Co-Toeplitz Quantization: A Simple Case

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Abstract. The author has introduced in a recent paper a new class of operators, called co-Toeplitz operators, with symbols in a co-algebra. This is the categorical dual to Toeplitz operators which have symbols in an algebra. The mapping from a symbol to its co-Toeplitz operator gives a quantization scheme, called co-Toeplitz quantization. A new, quite simple particular case of co-Toeplitz quantization is introduced in this note. Examples are given in order to show some of its properties.

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1. Introduction

In [2] I have defined co-Toeplitz operators in a dual way in terms of category theory to Toeplitz operators. The structures needed for this definition are a co-algebra \( C \) (see [1]) together with a sesqui-linear form \( \langle \cdot, \cdot \rangle \) defined on it. We let \( \Delta : C \to C \otimes C \) denote the co-multiplication of \( C \). Also, we suppose there is another co-algebra \( \mathcal{P} \) which injects into \( C \) by a map \( j : \mathcal{P} \to C \) and that there is a projection \( Q : C \to \mathcal{P} \), that is, \( Qj = id_{\mathcal{P}} \), the identity on \( \mathcal{P} \).
Then we define the co-Toeplitz operator $C_g : \mathcal{P} \to \mathcal{P}$ to be the linear operator defined by the composition

$$
\mathcal{P} \xrightarrow{j} \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{Q \otimes \text{id}} \mathcal{P} \otimes \mathcal{C} \xrightarrow{\pi_g} \mathcal{P}.
$$

(1.1)

The linear map $\pi_g : \mathcal{P} \otimes \mathcal{C} \to \mathcal{P}$, where $g \in \mathcal{C}$ is called the symbol of the co-Toeplitz operator $C_g$, is defined in [2] by

$$
\pi_g(\phi \otimes f) := \langle g, f \rangle \phi
$$

for $\phi \in \mathcal{P}$ and $f \in \mathcal{C}$. (The map $\pi_g$ is dual to $\alpha_g$ to be defined below.) The anti-linear map $g \mapsto C_g$ is called the co-Toeplitz quantization of $C$. This in general does not involve measure theory. For more details see [2].

The definition (1.1) is the dual diagram to that for a Toeplitz operator which is defined for a symbol $g \in \mathcal{A}$, an algebra, as the composition (right to left)

$$
\mathcal{P} \xleftarrow{\pi_g} \mathcal{A} \xrightarrow{\mu} \mathcal{A} \otimes \mathcal{A} \xleftarrow{\text{id} \otimes \alpha_g} \mathcal{P} \otimes \mathcal{A} \xrightarrow{P} \mathcal{P}.
$$

(1.2)

Here $\mathcal{P}$ is a sub-algebra of $\mathcal{A}$ with $\iota : \mathcal{P} \to \mathcal{A}$ being the inclusion map and $P : \mathcal{A} \to \mathcal{P}$ being a projection. Also, $\mu$ is the multiplication map of $\mathcal{A}$ and $\alpha_g(\phi) := \phi \otimes g$ for $\phi \in \mathcal{P}$.

The definition (1.1) has a particular case: $\mathcal{P} = \mathcal{C}$ and $j = Q = \text{id}_\mathcal{C}$. Then $C_g = \pi_g \Delta : \mathcal{C} \to \mathcal{C}$.

In this particular simple case, which is the new idea in this note, the only structures needed are a co-algebra and a sesqui-linear form on it. All the examples in this note fall within this case. This simple case does not have an interesting analogue for Toeplitz operators, since diagram (1.2) reduces to the right regular representation of $g$ acting on $\mathcal{A}$ if we put $\mathcal{P} = \mathcal{A}$.

If the sesqui-linear form is positive definite, then $C_g$ acts in a pre-Hilbert space and so may be considered as a densely defined operator acting in the Hilbert space completion of $\mathcal{C}$. Thus we can construct models for quantum physics, including creation and annihilation co-Toeplitz operators. (See [2].)

All objects in this paper are vector spaces over the complex numbers, and all arrows are linear maps, except as noted.

