General Relativity as a Genuine Connection Theory

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Abstract
The Palatini formulation is used to develop a genuine connection theory for general relativity, in which the gravitational field is represented by a Lorentz-valued spin connection. The existence of a tetrad field, given by the Fock–Ivanenko covariant derivative of the tangent-space coordinates, implies a coupling between the spin connection and the coordinate vector-field, which turns out to be the responsible for the onset of curvature. This connection-coordinate coupling can thus be considered as the very foundation of the gravitational interaction. The peculiar form of the tetrad field is shown to reduce both Bianchi identities of general relativity to a single one, which brings this theory closer to the gauge theories describing the other fundamental interactions of Nature. Some further properties of this approach are also examined.

1 Introduction

Differently from gauge theories [1] the fundamental field of general relativity is not a connection, but a metric tensor—or equivalently, a tetrad field. The difficulties in quantization have created a continuous interest in developing a connection-based formulation of the theory [2], which would bring it closer to the gauge theories describing the other fundamental interactions of Nature and to their successful quantization techniques [3]. The standard way of bringing connections to the forefront is to use Palatini’s variational method, by which the connection and the tetrad are independent variables. The field equations are, accordingly, obtained from independent variations with respect to both the tetrad and the spin connection. The first variation yields Einstein’s equation, whereas the second yields a constraint equation whose solution defines the connection in terms of the tetrad. Once written in terms of the tetrad, however, the connection loses its role as fundamental field, and one is led back to the usual metric-based formulation. In a genuine connection theory the spin connection should keep the role of fundamental field and should not be written in terms of the tetrad. It is the tetrad, as a derived field, that must be written in terms of the connection. The last point is the basic difference between the construction we are going to consider here and all existing approaches to the so called connection-based theories of gravity [4].

The reformulation of general relativity as a Lorentz-valued connection theory for gravitation requires a change in the traditional kinematic paradigm of the theory. More specifically,
instead of spacetime diffeomorphisms, the fundamental transformations behind a connection-based formulation of general relativity must be assumed to be the \textit{local} Lorentz group \cite{5}. In order to implement such a change it is necessary to devote special attention to the Minkowski tangent space \( M \), which is naturally attached to each point of spacetime \( \mathcal{R} \) and on which the Lorentz transformations will take place. In this way, general relativity can be reinterpreted as a theory emerging from the requirement of covariance under local Lorentz transformations \cite{6}, and accordingly the spin connection turns up as the basic field representing gravitation. It is important to notice that, as the Einstein-Hilbert Lagrangian stands on, this reinterpretation entails no change in \textit{dynamics}, but only in the underlying \textit{kinematics} of the theory. Conceptual changes, however, do show up which lead to fundamental differences with respect to the ordinary metric formulation. The basic purpose of this work is to study these differences.

2 Modified Palatini Formulation

2.1 The Spin Connection

We use the Greek alphabet (\( \mu, \nu, \rho, \ldots = 1, 2, 3, 4 \)) to denote indices related to spacetime, and the Latin alphabet (\( A, B, C, \ldots = 1, 2, 3, 4 \)) to denote indices related to each one of the Minkowski tangent spaces, whose metric tensor is chosen to be \( \eta_{AB} = \text{diag}(+1, -1, -1, -1) \). Using this notation, the spin connection \( A_\mu \), a field assuming values in the Lie algebra of the Lorentz group, is written as

\[
A_\mu = \frac{1}{2} A^{AB}_\mu J_{AB},
\]

where \( J_{AB} \) are the Lie algebra generators written in some appropriate representation. Since \( A^{AB}_\mu = -A^{BA}_\mu \), it automatically preserves the Minkowski metric:

\[
\partial_\mu \eta_{AB} - A^{C}_\mu \eta_{CB} - A^{C}_B \eta_{AC} = 0.
\]

The curvature of the connection \( A^{AB}_\mu \) is

\[
\Omega^{AB}_{\mu \nu} = \partial_\mu A^{AB}_\nu - \partial_\nu A^{AB}_\mu + A^{A}_E \eta^{EB}_\mu - A^{A}_B \eta^{EB}_\nu.
\]

Denoting by \( h_\mu = h^A_\mu \partial_A \) a general tetrad field, the torsion of \( A^{AB}_\mu \) is written in the form

\[
T^{A}_{\mu \nu} = \partial_\mu h^A_\nu - \partial_\nu h^A_\mu + A^{A}_E \eta^{EB}_\mu h^E_\nu - A^{A}_E \eta^{EB}_\nu h^E_\mu.
\]

It is important to remark that curvature and torsion are properties of a connection \cite{7}. Notice, however, that in the case of non-soldered bundles, as for example in Yang-Mills theories, no tetrad exists, and consequently torsion cannot be even defined. This is completely different from general relativity, whose spin connection, the so called Ricci coefficient of rotation, has vanishing torsion. Although vanishing, therefore, torsion is always present in general relativity.

