PHASE TRANSITIONS IN EDGE-WEIGHTED EXPONENTIAL RANDOM GRAPHS

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Abstract. The exponential family of random graphs represents an important and challenging class of network models. Despite their flexibility, conventionally used exponential random graphs have one shortcoming. They cannot directly model weighted networks as the underlying probability space consists of simple graphs only. Since many substantively important networks are weighted, this limitation is especially problematic. We extend the existing exponential framework by proposing a generic common distribution for the edge weights and rigorously analyze the associated phase transitions and critical phenomena. We then apply these general results to get concrete answers in exponential random graph models where the edge weights are uniformly distributed.

1. Introduction

Exponential random graphs are among the most widespread models for real-world networks as they are able to capture a broad variety of common network tendencies by representing a complex global structure through a set of tractable local features. These rather general models are exponential families of probability distributions over graphs, in which dependence between the random edges is defined through certain finite subgraphs, in imitation of the use of potential energy to provide dependence between particle states in a grand canonical ensemble of statistical physics. They are particularly useful when one wants to construct models that resemble observed networks as closely as possible, but without going into details of the specific process underlying network formation. For history and a review of developments, see e.g. Fienberg [14] [15], Frank and Strauss [16], Häggström and Jonasson [17], Newman [27], and Wasserman and Faust [33].

Many of the investigations into exponential random graphs employ the elegant theory of graph limits as developed by Lovász and coauthors (V.T. Sós, B. Szegedy, C. Borgs, J. Chayes, K. Vesztergombi, ...) [8] [9] [10] [23] [24]. Building on earlier work of Aldous [1] and Hoover [18], the graph limit theory connects sequences of graphs to a unified graphon space equipped with a cut metric. Though the theory itself is tailored to dense graphs, serious attempts have been made at formulating parallel results for sparse graphs [2] [4] [6] [7] [11]. Since networks are often very large in size, a pressing objective in studying exponential models is to understand their asymptotic tendencies. From the point of view of extremal combinatorics and statistical mechanics, emphasis has been made on the variational principle of the limiting normalization constant, concentration of the limiting probability distribution, phase transitions and asymptotic structures. See e.g. Aristoff and Zhu [3], Chatterjee and Diaconis [12], Kenyon et al. [20], Kenyon and Yin [21], Lubetzky and Zhao [25] [26], Radin and Sadun [28] [29], and Radin and Yin [30].

Despite their flexibility, conventionally used exponential random graphs admittedly have one shortcoming. They cannot directly model weighted networks as the underlying probability
space consists of simple graphs only, i.e., edges are either present or absent. Since many substantively important networks are weighted, this limitation is especially problematic. The need to extend the existing exponential framework is thus justified, and several generalizations have been proposed [22] [31] [34]. An alternative interpretation for simple graphs is such that the edge weights are iid and satisfy a Bernoulli distribution. This work will instead assume that the iid edge weights follow a generic common distribution and rigorously analyze the associated phase transitions and critical phenomena.

The rest of this paper is organized as follows. In Section 2 we provide basics of graph limit theory and introduce key features of edge-weighted exponential random graphs. In Section 3 we summarize some important general results, including a variational principle for the limiting normalization constant (Theorems 3.1 and 3.2) and an associated concentration of measure (Theorem 3.3) indicating that almost all large graphs lie near the maximizing set. Theorems 3.4 and 3.5 then give simplified versions of these theorems in the “attractive” region of the parameter space where the parameters $\beta_2, ..., \beta_k$ are all non-negative. In Section 4 we specialize to exponential models where the edge weights are uniformly distributed and show in Theorem 4.1 the existence of a first order phase transition curve ending in a second order critical point. Lastly, in Section 5 we investigate the asymptotic phase structure of a directed model where a large deviation principle is missing.

2. Edge-weighted exponential random graphs

Let $G_n \in \mathcal{G}_n$ be an edge-weighted undirected labeled graph on $n$ vertices, where the edge weights $x_{ij}$ between vertex $i$ and vertex $j$ are iid real random variables having a common distribution $\mu$. Any such graph $G_n$, irrespective of the number of vertices, may be represented as an element $h^{G_n}$ of a single abstract space $\mathcal{W}$ that consists of all symmetric measurable functions $h(x, y)$ from $[0, 1]^2$ into $\mathbb{R}$ (referred to as “graph limits” or “graphons”), by setting $h^{G_n}(x, y)$ as the edge weight between vertices $\lceil nx \rceil$ and $\lceil ny \rceil$ of $G_n$. For a finite simple graph $H$ with vertex set $V(H) = [k] = \{1, ..., k\}$ and edge set $E(H)$ and a simple graph $G_n$ on $n$ vertices, there is a notion of density of graph homomorphisms, denoted by $t(H, G_n)$, which indicates the probability that a random vertex map $V(H) \to V(G_n)$ is edge-preserving,

$$t(H, G_n) = \frac{|\text{hom}(H, G_n)|}{|V(G_n)||V(H)|}.$$  \hspace{1cm} (2.1)

For a graphon $h \in \mathcal{W}$, define the graphon homomorphism density

$$t(H, h) = \int_{0,1]^k} \prod_{(i,j) \in E(H)} h(x_i, x_j) dx_1 \cdots dx_k.$$  \hspace{1cm} (2.2)

Then $t(H, G_n) = t(H, h^{G_n})$ by construction, and we take (2.2) as the definition of graph homomorphism density $t(H, G_n)$ for an edge-weighted graph $G_n$. This graphon interpretation enables us to capture the notion of convergence in terms of subgraph densities by an explicit “cut distance” on $\mathcal{W}$:

$$d_{\square}(f, h) = \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} (f(x, y) - h(x, y)) \, dx \, dy \right|.$$ \hspace{1cm} (2.3)

for $f, h \in \mathcal{W}$. The common distribution $\mu$ for the edge weights yields probability measure $\mathbb{P}_n$ and the associated expectation $\mathbb{E}_n$ on $\mathcal{G}_n$, and further induces probability measure $\mathbb{Q}_n$ on the space $\mathcal{W}$ under the graphon representation.
Phase transitions in edge-weighted exponential random graphs

A non-trivial complication is that the topology induced by the cut metric is well defined only up to measure preserving transformations of [0, 1] (and up to sets of Lebesgue measure zero), which may be thought of vertex relabeling in the context of finite graphs. To tackle this issue, an equivalence relation ~ is introduced in \(\mathcal{W}\). We say that \(f \sim h\) if \(f(x, y) = h_\sigma(x, y) := h(\sigma x, \sigma y)\) for some measure preserving bijection \(\sigma\) of [0, 1]. Let \(h\) (referred to as a “reduced graphon”) denote the equivalence class of \(h\) in \((\mathcal{W}, d_\square)\). Since \(d_\square\) is invariant under \(\sigma\), one can then define on the resulting quotient space \(\tilde{\mathcal{W}}\) the natural distance \(\delta_\square\) by \(\delta_\square(f, \tilde{h}) = \inf_{\sigma_1, \sigma_2} d_\square(f_{\sigma_1}, h_{\sigma_2})\), where the infimum ranges over all measure preserving bijections \(\sigma_1\) and \(\sigma_2\), making \((\tilde{\mathcal{W}}, \delta_\square)\) into a metric space. With some abuse of notation we also refer to \(\delta_\square\) as the “cut distance.”

