FINE ASYMPTOTICS FOR THE MAXIMUM DEGREE IN WEIGHTED RECURSIVE TREES WITH BOUNDED RANDOM WEIGHTS

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Abstract. A weighted recursive tree is an evolving tree in which vertices are assigned random vertex-weights and new vertices connect to a predecessor with a probability proportional to its weight. Here, we study the maximum degree and near-maximum degrees in weighted recursive trees when the vertex-weights are almost surely bounded. We are able to specify higher-order corrections to the first order growth of the maximum degree established in prior work. The accuracy of the results depends on the behaviour of the weight distribution near the largest possible value and in certain cases we manage to find the corrections up to random order. Additionally, we describe the tail distribution of the maximum degree, the distribution of the number of vertices attaining the maximum degree and establish asymptotic normality of the number of vertices with near-maximum degree. Our analysis extends the results proved for random recursive trees (where the weights are constant) to the case of random weights. The main technical result shows that the degrees of several uniformly chosen vertices are asymptotically independent with explicit error corrections.

1. Introduction

The Weighted Recursive Tree model (WRT), first introduced by Borovkov and Vatutin [4], is a recursive tree process \((T_n, n \in \mathbb{N})\) and a generalisation of the random recursive tree model. Here we consider a variation, first studied by Hiesmayr and Işlak [10], where the first vertex does not necessarily have weight one. Let \((W_i)_{i \in \mathbb{N}}\) be a sequence of positive vertex-weights. Initialise the process with the tree \(T_1\), which consists of the vertex 1 (which denotes the root) and assign vertex-weight \(W_1\) to it. Recursively, at every step \(n \geq 2\), we obtain \(T_n\) by adding to \(T_{n-1}\) the vertex \(n\), assigning vertex-weight \(W_n\) to it and connecting \(n\) to a vertex \(i \in [n-1]\), which, conditionally on the vertex-weights \(W_1, \ldots, W_{n-1}\), is selected with a probability proportional to \(W_i\). In this paper, we consider edges to be directed towards the vertex with the smaller label. We note that allowing every vertex to connect to \(m \in \mathbb{N}\) many predecessors, each one selected independently, yields the more general Weighted Recursive Graph model (WRG) introduced in [14]. The focus of this paper is the WRT model in the case when the vertex-weights are almost surely bounded random variables.

Lodewijks and Ortgiese [14] established that, in the case of positive, bounded random vertex-weights, the maximum degree \(\Delta_n\) of the WRG model grows logarithmically and that \(\Delta_n / \log n \xrightarrow{a.s.} 1 / \log \theta_m\), where \(\theta_m := 1 + E[W] / m\) with \(E[W]\) the mean of the vertex-weight distribution and \(m \in \mathbb{N}\) the out-degree of each vertex. Note that setting \(m = 1\) yields the result for the WRT model. In this paper, we improve this result by describing the higher-order asymptotic behaviour of the maximum degree when the vertex-weights are almost surely bounded. In this case we are able to distinguish several classes of vertex-weight distributions for which different higher-order behaviour can be observed.

Beyond the initial work of Borovkov and Vatutin and also Hiesmayr and Işlak studying the height, depth and size of branches of the WRT model, other properties such as the degree distribution, large and maximum degrees, and weighted profile and height of the tree have been studied. Mailler and Uribe Bravo [16], as well as Sénizergues [18] and Sénizergues and Pain [17] study the weighted profile and height of the WRT model. Mailler and Uribe Bravo consider random vertex-weights with particular distributions, whereas Sénizergues and Pain allow for a more general model with both sequences of deterministic as well as random weights.

Iyer [11] and the more general work by Fountoulakis and Iyer [7] study the degree distribution of a large class of evolving weighted random trees, and Lodewijks and Ortgiese [14] study the degree distribution of the WRG model. In both cases, the WRT model is a particular example of the
models studied and all results prove the existence of an almost sure limiting degree distribution for the empirical degree distribution.

Finally, Lodewijks and Ortgiese [14] and Lodewijks [13] study the maximum degree and the labels of the maximum degree vertices of the WRG model for a large range of vertex-weight distributions. In particular, a distinction between distributions with unbounded support and bounded support is observed. In the former case the behaviour and size of the label of the maximum degree is mainly controlled by a balance of vertices being old (i.e. having a small label) and having a large vertex-weight. In the latter case, due to the fact that the vertex-weights are bounded, the behaviour is instead controlled by a balance of vertices being old and having a degree which significantly exceeds their mean degree.

A particular case of the WRT model is the Random Recursive Tree (RRT) model, which is obtained when each vertex-weight equals one almost surely. As a result, techniques used to study the maximum degree in the RRT model can be adapted to analyse the maximum degree in the WRT model. Lodewijks and Ortgiese [14] demonstrate this by adapting the approach of Devroye and Lu [5] for proving the almost sure convergence of the rescaled maximum degree in the Directed Acyclic Graphs model (DAG) (the multigraph case of the RRT model) and using it for the analysis of the maximum degree in the WRG model, as discussed above. Hence, we survey the development of the properties of the maximum degree of the RRT model.

Szymański was the first to study the maximum degree of the RRT model and proved its convergence of the mean; \( \mathbb{E}[\Delta_n / \log n] \to 1 / \log 2 \). Later, Devroye and Lu [5] extend this to almost sure convergence and extended this to the DAG model as well. Goh and Schmutz [9] showed that \( \Delta_n - [\log_2 n] \) converges in distribution along suitable subsequences and identified possible distributions for the limit. Adarrio-Berry and Eslava [1] provide a precise characterisation of the subsequential limiting distribution of rescaled large degrees in terms of a Poisson point process as well as a central limit theorem result for near-maximum degrees (of order \( \log_2 n - i_n \), where \( i_n \to \infty, i_n = o(\log n) \)). Eslava [6] extends this to the joint convergence of the degree and depth of high degree vertices.

In this paper we adapt part of the techniques developed by Adarrio-Berry and Eslava in [1]. They consist of two main components: First, they establish an equivalence between the RRT model and a variation of the Kingman \( n \)-coalescent and use this to provide a detailed asymptotic description of the tail distribution of the degrees of \( k \) vertices selected uniformly at random, for any \( k \in \mathbb{N} \). This variation of the Kingman \( n \)-coalescent is a process which starts with \( n \) trees, each consisting of only a single root. Then, at every step 1 through \( n - 1 \), a pair of roots is selected uniformly at random and independently of this selection, each possibility with probability \( 1/2 \), one of the two roots is connected to the other with a directed edge. This reduces the number of trees by one and, after \( n - 1 \) steps, yields a directed tree. It turns out that this directed tree is equal in law to the random recursive tree. In the \( n \)-coalescent all \( n \) roots in the initialisation are equal in law and the degrees of the vertices are exchangeable. This allows Adarrio-Berry and Eslava to obtain the degree tail distribution with a precise error rate. Second, this precise tail distribution is used to obtain joint factorial moments of the quantities

\[
X_i^{(n)} := |\{ j \in [n] : Z_n(j) = [\log_2 n] + i \}|, \quad i \in \mathbb{Z},
\]

\[
X_i^{(n)} := |\{ j \in [n] : Z_n(j) \geq [\log_2 n] + i \}|, \quad i \in \mathbb{Z},
\]

where \( Z_n(j) \) denotes the in-degree of vertex \( j \) in the tree of size \( n \). The joint factorial moments of these \( X_i^{(n)} \) are used to identify the limiting distribution of high degrees in the tree. The sub-sequential convergence, as mentioned above, is due to the floor function applied to \( \log_2 n \) and the integer-valued in-degrees \( Z_n(j) \).

For the WRT model, however, it provides no advantage to construct a ‘weighted’ Kingman \( n \)-coalescent to obtain precise asymptotic expression for the tail distribution of vertex degrees. As pairs of roots in the Kingman \( n \)-coalescent are selected uniformly at random and hence the roots are equal in law, it is not necessary to keep track of which roots are selected at what step. In a weighted version of the Kingman \( n \)-coalescent, pairs of roots would have to be selected with probabilities proportional to their weights, so that it is necessary to record which roots are selected at which step. As a result, a weighted Kingman \( n \)-coalescent is not (more) useful in analysing the tail distribution of vertex degrees.

Instead, we improve results on the convergence of the empirical degree distribution of the WRT model obtained by Iyer [11] and Lodewijks and Ortgiese [14]. We obtain a convergence rate to
the limiting degree distribution, the asymptotic empirical degree distribution for degrees \( k = k(n) \) which diverge with \( n \), as well as asymptotic independence of degrees of vertices selected uniformly at random. We combine this with the joint factorial moments of quantities similar to \((1.1)\) and use the techniques developed by Adarrio-Berry and Eslava [1] to derive fine asymptotics of the maximum degree in the WRT model.

**Notation.** Throughout the paper we use the following notation: we let \( \mathbb{N} := \{1,2,\ldots\} \) be the natural numbers, set \( \mathbb{N}_0 := \{0,1,\ldots\} \) to include zero and let \( [t] := \{i \in \mathbb{N} : i \leq t\} \) for any \( t \geq 1 \). For \( x \in \mathbb{R} \), we let \( [x] := \inf\{n \in \mathbb{Z} : n \geq x\} \) and \( \lfloor x \rfloor := \sup\{n \in \mathbb{Z} : n < x\} \), and for \( x \in \mathbb{R}, k \in \mathbb{N} \), let \((x)_k := x(x-1)(x-2)\cdots(x-(k-1))\), \( (x)^{(k)} := x(x+1)(x+2)\cdots(x+(k-1))\), and \((x)_0 = (x)^{(0)} := 1\). Moreover, for sequences \((a_n,b_n)_{n \in \mathbb{N}}\) such that \( b_n \) is positive for all \( n \) we say that \( a_n = o(b_n) \), \( a_n \sim b_n \), \( a_n = \mathcal{O}(b_n) \) if \( \lim_{n \to \infty} a_n/b_n = 0 \), \( \lim_{n \to \infty} a_n/b_n = 1 \) and if there exists a constant \( C > 0 \) such that \( |a_n| \leq Cb_n \) for all \( n \in \mathbb{N} \), respectively. For random variables \( X, (X_n)_{n \in \mathbb{N}} \) we denote \( X_n \xrightarrow{d} X, X_n \xrightarrow{p} X \) and \( X_n \xrightarrow{a.s.} X \) for convergence in distribution, probability and almost sure convergence of \( X_n \) to \( X \), respectively. Also, we write \( X_n \xrightarrow{op} 1 \) if \( X_n \xrightarrow{p} 0 \). Finally, we use the conditional probability measure \( \mathbb{P}_W(\cdot) := \mathbb{P}(\cdot | (W_i)_{i \in \mathbb{N}}) \) and conditional expectation \( \mathbb{E}_W[\cdot] := \mathbb{E}[\cdot | (W_i)_{i \in \mathbb{N}}) \), where the \((W_i)_{i \in \mathbb{N}}\) are the i.i.d. vertex-weights of the WRT model.

2. Definitions and main results

The weighted recursive tree (WRT) model is a growing random tree model that generalises the random recursive tree (RRT), in which vertices are assigned (random) weights and new vertices connect with existing vertices with a probability proportional to the vertex-weights.

The definition of the WRT model follows the one in [10]:

**Definition 2.1** (Weighted Recursive Tree). Let \((W_i)_{i \geq 1}\) be a sequence of i.i.d. copies of a positive random variable \( W \) such that \( \mathbb{P}(W > 0) = 1 \) and set

\[
S_n := \sum_{i=1}^{n} W_i.
\]

We construct the *weighted recursive tree* as follows:

1) Initialise the tree with a single vertex \( 1 \), denoted as the root, and assign to the root a vertex-weight \( W_1 \). Denote this tree by \( T_1 \).

2) For \( n \geq 1 \), introduce a new vertex \( n+1 \) and assign to it the vertex-weight \( W_{n+1} \). Conditionally on \( T_n \), connect to some \( i \in [n] \) with probability \( W_i/S_n \). Denote the resulting tree by \( T_{n+1} \).

We treat \( T_n \) as a directed tree, where edges are directed from new vertices towards old vertices.

**Remark 2.2.** *(i)* Note that the edge connection probabilities are invariant under a rescaling of the vertex-weights. In particular, we may without loss of generality assume for vertex-weight distributions with bounded support that \( x_0 := \sup\{x \in \mathbb{R} : \mathbb{P}(W \leq x) < 1\} = 1 \).

*(ii)* The model can be adapted to allow for a random out-degree: at every step the newly introduced vertex \( n+1 \) connects with every vertex \( i \in [n] \) independently with a probability equal to \( W_i/S_n \). The results presented below still hold for this model definition as well.

Lodewijks and Ortgiese studied certain properties of the Weighted Recursive Graph (WRG) model in [14]. This is a more general version of the WRT model that allows every vertex to connect to \( m \in \mathbb{N} \) vertices when introduced, yielding a multigraph when \( m > 1 \). This paper aims to recover and extend some of these results in the case \( m = 1 \) when the vertex-weights are almost surely bounded, i.e. \( x_0 < \infty \). As stated in Remark 2.2(ii), we can set \( x_0 = 1 \) without loss of generality. To formulate the results we need to assume that the distribution of the weights is sufficiently regular, allowing us to control their extreme value behaviour. In certain cases it is more convenient to formulate the assumptions in terms of the distribution of the random variable \((1 - W)^{-1}\):

**Assumption 2.3** (Vertex-weight distribution). The vertex-weights \( W_i, (W_i)_{i \in \mathbb{N}} \) are i.i.d. strictly positive random variables, which are:

- Bounded from above almost surely, such that \( x_0 := \sup\{x \in \mathbb{R} : \mathbb{P}(W \leq x) < 1\} = 1 \).
- Bounded away from zero almost surely: \( \exists w^* \in (0,1) \) such that \( \mathbb{P}(W \geq w^*) = 1 \).
Furthermore, the vertex-weights satisfy one of the following conditions:

**(Atom)** The vertex weights follow a distribution that has an atom at one, i.e., there exists a \( q_0 \in (0,1) \) such that \( \mathbb{P}(W = 1) = q_0 \). (Note that \( q_0 = 1 \) recovers the RRT model)

**(Weibull)** The vertex-weights follow a distribution that belongs to the Weibull maximum domain of attraction (MDA). This implies that there exist some \( \alpha > 1 \) and a positive function \( \ell \) which is slowly varying at infinity, such that
\[
\mathbb{P}(W \geq 1 - 1/x) = \mathbb{P}(1 - W)^{-1} \geq x = \ell(x)x^{-(\alpha - 1)}, \quad x \geq (1 - w^*)^{-1}.
\]

**(Gumbel)** The distribution belongs to the Gumbel maximum domain of attraction (MDA). This implies that there exist sequences \((a_n, b_n)_{n \in \mathbb{N}}, \) such that
\[
\max_{i \in [n]} \frac{W_i - b_n}{a_n} \xrightarrow{d} \Lambda,
\]
where \( \Lambda \) is a Gumbel random variable.

Within this class, we further distinguish the following two sub-classes:

**(RV)** There exist \( a, c, \tau > 0, \) and \( b \in \mathbb{R} \) such that
\[
\mathbb{P}(W > 1 - 1/x) = \mathbb{P}(1 - W)^{-1} > x \sim ax^b e^{-(x/c_1)^c} \quad \text{as } x \to \infty.
\]

**(RaV)** There exist \( a, c_1 > 0, b \in \mathbb{R}, \) and \( \tau > 1 \) such that
\[
\mathbb{P}(W > 1 - 1/x) = \mathbb{P}(1 - W)^{-1} > x \sim a(\log x)^b e^{-(\log(x)/c_1)^c} \quad \text{as } x \to \infty.
\]

**Remark 2.4.** The assumption that the vertex-weights are bounded away from zero is required only for a very specific part of the proof of Proposition 5.1. Though we were unable to omit this assumption, we believe it is a mere technicality that can be overcome or at the very least replaced by weaker conditions.

Throughout, we will write
\[
Z_n(i) := \text{in-degree of vertex } i \text{ in } T_n.
\]

Working with the in-degree allows us to (in principle) generalise our methods to graphs with random out-degree, as mentioned in Remark 2.2. Obviously, if the out-degree is fixed, we can recover the results for the degree from our results on the \( Z_n(i) \).

In [14], the following results are obtained for the WRG model: if we let \( \theta_m := 1 + \mathbb{E}[W]/m, \) and
\[
p_k(m) := \mathbb{E} \left[ \frac{\theta_m - 1}{\theta_m - 1 + W} \left( \frac{W}{\theta_m - 1 + W} \right)^k \right], \quad p_{\geq k}(m) := \sum_{j=k}^{\infty} p_k(m) = \mathbb{E} \left[ \left( \frac{W}{\theta_m - 1 + W} \right)^k \right], \quad (2.1)
\]
then almost surely for any \( k \in \mathbb{N} \) fixed,
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{Z_n(i) = k\} = p_k(m), \quad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{Z_n(i) \geq k\} = p_{\geq k}(m), \quad (2.2)
\]
whenever \( W \) follows a distribution with a finite mean. In particular, the above is satisfied for all cases in Assumption 2.3. Moreover, if the vertex-weights are bounded almost surely (without loss of generality \( x_0 = 1 \)),
\[
\max_{i \in [n]} \frac{Z_n(i)}{\log_{\theta_m} n} \xrightarrow{a.s.} 1.
\]

In this paper we improve these results when considering the WRT model with almost surely bounded weights. That is, we consider the case \( m = 1 \). For ease of writing, we let
\[
\theta := \theta_1 = 1 + \mathbb{E}[W] \quad \text{and} \quad p_k := p_k(1), p_{\geq k} := p_{\geq k}(1).
\]

First, we are able to extend the result in (2.2) to the case when \( k = k(n) \) that diverges with \( n \) in the sense that the difference between both sides converges to zero in mean, under certain constraints on \( k(n) \), and we obtain a convergence rate as well. Combining this result with techniques developed by Addario-Berry and Esclava in [1] for random recursive trees we are then able to identify the higher-order asymptotic behaviour of the maximum degree depending on the cases in Assumption 2.3. Additionally, in certain cases we are able to derive an asymptotic tail distribution for the maximum degree and obtain an asymptotic normality result for the number of vertices with ‘near-maximal’ degrees (in certain cases). These results can be extended to the model with a random out-degree as mentioned in Remark 2.2 as well.
Define $\theta := 1 + \mathbb{E}[W]$ and
\[
X^{(n)}_i := \{j \in [n] : Z_n(j) = \lfloor \log_\theta n \rfloor + i\}, \\
X^{(n)}_\geq i := \{j \in [n] : Z_n(j) \geq \lfloor \log_\theta n \rfloor + i\}.
\] (2.3)

For certain classes of vertex-weight distributions, we can prove the distributional convergence of these quantities along subsequences, as is the case for the RRT model in [1]. This result can be formulated in terms of convergence of point processes. Let $\mathbb{Z}^\ast := \mathbb{Z} \cup \{\infty\}$ and endow $\mathbb{Z}^\ast$ with the metric $d(i, j) = [2^{-i} - 2^{-j}]$ and $d(i, \infty) = 2^{-i}$, $i, j \in \mathbb{Z}$, and let $\mathcal{M}_\theta^\ast$ be the space of bounded finite measures on $\mathbb{Z}^\ast$. If we let $\mathcal{P}$ be a Poisson point process on $\mathbb{R}$ with intensity measure $\lambda(dx) := q_0 \theta^{-x} \log \theta \, dx$, $q_0 \in (0, 1]$, and define
\[
\mathcal{P}^\varepsilon := \sum_{x \in \mathcal{P}} \delta_{\lfloor x + \varepsilon \rfloor}, \\
\mathcal{P}^{(n)} := \sum_{i \in [n]} \delta_{Z_n(i) - \lfloor \log_\theta n \rfloor}, \\
\varepsilon_n := \log n - \lfloor \log_\theta n \rfloor,
\] (2.4)
then we can provide conditions such that $\mathcal{P}^{(n)}$ converges weakly to $\mathcal{P}^\varepsilon$, for subsequences $(n_\mathcal{E})_{\mathcal{E} \in \mathcal{N}}$ such that $\varepsilon_n \to \varepsilon$ as $\mathcal{E} \to \infty$. We abuse notation to write $\mathcal{P}^\varepsilon(\{i\}) = \{x \in \mathcal{P} : x = i\} = \{(x \in \mathcal{P} : x \in [1 + \varepsilon, i + 1 - \varepsilon)]\}$.

We now state our main results, which we split into several theorems based on the cases in Assumption 2.3.

**Theorem 2.5** (High degrees in WRTs (Atom) case). Consider the WRT model in Definition 2.1 with vertex-weights $(W_i)_{i \in [n]}$ that satisfy the (Atom) case in Assumption 2.3 for some $q_0 \in (0, 1]$. Fix $\varepsilon \in [0, 1]$. Let $(n_\mathcal{E})_{\mathcal{E} \in \mathcal{N}}$ be a positive integer sequence such that $\varepsilon_{n_\mathcal{E}} \to \varepsilon$ as $\mathcal{E} \to \infty$. Then $\mathcal{P}^{(n_\mathcal{E})}$ converges weakly to $\mathcal{P}^\varepsilon$ as $\mathcal{E} \to \infty$. Equivalently, for any $\varepsilon < \varepsilon' \in \mathbb{Z}$, jointly as $\mathcal{E} \to \infty$,
\[
(X^{(n_\mathcal{E})}_i, X^{(n_\mathcal{E})}_{i+1}, \ldots, X^{(n_\mathcal{E})}_{i-\varepsilon}, X^{(n_\mathcal{E})}_{2\varepsilon'}) \overset{d}{\to} (\mathcal{P}^\varepsilon(i), \mathcal{P}^\varepsilon(i+1), \ldots, \mathcal{P}^\varepsilon(i'-1), \mathcal{P}^\varepsilon(i'+\varepsilon)),
\]
for subsequences $(\mathcal{E})_{\mathcal{E} \in \mathcal{N}}$.

We note that this result only holds for the RRT model in which $q_0 := \mathbb{P}(W = 1) = 1$. In Theorem 2.5 we allow $q_0 \in (0, 1)$ as well, under the additional assumption that $\mathbb{P}(W \geq w^*) = 1$ for some $w^* \in (0, 1)$.

When the vertex-weight distribution belongs to the Weibull MDA, we can prove convergence in probability under a deterministic second-order scaling, but are unable to obtain what we conjecture to be a random third-order term similar to the result in Theorem 2.5:

**Theorem 2.6** (High degrees in WRTs, (Weibull) case). Consider the WRT model in Definition 2.1 with vertex-weights $(W_i)_{i \in [n]}$ that satisfy the (Weibull) case in Assumption 2.3 for some $\alpha > 1$. Then,
\[
\max_{i \in [n]} \frac{Z_n(i) - \log_\theta n}{\log_\theta \log_\theta n} \overset{p}{\to} -0(\alpha - 1).
\]

Finally, when the vertex-weight distribution belongs to the Gumbel MDA, we have similar results compared to the Weibull MDA case in the above theorem. Here we are also able to obtain a deterministic second-order scaling and we provide bounds for the third- and fourth-order behaviour of the maximum degree in a particular sub-case as well:

**Theorem 2.7** (High degrees in WRTs, (Gumbel) case). Consider the WRT model in Definition 2.1 with vertex-weights $(W_i)_{i \in [n]}$ that satisfy the (Gumbel) case in Assumption 2.3. In the (RV) sub-case, recall $\gamma := 1/(1 + \tau)$. Then,
\[
\max_{i \in [n]} \frac{Z_n(i) - \log_\theta n}{(\log_\theta n)^{1-\gamma}} \overset{p}{\to} \frac{\gamma}{(1-\gamma) \log_\theta \left(1 - \theta^{-1} / c_1 \right)^{1-\gamma}} := -C_{\theta, \tau, c_1}.
\] (2.5)

In the (RaV) sub-case,
\[
\max_{i \in [n]} \frac{Z_n(i) - \log_\theta n + C_1 (\log_\theta \log_\theta n)^{\tau} - C_2 (\log_\theta \log_\theta n)^{\tau-1} \log_\theta \log_\theta n}{(\log_\theta \log_\theta n)^{\tau-1}} \overset{p}{\to} C_3,
\] (2.6)
where
\[
C_1 := (\log_\theta)^{\tau-1} c_1^{-\tau}, \\
C_2 := (\log_\theta)^{\tau-1} \tau (\tau - 1) c_1^{-\tau}, \\
C_3 := (\log_\theta (\log_\theta (\tau - 1) \log_\theta - \log(\epsilon c_1^{-1} (1 - \theta^{-1})/\tau)) (\log_\theta)^{\tau-2} \tau c_1^{-\tau}.
\] (2.7)
We see that only in the \textbf{(Atom)} case we are able to obtain the higher-order asymptotics up to random order. This is due to the fact that, in this particular case, the vertices with high degree all have vertex-weight one. In the other classes covered in Theorems 2.6 and 2.7 vertices with high degrees have a vertex-weight close to one, which causes their degrees to grow slightly slower. This results in the higher-order asymptotics as observed in these theorems.

We are able to obtain more precise results related to the maximum and near-maximum degree vertices in the \textbf{(Atom)} case as well, which again recover and extend the results in [1].

**Theorem 2.8** (Asymptotic tail distribution for maximum degree in \textbf{(Atom)} case).

Consider the WRT model in Definition 2.1 with vertex-weights \((W_i)_{i \in [n]}\) that satisfy the \textbf{(Atom)} case in Assumption 2.3 for some \(q_0 \in (0, 1]\) and recall \(\varepsilon_n\) from (2.4). Then, for any \(i = i(n)\) with \(i + \log \theta n < (\theta/(\theta - 1)) \log n\) and \(\lim \inf_{n \to \infty} i > -\infty\),

\[
P\left(\max_{j \in [n]} Z_n(j) \geq \left(\log \theta n \right) + i \right) = \left(1 - \exp\left(-q_0 \theta^{-i+\varepsilon_n}\right)\right)(1 + o(1)).
\]

Moreover, let \(M_n \subseteq [n]\) denote the set of vertices that attain the maximum degree in \(T_n\), fix \(\varepsilon \in [0, 1]\) and let \((\nu_t)_{t \in \mathbb{N}}\) be a positive integer sequence such that \(\varepsilon_{n_t} \to \varepsilon\) as \(t \to \infty\). Then, \(|M_{n_t}| \xrightarrow{d} M_\varepsilon\), where \(M_\varepsilon\) has distribution

\[
P(M_\varepsilon = k) = \sum_{j \in \mathbb{Z}} \frac{1}{k!} (q_0(1-\theta^{-1})\theta^{-j+i})^k e^{-q_0\theta^{-j+i}}, \quad k \in \mathbb{N}.
\] (2.8)

Finally, we establish an asymptotic normality result for the number of vertices which have ‘near-maximum’ degrees. For a precise definition of ‘near-maximum’, we define sequences \((s_k, r_k)_{k \in \mathbb{N}}\) as

\[
s_k := \inf \left\{ x \in (0, 1) : \mathbb{P}(W \in (x, 1)) \leq \exp\left\{-(1-\theta^{-1})(1-x)k\right\} \right\},
\]

\[
r_k := \exp\left\{-(1-\theta^{-1})(1-s_k)k\right\}.
\] (2.9)

As a result, \(r_k\) can be used as the error term in the asymptotic expression of \(p_{\geq k}\) (as in (2.1)) when the weight distribution satisfies the \textbf{(Atom)} case (see Theorem 5.3) and is essential in quantifying how much smaller ‘near-maximum’ degrees are relative to the maximum degree of the graph in this case. We note that \(r_k\) is decreasing and converges to zero with \(k\) (see Lemma 8.3), and that in the definition of \(s_k, r_k\) we can allow the index to be continuous rather than just an integer (the proof of Lemma 8.3 can be adapted to still hold in this case). We can then formulate the following theorem:

**Theorem 2.9** (Asymptotic normality of near-maximum degree vertices, \textbf{(Atom)} case).

Consider the WRT model in Definition 2.1 with vertex-weights \((W_i)_{i \in [n]}\) that satisfy the \textbf{(Atom)} case in Assumption 2.3 for some \(q_0 \in (0, 1]\). Then, for \(i = i(n) \to -\infty\) such that \(i = o(\log n \land \log r_{\log_\theta n})\),

\[
X^{(n)}_i - q_0(1-\theta^{-1})\theta^{-i+\varepsilon_n} \xrightarrow{d} N(0, 1).
\]

**Remark 2.10.** The constraint \(i = o(\log n \land \log r_{\log_\theta n})\) can be simplified by providing more information on the tail of the weight distribution. Only when \(W\) has an atom at one and support bounded away from one do we have that \(o(\log n \land \log r_{\log_\theta n}) = o(\log n)\). That is, when there exists an \(s \in (0, 1)\) such that \(\mathbb{P}(W \in (s, 1)) = 0\). In that case, we can set \(s_k = s\) and \(r_k = \exp\{-(1-\theta^{-1})(1-s)k\}\) for all \(k\) large, so that

\[
\log r_{\log_\theta n} = -(1-\theta^{-1})(1-s)\log \theta n,
\]

so that indeed \(o(\log n \land \log r_{\log_\theta n}) = o(\log n)\). In all other cases it follows that \(s_k \uparrow 1\), so that \(\log r_{\log_\theta n} = o(\log n)\) and the constraint simplifies to \(i = o(\log r_{\log_\theta n})\).

**Outline of the paper**

In Section 3 we provide a short overview and intuitive idea of the proofs of Theorems 2.5, 2.6, 2.7, 2.8 and 2.9. In Section 4 we discuss two examples of vertex-weight distributions which satisfy the (\textbf{Weibull}) and (\textbf{Gumbel}) cases, respectively, for which more precise results can be obtained. We then provide the key concepts and results that are used in the proofs of the main theorems in Section 5. We use these results to prove the main theorems in Section 6. Finally, in Section 7 we provide the necessary techniques and results, comparable to what is presented in Section 5, to prove the statements regarding the examples of Section 4.
3. Intuitive idea of (the proof of) the main theorems

We provide a short intuitive idea as to why the results stated in Section 2 hold.

The main elements in obtaining a more precise understanding of the behaviour of the maximum degree of the WRT are the following:

(i) A precise expression of the tail distribution of the in-degree of uniformly at random selected vertices \((v_\ell)_{\ell \in [k]}\), for any \(k \in \mathbb{N}\). That is,

\[
P(Z_n(\ell) \geq m_\ell, \ell \in [k]) = \prod_{\ell=1}^{k} p_{\geq m_\ell}(1 + o(n^{-\beta})),
\]

for some \(\beta > 0\) and where \(m_\ell \in \mathbb{N}\) are such that \(m_\ell < c \log n\) for some \(c \in (0, \theta/(\theta - 1))\). This extends (2.2) in the sense of convergence in mean to \(k \in \mathbb{N}\) many uniformly at random selected vertices rather than just one, and allows the \(m_\ell\) to grow with \(n\) rather than being fixed. Moreover, the error term \(1 + o(n^{-\beta})\) extends previously known results as well, for which no convergence rate was known.

(ii) The asymptotic behaviour of \(p_{\geq k}\), as defined in (2.1), as \(k \to \infty\) for each case in Assumption 2.3.

