Manifestly T-Duality Symmetric Matrix Models

Tsunehide Kuroki\textsuperscript{1}, Yuji Okawa\textsuperscript{1}, Fumihiko Sugino\textsuperscript{2}, Tamiaki Yoneya\textsuperscript{1}
\textsuperscript{1} Institute of Physics, University of Tokyo, Komaba, Meguro-ku, Tokyo 153, Japan
kuroki, okawa, tam@hep1.c.u-tokyo.ac.jp
\textsuperscript{2}KEK Theory Group, Tsukuba, Ibaraki 305, Japan
sugino@theory.kek.jp

Abstract

We present a new class of matrix models which are manifestly symmetric under the T-duality transformation of the target space. The models may serve as a nonperturbative regularization for the T-duality symmetry in continuum string theory. In particular, it now becomes possible to extract winding modes explicitly in terms of extended matrix variables.

\textsuperscript{*}JSPS Research Fellow
I. INTRODUCTION

Matrix models can be regarded as a nonperturbative regularization of string theories. Recently, it became clear that various duality symmetries play vital roles in understanding the nonperturbative properties of string theories. Unfortunately, however, it does not seem that the standard matrix models are particularly suited for studying the duality symmetries. Even the T-duality which is valid to each order of genus expansion in continuum string theories is difficult to implement manifestly in matrix models, at least in their present interpretation. For example, as discussed in our previous paper [1], the Ising model described by a standard two-matrix model does not preserve T-duality symmetry, whose validity is naively expected from the Kramers-Wannier duality on a square lattice, once we take into account the higher genus effects. The reason for this failure is that there is no symmetry between global winding and momentum modes in such models. Although it is not difficult to define partition functions as sums over random surfaces preserving T-duality [1], it is quite nontrivial to write down corresponding matrix models preserving T-duality symmetry manifestly. Since at present matrix-model approaches seem to be the only known tractable way towards dynamical formulation of nonperturbative string theories, it is worthwhile to develop methods of treating duality symmetries exactly using matrix models from various possible standpoints. This is particularly so in view of a recent trend concerning new possibilities of using matrix models in seeking for fundamental theories of strings including D-branes.

The purpose of the present paper is to present a new class of extended matrix models which have exact and manifest T-duality symmetry under the usual random-surface interpretation of matrix models. Although the models in general are not exactly soluble using presently available techniques, we believe that the existence of such models is interesting by itself and will be useful for future investigations. To maintain the T-duality symmetry exactly, we need two key ingredients that have to be combined in a suitable way. One is that the global, as well as local, degrees of freedom corresponding to the winding and momentum modes must appear symmetrically for arbitrary topology of surfaces. The other is that the random triangulation of surfaces must be invariant under the duality transformation. For the first property, we adopt a variant of $Z_K$ spin systems ($K = an integer$) as the target space, generalizing the discussion given in the Appendix A of our previous paper [1]. For the second, we can use the result of reference [3] which is, to the best of our knowledge, the only known matrix model with exact self-duality. In this reference, the dual transformation for the random triangulation (namely, the measure of pure 2D gravity) was discussed, but the first problem was not addressed. In fact, a naive generalization of the method used for the one-matrix model does not work for systems coupled with matter. We will clarify how to combine self-dual matter systems with the method of [3].

The plan of the paper is as follows. In the next section, we briefly discuss a class of $Z_K$ spin systems on a fixed surface, which we call $Z_{P,Q}$ models ($K = P,Q$) and have manifest symmetry between “momentum” and “winding” modes analogously with the toroidal compactification of continuum string theory. In section III, we present our extended matrix models by coupling the $Z_{P,Q}$ models to 2D gravity with the self-dual measure, and exhibit

---

1 See, e.g., [3] in which the case of $c = 1$ has been discussed.
their exact T-duality symmetry. In section IV, some simple cases will be discussed for the purpose of concrete illustration. We will then discuss some relevant issues including the c = 1 limit.

II. $Z_{P,Q}$ Model on a Fixed Surface

As is well known, $Z_K$ spin systems in two dimensions [4] are self-dual under the Kramers-Wannier dual transformation. There is, however, a subtlety when we do not neglect global degrees of freedom. To make the system completely self-dual, we have to impose appropriate constraints in order to suppress the local vortex excitations which tend to violate the duality symmetry. In the case of the lattice $c = 1$ matter system, this was discussed in [2] and the Appendix A of [1]. Here we present its generalization to $Z_K$ systems. For clarity of notation, we assume the surface to be a fixed square lattice and denote the spin sites by $x$ and links by $x, \mu$ where $\mu$ is the direction of the link from a site $x$. The difference operation between the nearest neighbor sites $x$ and $x + \hat{\mu}$ where $\hat{\mu}$ is a unit lattice vector along the $\mu$ direction is denoted by $\Delta_\mu$.

For a given $K$, we introduce two integers $P, Q$ satisfying $K = PQ$. Spin variables $n_x = 0, 1, \ldots, P - 1 \mod P$ are then assumed to live in the space $Z_P$. In addition to the spin variables, we introduce link variables $m_{x,\mu} = 0, 1, \ldots, Q - 1 \mod Q$ living in the space $Z_Q$. Then the partition function of our $Z_{P,Q}$ model is defined as

$$Z = \left( \prod_{x,\mu} \sum_{n_x, m_{x,\mu}} \prod_{y,\mu} \sum_{\tilde{n}_y, \psi_{y,\mu}} e^{i \frac{2\pi}{Q} \tilde{n}_\epsilon_{\mu,\nu} \Delta_\mu m_{x,\nu} + i \frac{2\pi}{K} \epsilon_{\mu,\nu} \psi_\mu (\Delta_\nu n - P m_{x,\nu})} \prod_{y,\mu} \tilde{B}(\psi_{y,\mu}). \right) \tag{2.1}$$