Using the tetrad, a spin connection \( A^{AB}_\mu \) can be related with the corresponding spacetime connection \( \Gamma^A_{\nu \mu} \) through

\[
\Gamma^A_{\nu \mu} = h^A_\rho \partial_\mu h^A_\nu + h^A_\rho A^{AB}_B h^B_\nu.
\]

\footnote{The presence of nonmetricity would spoil the anti-symmetry in the first two indices, and consequently the connection would not be Lorentz-valued.}
The inverse relation is, consequently,
\[ A^A_{B\mu} = h^A_{\nu} \partial_{\mu} h^B_{\nu} + h^A_{\nu} \Gamma^B_{\mu \rho} h_{B\rho}. \] (5)

Equations (4) and (5) are simply different ways of expressing the property that the total— that is, acting on both indices—derivative of the tetrad vanishes identically:
\[ \partial_{\mu} h^A_{\nu} - \Gamma^A_{\nu \rho} h^B_{\rho} + A^A_{B\mu} h^B_{\nu} = 0. \] (6)

In what follows, we will denote the magnitudes related with general relativity with an over “\(\circ\)”. For example, the Ricci coefficient of rotation will be denoted by \(\circ A^C_{\ A\nu}\), whereas its curvature will be \(\circ \Omega^{A}_{B\mu\nu}\).

### 2.2 Ordinary Palatini Formulation

The basic gravitational variables in the Palatini framework is the pair \((h_{\mu}, A_{\mu})\) of independent 1-form fields. As is well known, the tetrad field provides an isomorphism between the tangent space \(\mathcal{M} = T_x \mathcal{R}\) at each \(x^\mu\) and the algebra of the translation group, which is also a vector space equipped with the metric \(\eta^{AB}\). It establishes, therefore, a relation between \(\eta^{AB}\) and the spacetime metric \(g^{\mu\nu}\):
\[ g^{\mu\nu} = \eta^{AB} h^{A\mu} h^{B\nu}. \] (7)

In the specific case of general relativity, the action of the gravitational field in the Palatini formulation can be written in the form
\[ S_P = \frac{1}{4k^2} \int_{\mathcal{R}} \epsilon_{\mu\nu\rho\sigma} \epsilon_{ABCD} h^A_{\mu} h^B_{\nu} \circ \Omega^{CD}_{\rho\sigma}, \] (8)
where \(k^2 = 8\pi G/c^4\), \(\epsilon_{ABCD}\) is the totally anti-symmetric Levi–Civita tensor on \(\mathcal{M}\) compatible with \(\eta_{AB}\), and
\[ \epsilon_{\mu\nu\rho\sigma} = h^A_{\mu} h^B_{\nu} h^C_{\rho} h^D_{\sigma} \epsilon_{ABCD}, \] (9)
with \(\epsilon_{0123} = h \equiv \det(h^A_{\mu})\). In contrast to the usual Einstein–Hilbert action, \(S_P\) depends on two independent variables. Variation of \(S_P\) with respect to the (inverse) tetrad \(h^{A\mu}\) yields Einstein’s equation
\[ \circ \Omega^A_{\mu} - \frac{1}{2} h_{\mu} \circ \Omega = 0, \] (10)
whereas variation with respect to the spin connection \(\circ A^A_{B\mu}\) yields
\[ \partial_{\mu} h^A_{\nu} - \partial_{\nu} h^A_{\mu} + [\circ A_{\mu}, h_{\nu}]^A = 0, \] (11)
which is a constraint equation determining the vanishing of torsion. It is then usually assumed that Eq. (11) can be solved for \(\circ A^A_{B\mu}\), in which case it becomes completely determined by the tetrad: \(\circ A_{\mu} = \circ A_{\mu}(h_{\nu})\). A further restriction to histories in which the connection is so determined reduces \(S_P\) to the ordinary Einstein–Hilbert action of general relativity,
\[ S_{EH} \equiv S_P(h_{\nu}, \circ A_{\mu}(h_{\nu})) = \frac{1}{2k^2} \int_{\mathcal{R}} d^4x \ h \circ R, \] (12)
where \(\circ R\) is the scalar curvature of \(\circ A_{\mu} = \circ A_{\mu}(h_{\nu})\).
2.3 Modifying the Palatini Formulation

The above procedure is not consistent with a genuine connection-based theory for gravitation, since in such a theory the connection is to be considered as a fundamental, and not a derived field like $\hat{A}_\mu(h_\nu)$. If we really want to obtain a formulation for gravity which is closer to a Yang–Mills theory, the spin connection is to be considered as a fundamental field, and accordingly the field equation (11) must be solved for the coframe $h_\nu$, and not for the connection $\hat{A}_\mu$. In other words, the coframe is to be completely determined by the connection:

$$h_\nu = h_\nu(\hat{A}_\mu).$$

Of course, as far as the theory is kept metric compatible, and torsion is assumed to vanish, the resulting theory is the same as general relativity, though written in a different set of field–coordinates. In this case, however, a restriction to histories on which the coframe is determined in terms of the spin connection $\hat{A}_\mu$ does not reduce $S_P$ to the Einstein–Hilbert action. It leads, actually, to a modified version of the Palatini action, which we indicate as

