TWISTOR LIFTS AND FACTORIZATION FOR CONFORMAL MAPS FROM A SURFACE TO THE EUCLIDEAN FOUR-SPACE

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Abstract. A conformal map from a Riemann surface to the Euclidean four-space is explained in terms of its twistor lift. A local factorization of a differential of a conformal map is obtained. As an application, the factorization of a differential provides an upper bound of the area of a super-conformal map around a branch point.

1. Introduction

In classical surface theory, we consider an oriented surface to be the image of an isometric immersion from a two-dimensional oriented Riemannian manifold. To investigate the Riemannian geometric properties, we frequently employ an orthogonal complex structure that is compatible with the orientation of a two-dimensional Riemannian manifold. We employ theory of holomorphic functions, Riemann surfaces and holomorphic vector bundles. This method is successful and has been investigated in various studies. For example, several important examples of minimal surfaces in Euclidean space are constructed by a meromorphic function and a holomorphic one-form on a Riemann surface by the Weierstrass representation formula [6], [20]. The Hopf’s theorem for constant mean curvature surfaces is proven by the holomorphic Hopf differential [11]. A holomorphic function is a (branched) conformal immersion and the theory of holomorphic functions is a successful theory. We obtain an idea for constructing a theory of conformal immersions so that it includes the theory of holomorphic functions.

The paper [17] seems to be one of initial significant achievements using by this idea. They refer to a branched conformal immersion from a Riemann surface to the four-dimensional Euclidean space $E^4$ as a conformal map. A conformal map is considered to be a holomorphic map from a Riemann surface to the four-dimensional Euclidean space with respect to an almost complex structure along $f$. The subsequent papers show that this approach is fruitful. For example, [7] introduces quaternionic holomorphic curves, which include holomorphic curves in complex projective space and obtains theorems that hold for holomorphic curves in complex projective space as special cases.

The almost complex structure along a conformal map is considered to be a map from a Riemann surface to the twistor space of $E^4$. A twistor lift is a pair that consists of a conformal map and an almost complex structure along the conformal map. An almost complex structure is invariant under conformal transforms of $E^4$.

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We can expect that the twistor space is useful for studying conformal maps. We implement this idea in this paper.

The twistor theory serves an important role in the study of surfaces in four-dimensional Riemannian manifolds, in particular, minimal surfaces (see [3] and [8], for example). Quaternionic holomorphic geometry [4] is another useful theory for studying the surfaces in the special case. As shown in Section 5 there is a close relation between the theory of twistor lifts and quaternionic holomorphic geometry. The use of twistor lifts has the advantage that it induces a factorization of the differential of a conformal map into a factor which describes intrinsic geometry of a surface and other factors which describe extrinsic geometry of a surface. More precisely, in Section 3 we show that the differential of a conformal map is factored by two maps into Sp(1) and a (1,0)-form locally. We refer to it as a canonical factorization. We note that the (1,0)-form gives the intrinsic Riemannian invariant of a conformal map. The maps into Sp(1) give the generalized Gauss map of a surface.

Our approach is motivated by researches of spinor structures for surfaces in three or four-space. If the ambient space is $\mathbb{E}^3$, Theorem 3.1.1 in [10] provides a canonical way for determining a spinor structure of $f^*\mathcal{T}\mathbb{E}^3$ for a given conformal immersion $f$. However, this way does not work for immersions into $\mathbb{E}^4$. In [1], fixing a spinor structure of the tangent bundle of a Riemann surface and that of the normal bundle, a representation formula for immersions into four-dimensional space form is obtained. In this paper, we fix a spinor structure of the tangent bundle of $\mathbb{E}^4$. Therefore we begin our discussion with recalling the twistor space of $\mathbb{E}^4$ after Salamon [18] through the spinor structure. This also leads that the relation between the theory of twistor lifts and quaternionic holomorphic geometry is clarified.

In Section 3 we define a conformal map using a map from a Riemann surface to the twistor space and explain that this definition coincides with the definition of a conformal map in [17]. Among twistor lifts of conformal maps, we distinguish a special lift that we refer to as a canonical lift. The canonical lift induces the canonical factorization of a differential of a conformal map. In section 4 we have a relation between the area of a conformal map and its canonical lifts. In Section 5 the relation between the theory of twistor lifts and quaternionic holomorphic geometry is given. In the last section, we give an application of the the canonical factorization to super-conformal maps.

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2. Preliminaries

Throughout this paper, all manifolds and maps are assumed to be smooth. We review the twistor space of $\mathbb{E}^4$ after Salamon [18].

2.1. Elementary representation theory. Let $V$ be a real four-dimensional vector space and let $\langle \cdot , \cdot \rangle$ be an inner product on $V$. We denote the norm of $v \in V$ by $|v|$. Let $(J_1, J_2, J_3)$ with $J_1 \circ J_2 = J_3$ be a hypercomplex structure of $V$ such that $\langle \cdot , \cdot \rangle$ is Hermitian with respect to $(J_1, J_2, J_3)$.

We consider $V$ to be a right quaternionic module by

$$v(a_0 + a_1i + a_2j + a_3k) = va_0 - (J_1v)a_1 - (J_2v)a_2 - (J_3v)a_3$$
for $v \in V$ and $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

Fix $v_0 \in V$ with $|v_0| = 1$. Define quaternionic linear automorphisms $\tilde{J}_1$, $\tilde{J}_2$, and $\tilde{J}_3$ of $V$ by $\tilde{J}_n v_0 = -J_n v_0$ ($n = 1, 2, 3$). Then $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$ is a hypercomplex structure of $V$ with $\tilde{J}_1 \circ \tilde{J}_2 = \tilde{J}_3$ such that $\langle \ , \rangle$ is Hermitian with respect to $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$. We consider $V$ to be a left quaternionic module by

$$(a_0 + a_1 i + a_2 j + a_3 k)v = a_0 v + a_1 \tilde{J}_1 v + a_2 \tilde{J}_2 v + a_3 \tilde{J}_3 v.$$ 

We note that $i v_0 = v_0 i$, $j v_0 = v_0 j$ and $k v_0 = v_0 k$. Then, $V$ is isomorphic to the non-commutative associative algebra of all quaternions $\mathbb{H}$. We often identify $V$ with $\mathbb{H}$ in this manner. The vector $v_0 \lambda = \lambda v_0 \in V$ with $\lambda \in \mathbb{H}$ is identified with $\lambda \in \mathbb{H}$. The set $U = \{v_0 \lambda : \lambda \in \mathbb{C}\}$ is identified with the set of all complex numbers $\mathbb{C}$.

We obtain an orthogonal decomposition of $V$ by real vector spaces

$$V = V_c \oplus V_c^\perp, \ V_c = \{v_0 r : r \in \mathbb{R}\}.$$ 

Then, $V_c$ is identified with the set $\text{Re} \, \mathbb{H}$ of all real parts of quaternions and $V_c^\perp$ is identified with the set $\text{Im} \, \mathbb{H}$ of all imaginary parts of quaternions. We denote the quaternionic conjugate of $v \in V \cong \mathbb{H}$ by $\overline{v}$.

If we consider $V$ to be a right complex vector space with the complex structure $-J_1$, then we denote it by $V_+$. We obtain $V_+ = U \oplus k U$. If we consider $V$ to be as a left complex vector space with complex structure $-\tilde{J}_1$, then we denote it by $V_-$. We obtain $V_- = U \oplus j U$.