After identifying graphs that are the same after vertex relabeling, the probability measure \(\tilde{\mathbb{P}}_n\) yields probability measure \(\tilde{\mathbb{P}}_n\) and the associated expectation \(\tilde{\mathbb{E}}_n\) (which coincides with \(\mathbb{E}_n\)). Correspondingly, the probability measure \(\mathbb{Q}_n\) induces probability measure \(\tilde{\mathbb{Q}}_n\) on the space \(\mathcal{W}\) under the measure preserving transformations.

By a \(k\)-parameter family of exponential random graphs we mean a family of probability measures \(\mathbb{P}_n^\beta\) on \(\mathcal{G}_n\) defined by, for \(G_n \in \mathcal{G}_n\),

\[
\mathbb{P}_n^\beta(G_n) = \exp \left( n^2 \left( \beta_1 t(H_1, G_n) + \cdots + \beta_k t(H_k, G_n) - \psi^\beta_n \right) \right) \mathbb{P}_n(G_n),
\]

(2.4)

where \(\beta = (\beta_1, \ldots, \beta_k)\) are \(k\) real parameters, \(H_1, \ldots, H_k\) are pre-chosen finite simple graphs (and we take \(H_1\) to be a single edge), \(t(H_i, G_n)\) is the density of graph homomorphisms, \(\mathbb{P}_n\) is the probability measure induced by the common distribution \(\mu\) for the edge weights, and \(\psi^\beta_n\) is the normalization constant,

\[
\psi^\beta_n = \frac{1}{n^2} \log \mathbb{E}_n \left( \exp \left( n^2 \left( \beta_1 t(H_1, G_n) + \cdots + \beta_k t(H_k, G_n) \right) \right) \right).
\]

(2.5)

We say that a phase transition occurs when the limiting normalization constant \(\psi^\beta_\infty = \lim_{n \to \infty} \psi^\beta_n\) has a singular point, as it is the generating function for the limiting expectations of other random variables,

\[
\lim_{n \to \infty} \mathbb{E}_n^\beta t(H_i, G_n) = \lim_{n \to \infty} \frac{\partial}{\partial \beta_i} \psi^\beta_n = \frac{\partial}{\partial \beta_i} \psi^\beta_\infty,
\]

(2.6)

\[
\lim_{n \to \infty} n^2 \left( \text{Cov}_n^\beta \left( t(H_i, G_n), t(H_j, G_n) \right) \right) = \lim_{n \to \infty} \frac{\partial^2}{\partial \beta_i \partial \beta_j} \psi^\beta_n = \frac{\partial^2}{\partial \beta_i \partial \beta_j} \psi^\beta_\infty.
\]

(2.7)

The exchange of limits in (2.6) and (2.7) is nontrivial, but may be justified using similar techniques as in Yang and Lee [35]. Since homomorphism densities \(t(H_i, G_n)\) are preserved under vertex relabeling, the probability measure \(\tilde{\mathbb{P}}_n^\beta\) and the associated expectation \(\tilde{\mathbb{E}}_n^\beta\) (which coincides with \(\mathbb{E}_n^\beta\)) may likewise be defined.

**Definition 2.1.** A phase is a connected region of the parameter space \(\{\beta\}\), maximal for the condition that the limiting normalization constant \(\psi^\beta_\infty\) is analytic. There is a \(j\)-th order transition at a boundary point of a phase if at least one \(j\)-th order partial derivative of \(\psi^\beta_\infty\) is discontinuous there, while all lower order derivatives are continuous.

More generally, we may consider exponential models where the terms in the exponent defining the probability measure contain functions on the graph space other than homomorphism densities. Let \(T : \tilde{\mathcal{W}} \to \mathbb{R}\) be a bounded continuous function. Let the probability measure \(\mathbb{P}_n^T\) and the normalization constant \(\psi^T_n\) be defined as in (2.4) and (2.5), that is,

\[
\mathbb{P}_n^T(G_n) = \exp \left( n^2 (T(\tilde{h} G_n) - \psi^T_n) \right) \mathbb{P}_n(G_n),
\]

(2.8)
Then the probability measure \( \tilde{P}^T_n \) and the associated expectation \( \tilde{E}^T_n \) (which coincides with \( E^T_n \)) may be defined in a similar manner.

We will assume that the common distribution \( \mu \) on the edge weights has finite support, which implies that the graphon space \( \mathcal{W} \) under consideration is a finite subset of \( \mathbb{R} \). These \( L^\infty \) graphons generalize graphons that take values in \([0,1]\) only and are better suited for generic edge-weighted graphs instead of just simple graphs. The “finite support” assumption also assures that the moment generating function

\[
M(\theta) = \int e^{\theta x} \mu(dx)
\]

is nicely defined. The domain of the function \( I \) can be extended to \( \tilde{\mathcal{W}} \) in the usual manner:

\[
I(\tilde{h}) = \int_{[0,1]^2} I(h(x,y)) dxdy,
\]

where \( h \) is any representative element of the equivalence class \( \tilde{h} \). It was shown in [13] that \( I \) is well defined on \( \tilde{\mathcal{W}} \) and is lower semi-continuous under the cut metric \( \delta_\square \). The space \( (\tilde{\mathcal{W}}, \delta_\square) \) enjoys many other important properties that are essential for the study of exponential random graph models. It is a compact space and homomorphism densities \( t(H, \cdot) \) are continuous functions on it. In fact, homomorphism densities characterize convergence under the cut metric: a sequence of graphs converges if and only if its homomorphism densities converge for all finite simple graphs, and the limiting homomorphism densities then describe the resulting graphon.

