THE GORENSTEIN PROPERTY FOR PROJECTIVE
COORDINATE RINGS OF RANK 2 PARABOLIC VECTOR
BUNDLES ON A SMOOTH CURVE

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Abstract. We study the Hilbert function of a projective coordinate ring of
a moduli space $M_{C,p}(\vec{\alpha})$ of rank 2 vector bundles with parabolic structure on
a marked projective curve. Using combinatorial methods we determine that
this algebra is not Gorenstein when the genus of $C$ is greater than 1 and the
number of marked points is greater than 2.

Contents

1. Introduction 1
2. Affine semigroups and the Gorenstein property 3
3. The polytope $P_T(\vec{r}, L)$ 4
4. The argument for generic $\vec{r}$ 7
5. The case of one marked point 9
References 9

1. Introduction

Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian positively graded algebra over an alge-
braically closed field $k = R_0$ of characteristic 0, and suppose that $R$ is the homo-
 morphic image of a polynomial ring $S = k[x_1, \ldots, x_n]$. Let $d = \text{dim}(R)$, then the
algebra $R$ is said to be Gorenstein when its canonical module $\Omega(R) = \text{Ext}_S^{n-d}(R, S)$
is isomorphic to a shifted copy $R(a)$ of $R$ as a graded module (see [BH93, Theorem
3.3.7]). If $X = \text{Proj}(R) \subset \mathbb{P}^n(k)$, and $R$ is Gorenstein, $X$ is said to be arith-
metically Gorenstein (AG). Arithmetically Gorenstein projective schemes enjoy a
number of nice geometric and combinatorial properties, in particular all of their
singularities are rational, and their Betti numbers (as an $S$ module) satisfy a type
of Poincare duality: $\beta_i(R) = \beta_{n-\text{dim}(R)-i}(R)$.

An elegant consequence of Richard Stanley’s work on formal power series [Sta78]
is that the Gorenstein property for a graded Noetherian domain is entirely captured
by properties of its Hilbert function:

\begin{equation}
\phi_R(n) = \text{dim}(R_n).
\end{equation}
Let $Φ_R(t) = \sum_{n≥0} Φ_R(n)t^n$ be the formal power series associated to $φ_R$; Stanley shows [Sta78, Theorem 6.1] that $R$ is Gorenstein if and only if

$$Φ_R(t^{-1}) = (-1)^d t^n Φ_R(t).$$

With this result in mind we study the Hilbert function $φ_{C,L}$ of a projective coordinate ring $R_{C,\vec{p}}(\vec{r}, L)$ for the moduli spaces of rank 2 parabolic vector bundles on a smooth projective curve $C$. The Hilbert function only depends on the genus of $C$ and the number of marked points $\vec{p}$ (see Section 3), hence we have omitted this information from the notation. Fix such a curve $C$, a non-negative integer $L$ (known as the level), a set of distinct points $\{p_1, \ldots, p_n\} \subset C$, and a corresponding set of non-negative integers $\vec{r} = \{r_1, \ldots, r_n\}$ with $0 ≤ r_i ≤ L$. This data specifies the moduli space $M_{C,\vec{p}}(\vec{a})$ of rank 2 vector bundles on $C$ with parabolic structure of weight $α_i = \frac{r_i}{T}$ at the marked point $p_i$; in particular this space is realized as $\text{Proj}(R_{C,\vec{p}}(\vec{r}, L))$ for a graded domain $R_{C,\vec{p}}(\vec{r}, L)$, see [Man09]. The following is our main theorem.

**Theorem 1.1.** Let $C$ have genus $≥ 2$ then $R_{C,\vec{p}}(\vec{r}, L)$ is Gorenstein if and only if $L = 1, 2$ or 4, and $\vec{r} = 0$, or $|\vec{p}| = 1$, and the marked point has weight equal to the level.

In particular, we show that if $0 < α_i < 1$ the algebra $R_{C,\vec{p}}(\vec{r}, L)$ is not Gorenstein. The case $\vec{r} = 0$ recovers the $SL_2(\mathbb{C})$ case of the well-known result of Kumar Narasimhan and Ramanathan, [KNR94] that the moduli spaces of (not parabolic) $G$ bundles on $C$ are Gorenstein for simply connected simple $G$.

