Generalizations of the Euler-Mascheroni constant associated with the hyperharmonic numbers

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September 6, 2021

Abstract

In this paper, we present two new generalizations of the Euler-Mascheroni constant arising from the Dirichlet series associated to the hyperharmonic numbers. We also give some inequalities related to upper and lower estimates, and evaluation formulas.

Keywords: Euler-Mascheroni constant, harmonic numbers, hyperharmonic numbers, digamma function, Euler-type sums, Stirling numbers of the first kind

MSC 2010
11Y60, 11B83, 33B15, 11M41, 11B73

1 Introduction

The Euler-Mascheroni constant γ = 0.5772156649... occurs in estimating the growth rate of the harmonic series:

\[ H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n + \gamma. \] (1.1)

The importance of the constant γ goes beyond its definition as it turns up in analysis, number theory, probability, and special functions. For instance, we have \( \psi(1) = \Gamma'(1) = -\gamma \), where

\[ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \ x > 0 \]

is Euler’s gamma function and \( \psi(x) = \Gamma'(x)/\Gamma(x) \) is the digamma function. γ is also related to the Riemann zeta function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ \text{Re} \ s > 1, \] (1.2)
in that (see [42] and [24])

\[
\gamma = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k) \quad \text{and} \quad \gamma = 1 - \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}.
\]

It might be worthful to record that the Euler-Mascheroni constant is nothing but the sum of the Mascheroni series (see [33])

\[
\gamma = \sum_{k=2}^{\infty} \frac{|b_k|}{k},
\]

where \(b_k\) are the Bernoulli numbers of the second kind (or the Gregory coefficients) [4, 26].

The definition and representations above provide several important generalizations for the Euler-Mascheroni constant. For instance, interpreting (1.1) as

\[
\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \int_{1}^{n} \frac{1}{x} \, dx \right)
\]

leads to a motivation to consider the limit

\[
\lim_{n \to \infty} \left( \sum_{k=1}^{n} f(k) - \int_{1}^{n} f(x) \, dx \right) \quad (1.3)
\]

(note that if \(f : (0, \infty) \to (0, \infty)\) is continuous, strictly decreasing and \(\lim_{x \to \infty} f(x) = 0\), then the limit (1.3) exists (cf. [43])). Certain functions are then found to be of special importance in connection with the study of \(\gamma\). Evidently, the function \(f(x) = 1/x\) in (1.3) corresponds to the constant \(\gamma\). The function \(f(x) = 1/x^s\) with \(0 < s \leq 1\) gives the so called generalized Euler-Mascheroni constants \(\gamma^{(s)}\):

\[
\gamma^{(s)} = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k^s} - \int_{1}^{n} \frac{1}{x^s} \, dx \right), \quad 0 < s \leq 1.
\]

The Stieltjes constant \(\gamma_m\) arises from the choice of \(f(x) = (\ln^m x)/x\), namely

\[
\gamma_m = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{\ln^m k}{k} - \int_{1}^{n} \frac{\ln^m x}{x} \, dx \right).
\]

The Stieltjes constant and the Riemann zeta function share a particular relationship in that

\[
\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k (s-1)^k,
\]

i.e., \(\gamma_m\) appears in the Laurent expansion of \(\zeta(s)\).
The constants $\gamma, \gamma(s)$, and $\gamma_m$ can in fact be obtained when we consider the multivariate function $f(x, s) = 1/x^s$. More precisely, the limit (1.3) corresponds to

\[
\begin{cases}
\gamma, & \text{for } f = f(x, 1), \\
\gamma(s), & \text{for } f = f(x, s) \text{ with } 0 < s \leq 1, \\
\gamma_m, & \text{for } f = \frac{\partial^m}{\partial s^m} f(x, s) \big|_{s=1}.
\end{cases}
\]

It is seen that these constants emerge in the interval $0 < s \leq 1$ where the series (1.2) is divergent. This observation suggests to consider generalizations of the Riemann zeta function for general versions of the Euler-Mascheroni constant. The classical example in this direction is the Hurwitz zeta function

\[
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)}, \quad \text{Re} (s) > 1, \quad a \neq 0, -1, -2, \ldots.
\]

This series diverges when $s \leq 1$, hence $f = f(x, a, s) = 1/ (x + a)^s$ in (1.3) with $s = 1$ brings out the generalized Euler-Mascheroni constant $\gamma(a)$, that is,

\[
\gamma(a) = \lim_{n \to \infty} \left( \sum_{k=0}^{n} \frac{1}{k + a} - \int_{0}^{n} \frac{1}{x + a} dx \right)
\]

(see [28, p. 453]). Accordingly, $f = \frac{\partial^m}{\partial s^m} \frac{1}{x + a} \big|_{s=1}$ in (1.3) introduces the generalized Stieltjes constants $\gamma_m(a)$, i.e.,

\[
\gamma_m(a) = \lim_{n \to \infty} \left( \sum_{k=0}^{n} \frac{\ln^m (k + a)}{k + a} - \int_{0}^{n} \frac{\ln^m (x + a)}{x + a} dx \right).
\]

We note that the constants $\gamma_m(a)$ occur in the Laurent series expansion of the Hurwitz zeta function (see [3, 44]).

Generalizations of the Euler-Mascheroni constant have been studied extensively. They appear in the evaluation of series and integrals containing some important functions (see, for example, [1, 6, 8–11, 13, 14, 16, 18, 21, 23, 30, 37, 42]). Besides their lower and upper bounds are studied (see, for example [3, 5, 19, 27, 29, 31, 32, 36, 39, 40, 47]) and some evaluation formulas are given (see, for example [4, 25, 41, 45]).

