Two scalar field cosmology: Conservation laws and exact solutions

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We consider the two scalar field cosmology in a FRW spatially flat spacetime where the scalar fields interact both in the kinetic part and the potential. We apply the Noether point symmetries in order to define the interaction of the scalar fields. We use the point symmetries in order to write the field equations in the normal coordinates and we find that the Lagrangian of the field equations which admits at least three Noether point symmetries describes linear Newtonian systems. Furthermore, by using the corresponding conservation laws we find exact solutions of the field equations. Finally, we generalize our results to the case of N scalar fields interacting both in their potential and their kinematic part in a flat FRW background.

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1. INTRODUCTION

An easy way to explain the acceleration expansion of the Universe (see [1–5]) is to consider an additional fluid which has a negative equation of state parameter. This new fluid counteracts the gravitational force and leads to the observed acceleration expansion.

Nevertheless, a plethora of alternative cosmological scenarios has been a result of the lack of a fundamental physical theory concerning the mechanism that induces the cosmic acceleration. Most of them are based either on the existence of new fields in nature (dark energy) or in some modification of the Einstein-Hilbert action [6–20].

One approach has been the consideration of two interacting scalar fields in a spatially flat FRW spacetime. In this approach the interaction of the scalar fields is usually limited to the potentials of the fields. This limitation is not necessary and one would like to know what happens if the interaction is extended to include as well the kinematics of the two fields. In this case the dynamical system becomes quite more complicated and the finding of analytic solutions is a major problem. Indeed in the literature one finds only a few successful attempts which find analytic solutions of the field equations for the extended interaction. In [31] the authors applied the superpotential method in order to determine a stable exact solution. Another exact solution with two scalar fields with exponential potential in which the one scalar field acts as a stiff matter given in [32]. Finally in [17, 33] it is shown that by using the deformation procedure it is possible to generate a two scalar field cosmological model from one scalar field, and use the solution of the single scalar field to determine analytic solutions of the two scalar field model. We note that in the case of a single scalar field there exists a large number of analytic solutions (for instance see [22–30]).

The action of two interacting scalar fields in their kinematic and potential part is [16–20]

\[
S = \int dx^4 \sqrt{-g} \left( R - \frac{1}{2} g_{ij} H_{AB} (\Phi^C) \Phi^A,i \Phi^B,j + V (\Phi^C) \right)
\]

where \( \Phi^A = (\phi, \psi) \) and \( H_{AB} = H_{AB} (\Phi^C) = H_{BA} \) is a symmetric tensor. The importance of action (1) is that a plethora of alternative theories of gravity can be written in this form under a conformal transformation, for details see [21].

The main purpose of the present work is to address the problem of finding analytic solutions of the two scalar field cosmology in a systematic way using the Noether symmetries of the field equations. As it will be shown the method we propose recovers the aforementioned solutions and produces, in addition, new ones which have not been considered.

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The idea to apply Noether symmetries in scalar field cosmology and on modified theories of gravity is not new and indeed it has gained a lot of attention in the literature \cite{34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58}. Not all approaches follow the same methods. In the present work we follow the approach of \cite{34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58} which is geometric hence, to our opinion, more fundamental.

The key to our approach is the recent result \cite{34} that the Noether point symmetries of Lagrange equations, for a first order Lagrangian, are the homothetic vectors\footnote{We recall that a homothetic vector (HV) $X$ of a metric $g_{ij}$ is a vector satisfying the identity $L_X g_{ij} = 2\psi g_{ij}$ where $\psi$ is a constant. In case $\psi = 0$ the vector $X$ is a Killing vector (KV).} of the metric of the space generated by the dynamic fields. In this way the determination of the Noether point symmetries becomes a problem of differential geometry. Fortunately this problem nowadays can be dealt easily with the use of appropriate software. For a general potential the field equations do not admit Noether point symmetries hence they are not Noether integrable. We demand that they admit extra Noether point symmetries which are linearly independent and in involution, and determine in each case the corresponding potentials using the results given in \cite{34}.

Following the above we consider the symmetric tensor $H_{AB}$ in \cite{1} as a metric in the space of the fields $\Phi^C$ and and apply the aforementioned result to determine the Noether vectors and consequently the corresponding Noether integrals which, provided that there are enough of them, lead to the solution of the field equations. Obviously the approach depends strongly on the metric $g_{ij}$ in \cite{1}. We show that in the flat FRW background the requirement that the Lagrangian admits at least two Noether point symmetries (apart from the trivial one $\partial_t$) limits the possible cases of interaction to two. One of them is the case considered in \cite{35} and the other, which is new, we consider and solve in detail in section \cite{1}. In the latter case we find that the dynamical system is equivalent to the motion of a particle in the $M^3$ space. To find the type of motion we use the Noether vectors to write the Lagrangian in normal coordinates and in these coordinates it is found that the corresponding potentials for which Noether symmetries are admitted, are equivalent to the 3d unharmonic hyperbolic oscillator and to the 3d forced oscillator. Subsequently in each case we determine easily the analytic solution.

The structure of the paper is as follows. In section \cite{2} we consider two scalar fields interacting both in their kinematic and potential parts in a spatially flat FRW spacetime. We give the basic properties of the dynamical system, we produce the field equations and we define the effective equation of state. In section \cite{3} we discuss briefly the basic theory of Noether point symmetries. We define the interaction of the scalar fields in the kinetic part by the requirement that the field equations admit at least two more Noether point symmetries. In section \cite{4} we determine the potentials of the fields for which this is the case and then solve analytically the resulting field equations. We find that for the first potential the late time behavior of the scale factor is that of deSitter solution. For the second potential the late time behavior is different. In order to determine the late time behavior in this case we write the Hubble function in terms of the redshifts and we find that that there exist a dust like fluid component in the Hubble function.

Finally in order to show the viability of the new solution we compare it at late time with the $\Lambda$–cosmology model using the supernova and the BAO data. We find that both models fit the data practically with the same statistic parameters. In section \cite{5} we extend our analysis to the case of $N$ scalar fields. In section \cite{6} we show how the interaction in the kinematic part of the scalar fields which we consider in section \cite{4} arises from the conformal equivalence in scalar tensor theory. Finally in section \cite{7} we draw our conclusions.