$$S'_P \equiv S_P(h_\nu(\hat{A}_\mu), \hat{A}_\mu).$$

Although presenting the same dynamics, the resulting theory will have different features in relation to general relativity. For example, the spacetime connection

$$\check{\Gamma}^\rho_{\mu\nu} = h_A^\rho \partial_\nu h^A_\mu + h_A^\rho A^A_{B\nu} h^B_\mu,$$

despite presenting zero torsion, will never, in this formulation, be written in terms of the metric or the tetrad. The crucial point of this modified Palatini formulation is then to solve the constraint equation (11) for $h^A_\mu$ in terms of the spin connection $\hat{A}^A_{B\mu}$. As we are going to see next, the requirement of local Lorentz covariance naturally yields such a solution.

3 Lorentz Transformations

Let us review some basic properties of the Lorentz transformations. Denoting the Cartesian Minkowski coordinates by $\{x^A\}$, the most general form of the generators of infinitesimal Lorentz transformations is

$$J_{AB} = L_{AB} + S_{AB},$$

where

$$L_{AB} = i(x_A \partial_B - x_B \partial_A)$$

is the orbital part of the generators, and $S_{ab}$ is the spin part of the generators, whose explicit form depends on the spin of the representation. The generators $J_{AB}$ satisfy the commutation relation

$$[J_{AB}, J_{CD}] = i(\eta_{BC} J_{AD} - \eta_{AC} J_{BD} - \eta_{BD} J_{AC} + \eta_{AD} J_{BC}),$$

which is to be identified with the Lie algebra of the Lorentz group. Each set of generators $L_{AB}$ and $S_{AB}$ satisfies the same commutation relation as $J_{AB}$, and these sets commute with each other.
A local—that is, position dependent—infinitesimal Lorentz transformation of Minkowski space coordinates is usually written with the orbital generators,

$$\delta_L x^A = -i \frac{1}{2} \epsilon^{CD} L_{CD} x^A \equiv -\epsilon^A_D x^D,$$

(18)

where $\epsilon^{CD} \equiv \epsilon^{CD}(x^\mu)$ is the parameter of the transformation. Now, due to the transitivity of Minkowski spacetime under translations, every two points related by a Lorentz transformation can be also related by a translation (though the converse is not true). In fact, by using the explicit form of $L_{CD}$, the transformation (18) can be rewritten as

$$\delta_L x^A = -i \xi^C P_C x^A,$$

(19)

which is a translation with

$$\xi^C = \epsilon^C_D x^D$$

(20)
as parameter, and $P_C = -i \partial_C$ as generator. This means essentially that an infinitesimal Lorentz transformation of the Minkowski coordinates is formally equivalent to a translation with $\xi^C$ as parameters.

On the other hand, because the coordinates $x^A$ behave collectively as a vector under Lorentz transformations, we can interpret their set \{\(x^A(x^\mu)\)\} as a vector field. In this case, the Lorentz generators are those of the (spin) vector representation [8],

$$\left(S_{CD}\right)^A_B = i \left(\delta_C^A \eta_{DB} - \delta_D^A \eta_{CB}\right),$$

(21)

which yields

$$\delta_S x^A = -i \frac{1}{2} \epsilon^{CD} \left(S_{CD}\right)^A_B x^B \equiv \epsilon^A_D x^D.$$

(22)

Therefore, we see that a Lorentz transformation of the Minkowski coordinates written with the complete generators $J_{CD}$ vanishes identically:

$$\delta_J x^A \equiv -i \frac{1}{2} \epsilon^{CD} J_{CD} x^A = 0.$$

(23)

Of course, the generators are defined up to a sign. However, provided $L_{ab}$ and $S_{ab}$ are chosen in such a way to satisfy the same commutation relation, they yield opposite Lorentz transformations, and consequently a vanishing total Lorentz transformation. This result—quite consistent by itself—comes from the fact that, concomitant with the Lorentz transformation (22) in its vector indices, a vector field $V^A(x)$, for example, necessarily undergoes the Lorentz transformation (18) in its arguments, yielding a fixed point transformation:

$$\delta V^A(x) \equiv V'^A(x) - V^A(x) = -i \frac{1}{2} \epsilon^{CD} J_{CD} V^A(x).$$

(24)

In the case of the coordinate itself, which is a Lorentz vector field, both transformations cancel each other, yielding a vanishing net result. For all other fields, $J_{AB}$ generates a Lorentz transformation at a fixed spacetime point, or equivalently, the change in the functional form of the field.
4 Lorentz Covariant Derivative and Coupling Prescription

Let us consider now a general matter field \( \Psi(x^\mu) \), which is a function of the spacetime coordinates \( \{x^\mu\} \). When considering the gravitational interaction, like in any gauge theory, the relevant transformation is that associated with the change in the functional form of the field. These transformations, as is well known, are generated by \( J_{AB} \):

\[
\delta J \Psi \equiv \Psi'(x) - \Psi(x) = -i \epsilon^{AB} J_{AB} \Psi(x).
\] (25)