In this study, we consider the following Dirichlet series (so called the Euler sums of the hyperharmonic numbers [20, 35] and also see [38])

\[
\sum_{k=1}^{\infty} \frac{h_k^{(r)}}{k^s} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{h_k^{(r)}}{k^x}, \quad (1.4)
\]

which are generalizations of $\zeta(s)$. Here $x^r = \Gamma(x + r) / \Gamma(x)$ for $x, r + x \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, and $h_n^{(r)}$ are the hyperharmonic numbers defined by [15]

\[
h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)} \quad \text{with} \quad h_n^{(0)} = \frac{1}{n}, \quad n, r \in \mathbb{N}.
\]
It is known that both series in (1.4) are convergent for \( s > r \) and divergent for \( 0 < s \leq r \) (see [35]). We mainly focus on the limit (1.3) for \( f = f(x, r) = h_x^{(r)} / x^r \) and \( f = f(x, r) = h_x^{(r)} / x^r \), where \( h_x^{(r)} \) is the analytic extension of \( h_n^{(r)} \), defined by [34]

\[
h_x^{(r)} = \frac{x^r}{\Gamma(r)} (\psi(x + r) - \psi(r)), \ r, x, r + x \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}.
\]  

We prove the following:

**Theorem 1** Let \( r \in [0, +\infty) \). Then,

a) The sequences \( \{y_n(r)\}_{n=1}^{\infty} \) and \( \{z_n(r)\}_{n=1}^{\infty} \) satisfy

\[
0 \leq z_n(r) < z_{n+1}(r) < \cdots < y_{n+1}(r) < y_n(r) \leq 1, \text{ for each } n \in \mathbb{N}.
\]

b) The sequences \( \{y_n(r)\}_{n=1}^{\infty} \) and \( \{z_n(r)\}_{n=1}^{\infty} \) converge to the common limit denoted by \( \gamma_h^{(r)} \), i.e.,

\[
\gamma_h^{(r)} = \lim_{n \to \infty} z_n(r) = \lim_{n \to \infty} y_n(r).
\]

c) For each \( n \in \mathbb{N} \), we have the following estimates:

\[
A(n, r) < \gamma_h^{(r)} - z_n(r) < A(n, r) + \frac{h_n^{(r)}}{(n+1)^r}
\]

and

\[
B(n, r) - \frac{h_{n+1}^{(r)}}{(n+1)^r} < y_n(r) - \gamma_h^{(r)} < B(n, r),
\]

where

\[
A(n, r) = h_n^{(r)} / n^r - \int_n^{n+1} \left( h_x^{(r)} / x^r \right) dx
\]

\[
B(n, r) = \int_1^{n+1} \left( h_x^{(r)} / x^r \right) dx.
\]

We next consider the function \( f = f(x, r) = h_x^{(r)} / x^r \) in the limit (1.3) for \( f = f(x, r) = h_x^{(r)} / x^r \). Let \( \{a_n(r)\}_{n=1}^{\infty} \) and \( \{b_n(r)\}_{n=1}^{\infty} \) be sequences defined by

\[
a_n(r) = \sum_{k=1}^{n-1} \frac{h_k^{(r)}}{k^r} - \int_1^{n+1} \frac{h_x^{(r)}}{x^r} dx \quad \text{and} \quad b_n(r) = \sum_{k=1}^{n} \frac{h_k^{(r)}}{k^r} - \int_1^{n} \frac{h_x^{(r)}}{x^r} dx.
\]

These sequences satisfy the following properties.
Theorem 2 Let \( r \in [0, +\infty) \). Then,
\( a) \) The sequences \( \{a_n (r)\}_{n=1}^{\infty} \) and \( \{b_n (r)\}_{n=1}^{\infty} \) satisfy
\[
0 \leq a_n (r) < a_{n+1} (r) < \cdots < b_n (r) < b_{n+1} (r) < b_n (r) \leq 1/ \Gamma (r+1), \text{ for all } n \in \mathbb{N}.
\]
\( b) \) The sequences \( \{a_n (r)\}_{n=1}^{\infty} \) and \( \{b_n (r)\}_{n=1}^{\infty} \) converge to the common limit denoted by \( \overline{\gamma}_{h(r)} \), i.e.,
\[
\overline{\gamma}_{h(r)} = \lim_{n \to \infty} a_n (r) = \lim_{n \to \infty} b_n (r).
\]
\( c) \) For each \( n \in \mathbb{N} \), we have the following estimates:
\[
C (n, r) < \overline{\gamma}_{h(r)} - a_n (r) < C (n, r) + \frac{h_{n+1} (r)}{(n+1)^r}
\]
and
\[
D (n, r) - \frac{h_{n+1} (r)}{(n+1)^r} < b_n (r) - \overline{\gamma}_{h(r)} < D (n, r),
\]
where \( C (n, r) = h_n (r)/n^r - \sum_{n+1}^{\infty} \left( \frac{h_x (r)}{x^r} \right) dx \) and \( D (n, r) = \sum_{n}^{\infty} \left( \frac{h_x (r)}{x^r} \right) dx \).
\( d) \) For \( r \geq 1 \) we have \( \overline{\gamma}_{h(r)} > \overline{\gamma}_{h(r+1)} \).