(i), which is proved in Proposition 5.1, allows us to obtain bounds on the probability of the event \(\{\max_{j \in [n]} Z_n(j) \geq k_n\}\) for any sequence \(k_n \to \infty\) as \(n \to \infty\). These probabilities can be expressed in terms of \(n p_{\geq k_n}\) (using union bounds and the Chung-Erdős inequality), as is shown in Lemma 5.8. By (ii) we can then precisely quantify \(k_n\) such that these bounds either tend to zero or one, which implies whether \(\{\max_{j \in [n]} Z_n(i) \geq k_n\}\) does or does not hold with high probability. This is the main approach for Theorems 2.6 and 2.7.

To obtain the random limits described in terms of the Poisson process \(P\), as in Theorem 2.5, we use a similar approach as in [1]. Both (i) and (ii) are essential, but are now used to obtain factorial moments of the quantities \(X^{(n)}_{i}\) and \(X^{(n)}_{i} \geq \ell\), defined in (2.3), as shown in Proposition 5.6. More specifically, for any \(i < i' \in \mathbb{Z}\) and \(a_i, \ldots, a_{i'} \in \mathbb{N}_0\), and recalling that \((x)_k := x(x-1)\ldots(x-(k-1))\),

\[
\mathbb{E}\left[\prod_{k=i}^{i'-1} \frac{(X^{(n)}_{k})^{a_k}}{a_k!} \right] = \prod_{k=i}^{i'-1} \left( \frac{q_0}{\theta^{k+\varepsilon_n}} \right)^{a_k} \left( \frac{q_0(1 - \theta^{-1})}{\theta^{-k+\varepsilon_n}} \right)^{a_k} (1 + o(1)).
\]

We stress that the specific form of the right-hand side is due to the underlying assumption in Theorem 2.5 that the vertex-weight distribution has an atom at one, as in the (Atom) case of Assumption 2.3. The error term can be specified in more detail, but we omit this as it serves no further purpose here. The result essentially follows directly from these estimates by observing that the right-hand side of (3.1) can be understood as the factorial moment of the Poisson random variables \(P([i-\varepsilon, i+1+\varepsilon]), \ldots, P([i'-\varepsilon, \infty]))\), when \(\varepsilon_n\) converges to some \(\varepsilon\).

The equality in (3.1) follows from the fact that \(X^{(n)}_{i}\) and \(X^{(n)}_{i} \geq \ell\) can be expressed as sums of indicator random variables of disjoint events, so that their factorial means can be understood via the probabilities in (i). Then, again using the asymptotic behaviour of \(p_{\geq k}\) (as in (ii)), allows us to obtain the right-hand side of (3.1).

Finally, Theorems 2.8 and 2.9 are also a result of (3.1). This is due to the fact that the events \(\{\max_{j \in [n]} Z_n(j) \geq [\log_{\theta} n] + i\}\) can be understood via the events \(\{X^{(n)}_{i} \geq \ell\}\). Again using ideas similar to ones developed in [1] then allow us to obtain the results.

4. Examples

In this section we discuss some particular choices of distributions for the vertex-weights for which more precise statements can be made compared to those stated in Section 2. The reason we can improve on these more general results is due to a better understanding of the asymptotic behaviour of \(p_{k}\) and \(p_{\geq k}\) (see (2.1)) as \(k \to \infty\). As discussed in Section 3, to understand the asymptotic behaviour of the (near)-maximum degree(s) up to random order a very precise asymptotic expression for \(p_{\geq k}\) is required. Though not possible in general in the (Weibull) and (Gumbel) cases of Assumption 2.3, certain choices of vertex-weight distributions do allow for a more explicit formulation of \(p_{\geq k}\), yielding improved asymptotics. The proofs of the results presented here are deferred to Section 7.
Example 4.1 ('Beta' distribution bounded away from zero). We consider a random variable $W$ with a tail distribution
\[
\Pr(W \geq x) = Z_{w^*} \int_x^\infty \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} s^{\alpha - 1} (1 - s)^{\beta - 1} ds, \quad x \in [w^*, 1),
\]
for some $\alpha, \beta > 0$, $w^* \in (0, 1)$ and where $Z_{w^*}$ is a normalising term to ensure that $\Pr(W \geq w^*) = 1$. $W$ can be interpreted as a beta random variable, conditionally on $\{W \geq w^*\}$. We set, for $\theta := 1 + \mathbb{E}[W] \in (1, 2),$
\[
X_i^{(n)} := |\{j \in [n] : Z_n(j) = [\log_\theta n - \beta \log_\theta \log_\theta n] + i\}|,
\]
\[
X_{\geq 1}^{(n)} := |\{j \in [n] : Z_n(j) \geq [\log_\theta n - \beta \log_\theta \log_\theta n] + i\}|,
\]
\[
\varepsilon_n := (\log_\theta n - \beta \log_\theta \log_\theta n) - [\log_\theta n - \beta \log_\theta \log_\theta n],
\]
\[
c_{\alpha, \beta, \theta} := Z_{w^*} (\Gamma(\alpha + \beta)/\Gamma(\alpha)) (1 - \theta^{-1})^{-\beta}.
\]

Then, we can formulate the following results.

**Theorem 4.2.** Consider the WRT model in Definition 2.1 with vertex-weights $(W_i)_{i \in [n]}$ whose distribution satisfies (4.1) for some $\alpha, \beta > 0$, $w^* \in [0, 1]$, fix $\varepsilon \in [0, 1]$. Let $(n_k)_{k \in \mathbb{N}}$ be an increasing integer sequence satisfying $\varepsilon_n \to \varepsilon$ as $k \to \infty$ and let $\mathcal{P}$ be a Poisson process on $\mathbb{R}$ with intensity measure $\lambda(x) = c_{\alpha, \beta, \theta} x^{-\alpha} \log_\theta x$. Define
\[
\mathcal{P}^x := \sum_{x < x + \varepsilon} \delta_{x}, \quad \mathcal{P}^{(n)} := \sum_{i \in [n]} \delta_{Z_n(i) - [\log_\theta n - \beta \log_\theta \log_\theta n]}.
\]

Then in $\mathcal{M}_\mathbb{E}^\theta$ (the space of bounded finite measures on $\mathbb{Z}^* = \mathbb{Z} \cup \{\infty\}$), $\mathcal{P}^{(n_k)}$ converges weakly to $\mathcal{P}^x$ as $k \to \infty$. Equivalently, for any $i < i' \in \mathbb{Z}$, jointly as $k \to \infty,$
\[
(X_i^{(n_k)}, X_{i+1}^{(n_k)}, \ldots, X_{i'-1}^{(n_k)}, X_{i'-1}^{(n_k)}) \xrightarrow{d} (\mathcal{P}^x(i), \mathcal{P}^x(i + 1), \ldots, \mathcal{P}^x(i' - 1), \mathcal{P}^x(i')), \infty). \]

We remark that the second-order term $\beta \log_\theta \log_\theta n$ is established in Theorem 2.6 as well and that the above theorem recovers this result and extends it to the random third-order term, which is shown in the proof of Theorem 2.5.

**Theorem 4.3.** Consider the WRT model in Definition 2.1 with vertex-weights $(W_i)_{i \in [n]}$ whose distribution satisfies (4.1) for some $\alpha, \beta > 0$, $w^* \in [0, 1)$, recall $\varepsilon_n$ and $c_{\alpha, \beta, \theta}$ from (4.2). Then, for any $i = i(n)$ with $i' = i + \delta \log_\theta n$ for some $\delta \in [0, \theta \log_\theta / (\theta - 1) - 1) (\delta = 0$ denotes $i = o(\log n)$) and $\lim \inf_{n \to \infty} i > -\infty$,
\[
\mathcal{P}(\max_{j \in [n]} Z_n(j) \geq [\log_\theta n - \beta \log_\theta \log_\theta n] + i) \to (1 - \exp \{- c_{\alpha, \beta, \theta} (1 + \delta) - \beta \theta^{-i + \varepsilon_n}\}) (1 + o(1)).
\]

Moreover, let $\mathcal{M}_n \subseteq [n]$ denote the set of vertices that attain the maximum degree in $T_n$, fix $\varepsilon \in [0, 1]$ and let $(n_k)_{k \in \mathbb{N}}$ be a positive integer sequence such that $\varepsilon_{n_k} \to \varepsilon$ as $k \to \infty$. Then, $|\mathcal{M}_{n_k}| \xrightarrow{d} \mathcal{M}_\varepsilon$, where $\mathcal{M}_\varepsilon$ has distribution
\[
\mathcal{P}(M \in \{k\}) = \sum_{j \in \mathbb{Z}} \frac{1}{j!} c_{\alpha, \beta, \theta} (1 - \theta^{-1}) \theta^{-j + \varepsilon}\) \exp \{c_{\alpha, \beta, \theta} \theta^{-j + \varepsilon}\},
\]
for $k \in \mathbb{N}$.

**Theorem 4.4.** Consider the WRT model in Definition 2.1 with vertex-weights $(W_i)_{i \in [n]}$ whose distribution satisfies (4.1) for some $\alpha, \beta > 0$, $w^* \in (0, 1)$ and recall $\varepsilon_n$ and $c_{\alpha, \beta, \theta}$ from (4.2). Then, for $i = i(n) \to -\infty$ such that $i = o(\log \log n)$,
\[
\frac{X_i^{(n)} - c_{\alpha, \beta, \theta} (1 - \theta^{-1}) \theta^{-i + \varepsilon_n}}{c_{\alpha, \beta, \theta} (1 - \theta^{-1}) \theta^{-i + \varepsilon_n}} \xrightarrow{d} N(0, 1).
\]

The three theorems are the analogue of Theorems 2.5, 2.8 and 2.9, respectively, where we now consider vertex-weights distributed according to a distribution as in (4.1) rather than a distribution with an atom at one.
Example 4.5 (Fraction of ‘gamma’ random variables). We consider a random variable $W$ with a tail distribution
\[ \mathbb{P}(W \geq x) = Z_w (1 - x)^{-b} e^{-x/(c_1 (1-x))}, \quad x \in [w^*, 1), \]  
for some $b \in \mathbb{R}, c_1 > 0, w^* \in (0, 1)$ and where $Z_w$ is a normalising term to ensure that $\mathbb{P}(W \geq w^*) = 1$. 

$X$ tends to $\mathbb{P}(W < x) = Z_w (1 - x)^{-b} e^{-x/c_1}, \quad x \geq (1 - w^*)^{-1},$

so that $W$ belongs to the Gumbel MDA as well by [14, Lemma 2.6], and satisfies the (Gumbel)-(RV) sub-case with $a = Z_w e^{1/c_1}, b \in \mathbb{R}, c_1 > 0, \tau = 1$. As a result $X := (1 - W)^{-1}$ is a ‘gamma’ random variable in the sense that its tail distribution is asymptotically equal to that of a gamma random variable, up to constants. $W$ can then be written as $W = (X - 1) / X$, a fraction of these ‘gamma’ random variables.

Recall $C_{\theta, r, c_1}$ from (2.5). We set, for $\theta := 1 + E[W] \in (1, 2)$,
\[ C := e^{c_i^{-1} (1-\theta^{-1})^2 \sqrt{\pi} e^{-1/4 + b/2} (1 - \theta^{-1})^{1/4 + b/2}}, \quad c_{c_{1, b, \theta}} := Z_w \theta^{c_{1, c_1}}/2, \]  
and
\[ X_i^{(n)} := \left\{ j \in n : Z_n(j) = \left[ \log_\theta n - C_{\theta, 1, c_1} \log_\theta n + (b/2 + 1/4) \log_\theta \log_\theta n \right] + i \right\}, \]
\[ X_i^{(n)} := \left\{ j \in n : Z_n(j) \geq \left[ \log_\theta n - C_{\theta, 1, c_1} \log_\theta n + (b/2 + 1/4) \log_\theta \log_\theta n \right] + i \right\}, \]
\[ \varepsilon_n := \left( \log_\theta n - C_{\theta, 1, c_1} \log_\theta n + (b/2 + 1/4) \log_\theta \log_\theta n \right) \]
\[ \left[ \log_\theta n - C_{\theta, 1, c_1} \log_\theta n + (b/2 + 1/4) \log_\theta \log_\theta n \right]. \]

Then, we can formulate the following results.

Theorem 4.6. Consider the WRT model in Definition 2.1 with vertex-weights $(W_i)_{i \in [n]}$ whose distribution satisfies (4.3) for some $b \in \mathbb{R}, c_1 > 0, w^* \in (0, 1)$ and recall $C_{\theta, r, c_1}$ from (2.6). Then,
\[ \max_{i \in [n]} Z_n(i) - \log_\theta n + C_{\theta, 1, c_1} \log \log \theta n \leq \frac{b - 1}{2} + \frac{4}{\log \log \theta n}. \]  

Furthermore, fix $\varepsilon \in [0, 1]$ and let $(n_i)_{i \in \mathbb{N}}$ be an increasing integer sequence satisfying $\varepsilon_{n_i} \to \varepsilon$ as $i \to \infty$. Let $P$ be a Poisson point process on $\mathbb{R}$ with intensity measure $\lambda(x) = c_{c_{1, b, \theta}} \theta^{-x} \log \theta dx$, where we recall $c_{c_{1, b, \theta}}$ from (4.4). Define
\[ P^\varepsilon := \sum_{x \in P \mid x \leq \varepsilon}, \quad P^{(n)} := \sum_{i \in [n]} \delta_{Z_n(i) - \log_\theta n - C_{\theta, 1, c_1} \log \log \theta n}. \]

Then in $\mathcal{M}_{\theta}^{\varepsilon}$, (the space of bounded finite measures on $\mathbb{R}^* = \mathbb{R} \cup \{ \infty \}$), $P^{(n)}$ converges weakly to $P^\varepsilon$ as $\varepsilon \to \infty$. Equivalently, for any $i, i' \in \mathbb{Z}$, jointly as $\varepsilon \to \infty$,
\[ (X_i^{(n)}, X_{i+1}^{(n)}, \ldots, X_{i'-1}^{(n)}, X_{i'}^{(n)}) \xrightarrow{d} (P^\varepsilon(i), P^\varepsilon(i+1), \ldots, P^\varepsilon(i'-1), P^\varepsilon(i')). \]

We remark that the second-order term in (4.6) is established in Theorem 2.7, (2.5), as well. The above theorem recovers this former result and extends it to the third-order rescaling and to the random fourth-order term, which is similar to the result in Theorem 2.5.

Theorem 4.7. Consider the WRT model in Definition 2.1 with vertex-weights $(W_i)_{i \in [n]}$ whose distribution satisfies (4.3) for some $b \in \mathbb{R}, c_1 > 0, w^* \in (0, 1)$ and recall $C_{\theta, r, c_1}, c_{c_{1, b, \theta}}$ and $\varepsilon_n$ from (2.6), (4.4) and (4.5), respectively. Then, for any $i = i(n)$ with $i \sim \delta \log \theta n$ for some $\delta \geq 0$ ($\delta = 0$ denotes $i = o(1)$ and $\log \log n$) and $\lim \inf_{n \to \infty} i > -\infty$,
\[ \mathbb{P}(\max_{j \in [n]} Z_n(j) \geq \log_\theta n - C_{\theta, 1, c_1} \log \log \theta n + (b/2 + 1/4) \log_\theta \log_\theta n) = (1 - \exp \left\{ - c_{c_{1, b, \theta}} \theta^{-i + \varepsilon_n - \delta C_{\theta, 1, c_1}/2} \right\} (1 + o(1)). \]

Moreover, let $\mathcal{M}_\varepsilon \subseteq [n]$ denote the set of vertices that attain the maximum degree in $T_n$, fix $\varepsilon \in [0, 1]$ and let $(n_i)_{i \in \mathbb{N}}$ be a positive integer sequence such that $\varepsilon_{n_i} \to \varepsilon$ as $i \to \infty$. Then, $|\mathcal{M}_\varepsilon| \xrightarrow{d} \mu_\varepsilon$, where $\mu_\varepsilon$ has distribution
\[ \mathbb{P}(M_\varepsilon = k) = \sum_{j \in \mathbb{Z}} \frac{1}{c_{c_{1, b, \theta}} (1 - \theta^{-i + \varepsilon})^k \exp \left\{ c_{c_{1, b, \theta}} \theta^{-i + \varepsilon} \right\}}, \]
for $k \in \mathbb{N}$. 

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Theorem 4.8. Consider the WRT model in Definition 2.1 with vertex-weights \((W_i)_{i \in [n]}\) whose distribution satisfies (4.3) for some \(b \in \mathbb{R}_{+} > 0, w^* \in (0, 1)\) and recall \(c_{1,b,\theta}\) and \(\varepsilon_n\) from (4.4) and (4.5), respectively. Then, for \(i = i(n) \to -\infty\) such that \(i = o(\log \log n)\),
\[
\frac{X^{(n)} - c_{1,b,\theta}(1 - \theta - 1)\theta^{i + \varepsilon_n}}{\sqrt{c_{1,b,\theta}(1 - \theta - 1)\theta^{i + \varepsilon_n}}} \xrightarrow{d} N(0, 1).
\]

The three theorems are the analogue of Theorems 2.5, 2.8 and 2.9, respectively, where we now consider vertex-weights distributed according to a distribution as in (4.3) rather than a distribution with an atom at one.

In both examples we see that a better understanding of the asymptotic behaviour of the tail of the degree distribution, \((\rho_{\geq k})_{k \in \mathbb{N}_{0}}\), allows us to identify the higher-order asymptotic behaviour of the (near-)maximum degree(s). It also shows that a higher order random limit as in the sense of Theorems 4.2 and 4.6 is not expressed just by vertex-weights whose distribution has an atom at one, and we conjecture that this result is in fact universal for all vertex-weights distributions with bounded support.

5. Degree tail distributions and factorial moments

In this section we state and prove the key elements required to prove the main results as stated in Section 2. We stress that the results presented and proved in this section cover all the classes introduced in Assumption 2.3 (in fact, they cover any vertex-weight \(W\) such that \(\sup\{x > 0 : P(W \leq x) < 1\} = 1, P(W \geq w^*) = 1\) for some \(w^* \in (0, 1)\)) and that the distinction between the classes of Assumption 2.3 follows in Section 6.

5.1. Statement of intermediate results and main ideas. As discussed in Section 3, to understand the asymptotic behaviour of the maximum degree and near-maximum degrees we require a more precise understanding of the convergence in mean of the empirical degree distribution. To that end, we present the following result:

Proposition 5.1 (Distribution of typical vertex degrees). Let \(W\) be a positive random variable such that \(x_0 := \sup\{x > 0 : P(W \leq x) < 1\} = 1\) and such that there exists a \(w^* \in (0, 1)\) so that \(P(W \geq w^*) = 1\). Consider the WRT model in Definition 2.1 with vertex-weights \((W_i)_{i \in [n]}\) which are i.i.d. copies of \(W\), fix \(k \in \mathbb{N}\) and let \((v_i)_{i \in [k]}\) be \(k\) vertices selected uniformly at random without replacement from \([n]\). For a fixed \(c \in (0, \theta/(\theta - 1))\), there exist \(\beta \geq \beta' > 0\) such that uniformly over non-negative integers \(m_\ell < c \log n, \ell \in [k]\),
\[
P(Z_n(v_\ell) = m_\ell, \ell \in [k]) = \sum_{\ell=1}^{k} E \left[ \frac{E[W]}{E[W] + W} \left( \frac{W}{E[W] + W} \right)^{m_\ell} \right] (1 + o(n^{-\beta})),
\]
and
\[
P(Z_n(v_\ell) \geq m_\ell, \ell \in [k]) = \sum_{\ell=1}^{k} E \left[ \left( \frac{W}{E[W] + W} \right)^{m_\ell} \right] (1 + o(n^{-\beta})).
\]

Remark 5.2. (i) In [5, Lemma 1], it is proved that the degrees \((Z_n(j))_{j \in [n]}\) are negative quadrant dependent when considering the RRT model (the WRT with deterministic weights, all equal to 1). That is, for any \(k \in \mathbb{N}\) and \(j_1 \neq \ldots \neq j_k \in [n], m_1, \ldots, m_k \in \mathbb{N}\),
\[
P \left( \bigcap_{i=1}^{k} Z_n(j_i) \geq m_\ell \right) \leq \prod_{j=1}^{k} P(Z_n(j_i) \geq m_\ell).
\]
This property only holds for the conditional probability measure \(P_W\) when considering the WRT (or, more generally, the WRG) model, as follows from [14, Lemma 7.1], and can be obtained ‘asymptotically’ for the probability measure \(P\), as in the proof of [14, Theorem 2.8, Bounded case]. Proposition 5.1 improves on this by establishing asymptotic independence under the non-conditional probability measure \(P\) of the degrees of typical vertices, which allows us to extend the results in [14] to more precise asymptotics.

(ii) We note that the result only requires the two main conditions in Assumption 2.3. Hence, results for other vertex-weight distributions that do not satisfy any of the particular cases outlined in this assumption can be obtained as well using the methods presented in this paper.
(iii) The result in Proposition 5.1 improves on known results, especially those in [8, 11]. In these papers similar techniques are used to prove a weaker result, in which the $m_k$ are not allowed to diverge with $n$ and where no convergence rate is provided.

To use this (tail) distribution of $k$ typical vertices $v_1, \ldots, v_k$, a precise expression for the expected values on the right-hand side in Proposition 5.1 is required. Recall $p_k$ from (2.1). The following theorem comes from [14, Theorem 2.7], in which the maximum degree of weighted recursive graphs is studied for a large class of vertex-weight distribution and in which asymptotic expressions of $p_k$ are presented.

**Theorem 5.3** ([14, Asymptotic behaviour of $p_k$]). Recall that $\theta := 1 + \mathbb{E}[W]$. We consider the different cases with respect to the vertex-weights as in Assumption 2.3, and can relax the assumption that $W$ is bounded away from zero, i.e. $w^* = 0$ is allowed.

(Atom) Recall that $q_0 = \mathbb{P}(W = 1) > 0$ and recall $r_k$ from (2.9). Then,

$$p_k = q_0 (1 - \theta^{-1}) \theta^{-k} (1 + O(r_k)).$$

(Weibull) Recall that $\alpha > 1$ is the power-law exponent. Then, for all $k > 1/\mathbb{E}[W]$,

$$\mathcal{L}(k) k^{-(\alpha - 1)} \theta^{-k} \leq p_k \leq \mathcal{L}(k) k^{-(\alpha - 1)} \theta^{-k},$$

where $\mathcal{L}(\cdot)$ are slowly varying at infinity.

(Gumbel) (i) If $W$ satisfies the (RV) sub-case with parameter $\tau > 0$, set $\gamma := 1/(\tau + 1)$. Then,

$$p_k = \exp \left\{ - \frac{\tau \gamma}{1 - \gamma} \left( \frac{1 - \theta^{-1}}{C_1} \right)^{1 - \gamma} \left( 1 + o(1) \right) \right\} \theta^{-k}.$$  

(ii) If $W$ satisfies the (RaV) sub-case with parameter $\tau > 1$,

$$p_k = \exp \left\{ - \left( \frac{\log k}{\tau} \left( 1 - \tau - 1 \right) \frac{\log \log k}{\log k} + \frac{K_{\tau,c_1,\theta}}{\log k} \left( 1 + o(1) \right) \right) \right\} \theta^{-k},$$

where $K_{\tau,c_1,\theta} := \tau \log \left( \exp \left( \frac{1}{(1 - \theta^{-1})/\tau} \right) \right)$.

**Remark 5.4.** Equivalent upper and lower bounds can be obtained for $p_{\geq k}$ as in (2.1) (with $m = 1$), by adjusting the multiplicative constants in front of the asymptotic expressions only. This is due to the fact that

$$(1 - \theta^{-1})^{-1} p_k = \mathbb{E} \left[ \frac{1}{\theta - 1 + W} \left( \frac{W}{\theta - 1 + W} \right)^k \right] \geq \mathbb{E} \left[ \left( \frac{W}{\theta - 1 + W} \right)^k \right] = p_{\geq k} \geq p_k.$$  

Equivalently, the proof of [14, Theorem 2.7] can be adapted to work for $p_{\geq k}$ in which case the same asymptotic behaviour is established for $p_{\geq k}$, but with different constants in the front.

We also provide less precise but more general bounds on the degree distribution.

**Lemma 5.5.** Let $W$ be a positive random variable with $x_0 := \sup \{ x > 0 : \mathbb{P}(W \leq x) < 1 \} = 1$. Then, for any $\xi > 0$ and $k$ sufficiently large,

$$(\theta + \xi)^{-k} \leq p_k \leq p_{\geq k} \leq \theta^{-k}.$$  

**Proof.** The upper bound on $p_{\geq k}$ directly follows from the fact that $x \mapsto (x/(\theta - 1 + x))^k$ is increasing in $x$, so that

$$p_{\geq k} = \mathbb{E} \left[ \left( \frac{W}{\theta - 1 + W} \right)^k \right] \leq \left( \frac{x_0}{\theta - 1 + x_0} \right)^k = \theta^{-k},$$

when $x_0 = 1$. For the lower bound, let us take some $\delta \in (0, \xi/(\theta - 1 + \xi))$ and define

$$f_k(\theta, x) := \frac{\mathbb{E}[W]}{\mathbb{E}[W] + \delta} \left( \frac{x}{\theta - 1 + x} \right)^k = \frac{1}{\delta} \left( \frac{\theta - 1}{\theta - \delta} \left( \frac{x}{\theta - 1 + x} \right)^k \right).$$

Note that $p_k = \mathbb{E}[f_k(\theta, W)]$. Then, since $f_k(\theta, x)$ is increasing in $x$ on $(0, 1]$ for $k$ sufficiently large,

$$\mathbb{E}[f_k(\theta, W)] \geq \mathbb{E} \left[ f_k(\theta, W) 1_{\{ W > 1 - \delta \}} \right] \geq \mathbb{P}(W > 1 - \delta) \left( \frac{\theta - 1}{\theta - \delta} \right) \left( \frac{\theta + \xi}{\theta - \delta} \right)^k.$$  

We note that, since $x_0 = 1$, $\mathbb{P}(W > 1 - \delta) > 0$ for any $\delta > 0$. Now, by the choice of $\delta$, $\xi/(\theta - \delta)/(\theta - \delta) > 1$, so we can find some $\gamma > 0$ sufficiently small so that

$$\mathbb{E}[f_k(\theta, W)] (\theta + \xi)^k \geq \mathbb{P}(W > 1 - \delta) \left( \frac{\theta - 1}{\theta - \delta} \right) \left( \frac{\theta + \xi}{\theta - \delta} \right)^k \geq (1 + \gamma)^k \geq 1,$$
as required. □

Recall the definition of $X_{\geq k}^{(n)}$, $X_{\geq i}^{(n)}$ and $\varepsilon_n$ from (2.3) and (2.4), respectively. Proposition 5.1 combined with Theorem 5.3 then allows us to obtain the following result.

**Proposition 5.6** (Factorial moments for vertex-weights satisfying the (Atom) case).

Consider the WRT model as in Definition 2.1 with vertex-weights $(W_i)_{i\in[n]}$ that satisfy the (Atom) case in Assumption 2.3 for some $q_0 \in (0, 1]$. Recall $r_k$ from (2.9), recall that $\theta := 1 + E[W]$ and that $(x)_k := x(x-1) \cdots (x-(k-1))$ for $x \in \mathbb{R}, k \in \mathbb{N}$, and $(x)_0 := 1$. For a fixed $K \in \mathbb{N}$, $c \in (0, \theta/(\theta-1))$, there exists a $\beta > 0$ such that the following holds. For any $i = i(n)$, $i' = i'(n)$ in $\mathbb{Z}$ such that $0 < i + \log_\beta n < i' + \log_\beta n < c \log n$ and $a_j \in \mathbb{N}_0, j \in \{i, \ldots, i'\}$ such that $\sum_{j=1}^{i'} a_j = K$,

$$
\mathbb{E} \left[ X_{\geq i}^{(n)} \prod_{k=1}^{i'-1} X_k^{(n)} \right] = \left( q_0 \theta^{-i'-\varepsilon_n} \right)^a \prod_{k=1}^{i'-1} \left( q_0 (1 - \theta^{-1}) \theta^{-k+\varepsilon_n} \right)^{a_k} \times (1 + \mathcal{O}(r_{[\log_\beta n]+i}^\beta n^{-\beta}))
$$

**Remark 5.7.** Related to Remark 2.10, the error term decays polynomially only if $W$ has an atom at one and support bounded away from one and $\log_\beta n + i > \eta \log n$ for some $\eta > 0$. That is, when there exists an $s \in (0, 1)$ such that $\mathbb{P}(W \in (s, 1)) = 0$. In that case, $s_k \leq s$ and $r_k \leq \exp(-(1-\theta^{-1})(1-s)k)$ for all large, so that

$$
r_{[\log_\beta n]+i}^\beta n^{-\beta} \leq \exp(-(1-\theta^{-1})(1-s)\eta \log n) \vee n^{-\beta} = n^{-\min\{\eta(1-\theta^{-1})(1-s)\beta\}}.
$$

In all other cases, the error term decays slower than polynomially.

**Proof of Proposition 5.6 subject to Proposition 5.1.** We closely follow the approach in [1, Proposition 2.1], where an analogue result is presented for the case $q_0 = 1$, i.e. for the random recursive tree. Set $K' := K - a_v$ and for each $i \leq k \leq i'$ and each $u \in \mathbb{N}$ such that $\sum_{\ell=1}^{k-1} a_{u, \ell} < u \leq \sum_{\ell=1}^{k} a_{u, \ell}$, let $m_u = [\log_\beta n] + k$. We note that $m_u < \log_\beta n + i' < c \log n$, so that the results in Proposition 5.1 can be used. Also, let $(v_u)_{u \in [K]}$ be $K$ vertices selected uniformly at random without replacement from $[n]$, and define $I := [K]\setminus[K']$. Then, as the $X_{\geq k}^{(n)}$ and $X_k^{(n)}$ can be expressed as sums of indicators,

$$
\mathbb{E} \left[ X_{\geq i}^{(n)} \prod_{k=1}^{i'-1} X_k^{(n)} \right] = \sum_{K'} \mathbb{E} \left[ (z_n(u) = m_u, z_n(v_u) \geq m_v, u \in [K'], w \in I) \right]
$$

where the second step follows from [1, Lemma 5.1] and is based on an inclusion-exclusion argument. We can now use Proposition 5.1. First, we note that there exists a $\beta > 0$ such that for non-negative integers $m_1, \ldots, m_K < c \log n$,

$$
\mathbb{P}(z_n(u) \geq m_u + \mathbb{1}_{\{u \in S\}}, u \in [K]) = \prod_{u \in S} \mathbb{E} \left[ \left( \frac{W}{W + W} \right)^{m_u+1} \right] (1 + \mathcal{O}(n^{-\beta})).
$$

Now, by Theorem 5.3 and the definition of $r_k$ in (2.9) and as $r_k$ is decreasing by Lemma 8.3 in the Appendix, when $|S| = \ell$,

$$
\prod_{u=1}^K \mathbb{E} \left[ \left( \frac{W}{W + W} \right)^{m_u+1} \right] = q_0^{|S|} \theta^{-\ell-\sum_{u=1}^K m_u} \left( 1 + \mathcal{O}(r_{[\log_\beta n]+i}^\beta n^{-\beta}) \right),
$$

as the smallest $m_u$ equals $[\log_\beta n] + i$. We have

$$
\sum_{\ell=0}^{K'} \sum_{S \subseteq [K]} (-1)^{K'} q_0^{|S|} \theta^{-\ell-\sum_{u=1}^K m_u} = (n) K_0^K \theta^{-K'} - \sum_{u=1}^K m_u \sum_{\ell=0}^{K'} \left( \begin{array}{c} K' \\ \ell \end{array} \right) (-1)^{\ell} \theta^{K'-\ell}
$$

(5.10)
We then observe that \((n)_K = \theta^K \log_a n \left(1 + O(1/n)\right)\). Moreover, we recall that \(K = \sum_{k=1}^{K'} a_k, K' = \sum_{k=1}^{K'-1} a_k\) and \(m_u = \lfloor \log_a n \rfloor + k\) if \(\sum_{i=1}^{k-1} a_i \leq u < \sum_{i=1}^{k} a_i\) for \(i \leq \ell \leq i'\), and recall \(\varepsilon_n\) from (2.4). Using (5.10) combined with (5.8) and (5.9) in (5.7), we obtain
\[
\mathbb{E}\left[\sum_{i=1}^{K'-1} \prod_{k=i}^{i'-1} X_k^{(n)}(a_k)\right] = q_0^K \left(1 - \theta^{-1}\right) \prod_{k=i}^{i'-1} \left(q_0(1 - \theta^{-1})^{-k + \varepsilon_n} a_k\right) \times (1 + O\left(r_{\log_a n}^2 + n^{-\beta}\right)),
\]
as desired. \(\square\)

The next lemma builds on [14, Lemma 7.1] and [5, Lemma 1] and provides bounds on the maximum degree that hold with high probability.

