Bound states in Yukawa theory

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June 13, 2005

Abstract

A generalization of the Gell-Mann–Low Theorem is applied to bound state calculations in Yukawa theory. The resulting effective Schrödinger equation is solved numerically for two-fermion bound states with the exchange of a massless boson. The complete low-lying bound state spectrum is obtained for different ratios of the constituent masses. No abnormal solutions are found. We show the consistency of the non-relativistic and one-body limits and discuss the special cases of identical fermions and fermion-antifermion states. To our knowledge, this is the first consistent calculation of bound states in pure Yukawa theory (without UV cutoff).

1 Introduction

During the more than 50 years that have passed since Bethe and Salpeter formulated their famous equation [1, 2], the calculation of relativistic bound states has proved to be one of the truly hard problems in quantum field theory. For most of the numerous proposals for its solution, a model theory of two scalar bosons interacting via the exchange of a third scalar has served as a first testing ground. In the case of the Bethe-Salpeter equation itself, it was shown by Wick and Cutkosky that this model theory has an analytical solution in

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the popular “ladder approximation” to the equation, in case that the exchanged boson is massless \[3\]. It is hence quite surprising that for one of the most natural generalizations of this model where two spin 1/2 fermions interact through the exchange of a scalar boson, not a single consistent formalism has been devised for its solution to date.

In this introduction, we will give a brief review of the most important intents to calculate relativistic two-fermion bound states in this model, i.e., in Yukawa theory. It was clear even before Wick and Cutkosky’s solution of the purely scalar model that the case of fermionic constituents means a lot more to the Bethe-Salpeter equation than just a technical complication due to the inclusion of spin degrees of freedom \[4\]: the fact that the kernel of the equation in the ladder approximation (after a Wick rotation) is not of Fredholm type in this case represents a fundamental difficulty for analytic as much as for numerical investigations.

In the case of equal constituent masses and zero boson mass, it was shown by Goldstein \[5\] (later corrected and improved upon by Green \[6\]), originally for the subspace of pseudoscalar bound states only, that the ladder approximation gives a continuum of coupling constants corresponding to a “tightly bound” state with zero mass, i.e., such that the binding energy completely compensates the masses of the constituents, while one expects a series of discrete values if a discrete spectrum of energy eigenvalues is to exist. The definite version of the argument is reviewed in Ref. \[7\] (see the references there). In the case of Yukawa theory, this is essentially all that is known about the solutions of the Bethe-Salpeter equation.

The exchange of a scalar boson between spin 1/2 fermions has come to play an important role in the description of the nucleon-nucleon interaction in the context of one-boson exchange (OBE) models. Although the scalar boson exchange is used as an effective and approximate description of an actual two-pion exchange in this case, it is believed to give the dominant attractive contribution to the intermediate-range potential. In the numerous numerical calculations for the deuteron system within OBE models, the scalar boson exchange is always accompanied by the exchange of other mesons, most importantly pions and rho and omega mesons. For this type of calculations, the Bethe-Salpeter equation as well as several of its three-dimensional reductions have been employed, notably the Blankenbecler-Sugar-Logunov-Tavkhelidze (BSLT) equation \[8\] and the Gross or spectator equation \[9\]. The latter equations avoid problems with abnormal solutions, the one-body limit, and others that arise in the Bethe-Salpeter equation (in the ladder approximation) \[4, 10\], see, however, Ref. \[11\]. Among the vast literature on OBE models, we mention the work of Tjon and collaborators \[12\] who use the Bethe-Salpeter equation, the work of Gross and collaborators \[13\] who base their analysis on the Gross equation, and the quite complete “Bonn model” by Holinde, Machleidt, et al. reviewed in Ref. \[14\], which employs the Bloch-Horowitz scheme \[15\] for the bound state calculations, a formalism that is not based on the Bethe-Salpeter equation.

In all these approaches, a momentum cutoff is introduced in order to make the equations well-defined. It is interpreted phenomenologically as representing the spatial extension of the nucleons and mesons. Although we have not found a corresponding calculation in the published literature, it appears plausible in view of the singular nature of the Bethe-Salpeter kernel in the ladder approximation that the introduction of a momentum cutoff would also
be necessary in the case of a pure Yukawa interaction, and that the corresponding numerical results depend on the value of the cutoff. Note, however, the work of Gari and collaborators [16] who dispense with a momentum cutoff and rather determine the nucleon and meson structures self-consistently through loop corrections within the same nucleon-meson model. It is perhaps not a coincidence that their approach employs the Okubo transformation technique [17], which is unrelated to the Bethe-Salpeter equation but very similar to the formalism proposed in the present paper.

More recently, calculations of bound states in Yukawa theory have been attempted in light-front quantization, both in a Tamm-Dancoff approximation [18] and in the so-called covariant light-front dynamics [19]. Just as in the equal-time quantized theory, an additional momentum cutoff has to be introduced for the one-boson exchange, and the solutions for the bound states turn out to depend on this cutoff. It was pointed out by van Iersel and Bakker [20] that the light front ladder approximation is incomplete as long as instantaneous terms are not taken into account. One possibility to include such terms was presented in Ref. [21] and may help to solve the renormalization problem in the future.

In this paper, we propose to use a generalization [22] of the Gell-Mann–Low theorem [2] for the calculation of relativistic bound states. After the obligatory application to the purely scalar model in a previous publication [23], we now turn to Yukawa theory. Surprisingly, the application of the generalized Gell-Mann–Low theorem turns out to be straightforward and, except for the necessary technical complications due to the spin degrees of freedom, completely analogous to the scalar case. No inconsistencies whatsoever have been found, and no necessity for the introduction of a cutoff (for the interaction between the fermions) arises. For the numerical calculation of the bound state spectrum, we will focus on the case of a massless exchanged boson, for the following reasons: first, the massless case is the most singular one, and if the formalism works in this case, it is expected to be applicable to the case of a massive exchanged boson as well. Second, the solutions in the non-relativistic limit are known analytically in this case (Coulomb potential) and our numerical solutions can be compared against them. In particular, one would like to find the degeneracies characteristic of the non-relativistic limit. Finally, we plan to extract the lowest-order fine and hyperfine structure from our effective Hamiltonian in the near future, and this can be done analytically only in the massless (Coulomb) case.

The organization of the paper is as follows: in the next section, we introduce the generalization of the Gell-Mann–Low theorem on which our approach is based, and comment on aspects relevant to the present work. We also present, in the same section, the effective Hamiltonians in the zero-, one-, and two-fermion sectors which result from the application of the generalized Gell-Mann–Low theorem to lowest non-trivial order. They describe the renormalization of the vacuum energy, the fermion mass renormalization, and the effective potential for the two-fermion dynamics, respectively. In Section 3, we perform a series of non-trivial analytical checks on the results obtained in Section 2. In particular, we investigate the non-relativistic and one-body limits of the effective Schrödinger equation, replace fermions by antifermions, and consider the case of identical constituents. All the properties of our effective Hamiltonian description turn out to be in accord with physical expectations.
We present numerical bound state solutions in Section 4, for fine structure constants between zero and one and arbitrary ratios of the constituent masses. The eight lowest-lying states are calculated in each case, including states with non-zero relative orbital angular momentum and mixtures of spin singlet and triplet states. The non-relativistic, one-body, and equal-mass limits are discussed in detail. Section 5 contains our conclusions. There are also four appendices. Appendices A and B are concerned with the appearing loop corrections and their regularization in different schemes from a Hamiltonian, not manifestly covariant perspective. They prepare the ground for future calculations at higher orders in the expansion of the effective Hamiltonian, where regularization and renormalization are expected to become central issues. Appendices C and D present several formulas which are used in the analytical separation of angular and spin variables in the effective Schrödinger equation.

2 The Bloch-Wilson Hamiltonian

The physical idea behind the Bloch-Wilson or effective Hamiltonian is very similar to the Born-Oppenheimer approximation: the integration over the “light” degrees of freedom generates the dynamics for the “heavy” degrees of freedom (the constituents in our case), where the “light” degrees of freedom are given by the so-called interacting “virtual clouds” around the constituents. The virtual clouds are, in turn, created by the constituents as described here through an adiabatic process. Technically, a generalization of the Gell-Mann–Low theorem is used which we will briefly describe in the following (for further details, see Ref. [22]).

The adiabatic evolution operator $U_\epsilon$ provides a map from an eigenstate of the free Hamiltonian to an exact eigenstate of the full interaction Hamiltonian, if its application to the free eigenstate, after a suitable normalization, is well defined. This is the content of the original theorem by Gell-Mann and Low [2]. To this end, the full Hamiltonian $H$ is split into a free part $H_0$ and an interacting part $H_1$. The adiabatic damping is implemented by replacing $H = H_0 + H_1$ through

$$H(t) = H_0 + e^{-\epsilon |t|} H_1.$$  \hspace{1cm} (1)

An eigenstate $|\phi\rangle$ of $H_0 = \lim_{t\to-\infty} H(t)$ is then evolved according to the time-dependent Schrödinger equation with Hamiltonian $H(t)$ from $t \to -\infty$ to the state $U_\epsilon |\phi\rangle$ at $t = 0$. According to the adiabatic theorem, in the limit $\epsilon \to 0$ the state $U_\epsilon |\phi\rangle$ is an eigenstate of $H(t=0) = H$. A possible infinite phase (see, for example, [24]) is taken care of by imposing the normalization condition

$$\langle \phi | \psi \rangle = 1,$$  \hspace{1cm} (2)

so that

$$|\psi\rangle = \frac{U_\epsilon |\phi\rangle}{\langle \phi | U_\epsilon | \phi \rangle}.$$  \hspace{1cm} (3)
As a generalization of this theorem, it can be shown \cite{22} that the adiabatic evolution operator also maps $H_0$-invariant subspaces $\Omega_0$, $H_0\Omega \subset \Omega_0$, to $H$-invariant subspaces $\Omega$, $H\Omega \subset \Omega$, if its application to the $H_0$-invariant subspace, after a suitable normalization, is well defined. When a subspace is invariant under a hermitian operator, this operator can be diagonalized in the subspace, in other words, the subspace is a direct sum of eigenspaces of the operator, in this case the Hamiltonian. Since the only important property of the adiabatic evolution operator in this context is its $H$-invariant image, there is still an infinity of possible “normalizations” of the operator that map between the same subspaces $\Omega_0$ and $\Omega$. Here we choose the normalization condition

$$P_0 U_{BW} = \mathbf{1}_{\Omega_0}, \quad (4)$$

where $P_0$ is the orthogonal projector to $\Omega_0$ and $\mathbf{1}_{\Omega_0}$ the identity operator in $\Omega_0$. $U_{BW}$ is the “Bloch-Wilson operator”, the normalized version of the adiabatic evolution operator given explicitly by

$$U_{BW} = U(\sigma P_0 U(\sigma))^{-1}. \quad (5)$$

Eq. (4) naturally generalizes the normalization condition \cite{22} in the original Gell-Mann–Low theorem. The subspace $\Omega_0$ has to be small enough in order that it is possible in practice to diagonalize the Hamiltonian in $\Omega = U_{BW}\Omega_0$, at least numerically. On the other hand, it has to be large enough for the Bloch-Wilson operator to be well defined. This is in principle a subtle issue and may depend on the normalization condition chosen. We will limit ourselves here to show that everything is well defined for a natural choice of the subspace (for the field theory and the perturbative order considered here, see the discussion below). In the case of a two-constituent bound state, a natural choice for the $H_0$-invariant subspace $\Omega_0$ is the space of all states of the two constituents as free particles. This will be mapped by the normalized adiabatic evolution operator to the $H$-invariant subspace $\Omega$, a direct sum of eigenstates of the full Hamiltonian, which is expected to coincide with the space of all physical two-particle states, scattering as well as bound states.

A further simplification is obtained by realizing that the Bloch-Wilson operator effects a similarity transformation between the subspaces $\Omega_0$ and $\Omega$. Since the free subspace $\Omega_0$ is usually much easier to work in, it is convenient to similarity transform the full Hamiltonian back to this subspace, thereby defining an effective or Bloch-Wilson Hamiltonian $H_{BW}$ in $\Omega_0$,

$$H_{BW} = P_0 H U_{BW}, \quad (6)$$

where we have taken into account that $P_0|\Omega$ is the inverse map to $U_{BW}$ as a consequence of the normalization \cite{4}. In the above example, the effective Hamiltonian acts on the subspace of all states of the free constituents. Its eigenvalues are exactly equal to the eigenvalues of the full Hamiltonian (in the corresponding subspace $\Omega$), and there is a one-to-one relation between the corresponding eigenstates. Mathematically, the time-independent Schrödinger equation for the effective Hamiltonian takes the form of a non-relativistic Schrödinger equation for the
two constituents, with the relativistic expression for the kinetic energies and a complicated (generally non-local and non-hermitian) interaction term. With the normalization chosen, the eigenstates $|\phi\rangle$ of the effective Hamiltonian have the immediate meaning of wavefunctions for the constituents:

$$|\phi\rangle = P_0(U_{BW}|\phi\rangle),$$ (7)

hence $|\phi\rangle$ is the two-particle component of the complicated exact eigenstate of $H$, $U_{BW}|\phi\rangle$.

Obviously, the Bloch-Wilson operator $U_{BW}$ cannot be determined exactly in any interesting practical application. However, the adiabatic evolution operator $U_\epsilon$, and hence $U_{BW} = U_\epsilon(P_0U_\epsilon)^{-1}$, have a well-known perturbative expansion, the Dyson series:

$$U_\epsilon = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{0} dt_1 \cdots \int_{-\infty}^{0} dt_n e^{-\epsilon(|t_1|+\cdots+|t_n|)} T[H_1(t_1) \cdots H_1(t_n)],$$ (8)

where

$$H_1(t) = e^{iH_0t}H_1 e^{-iH_0t}.$$ (9)

Every order in this expansion determines an approximation to the effective Hamiltonian and consequently of its eigenvalues and eigenstates. To compare with, in the Bethe-Salpeter approach an infinite part of the Dyson series has to be summed up to find an approximation to the bound states. The sum is performed in practice implicitly by solving an integral equation (the homogeneous Bethe-Salpeter equation) with a perturbative approximation to its kernel, and the counterpart in our Hamiltonian approach is the solution of the (approximated) effective Schrödinger equation. All of the above is exemplified in the application to a simple scalar model in [23].

Finally, let us comment on the sense in which the adiabatic evolution operator, or rather its normalized version $U_{BW}$, is well-defined in $\Omega_0$. In the proof of the generalized Gell-Mann–Low theorem, as well as in the proof of the original theorem, the adiabatic evolution operator is treated as a formal power series (in $H_1$) given by the Dyson series. Consequently, $U_{BW}$ is considered well-defined if every term of the corresponding series is well-defined (finite) disregarding convergence properties of the series as a whole. However, the UV divergencies of the usual covariant perturbation theory cannot be escaped: they appear in the context of the Bloch-Wilson Hamiltonian (in the most straightforward approach) as divergencies of the three-dimensional momentum integrals for large momenta. In the scalar model considered before it was shown that these divergencies can be dealt with (to the lowest non-trivial order considered there) rather easily, and, in fact, in the same way as in covariant perturbation theory. On the other hand, divergencies of the IR type are considered to potentially invalidate the existence of the Bloch-Wilson operator (for the specific subspace $\Omega_0$ chosen). In the time-independent version presented in [22], which is obtained by performing the time integrals in the Dyson series, the IR divergencies arise from vanishing energy denominators. They can usually be avoided by appropriately enlarging the subspace $\Omega_0$. However, the usefulness of the entire approach rests on the possibility of choosing a subspace $\Omega_0$ which is manageably
small, ideally the space of all states of the two constituents as free particles in the example mentioned above. Here, as for the scalar model before, we take a pragmatic approach: we will be content with the fact that no IR divergence arises to the perturbative order and for the choice of $\Omega_0$ considered.

We will now apply these ideas to Yukawa theory. To simplify matters, we consider two different fermion species, $A$ and $B$, which interact with a scalar boson. Hence the Hamiltonian of the theory consists of the free Hamiltonians for the fermions $A$ and $B$ and the scalar boson, and the (normal-ordered) interaction term

$$H_1 = g \int d^3 x : [\bar{\psi}_A(x)\psi_A(x) + \bar{\psi}_B(x)\psi_B(x)] \varphi(x) : . \quad (10)$$

We are interested in bound states of one fermion $A$ and one fermion $B$. Other bound states, of identical fermions or fermion and antifermion (e.g., in a theory with only one fermion species), are simply related to bound states of type $AB$ and can be treated by exactly the same method. They will be discussed in some detail in the next section. The application of the generalized Gell-Mann–Low theorem to lowest non-trivial order follows in very close analogy the purely scalar case [23]. In particular, for a subspace $\Omega_0$ with a fixed number of (free) particles $P_0H_1P_0 = 0$, and the effective Hamiltonian becomes to lowest non-trivial order

$$H_{BW} = H_0P_0 - i \int_{-\infty}^{0} dt e^{-\epsilon|t|} P_0H_1(0)H_1(t)P_0 + O(H_1^4) \quad (11)$$

(the limit $\epsilon \to 0$ being understood). For the rest of this chapter, we will discuss in detail the cases where $\Omega_0$ is the subspace of zero, one (fermionic), and two ($AB$) particles. We will go through the essential steps briefly, present the corresponding results, and take the opportunity to comment on several issues that have not received due attention in [23]. For illustrative purposes, it may be helpful to have a look at the diagrams presented there which are the same in the present case, only that spin labels have to be added to the external fermion lines.

### 2.1 Vacuum state and zero-point energy renormalization

Beginning with the vacuum state, we consider the Bloch-Wilson Hamiltonian for the subspace $\Omega_0 = \mathbb{C}[0]$, or, equivalently, for the orthogonal projector on $\Omega_0$, $P_0 = |0\rangle \langle 0|$. In this case, the application of the generalized version of the Gell-Mann–Low theorem is equivalent to the original Gell-Mann–Low theorem. The result for the Bloch-Wilson Hamiltonian is $H_{BW} = E_VP_0$, where (by Wick’s theorem)

$$E_V - E_0$$

$$= ig^2 \int_{-\infty}^{0} dt e^{-\epsilon|t|} \int d^3 x d^3 x' \Delta_F(0 - t, x - x') \text{tr} \left[ S_A^A(t - 0, x - x')S_A^A(t - 0, x' - x) \right] + ig^2 \int_{-\infty}^{0} dt e^{-\epsilon|t|} \int d^3 x d^3 x' \Delta_F(0 - t, x - x') \text{tr} \left[ S_B^B(t - 0, x - x')S_B^B(t - 0, x' - x) \right] . \quad (12)$$
Here $E_0$ is the vacuum energy of the free theory and $S_{F}^{A,B}$ and $\Delta_F$ are the fermionic (for particles $A,B$) and the bosonic Feynman propagators. In our conventions,

$$S_{F}^{A}(t, x) = i \int \frac{d^4p}{(2\pi)^4} \frac{p \cdot \gamma + m_A}{p^2 - m_A^2 + i\epsilon} e^{-i(p_0 t - p \cdot x)}$$ \hfill (13)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ \theta(t) (p \cdot \gamma + m_A) e^{-i(E_p^A t - p \cdot x)} - \theta(-t) (p \cdot \gamma - m_A) e^{i(E_p^A t - p \cdot x)} \right]$$ \hfill (14)$$

in the covariant and non-covariant representation, respectively. We are using the following shorthands for the relativistic kinetic energies:

$$E_{p}^{A,B} = \sqrt{m_{A,B}^2 + p^2}, \quad \omega_p = \sqrt{\mu^2 + p^2}$$ \hfill (15)$$

($\mu$ is the boson mass). The bosonic propagator $\Delta_F(t, x)$ is simply given by the $m_A$-coefficient of $S_{F}^{A}(t, x)$ (and the substitution $m_A \to \mu$). In Eq. (12), note the opposite sign as compared to the bosonic case [23].

We can use time translation invariance of the integrands on the r.h.s. of Eq. (12) to move the vertices from times $t < 0$ and $t = 0$ to $t = 0$ and $-t > 0$. Mathematically, this corresponds to replacing $t$ by $-t$ and exchanging $x$ and $x'$, just as we did for the one-particle states in the scalar case [23]. If we, furthermore, use the invariance of the bosonic propagator under the change of the direction of propagation and apply these manipulations to half the r.h.s. of Eq. (12), we arrive (in the limit $\epsilon \to 0$) at the usual expression for the one-loop correction to the vacuum energy in covariant perturbation theory [25],

$$E_V - E_0 = i \frac{g^2}{2} \int d^4x d^4x' \delta(x^0) \Delta_F(x - x') \text{tr} \left[ S_{F}^{A}(x - x') S_{F}^{A}(x' - x) \right]$$

$$+ i \frac{g^2}{2} \int d^4x d^4x' \delta(x^0) \Delta_F(x - x') \text{tr} \left[ S_{F}^{B}(x - x') S_{F}^{B}(x' - x) \right].$$ \hfill (16)$$

The appearance of $\delta(x^0)$ eliminates a factor

$$2\pi\delta(0) = \int_{-\infty}^{\infty} dt$$ \hfill (17)$$

(due to time translation invariance) in the result, and leaves us with

$$(2\pi)^3\delta^{(3)}(0) = \int d^3x$$ \hfill (18)$$

(due to spatial translation invariance), to be interpreted as the volume of space.

We can immediately express Eqs. (12) and (16) in momentum space, by use of the non-covariant and covariant expressions for the Feynman propagators in momentum space, Eqs. 8.
(14) and (13), respectively. The results are [26]

\[ E_V - E_0 = \left[ g^2 \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{2 E_p^A 2 E_p^A 2 \omega_{p+p'}} \frac{4 \left( -E_p^A E_{p'}^A + \mathbf{p} \cdot \mathbf{p}' + m_A^2 \right)}{E_{p'}^A + \omega_{p+p'}} \right] \]

\[ + g^2 \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{2 E_p^B 2 E_p^B 2 \omega_{p+p'}} \frac{4 \left( -E_p^B E_{p'}^B + \mathbf{p} \cdot \mathbf{p}' + m_B^2 \right)}{E_{p'}^B + \omega_{p+p'}} \right] \]

\[ = \left\{ \frac{g^2}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \frac{4(p \cdot p' + m_A^2)}{[(p - p')^2 - \mu^2 + i\epsilon][p^2 - m_A^2 + i\epsilon][p'^2 - m_A^2 + i\epsilon]} \right\} \]

\[ + \left\{ \frac{g^2}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \frac{4(p \cdot p' + m_B^2)}{[(p - p')^2 - \mu^2 + i\epsilon][p^2 - m_B^2 + i\epsilon][p'^2 - m_B^2 + i\epsilon]} \right\} \]

\[ (2\pi)^3 \delta^{(3)}(0) \]

The equivalence between the two expressions can, of course, be established directly by integrating over \( p_0 \) and \( p'_0 \) in the manifestly covariant expression. However, there is a subtlety involved in the integration which may be of interest for future calculations at higher orders and will be described in detail in Appendix A. The results (19) are highly UV divergent and will be treated as formal expressions only. It is, however, important to remark that they are IR finite in the limit \( \mu \to 0 \) (we are here only interested in massive constituents \( m_A, m_B \neq 0 \)).