In Section 2 we prove these results. In Section 3 we establish formulas to calculate \( \gamma_{h(r)} \) and \( \overline{\gamma}_{h(r)} \) (see Theorem 8 and Theorem 12). As a demonstration, we list the first few values of \( \gamma_{h(r)} \) and \( \overline{\gamma}_{h(r)} \) here:
\[
\begin{align*}
\gamma_{h(0)} &= \gamma \simeq 0.5772156649, & \overline{\gamma}_{h(0)} &= \gamma \simeq 0.5772156649, \\
\gamma_{h(1)} &\simeq 0.5290529699, & \overline{\gamma}_{h(1)} &\simeq \gamma_{h(1)} \simeq 0.5290529699, \\
\gamma_{h(2)} &\simeq 0.5551960549, & \overline{\gamma}_{h(2)} &\simeq 0.2586901244, \\
\gamma_{h(3)} &\simeq 0.5978616743, & \overline{\gamma}_{h(3)} &\simeq 0.08538807383, \\
\gamma_{h(4)} &\simeq 0.6439018350, & \overline{\gamma}_{h(4)} &\simeq 0.02123065279, \\
\gamma_{h(5)} &\simeq 0.6881913208, & \overline{\gamma}_{h(5)} &\simeq 0.004231410657, \\
\gamma_{h(6)} &\simeq 0.7284308281, & \overline{\gamma}_{h(6)} &\simeq 0.000703545796, \\
\gamma_{h(7)} &\simeq 0.7637312448, & \overline{\gamma}_{h(7)} &\simeq 0.000100330361, \\
\gamma_{h(8)} &\simeq 0.7939960448, & \overline{\gamma}_{h(8)} &\simeq 1.252451689 \times 10^{-5}, \\
\gamma_{h(9)} &\simeq 0.8195630509. & \overline{\gamma}_{h(9)} &\simeq 1.390145776 \times 10^{-6}.
\end{align*}
\]

We finally note that since
\[
\lim_{r \to 0} h^{(r)}_x = \lim_{r \to 0} \frac{x^r}{x^r} \left( \psi (x + r) - \psi (r) \right) = -\frac{1}{x} \lim_{r \to 0} \frac{\psi (r)}{\Gamma (r)} = \frac{1}{x},
\]
we use \( h^{(0)}_x = 1/x \) in the sequel.
2 Proofs of Theorems

For the proofs of Theorem 1 and Theorem 2 we need the following lemma.

Lemma 3 Let $r > 0$ and $0 < x < y$. Then,

$$\frac{\psi (y + r) - \psi (r)}{\psi (x + r) - \psi (r)} < \frac{y}{x}.$$  

Proof. We start by setting

$$\alpha (x, r) = \frac{\psi (x + r) - \psi (r)}{x}$$

for $r > 0$, $x > 0$. We show that $\alpha (x, r)$ is a decreasing function with respect to $x$. Differentiating with respect to $x$ gives

$$\frac{d}{dx} \alpha (x, r) = \frac{x \frac{d}{dx} \psi (x + r) - \psi (x + r) + \psi (r)}{x^2}.$$  

Let $\beta (x, r) = x \frac{d}{dx} \psi (x + r) - \psi (x + r) + \psi (r)$. Then,

$$\frac{d}{dx} \beta (x, r) = x \psi'' (x + r) < 0$$

implies that the function $\beta (x, r)$ is decreasing with respect to $x$. Since

$$\lim_{x \to 0^+} \beta (x, r) = \lim_{x \to 0^+} \left( x \frac{d}{dx} \psi (x + r) - \psi (x + r) + \psi (r) \right) = 0$$

and $\beta (x, r)$ is decreasing, we conclude that

$$\beta (x, r) = x \frac{d}{dx} \psi (x + r) - \psi (x + r) + \psi (r) < 0.$$  

This shows that the function $\alpha (x, r)$ is decreasing with respect to $x$, which completes the proof. ■

2.1 Proof of Theorem 1

a) We first show that the sequence $\{y_n (r)\}_{n=1}^{\infty}$ is decreasing. From the definition, we have

$$y_{n+1} (r) - y_n (r) = h^{(r)}_{n+1} \frac{(n+1)^r}{(n+1)^r} - \int_n^{n+1} \frac{h^{(r)}_x}{x^r} dx.$$  

The mean value theorem for definite integrals guarantees that there is an $N \in (n, n + 1)$ such that

$$\int_n^{n+1} \frac{h^{(r)}_x}{x^r} dx = \frac{h^{(r)}_N}{N^r}.$$
holds. Then, using (1.5) it is seen that
\[ y_{n+1}(r) - y_n(r) = \frac{h_{n+1}^{(r)}}{(n+1)^r} - \frac{h_N^{(r)}}{N^r} = \frac{N^{r+1} (n+1)^r (\psi(n+1+r) - \psi(r)) - (n+1)^r N^r (\psi(N+r) - \psi(r))}{\Gamma(r) (n+1)^{r+1} N^{r+1}}. \]

For \( r = 0 \), the inequality \( y_{n+1}(0) - y_n(0) < 0 \) is trivial since \( h_x^{(0)} = 1/x \). Let \( r > 0 \). Then,
\[ y_{n+1}(r) - y_n(r) < 0 \iff \psi(n+1+r) - \psi(r) < \frac{n+1}{N} \frac{(n+1)^r N^{r+1}}{N^r (n+1)^r}. \]

According to the Lemma 3 it remains to prove that
\[ 1 < \frac{(n+1)^r N^{r+1}}{N^r (n+1)^r} \text{ for } r > 0 \text{ and } n < N < n+1. \tag{2.1} \]

For this purpose, we set
\[ g(x,r) = \frac{\Gamma(x+r)}{x^r \Gamma(x)} = \frac{x^r}{x^r} \text{ for } r > 0, x > 0. \]

Differentiating with respect to \( x \) gives
\[ \frac{d}{dx} g(x,r) = \frac{\Gamma(x+r)}{\Gamma(x) x^r} \left( \psi(x+r) - \frac{r}{x} \right). \]

The substitutions \( x \to 1, r \to x \) and \( y \to r \) in Lemma 3 yields
\[ \frac{\psi(x+r) - \psi(x)}{r} < \frac{1}{x}. \]

Thus, we deduce that \( g(x,r) = \Gamma(x+r)/x^r \Gamma(x) \) is decreasing with respect to \( x \) for \( x, r > 0 \). This completes the proof of \( y_{n+1}(r) < y_n(r) \).