The Boltzmann factor $B(x)$ is assumed to be periodic under the translation $x \rightarrow x + K$, and the $\delta$-function constraint is understood modulo $Q$. Note that after the summation over the link variables, the Boltzmann factor can be regarded as a function on $Z_P$ with respect to the spin variables. Obviously, if the topology of the fixed surface is sphere, the constraint leads to

$$m_{x,\mu} = \Delta_\mu m_x, \ m_x \in Z_Q. \tag{2.2}$$

Therefore by redefining the spin variable by $n_x \rightarrow n'_x = n_x - P m_x \in Z_{PQ} = Z_K$, the system is reduced to an ordinary $Z_K$ spin system. If the fixed surface is not the sphere, we have instead of (2.2)

$$m_{x,\mu} = \Delta_\mu m_x + \overline{m}_{x,\mu}, \ (m_x, \overline{m}_{x,\mu}) \in Z_Q, \tag{2.3}$$

where $\overline{m}_{x,\mu}$ is a global vector field which cannot be reduced to the difference of the site variables and is associated with a nontrivial homology cycle of the surface. The degree of freedom represented by $\overline{m}_{x,\mu}$ is the analogue of the winding modes in the toroidal compactification of continuum string theory.

Now let us perform the dual transformation on the partition function (2.1). We introduce two auxiliary fields $\tilde{n}_y \in Z_Q$ and $\psi_{y,\mu} \in Z_K$ on dual sites $y$ and dual links, respectively, and rewrite (2.1), apart from a numerical proportional factor, as

$$Z = \left( \prod_{x,\mu} \sum_{n_x, m_{x,\mu}} \prod_{y,\mu} \sum_{\tilde{n}_y, \psi_{y,\mu}} e^{i \frac{2\pi}{Q} \tilde{n}_\epsilon_{\mu,\nu} \Delta_\mu m_{x,\nu} + i \frac{2\pi}{K} \epsilon_{\mu,\nu} \psi_\mu (\Delta_\nu n - P m_{x,\nu})} \prod_{y,\mu} \tilde{B}(\psi_{y,\mu}). \right) \tag{2.4}$$
Here we suppressed the subscripts for the sites on the exponent to avoid unnecessary notational complexity. The Boltzmann factor $\tilde{B}$ is the $Z_K$ Fourier transform of the original Boltzmann factor:

$$B(a) = \frac{1}{\sum_{b=0}^{K-1} \tilde{B}(b)} \sum_{b=0}^{K-1} e^{i\tilde{B}} \tilde{B}(b). \quad (2.5)$$

Reality of the Boltzmann factor leads $\tilde{B}(b) = \tilde{B}(K - b)$. We assume the normalization condition $B(0) = \tilde{B}(0) = 1$. Then solving the constraint coming from the summation over $m_\mu$, the general solution for $\psi_\mu$ is

$$\psi_\mu = \Delta_\mu \tilde{n} - Q \tilde{m}_\mu, \quad \tilde{m}_\mu \in Z_P. \quad (2.6)$$

After substituting this result and taking the summation over the original spin variables $n$, we have the constraint,

$$\epsilon_{\mu \nu} \Delta_\mu \tilde{m}_\nu = 0 \pmod{P}. \quad (2.7)$$

Thus the partition function after dual transformation takes, apart from a numerical overall normalization factor, the same form as the original one (2.1) with the interchange $P \leftrightarrow Q, \tilde{B} \leftrightarrow B$

$$Z = \prod_y \delta^{(P)}(\Delta_\mu \tilde{m}_{y,\nu} - \Delta_\nu \tilde{m}_{y,\mu}) \prod_y \tilde{B}(\Delta_\mu \tilde{n}_y - Q \tilde{m}_{y,\mu}). \quad (2.8)$$

The global modes represented by $\tilde{m}_{y,\mu}$ are interpreted as the analogue of the momentum mode of toroidally compactified strings, and hence the dual transformation interchanges the winding and momentum modes, precisely as required for T-duality symmetry. In particular, the systems with $P = Q$ are self-dual on surfaces of arbitrary genus.

Note that, although the models with a given $K = PQ$ are equivalent to the usual $Z_K$ spin models on the sphere, they are in general different on higher-genus surfaces, because of the different appearance of the global momentum and winding modes, depending on the choice of $P$ and $Q$. In a sense, the $Z_{PQ}$ model amounts to compactifying the $Z_K$ target space by a subgroup $Z_P$ of the isometry group $Z_K$. For example, the standard Ising model corresponds to $P = 2, Q = 1, Z_{2,1}$ model. Hence its dual is $Z_{1,2}$. Therefore the Ising model is not exactly self-dual on higher genus, as discussed in [1]. In this class of models, the simplest exactly self-dual model is $Z_{2,2}$, which is identical with the ($Z_4$) Ashkin-Teller model on the sphere. Finally, we note that taking the limit $P = Q \rightarrow \infty$ appropriately gives a lattice version of toroidally compactified strings discussed in the Appendix A of [1]. A brief discussion on this limit will be given in section IV.

