Rayleigh and Solberg criteria reversal near black holes: the optical geometry explanation

Marek A. Abramowicz

Nordita, Copenhagen, Denmark*

Summary
The familiar Newtonian version of the Rayleigh criterion demands that for dynamical stability the specific angular momentum should increase with the increasing circumferential radius of circular trajectories of matter. However, sufficiently close to a black hole the Rayleigh criterion reverses: in stable fluids, the angular momentum must decrease with the increasing circumferential radius. The geometrical reason for this reversal is that the space is so very strongly warped that it turns inside-out.

Key words: hydrodynamics: instabilities – relativity – optical reference geometry

1 INTRODUCTION
Abramowicz and Prasanna (1990) found that, surprisingly, in the region* $r < 3M$ around of a nonrotating (Schwarzschild) black hole with the mass $M$, the necessary condition for stability of axially symmetric, rotating isentropic and incompressible fluid against small, adiabatic and axially symmetric perturbations is that the specific (per unit mass) angular momentum of the fluid $j$ should decrease with the increasing circumferential radius of the circular trajectories of the fluid $r$,

$$\left(\frac{\partial j^2}{\partial r}\right) < 0. \quad (1.1)$$

The result was very surprising indeed, because it seemingly contradicted the old classical result of Rayleigh that, in Newton’s theory, the necessary condition for stability should be,

$$\left(\frac{\partial j^2}{\partial r}\right) > 0. \quad (1.2)$$
Already a long time ago the Rayleigh criterion (1.2) was generalized to isentropic, compressible fluids by Solberg, and to non-isentropic fluids by Høiland. Rayleigh and Solberg criteria, which both have identical form (1.2), can be derived from the more general Høiland criterion as special cases (see Tassoul, 1978 for a detailed description of these criteria). Relativistic version of the Høiland criterion has been derived with a great care, and in a very mathematically elegant way, in an important paper by Seguin (1975), but the effect of the Rayleigh and Solberg criteria reversal has not been noticed there.

The reversal of Rayleigh and Solberg stability criteria provides yet another nontrivial example of an ambiguity of the geometrical meaning of “inside” and “outside” directions. I pointed out a few years ago (Abramowicz, 1992) that several counter intuitive effects in dynamics of bodies closely orbiting black holes could be understood as resulting from a strong warping of geometry that literally turns the space inside out! In the present paper, I show that the surprising effect of the Rayleigh and Solberg criteria reversal may also be explained the same way.

2 THE LICHTENSTEIN THEOREM

My discussion will be restricted to the motion on the equatorial symmetry plane of the rotating fluid. This restriction is not relevant for the results, but it greatly simplifies some arguments.

Lichtenstein (1933) has proved that in Newton’s theory stationary rotating fluid bodies always display a mirror symmetry with respect to reflections in a plane orthogonal to the rotating axis and containing the barycenter of the body (the equatorial plane). For stationary, axially symmetric fluids, the equatorial plane is itself rotationally symmetric, and thus one may introduce on it a family of concentric circles around the symmetry centre. The circumferential radii \( r \) of these circles provide a convenient radial coordinate on the equatorial plane.

In the early 1970s I was a Ph.D. student of Andrzej Trautman in Warsaw. He used to give his students a lot of freedom in performing research, and several of us have been indeed involved in doing all kind of esoteric theoretical physics in addition to our official Ph.D. subjects. Chandrasekhar who was often coming to visit Trautman in Warsaw, asked me once what I was working on in addition to my main subject. I answered, proudly, that I was trying to prove Lichtenstein’s theorem in general relativity. Chandra could not hide his obviously disapproving surprise, and only after a moment of silence replied very politely, “This sounds a bit too formal to me”. Of course, when Chandra left, Trautman told me to stop with Lichtenstein and do something less formal.

It is still unknown whether the Lichtenstein theorem is valid in Einstein’s relativity, but all the known solutions of Einstein’s field equations describing stationary, axially symmetric spacetimes do show this discrete symmetry. For example, the Kerr solution in Boyer-Lindquist coordinates is obviously symmetric with respect to reflection \( \theta \rightarrow \pi - \theta \) with respect to the equatorial plane, \( \theta = \pi/2 \),
3 RAYLEIGH AND SOLBERG CRITERIA IN NEWTON’S THEORY

In this Section I remind the familiar heuristic proof of the Rayleigh and Solberg criteria, first given by Randers (1942), and in the next Section I use a slightly modified proof to derive the relativistic version of the criteria.