We address the Gorenstein property for $R_{C,\vec{p}}(\vec{r}, L)$ by bringing in two related integral domains: the algebra of conformal blocks $V_{C,\vec{p}}^{\uparrow}$ (see [Man09]) and the affine semigroup algebra $k[P_T(\vec{r}, L)]$ associated to a closely related convex integral polytope $P_T(\vec{r}, L)$ (see [Man09] and Section 3). The algebra $V_{C,\vec{p}}^{\uparrow}$ is multigraded by $\mathbb{Z}_{≥0}^{n+1}$ where $n = |\vec{p}|$, and contains each $R_{C,\vec{p}}(\vec{r}, L)$ as a graded subalgebra. In particular the $N$-th graded piece of $R_{C,\vec{p}}(\vec{r}, L)$ is $V_{C,\vec{p}}^{\uparrow}(N\vec{r}, NL)$, the $(N\vec{r}, NL)$-th graded piece of $V_{C,\vec{p}}^{\uparrow}$. Let $ψ_{g,n} : \mathbb{Z}_{≥0}^{n+1} → \mathbb{Z}_{≥0}$ be the multigraded Hilbert function of $V_{C,\vec{p}}^{\uparrow}$, then we have:

$$ψ_{g,n}(N\vec{r}, NL) = φ_{C,L}(N).$$

In Section 3 we give a proof that $V_{C,\vec{p}}^{\uparrow}$ is Gorenstein. To accomplish this we use a toric degeneration constructed in [Man09] to the normal affine semigroup algebra $k[P_T(1)]$ associated a convex polytope $P_T$ defined in [Man09] and Section 3. The Gorenstein property for normal affine semigroup algebras is captured combinatorially in [BH93, Theorem 6.35], see Section 2 for a description. Using this combinatorial approach, Theorem 1.1 is proved by analyzing a geometric and combinatorial relationship between $P_T(\vec{r}, L)$ and $P_T$. These techniques are similar to those employed by the 2nd author in [Man12], where a classification of arithmetically Gorenstein moduli of weighted points on the projective line is given. The projective coordinate rings of the latter moduli spaces are actually a special case of the projective coordinate rings on the moduli of rank 2 parabolic vector bundles on a projective line. They are the $R_{C,\vec{p}}(\vec{r}, L)$ distinguished by the property that $L$ is very large in comparison to the $r_i$ (see [Man09, Section
The algebra following is a graded rephrasing of [BH93, Theorem 6.3.5].

Remark 1.2. An important result of Mehta and Ramadas shows that the local rings of the $M_{G,R}(\bar{G})$ are all Cohen-Macaulay in positive characteristic. The toric degeneration techniques used here and in [Man09] establish that the $M_{G,R}(\bar{G})$ are all arithmetically Cohen-Macaulay with respect to the line bundles $L_{F,L}$ for any genus and number of marked points.

2. Affine semigroups and the Gorenstein property

As Stanley’s result in [Sta78] suggests, the combinatorics of a grading on an algebra can be used to determine if that algebra has the Gorenstein property. An extremal instance of this connection is present in the characterization of the Gorenstein property for a normal affine semigroup algebra. Let $C \subset \mathbb{R}^n$ be a positive polyhedral cone pointed at the origin, which is integral with respect to a lattice $M \subset \mathbb{R}^n$; this means the following:

1. $C$ is the intersection of a finite number of half spaces $H_{F_i} = \{ v \in \mathbb{R}^n | F_i(v) \geq 0 \}$, $F_i : \mathbb{R}^n \to \mathbb{R}$ is linear, and the extremal rays of $C$ all contain a point from $M$,

2. There is a linear function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F(w) \geq 0$ for all $w \in C$.

There is a natural semigroup $C$ associated to $C$ obtained as the intersection $C \cap M$. By formally adjoining scalars from the field $k$, we obtain the affine semigroup algebra $k[C]$. As $C = C \cap M$, the algebra $k[C]$ is normal and finitely generated by Gordon’s Lemma ([BH93, Proposition 6.1.2]). The algebra $k[C]$ is then Cohen-Macaulay ([BH93, 6.3.5]), and the Gorenstein property is captured entirely by the combinatorics of $C$. We let $\text{int}(C)$ denote the set of relative interior points of the cone $C$, the following is a consequence of [BH93, Theorem 6.3.5].

