Topological model for \(h\)-vectors of simplicial manifolds

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Abstract. Any manifold with boundary gives rise to a Poincare duality algebra in a natural way. Given a simplicial poset \(S\) whose geometric realization is a closed orientable homology manifold, and a characteristic function, we construct a manifold with boundary such that graded components of its Poincare duality algebra have dimensions \(h^n_k(S)\). This gives a clear topological evidence for two well-known facts about simplicial manifolds: the nonnegativity of \(h^n\)-numbers (Novik–Swartz theorem) and the symmetry \(h^n_k = h^{n-k}\) (generalized Dehn–Sommerville relations).

1. Introduction

Let \(S\) be a pure simplicial poset of dimension \(n - 1\) and \([m] = \{1, \ldots, m\} = \text{Vert}(S)\) be the set of its vertices. Let \(k\) be a ground ring which is either a field or \(\mathbb{Z}\). A map \(\lambda: [m] \to k^n\) is called a \((\text{homological})\ \text{characteristic function}\) if, for any maximal simplex \(I \in S\), the set of vertices of \(I\) maps to the basis of \(k^n\). We suppose that there is a fixed basis in \(k^n\), and, for any vertex \(i \in [m]\), the value \(\lambda(i)\) has coordinates \((\lambda_{i,1}, \ldots, \lambda_{i,n})\) in this basis.

Let \(k[S]\) be the face ring of \(S\) (see [11, 3]). By definition, \(k[S]\) is a commutative associative graded algebra over \(k\) generated by formal variables \(v_I\), one for each simplex \(I \in S\), with relations

\[v_{I_1} \cdot v_{I_2} = v_{I_1 \cap I_2} \cdot \sum_{J \in I_1 \vee I_2} v_J, \quad v_{\emptyset} = 1.\]

Here \(I_1 \vee I_2\) denotes the set of least upper bounds of \(I_1, I_2 \in S\), and \(I_1 \cap I_2 \in S\) is the intersection of simplices (it is well-defined and unique when \(I_1 \vee I_2 \neq \emptyset\)).

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summation over an empty set is assumed to be 0. For topological reasons we take
the doubled grading on the ring: the generator $v_i$ has degree $2|I|$, where $|I|$ denotes
the rank of $I$. The natural map $k[m] = k[v_1, \ldots, v_m] \to k[S]$ defines the structure
of $k[m]$-module on $k[S]$.

Any characteristic function $\lambda: [m] \to k^n$ determines the set of linear elements:
\[ \theta_1 = \sum_{i \in [m]} \lambda_{i,1} v_i, \quad \theta_2 = \sum_{i \in [m]} \lambda_{i,2} v_i, \quad \ldots, \quad \theta_n = \sum_{i \in [m]} \lambda_{i,n} v_i \in k[S] \]
(these elements have degree 2, but we will use the term “linear” when its meaning
is clear from the context). The definition of characteristic function implies that
$\theta_1, \ldots, \theta_n$ is a linear system of parameters in $k[S]$ (e.g. [4 Lm.3.5.8]). Moreover, any
linear system of parameters arise from some characteristic function in this way. Let
$\Theta$ be the ideal in $k[S]$ generated by the elements $\theta_1, \ldots, \theta_n$.

The quotient $k[S]/\Theta$ is a finite-dimensional vector space. The standard reasoning
in commutative algebra implies that, whenever $S$ is Cohen–Macaulay, the dimension
of the homogeneous component $(k[S]/\Theta)_{2k}$ is $h_k$, the $h$-number of $S$ [11].

When $S$ is Buchsbaum, the additive structure of $k[S]/\Theta$ is still independent of
the choice of characteristic function but dimensions of homogeneous components
have more complicated description. By Schenzel’s theorem [10, 8] the dimension of
$(k[S]/\Theta)_{2k}$ is
\[ h'_k \overset{\text{def}}{=} h_k + \binom{n}{k} \left( \sum_{j=1}^{k-1} (-1)^{k-j-1} \tilde{\beta}_{j-1}(S) \right), \]
where $\tilde{\beta}_k(S) = \dim \tilde{H}_k(S; k)$.

