CLASSIFICATION OF EXTREMAL ELLIPTIC $K3$ SURFACES AND FUNDAMENTAL GROUPS OF OPEN $K3$ SURFACES

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Abstract. We present a complete list of extremal elliptic $K3$ surfaces (Theorem 1.1). As an application, we give a sufficient condition for the topological fundamental group of complement to an $ADE$-configuration of smooth rational curves on a $K3$ surface to be trivial (Proposition 4.1 and Theorem 4.3).

1. Introduction

A complex elliptic $K3$ surface $f : X \rightarrow \mathbb{P}^1$ with a section $O$ is said to be extremal if the Picard number $\rho(X)$ of $X$ is 20 and the Mordell-Weil group $MW_f$ of $f$ is finite. The purpose of this paper is to present the complete list of all extremal elliptic $K3$ surfaces. As an application, we show that, if an $ADE$-configuration of smooth rational curves on a $K3$ surface satisfies a certain condition, then the topological fundamental group of the complement is trivial. (See Theorem 4.3 for the precise statement.)

Let $f : X \rightarrow \mathbb{P}^1$ be an elliptic $K3$ surface with a section $O$. We denote by $R_f$ the set of all points $v \in \mathbb{P}^1$ such that $f^{-1}(v)$ is reducible. For a point $v \in R_f$, let $f^{-1}(v)\#$ be the union of irreducible components of $f^{-1}(v)$ that are disjoint from the zero section $O$. It is known that the cohomology classes of irreducible components of $f^{-1}(v)\#$ form a negative definite root lattice $S_{f,v}$ of type $A_l$, $D_m$ or $E_n$ in $H^2(X; \mathbb{Z})$. Let $\tau(S_{f,v})$ be the type of this lattice. We define $\Sigma_f$ to be the formal sum of these types;

$$\Sigma_f := \sum_{v \in R_f} \tau(S_{f,v}).$$

The Néron-Severi lattice $NS_X$ of $X$ is defined to be $H^{1,1}(X) \cap H^2(X; \mathbb{Z})$, and the transcendental lattice $T_X$ of $X$ is defined to be the orthogonal complement of $NS_X$ in $H^2(X; \mathbb{Z})$. We call the triple $(\Sigma_f, MW_f, T_X)$ the data of the elliptic $K3$ surface $f : X \rightarrow \mathbb{P}^1$. When $f : X \rightarrow \mathbb{P}^1$ is extremal, the transcendental lattice $T_X$ is a positive definite even lattice of rank 2.

Theorem 1.1. There exists an extremal elliptic $K3$ surface $f : X \rightarrow \mathbb{P}^1$ with data $(\Sigma_f, MW_f, T_X)$ if and only if $(\Sigma_f, MW_f, T_X)$ appears in Table 2 given at the end of this paper.

In Table 2, the transcendental lattice $T_X$ is expressed by the coefficients of its Gram matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$ 

See Subsection 2.1 on how to recover the $K3$ surface $X$ from $T_X$.

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The classification of semi-stable extremal elliptic K3 surfaces has been done by Miranda and Persson \[7\] and complemented by Artal-Bartolo, Tokunaga and Zhang \[1\]. We can check that the semi-stable part of our list (No. 1 - No. 112) coincides with theirs. Nishiyama \[12\] classified all elliptic fibrations (not necessarily extremal) on certain K3 surfaces. On the other hand, Ye \[19\] has independently classified all extremal elliptic K3 surfaces with no semi-stable singular fibers by different methods from ours.

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2. Preliminaries

2.1. Transcendental lattice of singular K3 surfaces. Let \(Q\) be the set of symmetric matrices

\[
Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}
\]

of integer coefficients such that \(a\) and \(c\) are even and that the corresponding quadratic forms are positive definite. The group \(GL_2(\mathbb{Z})\) acts on \(Q\) from right by

\[
Q \mapsto g \cdot Q \cdot g^t,
\]

where \(g \in GL_2(\mathbb{Z})\). Let \(Q_1\) and \(Q_2\) be two matrices in \(Q\), and let \(L_1\) and \(L_2\) be the positive definite even lattices of rank 2 whose Gram matrices are \(Q_1\) and \(Q_2\), respectively. Then \(L_1\) and \(L_2\) are isomorphic as lattices if and only if \(Q_1\) and \(Q_2\) are in the same orbit under the action of \(GL_2(\mathbb{Z})\). On the other hand, each orbit in \(Q\) under the action of \(SL_2(\mathbb{Z})\) contains a unique matrix with coefficients satisfying

\[
-a < 2b \leq a \leq c, \quad \text{with} \quad b \geq 0 \quad \text{if} \quad a = c.
\]

(See, for example, Conway and Sloane \[3, p. 358\].) Hence each orbit in \(Q\) under the action of \(GL_2(\mathbb{Z})\) contains a unique matrix with coefficients satisfying

\[
0 \leq 2b \leq a \leq c.
\]

(2.1)

In Table 2, the transcendental lattice is represented by the Gram matrix satisfying the condition (2.1).

