Invariant Differential Operators for Quantum Symmetric Spaces, II

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Abstract

The two papers in this series analyze quantum invariant differential operators for quantum symmetric spaces in the maximally split case. In this paper, we complete the proof of a quantum version of Harish-Chandra’s theorem: There is a Harish-Chandra map which induces an isomorphism between the ring of quantum invariant differential operators and a ring of Laurent polynomial invariants with respect to the dotted action of the restricted Weyl group. We find a particularly nice basis for the quantum invariant differential operators that provides a new interpretation of difference operators associated to Macdonald polynomials. Finally, we set the stage for a general quantum counterpart to noncompact zonal spherical functions.

Introduction

Harmonic analysis on symmetric spaces studies invariant differential operators and their joint eigenspaces in connection with Lie groups. The discovery

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of quantum groups in the 1980’s inspired the growing subject of harmonic
analysis on quantum symmetric spaces. In particular, zonal spherical func-
tions on most quantum symmetric spaces have been identified with Mac-
donald or Macdonald-Koornwinder polynomials ([N], [NS], [S], [L3], [DN],
[NDS], and [DS]). In this paper, the focus is on quantum invariant differen-
tial operators. We complete the proof begun in [L4] of the quantum version
of Harish-Chandra’s fundamental result identifying invariant differential op-
erators with restricted Weyl group invariants. Moreover, we exhibit a nice
basis for the quantum invariant differential operators which correspond to
a special commuting family of difference operators associated to Macdonald
polynomials.

Let \( \mathfrak{g} \) be a semisimple Lie algebra and let \( \check{U} \) denote the simply connected
quantized enveloping algebra of \( \mathfrak{g} \) over the algebraic closure \( \overline{\mathbb{C}} \) of \( \mathbb{C}(q) \) ([Jo,
Section 3.2.10]). Quantum symmetric pairs are defined using a left coideal
subalgebra \( B \) which can be viewed as a quantum analog of the enveloping
algebra for the subalgebra of \( \mathfrak{g} \) fixed by an involution \( \theta \) ([L1, Section 7]). Our
investigation of quantum invariant differential operators focuses on the ring
\( \check{U}^B \) of \( B \) invariant elements of \( \check{U} \) with respect to the right adjoint action.
(Ultimately, it is more accurate to identify the ring of quantum invariant
differential operators with a quotient of the ring \( \check{U}^B \).)

Let \( \Sigma \) denote the restricted root system associated to \( \mathfrak{g} \) and \( \theta \). The Cartan
subalgebra of \( \check{U} \) is a group algebra of a multiplicative group \( \check{T} \) isomorphic
to the weight lattice of the root system of \( \mathfrak{g} \). Let \( \mathcal{A} \) be the subgroup of \( \check{T} \)
which is the image of the weight lattice of \( 2\Sigma \) under this isomorphism. The quantum Harish-Chandra projection map \( \mathcal{P}_B \) is a projection of \( \check{U} \) onto a slight enlargement of \( \mathcal{C}[\mathcal{A}] \) defined using a quantum Iwasawa decomposition
([L4]). It can be viewed as the function which computes the eigenvalues
of zonal spherical functions with respect to the action of elements in \( \check{U}^B \).
This paper is the second of two which analyze the image of \( \check{U}^B \) under the
Harish-Chandra map \( \mathcal{P}_B \).

The action of the restricted Weyl group \( W_\Theta \) on \( \Sigma \) induces a dotted action
of \( W_\Theta \) on the group algebra \( \mathcal{C}[\mathcal{A}] \). Building on work in [L4], we complete the
proof of a quantum version of Harish-Chandra’s theorem [HC, Section 4].

**Theorem A: (Corollary 4.2)** The Harish-Chandra map \( \mathcal{P}_B \) induces a sur-
jection from \( \check{U}^B \) onto the subring of invariants of \( \mathcal{C}[\mathcal{A}] \) with respect to the
dotted action of \( W_\Theta \).
Let $F_r(U)$ denote the locally finite part of $U$ with respect to the right adjoint action. We construct a linear map $L$ from $F_r(U)$ to $U^B$ which is morally the inverse of the Harish-Chandra map $P_B$. This map can be viewed as a generalization of the map used to analyze the center in [JL] (see also [Jo, Chapter 7]. Write $\tau$ for the isomorphism between $A$ and the weight lattice of $2\Sigma$ and let $P^+(2\Sigma)$ denote the semigroup of dominant integral restricted weights. We show that Theorem A is a consequence of fine information concerning $P_B(U^B)$ obtained in the following proposition.

**Theorem B:** (Theorem 4.1) For each $\lambda \in P^+(2\Sigma)$, there exists $t_\lambda$ in the intersection of $\tau(\lambda) + \text{Ker} \ P_B$ and $F_r(U)$ such that the top degree term of $P_B(L(t_\lambda))$ is a nonzero scalar multiple of $\tau(\lambda)$. Moreover, $\{P_B(L(t_\lambda)) | \lambda \in P^+(2\Sigma)\}$ forms a basis for $P_B(U^B)$.

Recall that a symmetric pair of Lie algebras can be written as a direct sum of irreducible symmetric pairs. Theorem A is established in [L4] for all symmetric pairs except those which contain a component of type EIII, EIV, EVII, or EIX. Thus the original motivation for this paper was to show that Theorem A holds for the remaining four exceptional types. However, Theorem B, which appears quite technical, has significance which goes beyond the particulars of this paper. We will ultimately use Theorem B to develop the theory of noncompact zonal spherical functions. Recall that compact quantum zonal spherical functions are elements of the quantized function algebra of the compact Lie group $G$ associated to $g$. Unfortunately, there is not yet a good definition of the quantized function algebra associated to a noncompact semisimple Lie group. As a result, the only noncompact zonal spherical functions that have been analyzed so far are those on the simplest noncompact symmetric space associated to $\mathfrak{sl}_2$ ([KS]). In a future paper, Theorem B will be the foundation of a general algebraic definition of zonal spherical functions; an outline follows. Theorem B implies that that $U^B \oplus \text{Ker} P_B = C\tau(P^+(2\Sigma)) \oplus \text{Ker} P_B$.

One shows that $F_r(U)$ is a subset of this direct sum of vector spaces. Given an algebra homomorphism $\Lambda$ from $C[A]$ to $C$, define a linear function $g_\Lambda$ on $U^B$ by $g_\Lambda(a) = \Lambda(P_B(a))$ for $a \in U$. Then $g_\Lambda$ extends to a function on the direct sum (and hence on $F_r(U)$) where $g_\Lambda(\text{Ker} \ P_B) = 0$. Better yet, $g_\Lambda$ extends to a function on $U$ by virtue of the basic local finiteness theorem ([JL, Theorem 6.4]) and an assumption that its restriction to $U^0$ is (dotted)
invariant with respect to $W_\Theta$. When $\mu$ in $P^+(2\Sigma)$, it turns out that $g_{\eta^\mu}$ is a compact quantum zonal spherical function at $\mu$ associated to $B$. Noncompact quantum zonal spherical functions correspond to choices of $\Lambda$ that are not of the form $q^\mu$ for $\mu \in P^+(2\Sigma)$.

The reader may wish to focus instead on completing the identification of compact quantum zonal spherical functions with orthogonal polynomials. Indeed, compact quantum zonal spherical functions associated to standard quantum symmetric pairs with reduced restricted root systems are identified with Macdonald polynomials in [L3] by computing radial components components of “small” elements in $\check{U}^B$. For most irreducible symmetric pairs, these small elements are actually contained in the center $Z(\check{U})$. However, this is not true for the four problematic exceptional types EIII, EIV, EVII, and EIX. In [L3], an elementwise computation was used to find the necessary elements of $\check{U}^B$ for the last three of these types. The arguments of this paper provide a simpler proof of their existence. Moreover, Theorem A guarantees the existence of $B$ invariant elements necessary to make the methods of [L3] extend to the remaining problematic type, EIII.

Let $\mathcal{X}$ be the radial component function on $\check{U}^B$. As a consequence of Theorem B and properties of $\mathcal{L}$, the set $\{\mathcal{X}(\mathcal{L}(t_\lambda))|\lambda \in P^+(2\Sigma)\}$ is a special commuting family of difference operators associated to Macdonald and related orthogonal polynomials. (See the discussion following Corollary 4.2 and [K, Theorem 6.6]). Thus, Theorem B leads to a natural quantum setting for these operators.

We briefly describe the contents of the rest of this paper. Section 1 sets notation and includes the definitions of the Harish-Chandra map $P_B$ as well as the map $\mathcal{X}$ which computes radial components. Results from [L4] are recalled in Theorem 1.3. In Section 2, we establish a criterion for $P_B(\check{U}^B)$ to be invariant under the dotted action of $W_\Theta$. This involves a careful filtration analysis of $P_B(Z(\check{U}))$ and its relation to $P_B(\check{U}^B)$.

The map $\mathcal{L}$ is introduced and studied in Section 3. A formula relating the value of a zonal spherical function at an element $a$ in $F_r(\check{U})$ and $P_B(\mathcal{L}(a))$ is presented in Lemma 3.5. A study of the possible highest degree term of $\mathcal{L}(a)$ for any $a$ in the set $F_r(\check{U}) \cap (\tau(\lambda) + \text{Ker } P_B)$ (Lemma 3.7) shows that Theorem A is a consequence of Theorem B (Theorem 3.8).

Theorem B is proved in Section 4. Using results of earlier sections, the proof is reduced to finding a set $\{t_\lambda|\lambda \in P^+(2\Sigma)\}$ such that each $t_\lambda$ lies in the intersection of $\tau(\lambda) + \text{Ker } P_B$ and $F_r(\check{U})$. This is immediate for some sym-
metric (see Section 4, Case (i)). For other symmetric pairs, the proof has two flavors. We use an induction argument exploiting fine information about the image of central elements under $P_B$ developed in [L3] for irreducible symmetric pairs containing a subpair of type AII (see Section 4, Case (ii)). The proof for the remaining four types is inspired by the “small element” arguments in [L3, Section 7]. Recall that $F_r(U)$ is a direct sum of modules $(ad_r U)t$, where $t$ is a group-like element of $U$ and $(ad_r U)t$ contains a unique (up to scalar) nonzero central element. Theorem B is established for irreducible symmetric pairs of type EIII, EIV, EVII, and EIX, by showing that the central element of $(ad_r U)t$ is not a scalar multiple of the $B$ invariant element $L(t)$ for suitable choices of $t$ (see Section 4, Cases (iii) and (iv)). Rather than dealing with element manipulation, we reduce the proof to computations in the commutative polynomial ring $C[A]^W$ which resemble character formula arguments.

1 Background and Notation

Let $C$ denote the complex numbers, $Q$ denote the rational numbers, $R$ denote the real numbers, and $q$ an indeterminate. Write $C$ for the algebraic closure of $C(q)$ and let $R$ be the real algebraic closure of $R(q)$.

Suppose that $\Phi$ is a root system with $\Phi^+$ equal to the subset of positive roots. Write $Q(\Phi)$ for the root lattice and $P(\Phi)$ for the weight lattice associated to $\Phi$. Set $Q^+(\Phi)$ equal to the span of the positive roots $\Phi^+$ over the nonnegative integers. Let $P^+(\Phi)$ denote the set of dominant integral weights associated to $\Phi$. If a symbol for the set of (positive) simple roots associated to $\Phi$ is specified, then we will often replace $\Phi$ with this symbol in the notation for the root and weight lattices and their subsets.

Let $g$ be a complex semisimple Lie algebra. Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $g$ and let $g = n^- \oplus \mathfrak{h} \oplus n^+$ be a fixed triangular decomposition. Let $\Delta$ denote the root system of $g$ and write $\pi = \{ \alpha_1, \ldots, \alpha_n \}$ for the set of positive simple roots. Let $( , )$ denote the Cartan inner product on $\mathfrak{h}^*$ associated to $\Delta$. Recall the standard partial order on $\mathfrak{h}^*: \lambda \geq \gamma$ if and only if $\lambda - \gamma \in Q^+(\pi)$.

Let $\theta$ be a maximally split involution with respect to the above triangular decomposition of $g$ (see [L1, (7.1), (7.2), and (7.3)]). Write $g^\theta$ for the corresponding fixed Lie subalgebra of $g$. We assume throughout the paper


that \( \mathfrak{g}, \mathfrak{g}^\theta \) is an irreducible symmetric pair in the sense of [A] (see also [L2, Section 7]). It should be noted that the results of this paper extend in a straightforward manner to general symmetric pairs \( \mathfrak{g}, \mathfrak{g}^\theta \).

The involution \( \theta \) induces an involution \( \Theta \) on \( \mathfrak{h}^\ast \) which restricts to an involution on \( \Delta \). Set \( \pi_\Theta = \{ \alpha_i \mid \Theta(\alpha_i) = \alpha_i \} \). Given \( \alpha \in \mathfrak{h}^\ast \), set \( \tilde{\alpha} = (\alpha - \Theta(\alpha))/2 \). The restricted root system \( \Sigma \) is the set

\[
\Sigma = \{ \tilde{\alpha} \mid \alpha \in \Delta \text{ and } \Theta(\alpha) \neq \alpha \}.
\]

Let \( W_\Theta \) denote the Weyl group associated to the root system \( \Sigma \).

Let \( \tilde{U} = U_q(\mathfrak{g}) \) denote the quantized enveloping algebra of \( \mathfrak{g} \) which is generated as an algebra over \( \mathbb{C} \) by \( x_i, y_i, t_i^\pm1 \) for \( 1 \leq i \leq n \). (See [L1, Section 1, (1.4)-(1.10)] or [Jo, 3.2.9] for the relations and Hopf algebra structure.) Let \( T \) denote the group generated by \( t_i \) for \( 1 \leq i \leq n \) and let \( U^0 \) denote the subalgebra of \( U \) generated by \( T \). Let \( U^+ \) denote the subalgebra of \( U \) generated by \( x_i, 1 \leq i \leq n \), and let \( G^- \) denote the subalgebra of \( U \) generated by \( y_i t_i, 1 \leq i \leq n \). Recall that there is a group isomorphism \( \tau \) from the additive group \( \mathbb{Q}(\pi) \) to the multiplicative group \( T \) defined by \( \tau(\alpha_i) = t_i \) for \( 1 \leq i \leq n \).

Sometimes it will be necessary to use larger Hopf algebras than \( U \) which are formed by enlarging \( T \). In particular, suppose that \( M \) is a multiplicative monoid isomorphic to an additive submonoid of \( \sum_{1 \leq i \leq n} \mathbb{Q} \alpha_i \) via the obvious extension of \( \tau \). Then \( UM \) is the Hopf algebra generated as an algebra by \( U \) and \( M \) as explained in [L4, Section 1]. The most common extension of \( U \) by this method is the simply connected quantized enveloping algebra, denoted by \( \tilde{U} \). As an algebra, \( \tilde{U} \) is generated by \( U \) and \( \tilde{T} \) where \( P(\pi) \) is isomorphic to \( \tilde{T} \) via \( \tau \) ([Jo, Section 3.2.10]). Set \( \tilde{U}^0 \) equal to the group algebra of \( \tilde{T} \) over \( \mathbb{C} \). Given a subalgebra \( A \) of \( \tilde{U} \), we write \( A_+ \) for the intersection of \( A \) with the augmentation ideal of \( \tilde{U} \).