Recall that the socle of a $k[m]$-module $\mathcal{M}$ is a $k$-subspace
\[ \text{Soc } \mathcal{M} \overset{\text{def}}{=} \{ y \in \mathcal{M} \mid k[m]^+ \cdot y = 0 \}, \]
where $k[m]^+$ is the maximal graded ideal of the ring $k[m]$. Since the products with
polynomials of positive degrees are trivial, the socle is a $k[m]$-submodule of $\mathcal{M}$.
In [8] Novik and Swartz proved the existence of certain submodules in $\text{Soc}(k[S]/\Theta)$
for any Buchsbaum simplicial poset. Namely, in degree $2k < 2n$ there exists a vector
subspace
\[ (I_{NS})_{2k} \subseteq \text{Soc}(k[S]/\Theta)_{2k}, \]
isomorphic to $\binom{n}{k} \tilde{H}^{k-1}(S; k)$ (this notation means the direct sum of $\binom{n}{k}$ copies of
$\tilde{H}^{k-1}(S; k)$). Let $I_{NS}$ denote the direct sum of $(I_{NS})_{2k}$ over all $k$, where we assumed
$(I_{NS})_{2n} = 0$. Since $I_{NS}$ lies in the socle, it is a $k[m]$-submodule. Moreover, $I_{NS}$ is an
ideal in $k[S]/\Theta$ (for simplicial complex this fact easily follows from the surjectivity
of the map $k[m] \to k[S]$, and for simplicial poset, whose geometrical realization is a
homology manifold, this was checked in [2 Rem. 8.3]). Thus we may consider
the quotient ring $(k[S]/\Theta)/I_{NS}$. The dimension of its homogeneous component of
degree 2k is equal to $h_k''$ where

$$h_k'' = h_k' - \binom{n}{k} \tilde{\beta}_{k-1}(S) = h_k + \binom{n}{k} \left( \sum_{j=1}^{k} (-1)^{k-j-1} \tilde{\beta}_{j-1}(S) \right),$$

for $0 \leq k \leq n-1$, and $h_n'' = h_n'$. In particular, $h_k'' \geq 0$ for any Buchsbaum simplicial poset.

Now we restrict to the case when the ground ring is either $\mathbb{Z}$ or $\mathbb{Q}$. The class of Cohen–Macaulay simplicial posets contains an important subclass of sphere triangulations. By abuse of terminology we call simplicial poset a homology sphere (resp. manifold) if its geometrical realization is a homology sphere (resp. manifold).

Every homology sphere is Cohen–Macaulay. For homology spheres the ring $\mathbb{k}[S]/\Theta$ is a Poincare duality algebra (this is not surprising in view of Danilov–Jurkiewicz and Davis–Januszkiewicz theorems). In general one can prove this by the following topological argument. Consider the cone over $|S|$ endowed with a dual simple face stratification and consider the identification space $X_S = (\text{Cone} \times T^n)/\sim$, similar to the construction of quasitoric manifold [6]. Using the same ideas as in [6], one can prove that the cohomology algebra of $X_S$ over $\mathbb{k}$ is isomorphic to $\mathbb{k}[S]/\Theta$ (see e.g. [7]). When $\mathbb{k} = \mathbb{Z}$, the space $X_S$ is a homology manifold over integers. In case $\mathbb{k} = \mathbb{Q}$, this space is a homology manifold over $\mathbb{Q}$. In both cases the Poincare duality over the corresponding ring implies that $\mathbb{k}[S]/\Theta$ is a Poincare duality algebra. In particular, this proves Dehn–Sommerville relations for homology spheres: $h_k = h_{n-k}$.

The goal of this paper is to construct a topological model for homology manifolds. Every closed homology manifold is a Buchsbaum simplicial poset, so the ideal $I_{NS} \subset \mathbb{k}[S]/\Theta$ is defined. There holds

**Theorem 1** (cf. [9], Th.1.4). Let $S$ be a simplicial poset whose geometrical realization is a closed connected orientable homology manifold. When the ground ring is either $\mathbb{Q}$ or $\mathbb{Z}$, the ring $(\mathbb{k}[S]/\Theta)/I_{NS}$ is a Poincare duality algebra.