### 3. Statement of results

The normalization constant plays a central role in statistical mechanics because it encodes essential information about the structure of the probability measure; even the existence of its limit bears important consequences. In the case of exponential random graphs, the computational tools currently used by practitioners to compute this constant become unreliable for large networks [5] [32]. This problem however can be circumvented if we know a priori that the limit of the normalization constant exists. One can then choose a “scaled down” network model with a smaller number of vertices and use the exact value of the normalization constant in the scaled down model as an approximation to the normalization constant in the larger model, and a computer program that can evaluate the exact value of the normalization constant for moderate sized networks would serve the purpose [19]. The following Theorem 3.1 is a generalization of the corresponding result (Theorem 3.1) in Chatterjee and Diaconis [12], where they assumed that the edge weights \( x_{ij} \) between vertices \( i \) and \( j \) are iid real random variables satisfying a special common distribution – Bernoulli having values 1 and 0 each with probability 1/2. For the sake of completeness and to motivate further discussions in Theorem 3.2, we present the proof details below.

**Theorem 3.1.** Let \( T : \tilde{\mathcal{W}} \to \mathbb{R} \) be a bounded continuous function. Let \( \psi^T_n \) and \( I \) be defined as before (see (2.9), (2.10) and (2.11)). Then the limiting normalization constant \( \psi^T_\infty := \lim_{n \to \infty} \psi^T_n \) exists, and is given by

\[
\psi^T_\infty = \sup_{\tilde{h} \in \tilde{\mathcal{W}}} \left( T(\tilde{h}) - \frac{1}{2} I(\tilde{h}) \right).
\]
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Proof. For each Borel set $\mathcal{A} \subseteq \tilde{\mathcal{W}}$ and each $n$, define

$$\tilde{\mathcal{A}}^n = \{ \tilde{h} \in \tilde{\mathcal{A}} : \tilde{h} = \tilde{h}^{G_n} \text{ for some } G_n \in \mathcal{G}_n \}. \tag{3.2}$$

Fix $\epsilon > 0$. Since $T$ is a bounded function, there is a finite set $R$ such that the intervals $\{(c, c + \epsilon) : c \in R\}$ cover the range of $T$. Since $T$ is a continuous function, for each $c \in R$, the set $\tilde{U}_c$ consisting of reduced graphons $\tilde{h}$ with $c < T(\tilde{h}) < c + \epsilon$ is an open set, and the set $\tilde{H}_c$ consisting of reduced graphons $\tilde{h}$ with $c \leq T(\tilde{h}) \leq c + \epsilon$ is a closed set. Let $\tilde{U}_c^n$ and $\tilde{H}_c^n$ be respectively defined as in (3.2). The large deviation principle, Theorem 1.2 of [13], implies that:

$$\limsup_{n \to \infty} \frac{\log \tilde{Q}_n(\tilde{H}_c^n)}{n^2} \leq -\inf_{\tilde{h} \in \tilde{H}_c} \frac{1}{2} I(\tilde{h}), \tag{3.3}$$

and that

$$\liminf_{n \to \infty} \frac{\log \tilde{Q}_n(\tilde{U}_c^n)}{n^2} \geq -\inf_{\tilde{h} \in \tilde{U}_c} \frac{1}{2} I(\tilde{h}), \tag{3.4}$$

where $\tilde{Q}_n$ denotes the family of probability measures on $\tilde{\mathcal{W}}$ that is inherited from the corresponding probability measures $\tilde{P}_n$ on $\mathcal{G}_n$.

We first consider $\limsup \psi_n^T$. For each $c \in R$,

$$\exp(n^2 \psi_n^T) \leq \sum_{c \in R} \exp(n^2(c + \epsilon)) \tilde{Q}_n(\tilde{H}_c^n) \leq |R| \sup_{c \in R} \exp(n^2(c + \epsilon)) \tilde{Q}_n(\tilde{H}_c^n). \tag{3.5}$$

This shows that

$$\limsup_{n \to \infty} \psi_n^T \leq \sup_{c \in R} \left( c + \epsilon - \inf_{\tilde{h} \in \tilde{H}_c} \frac{1}{2} I(\tilde{h}) \right). \tag{3.6}$$

Each $\tilde{h} \in \tilde{H}_c$ satisfies $T(\tilde{h}) \geq c$. Consequently,

$$\sup_{\tilde{h} \in \tilde{H}_c} (T(\tilde{h}) - \frac{1}{2} I(\tilde{h})) \geq \sup_{\tilde{h} \in \tilde{H}_c} (c - \frac{1}{2} I(\tilde{h})) = c - \inf_{\tilde{h} \in \tilde{H}_c} \frac{1}{2} I(\tilde{h}). \tag{3.7}$$

Substituting this in (3.6) gives

$$\limsup_{n \to \infty} \psi_n^T \leq \epsilon + \sup_{c \in R} \sup_{\tilde{h} \in \tilde{H}_c} (T(\tilde{h}) - \frac{1}{2} I(\tilde{h})) \tag{3.8}$$

$$= \epsilon + \sup_{\tilde{h} \in \tilde{\mathcal{W}}} (T(\tilde{h}) - \frac{1}{2} I(\tilde{h})).$$

Next we consider $\liminf \psi_n^T$. For each $c \in R$,

$$\exp(n^2 \psi_n^T) \geq \sup_{c \in R} \exp(n^2 c) \tilde{Q}_n(\tilde{U}_c^n). \tag{3.9}$$

Therefore for each $c \in R$,

$$\liminf_{n \to \infty} \psi_n^T \geq c - \inf_{\tilde{h} \in \tilde{U}_c} \frac{1}{2} I(\tilde{h}). \tag{3.10}$$

Each $\tilde{h} \in \tilde{U}_c$ satisfies $T(\tilde{h}) < c + \epsilon$. Therefore

$$\sup_{\tilde{h} \in \tilde{U}_c} (T(\tilde{h}) - \frac{1}{2} I(\tilde{h})) \leq \sup_{\tilde{h} \in \tilde{U}_c} (c + \epsilon - \frac{1}{2} I(\tilde{h})) = c + \epsilon - \inf_{\tilde{h} \in \tilde{U}_c} \frac{1}{2} I(\tilde{h}). \tag{3.11}$$
Together with \((3.10)\), this shows that
\[
\liminf_{n \to \infty} \psi_n^{T} \geq - \epsilon + \sup_{\tilde{c} \in \tilde{W}} \sup_{\tilde{h} \in \tilde{U}_{\tilde{c}}} (T(\tilde{h}) - \frac{1}{2} I(\tilde{h})).
\]

\[(3.12)\]

Since \(\epsilon\) is arbitrary, this completes the proof. \(\square\)

When \(T\) is not continuous, we lose control of the openness and closedness of the sets in \(\tilde{W}\), and hence can not resort to the large deviation principle of \([13]\) to find an explicit characterization of the limiting normalization constant \(\psi^{T}_{\infty}\), but we can still conclude that this important limit exists under mild conditions.

