AN \(A_{\infty}\)-COALGEBRA STRUCTURE ON A CLOSED COMPACT SURFACE

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Abstract. Let \(P\) be an \(n\)-gon with \(n \geq 3\). There is a formal combinatorial \(A_{\infty}\)-coalgebra structure on cellular chains \(C_{\ast}(P)\) with non-vanishing higher order structure when \(n \geq 5\). If \(X_g\) is a closed compact surface of genus \(g \geq 2\) and \(P_g\) is a polygonal decomposition, the quotient map \(q : P_g \rightarrow X_g\) projects the formal \(A_{\infty}\)-coalgebra structure on \(C_{\ast}(P_g)\) to a quotient structure on \(C_{\ast}(X_g)\), which persists to homology \(\text{H}_{\ast}(X_g; \mathbb{Z}_2)\), whose operations are determined by the quotient map \(q\), and whose higher order structure is non-trivial if and only if \(X_g\) is orientable or unorientable with \(g \geq 3\). But whether or not the \(A_{\infty}\)-coalgebra structure on homology observed here is topologically invariant is an open question.

To Tornike Kadeishvili on the occasion of his 70th birthday

1. Introduction

Let \(R\) be a commutative ring with unity and let \(P\) be an \(n\)-gon with \(n \geq 3\). In this paper we construct a formal combinatorial \(A_{\infty}\)-coalgebra structure on the cellular chains of \(P\), denoted by \(C_{\ast}(P)\), which is the graded \(R\)-module generated by the vertices, edges, and region of \(P\). For an application, let \(X_g\) be a closed compact surface of genus \(g \geq 2\) and let \(P_g\) be a polygonal decomposition. The quotient map \(q : P_g \rightarrow X_g\) sends the formal \(A_{\infty}\)-coalgebra structure on \(C_{\ast}(P_g)\) to a quotient structure on \(C_{\ast}(X_g)\), which persists to homology \(\text{H}_{\ast}(X_g; \mathbb{Z}_2)\), whose operations are determined by the quotient map \(q\), and whose higher order structure is non-trivial if and only if \(X_g\) is orientable or unorientable with \(g \geq 3\).

An \(A_{\infty}\)-coalgebra is the linear dual of an \(A_{\infty}\)-algebra defined by J. Stasheff \([6]\) in the setting of base pointed loop spaces. As motivation, we begin with a brief description of \(A_{\infty}\)-algebras.

Let \(S\) be a surface embedded in \(\mathbb{R}^3\) and let \(*\) be some specified base point on \(S\). A base pointed loop on \(S\) is a continuous map \(\alpha : I \rightarrow S\) such that \(\alpha(0) = \alpha(1) = *\). Let \(\Omega S\) denote the space of all base pointed loops on \(S\). Given \(\alpha, \beta \in \Omega S\), define their product \(\alpha \cdot \beta \in \Omega S\) to be

\[
(\alpha \cdot \beta)(t) = \begin{cases} 
\alpha(2t), & t \in [0, \frac{1}{2}], \\
\beta(2t - 1), & t \in [\frac{1}{2}, 1].
\end{cases}
\]

1. The main result (Theorem 4) was proved by the first author in his senior honors thesis \([4]\).
A homotopy from $\alpha$ to $\beta$ is a continuous map $H : I \to \Omega S$ such that $H(0) = \alpha$ and $H(1) = \beta$. Thus $\{H(s) : s \in I\}$ is a 1-parameter family of loops that continuously deforms $\alpha$ to $\beta$.

Let $\alpha, \beta, \gamma \in \Omega S$. Although $(\alpha \cdot \beta) \cdot \gamma \neq \alpha \cdot (\beta \cdot \gamma)$, the loops $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ are homotopic via a linear change of parameter homotopy $H$. Let $1 : \Omega S \to \Omega S$ be the identity map and define $m_2 : \Omega S \otimes \Omega S \to \Omega S$ by $m_2(\alpha \otimes \beta) = \alpha \cdot \beta$. Then $m_2(m_2 \otimes 1)(\alpha \otimes \beta \otimes \gamma) = (\alpha \cdot \beta) \cdot \gamma$ and $m_2(1 \otimes m_2)(\alpha \otimes \beta \otimes \gamma) = \alpha \cdot (\beta \cdot \gamma)$. Consider $m_2(m_2 \otimes 1), m_2(1 \otimes m_2) : \Omega S^{\otimes 3} \to \Omega S$ and think of the homotopy $H$ from $(\alpha \cdot \beta) \cdot \gamma$ to $\alpha \cdot (\beta \cdot \gamma)$ as a 3-ary operation $m_3 : \Omega S^{\otimes 3} \to \Omega S$. Identify $m_3$ with the interval $I$, its endpoint 0 with $m_2(m_2 \otimes 1)$, and its endpoint 1 with $m_2(1 \otimes m_2)$. Then the boundary $\partial m_3 = m_2(1 \otimes m_2) - m_2(m_2 \otimes 1)$ and the parameter space $I$ identified with $m_3$ is called the associahedron $K_3$. Thus $K_3$ controls homotopy associativity in three variables.