It was proved in [9] that $(\mathbb{k}[S]/\Theta)/I_{NS}$ is a Gorenstein ring, which implies the theorem over any field when $S$ is a simplicial complex. The theorem gives a straightforward evidence for the generalized Dehn–Sommerville relations for manifolds: $h_k'' = h_{n-k}''$.

The idea of our proof is the following. In Section 2 we associate a Poincare duality algebra with any manifold with boundary $(M, \partial M)$, either smooth, topological, or homological. This algebra will be denoted $\text{PD}_{(M, \partial M)}^*$. Given any homology manifold $S$, instead of taking the cone (as in the case of spheres) we consider the topological space $\hat{Q} = |S| \times [0, 1]$. This space is a manifold with two boundary components: $\hat{\partial}_0 \hat{Q}$ and $\hat{\partial}_1 \hat{Q}$. Consider the identification space $\hat{X} = (\hat{Q} \times T^n)/\sim$ with the relation collapsing certain torus subgroups over the points of $\hat{\partial}_0 \hat{Q}$ similar to a quasitoric case, and not touching the points over $\hat{\partial}_1 \hat{Q}$. The space $\hat{X}$ is a homology manifold
with boundary; its boundary consists of points over $\partial_1\hat{Q}$. Then Theorem \(\Box\) is an immediate consequence of

**Theorem 2.** The algebra $PD^*_\langle\hat{X},\hat{X}\rangle$ is isomorphic to $(\mathbb{k}[S]/\Theta)/I_{NS}$.

The only place in the arguments, where we need the restriction on a ground ring, is the construction of torus spaces. The relation $\sim$ collapses certain compact subgroups of the compact torus $T^n$, and this identification cannot be defined for general characteristic functions over general fields. Nevertheless, if the characteristic function $\lambda$ over $\mathbb{k}$ can be represented as $\lambda' \otimes \mathbb{k}$ for some characteristic function $\lambda'$ over $\mathbb{Z}$ (or $\mathbb{Q}$), then the statements hold true over a field $\mathbb{k}$ and this particular choice of characteristic function.

2. Poincare duality algebras

**Definition 2.1.** A finite-dimensional, graded, associative, graded-commutative, connected algebra $A^* = \bigoplus_{k=0}^d A^k$ over $\mathbb{k}$ is called Poincare duality algebra of formal dimension $d$, if

1. $A^d \cong \mathbb{k}$;
2. The product map $A^k \otimes A^{d-k} \rightarrow A^d$ is a non-degenerate pairing for all $k = 0, \ldots, d$. Over integers the finite torsion should be mod out.

While the motivating examples of Poincare duality algebras are cohomology of connected orientable closed manifolds, there exist another natural source of duality algebras.

**Construction 2.2.** Let $(M, \partial M)$ be a compact connected orientable manifold with boundary, $\dim M = d$. As a technical requirement we will also assume that $M$ contains a neighborhood of $\partial M$ of the form $\partial M \times [0, \varepsilon]$. Consider the $\mathbb{k}$-module $A^* = \bigoplus_{k=0}^d A^k$, where

$$A^k = \begin{cases} H^0(M), & \text{if } k = 0; \\ \text{image of } \iota^*: H^k(M, \partial M) \rightarrow H^k(M), & \text{if } 0 < k < d; \\ H^d(M, \partial M), & \text{if } k = d. \end{cases}$$

The homomorphism $\iota^*: H^k(M, \partial M) \rightarrow H^k(M)$ is induced by the inclusion $\iota: (M, \emptyset) \hookrightarrow (M, \partial M)$.