Let \(X\) be a K3 surface with \(\rho(X) = 20\); that is, \(X\) is a singular K3 surface in the terminology of Shioda and Inose \[16\]. The transcendental lattice \(T_X\) can be naturally oriented by means of a holomorphic two form on \(X\) (cf. \[16, p. 128\]). Let \(S\) denote the set of isomorphism classes of singular K3 surfaces. Using the natural orientation on the transcendental lattice, we can lift the map \(S \to Q/GL_2(\mathbb{Z})\) given by \(X \to T_X\) to the map \(S \to Q/SL_2(\mathbb{Z})\).

Proposition 2.1 (Shioda and Inose \[16\]). This map \(S \to Q/SL_2(\mathbb{Z})\) is bijective.

Moreover, Shioda and Inose \[16\] gave us a method to construct explicitly the singular K3 surface corresponding to a given element of \(Q/SL_2(\mathbb{Z})\) by means of Kummer surfaces. The injectivity of the map \(S \to Q/SL_2(\mathbb{Z})\) had been proved by Plaetskii-Shapiro and Shafarevich \[14\].

Suppose that an orbit \([Q]\) in \(Q/GL_2(\mathbb{Z})\) is represented by a matrix \(Q\) satisfying (2.1). Let \(\rho: Q/SL_2(\mathbb{Z}) \to Q/GL_2(\mathbb{Z})\) be the natural projection. Then we
have
\[ |\rho^{-1}([Q])| = \begin{cases} 
2 & \text{if } 0 < 2b < a < c \\
1 & \text{otherwise.}
\end{cases} \]

Therefore, if a data in Table 2 satisfies \( a = c \) or \( b = 0 \) or \( 2b = a \) (resp. \( 0 < 2b < a < c \)), then the number of the isomorphism classes of \( K3 \) surfaces that possess a structure of the extremal elliptic \( K3 \) surfaces with the given data is one (resp. two).

2.2. Roots of a negative definite even lattice. Let \( M \) be a negative definite even lattice. A vector of \( M \) is said to be a root of \( M \) if its norm is \(-2\). We denote by \( \text{root}(M) \) the number of roots of \( M \), and by \( \text{M}_{\text{root}} \) the sublattice of \( M \) generated by the roots of \( M \). Suppose that a Gram matrix \((a_{ij})\) of \( M \) is given. Then \( \text{root}(M) \) can be calculated by the following method. Let
\[ g_r(x) = - \sum_{i,j=1}^r a_{ij} x_i x_j \]
be the positive definite quadratic form associated with the opposite lattice \( M^{-} \) of \( M \), where \( r \) is the rank of \( M \). We consider the bounded closed subset \( E(g_r, 2) := \{ x \in \mathbb{R}^r ; g_r(x) \leq 2 \} \)
of \( \mathbb{R}^r \). Then we have
\[ \text{root}(M) + 1 = \left| E(g_r, 2) \cap \mathbb{Z}^r \right|, \]
where +1 comes from the origin. For a positive integer \( k \) less than \( r \), we write by \( p_k : \mathbb{R}^r \to \mathbb{R}^k \) the projection \( (x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_k) \). Then there exist a positive definite quadratic form \( g_k \) of variables \( (x_1, \ldots, x_k) \) and a positive real number \( \sigma_k \) such that
\[ p_k(E(g_r, 2)) = E(g_k, \sigma_k) := \{ y \in \mathbb{R}^k ; g_k(y) \leq \sigma_k \}. \]
The projection \( (x_1, \ldots, x_{k+1}) \mapsto (x_1, \ldots, x_k) \) maps \( E(g_{k+1}, \sigma_{k+1}) \) to \( E(g_k, \sigma_k) \). Hence, if we have the list of the points of \( E(g_k, \sigma_k) \cap \mathbb{Z}^k \), then it is easy to make the list of the points of \( E(g_{k+1}, \sigma_{k+1}) \cap \mathbb{Z}^{k+1} \). Thus, starting from \( E(g_1, \sigma_1) \cap \mathbb{Z} \), we can make the list of the points of \( E(g_r, 2) \cap \mathbb{Z}^r \) by induction on \( k \).

