EXPONENTIAL DECAY ESTIMATES AND SMOOTHNESS OF THE MODULI SPACE OF PSEUDOHOLOMORPHIC CURVES

KENJI FUKAYA, YONG-GEUN OH, HIROSHI OHTA, KAORU ONO

ABSTRACT. In this paper, we examine the dependence of standard gluing process for pseudoholomorphic curves under the change of the length $T$ of the neck-region with respect to the cylindrical metrics associated to the given analytic coordinates near the punctures in the setting of bordered open Riemann surface with boundary punctures. We establish exponential decay of the $T$-derivatives of the $T$-dependent family of glued solutions under the change of the length $T$ of the neck-region in a precise manner. This exponential decay estimate is an important ingredient to prove the smoothness of the Kuranishi structure constructed on the compactified moduli space of pseudoholomorphic curves given in the appendix of the authors’ book [FOOO1]. We also demonstrate the way how this smoothness follows from the exponential decay.

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1. Introduction

The theory of Kuranishi structures is introduced by the first and the third named authors in [FOn], and further amplified by the authors in [FOOO1], [FOOO2]. To implement the abstract theory of Kuranishi structure in the study of concrete moduli problem, especially that of the moduli space of pseudoholomorphic curves, one issue is to establish smoothness of the Kuranishi map \( s \) and of the coordinate change of the Kuranishi neighborhoods.

Let us first describe the problem in a bit more detail in the context of bordered stable pseudoholomorphic curves attached to Lagrangian submanifolds. Note that the boundary in our formulation the neck region is long strip \([-T, T] \times [0, 1]\), and the case where the source curve is singular corresponds to the case when \( T = \infty \). So a part of the coordinate of our Kuranishi neighborhood is naturally parameterized by the (infinite) semi-open interval \((T_0, \infty]\) or a product of several of them. Note \( \infty \) is included in \((T_0, \infty]\). As a topological space \((T_0, \infty]\) has unambiguous meaning. On the other hand there is no obvious choice of its smooth structure as a manifold with boundary. Moreover for several maps such as Kuranishi map in the definition of Kuranishi structure, it is not obvious whether it is smooth for the given coordinate of \((T_0, \infty]\) naturally arising from the standard gluing construction. This issue has recently been raised and asked to the present authors by several mathematicians.

There are several ways to resolve this problem. One approach is rather topological which uses the fact that the gluing chart is smooth in the \( T \)-slice on which the gluing parameter \( T \) above is fixed. This approach is strong enough to establish all the results of [FOn] for which the method used in [McSa] was good enough to work out the analytic details along this approach. However it is not clear to the authors whether this approach is good enough to establish smoothness of the Kuranishi map or of the coordinate changes at \( T = \infty \). This point was mentioned by the first and the fourth named authors themselves in [FOn, Remark 13.16]. To prove an existence of the Kuranishi structure that literally satisfies its axioms, we take a gluing method different from the ones employed in [McSa, FOn], called the alternating method, in this article. (See Remark 6.17 for a discussion about several other differences on the analytic points.)

Using the alternating method described in [FOOO1, Section A1.4], we can find an appropriate coordinate chart at \( T = \infty \) so that the Kuranishi map and the coordinate changes of our Kuranishi neighborhoods are of \( C^\infty \) class. For this purpose, we take the parameter \( s = 1/T \). As we mentioned in [FOOO1, page 771] this parameter \( s \) is different from the one taken in algebraic geometry when the

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1. We can generalize it to the case of pseudoholomorphic curves attached to totally real submanifolds for given almost complex structure \( J \) on \( X \). We do not discuss it here since we do not know its application.

2. Among others, Y.B. Ruan, C.C. Liu, J. Solomon, I. Smith and H. Hofer asked the question. We thank them for asking this question.
target $X$ is projective. The parameter used in algebraic geometry corresponds to $z = e^{-\text{const} \cdot T}$. See Section 8. It seems likely that in our situation where either almost complex structure is non-integrable and/or the Lagrangian submanifold enters as the boundary condition (the source being the bordered stable curve) the Kuranishi map or the coordinate changes are not smooth with respect to the parameter $z = e^{-\text{const} \cdot T}$. But it is smooth in our parameter $s = 1/T$, as is proved in [FOOO1, Proposition A1.56].

The proof of this smoothness relies on some exponential decay estimate of the solution of the equation (1.1) with respect to $T$, that is, the length of the neck. An outline of the proof of this exponential decay is given in [FOOO1, Section A1.4, Lemma A1.58]. Because of the popular demand of more details of this smoothness proof, we provide such detail in the present paper by polishing the presentation given in [FOOO2, Part 3]. The exponential decay estimate is proved as Theorem 6.4. We then show how it enables us to prove the smoothness of the coordinate change of Kuranishi structure in Section 8. See Theorems 8.25 and 8.32.

In the present paper, we restrict ourselves to the case where we glue two (bordered) stable maps whose domain curve (without considering the map) is already stable. By restricting ourselves to this case we can address all the analytic issues needed to work out the general case also without making the notations so much heavy.

In case when the element is a stable map from a curve which has single nodal singularity, its neighborhood still contains a stable map from a nonsingular curve. So we need to study the problem of gluing or of stretching the neck. Such a problem on gluing solutions of non-linear elliptic partial differential equation has been studied extensively in gauge theory and in symplectic geometry during the last decade of the 20th century. Several methods had been developed to solve the problem which are also applicable to our case. In this article, following [FOOO1, Section A1.4], we employ the alternating method, which was first exploited by Donaldson [D2] in gauge theory.

**Remark 1.1.** In [D2] Donaldson used alternating method by inductively solving nonlinear equation (ASD-equation) on each of the two pieces to which he divided given 4 manifold. In this paper we will solve linearized version of Cauchy-Riemann equation inductively on each of the divided pieces. So our proof is a mixture of alternating method and Newton’s iteration.

In this method, one solves the linearization of the given equation on each piece of the domain (that is the completion of the complement of the neck region of the source of our pseudoholomorphic curve.) Then we construct a sequence of families of maps that converges to a version of solutions of the Cauchy-Riemann equation, that is,

$$\overline{\partial} u' \equiv 0 \mod \mathcal{E}(u')$$

and which are parameterized by a manifold (or an orbifold). Here $\mathcal{E}(u')$ is a family of finite dimensional vector spaces of smooth sections of an appropriate vector bundle depending on $u'$.

We provide the relevant analytic details using the same induction scheme as [FOOO1] Section A1.4, page 773-776]. The only difference is that we use $L^2_m$ space (the space of maps whose derivative up to order $m$ are of $L^2$ class) here, while we used $L^p$ space following the tradition of symplectic geometry community.
It is well-known that the shift of $T$ causes loss of differentiability of the maps in terms of the order of Sobolev spaces. However, by considering various weighted Sobolev spaces with various $m$ simultaneously and using the definition of $C^\infty$-topology, which is a Fréchet topology, we can still get the differentiability of $C^\infty$ order and its exponential decay. See Remark 6.16.

In Section 5 we provide the details of the estimate and show that the induction scheme of [FOOO1, Section A1.4] provides a convergent family of solutions of our equation (1.1). This estimate is actually a fairly straightforward application of the (improved) Newton’s iteration scheme although its detail is tedious to write down.

We then prove the exponential decay estimate of the derivative of the gluing map with respect to the gluing parameter in Section 6. See Theorem 6.4.

We remark that in this paper we intentionally avoid using the framework of Hilbert or Banach manifold in our gluing analysis. Certainly one can use such a framework to interpret the proof given in Section 5 as a kind of implicit function theorem. (Recall that Newton’s iteration, which we use therein, is one of the ways of proving the implicit function theorem.) The reason why we avoid using such an infinite dimensional manifold framework is because in the proof of Section 6 where we study $T$-derivatives, we need to deal with the situation where domains of the maps vary. Handling with a family of Sobolev spaces of maps with varying domain raises a nontrivial issue especially when one tries to regard the total space of the whole family as a certain version of Hilbert or Banach manifolds. Such a fact has been recognized by many researchers in various branch of mathematics. For example, the domain rotation on the Sobolev loop space is just continuous not smooth [Kl]. For an infinite dimensional representation, say the regular representation, of a Lie group $G$, the representation space (which is a space of maps from $G$) is not acted by the Lie algebra of $G$, since $G$ action is not differentiable. This is the reason why the notion of “smooth vector” is introduced. Another example is the issue of smoothness of the determinant line bundle for a smooth family of operators of Dirac type, for which the relevant family of Sobolev spaces of sections do not make a smooth Banach bundle over the parameter space (see, e.g., [Q], [BF]). It may be also worth mentioning that the method of finite dimensional approximations is used in such situations.

In the study of pseudoholomorphic curves the non-smoothness of the action of the group of diffeomorphism of the domain to the Sobolev spaces is emphasized by H. Hofer, K. Wysocki and E. Zehnder [HWZ].

We also remark that this issue is the reason why we work with $L^2_m$ space rather than $L^p_1$ space. The proof of Theorem 6.4 given in this paper does not work if we replace $L^2_m$ by $L^p_1$. See Remark 6.16.

We bypass this problem by specifying the function spaces of maps we use (which are Hilbert spaces) and explicitly defining the maps between the function spaces, e.g., the cut-paste maps entering in the gluing construction. Keeping track of the function spaces we use at all the stages of the construction, rather than putting them in an abstract framework or in a black box of functional analysis, is crucial to perform such a cut-paste process precisely in the way that the relevant derivative estimates can be obtained. It is also useful for the study of the properties of the Kuranishi structure we obtain, like its relationship to the forgetful map, to the target space group action and etc.. (Note we use a Hilbert manifold
There are several appendices which collect some parts of the proofs of Sections 5 and 6. They comprise technical details which are based on some straightforward estimate that do not break the mainstream of the proofs. They mainly rely on the fact that for a given map smooth map $F : M \to N$ the assignment $u \mapsto F(u)$ for $u : \Sigma \to M$ defines a smooth map $L^2_m(\Sigma, M) \to L^2_m(\Sigma, N)$ if $m$ is sufficiently large.

The readers who have strong background in analysis may find those appendices something obvious or at least well-known to them. We include the details of those proofs only for completeness’ sake.

This paper grows up from [FOOO2, Part 3] (The contents of Section 8 is also taken from [FOOO2, Part 4]), which is a slightly modified version of the document which the authors uploaded to the google group Kuranishi in 2012 June. We would like to thank all the participants in the discussions of the ‘Kuranishi’ google group for motivating us to go through this painstaking labour. The authors also thank to M. Abouzaid who gave us important comments on [FOOO2, Part 3].

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2. Preliminaries

In this section, we collect various known results that will be used in relation to the gluing analysis, and provide basic setting of the study of gluing and exponential decay of the elements near the boundary of the moduli space of pseudoholomorphic maps from bordered Riemann surfaces.

2.1. Choice of Riemannian metric. We take the following choice of Riemannian metric for convenience. The proof of the next lemma can be borrowed from p.683 [Ye]. (See Remark A.2 for a relevant remark.)

**Lemma 2.1.** Let $(X, J)$ be an almost complex manifold and $L$ a (maximally) totally real submanifold with respect to $J$. There exists a Hermitian metric $g$ on $X$ such that $L$ is totally geodesic and satisfies

$$JT_pL \perp T_pL$$

for every $p \in L$.

**Proof.** Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on $L$. Since $L$ is totally real with respect to $J$, $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ is a local frame of $TX$ on $L$. By a partition of unity we obtain a metric $\tilde{g}$ on the vector bundle for which it satisfies $\tilde{g}(e_i, Je_j) = 0$ for all $i, j$. We then extend $\tilde{g}$ on $TX|_N$ to a tubular neighborhood of $N$. Then we set $\overline{g} = \tilde{g}(J, J) + \tilde{g}$.

Then we note that any metric on a vector bundle $E \to N$ (e.g., $E = TN$) can be extended to a metric on $E$ as a manifold so that the latter becomes reflection invariant on $E$. In particular the zero section of $E$ becomes totally geodesic. Using this, it is easy to enhance the above metric $g$ so that $L$ becomes totally geodesic in addition with respect to the resulting metric $g$. 

□
2.2. Exponential map and its inverse. Let $\nabla$ be the Levi-Civita connection of the above chosen metric $g$ in Lemma 2.1. We also use an exponential map. (The same map was used in [FOOO1, pages 410-411].) Denote by $	ext{Exp} : TX \to X \times X$ the (global) exponential map defined by

$$\text{Exp}(x, v) = (x, \exp_x v)$$

and by $E : U \to TX$ its inverse

$$E(x, y) = (x, \exp^{-1}_x(y))$$

defined on a neighborhood of the diagonal $\Delta \subset X \times X$. Here $U$ can be defined to be

$$U = \{(x, y) \in X \times X \mid d(x, y) < \iota_X\}$$

where $\iota_X$ is the injectivity radius of $(X, g)$. Here and hereafter $d(x, y)$ is the Riemannian distance between $x$ and $y$.

**Remark 2.2.** When the metric is flat and in flat coordinates, the maps introduced above also can be expressed as

$$\text{Exp}(x, v) = x + v, \quad E(x, y) = y - x.$$ 

Readers may find it useful to compare the invariant expression via the exponential maps in the present paper with these coordinate calculations used in [FOOO2] to get some help for visualization of exponential maps. Because of Lemma E.1, the difference between the two estimates will be exponentially small for the pseudoholomorphic maps $u(\tau, t)$ in the various circumstances we are looking at in the present paper.

2.3. Parallel transportation. We take and fix a Riemannian metric on $X$ that satisfies the conclusion of Lemma 2.1. We denote by $\iota_X$ the injectivity radius. Namely if $x, y \in X$ with $d(x, y) < \iota_X$ then there exists a unique geodesic of length $d(x, y)$ joining $x$ and $y$.

For two points $x, y \in X$ with $d(x, y) < \iota_X$ we denote by

$$\text{Pal}^y_x : T_x X \to T_y X$$

the parallel transport along the unique minimal geodesic. (We use Levi-Civita connection to define parallel transport.) Suppose $X$ has an almost complex structure $J$. We then denote by

$$(\text{Pal}^y_x)^J : T_x X \to T_y X,$$

the complex linear part of $\text{Pal}^y_x$. Namely we decompose (uniquely) $\text{Pal}^y_x$ to

$$\text{Pal}^y_x = (\text{Pal}^y_x)^J + ((\text{Pal}^y_x)^J)'$$

such that

$$(\text{Pal}^y_x)^J \circ J_x = J_y \circ (\text{Pal}^y_x)^J, \quad ((\text{Pal}^y_x)^J)' \circ J_x = -J_y \circ ((\text{Pal}^y_x)^J)'$$

In other words

$$(\text{Pal}^y_x)^J = \frac{1}{2} \left( \text{Pal}^y_x - J_y \circ \text{Pal}^y_x \circ J_x \right).$$

We choose and fix a constant $\iota'_X > 0$ such that the following holds.

**Condition 2.3.** If $d(x, y) \leq \iota'_X$ then $(\text{Pal}^y_x)^J$ is a linear isomorphism.
Let $\Sigma$ be a two dimensional complex manifold and $u, v : \Sigma \to X$. We assume
\[ \sup \{ d(u(z), v(z)) \mid z \in \Sigma \} \leq t'_X. \]
Then using pointwise maps $\text{Pal}_y^u$ and $(\text{Pal}_y^u)^J$ we obtain the maps of sections
\[ \text{Pal}_y^u : \Gamma(\Sigma, u^*TX) \to \Gamma(\Sigma, v^*TX), (\text{Pal}_y^u)^J : \Gamma(\Sigma, u^*TX) \to \Gamma(\Sigma, v^*TX). \]  
We also note that by composing with the $(0,1)$-projections with respect to $J : TM \to TM$, $(\text{Pal}_y^u)^J$ also induces the map
\[ (\text{Pal}_y^u)^{(0,1)} : Tu(x)X \otimes \Lambda^0_{-1} \to Tv(x)X \otimes \Lambda^0_{-1} \]
which in turn induces the map of sections
\[ (\text{Pal}_y^u)^{(0,1)} : \Gamma(\Sigma, u^*TX \otimes \Lambda^0_{-1}) \to \Gamma(\Sigma, v^*TX \otimes \Lambda^0_{-1}). \]

2.4. Exponential decay of pseudoholomorphic maps with small image.
We now consider a pseudoholomorphic map $u : [-S, S] \times [0, 1] \to (X, L)$
\[ \frac{\partial u}{\partial \tau} + J \frac{\partial u}{\partial t} = 0 \]
with finite energy $\mathcal{E}(u) < \infty$. We also recall the definition of energy
\[ \mathcal{E}(u) = \frac{1}{2} \int \left( \left| \frac{\partial u}{\partial \tau} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right) dt \, d\tau \]
where the norm $| \cdot |$ is measured in terms of a metric $g$ that is compatible with $\omega$, $g = \omega(\cdot, J \cdot)$ for a fixed compatible almost complex structure $J$. We use the following well-known uniform exponential decay estimate for pseudoholomorphic curve with small diameter.

Lemma 2.4. There exists $T_{\text{exp}} > 0$ and $\delta_1 > 0, \epsilon_1 > 0, C_{\text{exp}}>0$ depending only on $X, L$, and $\mathcal{E}_0$ with the following properties. If $T > T_{\text{exp}}, u : [-T - 1, T + 1] \times [0, 1] \to X$ is pseudoholomorphic, $\mathcal{E}(u) \leq \mathcal{E}_0$, $u([-T - 1, T + 1] \times \{0, 1\}) \subset L$ and Diam($\text{Im} u$) $\leq \epsilon_1$, then
\[ \left| \frac{\partial u}{\partial \tau}(\tau, t) \right| + \left| \frac{\partial u}{\partial t}(\tau, t) \right| \leq C_{\text{exp}} e^{-\delta_1 d(\tau, [\tau - T - 1, T + 1])} \]
for $(\tau, t) \in [-T, T] \times [0, 1]$.

We refer to [On, Lemma 11.2] or to [Oh2, Lemma B.1] for its proof, for example.

Using this $C^1$-exponential decay, $\epsilon$-regularity theorem and the uniform local a priori estimates, we also obtain

Lemma 2.5. There exists $C_{k, \epsilon} > 0$ depending only on the $(X, J, g)$, $L$, $\mathcal{E}_0$ and $k \geq 0$ such that if $u : [-T - 1, T + 1] \times [0, 1] \to X$ is as in Lemma 2.4 then
\[ \left| \frac{\partial u}{\partial \tau}(\tau, t) \right|_{\mathcal{C}^k} \leq C_{k, \epsilon} e^{-\delta_1 d(\tau, [\tau - T - 1, T + 1])}. \]
In particular, if $u : [0, \infty) \times [0, 1] \to X$ (or $u : (-\infty, 0] \times [0, 1] \to X$) is pseudoholomorphic $u([0, \infty) \times \{0, 1\}) \subset L$ (or $u((-\infty, 0] \times \{0, 1\}) \subset L$), Diam($\text{Im} u$) $\leq \epsilon_1$ and $\mathcal{E}(u) \leq \mathcal{E}_0$, and $k \geq 0$ then we have
\[ \left| \frac{\partial u}{\partial \tau}(\tau, t) \right|_{\mathcal{C}^k} \leq C_{k, \epsilon} e^{-\delta_1 |\tau|}. \]

Remark 2.6. Actually we can take $\delta_1 = \pi$. We do not need this explicit value in this paper.
Convention 2.7. Hereafter we assume that all pseudoholomorphic curves $u$ we consider satisfy the inequality
\[ \mathcal{E}(u) \leq \mathcal{E}_0, \]
for some fixed $\mathcal{E}_0$. So the $\mathcal{E}_0$ dependence of various constants will not be mentioned.

Remark 2.8. Since many constants appear in this paper we use the following enumeration convention. The constant such as, for example, $C_m$, (10.12) is the one appearing in formula (10.12) which may depend on $m$.

For some constants appearing repeatedly, we do not apply this rule of enumerating by the number of the associated formulae. The list of such constants is:
- $\epsilon_1'$, $X$ appears in Condition 2.7,
- $\epsilon_1$, $\delta_1$ appear in Lemma 2.4,
- $\delta$ appears in (3.11), $T_{1, \epsilon(1)}$, $T_{2, \epsilon(1), \epsilon(2)}$ appear in Theorem 3.13, $C_{1,m}$ appear Lemma 5.2, $\epsilon_2$ appears in Lemma 5.6, $C_{2,m}$ appears in Lemma 5.15, $C_{3,m}$ appears in Lemma 5.17, $C_{4,m}$ appears in (5.67), $\epsilon_{3,m}$ and $T_{3,m, \epsilon(4)}$, $T_{4,m}$ appear in Proposition 5.23, $C_{5,m}, C_{6,m}, C_{7,m}, C_{8,m}, C_{9,m}$ $T_{5,m, \epsilon(5)}$, $T_{6,m}, \epsilon_4$ appear in Proposition 6.9, $\epsilon_5$ appears in Proposition 7.1, $\delta_2$ appears in Theorem 8.17, $\delta_3$ appears in Proposition 8.19 and $\delta_4$ appears in Proposition 8.35, $\epsilon_6, \epsilon_7, \nu_1, \nu_2$ appear around (8.45), $\epsilon_8, \epsilon_9, \nu_{3,m}, \nu_{4,m}$ appear in Lemma 8.24.

The positive numbers $\epsilon(1), \epsilon(2), \ldots$ are not fixed but may vary.

3. Statement of the gluing theorem

For simplicity and clarity of the exposition on the main analytical points, we consider the case where we glue pseudoholomorphic maps from two stable bordered Riemann surfaces to $(X, L)$ in the present paper. The general case is proved in the same way in [FOOO2, Part IV].

Let $\Sigma_i$ be a bordered Riemann surface for $i = 1, 2$. We assume that each $\Sigma_i$ has strip-like ends such that $\Sigma_i \setminus K_i$ becomes semi-infinite strip. Recall that $\Sigma_i$ is conformally a bordered Riemann surface with one puncture.

More specifically, we fix a Kähler metric on $\Sigma_i$ whose end is isometric to the flat strip. We fix an isothermal coordinates $(\tau, t)$ so that we can decompose $\Sigma_i$ into
\[
\begin{align*}
\Sigma_1 &= K_1 \cup (\{-5T, \infty\} \times [0, 1]), \\
\Sigma_2 &= (\{-\infty, 5T\} \times [0, 1]) \cup K_2
\end{align*}
\]
(Written as (3.1))

where $K_i$ are compact subsets of $\Sigma_i$ respectively.

Here $(\tau, t)$ is the standard coordinates on the strip $\mathbb{R} \times [0, 1]$ and restricted to $[-5T, \infty) \times [0, 1]$ (resp. $(-\infty, 5T] \times [0, 1]$) on $\Sigma_1 \setminus K_1$ (resp. $\Sigma_2 \setminus K_2$). We would

\[ \text{Figure 1. } \Sigma_1 \text{ and } \Sigma_2. \]
like to note that the coordinate $\tau$ depends on $T$. For each $T > 0$, we put
\[ \Sigma_T = K_1 \cup ([-5T, 5T] \times [0, 1]) \cup K_2. \] (3.2)

**Figure 2.** $\Sigma_T$.

Let $X$ be a compact symplectic manifold with compatible almost complex structure and $L$ its Lagrangian submanifold.

Let $u_i : (\Sigma_i, \partial \Sigma_i) \to (X, L)$, $i = 1, 2$ be pseudoholomorphic maps of finite energy. Then, by the removable singularity theorem [Oh1], the maps $u_i$ smoothly extend to the associated puncture of $\Sigma_i$ and so the asymptotic limits
\[ \lim_{\tau \to \infty} u_1(\tau, t) \in L \] (3.3)
and
\[ \lim_{\tau \to -\infty} u_2(\tau, t) \in L \] (3.4)
are uniquely determined and do not depend on $t \in [0, 1]$.

We will consider a pair $(u_1, u_2)$ of the maps for which the limits (3.3), (3.4) coincide which we denote by $p_0 \in L$.

**Condition 3.1.** We also assume that
\[ \text{Diam } u_i(\Sigma_i \setminus K_i) \leq \epsilon_1 \] (3.5)
where $\epsilon_1$ is as in Lemma 2.4.

Then the exponential decay (2.10) holds by Lemma 2.5.

**Remark 3.2.** If $u_1$ (resp. $u_2$) satisfies (3.3) (resp. (3.4)) then we may replace $K_1$ (resp. $K_2$) by $K_1 \cup [-5T, -5T + S] \times [0, 1]$ (resp. $K_2 \cup [-S + 5T, 5T] \times [0, 1]$) and change the $T$ coordinate to $T - S$ (resp. $T + S$) for an appropriate $S$ so that (3.5) is satisfied. In other words, we can assume (3.5) without loss of generality.

For the clarity of our exposition, we consider the case of fixed complex structures of the sources $\Sigma_i$, in Sections 3-7. We then include the deformation parameter of the complex structures of $\Sigma_i$ in Section 8. In order to handle the case of genus 0, we also need to stabilize and so add marked points on the domain curve $\Sigma_i$. We denote the ordered set of marked points by $\vec{z}_i = (z_{i,1}, \ldots, z_{i,k_i})$ for $i = 1, 2$. For the simplicity of notations, we only consider boundary marked points.

We may add one point $\tau = \infty$ or $-\infty$ and compactify the source curve $\Sigma_i$ and extend our map $u_i$ thereto. We regard the added point as the 0-th marked point $z_{i,0}$. 
Assumption 3.3. In this paper we always assume that \((\Sigma_i, \vec{z}_i)\) is stable.

Note there is one more marked point at infinity other than \(\vec{z}_i\). We also remark that here \(\Sigma_i\) is assumed to have no interior or boundary node. So if \(\Sigma_i\) is not a disc then it is stable and if \(\Sigma_i\) is a disc, the condition \(k_i \geq 2\) is equivalent to the stability.

We will define a map:

\[
D_u, \overline{\partial}_i : W^{2}_{m+1,\delta}((\Sigma_i, \partial \Sigma_i); u^*_i TX, u^*_i TL) \to L^2_{m,\delta}(\Sigma_i; u^*_i TX \otimes \Lambda^{0,1}),
\]

which is the linearization of \(\overline{\partial}_i\) at \(u_i\). Here we denote \(\overline{\partial}_i = \overline{\partial}_j\) by omitting \(J\). The domain, that is, the function space

\[
W^{2}_{m+1,\delta}((\Sigma_i, \partial \Sigma_i); u^*_i TX, u^*_i TL),
\]

is defined below.

We assume that the constant \(\epsilon_1\) given in Condition 3.1 is at least smaller than \(\epsilon'_X\) given in Condition 2.3. Then to each \(v \in T_{p_0} L\), we define the section \(v^{\text{Pal}} \in \Gamma([-5T, 5T] \times [0, 1]; u^*_i TX)\) by

\[
v^{\text{Pal}}(t, v) \equiv \text{Pal}_{p_0}^{u_0}(\tau, t)(v).
\]

We note that the sections (3.7) satisfy the boundary condition at \(t = 0, 1\) because we use the Levi-Civita connection of a metric with respect to which the Lagrangian submanifold \(L\) is totally geodesic. (See Lemma 2.1)

More generally if \(x \in X, v \in T_{x} X\) and \(u : \Sigma_0 \to X\) a map to the \(\iota_X\) neighborhood of \(x\) from a 2 dimensional manifold, we define \(v^{\text{Pal}} \in \Gamma(\Sigma_0, u^*_i TX)\) by

\[
v^{\text{Pal}}(z) = \text{Pal}_{x}^{u}(z)(v).
\]

Definition 3.4. ([FOOO1]) Let \(p_0 \in L\) be the point given in (3.3) or in (3.4). Denote by \(L^{2}_{m+1,\text{loc}}((\Sigma_i, \partial \Sigma_i); u^*_i TX; u^*_i TL)\) the set of the sections \(s\) of \(u^*_i TX\) which are locally of \(L^{2}_{m+1}\)-class. (Namely its differential up to order \(m+1\) is of locally \(L^{2}\)-class. Here \(m\) is sufficiently large, say larger than 10.) We also assume \(s(z) \in u^*_i TL\) for \(z \in \partial \Sigma_i\). We define a Hilbert space

\[
W^{2}_{m+1,\delta}((\Sigma_i, \partial \Sigma_i); u^*_i TX, u^*_i TL)
\]

as the set of all pairs \((s, v)\) where \(s \in L^{2}_{m+1, \text{loc}}((\Sigma_i, \partial \Sigma_i); u^*_i TX; u^*_i TL), v \in T_{p_0} L\) such that

\[
\sum_{k=0}^{m+1} \int_{\Sigma_i \setminus K_i} e^{2\delta |r| \pm 5T} |\nabla^k (s - v^{\text{Pal}})|^2 < \infty.
\]

Here \(\pm = +\) for \(i = 1\) and \(\pm = -\) for \(i = 2\). We define its norm \(\|(s, v)\|_{W^{2}_{m+1,\delta}(\Sigma_i)}\) by

\[
\|(s, v)\|_{W^{2}_{m+1,\delta}(\Sigma_i)} = [3.9] + \sum_{i=1,2} \sum_{k=0}^{m+1} \int_{K_i} |\nabla^k s|^2 + |v|^2.
\]

Hereafter we omit \(\Sigma_i\) from the notation \(\|(s, v)\|_{W^{2}_{m+1,\delta}(\Sigma_i)}\) in case the domain is obvious from the context.

The space \(L^{2}_{m,\delta}(\Sigma_i; u^*_i TX \otimes \Lambda^{0,1})\) is defined similarly without boundary condition and without \(v\). (See (4.10).)

---

3In [FOOO1] \(L^p\) space is used in stead of \(L^{2}_{m}\) space.
Since the relevant $s$ for $T = \infty$ does not decay to 0 but converges to an element $v \in T_{p_0}X$, we denote the subscript of $\|s\|_{W^2_{m+1,\delta}}^2$ to be $W^2_{m+1,\delta}$ instead of $L^2_{m+1,\delta}$.

We let $D_u\overline{\partial}$ act trivially on the $v$-component and just act on the $s$-component of $(s, v) \in W^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u^*_i TX, u^*_i TL)$. We choose $\delta > 0$ with

$$\delta \leq \frac{\delta_1}{10}$$

(3.11)

where $\delta_1 > 0$ is the constant given in Lemma 2.4. We never change this constant $\delta$ in the rest of this paper. Lemma 2.5 implies

$$\left| (D_u\overline{\partial})(v^{\text{Pal}}) \right|_{C^k} < C_k(3.12)C_k(2.9) e^{-\delta_1|\tau|} \|v\|.$$  

Therefore the operator (3.6), especially with $L^2_{m+1,\text{loc}}((\Sigma_i, \partial\Sigma_i); u^*_i TX, u^*_i TL)$ as its target, is defined and bounded. It is a standard fact (see [LM]) that this operator is also Fredholm if we choose $\delta$ sufficiently small. In the current context, $\delta$ depends only on the geometry of $(X, g, J)$ and $L \subset X$.

**Remark 3.5.** We like to note that $W^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u^*_i TX, u^*_i TL)$ is a completion of the set of infinitesimal variations of $u_i : \Sigma_i \to X$ satisfying the boundary conditions

$$u_i(\partial\Sigma_i) \subset L, \quad \lim_{\tau \to +\infty} u(\tau, t) = \lim_{\tau \to -\infty} u(\tau, t) \in L$$

in particular allowing the point $p_0$ varies inside $L$. We have an exact sequence

$$0 \to L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u^*_i TX, u^*_i TL) \to W^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u^*_i TX, u^*_i TL) \to T_{p_0}L \to 0$$

where the first map is the inclusion $s \mapsto (s, 0)$ and the second map is the asymptotic evaluation of the section $s$ at $\pm \infty$ for $i = 1, 2$ respectively.

We next define our (family of) obstruction spaces $\mathcal{E}_i(u')$.

We fix a pair of pseudoholomorphic maps $u_i^{\text{ob}} : (\Sigma_i, \partial\Sigma_i) \to (X, L)$ for $i = 1, 2$. We assume

$$d(u_i(z), u_i^{\text{ob}}(z)) \leq \frac{t'_{\Sigma_i}}{2}$$  

(3.13)

where $t'_{\Sigma_i}$ is as in (2.3).

We take a finite dimensional linear subspace

$$\mathcal{E}_i^{\text{ob}} = \mathcal{E}_i(u_i^{\text{ob}}) \subset \Gamma(K_i; (u_i^{\text{ob}})^*TX \otimes \Lambda^{1,0}), \quad i = 1, 2$$

(3.14)

which consists of smooth sections of $(u_i^{\text{ob}})^*TX \otimes \Lambda^{1,0}$ supported in a compact subset $K_i^0$ of $\text{Int} K_i$.

**Remark 3.6.** When we use Theorems 3.13 and 6.4 for the construction of the Kuranishi structure on a moduli space of bordered pseudoholomorphic curves, we take several $u_i^{\text{ob}}$’s and take $\mathcal{E}_i^{\text{ob}}$ for each of them. Then in place of $\mathcal{E}(u') \oplus \mathcal{E}(u')$ we take a sum of finitely many of them. See [FOu] page 1003 and [FOOO2, Definition 18.12]. For the simplicity of exposition, we focus on the case when we have one set of $u_i^{\text{ob}}$, $\mathcal{E}_i^{\text{ob}}$ in this paper. The case of several such $u_i^{\text{ob}}$’s and $\mathcal{E}_i^{\text{ob}}$’s is treated in the same way as that of Theorems 3.13 and 6.4.
We state the first half of the main result as Theorem 3.13 below. The second half is Theorem 6.4. Actually Theorem 3.13 itself is classical. We state it since we write its proof in such a way that it is suitable for the proof of Theorem 6.4. Let
\[ u' : (\Sigma_T, \partial \Sigma_T) \to (X, L) \]  
(3.15)
be a smooth map.

We consider the following conditions for given constant \( \epsilon > 0 \).

**Condition 3.7.**

1. \( u'|_{K_i} \) is \( \epsilon \)-close to \( u_i|_{K_i} \) in \( C^1 \) sense. \(^4\)
2. The diameter of \( u'(X \setminus (K_1 \cup K_2)) \) is smaller than \( \epsilon \).

We define the map
\[ (\text{Pal}_{u'_i}^{(0,1)}) : \Gamma(K_i; (u'_i)^*TX \otimes \Lambda^{0,1}) \to \Gamma(\Sigma_T; (u')^*TX \otimes \Lambda^{0,1}) \]  
(3.16)
by composing the map \( (\text{Pal}_{u'_i}^{(0,1)}) : \Gamma(K_i; (u'_i)^*TX \otimes \Lambda^{0,1}) \to \Gamma(K_i; (u')^*TX \otimes \Lambda^{0,1}) \) in (2.7) with the map induced by the inclusion \( K_i \subset \Sigma_T \).

If we take \( \epsilon > 0 \) sufficiently small, then (3.13) and Condition 3.7 (1) imply that this map (3.16) is defined for such \( u' \).

We then define the map
\[ I_{u',i} : \mathcal{E}^c_i \to \Gamma(\Sigma_T; (u')^*TX \otimes \Lambda^{0,1}) \]  
(3.17)
to be the restriction of \( (\text{Pal}_{u'_i}^{(0,1)}) \) to \( \mathcal{E}^c_i \) and put
\[ \mathcal{E}_i(u') = I_{u',i}(\mathcal{E}^c_i). \]  
(3.18)
Recall that elements of \( \mathcal{E}^c_i \) are supported on \( K_i^0 \subset \text{Int} K_i \). Therefore for this definition, we first take the parallel transport of \( \eta \in \mathcal{E}^c_i \) along the shortest geodesic between \( u_i^c(z) \) and \( u'(z) \) for \( z \in K_i^0 \subset \Sigma_T \) and then extend the resulting section by zero to the rest of \( \Sigma_T \).

The equation we study in the rest of the paper is of the form
\[ \bar{\partial}u' \equiv 0, \quad \text{mod } \mathcal{E}_1(u') \oplus \mathcal{E}_2(u'). \]  
(3.19)

**Definition 3.8.** We denote by \( \mathcal{M} \mathcal{E}_i \otimes \mathcal{E}_c ((\Sigma_T, \bar{\partial}_i ; u_1, u_2)_\epsilon \) the set of solutions of (3.19) satisfying the Condition 3.7.

Theorem 3.13 will describe all the solutions of (3.19) ‘sufficiently close to the (pre-)glued map \( u_1 \# u_2 \).’ To make this statement precise, we need to prepare more notations.

Let \( u'_i : (\Sigma_i, \partial \Sigma_i) \to (X, L) \) be any smooth map, not necessarily pseudoholomorphic. We put the following conditions on \( u'_i \).

**Condition 3.9.**

1. \( u'_i|_{K_i} \) is \( \epsilon \)-close to \( u_i|_{K_i} \) in \( C^1 \) sense.
2. The diameter of \( u'_i(\Sigma_1 \setminus K_1) \), (resp. \( u'_i(\Sigma_2 \setminus K_2) \)) is smaller than \( \epsilon \).

Then we define
\[ I_{u'_i} : \mathcal{E}_c^c_i \to \Gamma(\Sigma_i; (u'_i)^*TX \otimes \Lambda^{0,1}) \]
in the same way as \( I_{u'} \) i.e.,
\[ I_{u'_i}(\eta) := (\text{Pal}_{u'_i}^{(0,1)})(\eta), \quad \eta \in \mathcal{E}_c^c_i. \]

\(^4\)We use a Riemannian metric on \( K_i \) and \( X \) to define \( C^1 \) norm we use.
(This makes sense if $u'_i$ satisfies Condition 3.9 for $\epsilon < \epsilon'_X/2$. In fact, then we have $d(u'_i(x), u^b_i(x)) < \epsilon'_X$.) We put

$$E_i(u'_i) = I_{u'_i}(E_i^b).$$

(3.20)

So we can define an equation

$$\bar{\nu}u'_i \equiv 0, \mod E_i(u'_i).$$

(3.21)

We write $E_i$ in place of $E_i(u'_i)$ sometimes.

**Remark 3.10.** Note that in the current situation where $\Sigma_i$ has no sphere or disk bubble and has nontrivial boundary with at least one boundary marked point (that is, either $\tau = \infty$ or $-\infty$ depending on $i = 1$ or $i = 2$), Assumption 3.3 implies that $\Sigma_i$ carries no nontrivial automorphism. In the case when there is a sphere bubble, the automorphism group can be nontrivial. We can use the standard trick to add marked points and then use codimension 2 submanifold to kill the extra freedom of moving added marked points. (See [FOOn Appendix] or [FOOO2 Part IV], for example.)

**Definition 3.11.** We denote by $M^E_i((\Sigma_i, \vec{z}_i); u_i)$ the set of all maps $u'_i : (\Sigma_i, \partial \Sigma_i) \to (X, L)$ whose domain is the marked bordered Riemann surface $(\Sigma_i, \vec{z}_i)$ as in (3.1) for $i = 1, 2$ such that

1. $u'_i$ satisfies (3.21).
2. Condition 3.9 is satisfied.

We work under the following assumption. This assumption is put on the pair $(u_1, u_2)$ which we glue.

**Assumption 3.12.** Let $u_1(\infty) = u_2(-\infty) = p_0 \in L$ and let $(D_{u_i} \bar{\nu})^{-1}(E_i(u_i))$ be the kernel of (3.22). We assume:

1. (Mapping transversality) $D_{u_i} : W_{m+1, \delta}^2((\Sigma_i, \partial \Sigma_i); u^*_i TX, u^*_i TL) \to L_{m, \delta}^2(\Sigma_i; u^*_i TX \otimes \Lambda^{0,1})/E_i(u_i)$ (3.22) is surjective.
2. (Evaluation transversality) For each $i = 1, 2$, define the linearized evaluation map

$$Dev_{i, \infty} : W_{m+1, \delta}^2((\Sigma_i, \partial \Sigma_i); u^*_i TX, u^*_i TL) \to T_{p_0} L$$

(3.23)

by

$$Dev_{i, \infty}(s, v) = v.$$

Then

$$Dev_{1, \infty} - Dev_{2, \infty} : (D_{u_1} \bar{\nu})^{-1}(E_1(u_1)) \oplus (D_{u_2} \bar{\nu})^{-1}(E_2(u_2)) \to T_{p_0} L$$

(3.24)

is surjective.

The surjectivity of (3.23), (3.24) and the implicit function theorem imply that, if $\epsilon$ is enough small, there exists a finite dimensional vector space $V_i$ and its neighborhood $V_i(\epsilon)$ of 0 which provides a local parametrization of $M^E_i((\Sigma_i, \vec{z}_i); u_i)$.

$$M^E_i((\Sigma_i, \vec{z}_i); u_i) \cong V_i(\epsilon)$$

(3.25)

near the given solution $u_i$. For any $\rho_i \in V_i(\epsilon)$, we denote by $u'_{\rho_i} : (\Sigma_i, \partial \Sigma_i) \to (X, L)$ the corresponding solution of (3.21).
We have an asymptotic evaluation map

\[ \text{ev}_{i,\infty} : \mathcal{M}^E_i((\Sigma_i, \tilde{z}_i); u_i) \to L. \]

Namely, we define

\[ \text{ev}_{i,\infty}(u'_i) = \lim_{\tau \to \pm \infty} u'_i(\tau, t). \]

(Here \( \pm = + \) for \( i = 1 \) and \( - \) for \( i = 2 \)).

We consider the fiber product:

\[ V_1(\epsilon) \times_L V_2(\epsilon) = \mathcal{M}^{E_1}((\Sigma_1, \tilde{z}_1); u_1)_\epsilon \times_L \mathcal{M}^{E_2}((\Sigma_2, \tilde{z}_2); u_2)_\epsilon. \]  

(3.26)

The surjectivity of (3.24) implies that the fiber product (3.26) is transversal and so \( V_1(\epsilon) \times_L V_2(\epsilon) \) is a smooth manifold. We write elements of \( V_1(\epsilon) \times_L V_2(\epsilon) \) as \( \rho = (\rho_1, \rho_2) \).

**Theorem 3.13.** For each \( \epsilon(1) > 0 \), there exist \( \epsilon(2) > 0 \), \( T_{1,\epsilon(1)} > 0 \) and a map

\[ \text{Glue}_T : \mathcal{M}^{E_1}((\Sigma_1, \tilde{z}_1); u_1)_{\epsilon(2)} \times_L \mathcal{M}^{E_2}((\Sigma_2, \tilde{z}_2); u_2)_{\epsilon(1)} \to \mathcal{M}^{E_1 \oplus E_2}((\Sigma_T, \tilde{z}); u_1, u_2)_{\epsilon(1)} \]

for \( T > T_{1,\epsilon(1)} \). Here \( \tilde{z} = \tilde{z}_1 \cup \tilde{z}_2 \).

The map \( \text{Glue}_T \) is a diffeomorphism onto its image.

There exist \( \epsilon_{1,\epsilon(1)}, \epsilon(2) > 0 \), \( T_{2,\epsilon(1),\epsilon(2)} > 0 \) depending on \( \epsilon(1) \), \( \epsilon(2) \) such that the image of \( \text{Glue}_T \) contains \( \mathcal{M}^{E_1 \oplus E_2}((\Sigma_T, \tilde{z}); u_1, u_2)_{\epsilon(1), \epsilon(2)} \) if \( T > T_{2,\epsilon(1),\epsilon(2)} \).

The main result about exponential decay estimate of this map will be given in Section 6. (Theorem 6.4.)

### 4. Proof of the Gluing Theorem: I - Cut-off Functions and Weighted Sobolev Norm

The proof of Theorem 3.13 was given in [FOOO1, Section 7.1.3]. Here we follow the scheme of the proof of [FOOO1, Section A1.4] and provide more of its details. As mentioned there, the scheme of the proof is originated from Donaldson’s paper [D2], and its Bott-Morse version in [Fu].

We first introduce certain cut-off functions. First let \( A_T \subset \Sigma_T \) and \( B_T \subset \Sigma_T \) be the domains defined by

\[ A_T = [-T - 1, -T + 1] \times [0, 1], \quad B_T = [T - 1, T + 1] \times [0, 1]. \]

We may regard \( A_T, B_T \) as subsets of \( \Sigma_i \). The third domain is

\[ \mathcal{X} = [-1, 1] \times [0, 1] \subset \Sigma_T. \]

We may also regard \( \mathcal{X} \subset \Sigma_i \). We recall readers the definitions of \( \Sigma_i \) from (3.1).

Let \( \chi_{\mathcal{A}}^+, \chi_{\mathcal{A}}^- \) be the smooth functions on \([-5T, 5T] \times [0, 1] \) defined by

\[ \chi_{\mathcal{A}}^+(\tau, t) = \begin{cases} 1 & \tau < -T - 1 \\ 0 & \tau > -T + 1. \end{cases} \]

(4.1)

\[ \chi_{\mathcal{A}}^- = 1 - \chi_{\mathcal{A}}^+. \]

See Figure 4. We also define

\[ \chi_{\mathcal{B}}^+(\tau, t) = \begin{cases} 1 & \tau < T - 1 \\ 0 & \tau > T + 1. \end{cases} \]

(4.2)

\[ \chi_{\mathcal{B}}^- = 1 - \chi_{\mathcal{B}}^+. \]
and

$$\chi^\rightarrow_X(\tau,t) = \begin{cases} 1 & \tau < -1 \\ 0 & \tau > 1. \end{cases}$$

(4.3)

$$\chi^\leftarrow_X = 1 - \chi^\rightarrow_X.$$ We extend these functions to $\Sigma_T$ and $\Sigma_i$ ($i = 1, 2$) so that they are locally constant outside $[-5T, 5T] \times [0, 1]$. We denote them by the same symbol.

We now introduce weighted Sobolev norms and their local versions for sections on $\Sigma_T$ or $\Sigma_i$. We first define weight functions $e_i, \delta : \Sigma_i \rightarrow [1, \infty)$, $i = 1, 2$ of $C^\infty$-class as follows:

$$e_{1,\delta}(\tau,t) = \begin{cases} e^\delta(\tau+5T) & \text{if } \tau > 1 - 5T \\ 1 & \text{on } K_1 \\ e^\delta(\tau+5T) & \text{if } -1 > \tau > 1 - 5T \\ 1 & \text{on } K_2 \\ e^\delta(\tau+5T) & \text{if } \tau < 5T - 1 \\ 1 & \text{on } K_2 \end{cases} \quad (4.4)$$

$$e_{2,\delta}(\tau,t) = \begin{cases} e^\delta(-\tau+5T) & \text{if } \tau < 5T - 1 \\ 1 & \text{on } K_2 \\ e^\delta(-\tau+5T) & \text{if } \tau > 5T - 1 \\ 1 & \text{on } K_2 \end{cases} \quad (4.5)$$

(We choose $\delta > 0$ so small that it satisfies $1 < e^\delta < 10$ above.) We also define $e_{T,\delta} : \Sigma_T \rightarrow [1, \infty)$ as follows. (See Figure 5.)

$$e_{T,\delta}(\tau,t) = \begin{cases} e^\delta(-\tau+5T) & \text{if } 1 < \tau < 5T - 1 \\ e^\delta(\tau+5T) & \text{if } -1 > \tau > 1 - 5T \\ 1 & \text{on } K_1 \cup K_2 \\ e^\delta(-\tau+5T) & \text{if } \tau < 5T - 1 \\ e^\delta(\tau+5T) & \text{if } \tau > 5T - 1 \\ 1 & \text{on } K_2 \end{cases} \quad (4.6)$$

$$\|s\|_{L^2_{m,\delta}}^2 = \sum_{k=0}^m \int_{\Sigma_i} e_{i,\delta}^2 |\nabla^k s|^2 \text{vol}_{\Sigma_i}.$$ (4.7)
For \((s, v) \in W_{m+1, \delta}^2((\Sigma_T, \partial \Sigma_T); u^*TX, u^*TL)\) we defined in (3.10)

\[
\|(s, v)\|_{W_{m+1, \delta}^2}^2 = \sum_{k=0}^{m+1} \int_{K_i} |\nabla^k s|^2 \text{vol}_{\Sigma_i}
\]

\[
+ \sum_{k=0}^{m+1} \int_{\Sigma_i \setminus K_i} e_i^2 \nabla^k (s - v^{\text{Pal}})^2 \text{vol}_{\Sigma_i} + \|v\|^2,
\]

where \(v^{\text{Pal}}\) is defined in (3.8).

We next define a weighted Sobolev norm for the sections on \(\Sigma_T\). Let \(s\) be a section of \(u^*TX\) with \(s|_{\partial \Sigma_T} \in \Gamma(u^*TL)\) such that

\[
s \in L_{m+1}^2((\Sigma_T, \partial \Sigma_T); u^*TX, u^*TL).
\]

Since we take \(m\) large, \(s\) is continuous. So the value \(s(0,1/2) \in T_{u(0,1/2)}X\) at \((0,1/2)\) is well defined.
We define the norm of $s$ by

$$
\|s\|_{W^2_{m+1,\delta}} = \sum_{k=0}^{m+1} \int_{K_1} |\nabla^k s|^2 \vol_{\Sigma_1} + \sum_{k=0}^{m+1} \int_{K_2} |\nabla^k s|^2 \vol_{\Sigma_2} + \sum_{k=0}^{m+1} \int_{[-5T,5T] \times [0,1]} e_{T,\delta}^2 |\nabla^k (s - s(0, 1/2)^{Pal})|^2 \vol_{\Sigma_i} + \|s(0, 1/2)\|^2.
$$

(4.9)

For $\eta \in L^2_m((\Sigma_T, \partial \Sigma_T); u^* TX \otimes \Lambda^{0,1})$, we define

$$
\|\eta\|_{L^2_{m,\delta}} = \sum_{k=0}^{m} \int_{\Sigma_T} e_{T,\delta}^2 |\nabla^k \eta|^2 \vol_{\Sigma_1}.
$$

(4.10)

These norms were used in [FOOO1, Section 7.1.3].

**Remark 4.1.** We remark that the norm $\|s\|_{W^2_{m+1,\delta}}$ is equivalent to the standard $L^2_{m+1}$-norm on $L^2_{m+1}((\Sigma_T, \partial \Sigma_T); u^* TX, u^* TL)$, when $T$ is fixed. However to study its relationship with the norm (4.8) as $T \to \infty$, making this choice of $T$-dependent norm in this way is necessary. We remark that the ratio between the norm $\|s\|_{W^2_{m+1,\delta}}$ and the standard $L^2_{m+1}$-norm diverges as $T$ goes to infinity.

For a subset $W$ of $\Sigma_i$ or $\Sigma_T$ we define $\|s\|_{W^2_{m+1,\delta}(W \subset \Sigma_i)}$, $\|\eta\|_{L^2_{m,\delta}(W \subset \Sigma_T)}$ by restricting the domain of the integration (4.9) or (4.10) to $W$. We use this notation in order to specify the weight etc. we use. In case there is no possibility of confusion of the domain and weight etc. we use, we omit the domain $\Sigma_i$ etc. from the notation $\|s\|_{W^2_{m,\delta}(\Sigma_i)}$ etc.
We also define the inner product on $W^{2m+1,\delta}_{m+1}((\Sigma_i, \partial\Sigma_i); u_i^*TX, u_i^*TL)$ by setting
\[
\langle\langle (s_1, v_1), (s_2, v_2) \rangle\rangle_{L^2} = \int_{\Sigma_i \setminus K_i} (s_1 - v_1^{Pal}, s_2 - v_2^{Pal}) + \int_{K_i} (s_1, s_2) + (v_1, v_2)
\] 
(4.11)
for $(s_j, v_j) \in W^{2m+1,\delta}_{m+1}((\Sigma_i, \partial\Sigma_i); u_i^*TX, u_i^*TL)$ for $j = 1, 2$.

5. Proof of the gluing theorem: II - Gluing by alternating method

We motivate the gluing construction we employ in this section by comparing it with the well-known Newton’s iteration scheme of solving the equation $f(x) = 0$ for a real-valued function, starting from an approximate solution $x_0$ that is sufficiently close to a genuine solution nearby. The iteration scheme is the inductive scheme via the recurrence relation
\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]
(5.1)
Therefore
\[
|x_{n+1} - x_n| = \frac{|f(x_n)|}{|f'(x_n)|}
\] 
(5.2)
and the expected solution is given by the infinite sum
\[
x_\infty = x_0 + \sum_{n=1}^{\infty} (x_{n+1} - x_n)
\]
provided that the series converges. A simple way of ensuring the required convergence is to establish existence of constant $C > 0$, $0 \leq \mu < 1$ independent of $n$ such that
\[
\left| \frac{f(x_n)}{f'(x_n)} \right| \leq C \mu^n
\] 
(5.3)
for all $n$.