**Lemma 5.8.** Consider the WRT model in Definition 2.1. Fix \(c \in (0, \theta/(\theta - 1))\) and let \((k_n)_{n \in \mathbb{N}}\) be a non-negative, diverging integer sequence such that \(k_n < c \log n\) and let \(v_1\) be a vertex selected uniformly at random from \([n]\). If \(\lim_{n \to \infty} n \mathbb{P}(Z_n(v_1) \geq k_n) = 0\), then
\[
\lim_{n \to \infty} \mathbb{P}\left(\max_{i \in [n]} Z_n(i) \geq k_n\right) = 0.
\]
Similarly, when instead \(\lim_{n \to \infty} n \mathbb{P}(Z_n(v_1) \geq k_n) = \infty\),
\[
\lim_{n \to \infty} \mathbb{P}\left(\max_{i \in [n]} Z_n(i) \geq k_n\right) = 1.
\]

**Remark 5.9.** Similar to what is discussed in Remark 5.2(i), the result in this lemma is stronger than the results presented in [14, Lemma 7.1] and [5, Lemma 1]. It extends the latter to the WRT model rather than just the RRT model, and improves the former as the result holds for the non-conditional probability measure \(\mathbb{P}\) rather than \(\mathbb{P}_{W^r}\), which is what is used in [14]. Due to the difficulties of working with the conditional probability measure, only a first order asymptotic result can be proved there. With the improved understanding of the degree distribution, as in Proposition 5.1, the above result can be obtained, which allows for finer asymptotics to be proved.

**Proof of Lemma 5.8 subject to Proposition 5.1.** The first result immediately follows from a union bound and the fact that
\[
n \mathbb{P}(Z_n(v_1) \geq k_n) = \sum_{i=1}^{n} \mathbb{P}(Z_n(i) \geq k_n).
\]
(5.11)

For the second result, let \(A_{n,i} := \{Z_n(i) \geq k_n\}, i \in [n]\). Then, by the Chung-Erdős inequality,
\[
\mathbb{P}\left(\max_{i \in [n]} Z_n(i) \geq k_n\right) = \mathbb{P}(\bigcup_{i=1}^{n} A_{n,i}) \geq \frac{\left(\sum_{i=1}^{n} \mathbb{P}(A_{n,i})\right)^2}{\sum_{i \neq j} \mathbb{P}(A_{n,i} \cap A_{n,j}) + \sum_{i=1}^{n} \mathbb{P}(A_{n,i})}.
\]
By (5.11) it follows that \(\sum_{i=1}^{n} \mathbb{P}(A_{n,i}) = n \mathbb{P}(Z_n(v_1) \geq k_n)\). Furthermore, by Proposition 5.1,
\[
\sum_{i \neq j} \mathbb{P}(A_{n,i} \cap A_{n,j}) = n(n - 1) \mathbb{P}(A_{n,v_1} \cap A_{n,v_2}) = (n \mathbb{P}(A_{n,v_1}))^2 (1 + o(1)),
\]
where \(v_2\) is another vertex selected uniformly at random, unequal to \(v_1\). Note that the condition that \(k_n < c \log n\) is required for this to hold. Together with the above lower bound, these two observations yield
\[
\mathbb{P}(\bigcup_{i=1}^{n} A_{n,i}) \geq \frac{(n \mathbb{P}(A_{n,v_1}))^2}{(n \mathbb{P}(A_{n,v_1}))^2 (1 + o(1)) + n \mathbb{P}(A_{n,v_1})} = \frac{n \mathbb{P}(A_{n,v_1})}{n \mathbb{P}(A_{n,v_1}) (1 + o(1)) + 1}.
\]
Hence, when \(n \mathbb{P}(A_{n,v_1}) = n \mathbb{P}(Z_n(v_1) \geq k_n)\) diverges with \(n\), we obtain the desired result. \(\square\)

What remains is to prove Proposition 5.1. As the proof is rather long and involved, we discuss the strategy of the proof first and take care of certain parts of the proof in separate lemmas. The main part of the proof is dedicated to proving (5.1). Once this is established, (5.2) follows without much effort. We thus focus on discussing the proof of (5.1).
The left-hand side of (5.1) can be expressed by conditioning on the values of the typical vertices, and splitting between cases of young and old vertices. That is,

\[
\mathbb{P}(Z_n(n) = m_\ell, \ell \in [k]) = \frac{1}{(n)_k} \sum_{1 \leq j_1 \neq \ldots \neq j_k \leq n} \mathbb{E}[P_W(Z_n(j)) = m_\ell, \ell \in [k])]
\]

\[
= \frac{1}{(n)_k} \sum_{n^{1-\varepsilon} \leq j_1 \neq \ldots \neq j_k \leq n} \mathbb{E}[P_W(Z_n(j)) = m_\ell, \ell \in [k])]
\]

\[
+ \frac{1}{(n)_k} \sum_{\ell \in I_n(\varepsilon)} \mathbb{E}[P_W(Z_n(j)) = m_\ell, \ell \in [k])],
\]

(5.12)

where for any \( \varepsilon \in (0, 1) \), \( I_n(\varepsilon) := \{j = (j_1, \ldots, j_k) : 1 \leq j_1 \neq \ldots \neq j_k \leq n, \exists i \in [k] j_i < n^{1-\varepsilon}\} \).

Splitting the sum on the first line into the two sums on the second and third line allows us to deal with them in a different way. In the sum on the second line, in which all indices are at least \( n^{1-\varepsilon} \), we can apply the law of large numbers on sums of vertex-weights to gain more control over the conditional probability of the event \( \{Z_n(j) = m_\ell, \ell \in [k]\} \). The aim is to show that this first sum has the desired form, as on the right-hand side of (5.1).

The sum on the third line, in which at least one of the indices takes on values strictly smaller than \( n^{1-\varepsilon} \) can be shown to be negligible compared to the first sum. Especially when \( m_\ell \) is large, this is non-trivial. To do this, consider the tail events \( \{Z_n(j) \geq m_\ell, \ell \in [k]\} \) and use the negative quadrant dependence of the degrees (see Remark 5.2 and [14, Lemma 7.1]), so that we can deal with the more tractable probabilities \( \mathbb{P}(Z_n(j) \geq m_\ell) \) for \( \ell \in [k] \), rather than the probability of all tail degree events. Depending on whether the indices in \( I_n(\varepsilon) \) are at most or at least \( n^{1-\varepsilon} \), we then use bounds similar to one developed in the proof of [14, Lemma 7.1] or use an approach similar to what we use to bound the sum on the second line of (5.12), respectively.

In the following lemma, we deal with the sum on the second line of (5.12).

**Lemma 5.10.** Let \( W \) be a positive random variable such that \( x_0 := \sup\{x > 0 : \mathbb{P}(W \leq x) < 1\} = 1 \). Consider the WRT model in Definition 2.1 with vertex-weights \( (W_i)_{i \in [n]} \) which are i.i.d. copies of \( W \) and fix \( k \in \mathbb{N}, c \in (0, \theta/(\theta - 1)) \). Then, there exist a \( \beta > 0 \) and \( \varepsilon \in (0, 1) \) such that uniformly over non-negative integers \( m_\ell < c \log n, \ell \in [k] \),

\[
((n)_k)^{-1} \sum_{n^{1-\varepsilon} \leq j_1 \neq \ldots \neq j_k \leq n} \mathbb{E}[P_W(Z_n(j)) = m_\ell, \ell \in [k]) = \prod_{\ell = 1}^k \mathbb{E}\left[\frac{E[W]}{E[W] + W\left(\frac{W}{E[W] + W}\right)}\right] (1 + o(n^{-\beta})).
\]

We note that the assumption \( \mathbb{P}(W \geq w^*) = 1 \) for some \( w^* \in (0, 1) \) is not required for this result to hold. To prove this lemma, we sum over all possible \( m_\ell \) vertices that connect to \( j_\ell \) for each \( \ell \in [k] \) and use the fact that the \( j_1, \ldots, j_k \) are at least \( n^{1-\varepsilon} \) to precisely control the connection probabilities and to evaluate the sums over all the possible \( m_\ell \) vertices, \( \ell \in [k] \), as well as the sum over the indices \( j_1, \ldots, j_k \).

In the following lemma, we show the sum on the third line of (5.12) is negligible compared to the sum on the second line.

**Lemma 5.11.** Let \( W \) be a positive random variable such that \( x_0 := \sup\{x > 0 : \mathbb{P}(W \leq x) < 1\} = 1 \) and such that there exists a \( w^* \in (0, 1) \) so that \( \mathbb{P}(W \geq w^*) = 1 \). Consider the WRT model in Definition 2.1 with vertex-weights \( (W_i)_{i \in [n]} \) which are i.i.d. copies of \( W \) and fix \( k \in \mathbb{N}, c \in (0, \theta/(\theta - 1)) \). Then, there exist an \( \eta > 0 \) and an \( \varepsilon \in (0, 1) \) such that uniformly over non-negative integers \( m_\ell < c \log n, \ell \in [k] \),

\[
\frac{1}{(n)_k} \sum_{\ell \in I_n(\varepsilon)} \mathbb{E}[P_W(Z_n(j)) = m_\ell, \ell \in [k]) = o\left(\prod_{\ell = 1}^k \mathbb{E}\left[\frac{E[W]}{E[W] + W\left(\frac{W}{E[W] + W}\right)}\right]^{m_\ell} n^{-\eta}\right),
\]

where \( I_n(\varepsilon) := \{j = (j_1, \ldots, j_k) : 1 \leq j_1 \neq \ldots \neq j_k \leq n, \exists i \in [k] j_i < n^{1-\varepsilon}\} \).

Note that the assumption that the vertex-weights are bounded away from zero is required in this lemma, where it was not necessary in Lemma 5.10. In fact, it is required for one inequality in the proof only, which convinces us that it could be omitted with more work or at the very least replaced by weaker assumptions.

It is clear that (5.2) in Proposition 5.1 immediately follows from using the results of Lemmas 5.10 and 5.11 in (5.12). In what follows we first prove Lemma 5.10 in Section 5.2, prove Lemma 5.11 in
Section 5.3 and finally complete the proof of Proposition 5.1 in Section 5.4. After each of the proofs we discuss the required adaptations to prove the results of Lemmas 5.10 and 5.11 for the model with random out-degree, as introduced in Remark 2.2(iii).

5.2. Proof of Lemma 5.10.

Proof of Lemma 5.10. To prove the result we provide a matching upper bound and lower bound (up to error terms) for

\[(n_k)^{-1} \sum_{n^{-1-\varepsilon} \leq j_k \leq n} \mathbb{E}[\mathbb{P}_W(Z_n(j) = m_\ell, \ell \in [k])].\]

Upper bound

Let us introduce the event

\[E_n := \left\{ \sum_{\ell=1}^j W_\ell \in \{(1 - \zeta_n)\mathbb{E}[W] j, (1 + \zeta_n)\mathbb{E}[W] j, \forall n^{-1-\varepsilon} \leq j \leq n \} \right\}, \tag{5.13}\]

where \(\zeta_n = n^{-\delta(1-\varepsilon)}/\mathbb{E}[W]\) for some \(\delta \in (0, 1/2)\). By noting that \(\tilde{S}_j := \sum_{\ell=1}^j \mathbb{E}[W] - j \mathbb{E}[W] / \mathbb{E}[W] = 1 + \mathbb{E}[W] = \theta\) and that \(\zeta_n \geq j^{-\delta} / \mathbb{E}[W]\) for \(j \geq n^{1-\varepsilon}\), we can use the Azuma–Hoeffding inequality to obtain

\[\mathbb{P}(E_n^c) \leq \sum_{j \geq n^{1-\varepsilon}} \mathbb{P}\left(\tilde{S}_j \geq \zeta_n j \mathbb{E}[W]\right) \leq 2 \sum_{j \geq n^{1-\varepsilon}} \exp\left\{ - \frac{j^{1-2\delta}}{2} \right\}. \tag{5.14}\]

Writing \(c_0 := 1/(2\theta^2)\), we further bound the sum from above by

\[2 \int_{n^{1-\varepsilon}}^{\infty} \exp\left\{ - c_0 x^{1-2\delta}\right\} dx = 2c_0^{-1(1-2\delta)} \frac{\Gamma\left(\frac{1}{1-2\delta}, c_0[n^{1-\varepsilon}]^{1-2\delta}\right)}{1-2\delta}, \tag{5.15}\]

where \(\Gamma(a, x)\) is the incomplete Gamma function, and we note that the right-hand side is \(o(n^{-\gamma})\) for any \(\gamma > 0\) (and thus of smaller order than \(\prod_{\ell=1}^d p_{m_\ell} n^{-\beta}\) for any \(\beta > 0\) and uniformly in \(m_1, \ldots, m_k < (\theta/(\theta - 1)) \log n\). This yields the upper bound

\[
\frac{1}{n_k} \sum_{n^{-1-\varepsilon} \leq j \leq n} \mathbb{E}[\mathbb{P}_W(Z_n(j) = m_\ell, \ell \in [k])]
\leq (n_k)^{-1} \sum_{n^{-1-\varepsilon} \leq j \leq n} \mathbb{E}[\mathbb{P}_W(Z_n(j) = m_\ell, \ell \in [k])] \mathbb{I}_{E_n} + O\left(\Gamma\left(\frac{1}{1-2\delta}, c_0[n^{1-\varepsilon}]^{1-2\delta}\right)\right).
\tag{5.16}\]

Now, to express the first term in (5.16) we consider ordered indices \(j_\ell, \ell \in [k]\), rather than unordered ones. We provide details for the case \(n^{-1-\varepsilon} \leq j_1 < j_2 < \ldots < j_k \leq n\) and discuss later on how the other permutations of \(j_1, \ldots, j_k\) can be dealt with.

Moreover, for every \(\ell \in [k]\), we introduce the ordered indices \(j_\ell < i_{1, \ell} < \ldots < i_{m_\ell, \ell} \leq n, \ell \in [k]\), which denote the steps at which vertex \(\ell\) increases its degree by one. Note that for every \(\ell \in [k]\) these indices are distinct by definition, but we also require that \(i_{s, \ell} \neq i_{i, j}\) for any \(\ell, j \in [k]\), which denote the steps at which vertex \(\ell\) increases its degree by one. We denote this constraint by adding a * on the summation symbol.

Finally, we define \(j_{k+1} := n\). Combining these additional steps, we arrive at

\[
\frac{1}{n_k} \sum_{n^{-1-\varepsilon} \leq j \leq n} \mathbb{E}[\mathbb{P}_W(Z_n(j) = m_\ell, \ell \in [k])] \mathbb{I}_{E_n}
= \frac{1}{n_k} \sum_{n^{-1-\varepsilon} \leq j \leq n} \sum_{j < i_{1, \ell} < \ldots < i_{m_\ell, \ell} \leq n} \mathbb{E}\left[\prod_{t=1}^{m_\ell} \frac{W_{i_t}}{\sum_{t=1}^{m_\ell} W_\ell} \mathbb{I}_{E_n}\right]
= \prod_{\ell=1}^{k} \prod_{s=j_{\ell}+1}^{\min\{m_\ell, \ell \in [k]\}} \left(1 - \frac{n_{-1-\varepsilon}}{\sum_{s=1}^{m_\ell} W_{i_s}}\right) \mathbb{I}_{E_n}.
\tag{5.17}\]

5.10
We then include the terms where \( s = i, t \) for \( \ell \in [m] \), \( t \in [k] \) in the second double product. To do this, we need to change the first double product to

\[
\prod_{s=1}^{k} \prod_{t=1}^{m_s} \frac{W_{j_{s,t}}}{W_{\ell} - \sum_{\ell=t}^{k} W_{j_{\ell,t}} \mathbf{1}_{\{i_s, t > j_{\ell,t}\}}} \leq \prod_{s=1}^{k} \prod_{t=1}^{m_s} \frac{W_{j_{s,t}}}{W_{\ell} - k}.
\]

(5.18)

that is, we subtract the vertex-weight \( W_{j_{\ell,t}} \) in the numerator when the vertex \( j_{\ell,t} \) has already been introduced by step \( i_{s,t} \). In the upper bound we use that the weights are bounded from above by one. We thus arrive at the upper bound

\[
\frac{1}{(n)_k} \sum_{n^{l-1} \leq j_1 < \ldots < j_k \leq n} \sum_{\ell \in [k]} \mathbb{E} \left[ \prod_{s=1}^{k} \prod_{t=1}^{m_s} \frac{W_{j_{s,t}}}{W_{\ell} - \sum_{\ell=t}^{k} W_{j_{\ell,t}} \mathbf{1}_{\{i_s, t > j_{\ell,t}\}}} \right].
\]

For ease of writing, we omit the first sum until we actually intend to sum over the indices \( j_1, \ldots, j_k \). We use the bounds from the event \( E_n \) to bound

\[
\sum_{\ell=1}^{i_{s,t} - 1} W_{\ell} \geq (i_{s,t} - 1) \mathbb{E}[W] (1 - \zeta_n), \quad \sum_{\ell=1}^{n} W_{\ell} \leq s \mathbb{E}[W] (1 + \zeta_n).
\]

For \( n \) sufficiently large, we observe that \((i_{s,t} - 1) \mathbb{E}[W] (1 - \zeta_n) - k \geq i_{s,t} \mathbb{E}[W] (1 - 2\zeta_n)\), so that we obtain

\[
\frac{1}{(n)_k} \sum_{j_k < i_1, \ldots, i_{m'}, t \leq n} \mathbb{E} \left[ \prod_{s=1}^{k} \prod_{t=1}^{m_s} \frac{W_{j_{s,t}}}{s \mathbb{E}[W] (1 + \zeta_n)} \prod_{u=1}^{j_{s+1}} \left( 1 - \sum_{\ell=t}^{s} W_{j_{\ell,u}} \right) \mathbf{1}_{E_n} \right].
\]

Moreover, relabelling the vertex-weights \( W_{j_{\ell,t}} \) to \( W_{\ell} \) for \( t \in [k] \) does not change the distribution of the terms within the expected value, so that the expected value remains unchanged. We can also bound the indicator from above by one, to arrive at the upper bound

\[
\frac{1}{(n)_k} \sum_{j_k < i_1, \ldots, i_{m'}, t \leq n} \mathbb{E} \left[ \prod_{s=1}^{k} \prod_{t=1}^{m_s} \frac{W_{\ell}}{s \mathbb{E}[W] (1 + \zeta_n)} \prod_{u=1}^{j_{s+1}} \left( 1 - \sum_{\ell=t}^{s} W_{\ell} \right) \mathbf{1}_{E_n} \right].
\]

We bound the final product from above by

\[
\prod_{s=1}^{j_{s+1}} \left( 1 - \frac{\sum_{\ell=t}^{s} W_{\ell}}{s \mathbb{E}[W] (1 + \zeta_n)} \right) \leq \exp \left\{ -\frac{1}{s \mathbb{E}[W] (1 + \zeta_n)} \sum_{s=j_{s+1}}^{j_{s+1}} \frac{\sum_{\ell=1}^{s} W_{\ell}}{s} \right\}
\]

\[
\leq \exp \left\{ -\frac{1}{s \mathbb{E}[W] (1 + \zeta_n)} \sum_{\ell=1}^{n} W_{\ell} \log \left( \frac{j_{s+1}}{j_{s+1} + 1} \right) \right\}
\]

(5.19)

As the weights are almost surely bounded by one, we thus find

\[
\prod_{s=j_{s+1}+1}^{j_{s+1}} \left( 1 - \frac{\sum_{\ell=t}^{s} W_{\ell}}{s \mathbb{E}[W] (1 + \zeta_n)} \right) \leq \left( \frac{j_{s+1}}{j_{s+1} + 1} \right)^{s \mathbb{E}[W] (1 + \zeta_n) / (1 + \zeta_n)} \left( 1 + \frac{1}{j_{s+1}} \right)^{k / (s \mathbb{E}[W] (1 + \zeta_n))}
\]

\[
= \left( \frac{j_{s+1}}{j_{s+1} + 1} \right)^{s \mathbb{E}[W] (1 + \zeta_n) / (1 + \zeta_n)} \left( 1 + \Theta(n^{-(1-\epsilon)}) \right).
\]
As a result, we obtain the upper bound
\[
(\frac{(n)_{k}}{k})^{-1} \sum_{\ell \leq n}^{\infty} \mathbb{E} \left[ \prod_{t=1}^{k} \left( \frac{W_{t}}{\mathbb{E}[W]} \right)^{m_{t}} \frac{1}{i_{s},t(1-2\zeta_{n})} \right]^{k} \left( \sum_{i=1}^{n} \frac{1}{j_{ni}} \right)^{-1} \sum_{i=1}^{n} \frac{1}{W_{i}/(\mathbb{E}[W](1+\zeta_{n}))} \right] \\
\times \left( 1 + \mathcal{O}(n^{-(1-\varepsilon)}) \right)
\]
\[
= (\frac{(n)_{k}}{k})^{-1} \sum_{\ell \leq n}^{\infty} (1-2\zeta_{n})^{-1} \sum_{i=1}^{n} \mathbb{E} \left[ \prod_{t=1}^{k} \left( \frac{W_{t}}{\mathbb{E}[W]} \right)^{m_{t}} \right] \\
\times \left( \sum_{i=1}^{n} \frac{1}{W_{i}/(\mathbb{E}[W](1+\zeta_{n}))} \right)^{k} \left( 1 + \mathcal{O}(n^{-(1-\varepsilon)}) \right),
\]
where in the last step we recall that \( j_{k+1} = n \). We then bound this from above even further by no longer constraining the indices \( i_{s},t \) to be distinct. That is, for different \( t_{1},t_{2} \in [k] \), we allow \( i_{s_{1},t_{1}} = i_{s_{2},t_{2}} \) to hold for any \( s_{1} \in [m_{t_{1}}], s_{2} \in [m_{t_{2}}] \). This yields
\[
(\frac{(n)_{k}}{k})^{-1} \sum_{\ell \leq n}^{\infty} (1-2\zeta_{n})^{-1} \sum_{i=1}^{n} \mathbb{E} \left[ \prod_{t=1}^{k} \left( \frac{W_{t}}{\mathbb{E}[W]} \right)^{m_{t}} \right] \\
\times \left( \sum_{i=1}^{n} \frac{1}{W_{i}/(\mathbb{E}[W](1+\zeta_{n}))} \right)^{k} \left( 1 + \mathcal{O}(n^{-(1-\varepsilon)}) \right).
\]
We set
\[
a_{t} := W_{t}/(\mathbb{E}[W](1+\zeta_{n})),
\]
and look at the terms
\[
\frac{n^{\sum_{t=1}^{k} a_{t}}}{(n)_{k}} \sum_{\ell \leq n}^{\infty} \prod_{t=1}^{k} \left( a_{t}(1+\zeta_{n}) )^{m_{t}} j_{t}^{a_{t}} \right)^{-1} \left( \sum_{s=1}^{n} \frac{1}{i_{s},t} \right). 
\]
We bound the sums from above by multiple integrals, almost surely, which yields
\[
\frac{n^{\sum_{t=1}^{k} a_{t}}}{(n)_{k}} \prod_{t=1}^{k} \left( a_{t}(1+\zeta_{n}) )^{m_{t}} j_{t}^{a_{t}} \right)^{-1} \left( \sum_{s=1}^{n} \frac{1}{i_{s},t} \right)^{k} \int_{x_{1},t}^{n} \cdots \int_{x_{m_{t}-1},t}^{n} \prod_{s=1}^{m_{t}} \frac{1}{i_{s},t} \, dx_{m_{t}} \cdots dx_{1}. 
\]
By repeated substitutions of the form \( u_{i,t} = \log(n/x_{i,t}), i \in [m_{t}-1] \), we obtain
\[
\frac{n^{\sum_{t=1}^{k} a_{t}}}{(n)_{k}} \prod_{t=1}^{k} \left( a_{t}(1+\zeta_{n}) )^{m_{t}} j_{t}^{a_{t}} \right)^{-1} \left( \sum_{s=1}^{n} \frac{1}{i_{s},t} \right)^{k} \int_{x_{1},t}^{n} \cdots \int_{x_{m_{t}-1},t}^{n} \prod_{s=1}^{m_{t}} \frac{1}{i_{s},t} \, dx_{m_{t}} \cdots dx_{1}. 
\]
We observe that switching the order of the indices \( j_{1}, \ldots, j_{k} \) achieves the same result as permuting the \( m_{1}, \ldots, m_{k} \) and \( a_{1}, \ldots, a_{k} \). Hence, if we let \( \pi : [k] \rightarrow [k] \) be a permutation, then considering the indices \( n^{1-\varepsilon} \leq j_{\pi(1)} < j_{\pi(2)} \leq \ldots \leq j_{\pi(k)} \leq n \) yields a similar result as in (5.23) but with a term \( j_{\pi(t)}^{a_{\pi(t)}} (\log(n/j_{\pi(t)}))^{m_{\pi(t)}} / m_{\pi(t)}! \) in the final product. Since this product is invariant to such permutations of the \( m_{t} \) and \( a_{t} \), the only conditions that would change is the summation order of the indices \( j_{1}, \ldots, j_{k} \). We will use this further on.

We then bound the sum over \( n^{1-\varepsilon} \leq j_{1} < \ldots < j_{k} \leq n \) from above by multiple integrals as well. First, we note that \( j_{t}^{a_{t}} (\log(n/j_{t}))^{m_{t}} \) is increasing up to \( j_{t} = n \exp\{-m_{t}/a_{t}\} \), at which it is maximised, and decreasing for \( n \exp\{-m_{t}/a_{t}\} < j_{t} \leq n \) for all \( t \in [k] \). To provide the optimal bound, we want to know whether this maximum is attained in \([n^{1-\varepsilon},n]\) or not. That is, whether \( n \exp\{-m_{t}/a_{t}\} \in [n^{1-\varepsilon},n] \) or not. To this end, we consider two cases:

1. \( m_{t} = c_{t} \log(n/(1+o(1))) \) with \( c_{t} \in \{0,1/(\theta-1)\}, t \in [k] \) (\( c_{t} = 0 \) denotes \( m_{t} = o(\log n) \)).
2. \( m_{t} = c_{t} \log(n/(1+o(1))) \) with \( c_{t} \in (1/(\theta-1),c), t \in [k] \).
Clearly, when \( c \leq 1/(\theta - 1) \) the second case can be omitted, so that without loss of generality we can assume \( c > 1/(\theta - 1) \). Moreover, we can assume without loss of generality that all terms \( m_1, \ldots, m_k \) satisfy the same case, as a mixture of different cases can be dealt with in the same way, as will become clear in what follows. In the second case, it directly follows that the maximum is almost surely attained at

\[
 n \exp\{-m_t/a_t\} \leq n \exp\{-c_t \log n/(\theta - 1)(1 + o(1))\} = n^{1-c_t(\theta - 1)(1+o(1))} = o(1),
\]

so that the summand \( j_t^{a_t}(\log(n/j_t))^{m_t} \) is almost surely decreasing in \( j_t \) when \( n^{1-c} \leq j_t \leq n \). In the first case, such a conclusion cannot be made in general and depends on the precise value of \( W_t \). Therefore, the first case requires a more involved approach. Throughout the rest of the proof, we assume case (1) holds and discuss as we go along what alterations to make when case (2) holds instead. In the first case, we use Corollary 8.2 (with \( g \equiv 1 \)) to obtain the upper bound

\[
(5.24)
\]

Here, we use that the integrand is maximised at \( x^* = n \exp\{-m_k/a_k\} \), that \( (x^*)^{m_k}/(mk)! = n^{\alpha_k}/(e\alpha_k)^\Gamma(x^*+1) \) and that \( x^*/(e\Gamma(x^*+1)) \leq 1 \) for any \( x^* > 0 \). In case (2) the summand on the left-hand side is decreasing in \( j_k \), so that we arrive at an upper bound without the additional error term \( n^{\alpha_k}/(mk)! \). Using a substitution \( y_k := \log(n/x_k) \), we obtain

\[
(5.25)
\]

where, conditionally on \( W_k, Y_k \) is a \( \Gamma(m_k + 1, 1 + a_k) \) random variable. As mentioned above, in the second case the second term in the square brackets can be omitted.

