Ala Amourah1*, Anas Aljarah2, M. Darus2, and Laith Abualigah3

1Department of Mathematics, Faculty of Science and Technology, Irbid National University, Irbid, Jordan.
2School of Mathematical Sciences Faculty of Science and Technology Universiti Kebangsaan Malaysia Bangi 43600 Selangor D. Ehsan, Malaysia.
3Faculty of Computer Sciences and Informatics, Amman Arab University, Amman, Jordan.

HANKEL DETERMINANT FOR STARLIKE AND CONVEX FUNCTIONS OF ORDER \( \frac{\alpha}{2} \)

ABSTRACT

The aim of this paper is to obtain an upper bound to the second Hankel Determinant \( |a_2a_4 - a_3^2| \) for starlike and convex functions of order \( \alpha \), \( 1 \leq \alpha \leq 2 \).

Key words: Analytic function, second Hankel functional, starlike and convex functions, upper bound.

1. INTRODUCTION

Let \( A \) denote the class of functions \( f \) of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

in the open unit disc \( U = \{ z \in \mathbb{D} : |z| < 1 \} \).

The Hankel determinant of \( f \) for \( q \geq 1 \) and \( n \geq 1 \) was defined by Pommerenke ([6], [7]) as

\[
H_q(n) = \begin{vmatrix}
a_2 & a_3 & \cdots & a_{n+1} \\
a_3 & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots \\
a_{n-q+1} & \cdots & \cdots & a_{n+2q-2}
\end{vmatrix}
\]  

(1.2)

Noor [9] determined the rate of growth of \( H_q(n) \) as \( n \to \infty \) for functions given by (1.1) with bounded boundary. Ehrenborg [10] studied the Hankel determinant of exponential polynomials. Janteng et al. discussed the Hankel determinant problem for the classes of starlike functions with respect to symmetric points and convex functions with respect to symmetric points in [11] and for the functions whose derivative has a positive real part in [12].

Easily, one can observe that the Fekete and Szego functional is \( H_2(1) \). Fekete and Szego [13] then further generalised the estimate \( |a_2 - \mu a_3^2| \), where \( \mu \) is real and \( f \in S \).

For our discussion in this paper, we consider the Hankel determinant in the case of \( q = 2 \) and \( n = 2 \): \( |a_2 - \mu a_3^2| \).

Let \( P \) denote the class of functions

\[
P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} c_n z^n,
\]

which are analytic in \( U \) and satisfy \( \text{Re} \{ P(z) \} > 0 \) for any \( z \in U \).

In this paper, we seek sharp upper bound to the functional \( |a_2a_4 - a_3^2| \) for the function \( f \) belonging to the class \( S_\alpha^* \) and \( C_\alpha^* \). The class \( S_\alpha^* \) and \( C_\alpha^* \) are defined as follows.
Definition 1.1. Let \( f \) be given by (1.1). Then \( f \in S^* \) if and only if
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{\alpha}{2} - 1, \quad (1 \leq \alpha \leq 2).
\]
(1.4)

To prove our main result in the next section, we shall require the following two Lemmas:

Lemma 1.3. ([1], [2]) If \( c \in P \), then \( |c_n| \leq 2 \), for each \( n \geq 1 \).

Lemma 1.4. ([3], [4], [5]) If \( c \in P \), then
\[
2c_2 = c_1^2 + (4 - c_1^2)x, \quad 4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - x^2),
\]
for some \( x \) and \( z \) satisfying \( |x| \leq 1 \) and \( |z| \leq 1 \) and \( c_1 \in [0, 2] \).

We employ techniques similar to those used earlier by Amourah et al. ([15], [16], [17], [18], [20]) and Al-Hawary et al. [19].