### 2.2 One-fermion states and mass renormalization

We will now turn to the one-fermion states, considering for concreteness fermions of type \( A \). The \( H_0 \)-invariant subspace considered is, correspondingly,

\[ \Omega_0 = \text{span} \{ |p_A, r\rangle | p_A \in \mathbb{R}^3, r = 1, 2 \} \]

where \( |p_A, r\rangle \) stands for a state of one fermion \( A \) with 3-momentum \( p_A \) and spin orientation \( r \) (we do not fix the basis to be used in spin space yet). The one-fermion states are normalized in a non-covariant fashion,

\[ \langle p_A, r | p'_A, s \rangle = (2\pi)^3 \delta(p_A - p'_A) \delta_{rs} \]

The application of the generalized Gell-Mann–Low theorem to lowest non-trivial order
yields the following matrix elements of the Bloch-Wilson Hamiltonian in $\Omega_0$, 
\begin{equation}
\langle p_A, r | H_{BW} | p'_A, s \rangle = (E_V + E_{p_A}^A) (2\pi)^3 \delta(p_A - p_A') \delta_{rs} \\
- ig^2 \int_{-\infty}^{0} dt e^{-\epsilon |t|} \int d^3 x d^3 x' \left[ \bar{\psi}_t^{A, r}(0, x) S_F^A(0 - t, x - x') \psi_t^{A, s}(t, x') \right] \Delta_F(0 - t, x - x') \\
- ig^2 \int_{-\infty}^{0} dt e^{-\epsilon |t|} \int d^3 x d^3 x' \left[ \bar{\psi}_{p_A, r}^{A}(t, x') S_F^A(t - 0, x' - x) \psi_{p_A, s}^{A}(0, x) \right] \Delta_F(t - 0, x' - x),
\end{equation}

where the fermion wave functions $\psi^{A, r}(t, x)$ are given by
\begin{equation}
\psi_{p_A, r}^{A}(t, x) = \frac{u_A(p_A, r)}{\sqrt{2E_p}} e^{-iE_p^A t + i\tilde{p} \cdot x}
\end{equation}

with the Dirac spinors normalized to
\begin{equation}
\bar{u}_A(p, r) u_A(p, s) = 2m_A \delta_{rs}.
\end{equation}

Using 3-momentum conservation at the vertices and, consequently, $E_{p_A}^A = E_{p_A'}^A$, the vertices in the last integral in Eq. (22) can be translated from times $t < 0$ and $t = 0$ to $t = 0$ and $-t > 0$, just as we did in the scalar case (23) and also for the corrections to the vacuum energy above. Equation (22) can then be written in the covariant form
\begin{equation}
\langle p_A, r | H_{BW} | p'_A, s \rangle = (E_V + E_{p_A}^A) (2\pi)^3 \delta(p_A - p_A') \delta_{rs} \\
- ig^2 \int d^4 x d^4 x' \delta(x^0) \left[ \bar{\psi}_t^{A, r}(x) S_F^A(x - x') \psi_t^{A, s}(x') \right] \Delta_F(x - x'),
\end{equation}

where the integral over $d^3 x$ serves to implement 3-momentum conservation.

Finally, then, the matrix elements of $H_{BW}$ take the form
\begin{equation}
\langle p_A, r | H_{BW} | p'_A, s \rangle = (E_V + E_{p_A}^A) (2\pi)^3 \delta(p_A - p_A') \delta_{rs} \\
+ \frac{1}{2E_{p_A}^A} \left[ \bar{u}_A(p_A, r) G(p_A) u_A(p_A, s) \right] (2\pi)^3 \delta(p_A - p_A'),
\end{equation}

where $G(p_A)$ is written in momentum space by use of the non-covariant and covariant expressions (14) and (13) for the Feynman propagators in momentum space in Eqs. (22) and (25), respectively, to give the following equivalent expressions (26)
\begin{equation}
G(p) = -ig^2 \int \frac{d^4 p'}{(2\pi)^4} \frac{1}{2E_{p'}^A} \frac{E_{p}^A \gamma_0 - p' \cdot \gamma + m_A}{E_{p'}^A + \omega_{p'-p} + E_{p}^A} \\
- ig^2 \int \frac{d^4 p'}{(2\pi)^4} \frac{1}{2E_{p'}^A} \frac{-E_{p}^A \gamma_0 - p' \cdot \gamma + m_A}{E_{p'}^A + \omega_{p'-p} + E_{p}^A} \\
= ig^2 \int \frac{d^4 p'}{(2\pi)^4} \left[ \left( p' \cdot \gamma + m_A \right) - \left( p' \cdot \gamma + m_A \right) \right] \bigg|_{p_0 = E_{p}^A}.
\end{equation}
The equivalence between the two expressions is shown directly in Appendix A by performing the integration over $p'_0$ in the manifestly covariant form.

We will now show that the contribution to $H_{BW}$ in the second line of Eq. (26) can be absorbed into a renormalization of the mass $m_A$. To this end, consider the covariant expression for $G(p)$ in Eq. (27): from covariance arguments, one has that $G(p)$ is of the form

$$G(p) = \left[ G_1(p^2) p \cdot \gamma + G_0(p^2) m_A \right] p_0 = E_A^p ,$$

hence by use of the Dirac equation

$$\bar{u}_A(p, r) G(p) u_A(p, s) = \bar{u}_A(p, r) \left[ G_1(m_A^2) m_A + G_0(m_A^2) m_A \right] u_A(p, s)$$

$$= 2m_A^2 \left[ G_1(m_A^2) + G_0(m_A^2) \right] \delta_{rs} .$$

Consequently, we define

$$\Delta m_A^2 = 2m_A^2 \left[ G_1(m_A^2) + G_0(m_A^2) \right]$$

in order to write

$$\langle p_A, r | H_{BW} | p_A', s \rangle = \left( E_V + E_{pA}^A + \frac{\Delta m_A^2}{2E_{pA}^A} \right) (2\pi)^3 \delta(p_A - p'_A) \delta_{rs} .$$

We are now in a position to perform the mass renormalization, completely within the Hamiltonian framework. First, define the renormalized or physical mass through

$$\left[ E_V + E_{pA}^A + \frac{\Delta m_A^2}{2E_{pA}^A} \right]_{p_A = 0} = E_V + M_A ,$$

then

$$M_A = m_A + \frac{\Delta m_A^2}{2m_A} + O(g^4) .$$

We can use Eq. (33) to express $m_A$ in terms of $M_A$ in Eq. (31), in particular in $E_{pA}^A$, taking into account that $\Delta m_A^2$ is of order $g^2$. Working consequently to order $g^2$, we arrive at

$$E_V + E_{pA}^A + \frac{\Delta m_A^2}{2E_{pA}^A} = E_V + \sqrt{M_A^2 + p_A^2} + O(g^4) .$$

Remarkably, through the mass renormalization (to the order presently considered) we have obtained an expression for the energy which is exactly covariant, in contradistinction to Eq. (31).

Now, the arguments to arrive at the manifestly covariant expression in Eq. (27) (see also Appendix A) and from there to the crucial equation (28), are formal in the sense that the (unregularized) expression for $G(p)$ is UV divergent. We show in Appendix B that a careful derivation using different regularization schemes leads to the same results. Observe, again, the absence of IR divergencies in the limit $\mu \to 0$ in the expressions (27) (or in the corresponding explicit expressions presented in Appendix B).
2.3 Two-fermion states and effective potential

Finally, in order to obtain the effective Schrödinger equation for $AB$ bound states, we consider the Bloch-Wilson Hamiltonian for the subspace

$$\Omega_0 = \text{span} \left\{ |p_A, r; p_B, s \rangle | p_A, p_B \in \mathbb{R}^3, r, s = 1, 2 \right\} \quad (35)$$

of all (free) states of one fermion $A$ and one fermion $B$, non-covariantly normalized as in Eq. [21]. Then the matrix elements of the Bloch-Wilson Hamiltonian to lowest non-trivial order turn out to be

$$\langle p_A, r; p_B, s | H_{BW} | p'_A, r'; p'_B, s' \rangle$$

$$= \left( E_V + \sqrt{M_A^2 + p_A'^2} + \sqrt{M_B^2 + p_B'^2} \right) (2\pi)^3 \delta(p_A - p'_A) \delta_{rr'} (2\pi)^3 \delta(p_B - p'_B) \delta_{ss'}$$

$$- ig^2 \int_{-\infty}^{0} dt e^{-i|t|} \int d^3x d^3x' \left[ \bar{\psi}_{p_B,s}(0, x) \psi_{p'_B,s'}(0, x) \right]$$

$$\times \Delta_F(0 - t, x - x') \left[ \bar{\psi}_{p_A,r}(t, x') \psi_{p'_A,r'}(t, x') \right]$$

$$- ig^2 \int_{-\infty}^{0} dt e^{-i|t|} \int d^3x d^3x' \left[ \bar{\psi}_{p_A,r}(0, x) \psi_{p'_A,r'}(0, x) \right]$$

$$\times \Delta_F(0 - t, x - x') \left[ \bar{\psi}_{p_B,s}(t, x') \psi_{p'_B,s'}(t, x') \right], \quad (36)$$

where $M_A$ and $M_B$ are the renormalized masses defined as in Eq. (32), and the wave functions $\psi_{p,r}^{A,B}(t, x)$ have been introduced in Eq. (23).

Due to the non-conservation of the perturbative energies, generally $E_{p_A}^A + E_{p_B}^B \neq E_{p'_A}^A + E_{p'_B}^B$ for $p_A + p_B = p'_A + p'_B$, and the corresponding lack of time translation invariance, the expressions with Feynman propagators in Eq. (36) cannot be converted to on-shell Feynman diagrams. It has not been much emphasized in [23] that the mixing of states with different perturbative energies is essential for the formation of bound states, because it is imperative for the localization of the constituents in relative position space: consider, e.g., states with total momentum $p_A + p_B = 0$, then a continuous superposition of states with different relative momenta $p = p_A$ is necessary to obtain a wavefunction of finite extension in relative position space.

Equation (36) leads to the following effective Schrödinger equation,

$$\left( \sqrt{M_A^2 + p_A'^2} + \sqrt{M_B^2 + p_B'^2} \right) \phi(p_A, r; p_B, s)$$

$$+ \sum_{r', s'=1}^2 \int \frac{d^3p_A'}{(2\pi)^3} \frac{d^3p_B'}{(2\pi)^3} \langle p_A, r; p_B, s | V | p'_A, r'; p'_B, s' \rangle \phi(p'_A, r'; p'_B, s')$$

$$= (E - E_V) \phi(p_A, r; p_B, s) \quad (37)$$
for the two-particle wave function in momentum space

\[ \phi(p_A, r; p_B, s) = \langle p_A, r; p_B, s | \phi \rangle, \quad |\phi\rangle \in \Omega_0. \tag{38} \]

Through the use of the non-covariant representations \([\text{II}]\) of the Feynman propagators in momentum space, the effective potential can be written as \([\text{II}]\)

\[ \langle p_A, r; p_B, s | V | p'_A, r'; p'_B, s' \rangle \]

\[ = - \frac{g^2}{\sqrt{2E_{p_A}^A 2E_{p_B}^B 2E_{p'_A}^A 2E_{p'_B}^B}} \frac{1}{2\omega_{p_A-p'_A}} \left( \frac{1}{E_{p_A}^A + \omega_{p_A-p'_A} - E_{p'_A}^A} + \frac{1}{E_{p_B}^B + \omega_{p_B-p'_B} - E_{p'_B}^B} \right) \]

\[ \times [\tilde{u}_A(p_A, r) u_A(p'_A, r')] [\tilde{u}_B(p_B, s) u_B(p'_B, s')] (2\pi)^3 \delta(p_A + p_B - p'_A - p'_B). \tag{39} \]

The masses \(m_{A,B}\) appearing in the kinetic energies \(E_{p}^{A,B}\) in the potential term can be replaced by their renormalized counterparts \(M_{A,B}\) to the present order in the perturbative expansion. Eq. \([39]\) differs from its scalar analogue only through the products of Dirac spinors (cf. Ref. \([23]\)).

The effective Hamiltonian commutes with the total 3-momentum operator \(P = p_A + p_B\), hence we consider total momentum eigenstates from now on. In particular, we specialize to the center-of-mass system \(P = 0\) where the effective Schrödinger equation becomes

\[ \left( \sqrt{M_A^2 + p^2} + \sqrt{M_B^2 + p^2} \right) \phi(p; r, s) - g^2 \sum_{r', s' = 1}^2 \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p}^A 2E_{p}^B 2E_{p'}^A 2E_{p'}^B}} \]

\[ \times \frac{1}{2\omega_{p-p'}} \left( \frac{1}{E_{p}^A + \omega_{p-p'} - E_{p'}^A} + \frac{1}{E_{p}^B + \omega_{p-p'} - E_{p'}^B} \right) \]

\[ \times [\tilde{u}_A(p, r) u_A(p', r')] [\tilde{u}_B(-p, s) u_B(-p', s')] \phi(p'; r', s') = (E - E_V) \phi(p; r, s), \tag{40} \]

with the relative momentum \(p = p_A - p_B\) and

\[ \phi(p_A, r; p_B, s) = \phi(p_A; r, s)(2\pi)^3 \delta(p_A + p_B). \tag{41} \]

Observe in the potential term in Eqs. \([39]\) and \([40]\) the square roots of the kinetic energies which are characteristic of the non-locality of the interaction, and the differences of energies in the denominators which are due to the retardation of the interaction. The latter lead to a further non-locality in the effective potential, and also introduce non-hermiticity in the effective Hamiltonian.

For the discussion of the non-relativistic and the one-body limits in the next section, and also for the numerical solution of the effective Schrödinger equation, it is convenient to cast Eq. \([40]\) into 2-spinorial form. To this end, the Dirac spinors are expressed in terms of Pauli spinors, most conveniently in the Dirac-Pauli representation,

\[ u_A(p, r) = \sqrt{E_{p}^A + M_A} \left( \begin{array}{c} \chi_A(p, r) \\ \frac{p \cdot \sigma}{E_{p}^A + M_A} \chi_A(p, r) \end{array} \right). \tag{42} \]
Here the Pauli spinors are normalized in the usual way,
\[ \chi_A^\dagger(p, r) \chi_A(p, s) = \delta_{rs}. \] (43)

The Pauli spinors may or may not depend on the momentum \( p \). The possibility of a momentum dependence is important if one wishes to employ helicity eigenspinors. In the present work, however, we will not make use of a momentum-dependent basis.

Eq. (40) can now be rewritten in spinorial form as
\[
\left( \sqrt{M_A^2 + p^2} + \sqrt{M_B^2 + p'^2} \right) \phi(p) \\
- g^2 \int \frac{d^3p'}{(2\pi)^3} \sqrt{\frac{E_A + M_A}{2E_p} \frac{E_B + M_B}{2E_{p'}} \frac{E_A + M_A}{2E_{p'}} \frac{E_B + M_B}{2E_{p'}}}
\times \frac{1}{2\omega_{p-p'}} \left( \frac{1}{E_p + \omega_{p-p'} - E_{p'}} + \frac{1}{E_{p'} + \omega_{p-p'} - E_p} \right)
\times \left[ 1 - \frac{p \cdot \sigma_A}{E_p + M_A} \frac{p' \cdot \sigma_A}{E_{p'} + M_A} \right] \left[ 1 - \frac{p \cdot \sigma_B}{E_p + M_B} \frac{p' \cdot \sigma_B}{E_{p'} + M_B} \right] \phi(p') = (E - E_V) \phi(p),
\] (44)

where the spinorial wave function \( \phi(p) \) is defined as
\[
\phi(p) = \sum_{r, s} \phi(p; r, s) [\chi_A(p, r) \otimes \chi_B(-p, s)].
\] (45)

As usual, \( \sigma_A \) is understood to act on \( \chi_A(p, r) \) only.

### 3 Limiting cases, identical fermions and antifermions

In the present section, we will perform a series of non-trivial analytical checks on the effective Schrödinger equation. We will begin with the non-relativistic limit: if the wave function \( \phi(p; r, s) \) is strongly suppressed for \( p^2 \gtrsim M_r^2 \), where
\[
M_r = \frac{M_A M_B}{M_A + M_B}
\] (46)
is the (renormalized) reduced mass, we can approximate the effective Schrödinger equation in the center-of-mass frame, Eq. (44), by
\[
\frac{p^2}{2M_r} \phi(p) - \int \frac{d^3p'}{(2\pi)^3} \frac{4\pi \alpha}{\mu^2 + (p - p')^2} \phi(p') = \left( E - E_V - M_A - M_B \right) \phi(p),
\] (47)
where we have introduced the “fine structure constant”
\[
\alpha = \frac{g^2}{4\pi}
\] (48)
(for details on the approximation of the energy denominators, see Ref. [23]). Observe in particular that the effective Hamiltonian acts trivially on the spin degrees of freedom in this limit, as expected from non-relativistic scattering processes where spin orientations remain unchanged. As a consequence, Eq. (47) has the same form as in the case of scalar constituents [23].

The non-relativistic limit of Eq. (44) is hence precisely what we expect on physical grounds, a non-relativistic Schrödinger equation with the usual Yukawa potential (after Fourier transforming to position space). As discussed in detail in Ref. [23], the limit is attained for $\alpha \ll 1$ and $\mu \ll M$, both conditions being necessary.

We will now consider the so-called one-body limit “$M_B \to \infty$” where $M_B^2 \gg M_A^2$ and the wave function $\phi(p; r, s)$ is negligibly small for $p^2 \gtrsim M_B^2$. In this case, we can approximate the effective Schrödinger equation (44) by

$$
\sqrt{M_A^2 + p^2} \phi(p) - g^2 \int \frac{d^3 p'}{(2\pi)^3} \frac{E_A^3 + M_A}{2E_p^A} \frac{E_A^3 + M_A}{2E_p'^A} \times \frac{1}{2\omega_{p-p'}} \frac{1}{\omega_{p-p'}} \left( \frac{1}{E_p^A + \omega_{p-p'} - E_p'^A} \right) \left[ 1 - \frac{p \cdot \sigma_A}{E_p^A + M_A} \frac{p' \cdot \sigma_A}{E_p'^A + M_A} \right] \phi(p')
$$

$$
= (E - E_V - M_B) \phi(p) . \quad (49)
$$

The effective Hamiltonian acts trivially on the spin of particle $B$ in this limit. Eq. (49) has the form of a relativistic equation for fermion $A$ in an external potential, independent of the mass (except for a constant shift in the energy) and the spin orientation of particle $B$, in accord with physical expectations [10].

We can, however, go one step further and compare Eq. (49) with the relativistic equation for particle $A$ in the external potential due to a fixed source which exchanges spinless bosons of mass $\mu$ with particle $A$. The latter physical situation can also be described within the same general formalism, providing an internal consistency check for the application of the generalized Gell-Mann–Low theorem.

To this end, we begin by defining the Hamiltonian $H'$ of the fixed source system, which consists of the Hamiltonians corresponding to free fermions $A$ and scalar bosons of mass $\mu$, and the interaction term

$$
H'_1 = g \int d^3 x : \bar{\psi}_A(x) \psi_A(x) \varphi(x) : + g \varphi(0) . \quad (50)
$$

The form of the second contribution to the interaction Hamiltonian results from replacing the dynamical fermion field $B$ in Eq. (10) with a fixed source at $x = 0$. To see that, we consider $\psi_B(x)$ as a classical Dirac field with probability density given by

$$
\rho(x) = \psi_B^\dagger(x) \psi_B(x) \approx \bar{\psi}_B(x) \psi_B(x) , \quad (51)
$$

the latter approximate equality holding when $\psi_B(x)$ describes a particle (not an anti-particle) and the relevant momenta satisfy $p^2 \ll M_B^2$. For a fermion $B$ localized at $x = 0$ we then
have

\[ \bar{\psi}_B(x)\psi_B(x) = \delta(x), \]  

(52)

from which Eq. (50) follows.

The Bloch-Wilson Hamiltonian in the one-fermion sector is represented diagrammatically to lowest non-trivial order [see Eq. (11)] in Ref. [23] (adding spin labels to the external lines for the present case). The corresponding algebraic expressions lead to a mass renormalization for fermion \( A \) which is identical to the one discussed in the previous section, and the following matrix elements of the effective Hamiltonian:

\[
\langle p_A, r | H'_{BW} | p'_A, r' \rangle = 
\left( E'_{V} + \sqrt{M^2_A + p^2_A} \right) (2\pi)^3 \delta(p_A - p'_A) \delta_{rr'}
- i g^2 \int_0^\infty dt e^{-|t|} \int d^3x \Delta_F(0 - t, 0 - x) \left[ \bar{\psi}^A_{p_A}(t, x) \psi^A_{p_A'}(t, x) \right]
- i g^2 \int_\infty^0 dt e^{-|t|} \int d^3x \left[ \bar{\psi}^A_{p_A}(0, x) \psi^A_{p_A'}(0, x) \right] \Delta_F(0 - t, -x),
\]  

(53)

where the (renormalized) vacuum energy \( E'_{V} \) differs from \( E_{V} \) as determined from Eq. (12) for the case of two dynamical fermions, precisely because fermion \( B \) has now been replaced by a fixed source which is formally considered to be part of the vacuum.

Using the non-covariant expression (14) for the propagators and Eq. (23) for the fermionic wave functions, Eq. (53) leads to the effective Schrödinger equation

\[
\sqrt{M^2_A + p^2} \phi(p) - g^2 \int d^3p' \left( \frac{1}{E^A_p + \omega_{p-p'}} + \frac{1}{E^A_{p'} + \omega_{p-p'} - E^A_p} \right) \left[ 1 - \frac{p \cdot \sigma}{E^A_p + M_A} - \frac{p' \cdot \sigma}{E^A_{p'} + M_A} \right] \phi(p')
= (E - E'_{V}) \phi(p),
\]  

(54)

where the wave function is now defined as

\[
\phi(p) = \sum_r \phi(p, r) \chi(p, r) \equiv \sum_r \langle p, r | \phi \rangle \chi(p, r).
\]  

(55)

Equation (54) is identical to Eq. (49) except for an irrelevant shift in the vacuum energy and the fact that the wave function in Eq. (49) includes the orientation of the spin of fermion \( B \) which, however, has no influence on the dynamics of particle \( A \). Self-consistency of the method in the one-body limit is hence established.
We will now investigate how the effective Schrödinger equation changes when we replace one of the constituents, say fermion $A$, by the corresponding antiparticle. First of all, consider the one-$\bar{A}$-antifermion sector where the matrix elements of the Bloch-Wilson Hamiltonian to lowest non-trivial order are given by

$$
\langle \mathbf{p}_A, r | H_{BW} | \mathbf{p}_A', s \rangle = \left( E_V + E_{\mathbf{p}_A}^A \right) (2\pi)^3 \delta(\mathbf{p}_A - \mathbf{p}_A') \delta_{rs}
$$

$$
+ ig^2 \int_{-\infty}^0 dt e^{-\epsilon t} \int d^3 x d^3 x' \left[ \psi_{\mathbf{p}_A', s}^A(t, x') S_F^A(t - 0, x - x') \psi_{\mathbf{p}_A, r}^A(0, x) \right] \Delta_F(0 - t, x - x')
$$

$$
+ ig^2 \int_{-\infty}^0 dt e^{-\epsilon t} \int d^3 x d^3 x' \left[ \psi_{\mathbf{p}_A', s}^A(0, x) S_F^A(0 - t, x - x') \psi_{\mathbf{p}_A, r}^A(0, x') \right] \Delta_F(t - 0, x' - x)
$$

[compare with Eq. (22) and note the change in sign]. Here we need the antifermion wave functions defined by

$$
\bar{\psi}_{\mathbf{p}, r}^A(t, x) = \bar{v}_A(\mathbf{p}, r) e^{-iE_A^t + i\mathbf{p} \cdot \mathbf{x}}.
$$

We can convert Eq. (56) to the form corresponding to a particle $A$ by introducing the charge conjugate wave functions

$$
\psi_{\mathbf{p}, r}^{A, C}(t, x) = C \left[ \bar{\psi}_{\mathbf{p}, r}^A(t, x) \right]^T
$$

(the superindex $T$ stands for transposition), with the charge conjugation matrix $C = -i\gamma^0\gamma^2$ in the Dirac-Pauli representation. Eq. (56) is then transformed into expression (22) for the subspace of one fermion $A$ (including the sign) with the wave function $\psi_{\mathbf{p}, r}^A(t, x)$ being replaced by $\psi_{\mathbf{p}, r}^{A, C}(t, x)$, which corresponds to the replacement of the particle spinor $u_A(\mathbf{p}, r)$ with the charge conjugate antiparticle spinor

$$
v_A^C(\mathbf{p}, r) = C \left[ \bar{v}_A(\mathbf{p}, r) \right]^T
$$

for the description of the spin orientation of the antifermion. Since $v_A^C(\mathbf{p}, r)$ is a positive-energy solution of the Dirac equation, the mass renormalization from $m_A$ to $M_A$ for antifermions is identical to the one for fermions $A$.

