The square of white noise as a Jacobi field

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Abstract
We identify the representation of the square of white noise obtained by L. Accardi, U. Franz and M. Skeide in [Comm. Math. Phys. 228 (2002), 123–150] with the Jacobi field of a Lévy process of Meixner’s type.

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1 Formulation of the result

The problem of developing a stochastic calculus for higher powers of white noise, i.e., “nonlinear stochastic calculus”, was first stated by Accardi, Lu, and Volovich in [4]. Since the white noise is an operator-valued distribution, in order to solve this problem one needs an appropriate renormalization procedure. In [5, 6], it was proposed to renormalize the commutation relations and then to look for Hilbert space representations of them. Let us shortly discuss this approach.

We will use $\mathbb{R}^d$, $d \in \mathbb{N}$, as an underlying space. Let $b(x)$, $x \in \mathbb{R}^d$, be an operator-valued distribution satisfying the canonical commutation relations:

$$[b(x), b(y)] = [b^\dagger(x), b^\dagger(y)] = 0,$$
$$[b(x), b^\dagger(y)] = \delta(x - y)1.$$  \hspace{1cm} (1)

Here, $[A, B] := AB - BA$ and $b^\dagger(x)$ is the dual operator of $b(x)$. Denote

$$B_x := b(x)^2, \quad B^\dagger_x := b^\dagger(x)^2, \quad N_x := b^\dagger(x)b(x), \quad x \in \mathbb{R}^d.$$ \hspace{1cm} (2)

One wishes to derive from (1) the commutation relations satisfied by the operators $B_x, B^\dagger_x, N_x$. To this end, one needs to make sense of the square of the delta function, $\delta(x)^2$. But it is known from the distribution theory that

$$\delta(x)^2 = c\delta(x),$$ \hspace{1cm} (3)
where \( c \in \mathbb{C} \) is an arbitrary constant (see [5] for a justification of this formula and bibliographical references).

Thus, using (1) and formula (3) as a renormalization, we get
\[
[B_x, B_y^\dagger] = 2c\delta(x-y)1 + 4\delta(x-y)N_y, \\
[N_x, B_y^\dagger] = 2\delta(x-y)B_y^\dagger, \\
[N_x, B_y] = -2\delta(x-y)B_y, \\
[N_x, N_y] = [B_x, B_y] = [B_x^\dagger, B_y^\dagger] = 0
\] (4)

(see [1, Lemma 2.1]).

Let \( \mathcal{S}(\mathbb{R}^d) \) denote the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^d \). For each \( \varphi \in \mathcal{S}(\mathbb{R}^d) \), we introduce
\[
B(\varphi) := \int_{\mathbb{R}^d} \varphi(x) B_x \, dx, \quad B^\dagger(\varphi) := \int_{\mathbb{R}^d} \varphi(x) B_x^\dagger \, dx, \quad N(\varphi) := \int_{\mathbb{R}^d} \varphi(x) N_x \, dx.
\] (5)

By (4),
\[
[B(\varphi), B^\dagger(\psi)] = 2c\langle \varphi, \psi \rangle 1 + 4N(\varphi\psi), \\
[N(\varphi), B^\dagger(\psi)] = 2B^\dagger(\varphi\psi), \\
[N(\varphi), B(\psi)] = -2B(\varphi\psi), \\
[N(\varphi), N(\psi)] = [B(\varphi), B(\psi)] = [B^\dagger(\varphi), B^\dagger(\psi)] = 0, \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^d).
\] (6)

Here, \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( L^2(\mathbb{R}^d, dx) \). The Lie algebra with generators \( B(\varphi), B^\dagger(\varphi), N(\varphi), \varphi \in \mathcal{S}(\mathbb{R}^d) \), and a central element \( 1 \) with relations (6) is called the square of white noise (SWN) algebra.

Now, one is interested in a Hilbert space representation of the SWN algebra with a cyclic vector \( \Phi \) satisfying \( B(\varphi)\Phi = 0 \) (which is called a Fock representation). In [5], it was shown that a Fock representation of the SWN algebra exists if and only if the constant \( c \) is strictly positive. In what follows, we will suppose, for simplicity of notations that \( c = 2 \).

Let us now recall the Fock representation of the SWN algebra constructed in [3] (see also references therein).

