Permanent index of matrices associated with graphs

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Abstract

A total weighting of a graph $G$ is a mapping $f$ which assigns to each element $z \in V(G) \cup E(G)$ a real number $f(z)$ as its weight. The vertex sum of $v$ with respect to $f$ is $\phi_f(v) = \sum_{e \in E(v)} f(e) + f(v)$. A total weighting is proper if $\phi_f(u) \neq \phi_f(v)$ for any edge $uv$ of $G$. A $(k,k')$-list assignment is a mapping $L$ which assigns to each vertex $v$ a set $L(v)$ of $k$ permissible weights, and assigns to each edge $e$ a set $L(e)$ of $k'$ permissible weights. We say $G$ is $(k,k')$-choosable if for any $(k,k')$-list assignment $L$, there is a proper total weighting $f$ of $G$ with $f(z) \in L(z)$ for each $z \in V(G) \cup E(G)$. It was conjectured in [T. Wong and X. Zhu, Total weight choosability of graphs, J. Graph Theory 66 (2011), 198-212] that every graph is $(2,2)$-choosable and every graph with no isolated edge is $(1,3)$-choosable. A promising tool in the study of these conjectures is Combinatorial Nullstellensatz. This approach leads to conjectures on the permanent indices of matrices $A_G$ and $B_G$ associated to a graph $G$. In this paper, we establish a method that reduces the study of permanent of matrices associated to a graph $G$ to the study of permanent of matrices associated to induced subgraphs of $G$. Using this reduction method, we show that if $G$ is a subcubic graph, or a 2-tree, or a Halin graph, or a grid, then $A_G$ has permanent index 1. As a consequence, these graphs are $(2,2)$-choosable.

Key words: Permanent index, matrix, total weighting

1 Introduction

A total weighting of a graph $G$ is a mapping $f$ which assigns to each element $z \in V(G) \cup E(G)$ a real number $f(z)$ as its weight. Given a total weighting $f$ of $G$, for a vertex $v$ of $G$, the...
vertex sum of \( v \) with respect to \( f \) is defined as \( \phi_f(v) = \sum_{e \in E(v)} f(e) + f(v) \). A total weighting is proper if \( \phi_f \) is a proper colouring of \( G \), i.e., for any edge \( uv \) of \( G \), \( \phi_f(u) \neq \phi_f(v) \). A total weighting \( \phi \) with \( \phi(v) = 0 \) for all vertices \( v \) is also called an edge weighting. A proper edge weighting \( \phi \) with \( \phi(e) \in \{1, 2, \ldots, k\} \) for all edges \( e \) is called a vertex colouring \( k \)-edge weighting of \( G \). Kalkowski, Łuczak and Thomason [7] first studied edge weighting of graphs. They conjectured that every graph with no isolated edges has a vertex colouring 3-edge weighting. This conjecture received considerable attention, and is called the 1-2-3 conjecture. Addario-Berry, Dalal, McDiarmid, Reed and Thomason [2] proved that every graph \( G \) is \((2,3)\)-choosable. They defined \( \tau(G) \) for \( G \) to be the evaluation of the polynomial at \( \phi \). Let \( \phi(v) = k \) for \( v \in V(G) \) and \( \phi(e) = k' \) for \( e \in E(G) \).

The list version of edge weighting of graphs was introduced by Bartnicki, Grytczuk and Niwczyk in [6], and the list version of total weighting of graphs was introduced independently by Wong and Zhu in [13] and by Przybyło and Woźniak [12]. Suppose \( \psi : V(G) \cup E(G) \to \{1, 2, \ldots, k\} \) is a mapping which assigns to each vertex and each edge of \( G \) a positive integer. A \( \psi \)-list assignment of \( G \) is a mapping \( L \) which assigns to \( z \in V(G) \cup E(G) \) a set \( L(z) \) of \( \psi(z) \) real numbers. Given a total list assignment \( L \), a proper \( L \)-total weighting is a proper total weighting \( \phi \) with \( \phi(z) \in L(z) \) for all \( z \in V(G) \cup E(G) \). We say \( G \) is total weight \( \psi \)-choosable if for any \( \psi \)-list assignment \( L \), there is a proper \( L \)-total weighting of \( G \). We say \( G \) is \((k,k')\)-choosable if \( G \) is \( \psi \)-total weight choosable, where \( \psi(v) = k \) for \( v \in V(G) \) and \( \psi(e) = k' \) for \( e \in E(G) \).