## 2. THE FIELD EQUATIONS

We assume that the fields interact in the spatially flat FRW spacetime

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2)$$

where $a(t)$ is the scale factor. In this spacetime the Lagrangian \cite{1} becomes

$$L \left(a, \dot{a}, \Phi^C, \dot{\Phi}^C \right) = -3a\ddot{a} + \frac{1}{2}a^3 H_{AB} \left(\Phi^C \right) \dot{\Phi}^A \dot{\Phi}^B - a^3 V \left(\Phi^C \right).$$

where the indices $A, B, C = 1, 2$ and $\Phi^A = (\phi, \psi)$. The field equations are the Friedmann equation:

$$-3a\ddot{a} + \frac{1}{2}a^3 H_{AB} \dot{\Phi}^A \dot{\Phi}^B + a^3 V \left(\Phi^C \right) = 0$$


and the Euler Lagrange equations
\[
\frac{\partial L}{\partial (\dot{a}, \Phi^C)} - \frac{\partial L}{\partial a} \left( \frac{\partial}{\partial t} \right) \dot{\Phi} = 0
\]
with respect to the variables \( a, \Phi^C \); that is,
\[
\ddot{a} + \frac{1}{2a} \dot{a}^2 + \frac{a}{4} H_{AB} \dot{\Phi}^A \dot{\Phi}^B - \frac{1}{2} \dot{a} V = 0
\]
(5)
\[
\ddot{\Phi}^A + \frac{3}{2a} \dot{a} \dot{\Phi}^A + \dot{\Gamma}^A_{BC} \dot{\Phi}^B \dot{\Phi}^C + H_{AB} V_B = 0
\]
(6)
where \( \dot{\Gamma}^A_{BC} \) are the connection coefficients of the two dimensional metric \( H_{AB} \).

In terms of the Hubble parameter \( H = \frac{\dot{a}}{a} \) the field equations (4) and (5) take the form
\[
H^2 = \frac{1}{3} \left[ \frac{1}{2} H_{AB} \dot{\Phi}^A \dot{\Phi}^B + V (\Phi^C) \right]
\]
(7)
\[
2 \dot{H} + 3H^2 = - \left[ \frac{1}{2} H_{AB} \dot{\Phi}^A \dot{\Phi}^B - V (\Phi^C) \right].
\]
(8)

We assume commoving observers \( u^a = \delta^a_0 \) and from the field equations we have the effective energy momentum tensor
\[
T_{ij} = \rho_{eff} u_i u_j + p_{eff} (g_{ij} + u_i u_j)
\]
(9)
where
\[
\rho_{eff} = \frac{1}{2} H_{AB} \dot{\Phi}^A \dot{\Phi}^B + V (\Phi^C)
\]
(10)
\[
p_{eff} = \frac{1}{2} H_{AB} \dot{\Phi}^A \dot{\Phi}^B - V (\Phi^C).
\]
(11)

It follows that the ”effective” equation of state is
\[
w_{eff} = \frac{\frac{1}{2} H_{AB} \dot{\Phi}^A \dot{\Phi}^B - V (\Phi^C)}{\frac{1}{2} H_{AB} \dot{\Phi}^A \dot{\Phi}^B + V (\Phi^C)}.
\]
(12)

Finally, the conservation equation \( T^{ij}_{;j} = 0 \) gives
\[
\dot{\rho}_{eff} + 3\rho_{eff} H (1 + w_{eff}) = 0
\]
(13)
from where we have the Klein Gordon equation (6) for the fields.

As we have already mentioned for a general interaction (metric) \( H_{AB} (\Phi^C) \) and potential \( V (\Phi^C) \) the dynamical system defined by the Lagrangian \( L \) admits only the Noether point symmetry \( \partial_t \) with Noether integral the Hamiltonian \( H \). In the following sections we apply the Noether symmetry approach in order to determine both the kinematic interaction of the two fields - that is the metric \( H_{AB} (\Phi^C) \) - and the corresponding potential \( V (\Phi^C) \) for which two more Noether symmetries are admitted, therefore the dynamical system is Liouville integrable.

3. NOETHER POINT SYMMETRIES

Before we proceed we review briefly the basic definitions concerning Noether point symmetries of systems of second order ordinary differential equations (ODEs) of the form
\[
\ddot{x}^i = \omega^i (t, x^j, \dot{x}^j).
\]
(14)
If the system of ODEs (14) results from a first order Lagrangian,
\[
L = L (t, x^j, \dot{x}^j)
\]
(15)
then the vector field

\[ X = \xi (t, x^i) \partial_t + \eta^i (t, x) \partial_i \]

in the augmented space \( \{t, x^i\} \) is the generator of a Noether point symmetry of the system \( (14) \) if the following condition is satisfied \( (16) \)

\[ X^{[1]} L + L \frac{d\xi}{dt} = \frac{df}{dt} \]

where \( f = f (t, x^i) \) is the Noether gauge function and \( X^{[1]} \) is the first prolongation of \( X \), i.e.

\[ X^{[1]} = X + \left( \frac{d\eta^i}{dt} - \dot{x}^i \frac{d\xi}{dt} \right) \partial_{\dot{x}^i}. \]

To every Noether point symmetry there corresponds a first integral (a Noether integral) of the system of equations \( (14) \) given by the formula:

\[ I = \xi E_H - \frac{\partial L}{\partial \dot{x}^i} \eta^i + f \]

where \( E_H \) is the Hamiltonian of \( L \).

The generator \( X \) of a Noether point symmetry of the Lagrangian \( (3) \) in the space \( \{t, a, \Phi^C\} \) is

\[ X = \xi (t, a, \Phi^C) \partial_t + \eta_a (t, a, \Phi^C) \partial_a + \eta_C^a (t, a, \Phi^C) \partial_{\Phi^C} \]

and the first prolongation vector is

\[ X^{[1]} = X + \left( \dot{\eta}_a - \dot{a} \xi \right) \partial_a + \left( \dot{\eta}_C^a - \dot{\Phi}^C \xi \right) \partial_{\Phi^C}. \]

Lagrangian \( (3) \) defines the kinetic metric

\[ ds^2_{(3)} = -6 a da^2 + a^3 H_{AB} (\Phi^C) d\Phi^A d\Phi^B \]

and the effective potential \( V_{eff} = a^3 V (\phi, \psi) \).

It has been shown \( (22) \) that for Lagrangians of the form \( T - V_{eff} \), where \( T \) is the kinetic energy, the Noether point symmetries are generated by the elements of the homothetic group of the kinetic metric \( T \). The metric \( (20) \) can be written

\[ ds^2_{(3)} = a^3 \left( - \frac{6}{a^2} da^2 + H_{AB} (\Phi^C) d\Phi^A d\Phi^B \right) \]

which shows that it is conformal to the 1+2 decomposable metric

\[ ds^2_{(3)} = \frac{6}{a^2} da^2 + H_{AB} (\Phi^C) d\Phi^A d\Phi^B \]

where the 2d metric \( H_{AB} \) is conformally flat (all 2d metrics are conformally flat).

The 3d metric \( (20) \) for a general metric \( H_{AB} (\Phi^C) \) admits the gradient \( HV H_V = \frac{3}{4} a \partial_a, \psi_{HV} = 1 \) \( (61) \). It is easy to show that the HV \( H_V \) does not generate a Noether point symmetry for the Lagrangian \( (3) \).