The aim is to continue this approach for the summation over the remaining indices \( j_{k-1}, \ldots, j_1 \). We deal with the two terms we have here in different ways. Let us start with the second term. We now use the exact same approach, but almost surely bound, for \( 2 \leq t \leq k - 1 \),

\[
P_W(Y_t \leq \log(n/j_{t-1})) \leq 1, \quad \text{and} \quad P_W(Y_t \leq c \log n) \leq 1, \quad \text{where, conditionally on } W_t, Y_t \text{ is a } \Gamma(m_t + 1, 1 + a_t) \text{ random variable for each } t \in [k - 1].
\]

Hence, for the second term we obtain

\[
(5.26)
\]

Using this in (5.23) thus yields a term

\[
\frac{1}{n} \prod_{t=1}^{k-1} \left( \frac{a_t^{m_t}}{(1 + a_t)^{m_t + 1}} + \frac{4}{n} \right) (1 + \zeta_n) \sum_{t=1}^{k-1} m_t (1 + O(1/n)).
\]

Again, if \( m_t \) satisfies case (2) for any \( t \in [k - 1] \), the term \( 4/n \) can be omitted for that specific value of \( t \) in the product. If \( m_k \) satisfies case (2), then this entire term can be omitted.
To continue the summation of the first term on the right-hand side of (5.25), we again use Corollary 8.2 (now with \( g(x) = \mathbb{P}_W(Y_k < \log(n/x)) \)), to obtain

\[
\sum_{j_k-1 = j_{k-2} + 1}^{n} \prod_{t=1}^{k-2} j_t^{n_t} \left( \frac{\log(n/j_t)}{m_t} \right)^{m_t} \frac{1}{(1 + a_t)^{m_t+1}} \mathbb{P}_W(Y_k \leq \log(n/j_k-1))
\]

\[
\leq \frac{n^{1+a_k}}{(1 + a_k)^{m_k+1}} \prod_{t=1}^{k-2} j_t^{n_t} \left( \frac{\log(n/j_t)}{m_t} \right)^{m_t} \left[ 4 \frac{n^{a_k-1}}{a_k^{m_k-1}} \right]
\]

\[
+ \int_{j_{k-2}}^{n} j_{k-1}^{a_{k-1}} \left( \frac{\log(n/x_{k-1})}{m_{k-1}} \right)^{m_{k-1}} \mathbb{P}_W(Y_k < \log(n/x_{k-1})) \, dx_{k-1}.
\]

Using a substitution \( y_{k-1} := \log(n/x_{k-1}) \) yields

\[
\frac{n^{1+a_k}}{(1 + a_k)^{m_k+1}} \prod_{t=1}^{k-2} j_t^{n_t} \left( \frac{\log(n/j_t)}{m_t} \right)^{m_t} \left[ 4 \frac{n^{a_k-1}}{a_k^{m_k-1}} \right]
\]

\[
+ \int_{y_{k-2}}^{n} y_{k-1} \prod_{t=k-1}^{k-2} \left( 1 + a_t \right)^{m_t+1} j_t^{n_t} e^{-(1+a_t)y_t} \, dp_d y_{k-1}
\]

\[
= 2 \frac{n^{1+a_k}}{(1 + a_k)^{m_k+1}} \prod_{t=1}^{k-2} j_t^{n_t} \left( \frac{\log(n/j_t)}{m_t} \right)^{m_t}
\]

\[
+ \frac{1}{(1 + a_k)^{m_k+1}} \prod_{t=k-1}^{k} \left( 1 + a_t \right)^{m_t+1} \mathbb{P}_W(Y_k < Y_{k-1} < \log(n/j_{k-2})) \prod_{t=1}^{k-2} j_t^{n_t} \left( \frac{\log(n/j_t)}{m_t} \right)^{m_t}.
\]

Using the same approach as in (5.26) for the first term on the right-hand side yields the upper bound

\[
n^{k-1 + \sum_{t=1}^{k} a_t} \prod_{t=1}^{k} \left( \frac{a_t^{m_t}}{(1 + a_t)^{m_t+1}} + \frac{4}{n} \right).
\]

Using this in (5.23) yields the term

\[
\frac{4}{n} \prod_{t \in [k], t \neq k-1} \left( \frac{a_t^{m_t}}{(1 + a_t)^{m_t+1}} + \frac{4}{n} \right) \left( 1 + \zeta_n \right)^{\sum_{t=1}^{k} m_t} \left( 1 + O(1/n) \right).
\]

It thus follows that we can continue this approach summing over \( j_{k-2}, \ldots, j_1 \) and use this in (5.23) to obtain

\[
\prod_{t=1}^{k} \frac{a_t^{m_t}}{(1 + a_t)^{m_t+1}} \mathbb{P}_W(Y_k < \ldots < Y_1 < \varepsilon \log n) \left( 1 + \zeta_n \right)^{\sum_{t=1}^{k} m_t}
\]

\[
+ O \left( \sum_{t=1}^{k} \prod_{t \in [k], t \neq k-1} \left( \frac{a_t^{m_t}}{(1 + a_t)^{m_t+1}} + \frac{4}{n} \right) \right),
\]

where we note that we can omit the terms \( (1 + \zeta_n)^{\sum_{t=1}^{k} m_t} \) with the big \( O \) notation as they are \( 1 + o(1) \) by the specific choice of \( \zeta_n \) and the bound on \( m_1, \ldots, m_k \). Finally, using his in (5.21) and then in (5.20) yields the upper bound

\[
\mathbb{E} \left[ \prod_{t=1}^{k} \frac{a_t^{m_t}}{(1 + a_t)^{m_t+1}} \mathbb{P}_W(Y_k < \ldots < Y_1 < \varepsilon \log n) \right] \left( 1 + \zeta_n \right)^{\sum_{t=1}^{k} m_t}
\]

\[
\times \left( 1 + O(n^{-(1-c)}) \right) + O \left( \sum_{t=1}^{k} \prod_{t \in [k], t \neq k-1} \left( \mathbb{E} \left[ \frac{a_t^{m_t}}{(1 + a_t)^{m_t+1}} \right] + \frac{4}{n} \right) \right).
\]

where the term on the second line contains all the error terms created throughout and the expected value can be included within the product by the independence of the \( a_t \).

As mentioned below (5.23), any different order of the indices \( j_1, \ldots, j_k \) results in a permutation of \( m_1, \ldots, m_k \) and \( a_1, \ldots, a_k \). So, if we consider summing over \( n^{1-\varepsilon} \leq j_1 \neq \ldots \neq j_k \) rather than the
ordered indices \( n^{1-\varepsilon} \leq j_1 < \ldots < j_k \), we obtain the above term for all permutations of the \( a_t, m_t \), and \( Y_t \). That is, if we let \( P_k \) be the set of permutations on \([k]\), we obtain

\[
\sum_{\pi \in P_k} \mathbb{E} \left[ \prod_{t=1}^{k} \frac{a_t^{m_t}}{(1 + a_t)^{m_t+1}} P_w(Y_{\pi(k)} < \ldots < Y_{\pi(1)} < \varepsilon \log n) \right] \left( 1 + \zeta_n \right) \left( \frac{1}{1 - 2\zeta_n} \right)^{\sum_{i=1}^{k} m_t} \\
\times \left( 1 + \mathcal{O}(n^{-(1-\varepsilon)}) \right) + \mathcal{O} \left( \sum_{t=1}^{k} \frac{1}{n} \prod_{t \neq \ell} \left( \mathbb{E} \left[ \frac{a_{\pi(t)}}{(1 + a_{\pi(t)})^{m_{\pi(t)}+1}} \right] + \frac{4}{n} \right) \right) \\
= \prod_{t=1}^{k} \mathbb{E} \left[ \frac{a_t^{m_t}}{(1 + a_t)^{m_t+1}} P_w(Y_t < \varepsilon \log n) \right] \left( 1 + \zeta_n \right) \left( \frac{1}{1 - 2\zeta_n} \right)^{\sum_{i=1}^{k} m_t} \left( 1 + \mathcal{O}(n^{1-\varepsilon}) \right) \\
+ \mathcal{O} \left( \sum_{\pi \in P_k} \sum_{t=1}^{k} \frac{1}{n} \prod_{t \neq \ell} \left( \mathbb{E} \left[ \frac{a_{\pi(t)}}{(1 + a_{\pi(t)})^{m_{\pi(t)}+1}} \right] + \frac{4}{n} \right) \right),
\]

where the last step follows from the conditional independence of the \( Y_t \), the independence of the \( a_t \), and the fact that \( Y_t \neq Y_s \) almost surely for \( t \neq s \). We can now simply bound the conditional probability from above by one almost surely.

Since \( m_t < c \log n \) for all \( t \in [k] \), the fraction on the right of the expected value in the last step is \( 1 + o(n^{-\delta(1-\varepsilon)(1-\ell)}) \) for any \( \varepsilon > 0 \). Furthermore, within the expected values, we can write

\[
\frac{a_t^{m_t}}{(1 + a_t)^{m_t+1}} = \frac{\mathbb{E}[W]}{\mathbb{E}[W] + W(\mathbb{E}[W] + W)^m_t} \left( 1 + o(n^{-\beta_1}) \right) + \mathcal{O} \left( \Gamma \left( \frac{1}{1 - 2\delta}, c_n \right) \right)
\]

almost surely. In total, combining this with (5.16) yields the final upper bound

\[
\prod_{t=1}^{k} \mathbb{E} \left[ \frac{\mathbb{E}[W]}{\mathbb{E}[W] + W(\mathbb{E}[W] + W)^m_t} \right] \left( 1 + o(n^{-\beta_1}) \right) + \mathcal{O} \left( \Gamma \left( \frac{1}{1 - 2\delta}, c_n \right) \right)
\]

for some \( \beta_1 > 0 \). We then finally observe that by Lemma 5.5 and since we assumed that \( m_t = c_t \log n(1 + o(1)) \) with \( c_t \in [0, 1/(\theta - 1)] \) for all \( t \in [k], 
\]

\[
\mathbb{E} \left[ \frac{\mathbb{E}[W]}{\mathbb{E}[W] + W(\mathbb{E}[W] + W)^m_t} \right] \geq \frac{1}{(\theta + \xi)^m_t} \geq n^{-\log(\theta + \xi)^{(1 + o(1))/(\theta - 1)}} \geq \frac{1}{n^{\eta}},
\]

for some small \( \gamma > 0 \). The final step can be made for \( \xi, \gamma \) sufficiently small, as \( \log(x)/(x - 1) < 1 \) for all \( x \in (1, 2) \). This implies that, for any \( \ell \in [k] \) and for some sufficiently small \( \eta > 0, 
\]

\[
\frac{1}{n} \prod_{t \in [k]} \left( \mathbb{E} \left[ \frac{\mathbb{E}[W]}{\mathbb{E}[W] + W(\mathbb{E}[W] + W)^m_t} \right] + \frac{4}{n} \right)
\]

As a result, the second big \( \mathcal{O} \) term in (5.29) can be incorporated in the \( o(n^{-\beta_1}) \) term when \( \beta_1 \) is taken sufficiently small and all \( m_t \) satisfy case (1). When all the \( m_t \) satisfy case (2), then the second big \( \mathcal{O} \) term can be omitted entirely. Moreover, since the term in the first big \( \mathcal{O} \) term is \( o(n^{-\gamma}) \) for any \( \gamma > 0 \), this term can also be incorporated in the \( o(n^{-\beta_1}) \) term as well, as \( m_t < c \log n \) for all
\[ \ell \in [k]. \] We thus obtain for both cases that
\[
\mathbb{P}(Z_n(v_\ell) = m_\ell, \ell \in [k]) \leq \prod_{\ell=1}^k \mathbb{E} \left[ \frac{\mathbb{E}[W]}{\mathbb{E}[W] + W} \left( \frac{W}{\mathbb{E}[W] + W} \right)^{m_\ell} \right] (1 + o(n^{-\beta_1})). \tag{5.31}
\]

When the \( m_\ell \) are such that some of the \( c_\ell \) satisfy case (1) and some satisfy case (2), that is, \( c_\ell \in [0, 1/(\theta - 1)] \) for some \( \ell \in [k] \) and \( c_\ell \in (1/(\theta - 1), c) \) for the other indices \( t \), then a combined approach can be used to yield (5.31).

**Lower bound**

We then focus on proving a similar lower bound. We define the event
\[
\tilde{E}_n := \left\{ \sum_{\ell=k+1}^j W_\ell \in (\mathbb{E}[W]) (1 - \zeta_n) j, \mathbb{E}[W] (1 + \zeta_n) j), \forall n^{1-\varepsilon} \leq j \leq n \right\}.
\]

With similar computations as in (5.14) it follows that \( \mathbb{P}(\tilde{E}_n) = (1 - o(n^{-\gamma})) \) for any \( \gamma > 0 \). We obtain a lower bound for the probability of the event \( \{Z_n(v_\ell) = m_\ell, \ell \in [k]\} \) by omitting the second term in (5.16). This yields
\[
\frac{1}{(n)_k} \sum_{n^{1-\varepsilon} \leq j_1 < \ldots < j_k \leq n} \mathbb{E}[W(Z_n(j_\ell) = m_\ell, \ell \in [k])]
\]
\[
\geq \frac{n}{(n)_k} \sum_{n^{1-\varepsilon} < j_1 < \ldots < j_k \leq n} \sum_{\ell_1, \ldots, \ell_k} \mathbb{E} \left[ \prod_{\ell=1}^k W_{j_\ell} \prod_{\ell \neq \ell_\ell, t \in [k]} \left( 1 - \sum_{i=1}^n \frac{1}{W_{i_\ell}} \right) \right].
\]

We again start by only considering the ordered indices \( n^{1-\varepsilon} < j_1 < \ldots < j_k \) and also omit this sum for now for ease of writing. We also omit the constraint \( s \neq i_{\ell_\ell}, \ell \in [m_\ell], t \in [k] \) in the final product. As this introduces more terms smaller than one, we obtain a lower bound. Then, in the two denominators, we bound the vertex-weights \( W_{j_1}, \ldots, W_{j_k} \) from above and below by one and zero, respectively, to obtain a lower bound
\[
\frac{(n)_k}{(n)_k} \sum_{j_\ell < i_{\ell_\ell}, t \in [k]} \mathbb{E} \left[ \prod_{\ell=1}^k W_{j_\ell} \prod_{\ell \neq \ell_\ell, t \in [k]} \left( 1 - \sum_{i=1}^n \frac{1}{W_{i_\ell}} \right) \right].
\]

As a result, we can now swap the labels of \( W_{j_\ell} \) and \( W_t \) for each \( t \in [k] \), which again does not change the expected value, but it changes the value of the two denominators to \( \sum_{t=k+1}^\infty \sum_{t=k+1}^\infty \sum_{\ell=1}^k \sum_{t=1}^n \sum_{i=1}^n \mathbb{E}[W_{i_t}] + k \) and \( \sum_{t=k+1}^\infty \sum_{t=k+1}^\infty \sum_{\ell=1}^k \sum_{t=1}^n \sum_{i=1}^n \mathbb{E}[W_{i_t}] + k \), respectively. After this we introduce the indicator \( \mathbb{I}_{\tilde{E}_n} \) and use the bounds in \( \tilde{E}_n \) on these sums to obtain a lower bound. Finally, we note that the (relabelled) weights \( W_t, t \in [k] \), are independent of \( \tilde{E}_n \) so that we can take the indicator out of the expected value. Combining all of the above steps, we arrive at the lower bound
\[
\frac{(n)_k}{(n)_k} \sum_{j_\ell < i_{\ell_\ell}, t \in [k]} \mathbb{E} \left[ \prod_{\ell=1}^k \frac{W_{j_\ell}}{\mathbb{E}[W]} \prod_{\ell \neq \ell_\ell, t \in [k]} \left( 1 - \frac{1}{\sum_{i=1}^n W_{i_\ell}} \right) \right] \mathbb{P}(\tilde{E}_n). \tag{5.32}
\]

The \( 1 + 2\zeta_n \) in the fraction on the first line arises from the fact that, for \( n \) sufficiently large, \( \sum_{t=k+1}^\infty (1 + \zeta_n) + k \leq \sum_{t=k+1}^\infty (1 + 2\zeta_n) \). As stated above, \( \mathbb{P}(\tilde{E}_n) = 1 - o(n^{-\gamma}) \) for any \( \gamma > 0 \). Similar to the calculations in (5.19) and using \( \log(1 - x) \geq -x - x^2 \) for \( x \) small, we obtain an almost sure
lower bound for the final product for \( n \) sufficiently large of the form

\[
\prod_{s=j_n+1}^{j_{n+1}} \left( 1 - \frac{\sum_{\ell=1}^{u} W_{\ell}}{(s-1)\mathbb{E}[W]} (1-\zeta_{n}) \right) \geq \exp \left\{ - \frac{1}{\mathbb{E}[W]} \frac{1}{(1-\zeta_{n})} \frac{1}{s-1} \sum_{s=j_n+1}^{j_{n+1}} \sum_{\ell=1}^{u} W_{\ell} \right\}
\]

\[
- \left( \frac{1}{\mathbb{E}[W]} (1-\zeta_{n}) \right)^2 \sum_{s=j_n+1}^{j_{n+1}} \sum_{\ell=1}^{u} \frac{1}{(s-1)^2} \right\}
\]

\[
\geq \left( \frac{j_{n+1}-j_n}{j_n} \right)^{-s} \mathbb{E}[W] \mathbb{E}[W] (1-\zeta_{n}) \left( 1 - \mathcal{O}(n^{-(1-\varepsilon)}) \right).
\]

Using this in (5.32) yields the lower bound

\[
\frac{1}{(n)_k} \sum_{j_k < i_1, \ldots, i_k \leq n} (1 + 2\zeta_n) \sum_{\ell=1}^{k} m_{\ell} \mathbb{E}\left[ \prod_{t=1}^{k} \left( \frac{W_{t}}{\mathbb{E}[W]} \right)^{m_{t}} \left( \frac{\tilde{a}_{t}}{\tilde{n}_{t}} \right)^{m_{t}} \prod_{s=1}^{\ell-1} i_{s, t} \right] \left( 1 - \mathcal{O}(n^{-\varepsilon}) \right),
\]

where \( \tilde{a}_{t} := W_{t}/\mathbb{E}[W] (1-\zeta_{n}) \). We now reintroduce the sum over the indices \( n^{1-\varepsilon} \leq j_1 < \ldots < j_k \leq n \) and bound the sum over the indices \( i_{s, t} \) from below. We note that the expression in the expected value decreases in \( i_{s, t} \) and we restrict the range of the indices to \( j_k + \sum_{\ell=1}^{k} m_{\ell} < i_1, \ldots < i_{s, t} \leq n, t \in [k] \), but no longer constrain the indices to be distinct (so that we can drop the * in the sum). In the distinct sums and the suggested lower bound, the number of values the \( i_{s, t} \) take on equal

\[
\prod_{t=1}^{k} \left( n - (j_t - 1) - \sum_{\ell=1}^{t-1} m_{\ell} \right) \quad \text{and} \quad \prod_{t=1}^{k} \left( n - (j_t - 1) - \sum_{\ell=1}^{k} m_{\ell} \right),
\]

respectively. It is straightforward to see that the former allows for more possibilities than the latter, as \( \binom{b}{c} > \binom{b}{c} \) when \( b > a \geq c \). As we omit the largest values of the expected value (since it decreases in \( i_{s, t} \) and we omit the largest values of \( i_{s, t} \), we thus arrive at the lower bound

\[
\frac{1}{(n)_k} \sum_{j_k < i_1, \ldots, i_k \leq n} (1 + 2\zeta_n) \sum_{\ell=1}^{k} m_{\ell} \mathbb{E}\left[ \prod_{t=1}^{k} \left( \frac{\tilde{a}_{t}(1-\zeta_{n})}{\tilde{n}_{t}} \right)^{m_{t}} \prod_{s=1}^{\ell-1} i_{s, t} \right] \left( 1 - \mathcal{O}(n^{-\varepsilon}) \right),
\]

where we also restrict the range of indices in the upper bound of the outer sum, as otherwise there would be a contribution of zero from these values of \( j_1, \ldots, j_k \). We use similar techniques compared to the upper bound of the proof to switch from summation to integration. However, due to the altered bounds on the range of the indices over which we sum and the fact that we require lower bounds rather than upper bound, we face some more technicalities.

For now, we omit the expected value and focus on the terms

\[
\sum_{n^{1-\varepsilon} < j_1 < \ldots < j_k \leq n} \sum_{j_k + \sum_{\ell=1}^{k} m_{\ell} < i_1, \ldots < i_{s, t} \leq n, t \in [k]} \prod_{s=1}^{k} i_{s, t}^{m_{t}} \prod_{s=1}^{n^{1-\varepsilon}} j_{s, t}^{m_{t}}.
\]

We start by restricting the upper bound on the outer sum to \( n - 2 \sum_{\ell=1}^{k} m_{\ell} \). This will prove useful later. We then bound the sum over the indices \( i_{s, t} \) from below by

\[
\sum_{j_k + \sum_{\ell=1}^{k} m_{\ell} < i_1, \ldots < i_{s, t} \leq n} \prod_{\ell=1}^{k} \prod_{s=1}^{m_{\ell}} i_{s, t}^{m_{t}} \prod_{s=1}^{n^{1-\varepsilon}} j_{s, t}^{m_{t}}
\]

\[
\geq \prod_{t=1}^{k} \int_{j_1 + \sum_{\ell=1}^{k} m_{\ell} + 1}^{n+1} \ldots \int_{j_{m_{t}-1,t} + 1}^{n+1} x_1^{m_{t}-1} \ldots x_{m_{t}-1,t} \prod_{s=1}^{m_{t}} x_{s,t}^{-1} \prod_{t=1}^{n^{1-\varepsilon}} x_{m_{t},1,t} \prod_{t=1}^{n^{1-\varepsilon}} x_{m_{t} - 1,t} \prod_{t=1}^{n^{1-\varepsilon}} x_{m_{t},1,t}^{m_{t}-1} \prod_{t=1}^{n^{1-\varepsilon}} x_{m_{t},1,t}^{m_{t}-1} \prod_{s=1}^{m_{t}} x_{s,t}^{m_{t}-1} \log \left( \frac{n+1}{x_{m_{t},1,t}+1} \right) \prod_{t=1}^{n^{1-\varepsilon}} x_{m_{t},1,t} \prod_{t=1}^{n^{1-\varepsilon}} x_{m_{t},1,t}^{m_{t}-1} \prod_{t=1}^{n^{1-\varepsilon}} x_{m_{t},1,t}^{m_{t}-1} \prod_{s=1}^{m_{t}} x_{s,t}^{m_{t}-1}. \]
The integrand can be bounded from below by using \( x_{m_{r-1}, t}^{-1} \geq (x_{m_{r-1}, t} + 1)^{-1} \). We also restrict the upper integration bound of the innermost integral to \( n \) and use a variable substitution \( y_{m_{r-1}, t} := x_{m_{r-1}, t} + 1 \) to obtain the lower bound
\[
\prod_{t=1}^{k} \int_{n+1}^{n+1} \int_{x_{1, t}+1}^{x_{m_{r-1}, t}+1} \frac{1}{m_t} \log \left( \frac{n + x_{m_{r-1}, t} + 2}{x_{m_{r-1}, t} + 2} \right) \, dx_{m_{r-1}, t} \, \cdots \, dx_{1, t}.
\]
Continuing this approach eventually leads to
\[
\prod_{t=1}^{k} \frac{1}{m_t} \log \left( \frac{n + x_{m_{r-1}, t} + 2}{x_{m_{r-1}, t} + 2} \right) m_t \geq \prod_{t=1}^{k} \frac{1}{m_t} \log \left( \frac{n}{j_t + 2 \sum_{t=1}^{k} m_t} \right)^{m_t}.
\]
Substituting this in (5.34) with the restriction on the outer sum discussed above yields
\[
\sum_{n^{1-\varepsilon} < j_1 < \cdots < j_k \leq n - 2 \sum_{t=1}^{k} m_t} \prod_{t=1}^{k} j_t^{\bar{a}_t} \frac{1}{m_t!} \log \left( \frac{n}{j_t + 2 \sum_{t=1}^{k} m_t} \right)^{m_t} \left( 1 - \frac{2 \sum_{t=1}^{k} m_t}{j_t + 2 \sum_{t=1}^{k} m_t} \bar{a}_t \right).
\]
To simplify the summation over \( j_1, \ldots, j_k \), we write the summand as
\[
\prod_{t=1}^{k} \left( j_t + 2 \sum_{t=1}^{k} m_t \right)^{\bar{a}_t} \frac{1}{m_t!} \log \left( \frac{n}{j_t + 2 \sum_{t=1}^{k} m_t} \right)^{m_t} \left( 1 - \frac{2 \sum_{t=1}^{k} m_t}{j_t + 2 \sum_{t=1}^{k} m_t} \bar{a}_t \right).
\]
Using that \( m_t < c \log n, j_t \geq n^{1-\varepsilon} \) and \( x^{\bar{a}_t} \geq x^{1/(E[W](1-\varepsilon))} \) for \( x \in (0, 1) \), we obtain the lower bound
\[
\prod_{t=1}^{k} \left( j_t + 2 \sum_{t=1}^{k} m_t \right)^{\bar{a}_t} \frac{1}{m_t!} \log \left( \frac{n}{j_t + 2 \sum_{t=1}^{k} m_t} \right)^{m_t} \left( 1 - O\left( \frac{\log n}{n^{1-\varepsilon}} \right) \right).
\]
We can then shift the bounds on the range of the sum to \( n^{1-\varepsilon} + 2 \sum_{t=1}^{k} m_t \) and \( n \) to obtain the lower bound
\[
\sum_{n^{1-\varepsilon} + 2 \sum_{t=1}^{k} m_t < j_1 < \cdots < j_k \leq n - 2 \sum_{t=1}^{k} m_t} \prod_{t=1}^{k} j_t^{\bar{a}_t} \frac{1}{m_t!} \log \left( n/j_t \right)^{m_t} \left( 1 - O\left( \frac{\log n}{n^{1-\varepsilon}} \right) \right).
\]
We can now use a similar approach as for the upper bound in (5.24) through (5.27) by considering the cases (1) and (2). Assuming case (1) holds for all \( m_1, \ldots, m_k \) and using Corollary 8.2, we obtain the lower bound
\[
\sum_{j_1 = \lfloor n^{1-\varepsilon} \rfloor + 2 \sum_{t=1}^{k} m_t}^{n - 2 \sum_{t=1}^{k} m_t} \cdots \sum_{j_{k-1} = j_{k-2} + 1}^{n - 2 \sum_{t=1}^{k} m_t} \prod_{t=1}^{k-1} j_t^{\bar{a}_t} \frac{1}{m_t!} \log \left( n/j_t \right)^{m_t} \times \left[ \int_{j_{k-1}+1}^{n} \frac{x^{\bar{a}_k}}{m_k!} \log \left( n/j_k \right)^{m_k} \, dx_k - \frac{n^{\bar{a}_k}}{m_k!} \right],
\]
and we again have for case (2) that the error term in the square brackets can be omitted. Following the same approach as in the upper bound, (5.24) through (5.27), but subtracting the error term rather than adding it, we thus obtain the lower bound
\[
\sum_{n^{1-\varepsilon} < j_1 < \cdots < j_k \leq n - 2 \sum_{t=1}^{k} m_t} \prod_{t=1}^{k} j_t^{\bar{a}_t} \frac{1}{m_t!} \log \left( \frac{n}{j_t + 2 \sum_{t=1}^{k} m_t} \right)^{m_t} \geq n^{k+\sum_{t=1}^{k} \bar{a}_t} \prod_{t=1}^{k} \left( \frac{1 - O\left( n^{1-\varepsilon} \log n \right)}{1 + \bar{a}_t} \right)^{m_t} \prod_{t=1}^{k} \left( 1 + \bar{a}_t \right)^{m_t+1} \log \left( \frac{n}{\left[ n^{1-\varepsilon} \right] + 2 \sum_{t=1}^{k} m_t} \right) + O\left( n^{k+\sum_{t=1}^{k} \bar{a}_t} \prod_{t=1}^{k} \left( \frac{1}{1 + \bar{a}_t} \right)^{m_t+1} \frac{4}{m_t!} \right).
\]
where, conditionally on \( W_t, \tilde{Y}_t \) is a \( \Gamma(m_t + 1, 1 + \tilde{a}_t) \) random variable for each \( t \in [k] \). Using this in (5.33) then finally yields the lower bound

\[
E \left[ \prod_{t=1}^{k} \frac{\tilde{a}_t^{m_t}}{(1 + \tilde{a}_t)^{m_{t+1}}} \mathbb{P}_W \left( \tilde{y}_k < \ldots < \tilde{y}_1 < \log \left( \frac{n}{n^{1-\varepsilon}} + 2 \sum_{t=1}^{k} m_t \right) \right) \right] \\
\times \left( 1 + O(n^{-\delta(1-\xi)(1-\varepsilon)}) \right) + O \left( \sum_{t=1}^{k} \prod_{t \in [k]} \left( \mathbb{E} \left[ \frac{\tilde{a}_t^{m_t}}{(1 + \tilde{a}_t)^{m_{t+1}}} \right] - \frac{4}{n} \right) \right),
\]

where we use, as in the upper bound, that \((1 - \zeta_0)/(1 + 2\zeta_0) \sum_{t=1}^{k} m_t = 1 - o(n^{-\delta(1-\xi)(1-\varepsilon)})\). If we then consider the summation over indices \( n^{1-\varepsilon} \leq j_1 \neq \ldots \neq j_k \) rather than \( n^{1-\varepsilon} \leq j_1 < \ldots < j_k \) we obtain, as in the upper bound,

\[
\prod_{t=1}^{k} E \left[ \frac{\tilde{a}_t^{m_t}}{(1 + \tilde{a}_t)^{m_{t+1}}} \mathbb{P}_W \left( \tilde{y}_t < \log \left( \frac{n}{n^{1-\varepsilon}} + 2 \sum_{t=1}^{k} m_t \right) \right) \right] \left( 1 - o(n^{-\delta(1-\xi)(1-\varepsilon)}) \right) \\
- O \left( \sum_{t=1}^{k} \prod_{t \in [k]} \left( \mathbb{E} \left[ \frac{\tilde{a}_t^{m_t}}{(1 + \tilde{a}_t)^{m_{t+1}}} \right] - \frac{4}{n} \right) \right).
\]

With a similar reasoning as in (5.28) and using that \( m_t < c \log n \) for all \( t \in [k] \), we can bound the expected value from below for large \( n \) as

\[
\prod_{t=1}^{k} \mathbb{E} \left[ \frac{\mathbb{E}[W^t]}{\mathbb{E}[W^t]} \mathbb{P}_W \left( \tilde{y}_t < \varepsilon(1 - \xi) \log n \right) \right] \left( 1 - o(n^{-\delta(1-\xi)(1-\varepsilon)}) \right) \\
+ O \left( \sum_{t=1}^{k} \prod_{t \in [k]} \left( \mathbb{E} \left[ \frac{\mathbb{E}[W^t]}{\mathbb{E}[W^t]} \right] - \frac{4}{n} \right) \right),
\]

for any \( \xi \in (0, 1) \). Unlike in the upper bound, we cannot trivially omit the conditional probability. Rather, it remains to show that it can be bounded from below by an indicator, at the cost of an additional error term. Since, conditionally on \( W_t, \tilde{Y}_t \sim \Gamma(m_t + 1, 1 + \tilde{a}_t) \), it follows that (again conditionally on \( W_t \)) \((1 + \tilde{a}_t) \tilde{Y}_t \overset{d}{=} X_t \), where \( X_t \sim \Gamma(m_t + 1, 1) \). We can thus write

\[
\mathbb{P}_W \left( \tilde{Y}_t \leq \varepsilon(1 - \xi) \log n \right) = \mathbb{P}_W \left( X_t \leq \varepsilon(1 - \xi)(1 + \tilde{a}_t) \log n \right) \geq \mathbb{P} \left( X_t \leq \varepsilon(1 - \xi) \log n \right),
\]

almost surely, where the final lower bound is obtained by bounding \( \tilde{a}_t \) from below by zero. As the event on the right-hand side no longer depends on the vertex-weights, we can also omit the conditional probability. In the case that \( m_t = o(\log n) \) for all \( t \in [k] \), by using the Chernoff inequality we then conclude that for \( n \) sufficiently large, almost surely,

\[
\mathbb{P} \left( X_t \leq \varepsilon(1 - \xi) \log n \right) \geq 1 - n^{-\varepsilon(1-\xi)/2}2^{m_{t+1}} = 1 - n^{-\varepsilon(1-\xi)(1+o(1))/2}.
\]

Using this in (5.36), we thus arrive at the lower bound

\[
\prod_{t=1}^{k} \mathbb{E} \left[ \frac{\mathbb{E}[W^t]}{\mathbb{E}[W^t]} \mathbb{P}_W \left( \tilde{y}_t \leq \varepsilon(1 - \xi) \log n \right) \right] \left( 1 - o(n^{-\delta(1-\xi)(1-\varepsilon)}) \right) \\
+ O \left( \sum_{t=1}^{k} \prod_{t \in [k]} \left( \mathbb{E} \left[ \frac{\mathbb{E}[W^t]}{\mathbb{E}[W^t]} \right] - \frac{4}{n} \right) \right).
\]

Then, via the same reasoning as in (5.30), the big \( O \) term can be included in the error term when the \( m_t \) are \( o(\log n) \). We thus establish (5.1) when the \( m_t \) are \( o(\log n) \) by combining (5.31), (5.38) and the above and by setting \( \beta := \beta_1 \wedge (\varepsilon(1 - \xi)/4) \wedge (\delta(1 - \varepsilon)(1 - \xi)) \).