**Theorem 3.2.** Let \(T : \tilde{W} \to \mathbb{R}\) be a bounded function. Let \(I\) and \(\psi_n^{T}\) be defined as before (see \((2.9)\), \((2.10)\) and \((2.11)\)). Then the limiting normalization constant \(\psi^{T}_{\infty}\) exists.

**Proof.** Since \(T\) is a bounded function, there is a finite set \(R\) such that the intervals \(\{(c, c + \epsilon) : c \in R\}\) cover the range of \(T\). Similarly as in the proof of Theorem 3.1, we have
\[
\exp(n^2 \psi_n^{T}) \leq |R| \sup_{c \in R} \exp(n^2 (c + \epsilon)) \tilde{Q}_n(\tilde{U}^n_c),
\]
and
\[
\exp(n^2 \psi_n^{T}) \geq \sup_{c \in R} \exp(n^2 c) \tilde{Q}_n(\tilde{U}^n_c).
\]
This implies that
\[
\limsup_{n \to \infty} \psi_n^{T} - \liminf_{n \to \infty} \psi_n^{T} \leq \frac{\log |R|}{n^2} + \epsilon.
\]
Since \(\epsilon\) is arbitrary, this completes the proof. \(\square\)

**Remark.** The boundedness condition on \(T\) can be relaxed even further. \((\log |R|)/(n^2) \to 0\) is sufficient.

Let \(\tilde{H}\) be the subset of \(\tilde{W}\) where \(\psi^{T}_{\infty}\) is maximized. By the compactness of \(\tilde{W}\), the continuity of \(T\) and the lower semi-continuity of \(I\), \(\tilde{H}\) is a nonempty compact set. The set \(\tilde{H}\) encodes important information about the exponential model \((2.8)\) and helps to predict the behavior of a typical random graph sampled from this model. The following Theorem 3.3 states that in the large \(n\) limit, the quotient image \(\tilde{h}^{G_n}\) of a random graph \(G_n\) drawn from \((2.8)\) must lie close to \(\tilde{H}\) with probability 1. Especially, if \(\tilde{H}\) is a singleton set, the theorem gives a law of large numbers for \(G_n\); while if \(\tilde{H}\) is not a singleton set, the theorem points to the existence of a first order phase transition.

**Theorem 3.3.** Let \(\tilde{H}\) be defined as in the above paragraph. Let \(\mathbb{P}_n^{T}\) \((2.8)\) be the probability measure on \(G_n\). Then
\[
\lim_{n \to \infty} \delta_{\tilde{H}^{G_n}}(\tilde{h}^{G_n}, \tilde{H}) = 0 \text{ almost surely.}
\]
\[(3.16)\]

**Proof.** The proof of this theorem follows a similar line of reasoning as in the proof of the corresponding result (Theorem 3.2) in Chatterjee and Diaconis \([12]\). The key observation is that the distance between the graph and the maximizing set decays exponentially as the number of vertices of the graph grows. For any \(\eta > 0\) there exist \(C, \gamma > 0\) such that for all \(n\) large enough,
\[
\mathbb{P}_n^{T}(\delta_{\tilde{H}^{G_n}}(\tilde{h}^{G_n}, \tilde{H}) > \eta) \leq Ce^{-n^{2}\gamma}.
\]
\[(3.17)\]
Theorem 3.5. Let $G$ be a convex function, must also be convex. □

The conclusion hence follows.

When the bounded continuous function $T$ on $\tilde{\mathcal{V}}$ is given by the sum of graph homomorphism densities $T = \sum_{k=1}^{K} \beta_k t(H_k, \cdot)$, the statement of Theorems 3.1 and 3.3 may be simplified in the "attractive" region of the parameter space where the parameters $\beta_2, \ldots, \beta_k$ are all non-negative, as seen in the following Theorems 3.4 and 3.5.

Theorem 3.4. Consider a general $k$-parameter exponential random graph model (2.4). Suppose $\beta_2, \ldots, \beta_k$ are non-negative. Then the limiting normalization constant $\psi^\beta_\infty$ exists, and is given by

$$
\psi^\beta_\infty = \sup_u \left( \beta_1 u e(H_1) + \cdots + \beta_k u e(H_k) - \frac{1}{2} I(u) \right),
$$

where $e(H_i)$ is the number of edges in $H_i$, $I$ is the Cramér function (2.10) (2.11), and the supremum is taken over all $u$ in the domain of $I$, i.e., where $I < \infty$.

Proof. The proof of this theorem follows a similar line of reasoning as in the proof of the corresponding result (Theorem 4.1) in Chatterjee and Diaconis [12], where the exact form of $I$ for the Bernoulli distribution was given. The crucial step is recognizing that $I$ is convex. So though the exact form of $I$ is not always obtainable for a generic common distribution $\mu$, the log of the moment generating function $M(\theta)$ is convex and $I$, being the Legendre transform of a convex function, must also be convex. □

Theorem 3.5. Let $G_n$ be an exponential random graph drawn from (2.4). Suppose $\beta_2, \ldots, \beta_k$ are non-negative. Then $G_n$ behaves like an Erdős-Rényi graph $G(n, u^*)$ in the large $n$ limit, where $u^*$ is picked randomly from the set $U$ of maximizers of (3.19).