As strengthening of the 1-2-3 conjecture and the 1-2 conjecture, it was conjectured in [13] that every graph with no isolated edges is \((1,3)\)-choosable and every graph is \((2,2)\)-choosable. Thes two conjectures received a lot of attention and are verified for some special classes of graphs. In particular, it was shown in [14] that every graph is \((2,3)\)-choosable. A promising tool in the study of these conjectures is Combinatorial Nullstellensatz. For each \( z \in V(G) \cup E(G) \), let \( x_z \) be a variable associated to \( z \). Fix an orientation \( D \) of \( G \). Consider the polynomial

\[
P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{e=uv \in E(D)} \left( \left( \sum_{e \in E(v)} x_e + x_v \right) - \left( \sum_{e \in E(u)} x_e + x_u \right) \right).
\]

Assign a real number \( \phi(z) \) to the variable \( x_z \), and view \( \phi(z) \) as the weight of \( z \). Let \( P_G(\phi) \) be the evaluation of the polynomial at \( x_z = \phi(z) \). Then \( \phi \) is a proper total weighting of \( G \) if and only if \( P_G(\phi) \neq 0 \). Note that \( P_G \) has degree \( |E(G)| \).
An index function of \( G \) is a mapping \( \eta \) which assigns to each vertex or edge \( z \) of \( G \) a non-negative integer \( \eta(z) \) and an index function \( \eta \) is valid if \( \sum_{z \in V(G) \cup E(G)} \eta(z) = |E(G)| \).

For a valid index function \( \eta \), let \( c_\eta \) be the coefficient of the monomial \( \prod_{z \in V \cup E} x_\eta(z) \) in the expansion of \( P_G \). It follows from Combinatorial Nullstellensatz [3, 5] that if \( c_\eta \neq 0 \), and \( L \) is a list assignment which assigns to each \( z \in V(G) \cup E(G) \) a set \( L(z) \) of \( \eta(z) + 1 \) real numbers, then there exists a mapping \( \phi \) with \( \phi(z) \in L(z) \) such that
\[
P_G(\phi) \neq 0.
\]

So to prove a graph \( G \) is \((k, k')\)-choosable, it suffices to show that there is a valid index function \( \eta \) with \( \eta(v) \leq k - 1 \) for \( v \in V(G) \), \( \eta(e) \leq k' - 1 \) for \( e \in E(G) \) and \( c_\eta \neq 0 \).

We write the polynomial \( P_G(\{x_z : z \in V(G) \cup E(G)\}) \) as
\[
P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{e \in E(D)} \sum_{z \in V(G) \cup E(G)} A_G[e, z] x_z.
\]

It is straightforward to verify that for \( e \in E(G) \) and \( z \in V(G) \cup E(G) \), if \( e = (u, v) \) (oriented from \( u \) to \( v \)), then
\[
A_G[e, z] = \begin{cases} 
1 & \text{if } z = v, \text{ or } z \neq e \text{ is an edge incident to } v, \\
-1 & \text{if } z = u, \text{ or } z \neq e \text{ is an edge incident to } u, \\
0 & \text{otherwise.}
\end{cases}
\]

Now \( A_G \) is a matrix, whose rows are indexed by the edges of \( G \) and the columns are indexed by edges and vertices of \( G \). Let \( B_G \) be the submatrix of \( A_G \) consisting of those columns of \( A_G \) indexed by edges. It turns out that \((k, k')\)-choosability of a graph \( G \) is related to the permanent indices of \( A_G \) and \( B_G \).

For an \( m \times m \) matrix \( A \) (whose entries are reals), the permanent of \( A \) is defined as
\[
\text{per}(A) = \sum_{\sigma \in S_m} \prod_{i=1}^{m} A[i, \sigma(i)]
\]
where \( S_m \) is the symmetric group of order \( m \), i.e., the summation is taken over all the permutations \( \sigma \) over \( \{1, 2, \ldots, m\} \). The permanent index of a matrix \( A \), denoted by \( \text{pind}(A) \), is the minimum integer \( k \) such that there is a matrix \( A' \) such that \( \text{per}(A') \neq 0 \), each column of \( A' \) is a column of \( A \) and each column of \( A \) occurs in \( A' \) at most \( k \) times (if such an integer \( k \) does not exist, then \( \text{pind}(A) = \infty \)).

Consider the matrix \( A_G \) defined above. Given a vertex or edge \( z \) of \( G \), let \( A_G(z) \) be the column of \( A_G \) indexed by \( z \). For an index function \( \eta \) of \( G \), let \( A_G(\eta) \) be the matrix, each of its column is a column of \( A_G \), and each column \( A_G(z) \) of \( A_G \) occurs \( \eta(z) \) times as a column of \( A_G(\eta) \). It is known [13, 13] and easy to verify that for a valid index function \( \eta \) of \( G \), \( c_\eta \neq 0 \) if and only if \( \text{per}(A_G(\eta)) \neq 0 \). Thus if \( \text{pind}(A_G) = 1 \), then \( G \) is \((2, 2)\)-choosable; if \( \text{pind}(B_G) \leq 2 \), then \( G \) is \((1, 3)\)-choosable. The following two conjectures are proposed in [13]:
Conjecture 1 [6] For any graph $G$ with no isolated edges, $\text{pind}(B_G) \leq 2$.