With this iteration scheme in our mind, we proceed with performing the scheme. In our construction we obtain an approximate solution of the linearized equation (which corresponds to $f(x_0)/f'(x_0)$ in (5.1)) by an alternating method. We start with a pair of maps
\[
u^\rho = (u_{11}^\rho, u_{22}^\rho) \in \mathcal{M}^{E_1}( (\Sigma_1, z_1); u_1) \times L \mathcal{M}^{E_2}( (\Sigma_2, z_2); u_2)
\]
for each sufficiently small $\epsilon_2 > 0$. Here $\rho_i \in V_i$ and $u_i^{\rho_i}$ is the corresponding map $(\Sigma_i, \partial\Sigma_i) \to (X, L)$ given by [3,25]. Let $\rho = (\rho_1, \rho_2)$. We put
\[
p^\rho_0 = \lim_{\tau \to -\infty} u_{11}^\rho(\tau, t) = \lim_{\tau \to -\infty} u_{22}^\rho(\tau, t).
\]

Pregluing: This step corresponds to pick the initial trial approximate solution $x_0$ in Newton’s scheme above.

Definition 5.1. We define
\[
u^\rho_{T,(0)} = \begin{cases} 
\text{Exp} \left( p^\rho_0, \chi^E \mathcal{E}(p^\rho_0, u_{11}^{\rho_1}) + \chi^{\mathcal{A}}\mathcal{E}(p^\rho_0, u_{22}^{\rho_2}) \right) & \text{on } [-5T, 5T] \times [0,1] \\
u_{11}^{\rho_1} & \text{on } K_1 \\
u_{22}^{\rho_2} & \text{on } K_2.
\end{cases}
\] 
(5.4)
This pre-glued map \( u_{T_i(0)}^e \) plays the role of \( x_0 \) in the Newton scheme for solving the equation

\[
\Pi_{\mathcal{E}_i}^\perp \overline{T} u = 0
\]  
(5.5)
on \( L^2_{m+1,\delta}(\Sigma_T, \partial \Sigma_T; (u_{T_i(0)}^e)^* T X, (u_{T_i(0)}^e)^* T L) \) over \( \Sigma_T \), where \( \Pi_{\mathcal{E}_i}^\perp \) is the projection to a subspace \( \mathcal{E}_i^\perp \). We will need to take the derivative of the map \( u \mapsto \Pi_{\mathcal{E}_i}^\perp \overline{T} u \) the analog to \( f'(x_n) \) in the Newton’s scheme. We will actually use an approximate derivative thereof whose explanation is now in order.

Note that the subspace \( \mathcal{E}_i(u'_i) \) varies on \( u'_i \). (See (3.18) for the definition.) So to obtain a linearized equation of (5.5) at the given map \( u'_i \), we need to take this dependence into account.

Let \( \Pi_{\mathcal{E}_i(u'_i)} \) be the \( L^2 \)-orthonormal projection of \( L^2_{m,\delta}(\Sigma_i; (u'_i)^* T X \otimes \Lambda^{0,1}) \) to \( \mathcal{E}_i(u'_i) \) or its associated idempotent as an endomorphism

\[
\Pi_{\mathcal{E}_i(u'_i)} : L^2_{m,\delta}(\Sigma_i; (u'_i)^* T X \otimes \Lambda^{0,1}) \to L^2_{m,\delta}(\Sigma_i; (u'_i)^* T X \otimes \Lambda^{0,1}).
\]

More explicitly, we represent the projection as follows: we choose a basis \( \{ \mathbf{e}_{i,a}(u'_i) \} \) of \( \mathcal{E}_i \) and set

\[
\mathbf{e}'_{i,a}(u'_i) := I_{u'_i}(\mathbf{e}_{i,a}), \quad a = 1, \ldots, \dim \mathcal{E}_i(u'_i).
\]

We apply standard Gram-Schmidt process to \( \mathbf{e}'_{i,a}(u'_i) \) and denote by \( \mathbf{e}_{i,a}(u'_i) \) the resulting orthonormal basis of \( \mathcal{E}_i(u'_i) \). Then, for \( A \in L^2_{m,\delta}(\Sigma_i; (u'_i)^* T X \otimes \Lambda^{0,1}) \) we put

\[
\Pi_{\mathcal{E}_i(u'_i)}(A) = \sum_{a=1}^{\dim \mathcal{E}_i} \langle A, \mathbf{e}_{i,a}(u'_i) \rangle_{L^2(\Sigma_i)} \mathbf{e}_{i,a}(u'_i),
\]

(5.6)where \( \mathbf{e}_{i,a}(u'_i) \) are supported in \( K_i \).

For given \( V \in \Gamma((\Sigma_i, \partial \Sigma_i), (u'_i)^* T X, (u'_i)^* T L) \), we put

\[
(D_{u'_i}\mathcal{E}_i)(A, V) := \frac{d}{ds} \bigg|_{s=0} (\text{Pal}_{i}^{[0,1]} - 1) \Pi_{\mathcal{E}_i(\text{Exp}(u'_i, s V))}(\text{Pal}_{i}^{[0,1]}(A)).
\]

(5.7)

Here \( \text{Pal}_{i}^{[0,1]} \) is the closure of the map \( \text{Pal}_{i}^{\text{Exp}(u'_i, s V)}(0,1) \) defined in (2.7). The right hand side is well defined in view of (5.6). We then put

\[
\Pi_{\mathcal{E}_i(u'_i)}^\perp(A) = A - \Pi_{\mathcal{E}_i(u'_i)}(A).
\]

The equation (3.21) is equivalent to \( \Pi_{\mathcal{E}_i(u'_i)}^\perp(\mathcal{J}) = 0 \). By evaluating the derivative

\[
\frac{\partial}{\partial s} \bigg|_{s=0} (\text{Pal}_{i}^{[0,1]} - 1) \big( \Pi_{\mathcal{E}_i(\text{Exp}(u'_i, s V))}^\perp \mathcal{J} \text{Exp}(u'_i, s V) \big)
\]

we obtain the linearization operator of (3.21).

As an operator into \( L^2_{m,\delta}(\Sigma_i; (u'_i)^* T X \otimes \Lambda^{0,1}) \) it is the projection to \( \mathcal{E}_i^\perp(u'_i) \) of

\[
(D_{u'_i}\mathcal{E}_i)^\perp(V) = (D_{u'_i}\mathcal{J})(V) - (D_{u'_i}\mathcal{E}_i)(\overline{T} u'_i, V).
\]

(5.8)

The approximate derivative of the map \( u \mapsto \Pi_{\mathcal{E}_i}^\perp \mathcal{J} u \) mentioned above then will be constructed by gluing the maps \( D_{u'_i}\mathcal{E}_i \) for \( i = 1, 2 \). (See (5.19) below for the precise expression of this approximate derivative.)

**Step 0-3 (Error estimates):** We recall \( \text{supp}(\mathcal{E}_i^\text{eb}) \subset \text{Int} K_i \) by definition of \( \mathcal{E}_i^\text{eb} \).
Lemma 5.2. There exists \( e^{\rho}_{t,T,(0)} \in E_i \) such that
\[
\| \overline{\partial} u^{\rho}_{t,T,(0)} - e^{\rho}_{1,T,(0)} - e^{\rho}_{2,T,(0)} \|_{L_{m,\delta}^2} < C_{1,m} e^{-\delta_1 T}.
\] (5.9)
Here \( C_{1,m} \) is a constant depending on \( X, L, u_i \) and \( m \) and \( u^{\rho}_{T,(0)} \) is as in Definition 5.1.
Moreover for any \( \epsilon(4) > 0 \) there exists \( \epsilon(4), \delta_1 \) such that
\[
\| e^{\rho}_{t,T,(0)} \|_{L^2(K_i)} < \epsilon(4)
\] (5.10)
if \( \rho = (\rho_1, \rho_2) \in V_1(\epsilon) \times L V_2(\epsilon) \) with \( \epsilon < \epsilon(4), \delta_1 \).
Furthermore \( e^{\rho}_{t,T,(0)} \) is supported in \( K_i \) and is independent of \( T \) as an element of \( L_{m}^2(K_i) \).

Remark 5.3. Note throughout the discussion of Sections 3-7 we fix \( X, L, u_i \). So we do not mention dependence of constants on them hereafter.

Proof. By definition, \( \overline{\partial} u_i^{\rho} \in E_i(u_i^{\rho}) \). We put
\[
e^{\rho}_{t,T,(0)} := \overline{\partial} u_i^{\rho} \in E_i(u_i^{\rho}) \cong E_i
\] (5.11)
extended to zero to \( \Sigma_T \setminus K_i \). Then, by definition, the support of \( \overline{\partial} u^{\rho}_{T,(0)} - e^{\rho}_{1,T,(0)} - e^{\rho}_{2,T,(0)} \) is contained in \( [-2T, 2T] \times [0,1] \). In fact on \( K_1 \cup [-5T, -2T] \times [0,1] \), \( u^{\rho}_{T,(0)} = u^{\rho}_{1} \) and on \( K_2 \cup [2T, 5T] \times [0,1] \), \( u^{\rho}_{T,(0)} = u^{\rho}_{2} \), so satisfies \( \overline{\partial} u^{\rho}_{T,(0)} = e^{\rho}_{1,T,(0)} + e^{\rho}_{2,T,(0)} \), there. Moreover by (2.10) and (5.4) the \( C^k \) norm of the map \( (\tau, t) \mapsto E(p^{\rho}_{T,(0)}, u^{\rho}_{T,(0)}(\tau, t)) : [-2T, 2T] \times [0,1] \to T_{\rho_0} X \) at \( (\tau, t) \) is smaller than \( C_k e^{-\delta_1 T} \), for any \( k \). Therefore we obtain the estimate (5.9).

\[\square\]

Remark 5.4. Note (5.11) (resp. (5.56), (C.12)) specifies the choice of \( e^{\rho}_{t,T,(0)} \) (resp. \( e^{\rho}_{t,T,(1)} \), \( e^{\rho}_{t,T,(k)} \)). This is the choice taken in this proof.

Step 0-4 (Separating error terms into two parts):

Definition 5.5. We put
\[
\text{Err}^{\rho}_{t,T,(0)} := \chi_{T}^{\rho} (\overline{\partial} u^{\rho}_{T,(0)} - e^{\rho}_{1,T,(0)}),
\]
\[
\text{Err}^{\rho}_{t,T,(0)} := \chi_{T}^{\rho} (\overline{\partial} u^{\rho}_{T,(0)} - e^{\rho}_{2,T,(0)}).
\]
We regard them as elements of the weighted Sobolev spaces \( L_{m,\delta}^2(\Sigma_1; (u_i^{\rho})^*TX \otimes \Lambda^{0,1}) \) and \( L_{m,\delta}^2(\Sigma_2; (u_i^{\rho})^*TX \otimes \Lambda^{0,1}) \) respectively. (We extend them by 0 outside \( \text{supp} \chi_T^{\rho} \), \( \text{supp} \chi_T^{\rho} \), respectively.)

Step 1-1 (Approximate solution for linearization): This step corresponds to solving the linearized equation
\[
f'(x_0)(v_0) = f(x_0), \quad \text{namely } v_0 = \frac{f(x_0)}{f'(x_0)}
\] (5.12)
in Newton’s iteration scheme.
We first cut-off \( u_{i,T,(0)}^\rho \) and extend it to obtain maps \( \hat{u}^\rho_{i,T,(0)} : (\Sigma_i, \partial \Sigma_i) \to (X, L) \) \((i = 1, 2)\) as follows. (This map is used to set the linearized operator \((5.14)\).)

\[
\hat{u}^\rho_{1,T,(0)}(z) = \begin{cases} 
\text{Exp} \left( p_0^\rho, \chi^\rho (\tau - T, t) E(p_0^\rho, u_{i,T,(0)}^\rho(\tau, t)) \right) & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0, 1] \\
\frac{\partial}{\partial s} \left|_{s=0} \left( (\text{Exp}(\hat{u}^\rho_{i,T,(0)}), sV) \right) \right) & \text{if } z \in [5T, \infty) \times [0, 1]. 
\end{cases}
\]

\[
\hat{u}^\rho_{2,T,(0)}(z) = \begin{cases} 
\text{Exp} \left( p_0^\rho, \chi^\rho (\tau + T, t) E(p_0^\rho, u_{i,T,(0)}^\rho(\tau, t)) \right) & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0, 1] \\
\frac{\partial}{\partial s} \left|_{s=0} \left( (\text{Exp}(\hat{u}^\rho_{i,T,(0)}), sV) \right) \right) & \text{if } z \in (-\infty, -5T) \times [0, 1]. 
\end{cases}
\]

Now we let

\[
D_{\hat{u}^\rho_{i,T,(0)}}(\overline{\partial}) : W^2_{m+1, \delta} \left( (\Sigma_i, \partial \Sigma_i); (\hat{u}^\rho_{i,T,(0)})^* TX, (\hat{u}^\rho_{i,T,(0)})^* TL \right) \rightarrow L^2_{m, \delta} \left( \Sigma_i; (\hat{u}^\rho_{i,T,(0)})^* TX \otimes \Lambda^{0,1} \right)
\]

be the (covariant) linearization of the Cauchy-Riemann equation at \( \hat{u}^\rho_{i,T,(0)} \), i.e.,

\[
(D_{\hat{u}^\rho_{i,T,(0)}}(\overline{\partial}))(V) = \left. \frac{\partial}{\partial s} \right|_{s=0} \left( (\text{Pal}_i^{0,1})^{-1}(\overline{\partial} \text{Exp}(\hat{u}^\rho_{i,T,(0)}), sV)) \right).
\]

Here \( \text{Pal}_i^{0,1} \) is the \((0, 1)\) part of the parallel translation along the map \( r \mapsto \text{Exp}(\hat{u}_{i,T,(0)}^\rho, rV) \), \( r \in [0, s] \) for some \( s \in [0, 1] \).

**Lemma 5.6.** There exist \( \varepsilon_2, T_{\text{Exp}} \) with the following properties. Let \( \mathcal{E}_i^{ob} \) be the subspace chosen in \((3.14)\). Define

\[
\mathcal{E}_i(\hat{u}^\rho_{i,T,(0)}) := I_{\hat{u}^\rho_{i,T,(0)}}(\mathcal{E}_i^{ob})
\]

as the subspace of \( L^2_{m, \delta} \left( \Sigma_i; (\hat{u}^\rho_{i,T,(0)})^* TX \otimes \Lambda^{0,1} \right) \). Then the following holds if \( T > T_{\text{Exp}} \) and \( \rho \in V_1(\varepsilon) \times L V_2(\varepsilon) \) with \( \varepsilon < \varepsilon_2 \).

\[
\text{Im}(D_{\hat{u}^\rho_{i,T,(0)}}(\overline{\partial})) + \mathcal{E}_i(\hat{u}^\rho_{i,T,(0)}) = L^2_{m, \delta} \left( \Sigma_i; (\hat{u}^\rho_{i,T,(0)})^* TX \otimes \Lambda^{0,1} \right).
\]

Moreover

\[
\text{Dev}_{1, \infty} - \text{Dev}_{2, \infty} = \text{Dev}_{1, \infty} - \text{Dev}_{2, \infty} v
\]

\[
: (D_{\hat{u}^\rho_{i,T,(0)}}(\overline{\partial}))^{-1}(\mathcal{E}_i(\hat{u}^\rho_{i,T,(0)})) \oplus (D_{\hat{u}^\rho_{2,T,(0)}}(\overline{\partial}))^{-1}(\mathcal{E}_2(\hat{u}^\rho_{2,T,(0)})) \rightarrow T_{\rho L}
\]

is surjective.

**Proof.** Since \( \hat{u}^\rho_{i,T,(0)} \) is close to \( u_i \) in exponential order, this is a consequence of Assumption 3.12. \( \square \)

We note that \( \overline{\partial} \hat{u}^\rho_{i,T,(0)} - e^\rho_{i,T,(0)} \) is exponentially small. Therefore the map

\[
V \mapsto (D_{\hat{u}^\rho_{i,T,(0)}}(\overline{\partial}))(V) - (D_{\hat{u}^\rho_{i,T,(0)}}(\mathcal{E}_i)(e^\rho_{i,T,(0)}), V),
\]
which is obtained by replacing \( \overline{\partial}u^p_{i,T,(0)} \) by \( e^p_{i,T,(0)} \), is exponentially close to the linearization operator \( (D_{u^p_{i,T,(0)}} \overline{\partial})^\perp \), defined in \( 5.8 \). We will consider this approximate linearization instead and denote it by

\[
D_{u^p_{i,T,(0)}}^{\text{app.}(0)} : W^2_{m+1,\delta}(\Sigma_i; (\hat{u}^p_{i,T,(0)})^*TX) \to L^2_{m,\delta}(\Sigma_i; (\hat{u}^p_{i,T,(0)})^*TX \otimes \Lambda^{0,1})
\]

with

\[
D_{u^p_{i,T,(0)}}^{\text{app.}(0)}(V) := (D_{\hat{u}^p_{i,T,(0)}} \overline{\partial})(V) - (D_{\hat{u}^p_{i,T,(0)}} E_i)(e^p_{i,T,(0)}, V).
\]

**Lemma 5.7.** There exists \( T_{22} \) with the following properties. If \( T > T_{22} \) and \( \rho \in V_1(\epsilon) \times L V_2(\epsilon) \) with \( \epsilon < \epsilon_2 \)

\[
\text{Im } D_{u^p_{i,T,(0)}}^{\text{app.}(0)} + E_i((\hat{u}^p_{i,T,(0)}) = L^2_{m,\delta}(\Sigma_i; (\hat{u}^p_{i,T,(0)})^*TX \otimes \Lambda^{0,1}).
\]

Moreover

\[
\text{Dev}_{1,\infty} - \text{Dev}_{2,\infty} : (D_{u^p_{i,T,(0)}}^{\text{app.}(0)})^{-1}(E_1) \oplus (D_{u^p_{i,T,(0)}}^{\text{app.}(0)})^{-1}(E_2) \to T_{\rho^p} L
\]

is surjective. Here we put \( E_i := E_i(\hat{u}^p_{i,T,(0)}) \).

**Proof.** The inequality \( (5.10) \) implies that \( (D_{u^p_{i,T,(0)}} E_i(\hat{u}^p_{i,T,(0)}, e) \text{ is small in the operator norm. The lemma then follows from Lemma } 5.6 \).}

**Remark 5.8.** Note that \( (5.10) \) and conclusions of Lemmas \( 5.6 \) and \( 5.7 \) are proved by taking a small neighborhood \( V_i(\epsilon) \) of \( 0 \) (in \( V_i \)) with respect to the \( C^m \) norm. (Note \( V_i(\epsilon) \subset \mathcal{M}_{\Sigma_i}((\Sigma_i, \bar{z}_i); \beta_i)(e(2) \) and \( V_i \) consists of smooth maps.) However we can take \( V_i(\epsilon) \) that is independent of \( m \) and the conclusion of Lemma \( 5.7 \) holds for all \( m \). In fact the elliptic regularity implies that if the conclusion of Lemma \( 5.7 \) holds for some \( m \) then it holds for all \( m' > m \). The inequality \( (5.10) \) holds for that particular \( m \) only. Only this inequality for \( m \) is used to show Lemma \( 5.7 \). In other words, \( \epsilon_2 \) is independent of \( m \).

We thus fix \( V_1 = V_1(\epsilon_2), V_2 = V_2(\epsilon_2) \) now and will never change it in Sections 5 and 6. In other words, the constant \( \epsilon_2 \) is fixed at this stage.

We consider the finite dimensional subspace

\[
\text{Ker}(\text{Dev}_{1,\infty} - \text{Dev}_{2,\infty}) \cap \{(D_{u_1} \overline{\partial})^\perp)^{-1}(E_1) \oplus (D_{u_2} \overline{\partial})^\perp)^{-1}(E_2)\}
\]

of

\[
\text{Ker}(\text{Dev}_{1,\infty} - \text{Dev}_{2,\infty}) \cap \bigoplus_{i=1}^2 W^2_{m+1,\delta}(\Sigma_i; \partial \Sigma_i); u^*_i TX, (u^*_i TL))
\]

which consists of smooth sections. Here we recall \( E_i := E_i(u_i) \). Note \( \overline{\partial}u_i = 0 \). Therefore \( (D_{u_i} \overline{\partial})^\perp = D_{u_i} \overline{\partial} \) by \( 5.8 \).

**Definition 5.9.** We define a linear subspace \( \mathcal{H}(E_1, E_2) \) of \( 5.23 \) by

\[
\mathcal{H}(E_1, E_2) \perp (5.22), \quad \mathcal{H}(E_1, E_2) \oplus (5.22) = (5.23).
\]

In other words, it is the \( L^2 \) orthogonal complement of \( 5.22 \) in \( 5.23 \). Here the \( L^2 \) inner product is defined by \( (4.11) \).

Note the support of elements of \( E_i(u_i) \) is contained in \( K_i \). Therefore the projection of \( C^k \) (resp. \( L^2_k \)) section to \( \mathcal{H}(E_1, E_2) \) is \( C^k \) (resp. \( L^2_k \)).
We use parallel transport to define an isomorphism from
\[ W^2_{m+1,\delta}((\Sigma_i, \partial \Sigma_i); u^*_i TX, u^*_i TL) \] (5.25)
to
\[ W^2_{m+1,\delta}((\Sigma_i, \partial \Sigma_i); (\tilde{u}_{i,T,(0)}^\rho)^*TX, (\tilde{u}_{i,T,(0)}^\rho)^*TL), \] (5.26)
as follows.

**Definition 5.10.** We define the isomorphism \( \Phi_{i(0)}(\rho, T) : (5.25) \rightarrow (5.26) \) by
\[ \Phi_{i(0)}(\rho, T) = \text{Pal}_{u_i^\rho,\mathbb{T}}. \] (5.27)

Here the right hand side is the closure of the map \( \text{Pal}_{u_i^\rho,\mathbb{T}} \) in (2.6).

**Lemma 5.11.**
1. \( \Phi_{i(0)}(\rho, T) \) sends \( (5.25) \) to \( (5.26) \).
2. The linear map \( (\Phi_{i(0)}(\rho, T), \Phi_{2(0)}(\rho, T)) \) sends the subspace \( (5.22) \) into the subspace
\[ \Ker(\text{Dev}_{1,\infty} - \text{Dev}_{2,\infty}) \]
\[ \cap \bigoplus_{i=1}^2 W^2_{m+1,\delta}((\Sigma_i, \partial \Sigma_i); (\tilde{u}_{i,T,(0)}^\rho)^*TX, (\tilde{u}_{i,T,(0)}^\rho)^*TL). \] (5.28)

Using Lemma 5.11, the proof is easy and is omitted.

**Definition 5.12.** We denote by \( \mathcal{H}_{i(0)}(\mathcal{E}_1, \mathcal{E}_2; \rho, T) \) the image of the restriction of the linear map \( (\Phi_{1(0)}(\rho, T), \Phi_{2(0)}(\rho, T)) \) to \( \mathcal{H}(\mathcal{E}_1, \mathcal{E}_2) \). It is a subspace of (5.28).

**Definition 5.13.** Let \( \text{Err}_{i,T,(0)}^\rho \) be the functions defined in Definition 5.5. We define the triple \( (V_{T,1,(1)}, V_{T,2,(1)}, \Delta p_{T,(1)}^\rho) \) to be the unique solution satisfying the requirements
\[ D_{\tilde{u}_{i,T,(0)}^\rho}^{\text{app},(0)}(V_{T,1,(1)}^\rho) + \text{Err}_{i,T,(0)}^\rho \in H_i(\tilde{u}_{i,T,(0)}^\rho), \] (5.29)
and
\[ ((V_{T,1,(1)}^\rho, \Delta p_{T,(1)}^\rho), (V_{T,2,(1)}^\rho, \Delta p_{T,(1)}^\rho)) \in \mathcal{H}_{i(0)}(\mathcal{E}_1, \mathcal{E}_2; \rho, T). \] (5.30)

Lemma 5.11 implies that such \( (V_{T,1,(1)}^\rho, V_{T,2,(1)}^\rho, \Delta p_{T,(1)}^\rho) \) exists and is unique if \( T > T_{28} \) with sufficiently large \( T_{28} \) and \( \rho \) is in \( V_1 \times L^2 \) chosen at Remark 5.8. In fact, when we consider the subspace
\[ \Ker(\text{Dev}_{1,\infty} - \text{Dev}_{2,\infty}) \cap \bigoplus_{i=1}^2 (D_{\tilde{u}_{i,T,(0)}^\rho}^{\text{app},(0)})^{-1}(\mathcal{E}_i) \] (5.31)
of (5.28) we still have a direct sum decomposition
\[ (5.31) \oplus \mathcal{H}_{i(0)}(\mathcal{E}_1, \mathcal{E}_2; \rho, T) = (5.28). \] (5.32)
Since (5.32) holds for \( T = \infty \) by definition, so it holds for \( T > T_{28} \).

The unique solution \( (V_{T,1,(1)}^\rho, V_{T,2,(1)}^\rho, \Delta p_{T,(1)}^\rho) \) corresponds to the solution \( v_0 \) to (5.12) in Newton’s scheme. Then the following estimate corresponds to estimating \(|x_1 - x_0| = |v_0|\) for the solution \( v_0 \) to the linearized equation (5.12) in Newton’s scheme.

**Remark 5.14.** Instead of \( \mathcal{H}_{i(0)}(\mathcal{E}_1, \mathcal{E}_2; \rho, T) \) we can use \( L^2 \) orthonormal complement of (5.31) in (5.28). This choice looks more natural and actually works for the proof of Theorems 3.13 and 6.4. We take the current choice since when we study \( T \) and \( \rho \) derivative in Subsection 6.2 the current choice makes calculation slightly shorter.
Lemma 5.15. There exists $C_{2,m}, T_m$ such that if $T > T_m$, then
\[ \|(V_{T,1,(1)}^p, \Delta p_{T,(1)}^p)\|_{W_{m+1,d}^2(\Sigma_c)} \leq C_{2,m} e^{-\delta_1 T}. \] (5.33)

This is immediate from construction and the uniform boundedness of the right inverse of the operator
\[ \Pi_{\tilde{E}_i} \circ D_{u_{i,T,(0)}}^{app} = \Pi_{\tilde{E}_i} \circ (D_{u_{i,T,(0)}}^{app} \bar{\mathcal{E}} - (D_{u_{i,T,(0)}}^{app} \mathcal{E})(\epsilon_{1,T,(0)}, \cdot)) \]
and the exponential decay estimates of the error term. (Lemma 5.2)

Step 1-2 (Gluing solutions): We use $(V_{T,1,(1)}^p, V_{T,2,(1)}^p, \Delta p_{T,(1)}^p)$ to find an approximate solution $u_{T,(1)}^p$ of the next level. This corresponds to writing down
\[ x_1 = x_0 - v_0 = x_0 - \frac{f(x_0)}{f'(x_0)} \]
in the Newton’s scheme.

Definition 5.16. We define $u_{T,(1)}^p(z)$ as follows.
1. If $z \in K_1$, we put
   \[ u_{T,(1)}^p(z) = \text{Exp}(\hat{u}_{1,T,(0)}^p(z), V_{T,1,(1)}^p(z)). \] (5.34)
2. If $z \in K_2$, we put
   \[ u_{T,(1)}^p(z) = \text{Exp}(\hat{u}_{2,T,(0)}^p(z), V_{T,2,(1)}^p(z)). \] (5.35)
3. If $z = (\tau, t) \in [-5T, 5T] \times [0, 1]$, we put
   \[ u_{T,(1)}^p(\tau, t) = \text{Exp} \left( u_{T,(0)}^p(\tau, t), \chi_{\bar{E}}(\tau, t)(V_{T,1,(1)}^p(\tau, t) - (\Delta p_{T,(1)}^p)^{\text{Pal}}) \right. \]
   \[ + \left. \chi_{\bar{A}}(\tau, t)(V_{T,2,(1)}^p(\tau, t) - (\Delta p_{T,(1)}^p)^{\text{Pal}} + (\Delta p_{T,(1)}^p)^{\text{Pal}}) \right). \] (5.36)

We recall that $\hat{u}_{1,T,(0)}^p(z) = u_{T,(0)}^p(z)$ on $K_1$ and $\hat{u}_{2,T,(0)}^p(z) = u_{T,(0)}^p(z)$ on $K_2$.

Step 1-3 (Error estimates): Let a constant $0 < \mu < 1$ be given, which we fix throughout the rest of the proof. For example we can take $\mu = 1/2$.

This step corresponds to establishing the inequality (5.3) for $n = 1$ in Newton’s iteration scheme.

Proposition 5.17. There exists $C_{3,m}$ such that for any $\epsilon(4) > 0$, there exists $T_m, \epsilon(4)$, such that for all $T > T_m, \epsilon(4)$, we can define $\epsilon_{1,T,(1)}^p \in \mathcal{E}_i$ that satisfies
\[ \|\bar{\mathcal{E}} u_{T,(1)}^p - (\epsilon_{1,T,(0)}^p + \epsilon_{1,T,(1)}^p) - (\epsilon_{2,T,(0)}^p + \epsilon_{2,T,(1)}^p)\|_{L_{m, \delta}^2(K_i)} < C_{1,m} \epsilon(4) \mu e^{-\delta_1 T} \] (5.37)
for all $T > T_m$. (Here $C_{1,m}$ is the constant given in Lemma 5.2.) Moreover
\[ \|\epsilon_{1,T,(1)}^p\|_{L_{m}^2(K_i)} < C_{3,m} e^{-\delta_1 T}. \] (5.38)

Note Remark 5.4 applies here.

Proof. The existence of $\epsilon_{1,T,(1)}^p$ satisfying
\[ \|\bar{\mathcal{E}} u_{T,(1)}^p - (\epsilon_{1,T,(0)}^p + \epsilon_{1,T,(1)}^p) - (\epsilon_{2,T,(0)}^p + \epsilon_{2,T,(1)}^p)\|_{L_{m, \delta}^2(K_i)} < C_{1,m} \epsilon(4) \mu e^{-\delta_1 T}/10 \]
is a consequence of the fact that $\bar{\mathcal{E}}$ is the linearized equation of (3.19) and the estimate (5.33). More explicitly, we can prove this by a standard quadratic estimate whose details are now in order.
We do the necessary estimates on the regions $K_1, K_2, ([−5T, 5T] \setminus [−T − 1, T + 1]) \times [0, 1]$ and region $[−T − 1, T + 1] \times [0, 1]$ separately.

We first estimate on $K_1$. We put $K_1^+ = K_1 \cup [−5T, −5T + 1] \times [0, 1]$.

**Remark 5.18.** The estimate on $K_1$ is rather lengthy. However, it consists only of standard and straightforward calculation. This estimate is not a part of the main idea of the proof. The main idea of the proof of this paper is the points mentioned in Remarks 5.21 and 6.12.

During this estimate on $K_1$ and on $[−5T, −T − 1] \times [0, 1]$ we use the following simplified notation.

$$
\begin{align*}
  u &= \hat{u}_{1,T,(0)}^\rho, \\
  V &= V_{T,1,(1)}^\rho, \\
  \mathcal{P} &= (\text{Pal}_{1}^{(0,1)})^{-1}, \\
  \epsilon &= \epsilon_{1,T,(0)}^\rho.
\end{align*}
$$

Here $\text{Pal}_{1}^{(0,1)}$ is the $(0, 1)$ part of the parallel translation along the curve $s \mapsto \text{Exp}(u, sV), s \in [0, s_0]$ for $s_0 \in [0, 1]$.

By Fundamental Theorem of Calculus

$$
\begin{align*}
g(1) &= g(0) + \int_0^1 g'(s) \, ds = g(0) + g'(0) + \int_0^1 ds \left( \int_0^s g''(r) \, dr \right)
\end{align*}
$$

applied to the function $g$ valued in $L^2_{m,\delta}(K_1^+; (\hat{u}_{1,T,(0)}^\rho)^*TX \otimes \Lambda^{0,1})$ given by $g(s) = \mathcal{P}\overline{\partial}(\text{Exp}(u, sV))$, we derive

$$
\begin{align*}
  \mathcal{P}\overline{\partial}(\text{Exp}(u, V)) &= \overline{\partial}(\text{Exp}(u, 0)) + \int_0^1 \frac{\partial}{\partial s} (\mathcal{P}\overline{\partial}(\text{Exp}(u, sV))) \, ds \\
  &= \overline{\partial}u + (Du)\overline{\partial}(V) + \int_0^1 ds \int_0^s \left( \frac{\partial}{\partial r} \right)^2 (\mathcal{P}\overline{\partial}(\text{Exp}(u, rV))) \, dr
\end{align*}
$$

on $L^2_{m,\delta}(\Sigma_T; u^*TX \otimes \Lambda^{0,1})$.

On the other hand, we have the following estimate.

**Lemma 5.19.** Let $m \geq 2$. Then

$$
\begin{align*}
  \left\| \int_0^1 ds \int_0^s \left( \frac{\partial}{\partial r} \right)^2 (\mathcal{P}\overline{\partial}(\text{Exp}(u, rV))) \, dr \right\|_{L^2_{m}(K_1)} &\leq C_{m, \text{5.42}} \|V\|^2_{L^2_{m+1,\delta}(K_1^+)} \\
  &\leq C'_{m, \text{5.42}} e^{-2\delta_{T}}.
\end{align*}
$$

**Proof.** This is rather an immediate consequence of the fact that the covariant derivative $\nabla^2 P = \left( \frac{2}{m} \right)^2 P$ do not differentiate $V$. Especially, this inequality is obvious for the case of flat metric and the standard complex structure on $\mathbb{C}^n$. For the readers’ convenience, we provide a full explanation on why the presence of exponential maps and the parallel transports do not hinder obtaining the required estimate in Appendix A. \qed
We also have
\[ \mathcal{P} \circ \Pi^+_1(\text{Exp}(u, V)) \circ \mathcal{P}^{-1} = \Pi^+_1(\text{Exp}(u, V)) + \int_0^1 \frac{d}{ds} \left( \mathcal{P} \circ \Pi^+_1(\text{Exp}(u, sV)) \circ \mathcal{P}^{-1} \right) ds \]
\[ = \Pi^+_1(\text{Exp}(u)) - (D_u \mathcal{E}_1)(\cdot, V) + \int_0^1 ds \int_0^s \frac{d^2}{dr^2} \left( \mathcal{P} \circ \Pi^+_1(\text{Exp}(u, rV)) \circ \mathcal{P}^{-1} \right) dr. \]  

By (5.43), (5.44) and (5.45), we have
\[ \parallel \mathcal{P} \circ \Pi^+_1(\text{Exp}(u, rV)) \circ \mathcal{P}^{-1} \parallel_{L^2_m(K_1)} \leq C_m e^{-2\delta_1 T}. \]  

We can estimate the third term of the right hand side of (5.43) in the same way as in (5.42) and also get
\[ \parallel \int_0^1 ds \int_0^s \frac{d^2}{dr^2} \left( \mathcal{P} \circ \Pi^+_1(\text{Exp}(u, rV)) \circ \mathcal{P}^{-1} \right) dr \parallel_{L^2_m(K_1)} \leq C_m e^{-2\delta_1 T}. \]  

Here the left hand side is the operator norm as an endomorphism of $L^2_m(K_1)$. 

On the other hand, (5.33), (5.34) and (5.41) imply that
\[ \parallel \mathcal{P}(\text{Exp}(u, V)) - \mathcal{P}^{-1} \parallel_{L^2_m(K_1)} \leq C_m e^{-2\delta_1 T}. \]  

(Note we regard $e = e^{0, T}(0) \in L^2(K_1, u^* T X)$.) We put
\[ \mathcal{Q} = \mathcal{P}(\text{Exp}(u, V)) - \mathcal{P}^{-1} \parallel e. \]  

By (5.43), (5.44) and (5.45), we have
\[ \parallel \Pi^+_1(\text{Exp}(u, V)) Q - \mathcal{P}^{-1} \Pi^+_1(\mathcal{P} Q) + \mathcal{P}^{-1} (D_u \mathcal{E}_1)(\mathcal{P} Q, V) \parallel_{L^2_m(K_1)} \]
\[ = \parallel \int_0^1 ds \int_0^s \frac{d^2}{dr^2} \left( \mathcal{P} \circ \Pi^+_1(\text{Exp}(u, rV)) Q \right) dr \parallel_{L^2_m(K_1)} \]
\[ \leq C_m e^{-2\delta_1 T}. \]

By (5.33) we have $\parallel V \parallel_{L^2_m(K_1)} \leq C_2 m e^{-2\delta_1 T}$. Therefore using (5.45), (5.46) and
\[ \parallel \mathcal{P}^{-1} (D_u \mathcal{E}_1)(\mathcal{P} Q, V) \parallel_{L^2_m(K_1)} \leq C_m \parallel V \parallel_{L^2_m(K_1)} \parallel Q \parallel_{L^2_m(K_1)} \]

we have
\[ \parallel \Pi^+_1(\text{Exp}(u, V)) Q - \mathcal{P}^{-1} \Pi^+_1(\mathcal{P} Q) \parallel_{L^2_m(K_1)} \leq C_m (5.48) e^{-2\delta_1 T}. \]

Substituting $Q = \mathcal{P}(\text{Exp}(u, V)) - \mathcal{P}^{-1} e$ in (5.48) back, we obtain
\[ \parallel \Pi^+_1(\text{Exp}(u, V)) \mathcal{P}(\text{Exp}(u, V)) - \mathcal{P}^{-1} \Pi^+_1(\mathcal{P} \mathcal{E}_1(u)) \parallel_{L^2_m(K_1)} \leq C_m e^{-2\delta_1 T}. \]  

Therefore applying (5.43) and (5.44) to the 3rd and 4th terms of (5.49), we obtain
\[ \parallel \Pi^+_1(\text{Exp}(u, V)) \mathcal{P}(\text{Exp}(u, V)) - \mathcal{P}^{-1} \Pi^+_1(\mathcal{P} \mathcal{E}_1(u)) \parallel_{L^2_m(K_1)} \leq C_m e^{-2\delta_1 T}. \]

We recall $u = \check{u}^o_{1, T}(0) = u^o_{1, t}$ on $K^1$. Therefore we derive
\[ \mathcal{U} u + (D_u \mathcal{E}_1)(V) - (D_u \mathcal{E}_1)(e, V) \in \mathcal{E}_1(u) \]
on $K_1^+$ from (5.19), (5.20) and Definition 5.5. Then (5.50) and (5.51) imply
\begin{equation}
\left\| \Pi_{E_1}^\perp (\Exp(u,V)) \overrightarrow{\partial}(\Exp(u,V)) \right\|_{L^2_m(K_1)} \\
\quad + \left\| \Pi_{E_1}^\perp (\Exp(u,V)) \overrightarrow{\partial}_u \right\|_{L^2_m(K_1)} \\
\quad + \left\| \Pi_{E_1}^\perp (\Exp(u,V)) (D_u \overrightarrow{\partial}) (V) \right\|_{L^2_m(K_1)} \\
\leq C_m, e^{-2\delta_1 T}.
\end{equation}
(5.52)

Combined with (5.41) and (5.42), this implies
\begin{equation}
\left\| \Pi_{E_1}^\perp (\Exp(u,V)) \overrightarrow{\partial}(\Exp(u,V)) \right\|_{L^2_m(K_1)} \leq C_m, e^{-\delta_1 T} \\
\leq C_{1,m} e^{-\delta_1 T} \epsilon(4) \mu/10,
\end{equation}
(5.53)
for $T > T_{\epsilon(4), \mu}$. If we choose $T_m(\epsilon(4))$ so that it satisfies
\begin{equation}
e^{-\delta_1 T_{\epsilon(4), \mu}} < \frac{C_{1,m} \epsilon(4) \mu}{10 C_m}.
\end{equation}
(5.54)

This gain of decay rate by the order of $\mu > 0$ independent of $T$ and $m$ is one of the crucial elements in the iteration scheme. (We refer readers to (5.79) to see how this gain is used.) We use (5.45) and (5.53) to show:

**Lemma 5.20.** There exists $C_m, \epsilon(4)$ such that:
\begin{equation}
\left\| \Pi_{E_1}^\perp (\Exp(u,V)) \overrightarrow{\partial}(\Exp(u,V)) \right\|_{L^2_m(K_1)} \leq C_m, e^{-\delta_1 T}.
\end{equation}
(5.55)

Here we regard $\epsilon = \epsilon_{1,T,(0)}$ as an element of $L^2_m(K_1; (\Exp(u,V)) \otimes TX \otimes \Lambda^{0,1})$.

**Proof.** This is a consequence of (5.45), $\epsilon \in E_1(\Exp(u,V))$ and certain estimate of parallel transport. We postpone the proof of this lemma till Appendix B. \(\square\)

Then if we set
\begin{equation}
\epsilon_{1,T,(1)} = \Pi_{E_1} \overrightarrow{\partial}(\Exp(u,V)) \left( \epsilon_{1,T,(0)} \right)
\end{equation}
(5.56)
(5.38) follows. (Recall $\epsilon = \epsilon_{1,T,(0)}$.)

The estimate of the left hand side of (5.37) on $K_1$ follows from (5.53). We have thus finished the estimate on $K_1$. The estimate on $K_2$ is the same.

We next consider the domain $[-5T, -T - 1] \times [0, 1]$. Let $S \in [-5T, -T - 1]$ and $\Sigma(S) = [S, S + 1] \times [0, 1]$.

We also put $v = \Delta p_{T,(1)}$.

In the same way as the proof of Lemma 5.19, we can use (5.33) to prove
\begin{equation}
\int_0^1 ds \int_0^s \left( \frac{\partial}{\partial r} \right)^2 (\overrightarrow{\partial}(\Exp(u,rV)) \right) dr \\
\leq C_m, e^{-\delta_1 T}
\end{equation}
(5.57)
(5.55)
\begin{equation}
\leq C_m, e^{-\delta_1 T}
\end{equation}
(5.57)
(5.55)
(See Appendix [A]) Here to prove the inequality of the last line we also use the boundedness of the domain \( \Sigma(S) \) and an equality \( \| v^{\Pai} \|_{L^2_{m+1}(\Sigma(S))} \leq C \| v \| \), which follows from the definition (3.7) of \( v^{\Pai} \).

We remark that the domain \( \Sigma(S) \) is disjoint from the support of elements of \( \mathcal{E}_i \). Therefore (5.41), (5.57) together with (5.29) imply

\[
\| \overline{\mathcal{F}}(\Exp(u, V)) \|_{L^2_{m}(\Sigma(S))} \leq C_m \| \overline{\mathcal{F}}(\Exp(u, V)) \|_{L^2_{m}(\Sigma(S))} e^{-2\delta_1 T} \tag{5.58}
\]

Note \( e_{T, \delta} \leq 10 e^{5T \delta} \leq 10 e^{T \delta_1 /2} \) by (3.11). Hence (5.58) implies

\[
\| \overline{\mathcal{F}}(\Exp(u, V)) \|_{L^2_{m, \delta}([-T,-T-1] \times [0,1])} \leq C_m \| \overline{\mathcal{F}}(\Exp(u, V)) \|_{L^2_{m, \delta}([0,1])} \tag{5.59}
\]

if \( T > T_{m, \epsilon}(4 \mu /10) \).

The inequality (5.59) is the estimate of the left hand side of (5.37) on \([-5T, -T-1] \times [0,1] \).

The estimate on \([T + 1, 5T] \times [0,1] \) is the same. (We use the notations given in (5.30) up to here.)

Now we do estimate \( \overline{\mathcal{U}} u^p_{T, (1)} \) on \([-T + 1, T - 1] \times [0,1] \). The inequality

\[
\| \overline{\mathcal{U}} u^p_{T, (1)} \|_{L^2_{m, \delta}([-T+1, T-1] \times [0,1])} \leq C_1 \mu (4 \mu /10) e^{-\delta_1 T} \tag{5.60}
\]

is also a consequence of the fact that (5.8) is the linearized equation of (3.19) and of the estimate (5.33). In fact, since the bump functions \( \chi^m_B \) and \( \chi^m_A \) are \( \equiv 1 \) there the proof is the same as the proof of (5.59).

On \( \mathcal{A}_T \), by definition of \( u^p_{T, (1)} \) we have

\[
\overline{\mathcal{U}} u_{T, (1)} = \overline{\mathcal{F}} \left( \Exp \left( u^p_{T, (0)}, \chi^m_A (V^{p}_{T, 2, (1)} - (\Delta p^p_{T, (1)})^{\Pai}) + V^{p}_{T, 1, (1)} \right) \right). \tag{5.61}
\]

Note

\[
\| \chi^m_A (V^{p}_{T, 2, (1)} - (\Delta p^p_{T, (1)})^{\Pai}) \|_{L^2_{m+1}(\mathcal{A}_T)} \leq C_m \| \chi^m_A (V^{p}_{T, 2, (1)} - (\Delta p^p_{T, (1)})^{\Pai}) \|_{L^2_{m+1, \delta}(\mathcal{A}_T \subset \Sigma_2)} \tag{5.62}
\]

(5.61)

The first inequality follows from the fact that the weight function \( e_{2, \delta} \) is around \( e^{6T \delta} \) on \( \mathcal{A}_T \). The second inequality follows from (5.33). On the other hand the weight function \( e_{T, \delta} \) is around \( e^{4 \delta T} \) at \( \mathcal{A}_T \). See Figure 6. Therefore

\[
\| \overline{\mathcal{U}} u^p_{T, (1)} \| = \overline{\mathcal{F}}(\Exp(u^p_{T, (0)}, V^{p}_{T, 1, (1)})) \|_{L^2_{m, \delta}(\mathcal{A}_T \subset \Sigma_2)} \leq C_m \| \chi^m_A (V^{p}_{T, 2, (1)} - (\Delta p^p_{T, (1)})^{\Pai}) \|_{L^2_{m+1, \delta}(\mathcal{A}_T \subset \Sigma_2)} \tag{5.62}
\]

Therefore

\[
\leq C_m \| \chi^m_A (V^{p}_{T, 2, (1)} - (\Delta p^p_{T, (1)})^{\Pai}) \|_{L^2_{m+1, \delta}(\mathcal{A}_T \subset \Sigma_2)} e^{-2\delta T}. \tag{5.63}
\]

See Appendix [A] for the proof of the first inequality.

**Remark 5.21.** This drop of the weight is the main part of the idea. It was used in [FOOO1] page 414 (and also in [Fu] (8.7.2)). This is the place where the choice of
our weight functions enables us to implement the alternating method by exploiting the exponential decay of the Green kernel.

By Definition 5.5, \( \text{Err}_{\rho, T, (0)} = 0 \) on \( \mathcal{A}_T \). Using this in the same way as we did to obtain (5.58) we derive

\[
\| \partial (\text{Exp}(u_{T, (0)}^\rho, V_{T, 1, (1)}^\rho)) \|_{L^2_{m, \delta} (\mathcal{A}_T \subset \Sigma_T)} \leq C_{1, m} e^{-\delta T} e(4) \mu/20 \quad (5.63)
\]

for \( T > T_{m, \epsilon (4), 5.62} \).

Therefore combining (5.60)-(5.63), we can find a constant \( T_{m, \epsilon (4), 5.64} \) such that

\[
\| \partial u_{T, (1)}^\rho \|_{L^2_{m, \delta} (\mathcal{A}_T \subset \Sigma_T)} < C_{1, m} \mu \epsilon (4) e^{-\delta T}/10 \quad (5.64)
\]

for all \( T > T_{m, \epsilon (4), 5.64} \).

The estimate on \( B_T \) is similar and so omitted. The proof of Proposition 5.17 is now complete. \( \square \)

**Step 1-4 (Separating error terms into two parts):** We start with writing down the (approximate) error in the first induction step.

**Definition 5.22.** We put

\[
\text{Err}^\rho_{1, T, (1)} = \chi_X^{-} (\partial u_{T, (1)}^\rho) - (\epsilon_{1, T; (0)}^\rho + \epsilon_{1, T; (1)}^\rho), \\
\text{Err}^\rho_{2, T, (1)} = \chi_X^{-} (\partial u_{T, (1)}^\rho) - (\epsilon_{2, T; (0)}^\rho + \epsilon_{2, T; (1)}^\rho).
\]

We regard them as elements of the weighted Sobolev spaces \( L^2_{m, \delta} (\Sigma_1; (\hat{u}_{1, T, (1)}^\rho)^* TX \otimes \Lambda^{0,1}) \) and \( L^2_{m, \delta} (\Sigma_2; (\hat{u}_{2, T, (1)}^\rho)^* TX \otimes \Lambda^{0,1}) \) respectively, by extending them to be 0 outside the support of \( \chi \).

We put

\[
p_{T; (1)}^\rho = \text{Exp}(p^\rho, \Delta p_{T, (1)}^\rho), \quad (5.65)
\]
We then come back to Step $\kappa$-1 for $\kappa = 1$ to establish the following and continue our inductive steps for $\kappa \geq 1$.

\[
\left\| (V^\rho_{T,i,\kappa}(\Delta p^\rho_{T,\kappa})) \right\|_{W^{2,+1}_{m+1,\delta}(\Sigma_i)} < C_{2,m} \mu^{\kappa-1} e^{-\delta_1 T}, \quad (5.66)
\]

\[
\left\| E(u^\rho_{T,0}, u^\rho_{T,\kappa}) \right\|_{W^{2,+1}_{m+1,\delta}(\Sigma_T)} < C_{4,m}(2 - \mu^\kappa) e^{-\delta_1 T}, \quad (5.67)
\]

\[
\left\| \text{Err}_{1,T,\kappa} \right\|_{L^2_{m,\delta}(\Sigma_i)} < C_{1,m} \epsilon(5) \mu^\kappa e^{-\delta_1 T}, \quad (5.68)
\]

\[
\left\| \epsilon_{2,T,\kappa} \right\|_{L^2_{m,\delta}(K_i)} < C_{3,m} \mu^{\kappa-1} e^{-\delta_1 T}, \quad \text{for } \kappa \geq 1. \quad (5.69)
\]

Note we also have

\[
\partial_T \epsilon_{T,\kappa} - \text{Err}_{1,T,\kappa} - \text{Err}_{2,T,\kappa} = \sum_{i=1}^{2} \sum_{j=0}^{\kappa} \epsilon_{i,T,(j)}. \quad (5.70)
\]

See Definitions 5.22 and 5.34.

More specifically, we will prove the next proposition. Note we have already chosen the constants $C_{1,m}, C_{2,m}, C_{3,m}$.

**Proposition 5.23.**

1. For any $\epsilon(5) > 0$ and $C_{4,m}$ there exists $T_{3,m,\epsilon}(5) > 0$ such that if $T > T_{3,m,\epsilon}(5)$ then (5.66), (5.67) for $\kappa < 1$ imply (5.68) and (5.69) for $\kappa < 1$. (Here $T_{3,m,\epsilon}(5)$ may depend on $C_{1,m}, C_{2,m}, C_{3,m}, C_{4,m}$.)

2. For any $C_{4,m}$ there exists $\epsilon_{3,m} > 0$ independent of $T, \kappa$ such that if $\epsilon(5) < \epsilon_{3,m}$ then (5.68) and (5.69) for $\kappa < 1$ imply (5.66) for $\kappa$. (Here $\epsilon_{4,m}$ may depend on $C_{1,m}, C_{2,m}, C_{3,m}, C_{4,m}$.)

3. We can choose $C_{4,m}$ such that the following holds. There exists $T_{4,m} > 0$ depending on $C_{1,m}, C_{2,m}, C_{3,m}, C_{4,m}$ with the following properties. Suppose (5.66) for $\kappa < 1$ and that $u^\rho_{T,\kappa}$ is defined from $u^\rho_{T,(\kappa-1)}$ and $V^\rho_{T,i,\kappa}$ as in Definition 5.32 and (5.67) for $\kappa < 1$. Then we have (5.67) for $\kappa$, if $T > T_{4,m}$.