For a real separable Hilbert space \( \mathcal{H} \), denote by \( \mathcal{F}(\mathcal{H}) \) the symmetric Fock space over \( \mathcal{H} \):
\[
\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} n!,
\]
where \( \otimes \) stands for the symmetric tensor product. Thus, each \( f \in \mathcal{F}(\mathcal{H}) \) is of the form \( f = (f^{(n)})_{n=0}^{\infty} \), where \( f^{(n)} \in \mathcal{H}^{\otimes n} \) and \( \|f\|^2_{\mathcal{F}(\mathcal{H})} = \sum_{n=0}^{\infty} \|f^{(n)}\|^2_{\mathcal{H}^{\otimes n} n!}. \) Now take
\( \mathcal{H} \) to be \( L^2(\mathbb{R}^d, dx) \otimes \ell_2 \), where the \( \ell_2 \) space has the orthonormal basis \( (e_n)_{n=1}^{\infty}, e_n = (0, \ldots, 0, 1_{\text{\textit{nth place}}}, 0, \ldots) \).

Denote by \( \mathfrak{F} \) the linear subspace of \( \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \) that is the linear span of the vacuum vector \( \Omega = (1, 0, 0, \ldots) \) and vectors of the form \( (\varphi \otimes \xi)^{\otimes n} \), where \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) and \( \xi \in \ell_{2,0} \), \( n \in \mathbb{N} \). Here, \( \ell_{2,0} \) denotes the linear subspace of \( \ell_2 \) consisting of finite vectors, i.e., vectors of the form \( \xi = (\xi_1, \xi_2, \ldots, \xi_m, 0, 0, \ldots) \), \( m \in \mathbb{N} \). The set \( \mathfrak{F} \) is evidently a dense subset of \( \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \).

Denote by \( J^+, J^0, J^- \) the linear operators in \( \ell_2 \) with domain \( \ell_{2,0} \) defined by the following formulas:

\[
J^+ e_n = \sqrt{n(n + 1)} e_{n+1}, \\
J^0 e_n = ne_n, \\
J^- e_n = \sqrt{(n-1)n} e_{n-1}, \quad n \in \mathbb{N}.
\] (7)

Now, for each \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^d) \) and \( \xi \in \ell_{2,0} \), we set

\[
B^\dagger(\varphi)(\psi \otimes \xi)^{\otimes n} = 2(\varphi \otimes e_1) \hat{\otimes} (\psi \otimes \xi)^{\otimes n} + 2n((\varphi \psi) \otimes (J^+ \xi))^{\otimes n}, \\
N(\varphi)(\psi \otimes \xi)^{\otimes n} = 2n((\varphi \psi) \otimes J^0 \xi)^{\otimes n}, \\
B(\varphi)(\psi \otimes \xi)^{\otimes n} = 2n((\varphi \psi) \otimes (J^- \xi))^{\otimes n}, \quad n \in \mathbb{N},
\] (8)

where \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^d) \) and \( \xi \in \ell_{2,0} \). Thus,

\[
B^\dagger(\varphi) = 2A^+(\varphi \otimes e_1) + 2A^0(\varphi \otimes J^+), \\
N(\varphi) = 2A^0(\varphi \otimes J^0), \\
B(\varphi) = 2A^-(\varphi \otimes e_1) + 2A^0(\varphi \otimes J^-),
\] (9)

where \( A^+(\cdot), A^0(\cdot), \) and \( A^-(\cdot) \) are the creation, neutral, and annihilation operators in \( \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \), respectively. The operator \( B^\dagger(\varphi) \) is the restriction of the adjoint operator of \( B(\varphi) \) to \( \mathfrak{F} \), while the operator \( N(\varphi) \) is Hermitian. It is easy to see that the operators \( B^\dagger(\varphi), N(\varphi), B(\varphi), \varphi \in \mathcal{S}(\mathbb{R}^d) \), constitute a representation of the SWN algebra.

In what follows, the closure of a closable operator \( A \) will be denoted by \( \tilde{A} \). Since the adjoint operators of \( B^\dagger(\varphi), N(\varphi), B(\varphi) \) are densely defined, they are closable.

The last part of [3] is devoted to studying those classical infinitely divisible processes which are built from the SWN in a similar way as the Wiener and Poisson processes are built from the usual white noise. So, for each parameter \( \beta \geq 0 \), we define

\[
X_\beta(x) := B^\dagger_1 + B_x + \beta N_x, \quad x \in \mathbb{R}^d.
\] (10)

Notice that we want a formally self-adjoint process, so the parameter \( \beta \) must be real (we also exclude from consideration the case \( \beta < 0 \), since it may be treated by a trivial transformation of the case \( \beta > 0 \)).
In view of (1) and (2), the only privileged parameter is $\beta = 2$, when $X_\beta(x)$ becomes the renormalized square of the classical white noise $b^I(x) + b(x)$, see [1, Section 3].

Analogously to (5), we introduce, for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$X_\beta(\varphi) := \int_{\mathbb{R}^d} \varphi(x)X_\beta(x) \, dx = B^1(\varphi) + B(\varphi) + \beta N(\varphi). \quad (11)$$

As easily seen, $\tilde{X}_\beta(\varphi)$ is a self-adjoint operator.