2. MAIN RESULT

Theorem 2.1. If \( f \in S^* \), then
\[
|a_2a_4 - a_3^2| \leq \frac{2 - \alpha}{2} \left[ 4 \left( \frac{2 - \alpha}{2} \right)^{-1} + 3 \right], \quad (1 \leq \alpha \leq 2).
\]
(2.1)

Proof. Since \( f \in S^* \), by Definition 1.1 we have
\[
\frac{1 - \frac{\alpha}{2} + \frac{zf'(z)}{f(z)}}{2 - \frac{\alpha}{2}} = P(z).
\]
(2.2)
Replacing \( f(z) \) and \( q(z) \) with their equivalent series expressions in (2.2), we have
\[
\left( 1 - \frac{\alpha}{2} + \frac{zf'(z)}{f(z)} \right) = \left( 2 - \frac{\alpha}{2} \right) P(z).
\]
Using the Binomial expansion in the left hand side of the above expression, upon simplification, we obtain
\[
\left( 2 - \frac{\alpha}{2} \right) z + \left( 2 - \frac{\alpha}{2} \right)(c_1 + a_2) z^2 + \left[ c_2 + a_2 c_1 + a_3 \right] \left( 2 - \frac{\alpha}{2} \right) z^3 + \left( c_3 + a_2 c_2 + a_3 c_1 + a_4 \right) \left( 2 - \frac{\alpha}{2} \right) z^4 + \cdots
\]
\[
= \frac{1}{2 - \frac{\alpha}{2}} z + \frac{\alpha}{2} \left( a_2 z^2 + \frac{4 - \alpha}{2} a_3 z^3 + \cdots \right).
\]
On equating coefficients in (2.3), we get
\[
\begin{align*}
a_2 &= \frac{2 - \alpha}{2} c_1 \quad c_3 = \frac{2 - \alpha}{2} c_2 + \frac{2 - \alpha}{2} c_1^2, \\
a_4 &= \frac{2 - \alpha}{6} \left( 2c_3 + 3 \left( 2 - \frac{\alpha}{2} \right) c_2 c_1 + \frac{2 - \alpha}{2} c_1^3 \right),
\end{align*}
\]
in the second Hankel functional
\[
\left| a_2 a_4 - a_3^2 \right| = \frac{2 - \alpha}{12} \left( 4c_3 c_2 - 3c_2^2 - c_1^2 \left( 2 - \frac{\alpha}{2} \right)^2 \right).
\]
Using Lemma 1.4, it gives
\[
\left| a_2 a_4 - a_3^2 \right| = \frac{2 - \alpha}{12} \left( 4c_3 c_2 - 3c_2^2 - c_1^2 \left( 2 - \frac{\alpha}{2} \right)^2 \right).
\]
(2.3)

Assume that \( \delta = |x| \leq 1, \ c_1 = c \) and \( c \in [0, 2] \), using triangular inequality and we have
\[
\left| a_2 a_4 - a_3^2 \right| \leq \frac{2 - \alpha}{2} \left[ \left( 4c_3^2 - 1 \right) c_1^2 \delta + \left( 4 - c_1^2 \right) \delta^2 (12 + c_1^2) \right].
\]
(2.5)
$$\left(2-\alpha\right)^2 \left[4 \left(2-\frac{\alpha}{2}\right)^2 - 1\right] c^4 + 8 \left(4-c^2\right) c + 2 \left(4-c^2\right) c^2 \delta + a_d a_d - a_d^2 \leq \frac{\left(2-\frac{\alpha}{2}\right)^2}{3} \left[4 \left(2-\frac{\alpha}{2}\right)^2 - 1\right] + 3.$$

(2.11)

This completes the proof of our theorem 2.1.

In particular, considering \( \alpha = 2 \) in Theorem 2.1, we have the following result.

Remark 2.2. If \( f \in S^* \) then
\[ |a_4 a_d - a_d^2| \leq 1. \]

This inequality is sharp and coincides with that of Janteng, Halim and Darus [14].