**Proof.** The rest of this section will be occupied with the proof of this proposition by describing each of Steps $\kappa$-1, ..., $\kappa$-4. Note Step $\kappa$-1 corresponds to Proposition 5.23 (2). Steps $\kappa$-2, $\kappa$-3, $\kappa$-4 corresponds to Proposition 5.23 (1). The proof of Proposition 5.23 (3) is straightforward. We provide its detail in Appendix [E] together with versions involving $T$ and $\rho$ derivatives. See Proposition 6.9 (3).

Since Steps $\kappa$-1, ..., $\kappa$-4 are largely repetitions of the same kind of tedious but straightforward estimates as in $\kappa = 1$, we will only state the main definitions and statements required to perform the inductive schemes that will be needed for the exponential estimate of the $T$-derivatives in the next section, and leave the details of the relevant estimates to Appendix [C].

**Remark 5.24.** Various constants $C_{m,*(\cdot,*)}$ and $T_{m,\epsilon(5),*(\cdot,*)}$ etc. will appear in the course of the proof of Proposition 5.23 (including the appendices quoted there.) Those constants depend on $m$ as well as the numbers explicitly appearing in the subscript and the previously given constants $C_{1,m}, C_{4,m}$. (Especially they depend on $W_{m+1,1,\delta}$ norm of $u^\rho_{T,\kappa}$ which is estimated by (5.67).) The important point is that they are independent of $\kappa$ and $T$. 
Therefore, even though we have infinitely many steps \( \kappa = 0, 1, 2, \ldots \) to work out, we need to make choices of those constants only finitely many times.

**Step \( \kappa \)-1:** In this step we prove Proposition 5.23 (1). We first cut-off \( u^\rho_{T,1(\kappa - 1)} \) and extend by zero to obtain maps \( \hat{u}^\rho_{i,T,1(\kappa - 1)} : (\Sigma_i, \partial \Sigma_i) \rightarrow (X, L) \ (i = 1, 2) \) as follows: We first define \( p^\rho_{T,1(\kappa - 1)} \) inductively by

\begin{equation}
\tag{5.71}
p^\rho_{T,1(\kappa - 1)} = \text{Exp}(p^\rho_{T,1(\kappa - 2)}, \Delta p^\rho_{T,1(\kappa - 1)}),
\end{equation}

starting from (5.65). Then we put

\begin{equation}
\begin{cases}
\hat{u}^\rho_{1,T,1(\kappa - 1)}(z) = \text{Exp} \left( p^\rho_{T,1(\kappa - 1)}; \chi^T_B (\tau - T, t) E(p^\rho_{T,1(\kappa - 1)}, u^\rho_{T,1(\kappa - 1)}(\tau, t)) \right) & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0, 1] \\
& \text{if } z \in K_1 \\
\hat{u}^\rho_{1,T,1(\kappa - 1)}(z) = u^\rho_{T,1(\kappa - 1)}(z) & \text{if } z \in [5T, \infty) \times [0, 1].
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
\hat{u}^\rho_{2,T,1(\kappa - 1)}(z) = \text{Exp} \left( p^\rho_{T,1(\kappa - 1)}; \chi^T_A (\tau + T, t) E(p^\rho_{T,1(\kappa - 1)}, u^\rho_{T,1(\kappa - 1)}(\tau, t)) \right) & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0, 1] \\
& \text{if } z \in K_2 \\
\hat{u}^\rho_{2,T,1(\kappa - 1)}(z) = u^\rho_{T,1(\kappa - 1)}(z) & \text{if } z \in (-\infty, -5T] \times [0, 1].
\end{cases}
\end{equation}

We have the following estimate of \( \hat{u}^\rho_{i,T,1(\kappa - 1)} \) on the neck region.

**Lemma 5.25.**

\begin{align}
\| \text{Exp}(p^\rho_{\kappa,0}, \hat{u}^\rho_{1,T,1(\kappa - 1)}) \|_{L^2_{m+1}([-T, T] \times [0, 1])} & \leq C_{m,5.73} e^{-\delta_1 T}, \\
\| \text{Exp}(p^\rho_{\kappa,0}, \hat{u}^\rho_{2,T,1(\kappa - 1)}) \|_{L^2_{m+1}([-T, T] \times [0, 1])} & \leq C_{m,5.73} e^{-\delta_1 T}.
\end{align}

(5.73)

We remark that in the left hand side of (5.73) we take \( L^2_{m+1} \) norm without weight.

**Proof.** For \( \kappa = 1 = 0 \) this is a consequence of Condition 3.1 and Lemma 2.5. Then using (5.66) we can prove the lemma by induction on \( \kappa \). Its version with \( T \) and \( T \rho \) derivatives included, is (6.10) proven in Appendix E.

Now we consider the linearization of (3.19) at \( \hat{u}^\rho_{i,T,1(\kappa - 1)} \)

\begin{equation}
D \hat{u}^\rho_{i,T,1(\kappa - 1)} \overline{\partial} : W^2_{m+1,\delta}((\Sigma_i, \partial \Sigma_i);(\hat{u}^\rho_{i,T,1(\kappa - 1)})^*TX, (\hat{u}^\rho_{i,T,1(\kappa - 1)})^*TL) \\
\rightarrow L^2_{m,\delta}(\Sigma_i; (\hat{u}^\rho_{i,T,1(\kappa - 1)})^*TX) \otimes \Lambda^{0,1}).
\end{equation}

(5.74)

We denote

\begin{equation}
\tag{5.75}
(\mathfrak{e}^\rho_{i,T,1(\kappa - 1)}) = \sum_{a=0}^{n-1} e^\rho_{i,T,1(a)}.
\end{equation}
Similarly as for the operator $D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(0)}$ in (5.19), we define the approximate linearization of (3.19)
\[ D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(k-1)} := D_{\bar{u}^e_{1,T,(0)}} \partial - (D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(0)}) (E_1)((se)^e_{1,T,(0)}), \]
(5.76)
by replacing $\bar{u}^e_{1,T,(0)}$ by $(se)^e_{1,T,(0)}$ in the expression of the linearization operator
\[ (D_{\bar{u}^e_{1,T,(0)}} \partial)^{-1} = D_{\bar{u}^e_{1,T,(0)}} \partial - (D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(0)}) (E_1)((\partial \bar{u}^e_{1,T,(0)}))^{-1}. \]

**Lemma 5.26.** Denote $E_1 = E_1((\bar{u}^e_{1,T,(0)}))$. Then $T_m, (5.77)$ such that if $T > T_m, (5.77)$ then we have
\[ \text{Im}(D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(k-1)} + E_1) = L^2_{m,\delta}(\Sigma_i; (\bar{u}^e_{1,T,(0)})^T X \otimes \Lambda^{0,1}). \]
(5.77)
Moreover the map
\[ \text{Dev}_1, \infty - \text{Dev}_2, \infty : (D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(k-1)})^{-1}(E_1) \oplus (D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(k-1)})^{-1}(E_2) \to T_{e^e_{1,T,(0)}} L \]
is surjective.

**Proof.** Using the inequality (5.69), we estimate
\[
\left\| e^e_{1,T,(0)} - \sum_{a=0}^{k-1} e^e_{1,T,(a)} \right\|_{L^2_{m,\delta}(K_i)} \leq \sum_{a=1}^{k-1} \| e^e_{1,T,(a)} \|_{L^2_{m,\delta}(K_i)} < C_3 \frac{\epsilon^{-\delta}}{1 - \mu}.
\]
(5.79)
This in particular implies
\[ Y \mapsto (D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(k-1)} (E_1)((se)^e_{1,T,(0)}), \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} (Y)) \]
\[ - \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} ((D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(0)}) \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} (Y)) \]
is small in the operator norm. We prove this fact in Appendix D. Then this smallness implies that of the operator norm of the difference operator
\[ D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(k-1)} \circ \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} - \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} \circ D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(0)} \circ \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} \]
is small. This can be easily seen by rewriting the value of the difference operator as
\[
(D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(k-1)} (\text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} (Y)) - \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} (D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(0)} (\text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} (Y))))
\]
\[ = (D_{\bar{u}^e_{1,T,(0)}} \partial) (\text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} (Y)) - \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} ((D_{\bar{u}^e_{1,T,(0)}} \partial) (\text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} (Y))))
\]
\[ - (D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(k-1)} (E_1)((se)^e_{1,T,(0)}), \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} (Y))
\]
\[ - \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} ((D_{\bar{u}^e_{1,T,(0)}}^{\text{app.}(0)} (E_1)((se)^e_{1,T,(0)}), \text{Pal}_{u^e_{1,T,(0)}} \bar{u}^e_{1,T,(0)} (Y)) \).
\]
The first summand of the right hand side clearly small by the smallness of $L^2_{m,\delta}$-norm of $E((\bar{u}^e_{1,T,(0)}), \bar{u}^e_{1,T,(0)})$ arising from the induction hypothesis (5.67). Then the lemma follows by combining Lemma 5.7 and the above mentioned smallness.

Note that Remark 5.8 still applies to Lemma 5.26.
Definition 5.27. We define a linear map
\[ \Phi_{\varepsilon(T,\rho)} : W^2_{m+1,\delta}(\Sigma_i, \partial \Sigma_i; u_i^T X, u_i^T TL) \]
\[ \rightarrow W^2_{m+1,\delta}(\Sigma_i, \partial \Sigma_i; (\hat{u}_{i,T,(\varepsilon-1)})^T X, (\hat{u}_{i,T,(\varepsilon-1)})^T TL), \]
as the closure of the map \( \text{Pal}_{\varepsilon,T,(\varepsilon-1)} \) in (2.7).

Lemma 5.28. (1) \( \Phi_{\varepsilon(T,\rho)} \) sends (5.25) to
\[ W^2_{m+1,\delta}(\Sigma_i, \partial \Sigma_i; (\hat{u}_{i,T,(\varepsilon-1)})^T X, (\hat{u}_{i,T,(\varepsilon-1)})^T TL). \] (5.81)
(2) The map \( \Phi_{\varepsilon(T,\rho)} \) sends the subspace (5.23) to the subspace
\[ \text{Ker}(\text{Dev}_{1,\infty} - \text{Dev}_{2,\infty}) \]
\[ \cap \bigoplus_{i=1}^2 W^2_{m+1,\delta}(\Sigma_i, \partial \Sigma_i; (\hat{u}_{i,T,(\varepsilon-1)})^T X, (\hat{u}_{i,T,(\varepsilon-1)})^T TL). \] (5.82)
The proof is easy and is omitted.

Definition 5.29. We denote by \( \delta_{\varepsilon(T,\rho)}(\varepsilon_1, \varepsilon_2; \rho, T) \) the image of the subspace \( \delta(\varepsilon_1, \varepsilon_2) \) (See Definition 5.9) by the map \( \Phi_{\varepsilon(T,\rho)} \). It is a subspace of (5.82).

Definition 5.30. We define \( (V^\rho_{T,1,\varepsilon}, V^\rho_{T,2,\varepsilon}, \Delta p^\rho_{T,\varepsilon}) \) by
\[ D^\rho_{\varepsilon,T,(\varepsilon-1)}(V^\rho_{T,1,\varepsilon}) + \text{Err}_{\varepsilon,T,(\varepsilon-1)} \in \varepsilon_i(\hat{u}_{i,T,(\varepsilon-1)}) \] (5.83) and
\[ ((V^\rho_{T,1,\varepsilon}, \Delta p^\rho_{T,\varepsilon}), (V^\rho_{T,2,\varepsilon}, \Delta p^\rho_{T,\varepsilon})) \in \delta_{\varepsilon(T,\rho)}(\varepsilon_1, \varepsilon_2; \rho, T). \] (5.84)
Lemma 5.26 implies that such \( (V^\rho_{T,1,\varepsilon}, V^\rho_{T,2,\varepsilon}, \Delta p^\rho_{T,\varepsilon}) \) exists and is unique if \( T > T_m \).

Lemma 5.31. There exist \( T_m \) and \( \varepsilon_{3,m} > 0 \) such that if \( \varepsilon(5) < \varepsilon_{3,m} \), then (5.68) for \( \kappa - 1 \) imply that the next inequality for \( T > T_m \) holds.
\[ \| (V^\rho_{T,1,\varepsilon}, \Delta p^\rho_{T,\varepsilon}) \|_2 W^2_{m+1,\delta}(\Sigma_i) \leq C_{2,m} e^{-d_1 T}. \] (5.85)
Proof. We take \( T_m \) large so that the inverse of
\[ \Pi_{\varepsilon,T,(\varepsilon-1)} \circ D^\rho_{\varepsilon,T,(\varepsilon-1)} \]
is uniformly bounded and choose \( \varepsilon_{3,m} \) small such that \( \varepsilon_{3,m}^{-1} \) is much larger than the norm of the above mentioned inverse. Then (5.85) follows from uniform boundedness of the above mentioned inverse together with (5.68) for \( \kappa - 1 \).

This lemma implies the inequality (5.66). We finished the proof of Proposition 5.23 (2).

Step \( \kappa \)-2: We start the proof of Proposition 5.23 (1).
We use \( (V^\rho_{T,1,\varepsilon}, V^\rho_{T,2,\varepsilon}, \Delta p^\rho_{T,\varepsilon}) \) to find an approximate solution \( u^\rho_{T,\varepsilon} \) of the next level. We remark that \( (V^\rho_{T,1,\varepsilon}, V^\rho_{T,2,\varepsilon}, \Delta p^\rho_{T,\varepsilon}) \) is the counterpart \( v_{\varepsilon} \) given by
\[ v_{\varepsilon} = \frac{f(x_{\varepsilon})}{f'(x_{\varepsilon})} \] and the next definition corresponds to the next iteration \( x_{\varepsilon+1} = x_{\varepsilon} + v_{\varepsilon} \) in Newton's scheme.
**Definition 5.32.** We define $u^\rho_{T,(\kappa)}(z)$ as follows.

1. If $z \in K_1$, we put
   
   $$u^\rho_{T,(\kappa)}(z) = \text{Exp}(\hat{u}^\rho_{T,(\kappa-1)}(z), V^\rho_{T,1,(\kappa)}(z)).$$  \hspace{1cm} (5.86)

2. If $z \in K_2$, we put
   
   $$u^\rho_{T,(\kappa)}(z) = \text{Exp}(\hat{u}^\rho_{2,T,(\kappa-1)}(z), V^\rho_{T,2,(\kappa)}(z)).$$  \hspace{1cm} (5.87)

3. If $z = (\tau, t) \in [-5T, 5T] \times [0, 1]$, we put
   
   $$u^\rho_{T,(\kappa)}(\tau, t) = \text{Exp} \left( u^\rho_{T,(\kappa-1)}(\tau, t), \chi^\rho_{T} (\tau, t) (V^\rho_{T,1,(\kappa)}(\tau, t) - (\Delta p^\rho_{T,(\kappa)})^\text{Pal}) \right)$$
   
   $$+ \chi^\rho_{T} (\tau, t) (V^\rho_{T,2,(\kappa)}(\tau, t) - (\Delta p^\rho_{T,(\kappa)})^\text{Pal}) + (\Delta p^\rho_{T,(\kappa)})^\text{Pal} \right).$$  \hspace{1cm} (5.88)

We note that $\hat{u}^\rho_{T,(\kappa-1)}(z) = u^\rho_{T,(\kappa-1)}(z)$ on $K_1$ and $u^\rho_{2,T,(\kappa-1)}(z) = u^\rho_{T,(\kappa-1)}(z)$ on $K_2$.

**Step $\kappa$-3:** The proof of the following proposition is largely a duplication of that of Proposition 5.17 plus (5.59) and so its details will be postponed till Appendix C.

**Proposition 5.33.** For any $\epsilon(5) > 0$, there exists $T_{m,\epsilon(5)}$, $C_{\epsilon(5)} > 0$ with the following properties. If $T > T_{m,\epsilon(5)}$, we can define $e^\rho_{T,(\kappa)} \in L_i$ that satisfies

$$\left\| \partial u^\rho_{T,(\kappa)} - \sum_{a=0}^\kappa e^\rho_{1,T,(a)} - \sum_{a=0}^\kappa e^\rho_{2,T,(a)} \right\|_{L^2_{m,\delta}(\Sigma_T)} < C_{1,m} \mu^\kappa \epsilon(5) e^{-\delta_1 T},$$

and

$$\left\| e^\rho_{T,(\kappa)} \right\|_{L^2_{m,\delta}(K_1)} < C_{3,m} \mu^{\kappa-1} e^{-\delta_1 T}.$$  \hspace{1cm} (5.89)

Note Remark 5.4 applies here.

**Step $\kappa$-4:** Similarly as before, we introduce the following definition.

**Definition 5.34.** We put

$$\text{Err}^\rho_{1,T,(\kappa)} = \chi^\rho_{X} \left( \partial u^\rho_{T,(\kappa)} - \sum_{a=0}^\kappa e^\rho_{1,T,(a)} \right),$$

$$\text{Err}^\rho_{2,T,(\kappa)} = \chi^\rho_{X} \left( \partial u^\rho_{T,(\kappa)} - \sum_{a=0}^\kappa e^\rho_{2,T,(a)} \right).$$

We regard them as elements of the weighted Sobolev spaces

$$L^2_{m,\delta}(\Sigma_i; (\hat{u}^\rho_{i,T,(\kappa)})^* TX \otimes \Lambda^{0,1})$$

for $i = 1, 2$ respectively. (We extend them by 0 outside a compact set.)

We put $p^\rho_{(\kappa-1)} = \text{Exp}(p^\rho_{(\kappa-1)}, \Delta p^\rho_{(\kappa)}).$ See (5.71). Then Proposition 5.33 implies (5.68) and (5.69).

We have thus finished the proof of Proposition 5.23 (1). The proof of Proposition 5.23 is complete.
We prepare some notations to state the result. The main result of this section is an estimate of $u_T$. We recall that for each positive constant $S > 0$, there exists $T > T_1$ such that if $T > T_1$, we defined the surface $\Sigma_T$ via the coordinates $(\tau, t)$ such that the relationship between the three coordinates $(\tau, t)$, $(\tau', t)$ and $(\tau'', t)$ are given by

$$\tau' = \tau + 5T$$

and

$$\tau'' = \tau - 5T.$$  

We may use either $(\tau', t)$ or $(\tau'', t)$ as the coordinate of $\Sigma_T \setminus (K_1 \cup K_2)$. Namely

$$\Sigma_T = K_1 \cup (0, 10T]_\tau \times [0, 1] \cup K_2 = K_1 \cup ([0, 0, \infty]_\tau \times [0, 1]) \cup K_2,$$

We introduce a new coordinate $(\tau', t)$ and $(\tau'', t)$ such that the relationship between the three coordinates $(\tau, t)$, $(\tau', t)$ and $(\tau'', t)$ are given by

and

$$\tau'' = \tau - 5T.$$

We may use either $(\tau', t)$ or $(\tau'', t)$ as the coordinate of $\Sigma_T \setminus (K_1 \cup K_2)$. Namely

$$\Sigma_T = K_1 \cup (0, 10T]_\tau \times [0, 1] \cup K_2 = K_1 \cup ([0, 0, \infty]_\tau \times [0, 1]) \cup K_2.$$

Remark 6.2. We remark that $(\tau', t)$ coordinate of $\Sigma_1$ and $(\tau'', t)$ coordinate of $\Sigma_2$ are independent of $T$ but $(\tau, t)$ coordinates of them are $T$ dependent.

We define

$$K_1^S = K_1 \cup ([0, S]_\tau \times [0, 1]),$$

$$K_2^S = ([S, \infty]_\tau \times [0, 1]) \cup K_2,$$

for each positive constant $S > 0$. We have a natural embedding $K_1^S \hookrightarrow \Sigma_1$ (resp. $K_2^S \hookrightarrow \Sigma_2$) via the coordinates $\tau'$ (resp. via the coordinates $\tau''$).

If $S > 0$, $T > \frac{S}{10}$, we have embeddings of $K_1^S \to \Sigma_T$ by setting $\tau = \tau' - 5T$ and $K_2^S \to \Sigma_T$ by $\tau = \tau'' + 5T$.

6. Exponential decay of $T$ derivatives

We first state the result of this section which is also the main result of this paper. We recall that for $T$ sufficiently large and $\rho = (\rho_1, \rho_2) \in V_1 \times LV_2$ we have defined $u_{T,(\kappa)}$ for each $\kappa = 0, 1, 2, \ldots$. We denote its limit by

$$u_{T} = \lim_{\kappa \to \infty} u_{T,(\kappa)} : (\Sigma_T, \partial \Sigma_T) \to (X, L).$$

The main result of this section is an estimate of $T$ and $\rho$ derivatives of this map. We prepare some notations to state the result.

For $T > 0$, we defined the surface $\Sigma_T$ as the union

$$\Sigma_T = K_1 \cup ([-5T, 5T]_\tau \times [0, 1]) \cup K_2$$

by identifying $\partial K_1 \cong [-5T] \times [0, 1]$ and $\partial K_2 \cong [5T] \times [0, 1]$ where we denote by $(\tau, t)$ the coordinates on $[-5T, 5T]_\tau \times [0, 1]$. Therefore we have the natural inclusion $K_1 \subset \Sigma_T$.

We introduce a new coordinate $(\tau', t)$ and $(\tau'', t)$ such that the relationship between the three coordinates $(\tau, t)$, $(\tau', t)$ and $(\tau'', t)$ are given by

$$\tau' = \tau + 5T$$

and

$$\tau'' = \tau - 5T.$$
We then obtain a map
\[ \text{Glures}_{i,S} : [T_{\max}, \infty) \times V_1 \times L \to \text{Map}_{L^2_{m+1}} ((K_i^S, K_i^S \cap \partial \Sigma_i), (X, L)) \]
by
\[
\begin{align*}
\text{Glures}_{1,S}(T, \rho)(x) &= u^0_T(x) \quad x \in K_1 \\
\text{Glures}_{1,S}(T, \rho)(\tau', t) &= u^0_T(\tau', t) \quad (= u^0_T(\tau + 5T, t)) \\
\text{Glures}_{2,S}(T, \rho)(x) &= u^0_T(x) \quad x \in K_2 \\
\text{Glures}_{2,S}(T, \rho)(\tau'', t) &= u^0_T(\tau'', t) \quad (= u^0_T(\tau - 5T, t)).
\end{align*}
\] (6.5)
\[
\begin{align*}
\text{Glures}_{1,S}(T, \rho)(x) &= u^0_T(x) \quad x \in K_1 \\
\text{Glures}_{1,S}(T, \rho)(\tau', t) &= u^0_T(\tau', t) \quad (= u^0_T(\tau + 5T, t)) \\
\text{Glures}_{2,S}(T, \rho)(x) &= u^0_T(x) \quad x \in K_2 \\
\text{Glures}_{2,S}(T, \rho)(\tau'', t) &= u^0_T(\tau'', t) \quad (= u^0_T(\tau - 5T, t)).
\end{align*}
\] (6.6)

Here the map ‘Glures’ stands for the phrase ‘gluing followed by restriction’, and \( \text{Map}_{L^2_{m+1}} ((K_i^S, K_i^S \cap \partial \Sigma_i), (X, L)) \) is the space of maps of \( L^2_{m+1} \) class \( (m \) is sufficiently large, say \( m > 10. \) \) It has a structure of Hilbert manifold in an obvious way. This Hilbert manifold is independent of \( T. \) So we can define \( T \) derivative of a family of elements of \( \text{Map}_{L^2_{m+1}} ((K_i^S, K_i^S \cap \partial \Sigma_i), (X, L)) \) parameterized by \( T. \)

**Remark 6.3.** The domain and the target of the map \( \text{Glures}_{i,S} \) depend on \( m. \) However its image actually lies in the set of smooth maps. Also none of the constructions of \( u^0_T \) depend on \( m. \) (The proof of the convergence of \( 6.1 \) depends on \( m. \) So the numbers \( T_{3, m, \epsilon(5)}, T_{4, m, \epsilon(6)} \) in Proposition 5.23 (and \( T_{5, m, \epsilon(6)}, T_{6, m, \epsilon(6)} \) in Proposition 6.9) depend on \( m. \) ) The map \( \text{Glures}_{i,S} \) is independent of \( m \) on the intersection of the domains. Namely the map \( \text{Glures}_{i,S} \) constructed by using \( L^2_{m_1} \) norm coincides with the map \( \text{Glures}_{i,S} \) constructed by using the \( L^2_{m_2} \) norm on \( \max \{ T_{3, m_1, \epsilon(5)}, T_{4, m_1, \epsilon(6)}, T_{5, m_1}, T_{6, m_1}, T_{3, m_2, \epsilon(5)}, T_{4, m_2, \epsilon(6)} \} \times V_1 \times L \)

\[
\text{Map}_{L^2_{m+1}} ((K_i^S, K_i^S \cap \partial \Sigma_i), (X, L)).
\]

\[5\text{We can also use } C^k((K_i^S, K_i^S \cap \partial \Sigma_i), (X, L)), \text{ the Banach manifold of } C^k \text{ maps, in place of } \text{Map}_{L^2_{m+1}} ((K_i^S, K_i^S \cap \partial \Sigma_i), (X, L)).\]
Theorem 6.4. For each given $m$ and $S$, there exist $T_{m,S}$ such that
\[ \left\| \nabla^n_{\rho} \frac{d^T}{dT} \text{Glures}_{i,S} \right\|_{L^2_{m+1-\ell}} < C_{m,S} e^{-\delta_1 T} \quad (6.7) \]
holds for all $T > T_{m,S}$ and for $(n, \ell)$ with $m - 2 \geq n, \ell \geq 0$ and $\ell > 0$. Here $\nabla^n_{\rho}$ is the $n$-th derivative in the $\rho$-direction.

Remark 6.5. Theorem 6.4 is equivalent to [FOOO1, Lemma A1.58]. The proof below uses the same inductive scheme as the one in [FOOO1, page 776]. We use $L^2_m$ norm in place of $L^2_L$ norm and we add thorough detail.

Remark 6.6. We remark that the map
\[ \mathcal{M}^{E_1 \oplus E_2}(\Sigma_T, \bar{z})_0; u_1, u_2) \rightarrow \prod_{i=1}^{2} \text{Map}_{L^2_{m+1}}((K_i^S, K_i^T \cap \partial \Sigma_i), (X, L)), \]
which is obtained by restricting the domain, is an embedding for each fixed $T$. This is a consequence of unique continuation. We note that the target of the map does not depend on $T$. Therefore we can use Theorem 6.4 to study $T$-dependence of the moduli space $\mathcal{M}^{E_1 \oplus E_2}(\Sigma_T, \bar{z})_0; \beta)$. We used this fact and Theorem 6.4 to show smoothness of the coordinate change of the Kuranishi structure on the moduli space of bordered pseudoholomorphic curves in [FOOO1, II, Appendix A, page 764-773]. See Section 8 for more details.

The remaining section will be occupied by the proof of this theorem.

The construction of $u^p_{T,(\kappa)}$ was given by induction on $\kappa$. We divide the inductive step of the construction of $u^p_{T,(\kappa)}$ from $u^p_{T,(\kappa-1)}$ into two:

(Part A) Start from $(V_{T,1,(\kappa)}, V_{T,2,(\kappa)}, \Delta p^p_{T,(\kappa)})$ and end with $\text{Err}_{1,T,(\kappa)}$ and $\text{Err}_{2,T,(\kappa)}$. This is the step of the error estimates for the $\kappa$-th iteration map $u^p_{T,(\kappa)}$, which is essentially a computational step.

(Part B) Start from $\text{Err}_{1,T,(\kappa-1)}$ and $\text{Err}_{2,T,(\kappa-1)}$ and end with $(V_{T,1,(\kappa)}, V_{T,2,(\kappa)}, \Delta p^p_{T,(\kappa)})$. This is the step of the error estimates for the $\kappa$-th iteration map $u^p_{T,(\kappa)}$. We will prove the following inequalities inductively over $\kappa \geq 0$,
\[ \left\| \nabla^n_{\rho} \frac{\partial^T}{\partial T} (V_{T,i,(\kappa)}, \Delta p^p_{T,(\kappa)}) \right\|_{W^2_{m+1-\ell, \beta}(\Sigma_i)} < C_{5,m} \mu^{\kappa-1} e^{-\delta_1 T}, \quad (6.8) \]
\[ \left\| \nabla^n_{\rho} \frac{\partial^T}{\partial T} (E(u_i, u^p_{T,(\kappa)}), E(p_0, p^p_{T,(\kappa)})) \right\|_{W^2_{m+1-\ell, \beta}(K_i^{T+1} \subset \Sigma_i)} < C_{6,m} (2 - \mu^\kappa) e^{-\delta_1 T}, \quad (6.9) \]
\[ \left\| \nabla^n_{\rho} \frac{\partial^T}{\partial T} (p^p_{T,(\kappa)}, \tilde{u}^p_{T,(\kappa)}) \right\|_{L^2_{m+1-\ell}(K_i^{T+1} \setminus K_i^S)} < C_{7,m} (2 - \mu^\kappa) e^{-\delta_1 T}, \quad (6.10) \]
\[ \left\| \nabla^n_{\rho} \frac{\partial^T}{\partial T} \text{Err}_{T,(\kappa)} \right\|_{L^2_{m+1-\ell, \beta}(\Sigma_i)} < C_{8,m} \mu(6(\kappa) \mu^\kappa) e^{-\delta_1 T}, \quad (6.11) \]
\[ \left\| \nabla^n_{\rho} \frac{\partial^T}{\partial T} e^p_{T,(\kappa)} \right\|_{L^2_{m+1-\ell}(K_i^{T+1})} < C_{9,m} \mu^{\kappa-1} e^{-\delta_1 T}. \quad (6.12) \]
Here $0 \leq \ell, n \leq m - 2$. In Formula (6.10) we assume $\ell > 0$ in addition.
Remark 6.7. Note we use the $T$-independent coordinates $(\tau', t)$ on $K_1^{5T+1} \setminus K_1$, $K'^{5T} \setminus K_1^T$ and $(\tau'', t)$ on $K_2^{5T+1} \setminus K_2$, $K'^{5T} \setminus K_2^T$.

In (6.11) we use $\tau'$ coordinate for $i = 1$ and $\tau''$ coordinate for $i = 2$.

We also remark that $\Sigma_T = K_1^{5T+1} \cup K_2^{5T+1}$. (See the definition of $\Sigma_T$ at the beginning of this section and (6.4).)

Recall from Definition 5.30 that the pair $(V_{T,i,(\kappa)}^\rho, \Delta p_{T,(\kappa)}^\rho)$ appearing in (6.8) is an element of the weighted Sobolev space $W_{m+1, \delta}^2((\Sigma_i ; \partial \Sigma_i); (\tilde{u}_{i,T,(\kappa-1)}^\rho)^* TX, (\tilde{u}_{i,T,(\kappa-1)}^\rho)^* TL)$ which depends on $T$ and $\rho$. We use the inverse of $\Phi_{i,(\kappa-1)}(\rho, T)$ (Definition 5.27), to identify

$$W_{m+1, \delta}^2((\Sigma_i ; \partial \Sigma_i); (\tilde{u}_{i,T,(\kappa-1)}^\rho)^* TX, (\tilde{u}_{i,T,(\kappa-1)}^\rho)^* TL) \cong W_{m+1, \delta}^2((\Sigma_i, \partial \Sigma_i); u_i^* TX, u_i^* TL).$$

Namely we put

$$\tilde{u}_{i,T,(\kappa-1)}^\rho(\rho, T; (s, v)) = \left( (\Phi_{i,(\kappa-1)}(\rho, T)^{-1}(s), (\tilde{p}_{\rho_{i-1}}(\rho))^{-1}(v) \right).$$

(6.13)

Using $\tilde{u}_{i,T,(\kappa-1)}^\rho(\rho, T; \cdot)$, the formula (6.8) is:

$$\left\| \nabla_h^h \frac{\partial}{\partial T} \left( \tilde{u}_{i,T,(\kappa-1)}^\rho(\rho, T; (V_{T,i,(\kappa)}^\rho, \Delta p_{T,(\kappa)}^\rho)) \right) \right\|_{W_{m+1-\ell, \delta}^2(\Sigma_i)} \leq C_{5,m} \kappa^{-1} e^{-\delta_i T}.$$  

(6.14)

We use $\tau'$ coordinate in case $i = 1$.

We can make sense of (6.9) and (6.11) in the same way as (6.14). We use the isomorphism

$$L^2_{m, \delta}(\Sigma_i; (\tilde{u}_{i,T,(\kappa-1)}^\rho)^* TX \otimes \Lambda^{0,1}) \cong L^2_{m, \delta}(\Sigma_i; u_i^* TX \otimes \Lambda^{0,1}),$$

which is the closure of

$$\left( (\tilde{p}_{\rho_{i-1}}(\rho))^{0,1} \right)^{-1}.$$  

(6.15)

where $(\tilde{p}_{\rho_{i-1}}(\rho))^{0,1}$ is as in (2.7), to formulate (6.10) and (6.12).

A similar remark applies to (6.10) and (6.12). (6.10) means

$$\left\| \nabla_h^h \frac{\partial}{\partial T} \tilde{p}_{\rho_{i-1}}(\rho)(E(\tilde{p}_{\rho_{i-1}}^\rho, \tilde{u}_{i,T,(\kappa)}^\rho)) \right\|_{L^2_{m+1-\ell, \delta}(K_i^{5T} \setminus K_i^T)} \leq C_{7,m} (2 - \mu \kappa) e^{-\delta_i T}.$$  

(6.16)

and in (6.12) we regard $e_{i,T,(\kappa)}^\rho \in E_i^\rho$.

In this way we can safely work with (6.8)-(6.12).

Remark 6.8. Similar remarks also apply to the case $i = 2$ using the $\tau''$ coordinate.

The inductive proof of (6.8)-(6.12) is written as the proof of the next proposition.

Proposition 6.9. We can choose $C_{5,m}, C_{8,m}, C_{9,m}$ so that the following holds.

1. For any $\epsilon(6) > 0$ and $C_{6,m}, C_{7,m}$, there exists $T_{5,m,\epsilon(6)} > 0$ such that (6.8), (6.9) and (6.10) for $\leq \kappa$ imply (6.11) and (6.12) for $\leq \kappa$, if $T > T_{5,m,\epsilon(6)}$.

2. For any $C_{6,m}, C_{7,m}$, we can choose $\epsilon_{4,m}$ such that if $\epsilon(6) < \epsilon_{4,m}$ then (6.8), (6.12) for $\leq \kappa - 1$ imply (6.8) for $\kappa$. $\epsilon_{4,m}$ may depend on $C_{5,m}, C_{6,m}, C_{7,m}, C_{8,m}, C_{9,m}$. 


(3) We can choose $C_{6,m}$, $C_{7,m}$ with the following properties. There exists $T_{6,m}$ such that the inequalities \[6.9\], \[6.10\] follow from \[6.8\] for $\leq \kappa$ and \[6.9\], \[6.10\] for $\leq \kappa - 1$ if $T > T_{6,m}$. Here $T_{6,m}$ may depend on $C_{5,m}$, $C_{6,m}$, $C_{7,m}$, $C_{8,m}$, $C_{9,m}$.

The rest of this section will be occupied by the proof of Proposition \[6.9\] (3) is elementary. We provide its proof in Appendix E for completeness’ sake. We will prove (1) and (2) in this section. We divide our proof into two parts, the proof of (1) and (2).

Remark 6.10. We choose the constants $C_{5,m}$, $C_{8,m}$, $C_{9,m}$ so that \[6.8\], \[6.11\] and \[6.12\] hold for $\kappa = 0, 1$. We do not need to change them in later steps. The constants $C_{6,m}$, $C_{7,m}$ are chosen during the proof of Proposition \[6.9\] (3) given in Appendix E.

We also have:

**Lemma 6.11.** The inequalities \[6.8\], \[6.9\], \[6.10\], \[6.11\] and \[6.12\] for $\leq \kappa$ imply

$$\left\| \nabla^{p}_{\rho} E(u_{i}, u_{T,(\kappa)}) \right\|_{W^{2}_{m+1,4}(K_{\text{int}} \subset \Sigma)} < C_{m,6.17}$$

(6.17)

for $0 \leq n \leq m - 2$.

**Proof.** Note \[6.8\] implies

$$\lim_{T \to \infty} u_{T,(\kappa)}^{p}_{i} \big|_{\Sigma_{i}} = u_{i}^{p} \big|_{\Sigma_{i}}$$

Therefore using

$$\left\| \nabla^{p}_{\rho} E(u_{i}, u_{T,(\kappa)}) \right\|_{W^{2}_{m+1,4}(K_{\text{int}} \subset \Sigma)} < C_{m,6.17}$$

(6.18)

the lemma can be proved by integrating the $\ell = 1$ case of \[6.9\] on $T \in [T_{0}, \infty]$. \(\square\)

6.1. Part A: error estimates. In this subsection we prove Proposition \[6.9\] (1). This subsection corresponds to the discussion in [FOOO1, page 776 paragraph (A) and (B)].

Suppose that the triple $(V_{T,1,(\kappa)}^{p}, V_{T,2,(\kappa)}^{p}, \Delta p_{T,(\kappa)}^{p})$ satisfies \[6.8\]. Then noting that $\text{supp} \, \epsilon_{i,(\kappa)} \subset \text{Int} K_{i}$, we find that

(1) \begin{equation*}
\text{Err}_{1,T,(\kappa)}^{p}(z) = \Pi_{i}^{1}(\tilde{a}_{i,T,(\kappa-1)}^{p}) \mathcal{B} \left( \text{Exp} \left( \tilde{a}_{1,T,(\kappa-1)}^{p}(z), V_{T,1,(\kappa)}^{p}(z) \right) \right)
\end{equation*}

(6.19)

for $z \in K_{1}$.

(2) \begin{equation*}
\text{Err}_{1,T,(\kappa)}^{p}(\tau', t)
= (1 - \chi(\tau' - 5T)) \times 
\mathcal{B} \left( \text{Exp} \left( u_{T,(\kappa-1)}^{p}(\tau', t), \chi(\tau' - 4T) \left( V_{T,2,(\kappa)}^{p}(\tau' - 10T, t) - (\Delta p_{T,(\kappa)}^{p})^{\text{Pal}}(\tau', t) \right) + V_{T,1,(\kappa)}^{p}(\tau', t) \right) \right)
\end{equation*}

(6.20)

for $(\tau', t) \in [0, \infty)_{\tau} \times [0, 1]$. (Recall $\tau = \tau' - 5T$, $\tau' = \tau'' + 10T$ and $V_{T,2,(\kappa)}^{p}$ is defined in terms of the variable of $(\tau'', t)$.) See Figure 8.
Here $\chi : \mathbb{R} \to [0, 1]$ is a smooth function such that

$$\chi(\tau) = \begin{cases} 
0 & \tau < -1 \\
1 & \tau > 1 
\end{cases}$$

$$\chi'(\tau) > 0 \quad \text{for} \quad \tau \in (-1, 1).$$

(6.21)

The same kind of statements also hold for $i = 2$. Since the latter case can be dealt very much exactly in the same way, we omit its details.

**Remark 6.12.** Note that in Formulae (6.8)-(6.12) the Sobolev norms in the left hand side are $W_{m+1,\ell,\delta}(\Sigma_i)$ etc. and are not $W_{m+1,\ell,\delta}(\Sigma_i)$ etc. The origin of this loss of differentiability (in the sense of Sobolev space) comes from the term $V_{\tau,2,(\kappa)}^p(\tau' - 10T)$. In fact, we have

$$\frac{\partial}{\partial T} V_{\tau,2,(\kappa)}^p(\tau' - 10T) = -10 \frac{\partial V_{\tau,2,(\kappa)}^p}{\partial \tau''}(\tau' - 10T)$$

for a fixed $T_1$. Hence $\partial / \partial T$ is continuous as $L_{m+1}^2 \to L_m^2$. We remark in (6.8) for $i = 2$ we use the coordinate $(\tau'', t)$ on $(-\infty, 0] \times [0, 1]$ to define $T$ derivative of $V_{\tau,2,(\kappa)}^p$.

Taking this remark into account, we continue with the proof. The proof is divided into three parts.
Lemma 6.13. For any $\epsilon(7) > 0$, there exists $T_{m,\epsilon(7)}(\ref{6.23}) > 0$ with the following properties. If $T > T_{m,\epsilon(7)}(\ref{6.23})$, then the element $e_{i,T,(\kappa)} \in \mathcal{E}_{i}$ in Proposition \ref{5.33} satisfies the following for $0 \leq \ell, n \leq m - 2$.

\[
\left\| \nabla_{\rho} \frac{d^\ell}{dt^\ell} \left( (\text{Pal}_{u_i})(0,1) \right)^{-1} \left( \delta u_{T,(\kappa)} - \sum_{a=0}^{\kappa} e_{1,T,(\kappa)} - \sum_{a=0}^{\kappa} e_{2,T,(\kappa)} \right) \right\|_{L^2_{m-\ell-1}(K^{4T-1})} \leq C_{5,m} \mu^\epsilon \epsilon(7)^{e^{-\delta_1 T}}.
\]

and

\[
\left\| \nabla_{\rho} \frac{d^\ell}{dt^\ell} (e_{i,T,(\kappa)}) \right\|_{L^2_{m-\ell-1}(K^{4T-1})} \leq C_{5,m} \frac{\mu^\epsilon}{10} \epsilon(7)^{e^{-\delta_1 T}}.
\]

We provide its proof in Appendix \ref{C} for completeness.

We next study the neck region. The point explained in Remark \ref{6.12} appears here. We will do the corresponding estimates for $\text{Err}^\rho_{1,T,(\kappa)}$ given in \ref{6.20} by considering them for $\tau' \in [4T + 1, \infty)_{\tau'}$ and for $\tau' \in [4T - 1, 4T + 1]_{\tau'}$, separately.

(\textbf{Estimate 2}): We first consider the domain $\tau' \in [4T + 1, \infty)_{\tau'}$. (Note this domain contain $\mathbb{X}$.) There the formula \ref{6.20} is reduced to

\[
\text{Err}^\rho_{1,T,(\kappa)}(\tau',t) = (1 - \chi(\tau' - 5T)) \times \mathcal{D}\left( \text{Exp} \left( u^\rho_{T,(\kappa-1)}(\tau',t), 
\left( V^{\rho}_{T,2,(\kappa)}(\tau' - 10T,t) - (\Delta p^{i}_{T,(\kappa)})^{\text{Pal}} + V^{\rho}_{T,1,(\kappa)}(\tau',t) \right) \right) \right)
\]

because $\chi(\tau' - 4T) = 1$ on $[4T + 1, \infty)$ as $\tau' - 4T \geq 1$.

We remark that $\text{Err}^\rho_{1,T,(\kappa)}(\tau',t) = 0$ on $\tau' > 5T + 1$ since $1 - \chi(\tau' - 5T) = 0$ there. So we need to study only on $\tau' \in [4T + 1, 5T + 1]_{\tau'}$.

We apply \ref{5.40} to the function

\[
g(s) = \mathcal{P}^{-1} \mathcal{D}\left( \text{Exp} \left( u^\rho_{T,(\kappa-1)}(\tau',t), s \left( V^{\rho}_{T,2,(\kappa)}(\tau' - 10T,t) - (\Delta p^{i}_{T,(\kappa)})^{\text{Pal}} + V^{\rho}_{T,1,(\kappa)}(\tau',t) \right) \right) \right)
\]

where $\mathcal{P}$ is induced from the parallel transport along the curve

\[
r \mapsto \text{Exp} \left( u^\rho_{T,(\kappa-1)}(\tau',t), r \left( V^{\rho}_{T,2,(\kappa)}(\tau' - 10T,t) - (\Delta p^{i}_{T,(\kappa)})^{\text{Pal}} + V^{\rho}_{T,1,(\kappa)}(\tau',t) \right) \right)
\]

$r \in [0,s]$. We remark that

\[
g(0) = \mathcal{D}u^\rho_{T,(\kappa-1)}(\tau',t),
\]

\[
g'(0) = (\mathcal{D}u^\rho_{T,(\kappa-1)})(\text{Exp} \left( V^{\rho}_{T,2,(\kappa)}(\tau' - 10T,t) - (\Delta p^{i}_{T,(\kappa)})^{\text{Pal}} + V^{\rho}_{T,1,(\kappa)}(\tau',t) \right) \right)
\]

\footnote{The notation $[0,4T - 1]_{\tau'}$ is introduced in Remark \ref{6.1}}
Then we find that 
\[
\frac{1}{(1 - \chi(t' - 5T))} \mathcal{P}^{-1}(\text{Err}_{1,T,\kappa}^p(t', t))
\]
is equal to 
\[
\overline{\partial}u_{T,(\kappa-1)}^p(t', t) + (Du_{T,(\kappa-1)}^p) \overline{\partial}
\frac{(V_{T,2,\kappa}^p(t') - 10T, t) - (\Delta p_{T,\kappa}^p)^{\text{Pal}} + V_{T,1,\kappa}^p(t', t)}{	ext{Exp} \left( u_{T,(\kappa-1)}^p(t', t) \right),
\int_0^1 \frac{d^2}{dt^2} \mathcal{P}^{-1}(\overline{\partial}(\text{Exp} \left( u_{T,(\kappa-1)}^p(t', t) \right),
\left( V_{T,2,\kappa}^p(t') - 10T, t \right) - (\Delta p_{T,\kappa}^p)^{\text{Pal}} + V_{T,1,\kappa}^p(t', t)) \right) dt.
\]
(Note that we are away from the support of \(E_i(u_{T,(\kappa-1)}^p)\).

We denote the parallel transport along the shortest path by 
\[
\mathcal{P} = \left( \text{Pal}_{u_{T,(\kappa-1)}}^{u_{T,\kappa}^p} \right)^{(0,1)}
\]
on the domain \([0, 5T]_{t', \bar{t}} \times [0, 1] \).

**Lemma 6.14.** For any \(\epsilon(8) > 0\) there exists \(T_m,\epsilon(8), [4T + 1, 5T + 1]\) such that 
\[
\left\| \nabla_{\rho} \frac{d\tilde{\xi}}{dT^\rho} \mathcal{P}^{-1} \int_0^1 ds \int_0^s \left( \frac{d^2}{dt^2} \mathcal{P}^{-1}(\overline{\partial}(\text{Exp} \left( u_{T,(\kappa-1)}^p(t', t) \right),
\left( V_{T,2,\kappa}^p(t') - 10T, t \right) - (\Delta p_{T,\kappa}^p)^{\text{Pal}} + V_{T,1,\kappa}^p(t', t)) \right) dt \right\|_{L^2_{m-\ell,\delta}([4T + 1, 5T + 1] \times [0, 1])}
\]
\[
\leq \mu\epsilon(8)e^{-\delta_1T}
\]
holds if \(T > T_m,\epsilon(8)\).

**Proof.** We put 
\[
V(t', t) = (\mathcal{P}')^{-1} \left( (V_{T,2,\kappa}^p(t') - 10T, t) - (\Delta p_{T,\kappa}^p)^{\text{Pal}} + V_{T,1,\kappa}^p(t', t) \right),
\]
where 
\[
\mathcal{P}' = \text{Pal}_{u_{T,(\kappa-1)}}^{u_{T,\kappa}^p}.
\]
(Note on the domain \([4T + 1, 5T + 1]_{t', \bar{t}} \times [0, 1] \), we have \(u_{T,(\kappa-1)}^p = u_{T,\kappa}^p\).) So we may regard \(V_{T,2,\kappa}^p\) as a section of \((u_{T,(\kappa-1)}^p)^{T,\bar{t}}X)\).

We take \(L_{m-\ell}^2\) norm in place of \(L_{m-\ell,\delta}^2\) norm of the left hand side of (6.26). Then in the same way as in the proof of Lemma 6.13 given in Appendix G we can estimate the norm by 
\[
C_m,\epsilon(8) \sum_{\ell_1 + \ell_2 \leq \ell} \left\| \nabla_{\rho} \frac{d\tilde{\xi}}{dT^\rho} V \right\|_{L^2_{m+1-\ell_1}([4T + 1, 5T + 1] \times [0, 1])}
\times \left\| \nabla_{\rho} \frac{d\tilde{\xi}}{dT^\rho} V \right\|_{L^2_{m+1-\ell_2}([4T + 1, 5T + 1] \times [0, 1])}.
\]

Then applying the formula
\[
\frac{\partial^\ell}{\partial T^\ell} (\mathfrak{A}^{-1} (V_{T,2,\kappa}^\rho(\tau^T - 10T)))_{|T=T_2} = \sum_{\ell_1 + \ell_2 = \ell} (-1)^{\ell_2} \frac{\partial^{\ell_1}}{\partial T^{\ell_1}} \frac{\partial^{\ell_2}}{\partial T^{\ell_2}} (\mathfrak{A}^{-1} (V_{T,2,\kappa}^\rho)) (\tau^T - 10T),
\]
(6.28)
we obtain:
\[
\left\| \nabla_{\rho} \frac{d^{\ell_i}}{dT^{\ell_i}} V \right\|_{L^2_{m+1-\ell_i}([4T+1.5T+1],r \times [0,1])}
\leq C_m \left( \mathfrak{A}^{-1} (V_{T,1,\kappa}^\rho) \right)_{|T=T_2} \left\| \nabla_{\rho} \frac{d^{\ell_i}}{dT^{\ell_i}} (\mathfrak{A}^{-1}) (V_{T,2,\kappa}^\rho) \right\|_{L^2_{m+1-\ell_i}([4T+1.5T+1],r \times [0,1])}
+ C_m \left( \mathfrak{A}^{-1} (V_{T,2,\kappa}^\rho) \right)_{|T=T_2} \left\| \nabla_{\rho} \frac{d^{\ell_i}}{dT^{\ell_i}} (\mathfrak{A}^{-1}) (\Delta p_{T,\kappa}^\rho)_{\text{Par}} \right\|_{L^2_{m+1-\ell_i}([4T+1.5T+1],r \times [0,1])}
\leq C_m \mu^{-1} e^{-\delta_1 T}
\]
(6.29)
Note the weight function on our domain $[4T+1.5T+1] \times [0,1]$ is not greater than $10e^{5T \theta}$. Therefore substituting (6.29) into (6.27), we obtain
\[
\text{LHS of (6.26)} \leq C_m T_m e^{5T \delta} e^{-2T \delta_1 \mu^{2\kappa-2}} \leq \mu^\delta e(8) e^{-\delta_1 T}
\]
(6.30)
by taking $T_m e(8), 6.26$ so that $C_m e^{-5T_m e(8)} e^{-\delta} \mu^{2\kappa-2} \leq e(8)$. Here we also used (3.11).

Using (6.9), (6.11) we obtain
\[
\left\| \nabla_{\rho} \frac{d^{\ell}}{dT^{\ell}} (\mathfrak{A}^{-1} (D_{u_{\ell, \kappa}})) (\Delta p_{T,\kappa}^\rho)_{\text{Par}} \right\|_{L^2_{m+1-\ell}([T+9T],r \times [0,1])}
\leq \sum_{\ell' \leq \ell, n \leq N} C_m T \text{e}^{-\delta T} \left\| \nabla_{\rho} \frac{d^{\ell'}}{dT^{\ell'}} \left( \Delta p_{T,\kappa}^\rho \right)_{\text{Par}} \right\|_{L^2_{m-\ell'}([T+9T],r \times [0,1])}
\leq C_m T \text{e}^{-\delta T}
\]
(6.31)
To show this inequality for $\ell' = \ell$ we use the fact that $(\Delta p_{T,\kappa}^\rho)_{\text{Par}}$ is almost a ‘constant’ and its first derivative is small. See the last part of Section G.

We use Lemma 6.14 and (6.31) to show
\[
\left\| \nabla_{\rho} \frac{d^{\ell}}{dT^{\ell}} (\mathfrak{A}^{-1} (D_{u_{\ell, \kappa}} - \bar{D}_{u_{\ell, \kappa}})) \right\|_{L^2_{m+1-\ell}([4T+1.5T+1],r \times [0,1])}
\leq 2\mu^\delta e(8) e^{-\delta_1 T}
\]
(6.32)
for $T > T_m e(8)$. 6.33.
By (5.70) we have
\[ \mathcal{E}u_{i,T,(\kappa-1)}(\tau', t) = \text{Err}_{i,T,(\kappa-1)}^0(\tau', t) + \text{Err}_{i,T,(\kappa-1)}^1(\tau', t) \]
outside the support of \( \mathcal{E}_1, \mathcal{E}_2 \), which contains the subset \([4T + 1, 5T + 1]_{\tau'} \times [0, 1]\) we are studying.

