Origin of $2^+_1$ Excitation Energy Dependence on Valence Nucleon Numbers

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Abstract

It has been shown recently that a simple formula in terms of the valence nucleon numbers and the mass number can describe the essential trends of excitation energies of the first $2^+_1$ states in even-even nuclei. By evaluating the first order energy shift due to the zero-range residual interaction, we find that the factor which reflects the effective particle number participating in the interaction from the Fermi orbit governs the main dependence of the first $2^+_1$ excitation energy on the valence nucleon numbers.

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Valence nucleon numbers $N_p$ and $N_n$ have been extensively adopted to parameterize various nuclear properties phenomenologically. The valence proton (neutron) number $N_p$ ($N_n$) is defined as the number of proton (neutron) particles below the mid-shell or the number of proton (neutron) holes past the mid-shell within the given major shell. Hamamoto was the first to show that the square root of the ratio of the measured and the single-particle $B(E2)$ values, $[B(E2)_{\text{exp}}/B(E2)_{\text{sp}}]^{1/2}$, is roughly proportional to the product $N_p N_n$ [1]. Casten extended the idea and suggested the $N_p N_n$ scheme where he showed that if we parameterize the collective variables or operators in even-even nuclei in terms of the product $N_p N_n$ then we get a substantial reduction in the number of parameters without serious loss of accuracy [2, 3]. It was also demonstrated that the $N_p N_n$ scheme could be applied not only to even-even nuclei but also to odd-$A$ and odd-odd nuclei [4].

Recently, we proposed a simple empirical formula in terms of the valence nucleon numbers $N_p$ and $N_n$, and the mass number $A$ for the excitation energies $E_x(2^+_1)$ of the first $2^+$ states

\[ E_x(2^+_1) = a_1 N_p + a_2 N_n + a_3 A + a_4. \]

FIG. 1: Excitation energies of the first $2^+$ states in even-even nuclei. The solid lines represent isotopic chains. Part (a) shows the data quoted from Ref. 6, and part (b) shows the results obtained by Eq. (1).
in even-even nuclei \cite{5}. The formula is given by

\[ E_x = \alpha A^{-\gamma} + \beta [\exp(-\lambda N_p) + \exp(-\lambda N_n)] \]  

(1)

where the parameters \( \alpha, \beta, \gamma, \) and \( \lambda \) are fitted from the data. From Fig. 1 which is reproduced from Ref. 5, we find that the essential trends of the excitation energies \( E_x(2^+ \uparrow) \) are well reproduced throughout the periodic table. The data displayed in part (a) of Fig. 1 is quoted from the recent compilation by Raman et al. \cite{6} and the results shown in part (b) of Fig. 1 were obtained by applying Eq. (1) where \( \alpha = 34.9 \text{ MeV}, \beta = 1.00 \text{ MeV}, \gamma = 1.19, \) and \( \lambda = 0.36 \) \cite{5}.

In this brief report, we want to examine what makes such an odd dependence of the excitation energies \( E_x(2^+ \uparrow) \) as given by Eq. (1) on the valence nucleon numbers \( N_p \) and \( N_n \). Since we are dealing with the systematics observed throughout the whole periodic table instead of inspecting the detailed structure of a single nucleus, we want to keep the physics involved as simple as possible. For that purpose, we first find the difference between the energy of the lowest two quasi-particle state and the excitation energy of the first \( 2^+ \) state. Then we compare it with the first order energy shift due to the residual interaction. Here, we obtain the former energy difference from the measured quantities and the latter energy shift we evaluate by using simple perturbation theory.