In a similar way, homotopy associativity in four variables is controlled by the associahedron $K_4$, which is a pentagon. The vertices of $K_4$ are identified with the five ways one can parenthesize four variables, its edges are identified with the homotopies that preform a single shift of parentheses, and its 2-dimensional region is identified with a 4-ary operation $m_4 : \Omega S^{\otimes 4} \to \Omega S$. Thus $\partial m_4 = m_2(m_3 \otimes 1) - m_3(m_2 \otimes 1 \otimes 1) + m_3(1 \otimes m_2 \otimes 1) - m_3(1 \otimes 1 \otimes m_2) + m_2(1 \otimes m_3)$. In general, homotopy associativity in $n$ variables is controlled by the associahedron $K_n$, which is an $n - 2$ dimensional polytope whose vertices are identified with the various ways to parenthesize $n$ variables. While associahedra are independently interesting geometric objects, they also organize the data in the definition of an $A_\infty$-(co)algebra.

![Figure 1. The associahedron $K_4$.](image)

But before we can define an $A_\infty$-(co)algebra, we need some preliminaries. A differential graded (d.g.) $R$-module is a graded $R$-module $V = \oplus_{i \geq 0} V_i$ equipped with a differential operator $\partial : V_\ast \to V_{\ast - 1}$ such that $\partial \circ \partial = 0$. Let $(V, \partial V)$ and $(W, \partial W)$ be d.g. $R$-modules. A linear map $f : V \to W$ has degree $p$ if $f : V_i \to W_{i+p}$; the map $f$ is a chain map if $f \circ \partial V = (-1)^p \partial W \circ f$. Denote the degree of $f$ by $|f|$ and the $R$-module of all linear maps of degree $p$ by $\text{Hom}_p(V, W)$.

**Proposition 1.** $\text{Hom}_\ast(V, W)$ is a d.g. $R$-module with differential $\delta$ given by $\delta(f) = f \circ \partial V - (-1)^{|f|} \partial W \circ f$.

**Proof.** The proof is straight-forward and omitted. \qed
Note that $f$ is a chain map if and only if $\delta(f) = 0$. Now $m_2 \in \text{Hom}_*(C_*(P)^{\otimes 2}, C_*(P))$ and $m_3 \in \text{Hom}_*(C_*(P)^{\otimes 3}, C_*(P))$. Since $|m_3| = 1$ we have

$$\delta(m_3) = m_3 \circ \partial^{\otimes 3} - (-1)^1 \partial \circ m_3 = \partial \circ m_3 = m_2(1 \otimes m_2) - m_2(m_2 \otimes 1),$$

where $m_3 \circ \partial^{\otimes 3} = 0$ because loops have empty boundary. Then $\delta(m_3)$ measures the deviation of $m_2$ from associativity, and in certain situations one can express this deviation in terms of a degree 0 chain map $\alpha_3 : C_*(K_3) \rightarrow \text{Hom}_*(C_*(P)^{\otimes 3}, C_*(P))$. Let $\theta_n$ denote the top dimensional cell of $K_n$.

**Definition 2.** Let $(V, \partial)$ be a d.g. $R$-module. For each $n \geq 2$, choose a map $\alpha_n : C_*(K_n) \rightarrow \text{Hom}(V^{\otimes n}, V)$ of degree 0, and let $m_n = \alpha_n(\theta_n)$. Then $(V, \partial, m_2, m_3, \ldots)$ is an $A_\infty$-coalgebra if $\delta \alpha_n = \alpha_n \partial$ for each $n \geq 2$.

The definition of an $A_\infty$-coalgebra mirrors the definition of an $A_\infty$-algebra.

**Definition 3.** Let $(V, \partial)$ be a d.g $R$-module. For each $n \geq 2$, choose a map $\alpha_n : C_*(K_n) \rightarrow \text{Hom}(V^{\otimes n})$ of degree 0, and let $\Delta_n = \alpha_n(\theta_n)$. Then $(V, \partial, \Delta_2, \Delta_3, \ldots)$ is an $A_\infty$-coalgebra if $\delta \alpha_n = \alpha_n \partial$ for each $n \geq 2$.

Evaluating both sides of the equation in Definition 3 at $\theta_n$ produces the classical structure relation

$$\Delta_n \partial - (-1)^{n-2} \sum_{i=0}^{n-1} (1^{\otimes i} \otimes \partial \otimes 1^{\otimes n-i-1}) \Delta_n$$

$$= \sum_{i=1}^{n-2} \sum_{j=0}^{n-i-1} (-1)^{i(j+n+1)} (1^{\otimes j} \otimes \Delta_{i+1} \otimes 1^{\otimes n-i-j-1}) \Delta_{n-i},$$

which expresses $\Delta_n$ as a chain homotopy among the quadratic compositions encoded by the codimension 1 cells of $K_n$ (see Figure 2). The sign $(-1)^{i(j+n+1)}$ is the combinatorial sign derived by Saneblidze and Umble in [5]. Note that when $n = 2$, Relation 1 has the form $\Delta_2 \partial = (\partial \otimes 1 + 1 \otimes \partial) \Delta_2$, which is dual to the Leibniz Rule $dm = m(d \otimes 1 + 1 \otimes d)$ in calculus and says that $\partial$ is a coderivation of $\Delta_2$.