It thus remains to prove (5.1) when (some of) the \( m_t \) grow faster. The upper bound in (5.31) suffices in this case as well. For the lower bound, (5.36) still holds and we can deal with all \( m_t \) such that \( m_t = o(\log n) \) as in (5.37), so that we can assume without loss of generality that \( m_t \geq \eta \log n \) for some \( \eta \in (0, c) \) for all \( t \in [k] \). Or, more specifically, we assume that \( m_t \sim c_t \log n \) for some \( c_t \in (0, c) \) for all \( t \in [k] \) (so that taking \( \eta < \min_{t \in [k]} c_t \) yields the same result). Then, we bound

\[
\mathbb{P}_W \left( X_t \leq \varepsilon(1 - \xi)(1 + \tilde{a}_t) \log n \right) \geq \mathbb{P} \left( X_t \leq \frac{c_t \varepsilon}{1 - \mu} \log n \right) \mathbb{I}_{\{1 + \tilde{a}_t > c_t / ((1 - \mu)(1 - \xi))\}},
\]
where $\mu \in (0, 1)$ is a small constant and where we again can switch to the non-conditional probability measure $P$ in the last step as there are no vertex-weights involved in the probability. Now, by choosing $\varepsilon \in (1 - \mu, 1)$ we can obtain a bound on the rate of convergence to one by applying a standard large deviation bound. Let $(V_i)_{i \in \mathbb{N}}$ be i.i.d. exponential rate 1 random variables and let $I(a) := a - 1 - \log(a)$ be their rate function. Then, as we can think of $X_t$ as the sum of $V_1, \ldots, V_{m_t + 1}$,

$$P\left(X_t \geq \frac{c_2 \varepsilon}{1 - \mu} \log n\right) = P\left(\sum_{i=1}^{m_t + 1} V_i \geq (m_t + 1) \frac{c_2 \varepsilon \log n}{(1 - \mu)(m_t + 1)}\right)$$

$$\leq \exp\left\{ - (m_t + 1) I\left(\frac{c_2 \varepsilon \log n}{(1 - \mu)(m_t + 1)}\right)\right\}.$$ 

In the first step, we express the upper bound within the probability in terms of the mean of the sum of random variables, which equals $m_t + 1$. We then use the large deviations bound in the second step, which we can do as the argument of $I$ is strictly greater than 1 when $n$ is sufficiently large (as $m_t + 1 \sim c_1 \log n$ and $\varepsilon \in (1 - \mu, 1)$ is sufficiently close to one. Since $I((c_2 \varepsilon \log n)/(1 - \mu)(m_t + 1)) = (\varepsilon/(1 - \mu) - 1 - \log(\varepsilon/(1 - \mu)))/((1 + o(1))$, we thus arrive at

$$P\left(X_t \geq \frac{c_2 \varepsilon}{1 - \mu} \log n\right) \leq e^{-c_2 \log n\varepsilon/(1 - \mu) - 1 - \log(\varepsilon/(1 - \mu))}(1 + o(1)) = n^{-c_2 \varepsilon (1 + \theta)/(1 + o(1))},$$

where $c_{t, \mu, \varepsilon} := c_t(\varepsilon/(1 - \mu) - 1 - \log(\varepsilon/(1 - \mu))) > 0$ as $\varepsilon \geq 1 - \mu$. When combining this with (5.39) in (5.36) and recalling that $\tilde{a}_t = W_t/(E[W](1 - c_0)) \geq W_t/E[W]$, we obtain the lower bound

$$\frac{1}{(n \varepsilon)^{1 - \varepsilon \gamma}} \sum_{1 \leq n \leq n \leq n} P_W(Z_n(j_t) = m_{\varepsilon}, \ell \in [k])$$

$$\geq k \sum_{t=1}^{k} E\left[ E[W] + W \left( \frac{W}{E[W]} + W \right)^{m_t} \right] (1 + o(n^{-\varepsilon \gamma}))$$

$$+ O\left( \sum_{\pi \in P_k} \sum_{1 \leq n \leq n} \prod_{\ell \neq \ell} \left( E \left[ \frac{E[W]}{E[W]} + W \left( \frac{W}{E[W] + W} \right)^{m_{\varepsilon}(\ell)} \right] - 4 n\right) \right),$$

for some $\beta_2 < \min_{t \in [k]} c_{t, \mu, \varepsilon} \cdot \varepsilon (1 - 1 - \varepsilon) \leq \varepsilon (1 - 1 - \varepsilon)/4$. We can then replace $c_t$ in the indicator by $\tilde{c}_t := \max(\pi \in [k]) \tilde{c}_t$ to obtain a further lower bound. The indicator in the expected value then is non-zero with positive probability when $(\tilde{c}_t(1 - 1 - \varepsilon) - 1)E[W] < 1$, or, equivalently, $\tilde{c} < (1 - 1 - \varepsilon)\theta/(\theta - 1)$. We can therefore take any $c_1, \ldots, c_k < \theta/(\theta - 1)$ and set $\mu$ and $\varepsilon$ small enough for this inequality to be satisfied.

We now argue that at the cost of an additional error term, we can omit the indicator in the expected value. We recall that we assumed that $\eta \log n \leq m_t \leq c \log n$ for all $\ell \in [k]$ for some $\eta \in (0, c)$. Recall $f_k(\theta, W)$ from (5.6) and note that $p_k = E[f_k(\theta, W)]$. Let us also set $a := \tilde{c}_t(1 - 1 - \varepsilon) - 1)E[W]$ and note that by the above, $a \in (0, 1)$. Since $m_{\varepsilon} > 1/E[W]$ (this implies $f_m(\theta, x)$ is increasing in $x$), for $n$ large,

$$E[f_{m_{\varepsilon}}(\theta, W)] = E[f_{m_{\varepsilon}}(\theta, W) \mathbb{1}_{\{W > \gamma\}} + \mathbb{1}_{\{W \leq \gamma\}}] \leq E[f_{m_{\varepsilon}}(\theta, W) \mathbb{1}_{\{W \leq \gamma\}}] + f_{m_{\varepsilon}}(\theta, a).$$

We then bound

$$f_{m_{\varepsilon}}(\theta, a) \leq \left( \frac{a}{\theta - 1 + a} \right)^{m_{\varepsilon}} \leq (\theta + \varepsilon)^{-m_{\varepsilon}},$$

for some sufficiently small $\varepsilon > 0$, as $a \in (0, 1)$ and the fraction is strictly increasing in $a$. Since it follows from the proof of Lemma 5.5 that $E[f_k(\theta, W)](\theta + \varepsilon)^k$ diverges exponentially fast as $k$ tends to infinity when $W$ is bounded by one, it then follows that

$$E[f_{m_{\varepsilon}}(\theta, W) \mathbb{1}_{\{W \leq \gamma\}}] \geq E[f_{m_{\varepsilon}}(\theta, W)] (\theta + \varepsilon)^{-m_{\varepsilon}} = E[f_{m_{\varepsilon}}(\theta, W)] (1 - o((1 + \gamma)^{-m_{\varepsilon}})),$$

for some small $\gamma > 0$. As $m_{\varepsilon} \geq \eta \log n$ for all $\ell \in [k]$ it then follows that $1 - o((1 + \gamma)^{-m_{\varepsilon}}) = 1 - o(n^{-\beta_2})$ for some $\beta_2 > 0$. We thus obtain the lower bound

$$\prod_{t=1}^{k} E\left[ E[W] + W \left( \frac{W}{E[W] + W} \right)^{m_{\varepsilon}(\ell)} \right] (1 + o(n^{-\beta_2}))$$

$$+ O\left( \sum_{\pi \in P_k} \sum_{1 \leq n \leq n} \prod_{\ell \neq \ell} \left( E \left[ \frac{E[W]}{E[W]} + W \left( \frac{W}{E[W] + W} \right)^{m_{\varepsilon}(\ell)} \right] - 4 n\right) \right).$$
Finally, with a similar reasoning as in (5.30), we can either include the big $\mathcal{O}$ term in the error term $1 + o(n^{-\beta_2\wedge\beta_2^2})$ in case (1) or omit it completely in case (2). Again, a combination of $m_t$ which satisfy either case (1) or case (2) can be dealt with by combining both approaches. The proof is then concluded by combining the upper bound in (5.31) and the lower bound above and setting $\beta := \beta_1 \wedge \beta_2 \wedge \beta_2^2 \wedge (\varepsilon(1 - \xi)/4) \wedge \delta(1 - \varepsilon)(1 - \xi)$.

\[ \square \]

We now discuss the necessary changes to the proof for it to hold for the model with random out-degree, as introduced in Remark 2.2(ii), as well. The main difference between these two models is the fact that the events $\{n \to i\}$ and $\{n \to j\}$ are no longer disjoint, where $u \to v$ denotes that vertex $u$ connects to vertex $v$ with an edge directed towards $v$. As a result, the probability, conditional on the vertex-weights, that vertex $n$ does not connect to vertices $j_1, \ldots, j_k \in [n - 1]$ now equals

\[
\prod_{\ell=1}^{k} \left(1 - \frac{W_{j_\ell}}{S_{n-1}}\right) \quad \text{instead of} \quad \left(1 - \frac{\sum_{\ell=1}^{k} W_{j_\ell}}{S_{n-1}}\right). \tag{5.40}
\]

Moreover, as a new vertex can connect to multiple vertices at once, it is no longer necessary to restrict ourselves to indices $j_\ell < i_{1,\ell} < \ldots < i_{m_t,\ell} \leq n, \ell \in [k]$ with $i_{s,\ell} \neq i_{t,\ell}$ for any $\ell, j \in [k], s \in [m_t], t \in [m_j]$. Instead, when $\ell \neq j$, $i_{s,\ell} = i_{t,\ell}$ is allowed for any $s \in [m_t], t \in [m_j]$. That is, the steps in which a vertex increase their degree by one no longer need to be disjoint.

In the proof of the upper bound, this changes the right-hand side of (5.17) into

\[
\frac{1}{(n)_k} \sum_{n^{-1-\varepsilon} \leq j_1 < \ldots < j_n \leq n} \sum_{\ell \in [k]} \mathbb{E} \left[ \prod_{\ell=1}^{k} \prod_{s=1}^{m_t} \frac{W_{j_s}}{\sum_{r=1}^{n-1} W_r} \prod_{u=1}^{k} \prod_{s=1}^{j_{u+1}} \prod_{r=1}^{u} \left(1 - \frac{W_{j_r}}{\sum_{r=1}^{n-1} W_r}\right) \mathbb{I}_{E_n} \right],
\]

where we change the last term in the expected value due to (5.40) and omit the $*$ in the inner sum due to the degree increments no longer being disjoint. We can then again use the event $E_n$ to bound the denominators with deterministic quantities. Then, the exact same steps to obtain the upper bound as in (5.19) can be performed. Most importantly, this yields that the difference in the upper bound due to (5.40) is no longer present. Furthermore, as we omit the constraint of disjoint indices $i_{s,\ell}$ in (5.20), we obtain the exact same upper bound as we would here. As a result, the remaining steps of the proof of the upper bound carry through as well.

The same is true for the lower bound. In fact, in the original proof we argue why we can omit the constraint of disjoint indices $i_{s,\ell}$ in a certain way and still obtain a lower bound in (5.33), which is no longer necessary here. Hence, we obtain (5.35) without the factor two in front of $\sum_{\ell=1}^{k} m_t$. This change carries through all the steps, so that we arrive at the same lower bound as in (5.36). The remaining steps for the lower bound can then be followed as well, which, together with the upper bound, establishes the desired result.

5.3. Proof of Lemma 5.11.

**Proof of Lemma 5.11.** We aim to bound

\[
\frac{1}{(n)_k} \sum_{j \in J_n(\varepsilon)} \mathbb{E} \left[ P_W(Z_n(j)) = m_\ell, \ell \in [k] \right], \tag{5.41}
\]

where we recall that $I_n(\varepsilon) := \{ j = (j_1, \ldots, j_k) : 1 \leq j_1 \neq \ldots \neq j_k \leq n, \exists i \in [k] j_i < n^{1-\varepsilon} \}$. We first assume that $m_\ell = c_\ell \log n(1 + o(1))$ for some $c_\ell \in [0, 1/\log \theta]$ for all $\ell \in [k]$, where $c_\ell = 0$ denotes that $m_\ell = o(\log n)$. We define

\[
I_n(\varepsilon, i) := \{ j \in I_n(\varepsilon) : \|\ell \in [k] : \ell < n^{1-\varepsilon}\| = i \}, \quad i \in [k],
\]
that is, \( I_n(\varepsilon, i) \) denotes the set of indices \( j = (j_1, \ldots, j_k) \) such that exactly \( i \) of the indices are smaller than \( n^{1-\varepsilon} \), and note that \( I_n(\varepsilon) = I_n(\varepsilon, 1) \cup \cdots \cup I_n(\varepsilon, k) \). We then write

\[
\frac{1}{(n)^k} \sum_{j \in I_n(\varepsilon)} \mathbb{E}[\mathbb{P}(Z_n(j^\ell) = m_\ell, \ell \in [k])] = \sum_{i=1}^k \frac{1}{(n)^k} \sum_{j \in I_n(\varepsilon, i)} \mathbb{E}[\mathbb{P}_W(Z_n(j^\ell) = m_\ell, \ell \in [k])],
\]

and bound the probability on the right-hand side from above by omitting all events \( \{Z_n(j^\ell) = m_\ell\} \) whenever \( j^\ell < n^{1-\varepsilon} \). This leaves us with

\[
\sum_{i=1}^{k-1} \frac{1}{(n)^k} n^{(1-\varepsilon)} \sum_{S \subseteq [k], |S| = k-i} \sum_{\ell \in S} \mathbb{E}[\mathbb{P}_W(Z_n(j^\ell) = m_\ell, \ell \in S)] + \frac{n^{k(1-\varepsilon)}}{(n)^k}, \tag{5.42}
\]

where we recall that the \( \ast \) on the second sum symbol denotes that we only consider distinct values of \( j_i, \ell \in S \). We isolated the case \( i = k \) here as in this case no indices are larger than \( n^{1-\varepsilon} \) and we hence bound the probability from above by one, whereas \( i = k \) would yield a contribution of zero in the triple sum. The inner sum can then be dealt with in the same manner as in the derivation of the upper bound in (5.31), to yield an upper bound

\[
\sum_{i=1}^{k-1} \frac{n^{-\varepsilon}}{n^{k-i}} \sum_{S \subseteq [k], |S| = k-i} \prod_{\ell \in S} \mathbb{E}\left[ \frac{\mathbb{E}[W]}{\mathbb{E}[W] + W}\left(\frac{W}{\mathbb{E}[W] + W}\right)^{m_\ell}\right] (1 + o(n^{-\beta})) + 2n^{-k\varepsilon},
\]

for some \( \beta > 0 \). It thus remains to show that for any \( m_\ell = c_\ell \log n(1 + o(1)) \) with \( c_\ell \in [0, 1/\log \theta] \) we can take \( \varepsilon \) sufficiently close to one and a small \( \eta > 0 \), such that

\[
n^{-\varepsilon} = o\left( \mathbb{E}\left[ \frac{\mathbb{E}[W]}{\mathbb{E}[W] + W}\left(\frac{W}{\mathbb{E}[W] + W}\right)^{m_\ell}\right] n^{-\eta} \right).
\]

By Lemma 5.5, we have for any \( \xi > 0 \) and \( n \) sufficiently large, that

\[
\mathbb{E}\left[ \frac{\mathbb{E}[W]}{\mathbb{E}[W] + W}\left(\frac{W}{\mathbb{E}[W] + W}\right)^{m_\ell}\right] \geq (\theta + \xi)^{-m_\ell} n^{-\varepsilon} = o\left(n^{-\eta - c_\ell \log(\theta + \xi)(1 + o(1))}\right),
\]

and

\[
n^{-\varepsilon} = o\left(n^{-\eta - c_\ell \log(\theta + \xi)(1 + o(1))}\right) \text{ when we choose } \eta \text{ and } \xi \text{ sufficiently small and } \varepsilon \text{ sufficiently close to 1, since } c_\ell \log \theta < 1 \text{ for any } \ell \in [k].
\]

As a result,

\[
\frac{1}{(n)^k} \sum_{j \in I_n(\varepsilon, i)} \mathbb{E}[\mathbb{P}_W(Z_n(j^\ell) = m_\ell, \ell \in [k])] = o\left( \prod_{\ell=1}^k \mathbb{E}\left[ \frac{\mathbb{E}[W]}{\mathbb{E}[W] + W}\left(\frac{W}{\mathbb{E}[W] + W}\right)^{m_\ell}\right] n^{-\eta} \right).
\]

We now assume that \( m_\ell = c_\ell \log n(1 + o(1)) \) with \( c_\ell \in [1/\log \theta, \theta/(\theta - 1)] \) for all \( \ell \in [k] \). In this case, the crude bound used above no longer suffices. Now, the aim is to combine the above with a similar approach as at the start of the proof of [14, Theorem 2.9, Bounded case] and also use the assumption that \( W \geq w^* > 0 \) almost surely for some \( w^* \in (0, 1) \). First, we consider the set of indices \( I_n(\varepsilon, k) \). To make use of the negative quadrant dependence of the degrees \( \{Z_n(i)\}_{i \in [n]} \) (see Remark 5.2 and [14, Lemma 7.1]), we create an upper bound by considering the event \( \{Z_n(j^\ell) \geq m_\ell, \ell \in [k]\} \). Then, using the tail distribution and the negative quadrant dependency of the degrees under the conditional probability measure \( \mathbb{P}_W \) yields

\[
\frac{1}{(n)^k} \sum_{j \in I_n(\varepsilon, k)} \mathbb{P}(Z_n(j^\ell) = m_\ell, \ell \in [k]) \leq \frac{1}{(n)^k} \sum_{1 \leq j_1 \neq \cdots \neq j_k < n^{1-\varepsilon}} \mathbb{E}\left[ \prod_{\ell=1}^k \mathbb{P}_W(Z_n(j^\ell) \geq m_\ell) \right].
\]

We then also allow the indices \( j_1, \ldots, j_k \) to take any value between 1 and \( n^{1-\varepsilon} \), to obtain the upper bound

\[
\frac{1}{(n)^k} \mathbb{E}\left[ \prod_{\ell=1}^k \left( \sum_{i < n^{1-\varepsilon}} \mathbb{P}_W(Z_n(i) \geq m_\ell) \right) \right].
\]

As in the proof of [14, Theorem 2.9, Bounded case], we apply a Chernoff bound to the conditional probability measure \( \mathbb{P}_W \) to obtain

\[
\frac{1}{(n)^k} \mathbb{E}\left[ \prod_{\ell=1}^k \left( \sum_{i < n^{1-\varepsilon}} \mathbb{P}_W(Z_n(i) \geq m_\ell) \right) \right] \leq \frac{1}{(n)^k} \mathbb{E}\left[ \prod_{\ell=1}^k \left( \exp\{m_\ell(1 - u_{i,\ell} + \log u_{i,\ell})\} \right) \right],
\]
where \( u_{i,\ell} := W_i(H_n - H_i)/m_\ell \) and \( H_n := \sum_{j=1}^{n-1} 1/S_j \). We then introduce the constants \( \delta \in (0, 1/2), C > kc_\theta^{-1} \theta \log(\theta)/(\theta - 1) \) (with \( c_\theta := 1/(2\theta^2) \)), the sequence 
\( \zeta_n := (C \log n)^{-\delta/(1-2\delta)}/E[W], n \in \mathbb{N} \), and introduce the event

\[ E'_n := \left\{ \sum_{\ell=1}^j W_\ell \geq jE[W] (1 - \zeta_n), \forall (C \log n)^{1/(1-2\delta)} \leq j \leq n \right\}. \]

The event \( E'_n \) is similar to the event \( E_n \) introduced in (5.13), but considers a larger range of indices \( j \). The particular choice of the lower bound on the indices \( j \) follows from the fact that we want as much control over the partial sums of the vertex-weights as possible, but need to ensure that \( \mathbb{P}( (E'_n)^c) \) decays sufficiently fast, which we can achieve via this choice.

We can use the event \( E'_n \) in the expected value to arrive at the upper bound

\[ \frac{1}{(n)_k} \mathbb{E} \left[ \prod_{\ell=1}^k \left( \sum_{i<n^{1-\varepsilon}} \exp\{m_\ell (1 - u_{i,\ell} + \log u_{i,\ell})\} \right) \mathbb{I}_{E'_n} \right] + \mathbb{P}( (E'_n)^c). \tag{5.43} \]

We defer the proof that \( \mathbb{P}( (E'_n)^c) \) decays sufficiently fast for now and focus on the first term. We bound \( u_{i,\ell} \) from above by

\[ u_{i,\ell} \leq \frac{H_n}{m_\ell} = \frac{1}{m_\ell} \left[ \sum_{j<(C \log n)^{1/(1-2\delta)}} \frac{1}{S_j} + \sum_{j=(C \log n)^{1/(1-2\delta)}}^{n} \frac{1}{S_j} \right], \]

and using \( W_i \geq w^* \) almost surely for all \( i \in \mathbb{N} \) as well as the bound in the event \( E'_n \) then yields

\[ u_{i,\ell} \leq \frac{1}{m_\ell} \left[ (1 - 2\delta \log(C \log n) + \frac{1}{E[W]} \log \left( \frac{n}{(C \log n)^{1/(1-2\delta)}} \right) \right] (1 + o(1)) = \frac{1 + o(1)}{c_\ell E[W].} \]

Since \( E[W] = \theta - 1 \) and \( c_\ell \geq 1/\log \theta \), it follows that \( 1/(c_\ell E[W]) \leq \log(\theta)/(\theta - 1) < 1 \) for all \( \theta \in (1, 2) \). Since \( x \mapsto 1 - x + \log x \) is increasing for \( x \in (0, 1) \) we can thus use this upper bound in the first term of (5.43) to bound it from above by

\[ \frac{1}{(n)_k} \prod_{\ell=1}^k \left( \sum_{i<n^{1-\varepsilon}} \exp \left\{ c_\ell \log n \left( 1 - \frac{1}{c_\ell E[W]} + \log \left( \frac{1}{c_\ell E[W]} \right) \right) (1 + o(1)) \right\} \right) \]

\[ \leq \frac{1}{(n)_k} \prod_{\ell=1}^k \exp \left\{ \log n \left( 1 - \varepsilon + c_\ell \left( 1 - \frac{1}{c_\ell E[W]} + \log \left( \frac{1}{c_\ell E[W]} \right) \right) (1 + o(1)) \right\} \]

\[ \leq \exp \left\{ \log n(1 + o(1)) \sum_{\ell=1}^k ( - \varepsilon + c_\ell \left( 1 - \frac{1}{c_\ell E[W]} + \log \left( \frac{1}{c_\ell E[W]} \right) \right) ) \right\}. \]

We then require that

\[ \exp \left\{ \log n(1 + o(1)) \sum_{\ell=1}^k ( - \varepsilon + c_\ell \left( 1 - \frac{1}{c_\ell E[W]} + \log \left( \frac{1}{c_\ell E[W]} \right) \right) ) \right\} = o\left(n^{-\eta} \prod_{\ell=1}^k p_{m_\ell}\right), \tag{5.44} \]

for some \( \eta > 0 \). As, by Lemma 5.5, \( p_{m_\ell} \geq (\theta + \xi)^{-m_\ell} = \exp\{-\log n(1 + o(1))c_\ell \log(\theta + \xi)\} \), it suffices to show that

\[ \sum_{\ell=1}^k ( - \varepsilon + c_\ell \left( 1 - \frac{1}{c_\ell E[W]} + \log \left( \frac{1}{c_\ell E[W]} \right) \right) ) < - \sum_{\ell=1}^k c_\ell \log(\theta + \xi), \tag{5.45} \]

when \( \xi \) is sufficiently small and \( \varepsilon \) sufficiently close to one. We show that this strict inequality can be achieved for each term individually, by arguing that we can choose \( \varepsilon \in (0, 1) \) such that

\[ \varepsilon > c_\ell \left( 1 - \frac{1}{c_\ell (\theta - 1)} + \log \left( \frac{\theta + \xi}{c_\ell (\theta - 1)} \right) \right), \quad \ell \in [k], \]

where we note that we have written \( E[W] \) as \( \theta - 1 \). The right-hand side is increasing in \( c_\ell \) when \( c_\ell \in [1/\log \theta, \theta/(\theta - 1)] \), so that all \( k \) inequalities are satisfied when we solve

\[ \varepsilon > \tilde{c} \left( 1 - \frac{1}{\tilde{c}(\theta - 1)} + \log \left( \frac{\theta + \xi}{\tilde{c}(\theta - 1)} \right) \right), \]
with \( \tilde{c} := \max_{\ell \in [k]} c_{\ell} \). We now show that the right-hand side is strictly smaller than one when \( \xi \) is sufficiently small. We write
\[
\tilde{c}\left(1 - \frac{1}{\tilde{c}(\theta - 1)} + \log \left(\frac{\theta + \xi}{\tilde{c}(\theta - 1)}\right)\right) = \tilde{c}\left(1 - \frac{1}{\tilde{c}(\theta - 1)} + \log \left(\frac{\theta}{\tilde{c}(\theta - 1)}\right)\right) + \tilde{c}\log \left(1 + \frac{\xi}{\theta}\right) \leq \tilde{c}\left(1 - \frac{1}{\tilde{c}(\theta - 1)} + \log \left(\frac{\theta}{\tilde{c}(\theta - 1)}\right)\right) + \frac{\xi}{\theta - 1}.
\]
where the final upper bound follows from the fact that \( \log(1 + x) \leq x \) for \( x > -1 \) and \( \tilde{c} < \theta/(\theta - 1) \).

We denote the first term on the right-hand side by \( \kappa = \kappa(\tilde{c}, \theta) \). As \( \kappa \) is increasing in \( \tilde{c} \) when \( \tilde{c} < \theta/(\theta - 1) \) (which is the case when \( c_{\ell} < \theta/(\theta - 1) \) for all \( \ell \in [k] \)), we have \( \kappa < 1 \), as \( \tilde{c} < \theta/(\theta - 1) \). Thus, setting \( \xi < (1 - \kappa)(\theta - 1)/2 \) we achieve the desired result. Now, taking \( \varepsilon \in (\kappa + \xi/(\theta - 1), 1) \), we arrive at (5.44) for some small \( \eta > 0 \). It thus follows that
\[
\frac{1}{(n)_k} \sum_{j \in I_n(\varepsilon, k)} \mathbb{E}[\mathbb{P}_W(Z_n(j) = m_\ell, \ell \in [k])] = o\left(n^{-\eta} \prod_{\ell=1}^k p_{m_\ell}\right), \quad (5.46)
\]
for some small \( \eta > 0 \).

We now consider the remaining sets \( I_n(\varepsilon, 1), \ldots, I_n(\varepsilon, k - 1) \) and aim to bound
\[
\frac{1}{(n)_k} \sum_{i=1}^{k-1} \sum_{j \in I_n(\varepsilon, i)} \mathbb{P}(Z_n(j) \geq m_\ell, \ell \in [k])\]
again, using the negative quadrant dependence and introducing the events \( E'_n \) and \( E_n \) (recall \( E_n \) from (5.13)) yields the upper bound
\[
\frac{1}{(n)_k} \sum_{i=1}^{k-1} \sum_{j \in I_n(\varepsilon, i)} \mathbb{E}\left[\mathbb{1}_{E'_n} \prod_{\ell=1}^k \mathbb{P}_W(Z_n(j) \geq m_\ell)\right] + \mathbb{P}((E'_n)^c) + \mathbb{P}(E_n^c).
\]
The aim is to treat the probabilities of indices which are at most \( n^{1-\varepsilon} \) in the same way as when dealing with the indices in \( I_n(\varepsilon, k) \) to reach a bound as in (5.44), for which we use the event \( E'_n \). For the indices which are larger than \( n^{1-\varepsilon} \) such an upper bound will not suffice. Instead, we aim to bound \( \mathbb{P}_W(Z_n(j) \geq m_\ell) \) when \( n^{1-\varepsilon} \leq j_\ell \leq n \) in a similar way as we bounded \( \mathbb{P}_W(Z_n(j) = m_\ell) \) from above in the proof of Lemma 5.10, for which we use \( E_n \).