The explicit form of \( L_{AB} \) is the same for all fields, whereas that of \( S_{AB} \) depends on the Lorentz representation \( \Psi \) belongs to. Notice that the orbital generators \( L_{AB} \) are able to act on the spacetime argument of \( \Psi(x^\mu) \), due to the relations

\[
\partial_A = (\partial_A x^\mu) \partial_\mu \quad \text{and} \quad \partial_\mu = (\partial_\mu x^A) \partial_A.
\]

By using the explicit form of \( L_{AB} \), the Lorentz transformation (25) can be rewritten in the form

\[
\delta J \Psi = -i \xi^C P_C \Psi - \frac{i}{2} \epsilon^{AB} S_{AB} \Psi,
\] (26)

with \( \xi^C \) given by Eq. (20). Again, we see that the orbital part of the transformation reduces to a translation, and consequently the Lorentz transformation of a general field \( \Psi \) can be rewritten as a “translation” plus a pure spin transformation. Despite presenting such a particular form, it is important to notice that, because \([P_C, S_{AB}] = 0\), it is not a Poincaré, but a genuine Lorentz transformation.

In a connection-based approach to general relativity, the fundamental field representing gravitation is the spin connection \( \overset{\circ}{\partial}_\mu \). Accordingly, the Lorentz covariant derivative of the matter field \( \Psi \), that is, the derivative which is covariant under the Lorentz transformations generated by the \( J_{AB} \)’s is [9]

\[
\overset{\circ}{\partial}_C \Psi = \partial_C \Psi + \frac{1}{2} \overset{\circ}{A}^{AB}_C \frac{\delta J \Psi}{\delta \epsilon^{AB}}.
\] (27)

where \( \overset{\circ}{A}^{AB}_C = \overset{\circ}{A}^{AB}_\mu h_C^\mu \). Substituting the transformation (25), it becomes

\[
\overset{\circ}{\partial}_C \Psi = \partial_C \Psi - \frac{i}{2} \overset{\circ}{A}^{AB}_C J_{AB} \Psi.
\] (28)

Using the identity

\[
\frac{i}{2} \overset{\circ}{A}^{AB}_\mu J_{AB} = \frac{i}{2} \overset{\circ}{A}^{AB}_\mu S_{AB} + \overset{\circ}{B}^A_\mu P_A,
\] (29)

where \( \overset{\circ}{B}^A_\mu \equiv \overset{\circ}{A}^A_{B\mu} x^B \), the covariant derivative (28) can be rewritten in the form

\[
\overset{\circ}{\partial}_C \Psi = h_C^\mu \overset{\circ}{\partial}_\mu \Psi,
\] (30)

where

\[
\overset{\circ}{\partial}_\mu = \partial_\mu - \frac{i}{2} \overset{\circ}{A}^{AB}_\mu S_{AB}
\] (31)
is the usual Fock–Ivanenko covariant derivative operator \([10]\), and \(h C^\mu\) is the inverse of the tetrad \([5]\):

\[
h C^\mu = \partial_\mu x^C + \tilde{A}^C_{\mu} x^D.
\]

(32)

Our initial contention, that the tetrad field should be a derived quantity written in terms of the spin connection \(A^C_{\mu}\), is in this way vindicated.

On account of the above results, the coupling of a general matter field to gravitation can then be accomplished by replacing all ordinary by covariant derivatives:

\[
\partial_C \rightarrow \tilde{\partial}_C \equiv h C^\mu \tilde{D}_\mu.
\]

(33)

The gravitational coupling prescription, therefore, is composed of two parts. The Fock–Ivanenko derivative accounts for the coupling of the spin of the matter field to gravitation. This coupling is not universal as it depends on the spin content of the matter field. On the other hand, the tetrad, which appears in the coupling prescription multiplying the Fock-Ivanenko derivative, accounts for the coupling of the energy and momentum of the matter field to gravitation. This part of the coupling prescription is universal in the sense that all fields in Nature will respond equally to its action. As the nontrivial part of the tetrad comes from the orbital Lorentz generators, we can say that these generators are the responsible for the universality of the gravitational interaction.