We now turn to the two-particle sector with an antifermion $\bar{A}$ and a fermion $B$. The matrix elements of the corresponding Bloch-Wilson Hamiltonian to lowest non-trivial order...
read

\[
\langle p_A, r; p_B, s | H_{BW} | p'_A, r'; p'_B, s' \rangle
\]

\[
= \left( E_V + \sqrt{M_A^2 + p_A^2} + \sqrt{M_B^2 + p_B^2} \right) (2\pi)^3 \delta(p_A - p'_A) \delta(r, r') (2\pi)^3 \delta(p_B - p'_B) \delta(
\]

\[
+ ig^2 \int_{-\infty}^{0} dt e^{-\epsilon t} \int d^3x d^3x' \left[ \bar{\psi}_{p_B, s}^B(0, x) \psi_{p_B, s'}^B(0, x) \right]
\]

\[
\times \Delta_F(0 - t, x - x') \left[ \bar{\psi}_{p'_A, r'}^A(t, x') \psi_{p'_A, r}^A(t, x) \right]
\]

\[
+ ig^2 \int_{-\infty}^{0} dt e^{-\epsilon t} \int d^3x d^3x' \left[ \bar{\psi}_{p'_A, r'}^A(0, x) \psi_{p'_A, r}^A(0, x) \right]
\]

\[
\times \Delta_F(0 - t, x - x') \left[ \bar{\psi}_{p_B, s}^B(t, x') \psi_{p_B, s'}^B(t, x) \right],
\] 

(60)

to be compared with Eq. (36). Again, the introduction of the charge conjugate wave functions converts Eq. (60) to the form of Eq. (36), with the wave function \( \psi_{p, r}^A(t, x) \) replaced by \( \psi_{p, r}^A(t, x) \), or the spinor \( u_A(p, r) \) by \( v_A^C(p, r) \). The effective Schrödinger equation for constituents \( \bar{AB} \) can then be written in the spinorial form of Eq. (44), where in the definition (13) of the wave function the Pauli spinor \( \chi_A(p, r) \) has to be replaced by the charge conjugate spinor

\[
\xi_A^C(p, r) = i\sigma^2 \xi_A^A(p, r),
\] 

(61)

\( \xi_A(p, r) \) being the Pauli spinor that describes the spin orientation of the corresponding negative-energy solution of the Dirac equation. The most important result is that, just as for the mass renormalization, the interaction via a scalar boson is “charge conjugation blind”, i.e., does not distinguish between a fermion \( A \) and its antifermion \( \bar{A} \). This is, in fact, the expected behaviour.

The same arguments are used to describe an antifermion \( \bar{A} \) interacting with a static source which leads to a Bloch-Wilson Hamiltonian and an effective Schrödinger equation analogous to Eqs. (53) and (54). Consequently, the one-body limit is self-consistent also in the case of a (light) antifermion \( \bar{A} \). More interesting is the case of an antisource corresponding to the one-body limit \( M_B \rightarrow \infty \) for an antifermion \( \bar{B} \): a simple-minded argument replaces \( \psi_B(x) \) in the interaction Hamiltonian with a classical negative-energy Dirac field, which leads to a probability density

\[
\rho(x) = \psi_B^\dagger(x) \psi_B(x) \approx -\bar{\psi}_B(x) \psi_B(x),
\] 

(62)

if we suppose that the relevant momenta satisfy \( p^2 \ll M_B^2 \). Equation (62) with \( \rho(x) = \delta(x) \) would be in conflict with the one-body limit \( M_B \rightarrow \infty \) for constituents \( \bar{AB} \) or \( \bar{AB} \). The reason is, of course, that \( \rho(x) \) is to be interpreted physically as a charge density (when multiplied with the charge of fermion \( B \)), and is negative for antifermions, hence it is \( \bar{\psi}_B(x) \psi_B(x) \)
which turns out to be positive and is to be replaced by \( \delta(x) \). The latter results are related to the use of anticommutators in the quantization of the Dirac field.

In order to have a formally satisfactory description, we define an approximately localized antifermion state with spin orientation \( s \) as

\[
| x = 0, \Delta x; s \rangle = \int \frac{d^3p_B}{(2\pi)^3} \left( \frac{8\pi}{3} \Delta x^2 \right)^{3/4} e^{-\Delta x^2 p_B^2/3} | p_B, s \rangle ,
\]

where \( | p_B, s \rangle \) denotes a \( \bar{B} \)-antifermion 3-momentum eigenstate. If we choose \( M_B \) large enough for

\[
\Delta x^2 M_B^2 \gg 1
\]

to hold, we obtain

\[
\langle x = 0, \Delta x; s | : \bar{\psi}_B(x) \psi_B(x) : | x = 0, \Delta x; s \rangle = \left( \frac{3}{2\pi^2 \Delta x^2} \right)^{3/2} e^{-3x^2/(2\Delta x^2)} .
\]

In the limit \( \Delta x \to 0 \) the right-hand side of Eq. (65) tends towards \( \delta(x) \). Equation (64) implies that we need \( M_B \to \infty \) (even faster) in this limit. We take Eq. (65) as justification to replace \( [ : \bar{\psi}_B(x) \psi_B(x) :] \) with \( \delta(x) \) for a fixed antisource, leading to the interaction Hamiltonian (50).

We will now consider bound states of identical fermions, \( A \)-fermions to be concrete. To this end, we calculate the Bloch-Wilson Hamiltonian to lowest non-trivial order for the subspace

\[
\Omega_0 = \text{span} \{ | p_A, r; p_A', s \rangle | p_A, p_A' \in \mathbb{R}^3, r, s = 1, 2 \}
\]

of two-\( A \)-fermion states. As a consequence of the identity of the constituents, “crossed” diagrams appear which carry a relative minus sign due to the antisymmetry

\[
| p_A, r; p_A', s \rangle = -| p_A', s; p_A, r \rangle .
\]

However, mass renormalization works exactly as before, and also the effective Schrödinger equation is the same as in the case of \( AB \) bound states (with \( M_B = M_A \)) when we take into account the antisymmetry of the wave function

\[
\phi(p_A, r; p_A', s) = -\phi(p_A', s; p_A, r) .
\]

In the center-of-mass system, we have consequently

\[
\phi(p) = -\phi(-p)^t
\]

with the definitions (41) and (45), where the transposition \( t \) refers to the tensor product,

\[
[\chi_A(-p, r) \otimes \chi_A(p, s)]^t = \chi_A(p, s) \otimes \chi_A(-p, r) .
\]
The consequences of antisymmetry for the solutions of the effective Schrödinger equation will be discussed in the next section. In the non-relativistic limit, however, the situation is particularly simple because the spin degrees of freedom do not participate in the dynamics: the solutions of Eq. (17) with even orbital angular momentum have symmetric spatial wave functions and hence antisymmetric spin states (total spin zero), while solutions with odd orbital angular momentum have antisymmetric orbital wave functions and hence necessarily symmetric spin states (total spin one).

In a theory which only contains $A$-fermions and scalar bosons, the results for the mass renormalization and the $AA$ bound states are the same as the ones presented above for a theory with $A$- and $B$-fermions, only the (irrelevant) corrections to the free vacuum energy, $E_V - E_0$, are different (the $B$-fermion vacuum loops are missing).

The last case we will consider in this section is the one of $AA$ bound states, of one fermion and the corresponding antifermion. In this case, there are additional contributions from the virtual annihilation diagrams to the effective Hamiltonian, so that the effective potential reads

$$
\langle \mathbf{p}_A, r; \mathbf{p}_A, s | V | \mathbf{p}_A', r'; \mathbf{p}_A', s' \rangle
= i g^2 \int_{-\infty}^{0} dt e^{-\epsilon|t|} \int d^3 x d^3 x' \left[ \tilde{\psi}_{\mathbf{p}_A, s}^A(0, \mathbf{x}) \psi_{\mathbf{p}_A, s}^A(0, \mathbf{x}) \right] \times \Delta_F(0 - t, \mathbf{x} - \mathbf{x}') \left[ \tilde{\psi}_{\mathbf{p}_A, r}^A(t, \mathbf{x}') \psi_{\mathbf{p}_A, r}^A(t, \mathbf{x}') \right]
$$

$$
+ i g^2 \int_{-\infty}^{0} dt e^{-\epsilon|t|} \int d^3 x d^3 x' \left[ \tilde{\psi}_{\mathbf{p}_A, r}^A(t, \mathbf{x}) \psi_{\mathbf{p}_A, r}^A(t, \mathbf{x}) \right] \times \Delta_F(0 - t, \mathbf{x} - \mathbf{x}') \left[ \tilde{\psi}_{\mathbf{p}_A, s}^A(t, \mathbf{x}') \psi_{\mathbf{p}_A, s}^A(t, \mathbf{x}') \right]
$$

$$
- i g^2 \int_{-\infty}^{0} dt e^{-\epsilon|t|} \int d^3 x d^3 x' \left[ \tilde{\psi}_{\mathbf{p}_A, s}^A(0, \mathbf{x}) \psi_{\mathbf{p}_A, s}^A(0, \mathbf{x}) \right] \times \Delta_F(0 - t, \mathbf{x} - \mathbf{x}') \left[ \tilde{\psi}_{\mathbf{p}_A, r}^A(t, \mathbf{x}') \psi_{\mathbf{p}_A, r}^A(t, \mathbf{x}') \right]
$$

$$
- i g^2 \int_{-\infty}^{0} dt e^{-\epsilon|t|} \int d^3 x d^3 x' \left[ \tilde{\psi}_{\mathbf{p}_A', r}^A(0, \mathbf{x}) \psi_{\mathbf{p}_A', r}^A(0, \mathbf{x}) \right] \times \Delta_F(0 - t, \mathbf{x} - \mathbf{x}') \left[ \tilde{\psi}_{\mathbf{p}_A, s}^A(t, \mathbf{x}') \psi_{\mathbf{p}_A, s}^A(t, \mathbf{x}') \right]
$$

$$
= \frac{g^2}{\sqrt{2E_{p_A}^A 2E_{p_A}^A 2E_{p_A}^A 2E_{p_A}^A}} \left\{ \frac{1}{2\omega_{p_A-p_A}} \left( \frac{1}{E_{p_A}^A + \omega_{p_A-p_A} - E_{p_A}^A} + \frac{1}{E_{p_A}^A + \omega_{p_A-p_A} - E_{p_A}^A} \right) \times [\tilde{u}_A(p_A, r) u_A(p_A', r')] [\bar{\psi}_A(p_A', s') \bar{v}_A(p_A, s)] \right\}
$$

$$
- \frac{1}{2\omega_{p_A-p_A}} \left( \frac{1}{\omega_{p_A-p_A} + E_{p_A}^A + E_{p_A}^A} + \frac{1}{\omega_{p_A-p_A} + E_{p_A}^A + E_{p_A}^A} \right) \right\}
$$

$$
- \frac{1}{2\omega_{p_A-p_A}} \left( \frac{1}{\omega_{p_A-p_A} + E_{p_A}^A + E_{p_A}^A} + \frac{1}{\omega_{p_A-p_A} + E_{p_A}^A + E_{p_A}^A} \right) \right\}
$$

20
\[
\times \left[ \bar{u}_A(p_A, r) v_A(p_A, s) \right] \left[ \bar{v}_A(p_A', s') u_A(p_A', r') \right] \right) (2\pi)^3 \delta(p_A + p_A - p_A' - p_A').
\] (71)

We will leave the discussion of the effect of virtual annihilation on the bound state energies in different theories and the related instability of these states for a future investigation.

To close this section, we will briefly comment on the corresponding results in a purely bosonic theory, where the constituents are chosen to be charged scalar bosons. The consistency of the non-relativistic and one-body limits in this case has been shown in detail in Ref. [23]. If we substitute one or both of the constituents by antibosons, there are, compared to the fermionic case, no additional minus signs from anticommutation relations to take into account and, of course, no spinor structures, consequently the whole argument is much simpler than for fermionic constituents. The results are, however, finally the same: mass renormalization is identical for antiparticles and for particles, and the interaction due to scalar boson exchange is universally attractive and does not distinguish particles from antiparticles.

As for a static antisource, consider the charge density (properly to be multiplied by the charge of boson \( B \))

\[
\rho(x) = \phi_B^*(x) i \frac{\partial}{\partial t} \phi_B(x) - \phi_B(x) i \frac{\partial}{\partial t} \phi_B^*(x) \approx -2M_B \phi_B^*(x) \phi_B(x)
\] (72)

for a classical negative-energy solution of the Klein-Gordon equation, in case that the relevant momenta fulfill \( p^2 \ll M_B^2 \). The probability density is the negative of \( \rho(x) \), hence we would replace

\[
: \phi_B^\dagger(x) \phi_B(x) := \frac{1}{2M_B} \delta(x)
\] (73)

in the interaction Hamiltonian for a localized antisource. A more formal argument proceeds in analogy with Eqs. (63)–(65) for the fermionic case, with the result (73). The one-boson limit \( M_B \to \infty \) is then consistent also in the case of an antiboson \( \bar{B} \).

Finally, in the case of identical bosonic constituents, the effective Schrödinger equation is the same as for bosonic \( AB \) bound states, only that the wave function has to be symmetric under particle exchange in this case. Since there are no spin degrees of freedom in the scalar bosonic case, the spatial wave function has to be symmetric, hence only even angular momenta are allowed.

### 4 Numerical solution

In order to actually solve the effective Schrödinger equation in the form (44), it is convenient first to separate off the angular and spin degrees of freedom. A direct numerical solution of Eq. (44) would lead to numerical instabilities for equal and opposite momenta, due to the presence of a singularity in the integrand. The effective Hamiltonian is rotationally invariant, hence it is natural to consider total angular momentum eigenstates. To make contact to the
usual spectroscopy, we choose to couple first the individual spins to a total spin $S$ and then couple this spin with the relative orbital angular momentum $L$ to the total angular momentum $J$. The usual construction with Clebsch-Gordan coefficients yields simultaneous eigenstates of $J^2$, $J_z$, $S^2$, and $L^2$ which we will denote as $^{2S+1}Y_{JM}(\hat{p})$, $\hat{p} \equiv p/|p|$. Explicit expressions are given in Appendix [C].

The Hamiltonian (44) contains the helicity operators $\hat{p} \cdot \sigma_A$ and $\hat{p} \cdot \sigma_B$. These operators are hermitian and unitary, and in particular

\[
(\hat{p} \cdot \sigma_A)^2 = (\hat{p} \cdot \sigma_B)^2 = 1. \tag{74}
\]

The helicity operators are invariant under spatial rotations, however, they are odd under spatial parity transformations which maintain the spin directions unchanged. Since the Schrödinger equation (44) contains only even powers of helicity operators, the effective Hamiltonian is parity even (the intrinsic parities of the constituent fermions have no use in the present context, and we will not consider them in the following).

We hence have the conservation of total angular momentum $J$ and spatial parity $(-1)^l$, but a priori not of relative orbital angular momentum $l$ or total spin $S$. For given $J$, $l$ can take the values $J$ (for $S = 0$) and $J, J \pm 1$ (for $S = 1$), $l = J, J - 1$ being excluded for $J = 0$, $S = 1$. Taking into account the conservation of $(-1)^l$, we can hence conclude without any explicit calculation that the effective Hamiltonian may mix states with $l = J, S = 0$ and with $l = J, S = 1$ on the one hand (we will call this “S-coupling”), and states $l = J - 1, S = 1$ and $l = J + 1, S = 1$ (“L-coupling”) on the other. The effective Schrödinger equation will then decay into pairs of coupled one-dimensional equations. In the special case $J = 0$, neither of the two mixings is possible. For future use we remark that the use of helicity eigenstates is expected to diagonalize the effective Hamiltonian in the S-coupled sector, thus slightly simplifying the calculations, although there is no reason why L-coupling should not occur in this case.

For the actual solution of the effective Schrödinger equation (44), we need explicit expressions for the application of the helicity operators. Again, it is clear from the fact that the helicity operators preserve total angular momentum and change parity, that the application of a helicity operator maps S-coupled states to L-coupled states and vice versa. The explicit expressions, as well as a rather pedestrian way to derive them, are presented in Appendix [C]. Finally, the integration over the angles, i.e., over $\hat{p}'$, in Eq. (44) can be performed with the help of a partial wave decomposition combined with the spherical harmonics addition theorem,

\[
V(p, p', \cos \theta) = \sum_{l=0}^{\infty} \frac{4\pi}{2l + 1} a_t(p, p') \sum_{m=-l}^{l} Y_{lm}(\hat{p})Y_{lm}^*(\hat{p}'), \tag{75}
\]

where $p \equiv |p|$, $\theta$ denotes the angle between $p$ and $p'$, and

\[
a_t(p, p') = \frac{2l + 1}{2} \int_{-1}^{1} d \cos \theta P_l(\cos \theta) V(p, p', \cos \theta). \tag{76}
\]
The whole procedure outlined above can be carried through independently of the boson mass $\mu$. In what follows, I will focus on the case of massless bosons $\mu = 0$, as discussed in the introduction. The explicit form of the effective Schrödinger equation in this case is, again, given in Appendix C. No additional difficulties are expected in the massive case in principle, even though the existence of a critical coupling constant changes the qualitative features of the spectrum.

The (pairs of) one-dimensional integral equations can now be solved numerically. To this end, the equations were converted to (continuous) matrix form and the corresponding two-dimensional integrals approximated by finite sums over a discrete two-dimensional grid. The distribution of abscissas took the logarithmic singularity of the integrand and the long range of the wave functions in configuration space into account. We have approximated the solutions by a finite linear combination of an appropriately chosen set of basis functions, the same we had used before in the scalar case [23]. Two parameters that determine the shape of the basis functions were optimized variationally. The orthogonality of the basis functions could be retained numerically to 11 to 14 decimal places. Both energies and wave functions converged with increasing number of integration points and basis functions. However, in our experience [23] convergence does not guarantee the correctness of the solutions if the choice of basis is inappropriate. For this reason, we have also checked the residual $r_i(p)$ of the solutions $\phi_i(p)$ defined as $(E - H_{BW})\phi_i(p) = r_i(p)$. The determination of the residual $r_i(p)$ and the “point-wise” convergence of the wave functions were limited essentially by the redundancy of the grid points (up to 400) with respect to the number of basis functions (up to 40). The analogue of the Gibbs phenomenon in Fourier series was, in the worst case, of the order of two percent.

The numerical solutions (for $\mu = 0$) are shown in Figs. 1 and 2 for two extreme mass ratios, $M_A = M_B$ and $M_B \to \infty$ (with $M_A$ fixed), as functions of the (Yukawa theory) fine structure constant $\alpha = g^2/4\pi$. Between these two extremes, the eigenvalues for fixed $\alpha$ can be seen to vary smoothly with the mass ratio in Fig. 3 for $\alpha = 1$. In all figures, the energy eigenvalues are represented normalized to twice the non-relativistic ionization energy in a Coulomb potential, $M_r \alpha^2$, where $M_r$ denotes the reduced mass.

In Figs. 1 and 2 it is seen that in the small-coupling limit $\alpha \to 0$, the energies tend to the non-relativistic Coulomb values

$$-\frac{M_r \alpha^2}{2n^2}, \quad n = 1, 2, 3, \ldots,$$

as expected for $\mu = 0$. In particular, we observe the characteristic degeneracies in this limit. For instance, we expect and find that the following states tend to the same energy eigenvalue $-M_r \alpha^2/8$ (principal quantum number $n = 2$) for $\alpha \to 0$: the ground states in the L-coupled $J = 0$ and $J = 2$ sectors (states $2^3P_0$ and $2^3P_2$ in the usual spectroscopic notation $n^{2S+1}L_J$) and in the S-coupled $J = 1$ sector ($2^1P_1$), as well as the first excited states in the S-coupled $J = 0$ and $J = 1$ sectors ($2^1S_0$ and $2^3P_1$) and in the L-coupled $J = 1$ sector ($2^3S_1$). To be precise, the lowest-energy eigenstates in the S-coupled $J = 1$ sector are linear combinations
Figure 1: The lowest energy eigenvalues (principal quantum numbers \( n = 1 \) and \( n = 2 \)) of the effective Hamiltonian as functions of the fine structure constant \( \alpha = g^2/(4\pi) \), for the case of equal fermion masses.
Figure 2: The lowest energy eigenvalues \((n = 1 \text{ and } n = 2)\) in the one-body limit \(M_B \to \infty\).
Figure 3: The variation of the lowest energy eigenvalues with the mass ratio $M_r/M_A$ between the extreme cases $M_r = M_A/2$ and $M_r = M_A$ depicted in Figs. 1 and 2, respectively, for fixed $\alpha = 1$. 

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of the $2^1P_1$ and $2^3P_1$ states, the coefficients depending on the mass ratio. In Fig. 2 only three of these degenerate states are visible because there are always two curves lying on top of each other (see the discussion of the one-body limit below).

Let us now discuss the equal-mass case of Fig. 1 in detail. The binding is weaker in this case than predicted by the non-relativistic formula. Compared to an electromagnetic interaction (exchange of photons) as in positronium, where we eliminate the contribution of the virtual annihilation diagram, the sign of the relativistic corrections is just the opposite (with the exception of the $1^3S_1$ state). Also, the ordering of the levels is different.

In the case of equal masses, the effective Hamiltonian possesses an additional symmetry under the exchange of the fermions $A$ and $B$,

$$\phi(p) \rightarrow \phi(-p)^t$$

in terms of the spinorial wave function in the center-of-mass system (see Eq. (70)). This symmetry is implicit in our discussion of bound states of identical fermions in Section 3, where we saw that in the case of identical constituents the effective Schrödinger equation is the same as for $AB$ bound states, hence it must possess exchange symmetry. The exchange parity for the angular momentum eigenstates is $(-1)^l(-1)^{S+1}$. Together with the symmetry under spatial parity $(-1)^l$ (remember that we do not take the intrinsic parities of the fermions into account), this new symmetry forbids the $S$-coupling, hence $S$ becomes a good quantum number in this case. This can also be seen explicitly in the expressions for the matrix elements of the effective potential in the $S$-coupled sector, Eq. (169) in Appendix C, for $M_A = M_B$. In the case of identical fermionic constituents, the wave function must be antisymmetric with respect to particle exchange, hence, for $J$ even, in the $S$-coupled sector only $S = 0$ states are possible, while we have both $l = J - 1$ and $l = J + 1$ states in the $L$-coupled sector, in particular the coupling between these states remains. On the other hand, for $J$ odd, we have only $S = 1$ states in the $S$-coupled sector, and no possible state in the $L$-coupled sector. Among the eight states shown in Fig. 1 five are antisymmetric under particle exchange, hence, for $J$ even, in the $S$-coupled sector only $S = 0$ states are possible, while we have both $l = J - 1$ and $l = J + 1$ states in the $L$-coupled sector, in particular the coupling between these states remains. On the other hand, for $J$ odd, we have only $S = 1$ states in the $S$-coupled sector, and no possible state in the $L$-coupled sector. Among the eight states shown in Fig. 1 five are antisymmetric under particle exchange, namely the states $1^1S_0$, $2^1S_0$, $2^3P_0$, $2^3P_1$, and $2^3P_2$. We can see explicitly in these examples that the absence of $S$-coupling is necessary for the antisymmetry of the states.

In the one-body limit $M_B \rightarrow \infty$ depicted in Fig. 2 the sign of the relativistic corrections changes for several states with respect to the equal-mass case. Within numerical accuracy, there are always two exactly degenerate states. The reason for the degeneracy can be seen from the effective Schrödinger equation in the one-body limit, Eq. (49): the effective Hamiltonian is invariant under rotations which involve the spatial coordinates and the spin $s_A$ of particle $A$ only, and independently under rotations of spin $s_B$ (which does not affect the dynamics). The total angular momentum of fermion $A$, $j_A = \mathbf{L} + s_A$, is then a conserved quantity, and it is natural to consider simultaneous eigenstates of $j_A^2$, $j_{A,z}$, $\mathbf{L}^2$, and $s_{B,z}$. Since $j_A = l \pm 1/2$ and spatial parity $(-1)^l$ is conserved as before, it follows that $l$ is a good quantum number in this limit.