Proposition 2.1. With the assumptions above, the affine semigroup algebra $k[C]$ is Gorenstein if and only if the set $\text{int}(C) \cap M$ is equal to the set $\omega + C$ for some $\omega \in C$.

As $0 \in C$, we have automatically $\omega \in \text{int}(C) \cap M$. Rephrased, Proposition 2.1 says that in order to show that $k[C]$ is Gorenstein we must be able to find some interior point $\omega$ such that $v = \omega \in C$ for any $v \in \text{int}(C) \cap M$.

A convex polytope $P \subset \mathbb{R}^n$ is said to be a lattice polytope with respect to $M \subset \mathbb{R}^n$ if all of the vertices of $P$ are in $M$. There is a pointed integral polyhedral cone $C_P$ associated to $P$ obtained by taking the $\mathbb{R}_{\geq 0}$ span of the vertices of $P \times \{1\} \subset \mathbb{R}^{n+1}$. Identifying $\mathbb{R}^n \times \{L\} \subset \mathbb{R}^{n+1}$ with $\mathbb{R}^n$, the level set $C_P \cap \mathbb{R}^n \times \{L\}$ of this cone is isomorphic to the Minkowski sum $L \circ P = P + \cdots + P$. The intersection $C_P = C_P \cap M \times \mathbb{Z} \subset \mathbb{R}^{n+1}$ is then a normal affine semigroup. We let $k[P]$ denote the corresponding affine semigroup algebra. By construction $k[P]$ comes with a grading where elements from $C_P \cap M \times \{L\}$ are given homogeneous degree $L$. The following is a graded rephrasing of [BH93, Theorem 6.3.5].

Proposition 2.2. The algebra $k[P]$ is Gorenstein if and only if some Minkowski sum $a \circ P$ contains a unique interior lattice point $\omega_a \in \text{int}(aP) \cap M$ such that for any interior point $v \in \text{int}(L \circ P) \cap M$ there is a lattice point $u \in (a - L) \circ P \cap M$ such that $u + \omega_a = v$. 
The negative $-a$ of the degree of the unique interior generator $\omega_a$ is called the $a$-invariant of $k[P]$.

For any pointed integral polyhedral cone $C$, the submodule of $k[C]$ generated by the set of interior points $\text{int}(C) \cap \mathcal{M}$ is known to be isomorphic to the canonical module $\Omega(k[C])$, see [BH93]. The conditions described in Propositions 2.1 and 2.2 are then translations of the requirement that $\omega(k[C])$ be a principal module.

**Example 2.3.** The canonical module is the module of top differential forms defined on the non-singular points of $\text{Spec}(k[C])$. Taking the polynomial ring $S = k[x_1, \ldots, x_m]$, the canonical module should be generated by the differential form $dx_1 \wedge \cdots \wedge dx_m$, and indeed we see that the interior $\mathbb{Z}^n$ points in the simplicial cone $\mathbb{R}_{\geq 0}$ are all translates by $(1, \ldots, 1)$.

3. The polytope $P_T(\vec{r}, L)$

In [Man09] the polytope $P_T(\vec{r}, L)$ is defined as follows.

**Definition 3.1.** Let $\Gamma$ be a $a$ a trivalent graph with $n$ leaves, and let $\vec{r}$ be an $n$-tuple of non-negative integers. For a non-negative integer $L$, $P_T(\vec{r}, L)$ is the set of functions $w : E(\Gamma) \to \mathbb{R}$ which satisfy the following conditions:

1. $w(e) \geq 0$ for all $e \in E(\Gamma)$,

2. For any three edges $e, f, g$ meeting at a vertex $|w(e) - w(g)| \leq w(f) \leq w(e) + w(g)$,

3. For any three edges $e, f, g$ meeting at a vertex $w(e) + w(f) + w(g) \leq 2L$,

4. For $\ell_i$ the $i$-th leaf-edge of $\Gamma$, $w(\ell_i) = r_i$.

The polytope $P_T(L)$ is defined by the first three conditions above.