2.3. Root lattices of type ADE. A root type is, by definition, a finite formal sum \( \Sigma \) of \( A_l \), \( D_m \) and \( E_n \) with non-negative integer coefficients;
\[ \Sigma = \sum_{l \geq 1} a_l A_l + \sum_{m \geq 4} d_m D_m + \sum_{n=6}^8 e_n E_n. \]
We denote by \( L(\Sigma) \) the negative definite root lattice corresponding to \( \Sigma \). The rank of \( L(\Sigma) \) is given by
\[ \text{rank}(L(\Sigma)) = \sum_{l \geq 1} a_l l + \sum_{m \geq 4} d_m m + \sum_{n=6}^8 e_n n, \]
and the number of roots of \( L(\Sigma) \) is given by
\[ (2.2) \text{root}(L(\Sigma)) = \sum_{l \geq 1} a_l (l^2 + l) + \sum_{m \geq 4} d_m (2m^2 - 2m) + 72 e_6 + 126 e_7 + 240 e_8. \]
(See, for example, Bourbaki [5] 3. Because of $L(\Sigma)_{\text{root}} = L(\Sigma)$, we have
\begin{equation}
L(\Sigma_1) \cong L(\Sigma_2) \iff \Sigma_1 = \Sigma_2. 
\end{equation}

We also define $eu(\Sigma)$ by
\[ eu(\Sigma) := \sum_{l \geq 1} a_l(l + 1) + \sum_{m \geq 4} d_m(m + 2) + \sum_{n=6}^{8} e_n(n + 2). \]

**Lemma 2.2.** Let $f : X \to \mathbb{P}^1$ be an elliptic K3 surface. Then $eu(\Sigma_f)$ is at most 24. Moreover, if $eu(\Sigma_f) < 24$, then there exists at least one singular fiber of type $I_1, II, III$ or $IV$.

**Proof.** Let $e(Y)$ denote the topological euler number of a CW-complex $Y$. Then $e(X) = 24$ is equal with the sum of topological euler numbers of singular fibers of $f$. Every singular fiber has a positive topological euler number. We have defined $eu(\Sigma)$ in such a way that, if $v \in R_f$, then $eu(\tau(S_{f, v})) \leq e(f^{-1}(v))$ holds, and if $eu(\tau(S_{f, v})) < e(f^{-1}(v))$, then the type of the fiber $f^{-1}(v)$ is either III or IV. Hence $eu(\Sigma_f)$ does not exceed the sum of the topological euler numbers of reducible singular fibers, and if $eu(\Sigma_f) < 24$, then there is an irreducible singular fiber or a singular fiber of type III or IV.

**2.4. Discriminant form and overlattices.** Let $L$ be an even lattice, $L'$ the dual of $L$, $D_L$ the discriminant group $L'/L$ of $L$, and $q_L$ the discriminant form on $D_L$. (See Nikulin [11] n. 4 for the definitions.) An overlattice of $L$ is, by definition, an integral sublattice of the $Q$-lattice $L'$ containing $L$.

**Lemma 2.3** (Nikulin [11] Proposition 1.4.2). (1) Let $A$ be an isotopic subgroup of $(D_L, q_L)$. Then the pre-image $M := \phi_L^{-1}(A)$ of $A$ by the natural projection $\phi_L : L' \to D_L$ is an overlattice of $L$, and the discriminant form $(D_M, q_M)$ of $M$ is isomorphic to $(A/Q, q_L|_{A/Q})$, where $A/Q$ is the orthogonal complement of $A$ in $D_L$, and $q_L|_{A/Q}$ is the restriction of $q_L$ to $A/Q$. (2) The correspondence $A \mapsto M$ gives a bijection from the set of isotopic subgroups of $(D_L, q_L)$ to the set of even overlattices of $L$.

**Lemma 2.4** (Nikulin [11] Corollary 1.6.2). Let $S$ and $K$ be two even lattices. Then the following two conditions are equivalent. (i) There is an isomorphism $\gamma : D_S \cong D_K$ of abelian groups such that $\gamma^* q_K = -q_S$. (ii) There is an even unimodular overlattice of $S \oplus K$ into which $S$ and $K$ are primitively embedded.

**2.5. Néron-Severi groups of elliptic K3 surfaces.** Let $f : X \to \mathbb{P}^1$ be an elliptic K3 surface with the zero section $O$. In the Néron-Severi lattice $NS_X$ of $X$, the cohomology classes of the zero section $O$ and a general fiber of $f$ generate a sublattice $U_f$ of rank 2, which is isomorphic to the hyperbolic lattice
\[ H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Let $W_f$ be the orthogonal complement of $U_f$ in $NS_X$. Because $U_f$ is unimodular, we have $NS_X = U_f \oplus W_f$. Because $U_f$ is of signature $(1, 1)$ and $NS_X$ is of signature $(1, \rho(X) - 1)$, $W_f$ is negative definite of rank $\rho(X) - 2$. Note that $W_f$ contains the sublattice
\[ S_f := \bigoplus_{v \in R_f} S_{f, v} \]
generated by the cohomology classes of irreducible components of reducible fibers of \( f \) that are disjoint from the zero section. By definition, \( S_f \) is isomorphic to \( L(\Sigma_f) \).

**Lemma 2.5** (Nishiyama [12] Lemma 6.1). The sublattice \( S_f \) of \( W_f \) coincides with \((W_f)_{\text{root}}\), and the Mordell-Weil group \( MW_f \) of \( f \) is isomorphic to \( W_f/S_f \). In particular, \( \text{root}(L(\Sigma_f)) \) is equal with \( \text{root}(W_f) \).