In a reference frame which corotates with the small fluid element under consideration, three forces are acting on the element: gravitational force $G$, centrifugal force $Z$, and applied force (e.g. pressure) $T$. On a particular circle $r = r_0$ it is,

$$G(r_0) + Z(r_0, j_0^2) + T(r_0) = 0. \quad (3.1)$$

Consider an adiabatic, axially symmetric perturbation in which the small element initially located at $r_0$ is displaced to $r = r_0 + \delta r$. As the perturbation is adiabatic and axially symmetric, it does not change the specific angular momentum of the element. The (squared) angular momentum at the perturbed position $r = r_0 + \delta r$ is therefore $j_0^2$. On the other hand, the (squared) specific angular momentum needed at this very place for the balance of the forces (3.1) should be,

$$j^2 = j^2(r_0 + \delta r) = j_0^2 + \delta j^2 = j_0^2 + \left( \frac{\partial j^2}{\partial r} \right) \delta r. \quad (3.2)$$

Because $j_0^2 \neq j^2$, there is unbalanced force $\delta T$ at $r_0 + \delta r$,

$$T(r) + G(r) + Z(r, j^2 - \delta j^2) = \delta T. \quad (3.3)$$

Subtracting (3.1) from (3.3) and using (3.2) one gets,

$$\delta T = - \left( \frac{\partial Z}{\partial j^2} \right) \left( \frac{\partial j^2}{\partial r} \right) \delta r. \quad (3.4)$$

Of course, the perturbation could be stable only if the unbalanced force brings the perturbed element back to its original position, i.e. only if $\delta T \delta r < 0$. Thus, the necessary condition for stability yields,

$$\left( \frac{\partial Z}{\partial j^2} \right) \left( \frac{\partial j^2}{\partial r} \right) > 0. \quad (3.5)$$

In the corotating frame, according to Newton’s theory, $Z(r, j^2) = j^2/r^3$. Therefore, in Newton’s theory, $\partial Z/\partial j^2$ is unconditionally positive,

$$\left( \frac{\partial Z}{\partial j^2} \right) = \frac{1}{r^3} > 0, \quad (3.6)$$

and this is why the Rayleigh and Solberg criteria take the familiar form (1.2). However, Abramowicz and Prasanna (1990) have shown that, according to Einstein’s theory of general relativity, it could be that $\partial Z/\partial j^2 < 0$. In this case the Rayleigh and Solberg criteria demand that for stability $\partial j^2/\partial r < 0$. 

The Rayleigh, Solberg and Høiland criteria are valid for stationary and axially symmetric equilibria of rotating fluids. Thus, their generalization to Einstein’s theory should be derived in stationary and axially symmetric spacetimes. In such spacetimes two commuting Killing vector fields exist: asymptotically timelike Killing vector \( \eta^i \) (corresponding to stationarity), and spacelike vector \( \xi^i \) with closed trajectories (corresponding to axial symmetry). The stationary observers are defined as those with velocities \( N^i = \eta^i(\eta\eta)^{-1/2} \). More convenient are locally non rotating observers with four velocities, \( n^i = e^\Phi(\eta^i + \omega\xi^i) \),

\[
e^{-2\Phi} = (\eta\eta) - \omega(\xi\xi), \quad \omega = -\frac{(\eta\eta)}{(\xi\xi)}. \tag{4.2}
\]

One may check that \( n^i \) is a unit timelike vector which has zero vorticity and is hypersurfaces orthogonal. The hypersurfaces orthogonal to trajectories of \( n^i \) are three dimensional spaces \( t = \text{const} \) with the metric,

\[
h_{ik} = g_{ik} - n_in_k. \tag{4.3}
\]

Because \( n\xi = 0 \), the three dimensional space is axially symmetric, and a circle in space can be defined as a trajectory of the projection of \( \xi^i \) into space. A particle which moves along a circle in space has its four velocity in the spacetime given by,

\[
u^i = \gamma(n^i + \nu\tau^i). \tag{4.4}
\]