It can be verified from Definition 3.1 that the Minkowski sum $L \circ P_T(1)$ is $P_T(L)$. In the special case that $\Gamma$ is the trinode we let $P_3(\vec{r})$ denote this polytope. It is straightforward to check that $P_3(\vec{r})$ is the convex hull of $(0, 0, 0)$, $(L, L, 0)$, $(L, 0, L)$, and $(0, L, L)$ in $\mathbb{R}^3$. These convex bodies are considered in relation to the lattice $\mathcal{M}_T \subset \mathbb{R}^{E(\Gamma)}$ defined by the requirement that $w(e) \in \mathbb{Z}$ for all $e \in E(\Gamma)$ and $w(e) + w(f) + w(g) \in 2\mathbb{Z}$ for any three edges meeting at a vertex. Let $P_T(L) = P_T(L) \cap \mathcal{M}_T$ and $P_T(\vec{r}, L) = P_T(\vec{r}, L) \cap \mathcal{M}_T$ be the sets of lattice points in these polytopes $P_T(L)$ and $P_T(\vec{r}, L)$, respectively.

A main result of [Man09] states that for any curve $C$ with marked points $\vec{p}$, and any graph $\Gamma$ with first Betti number equal to the genus of $C$ and $|L(\Gamma)| = |\vec{p}|$, the graded domain $R_{C, \vec{p}}(\vec{r}, L)$ can be flatly degenerated to the affine semigroup algebra $k[P_T(\vec{r}, L)]$. The next proposition is a consequence of the method used to prove this theorem.

**Proposition 3.2.** The number of lattice points $|P_T(\vec{r}, L)|$ and $|P_T(L)|$ only depends on $\beta_1(\Gamma)$ and $|L(\Gamma)|$. Furthermore, for any two graphs $\Gamma, \Gamma'$ with the same first Betti number $g$ and the same number of leaves $n$, $k[P_T(\vec{r}, L)]$ is Gorenstein if and only if $k[P_{\Gamma'}(\vec{r}, L)]$ is Gorenstein if and only if $R_{C, \vec{p}}(\vec{r}, L)$ is Gorenstein, where $C$ is a generic marked genus $g$ curve with $|\vec{p}| = n$. 
3.1. Interior points. In light of Proposition 2.1 we determine a necessary and sufficient condition that a point be interior to one of these convex bodies.

Proposition 3.3. A point \( w \in \mathcal{P}_T(L) \) is an interior point if and only if all inequalities in 3.1 are strict on \( w \).

Proof. The polytope \( \mathcal{P}_T(L) \) can be constructed as an intersection of the polytope \( \mathcal{P}_3(L)^{V(T)} \subset \mathbb{R}^{3V(T)} \) with a linear space. We construct \( \mathcal{P}_T(L) \) in this way by first splitting each edge \( e \in E(\Gamma) \) to create a forest of \( |V(\Gamma)| \) trinodes, and we let \( e_1, e_2 \) denote the pair of edges created by splitting \( e \in E(\Gamma) \). The polytope \( \mathcal{P}_T(L) \) is then the subset of those \( \bar{w} \in \mathcal{P}_3(L)^{V(T)} \subset \mathbb{R}^{3V(T)} \) satisfying \( \bar{w}(e_1) = \bar{w}(e_2) \).

The interior points of \( \mathcal{P}_3(L)^{V(T)} \) are precisely those \( \bar{w} \) which make the above inequalities 1–3 strict at each trinode. In order to establish that interior points of \( \mathcal{P}_T(L) \) have the same description it suffices to show that \( \mathcal{P}_T(L) \) contains an interior point of \( \mathcal{P}_3(L)^{V(T)} \). We construct such a point \( \omega \) by choosing \( \epsilon \) such that \( 0 < 3\epsilon < 2L \), and defining \( \omega(e) = \epsilon \).

A similar argument can be used to classify interior points of \( \mathcal{P}_{\Gamma_{g,n}}(\bar{r},L) \) for a special trivalent graph \( \Gamma_{g,n} \), provided the parameters \( \bar{r} \) are chosen strictly between 0 and \( L \). We let \( \Gamma_{g,n} \) denote the graph version of the depicted in Figure 1 with \( n \) leaves and first Betti number \( g \).

![Figure 1. The graph \( \Gamma_{g,n} \)](image)

Proposition 3.4. If \( \bar{r} \) are any real numbers chosen such that \( 0 < r_i < L \) then a point \( w \in \mathcal{P}_{\Gamma_{g,n}}(\bar{r},L) \) is an interior point if and only if all inequalities in Definition 3.1 are strict.