Conjecture 2 [13] For any graph $G$, $\text{pind}(A_G) = 1$.

The discussion above shows that Conjecture 1 implies that any graph without isolated edges is $(1,3)$-choosable, and Conjecture 2 implies that every graph is $(2,2)$-choosable.

We say an index function $\eta$ is non-singular if there is a valid index function $\eta' \leq \eta$ with $\text{per}(A_G(\eta')) \neq 0$. In this paper, we are interested in non-singularity of index functions $\eta$ for which $\eta(e) = 1$ for every edge $e$ and $\eta(v)$ can be any non-negative integers for any every vertex $v$. Assume $\eta$ is such an index function of $G$. We delete a vertex $v$, and construct an index function $\eta'$ for $G - v$ from the restriction of $\eta$ to $G - v$ by doing the following modification: $\eta(v)$ of the neighbours $u$ of $v$ have $\eta'(u) = \eta(u) + 1$, and all the other neighbours $u$ of $v$ (if any) have $\eta'(u) = \eta(u) - 1$. We prove that if $\eta'$ is a non-singular index function of $G - v$, then $\eta$ is a non-singular index function of $G$. Applying this reduction method, we prove that Conjecture 2 holds for subcubic graphs, 2-trees, Halin graphs and grids. Consequently, subcubic graphs, 2-trees, Halin graphs and grids are $(2,2)$-choosable.

2 Reduction to induced subgraphs

To study non-singularity of index functions of $G$, we shall consider matrices whose columns are linear combinations of columns of $A_G$. Assume $A$ is a square matrix whose columns are linear combinations of columns of $A_G$. Define an index function $\eta_A : V(G) \cup E(G) \to \{0,1,\ldots\}$ as follows:

For $z \in V(G) \cup E(G)$, $\eta_A(z)$ is the number of columns of $A$ in which $A_G(z)$ appears with nonzero coefficient.

It is known [13] that columns of $A_G$ are not linearly independent. In particular, if $e = uv$ is an edge of $G$, then

$$A_G(e) = A_G(u) + A_G(v)$$

Thus a column of $A$ may have different ways to be expressed as linear combinations of columns of $A_G$. So the index function $\eta_A$ is not uniquely determined by $A$. Instead, it is determined by the way we choose to express the columns of $A$ as linear combinations of columns of $A_G$. For simplicity, we use the notation $\eta_A$, however, whenever the function $\eta_A$ is used, an explicit expression of the columns of $A$ as linear combinations of columns of $A_G$ is given, and we refer to that specific expression.

It is well-known (and follows easily from the definition) that the permanent of a matrix is multi-linear on its column vectors and row vectors: If a column $C$ of $A$ is a linear combination of two columns vectors $C = \alpha C' + \beta C''$, and $A'$ (respectively, $A''$) is obtained from $A$ by replacing the column $C$ with $C'$ (respectively, with $C''$), then

$$\text{per}(A) = \alpha \text{ per}(A') + \beta \text{ per}(A'').$$
By using (2) repeatedly, one can find matrices $A_1, A_2, \ldots, A_q$ and real numbers $a_1, a_2, \ldots, a_q$ such that

$$\operatorname{per}(A) = \sum_{j=1}^{q} a_j \operatorname{per}(A_j)$$

where each $A_j$ is a square matrix consisting of columns of $A_G$, with each column $A_G(z)$ appears at most $\eta(z)$ times. Thus if $\operatorname{per}(A) \neq 0$, then one of the $\operatorname{per}(A_j) \neq 0$. Thus if $\operatorname{per}(A) \neq 0$, then $\eta_A$ is a non-singular index function of $G$.

**Theorem 1** Suppose $G$ is a graph, $\eta$ is an index function of $G$ for which $\eta(e) = 1$ for every edge $e$. Let $v$ be a vertex of $G$. Let $\eta'$ be obtained from the restriction of $\eta$ to $G - v$ by the following modification: Choose $d_G(v) - \eta(v)$ neighbours $u$ of $v$ with $\eta(u) \geq 1$, and let $\eta'(u) = \eta(u) - 1$. For the other $\eta(v)$ neighbours $u$ of $v$, let $\eta'(u) = \eta(u) + 1$. If $\eta'$ is a non-singular index function of $G - v$, then $\eta$ is a non-singular index function of $G$.

Theorem 1 follows from the following more general statement.

**Theorem 2** Suppose $G$ is a graph, $v$ is a vertex of $G$ and $E(v) = \{e_1, e_2, \ldots, e_k\}$, with $e_i = \overrightarrow{v_i}$ for $i = 1, 2, \ldots, k$. Assume $\eta$ is an index function of $G$. Here $\eta(e)$ can be any non-negative integer. Choose a subset $J$ of $\{1, 2, \ldots, k\}$ and integers $1 \leq k_i \leq \min\{\eta(e_i), \eta(v_i)\}$ such that $\eta(v) + \sum_{i \in J} k_i = k$. Let $\eta'$ be the index function of $G' = G - v$ which is equal to the restriction of $\eta$ to $G - v$, except that

1. For $i \in J$, $\eta'(v_i) = \eta(v_i) - k_i$.
2. For $i \in \{1, 2, \ldots, k\} \setminus J$, $\eta'(v_i) = \eta(v_i) + \eta(e_i)$.