Proof. The assertions in this theorem are direct consequences of Theorems 3.3 and 3.4. □

4. An application: Uniformly distributed edge weights

Consider the set $G_n$ of all edge-weighted undirected labeled graphs on $n$ vertices, where the edge weights $x_{ij}$ between vertex $i$ and vertex $j$ are iid real random variables uniformly distributed on $(0, 1)$. The common distribution for the edge weights yields probability measure $\mathbb{P}_n$ on the set $G_n$. Give the set of such graphs the probability

$$
\mathbb{P}_n^\beta(G_n) = \exp \left( n^2 \left( \beta_1 t(H_1, G_n) + \beta_2 t(H_2, G_n) - \psi^\beta_n \right) \right) \mathbb{P}_n(G_n),
$$

where $\beta = (\beta_1, \beta_2)$ are 2 real parameters, $H_1$ is a single edge, $H_2$ is a finite simple graph with $p \geq 2$ edges, and $\psi^\beta_n$ is the normalization constant,

$$
\psi^\beta_n = \frac{1}{n^2} \log \mathbb{E}_n \left( \exp \left( n^2 \left( \beta_1 t(H_1, G_n) + \beta_2 t(H_2, G_n) \right) \right) \right).
$$

The associated expectation $\mathbb{E}_n^\beta$ may be defined accordingly, and a phase transition occurs when the limiting normalization constant $\psi^\beta_\infty = \lim_{n \to \infty} \psi^\beta_n$ has a singular point.
The V-shaped region (with phase transition curve $r(\beta_1)$ inside) in the $(\beta_1, \beta_2)$ plane. Graph drawn for $p = 2$.

**Theorem 4.1.** For any allowed $H_2$, the limiting normalization constant $\psi^\beta_\infty$ is analytic at all $(\beta_1, \beta_2)$ in the upper half-plane ($\beta_2 \geq 0$) except on a certain decreasing curve $\beta_2 = r(\beta_1)$ which includes the endpoint $(\beta^c_1, \beta^c_2)$. The derivatives $\frac{\partial}{\partial \beta_1} \psi^\beta_\infty$ and $\frac{\partial}{\partial \beta_2} \psi^\beta_\infty$ have (jump) discontinuities across the curve, except at the end point where, however, all the second derivatives $\frac{\partial^2}{\partial \beta_1 \partial \beta_2} \psi^\beta_\infty$, $\frac{\partial^2}{\partial \beta_1^2} \psi^\beta_\infty$ and $\frac{\partial^2}{\partial \beta_2^2} \psi^\beta_\infty$ diverge.

**Corollary 4.2.** For any allowed $H_2$, the parameter space $\{(\beta_1, \beta_2) : \beta_2 \geq 0\}$ consists of a single phase with a first order phase transition across the indicated curve $\beta_2 = r(\beta_1)$ and a second order phase transition at the critical point $(\beta^c_1, \beta^c_2)$.

**Proof.** The moment generating function for the uniform $(0,1)$ distribution is given by $M(\theta) = (e^\theta - 1)/\theta$. The associated Cramér function $I$ is finite on $(0,1)$,

$$I(u) = \sup_{\theta \in \mathbb{R}} \left( \theta u - \log e^\theta - 1 \right),$$

but does not admit a closed-form expression. As shown in Theorem 3.4, the limiting normalization constant $\psi^\beta_\infty$ exists and is given by

$$\psi^\beta_\infty = \sup_{0 \leq u \leq 1} \left( \beta_1 u + \beta_2 u^p - \frac{1}{2} I(u) \right).$$

A significant part of computing phase boundaries for the 2-parameter exponential model is then a detailed analysis of the calculus problem (4.4). However, as straightforward as it sounds, since
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Figure 2. Outside the V-shaped region, \( l(u) \) has a unique local maximizer (hence global maximizer) \( u^* \). Graph drawn for \( \beta_1 = -5 \), \( \beta_2 = 3.5 \), and \( p = 2 \).

the exact form of \( I \) is not obtainable, getting a clear picture of the asymptotic phase structure is not that easy and various tricks need to be employed.

Consider the maximization problem for
\[
l(u; \beta_1, \beta_2) = \beta_1 u + \beta_2 u^{p-1} - \frac{1}{2} I(u),
\]

where \(-\infty < \beta_1 < \infty\) and \(0 \leq \beta_2 < \infty\) are parameters. The location of maximizers of \( l(u) \) on the interval \([0,1]\) is closely related to properties of its derivatives \( l'(u) \) and \( l''(u) \),

\[
l'(u) = \beta_1 + p \beta_2 u^{p-1} - \frac{1}{2} I'(u),
\]

\[
l''(u) = p(p-1) \beta_2 u^{p-2} - \frac{1}{2} I''(u).
\]

Following the duality principle for the Legendre transform between \( I(u) \) and \( \log M(\theta) \) [30], we first analyze properties of \( l''(u) \) on the interval \([0,1]\). Recall that

\[
I(u) + \log \frac{e^\theta - 1}{\theta} = \theta u,
\]

where \( \theta \) and \( u \) are implicitly related. Taking derivatives, we have

\[
u = \left( \log \frac{e^\theta - 1}{\theta} \right)' |_{\theta},
\]

\[
I''(u) \cdot \left( \log \frac{e^\theta - 1}{\theta} \right)'' |_{\theta} = 1,
\]

Consider the function

\[
m(u) = \frac{I''(u)}{2p(p-1)u^{p-2}}
\]
on $[0,1]$. By (4.7), analyzing properties of $m(u)$ translates to analyzing properties of the function
\[
n(\theta) = 2p(p-1) \left( \log \frac{e^{\theta} - 1}{\theta} \right) \left. \frac{\theta}{\log e^{\theta} - 1} \frac{\theta}{(e^{\theta} - 1) \theta} \right| = 0 \left. \theta \right| = 0,
\]
(4.9)
of $n(\theta)$ over $\mathbb{R}$, where $m(u)n(\theta) = 1$ and $u$ and $\theta$ satisfy the dual relationship. We recognize that
\[
\lim_{\theta \to -\infty} n(\theta) = 0,
\]
(4.10)
\[
\lim_{\theta \to 0} n(\theta) = \frac{p(p-1)}{3} \left( \frac{1}{2} \right)^{p-1},
\]
which implies that $n(\theta)$ achieves a finite global maximum. We claim that the global maximum can only be attained at $\theta_0 \geq 0$. This would further imply that there is a finite global minimum for $m(u)$, and it can only be attained at $u_0 \geq \frac{1}{2}$. First suppose $p = 2$. Then $n'(\theta_0) = 0$ easily implies that $\theta_0 = 0$. Now suppose $p > 2$. Then since $n'(\theta_0) = 0$, we have
\[
\left( \frac{e^{2\theta} - e^{\theta}(\theta^2 + 2) + 1}{(e^{\theta} - 1)^2} \right) \left. \theta \right| = 0 \left. \theta \right| = 0 \left. \theta \right| = 0 \left. \theta \right| = 0 < 0.
\]
(4.11)
This says that $\theta_0$ is on the portion of $\mathbb{R}$ where $(e^{2\theta} - e^{\theta}(\theta^2 + 2) + 1) / ((e^{\theta} - 1)^2 \theta^2)$ is decreasing, i.e., $\theta_0 > 0$. As $\theta$ goes from $0$ to $\infty$, the left hand side of (4.11) first decreases from $0$ then increases to $0$ and is always negative, whereas the right hand side of (4.11) monotonically increases from $-(p-2)/144$ to $0$ and approaches $0$ at a rate faster than the left hand side, and so there exists a $\theta_0$ which makes both sides equal. Our numerical computations show that $n'(\theta_0) = 0$ is uniquely defined, and $n(\theta)$ increases from $-\infty$ to $\theta_0$ and decreases from $\theta_0$ to $\infty$. See Table 1 for some of these critical values.