As a further consequence, states that only differ in the value of $j_{A,z}$ or $s_{B,z}$ are degenerate, and one may as well consider simultaneous eigenstates of $\mathbf{J}^2$, $J_z$, $J_A^2$, and $\mathbf{L}^2$, where $\mathbf{J} = j_A + s_B$ with eigenvalues $J = j_A \pm 1/2$. Now states which only differ in the eigenvalue of $\mathbf{J}^2$ or $J_z$
are degenerate. We can express the latter states in terms of the simultaneous eigenstates of $J^2$, $J_z$, $L^2$, and $S^2$ that we are using in the numerical calculations. From the foregoing discussion, we expect the absence of $L$-coupling and the degeneracy of one of the $S$-coupled states which is eigenstate of $J^2$ with eigenvalue $j_A = J + 1/2$ and $l = J$, with an $L$-coupled state with $J^2$-eigenvalue $J + 1$ and $l = J$ (hence necessarily $j_A = J + 1/2$). The other $S$-coupled state with $l = J$ is an $j_A^2$-eigenstate with $j_A = J - 1/2$, and is degenerate with an $L$-coupled state with $J^2$-eigenvalue $J - 1$, $l = J$, and $j_A = J - 1/2$. Explicit expressions for the eigenstates in the different coupling schemes and their relations are given in Appendix C, where we follow a different line of reasoning starting from the explicit expressions for the matrix elements of the effective potential in the $S$- and $L$-coupled sectors, Eqs. (169) and (172).

These expectations are fully borne out in the results of the numerical calculations. Among the eight states calculated in Fig. 2 we expect and find the following to be degenerate: $1^1S_0$ and $1^3S_1$, $2^1S_0$ and $2^3S_1$, $2^3P_0$ with one linear combination of $2^1P_1$ and $2^3P_1$, and $2^3P_2$ with the orthogonal linear combination of $2^1P_1$ and $2^3P_1$. The coefficients of these linear combinations are predicted and found to be $-\sqrt{1/3}$ and $\sqrt{2/3}$, and $\sqrt{2/3}$ and $\sqrt{1/3}$, respectively (see Eq. (180) in Appendix C), and correspond to eigenstates of $j_A^2$ with eigenvalues $j_A = 1/2$ and $3/2$. Unlike for the Dirac equation with an electromagnetic Coulomb potential, states with the same $j_A$ but different $l$, here $j_A = 1/2$ and $l = 0$ (states $2^1S_0$ and $2^3S_1$) or $l = 1$ (states $2^3P_0$ and the first linear combination of $2^1P_1$ and $2^3P_1$), are not degenerate. In addition, the ordering of the $l = 1$ states is opposite to the electromagnetic case.

Figure 3 shows the smooth transition between the two extreme mass ratios $M_A = M_B$ and $M_A/M_B = 0$ for fixed $\alpha = 1$, and the appearance of the characteristic degeneracies in the limit $M_A/M_B \to 0$. One would like to compare the coefficients of the mixing of the $S$-coupled states $2^1P_1$ and $2^3P_1$ against theoretical predictions. However, the diagonalization of the effective potential matrix (169) is generally not possible without solving the entire equation, i.e., it cannot be isolated from the $p$-dependence of the wave function. To the order $\alpha^4$ of the first relativistic corrections, the diagonalization can still be performed analytically and turns out to factorize from the “radial” $p$-dependence. The results are presented in Appendix D and provide very satisfactory approximations to the corresponding results of the numerical calculations. In particular, the analytical results indicate that there is no hyperfine splitting to the order $\alpha^4$ and $M_A/M_B$, for the energy levels that are degenerate in the one-body limit. In fact, even for $\alpha = 1$ no hyperfine splitting of the order $M_A/M_B$ is visible in Fig. 4.

5 Conclusions

We have presented what appears to be the first consistent treatment of bound states in Yukawa theory. It is the result of a straightforward application of the generalized Gell-Mann–Low theorem. The consistency of the method has been checked thoroughly. In particular, we have shown that mass renormalization can be performed exactly as in a manifestly covariant formulation, even though the renormalization conditions were imposed entirely within our Hamiltonian framework. We have checked the non-relativistic and one-body limits, replaced
the fermionic constituents by antifermions, and considered the case of identical constituents. In all these cases, the formalism generates the correct results in a very natural way. In the numerical calculations, no abnormal solutions have been found (nor were there expected to be any, due to the absence of relative time or energy as a dynamical variable), and all the characteristic degeneracies in the non-relativistic and one-body limits show up in the numerical results with very good accuracy.

In general terms, the framework presented here has several advantages over other formulations of quantum field theoretic bound state equations. As we have shown, the derivation of the effective Schrödinger equation is straightforward and presents no essential complications in the case of fermionic constituents as compared to scalar bosons. In principle, the complete bound state spectrum can be obtained as we have demonstrated by numerically determining the eight lowest-lying states (corresponding to the non-relativistic principal quantum numbers $n = 1$ and $n = 2$). The wave functions for these states are also obtained in the course of the (approximate) diagonalization of the effective Hamiltonian.

Several rather technical issues, which are nonetheless expected to be important for related work in the near future, have been treated in detail in the appendices. In particular, the relation between manifestly covariant and non-covariant representations of the relevant loop integrals has been established (Appendix A). We have discussed dimensional, Pauli-Villars, Schwinger proper time and covariant and non-covariant cutoff regularizations for the appearing one-loop integrals from a non-covariant Hamiltonian perspective (Appendix B). Finally, we have presented explicit expressions for the angular momentum eigenstates in different coupling schemes, for the application of helicity operators and the coefficient functions in a partial wave expansion, all necessary ingredients for the separation of angular and spin variables in the effective Schrödinger equation (Appendix C).

The results of this work, if only as a point of departure for the application to more realistic physical situations in the future, bear on fundamental issues in nuclear and high energy physics, as for example the nucleon-nucleon interaction. In this respect, one interesting particular result of the numerical computations is the qualitative difference between the relativistic bound state spectra for scalar (boson exchange) and electromagnetic interactions.

Acknowledgments

One of us (A.W.) gratefully acknowledges support by CIC-UMSNH and Conacyt grant 32729-E. Part of the research (by N.L.) was done at the Department of Physics and Astronomy of the University of Pittsburgh in the group of Eric Swanson and Steve Dytman.

A Covariant and non-covariant representations of loop integrals

In this appendix, we will relate different expressions for loop integrals in momentum space, where the manifestly covariant representations arise directly from the momentum space
Feynman rules, while the non-covariant representations result naturally from the application of the Gell-Mann–Low theorem.

Let us begin with the lowest-order correction to the vacuum energy density Eq. (19). The equivalence will be established by performing the integrations over \( p_0 \) and \( p_0' \) through the use of complex integration theory. For greater clarity, we will first discuss the analogous problem in a purely scalar theory [23]. The corresponding formula differs from Eq. (19) by the global sign (for both expressions) and the numerators of the integrands which are simply equal to one. Considering the first term (for particle \( A \)) for concreteness, we begin with the integral over \( p_0' \),

\[
\int_{-\infty}^{\infty} \frac{dp_0'}{2\pi} \frac{1}{((p-p')^2 - \mu^2 + i\epsilon)((p')^2 - m_A^2 + i\epsilon)}.
\]

(79)

Standard application of the residue theorem, closing the integration contour through the usual large semicircle in the upper half plane, gives

\[
-i\left\{ \frac{1}{2E_{p'}^A ((p_0 + E_{p'}^A)^2 - \omega_{p-p'}^2 + i\epsilon)} + \frac{1}{2\omega_{p-p'} ((p_0 - \omega_{p-p'} - E_{p'}^A)^2 - (E_{p'}^A)^2 + i\epsilon)} \right\}.
\]

(80)

This expression becomes much more transparent after a decomposition in partial fractions with respect to \( p_0 \),

\[
\frac{i}{2\omega_{p-p'} 2E_{p'}^A} \left[ \frac{1}{p_0 + \omega_{p-p'} + E_{p'}^A - i\eta} - \frac{1}{p_0 - \omega_{p-p'} - E_{p'}^A + i\eta} \right. \\
\left. + 2\pi i \delta (p_0 - \omega_{p-p'} + E_{p'}^A) \right].
\]

(81)

where we have used the formula

\[
\frac{1}{\omega - i\eta} - \frac{1}{\omega + i\eta} = 2\pi i \delta (\omega)
\]

(82)

(for \( \eta \to 0 \)). Equation (81) cannot be correct as it stands: the original integral (79) is even under \( p_0 \to -p_0 \) (by the substitution \( p_0' \to -p_0' \) of the integration variable), and this symmetry is manifestly broken by the delta function in Eq. (81).

However, the term with the delta function only contributes for \( p_0 = \omega_{p-p'} - E_{p'}^A \), and this is precisely the value of \( p_0 \) where the two poles in the upper half plane coincide. The correct evaluation of the residue at the double pole gives for the integral in this case

\[
\frac{i}{2\omega_{p-p'} 2E_{p'}^A} \left[ \frac{1}{p_0 + \omega_{p-p'} + E_{p'}^A - i\eta} - \frac{1}{p_0 - \omega_{p-p'} - E_{p'}^A + i\eta} \right]_{p_0=\omega_{p-p'} - E_{p'}^A},
\]

(83)

so the result (81) is wrong for this value of \( p_0 \), and the term with the delta function has to be omitted in (81). The integration over \( p_0 \) can then be performed easily, using the residue
Theorem again. The result, after substituting $p' \rightarrow -p'$, is the one expected from Eq. (19) or rather its scalar analogue. Note that the use of Eq. (81) including the delta function would lead to additional (incorrect) terms in Eq. (19).

In the Yukawa case, Eq. (19) proper, the argument is nearly identical, only the expressions are slightly more complicated. The result for the naive $p_0'$-integration, after the decomposition in partial fractions, reads

$$
\int_{-\infty}^{\infty} \frac{dp'}{2\pi} \frac{4(p \cdot p' + m_A^2)}{[(p - p')^2 - \mu^2 + i\epsilon][p'^2 - m_A^2 + i\epsilon]}
= \frac{4i}{2\omega_{p-p'} \, 2E_{p'}^A} \left\{ -2E_{p'}^A + \left[ E_{p'}^A (E_{p'}^A + \omega_{p-p'}) - p \cdot p' + m_A^2 \right] \left[ \frac{1}{p_0 + \omega_{p-p'} + E_{p'}^A - i\eta} \right.
\right.
\left. \left. \left. - \frac{1}{p_0 - \omega_{p-p'} - E_{p'}^A + i\eta} \right] + \left[ E_{p'}^A (E_{p'}^A - \omega_{p-p'}) - p \cdot p' + m_A^2 \right] 2\pi i \delta \left( p_0 - \omega_{p-p'} + E_{p'}^A \right) \right\}.
\right.
\right.
\right.
(84)

By a calculation of the residue at the double pole for the special case $p_0 = \omega_{p-p'} - E_{p'}^A$, one can again show that the term with the delta function in Eq. (81) is spurious. Integration over $p_0$ of the rest gives the desired result.

We now turn to the lowest-order corrections to the mass, Eq. (27). As before, we begin with the scalar case where all the numerators in Eq. (27) are replaced by one $[23]$. Then the result of the $p_0'$-integration in the manifestly covariant expression is given precisely by Eq. (81) above, with $p_0$ to be replaced by $E_{p'}^A$. The delta function in Eq. (81) is, of course, again spurious, although it cannot give any contribution anyway as long as $\mu^2 < 4m_A^2$.

In Yukawa theory, where Eq. (27) properly applies, the $p_0'$-integration gives the following result $[26]$, after a decomposition in partial fractions (with respect to $p_0$),

$$
\int_{-\infty}^{\infty} dp_0' \frac{p' \cdot \gamma + m_A}{2\pi \, 2\omega_{p-p'} \, 2E_{p'}^A \left[(p - p')^2 - \mu^2 + i\epsilon][p'^2 - m_A^2 + i\epsilon] \right]}
= \frac{i}{2\omega_{p-p'} \, 2E_{p'}^A} \left[ -E_{p'}^A \gamma_0 - p' \cdot \gamma + m_A \right] \left[ \frac{1}{p_0 + \omega_{p-p'} + E_{p'}^A - i\eta} \right.
\left. \left. \left. - \frac{1}{p_0 - \omega_{p-p'} - E_{p'}^A + i\eta} \right) + \left( E_{p'}^A \gamma_0 - p' \cdot \gamma + m_A \right) 2\pi i \delta \left( p_0 - \omega_{p-p'} + E_{p'}^A \right) \right],
\right.
\right.
\right.
(85)

putting $p_0 = E_{p'}^A$. Again, the delta function turns out to be spurious (by explicitly considering the case of a double pole in the upper half plane), which leads to the non-covariant expression in Eq. (27).

B Regularization of one-loop integrals

The aim of this appendix is to derive Eq. (28) starting from the non-covariant expression for $G(p)$ in Eq. (27) in a suitably regularized form, so that all integrals appearing in the
derivation are well-defined. By far the simplest way is to go through the manifestly covariant form also presented in Eq. (27), regularized correspondingly for the present purpose. For the complications that arise in a direct derivation in the non-covariant formulation for the simpler case of a purely scalar theory where it has to be shown that \( G(p) \) is actually independent of \( p (\text{and only depends on } p^2|_{p_0=E_\phi^A}=m_A^2) \), see Ref. [27].

### B.1 Dimensional regularization

The technically simplest regularization scheme (although not the most natural one in the present context, see below) is dimensional regularization. The idea is to continuously change the dimension of space to smaller values where all the integrals are well-defined, to be save to spatial dimensions smaller than two, i.e., space-time dimensions \( D < 3 \). We can then establish the relation between the non-covariant and covariant expression in Eq. (27) for these dimensions and show that the dependence on \( D \) is analytical, with a simple pole appearing at \( D = 4 \). As a consequence, Eq. (28) can be shown to hold true for space-time dimensions arbitrarily close to (but smaller than) four by analytical continuation. However, the definition of the integrals in arbitrary (non-integer) dimensions is made precise only in the (Euclidean) covariant formulation, which makes this form of regularization somewhat unnatural for the non-covariant expressions.

In detail, we begin by establishing the relation between the non-covariant and covariant expressions for \( G(p) \) in Eq. (27), but for \( (D-1) \) spatial dimensions instead of three where, to begin with, \( D < 3 \) in order that all integrals are well-defined. The analogue of Eq. (27) in \( D \) dimensions can then be shown by integrating over \( p_0' \) in the covariant form exactly as detailed in Appendix A [see Eq. (35) and the following remarks]. The \( dD-1p' \)-integration is not touched in this process. With the covariant form at hand, we can introduce Feynman parameters in the usual way:

\[
G^{\text{DR}}_\varepsilon(p) = i g^2 \int \frac{d^D p'}{(2\pi)^D} \int_0^1 dx \frac{p' \cdot \gamma + m_A}{[(p-p')^2 - \mu^2 x + p'^2(1-x) - m_A^2(1-x) + i\varepsilon]^2} \bigg|_{p_0=E_\phi^A} \\
= i g^2 \int_0^1 dx \int \frac{d^D p'}{(2\pi)^D} \frac{p' \cdot \gamma + m_A}{[(p' - xp)^2 + x(1-x)p^2 - x\mu^2 - (1-x)m_A^2 + i\varepsilon]^2} \bigg|_{p_0=E_\phi^A} \\
= i g^2 \int_0^1 dx \int \frac{dD q}{(2\pi)^D} \frac{x p \cdot \gamma + m_A}{[q^2 + x(1-x)p^2 - x\mu^2 - (1-x)m_A^2 + i\varepsilon]^2} \bigg|_{p_0=E_\phi^A},
\]

denoting the dimensionally regularized form of \( G(p) \) as \( G^{\text{DR}}_\varepsilon(p) \), where \( \varepsilon = 4 - D \). In the last step, we have shifted the integration variable to \( q = p' - xp \) and used that for the term \( q \cdot \gamma \) emerging in the numerator, the integrand is odd. These manipulations are unproblematic as long as we stick to dimensions \( D < 3 \) where the integrals are well-defined.

It is now convenient to decompose \( G^{\text{DR}}_\varepsilon(p) \) in two parts in analogy with Eq. (28),

\[
G^{\text{DR}}_\varepsilon(p) = [G^{\text{DR}}_{1,\varepsilon}(p^2)p \cdot \gamma + G^{\text{DR}}_{0,\varepsilon}(p^2)\gamma_{m_A}]_{p_0=E_\phi^A}.
\]

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However, all space-time vectors in this equation are still $D$-dimensional. Replacing $p^2|_{p_0=E_0}$ by $m_A^2$ and Wick rotating to Euclidean space gives

$$G_{0,\epsilon}(m_A^2) = -g^2 \int_0^1 dx \int \frac{d^Dq_E}{(2\pi)^D} \frac{1}{\left[q_E^2 + (1-x)m_A^2 + x\mu^2\right]^2} \tag{88}$$

The corresponding expression for $G_{1,\epsilon}(m_A^2)$ is equal to Eq. \[(88)\] except for an additional factor of $x$ in the integrand. It is these latter integrals which are rigorously defined for arbitrary, continuous values of $D$. They are actually defined in such a way as to make them analytical functions of $D$, with a simple pole at $D = 4$. We can use analytic continuation to define all the foregoing integrals for $3 \leq D < 4$, leaving the established relations intact, in particular Eq. \[(87)\]. In the limit $\epsilon \to 0$ or $D \to 4$, via the standard formulae of dimensional regularization,

$$G_{0,\epsilon}(m_A^2) = -\frac{g^2}{(4\pi)^2} \int_0^1 dx \left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln \left(\frac{1-x}{\kappa^2}m_A^2 + x\mu^2\right)\right] \tag{89},$$

plus terms which tend to zero in this limit. In Eq. \[(89)\], $\kappa$ is the renormalization scale, and $g$ is left dimensionless also for $D \neq 4$. The corresponding expression for $G_{1,\epsilon}^{\text{DR}}(m_A^2)$ is, again, equal to Eq. \[(89)\] except for an additional factor of $x$ in the integrand.

After an integration by parts, the evaluation of the integrals over the Feynman parameter $x$ is straightforward and yields \[26\]

\begin{align*}
G_{0,\epsilon}^{\text{DR}}(m_A^2) &= -\frac{g^2}{(4\pi)^2} \left\{ \frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln \frac{m_A^2}{\kappa^2} + 2 - \frac{1}{2} \frac{\mu^2}{m_A^2} \ln \frac{\mu^2}{m_A^2} \right. \\
&\quad - 2 \sqrt{\frac{\mu^2}{m_A^2} \left( 1 - \frac{\mu^2}{4m_A^2} \right)} \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \}, \tag{90} \\
G_{1,\epsilon}^{\text{DR}}(m_A^2) &= -\frac{g^2}{(4\pi)^2} \left\{ \frac{1}{2} \left[ \frac{2}{\epsilon} - \gamma_E + \ln(4\pi) \right] - \frac{1}{2} \ln \frac{m_A^2}{\kappa^2} + 3 - \frac{1}{2} \frac{\mu^2}{m_A^2} \right. \\
&\quad - \left( \frac{\mu^2}{m_A^2} - \frac{1}{4} \frac{\mu^4}{m_A^4} \right) \ln \frac{\mu^2}{m_A^2} - \left( 2 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2}{m_A^2} \left( 1 - \frac{\mu^2}{4m_A^2} \right)} \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\} \\
&\quad - \left( \frac{\mu^2}{m_A^2} - \frac{1}{4} \frac{\mu^4}{m_A^4} \right) \ln \frac{\mu^2}{m_A^2} - \left( 2 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2}{m_A^2} \left( 1 - \frac{\mu^2}{4m_A^2} \right)} \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\} \\
&\quad - \left( \frac{\mu^2}{m_A^2} - \frac{1}{4} \frac{\mu^4}{m_A^4} \right) \ln \frac{\mu^2}{m_A^2} - \left( 2 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2}{m_A^2} \left( 1 - \frac{\mu^2}{4m_A^2} \right)} \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\} \\
&\quad - \left( \frac{\mu^2}{m_A^2} - \frac{1}{4} \frac{\mu^4}{m_A^4} \right) \ln \frac{\mu^2}{m_A^2} - \left( 2 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2}{m_A^2} \left( 1 - \frac{\mu^2}{4m_A^2} \right)} \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\} \\
(91)
\end{align*}

(for $\epsilon \to 0$). Observe the simple pole at $\epsilon = 0$ or $D = 4$ and the absence of IR divergences for $\mu \to 0$.

The result for $G_{0,\epsilon}^{\text{DR}}(m_A^2)$ gives immediately the explicit dimensionally regularized expression for $\Delta m_A^2$ in the purely scalar case considered in Ref. \[23\] (where the coupling constant $g$ has the dimension of mass). For Yukawa theory, we define $\Delta m_A^2$ by

$$\Delta m_A^2 = 2m_A^2 \left[ G_{0,\epsilon}^{\text{DR}}(m_A^2) + G_{1,\epsilon}^{\text{DR}}(m_A^2) \right] \tag{92}$$

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so that
\[ \bar{u}_A^r(p, r) G_{\xi}^{DR}(p) u_A^s(p, s) = \Delta m_A^2 \delta_{rs} \] (93)
from the analogue of Eq. (29), where the Dirac equation in \( D \) dimensions is used. Explicitly, from Eqs. (90) and (91),
\[ \frac{\Delta m_A^2}{2m_A^2} = \frac{g^2}{(4\pi)^2} \left[ \frac{3}{2} \left( \frac{2}{\varepsilon} - \frac{\gamma_E}{2} + \ln(4\pi) \right) - \frac{3}{2} \ln \frac{m_A^2}{\kappa^2} + \frac{7}{2} - \frac{1}{2} \frac{\mu^2}{m_A^2} - \left( \frac{3}{2} \frac{\mu^2}{m_A^2} - \frac{1}{4} \frac{\mu^4}{m_A^4} \right) \ln \frac{\mu^2}{m_A^2} \right. \]
\[ \left. - \left( 4 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2}{m_A^2} \left( 1 - \frac{1}{4} \frac{\mu^2}{m_A^2} \right)} \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right] , \] (94)
which can now be used to redefine \( m_A \) in terms of \( M_A \) as in Eq. (33) and the limit \( \varepsilon \to 0 \) be taken with the (finite) physical mass \( M_A \) held fixed.

### B.2 Pauli-Villars regularization

We now consider Pauli-Villars regularization which turns out to be the most natural regularization scheme in the context of Hamiltonian (non-covariant) perturbation theory. It is effected by subtracting from the expression in Eq. (27), denoted for the time being as \( G(p, \mu) \) to make its dependence on the boson mass \( \mu \) explicit, a similar contribution for a fictitious “heavy” boson of mass \( \Lambda \) to define
\[ G_{PV}^{\Lambda}(p) = G(p, \mu) - G(p, \Lambda) . \] (95)

It is then easy to verify by power counting that \( G_{PV}^{\Lambda}(p) \) is UV finite, both from the non-covariant and the manifestly covariant expression.

The \( p_0' \)-integration in the covariant expression can then be performed again as in Appendix A, only that the following integration over \( d^4p' \) is now well-defined. This establishes the equivalence of the non-covariant and manifestly covariant expressions for \( G_{PV}^{\Lambda}(p) \) [i.e., the regularized form of Eq. (27)] rigorously. Starting from the covariant form, we now introduce Feynman parameters and shift the integration variable as in Eq. (86) to obtain
\[ G(p, \mu) = ig^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \left[ \frac{x p \cdot \gamma + m_A}{q^2 + x(1-x)p^2 - x\mu^2 - (1-x)m_A^2 + i\epsilon} \right] \bigg|_{p_0 = E^A_{p}} \] (96)
and the analogous expression for \( G(p, \Lambda) \) [remembering that its difference \( G_{PV}^{\Lambda}(p) \) is well-defined in 4 dimensions]. From Eq. (96) and the corresponding expression for \( G(p, \Lambda) \), it is clear that \( G_{PV}^{\Lambda}(p) \) is of the form described in Eq. (28). The mass renormalization can then already be effected at this stage as detailed following Eq. (28), reading \( G_{PV}^{\Lambda}(p) \) instead of \( G(p) \).