First, we split the summation over \( I_n(\varepsilon, i) \) over all possible configurations of indices with are at most and at least \( n^{1-\varepsilon} \), similar to (5.42). That is,
\[
\frac{1}{(n)_k} \sum_{i=1}^{k-1} \sum_{j \in I_n(\varepsilon, i)} \mathbb{E}\left[\mathbb{1}_{E'_n} \prod_{\ell=1}^k \mathbb{P}_W(Z_n(j) \geq m_\ell)\right] = \frac{1}{(n)_k} \sum_{i=1}^{k-1} \sum_{j \in I_n(\varepsilon, i)} \sum_{S \subseteq [k]}^{S \neq \emptyset} \sum_{|S| = i}^{S \neq \emptyset} \sum_{\ell \in S}^{S \neq \emptyset} \mathbb{1}_{E'_n} \prod_{\ell \in S}^{S \neq \emptyset} \mathbb{P}_W(Z_n(j_\ell) \geq m_\ell) \prod_{\ell \in [k]\backslash S} \mathbb{P}_W(Z_n(j_\ell) \geq m_\ell).
\]
Using the event \( E'_n \), we can follow similar steps as above to bound the sum over the indices \( j_\ell \) and the product of probabilities \( \mathbb{P}_W(Z_n(j_\ell) \geq m_\ell) \) for \( \ell \in S \) from above by the deterministic upper bound
\[
\exp\left\{ \log n(1 + o(1)) \left( -\varepsilon + c_\ell \left( 1 - \frac{1}{c_\ell \mathbb{E}[W]} + \log \left(\frac{1}{c_\ell \mathbb{E}[W]}\right)\right) \right) \right\} =: n^{C(S)(1 + o(1))},
\]
which yields
\[
\sum_{i=1}^{k-1} \frac{n^i}{(n)_k} \sum_{S \subseteq [k]}^{S \neq \emptyset} n^{C(S)(1 + o(1))} \sum_{\ell \in [k]\backslash S}^{n^{1-\varepsilon} \leq j_\ell \leq n} \prod_{\ell \in [k]\backslash S} \mathbb{P}_W(Z_n(j_\ell) \geq m_\ell) \left[ E_n \prod_{\ell \in S} \mathbb{P}_W(Z_n(j_\ell) \geq m_\ell) \right]. \quad (5.47)
\]
We now proceed to bound each individual probability \( \mathbb{P}_W(Z_n(j_\ell) \geq m_\ell) \) when \( \ell \in [k]\backslash S \). This follows a similar approach to the upper bound of \( \mathbb{P}_W(Z_n(j_\ell) = m_\ell) \) in the proof of Lemma 5.10,
with a couple of modifications. Introducing indices \( j_\ell < i_1 < \ldots < i_{m_\ell} \leq n \), which denote the steps at which vertex \( j_\ell \) increases its degree, we can write

\[
P_W(Z_n(j_\ell) \geq m_\ell) = \sum_{j_\ell < i_1 < \ldots < i_{m_\ell} \leq n} \prod_{t=1}^{m_\ell} \frac{\Pi W_{j_t}}{\prod_{s=1}^{t-1} \Pi W_r \prod_{s \neq t, t \in [m_\ell]} \Pi W_r} \left( 1 - \frac{\Pi W_{j_t}}{\sum_{s=t}^{n} \Pi W_r} \right).
\]

The second product, in comparison to dealing with the event \( \{ Z_n(j_\ell) = m_\ell \} \), goes up to \( i_{m_\ell} - 1 \) instead of \( n \). This is due to the fact that we now only need to control the connections vertex \( j_\ell \) does and does not make up to its \( m_\ell \)th connection. Using the same idea as in (5.18) and using the event \( E_n \), we obtain the upper bound

\[
\sum_{j_\ell < i_1 < \ldots < i_{m_\ell} \leq n} \prod_{t=1}^{m_\ell} \left( \frac{\Pi W_{j_t}}{(i_t - 1)E[W]} \right) \left( 1 - \frac{\Pi W_{j_t}}{\sum_{s=1}^{i_{m_\ell} - 1} \Pi W_s} \right) \left( 1 - \frac{\Pi W_{j_t}}{\sum_{s=t}^{n} \Pi W_r} \right).
\]

The last step follows from the fact that \( (i_t - 1)(1 - \epsilon_i)E[W] - 1 \geq i_t(1 - 2\epsilon_i)E[W] \) for \( n \) sufficiently large. Using this in the expected value of (5.18) yields

\[
\sum_{n^{1-t} \leq j \leq n} E \left[ \prod_{t \in [k] \setminus S} \left( \sum_{j_t < i_1 < \ldots < i_{m_\ell} \leq n} \prod_{t=1}^{m_\ell} \left( \frac{\Pi W_{j_t}}{\Pi i_t E[W]} \right) \left( 1 - \frac{\Pi W_{j_t}}{\sum_{s=t}^{n} \Pi W_r} \right) \right) \right].
\]

We can now relabel the vertex-weights \( W_{j_t} \) as \( W_t, \ell \in [k] \setminus S \). This does not change the expected value and is possible since the indices \( j_\ell, \ell \in [k] \setminus S \) are distinct. Directly after this, we omit the condition that the indices \( j_\ell \) are distinct, which is one of no consequence as the weights have been relabelled already. We hence arrive at the upper bound

\[
\prod_{\ell \in [k] \setminus S} E \left[ \sum_{n^{1-t} \leq j \leq n} \sum_{j_t < i_1 < \ldots < i_{m_\ell} \leq n} \prod_{t=1}^{m_\ell} \left( \frac{\Pi W_{j_t}}{\Pi i_t E[W]} \right) \left( 1 - \frac{\Pi W_{j_t}}{\sum_{s=t}^{n} \Pi W_r} \right) \right].
\]

where the product can be taken out of the expected value due to the independence of the vertex-weights \( W_1, \ldots, W_n \). As a result, we can deal with each of expected values individually. Following the same approach as in (5.19) and setting \( a_\ell := W_\ell / (E[W] + \epsilon_i) \), we obtain the upper bound

\[
E \left[ \sum_{n^{1-t} \leq j \leq n} \sum_{j_t < i_1 < \ldots < i_{m_\ell} \leq n} \prod_{t=1}^{m_\ell} \left( \frac{\Pi W_{j_t}}{\Pi i_t E[W]} \right) \left( 1 + O(n^{-1/2}) \right) \right] = E \left[ a_\ell^{m_\ell} \sum_{n^{1-t} \leq j \leq n} \sum_{j_t < i_1 < \ldots < i_{m_\ell} \leq n} \prod_{t=1}^{m_\ell} \left( \frac{\Pi W_{j_t}}{\Pi i_t E[W]} \right) \left( 1 + O(n^{-1/2}) \right) \right].
\]

We then observe that the summand of the inner sum over the indices \( i_1, \ldots, i_{m_\ell} \) is decreasing, so that we can bound it from above almost surely by the multiple integrals

\[
\int_{j_t}^{n} \int_{x_1}^{n} \cdots \int_{x_{m_\ell-1}}^{n} \prod_{t=1}^{m_\ell-1} x_t^{-1} \left( 1 + a_\ell \right) dx_m \cdots dx_1 =: I_{m_\ell}.
\]

Calculating the value of the innermost integral yields the recursion

\[
I_{m_\ell} = I_{m_\ell-1} - \frac{1}{n^{a_\ell a_\ell}} \int_{j_t}^{n} \int_{x_1}^{n} \cdots \int_{x_{m_\ell-1}}^{n} \prod_{t=1}^{m_\ell-1} dx_{m_\ell-1} \cdots dx_1 = \frac{I_{m_\ell-1} - \log(n/j_{\ell})}{n^{a_\ell a_\ell} (m_\ell - 1)!} - \frac{1}{n^{a_\ell a_\ell}} \sum_{s=0}^{m_\ell-1} a_\ell^{m_\ell - s} \frac{\log(n/j_{\ell})}{s!}.
\]

where the last step follows from (5.22). By continuing the recursion we find that

\[
I_{m_\ell} = \frac{I_1}{a_\ell^{m_\ell-1}} - n^{a_\ell} \sum_{s=0}^{m_\ell-1} a_\ell^{m_\ell - s} \frac{\log(n/j_{\ell})}{s!} = a_\ell^{m_\ell - a_\ell} \left( 1 - \frac{1}{n^{a_\ell a_\ell}} \sum_{s=0}^{m_\ell-1} a_\ell^{m_\ell - s} \frac{\log(n/j_{\ell})}{s!} \right).
\]

Multiplying this with the \( a_\ell^{m_\ell - a_\ell} \) in the expected value in (5.49), we arrive at

\[
1 - \frac{1}{s!} \left( \frac{a_\ell \log(n/j_{\ell})}{s!} \right)^s = P_W(P(a_\ell) \geq m_\ell),
\]
where, conditionally on $W_\ell$, $P(a_\ell) \sim \text{Poi}(a_\ell \log(n/j_\ell))$. We now use the following duality between Poisson and gamma random variables. Let $X \sim \Gamma(m_\ell,1)$ be a gamma random variable. We can also interpret $X$ as a sum of $m_\ell$ rate one exponential random variables. Then, conditionally on $W_\ell$, the event $\{P(a_\ell) \geq m_\ell\}$ can be thought of as the event that in a rate one Poisson process at least $m_\ell$ particles have arrived before time $a_\ell \log(n/j_\ell)$. This is equivalent to the sum of the first $m_\ell$ inter-arrival times (which are rate one exponentially distributed) being at most $a_\ell \log(n/j_\ell)$. As we mentioned, this sum of $m_\ell$ rate one exponential random variables is, in law, identical to $X$

\[
P_W(P(a_\ell) \geq m_\ell) = P_W(X \leq a_\ell \log(n/j_\ell)) = P_W(Y \leq \log(n/j_\ell)),
\]

where, conditionally on $W_\ell$, $Y \sim \Gamma(m_\ell, a_\ell)$. Then, by the choice of $\zeta_n$, $(1 + \zeta_n)/(1 - 2\zeta_n)^{m_\ell} = 1 + O(n^{-\delta(1-\varepsilon)} \log n)$. Using both these results in (5.49), we arrive at

\[
E \left[ \sum_{n^{1-\varepsilon} \leq k \leq n} P_W(Y \leq \log(n/j_\ell)) \right] = O(a_\ell \log(n/j_\ell)).
\]

As the conditional probability is decreasing in $j_\ell$, we can bound the sum from above by an integral almost surely to obtain

\[
\int_{[n^{1-\varepsilon}]} P_W(Y \leq \log(n/x)) \, dx = \int_{[n^{1-\varepsilon}]} \int_{0}^{\log(n/x)} a_\ell^{m_\ell} \frac{(m_\ell - 1)!}{y^{m_\ell} e^{-y}} \, dy \, dx
\]

\[
= \int_{0}^{\log(n/[n^{1-\varepsilon}])} \int_{[n^{1-\varepsilon}]} a_\ell^{m_\ell} \frac{(m_\ell - 1)!}{y^{m_\ell} e^{-y}} \, dy \, dx
\]

\[
\leq n \int_{0}^{\log(n/[n^{1-\varepsilon}])} \frac{a_\ell^{m_\ell}}{(m_\ell - 1)!} \frac{y^{m_\ell} e^{-((1 + a_\ell)y)}}{y^{m_\ell} e^{-y}} \, dy
\]

\[
= n \left( \frac{a_\ell}{1 + a_\ell} \right)^{m_\ell} P_W(Y' \leq \log(n/[n^{1-\varepsilon}])).
\]

Here, we switch the integration over $x$ and $y$ in the second step and let $Y'$, conditionally on $W_\ell$, be a $\Gamma(m_\ell, 1 + a_\ell)$ random variable. We can then bound the conditional probability from above by one almost surely. Combining this almost sure upper bound with (5.50) in (5.48), we arrive at

\[
n^{k-|S|} \prod_{\ell \in [k] \setminus S} E \left[ \left( \frac{a_\ell}{1 + a_\ell} \right)^{m_\ell} \right] \left( 1 + O(n^{-\delta(1-\varepsilon)} \log n) \right).
\]

Finally, with the same steps as in (5.28), we obtain

\[
n^{k-|S|} \prod_{\ell \in [k] \setminus S} E \left[ \left( \frac{W}{E[W] + W} \right)^{m_\ell} \right] \left( 1 + o(n^{-\delta(1-\varepsilon)} \log n) \right),
\]

for any $\xi > 0$. We then use this bound in (5.47) to find, for some positive constant $K$, the upper bound

\[
K \sum_{i=1}^{K-1} \sum_{S \subseteq [k] \setminus \{i\}} \prod_{\ell \in [k] \setminus S} E \left[ \left( \frac{W}{E[W] + W} \right)^{m_\ell} \right] = K \sum_{i=1}^{K-1} \sum_{S \subseteq [k] \setminus \{i\}} \eta(C(S)(1 + o(1))) \prod_{\ell \in [k] \setminus S} p_{\geq m_\ell}.
\]

By Remark 5.4, the tail probability $p_{\geq m_\ell} = O(p_{m_\ell})$ and by (5.45) we have $n^{C(S)(1 + o(1))} = o\left(n^{-\eta(S)} \prod_{\ell \in S} p_{m_\ell}\right)$ for some $\eta(S) > 0$. Combined, this yields

\[
\frac{1}{(n)^k} \sum_{i=1}^{k-1} \sum_{j \in I_{\gamma}(\ell, i)} E_n \prod_{\ell = 1}^{k} \prod_{\ell \in [k] \setminus S} P_W(Z_n(j_\ell) \geq m_\ell) \leq K \sum_{i=1}^{k-1} \sum_{S \subseteq [k] \setminus \{i\}} \eta(C(S)(1 + o(1))) \prod_{\ell \in [k] \setminus S} p_{\geq m_\ell}
\]

\[
= o(n^{-\tilde{\eta}} \prod_{\ell = 1}^{k} p_{m_\ell}),
\]

with

\[
\tilde{\eta} := \min_{S \subseteq [k] \setminus S \subseteq k-1} \left( C(S) - \sum_{\ell \in S} \log(\theta + \xi) \right),
\]

which is strictly positive when $\xi$ is sufficiently small and $\varepsilon$ is set sufficiently close to one, similar to what is discussed above. Combining this with the fact that $P(I_{\gamma}(E_n))$ and $P(E_n)$ are $o(n^{-\eta} \prod_{\ell = 1}^{k} p_{m_\ell})$ uniformly in $m_1, \ldots, m_k < e \log n$ for some $\eta > 0$ (we will prove this for the former probability at
the end, and for the latter probability this follows from (5.15)), and the result in (5.46), we finally conclude that
\[
\frac{1}{(n)k} \sum_{j \in I_n(c)} \mathbb{E} [\mathbb{P}_W (Z_n(j \ell) = m \ell, \ell \in [k])] = \frac{1}{(n)k} \sum_{j \in I_n(\varepsilon)} \mathbb{E} [\mathbb{P}_W (Z_n(j \ell) = m \ell, \ell \in [k])]
+ \frac{1}{(n)k} \sum_{i=1}^{k-1} \sum_{j \in I_n(\varepsilon)} \mathbb{E} [\mathbb{P}_W (Z_n(j \ell) = m \ell, \ell \in [k])]
= o\left(n^{-\eta} \prod_{\ell=1}^{k} p_{m_{\ell}}\right),
\]
for some \(\eta > 0\) in the case that \(m_{\ell} = c_{\ell} \log n(1 + o(1))\) with \(c_{\ell} \in [1/\log \theta, \theta/(\theta - 1)]\) for all \(\ell \in [k]\) as well.

When the \(m_{\ell}\) do not all behave the same, the same is true for some \(\ell \in [k]\) \(c_{\ell} \in [0, 1/\log \theta]\) and for some \(c_{\ell} \in [1/\log \theta, \theta/(\theta - 1)]\), we can use a combination of the approaches outlined for either of the cases.

It remains to prove that \(\mathbb{P}((E'_n)^c)\) decays sufficiently fast. By a union bound and using the same approach as in (5.14) and (5.15), we find that for some positive constant \(C_{\theta, \delta}\) and with \(c_\theta := 1/(2\theta^2)\),
\[
\mathbb{P}((E'_n)^c) \leq \sum_{j = [\lceil(C\log n)^{1/(1-2\delta)}\rceil]}^{\infty} \exp \left\{ -c_{\theta}(j - 1)^{1-2\delta}\right\} \leq C_{\theta, \delta} \Gamma\left(\frac{1}{1-2\delta}, c_{\theta}([C\log n]^{1/(1-2\delta)})^{1-2\delta}\right).
\]
Using that \(\Gamma(s, x) = x^{s-1}e^{-x}(1 + o(1))\) for a fixed \(s \in \mathbb{R}\) and as \(x\) tends to infinity, we obtain
\[
\mathbb{P}((E'_n)^c) \leq C_{\theta, \delta} \left(c_{\theta}C\log n\right)^{2\delta/(1-2\delta)} \exp \left\{ -c_{\theta}\left([C\log n]^{1/(1-2\delta)}\right)^{1-2\delta}\right\}(1 + o(1))
= \tilde{C}_{\theta, \delta}(C\log n)^{2\delta/(1-2\delta)} \exp\{-c_{\theta}C\log n\}(1 + o(1))
= n^{-c_{\theta}C(1 + o(1))}.
\]

As mentioned when introducing the event \(E'_n\) in (5.43), the choice of \(C\) yields \(\mathbb{P}((E'_n)^c) \leq n^{-(k\delta\log(\theta)/(\theta - 1) + \delta)}\) for \(n\) large and \(\eta\) sufficiently small, so that \(\mathbb{P}((E'_n)^c) = o(\prod_{\ell=1}^{k} p_{m_{\ell}}n^{-\eta})\) for any choice of \(m_{\ell} < (\theta/(\theta - 1))\log n(1 + o(1)), \ell \in [k]\), which concludes the proof. \(\square\)

We now discuss the required adaptations so that the proof holds for the model with random out-degree as well. This follows from the fact that Lemma 5.10 holds for this model, together with the fact that the degrees are still negatively quadrant dependent when the out-degree is random. As a result, all probabilities related to the degrees of multiple vertices can either be dealt with using Lemma 5.10 or can be split as a product of probabilities of individual vertices. From the perspective of the in-degree of an individual vertex \(i \in [n]\), the model with out-degree one and the model with random out-degree are equivalent, as in every step the in-degree of vertex \(i\) increases by one with the same probability. Hence, the proof follows through in exactly the same way and Lemma 5.11 holds for the model with random out-degree as well.

5.4. Proof of Proposition 5.1. We finally prove Proposition 5.1, using Lemmas 5.10 and 5.11. We remark that the proof does not use that the out-degree is deterministic but uses (5.1) only, so that the proof and hence (5.2) hold for both the model with fixed and random out-degree.

Proof of Proposition 5.1. As discussed before, (5.1) directly follows from (5.12) combined with Lemmas 5.10 and 5.11. Using (5.1), we then prove (5.2). For ease of writing, we recall that
\[
p_k := \mathbb{E} \left[ \frac{\theta - 1}{\theta - 1 + W} \left(\frac{W}{\theta - 1 + W}\right)^k \right], \quad p_{\geq k} := \mathbb{E} \left[ \left(\frac{W}{\theta - 1 + W}\right)^k \right].
\]
We start by assuming that \(m_{\ell} = c_{\ell} \log n(1 + o(1))\) with \(c_{\ell} \in (0, c)\) for each \(\ell \in [k]\). We discuss how to adjust the proof when \(m_{\ell} = o(\log n)\) for some or all \(\ell \in [k]\) at the end.
For each $\ell \in [k]$ take an $\eta_\ell \in (0, c - c_\ell)$ so that $\lceil (1 + \eta_\ell) m_\ell \rceil < c \log n$. Then, we use the upper bound
\[
P(\mathcal{Z}_n(v_\ell) \geq m_\ell, \ell \in [k]) \leq \sum_{j_1 = m_1}^{\lceil (1 + \eta_1) m_1 \rceil} \cdots \sum_{j_k = m_k}^{\lceil (1 + \eta_k) m_k \rceil} P(\mathcal{Z}_n(v_\ell) = j_\ell, \ell \in [k])
\]
\[+ \sum_{i=1}^{k} P(\mathcal{Z}_n(v_i) \geq \lceil (1 + \eta_i) m_i \rceil, \mathcal{Z}_n(v_\ell) \geq m_\ell, \ell \neq i). \tag{5.53} \]

We first discuss the first term on the right-hand side. As (5.1) holds uniformly in $m_1, \ldots, m_k < c \log n$, we find
\[
\sum_{j_1 = m_1}^{\lceil (1 + \eta_1) m_1 \rceil} \cdots \sum_{j_k = m_k}^{\lceil (1 + \eta_k) m_k \rceil} P(\mathcal{Z}_n(v_\ell) = j_\ell, \ell \in [k]) = \sum_{j_1 = m_1}^{\lceil (1 + \eta_1) m_1 \rceil} \cdots \sum_{j_k = m_k}^{\lceil (1 + \eta_k) m_k \rceil} \prod_{\ell = 1}^{k} P_j(1 + o(n^{-\beta}))
\]
\[= \prod_{\ell = 1}^{k} (p_{\geq m_\ell} - p_{\geq \lceil (1 + \eta_\ell) m_\ell \rceil})(1 + o(n^{-\beta})) \tag{5.54} \]
\[\leq \prod_{\ell = 1}^{k} p_{m_\ell}(1 + o(n^{-\beta})). \]

To finish the upper bound, it remains to show the the term on the second line of (5.53) can be incorporated in the $o(n^{-\beta})$ term, and it suffices to show this can be done for each term in the sum, independent of the value of $i$. Using the negative quadrant dependence of the degrees under the conditional probability measure $P_W$ (see Remark 5.2 and [14, Lemma 7.1]), we find
\[
P(\mathcal{Z}_n(v_i) \geq \lceil (1 + \eta_i) m_i \rceil, \mathcal{Z}_n(v_\ell) \geq m_\ell, \ell \in [k] \setminus \{i\})
\]
\[= \frac{1}{(n_k)} \sum_{1 \leq j_1 \neq \ldots \neq j_k \leq n} \mathbb{E}[P_W(\mathcal{Z}_n(j_i) \geq \lceil (1 + \eta_i) m_i \rceil) \prod_{\ell \in [k] \setminus \{i\}} P_W(\mathcal{Z}_n(j_\ell) \geq m_\ell)]
\]
\[= \frac{1}{(n_k)} \sum_{j_i \in I_i(v_i)} \mathbb{E}[P_W(\mathcal{Z}_n(j_i) \geq \lceil (1 + \eta_i) m_i \rceil) \prod_{\ell \in [k] \setminus \{i\}} P_W(\mathcal{Z}_n(j_\ell) \geq m_\ell)]
\[+ \frac{1}{(n_k)} \sum_{n^{1 - \varepsilon} \leq j_i \neq \ldots \neq j_k \leq n} \mathbb{E}[P_W(\mathcal{Z}_n(j_i) \geq \lceil (1 + \eta_i) m_i \rceil) \prod_{\ell \in [k] \setminus \{i\}} P_W(\mathcal{Z}_n(j_\ell) \geq m_\ell)]. \]

The first term in the last step can be included in the little $o$ term in (5.54) (even when considering $m_i$ rather than $\lceil (1 + \eta_i) m_i \rceil$ in the probability), as follows from computations similar to the ones in (5.41) through (5.52), combined with Remark 5.4 (which states that $p_{\geq m} = O(p_{m_\ell})$). It remains to show that the same holds for the second term in the last step. Again, we use an argument similar to the steps performed in (5.47) through (5.51) to arrive at
\[
\frac{1}{(n_k)} \sum_{n^{1 - \varepsilon} \leq j_i \neq \ldots \neq j_k \leq n} \mathbb{E}[P_W(\mathcal{Z}_n(j_i) \geq \lceil (1 + \eta_i) m_i \rceil) \prod_{\ell \in [k] \setminus \{i\}} P_W(\mathcal{Z}_n(j_\ell) \geq m_\ell)]
\]
\[\leq K p_{\geq \lceil (1 + \eta_i) m_i \rceil} \prod_{\ell \in [k] \setminus \{i\}} p_{\geq m_\ell}, \]
for some positive constant $K$. By Lemma 5.5 we have the inequalities
\[
p_{\geq \lceil (1 + \eta_i) m_i \rceil} \leq \theta^{-\lceil (1 + \eta_i) m_i \rceil} \leq \theta^{-m_i} \theta^{-m_i}, \quad p_{\geq m_i} \geq (\theta + \xi)^{-m_i},
\]
for any $\xi > 0$. As a result, taking $\xi \in (0, \theta(\theta^{m_i} - 1))$ and setting $\phi_i := 1 - (1 + \xi/\theta)\theta^{-m_i} > 0$, we obtain
\[
p_{\geq \lceil (1 + \eta_i) m_i \rceil} \leq (\theta + \xi)^{-m_i} ((1 + \xi/\theta)\theta^{-m_i})^{m_i} \leq p_{\geq m_i}(1 - \phi_i)^{m_i}. \tag{5.55} \]
As $m_i = c_i \log n(1 + o(1))$, it follows that $(1 - \phi_i)^{m_i} = n^{-c_i \log(1/(1-\phi_i))(1+o(1))}$, so that
\[
\frac{1}{(n_k)} \sum_{n^{1 - \varepsilon} \leq j_i \neq \ldots \neq j_k \leq n} \mathbb{E}[P_W(\mathcal{Z}_n(j_i) \geq \lceil (1 + \eta_i) m_i \rceil) \prod_{\ell \in [k] \setminus \{i\}} P_W(\mathcal{Z}_n(j_\ell) \geq m_\ell)]
\]
\[\leq K p_{\geq \lceil (1 + \eta_i) m_i \rceil} \prod_{\ell \in [k] \setminus \{i\}} p_{\geq m_\ell}.
\]
can be incorporated in the little o term in (5.54) for each $i \in [k]$ when we take a $\beta' < \beta \wedge \min_{i \in [k]} c_i \log(1/(1 - \phi_i))$. This yields
\[ P(Z_n(v_i) \geq m_\ell, \ell \in [k]) \leq \prod_{\ell = 1}^{k} \mathbb{E} \left[ \left( \frac{W}{\theta - 1 + W} \right)^{m_\ell} \right] (1 + o(n^{-\beta})). \] (5.56)

For a lower bound, we can omit the second line of (5.53) and use (5.54) to immediately obtain
\[ P(Z_n(v_i) \geq m_\ell, \ell \in [k]) \geq \sum_{j_1 = m_1}^{[(1 + n_{m_1})]} \cdots \sum_{j_k = m_k}^{[(1 + n_{m_k})]} P(Z_n(v_i) = i, \ell \in [k]) \]
\[ = \prod_{\ell = 1}^{k} (p_{\geq m_\ell} - p_{\geq [(1 + n_{m_\ell})]})(1 + o(n^{-\beta})). \]

Again using (5.55) yields $p_{\geq m_\ell} - p_{\geq [(1 + n_{m_\ell})]} = p_{\geq m_\ell}(1 + o(n^{-\beta}))$ when we set $\beta' < \beta \wedge \min_{i \in [k]} c_i \log(1/(1 - \phi_i))$. Combined with (5.56) this yields (5.2) and concludes the proof. \hfill $\Box$

6. PROOFS OF THE MAIN THEOREMS

With the tools developed in Section 5, in particular Propositions 5.1 and 5.6 and Lemma 5.8, we now prove the main results formulated in Section 2.

First, we prove the main result for high degree vertices when the vertex-weight distribution has an atom at one, as in the (Atom) case.

**Proof of Theorem 2.5.** The proof follows the same argument as [1, Theorem 1.2]. For an integer subsequence $(n_\ell)_{\ell \in \mathbb{N}}$ such that $n_\ell \to \varepsilon$ as $\ell \to \infty$, it suffices to prove that for any $i < i' \in \mathbb{Z}$,
\[ (X^{(n_\ell)}_i, X^{(n_\ell)}_{i+1}, \ldots, X^{(n_\ell)}_{i' - 1}, X^{(n_\ell)}_{i' + d}) \to (P^\varepsilon(i), P^\varepsilon(i + 1), \ldots, P^\varepsilon(i' - 1), P^\varepsilon([i', \infty)]) \] as $\ell \to \infty$
holds. We obtain this via the convergence of the factorial moments of $X^{(n_\ell)}_i, \ldots, X^{(n_\ell)}_{i' - 1}, X^{(n_\ell)}_{i' + d}$. Recall $r_k$ in (2.9). By Proposition 5.6, for any non-negative integers $a_i, \ldots, a_{i'}$,
\[ \mathbb{E} \left[ \left( X^{(n_\ell)}_{i'} \right)_{a_{i'}} \prod_{k = 1}^{i' - 1} \left( X^{(n_\ell)}_k \right)_{a_k} \right] \]
\[ = \left( q_0 \theta^{-i' + \varepsilon_{n_\ell}} \right)_{a_{i'}} \prod_{k = 1}^{i' - 1} \left( q_0 (1 - \theta^{-1}) \theta^{-k + \varepsilon_{n_\ell}} \right)^{a_k} \]
\[ \times (1 + O(r_{[\log_\theta n_{\ell}] + i} \vee n_{\ell}^{-\beta})) \]
\[ \to \left( q_0 \theta^{-i' + \varepsilon} \right)_{a_{i'}} \prod_{k = 1}^{i' - 1} \left( q_0 (1 - \theta^{-1}) \theta^{-k + \varepsilon} \right)^{a_k}, \]
as $\ell \to \infty$. By using the properties of Poisson processes, it follows that the limit equals
\[ \mathbb{E} \left[ \left( P^\varepsilon([i', \infty]) \right)_{a_{i'}} \prod_{k = 1}^{i' - 1} \left( P^\varepsilon(k) \right)_{a_k} \right], \]
due to the particular form of the intensity measure of the Poisson process $P$ (which is used in the definition of the Poisson process $P^\varepsilon$). The result then follows from [12, Theorem 6.10]. \hfill $\Box$

For the results for the (Weibull) and (Gumbel) cases, as outlined in Theorems 2.6 and 2.7, respectively, we combine the asymptotic behaviour of $p_{\geq k}$ in Theorem 5.3 with Proposition 5.1 and Lemma 5.8.

**Proof of Theorem 2.6.** To establish the convergence in probability, it follows from Lemma 5.8 that we need only consider $n \mathbb{P}(Z_n(v_k) \geq k_n)$ for some adequate integer-valued $k_n$ such that $k_n < c \log n$ for some $c \in (0, \theta/(\theta - 1))$ and where $v_k$ is a vertex selected from $[n]$ uniformly at random. By Proposition 5.1, this quantity equals $n p_{\geq k_n}(1 + o(1))$. Then, we use Theorem 5.3 and Remark 5.4 to obtain that, when $W$ satisfies the (Weibull) case in Assumption 2.3, this quantity is at most
\[ n\mathbb{E} T(k_n) k_n^{-(\alpha - 1) \theta^{-k_n}}, \]
where \( \overline{C} > 1 \) is a constant. Now fix an arbitrary \( \eta > 0 \) and let \( k_n := \lfloor \log_2 n - (\alpha - 1)(1 - \eta) \log_2 \log_2 n \rfloor \). This yields
\[
\begin{align*}
&\frac{n\overline{C}(\log_2 n(1 + c_1))(\log_2 n)^{-\gamma}}{\log_2 \log_2 n} (1 - 1) \gamma - \log_2 n - (\alpha - 1)(1 - \eta) \log_2 \log_2 n (1 + o(1)) \\
&\leq \frac{\overline{C}(\log_2 n)(\log_2 n)^{-\gamma}}{\log_2 \log_2 n} (1 - 1) \gamma - \log_2 n (1 - 1) \gamma (1 - 1) \eta ) \\
&= \frac{\overline{C}(\log_2 n)(\log_2 n)^{-\gamma}}{\log_2 \log_2 n} (1 - 1) \eta .
\end{align*}
\]

Here, \( \overline{C} > 0 \) is a suitable constant and we use that \( k_n = \log_2 n(1 + o(1)) \) in the first step. Furthermore, we use [2, Theorem 1.5.2], which states that for a slowly-varying function \( T(x) \), \( T(\log_2 n(1 + o(1))) \leq c \overline{C}(\log_2 n) \) for some constant \( c > 1 \) and \( n \) sufficiently large. Finally, we use [2, Proposition 1.3.6 (v)] to obtain that for any \( \eta > 0 \), the final line of (6.1) tends to zero with \( n \). This shows that for any \( \eta > 0 \), with high probability,
\[
\max_{j \in [n]} \frac{Z_n(j) - \log_2 n}{\log_2 \log_2 n} \leq -(\alpha - 1)(1 - \eta)
\]
holds, due to the first result in Lemma 5.8. A similar approach, when setting \( k_n := \lfloor \log_2 n - (\alpha - 1)(1 + \eta) \log_2 \log_2 n \rfloor \), yields
\[
nP(Z_n(v_1) \geq k_n) \to \infty,
\]
so that for any \( \eta > 0 \), with high probability,
\[
\max_{j \in [n]} \frac{Z_n(j) - \log_2 n}{\log_2 \log_2 n} \geq -(\alpha - 1)(1 + \eta)
\]
holds. Together, these two bounds prove the desired result. \( \square \)

**Proof of Theorem 2.7.** The proof of this theorem follows a similar approach to the proof of Theorem 2.6. That is, we again apply the results from Theorem 5.3 together with the fact that
\[
nP(Z_n(v_1) \geq k_n) = np_{k_n} (1 + o(1)),
\]
for some adequate integer-valued \( k_n \) such that \( k_n < c \log_2 n \) for some \( c \in (0, \theta/(\theta - 1)) \), as follows from Proposition 5.1 and Lemma 5.8. In the (Gumbel)-(RV) sub-case, we know that
\[
p_{k_n} = \exp \left\{ -\frac{\tau \gamma}{1 - \gamma} \left( \frac{1 - \theta - 1}{c_1} \right)^{k_n} (1 + o(1)) \right\} \theta^{-k_n},
\]
where we recall that \( \gamma = 1/(\tau + 1) \). To prove the desired results, we first set \( k_n = \lfloor \log_2 n - (1 - \eta) \log_2 \log_2 n \rfloor \) for any \( \eta > 0 \), where we recall \( C, \theta, \tau \) from (2.5). Using this in (6.2) then yields
\[
np_{k_n} = n \theta^{-k_n} e^{-\log_2 \log_2 n (1 + o(1)) (1 - 1) \gamma (1 + o(1))} \geq e^{-\log_2 \log_2 n (1 + o(1)) (1 - 1) \gamma (1 + o(1))},
\]
where we use that \( k_n = \log_2 n (1 + o(1)) \) in the last step. Hence, \( np(Z_n(v_1) \geq k_n) \) diverges. We thus conclude from Lemma 5.8 that
\[
\max_{j \in [n]} \frac{Z_n(j) - \log_2 n}{\log_2 n^{1 - \gamma}} \geq -(1 + \eta) C, \theta, \tau, \tau
\]
holds with high probability. A similar approach, setting
\[
k_n := \lfloor \log_2 n - (1 - \eta) C, \theta, \tau, \tau \log_2 n \rfloor \) and combining this with the first result of Lemma 5.8 yields
\[
\max_{j \in [n]} \frac{Z_n(j) - \log_2 n}{\log_2 n^{1 - \gamma}} \leq -(1 - \eta) C, \theta, \tau, \tau
\]
holds with high probability. Together, these two bounds prove (2.5).