There is a well-defined product on $A^*$ induced by the cup-products in cohomology. Indeed, let $a_1 \in A^{k_1}$ and $a_2 \in A^{k_2}$. If either $k_1$ or $k_2$ is zero, then there is nothing to define, since $A^0$ is spanned by the unit of the ring. If $k_1 + k_2 < d$ then $a_1 \cdot a_2$ is just the product of two elements in the ring $H^*(M)$. This product lies in the image of $H^*(M, \partial M)$. If $k_1 + k_2 = d$, then we may consider the elements $b_1, b_2 \in H^*(M, \partial M)$ such that $\iota^*(b_i) = a_i$, and take their product in the ring $H^*(M, \partial M)$. This gives an element in $H^d(M, \partial M) = A^d$ which we call the product of $a_1$ and $a_2$. It is easily
seen that this element does not depend on the choice of representatives $b_1, b_2$ for the elements $a_1, a_2$.

The Poincare–Lefschetz duality implies that the pairing between $A^k$ and $A^{d-k}$ is non-degenerate [5]. Thus $A^*$ is a Poincare duality algebra. We denote it by $\text{PD}_{(M,\partial M)}$ and call the Poincare duality algebra of a manifold with boundary.

Remark 2.3. By Poincare–Lefchetz duality, instead of cohomology we can work with homology. We have

$$\text{PD}^k_{(M,\partial M)} \cong \begin{cases} 
H_d(M, \partial M), & \text{if } k = 0; \\
\text{image of } t_*: H_{d-k}(M) \to H_{d-k}(M, \partial M), & \text{if } 0 < k < d; \\
H_0(M), & \text{if } k = d,
\end{cases}$$

and the product is given by the intersection product in homology.

3. Collar model

3.1. Buchsbaum simplicial posets. Recall that a finite partially ordered set (poset) $S$ is called simplicial if: (1) there exists a unique minimal element $\hat{0} \in S$; (2) for each element $J \in S$, the lower order ideal $\{I \in S \mid I \leq J\}$ is isomorphic to the poset of faces of a $k$-simplex for some number $k$. Elements of $S$ are called simplices, and the atoms of $S$ are called vertices. The number $k$ is called the dimension of $I$ and the number $|I| = \dim I + 1$, which is equal to the number of vertices of $I$, is called the rank of $I$. In the following we assume that $S$ is pure of dimension $n-1$, which means that all maximal simplices of $S$ have the same rank $n$. Let $S'$ be the barycentric subdivision of $S$. For each proper simplex $I \in S \setminus \hat{0}$ consider the following subsets of the geometrical realization $|S| = |S'|$:

$$G_I = |\{(I_0 < I_1 < \ldots) \in S' \text{ such that } I_0 \geq I\}|,$$

$$\partial G_I = |\{(I_0 < I_1 < \ldots) \in S' \text{ such that } I_0 > I\}|.$$

The subset $G_I$ is called the face of $|S|$ dual to $I$. A simplicial poset $S$ is called Buchsbaum (over $\mathbb{K}$) if $H_j(G_I, \partial G_I; \mathbb{K}) = 0$ for any $I \in S \setminus \hat{0}$ and $j \neq \dim G_I$. In particular, any homology manifold is Buchsbaum, since in this case $G_I$ are homological cells.

3.2. Collar model. Consider the compact $n$-torus with a fixed coordinate representation $T^n = \{(t_1, \ldots, t_n) \mid t_\lambda \in \mathbb{C}, |t_\lambda| = 1\}$. If $\mathbb{K}$ is either $\mathbb{Z}$ or $\mathbb{Q}$, the vector $w = (w_1, \ldots, w_n) \in \mathbb{K}^n$ determines a compact 1-dimensional subgroup $t^w = \{e^{2\pi \sqrt{-1}w_1 t}, \ldots, e^{2\pi \sqrt{-1}w_n t} \mid t \in \mathbb{R}\}$. Let $[m] = \text{Vert}(S)$ be the set of vertices of $S$ and let $\lambda: [m] \to \mathbb{K}^n$ be a characteristic function over $\mathbb{Z}$ or $\mathbb{Q}$. Let $T_I \subset T^n$ denote the one dimensional subgroup $t^{\lambda(I)}$. For a simplex $I \in S$, let $T_I$ denote the product of the one-dimensional subgroups $T_i$ corresponding to the vertices of $I$ (the product is taken inside $T^n$). The definition of characteristic function implies that $T_I$ is a compact subtorus of $T^n$ of dimension $|I|$.
Consider the space \( \hat{Q} = |S| \times [0, 1] \) which will be called the \textit{collar} of \(|S|\). Let \( \partial_0 \hat{Q} \) denote the subset \(|S| \times \{ \varepsilon \} \) for \( \varepsilon = 0, 1 \). The faces \( G_I \) can be considered as the subsets of \( \partial_0 \hat{Q} \subset \hat{Q} \). To make the notation uniform, we set \( G_0 = \hat{Q} \) and \( T_0 = \{ 1 \} \subset T^n \).