In the case $d = 1$, it was shown in [3] that the quantum process $(\tilde{X}_\beta(\chi_{[0,t]}))_{t \geq 0}$ (\chi\Delta
denoting the indicator function of a set $\Delta$) is associated with a classical Lévy process $(Y_\beta(t))_{t \geq 0}$, which is a gamma process for $\beta = 2$, a Pascal process for $\beta > 2$, and a Meixner process for $0 \leq \beta < 2$. (One has, of course, to extend the SWN algebra in order to include the operators indexed by the indicator functions, for example, to take the set $L^2(\mathbb{R}, dx) \cap L^\infty(\mathbb{R}, dx)$ instead of $\mathcal{S}(\mathbb{R})$.)

We also refer to [1, 2, 3] and references therein for a discussion of other aspects of the SWN.

On the other hand, in papers [16, 19, 20, 11] (see also [17, 12, 10, 13]), the Jacobi field of the Lévy processes of Meixner’s type, i.e., the gamma, Pascal, and Meixner processes, was studied. Let us shortly explain this approach.

Let $\mathcal{S}'(\mathbb{R}^d)$ be the Schwartz space of tempered distributions. The $\mathcal{S}'(\mathbb{R}^d)$ is the dual space of $\mathcal{S}(\mathbb{R}^d)$ and the dualization between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ is given by the scalar product in $L^2(\mathbb{R}^d, dx)$. We will preserve the symbol $\langle \cdot, \cdot \rangle$ for this dualization. Let $\mathcal{C}(\mathcal{S}'(\mathbb{R}^d))$ denote the cylinder $\sigma$-algebra on $\mathcal{S}'(\mathbb{R}^d)$.

For each $\beta \geq 0$, we define a probability measure $\mu_\beta$ on $(\mathcal{S}'(\mathbb{R}^d)), \mathcal{C}(\mathcal{S}'(\mathbb{R}^d))$ by its Fourier transform

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\omega(x) \varphi} \mu_\beta(d\omega) = \exp \left[ \int_{\mathbb{R}^d} (e^{is\varphi(x)} - 1 - is\varphi(x)) \nu_\beta(ds) \, dx \right], \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \quad (12)$$

where the measure $\nu_\beta$ on $\mathbb{R}$ is specified as follows.

Let $\tilde{\nu}_\beta$ denote the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ whose orthogonal polynomials $(\tilde{P}_{\beta,n})_{n=0}^\infty$ with leading coefficient 1 satisfy the recurrence relation

$$s\tilde{P}_{\beta,n}(s) = \tilde{P}_{\beta,n+1}(s) + \beta(n + 1)\tilde{P}_{\beta,n}(s) + n(n + 1)\tilde{P}_{\beta,n-1}(s), \quad \beta \in [0, 2), \quad n \in \mathbb{Z}_+, \tilde{P}_{\beta,-1}(s):=0. \quad (13)$$

By [14, Ch. VI, sect. 3], $(\tilde{P}_{\beta,n})_{n=0}^\infty$ is a system of polynomials of Meixner’s type, the measure $\tilde{\nu}_\beta$ is uniquely determined by the above condition and is given as follows. For $\beta \in [0, 2)$,

$$\tilde{\nu}_\beta(ds) = \frac{\sqrt{4 - \beta^2}}{2\pi} |\Gamma(1+i(4-\beta^2)^{-1/2}s)|^2 \exp \left[ -s(4-\beta^2)^{-1/2} \arctan \left(\beta(4-\beta^2)^{-1/2}\right) \right] ds$$

4
(\nu_\beta is a Meixner distribution), for \beta = 2

\nu_2(ds) = \chi_{(0,\infty)}(s)e^{-s} ds

(\nu_2 is a gamma distribution), and for \beta > 2

\nu_\beta(ds) = (\beta^2 - 4) \sum_{k=1}^{\infty} p_{\beta,k}^k k \delta_{\sqrt{\beta^2 - 4} k}, \quad p_{\beta} := \frac{\beta - \sqrt{\beta^2 - 4}}{\beta + \sqrt{\beta^2 - 4}}

(\nu_\beta is now a Pascal distribution).

Notice that, for each \beta \geq 0, \nu(\{0\}) = 0, and hence, we may define

\nu_\beta(ds) := \frac{1}{s^2} \nu_\beta(ds).