Here \( \tau^i = \xi^i/r \), with \( r^2 = -(\xi\xi) \) is a unit vector orthogonal to \( n^i \), and \( \nu \) is the orbital speed. From \( uu = nn = 1 \), \( \tau\tau = -1 \) one deduces that \( \gamma = 1/(1 - \nu^2)^{1/2} \), so that \( \gamma \) is the Lorentz redshift factor. Note that the projection of the four velocity \( u^i \) into the 3-D space gives, \( \tilde{v}^k = u^ih^k_i = \gamma\nu\tau^k \) and therefore \( \tilde{v}^2 = -(\tilde{v}\tilde{v}) = \gamma^2\nu^2 \). While the velocity \( \nu \) changes between \( \pm 1 \), the “velocity” \( \tilde{v} \) changes between \( \pm\infty \).

Abramowicz, Nurowski and Wex (1993) demonstrated that in the corotating reference frame of matter that moves according to (4.4), the acceleration \( a_i = u^k\nabla_ku_i \) equals,

\[
a_i = \nabla_i\Phi + \frac{\gamma^2\nu^2}{R}\nabla_iR - \gamma^2\nu R\nabla_i\omega, \tag{4.5}
\]

where \( R \) is the radius of gyration \( R \) defined as (Abramowicz, Miller and Stuchlík, 1993),

\[
R = re^\Phi = [-(\xi\xi)]^{1/2}e^\Phi. \tag{4.6}
\]

Abramowicz, Nurowski and Wex explained that the acceleration formula (4.5) suggests a very natural and convenient way to covariantly define gravitational, centrifugal and Lense-Thirring (“Coriollis”) forces,

\[
\mathcal{G}_i = \nabla_i\Phi, \quad \mathcal{Z}_i = \frac{\gamma^2\nu^2}{R}\nabla_iR, \quad \mathcal{C}_i = -\gamma^2\nu R\nabla_i\omega. \tag{4.7}
\]

In order to repeat derivation of the stability criterion in a way similar to that discussed in the previous Section, one needs to know how to express the orbital speed by angular
momentum which is conserved during an axially symmetric, non-stationary perturbation. In relativity, exactly as in Newton’s theory, orbital speed \( v \), and specific angular momentum per unit energy \( \ell \),

\[
\ell = -\frac{(u\xi)}{(u\eta)},
\]

are connected by the “gyration equation”,

\[
v = \frac{\ell}{R}.
\]

The specific angular momentum per unit energy \( \ell \) is \textit{not}, in general, conserved by axially symmetric perturbations, because such perturbations need not to be stationary. The quantity conserved in axially symmetric perturbations (including nonstationary ones) is the specific angular momentum per unit mass,

\[
j = \frac{\ell}{E},
\]

where \( E \) is the specific per unit mass energy. In Newton’s theory \( j = \ell \) for any fluid, but in relativity \( j \neq \ell \). However, in the special case of isentropic fluid considered here, the relativistic von Zeipel theorem (Abramowicz, 1974) guarantees that \( E = \mathcal{E}(\ell) \), \( j = j(\ell) \), and

\[
\frac{dj^2}{d\ell^2} = \frac{\mathcal{E}^2}{1 - \Omega \ell} > 0.
\]

Here \( \Omega \) is the orbital angular velocity measured by stationary observers. It is connected to the orbital angular velocity measured by locally non rotating observers \( \tilde{\Omega} \) by, \( \tilde{\Omega} = \Omega - \omega \), where

\[
\tilde{\Omega} = \frac{v^2}{\ell} = \frac{v}{R} = \frac{\ell}{R^2}.
\]

Using general expressions (4.7) for inertial forces, gyration equations (4.9), and assuming that the fluid is isentropic, I write on the equatorial plane,

\[
\mathcal{G}(r) = \frac{\partial \Phi}{\partial r},
\]

\[
\mathcal{Z}(r, \ell^2(j)) = \frac{\ell^2(j)}{R^2 - \ell^2(j)} \frac{1}{R^3} \frac{\partial R}{\partial r},
\]

\[
\mathcal{C}(r, \ell(j)) = -\frac{\ell(j) R^2}{R^2 - \ell^2(j)} \frac{\partial \omega}{\partial r}.
\]

