Correlation Cube Attack Revisited

Improved Cube Search and Superpoly Recovery Techniques

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Asiacrypt 2023
An output bit of symmetric cipher could be written as a Boolean function of IV (plaintext) $x \in \mathbb{F}_2^n$ and key $k \in \mathbb{F}_2^m$. Given $I = \{i_0, \cdots, i_{d-1}\} \subset \{0, 1, \cdots, n-1\}$, one can write $f$ as

$$f(x, k) = f_i(x^I, k) \cdot x_I^1 + q_I(x, k).$$

Summing $f$ over all $2^d$ possible values of $x_I$, one has

$$\bigoplus_{C_I = \{x | x_I \in \mathbb{F}_2^d\}} f(x, k) = f_I(x^I, k).$$

**Cube attack**

**Preprocessing phase**: Recover the expressions of $f_I$ for multiple $I$.

**Online phase**: Calculate the values of $f_I$s, and solve the system of equations about key.
Let $f_i(x_J, k) = \bigoplus_{i=1}^{r} h_i q_i$, and $Q_i = \{h_i\}_i$ is called the basis of $f_i$.

- **Preprocessing Phase**
  1. Obtain the basis $Q_i$s for $f_i$s.
  2. Add tuples $(I, h_i, b)$ to $\Omega$ where $\Pr(h_i = b \mid f_i) > p$.

- **Online Phase**
  1. Randomly selects $\alpha$ values of $x_J$, checks if $f_i$ is zero constant
  2. Construct equations according to the element in $\Omega$. 

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Motivation

Assume $f_i(x_J, k) = \bigoplus_{i=1}^r h_i q_i$.

The case of constructing an erroneous equation: (for a fixed key)

1. $(I, h_i, 1) \in \Omega$: If $h_i = 0$, $\bigoplus_{j \neq i} h_j q_j = 1$ hold for certain values of $x_J$.
2. $(I, h_i, 0) \in \Omega$: If $h_i = 1$, $q_i = \bigoplus_{j \neq i} h_j q_j$ hold for all values of $x_J$.

Note that the occurrence of the first case is possible only when $r > 1$.

strategies:

1. Only use "special" ISoC $I$ that satisfy $f_I = hq$.
2. Infer the value of $h$ using multiple "special" ISoC $I_i$ that satisfy $f_{I_i} = h q_i$. 
New correlation cube attack

1 Preprocessing phase:
   a. Identify special ISoCs.
   b. For each $h$, let $T_h = \{ I : h | f_I \}$. 
   c. Let $T_1 = \{ T_h : \Pr(h = 0 | \forall I \in T_h : f_I = 0) \leq p \}$. 
   d. Let $T = \{ T_h : \Pr(h = 0 | \forall I \in T_h : f_I = 0) > p \}$. 

2 Online phase:
   a. Computes the value of $f_I$ for each ISoC $I$.
   b. For every $T_h$ in $T$, make a guess on the value of $h$ based on $f_I$’s value for all $I$ in $T_h$.
   c. For any $T_h$ in $T_1$, if $\exists I \in T_h$ satisfies $f_I = 1$, then $h = 1$. Otherwise, no guess is made for $h$.
   d. Store the equations $h = 1$ in to a set $G_1$, while store the other equations into a set $G_0$.
   e. Using these derived equations along with partial key guesses, we can try to obtain a candidate of the key.
      » If verifications for all partial key guesses do not yield a valid key, modify some equations from $G_0$ and solve again until a valid key is obtained.
Challenges

1. To acquire a significant number of special ISoCs.
   - Introduce a "vector numeric mapping" technique.
   - Propose an algorithm for fast search of lots of good ISoCs.

2. To decompose a complicated Boolean polynomial.
   - Propose "variable substitution" technique to recover superpolys.
Related work

- Search good \textit{ISoC}.
  
  1. Numeric mapping technique [Liu17]
  2. Division property + heuristic algorithms [YT21, CT22]

- Recover superpolys.
  