Because \( W_f \oplus U_f \oplus T_X \) has an even unimodular overlattice \( H^2(X;\mathbb{Z}) \) into which \( NS_X = W_f \oplus U_f \) and \( T_X \) are primitively embedded, and because the discriminant form of \( NS_X \) is equal with the discriminant form of \( W_f \) by \( DU_f = (0) \), Lemma 2.4 implies the following:

**Corollary 2.6.** There is an isomorphism \( \gamma : D_{W_f} \cong D_{T_X} \) of abelian groups such that \( \gamma^* q_{T_X} \) coincides with \( -q_{W_f} \).□

### 2.6. Existence of elliptic K3 surfaces

Let \( \Lambda \) be the K3 lattice \( L(2E_8) \oplus H^{\oplus 3} \).

**Lemma 2.7** (Kondō [5] Lemma 2.1). Let \( T \) be a positive definite primitive sublattice of \( \Lambda \) with \( \text{rank}(T) = 2 \), and \( T^\perp \) the orthogonal complement of \( T \) in \( \Lambda \). Suppose that \( T^\perp \) contains a sublattice \( H_T \) isomorphic to the hyperbolic lattice. Let \( M_T \) be the orthogonal complement of \( H_T \) in \( T^\perp \). Then there exist an elliptic K3 surface \( f : X \to \mathbb{P}^1 \) such that \( T_X \cong T \) and \( W_f \cong M_T \).

**Proof.** By the surjectivity of the period map of the moduli of K3 surfaces (cf. Todorov [17]), there exist a K3 surface \( X \) and an isomorphism \( \alpha : H^2(X;\mathbb{Z}) \cong \Lambda \) of lattices such that \( \alpha^{-1}(T) = T_X \). By Kondō [5] Lemma 2.1, the K3 surface \( X \) has an elliptic fibration \( f : X \to \mathbb{P}^1 \) with a section such that \( \mathbb{Z}[F]^\perp/\mathbb{Z}[F] \cong M_T \), where \( [F] \in U_f \) is the cohomology class of a fiber of \( f \), and \( \mathbb{Z}[F]^\perp \) is the orthogonal complement of \( [F] \) in the Néron-Severi lattice \( NS_X \). Because \( NS_X \) is equal with \( U_f \oplus W_f \), and because \( \mathbb{Z}[F]^\perp \cap U_f \) coincides with \( \mathbb{Z}[F] \), we see that \( \mathbb{Z}[F]^\perp/\mathbb{Z}[F] \) is isomorphic to \( W_f \). □

### 2.7. Datum of extremal elliptic K3 surfaces

**Proposition 2.8.** A triple \((\Sigma, MW, T)\) consisting of a root type \( \Sigma \), a finite abelian group \( MW \) and a positive definite even lattice \( T \) of rank 2 is a data of an extremal elliptic K3 surface if and only if the following hold:

- \((D1)\) \( \text{length}(MW) \leq 2 \), \( \text{rank}(L(\Sigma)) = 18 \) and \( cu(\Sigma) \leq 24 \).
- \((D2)\) There exists an overlattice \( M \) of \( L(\Sigma) \) satisfying the following:
  - \((D2-a)\) \( M/L(\Sigma) \cong MW \),
  - \((D2-b)\) there exists an isomorphism \( \gamma : D_M \cong D_T \) of abelian groups such that \( \gamma^* q_T = -q_M \), and
  - \((D2-c)\) \( \text{root}(L(\Sigma)) = \text{root}(M) \).

**Proof.** Suppose that there exists an extremal elliptic K3 surface \( f : X \to \mathbb{P}^1 \) with data equal with \((\Sigma, MW, T)\). It is obvious that \( \Sigma \) and \( MW \) satisfy the condition \((D1)\). Via the isomorphism \( S_f \cong L(\Sigma) \), the overlattice \( W_f \) of \( S_f \) corresponds to an overlattice \( M \) of \( L(\Sigma) \), which satisfies the conditions \((D2-a)-(D2-c)\) by Lemma 2.3 and Corollary 2.6. Conversely, suppose that \((\Sigma, MW, T)\) satisfies the conditions \((D1)\) and \((D2)\). By Lemma 2.4, the condition \((D2-b)\) and \( D_H = 0 \) imply that there exists an even unimodular overlattice of \( M \oplus H \oplus T \) into which \( M \oplus H \) and \( T \) are primitively embedded. By the theorem of Milnor (see, for example, Serre [13]) on the classification of even unimodular lattices, any even unimodular lattice of
signature (3, 19) is isomorphic to the $K3$ lattice $\Lambda$. Then Lemma 2.7 implies that there exists an elliptic $K3$ surface $f : X \to \mathbb{P}^1$ satisfying $W_f \cong M$ and $T_X \cong T$. The condition $(D2 - c)$ implies $M_{\text{root}} = L(\Sigma)$. Combining this with Lemma 2.5, we see that $S_f \cong L(\Sigma)$. Then (2.2) implies that $\Sigma_f = \Sigma$. Using Lemma 2.5 and the condition $(D2 - a)$, we see that $MW_f \cong MW$. Thus the data of $f : X \to \mathbb{P}^1$ coincides with $(\Sigma, MW, T)$.