Moreover by (5.83) in Definition 5.30 and Definition 5.34, we have
\[ -(D_{\tau'} u_{i,T,(\kappa-1)}) \mathcal{F}(V_{T,i,(\kappa)}) = \text{Err}_{i,T,(\kappa-1)}^0(\tau', t) \]
for \( i = 1, 2 \), on \([4T + 1, 5T + 1]_{\tau'} \times [0, 1]\), which lies outside the support of \( \mathcal{E}_1, \mathcal{E}_2 \).

Note \( u_{i,T,(\kappa-1)} = u_{i,T,(\kappa)} \) on the domain \([4T + 1, 5T + 1]_{\tau'} \times [0, 1] \), since \( \chi(\tau - T) = 0 \) therein. (See (5.72).) Therefore (6.32) implies
\[ \left\| \nabla_\rho \frac{d}{d\tau} \mathcal{F}^{-1} \text{Err}_{i,T,(\kappa)}(\tau', t) \right\|_{L^2_{m-\ell, \delta}([4T + 1, 5T + 1]_{\tau'} \times [0, 1])} \leq C_{m, 6.33} \mu^\epsilon(8)e^{-\delta T}. \quad (6.33) \]

(\textbf{Estimate 3): We next consider } \tau' \in [4T - 1, 4T + 1]_{\tau'} \text{. In other words we study the estimate on the domain } \mathcal{A}_T. \text{ There the formula (6.20) is reduced to}
\[ \text{Err}_{i,T,(\kappa)}(\tau', t) = \mathcal{F}^{-1} \left( \text{Exp} \left( u_{i,T,(\kappa-1)}^0(\tau', t), V_{T,1,(\kappa)}^0(\tau', t) + \chi(\tau' - 4T) \left(V_{T,2,(\kappa)}^0(\tau' - 10T, t) - (\Delta p_{T,(\kappa)})^{\text{Pal}} \right) \right) \right) \quad (6.34) \]
\[ \text{since } 1 - \chi(\tau' - 5T) \equiv 1 \text{ as } \tau' - 5T \leq -T + 1 < -1. \] (Here \( \mathcal{F} \) is parallel transport along the minimal geodesic which sends a section of \((u')^*TX\) to a section of \((u')^*TX\) for an appropriate \( u' \).

The next lemma claims that the term containing \( V_{T,2,(\kappa)}^0(\tau' - 10T, t) \) in (6.34) is small.

\textbf{Lemma 6.15.} For any positive number \( \epsilon(9) \), there exists \( T_{m,\epsilon(9)} \) such that the next inequality holds for \( T > T_{m,\epsilon(9)} \).
\[ \left\| \nabla_\rho \frac{d}{d\tau} \mathcal{F}^{-1} \left( \text{Exp} \left( u_{i,T,(\kappa-1)}^0(\tau', t), V_{T,1,(\kappa)}^0(\tau', t) \right) - \text{Err}_{i,T,(\kappa)}^0(\tau', t) \right) \right\|_{L^2_{m-\ell, \delta}([4T - 1, 4T + 1]_{\tau'} \times [0, 1] \subset \Sigma_T)} \leq \mu^\epsilon(9) e^{-\delta_T}. \quad (6.35) \]

\textbf{Proof.} This is a consequence of ‘drop of the weight’ we mentioned in Remark 5.21.

The left hand side of (6.35) is the \( L^2_{m-\ell, \delta} \) norm of the next formula
\[ \nabla_\rho \frac{d}{d\tau} \mathcal{F}^{-1} \left( \int_0^1 \left( \frac{\partial}{\partial s} \right) \mathcal{P} \left( \text{Exp} \left( u_{i,T,(\kappa-1)}^0(\tau', t), V_{T,1,(\kappa)}^0(\tau', t) \right) + r\chi(\tau' - 4T) \left(V_{T,2,(\kappa)}^0(\tau' - 10T, t) - (\Delta p_{T,(\kappa)})^{\text{Pal}} \right) \right) \right) \right) \right) \quad (6.36) \]
\[ \text{Here } \mathcal{P} \text{ is the inverse of the (0,1) part of the parallel transport along the curve} \]
\[ s \mapsto \text{Exp} \left( u_{i,T,(\kappa-1)}^0(\tau', t), V_{T,1,(\kappa)}^0(\tau', t) + s\chi(\tau' - 4T) \left(V_{T,2,(\kappa)}^0(\tau' - 10T, t) - (\Delta p_{T,(\kappa)})^{\text{Pal}} \right) \right). \]
We can estimate the integrand of (6.36) as
\[
\int_{C_m} \left\| \nabla \frac{d^t}{dT} \mathcal{P}^{-1} \right\|^2 L^2_{m+\ell} \leq C_m \sum_{\ell' \leq \ell} \sum_{n' \leq n} \left\| \nabla \frac{d^t}{dT} \mathcal{P}^{-1} \left( V^p_{T,2,(\kappa)}(\tau'', t) \right) \right\|^2 L^2_{m+1-\ell'} \quad (6.37)
\]
(We use (6.28) here.) Note the norm in (6.37) is \( L^2_{m+1-\ell'} \) norm without weight.

The proof of (6.37) is similar to the proof of (5.62) given at the end of Section A.

By the induction hypothesis the \( L^2_{m+1-\ell'} \) norm of
\[
\nabla \frac{d^t}{dT} \mathcal{P}^{-1} \left( V^p_{T,2,(\kappa)}(\tau'', t) - (\Delta p^\mu_{T,(\kappa)})^{\mathcal{P}^{-1}} \right)
\]
with weight \( e_{2,\delta} \) is estimated by \( C_{2,m}\mu^{-1}e^{-\delta_1 T} \).

The proof of (6.37) is similar to the proof of (6.38) given at the end of Section A.

We take \( T_{m,\epsilon}(9) \) such that \( C_m \mu^{-1}e^{-\delta_2 T_{m,\epsilon}(9)} \leq \epsilon(9) \mu \). The lemma follows. \( \square \)

Using (6.40) as in the proof of Proposition 5.33 we calculate
\[
\mathcal{D} \left( \right) \left( \text{Exp}(u^p_{T,(\kappa-1)}, rV^p_{T,1,(\kappa)}(\tau', t)) \right)
= \mathcal{D} u^p_{T,(\kappa-1)}(\tau', t) + (D u^p_{T,(\kappa-1)})^{\mathcal{D}}(V^p_{T,1,(\kappa)}(\tau', t))
\]
\[
+ \int_0^1 ds \int_0^s \left( \frac{\partial^2}{\partial s^2} \right) \mathcal{P}^{-1} (\mathcal{D} \left( \text{Exp}(u^p_{T,(\kappa-1)}, rV^p_{T,1,(\kappa)}(\tau', t)) \right)) \right) dr.
\]
Here \( \mathcal{P} \) is the linear map
\[
(\mathcal{P} u^p_{T,(\kappa-1)} + V^p_{T,1,(\kappa)}(\tau', t))^{(0,1)}.
\]
We can again estimate the third term of the right hand side of (6.40) in the same way as the proof of inequality (G.3) given in Appendix G and obtain
\[
\int_{C_m} \left\| \nabla \frac{d^t}{dT} \mathcal{P}^{-1}(3\text{rd term of } (6.40)) \right\|^2 L^2_{m+1-\ell'}([4T-1,4T+1], \times [0,1]) 
\leq C_m \sum_{\ell' \leq \ell} \sum_{n' \leq n} \left\| \nabla \frac{d^t}{dT} \mathcal{P}^{-1}(V^p_{T,1,(\kappa)}) \right\|^2 L^2_{m+1-\ell'}([4T-1,4T+1], \times [0,1]).
\]

EXPO.
Since
\[
\left\| \nabla_{\rho}^n \frac{\partial^\ell}{\partial t^\ell} (\mathcal{P}')^{-1}V_{T,1,(\kappa)}^p \right\|_{L^2_{m+1-\epsilon}([4T-1,4T+1], \mathbb{R}^n)} 
\leq \left\| \nabla_{\rho}^n \frac{\partial^\ell}{\partial t^\ell} (\mathcal{P}')^{-1}(V_{T,1,(\kappa)}^p - (\Delta_{T,1,(\kappa)}^p)_{(\kappa)}) \right\|_{L^2_{m+1-\epsilon}([4T-1,4T+1], \mathbb{R}^n)} + \left\| \nabla_{\rho}^n \frac{\partial^\ell}{\partial t^\ell} \text{Pal}_{p_0}^p \text{Pal}_{p_0}^p (\Delta_{T,1,(\kappa)}^p) \right\|_{L^2_{m+1-\epsilon}([4T-1,4T+1], \mathbb{R}^n)} \leq C_{m,6.42} \mu^{\kappa-1} e^{-\delta_1 T}
\]
by (6.8) we have
\[
(6.41) \leq C_{m,6.43} \mu^{2(\kappa-1)} e^{-2\delta_1 T}. \tag{6.43}
\]
Finally we observe that
\[
\bar{\partial}u_{T,1,(\kappa-1)}^p + (D_{n_0}^p \bar{\partial})(V_{T,1,(\kappa)}^p (\tau', t)) = 0, \tag{6.44}
\]
on \([4T-1, 4T+1], \mathbb{R}^n \times [0, 1]. \) This follows from (5.70) and Definition 5.30 (5.83) together with Err_{T,1,(\kappa)}^p (\tau', t) = 0 on \([4T-1, 4T+1], \mathbb{R}^n \times [0, 1]. \)
In sum, by Lemma 6.15 and (6.40), (6.43) and (6.44), we obtain
\[
\left\| \nabla_{\rho}^n \frac{\partial^\ell}{\partial t^\ell} (\mathcal{P}')^{-1} \text{Err}_{T,1,(\kappa)}^p \right\|_{L^2_{m-\delta, \epsilon}([4T-1,4T+1], \mathbb{R}^n)} \leq C_{m,6.45} \mu^\kappa \epsilon(9) e^{-\delta_1 T}. \tag{6.45}
\]
for \(T > T_{m,\epsilon}(9), 6.45. \)

We can now complete the proof of Proposition 6.9 (1). We take \(\epsilon(7), \epsilon(8), \epsilon(9)\) such that \(\epsilon(7) < C_{8,m}\epsilon(6)/10, C_{m,6.13} \epsilon(8) < C_{8,m}\epsilon(6)/10\) and \(\epsilon(9) < C_{8,m}\epsilon(6)/10. \)
Then if \(T > \max\{T_{m,\epsilon}(7), 6.23, T_{m,\epsilon}(8), 6.26, T_{m,\epsilon}(9), 6.35, T_{m,\epsilon}(9), 6.45\} \) (6.13) 6.15 and Formulae (6.33), 6.45 imply (6.11). 6.12) then follows from Lemma 6.13 (6.24).

**Remark 6.16.** In [Ab] Abouzaid used \(L^p\) norm for the maps \(u. \) He then proved that the gluing map is continuous with respect to \(T\) (that is \(S\) in the notation of [Ab]) but does not prove its differentiability with respect to \(T. \) (Instead he used the technique to remove the part of the moduli space with \(T > T_0. \) This technique certainly works for the purpose of [Ab].) In fact if we use \(L^p\) norm instead of \(L^m\) norm then the left hand side of (6.10) becomes \(L^p\) norm which is hard to use.

Abouzaid mentioned in [Ab], Remark 5.1 that this point is related to the fact that quotients of Sobolev spaces by the diffeomorphisms in the source are not naturally equipped with the structure of smooth Banach manifold. Indeed in the situation where there is an automorphism on \(\Sigma_2, \) for example when \(\Sigma_2\) is the disk with one boundary marked point at \(\infty, \) then the \(T\) parameter is killed by a part of the automorphism. So the shift of \(V_{T,2,(\kappa)}^p \) by \(T\) that appears in the second term of (6.20) will be equivalent to the action of the automorphism group of \(\Sigma_2\) in such a situation. The shift of \(T\) causes the loss of differentiability in the sense of Sobolev space in the formulas (6.8) - (6.12). However at the end of the day we can still get the differentiability of \(C^\infty\) order and its exponential decay by using various weighted Sobolev spaces with various \(m\) simultaneously using the fact that \(C^\infty\) topology is a Frechét topology, as we show during the proof of Lemma 8.28. (See Remark 6.3 also.)
**Remark 6.17.** In [DK] Subsection 7.2.3 Donaldson-Kronheimer mentioned that there is exactly one place where their construction of the basic package on the moduli space of ASD-connections uses the $L^p_m$ space for $p \neq 2$ which is not conformally invariant. It is exactly the place of gluing construction, the place similar to what we are studying in this paper. There Donaldson-Kronheimer used the $L^p_1$ space. The reason why they do need $L^p_m$ norm for $p \neq 2$ is that in [DK] Donaldson-Kronheimer do not use weighted Sobolev or Banach norm.

In the framework of [DK] (which is adapted to the pseudoholomorphic curve by [McSa]) the ‘neck region’ of 4 manifold is regarded as $D^4(1) \setminus D^4(1/R^2)$ with standard Riemannian metric on $D^4(1) \setminus D^4(1/R)$ and the metric induced from the standard metric by $x \mapsto R|x|^2$ on $D^4(1/R) \setminus D^4(1/R^2)$. (In [McSa] the ‘neck region’ of the source curve is $D^2(1) \setminus D^2(1/R^2)$ with a similar metric.) So their metric is different from the cylindrical metric on $[-5T, 5T] \times S^1$, which is one we use in this paper.

If we change the variables from $D^2(1) \setminus D^2(1/R^2)$ to $[-5T, 5T] \times S^1$ (with $R = e^{5\pi T}$) but still use the above mentioned Riemannian metric of $D^2(1) \setminus D^2(1/R^2)$ then it is equivalent to using cylindrical metric of $[-5T, 5T] \times S^1$ together with some weight. Note this weight function is 1 if and only if the Sobolev space involved is conformally invariant. So in the situation such as the one appearing in [DK] Subsection 7.2.3 this weight function is nontrivial. Actually one can observe that the weight function appears in that way is similar to the weight $e_{\delta,T}$, which we use in this paper. In other words, if we use an appropriate $L^p_m$ norm, using cylindrical metric with weight is not very different from using the the above mentioned metric on $D^2(1) \setminus D^2(1/R^2)$.

However if we consider $L^2_m$ norm with $m$ large enough then our weighted norm (after changing the variables to $D^2(1) \setminus D^2(1/R^2)$) does not coincide with the $L^p_m$ norm with respect to the above mentioned metric. Thus it seems important to use a weighted norm for the study of higher derivative with respect to the gluing parameter $T$.

We remark that in [FU] Freed-Uhlenbeck worked out the gluing analysis of ASD connections in the frame work of $L^2$ theory (that is, without using $L^p_k$ spaces but using only $L^2_k$ spaces). Freed-Uhlenbeck used cylindrical metric on $\mathbb{R} \times S^3$. So the method of [FU] is closer to ours. It seems that Freed-Uhlenbeck do not need to use weighted Sobolev norm since in their case they can use the fact that their 3 manifold is $S^3$ and show exponential decay without using weighted Sobolev norm. Actually, in their situation, Chern-Simons functional on $S^3$ is not only a Bott-Morse function but also a Morse function. In our situation, the non-linear Cauchy-Riemann equation on $\mathbb{R} \times S^1$ or on $\mathbb{R} \times [0, 1]$ with Lagrangian boundary condition is degenerate at infinity. In other words we are in Bott-Morse situation. By this reason, it seems inevitable to use weighted Sobolev norm when we work with cylindrical metric.

**Remark 6.18.** Another difference between our construction and the construction of [DK],[McSa] is the choice of cut off function. In the formula at the end of [McSa] page 172 they used the cut off function $\beta$ appearing in [DK] Lemma 7.2.10, [McSa] Lemma A.1.1 to obtain the right inverse. If we rewrite their formula in terms of the cylindrical coordinate the cut off function appearing there has mostly of constant slope $|1/\log \delta|$ and the support of its first derivative has length $\sim |\log \delta|$, here $\delta$ is a small number. So the size of the error term caused by the derivative of the cut off function is (pointwise) $\sim |1/\log \delta|$, which is small.
Our choice of cut off function is, for example, $\chi_{\frac{1}{\epsilon}}$. In the cylindrical coordinate the size of its derivative is $\sim 1$ and the length of the support of its first derivative is also $\sim 1$. So the size of the error term caused by the derivative of the cut off function is (pointwise) $\sim 1$, which is not small. We use the ‘drop of the weight argument’ mentioned in Remark 6.21 to show that this error term is small in our weighted Sobolev norm. See Figure 9.

Since we use weighted Sobolev space, the estimate is easier to carry out in case the support of the derivative of the cut off function has bounded length. (This is because then the ratio between maximum and minimum of the cut off function on the above mentioned support is bounded.)

\[ \begin{array}{c}
\text{Figure 9. The choices of cut off functions.}
\end{array} \]

### 6.2. Part B: estimates for the approximate inverse

In this subsection we prove Proposition 6.9 (2). This subsection corresponds to the discussion given in [FOOO1, page 776 the paragraph next to (B)].

We assume (6.8)–(6.12) for $\kappa$ and will prove (6.8) for $\kappa + 1$. (So we are doing Step $\kappa + 1$.)

This part is nontrivial only because the construction here is global. (Solving linear equation.) So we first review the set-up of the function space that is independent of $T, \rho, \kappa$.

In Definition 5.29 we defined a function space $\mathcal{B}(\kappa)(E_1, E_2; \rho, T)$, that is a subspace of (5.82). We solved linearized equation on it. (See (5.84).) The space (5.82) is $T, \rho, \kappa$-dependent. However $\mathcal{B}(\kappa)(E_1, E_2; \rho, T)$ is the image of $\mathcal{B}(E_1, E_2)$, which is independent of $T, \rho, \kappa$, by the map $(\Phi_{1;\kappa}(\rho, T), \Phi_{2;\kappa}(\rho, T))$. We recall $\mathcal{B}(E_1, E_2)$ is defined in Definition 5.9. We put

\[ I_{0;\kappa, \rho, T} = (\Phi_{1;\kappa}(\rho, T), \Phi_{2;\kappa}(\rho, T)) : \mathcal{B}(E_1, E_2) \to \mathcal{B}(\kappa)(E_1, E_2; \rho, T). \]

By composing it we can take the domain independent of $T, \rho, \kappa$. 
We next consider the target. Using the map in (2.7) we define
\[ I^1_{i,(\kappa),\rho,T} : L^2_{m,\delta}(\Sigma_i; u^\rho_i TX \otimes \Lambda^{0,1}) \to L^2_{m,\delta}(\Sigma_i; (\hat{u}^\rho_i)_{T,(\kappa)})^*TX \otimes \Lambda^{0,1} \]
as the closure of \((\text{Pal}_{u^\rho_i}^\delta T,(\kappa)))^{(0,1)}.\]

Then we define
\[ I^1_{i,(\kappa),\rho,T} : L^2_{m,\delta}(\Sigma_1; u^\rho_1 TX \otimes \Lambda^{0,1}) \oplus L^2_{m,\delta}(\Sigma_2; u^\rho_2 TX \otimes \Lambda^{0,1}) \]
\[ \to L^2_{m,\delta}(\Sigma_1; (\hat{u}^\rho_1)_{T,(\kappa)})^*TX \otimes \Lambda^{0,1}) \oplus L^2_{m,\delta}(\Sigma_2; (\hat{u}^\rho_2)_{T,(\kappa)})^*TX \otimes \Lambda^{0,1}) \]
as the direct sum \(I^1_{i,(\kappa),\rho,T} = I^1_{1,(\kappa),\rho,T} \oplus I^1_{2,(\kappa),T} \). Thus the composition
\[ (I^1_{i,(\kappa),\rho,T})^{-1} \circ (D_{\rho,T})_{\kappa} \circ (D_{\rho,T})_{\kappa} \circ I^0_{i,(\kappa),\rho,T} \]
defines an operator, which we denote by
\[ D_{(\kappa),\rho,T} : \mathcal{E}_1(\Sigma_1,\Sigma_2) \to L^2_{m,\delta}(\Sigma_1; u^\rho_1 TX \otimes \Lambda^{0,1}) \oplus L^2_{m,\delta}(\Sigma_2; u^\rho_2 TX \otimes \Lambda^{0,1}) \].

Here both the target and the domain are independent of \(\kappa, \rho, T\).

We need to invert the operator \(D_{\rho,T}^\text{app,(\kappa)} \circ D_{\rho,T}^\text{app,(\kappa)} \circ D_{\rho,T}^\text{app,(\kappa)} \circ \mathcal{E}_1(\hat{u}^\rho_1)_{T,(\kappa)} \oplus \mathcal{E}_2(\hat{u}^\rho_2)_{T,(\kappa)} \). (See (5.29) and (5.83).) We remark the subspace
\[ \mathcal{E}_{1,(\kappa),\rho,T} := (I^1_{i,(\kappa),\rho,T})^{-1}(\mathcal{E}_1(\hat{u}^\rho_1)_{T,(\kappa)} \oplus \mathcal{E}_2(\hat{u}^\rho_2)_{T,(\kappa)}) \]
is \((\kappa, \rho, T)\)-dependent. In fact, by definition \(\mathcal{E}_1(\hat{u}^\rho_1)_{T,(\kappa)} \) is the image of \(\mathcal{E}_i^{\rho,b} \) by the parallel transport \((\text{Pal}_{u^{\rho,b}_i}^\delta T,(\kappa)))^{(0,1)}\). (See (3.18).) Therefore
\[ (I^1_{i,(\kappa),\rho,T})^{-1}(\mathcal{E}_1(\hat{u}^\rho_1)_{T,(\kappa)} \oplus \mathcal{E}_2(\hat{u}^\rho_2)_{T,(\kappa)}) \]
\[ = \bigoplus_{i=1}^2 \left((\text{Pal}_{u^{\rho,b}_i}^\delta T,(\kappa)))^{(0,1)} \right)^{-1}(\text{Pal}_{u^{\rho,b}_i}^\delta T,(\kappa))) \right)((\mathcal{E}_i^{\rho,b}) \]
which is different from \(\mathcal{E}_1(u^b_1) \oplus \mathcal{E}_2(u^b_2) \) since
\[ (\text{Pal}_{u^{\rho,b}_i}^\delta T,(\kappa)))^{(0,1)} \neq \left((\text{Pal}_{u^{\rho,b}_i}^\delta T,(\kappa)))^{(0,1)} \right)^{-1}(\text{Pal}_{u^{\rho,b}_i}^\delta T,(\kappa))) \right)((\mathcal{E}_i^{\rho,b}) \]
in general.

**Remark 6.19.** In our situation where we consider only one \(u^b_i\) we can trivialize the bundle \(u^b \to L^2_{m}(\Sigma_T, (u^b)^*TX \otimes \Lambda_{0,1}) \) by sending them to
\[ \bigoplus_{i=1}^2 L^2_{m}(\Sigma_i, (u^b_i)^*TX \otimes \Lambda_{0,1}) \]
using parallel transport. Then the image of \(\mathcal{E}_i(u^b) \) in (6.49) will not vary.

However for our application, we need to consider several different \(u^b_i\)'s, in which case there is no choice of the trivialization of \(u^b \to \mathcal{E}_i(u^b) \subset L^2_{m}(\Sigma_T, (u^b)^*TX \otimes \Lambda_{0,1}) \) so that the direct sum of the obstruction spaces do not vary.

However the way to estimate this discrepancy in the case when we have several \(u^b_i\)'s is the same as we do in this subsection.
We can estimate the \((T, \rho)\) and \(\kappa\) dependence of (6.47) and use certain elementary functional analysis to go around the problem of this dependence as follows.

We first observe the next lemma. Let \(\{e_{i,a} \mid a = 1, \ldots, \dim \mathcal{E}_{i}^{\text{ob}}\}\) be a basis of \(\mathcal{E}_{i}^{\text{ob}}\). We put

\[
e'_{i,a;\kappa}(\rho, T) = \left( \left( \text{Pal}_{u^{\rho}_{i,T,\kappa}} \right)^{0,1} \circ \left( \text{Pal}_{u^{\rho}_{i,T,\kappa}} \right)^{0,1} \right) (e_{i,a}) \in L_{m,\delta}^{2}(\Sigma_{i}^{\prime}; u_{i}^{\ast} TX \otimes \Lambda^{0,1}).
\]

(6.50)

Note \(\{e'_{i,a;\kappa}(\rho, T) \mid a = 1, \ldots, \dim \mathcal{E}_{i}\}\) is a basis of \(\mathcal{E}_{i;\kappa,\rho,T}^{\text{ob}}\).

We denote by \(e_{i,a;\kappa}(\rho, T) (a = 1, \ldots, \dim \mathcal{E}_{i})\) the basis obtained from \(e'_{i,a;\kappa}(\rho, T)\) by applying Gram-Schmidt orthogonalization to \(e'_{i,a;\kappa}(\rho, T)\). (We use the \(L^{2}\) inner product (4.11) on \(L_{m,\delta}^{2}(\Sigma_{i}^{\prime}; u_{i}^{\ast} TX \otimes \Lambda^{0,1}).\)

**Lemma 6.20.** There exists \(C_{m, \delta}(6.51)\) such that

\[
\left\| \nabla_{\rho} \frac{\partial^{\prime}}{\partial T} e_{i,a;\kappa}(\rho, T) \right\|_{L_{m+1-\ell}^{2}} \leq C_{m, \delta}(6.51) e^{-\delta_{i}T},
\]

(6.51)

for \(m - 2 \geq n, \ell \geq 0, \ell > 0\).

We remark that since the support of \(e_{i,a;\kappa}(\rho, T)\) is in \(K_{i}\), we do not need to use weighted norm.

**Proof.** We prove this lemma in Section [F].

Let \(\mathcal{E}_{i;\kappa,\rho,T}^{\perp} \) be the \(L^{2}\) orthonormal complement of \(\mathcal{E}_{i;\kappa,\rho,T}\) in \(L_{m,\delta}^{2}(\Sigma_{i}^{\prime}; u_{i}^{\ast} TX \otimes \Lambda^{0,1})\). In other words, we define

\[
\mathcal{E}_{i;\kappa,\rho,T}^{\perp} = \{ W \in L_{m,\delta}^{2}(\Sigma_{i}^{\prime}; u_{i}^{\ast} TX \otimes \Lambda^{0,1}) \mid \langle W, e_{i,a;\kappa}(\rho, T) \rangle_{L^{2}} = 0, a = 1, \ldots, \dim \mathcal{E}_{i}^{\text{ob}} \}.
\]

Here we use \(L^{2}\) inner product defined in (4.11). Since elements \(e_{i,a;\kappa}(\rho, T)\) are supported in \(K_{i}\) and are smooth, we can safely use the \(L^{2}\) inner product, without weight. We also define the projections

\[
\Pi_{\mathcal{E}_{i;\kappa,\rho,T}^{\perp}} : L_{m,\delta}^{2}(\Sigma_{i}^{\prime}; u_{i}^{\ast} TX \otimes \Lambda^{0,1}) \to \mathcal{E}_{i;\kappa,\rho,T}^{\perp},
\]

by

\[
\Pi_{\mathcal{E}_{i;\kappa,\rho,T}^{\perp}} (W) = \sum_{a=1}^{\dim \mathcal{E}_{i}^{\text{ob}}} \langle W, e_{i,a;\kappa}(\rho, T) \rangle_{L^{2}} e_{i,a;\kappa}(\rho, T).
\]

(6.52)

We put

\[
\mathcal{E}_{\kappa,\rho,T} = \bigoplus_{i=1}^{2} \mathcal{E}_{i;\kappa,\rho,T}, \quad \mathcal{E}_{\kappa,\rho,T}^{\perp} = \bigoplus_{i=1}^{2} \mathcal{E}_{i;\kappa,\rho,T}^{\perp}
\]

and \(\Pi_{\mathcal{E}_{\kappa,\rho,T}} = \Pi_{\mathcal{E}_{1;\kappa,\rho,T}} \oplus \Pi_{\mathcal{E}_{2;\kappa,\rho,T}}\). The operator we need to invert is

\[
(\text{id} - \Pi_{\mathcal{E}_{\kappa,\rho,T}}) \circ D_{(\kappa),\rho,T} : \mathcal{Y}(\mathcal{E}_{1}, \mathcal{E}_{2}) \to \mathcal{E}_{\kappa,\rho,T}^{\perp}.
\]

Let \(\mathcal{E}_{i}^{\perp}\) be the \(L^{2}\) orthogonal complement of \(\mathcal{E}_{i} = \mathcal{E}_{i}(u_{i})\) in \(L_{m,\delta}^{2}(\Sigma_{i}^{\prime}; u_{i}^{\ast} TX \otimes \Lambda^{0,1})\), \(\Pi_{\mathcal{E}_{i}^{\perp}} : L_{m,\delta}^{2}(\Sigma_{i}^{\prime}; u_{i}^{\ast} TX \otimes \Lambda^{0,1}) \to \mathcal{E}_{i}^{\perp}\) the associated \(L^{2}\) projection. We put \(\Pi = \Pi_{\mathcal{E}_{1}^{\perp}} \oplus \Pi_{\mathcal{E}_{2}^{\perp}}\). By Lemma 6.20 the restriction of \(\Pi\) induces an isomorphism

\[
\Pi : \mathcal{E}_{\kappa,\rho,T}^{\perp} \to \mathcal{E}_{1}^{\perp} \oplus \mathcal{E}_{2}^{\perp}.
\]

(6.53)
The map \( D_{(\kappa),\rho,T} \) induces a map
\[
\overline{D}_{(\kappa),\rho,T} = \Pi \circ (\text{id} - \Pi \xi_{(\kappa),\rho,T}) \circ D_{(\kappa),\rho,T} : \mathcal{H}_1, \mathcal{H}_2 \to \mathcal{H}_1^\perp + \mathcal{H}_2^\perp. \tag{6.54}
\]
Both the source and the target of \( \overline{D}_{(\kappa),\rho,T} \) are independent of \( \kappa, T, \rho \). We will invert this operator by using the next lemma.

Note \( \mathcal{H}_1, \mathcal{H}_2 \) is a subspace of
\[
\bigoplus_{i=1}^{2} W^2_{m+1,\delta}(\xi_i, \partial \xi_i); u_i^1 T X, u_i^2 T L).
\]
Its element is a pair \((V_1, V_2)\) where \( V_i = (s_i, v_i) \) is a pair of a section \( s_i \) and its asymptotic value \( v_i \). (Note \( v_1 = v_2 \) for an element of \( \mathcal{H}_1, \mathcal{H}_2 \).)

**Lemma 6.21.** For \( V \in \mathcal{H}_1, \mathcal{H}_2 \) the following holds:

1. There exist \( C_m, C_m', C_m'' > 0 \) such that
\[
C_m, C_m' \| V \|_{W^2_{m+1,\delta}} \leq \| \overline{D}_{(\kappa),\rho,T}(V) \|_{L^2_{m,\delta}} \leq C_m' \| V \|_{W^2_{m+1,\delta}}. \tag{6.55}
\]
2. \[
\| \overline{D}_{(\kappa),\rho,T}(V) - \overline{D}_{(0),\rho,T}(V) \|_{L^2_{m,\delta}} \leq C_m, C_m' \| V \|_{W^2_{m+1,\delta}}. \tag{6.56}
\]

Moreover
\[
\left\| \overline{D}_{(\kappa),\rho,T}(V) \right\|_{L^2_{m-\ell,\delta}} \leq C_m, C_m' \| V \|_{W^2_{m+1,\delta}}. \tag{6.57}
\]

for \( m \geq n, \ell \geq 0, \ell > 0 \).

**Proof.** The inequality \(6.56\) follows by integrating \(6.57\) on \( T \in [0, \infty) \) and using the fact that \( \overline{D}_{(\kappa),\rho,\infty} = \overline{D}_{(0),\rho,\infty} \). \(6.55\) follows from \(6.56\) and the invertibility of \( \overline{D}_{(\kappa),\rho,T} \).

The proof of \(6.57\) will occupy the rest of the proof.

First, we remark that, by Lemma \( 6.20 \) and \( 6.52 \) we have:
\[
\left\| \nabla^\rho \frac{\partial^\ell}{\partial T^\ell} (\Pi \circ (\text{id} - \Pi \xi_{(\kappa),\rho,T})) \right\|_{L^2_{m}} \leq C_m, C_m' e^{-\delta_1 T}. \tag{6.58}
\]

We next study \( D_{(\kappa),\rho,T} \). By \(6.52\) and \(6.88\), we obtain the next inequalities by induction hypothesis:
\[
\left\| \nabla^\rho \xi_{(i,T,(\kappa))} \right\|_{L^2_{m-\ell}(K^{\tau_T})} \leq C_m, C_m', \tag{6.59}
\]
\[
\left\| \nabla^\rho \xi_{(0,T,(\kappa))} \right\|_{L^2_{m-\ell}(K^{\tau_T})} \leq C_m, C_m', \tag{6.60}
\]
and
\[
\left\| \nabla^\rho \frac{\partial^\ell}{\partial T^\ell} \xi_{(i,T,(\kappa))} \right\|_{L^2_{m-\ell}(K^{\tau_T})} \leq C_m, C_m', \tag{6.61}
\]
\[
\left\| \nabla^\rho \frac{\partial^\ell}{\partial T^\ell} \xi_{(0,T,(\kappa))} \right\|_{L^2_{m-\ell}(K^{\tau_T})} \leq C_m, C_m'. \tag{6.62}
\]

Here \( m - 2 \geq n, \ell \geq 0, \ell > 0 \). Note we use \( \tau' \) as the coordinate of \([0, \infty), \tau'' \) as the coordinate of \((0, \infty), \) in the left hand sides of \(6.60\) and \(6.62\). Hereafter we assume \( n, \ell \) satisfies \( m - 2 \geq n, \ell \geq 0, \ell > 0 \) until the end of the proof of Lemma
In fact (6.60), (6.62) follows from (6.10), (6.59) follows from Lemma 6.11.

We also remark that
domain \( \Sigma \) follows from (6.9).

6.21. In fact (6.60), (6.62) follows from (6.10). (6.59) follows from Lemma 6.11.

We will derive (6.56), (6.57) with \( T_k \) replaced by \( D_{(\kappa),\rho,T} \) from these four inequalities as follows.

We put \( V = (V_1, V_2) \) and \( V_i = (s_i, v_i) \).

We consider the case of \( \Sigma_1 \) only since the case of \( \Sigma_2 \) is the same. We divide the domain \( \Sigma_1 \) into pieces \( K_1 \) and \( [k - 1, k + 1]_{\tau'} \times [0, 1], (k \in [0, \infty) \cap \mathbb{Z}) \).

We denote each piece of these domains by \( \Sigma(a) \). For each such piece \( \Sigma(a) \), we define \( \Sigma(a, +) \) as follows. (We use the \( \tau' \) coordinate.)

(a) If \( \Sigma(a) = K_1 \) then \( \Sigma(a, +) = K_1 \cup [0, 1]_{\tau'} \times [0, 1] \).

(b) If \( \Sigma(a) = [k - 1, k + 1]_{\tau'} \times [0, 1] \) then \( \Sigma(a, +) = [k - 2, k + 2]_{\tau'} \times [0, 1] \).

We first consider the case when \( a \) is as in (a) above. Then from (6.59), (6.61) we derive
\[
\left\| \nabla_{\rho} \frac{\partial^\ell}{\partial T^\ell} D_{(\kappa),\rho,T}(V) \right\|_{L^2_{m-\ell}(\Sigma(a))} \leq C_m,6.63 e^{-\delta_1 T} \| s_1 \|_{L^2_{m+1}(\Sigma(a, +))}. \tag{6.63}
\]
(Here we use the fact that \( \ell > 0 \) and that \( V \) is independent of \( T, \rho \)).

We next consider the case when \( a \) is as in (b), that is, \( \Sigma(a) = [k - 1, k + 1]_{\tau'} \times [0, 1] \). Then
\[
\begin{align*}
\left\| \nabla_{\rho} \frac{\partial^\ell}{\partial T^\ell} D_{(\kappa),\rho,T}(V) \right\|_{L^2_{m-\ell}(\Sigma(a))} &\leq \left\| \nabla_{\rho} \frac{\partial^\ell}{\partial T^\ell} D_{(\kappa),\rho,T}(s_1 - v_1^{\text{pal}}) \right\|_{L^2_{m-\ell}(\Sigma(a, +))} \\
&\quad + \left\| \nabla_{\rho} \frac{\partial^\ell}{\partial T^\ell} D_{(\kappa),\rho,T}(v_1^{\text{pal}}) \right\|_{L^2_{m-\ell}(\Sigma(a, +))} \\
&\leq C_m,6.65 e^{-\delta_1 T} \| s_1 - v_1^{\text{pal}} \|_{L^2_{m+1}(\Sigma(a, +))} + C_m,6.65 e^{-\delta_1 T} \| v_1 \|.
\end{align*}
\tag{6.64}
\]
Here we use (6.59)-(6.62) to estimate (6.64).

Moreover for those \( \tau' \) with \( \Sigma(a, +) \cap [0, 7 T + 1]_{\tau'} \times [0, 1] = \emptyset \) we may improve (6.65) to be \( C_m,6.65 e^{-\delta_1 \| s_1 - v_1^{\text{pal}} \|_{L^2_{m+1}(\Sigma(a, +))}} \). (Namely the second term of (6.65) drops out.) This is because \( \tilde{\alpha}_{1,T,1}^{\text{pal}}(x) \) is then constant on \( \Sigma(a, +) \) and therefore
\[
D_{(\kappa),\rho,T}(v_1^{\text{pal}}) = 0 \text{ there}.
\]

The norm
\[
\sum_a e_{1,\delta}(p(a))^2 \| Y \|_{L^2_{m+1}(\Sigma(a))}^2,
\]
is equivalent to the \( L^2_{m+1,\delta} \) norm \( \| Y \|_{L^2_{m+1,\delta}(\Sigma_1)}^2 \). (Here \( e_{1,\delta} \) is the weight function in (4.4) and \( p(a) \in \Sigma(a) \) is any chosen point for each \( a \)).

Moreover the norm
\[
\sum_a \| s_1 \|_{L^2_{m+1}(\Sigma(a))}^2 + \sum_a e_{1,\delta}(p(a))^2 \| s_1 - v_1^{\text{pal}} \|_{L^2_{m+1}(\Sigma(a))}^2 + \| v_1 \|^2,
\]
is equivalent to the \( W^2_{m+1,\delta} \) norm \( \| (s_1, v_1) \|_{W^2_{m+1,\delta}(\Sigma_1)}^2 \).
Therefore taking the weighted sum of the square of the inequalities (6.63), (6.65) with weight $e_{T, \delta}(p(a))^2$ and using the above mentioned equivalence of norms, we derive the estimate

$$\left\| \nabla_{\rho} \frac{\partial}{\partial T} D(\kappa, \rho, T) \right\|_{L^2_{m-\delta}(\Sigma_1)}^2 \leq C_m e^{-\delta T} \sum_{\alpha: \text{Case } (a)} \left\| s_1 \right\|_{L^2_{m+1}(\Sigma(a, +))}^2 + C_m' e^{-\delta T} \left\| s_1 - v_{11}^p \right\|_{L^2_{m+1}(\Sigma(a))}^2 + 2e^{-\delta T} T e^{14\delta T} C_m e^{-\delta T/10} \left\| v_1 \right\|_{W^2_{m+1, \delta}(\Sigma_1)}^2.$$  \hspace{1cm} (6.66)

Here we use the fact that if $\Sigma(a, +) \cap [0, 7T + 1] \times [0, 1] \neq \emptyset$ then $e_{1, \delta}(p(a)) \leq C e^{7\delta T}$. Using $\delta < \delta_1/10$, (6.66) implies

$$\left\| \nabla_{\rho} \frac{\partial}{\partial T} D(\kappa, \rho, T) \right\|_{L^2_{m-\delta}(\Sigma_1)} \leq C_m e^{-\delta T/10} \left\| (s_1, v_1) \right\|_{W^2_{m+1, \delta}(\Sigma_1)}. \hspace{1cm} (6.67)$$

This inequality and the same inequality for $\Sigma_2$ together with (6.54) and (6.58) imply (6.57).

We recall from the definition (6.54) and (6.46) that $D(0, \rho_0, T)$ is independent of $T$ for any $\rho_0 \in V(\varepsilon_2) \times L^2 \varepsilon_2)$. So we denote the common operator by $D_0$. Since the operators $D(\kappa, \rho, T)$ are invertible, we can expand the operator

$$D^{-1}_{(\kappa, \rho, T)} : \mathcal{E}_1^+ \oplus \mathcal{E}_2^+ \rightarrow \mathcal{H}(\mathcal{E}_1, \mathcal{E}_2)$$

into

$$D^{-1}_{(\kappa, \rho, T)} = (1 + (D_{(\kappa, \rho, T)} - D_0)^{-1} D_0)^{-1} = D_0^{-1} \sum_{k=0}^{\infty} (-1)^k ((D_{(\kappa, \rho, T)} - D_0)^{-1})^k. \hspace{1cm} (6.68)$$

Note the right hand side converges as far as $\rho$ is in a sufficiently small neighborhood $V(\rho_0)$ of $\rho_0$ and $T > T_m(6.68).$

Therefore by differentiating this by $T$ using the Leibnitz rule, we derive

$$\frac{\partial}{\partial T} D^{-1}_{(\kappa, \rho, T)} = D_0^{-1} \sum_{k=0}^{\infty} (-1)^{k_1 + k_2 + 1} k_1 (D_{(\kappa, \rho, T)} - D_0)^{-1} K_{k_1} K_{k_2} \frac{\partial}{\partial T} D^{-1}_{(\kappa, \rho, T)} \sum_{k_0=0}^{\infty} k_0 D_0^{-1}.$$  \hspace{1cm} (6.68)

Along the way, we lose 1 differentiability. Here we note that the operator $(D_{(\kappa, \rho, T)} - D_0)^{-1}$ is uniformly bounded as $\kappa, \rho, T$ vary with $\rho \in V(\rho_0)$, $T > T_m(6.68).$ In the same way, we can differentiate with respect to $\rho$ without loss of derivative.

In a similar way we can derive

$$\left\| \nabla_{\rho} \frac{\partial}{\partial T} D_{(\kappa, \rho, T)} \right\|_{W^2_{m+1, \delta, \rho}} \leq C_m e^{-\delta T/10} \left\| W \right\|_{L^1_{m, \delta}} \hspace{1cm} (6.69)$$
for $\ell > 0$ and $0 \leq \ell, n \leq m - 2$. (Here we assume $W$ is $T, \rho$ independent.) Note \[6.69\] holds for $\rho \in V(\rho_0^0)$. Covering the compact set $V_1(\epsilon_2) \times \nu V_2(\epsilon_2)$ by finitely many such open sets $V(\rho_0^0)$, the same holds for any $\rho \in V_1(\epsilon_2) \times \nu V_2(\epsilon_2)$.

By definition we have
\[
(V_{T,1,(\kappa+1)}, V_{T,2,\rho,T}, \Delta_{\rho, T}(\kappa+1)) = (I_{\rho,T}^{-1}(\rho,T) \circ D_{(\kappa),\rho,T} \circ (I_{\rho,T}^{-1}(\rho,T,\kappa)))^{-1}(\rho,T) \cdot \rho(\kappa+1).)
\]

We remark that $(I_{\rho,T}^{-1}(\rho,T,\kappa+1))^{-1}(\rho,T) \cdot \rho(\kappa+1)$ is estimated by \[6.9\] since the isomorphism \[6.15\] nothing but $(I_{\rho,T}^{-1}(\rho,T,\kappa))^{-1}$.

Therefore, using Lemma \[6.21\] we derive \[6.8\] for $\kappa + 1$, by choosing $T_{2,m,\epsilon(0)}$ such that $C_{m,\epsilon(0)} \leq C_{5,m,\mu}$.

The proof of Theorem \[6.4\] is now complete. \[\square\]

7. Surjectivity and Injectivity of the Gluing Map

In this section we prove surjectivity and injectivity of the map $\text{Glue}_T$ in Theorem \[3.13\] and complete the proof of Theorem \[3.13\]. The proof goes along the line of \[D1\]. (See also \[F1\].) The surjectivity proof along the line of this section is written in \[F0n\] Section 14] and injectivity is proved in the same way. \[F0n\] Section 14] studies the case of pseudoholomorphic curve without boundary. It however can be easily adapted to the bordered case as we mentioned in \[F0001\] page 417 lines 21-26. Here we explain the argument in our situation in more detail.

We begin with the following a priori estimate.

**Proposition 7.1.** There exist $\epsilon_5, C_{m,\epsilon(7.1)}$ such that if $u : (\Sigma_T, \partial \Sigma_T) \to (X, L)$ is an element of $M^{\epsilon_5 \oplus \epsilon_2}((\Sigma_T, \partial \Sigma_T); (\Sigma_T, \partial \Sigma_T))$ for $0 < \epsilon < \epsilon_5$ then we have
\[
\left\| \frac{\partial u}{\partial T} \right\|_{C^0((\Sigma_T, \partial \Sigma_T) \times [0,1])} \leq C_{m,\epsilon(7.1)} e^{-\delta_1(5\tau - |\tau|)}. \tag{7.1}
\]

This is a consequence of Lemma \[2.5\] We also have the following:

**Lemma 7.2.** $M^{\epsilon_5 \oplus \epsilon_2}((\Sigma_T, \partial \Sigma_T); (\Sigma_T, \partial \Sigma_T))$ is a smooth manifold of dimension $\dim V_1 + \dim V_2 - \dim L$.

**Proof.** Assumption \[3.12\] and the Mayer-Vietoris principle \[M2\] imply that the linearized operator:
\[
W_{m+1,0}(\Sigma_T, \partial \Sigma_T; u^*TX, u^*TL) \to L^2_{m,0}(\Sigma_T; u^*TX \otimes \Lambda^{0,1})(E_1(u) \oplus E_2(u))
\]
of the equation \[3.19\], which is defined by
\[
V \mapsto (D_u)(\Sigma_T) - (D_u)(\epsilon_1, V) - (D_u)(\epsilon_2, V) \mod E_1(u) \oplus E_2(u)
\]
is surjective. (See \[F0001\] Proposition 7.1.27.) Here $D_u = (\epsilon_1, \epsilon_2) \in E_1(u) \oplus E_2(u)$. The lemma then follows from the implicit function theorem to solve the equation $\mathcal{D}u' = E_1(u) \oplus E_2(u)$, for $u'$.

7 Here surjectivity means the second half of the statement of Theorem \[3.13\] that is ‘The image contains $M^{\epsilon_5 \oplus \epsilon_2}((\Sigma_T, \partial \Sigma_T); (u_1, u_2))_{(\epsilon_1), (\epsilon_2), (\Sigma_T)}$.’
Proof of surjectivity. Let \( u : (\Sigma_T, \partial \Sigma_T) \to (X, L) \) be an element of \( \mathcal{M}^{\xi_1 \oplus \xi_2}((\Sigma_T, \tilde{z}); u_1, u_2)_\epsilon \).
The purpose here is to show that \( u \) is in the image of \( \text{Glue}_T \). We define \( u'_i : (\Sigma_i, \partial \Sigma_i) \to (X, L) \) as follows. We put \( p_0^i = u(0, 0) \in L \).

\[
\begin{align*}
u'_1(z) &= \begin{cases} 
\text{Exp}(p_0^i, \chi_B^-(\tau - T, t)E(p_0^i, u(\tau, t))) & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0, 1] \\
u(z) & \text{if } z \in K_1 \\
p_0^i & \text{if } z \in [5T, \infty) \times [0, 1].
\end{cases} \\
u'_2(z) &= \begin{cases} 
\text{Exp}(p_0^i, \chi_B^+(\tau + T, t)E(p_0^i, u(\tau, t))) & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0, 1] \\
u(z) & \text{if } z \in K_2 \\
p_0^i & \text{if } z \in (-\infty, -5T) \times [0, 1].
\end{cases}
\end{align*}
\]

(7.2)

Proposition 7.1 implies

\[
\| \Pi_{E_i^{\epsilon_i}(u_i)} \partial u'_i \|_{L^2_{m,\epsilon}(\Sigma_i)} \leq C_m(7.3) e^{-\delta_i T}.
\]

On the other hand, by assumption and elliptic regularity we have

\[
\| u'_i - u_i \|_{W^2_{m+1, \epsilon}(\Sigma_i)} \leq C_m(7.4) \epsilon.
\]

(7.4)

Here and henceforth in this section we will abuse the notation \( u'_i - u_i = E(u_i, u'_i) \) as we mentioned in Remark 2.2.

Therefore applying an implicit function theorem we obtain the following:

**Lemma 7.3.** For each \( \epsilon < (10) \) there exist \( \epsilon_m, \epsilon(10). (7.5) \) and \( C_m. (7.6) \) with the following properties. If \( u \in \mathcal{M}^{\xi_1 \oplus \xi_2}((\Sigma_T, \tilde{z}); u_1, u_2)_\epsilon \) with \( \epsilon < \epsilon_m, \epsilon(10). (7.5) \) then there exist \( \rho_i \in V_i \) (\( i = 1, 2 \)) such that

\[
\| u'_i - u_i^\rho \|_{W^2_{m+1, \epsilon}(\Sigma_i)} \leq C_m(7.6) e^{-\delta_i T}.
\]

(7.5)

We put \( \rho = (\rho_1, \rho_2) \in V_1 \times_L V_2 \). Then \( u \in \mathcal{M}^{\xi_1 \oplus \xi_2}((\Sigma_T, \tilde{z}); u_1, u_2)_\epsilon \) also implies

\[
|\rho_i| \leq \epsilon.
\]

(7.6)

By (7.5) we have

\[
\| u - u_T^\rho \|_{W^2_{m+1, \epsilon}(\Sigma_T)} \leq C_m(7.7) e^{-\delta_i T}.
\]

(7.7)

Here \( u_T^\rho = \text{Glue}_T(\rho) \).

We put \( V = E(u_T^\rho, u) \in \Gamma((\Sigma_T, \partial \Sigma_T); (u_T^\rho)^*TX; (u_T^\rho)^*TL) \). Then

\[
u(z) = \text{Exp}(u_T^\rho(z), V(z)).
\]

For \( s \in [0, 1] \) we define \( u^s : (\Sigma_T, \partial \Sigma_T) \to (X, L) \) by

\[
u^s(z) = \text{Exp}(u_T^\rho(z), sV(z)).
\]

(7.8)

(7.7) implies

\[
\| \Pi_{E_i^{\epsilon_i}(u_i)} \partial u^s \|_{L^2_{m,\epsilon}(\Sigma_T)} \leq C_m(7.9) e^{-\delta_i T}
\]

(7.9)

and

\[
\| \frac{\partial}{\partial s} u^s \|_{W^2_{m+1, \epsilon}(\Sigma_T)} \leq C_m(7.10) e^{-\delta_i T}
\]

(7.10)

for each \( s \in [0, 1] \).
Lemma 7.4. If $T$ is sufficiently large, then there exists $\tilde{u}^s : (\Sigma_T, \partial \Sigma_T) \to (X, L)$ \linebreak $(s \in [0, 1])$ with the following properties.

1. \[ \overline{\partial} \tilde{u}^s \equiv 0 \mod (E_1 \oplus E_2)(\tilde{u}^s). \]
2. \[ \left\| \frac{\partial}{\partial s} \tilde{u}^s \right\|_{W^{2, m+1, \delta}(K^S_1)} \leq C_{m,S} e^{-\delta_1 T}. \quad (7.11) \]
3. $\tilde{u}^s = u^s$ for $s = 0, 1$.

Proof. Run the alternating method described in Section 5 in the one-parameter family version. Since $u^s$ is already a solution for $s = 0, 1$, it does not change. More precisely, we regard $u^s$ in (7.8) as $u^s_{T,(0)} = u^s_{T,(0)}$ and start our inductive construction at Step 0-3 (Lemma 5.2).

Then for $s = 0, 1$,
\[ \text{Err}^s_{1,T,(0)} = \chi_X (\overline{\partial} u^s_{T,(0)} - \epsilon^s_{1,T,(0)}) = 0. \]

We can show $\text{Err}^s_{1,T,(0)} = 0$ in the same way. (Here $\text{Err}^s_{1,T,(0)}$ is as in Definition 5.5.) Hence $V^s_{1,T,(1)} = V^s_{T,2,(1)} = 0$ for $s = 0, 1$ by (5.29). Therefore $u^s_{T,(1)} = u^s_{T,(0)}$ by Definition 5.16. Thus $u^s_{T,(\kappa)} = u^s_{T,(0)} = u^s$ for all $\kappa$ and $s = 0, 1$.