We associate to \( \mathfrak{g}, \mathfrak{g}^\theta \) a set \( \mathcal{B} \) of left coideal subalgebras of \( U \). More precisely, \( \mathcal{B} \) equals the orbit of real analogs of \( U(\mathfrak{g}^\theta) \) in \( U_q(\mathfrak{g}) \) under the Hopf algebra automorphisms of \( U \) fixing elements of \( T \). The reader is referred to [L4, Section 1] for more details. Notation concerning \( \mathcal{B} \) will be defined as needed.

Given a multiplicative group \( G \), we write \( \mathcal{C}[G] \) for the group algebra generated by \( G \) over \( \mathcal{C} \). For any additive group \( H \), write \( \mathcal{C}[H] \) for the group ring generated by \( z^\lambda \) for \( \lambda \in H \). The first notation will be applied to groups
related to $\tilde{T}$ while the second notation will be used for groups related to $P(\pi)$.

Let $\text{ad}_r$ denote the right adjoint action of $U$ on $\tilde{U}$ and let $\text{ad}$ denote the left adjoint action of $U$ on $\tilde{U}$. Given $B$ in $\mathcal{B}$, write $\tilde{U}^B$ for the subset of $B$ invariants in $\tilde{U}$ with respect to the right adjoint action. There are two important maps which are used to study $\tilde{U}^B$. The first is the quantum Harish-Chandra projection map and the second is the function which computes the quantum radial components. We review the definition of both of these maps here.

Set $\tilde{\mathcal{A}} = \{\tau(\tilde{\mu})|\mu \in P(\pi)\}$. Let $\tilde{T}_\Theta = \{\tau((\mu + \Theta(\mu))/2)|\mu \in P(\pi)\}$ and let $\mathcal{M}$ denote the subalgebra of $U$ generated by $x_i, y_i, t_i^{\pm 1}$ for $\alpha_i \in \pi_\Theta$. Let $N^+$ be the subalgebra of $U^+$ generated by the elements $(ad x) x_j$ with $\alpha_j \in \pi \setminus \pi_\Theta$ and $x \in \mathcal{M} \cap U^+$. As in [L4, (2.4)], we have the following inclusion:

$$\tilde{U} \subseteq ((\tilde{B}\tilde{T}_\Theta) + \tilde{U} + N^+_+ \tilde{A}) \oplus \mathcal{C}[\tilde{A}].$$

(1.1)

**Definition 1.1** The (quantum) Harish-Chandra map with respect to the symmetric pair $g, g^\Theta$ and subalgebra $B$ in $\mathcal{B}$ is the projection $\mathcal{P}_B$ of $\tilde{U}$ onto $\mathcal{C}[\tilde{A}]$ using the direct sum decomposition (1.1).

Set $\mathcal{A}$ equal to the group consisting of the elements $\tau(2\tilde{\mu})$ for $\mu \in P(\Sigma)$. Write $\mathcal{C}[Q(\Sigma)] \mathcal{A}$ for the subring of $\text{End}_r \mathcal{C}[P(\pi)]$ generated by $\mathcal{C}[Q(\Sigma)]$ and $\mathcal{A}$ as explained in [L4, Section 4]. Given $f \in \mathcal{C}[Q(\Sigma)] \mathcal{A}$ and $a \in \mathcal{C}[Q(\Sigma)]$, we write $a * f$ for the right action of $f$ on $a$ ([L4, Section 4]). Let $\mathcal{C}[Q(\Sigma)] \mathcal{A}$ denote the localization of $\mathcal{C}[Q(\Sigma)] \mathcal{A}$ at the Ore set $\mathcal{C}[Q(\Sigma)] \setminus \{0\}$.

Let $\{\varphi_\lambda|\lambda \in P^+(2\Sigma)\}$ be a $W_\Theta$ invariant zonal spherical family associated to $g, g^\Theta$ (See [L2], [L3], and [L4, Section 4]). Recall that $\varphi_\lambda \in \mathcal{C}[P(2\Sigma)]$ for each $\lambda$. Moreover, elements of of the ring $\mathcal{C}[P(\pi)]$ (which contains $\mathcal{C}[P(2\Sigma)]$) can be thought of as functions on $U^0$ where

$$z^\lambda(\tau(\mu)) = q^{(\lambda, \mu)}$$

for all $\lambda \in P(\pi)$ and $\tau(\mu) \in T$.

**Definition 1.2** ([L4, Theorem 4.1]) Fix $B \in \mathcal{B}$. The radial component map $\mathcal{X}$ is a function $\mathcal{X} : \tilde{U}^B \mapsto \mathcal{C}[Q(\Sigma)] \mathcal{A}$ such that

$$(\varphi_\lambda * \mathcal{X}(u))(t) = z^\lambda(\mathcal{P}_B(u)) \varphi_\lambda(t)$$

(1.2)
for all \( \lambda \in P^+(2\Sigma) \), \( t \in \mathcal{A} \), and \( u \in \check{U}^B \).

Note that (1.2) ensures that \( \varphi_{\lambda} \) is an eigenvector for the action of \( \mathcal{X}(u) \) with eigenvalue equal to \( z^\lambda(\mathcal{P}_B(u)) \).

We have the following results from [L4] concerning the images of \( \check{U}^B \) under these two maps. (See [L4, Theorem 2.4, Theorem 2.6 and Corollary 4.2].)

**Theorem 1.3** Let \( B \in \mathcal{B} \). The map \( \mathcal{P}_B \) restricts to an algebra homomorphism from \( \check{U}^B \) into \( C[A] \) with kernel equal to \( (B\mathcal{T}_\Theta)_+\check{U} \cap \check{U}^B \). Moreover there is an algebra isomorphism from \( \mathcal{X}(\check{U}^B) \) onto \( \mathcal{P}_B(\check{U}^B) \) given by \( \mathcal{X}(u) \mapsto \mathcal{P}_B(u) \) for all \( u \in \check{U}^B \).

Unless state otherwise, we assume that \( B \) is an arbitrary fixed subalgebra of \( \mathcal{B} \). At the end of the paper, we include an appendix with definitions of symbols introduced later in the paper. For more information about notation and for undefined notions, the reader is referred to [Jo], [L3], and [L4].

## 2 Weyl group invariance

In [L4], \( \mathcal{P}_B(\check{U}^B) \) is shown to be invariant under a dotted action of the restricted Weyl group \( W_\Theta \) whenever \( g, g^\theta \) is not of type EIII, EIV, EVII, or EIX. The purpose of this section is to provide criteria for the extension of this result to the remaining four cases. Recall that the center \( Z(\check{U}) \) of \( \check{U} \) is a subalgebra of \( \check{U}^B \) ([L3, Lemma 3.5]). Furthermore, it is shown in [L4, Lemma 5.3] that \( \mathcal{P}_B(Z(\check{U})) \) is \( W_\Theta \) invariant. Exploiting the fact that \( \mathcal{P}_B(Z(\check{U})) \) is “large” inside of \( \mathcal{P}_B(\check{U}^B) \), we find necessary conditions in Theorem 2.5 which guarantee the \( W_\Theta \) invariance of \( \mathcal{P}_B(\check{U}^B) \). Then, in Section 4, we show that \( \mathcal{P}_B(\check{U}^B) \) satisfies these conditions.

We recall some notation from [L4]. Let \( \rho \) denote the half sum of the positive roots in \( \Delta \). The dotted action of \( W_\Theta \) on \( C[A] \) is defined by

\[
w.\varphi^{(\rho,\lambda)}(\lambda) = \varphi^{(\rho,w\lambda)}(w\lambda)\]

for all \( w \in W_\Theta \) and \( \lambda \in P(\Sigma) \). Given \( \eta \in P^+(\Sigma) \), set

\[
\check{m}(2\eta) = \sum_{\gamma \in W_\Theta \eta} \varphi^{(\rho,2\gamma)}(2\gamma).
\]
Note that the set \( \{ \hat{m}(2\eta) | \eta \in P^+(\Sigma) \} \) is a basis for \( \mathcal{C}[\mathcal{A}]^{W_{\Theta}} \).

One of the important tools used to analyze the radial components in [L3] and [L4] is a degree function and related filtration (see [L4, (4.1), (4.2), and (4.3)]). We briefly recall how this degree function behaves on \( \mathcal{C}[\mathcal{A}] \). Suppose that \( \tau(\mu) \in \mathcal{A} \). We can write \( \mu = \sum m_i \tilde{\alpha}_i \) where the \( m_i \) are rational numbers. Let \( \text{deg} \tau(\mu) = -\sum m_i \).

Set \( \text{deg} \tau(\mu) \). Let \( \text{tip} \) be the function on \( \mathcal{C}[\mathcal{A}] \) which computes the highest degree homogeneous term. Thus

\[
\text{tip}(g) = \sum \{ \gamma | \text{deg}(\tau(\gamma)) = \text{deg} g \} \]

for all Laurent polynomials \( g = \sum \gamma a_\gamma \tau(\gamma) \) in \( \mathcal{C}[\mathcal{A}] \). By [L4, Lemma 4.3],

\[
\{ \text{tip}(X) | X \in \mathcal{P}_B(\mathcal{U}^B) \} \subseteq \text{span}\{ \tau(-2\eta) | \eta \in P^+(\Sigma) \}. \tag{2.1}
\]

Setting \( \text{deg} f = 0 \) for all \( f \in \mathcal{C}(Q(\Sigma)) \) allows us to extend the degree function on \( \mathcal{C}[\mathcal{A}] \) to a function on \( \mathcal{C}(Q(\Sigma)) \mathcal{A} \). It follows from [L4, (4.4)] that

\[
\mathcal{X}(a) = \text{tip}(\mathcal{P}_B(a)) f + \text{lower degree terms} \tag{2.2}
\]

for some \( f \in \mathcal{C}(Q(\Sigma)) \).

Let \( \mathcal{C}[[Q(\Sigma)]] \) denote the power series ring consisting of possibly infinite sums of the form \( \sum_{\gamma \in Q^+(\Sigma)} a_\gamma z^{-\gamma} \) for \( a_\gamma \in \mathcal{C} \). The next lemma provides another connection between the radial components and the Harish-Chandra projection of \( B \) invariant elements.

**Lemma 2.1** For all \( u \in \mathcal{U}^B \),

\[
\mathcal{X}(u) = \mathcal{P}_B(u) + \sum_{\alpha_i \in \pi \setminus \pi_{\Theta}} z^{-\tilde{\alpha}_i} \mathcal{C}[[Q(\Sigma)]] \mathcal{A}.
\]

**Proof:** Suppose \( u \in \mathcal{U}^B \). As in the proof of [L3, Theorem 3.4], we can write

\[
\mathcal{X}(u) = \sum_{i=1}^m \sum_{\gamma \leq \beta_i} a_{\gamma} z^{\gamma}
\]

where \( \gamma \) and the \( \beta_i \), for \( 1 \leq i \leq m \), are elements of \( Q(\Sigma) \) and the \( a_\gamma \) are Laurent polynomials in \( \mathcal{C}[\mathcal{A}] \). Moreover, we may assume that the \( \beta_i \), \( 1 \leq i \leq m \), are not comparable via the standard partial ordering on \( \mathfrak{h}^* \).
Since \( \{a_{\beta_i} \mid 1 \leq i \leq m \} \) is a finite set of Laurent polynomials in \( C[A] \), it follows that there exists \( \lambda \in P^+(2\Sigma) \) such that \( z^\lambda(a_{\beta_i}) \neq 0 \) for each \( i \). By [L2, Lemma 4.1] the zonal spherical function \( \varphi_\lambda \) is contained in \( z^\lambda + \sum_{\gamma<\lambda} Cz^\gamma \).

Thus

\[
\varphi_\lambda \ast \mathcal{X}(u) = \varphi_\lambda \ast \sum_{i=1}^m \sum_{\gamma \leq \beta_i} a_\gamma z^\gamma \\
\in \sum_{i=1}^m (z^\lambda(a_{\beta_i}))z^{\lambda+\beta_i} + \sum_{\gamma<\beta_i} Cz^\gamma.
\]

The fact that \( \varphi_\lambda \) is an eigenvector for the action of \( \mathcal{X}(u) \) forces \( m = 1 \), \( \beta_1 = 0 \), and \( z^{\beta_1} = 1 \). The lemma now follows from the fact that for each \( \lambda \in P^+(2\Sigma) \), the eigenvalue of \( \varphi_\lambda \) is \( z^\lambda(P_B(u)) \) (see Definition 1.2).

Given \( \eta \in P^+(\Sigma) \), set

\[
N_\eta = \sum_{\gamma' \leq \eta} \sum_{\beta \in W_{\gamma'}} C\tau(-2\beta) \quad \text{and} \quad N_\eta^+ = \sum_{\gamma' < \eta} \sum_{\beta \in W_{\gamma'}} C\tau(-2\beta).
\]

Consider an element \( a \in \tilde{U}^B \) such that \( \text{tip}(a) = \tau(-2\eta) \) for some \( \eta \in P^+(\Sigma) \).

By [L4, Theorem 4.1], \( \mathcal{X}(a) \) is \( W_\theta \) invariant. Hence Lemma 2.1 implies that \( P_B(a) \in N_\eta \).

Let \( w_0' \) denote the longest element of the Weyl group \( W_\theta \). To make the notation of the rest of this section easier to read, we assume that \( w_0' = -1 \). It should be noted that this is true for the four types of irreducible symmetric pairs which are the primary interest of this section. Moreover, the results are easily extended to the case when \( w_0' \) is not equal to \(-1\).

Suppose that \( \mu \in P^+(\pi) \). Recall that there exists a unique central element \( z_{2\mu} \) in \( \tau(2\mu) + (\text{ad}_r U_+) \tau(2\mu) \) (see [L4, Section 5]). By [L4, Lemma 5.1] and subsequent discussion, we have

\[
P_B(z_{2\mu}) \in \hat{m}(2\bar{\mu}) + N_{\bar{\mu}}
\]

up to a nonzero scalar. Note that \( \text{tip}(z_{2\mu}) = \hat{m}(2\bar{\mu}) = q^{(\rho,-2\bar{\mu})}\tau(-2\bar{\mu}) \).

Moreover, by [L4, Lemma 5.3], \( P_B(z_{2\mu}) \) is invariant under the dotted action of \( W_\theta \).

**Lemma 2.2** If \( a \in \tilde{U}^B \) such that \( \text{tip}(P_B(a)) = q^{(\rho,-2\eta)}\tau(-2\eta) \) then

\[
P_B(a) = \hat{m}(2\eta)) + N_{\eta}^+.
\]
Moreover, if \( \text{tip}(\mathcal{P}_B(a)) = q^{(\rho, -2\eta)}\tau(-2\eta) \) where \( \eta \) is a minuscule or pseudominuscule restricted fundamental weight, then \( \mathcal{P}_B(a) \in \mathcal{C}[\mathcal{A}]^{W_\Theta} \).

**Proof:** Assume that \( u \in \hat{U}^B \) so that \( \text{tip}(u) = q^{(\rho, -2\eta)}\tau(-2\eta) \). By the discussion preceding the lemma, \( \mathcal{P}_B(u) \in \mathcal{N}_\eta \). Thus, there exist scalars \( a_\beta \) such that

\[
\mathcal{P}_B(u) \in \sum_{\beta \in W_\Theta \eta} a_\beta \tau(-2\beta) + \mathcal{N}_\eta^+.
\]

(2.5)

The fact that \( \text{tip}(u) = q^{(\rho, -2\eta)}\tau(-2\eta) \) ensures that \( a_\eta \) is nonzero. Assume that \( \sum_{\beta \in W_\Theta \eta} a_\beta \tau(-2\beta) \) is not \( W_\Theta \) invariant.