If $\eta'$ is a non-singular index function for $G'$, then $\eta$ is a non-singular index function for $G$.

**Proof.** Assume $\eta'$ is non-singular. Let $\eta'' \leq \eta'$ be a valid index function with $\operatorname{per}(A_{G'}(\eta'')) \neq 0$.

Assume $|E(G)| = m$ and $|E(G')| = m' = m - k$. By viewing each vertex and each edge of $G'$ as a vertex and an edge of $G$, $A_G(\eta'')$ is an $m \times m'$ matrix, consisting of $m'$ columns of $A_G$. First we extend $A_G(\eta'')$ into an $m \times m$ matrix $A$ by adding $k$ copies of the column $A_G(v)$. The added $k$ columns has $k$ rows (the rows indexed by edges incident to $v$) that are all 1’s (with all these edges oriented towards $v$), and all the other entries of these $k$ columns are 0. Therefore $\operatorname{per}(M) = \operatorname{per}(A_G(\eta''))k!$, and hence $\operatorname{per}(M) \neq 0$.

Starting from the matrix $M$, for each $i \in \{1, 2, \ldots, k\} \setminus J$, remove $\min\{\eta(e_i), \eta''(v_i)\}$ copies of the column $A_G(v_i)$ and add $\min\{\eta(e_i), \eta''(v_i)\}$ copies of the column $A_G(e_i)$. Denote by $M'$ the resulting matrix.
Claim 1  For the matrix $M'$ constructed above, we have $\text{per}(M') = \text{per}(M)$.

Proof. Since by (1), $A_G(e_i) = A_G(v_i) + A_G(v)$, we re-write $\min\{\eta(e_i), \eta''(v_i)\}$ copies of the column $A_G(e_i)$ of $M'$ as $A_G(v) + A_G(v_i)$. Then we expand the permanent using its multilinear property (i.e. using (2) repeatedly), to obtain the following equation:

$$\text{per}(M') = \text{per}(M) + \sum_{M''} \text{per}(M'')$$

where $M''$ are those matrices which contain at least $k+1$ copies of the column $A_G(v)$. Since these $k+1$ columns has all 1’s in $k$ rows and 0 in all other entries, we have $\text{per}(M'') = 0$ for all $M''$, and so $\text{per}(M') = \text{per}(M)$.

For each $i \in J$, write $k_i$ copies of $A_G(v_i)$ in $M'$ as $A_G(e_i) - A_G(v_i)$. Note that this step does not change the matrix, since $A_G(v) = A_G(e_i) - A_G(v_i)$ (by (1)). Now each column of $M'$ is a linear combination of columns of $A_G$.

We shall show that, with the linear combination of columns of $M'$ given in the above paragraph, $\eta_{M'}(z) \leq \eta(z)$ for $z \in V(G) \cup E(G)$.

If $z \notin \{e_i, v_i : i = 1, 2, \ldots, k\} \cup \{v\}$, $\eta_{M'}(z) = \eta_M(z) \leq \eta''(z) \leq \eta'(z) = \eta(z)$. If $i \in \{1, 2, \ldots, k\} - J$, then $\eta_{M'}(e_i) = \min\{\eta(e_i), \eta''(v_i)\} \leq \eta(e_i)$, and $\eta_{M'}(v_i) = \eta_M(v_i) - \min\{\eta(e_i), \eta''(v_i)\} \leq \eta'(v_i) = \eta(v_i)$. If $i \in J$, then $\eta_{M'}(e_i) = k_i \leq \eta(e_i)$ and $\eta_{M'}(v_i) = \eta''(v_i) + k_i \leq \eta'(v_i) + k_i = \eta(v_i)$. Finally, $\eta_{M'}(v) = k - \sum_{i \in J} k_i = \eta(v)$. As $\text{per}(M') \neq 0$, we conclude that $\eta$ is a non-singular index function for $G$. This completes the proof of Theorem 2.

Theorem 1 follows from Theorem 2 by choosing $k_i = 1$ and $|J| = d(v) - \eta(v)$. By definition, if $\eta''$ is non-singular and $\eta' \geq \eta''$, then $\eta'$ is also non-singular. So the following is equivalent to Theorem 1.