![Figure 2. The quadratic compositions encoded by the codim 1 cells of $K_4$.](image)

2. **Statement of the Main Result**

Let $P$ be a counterclockwise oriented $n$-gon, $n \geq 3$. Label the vertices $v_1, v_2, \ldots, v_n$ and the edges $e_1, e_2, \ldots, e_n$. Define $v_1$ to be the initial vertex and $v_n$ to be the terminal vertex, and direct the edges from $v_1$ to $v_n$. This assignment partially orders
the vertices as indicated in Figure 3. Edges whose direction is consistent with orientation are positive. Let \( \partial : C_*(P) \to C_{*-1}(P) \) be the boundary operator induced by geometric boundary. Then \( \partial(v_i) = 0 \), \( \partial(e_i) = v_{i+1} - v_i \) if \( i < n \), \( \partial(e_n) = v_n - v_1 \), and \( \partial(p) = e_1 + e_2 + \cdots + e_{n-1} - e_n \).

![Figure 3. An n-gon for n = 6](image)

We will use the diagonal approximation \( \Delta_2 : C_*(P) \to C_*(P) \otimes C_*(P) \) defined by D. Kravatz in [2] and given by

\[
\Delta_2(P) = v_1 \otimes P + P \otimes v_n + \sum_{0<i_1<i_2<n} e_{i_1} \otimes e_{i_2},
\]

\[
\Delta_2(e_i) = v_i \otimes e_i + e_i \otimes v_{i+1} \text{ if } i < n,
\]

\[
\Delta_2(e_n) = v_1 \otimes e_n + e_n \otimes v_n,
\]

\[
\Delta_2(v_i) = v_i \otimes v_i.
\]

A more general exposition of Kravatz’s diagonal appears in [1]. For \( k > 2 \), define the \( k \)-ary \( A_\infty \)-coalgebra operation \( \Delta_k : C_*(P) \to C_*(P) \otimes_k \) by

\[
\Delta_k(\sigma) = \begin{cases} 
\sum_{0<i_1<i_2<\cdots<i_k<n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, & \text{if } \sigma = P, \\
0, & \text{otherwise.}
\end{cases}
\]

Then by definition, \( \Delta_k = 0 \) for all \( k \geq n \). We can now state our main result.

**Theorem 4.** Let \( P \) be an \( n \)-gon. The structure operations \( \{ \Delta_k \}_{2 \leq k \leq n} \) defined above impose an \( A_\infty \)-coalgebra structure on \((C_*(P), \partial)\). Furthermore, \( \Delta_k \) vanishes for all \( k \geq n \).

**3. Proof of the Main Result**

We must verify Relation 4 for all \( k \geq 2 \). When \( k = 2 \), verification is easy and left to the reader. We first verify Relation 4 for all \( k > 3 \), then consider the special case \( k = 3 \). To simplify notation we establish following notational devices:

- The symbol \( \partial \Delta_k(P) \) denotes \( \sum_{i=0}^{k-1} (1 \otimes \partial \otimes 1^{k-i-1}) \Delta_k(P) \).
- The symbol \( \Delta_j \Delta_k(P) \) denotes \( \sum_{i=0}^{k-1} (1 \otimes \Delta_j \otimes 1^{k-i-1}) \Delta_k(P) \).

The fact that \( \Delta_j \) and \( \Delta_k \) vanish on edges and vertices when \( j, k \geq 3 \) implies:

**Proposition 5.** \( \Delta_j \Delta_k(P) = \partial \Delta_k(P) = 0 \) whenever \( i, j \geq 3 \).
In view of Proposition \(5\), all non-vanishing terms in Relation \(1\) apply some \(\Delta_k\) to the 2-cell \(P\). Therefore (up to sign) Relation \(1\) reduces to
\[
(\Delta_{k-1} \Delta_2 + \Delta_2 \Delta_{k-1} + \partial \Delta_k)(P) = 0.
\]
The signs in Relation \(2\) follow from the Sign Commutation Rule: If an object of degree \(p\) passes an object of degree \(q\), affix the sign \((-1)^{pq}\). First, consider a term \(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}\) of \(\Delta_k(P)\). Since \(|\partial| = -1\), multiplying by the sign in Relation \(1\) and simplifying gives
\[
(-1)^{k-2} \left(1 \otimes \cdots \otimes \partial \otimes \cdots \otimes 1\right) (e_{j_1} \otimes \cdots \otimes e_{j_k}) = (-1)^{i+k+1} e_{j_1} \otimes \cdots \otimes e_{j_{i+1}} \otimes \cdots \otimes e_{j_k}.
\]
Second, since \(|\Delta_2| = 0\), the sign of \(\Delta_2 \Delta_{k-1}\) is the sign \((-1)^{i+k}\), which is opposite the sign in Relation \(1\) where \(\Delta_2\) is applied in the \(i^\text{th}\) position. Third, the signs of the terms \((\Delta_{k-1} \otimes 1) \Delta_2\) and \((1 \otimes \Delta_{k-1}) \Delta_2\) given by Relation \(1\) simplify to \(-1\) and \((-1)^{k+1}\), respectively. Since \(|\Delta_{k-1}| = k - 3\), the Sign Commutation Rule can only introduce a sign when \(1 \otimes \Delta_{k-1}\) is applied to a pair of edges. However, this particular situation never occurs in our proof.