2. Manin Quantum Plane

We define $\mathcal{C}$ to be the algebra generated by two elements $a, c$ with the relation $ac = qca$ for some non-zero $q \in \mathbb{C}$, the complex numbers. This is called the Manin quantum plane. The notation follows that used in [2]. We define the co-multiplication $\Delta$ to be the algebra morphism determined by

$$
\Delta(a) = a \otimes a \quad \text{and} \quad \Delta(c) = c \otimes a.
$$

This is well defined on $\mathcal{C}$, since $\Delta(ac - qca) = 0$ as the reader can verify. We note that $\mathcal{C}$ does not have a co-unit, though this has no great importance for our purposes. Clearly, $\mathcal{B} := \{a^i c^j \mid i, j \in \mathbb{N}\}$, is a Hamel basis of $\mathcal{C}$, where $\mathbb{N}$ denotes the non-negative integers. Since $C_g$ is anti-linear in the symbol $g \in \mathcal{C}$, it suffices to calculate $C_g$ for the basis elements $a^i c^j$. And since $C_g$
We define the degree of the monomial \( a^i c^j \). So \( C_i = 0 \) except on the set of monomials of degree \( i \). With this inner product we see that \( C_i \neq 0 \) on monomials with degree \( i \).

This is based on Example 2.4.8 in [1]. We let \( \mathcal{C} \) be the vector space with basis \( \{ x_i \mid i \in \mathbb{N} \} \). The co-multiplication \( \Delta \) is the linear map determined by

\[
\Delta(x_n) := \sum_{i+j=n} x_i \otimes x_j.
\]
The degree of each basis element is defined by \( \deg x_n := n \). We also define a sesqui-linear form by
\[
\langle x_i, x_j \rangle := w(i) \delta_{i,j},
\]
where \( w : \mathbb{N} \to (0, \infty) \) is a strictly positive weight function. So this is an inner product making \( C \) into a pre-Hilbert space. Again, it suffices to compute the co-Toeplitz operators \( T_g \) for \( g \) in the basis. So we compute as follows:
\[
C_{x_k}(x_n) = \pi_{x_k} \Delta(x_n) = \pi_{x_k} \left( \sum_{i+j=n} x_i \otimes x_j \right)
= \sum_{i+j=n} \langle x_k, x_j \rangle x_i = \sum_{i+j=n} \delta_{k,j} w(k) x_i.
\]
Now if \( k > n \) we have \( \delta_{k,j} = 0 \) for all the terms in the last sum, since \( j \leq n \). So \( C_{x_k}(x_n) = 0 \) if \( k > n \). For the opposite case \( 0 \leq k \leq n \) we have
\[
C_{x_k}(x_n) = \sum_{i+j=n} \delta_{k,j} w(k) x_i = w(k) x_{n-k}.
\]
If we define \( x_i := 0 \) and \( w(i) := 0 \) for all integers \( i < 0 \), then we can write this result as one formula for all \( k, n \in \mathbb{N} \):
\[
C_{x_k}(x_n) = w(k) x_{n-k}.
\]
So for \( k > 0 \) we have that \( C_{x_k} \) decreases degree by \( k \) and so is an annihilation operator. On the other hand \( C_{x_0} \) is a preservation operator.

4. Negative Degrees

Here we give a modification of the previous example that includes negative degrees. We let \( M \geq 1 \) be an integer and define \( C \) to be the complex vector space with basis \( \{ x_i \} \) for integers \( i \in [-M, M] \). So \( \dim C = 2M + 1 \).

We define \( \deg x_i := i \) for \( i \in [-M, M] \). For convenience we also define \( x_i := 0 \) for all integers \( i \) with \( |i| > M \). We use the same formulas as in the previous example, but with new interpretations. So, the co-multiplication \( \Delta \) is defined by (3.1), but now for integers \( |n| \leq M \). With our definitions only finitely many terms in the (now) infinite sum (3.1) are non-zero. We also define a sesqui-linear form by (3.2) but now for integers \( i, j \in [-M, M] \). Again, for convenience we put \( w(i) := 0 \) for \( |i| > M \). The same calculation as in the previous example gives
\[
C_{x_k}(x_n) = w(k) x_{n-k}
\]
but now for all integers \( k, n \in [-M, M] \). There are three cases:

1. \( C_{x_k} \) increases degree by \( |k| \) if \( k < 0 \) and is a creation operator.
2. \( C_{x_k} \) decreases degree by \( k \) if \( k > 0 \) and is an annihilation operator.
3. \( C_{x_k} \) preserves degree if \( k = 0 \) and is a preservation operator.