To evaluate \( G_{PV}^{\Lambda}(p) \) analytically, it is easiest to apply dimensional regularization to \( G(p, \mu) \) and \( G(p, \Lambda) \) separately and analytically continue the result for the difference \( G_{PV}^{\Lambda}(p) \).
back to four space-time dimensions. We start with the form obtained in Eq. (96) above, replace \( p^2|_{p_0=E_p} \) by \( m_A^2 \), and separate \( G^\text{PV}_\Lambda(p) \) in two parts \( G^\text{PV}_{0,\Lambda}(m_A^2) \) and \( G^\text{PV}_{1,\Lambda}(m_A^2) \), analogous to Eq. (28). The continuation to \( D \) space-time dimensions and a Wick rotation to Euclidean space leads to the expression (88) for the contribution \( G \). The integrations over \( d^D q_E \) and the Feynman parameter then yield the results given in Eqs. (90) and (91) for the contributions \( G^0(m_A^2, \mu) \) and \( G^1(m_A^2, \mu) \), respectively. The expressions for \( G^0(m_A^2, \Lambda) \) and \( G^1(m_A^2, \Lambda) \) are obtained by replacing \( \mu \rightarrow \Lambda \). In the (relevant) limit of large \( \Lambda \), one analytically continues the last terms in Eqs. (90) and (91) to \( \Lambda^2 > 4m_A^2 \) and expands the complete expression in powers of \( m_A^2/\Lambda^2 \) (up to second order). This gives, after several cancellations,

\[
G^0(m_A^2, \Lambda) \rightarrow - \frac{g^2}{(4\pi)^2} \left\{ \frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln \frac{\Lambda^2}{\kappa^2} + 1 \right\},
\]

\[
G^1(m_A^2, \Lambda) \rightarrow - \frac{g^2}{(4\pi)^2} \left\{ \frac{1}{2} \left[ \frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) \right] - \frac{1}{2} \ln \frac{\Lambda^2}{\kappa^2} + \frac{1}{4} \right\}
\]

(97)

for \( \Lambda^2 \gg m_A^2 \). The same results can be obtained somewhat easier by replacing \((1-x)^2m_A^2 + x\Lambda^2 \rightarrow x\Lambda^2 \) in the argument of the logarithm in Eq. (88) [and analogously for \( G^1(m_A^2, \Lambda) \)] from the start.

We hence obtain, finally, for \( G^0(m_A^2) \) and \( G^1(m_A^2) \) in Pauli-Villars regularization

\[
G^\text{PV}_{0,\Lambda}(m_A^2) = - \frac{g^2}{(4\pi)^2} \left\{ \ln \frac{\Lambda^2}{m_A^2} + 1 - \frac{1}{2} \ln \frac{\mu^2}{m_A^2} \right\},
\]

\[
- 2 \left( \frac{\mu^2}{m_A^2} \right) \left( 1 - \frac{1}{4} \frac{\mu^2}{m_A^2} \right) \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\},
\]

(98)

\[
G^\text{PV}_{1,\Lambda}(m_A^2) = - \frac{g^2}{(4\pi)^2} \left\{ \frac{1}{2} \ln \frac{\Lambda^2}{m_A^2} + 5 \right\} - \frac{1}{2} \frac{\mu^2}{m_A^2} - \frac{1}{4} \frac{\mu^4}{m_A^4} \ln \frac{\mu^2}{m_A^2}
\]

\[
- \left( 2 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\} \right\} \right\}.
\]

(99)

(\( \Lambda^2 \gg m_A^2 \)). We obtain a logarithmic UV divergence with \( \Lambda \rightarrow \infty \) and, again, the absence of IR divergences for \( \mu \rightarrow 0 \). We emphasize that we have used dimensional regularization only as a convenient calculational tool here, and that \( G^\text{PV}_{0,\Lambda}(m_A^2) \) and \( G^\text{PV}_{1,\Lambda}(m_A^2) \) are well-defined in four space-time dimensions from the start.

The result for \( G^\text{PV}_{0,\Lambda}(m_A^2) \) gives directly the expression for \( \Delta m_A^2 \) in Pauli-Villars regularization for the purely scalar case of Ref. [25]. For Yukawa theory, we have from Eq. (30)

\[
\frac{\Delta m_A^2}{2m_A^2} = - \frac{g^2}{(4\pi)^2} \left\{ \frac{3}{2} \ln \frac{\Lambda^2}{m_A^2} + 9 \right\} - \frac{1}{2} \frac{\mu^2}{m_A^2} - \left( \frac{3}{2} \frac{\mu^2}{m_A^2} - \frac{1}{4} \frac{\mu^4}{m_A^4} \right) \ln \frac{\mu^2}{m_A^2}
\]

(100)
\[- \left( 4 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2}{m_A^2} \left( 1 - \frac{\mu^2}{4 m_A^2} \right)} \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right). \quad (101)

### B.3 Schwinger proper time regularization

The other two regularization schemes that we will discuss, Schwinger proper time and momentum cutoff regularization, are only effective in Euclidean space. Hence we have to write the non-covariant expression in Eq. (27) in Euclidean space in order to regularize. After evaluation, it can then be analytically continued to physical values of the external variables.

Let us begin with the diagrammatic expressions in Eq. (22). We rewrite the first contribution to the mass renormalization by use of the covariant representations (13) of the propagators and Eq. (23) for the fermionic wave functions as

\[-ig^2 \int_{-\infty}^{0} dt e^{-\epsilon|t|} \int d^3x \int d^3x' \left[ \bar{\psi}^{A}_{p_{A},r}(0,x) S_{F}^{A}(0-t,x-x') \psi_{p_{A},s}^{A}(t,x') \right] \Delta_{F}(0-t,x-x') \]

\[= ig^2 \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \int d^3x \int d^3x' \frac{e^{-ip_{A}\cdot x}}{2E_{p_{A}}^A} \int dp' \left[ \bar{u}_{A}(p_{A},r) \left( p' \cdot \gamma + m_{A} \right) u_{A}(p_{A}',s) \right] e^{ip'\cdot(x-x')} \]

\[\times \int \frac{dk_0}{2\pi} \frac{e^{-ik\cdot(x-x')}}{k_0^2 - k^2 - \mu^2 + i\epsilon} \frac{e^{ip_{A}'\cdot x'}}{\sqrt{2E_{p_{A}'}^A}} \int_{-\infty}^{0} dt e^{i\epsilon t} \int_{-\infty}^{0} dt e^{i(p_0+k_0-p_0)t} \bigg|_{p_0=E_{p_{A}'}^A} \]

In the latter expression we Wick rotate in the mathematically positive sense

\[p'_{0E} \to p_{0} = ip_{0E}, \quad k_{0} \to k_{0} = ik_{0E}, \quad (103)\]

where \(p'_{0E}\) and \(k_{0E}\) are real after the rotation. The sense of the rotation is determined by the position of the poles in the integrand (by the \(i\epsilon\)-prescription). Since \(p'_{0E}\) and \(k_{0E}\) take both positive and negative values, it is imperative to Wick rotate \(t\), too, keeping the exponents imaginary in order that the integrand do not blow up. This implies the rotation

\[t \to t = -it_{E}^{F} \quad (104)\]

in the negative sense, where \(-\infty < t_{E}^{F} \leq 0\). Keeping \(p_{0}\) fixed at \(p_{0} = E_{p_{A}'}^{A}\) would then lead to a divergence in the \(t_{E}^{F}\)-integration, so that we have to rotate, in addition,

\[p_{0} \to p_{0} = ip_{0E}^{F}. \quad (105)\]

After performing the integration, we then analytically continue the result to \(p_{0E}^{F} = -iE_{p_{A}'}^{A}\).
As a result of this Wick rotation, the expression \( \text{(102)} \) is written as

\[
- \frac{g^2}{\sqrt{2E_{p_A}^A}} \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \int d^3x d^3x' e^{-ip_A \cdot x} \times \\
\times \int \frac{dp_0^E}{2\pi} \left[ \frac{\bar{u}_A(p_A, r) \left( (p_0^E + p' \cdot \gamma^E + m_A) u_A(p_A', s) \right)}{(p_0^E)^2 + p'^2 + m_A^2} \right] e^{ip' \cdot (x-x')} \times \\
\times \int \frac{dk_0^E}{2\pi} \frac{e^{ik \cdot (x-x')}}{(k_0^E)^2 + k^2 + \mu^2} \int_0^0 dt^E e^{iE^\gamma E t} e^{i(p_0^E + k_0^E - p_0^E)t^E} \bigg|_{p_0^E \to -iE_{p_A}^A}, \tag{106}
\]

where we have defined \( \epsilon^E = -i\epsilon \) to assure the convergence of the \( t^E \)-integration. Alternatively, we can assume \( \epsilon \) to have a positive imaginary part from the beginning. As for the \( \gamma \) matrices, we have chosen the common convention where

\[
\gamma_0^E = i\gamma_0 \quad \text{and} \quad \gamma_i^E = \gamma_i = -\gamma_i.
\tag{107}
\]
i.e., \( \gamma_0^E = i\gamma_0 \) and \( \gamma_i^E = \gamma_i = -\gamma_i \). Finally, for the momentum-space representation, we integrate over \( d^3x \), \( d^3x' \) and \( d^3k \) (taking advantage of the three-dimensional \( \delta \)-function resulting from the \( x \)- and \( x' \)-integrations). The resulting contribution to \( G(p) \) [compare with Eq. \( \text{(26)} \)] is

\[
- i g^2 \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \int d^3x d^3x' \left[ \bar{u}_{p_A, r}(t, x') S_F^A(t - 0, x' - x) \psi_{p_A, s}(0, x) \right] \Delta_F(t - 0, x' - x)
\]

\[
= i g^2 \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \int d^3x d^3x' e^{-i p_A \cdot x'} \sqrt{2E_{p_A}^A} \int \frac{dp_0^E}{2\pi} \left[ \frac{\bar{u}_A(p_A, r) \left( (p_0^E - p_0^E) + m_A \right) u_A(p_A', s) \right]}{p_0^2 - p'^2 - m_A^2 + i\epsilon} e^{ip' \cdot (x-x')} \times \\
\times \int \frac{dk_0^E}{2\pi} \frac{e^{-ik \cdot (x-x')}}{k_0^2 - k^2 - \mu^2 + i\epsilon} \sqrt{2E_{p_A}^A} \int_0^0 dt e^{iE t} e^{-i(p_0^E + k_0 - p_0^E)t} \bigg|_{p_0 = E_{p_A}^A}, \tag{109}
\]

and going through the steps following Eq. \( \text{(102)} \), we arrive at the following contribution to
G(p):

\[-g^2 \int \frac{d^3p'}{(2\pi)^3} \int \frac{dp'^E}{2\pi} \frac{\mu'^E \gamma'^E + p' \cdot \gamma^E + m_A}{(p'^E)^2 + p'^2 + m_A^2} \times \int \frac{dk^E}{2\pi} \frac{1}{(k^E)^2 + (p - p')^2 + \mu^2} \int_{-\infty}^{0} dt^E e^{E_t^E} e^{-i(p'_0 + k^E_0 - p^E_0)t^E} \bigg|_{p^E_0 \to -iE_{pA}^E}. \tag{110}\]

It is easy to check the correctness of the results \((108)\) and \((110)\) by performing the integrations over \(p'^E_0\) and \(k^E_0\) (using the residue theorem) and finally over \(t^E\), leading to the non-covariant expression in Eq. \((27)\).

After this preparation, we are now in a position to introduce the Schwinger proper time regularization by replacing

\[\frac{1}{(p'^E)^2 + (E_{p'}^A)^2} = \int_0^\infty d\alpha e^{-\alpha((p'^E_0)^2 + (E_{p'}^A)^2)} \rightarrow \int_{1/\Lambda^2}^\infty d\alpha e^{-\alpha((p'^E_0)^2 + (E_{p'}^A)^2)}, \tag{111}\]

and analogously for the other (scalar) propagator. We emphasize that only in Euclidean space the modification of the lower limit for the parameter integration corresponds to the exponential suppression of the propagator for large momenta. Consequently, the proper time regularization of the first contribution Eq. \((108)\) reads

\[-g^2 \int \frac{d^3p'}{(2\pi)^3} \int_{1/\Lambda^2}^{\infty} \frac{d\alpha d\beta e^{-\alpha(E_{p'}^A)^2 - \beta \omega^2_{p' - p'}} \int_{-\infty}^{0} dt^E e^{E_t^E} e^{-i\mu^E_0 t^E}}{4 \pi \sqrt{\alpha \beta}} \times \int \frac{dp'^E}{2\pi} \left[ (p'^E_0 \gamma^E + p' \cdot \gamma^E + m_A) e^{-\alpha(p'^E_0)^2 + \mu^E_0 t^E} \int \frac{dk^E}{2\pi} e^{-\beta(k^E_0)^2 + iE_{pA}^E t^E} \right]_{p^E_0 \to -iE_{pA}^E}. \tag{112}\]

The integrations over \(p'^E_0\) and \(k^E_0\) in Eq. \((112)\) are readily performed. After shifting the Euclidean time variable \(t^E \to \tau^E = t^E + 2i\alpha\beta p^E_0/(\alpha + \beta)\), one may analytically continue \(p^E_0\) to \(-iE_{pA}^E\) directly in the integrand to obtain

\[-g^2 \int \frac{d^3p'}{(2\pi)^3} \int_{1/\Lambda^2}^{\infty} \frac{d\alpha d\beta e^{-\alpha(E_{p'}^A)^2 - \beta \omega^2_{p' - p'} + \alpha \beta (E_{pA}^A)^2/(\alpha + \beta)}}{4 \pi \sqrt{\alpha \beta}} \times \int_{-\infty}^{2\alpha \beta E_{pA}^A/(\alpha + \beta)} d\tau^E \left[ \left( \frac{\beta E_{pA}^A}{\alpha + \beta} - \frac{\mu^E_0}{2\alpha} \right) \gamma_0 - p' \cdot \gamma + m_A \right] e^{-(\alpha + \beta)(\tau^E)^2/(4\alpha \beta)}. \tag{113}\]

A little algebra shows the first exponent in this expression to be negative. Integration over \(\tau^E\) gives \((28)\)

\[-g^2 \int \frac{d^3p'}{(2\pi)^3} \int_{1/\Lambda^2}^{\infty} \frac{d\alpha d\beta e^{-\alpha(E_{p'}^A)^2 - \beta \omega^2_{p' - p'}}}{4 \pi \sqrt{\alpha \beta}} \left\{ \frac{\beta E_{pA}^A}{\alpha + \beta} \gamma_0 \right. \left. + \sqrt{\frac{\pi \alpha \beta}{\alpha + \beta}} \frac{\beta E_{pA}^A}{\alpha + \beta} \gamma_0 - p' \cdot \gamma + m_A \right\} e^{\alpha \beta (E_{pA}^A)^2/(\alpha + \beta)} \text{erfc} \left( -\sqrt{\frac{\alpha \beta}{\alpha + \beta}} E_{pA}^A \right), \tag{114}\]
where the complementary error function is defined as

\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} dt \, e^{-t^2} = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{-x} dt \, e^{-t^2}. \]  

(115)

For the second contribution to the mass renormalization we start from Eq. (110), implement the Schwinger proper time regularization as in Eq. (111) and integrate over \( p_0^E, k_0^E, \) and \( t^E \) (after a shift and the continuation \( p_0^E \rightarrow -iE^A_p \)) to arrive at

\[ -g^2 \int \frac{d^3p'}{(2\pi)^3} \int_{1/\Lambda^2}^{\infty} \frac{d\alpha d\beta}{4\pi \sqrt{\alpha \beta}} e^{-\alpha (E_p^A)^2 - \delta \omega^2 p' \cdot p'} \left\{ -\frac{\beta}{\alpha + \beta} \gamma_0 + \frac{\sqrt{\pi \alpha \beta}}{\alpha + \beta} \left[ \frac{\beta E_p^A}{\alpha + \beta} \gamma_0 - p' \cdot \gamma + m_A \right] e^{\alpha \beta (E_p^A)^2 / (\alpha + \beta)} \text{erfc} \left( \frac{\sqrt{\alpha \beta} E_p^A}{\alpha + \beta} \right) \right\}. \]  

(116)

The results (114) and (116) represent the proper time regularization of the non-covariant expressions in Eq. (27). When we add them up, we arrive at the proper time regularized equivalent to Eq. (27),

\[ G_{PT}^A(p) = -g^2 \int \frac{d^4p'}{(2\pi)^4} \int_{1/\Lambda^2}^{\infty} \frac{d\alpha d\beta}{4\pi \sqrt{\alpha \beta}} \left[ \frac{\beta E_p^A}{\alpha + \beta} \gamma_0 - p' \cdot \gamma + m_A \right] \times e^{-\alpha (E_p^A)^2 - \delta \omega^2 p' \cdot p' + \alpha \beta (E_p^A)^2 / (\alpha + \beta)}, \]  

(117)

where we have used that [see Eq. (115)]

\[ \text{erfc}(x) + \text{erfc}(-x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \, e^{-t^2} = 2. \]  

(118)

Next, we show that the sum (117) of the non-covariant expressions coincides with the Schwinger proper time regularized equivalent of the covariant form in Eq. (27). To this end, first perform a Wick rotation to rewrite the covariant expression in Eq. (27) in Euclidean space,

\[ G(p) = -g^2 \int \frac{d^4p^E}{(2\pi)^4} \frac{p^E \cdot \gamma^E + m_A}{[(p^E - p^E)^2 + \mu^2] \left[(p^E)^2 + m_A^2\right]} \bigg|_{p_0^E \rightarrow -iE^A_p}, \]  

(119)

where all the products of Euclidean 4-vectors (including squares) are understood to be Euclidean scalar products, i.e., taken with the positive Euclidean metric. The proper time regularization is introduced as in Eq. (111) and yields

\[ G_{PT}^A(p) = -g^2 \int_{1/\Lambda^2}^{\infty} d\alpha d\beta \int \frac{d^4p^E}{(2\pi)^4} \left[ p_0^E \gamma_0^E + p' \cdot \gamma^E + m_A \right] \times e^{-\alpha [(p_0^E)^2 + \mu^2] - \beta [(p_0^E - p_0^E)^2 + (p - p')^2 + \mu^2]} \bigg|_{p_0^E \rightarrow -iE^A_p}. \]  

(120)
Integrating over $p_0^E$ and continuing $p_0^E$ to $-iE_0^A$ results in Eq. (117), thus establishing the equivalence of the non-covariant and covariant proper time regularized expressions.

Note that there is a much faster way to establish this equivalence by adding up Eq. (112) and its counterpart from Eq. (110), substituting $t^E \rightarrow -t^E$ in the latter expression, and using the formula

$$\int_{-\infty}^{\infty} dt^E e^{-\epsilon t^E} e^{i\omega t^E} = 2\pi \delta(\omega).$$  \hspace{1cm} (121)$$

This procedure directly leads to the form (120), so it indeed represents a considerable shortcut. However, we were interested in writing down the proper time regularized equivalent of Eq. (27), which is why we followed the calculation through to expressions (114) and (116).

To establish the form (28), we start again from the covariant expression (120), but integrate over all four components of $p^E$ this time with the result

$$G_{0\Lambda}^{PT}(m_A^2) = -\frac{g^2}{(4\pi)^2} \int_{1/\Lambda^2}^{\infty} d\alpha d\beta \left[ \frac{\beta}{\alpha + \beta} p \cdot \gamma + m_A \right] e^{-\alpha m_A^2 - \beta \mu^2 + \alpha \beta p^2 / (\alpha + \beta)} \left|_{p_0 = E_0^A} \right.,$$  \hspace{1cm} (122)$$

from where relation (28) can be established. Note that the exponent in Eq. (122) is manifestly negative for $p_0^2 = E_0^A = m_A^2$.

In order to evaluate the functions $G_{0\Lambda}(m_A^2)$ and $G_{1\Lambda}(m_A^2)$ in proper time regularization explicitly, we change variables $(\alpha, \beta)$ to $(\rho, \sigma)$, where

$$\rho = \alpha + \beta, \quad \sigma = \frac{\beta}{\alpha + \beta},$$

$$0 < \sigma < 1, \quad \rho \geq P(\sigma, \Lambda^2) \equiv \max \left( \frac{1}{\sigma \Lambda^2}, \frac{1}{(1 - \sigma) \Lambda^2} \right).$$  \hspace{1cm} (123)$$

The integration over $\rho$ can now be performed and leads to

$$G_{\Lambda}^{PT}(p) = -\frac{g^2}{(4\pi)^2} \int_0^1 d\sigma \left[ \sigma p \cdot \gamma + m_A \right] E_1 \left( [(1 - \sigma)^2 m_A^2 + \sigma \mu^2] P(\sigma, \Lambda^2) \right),$$  \hspace{1cm} (124)$$

where we have introduced the exponential integral function $E_1$,

$$E_1(x) = \int_{x}^{\infty} dt \frac{e^{-t}}{t} = \int_{1}^{\infty} dt \frac{e^{-xt}}{t}, \quad x > 0.$$  \hspace{1cm} (125)$$

Using the expansion of $E_1(x)$ for small values of the argument, we can approximate $G_{\Lambda}^{PT}(p)$ by

$$\frac{g^2}{(4\pi)^2} \int_0^1 d\sigma \left[ \sigma p \cdot \gamma + m_A \right] \left[ \gamma_E + \ln \left( [(1 - \sigma)^2 m_A^2 + \sigma \mu^2] P(\sigma, \Lambda^2) \right) \right] ,$$  \hspace{1cm} (126)$$

all higher terms in the expansion being suppressed by powers of $1/\Lambda$. For the sake of readability, the demonstration of this latter assertion is relegated to the end of this section.
The results (90) and (91) and the integral of \( \ln(\Lambda^2 P(\sigma, \Lambda^2)) \) which is elementary, can then be used to establish that

\[
G_{P^T 0, \Lambda}(m_A^2) = -\frac{g^2}{(4\pi)^2} \left\{ \ln \frac{\Lambda^2}{m_A^2} - \gamma_E - \ln 2 + 1 - \frac{1}{2} \frac{\mu^2}{m_A^2} \ln \frac{\mu^2}{m_A^2} + 2 \right. \\
- \left. 2 \sqrt{\frac{\mu^2}{m_A^2} \left( 1 - \frac{1}{4} \frac{\mu^2}{m_A^2} \right)} \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\} ,
\]

(127)

\[
G_{P^T 1, \Lambda}(m_A^2) = -\frac{g^2}{(4\pi)^2} \left\{ \frac{1}{2} \left[ \ln \frac{\Lambda^2}{m_A^2} - \gamma_E - \ln 2 \right] + 1 - \frac{1}{2} \frac{\mu^2}{m_A^2} - \left( \frac{\mu^2}{m_A^2} - \frac{1}{4} \frac{\mu^4}{m_A^4} \right) \ln \frac{\mu^2}{m_A^2} \\
- \left( 2 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2}{m_A^2} \left( 1 - \frac{1}{4} \frac{\mu^2}{m_A^2} \right)} \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\} .
\]

(128)

As before, \( G_{P^T 0, \Lambda}(m_A^2) \) directly gives \( \Delta m_A^2 \) in the purely scalar theory, while in the Yukawa case we have

\[
\frac{\Delta m_A^2}{2m_A^2} = -\frac{g^2}{(4\pi)^2} \left\{ \frac{3}{2} \left[ \ln \frac{\Lambda^2}{m_A^2} - \gamma_E - \ln 2 \right] + 2 - \frac{1}{2} \frac{\mu^2}{m_A^2} - \left( \frac{3}{2} \frac{\mu^2}{m_A^2} - \frac{1}{4} \frac{\mu^4}{m_A^4} \right) \ln \frac{\mu^2}{m_A^2} \\
- \left( 4 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2}{m_A^2} \left( 1 - \frac{1}{4} \frac{\mu^2}{m_A^2} \right)} \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\} .
\]

(129)

Finally, we will give the demonstration of Eq. (126) which was relegated to this point. To begin with, let us note that the approximation (126) is not trivial because the function

\[
P(\sigma, \Lambda^2)
\]

takes large values for \( \sigma \) close to the limits of integration, 0 and 1. We further remark that the extension of the integration domain to \( 0 < \sigma < 1, \rho > 2/\Lambda^2 \) [compare with Eq. (123)], in which case the approximation corresponding to Eq. (126) is straightforward, leads to a different result.