**Lemma 6.** All non-vanishing terms contain exactly one tensor factor \(v_i\). Furthermore, if \(e_i \otimes v_j\) or \(v_j \otimes e_i\) appears within some term, then \(i < j\) or \(j < i \leq k\).

**Proof.**

**Case 1:** Consider \(\partial \Delta_k(P)\). Note that \(\Delta_k(P)\) will either vanish if \(k \geq n\), or it will produce terms of the form \(e_{i_1} \otimes \cdots \otimes e_{i_k}\), where \(0 < i_1 < \cdots < i_k < n\). Applying \(\partial\) to any \(e_i\) will create two new terms by replacing it with \(v_i\) or \(v_{i+1}\). Since the subscript \(i\) is between the subscripts on either side, either choice gives the desired result.

**Case 2:** Consider \(\Delta_{k-1} \Delta_2(P)\). Since \(\Delta_{k-1}\) acts non-trivially only on \(P\), the terms of interest produced by \(\Delta_2\) are \(v_i \otimes P\) and \(P \otimes v_n\). When \(\Delta_{k-1}\) acts on the \(P\), it produces a term of the form \(e_{i_1} \otimes \cdots \otimes e_{i_{k-1}}\), where \(0 < i_1 < \cdots < i_{k-1} < n\). Since \(0 < i < n\) for all \(i\), all terms either begin with \(v_1\) or end with \(v_n\) and have the desired form.

**Case 3:** Consider \(\Delta_2 \Delta_{k-1}(P)\). Notice that \(\Delta_{k-1}(P)\) produces terms of the form \(e_{i_1} \otimes \cdots \otimes e_{i_{k-1}}\), where \(0 < i_1 < \cdots < i_{k-1} < n\). When \(\Delta_2\) is applied to an \(e_i\) factor, it has the effect of either inserting \(v_i\) to the left, or \(v_{i+1}\) to the right of the \(e_i\). Since the subscript \(i\) is between the subscripts on either side, either choice gives the desired result.

This information allows us to classify the terms in Relation \(2\) relative to the position of \(v_i\) with respect to \(e_{i-1}\) and \(e_{i+1}\). We say that \(v_i\) is left adjacent if \(e_{i-1}\) is immediately to its left; \(v_i\) is right adjacent if \(e_{i+1}\) is immediately to its right.

**Definition 7.** A term in which \(v_i\) is left adjacent is left attached; a term in which \(v_i\) is right adjacent is right attached. A term that is both left and right attached is doubly attached; a term that is left attached or right attached but not both is singly attached; and a term that is neither left attached nor right attached is unattached.

We can sub-classify within these sets as follows:

**Definition 8.** A term that begins with \(v_1\) is extreme minimal. A term that ends with \(v_n\) is extreme maximal.
Since doubly attached terms cannot be extreme, there are five classes of terms, and the proof reduces to showing that each class cancels itself in Relation \[2\]. But before we can do this, we need some lemmas.

**Lemma 9.** Let \(2 < k < n\). If \(v_i\) appears in the \(i^{th}\) position of a term in \(\partial\Delta_k(P)\), that term is generated one of the following two ways, each way contributing at most one instance:

- **Way 1:** If the term is unattached on the left and is not minimally extreme, an instance of that term is generated with sign \((-1)^{i+k+1}\).
- **Way 2:** If the term is unattached on the right and is not maximally extreme, an instance of that term is generated with sign \((-1)^{i+k}\).

**Proof.**

**Case 1:** Consider a term that contains \(e_{i_1} \otimes v_{i_2} \otimes e_{i_3}\), where \(0 < i_1 < i_2 \leq i_3 < n\). Since \(\Delta_k(P)\) cannot generate \(v_{i_2}\), it was produced by \(\partial\). Since \(v_{i_2}\) appears between \(e_{i_1}\) and \(e_{i_3}\), we know that \(\partial\) was applied to a term of the form \(e_{i_1} \otimes e_x \otimes e_{i_3}\) with \(0 < i_1 < x < i_3 < n\). Furthermore, there are only two possible edges \(e_x\) that \(\partial\) could have acted upon to generate \(v_{i_2}\), namely when \(x = i_2 - 1\) or \(x = i_2\). Therefore, since there are only two possible values for \(x\), there are at most two ways to generate any given term.

Suppose that \(x = i_2 - 1\). Since \(i_1 < x\), we have \(i_1 < i_2 - 1\) forcing the original term to be unattached on the left. To see that all left unattached terms can be generated, simply note that the subscript of any left unattached term will satisfy the inequality \(i_1 < i_2 - 1\) by definition. Then \(\Delta_k(P)\) produces the term \(e_{i_1} \otimes e_{i_2-1} \otimes e_{i_3}\), where \(0 < i_1 < i_2 \leq i_3 < n\), and produces the desired term after \(\partial\) is applied to \(e_{i_2-1}\). Now applying \(\partial\) this way produces a positive \(v_{i_2}\) in the same position in which \(\partial\) was applied, and by the discussion above regarding the Sign Commutation Rule, the term’s sign is \((-1)^{i+k+1}\). Similarly, suppose \(x = i_2\). Then all right unattached terms, and only right unattached terms, can be generated. Similar analysis reveals that the term’s sign is \((-1)^{i+k}\).