(14)

Then, \mu_\beta is the measure of gamma noise for \beta = 2, Pascal noise for \beta > 2, and Meixner noise for \beta \in [0, 2). Indeed, for each \beta \geq 0, \mu_\beta is a generalized process on \mathcal{S}'(\mathbb{R}^d) with independent values (cf. [15]). Next, for each \varphi \in \mathcal{S}(\mathbb{R}^d), we have

\int_{\mathcal{S}'(\mathbb{R}^d)} \langle \omega, \varphi \rangle^2 \mu_\beta(d\omega) = \int_{\mathbb{R}^d} \varphi(x)^2 dx.

(15)

Hence, for each \( f \in L^2(\mathbb{R}^d, dx) \), we may define, in a standard way, the random variable \langle \cdot, f \rangle from \( L^2(\mathcal{S}'(\mathbb{R}^d), d\mu_\beta) \) satisfying (15) with \( \varphi = f \).

Then, for each open, bounded set \( \Delta \subset \mathbb{R}^d \), the distribution \mu_{\beta,\Delta} of the random variable \langle \cdot, \chi_\Delta \rangle under \mu_\beta is given as follows. For \beta > 2, \mu_{\beta,\Delta} is the negative binomial (Pascal) distribution

\mu_{\beta,\Delta}(ds) = (1 - p_\beta)|\Delta| \sum_{k=0}^{\infty} \frac{(|\Delta|)_k}{k!} p_{\beta,k}^k k \delta_{\sqrt{\beta^2 - 4} k - 2|\Delta|/(\beta + \sqrt{\beta^2 - 4})},

where for \( r > 0 \) \( r_0 := 1, (r)_k := r(r+1) \cdots (r+k-1), k \in \mathbb{N} \). For \beta = 2, \mu_{2,\Delta} is the Gamma distribution

\mu_{2,\Delta}(ds) = \frac{(s + |\Delta|)^{|\Delta| - 1}e^{-(s+|\Delta|)}}{\Gamma(|\Delta|)} \chi_{(0,\infty)}(s + |\Delta|) ds.

Finally, for \beta \in [0, 2),

\mu_{\beta,\Delta}(ds) = \frac{(4 - \beta^2)^{|\Delta| - 1}/2}{2\pi \Gamma(|\Delta|)} |\Gamma(|\Delta|/2 + i(4 - \beta^2)^{-1/2}(s + \beta|\Delta|/2)|^2}

\times \exp \left[ - (2s + \beta|\Delta|)(4 - \beta^2)^{-1/2} \arctan \left( \beta(4 - \beta^2)^{-1/2} \right) \right] ds.

Here, \( |\Delta| := \int_{\Delta} dx \).
We denote by \( P(S'(\mathbb{R}^d)) \) the set of continuous polynomials on \( S'(\mathbb{R}^d) \), i.e., functions on \( S'(\mathbb{R}^d) \) of the form
\[
F(\omega) = \sum_{i=0}^{n} \langle \omega^{\otimes i}, f^{(i)} \rangle, \quad \omega^{\otimes 0} := 1, \quad f^{(i)} \in S(\mathbb{R}^d)^{\otimes i}, \quad i = 0, \ldots, n, \quad n \in \mathbb{Z}_+.
\]

The greatest number \( i \) for which \( f^{(i)} \neq 0 \) is called the power of a polynomial. We denote by \( P_n(S'(\mathbb{R}^d)) \) the set of continuous polynomials of power \( \leq n \).

The set \( P(S'(\mathbb{R}^d)) \) is a dense subset of \( L^2(S'(\mathbb{R}^d), d\mu_\beta) \). Let \( P_\alpha(S'(\mathbb{R}^d)) \) denote the closure of \( P_\alpha(S'(\mathbb{R}^d)) \) in \( L^2(S'(\mathbb{R}^d), d\mu_\beta) \), let \( P_n(S'(\mathbb{R}^d)) \), \( n \in \mathbb{N} \), denote the orthogonal difference \( P_\lambda(S'(\mathbb{R}^d)) \oplus P_{\lambda-1}(S'(\mathbb{R}^d)) \), and let \( P_0(S'(\mathbb{R}^d)) =: P_0'(S'(\mathbb{R}^d)) \). We evidently have the orthogonal decomposition
\[
L^2(S'(\mathbb{R}^d), d\mu_\beta) = \bigoplus_{n=0}^{\infty} P_n(S'(\mathbb{R}^d)). \tag{16}
\]

For a monomial \( \langle \omega^{\otimes n}, f^{(n)} \rangle \), \( f^{(n)} \in S(\mathbb{R}^d)^{\otimes n} \), we denote by \( \langle \omega^{\otimes n}, f^{(n)} \rangle \) the orthogonal projection of \( \langle \omega^{\otimes n}, f^{(n)} \rangle \) onto \( P_n(S'(\mathbb{R}^d)) \). The set \( \{ \langle \omega^{\otimes n}, f^{(n)} \rangle : f^{(n)} \in S(\mathbb{R}^d)^{\otimes n} \} \) is dense in \( P_n(S'(\mathbb{R}^d)) \).