Remark 2.9. In the light of Lemma 2.3, the condition $(D2)$ is equivalent to the following:

$(D3)$ There exists an isotopic subgroup $A$ of $(D_L(\Sigma), q_L(\Sigma))$ satisfying the following:

$(D3 - a)$ $A$ is isomorphic to $MW$,

$(D3 - b)$ there exists an isomorphism $\gamma : A^\perp / A \cong D_T$ of abelian groups such that $\gamma^* q_T = -q_L(\Sigma)|A^\perp / A$, and

$(D3 - c)$ $\text{root}(\phi_L(\Sigma)(A))$ is equal with $\text{root}(L(\Sigma))$, where $\phi_L(\Sigma) : L(\Sigma) \cong D_L(\Sigma)$ is the natural projection.

Remark 2.10. We did not use the conditions $\text{length}(MW) \leq 2$ and $\text{eu}(\Sigma) \leq 24$ in the proof of the “if” part of Proposition 2.8. It follows that, if $(\Sigma, MW, T)$ satisfies $\text{rank}(L(\Sigma)) = 18$ and the condition $(D2)$, then $\text{length}(MW) \leq 2$ and $\text{eu}(\Sigma) \leq 24$ follow automatically. This fact can be used when we check the computer program described in the next section.

3. Making the list

First we list up all root types $\Sigma$ satisfying $\text{rank}(L(\Sigma)) = 18$ and $\text{eu}(\Sigma) \leq 24$. This list $\mathcal{L}$ consists of 712 elements.

Next we run a program that takes an element $\Sigma$ of the list $\mathcal{L}$ as an input and proceeds as follows.

Step 1. The program calculates the intersection matrix of $L(\Sigma)$. Using this matrix, it calculates the discriminant form of $L(\Sigma)$, and decomposes it into $p$-parts:

$$(D_L(\Sigma), q_L(\Sigma)) = \bigoplus_p (D_L(\Sigma), q_L(\Sigma))_p,$$

where $p$ runs through the set $\{p_1, \ldots, p_k\}$ of prime divisors of the discriminant $|D_L(\Sigma)|$ of $L(\Sigma)$. We write the $p_i$-part of $(D_L(\Sigma), q_L(\Sigma))$ by $(D_L(\Sigma), i, q_L(\Sigma), i)$.

Step 2. For each $p_i$, it calculates the set $I(p_i)$ of all pairs $(A, A^\perp)$ of an isotopic subgroup $A$ of $(D_L(\Sigma), i, q_L(\Sigma), i)$ and its orthogonal complement $A^\perp$ such that $\text{length}(A) \leq 2$.

Step 3. For each element $A := ((A_1, A^\perp_1), \ldots, (A_k, A^\perp_k)) \in I(p_1) \times \cdots \times I(p_k)$, it calculates the $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form

$q_A := q_L(\Sigma)|A^\perp / A_1 \times \cdots \times q_L(\Sigma)|A^\perp / A_k$

on the finite abelian group $D_A := A_1^\perp / A_1 \times \cdots \times A_k^\perp / A_k$.

Let $d(A)$ be the order of $D_A$. 
Step 4. It generates the list $T(d(A))$ of positive definite even lattices of rank 2 with discriminant equal with $d(A)$. For each $T \in T(d(A))$, it calculates the discriminant form of $T$ and decomposes it into $p$-parts. If $D_T$ is isomorphic to $D_A$ and $q_T$ is isomorphic to $-q_A$, then it proceeds to the next step. Note that the automorphism group of a finite abelian $p$-group of length $\leq 2$ is easily calculated, and hence it is an easy task to check whether two given quadratic forms on the finite abelian $p$-group of length $\leq 2$ are isomorphic or not.

Step 5. It calculates the Gram matrix of the sublattice $\tilde{L}(A)$ of $L(\Sigma)^{\vee}$ generated by $L(\Sigma) \subset L(\Sigma)^{\vee}$ and the pull-backs of generators of the subgroups $A_i \subset D_L(\Sigma)$ by the projection $L(\Sigma)^{\vee} \to D_L(\Sigma) \to D_L(\Sigma)$. Then it calculates root($\tilde{L}(A)$) by the method described in the subsection 2.2. If root($\tilde{L}(A)$) is equal with root($L(\Sigma)$), then it puts out the pair of the finite abelian group $MW := A_1 \times \cdots \times A_k$ and the lattice $T$.