By [L4, Theorem 3.7], there exists \( \gamma \) and \( \lambda \) in \( P^+(\pi) \) such that \( \lambda = \bar{\lambda} = \bar{\gamma} + \eta \). Since \( \mathcal{P}_B \) is an algebra map (Theorem 1.3), we have \( \mathcal{P}_B(uz_{2\gamma}) = \mathcal{P}_B(u)\mathcal{P}_B(z_{2\gamma}) \). Hence, by (2.4) and (2.5) we can write

\[
\mathcal{P}_B(uz_{2\gamma}) = \sum_{\beta \in W_\Theta \lambda} a_\beta \tau(-2\beta) + \mathcal{N}_\lambda^+.
\]

(2.6)

Furthermore, our assumptions on the \( a_\beta \) ensure that \( \sum_{\beta \in W_\Theta \lambda} a_\beta \tau(-2\beta) \) is not \( W_\Theta \) invariant.

Set \( X = \mathcal{P}_B(uz_{2\gamma}) - a_{\bar{\lambda}}' q^{(\rho, 2\bar{\lambda})} \mathcal{P}_B(z_{2\lambda}) \). By (2.4) and (2.6), we have

\[
X \in \sum_{\beta \in W_\Theta \lambda, \beta \neq \bar{\lambda}} b_\beta \tau(-2\beta) + \mathcal{N}_{\bar{\lambda}}^+
\]

(2.7)

for some set of scalars \( b_\beta \) which are not all equal to zero. By [L4, Lemma 4.3], \( \text{tip}(X) \) is a linear combination of elements in the set \{\( \tau(-2\gamma) | \gamma \in P^+(\Sigma) \)\}. Since \( \lambda \in P^+(\pi) \), it follows from [L4, Lemma 3.3] that \( \bar{\lambda} \in P^+(\Sigma) \). Hence if \( \beta \in W_\Theta \lambda \setminus \{\bar{\lambda}\} \), then \( \beta \) is not an element of \( P^+(\Sigma) \). Thus \( \text{tip}(X) \) is contained in \( \mathcal{N}_{\bar{\lambda}}^+ \). It now follows from the \( W_\Theta \) invariance of \( \mathcal{X}(\hat{U}^B) \) ([L4, Theorem 4.1]), (2.2), Lemma 2.1, and (2.7) that \( \mathcal{P}_B(X) \in \mathcal{N}_{\bar{\lambda}}^+ \). However, \( \tau(-2\beta) \notin \mathcal{N}_{\bar{\lambda}}^+ \) for any \( \beta \) in \( W_\Theta \bar{\lambda} \). This contradiction proves the first assertion of the lemma.

Assume now that \( \eta \) is either a minuscule or pseudominuscule restricted fundamental weight (see [M] or [L3, Section 7]). It follows that the set \( \{\gamma \in P^+(\Sigma) | \gamma < \eta \} \) is a subset of \( \{0\} \). Thus \( \mathcal{N}_{\eta}^+ \) is a subset of \( \mathcal{C} \). Therefore \( \mathcal{P}_B(u) \) equals \( m(2\eta) + c \) for some scalar \( c \). \( \square \)

Now suppose that \( a \in \hat{U}^B \) and

\[
\text{tip}(\mathcal{P}_B(a)) = \sum_j a_j q^{(\rho, -2\gamma_j)}\tau(-2\gamma_j)
\]
with $\gamma_j \in P^+(\Sigma)$. It follows that each $\tau(\gamma_j)$ must have the same degree as $P_B(a)$. This implies that the $\gamma_j$ are pairwise incomparable with respect to the standard partial order on $P(\Sigma)$. The proof of Lemma 2.2 can be easily generalized to show that

$$P_B(a) = \sum_j a_j \hat{m}(2\gamma_j) + \sum_j N^+_{\gamma_j}.$$ 

It is sometimes useful to consider the “opposite” degree function on $C[A]$. In particular, set $odeg(\tau(\eta)) = m$ where $\eta = \sum_i m_i \tilde{\alpha}_i$ and $m = \sum_i m_i$. We define a function “top” on $C[A]$ in a manner analogous to the tip function. Given $g = \sum a_\gamma \tau(\gamma)$ in $C[A]$, set

$$\text{top}(g) = \sum_{\{\gamma | odeg(\tau(\gamma)) = odeg(g)\}} a_\gamma \tau(\gamma).$$

Recall that $tip(\hat{m}(2\eta)) = q^{(\rho,-2\eta)} \tau(-2\eta)$. On the other hand, $top(\hat{m}(2\eta)) = q^{(\rho,2\eta)} \tau(2\eta)$. The following description of the set of “tops” of the images of elements in $\check{U}^B$ under $P_B$ is analogous to the description of the set of “tips” given in [L4, Lemma 4.3].

**Lemma 2.3** The set $\{\text{top}(P_B(a) | a \in \check{U}^B)\}$ is a subset of $\text{span}\{\tau(2\gamma) | \gamma \in P^+(\Sigma)\}$. Moreover, $\text{top}(P_B(a)) = w'_b(\text{tip}(P_B(a)))$ for all $a \in \check{U}^B$.

**Proof:** Suppose that $a \in \check{U}^B$. There exist elements $\gamma_1, \ldots, \gamma_s$ in $P^+(\Sigma)$ and scalars $a_1, \ldots, a_s$ such that

$$\text{tip}(P_B(a)) = \sum_{j=1}^s a_j q^{(\rho,-2\gamma_j)} \tau(-2\gamma_j).$$

By the discussion preceding the lemma,

$$P_B(a) = \sum_{j=1}^s a_j \hat{m}(2\gamma_j) + \sum_{j=1}^s N^+_{\gamma_j}.$$ 

Suppose that $\deg P_B(a) = m$. It follows that $\deg(\tau(-2\gamma_j)) = m$ for each $j$. Hence $odeg(\tau(2\gamma_j)) = m$ for all $1 \leq j \leq s$. Note that $\gamma' < \gamma$ implies that $odeg(\tau(\gamma')) < odeg(\tau(\gamma))$ for all $\gamma$ and $\gamma'$ in $P(\Sigma)$. Thus $odeg \ m(2\gamma_j) > odeg \ c$
for all $c \in \mathcal{N}_{\gamma_j}^+$ and all $j$. Hence $m = \text{odeg}(\mathcal{P}_B(a)) = \text{deg}(\mathcal{P}_B(a))$. Since 
\[ \text{top}(\hat{m}(2\eta)) = q^{(\rho,2\eta)}\tau(2\eta) \] for each $\eta \in P^+(\Sigma)$, it follows that 
\[ \text{top}(\mathcal{P}_B(a)) = \sum_j a_j q^{(\rho,2\gamma_j)}\tau(2\gamma_j) \]
\[ = w'_0, (\sum_j a_j (q^{(\rho,-2\gamma_j)}\tau(-2\gamma_j)) \].

By Lemma 2.3 and (2.1) we have that 
\[ \{\text{top}(X) | X \in \mathcal{P}_B(\hat{U}^B)\} \subseteq \text{span}\{\tau(2\eta) | \eta \in P^+(\Sigma)\}. \quad (2.8) \]
In the Section 4, we show that the above inclusion is actually an equality. First, we show that equality in (2.8) is a sufficient condition for $\mathcal{P}_B(\hat{U}^B)$ to be invariant under the dotted $W_\Theta$ action. The previous lemma is all that is necessary for the three types of irreducible symmetric pairs: EI, II, and EVII. However, the situation is more delicate when $g, g^\theta$ is of EIX. The following technical result handles this one special case.

**Lemma 2.4** Assume that $g, g^\theta$ is of type EIX. Assume further that 
\[ \{\text{top}(X) | X \in \mathcal{P}_B(\hat{U}^B)\} = \text{span}\{\tau(2\eta) | \eta \in P^+(\Sigma)\}. \]
Then $\mathcal{P}_B(\hat{U}^B) = \mathcal{C}[A]^{W_\Theta}$.

**Proof:** Checking the list of irreducible symmetric pairs in [A], we see that $g$ is of type E8 and $\Sigma$ is of type F4. Note that $\mathcal{N}_{\gamma} = \sum_{\gamma \leq \eta} \sum_{\beta \in W_{\gamma}} C\tau(2\beta)$ and $\mathcal{N}_{\eta}^+ = \sum_{\gamma \leq \eta} \sum_{\beta \in W_{\gamma}} C\tau(2\beta)$ for all $\eta \in P^+(\Sigma)$. Write $\omega_1, \ldots, \omega_8$ for the fundamental weights for the root system of $g$. Write $\eta_1, \eta_2, \eta_3, \eta_4$ for the fundamental weights of F4. Here we are assuming that both sequences are ordered using the ordering in [H, Chapter III].

Since $\mathcal{P}_B$ is an algebra homomorphism ([L4, Theorem 2.4]), we have 
\[ \text{top}(\mathcal{P}_B(ab)) = \text{top}(\mathcal{P}_B(a))\text{top}(\mathcal{P}_B(b)) \]
for all $a$ and $b$ in $\hat{U}^B$. By the assumptions of the lemma, there exist $u_1, u_2, u_3$, and $u_4$ in $\hat{U}^B$ such that $\text{top}(\mathcal{P}_B(u_i)) = \tau(2\eta_i)$ for $1 \leq i \leq 4$. It follows that $\mathcal{P}_B(u_1), \mathcal{P}_B(u_2), \mathcal{P}_B(u_3),$ and $\mathcal{P}_B(u_4)$ generates $\mathcal{P}_B(\hat{U}^B)$. Thus it is sufficient to show that $\mathcal{P}_B(u_i) \in \mathcal{C}[P(\Sigma)]^{W_\Theta}$ for $1 \leq i \leq 4$. 

13
It is straightforward to check that \( \tilde{\omega}_1 = 2\eta_1, \tilde{\omega}_2 = \eta_1 + \eta_3, \tilde{\omega}_3 = \eta_1 + 2\eta_3, \tilde{\omega}_4 = 2\eta_4 + 2\eta_3, \tilde{\omega}_5 = 2\eta_4 + \eta_3, \tilde{\omega}_6 = 2\eta_3, \tilde{\omega}_7 = \eta_2, \) and \( \tilde{\omega}_8 = \eta_1. \) Set \( c_1 = 2z_{2\omega_8} \) and \( c_2 = z_{2\omega_7}. \) By (2.4), \( \text{top}(\mathcal{P}_B(c_i)) = q^{(\rho,2\eta_i)}(2\eta_i) \) for \( i = 1, 2 \) and by \([L_4, \text{Lemma 5.3}], \) both \( \mathcal{P}_B(c_1) \) and \( \mathcal{P}_B(c_2) \) are \( W_{\Theta} \) invariant. Note that \( \eta_4 \) is pseudominuscule. Hence, \( \text{Lemma 2.2} \) ensures that there exists \( c_4 \) in \( \tilde{U}^B \) such that \( \mathcal{P}_B(c_4) = \hat{m}(2\eta_4) \). In particular, \( \mathcal{P}_B(c_4) \) is \( W_{\Theta} \) invariant and \( \text{top}(\mathcal{P}_B(c_4)) = q^{(\rho,2\eta_4)}(2\eta_4). \)

Now choose \( c_3 \in \tilde{U}^B \) such that \( \text{top}(\mathcal{P}_B(c_3)) = q^{(\rho,2\eta_3)}(2\eta_3) \). Note that the only nonzero dominant integral weights less than \( \eta_3 \) are \( \eta_1 \) and \( \eta_4 \). Moreover, \( \eta_4 < \eta_1. \) It follows that \( N^{\tau}_{\eta_3} = N^{-}_{\eta_1}. \) Thus by \( \text{Lemma 2.2}, \) \( \mathcal{P}_B(c_3) \in \hat{m}(2\eta_3) + N_{\eta_1}. \) Furthermore, there exist scalars \( a_\beta, \) for \( \beta \in W_{\Theta} \eta_1, \) such that

\[
\mathcal{P}_B(c_3) \in \hat{m}(2\eta_3) + \sum_{\beta \in W_{\Theta} \eta_1} a_\beta \tau(2\beta) + N_{\eta_1}. \tag{2.9}
\]

Assume that \( \sum_{\beta \in W_{\Theta} \eta_1} a_\beta \tau(2\beta) \) is not \( W_{\Theta} \) invariant. Consider the element \( c_3c_4 \) of \( \tilde{U}^B. \) Note that \( \eta_2 \) is the only dominant integral weight less than \( \eta_3 + \eta_4 \) and greater than \( \eta_1 + \eta_4. \) Since \( \mathcal{P}_B \) is an algebra homomorphism, we have that \( \mathcal{P}_B(c_3c_4) = \mathcal{P}_B(c_3) \mathcal{P}_B(c_4). \) It follows that

\[
\mathcal{P}_B(c_3c_4) \in \text{span}\{\hat{m}(2\eta_3 + 2\eta_4), \hat{m}(2\eta_2)\} + \sum_{\beta \in W_{\Theta} \eta_1} b_\beta \tau(2\beta) + N^{+}_{\eta_3 + \eta_4}.
\]

Furthermore, the assumption on the \( a_\beta \) ensures that \( \sum_{\beta \in W_{\Theta} \eta_1} b_\beta \tau(2\beta) \) is not \( W_{\Theta} \) invariant. Now \( \tilde{\omega}_2 = \eta_3 + \eta_4 \) and \( \tilde{\omega}_7 = \eta_2. \) By (2.4) there exists \( X \in Z(\tilde{U}) \) such that

\[
\mathcal{P}_B(c_3c_4 + X) \in \sum_{\beta \in W_{\Theta} \eta_1} b'_\beta \tau(2\beta) + N^{+}_{\eta_3 + \eta_4}
\]

and \( \sum_{\beta \in W_{\Theta} \eta_1} b'_\beta \tau(2\beta) \) is not \( W_{\Theta} \) invariant. This contradicts \( \text{Lemma 2.2}. \) Hence

\[
\mathcal{P}_B(c_3) \in \hat{m}(2\eta_3) + a\hat{m}(2\eta_1) + N_{\eta_3}
\]

for some scalar \( a. \)

Adding a constant to \( c_3 \) if necessary, we may assume that

\[
\mathcal{P}_B(c_3) = \hat{m}(2\eta_3) + a\hat{m}(2\eta_1) + \sum_{\beta \in W_{\Theta} \eta_1} e_\beta \tau(2\beta)
\]

for some scalar \( e_\beta. \)
for some scalars $e_{\beta}$. Assume further that $\sum_{\beta \in W_{\Theta} \eta_1} e_{\beta} \tau(2\beta)$ is not $W_{\Theta}$ invariant. Examining the dominant integral weights of $F_4$ less than or equal to $\eta_3 + \eta_4$ yields

$$\mathcal{P}_B(c_3c_4) \in \mathcal{C}\tilde{m}(2\eta_3 + 2\eta_4) + \mathcal{C}\tilde{m}(2\eta_1 + 2\eta_4) + \mathcal{C}\tilde{m}(2\eta_2) + \sum_{\beta \in W_{\Theta} \eta_1} e'_{\beta} \tau(2\beta) + N_{\eta_3}.$$ 

Moreover, our assumptions on $c_4$ and $c_3$ ensure that $\sum_{\beta \in W_{\Theta} \eta_3} e'_{\beta} \tau(2\beta)$ is not $W_{\Theta}$ invariant. Note that $\text{top}(\mathcal{P}_B(c_3c_4))$ is a scalar multiple of $\tau(2\eta_1 + 2\eta_4)$ and $\mathcal{P}_B(c_1c_4)$ is $W_{\Theta}$ invariant. On the other hand, both $\eta_3 + \eta_4$ and $\eta_2$ are in $P^+(\pi)$. Hence using (2.4) we can find $X \in \tilde{U}^B$ and scalars $e''_{\beta}$ such that

$$\mathcal{P}_B(c_3c_4 + X) \in \sum_{\beta \in W_{\Theta} 2\eta_3} e''_{\beta} \tau(2\beta) + N_{\eta_3}$$

and $\sum_{\beta \in W_{\Theta} \eta_3} e''_{\beta} \tau(2\beta)$ is not $W_{\Theta}$ invariant. Once again this contradicts the previous lemma. It follows that $\mathcal{P}_B(c_3)$ is $W_{\Theta}$ invariant. \(\square\)

The next result provides the essential criterion which will be used to show that $\mathcal{P}_B(\tilde{U}^B)$ is $W_{\Theta}$ invariant.