  1. Linearity tests [DS09]
  2. Degree tests [FV14]
  3. Division property [TIHM17, WHT$^+$18, WHG$^+$19, HLM$^+$20, HSWW20, HST$^+$21, HHPW22]
Vector degree

\[ f(x) = \bigoplus_{u \in \mathbb{F}_2^d} g_u(x_{I_c}) x_I^u \]

\[ \text{vdeg}_{[I,x]}(f) = \deg(g_{u_0}, g_{u_1}, \ldots, g_{u_{2^d-1}}) x_{I_c} = \left( \deg(g_{u_0}) x_{I_c}, \ldots, \deg(g_{u_{2^d-1}}) x_{I_c} \right) \]

- \( \deg(f) = \max_{0 \leq j < 2^{|I|}} \{ \text{vdeg}_j(f)[j] + \text{wt}(j) \} \).
- \( \text{vdeg}_{[I,x]}(f) \preceq v \Rightarrow \deg(f) \leq \max_{0 \leq j < 2^{|I|}} \{ \min \{ v[j], n - |I| \} + \text{wt}(j) \} \).
- If \( I_1 \subset I_2, \text{vdeg}_{I_1}(f)[j] = \max_{0 \leq j' < 2^{|I_2|} - |I_1|} \{ \text{vdeg}_{I_2}(f)[j' \cdot 2^{|I_1|} + j] + \text{wt}(j') \} \).
Vector degree

Example

\[ f = x_0 + x_0x_2 + x_1x_2x_3 + x_0x_1 \]

- \( I_2 = \{0, 1\}, f = 0 \cdot 1 + (1 + x_2) \cdot x_0 + x_2x_3 \cdot x_1 + 1 \cdot x_0x_1 \)
  \[ \text{vdeg}_{[I_2, x]}(f) = [-\infty, 1, 2, 0] \Rightarrow \deg(f) = \max\{-\infty + 0, 1 + 1, 2 + 1, 0 + 2\} = 3 \]

- \( I_1 = \{0\}, f = x_1x_2x_3 \cdot 1 + (1 + x_1 + x_2) \cdot x_0 \)
  \[ \text{vdeg}_{[I_1, x]}(f) = [3, 1] \Rightarrow \deg(f) = \max\{3 + 0, 1 + 1\} = 3 \]

- \( \text{vdeg}_{[I_1, x]}(f)[0] = \max\{\text{vdeg}_{[I_2, x]}[0] + 0, \text{vdeg}_{[I_2, x]}[2] + 1\} = \max\{\infty + 0, 2 + 1\} = 3 \)
  \[ \text{vdeg}_{[I_1, x]}(f)[1] = \max\{\text{vdeg}_{[I_2, x]}[1] + 0, \text{vdeg}_{[I_2, x]}[3] + 1\} = \max\{1 + 0, 0 + 1\} = 1 \]
A new method for vector degree evaluation

Let $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$, $f = \bigoplus_u a_u v^u$, $g : \mathbb{F}_2^m \rightarrow \mathbb{F}_2^n$

**Vector numeric mapping**

$VDEG_d : \mathbb{B}_n \times \mathbb{Z}^{n \times 2^d} \rightarrow \mathbb{Z}^{2^d}$

$(f, V) \mapsto v$

where $v[j] = \max_{a_u \neq 0} \max_{j_0, \ldots, j_{n-1}} \left\{ \sum_{i=0}^{n-1} u[i] V[i][j_i] \right\}$

$0 \leq j_i \leq u[i](2^d - 1)$

$j = \sqrt[n-1]{u[i]j_i}$

**Vector degree evaluation**

$vdeg_I(g) \leq V \quad \Rightarrow \quad vdeg_I(f \circ g) \leq VDEG_{|I|}(f, V)$
A new method for degree evaluation

Let \( f(x) = f_{r-1} \circ f_{r-2} \circ \cdots \circ f_0(x) \). We denoted the upper bound of the vector degree of \( f \) w.r.t. \( x \) and \( I \) by

\[
\hat{\nu}_{deg}[I,x](f) = \text{VDEG}(f_{r-1}, V_{r-2}),
\]

where \( V_i = \text{VDEG}(f_i, V_{i-1}), 0 < i \leq r - 2 \), and \( V_0 = \nu_{deg}[I,x](f_0) \).

Mode 1. \( \hat{\nu}_{deg}[I,x](f) = \max_{0 \leq j < 2^{|I|}} \{ \min \{ \hat{\nu}_{deg}[I,x](f)[j], n - |I| \} + \text{wt}(j) \} \).

Mode 2. \( \hat{\nu}_{deg}[I,x](f) = \hat{\nu}_{deg}[I,x](f)[2^{|I|} - 1] + |I| \).