To prove (2.6) we apply the same methodology but use the asymptotic expression of \( p_k \) (and \( p_{\geq k} \) by adjusting constants), as in (5.5). We then consider the constants \( C_1, C_2, C_3 \) from (2.7) and set \( k_n := \lfloor \log_2 n - 1 \log_2 \log_2 n \rfloor \) and combining this with the first result of Lemma 5.8 yields
\[
\max_{j \in [n]} \frac{Z_n(j) - \log_2 n}{\log_2 n^{1 - \gamma}} \leq -(1 - \eta) C, \theta, \tau, \tau
\]
holds with high probability.
Using Taylor expansions, we obtain
\[
\frac{\tau(\tau - 1)}{c_1} (\log k_n)^{\tau - 1} \log k_n = -\left(\frac{\log k_n}{c_1}\right)^{\tau} + o(1) = -\log(\theta) C_1 (\log_5 \log_9 n)^{\tau} + o(1),
\]
\[
\frac{\tau(\tau - 1)}{c_1} (\log k_n)^{\tau - 1} \log k_n = \log(\theta) C_2 (\log_5 \log_9 n)^{\tau - 1} \log_5 \log_9 n + o(1),
\]
\[
-K_{\tau,c_1,\theta} (\log k_n)^{\tau - 1} = -(\log(\theta))^{\tau - 1} K_{\tau,c_1,\theta} (\log_5 \log_9 n)^{\tau - 1} + o(1),
\]
where we recall \(K_{\tau,c_1,\theta}\) from (5.5) in Theorem 5.3, so that
\[
n \exp \left\{ -\left(\frac{\log k_n}{c_1}\right)^{\tau} + \frac{\tau(\tau - 1)}{c_1} (\log k_n)^{\tau - 1} \log k_n - K_{\tau,c_1,\theta} (\log_5 \log_9 n)^{\tau - 1} (1 + o(1)) \right\} \theta^{-k_n}
\]
\[
= n \exp \left\{ -\log(\theta) C_1 (\log_5 \log_9 n)^{\tau} + \log(\theta) C_2 (\log_5 \log_9 n)^{\tau - 1} \log_5 \log_9 n + \log(\theta) C_3 (\log_5 \log_9 n)^{\tau - 1} (1 + o(1)) \right\} \theta^{-k_n}
\]
\[
\leq n \exp \left\{ -(\eta - o(1))(\log_5 \log_9 n)^{\tau - 1} \right\},
\]
where in the last step we use that \(k_n \geq \log n - C_1 (\log_5 \log_9 n)^{\tau} + C_2 (\log_5 \log_9 n)^{\tau - 1} \log_5 \log_9 n + (C_3 + \eta)(\log_5 \log_9 n)^{\tau - 1}\). As the right-hand side tends to zero with \(n\), Lemma 5.8 yields for any fixed \(\eta > 0\), with high probability,
\[
\max_{i \in [n]} Z_n(i) - \left(\log_5 \log_9 n - C_1 (\log_5 \log_9 n)^{\tau} + C_2 (\log_5 \log_9 n)^{\tau - 1} \log_5 \log_9 n \right) \leq C_3 + \eta.
\]
With a similar approach, setting
\[
k_n = \left[ \log_5 \log_9 n - C_1 (\log_5 \log_9 n)^{\tau} + C_2 (\log_5 \log_9 n)^{\tau - 1} \log_5 \log_9 n + (C_3 - \eta)(\log_5 \log_9 n)^{\tau - 1} \right],
\]
we can obtain that for any fixed \(\eta > 0\), with high probability,
\[
\max_{i \in [n]} Z_n(i) - \left(\log_5 \log_9 n - C_1 (\log_5 \log_9 n)^{\tau} + C_2 (\log_5 \log_9 n)^{\tau - 1} \log_5 \log_9 n \right) \geq C_3 - \eta.
\]
Together these two bounds yield (2.6), which concludes the proof. \(\square\)

**Proof of Theorem 2.8.** We first prove the asymptotic distribution of the maximum degree, whose proof follows the same approach as the proof of [1, Theorem 1.3]. We need to consider two cases: \(i = O(1)\) and \(i \rightarrow \infty\) such that \(i + \log_3 n < (\theta/(\theta - 1)) \log n \) and \(\lim inf_{n \rightarrow \infty} i > -\infty\). For the former case, as \(\exp\{-q_0 \theta^{-i + \varepsilon_n}\} = O(1)\), it suffices to prove
\[
\mathbb{P}\left(\max_{j \in [n]} Z_n(j) \geq \left[ \log_5 \log_9 n \right] + i \right) - (1 - \exp\{-q_0 \theta^{-i + \varepsilon_n}\}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
By the definition of \(X^{(n)}_{\geq i}\) in (2.3), this is equivalent to
\[
\mathbb{P}\left(\max_{j \in [n]} X^{(n)}_{\geq i} = 0\right) - \exp\{-q_0 \theta^{-i + \varepsilon_n}\} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (6.3)
\]
This follows from Theorem 2.5 and the subsequence principle. That is, if we assume the convergence in (6.3) does not hold, then there exists a subsequence \((n_k)_{k \in \mathbb{N}}\) and a \(\delta > 0\) such that
\[
\mathbb{P}\left(\max_{j \in [n_k]} X^{(n_k)}_{\geq i} = 0\right) - \exp\{-q_0 \theta^{-i + \varepsilon_n}\} > \delta \quad \forall \ell \in \mathbb{N}. \quad (6.4)
\]
However, as \(\varepsilon_n\) is bounded, there exists a subsequence \(\varepsilon_{n_k}\) such that \(\varepsilon_{n_k} \rightarrow \varepsilon\) for some \(\varepsilon \in [0, 1]\). Then, by Theorem 2.5, the statement in (6.4) is false, from which the result follows.

In the latter case, we need only consider \(\mathbb{E}\left[X^{(n)}_{\geq i}\right]\) and \(\mathbb{E}\left[(X^{(n)}_{\geq i})^2\right]\), as
\[
\mathbb{E}\left[X^{(n)}_{\geq i}\right] - \frac{1}{2} \mathbb{E}\left[(X^{(n)}_{\geq i})^2\right] \leq \mathbb{P}\left(X^{(n)}_{\geq i} > 0\right) \leq \mathbb{E}\left[X^{(n)}_{\geq i}\right], \quad (6.5)
\]
again see [1, Theorem 1.3] and its proof for more details. By Proposition 5.6, we have that
\[ E[X(n)_{\geq i}] = q_0 \theta^{-i+\varepsilon_n}(1 + o(1)), \quad E[(X(n)_{\geq i})^2] = (q_0 \theta^{-i+\varepsilon_n})^2(1 + o(1)). \]
Hence, as \( i \to \infty \) and \( \varepsilon_n \) is bounded,
\[ E[X(n)_{\geq i}] = (1 - \exp\{q_0 \theta^{-i+\varepsilon_n}\})(1 + o(1)), \]
\[ E[X(n)_{\geq i}] - \frac{1}{2} E[(X(n)_{\geq i})^2] = (1 - \exp\{q_0 \theta^{-i+\varepsilon_n}\})(1 + o(1)). \]
Combining this with (6.5) yields the desired result.

Recall \( \varepsilon_n := \log \eta n - [\log \eta n] \). We now prove the limiting distribution of \(|M_{n_k}|\) has the desired distribution, as in (2.8), when the subsequence \((n_k)_{k \in \mathbb{N}}\) is such that \( \varepsilon_{n_k} \to \varepsilon \in [0,1] \), which follows the same approach as [6, Theorem 1.1]. Consider the event \( \mathcal{E}_{j,k} := \{X(n)_j = k, X(n)_{j+1} = 0\} \) for \( j \in \mathbb{Z}, k \in \mathbb{N} \). \( \mathcal{E}_{j,k} \) implies that there are exactly \( k \) vertices which attain the maximum degree \([\log \eta^n] + j\) in the tree \( T_{n_k} \). We observe that the events \( \mathcal{E}_{j,k} \) are pairwise disjoint for different values of \( j \). As a result, we obtain the following inequalities: For any \( M \in \mathbb{N} \),
\[ \mathbb{P}(|M_{n_k}| = k) = \mathbb{P}\left( \bigcup_{j \in \mathbb{Z}} \mathcal{E}_{j,k} \right) \leq \mathbb{P}\left( \bigcup_{j \leq -(M+1)} \mathcal{E}_{j,k} \right) + \sum_{j=-M}^{M-1} \mathbb{P}(\mathcal{E}_{j,k}) + \mathbb{P}\left( \bigcup_{j \geq M} \mathcal{E}_{j,k} \right) \]
\[ \leq \mathbb{P}(X(n)_{\geq -M} = 0) + \sum_{j=-M}^{M-1} \mathbb{P}(\mathcal{E}_{j,k}) + \mathbb{P}(X(n)_{\geq M} > 0), \]
and
\[ \mathbb{P}(|M_{n_k}| = k) = \mathbb{P}\left( \bigcup_{j \in \mathbb{Z}} \mathcal{E}_{j,k} \right) \geq \sum_{j=-M}^{M-1} \mathbb{P}(\mathcal{E}_{j,k}). \]
By Theorem 2.5 it thus follows that
\[ \limsup_{n_k \to \infty} \mathbb{P}(|M_{n_k}| = k) \leq \liminf_{M \to \infty} \left( \mathbb{P}(X(n)_{\geq -M} = 0) + \sum_{j=-M}^{M-1} \mathbb{P}(X(n)_j = k, X(n)_{j+1} = 0) + \mathbb{P}(X(n)_{\geq M} > 0) \right), \]
\[ \liminf_{n_k \to \infty} \mathbb{P}(|M_{n_k}| = k) \geq \limsup_{M \to \infty} \sum_{j=-M}^{M} \mathbb{P}(X(n)_j = k, X(n)_{j+1} = 0), \]
where \( X(n)_j \sim \text{Poi}(q_0(1 - \theta^{-1})\theta^{-j+\varepsilon}) \) and \( X(n)_{\geq j} \sim \text{Poi}(q_0(1 - \theta^{-1})\theta^{-j+\varepsilon}) \) are independent Poisson random variables, for \( j \in \mathbb{Z} \). As a result,
\[ \mathbb{P}(X(n)_{\geq -M} = 0) = e^{-q_0 \theta^{M+\varepsilon}}, \quad \mathbb{P}(X(n)_{\geq M} > 0) = 1 - e^{-q_0 \theta^{-M+\varepsilon}}, \]
\[ \mathbb{P}(X(n)_j = k, X(n)_{j+1} = 0) = \mathbb{P}(X(n)_j = k) \mathbb{P}(X(n)_{j+1} = 0) = \frac{1}{k!}(q_0(1 - \theta^{-1})\theta^{-j+\varepsilon})^k e^{-q_0 \theta^{-j+\varepsilon}}. \]
Hence, we obtain
\[ \limsup_{n_k \to \infty} \mathbb{P}(|M_{n_k}| = k) \leq \liminf_{M \to \infty} \left( e^{-q_0 \theta^{M+\varepsilon}} + \sum_{j=-M}^{M-1} \frac{1}{k!}(q_0(1 - \theta^{-1})\theta^{-j+\varepsilon})^k e^{-q_0 \theta^{-j+\varepsilon}} + 1 - e^{-q_0 \theta^{-M+\varepsilon}} \right) \]
\[ \leq \sum_{j \in \mathbb{Z}} \frac{1}{k!}(q_0(1 - \theta^{-1})\theta^{-j+\varepsilon})^k e^{-q_0 \theta^{-j+\varepsilon}}, \]
\[ \liminf_{n_k \to \infty} \mathbb{P}(|M_{n_k}| = k) \geq \limsup_{M \to \infty} \sum_{j=-M}^{M} \frac{1}{k!}(q_0(1 - \theta^{-1})\theta^{-j+\varepsilon})^k e^{-q_0 \theta^{-j+\varepsilon}} \]
\[ = \sum_{j \in \mathbb{Z}} \frac{1}{k!}(q_0(1 - \theta^{-1})\theta^{-j+\varepsilon})^k e^{-q_0 \theta^{-j+\varepsilon}}. \]
It is then readily checked that the limit is indeed finite and that summing over all \( k \in \mathbb{N} \) yields one, which concludes the proof. \qed
Proof of Theorem 2.9. The proof follows the same argument as the proof of [1, Theorem 1.4], which is based on [3, Theorem 1.24]. Let 1 \leq a \leq b be integers. Then, by Proposition 5.6 and since i = o(\log n),
\[ E \left[ \left( X_i^{(a)} \right)^a \right] = o(\theta^{-i-1} \theta^{-i+1} \varepsilon). \]
It remains to show that the right-hand side is in fact \( o(\theta^b) \). We note that i = o(\log r_{\log n} \land \log n), so that we can write the right-hand side as
\[ O((r_{\log n})^{1-o(\log n)} \theta^{-o(1)} \log n) = O((r_{\log n})^{1-o(1)} \log n) = o(\theta^b), \]
by the constraints on i, from which the result follows. \( \square \)

7. Technical details of examples

In this section we discuss some technical details of the examples discussed in Section 4. In particular, for each example we provide a precise asymptotic expression of \( p_k \) and \( p_{\geq k} \) as well as a key element that leads to the results in Section 4. That is, for each of the examples we state and prove the analogue of Proposition 5.6. The three theorems presented in each of the examples in Section 4 mimic three of the theorems presented in Section 2. That is, Theorems 4.2 and 4.6 are the analogue of Theorems 2.5. Theorems 4.3 and 4.7 are the analogue of Theorem 2.8 and Theorems 4.4 and 4.8 are the analogue of Theorem 2.9. As a result, their proofs are very similar to the proofs of Theorems 2.5, 2.8 and 2.9, so we omit proving the theorems stated in Section 4.

7.1. Example 4.1, beta distribution bounded away from zero. Let the distribution of \( W \) satisfy (4.1) for some \( \alpha, \beta > 0, w^* \in (0, 1) \). We prove a result on (the tail of) the limiting degree distribution and provide additional building blocks required to prove the results in Example 4.1.

Lemma 7.1. Let the distribution of \( W \) satisfy (4.1) for some \( \alpha, \beta > 0, w^* \in (0, 1) \) and recall \( p_k, p_{\geq k} \) from (2.1). Then,
\[
\begin{align*}
p_k &= Z_{w^*} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} (1 - \theta^{-1})^{1-\beta} k^{-\beta} \theta^{-k} (1 + O(1/k)), \\
p_{\geq k} &= Z_{w^*} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} (1 - \theta^{-1})^{-\beta} k^{-\beta} \theta^{-k} (1 + O(1/k)).
\end{align*}
\]

Note that this Lemma improves on the bounds in (5.3) by providing a precise multiplicative constant, rather than two slowly-varying functions that are (possibly) of different order.

Proof. By the distribution of \( W \), we immediately obtain that
\[
p_k = \int_{w^*}^1 (\theta - 1)x^k (\theta - 1 + x)^{-(k+1)} Z_{w^*} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx,
\]
where \( \Gamma \) is the gamma function. Let us denote the integrand by \( f(x, k) \). We first consider the asymptotic behaviour of the integral when its lower bound on the integration variable \( x \) is zero, and show that it is equal to the right-hand side of the first equation in (7.1). Then, we show that
\[
\int_0^{w^*} f(x, k) \, dx = o \left( \int_{w^*}^1 f(x, k) \, dx \right),
\]
which combined yields the desired result. So, we start with
\[
\begin{align*}
\int_0^1 (\theta - 1)x^k (\theta - 1 + x)^{-(k+1)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx \\
= (\theta - 1)^{-k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{k+\alpha-1} (1-x)^{\beta-1} (1+x/(\theta - 1))^{-(k+1)} dx,
\end{align*}
\]
where we omit \( Z_{w^*} \) for ease of writing. We now use Euler’s integral representation of the hypergeometric function. That is, for \( a, b, c, z \in \mathbb{C} \) such that \( \text{Re}(c) > \text{Re}(b) > 0 \) and \( z \) is not a real number greater than one,
\[
\int_0^1 x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a} dx = \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)} {}_2F_1(a, b, c, z),
\]
where $2F_1$ is the hypergeometric function. Applying this in (7.2), we thus obtain

$$(\theta - 1)^{-k} \frac{\Gamma(\alpha + \beta) \Gamma(k + \alpha)}{\Gamma(k + \alpha + \beta)} 2F_1(k + 1, k + \alpha, k + \alpha + \beta, -1/(\theta - 1)).$$

We then use one of the Euler transformations of the hypergeometric function,

$$2F_1(a, b, c, z) = (1 - z)^{c-a-b} 2F_1(c - a, c - b, c, z),$$

to arrive at

$$\theta^{-k} \frac{\Gamma(\alpha + \beta) \Gamma(k + \alpha)}{\Gamma(k + \alpha + \beta)} \left(\frac{\theta}{\theta - 1}\right)^{\beta - 1} 2F_1(\alpha + \beta - 1, \beta, k + \alpha + \beta, -1/(\theta - 1)).$$

(7.3)

We directly find a particular case in which we can find the value of the hypergeometric function explicitly, namely when $\alpha + \beta = 1$. When $\alpha + \beta = 1$, we find that the hypergeometric function on the right-hand side of (7.3) equals one as the first argument equals zero, independent of the other arguments, so that we arrive at

$$\left(1 - \frac{1}{(\theta - 1)^{\beta - 1}}\right) \frac{\Gamma(k + \alpha)}{\Gamma(k + \alpha + \beta)} = \left(1 - \frac{1}{\theta - 1}\right)^{\beta - 1} k^{-\beta} \theta^{-k} \left(1 + O(1/k)\right),$$

since $\Gamma(x)/\Gamma(x) = x^\beta (1 + O(1/x))$ as $x \to \infty$ and $\alpha = 1 - \beta$ in this particular case. When $

\alpha + \beta \neq 1$, we can obtain a similar expression. First, we use one of Pfaff’s transformations for the hypergeometric function,

$$2F_1(a, b, c, z) = (1 - z)^{-b} 2F_1(b, c - a, c, z/(z - 1)).$$

Then, applying this to the right-hand side of (7.3), so that $z/(z - 1) = 1/\theta \in (-1, 1)$, we can express the hypergeometric function as a power series. Namely, for $z$ such that $|z| < 1$,

$$2F_1(a, b, c, z) = \sum_{j=0}^{\infty} \frac{a^{(j)} b^{(j)}}{c^{(j)} \Gamma(j)} z^j,$$

where $a^{(j)} := \prod_{\ell=1}^{j-1} (a + (\ell - 1))$ (and similarly for $b^{(j)}, c^{(j)}$). Thus, combining the Pfaff transformation and the power series representation yields

$$2F_1(\alpha + \beta - 1, \beta, k + \alpha + \beta, -1/(\theta - 1)) = \left(\frac{\theta}{\theta - 1}\right)^{\beta} \sum_{j=0}^{\infty} \frac{\beta^{(j)} (k + 1)^{\beta - j}}{(k + \alpha + \beta)^{\beta + j}} \theta^{-j}.$$  

(7.4)

From the $\alpha + \beta = 1$ case, we immediately distil that

$$\sum_{j=0}^{\infty} \frac{\beta^{(j)} j^{\beta - j}}{\Gamma(j)^{\beta}} = \left(\frac{\theta}{\theta - 1}\right)^{\beta}.$$  

(7.5)

The aim is to show that for $k$ large, the series in (7.4) is close to $\theta/((\theta - 1)^{\beta})$ for any choice of $\alpha, \beta > 0$, so that the entire term in (7.4) is close to one. We consider two cases, namely $\alpha + \beta < 1$ and $\alpha + \beta > 1$. Let us start with the latter. We immediately obtain the upper bound $(k + \alpha + \beta)^{(\beta)} > (k+1)^{(\beta)}$, so that using (7.5) yields that the right-hand side of (7.4) is at most one. For a lower bound, we note that

$$\frac{(k+1)^{(\beta)}}{(k + \alpha + \beta)^{(\beta)}} = \prod_{\ell=1}^{j} \left(1 - \frac{\alpha + \beta - 1}{k + \alpha + \beta + (\ell - 1)}\right) \geq \left(1 - \frac{\alpha + \beta - 1}{k + \alpha + \beta}\right)^{j},$$

as the fraction in the second step in decreasing in $\ell$, since $\alpha + \beta - 1 > 0$. We thus obtain the lower bound

$$\left(\frac{\theta}{\theta - 1}\right)^{\beta} \sum_{j=0}^{\infty} \frac{\beta^{(j)} (k + 1)^{\beta - j}}{(k + \alpha + \beta)^{\beta + j}} \theta^{-j} \geq \left(\frac{\theta}{\theta - 1}\right)^{\beta} \sum_{j=0}^{\infty} \frac{\beta^{(j)} j!}{\Gamma(j)^{\beta}} \left(1 - \frac{\alpha + \beta - 1}{k + \alpha + \beta}\right)^{j},$$

which, as in (7.5), equals

$$\left(\frac{\theta - 1}{\theta - 1 + k + \alpha + \beta}\right)^{\beta} = \left(1 - \frac{\alpha + \beta - 1}{(\theta - 1)(k + \alpha + \beta) + (\alpha + \beta - 1)}\right)^{\beta} = 1 - O(1/k).$$

A similar approach can be used for $\alpha + \beta < 1$, where one would have an elementary lower bound and an upper bound that is $1 + O(1/k)$. In total, combining the above in (7.4) and then in (7.3) yields

$$\int_{0}^{1} f(x, k) \, dx = Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(k + \alpha + \beta)} \left(1 - \frac{1}{(\theta - 1)^{\beta - 1}}\right) k^{-\beta} \theta^{-k} (1 + O(1/k)).$$
We then consider
\[ \int_0^{w^*} (\theta - 1)x^k(\theta - 1 + x)^{-(k+1)} Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \, dx. \]

We bound the term \((1 - x)^{\beta-1}\) from above by \((1 - w^*)^{(\beta-1)\cdot 0}\). Since \(x^{k+\alpha-1}(1 + x/(\theta - 1))^{-(k+1)}\) is increasing for \(x \in (0, 1)\) when \(k\) is sufficiently large, we obtain the upper bound
\[ (\theta - 1)^{-k} Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} (1 - w^*)^{(\beta-1)\cdot 0} \left( \frac{w^*}{\theta - 1} \right)^{(k+1)} = Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} (1 - w^*)^{(\beta-1)\cdot 0} \left( \frac{w^*}{\theta - 1} \right)^{(k+1)} \]

Since \(w^*/(\theta - 1 + w^*)\) is increasing in \(w^*\), it is strictly smaller than \(1/\theta\). Hence,
\[ \int_0^{w^*} (\theta - 1)x^k(\theta - 1 + x)^{-(k+1)} Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \, dx = o(\theta^{-k} k^{-(1+\beta)}), \]

independent of the value of \(\beta\), so that
\[ p_k = \int_0^1 (\theta - 1)x^k(\theta - 1 + x)^{-(k+1)} Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \, dx - \int_0^{w^*} (\theta - 1)x^k(\theta - 1 + x)^{-(k+1)} Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \, dx \]
\[ = Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} (1 - \theta^{-1})^{-\beta} k^{-\beta} \theta^{-k} (1 + O(1/k)) = o(\theta^{-k} k^{-(1+\beta)}). \]

which proves the first line of (7.1).

An equivalent computation can be performed for
\[ \int_0^1 x^k(\theta - 1 + x)^{-k} Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \, dx, \]

(7.6) to obtain that it equals
\[ \theta^{-k} Z_w \frac{\Gamma(\alpha + \beta)\Gamma(k + \alpha)}{\Gamma(\alpha)\Gamma(k + \alpha + \beta)} \left( \frac{\theta}{\theta - 1} \right)^{(k+1)} F(1, 1, k + \alpha + \beta, -1/(\theta - 1)) \]
\[ = \theta^{-k} Z_w \frac{\Gamma(\alpha + \beta)\Gamma(k + \alpha)}{\Gamma(\alpha)\Gamma(k + \alpha + \beta)} \sum_{j=0}^{\infty} \frac{\beta^j (1 + \alpha + \beta)^{(1+\beta)(j)}}{(k + \alpha + \beta)^{(j+1)}} \theta^{-j}. \]

In this case an equivalent approach for bounding the sum on the right-hand side works for all \(\alpha, \beta > 0\). Hence, for (7.6) we obtain the asymptotic expression
\[ Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} (1 - \theta^{-1})^{-\beta} k^{-\beta} \theta^{-k} (1 + O(1/k)). \]

Via the same we can show that the integral from zero to \(w^*\) is asymptotically negligible, which finishes the proof. \(\square\)

Recall that in this example we set
\[ X^{(n)}_{\leq} := \{ j \in [n] : Z_n(j) = [\log_\theta n - \beta \log_\theta \log_\theta n + i] \}, \]
\[ X^{(n)}_\geq := \{ j \in [n] : Z_n(j) = [\log_\theta n - \beta \log_\theta \log_\theta n + i] \}, \]
\[ \varepsilon_n := (\log_\theta n - \beta \log_\theta \log_\theta n) - [\log_\theta n - \beta \log_\theta \log_\theta n]. \]

We then state the analogue of Proposition 5.6.

**Proposition 7.2.** Consider the WRT model as in Definition 2.1 with vertex-weights \((W_i)_{i \in [n]}\) whose distribution satisfies (4.1) for some \(\alpha, \beta > 0, w^* \in (0, 1), \) and recall \(c_{\alpha, \beta, \theta}\) from (4.2). For a fixed \(K \in \mathbb{N}, c \in (0, \theta/(\theta - 1))\), the following holds. For any \(i, i' = i(n), i'(n)\) in \(\mathbb{Z}\) such that
\[ 0 < \log \theta n + i < \log \theta n + i' < c \log n \text{ and } i, i' \sim \delta \log \theta n, \text{ for some } \delta \in (-1, c \log \theta - 1) \cup \{0\} \] (\( \delta = 0 \) denotes \( i, i' = o(\log n) \)) and for \( a_1, \ldots, a_{\nu'} \in \mathbb{N}_0 \) satisfying \( a_1 + \cdots + a_{\nu'} = K \),

\[
\mathbb{E}
\left[
\left(X^{(n)}_{\geq i'}\right)_{a_1'}
\prod_{k=1}^{i'-1}
\left(X^{(n)}_{k}\right)_{a_k'}
\right]
= \left(\frac{c_{\alpha, \beta, \theta}}{(1 + \delta)^3}\right)^{a_{\nu'}}\prod_{k=1}^{i'-1}
\left(\frac{c_{\alpha, \beta, \theta}(1 - \theta^{-1})}{(1 + \delta)^3}\right)^{a_k}
\times
\left(1 + \mathcal{O}\left(\frac{\log \log n}{\log n} \sqrt{\frac{1 - \delta - \log \theta n}{\log n}}\right)\right).
\]

**Proof.** Set \( K' := K - a_{\nu'} \) and for each \( i \leq k \leq i' \) and for each \( u \) such that \( \sum_{\ell=1}^{k-1} a_\ell < u \leq \sum_{\ell=i}^{k} a_\ell \), let \( m_u = |\log \theta n - \log \log \theta n| + k \). Also, let \( (v_u)_{u \in [K]} \) be \( K \) vertices selected uniformly at random without replacement from \([n]\). Then, as the \( X^{(n)}_{\geq i'} \) and \( X^{(n)}_{k} \) can be expressed as sums of indicators, following the same steps as in the proof of Proposition 5.6,

\[
\mathbb{E}
\left[
\left(X^{(n)}_{\geq i'}\right)_{a_1'}
\prod_{k=1}^{i'-1}
\left(X^{(n)}_{k}\right)_{a_k'}
\right]
= (n)_K \sum_{|S| = \ell} \prod_{\ell=0}^{K} \mathbb{P}(\deg(v_u) \geq m_u + 1_{\{u \in S\}}), u \in [K].
\]

By Proposition 5.1,

\[
\mathbb{P}(\deg(v_\ell) \geq m_u + 1_{\{u \in S\}}) = \prod_{u=1}^{K} \mathbb{E}
\left[
\left(\frac{W}{W + \log n} + \log n\right)^{m_u}\right]
= \left(Z_w \frac{\Gamma(\alpha + \beta)}{\Gamma(\ell)}(1 - \theta^{-1})^{-\beta} \right)^{K} \theta^{-\sum_{u=1}^{K} m_u - \ell} \prod_{u=1}^{K} (m_u + 1_{\{u \in S\}})^{-\beta}(1 + \mathcal{O}(1/\log n)).
\]

Here, we are able to obtain the error term \( 1 - \mathcal{O}(1/\log n) \) due to the fact that \( c \log n + i > \eta \log n \) for some \( \eta \in (0, 1 + \delta) \) when \( n \) is large. Moreover, as \( i, i' \sim \delta \log \theta n \) and thus \( m_u \sim (1 + \delta) \log \theta n \) for each \( u \in [K] \),

\[
\prod_{u=1}^{K} (m_u + 1_{\{u \in S\}})^{-\beta} = ((1 + \delta) \log \theta n)^{-\beta K} \left(1 + \mathcal{O}\left(\frac{\log \log n}{\log n} \sqrt{\frac{1 - \delta - \log \theta n}{\log n}}\right)\right),
\]

uniformly in \( S \) (and \( \ell \)). Recalling that \( c_{\alpha, \beta, \theta} = Z_w \Gamma(\alpha + \beta)/(\Gamma(\ell)) (1 - \theta^{-1})^{-\beta} \), we thus arrive at

\[
(n)_K \sum_{\ell=0}^{K} \prod_{|S| = \ell} \left(-1\right)^\ell \left(c_{\alpha, \beta, \theta}(1 + \delta)^{-\beta} \log \theta n\right)^{-\beta K} \theta^{-\sum_{u=1}^{K} m_u - \ell}
\times \left(1 + \mathcal{O}\left(\frac{\log \log n}{\log n} \sqrt{\frac{1 - \delta - \log \theta n}{\log n}}\right)\right)
= \left(c_{\alpha, \beta, \theta}(1 + \delta)^{-\beta} \theta^{-\iota + \epsilon_n}\right)^{a_{\nu'}} \prod_{k=1}^{i'-1} \left(c_{\alpha, \beta, \theta}(1 + \delta)^{-\beta} (1 - \theta^{-1})^{\theta^{-k+\epsilon_n}}\right)^{a_k}
\times \left(1 + \mathcal{O}\left(\frac{\log \log n}{\log n} \sqrt{\frac{1 - \delta - \log \theta n}{\log n}}\right)\right),
\]

where the last step follows from a similar argument as in the proof of Proposition 5.6.

\[ \square \]

With Proposition 7.2 at hand, the proofs of Theorems 4.2, 4.3 and 4.4 follow the same approach as the proofs of Theorems 2.5, 2.8 and 2.9, respectively.