The above analysis shows that for $\beta_2 \leq m(u_0)$, $l''(u) \leq 0$ over the entire interval $[0,1]$; while for $\beta_2 > m(u_0)$, $l''(u)$ takes on both positive and negative values, and we denote the transition points by $u_1$ and $u_2$ ($u_1 < u_0 < u_2$). Properties of $l''(u)$ on $[0,1]$ entails properties of $l'(u)$ over the same interval. For $\beta_2 \leq m(u_0)$, $l'(u)$ is monotonically decreasing. For $\beta_2 > m(u_0)$, $l'(u)$ is decreasing from $0$ to $u_1$, increasing from $u_1$ to $u_2$, and then decreasing again from $u_2$ to $1$.

The analytic properties of $l''(u)$ and $l'(u)$ help us investigate properties of $I(u)$ on the interval $[0,1]$. Being the Legendre transform of a smooth function, $I(u)$ is a smooth function, and grows

| $p$ | $\theta_0$ | $n(\theta_0)$ | $u_0$ | $m(u_0)$ |
|-----|-------------|----------------|-------|----------|
| 2   | 0           | 0.3333         | 0.5   | 3        |
| 3   | 1.3251      | 0.5575         | 0.6073| 1.7937   |
| 5   | 2.9869      | 0.8324         | 0.7183| 1.2014   |
| 10  | 5.6256      | 1.0894         | 0.8259| 0.9180   |

Table 1. Critical values for $m(u)$ and $n(\theta)$ as a function of $p$. 
unbounded when $u$ approaches 0 or 1. By the duality principle for the Legendre transform, $I'(u) = \theta$ where $\theta$ and $u$ are linked through (4.7), so $I'(0) = -\infty$ and $I'(1) = \infty$. This says that $l(u)$ is a smooth function, $l(0) = l(1) = -\infty$, $l'(0) = \infty$ and $l'(1) = -\infty$, so $l(u)$ cannot be maximized at 0 or 1. For $\beta_2 \leq m(u_0)$, $l(u)$ crosses the $u$-axis only once, going from positive to negative. Thus $l(u)$ has a unique local maximizer (hence global maximizer) $u^*$. For $\beta_2 > m(u_0)$, the situation is more complicated. If $l'(u_1) \geq 0$ (resp. $l'(u_2) \leq 0$), $l(u)$ has a unique local maximizer (hence global maximizer) at a point $u^* > u_2$ (resp. $u^* < u_1$). See Figures 2 and 3. If $l'(u_1) < 0 < l'(u_2)$, then $l(u)$ has two local maximizers $u_1^*$ and $u_2^*$, with $u_1^* < u_1 < u_0 < u_2 < u_2^*$.

Let

$$f(u) = \frac{uI''(u)}{2(p - 1)} - \frac{1}{2} I'(u)$$

(4.12)

so that $l'(u_1) = \beta_1 + f(u_1)$ and $l'(u_2) = \beta_1 + f(u_2)$. Again by the duality principle for the Legendre transform, analyzing properties of the function $f(u)$ on $[0, 1]$ translates to analyzing properties of the function

$$g(\theta) = \frac{\theta (e^\theta - 1) (e^\theta (\theta - 1) + 1)}{2(p - 1) (e^{2\theta} - e^\theta (\theta^2 + 2) + 1)} - \frac{1}{2} \theta$$

(4.13)

over $\mathbb{R}$, where $f(u) = g(\theta)$ and $u$ and $\theta$ satisfy the dual relationship. We recognize that

$$\lim_{\theta \to -\infty} g(\theta) = \infty,$$

(4.14)

$$\lim_{\theta \to 0} g(\theta) = \frac{3}{p - 1},$$

$$\lim_{\theta \to \infty} g(\theta) = \infty,$$

which implies that $g(\theta)$ achieves a finite global minimum. We claim that the global minimum can only be attained at the same unique $\theta_0$ that makes $n'(\theta) = 0$. This can be checked by

Figure 3. Outside the V-shaped region, $l(u)$ has a unique local maximizer (hence global maximizer) $u^*$. Graph drawn for $\beta_1 = -2.5$, $\beta_2 = 4$, and $p = 2$. 
identifying the condition for \(g'(\theta) = 0\). For \(p = 2\), \(g'(\theta_0) = 0\) easily implies that \(\theta_0 = 0\). For \(p > 2\), since \(g'(\theta_0) = 0\), we have

\[
\left( \frac{e^\theta (\theta - 1) + 1}{e^\theta - 1} \right) \frac{d}{d\theta} \left( \frac{e^\theta (\theta^2 + 2) + 1}{e^\theta - 1} \right) \bigg|_{\theta_0} - \left( \frac{e^\theta (\theta - 1) + 1}{e^\theta - 1} \right) \frac{d}{d\theta} \left( \frac{e^\theta (\theta^2 + 2) + 1}{e^\theta - 1} \right) \bigg|_{\theta_0}
\]

which yields the same solution as \(|4.11|\) after a simple transformation. Thus \(g(\theta)\) decreases from \(-\infty\) to \(\theta_0\) and increases from \(\theta_0\) to \(\infty\). This further implies that there is a finite global minimum for \(f(u)\) attained at \(u_0\), and \(f(u)\) decreases from \(0\) to \(u_0\) and increases from \(u_0\) to \(1\). Both \(\theta_0\) and \(u_0\) coincide with the critical values listed in Table 1. See Table 2. We conclude that \(l'(u_1) \geq 0\) for \(\beta_1 \geq -f(u_0)\) and \(\beta_2 = m(u_0)\). The only possible region in the \((\beta_1, \beta_2)\) plane where \(l'(u_1) < 0 < l'(u_2)\) is bounded by \(\beta_1 < -f(u_0) := \beta_1^*\) and \(\beta_2 > m(u_0) := \beta_2^*\).