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Theorem 2.1. If \( f \in C^* \), then
\[ |a_4 a_d - a_d^2| \leq \frac{(2-\frac{\alpha}{2})^2}{144} \left[17 \left(2-\frac{\alpha}{2}\right)^2 + 2 \left(2-\frac{\alpha}{2}\right) + 17\right], (1 \leq \alpha \leq 2). \]

(2.12)

Proof. Since \( f \in C^* \), by Definition 1.2 we have
\[ 2 - \frac{\alpha}{2} + \frac{zf''(z)}{f'(z)} = P(z). \]

(2.2)

Replacing \( f(z) \) and \( q(z) \) with their equivalent series expressions in (2.13), we have
\[ \left(-\frac{\alpha}{2} + \frac{zf''(z)}{f'(z)}\right) = \left(2 - \frac{\alpha}{2}\right) P(z). \]

Using the Binomial expansion in the left hand side of the above expression, upon simplification, we obtain
\[ \left(2 - \frac{\alpha}{2}\right) c + \left[c + 2 a_2 c + a_4\right] \left(2 - \frac{\alpha}{2}\right) z^2 \]
\[ + \left[c_3 + 2 a_2 c_2 + 3 a_4 c_1\right] \left(2 - \frac{\alpha}{2}\right) z^3 + \cdots \]
\[ = 2 a_2 z + 6 a_3 z^2 + 12 a_4 z^3 + \cdots. \]

(2.14)

On equating coefficients in (2.14), we get
\[ a_2 = \frac{\left(2 - \frac{\alpha}{2}\right)}{2} c_1, \quad a_3 = \frac{\left(2 - \frac{\alpha}{2}\right)}{6} \left(c_2 + \left(2 - \frac{\alpha}{2}\right) c_1^2\right), \]
\[ a_4 = \frac{\left(2 - \frac{\alpha}{2}\right)}{12} \left(3 \frac{2 - \frac{\alpha}{2}}{2} c_2 + \frac{2 - \frac{\alpha}{2}}{2} c_3^2 + \frac{2 - \frac{\alpha}{2}}{2} c_1^3\right), \]
\[ (2) \]

in the second Hankel functional

\[
\left| a_2 a_4 - a_3^2 \right| = \frac{\left(2 - \frac{\alpha}{2}\right)^2}{144} \left[ 6c_1c_3 - 4c_2^2 + \left(2 - \frac{\alpha}{2}\right) c_1^2 c_2 - c_1^4 \left(2 - \frac{\alpha}{2}\right)^2 \right].
\]

Using Lemma 1.4, it gives

\[
\left| a_2 a_4 - a_3^2 \right| = \frac{\left(2 - \frac{\alpha}{2}\right)^2}{144} \left[ \frac{1 + \left(2 - \frac{\alpha}{2}\right) - 2 \left(2 - \frac{\alpha}{2}\right)^2 c_1^4}{2} + \frac{2 + \left(2 - \frac{\alpha}{2}\right)}{2} \left(4 - c_2^2\right) c_1^2 x^2 + 3 \left(4 - c_1^2\right) \left(1 - x^2\right) c_1^2 x + \frac{\left(4 - c_2^2\right) c_1^2}{2} \right]
\]

\[
= \frac{\left(8 + c_2^2\right)}{2} \left(4 - c_1^2\right) x^2 + 3 \left(4 - c_1^2\right) \left(1 - x^2\right) c_1^2 x + \frac{\left(4 - c_2^2\right) c_1^2}{2}
\]

Assume that \( \delta = \|x\| \leq 1, \ c_1 = c \) and \( c \in [0, 2] \), using triangular inequality and \( \|x\| \leq 1 \), we have

\[
\left| a_2 a_4 - a_3^2 \right| \leq \frac{\left(2 - \frac{\alpha}{2}\right)^2}{144} \left[ \frac{1 + \left(2 - \frac{\alpha}{2}\right) - 2 \left(2 - \frac{\alpha}{2}\right)^2 c_1^4}{2} + \frac{2 + \left(2 - \frac{\alpha}{2}\right)}{2} \left(4 - c_2^2\right) c_1^2 \delta + \frac{\left(4 - c_2^2\right) c_1^2}{2} \right]
\]

\[ (2.16) \]

\[ F(c, \delta). \]