Theorem 3 Suppose $G$ is a graph, $\eta$ is an index function of $G$ for which $\eta(e) = 1$ for every edge $e$. Let $v$ be a vertex of $G$. Let $\eta'$ be obtained from the restriction of $\eta$ to $G - v$ by the following modification: Choose at least $d_G(v) - \eta(v)$ neighbours of $v$ with $\eta(u) \geq 1$, and let $\eta'(u) = \eta(u) - 1$. For the other neighbours $u$ of $v$, let $\eta'(u) = \eta(u) + 1$. If $\eta'$ is a non-singular index function of $G - v$, then $\eta$ is a non-singular index function of $G$.

We shall apply Theorem 3 repeatedly and delete a sequence of vertices in order. We need to record which vertices are deleted, and when a vertex is deleted, for which neighbours $u$ we have $\eta'(u) = \eta(u) + 1$. For this purpose, instead of really removing the deleted vertices, we indicate the deletion of $v$ by orient all the edges incident to $v$ from $v$ to its neighbours, and then choose a subset of these oriented edges (to indicate those neighbours $u$ for which $\eta'(u) = \eta(u) + 1$).

The index function $\eta$ is changing in the process of the deletion. For convenience, we denote by $\eta_i$ the index function after the deletion of the $i$th vertex. In particular, $\eta_0 = \eta$. 

Assume a vertex \( v \) is deleted in the \( i \)th step, for each neighbour \( u \) of \( v \) (at the time \( v \) is deleted), orient the edge as an arc from \( v \) to \( u \). After a sequence of vertices are deleted, we obtain a digraph \( D \) formed by edges incident to the “deleted” vertices. Let \( D' \) be the sub-digraph of \( @D \) formed by those arcs \((v,u)\) with \( u \) be the neighbour of \( v \) (at the time \( v \) is deleted) and for which we have \( \eta'(u) = \eta(u) + 1 \).

If \( u \) is deleted in the \( i \)th step, then \( d^+_D(u) \leq \eta_{i-1}(u) \). After the \( i \)th step, all edges incident to \( u \) are oriented. On the other hand, \( d^-_{D'}(u) \) is the number of indices \( j < i \) for which \( \eta_j(u) = \eta_{j-1}(u) + 1 \), and \( d^-_{D'}(u) - d^-_{D}(u) \) is the number of indices \( j < i \) for which \( \eta_j(u) = \eta_{j-1}(u) - 1 \). Thus \( d^+_D(u) \leq \eta(u) + d^-_{D'}(u) - (d^-_{D}(u) - d^-_{D'}(u)) \).

If after the \( i \)th step, \( u \) is not deleted, then \( d^+_D(u) = 0 \) and \( \eta_i(u) = \eta(u) + d^-_{D'}(u) - (d^-_{D}(u) - d^-_{D'}(u)) \geq 0 \).

The following corollary summarize the final effect of the repeated application of Theorem

Corollary 1 Suppose \( G \) is a graph, \( \eta \) is an index function of \( G \) with \( \eta(e) = 1 \) for all edges \( e \), and \( X \) is a subset of \( V(G) \). Let \( G' = G - E[X] \) be obtained from \( G \) by deleting edges in \( G[X] \). Let \( D \) be an acyclic orientation of \( G' \), in which each vertex \( v \in X \) is a sink. Assume \( D' \) is a sub-digraph of \( D \) such that for all \( v \in V(D) \),

\[
\eta(v) + 2d^-_{D'}(v) - d^-_{D}(v) \geq d^+_{D'}(v),
\]

(*)

Let \( \eta' \) be the index function defined as \( \eta'(e) = 1 \) for every edge \( e \) of \( G[X] \) and \( \eta'(v) = \eta(v) + 2d^-_{D'}(v) - d^-_{D}(v) \) for \( v \in X \). If \( \eta' \) is a non-singular index function for \( G[X] \), then \( \eta \) is a non-singular index function for \( G \).

Proof. Assume \( \eta' \) is non-singular for \( G[X] \). We shall prove that \( \eta \) is non-singular for \( G \). We prove this by induction on \( |V - X| \). If \( V - X = \emptyset \), then \( \eta = \eta' \) and there is nothing to prove.

Assume \( V - X \neq \emptyset \). Since the orientation \( D \) is acyclic, there is a source vertex \( v \notin X \). Let \( e_1, e_2, \ldots, e_k \) be the set of edges incident to \( v \) and \( e_i = vv_i \).

Consider the graph \( G - v \). Let \( \eta'' \) be the index function on \( G - v \) defined as \( \eta'' = \eta \) on \( G - v \), except that for \( i = 1, 2, \ldots, k \), if \( e_i \notin D' \), then \( \eta''(v_i) = \eta(v_i) - 1 \), and if \( e_i \in D' \), then \( \eta''(v_i) = \eta(v_i) + 1 \).