It should be obvious that the acceleration formula (4.9) may now be written on the equatorial plane in the form similar to the Newtonian equation (3.1),

\[
\mathcal{G}(r_0) + \mathcal{Z}(r_0, \ell^2(j_0)) + \mathcal{C}(r_0, \ell(j_0)) + T(r_0) = 0.
\]

The additional Lense-Thirring force \( \mathcal{C} \) is not present in Newtonian case, and one must remember that the quantity conserved in the perturbation is not \( \ell_0 \) but \( j_0 \). Taking into account that \( \partial j^2 / \partial \ell^2 > 0 \) according to (4.11), one writes the general form of the criterion in the form,

\[
\left[ \frac{\partial \mathcal{Z}}{\partial \ell^2} + \left( \frac{1}{2\ell} \right) \left( \frac{\partial \mathcal{C}}{\partial \ell} \right) \right] \left( \frac{\partial \ell^2}{\partial r} \right) > 0.
\]
After a tedious calculation that utilises gyration equation one can rearrange this and write,

\[
\left( \frac{\partial \ln R}{\partial r} - \frac{1 + v^2}{2\Omega} \frac{\partial \omega}{\partial r} \right) \left( \frac{\partial \ell^2}{\partial r} \right) > 0.
\]  (4.18)

Seguin (1975) wrote the criterion in a different, but equivalent form, which follows from (4.17) also after tedious calculations. In the notation adopted here it yields,

\[
\left( \frac{\partial \ell}{\partial r} - R^2 \frac{\partial \Omega}{\partial r} \right) \left( \frac{\partial \ell}{\partial r} \right) > 0.
\]  (4.19)

Although (4.18) and (4.19) are fully equivalent, it is convenient to discuss physical and geometrical meaning of the criterion using (4.18).

5 REVERSAL OF THE RAYLEIGH AND SOLBERG CRITERIA

The reversal of the Rayleigh and Solberg criteria occurs when the quantity

\[
Q(r, v) \equiv v \frac{\partial R}{\partial r} - \frac{1}{2}(1 + v^2)R^2 \frac{\partial \omega}{\partial r}
\]  (5.1)

changes its sign, i.e. at the circle \( Q(r, v) = 0 \). I shall explain the physical and geometrical meaning of \( Q = 0 \) by considering precession of gyroscopes moving on circular orbits, and circular free motion of photons.

The spin of an orbiting gyroscope does precess. The precession rate with respect to the vector \( \tau^i \) tangent to the circle is given by the Fermi-Walker derivative, \( \omega_i^* = u^j \nabla_j u_i - (u_i a_j - u_j a_i)\tau_j \). On the equatorial plane, this vector has only one nonzero component which equals (Abramowicz, 1993),

\[
\omega^* = \gamma^3 \left[ -\tilde{\Omega} \frac{\partial R}{\partial r} + \frac{1}{2}(1 + v^2)R \frac{\partial \omega}{\partial r} \right].
\]  (5.2)

The factor \( \gamma^3 \) in the front of the bracket is due to special relativistic effects and represents the well-known Thomas precession. The first term in the bracket represents the geodesic (de Sitter) precession and the second term is the gravomagnetic (Lense-Thirring) precession\(^\dagger\). In a weak gravitational field, \( \partial R/\partial r \approx 1 > 0 \), and \( \partial R/\partial r \gg |\partial \omega/\partial r| \), so that \( \omega^* \) and \( \tilde{\Omega} \) have opposite signs. This is exactly what is expected by intuition which tells that an orbiting gyroscope must precess backward with respect to its orbital motion in order to always point to the same direction in space. Thus, gyroscopes orbiting clockwise precess anti-clockwise and vice versa. However, surprisingly, the bracket in (5.2) may change its sign. When this happens, the sense of the precession is exactly opposite to what is expected: gyroscopes are precessing forward — those on clockwise orbits are precessing clockwise, and those on anti-clockwise orbits are precessing anti-clockwise. (Rindler and Perlick, 1990; Abramowicz

\(^\dagger\) There is a considerable confusion in the literature about how one should interpret precession of an orbiting gyroscope in terms of different kinds of simple precessions. In particular, some authors have previously been discussing a rather unfortunate concept of the “gravitational Thomas precession”. This was because in approximate formulae used previously by several authors, the simple and unique division provided by (5.2) was not apparent.
1990, 1993; Nayak and Vishveshwara 1998). It is not difficult to see that the reversal of the Rayleigh criterion and the reversal of the sense of gyroscope precession go hand in glove. Indeed, (5.2) may be written as,

$$\omega^* = -\gamma^2 \frac{1}{R^2} Q,$$

which means that both reversals are governed by exactly the same condition $Q = 0$.