### III. SELF-DUAL MATRIX MODELS: GENERAL THEORY

We now show how to construct the matrix model corresponding to the $Z_{P,Q}$ model. In the standard method of coupling the $Z_K$ spin system to 2D gravity, we introduce $K$ different Hermitian $N \times N$ matrices $M_a, \; a = 1, 2, \ldots, K$ and assume the action
\[ S = \text{NTr} \left[ \frac{1}{2} \sum_{a,b=1}^{K} C_{ab} M_a M_b + \sum_{a=1}^{K} V(M_a) \right]. \tag{3.1} \]

In the Feynman graph expansion, the potential \( V(M_a) \) represents a spin site which is on the center of a discretized surface element. The propagator \( C_{ab}^{-1} \), on the other hand, corresponds to the Boltzmann factor assigned to a link connecting the nearest neighbor spin sites with spin variables \( a \) and \( b \). The \( Z_K \) symmetry requires that the kinetic operator (and the propagator) satisfies translation invariance modulo \( K \) with respect to the indices \( a, b \),

\[ C_{ab} = C(a - b) = C(a - b \pm K), \quad C_{ab}^{-1} = D(a - b) = D(a - b \pm K). \]

In our terminology, this construction only represents the \( Z_{K,1} \) model. To extend the construction to general \( Z_{P,Q} \) matrix models, we first introduce a set of \( P \) different Hermitian \( QN \times QN \) matrices, denoted as \( M_a, a = 1, 2, \ldots, P \). The index \( a \) ranging from 1 to \( P \) corresponds to the \( Z_P \) spin variables as above, while the additional \( Q \times Q \) matrix elements of each \( M_a \) are supposed to be associated with the \( Z_Q \) link variables. To prevent unnecessary complication in notations, we will always suppress the original \( U(N) \) indices in the following and only indicate the additional \( Q \times Q \) matrix indices by \( i, j, \ldots \in \{1, 2, \ldots, Q\} \). The choice of an appropriate propagator will then enable us to construct the \( Z_{P,Q} \) matrix model. The correct choice is, suppressing the \( U(N) \) indices, as follows:

\[ D_{ij,kl}^{a,b} = \frac{1}{N} \sum_{m=0}^{Q-1} B(a - b - Pm) (L^m)_{il} (L^m)^{kj}, \tag{3.2} \]

where \( B \) is nothing but the Boltzmann factor used in the previous section, and \( L \) is an arbitrary \( Q \times Q \) unitary matrix satisfying the conditions

\[ L^Q = I, \quad L^{-1} = L^\dagger, \tag{3.3} \]

\[ \text{tr}_Q L^i = 0 \text{ if } i \neq 0 \text{ (modulo } Q). \tag{3.4} \]

Here the trace notation \( \text{tr}_Q \) means taking the trace only with respect to the indices \( i, j, \ldots \). The notation \( \text{Tr} \) will be used for the total trace operation including both the \( U(N) \) and the \( U(Q) \) indices. On the other hand, the trace operation with respect only to the ordinary \( U(N) \) indices will be denoted by \( \text{tr}_N \). A formula for the inverse of the propagator \( D_{ij,kl}^{a,b} \) will be given later.

The potential function corresponding to the spin sites are assumed to be the following special form using the result of \( [3] \)

\[ \text{NTr} \sum_{a=1}^{P} \ln(1 - gM_a). \tag{3.5} \]

This is necessary to ensure that the measure for the random triangulation is invariant under the T-duality transformation even before taking the (double) scaling limit.

The model is invariant under the group \( U(N) \times U(Q), M_a \rightarrow U M_a U^{-1}, \quad U \in U(N) \times U(Q) \), provided the matrix \( L \) is transformed by the \( U(Q) \) part of \( U \). It is easy to check that
the conditions (3.3), (3.4) precisely impose the constraint corresponding to the $\delta^{(Q)}$ function in (3.1) for general triangulation of arbitrary surfaces, at each elementary closed circle of links (i.e., plaquette). Hence, the matrix $L$ is arbitrary under the $U(N) \times U(Q)$ invariant conditions (3.3), (3.4). Also it is not difficult to prove explicitly that this model restricted to the sphere approximation is equivalent to the standard $Z_K$ matrix model as briefly discussed in section IV.

Now that we have given the definitions of the matrix models, let us next proceed to rewriting of the models in a manifestly T-duality symmetric form. We will extend the method given in [3] for the case of pure gravity to our $Z_{P,Q}$ model. For this purpose, it is convenient to go to a particular representation for the matrix $L$ using the $U(Q)$ symmetry, namely the diagonalized representation given by

$$L_{ij} = \delta_{ij} e^{\frac{2\pi i}{Q}(i-1)}.$$  \hspace{1cm} (3.6)

Thus the propagator is now

$$D^{ac,bd}_{ij,kl} = \frac{1}{N} \delta_{ac} \delta_{bd} \sum_{m=0}^{Q-1} B(a-b-Pm) \delta_{il} \delta_{kj} e^{\frac{2\pi i}{Q}(i-j)m}.$$  \hspace{1cm} (3.7)

Here, by putting additional Kronecker $\delta$’s, $\delta_{ac}, \delta_{bd}$, we duplicated the indices $a$ and $b$ in order to make the appearance of the $i, k \in (Z_Q)$ and $a, b \in (Z_P)$ indices symmetric. We note that in this representation the propagator has manifest $Z_Q$ periodicity with respect to the indices $i, j, \ldots$ while the periodicity with respect to the indices $a, b, \ldots$ is not manifest. This is expected from the situation of the fixed-surface model of the previous section, since the $Z_P$ periodicity appears only after summing over the link variables. In the matrix model, the latter operation appears only for Feynman amplitudes after taking the trace operation.

We first introduce $PN$ auxiliary vector fields $\psi_i^a$ which transform as a complex vector under the group $U(N) \times U(Q)$. Here as before we have suppressed the $U(N)$ vector (“color”) indices and also the “flavor” $U(N)$ vector indices, while both of the matter indices $a(\in U(P)$ vector) and $i(\in U(Q)$ vector) are explicitly indicated. Thus the auxiliary fields $\psi_i^a$ can actually be treated as $NQ \times NP$ matrix fields which transform as

$$\psi_i^a \rightarrow (U_c \psi V_f)_i^a,$$

where $U_c \in U(N) \times U(Q)$ and $V_f \in U(N) \times U(P)$. Using these auxiliary fields, rewrite the potential term as

$$e^{-N\text{tr}_{N} \sum_{a=1}^{P} \ln(1-gM_a)} = \int d\psi d\psi^\dagger e^{-\sum_{a=1}^{P} \sum_{i,j=1}^{Q} \text{tr}_{N} \psi_i^a \psi_j^a (1-gM_a)_{ij} \psi_i^a \psi_j^a}.$$  \hspace{1cm} (3.8)