**Case 2:** Consider a term of the form \(v_{i_2} \otimes e_{i_3} \otimes \cdots\), where \(0 < i_2 \leq i_3 < n\). By an argument similar to the one above, \(\partial\) must have been applied to a term of the form \(e_x \otimes e_{i_3} \otimes \cdots\), where \(x = i_2 - 1\) or \(x = i_2\). The proof is similar to the case above and left to the reader; the only significant difference is that when \(x = i_2 - 1\) we cannot immediately conclude that all possible left unattached terms will be generated. This is because \(x > 0\), and since \(x = i_2 - 1\), we have \(i_2 > 1\). Hence only non-minimally extreme left unattached terms can be formed, and an analysis similar to that in Case 1 shows that the terms produced have the sign \((-1)^{i+k+1}\).

**Case 3:** Consider a term of the form \(\cdots \otimes e_{i_1} \otimes v_{i_2}\), where \(0 < i_1 < i_2 < n\). This case is can be proved using an argument very similar to case 2, the details of which are left to the reader. \(\square\)

**Lemma 10.** Let \(3 < k \leq n\). If \(v_i\) appears in the \(i^{th}\) position of a term in \(\Delta_2\Delta_{k-1}(P)\), that term is generated one of the following two ways, each way contributing at most one instance:

- **Way 1:** If the term is attached on the left, an instance of that term is generated with sign \((-1)^{i+k+1}\).
- **Way 2:** If the term is attached on the right, an instance of that term is generated with sign \((-1)^{i+k}\).
Proof. **Case 1:** Consider a term that contains $e_{i_1} \otimes v_{i_2} \otimes e_{i_3}$, where $0 < i_1 < i_2 \leq i_3 < n$. Since $\Delta_{k-1}(P)$ cannot produce $v_{i_2}$, it is produced by $\Delta_2$, and because $\Delta_2$ inserts a $v$ next to the $e$ to which it was applied, $\Delta_2$ was applied to either $e_{i_1}$ or $e_{i_3}$. Additionally, because of how $\Delta_2$ is defined, the only way $\Delta_2(e_{i_1})$ can generate $v_{i_2}$ is if $i_1 = i_2 - 1$, and the only way $\Delta_2(e_{i_3})$ can generate $v_{i_2}$ is if $i_2 = i_3$. Therefore, there are at most two ways to create the desired term.

Now suppose $\Delta_2$ was applied to $e_{i_1}$ and that $i_1 = i_2 - 1$. Then by definition, the resulting term will be left attached. Furthermore, any given left attached term of the form $v_{i_2} \otimes e_{i_3} \otimes \cdots$, where $0 < i_2 - 1 \leq i_3 < n$, can be generated by simply applying $\Delta_2$ to the term $e_{i_2} \otimes e_{i_3}$ generated by $\Delta_k(P)$. Now in order to create a term with $v_{i_2}$ in the $i^{th}$ position, we must apply $\Delta_2$ to the $(i - 1)^{st}$ position, and so by the discussion above on the Sign Commutation Rule, the term’s sign is $(-1)^{i+k+1}$. If we suppose that $\Delta_2$ was applied to $e_{i_3}$ and that $i_2 = i_3$, then a very similar argument shows that all right attached terms of the form $e_{i_1} \otimes v_{i_2} \otimes e_{i_3}$, where $0 < i_2 < n$ are generated with sign $(-1)^{i+k}$.

**Case 2:** Consider a term of the form $v_{i_2} \otimes e_{i_3} \otimes \cdots$, where $0 < i_2 < i_3 < n$. By an argument similar to the one above, this term can only be generated if $\Delta_2$ is applied to $e_{i_3}$ and if $i_2 = i_3$. The same reasoning as in case one shows that all right attached terms, and only those terms, will be generated with sign $(-1)^{i+k}$.

**Case 3:** Consider a term of the form $\cdots \otimes e_{i_1} \otimes v_{i_2}$, where $0 < i_1 < i_3 < n$. This case is nearly identical to case 2, and we leave it to the reader to show that all left attached terms, and only those terms, can be generated with sign $(-1)^{i+k+1}$. □

**Proposition 11.** Let $2 < k < n$. If $v_i$ appears in the $i^{th}$ position of a term in $\partial \Delta_k(P)$, after cancellations the terms that remain are:

- All maximally extreme unattached terms with positive sign.
- All minimally extreme unattached terms with sign $(-1)^k$.
- All left attached terms with sign $(-1)^{i+k}$.
- All right attached terms with sign $(-1)^{i+k+1}$.

**Proof.** The proof requires several straightforward applications of Lemma 9, some of which show that no doubly attached terms are generated, all unattached terms generated cancel, and all extreme terms generated are unattached. The details are left to the reader. □

**Proposition 12.** Let $3 < k \leq n$. Then $\Delta_{k-1} \Delta_2(P)$ contains:

- All minimally extreme terms with sign $(-1)^{k+1}$.
- All maximally extreme terms with negative sign.