So we get the three types of operators relevant to physics by using the basis elements with positive, negative and zero degrees. We also can define a \( * \)-operation (\( \equiv \) conjugation) on \( C \) by putting \( x_i^* := x_{-i} \). This definition is
motivated by the theory of complex variables. Using this as motivation, for each \( i > 0 \) we then define the elements \( x_i \) to be holomorphic and the elements \( x_i^* \) to be anti-holomorphic. (The element \( x_0 \) could be defined as being both holomorphic and anti-holomorphic, if one wished. But I opt not to do that.) Then the anti-holomorphic elements are symbols of creation operators while the holomorphic elements are symbols of annihilation operators.

5. Matrix Coalgebra

This example comes from Example 2.4.1 in [1]. Let \( \mathcal{C} \) be the vector space with basis \( \{ E_{i,j} \mid 1 \leq i, j \leq n \} \), where \( n \geq 1 \) is an integer. So, \( \text{dim } \mathcal{C} = n^2 \).

Of course, the motivation is that \( E_{i,j} \) is analogous to the \( n \times n \) matrix with all entries 0, except for row \( i \) and column \( j \) which has the entry 1.

Let the co-multiplication be determined by

\[
\Delta(E_{i,j}) := \sum_{k=1}^{n} E_{i,k} \otimes E_{k,j}.
\]

(As a curious parenthetical remark, let us note that this vector space has a natural algebra structure motivated by matrix multiplication. However, this does not combine with this co-multiplication to give us a bi-algebra. See [1].)

We define an inner product on \( \mathcal{C} \) by making the basis \( \{ E_{i,j} \} \) orthonormal.

Then we calculate

\[
C_{E_{r,s}}(E_{i,j}) = \pi_{E_{r,s}} \Delta(E_{i,j}) = \pi_{E_{r,s}} \left( \sum_{k=1}^{n} E_{i,k} \otimes E_{k,j} \right)
\]

\[
= \sum_{k=1}^{n} \langle E_{r,s}, E_{k,j} \rangle E_{i,k} = \sum_{k=1}^{n} \delta_{r,k} \delta_{s,j} E_{i,k} = \delta_{s,j} E_{i,r}.
\]

So, \( C_{E_{r,s}}(E_{i,j}) \) is either zero or another basis element.

Another sesqui-linear form is given by \( \langle E_{i,j}, E_{r,s} \rangle := w(i + s) \delta_{i-j,r-s} \) with a weight function \( w : \mathbb{N} \to (0, \infty) \). Then

\[
C_{E_{r,s}}(E_{i,j}) = \pi_{E_{r,s}} \Delta(E_{i,j}) = \pi_{E_{r,s}} \left( \sum_{k=1}^{n} E_{i,k} \otimes E_{k,j} \right)
\]

\[
= \sum_{k=1}^{n} \langle E_{r,s}, E_{k,j} \rangle E_{i,k} = \sum_{k=1}^{n} w(r + j) \delta_{r-s,k-j} E_{i,k} = w(r + j) E_{i,j+r-s},
\]

where we put \( E_{i,j} = 0 \) if \( j \leq 0 \) or \( j > n \). If we define \( \text{deg } E_{i,j} := i + j \), then we see that \( C_{E_{r,s}} \) changes degree by \( r - s \). So, \( C_{E_{r,s}} \) is a creation operator if \( r > s \), it is an annihilation operator if \( r < s \) and finally it is a preservation operator if \( r = s \).

Using the adjoint operation of matrices as motivation, we also define a \( * \)-operation by \( E_{i,j}^* := E_{j,i} \). We also say that \( E_{i,j} \) is holomorphic if \( i < j \) (‘upper triangular’) and is anti-holomorphic if \( i > j \) (‘lower triangular’). As previously, the anti-holomorphic \( E_{i,j} \) are the symbols of (degree increasing)
creation operators and, on the other hand, the holomorphic $E_{i,j}$ are the symbols of (degree decreasing) annihilation operators. Also the ‘diagonal’ elements $E_{i,i}$, which are self-adjoint (or real) with respect to the $\ast$-operation, are symbols of (degree preserving) preservation operators.

6. Concluding Remarks

The quantization of co-algebras is a new field of research with co-Toeplitz quantization being the first theory that achieves this. It is remarkable that any sesqui-linear form defined on a co-algebra $C$ is sufficient extra structure to give us a co-Toeplitz quantization of $C$. It is noteworthy that in some of these examples a $\ast$-operation can be defined thereby giving holomorphic and anti-holomorphic elements, which are symbols whose co-Toeplitz operators are annihilation and creation operators, respectively.

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References

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