To show that the sequence \( \{z_n(r)\}_{n=1}^{\infty} \) is increasing we again appeal to the mean value theorem for definite integrals and deduce that
\[ z_{n+1}(r) - z_n(r) = \frac{h_n^{(r)}}{n^r} - \frac{h_N^{(r)}}{N^r}, N \in (n,n+1) \]
\[ = \frac{N^{r+1} \psi(n+r) - \psi(r)) - n^{r+1} N^r (\psi(N+r) - \psi(r))}{\Gamma(r) n^{r+1} N^{r+1}}. \]

Since \( h_x^{(0)} = 1/x \) we have \( z_{n+1}(r) - z_n(r) < 0 \) for \( r = 0 \). Let \( r > 0 \). Then,
\[ z_{n+1}(r) - z_n(r) > 0 \iff \frac{\psi(n+r) - \psi(r)}{\psi(N+r) - \psi(r)} > \frac{n^{r+1} N^r}{N^{r+1} n^{r+1}}. \]
It is seen from Lemma 3 and (2.1) that the sequence \( \{z_n(r)\}_{n=1}^{\infty} \) is increasing. Now, above facts and

\[
y_n(r) - z_n(r) = \frac{h_n^{(r)}}{n^r} > 0, \text{ for each } n \in \mathbb{N}
\]

imply that

\[
z_1(r) < \cdots < z_n(r) < z_{n+1}(r) < \cdots < y_{n+1}(r) < y_n(r) < \cdots < y_1(r).
\]

b) The sequence \( \{z_n(r)\}_{n=1}^{\infty} \) is increasing and bounded above, hence the limit \( \lim_{n \to \infty} z_n(r) \) exists. Similarly the limit \( \lim_{n \to \infty} y_n(r) \) exists. We observe that

\[
y_n(r) - z_n(r) = \frac{h_n^{(r)}}{n^r} \sim \frac{1}{n^r} \frac{n^{r-1} \ln n}{\Gamma(r)} \to 0, n \to \infty,
\]

so \( \lim_{n \to \infty} z_n(r) = \lim_{n \to \infty} y_n(r) \). Thus,

\[
0 = z_1(r) < \cdots < z_n(r) < \cdots < z_{n+1}(r) < \cdots < y_{n+1}(r) < y_n(r) < \cdots < y_1(r) = 1. \quad (2.2)
\]

c) The inequalities can be easily seen from (2.2).

2.2 Proof of Theorem 2

We sketch the proof which is similar to the proof of Theorem 1.

a) From the definition of \( \{b_n(r)\}_{n=1}^{\infty} \) and the mean value theorem for definite integrals, it is seen that

\[
b_{n+1}(r) - b_n(r) = \frac{h_{n+1}^{(r)}}{(n+1)^r} - \frac{h_n^{(r)}}{N^r} \quad N \in (n, n+1).
\]

Thus, the validity of \( b_{n+1}(r) < b_n(r) \) follows from (1.5) and Lemma 3 when \( r > 0 \), and from \( h_0^{(r)} = 1/x \) when \( r = 0 \). In a similar way it can be seen that the sequence \( \{a_n(r)\}_{n=1}^{\infty} \) is increasing, and

\[
a_1(r) < \cdots < a_n(r) < a_{n+1}(r) < \cdots < b_{n+1}(r) < b_n(r) < \cdots < b_1(r).
\]

b) The sequence \( \{a_n(r)\}_{n=1}^{\infty} \) is increasing and bounded above, hence the limit \( \lim_{n \to \infty} a_n(r) \) exists. Similarly the limit \( \lim_{n \to \infty} b_n(r) \) exists. We observe that

\[
b_n(r) - a_n(r) = \frac{h_n^{(r)}}{n^r} \sim \frac{\ln (n + r)}{n \Gamma(r)} \to 0, n \to \infty,
\]

so

\[
\lim_{n \to \infty} a_n(r) = \lim_{n \to \infty} b_n(r) = \varphi_h(r).
\]
Thus,

\[ 0 = a_1 (r) < \cdots < a_n (r) < \cdots < \gamma_h \( r \) < \cdots < b_n (r) < \cdots < b_1 (r) = \frac{1}{\Gamma (r)}. \]  

(2.3)

c) The estimates follow from (2.3).

d) By the definition, we have

\[ a_n (r + 1) - a_n (r) = \int_1^n f (x, r) \, dx - \sum_{k=1}^{n-1} f (k, r), \]

where

\[ f (x, r) = \frac{h^{(r)}_y}{x^r} - \frac{h^{(r+1)}_x}{x^{r+1}}. \]

We would like to show that \( a_n (r + 1) - a_n (r) < 0 \). For a decreasing function \( f \), the sum \( \sum_{k=1}^{n-1} f (k, r) \) is greater than the integral \( \int_1^n f (x, r) \, dx \) because \( \sum_{k=1}^{n-1} f (k, r) \) corresponds to the upper Darboux sum of \( f \) with respect to the partition \( P = \{ x_k = k : k = 1, 2, \ldots, n - 1 \} \). Therefore, we need to show that \( f (x, r) \) is a decreasing function with respect to \( x \).