**Construction 3.1.** Consider the identification space \( \hat{X} = (\hat{Q} \times T^n)/\sim \), where the points \((x, t), (x', t')\) are identified whenever \( x = x' \in G_I \) and \( t^{-1}t' \in T_I \) for some simplex \( I \in S \). Let \( f: \hat{Q} \times T^n \to \hat{X} \) denote the quotient map, and \( \mu \) denote the projection to the first factor, \( \mu: \hat{X} \to \hat{Q} \). The preimage \( \mu^{-1}(G_I) \) is denoted by \( X_I \). Let \( \partial_1 \hat{X} \) denote the subset \( \partial_1 \hat{Q} \times T^n \subset \hat{X} \).

### 3.3. Absolute and relative spectral sequences.

The dual face structure on \(|S|\) induces a topological filtration
\[
Q_0 \subset Q_1 \subset \ldots \subset Q_{n-1} = \partial_0 \hat{Q} \subset Q_n = \hat{Q},
\]
which lifts to the orbit type filtration on \( \hat{X} \):
\[
X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_n = \hat{X}.
\]
Let \( (E_{\hat{Q}})_p^1 = H_{p+q}(Q_p; Q_{p-1}) \Rightarrow H_{p+q}(\hat{Q}) \) and \( (E_{\hat{X}})_p^1 = H_{p+q}(X_p; X_{p-1}) \Rightarrow H_{p+q}(\hat{X}) \) be the homological spectral sequences associated with these filtrations. By the result of [1], whenever \( S \) is Buchsbaum, the map
\[
f^*_q: \bigoplus_{q_1 + q_2 = q} (E_{\hat{Q}})^2_{p, q_1} \otimes H_{q_2}(T^n) \to (E_{\hat{X}})^2_{p, q}
\]
is an isomorphism for \( p > q \) and injective for \( p = q \). The rightmost column of \((E_{\hat{Q}})^{*, *}_{p, q}\) (which is the source of all higher differentials) vanish: \((E_{\hat{Q}})^{1, *}_{p, q} \cong H_{n+p}(\hat{Q}, \partial_0 \hat{Q}) = 0\), since the collar \( \hat{Q} \) collapses to \( \partial_0 \hat{Q} \). Similar for \((E_{\hat{X}})^{*, *}_{p, q}\). Thus there are no higher differentials \( d^{p, 2} \) in both spectral sequences.