Proof. It suffices to produce a point \( w \in \text{int}(\mathcal{P}_{\Gamma_{g,n}}(L)) \) which is also in \( \mathcal{P}_{\Gamma_{g,n}}(\bar{r},L) \). Given two fixed lengths \( r, s < L \), it is possible to find \( d < L \) such that all triangle inequalities on \( d, r, s \) are strict and \( d + r + s < 2L \). By induction we can then find \( d_1, \ldots, d_{n-2} \) so that \( (r_1, r_2, d_1), \ldots, (d_{n-3}, r_n, d_{n-2}) \) are all in the interior of \( \mathcal{P}_3(L) \). Now we find any \( 0 < \epsilon < d \) with \( 3\epsilon < 2L \), and assign \( \epsilon \) to every other edge in \( \Gamma_{g,n} \).

Using Propositions 3.3 and 3.4, we give a classification of the interior lattice points of \( \mathcal{P}_T(L) \) and \( \mathcal{P}_{\Gamma_{g,n}}(\bar{r},L) \). Let \( \omega_T \in \mathcal{P}_T(4) \) be the point which assigns 2 to each edge of \( \Gamma \).

Proposition 3.5. Suppose that \( 0 < r_i < L \). A lattice point \( w \in \mathcal{P}_T(L) \) is in \( \text{int}(\mathcal{P}_T(L)) \) if and only if \( w = \omega_T + u \) for some \( u \in \mathcal{P}_T(L-4) \). A point \( w \in \mathcal{P}_{\Gamma_{g,n}}(\bar{r},L) \) is in \( \text{int}(\mathcal{P}_{\Gamma_{g,n}}(\bar{r},L)) \) if and only if \( w = \omega_T + u \) for some \( u \in \mathcal{P}_{\Gamma_{g,n}}(\bar{r} - \bar{2},L - 4) \).
Proof. Given Proposition 3.4, the second claim follows from the first. Suppose that \( w \in P_1(L) \) is an interior lattice point. It follows that for any trinode \( v \in V(\Gamma) \) with incident edges \( e, f, g \), we must have \( |w(f) + w(g)| < w(e) < w(f) + w(g) \) and \( w(e) + w(f) + w(g) < 2L \). The weak versions of these inequalities are still satisfied if 2 is taken from each edge value and 4 is taken from the level.

Proposition 3.5 is enough to establish the Gorenstein property for the affine semigroup algebra \( k[P_{\mathfrak{T},g,n}(L)] \) for certain values of \( L \).

Theorem 3.6. The algebra \( k[P_1(L)] \) is Gorenstein if and only if \( L = 1, 2, 4 \). As a consequence, the algebra of conformal blocks \( V^\dagger_{C,\bar{p}} \) is Gorenstein for generic \((C, \bar{p})\).

Proof. The set \( P_1(1) \) always has cardinality bigger than 1, so it follows that \( P_1(L) \) contains at least \( P_1(1) \) interior points for any \( L \geq 4 \). The polytope \( P_1(3) \) contains no interior points, however its 2-nd Minkowski sum \( P_1(6) \) has \( L \geq 4 \), so it must contain more than one interior point. The cases \( L = 1, 2, 4 \) all contain \( \omega_\Gamma \) in their 4-th, 2-nd and 1-st Minkowski sums respectively. It follows that this point is a summand of all the interior points in the cones over these polytopes.

The proof of Theorem 3.6 implies that the \( a \)-invariants of \( k[P_1(1)], k[P_1(2)] \) and \( k[P_1(4)] \) are \(-4, -2,\) and \(-1\) respectively. In particular, as it shares the same Hilbert function as \( k[P_1(1)] \), the algebra of conformal blocks \( V^\dagger_{C,\bar{p}} \) always has \( a \)-invariant equal to \(-4\) when considered with respect to the grading by the so-called level, regardless of the genus of \( C \) or number of marked points. In particular, if no markings are chosen then we conclude that \( R_C(L) \) is Gorenstein for generic \( C \) if and only if \( L = 1, 2 \) or 4. This proves part of Theorem 1.1.

3.2. Polyhedral symmetries. Next we observe how a class of symmetries of the polytope \( P_1(L) \) affect the Hilbert function \( \psi_{g,n}(\vec{r}, L) \). Let \( C_L : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the affine map which sends \((x, y, z) \rightarrow (L - x, L - y, z)\).