For any $v \in V$ with $|v| = 1$, a quadruplet $(v, -J_1 v, -J_3 v, -J_2 v)$ is an orthonormal basis of $V$. The ordered orthonormal basis

$$\bar{v}_0 := (v_0, -J_1 v_0, -J_3 v_0, -J_2 v_0)$$

determines an orientation. When we identify $V$ with $\mathbb{H}$, the ordered basis $\bar{v}_0$ is identified with $(1, i, k, j)$. The set of all orthonormal ordered bases $(v_1, v_2, v_3, v_4)$ with the same orientation as $\bar{v}_0$ constitutes the special orthogonal group $\text{SO}(4)$ by the relation

$$(v_1, v_2, v_3, v_4) = (v_0, -J_1 v_0, -J_3 v_0, -J_2 v_0) \beta, \ \beta \in \text{SO}(4).$$

Let $I$ be an orthogonal complex structure of $V$. The set of all orthonormal ordered bases of the form $(v_1, -I v_1, v_2, -I v_2)$ with the same orientation as $\bar{v}_0$ constitutes a subgroup of $\text{SO}(4)$, which is isomorphic to the unitary group $U(2)$ by

$$(v_1, -I v_1, v_2, -I v_2) = (v_0, -J_1 v_0, -J_3 v_0, -J_2 v_0) \beta, \ \beta \in U(2).$$

The set of all orthonormal ordered bases of the form $(v_1, -J_1 v_1, -J_3 v_0, -J_2 v_0)$ with the same orientation as $\bar{v}_0$ constitutes a subgroup of $U(2)$, which is isomorphic to the unitary group $U(1)$ by

$$(v_1, -J_1 v_1, -J_3 v_0, -J_2 v_0) = (v_0, -J_1 v_0, -J_3 v_0, -J_2 v_0) \beta, \ \beta \in U(1).$$

The group $U(1)$ is isomorphic to the set of all unit complex numbers. Similarly, the set of all orthonormal ordered bases of the form $(v_0, -J_1 v_0, v_2, -J_1 v_2)$ with the same orientation as $\bar{v}_0$ constitutes a subgroup of $U(2)$, which is isomorphic to the unitary group $U(1)$ by

$$(v_0, -J_1 v_0, v_2, -J_1 v_2) = (v_0, -J_1 v_0, -J_3 v_0, -J_2 v_0) \beta, \ \beta \in U(1).$$
The set of all orthonormal ordered bases of the form \((v, -J_1v, -J_3v, -J_2v)\) with the same orientation as \(\tilde{v}_0\) constitutes a subgroup of \(SO(4)\), which is isomorphic to the symplectic group \(Sp(1)\) by
\[
(v, -J_1v, -J_3v, -J_2v) = (v_0, -J_1v_0, -J_3v_0, -J_2v_0) \beta, \quad \beta \in Sp(1).
\]
The symplectic group \(Sp(1)\) is isomorphic to the group of all unit quaternions. A double-covering \(\phi: Sp(1) \times Sp(1) \to SO(4)\) is defined by
\[
(\alpha_1, \alpha_2) = a(-J_1v_0)b_{-1}^{-1}, \quad a(-J_3v_0)b_{-1}^{-1}, \quad a(-J_2v_0)b_{-1}^{-1}
\]
where \(a, b \in Sp(1)\). We fix a complex line \(L\) into \(Sp(1)\). Then, \(Sp(1)\) is a double-covering \(Sp(1) \to Sp(1)\) isomorphic to \(SO(3)\). The map \(\phi\) composed with the inclusion \(a \mapsto (a, a)\) of \(Sp(1)\) into \(Sp(1) \times Sp(1)\) is a double-covering \(Sp(1) \to SO(3)\).

The maps \(\phi_{U(1) \times Sp(1)} \) and \(\phi_{Sp(1) \times U(1)}\) are double-coverings of \(U(2)\). Selecting the double-covering \(\phi_{U(1) \times Sp(1)}\), we obtain
\[
SO(4)/U(2) \cong (Sp(1) \times Sp(1))/(U(1) \times Sp(1)) = Sp(1)/U(1).
\]
We fix a complex line \(L = \{v_0 \lambda : \lambda \in \mathbb{C}\}\) in \(V_+\). Let \(a = a_0 + a_1i + a_2k + a_3j\). Then, \(aL = \{v_0(a_0 + a_1i) - (J_3v_0)(a_2 + a_3j)\lambda : \lambda \in \mathbb{C}\}\) is a complex line. Let \((W_0, W_1)\) be a holomorphic coordinate of \(V_+\) such that \(V_+ = \{v_0W_0 - (J_3v_0)W_1 : W_0, W_1 \in \mathbb{C}\}\) and let \([W_0, W_1]\) be the homogeneous coordinate of \(\mathbb{P}(V_+).\) Then, \(aL = [a_0 + a_1i, a_2 + a_3j]\). For \(a \in Sp(1)\), we denote \(aU(1) \in Sp(1)/U(1)\) by \(a^\beta\). The correspondence \(a^\beta \mapsto aL\) for any \(a \in Sp(1)\) identifies \(Sp(1)/U(1)\) with \(\mathbb{P}(V_+).\)

Consider \(Sp(1)\) as the three-dimensional sphere \(S^3 = \{a \in \mathbb{H} : |a| = 1\}\). Let \(S^2\) be the two-dimensional sphere \(\{a \in \text{Im} \mathbb{H} : |a| = 1\}\). We obtain the Hopf map \(H: S^3 \to S^2, H(a) = aia^{-1}\) of the Hopf fibration. The map \(\Phi_+: Sp(1)/U(1) \to S^2\) defined by \(\Phi_+(a^\beta) = aia^{-1}\) identifies \(Sp(1)/U(1)\) with \(S^2\).

There is a bijective map \(I_+\) from \(Sp(1)/U(1)\) to the set of all orthogonal complex structures of \(V\) such that
\[
(v_1, -I_+(a^\beta)v_1, v_2, -I_+(a^\beta)v_2) = (v_0, -J_1v_0, -J_3v_0, -J_2v_0) \phi(a, b).
\]
Similarly, selecting the double-covering \(\phi_{U(1) \times Sp(1)}\), we obtain
\[
SO(4)/U(2) \cong (Sp(1) \times Sp(1))/(Sp(1) \times U(1)) = Sp(1)/U(1).
\]
For \(b^{-1} \in Sp(1)\), we denote \(U(1)b^{-1} \in Sp(1)/U(1)\) by \((b^{-1})^\sharp\). The correspondence \((b^{-1})^\sharp \mapsto \text{Lab}^{-1}\) for any \(b^{-1} \in Sp(1)\) identifies \(Sp(1)/U(1)\) with \(\mathbb{P}(V_-).\) The map \(\Phi_-: Sp(1)/U(1) \to S^2\) defined by \(\Phi_-((b^{-1})^\sharp) = bb^{-1}\) identifies \(Sp(1)/U(1)\) with \(S^2\). The bijective map \(I_-\) from \(Sp(1)/U(1)\) to the set of all orthogonal complex structures of \(V\) exists such that
\[
(v_1, -I_-((b^{-1})^\sharp)v_1, v_2, -I_-((b^{-1})^\sharp)v_2) = (v_0, J_1v_0, J_3v_0, J_2v_0) \phi(a, b).
\]
Then,
\[
(v_1, -Iv_1, v_2, -Iv_2) = (v_0, J_1v_0, J_3v_0, J_2v_0) \phi(a, b),
\]
\[
-I = -I_+(a^\beta) = -I_-((b^{-1})^\sharp).
\]
We note that
\[
-I_-((b^{-1})^\sharp)v = vbb^{-1}, \quad v \in V.
\]
For $\beta \in \text{Sp}(1)/U(1)$ with $\beta = (b^{-1})^4$, we exchange the notation $I_-(\beta)$ with $I^-_\beta$:

$$-I^-_\beta v = v\Phi_-(\beta).$$

For $\alpha \in \text{Sp}(1)/U(1)$, define the orthogonal complex structure $I^\alpha$ by

$$-I^\alpha_+ v = -\Phi_+(\alpha)v.$$

Then,

$$-I^\alpha_+ v_1 = Iv_1, \quad -I^\alpha_+ v_2 = -Iv_2.$$

Let $V_1$ be the subspace of $V$ spanned by $v_1$ and $-Iv_1$ and let $V_2$ be the subspace of $V$ spanned by $v_2$ and $-Iv_2$. Then,

$$V_1 = \{v \in V : I^-_\beta v = -I^\alpha_+ v\}, \quad V_2 = \{v \in V : I^\alpha_+ v = I^\beta_+ v\},$$

$$V = V_1 \oplus V_2.$$

2.2. Twistor space. Let $TV$ be the tangent bundle of $V$ and let $T_0V$ be the tangent space of $V$ at $v$. We identify $T_0V$ with $V$ in the usual manner. We denote the integrable hypercomplex structures and the Riemannian metric induced from $V$ by the same symbols: $(J_1, J_2, J_3), (J_1, J_2, J_3)$ and $(\langle , \rangle)$ respectively. Then, $(\langle , \rangle)$ is Hermitian with respect to $(J_1, J_2, J_3)$.