Then $(\Sigma, MW, T)$ satisfies the conditions $(D1)$ and $(D3)$, and all triples $(\Sigma, MW, T)$ satisfying $(D1)$ and $(D3)$ are obtained by this program.

4. Fundamental groups of open K3 surfaces

A simple normal crossing divisor $\Delta$ on a K3 surface $X$ is said to be an ADE-configuration of smooth rational curves if each irreducible component of $\Delta$ is a smooth rational curve and the intersection matrix of the irreducible components of $\Delta$ is a direct sum of the Cartan matrices of type $A_l$, $D_m$ or $E_n$ multiplied by $-1$. It is known that $\Delta$ is an ADE-configuration of smooth rational curves if and only if each connected component of $\Delta$ can be contracted to a rational double point. We consider the following quite plausible hypothesis. Let $\Delta$ be an ADE-configuration of smooth rational curves on a K3 surface $X$.

**Hypothesis.** If $\pi_1^{alg}(X \setminus \Delta)$ is trivial, then so is $\pi_1(X \setminus \Delta)$.

Here $\pi_1^{alg}(X \setminus \Delta)$ is the algebraic fundamental group of $X \setminus \Delta$, which is the pro-finite completion of the topological fundamental group $\pi_1(X \setminus \Delta)$.

**Proposition 4.1.** Suppose that Hypothesis is true for any ADE-configuration of smooth rational curves on an arbitrary K3 surface. Let $\Delta$ be an ADE-configuration of smooth rational curves on a K3 surface $X$. Then $\pi_1(X \setminus \Delta)$ satisfies one of the following:

(i) $\pi_1(X \setminus \Delta)$ is trivial.
(ii) There exist a complex torus $T$ of dimension 2 and a finite automorphism group $G$ of $T$ such that $T/G$ is birational to $X$ and that $\pi_1(X \setminus \Delta)$ fits in the exact sequence

$$1 \to \pi_1(T) \to \pi_1(X \setminus \Delta) \to G \to 1.$$ 

(iii) $\pi_1(X \setminus \Delta)$ is isomorphic to a symplectic automorphism group of a K3 surface.

**Remark 4.2.** Fujiki [4] classified the automorphism groups of complex tori of dimension 2. In particular, the $G$ in (ii) is either one of $\mathbb{Z}/n$ ($n = 2, 3, 4, 6$), $Q_8$ (Quaternion of order 8), $D_{12}$ (Dihedral of order 12) and $T_{24}$ (Tetrahedral of order 24).
24), whence the \( \pi_1(X \setminus \Delta) \) in (ii) is a soluble group. Mukai [13] presented the complete list of symplectic automorphism groups of K3 surfaces. (See also Kondō [12] and Xiao [18].) Under Hypothesis, therefore, we know what groups can appear as \( \pi_1(X \setminus \Delta) \).

**Proof of Proposition 4.1.** Suppose that \( \pi_1(X \setminus \Delta) \) is non-trivial. By Hypothesis, \( \pi^{alg}_1(X \setminus \Delta) \) is also non-trivial. For a surjective homomorphism \( \phi : \pi_1(X \setminus \Delta) \to G \) from \( \pi_1(X \setminus \Delta) \) to a finite group \( G \), we denote by

\[
\psi_\phi : \tilde{Y}_\phi \to X
\]

the finite Galois cover of \( X \) corresponding to \( \phi \), which is étale over \( X \setminus \Delta \) and whose Galois group is canonically isomorphic to \( G \). Let \( \rho : \tilde{Y}_\phi \to \tilde{Y}_\phi' \) be the resolution of singularities, and \( \gamma : \tilde{Y}_\phi' \to Y_\phi \) the contraction of \((-1)\)-curves. We denote by \( \Delta_\phi \) the union of one-dimensional irreducible components of \( \gamma(\rho^{-1}(\psi^{-1}_\phi(\Delta))) \). Then it is easy to see that \( Y_\phi \) is either a K3 surface or a complex torus of dimension 2, and that the Galois group \( G \) of \( \psi_\phi \) acts on \( Y_\phi \) symplectically. Moreover, \( \Delta_\phi \) is an empty set or an ADE-configuration of smooth rational curves. We have an exact sequence

\[
1 \to \pi_1(Y_\phi \setminus \Delta_\phi) \to \pi_1(X \setminus \Delta) \to G \to 1,
\]

because \( \pi_1(\tilde{Y}_\phi \setminus \psi^{-1}_\phi(\Delta)) \) is isomorphic to \( \pi_1(Y_\phi \setminus \Delta_\phi) \). Suppose that there exists \( \phi : \pi_1(X \setminus \Delta) \to G \) such that \( Y_\phi \) is a complex torus of dimension 2. Then \( \Delta_\phi \) is empty, and hence (ii) occurs. Suppose that no complex tori of dimension 2 appear as a finite Galois cover of \( X \) branched in \( \Delta \). Then any finite quotient group of \( \pi_1(X \setminus \Delta) \) must appear in Mukai’s list of symplectic automorphism groups of K3 surfaces. Because this list consists of finite number of isomorphism classes of finite groups, there exists a maximal finite quotient \( \phi_{max} : \pi_1(X \setminus \Delta) \to G_{max} \) of \( \pi_1(X \setminus \Delta) \). Then \( \pi_1(Y_{\phi_{max}} \setminus \Delta_{\phi_{max}}) \) has no non-trivial finite quotient group, and hence it is trivial by Hypothesis. Thus (iii) occurs.