This being said, we start from Eq. (124) and integrate by parts in order to get rid of the exponential integral function. The result can be written as

\[
G_{P^T}(p) = -\frac{g^2}{(4\pi)^2} \int_0^1 d\sigma \left[ \frac{\sigma^2}{2} p \cdot \gamma + \sigma m_A \right] e^{-[(1-\sigma)^2 m_A^2 + \sigma \mu^2] P(\sigma, \Lambda^2)} \\
\times \frac{d}{d\sigma} \ln \left( [(1-\sigma)^2 m_A^2 + \sigma \mu^2] P(\sigma, \Lambda^2) \right) .
\]

(130)

In the exponent in this expression, we keep the divergent terms (for \( \sigma \rightarrow 0 \) and \( \sigma \rightarrow 1 \)) and
expand the rest in powers of $\sigma$ and $(1 - \sigma)$, respectively, leading to

$$
- \frac{g^2}{(4\pi)^2} \left\{ \int_0^{1/2} d\sigma \left[ \frac{\sigma^2}{2} p \cdot \gamma + \sigma m_A \right] e^{-m_A^2/(\Lambda^2 \sigma)} \frac{d}{d\sigma} \left[ \ln \left( \frac{(1 - \sigma)^2 m_A^2 + \sigma \mu^2}{\Lambda^2} \right) - \ln \sigma \right] e^{-m_A^2/(\Lambda^2 \sigma)} \right\}
+ \int_{1/2}^1 d\sigma \left[ \frac{\sigma^2}{2} p \cdot \gamma + \sigma m_A \right] e^{-\mu^2/(\Lambda^2 (1 - \sigma))} \frac{d}{d\sigma} \left[ \ln \left( \frac{(1 - \sigma)^2 m_A^2 + \sigma \mu^2}{\Lambda^2} \right) - \ln(1 - \sigma) \right] e^{-\mu^2/(\Lambda^2 (1 - \sigma))} \right\},
$$

(131)

the higher terms in the expansion of the exponential of the finite terms being suppressed by powers of $\Lambda$, as we shall see shortly. First, note that we can, by way of the substitutions

$$
y = \frac{1}{2\sigma} \quad \text{and} \quad y = \frac{1}{2(1 - \sigma)}
$$

(132)

in the first and second integral in Eq. (131), respectively, express the contributions that contain the derivatives of $\ln \sigma$ and $\ln(1 - \sigma)$ in terms of the exponential integral functions [29]

$$
E_n(x) = \int_1^x dt \frac{e^{-xt}}{t^n}, \quad x > 0, \quad n = 1, 2, 3.
$$

(133)

For small $x$, we have the expansions [29]

$$
E_1(x) = -\gamma_E - \ln x + \mathcal{O}(x),
$$
$$
E_2(x) = 1 + \mathcal{O}(x \ln x),
$$
$$
E_3(x) = 1/2 + \mathcal{O}(x),
$$

(134)

which lead to

$$
- \frac{g^2}{(4\pi)^2} \left\{ - \int_0^{1/2} d\sigma \left[ \frac{\sigma^2}{2} p \cdot \gamma + \sigma m_A \right] \frac{d}{d\sigma} \left[ \ln \left( \frac{(1 - \sigma)^2 m_A^2 + \sigma \mu^2}{\Lambda^2} \right) - \ln \sigma \right] + \int_{1/2}^1 d\sigma \left[ \frac{\sigma^2}{2} p \cdot \gamma + \sigma m_A \right] \frac{d}{d\sigma} \left[ \ln \left( \frac{(1 - \sigma)^2 m_A^2 + \sigma \mu^2}{\Lambda^2} \right) - \ln(1 - \sigma) \right] \right\}
$$

$$
= - \frac{g^2}{(4\pi)^2} \left[ \frac{1}{2} p \cdot \gamma + m_A \right] \left[ \ln \left( \frac{\Lambda^2}{\mu^2} \right) - \gamma_E - \ln 2 - 1 \right]
$$

(135)

in the limit of large $\Lambda$. In an analogous way, we can see that the higher orders in the expansion of the exponential of the finite terms (for $\sigma \to 0$ and $\sigma \to 1$) in Eq. (130) are suppressed by powers of $\Lambda$, taking into account that the expressions resulting from this expansion and the first terms in the integrals (131) are continuous bounded functions over the intervals in question, and that furthermore the higher orders in the expansion carry inverse powers of $\Lambda$. 
As far as the first terms in the integrals \((1331)\) are concerned, the limit \(\Lambda \to \infty\) can be taken naively there (note that the \(\Lambda\)-dependence in the argument of the logarithm is spurious since its derivative with respect to \(\sigma\) gives zero), as we will show in a moment. If we suppose, for the time being, that this is indeed correct, the exponentials can be replaced by one, and an integration by parts yields

\[
- \frac{g^2}{(4\pi)^2} \left\{ \int_0^{1/2} d\sigma \left[ \frac{\sigma^2}{2} p \cdot \gamma + \sigma m_A \right] e^{-m_A^2/(\Lambda^2 \sigma)} \frac{d}{d\sigma} \ln \left( \frac{(1 - \sigma)^2 m_A^2 + \sigma \mu^2}{\Lambda^2} \right) + \int_{1/2}^1 d\sigma \left[ \frac{\sigma^2}{2} p \cdot \gamma + \sigma m_A \right] e^{-\mu^2/(\Lambda^2 (1 - \sigma))} \frac{d}{d\sigma} \ln \left( \frac{(1 - \sigma)^2 m_A^2 + \sigma \mu^2}{\Lambda^2} \right) \right\} 
\]

\[= - \frac{g^2}{(4\pi)^2} \left\{ \frac{1}{2} p \cdot \gamma + m_A \right\} \ln \left( \frac{\Lambda^2}{\mu^2} \right) + \int_0^1 d\sigma \left[ \sigma p \cdot \gamma + m_A \right] \ln \left( \frac{(1 - \sigma)^2 m_A^2 + \sigma \mu^2}{\Lambda^2} \right) \]

\((136)\)

for \(\Lambda \to \infty\) (the \(\Lambda\)-dependence cancels between the two terms). Using Eq. \((135)\) and the results \((90)\) and \((91)\) for the remaining integration, we finally arrive at the expressions \((127)\) and \((128)\), thus confirming the approximation \((126)\).

Let us now show in detail that it is, indeed, correct to replace the exponential on the left-hand side of Eq. \((136)\) by one. To this end, take the integrals on the left-hand side of Eq. \((136)\) as they stand, perform the substitutions Eq. \((132)\) and decompose the result in partial fractions to find, after a somewhat lengthy calculation,

\[
- \frac{g^2}{(4\pi)^2} \int_1^{\infty} dy \left\{ \frac{1}{2} p \cdot \gamma \left[ \frac{1}{2y^3} - \frac{M_+ + M_-}{2y^2} + \frac{M_+^2 + M_-^2}{y^2} + \frac{M_+^2}{y - 1/(2 + 2M_+)} - \frac{M_-^2}{y - 1/(2 + 2M_-)} \right] \right\} e^{-2(\mu^2/\Lambda^2) y} 
\]

\[
+ \frac{g^2}{(4\pi)^2} \int_1^{\infty} dy \left\{ \frac{1}{2} p \cdot \gamma \left[ \frac{1}{2y^3} - \frac{M_+ + M_-}{2y^2} + \frac{M_+^2 + M_-^2}{y^2} + \frac{M_+^2}{y - 1/(2 + 2M_+)} + \frac{M_-^2}{y - 1/(2 + 2M_-)} \right] \right\} e^{-2(\mu^2/\Lambda^2) y} , \quad (137)
\]

where we have introduced the notations

\[
M_\pm = \frac{\mu^2}{2m_A^2} - 1 \pm \sqrt{\frac{\mu^2}{2m_A^2} \left( \frac{\mu^2}{2m_A^2} - 2 \right)} . \quad (138)
\]

For simplicity, we have written this expression for the case \(\mu^2 \geq 4m_A^2\) where the square roots are real. To compare with Eqs. \((127)\) and \((128)\) in the end, one has to analytically continue the result to smaller values of \(\mu^2\).
The integrals in Eq. (137) are well-defined with and without the exponentials. However, this is not true (in all cases) for the integrals over the partial fractions individually. It is clear then, that the exponentials can be considered as regulating factors for the integrals over the partial fractions. After summing up the individual results, the limit \( \Lambda \to \infty \) can safely be taken, thus removing the regulator. In the end, this is equivalent to replacing the exponentials by one in Eq. (136). The integrals in (137) can also be explicitly calculated with the help of Eqs. (133) and (134) in the limit \( \Lambda \to \infty \). As a result, Eq. (136) is recovered [using Eqs. (91) and (91) to evaluate the integral in Eq. (136)].

These comments conclude the demonstration of the correctness of the approximation (126) and hence of the results (127), (128) and (129) for the Schwinger proper time regularization. It is clear from the above that the proper time regularization will not be the method of choice in a Hamiltonian (not explicitly covariant) approach, considering the difficulties in establishing the regularized form of the non-covariant expressions and the evaluation of the corresponding (one-loop) integral as compared to the other regularizations discussed before.

### B.4 Momentum cutoff regularization

At last, we discuss the probably simplest regularization scheme available, the use of a momentum cutoff. As it turns out, it does not have very simple properties when used in the present context. We start with the non-covariant expressions written in Euclidean space after a Wick rotation in Eqs. (108) and (110). The momentum cutoff is implemented by restricting the integration domain (in Euclidean space) to \((p^E)^2 \leq \Lambda^2\), hence the complete regularized expression is

\[
G_{\Lambda}^{MC}(p) = -g^2 \int_{-\infty}^{\infty} \frac{d^4 p^E}{(2\pi)^4} \left[ \frac{1}{(p_0^E)^2 + (p^E)^2 + \mu^2} \right] \int_{-\infty}^{0} dt^E e^{E^A_{0}E} e^{i(p_0^E + k_0^E - p^E)v^E + \int_{-\infty}^{0} dt^E e^{E^A_{0}E} e^{-i(p_0^E + k_0^E - p^E)v^E}}. \tag{139}
\]

We can perform the integrations over \(k_0^E\) (with the help of the residue theorem) and \(t^E\) in Eq. (139), separately for the two contributions, to obtain the result

\[
G_{\Lambda}^{MC}(p) = -g^2 \int_{-\infty}^{\infty} \frac{d^4 p^E}{(2\pi)^4} \frac{i}{2\omega_{p-p'}} \left[ \frac{p_0^E \gamma_{0}^E + p' \cdot \gamma^E + m_A}{(p_0^E)^2 - (p^E)^2 + \mu^2} \right] \bigg|_{p_0^E \to -iE_p^A}.
\]

Adding up the integrands in Eq. (140) gives

\[
G_{\Lambda}^{MC}(p) = -g^2 \int_{-\infty}^{\infty} \frac{d^4 p^E}{(2\pi)^4} \left[ \frac{p_0^E \gamma_{0}^E + p' \cdot \gamma^E + m_A}{(p_0^E)^2 - (p^E)^2 + \mu^2} \right] \bigg|_{p_0^E \to -iE_p^A}. \tag{141}
\]

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which obviously coincides with the cutoff regularization of the Euclidean version \((119)\) of the covariant form in Eq. \((27)\), thus establishing the equivalence of the non-covariant and covariant expressions in Eq. \((27)\) in the momentum cutoff regularized form. One can use rotations in four-dimensional (Euclidean) space, taking into account the form of the integrand in Eq. \((111)\) as well as the invariance of the integration measure and the integration domain, to show that \(G^{\text{MC}}_{\Lambda}(\mathbf{p})\) is in fact of the form of Eq. \((28)\).

In order to obtain the three-dimensional form of the non-covariant expression in cutoff regularization, we start with Eq. \((140)\) and perform the \(p_0^E\)-integration in

\[
\int_{\Lambda} d^4p^E \frac{d^3p'}{(2\pi)^3} \frac{\Lambda - p'^2}{2\pi} \frac{dp_0^E}{2\pi} = \int_{\Lambda} d^3p' \frac{\Lambda - p'^2}{2\pi} \frac{dp_0^E}{2\pi}
\]

\[(142)\]

(with the \(d^3p^E\)-integration restricted to \(p^2 \leq \Lambda^2\)). To this end, it is easiest to decompose the integrand in partial fractions with respect to \(p_0^E\), for the first non-covariant contribution

\[
\frac{i}{2\omega_{\mathbf{p}'-\mathbf{p}}} \left[ \frac{1}{E_{\mathbf{p}'-\omega_{\mathbf{p}'-\mathbf{p}}-i\mathbf{p}^E_0}} \left( \frac{(\mathbf{p}^E_0 + i\omega_{\mathbf{p}'-\mathbf{p}})\gamma_0^E + \mathbf{p}' \cdot \gamma^E + m_A}{\mathbf{p}^E_0 - \mathbf{p}^E - i\omega_{\mathbf{p}'-\mathbf{p}}} - \frac{iE^A_{\mathbf{p}'-\gamma_0^E} + \mathbf{p}' \cdot \gamma^E + m_A}{\mathbf{p}^E_0 - \mathbf{p}^E + iE^A_{\mathbf{p}'}} \right) + \right.
\]

\[
\left. \frac{1}{E_{\mathbf{p}'2} + \omega_{\mathbf{p}'-\mathbf{p}} - i\mathbf{p}^E_0} \left( \frac{(\mathbf{p}^E_0 + i\omega_{\mathbf{p}'-\mathbf{p}})\gamma_0^E + \mathbf{p}' \cdot \gamma^E + m_A}{\mathbf{p}^E_0 - \mathbf{p}^E - i\omega_{\mathbf{p}'-\mathbf{p}}} - \frac{-iE^A_{\mathbf{p}'-\gamma_0^E} + \mathbf{p}' \cdot \gamma^E + m_A}{\mathbf{p}^E_0 + iE^A_{\mathbf{p}'}} \right) \right].
\]

\[(143)\]

The decomposition in partial fractions for the second contribution in Eq. \((140)\) can be obtained immediately from the one above by realizing that the two contributions are complex conjugate to each other (if we consider \(p^E_0\) and the Euclidean \(\gamma\)-matrices as real, for example by using the Majorana representation).

The \(p_0^E\)-integration is then straightforward, although the result is rather lengthy, indicating that the momentum cutoff would not usually be the regularization of choice in the present not manifestly covariant context, either. However, if we extended the \(p_0^E\)-integration to the whole real axis while sticking somewhat arbitrarily to \(p^2 \leq \Lambda^2\), we would obtain the following simple result for the sum of the two contributions:

\[
G^{\text{NC}}_{\Lambda}(\mathbf{p}) = -g^2 \int_{\Lambda} \frac{d^3p'}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}'-\omega_{\mathbf{p}'-\mathbf{p}}}} \left. \frac{-iE^A_{\mathbf{p}'-\gamma_0^E} + \mathbf{p}' \cdot \gamma^E + m_A}{E_{\mathbf{p}'2} + \omega_{\mathbf{p}'-\mathbf{p}} - i\mathbf{p}^E_0} \right|_{\mathbf{p}^E_0 \rightarrow -iE^A_{\mathbf{p}'}}
\]

\[
- g^2 \int_{\Lambda} \frac{d^3p'}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}'-\omega_{\mathbf{p}'-\mathbf{p}}}} \left. \frac{iE^A_{\mathbf{p}'-\gamma_0^E} + \mathbf{p}' \cdot \gamma^E + m_A}{E_{\mathbf{p}'2} + \omega_{\mathbf{p}'-\mathbf{p}} + i\mathbf{p}^E_0} \right|_{\mathbf{p}^E_0 \rightarrow -iE^A_{\mathbf{p}'}}.
\]

\[(144)\]

Eq. \((144)\) results directly from the non-covariant expression in Eq. \((27)\) by restricting the 3-momentum integration to \(p^2 \leq \Lambda^2\). It is hence, in a not manifestly covariant approach, a
very natural regularization scheme. Although it is apparent that four-dimensional Euclidean rotational invariance is broken in Eq. (144) and that hence \( G^\text{NC}_\Lambda(p) \) will not be of the form (28) for any finite value of \( \Lambda \), it is not clear a priori whether the form (28) could not be recovered in the limit \( \Lambda \to \infty \). In the special case of a purely scalar theory, it was shown in Ref. [27] that covariance is indeed reestablished for \( \Lambda \to \infty \) in the sense that \( G^\text{NC}_\Lambda(p) \) becomes independent of \( p \) in this limit (and only depends on the square of the four-vector, \( p^2|_{p_0=E^A_p} = m^2_\Lambda \)).

In the following, we will consider the integration (142) of the integrand (143) and its complex conjugate counterpart arising from the second non-covariant contribution in the precise, we divide the integration over the spatial momentum becomes independent of \( p \) in this limit (and only depends on the square of the four-vector, \( p^2|_{p_0=E^A_p} = m^2_\Lambda \)).

In the following, we will consider the integration (142) of the integrand (143) and its complex conjugate counterpart arising from the second non-covariant contribution in the limit of large \( \Lambda \) and compare it with \( G^\text{NC}_\Lambda(p) \). To make this comparison mathematically precise, we divide the integration over the spatial momentum \( p' \) into the two regions \( p'^2 < K^2 \) and \( K^2 < p'^2 < \Lambda^2 \) with an intermediate scale \( K \) which fulfills \( p^2, m^2_\Lambda, \mu^2 \ll K^2 \ll \Lambda^2 \). Then, in the limit \( \Lambda \to \infty \), we can approximate the integral over the first region,

\[
\int^K d^3p' \int_{\sqrt{\Lambda^2-p'^2}}^{\Lambda} \frac{dp_0'^E}{2\pi} \to \int^K d^3p' \int_{-\infty}^{\infty} \frac{dp_0'^E}{2\pi}, \quad (145)
\]

hence the integral over \( p'^2 < K^2 \) of Eq. (143) and its complex conjugate, taking special care of the analytic continuation of the logarithms resulting from the \( p_0'^E \)-integration, tend to

\[
-g^2 \int^K \frac{d^3p'}{(2\pi)^3} 2E^A_{p'} \frac{\gamma^E_{0} + p' \cdot \gamma^E + m_A}{E^A_{p'} + \omega_{p-p'} - ip_0'^E} \bigg|_{p_0'^E \to -iE^A_p} 
-g^2 \int^K \frac{d^3p'}{(2\pi)^3} 2E^A_{p'} \frac{\gamma^E_{0} + p' \cdot \gamma^E + m_A}{E^A_{p'} + \omega_{p-p'} + ip_0'^E} \bigg|_{p_0'^E \to -iE^A_p}. \quad (146)
\]

The first corrections to this result, of relative order \((K/m_A)(K/\Lambda)\) [which can be suppressed in the limit of large \( \Lambda \) through a suitable choice of \( K \), e.g., \( K = (m^2_\Lambda/\Lambda)^{1/4} \)], cancel among the two contributions. Incidentally, the second corrections, of order \((K/m_A)(K/\Lambda)^2\), also vanish.

Now, for the other integration region \( K^2 < p'^2 < \Lambda^2 \), we have \( p^2, m^2_\Lambda, \mu^2 \ll p'^2, \Lambda^2 \). A lengthy calculation leads to the following result: the leading terms in a systematic expansion vanish for both contributions, the subleading terms cancel among the two contributions for the \( \gamma^E_0 \)-coefficients or as a result of the \((p' \to -p')\)-symmetry of integrand, integration measure and domain for the \( \gamma^E \)-coefficient, and it is hence the subsubleading terms that give the dominant contributions which turn out to be

\[
-g^2 \int^\Lambda_K \frac{d^3p'}{(2\pi)^3} \frac{p_0'^E \gamma^E_0 + p \cdot \gamma^E + 2m_A}{4\pi p^2} \left[ \frac{\pi}{2|p'|} - \frac{\arcsin(|p'|/\Lambda)}{|p'|} + \sqrt{1 - p'^2/\Lambda^2} \right] 
+ g^2 \int^\Lambda_K \frac{d^3p'}{(2\pi)^3} \frac{3p_0'^E \gamma^E_0 - p \cdot \gamma^E}{6\pi \Lambda^2} \sqrt{1 - p'^2/\Lambda^2} \Lambda. \quad (147)
\]
The first term in square brackets in Eq. (147) coincides with the dominant contribution from the part of \( G^NC_\Lambda (\p) \) that stems from the second integration region \( K^2 < \p^2 < \Lambda^2 \) [cf. Eq. (144)], and hence, when added to Eq. (146), combines to \( G^NC_\Lambda (\p) \). The other terms in Eq. (147) are readily integrated [28] in the limit \( \Lambda \to \infty \). The final result is

\[
G^NC_\Lambda (\p) = -g^2 \int_0^\Lambda \frac{d^4 p^E}{(2\pi)^4} \frac{p_0^E \gamma_0 + \p' \cdot \gamma^E + m_A}{[(p_0^E - p_0^E)^2 + (\p - \p')^2 + \mu^2][(p_0^E)^2 + \p^2 + m_A^2]} \bigg|_{\p^E \to -iE_p^E} + \frac{g^2}{(4\pi)^2} \left( \frac{1}{2} - \ln 2 \right) \left( E_p^A \gamma_0 - \p \cdot \gamma + 2m_A \right) - \frac{g^2}{(4\pi)^2} \frac{1}{12} \left( 3E_p^A \gamma_0 + \p \cdot \gamma \right) \tag{148}
\]

in the limit \( \Lambda \to \infty \).

The result (148) has two important consequences: first, together with the expressions (157) and (158) for \( G^MC_\Lambda (\p) \) below, it provides an explicit expression for \( G^NC_\Lambda (\p) \) in the limit of large \( \Lambda \). Second, and more importantly, it shows that Lorentz invariance is broken, even in the limit \( \Lambda \to \infty \), through the last term in Eq. (148). For example, it can explicitly be shown that this term leads to a non-covariant contribution to the energies of the one-particle states by following the procedure detailed in Eqs. (31)–(34). It is then clear that the non-covariant cutoff regularization is not suitable for a fermionic theory. On the other hand, in a purely scalar theory, given by the \( m_A \)-coefficient in Eq. (148), Lorentz invariance is recovered in the limit \( \Lambda \to \infty \), in agreement with the result in Ref. [27] which was obtained by a different method.

Let us now return to the expression for \( G^MC_\Lambda (\p) \) given in Eq. (141). In order to calculate \( G_0(m_A^2) \) and \( G_1(m_A^2) \) in cutoff regularization, one could use four-dimensional spherical coordinates for the \( d^4 p^E \)-integration (see also below), which would also explicitly confirm that \( G^MC_\Lambda (\p) \) is of the form of Eq. (28) for any finite value of \( \Lambda \). However, as long as one is only interested in the result for large enough \( \Lambda \), it is much quicker in the present situation to introduce Feynman parameters and proceed in analogy with Eq. (86) [in \( (D = 4) \)-dimensional Euclidean space and for the integration domain \( (p^E)^2 \leq \Lambda^2 \)]. We thus arrive at

\[
G^MC_\Lambda (\p) = -g^2 \int_0^1 dx \int_0^\Lambda \frac{d^4 q^E}{(2\pi)^4} \frac{(q^E + x p^E) \cdot \gamma^E + m_A}{[(q^E)^2 + x(1-x)(p^E)^2 + x\mu^2 + (1-x)m_A^2]^2} \bigg|_{p_0^E \to -iE_p^E} \tag{149}
\]

with \( q^E \) to be integrated over the four-dimensional Euclidean domain \( (q^E + x p^E)^2 \leq \Lambda^2 \) (as a result of the shift of the integration variable to \( q^E = p^E - x p^E \)). To evaluate this expression it is easiest to calculate the difference

\[
\Delta G^MC_{\Lambda,\Lambda'}(\p) = G^MC_\Lambda (\p) - G^MC_{\Lambda'}(\p) \tag{150}
\]

between \( G^MC_\Lambda (\p) \) and

\[
G^MC_{\Lambda'}(\p) = -g^2 \int_0^1 dx \int_0^\Lambda \frac{d^4 q^E}{(2\pi)^4} \frac{x p^E \cdot \gamma^E + m_A}{[(q^E)^2 + x(1-x)(p^E)^2 + x\mu^2 + (1-x)m_A^2]^2} \bigg|_{p_0^E \to -iE_p^E} \tag{151}
\]
with the integration domain restricted to \((q^E)^2 \leq \Lambda^2\) [observe that in this case the first term in the numerator in Eq. (149) does not contribute because of the symmetry of the integration domain]. The \(d^4q^E\)-integral in \(G_{\Lambda}^{MC}(p)\) can be evaluated by standard methods.