Lemma 3.7. The map \( C_L \) defines a symmetry of \( P_3(L) \) and permutes the members of \( P_3(L) \).

As a consequence of Lemma 3.7 a point \((x, y, z)\) is in \( P_3(L) \) if and only if \((L - x, L - y, z)\) is as well. The function \( \psi_{0,3}(r_1, r_2, r_3, L) \) is either 1 or 0 depending on whether or not \((r_1, r_2, r_3) \in P_3(L)\), so we must have \( \psi_{0,3}(r_1, r_2, r_3, L) = \psi_{0,3}(L - r_1, L - r_2, r_3, L) \). We recall three facts about \( \psi_{g,n}(\vec{r}, L) \) which follow from properties of the algebra of conformal blocks \( V^\dagger_{C,\bar{p}} \) (see [Man09]).

Proposition 3.8. Three identities hold for the Hilbert functions \( \psi_{g,n}(\vec{r}, L) \). They satisfy so-called “vacuum propagation:”

\[
\psi_{g,n+1}(\vec{r}, 0, L) = \psi_{g,n}(\vec{r}, L) .
\]

Any permutation \( \sigma \) applied to \( \vec{r} \) leaves the Hilbert function unchanged:

\[
\psi_{g,n}(\vec{r}, L) = \psi_{g,n}(\sigma(\vec{r}), L) .
\]

Finally, the Hilbert functions \( \psi_{g,n}(\vec{r}, L) \) have a so-called “factorization” property:

\[
\psi_{g,n}(r_1, r_2, \ldots, r_n, L) = \sum_{0 \leq a \leq L} \psi_{g,n-1}(a, \ldots, r_n, L)\psi_{0,3}(a, r_1, r_2, L) .
\]
Note that Proposition 3.8 implies that the function $\phi_{r,L}^{g,n}(N)$ does not depend on the order of $\vec{r}$, and $\phi_{r,0,L}^{g,n+1}(N) = \phi_{r,L}^{g,n}(N)$. We will need one further symmetry operation, which is captured in the following proposition.

**Remark 3.9.** “Factorization” as in Proposition 3.8 above was used to define an interesting convolution operation on the Hilbert functions we consider here in [BW07].

**Proposition 3.10.** For any $\vec{r} = (r_1, r_2, \ldots, r_n)$ and $L$, we have

\[
\psi_{g,n}(r_1, r_2, \ldots, r_n, L) = \psi_{g,n}(L - r_1, L - r_2, \ldots, r_n, L).
\]

\[
\phi_{r_1, r_2, \ldots, r_n, L}^{g,n}(N) = \phi_{L-r_1, L-r_2, \ldots, r_n, L}^{g,n}(N).
\]

**Proof.** The second equation follows from the first. By Propositions 3.8 and 3.10 we can compute:

\[
\psi_{g,n}(r_1, r_2, \ldots, r_n, L) = \sum_{0 \leq a \leq L} \phi_{g,n-1}(a, \ldots, r_n, L) \psi_{0,3}(a, r_1, r_2, L) = \sum_{0 \leq a \leq L} \phi_{g,n-1}(a, \ldots, r_n, L) \phi_{0,3}(a, L - r_1, L - r_2, L) = \psi_{g,n}(L - r_1, L - r_2, \ldots, r_n, L).
\]

**Remark 3.11.** Using [SX10, Equation 8], it’s possible to deduce Proposition 3.10 in the case $g = 0$ by applying permutations and so-called Cremona transformations from the birational geometry of moduli spaces of rank 2 parabolic vector bundles. It is remarkable that these conclusions can also be deduced from symmetry properties of a tetrahedron.

These symmetries, and the fact that the Gorenstein property depends only on the Hilbert function in this context, allows us to reduce the proof of Theorem 1.1 to two cases.

**Proposition 3.12.** Let $g \geq 2$ and $n \geq 1$, then the Hilbert function $\psi_{g,n}(\vec{r}, L)$ coincides with $\psi_{g,n}(\vec{r}', L')$ where either all entries satisfy $0 < r_i' < L$ or $n' = 1$ and $r_1' = L$.

The rest of the paper is dedicated to handling these two cases.