Then $\tilde{u}^s = \lim_{\kappa \to \infty} u^s_{T,(\kappa)} = u^s$ for $s = 0, 1$ follows.

(7.11) is proved in the same way as the the estimate of $\rho$ derivatives in Section 6.

Lemma 7.5. The map $\text{Glue}_T : V_1(\epsilon) \times_L V_2(\epsilon) \to \mathcal{M}^{E_1 \oplus E_2}((\Sigma_T, \vec{z}); u_1, u_2)_\epsilon$ is an immersion if $\epsilon < \epsilon_2$ and $T$ is sufficiently large.

Proof. We consider the composition of $\text{Glue}_T$ with
\[ \mathcal{M}^{E_1 \oplus E_2}((\Sigma_T, \vec{z}); u_1, u_2)_\epsilon \to L_{m+1}^{2}((K^S_1, K^S_1 \cap \partial \Sigma_T), (X, L)) \]
defined by restriction. In the case $T = \infty$ this composition is obtained by restriction of maps. By the unique continuation, this is certainly an immersion for $T = \infty$. Then the exponential decay property stated in Theorem 6.4 implies that it is an immersion for sufficiently large $T$.

We will prove $u$ is in the image of $\text{Glue}_T$ by showing that the following set
\[ A = \{ s \in [0, 1] \mid \tilde{u}^s \in \text{image of } \text{Glue}_T \} \]
is nonempty, open and closed in $[0, 1]$. Obviously $0 \in A$ since $\tilde{u}^0 = u^0$. Lemma 7.2 implies that $\mathcal{M}^{E_1 \oplus E_2}((\Sigma_T, \vec{z}); u_1, u_2)_\epsilon$ is a smooth manifold and has the same dimension as $V_1 \times_L V_2$. Therefore Lemma 7.5 implies that $A$ is open. The closedness of $A$ follows from (7.11).

Therefore $1 \in A$. Namely $u$ is in the image of $\text{Glue}_T$ as required.

Proof of injectivity. Let $\rho^j = (\rho_1^j, \rho_2^j) \in V_1 \times_L V_2$ for $j = 0, 1$. We assume
\[ \text{Glue}_T(\rho^0) = \text{Glue}_T(\rho^1) \]
and
\[ \| \rho^j \| < \epsilon. \quad (7.13) \]
We will prove that $\rho^0 = \rho^1$ if $T$ is sufficiently large and $\epsilon$ is sufficiently small. We may assume that $V_1 \times_L V_2$ is connected and simply connected. Then, we have a path $s \mapsto \rho^s = (\rho_1^s, \rho_2^s) \in V_1 \times_L V_2$ such that
We put $V(s) = E(w^0_1, w^0_2) \in \Gamma((\Sigma_T, \partial \Sigma_T); (w^0_1)^*TX; (w^0_2)^*TL)$. Then

$$w^0_1(z) = \text{Exp}(w^0_1(z), V(s)(z)).$$

(By (2) $w^0_1(z)$ is $C^0$-close to $w^0_2(z)$, as $\epsilon \to 0$. Therefore $V(s)$ is well defined if $\epsilon$ is small.) Note $V(1) = V(0)$ since $w^1 = w^0$. Then for $w \in D^2 = \{w \in \mathbb{C} \mid |w| \leq 1\}$, there exists $V(w)$ such that

1. $V(s) = V(w)$ if $w = e^{2\pi \sqrt{-1}s}$.
2. $\frac{\partial}{\partial x} V(w) + \frac{\partial}{\partial y} V(w) \leq \Phi_2(\epsilon) + \Psi_2(T)$ (7.14)

where $\lim_{s \to 0} \Phi_2(\epsilon) = 0, \lim_{s \to \infty} \Psi_2(T) = 0$.

We put $u^w(z) = \text{Exp}(w^0_2(z), V(w)(z))$.

**Lemma 7.6.** If $T$ is sufficiently large and $\epsilon$ is sufficiently small then there exists $\tilde{u}^w : (\Sigma_T, \partial \Sigma_T) \to (X, L) (s \in [0, 1])$ with the following properties.

1. $\bar{\partial} \tilde{u}^w \equiv 0 \mod (\mathcal{E}_1 \oplus \mathcal{E}_2)(\tilde{u}^w)$.
2. $\left\| \frac{\partial}{\partial x} \tilde{u}^w \right\|_{W^2_{m+1,\delta}(\Sigma_T)} + \left\| \frac{\partial}{\partial y} \tilde{u}^w \right\|_{W^2_{m+1,\delta}(\Sigma_T)} \leq \Phi_3(\epsilon) + \Psi_3(T)$ (7.15)

with $\lim_{s \to 0} \Phi_3(\epsilon) = 0, \lim_{s \to \infty} \Psi_3(T) = 0$.
3. $\tilde{u}^w = u^w$ for $w \in \partial D^2$.

**Proof.** Run the alternating method described in Subsection 5 in the two-parameter family version. (3) is proved in the same way as Lemma 7.4.

**Lemma 7.7.** If $T$ is sufficiently large and $\epsilon$ is sufficiently small, there exists a smooth map $F : D^2 \to V_1 \times_L V_2$ such that

1. $\text{Glue}_T(F(w)) = \tilde{u}^w$.
2. If $s \in [0, 1]$ then we have:

$$F(e^{2\pi \sqrt{-1}s}) = \rho^s.$$

**Proof.** Note that $\rho \mapsto \text{Glue}_T(\rho)$ is a local diffeomorphism. So we can apply the proof of homotopy lifting property as follows. Let $D^2(r) = \{z \in \mathbb{C} \mid |z - (r - 1)| \leq r\}$. We put

$$A = \{r \in [0, 1] \mid \exists F : D^2(r) \to V_1 \times L V_2 \text{ satisfying (1) above and } F(-1) = \rho^{1/2}\}.$$  

Since $\text{Glue}_T(\rho)$ is a local diffeomorphism, $A$ is open. We can use (7.15) to show closedness of $A$. Then, since $0 \in A$, it follows that $1 \in A$. The proof of Lemma 7.7 is complete.

Lemma 7.7 implies $\rho^0 = \rho^1$. The proof of Theorem 3.13 is now complete.
8. Exponential decay estimate implies smoothness of coordinate change

In this section we demonstrate the way how we use Theorems 3.13 and 6.4 to prove smoothness of coordinate change of the Kuranishi structure of the moduli space of bordered stable maps. Here we provide an argument in the case when we glue two source curves which are non-singular and stable. The proof of the general case is given in [FOOO2, Part IV]. (See especially its Section 21.)

8.1. Including deformation of source curve. We first generalize Theorems 3.13 and 6.4 and include the deformation of complex structure of the source curves \((\Sigma, z)\). We consider the situation of Theorem 3.13. Note we assumed that \((\Sigma, z)\) is stable. We also remark that there is no automorphism of \((\Sigma, z)\). (See Remark 3.10.)

Let \(g\) be the genus of \(\Sigma\). For simplicity of notation we assume that the boundary of \(\Sigma\) is connected. Let \(k_i + 1\) be the number of boundary marked points. (For the sake of simplicity of notations we consider the case when we have only boundary marked points. The case when we also have interior marked points can be studied in the same way.) We denote by

\[ M_{g, k_i + 1} \] (8.1)

the moduli space of bordered and marked Riemann surfaces with the same topological type as \((\Sigma, z)\). Let \(\mathcal{CM}_{g, k_i + 1}\) be its compactification which consists of stable marked bordered curves. (See [FOOO1, Subsection 2.1.2] for its definition.)

Let \(V_i\) be a neighborhood of \([\Sigma, z]\) in \(M_{g, k_i + 1}\) and

\[ \pi_i : C_i \to V_i, \quad \tilde{\pi}_i : V_i \to \mathcal{CM}_{g, k_i + 1} \]

be the universal family. Namely \(C_i\) has a fiberwise complex structure such that \((\pi_i^{-1}(\sigma), (z_{0}(\sigma), \ldots, z_{k_i}(\sigma)))\) is a representative of \(\sigma \in V_i\).

We may choose \(V_i\) so small that the bundle (8.2) is topologically trivial. We fix a trivialization \(V_i \cong |\Sigma_i| \times V_i\) such that \(\pi_i\) is the projection to the second factor and that

\[ \tilde{\pi}_i : V_i \to \mathcal{CM}_{g, k_i + 1} \]

are constant maps. (Here \(|\Sigma_i|\) is the bordered surface \(\Sigma_i\) with its complex structure forgotten. Hereafter we write \(\Sigma_i\) in place of \(|\Sigma_i|\) by a slight abuse of notation.) We denote

\[ \Sigma_i(\sigma) = \pi_i^{-1}(\sigma), \quad \tilde{z}_i(\sigma) = (z_{0}(\sigma), \ldots, z_{k_i}(\sigma)). \]

We will next define \(\Sigma_1(\sigma_1) \# \Sigma_2(\sigma_2) = \Sigma_T\), which is obtained from \(\Sigma_1(\sigma_1)\) and \(\Sigma_2(\sigma_2)\) by gluing. To specify the way to glue we need to fix families of coordinates at the 0-th marked points \(z_{i, 0} \in \Sigma_i\).

For our purpose it is useful to take an analytic family of coordinates, which we define below. (Definition 8.5.) To define it we start with an analogue of closed Riemann surface. Let \(M_{g, \ell + 1}\) be the moduli space of Riemann surface of genus \(g\) with \(\ell + 1\) marked points and \(\mathcal{CM}^{cl}_{g, \ell + 1}\) its compactification consisting of stable curves. (See [DM,].)

Let

\[ \pi : \mathcal{CM}^{cl}_{g, \ell + 1} \to \mathcal{CM}^{cl}_{g, \ell + 1} \] (8.3)

be the universal family. Let \([\Sigma, \bar{z}] \in \mathcal{CM}^{cl}_{g, \ell + 1}\) and \(\Gamma\) the automorphism group of \([\Sigma, \bar{z}]\). Then a neighborhood of \([\Sigma, \bar{z}]\) in \(\mathcal{CM}^{cl}_{g, \ell + 1}\) may be regarded as \(\mathcal{V}/\Gamma\), where \(\mathcal{V}\) is a complex manifold. \(\pi^{-1}(\mathcal{V}/\Gamma)\) is identified with \(\mathcal{C}^{cl}(\mathcal{V})/\Gamma\) where \(\mathcal{C}^{cl}(\mathcal{V})\) is a
Let as follows. (Here Glusoc stands for “gluing source”.)

\[(\Sigma(\sigma), \tilde{z}(\sigma)) = (r^{-1}(\sigma), (z_0(\sigma), \ldots, z_\ell(\sigma)))\]

is a representative of \(\sigma\). We put \(z_i(\sigma) = \tilde{z}_i(\sigma)\).

**Definition 8.1.** A complex analytic family of coordinates at \(j\)-th marked point on \(\mathcal{V}\) is a map \(\tilde{\varphi} : D^2 \times \{\sigma\} \to \mathcal{C}^{cl}(\mathcal{V})\) with the following properties.

1. \(\tilde{\varphi}\) is a biholomorphic map onto its image, which is an open subset.
2. \(\pi \circ \tilde{\varphi} : D^2 \times \{\sigma\} \to \mathcal{V}\) coincides with the projection to the second factor.
3. \(\tilde{\varphi}(0, \sigma) = z_j(\sigma)\).
4. We consider the \(\Gamma\) action on \(T_{z_j} \Sigma \cong \mathbb{C}\) and use it to define a \(\Gamma\) action on \(D^2\). Then \(\tilde{\varphi}\) is \(\Gamma\) invariant.

Definition 8.1 implies that the restriction of \(\tilde{\varphi}\) to \(D^2 \times \{\sigma\}\) becomes a complex coordinate of \(\Sigma(\sigma)\) at \(z_j(\sigma)\).

The existence of complex analytic family of coordinates is a consequence of \(\Gamma\) equivariant version of implicit function theorem in complex analytic category and is standard. (See Appendix H.)

Let \(\mathcal{V}_i\) be neighborhoods of \([\Sigma_{i1}^{cl}, \tilde{z}_i] \in \mathcal{M}_{g_i, \ell_i+1}^{cl}\). We assume that \([\Sigma_{i1}^{cl}, \tilde{z}_1] \neq [\Sigma_{i2}^{cl}, \tilde{z}_2]\). We put \(\Gamma_i = \text{Aut}(\Sigma_{i1}^{cl}, \tilde{z}_i)\). Let \(\tilde{\varphi}_i : D^2 \times \mathcal{V}_i \to \mathcal{C}^{cl}(\mathcal{V}_i)\) be complex analytic families of coordinates at 0-th marked points on neighborhoods of \([\Sigma_{i1}^{cl}, \tilde{z}_i]\).

We identify \(z_{1,0} \in \Sigma_1^{cl}\) with \(z_{2,0} \in \Sigma_2^{cl}\). Then we obtain an element

\([\Sigma_{\infty}^{cl}, \tilde{z}_{\infty}] \in \mathcal{CM}^{cl}_{g_1+g_2, \ell_1+\ell_2}\).

Using our complex analytic families of coordinates we define a map

\[\text{Glusoc} : \mathcal{V}_1 \times \mathcal{V}_2 \times D^2(\epsilon) \to \mathcal{CM}^{cl}_{g_1+g_2, \ell_1+\ell_2}\] (8.4)

as follows. (Here Glusoc stands for “gluing source”.)

Let \(\sigma_i \in \mathcal{V}_i\) and \((\Sigma_i(\sigma_i), \tilde{z}_i(\sigma_i))\) be a marked Riemann surface representing it.

We define \(\varphi_{i,\sigma_i} : D^2 \to \Sigma_i(\sigma_i)\) by

\[\varphi_{i,\sigma_i}(z) = \tilde{\varphi}_i(z, \sigma_i)\]. (8.5)

Let \(r \in D^2(\epsilon)\). We consider the disjoint union

\[(\Sigma_1(\sigma_1) \setminus \varphi_{1,\sigma_1}(D^2(|r|))) \cup (\Sigma_2(\sigma_2) \setminus \varphi_{2,\sigma_2}(D^2(|r|)))\].

(Here and hereafter \(D^2(r) = \{z \in \mathbb{C} \mid |z| < r\}\).) We identify \(\varphi_{1,\sigma_1}(z) \in \Sigma_1(\sigma_1)\) \(\varphi_{2,\sigma_2}(w) \in \Sigma_2(\sigma_2)\) \(D^2(|r|)\) if \(zw = r\).

See Figure 10

By this identification we obtain a Riemann surface, which we denote by \(\Sigma_1(\sigma_1) \#_r \Sigma_2(\sigma_2)\).

We put

\[\text{Glusoc}(\sigma_1, \sigma_2, r) = (\Sigma_1(\sigma_1) \#_r \Sigma_2(\sigma_2), \tilde{z}_1'(\sigma_1) \cup \tilde{z}_2'(\sigma_2)),\] (8.6)

where \(\tilde{z}_i'(\sigma) = \tilde{z}_i(\sigma_i) \setminus \{z_{i,0}(\sigma_i)\}\).

We define \(\Gamma_i\) action on \(D^2(\epsilon)\) by identifying \(D^2(\epsilon)\) with the ball of radius \(\epsilon\) centered at origin in the tangent space \(T_{z_{i,0}} \Sigma_i\). Then Glusoc is \(\Gamma_1 \times \Gamma_2\) equivariant.

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8In case \([\Sigma_{i1}^{cl}, \tilde{z}_1] = [\Sigma_{i2}^{cl}, \tilde{z}_2]\) we may have extra \(\mathbb{Z}_2\) symmetry. Other than that the argument is the same as the case \([\Sigma_{i1}^{cl}, \tilde{z}_1] \neq [\Sigma_{i2}^{cl}, \tilde{z}_2]\).
Lemma 8.2. The map $\text{Glusoc}$ induces a biholomorphic map

$$\frac{\mathcal{V}_1 \times \mathcal{V}_2 \times D^2(\epsilon)}{\Gamma_1 \times \Gamma_2} \to \mathcal{CM}^{cl}_{g_1+g_2,\ell_1+\ell_2}$$

onto an open neighborhood of $[S^{cl}, \vec{z}]$.

Proof. We define

$$\Omega_1 = \mathcal{V}_2 \times \bigcap_{r \in D^2(\epsilon), \sigma_1 \in \mathcal{V}_1} (\Sigma_1(\sigma_1) \setminus \varphi_1, \sigma_1(D^2(|r|))) \times \{r\}, \quad (8.7)$$

$$\Omega_2 = \mathcal{V}_1 \times \bigcap_{r \in D^2(\epsilon), \sigma_2 \in \mathcal{V}_2} (\Sigma_2(\sigma_2) \setminus \varphi_2, \sigma_2(D^2(|r|))) \times \{r\}. \quad (8.8)$$

We regard $\Omega_1$ (resp. $\Omega_2$) as an open subset of $\mathcal{V}_2 \times C^{cl}(\mathcal{V}_1)$ (resp. $\mathcal{V}_1 \times C^{cl}(\mathcal{V}_2)$). So they are complex manifolds. We also put

$$\Omega_3 = \{ (z, w) \mid |z|, |w| < 1, \ zw < \epsilon \} \times \mathcal{V}_1 \times \mathcal{V}_2. \quad (8.9)$$

We identify $(z, w, \sigma_1, \sigma_2) \in \Omega_3$ with $(\sigma_2, \varphi_1(z), zw) \in \Omega_1$ and with $(\sigma_1, \varphi_2(w), zw) \in \Omega_2$. We obtain a complex manifold by this identification, which we denote by $\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \epsilon)$. We define $\pi : \mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \epsilon) \to \mathcal{V}_1 \times \mathcal{V}_2 \times D^2(\epsilon)$ by

$$\pi(\sigma_2, \tilde{z}, r) = (\pi_1(\tilde{z}), \sigma_2, r), \quad \text{on } \Omega_1,$$

$$\pi(\sigma_1, \tilde{w}, r) = (\sigma_1, \pi_2(\tilde{w}), r), \quad \text{on } \Omega_2,$$

$$\pi((z, w), \sigma_1, \sigma_2) = (\sigma_1, \sigma_2, zw), \quad \text{on } \Omega_3. \quad (8.10)$$

Here $\tilde{z} \in \mathcal{C}(\mathcal{V}_1)$, $\tilde{w} \in \mathcal{C}(\mathcal{V}_2)$, $r \in D^2(\epsilon)$ and $\pi_i : \mathcal{C}(\mathcal{V}_i) \to \mathcal{V}_i$ is the projection. $\pi$ is a well defined holomorphic map.

Sections $\delta_{i,j} : \mathcal{V}_i \to \mathcal{C}(\mathcal{V}_i)$, $j = 0, \ldots, k_i$ (which gives $j$-th marked point of the fiber) induce the sections $\gamma_j : \mathcal{V}_1 \times \mathcal{V}_2 \times D^2(\epsilon) \to \mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \epsilon)$ for $j = 1, \ldots, k_1 + k_2$ in an obvious way.

For $(\sigma_1, \sigma_2, r) \in \mathcal{V}_1 \times \mathcal{V}_2 \times D^2(\epsilon)$ it is easy to see from the definition that

$$(\pi^{-1}(\sigma_1, \sigma_2, r), (\gamma_j(\sigma_1, \sigma_2, r))_{j=1,\ldots,k_1+k_2})$$

is a representative of $\text{Glusoc}(\sigma_1, \sigma_2, \epsilon)$.

We furthermore observe that $\pi : \mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \epsilon) \to \mathcal{V}_1 \times \mathcal{V}_2 \times D^2(\epsilon)$ is $\Gamma_1 \times \Gamma_2$ equivariant.
Therefore the biholomorphicity of the map $\text{Glusoc}$ is a consequence of the definition of universal family.

Now we go back to the case of bordered surface. Let $[\Sigma, \vec{z}] \in M_{g,k+1}$. We take its double as in [FOOO1, page 44] to obtain a closed Riemann surface, $[\Sigma^\text{cl}, \vec{z}^\text{cl}] \in M^\text{cl}_{2g,k+1}$. There exits an anti-holomorphic involution $\tau: \Sigma^\text{cl} \to \Sigma^\text{cl}$ such that:

(a) The fixed point set $\Sigma^\text{cl}_R$ of $\tau$ is $S^1$, which contains all the marked points.
(b) The complement $\Sigma^\text{cl} \setminus \Sigma^\text{cl}_R$ consists of two connected components. The closure of one of them with marked points is biholomorphic to $(\Sigma, \vec{z})$.

See Figure 11. Let $\pi: \mathcal{C}(V) \to V$ be a universal family in a neighborhood of $[\Sigma^\text{cl}, \vec{z}^\text{cl}]$.

**Lemma 8.3.** There exist anti-holomorphic involutions $\tau: \mathcal{C}(V) \to \mathcal{C}(V)$, $\tau: V \to V$ with the following properties.

1. $\pi \circ \tau = \tau \circ \pi$.
2. The real dimension of the fixed point set $V^R$ of $\tau: V \to V$ is equal to the complex dimension of $V$. $V^R$ is identified with an open neighborhood of $[\Sigma, \vec{z}]$ in $\mathcal{M}_{g,k+1}$.
3. $\sigma_0 = [\Sigma^\text{cl}, \vec{z}^\text{cl}] \in V$ is a fixed point of $\tau$.
4. The restriction of $\tau: V \to V$ to the fiber of $\sigma_0$ coincides with the map $\tau: \Sigma^\text{cl} \to \Sigma^\text{cl}$.
5. We restrict the universal family to the fixed point set $V^R$ of $\tau: V \to V$.

We obtain a family of bordered marked Riemann surfaces in the same way as (b) applied to each of the fiber of $\sigma \in V^R$. This family is the universal family on $V^R \subset \mathcal{M}_{g,k+1}$ of bordered Riemann surfaces.

**Proof.** By [DM], $\mathcal{C}^\text{cl} \to \mathcal{M}^\text{cl}_{g,k+1}$ is a morphism in the category of the stacks defined over $\mathbb{R}$. The marked curve $[\Sigma^\text{cl}, \vec{z}^\text{cl}]$ together with $\tau: \Sigma^\text{cl} \to \Sigma^\text{cl}$ defines an $\mathbb{R}$-valued point of $\mathcal{M}^\text{cl}_{g,k+1}$. The lemma is a consequence of this fact. \qed
Lemma 8.4. In the situation of Lemma 8.3 there exists a complex analytic family of coordinates at the $j$-th marked point, $\varphi : D^2 \times \mathcal{V} \to C^c(\mathcal{V})$, in the sense of Definition 8.1 such that

$$\varphi(z, \tau(\sigma)) = \tau(\varphi(z, \sigma))$$  \hspace{1cm} (8.11)

in addition.

Proof. In place of $\Gamma = \text{Aut}(\Sigma, z^i)$ we consider its $\mathbb{Z}_2$ extension $\Gamma_+ = \Gamma \cup \{\gamma \tau \mid \gamma \in \Gamma\}$. Then the proof is the same as the proof of existence of complex analytic family of coordinates, which uses an equivariant version of implicit function theorem. For completeness’ sake, we provide the detail of the proof in Appendix H.

We take a complex analytic family of coordinates $\varphi : D^2 \times \mathcal{V} \to C^c(\mathcal{V})$ as in Lemma 8.4. Note $\Sigma = \Sigma(\sigma_0)$ is the closure of one of the connected components of $\Sigma^c \setminus \Sigma^c_R$. Replacing $\varphi$ by $(z, \sigma) \to \varphi(-z, \sigma)$ if necessary we may assume

$$\varphi(D^2_{\leq 0}(1) \times \{\sigma_0\}) \subset \Sigma.$$  

(Here and hereafter $D^2_{\leq 0}(r) = \{z \in D^2(r) \mid \text{Im} z \leq 0\}$.) Then for any $\sigma \in \mathcal{V}_R$ we have

$$\varphi(D^2_{\leq 0}(1) \times \{\sigma\}) \subset \Sigma(\sigma).$$

We identify the bordered Riemann surface $\Sigma(\sigma)$ as a connected component of $\Sigma^c(\sigma) \setminus \Sigma^c_R(\sigma)$ by Lemma 8.3 (5).

We thus obtain

$$\varphi^R : D^2_{\leq 0}(1) \times \mathcal{V}^R \to C(\mathcal{V}^R).$$  \hspace{1cm} (8.12)

Here $\pi : C(\mathcal{V}^R) \to \mathcal{V}^R$ is the universal family of bordered Riemann surfaces. For each $\sigma \in \mathcal{V}^R$ the map $\varphi^R$ determines a (complex) coordinate at the $j$-th marked point of the bordered surface $\Sigma(\sigma)$.

Definition 8.5. We define an analytic family of coordinates at the $j$-th marked point on $\mathcal{V}^R$ to be a map $\varphi^R$ as in (8.12) obtained from the complex analytic family of coordinates satisfying the conclusion of Lemma 8.4.

Let $[\Sigma_i, \bar{z}_i] \in \mathcal{M}_{g_i}; k_i+1 \times [\Sigma^c_i, \bar{z}_i^c] \in \mathcal{M}^c_{g_i}; k_i+1$. Let $\mathcal{C}(\mathcal{V}_i) \to \mathcal{V}_i$ be the universal family on a neighborhood $\mathcal{V}_i$ of $[\Sigma^c_i, \bar{z}_i^c]$ which satisfies the conclusion of Lemma 8.3. We take $\varphi_i : D^2(1) \times \mathcal{V}_i \to \mathcal{C}(\mathcal{V}_i)$ which is a complex analytic family of coordinates at the 0-th marked points. We assume that $\varphi_i$ satisfies the conclusion (8.11) of Lemma 8.4. Then we obtain analytic families of coordinates at 0-th marked points on $\mathcal{V}_i^R$,

$$\varphi_i^R : D^2_{\leq 0}(1) \times \mathcal{V}^R_i \to C(\mathcal{V}^R_i).$$  \hspace{1cm} (8.13)

We define $\varphi_{i,\sigma_i}^R : D^2_{\leq 0} \to \Sigma_i(\sigma_i)$ by

$$\varphi_{i,\sigma_i}^R(z) = \varphi^R_i(z, \sigma_i).$$  \hspace{1cm} (8.14)

Let $r \in [0, \epsilon)$, $(\sigma_1, \sigma_2) \in \mathcal{V}^R \times \mathcal{V}^R$. We define $\Sigma_1(\sigma_1) \# \Sigma_2(\sigma_2)$ as follows. Let $z_{i,0}(\sigma_i) \in \Sigma_i(\sigma_i)$ be the 0-th marked point. We consider the disjoint union

$$\left(\Sigma_1(\sigma_1) \setminus \varphi_{1,\sigma_1}^R(D^2_{\leq 0}(r))\right) \sqcup \left(\Sigma_2(\sigma_2) \setminus \varphi_{2,\sigma_2}^R(D^2_{\leq 0}(r))\right).$$

We define an equivalence relation on this set such that $\varphi_{1,\sigma_1}^R(z) \in \Sigma_1(\sigma_1) \setminus \varphi_{1,\sigma_1}^R(D^2_{\leq 0}(r))$, is equivalent to $\varphi_{2,\sigma_2}^R(w) \in \Sigma_2(\sigma_2) \setminus \varphi_{2,\sigma_2}^R(D^2_{\leq 0}(r))$ if and only if

$$zw = -r.$$  \hspace{1cm} (8.15)
Let $\Sigma_1(\sigma_1) \#_r \Sigma_2(\sigma_2)$ be the set of the equivalence classes of the above equivalence relation. It becomes a bordered curve. Thus we have defined

$$Glusoc^R : V_1^R \times V_2^R \times [0, \epsilon) \to \mathcal{CM}_{g_1+g_2, \ell_1+\ell_2}. \quad (8.16)$$

The next diagram commutes:

$$V_1^R \times V_2^R \times [0, \epsilon) \xrightarrow{Glusoc^R} \mathcal{CM}_{g_1+g_2, \ell_1+\ell_2} \xrightarrow{\varphi_{\sigma_1}} \mathcal{CM}_{g_1+g_2, \ell_1+\ell_2}. \quad (8.17)$$

Here $[0, \epsilon) \to D^2(\epsilon)$ in the first vertical arrow is $r \mapsto -r$. The other parts of the vertical arrows are obvious inclusions.

The source curve gluing map $Glusoc^R$ can be identified with the one we described at the beginning of Section 3 as follows. We put

$$\exp(-10\pi T) = r, \quad K_1(\sigma_1) = \Sigma_1(\sigma_1) \setminus \varphi_{\sigma_1}(D^2_{\leq 0}). \quad (8.18)$$

We define

$$\Sigma_1(\sigma_1) \setminus K_1(\sigma_1) \cong [0, \infty)_{t'} \times [0, 1], \quad \Sigma_2(\sigma_2) \setminus K_2(\sigma_2) \cong (-\infty, 0]_{t''} \times [0, 1].$$

by

$$\varphi_{\sigma_1}(e^{\pi(x+\sqrt{-1}y)}) \mapsto (-x, -y) = (\tau', t), \quad \varphi_{\sigma_2}(e^{\pi(x+\sqrt{-1}(y+1))}) \mapsto (x, y) = (\tau'', t).$$

Then $\Sigma_1(\sigma_1) \#_T \Sigma_2(\sigma_2)$ as in (3.2) is isomorphic to $\Sigma_1(\sigma_1) \#_R \Sigma_2(\sigma_2)$.

In fact $z, w$ appearing in (8.15) are related to the coordinate $\tau', \tau'', t$ of the neck region we used in Section 6 by

$$z = e^{-\pi(\tau'+\sqrt{-1}T)}, \quad w = -e^{\pi(\tau''+\sqrt{-1}T)}.$$

Note $zw = -r = -e^{-10\pi T}$ is equivalent to $\tau'' = \tau' - 10T$. See Figure 12.

![Figure 12. Coordinate at infinity of bordered curve.](image)

**Definition 8.6.** For $[\Sigma_i, \vec{z}_i]$ in $\mathcal{M}_{g_i;k_i+1}$ we consider the following set $\Xi_i$ of data:

1. A neighborhood $V_i^R$ of $[\Sigma_i, \vec{z}_i]$ in $\mathcal{M}_{g_i;k_i+1}$.
2. An analytic family of coordinates $\varphi_i^R$ at the 0-th marked points as in (8.13).
Let $\Xi = (\Xi^1, \Xi^2)$.

**Definition-Lemma 8.7.** Let $\Xi = (\Xi^1, \Xi^2)$ be a gluing data centered at $([\Sigma_1, z_1], [\Sigma_2, z_2])$.

1. It induces the map (8.16). We call it the source gluing map associated to $\Xi$ and write $\text{Glusoc}^\Sigma_{\Xi}$.
2. For any $\sigma = (\sigma_1, \sigma_2) \in V_1^R \times V_2^R$, it induces a holomorphic embedding
   \[ \tilde{\eta}_{\Xi, \sigma} : K_i(\sigma) \rightarrow \Sigma_1(\sigma_1) \#_r \Sigma_2(\sigma_2). \]
   We call it the canonical holomorphic embedding associated to $\Xi$. Here $\Sigma_1(\sigma_1) \#_r \Sigma_2(\sigma_2) \subset X$ together with marked points represents $\text{Glusoc}^\Sigma_{\Xi}(\sigma_1, \sigma_2, r)$.
3. For any $\sigma = (\sigma_1, \sigma_2) \in V_1^R \times V_2^R$, it also induces a smooth embedding
   \[ \eta_{\Xi, \sigma} : K_i \rightarrow \Sigma_1(\sigma_1) \#_r \Sigma_2(\sigma_2). \]
   We call it the canonical embedding associated to $\Xi$. Note $K_i \subset \Sigma_i$.

**Proof.** We discussed (1) already. (2) is a consequence of (8.18). (3) follows from (2) and the trivialization of the universal family bundle given in Definition 8.6(3). \qed

We recall the definitions of stable map and of its moduli spaces.

**Definition 8.8.** Two marked pseudoholomorphic maps $((\Sigma, z), u), ((\Sigma', z'), u')$ are said to be isomorphic, if there exists an isomorphism $\varphi : (\Sigma, z) \rightarrow (\Sigma', z')$ such that $u' = u \circ \varphi^{-1}$. A self-isomorphism $\varphi : (\Sigma, z) \rightarrow (\Sigma, z)$ is called an automorphism of $(\Sigma, z)$ if $u = u \circ \varphi^{-1}$. We denote
   \[ \text{Aut}((\Sigma, z), u) = \{ \varphi \in \text{Aut}(\Sigma, z) \mid u \circ \varphi = u \}. \]
   We call the pair $((\Sigma, z), u)$ a stable map if $\# \text{ Aut}((\Sigma, z), u)$ is finite. We define the moduli space of the isomorphism classes of bordered stable curves $u : (\Sigma, \partial \Sigma) \rightarrow (X, L)$ of genus $g$ with $k + 1$ boundary marked points and homology class $\beta \in H_2(X, L; \mathbb{Z})$ and denote it by
   \[ M_{g,k+1}(X, L; \beta). \]
   See [FOOO:1 Definition 2.1.27].

We next include maps and define obstruction spaces. Let $(\Sigma_i^{\text{eb}}, \varphi_i) \in M_{g_i, k_i+1}$, and let $u_i^{\text{eb}} : (\Sigma_i^{\text{eb}}, \partial \Sigma_i^{\text{eb}}) \rightarrow (X, L)$ be a pseudoholomorphic map.

**Definition 8.9.** An obstruction bundle data centered at $((\Sigma_1^{\text{eb}}, z_1^{\text{eb}}), u_1^{\text{eb}}), ((\Sigma_2^{\text{eb}}, z_2^{\text{eb}}), u_2^{\text{eb}}))$ consist of the objects $(\Xi^{\text{eb}}, (\sigma_1^{\text{eb}}, \sigma_2^{\text{eb}}))$ such that:
1. $\Xi^{\text{eb}} = (\Xi_1^{\text{eb}}, \Xi_2^{\text{eb}})$ is a gluing data centered at $((\Sigma_1^{\text{eb}}, z_1^{\text{eb}}), (\Sigma_2^{\text{eb}}, z_2^{\text{eb}}))$. 
(2) $E^{gb}$ is a finite dimensional subspace of $\Gamma(\Sigma_i^{gb} : (u_i^{gb})^*TX \otimes \Lambda^{0,1})$, the space of smooth sections. We assume that the support of elements of $E^{gb}$ are contained in Int $K_i^{gb}$. Here $K_i^{gb} = \Sigma_i^{gb} \setminus \phi_{i,gb}^R(D_{\Sigma_i^{gb}}^{2,1}(1))$. Note the map $\phi_{i,gb}^R$ is the analytic family of coordinates at the $0$-th marked point which is a part of $\Xi^{gb}$.

(3) $u_1^{gb}$, $u_2^{gb}$ satisfy Assumption \ref{assumption:3.12}. Here we replace $\Sigma_i$, $z_i$, $u_i$, $ev_{i,\phi}$ in Assumption \ref{assumption:3.12} by $\Sigma_i^{gb}$, $z_i^{gb}$, $u_i^{gb}$, $ev_{i,\phi,0}$.

We call $(((\Sigma_1^{gb}, z_1^{gb}), u_1^{gb})), ((\Sigma_2^{gb}, z_2^{gb}), u_2^{gb})))$ the obstruction center.

We define obstruction bundle $E_i(u')$ for an element $u' : (\Sigma', \partial\Sigma') \to (X, L)$ satisfying the next condition for $u'$ and $\nu$.

**Condition 8.10.** (1) $[\Sigma', \Sigma']$ is an element of the image of the source gluing map $Glusoc_{\Xi^{gb}}^R : \Sigma_1^{gb,\nu} \times \Sigma_2^{gb,\nu} \times [0, \nu) \to \mathcal{CM}_{g_1+g_2, \ell_1+\ell_2}$ associated with $\Xi^{gb}$.

(2) For $z \in K_i^{gb}$ we have
\[
d(u_i^{gb}(z), u'(\mathcal{G}_{\Xi^{gb},i}(z))) \leq \frac{\ell_X}{2}.
\]
Here $\ell_X$ is the constant defined as in Condition \ref{condition:2.3} and $\mathcal{G}_{\Xi^{gb},i}$ is the canonical embedding associated to $\Xi^{gb}$.

Now we define
\[
I_{u',i} : E_{i}^{gb} \to C^\infty(\Sigma', (u')^*TX \otimes \Lambda^{0,1}(\Sigma'))
\]
as follows. Let $z \in K_i^{gb}$. We have a parallel transport
\[
(Pal_i^{u'gb}(z))^j \cdot T_{u_i^{gb}}^b(z) \to T_{u'(z)}X
\]
defined in \ref{definition:2.5}. On the other hand we have a projection
\[
(\Lambda^{0,1}(\Sigma_i^{gb}))_z \to (\Lambda^{0,1}(\Sigma'))_z.
\]
We remark that the map \ref{definition:8.21} is complex linear. The linear map $I_{u',i}$ is induced by the tensor product (over $\mathbb{C}$) of the two maps \ref{definition:8.20} and \ref{definition:8.21}.

**Definition 8.11.** $E_i(u')$ is the image of the map $I_{u',i}$ in \ref{condition:8.10}.

**Remark 8.12.** We use a gluing data centered at $(\Sigma_1^{gb}, z_1^{gb})$ to define $I_{u',i}$ and $E_i(u')$. In other words we do not use one centered at $(\Sigma_i, \bar{z}_i)$. This is an important point which enables us to define coordinate change of Kuranishi structure. See Remark \ref{remark:8.26}.

We now define the moduli space we study. Consider an obstruction bundle data centered at $(((\Sigma_1^{gb}, z_1^{gb}), u_1^{gb})), ((\Sigma_2^{gb}, z_2^{gb}), u_2^{gb})))$, which we denote by $(\Xi^{gb}, (E_1^{gb}, E_2^{gb}))$. We also consider $(((\Sigma_1, \bar{z}_1), u_1), ((\Sigma_2, \bar{z}_2), u_2))$ and a gluing data $\Xi$ centered at $((\Sigma_1, \bar{z}_1), (\Sigma_2, \bar{z}_2))$.

**Condition 8.13.** We assume that the pair $(((\Sigma_1, \bar{z}_1), u_1), ((\Sigma_2, \bar{z}_2), u_2))$ is close to $(((\Sigma_1^{gb}, z_1^{gb}), u_1^{gb})), ((\Sigma_2^{gb}, z_2^{gb}), u_2^{gb})))$ in the following sense. We also assume that the neighborhoods $\mathcal{V}_i$ of $(\Sigma_i, \bar{z}_i)$ in $\mathcal{M}_{g_i, k_i, 1}$ which is a part of data $\Xi$ is small in the following sense.

(1) $(\Sigma_i, \bar{z}_i)$ is contained in the neighborhoods $\mathcal{V}_i^{gb}$ of $(\Sigma_i^{gb}, z_i^{gb})$ in $\mathcal{M}_{g_i, k_i, 1}$ which is a part of $\Xi^{gb}$. Moreover $\mathcal{V}_i \subset \mathcal{V}_i^{gb}$.
(2) Let \((\Sigma_i, \tilde{z}_i') \cong (\Sigma_i^b(\sigma_i^0), \tilde{z}_i'(\sigma_i^0))\) with \(\sigma_i^0 \in \mathcal{V}_i^b\). The two gluing data \(\Xi_i\) and \(\Xi_i^b\) determine a diffeomorphism

\[\gamma_i^0 : \Sigma_i^b \cong \Sigma_i^b(\sigma_i^0) \to \Sigma_i\]

(Here the first diffeomorphism is induced by \(\Xi_i^b\) and the second diffeomorphism is induced by \(\Xi_i\).) We require

\[\sup\{d(u_i(\gamma_i^0(z)), u_i^b(z)) \mid z \in \Sigma_i^b, i = 1, 2\} \leq \frac{i_X}{4}.\]  \hfill (8.22)

(3) We also require

\[\gamma_i^0(K_i^b) \subset \text{Int} K_i.\]  \hfill (8.23)

We recall that \((\Sigma_{\infty}, \tilde{z}_{\infty})\) is a union of \(\Sigma_1\) and \(\Sigma_2\), which are glued to each other at their 0-th marked points, which may also carry their marked points other than the 0-th ones.

**Definition 8.14.** We assume that \(((\Sigma_1^b, \tilde{z}_1'), u_1^b), ((\Sigma_2^b, \tilde{z}_2'), u_2^b)), (\Xi^b, (\mathcal{E}_1^b, \mathcal{E}_2^b)),\)

\(((\Sigma_1, \tilde{z}_1'), u_1), ((\Sigma_2, \tilde{z}_2), u_2))\) and \(\Xi\) satisfy Condition \[8.13\]

We define the moduli space \(\mathcal{M}_{+}^{\Xi_i^b, \Xi}(\Sigma_{\infty}, \tilde{z}; u_1, u_2)_{\epsilon, \nu}\) as the set of all the isomorphism classes of \((\Sigma', \tilde{z}', u')\) such that the following three conditions are satisfied.

1. \((\Sigma', \tilde{z}') = \text{GluSoc}_S^\nu(\sigma_1, \sigma_2, T) \in \text{Im}(\text{GluSoc}_S^\nu).\) \((\sigma_1, \sigma_2)\) are in the \(\epsilon\) neighborhood of \(\sigma_0\) in \(\mathcal{V}_1 \times \mathcal{V}_2\). \((\mathcal{V}_1\text{ and } \mathcal{V}_2\text{ are parts of } \Xi.\) Here \(\sigma_0\) corresponds to \((\Sigma_1, \tilde{z}_1)\) and \((\Sigma_2, \tilde{z}_2)\). \(T > 1/\nu.\)

2. Condition \[8.7\] is satisfied. Namely:
   (a) We assume \(u_i|_{K_i}\) is \(\epsilon\) close to \(u' \circ \mathcal{J}_{\Xi_i}|_{K_i}\) in \(C^1\) sense. Here \(\mathcal{J}_{\Xi_i}\) is defined in Definition-Lemma \[8.7\]
   (b) \(\text{Diam}\{u'(z) \mid z \in \Sigma' \setminus \{K_1 \cup K_2\}\} < \epsilon.\)

3. \(\mathcal{J}u' \equiv 0 \mod \mathcal{E}_1(u') \oplus \mathcal{E}_2(u')\).  \hfill (8.24)

Note (1), (2) and \[8.19\], \[8.22\] imply that \(\mathcal{E}_1(u'), \mathcal{E}_2(u')\) are defined if \(\epsilon\) is sufficiently small.

Note that if \((\Sigma_{\infty}^b, \partial \Sigma_{\infty}^b) \cong (\Sigma_i, \partial \Sigma_i)\) as bordered Riemann surfaces and \(\Xi_{\infty} = \Xi\)

the moduli space \(\mathcal{M}_{+}^{\Xi_i^b, \Xi}(\Sigma_{\infty}^b, \tilde{z}; u_1, u_2)_{\epsilon, \nu}\) we defined in Definition \[8.8\] is a subset of the moduli space \(\mathcal{M}_{+}^{\Xi_1^b, \Xi}(\Sigma_{\infty}, \tilde{z}_{\infty}; u_1, u_2)_{\epsilon, \nu}\) we defined here, for \(T > 1/\nu.\)

We also define

\[I_{u'} : \mathcal{E}_i^b \to \Gamma(\Sigma_i; (u'_i)^*TX \otimes \Lambda^{0,1}(\Sigma_i))\]  \hfill (8.25)

in the same way as \(I_{u'}.\) Namely it is induced from the tensor product over \(\mathbb{C}\) of the projection \((\Lambda^{0,1}(\Sigma_{\infty}^b))_z \to (\Lambda^{0,1}(\Sigma_i(\sigma_i)))_z)\), and the complex linear part of the parallel transport

\[\text{Pal}^u_{\bar{u}_i}((z)): T_{u'_{\bar{u}_i}(z)}X \to T_{u'_{\bar{u}_i}(z)}X.\]

We put

\[\bar{u}'_{u_i} = 0, \quad \mod \mathcal{E}_i(u'_{u_i}).\]  \hfill (8.27)

**Definition 8.15.** \(\mathcal{M}_{+}^{\Xi_i^b}(\Sigma_i, \tilde{z}_i; u_i)_\epsilon\) is the set of isomorphism classes of \((\Sigma_i', \tilde{z}_i'), u_i')\) with the following properties.
(1) \((\Sigma_i', \tilde{z}_i')\) represents an element \(\sigma_i\) of \(\mathcal{V}_i^B\) and is in an \(\epsilon\) neighborhood of \([\Sigma_i, \tilde{z}_i]\).

(2) \(u_i'\) is \(\epsilon\) close to \(u_i\) in \(C^1\) topology.

(3) \(u_i'\) satisfies equation (8.27).

We assumed Assumption 8.12 in Definition 8.9 (4), where \(\text{ev}_{i,\infty}\) in (3.23), (3.24) are \(\text{ev}_0\) here. Then we can take \(\epsilon\) small so that \(\mathcal{M}^{\Xi}_+((\Sigma_i, \tilde{z}_i); u_i)\) is a smooth manifold. Moreover the fiber product

\[
\mathcal{M}^{\Xi}_+ = \mathcal{M}^{\Xi}_+((\Sigma_1, \tilde{z}_1); u_1) \times_{\mathcal{V}_1^B} \mathcal{M}^{\Xi}_+((\Sigma_2, \tilde{z}_2); u_2),
\]

is transversal. (Note the evaluation map \(\text{ev}_0\) here corresponds to \(\text{ev}_{i,\infty}\) \((i = 1, 2)\) in Section 3.)

Now Theorems 3.13 and 6.4 are generalized as follows.

**Theorem 8.16.** For any \(\epsilon(11), \nu(1) > 0\) there exists \(\epsilon(12) > 0\), \(T_{7, m, \epsilon(11), \nu(1)} > 0\) and a map

\[
\text{Glue}_+ : \mathcal{M}^{\Xi}_+((\Sigma_1, \tilde{z}_1); u_1)_{\epsilon(12)} \times_{\mathcal{V}_1^B} \mathcal{M}^{\Xi}_+((\Sigma_2, \tilde{z}_2); u_2)_{\epsilon(12)} \times (T_{7, m, \epsilon(11), \nu(1)}, \infty)
\]

\[
\rightarrow \mathcal{M}^{\Xi}_+((\Sigma_\infty, \tilde{z}_{\infty}); u_1, u_2)_{\epsilon(11), \nu(1)}
\]

with the following properties.

1. The map \(\text{Glue}_+\) is a homeomorphism onto its image. The image contains

\[
\mathcal{M}^{\Xi}_+ = \mathcal{M}^{\Xi}_+((\Sigma_\infty, \tilde{z}_{\infty}); u_1, u_2)_{\epsilon(11), \nu(1)}_{\text{ev}_0},
\]

where \(\epsilon(11), \nu(1)\) are positive number depending on \(\epsilon(11), \nu(1)\).

2. The next diagram commutes

\[
\xymatrix{ \mathcal{M}^{\Xi}_+((\Sigma_1, \tilde{z}_1); u_1)_{\epsilon(12)} \times_{\mathcal{V}_1^B} \mathcal{M}^{\Xi}_+((\Sigma_2, \tilde{z}_2); u_2)_{\epsilon(12)} \times (T_{7, m, \epsilon(11), \nu(1)}, \infty) \ar[d]_{\text{Glue}_+} \ar[r] & \mathcal{V}_1^B \times \mathcal{V}_2^B \times (T_{7, m, \epsilon(11), \nu(1)}, \infty) \ar[d]_{\text{Glue}_+} \\
\mathcal{M}^{\Xi}_+((\Sigma_\infty, \tilde{z}_{\infty}); u_1, u_2)_{\epsilon(11), \nu(1)} \ar[r] & \mathcal{CM}_{9_1+9_2, \epsilon_i+\epsilon_2} }
\]

where the horizontal arrows are defined by forgetting the map part.

3. The map \(\text{Glue}_+\) defines a fiberwise diffeomorphism onto its image. Here fiber means the fiber of the horizontal arrows of (8.29).

This is a family version of Theorem 3.13 incorporating the variation of complex structures on the source curve into the gluing. A generalization of Theorem 6.4 is the following Theorem 8.17. We recall

\[
K_i^S = K_1 \cup ([0, S]_{\tau} \times [0, 1]), \quad K_2^S = K_2 \cup ([S, 0]_{\tau} \times [0, 1]).
\]

We define

\[
\text{Glue}_{i,+, S} : \mathcal{M}^{\Xi}_+((\Sigma_1, \tilde{z}_1); u_1)_{\epsilon(12)} \times_{\mathcal{V}_1^B} \mathcal{M}^{\Xi}_+((\Sigma_2, \tilde{z}_2); u_2)_{\epsilon(12)} \times (T_{7, m, \epsilon(11), \nu(1)}, \infty) \rightarrow \text{Map}_{1+}\mathcal{T}_{m+1}^{\Xi}(\mathcal{S}_{1, \Sigma}^S, \mathcal{S}^S_{1, \Sigma} \cap \partial\mathcal{S}_1), (X, L))
\]

as the composition of \(\text{Glue}_+\) with the restriction map

\[
\mathcal{M}^{\Xi}_+ = \mathcal{M}^{\Xi}_+((\Sigma_\infty, \tilde{z}_{\infty}); u_1, u_2)_{\epsilon(11), \nu(1)} \rightarrow \text{Map}_{1+}\mathcal{T}_{m+1}^{\Xi}(\mathcal{S}_{1, \Sigma}^S, \mathcal{S}^S_{1, \Sigma} \cap \partial\mathcal{S}_1), (X, L))
\]

Here we use the map \(\mathcal{Y}_{2, i}^{\Xi}\) in Lemma-Definition 8.7 to regard

\[
K_i^S \cong K_i^S(\sigma_i) \subset \Sigma_1(\sigma_i) \# T\Sigma_2(\sigma_2).
\]

Note the diffeomorphism \(K_i^S \cong K_i^S(\sigma_i)\) is induced by the gluing data \(\Xi\) centered at \((\Sigma_1, \tilde{z}_1), (\Sigma_2, \tilde{z}_2)\).
Theorem 8.17. There exists \( \delta_2 > 0 \) with the following property. For each \( m \) and \( S \), there exist \( T_{m,S} > \frac{\delta_2}{m} \) and \( C_{m,S} > 0 \) such that
\[
\left\| \nabla_{\sigma,\rho} \frac{d^n}{dt^n} \text{Glures}_{i,S} \right\|_{L^2_{m+1-\ell}} < C_{m,S} e^{-\delta_2 T} \tag{8.30}
\]
for all \( n, \ell \leq m \). Here \( \nabla_{\sigma,\rho} \) is a differentiation with respect to \((\sigma_1, \rho_1), (\sigma_2, \rho_2)\) the fiber product (8.28).

Proof of Theorems 8.16 and 8.17. We take and fix a section
\[
V_i \rightarrow \mathcal{M}^i_+((\Sigma_i, \vec{z}_i); u_i), \quad \sigma_i \rightarrow (\sigma_i, \rho^0_i(\sigma_i))
\]
of the projection
\[
\mathcal{M}^i_+((\Sigma_i, \vec{z}_i); u_i) \rightarrow V_i^R.
\]
We denote the map : \( \Sigma_i(\sigma_i) \rightarrow X \) corresponding to \((\sigma_i, \rho^0_i(\sigma_i)) \) in \( V_i \) by \( u_i^{\sigma_i} \).

For each \((\sigma_1, \sigma_2) \in V_1^R \times V_2^R \) and \( T \) we run the alternating method developed in Sections 5 and 6.

Namely we start with \((\Sigma_1(\sigma_1), \vec{z}_1(\sigma_1)), (\Sigma_2(\sigma_2), \vec{z}_2(\sigma_2)), u_i^{\sigma_i})\) in place of \((\Sigma_1, \vec{z}_1), u_1\), \((\Sigma_2, \vec{z}_2), u_2)\) and use the coordinates at 0-th marked points which we determined as a part of the gluing data \( Z \) centered at \((\Sigma_1, \vec{z}_1), (\Sigma_2, \vec{z}_2), u_2)\).