Mode 3. \( \hat{\nu}_{deg}[I,x](f) = \max_{0 \leq j < 2^{|I|}} \{ \hat{\nu}_{deg}[I,x](f)[j] + \text{wt}(j) \} \).

Degree evaluation [Mode 1]

\[ \nu_{deg}(f) \leq \hat{\nu}_{deg}[I,x](f) \Rightarrow \nu_{deg}(f) \leq \nu_{deg}[I,x](f) \]
Estimation comparison between inclusion-based index set

\[ I_1 \subset I_2 \implies \hat{\text{deg}}_{[I_2,x]}(f) \leq \hat{\text{deg}}_{[I_1,x]}(f) \]

**Example**

Let \( f = y_0y_1, \ g = [x_0x_2 + x_1, x_0x_1 + x_3]. \) \( \text{deg}(f \circ g)? \) \( (f \circ g = x_0x_1 + x_0x_1x_2 + x_0x_2x_3 + x_1x_3) \)

- \( I_1 = \{1\}, \ V = \begin{bmatrix} \text{vdeg}_{I_1}(g_0) \\ \text{vdeg}_{I_1}(g_1) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \)
  \[ \text{vdeg}_{I_1}(f \circ g) = [3, 3] \implies \]
  
  Mode 1. \( \text{deg}(f \circ g) = 4, \)  
  Mode 2. \( \text{deg}(f \circ g) = 4, \)  
  Mode 3. \( \text{deg}(f \circ g) = 4 \)

- \( I_2 = \{0, 1\}, \ V = \begin{bmatrix} \text{vdeg}_{I_2}(g_0) \\ \text{vdeg}_{I_2}(g_1) \end{bmatrix} = \begin{bmatrix} -\infty \\ 1 \\ -\infty \end{bmatrix} \)
  \[ \text{vdeg}_{I_2}(f \circ g) = [\infty, 2, 1, 1] \implies \]
  
  Mode 1. \( \text{deg}(f \circ g) = 3, \)  
  Mode 2. \( \text{deg}(f \circ g) = 3, \)  
  Mode 3. \( \text{deg}(f \circ g) = 3 \)
Theorem 5. [This work]

Let $J \subset K \subset I$. Then we have

$$\hat{\text{vdeg}}_{[J,x_K]}(f|_{x_K^c} = 0) \preceq \hat{\text{vdeg}}_{[J,x_I]}(f|_{x_I^c} = 0).$$

If $\hat{\text{deg}}_{[J,x_K]}(f|_{x_K^c} = 0) \geq d$, then $\hat{\text{deg}}_{[J,x_I]}(f|_{x_I^c} = 0) \geq d$ for all ISoCs $I$ satisfying $K \subset I$.

- If ISoC $I$ satisfies that $\hat{\text{deg}}_{[J,x_I]}(f|_{x_I^c} = 0) \geq d$, iteratively choose a series of ISoCs $I \supsetneq I_1 \supsetneq \cdots \supsetneq I_q \supset J$ such that $\hat{\text{deg}}_{[J,x_{I_i}]}(f|_{x_{I_i}^c} = 0) \geq d$ for all $1 \leq i \leq q$ and $\hat{\text{deg}}_{[J,x_{I_i}']} (f|_{x_{I_i'}^c} = 0) < d$ for any $I' \subsetneq I_q$.

- Delete all the supersets of $I_q$. 

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Process of searching good ISoCs

Let $J$ be a given index set, $\Omega$ be the set of all subsets of $[n]$ containing $J$ and with size $k$, $d$ be a threshold of degree, and $a$ be the number of repeating times. The main steps are:

1. Prepare an empty set $\mathcal{I}$.
2. Select an element $I$ from $\Omega$ as an ISoC.
3. Compute $\hat{\deg}_{[J, x_i]}(f|_{x_i=0})$;
   a. If $\hat{\deg}_{[J, x_i]}(f|_{x_i=0}) < d$, then add $I$ to $\mathcal{I}$ and goto Step 5;
   b. otherwise, set $count = 0$ and goto Step 4.
4. $count = count + 1$. Let $I' = I$, randomly remove an element $i \in I' \setminus J$ from $I'$ and let $x_i = 0$. Compute $\hat{\deg}_{[J, x'_i]}(f|_{x'_i=0})$.
   a. If $\hat{\deg}_{[J, x'_i]}(f|_{x'_i=0}) < d$ and $count < a$, then goto Step 4;
   b. If $\hat{\deg}_{[J, x'_i]}(f|_{x'_i=0}) < d$ and $count \geq a$, then goto Step 5;
   c. If $\hat{\deg}_{[J, x'_i]}(f|_{x'_i=0}) \geq d$, then let $I = I'$ and goto Step 3.b;
5. Remove all the sets containing $I$ from $\Omega$. If $\Omega \neq \emptyset$, goto Step 2; otherwise, output $\mathcal{I}$.
MILP model for searching good ISoCs.

\[
b_i = \begin{cases} 
1, & i \in I \\
0, & \text{otherwise}
\end{cases}
\]

- To describe that the size of each element of \( \Omega \) is equal to \( k \), we use
  \[
  \sum_{i=0}^{n-1} b_i = k.
  \]

- To describe that each element of \( \Omega \) includes the set \( J \), we use
  \[
  b_j = 1 \text{ for } \forall j \in J.
  \]

- To describe removing all the sets that contain \( I \) from \( \Omega \), we use
  \[
  \sum_{i \in I} b_i < |I|.
  \]

**callback function** in Gurobi + **degree evaluation**
The framework of superpoly recovery [HST + 21]

Let $f(x, k) = f_{r-1} \circ f_{r-2} \circ \cdots \circ f_0(x, k)$ and denote the input and output of $f_i$ by $y_i$ and $y_{i+1}$, respectively.

$$\text{Coe}(f, x^u) = \bigoplus_{\pi_{u_{r_m}}(y_{r_m}) \in V_{r_m}} \text{Coe}(\pi_{u_{r_m}}(y_{r_m}), x^u).$$

The specific steps of recovering a superpoly requires two steps:

1. Try to obtain $V_{r_m}$. If the model is solved within an acceptable time, goto Step 2.

2. For each term $\pi_{u_{r_m}}(y_{r_m})$ in $V_{r_m}$, compute $\text{Coe}(\pi_{u_{r_m}}(y_{r_m}), x^u)$ with our new techniques and sum them.
Variable substitution technique for coefficient recovery

Let $f(x, k) = f_{r-1} \circ f_{r-2} \circ \cdots \circ f_0(x, k)$ Let $\tilde{f}_{r_m}$ denote $f_{r_m-1} \circ \cdots \circ f_0$, i.e., $y_{r_m} = \tilde{f}_{r_m}(x, k)$. Assume the algebraic normal form of $\tilde{f}_{r_m}$ in $x$ is

$$\tilde{f}_{r_m} = \bigoplus_{v \in \mathbb{F}_2^n} h_v(k)x^v.$$ 

Introduce new intermediates $z$ to substitute these nonzero $h_v[j]$’s. From the ANF of $\tilde{f}_{r_m}$, it is natural to derive the new representation $g_{r_m}$ such that $y_{r_m} = g_{r_m}(x, z)$, whose ANF in $x$ and $z$ can be written as

$$g_{r_m}[j] = \bigoplus_{v} a_{v,j} z^{e_{v,j}} x^v.$$ 

The process of recovering $\text{Coe}(\pi_{u_{r_m}}(y_{r_m}), x''')$ is as follows:

1. Compute the ANF of $y_{r_m}$ in $x$.
2. Substitute all different non-constant $h_v[j]$ for all $v$ and $j$ by new variables $z$.
3. Recover $\text{Coe}(\pi_{u_{r_m}}(y_{r_m}), x''')$ in $z$ by monomial prediction.
An example to illustrate the process

Example

Assume $y_{rm} = f_{rm}(x, k) = [(k_0k_1 \oplus k_2k_5 \oplus k_9 + k_{10})x_0x_2 \oplus (k_3 \oplus k_6)x_5, (k_2k_7 \oplus k_8)x_3 \oplus x_6x_7]$.

**Variable substitution:** $k_0k_1 \oplus k_2k_5 \oplus k_9 + k_{10} \rightarrow z_0, \quad k_3 \oplus k_6 \rightarrow z_1, \quad k_2k_7 \oplus k_8 \rightarrow z_2$

$\Rightarrow y_{rm} = g_{rm}(x, z) = [z_0x_0x_2 \oplus z_1x_5, z_2x_3 \oplus x_6x_7]$.