Here we used the notation $\psi^\dagger$ for the complex conjugate of the $\psi$’s treating them as $N \times N$ (color $\times$ flavor) matrices. We can then perform the integration over the matrix $M_a$. Using the propagator (3.7), the partition function takes the form

$$Z = N_{P,Q}(B) \int d\psi d\psi^\dagger \exp \left[ -\sum_{i=1}^{Q} \sum_{a=1}^{P} \text{tr}_{N} \psi_i^a \psi_i^a + \frac{1}{2} g^2 \sum_{i,j,k,l=1}^{Q} \sum_{a,b,c,d=1}^{P} \text{tr}_{N} \psi_i^a \psi_j^a D_{ij,kl} \psi_k^d \psi_k^d \right].$$  \hspace{1cm} (3.9)
The normalization constant $N_{P,Q}(B)$ originates from the Gaussian integration,

$$N_{P,Q}(B) = \det^{1/2}_{(NPQ)^2} D,$$

where $D$ is the matrix of the propagator (3.7).

The T-dual transformation amounts to rewriting the partition function in terms of the dual Boltzmann factor $\tilde{B}(a)$, defined by (2.5). By directly substituting (2.5) to the propagator (3.7), we find

$$D_{ij,kl}^{ab,cd} = \frac{Q}{N \sum_{b=0}^{N-1} B(b)} \sum_{m=0}^{P-1} \tilde{B}(i - j - Q\tilde{m}) e^{i \frac{2\pi}{N} (a-c)(i-j-Q\tilde{m})} \delta_{ab} \delta_{cd} \delta_{il} \delta_{jk}. \quad (3.10)$$

A little examination of this expression shows that the partition function in terms of the $\tilde{B}$ takes the same general form as the original one apart from the normalization factor,

$$Z = N_{P,Q}(B) \int d\phi d\phi^\dagger \exp \left[ -\sum_{a=1}^{P} \sum_{i=1}^{Q} \text{tr}_N \phi^i_a \phi^\dagger_a i + \frac{1}{2} \tilde{g}^2 \sum_{a,b,c,d=1}^{P} \sum_{i,j,k,l=1}^{Q} \text{tr}_N \phi^i_a \phi^\dagger_d \tilde{D}^{li,jk}_{da,bc} \phi^k_c \phi^\dagger_b \right],$$

(3.11)

after interchanging the $Z_P$ and $Z_Q$ indices by making the following redefinitions of the auxiliary vectors, propagator and the coupling constant, respectively, as

$$\phi^i_a \equiv e^{i \frac{2\pi}{Q} a_i \psi^i_1}, \quad (3.12)$$

$$\tilde{D}^{li,jk}_{da,bc} \equiv \frac{1}{N} \sum_{m=0}^{P-1} \tilde{B}(l - j - Q\tilde{m}) e^{i \frac{2\pi}{N} (d-a)\tilde{m}} \delta_{ab} \delta_{cd} \delta_{il} \delta_{jk}, \quad (3.13)$$

$$\tilde{g}^2 \rightarrow \tilde{g}^2 = \frac{\frac{g^2 Q}{\sum_{b=0}^{K-1} B(b)}}{P}. \quad (3.14)$$

Here we have interchanged the color and flavor indices by performing a cyclic permutation for the auxiliary fields $\psi$’s in the quartic term of the action [2] corresponding to the replacement,

$$D_{ij,kl}^{ab,cd}(B) \rightarrow e^{i \frac{2\pi}{NP}(a-c)(i-j)} \tilde{D}^{li,jk}_{da,bc}(\tilde{B}),$$

of which the phase factor is absorbed by (3.12). We note that in this dual form, the $Z_P$ periodicity instead of the $Z_Q$ periodicity is now manifest. This comes about due to the redefinition (3.12) of the auxiliary fields: The field $\psi^i_a$ is supposed to be manifestly $Z_Q$ periodic with respect to the index $i$, while it acquires a phase $e^{i \frac{2\pi}{NP} i}$ under the translation $a \rightarrow a + P$. This property is interchanged after the above redefinition from $\psi$ to $\phi$. It should also be remarked that the auxiliary fields $\psi$ and $\phi$ carry both the matter ($Z_P$) and dual-matter ($Z_Q$) indices in addition to the color and flavor $U(N)$ indices. This is an interesting

---

2 Note that we have used the same trace notation $\text{tr}_N$ for $U(N)$ flavor indices.
aspect of the present formalism, which is something not appeared in the familiar formulation of T-duality in string theory.

The above result is precisely the dual transformed structure of the $Z_{P,Q}$ models, apart from the redefinition of the coupling constant which is connected to the overall normalization factor suppressed in the previous discussion for the fixed surface. It is evident that by reversing the route from the $M_a$ ($a = 1, 2, \ldots, P$) representation to the $\psi^a_i$ representation, we can construct the dual transformed matrix model, now with $Q$ Hermitian $NP \times NP$ matrices $\tilde{M}_i$ ($i = 1, 2, \ldots, Q$) from the $\phi^i_a$ representation. Thus we have established that our matrix models have desired T-duality properties. Formally, the T-duality symmetry corresponds to the following identity for the partition function,

$$
\frac{Z_{P,Q}(B, g)}{\mathcal{N}_{P,Q}(B)} = \frac{Z_{Q,P}(\tilde{B}, \tilde{g})}{\mathcal{N}_{Q,P}(\tilde{B})}.
$$

Note that the normalization constants $\mathcal{N}_{P,Q}$ and $\mathcal{N}_{Q,P}$ whose ratio is easily calculable using (3.10) are independent of the coupling constant $g$. As a consequence, the normalization constants can be neglected in the scaling limit.