Therefore in order the Lagrangian \( (3) \) to admit additional Noether point symmetries we have to consider special forms of the two dimensional metric \( H_{AB} (\Phi^C) \) for which the 3d metric \( (20) \) admits a greater homothetic algebra. Because we require two more first integrals, apart from the Hamiltonian, the 2d metric \( H_{AB} \) must be such that the 3d metric \( (20) \) admits a homothetic algebra \( G_H \) with \( \dim G_H \geq 3 \). This happens in two cases:

Case 1: The 2d metric \( H_{AB} \) admits three Killing vectors (KVs) which span the E(2) group, i.e., \( H_{AB} \) is flat.

Case 2: The 2d metric \( H_{AB} \) admits three KVs which span the SO(3) group, i.e., \( H_{AB} \) is a space of constant non-vanishing curvature.

Before we proceed, we rewrite the Lagrangian \( (3) \) in a more convenient form. We apply the coordinate transformation

\[ a = \left( \frac{3}{8} \right)^{\frac{1}{4}} a^\frac{3}{4} \]
and the Lagrangian (3) becomes

\[ L = -\frac{1}{2} \dot{u}^2 + \frac{1}{2} u^2 h_{AB} \dot{\Phi}^A \dot{\Phi}^B - u^2 U(\Phi^C) \]  
(24)

where we have set \( h_{AB} = \frac{3}{8} H_{AB} \) and \( U = 3^4 V \).

In Case 1 the two dimensional metric \( h_{AB} \) is \( h_{AB} = \delta_{AB} \). In the case where \( h_{AB} = \eta_{AB} \) the action (1) describes the Quintom models [63, 64]; this case has been considered in [35]. Furthermore, this case is equivalent to that of a single complex scalar field i.e.

\[ S = \int dx^4 \sqrt{-g} \left( -\frac{R}{2} + \frac{1}{2} g_{ij} \Psi^i \Psi^j + V(\Psi, \Psi^*) \right) \]  
(25)

where \( \Psi = \phi + i\psi \) with product \( \Psi \Psi^* = |\Psi|^2 \) [62]. We remark that in the case where \( h_{AB} \) is flat, we can always find a 'coordinate' transformation in order to write the kinetic term in the simplest form. For instance the mixed kinetic term in [63] can be reduced in this manner.

In Case 2 the 2d metric \( h_{AB} \) is

\[ h_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\phi} \end{pmatrix} \]  
(26)

which is the metric of a maximally symmetric space of negative curvature. This case is also equivalent to the one of a single complex scalar field \( \Psi = \phi + i\psi \) with the non Euclidian product

\[ \Psi \Psi^* = \left( 1 - \frac{1}{4} |\Psi|^2 \right)^{-2} |\Psi|^2 \]  
(27)

where \( |\Psi| \) is the Euclidian norm. In the following we discuss in detail the new Case 2.

4. THE METRIC \( H_{AB} \) ADMITS THE \( SO(3) \) LIE ALGEBRA

As it has been remarked in this case \( h_{AB} \) is the metric of a space of constant non vanishing curvature. From (26) the kinetic metric (31) becomes

\[ ds^2_{(3)} = -du^2 + u^2 \left( d\phi^2 + e^{2\phi} d\psi^2 \right) . \]  
(28)

It is easy to show that this metric is flat, therefore admits a seven dimensional homothetic Lie algebra consisting of the following vectors:

- Three gradient KVs (translations) \( K^I, I = 1, 2, 3 \) :

\[ K^1 = -\frac{1}{2} \left( e^\phi (1 + \psi^2) + e^{-\phi} \right) \partial_u \]
\[ + \frac{1}{2u} \left( e^\phi (1 + \psi^2) - e^{-\phi} \right) \partial_\phi + \frac{1}{u} \psi e^{-\phi} \partial_\psi \]

\[ K^2 = -\frac{1}{2} \left( e^\phi (1 - \psi^2) - e^{-\phi} \right) \partial_u \]
\[ + \frac{1}{2u} \left( e^\phi (1 - \psi^2) + e^{-\phi} \right) \partial_\phi - \frac{1}{u} \psi e^{-\phi} \partial_\psi \]

\[ K^3 = -\psi e^\phi \partial_u + \frac{1}{u} \psi e^\phi \partial_\phi + \frac{1}{u} e^{-\phi} \partial_\psi \]

where the corresponding gradient Killing functions \( S_{(I)} \) are

\[ S_{(1)} = \frac{1}{2} u \left( e^\phi (1 + \psi^2) + e^{-\phi} \right) \]
\[ S_{(2)} = \frac{1}{2} u (e^\phi (1 - \psi^2) - e^{-\phi}) \]

\[ S_{(3)} = u \psi e^\phi. \]

- Three non-gradient KVs (the rotations) which span the \( SO(3) \) algebra

\[ X_{12} = \partial_\psi, \quad X_{23} = \partial_\phi + \psi \partial_\psi \]

\[ X_{13} = \psi \partial_\phi + \frac{1}{2} (\psi^2 - e^{2\phi}) \partial_\phi. \] (29)

- The gradient HV \( H_V = u \partial_u, \quad \psi H_V = 1 \).

For this choice of \( H_{AB} \) the Lagrangian (24) becomes

\[ L = -\frac{1}{2} \ddot{u} + \frac{1}{2} u^2 \left( \dot{\phi}^2 + e^{2\phi} \dot{\psi}^2 \right) - u^2 U(\Phi^C). \] (31)

The field equations are the Hamiltonian

\[ E = -\frac{1}{2} \ddot{u} + \frac{1}{2} u^2 \left( \dot{\phi}^2 + e^{2\phi} \dot{\psi}^2 \right) + u^2 U(\Phi^C) = 0 \] (32)

and the Euler Lagrange equations

\[ \ddot{u} + \frac{1}{2} \ddot{\phi}^2 + u e^{2\phi} \ddot{\psi}^2 - 2uU = 0 \] (33)

\[ \dot{\phi} - e^{2\phi} \dot{\psi}^2 + U, \phi = 0 \] (34)

\[ \ddot{\psi} + \frac{2}{u} \dot{u} \dot{\psi} + 2 \dot{\phi} \dot{\psi} + e^{-2\phi} U, \psi = 0. \] (35)

In the following we demand the field equations to admit two extra Noether point symmetries which are linearly independent and in involution, so that the dynamical system is Liouville integrable.

Because the Lagrangian (31) describes the motion of a particle in a three dimensional flat space we need to know all potentials \( U(\Phi^C) \) for which the Lagrangian admits the extra two Noether point symmetries. The answer to this problem has been given in \([59]\). Indeed in \([59]\) all 3d dynamical systems (that is potentials) have been determined which admit extra Noether point symmetries. These potentials as well as the corresponding Noether vectors and the subsequent Noether integrals are given for each case in the form of tables. In the following we use these tables to get directly the results we need in our problem.

From the tables of \([59]\) we read that there are only two possible cases that the Noether generators span the \( so(3) \) algebra, one is the case of the unharmonic oscillator and the other is the case of the forced oscillator.