Within the standard BCS theory \cite{7}, the single quasi-particle energy \( E_k \) of the \( k \)-th orbit is given by

\[ E_k = \sqrt{(\epsilon_k - \mu)^2 + \Delta^2} \]  

(2)

where \( \epsilon_k, \mu, \) and \( \Delta \) are the single particle energy, the Fermi energy, and the gap energy, respectively. We can obtain the gap energy \( \Delta \) empirically from the binding energies of the adjacent nuclei. Let \( B(Z, N) \) be the negative binding energy of a nucleus whose atomic number and neutron number are \( Z \) and \( N \), respectively. Then the proton gap energy \( \Delta_p \) and the neutron gap energy \( \Delta_n \) can be approximated by the following three point expressions \cite{8}:

\[ \Delta_p \approx \frac{1}{2} |B(Z + 1, N) - 2B(Z, N) + B(Z - 1, N)|, \]  

(3)

\[ \Delta_n \approx \frac{1}{2} |B(Z, N + 1) - 2B(Z, N) + B(Z, N - 1)|. \]  

(4)

Let us write the excitation energy \( E_x(2^+ \uparrow) \) of the first \( 2^+ \) state by simple perturbation
FIG. 2: Upper part shows twice the gap energy $2\Delta$, which is the lesser of $2\Delta_p$ (Eq. (3)) and $2\Delta_n$ (Eq. (4)). Binding energies are taken from Ref. 10. Lower part shows the measured excitation energies of the first $2^+$ states (the same as those shown in Fig. (a)).

theory as

$$E_x(2^+_1) = E^{(0)} + E^{(1)} + \cdots.$$  (5)

Then the unperturbed energy $E^{(0)}$ is equal to the energy of the lowest two quasi-particle state, $E_k + E_{k'}$ where $k$ and $k'$ indicate the orbits which belong to the lowest two quasi-particle state. Also the first order energy shift $E^{(1)}$ is given by

$$E^{(1)} = \langle j_k j_{k'} | \hat{V} | j_k j_{k'} \rangle_{J^\pi=2^+}.$$  (6)

which is the expectation value of the residual interaction $\hat{V}$ on the lowest two quasi-particle state which is coupled to the total angular momentum $J^\pi = 2^+$. 

From Eq. (2), we have that $E_k \approx \Delta$ for the single particle orbits $k$ near the Fermi level. Therefore, we take twice the gap energy $2\Delta$ in place of the lowest two quasi-particle energy $E_k + E_{k'}$ [9]. At the upper part of Fig. 2, twice the gap energy, $2\Delta$, is shown for even-even nuclei. Here $2\Delta$ is determined by the lesser of $2\Delta_p$ and $2\Delta_n$ which is estimated empirically either by Eq. (3) or by Eq. (4) [10]. We can observe that all of the $2\Delta$ are located higher in energy than the measured $E_x(2^+_1)$ data, which is shown at the lower part of Fig. 2. We, therefore, want to compare the difference between $2\Delta$ at the upper part and $E_x(2^+_1)$ at the lower part of Fig. 2 with the first order energy shift $E^{(1)}$ which will be given shortly.

In order to estimate $E^{(1)}$, we take only the central part of the residual interaction in the
zero-range approximation,
\[ \hat{V}(1, 2) = -v_0 \delta^3(\vec{r}_1 - \vec{r}_2), \] (7)
where \( v_0 \), which is the strength of the zero-range interaction, will be treated as the only model parameter in our calculations. Analytic expressions for the matrix elements of the zero-range interactions are readily available from the literature \[11\]. The expectation value appearing in Eq. (6) is given by
\[ E(1) = \langle j_k j_{k'} | \hat{V} | j_k j_{k'} \rangle_{J=2^+} = -v_0 \frac{R^4}{4\pi} \left\{ \frac{(2j_k + 1)(2j_{k'} + 1)}{(1 + \delta_{j_k j_{k'}})} \right\} \times \left\{ [1 + (-)^{j_k + j_{k'}}] \left( u^2_{j_k} v^2_{j_{k'}} + v^2_{j_k} u^2_{j_{k'}} \right) + [2 - (-)^{j_k + j_{k'}}] \left( u_{j_k} v_{j_{k'}} + v_{j_k} u_{j_{k'}} \right)^2 \right\} \]
\[ + (-)^{j_k + j_{k'}} < \frac{1}{2} j_k \frac{1}{2} j_{k'} | J1 >^2 (u_{j_k} v_{j_{k'}} + (-)^{j_k + j_{k'}} v_{j_k} u_{j_{k'}})^2 \}, \] (8)
where \( < R^4 > \), \( v^2_{j_k} \), and \( u^2_{j_k} \) are the integral of the radial wave functions
\[ < R^4 > = \int_0^{\infty} dr r^2 \left[ R_{j_k}(r) R_{j_{k'}}(r) \right]^2, \] (9)
the occupation probability of the \( k \)-th orbit, and \( u^2_{j_k} = 1 - v^2_{j_k} \), respectively.