**Theorem 2.5** Assume that $g, g^\theta$ is of type EIII, EIV, EVII, and EIX. Assume further that

$$\{\text{top}(X)|X \in \mathcal{P}_B(\tilde{U}^B)\} = \text{span}\{\tau(2\eta)|\eta \in P^+(\Sigma)\}.$$

Then $\mathcal{P}_B(\tilde{U}^B) = \mathcal{C}[A]^{W_{\Theta}}$.

**Proof:** Write $\eta_1, \ldots, \eta_t$ for the fundamental weights of the restricted root system $\Sigma$. Since $\mathcal{P}_B$ is an algebra homomorphism ([L4, Theorem 2.4]), it is sufficient to find $u_1, \ldots, u_t$ in $\tilde{U}^B$ such that $\text{top}(\mathcal{P}_B(u_i)) = \tau(2\eta_i)$ and $\mathcal{P}_B(u_i) \in \mathcal{C}[A]^{W_{\Theta}}$ for $1 \leq i \leq t$. This is exactly what is done for EIX in the previous lemma.

If $\eta_i \in P^+(\pi)$, then by (2.3) there exists $u_i \in Z(\tilde{U})$ such that $\text{top}(\mathcal{P}_B(u_i)) = \tau(2\eta_i)$. Moreover, $\mathcal{P}_B(u_i) \in \mathcal{C}[A]^{W_{\Theta}}$. Now suppose that $g, g^\theta$ is of type EIV or EVII. A straightforward calculation shows that the only fundamental restricted weights not contained in $P^+(\pi)$ are $\eta_1$ and $\eta_2$. Furthermore, both $\eta_1$ and $\eta_2$ are either minuscule or pseudominuscule. On the other hand suppose
that \( g, g^\theta \) is of type EIII. Then \( \Sigma \) is of type \( B_2 \) and the only fundamental restricted weight not contained in \( P^+(\pi) \) is \( \eta_1 \). It is straightforward to check that \( \eta_1 \) is a pseudominuscule weight. The theorem now follows from Lemma 2.2 and Lemma 2.3. \( \square \)

3 An inverse to the Harish-Chandra map

Recall that the ordinary Harish-Chandra map is a projection of \( \check{U} \) onto \( \check{U}^0 \). Furthermore, this Harish-Chandra map induces an isomorphism between \( Z(\check{U}) \) and a particular subring of \( \check{U}^0 \). There are a number of different proofs of this result in the literature (see [B]). The approach taken by [JL] (see also [Jo, Section 7]) establishes the surjectivity of this isomorphism by using a map which lifts certain elements of \( T \) to \( Z(\check{U}) \). In this section, we study an analog \( \mathcal{L} \) of this map associated to quantum symmetric pairs. Some of the material presented here is a generalization of parts of [L3, Section 7].

Let \( \phi \) be the Hopf algebra automorphism of \( U \) which fixes elements in \( T \) such that

\[
\phi(x_i) = q^{(-2\rho, \tilde{\alpha}_i)} x_i \quad \phi(y_i) = q^{(2\rho, \tilde{\alpha}_i)} y_i
\]

for all \( 1 \leq i \leq n \). Note that \( \tilde{\alpha}_i = 0 \) whenever \( \Theta(\alpha_i) = \alpha_i \). Thus \( \phi \) acts as the identity on the subalgebra \( \mathcal{M} \) of \( U \).

Set \( T_\Theta = \{ \tau(\beta)|\Theta(\beta) = \beta \text{ and } \beta \in Q(\pi) \} \). Recall that \( B \) is generated by \( \mathcal{M}, T_\Theta \) and elements of the form \( y_i t_i + d_i \theta(y_i) t_i + s_i t_i \) for \( \alpha_i \in \pi \setminus \pi_\Theta \) and suitably chosen scalars \( d_i \) and \( s_i \) in \( \mathbb{R} \) ([L4, Section 1]). Here \( \theta \) is a lift of the involution \( \theta \) to a \( \mathbb{C} \) algebra automorphism of \( U \). Moreover, \( \mathcal{M} \) and \( T_\Theta \) are subsets of each coideal subalgebra in \( \mathcal{B} \). (See [L2, Section 7] and [L4, Section 1] for more details.)

Let \( \sigma \) denote the antipode and \( \Delta \) the comultiplication of \( U \) (see [Jo, 3.2.9] or [L1, Section 7]). The following lemma generalizes [L3, Lemma 7.3].

**Lemma 3.1** Suppose that \( a \in \check{U} \) and \( b \in B_+ \). Then \( (\text{ad}_r b)a \) is contained in \( \phi(B_+)\check{U} + \check{U}B_+ \).

**Proof:** Note that \( \mathcal{M}T_\Theta \) is a Hopf subalgebra of \( U \) and a subalgebra of both \( \phi(B)\check{T}_\Theta \) and \( B\check{T}_\Theta \). Hence

\[
(\text{ad}_r (\mathcal{M}T_\Theta)_+)a \subset \mathcal{M}T_\Theta a \mathcal{M}T_\Theta \subset (\phi(B))_+a + \check{U}B_+.
\]
Set \( C_i = y_i t_i + d_i \tilde{\theta}(y_i) t_i + s_i (t_i - 1) \) for \( \alpha_i \in \pi \setminus \pi_\Theta \) where the \( s_i \) and \( d_i \) are chosen so that \( C_i \in B \). Note that \( C_i \) is in \( B_+ \) for each \( i \). Now \( \sigma(s_i(t_i - 1)) = s_i(t_i^{-1} - 1) \). Hence the proof of [L3, Lemma 7.3] shows that

\[
\sigma(C_i) \in -q(2\rho,\alpha_i) y_i - d_i q(2\rho,\Theta(\alpha_i)) \tilde{\theta}(y_i) + s_i(t_i^{-1} - 1) + (MT_\Theta)_+ U
\]

Furthermore, by the arguments in [L3, Lemma 7.3] we have that

\[
\Delta(y_i t_i + d_i \tilde{\theta}(y_i) t_i) \in (y_i t_i + d_i \tilde{\theta}(y_i) t_i) \otimes 1 + t_i \otimes (y_i t_i + d_i \tilde{\theta}(y_i) t_i) + U \otimes (MT_\Theta)_+.
\]

Now \( \Delta(t_i - 1) = t_i \otimes t_i - 1 \otimes 1 = (t_i - 1) \otimes 1 + t_i \otimes (t_i - 1) \). Hence

\[
\Delta(C_i) \in C_i \otimes 1 + t_i \otimes C_i + U \otimes (MT_\Theta)_+.
\]

It follows that

\[
(ad, C_i)a = -\sigma(C_i)a + t_i^{-1} aC_i + U a (MT_\Theta)_+
\]

\[
\subset -q(\rho,\Theta(\alpha_i) + \alpha_i) \phi(C_i) t_i^{-1} a + (MT_\Theta)_+ U a + t_i^{-1} a C_i + \tilde{U}(MT_\Theta)_+ 
\]

\[
\subset \phi(B_+) \tilde{U} + \tilde{U} B_+
\]

for all \( a \in \tilde{U} \). The lemma now follows from the fact that \( B \) is generated by the \( C_i, \alpha_i \in \pi \setminus \pi_\Theta \) and \( MT_\Theta \). \( \Box \)

Let \( \chi \) denote the Hopf algebra automorphism of \( U \) defined by \( \chi(x_i) = q^{(\rho,\alpha_i)} x_i \) and \( \chi(y_i) = q^{- (\rho,\alpha_i)} y_i \) for all \( 1 \leq i \leq n \). Note that \( \phi = \chi^{-2} \). Note further that \( (\rho,\alpha_i) = (\tilde{\rho},\tilde{\alpha}_i) = (\tilde{\rho},\alpha_i) \). Hence \( \tau(\tilde{\rho}) x_i \tau(-\tilde{\rho}) = \chi(x_i) \) and \( \tau(\tilde{\rho}) y_i \tau(-\tilde{\rho}) = \chi(y_i) \) for all \( 1 \leq i \leq n \). It follows that \( \tau(\tilde{\rho}) u \tau(-\tilde{\rho}) = \chi(u) \) for all \( u \in U \).

**Lemma 3.2** We have the following equality of sets:

\[
\tau(\tilde{\rho})(\phi(B_+) \tilde{U} + \tilde{U} B_+) = \chi^{-1}(B_+) \tau(\tilde{\rho}) \tilde{U} + \tau(\tilde{\rho}) \tilde{U} B_+.
\]

**Proof:** By (3.1), we have that \( \phi(u) = \tau(-2\tilde{\rho}) u \tau(2\tilde{\rho}) \) for all \( u \in U \). Hence \( \tau(\tilde{\rho}) \phi(B) \tau(-\tilde{\rho}) = \tau(-\tilde{\rho}) B \tau(\tilde{\rho}) = \chi^{-1}(B) \). It follows that \( \tau(\tilde{\rho})(\phi(B)) \tilde{U} = \chi^{-1}(B) \tau(\tilde{\rho}) \tilde{U}. \) \( \Box \)
It is sometimes helpful to use a slightly different form of the Harish-Chandra map associated to $B$. Let $N^-$ be the subalgebra of $G^-$ generated by the set $(\text{ad } M \cap G^-)C[y_i t_i | \alpha_i \notin \pi_0]$. In particular, by [L4, Theorem 2.2 and (2.3)], we have an inclusion

$$ \check{U} \subset (\check{U} (B \check{T}_\Theta)_+ + N^- \check{A}) \oplus C[\check{A}]. $$

(3.2)

Let $P'_B$ be the projection of $\check{U}$ onto $C[\check{A}]$ using the direct sum decomposition of (3.2).

Let $L(\lambda)$ denote the finite-dimensional simple $U$ module of highest weight $\lambda$ where $\lambda \in P^+ (\pi)$.

**Lemma 3.3** For all $c \in \check{U}^B$, we have $P_B(c) = P'_B(c)$. Moreover, if $\lambda \in P^+ (2\Sigma)$ and $\xi^*_\lambda$ is a nonzero $B$ invariant element of $L(\lambda)$, then

$$ u \xi^*_\lambda = z^\lambda (P_B(u)) \xi^*_\lambda $$

for all $u \in \check{U}^B$.

**Proof:** Let $\kappa$ be the antiautomorphism of $U$ which restricts to an algebra antiautomorphism of $B$ as described in ([L2, Theorem 3.1]). Note that $\kappa(t) = t$ for all $t \in \check{T}$. Moreover, $\kappa$ can be extended to $\check{U} \check{A} \check{T}_\Theta$ so that $\kappa(t) = t$ for all $t$ in $\check{A} \check{T}_\Theta$. As explained in the proof of [L3, Theorem 2.2], we have $\kappa(N^+) = N^-$. Hence applying $\kappa$ to (1.1) yields (3.2). The first assertion of the lemma now follows from the fact that $\kappa$ restricts to the identity on $C[\check{A}]$.

Let $\xi^*_\lambda$ denote the $B$ invariant vector of $L(\lambda)^*$. In [L3, preceding Theorem 3.6], it is shown that $\xi^*_\lambda c = z^\lambda (P_B(c)) \xi^*_\lambda$. Applying $\kappa$ and switching the roles of $\xi^*_\lambda$ and $\xi^*_\lambda$ in the argument in [L3] yields $u \xi^*_\lambda = z^\lambda (P'_B(u)) \xi^*_\lambda$. The second assertion of the lemma now follows from the first. $\square$

Let $F_r(\check{U})$ denote the locally finite part of $\check{U}$ with respect to the right adjoint action. Recall that the action of $(\text{ad}_r B)$ on $F_r(\check{U})$ is semisimple ([L4, Theorem 1.1]). As explained in [L3, Section 7, before Lemma 7.5]), we have that

$$ F_r(\check{U}) = \check{U}^B \oplus (\text{ad}_r B_+) F_r(\check{U}). $$

(3.3)

**Definition 3.4** The map $\mathcal{L}$ is the projection map from $F_r(\check{U})$ onto $\check{U}^B$ using the direct sum decomposition (3.3). Moreover, given $a \in F_r(\check{U})$ such that $a \notin (\text{ad}_r B_+) F_r(\check{U})$, we have that $\mathcal{L}(a)$ is the unique element of $\check{U}^B$ which is contained in $a + (\text{ad}_r B_+) a$ (see [L3, Section 7, before Lemma 7.5]).
Given \( \lambda \in P^+(2\Sigma) \), let \( g_\lambda \) denote the zonal spherical function \( g_\lambda^{\chi^{-1}(B),B} \) (see [L2, Section 4] or [L4, Section 4]). Recall that elements of \( R_q[G] \) can be viewed as functions on \( \tilde{U} \). Write \( \varphi_\lambda \) for the image of \( g_\lambda \) in \( C[P(\pi)] \) obtained by restricting to \( U^0 \). By [L2, Corollary 5.4 and Theorem 6.3], each \( \varphi_\lambda \) is in \( C[P(2\Sigma)] \) and is invariant under the action of \( W_\Theta \). Moreover, the zonal spherical family \( \{ \varphi_\lambda | \lambda \in P^+(2\Sigma) \} \) is a basis for \( C[P(2\Sigma)]^{W_\Theta} \).

Let \( \xi_\lambda \) be a nonzero \( B \) invariant vector of \( L(\lambda) \). Let \( \zeta^* \lambda \) be a nonzero \( \chi^{-1}(B) \) invariant vector of \( L(\lambda)^* \). By [L2, Section 4], rescaling if necessary, \( g_\lambda \) can be identified with the function on \( \tilde{U} \) which sends \( u \) to \( \zeta^* \lambda(u \xi_\lambda) \) for all \( u \in \tilde{U} \).

**Lemma 3.5** Suppose there exists \( b \in \tilde{U}(B\tilde{T}_\Theta)^+_+ \) and \( a \in \tilde{U}^o \) such that \( a + b \in F_\nu(\tilde{U}) \). Then

\[
z^\lambda(P_B(L(a + b))\varphi_\lambda(\tau(\tilde{p})) = \varphi_\lambda(a \tau(\tilde{p}))
\]

for all \( \lambda \in P^+(2\Sigma) \).