In order to write down the actions of the matrix models explicitly in terms of the Boltzmann factor $B$, we need a general formula for the inverse of the propagator (3.2). This is easily obtained by applying $Z_P$ Fourier transformation appropriately. The result is

$$
C_{ij,kl}^{ab} \equiv (D^{-1})_{ij,kl}^{ab} = N \delta_{il} \delta_{jk} \frac{1}{P} e^{-i \frac{2\pi}{P} (a-b)(i-k)} \sum_{m=0}^{P-1} \frac{1}{B(m;i-k)} e^{i \frac{2\pi}{P} (a-b)m},
$$

where

$$
\tilde{B}(m;i-k) = \sum_{a=0}^{P-1} \sum_{l=0}^{Q-1} B(a - Pl)e^{-i \frac{2\pi}{P}(a-Pl)(i-k)} e^{-i \frac{2\pi}{P} ma}.
$$

Corresponding to the $Z_Q$ periodicity of the propagator, this form of the kinetic term is $Z_Q$ periodic with respect to the indices $i,j,\ldots$, while under the $Z_P$ translation it is periodic only up to a phase $e^{-i \frac{2\pi}{P}(i-k)}$. Of course, by inverting this formula, we could have started from the general form of the kinetic term having the same periodicity and expressed the Boltzmann factor in terms of the kinetic term.

Finally, we briefly touch upon the question of observables in our models. Since we have to preserve the $U(N) \times U(Q)$ symmetry, the set of the most general invariants consists of the traces of arbitrary polynomials consisting of the matrices $M_a$ and $L$ where $L$ is the $U(N) \times U(Q)$ matrix acting as the identity in $U(N)$ and as $L$ in $U(Q)$:

$$
A_{ab\ldots}(n,l,\ldots;m,p,\ldots) = \frac{1}{NQ} \text{Tr}(M_a^nL^mM_p^lL^p\ldots).
$$

If the total power of the matrix $L$ is $q \pmod{Q}$, $A_{ab\ldots}(n,l,\ldots;m,p,\ldots)$ represents a loop state with “winding number $q$”. In the standard models, it has been impossible to explicitly extract the winding modes in terms of the matrix variables. Our construction thus indicates how to remedy this deficiency of the usual matrix models by extending the $U(N)$ symmetry to $U(N) \times U(Q)$. 

8
IV. EXAMPLES AND DISCUSSIONS

The purpose of this section is to first present a few simple special cases of our extended models in more concrete forms as an illustration of the general theory, and then further discuss some of the important aspects of the models, in particular, the reduction of the degrees of freedom and the \( c = 1 \) limit.

A. Examples

We give concrete formulas for three cases, \( Z_{1,2} \), \( Z_{1,3} \) and \( Z_{2,2} \).

(i) Duality between \( Z_{1,2} \) and \( Z_{2,1} \) matrix models

This corresponds to the case treated in \([5,1]\) with cubic potential. The \( Z_{1,2} \) model has only link variables. The dual version, on the other hand, is the \( Z_{2,1} \) model which is nothing but the ordinary two-matrix model with spin variables, but without any link variables. Let us confirm this by an explicit computation. The propagator of the \( Z_{1,2} \) matrix model is

\[
\langle M_{ij} M_{kl} \rangle = \frac{1}{N} (\delta_{il} \delta_{kj} + e^{-2\beta L_{il} L_{kj}}),
\]

where \( \beta \) is an inverse temperature of the \( Z_{1,2} \) spin system. \( L \) has the properties in Eqs. (3.3) and (3.4) with \( Q = 2 \). Correspondingly, the action is

\[
S_M = N \frac{1}{1 - e^{-4\beta}} \frac{1}{2} \text{Tr}(M^2 - e^{-2\beta} L M L M) + N \text{Tr} \ln(1 - g M).
\]

Introducing the auxiliary variables \( \psi \) and performing the \( M \)-integral, we have the action

\[
S_\psi = N \sum_{i=1}^{2} \text{tr}_N \psi_i^\dagger \psi_i - N \frac{1}{2} g^2 \text{tr}_N \left\{ \left( \sum_{i=1}^{2} \psi_i^\dagger \psi_i \right)^2 + e^{-2\beta} \left( \sum_{i,j=1}^{2} \psi_i^\dagger L_{ij} \psi_j \right)^2 \right\}.
\]

For the purpose of direct transition to the dual-matrix representation in the simple example treated here, it is convenient to introduce two auxiliary matrices \( U \) and \( V \), which are Hermitian and have \( U(N) \) flavor indices, instead of faithfully following the procedure adopted in the general theory. Then the action \( S_\psi \) can be rewritten as

\[
S_{UV\psi} = N \text{tr}_N \psi_i^\dagger \psi_i + N \frac{1}{2} \text{tr}_N (U^2 + e^{2\beta} V^2) - N g \text{tr}_N \left( \sum_{i} \psi_i U \psi_i^\dagger \right) - \sum_{i,j} L_{ij} \psi_j^\dagger V \psi_i^\dagger).
\]

Integrating \( \psi \) and changing the basis as

\[
\tilde{M}_1 = U + V, \quad \tilde{M}_2 = U - V,
\]

we get the dual action

\[
\tilde{S}_M = N \frac{1}{1 - e^{-4\beta}} \frac{1}{2} \text{tr}_N \left[ \tilde{M}_1^2 + \tilde{M}_2^2 - 2e^{-2\beta} \tilde{M}_1 \tilde{M}_2 \right] + N \sum_{a=1}^{2} \text{tr}_N \ln(1 - \tilde{g} \tilde{M}_a),
\]
where we rescaled \( \tilde{M}_a \) as \( \tilde{M}_a \rightarrow \sqrt{1 + e^{-2\tilde{\beta}} \tilde{M}_a} \) corresponding to the redefinition (3.14). The dual temperature \( \tilde{\beta} \) is defined by the Fourier transformation (2.5) as

\[
e^{-2\tilde{\beta}} = \tanh \beta,
\]

which is of course the famous Kramers-Wannier relation.