Except the point which we will explain below the proof then goes in the same way as before and we obtain a map
\[
V_i(\sigma_i, \epsilon) \times L V_2(\sigma_2, \epsilon) \times (T_0, \infty) \rightarrow \mathcal{M}^i_+ \oplus \mathcal{E}_2((\Sigma_\epsilon, \vec{z}); u_1, u_2; \epsilon, \nu).
\]
Here \( V_i(\sigma_i, \epsilon) \) is the fiber of the map \( \mathcal{M}^i_+((\Sigma_i, \vec{z}_i); u_i) \rightarrow V_i \) and \( T_0 > 1/\nu \). The union of these maps over \( \sigma_1, \sigma_2 \) will be our map Glue_+.

The point we need to clarify to adapt the proof of Sections 5 and 6 to our situation is the following: The way we define \( E_i(u') \) in this section is slightly different from that of Sections 5 and 6. Namely we start with \( E_{i}^{eb} \) defined on \( (\Sigma_i^{eb}, u_i^{eb}) \) in this section with \( \Sigma_i^{eb} \neq \Sigma_i \), while in Sections 5 and 6 \( E_{i}^{eb} \) was defined on \( (\Sigma_i, u_i^{eb}) \), i.e., \( \Sigma_i^{eb} = \Sigma_i \).

We use the next Proposition 8.19 to obtain required estimate of our \( E_i(u') \). Then the arguments in Sections 5 and 6 also go through for the proofs of Theorems 8.16 and 8.17.

In fact Theorem 8.16 (1),(3) are proved in the same way as Sections 5, 6 and 7. The estimate of \( \sigma \) derivative is the same as that of \( \rho \) derivative. So the proof of Theorem 8.17 is the same as that of Section 6. Theorem 8.16 (2) follows from the fact that the alternating method we use does not change the complex structure of the source.

To state Proposition 8.19 we need to prepare some notations.

Let \( (\Sigma', \vec{z}') = \text{Glures}_Z^{\Sigma}((\sigma_1, \sigma_2), T) \) and \( u' : (\Sigma', \partial \Sigma') \rightarrow (X, L) \) a smooth map satisfying
\[
\begin{align*}
&d(u'(z), u(z)) \leq \epsilon, \quad \text{for } z \in K_i \\
&\text{Diam}(u'(\Sigma' \setminus (K_1 \cup K_2))) \leq \epsilon.
\end{align*}
\tag{8.31}
\]
We take a basis \( e_{i,a} \) (\( a = 1, \ldots, \dim E_{i}^{eb} \)) of \( E_{i}^{eb} \subset \Gamma(K_i^{eb} ; (u_i^{eb})^*TX \otimes \Lambda^{0,1}(\Sigma_i^{eb})) \).

We put
\[
e_{i,a}^0(u') = (I_{u_i^{eb}} \circ I_{u'}^{\Sigma_i} \circ I_{u,a}^{\Sigma_i})(e_{i,a}) \in \Gamma(K_i ; (u_i)^*TX \otimes \Lambda^{0,1}(\Sigma_i)).
\tag{8.32}
\]

\( \sigma_1 \) and \( \sigma_2 \) parameterize the source curve and \( \rho_1 \) and \( \rho_2 \) parameterize the map.
Here the map $I_{u',i}$ is \((8.19)\) and the maps

\[
I_{u',i}^\sigma: \Gamma(\mathcal{J}_{\Xi}^\sigma(K_{i}^{ob}); (u')^*TX \otimes \Lambda^{0,1}(\Sigma')) \rightarrow \Gamma(\Sigma_i(\sigma_i); (u_i^\sigma)^*TX \otimes \Lambda^{0,1}(\Sigma_i))
\]

\[
I_{u_i}^\sigma: \Gamma(\Sigma_i(\sigma_i); (u_i^\sigma)^*TX \otimes \Lambda^{0,1}(\Sigma_i)) \rightarrow \Gamma(K_i; (u_i)^*TX \otimes \Lambda^{0,1}(\Sigma_i))
\]

are defined as follows. Note since $(\Sigma', \vec{z}) = \text{GluAsc}_{\Xi}^R((\sigma_1', \sigma_2'), T')$ we have a smooth embedding

\[
\mathcal{J}_{\Xi}^\sigma: K_i^{ob} \rightarrow \Sigma'
\]

by Definition-Lemma \ref{def:glue_embedding}. Its image appears in the domain of $I_{u',i}^\sigma$.

We use an embedding

\[
\mathcal{J}_{\Xi}^\sigma: K_i \rightarrow \Sigma'
\]

which is also obtained by Definition-Lemma \ref{def:glue_embedding} in the definition of $I_{u',i}^\sigma$.

**Remark 8.18.** We remark that in (8.33) we use the gluing data $\Xi_{ob}$ centered at $((\Sigma_1^{ob}, \vec{z}_1^{ob}), (\Sigma_2^{ob}, \vec{z}_2^{ob}))$. In (8.34) we use the gluing data $\Xi = (\Xi_1, \Xi_2)$ centered at $((\Sigma_1, \vec{z}_1), (\Sigma_2, \vec{z}_2))$.

We assume the next relation so that the composition is well-defined.

\[
\mathcal{J}_{\Xi}^\sigma: K_i^{ob} \subset \mathcal{J}_{\Xi}^\sigma(K_i)
\]

By Condition \ref{cond:glue_embedding} we may choose $\mathcal{V}_i$, a neighborhood of $[\Sigma_i, \vec{z}_i]$ in the moduli space of marked bordered curve, small such that if $(\sigma_1, \sigma_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ then (8.35) is satisfied.

**Figure 13.** Two gluing data gives different embedding.

The smooth embedding $\mathcal{J}_{\Xi}^\sigma$ induces a complex linear map

\[
\Lambda_{\Xi}^{0,1}(\Sigma') \rightarrow \Lambda_{\Xi}^{1,0}(\Sigma_i(\sigma_i)) \rightarrow \Lambda_{\Xi}^{0,1}(\Sigma_i(\sigma_i)).
\]
Here the second map is the projection and the first map is the complex linear part of the map induced by $\mathcal{D}_{\Xi_i}^2$. Namely (8.36) is given by

$$\frac{1}{2} ((\mathcal{D}_{\Xi_i}^2)^* - j_{\Sigma_i(\sigma_i)} \circ ((\mathcal{D}_{\Xi_i}^2)^{-1} \circ j_{\Sigma^u})).$$

We also consider the map:

$$\left( \left( (\text{Pal}_{u_i^{\ast}(z)}^{u_i^{\ast}}(z)) \right)^J \right)^{-1} : T_{u_i(z)} \rightarrow T_{u_i^{\ast}(z)} X. \quad (8.37)$$

Then $I_{u_i^{\ast}}$ is obtained from the tensor product of (8.36) and (8.37). $I_{u_i^0}$ is induced by the tensor product of the projection $\Lambda^0_i(\sigma_i) \rightarrow \Lambda^0_i(\Xi_i)$ and a complex linear part of the parallel transformation

$$\left( \left( (\text{Pal}_{u_i^{\ast}(z)}^{u_i^{\ast}}(z)) \right)^J \right)^{-1} : T_{u_i^{\ast}(z)} X \rightarrow T_{u_i(z)} X. \quad (8.39)$$

Let $(\sigma,\rho) = ((\sigma_1,\rho_1),(\sigma_2,\rho_2)) \in V_1 \times L \times V_2$ and $(\Sigma_T^{\sigma,\rho},\Sigma_T^{\rho,\sigma}) = \text{Glusoc}^R(\sigma_1,\sigma_2,T)$.

We denote by $\tilde{u}_{\sigma,T(\kappa)}^\rho : \Sigma_i^\sigma \cong \Sigma_i \rightarrow X$ the map obtained at the $\kappa$-th inductive step of our alternating method starting from $((\Sigma_1(\sigma_1),\bar{z}_1(\sigma_1)),u_1^{\ast})$, $((\Sigma_2(\sigma_2),\bar{z}_2(\sigma_2)),u_2^{\ast})$. The diffeomorphism $\Sigma_i^\sigma \cong \Sigma_i$ is a part of datum $\Xi_i$.

In other words $\tilde{u}_{\sigma,T(\kappa)}^\rho$ corresponds to the map $\tilde{u}_{\sigma,T(\kappa)}^\rho$ in Definition 5.32. We put

$$e_{i,\alpha;T(\kappa)}'(\sigma,\rho,T) = e_{i,\alpha}^0(u_{\sigma,T(\kappa)}^\rho) \in \Gamma(K_i;u_i^T X \otimes \Lambda^0(\Sigma_i)). \quad (8.40)$$

Here $e_{i,\alpha}^0(u_{\sigma,T(\kappa)}^\rho)$ is as in (8.32).

We remark that $e_{i,\alpha;T(\kappa)}'(\sigma,\rho,T)$ is a basis of

$$\left( I_{u_i^{\ast}}^0 \circ I_{u_{\sigma,T(\kappa)}^\rho}^0 \right) (\mathcal{E}_{i}(\tilde{u}_{\sigma,T(\kappa)}^\rho)).$$

Moreover there exists a canonical isomorphism

$$\mathcal{E}_{i}(\tilde{u}_{\sigma,T(\kappa)}^\rho) \cong \mathcal{E}_{i}(u_{\sigma,T(\kappa)}^\rho) \quad (8.41)$$

where $u_{\sigma,T(\kappa)}^\rho$ corresponds to $u_{\sigma,T(\kappa)}^\rho$ in Definition 5.32 (8.41) is a consequence of the equality $u_{\sigma,T(\kappa)}^\rho = u_{\sigma,T(\kappa)}^\rho$ on $K_i \subset \Sigma_i(\sigma_i)$ which we regard as a subset $K_i \subset \Sigma_T^2$. This equality follows from the definition (5.72).

**Proposition 8.19.** There exists $\delta_3 > 0$ such that we can estimate $e_{i,\alpha;T(\kappa)}'(\rho,\sigma,T)$ and its $T$, $\rho$, $\sigma$ derivatives as

$$\left\| \nabla_{\sigma,\rho}^n \frac{\partial^\ell}{\partial T^\ell} e_{i,\alpha;T(\kappa)}'(\sigma,\rho,T) \right\|_{L_2^{m+1-\ell}} \leq C_m \cdot (8.42) e^{-\delta_3 T}$$

for $m - 2 \geq n, \ell \geq 0, \ell > 0$, if $T > T_m,8.32$. 


This proposition corresponds to Lemma 6.20. The proof will be given in Subsection 8.3.

We can use Proposition 8.19 to control the obstruction bundle $E(u')$ together with its derivatives.

More explicitly: we can use the proposition to show (6.44): we use it to show a version of Lemma 6.20, where $\sigma$ derivative as well as $\rho$ and $T$ derivative is included and the definition of $E(u')$ is slightly modified as we mentioned at the beginning of Theorem 8.17.

We can use Proposition 8.19 also in all the other places where we need to control $E(u')$. Note we take $\delta$ such that not only $\delta < \delta_1/10$ but also $\delta < \delta_3/10$ holds. The constant $\delta_3$ in Theorem 8.17 is determined by $\delta_1$ and $\delta_3$.

The proofs of Theorems 8.16 and 8.17 are now complete except the proof of Proposition 8.19. 

\[ \square \]

8.2. Smoothness of coordinate change. We now use Theorem 8.16 to prove the smoothness of the coordinate change. We begin with explaining the situation where we study coordinate change.

**Situation 8.20.** Let $(\Sigma_i^b, z_i^b) \in \mathcal{M}_{g_i, k_i, 1}$. We take $\Xi^b = (\Xi_1^b, \Xi_2^b)$ a gluing data centered at $((\Sigma_1^b, z_1^b), (\Sigma_2^b, z_2^b))$. We consider $u_i^b : (\Sigma_i^b, \partial \Sigma_i^b) \to (X, L)$, a pseudoholomorphic map of homology class $\beta_i$. For $j = 1, 2$, let $(\Xi^b, (\xi^b_1, \xi^b_2))$ be an obstruction bundle data centered at $(((\Sigma_1^b, z_1^b), u_1^b), ((\Sigma_2^b, z_2^b), u_2^b))$. Note we use the same gluing data $\Xi^b$ for $j = 1, 2$.

We assume that the next inclusion holds for $i = 1, 2$.

\[ E^{(1)}_i \subseteq E^{(2)}_i. \]  

(8.43)

In sum we take two obstruction bundle data with the same gluing data and satisfying (8.43).

**Situation 8.21.** We next consider $((\Sigma_i^{(j)}, z_i^{(j)}), u_i^{(j)})$ for $i, j = 1, 2$. Here:

1. $(\Sigma_i^{(j)}, z_i^{(j)}) \in \mathcal{M}_{g_i, k_i, 1}$.
2. For each $i = 1, 2$, $j = 1, 2$, $u_i^{(j)} : (\Sigma_i^{(j)}, \partial \Sigma_i^{(j)}) \to (X, L)$ is a pseudoholomorphic map of homology class $\beta_i$.

We call $((\Sigma_1^{(j)}, z_1^{(j)}), u_1^{(j)}), ((\Sigma_2^{(j)}, z_2^{(j)}), u_2^{(j)}))$ for $j = 1, 2$ a chart center.

Let $\Xi^{(j)} = (\Xi_1^{(j)}, \Xi_2^{(j)})$ be a gluing data centered at $(\Sigma_i^{(j)}, z_i^{(j)})$. We assume

\[ \mathcal{V}^{(1),R}_i \subset \mathcal{V}^{(2),R}_i. \]  

(8.44)

where $\mathcal{V}^{(j),R}_i$ is a neighborhood of $[\Sigma_i^{(j)}, z_i^{(j)}]$ in $\mathcal{M}_{g_i, k_i, 1}$ which is a part of $\Xi_i^{(j)}$.

Note (8.44) implies $[\Sigma_i^{(1)}, z_i^{(1)}] \in \mathcal{V}^{(2),R}_i$.

We also assume Conditions 8.22 and 8.23 below.

When $u_i : (\Sigma_i, \partial \Sigma_i) \to (X, L)$ is given and satisfies $ev_{1, \infty}(u_1) = ev_{1, \infty}(u_2)$, we denote by $u_\infty : (\Sigma_\infty, \partial \Sigma_\infty) \to (X, L)$ the map which is restricted to $u_i$ on $\Sigma_i$.

We take and fix $\epsilon_6, \nu_1 > 0$. We put $\epsilon(1) = \epsilon_6$ and $\nu(1) = \nu_1$ in Theorem 8.16 (Here we apply Theorem 8.16 to Glue$^{(2)}_\rho$). We put

\[ \epsilon_7 = \epsilon(1), \nu(1), (8.29); \quad \nu_2 = \nu(1), (8.29). \]  

(8.45)

\[ \text{Actually Lemma 6.20 gives an estimate of the orthonormal frame obtained from } e_{i, a}(\rho, T, \kappa) \text{ by the Gram-Schmidt process. However the argument to study the Gram-Schmidt process in our situation is the same as the one in the proof of Lemma 6.20.} \]
where the right hand sides are obtained in Theorem 8.16.

**Condition 8.22.** We identify \((\Sigma_1^{(j)}, \varphi_1^{(j)})\) with \((\Sigma_2^{(j)}, \varphi_2^{(j)})\) at their 0-th marked points to obtain \((\Sigma_0^{(j)}, \varphi_0^{(j)})\). Together with \(u_1^{(j)}\) and \(u_2^{(j)}\) it gives an element of \(M_{g_1+g_2,k_1+k_2}(X, L; \beta)\) which we denote by \(((\Sigma_0^{(j)}, \varphi_0^{(j)}), u_0^{(j)})\). We require

\[
((\Sigma_1^{(1)}, \varphi_1^{(1)}), u_1^{(1)}) \in M_+^{(1)} \oplus \mathcal{E}_2^{(1)} - \big( ((\Sigma_0^{(2)}, \varphi_0^{(2)}), u_0^{(2)})_{\epsilon/2, \nu_2/2} \big). \quad (8.46)
\]

Roughly speaking Condition 8.22 means that \(u_1^{(1)}\) is close to \(u_1^{(2)}\).

**Condition 8.23.** Let \(\sigma = (\sigma_1, \sigma_2) \in \mathcal{V}_1^{(1), \mathbb{R}} \times \mathcal{V}_2^{(2), \mathbb{R}}\). By (8.44) \(\sigma \in \mathcal{V}_1^{(2), \mathbb{R}} \times \mathcal{V}_2^{(2), \mathbb{R}}\).

Using the gluing data \(\Xi^{(1)}\), \(\Xi^{(2)}\) taken in Situation 8.21 we obtain an embedding

\[
K_1^{(1)} \subset \Sigma_1^{(1)} \cong \Sigma_1^{(1)}(\sigma_1) \cong \Sigma_1^{(2)}(\sigma_1), \quad (8.47)
\]

where the first inclusion is by definition, the first \(\cong\) is the diffeomorphism which is a part of \(\Xi^{(1)}\) the second \(\cong\) is the unique biholomorphic map.

We require

\[
K_1^{(1)} \subseteq \text{Int } K_1^{(2)} \quad (8.48)
\]

via the inclusion (8.47).

Note that for given \(\Xi^{(1)}, \Xi^{(2)}\) we can always modify the analytic family of coordinates \(\varphi_{1,2}\), which is a part of \(\Xi^{(2)}\) so that Condition 8.23 is satisfied. In fact we may replace \(\varphi_{1,2}\) by conformal change \(z \mapsto \varphi_{1,2}(\epsilon z)\) for sufficiently small \(\epsilon\). In other words we may assume Condition 8.23 without loss of generality.

Theorem 8.25, the main result of this section, concerns the first vertical arrow of the next Diagram (8.49).

**Lemma 8.24.** There exist constants \(\varepsilon_8, \varepsilon_9, \varepsilon_{10}, \nu_3, \nu_4, \nu_{4,m}\), and \(T_m, T_0\) such that we have the following commutative diagram for any positive numbers \(\varepsilon \leq \varepsilon_{10}\) and \(T_0 \geq T_m, T_0\).

\[
\begin{array}{ccc}
M_+^{(1)} & \text{Glue}^{(1)} & M_+^{(1)} \oplus \mathcal{E}_2^{(1)} \\
| \downarrow \delta | & | \downarrow \delta | & | \downarrow \delta |
\end{array}
\]

\[
\begin{array}{ccc}
M_+^{(2)} & \text{Glue}^{(2)} & M_+^{(2)} \oplus \mathcal{E}_2^{(2)} \\
| \downarrow \delta | & | \downarrow \delta | & | \downarrow \delta |
\end{array}
\]

\[
((\Sigma_1^{(1)}, \varphi_1^{(1)}), u_1^{(1)})_{\varepsilon_9, \nu_3, \nu_m} \quad (8.49)
\]

**Proof.** Theorem 8.16 implies that we can choose \(\varepsilon_8, \nu_4, \nu_{4,m}\) so that the lower horizontal arrow \(\text{Glue}^{(2)}_+\) exists for a given \(\varepsilon_6\) and \(\nu_1\).

Since \(((\Sigma_1^{(1)}, \varphi_1^{(1)}), u_1^{(1)})\) is an element of \(M_+^{(1)} \oplus \mathcal{E}_2^{(1)}\) by Condition 8.22 we use (8.43) and Theorem 8.16 (1) to show that we may take the constants \(\varepsilon_9, \nu_3, \nu_m\) so small that the natural inclusion induces the right vertical
arrow of (8.49) and

\[
\text{Im} \left( \mathcal{M}_+^{e(1) \oplus e(2)} \left( (\Sigma_1^1, z_1^1); u_1^{(1)}, u_2^{(1)} \right)_{\nu_1, \nu_3} \right) 
\subset \text{Im} \left( \begin{array}{c}
\mathcal{M}_+^{e(2)} \left( (\Sigma_1^2, z_1^2); u_1^{(2)} \right)_{\nu_0} \\
\mathcal{M}_+^{e(2)} \left( (\Sigma_2^2, z_2^2); u_2^{(2)}; \beta_2 \right)_{\nu_0} \times (1/\nu_4, \infty) 
\end{array} \right)
\]

in \( \mathcal{M}_+^{e(2) \oplus e(2)} \left( (\Sigma_1^2, z_2^2); u_1^{(2)}, u_2^{(2)} \right)_{\nu_0, \nu_3} \).

Then, using Theorem 8.16 (1) (the surjectivity and injectivity of the gluing map), we may take \( \epsilon_0 \) so small and \( T_m, (8.49) \) so large that there exists a unique map \( \tilde{\mathfrak{s}} \) so that Diagram (8.49) commutes. \( \square \)

**Theorem 8.25.** There exists \( T_m, (8.25) > 0 \) with the following property.

We identify \( (T(1), \infty) \) (resp. \( (1/\nu_4, \infty) \)) with \( [0, 1/T(1)] \) (resp. \( [0, \nu_4] \)) by \( T \mapsto s = 1/T \).

Then the map \( \mathfrak{s} \) in (8.49) is a smooth embedding on

\[
\mathcal{M}_+^{e(1)} \left( (\Sigma^1, z_1^1); u_1^{(1)} \right)_{\nu_0} \times \mathcal{M}_+^{e(1)} \left( (\Sigma_2^2, z_2^2); u_2^{(1)} \right) \times (T(1), \infty)
\]

if \( T(1) > T_m, (8.25) \).

**Remark 8.26.** We remark that the obstruction bundle data contains analytic family of coordinates at the 0-th marked point, which we use it to define obstruction bundle \( \mathcal{E}^{(j)}(u') \). This is essential for the right vertical arrow of (8.49) to exist. In fact because of this choice, the obstruction bundle \( \mathcal{E}^{(j)}(u') \) depends only on \( u' \), its source (marked bordered) curve, and the obstruction bundle data. In particular it is independent of the analytic family of coordinates \( \varphi_{(j), R} \), which we use to perform the inductive construction of the gluing map by an alternating method.

On the other hand the analytic families of coordinates at two chart centers are in general different from each other. Proposition 8.27 below is used to estimate the discrepancy between these two choices.

**Proof.** We observe that a neighborhood of \( (\Sigma^j, z^j) \in \mathcal{M}_{g_1+g_2, k_1+k_2} \) is parameterized by the map (8.16), that is,

\[
\text{Glusoc}^{(j), R} : \mathcal{V}^{(j), R}_1 \times \mathcal{V}^{(j), R}_2 \times (T, \infty) \to \mathcal{CM}_{g_1+g_2, k_1+k_2}.
\]

So we obtain a map

\[
\mathcal{M}_+^{e(1) \oplus e(2)} \left( (\Sigma^j, z^j); u^{(j)}, u^{(j)} \right)_{\epsilon, \nu} \to \mathcal{V}^{(j), R}_1 \times \mathcal{V}^{(j), R}_2 \times (1/\nu, \infty)
\]

by forgetting the map part of the stable map and compose it with the inverse of (8.50), for \( j = 1, 2 \).

**Proposition 8.27.** There exist \( \delta_3 > 0 \), and a (strata-wise) smooth map

\[
\Phi : \mathcal{V}^{(1), R}_1 \times \mathcal{V}^{(1), R}_2 \times (1/\nu_3, \infty) \to \mathcal{V}^{(2), R}_1 \times \mathcal{V}^{(2), R}_2 \times (1/\nu_1, \infty)
\]

such that:
We put its component of the factor in the third line is the restriction map and where the $V_3$ where (8.53)
the fact that during the inductive step of the construction of gluing map in Sections 5 and 6, we never change the complex structure of the source. The estimate (8.53)
the right vertical arrow of Diagram (8.49).

(2) We put $\Phi(\sigma,T) = (\sigma'(\sigma,T), T'((\sigma,T)))$. Then we have the following estimate.

\[
\left| \frac{\partial^n}{\partial T^\ell} \sigma'(\sigma,T) \right| \leq C_{m,\delta_3} e^{-\delta_3 T}, \\
\left| \frac{\partial^n}{\partial T^\ell} (T'(\sigma,T) - T) \right| \leq C_{m,\delta_3} e^{-\delta_3 T}
\]

for $m - 2 \geq n, \ell \geq 0, \ell > 0$.

Note $\delta_3$ in this proposition is the same constant as in Proposition 8.19.

**Proof.** The existence of the map $\Phi$ satisfying (1) is an immediate consequence of the fact that during the inductive step of the construction of gluing map in Sections 5 and 6 we never change the complex structure of the source. The estimate (8.53) will be proved in Subsection 8.3.

We define

\[
\text{Resfor}^{(j)} : M_+^{e_{(j)}} \times \Sigma_\infty^j \times ((\Sigma_1^j, \Sigma_\infty^j) \times u_1^{(j)}, u_2^{(j)}), \nu \rightarrow V_1^{(j)}, V_2^{(j)}, (1/\nu, \infty) \]

\[\times \prod_{i=1}^2 \text{Map}_{\nu_3^{(j)}} ((K_1^{(j),S}, K_1^{(j),S} \cap \partial \Sigma_i), (X,L))\]

where the $V_1^{(j)}, V_2^{(j)}, (1/\nu, \infty)$ component of Resfor$^{(j)}$ is the map (8.51) and its component of the factor in the third line is the restriction map and $K_1^{(j),S} = \Sigma_1^{(j)} \setminus \nu_1^{(j),S} (D_0^{(j)}(1))$.

We put

\[
U^{(j)}_{e^{(j)}, \nu^{(j)}, T(2)} = (\text{Glue}_+^{(j)})^{-1} \left( M_+^{e^{(j)}} \times \Sigma_\infty^j \times ((\Sigma_1^j, \Sigma_\infty^j) \times u_1^{(j)}, u_2^{(j)}), \nu^{(j)} \right) \cap \pi_3^{-1}((T(2), \infty))
\]

\[\subset M_+^{e^{(j)}} \times ((\Sigma_1^j, \Sigma_\infty^j) \times u_1^{(j)}, u_2^{(j)}), \nu^{(j)} \times_{\nu_3^{(j)}} M_+^{e^{(j)}} \times ((\Sigma_1^j, \Sigma_\infty^j) \times u_1^{(j)}, u_2^{(j)}), \nu^{(j)} \times T(2), \infty),\]

where $(e^{(1)}, \nu^{(1)}, e^{(1)}) = (e_9, \nu_3, \epsilon)$ with $\epsilon < \epsilon_{10}$, and $(e^{(2)}, \nu^{(2)}, e^{(2)}) = (\epsilon_6, \nu_1, \epsilon_8)$ and $\pi_3$ is the projection to $(1/\nu, \infty)$ factor.

**Lemma 8.28.** We identify $(T(2), \infty)$ with $[0, \nu^{(j)}]$ by $T \mapsto s = 1/T$. Then

\[
\text{Resfor}^{(j)} \circ \text{Glue}_+^{(j)} : U^{(j)}_{e^{(j)}, \nu^{(j)}, T(2)} \rightarrow (\text{RHS of (8.54)})
\]

is a smooth embedding if $T(2) > T_{0.56}$.
**Proof.** The composition of \( \text{Resfor}^{(j)} \circ \text{Glue}^{(j)}_+ \) with the projection to the factor \( \mathcal{V}^{(j),\mathbb{R}}_1 \times \mathcal{V}^{(j),\mathbb{R}}_2 \times (1/\nu^{(j)}, \infty] \) is smooth since it coincides with the projection to this factor. In other words the next diagram commutes.

\[
\begin{array}{ccc}
\mathcal{U}^{(j),\nu(j),\mathbb{R},T(2)} & \longrightarrow & \mathcal{V}^{(j),\mathbb{R}}_1 \times \mathcal{V}^{(j),\mathbb{R}}_2 \times (T(2), \infty]
\\
\downarrow \text{Resfor}^{(j)} \circ \text{Glue}^{(j)}_+ & & \downarrow
\\
\mathcal{V}^{(j),\mathbb{R}}_1 \times \mathcal{V}^{(j),\mathbb{R}}_2 \times (T(2), \infty] \times \prod_{i=1}^2 \text{Map}_{\mathbb{R}^{2m+1}}(\mathcal{K}^{(j),S,i}_1, \mathcal{K}^{(j),S,i}_2 \cap \partial \Sigma_i, (X, L)) & \longrightarrow & \mathcal{V}^{(j),\mathbb{R}}_1 \times \mathcal{V}^{(j),\mathbb{R}}_2 \times (T(2), \infty],
\end{array}
\]

(8.57)

where the second vertical arrow is the identity map and horizontal arrows are projections. The commutativity of Diagram (8.57) follows from the fact that the construction of our map \( \text{Glue}^{(j)}_+ \) does not change the complex structure of the source curve.

The stratawise smoothness, (that is, smoothness of the restriction to \( T = \infty \) and to \( T \neq \infty \)) of \( \text{Resfor}^{(j)} \circ \text{Glue}^{(j)}_+ \) is a consequence of standard elliptic regularity.

We consider the composition of \( \text{Resfor}^{(j)} \circ \text{Glue}^{(j)}_+ \) with the projection to the factor in the third line of the right hand side of (8.54). It becomes a map

\[
\mathcal{U}^{(j),\nu(j),\mathbb{R},T(2)} \rightarrow \prod_{i=1}^2 \text{Map}_{\mathbb{R}^{2m+1}}(\mathcal{K}^{(j),S,i}_1, \mathcal{K}^{(j),S,i}_2 \cap \partial \Sigma_i, (X, L)).
\]

(8.58)

Here we use \( s = 1/T \) instead of \( T \) for the coordinate of \([0, \nu^{(j)}] \) factor.

**Sublemma 8.29.** We have

\[
\lim_{s_0 \to 0} \left\| \nabla_{\sigma,\rho} \frac{d^f}{ds^f} \right\|_{L^2_{m+1,m+1}}^{(8.58)} = 0,
\]

for \( m - 2 \geq n, \ell \geq 0, \ell > 0. \)

**Proof.** Note

\[
\frac{d^f}{ds^f} = \frac{1}{s^{2\ell}} Q(s, \frac{d}{dT}) \frac{d}{dT},
\]

where \( Q(x, \xi) = \sum_{i=0}^{\ell-1} Q_i(x) \xi^i \) and \( Q_i \) are polynomials. Therefore by the exponential decay provided in Theorem 8.17 we derive

\[
\left\| \nabla_{\sigma,\rho} \frac{d^f}{ds^f} \right\|_{L^2_{m+1,m+1}}^{(8.58)} \leq C_{m,n,\ell} s_0^{-2\ell} e^{-\delta_2/s_0}.
\]

(8.59)

The sublemma follows. \( \square \)

Thus the map (8.56) is smooth, that is, of \( C^\infty \) class.

To prove that it is a smooth *embedding*, we use Diagram (8.57). In view of its commutativity, it suffices to show that \( \pi_T \circ \text{Resfor}^{(j)} \circ \text{Glue}^{(j)}_+ \) is a smooth embedding when we restrict it to the fiber of an arbitrary point \((\sigma_1, \sigma_2, T)\) in \( \mathcal{V}^{(j),\mathbb{R}}_1 \times \mathcal{V}^{(j),\mathbb{R}}_2 \times (1/\nu^{(j)}, \infty] \). Here \( \pi_T \) is the projection to the factor in the third line of (8.54).

At \( T = \infty \) (or \( s = 0 \)), the map \( \pi_T \circ \text{Resfor}^{(j)} \circ \text{Glue}^{(j)}_+ \) is actually the restriction map, since, there, \( \text{Glue}^{(j)}_+ \) is obtained by identifying two stable maps at 0-th marked
points. Therefore it is an embedding by unique continuation of pseudoholomorphic curve.

On the other hand, \[8.59\] implies that the restriction of the map \(\text{Resfor}^{(j)} \circ \text{Glue}^{(j)}\) to the fiber of \((\sigma_1,\sigma_2,T)\) converges to its restriction to \((\sigma_1,\sigma_2,\infty)\) in \(C^1\) sense as \(T\) goes to infinity. Therefore we obtain required properties for sufficiently large \(T(2)\).

Now we are in the position to complete the proof of Theorem 8.25. Using Proposition 8.27 (especially the estimate (8.53)) and Lemma 8.28 applied to \(\text{Resfor}^{(2)} \circ \text{Glue}^{(2)}\) we find that it suffices to show the following:

**Lemma 8.30.** We consider the composition of the next three maps. The first map to be composed is:

\[
\text{Glue}^{(1)} : \mathcal{M}^{(1)}_+((\Sigma_1^1, z_1^1); u_1^1)_{\epsilon_1} \times_{\text{ev}_0} \mathcal{M}^{(1)}_+((\Sigma_2^1, z_2^1); u_2^1)_{\epsilon} \times (T(3), \infty) \to \mathcal{M}^{(1)}_+((\Sigma_1^1, z_1^1); u_1^1)_{\epsilon_1} \times (T(3), \infty) \to \mathcal{M}^{(1)}_+((\Sigma_2^1, z_2^1); u_2^1)_{\epsilon} \times (T(3), \infty)
\]

the second map is

\[
\mathcal{M}^{(2)}_+((\Sigma_1^2, z_1^2); u_1^2)_{\epsilon_1} \times (T(3), \infty) \to \mathcal{M}^{(2)}_+((\Sigma_2^2, z_2^2); u_2^2)_{\epsilon} \times (T(3), \infty)
\]

and the third map is

\[
\text{pr} \circ \text{Resfor}^{(2)} : \mathcal{M}^{(2)}_+((\Sigma_1^2, z_1^2); u_1^2)_{\epsilon} \times (T(3), \infty) \to \mathcal{M}^{(2)}_+((\Sigma_2^2, z_2^2); u_2^2)_{\epsilon} \times (T(3), \infty)
\]

Then the restriction of this composition to the fiber of

\[
\mathcal{M}^{(1)}_+((\Sigma_1^1, z_1^1); u_1^1)_{\epsilon} \times (T(3), \infty) \to \mathcal{M}^{(1)}_+((\Sigma_2^1, z_2^1); u_2^1)_{\epsilon} \times (T(3), \infty)
\]

is a smooth embedding if \(T(3) \geq T_{m(8.30)}\).

**Proof.** This composition is nothing but the composition \(\text{pr} \circ \text{Resfor}^{(1)} \circ \text{Glue}^{(1)}\). Therefore applying Lemma 8.28 to \(j = 1\), we obtain the lemma.

The proof of Theorem 8.25 is now complete.

In a similar way we can prove the smoothness of the Kuranishi map as follows. Hereafter we write

\[
V(j) = \mathcal{M}^{(2)}_+((\Sigma_1^j, z_1^j); u_1^j)_{\epsilon_1} \times_{\text{ev}_0} \mathcal{M}^{(1)}_+((\Sigma_2^j, z_2^j); u_2^j)_{\epsilon} \times (T(3), \infty)
\]

where \((\epsilon_1, \nu_1^1, \epsilon_1) = (\epsilon_9, \nu_3, \epsilon)\) with \(\epsilon < \epsilon_{10}\), and \((\epsilon_2, \nu_2^2, \epsilon_2) = (\epsilon_6, \nu_1, \epsilon_8)\).
We consider the space $M^s_j((\Sigma_\infty, z_\infty); u_1^j, u_2^j) e^{(j), \nu(j)}$. We regard the image of
\[ \text{Glue}^j : V^j \times [0, \nu^j) \to M^s_j((\Sigma_\infty, z_\infty); u_1^j, u_2^j) e^{(j), \nu(j)} \]
as a smooth manifold so that $\text{Glue}^j$ is a diffeomorphism.

Let $\mathfrak{r} = ((\Sigma_T, \overline{z}), u') \in M^s_j((\Sigma_\infty, z_\infty); u_1^j, u_2^j) e^{(j), \nu(j)}$. The definition of the moduli space implies:
\[ \overline{\sigma}u' \equiv 0, \mod e^{s_1^j}(u') \oplus e^{s_2^j}(u'). \quad (8.61) \]
We write the left hand side as $\overline{\sigma}(\mathfrak{r})$. By the inverse of map $I_{u', 1}$ in (8.19) we obtain
\[ \mathfrak{g}(\mathfrak{r}) = (I_{u', 1} \oplus I_{u', 2})^{-1}(\overline{\sigma}(\mathfrak{r})) \in e^{s_1^j} \oplus e^{s_2^j}. \]
Note $e^{s_1^j} \oplus e^{s_2^j}$ is independent of $\mathfrak{r}$.

We thus obtain a map
\[ \mathfrak{g}(\mathfrak{r}) : \text{Glue}^j : V^j \times [0, \nu^j) \to e^{s_1^j} \oplus e^{s_2^j}. \quad (8.62) \]
This is by definition the Kuranishi map.

**Proposition 8.31.** The map $\mathfrak{g}(\mathfrak{r})$ in (8.62) is smooth.

**Proof.** The smoothness at each of the stratum ($T = \infty$ and $T \neq \infty$) follows from elliptic regularity. Note the support of elements of $e^{s_1^j}(u')$ is in the image of $R_{\overline{\sigma}}$. Therefore we can use (8.30) to show that all the derivative of $\mathfrak{g}$, including at least one $s = 1/T$ derivative vanishes at $s = 0$ in the same way as the proof of Sublemma 8.29 Furthermore (8.30) implies that the restriction of $\mathfrak{g}(\mathfrak{r})$ to $s = s_0$ converges to its restriction to $s = 0$ in $C^m$ sense for any $m$ as $s_0$ goes to 0. The proof of Proposition 8.31 is complete. \[ \square \]

We put $\hat{V}(\mathfrak{r}) = V^j \times [0, \nu^j) \oplus e^{s_2^j}(u')$ and $\hat{E}(\mathfrak{r}) = e^{s_1^j} \oplus e^{s_2^j}$. Then we can find $\psi(\mathfrak{r})$ so that the quintic $(\hat{V}(\mathfrak{r}), \hat{E}(\mathfrak{r}), \psi(\mathfrak{r}), \mathfrak{g}(\mathfrak{r}))$ becomes a Kuranishi neighborhood of the moduli space of stable bordered curves $M_{g_1+g_2,k_1+k_2}(X, L; \beta_1 + \beta_2)$ in the sense of [FOO01] Definition A1.1.

In fact, the moduli space $M_{g_1+g_2,k_1+k_2}(X, L; \beta_1 + \beta_2)$ (See Definition 8.8) is locally identified with the zero set of $\mathfrak{g}(\mathfrak{r})$. This fact follows from the ‘injectivity’ and ‘surjectivity’ we proved in Section 7. Therefore we obtain our parametrization map
\[ \psi(\mathfrak{r}) : (\hat{V}(\mathfrak{r}))^{-1}(0) \to M_{g_1+g_2,k_1+k_2}(X, L; \beta_1 + \beta_2). \]
Moreover the group of automorphisms of the objects in our neighborhood in the moduli space $M_{g_1+g_2,k_1+k_2}(X, L; \beta_1 + \beta_2)$ is trivial. Hence we can put $\Gamma = \{1\}$.

Now we prove:

**Theorem 8.32.** There exists a smooth coordinate change
\[ (\phi_{21}, \phi_{21}, \text{id}) : (\hat{V}(1), \hat{E}(1), \psi(1), \mathfrak{g}(1)) \to (\hat{V}(2), \hat{E}(2), \psi(2), \mathfrak{g}(2)) \]
of Kuranishi neighborhoods in the sense of [FOO01] Definition A1.3,
Proof. The map $\phi_{21} : \hat{\nu}^{(1)} \to \hat{\nu}^{(2)}$ is the embedding $\mathcal{F}$ in Diagram (8.49). This map is a smooth embedding by Theorem 8.25.

The bundle map $\hat{\phi}_{21} : \mathcal{E}^{(1)} \to \mathcal{E}^{(2)}$ is obtained from (8.43), that is, $\mathcal{E}^{(1)}_1(1) \oplus \mathcal{E}^{(1)}_2(2) \subset \mathcal{E}^{(2)}_1(1) \oplus \mathcal{E}^{(2)}_2(2)$, as follows.

If $\mathfrak{f} \in \hat{\nu}^{(1)}$ and $u'$ is the map part of $\text{Glue}^{(1)}_1(\mathfrak{f}) \in \mathcal{M}^{(1)}_1(\Sigma_{\infty}, \hat{\nu}^{(1)}; u^{(1)}_1, u^{(1)}_2)$ then

$$\mathcal{E}^{(1)}_1(\mathfrak{f}) = \mathcal{E}^{(1)}_2(1)(u') + \mathcal{E}^{(1)}_2(1)(u') \subset \mathcal{E}^{(2)}_1(1)(u') + \mathcal{E}^{(2)}_2(2)(u').$$

Note since $u'$ is also a map part of $\text{Glue}^{(2)}_1(\phi_{21}(\mathfrak{f}))$

$$\mathcal{E}^{(2)}_2(\text{Glue}^{(2)}_1(\phi_{21}(\mathfrak{f}))) = \mathcal{E}^{(2)}_2(1)(u') + \mathcal{E}^{(2)}_2(2)(u').$$

The fiber of $\hat{\phi}_{21}$ at $\mathfrak{f}$ is the obvious inclusion $\mathcal{E}^{(1)}_1(\mathfrak{f}) \subset \mathcal{E}^{(2)}_2(\phi_{21}(\mathfrak{f}))$.

The smoothness of $\hat{\phi}_{21}$ is proved in the same way as Theorem 8.25 and Proposition 8.31 using the fact that the support of elements of $\mathcal{E}^{b}_i$ is in $\mathcal{K}^{b}_i$.

The group homomorphism $\text{id} : \{1\} \to \{1\}$ is the identity map. Various commutativities of compositions of maps required in [FOOO1] Definition A.1.3 are trivial to check in our case.

Remark 8.33. In this paper we assume that $(\Sigma_i, z_i)$ is stable. For the unstable case we need to add marked points $z_i^f$ and use the slices of codimension 2 submanifolds to reduce the construction of the Kuranishi chart to the case when the source is stable. (We use the slices that are transversal to the map $u_i : \Sigma_i \to X$ at $z_i^f$, to cut down the moduli space to one of correct dimension. See [FOn] Appendix and [FOOO2] Part IV.)

We also need to show that the change of the choices of extra marked points and codimension 2 submanifolds, induces a smooth coordinate change. Once the estimates (8.30) and (8.53) are established the rest of the proof of this statement is rather geometric than analytic. So we do not discuss this point in this paper, whose focus lies in analytic part of the story. See [FOOO1] page 772 and [FOOO2] Part IV for the argument of this point.

8.3. Comparison between two choices of analytic families of coordinates.

In this subsection we prove Propositions 8.19 and 8.27.

The main part of the proof is Proposition 8.35 below. We first need to set up notations to state it. Let $\mathcal{E}^{(j)}_i = \{(\Sigma_i^{(j)}, z_i^{(j)}) \in \mathcal{M}_{g_i,k_i+1}, i = 1, 2, j = a, b\}$. We choose the gluing data $\Xi^{(j)} = (\Sigma_1^{(j)}, \Sigma_2^{(j)})$ centered at $((\Sigma_1^{(j)}, z_1^{(j)}), (\Sigma_2^{(j)}, z_2^{(j)}))$ for $j = a$ and $j = b$. They induce the source gluing maps:

$$\text{Glue}^{(j)}_i : V_i^{(j), R} \times V_2^{(j), R} \times [T_0, \infty) \to \mathcal{C}M_{g_i+g_2,k_i+k_2}, \quad (8.63)$$

by Definition-Lemma 8.7. Here $V_i^{(j), R}$ is an open neighborhood of $[\Sigma_i^{(j)}, z_i^{(j)}]$ in $\mathcal{M}_{g_i,k_i+1}$, which is a part of $\Xi_i^{(j)}$.

Remark 8.34. In (8.16) the third factor of the domain is $[0, \epsilon)$. It is related to $(T_0, \infty)$ by $T \mapsto r = e^{-1/T}$. See (8.18).

We assume

$$V_i^{(a), R} \subset V_i^{(b), R}. \quad (8.64)$$
Then we may choose $T_j$ and a map $\Phi$ so that the next diagram commutes.

$$
\begin{array}{ccc}
\mathcal{V}_1^{(a),R} \times \mathcal{V}_2^{(a),R} \times (T_a, \infty) & \xrightarrow{\text{Glusoc}^R_{\Xi^{(a)}}} & \mathcal{C} \mathcal{M}_{g_1+g_2,k_1+k_2} \\
\Phi \downarrow & & \downarrow \text{id} \\
\mathcal{V}_1^{(b),R} \times \mathcal{V}_2^{(b),R} \times (T_b, \infty) & \xrightarrow{\text{Glusoc}^R_{\Xi^{(b)}}} & \mathcal{C} \mathcal{M}_{g_1+g_2,k_1+k_2}
\end{array}
$$

(8.65)

Let $K^{(j)}_i \subset \Sigma_i \setminus \text{Im}(\varphi_i^{(j),R})$. For $\sigma_a \in \mathcal{V}_i^{(a),R}$ the canonical embedding associated to $\Xi^{(i)}$ induces

$$
\mathcal{V}^{\sigma_a}_{\Xi^{(a)},i} : K^{(a)}_i \to \Sigma_1^{(a),\sigma_a} \# T \Sigma_2^{(a),\sigma_a}.
$$

(8.66)

On the other hand, the canonical embedding associated to $\Xi^{(b)}$ induces

$$
\mathcal{V}^{\sigma_b}_{\Xi^{(b)},i} : K^{(b)}_i \subset \Sigma_1^{(b),\sigma_b} \# T \Sigma_2^{(b),\sigma_b}.
$$

(8.67)

Let $(\sigma_b, T_b) = \Phi(\sigma_a, T_a)$, that is

$$
\Sigma_1^{(a),\sigma_a} \# T_a \Sigma_2^{(a),\sigma_a} \subset \Sigma_1^{(b),\sigma_b} \# T_b \Sigma_2^{(b),\sigma_b}.
$$

We assume

$$
\mathcal{V}^{\sigma_a}_{\Xi^{(a)},i}(K^{(a)}_i) \subset \mathcal{V}^{\sigma_b}_{\Xi^{(b)},i}(K^{(b)}_i).
$$

(8.68)

We then obtain a smooth embedding

$$
\Psi_{\sigma,T} = (\mathcal{V}^{\sigma_a}_{\Xi^{(a)},i})^{-1} \circ \mathcal{V}^{\sigma_b}_{\Xi^{(b)},i} : K^{(a)}_i \to K^{(b)}_i.
$$

(8.69)

It induces

$$
\Psi : \mathcal{V}_1^{(a),R} \times \mathcal{V}_2^{(a),R} \times (T_a, \infty) \times K^{(a)}_i \to K^{(b)}_i.
$$

(8.70)

**Proposition 8.35.** We assume (8.64) and (8.68). Then, there exists $\delta_4 > 0$ with the following properties.

1. We can estimate $\Phi$ as follows. We put $\Phi(\sigma; T) = (\sigma'(\sigma; T), T'(\sigma; T))$. Then we have the following estimate.

$$
\begin{align*}
\left| \nabla^n \frac{d^\ell}{d T^\ell} \sigma'(\sigma; T) \right| & \leq C_{\delta, (8.71)} e^{-\delta_4 T}, \\
\left| \nabla^n \frac{d^\ell}{d T^\ell} (T'(\sigma; T) - T) \right| & \leq C_{\delta, (8.71)} e^{-\delta_4 T}
\end{align*}
$$

for $\ell > 0$.

2. We can estimate $\Psi$ as follows.

$$
\left\| \nabla^n \frac{d^\ell}{d T^\ell} \Psi \right\|_{C^\ell(K^{(a)}_i, K^{(b)}_i)} \leq C_{\delta, (8.72)} e^{-\delta_4 T}
$$

for $\ell > 0$.

**Proof.** We first prove (8.71). By taking a double of (8.65) we obtain the next diagram.

$$
\begin{array}{ccc}
\mathcal{V}_1^{(a)} \times \mathcal{V}_2^{(a)} \times D^2(\epsilon) & \xrightarrow{\text{Glusoc}^C_{\Xi^{(a)}}} & \mathcal{C} \mathcal{M}_{g_1+g_2,k_1+k_2} \\
\Phi^c \downarrow & & \downarrow \text{id} \\
\mathcal{V}_1^{(b)} \times \mathcal{V}_2^{(b)} \times D^2(\epsilon) & \xrightarrow{\text{Glusoc}^C_{\Xi^{(b)}}} & \mathcal{C} \mathcal{M}_{g_1+g_2,k_1+k_2}
\end{array}
$$

(8.73)
Since $\varphi_{(j),R}^i$ are analytic coordinates Glusoc $C_{Gluc}$ and Glusoc $C_{Gluc}$ are holomorphic by Lemma 8.28. Therefore $\Phi^C$ is also holomorphic. We remark that
\[
\Phi^C(V_1^{(a)} \times V_2^{(a)} \times \{0\}) \subseteq (V_1^{(b)} \times V_2^{(b)} \times \{0\}).
\]

Therefore there exist holomorphic maps $\mathcal{F} : V_1^{(a)} \times V_2^{(a)} \rightarrow V_1^{(b)} \times V_2^{(b)}$ and $\mathcal{G} = (G_1, G_2) : V_1^{(a)} \times V_2^{(a)} \times D^2(1) \rightarrow C^{\dim V_1^{(b)} + \dim V_2^{(b)}} \times C$ such that
\[
\Phi^C(\sigma_1, \sigma_2, r) = (\mathcal{F}(\sigma_1, \sigma_2), 0) + \mathcal{G}(\sigma_1, \sigma_2, r)
\]
Moreover $G_2(\sigma_1, \sigma_2, 0) \neq 0$. Here we regard $\psi^{(j)}$ as an open set of $C^{\dim V_1^{(j)}}$.

Note $\Phi$ is a restriction of $\Phi^C$ where $T$ coordinate is identified with a part of the standard coordinate $r$ of $D^2(\epsilon)$ by $T = e^{-10\pi T} \in D^2(\epsilon)$. (See Diagram (8.17) and Formula (8.18).) Therefore
\[
\sigma'(\sigma, T) = \mathcal{F}(\sigma) + e^{-10\pi T} \mathcal{G}_1(\sigma, e^{-10\pi T}),
\]
\[
T'(\sigma, T) = T - \log \mathcal{G}_2(\sigma, e^{-10\pi T})/10\pi.
\]
Here $\sigma = (\sigma_1, \sigma_2)$. Now, the holomorphicity of $\mathcal{F}$ and $\mathcal{G}_1, \mathcal{G}_2$ implies the estimate (8.71).

We next prove (2). By taking the double of (8.70) we obtain a map:
\[
\Psi^C : V_1^{(a)} \times V_2^{(a)} \times D^2(\epsilon) \times K_1^{(a)} \rightarrow K_1^{(b)}.
\]
Using the trivialization of the universal bundle which is a part of the data $\Xi_1^{(j)} (j = a, b)$ we identify
\[
V_i^{(j)} \times K_1^{(j)} \subset C(V_1^{(j)}).
\]

We use this embedding to define a complex structure of $V_1^{(j)} \times V_2^{(j)} \times K_1^{(j)}$.

We consider the map
\[
\Phi^C \times \Psi^C : V_1^{(a)} \times V_2^{(a)} \times D^2(\epsilon) \times K_1^{(a)} \rightarrow V_1^{(b)} \times V_2^{(b)} \times D^2(\epsilon) \times K_1^{(b)}.
\]
The map $\Phi^C \times \Psi^C$ is holomorphic with respect to the above complex structures because the left (resp. right) hand side is an open set of the universal family $C(V_1^{(a)} \times V_2^{(a)} \times D^2(\epsilon))$ (resp. $C(V_1^{(b)} \times V_2^{(b)} \times D^2(\epsilon))$). Note neither left nor the right hand side is the direct product as a complex manifold with respect to this complex structure.

On the other hand, the projection
\[
\Pr : V_1^{(b)} \times V_2^{(b)} \times D^2(\epsilon) \times K_1^{(b)} \rightarrow K_1^{(b)}
\]
is certainly of $C^\infty$ class. Now we have
\[
\Psi(\sigma, T, z) = \Pr(\Phi^C(\sigma, e^{-10\pi T}), \Psi^C(\sigma, e^{-10\pi T}, z))
\]
Using this holomorphicity of $(\Phi^C, \Psi^C)$ and smoothness of $\Pr$ we obtain the estimate (8.72).