- To compute $\text{Coe}(y_{rm}[0]y_{rm}[1], x_0x_2x_3)$, at least $4 \times 2 = 8$ monomial trails $k^w x_0x_2x_3 \leadsto y_{rm}[0]y_{rm}[1]$ to form $(k_0k_1 \oplus k_2k_5 \oplus k_9 + k_{10})(k_2k_7 \oplus k_8)x_0x_2x_3$.

- After variable substitution, there remains only one trail $z_0z_2x_0x_2x_3$, which means we have consolidated 8 monomial trails into a single one.

- **Reduce the number of monomial trails.**

- **Make the superpoly more concise and easy to factorize.**
Trivium stream cipher [De 06]

Padding:

\[(s_0, s_1, \ldots, s_{92}) \leftarrow (k_0, k_1, \ldots, k_{79}, 0, \ldots, 0)\]

\[(s_{93}, s_{94}, \ldots, s_{176}) \leftarrow (v_0, v_1, \ldots, v_{79}, 0, \ldots, 0)\]

\[(s_{177}, s_{178}, \ldots, s_{287}) \leftarrow (0, 0, \ldots, 0, 1, 1, 1)\].

Update:

\[s_{92} \leftarrow s_{65} \oplus s_{90} \cdot s_{91} \oplus s_{92} \oplus s_{170}\]

\[s_{176} \leftarrow s_{161} \oplus s_{174} \cdot s_{175} \oplus s_{176} \oplus s_{263}\]

\[s_{287} \leftarrow s_{242} \oplus s_{285} \cdot s_{286} \oplus s_{287} \oplus s_{68}\]

Output:

\[z = s_{65} \oplus s_{92} \oplus s_{161} \oplus s_{176} \oplus s_{242} \oplus s_{287}\]
Parameter settings:

- **Search ISoCs**: Mode = 2;
  1. 820 rounds: \( J = \{0, 1, 2, i, i + 1\} \), where \( 3 \leq i \leq 26 \); \( \Omega = \{I \supset J : |I| = 38\} \); \( d = 41 \).
  2. 825 rounds: \( J = \{0, 1, \ldots, 10\} \setminus \{j_0, j_1, j_2\} \), where \( j_0 > 1, j_1 > j_0 + 1 \) and \( j_1 + 1 < j_2 < 11 \);
     \( \Omega = \{I \supset J : |I| = 41\} \); \( d = 44 \).
  3. 830 rounds: \( J = \{0, 1, \ldots, 10\} \setminus \{j_0, j_1, j_2\} \), where \( j_0 > 2, j_1 > j_0 + 1 \) and \( j_1 + 1 < j_2 < 11 \);
     \( \Omega = \{I \supset J : |I| = 41\} \); \( d = 45 \).

- Recover superpolys: \( r_m = 200 \).
- New correlation cube attack: \( p = 0.77 \)

| # of Rounds | size of ISoC | # of ISoCs | Total time | # of keys | Ref. |
|-------------|-------------|------------|------------|----------|-----|
| 820         | 38          | \( 2^{13} \) | \( 2^{52} \) | \( 2^{79.2} \) | This work |
| 820         | 38          | \( 2^{13} \) | \( 2^{60} \) | \( 2^{79.2} \) | This work |
| 825         | 41          | \( 2^{12} \) | \( 2^{54} \) | \( 2^{79.3} \) | This work |
| 825         | 41          | \( 2^{12} \) | \( 2^{60} \) | \( 2^{79.7} \) | This work |
| 830         | 41          | \( 2^{13} \) | \( 2^{55} \) | \( 2^{78.9} \) | This work |
| 830         | 41          | \( 2^{13} \) | \( 2^{60} \) | \( 2^{79.4} \) | This work |
We give a generalized definition of degree of Boolean function and give out a degree evaluation method with the vector numeric mapping technique.

We introduce a pruning technique to fast filter the ISoCs and describe it into an MILP model to search automatically.

Propose a variable substitution technique for cube attacks, which makes great improvement to the computational complexity of superpoly recovery and can provide more concise expression in new variables.

We perform practical key recovery attacks on 820-, 825- and 830-round Trivium cipher, promoting up to 10 more rounds than previous best practical attacks as we know.
Thanks for your attention!
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