**Proof:** By Lemma 3.3 and the definition of \( \varphi_\lambda \) we have that

\[
g_\lambda(\tau(\tilde{p})L(a + b)) = z^\lambda(P_B(L(a + b))g_\lambda(\tau(\tilde{p}))
\]

\[
= z^\lambda(P_B(L(a + b))\varphi_\lambda(\tau(\tilde{p})).
\]

By Lemma 3.1 and Lemma 3.2, \( g_\lambda(\tau(\tilde{p})c) = 0 \) for all \( c \in (ad_\pi B_+)\tilde{U} \). Note further that \( g_\lambda(\tau(\tilde{p})b) = 0 \) since \( \tau(\tilde{p})b \in \tilde{U}(B\tilde{T}_\Theta)^+_+ \). It follows from Definition 3.4 that

\[
g_\lambda(\tau(\tilde{p})L(a + b)) = g_\lambda(\tau(\tilde{p})(a + b)) = \varphi_\lambda(\tau(\tilde{p})a) \Box
\]

The decomposition (3.2) is established using [L4, Lemma 2.1] which is particularly well suited to computing the tip of elements. On the other hand, it is easier to compute the top of elements constructed using the right adjoint action. Thus it is necessary to transform the information contained in [L3, Lemma 2.1] and [L4, Lemma 2.1]. This is done in the next lemma after we introduce some notation, mostly from [L4].

Recall that \( N^+ \) is a subalgebra of \( U^+ \) and \( N^- \) is a subalgebra of \( G^- \). Set \( \tilde{N}^+ = \sum_{\gamma \in Q^+_{q}(\pi)} N^+_\gamma \tau(-\gamma) \) and \( \tilde{N}^- = \sum_{\gamma \in Q^+_{q}(\pi)} N^-_\gamma \tau(-\gamma) \). Let \( \kappa \) be the antiautorphism defined in [L2, Theorem 3.1] and used in the proof of Lemma 3.3. Recall that \( \kappa \) fixes elements of \( T \) and \( \kappa(N^+) = N^- \). Hence \( \kappa(\tilde{N}^+) = \tilde{N}^- \).
Moreover, there exist elements \( \tau_1, \ldots, \tau_n \) of \( P(2\Sigma) \) such that \( \beta_1 \leq \gamma \) for each \( 1 \leq i \leq n \) and

\[
\top(\mathcal{P}_B(\mathcal{L}(2\gamma) + b)) = \sum_{i=1}^{m} a_{\beta_i} \tau(2\beta_i).
\]
Proof: By Lemma 3.3, \( \mathcal{P}_B(\mathcal{L}(\tau(2\gamma)+b)) = \mathcal{P}'_B(\mathcal{L}(\tau(2\gamma)+b)) \). We work with this second version of the Harish-Chandra map. By Definition 3.4 there exists \( c \in B_+ \) so that \( \mathcal{L}(\tau(2\gamma)+b) = \tau(2\gamma)+b+(\text{ad}_r c)(\tau(2\gamma)+b) \). By assumption, \( b \in \hat{U}(B\hat{T}_\Theta)_+ \). Moreover, Lemma 3.1 ensures that \( (\text{ad}_r c)b \in \hat{U}(B\hat{T}_\Theta)_+ \). Hence \( \mathcal{P}'_B(\mathcal{L}(\tau(2\gamma)+b)) = \mathcal{P}'_B(\tau(2\gamma)+(\text{ad}_r c)\tau(2\gamma)) \).

Now \( \tau(2\gamma)+(\text{ad}_r c)\tau(2\gamma) \in (\text{ad}_r U)\tau(2\gamma) \). Examining the right adjoint action of the generators of \( \hat{U} \) ([L2, (1.2)] and the relations of \( \hat{U} \) yield

\[
(\text{ad}_r U)\tau(2\gamma) = \sum_{\beta \in Q}(\pi U)^{-}\tau(2\gamma - 2\beta).
\]

Thus, the first assertion follows from (3.5). The second assertion is now a consequence of (2.8). \(\square\)

In Theorem 2.5, a criterion was found concerning the set of tops of elements in \( \mathcal{P}_B(\hat{U}^B) \) which ensures surjectivity of the Harish-Chandra map. The next result provides another, more useful, criterion. In particular, we show that the image of \( \hat{U}^B \) under \( \mathcal{P}_B \) is all of \( \mathcal{C}[\mathcal{A}]^{W_\Theta} \) if \( F_r(\hat{U}) \) contains a set of elements of a particular form. In the next section, we find this desired set of elements in \( F_r(\hat{U}) \).

Theorem 3.8 Suppose that for each \( \gamma \in P^+(\Sigma) \), there exists \( b_\gamma \in \hat{U}(B\hat{T}_\Theta)_+ \) such that \( \tau(2\gamma)+b_\gamma \in F_r(\hat{U}) \). Then

\[
\{\mathcal{P}_B(\mathcal{L}(\tau(2\gamma)+b_\gamma))|\gamma \in P^+(\Sigma)\} \text{ is a basis for } \mathcal{P}_B(\hat{U}^B).
\]

Moreover, \( \mathcal{P}_B(\hat{U}^B) = \mathcal{C}[\mathcal{A}]^{W_\Theta} \).

Proof: Suppose that \( a \) is a linear combination of elements in the set \( \{\mathcal{L}(\tau(2\beta)+b_\beta)|\beta \in P^+(\Sigma) \text{ and } \beta \leq \gamma\} \). Arguing as in [L3, Lemma 7.5] using Lemma 3.5, we see that \( \mathcal{P}_B(a) = 0 \) if and only if \( a = 0 \). Thus, the set \( \{\mathcal{P}_B(\mathcal{L}(\tau(2\beta)+b_\beta))|\gamma \in P^+(\Sigma)\} \) is linearly independent over \( \mathcal{C} \).

By the previous lemma, \( \mathcal{P}_B(\mathcal{L}(\tau(2\gamma)+b_\gamma)) \in \tau(2\gamma)\mathcal{A}_\Theta \). Let

\[
S_\gamma = \text{span}\{\mathcal{P}_B(\mathcal{L}(\tau(2\beta)+b_\beta))|\beta \in P^+(\Sigma) \text{ and } \beta \leq \gamma\}.
\]

It follows that the dimension of \( S_\gamma \) over \( \mathcal{C} \) is just the cardinality of the set \( \{\beta \in P^+(\Sigma) \text{ and } \beta \leq \gamma\} \). On the other hand, Lemma 3.7 ensures that \( \{\text{top}(a)|a \in S_\gamma\} \) is a subspace of \( \text{span}\{\tau(2\beta)|\beta \in P^+(\Sigma) \text{ and } \beta \leq \gamma\} \). Since
the dimension of \( \{ \text{top}(\mathcal{P}_B(a)) | a \in S_\gamma \} \) is equal to the dimension of \( S_\gamma \) and both are finite dimensional, it follows that

\[
\{ \text{top}(a) | a \in S_\gamma \} = \text{span}\{ \tau(2\beta) | \beta \in P^+(\Sigma) \text{ and } \beta \leq \gamma \}.
\]

This forces \( \text{top}(\mathcal{P}_B(\mathcal{L}(\tau(2\beta) + b_\beta))) = \tau(2\beta) \) up to a nonzero scalar for all \( \beta \in P^+(\Sigma) \). Hence (2.8) and the above discussion yields

\[
\{ \text{top}(\mathcal{P}_B(a)) | a \in \check{U}^B \} \subseteq \text{span}\{ \tau(2\beta) | \beta \in P^+(\Sigma) \} = \text{span}\{ \text{top}(\mathcal{P}_B(\mathcal{L}(\tau(2\beta) + b_\beta))) | \beta \in P^+(\Sigma) \}
\]

\[
\subseteq \text{span}\{ \text{top}(\mathcal{P}_B(a)) | a \in \check{U}^B \}.
\]

The theorem now follows from Theorem 2.5. \( \square \)

### 4 Surjectivity of the Harish-Chandra map

In this section, we prove that the image of \( \check{U}^B \) under \( \mathcal{P}_B \) is the entire invariant ring \( \mathcal{C}[\mathcal{A}]^W_\Theta \). The approach is to show that \( F_r(\check{U}) \) contains a set of elements

\[
\{ \tau(2\gamma) + b_\gamma | \gamma \in P^+(\Sigma) \}
\]

where \( b_\gamma \in \check{U}(BT^\Theta)_+ \) and then apply Theorem 3.8. In particular, we prove the following theorem and corollary, the main results of this paper.

**Theorem 4.1** For each \( \gamma \in P^+(\Sigma) \), there exists \( b_\gamma \in \check{U}(BT^\Theta)_+ \) such that \( \tau(2\gamma) + b_\gamma \in F_r(\check{U}) \). Moreover

\[
\{ \mathcal{P}_B(\mathcal{L}(\tau(2\gamma) + b_\gamma)) | \gamma \in P^+(\Sigma) \}
\]

is a basis for \( \mathcal{P}_B(\check{U}^B) \).

The next corollary is an immediate consequence of Theorem 4.1, [L4, Corollary 4.2 and Theorem 5.5], and Theorem 3.8.

**Corollary 4.2** For each irreducible symmetric pair \( \mathfrak{g}, \mathfrak{g}^\theta \) and each \( B \in \mathcal{B} \), the Harish-Chandra map \( \mathcal{P}_B \) maps \( \check{U}^B \) onto \( \mathcal{C}[\mathcal{A}]^W_\Theta \). The kernel of the restriction of \( \mathcal{P}_B \) to \( \check{U}^B \) is \( (BT^\Theta)_+ \check{U} \cap \check{U}^B \). Moreover, \( \mathcal{P}_B(\check{Z}(\check{U})) = \mathcal{P}_B(\check{U}^B) \) if and only if \( \mathfrak{g}, \mathfrak{g}^\theta \) is not of type EIII, EIV, EVII, or EIX.
Suppose that we have found a set of elements \( \{ b_\gamma \mid \gamma \in P^+(\Sigma) \} \) which satisfy the conditions of Theorem 4.1. Recall that the compact zonal spherical functions have been identified with Macdonald polynomials for a large class of quantum symmetric pairs (see for example [L3]). It follows from Lemma 3.5 the set \( \{ X(\mathcal{L}(\tau(2\gamma) + b_\gamma)) \mid \gamma \in P^+(\Sigma) \} \) is precisely the family of commuting difference operators associated to these Macdonald polynomials described in [K, Theorem 6.6]. Thus Theorem 4.1 and the results of Section 3 provide a natural quantum interpretation of these difference operators.

Recall the antiautomorphism \( \kappa \) defined in [L2, Theorem 3.1] and used in the proof of Lemma 3.3. Since \( \kappa(t) = t \) for all \( t \in \hat{T} \), it follows that \( \kappa((\ad r_t)a) = (\ad r_t)\kappa(a) \) for all \( a \in \hat{U} \). Moreover, up to nonzero scalars, \( \kappa((\ad r_{x_i})a) = (\ad r_{y_i})\kappa(a) \), and \( \kappa((\ad r_{y_i})a) = (\ad r_{y_i})\kappa(a) \), for all \( 1 \leq i \leq n \) (see the proof of [L3, Theorem 2.2]). It follows that \( \kappa(F_r(\hat{U})) = F_r(\hat{U}) \). Since \( \kappa \) restricts to an antiautomorphism of \( B \), we see further that \( \kappa((\ad r_{x_i})a) = (\ad r_{y_i})\kappa(a) \), and \( \kappa((\ad r_{y_i})a) = (\ad r_{y_i})\kappa(a) \), for all \( 1 \leq i \leq n \). The strategy in this section is to show that \( F_r(\hat{U}) \) contains a set of elements

\[
\{ \tau(2\gamma) + b'_\gamma \mid \gamma \in P^+(\Sigma) \}
\]

where each \( b'_\gamma \in (B\tilde{T}_\Theta)_+\hat{U} \). Applying \( \kappa \) to this set yields the desired set of the form (4.1). The rest of Theorem 4.1 then follows from Theorem 3.8.

Recall that \( F_r(\hat{U}) \cap \hat{T} \) is equal to \( \{ \tau(2\mu) \mid \mu \in P^+(\pi) \} \) (see [Jo, Section 7] and discussion preceding [L3, Lemma 7.2]). The next lemma shows that it is very easy to find \( b'_\gamma \) when \( \gamma \in \tilde{P}^+(\pi) \).

**Lemma 4.3** Suppose that \( \gamma \in P^+(\Sigma) \) and \( \gamma = \tilde{\beta} \) for some \( \beta \in P^+(\pi) \). Then

\[
\tau(2\gamma) + b'_\gamma \in F_r(\hat{U})
\]

where \( b'_\gamma = \tau(2\beta) - \tau(2\gamma) \). Moreover, \( b'_\gamma \in \mathcal{C}[\hat{T}_\Theta]_+\hat{T} \).

**Proof:** Since \( \beta \in P^+(\pi) \), it follows that \( \tau(2\beta) \in F_r(\hat{U}) \). Therefore \( \tau(2\gamma) + b'_\gamma = \tau(2\beta) \) is in \( F_r(\hat{U}) \). Since \( \gamma = \tilde{\beta} = (\beta - \Theta(\beta))/2 \), we have

\[
b'_\gamma = \tau(2\beta) - \tau(2\gamma) = \tau(\beta + \Theta(\beta))\tau(2\gamma) - \tau(2\gamma) = [(\tau(\beta + \Theta(\beta)) - 1] \tau(2\gamma).
\]

The lemma follows from the fact that \( (\tau(\beta + \Theta(\beta)) - 1 \in \mathcal{C}[\hat{T}_\Theta]_+ \). □

We break the proof of Theorem 4.1 into four cases.
(i) \( g, g^\theta \) is not of type EIII, EIV, EVII, EIX, or CII(ii) and \( g \) does not contain a \( \theta \) invariant Lie subalgebra \( \mathfrak{r} \) of rank greater than or equal to 7 such that \( \mathfrak{r}, \mathfrak{r}^\theta \) is of type AII.

(ii) \( g, g^\theta \) is of type CII(ii) or \( g \) contains a \( \theta \) invariant Lie subalgebra \( \mathfrak{r} \) of rank greater than or equal to 7 such that \( \mathfrak{r}, \mathfrak{r}^\theta \) is of type AII.

(iii) \( g, g^\theta \) is of type EIV, EVII, or EIX.

(iv) \( g, g^\theta \) of of type EIII.

It should be noted that in the first two cases, Corollary 4.2 follows from [L4, Theorem 6.1]. Indeed, Cases (i) and (ii) correspond precisely to the situation when the image of the center \( Z(\check{\mathfrak{u}}) \) under \( \mathcal{P}_B \) is equal to the image of \( \check{\mathfrak{u}} \) under \( \mathcal{P}_B \). On the other hand, Cases (iii) and (iv) rely on an analysis of symmetric pairs of type DI. Thus after completing Case (ii), we take a detour and focus on type DI pairs.

**Case (i):** By [L4, Theorem 3.5], \( \tau(2\gamma) + b' \in (B\check{T})_+\check{U} \) so that \( \tau(2\gamma) + b' \) is in \( F_r(\check{U}) \) is more delicate. Set \( t \) equal to the rank of \( \Sigma \).

Let \( \mu_1, \ldots, \mu_t \) denote the simple roots for the restricted root system \( \Sigma \). Let \( \eta_1, \ldots, \eta_t \) denote the corresponding fundamental weights in \( P^+(\Sigma) \). The next lemma reduces the work to finding just a finite number of elements associated to the weights \( \eta_1, \ldots, \eta_t \).