**Proof of Proposition 8.27** (1) is already proved. Replacing $a, b$ by 1, 2 we apply Proposition 8.35 (1). In fact, (8.64) and (8.68) follow from (8.44) and (8.48), respectively. Then (8.53) follows from (8.71).
Proof of Proposition 8.19. The map $I^{w_i}_u \circ I^{a_i}_u \circ I^{a_i}_u$ (see (8.32)), which we use to define $e'_{i,a,\kappa}(\sigma, \rho, T)$, is a tensor product of two $C$-linear maps: one is a map between $\Lambda^{0,1}$ bundles; the other is a map between sections of the pull back bundles of $TX$. We denote the former by $I_{\Sigma}$ and the later by $I_{X}$.

$I_{\Sigma}$ is a composition of (8.21), (8.36) and (8.38). Note this map does not depend on $u'$ but depends only on $\sigma$ and $T$, (which determine the source curve of $u'$). (8.21) and (8.38) are independent of $T$ and depend smoothly on $\sigma$. We apply Proposition 8.35 to $\Sigma_i^{(2)} = \Sigma_i^{ab}$, etc. and $\Sigma_i^{(5)} = \Sigma_i$, etc.. We can then apply (8.72) to estimate the map $\Psi_{\rho,T}$ below.

$$\Psi_{\rho,T} = (\mathcal{J}_{\Sigma_i}^\sigma)^{-1} \circ \mathcal{J}_{\Sigma_i}^{a_i} : K_i \rightarrow K_i.$$ We then obtain

$$\|\nabla_{\sigma,\rho} \frac{\partial}{\partial T} \Psi_{\rho,T}\| \leq C_{m, \rho, \sigma} e^{-\delta_i T},$$

for $m - 2 \geq n, \ell \geq 0, \ell > 0$. Note (8.68) and (8.64) follows from Condition 8.13 (1) and (8.35), respectively.

We remark that in the definition of (8.36) the process to pull back the differential form by $\Psi_{\rho,T}$ is included. This is usually a difficult process to study in Sobolev spaces. However in our situation we apply it to smooth forms $e_{i,a}$ which is fixed during the construction. So we can use (8.76) to deduce the next inequality

$$\left\|\nabla_{\sigma,\rho} \frac{\partial}{\partial T} I_{\Sigma_i}(e_{i,a})\right\| \leq C_{m, 8.77} e^{-\delta_i T}.$$ We next discuss $I_X$. By definition $I_X$ is induced by the composition of the parallel transportations (8.20), (8.37) and (8.39). In the case when $u' = u^a_{\rho, \kappa}$ we can estimate it by using induction hypothesis (that is, the version of (6.9) including $\sigma$ derivatives). We obtain this estimate in the same way as the proof of Lemma 6.20 given in Section F.

Thus together with (8.77) we obtain the required estimate (8.42). □

Remark 8.36. Proposition 8.35 (1) is a version of [FOOO2 Proposition 16.11] and Proposition 8.35 (2) is a version of [FOOO2 Proposition 16.15]. Actually the assumption of [FOOO2 Proposition 16.11] and of [FOOO2 Proposition 16.15] are weaker than that of Proposition 8.35. Namely in [FOOO2] we studied a smooth family of coordinates at the 0-th marked point. Here we consider an analytic family of coordinates at the 0-th marked point.

The proof of [FOOO2 Propositions 16.11 and 16.15] is given in [FOOO2 Section 25] and uses a method similar to the proof of Theorems 3.13 and 6.4 of this paper to find a biholomorphic map between Riemann surfaces with appropriate estimate. In other words it is based on a study of non-linear partial differential equation. The proof of Proposition 8.35 we provide in this section is based on complex geometry and is shorter than [FOOO2 Section 25].

In case when the almost complex structure of the target space $X$ is integrable, we may prove a similar estimate as Theorem 6.4 for the moduli space of stable maps without boundary, by using complex geometry in a similar way as the proof of Proposition 8.35. Namely we may use existence of universal family of stable
maps (with obstruction bundles) in complex analytic category and translate the complex analyticity of this moduli space into an exponential decay estimate.\footnote{Because we need to study the case when obstruction bundle is present, it is nontrivial to work out the proof of exponential decay in this way.}

In the situation of Proposition 8.35, the target space is not involved. So all the complex structures involved are integrable. This is the reason why we can use complex geometry to find a shorter proof. Since for the purpose of this paper (and all the applications we can see at this stage), we can always restrict ourselves to an analytic family of coordinates, we provide this shorter proof in this paper.

The existence of smooth coordinate change between Kuranishi chart in this section and those in [FOOO2] can be proved by using [FOOO2, Propositions 16.11 and 16.15].

In the genus 0 case, a result corresponding to Proposition 8.35 is proved in [FOOO1, Lemma A1.59]. The proof there uses hyperbolic geometry and is different from both of this paper and [FOOO2].

Remark 8.37. As we mentioned already the proof of Proposition 8.19 is the main extra point in the proof of Theorem 8.16 other than those appearing in the proof of Theorems 3.13 and 6.4. In [FOOO2], this point was discussed in detail as the proof of [FOOO2, Lemma 19.14].

The contents of this section is taken from [FOOO2, Part 4]. The contents of other sections of this paper are taken from [FOOO2, Part 3].

**Appendix A. Error term estimate of non-linear Cauchy-Riemann equation I**

Let $\Omega$ be an open subset of a bordered Riemann surface $\Sigma$ and $u_1: (\Omega, \Omega \cap \partial \Sigma) \to (X, L)$ a pseudoholomorphic map. Consider two smooth sections $V^0, W^0 \in \Gamma(\Omega, u_1^*TX)$ such that their restrictions to $\Omega \cap \partial \Sigma$ are in $\Gamma(\Omega \cap \partial \Sigma, u_1^*TX)$. We study the maps

\[ u(z) = \text{Exp}(u_1(z), W^0(z)) \]

and

\[ v_r(z) = \text{Exp}(u(z), r^{\text{Pal}}_{u_1}(V^0)(z)). \]  
(A.1)

We take a trivialization of $u_1^*TX$ on $\Omega$ and identify $u_1^*TX \cong \Omega \times \mathbb{R}^n$. We write an element (of the total space of) $u_1^*TX$ as $(z, (\xi_1, \ldots, \xi_n))$. So $V^0$, $W^0$ are regarded as $(V^0)_j : \Omega \to \mathbb{R}^n$, $(W^0)_j : \Omega \to \mathbb{R}^n$. Let $z = x + \sqrt{-1}y$ be a complex coordinate of $\Omega$.

Let $R > 0$ be a number smaller than $\epsilon_X/10$. We denote by $D^n(R)$ the ball of radius $R$ centered at 0 in $\mathbb{R}^n$.

We define $\hat{F}: \Omega \times D^n(R) \times D^n(R) \to X$ by

\[ \hat{F}(z, v, w) = \text{Exp} \left( \text{Exp}(u_1(z), w), \text{Pal}_{u_1^*(TX)}^{\text{Exp}(u_1(z), w)}(v) \right) \]

where $\Omega \times \mathbb{R}^n \times \mathbb{R}^n \supset \Omega \times D^n(R) \times D^n(R)$ is identified with the total space of the direct sum bundle $u_1^*(TX \oplus TX)$. See Figure 14.

We also define $F: \Omega \times D^n(R) \times D^n(R) \to \mathbb{R}^n$ by

\[ F(z, v, w) = E(u_1(z), \hat{F}(z, v, w)). \]
\[
\hat{F}(z,v,w) = \text{Ext}(\text{Ext}(u_1(z),w),\text{Pal}(v))
\]

**Figure 14.** \(F(z,v,w)\) and \(\hat{F}(z,v,w)\)

We denote

\[
\mathcal{PP}(z,v,w) = ((\text{Pal}^{\text{Exp}}_{u_1(z)}(v,w))^{(0,1)} - 1)\circ \left((\text{Pal}^{\hat{F}(z,v,w)}_{u_1(z),w})^{(0,1)} - 1\right) : T\hat{F}(z,v,w) \otimes \Lambda^{0,1} \Omega \rightarrow T_{u_1(z)} X \otimes \Lambda^{0,1} \Omega.
\]

We remark

\[v_r(z) = \hat{F}(z,rV^0(z),W^0(z)).\]

We study

\[\mathcal{PP}(z,rV^0(z),W^0(z))(\partial_{v_r}).\]  

**Lemma A.1.** There exist smooth maps

\[
\mathfrak{G} : \Omega \times D^n(R) \times D^n(R) \rightarrow \mathbb{R}^n,
\]

\[
\delta_{v_{o,j}}, \delta_{v_{m,j}}, \delta_{w_{o,j}}, \delta_{w_{m,j}} : \Omega \times D^n(R) \times D^n(R) \rightarrow \mathbb{R}^n
\]

\(j = 1, \ldots, n\) such that

\[
\mathfrak{A.2} = \mathfrak{G}(z,rV^0(z),W^0(z)) + r \sum_j \frac{\partial V^0_j}{\partial x}(z)\delta_{v_{o,j}}(z,rV^0(z),W^0) + \sum_j \frac{\partial W^0_j}{\partial x}(z)\delta_{w_{o,j}}(z,rV^0(z),W^0(z)) + r \sum_j \frac{\partial V^0_j}{\partial y}(z)\delta_{w_{m,j}}(z,rV^0(z),W^0(z)) + \sum_j \frac{\partial W^0_j}{\partial y}(z)\delta_{w_{o,j}}(z,rV^0(z),W^0(z)).
\]

We remark that the maps \(\mathfrak{G}, \delta_{v_{o,j}}, \delta_{v_{m,j}}, \delta_{w_{o,j}}, \delta_{w_{m,j}}\) depend only on \(\Omega, u_1, X\) and are independent of \(V^0, W^0\).

**Proof.** We emphasize that we do not need to obtain an explicit form of the smooth maps appearing in \(\mathfrak{A.3}\).
We first observe that there exists an \( n \times n \) matrix valued smooth map \( H \) on \( \Omega \times D^n(R) \times D^n(R) \) such that
\[
\frac{\partial F(z, v, w)}{\partial x} = H(z, v, w) \left( \mathcal{P}(z, v, w) \left( \frac{\partial \hat{F}(z, v, w)}{\partial x} \right) \right)
+ \frac{\partial}{\partial x} \mathcal{E}(u_1(z), \xi) \bigg|_{\xi = \hat{F}(z, v, w)},
\]
\[
\frac{\partial F(z, v, w)}{\partial y} = H(z, v, w) \left( \mathcal{P}(z, v, w) \left( \frac{\partial \hat{F}(z, v, w)}{\partial y} \right) \right)
+ \frac{\partial}{\partial y} \mathcal{E}(u_1(z), \xi) \bigg|_{\xi = \hat{F}(z, v, w)},
\]
\[
\frac{\partial F(z, v, w)}{\partial v_j} = H(z, v, w) \left( \mathcal{P}(z, v, w) \left( \frac{\partial \hat{F}(z, v, w)}{\partial v_j} \right) \right)
\]
\[
\frac{\partial F(z, v, w)}{\partial w_j} = H(z, v, w) \left( \mathcal{P}(z, v, w) \left( \frac{\partial \hat{F}(z, v, w)}{\partial w_j} \right) \right).
\]

In fact we have
\[
H(z, v, w) = (D\hat{F}(z, v, w) \mathcal{E}(u_1(z), \cdot))^{-1} \circ \mathcal{P}(z, v, w)^{-1}.
\]

Here \((D\hat{F}(z, v, w) \mathcal{E}(u_1(z), \cdot))\) is the differential of the map \( \xi \mapsto \mathcal{E}(u_1(z), \xi) \) at \( \xi = \hat{F}(z, v, w) \in X \).

We next observe that there exists a matrix valued smooth function \( \mathcal{J} \) on \( \Omega \times D^n(R) \times D^n(R) \) such that
\[
\mathcal{J}(z, v, w)(\mathcal{P}(z, v, w)(\xi)) = \mathcal{P}(z, v, w)(\mathcal{J}(\hat{F}(z, v, w)) (\xi))
\]
for any \( \xi \in T_{\hat{F}(z, v, w)} X \). Here \( \mathcal{J}(\hat{F}(z, v, w)) : T_{\hat{F}(z, v, w)} X \to T_{\hat{F}(z, v, w)} X \) is the almost complex structure of \( X \).

By definition
\[
\mathcal{J}(z, v, w) = \frac{\partial}{\partial x} \hat{F}(z, rV^0(z), W^0(z)) + J_{\hat{F}(z, rV^0(z), W^0(z))} \left( \frac{\partial}{\partial y} \hat{F}(z, rV^0(z), W^0(z)) \right).
\]

Note
\[
\frac{\partial}{\partial x} \hat{F}(z, rV^0(z), W^0(z)) = \frac{\partial \hat{F}}{\partial x} + \sum_j r \frac{\partial V^0_j}{\partial x} \frac{\partial \hat{F}}{\partial v_j} + \sum_j \frac{\partial W^0_j}{\partial x} \frac{\partial \hat{F}}{\partial w_j},
\]
\[
\frac{\partial}{\partial y} \hat{F}(z, rV^0(z), W^0(z)) = \frac{\partial \hat{F}}{\partial y} + \sum_j r \frac{\partial V^0_j}{\partial y} \frac{\partial \hat{F}}{\partial v_j} + \sum_j \frac{\partial W^0_j}{\partial y} \frac{\partial \hat{F}}{\partial w_j}.
\]

Now it is fairly obvious that we can find \( \mathcal{J}, \mathcal{J}^x_{p,j}, \mathcal{J}^x_{m,j}, \mathcal{J}^x_{p,j}, \mathcal{J}^x_{m,j} \) from \( H, \mathcal{J} \), the \( x,y \) derivatives of \( \mathcal{E}(u_1(z), \xi) \) with \( \xi = \hat{F} \), and
\[
\mathcal{P}(z, v, w) \left( \frac{\partial \hat{F}}{\partial x} \right), \quad \mathcal{P}(z, v, w) \left( \frac{\partial \hat{F}}{\partial y} \right), \quad \mathcal{P}(z, v, w) \left( \frac{\partial \hat{F}}{\partial v_j} \right), \quad \mathcal{P}(z, v, w) \left( \frac{\partial \hat{F}}{\partial w_j} \right). \quad (A.6)
\]
are also \( \mathbb{R}^n \) valued smooth functions of \((z,v,w)\) because of (A.4).

The proof of Lemma [A.1] is complete.  

By (A.3) we have

\[
\frac{d^2}{dr^2}(A.2) = \sum_{ij} V_j^0 V_i^0 \delta_{ij} \left( r, z, W^0, V^0, \frac{\partial W^0}{\partial x}, \frac{\partial V^0}{\partial x}, \frac{\partial W^0}{\partial y}, \frac{\partial V^0}{\partial y} \right)
+ \sum_{ij} V_j^0 \frac{\partial V_i^0}{\partial x} \delta^x_{ij} (r, z, W^0, V^0) + \sum_{ij} V_j^0 \frac{\partial V_i^0}{\partial y} \delta^y_{ij} (r, z, W^0, V^0)
\]  

(A.7)

where \( F_{ij}, F^x_{ij}, F^y_{ij} \) are smooth maps independent of \( V^0, W^0 \).

**Proof of Lemma 5.19.** We take: \( \Sigma = \Sigma_1 \), and \( \Omega_\alpha (\alpha = 1, \ldots, A) \) open subsets of \( \Sigma_1 \) such that \( \bigcup_{\alpha = 1}^A \Omega_\alpha \supset K_1 \). We also put \( u_1 = u_1 \) and

\[
\begin{align*}
W^0(z) &= E(u_1(z), u(z)) \in T_{u_1(z)}(X), \\
V^0(z) &= \text{Pal}^{u_1(z)}_{u_1(z)}(V(z)) \in T_{u_1(z)}(X).
\end{align*}
\]

Then by (A.7), we obtain:

\[
\left\| \int_0^1 ds \int_0^s \frac{d^2}{dr^2}(A.2) \right\|_{L^2_{m+1}(\Omega_\alpha)} \leq C \left( \| V^0 \|_{L^2_{m+1}(\Omega_\alpha)} \right) \quad (A.8)
\]

where \( C \) depends on \( \max\{\| V^0 \|_{L^2_{m+1}(\Omega_\alpha)}, \| W^0 \|_{L^2_{m+1}(\Omega_\alpha)}\} \).

By taking the sum over \( \alpha \), we obtain the inequality (5.42) to be proven.  

**Proof of (5.57).** By putting \( \Sigma = \Sigma(S) \) the proof is the same as the proof of Lemma 5.19.

**Proof of the first inequality of (5.62).** We put \( \Sigma = \Sigma_1 \), \( \Omega \) a neighborhood of \( \mathcal{A} \), \( u_1 = u_1 \),

\[
\begin{align*}
u_{r}(z) &= \text{Exp}(u_{T,0}^p(z), V_{T,1,(1)}^p), \\
u_{r}(z) &= \text{Exp}(u_{T,0}^p(z), V_{T,1,(1)}^p + r \chi_{\mathcal{A}}(V_{T,2,(1)}^p) - (\Delta p_{T,1,(1)}^p)^{\text{Pal}})).
\end{align*}
\]

We then define

\[
\begin{align*}
W^0(z) &= E(u_1(z), u(z)) \in T_{u_1(z)}(X), \\
V^0(z) &= \text{Pal}^{u_1(z)}_{u_1(z)}(\chi_{\mathcal{A}}(V_{T,2,(1)}^p) - (\Delta p_{T,1,(1)}^p)^{\text{Pal}})) \in T_{u_1(z)}(X).
\end{align*}
\]

We have

\[
P \mathcal{V}_r - \mathcal{V} = \int_0^{r_0} \left( \frac{\partial}{\partial r} \right) P(\mathcal{V}_r) dr
\]

(A.9)

Here \( P \) is the inverse of the complex linear part of the parallel transport along the path \( r \mapsto v_r(z) \). This path is not a geodesic. However we can apply the same
argument as the proof of Lemma [A.1] to obtain
\[
P(\partial \frac{\partial}{\partial r}) = \sum V_i^0 \tilde{\delta}_i (r, z, W^0, \frac{\partial W^0}{\partial x}, \frac{\partial W^0}{\partial y}) + \sum \frac{\partial V_i}{\partial x} \tilde{\delta}_i^x (r, z, W^0, V^0) + \sum \frac{\partial V_i}{\partial y} \tilde{\delta}_i^y (r, z, W^0, V^0),
\]
where \(\tilde{\delta}_i, \tilde{\delta}_i^x, \tilde{\delta}_i^y\) are smooth maps. Here \(P = ((\text{Pal}_u u_1)^{0.1})^{-1}\). The first inequality of (5.62) follows from (A.9) and (A.10).

\[\Box\]

Remark A.2. In the argument of this section or anywhere in this paper we never use the fact that we take parallel transport with respect to the Levi-Civita connection. We can actually use any linear connection which preserves \(TL \subset TX\) in place of Levi-Civita connection of the metric given in Lemma 2.1, as we did in [FOOO1, Chapter 7].

In the case when there are several (finitely many) Lagrangian submanifolds \(L_\alpha\) so that they have mutually clean intersection, we cannot generalize Lemma 2.1 to find a Hermitian metric such that all the \(L_\alpha\) are totally geodesic. However it is easy to see that there exists a linear connection which preserves all the subspaces \(TL_\alpha \subset TX\).

Appendix B. Estimate of Parallel transport 1

Let \(X\) be a Riemannian manifold. We put
\[
\mathcal{U} = \{(x, y, v, w) \mid x, y \in X, d(x, y) \leq \epsilon_X/2, v, w \in T_x X, |v|, |w| < \epsilon_X/10\},
\]
which is a smooth manifold. We consider a smooth vector bundle \(\mathcal{L}\) on \(\mathcal{U}\) whose fiber at \((x, y, v, w)\) is
\[
\mathcal{L}_{(x,y,v,w)} = \text{Hom}_\mathbb{R}(T_y X, T_x X).
\]
We define its smooth section \(\mathfrak{P}\) whose value at \((x, y, v, w)\) is as follows. We put
\[
p = \text{Exp}(x, w), \quad q = \text{Exp}(p, \text{Pal}_p^x (v)),
\]
then
\[
\mathfrak{P}(x, y, v, w) = ((\text{Pal}_p^x)^{j})^{-1} \circ ((\text{Pal}_p^y)^{j})^{-1} \circ (\text{Pal}_p^y)^{j}.
\]
The smoothness of \(\mathfrak{P}\) is obvious from the definition.

Lemma B.1. Let \(\mathcal{S}\) be a 2 dimensional manifold and \(v(s)\) and \(w(s)\) be an \(s \in \mathcal{S}\) parameterized family of smooth maps \(\mathcal{S} \to T_x X\) with \(|v(s)|, |w(s)| \leq \epsilon_X/10\). If
\[
\left\| \frac{\partial^{n_1}}{\partial s^{n_1}} v \right\|_{L^2_m(\mathcal{S})} \leq 1, \quad \left\| \frac{\partial^{n_2}}{\partial s^{n_2}} w \right\|_{L^2_m(\mathcal{S})} \leq 1
\]
for \(n_1, n_2 \leq n\), then
\[
\left\| \frac{\partial^n}{\partial s^n} (\mathfrak{P}(x, y, v, w) - \mathfrak{P}(x, y, v, 0)) \right\|_{L^2_m(\mathcal{S})} \leq C_m \sum_{n' \leq n} \left\| \frac{\partial^{n'}}{\partial s^{n'}} w \right\|_{L^2_m(\mathcal{S})}.
\]
Proof. The lemma is an immediate consequence of the smoothness of $\mathfrak{P}$. 

Proof of Lemma 5.20. We consider $\epsilon = \epsilon_1^{\rho}_{1,T,(0)} \in \mathcal{E}_1$. When we defined $\mathcal{E}_1$ it was a subset $\mathcal{E}_1 \subset L^2_m(\Sigma_1; (u^{\rho}_{1T})^* TX \otimes \Lambda^{0,1})$. We used parallel transport to regard 

$\epsilon \in L^2_m(\Sigma_1; (\tilde{u}^{\rho}_{1,T,(0)})^* TX \otimes \Lambda^{0,1}) = L^2_m(\Sigma_1; u^* TX \otimes \Lambda^{0,1})$. 

It appears in (5.45). So more precisely $\epsilon$ in (5.45) is:

$$z \mapsto (\text{Pal}_{u_1^{\rho}}^{\hat{A}_{1,T,(0)}}(z))^{(0,1)}(\epsilon_1^{\rho}_{1,T,(0)}(z)).$$

(Here $z \in K_1^{\rho}$.) The expression $\mathcal{P}^{-1}\epsilon$ in (5.45) is then equal to

$$z \mapsto (\text{Pal}_{u_1^{\rho}}^{\text{Exp}(\hat{u}^{\rho}_{1,T,(0)})(z),V_{1,T,(1)}^{\rho}}(z))^{(0,1)}(\epsilon_1^{\rho}_{1,T,(0)}(z)). \quad (B.5)$$

On the other hand $\epsilon \in L^2_m(\Sigma_1; (\text{Exp}(\hat{u}^{\rho}_{1,T,(0)}),V_{1,T,(1)}^{\rho}))^* TX \otimes \Lambda^{0,1}$ appearing in Lemma 5.20 is, by definition,

$$z \mapsto (\text{Pal}_{u_1^{\rho}}^{\text{Exp}(\hat{u}^{\rho}_{1,T,(0)}),V_{1,T,(1)}^{\rho}}(z))^{(0,1)}(\epsilon_1^{\rho}_{1,T,(0)}(z)). \quad (B.6)$$

Using the section $\mathfrak{P}$ in (B.3), the sections (B.5) and (B.6) are written as follows.

(B.5) $= \mathfrak{P}(\hat{u}^{\rho}_{1,T,(1)}(z),u_1^{\rho}(z),0,V_{1,T,(1)}^{\rho}(z))(\epsilon_1^{\rho}_{1,T,(0)}(z))$

(B.6) $= \mathfrak{P}(\hat{u}^{\rho}_{1,T,(1)}(z),u_1^{\rho}(z),0,0)(\epsilon_1^{\rho}_{1,T,(0)}(z))$

Therefore (B.5), (B.6) are estimated by $Ce^{-\delta_1 T}$ by Lemma B.1. Then the estimate (5.55) easily follows from this. 

\[\square\]

Appendix C. Error Term Estimate of Non-linear Cauchy-Riemann Equation II

In this appendix, we give a proof of Proposition 5.33. Certain estimates of the parallel transport are postponed to the next section.

Proof of Proposition 5.33. The proof is similar to that of Proposition 5.17 and proceed as follows.

We first perform estimate on $K_1$. We use the simplified notation:

$$u = \hat{u}^{\rho}_{1,T,(\kappa-1)}, \quad V = V_{1,T,(\kappa)}^{\rho}, \quad \mathcal{P} = (\text{Pal}_1^{(0,1)})^{-1}, \quad \epsilon = \epsilon^{\rho}_{1,T,(\kappa-1)}. \quad (C.1)$$

Here Pal_1^{(0,1)} is the $(0,1)$ part of the parallel translation along the map $s \mapsto \text{Exp}(u,sV)$, $s \in [0,s_0]$ for some $s_0 \in [0,1]$.

We obtain the same formula as (5.41)

$$\mathcal{P}\overline{\partial}(\text{Exp}(u,V)) = \overline{\partial}(\text{Exp}(u,0)) + \int_0^1 \frac{\partial}{\partial s} (\mathcal{P}\overline{\partial}(\text{Exp}(u,sV))) \, ds \quad (C.2)$$

$$= \overline{\partial}(\text{Exp}(u,0)) + (D_u\overline{\partial})(V) + \int_0^1 ds \int_0^s \left(\frac{\partial}{\partial r}\right)^2 (\mathcal{P}\overline{\partial}(\text{Exp}(u,rV))) \, dr.$$
Then we also obtain
\[ \left\| \int_0^1 ds \int_0^s \left( \frac{\partial}{\partial r} \right)^2 (P \overline{\Phi}(\text{Exp}(u, rV))) \ dr \right\|_{L^2_{m}(K_1)} \leq C_m \|V\|_{L^2_{m+1, \delta}} \leq C'_{m, \delta} e^{-2\delta T} \mu^{2(\alpha-1)} \] (C.3)

provided \( m \geq 2 \). One can prove (C.3) by applying Lemma A.1 to
\[ W^0(z) = E(u_1(z), \hat{u}^0_{1, T, (\alpha-1)}(z)), \quad V^0(z) = \text{Pal}_{u^1_{\hat{u}^0_{1, T, (\alpha-1)}}}(z) \left( V^0_{T, 1, (\alpha-1)}(z) \right) \] (C.4)
and applying Sobolev inequality etc. to the right hand side of (A.7).

**Remark C.1.** We remark that we are working by induction on \( \kappa \) and take infinitely many steps \( \kappa = 1, 2, \ldots \). For this proof to work we need the constants \( C_m, C_{m, \delta} \) and \( C'_{m, \delta} \) in (C.3) to be independent of \( \kappa \).

The reason why we can take \( C_m, C_{m, \delta} \) and \( C'_{m, \delta} \) to be independent of \( \kappa \) is as follows. As we mentioned right after (A.8), the constant \( C_{A, \delta} \) depends only on \( \max\{\|V\|_{L^2_{m+1}(\Omega)}, \|W\|_{L^2_{m+1}(\Omega)}\} \). In our situation where \( W^0 \) and \( V^0 \) are given by (C.4), their local \( L^2_{m+1} \) norms are bounded by the induction hypothesis [(6.8) and (6.9)] and Lemma 6.11 by a number independent of \( \kappa \).

Therefore \( C_m, C_{m, \delta} \) and \( C'_{m, \delta} \) can be taken to be independent of \( \kappa \). It can be taken to be independent of \( T \) since we are working on \( K_1 \) which is independent of \( T \).

Independence of the constants of \( \kappa \) or \( T \) appears in other part of the proof, which can be proved in the same way. So we do not mention it usually.

Next we have
\[ \mathcal{P} \circ \Pi^\perp_{\mathcal{E}_1(\text{Exp}(u, V))} \circ \mathcal{P}^{-1} = \Pi^\perp_{\mathcal{E}_1(u)} + \int_0^1 \frac{d}{ds} \left( \mathcal{P} \circ \Pi^\perp_{\mathcal{E}_1(\text{Exp}(u, sV))} \circ \mathcal{P}^{-1} \right) \ ds \] (C.5)
\[ = \Pi^\perp_{\mathcal{E}_1(u)} - (D_u \mathcal{E}_1)(\cdot, V) + \int_0^1 ds \int_0^s ds' \left( \mathcal{P} \circ \Pi^\perp_{\mathcal{E}_1(\text{Exp}(u, rV))} \circ \mathcal{P}^{-1} \right) \ dr. \]

in which we can estimate the third term in the same way as (C.3). (See Section D)

On the other hand, (5.89) for \( \leq \kappa - 1 \), (5.66) for \( \leq \kappa \) and (C.2), (C.3) imply that
\[ \left\| \overline{\Phi}(\text{Exp}(u, V)) - \mathcal{P}^{-1} \epsilon \right\|_{L^2_{m}(K_1)} \leq C_{m, \delta} e^{-2\delta T} \mu^{\alpha-1}. \] (C.6)

We also use the next inequality
\[ \left\| \mathcal{P}^{-1}(D_u \mathcal{E}_1)(\mathcal{P}Q, V) \right\|_{L^2_{m}(K_1)} \leq C_{m, \delta} \left\| Q \right\|_{L^2_{m}(K_1)} \left\| V \right\|_{L^2_{m}(K_1)}. \] (C.7)

Here \( Q \) is a section of \( u^*TX \otimes \Lambda^{0, 1} \) of \( L^2_{m} \) class. (C.7) is a version of (5.47) and is proved in Section D. Using (C.6) and (C.7) we can show
\[ \left\| \Pi^\perp_{\mathcal{E}_1(\text{Exp}(u, V))} \overline{\Phi}(\text{Exp}(u, V)) - \mathcal{P}^{-1} \Pi^\perp_{\mathcal{E}_1(u)}(\mathcal{P} \overline{\Phi}(\text{Exp}(u, V))) + \mathcal{P}^{-1}(D_u \mathcal{E}_1)(\epsilon, V) \right\|_{L^2_{m}(K_1^+)} \leq C_{m, \delta} e^{-2\delta T} \mu^{\alpha-1}, \] (C.8)
in the same way as (5.50).
By (5.83) we have:
\[ \partial u + (D_u \mathcal{E})(V) - (D_u \mathcal{E}_1) (\varepsilon, V) \in \mathcal{E}_1(u) \] (C.9)
on $K_1$.

Summing up we have derived
\[ \| \Pi^1_{E_1}(\exp\{u, V\}) (D\exp\{u, V\}) \|_{L^2_m(K_1)} \leq C_m \epsilon^{-2\delta_1} T \mu^{-1} e^{-\delta_1 T \epsilon(5) \mu^\kappa / 10} \] (C.10)
for $T > T_{m, \epsilon(5), C.10}$, as long as we choose $T_{m, \epsilon(5), C.10}$ so that
\[ C_m e^{-\delta_1 T \epsilon(5) \mu^\kappa / 10} \leq C_{1, m} \epsilon(5) \mu. \]

We emphasize that this choice of $T_{m, \epsilon(5), C.10}$ does not depend on $\kappa$ but depend only on $m$, $\epsilon(5)$ and $\mu$.

It follows from (C.6) that
\[ \| \Pi_{E_1}(\exp\{u, V\}) (D\exp\{u, V\}) \| - \epsilon \| L^2_m(K_1) \| \leq C_m \epsilon^{-2\delta_1} T \mu^{-1} e^{-\delta_1 T \epsilon(5) \mu^\kappa / 10} \] (C.11)
in the same way as the proof of Lemma 5.20 given in Section 3.

Then we can prove (5.24) by putting
\[ \epsilon^0_{1, T, \epsilon(\kappa)} = \Pi_{E_1}(\exp\{u, V\}) (D\exp\{u, V\}) - \epsilon \in E_1(\exp\{u, V\}) \cong \mathcal{E}_1^1. \] (C.12)

We can perform the estimate on $[-5T, -T - 1] \times [0, 1]$ modifying the proof of (5.59) in the same way as follows. For $[S, S + 1] \times [0, 1] \subset [-5T, -T - 1] \times [0, 1]$ the inequality
\[ \| \partial u_{T, \epsilon(\kappa)} \|_{L^2_m([-5T, -T - 1] \times [0, 1] \subset \Sigma_T)} < C_m \mu^{-1} e^{-2\delta_1 T} \] (C.13)
can be proved in the same way as (C.10). Therefore
\[ \| \partial u_{T, \epsilon(\kappa)} \|_{L^2_m([-5T, -T - 1] \times [0, 1] \subset \Sigma_T)} \leq 40T e^{4T T} C_m \mu^{-1} e^{-2\delta_1 T} \] (C.14)
\[ \leq C_1, m e^{-\delta_1 T \epsilon(5) \mu^\kappa / 10}, \]
for $T > T_{m, \epsilon(5), C.10}$, (Here we use the fact that the weight function $e_{T, \delta}^0$ is smaller than $10e^{4T T}$ on our domain.)

The estimate on $K_2$ and $[T + 1, 5T] \times [0, 1]$ are the same. Notation (C.11) is used up to here.

The estimate $\partial u_{T, \epsilon(\kappa)}$ on $[-T + 1, T - 1] \times [0, 1]$ is as follows. Note the bump functions $\chi_{B}^+$ and $\chi_{A}^-$ are $\equiv 1$ there and
\[ \partial u_{T, \epsilon(\kappa - 1)} + (D_{u_{T, \epsilon(\kappa - 1)}} \mathcal{E}) (V_{T, 1, \epsilon(\kappa)} + V_{T, 2, \epsilon(\kappa)}) = 0. \]

Therefore the inequality
\[ \| \partial u_{T, \epsilon(\kappa)} \|_{L^2_m([-T + 1, T - 1] \times [0, 1] \subset \Sigma_T)} < C_m \epsilon^{-2\delta_1 T} T \mu^{-1} \] (C.15)
can be proved in the same way as (C.10). Since $e_{T, \delta} \leq 10e^{5\delta T} \leq 10e^{5\delta T / 2}$, it implies
\[ \| \partial u_{T, \epsilon(\kappa)} \|_{L^2_m([-T + 1, T - 1] \times [0, 1] \subset \Sigma_T)} < e^{-\delta_1 T \epsilon(5) \mu^\kappa / 10}. \] (C.16)
for $T > T_{m, \epsilon(5), C.10}$.

On $A_T$ we have
\[ u_{T, \epsilon(\kappa)} = \exp(u_{T, \epsilon(\kappa - 1)} \chi_{A}^-(V_{T, 2, \epsilon(\kappa)} - \Delta p_{T, \epsilon(\kappa)}^0 + V_{T, 1, \epsilon(\kappa)}). \] (C.17)
Note
\[
\left\| \chi_A^\circ (V_{T,2,(\kappa)}^p - \Delta p_T^\rho_{T,(\kappa)}) \right\|_{L^2_{m+1}(A_T)} \\
\leq C_m \epsilon e^{-6T\delta} \left\| V_{T,2,(\kappa)}^p - \Delta p_T^\rho_{T,(\kappa)} \right\|_{L^2_{m+1,\delta}(A_T \cap \Sigma_T)} (C.18)
\]
\[
\leq C'_m \epsilon e^{-6T\delta - T \delta_1} \mu^{\kappa - 1}.
\]

The first inequality follows from the fact that the weight function \( e_{2,\delta} \) is around \( e^6T\delta \) on \( A_T \). The second inequality follows from \( (5.55) \). On the other hand the weight function \( e_{T,\delta} \) is around \( e^{4T\delta} \) at \( A_T \). \( (C.18) \) implies
\[
\left\| \chi_A^\circ (V_{T,2,(\kappa)}^p - \Delta p_T^\rho_{T,(\kappa)}) \right\|_{L^2_{m+1,\delta}(A_T \cap \Sigma_T)} \leq C_m \epsilon e^{-2T\delta - T \delta_1} \mu^{\kappa - 1}.
\]

Therefore in the same way as we did on \( K \) we have
\[
\left\| \partial (V_{T,2,(\kappa)}^p - \Delta p_T^\rho_{T,(\kappa)}) \right\|_{L^2_{m+1,\delta}(A_T \cup \Sigma_T)} \leq C_m \epsilon (5) e^{-\delta T} / 20
\]

for \( T > T_{m,e(5)}, (C.20) \).

Since \( \text{Ent}^\rho_{T,z,(\kappa-1)} = 0 \) on \( A_T \) we have
\[
\partial u_{T,(\kappa-1)}^\rho + (D u_{T,(\kappa-1)}^\rho) \partial (V_{T,1,(\kappa)}^p) = 0.
\]

Therefore in the same way as we did on \( K_1 \) we can show
\[
\left\| \partial (\text{Exp}(u_{T,1,(\kappa)}^p), V_{T,1,(\kappa)}^p) \right\|_{L^2_{m+1,\delta}(A_T \cup \Sigma_T)} \leq C_m \epsilon (5) e^{-\delta T} \mu^{\kappa - 1} / 20
\]

for \( T > T_{m,e(5)}, (C.22) \).

\( (C.20) \) and \( (C.22) \) imply
\[
\left\| \partial u_{T,(\kappa)}^\rho \right\|_{L^2_{m+1,\delta}(A_T \cup \Sigma_T)} \leq C_m \epsilon (5) e^{-\delta T} \mu^{\kappa - 1} / 10
\]

The estimate on \( B_T \) is similar. The proof of Proposition 5.33 is complete except the estimate of parallel transport given in the next section. □

**Appendix D. Estimate of Parallel Transport 2**

**Proof of \( (C.7) \).** Let \( \Omega \) is a small neighborhood of \( K_1 \). We consider the vector bundle \( \mathcal{L} \) on \( \mathcal{U} \) defined by \( (B.1) \) and \( (B.2) \). We put
\[
\mathcal{V} = \{(z, \nu, \mathbf{w}) | z \in \Omega, \nu, \mathbf{w} \in T_{u_1(z)}X, |\nu|, |\mathbf{w}| < R\},
\]
with \( R < \epsilon_X / 10 \). We pull \( \mathcal{L} \) back by the map \( \mathcal{V} \rightarrow \mathcal{U} \) defined by
\[
(z, \nu, \mathbf{w}) \mapsto (u_1(z), u_1^\rho(z), \nu, \mathbf{w})
\]
to obtain a vector bundle \( \mathcal{L} \) on \( \mathcal{V} \).

We pull back the section \( \mathcal{V} \) in \( (B.3) \) by this map and tensor it with the identity in \( \text{End}(\Lambda^{0,1}(\Omega)) \). We then obtain a section \( P \) of \( \mathcal{L} \otimes \text{End}(\Lambda^{0,1}(\Omega)) \). \( P \) is written as:
\[
P(z, \nu, \mathbf{w}) = \left( \text{Pal}_{u_1(z)}^{\text{Exp}(u_1(z), \mathbf{w})}(0,1) \right)^{-1} \circ \left( \text{Pal}_{u_1(z)}^{\hat{v}(z, \nu, \mathbf{w})}(0,1) \right)^{-1} \circ \left( \text{Pal}_{u_1(z)}^{\hat{v}(z, \nu, \mathbf{w})}(0,1) \right)
\]
where
\[
\hat{v}(z, \nu, \mathbf{w}) = \text{Exp}(\text{Exp}(u_1(z), \mathbf{w}), \text{Pal}_{u_1(z)}^{\text{Exp}(u_1(z), \mathbf{w})}(\nu)).
\]

See Figure 15.
Let $\hat{e}_i(z)$ $(i = 1, \ldots, \dim \mathcal{E}_1)$ be a basis of $\mathcal{E}_1 = \mathcal{E}_1(u_{10})$. We define

$$\hat{e}_i(z, v, w) = P(z, v, w)(\hat{e}_i(z)), $$

which is a smooth section of the bundle $u_{10} TX \otimes \Lambda^{0,1}(\Omega)$ on $\mathfrak{U}$. (Here we denote the map of $(z, v, w) \rightarrow u_1(z)$ by $u_1$ by a slight abuse of notation.) We consider

$$e'_i(z, r) = P(z, rV^0(z), W^0(z))(\hat{e}_i(z))$$

$$= e_i(z, rV^0(z), W^0(z)) \in T_{u_1(z)}X \otimes \Lambda^{0,1}(\Omega).$$

The sections $e'_i \in \Gamma(T_{u_1(z)}X \otimes \Lambda^{0,1}(\Omega))$, $(i = 1, \ldots, \dim \mathcal{E}_1)$ form a basis of $P(\mathcal{E}_1)$.

In the next step, we will use the Gram-Schmidt process to modify it to an orthonormal basis with respect to the definite quadratic form on $T_{u_1(z)}X$. We define its section $g$ as follows. We define $\mathcal{P}\mathcal{P} : T_{u_1(z)}X \otimes \Lambda^{0,1}_z \rightarrow T_{\hat{v}(z, v, w)} \otimes \Lambda^{0,1}_z$ by

$$\mathcal{P}\mathcal{P} = \left(\text{Pal}_{\text{Exp}(u_{10}(z), \mathfrak{U})}^{\hat{v}(z, v, w)}\right)^{(0,1)} \circ \left(\text{Pal}_{u_{10}(z)}^{\text{Exp}(u_1(z), \mathfrak{U})}\right)^{(0,1)}.$$ 

Then

$$g(z, v, w)(\tilde{v}, \tilde{w}) = g_{\hat{v}(z, v, w)}(\mathcal{P}\mathcal{P}(\tilde{v}), \mathcal{P}\mathcal{P}(\tilde{w})).$$

Here $\hat{v}$ is as in (D.1), and $g_{\hat{v}(z, v, w)}$ is the Riemann metric tensor at $\hat{v}(z, v, w) \in X$. It is also obvious that $g$ is a smooth section.

We define $e_i(z, r)$ by induction on $i$ as follows. Suppose $e_j(z, r)$ is defined for $j < i$. We put

$$e'_i(z, r) = e'_i(z, r)$$

$$- \sum_{j=1}^{i-1} \int_{z \in \Omega} g(z, rV^0(z), W^0(z))(e'_i(z, r), e_j(z, r)) d\text{vol} \times e_j(z, r),$$

and

$$e_i(z, r) = \frac{e''_i(z, r)}{\left(\int_{z \in \Omega} g(z, rV^0(z), W^0(z))(e''_i(z, r), e''_i(z, r)) d\text{vol}\right)^{1/2}}.$$

Figure 15. Map $P$
Then $e_i(z, r)$ ($i = 1, \ldots, \dim E_1$) is the orthonormal basis we look for.

**Remark D.1.** Since $L^2$ norm is not defined pointwise, we can not write $e_i(z, r)$ in a form $\tilde{e}_i(z, rV^0(z), W^0(z))$ with some smooth map $\tilde{e}_i(z, v, w)$. In fact (D.4) and (D.5) contain integration. On the other hand, as far as smoothness in the sense of Sobolev space concerns, integration behaves nicely.

Now the linear map

$$
\left((\text{Pal}_{u_1^1})^{(0,1)}\right)^{-1} \circ (\text{Pal}_{u_1^1})^{(0,1)} \circ \Pi_{E_1}^{(0,1)} \circ (\text{Pal}_{u_1^1})^{(0,1)} \circ (\text{Pal}_{u_1^1})^{(0,1)}
$$

is written as

$$
\Gamma(\Omega; u_1^1TX \otimes \Lambda^{0,1}) \rightarrow \Gamma(\Omega; u_1^1TX \otimes \Lambda^{0,1})
$$

(D.6)

We write the right hand side of (D.7) by

$$
H(r, Q)(\cdot)
$$

where $\cdot \in \Omega$. Namely $(r, w) \mapsto H(r, Q)(w)$ is a section of the pullback of $u_1^1TX \otimes \Lambda^{0,1}$ to $[0, \epsilon) \times \Omega$.

By definition of $D_uE_1$, we have

$$
H(r, Q)(\cdot)
$$

(D.8)

then using (D.9) and (D.7) we obtain (C.7).

**Estimate of the third term of (C.5).** Using (D.10), (D.9) in the same way, we can prove

$$
\left\| \frac{d^2}{dr^2} e_i \right\|_{L^2_m(\Omega)} \leq C_m \| V^0 \|^2_{L^2_m},
$$

(D.10)

Then using also (D.7) and (D.8) we obtain (C.7).

Then using also (D.10) and (D.7) also we estimate

$$
\left\| \frac{d^2}{dr^2} H(r, Q) \right\|_{L^2_m(\Omega)} \leq C_m \| V^0 \|^2_{L^2_m} \| Q \|_{L^2_m}
$$

(D.11)

for any $Q \in L^2_m(\Omega; u_1^1YX \otimes \Lambda^{0,1})$. This gives the required estimate of the third term of (C.5).
Estimate of (5.80). We put
\[ W_0 = E(u_1, \hat{u}_{1,T,(0)}) \]

Then using (D.10) and (D.8) we obtain
\[ \| (D_{\hat{u}_{1,T,(0)}} E_1)(V, Q) \|_{L^2_m} \leq C_{m, (D.12)} \| V \|_{L^2_m} \| Q \|_{L^2_m} \quad (D.12) \]

for any \( V, Q \in L^2_m (\Omega; (\hat{u}_{1,T,(0)})^* TX \otimes \Lambda^{0,1}) \).

Using (5.79) also we can estimate
\[ \left\| (D_{\hat{u}_{1,T,(0)}} E_1)((\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}((\text{se})^0)_{u_1,T,(\kappa-1)}), Q) - (D_{\hat{u}_{1,T,(0)}} E_1)((\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(\epsilon^0)_{1,T,(0)}), Q) \right\|_{L^2_m} \leq C_{m, (D.13)} e^{-\delta_1 T} \mu \| Q \|_{L^2_m} \quad (D.13) \]

for \( L^2_m \) section \( Q \) of \( (\hat{u}_{1,T,(0)})^* TX \otimes \Lambda^{0,1}(\Omega) \).

We put \( (\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}((\text{se})^0)_{u_1,T,(\kappa-1)}) = \text{se} \).

Then using (D.10) to \( V_0 = E(\hat{u}_{1,T,(0)}^{\hat{u}_{1,T,(0)}}, \hat{u}_{1,T,(\kappa-1)}) \) and integrating along the curve \( r \mapsto \text{Exp}(\hat{u}_{1,T,(0)}, r \check{V}) \) we obtain
\[ \left\| (\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(D_{\hat{u}_{1,T,(0)}} E_1)(\text{se}, Q) - (D_{\hat{u}_{1,T,(0)}} E_1)((\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(\text{se})_{u_1,T,(\kappa-1)}), ((\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(\epsilon^0)_{1,T,(0)}), Q) \right\|_{L^2_m} \leq C_{m, (D.14)} \| V \|_{L^2_m} \| Q \|_{L^2_m} \quad (D.14) \]

Note we use induction hypothesis (5.67) here.

In the same way as the proof of Lemma 5.20 we can show
\[ \left\| ((\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(\text{se})_{u_1,T,(\kappa-1)}) - (\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(\text{se})_{u_1,T,(\kappa-1)}) \right\|_{L^2_m} \leq C'_{m, (D.15)} e^{-\delta_1 T} \| \check{V} \|_{L^2_m} \quad (D.15) \]

and
\[ \left\| ((\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(Y) - (\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(Y)) \right\|_{L^2_m} \leq C'_{m, (D.16)} e^{-\delta_1 T} \| Y \|_{L^2_m} \quad (D.16) \]

Combining (D.13), (D.14), (D.15), and (D.16) we obtain the required formula:
\[ \left\| (D_{\hat{u}_{1,T,(\kappa-1)}} E_1) ((\text{Pal}_{u_1}^{\hat{u}_{1,T,(\kappa-1)}})^{(0,1)}((\text{se})^0)_{u_1,T,(\kappa-1)}), ((\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(Y)) - (\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(D_{\hat{u}_{1,T,(0)}} E_1) ((\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(\epsilon^0)_{1,T,(0)}), ((\text{Pal}_{u_1}^{\hat{u}_{1,T,(0)}})^{(0,1)}(Y)) \right\|_{L^2_m} \leq C_{m, (D.17)} e^{-\delta_1 T} \| Y \|_{L^2_m} \quad (D.17) \]
APPENDIX E. ESTIMATE OF THE NON-LINEARITY OF EXPONENTIAL MAP

We use the next lemma for the proof of Proposition 6.9 (3).

Lemma E.1. Let $\mathcal{S}$ be a 2-dimensional manifold and $u : \mathcal{S} \to X$ a smooth map. Let $v : \mathcal{S} \to X$ be a map of $L^2_{m+1}$ class with $m > 2$. We assume $d(u(z), v(z))$ is smaller than $\iota_X/2$ for all $z \in \mathcal{S}$. We define a map

$$E_P : L^2_{m+1}(\mathcal{S}, u^*TX) \to L^2_{m+1}(\mathcal{S}, u^*TX)$$

by the next formula

$$E_P(v)(z) = E\left(u(z), \text{Exp}(z, v(z), \text{Pal}^v_{u(z)}(V(z)))\right).$$  \hfill (E.1)

We assume $\|v\|_{L^2_{m+1}}^2, \|E(u, v)\|_{L^2_{m+1}}^2 \leq 1$. Here $E(u, v)(z) = E(u(z), v(z))$. Then

$$\|E_P(v) - E(u, v) - V\|_{L^2_{m+1}}^2 \leq C_m \|E(u, v)\|_{L^2_{m+1}}^2 \|V\|_{L^2_{m+1}}^2. \hfill (E.2)$$

Remark E.2. In the simplified notation mentioned in Remark 2.2, we have $E_P(v)(V) = (v - u) + V$. This way of writing is a bit confusing here since (E.2) implies that this ‘equality’ holds modulo quadratic order term.

Proof. Let

$$\Omega = \{(V, W) \in TX \oplus TX \mid \|V\|, |W| \leq \iota_X/10\}.$$

We define a map $F : \Omega \to TX$ as follows. Let $(V, W) \in T_xX \oplus T_{x'}X$. We put $y = \text{Exp}(x, W)$ and

$$F(V, W) = E(x, \text{Exp}(x, \text{Pal}^x_{V}(W))).$$

This map is obviously smooth. So by the compactness of $\Omega$ its $C^k$ norm is uniformly bounded for any $k$. Moreover we have

$$F(0, 0) = 0, \quad F(V, 0) = V, \quad F(0, W) = W.$$

We observe

$$E_P(u)(z) = F(V(z), E(u(z), v(z))).$$

Therefore the lemma follows from the standard fact that the left composition with smooth map defines a smooth map between $L^2_{m+1}$ spaces. In fact

$$E_P(u)(V) - E(u, v) - V = F(V, E(u, v)) - F(0, E(u, v)) - F(0, 0) - F(0, E(u, v))$$

$$= \int_0^1 \frac{\partial}{\partial r}(F(V, rE(u, v)) - F(0, rE(u, v)))dr.$$

$$= \int_0^1 \int_0^1 \frac{\partial^2}{\partial r \partial t}F(tV, rE(u, v))drdt. \hfill (E.3)$$

Therefore we can estimate $E_P(u)(V) - E(u, v) - V$ as (E.2), by using

$$\frac{\partial^2}{\partial r \partial t}F(tV, rE(u, v)) = \sum_{a, b} V_a E_b(u, v) G_{a, b}(r, t, V, rE(u, v)), \hfill (E.4)$$

where $V_a, E_b(u, v)$ are components of $V, E(u, v)$, respectively, and $G_{a, b}$ are smooth.

The following variant of Lemma E.1 can be proved in the same way.
Lemma E.3. In the situation of Lemma E.1 we assume \( v = v(s), \) \( V = V(s) \) are families of \( C^1 \) class parameterized by \( s \in S \) for some parameter space \( S \subset \mathbb{R}^N \). We assume \( \left\| \frac{d^n}{ds^n} v \right\|_{L^2_{m+1}} \cdot \left\| \frac{d^n}{ds^n} E(u, v) \right\|_{L^2_{m+1}} \leq 1 \) for \( n \leq n' \leq n \). Then

\[
\left\| \frac{d^n}{ds^n} \left( \mathcal{E} \mathcal{P}_{v(s)}(V(s)) - E(u, v(s)) - V(s) \right) \right\|_{L^2_{m+1}} \leq C_{m, \text{E.3}} \sum_{k_1 + k_2 \leq n} \left\| \frac{d^{k_1}}{ds^{k_1}} V(s) \right\|_{L^2_{m+1}} \left\| \frac{d^{k_2}}{ds^{k_2}} E(u, v) \right\|_{L^2_{m+1}}
\]

(E.5)

Proof. The lemma follows easily by taking \( s \) differentiation of (E.3) and (E.4).