(ii) \( Z_{1,3} \) matrix model

We repeat a similar calculation for the \( Z_{1,3} \) matrix model. The propagator and the action have the following forms:

\[
\langle M_{ij} M_{kl} \rangle = \frac{1}{N} \left[ \delta_{il} \delta_{kj} + e^{-\frac{3}{2} \beta} L_{il} (L^2)_{kj} + e^{-\frac{3}{2} \beta} (L^2)_{il} L_{kj} \right],
\]

\[
S_M = N \coth \frac{3}{4} \beta \frac{1}{1 + 2 e^{-\frac{3}{2} \beta} \frac{1}{2}} \text{Tr} \left[ M^2 - \frac{2}{1 + e^{\frac{3}{2} \beta} CL^2 M} \right] + N \text{Tr} \ln(1 - g M). \tag{4.8}
\]

Through the same steps as before, we obtain the action \( S_\psi \) of the variables \( \psi_i \)'s. Then, introducing a Hermitian matrix \( U \) and a complex matrix \( X \), we rewrite \( S_\psi \) as

\[
S_{UX\psi} = N \sum_{i=1}^3 \text{tr}_N \psi_i^\dagger \psi_i + N \frac{1}{2} \text{tr}_N (U^2 + 2 e^{\frac{3}{2} \beta} X^\dagger X)
- N g \sum_{i=1}^3 \text{tr}_N (\psi_i U \psi_i^\dagger) - N g \sum_{i,j=1}^3 \text{tr}_N (X^\dagger \psi_i^\dagger L_{ij} \psi_j + \psi_i^\dagger (L^2)_{ij} \psi_j X). \tag{4.9}
\]

After the \( \psi \)-integral and the replacement

\[
\tilde{M}_1 = U + X + X^\dagger, \quad \tilde{M}_2 = U + \omega^2 X + \omega X^\dagger, \quad \tilde{M}_3 = U + \omega X + \omega^2 X^\dagger,
\]

with \( \omega = e^{i \frac{\pi}{3}} \), we obtain the action of the \( Z_{3,1} \) matrix model

\[
\tilde{S}_M = N \coth \frac{3}{4} \beta \frac{1}{1 + 2 e^{-\frac{3}{2} \beta} \frac{1}{2}} \text{tr}_N \left[ \sum_{a=1}^3 \tilde{M}_a^2 - \frac{2}{1 + e^{\frac{3}{2} \beta} (\tilde{M}_1 \tilde{M}_2 + \tilde{M}_2 \tilde{M}_3 + \tilde{M}_3 \tilde{M}_1)} \right]
+ N \sum_{a=1}^3 \text{tr}_N \ln(1 - \tilde{g} \tilde{M}_a), \tag{4.11}
\]

where the rescaling \( \tilde{M}_a \rightarrow \sqrt{1 + 2 e^{-\frac{3}{2} \beta} \tilde{M}_a} \) was done, and the dual temperature is given by

\[
e^{\frac{3}{2} \tilde{\beta}} = \frac{e^{\frac{3}{2} \beta} + 2}{e^{\frac{3}{2} \beta} - 1}.
\]

(iii) \( Z_{2,2} \) matrix model

This is the simplest example of self-dual models. If we adopt the standard Boltzmann factor for the \( Z_4 \) model,

\[
B(\Delta_\mu n_\mu - 2 m_\mu) = e^{\beta [\cos \frac{2\pi}{4} (\Delta_\alpha n_\alpha - 2 m_\alpha, \nu)]},
\]

10
having a self-dual structure with
\[ e^{\tilde{\beta}} = \coth \frac{\beta}{2}, \tag{4.12} \]
the propagator and the corresponding action are given, respectively, as
\[
\langle (M_1)_{ij}(M_1)_{kl} \rangle = \langle (M_2)_{ij}(M_2)_{kl} \rangle = \frac{1}{N} (\delta_{il} \delta_{kj} + e^{-2\beta} L_{il} L_{kj}),
\]
\[
\langle (M_1)_{ij}(M_2)_{kl} \rangle = \frac{1}{N} e^{-\beta} (\delta_{il} \delta_{kj} + L_{il} L_{kj}), \tag{4.13}
\]
\[
S_M = N \frac{e^{2\beta}}{4 \sinh^2 \frac{\beta}{2}} \frac{1}{\beta} \text{Tr} \left\{ \sum_{a=1}^{2} (M_a)^2 + e^{-2\beta} \mathcal{L} M_a \mathcal{L} M_a - 2e^{-\beta} M_1 M_2 - 2e^{-\beta} \mathcal{L} M_1 \mathcal{L} M_2 \right\} \\
+ N \sum_{a=1}^{2} \text{Tr} \ln[1 - g M_a]. \tag{4.14}
\]
Following the steps of the general theory, we can easily confirm that the model is self-dual with respect to (4.12) and \( \tilde{\beta} = g \frac{1 + e^{-\beta}}{\sqrt{2}} \).

**B. Reduction of degrees of freedom**

From the discussion of section II, it is clear that the \( Z_{P,Q} \) matrix models must be equivalent to the standard \( Z_K \) matrix models when restricted to the sphere approximation. Apparently, however, the \( Z_{P,Q} \) matrix models have more degrees of freedom than the \( Z_K \) models with \( K = PQ \). If counted as \( N \times N \) Hermitian matrices, we have \( PQ^2 \) matrices instead of \( PQ \). Let us therefore examine how the reduction of degrees of freedom occurs. It will turn out that there is a sort of local gauge symmetry in Feynman diagrams which accounts for the superfluous degrees of freedom.