We also need the homological spectral sequences for the relative homology:
\[
(E_{(\hat{Q}, \partial_1 \hat{Q})})_{p, q}^r \Rightarrow H_{p+q}(\hat{Q}, \partial_1 \hat{Q}) = 0, \quad (E_{(\hat{X}, \partial_1 \hat{X})})_{p, q}^r \Rightarrow H_{p+q}(\hat{X}, \partial_1 \hat{X}).
\]
The first pages are the following:
\[
(E_{(\hat{Q}, \partial_1 \hat{Q})})_{p, q}^1 = \begin{cases} 
H_{p+q}(Q_p; Q_{p-1}), & \text{if } p < n; \\
H_{n+q}(\hat{Q}, Q_{n-1} \sqcup \partial_1 \hat{Q}), & \text{if } p = n,
\end{cases}
\]
\[
(E_{(\hat{X}, \partial_1 \hat{X})})_{p, q}^1 = \begin{cases} 
H_{p+q}(X_p; X_{p-1}), & \text{if } p < n; \\
H_{n+q}(\hat{X}, X_{n-1} \sqcup \partial_1 \hat{X}), & \text{if } p = n.
\end{cases}
\]
Note that the rightmost terms \((E_{(\hat{Q}, \partial_1 \hat{Q})})_{n, q}^1\) have the form:
\[
H_{n+q}(\hat{Q}, \partial_0 \hat{Q} \sqcup \partial_1 \hat{Q}) \cong H_{n+q}(|S| \times [0, 1], |S| \times \{ 0, 1 \}) \cong H_{n+q-1}(S),
\]
and the higher differentials
\[
(d_{(\hat{Q}, \partial_1 \hat{Q})})^*: (E_{(\hat{Q}, \partial_1 \hat{Q})})_{n, -r+1}^* \to (E_{(\hat{Q}, \partial_1 \hat{Q})})_{n, -r, 0}^* \cong H_{n-r}(S)
\]
are isomorphisms (so that the spectral sequence for \((\hat{Q}, \partial_1 \hat{Q})\) collapses to zero). Similar to the non-relative case, the induced map

\[
f^*_s : \bigoplus_{q_1 + q_2 = q} (E_{(\hat{Q}, \partial_1 \hat{Q})}/p_{q_1}) \otimes H_{q_2}(T^n) \to (E_{(\hat{X}, \partial_1 \hat{X})}/p_{q})^2
\]

is an isomorphism for \(p > q\) and injective for \(p = q\) (this follows from the general method developed in \([1]\)).

### 3.4. Proof of Theorem 2

The proof essentially relies on calculations made in \([2]\). If \(S\) is a connected orientable homology manifold, then \(\hat{X}\) is a connected orientable homology manifold with the boundary \(\partial_1 \hat{X} \cong |S| \times T^n\). The boundary admits a collar neighborhood as required in Construction \([2]\). For \(I \neq 0\) the subset \(X_I\) is a closed submanifold of codimension \(2|I|\) lying in the interior of \(\hat{X}\). It is called the face submanifold, and its homology class \([X_I] \in H_{2n-2|I|}(\hat{X})\) is called the face class. Note that for \(|I| \neq 0\) the classes \([X_I]\) appear in the spectral sequence \((E_\hat{X})^*_{s,*}\) as the free generators of the group: \((E_\hat{X})^1_{q,q}\) with \(q = n - |I|\). The relations on these classes in \(H_{2q}(\hat{X})\) are precisely the images of first differentials, hitting the group \((E_\hat{X})^1_{q,q}\). In \([2]\) Prop.4.3 and Lm.8.2 we checked that these relations are the same as the linear relations on \(v_I\) in the quotient ring \((k[S]/\Theta)_2(n-q)\) when \(q \leq n - 2\). If \(q = n - 1\), there are no relations on \([X_I] \in H_{2n-2}(\hat{X})\) since there are no differentials hitting the group \((E_\hat{X})^1_{n-1,n-1}\).

In addition to face classes, there exist other homology classes in \(H_s(\hat{X})\), namely the classes coming from the part of the spectral sequence below the diagonal. They lie in the groups \((E_\hat{X})^2_{p,q} \cong H_p(\hat{Q}) \otimes H_q(T^n)\) for \(q < p < n\). In \([2]\) we called them spine classes.

Let us keep track on the behavior of homology classes, when they map to the relative homology by the homomorphism \(\iota_* : H_s(\hat{X}) \to H_s(\hat{X}, \partial_1 \hat{X})\). Again, we may look at their representatives in the spectral sequence \((E_{(\hat{X}, \partial_1 \hat{X})})^*\). At this time, higher differentials are nontrivial. All spine classes of \(H_s(\hat{X})\) are killed by higher differentials. Indeed, they lie in the part of the relative spectral sequence which is isomorphic to \((E_{(\hat{Q}, \partial_1 \hat{Q})})^*_{s,*} \otimes H_s(T^n)\), and the latter sequence collapses to 0.