5 Connection and Tetrad Transformations

A general element of the Lorentz group is written as

\[
U_J = U_L U_S = U_S U_L \equiv \exp \left[ -i \frac{1}{2} \epsilon^{AB} J_{AB} \right],
\]

(34)

with

\[
U_S = \exp \left[ -i \frac{1}{2} \epsilon^{AB} S_{AB} \right] \quad \text{and} \quad U_L = \exp \left[ -i \frac{1}{2} \epsilon^{AB} L_{AB} \right].
\]

(35)

By construction, under a local Lorentz transformation generated by \(U_J\), the gauge covariant derivative \(\tilde{D}_C\) transforms according to

\[
\tilde{D}_C' \Psi(x) = U_J \tilde{D}_C \Psi(x).
\]

(36)

Notice that, in addition to the Lorentz rotation in the matrix (spin) indices of \(\tilde{D}_C\), which is the only transformation occurring in (internal) Yang-Mills theories, in the case of the (external) Lorentz gauge group the spacetime index of \(\tilde{D}_C\) is also necessarily transformed. Using the expressions \([\text{\ref{expr:U}}]\)

\[
\Psi(x) = U_{-S}^{-1} \Psi'(x') \quad \text{and} \quad \Psi'(x) = U_{L} \Psi'(x')
\]

in the transformation \(\text{\ref{expr:U}}\), it acquires the form

\[
U_{-L} \tilde{D}_C U_L = h C^\mu U_S \tilde{D}_\mu U_{-S}^{-1}.
\]

(37)
Now, the Fock–Ivanenko derivative $\hat{\mathcal{D}}_\mu$ is defined by

$$\hat{\mathcal{D}}_\mu \Psi = \partial_\mu \Psi + \frac{1}{2} A^{AB}_\mu \frac{\delta \Psi}{\delta \epsilon_{AB}},$$

where

$$\delta \Psi \equiv \Psi'(x') - \Psi(x) = -\frac{i}{2} \epsilon_{AB} S_{AB} \Psi$$

is the total change in $\Psi(x)$. It transforms, consequently, as

$$\hat{\mathcal{D}}'_\mu = U_S \hat{\mathcal{D}}_\mu U_S^{-1},$$

from where we can obtain the typical connection gauge transformation

$$\hat{A}'_\mu = U_S \hat{A}_\mu U_S^{-1} + i U_S \partial_\mu U_S^{-1}.$$  \hspace{1cm} (40)

Its infinitesimal version is given by

$$\delta S \hat{A}^{CD}_\mu = -\left( \partial_\mu \epsilon^{CD} + \hat{A}^C_{A\mu} \epsilon^{AD} + \hat{A}^D_{A\mu} \epsilon^{CA} \right) \equiv -\hat{\mathcal{D}}_\mu \epsilon^{CD}.$$ \hspace{1cm} (41)

We see in this way that the Lorentz gauge transformations are generated by the spin (matrix) part of the representation, that is, by the generators $S_{AB}$.

Let us then return to the transformation law (37). Substituting Eq. (39), it becomes

$$U_L^{-1} \hat{\mathcal{D}}'_C U_L = hc^\mu \hat{\mathcal{D}}'_\mu \equiv \hat{\mathcal{D}}'_C.$$ \hspace{1cm} (42)

It is clear from this expression that the orbital part of the generators are responsible for the Lorentz transformation in the spacetime index of the covariant derivative $\hat{\mathcal{D}}_C$. In fact, denoting by $\Lambda'_{AC} \equiv (U_S)^{AC}_{\prime}$ the element of the Lorentz group in the vector representation, the tetrad transformation can be written in the usual form

$$hc^\mu = \Lambda'_{AC} h_{A\mu},$$

and we easily see that

$$\hat{\mathcal{D}}'_C = \Lambda'_{AC} \hat{\mathcal{D}}'_A \equiv U_L \hat{\mathcal{D}}'_C U_L^{-1}.$$ \hspace{1cm} (43)

The transformation law of the covariant derivative can then be written as

$$\hat{\mathcal{D}}'_C = \Lambda'_{AC} U_S \hat{\mathcal{D}}_A U_S^{-1}.$$ \hspace{1cm} (44)

It is important to observe the different roles played by each one of the Lorentz generators: whereas the spin part $S_{AB}$ generates the (internal) Lorentz gauge transformation, the orbital part $L_{AB}$ is responsible for transformation of the (external) spacetime index of the covariant derivative.

Let us obtain now the infinitesimal Lorentz transformation of the tetrad field (32). The transformation generated by $S_{AB}$ corresponds to the total change in the tetrad, that is, $\delta_S h^A_{\mu} = h^A_{\mu}(x') - h^A_{\mu}(x)$. From Eq. (32) we see that

$$\delta_S h^A_{\mu} = \partial_\mu (\delta_S x^A) + (\delta_S \hat{A}^A_{\mu} ) x^D + \hat{A}^A_{D\mu} (\delta_S x^D).$$ \hspace{1cm} (45)
Using Eqs. (22) and (41), it is easy to see that

\[ \delta_S h^A \mu = \epsilon^A C h^C \mu \equiv -\frac{i}{2} \epsilon^{CD} (S_{CD})^A B h^B \mu, \]  

(46)
as it should be, since \( h^A \mu \) is a Lorentz vector field in the algebraic index. On the other hand, the transformation generated by \( L_{CD} \) corresponds to a change in the coordinate only, that is, \( \delta_L h^A \mu \equiv h^A \mu (x') - h^A \mu (x). \) Since \( \hat{A}^{CD} \mu \) responds only to the spin representation \( S_{CD} \), we see from Eq. (32) that

\[ \delta_L h^A \mu = \partial_\mu (\delta_L x^A) + \hat{A}^A D_\mu (\delta_L x^D). \]  