4.1. Case A: The unharmonic oscillator

In this case the potential \( U(\phi, \psi) \) is

\[ u^2 U(\phi, \psi) = \frac{\omega^2_1}{2} S_{(1)}^2 - \frac{\omega^2_2}{2} S_{(2)}^2 - \frac{\omega^2_3}{2} S_{(3)}^2 \] (36)

where \( S_{(I)}^2, \quad I = 1, 2, 3 \) are the gradient KV functions of the flat space. The extra Noether point symmetries are (see Table 6 line 1 of \([59]\) with \( p = 0) \)

\[ T_I (t) K^I, \quad T_2 (t) K^2, \quad T_3 (t) K^3 \]

where

\[ T_{tt}^I = \omega_{IJ} T^J, \quad \omega_{IJ} = \text{diag} \left( (\omega_1)^2, (\omega_2)^2, (\omega_3)^2 \right) \] (37)

with gauge functions \( f_{(I)} = T_{I, t} S_{(I)} \) and corresponding Noether integrals

\[ I_C^I = T_{I, \xi} S_{(I)} - T_{I, t} S_{(I)}. \] (38)
This dynamical system is the 3d ‘unharmonic oscillator’ which is a well known integrable system. We observe that when \( \psi = \psi_0 = \text{const} \) (that is \( \dot{\psi} = 0 \)), the potential reduces to the well known UDM potential \([65, 66]\), i.e. the Lagrangian \((31)\) becomes

\[
L = -\frac{1}{2} \dot{u} + \frac{1}{2} u^2 \dot{\phi}^2 - \frac{u^2}{8} \left( \tilde{\omega}_1^2 e^{2\phi} + \tilde{\omega}_2^2 e^{-2\phi} + \tilde{\omega}_3^2 \right)
\]

(39)

where \( \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3 \) are

\[
\tilde{\omega}_1^2 = \omega_1^2 \left( 1 + \psi_0^2 \right)^2 + \omega_2^2 \left( 1 - \psi_0^2 \right) + 4 \omega_3^2 \psi_0^2
\]

\[
\tilde{\omega}_2^2 = \omega_1^2 + \omega_2^2
\]

\[
\tilde{\omega}_3^2 = 2\omega_1^2 \left( 1 + \psi_0^2 \right) - 2\omega_2^2 \left( 1 - \psi_0^2 \right).
\]

In \([53]\) it has been shown that the Lagrangian \((39)\) describes the two dimensional hyperbolic oscillator.

4.1.1. Normal Coordinates

Under the coordinate transformation

\[
x = \frac{1}{2} \left( e^\phi (1 + \psi^2) + e^{-\phi} \right)
\]

(40)

\[
y = \frac{1}{2} \left( e^\phi (1 - \psi^2) - e^{-\phi} \right)
\]

(41)

\[
z = u\psi e^\phi
\]

(42)

with inverse

\[
u^2 = x^2 - y^2 - z^2, \quad \phi = \ln \frac{x + y}{\sqrt{x^2 - y^2 - z^2}}, \quad \psi = \frac{z}{x + y}
\]

(43)

the Lagrangian \((31)\) becomes

\[
L = -\frac{1}{2} x^2 + \frac{1}{2} y^2 + \frac{1}{2} z^2 - \frac{\omega_1^2}{2} x^2 + \frac{\omega_2^2}{2} y^2 + \frac{\omega_3^2}{2} z^2.
\]

In these coordinates the field equations \((32)-(35)\) and the constraint equation are reduced as follows

\[
\ddot{x} - (\omega_1)^2 x = 0
\]

(44)

\[
\ddot{y} - (\omega_2)^2 y = 0
\]

(45)

\[
\ddot{z} - (\omega_3)^2 z = 0
\]

(46)

\[
0 = -\frac{1}{2} x^2 + \frac{1}{2} y^2 + \frac{1}{2} z^2 - \frac{\omega_1^2}{2} x^2 - \frac{\omega_2^2}{2} y^2 - \frac{\omega_3^2}{2} z^2.
\]

(47)

The analytic solution for the scale factor is found easily to be

\[
a(t) = \left( \frac{3}{8} \right)^{\frac{1}{3}} \left[ B_{IJ} X^I X^J \right]^{\frac{1}{2}}
\]

(48)

where \( B_{IJ} = \text{diag} (1, -1, -1) \) and

\[
X^I = \omega^I_J X^J, \quad \text{where} \quad \omega^I_J = \text{diag} \left( (\omega_1)^2, (\omega_2)^2, (\omega_3)^2 \right) ; \quad I, J = 1, 2, 3
\]

(49)
We note that at late time the scale factor follows an exponential law, i.e.
\[ \omega \]

For instance, if Hamiltonian constraint
\[ a \]

and the scale factor takes the following form
\[ a^3(t) = \frac{3}{8} \left[ \frac{x_1^2}{t^2} \sinh^2 (\omega t + x_2) + \frac{x_2^2}{t^2} \sinh^2 (\omega t + y_2) - \frac{x_2^2}{t^2} \sinh^2 (\omega t + z_2) \right]. \] (51)

We note that at late time the scale factor follows an exponential law, i.e. \( a(t) \propto e^{\omega t} \). Furthermore from the singularity condition \( a(t \to 0) = 0 \) we have the additional constraint equation
\[ x_1^2 \sinh^2 x_2 - x_2^2 \sinh^2 y_2 - x_2^2 \sinh^2 (z_2) = 0. \] (52)

Due to the constraints \( 50 \) and \( 52 \) the free parameters of the model are four.

When \( \omega_{\mu} = \omega_{\nu} \), i.e.
\[ \det \omega_{IJ} = \omega_{iJ}^3, \ \ i \neq J; I, J = 1, 2, 3 \]

the dynamical system admits one extra Noether symmetry. This is the rotation normal to the plane defined by the \( x^1, x^2 \) axes and it is generated by the vector \( X = x^I \partial_I - \varepsilon x^J \partial_J \) where \( \varepsilon = -1 \) if \( x^I/x^J = 1 \) and \( \varepsilon = 1 \) if \( x^I/x^J = -1 \). Finally when \( \det \omega_{IJ} = \omega_{iJ}^3 \), the potential \( V(\phi, \psi) = V_0 \) and the dynamical system is the 3d hyperbolic oscillator (or the free particle if all \( \omega_J = 0 \)) and admits 12 Noether point symmetries (including the \( \partial_\phi \) \[ 59, 68 \]. That means that the Noether point symmetries can also be used in order to reduce the number of free parameters.