**Proof.** Since $\Delta_{k-1}$ only acts non-trivially on $P$, the only terms produced by $\Delta_2$ that do not immediately vanish are $v_1 \otimes P + P \otimes v_n$. Additionally, since $k > 3$, $\Delta_{k-1}$ produces no primitive terms when it is applied by the definition of $\Delta_k$ for $k > 2$.

When $\Delta_{k-1}$ is applied to the first term, it generates terms of the form $v_1 \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots$, which are all minimally extreme. Since the sign commutation rule introduces nothing new, we have that the term has the final sign $(-1)^{k+1}$.

When $\Delta_{k-1}$ is applied to the second term, it generates terms of the form $\cdots \otimes e_{i_{k-3}} \otimes e_{i_{k-2}} \otimes v_n$, which are maximally extreme. Since the sign commutation rule introduces nothing new, we have that the sign will always be negative. □
Proposition 13. Let $3 < k \leq n$. If $v_i$ appears in the $i^{th}$ position of a term in $\Delta_2 \Delta_{k-1}(P)$, after cancellations the terms that remain are:

- All maximally extreme singly attached terms with positive sign.
- All minimally extreme singly attached terms with sign $(-1)^k$.
- All left attached terms with sign $(-1)^{i+k+1}$.
- All right attached terms with sign $(-1)^{i+k}$.

**Proof.** The proof requires several straightforward applications of Lemma 10, some of which show that no unattached terms are generated, all doubly attached terms cancel, and all extreme terms are singly attached. The details are left to the reader. \[ \square \]

Now when $k > 3$, all three propositions apply and we see that in Relation 2 the singly attached terms in $\partial \Delta_1(P)$ and $\Delta_2 \Delta_{k-1}(P)$ cancel, the extreme unattached terms in $\partial \Delta_k(P)$ and $\Delta_{k-1} \Delta_2(P)$ cancel, and the extreme singly attached terms in $\Delta_{k-1} \Delta_2(P)$ and $\Delta_2 \Delta_{k-1}(P)$ cancel. Therefore, all terms cancel and the relation is satisfied.

Having proved the theorem for $k > 3$, we consider the special case $k = 3$. We must show that:

$$[(\partial \otimes 1 \otimes 1 + 1 \otimes \partial \otimes 1 + 1 \otimes 1 \otimes \partial) \Delta_3 + (\Delta_2 \otimes 1 + 1 \otimes \Delta_2) \Delta_2](P) = 0$$

**Proposition 14.** The terms generated by $(\partial \Delta_3 + \Delta_2 \Delta_2)(P)$ form three classes, which independently satisfy the conclusions of Propositions 11, 12, and 13.

**Proof.** Note that the proof of Proposition 11 still applies to $\partial \Delta_3(P)$ as before since it never used the restriction that $k \geq 3$. Therefore, we must show that $(\Delta_2 \otimes 1 - 1 \otimes \Delta_2) \Delta_2$ can be decomposed into parts that respectively satisfy the conclusions of Proposition 12 and Proposition 13. Now expanding we have

$$(\Delta_2 \otimes 1 + 1 \otimes \Delta_2) \Delta_2(P) = (\Delta_2 \otimes 1 + 1 \otimes \Delta_2)(v_1 \otimes P + P \otimes v_n)$$

$$= (\Delta_2 \otimes 1 + 1 \otimes \Delta_2) \sum_{0 < i_1 < i_2 < n} e_{i_1} \otimes e_{i_2}.$$
which generates every possible extreme term and only extreme terms, all with the
correct sign for \( k = 3 \). Therefore, \((\Delta_2 \otimes 1 + 1 \otimes \Delta_2) (v_1 \otimes P + P \otimes v_{n+1})\) satisfies
the hypothesis of Proposition \( \ref{prop:12} \)
\[\square\]
Since the terms generated by \((\partial \Delta_3 + \Delta_2 \Delta_2)(P)\) fall into classes that individually
satisfy the conclusions of Proposition \( \ref{prop:11} \) Proposition \( \ref{prop:12} \) and Proposition \( \ref{prop:13} \) they
cancel each other out as before. Therefore \( \Delta_3 \) also satisfies Relation \( \ref{rel:1} \) and the proof
of Theorem \( \ref{thm:4} \) is complete.

4. Generalization of Result

Until now, we have been working with an \( n \)-gon \( P \) whose initial and terminal
vertices are adjacent. Our result extends to situations in which the initial and
terminal vertices are non-adjacent. Suppose \( v_1 \) is the initial vertex and \( v_t \) is the
terminal vertex, where \( t > 1 \). Define the generalized \( \Delta'_2 \) to be the same as in the
introduction, except that
\[
\Delta'_2(P) = v_1 \otimes P + P \otimes v_t + \sum_{0 < i_1 < i_2 < t} e_{i_1} \otimes e_{i_2} - \sum_{n \geq i_1 > i_2 \geq t} e_{i_1} \otimes e_{i_2}.
\]
Additionally, for \( k > 2 \), define the generalized \( k \)-ary \( A_\infty \)-coalgebra operation \( \Delta'_k \) by
\[
\Delta'_k(\sigma) = \begin{cases} 
\sum_{0 < i_1 < i_2 < \ldots < i_k < t} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \\
-\sum_{n \geq i_1 > i_2 > \ldots > i_k \geq t} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, & \text{if } \sigma = P, \\
0, & \text{otherwise.}
\end{cases}
\]
Then by definition, \( \Delta'_k = 0 \) for all \( k \geq \max\{t, n - t + 2\} \). Now all that remains is
to show that the operations \( \{\Delta'_k\}_{n \geq 2} \) extend to this general setting.