Let $\phi = (A_0, -J_1A_0, -J_2A_0, -J_3A_0)$ be an orthonormal ordered frame that corresponds to $v_0$. Then, $iA_0 = A_0i, jA_0 = A_0j$ and $kA_0 = A_0k$. The set of all orthonormal ordered frames $(A_1, A_2, A_3, A_4)$ with the same orientation as $\phi$ constitutes a principal $SO(4)$-bundle $P$ over $V$. The set of all orthonormal ordered frames of the form $(A_1, -IA_1, A_2, -IA_2)$ with orthogonal almost complex structure $I$ of $TV$ and the same orientation as $\phi$ constitutes a principal $U(2)$-bundle $Q$ over $V$. Then, $Q$ is identified with a section of the fiber bundle

$$\pi^V: Z \rightarrow V, \quad Z = P \times_{SO(4)} SO(4)/ U(2) = V \times SO(4)/ U(2).$$

The bundle $Z$ is referred to as the twistor space of $V$. The set of all sections of $\pi^V$ is considered to be the set of all almost complex structures of $V$. For an orthogonal almost complex structure $I$ of $V$, we obtain the orthonormal ordered frame $(A_1, -IA_1, A_2, -IA_2)$, which corresponds to the map $(a, b): V \rightarrow \text{Sp}(1) \times \text{Sp}(1)$ by the equation

$$(A_1, -IA_1, A_2, -IA_2) = (A_0, -J_1A_0, -J_3A_0, -J_2A_0)\phi(a, b),$$

$$-I = -I_+(\alpha) = -I_-(\beta), \quad \alpha = a^2, \quad \beta = (b^{-1})^2.$$

As we stated in Section 1, we consider the twistor space of $\mathbb{E}^4$ through the spinor structure. we consider the spinor structure of $\mathbb{E}^4$. Let $P$ be the spinor structure of $P$. Then, $\hat{P}$ is the $\text{Sp}(1) \times \text{Sp}(1)$-bundle, which is the lift of $P$ by $\phi$. Selecting $\phi_{U(1) \times \text{Sp}(1)}$ for the double covering of $U(2)$ and considering $V$ to be the right complex vector space $V_+$, the twistor space is identified with the fiber bundle

$$\hat{\pi}^V_+: \hat{Z}_+ \rightarrow V,$$

$$\hat{Z}_+ = \hat{P} \times_{\text{Sp}(1) \times \text{Sp}(1)} (\text{Sp}(1) \times \text{Sp}(1))/(U(1) \times \text{Sp}(1))$$

$$= V \times \text{Sp}(1)/ U(1) \cong V \times \mathbb{P}(V_+).$$
Let \( J_{\mathcal{P}(V_+)} \) be the complex structure of \( \mathcal{P}(V_+) \). An integrable complex structure \( J_{\mathcal{Z}^+} \) of \( \mathcal{Z}^+ \) is defined by
\[
J_{\mathcal{Z}^+}(A, S) = (-I_+(\alpha)A, J_{\mathcal{P}(V_+)}S),
\]
for \((v, \alpha) \in V \times \mathcal{P}(V_+), (A, S) \in T_{(v, \alpha)}(V \times \mathcal{P}(V_+)) \cong T_v V \times T_\alpha \mathcal{P}(V_+).\]

Selecting \( \phi|_{Sp(1) \times U(1)} \) for the double covering of \( U(2) \) and considering \( V \) to be the left complex vector space \( V_- \), the twistor space is also identified with the fiber bundle
\[
\hat{\pi}^V: \mathcal{Z}_- \to V, \]
\[
\tilde{\mathcal{Z}}_- = \tilde{P} \times_{Sp(1) \times Sp(1)} (Sp(1) \times Sp(1))/(Sp(1) \times U(1)) = V \times Sp(1)/U(1) \cong V \times \mathcal{P}(V_-).
\]
Let \( J_{\mathcal{P}(V_-)} \) be the complex structure of \( \mathcal{P}(V_-) \). Using a similar discussion, we obtain the complex structure \( J_{\mathcal{Z}^-} \) of \( \mathcal{Z}^- \), which is defined by
\[
J_{\mathcal{Z}^-}(A, S) = (-I_-(\beta)A, J_{\mathcal{P}(V_-)}S),
\]
for \((v, \beta) \in V \times \mathcal{P}(V_-), (A, S) \in T_{(v, \beta)}(V \times \mathcal{P}(V_-)) \cong T_v V \times T_\beta \mathcal{P}(V_-).\]

### 3. Conformal maps

We explain conformal maps by the twistor setting. Let \( \Sigma \) be a Riemann surface with the complex structure \( J_\Sigma \). Recall that a holomorphic function on \( \Sigma \) is a conformal map \( h: \Sigma \to \mathbb{C} \) such that \( dh \circ J_\Sigma = i dh = dh i \). We consider a conformal map from \( \Sigma \) to \( V \) as an analog of a holomorphic function. For the map \( f: \Sigma \to V \), denote by \( T_f \) the set of all maps \( \mu \) from \( \Sigma \) such that \( \mu(p) \) is an orthogonal complex structure of \( T_{f(p)} V \) for each \( p \in \Sigma \).

**Definition 1.** We refer to a non-constant map \( f: \Sigma \to V \) a **conformal map** if there exists a map \( I^\Sigma \in T_f \) such that \( df \circ J_\Sigma = -I^\Sigma df \).

If a conformal map \( f \) is not an immersion at \( p \), then \( df \) is the zero map at \( p \) and \( p \) is a branch point of \( f \).

Let \( T\Sigma \) be the tangent bundle of \( \Sigma \) and let \( T_p \Sigma \) be the tangent space of \( \Sigma \) at \( p \). Then, the tangent space of \( f \) at \( p \) is \( df(T_p \Sigma) \). Denote the normal space of \( f \) at \( p \) by \( (df(T_p \Sigma))^\perp \). The twistor space of \( V \) explains conformal maps as follows.

**Theorem 1.** If \( f: \Sigma \to V \) is a conformal map with \( df \circ J_\Sigma = -I^\Sigma df \) for a map \( I^\Sigma \in T_f \), then maps \( \alpha: \Sigma \to Sp(1)/U(1) \) and \( \beta: \Sigma \to Sp(1)/U(1) \) exist such that \( I^\Sigma = I_+(\alpha) = I_-(\beta) \).

At each point \( p \) of \( \Sigma \), the open set \( U \) including \( p \), local lifts \( \alpha: U \to Sp(1) \) and \( b: U \to Sp(1) \) for \( \alpha \) and \( \beta \), respectively, and a complex \((1,0)\)-form \( \eta \) on \( U \) exist such that
\[
df = ak \eta b^{-1}.
\]

**Proof.** We only have to prove the theorem for a point in which \( f \) is an immersion.

Assume that \( f \) is an immersion at \( p \in \Sigma \). Because \( I^\Sigma \) is an orthogonal complex structure of \( V \), the maps \( \alpha: \Sigma \to Sp(1)/U(1) \) and \( \beta: \Sigma \to Sp(1)/U(1) \) exist such that \( I^\Sigma = I_+(\alpha) = I_-(\beta) \) per the discussion in Section 2. Because \( df(T_p \Sigma) \) is
preserved by $I^\Sigma(p)$, we may assume that the existence the open set $U$, including $p$, and the ordered orthogonal local frame of $TV$ on $U$ of the form

$$(A_1, -I^\Sigma A_1, A_2, -I^\Sigma A_2)$$

such that $df(T\Sigma)$ is framed by $A_2$ and $-I^\Sigma A_2$. The maps $a: U \to \text{Sp}(1)$ and $b: U \to \text{Sp}(1)$ exist such that

$$(A_1, -I^\Sigma A_1, A_2, -I^\Sigma A_2) = (A_0, -J_1 A_0, -J_3 A_0, -J_2 v_0) \phi(a, b),
\quad a^3 = \alpha, \quad (b^{-1})^t = \beta.$$

Because $I^\Sigma = I^\Sigma_+ = I^\Sigma_-$ on $df(T_q \Sigma)$ for each $q \in U$, we obtain

$$df \circ J_\Sigma = -aia^{-1} df = df b b^{-1}.$$

Because

$$a^{-1} (df \circ J_\Sigma) b = -ia^{-1} df b = a^{-1} df bi,$$

the complex $(1,0)$-form $\eta$ exists such that $a^{-1} df b = k \eta$. Therefore, $df = ak \eta b^{-1}$. □

\[\begin{array}{c}
\text{Sp}(1) \overset{b}{\longrightarrow} \text{Sp}(1)/U(1) \\
H \longrightarrow \Sigma \quad \alpha \quad \beta \\
S^2 \quad (N = -aia^{-1}) \quad \Sigma \\
\longrightarrow \hat{Z}_+ \subset \mathbb{R}^4 \times (\text{Sp}(1)/U(1)) \quad \hat{f}_+ \\
\longrightarrow \mathbb{R}^4 \quad f
\end{array}\]

Fig 1: conformal map and twistor space

Because $\phi(a, a)$ preserves $V^\perp_c$ for each $a$, we immediately obtain a three-dimensional version of Theorem 1.