### 4.2. Case B: The forced oscillator

In this case the potential is
\[ u^2 U(\phi, \psi) = \frac{\omega_0^2}{2} u^2 + \frac{\mu^2}{2 (1 - a_0)} (S_J + a_0 S_J)^2 - \frac{\omega_3^2}{2} S_K^2 \] (53)

where \( a_0 \neq 1 \); from Table A.1 line 1 of \[ 59 \] we read that the Noether point symmetries are
\[ X_1 = T^I (t) \left( K^I + a_0 K^J \right), \ \ X_2 = T_K (t) K^K \] (54)
\[ X_3 = T^* (t) (a_0 K^I + K^I) \]

where the functions \( T, T^*, T_I \) are the solutions of the system
\[ \begin{align*}
T_{tt} &= (\mu^2 + \omega_0^2) T, \ \ T_{tt} = (\omega_3^2 + \omega_0^2) T_I \\
T_{tt}^* &= \omega_0^2 T.
\end{align*} \] (55)

The gauge functions are,
\[ f_1 = T_\mu \left( S_\mu + a_0 S_\nu \right), \ \ f_2 = T_\nu \left( S_\mu + S_\nu \right) \]
and
\[ f_3 = T^*_tt \left( a_0 S_\mu + S_\nu \right). \]
Hence the corresponding Noether integrals are

\[ I_1 = T \frac{d}{dt} (S(I) + a_0 S(J)) - T, (S(I) + a_0 S(J)) \]  

\[ I_2 = T \frac{d}{dt} S(K) - T, S(K) \]  

\[ I_3 = T \frac{d}{dt} (a_0 S(I) + S(J)) - T, (a_0 S(I) + S(J)) \]  

In order to continue we select \( I = 1, J = 2, K = 3 \).

4.2.1. Normal Coordinates

In case \( I = 1, J = 2, K = 3 \) the potential becomes

\[ u^2 U(\phi, \psi) = \frac{\omega_0^2 u^2}{2} + \frac{\mu^2}{2 (1 - a_0^2)} (S(1) + a_0 S(2))^2 - \frac{\omega_0^2 S^2}{2} \]  

where \( a_0 \neq 1 \). Under the coordinate transformation

\[ x = (w + v), y = \frac{1}{a_0} (w - v), z = z \]  

where the variables \((x, y, z)\) follow from \[40]-\[42\] the Lagrangian becomes

\[ L = T_{NC} - V_{NC} \]  

where \( T_{NC} \) is the kinetic energy in the coordinates \((w, v, z)\)

\[ T_{NC} = \frac{1}{2} \left[ \left( \frac{1}{a_0^2} - 1 \right) \dot{w}^2 - \left( \frac{1}{a_0^2} + 1 \right) dwdv + \left( \frac{1}{a_0^2} - 1 \right) dv^2 + \frac{1}{2} z^2 \right] \]  

and \( V_{NC} \) is the effective potential

\[ V_{NC} = -\frac{2\mu^2}{(a_0^2 - 1)} w^2 - \frac{1}{2} (\omega_3^2 + \omega_0^2) z^2 + \frac{\omega_0^2}{2} \left( (w + v)^2 - \frac{1}{a_0^2} (w - v)^2 \right) \]  

From this Lagrangian the field equations \[32]-\[35\] and the Hamiltonian constraint become

\[ \ddot{w} - (\mu^2 + \omega_0^2) w = 0 \]  

\[ \ddot{v} + \frac{a_0^2 + 1}{a_0^2 - 1} \mu^2 w - \omega_0^2 \nu = 0 \]  

\[ \ddot{z} - (\omega_3^2 + \omega_0^2) z = 0 \]  

\[ T_{NC} + V_{NC} = 0 \]  

This system can be solved easily. For instance in the case where \( \mu^2 \omega_0^2 \neq 0 \) the exact solution is

\[ w(t) = w_1 \exp \left( \sqrt{\mu^2 + \omega_0^2} t \right) + w_2 \exp \left( -\sqrt{\mu^2 + \omega_0^2} t \right) \]  

\[ z(t) = z_1 \exp \left( \sqrt{\omega_3^2 + \omega_0^2} t \right) + z_2 \exp \left( -\sqrt{\omega_3^2 + \omega_0^2} t \right) \]  

\[ v(t) = \frac{1 + a_0^2}{1 - a_0^2} w(t) + v_1 \exp \omega_0 t + v_2 \exp (-\omega_0 t) \]  

with Hamiltonian constraint

\[ -2 (\omega_3^2 + \omega_0^2) z_1 z_2 + \frac{8 (\mu^2 + \omega_0^2)}{1 - a_0^2} w_1 w_2 + \frac{2}{a} (a_0^2 - 1) \omega_0^2 v_1 v_2 = 0. \]  

We note that for this potential the scale factor at late time follows also an exponential law.
4.2.2. Subcase B.1

In this case the potential is
\[ u^2 U (\phi, \psi) = \frac{\omega_0^2}{2} u^2 + \frac{\mu^2}{2} (S_I(t) + S_J(t))^2 + \frac{\omega_3^2}{2} S_K(t). \] (69)

The dynamical system admits the Noether point symmetries
\[ \bar{X}_1 = \bar{T}(t) (K_I + K_J), \bar{X}_2 = T_K(t) K_K. \] (70)

The functions \( \bar{T}(t), T_K(t) \) follow from (54) and the corresponding Noether integrals are given in (56), (57).

In the case where \( I = 1, J = 2 \) and \( K = 3 \) under the coordinate transformation (60) (for \( a_0 = 1 \)) the Lagrangian (31) becomes
\[ L = -2 \dot{w} \dot{v} + \frac{1}{2} \dot{z}^2 - 4\mu^2 w^2 - 2\omega_0^2 w v + \frac{1}{2} \left( \omega_3^2 + \omega_0^2 \right) z^2. \] (74)

and the field equations (52) - (55) and the Hamiltonian constraint become
\[ \dot{w} - \omega_0^2 w = 0 \] (71)
\[ \ddot{v} - 4\mu^2 w - \omega_0^2 v = 0 \] (72)
\[ \ddot{z} - (\omega_3^2 + \omega_0^2) z = 0 \] (73)
\[ 0 = -2 \dot{w} \dot{v} + \frac{1}{2} \dot{z}^2 - 4\mu^2 w^2 + 2\omega_0^2 w v - \frac{1}{2} \left( \omega_3^2 + \omega_0^2 \right) z^2. \] (74)

The analytic solution of this system of equations is
\[ w(t) = w_1 \exp (\omega_0 t) + w_2 \exp (-\omega_0 t) \] (75)
\[ z(t) = z_1 \exp \left( \sqrt{\omega_3^2 + \omega_0^2 t} \right) + z_2 \exp \left( -\sqrt{\omega_3^2 + \omega_0^2 t} \right) \] (76)
\[ v(t) = (2\omega_0 t - v_1) \mu \frac{w_1}{\omega_0} \exp (\omega_0 t) + \] (77)
\[ - (2\omega_0 t + v_2) \mu \frac{w_2}{\omega_0} \exp (-\omega_0 t) \] (78)

with Hamiltonian constraint
\[ 2\mu^2 w_1 w_2 (v_1 + v_2 - 4) + z_1 z_2 \left( \omega_3^2 + \omega_0^2 \right) = 0. \]

As it is the case with case A for special values of the parameters \( \omega_0, \mu, \omega_3 \) it is possible that the dynamical system admits more Noether point symmetries, which are produced by the elements of \( SO(3) \). However this adds nothing to the integrability of the system and there is no point to consider these cases further.