Corollary 15. Let \( P \) be an \( n \)-gon with initial vertex \( v_1 \) and terminal vertex \( v_t \),
where \( t > 1 \). The operations \( \{\Delta'_k\}_{2 \leq k < \max\{t, n - t + 2\}} \) defined above impose an \( A_\infty \)-
coalgebra structure on \((C_\ast(P), \partial)\).

Proof. Draw an additional edge from \( v_1 \) to \( v_t \) and denote it by \( e_0 \). Define \( P_1 \) to be
the polygon with vertices \( v_1, v_2, \ldots, v_t \) oriented counterclockwise and let \( P_2 \) to be
the polygon with vertices \( v_t, v_{t+1}, \ldots, v_n, v_1 \) oriented counterclockwise as illustrated
in Figure 4.

![Figure 4. A 7-gon with \( v_t = v_5 \).](image-url)
Then by the way edges are directed with respect to the orientation, we have
\[ \partial(P_1) = e_1 + e_2 + \cdots + e_{t-1} - e_0 \]
and
\[ \partial(P_2) = -e_1 - e_{t+1} - \cdots - e_n + e_0. \]
Note that
\[ \partial(P_1) + \partial(P_2) = \partial(P). \]
Furthermore, define \( v_1 \) to be the initial vertex and \( v_t \) to be the terminal vertex in both \( P_1 \) and \( P_2 \); then \( P_1 \) and \( P_2 \) satisfy the hypothesis of Theorem 4 and it is straightforward to show that \( \Delta_2(P_1) + \Delta_2(P_2) = \Delta_2(P) \) and \( \Delta_k(P_1) + \Delta_k(P_2) = \Delta_k(P) \) for \( k > 2 \). All that remains is to verify that Relation 1 holds on \( P \) for all \( k \geq 2 \), which can be done using the relations above and applying the main theorem to each part. The details are left to the reader. Therefore, the operations \( \{\Delta_n\}_{n \geq 2} \) defined on \( P \) above satisfy all \( A_\infty \)-coalgebra relations on cellular chains of \( P \). □

5. Application to closed compact surfaces

The celebrated classification of closed compact surfaces (cf. [3], for example) states that a closed compact surface of genus \( g \), denoted by \( X_g \), is homeomorphic to a sphere with \( g \geq 0 \) handles when orientable or a connected sum of \( g \geq 1 \) real projective planes when unorientable.

To obtain the connected sum \( X \# Y \) of two surfaces \( X \) and \( Y \), remove the interior of a disk from \( X \) and from \( Y \) then glue the two surfaces together along their boundaries. Of course, a sphere with \( g \geq 1 \) handles is the connected sum of \( g \) tori, and a Klein bottle is the connected sum of two real projective planes.

![Figure 5. A polygonal decomposition of the connected sum of three and four real projective planes.](image-url)
Figure 6. A polygonal decomposition of the connected sum of three tori.

Furthermore, when \( g \geq 1 \), \( X_g \) can be expressed as the quotient of a \( 4g \)-gon when orientable or a \( 2g \)-gon when unorientable as pictured in Figures 5 and 6 (the dotted lines represent common boundaries in the connected sums). To recover \( X_g \) from a polygonal decomposition \( P_g \), glue the edges with the same label together so that the arrows directing the edges align. This gluing operation defines the projection \( p : P_g \rightarrow X_g \), which encodes the topology of \( X_g \).

The cutting and pasting procedures indicated in Figures 5 and 6 can be continued indefinitely, and the directed edges of a particular polygonal decomposition \( P_g \) so obtained define a vertex poset with initial and terminal vertices labeled \( v \) (all vertices are identified in \( X_g \)). These are exactly the configurations to which Corollary 15 applies.

Indeed, the formal \( A_\infty \)-coalgebra structure on \( C_\ast (P_g) \) given by Corollary 15 projects to a quotient structure on \( C_\ast (X_g) \) in the obvious way. If \( X_3 \) is the connected sum of three real projective planes, for example, using the decomposition in Figure 5 we obtain an \( A_\infty \)-coalgebra structure on \( C_\ast (X_3) \) with non-trivial operations

\[
\Delta_2 (X_3) = v \otimes X_3 + X_3 \otimes v + e_1 \otimes e_1 - e_2 \otimes e_2 + e_3 \otimes e_3 \\
\Delta_4 (X_3) = e_1 \otimes e_1 \otimes e_3 \otimes e_3.
\]

Furthermore, each cellular chain in \( C_\ast (P_g) \otimes \mathbb{Z}_2 \) projects to a non-bounding cycle in \( C_\ast (X_g) \otimes \mathbb{Z}_2 \) so that \( H_\ast (X_g; \mathbb{Z}_2) = C_\ast (X_g) \otimes \mathbb{Z}_2 \).

