Conical Perspective Image of an Architectural Object Close to Human Perception

Jolanta Dzwierzynska 1

1 Department of Architectural Design and Engineering Graphics, Rzeszow University of Technology, Poznanska 2, 35-084 Rzeszow, Poland
joladz@prz.edu.pl

Abstract. The aim of the study is to develop a method of computer aided constructing conical perspective of an architectural object, which is close to human perception. The conical perspective considered in the paper is a central projection onto a projection surface being a conical rotary surface or a fragment of it. Whereas, the centre of projection is a stationary point or a point moving on a circular path. The graphical mapping results of the perspective representation is realized directly on an unrolled flat projection surface. The projective relation between a range of points on a line and the perspective image of the same range of points received on a cylindrical projection surface permitted to derive formulas for drawing perspective. Next, the analytical algorithms for drawing perspective image of a straight line passing through any two points were formulated. It enabled drawing a perspective wireframe image of a given 3D object. The use of the moving view point as well as the application of the changeable base elements of perspective as the variables in the algorithms enable drawing conical perspective from different viewing positions. Due to this fact, the perspective drawing method is universal. The algorithms are formulated and tested in Mathcad Professional software, but can be implemented in AutoCAD and majority of computer graphical packages, which makes drawing a perspective image more efficient and easier. The presented conical perspective representation, and the convenient method of its mapping directly on the flat unrolled surface can find application for numerous advertisement and art presentations.

1. Introduction

The concept of "conical perspective" was introduced by different research in various ways depending on what the research is related to.

Conical projection, that is a central projection onto a conical surface is a commonly used map projection method [1]. Hundreds of map projections have been created so far and a lot of methods for drawing the round Earth’s surface on a flat surface have been proposed [2], [3]. Although none of them is perfectly accurate, the central projections onto a conical surface are relatively simple to generate maps with good accuracy [4].

The terms “wide conical perspective” and “conical panorama” were introduced with development of digital panoramic photography and digital panorama creation methods [5,6,7]. Conical perspective as a method of presenting architectural environment is hardly mentioned in the traditional textbooks on geometry. It only receives some attention as a kind of cartographic projection [8].

Due to the fact that every conical perspective representation is a kind of geometric projection, it is proposed an analysis of conical perspective from a geometrical point of view. The idea of perspective creation from the centre moving on a circular path onto a conical rotational surface or on a fragment of
it is developed. Such an approach was presented in the case of the classical panorama onto a cylindrical surface in [9] and in the case of an inverse cylindrical panorama where the centres of projection were dispersed on a circle or a line in [10]. Additionally, in the paper it is proposed graphical mapping of the results of the conical perspective representation with computer aid, directly on the unrolled conical projection surface.

2. Geometrical background of conical perspective

The projection apparatus of conical perspective is defined similarly to a projection apparatus of linear perspective onto a flat projection plane. However, in the case of conical perspective, the projection surface $\tau$ is a rotational conical surface or a fragment of this surface. Due to this fact, the apparatus of the considered conical perspective is composed of the rotational conical surface $\tau$, a centre of projection $S/S_F$ and a so called base plane $\pi$ perpendicular to the axis $l$ of the projection surface $\tau$. The centre of projection can be a single stationary point $S$ belonging to the axis $l$ or any point moving on the circular path $s$ included in a horizon plane $\chi$ (figure 1a, b).

![Figure 1. The structure of the apparatus of conical perspective: a- from a stationary centre, b- from a moving centre](image)

Similarly as in the case of vertical perspective onto a flat projection plane, the given point $F$ is projected and additionally the orthogonal projection $F^O$ of this point $F$ received on the base plane $\pi$. Due to this fact, the image of any point $F$ in conical perspective is a pair of two projections ($F^S$, $F^{OS}$); a main one and an auxiliary one. The main projection $F^S$ is the central projection onto a rotational cylindrical surface $\tau$ from the stationary centre $S \in l$ or the centre $S_F$ moving on the circular path $s$ included in a horizon plane $\chi$. The auxiliary projection $F^{OS}$ is a main projection of $F^O$ onto $\tau$, where $F^O$ is the projection of $F$ from the centre $O_\infty \in l$ onto the projection plane $\pi \perp l$ (figure 2a, 2b). Both projections are contained in one ruling $t_\nu$ of the surface $\tau$. A moving centre $S_F$ of perspective projection is attributed to the given real point $F$ cutting the circle $s$ by half-plane $\lambda$ as it is shown in the figure 2b.