**Lemma 4.4** Suppose that for each \( B \in \mathcal{B} \) there exists a subset \( \{b_i^B| 1 \leq i \leq r\} \) of \( (B\check{T})_+\check{U} \) such that \( \tau(2\eta_i) + b_i^B| 1 \leq i \leq r\} \) is a subset of \( F_r(\check{U}) \). Then for each \( B \in \mathcal{B} \), \( F_r(\check{U}) \) contains a subset of the form (4.2).

**Proof:** Let \( S \) be the subset of \( \{\tau(2\gamma)| \gamma \in P^+(\Sigma)\} \) such that \( \tau(2\gamma) \in F_r(\check{U}) + (B\check{T})_+\check{U} \) for each \( \tau(\gamma) \in S \) and every \( B \in \mathcal{B} \). To prove the lemma, it is sufficient to show that \( S \) equals \( \{\tau(2\gamma)| \gamma \in P^+(\Sigma)\} \). By assumption, \( S \) contains the subset \( \{\tau(2\eta_i)| 1 \leq i \leq t\} \). Hence, it is sufficient to show that \( S \) is multiplicatively closed.
Fix \( \eta \) and \( \beta \) so that \( \tau(2\eta) \) and \( \tau(2\beta) \) are both in \( S \). The map \( u \mapsto \tau(\eta)u\tau(-\eta) \) defines a Hopf algebra automorphism of \( \hat{U} \), which we denote by \( \psi \). Note that elements of \( U^0 \) are fixed under the action of \( \psi \). It follows that \( \psi \) permutes the elements of \( B \). Fix \( B \in B \) and let \( B' \in B \) be chosen so that \( \psi(B') = B \). Choose \( b \in (B\hat{T}_\Theta)_+\hat{U} \) and \( c \in (B'\hat{T}_\Theta)_+\hat{U} \) so that \( \tau(2\eta) + b \) and \( \tau(2\beta) + c \) are elements of \( F_\tau(\hat{U}) \). Note that \( \psi(c) \) is in \( (B\hat{T}_\Theta)_+\hat{U} \). Hence

\[
(\tau(2\eta) + b)(\tau(2\beta) + c) = \tau(2\eta + 2\beta) + b(\tau(2\beta) + c) + \tau(2\eta)c
\]

\[
= \tau(2\eta + 2\beta) + b(\tau(2\beta) + c) + \psi(c)\tau(2\eta)
\]

\[
\in \tau(2\eta + 2\beta) + (B\hat{T}_\Theta)_+\hat{U}.
\]

Thus \( S \) contains \( \tau(2\beta)\tau(2\eta) = \tau(2\eta + 2\beta) \). \( \square \)

**Case (ii):** Assume that \( g, g^\theta \) satisfies the conditions of Case (ii). A list of the possibilities for \( g, g^\theta \) can be found in [L4, (3.11)-(3.15)]. Note that \( t \geq 4 \) in all cases (and \( t \geq 7 \) if \( g, g^\theta \) is not of type CII(ii)). From the list in [L4, Section 3] and the classification in [A], we see that \( \Sigma \) must be of type \( A_t \), \( B_t \), or \( C_t \). Note further that we may assume that the restricted root system associated to the smaller symmetric pair \( \tau, \tau^\theta \) has rank \( t - 1 \).

The next lemma will be used to relate the fundamental weights of the root system of type \( A_j \) generated by \( \mu_1, \ldots, \mu_j \) to the fundamental weights associated to \( \Sigma \).

**Lemma 4.5** Assume that \( \phi \) is a root system of type \( A_m, C_m \) or \( D_m \) with set of simple roots \( \{\beta_1, \ldots, \beta_m\} \). Let \( \lambda_1, \ldots, \lambda_m \) denote the corresponding fundamental weights in the weight lattice of \( \phi \). Fix \( j \) such that \( 1 \leq j < m \). Write \( \lambda'_1, \ldots, \lambda'_j \) for the fundamental weights in the root system of type \( A_j \) generated by \( \beta_1, \ldots, \beta_j \) (in the obvious order). Then

\[
\lambda'_k + k(j + 1)^{-1}\lambda_{j+1} = \lambda_k
\]

for all \( 1 \leq k \leq j \).

**Proof:** The assumptions on \( \phi \) ensure that \( (\beta_i, \beta_{i+1}) = -(\lambda_{i+1}, \beta_{i+1}) \) for all \( 1 \leq i \leq m - 1 \). Now \( (\lambda'_k, \beta_i) = \delta_{ik}(\beta_i, \beta_i)/2 \) for all \( 1 \leq i \leq j \). It follows from [H, Section 13.2, Table 1] that

\[
\lambda'_k = [(j - k + 1)\beta_1 + 2(j - k + 1)\beta_2 + \ldots + (k - 1)(j - k + 1)\beta_{j-1} + r(j - k + 1)\beta_k + r(j - k)\mu_{k+1} + \ldots + k\beta_j](j + 1)^{-1}.
\]

(4.3)

25
Hence \((\lambda_k', \beta_{j+1}) = k(j + 1)^{-1}(\beta_j, \beta_{j+1}) = -k(j + 1)^{-1}(\lambda_j, \beta_{j+1})\). Note that (4.3) also implies that \((\lambda_k', \beta_s) = 0\) for all \(s > j + 1\). This proves the lemma. 

Define subsets \(\pi_i\) of \(\pi\) for \(1 \leq i \leq t - 1\) by \(\pi_i = \{\alpha_j | 1 \leq j \leq 2i + 1\}\). Let \(g_i\) denote the semisimple Lie subalgebra of \(g\) generated by the positive and negative root vectors associated to the simple roots in \(\pi_i\). Note that the root system of \(g_i\) is of type \(A_{2i+1}\). We have \(g_i \subseteq g_{i+1}\) for \(1 \leq i \leq t - 2\). Moreover, \(\theta\) restricts to an involution on \(g_i\) such that the symmetric pair \(g_i, g_i^\theta\) is of type \(\text{AII}\) and the rank of the corresponding restricted root system is \(i\). Given \(i\) such that \(1 \leq i \leq t - 1\), let \(U_i\) denote the quantized enveloping algebra of \(g_i\) considered as a subalgebra of \(U_q(g)\). In particular, \(U_i\) is generated by \(x_j, y_j, t_j^{\pm 1}\) for all \(j\) such that \(\alpha_j \in \pi_i\). Set \(U_0 = \mathcal{C}\) and \(U = U_t\).

Let \(\tilde{U}_i\) be the simply connected quantized enveloping algebra of \(g_i\). Let \(\tilde{U}^i\) be the subalgebra of \(U\) generated by \(U_i\) and \(\tilde{T}\) for \(0 \leq i \leq t\). Note that when \(i = 0\), this definition agrees with the earlier definition of \(\tilde{U}^0\). Now \(\tilde{U}_i \subseteq \tilde{U}^i\) for all \(i\) and \(\tilde{U}^i = \tilde{U}\). Write \(\omega_j\) for the fundamental weight corresponding to the simple root \(\alpha_j\) in \(P^+(\pi)\). Note that \(\tau(\omega_j)\) is in the center of \(\tilde{U}_i\) for each \(j > 2i + 1\). In particular, \(\tilde{U}^i\) is isomorphic to the tensor product \(\tilde{U}_i \otimes_{\mathcal{C}} \mathcal{C}[\tau(\omega_{2i+1})^{\pm 1}, \ldots, \tau(\omega_n)^{\pm 1}]\) as an algebra. Furthermore, \(F_r(\tilde{U}^i) = F_r(\tilde{U}_i)\mathcal{C}[\tau(\omega_{2i+1})^{\pm 1}, \ldots, \tau(\omega_n)^{\pm 1}]\).

Recall the map \(L\) which lifts elements of \(F_r(\tilde{U})\) to elements of \(\tilde{U}^B\) defined in Definition 3.4. Given \(i\) such that \(1 \leq i < t\), we define the same type of map, denoted by \(L_i\), for the quantum symmetric pair \(U_i, U_i \cap B\). In particular, given \(a \notin (\text{ad}_r (B \cap U_i)_+)\tilde{U}_i\), we have that \(L_i(a)\) is the unique element in \(a + (\text{ad}_r (B \cap U_i)_+)a\).

Note that \(\mathcal{C}[\tau(\omega_{2i+1})^{\pm 1}, \ldots, \tau(\omega_n)^{\pm 1}]\) is contained in the center of \(\tilde{U}^i\). Thus we can extend \(L_i\) to \(\tilde{U}_i\) by insisting that \(L_i(ab) = L_i(a)b\) for all \(b\) in the ring \(\mathcal{C}[\tau(\omega_{2i+1})^{\pm 1}, \ldots, \tau(\omega_n)^{\pm 1}]\).

Let \(P_{B \cap U_j}\) denote the Harish-Chandra map on \(\tilde{U}_j\) with respect to the quantum symmetric pair \(B \cap U_j, U_j\). Note that \(P_{B \cap U_j}(a) = P_B(a)\) for all \(a \in \tilde{U}_j\). Moreover, \(P_B(ab) = P_{B \cap U_j}(a)b\) for all \(a \in \tilde{U}_j\) and \(b\) in the center of \(\tilde{U}_j\).

The next lemma shows that \(F_r(\tilde{U})\) contains a set of the form (4.2). In particular, Theorem 4.1 for Case (ii) symmetric pairs is a consequence of the following result.
Lemma 4.6 For each $1 \leq i \leq t$, there exists $b_i \in ((B \cap U_{i-1})\tilde{T}_\Theta)_{\ast} \bar{U}^{i-1}$ such that \{\tau(2\eta_i) + b_i| 1 \leq i \leq t\} is a subset of $F_r(U)$.

Proof: By [L4, Lemma 3.5], we have that $\tilde{\omega}_1 = \eta_1$. Thus by Lemma 4.3, there exists $b_1 \in ((B \cap U_0)\tilde{T}_\Theta)_{\ast} \bar{U}^0$ such that $\tau(2\eta_1) + b_1 \in F_r(U)$. If $g_0^\theta$ is not of type CII(ii), then by [L4, Lemma 3.5] $\tilde{\omega}_n = \eta_n$. Thus, if $g_0^\theta$ is not of type CII(ii), then there also exists an appropriate choice for $b_n$. Note that if $g_0^\theta$ is of type CII(ii), then $\Sigma$ is of type $C_t$. Set $t' = t$ if $g_0^\theta$ is not of type CII(ii) and set $t' = t - 1$ otherwise. The proof is by induction on $t'$.

Suppose that $1 \leq j < t'$. Assume that we have found $b_1, \ldots, b_j$ so that $b_i \in ((B \cap U_{i-1})\tilde{T}_\Theta)_{\ast} \bar{U}^{i-1}$ for $1 \leq i \leq j$ and \{\tau(2\eta_i) + b_i| 1 \leq i \leq j\} is a subset of $F_r(U)$. It follows that $b_i \in \bar{U}^i$ for each $i$ and $\tau(2\eta_i) + b_i \in F_r(\bar{U}^i)$ for $1 \leq i \leq j$.

Let $\eta_1', \ldots, \eta_j'$ denote the fundamental weights in the root system of type $A_j$ generated by $\mu_1, \ldots, \mu_j$. Lemma 4.5 and the discussion preceding this lemma imply that

$$L_j(\tau(2\eta_k)) = L_j(\tau(2\eta_k'))\tau(k(j + 1)^{-1}2\eta_{j+1})$$

for each $1 \leq k \leq j$. Let $W_0'$ denote the Weyl group of the root system generated by $\mu_1, \ldots, \mu_j$. Given a dominant integral weight $\beta$ in the weight lattice of this root system, set

$$\hat{m}'(2\beta) = \sum_{\gamma \in W_0'} q^{(\rho, 2\gamma)}\tau(2\gamma).$$

Note that $\eta_k'$ is a minuscule fundamental weight in the weight lattice of the root system generated by $\mu_1, \ldots, \mu_j$. By Lemma 2.2, it follows that $P_{B \cap U_j}(L_j(\tau(2\eta_k'))) = \hat{m}'(2\eta_k')$. Hence

$$P_B(L_j(\tau(2\eta_k)) = \hat{m}'(2\eta_k')\tau(k(j + 1)^{-1}2\eta_{j+1})$$

(4.4)

up to a nonzero scalar. By [L4, Theorem 2.4] and (4.4), we have

$$P_B(L_j(\tau(2\eta_k)L_j(\tau(2\eta_{j+1-k}))) = \hat{m}'(2\eta_k')\hat{m}'(2\eta_{j+1-k}')\tau(2\eta_{j+1})$$

(4.5)

up to a nonzero scalar for all $1 \leq k \leq j$.

Now let $\omega_{ij}, \ldots, \omega_{2j+1,j}$ denote the fundamental weights corresponding to the simple roots in the root system generated by $\pi_j$. Since $j < t'$, we have
that $\pi_j$ generates a root system of type $A_{2j+1}$. Moreover, checking the list [L4, (3.11)-(3.15)], we see that $\pi$ is of type $A_n, C_n,$ or $D_n$. Thus Lemma 4.5 implies that
\[ \omega_{kj} + k(2j + 2)^{-1}\omega_{2j+2} = \omega_j \] (4.6)
for $1 \leq k \leq 2j + 1$. By [L4, Lemma 3.1 and Lemma 3.5], we further have that $\bar{\omega}_{2k} = 2\eta_k$ for all $1 \leq k \leq t'$. Recall ([Jo, Chapter 7] and [L4, Section 5]) that there exists a unique central element $z'_{j+1}$ of $\mathcal{U}_j$ contained in $\tau(2\omega_{j+1,j}) + (\text{ad}_r(U_j))\tau(2\omega_{j+1,j})$. Set $z_{j+1} = z'_{j+1}\tau(\omega_{2j+2})$ It follows that $z_{j+1}$ is in $(\text{ad}_r U_j)\tau(2\omega_j)$ and is central in $\mathcal{U}_j$. Moreover,
\[ \mathcal{P}_B(z_{j+1}) = \mathcal{P}_{Br}(z'_{j+1})\tau(\omega_{2j+2}) = \mathcal{P}_B(z'_{j+1})\tau(2\eta_{j+1}). \] (4.7)

It follows from [L4, Lemma 6.7] that 1 is in the $\mathcal{C}$ span of the set
\[ \{\mathcal{P}_{Br}(z'_{j+1})\} \cup \{\hat{m}'(2\eta'_k)\hat{m}'(2\eta'_{j+1-k}) | 1 \leq k \leq j\}. \]
In particular, there exists $c$ in $\mathcal{U}_j^{(B \cap U_i)}$ such that $c$ is a linear combination of elements in the set
\[ \{z'_{j+1}\} \cup \{L_j(2\eta'_k)L_j(2\eta'_{j+1-k}) | 1 \leq k \leq j\} \]
and $\mathcal{P}(BrU_j)(c) = 1$. By [L4, Corollary 4.2], we see that
\[ c - 1 \in (B\hat{T}_\Theta \cap U_j) + \mathcal{U}_j. \] (4.8)

Now consider the element $c\tau(2\eta_{j+1})$. It follows from the choice of $c$, (4.5), and (4.7), that $\tau(2\eta_{j+1})$ is a linear combination of elements in the set
\[ \{\mathcal{P}_B(z_{j+1})\} \cup \{\hat{m}'(2\eta'_k)\hat{m}'(2\eta'_{j+1-k})\tau(2\eta_{j+1}) | 1 \leq k \leq j\}. \]
Moreover, (4.8) implies that
\[ c \in \tau(2\eta_{j+1}) + ((B \cap U_j)\hat{T}_\Theta) + \mathcal{U}. \]

**Symmetric pairs of type DI:** The next few lemmas will provide a method for finding appropriate $B$ invariant elements inside of $F_r(\mathcal{U})$ when $g, g^\theta$ is of type EIII, EIV, EVII, or EIX. Note that in each of these cases, $g$ contains a semisimple Lie subalgebra $r$ such that $r, r^\theta$ is of type DI, case (i). (In fact, for
the latter three cases, \( \mathfrak{g} \) contains more than one such Lie subalgebra.) Thus in this subsection, we analyze symmetric pairs of type \( DI \), case (i). This information is then pulled back to the above four symmetric pair types to finish the proof of Theorem 4.1.