**Corollary 1.** If $f: \Sigma \to V^\perp_c$ is a conformal map with $df \circ J_\Sigma = -I^\Sigma df$ for the map $I^\Sigma \in \Upsilon_f$, the maps $\alpha: \Sigma \to \text{Sp}(1)/U(1)$ and $\beta: \Sigma \to \text{Sp}(1)/U(1)$ exist such that

$$I^\Sigma = I_+ (\alpha) = I_- (\beta).$$

At each point $p$ of $\Sigma$, the open set $U$, including $p$, a map $a: U \to \text{Sp}(1)$ with $a^3 = \alpha$ and $(a^{-1})^t = \beta$ and the complex $(1,0)$-form $\eta$ on $U$ exist such that

$$df = ak \eta a^{-1}.$$

We review a holomorphic map by this formulation. A holomorphic map $f: \Sigma \to V$ is a conformal map with $df \circ J_\Sigma = -J_1 df = df i$ and a holomorphic map $f: \Sigma \to V$ is a conformal map with $df \circ J_\Sigma = -J_1 df = -i df$.

An orthogonal almost complex structure of $V$ is preserved by a conformal transformation of $V$. Thus to analyze a conformal map by its lift to the twistor space is a natural idea. We distinguish the following lifts:

**Definition 2.** Let $f: \Sigma \to V$ be a conformal map with $df \circ J_\Sigma = -I^\Sigma df$ and $\alpha: \Sigma \to \text{Sp}(1)/U(1)$ and $\beta: \Sigma \to \text{Sp}(1)/U(1)$ be maps with $I^\Sigma = I_+ (\alpha) = I_- (\beta)$. We refer to $\hat{f}_+ = (f, \alpha): \Sigma \to \hat{Z}_+$ as a left canonical lift of $f: \Sigma \to V$ and $\hat{f}_- = (f, \beta): \Sigma \to \hat{Z}_-$ as a right canonical lift of $f: \Sigma \to V$.

A left or right canonical lift is referred to a twistor lift in [5] and [8].
Definition 3. Let $f: \Sigma \to V$ be a conformal map with left canonical lift $(f, \alpha)$ and right canonical lift $(f, \beta)$. Assume that $a: \Sigma \to \text{Sp}(1)$ and $b: \Sigma \to \text{Sp}(1)$ are maps with $a^* = \alpha$ and $(b^{-1})^\sharp = \beta$. We refer to $df = ak \eta b^{-1}$ with the complex $(1,0)$-form $\eta$ on $\Sigma$ as a canonical factorization of $df$ by $a$, $b^{-1}$ and $\eta$.

4. Local conformal maps

In this section, we investigate properties of local conformal maps by a canonical factorization.

At first, we assume that $\Sigma$ is a simply-connected open subset of $\mathbb{C}$. We denote the standard holomorphic coordinate of $\mathbb{C}$ by $z$. Then, a $(1,0)$-form is $cdz$ for a complex function $c$. Then, $\eta = -ka^{-1} df b = cdz$ for a complex function $c$. Theorem 1 delivers a method of construction for a conformal map.

Lemma 1. If the maps $a, b: \Sigma \to \text{Sp}(1)$ and the complex $(1,0)$-form $\eta$ on $\Sigma$ satisfies

\begin{equation}
(1) \quad da \wedge k \eta b^{-1} + ak d\eta b^{-1} - ak \eta \wedge db^{-1} = 0,
\end{equation}

a conformal map $f: \Sigma \to V$ exists with $df = ak \eta b^{-1}$.

Proof. Differentiating the one-form $ak \eta b^{-1}$, we obtain

\[ d(ak \eta b^{-1}) = da \wedge k \eta b^{-1} + ak d\eta b^{-1} - ak \eta \wedge db^{-1}. \]

If the maps $a, b: \Sigma \to \text{Sp}(1)$ and the complex $(1,0)$-form $\eta$ satisfies the equation (1), then the map $f: \Sigma \to V$ exists with $df = ak \eta b^{-1}$. Because $df \circ J_{\Sigma} = -aia^{-1} df = df b b^{-1}$, the map $f$ is conformal.

In the following section, we assume that the conformal map $f$ has a canonical factorization $df = ak \eta b^{-1}$. The maps $a$, $b^{-1}$ and the one-form $\eta$ of a canonical factorization $df = ak \eta b^{-1}$ are not uniquely determined. If $u$ and $v$ are maps from $\Sigma$ to $U(1)$, then $(au)^\sharp = a^\sharp$ and $(vb^{-1})^\sharp = (b^{-1})^\sharp$. Then,

\[ df = ak \eta b^{-1} = (au)(u^{-1}k \eta v^{-1})(bv^{-1})^{-1} = (au)k(u \eta v^{-1})(bv^{-1})^{-1}. \]

Let $\Omega^{(1,0)}$ be the set of all complex one-forms of type $(1,0)$ on $\Sigma$. Then, $U(1)$ acts on $\Omega^{(1,0)}$ by multiplication. For a conformal map $f$ with canonical factorization $df = ak \eta b^{-1}$, we obtain the unique triplet $(a^\sharp, (b^{-1})^\sharp, [\eta])$, which consisting of $a^\sharp$, $(b^{-1})^\sharp: \Sigma \to \text{Sp}(1)/U(1)$ and $[\eta] \in \Omega^{(1,0)}/U(1)$.

By Lemma 1 we obtain a representation formula for a conformal map $f: \Sigma \to V$ with the canonical factorization $df = ak \eta b^{-1}$:

\[ f(p) = \int_{\gamma} ak \eta b^{-1} + f(p_0), \]

\[ da \wedge k \eta b^{-1} + ak d\eta b^{-1} - ak \eta \wedge db^{-1} = 0, \]

$a, b: \Sigma \to \text{Sp}(1), \ \eta \circ J_{\Sigma} = i \eta = \eta i$.

Here, $\gamma$ is a path from $p_0$ to $p$. The zeros of $\eta$ are the branch points of $f$.

We fix a canonical factorization $df = ak \eta b^{-1}$. If a $(1,0)$-form $\eta$ is nowhere vanishing, then $\eta$ is a global section of a real line bundle $l(\eta) = \cup_{p \in \Sigma} \{ r \eta_p : r \in \mathbb{R} \}$ with the projection $\pi_{l(\eta)}: l(\eta) \to \Sigma$, $\pi_{l(\eta)}(r \eta_p) = p$. 

Lemma 2. Let $f: \Sigma \to V$ be a conformal immersion with canonical factorization $df = ak\eta b^{-1}$. Let $\tilde{f}$ be an orientation-preserving conformal transform of $f$ in $V$. Then, the canonical factorization $d\tilde{f} = \tilde{a}k\tilde{\eta}b^{-1}$ exists such that $\tilde{a}^{-1}d\tilde{a} = a^{-1}da$, $\tilde{b}^{-1}d\tilde{b} = b^{-1}db$ and $l(\eta) = l(\tilde{\eta})$.

If $\tilde{f}$ is a Euclidean motion of $f$, then the canonical factorization $d\tilde{f} = \tilde{a}k\tilde{\eta}b^{-1}$ exists with $\tilde{\eta} = \eta$.

Proof. The map

$$\tilde{f} = \lambda f \mu^{-1} + \nu$$

is a conformal transform of $f$. The differential of $\tilde{f}$ is

$$d\tilde{f} = \lambda df \mu^{-1} = \lambda ak \eta b^{-1} \mu^{-1} = \frac{\lambda}{|\lambda|} ak \frac{|\lambda|}{|\mu|} \eta b^{-1} \frac{\mu^{-1}}{|\mu|},$$

Thus the canonical factorization $d\tilde{f} = \tilde{a}k\tilde{\eta}b^{-1}$ with

$$\tilde{a} = \frac{\lambda}{|\lambda|} a, \tilde{b} = \frac{\mu}{|\mu|} b, \tilde{\eta} = \frac{|\lambda|}{|\mu|} \eta$$

satisfies $\tilde{a}^{-1}d\tilde{a} = a^{-1}da$, $\tilde{b}^{-1}d\tilde{b} = b^{-1}db$ and $l(\tilde{\eta}) = l(\eta)$.

A Euclidean motion $\tilde{f}$ of $f$ is

$$\tilde{f} = \lambda f \mu^{-1} + \nu$$

Then, we obtain the factorization

$$d\tilde{f} = \lambda df \mu^{-1} = (\lambda a)k \eta (\mu b)^{-1}.$$

Because $|\lambda a| = |\mu b| = 1$, this result is a canonical factorization with $\tilde{\eta} = \eta$. \hfill \Box

Because we can fix the complex $(1,0)$-from $\eta$ under Euclidean motions, the one-form $\eta$ includes Riemannian geometric information of $f$. Because the first fundamental form of $f$ is

$$\frac{1}{2} (df \otimes_R df + df \otimes_R df) = \frac{1}{2} (\eta \otimes_R \eta a^{-1} + b \eta \otimes_R \eta b^{-1}),$$

the $(1,0)$-one-form $\eta$ can generally describe a part of the Riemannian geometric properties. If the image of $f$ is included in $V_c^\perp$, then the $(1,0)$ form completely explains the Riemannian geometric properties.