To see what does this condition really mean, one should consider the free (geodesic) motion of photons which is described by general equations $a_i = 0$, and $v = \pm 1$. It follows directly from (4.9), that they are equivalent to

$$\frac{\partial R}{\partial r} \pm R^2 \frac{\partial \omega}{\partial r} = 0.$$ (5.3)

One should note that (5.3) is the same as the ultra-relativistic limit, $v = \pm 1$, of both (5.1) and (5.2). Thus, for the ultra-relativistic fluid, both reversals — of the Rayleigh and Solberg criteria and of the sense of the precession of gyroscopes — occur at the location of the circular photon orbit. The conclusion that the reversal of the sense of gyroscope precession occurs at the location of a circular photon orbit has been previously reached by Nayak and Vishveshwara, 1998.

6 DISCUSSION AND CONCLUSIONS

Abramowicz, Carter and Lasota (1988) have noticed that dynamics of particles and photons looks simple when described in a conformally rescaled three dimensional geometry of space with the metric,

$$\tilde{h}_{ik} = e^{-2\Phi} h_{ik}.$$ (6.1)

The conformally rescaled metric (6.1) is called the optical reference geometry§.

Consider a static spacetime, $\omega = 0$. Its metric can be written in the form,

$$ds^2 = e^{-2\Phi} \left( dt^2 + \tilde{h}_{ik} dx^i dx^k \right),$$ (6.2)

where the time coordinate $t$ is invariantly defined as the synchronized time of the static observers, who have four velocities given by $N^i = n^i$. It is, $\delta_i^{(t)} = \eta^i$. Relativistic Fermat’s principle assures that $\int dt$ on the light trajectory has an extremal value, $\delta \int dt = 0$. This, together with $dt^2 = -\tilde{h}_{ik} dx^i dx^k$ that follows from (6.2) and the fact that $ds = 0$ for light, is equivalent to say that for any light trajectory $x^i = x^i(p)$ in the space of optical reference geometry,

$$\delta \int \left( -\tilde{h}_{ik} \frac{dx^i}{dp} \frac{dx^k}{dp} \right)^{\frac{1}{2}} dp = 0.$$ (6.3)

This means that light trajectories are geodesic lines in the optical reference geometry, and this is the reason for its name.

§ The optical reference geometry (6.1) has been introduced by Dowker and Kennedy (1978) and further discussed by Gibbons and Perry (1978), and Kennedy, Critchely, and Dowker (1980). It should be not confused with the optical geometry introduced by Trautman (1984) and Robinson and Trautman (1984) which is a totally different concept.
The circumference radius of a $r = \text{const}$ circle equals, in the optical reference geometry, to the radius of gyration of the circle, because both radii are given by the same formula, $R = re^\Phi$. I shall now calculate the radius of the curvature of the circle in optical geometry, $\tilde{R}$, using the well-known Frenet construction (see e.g. Synge and Schild, 1959),

$$\tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_k = -\tilde{\lambda}_k \frac{1}{\tilde{R}}. \quad (6.4)$$

Here $\tilde{\lambda}$ is the unit, outside pointing, vector orthogonal to the circle, $\tilde{\tau}^i$ is the unit vector tangent to the circle,

$$\tilde{\tau}^i = e^{-\Phi}, \quad \tilde{\tau}_i = e^\Phi, \quad (6.5)$$

and the covariant derivative operator $\tilde{\nabla}_i$ obeys,

$$\tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_k = \tau^i \nabla_i \tau_k + \nabla_k \Phi + \tau_k \tau^i \nabla_i \Phi. \quad (6.6)$$

From (6.4) I derive,

$$\frac{\partial R}{\partial r} = \varepsilon \frac{R}{R} \quad (6.7)$$

where $\varepsilon = \text{sign}(\partial R/\partial r)$. The formula for the gyroscope precession takes now the form,

$$\omega^* = \gamma^3 \left[-\varepsilon \frac{R}{R} \tilde{\Omega} + \frac{1}{2} (1 + v^2) R \frac{\partial \omega}{\partial r} \right]. \quad (6.8)$$