(47)

Using the transformation (19), we get

\[ \delta_L h^A \mu = -\hat{D}_\mu \xi^A. \]  

(48)

It is then easy to see that \( \delta_L h^A \mu \) induces on the metric tensor (7) the transformation

\[ \delta_L g_{\mu \nu} = -\hat{\nabla}_\mu \xi_\nu - \hat{\nabla}_\nu \xi_\mu, \]  

(49)

where \( \xi_\nu = \xi_A h^A \nu \), and \( \hat{\nabla}_\mu \) is the covariant derivative in the spacetime connection (14). As is well known, this equation represents the response of \( g_{\mu \nu} \) to spacetime diffeomorphisms \( x^\mu = x^\mu + \xi^\mu (x) \) [11]. In other words, the Lorentz transformations generated by \( L_{AB} \) are equivalent to a spacetime general coordinate transformation. This means essentially that the ordinary formulation of general relativity is included as a particular case of this more general approach, which allows from the very beginning the inclusion of both integer and half-integer spin fields. Furthermore, we see that the energy-momentum tensor of any matter field must follow, through Noether’s theorem [12], from the invariance of the corresponding Lagrangian under a Lorentz transformation generated by the orbital generators only [13].

Concerning this last point, it is interesting to notice the following property. Usually, the functional derivative of a matter field lagrangian \( \mathcal{L} \) in relation to the spin connection gives the spin tensor. However, due to the dependence of the tetrad on the spin connection,

\[ \mathcal{J}^\mu_{AB} = \frac{1}{h} \frac{\delta \mathcal{L}}{\delta \hat{A}^A_{AB} \mu} \]  

(50)

will represent now the total angular momentum, that is, spin plus orbital. In fact, as the tetrad depends on the spin connection, the above expression can be rewritten in the form

\[ \mathcal{J}^\mu_{AB} = \frac{1}{h} \frac{\delta \mathcal{L}}{\delta h^C \rho} \frac{\delta h^C \rho}{\delta \hat{A}^A_{AB} \mu}. \]  

(51)

However, from Eq. (32) we see that

\[ \frac{\delta h^C \rho}{\delta \hat{A}^A_{AB} \mu} = \delta^\mu_\rho (\delta^C A x_B - \delta^C B x_A). \]  

(52)
Substituting this relation in Eq. (51), we get

$$J_{AB}^\mu = x_A \Theta_B^\mu - x_B \Theta_A^\mu, \quad (53)$$

where

$$\Theta_A^\mu = \frac{1}{h} \frac{\delta \mathcal{L}}{\delta h^A_{\mu}} \quad (54)$$

is the so-called dynamical energy-momentum tensor. As far as $\mathcal{L}$ is local Lorentz invariant, $\Theta^{\mu\nu} = \Theta_A^{\mu} h^A_{\nu}$ will be the symmetric energy-momentum tensor (see Ref. [14], page 371), and $J_{AB}^\mu$ will represent the total angular momentum tensor [15].

6 The Tangent-Space Coordinates as a Vector Field

6.1 General Properties

As we have seen, the Minkowski coordinates $x^A$ can be interpreted both as a set of four scalar functions—in which case its Lorentz transformation is that generated by the orbital generators $L_{AB}$—and as a vector field $x^A(x^\mu)$, in which case its Lorentz transformation is that generated by the vector representation $S_{AB}$. When we consider it as a vector field, its Fock–Ivanenko covariant derivative

$$\overset{\circ}{D}_\mu x^C = \partial_\mu x^C + \frac{1}{2} A_{AB}^\mu \frac{\delta x^C}{\delta e_{AB}} \quad (55)$$

turns out to be, through the use of the transformation (22),

$$\overset{\circ}{D}_\mu x^C = \partial_\mu x^C + \overset{\circ}{A}_{CB\mu} x^B \equiv h^C_{\mu}. \quad (56)$$

This is to say that the Fock–Ivanenko covariant derivative of the vector field $x^C(x^\mu)$ exactly coincides with the tetrad field. We can consequently conclude, taking into account the usual concepts underlying the minimal coupling prescription, that the gravitational field $\overset{\circ}{A}_{C\mu}$ couples to the Minkowski tangent space coordinates, that is, to the vector field $x^A$, and that the corresponding coupling constant is equal to 1. On their side, the coordinates $\{x^A\}$ respond to the Lorentz gauge interaction. This is an essential point: it is this coupling the responsible for the non-triviality of the tetrad, that is, for its deviation from an exact differential form. A trivial, exact tetrad would represent a mere coordinate transformation, and not a gravitational field. By the standard expression $g_{\mu\nu} = \eta_{AB} h_A^\mu h_B^\nu$, therefore, it would give simply the Minkowski metric written in arbitrary coordinates, with vanishing Riemannian curvature. It is the coordinate–connection coupling, with the ensuing tetrad non-triviality, that leads to the non-trivial Riemannian metric on which the general relativity description of gravitation is based. We can say, therefore, that the very foundation of that description is the coupling between the spin connection and the tangent space coordinates. In addition, since the non-trivial part $\overset{\circ}{B}_{A\mu} = A_{AB\mu} x^B$ of the tetrad field comes from the orbital part of the Lorentz generators (see Eq. (23)), and as its action takes place on the arguments of every field, the orbital generators $L_{AB}$ appear as the responsible for the universality of gravitation.
6.2 Modified Palatini Action