Let \( y > x > 0 \). Using (1.5) we have

\[ f (y, r) - f (x, r) = \left( \frac{\psi (y + r) - \psi (r)}{y} - \frac{\psi (x + r) - \psi (r)}{x} \right) \left( 1 - \frac{1}{r} \right) \Gamma (r) \]

\[ + \frac{1}{r \Gamma (r + 1)} \frac{x - y}{(y + r) (x + r)}. \]

Thus, Lemma 3 and the fact that

\[ \frac{x - y}{(y + r) (x + r)} < 0 \]

imply \( f (y, r) - f (x, r) < 0 \), i.e., the function \( f (x, r) \) is decreasing.

Since \( \gamma_h \( r \) = \lim_{n \to \infty} a_n (r) \), we immediately conclude that

\[ \lim_{n \to \infty} a_n (r + 1) = \gamma_h \( r + 1 \) \leq \gamma_h \( r \) = \lim_{n \to \infty} a_n (r), \]

or equivalently

\[ \gamma_h \( r \) \geq \cdots \geq \gamma_h \( r \) \geq \gamma_h \( r + 1 \) \geq \cdots \geq 0. \]

3 On evaluations of \( \gamma_h \( r \) \) and \( \gamma_h \( r \) \)

In this section, we give representations for the constants \( \gamma_h \( r \) \) and \( \gamma_h \( r \) \) when \( r \) is a non-negative integer. We recall that the Stirling numbers of the first kind \( \left[ \frac{r}{j} \right] \) are defined by

\[ x^r = \sum_{j=0}^{r} \left[ \frac{r}{j} \right] x^j, \]  

(3.1)
and the generalized harmonic numbers $H_n^{(r)}$ are defined by

$$H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}.$$ 

Note that throughout this section, an empty sum is assumed to be zero.

### 3.1 The constants $\gamma_{h^{(r)}}$

We first analyze the sum and integral in the definition of $y_n^{(r)}$, respectively. The aforementioned sum may be evaluated as follows:

**Lemma 4** Let $r$ be a non-negative integer. Then,

$$\Gamma (r) \sum_{k=1}^{n} \frac{h_k^{(r)}}{k^r} = \frac{(H_n)^2 + H_n^{(2)}}{2} - (\psi (r) + \gamma) H_n$$

$$+ \sum_{j=0}^{r-1} \left[ \sum_{k=1}^{n} \frac{H_{k+r-1}}{k^{r+1-j}} - (\psi (r) + \gamma) H_n^{(r+1-j)} \right]$$

$$+ \sum_{j=1}^{r-1} H_j - H_{r-1} (H_{n+r-1} - H_n) + \sum_{j=1}^{r-1} \frac{H_{j-1}}{j + n}. \quad (3.2)$$

**Proof.** Using (1.5), (3.1) and $\psi (r) + \gamma = H_{r-1}$ we have

$$h_k^{(r)} = \frac{1}{\Gamma (r)} \sum_{j=0}^{r} \left[ k^j - 1 \right] H_{k+r-1} - (\psi (r) + \gamma),$$

from which we obtain that

$$\Gamma (r) \sum_{k=1}^{n} \frac{h_k^{(r)}}{k^r} = \sum_{j=0}^{r-1} \left[ \sum_{k=1}^{n} \frac{H_{k+r-1}}{k^{r+1-j}} + \sum_{k=1}^{n} \frac{H_{k+r-1}}{k} \right]$$

$$- (\psi (r) + \gamma) \sum_{j=0}^{r-1} \left[ H_n^{(r+1-j)} - (\psi (r) + \gamma) H_n. \right. \quad (3.3)$$

It is easy to see that

$$\sum_{k=1}^{n} \frac{H_{k+r-1}}{k} = \sum_{k=1}^{n} \frac{H_k}{k} + \sum_{j=1}^{r-1} \frac{H_j}{j} + H_{r-1} H_n - \sum_{j=1}^{r-1} \frac{H_{n+j}}{j}$$

and

$$\sum_{j=1}^{r-1} \frac{H_{n+j}}{j} = H_{r-1} H_n - H_{r-1} H_n - H_{r-1} (H_{n+r-1} - H_n) + \sum_{j=1}^{r-1} \frac{H_{j-1}}{j + n}.$$
Using these and the identity \([2, \text{Lemma 1}]\)
\[
\sum_{k=1}^{n} H_k = \frac{(H_n)^2 + H_n^{(2)}}{2},
\]
we deduce that
\[
\sum_{k=1}^{n} \frac{H_{k+r-1}}{k} = \frac{(H_n)^2 + H_n^{(2)}}{2} + \sum_{j=1}^{r-1} \left( \frac{H_j}{j} + \frac{H_{j-1}}{j+n} \right) - H_{r-1} (H_{n+r-1} - H_n).
\]

Hence, (3.2) follows from (3.3) and (3.4).