We can prove Proposition 5.23 (3) by using Lemma E.3. Actually Proposition 5.23 (3) follows from (6.9) in the same way as the proof of Lemma 6.11, that is, by integration over \( T \).

Proof of Proposition 6.9 (3). Using the simplified notation as in Remark 2.2 we have

\[ u_{T, (\kappa)}^p = \nu u_1 + \sum_{j=0}^{\kappa} V_{T, 1, (j)}^p \]

on \( K_1 \). Therefore

\[ \left\| u_{T, (\kappa)}^p - u_1^p \right\|_{L^2_{m+1}} \leq \nu \sum_{j=0}^{\kappa} \left\| V_{T, 1, (j)}^p \right\|_{L^2_{m+1}} \]

\[ \leq \nu C_{5, m} \sum_{j=0}^{\kappa} \mu^j \mu^{-1} e^{-\delta_i T} \leq C_{5, m} \mu^{-1} \frac{1 - \mu^{\kappa + 1}}{1 - \mu} e^{-\delta_i T}. \]

This would prove (6.9) on \( K_1 \). However here the above notation \( + \) is rather imprecise and we need to use exponential map and parallel transport. In other words we need to estimate the effect of the nonlinearity of the exponential map to work out the above inequality. We use Lemma E.3 for this purpose. The above inequality suggests that \( C_{6, m} \) depends on \( C_{5, m} \).

In the proof of Proposition 6.9 (3), we need to decide \( C_{6, m} \) and \( C_{7, m} \) depending on \( C_{5, m}, C_{8, m}, C_{9, m} \). So the dependence of the constants appearing during the proof on \( C_{5, m}, C_{6, m}, C_{7, m}, C_{8, m}, C_{9, m} \) need to be carefully examined.

Hereafter during the proof of Proposition 6.9 (3) we use the notation of the constants \( C_{m, *}, D_{m, *}, \) as follows. (Here * is the number of the formula where the constants appear.) \( C_{m, *} \) is a constant depending on all of \( C_{5, m}, C_{6, m}, C_{7, m}, C_{8, m}, C_{9, m} \), and on \( m \). \( D_{m, *} \) is a constant depending \( C_{5, m}, C_{8, m}, C_{9, m}, m \) but is independent of \( C_{6, m} \) and \( C_{7, m} \). \( T_{m, *}, D_{m, *}, \) is a constant depending on all of \( C_{5, m}, C_{6, m}, C_{7, m}, C_{8, m}, C_{9, m} \) and on \( m \).

We now start the proof. Let \( m - 2 \geq n, \ell \geq 0 \) and \( \ell > 0 \).

On \( K_1 \) we have \( u_{T, (\kappa)}^p = \text{Exp}(u_{T, (\kappa-1)}^p, V_{T, 1, (j)}^p) \). Therefore by (6.8) for \( \leq \kappa \), (6.9) for \( \leq \kappa - 1 \), and Lemma E.3 we have

\[
\left\| \nabla \left. \frac{d^\ell}{dt} \right|_{L^{m+1-\ell}} E(u_1, u_{T, (\kappa)}^p) \right\|_{L_{m+1-\ell}} \leq \left\| \nabla \left. \frac{d^\ell}{dt} \right|_{L^{m+1-\ell}} E(u_1, u_{T, (\kappa-1)}^p) \right\|_{L_{m+1-\ell}} + C_{5, m} \mu^{-1} e^{-\delta_i T} + C_{m, \text{E.6}} \mu^{-1} e^{-2s_i T}.
\]

(6.6)
Taking $T_{m,E.7}$ so that $C_{m,E.7} \leq 1$ we have
\[ \text{LHS of } \mathcal{E.6} \leq \left\| \nabla^n_{\rho} \frac{\partial^f}{\partial T^4} E(u_1, u_{\rho,T,2,10}(\kappa-1)) \right\|_{L_{m-1-\epsilon,\delta(K_1)}^2} + D_{m,E.7} \mu^{\kappa-1} e^{-\delta_1 T} \quad (E.7) \]
for $T \geq T_{m,E.7}$.

We next study at the neck region. We consider $m-2 \geq n, \ell \geq 0$. (The case $\ell = 0$ is included.) We put
\[ S_{m} = \text{Pal}_{u_{\rho,T,2,10}(\kappa-1)} \quad (\text{resp. } \text{Pal}_{u_{\rho,T,2,10}(\kappa-1)}). \]
Suppose \[ S_{m} = \text{Pal}_{u_{\rho,T,2,10}(\kappa-1)} \quad (\text{resp. } \text{Pal}_{u_{\rho,T,2,10}(\kappa-1)}). \]

When we use $\tau'$ (resp. $\tau''$) coordinate we write $\Sigma(S')_{\tau'}$ (resp. $\Sigma(S')_{\tau''}$). Using (6.28) we can estimate
\[ \left\| \nabla^n_{\rho} \frac{\partial^f}{\partial T^4} \mathcal{P}_1 U(\tau', t) \right\|_{L_{m-1-\epsilon,\delta(K_1)}^2(\Sigma(S')_{\tau'})} \]
\[ \leq \sum_{l=1}^n \sum_{n'=0}^n D_{m,E.11} \left( \left\| \nabla^n_{\rho} \frac{\partial^f}{\partial T^4} \mathcal{P}_1 U_{\rho,T,1,10}(\kappa-1)(\tau', t) \right\|_{L_{m-1-\epsilon,\delta}(\Sigma(S')_{\tau'})} \right) \]
\[ + \left\| \nabla^n_{\rho} \frac{\partial^f}{\partial T^4} \mathcal{P}_1 U_{\rho,T,2,10}(\kappa-1)(\tau'', t) \right\|_{L_{m-1-\epsilon,\delta}(\Sigma(S')_{\tau''})} \quad (E.11) \]

We remark that the second term of (E.11) drops if $S' \leq T$. (This is because $\chi(\tau' - 4T) = 0$ on $\Sigma(S')_{\tau'}$ in that case.)

By (6.8) we have
\[ \sum_{S = -5T}^{0} \left\| \nabla^n_{\rho} \frac{\partial^f}{\partial T^4} \mathcal{P}_1 U_{\rho,T,1,10}(\kappa-1)(\tau', t) \right\|_{L_{m-1-\epsilon,\delta}(\Sigma(S')_{\tau'})} \leq C_{5,m} \mu^{\kappa-1} e^{-\delta_1 T}, \quad (E.12) \]
for $i = 1, 2$. In case $i = 2$ we need to replace $\mathcal{P}_2$ by $\mathcal{P}_1$ to apply (E.12) to estimate (E.11).
Sublemma E.4. We have
\[
\left\| \left( \nabla^{n}_{\rho} \frac{\partial}{\partial T} \right) \left( \Psi_{1} U^{p}_{T,2,(\kappa)}(\tau'', t) \right) \right\| \leq D_{m} \left( \Sigma(S'), \delta \right) \sum_{l' \leq \ell} \sum_{n'' \leq n'} \left\| \left( \nabla^{n''}_{\rho} \frac{\partial^{l'}}{\partial T^{l'}} \right) \left( \Psi_{2} U^{p}_{T,2,(\kappa)}(\tau'', t) \right) \right\|_{L^{2}_{m+1-\ell'}(\Sigma(S'), \delta)}
\]
for \( T > T_{m}, \) \( \ell \), and \( \delta \).

Postponing the proof of the sublemma we continue the proof. Note the ratio between the weights \( \epsilon_{1, \delta} \) and \( \epsilon_{T, \delta} \) are bounded on \( \tau' \in [0, 5T], \kappa \times [0, 1] \) and the weight \( \epsilon_{2, \delta} \) is larger than \( \epsilon_{T, \delta} \). Therefore taking the weighted sum of the square of \( \Psi_{1} \), \( \Psi_{2} \) imply
\[
\left\| \left( \nabla^{n}_{\rho} \frac{\partial}{\partial T} \right) \left( \Psi_{1} U^{(\tau', t)} \right) \right\|_{L^{2}_{m+1-\ell, \delta}(\Sigma, \delta)} \leq D_{m} \left( \Sigma(S'), \delta \right) \mu^{T_{m} - \delta_{T} T}.
\]
We put \( q = u_{1}(0, 1/2) \) and
\[
\Psi_{1}' = \text{Pal}^{\mu_{1}}_{q} \circ \text{Pal}^{\delta}_{u^{p}_{T, \kappa-1}}
\]

Sublemma E.5. We have
\[
\left\| \left( \nabla^{n}_{\rho} \frac{\partial}{\partial T} \right) \left( \Psi_{1} \left( \left( \Delta p^{p}_{T, \kappa, \ell} \right)^{\text{Pal}} \right) \right) \right\|_{L^{2}_{m+1-\ell, \delta}(\Sigma, \delta)} \leq D_{m} \left( \Sigma(S'), \delta \right) \mu^{T_{m} - \delta_{T} T}.
\]
Here we regard \( \left( \Delta p^{p}_{T, \kappa, \ell} \right)^{\text{Pal}} \) as a section on \( (u^{p}_{T, \kappa-1})^{*} TX \) (as we do in \( E.10 \)).

So \( E.15 \) is
\[
\left\| \left( \nabla^{n}_{\rho} \frac{\partial}{\partial T} \right) \left( \text{Pal}^{\mu_{1}}_{u^{p}_{T, \kappa-1}} \circ \text{Pal}^{\delta}_{u^{p}_{T, \kappa-1}} \left( \Delta p^{p}_{T, \kappa} \right) \right) \right\|_{L^{2}_{m+1-\ell, \delta}(\Sigma, \delta)} \leq D_{m} \left( \Sigma(S'), \delta \right) \mu^{T_{m} - \delta_{T} T}.
\]
Postponing the proof of the sublemma we continue the proof.

Lemma E.3 and \( E.10 \), \( E.14 \), \( E.15 \), imply
\[
\left\| \left( \nabla^{n}_{\rho} \frac{\partial}{\partial T} \right) \left( \Psi_{1} E \left( u^{p}_{T, \kappa-1}, u^{p}_{T, \kappa} \right) \right) \right\|_{L^{2}_{m+1-\ell, \delta}(\Sigma, \delta)} \leq D_{m} \left( \Sigma(S'), \delta \right) \mu^{T_{m} - \delta_{T} T}.
\]
Using also Lemma E.3 we can show the next sublemma.
Sublemma E.6. The next inequality holds.
\[
\left\| \nabla_{\rho} \frac{\partial}{\partial t} E(u_1, u_{T, \kappa}) \right\|_{W^{2}_{m+1-\ell, \delta}([0, T+1], \mathbb{R}^+ \times [0, 1] \subset \Sigma_T)} \\
\leq \left\| \nabla_{\rho} \frac{\partial}{\partial t} E(u_1, u_{T, \kappa-1}) \right\|_{W^{2}_{m+1-\ell', \delta}([0, T+1], \mathbb{R}^+ \times [0, 1] \subset \Sigma_T)} \\
+ D_m, [E.18] \mu^{k-1} e^{-\delta_1 T}.
\]

Note the term corresponding to \( \mathcal{Q}_I^T((\Delta \rho)_{T, \kappa})^{Pal} = (\text{Pal}_{q_1} \circ \text{Pal}_{q_{T, \kappa-1}})((\Delta \rho)_{T, \kappa})^{Pal} \) disappears in \( [E.18] \) since we take \( W^{2}_{m+1-\ell', \delta} \) norm in place of \( L^{2}_{m+1-\ell', \delta} \) norm. See \( [4.9] \) and we recall \( q = u_1(0, 1/2) \). Postponing the detail of the proof of the sublemma we continue the proof.

We choose \( C_{6, m} \) such that
\[
D_{m, [E.18]} \leq C_{6, m}(1 - \mu)/10, \quad D_{m, [E.7]} \leq C_{6, m}(1 - \mu)/10.
\]
Then \( [E.7], [E.18] \) and the \( \kappa - 1 \) case of \( [6.9] \) implies that for \( \ell > 0 \) we have
\[
\left\| \nabla_{\rho} \frac{\partial}{\partial t} E(u_1, u_{T, \kappa}) \right\|_{W^{2}_{m+1-\ell, \delta}([0, T+1])} \\
\leq \left\| \nabla_{\rho} \frac{\partial}{\partial t} E(u_1, u_{T, \kappa-1}) \right\|_{W^{2}_{m+1-\ell, \delta}([0, T+1])} \\
+ D_m, [E.18] \mu^{k-1} e^{-\delta_1 T} + D_m, [E.7] \mu^{k-1} e^{-\delta_1 T} \\
\leq C_{6, m}(2 - \mu^{k-1}) e^{-\delta_1 T} + C_{6, m}(2 - \mu^{k}) e^{-\delta_1 T}/5
\]
\[
\leq C_{6, m}(2 - \mu^{k}) e^{-\delta_1 T}.
\]

We thus proved \( \kappa \) case of \( [6.9] \).

Remark E.7. In case \( \ell = 0 \) \( [6.9] \) fails for \( \kappa = 0 \).

We next prove \( [6.10] \). We consider \( m - 2 \geq n, \ell \geq 0 \). (The case \( \ell = 0 \) is included.) We denote
\[
d_{\ell, n}(u, v) = \sum_{n' \leq n, \ell' \leq \ell} \left\| \nabla_{\rho} \frac{\partial}{\partial t} \text{Pal}_{u, \ell, \kappa}^{n'} E(u, v) \right\|_{L^{2}_{m+1-\ell', \delta}((K^T_{\ell, 1}) \subset \Sigma_T)}.
\]
Here \( u, v \) are \( (T, \rho) \) dependent family of maps on \( \Sigma_1 \). (We assume \( d(u_1(z), u(z)) \leq \epsilon_\lambda \) etc. so that \( \text{Pal}_{u, \ell, \kappa}^{n'} \) etc. is defined.) We remark that we take \( L^{2}_{m+1-\ell', \delta} \) norm without weight in the right hand side.

Note \( d_{\ell, n} \) is not a metric. In particular it may not satisfy triangle inequality. However Lemma \( E.3 \) implies
\[
d_{\ell, n}(v_1, v_3) \leq d_{\ell, n}(v_1, v_2) + d_{\ell, n}(v_2, v_3) + C_{6, [E.20]} d_{\ell, n}(v_1, v_2) d_{\ell, n}(v_2, v_3) \quad [E.20]
\]
if
\[
\left\| \nabla_{\rho} \frac{\partial}{\partial t} E(u_1, v_i) \right\|_{L^{2}_{m+1-\ell, \delta}((K^T_{\ell, 1}) \subset \Sigma_T)} \leq 1
\]
for \( i = 1, 2, 3, n', \ell' \leq n, \ell \leq \ell \).
Note (E.17) holds when we replace $\mathcal{F}_1$ there by $\mathcal{F}_1$. It then follows
\[ d_{t,n}(u^p_{T,(\kappa)}, \text{Exp}(u^p_{T,(\kappa-1)}, (\Delta p^p_{T,(\kappa)}))) \leq D_m(2 - \mu^{-1})e^{-\delta_1 T}. \] (E.21)

We put
\[ \square_{\kappa-1} = E(p^p_0, u^p_{T,(\kappa-1)}). \]
The induction hypothesis, which is $\kappa - 1$ case of (6.10), implies
\[ \left\| \nabla^n \frac{\partial t}{\partial t} \text{Pal}^{p^p_0}(\square_{\kappa-1}) \right\|_{L^2_{m+1-\ell}(\mathcal{K}_1^{\mathcal{T}T})} \leq C_{T,m}(2 - \mu^{-1})e^{-\delta_1 T}. \] (E.22)

Note (6.10) is actually (6.16).

Then Lemma E.3 implies
\[ d_{t,n}(\text{Exp}(u^p_{T,(\kappa-1)}, (\Delta p^p_{T,(\kappa)})), \text{Exp}(p^p_0, \square_{\kappa-1} + \text{Pal}^{p^p_0}_{p^p_{T,(\kappa-1)}}(\Delta p^p_{T,(\kappa)}))) \leq C_m \mu^{-1}e^{-\delta_1 T}. \] (E.23)

On the other hand, when we regard $(\Delta p^p_{T,(\kappa)})$ as an element of $T_{\mathcal{F}^0}X$ we have
\[ \left\| \nabla^n \frac{\partial t}{\partial t} \text{Pal}^{p^p_0}(E(p^p_0, p^p_{T,(\kappa)})) \right\| \leq C_m \mu^{-1}e^{-\delta_1 T}. \] (E.24)

This follows from (6.8), Lemma B.1 and
\[ \text{(E.21), (E.23) and (E.20)} \]
implies
\[ d_{t,n}(u^p_{T,(\kappa)}, \text{Exp}(p^p_0, \square_{\kappa-1} + \text{Pal}^{p^p_0}_{p^p_{T,(\kappa-1)}}(\Delta p^p_{T,(\kappa)}))) \leq D_m \mu^{-1}e^{-\delta_1 T}. \] (E.26)

if $T > T_m$. We use the fact that the exponent of $e$ in (E.23) is $-2\delta_1 T$, to change $C_m$ there to $D_m$. The third term of (E.20) is estimated by $C\mu^2\kappa^{-1}e^{-2\delta_1 T}$ in our case. So it is estimated by $D\mu^{-1}e^{-\delta_1 T}$ if $T$ is sufficiently large.

Using Lemma E.3, (E.24), (E.22), (E.26) imply
\[ \left\| \nabla^n \frac{\partial t}{\partial t} \text{Pal}^{p^p_0}(E(p^p_0, u^p_{T,(\kappa)})) \right\| \leq C_{T,m}(2 - \mu^{-1})e^{-\delta_1 T} + D_m \mu^{-1}e^{-\delta_1 T}. \] (E.27)

We choose $C_{T,m}$ such that
\[ C_{T,m}(1 - \mu) > D_m. \]

Then
\[ \left\| \nabla^n \frac{\partial t}{\partial t} \text{Pal}^{p^p_0}(E(p^p_0, u^p_{T,(\kappa)})) \right\| \leq C_{T,m}(2 - \mu^{-1})e^{-\delta_1 T}. \] (E.28)

Remark E.8. We remark that when we consider the domain $K_1^{\mathcal{T}T} \setminus K_1$ in place of $K_1^{\mathcal{T}T} \setminus K_1$ and $\ell = 0$ then (E.28) fails in the first step, that is, $\kappa = 0$.

To complete the proof of Proposition 6.9 (3) it remains to prove sublemmata. □
Proof of Sublemma E.4 We take and fix $\rho_0'$ and prove this estimate under the assumption that $d(\rho, \rho_0') \leq e^{-\delta_1 T}$. As usual we use $\tau'$ (resp. $\tau''$) coordinate in the definition of $L^2_{m+1}$ norm for $i = 1$ (resp. $i = 2$). We consider maps

$$\Psi'' = \text{Pal}_{\rho_0'} : \Gamma(\Sigma(S'); T_{\rho_0'}X) \to \Gamma(\Sigma(S'); u_1^*TX).$$

Let $W$ be a $(T, \rho)$ parameterized family of sections of $\Gamma(\Sigma(S'); T_{\rho_0'}X)$. By using $(T, \rho)$ independence of $\Psi''$ we can prove

$$D_{m+1}^{-1} \| \frac{\partial}{\partial \ell} \Psi''_i(W) \|_{L^2_{m+1-\ell}(\Sigma(S'))} \leq D_m \| \frac{\partial}{\partial \ell} \Psi''_i(W) \|_{L^2_{m+1-\ell}(\Sigma(S'))} \leq D_m$$

for $i = 1, 2$.

Since $S'$ is $T$ we have

$$\| E(p_0^{\rho}, u_{T,(\kappa-1)}) \|_{L^2_{m+1}(S', S'+1, \ell\times[0,1])} \leq C_{m, (E.30)} e^{-\delta_1 T}. \quad (E.29)$$

Here $C_{m, (E.30)}$ depends on $C_{\tau, m}$.

We put:

$$\Psi''' = \text{Pal}_{\rho_0'} : \Gamma(\Sigma(S'); T_{\rho_0'}X) \to \Gamma(\Sigma(S'); (u_{T,(\kappa-1)}^\rho)^*TX).$$

If $W$ is a $(T, \rho)$ independent section of $u_2^*TX$ on $\Sigma(S')$, Lemma B.1 implies

$$\| \frac{\partial}{\partial \ell} (\Psi''' \circ (\Psi''_2) - \Psi_1 \circ (\Psi''_2) - (\Psi_1 \circ (\Psi''_2)) - (\Psi_1 \circ (\Psi''_2)) - (\Psi_1 \circ (\Psi''_2)) \|_{L^2_{m+1-\ell}(\Sigma(S'))} \leq C_{m, (E.31)} e^{-\delta_1 T} \| W \|_{L^2_{m+1-\ell}(\Sigma(S'))}$$

See Figure 16.

![Figure 16. $\Psi'''$, $\Psi''_i$ and $\Psi_i$](image-url)
We put

\[ \dot{W}(T, \rho) = \mathcal{P}_2(U^n_{T,2,(\kappa)}). \]

Then using \ref{E.12}, \ref{E.31} we can derive

\[
\left\| \nabla^n_n \frac{\partial l}{\partial T^l} \mathcal{P}_1(U^n_{T,2,(\kappa)}) - \mathcal{P}_T^n(\mathcal{P}_T^n)^{-1}(\dot{W}(T, \rho)) \right\|_{L^2_{m+\ell-1}(\Sigma^{(S')}_{\tau^\rho})} = \left\| \nabla^n_n \frac{\partial l}{\partial T^l} (\mathcal{P}_1 \circ \mathcal{P}_2^{-1} \circ (\mathcal{P}_2^{-1})(\dot{W}(T, \rho))) \right\|_{L^2_{m+\ell-1}(\Sigma^{(S')}_{\tau^\rho})} \leq \sum_{n' \leq n} \sum_{l' \leq l} C_m \left( \sum_{\tau^\rho} e^{-\delta_1 T} \right) \left\| \nabla^n_n' \frac{\partial l'}{\partial T^l'} \dot{W}(T, \rho) \right\|_{L^2_{m+\ell-1}(\Sigma^{(S')}_{\tau^\rho})}. \tag{E.32}
\]

Therefore using \ref{E.29} also we have

\[
\left\| \left( \nabla^n_n' \frac{\partial l'}{\partial T^l'} \right) (\mathcal{P}_1 U^n_{T,2,(\kappa)}(\tau^n, t)) \right\|_{L^2_{m+\ell-1}(\Sigma^{(S')}_{\tau^\rho})} \leq (D_m \ref{E.33} + C_m \ref{E.33} e^{-\delta_1 T}) \sum_{n' \leq n} \sum_{l' \leq l} \left\| \nabla^n_n' \frac{\partial l'}{\partial T^l'} \dot{W}(T, \rho) \right\|_{L^2_{m+\ell-1}(\Sigma^{(S')}_{\tau^\rho})}. \tag{E.33}
\]

We take \( T_m \ref{E.13} \) such that \( e^{-\delta_1 T} C_m \ref{E.33} \) \( \leq 1 \) if \( T > T_m \ref{E.13} \). The sublemma is proved. \( \square \)

**Proof of Sublemma \ref{E.3}** The proof is similar to the proof of Sublemma \ref{E.4}. We take and fix \( \rho_0^{\kappa} \) and prove this estimate under the assumption that \( d(\rho, \rho_0^{\kappa}) \leq e^{-\delta_1 T}. \)

We observe

\[
\left\| \nabla^n_n \frac{\partial l}{\partial T^l} E(u^n_{1,0}, u^n_{1,(\kappa-1)}) \right\|_{L^2_{m+\ell-1}(\Sigma^{(S')}_{\tau^\rho})} \leq C_m \ref{E.34} e^{-\delta_1 T}, \tag{E.34}
\]

\[
\left\| \nabla^n_n \frac{\partial l}{\partial T^l} E(p^n_{0,0}, p^n_{1,(\kappa-1)}) \right\|_{L^2_{m+\ell-1}(\Sigma^{(S')}_{\tau^\rho})} \leq C_m \ref{E.34} e^{-\delta_1 T}.
\]

Therefore using an obvious variant of Lemma \ref{B.1} we can prove

\[
\left\| \nabla^n_n \frac{\partial l}{\partial T^l} \left( \mathcal{P}_1 \circ \text{Pal}_{u^n_{1,0}^{\kappa-1}} \circ \text{Pal}_{p^n_{0,0}^{\kappa-1}} \right) \right\|_{L^2_{m+\ell-1}(\Sigma^{(S')}_{\tau^\rho})} \leq C_m \ref{E.35} e^{-\delta_1 T} \|W\| \tag{E.35}
\]

and

\[
\left\| \nabla^n_n \frac{\partial l}{\partial T^l} \left( \mathcal{P}_1 \circ \text{Pal}_{u^n_{1,0}^{\kappa-1}} \circ \text{Pal}_{p^n_{0,0}^{\kappa-1}} \right) \right\|_{L^2_{m+\ell-1}(\Sigma^{(S')}_{\tau^\rho})} \leq C_m \ref{E.36} e^{-\delta_1 T} \|W\| \tag{E.36}
\]
for $W \in T_{p_0}X$. See Figure 17.

\[ \parallel \nabla^n \frac{\partial}{\partial T^\ell} (\text{Pal}_{p_0} (\Delta p_{T,(\kappa)})_{(\kappa)}) \parallel \leq C_m, \tag{E.37} \]

Note (6.8) and (6.16) imply

\[ \parallel \nabla^n \frac{\partial}{\partial T^\ell} (\text{Pal}_{p_0} (\Delta p_{T,(\kappa)})_{(\kappa)}) \parallel \leq C_m, \mu^{\kappa_1} e^{-\delta_1 T}. \tag{E.37} \]

On the other hand a similar formula as (E.29) holds with $\mathcal{P}_{i}$ replaced by $\text{Pal}_{p_0}$ or $\text{Pal}_{u}$. Therefore we obtain

\[ \parallel \nabla^n \frac{\partial}{\partial T^\ell} (\text{Pal}_{u} (\Delta p_{T,(\kappa)})) \parallel \leq C_m, \mu^{\kappa_1} e^{-\delta_1 T}. \tag{E.38} \]

By Lemma 2.5 and Condition 3.1, we derive

\[ \parallel E(q,u_1) \parallel_{L^2_{m+1-\ell}(\Sigma(S'),\tau')} \leq D_m, \tag{E.39} \]

Therefore by Lemma 3.1

\[ \parallel (\text{Pal}_{u_1} - \text{Pal}_{u_0}) (W) \parallel_{L^2_{m+1-\ell}(\Sigma(S'),\tau')} \leq D_m, \tag{E.40} \]

for $W \in \Gamma(\Sigma(S'),(u_1^{(0)})^{*}TX)$.

Now (E.35), (E.36), (E.38), (E.40) imply

\[ \parallel \nabla^n \frac{\partial}{\partial T^\ell} (\mathcal{P}_1 (\Delta p_{T,(\kappa)})_{(\kappa)}) - \mathcal{P}_1 ((\Delta p_{T,(\kappa)})_{(\kappa)}) \parallel \leq (C_m, (E.41) e^{-\delta_1 T} + D_m, (E.41) e^{-\delta_1 S'}) \mu^{\kappa_1} e^{-\delta_1 T}. \]

By taking weighted sum of the square of (E.41), Sublemma E.5 follows easily. \(\square\)
Proof of Sublemma E.6 We take and fix $\rho_0'$ and prove this estimate under the assumption that $\delta_l(\rho, \rho_0') \leq e^{-\delta_1 T}$.

We put

$$(\Delta q^\rho_{T,(\kappa)})' = (\text{Pal}^q_{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^0_{\rho_0'_{T,(\kappa)}')})(\Delta p^\rho_{T,(\kappa)}) \in \Gamma(\Sigma(S'), T_q X)$$

and

$$\Delta q^\rho_{T,(\kappa)} = (\text{Pal}^q_{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^0_{\rho_0'_{T,(\kappa)}')})(\Delta p^\rho_{T,(\kappa)}) \in T_q X.$$ (See Figure 17)

Note $(\Delta q^\rho_{T,(\kappa)})'$ is $z \in \Sigma(S')$ dependent but $\Delta q^\rho_{T,(\kappa)}$ is a $(T, \rho)$ dependent family of constant maps on $\Sigma(S')$.

By (E.36) we obtain:

$$\left\| \frac{\partial^k}{\partial T^k} \left( (\text{Pal}^q_{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^0_{\rho_0'_{T,(\kappa)}')})(\Delta p^\rho_{T,(\kappa)}) \right) \right\|_{L^2_n(\Sigma(S'))} \leq C_m, (E.42) \mu^{k-1} e^{-\delta_1 T}.$$ (E.43)

We use Lemma B.1 and

$\left\| E(p^\rho_{0'}, u^\rho_{0'}) \right\|_{L^2_n(\Sigma(S'))} \leq D_m, (E.44) e^{-\delta_1 S'}.$ (E.45)

to derive

$$\left\| \frac{\partial^k}{\partial T^k} \left( \Delta q^\rho_{T,(\kappa)} \right) \right\|_{L^2_n(\Sigma(S'))} \leq D_m, (E.44) \mu^{k-1} e^{-\delta_1 (T+S')}.$$ (E.46)

On the other hand, (E.17) and Lemma E.3 imply

$$\left\| \frac{\partial^k}{\partial T^k} \left( E(u_1, u^\rho_{T,(\kappa)}) - E(u_1, u^\rho_{T,(\kappa)}) \right) - (\text{Pal}^q_{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^0_{\rho_0'_{T,(\kappa)}'}) \right\|_{L^2_n(\Sigma(S'))} \leq D_m, (E.44) \mu^{k-1} e^{-\delta_1 T}.$$ (E.47)

By (E.42), (E.44) and (E.45) we obtain

$$\left\| \frac{\partial^k}{\partial T^k} \left( E(u_1, u^\rho_{T,(\kappa)}) - E(u_1, u^\rho_{T,(\kappa)}) \right) - (\text{Pal}^q_{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^0_{\rho_0'_{T,(\kappa)}'}) \right\|_{L^2_n(\Sigma(S'))} \leq D_m, (E.44) \mu^{k-1} e^{-\delta_1 T}.$$ (E.46)

Using Lemma E.3 we can also show

$$\left\| \frac{\partial^k}{\partial T^k} \left( E(u_1, u^\rho_{T,(\kappa)}) - E(u_1, u^\rho_{T,(\kappa)}) \right) - (\text{Pal}^q_{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^{\rho_0'_{T,(\kappa)}') \circ \text{Pal}^0_{\rho_0'_{T,(\kappa)}'}) \right\|_{L^2_n(\Sigma(S'))} \leq C_m, (E.47) \mu^{k-1} e^{-2\delta_1 T}.$$ (E.47)
for $S' \geq T$. In fact $u_{T,(\kappa-1)}^\rho$, $u_{T,(\kappa)}^\rho$, $p_{T,(\kappa-1)}^\rho$ are all close to each other by the order of $e^{-\delta_1 T}$ there, including their $(T,\rho)$ derivatives. Also $u_1$ and $q$ are close by the order of $e^{-\delta_1 T}$, there. Therefore all the error terms appearing while applying Lemma [E.3] are of the order $\mu^{\kappa-1} e^{-2\delta_1 T}$.

(E.42), (E.44) and (E.47) imply

$$\|\nabla^u \partial_t \left( E(u_1, u_{T,(\kappa)}^\rho)(0, 1/2) - E(u_1, u_{T,(\kappa-1)}^\rho)(0, 1/2) - \Delta q_{T,(\kappa)}^\rho \right) \| \leq C_{m,E.48} \mu^{\kappa-1} e^{-\delta_1 T}. \tag{E.48}$$

Here the second inequality follows from (6.9) and (6.10).

Then (D.4) and (D.5) imply the required inequality (6.51). \hfill \Box

APPENDIX F. ESTIMATE OF PARALLEL TRANSPORT 3

Proof of Lemma 6.20 The proof is similar to the argument of Subsection D. We put

$$W_0(z) = E(u_1(z), \hat{u}_{1,T,(\kappa)}^\rho)$$

then we estimate $e'(z, 0)$ in (D.2) and obtain

$$\left\| \nabla^u \partial_t \left( \frac{\partial t}{\partial t} \right) e'_1(z, 0) \right\|_{L^2_{m+1-t}} \leq C_{m,E.1} \left\| \nabla^u \partial_t \left( \frac{\partial t}{\partial t} \right) (W_0) \right\|_{L^2_{m+1-t}} \leq C_{m,E.1} e^{-\delta_1 T}. \tag{F.1}$$

We apply Lemma A.1 to $W_0(z) = E(u_1(z), u(z))$, $V_0(z) = \text{Pal}_{u_1(z)}^1 (V(z))$ (G.2) and use induction hypothesis. We remark that $V_0$, $W_0$, $e$ are $T$ dependent. We divide $K_1 \subset \bigcup \Omega_a$ such that $u_1'TX$ etc. are trivial on $\Omega_a$. (See the beginning of Section A.)

We put

$$\Psi = \left( \text{Pal}_{u_1}^n \right)^{(0,1)}.$$

Then (D.4) and (D.5) imply the required inequality (6.51). \hfill \Box

APPENDIX G. ESTIMATE OF $T$ DERIVATIVE OF THE ERROR TERM OF NON-LINEAR CAUCHY-RIEMANN EQUATION

Proof of Lemma 6.13 We discuss estimate on $K_1$. Estimate on $K_1^{4T-1} \setminus K_1$ is similar. We put $K_1 = K_1^{4T-1} \setminus K_1 \subset [T,-5T+1, \times [0,1].$

We use the simplified notation:

$$u = \hat{u}_{1,T,(\kappa-1)}^\rho, \quad V = V_{T,(\kappa)}^\rho,$$

$$P = (\text{Pal}_1^{(0,1)})^{-1}, \quad e = se_{1,T,(\kappa-1)}^\rho.$$

where $\text{Pal}_1^{(0,1)}$ is 0,1 part of the parallel transport $r \rightarrow \text{Exp}(u(z), rV)$. We apply Lemma A.1 to

$$W^0(z) = E(u_1(z), u(z)), \quad V^0(z) = \text{Pal}_{u_1(z)}^1 (V(z)) \tag{G.2}$$

and use induction hypothesis. We remark that $V^0$, $W^0$, $e$ are $T$ dependent. We divide $K_1 \subset \bigcup \Omega_a$ such that $u_1'TX$ etc. are trivial on $\Omega_a$. (See the beginning of Section A.)

We put

$$\Psi = \left( \text{Pal}_{u_1}^n \right)^{(0,1)}.$$

We then can show:
Lemma G.1.

\[
\left\| \nabla^\ell_0 \frac{d^\ell}{dT^\ell} \mathcal{D}^{-1} \int_0^1 ds \int_0^s \frac{d^2}{dr^2} (\mathcal{D}(\text{Exp}(u, rV)) \, dr \right\|_{L^2_m} \leq C_m e^{-2\delta_T} \mu^{2(k-1)}.
\]

Proof. For simplicity of notation we consider the case \( n = 0 \). In the case \( n \neq 0 \) (that is, the case when the \( \rho \) derivative is included) the proof is the same.

We take \( \ell \)-th \( T \) derivative of (A.7) and find that

\[
\frac{d^\ell}{dT^\ell} (A.7) = \sum_{ij, \ell_1, \ell_2, \ell_3} \frac{\partial^{\ell_3} V_0}{\partial T^{\ell_3}} \frac{\partial^{\ell_2} V_0}{\partial T^{\ell_2}} \mathcal{D}_{ij, \ell_3} \left( r, z, V_0, \ldots, \frac{\partial^{\ell_3+1} V_0}{\partial T^{\ell_3}}, V_0, \ldots, \frac{\partial^{\ell_3+1} V_0}{\partial T^{\ell_3}} \right)
\]

where \( \mathcal{D}_{ij, \ell_3} \) are smooth maps of the variables in the parentheses. Note \( V_0 \) there, for example, means \( V_0(z) \in \mathbb{R}^n \).

By induction hypothesis, that is an estimate of \( \frac{\partial^{\ell_3} V_0}{\partial x}, \frac{\partial^{\ell_3} V_0}{\partial y}, \frac{\partial^{\ell_3} W_0}{\partial x}, \frac{\partial^{\ell_3} W_0}{\partial y} \) and their \( \ell' \)-derivatives (for \( \ell' \leq \ell \)), we can show that \( L^2_m(\Omega_0) \) norms of the maps \( \mathcal{D}_{ij, \ell_3} \) are bounded.

Therefore

\[
\left\| \frac{d^\ell}{dT^\ell} (A.7) \right\|_{L^2_m(\Omega_0)} \leq C_m \left( \sum_{j=0}^{\ell} \left\| \frac{d^j}{dT^j} V \right\|_{L^2_m} \right)^2.
\]

Taking the sum over \( a \), we obtain the first inequality.

The second inequality is the consequence of induction hypothesis. \( \square \)
Lemma G.2. We have the next two inequalities.

\[ \left\| \nabla^n \frac{d^\ell}{dT^\ell} \mathcal{Q}^{-1}(D_u \mathcal{E}_1)(PQ, V) \right\|_{L^2_{m-\ell}(K_1)} \leq C_m, \quad (G.5) \]

\[ \times \left( \sum_{n'=0}^n \sum_{\ell'=0}^\ell \left\| \nabla^{n'} \frac{d^{\ell'}}{dT^{\ell'}} Q \right\|_{L^2_{m-k}(K_1^+)} \right)^2. \]

\[ \left\| \nabla^n \frac{d^\ell}{dT^\ell} \mathcal{Q}^{-1} \int_0^1 ds \int_0^s \left( \frac{\partial}{\partial r} \right)^2 (P \overline{\mathcal{D}}(\text{Exp}(u, rV))) dr \right\|_{L^2_m(K_1)} \leq C_m, \quad (G.6) \]

\[ \times \left( \sum_{n'=0}^n \sum_{\ell'=0}^\ell \left\| \nabla^{n'} \frac{d^{\ell'}}{dT^{\ell'}} V \right\|_{L^2_{m-k}(K_1^+)} \right)^2. \]

Proof. We prove this lemma in a similar way to (G.3) and also to Lemma E.3 as follows. For simplicity of formula we consider the case \( n = 0 \).

We divide \( K_1 \subset \bigcup \Omega_a \) as above. We define \( W_0, V_0 \) as in (G.2) and apply the calculation in Subsection D to \( e'_i(z, r) \) and \( e_i(z, r) \).

By the smoothness of \( \hat{e}_i \) and (D.2) the next inequality holds for \( \ell > 0 \).

\[ \left\| \left( \frac{d}{dT} \right)^\ell e'_i \right\|_{L^2_{m-\ell}(\Omega_a)} \leq C_m, \quad (G.7) \]

\[ \sum_{k=1}^\ell \left( \left\| \frac{d^k V^0}{dT^k} \right\|_{L^2_{m-k}(K_1^+)} + \left\| \frac{d^k W^0}{dT^k} \right\|_{L^2_{m-k}(K_1^+)} \right) \leq C'_m, \quad (G.7) \]

We also have the next formula for \( \ell \geq 0 \).

\[ \left\| \left( \frac{d}{dT} \right)^\ell \frac{d}{dr} e'_i \right\|_{L^2_{m-\ell}(\Omega_a)} \leq C_m, \quad (G.8) \]

Then using (D.4) and (D.5) we can prove

\[ \left\| \left( \frac{d}{dT} \right)^\ell e_i \right\|_{L^2_{m-\ell}(\Omega_a)} \leq C_m, \quad (G.9) \]

by induction on \( i \). Using this formula and (D.7), (D.8) it is easy to show (G.5). The proof of (G.6) is similar.

We use the inequality (G.3), (G.5) in place of (C.3), (C.7), respectively, for our estimate of \( T \)-derivatives. We use (G.6) to estimate the \( T \) derivatives of the third
Lemma G.3. We can then prove
\[
\left\| \frac{d^t}{dt^t} (\mathcal{P}')^{-1} \left( \overline{\mathcal{D}}(\text{Exp}(u, V)) - \mathcal{P}^{-1} \epsilon \right) \right\|_{L^2_{\kappa}(K_i)} \leq C_{m, \kappa} e^{-\delta T} \mu^{-1}
\] (G.10)
in the same way as (C.6). (Here we put \( \mathcal{P}' = (\text{Pal}_{u_1}^{\text{Exp}(u, V)})^{(0,1)} \).

We then deduce
\[
\left\| \frac{d^t}{dt^t} (\mathcal{P}')^{-1} \left( \Pi_{E_1}(\text{Exp}(u, V)) - \mathcal{P}^{-1} \Pi_{E_1}(\mathcal{P}(\text{Exp}(u, V))) + \mathcal{P}^{-1}(D_\epsilon E_1)(\mathcal{P}_\epsilon, V) \right) \right\|_{L^2_{\kappa}(K_i^+)} \leq C_{m, \kappa} e^{-\delta T} \mu^{-1},
\] (G.11)
in a similar way as (C.5).

The rest of the proof of Lemma 6.13 is a straightforward modification of the proof of (C.10) in the same way as above and so is omitted. \( \square \)

Proof of (6.31). We take \( \rho_0' \) and prove the inequality for \( \rho \) with \( d(\rho, \rho_0') \leq e^{-\delta T} \).

Let \( \Omega = [S', S' + 1] \times [0, 1] \subset [T, 9T] \times [0, 1] \). We consider \( u'_1 = \rho_0' \) (constant map) and
\[
W^0(z) = E(\rho_0'^0, u_{T,(\kappa-1)}^0(z)),
\]
\[
V^0(z) = (\text{Pal}_{u_{T,(\kappa-1)}^0}^{\rho_0'^0} \circ \text{Pal}_{u_{T,(\kappa-1)}^0}^{\rho_0'^0}(z)) (\Delta p_{T,(\kappa)})
\] (G.12)
We then apply Lemma A.3. Note
\[
v_r = \text{Exp}(u_{T,(\kappa-1)}^0, r(\Delta p_{T,(\kappa)}) \text{Pal})
\]
We put
\[
\mathcal{P}' = ((\text{Pal}_{u_{T,(\kappa-1)}^0}^{\rho_0'^0})^{(0,1)})^{-1} \circ ((\text{Pal}_{u_{T,(\kappa-1)}^0}^{\rho_0'^0})^{(0,1)})^{-1}.
\]

Lemma G.3.
\[
\left\| \nabla_{\rho}^n \frac{\partial}{\partial t} \frac{\partial}{\partial r} \right\|_{L^2_{m-\kappa}(\Omega)} \leq C_{m, \kappa} e^{-2T\delta_1} \mu^{-1}
\] (G.13)

Proof. By differentiating (A.3) once by \( r \) we obtain
\[
\frac{\partial}{\partial r} \mathcal{P}' \mathcal{P}_r = \sum_i V^0_i \tilde{F}_i \left( z, W^0, \frac{\partial W^0}{\partial x}, \frac{\partial W^0}{\partial y} \right)
\]
\[
+ \sum_i \frac{\partial V^0_i}{\partial x} \tilde{F}_i \left( z, W^0 \right) + \sum_i \frac{\partial V^0_i}{\partial y} \tilde{F}_i \left( z, W^0 \right),
\] (G.14)

Note
\[
\left\| \nabla_{\rho}^n \frac{\partial}{\partial t} \left( \text{Pal}_{u_{T,(\kappa-1)}^0}^{\rho_0'^0} \circ \text{Pal}_{u_{T,(\kappa-1)}^0}^{\rho_0'^0}(z) \right) - \text{Pal}_{u_{T,(\kappa-1)}^0}^{\rho_0'^0}(z) \right\|_{L^2_{m-\kappa}(\Omega)} \leq C_{m, \kappa} e^{-\delta T} \mu^{-1}
\] (G.15)
follows from (6.10) and (6.8) in the same way as Section B. Using also the fact that \( \text{Pal}_{u_{T,(\kappa-1)}^0}^{\rho_0'^0}(z) \) is constant on \( \Omega \), we can use Lemma B.1 to estimate the 2nd and 3rd terms of (G.14) by \( C_{m, \kappa} e^{-2\delta T} \mu^{-1} \).
We also remark that when we substitute \( W^0 = 0 \) then (G.14) vanishes. In fact if \( W^0 = 0 \) then \( v_r(z) = \text{Exp}(p^0, T, p^0) (\Delta p^0_T(x, z)) \), which is a constant map. So \( \overline{\partial} v_r = 0 \).

Therefore

\[
\left\| \frac{\partial}{\partial r} \right\|_{r=0}^{\mathcal{P}' \mathcal{P} \overline{\partial} v_r}_{L^2_{m+1}(\Omega)} \leq C_{m, G.15} \mu^{k-1} e^{-2\delta_1 T}
\]

\[
\leq C_{m, G.16} \| W_0 \|_{L^2_0(\Omega)} \left( \| W_0 \|_{L^2_0(\Omega)} + \left\| \frac{\partial W_0}{\partial t} \right\|_{L^2_0(\Omega)} + \left\| \frac{\partial W_0}{\partial y} \right\|_{L^2_0(\Omega)} \right)
\]

\[
\leq C_{m, G.16}' \| W_0 \|_{L^2_0(\Omega)} \| W_0 \|_{L^2_m(\Omega)}
\]

\[
\leq C_{m, G.16}' \mu^{k-1} e^{-2\delta_1 T}.
\]

Here the last inequality follows from (6.8) and (6.10). (Note we are working on \([T, gT] \times [0, 1].\)

The case when \( T \) and \( \rho \) derivatives are included is similar. \(\square\)

We put

\[
\mathcal{P}(\mathcal{P}')^{-1} = (\text{Pal}_{u_1}(0, 1)) - 1 \circ (\text{Pal}_{u_0}(0, 1)).
\]

Here \( u_1 : \Sigma_1 \to X \) is the map we start with. (It is different from \( u_1' \)) \( v_0 = u_1'_{T, (k-1)} \). This is a \((\rho, T)\) parameterized family of sections of the bundle \( \text{Hom}(T, p_0, X, u_1 T X) \) on our domain \( \Omega \).

Let \( W \in T, p_0, X \). Then, by (6.10) we can show the inequality

\[
\left\| \nabla^m_\rho \right\|_{r=0}^{\mathcal{P}' \mathcal{P} \overline{\partial} v_r}_{L^2_0(\Omega)} \leq C_{m, G.17} e^{-\delta_1 T} |W|
\]

by using Lemma B.1.

Note \((\text{Pal}_{u_1}(0, 1))^{-1}\) is \((T, \rho)\) independent and is bounded. Therefore by putting \( W = \mathcal{P}' \mathcal{P} \overline{\partial} v_r \), the formulae \( \mathcal{P}(\mathcal{P}')^{-1} \circ \mathcal{P}' \mathcal{P} = \mathcal{P} \mathcal{P} \), (G.13) and (G.17) imply

\[
\left\| \nabla^m_\rho \right\|_{r=0}^{\mathcal{P}' \mathcal{P} \overline{\partial} v_r}_{L^2_0(\Omega)} \leq C_{m, G.18} \mu^{k-1} e^{-2\delta_1 T}.
\]

where

\[
\mathcal{P} \mathcal{P} = (\text{Pal}_{u_1}(0, 1))^{-1} \circ (\text{Pal}_{u_0}(0, 1))^{-1}.
\]

(6.31) follows immediately from this formula. In fact

\[
\left. \frac{\partial}{\partial r} \right|_{r=0}^{\mathcal{P}' \mathcal{P} \overline{\partial} v_r} = (\text{Pal}_{u_1'_{T, (k-1)}}(0, 1))^{-1} (D_{u_1'_{T, (k-1)}}(\overline{\partial})((\Delta p_{T, (k-1)}) \text{Pal})).
\]

\(\square\)
Appendix H. Proof of Lemma 8.4

Let $\Gamma_+$ be a finite group and $\Gamma$ be its index 2 subgroup. We assume that there exists $\tau \in \Gamma_+$ of order 2 such that $\Gamma_+ = \Gamma \cup \tau \Gamma$.

A smooth action of $\Gamma_+$ to a complex manifold $M$ is said to be holomorphic if all the action of elements of $\Gamma$ is holomorphic and $\tau$'s action is by anti-holomorphic involution.

Lemma H.1. Let $p \in M$ be a fixed point of holomorphic action of $\Gamma_+$ then there exists a complex coordinate of $M$ at $p$ such that $\Gamma_+$ action is linear in this coordinate.

Proof. We fix a coordinate of $M$ at $p$. Then $\Gamma_+$ induces a holomorphic action $\rho$ on $\Omega \subset \mathbb{C}^n$ such that 0 is a fixed point. We consider the $\Gamma_+$ action at the tangent space of 0 which we denote by $\rho_0$. For $t \in (0, 1]$ we define the action $\rho_t$ by

$$\rho_t(g, x) = \rho(g, tx)/t.$$ 

Together with $\rho_0$ it defines a smooth one parameter family of holomorphic actions. Let

$$g \mapsto \left. \frac{d}{dt} \right|_{t=t_0} \rho_t(g, x) \in \text{Hol}(U, T\mathbb{C}^n).$$

Here $\text{Hol}(U, TX)$ is the vector space of holomorphic vector fields on $U$. For each $t_0$ it defines a 1 cycle of the complex $C^*(\Gamma_+, \text{Hol}(U, T\mathbb{C}^n))$ which calculates the group cohomology with local coefficient induced by the conjugate action associated to $\rho_{t_0}$.

(Note the conjugate action by anti-holomorphic involution $\tau$ preserves the set of holomorphic vector fields.)

Using the fact that $\Gamma_+$ is a finite group we can show that the first cohomology vanishes. In fact the Eilenberg-MacLane space $K(\Gamma_+, 1)$ has finite cover $\tilde{K}(\Gamma_+, 1)$ which is contractible. All the local systems on $K(\Gamma_+, 1)$ pulled back to $\tilde{K}(\Gamma_+, 1)$ are trivial and the first cohomologies of the pulled back are trivial. Since $\mathbb{Q}$ is contained in the coefficient ring of our local system, Gysin map induces isomorphism between cohomology on $K(G, 1)$ and on $\tilde{K}(\Gamma_+, 1)$.

So there exists one parameter family of holomorphic vector fields $V_t$ such that

$$\left. \frac{d}{dt} \right|_{t=t_0} \rho_t(g, x) = \rho_{t_0}(g, \cdot)_* V_t - V_t\rho_{t_0}(g, \cdot)_*.$$ 

By integrating $V_t$ we obtain a biholomorphic map which interpolates $\rho_0$ and $\rho$. The lemma follows. \qed

Lemma H.2. Let $M$, $N$ be complex manifolds on which $\Gamma_+$ has holomorphic actions. Let $p \in M$ and $q \in N$ be fixed points of $\Gamma_+$. Let $F : M \to N$ be a holomorphic map such that $F(p) = q$ and $d_pF : TM \to TN$ is surjective.

We decompose $T_pM = T_qN \oplus V$ as $\Gamma_+$ complex vector space such that $d_pF$ is identity on $T_qN$ and is 0 on $V$.

Then there exists a $\Gamma_+$ equivariant biholomorphic map

$$\varphi : B_\varepsilon(T_qN) \times B_r(V) \to M$$

which sends $(0, 0)$ to $p$ such that $F \circ \varphi$ is constant in $V$ direction.
Proof. Using Lemma H.1 we may assume that $M$, $N$ are open subsets $V$, $W$ of $\mathbb{C}^M$, $\mathbb{C}^N$ and $\Gamma_+$ action is linear. We may assume $p = 0$, $q = 0$. We take $\Gamma_+$ invariant complex linear subspaces $U_1, U_2$ of $\mathbb{C}^M$ such that $U_1 \oplus U_2 = \mathbb{C}^M$, $(d_0 F)(U_1) = 0$ and $d_0 F$ induce an isomorphism between $U_2$ and $\mathbb{C}^N$. We define $G : V \to U_1 \times W$ by $G(x + y) = (x, F(x + y))$, where $x \in U_1, y \in U_2$. By inverse mapping theorem $G$ has a local inverse $\varphi$. Then $F \circ \varphi$ is the projection to the $W$ factor. Therefore $\varphi$ gives a required coordinate. (The $\Gamma_+$ equivariance and holomorphicity of $\varphi$ is obvious from construction.) □

Proof of Lemma 8.4. Applying Lemma H.2 to $\mathbb{C}^{cl}(V) \to V$ we can prove Lemma 8.4 easily. □

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