From (75) - (78) and (60) we have that at late time the scale factor has the following functional form
\[ a(t) = \left( 3 \frac{\omega_3^2}{\omega_0} \mu^2 \right)^{1/3} t^{1/3} e^{\frac{2}{3} \omega_0 t}. \] (79)

We can reconstruct this solution for the scale factor by selecting \((w_2, z_2, \omega_3) = 0\) in (59) - (57) and by applying the singularity condition \( a(t \to 0) = 0 \). Moreover for the scale factor (79) the Hubble function takes the form
\[ H(t) = \frac{2}{3} \omega_0 + \frac{1}{3} t. \] (80)
This solution is of interest because it is also the late time behavior of the scale factor for values \((w_2, z_2, \omega_3) \neq 0\). However solution \([79]\) identifies the second scalar field to be a constant, i.e. \(\psi(t) = \psi_0\).

In order to study the late time behaviour of \([79]\) we write the Hubble function \([80]\) in terms of the redshift \(z\), \(a_0 = 1 + z\) where \(a_0\) is the renormalized parameter so that \(a(t_{\text{today}}) = 1\). From equation \([79]\) it follows that the parameter \(t\) in terms of the scale factor is expressed as follows

\[
t = \frac{1}{2\omega_0} W\left(\frac{2\omega_0^2}{3 w_f^2 \mu^2} a^3\right)
\]

where \(W(x)\) is the Lambert-W function. Then the Hubble function \([81]\) becomes

\[
H(z) = \frac{2}{3} \omega_0 \left(1 + \left(W\left(\frac{c}{(1+z)^\mu}\right)\right)^{-1}\right)
\]

where \(c = \frac{2 a_0 \omega_0^2}{3 w_f^2 \mu^2}, \omega_0 = \tilde{\omega} H_0.\) \(H_0\) is the Hubble constant for the present time \(H(0) = H_0\), hence from \([81]\) we have the constrain condition

\[
c = \frac{2\tilde{\omega}_0}{3 - 2\tilde{\omega}_0} \exp\left(\frac{2\tilde{\omega}_0}{3 - 2\tilde{\omega}_0}\right).
\]

Finally the free parameters of the model are the \(\tilde{\omega}_0\) and the Hubble constant \(H_0\). We consider the Taylor expansion of the Hubble function near the present time, i.e. \(z = 0\)

\[
H(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + O(z^4)
\]

Using \([81]\) we find that the constants \(a_0, a_1, a_2\) and \(a_3\) are

\[
a_0 = \frac{2}{3} \tilde{\omega}_0 H_0 \left(1 + \frac{1}{W_c}\right), \quad a_1 = \frac{2\tilde{\omega}_0 H_0}{(1 + W_c) W_c}
\]

\[
a_2 = \tilde{\omega}_0 H_0 \frac{4 W_c + 2 - W_c^2}{(1 + W_c)^3 W_c}
\]

\[
a_3 = \tilde{\omega}_0 H_0 \frac{2W_c^4 - 10W_c^3 + 21W_c^2 + 8W_c + 2}{3 (1 + W_c)^5 W_c}
\]

and \(W_c = W(c)\). Therefore the Hubble function \([82]\) can be written in the form

\[
H(z) = H_0 \sqrt{\Omega_m (1 + z)^3 + (1 - \Omega_m) f_{DE}(z, \tilde{\omega}_0)}
\]

where we have defined \(\Omega_m = \frac{8\tilde{\omega}_0}{2\tilde{\omega}_0} \frac{(W_c^3 - 6W_c^2 + 12W_c + 10)}{(1 + W_c)^3 W_c}\) to be the density at the present time of the dust like fluid (dark matter) and the function \(f_{DE}(z, \tilde{\omega}_0)\) describes the effective dark energy fluid.

### 4.2.3. Cosmological Constrains of the late time solution

We proceed with a joint likelihood analysis of the cosmological solution \([79]\) by using the Type Ia supernova data set of Union 2.1 \([69]\) and the 6dF, the SDSS and WiggleZ BAO data \([70, 71]\). The likelihood function is defined as follows

\[
\mathcal{L}(p) = \mathcal{L}_{SNIa} \times \mathcal{L}_{BAO}
\]

where \(p\) are the constrain parameters and \(\mathcal{L}_A \propto e^{-\chi^2_A/2}\); that is, \(\chi^2 = \chi^2_{SNIa} + \chi^2_{BAO}\). The Union 2.1 data set provide us with 580 SNIa distance modulus at observed redshift. The chi-square parameter is given by the expression\(^2\)

\[
\chi^2_{SNIa} = \sum_{i=1}^{N_{SNIa}} \left(\frac{\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i; p)}{\sigma_i}\right)^2
\]

\(^2\) For the SNIa test we have applied the diagonal covariant matrix without the systematic errors
where $z_i$ is the observed redshift in the range $0.015 \leq z_i \leq 1.414$ and $\mu$ is the distance modulus

$$\mu = m - M = 5 \log D_L + 25$$

(86)

and $D_L$ is the luminosity distance. For the constraint with the BAO data the corresponding chi-square parameter is defined as follows

$$\chi^2_{BAO} = \sum_{i=1}^{N_{BAO}} \sum_{j=1}^{N_{BAO}} \left[ d_{\text{obs}} (z_i) - d_{\text{th}} (z_i; p) \right] C^{-1}_{ij} \left[ d_{\text{obs}} (z_j) - d_{\text{th}} (z_j; p) \right]$$

(87)

where $N_{BAO} = 6$, $C^{-1}_{ij}$ is the inverse of the covariant matrix in terms of $d_z$ (see [72]), and the parameter $d_z$ follows from the relation $d_z = \frac{l_{\text{BAO}}}{D_V(z)}$, $l_{\text{BAO}} (z_{\text{drag}})$ is the BAO scale at the drag redshift and $D_V(z)$ is the volume distance [71].