Lemma 3. If $df = ak\eta b^{-1}$ is a canonical factorization of the conformal map $f: \Sigma \to V_c^\perp$, then the first fundamental form is

$$\frac{1}{2} (\eta \otimes_R \eta + \eta \otimes_R \eta).$$

Proof. If the codomain of $f$ is contained in $V_c^\perp$, we may assume that $a = b$. Then, the first fundamental form is

$$\frac{1}{2} (a \eta \otimes_R \eta a^{-1} + a \eta \otimes_R \eta a^{-1}) = \frac{1}{2} a (\eta \otimes_R \eta + \eta \otimes_R \eta a^{-1}) = \frac{1}{2} (\eta \otimes_R \eta + \eta \otimes_R \eta).$$

\hfill \Box
If the codomain of \( f \) is not contained in \( V_c^\perp \), then \( \eta \) is insufficient for explaining the Riemannian geometric properties of \( f \). However, we observe that the area of \( f \) is described by \( \eta \) as follows:

We denote the \( L^2 \)-norm of a one-form \( \omega \) by \( \| \omega \|_\Sigma \):

\[
\| \omega \|_\Sigma = \left( -\int_\Sigma \omega \wedge (\overline{\omega} \circ J_\Sigma) \right)^{1/2}.
\]

In the space of all square integrable one-forms, an inner product is defined as

\[
\langle \langle \omega_1, \omega_2 \rangle \rangle_\Sigma = -\frac{1}{2} \int_\Sigma (\omega_1 \wedge \overline{\omega_2} \circ J_\Sigma + \omega_2 \wedge \overline{\omega_1} \circ J_\Sigma).
\]

For the conformal map \( f : \Sigma \to V \), we denote the area element of \( f \) by \( dA \) and denote the area of \( f \) by \( A(f) \). Let \( z = x + iy \) be a local holomorphic coordinate of \( \Sigma \) such that \((x, y)\) is a local real coordinate. Then,

\[
dA = \sqrt{|f_x|^2|f_y|^2 - \langle f_x, f_y \rangle} \ dx \wedge dy = |f_x||f_y| \ dx \wedge dy = -\frac{1}{2} df \wedge (df \circ J_\Sigma),
\]

\[
2A(f) = \|df\|_\Sigma^2.
\]

We recall quaternionic holomorphic geometry (see also Corollary \( \text{2} \) in the next section). Assume that there exist maps \( N, \tilde{N} : \Sigma \to \text{Im} \mathbb{H} \cap \text{Sp}(1) \) such that \( df \circ J_\Sigma = N \ dx = -df \tilde{N} \). Then

\[
dA = \frac{1}{2} df \wedge df \tilde{N} = -\frac{1}{2} df \wedge \tilde{N} \ df = \frac{1}{2} N \ df \wedge df \tilde{N}.
\]

We see that the area involves the maps \( N \) and \( \tilde{N} \). The maps \( N \) and \( \tilde{N} \) are written locally as \( N = -a_i a^{-1}_i \) and \( \tilde{N} = -b_i b^{-1}_i \). Hence it is natural to expect that the maps \( a \) and \( b \) are involved in the area. However, we have the following formula for the area which does not includes \( a \) and \( b \).

**Lemma 4.** Let \( f : \Sigma \to V \) be a conformal map with the canonical factorization \( df = ak \eta b^{-1} \). Then,

\[
2A(f) = \|\eta\|_\Sigma^2.
\]

**Proof.** The area element of \( f \) is

\[
dA = -\frac{1}{2} ak \eta b^{-1} \wedge (ak \eta b^{-1} \circ J_\Sigma) = -\frac{1}{2} \eta \wedge (\eta \circ J_\Sigma).
\]

Thus, the lemma holds. \( \Box \)

**5. Quaternionic Holomorphic Geometry**

We collect the relation between twistor lifts and quaternionic holomorphic geometry. We have a relation among the area of a conformal map, that of its Darboux transform and that of its canonical lift. Considering the spinor structure in Section \( \text{2.2} \), we identify the twistor space of \( V \) with \( \bar{Z}_\perp \). We arrive to the definition of conformal maps in Pedit and Pinkall [17] by Theorem \( \text{1} \) (see also Fig 1):

**Corollary 2.** A non-constant map \( f : \Sigma \to V \) is a conformal map if and only if the maps \( N, \tilde{N} : \Sigma \to \text{Im} \mathbb{H} \cap \text{Sp}(1) \) exist such that \( df \circ J_\Sigma = N \ df = -df \tilde{N} \).
Proof. Assume that \( f: \Sigma \to V \) is a conformal map with \( df \circ J_\Sigma = -I_J \circ df = -I_+ (\alpha) df = -I_- (\beta) df \). Let \( N = -\Phi_+ (\alpha) \) and \( \tilde{N} = -\Phi_- (\beta) \). By Theorem 1, we obtain \( df \circ J_\Sigma = N \circ df = -df \tilde{N} \).

For the maps \( N, \tilde{N}: \Sigma \to \text{Im} \mathbb{H} \cap \text{Sp} (1) \), we obtain the maps \( \alpha = \Phi_+^{-1} (N) \) and \( \beta = \Phi_-^{-1} (\tilde{N}) \). Let \( a \) and \( b \) be local maps such that \( a^b = \alpha \) and \( (b^{-1})^t = \beta \). Then \( N = -aia^{-1} \) and \( \tilde{N} = -bib^{-1} \). In addition, \( \phi (a, b) \) defines the map \( I^2 \in \Upsilon_f \) such that \( df \circ J_\Sigma = -I_J \circ df \).

We connect the canonical factorization with the Weierstrass representation by Pedit and Pinkall ([17], Theorem 4.3) and obtain a global representation of a differential of a conformal map.

Let \( f: \Sigma \to V \) be a conformal map with left canonical lift \( (f, \alpha) \) and right canonical lift \( (f, \beta) \). Assume that \( df = ak \eta b^{-1} \) is a canonical factorization. Let \( L \) and \( \tilde{L} \) be the trivial right quaternionic line bundles over \( \Sigma \) with fiber \( V \). Define a real bilinear pairing \( \l (\cdot, \cdot) : L \otimes_\mathbb{R} \tilde{L} \to T^* \Sigma \otimes_\mathbb{R} V \) by

\[
(v_\lambda, v_\mu) = \frac{1}{2} df \mu \lambda, \mu \in \mathbb{H},
\]

where \( \otimes_\mathbb{R} \) indicates the tensor product over \( \mathbb{R} \). The quaternionic linear complex structures \( J_L \) and \( J_{\tilde{L}} \) exist for \( L \) and \( \tilde{L} \), respectively, such that

\[
(v_0, v_0) \circ J_\Sigma = df \circ J_\Sigma = -\Phi_+ (\alpha) df = df \Phi_- (\beta) = -\Phi_+ (\alpha) (v_0, v_0) = (v_0, v_0) \Phi_- (\beta) = (v_0, v_0) (v_0, v_0) = (v_0, v_0) (v_0, v_0).
\]

Define the quaternionic holomorphic structures \( D_L \) and \( D_{\tilde{L}} \) for \( L \) and \( \tilde{L} \), respectively, by

\[
D_L (v_\lambda) = v_0 \frac{1}{2} (d\lambda + \Phi_+ (\alpha) d\lambda \circ J_\Sigma),
\]

\[
D_{\tilde{L}} (v_\mu) = v_0 \frac{1}{2} (d\mu + \Phi_- (\beta) d\mu \circ J_\Sigma)
\]

\[
(\lambda, \mu: \Sigma \to \mathbb{H}).
\]

Then,

\[
d(v_\lambda, v_\mu) = \frac{1}{2} (d\lambda - d\lambda \circ J_\Sigma \Phi_+ (\alpha)) \wedge (v_0, v_0) \mu
\]

\[
-(v_0 \lambda, v_0) \wedge \frac{1}{2} (d\mu + \Phi_- (\beta) d\mu \circ J_\Sigma)
\]