Using the fact that the tetrad is the covariant derivative of the tangent space coordinates, as given by Eq. (56), the modified action functional of the gravitational field can be rewritten in the form

\[ S'_P = \frac{1}{4k^2} \int_R \epsilon^{\mu\nu\rho\sigma} \epsilon_{ABCD} D^\mu x^A D^\nu x^B \Omega^{CD}_{\rho\sigma}. \]  

(57)

Variation of \( S'_P \) in relation to the spin connection \( A^{AB}_{\mu} \) yields, up to a factor of \( x^A \), Einstein’s equation. Now, using the tetrad \( \text{(56)} \), as well as expression \( \text{(2)} \) for the specific case of the Ricci coefficient of rotation, the constraint equation \( \text{(11)} \) acquires the form

\[ T^A_{\mu\nu} \equiv \Omega^{A}_{\mu\nu} x^B = 0. \]  

(58)

In a sense, torsion appears as a measure of the non-orthogonality between curvature and the Minkowski coordinate field. When torsion vanishes, like in general relativity, curvature becomes orthogonal to \( x^A \).

On the other hand, as a consequence of the above described coupling between the spin connection \( A^{AB}_{\mu} \) and the coordinates \( x^A \), the gravitational action results to depend also on the vector field \( x^A \). It is then interesting to observe that a variation of \( \text{(57)} \) in relation to \( x^A \) yields

\[ \nabla_\mu \left[ \Omega^\mu_\nu - \frac{1}{2} \partial^\mu \Omega \right] = 0, \]  

(59)

which is the contracted form of the curvature Bianchi identity of general relativity.

6.3 Lagrangian of the Tangent-Space Coordinates

Now, if the coordinate \( x^a(x^\mu) \) is considered to be a vector field, it must have a Lagrangian density. Assuming that it is massless, the natural lagrangian for a vector field on a Riemannian manifold, and responding to a gauge interaction, is

\[ L_x = -\epsilon_\Lambda \frac{h}{2} \eta_{CD} \eta^{AB} D_A x^C D_B x^D, \]  

(60)

where \( \epsilon_\Lambda \) is a positive constant with dimension of energy density, introduced to give the Lagrangian the appropriate dimension. Using that

\[ D_A x^C = h^\mu_A D_\mu x^C \equiv h^\mu_A h^C_\mu = \delta^C_A, \]  

(61)

which follows from the orthogonality properties \( h^{\lambda_A}_\mu h^A_\nu = \delta^\nu_\nu \) and \( h^{A}_\mu h^B_\mu = \delta^B_A \), we find

\[ L_x = -2 h \epsilon_\Lambda. \]  

(62)

The Lagrangian of the vector field \( x^A \), therefore, corresponds to a cosmological term for the gravitational field equations, with

\[ \Lambda = \frac{16\pi G}{c^3} \epsilon_\Lambda. \]  

(63)
playing the role of cosmological constant. In other words, a cosmological term is nothing but a Lagrangian for the tangent space coordinates, seen as a vector field. This result provides a new interpretation for the origin of the dark energy of the universe [16], which would then come from considering the tangent-space coordinates as a vector field. Observe that the value of $\Lambda$ does not depend on the gravitational field, and is left as a free parameter. In fact, even in the absence of gravitation, where the tetrads become trivial, provided the tangent-space coordinates are considered as a vector field, its Lagrangian will always give rise to a cosmological term.

7 Final Remarks

When general relativity is conceived as a genuine Lorentz-valued connection theory, the local Lorentz group emerges as the kinematic symmetry behind gravitation, and the Ricci coefficient of rotation $\bar{A}^C_{D\mu}$, the spin connection of general relativity, becomes the fundamental field describing gravitation. In this theory, a tetrad given by the Fock-Ivanenko covariant derivative of the tangent space coordinates,

$$h^C_\mu \equiv \bar{\partial}_\mu x^C = \partial_\mu x^C + \bar{A}^C_{B\mu} x^B,$$

shows up naturally. Since it depends on the spin connection, it is not a fundamental but a derived field, as it should be in a true connection-based theory. A fundamental consequence of this approach refers to the Bianchi identities. To see that, it is important to observe that absence of torsion, as in Yang–Mills theories, is completely different from a present, but vanishing torsion, as in general relativity. This difference is revealed by the fact that, whereas in Yang–Mills theories there is only one Bianchi identity, in general relativity there are two: one for torsion, given by

$$\bar{\partial}^A_{\rho\mu\nu} + \bar{\partial}^A_{\nu\rho\mu} + \bar{\partial}^A_{\mu\nu\rho} = 0,$$

and one for curvature, which reads

$$\bar{D}_\rho \bar{\partial}^A_{B\mu\nu} + \bar{D}_\nu \bar{\partial}^A_{B\rho\mu} + \bar{D}_\mu \bar{\partial}^A_{B\nu\rho} = 0.$$