Now we evaluate the integral in the definition of \(y_n(r)\).

\[\text{Lemma 5} \quad \text{Let } r \text{ be a non-negative integer. Then,}\]
\[
\Gamma (r) \int_{1}^{n} \frac{h_x (r)}{x^r} dx = \frac{1}{2} (\ln n)^2 - \psi (r) \ln n + \frac{(n+1)^{r-1}}{n^r} - r! + \left( r + \frac{1}{2} \right) \left( 1 - \frac{1}{n} \right)
\]
\[+ \sum_{j=0}^{r-1} \left( \frac{r}{r+1-j} \left( 1 - \frac{1}{n^{r+1-j}} \right) \psi \left( \frac{r}{r-j} \left( 1 - \frac{1}{n^{-j}} \right) \right) \right) \]
\[- \int_{0}^{\infty} \ln \left( 1 + t^2 \right) - \ln \left( 1 + \left( t/n \right)^2 \right) \left( e^{2\pi t} - 1 \right) \left( t^2 + z^2 \right) dt \]
\[+ \sum_{j=0}^{r-1} \left[ \frac{r}{r+1-j} \right] \int_{1}^{n} \psi \left( x + 1 \right) \frac{x}{x^{r+1-j}} dx.\]

\(3.5\)

\[\text{Proof.} \quad \text{Using (1.5) and the well-known identity } \psi (x + 1) = \psi (x) + 1/x \text{ we find that}\]
\[
\Gamma (r) \int_{1}^{n} \frac{h_x (r)}{x^r} dx = \int_{1}^{n} \frac{x^\psi}{x^{r+1}} \psi (x + 1) dx
\]
\[+ \int_{1}^{n} \frac{\psi (x) + 1}{x^{r+1}} \left( 1 - \frac{1}{n^r} \right) \psi \left( \frac{r}{r-j} \left( 1 - \frac{1}{n^{-j}} \right) \right) dx.
\]
We now consider each of the integrals on the right hand side one by one.

In view of (3.1) we have
\[
\int_{1}^{n} \frac{x^\psi}{x^{r+1}} \psi (x + 1) dx = \int_{1}^{n} \frac{\psi (x + 1)}{x} dx + \sum_{j=0}^{r-1} \left[ \frac{r}{r+1-j} \right] \int_{1}^{n} \frac{\psi (x + 1)}{x^{r+1-j}} dx.
\]

Thanks to the expression [1, Eq. 6.3.21]
\[
\psi (z) = \ln z - \frac{1}{2z} - 2 \int_{0}^{\infty} \frac{t}{(e^{2\pi t} - 1) (t^2 + z^2)} dt, \quad |\arg z| < \frac{\pi}{2},
\]
we find
\[
\int_{1}^{n} \frac{\psi (x + 1)}{x} dx = \frac{1}{2} (\ln n)^2 - \frac{1}{2n} + \frac{1}{2} - 2 \int_{0}^{\infty} \int_{1}^{n} \frac{t}{(e^{2\pi t} - 1) (t^2 + x^2)} dx dt.
\]
Hence we deduce that
\[
\int_1^n \frac{\psi(x + 1)}{x} dx = \frac{1}{2} \left( \ln n \right)^2 - \frac{1}{2n} + \frac{1}{2} - \int_0^\infty \frac{\ln \left( 1 + t^2 \right) - \ln \left( 1 + \left( t/n \right)^2 \right)}{(e^{2\pi t} - 1)t} dt.
\]
It is clear that
\[
\int_1^n \frac{1}{x^r} \frac{d}{dx} (x + 1)^{-1} dx = \frac{1}{n^r} (n + 1)^{-1} - r! + r \sum_{j=0}^{r-1} \left[ \frac{1}{j} \frac{1}{(r+1-j)} \right]
\]
and
\[
\psi(r) \int_1^n \frac{x^r}{x^{r+1}} dx = \psi(r) \ln n + \psi(r) \sum_{j=0}^{r-1} \left[ \frac{1}{r-j} \left( 1 - \frac{1}{n^{r-j}} \right) \right].
\]
The results above yield (3.5).

We now consider the limit case of the integral
\[
\int_1^n \frac{\psi(x + 1)}{x^p} dx
\]
in (3.5), in which we require the constant \( \sigma_p \):
\[
\sigma_p = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} \zeta(p + k), \quad p \geq 1.
\]
The constant \( \sigma_p \) occurs in several series and integral evaluations (see [7, 12, 17, 21]).

**Lemma 6** For an integer \( p \geq 2 \), we have
\[
\int_1^\infty \frac{\psi(x + 1)}{x^p} dx = -\frac{\gamma}{p-1} + (-1)^p \left( \sigma_p - \zeta'(p) \right) + \sum_{j=2}^{p-1} \frac{(-1)^{p-1-j}}{j-1} \zeta(p + 1 - j).
\]

**Proof.** In view of the series representation [1, Eq. 6.3.16]
\[
\psi(x + 1) = -\gamma + \sum_{k=1}^\infty \frac{x}{k(k + x)}
\]
we obtain
\[
\int_1^n \frac{\psi(x + 1)}{x^p} dx = \frac{\gamma}{p-1} \left( \frac{1}{n^p+1} - 1 \right) + \sum_{k=1}^\infty \frac{1}{k} \int_1^n \frac{dx}{(k + x)x^{p-1}}.
\]
Considering the following partial fraction decomposition
\[
\frac{1}{x^{p-1}(k + x)} = \sum_{j=1}^{p-1} \frac{A_j}{x^j} + \frac{1}{(-k)^{p-1}} \frac{1}{k + x},
\]
where \( A_j = (-1)^{p-1-j}/k^{p-j} \), \( 1 \leq j \leq p-1 \), we see that

\[
\int_1^n \frac{\psi(x+1)}{x^p} \, dx = \frac{\gamma}{p-1} \left( \frac{1}{n^{p+1}} - 1 \right) + \sum_{k=1}^{\infty} \sum_{j=2}^{p-1} \frac{(-1)^{p-1-j}}{k^{p+1-j}} \frac{1}{j-1} \left( 1 - \frac{1}{n^{j-1}} \right)
\]

\[
+ \sum_{k=1}^{\infty} \frac{(-1)^p}{k^p} \left( \ln \frac{n}{k+n} - \ln \frac{1}{k+1} \right).
\]