Then, for any nowhere-vanishing holomorphic section \( v_0 \lambda \) of \( L \) and \( v_0 \mu \) of \( \tilde{L} \), the pairing \( (v_0 \lambda, v_0 \mu) \) is a closed one-form such that

\[
(v_0 \lambda, v_0 \mu) \circ J_\Sigma = -\Phi_+ (\tilde{\alpha}) (v_0 \lambda, v_0 \mu) = (v_0 \lambda, v_0 \mu) \Phi_- (\tilde{\beta}),
\]

\[
\tilde{\alpha} = \left( \frac{\lambda a}{|\lambda|} \right)^b, \quad \tilde{\beta} = \left( \frac{\mu b^{-1}}{|\mu|} \right)^t.
\]

If \( (v_0 \lambda, v_0 \mu) \) is exact, then a conformal map \( g: \Sigma \to V \) exists with canonical lifts \( (g, \tilde{\alpha}) \) and \( (g, \tilde{\beta}) \):

\[
dg = (v_0 \lambda, v_0 \mu), \quad dg \circ J_\Sigma = -\Phi_+ (\tilde{\alpha}) dg = dg \Phi_- (\tilde{\beta}).
\]

The branch points of \( g \) are the branch points of \( f \).
Let $E_L = \{ \psi \in L : J_L \psi = \psi i \}$ and let $E_L^c = \{ \psi \in \tilde{L} : J_L \psi = \psi i \}$. The bundles $E_L$ and $E_L^c$ are the eigenbundles of $J_L$ and $J_L^c$, respectively.

**Lemma 5.** Let $f : \Sigma \rightarrow V$ be a conformal map with the canonical factorization $df = ak \eta b^{-1}$. Then, $\nu a$ and $\nu b$ are sections of $E_L$ and $E_L^c$ respectively.

**Proof.** Because $df = ak \eta b^{-1}$, we obtain
\[
\eta = -ka^{-1} df b = -ka^{-1} (\nu_0, \nu_0) b.
\]
Then,
\[
\eta \circ J_\Sigma = -k(J_L(\nu_0 a), \nu_0)b = -ka^{-1}(\nu_0, J_L(\nu_0 b))
\]
\[
= -k(-i)(\nu a, \nu_0)b = -ka^{-1}(\nu_0, \nu_0 b)i.
\]
Thus, $J_L(\nu_0 a) = \nu a i$ and $J_L^c(\nu_0 b) = \nu b i$. \qed

If the image of $f$ is contained in $V_c^\perp$, then we obtain $L = \tilde{L}$, $J_L = J_L$ and $a = b$. The eigenbundle $E$ is a spinor bundle of $\Sigma$.

We assume that $\Sigma$ is a simply-connected open subset of $\mathbb{C}$. We recall procedures to construct a conformal map by two given conformal maps. Assume that $f : \Sigma \rightarrow V$ is a nowhere-vanishing conformal map and $g : \Sigma \rightarrow V$ is a conformal map. Pedit and Pinkall showed in [17] that, if $df \circ J_\Sigma = N df$ and $dg \circ J_\Sigma = N dg$, then the map $h = f^{-1}g$ is a conformal map with $dh \circ J_\Sigma = (d^{-1} N f dh$, and if $df \circ J_\Sigma = -df N$ and $dg \circ J_\Sigma = -dg N$, then the map $h = gf^{-1}$ is a conformal map with $dh \circ J_\Sigma = -h f N f^{-1}$. This result is used to construct a (Hamiltonian stationary) Lagrangian surface in [13]. In terms of the canonical lift, we obtain the following lemma:

**Lemma 6.** If the left canonical lift of $f$ is $(f, a^\flat)$ and the left canonical lift of $g$ is $(g, a^\flat)$, then the map $h = f^{-1}g$ is a conformal map with left canonical lift $(h, ([f|f^{-1}a]^\flat)^{\sharp})$. If the right canonical lift of $f$ is $(f, (b^{-1})^\sharp)$ and the right canonical lift of $g$ is $(g, (b^{-1})^\sharp)$, then the map $h = gf^{-1}$ is a conformal map with right canonical lift $(h, ((fb/|f|)^{-1})^\sharp)$.

**Proof.** If the left canonical lift of $f$ is $(f, a^\flat)$, then $df \circ J_\Sigma = -aia^{-1} df$. The differential of $h = f^{-1} g$ is
\[
dh = f^{-1} (-df f^{-1}g + dg).
\]
Thus,
\[
dh \circ J_\Sigma = -f^{-1}aia^{-1} f (f^{-1} (-df f^{-1}g + dg)).
\]
Therefore, the map $h = f^{-1} g$ is a conformal map with left canonical lift $(h, ([f|f^{-1}a]^\flat)^{\sharp})$.

If the right canonical lift of $f$ is $(f, (b^{-1})^\sharp)$, then $df \circ J_\Sigma = df b b^{-1}$. The differential of $h = gf^{-1}$ is
\[
dh = (dg - gf^{-1} df) f^{-1}.
\]
Thus,
\[
dh \circ J_\Sigma = ((dg - gf^{-1} df) f^{-1}) fb b^{-1} f^{-1}.
\]
Therefore, the map $h = gf^{-1}$ is a conformal map with right canonical lift $(h, ((fb/|f|)^{-1})^\sharp)$. \qed

We cite the following lemma, which is subsequently applied.
Lemma 7. Let \( \omega \) be a one-form with values in \( V \) such that \( \omega \circ J_\Sigma = N \omega = -\omega \tilde{N} \) for maps \( N, \tilde{N} : \Sigma \rightarrow \text{Im} \mathbb{H} \cap \text{Sp}(1) \).

If \( \eta \) is a one-form with values in \( V \) such that \( \eta \circ J_\Sigma = \eta N \), then \( \eta \wedge \omega = 0 \). If \( \eta \) is a one-form with values in \( V \) such that \( \eta \circ J_\Sigma = -\tilde{N} \eta \), then \( \omega \wedge \eta = 0 \).

Assume that \( \omega \wedge \eta = 0 \). Then the map \( \hat{\eta} \) is a one-form with values in \( V \) such that \( \eta \wedge \hat{\eta} = 0 \).

We translate Lemma 7 into the language of conformal maps and their canonical factorization.

Lemma 8. Let \( f : \Sigma \rightarrow V \) be a conformal map with left canonical lift \((f, \omega^*)\) and right canonical lift \((f, (b^{-1})^*)\).

If \( h_L : \Sigma \rightarrow V \) is a conformal map with right canonical lift \((h_L, ((ak)^{-1})^*)\), then \( dh_L \wedge df = 0 \). If \( h_R : \Sigma \rightarrow V \) is a conformal map with left canonical lift \((h_R, (bk)^*)\), then \( df \wedge dh_R = 0 \).

Assume that \( f \) is an immersion. If \( h_L : \Sigma \rightarrow V \) is a map such that \( dh_L \wedge df = 0 \), then \( h_L \) is a conformal map with right canonical lift \((h_L, ((ak)^{-1})^*)\). If \( h_R : \Sigma \rightarrow V \) is a map such that \( df \wedge dh_R = 0 \), then \( h_R \) is a conformal map with left canonical lift \((h_R, (bk)^*)\).

Proof. Because \( df \circ J_\Sigma = (-aia^{-1}) df = df (bib^{-1}) \), Lemma 8 is based on Lemma 7.

Definition 4. We refer to the conformal map \( h_R : \Sigma \rightarrow V \) with \( df \wedge dh_R = 0 \) as the right Bäcklund transform of \( f \) and a conformal map \( dh_L : \Sigma \rightarrow V \) with \( dh_L \wedge df = 0 \) as the left Bäcklund transform of \( f \).

In [4], the right Bäcklund transform and the left Bäcklund transform for a Willmore surface are given and called the forward Bäcklund transform and the backward Bäcklund transform respectively. The Bäcklund transforms for a conformal map of a Riemann surface to \( S^4 \) is defined in [12]. Restricting the codomain of a conformal map to \( S^4 \) with one point removed and fixing the stereographic projection from the point, the Bäcklund transforms are reduced to Definition 4 (see [15]). The Darboux transforms of a conformal map of a Riemann surface into \( S^4 \) is defined in [2]. In a similar manner, as the Bäcklund transforms, we obtain a Darboux transform of a conformal map of a Riemann surface into \( E^4 \) (see [15]). In terms of canonical lifts, a Darboux transform is explained as follows.

Lemma 9. Let \( f : \Sigma \rightarrow V \) be a conformal map with left canonical lift \((f, \omega^*)\) and right canonical lift \((f, (b^{-1})^*)\).

If \( h_L : \Sigma \rightarrow V \) is a left Bäcklund transform of \( f \) with \( dh_L = h_L df \) and nowhere vanishing, then the map \( \hat{f}_L := -h_L^{-1} g_L + f : \Sigma \rightarrow V \) is a conformal map with right canonical lift \((\hat{f}_L, ((ah_L^{-1} g_L)^{-1})^*)\).