Denote by \( \mathbb{Z}^{\infty}_{+,0} \) the set of all sequences \( \alpha \) of the form \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots) \), \( \alpha_i \in \mathbb{Z}_+ \), \( n \in \mathbb{N} \). Let \( |\alpha| := \sum_{i=1}^{\infty} \alpha_i \), evidently \( |\alpha| \in \mathbb{Z}_+ \). For each \( \alpha \in \mathbb{Z}^{\infty}_{+,0} \), \( 1\alpha_1 + 2\alpha_2 + \cdots = n, \quad n \in \mathbb{N} \), and for any function \( f^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R} \) we define a function \( D_\alpha f^{(n)} : (\mathbb{R}^d)^{|\alpha|} \to \mathbb{R} \) by setting
\[
(D_\alpha f^{(n)})(x_1, \ldots, x_{|\alpha|}) := f^{(n)}((x_1, \ldots, x_{\alpha_1}, x_{\alpha_1+1}, x_{\alpha_1+1}, x_{\alpha_1+2}, x_{\alpha_1+2}, \ldots, x_{\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}),

x_{\alpha_1+\alpha_2+1}, x_{\alpha_1+\alpha_2+1}, x_{\alpha_1+\alpha_2+1}, \ldots). \tag{17}
\]

We define a scalar product on \( S(\mathbb{R}^d)^{\otimes n} \) by setting for any \( f^{(n)}, g^{(n)} \in S(\mathbb{R}^d)^{\otimes n} \)
\[
(f^{(n)}, g^{(n)})_{F_\alpha(\mathbb{R}^d)} := \sum_{\alpha \in \mathbb{Z}^{\infty}_{+,0} : 1\alpha_1 + 2\alpha_2 + \cdots = n} K_\alpha \int_{X_{|\alpha|}} (D_\alpha f^{(n)})(x_1, \ldots, x_{|\alpha|}) \times (D_\alpha g^{(n)})(x_1, \ldots, x_{|\alpha|}) dx_1 \cdots dx_{|\alpha|}, \tag{17}
\]
where
\[
K_\alpha = \frac{n!}{\alpha_1! \alpha_2! \alpha_3! \cdots}. \tag{18}
\]

Let \( F_\alpha(L^2(\mathbb{R}^d, dx)) \) be the closure of \( S(\mathbb{R}^d)^{\otimes n} \) in the norm generated by (17), (18). The extended Fock space \( F_\alpha(L^2(\mathbb{R}^d, dx)) \) over \( L^2(\mathbb{R}^d, dx) \) is defined as
\[
F_\alpha(L^2(\mathbb{R}^d, dx)) := \bigoplus_{n=0}^{\infty} F_\alpha(L^2(\mathbb{R}^d, dx)) n!, \tag{19}
\]
where $\mathcal{F}_{\text{Ext}}^{(n)}(L^2(\mathbb{R}^d, dx)) := \mathbb{R}$. We also denote by $\Omega$ the vacuum vector in $\mathcal{F}_{\text{Ext}}(L^2(\mathbb{R}^d, dx))$: $\Omega = (1, 0, 0, \ldots)$. Therefore, for each $f^{(n)}, g^{(n)} \in S(\mathbb{R}^d) \hat{\otimes} n$, $n \in \mathbb{N}$, we have

$$\int_{S'(\mathbb{R}^d)} :\omega^{\otimes n}, f^{(n)}: :\omega^{\otimes n}, g^{(n)}: \mu_\beta(d\omega) = (f^{(n)}, g^{(n)})_{\mathcal{F}_{\text{Ext}}^{(n)}(L^2(\mathbb{R}^d, dx))} n!.$$  

(20)

Therefore, for each $f^{(n)} \in \mathcal{F}_{\text{Ext}}^{(n)}(L^2(\mathbb{R}^d, dx))$, we can define, a random variable $:\omega^{\otimes n}, f^{(n)}$: from $L^2(S'(\mathbb{R}^d), d\mu_\beta)$ such that equality (20) remains true for any $f^{(n)}, g^{(n)} \in \mathcal{F}_{\text{Ext}}^{(n)}(L^2(\mathbb{R}^d, dx))$, and furthermore

$$\mathcal{F}_{\text{Ext}}(L^2(\mathbb{R}^d, dx)) \ni f = (f^{(n)})_{n=0}^\infty \mapsto$$

$$\mapsto U_\beta f = (U_\beta f)(\omega) = \sum_{n=0}^\infty :\omega^{\otimes n}, f^{(n)}: \in L^2(S'(\mathbb{R}^d), d\mu_\beta) \quad (21)$$

is unitary.