We next maximize the function \( F(c, \delta) \) on the closed region \([0, 2] \times [0, 1]\). Differentiating \( F(c, \delta) \) in (2.16) partially with respect to \( \delta \), we get

\[
\frac{\partial F}{\partial \delta} = \frac{\left(2 - \frac{\alpha}{2}\right)^2}{144} \left[ \frac{2 + \left(2 - \frac{\alpha}{2}\right)}{2} \right] c^2 (4 - c^2) + (c - 4) (c - 2) (4 - c^2) \delta.
\]

\[ \frac{\partial F}{\partial \delta} > 0 \]

We have \( \frac{\partial F}{\partial \delta} > 0 \). Thus \( F(c, \delta) \) cannot have a maximum in the interior of the closed square \([0, 2] \times [0, 1]\). Moreover, for fixed \( c \in [0, 2] \) \( \max_{0 \leq \delta \leq 1} F(c, \delta) = F(c, 1) = G(c). \)

\[ G(c) = \frac{\left(2 - \frac{\alpha}{2}\right)^2}{144} \left[ \frac{2 + \left(2 - \frac{\alpha}{2}\right)}{2} \right] (c - 2) (c - 4) (4 - c^2) \]

\[ = \frac{\left(8 + c_2^2\right)}{2} \left(4 - c_1^2\right) x^2 + 3 \left(4 - c_1^2\right) \left(1 - x^2\right) c_1^2 x + \frac{\left(4 - c_2^2\right) c_1^2}{2}
\]

\[ (2.17) \]

For optimum value of \( G(c) \), consider \( G'(c) = 0 \). From (2.18), we get

\[ c \left[ - \left(2 - \frac{\alpha}{2}\right)^2 - 1 \right] c^2 + \left(2 - \frac{\alpha}{2}\right) + 1 = 0. \]

\[ (2.20) \]

We now discuss the following Cases.
Case 1) If $c = 0$, then, from (2.19), we obtain

$$G''(c) = \left(\frac{2 - \alpha}{2}\right)^2 \left[\left(1 - \frac{2 - \alpha}{2}\right) + 1\right] > 0.$$ 

From the second derivative test, $G(c)$ has minimum value at $c = 0$.

Case 2) If $c = 0$, then, from (2.19), we get

$$c^2 = \frac{1 + \left(\frac{2 - \alpha}{2}\right)^2}{1 + \left(\frac{2 - \alpha}{2}\right)^2}.$$

Using the value of $c^2$ given in (2.21) in (2.19), after simplifying, we obtain

$$G''(c) = \frac{-60 \left(\frac{2 - \alpha}{2}\right)^2}{144} \left[\left(2 - \frac{\alpha}{2}\right) + 1\right] < 0.$$ 

By the second derivative test, $G(c)$ has maximum value at $c$, where $c^2$ given in (2.21).

Using the value of $c^2$ given by (2.21) in (2.17), upon simplification, we obtain

$$\max_{0 \leq c \leq 2} G(c) = \frac{\left(2 - \frac{\alpha}{2}\right)^2}{144} \left(\frac{17\left(2 - \frac{\alpha}{2}\right)^2 + 2\left(2 - \frac{\alpha}{2}\right) + 17}{1 + \left(2 - \frac{\alpha}{2}\right)^2}\right).$$

(2.22)

Considering, the maximum value of $G(c)$ at $c$, where $c^2$ is given by (2.21), from (2.16) and (2.22), we obtain

$$|a_2a_4 - a_3^2| \leq \frac{\left(2 - \frac{\alpha}{2}\right)^2}{144} \left(\frac{17\left(2 - \frac{\alpha}{2}\right)^2 + 2\left(2 - \frac{\alpha}{2}\right) + 17}{1 + \left(2 - \frac{\alpha}{2}\right)^2}\right).$$

(2.23)

This completes the proof of our Theorem 2.3.

Remark 2.4. If $f \in C$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$ 

This inequality is sharp and coincides with that of Janteng, Halim and Darus [14].

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