### 7.2. Example 4.5, fraction of ‘gamma-like’ random variables

Let the distribution of \( W \) satisfy (4.3) for some \( \alpha, \beta > 0, w^* \in (0, 1) \). We prove a result on (the tail of) the limiting degree distribution and provide additional building blocks required to prove the results in Example 4.5.
Lemma 7.3. Let the distribution of $W$ satisfy (4.3) for some $b \in \mathbb{R}, c_1 > 0, w^* \in (0, 1)$ and recall $p_k, p_{\geq k}$ and $C$ from (2.1) and (4.4), respectively. Then,

$$p_k = Z_{w^*}(1 - \theta^{-1})Ck^{b/2+1/4}e^{-2\sqrt{c_1^{-1}(1-\theta^{-1})}}\theta^{-k}(1 + O(1/\sqrt{k})),
$$

$$p_{\geq k} = Z_{w^*}Ck^{b/2+1/4}e^{-2\sqrt{c_1^{-1}(1-\theta^{-1})}}\theta^{-k}(1 + O(1/\sqrt{k})).$$

Note that this Lemma improves on the bounds in (5.4) by providing a polynomial correction term and a precise multiplicative constant.

Proof. We start by proving the equality for $p_{\geq k}$ and then show the similar result for $p_k$. By (4.3), we obtain the following expression for $p_{\geq k}$.

$$p_{\geq k} = \int_{w^*}^{1} x^{k}(\theta - 1 + x)^{-k}Z_{w^*}c_1^{-1}(1-x)^{-2(b+2)}e^{-c_1^{-1}x/(1-x)} dx$$

$$= \int_{w^*}^{1} x^{k}(\theta - 1 + x)^{-k}Z_{w^*}b(1-x)^{-(1+b)}e^{-c_1^{-1}x/(1-x)} dx.$$  

(7.7)

The second integral is of a similar form as the first, with a different constant in front and a different polynomial exponent. We hence only provide an explicit analysis of the first integral. As in the proof of Lemma 7.1, we start by considering the integral

$$\int_{0}^{1} x^{k}(\theta - 1 + x)^{-k}c_1^{-1}(1-x)^{-2(b+2)}e^{-c_1^{-1}x/(1-x)} dx,$$

(7.8)

where we omit the constant $Z_{w^*}$ for ease of writing, and then show that the integral from zero up to $w^*$ is asymptotically negligible compared to the integral from zero to one. Using a variable transform $u = x/(1-x)$, we find that (7.8) equals

$$\theta^{-k}c_1^{-1} \int_{0}^{\infty} u^k(1+u)^{b-k}(1 - \frac{1}{\theta(1+u)})^{-k} e^{-c_1^{-1}u} du.$$

We now define $X_u$ to be a negative binomial random variable with parameters $k$ and $p_u := (\theta(1+u))^{-1}$, for any $u > 0$. As the sum over the probability mass function of $X_u$ is one irrespectively of the value of $u$, we obtain that the above equals

$$\theta^{-k}c_1^{-1} \int_{0}^{\infty} \sum_{j=0}^{\infty} \binom{j+k-1}{j} p_u^j(1-p_u)^k u^k(1+u)^{b-k}(1 - \frac{1}{\theta(1+u)})^{-k} e^{-c_1^{-1}u} du$$

$$= \theta^{-k}c_1^{-1} \int_{0}^{\infty} \sum_{j=0}^{\infty} \binom{j+k-1}{j} \theta^{-j} u^k(1+u)^{b-j} e^{-c_1^{-1}u} du$$

$$= \theta^{-k}c_1^{-1} \sum_{j=0}^{\infty} \binom{j+k-1}{j} \theta^{-j} \Gamma(k+1)U(k+1, 2+b-j, c_1^{-1}),$$

where $U(a, b, z)$ is the confluent hypergeometric function of the second kind, defined as

$$U(a, b, z) := \frac{1}{\Gamma(a)} \int_{0}^{\infty} x^{a-1}(1+x)^{b-a-1}e^{-zx} dx,$$

whenever Re($a$) > 0. Using the Kummer transform $U(a, b, z) = z^{1-b}U(1+a-b, 2-b, z)$ yields

$$\theta^{-k}c_1^{-1} \sum_{j=0}^{\infty} \binom{j+k-1}{j} \theta^{-j} \Gamma(k+1) c_1^{b-j} U(j+k-b, j-b, c_1^{-1}).$$

Again using the definition of the confluent hypergeometric function of the second kind, we obtain

$$\c_1^{b-1} \theta^{-k} \sum_{j=0}^{\infty} \frac{\Gamma(j+k+1)}{(k+1)(j+1)(k+j+b)}(c_1 \theta)^{-j} \int_{0}^{\infty} u^{j+k-b-1}(1+u)^{-(k+1)}e^{-c_1^{-1}u} du$$

$$= \c_1^{b} k^{b-k} \frac{\Gamma(k)}{\Gamma(k-b)} \sum_{j=0}^{\infty} \frac{\Gamma(j+k+1)}{(k+1)(j+1)(k+j+b)}(c_1 \theta)^{-j} \int_{0}^{\infty} u^{j+k-b-1}(1+u)^{-(k+1)}e^{-c_1^{-1}u} du$$

$$= \c_1^{b} k^{b-k} \frac{\Gamma(k)}{\Gamma(k-b)} \sum_{j=0}^{\infty} \frac{(k)^{(j)}}{(k-b)^{(j)}} \frac{1}{j!}(c_1 \theta)^{-j} \int_{0}^{\infty} u^{j+k-b-1}(1+u)^{-(k+1)}e^{-c_1^{-1}u} du,$$
where \((x)^{(j)} := x(x + 1) \cdots (x + (j - 1)), x \in \mathbb{R}, j \in \mathbb{N}_0\). When can then bound
\[
\frac{(k)^{(j)}}{(k - b)^{(j)}} \geq \begin{cases} 
1 \left( \frac{k}{k - b} \right)^j & \text{if } k > b \geq 0, \\
1 & \text{if } b < 0.
\end{cases}
\]
\begin{equation}
(7.9)
\end{equation}

As the bounds are symmetric, we can assume that \(b \geq 0\) without loss of generality; the other case follows similarly. We deal with the lower bound first. This yields
\[
\begin{aligned}
c_1^b k \theta^{k - b} \frac{\Gamma(k)}{\Gamma(k - b)} & \sum_{j=0}^{\infty} \frac{1}{j!} (c_1 \theta)^{-j} \int_0^{\infty} u^{k - b - 1} (1 + u)^{-(k + 1)} e^{-c_1 u} du \\
& = c_1^b k \theta^{k - b} \frac{\Gamma(k)}{\Gamma(k - b)} \int_0^{\infty} u^{k - b - 1} (1 + u)^{-(k + 1)} e^{-c_1 u} du \\
& = c_1^b k \theta^{k - b} \frac{\Gamma(k)}{\Gamma(k - b)} \Gamma(k - b) U(k - b, -b, c_1^{-1}(1 - \theta^{-1})).
\end{aligned}
\]

It follows from [19, (3.12) and (3.15)] that, when \(a > d/2\) is large and \(a, z\) are bounded,
\[
\Gamma(a) U(a, d, z^2) = 2e^{z^2/2} \left( \frac{2 z}{a} \right)^{1-d} K_{1-d}(uz)(1 + O(1/u)),
\]
where \(u = 2 \sqrt{a-d/2}\) and \(K_{1-d}(uz)\) is a modified Bessel function. Combined with the asymptotic expression for the modified Bessel function as in [15, (10.40.2)], we obtain
\[
\Gamma(a) U(a, d, z^2) = 2 \sqrt{\frac{\pi}{2uz}} e^{z^2/2-uz} \left( \frac{2z}{a} \right)^{1-d} (1 + O(1/u)).
\]

In this particular case, it yields
\[
\Gamma(k - b) U(k - b, -b, c_1^{-1}(1 - \theta^{-1}))
\]
\[
eq e^{c_1^{-1}(1-\theta^{-1})/2} \sqrt{\pi c_1^{-1}(1-\theta^{-1})} (1 + O(1/\sqrt{k})).
\]

Using this, as well as \(\Gamma(k)/\Gamma(k - b) = k^b (1 + O(1/k))\), in (7.10), we arrive at
\[
eq e^{c_1^{-1}(1-\theta^{-1})/2} \sqrt{\pi c_1^{-1}(1-\theta^{-1})} (1 + O(1/\sqrt{k})).
\]

We then tend to the upper bound in (7.9) for \(b \geq 0\), which yields
\[
\begin{aligned}
c_1^b k \theta^{k - b} \frac{\Gamma(k)}{\Gamma(k - b)} & \sum_{j=0}^{\infty} \frac{1}{j!} (c_1 \theta)^{-j} \int_0^{\infty} u^{k + b - 1} (1 + u)^{-(k + 1)} e^{-c_1 u} du \\
& = c_1^b k \theta^{k - b} \frac{\Gamma(k)}{\Gamma(k - b)} \int_0^{\infty} u^{k + b - 1} (1 + u)^{-(k + 1)} e^{-c_1 u} du \\
& = c_1^b k \theta^{k - b} \frac{\Gamma(k)}{\Gamma(k - b)} \Gamma(k - b) U(k - b, -b, c_1^{-1}(1 - \theta^{-1})).
\end{aligned}
\]

From the asymptotic results in (7.11) we find that
\[
U(k - b, -b, \frac{1}{c_1}(1 - \theta^{-1})) = U(k - b, -b, \frac{1}{c_1}(1 - \theta^{-1})) (1 + O(1/\sqrt{k})),
\]
so that the lower and upper bound match up to error terms (of the same order). By (7.11), we thus arrive at
\[
\begin{aligned}
\int_0^{1} x^k (\theta - 1 + x)^{-k} Z_{w^*} c_1^{-1}(1 - x)^{-c_1^{-1} x/(1-x)} dx \\
& = Z_{w^*} c_1^{-1}(1-\theta^{-1})/2 \sqrt{\pi c_1^{-1}(1-\theta^{-1})} (1 + O(1/\sqrt{k})).
\end{aligned}
\]

Then, we bound
\[
\int_0^{w^*} x^k (\theta - 1 + x)^{-k} Z_{w^*} c_1^{-1}(1 - x)^{-c_1^{-1} x/(1-x)} dx \\
\leq \frac{Z_{w^*}}{c_1} (1 - w^*)^{-((2+b)v_0)} \int_0^{w^*} x^k (\theta - 1 + x)^{-k} dx \\
\leq \frac{Z_{w^*}}{c_1} (1 - w^*)^{-((2+b)v_0)} w^* \left( \frac{w^*}{\theta - 1 + w^*} \right)^k.
\]
where we use that \( x \mapsto (x/(\theta-1+x))^k \) is increasing in \( x \). This also implies that \( w^*/(\theta-1+w^*) < 1/\theta \), so that
\[
\int_0^{w^*} x^k(\theta-1+x)^{-k}Z\,z_{c_1}\,c_1^{-1}(1-x)^{(2+b)}e^{-c_1^{-1}x/(1-x)}\,dx = o(\theta^{-k}e^{-2\sqrt{c_1^{-1}(1-\theta^{-1})k}\sqrt{1+b/2})
\]

independent of the values of \( c_1, b \) and \( \theta \). As a result,
\[
\int_0^1 x^k(\theta-1+x)^{-k}Z\,z_{c_1}\,c_1^{-1}(1-x)^{(2+b)}e^{-c_1^{-1}x/(1-x)}\,dx
= Z\,z_{c_1}Ck^{1/4+b/2}e^{-2\sqrt{c_1^{-1}(1-\theta^{-1})k}\theta^{-k}(1+O(1/\sqrt{k}))}
\tag{7.12}
\]

Finally, when considering the second integral in (7.7), we observe its integrand is similar to that of the first integral but with a different constant in front and with a constant \( \tilde b = b - 1 \) in the polynomial exponent. We can thus follow the exact same steps as for the first integral in (7.7) to conclude that it can be included in the \( O(1/\sqrt{k}) \) term in (7.12). In total,
\[
p_{\geq k} = Z\,z_{c_1}Ck^{1/4+b/2}e^{-2\sqrt{c_1^{-1}(1-\theta^{-1})k}\theta^{-k}(1+O(1/\sqrt{k}))}
\]
as required.

We now show the result for \( p_k \), which uses the above steps with several minor adjustments. First,
\[
p_k = Z\,z_{c_1}(\theta-1)\int_0^1 x^k(\theta-1+x)^{-k}(-k+1)c_1^{-1}(1-x)^{(2+b)}e^{-c_1^{-1}x/(1-x)}\,dx
- Z\,z_{c_1}(\theta-1)\int_0^1 x^k(\theta-1+x)^{-k+1}b(1-x)^{-(b+1)}e^{-c_1^{-1}x/(1-x)}\,dx
\tag{7.13}
\]

As for the proof of the asymptotic expression of \( p_{\geq k} \), we consider the first integral only as the second one is of lower order. Moreover, we again consider the first integral with a lower bound of zero for the integration variable \( x \). So, omitting \( Z\,z_{c_1} \) for now, we have
\[
(\theta-1)\int_0^1 x^k(\theta-1+x)^{-k}(-k+1)c_1^{-1}(1-x)^{(2+b)}e^{-c_1^{-1}x/(1-x)}\,dx
= (\theta-1)c_1^{-1}\theta^{-k}\int_0^\infty u^k(1+u)^{b-k}(1-\frac{1}{\theta(1+u)})\,(1-k+1)c_1^{-1}u\,du
= (\theta-1)c_1^{-1}\theta^{-k}\sum_{j=0}^{\infty} \binom{j+k}{j}\theta^{-j}\int_0^\infty u^k(1+u)^{b-(j+k)}e^{-c_1^{-1}u}\,du
= (\theta-1)c_1^{-1}\theta^{-k}\sum_{j=0}^{\infty} \binom{j+k}{j}\theta^{-j}\Gamma(k+1)U(k+1,2+b-j,c_1^{-1})
= (\theta-1)c_1^{-1}\theta^{-k}\sum_{j=0}^{\infty} \binom{j+k}{j}\theta^{-j}\Gamma(k+1)U(k+1,2+b-j,c_1^{-1})
= (\theta-1)c_1^{-1}\theta^{-k}\sum_{j=0}^{\infty} \binom{j+k}{j}\theta^{-j}\Gamma(k+1)\frac{(k+1)(c_1\theta)^{-j}}{(k-b)(c_1\theta)^{-j}}\int_0^\infty u^{k+j-b-1}(1+u)^{-(k+1)c_1^{-1}u}\,du.
\]

Similar to (7.9), we bound
\[
\frac{(k+1)^{(j)}}{(k-b)^{(j)}} \geq \begin{cases} 
1 & \text{if } k > b \geq -1, \\
\left(\frac{k+1}{k-b}\right)^{j} & \text{if } b < -1.
\end{cases}
\]

and
\[
\frac{(k)^{(j)}}{(k-b)^{(j)}} \leq \begin{cases} 
1 & \text{if } b < -1, \\
\left(\frac{k+1}{k-b}\right)^{j} & \text{if } k > b \geq -1.
\end{cases}
\]

Again, we assume without loss of generality that \( b \geq -1 \). Moreover, we only concern ourselves with the lower bound on \((k+1)^{(j)}/(k-b)^{(j)}\) when \( b \geq -1 \), since we obtain a matching upper bound with the required error term when using the upper bound on \((k+1)^{(j)}/(k-b)^{(j)}\) when \( b \geq -1 \), as in the proof for \( p_{\geq k} \). Thus, we obtain the lower bound
\[
(1-\theta^{-1})c_1^{-1}\theta^{-k}\frac{\Gamma(k+1)}{(k-b)}\sum_{j=0}^{\infty} \binom{(c_1\theta)^{-j}}{j!}\int_0^\infty u^{k+j-b-1}(1+u)^{-(k+1)c_1^{-1}u}\,du
= (1-\theta^{-1})c_1^{-1}\theta^{-k}\frac{\Gamma(k)}{(k-b)}\int_0^\infty u^{k+j-b-1}(1+u)^{-(k+1)c_1^{-1}(1-\theta^{-1})}\,du.
\]
which, up to the constant \((1 - \theta^{-1})\), is the exact same expression as in (7.10). As discussed above, using the upper bound on \((k+1)^{\delta}/(k-b)^{\delta}\) yields a matching upper bound (up to error terms). Then, the same approach as in the proof of \(P_{2,k}\) can be used to show that

\[
\int_0^{w^*} x^k (\theta - 1 + x)^{-(k+1)c_1^{-1}} (1-x)^{b-2} e^{c_1^{-1} x/(1-x)} \, dx = o(\theta^{-k} e^{2\sqrt{c_1^{-1}(1-\theta^{-1})^2 k^{-1/4} + b/2})},
\]

so that the integral from \(w^*\) to 1 is asymptotically equivalent to the integral from 0 to 1. Following the same steps as above for the second integral in (7.13), we find it can be included in the error term as well. Hence, the result follows.

Recall that in this example we set

\[
X_{t}^{(n)} := \{|j \in [n] : Z_{t}(j) = [\log_\theta n - C_{1,c_1} \sqrt{\log \log n + (b/2 + 1/4) \log \log \log n} + i]\},
\]

\[
X_{\leq t}^{(n)} := \{|j \in [n] : Z_{t}(j) \geq [\log_\theta n - C_{1,c_1} \sqrt{\log \log n + (b/2 + 1/4) \log \log \log n} + i]\},
\]

\[
\varepsilon_n := (\log_\theta n - C_{1,c_1} \sqrt{\log \log n + (b/2 + 1/4) \log \log \log n})
- [\log_\theta n - C_{1,c_1} \sqrt{\log \log n + (b/2 + 1/4) \log \log \log n}].
\]

We then state the analogue of Proposition 5.6.

**Proposition 7.4.** Consider the WRT model as in Definition 2.1 with vertex-weights \((W_i)_{i \in [n]}\) whose distribution satisfies (4.3) for some \(b \in \mathbb{R}, c_1 > 0, w^* \in (0,1)\), and recall \(c_{c_1,b,\theta}\) from (4.4). For a fixed \(K \in \mathbb{N}, c \in (1, \theta/(\theta - 1))\) the following holds. For any \(i, i' = i(n), i'(n) \in \mathbb{Z}\) such that \(0 < \log_\theta n + i < \log_\theta n + i' < c_1 \log_\theta n\) and \(i, i' \sim \delta \sqrt{\log_\theta n}\) for some \(\delta \in \mathbb{R} (\delta = 0\) denotes \(i, i' = o(\sqrt{\log_\theta n})\) and for \(a_1, \ldots, a_{c_1} \in \mathbb{N_0}\) satisfying \(a_1 + \cdots + a_{c_1} = K\),

\[
\mathbb{E} \left[ X_{\geq t}^{(n)}_{i'} \prod_{k=1}^{i'-1} X_{< k}^{(n)}_{a_k} \right] = (n)_K \sum_{\ell=0}^{K'} \sum_{\mathcal{S} \subseteq [K']} (-1)^i \mathbb{P}(\deg(v_u) \geq m_u + \mathbb{1}_{\{u \in \mathcal{S}\}}, u \in [K]).
\]

By Proposition 5.1,

\[
\mathbb{P}(\deg(v_{\ell}) \geq m_u + \mathbb{1}_{\{u \in \mathcal{S}\}}, u \in [K]) = \prod_{u=1}^{K} \mathbb{E} \left[ \left( \frac{W}{{\mathbb{E}[W]+W}^{m_u+1(\mathbb{1}_{\{u \in \mathcal{S}\}})} \right)^{\alpha_c} \right]\bigg| |\mathcal{S}| = \ell,
\]

for some \(\beta > 0\). By Lemma 7.3 (and recalling the constant \(C\) in (4.4)), when \(|\mathcal{S}| = \ell\),

\[
\prod_{u=1}^{K} \mathbb{E} \left[ \left( \frac{W}{{\mathbb{E}[W]+W}^{m_u+1(\mathbb{1}_{\{u \in \mathcal{S}\}})} \right)^{\alpha_c} \right] = (Z_{w,c})^K \theta^{-\sum_{u=1}^{K} m_u - \ell} \exp \left\{ -2 \sum_{u=1}^{K} \frac{1 - \theta^{-1}}{c_1} (m_u + \mathbb{1}_{\{u \in \mathcal{S}\}}) \prod_{u=1}^{K} (m_u + \mathbb{1}_{\{u \in \mathcal{S}\}})^{b/2+1/4} \right\}
\]

\[
\times (1 + O(1/\sqrt{\log n})).
\]

Here, we are able to obtain the error term \(1 + O(1/\sqrt{\log n})\) due to the fact that \(\log_\theta n + i > \eta \log n\) for some \(\eta > 0\) when \(n\) is large. We note that \(C_{1,c_1} \log \theta = 2 \sqrt{c_1^{-1}(1 - \theta^{-1})}\). As \(i, i' \sim \delta \sqrt{\log_\theta n}\),

\[
\prod_{u=1}^{K} (m_u + \mathbb{1}_{\{u \in \mathcal{S}\}})^{b/2+1/4} = (\log_\theta n)^{b/2+1/4} (1 + O(1/\sqrt{\log_\theta n})).
\]
uniformly in $S$ (and $\ell$). Moreover, again uniform in $S$ and $\ell$,

$$\exp \left\{ -C_{\theta,1,c_1} \log \theta \sum_{u=1}^{K} \sqrt{m_u + \mathbb{I}\{u \in S\}} \right\}
= \exp \left\{ - \left( C_{\theta,1,c_1} \log \theta \sqrt{\log_\theta n} - \frac{C_{\theta,1,c_1} - \delta}{2} \right)^K \right\}
\times \left( 1 + O \left( \frac{\log_\theta \log_\theta n \sqrt{|i - \sqrt{\log_\theta n}| \vee |i' - \sqrt{\log_\theta n}|}}{\sqrt{\log_\theta n}} \right) \right)$$

This last step follows from the fact that the first-order term of $m_u$ is $\log_\theta n$ and its second-order term is $-(C_{\theta,1,c_1} - \delta)\sqrt{\log_\theta n}$. Finally, its lower-order terms are $\log_\theta \log_\theta n + (|i - \sqrt{\log_\theta n}| \vee |i' - \sqrt{\log_\theta n}|)$. Then using a Taylor expansion for the square root yields the result. Combining all of the above and recalling that $c_{1,b,\theta} = Z_w \cdot C \theta \sqrt{c_{1,c_1}} / 2$, we thus arrive at

$$(nK)^{K'} \sum_{\ell=0}^{K'} \sum_{S \subseteq [K']} \frac{(-1)^{\ell}}{|S|=\ell} \frac{\left( Z_w \cdot C \theta \sqrt{\log_\theta n} \right)^{\ell/2+1/4}}{\exp \left\{ - \left( C_{\theta,1,c_1} \log \theta \left( \sqrt{\log_\theta n} - \frac{C_{\theta,1,c_1} - \delta}{2} \right) \right)^K \right\}}
\times \theta^{-\sum_{k=1}^{K'} m_k - \ell} \left( 1 + O \left( \frac{\log_\theta \log_\theta n \sqrt{|i - \sqrt{\log_\theta n}| \vee |i' - \sqrt{\log_\theta n}|}}{\sqrt{\log_\theta n}} \right) \right)
= \left( c_{1,b,\theta} \theta^{-1+\varepsilon - C_{\theta,1,c_1}} \delta / 2 \right)^{\ell/2} \prod_{k=1}^{K'} \left( c_{1,b,\theta} (1 - \theta^{-1})^{\ell_k - k + \varepsilon - C_{\theta,1,c_1}} \delta / 2 \right)^{a_k}
\times \left( 1 + O \left( \frac{\log_\theta \log_\theta n \sqrt{|i - \sqrt{\log_\theta n}| \vee |i' - \sqrt{\log_\theta n}|}}{\sqrt{\log_\theta n}} \right) \right),$$

where the last step follows from a similar argument as in the proof of Proposition 5.6.

With Proposition 7.4 at hand, the proofs of Theorems 4.6, 4.7 and 4.8 follow the same approach as the proofs of Theorems 2.5, 2.8 and 2.9.

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Lemma 8.3. Consider the sequences $g(k)$. It is bounded by $1$ to obtain the upper bound $2f(n)g(\ell - 1) \leq \sum_{k=\ell}^{n} f(k)g(k) \leq \int_{\ell-1}^{n} f(x)g(x) \, dx + 2f(n+1)g(\ell - 1)$.

Proof. We only prove the upper bound, the lower bound follows from an analogous approach. By definition for $k \in \{\ell, \ldots, n\}$ and $x \in [k - 1, k]$, $f(k) \leq f(x + 1)$ and $g(k) \leq g(x)$.

Integrating both sides gives

$$f(k)g(k) \leq \int_{k-1}^{k} f(x + 1)g(x) \, dx$$

$$\leq \int_{k-1}^{k} f(x)g(x) \, dx + g(\ell - 1) \int_{k-1}^{k} f(x + 1) - f(x) \, dx$$

$$\leq \int_{k-1}^{k} f(x)g(x) \, dx + g(\ell - 1)(f(k + 1) - f(k - 1)).$$

Hence, summing from $\ell$ to $n$ gives

$$\sum_{k=\ell}^{n} f(k)g(k) \leq \int_{\ell-1}^{n} f(x)g(x) \, dx + g(\ell - 1) \sum_{k=\ell}^{n} (f(k + 1) - f(k - 1))$$

$$\leq \int_{\ell-1}^{n} f(x)g(x) \, dx + 2f(n + 1)g(\ell - 1),$$

as required. \qed

Corollary 8.2. Fix $\ell, n \in \mathbb{N}$ such that $\ell < n$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a positive integrable function, increasing on $[\ell, x^*]$ and decreasing on $[x^*, n]$, where $x^*$ is not necessarily an integer. Suppose $g : \mathbb{R} \to [0, 1]$ is a positive, integrable function, decreasing on $[\ell, n + 1]$ and bounded by $1$. Then,

$$\int_{\ell}^{n} f(x)g(x) \, dx \leq \sum_{k=\ell+1}^{n} f(k)g(k) \leq \int_{\ell}^{n} f(x)g(x) \, dx + 4f(x^*).$$

Proof. We only prove the upper bound, the lower bound follows from an analogous approach. Note that $f$ is increasing on $[\ell, [x^*]]$, so that by Lemma 8.1

$$\sum_{k=[x^*]+1}^{[x^*]^{-1}} f(k)g(k) \leq \int_{\ell}^{[x^*]^{-1}} f(x)g(x) \, dx + 2f(x^*),$$

where we used that $g$ is decreasing and bounded by $1$. Also, note that by the fact that $f$ and $g$ are both decreasing on $[x^*, n]$,

$$\sum_{k=[x^*]+1}^{n} f(k)g(k) \leq \int_{[x^*]}^{n} f(x)g(x) \, dx.$$

It remains to bound $f([x^*])g([x^*]) + f([x^*])g([x^*])$. We use that $f$ is maximised at $x^*$ and that $g$ is bounded by one to obtain the upper bound $2f(x^*)$. Combining all of the above and including in the integrals in the upper bounds the range $[[x^*] - 1, [x^*]]$, we thus obtain

$$\sum_{k=\ell+1}^{n} f(k)g(k) \leq \int_{\ell}^{n} f(x)g(x) \, dx + 4f(x^*),$$

as required. \qed

Lemma 8.3. Consider the sequences $(s_k, r_k)_{k \in \mathbb{N}}$ in (2.9). These sequences have the following properties:
(i) $s_k$ is increasing,

(ii) $r_k$ is decreasing and $\lim_{k \to \infty} r_k = 0$.

Proof. (i) Assume that $s_{k+1} < s_k$ for some $k \in \mathbb{N}$ and take $x \in (s_{k+1}, s_k)$. By the definition of $s_{k+1}, s_k$ and the choice of $x$,

$$P(W \in (x, 1)) \leq e^{-(1-\theta^{-1})(1-x)(k+1)} < e^{-(1-\theta^{-1})(1-x)k} < P(W \in (x, 1)),$$

which leads to a contradiction.

(ii) Assume that $s_k < s_{k+1}$ (otherwise the claim is immediately clear). Note that since the function $P(W \in (x, 1))$ is càdlàg, we have for any $x < s_k$,

$$P(W \in (s_k, 1)) \leq r_k \leq \lim_{y \uparrow s_k} P(W \in (y, 1)) \leq P(W \in (x, 1)). \quad (8.1)$$

Hence, we have that for any $x \in (s_k, s_{k+1})$,

$$r_k \geq P(W \in (s_k, 1)) \geq P(W \in (x, 1)) \geq r_{k+1}.$$

For the second part, since $s_k$ is increasing by (i), we have that $s_k \to s^* \in (0, 1]$. Suppose that $s^* \in (0, 1)$. Then, for $k$ sufficiently large, we have $s_k \leq (1 + s^*)/2$ and so $r_k \leq e^{-(1-\theta^{-1})(1-s^*)k/2}$ and so $r_k$ converges to 0.

Therefore, we can assume that $s_k \uparrow 1$. Let $k_0$ be such the smallest $k$ such that $s_k < s_{k+1}$. Such a $k_0$ exists, otherwise $s^* < 1$ since each $s_k < 1$. Then, for $k \geq k_0$, let $\ell_k$ be the largest integer such that $s_{\ell_k} < s_k$.

The assumption that $s_k \uparrow 1$ also excludes that $s_{\ell_k}$ is eventually constant and so $\ell_k \to \infty$. In particular, we can argue as in (8.1) to see that

$$r_k \leq P(W \in (s_{\ell_k}, 1)).$$

Moreover, as $s_{\ell_k} \to 1$ as $k \to \infty$, we deduce that $r_k \to 0$. \hfill \qed

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