If \( h_R : \Sigma \rightarrow V \) is a right Bäcklund transform of \( f \) with \( dh_R = h_R df \) and nowhere vanishing, then the map \( \hat{f}_R := -g_R h_R^{-1} + f : \Sigma \rightarrow V \) is a conformal map with left canonical lift \((\hat{f}_R, (g_R h_R^{-1} bk/|g_R h_R^{-1}|)^*)\).

Proof. Because
\[
\begin{align*}
\hat{f}_L &= -h_L^{-1} g_L - h_L^{-1} d_L + df = -dh_L^{-1} g_L, \\
\hat{f}_R &= -d_R h_R^{-1} - g_R dh_R^{-1} + df = -g_R dh_R^{-1},
\end{align*}
\]
the lemma holds.

**Definition 5.** We refer to \( \hat{f}_L \) in Lemma 9 as the left Darboux transform of \( f \) by a left Bäcklund transform \( h_L \) and \( \hat{f}_R \) as the right Darboux transform of \( f \) by a right Bäcklund transform \( h_R \).

We obtain the following relation between the area of a conformal map and the area of its Darboux transform.

**Theorem 2.** Let \( f : \Sigma \to V \) be a conformal map, let \( h_L \) be the right Bäcklund transform of \( f \), let \( \hat{f}_L \) be the right Darboux transform by \( h_L \), let \( h_R \) the right Bäcklund transform of \( f \) and let \( \hat{f}_R \) be the right Darboux transform by \( h_R \). Assume that \( dg_L = h_L df \) and \( dg_R = df h_R \). If \( f, \hat{f}_L, df, d\hat{f}_L, d(h_L^{-1}g_L) \) and \( d(g_R h_R^{-1}) \) are square integrable, then

\[
A(f) + A(\hat{f}_L) - \langle \langle df, d\hat{f}_L \rangle \rangle_\Sigma = \frac{\|d(h_L^{-1}g_L)\|_\Sigma^2}{2},
\]

\[
A(f) + A(\hat{f}_R) - \langle \langle df, d\hat{f}_R \rangle \rangle_\Sigma = \frac{\|d(g_R h_R^{-1})\|_\Sigma^2}{2}.
\]

**Proof.** By the definition of the left Darboux transform, we obtain

\[
d(f - \hat{f}_L) = df - d\hat{f}_L = d(h_L^{-1}g_L).
\]

Thus,

\[
\|df - d\hat{f}_L\|_\Sigma^2 = \|d(h_L^{-1}g_L)\|_\Sigma^2.
\]

Then,

\[
2A(f) + 2A(\hat{f}_L) - 2\langle \langle df, d\hat{f}_L \rangle \rangle_\Sigma = \|d(h_L^{-1}g_L)\|_\Sigma^2.
\]

Then, we obtain the former equality. In a similar manner, we have the latter equality. \( \square \)

### 6. Super-conformal maps

We apply the canonical factorization to super-conformal maps. In our canonical factorization, the \((1, 0)\)-form explains the intrinsic Riemannian geometry of a conformal map and the maps into \( \text{Sp}(1) \) give the generalized Gauss map of a surface. Moreover these are expressed in terms of the multiplication of \( \mathbb{H} \). We obtain an estimate of the area of a super-conformal map.

Prior to our discussion of super-conformal maps, we investigate the map \( N = -aia^{-1} : \Sigma \to S^2 = \text{Im} \mathbb{H} \cap \text{Sp}(1) \) with \( a : \Sigma \to \text{Sp}(1) \). We consider \( S^2 = \text{Im} \mathbb{H} \cap \text{Sp}(1) \) to be the Riemann sphere \( \mathbb{C}P^1 \). Let \( w \) the stereographic projection from the north pole. Then

\[
w \mapsto \frac{2 \text{Re} w}{|w|^2 + 1} + i + \frac{2 \text{Im} w}{|w|^2 + 1} j + \frac{|w|^2 - 1}{|w|^2 + 1} k
\]

is a holomorphic parametrization of \( S^2 \setminus \{k\} \). The following lemma is proven in [16]. We provide an alternate short proof.

**Lemma 10.** The map \( N : \Sigma \to \text{Im} \mathbb{H} \cap \text{Sp}(1) \cong \mathbb{C}P^1 \) is holomorphic if and only if \( dN \circ J_\Sigma = -N dN = dN N \).

The map \( N : \Sigma \to \text{Im} \mathbb{H} \cap \text{Sp}(1) \cong \mathbb{C}P^1 \) is anti-holomorphic if and only if \( dN \circ J_\Sigma = N dN = -dN N \).
Proof. Let \((x, y)\) be a local conformal coordinate of \(\Sigma\) with

\[ J_\Sigma \frac{\partial}{\partial x} = \frac{\partial}{\partial y}. \]

The map \(N: \Sigma \to \text{Im}\mathbb{H} \cap \text{Sp}(1)\) is holomorphic if and only if the vector product \(N \times N_x\) is equal to \(-N_y\). Differentiating \(N^2 = -1\), we obtain \(N dN = -(dN)N\).

Then, \(N\) is holomorphic if and only if \(dN \circ J_\Sigma = -N dN = (dN)N\). Similarly, \(N\) is anti-holomorphic if and only if \(dN \circ J_\Sigma = N dN = -(dN)N\). \(\square\)

This lemma is translated as follows:

**Lemma 11.** Let \(\alpha: \Sigma \to \text{Sp}(1)/U(1)\) be a map and let \(a: \Sigma \to \text{Sp}(1)\) be a map with \(a^3 = \alpha\). Let \(N = -\Phi_+(\alpha)\). The map \(N: \Sigma \to \text{Im}\mathbb{H} \cap \text{Sp}(1) \cong \mathbb{C}P^1\) is holomorphic if and only if the map \(\alpha: \Sigma \to \text{Sp}(1)/U(1) \cong \mathbb{P}(V_+)\) is anti-holomorphic. The map \(N: \Sigma \to \text{Im}\mathbb{H} \cap \text{Sp}(1) \cong \mathbb{C}P^1\) is anti-holomorphic if and only if the map \(\alpha: \Sigma \to \text{Sp}(1)/U(1) \cong \mathbb{P}(V_-)\) is holomorphic.

Let \(\beta: \Sigma \to \text{Sp}(1)/U(1)\) be a map and \(b: \Sigma \to \text{Sp}(1)\) be a map with \((b^{-1})^3 = \beta\). Let \(\tilde{N} = -\Phi_-(\beta)\). A map \(\tilde{N}: \Sigma \to \text{Im}\mathbb{H} \cap \text{Sp}(1) \cong \mathbb{C}P^1\) is holomorphic if and only if \(\beta: \Sigma \to \text{Sp}(1)/U(1) \cong \mathbb{P}(V_-)\) is anti-holomorphic. A map \(\tilde{N}: \Sigma \to \text{Im}\mathbb{H} \cap \text{Sp}(1) \cong \mathbb{C}P^1\) is anti-holomorphic if and only if \(\beta: \Sigma \to \text{Sp}(1)/U(1) \cong \mathbb{P}(V_-)\) is holomorphic.

A variant of this lemma is also proven in \([16]\). We provide an improved proof by the form \(N = -aia^{-1}\).

Proof. The map \(a^3\) is holomorphic if and only if the map \(c: \Sigma \to \mathbb{C} \setminus \{0\}\) exists such that the map \(ac: \Sigma \to \mathbb{P}(V_+)\) is holomorphic. Thus, \(a^3\) is holomorphic if and only if the map \(c: \Sigma \to \mathbb{C} \setminus \{0\}\) exists such that the map \(ac: \Sigma \to V_+\) is holomorphic.

The map \(ac: \Sigma \to V_+\) is holomorphic if and only if \(d(ac) \circ J_\Sigma = d(ac)i\). The differential of \(N = -aia\) is

\[ dN = d(-(ac)i(ac)^{-1}) = (ac)(-(ac)^{-1} d(ac)i + i(ac)^{-1} d(ac))(ac)^{-1}. \]

Thus, if \(ac\) is holomorphic, then \(dN \circ J_\Sigma = -dN N = N dN\). If \(a^3\) is holomorphic, then \(N\) is anti-holomorphic. If \(N\) is anti-holomorphic, then \(dN \circ J_\Sigma = N dN = -dN N\). Thus,

\[ (- (ac)^{-1} d(ac)i + i(ac)^{-1} d(ac)) \circ J_\Sigma = - (ac)^{-1} d(ac)i + i(ac)^{-1} d(ac)i = -i(- (ac)^{-1} d(ac)i + i(ac)^{-1} d(ac)). \]

Therefore, we can select \(c\) such that \(ac\) is holomorphic. Then, \(a^3\) is holomorphic.