Letting \( n \to \infty \), with the use of [21, Theorem 4]

\[
\sum_{k=1}^{\infty} \frac{\ln (k+1)}{k^p} = \sigma_p - \zeta'(p),
\]

we arrive at the desired result. □

**Remark 7** The formula given in Lemma 6 has also been recorded in the recent paper [17, Lemma 1], with a slightly different proof.

We now recall the formula [46, Theorem 2.1],

\[
\sum_{n=r}^{\infty} H_n \frac{H_n}{(n+1-r)^p} = \frac{1}{2} (p+2) \zeta (p+1) - \frac{1}{2} \sum_{j=1}^{p-2} \zeta (p-j) \zeta (j+1)
\]

\[
- \sum_{m=1}^{p-1} (-1)^m \zeta (p+1-m) H_{r-1}^{(m)} - (-1)^p \sum_{m=1}^{r} \frac{H_m}{mp^p} \quad (3.6)
\]

for \( p \in \mathbb{N} \setminus \{1\} \), which appears in the following evaluation formula for the constant \( \gamma_{h(r)} \).

**Theorem 8** Let \( r \) be a non-negative integer. Then,

\[
\Gamma (r) \gamma_{h(r)} = \frac{1}{2} \gamma^2 + \frac{1}{2} \zeta (2) - \frac{1}{2} + \int_0^{\infty} \ln \left( \frac{1+e^t}{e^t-1} \right) dt - \left( \psi (r) + \gamma + r! - r \right)
\]

\[
+ \sum_{j=1}^{r-1} \frac{H_j}{j} + \sum_{j=1}^{r-1} \frac{r}{j} \left( E (r,j) + \frac{\psi (r)}{r-j} - \frac{r}{r+1-j} - H_{r-1} \zeta (r+1-j) \right),
\]

where

\[
E (r,j) = \sum_{k=r}^{\infty} \frac{H_k}{(k+1-r)^{r-j+1}} - \int_1^{\infty} \frac{\psi (x+1)}{x^{r+1-j}} \, dx.
\]
Proof. From Lemma 4 and Lemma 5, we have
\[
\Gamma(r) \left( \sum_{k=1}^{n} \frac{h_k^{(r)}}{k^r} - \int_{1}^{n} \frac{h_x^{(r)}}{x^r} dx \right) = \frac{(H_n)^2 + H_n^{(2)}}{2} - (\psi(r) + \gamma) H_n - \frac{1}{2} (\ln n)^2
\]
\[
+ \psi(r) \ln n - H_{r-1} (H_{n+r-1} - H_n) + \sum_{j=1}^{r-1} \frac{H_{j-1}}{j + n} - \frac{(n + 1)^{r-1}}{n^r}
\]
\[
+ \sum_{j=0}^{r-1} \left[ \sum_{k=1}^{n} \frac{H_{k+r-1}}{k^{r+1-j}} - (\psi(r) + \gamma) H_n^{(r+1-j)} \right] + \sum_{j=1}^{r-1} \frac{H_j}{j} - \left( r + \frac{1}{2} \right) \left( 1 - \frac{1}{n} \right)
\]
\[
+ r! \sum_{j=0}^{r-1} \left( \frac{r}{r + 1 - j} \left( 1 - \frac{1}{n^{r+1-j}} \right) - \frac{\psi(r)}{r - j} \left( 1 - \frac{1}{n^{r-j}} \right) \right)
\]
\[
+ \int_{0}^{\infty} \frac{\ln (1 + t^2) - \ln (1 + (t/n)^2)}{(e^{2\pi t} - 1) t} dt - \sum_{j=0}^{r-1} \left[ r \int_{1}^{n} \frac{\psi(x + 1)}{x^{r+1-j}} dx \right].
\] (3.7)

Considering the well-known properties
\[
H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O(n^{-4}) \quad \text{and} \quad \lim_{n \to \infty} (H_{n+r-1} - H_n) = 0, \quad (3.8)
\]
and then letting \( n \to \infty \) in (3.7) give the desired formula after some manipulations. ■

With the help of Lemma 6 and (3.6), the statement of Theorem 8 can be equivalently written as
\[
\Gamma(r) \gamma_h^{(r)} = \frac{1}{2} \gamma^2 + \frac{1}{2} \zeta(2) - \frac{1}{2} + \int_{0}^{\infty} \frac{\ln (1 + t^2)}{(e^{2\pi t} - 1) t} dt + \sum_{j=1}^{r-1} \frac{H_j}{j} - (\psi(r) + \gamma) \gamma + r!
\]
\[
- r + \sum_{j=1}^{r-1} \left[ \frac{r+3-j}{2} \zeta(r + 2 - j) - \frac{r-j-1}{2} \sum_{v=1}^{r-j-1} \zeta(v+1) \right]
\]
\[
- \sum_{v=2}^{r-j} (-1)^v \zeta(r + 2 - j - v) \left( H_v^{(r)} + \frac{(-1)^{r-j}}{v - 1} \right) + \frac{H_{r-1}}{r-1}
\]
\[
+ (-1)^{r-j} \left( \sigma_{r+1-j} - \zeta'(r + 1 - j) + \sum_{v=1}^{r-1} \frac{H_{v+1}}{r+j-v} \right)
\]
\[
+ 2 \zeta(3) + \frac{1}{72} \pi^2 (\pi^2 - 27) + \frac{5}{4} - \zeta'(3) + \zeta'(2) - \sigma_2 + \sigma_3.
\]