On the other hand, the diagonal cells of the relative spectral sequence are hit by higher differentials as well. Thus there are more relations on \([X_I]\) in the group \(H_s(\hat{X}, \partial_1 \hat{X})\) than in the group \(H_s(\hat{X})\). The higher differential

\[
(d_{(\hat{X}, \partial_1 \hat{X})}^\gamma) : (E_{(\hat{X}, \partial_1 \hat{X})}/n,n-2r+1,1) \to (E_{(\hat{X}, \partial_1 \hat{X})}/n-r,n-r,1)
\]

is injective, and gives an inclusion of \(H_{n-r}(S) \otimes H_{n-r}(T^n)\) into \((E_{(\hat{X}, \partial_1 \hat{X})}/n-n-r,1)\). Under the degree reversing identification \([X_I] \leftrightarrow v_I\) (and by Poincare duality in \(S\)), this inclusion gives the Novik–Swartz submodule \((I_{NS})_{2r} \cong \hat{H}^{n-r-1}(S)\) inside \((k[S]/\Theta)_{2r}\), for \(r \geq 2\) (see details in \([2]\)). When \(r = 1\), only the first differential \((d_{(\hat{X}, \partial_1 \hat{X})}^\gamma)_{1}\) hits
the cell $(E_{(\hat{X}, \partial_1 \hat{X})})_{n-1, n-1}^1$. Its image corresponds to the linear span of $\theta_1, \ldots, \theta_n$ in $k[S]_2$. There is no Novik–Swartz submodule $I_{NS}$ in $(k[S]/\Theta)_2$ since $S$ is connected.

These considerations prove that the image of the map

$$\nu_* : H_{2n-2r}(\hat{X}) \to H_{2n-2r}(\hat{X}, \partial_1 \hat{X})$$

is isomorphic to the homogeneous component of $(k[S]/\Theta)/I_{NS}$ of degree $2r$ for each $0 < r < n$.

When $r = n$, the submodule $(I_{NS})_{2n}$ is trivial. Thus $H_0(\hat{X})$ coincides with $(k[S]/\Theta)_{2n} = ((k[S]/\Theta)/I_{NS})_{2n}$. The group $H_{2n}(\hat{X}, \partial_1 \hat{X}) \cong k$ is obviously identified with $((k[S]/\Theta)/I_{NS})_0 \cong k$.

Theorem 2 now follows from Remark 2.3 and the fact that the correspondence $[X_I] \leftrightarrow \nu_I$ translates the intersection product on $\hat{X}$ to the product in the face ring.

References

[1] A. Ayzenberg, Locally standard torus actions and sheaves over Buchsbaum posets, preprint arXiv:1501.04768.
[2] A. Ayzenberg, Homology cycles in manifolds with locally standard torus actions, preprint arXiv:1502.01130v2.
[3] V. M. Buchstaber, T. E. Panov, Combinatorics of Simplicial Cell Complexes and Torus Actions, Proc. of the Steklov Institute of Mathematics, Vol. 247, 2004, pp. 1–17.
[4] V. Buchstaber, T. Panov, Toric Topology, preprint arXiv:1210.2368.
[5] G. E. Bredon, Topology and Geometry. Grad. Texts in Math. 139 (1993).
[6] M. Davis, T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. J., 62:2 (1991), 417–451.
[7] M. Masuda, T. Panov, On the cohomology of torus manifolds, Osaka J. Math. 43 (2006), 711–746.
[8] I. Novik, Ed Swartz, Socles of Buchsbaum modules, complexes and posets, Adv. Math., 222 (2009), 2059–2084.
[9] I. Novik and E. Swartz, Gorenstein rings through face rings of manifolds, Composit. Math. 145 (2009), 993–1000.
[10] P. Schenzel, On the Number of Faces of Simplicial Complexes and the Purity of Frobenius, Math. Zeitschrift 178, 125–142 (1981).
[11] R. P. Stanley, f-vectors and h-vectors of simplicial posets, J. Pure Appl. Algebra 71 (1991), 319–331.

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