Let \( H = D - v \) and \( H' = D' - v \). We shall show that

\[
\eta''(u) + 2d^-_{H'}(u) - d^-_{H}(u) \geq d^+_{H'}(u) \text{ for all } u \in V(H) \]

(**)

If \( u \notin \{v_1, v_2, \ldots, v_k\} \), then (**) is the same as (*). If \( u = v_i \) and \( e_i \in D' \), then \( \eta''(v_i) = \eta(v_i) + 1, d^-_{H'}(v_i) = d^-_{D'}(v_i) - 1, d^-_{H}(v_i) = d^-_{D}(v_i) - 1 \) and \( d^+_{H'}(v_i) = d^+_{D'}(v_i) \). So (**) follows from (*). If \( u = v_i \) and \( e_i \notin D' \), then \( \eta''(v_i) = \eta(v_i) - 1, d^-_{H'}(v_i) = d^-_{D'}(v_i), d^-_{H}(v_i) = d^-_{D}(v_i) - 1 \) and \( d^+_{H'}(v_i) = d^+_{D'}(v_i) \). Again (**) follows from (*).
Therefore, by induction hypothesis, $\eta''$ is non-singular for $G - v$. Apply Theorem \ref{thm:main} to $\eta''$ and $\eta$, with $J = \{i : 1 \leq i \leq k, e_i \notin D'\}$ and $k_i = 1$ for $i \in J$, we conclude that $\eta$ is non-singular for $G$.

3 Application of the reduction method

**Lemma 1** Suppose $G$ is a connected graph, and $\eta$ is an index function with $\eta(e) = 1$ for all $e \in E(G)$. Assume one of the following holds:

- $\eta(v) \geq \max\{1, d_G(v) - 2\}$ for every vertex $v$.
- Each vertex $v$ has $\eta(v) \geq d_G(v) - 2$ and at least one vertex $v$ has $\eta(v) \geq d_G(v)$.

Then $\eta$ is a non-singular index function of $G$.

**Proof.** Assume the lemma is not true and $G$ is a counterexample with minimum number of vertices.

Assume first that $\eta(v) \geq \max\{1, d_G(v) - 2\}$ for all $v$. By reducing the value of $\eta$ if needed, we may assume that $\eta(v) = \max\{1, d_G(v) - 2\}$. Let $v$ be a non-cut vertex of $G$ and let $v_1, \ldots, v_k$ be the neighbours of $v$. Consider the graph $G - v$. Let $\eta'$ be the index function of $G - v$ defined as $\eta'(v_i) = \eta(v_i)$ for $i = 1, 2, \ldots, k - 1$ and $\eta'(v_k) = \eta(v_k) + 1$. For each $i \in \{1, 2, \ldots, k - 1\}$, we have $\eta'(v_i) \geq d_{G - v}(v_i) - 2$, and $\eta'(v_k) \geq d_{G - v}(v_k)$. As $G - v$ is connected, the condition of the lemma is satisfied by $G - v$ and $\eta'$. By the minimality of $G$, $\eta'$ is a non-singular index function for $G - v$. By Theorem \ref{thm:main} $\eta$ is a non-singular index function for $G$.

Assume each vertex $u$ has $\eta(u) \geq d_G(u) - 2$ and one vertex $v$ has $\eta(v) \geq d_G(v)$. Let $\eta'$ be the index function of $G - v$ defined as $\eta'(u) = \eta(u)$ except that $\eta'(u) = \eta(u) + 1$ for all neighbours $u$ of $v$. Note that for all the neighbours $u$ of $v$, $\eta'(u) \geq d_{G - v}(u)$. Thus each component of $G - v$, together with $\eta'$, satisfies the condition of the lemma. By the minimality of $G$, $\eta'$ is a non-singular index function for $G - v$. Apply Theorem \ref{thm:main} again, we conclude that $\eta$ is a non-singular index function for $G$.

A graph $G$ is called *subcubic* if $G$ has maximum degree at most 3.

**Corollary 2** Conjecture \ref{conj:main} holds for subcubic graphs, i.e., if $G$ is a subcubic graph, then $\text{pind}(AG) = 1$. As a consequence, subcubic graphs are $(2,2)$-choosable.

**Proof.** If $G$ has maximum degree at most 3, then it follows from Lemma \ref{lem:subcubic} that $\eta(z) = 1$ for all $z \in V(G) \cup E(G)$ is a non-singular index function.
A graph $G$ is a 2-tree if there is an acyclic orientation of $G$ (also denoted by $G$) such that the following hold: (1) there are two adjacent vertices $v_0, v_1$ with $d^+_G(v_i) = i$ ($i = 0, 1$). (2) every other vertex $v$ has $d^+_G(v) = 2$, and the two out-neighbours of $v$ are adjacent. If $N^+_G(v) = \{u, w\}$ and $(u, w)$ is an arc, then $v$ is called a son of the arc $e = (u, w)$. For an acyclic oriented graph $G$, for $v \in V(G)$, let $\rho_G(v)$ be the length of the longest directed path ending at $v$. So if $v$ is a source, then $\rho_G(v) = 0$.