We examine the behavior of \(l'(u_1)\) and \(l'(u_2)\) more closely when \(\beta_1\) and \(\beta_2\) are chosen from this region. Recall that \(u_1 < u_0 < u_2\). By monotonicity of \(f(u)\) on the intervals \((0, u_0)\) and \((u_0, 1)\), there exist continuous functions \(a(\beta_1)\) and \(b(\beta_1)\) of \(\beta_1\), such that \(l'(u_1) \leq a(\beta_1)\) and \(l'(u_2) \geq b(\beta_1)\) for every \((\beta_1, \beta_2)\) where \(l'(u_1) = 0\) and \(l'(u_2) = 0\). As \(\beta_1 \to -\infty, a(\beta_1) \to 0\) and \(b(\beta_1) \to 1\). \(a(\beta_1)\) is an increasing function of \(\beta_1\), whereas \(b(\beta_1)\) is a decreasing function, and they satisfy \(f(a(\beta_1)) = f(b(\beta_1)) = -\beta_1\). The restrictions on \(u_1\) and \(u_2\) yield restrictions on \(\beta_2\), and we have \(l'(u_1) < 0\) for \(\beta_2 < m(a(\beta_1))\) and \(l'(u_2) > 0\) for \(\beta_2 > m(b(\beta_1))\). As \(\beta_1 \to -\infty, m(a(\beta_1)) \to \infty\) and \(m(b(\beta_1)) \to \infty\). \(m(a(\beta_1))\) and \(m(b(\beta_1))\) are both decreasing functions of \(\beta_1\), and they satisfy \(l'(u_1) = 0\) when \(\beta_2 = m(a(\beta_1))\) and \(l'(u_2) = 0\) when \(\beta_2 = m(b(\beta_1))\). As \(l'(u_2) > l'(u_1)\) for every \((\beta_1, \beta_2)\), the curve \(m(b(\beta_1))\) must lie below the curve \(m(a(\beta_1))\), and together they generate the bounding curves of the V-shaped region in the \((\beta_1, \beta_2)\) plane with corner point \((\beta_1^*, \beta_2^*)\) where two local maximizers exist for \(l(u)\). See Figures 1, 4, and 5.

Fix an arbitrary \(\beta_1 < \beta_1^*\), we examine the effect of varying \(\beta_2\) on the graph of \(l'(u)\). It is clear that \(l'(u)\) shifts upward as \(\beta_2\) increases and downward as \(\beta_2\) decreases. As a result, as \(\beta_2\) gets large, the positive area bounded by the curve \(l'(u)\) increases, whereas the negative area decreases. By the fundamental theorem of calculus, the difference between the positive and negative areas is the difference between \(l(u_2^*)\) and \(l(u_1^*)\), which goes from negative \((l'(u_2) = 0, u_1^*\) is the global maximizer) to positive \((l'(u_1) = 0, u_2^*\) is the global maximizer) as \(\beta_2\) goes from \(m(b(\beta_1))\) to \(m(a(\beta_1))\). Thus there must be a unique \(\beta_2: m(b(\beta_1)) < \beta_2 < m(a(\beta_1))\) such that \(u_1^*\) and \(u_2^*\) are both global maximizers, and we denote this \(\beta_2\) by \(r(\beta_1)\). See Figures 1 and 6.

The parameter values of \((\beta_1, r(\beta_1))\) are exactly the ones for which positive and negative areas bounded by \(l'(u)\) equal each other. An increase in \(\beta_1\) induces an upward shift of \(l'(u)\), and may

\[
\begin{array}{cccccc}
\hline
p & \theta_0 & g(\theta_0) & u_0 & f(u_0) \\
\hline
2 & 0 & 3 & 0.5 & 3 \\
3 & 1.3251 & 1.3222 & 0.6073 & 1.3222 \\
5 & 2.9869 & 0.1059 & 0.7183 & 0.1059 \\
10 & 5.6256 & -1.1723 & 0.8259 & -1.1723 \\
\hline
\end{array}
\]

Table 2. Critical values for \(f(u)\) and \(g(\theta)\) as a function of \(p\).
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Figure 4. Along the lower bounding curve $m(b(\beta_1))$ of the V-shaped region, $l'(u)$ has two zeros $u_1^*$ and $u_2^*$, but only $u_1^*$ is the global maximizer for $l(u)$. Graph drawn for $\beta_1 = -5.7$, $\beta_2 = 5$, and $p = 2$.

be balanced by a decrease in $\beta_2$. Similarly, a decrease in $\beta_1$ induces a downward shift of $l'(u)$, and may be balanced by an increase in $\beta_2$. This justifies that $r(\beta_1)$ is monotonically decreasing in $\beta_1$.

The rest of the proof follows as in the proof of the corresponding result (Theorem 2.1) in Radin and Yin [30], where some probability estimates were used. A (jump) discontinuity in the first derivatives of $\psi^{\beta}$ across the curve $\beta_2 = r(\beta_1)$ indicates a discontinuity in the expected local densities (2.6), while the divergence of the second derivatives of $\psi^{\beta}$ at the critical point $(\beta_1^c, \beta_2^c)$ implies that the covariances of the local densities go to zero more slowly than $1/n^2$ (2.7). We omit the proof details. □

Remark. The maximization problem (4.4) is solved at a unique value $u^*$ off the phase transition curve $\beta_2 = r(\beta_1)$, and at two values $u_1^*$ and $u_2^*$ along the curve. As $\beta_1 \to -\infty$ (resp. $\beta_2 \to \infty$), $u_1^* \to 0$ and $u_2^* \to 1$. The jump from $u_1^*$ to $u_2^*$ is quite noticeable even for small parameter values of $\beta$. For example, taking $p = 2$, $\beta_1 = -5$, and $\beta_2 = 5$, numerical computations yield that $u_1^* \approx 0.137$ and $u_2^* \approx 0.863$ with $l(u_1^*) \approx l(u_2^*) \approx -1.0854$.

Proposition 4.3. The Cramér function $I(u)$ (4.3) for the uniform distribution on $(0,1)$ is symmetric about the line $u = \frac{1}{2}$.