We evaluate the matrix element in Eq. (8) by using the following procedure. First, the single-particle wave functions and energies are generated by diagonalizing the Woods-Saxon (WS) potential
\[ V_{WS}(r) = -\frac{V_0}{1 + \exp \left[ \frac{r - R_0}{a} \right]} \] (10)
where for the WS parameters we use \( V_0 = 50 \) MeV, \( R_0 = 1.27 \) fm, and \( a = 0.67 \) fm. Then the radial integral \( < R^4 > \) of Eq. (9) is evaluated using the WS wave functions. Also the occupation probabilities \( v^2_{j_k} \) in Eq. (8) are determined by solving the BCS equations with the simple monopole pairing interaction \[12\]
\[ < (j_1 j_2) | v_{pair} | (j_3 j_4) > = -\frac{1}{2} G \delta_{j_0} \sqrt{(2j_1 + 1)(2j_3 + 1)} \] (11)
where the strength \( G \) of the pairing interaction is adjusted to reproduce the measured gap energy by Eq. (3) or Eq. (4) for each nucleus in our calculations.

In Fig. 3, we compare our calculated results (solid squares) for \( -E^{(1)} \) with the measured differences (solid circles) \( 2\Delta - E_x(2^+ \rightarrow 0^+) \) for the following four cases: (a) \( Z = 52 \) isotopic
FIG. 3: Calculated results (solid squares) for the energy shift $-E^{(1)}$ are compared with the measured differences (solid circles) $2\Delta - E_x(2^+_1)$ for the following four cases: (a) $Z = 52$ isotopic chain, (b) $Z = 70$ isotopic chain, (c) $N = 70$ isotonic chain, and (d) $N = 102$ isotonic chain.

In our calculations, the strength $v_0$ of the zero-range interaction in Eq. (7) is adjusted so that the calculated $-E^{(1)}$ coincides exactly with the measured difference $2\Delta - E_x(2^+_1)$ for the heaviest lead isotope $^{184}_{82}$Pb in Fig. 3(d). The resulting adopted value of $v_0$ in all of our subsequent calculations for $E^{(1)}$ is $v_0 = 1,700\text{ MeV fm}^3$. By observing the results displayed in Fig. 3, we find that the first order energy shift $-E^{(1)}$ calculated by such a simple model like perturbation theory describes the measured difference $2\Delta - E_x(2^+_1)$ unbelievably well. Thus, this results enable us to examine $E^{(1)}$ in place of $E_x(2^+_1)$ to find the origin of its dependence on the valence nucleon numbers $N_p$ and $N_n$.

If the total angular momentum $j_f$ of the Fermi orbit is larger than $1/2$, which holds practically in most cases of our calculations, then both of the two quasi-particles which form the lowest two quasi-particle state belong to the Fermi orbit. Therefore $j_k$ and $j_{k'}$ in Eq. (8) can be replaced by $j_f$. This, in turn, reduces the expression for the energy shift $E^{(1)}$ in Eq. (8) to a very simple form which is written as

$$E^{(1)} = \langle j_f j_f | \hat{V} | j_f j_f \rangle_{J^\pi = 2^+}$$
$$= \left[ -\frac{3V_0}{20\pi} < R^4 > \langle j_f | \frac{1}{2} j_f - \frac{1}{2} | 2, 0 >^2 \right] \left[ (2j_f + 1)^2 u_{j_f}^2 v_{j_f}^2 \right] \tag{12}$$

where we use the known value of the Clebsch-Gordan coefficient

$$\langle j \frac{1}{2} j \frac{1}{2} | J, 1 > = 0 \tag{13}$$
FIG. 4: Calculated results (solid squares) for the energy shift $-E^{(1)}$ are compared with the effective particle number $(2j_f + 1)^2 u_{j_f}^2 v_{j_f}^2$ for the same four cases as those in Fig. 3. Note that the scale for the energy shift is shown at the far left axis while that for the effective particle number is shown at the far right axis.

which holds identically for any even $J$ and half integer values of $j$.