In a curved space, consider a circle with circumferential radius $R$ and curvature radius $\mathcal{R}$. If $R \neq \mathcal{R}$, a simple geometrical construction reveals that there is a nonzero deficit angle,

$$\delta \phi = 2\pi \frac{\mathcal{R} - R}{\mathcal{R}}. \quad (6.9)$$

In the case of a very slow orbital motion, $\gamma = 1$, in the static space, $\omega(r) = 0$, the precession of a gyroscope orbiting the circle with a steady angular velocity $\Omega = 2\pi/T$ ($T$ is the orbital period) is obviously given by,

$$\omega^* = \frac{2\pi - \delta \phi}{T} \Omega \frac{R}{\mathcal{R}}. \quad (6.10)$$

This is exactly the first term in formula (6.8). Thus, this term is indeed just exactly the geodesic precession — but “geodesic” in the sense of the optical reference geometry $\tilde{h}_{ik}$, not the directly projected geometry $h_{ik}$.

In this article, I have described a rather counter intuitive effect of the reversal of the Rayleigh and Solberg stability criteria, and explained that it could be understood in terms of a very simple intuitive argument with a help of the optical reference geometry. In addition to this effect, the optical reference geometry has already successfully explained a score of similarly strange other effects. One should ask therefore the question: why is the optical reference geometry, and not the “real” geometry, so excirdingly well fitted to behaviour of particles, fluids, charges, photons, gravitational waves, electromagnetic fields and gyroscopes? Are we missing something? Is the optical geometry just a particularly good map of the real curved geometry, or is the optical geometry itself the real geometry of space?
ACKNOWLEDGEMENTS

This research was supported by Nordita through the Nordic Project, *Non-linear Phenomena in Accretion Discs Orbiting Black Holes*. I would like to thank Sebastiano Sonogho for many very helpful suggestions.

REFERENCES

Abramowicz M.A., 1974, *Acta Astr.*, 24, 45
Abramowicz M.A., 1990, *Mon. Not. R. astr. Soc.*, 245, 733
Abramowicz M.A., 1992, *Mon. Not. R. astr. Soc.*, 256, 710
Abramowicz M.A., 1993, in *The Renaissance of General Relativity and Cosmology*, eds. G. Ellis, A. Lanza and J. Miller, Cambridge University Press, (Cambridge)
Abramowicz M.A., Carter B. and Lasota J.-P., 1988, *Gen. Relat. Grav.*, 20, 1173
Abramowicz M.A., Miller J.C. and Stuchlík Z., 1993, *Phys. Rev. D.*, 47, 1440
Abramowicz M.A., Nurowski P., and Wex N., 1993, *Class. Quantum Grav.*, 10, L183
Abramowicz M.A. and Prasanna A.R., 1989, *Mon. Not. R. astr. Soc.*, 245, 270
Dowker J.S., and Kennedy G., 1978, *J. Phys. A*, 11, 895
Gibbons G.W., and Perry J.M., 1978, *Proc. Roy. Soc. Lond. A*, 358, 467
Kennedy G., Critchely R., Dowker J.S., 1980, *Ann. Phys. NY*, 125, 346
Lichtenstein L., 1933, *Gleichgewichtsfiguren Rotierender Flüssigkeiten*, Springer Verlag, (Berlin)
Nayak K.R., and Vishveshwara C.V., 1998, *Gen. Rel. Grav.*, 30, 593
Randers G., 1942, *Astrophys. J.*, 95, 454
Rindler W., and Perlick C., 1990, *Gen. Rel. Grav.*, 22, 1067
Robinson I. and Trautman A., 1986, in *Les Théories de la Gravitation*, Éditions du CNRS (Paris)
Seguin F.H., 1975, *Astrophys. J.*, 197, 745
Semerak O., 1997, *Gen. Rel. Grav.*, 13, 2987
Synge J.L. and Schild A., 1959, *Tensor Calculus*, University of Toronto Press, (Toronto)
Tassoul J.-L., 1978, *Theory of Rotating Stars*, Princeton University Press (Princeton)
Trautman A., 1984, *J. Geom. Phys.*, 1, 85