For simplicity, we here consider a concrete example of the \( Z_{P,2} \) (\( Q = 2 \)) models. Once we understand this special case, extension to the general case will be straightforward. For this purpose, it is more advantageous to use the real representation of the matrix \( L \) given by
\[
L_{ij} = \delta_{i+1,j}^{(Q)}, \tag{4.15}
\]
than the diagonal representation used in the general theory in section III. The nonzero components of the propagator then satisfy the following symmetry properties with respect to the \( Z_2 \) indices \( (i, j \text{ type}) \)
\[
D_{11,11} = D_{22,22} = D_{12,21} = D_{21,12}, \nonumber \\
D_{11,22} = D_{22,11} = D_{12,12} = D_{21,21}, \tag{4.16}
\]
where we have suppressed the \( Z_P \) indices \( (a, b \text{ type}) \) since the identities are valid for arbitrary \( a, b \). All other components of the propagator vanish. In particular, there is no coupling between the diagonal and off-diagonal matrix elements with respect to the \( Z_2 \) indices. This
shows that the links assigned to the off-diagonal matrix elements must always form $Z_2$ closed loops. Furthermore, an amplitude with the off-diagonal elements is identical with a corresponding diagram in which all the off-diagonal matrix elements are replaced by the corresponding diagonal elements as indicated in (4.16). Note here that the vertices behave as identities with respect the $Z_2$ indices and hence do not have any contribution violating the symmetry between the diagrams with diagonal and off-diagonal elements. In the case of sphere topology, the correspondence between the diagrams is established by making the interchange $1 \leftrightarrow 2$ globally for all $Z_2$ indices inside the domain enclosed by the closed curve of off-diagonal links. This replaces the off-diagonal links along the closed curve to links with diagonal matrix elements, without changing the amplitude. We can start this procedure from the smallest domains and go to larger ones successively to eliminate all off-diagonal links in this way. Thus the Feynman diagram contributions have a kind of gauge degeneracy $2^L$ which is determined by the number $L$ of independent $Z_2$ closed loops of off-diagonal links on the surface. Using the fact that $L$ coincides with the number of dual sites, it is easy to see that the free energy becomes essentially identical with the model with only the diagonal matrix elements $(M_\alpha)_{ii}$, $(i = 1, 2)$ under the rescaling of $N$ and the coupling constant as $N \to 2N, g_n \to 2^{n-n} g_n$ where $g_n$ is the coupling constant for the term $\frac{g_n}{n} \text{Tr} M^n$ in the potential. Thus the matrix spin degree of freedom is reduced to $2^P$ which is the same as for the standard $Z_2$ model. Obviously, the present argument cannot be extended to higher genus, since the above correspondence between diagonal and off-diagonal amplitudes cannot be established when the closed loop of off-diagonal links wraps around a non-trivial homology cycle.

Extension to the general case is straightforward. The links with off-diagonal elements form $Z_Q$ closed loops and the propagator satisfies the identities $D_{ij,kl} = D_{i+m,j+n,k+n+l+m}$. Repeating the above arguments, we arrive at the rescaling $N \to QN, \ g_n \to Q^\frac{n-2}{2} g_n$ after the system is reduced to matrices with diagonal (with respect to the $Z_Q$ indices) elements. The same result can also be obtained by a simple counting of the Feynman diagrams by focusing on their dependence with respect to $N, Q$ and the coupling constant.

C. $c = 1$ limit

We next turn to the problem of the $c = 1$ limit of our models which corresponds to taking the limit $P = Q \to \infty$. This problem is important if one regards the present matrix models as a nonperturbative regularization of the T-duality symmetry of string theory.

First of all, take the $Z_{Q,Q}$ model assuming the most general form of the Boltzmann factor \[ B(\alpha) = \exp \left( \sum_{\delta=0}^{Q^2-1} K_{\delta} \left( \cos \frac{2\pi \alpha \delta}{Q^2} - 1 \right) \right), \] where $K_{\delta}$’s are constants. In the limit $Q \to \infty$, we obtain the standard Gaussian Boltzmann factor as follows: when $K_{\delta}$ is large with $\frac{K_{\delta}}{Q^2} = \beta_\delta$ being fixed, we can expand the cosine to obtain

\[ B(\alpha) \sim \prod_{\delta=0}^{Q^2-1} \sum_{n^\delta_\mu \in Z} \exp \left[ -\frac{\beta_\delta}{2} \left( \frac{2\pi}{Q} \alpha - 2\pi Q n^\delta_\mu \right)^2 \right] \]
\[
\sum_{n_{\mu} \in \mathbb{Z}} \exp \left[ -\frac{R^2}{4\pi} \left( \frac{2\pi}{Q} (\alpha - 2\pi Q n_{\mu}) \right)^2 \right].
\] (4.17)

In the last expression, we left one of \(\beta_{\delta}'s\) with \(\delta = O(1)\) nonzero and redefined \(\frac{R^2}{2\pi} \equiv \beta_{\delta}\delta^2\) to make \(B(\alpha)\) fit to the ordinary Gaussian Boltzmann factor of the \(c = 1\) model.