We denote by $\mathcal{F}_{\text{fin}}(S(\mathbb{R}^d))$ the dense subset of $\mathcal{F}_{\text{Ext}}(L^2(\mathbb{R}^d, dx))$ consisting of vectors of the form $(f^{(0)}, f^{(1)}, \ldots, f^{(n)}, 0, 0, \ldots)$, where $f^{(i)} \in S(\mathbb{R}^d) \hat{\otimes} i$. For each $\beta \geq 0$ and each $\varphi \in S(\mathbb{R}^d)$, we define an operator $a_\beta(\varphi)$ on $\mathcal{F}_{\text{fin}}(S(\mathbb{R}^d))$ by the following formula:

$$a_\beta(\varphi) = a^+(\varphi) + \beta a^0(\varphi) + a^-(\varphi).$$

Here, $a^+(\xi)$ is the standard creation operator:

$$a^+(\varphi)f^{(n)} := \varphi \hat{\otimes} f_n, \quad f^{(n)} \in S(\mathbb{R}^d) \hat{\otimes} n, \quad n \in \mathbb{Z}_+,$$  

(22)

$a^0(\varphi)$ is the standard neutral operator:

$$(a^0(\varphi)f^{(n)})(x_1, \ldots, x_n) = (\varphi(x_1) + \cdots + \varphi(x_n)) f_n(x_1, \ldots, x_n),$$  

(23)

and

$$a^-(\varphi) = a_1^-(\varphi) + a_2^-(\varphi),$$  

(24)

where $a_1^-(\varphi)$ is the standard annihilation operator:

$$(a_1^-(\varphi)f^{(n)})(x_1, \ldots, x_{n-1}) = n \int_{\mathbb{R}^d} \varphi(x)f^{(n)}(x, x_1, \ldots, x_{n-1}) \, dx,$$  

(25)

and

$$(a_2^-(\varphi)f^{(n)})(x_1, \ldots, x_{n-1}) = n(n-1)(\varphi(x_1)f^{(n)}(x_1, x_2, x_3, \ldots, x_{n-1})) \sim,$$  

(26)

$$(\cdot)^\sim$$

denoting symmetrization of a function.
Denote by $\partial_x^I$, $\partial_x$ the standard creation and annihilation operators at point $x \in \mathbb{R}^d$:

$$\partial_x^I f^{(n)} = \delta_x \otimes f^{(n)}, \quad \partial_x f^{(n)}(x_1, \ldots, x_{n-1}) = nf^{(n)}(x, x_1, \ldots, x_{n-1}).$$

Then, at least formally, we have the following representation:

$$a^+(\varphi) = \int_{\mathbb{R}^d} \varphi(x)\partial_x^I dx, \quad a^0(\varphi) = \int_{\mathbb{R}^d} \varphi(x)\partial_x^I \partial_x dx, \quad a^-(\varphi) = \int_{\mathbb{R}^d} \varphi(x)(\partial_x + \partial_x^I \partial_x^2) dx,$$

so that

$$a_\beta(\varphi) = \int_{\mathbb{R}^d} \varphi(x)(\partial_x + \beta \partial_x^I \partial_x + \partial_x^I \partial_x^2) dx. \quad (28)$$

(In fact, equalities (27), (28) may be given a precise meaning, cf. [16, 19].)

The operators $a_\beta(\varphi)$, $\varphi \in S(\mathbb{R}^d)$, are essentially self-adjoint on $F^\text{fin}(S(\mathbb{R}^d))$ and the image of any $\tilde{a}_\beta(\varphi)$, $\varphi \in S(\mathbb{R}^d)$, under the unitary $U_\beta$ is the operator of multiplication by the random variable $\langle \cdot, \varphi \rangle$. Thus, $(\tilde{a}(\varphi))_{\varphi \in S(\mathbb{R}^d)}$ is the Jacobi field of $\mu_\beta$, see [8, 9, 18, 11] and the references therein.

The functional realization of the operators $a^+(\varphi)$, $a^0(\varphi)$, $a^-(\varphi)$, i.e., the explicit action of the the image of these operators under the unitary $U_\beta$ is discussed in [16, 19].