By straightforward computations we observe that
\[
\gamma_h^{(0)} = -\gamma \lim_{r \to 0} \frac{\psi(r)}{\Gamma(r)} = \gamma;
\]
\[
\gamma_h^{(1)} = \frac{1}{2} \gamma^2 + \frac{1}{2} \zeta(2) - \frac{1}{2} + \int_{0}^{\infty} \frac{\ln (1 + t^2)}{(e^{2\pi t} - 1) t} dt,
\]
\[
\gamma_h^{(2)} = \gamma_h^{(1)} - \gamma - \sigma_2 + \zeta'(2) + 2 \zeta(3),
\]
\[
\gamma_h^{(3)} = \frac{1}{2} \gamma_h^{(2)} - \frac{1}{4} \gamma + 2 \zeta(3) + \frac{1}{72} \pi^2 (\pi^2 - 27) + \frac{5}{4} - \zeta'(3) + \zeta'(2) - \sigma_2 + \sigma_3.
\]
Moreover, according to table on page 5 it seems that \( \gamma_{h(r)} \leq \gamma_{h(r+1)} \) for \( r \geq 1 \). Our attempts to prove this observation were not successful, therefore it remains an open question.

**Remark 9** Combining Connon’s [14] results (3.9) and (3.46) gives (with a misprint corrected)

\[
\int_0^\infty \frac{\ln (1 + t^2)}{(e^{2\pi t} - 1)t} dt = \sigma_1 + \frac{1}{2} - \zeta(2) + \gamma_1.
\]

### 3.2 The constants \( \overline{\gamma}_{h(r)} \)

The following lemmas are the analogues of Lemma 4 and Lemma 5.

**Lemma 10** Let \( r \) be a non-negative integer. Then,

\[
\begin{align*}
\Gamma (r) \sum_{k=1}^{n} \frac{h_k^{(r)}}{k^r} &= \frac{(H_n)^2 + H_n^{(2)}}{2} - H_{r-1} (H_{n+r-1} - H_n) - (\psi(r) + \gamma) H_n \\
&+ \sum_{j=1}^{r-1} \frac{H_j}{j} + \sum_{j=1}^{r-1} \frac{H_{j-1}}{j+n}.
\end{align*}
\]

**Proof.** The proof follows from (1.5), \( \psi(r) + \gamma = H_{r-1} \) and (3.4). ■

The proof of the following lemma is similar to the proof of Lemma 5. Thus we omit it.

**Lemma 11** Let \( r \) be a non-negative integer. Then,

\[
\begin{align*}
\Gamma (r) \int_1^n \frac{h_k^{(r)}}{x^r} dx &= \frac{1}{2} \ln(n)^2 - \frac{1}{2} + \frac{1}{2} - \int_0^\infty \frac{\ln (1 + t^2) - \ln \left(1 + \left(\frac{t}{n}\right)^2\right)}{(e^{2\pi t} - 1)t} dt \\
&+ \sum_{j=1}^{r-1} \frac{\ln n + \ln (1 + j) - \ln (n + j)}{j} - \psi(r) \ln n.
\end{align*}
\]

To present the counterpart of Theorem 8 we use Lemma 10 and Lemma 11. Thus,

\[
\begin{align*}
\Gamma (r) \left( \sum_{k=1}^{n} \frac{h_k^{(r)}}{k^r} - \int_1^n \frac{h_k^{(r)}}{x^r} dx \right) &= \frac{(H_n)^2 + H_n^{(2)}}{2} - \frac{1}{2} \ln(n)^2 + \psi(r) \ln n - (\psi(r) + \gamma) H_n \\
&- \frac{1}{2} + \int_0^\infty \frac{\ln (1 + t^2) - \ln \left(1 + \left(\frac{t}{n}\right)^2\right)}{(e^{2\pi t} - 1)t} dt + \sum_{j=1}^{r-1} \frac{H_j - \ln (1 + j)}{j} \\
&+ \sum_{j=1}^{r-1} \frac{H_{j-1}}{j + n} - H_{r-1} (H_{n+r-1} - H_n) - \sum_{j=1}^{r-1} \frac{\ln n - \ln (n + j)}{j} + \frac{1}{2n}.
\end{align*}
\]
We now let \( n \to \infty \), Using (3.8) we deduce that

\[
\Gamma (r) \gamma_h(r) = \frac{\gamma^2}{2} + \frac{\zeta(2)}{2} - \frac{1}{2^r} \int_0^\infty \ln \left( \frac{1 + t^2}{e^{2\pi t} - 1} \right) dt - (\psi(r) + \gamma) \gamma + \frac{r-1}{2} \sum_{j=1}^{r-1} \frac{H_j - \ln (1 + j)}{j}.
\]

Thus, we have obtained the following theorem.

**Theorem 12** Let \( r \) be a non-negative integer. Then,

\[
\gamma_h(r) = \frac{1}{\Gamma (r)} \left( \gamma_h^{(1)} - (\psi(r) + \gamma) \gamma + \frac{r-1}{2} \sum_{j=1}^{r-1} \frac{H_j - \ln (1 + j)}{j} \right).
\]

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