For the Hubble constant we consider the value $H_0 = 69.6 \text{ km s}^{-1}\text{Mpc}^{-1}$ (see [73]) hence the free parameter of the Hubble function [83] is $p = \Omega_{m0}$. Therefore, we find that the best fit value $\bar{\omega}_{m0}$ of the model [83] we derived is $\Omega_{m0} = 0.31_{-0.024}^{+0.025}$, with $\chi^2_{\text{total}} = 564.8$; the corresponding value of the constant $\bar{\omega}_0$ is $\bar{\omega}_0 = 0.925$. When $f_{DE}(z, \bar{\omega}_0) = 1$ in [83] the Hubble function [83] reduces to that of the $\Lambda$–cosmology. Therefore, by constraining the $\Lambda$–cosmology with the SNIa and the BAO data we find the minimum chi-square parameter $\min \Delta \chi^2_{\text{total}} = 564.5$ with matter density $\Omega_{m0} = 0.28_{-0.015}^{+0.017}$. We note that the difference between the minimum chi-square parameters is $\min (\chi^2_{\text{total}} - \Lambda \chi^2_{\text{total}}) = 0.3$ which leads to the conclusion that both the model [83] and the $\Lambda$–cosmology model fit the SNIa and the BAO data with similar statistic parameters. This proves the viability of the solution we have found.

5. THE CASE OF $N$ INTERACTING SCALAR FIELDS

As we have seen in section 4.1 the UDM model [63, 66] at the level of Noether symmetries (but also as a dynamical system) is equivalent to the unharmonic oscillator [53]. In the case of two scalar fields this happens if the fields interact in their kinematic part so that $2d$ the interaction metric $H_{AB}$ admits $so(3)$ as the Killing algebra. In this section, we consider the case of $N$ scalar fields which interact in their kinematic and potential parts with action [74]

$$S = \int dx^4 \sqrt{-g} \left( R - \frac{1}{2} g_{ij} G_{a\beta} \Phi^{a,i} \Phi^{\beta,i} + V (\Phi^i) \right)$$

(88)

where $G_{a\beta}$ is a second order symmetric tensor and $a, \beta = 1, 2, \ldots, N$. We consider again $G_{a\beta}$ as a metric in the space spanned by the $N$ scalar fields. If we assume that $G_{a\beta}$ admits the algebra $so(N + 1)$ as KVs, then $G_{ab}$ is the metric of a space of constant curvature and the fundamental length $ds^2_G = G_{ab} d\Phi^a d\Phi^b$ can be written as follows

$$ds^2_G = d\Phi_1^2 + e^{2\Phi_1} \left[ d\Phi_2^2 + d\Phi_3^2 + \ldots + d\Phi_{N-1}^2 \right].$$

(89)

Assuming that the interaction takes place in the FRW spatially flat spacetime [2] the Lagrangian [88] is

$$L \left( a, \dot{a}, \Phi^i, \dot{\Phi}^i \right) = -3a \ddot{a} + \frac{1}{2} a^3 G_{a\beta} \ddot{\Phi}^{a,i} \Phi^{\beta,i} - a^3 V (\Phi^i).$$

(90)

We introduce the new variable $u$ (see [28]) and the Lagrangian [89] becomes

$$L \left( u, \dot{u}, \Phi^i, \dot{\Phi}^i \right) = -\frac{1}{2} u^2 + \frac{1}{2} u^2 G_{a\beta} \ddot{\Phi}^{a,i} \Phi^{\beta,i} - u^2 V (\Phi^i)$$

in which we have introduced the effective potential

$$u^2 V (\Phi) = \frac{1}{2} A_{IJ} S^I S^J, \ J = 1..N + 1$$

(91)
where $A_{IJ} = -\text{diag} \left((\omega_1)^2, (\omega_2)^2, ..., (\omega_N)^2, - (\omega_{N+1})^2\right)$ and

$$S_{N+1} = \frac{1}{2} u \left( e^{\Phi_1} \left( 1 + \Phi_2^2 + \Phi_3^2 + ... + \Phi_{N-1}^2 \right) + e^{-\Phi_1} \right)$$

$$S_1 = \frac{1}{2} u \left( e^{\Phi_1} \left( 1 - \left( \Phi_2^2 + \Phi_3^2 + ... + \Phi_{N-1}^2 \right) \right) - e^{-\Phi_1} \right)$$

$$S_2 = u e^{\Phi_2}$$

$$S_3 = u e^{\Phi_1}$$

... 

$$S_N = u e^{\Phi_1} \Phi_N.$$

where $S_{N+1}$ are the gradient KVs of the flat space. Under the coordinate transformation $Z_J = S_J$ the Lagrangian becomes

$$L \left( Z^I, \dot{Z}^I \right) = -\frac{1}{2} \eta_{IJ} \dot{Z}^I \dot{Z}^J - \frac{1}{2} A_{IJ} Z^I Z^J$$

(92)

where $\eta_{IJ} = \text{diag} (1, 1, ..., 1, -1)$. Therefore the exact solution of the scale factor is

$$a(t) = \left( \frac{3}{8} \right)^{\frac{1}{3}} (\eta_{IJ} Z^I Z^J)^{\frac{1}{3}}$$

(93)

where $Z^I(t)$ satisfies

$$\ddot{Z}^I - \eta^{IJ} A_{JK} Z^K = 0$$

(94)

and

$$\frac{1}{2} \eta_{IJ} \ddot{Z}^I \ddot{Z}^J - \frac{1}{2} A_{IJ} Z^I Z^J = 0.$$  

(95)

Note that equations (94) describe the $(N + 1)$ anisotropic oscillator. Another attempt to apply the Noether point symmetries in the case of $N$ scalar fields can be found in [36]. However in [36] the authors consider that the metric $G_{\alpha\beta}$ is invariant under the $E(N)$ Lie algebra; that is, $G_{\alpha\beta}$ is a flat space, and the scalar fields have the same potentials and the same initial conditions, therefore the problem reduces to the one scalar field cosmology.

6. CONFORMAL EQUIVALENCE

In this section we study the interaction of two scalar fields under a conformal transformation and we show how the cases of section 4 follow from the conformal equivalence in scalar tensor theory.

We assume that we have a model consisting of one non-minimally coupled scalar field $\Phi$ and one minimally coupled scalar field $\psi$. Then the action is

$$S = \int dx^4 \sqrt{-g} \left[ F(\Phi) R + \frac{1}{2} g^{ij} \Phi_i \Phi_j - \frac{1}{2} g_{ij} \psi_i \psi_j - V(\Phi, \psi) \right].$$

(96)

Under the conformal transformation $\bar{g}_{ij} = N^2 g_{ij}$ where $N = \frac{1}{\sqrt{-2F(\Phi)}}$ the action becomes (see [54, 67, 73])

$$S = \int dx^4 \sqrt{-\bar{g}} \left[ \frac{R}{2} + \frac{1}{2} \left( \frac{3 F_{\Phi}^2 - F}{2 F^2} \right) g_{ij} \Phi^i \Phi^j + \frac{1}{2} \frac{1}{4 F(\Phi)} g_{ij} \psi^i \psi^j - V(\Phi, \psi) \right]$$

(97)

where we have set

$$V(\Phi, \psi) = \frac{\bar{V}(\Phi, \psi)}{4 F^2}.$$  

(98)

If we consider the transformation $\Phi \rightarrow \phi$ by the formula

$$d\phi = \sqrt{\left( \frac{3 F_{\Phi}^2 - F}{2 F^2} \right)} d\Phi$$

(99)
the action \( S \) takes the form:

\[
S = \int dx^4 \sqrt{-g} \left[ -\frac{\mathcal{R}}{2} + \frac{1}{2} g_{ij} \phi^i \phi^j + \frac{1}{8} \frac{1}{F(\phi)} g_{ij} \psi^i \psi^j - V(\phi, \psi) \right]
\]  

(100)

which shows that the two scalar fields \( \phi, \psi \) interact in their kinematic part with the metric \( g_{ij} = -2F(\phi) \bar{g}_{ij} \).