**Theorem 4** Let $G$ be a 2-tree and let $\eta$ be an index function of $G$. Assume $\eta(z) \geq 1$ for all $z \in E(G) \cup V(G)$, except that possibly there is one arc $(u, w)$ with $\rho_G(u) \leq 1$, for which $\eta(w) \geq 0$ and $\eta(u) \geq 2$. Then $\eta$ is non-singular for $G$.

**Proof.** Assume the theorem is not true and $G$ is a counterexample with minimum number of vertices. If the special arc $(u, w)$ specified in the theorem does not exist, then let $e = (u, w)$ be an arc which has at least one son, and with $\rho_G(u) = 1$. Note that all the sons of $e$ are sources. Let $v$ be a son of $(u, w)$ and let $\eta'$ be the index function of $G' = G - v$ which is equal to $\eta$, except that $\eta'(u) = \eta(u) + 1 \geq 2$ and $\eta'(w) = \eta(w) - 1 \geq 0$. Then $G'$ and $\eta'$ satisfying the condition of the theorem, with $e$ be the special edge (note that $\rho_{G-v}(u) \leq \rho_G(u) = 1$). Hence $\eta'$ is non-singular for $G'$. It follows from Theorem 1 that $\eta$ is non-singular for $G$.

Assume the special arc $e = (u, w)$ exists. If $u$ is a source, then delete $u$, and let $\eta'$ be the index function of $G' = G - u$ which is equal to $\eta$, except that $\eta'(v) = \eta(v) + 1$ for neighbours $v$ of $u$. Then $\eta'(v) \geq 1$ for each vertex of $G'$, hence $G'$ and $\eta'$ satisfying the condition of the theorem. So $\eta'$ is non-singular for $G'$, and it follows from Theorem 1 that $\eta$ is non-singular for $G$.

If $u$ is not a source vertex and $e$ has a son $v$, then $v$ is a source vertex. We delete $v$ and let $\eta'$ be the index function of $G' = G - v$ which is equal to $\eta$, except that $\eta'(u) = \eta(u) - 1$ and $\eta'(w) = \eta(w) + 1$. Then $G'$ and $\eta'$ satisfying the condition of the theorem, and hence $\eta'$ is non-singular for $G'$. It follows from Theorem 1 that $\eta$ is non-singular for $G$.

If $u$ is not a source vertex and $e$ has no son, then there is an arc $e' = (u, w')$ which has a son $a$. Since $\rho_G(u) \leq 1$, all the sons of $e'$ are sources. If $e'$ has more than one son, say $a, b$ are both sons of $e'$, then let $\eta'$ be the restriction of $\eta$ to $G - \{a, b\}$. By the minimality of $G$, $\eta'$ is non-singular for $G - \{a, b\}$. By Corollary 1 (with $D$ consists of the four arcs incident to $a, b$ and $D'$ consists of arcs $au, bw'$), $\eta$ is non-singular for $G$. Assume $e'$ has only one son $a$. Let $\eta'$ be the restriction of $\eta$ to $G - \{a, u\}$, except that $\eta'(w) = 1$. By the minimality of $G$, $\eta'$ is non-singular for $G - \{a, u\}$. By Corollary 1 (with $D$ consists of the four arcs incident to $a, u$ and $D'$ consists of arcs $au', uw$), $\eta$ is non-singular for $G$.  

**Corollary 3** Conjecture 2 holds for 2-trees, i.e., if $G$ is a 2-tree, then $\text{pind}(A_G) = 1$, and hence is $(2, 2)$-choosable.

**Theorem 5** If $T$ is a tree with leaves $v_1, v_2, \ldots, v_n$, and $G$ is obtained from $T$ by adding edges $v_i v_{i+1}$ ($i = 1, 2, \ldots, n$, with $v_{n+1} = v_1$), then $\text{pind}(A_G) = 1$, and hence $G$ is $(2, 2)$-choosable.
**Proof.** First we construct an acyclic orientation of $G$ as follows: We choose a non-leaf vertex $u$ of $T$ as the root of $T$. Orient the edges of the tree from father to son. Then orient the added edges from $v_i$ to $v_{i+1}$ for $i = 1, 2, \ldots, n - 1$, and orient the edge $v_1v_n$ from $v_1$ to $v_n$. The resulting digraph is $D$. Now we choose a sub-digraph $D'$ of $D$ as follows: $D'$ consists of a directed path $P$ from the root vertex $u$ to $v_1$, and all the edges $v_iv_{i+1}$ for $i = 1, 2, \ldots, n - 1$, and the edge $v_1v_n$. Let $\eta$ be the constant function $\eta \equiv 1$, let $X = \{v_n\}$, and let $\eta'(v_n) = 0$, which is an index function of $G[X]$. Then $\eta'$ is a non-singular index function of $G[X]$. To prove that $\text{pind}(A_G) = 1$, i.e., $\eta$ is a non-singular index function of $G$, it suffices, by Corollary 1, to show that for each vertex $v$,

$$1 + 2d^-_{D'}(v) - d^-_{D}(v) \geq d^+_{D'}(v).$$

This is a routine check. Assume first that $v$ is not a leaf of $T$.