The expression for the first order energy shift $E^{(1)}$ can be divided into two major factors as shown by the large square brackets in Eq. (12). One is the radial integral $\langle R^4 \rangle$ times the Clebsch-Gordan coefficient squared and the other is $(2j_f + 1)^2 u_{j_f}^2 v_{j_f}^2$ which can be interpreted to reflect the number of effective particles that participate in the interaction from the Fermi orbit. Out of the two factors, we expect that, the former is kept more or less the same over different isotopes and isotones. Therefore, the variation in values of $E^{(1)}$ follows that of the effective particle number $(2j_f + 1)^2 u_{j_f}^2 v_{j_f}^2$. In order to check this point, we compare the negative of the first order energy shift $-E^{(1)}$ (solid squares) with the effective particle number $(2j_f + 1)^2 u_{j_f}^2 v_{j_f}^2$ (solid circles) in Fig. 4 for the same four cases as those in Fig. 3. Note that the scale for the energy shift is shown at the far left axis while that for the effective particle number is shown at the far right axis in Fig. 4. We can confirm undoubtedly from Fig. 4 that the behavior of $E^{(1)}$ follows that of the effective particle number quite closely.

Finally, we compare the measured excitation energies (solid circles) $E_x(2^+_1)$ of the first $2^+$ states in Fig. 5 for the same four cases as those in Fig. 3 with the one calculated by perturbation theory (solid squares), namely $2\Delta + E^{(1)}$, and with the one obtained from the empirical formula (solid triangles) given by Eq. (1). By observing these plots, we can ascertain, first of all, that the empirical formula, Eq. (1), expressed in terms of the valence nucleon numbers
FIG. 5: Measured excitation energies (solid circles) $E_x(2^+_1)$ of the first $2^+$ states for the same four cases as those in Fig. 3 are compared with the one calculated by the perturbation theory (solid squares), namely $2\Delta + E^{(1)}$, and with the one obtained from the empirical formula (solid triangles) given by Eq. (1). Measured excitation energies $E_x(2^+_1)$ are quoted from Ref. 6.

describes the measured excitation energies $E_x(2^+_1)$ quite well, even quantitatively to some extent. Furthermore, we can also confirm that simple perturbation theory can be employed in examining the possible dependence of $E_x(2^+_1)$ on the valence nucleon numbers at least qualitatively.

In summary, we have examined the recently proposed empirical formula, Eq. (1), for the possible origin of its dependence on valence nucleon numbers. Recently, it has been shown that Eq. (1), which depends on the mass number $A$ and the valence nucleon numbers $N_p$ and $N_n$ in a very simple fashion, can describe the essential trends of excitation energies $E_x(2^+_1)$ of the first $2^+$ states in even-even nuclei throughout the whole periodic table [5]. In order to find out what makes such a dependence of $E_x(2^+_1)$ on $N_p$ and $N_n$, we calculated the first order energy shift $E^{(1)}$ resulting from the zero-range residual interaction by using simple perturbation theory. Then the shift was compared with the difference between the energy of the lowest two quasi-particle state and the measured excitation energy $E_x(2^+_1)$. The lowest two quasi-particle energy was approximated by twice the gap energy determined from the binding energies of the adjacent nuclei through the three point expression given by Eq. (3) or (4). In our calculations, the strength $v_0$ of the zero-range residual interaction given by Eq. (7) was the only free parameter. It was kept fixed in all of our calculations at $v_0 = 1,700\text{ MeVfm}^3$ for which the calculated $-E^{(1)}$ coincided exactly with the measured
difference \(2\Delta - E_x(2^+_1)\) for the heaviest lead isotope \(^{184}_{82}\text{Pb}\). We found that the variation in values of \(E^{(1)}\) followed that of the factor \((2j_f + 1)^2u^2_{j_f}v^2_{j_f}\) which can be interpreted as the effective particle number participating in the interaction from the Fermi orbit. Therefore, we concluded that the effective particle number governed the main dependence of the first \(2^+\) excitation energy on the valence nucleon numbers as given by Eq. (1).

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