Incidentally, it would in principle be possible to define cutoff regularization via Eq. (151) [instead of Eq. (149)], a choice that would obviously fulfill Eq. (28), too. The non-covariant expression corresponding to Eq. (139) in this alternative regularization is

\[
G_{\Lambda}^{MC}(p) =
\]

\[
- g^2 \int_0^1 dx \int \frac{d^4p^E}{(2\pi)^4} \frac{p_0^E \gamma_0 + p' \cdot \gamma^E + m_A}{2\pi \left[ (k_0^E)^2 x + \omega_{p-p'}^2 x + (p_0^E)^2 (1 - x) + (E_p^A)^2 (1 - x) \right]^2}
\]

\[
\times \left[ \int_{-\infty}^0 dt E e^{t E} e^{i(p_0^E + k_0^E - p_0^E) t^E} + \int_{-\infty}^0 dt E e^{t E} e^{-i(p_0^E + k_0^E - p_0^E) t^E} \right]_{p^E \rightarrow -iE_p^A},
\]

(152)

where the \(d^4p^E\)-integration is now over \((p^E - xp^E)^2 \leq \Lambda^2\). When comparing this expression with Eq. (139), it becomes clear that this regularization is not too natural in the present non-covariant context. Still, we can integrate over \(k_0^E\) (again with the help of the residue theorem) and \(t_E\) in Eq. (152). The result for the first contribution is

\[
- g^2 \int_0^1 dx \int \frac{d^4p^E}{(2\pi)^4} \frac{p_0^E \gamma_0 + p' \cdot \gamma^E + m_A}{2\pi \left[ (p_0^E - p_0^E)^2 x + \omega_{p-p'}^2 x + (p_0^E)^2 (1 - x) + (E_p^A)^2 (1 - x) \right]^2}
\]

\[
\times \left\{ \frac{1}{2} + i \frac{(p_0^E - p_0^E)^2}{4 \left[ (p_0^E - p_0^E)^2 x + 3 \omega_{p-p'}^2 (1 - x) + (E_p^A)^2 (1 - x) \right]^{3/2}} \right\},
\]

(153)

to be analytically continued to \(p_0^E \rightarrow -iE_p^A\). The result for the second contribution can again be obtained by complex conjugation from the above (considering \(p_0^E\) and the Euclidean \(\gamma\)-matrices as real). The integrations over both \(x\) and \(p_0^E\) in Eq. (153) look forbidding. However, the sum of the two non-covariant contributions is readily seen to give \(G_{\Lambda}^{MC}(p)\) as defined in Eq. (151), after shifting the four-momentum integration variable to \(q^E = p^E - xp^E\).

Returning to the explicit calculation of \(G_{\Lambda}^{MC}(p)\), we note that the difference \(\Delta G_{\Lambda,\Lambda'}^{MC}(p)\) is an integral over the difference of the four-balls \((q^E + xp^E)^2 \leq \Lambda^2\) and \((q^E)^2 \leq \Lambda^2\). We introduce four-dimensional (Euclidean) spherical coordinates with the fourth axis oriented in direction of \(p^E\) and the corresponding polar angle denoted as \(\chi\). Then for the four-ball \((q^E + xp^E)^2 \leq \Lambda^2\) the integration along the radius runs up to \(R(\chi)\),

\[
R(\chi) = \Lambda - x|p^E| \cos \chi + \mathcal{O}(1/\Lambda)
\]

(154)
in the limit of large \(\Lambda\). Furthermore, the denominator of the integrand in the region between
the two four-balls can be approximated by $\Lambda^4$ in this limit. We then have

$$\Delta G_{A,A'}^{MC}(\mathbf{p}) = -g^2 \int_0^1 dx \int_0^\pi d\chi \sin^2 \chi \int_\Lambda^{R(\chi)} r^3 \cos \chi \gamma_4 E + x(p^E \cdot \gamma^E) + m_A,$$

while the other components of $q^E$ in the numerator do not contribute for symmetry reasons (integration over the other angular coordinates). The integrations in Eq. (155) yield, in the limit of large $\Lambda$,

$$\Delta G_{A,A'}^{MC}(\mathbf{p}) = \frac{g^2}{(4\pi)^2} \frac{|p^E|\gamma_4}{4} = \frac{g^2}{(4\pi)^2} \frac{p \cdot \gamma}{4},$$

where we have used $|p^E|\gamma_4 = p^E \cdot \gamma^E$ and replaced $p_0^E \rightarrow -iE_p^A \equiv -ip^0$. In particular, for a purely scalar theory, the difference $\Delta G_{A,A'}^{MC}(\mathbf{p})$ tends to zero for $\Lambda \rightarrow \infty$.

Now from Eqs. (150) and (151), the above result (156) and the result of the integration over the Feynman parameter $x$ in Eqs. (150) and (151), we finally get the explicit expressions

$$G_{0,A}(m_A^2) = -\frac{g^2}{(4\pi)^2} \left\{ \ln \frac{\Lambda^2}{m_A^2} + 1 - \frac{\mu^2}{2m_A^2} \ln \frac{\mu^2}{m_A^2} \right\} - 2 \sqrt{\frac{\mu^2}{m_A^2}} \left( 1 - \frac{\mu^2}{4m_A^2} \right) \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\},$$

$$G_{1,A}(m_A^2) = -\frac{g^2}{(4\pi)^2} \left\{ \frac{1}{2} \ln \frac{\Lambda^2}{m_A^2} + \frac{3}{4} - \frac{\mu^2}{2m_A^2} - \left( \frac{\mu^2}{m_A^2} - 1 \frac{\mu^4}{4m_A^2} \right) \ln \frac{\mu^2}{m_A^2} \right\}$$

$$\left( 2 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2}{m_A^2}} \left( 1 - \frac{\mu^2}{4m_A^2} \right) \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\},$$

($\Lambda^2 \gg m_A^2$). Again, the result for $G_{0,A}(m_A^2)$ gives directly the expression for $\Delta m_A^2$ in momentum cutoff regularization for the purely scalar case, while for Yukawa theory we have

$$\frac{\Delta m_A^2}{2m_A^2} = -\frac{g^2}{(4\pi)^2} \left\{ \frac{3}{2} \ln \frac{\Lambda^2}{m_A^2} + \frac{7}{4} - \frac{\mu^2}{2m_A^2} - \left( \frac{3 \mu^2}{2m_A^2} - 1 \frac{\mu^4}{4m_A^2} \right) \ln \frac{\mu^2}{m_A^2} \right\}$$

$$- \left( 4 - \frac{\mu^2}{m_A^2} \right) \sqrt{\frac{\mu^2}{m_A^2}} \left( 1 - \frac{\mu^2}{4m_A^2} \right) \arctg \sqrt{\frac{\mu^2/(4m_A^2)}{1 - \mu^2/(4m_A^2)}} \right\}.\quad (159)$$

### C Separation of angular variables and spin

In terms of the well-known eigenstates $\chi_{S,\mathbf{s}}$ of total spin $\mathbf{S} = \mathbf{s}_A + \mathbf{s}_B$ ($S = 0$ or $S = 1$), one has the following explicit expressions for the eigenstates of $\mathbf{J}^2$ and $J_z$ with eigenvalues
$J(J+1)$ and $M$:

$$1^J_{JM}(\hat{p}) = Y_{JM}(\hat{p}) \chi_{00},$$

$$3^J_{J-1,M}(\hat{p}) = \frac{1}{\sqrt{2J(2J-1)}} \left[ \sqrt{(J-M-1)(J-M)} Y_{J-1,M+1}(\hat{p}) \chi_{11} + \sqrt{2(J-M)(J+M)} Y_{J-1,M}(\hat{p}) \chi_{10} + \sqrt{(J+M-1)(J+M)} Y_{J-1,M-1}(\hat{p}) \chi_{11} \right] (J \geq 1),$$

$$3^J_{JM}(\hat{p}) = \frac{1}{\sqrt{2J(J+1)}} \left[ \sqrt{(J-M)(J+M+1)} Y_{J,M+1}(\hat{p}) \chi_{11} + \sqrt{2M} Y_{JM}(\hat{p}) \chi_{10} - \sqrt{(J-M+1)(J+M)} Y_{J,M-1}(\hat{p}) \chi_{11} \right] (J \geq 1),$$

$$3^J_{J+1,M}(\hat{p}) = \frac{1}{\sqrt{2J(J+1)(2J+3)}} \left[ \sqrt{(J+M+1)(J+M+2)} Y_{J+1,M+1}(\hat{p}) \chi_{11} + \sqrt{2(J-M+1)(J+M+1)} Y_{J+1,M}(\hat{p}) \chi_{10} - \sqrt{(J-M+1)(J+M+2)} Y_{J+1,M-1}(\hat{p}) \chi_{11} \right] .$$

To determine the action of the helicity operators on these eigenfunctions, the explicit spinor representation

$$\hat{p} \cdot \sigma = \begin{pmatrix} \cos \vartheta & \sin \vartheta e^{-i\varphi} \\ \sin \vartheta e^{i\varphi} & -\cos \vartheta \end{pmatrix}$$

(162)

can be used, together with the following special instances of the spherical harmonics addition relation [30]:

$$\cos \vartheta Y_{lm}(\hat{p}) = \sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}} Y_{l-1,m}(\hat{p}),$$

$$\sin \vartheta e^{i\varphi} Y_{lm}(\hat{p}) = \sqrt{\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}} Y_{l-1,m+1}(\hat{p}).$$

$$-\sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}} Y_{l+1,m+1}(\hat{p}),$$

(161)
\[
\sin \vartheta e^{-i\varphi} Y_{lm}(\hat{p}) = -\sqrt{\frac{(l + m - 1)(l + m)}{(2l - 1)(2l + 1)}} Y_{1-1,m-1}(\hat{p}) + \sqrt{\frac{(l - m + 1)(l - m + 2)}{(2l + 1)(2l + 3)}} Y_{1+1,m-1}(\hat{p}) .
\]

It is then a straightforward, though somewhat lengthy exercise to derive the following formulas for the application of the helicity operators to the total angular momentum eigenstates:

\[
(\hat{p} \cdot \sigma_A) Y_{JM}^J(\hat{p}) = \sqrt{\frac{J}{2J + 1}} Y_{J-1,M}^J(\hat{p}) - \sqrt{\frac{J + 1}{2J + 1}} Y_{J+1,M}^J(\hat{p}) ,
\]

\[
(\hat{p} \cdot \sigma_A) Y_{J-1,M}^J(\hat{p}) = -\sqrt{\frac{J}{2J + 1}} Y_{J-1,M}^J(\hat{p}) - \sqrt{\frac{J + 1}{2J + 1}} Y_{J+1,M}^J(\hat{p}) ,
\]

\[
(\hat{p} \cdot \sigma_A) Y_{J+1,M}^J(\hat{p}) = -\sqrt{\frac{J}{2J + 1}} Y_{J+1,M}^J(\hat{p}) - \sqrt{\frac{J + 1}{2J + 1}} Y_{J+1,M}^J(\hat{p}) ,
\]

and

\[
(\hat{p} \cdot \sigma_B) Y_{JM}^J(\hat{p}) = -\sqrt{\frac{J}{2J + 1}} Y_{J+1,M}^J(\hat{p}) + \sqrt{\frac{J + 1}{2J + 1}} Y_{J+1,M}^J(\hat{p}) ,
\]

\[
(\hat{p} \cdot \sigma_B) Y_{J-1,M}^J(\hat{p}) = -\sqrt{\frac{J}{2J + 1}} Y_{J-1,M}^J(\hat{p}) - \sqrt{\frac{J + 1}{2J + 1}} Y_{J+1,M}^J(\hat{p}) ,
\]

\[
(\hat{p} \cdot \sigma_B) Y_{J+1,M}^J(\hat{p}) = -\sqrt{\frac{J}{2J + 1}} Y_{J+1,M}^J(\hat{p}) - \sqrt{\frac{J + 1}{2J + 1}} Y_{J+1,M}^J(\hat{p}) .
\]

In the special case \( J = 0 \), the states \( Y_{J-1,M}^J(\hat{p}) \) and \( Y_{J+1,M}^J(\hat{p}) \) do not exist, and on the right–hand sides for the application of one of the helicity operators to \( Y_{JM}^J(\hat{p}) \) and \( Y_{J+1,M}^J(\hat{p}) \), only one term remains.

By use of Eq. (165), where now

\[
a_l(p,p') = \frac{2l + 1}{2} \int_{-1}^{1} d\cos\theta \ P_l(\cos\theta) \frac{1}{2|\hat{p} - \hat{p}'|} \ 
\times \left( \frac{1}{E_p^A + |\hat{p} - \hat{p}'| - E_{p'}^A} + \frac{1}{E_p^B + |\hat{p} - \hat{p}'| - E_{p'}^B} \right),
\]
(for \( \mu = 0 \)), the effective Schrödinger equation Eq. (14) decouples into pairs of coupled one-dimensional integral equations. For the S-coupled states, we introduce the wave function

\[
\phi(p) = s_{\phi_0}^J(p) \mathcal{Y}_{JM}(\hat{p}) + s_{\phi_1}^J(p) \mathcal{Y}_{JM}(\hat{p}) .
\]  

(167)

The effective Schrödinger equation for the coefficient functions becomes

\[
\left( \sqrt{M_A^2 + p^2} + \sqrt{M_B^2 + p^2} \right) \begin{pmatrix}
  s_{\phi_0}^J(p) \\
  s_{\phi_1}^J(p)
\end{pmatrix}
- \frac{g^2}{2\pi^2} \int_0^\infty dp' p'^2 \sqrt{\frac{E_p^A + M_A}{2E_p^A} \frac{E_p^B + M_B}{2E_p^B} \frac{E_{p'}^A + M_A}{2E_{p'}^A} \frac{E_{p'}^B + M_B}{2E_{p'}^B}}
\times \
\frac{1}{2J+1} \begin{pmatrix}
  s_{V_{00}}^J(p,p') & s_{V_{10}}^J(p,p') \\
  s_{V_{01}}^J(p,p') & s_{V_{11}}^J(p,p')
\end{pmatrix}
\begin{pmatrix}
  s_{\phi_0}^J(p') \\
  s_{\phi_1}^J(p')
\end{pmatrix}
= (E - E_V) \begin{pmatrix}
  s_{\phi_0}^J(p) \\
  s_{\phi_1}^J(p)
\end{pmatrix} ,
\]

(168)

with

\[
s_{V_{00}}^J(p,p') = \left[ 1 + \frac{p}{E_p^A + M_A} \frac{p}{E_p^B + M_B} \frac{p'}{E_{p'}^A + M_A} \frac{p'}{E_{p'}^B + M_B} \right] a_J(p,p')
- \left[ \frac{p}{E_p^A + M_A} \frac{p'}{E_{p'}^A + M_A} + \frac{p}{E_p^B + M_B} \frac{p'}{E_{p'}^B + M_B} \right]
\times \left[ J \frac{a_{J-1}(p,p')}{2J-1} + (J+1) \frac{a_{J+1}(p,p')}{2J+3} \right] ,
\]

\[
s_{V_{11}}^J(p,p') = \left[ 1 + \frac{p}{E_p^A + M_A} \frac{p}{E_p^B + M_B} \frac{p'}{E_{p'}^A + M_A} \frac{p'}{E_{p'}^B + M_B} \right] a_J(p,p')
- \left[ \frac{p}{E_p^A + M_A} \frac{p'}{E_{p'}^A + M_A} + \frac{p}{E_p^B + M_B} \frac{p'}{E_{p'}^B + M_B} \right]
\times \left[ (J+1) \frac{a_{J-1}(p,p')}{2J-1} + J \frac{a_{J+1}(p,p')}{2J+3} \right] ,
\]

\[
s_{V_{01}}^J(p,p') = s_{V_{10}}^J(p,p')
\]

\[
= \left[ \frac{p}{E_p^A + M_A} \frac{p'}{E_{p'}^A + M_A} - \frac{p}{E_p^B + M_B} \frac{p'}{E_{p'}^B + M_B} \right]
\times \sqrt{J(J+1)} \left[ \frac{a_{J-1}(p,p')}{2J-1} - \frac{a_{J+1}(p,p')}{2J+3} \right] .
\]  

(169)
On the other hand, with the wave function

$$\phi(p) = L\phi_{J-1}^J(p) \mathcal{Y}_{J-1,M}^J(\hat{p}) + L\phi_{J+1}^J(p) \mathcal{Y}_{J+1,M}^J(\hat{p}) ,$$  \hspace{1cm} (170)

the effective Schrödinger equation for the L-coupled states becomes

$$ \left( \sqrt{M_A^2 + p^2} + \sqrt{M_B^2 + p^2} \right) \left( \begin{array}{c} L\phi_{J-1}^J(p) \\ L\phi_{J+1}^J(p) \end{array} \right) $$

$$ - \frac{g^2}{2\pi^2} \int_0^\infty dp' p'^2 \sqrt{\frac{E^A_{p'} + M_A}{2E^A_p} \frac{E^B_{p'} + M_B}{2E^B_p}} \left( \begin{array}{c} \mathcal{V} J_{J-1,J-1}(p, p') \\ \mathcal{V} J_{J+1,J+1}(p, p') \end{array} \right) \left( \begin{array}{c} L\phi_{J-1}^J(p') \\ L\phi_{J+1}^J(p') \end{array} \right) $$

$$ = (E - E_V) \left( \begin{array}{c} L\phi_{J-1}^J(p) \\ L\phi_{J+1}^J(p) \end{array} \right) ,$$  \hspace{1cm} (171)

where

$$ \mathcal{V} J_{J-1,J-1}(p, p') = (2J + 1) \frac{a_{J-1}(p, p')}{2J - 1} $$

$$ - \left[ \frac{p}{E^A_p + M_A} \frac{p'}{E^A_{p'} + M_A} + \frac{p}{E^B_p + M_B} \frac{p'}{E^B_{p'} + M_B} \right] a_J(p, p') $$

$$ + \frac{p}{E^A_p + M_A} \frac{p'}{E^B_{p'} + M_B} \frac{p'}{E^A_{p'} + M_A} \frac{p}{E^B_p + M_B} $$

$$ \times \frac{1}{2J + 1} \left[ a_{J-1}(p, p') \frac{2J + 1}{2J - 1} + 4J(J + 1) \frac{a_{J+1}(p, p')}{2J + 3} \right] ,$$

$$ \mathcal{V} J_{J+1,J+1}(p, p') = (2J + 1) \frac{a_{J+1}(p, p')}{2J + 3} $$

$$ - \left[ \frac{p}{E^A_p + M_A} \frac{p'}{E^A_{p'} + M_A} + \frac{p}{E^B_p + M_B} \frac{p'}{E^B_{p'} + M_B} \right] a_J(p, p') $$

$$ + \frac{p}{E^A_p + M_A} \frac{p}{E^B_{p'} + M_B} \frac{p'}{E^A_{p'} + M_A} \frac{p'}{E^B_p + M_B} $$

$$ \times \frac{1}{2J + 1} \left[ 4J(J + 1) \frac{a_{J-1}(p, p')}{2J - 1} + \frac{a_{J+1}(p, p')}{2J + 3} \right] ,$$

$$ \mathcal{V} J_{J-1,J+1}(p, p') = \mathcal{V} J_{J+1,J-1}(p, p') $$

$$ = \frac{p}{E^A_p + M_A} \frac{p}{E^B_p + M_B} \frac{p'}{E^A_{p'} + M_A} \frac{p'}{E^B_{p'} + M_B} $$

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In the special case $J = 0$, $S(l) \equiv L_{j-1}(p) \equiv 0$, and the equations for $S(l)$ and $L_{j+1}(p)$ decouple.

The functions $a_i(p, p')$ are quite straightforwardly calculated from Eq. (163), for any given value of $l$. The results for the functions which are relevant for the numerical calculations presented here ($l \leq 3$) are:

\[
\begin{align*}
a_0(p, p') &= \frac{1}{4pp'} \left\{ \ln \left( \frac{E_p^A + (p + p') - E_{p'}^A}{E_p^A + |p - p'| - E_{p'}^A} \right) + \ln \left( \frac{E_p^B + (p + p') - E_{p'}^B}{E_p^B + |p - p'| - E_{p'}^B} \right) \right\}, \\
a_1(p, p') &= \frac{3}{4pp'} \left\{ \frac{(E_p^A - E_{p'}^A + E_p^B - E_{p'}^B)(p + p' - |p - p'|)}{2pp'} - 2 \\
&\quad - \frac{p^2 + p'^2 - (E_p^A - E_{p'}^A)^2}{2pp'} \ln \left( \frac{E_p^A + (p + p') - E_{p'}^A}{E_p^A + |p - p'| - E_{p'}^A} \right) \\
&\quad + \frac{p^2 + p'^2 - (E_p^B - E_{p'}^B)^2}{2pp'} \ln \left( \frac{E_p^B + (p + p') - E_{p'}^B}{E_p^B + |p - p'| - E_{p'}^B} \right) \right\}, \\
a_2(p, p') &= \frac{5}{8pp'} \left\{ \frac{3[2(p^2 + p'^2) - (E_p^A - E_{p'}^A)^2]}{4p^2p'^2} \right. \\
&\quad - \frac{3[2(p^2 + p'^2) - (E_p^B - E_{p'}^B)^2]}{4p^2p'^2} \right. \\
&\quad - \frac{(E_p^A - E_{p'}^A + E_p^B - E_{p'}^B)[(p + p')^3 - |p - p'|^3]}{4p^2p'^2} \\
&\quad - \frac{3[2(p^2 + p'^2) - (E_p^A - E_{p'}^A)^2 - (E_p^B - E_{p'}^B)^2]}{2pp'} \\
&\quad + \left[ \frac{3[p^2 + p'^2 - (E_p^A - E_{p'}^A)^2]}{4p^2p'^2} - 1 \right] \ln \left( \frac{E_p^A + (p + p') - E_{p'}^A}{E_p^A + |p - p'| - E_{p'}^A} \right) \\
&\quad + \left[ \frac{3[p^2 + p'^2 - (E_p^B - E_{p'}^B)^2]}{4p^2p'^2} - 1 \right] \ln \left( \frac{E_p^B + (p + p') - E_{p'}^B}{E_p^B + |p - p'| - E_{p'}^B} \right) \right\}, \\
a_3(p, p') &= \frac{7}{8pp'} \left\{ \frac{5[3(p^2 + p'^2)^2 - 3(p^2 + p'^2)(E_p^A - E_{p'}^A)^2 + (E_p^A - E_{p'}^A)^4(E_p^A - E_{p'}^A)]}{8p^3p'^3} \\
&\quad + \frac{5[3(p^2 + p'^2)^2 - 3(p^2 + p'^2)(E_p^B - E_{p'}^B)^2 + (E_p^B - E_{p'}^B)^4(E_p^B - E_{p'}^B)]}{8p^3p'^3} \right\}.
\end{align*}
\]
used a quasi-algebraic method which increases speed and accuracy. To this end, the integrand

\[ \frac{3(E_p^A - E_{p'}^A + E_p^B - E_{p'}^B)}{2pp'} \left( p + p' - |p - p'| \right) \]

\[ + \frac{5(E_p^A - E_{p'}^A)[3(p^2 + p'^2) - (E_p^A - E_{p'}^A)^2]}{24p^3p'^3} \]

\[ + \frac{5(E_p^B - E_{p'}^B)[3(p^2 + p'^2) - (E_p^B - E_{p'}^B)^2]}{24p^3p'^3} \left[ (p + p')^3 - |p - p'|^3 \right] \]

\[ + \frac{(E_p^A - E_{p'}^A + E_p^B - E_{p'}^B)[(p + p')^5 - |p - p'|^5]}{8p^3p'^3} \]

\[ - \frac{5[p^2 + p'^2 - (E_p^A - E_{p'}^A)^2]^2 + 5[p^2 + p'^2 - (E_p^B - E_{p'}^B)^2]^2}{4p^2p'^2} \]

\[ + \frac{5[p^2 + p'^2 - (E_p^A - E_{p'}^A)^2]^3}{8p^3p'^3} - \frac{3[p^2 + p'^2 - (E_p^A - E_{p'}^A)^2]^2}{2pp'} \]

\[ \times \ln \left( \frac{E_p^A + (p + p') - E_{p'}^A}{E_p^A + |p - p'| - E_{p'}^A} \right) \]

\[ + \frac{5[p^2 + p'^2 - (E_p^B - E_{p'}^B)^2]^3}{8p^3p'^3} - \frac{3[p^2 + p'^2 - (E_p^B - E_{p'}^B)^2]^2}{2pp'} \]

\[ \times \ln \left( \frac{E_p^B + (p + p') - E_{p'}^B}{E_p^B + |p - p'| - E_{p'}^B} \right) \}

(173)

In the actual numerical calculations we did not rely on these explicit expressions, but rather used a quasi-algebraic method which increases speed and accuracy. To this end, the integrand in Eq. (169) is written, after changing variables from \( \cos \theta \) to \( x = |p - p'| \), as a (finite) Laurent series in \( x \) around \( J = 0 \) (or \( E_{p'} - E_p \) for the second term). This requires expressing \( P_l(\cos \theta) \) as a polynomial in \( x \), a task left to the computer. Each term in the Laurent series has a known integral which is simply inserted. Any value of \( l \) can be handled easily by this method.