Multiplying the curvature Bianchi identity (66) by $x^B$, and using the identity (68), the result is easily seen to be the torsion Bianchi identity (65). According to this approach, therefore, similarly to the Yang–Mills theories, general relativity turns out to present only one independent Bianchi identity.

It is interesting to observe that, even in a general case, characterized by the simultaneous presence of curvature and torsion, the two Bianchi identities are also reduced to a single one. In this case, torsion satisfies the relation

$$\mathcal{T}^A_{\mu\nu} \equiv \bar{\Omega}^A_{B\mu\nu} x^B,$$

and is consequently always orthogonal to $x_A$:

$$x_A \mathcal{T}^A_{\mu\nu} = 0.$$
Now, in this general case, the two Bianchi identities \([17]\) can be written in the form
\[
D_\rho T^{A}{}_{\mu\nu} + D_\nu T^{A}{}_{\rho\mu} + D_\mu T^{A}{}_{\nu\rho} = \Omega^{A}{}_{\rho\mu\nu} + \Omega^{A}{}_{\nu\rho\mu} + \Omega^{A}{}_{\mu\nu\rho}
\] (69)
and
\[
D_\rho \Omega^{A}{}_{B\mu\nu} + D_\nu \Omega^{A}{}_{B\rho\mu} + D_\mu \Omega^{A}{}_{B\nu\rho} = 0.
\] (70)
As before, multiplying the curvature Bianchi identity (70) by \(x^B\), and using the relation (67), the result will be the torsion Bianchi identity (69).

Summing up, we can say that, when the fundamental field of gravitation is assumed to be the spin connection, the two Bianchi identities of gravitation are reduced to a single one. This result may be interpreted as an indication that, according to this approach, curvature and torsion might be equivalent ways of describing the gravitational field, and consequently related with the same degrees of freedom of gravity \([18]\). In this case, gravitation becomes closer to the Yang–Mills theories which, as already pointed out, have only one Bianchi identity. Whether the successful canonical quantization techniques of the Yang-Mills theories will become applicable to such a theory is an open question yet to be explored.

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**References**

[1] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[2] For a recent overview, see A. Ashtekar and J. Lewandowski, Class. Quant. Grav. **21**, R5 (2004) [gr-qc/0404018].

[3] See, for example, R. Gambini and J. Pullin, *Loops, Knots, Gauge Theories and Quantum Gravity*, (Cambridge University Press, Cambridge, 1996).

[4] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); A. Ashtekar, Phys. Rev. D**36**, 1587 (1987).

[5] M. Calçada and J. G. Pereira, Int. J. Theor. Phys. **41**, 729 (2002) [gr-qc/0201059].

[6] The first attempt to reinterpret general relativity as a gauge theory for the Lorentz group was made by R. Utiyama, Phys. Rev. **101**, 1597 (1956); the case of the Poincaré group was first considered by T. W. B. Kibble, J. Math. Phys. **2**, 212 (1961).

[7] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 1 (Wiley, New York, 1963).

[8] P. Ramond, *Field Theory: A Modern Primer*, 2nd edition (Addison-Wesley, Redwood, 1989).
[9] R. Aldrovandi and J. G. Pereira, An Introduction to Geometrical Physics (World Scientific, Singapore, 1995).

[10] V. A. Fock and D. Ivanenko, Z. Phys. 54, 798 (1929); V. A. Fock, Z. Phys. 57, 261 (1929).

[11] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Pergamon, Oxford, 1975).

[12] N. P. Konopleva and V. N. Popov, Gauge Fields (Harwood, New York, 1980).

[13] M. Calçada and J. G. Pereira, Phys. Rev. D66, 044001 (2002) gr-qc/0201076.

[14] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972), p. 371.

[15] K. Hayashi, Lett. Nuovo Cimento 5, 529 (1972).

[16] Some recent review are S. M. Carroll, Why is the Universe Accelerating?, in Measuring and Modeling the Universe, ed. by W. L. Freedman (Cambridge University Press, Cambridge, 2003) astro-ph/0310342; T. Padmanabhan, Phys. Rep. 380, 235 (2003) hep-th/0212290; V. Sahni, Int. J. Mod. Phys. D9, 373 (2000) astro-ph/9904398.

[17] R. Weitzenböck, Invariantentheorie (Noordhoff, Gronningen, 1923), p. 356.

[18] H. I. Arcos and J. G. Pereira, Class. Quant. Grav 21, 5193 (2004) gr-qc/0408096.