A direct computation shows that the operators $2a^+(\varphi)$, $2a^0(\varphi)$, $2a^-(\varphi)$, $\varphi \in S(\mathbb{R}^d)$, satisfy the commutation relations (6), and hence generate a SWN algebra. In fact, we have the following result:

**Theorem 1**  For each $\beta \geq 0$, there exists a unitary operator

$$I_\beta : F(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \to F^\text{ext}(L^2(\mathbb{R}^d, dx))$$

such that $I_\beta \Omega = \Omega$ and the operators $\tilde{X}_\beta(\varphi)$, $\tilde{B}^I(\varphi)$, $\tilde{N}(\varphi)$, $\tilde{B}(\varphi)$, $\varphi \in S(\mathbb{R}^d)$, are unitarily isomorphic under $I_\beta$ to two times the operators $\tilde{a}(\varphi)$, $\tilde{a}^+(\varphi)$, $\tilde{a}^0(\varphi)$, $\tilde{a}^-(\varphi)$, respectively.

Notice that the unitary operator

$$U_\beta := U_\beta I_\beta : F(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \to L^2(S'(\mathbb{R}^d), d\mu_\beta)$$

has the following properties: $U_\beta \Omega = 1$ and

$$U_\beta \tilde{X}_\beta(\varphi) U_\beta^{-1} = 2\langle \cdot, \varphi \rangle \cdot, \quad \varphi \in S(\mathbb{R}^d)$$

(compare with [3])

By virtue of (5), (10), (27), and (28), we get from Theorem 1:

$$B_x = 2(\partial_x + \partial_x^I \partial_x^2), \quad N_x = 2\partial_x \partial_x^I, \quad B_x^I = 2\partial_x^I, \quad (29)$$

and

$$X_\beta(x) = 2(\partial_x^I + \beta \partial_x^I \partial_x + \partial_x^I \partial_x^2), \quad x \in \mathbb{R}^d$$

(where the equalities are to be understood in the sense of the unitary isomorphism). The reader is advised to compare (29) with the informal representation (2).
2 Proof of the theorem

The proof of Theorem 1 is essentially based on the results of [20]. By (9) and (11), we get, for each \( \varphi \in S(\mathbb{R}^d) \),

\[
X_\beta(\varphi) = 2(A^+(\varphi \otimes e_1) + A^0(\varphi \otimes J_\beta) + A^-(\varphi \otimes e_1)),
\]

where

\[
J_\beta := J^+ + \beta J^0 + J^-.
\]

By (7), the operator \( J_\beta \) is given by a Jacobi matrix (see e.g. [7]). Furthermore, \( J_\beta \) is essentially self-adjoint on \( \ell^2 \) and, by (13), \( \tilde{\nu}_\beta \) is the spectral measure of \( \tilde{J}_\beta \). The latter means that there exists a unitary operator

\[
I_\beta^{(1)} : \ell_2 \to L^2(\mathbb{R}, d\tilde{\nu}_\beta)
\]

such that \( I_\beta^{(1)} e_1 = 1 \) and, under \( I_\beta^{(1)} \), the operator \( \tilde{J}_\beta \) goes over into the operator of multiplication by \( s \).

Next, by (14), the operator

\[
L^2(\mathbb{R}, d\tilde{\nu}_\beta) \ni f \mapsto I_\beta^{(2)} f = (I_\beta^{(2)} f)(s) := f(s) s \in L^2(\mathbb{R}, d\nu_\beta)
\]

is unitary. Setting

\[
I_\beta^{(3)} := I_\beta^{(2)} I_\beta^{(1)} : \ell_2 \to L^2(\mathbb{R}, d\nu_\beta),
\]

we get a unitary operator such that \( I_\beta^{(3)} e_1 = (I_\beta^{(3)} e_1)(s) = s \) and, under \( I_\beta^{(3)} \), \( \tilde{J}_\beta \) goes over into the operator of multiplication by \( s \).

Using \( I_\beta^{(3)} \), we can naturally construct a unitary operator

\[
I_\beta^{(4)} : \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \to \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes L^2(\mathbb{R}, d\nu_\beta))
\]

such that \( I_\beta^{(4)} \Omega = \Omega \) and, under \( I_\beta^{(4)} \), the operator \( X_\beta(\varphi) \) goes over into the operator

\[
X_\beta(\varphi) = 2(A^+(\varphi \otimes s) + A^0(\varphi \otimes s) + A^-(\varphi \otimes s)).
\]

It follows from [20] that there exists a unitary operator

\[
I_\beta^{(5)} : \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes L^2(\mathbb{R}, d\nu_\beta)) \to L^2(\mathcal{S}'(\mathbb{R}^d), d\mu_\beta)
\]

such that \( I_\beta^{(5)} \Omega = 1 \) and, under \( I_\beta^{(5)} \), the operator \( \tilde{X}_\beta(\varphi) \) goes over into the operator of multiplication by \( 2\langle \cdot, \varphi \rangle \).