In the one-body limit \( M_B \to \infty \), the matrix for the effective potential in the S-coupled sector, Eq. (169), tends to

\[
\begin{pmatrix}
S_{00}(p, p') & S_{01}^J(p, p') \\
S_{10}^J(p, p') & S_{11}^J(p, p')
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \alpha_J(p, p')
\]

\[- \frac{p}{E_p^A + M_A} \frac{p'}{E_{p'}^A + M_A} \begin{pmatrix} J & -\sqrt{J(J+1)} \\ -\sqrt{J(J+1)} & J + 1 \end{pmatrix} \frac{\alpha_{J-1}(p, p')}{2J - 1} \]

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This matrix is diagonalized by the orthogonal linear combinations

\[
\begin{pmatrix}
    S\phi_0^J(p) \\
    S\phi_1^J(p)
\end{pmatrix} = \begin{pmatrix}
    -\sqrt{J} \\
    \sqrt{J + 1}
\end{pmatrix} \frac{S\phi_{J-1/2}^J(p)}{\sqrt{2J + 1}},
\]

and

\[
\begin{pmatrix}
    S\phi_0^J(p) \\
    S\phi_1^J(p)
\end{pmatrix} = \begin{pmatrix}
    \sqrt{J + 1} \\
    \sqrt{J}
\end{pmatrix} \frac{S\phi_{J+1/2}^J(p)}{\sqrt{2J + 1}},
\]

with the respective eigenvalues

\[
(2J + 1) \left( \frac{a_J(p, p')}{2J + 1} - \frac{p}{E_p^A + M_A} \frac{p'}{E_{p'}^A + M_A} \frac{a_{J-1}(p, p')}{2J - 1} \right),
\]

\[
(2J + 1) \left( \frac{a_J(p, p')}{2J + 1} - \frac{p}{E_p^A + M_A} \frac{p'}{E_{p'}^A + M_A} \frac{a_{J+1}(p, p')}{2J + 3} \right).
\]

Alternatively, we can write the wave function as

\[
\phi(p) = S\phi_{J-1/2}^J(p) J_{-1/2} Y_{JM}^J(\hat{p})
\]

or

\[
\phi(p) = S\phi_{J+1/2}^J(p) J_{+1/2} Y_{JM}^J(\hat{p})
\]

with

\[
J_{-1/2} Y_{JM}^J(\hat{p}) = -\sqrt{\frac{J}{2J + 1}} Y_{JM}^J(\hat{p}) + \sqrt{\frac{J + 1}{2J + 1}} Y_{JM}^J(\hat{p}),
\]

\[
J_{+1/2} Y_{JM}^J(\hat{p}) = \sqrt{\frac{J + 1}{2J + 1}} Y_{JM}^J(\hat{p}) + \sqrt{\frac{J}{2J + 1}} Y_{JM}^J(\hat{p}),
\]

anticipating the notation \( j_A Y_{JM}^J(\hat{p}) \). In the special case \( J = 0 \), of course, \( S\phi_1^J(p) \equiv 0 \), \( S\phi_{J-1/2}^J(p) \equiv 0 \), and \( J_{-1/2} Y_{JM}^J(\hat{p}) \) does not exist, since there is only one state in the S-coupled \( J = 0 \) sector.

The matrix \([172]\) of the effective potential for L-coupled states, on the other hand, becomes diagonal in the limit \( M_B \to \infty \), hence there is no L-coupling in this limit. The diagonal matrix elements tend to

\[
_{L}V_{J-1,J-1}(p, p') = (2J + 1) \left( \frac{a_{J-1}(p, p')}{2J - 1} - \frac{p}{E_p^A + M_A} \frac{p'}{E_{p'}^A + M_A} \frac{a_J(p, p')}{2J + 1} \right),
\]

\[
_{L}V_{J+1,J+1}(p, p') = (2J + 1) \left( \frac{a_{J+1}(p, p')}{2J + 3} - \frac{p}{E_p^A + M_A} \frac{p'}{E_{p'}^A + M_A} \frac{a_J(p, p')}{2J + 1} \right).
\]
Then, from Eqs. (168) and (171), \( S_{j-1/2}(p) \) in Eq. (175) and \( L_{j-1/2}(p) \equiv L_{j}^{1}(p) \) in Eq. (170),

\[
\phi(p) = L_{j-1/2}^{1}(p) S_{j-1/2}(p) \beta_{JM}(\hat{p}) ,
\]

fulfill exactly the same one-dimensional equation in the limit \( M_{B} \to \infty \) (for \( J \geq 1 \)),

\[
\sqrt{M_{A}^{2} + p^{2}} \left( \frac{S_{j-1/2}(p)}{L_{j-1/2}(p)} \right) - \frac{g^{2}}{2\pi^{2}} \int_{0}^{\infty} dp' p'^{2} \sqrt{\frac{E_{p}^{A} + M_{A}}{2E_{p}} \frac{E_{p'}^{A} + M_{A}}{2E_{p'}}} \left( \frac{a_{j}(p, p')}{2J + 1} - \frac{p}{E_{p}^{A} + M_{A}} \frac{p'}{E_{p'}^{A} + M_{A}} \frac{a_{j-1}(p, p')}{2J - 1} \right) \left( \frac{S_{j-1/2}(p')}{L_{j-1/2}(p')} \right) = (E - E_{V} - M_{B}) \left( \frac{S_{j-1/2}(p)}{L_{j-1/2}(p)} \right) .
\]

Likewise, \( S_{j+1/2}(p) \) in Eq. (176) and \( L_{j+1/2}(p) \equiv L_{j}^{1}(p) \) in Eq. (170),

\[
\phi(p) = L_{j+1/2}^{1}(p) S_{j+1/2}(p) \beta_{JM}(\hat{p}) ,
\]

both fulfill the equation

\[
\sqrt{M_{A}^{2} + p^{2}} \left( \frac{S_{j+1/2}(p)}{L_{j+1/2}(p)} \right) - \frac{g^{2}}{2\pi^{2}} \int_{0}^{\infty} dp' p'^{2} \sqrt{\frac{E_{p}^{A} + M_{A}}{2E_{p}} \frac{E_{p'}^{A} + M_{A}}{2E_{p'}}} \left( \frac{a_{j}(p, p')}{2J + 1} - \frac{p}{E_{p}^{A} + M_{A}} \frac{p'}{E_{p'}^{A} + M_{A}} \frac{a_{j+1}(p, p')}{2J + 3} \right) \left( \frac{S_{j+1/2}(p')}{L_{j+1/2}(p')} \right) = (E - E_{V} - M_{B}) \left( \frac{S_{j+1/2}(p)}{L_{j+1/2}(p)} \right) .
\]

As a result, every state is (at least) twofold degenerate in the one-body limit.

The states \( j_{-1/2} Y_{JM}^{1}(\hat{p}) \) and \( j_{+1/2} Y_{JM}^{1}(\hat{p}) \) defined in Eq. (180), as well as \( j_{-1/2} Y_{JM}^{J-1}(\hat{p}) \) and \( j_{+1/2} Y_{JM}^{J+1}(\hat{p}) \) defined in Eq. (181), can be rewritten in terms of the eigenstates \( Y_{lm}^{j_{A}}(\hat{p}) \) of \( J_{A}, J_{A}, \) and \( L_{A}^{2} \) (where \( j_{A} = L + s_{A} \)),

\[
Y_{lm}^{l-1/2}(\hat{p}) = \frac{1}{\sqrt{2l + 1}} \left( \begin{array}{c}
-\sqrt{l - m + 1/2} Y_{l,m-1/2}(\hat{p}) \\
\sqrt{l + m + 1/2} Y_{l,m+1/2}(\hat{p})
\end{array} \right) \quad (l \geq 1) ,
\]

\[
Y_{lm}^{l+1/2}(\hat{p}) = \frac{1}{\sqrt{2l + 1}} \left( \begin{array}{c}
\sqrt{l + m + 1/2} Y_{l,m-1/2}(\hat{p}) \\
-\sqrt{l - m + 1/2} Y_{l,m+1/2}(\hat{p})
\end{array} \right) ,
\]

\( 57 \).
as

\[ J_{-1/2}Y_{JM}^J(\hat{p}) = \sqrt{\frac{J-M}{2J}} Y_{JM+1/2}^{J-1/2}(\hat{p}) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \sqrt{\frac{J+M}{2J}} Y_{JM-1/2}^{J+1/2}(\hat{p}) \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \]

\[ J_{-1/2}Y_{JM}^{J-1}(\hat{p}) = \sqrt{\frac{J+M}{2J}} Y_{JM+1/2}^{J-1/2}(\hat{p}) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) - \sqrt{\frac{J-M}{2J}} Y_{JM-1/2}^{J+1/2}(\hat{p}) \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \]

\[ J_{+1/2}Y_{JM}^J(\hat{p}) = \sqrt{\frac{J+M+1}{2(J+1)}} Y_{JM+1/2}^{J+1/2}(\hat{p}) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) - \sqrt{\frac{J-M+1}{2(J+1)}} Y_{JM-1/2}^{J-1/2}(\hat{p}) \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \]

\[ J_{+1/2}Y_{JM}^{J+1}(\hat{p}) = \sqrt{\frac{J-M+1}{2(J+1)}} Y_{JM+1/2}^{J+1/2}(\hat{p}) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \sqrt{\frac{J+M+1}{2(J+1)}} Y_{JM-1/2}^{J-1/2}(\hat{p}) \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \]

(187)

hence they are simultaneous eigenstates of \( J^2 \), \( J_z \), \( j_A^3 \), and \( L^2 \).

Using the formulas

\[ (\hat{p} \cdot \sigma) Y_{l,m}^{l-1/2}(\hat{p}) = -Y_{l-1,m}(\hat{p}), \]

\[ (\hat{p} \cdot \sigma) Y_{l,m}^{l+1/2}(\hat{p}) = -Y_{l+1,m}(\hat{p}), \]

(188)

proven, e.g., with the help of Eqs. (162) and (163), and consistent with the fact that the application of helicity operators conserves angular momentum \( j_A = L + s_A \) and changes spatial parity \((-1)^J\), it is easily seen that for the wave functions (178) and (182) the effective Schrödinger equation in the one-body limit, Eq. (49), is equivalent to Eq. (183), while for the wave functions (179) and (184), Eq. (49) is equivalent to Eq. (188).

The coefficient functions \( a_l(p,p') \) of the partial wave expansion in the one-body limit \( M_B \to \infty \) are defined by

\[ a_l(p,p') = \frac{2l+1}{2} \int_{-1}^1 d\cos \theta \, P_l(\cos \theta) \frac{1}{2|p-p'|} \left( \frac{1}{|p-p'|} + \frac{1}{E_p^A + |p-p'| - E_{p'}^A} \right), \]

(189)

see Eq. (49). The explicit expressions in the cases \( l = 0, 1, 2 \), which are the ones relevant for the numerical results presented here, are

\[ a_0(p,p') = \frac{1}{4pp'} \left\{ \ln \frac{p+p'}{|p-p'|} + \ln \left( \frac{E_p^A + (p+p') - E_{p'}^A}{E_p^A + |p-p'| - E_{p'}^A} \right) \right\}, \]

\[ a_1(p,p') = \frac{3}{4pp'} \left\{ \frac{(E_p^A - E_{p'}^A)(p+p' - |p-p'|)}{2pp'} - 2 \right. \\
+ \frac{p^2 + p'^2}{2pp'} \ln \frac{p+p'}{|p-p'|} + \frac{p^2 + p'^2 - (E_p^A - E_{p'}^A)^2}{2pp'} \ln \left( \frac{E_p^A + (p+p') - E_{p'}^A}{E_p^A + |p-p'| - E_{p'}^A} \right) \right\}, \]

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\[a_2(p, p') = \frac{5}{8pp'} \left\{ \frac{3[2(p^2 + p'^2) - (E_p^A - E_{p'}^A)^2]}{4p^2p'^2}(E_p^A - E_{p'}^A)(p + p' - |p - p'|) \right. \\
\left. - \frac{(E_p^A - E_{p'}^A)(p + p')^3 - |p - p'|^3}{4p^2p'^2} - \frac{3[2(p^2 + p'^2) - (E_p^A - E_{p'}^A)^2]}{2pp'} \right. \\
\left. + \left[ \frac{3(p^2 + p'^2)^2}{4p^2p'^2} - 1 \right] \ln \frac{p + p'}{|p - p'|} \right. \\
\left. + \left[ \frac{3[p^2 + p'^2 - (E_p^A - E_{p'}^A)^2]}{4p^2p'^2} - 1 \right] \ln \left( \frac{E_p^A + (p + p') - E_{p'}^A}{E_p^A + |p - p'| - E_{p'}^A} \right) \right\} . \quad (190)\]

They coincide with the limit \(M_B \to \infty\) of the corresponding expressions in Eq. (173).

The consistency of the one-body limit is hence fully established at the level of the effective Schrödinger equation after the separation of angular and spin variables.

### D Approximate diagonalization of the effective potential matrices

To order \(\alpha^4\), i.e., for the lowest-order relativistic corrections, we can approximate

\[\frac{p}{E_p^A + M_A} = \frac{p}{2M_A}, \quad \frac{p}{E_p^B + M_B} = \frac{p}{2M_B}, \quad \text{and analogously for } p' \text{ instead of } p, \text{ in the potential terms of the effective Schrödinger equations (168) and (171).} \]

Furthermore, terms containing

\[\frac{p}{E_p^A + mM_A} \quad \frac{p}{E_p^B + M_B} \quad \frac{p'}{E_{p'}^A + M_A} \quad \frac{p'}{E_{p'}^B + M_B}\]

\(\text{can be neglected. The matrix (172) in the L-coupled sector can then be approximated by the following diagonal matrix}\)

\[\frac{1}{2J + 1} \left( \begin{array}{cc} \mathcal{L}_{J-1,J-1}(p, p') & \mathcal{L}_{J-1,J+1}(p, p') \\ \mathcal{L}_{J+1,J-1}(p, p') & \mathcal{L}_{J+1,J+1}(p, p') \end{array} \right) \]

\[= \left( \begin{array}{cc} a_{J-1}(p, p') & 0 \\ 0 & a_{J+1}(p, p') \end{array} \right) - \left( \begin{array}{cc} pp' & pp' \\ 4M_A^2 & 4M_B^2 \end{array} \right) \left( \begin{array}{cc} a_J(p, p') & 0 \\ 0 & a_J(p, p') \end{array} \right) . \quad (193)\]

In particular, to order \(\alpha^4\) there is no L-coupling. We hence expect, at least for moderate values of \(\alpha\), the mixing for the solutions in the L-coupled sector (with \(J \geq 1\)) to be rather small.
The matrix elements $S_{00}^J(p, p')$, $S_{11}^J(p, p')$, and $S_{01}^J(p, p')$ in the S-coupled sector can be approximated by

$$S_{00}^J(p, p') = a_J(p, p') - \left[ \frac{pp'}{4M_A^2} + \frac{pp'}{4M_B^2} \right] \left[ J \frac{a_{J-1}(p, p')}{2J-1} + (J + 1) \frac{a_{J+1}(p, p')}{2J+3} \right],$$

$$S_{11}^J(p, p') = a_J(p, p') - \left[ \frac{pp'}{4M_A^2} + \frac{pp'}{4M_B^2} \right] \left[ (J + 1) \frac{a_{J-1}(p, p')}{2J-1} + J \frac{a_{J+1}(p, p')}{2J+3} \right],$$

$$S_{01}^J(p, p') = S_{10}^J(p, p') = \left[ \frac{pp'}{4M_A^2} - \frac{pp'}{4M_B^2} \right] \sqrt{J(J+1)} \left[ a_{J-1}(p, p') - \frac{a_{J+1}(p, p')}{2J+3} \right].$$

This approximate matrix, somewhat fortunately, can be diagonalized through a $p$- and $p'$-independent transformation. After some algebraic labor, one obtains the following eigenvectors:

$$v_{J,x}^{(+)} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} - \left( 1 - \frac{1}{\sqrt{1 + 4J(J+1)x^2}} \right)^{1/2} \\ \left( 1 + \frac{1}{\sqrt{1 + 4J(J+1)x^2}} \right)^{1/2} \end{array} \right),$$

$$v_{J,x}^{(-)} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \left( 1 + \frac{1}{\sqrt{1 + 4J(J+1)x^2}} \right)^{1/2} \\ 1 - \left( \frac{1}{\sqrt{1 + 4J(J+1)x^2}} \right)^{1/2} \end{array} \right).$$

($J \geq 1$), where we have expressed the mass dependence through the parameter

$$x = \frac{M_B^2 - M_A^2}{M_B^2 + M_A^2}.$$

The corresponding eigenvalues are

$$\frac{1}{2J+1} S_{J,x}^{(+)}(p, p') = a_J(p, p') \frac{pp'}{2J+1} - \frac{1}{4M_A^2} \left[ \left( 1 + \sqrt{1 + 4J(J+1)x^2} \right) \frac{a_{J-1}(p, p')}{2J-1} + \left( 1 - \frac{\sqrt{1 + 4J(J+1)x^2}}{2J+1} \right) \frac{a_{J+1}(p, p')}{2J+3} \right],$$

$$\frac{1}{2J+1} S_{J,x}^{(-)}(p, p') = a_J(p, p') \frac{pp'}{2J+1} - \frac{1}{4M_A^2} \left[ \left( 1 - \sqrt{1 + 4J(J+1)x^2} \right) \frac{a_{J-1}(p, p')}{2J-1} \right].$$
\[
\left. + \left(1 + \frac{\sqrt{1 + 4J(J + 1)x^2}}{2J + 1} \right) \frac{a_{J+1}(p, p')}{2J + 3} \right].
\]

These formulas are valid for \( x \geq 0 \) or \( M_B \geq M_A \). Comparing these results with the approximate diagonal matrix elements for L-coupling,

\[
\frac{1}{2J - 1} L_{JJ}^{J-1}(p, p') = \frac{a_J(p, p')}{2J + 1} - \frac{pp'}{4M_A^2} \frac{a_{J-1}(p, p')}{1 + x} \frac{2}{2J - 1},
\]

\[
\frac{1}{2J + 3} L_{JJ}^{J+1}(p, p') = \frac{a_J(p, p')}{2J + 1} - \frac{pp'}{4M_A^2} \frac{a_{J+1}(p, p')}{1 + x} \frac{2}{2J + 3},
\]

(198)

one notes the coincidence in the one-body limit \( x \to 1 \). For \( x \) close to one, Eq. (197) can be expanded in \( x \) around one. Consequently, the energy levels that are degenerate at \( x = 1 \) split for \( M_A \ll M_B \) by terms of the order of (at least) \( (M_A/M_B)^2 \), which points to the fact that there is no hyperfine splitting of the levels in the strict sense, i.e., to orders \( \alpha^4 \) and \( M_A/M_B \).

References

[1] E.E. Salpeter and H.A. Bethe, Phys. Rev. 84, 1232 (1951).
[2] M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).
[3] G.C. Wick, Phys. Rev. 96, 1124 (1954); R.E. Cutkosky, ibid. 96, 1135 (1954).
[4] for a review of early developments, see N. Nakanishi, Prog. Theor. Phys. (Suppl.) 43, 1 (1969).
[5] J.S. Goldstein, Phys. Rev. 91, 1516 (1953).
[6] H.S. Green, Phys. Rev. 97, 540 (1955).
[7] N. Setô, Prog. Theor. Phys. (Suppl.) 95, 25 (1988).
[8] A.A. Logunov and A.N. Tavkhelidze, Nuovo Cim. 29, 380 (1963); R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966).
[9] F. Gross, Phys. Rev. 186, 1448 (1969).
[10] F. Gross, Phys. Rev. C 26, 2203 (1982).
[11] V. Pascalutsa and J.A. Tjon, Phys. Lett. B 435, 245 (1998); Phys. Rev. C 60, 034005 (1999).
[12] J. Fleischer and J.A. Tjon, Nucl. Phys. B84, 375 (1975); M.J. Zuilhof and J.A. Tjon, Phys. Rev. C 22, 2369 (1980).
[13] F. Gross, Phys. Rev. D 10, 223 (1974); W.W. Buck and F. Gross, *ibid.* 20, 2361 (1979); F. Gross, J.W. Van Orden, and K. Holinde, Phys. Rev. C 45, 2094 (1992).

[14] R. Machleidt, K. Holinde, and C. Elster, Phys. Rep. 149, 1 (1987).

[15] C. Bloch and J. Horowitz, Nucl. Phys. 8, 91 (1958).

[16] U. Kaulfuss and M. Gari, Nucl. Phys. A408, 507 (1983); J. Flender and M.F. Gari, Phys. Rev. C 51, R1 (1995).

[17] S. Okubo, Prog. Theor. Phys. 12, 603 (1954).

[18] S. Glazek, A. Harindranath, S. Pinsky, J. Shigemutsu, and K. Wilson, Phys. Rev. D 47, 1599 (1993).

[19] M. Mangin-Brinet, J. Carbonell, and V.A. Karmanov, Phys. Rev. C 68, 055203 (2003).

[20] M. van Iersel and B.L.G. Bakker, [hep-ph/0407318](http://arxiv.org/abs/hep-ph/0407318)

[21] J.H.O. Sales, T. Frederico, B.V. Carlson, and P.U. Sauer, Phys. Rev. C 63, 064003 (2001).

[22] A. Weber, in *Particles and Fields — Seventh Mexican Workshop*, edited by A. Ayala, G. Contreras, and G. Herrera, AIP Conf. Proc. No. 531 (AIP, New York, 2000), p. 305, [hep-th/9911198](http://arxiv.org/abs/hep-th/9911198)

[23] A. Weber and N.E. Ligterink, Phys. Rev. D 65, 025009 (2002).

[24] A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).

[25] see, e.g., M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory* (Perseus Books, Cambridge, MA, 1995).

[26] F. Estrada Chávez, M.Sc. thesis, Universidad Michoacana de San Nicolás de Hidalgo, 2003.

[27] A. Krüger and W. Glöckle, Phys. Rev. C 60, 024004 (1999).

[28] see, e.g., I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 2000), sixth edition.

[29] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1965).

[30] see, e.g., C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (Wiley-Interscience, Paris, 1977).