For an ADE-configuration \( \Delta \) of smooth rational curves on a K3 surface \( X \), we denote by \( \mathbb{Z}[\Delta] \) the sublattice of \( H^2(X; \mathbb{Z}) \) generated by the cohomology classes of the irreducible components of \( \Delta \), which is isomorphic to a negative definite root lattice of type ADE. We denote by \( \Sigma_\Delta \) the root type such that \( \mathbb{Z}[\Delta] \) is isomorphic to \( L(\Sigma_\Delta) \). Using the list of extremal elliptic K3 surfaces, we prove the following theorem. We first consider the following conditions on a root type \( \Sigma \) (see (2.4) for the definition of \( D_{L(\Sigma)} \)).

\( (N1) \) \( \text{rank}(L(\Sigma)) \leq 18 \), and
\( (N2) \) \( \text{length}(D_{L(\Sigma)}) \leq 20 - \text{rank}(L(\Sigma)) \).

**Theorem 4.3.** Let \( X \) be a K3 surface and \( \Delta \) an ADE-configuration of smooth rational curves on \( X \). Suppose that the root type \( \Sigma_\Delta \) satisfies conditions (N1) and (N2). If \( \mathbb{Z}[\Delta] \) is primitive in \( H^2(X; \mathbb{Z}) \) then \( \pi_1(X \setminus \Delta) \) is trivial.

In virtue of Lemma 4.6 below, we can easily derive the following:

**Corollary 4.4.** Let \( X \) be a K3 surface and \( \Delta \) an ADE-configuration of smooth rational curves on \( X \). Suppose that \( \Sigma_\Delta \) satisfies the conditions (N1) and (N2). Then Hypothesis is true. \( \square \)
Remark 4.5. The conditions (N1) and (N2) come from Nikulin [11, Theorem 1.14.1] (see also Morrison [8, Theorem 2.8]), which gives a sufficient condition for the uniqueness of the primitive embedding of $L(\Sigma)$ into the K3 lattice $\Lambda$.

First we prepare some lemmas. Let $\mathbb{Z}[\Delta]$ be the primitive closure of $\mathbb{Z}[\Delta]$ in $H^2(X;\mathbb{Z})$.

Lemma 4.6 (Xiao [13] Lemma 2). The dual of the abelianisation of $\pi_1(X \setminus \Delta)$ is canonically isomorphic to $\mathbb{Z}[\Delta]/\mathbb{Z}[\Delta]$. In particular, if $\pi_1(\Delta)$ is primitive in $H^2(X;\mathbb{Z})$. \hfill $\Box$

Let $\Gamma_1$ and $\Gamma_2$ be graphs with the set of vertices denoted by $\text{Vert}(\Gamma_1)$ and $\text{Vert}(\Gamma_2)$, respectively. An embedding of $\Gamma_1$ into $\Gamma_2$ is, by definition, an injection $f : \text{Vert}(\Gamma_1) \to \text{Vert}(\Gamma_2)$ such that, for any $u, v \in \text{Vert}(\Gamma_1)$, $f(u)$ and $f(v)$ are connected by an edge of $\Gamma_2$ if and only if $u$ and $v$ are connected by an edge of $\Gamma_1$.

Let $\Gamma(\Sigma)$ denote the Dynkin graph of $\Sigma$.

Lemma 4.7. Suppose that $\Sigma$ satisfies the conditions (N1) and (N2). Then there exists $\Sigma'$ satisfying $\text{rank}(L(\Sigma')) = 18$ and the condition (N2) such that $\Gamma(\Sigma)$ can be embedded in $\Gamma(\Sigma')$.

Proof. This is checked by listing up all $\Sigma$ satisfying the conditions (N1) and (N2) using computer. \hfill $\Box$

Lemma 4.8. Let $f : X \to \mathbb{P}^1$ be an elliptic surface with the zero section $O$. Suppose that a fiber $f^{-1}(v)$ over $v \in \mathbb{P}^1$ is a singular fiber of type $\text{III}$ or $\text{IV}$. Let $\Xi$ be a union of some irreducible components of $f^{-1}(v)$ that does not coincide with the whole fiber $f^{-1}(v)$. If $U$ is a small open disk on $\mathbb{P}^1$ with the center $v$, then $f^{-1}(U) \setminus (\Xi \cup (f^{-1}(U) \cap O))$ has an abelian fundamental group.