We define the unitary

\[
I_\beta := U_\beta^{-1} I_\beta^{(5)} I_\beta^{(4)} : \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \to \mathcal{F}_{\text{Ext}}(L^2(\mathbb{R}^d, dx)),
\]
where $U_\beta$ is given by (21). We evidently get $I_\beta \Omega = \Omega$ and $\tilde{a}(\varphi) = I_\beta^{-1}\tilde{X}_\beta(\varphi)I_\beta^{-1}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Next, we denote by $\mathfrak{G}$ the subset of $\mathcal{F}_{\text{Ext}}(L^2(\mathbb{R}^d, dx))$ defined as the linear span of $\Omega$ and the vectors of the form $\varphi^{\otimes n}$, where $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $n \in \mathbb{N}$. We note:

$$(I_\beta^{(3)} e_n)(s) = P_{\beta,n}(s), \quad n \in \mathbb{N},$$

where

$$P_{\beta,n}(s) := s\tilde{P}_{\beta,n-1}(s), \quad n \in \mathbb{N},$$

and $(\tilde{P}_{\beta,n})_{n=0}^\infty$ are defined by (13). Hence, by [20, Sect. 4 and Corollary 5.1],

$$\mathfrak{G} \subset I_\beta \mathfrak{F}.$$

Furthermore, by (7), (8), (22)–(26) and by [20, Corollary 5.1], we get:

$$I_\beta B^\dagger(\varphi)I_\beta^{-1} \upharpoonright \mathfrak{G} = a^+(\varphi) \upharpoonright \mathfrak{G},$$

$$I_\beta N(\varphi)I_\beta^{-1} \upharpoonright \mathfrak{G} = a^0(\varphi) \upharpoonright \mathfrak{G},$$

$$I_\beta B(\varphi)I_\beta^{-1} \upharpoonright \mathfrak{G} = a^-(\varphi) \upharpoonright \mathfrak{G}, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

(30)

We now endow $\mathcal{F}_{\text{fin}}(\mathcal{S}(\mathbb{R}^d))$ with the topology of the topological direct sum of the spaces $\mathcal{F}_n(\mathcal{S}(\mathbb{R}^d))$. Thus, the convergence in $\mathcal{F}_{\text{fin}}(\mathcal{S}(\mathbb{R}^d))$ means the uniform finiteness and the coordinate-wise convergence in each $\mathcal{F}_n(\mathcal{S}(\mathbb{R}^d))$. As easily seen, $\mathfrak{G}$ is a dense subset of $\mathcal{F}_{\text{fin}}(\mathcal{S}(\mathbb{R}^d))$. Since the operators $a^+(\varphi)$, $a^0(\varphi)$, and $a^-(\varphi)$ act continuously on $\mathcal{F}_{\text{fin}}(\mathcal{S}(\mathbb{R}^d))$ and since $\mathcal{F}_{\text{fin}}(\mathcal{S}(\mathbb{R}^d))$ is continuously embedded into $\mathcal{F}_{\text{Ext}}(L^2(\mathbb{R}^d, dx))$ (cf. [16, p. 37]), the closure of the operators $a^+(\varphi)$, $a^0(\varphi)$, and $a^-(\varphi)$ restricted to $\mathfrak{G}$ coincides with $\tilde{a}^+(\varphi)$, $\tilde{a}^0(\varphi)$, and $\tilde{a}^-(\varphi)$, respectively. Hence, by (30), $\tilde{B}^\dagger(\varphi)$, $\tilde{N}(\varphi)$, and $\tilde{N}(\varphi)$ are extensions of the operators $\tilde{a}^+(\varphi)$, $\tilde{a}^0(\varphi)$, and $\tilde{a}^-(\varphi)$, respectively.

Finally, analogously to the proof of [20, Theorem 6.1], we conclude that $I_\beta \mathfrak{F}$ is a subset of the domain of $\tilde{a}^+(\varphi)$, respectively $\tilde{a}^0(\varphi)$, respectively $\tilde{a}^-(\varphi)$, and furthermore

$$I_\beta B^\dagger(\varphi)I_\beta^{-1} = \tilde{a}^+(\varphi) \upharpoonright I_\beta \mathfrak{F},$$

$$I_\beta N(\varphi)I_\beta^{-1} = \tilde{a}^0(\varphi) \upharpoonright I_\beta \mathfrak{F},$$

$$I_\beta B(\varphi)I_\beta^{-1} = \tilde{a}^-(\varphi) \upharpoonright I_\beta \mathfrak{F}, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

This yields:

$$I_\beta \tilde{B}^\dagger(\varphi)I_\beta^{-1} = \tilde{a}^+(\varphi),$$

$$I_\beta \tilde{N}(\varphi)I_\beta^{-1} = \tilde{a}^0(\varphi),$$

$$I_\beta \tilde{B}(\varphi)I_\beta^{-1} = \tilde{a}^-(\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^d),$$

which concludes the proof.

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