1. If $v$ is not on path $P$, then $d^-_{D'}(v) = 0$, $d^-_{D}(v) = 1$ and $d^+_{D'}(v) = 0$. So $1 + 2d^-_{D'}(v) - d^-_{D}(v) = 0 \geq d^+_{D'}(v)$.

2. If $v$ is on $P$, but is not the root $u$, then $d^-_{D'}(v) = 1$, $d^-_{D}(v) = 1$ and $d^+_{D'}(v) = 1$. So $1 + 2d^-_{D'}(v) - d^-_{D}(v) = 2 \geq d^+_{D'}(v)$.

3. If $v = u$, then $d^-_{D'}(v) = 0$, $d^-_{D}(v) = 0$ and $d^+_{D'}(v) = 1$. So $1 + 2d^-_{D'}(v) - d^-_{D}(v) = 1 \geq d^+_{D'}(v)$.

Next, consider the case that $v$ is a leaf of $T$.

1. If $v = v_1$, then $d^-_{D'}(v) = 1$, $d^-_{D}(v) = 1$ and $d^+_{D'}(v) = 2$. So $1 + 2d^-_{D'}(v) - d^-_{D}(v) = 2 \geq d^+_{D'}(v)$.

2. If $v = v_i$, for $1 < i < n$, then $d^-_{D'}(v) = 1$, $d^-_{D}(v) = 2$ and $d^+_{D'}(v) = 1$. So $1 + 2d^-_{D'}(v) - d^-_{D}(v) = 1 \geq d^+_{D'}(v)$.

3. If $v = v_n$, then $d^-_{D'}(v) = 2$, $d^-_{D}(v) = 3$ and $d^+_{D'}(v) = 0$. So $1 + 2d^-_{D'}(v) - d^-_{D}(v) = 2 \geq d^+_{D'}(v)$.

\[\square\]

A **Halin graph** is a planar graph obtained by taking a plane tree (an embedding of a tree on the plane) without degree 2 vertices by adding a cycle connecting the leaves of the tree cyclically.

**Corollary 4** Conjecture 2 holds for Halin graphs, i.e., if $G$ is a Halin graph, then $\text{pind}(A_G) = 1$, and hence is $(2,2)$-choosable.
A grid is the Cartesian product of two paths, \( P_n \Box P_m \), with vertex set
\[
V = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m \}
\]
and edge set
\[
E = \{(i, j)(i', j') : i = i', j' = j + 1 \text{ or } i' = i + 1, j' = j \}.
\]

**Lemma 2** Assume \( m, n \geq 1 \). Let \( \eta \) be an index function of \( P_n \Box P_m \), with \( \eta(e) = 1 \) for edges \( e \), and one of the following holds:

1. \( \eta(v) = 1 \) for all vertices \( v \).
2. \( \eta(v) = 1 \) for all vertices \( v \), except that \( \eta(n, 1) = 0 \) and \( \eta((n, j)) = 2 \) for \( 2 \leq j \leq m \).

Then \( \eta \) is non-singular for \( G \).

**Proof.** We prove it by induction on the number of vertices of \( G \). The case \( n = 1 \) or \( m = 1 \) is easy and omitted. Assume \( n, m \geq 2 \). If \( \eta(v) = 1 \) for all vertices \( v \), then we delete vertices \( (n, 1), (n, 2), \ldots, (n, m) \) in this order. When deleting \( (n, 1) \), we increase \( \eta(n, 2) \) by 1 and decrease \( \eta(n - 1, 1) \) by 1. When deleting \( (n, j) \) for \( j \geq 2 \), we increase \( \eta(n, j + 1) \) by 1 and increase \( \eta(n - 1, j) \) by 1. After all the vertices \( (n, 1), (n, 2), \ldots, (n, m) \) are deleted, we obtain a grid \( P_{n-1} \Box P_m \) and an index function \( \eta' \) which satisfies the condition of the lemma and hence is non-singular. By Theorem \( \square \) \( \eta \) is non-singular.

Assume \( \eta(n, 1) = 0 \) and \( \eta(n, j) = 2 \) for \( 2 \leq j \leq m \). We delete vertices \( (n, m), (n, m - 1), \ldots, (n, 1) \) in this order, and need not to change \( \eta \) except for while deleting \( (n, 2) \), we increase \( \eta(n, 1) \) by 1. It follows from induction hypothesis that the resulting index function is non-singular for \( P_{n-1} \Box P_m \), and by Theorem \( \square \) that the original index function \( \eta \) is non-singular for \( G \). \( \blacksquare \)

**Corollary 5** Conjecture \( \square \) holds for grids, and hence grids are \((2, 2)\)-choosable.

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