Let \( \mathcal{P} \) denote the ordinary Harish-Chandra projection map of \( \bar{U} \) onto \( \bar{U}^0 \) using the direct sum decomposition [L4, (5.1)]. Let \( \mathcal{W} \) denote the Weyl group of the root system \( \Delta \) of \( \mathfrak{g} \). Recall the central element \( z_2 \mu \) introduced before Lemma 2.2.

**Lemma 4.7** Let \( \mathfrak{g}, \mathfrak{g}^\theta \) be a symmetric pair of type \( DI \), case (i). Assume further that \( \pi_\Theta = \{ \alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_{n-1}, \alpha_n \} \) where \( n - 1 \geq m \geq 3 \). Then

\[
\mathcal{P}_B(z_{2\omega_1}) = a^{-1}(\sum_{\mu \in \mathcal{W}_{\mathfrak{g}\omega_1}} q^{(\bar{\beta}, 2\mu)}\tau(2\mu) + \sum_{i=0}^{m-1} (q^{2i} + q^{-2i}))
\]

where

\[
a = (\sum_{\mu \in \mathcal{W}_{\mathfrak{g}\omega_1}} q^{(\bar{\beta}, 2\mu)} + \sum_{i=0}^{m-1} (q^{2i} + q^{-2i})).
\]

**Proof:** First we compute \( \mathcal{P}(z_{2\omega_1}) \). Note that \( \omega_1 \) is a minuscule weight. In particular, there does not exist \( \beta \in P^+(\pi) \) with \( \beta < \omega_1 \). It follows from [L4, Lemma 5.1] that

\[
\mathcal{P}(z_{2\omega_1}) = a^{-1}(\sum_{\mu \in \mathcal{W}_{\omega_1}} q^{(\bar{\beta}, 2\mu)}\tau(2\mu))
\]

where \( a = \sum_{\mu \in \mathcal{W}_{\omega_1}} q^{(\bar{\beta}, 2\mu)} \).

By [H, Section 13.2, Table 1],

\[
\omega_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-2} + (1/2)(\alpha_{n-1} + \alpha_n).
\]

Recall that we are assuming \( \pi_\Theta = \{ \alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_{n-1}, \alpha_n \} \). It is straightforward to check using [A] or the list in [L2, Section 7] that

\[
\mu_{n-m} = \tilde{\alpha}_{n-m} = \alpha_{n-m} + \alpha_{n-m+1} + \ldots + \alpha_{n-2} + (\alpha_{n-1} + \alpha_n)/2 \quad (4.9)
\]

\[
\mu_i = \tilde{\alpha}_i = \alpha_i \quad \text{for} \quad 1 \leq i \leq n - m - 1 \quad (4.10)
\]

and \( \tilde{\alpha}_j = 0 \) for \( \alpha_j \in \pi_{\Theta} \). Furthermore the rank \( t \) of \( \Sigma \) equals \( n - m \) and \( \Sigma \) is of type \( B_t \). It follows from [L4, Lemma 3.1] and [H, Section 13.2] that

\[
\tilde{\omega}_1 = \eta_1 = \sum_{1 \leq i \leq n-m} \tilde{\alpha}_i. \quad (4.11)
\]
From the description of the fundamental weights corresponding to a root system of type $B_t$, we see that the only restricted root in $P^+(\Sigma)$ which is strictly less than $\eta_1$ is 0. Thus the $W$ orbit of $\omega_1$ can be written as a union of two sets

$$S_1 = \{ \pm (\alpha_i + \cdots + \alpha_{n-2} + (\alpha_{n-1} + \alpha_n)/2) | 1 \leq i \leq n-m \}$$

and

$$S_2 = \{ \pm (\alpha_i + \alpha_{i+1} + \cdots + \alpha_n - (\alpha_{n-1} + \alpha_n)/2) | n-m+1 \leq i \leq n-1 \}$$

It is straightforward to check that $\mu = \tilde{\mu}$ and hence $(\rho, \mu) = (\tilde{\rho}, \mu)$ for all $\mu \in S_1$. Moreover, $S_1$ corresponds to the $W_\Theta$ orbit of $\eta_1$. On the other hand, $S_2$ is a subset of $Q(\pi_\Theta)$. Hence, the image under $\tilde{\cdot}$ of elements in the second set $S_2$ are all equal to 0.

We have

$$\mathcal{P}_B(z_{2\omega_1}) = a^{-1}(\sum_{\mu \in S_1} q^{(\tilde{\rho}, 2\mu)} T(2\mu) + \sum_{\mu \in S_2} q^{(\rho, 2\mu)}).$$

The lemma now follows from the fact that

$$\sum_{\mu \in S_2} q^{(n, 2\mu)} = \sum_{i=0}^{m-1} (q^{2i} + q^{-2i}). \Box$$

Continue the assumption that $g$, $g^\theta$ is a symmetric pair of type $DI$ case (i) and that $\pi_\Theta = \{\alpha_{n-m+1}, \alpha_{m-m+2}, \ldots, \alpha_n\}$ with $n-1 \geq m \geq 3$. It follows that $\omega_n$ is a fundamental minuscule weight in $P(\pi)$. Furthermore, $\tilde{\omega}_n = \eta_t$ ([L4, Lemma 3.2 and (3.5)]). Since $\Sigma$ is of type $B_t$, we see that $\eta_t$ is a minuscule fundamental weight with respect to the root system $\Sigma$. Let $\varphi_{2\eta_t}$ denote the element of a $W_\Theta$ invariant zonal spherical function associated to $2\eta_t$. Multiplying by a nonzero scalar if necessary, we may assume that $\varphi_{2\eta_t}(\tau(\tilde{\rho})) = 1$. Since $\eta_t$ is minuscule and $\varphi_{2\eta_t}$ is a $W_\Theta$ invariant element of $\mathcal{C}[P(2\Sigma)]$, it follows that

$$\varphi_{2\eta_t} = \frac{\sum_{\mu \in W_\Theta \eta_t} z_{2\mu}}{\sum_{\mu \in W_\Theta \eta_t} q^{(\tilde{\rho}, 2\mu)}}. \quad (4.12)$$

Set $c_{2\omega_1} = \mathcal{L}(\tau(2\omega_1))$ and recall that $c_{2\omega_1}$ is the unique element in $\hat{U}^B$ which is also contained in $\tau(2\omega_1) + (ad_r B_+ \tau(2\omega_1))$. In the next lemma, we use this zonal spherical function to distinguish between the $B$ invariant element $c_{2\omega_1}$ and the central element $z_{2\omega_1}$.
**Lemma 4.8** Suppose that \( g, g^\theta \) is of type DI case (i). Assume further that \( \pi = \{\alpha_{n-m+1}, \alpha_{m-m+2}, \ldots, \alpha_n\} \) with \( n - 1 \geq m \geq 3 \). Then \( \mathcal{P}_B(c_2\omega_1) \neq \mathcal{P}_B(z_2\omega_1) \).

**Proof:** By (1.2) we have
\[
\varphi_{2\eta_t}(\tau(2\omega_1)\tau(\rho)) = z^{2m}(\mathcal{P}_B(c_2\omega_1)).
\]
Recall that \( \Sigma \) is a root system of type \( B_t \) with \( t = n - m \) and set of positive simple roots given by (4.9) and (4.10). By [H, Section 13.2, Table 1], we have
\[
\bar{\omega}_n = \eta_t = 1/2(\mu_1 + 2\mu_2 + \cdots + (n-m)\mu_{n-m}). \tag{4.13}
\]
Furthermore, one checks that \( \sum_{\mu \in W_\Theta n \eta_{m-n}} z^{2\mu} \) is equal to the product
\[
\prod_{m \leq s \leq n-1} (z^{(\mu_{n-s} + \mu_{n-s+1} + \cdots + \mu_{n-m})} - z^{-(\mu_{n-s} + \mu_{n-s+1} + \cdots + \mu_{n-m})}). \tag{4.14}
\]
Indeed, writing (4.14) as a sum of elements of the form \( z^\beta \), we see that (4.14) is an element of the set
\[
z^{2\eta_t} + \sum_{\gamma \in Q^+(\Sigma)} N z^{2\eta_t - 2\gamma}.
\]
On the other hand, it is not hard to see that (4.14) is \( W_\Theta \) invariant. Since \( \eta_t \) is minuscule fundamental weight, the desired equality follows.

By (4.9) and (4.10) we have \( (\bar{\rho}, \mu_i) = (\rho, \mu_i) = m \) while \( (\bar{\rho}, \alpha_i) = (\rho, \alpha_i) = 1 \) for \( 1 \leq i \leq n - m - 1 \). It follows from (4.12) and the previous paragraph that
\[
\varphi_{2\eta_t} = a^{-1} \prod_{m \leq s \leq n-1} (z^{(\mu_{n-s} + \mu_{n-s+1} + \cdots + \mu_{n-m})} - z^{-(\mu_{n-s} + \mu_{n-s+1} + \cdots + \mu_{n-m})})
\]
where
\[
a = \prod_{m \leq s \leq n-1} (q^s + q^{-s}).
\]
Now (4.9) and (4.10) and the fact that \( \pi \) generates a root system of type \( D_n \) ensure that \( (\omega_1, \mu_i) = \delta_{i1}(\omega_1, \alpha_1) = \delta_{ij} \). Hence
\[
\varphi_{2\eta_t}(\tau(2\omega_1 + \bar{\rho})) = a^{-1}(q^{n+1} + q^{-n-1}) \prod_{m \leq s \leq n-2} (q^s + q^{-s})
\]
\[
= (q^{n+1} + q^{-n-1})(q^{n-1} - q^{n+1})^{-1}.
\]
Furthermore, it is straightforward to check using (4.11) that the $W_\Theta$ orbit of $\eta_1$ is the set \( \{ \pm(\mu_{n-s} + \mu_{n-s+1} + \cdots + \mu_{n-m}) \mid m \leq s \leq n-1 \} \). By Lemma 4.7, \( b\mathcal{P}_B(z_{2\omega_1}) - \sum_{i=0}^{m-1} (q^{m_i} + q^{-m_i}) \) equals
\[
\sum_{m \leq s \leq n-1} q^{2s} \tau(2(\mu_{n-s} + \cdots + \mu_{n-m})) + q^{-2s} \tau(-2(\mu_i + \cdots + \mu_{n-m}))
\]
where \( b = (\sum_{m \leq s \leq n-1} q^{2s} + q^{-2s}) + \sum_{i=0}^{m-1} (q^{2i} + q^{-2i}) \). Hence using (4.13) we have
\[
z^{2n} (\mathcal{P}_B(z_{2\omega_1})) = b^{-1}(q^{2n} + q^{-2n} - (q^{2(n-1)} + q^{-2(n-1)}) + b).
\]
Thus \( \mathcal{P}_B(z_{2\omega_1}) = \mathcal{P}_B(c_{2\omega_1}) \) implies that
\[
(q^{n+1} + q^{-n-1})b = (q^{2n} + q^{-2n} - (q^{2(n-1)} + q^{-2(n-1)}) + b)(q^{n-1} + q^{-n+1}). \quad (4.15)
\]
Note that the coefficient of $q^{n-1}$ is 0 in the left hand side of (4.15). On the other hand, the coefficient of $q^{n-1}$ in the right hand side of (4.15) is 2. This contradiction proves the lemma. \( \square \)

For Cases (iii) and (iv) below, it is convenient to use different notation for the fundamental restricted weights in \( P^+(\Sigma) \). Given \( \alpha_i \notin \pi_\Theta \), let \( \omega'_i \) denote the fundamental weight associated to the simple restricted root \( \tilde{\alpha}_i \).

**Case (iii):** We now consider the three exceptional types of symmetric pairs, EIV, EVII, and EIX. In each case, we have that the set of fundamental weights in \( P^+(\Sigma) \) not contained in the image of \( P^+(\pi) \) under \( \tau \) is precisely the set \( \{ \omega'_1, \omega'_6 \} \). Furthermore, one checks that \( \omega'_6 \) is the unique nonzero element of \( P^+(\Sigma) \) strictly less than \( \tilde{\omega}_1 = 2\omega'_1 \) while \( \omega'_1 \) is the unique nonzero element of \( P^+(\Sigma) \) strictly less than \( \tilde{\omega}_n \). Here \( \tilde{\omega}_n = \omega'_n \) in the latter two cases (i.e. \( n = 7 \) or \( n = 8 \)) while \( \tilde{\omega}_n = 2\omega'_6 \) when \( g, g^\theta \) is of type EIV.

Theorem 4.1 follows from Lemma 4.3 and the next lemma for Case (iii) symmetric pairs.

**Lemma 4.9** Assume that \( g, g^\theta \) is of type EIV, EVII, or EIX. Then there exists \( f \in (\text{ad}_r U)\tau(2\omega_1) \) such that
\[
f \in \tau(2\omega'_6) + (B\tilde{T}_\Theta)_+ \tilde{U}
\]
and \( g \in (\text{ad}_r U)\tau(2\omega_n) \) such that
\[
g \in \tau(2\omega'_1) + (B\tilde{T}_\Theta)_+ \tilde{U}.
\]

32
Proof: Let \( \pi' \) be the subset of \( \pi \) equal to \( \{ \alpha_i | i > 1 \} \). Note that \( \pi' \) generates a root system of type \( D_{n-1} \). Furthermore, \( \alpha_{n-i} \) is the \((i + 1)\)th simple root in this root system with respect to the ordering of the simple roots given in [H, Chapter III]. Now \( \Theta \) restricts to an involution on the root system generated by \( \pi' \). Moreover, \( \pi' \cap \pi_\Theta = \pi_\Theta = \{ \alpha_2, \alpha_3, \alpha_4, \alpha_5 \} \). Let \( \tau \) be the semisimple Lie subalgebra of \( g \) generated by the positive and negative root vectors corresponding to the simple roots in \( \pi' \). The Lie algebra \( \tau \) has rank \( n - 1 \) and \( \tau, \tau^\theta \) is a symmetric pair of type DI, case (i). We write \( U_q(\tau) \) for the quantized enveloping algebra of \( \tau \) identified in the obvious way with a subalgebra of \( U \). Let \( \check{U}_q(\tau) \) denote the simply connected quantized enveloping algebra of \( \tau \). Let \( \Sigma' \) be the restricted root system associated to the symmetric pair \( \tau, \tau^\theta \) and let \( W_\Theta \) denote the corresponding restricted Weyl group.

Let \( \nu_n \) denote the fundamental weight associated to the simple root \( \alpha_n \) considered as an element in the root system associated to \( \pi' \). Let \( z'_{2\nu_n} \) be the unique central element of \( \check{U}_q(\tau) \) such that

\[
z'_{2\nu_n} \in \tau(2\nu_n) + (\text{ad}_\tau U_q(\tau)\tau(2\nu_n)).
\]

Similarly, Let \( c'_{2\nu_n} \) be the unique \( (U_q(\tau) \cap B) \) invariant element of \( \check{U}_q(\tau) \) such that

\[
c'_{2\nu_n} \in \tau(2\nu_n) + (\text{ad}_\tau (U_q(\tau) \cap B)\tau(2\nu_n)).
\]

Note that both \( z'_{2\nu_n} \) and \( c'_{2\nu_n} \) are elements of \( (\text{ad}_\tau U_q(\tau))\tau(2\nu_n) \).