For a general unorientable \( X_g \), label the edge of the \( i^{th} \) real projective plane \( e_i \), and label the vertex \( v \). Let \( \lfloor x \rfloor \) denote the floor of \( x \), and for a given \( s \) define the sequences

\[
\{i_p = 2 \left\lfloor \frac{p+1}{2} \right\rfloor - 1\}^{2s}_{p=1} \text{ and } \{j_q = 2 \left\lfloor \frac{q+1}{2} \right\rfloor \}^{2s}_{q=1}.
\]

**Corollary 16.** Let \( X_g \) be a closed compact unorientable surface of genus \( g \geq 2 \) and let \( P_g \) be the polygonal decomposition of \( X_g \) indicated in Figure 5. The formal \( A_\infty \)-coalgebra structure on \( C_\ast (P_g) \) projects to a non-trivial \( A_\infty \)-coalgebra structure on \( C_\ast (X_g) \) with operations \( \{\Delta_k\}_{k \geq 2} \) determined by the quotient map \( q : P_g \rightarrow X_g \) and defined by
\[ \Delta_2(v) = v \otimes v \]
\[ \Delta_2(e_i) = v \otimes e_i + e_i \otimes v \]
\[ \Delta_2(X_g) = v \otimes X_g + X_g \otimes v + \sum_{g \in \{2s-1, 2s\}} e_{2i-1} \otimes e_{2i-1} - \sum_{g \in \{2s+1\}} e_{2j} \otimes e_{2j} \]
\[ \Delta_k(X_g) = \sum_{0 < p_1 < \cdots < p_k \leq 2s} e_{i_{p_1}} \otimes \cdots \otimes e_{i_{p_k}} - \sum_{0 < q_1 < \cdots < q_k \leq 2s} e_{j_{q_1}} \otimes \cdots \otimes e_{j_{q_k}}, \ k \geq 3 \]
\[ \Delta_k(\sigma) = 0, \text{ for all } \sigma \neq X_g \text{ and } k \geq 3. \]

For a general orientable \( X_g \), label the edges along one edge-path from initial to terminal vertex \( e_1, e_2, \ldots, e_{2g} \). Let \( \hat{e}_{2k-1} = e_{2k} \) and \( \hat{e}_{2k} = e_{2k-1} \), and label the edges along the other edge-path \( \hat{e}_1, e_2, \ldots, \hat{e}_{2g} \).

**Corollary 17.** Let \( X_g \) be a closed compact orientable surface of genus \( g \geq 2 \) and let \( P_g \) be the polygonal decomposition of \( X_g \) indicated in Figure 6. The formal \( A_\infty \)-coalgebra structure on \( C_*(P_g) \) projects to a non-trivial \( A_\infty \)-coalgebra structure on \( C_*(X_g) \) with operations \( \{\Delta_k\}_{k \geq 2} \) determined by the quotient map \( q : \hat{P}_g \to X_g \) and defined by

\[ \Delta_2(v) = v \otimes v \]
\[ \Delta_2(e_i) = v \otimes e_i + e_i \otimes v \]
\[ \Delta_2(X_g) = v \otimes X_g + X_g \otimes v + \sum_{i=1}^g e_{2i-1} \otimes e_{2i-1} - e_{2i} \otimes e_{2i-1} \]
\[ \Delta_k(X_g) = \sum_{0 < i_1 < \cdots < i_k \leq 2g} e_{i_1} \otimes \cdots \otimes e_{i_k} - \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_k}, \ k \geq 3 \]
\[ \Delta_k(\sigma) = 0, \text{ for all } \sigma \neq X_g \text{ and } k \geq 3. \]

**Remark 18.** It is interesting to note that our definitions of \( \Delta_2 \) in Corollaries 16 and 17 allow us to read off the cup product on \( H_*(X_g; \mathbb{Z}_2) \) directly from the components of \( \Delta_2 \) without performing additional calculations. In general, one has a choice: Compute cup products using a standard diagonal at the expense of long calculations or construct an application-specific diagonal that minimizes the calculations at the expense of the accompanying combinatorial difficulties.

Finally, when \( X_g \) is the sphere \( S^2 \), the \( A_\infty \)-coalgebra structure on \( C_*(X_g) \) is clearly degenerate; if \( X_g \) is a real projective plane, a torus, or a Klein bottle, there is one non-vanishing \( A_\infty \)-coalgebra operation on \( C_*(X_g) \), namely \( \Delta_2 \), which induces the non-trivial cup product in cohomology (the formula for \( \Delta_2 \) on a real projective plane requires independent verification; the proof is straight-forward and omitted). But if \( X_g \) is a closed compact surface that is orientable with \( g \geq 2 \) or unorientable with \( g \geq 3 \), the combinatorial homotopy coassociative diagonal on \( C_*(P_g) \) induces a formal \( A_\infty \)-coalgebra structure that projects to a strictly coassociative \( A_\infty \)-coalgebra structure on \( C_*(X_g) \) with non-trivial higher order structure determined by the quotient map \( q \). But whether or not the \( A_\infty \)-coalgebra structure \( (H = H_*(X_g; \mathbb{Z}_2), \Delta_k) \) observed here is topologically invariant is an open question.
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