Proof. Recall that $I(u)$ is the Legendre transform of the log moment generating function $\log M(\theta)$ for the uniform $(0,1)$ distribution. Take $0 < u < \frac{1}{2}$. Though the exact form of $I$ is not obtainable, following the duality principle, we have

\begin{equation}
I(u) = \theta_1 u - \log \frac{e^{\theta_1} - 1}{\theta_1},
\end{equation}

\begin{equation}
I(1-u) = \theta_2 (1-u) - \log \frac{e^{\theta_2} - 1}{\theta_2},
\end{equation}

13
where $\theta_1$ and $\theta_2$ are both finite and uniquely defined, and satisfy $I'(u) = \theta_1$ and $I'(1 - u) = \theta_2$. We show that $I(u) = I(1 - u)$ when $\theta_2 = -\theta_1$. The calculation is straightforward:

$$I(1 - u) = -\theta_1 (1 - u) - \log \frac{e^{-\theta_1} - 1}{-\theta_1} = \theta_1 u - \log \frac{e^{\theta_1} - 1}{\theta_1} = I(u). \quad (4.17)$$

This verifies our claim. \hfill \square

**Corollary 4.4.** Take $H_1$ a single edge and $H_2$ a finite simple graph with $p \geq 2$ edges. The associated phase transition curve $\beta_2 = r(\beta_1)$ lies above the straight line $\beta_2 = -\beta_1$ when $p \geq 3$, and is exactly portion of the straight line $\beta_2 = -\beta_1$ ($\beta_1 \leq -3$) when $p = 2$.

**Proof.** From the proof of Theorem 4.1 there are two global maximizers $u^*_1$ and $u^*_2$ for $l(u)$ along the phase transition curve $\beta_2 = r(\beta_1)$, with $u_0 \geq \frac{1}{2}$ and $0 < u^*_1 < u_0 < u^*_2 < 1$. Furthermore, the $y$-coordinate $\beta^*_2$ of the critical point $(\beta^*_1, \beta^*_2)$ is always positive. On the straight line $\beta_1 + \beta_2 = 0$, we rewrite $l(u) = \beta_1 (u - u^p) - \frac{1}{2} I(u)$. First suppose $p = 2$. Since $I(u)$ and $u - u^2$ are both symmetric about the line $u = \frac{1}{2}$, two global maximizers $u^*_1$ and $u^*_2$ exist for $l(u)$. Next consider the generic case $p > 2$. Analytical calculations give that $u - u^p < (1 - u) - (1 - u)^p$ for $0 < u < \frac{1}{2}$. Since $I(u)$ is symmetric, this says that for $\beta_1 < 0$ (resp. $\beta_2 > 0$), the global maximizer $u^*$ of $l(u)$ satisfies $u^* \leq \frac{1}{2}$. This implies the desired result. \hfill \square

5. **Further Discussion**

But what if the common distribution $\mu$ on the edge weights does not have finite support? Unfortunately, we do not have a generic large deviation principle in this case. As an example, we will examine the asymptotic phase structure for a completely solvable exponential model and hope the analysis would provide some inspiration. Let $G_n \in \mathcal{G}_n$ be an edge-weighted directed labeled graph on $n$ vertices, where the edge weights $x_{ij}$ from vertex $i$ to vertex $j$ are iid real random variables whose common distribution $\mu$ is standard Gaussian. As in the undirected case,
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Figure 6. Along the phase transition curve \( r(\beta_1), l(u) \) has two local maximizers \( u^*_1 \) and \( u^*_2 \), and both are global maximizers for \( l(u) \). Graph drawn for \( \beta_1 = -5 \), \( \beta_2 = 5 \), and \( p = 2 \).

The common distribution for the edge weights yields probability measure \( P_n \) and the associated expectation \( E_n \) on \( G_n \). Give the set of such graphs the probability

\[
P^n(\beta)(G_n) = \exp \left( n^2 \left( \beta_1 e(G_n) + \beta_2 s(G_n) - \psi^n(\beta) \right) \right) P_n(G_n), \tag{5.1}
\]

where \( \beta = (\beta_1, \beta_2) \) are 2 real parameters, \( H_1 \) is a directed edge, \( H_2 \) is a directed 2-star, \( e(G_n) \) and \( s(G_n) \) are respectively the directed edge and 2-star homomorphism densities of \( G_n \),

\[
e(G_n) = \frac{1}{n^2} \sum_{1 \leq i,j \leq n} x_{ij}, \quad s(G_n) = \frac{1}{n^3} \sum_{1 \leq i,j,k \leq n} x_{ij}x_{ik}, \tag{5.2}
\]

and \( \psi^n(\beta) \) is the normalization constant,

\[
\psi^n(\beta) = \frac{1}{n^2} \log E_n \left( \exp \left( n^2 \left( \beta_1 e(G_n) + \beta_2 s(G_n) \right) \right) \right). \tag{5.3}
\]

The associated expectation \( E^n(\beta) \) may be defined accordingly, and a phase transition occurs when the limiting normalization constant \( \psi^\beta = \lim_{n \to \infty} \psi^n(\beta) \) has a singular point.

Plugging the formulas for \( e(G_n) \) and \( s(G_n) \) into (5.3) and using the iid property of the edge weights, we have

\[
\psi^n(\beta) = \frac{1}{n^2} \log E_n \left( \exp \left( \beta_1 \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \right) + \beta_2 \frac{1}{n^2} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} x_{ij} \right)^2 \right) \tag{5.4}
\]

\[
= \frac{1}{n} \log E \left( \exp \left( \beta_1 Y + \frac{\beta_2}{n} Y^2 \right) \right),
\]

where \( Y = \sum_{j=1}^{n} x_{ij} \) satisfies a Gaussian distribution with mean 0 and variance \( n \), and \( E \) is the associated expectation. We compute, for \( \beta_2 < \frac{1}{2} \),

\[
E \left( \exp \left( \beta_1 Y + \frac{\beta_2}{n} Y^2 \right) \right) = \int_{-\infty}^{\infty} e^{\beta_1 y + \frac{\beta_2}{n} y^2} \frac{1}{\sqrt{2\pi n}} e^{-\frac{y^2}{2n}} dy = \frac{1}{\sqrt{1 - 2\beta_2}} e^{\frac{n\beta_1^2}{2(1-2\beta_2)}}, \tag{5.5}
\]
which implies that
\[ \psi_{\beta} = \frac{\beta_1^2}{2(1 - 2\beta_2)}. \]  
This is a smooth function in terms of the parameters \( \beta_1 \) and \( \beta_2 \), and so \( \psi_{\beta} \) does not admit a phase transition.

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