In the conical perspective projection, like in the case of the projection onto a flat projection plane, we represent all points which are situated behind the projection surface $\tau$ that is all points located on the other side of the projection surface than the centre of projection and the points which are situated above or in the base plane $\pi$, as well as the points included in $\tau$. When the projection surface $\tau$ is a full conical surface we obtain an apparatus of conical panoramic projection. In the case of conical
perspective projection the structure of the projection apparatus can be various dependently on the location of the surface’s vertex \(W\) towards the base plane \(\pi\). If the surface’s vertex \(W\) is above \(\pi\) it is a variant A of the apparatus, if the vertex \(W\) is below the base plane \(\pi\) it is variant B.

![Figure 2. Conical perspective of the real point \(F\): a- from a stationary centre \(S\), b- from a moving centre \(S_F\)](image)

3. Mapping conical perspective directly on an unrolled projection surface

Due to the fact that each conical surface is a developable surface, the images of the conical perspective representation are presented on a flat unrolled surface. In order to do that, it is necessary to transform the images contained in the projection surface \(\tau\) into their counterparts included in the unrolled surface \(\tau^R\). Such aim can be achieved by projecting conical surface \(\tau\) from the centre \(S/S_F\) onto the base plane \(\pi\) and establishing projective relations between points on rulings of this degenerate flat surface obtained as a result of projection and their counterparts on rulings contained in unrolled surface. We use similar projection to develop analytical algorithms for drawing conical perspective from a stationary and moving centre with computer aid. Such an approach is presented in [11], where the graphical construction of a conical panorama from a single view point is shown.

3.1. Establishing equations displaying geometrical relations occurring during projection

3.1.1. Conical perspective of version A. Let us consider a conical perspective projection \(^S_tF\) of a ruling \(t_F\) from a centre \(S_F \in s\) onto a base plane \(\pi\) (figure 3).

Four characteristic points included in the ruling \(t_F\): \(W_F, P_F, F^{0S}, H_F\) are considered. A point \(P_F\) is included in the base circle \(\rho\), a point \(H_F\) is included in the horizon circle \(h\) and the point \(W_F = W\) is a vertex of the projection surface. After this projection four points: \(^S\hat{W}_F, ^S\hat{P}_F, ^S\hat{F}^{0S}, ^S\hat{H}\) are received, which are included in the line \(^S_tF\).

These series of points included in \(t_F\) and the series of points included in \(^S_tF\) are homologous. Due to this fact, the cross ratio of the quadruple of these points is invariant in this projection, which can be shown by the equation below [12]:
\[
\frac{s_F^{-OS}P_F}{s_W^{-OS}P_F} = \frac{s_F^{-OS}H_F}{s_W^{-OS}H_F} = \frac{F^{-OS}P_F}{W_F} = \frac{F^{-OS}H_F}{W_F}
\]  

(1)

Also the series of points: \(S_F, S_H, S_W, S^F, S^O\) on the ruling \(s_t F\) and the series of points: \(P^R, H^R, W^R, F^O, S^R\) on the ruling \(t^R\) of the unfolded surface \(\tau^R\) are related by the projective transformation, see figure 4.

Figure 3. Central projection of the ruling \(t_F\) with its characteristic points onto \(\pi\) in order to realize transformation

Figure 4. Graphical connection between points on the ruling \(s_t F\) and points on the ruling \(t^R\) of the enveloped surface \(\tau^R\).
Due to this fact, the cross ratio of the quadruple of these points is preserved during transformation, which can be expressed by the equation below:

\[
\frac{P^R_F F^O S^R}{W^R_F F^O S^R} = \frac{P^R_F H^R_F}{W^R_F H^R_F} = \frac{s P^S_F F^O S^S}{s W^S_F F^O S^S} = \frac{s P^S_F H^S_F}{s W^S_F H^S_F}
\]  

(2)

Determining (figure 4):