To give an example of the above equivalence let us consider the simple case where \( F(\Phi) = f_0 \Phi^2, \ f_0 \neq \frac{1}{12} \) (see [67]). Then from [90] we find \( \phi = C \ln \Phi \) where \( C = \sqrt{\frac{2f_0}{\sqrt{2}f_0}} \). Replacing we find that the action becomes

\[
S = \int dx^4 \sqrt{-\bar{g}} \left[ -\frac{\mathcal{R}}{2} + \frac{1}{2} \bar{g}_{ij} \phi^i \phi^j + \frac{1}{8} e^{-C\phi} \bar{g}_{ij} \psi^i \psi^j - V(\phi, \psi) \right]
\]  

(101)

which implies that the two scalar fields \( \phi, \psi \) interact with the two dimensional metric 77

\[
ds^2_{(2)} = d\phi^2 + \frac{1}{4} e^{-C\phi} d\psi^2.
\]  

(102)

In the case where \( C = 2 \), i.e. \( f_0 = -\frac{1}{6} \), the \( ds^2_{(2)} \) is a space of constant curvature, and then the action (101) is the one we considered in section 4.

We note that by replacing \( F(\Phi) = -\frac{1}{6} \Phi^2 \) and \( \Phi = \sqrt{6\zeta} \) in (99) the action becomes

\[
S = -\int dx^4 \sqrt{-g} \left[ \zeta \mathcal{R} - \frac{3}{\zeta} \bar{g}_{ij} \zeta^i \zeta^j + \frac{1}{2} \bar{g}_{ij} \psi^i \psi^j + \mathcal{V}(\zeta, \psi) \right].
\]  

(103)

and \( \zeta \) is a Brans-Dicke scalar field.

7. CONCLUSION

We have considered two scalar fields interacting both in their kinematic and potential parts in a spatially flat FRW spacetime and determined those interactions for which the dynamical system of the two scalar fields is Liouville integrable. The system has three variables therefore for this to be the case two more first integrals are required (in addition to the Hamiltonian). One systematic way to find these integrals is to assure that the Lagrangian admits an integrable. The system has three variables therefore for this to be the case two more first integrals are required (in addition to the Hamiltonian).

In order to examine the viability of the solution we considered the late time behavior of the scale factor and found that the scalar field introduces a dust like component in the Hubble function. We performed a joint likelihood analysis with the Supernova data of Union 2.1 and the 6dF, SDSS and WiggleZ BAO data, and we found that the model fits the cosmological data with a minimum \( \chi^2_{\text{total}} = 564.8 \) and today’s value of the dark energy density \( \Omega_{m0} = 0.31^{+0.023}_{-0.024} \). Comparing these with the corresponding values of the \( \Lambda \)-cosmology model we find that the difference between the two statistical parameters \( \chi^2_{\text{total}} \) of the two models is \( \Delta \chi^2_{\text{total}} = 0.3 \). This implies that the analytic solution we have obtained mimics the \( \Lambda \)-cosmology at late time.

We generalized these considerations to the case of \( N \) scalar fields interacting both in their potential as well as in their kinematic part in a flat FRW background and we computed again the analytic form of the scale factor. Finally, we have shown that this type of interaction also follows from a conformal transformation in the Brans-Dicke action.

Concluding we remark that it is possible to extend the symmetry method for the action in order to determine invariant solutions of the Wheeler-DeWitt equation in quantum cosmology. Such an analysis is in progress and it will be published in a forthcoming paper.
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Appendix A: Representations of the \( so(3) \) Lie algebra

The form of the metric \( h_{AB} \) we considered in section 4 corresponds to the representation of the \( so(3) \) Lie algebra with elements \([29,30]\).

However it is possible to consider a different representation of the \( so(3) \) Lie algebra, therefore another form of the metric \( h_{AB} \). For instance another form of \( h_{AB} \) is

\[
d\bar{s}^2 = d\tilde{\phi}^2 + \sinh^2 \tilde{\phi} d\tilde{\psi}^2. \tag{A1}
\]

Obviously the two representations are related with a coordinate transformation \((\phi, \psi) \rightarrow (\bar{\phi}, \bar{\psi})\) therefore our results remain true for the new representation. In order to show this we consider the Lagrangian in which \( h_{AB} \) is of the form \((A1)\). The Lagrangian is:

\[
L = -\frac{1}{2} \dot{u}^2 + \frac{1}{2} u^2 \left( \tilde{\phi}'^2 + \sinh^2 \tilde{\phi} \tilde{\psi}'^2 \right) - u^2 U (\tilde{\Phi}^C). \tag{A2}
\]

For this Lagrangian the field equations and the Klein Gordon equations are

\[
-\frac{1}{2} \ddot{u} + \frac{1}{2} u^2 \left( \tilde{\phi}'^2 + \sinh^2 \tilde{\phi} \tilde{\psi}'^2 \right) - u^2 U (\tilde{\Phi}^C) = 0 \tag{A3}
\]

\[
\ddot{\phi} + 2 \frac{\dot{u}}{u} \tilde{\phi}' + \sinh \tilde{\phi} \cosh \tilde{\phi} \tilde{\psi}'^2 + U,\tilde{\phi} = 0 \tag{A4}
\]

\[
\ddot{\psi} + \frac{2}{u} \dot{u} \tilde{\psi} + 2 \coth \tilde{\phi} \tilde{\phi}' \tilde{\psi}' + \sinh^{-2} \tilde{\phi} U,\tilde{\psi} = 0. \tag{A5}
\]

The kinetic metric of \((A2)\) is

\[
dS^2_{(3)} = -du^2 + u^2 \left[ d\tilde{\phi}^2 + \sinh^2 \tilde{\phi} d\tilde{\psi}^2 \right]
\]

and the corresponding gradient KVs are

\[
\tilde{S}_{(1)} = u \cos \tilde{\psi} \sinh \tilde{\phi},
\tilde{S}_{(2)} = u \sin \tilde{\psi} \sinh \tilde{\phi},
\tilde{S}_{(3)} = u \cosh \tilde{\phi}.
\]

By replacing the functions \( \tilde{S}_{(1-3)} \) instead of \( S_{(1-3)} \) in the potentials of section 4 we find the same exact solutions for the scale factor. Working similarly we show that the result holds for the case of the \( N \) scalar fields of section 5.

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