Now \( \Sigma' \) is a root system of type \( B_{n-5} \). Moreover, if we order the roots of \( \Sigma' \) as in [H], then \( \tilde{\alpha}_n \) corresponds to the first simple root of \( \Sigma' \). By (4.11), \( \tilde{\nu}_n \) is the fundamental weight corresponding to \( \tilde{\alpha}_n \). It follows that \( \tilde{\nu}_n \) is a pseudomuscle weight in \( P(\Sigma') \). Hence, Lemma 2.2 implies that both \( \mathcal{P}_{B \cap U_q(\tau)}(z'_{2\nu_n}) \) and \( \mathcal{P}_{B \cap U_q(\tau)}(c'_{2\nu_n}) \) are linear combinations of 1 and \( \sum_{\gamma \in W_{\tilde{n}}^{-\nu_n} q(\rho, 2\gamma)} \tau(2\gamma) \).

Lemma 4.8 ensures that \( \mathcal{P}_{B \cap U_q(\tau)}(z'_{2\nu_n}) \neq \mathcal{P}_{B \cap U_q(\tau)}(c'_{2\nu_n}) \). Hence there is a linear combination \( X \) of \( z'_{2\nu_n} \) and \( c'_{2\nu_n} \) such that \( \mathcal{P}_{B \cap U_q(\tau)}(X) = 1 \). It follows that \( \mathcal{P}_{B \cap U_q(\tau)}(X - 1) = 0 \). Thus by [L4, Corollary 4.2], \( X - 1 \in (B \hat{T}_\Theta \cap \check{U}_q(\tau))_+ U \). In particular,

\[
X \in (\text{ad}_\tau U_q(\tau))\tau(2\nu_n) \quad \text{and} \quad X \in 1 + (B \hat{T}_\Theta)_+ \check{U}.
\]

(4.16)

Since \( \alpha_n \) is the first root in the root system of type \( D_{n-2} \) generated by \( \pi' \), we have

\[
\nu_n = 1/2(\alpha_2 + \alpha_3) + \alpha_4 + \ldots + \alpha_n.
\]
Note that \((\omega_n - \nu_n, \alpha_i) = 0\) for all \(\alpha_i \in \pi'\). On the other hand
\[
(\omega_n - \nu_n, \alpha_1) = (-\nu_n, \alpha_1) = -(\alpha_3, \alpha_1)/2 = (\omega_1, \alpha_1)/2.
\]
It follows that \(\omega_n = \nu_n + \omega_1/2\). Now \(\tau(\omega_1)\) commutes with elements of \(U_q(\tau)\).
Hence
\[
\text{(ad}_r U_q(\tau))\tau(2\omega_n) = ((\text{ad}_r U_q(\tau))\tau(2\nu_n))\tau(\omega_1). \tag{4.17}
\]
It follows from (4.16) and (4.17) that
\[
X\tau(\omega_1) \in (\text{ad}_r U)\tau(2\omega_n) \quad \text{and} \quad X\tau(\omega_1) \in \tau(\bar{\omega}_1) + (B\bar{T}_\Theta)_+ \bar{U}.
\]
The second assertion now follows from the fact that \(\bar{\omega}_1 = 2\omega'_1\) ([L4, Lemma 3.1]). The first assertion is proved in exactly the same way where we replace \(\pi'\) with the set \(\{\alpha_i | i < 5\}\). \(\square\)

**Case (iv):** We next turn our attention to the symmetric pair of type EIII. In this case, \(\tilde{\alpha}_1 = (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)/2\) and \(\tilde{\alpha}_2 = \alpha_2 + \alpha_4 + (\alpha_3 + \alpha_5)/2\). Moreover, both \(\tilde{\alpha}_1\) and \(2\tilde{\alpha}_1\) are elements of \(\Sigma\). In particular, the restricted root system \(\Sigma\) is nonreduced of type BC2. Now \(\Sigma\) contains a root system with set of positive simple roots \(\{2\tilde{\alpha}_1, \tilde{\alpha}_2\}\) of type B2. Here \(\tilde{\alpha}_2\) is the short simple root and \(2\tilde{\alpha}_1\) is the long simple root. As explained in [L4, Section 3], the fundamental weight \(\omega'_1\) associated to \(\tilde{\alpha}_1\) satisfies \((\omega'_1, 2\tilde{\alpha}_1) = (2\tilde{\alpha}_1, 2\tilde{\alpha}_1)/2\).

In particular, the weight lattice \(P(\Sigma)\) is the same as the weight lattice of the underlying root system of type B2. Thus \(\omega'_1 = 2\tilde{\alpha}_1 + \tilde{\alpha}_2\) and \(\omega'_2 = \tilde{\alpha}_1 + \tilde{\alpha}_2\). It is straightforward to check that \(\bar{\omega}_1 = 2\tilde{\alpha}_1 + \tilde{\alpha}_2\) and \(\bar{\omega}_2 = 2\tilde{\alpha}_1 + 2\tilde{\alpha}_2\). Note that \(\omega'_1 \in P^+(\pi)\) while \(\omega'_2\) is not an element of this set. (This last fact is also used in the proof of [L4, Theorem 2.5]).

By Lemma 4.3, we have that \(\tau(2\omega'_1) + (\tau(2\omega_1) - \tau(2\omega'_1)) \in F_r(\bar{U})\) and \(\tau(2\omega_1) - \tau(2\omega'_1) \in (B\bar{T}_\Theta)_+ \bar{U}\). Thus Theorem 4.1 for this last case follows from the next lemma.

**Lemma 4.10** Assume that \(g, g^\theta\) is of type EIII. Then there exists \(f \in F_r(\bar{U})\) such that
\[
f \in \tau(2\omega'_2) + (B\bar{T}_\Theta)_+ \bar{U}.
\]

**Proof:** Let \(\pi'\) be the subset of \(\pi\) equal to \(\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}\). Note that \(\pi'\) generates a root system of type D4. Furthermore, \(\alpha_i\) is the \((i - 1)^{th}\) simple root with respect to the ordering of the simple roots given in [H]. Now \(\Theta\)
restricts to an involution on the root system generated by $\pi'$. Note that $\pi' \cap \pi = \{\alpha_3, \alpha_4, \alpha_5\}$. Let $\nu_2$ denote the fundamental weight associated to the simple root $\alpha_2$ with respect to the root system generated by $\pi'$. We have $\nu_2 = (\alpha_3 + \alpha_5)/2 + \alpha_4 + \alpha_2$. Now $(\omega_2 - \nu_2, \alpha_i) = (\omega, \alpha_i)/2$ for $i = 1$ and $i = 6$. Also, $(\omega_2 - \nu_2, \alpha_i) = 0$ for $i \notin \{1, 6\}$. Therefore

$$\omega_2 = \nu_2 + (\omega_1 + \omega_6)/2.$$ 

By [L4, Lemma 3.1], $\bar{\omega}_1 + \bar{\omega}_6 = 2\omega'_1$. The rest of the argument follows as in the proof of Lemma 4.9. □

5 Appendix: Commonly used notation

Here is a list of notation defined in Section 1 (in the following order):

$\mathbb{C}$, $\mathbb{Q}$, $\mathbb{R}$, $q$, $\mathcal{C}$, $\mathcal{R}$, $Q(\Phi)$, $P(\Phi)$, $Q^+(\Phi)$, $P^+(\Phi)$, $g$, $n^-$, $h$, $n^+$, $\Delta$, $\pi = \{\alpha_1, \ldots, \alpha_n\}$, $(\ , \ )$, $\leq$, $\theta$, $g^\theta$, $\Theta$, $\pi_\Theta$, $\tilde{\alpha}$, $\Sigma$, $U$, $x_i$, $y_i$, $t_i^{\pm 1}$, $T$, $U^0$, $U^+$, $G^-$, $\tau$, $\bar{U}$, $\bar{U}^0$, $A_+$, $B$, $\mathcal{C}[G]$, $\mathcal{C}[H]$, $\text{ad}_r$, $\text{ad}$, $\bar{U}^B$, $\tilde{A}$, $\mathcal{T}_\Theta$, $\mathcal{M}$, $N^+$, $\mathcal{P}_B$, $\mathcal{A}$, $*$, $\varphi_\lambda$, $\mathcal{X}$, $\mathcal{C}(Q(\Sigma))\mathcal{A}$.

Defined in Section 2:

$Z(\bar{U})$ the center of $\bar{U}$

$\rho$ half sum of positive roots in $\Delta$

$w, q^{(\rho, \lambda)}\tau(\lambda)$

$\hat{m}(2\eta)$ highest degree term w.r.t. “deg” degree function

$\mathcal{C}[Q(\Sigma)]$ power series ring in the $z^{-\gamma}$ for $\gamma \in Q^+(\Sigma)$

$\mathcal{N}_\eta,$ see (2.3)

$\mathcal{N}^+_\eta$ see (2.3)

$w'_o$ the longest element of $W_\Theta$

$z_{2\mu}$ unique central element in $\tau(2\mu) + (\text{ad}_r + U_+)\tau(2\mu)$

$\text{top}$ highest degree term w.r.t. “odeg” degree function

Defined in Section 3:

$\phi$ Hopf algebra automorphism of $U$ (see (3.1))

$T_\Theta \{\tau(\beta)|\Theta(\beta) = \beta$ and $\beta \in Q(\pi)\}$

$y_i t_i + d_i \tilde{\theta}(y_i) t_i + s_i t_i$ one of the generators of $B$
lift of the involution \( \theta \) to \( U \)

the antipode of \( U \)

the coproduct of \( U \)

see definition preceding Lemma 3.2

algebra generated by \( (\text{ad} \ M \cap G^-)[y_i t_i, \alpha_i \notin \pi_\Theta] \)

projection onto \( C[A] \) defined using (3.2)

antiautomorphism of \( U \) preserving \( B \)

highest weight simple \( U \) module of highest weight \( \lambda \)

locally finite part of \( \bar{U} \) w.r.t \( \text{ad}_r \)

see Definition 3.4

zonal spherical function at \( \lambda \) associated to \( \chi^{-1}(B), B \)

nonzero \( B \) invariant vector of \( L(\lambda) \)

nonzero \( \chi^{-1}(B) \) invariant vector of \( L(\lambda)^* \)

\[
\begin{align*}
\sum_{\gamma \in Q^+(\pi)} \gamma \tau(-\gamma) \\
\sum_{\gamma \in Q^-(\pi)} \gamma \tau(-\gamma)
\end{align*}
\]

\( \{\tau(-\gamma) | \gamma \in Q^+(\pi) \text{ and } \tilde{\gamma} \in P(2\Sigma) \} \)

\( \{\tau(-\gamma) | \gamma \in Q^+(\Sigma) \cap P(2\Sigma) \} \)

rank of the restricted root system \( \Sigma \)

simple roots in \( \Sigma \)

fundamental weights in \( P^+(\Sigma) \)

\( \{\alpha_j | 1 \leq j \leq 2i + 1 \} \)

semisimple Lie subalgebra of \( g \) with simple roots \( \pi_i \)

subalgebra of \( U \) equal to \( U_q(g_i) \)

algebra generated by \( y_i \), \( 1 \leq i \leq n \)

algebra generated by \( x_i t_i^{-1}, 1 \leq i \leq n \)

\( \{\tau(-\gamma) | \gamma \in Q^+(\Sigma) \cap P(2\Sigma) \} \)

Defined in Section 4:

\( t \)

\( \mu_1, \ldots, \mu_t \)

\( \eta_1, \ldots, \eta_t \)

\( \pi_i \)

\( g_i \)

\( U_i \)

\( U_0 \)

\( U_t \)

\( \bar{U}_i \)

\( \bar{U}^i \)

\( \omega_j \)

\( \mathcal{P} \)

\( W \)

\( c_{2\omega_1} \)
\[ \omega_i' \] restricted fundamental weight associated to \( \tilde{\alpha}_i \)

REFERENCES

[A] S. Araki, On root systems and an infinitesimal classification of irreducible symmetric spaces, *Journal of Mathematics, Osaka City University* **13** (1962), no. 1, 1-34.

[B] P. Baumann, On the center of quantized enveloping algebras, *Journal of Algebra* **203** (1998), 244-260.

[DN] M.S. Dijkhuizen and M. Noumi, A family of quantum projective spaces and related \( q \)-hypergeometric orthogonal polynomials, *Transactions of the A.M.S.* **350** (1998), no. 8, 3269-3296.

[DS] M.S. Dikhuizen and J.V. Stokman, Some limit transitions between BC type orthogonal polynomials interpreted on quantum complex Grassmannians, *Publ. Res. Inst. Math. Sci.* **35** (1999), 451-500.

[H] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York (1972).

[HC] Harish-Chandra, Spherical functions on a semisimple Lie group, I, *American Journal of Mathematics* **80** (1958) 241-310.

[JL] A. Joseph and G. Letzter, Local finiteness of the adjoint action for quantized enveloping algebras, *Journal of Algebra* **153** (1992), 289-318.

[Jo] A. Joseph, *Quantum Groups and Their Primitive Ideals*, Springer-Verlag, New York (1995).

[KS] E. Koelink, J. Stokman, Fourier transforms on the quantum SU(1,1) group. With an appendix by Mizan Rahman *Publ. Res. Inst. Math. Sci.* **37** (2001), no. 4, 621-715.

[K] A.A. Kirillov, Jr., Lectures on affine Hecke algebras and Macdonald’s conjectures, *Bulletin of the American Mathematical Society* **34** (1997), No. 3, 251-292.

[L1] G. Letzter, Coideal subalgebras and quantum symmetric pairs, In: *New Directions in Hopf Algebras, MSRI publications* **43**, Cambridge University Press (2002), 117-166.
[L2] G. Letzter, Quantum symmetric pairs and their zonal spherical functions, *Transformation Groups* 8 (2003), no. 3, 261-292.

[L3] G. Letzter, Quantum zonal spherical functions and Macdonald polynomials, Advances in Mathematics, in press (corrected proof available online).

[L4] G. Letzter, Invariant differential operators for quantum symmetric spaces, I, preprint (arXiv:math.QA/0406193).

[M] I.G. Macdonald, Orthogonal polynomials associated with root systems, *Séminaire Lotharingien de Combinatoire* 45 (2000/01) 40 pp.

[N] M. Noumi, Macdonald’s symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces, *Advances in Mathematics* 123 (1996), no. 1, 16-77.

[NDS] M. Noumi, M.S. Dijkhuizen, and T. Sugitani, Multivariable Askey-Wilson polynomials and quantum complex Grassmannians, *Fields Institute Communications* 14 (1997), 167-177.

[NS] M. Noumi and T. Sugitani, Quantum symmetric spaces and related q-orthogonal polynomials, in: *Group Theoretical Methods in Physics (ICGTMP)* (Toyonaka, Japan, 1994), World Science Publishing, River Edge, New Jersey (1995), 28-40.

[S] T. Sugitani, Zonal spherical functions on quantum Grassmann manifolds, *J. Math. Sci. Univ. Tokyo* 6 (1999), no. 2, 335-369.