- the radius of base circle by \( r \)
- the distance of the point \( W_\ell \) from the centre of the base circle \( p \) by \( r_w \)
- the distance of the point \( F^O, S^O \) from the centre of the base circle \( p \) by \( k \),
- the distance of the point \( W^R_F \) by \( d_o \),
- the distance of the point \( H^R_F \) from the point \( P^R_F \) by \( h_n \),
- the distance of the point \( P^R_F \) from the point \( W^R_F \) by \( t \)

and according to figure 6 equations (2) looks as follows:

\[
\frac{t - d_o}{d_o} = \frac{t - h_n}{h_n} = \frac{k - r}{k - r_w};
\]

\[
d_o = \frac{t \cdot (t - h_n) \cdot (k - r_w)}{h_n \cdot (r_w - r) + t \cdot (k - r_w)}
\]  

(3)

Let us consider a vertical straight line \( m \) going through point \( F \) with a series of characteristic points \( M^\infty, H, F, F^O \) included in it (figure 5).

Figure 5. Graphical relation between points: \( M^\infty, H, F, F^O \) included in the vertical line \( m \) and their projections: \( M^\infty, H_F, F^S, F^O_S \) onto \( \tau \)
According to figure 5 we can state that this series of points is homologous with an appropriate series of points \( M_\infty, H_F, F^5, F^{O_S} \) included in the ruling \( t_F \), which can be expressed in a following way:

\[
\frac{F^0 H}{FH} \cdot \frac{F^0 M_\infty}{FM_\infty} = \frac{F^{O_S} H_F}{F^5 H_F} \cdot \frac{F^{O_S} N}{F^5 N}
\]

Determining:

- the height of horizon \( h \)
- the radius of the circle of viewpoints \( r_s \)
- the distance of the point \( F^5 \) from the point \( W \) by \( d \),
- the distance of the point \( P_F \) from the point \( W \) by \( t \),
- the distance of the point \( N \) from the point \( W \) by \( n \),

and according to figure 7 equation (4) looks as follows:

\[
d = \frac{h \cdot (d_0 - n) \cdot (t - h_i) + n \cdot (h - w) \cdot (-h_i + t - d_0)}{h \cdot (d_0 - n) + (h - w) \cdot (t - d_0 - h_i)}
\]

3.1.2. Conical perspective of version B.

Analogously to the case of conical perspective of version A, we project the ruling \( t_F \) from a centre \( S \in \Sigma \) onto a base plane \( \pi \) and establish a geometrical relation between the proper points of this ruling during projection.

Figure 6. Central projection of the ruling \( t_F \) with its characteristic points onto \( \pi \) in order to realize transformation

Due to the fact that cross ratio of the quadruple of the series of points: \( P_F, H_F, W, F^{O_S} \) is invariant during projection, the equation (1) can be applied in this case too.
Similarly to the case of the version A, the points: $S_{PF}$, $S_{HF}$, $S_{WF}$, $S_{FO}$, $S_{OF}$ on the ruling $t^R_F$ and the points: $P^R_{FR}$, $H^R_{FR}$, $W^R_{FR}$, $F^O_{OS}$ on the ruling $t^R_F$ on the developed surface $\tau^R$ are related by the projective transformation, see figure 7.

Figure 7. Graphical connection between points on the ruling $S_{tF}$ and points on the ruling $t^R_F$ of the developed surface $\tau^R$

Therefore, determining:

- the radius of base circle by $r_p$,
- the distance of the point $S_{WF}$ from the center of the base circle $p$ by $r_w$,
- the distance of the point $S_{FO}$ from the center of the base circle $p$ by $k$,
- the distance of the point $F^O_{OS}$ from the point $W^R_{FR}$ by $d_0$,
- the distance of the point $H^R_{FR}$ from the point $P^R_{FR}$ by $h_o$,
- the distance of the point $W^R_{FR}$ from the point $P^R_{FR}$ by $e_o$,

the equation (2) looks as follows:

$$d_0 = \frac{-e_o \cdot (h_o + e_o) \cdot (k - r_w)}{h_o (k - r_p) - (h_o + e_o) \cdot (k - r_w)}$$

(6)

In turn, perspective relations between the series of points on the vertical line $m$ and appropriate points on the ruling $t_F$ permit derivation of the following equation:

$$d = \frac{h \cdot (h_o + e_o) \cdot (d_0 - n) + n \cdot (h - w) \cdot (h_o + e_o - d_0)}{(h - w) \cdot (h_o + e_o - d_0) + h \cdot (d_0 - n)}$$