Similarly, \(a^3\) is anti-holomorphic if and only if \(N\) is holomorphic.

The map \((b^{-1})^3\) is holomorphic if and only if the map \(c: \Sigma \to \mathbb{C} \setminus \{0\}\) exists such that a map \(c^{-1}b^{-1} = (bc)^{-1}: \Sigma \to V_-\) is holomorphic. The map \((bc)^{-1}: \Sigma \to V_-\) is holomorphic if and only if \(d(bc)^{-1} = -i d(bc)^{-1}\). The differential of \(N = -bib^{-1}\) is

\[ d\tilde{N} = d(-(bc)i(bc)^{-1}) = (bc)(d(bc)^{-1} (bc)i - i d(bc)^{-1} (bc))(bc)^{-1}. \]

Thus, if \((bc)^{-1}\) is holomorphic, then \(d\tilde{N} \circ J_\Sigma = -d\tilde{N} \tilde{N} = \tilde{N} d\tilde{N}\). Therefore, if \((b^{-1})^3\) is holomorphic, then \(\tilde{N}\) is anti-holomorphic. If \(\tilde{N}\) is anti-holomorphic, then \(d\tilde{N} \circ J_\Sigma = \tilde{N} d\tilde{N} = -d\tilde{N} \tilde{N}\). Thus,

\[ (d(bc)^{-1} (bc)i - i d(bc)^{-1} (bc)) \circ J_\Sigma. \]
\[(d(bc)^{-1}(bc)i - i d(bc)^{-1}(bc))i = -i d(bc)^{-1}(bc)i - i d(bc)^{-1}(bc)).\]

Therefore, we can choose \(c\) such that \((bc)^{-1}\) is holomorphic. Then, \((b^{-1})^t\) is holomorphic. \(\square\)

A conformal map is referred to as a super-conformal map if its curvature ellipse is a circle at each immersed point. As shown in [14], a super-conformal map is a Bäcklund transform of a minimal surface. A holomorphic function is a super-conformal.

**Theorem 3.** The left canonical lift or the right canonical lift of a conformal map is holomorphic if and only if the conformal map is super-conformal.

**Proof.** If \(f\) is conformal, then \(f\) is always holomorphic with respect to \(T^2\). If the left canonical lift \((f, \alpha)\) is holomorphic, then \(\alpha\) is holomorphic. If \((f, \alpha)\) is holomorphic, then \(f\) is super-conformal. Similarly, if the right canonical left \((f, \beta)\) is holomorphic, then \(f\) is super-conformal.

Conversely, if \(f\) is super-conformal with \(df \circ J_\Sigma = N df = -df \tilde{N}\), then \(N\) or \(\tilde{N}\) is anti-holomorphic by Lemma 11. If \(N = -\Phi_+(\alpha)\), then \(\alpha\) is holomorphic by Lemma 11. Similarly, if \(\tilde{N} = -\Phi_-(\beta)\), then \(\beta\) is holomorphic by Lemma 11. \(\square\)

**Lemma 12.** A conformal map \(f\) is super-conformal if and only if \(N\) or \(\tilde{N}\) is anti-holomorphic.

If \(f\) is super-conformal, then a holomorphic lift of \(f\) to the twistor space exists (see, for example, [4], Theorem 5). We have distinguished this holomorphic lift.

**Theorem 4** ([19], [9]). Let \(f: \Sigma \to V\) be a conformal map with canonical factorization \(df = a \eta b^{-1}\). We may assume that \(a: \Sigma \to V_+\) is holomorphic. Then, the local complex function \(c\) exists such that \(d(ac) \circ J_\Sigma = d(ac)i\). We obtain the factorization \(df = \dot{a}\zeta\) with \(\dot{a} = ac\) and \(\zeta = k\dot{c}^{-1}\eta b^{-1}\). Differentiating \(df = \dot{a}\zeta\), we obtain

\[0 = d(df) = \dot{a} \zeta + \dot{a} d\zeta.\]

The branch points of \(f\) is exactly the zeros of \(ac\) or the zeros of \(\zeta\). We employ this factorization for an estimate of the area. Let \(D = \{z \in \mathbb{C} : |z| < 1\}\), and \(D_r = \{z \in \mathbb{C} : |z| < r\}\). We recall the Schwarz lemma:

**Theorem 5.** Let \(f: D \to D\) be a holomorphic function such that \(f(0) = 0\). Then, \(|f(z)| \leq |z|\) on \(D\) and \(|f_z(0)| \leq 1\). The equality holds if and only if \(|f_z(0)| = 1\) or there exists \(z_0 \in D \setminus \{0\}\) such that \(|f(z_0)| = |z_0|\).

We have the following area estimate for a super-conformal map by the factorization and the Schwarz lemma.

**Theorem 6.** Let \(f: D \to V\) be a super-conformal map of order \(m\). Assume that \(f\) has the factorization \(df = \dot{a}\zeta\) by a one-form \(\zeta\) and a holomorphic map \(\dot{a}: D \to V_+\). Let \(a_0\) and \(a_1\) be holomorphic functions such that \(\dot{a} = a_0 + ka_1\). Assume that \(0\) is a zero of \(a_0\) and \(a_1\) of order \(m_0 - 1\) and \(m_1 - 1\) respectively.
Assume that positive numbers $C_{a_0}$, $C_{a_1}$ and $C_\zeta$ exist such that $|a_0(z)/z^{m_0-2}| \leq C_{a_0}$, $|a_1(z)/z^{m_1-2}| \leq C_{a_1}$ and $\zeta \wedge (\overline{\zeta} \circ J_\Sigma) \geq C_\zeta dz \wedge (d\overline{\zeta} \circ J_\Sigma)$ on $D$. Then
\[
A(f|_{D_r}) \leq \pi C_\zeta \left( \frac{C_{a_0}^2}{m_0} r^{2m_0} + \frac{C_{a_1}^2}{m_1} r^{2m_1} \right) \quad (0 < r < 1).
\]

Assume that $\zeta \wedge (\overline{\zeta} \circ J_\Sigma) = C_\zeta dz \wedge (d\overline{\zeta} \circ J_\Sigma)$ and $z_0 \in D \setminus \{0\}$ exists such that
- $|a_0(z_0)| = C_{a_0}|z_0|^{m_0-1}$ or $||(a_0)\zeta/z^{m_0-2}\zeta(0)|| = C_{a_0}$,
- $|a_1(z_0)| = C_{a_1}|z_0|^{m_1-1}$ or $||(a_1)\zeta/z^{m_1-2}\zeta(0)|| = C_{a_1}$.

Then, equality holds.

Proof. By the Schwarz lemma, we obtain $|a_0(z)| \leq C_{a_0}|z|^{m_0-1}$ and $|a_1(z)| \leq C_{a_1}|z|^{m_1-1}$. The equality simultaneously holds if and only if $z_0 \in D \setminus \{0\}$ exists such that the following equalities hold:
- $|a_0(z_0)| = C_{a_0}|z_0|^{m_0-1}$ or $||(a_0)\zeta/z^{m_0-2}\zeta(0)|| = C_{a_0}$,
- $|a_1(z_0)| = C_{a_1}|z_0|^{m_1-1}$ or $||(a_1)\zeta/z^{m_1-2}\zeta(0)|| = C_{a_1}$.

By the Schwarz inequality, the area of $f|_{D_r}$ is
\[
A(f|_{D_r}) = -\frac{1}{2} \int_{D_r} df \wedge (d\overline{f} \circ J_\Sigma) = -\frac{1}{2} \int_{D_r} |\check{a}|^2 \zeta \wedge (\overline{\zeta} \circ J_\Sigma)
\leq -\frac{1}{2} C_\zeta \int_{D_r} (|a_0|^2 + |a_1|^2) dz \wedge (d\overline{\zeta} \circ J_\Sigma)
\leq -\frac{1}{2} C_\zeta \int_{D_r} (C_{a_0}^2|z|^{2m_0-2} + C_{a_1}^2|z|^{2m_1-2}) dz \wedge (d\overline{\zeta} \circ J_\Sigma)
= \pi C_\zeta \left( \frac{C_{a_0}^2}{m_0} r^{2m_0} + \frac{C_{a_1}^2}{m_1} r^{2m_1} \right).
\]

Because $a$ is holomorphic, the condition for equality is based on the condition for equality in the Schwarz lemma.

\[\square\]

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