For taking the \(c = 1\) limit on a fixed lattice, we make the following replacement

\[
\frac{2\pi n_{\mu}}{Q} \rightarrow X, \quad \frac{2\pi}{Q} \sum_{n_{\mu} \in \mathbb{Z}_Q} \rightarrow \int_0^{2\pi} dX.
\]

Then the partition function of the \(Z_{Q,\bar{Q}}\) model becomes, apart from a numerical proportional factor,

\[
Z = \int_0^{2\pi} dX \sum_{N_{\mu} \in \mathbb{Z}} \delta_{\epsilon_{\mu\nu}\Delta_{\mu}N_{\nu}} \exp \left[ -\frac{R^2}{4\pi} (\Delta_{\mu}X - 2\pi N_{\mu})^2 \right],
\] (4.18)

where we rewrite \(m_{\mu} + Q n_{\mu} \rightarrow N_{\mu}\). Here and below, we suppress the subscripts for sites and correspondingly the product symbols for sites for notational simplicity. In the dual representation, corresponding to \(\tilde{n} \in \mathbb{Z}_Q\) and \(\psi \in \mathbb{Z}_K\) in section II, it is natural to take the limit as,

\[
\frac{2\pi \tilde{n}_{\mu}}{Q} \rightarrow \tilde{X}, \quad \frac{2\pi}{Q} \sum_{\tilde{n}_{\mu} \in \mathbb{Z}_Q} \rightarrow \int_0^{2\pi} d\tilde{X}, \quad \frac{2\pi}{Q} \sum_{\psi_{\mu} \in \mathbb{Z}_K} \rightarrow \int_{-\infty}^\infty d\psi_{\mu}.
\]

Similarly to the section II, the summation over \(N_{\mu}\) leads to \(\psi_{\mu} = \Delta_{\mu}\tilde{X} - 2\pi \tilde{m}_{\mu}, \quad \tilde{m}_{\mu} \in \mathbb{Z}\), and the summation over \(m_{\mu}\) imposes the constraint on \(\tilde{m}_{\mu}\). Thus the partition function after the dual transformation takes the same form as (4.18),

\[
Z \propto \int_0^{2\pi} d\tilde{X} \sum_{\tilde{m}_{\mu} \in \mathbb{Z}} \delta_{\epsilon_{\mu\nu}\Delta_{\mu}\tilde{m}_{\nu}} \exp \left[ -\frac{1}{4\pi R^2} (\Delta_{\mu}\tilde{X} - 2\pi \tilde{m}_{\mu})^2 \right],
\] (4.19)

with the correspondence \(R \leftrightarrow 1/R\). As has been discussed in [2] and the Appendix A of [1], it is also straightforward to show self-duality by a direct dual transformation (4.18) \(\leftrightarrow\) (4.19) in the continuous target space.

Let us next consider the \(Q \rightarrow \infty\) limit of the matrix model corresponding to the \(Z_{Q,\bar{Q}}\) model. Since the model is manifestly self-dual for an arbitrary integer \(Q\), all we need to check is whether the dual transformation laws of physical parameters are well-defined in the limit. There are two such parameters, namely, the compactification radius \(R\) and the coupling constant \(g\). First, as for \(R\), it follows from (4.18) and (4.19) that \(R\) is transformed as \(R \leftrightarrow 1/R\), since the transformation law of the Boltzmann factor is of course identical with the case of the fixed lattice. The transformation of the coupling constant is given by (3.14) which in the present limit leads to

\[
g^2 \rightarrow \tilde{g}^2 = \frac{g^2}{R}.
\] (4.20)

Thus both are well-defined in the \(c = 1\) limit.
In particular, nonsingular transformation property of the coupling constant would allow a well-defined double scaling limit in the sense of world sheet. From the relation (4.20), we expect that the critical point $g_c$ as a function of $R$ would behave as

$$g_c = f(R)R^{1/4},$$

where $f(R)$ is a function invariant under the dual transformation $R \leftrightarrow 1/R$. Unfortunately, however, self-duality alone is not powerful enough for obtaining exact results in the scaling limit. For example, it is possible that the function $f(R)$ actually vanishes in the $c = 1$ limit due to the degeneracy, as suggested from the counting argument of the previous subsection. In that case, the interplay between the $c = 1$ limit and the scaling limit is rather subtle, since taking the formal zero-coupling limit $g \to 0$ directly in the action is not meaningful and we have to examine the physical correlators as in the usual situation in general lattice field theories. In any case, however, our discussion is sufficient to show that the present models can serve as a nonperturbative regularization of the T-duality symmetry in the continuum critical string theories. In particular, it is guaranteed that the properly defined continuum partition function is duality symmetric, as has been explicitly computed in the standard $c = 1$ model of finite radius.

D. Conclusion

We have established a new class of extended matrix models which have manifest symmetry under the T-duality transformation. We can think of several possible extensions and applications of our work. Among others, we would like to mention the following.

1. Construction of manifestly T-duality symmetric string field theories: Using the macroscopic loop variables including the $L$ matrix, we can try to derive stochastic Hamiltonians following our previous works.

2. Extension to open strings: In particular, it is interesting to formulate some toy models for Dirichlet branes using the present method. It might be useful for studying the nonperturbative dynamics of the D-branes.

3. Inclusion of fermions: In particular, extension to supersymmetric models.

4. Application to critical strings by extending the target space to higher dimensional spaces: For this, however, we have to deepen our understanding of the matrix-model formulation of critical strings.

In all of these possibilities, it would be more or less crucial to develop some powerful methods for treating the scaling limit in order to perform useful nonperturbative analysis on the basis of our models. We hope to return to some of the above issues in future works.

The works of Y.O. and T.Y. are partially supported by the Japan Society for the Promotion of Science under the Predoctoral Research Program (No.08-4158) and the US-Japan Collaborative Program for Scientific Research, respectively.
REFERENCES

[1] T. Asatani, T. Kuroki, Y. Okawa, F. Sugino and T. Yoneya, preprint hep-th/9607218.
[2] D. Gross and I. Klebanov, Nucl. Phys. B344, 475 (1990).
[3] W. Siegel, Phys. Rev. D54, 2797 (1996).
[4] J. L. Cardy, J. Phys. A: Math. Gen. 13, 1507 (1980).
   F. C. Alcaraz and R. Köberle, J. Phys. A: Math. Gen. 13, L153 (1980).
   F. C. Alcaraz, J. Phys. A: Math. Gen. 20, 2511 (1987).
[5] S. M. Carroll, M. E. Ortiz and W. Taylor IV, Nucl. Phys. B468 [FS], 420 and 383 (1996).
[6] F. Sugino and T. Yoneya, Phys. Rev. D53, 4448 (1996).