(7)

, where $n$ it is the distance of the point $N$ from $W$ (figure 8).
3.2. Developing algorithms for drawing conical perspective of a line

3.2.1. Methods and methodology

The hereby considered perspective representation is a nonlinear projection. Due to this fact, it cannot be expressed by any linear transformation [13], [14], however, equations (4), (5), (6), (7) define the position of the image \( F^s, F^{os} \) of any point \( F \) on the unrolled surface, which enables developing algorithms for drawing a perspective projection of any straight line with computer aid. We develop the algorithms and check their correctness in Mathcad Professional software.

For this purpose, we place a spatial right-hand Cartesian coordinate system of axes \( x, y, z \) in such a way that \( x \) and \( y \) are included in the base plane \( \Pi \), the origin of the system is a centre of the base circle and the axis \( z \) is of different direction than the axis \( l \) of the surface \( \tau \). Next, we derive analytical algorithms in Mathcad Professional software for drawing a conical perspective projection of a straight line passing through any two points determined by their Cartesian coordinates. Therefore, the projection of the line is drawn as a plot of function \( d(\Phi) \) in polar coordinate system of axes directly on the unrolled conical surface \( \tau^R \). The vertex \( W^R \) contained in it is chosen as a pole, whereas the border ruling \( t^R \) is taken as a polar axis (figure 4, 7).

The capability of drawing lines in conical perspective enables drawing edge models of spatial figures.

3.2.2. Some examples of applications of the algorithms. Let us show some examples of the application of the algorithms for drawing conical perspective of the simple forms. In the figure 9 we show location of the objects towards the representation apparatus in both A and B cases.
We establish the base elements of perspective, that is: the height of horizon $h$, the radius of circle of viewpoints $r_S$, as well as the parameters determining the projection surface: the radius of base circle $r$, the distance of the vertex from the base plane $l$ or $e$ as shown in the figure 10. They are defined in such a way that each object can be included in a cone of good vision during the cone’s rotation around the axis $l$ (figure 10).

In the figure 11 we show the result of conical perspective projection from a moving viewpoint of the considered objects. The perspective image has been generated in Mathcad Professional software for both A and B versions of the projection apparatus.
In the case of conical perspective from a single stationary view point we apply the same algorithms as in the case of the conical perspective creation from the moving viewpoint. However, in order to draw conical perspective from the stationary point we assume that the radius of the circle of viewpoints equals zero. In the figure 12b and figure 13b we present conical perspective of a single building from a stationary view point which position towards the building is shown respectively in figure 12a and figure 13a.
It is worth noting that applying version A of the apparatus we can achieve a perspective image from frog’s eye view, whereas applying version B we can obtain a bird eye perspective.

4. Results and discussion
We attempted to elaborate on the method of conical perspective mapping in which the results of conical projection are close to human perception.

In general, the image of any architectural form depends on the structure of the representation apparatus, as well as on the location of the figure towards it. As far as the structure is concerned we can apply two versions of the apparatus - A and B. The projection surface can be assumed as a fragment of the conical surface or a full surface, whereas the centre of projection can move or not. Owing to the application in our algorithms of the changeable base elements of perspective such as: the radius of the base circle $r$, the radius of the circle of viewpoints $r_s$, the height of horizon $h$; we can achieve conical perspective images of a given object from a variety of viewing positions, as well as on a conical projection surface determined by different metric characteristics. Moreover, due to the fact that straight lines appear very often in any architectural form, we use information about them and develop algorithms for drawing them. Curved lines, in order to be represented, should be approximated by some segments of straight lines.

5. Conclusions
In this paper, a new geometrical approach to conical perspective construction from a stationary and moving viewpoint is presented. The use of a moving centre of the perspective representation aims at a better approximation of the real images experienced while watching the objects by the obtained projections. Mapping of the results of the perspective representation is realized directly on a flat unrolled surface with computer aid. The analytical algorithms with changeable base elements of perspective proposed in the paper enable drawing conical perspective images from different viewing positions. The algorithms elaborated and tested in Mathcad Professional software can be implemented in the majority of graphical packages, which can make drawing perspective efficient and convenient.

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