**4. The Argument for Generic $\vec{r}$**

Now we prove that the semigroup algebra $k[P_{\Gamma_{g,n}}(\vec{r}, L)]$ cannot be Gorenstein if $g \geq 2$, $n \geq 1$, and $0 < r_i < L$. The main idea of the proof is to show that $P_{\Gamma_{g,n}}(\vec{r}, L)$ must contain at least two points. By Proposition 3.5, if some $P_{\Gamma_{g,n}}(\vec{r}, L)$ contains interior lattice points, then they have $\omega_{\Gamma_{g,n}}$ as a summand. As a consequence, if $P_{\Gamma_{g,n}}(\vec{r} - \vec{2}, L - 4)$ contains more than one point, $P_{\Gamma_{g,n}}(\vec{r}, L)$ must contain two or more lattice points.

**Proposition 4.1.** Let $n \geq 1$ and $g \geq 2$. Then if $P_{\Gamma_{g,n}}(\vec{r}, L)$ contains a lattice point $w$, it contains strictly greater than 1 lattice point, i.e. there is a $w' \in P_{\Gamma_{g,n}}(\vec{r}, L)$ with $w \neq w'$.
Proof. Let $e$ be the end in $\Gamma_{g,n}$ which separates the tree portion from the portion with loops (see Figure 1). Let $w$ be in $P_{\Gamma_{g,n}}$, and consider $w(e)$. If $w(e)$ is odd, then we are done, since the two edges adjacent to $e$ must have different parities (since $w(e) + x + y \in 2\mathbb{Z}$), hence we can swap them and create a distinct lattice point (as shown in Figure 2).

If $w(e) > 0$ is even, then $w(e) \leq L$ by the triangle inequalities. Hence, we can construct the following two graphs (Figure 3):

which clearly define distinct lattice points.

If $w(e) = 0$, then we can create the following graphs (seen in Figure 4), which again describe distinct lattice points.
Hence, given a lattice point \( w \in P_{\Gamma, n}(\vec{r}, L) \), we can always find a distinct \( w' \in P_{\Gamma, n}(\vec{r}, L) \).

\[ \square \]

Corollary 4.2. For any \( \Gamma, n \) with \( n \geq 1 \), \( g \geq 2 \), \( \text{int}(P_{\Gamma, n}(\vec{r}, L)) \cap M_{\Gamma, n} \) is empty or contains at least 2 points.

5. The case of one marked point

It remains only to check the part of Theorem 1.1 captured in the following proposition.

Proposition 5.1. Let \( C \) be a curve of genus \( \geq 2 \), then the algebra \( R_{C, p}(L, L) \) is Gorenstein if and only if \( L = 1, 2, \) or 4.

Proof. We employ the graph \( \Gamma_{g, 1} \) and we note that \( P_{\Gamma_{g, 1}}(L, L) \) is isomorphic to the polytope \( P_{\Gamma_{g-1, 2}}(L) \) intersected with the hyperplane \( a + b = L \), where \( a \) and \( b \) are the labels on the two edges in \( \Gamma_{g-1, 2} \). By taking any point \( u \) with \( u(a) \) and \( u(b) \) nonzero and all other edges assigned \( \epsilon \) with \( 3\epsilon \leq 2L \), we construct an interior point of \( P_{\Gamma_{g-1, 2}}(L) \) which is also in \( P_{\Gamma_{g, 1}}(L, L) \). It follows that interior points of the latter polytope are those which make all inequalities strict, and lattice interior points must have the point corresponding to \( \omega_{\Gamma_{g-1, 2}} \) as a summand. An argument similar to the proof of Theorem 3.6 the implies that this algebra is Gorenstein for \( L = 1, 2 \) and 4.

If \( L \) is odd, we are forced to assign \( a \) and \( b \) different numbers \( u(a) \neq u(b) \). If \( u \) were interior, then we could define another interior point \( u' \) by setting \( u'(\epsilon) = u(\epsilon) \) for \( \epsilon \neq a, b \) and \( u(a) = u'(b) \) and \( u(b) = u'(a) \).

If \( L \) is even and \( \geq 4 \), then we may change the label assigned to the end loop of \( \Gamma_{g, 1} \) by \( \omega_{\Gamma_{g-1, 2}} \) from 2 to 3 to obtain a second interior point.

\[ \square \]

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