Proof. This can be proved easily by the van-Kampen theorem. \hfill $\Box$

Lemma 4.9. Let $\Sigma$ be satisfying the conditions (N1) and (N2). Suppose that $(X, \Delta)$ and $(X', \Delta')$ satisfy the following:

(a) $\Sigma_{\Delta} = \Sigma_{\Delta'} = \Sigma$,
(b) $\mathbb{Z}[\Delta] = \mathbb{Z}[\Delta]$ and $\mathbb{Z}[\Delta'] = \mathbb{Z}[\Delta']$.

Then there exists a connected continuous family $(X_t, \Delta_t)$ parameterized by $t \in [0, 1]$ such that $(X_0, \Delta_0) = (X, \Delta)$, $(X_1, \Delta_1) = (X', \Delta')$ and that $(X_t, \Delta_t)$ are diffeomorphic to one another. In particular, $\pi_1(X \setminus \Delta)$ is isomorphic to $\pi_1(X' \setminus \Delta')$.

Proof. By Nikulin [11, Theorem 1.14.1], the primitive embedding of $L(\Sigma)$ into the K3 lattice $\Lambda$ is unique up to $\text{Aut}(\Lambda)$. Hence the assertion follows from Nikulin’s connectedness theorem [10, Theorem 2.10]. \hfill $\Box$

Proof of Theorem 4.3. Let us consider the following:

Claim 1. Suppose that $\Sigma$ satisfies $\text{rank}(L(\Sigma)) = 18$ and the condition (N2). Then there exists an $ADE$-configuration of smooth rational curves $\Delta_{\Sigma}$ on a K3 surface $X_\Sigma$ such that $\Sigma_{\Delta_{\Sigma}} = \Sigma$ and $\pi_1(X_\Sigma \setminus \Delta_{\Sigma}) = \{1\}$.

We deduce Theorem 4.3 from Claim 1. Suppose that $\Delta$ is an $ADE$-configuration of smooth rational curves on a K3 surface $X$ such that $\Sigma_{\Delta}$ satisfies the conditions (N1) and (N2), and that $\mathbb{Z}[\Delta]$ is primitive in $H^2(X;\mathbb{Z})$. By Lemma 4.7, there
exists $\Sigma_1$ satisfying $\text{rank}(L(\Sigma_1)) = 18$ and the condition (N2) such that $\Gamma(\Sigma) \subset \Gamma(\Sigma_1)$. By Claim 1, we have $(X_1, \Delta_1)$ such that $\Sigma_{\Delta_1} = \Sigma_1$ and $\pi_1(X_1 \setminus \Delta_1) = \{1\}$. Let $\Delta' \subset \Delta_1$ be the sub-configuration of smooth rational curves on $X_1$ corresponding to the subgraph $\Gamma(\Sigma_{\Delta'}) \rightarrow \Gamma(\Sigma_1) = \Gamma(\Sigma_{\Delta_1})$. There is a surjection from $\pi_1(X_1 \setminus \Delta_1)$ to $\pi_1(X_1 \setminus \Delta')$, and hence $\pi_1(X_1 \setminus \Delta')$ is trivial. In particular, $\mathbb{Z}[\Delta']$ is primitive in $H^2(X_1; \mathbb{Z})$. Because of $\Sigma_{\Delta'} = \Sigma_{\Delta_1}$, Lemma 4.3 implies that $\pi_1(X \setminus \Delta)$ is isomorphic to $\pi_1(X_1 \setminus \Delta')$. Thus $\pi_1(X \setminus \Delta)$ is trivial.

Let $f : X \rightarrow \mathbb{P}^1$ be an extremal elliptic $K3$ surface. For a point $v \in R_f$, we denote the total fiber of $f$ over $v$ by

$$\sum_{i=1}^{r_v} m_{v,i} C_{v,i},$$

where $m_{v,i}$ is the multiplicity of the irreducible component $C_{v,i}$ of $f^{-1}(v)$. We denote by $\Gamma_f$ the union of the zero section and all irreducible fibers $f^{-1}(v) (v \in R_f)$.

**Claim 2.** Suppose that $MW_f = (0)$. Suppose that a sub-configuration $\Delta$ of $\Gamma_f$ satisfies the following two conditions.

(Z1) The number of $v \in R_f$ such that $m_{v,i} = 1 \Rightarrow C_{v,i} \subset \Delta$ holds is at most one.

(Z2) Either one of the following holds:

(Z2·a) The configuration $\Delta$ does not contain the zero section,

(Z2·b) there is a point $v_1 \in R_f$ such that the type $\tau(S_{f,v_1})$ is $A_1$ and that $F_1 := f^{-1}(v_1)$ and $\Delta$ have no common irreducible components, or

(Z2·c) $e_u(\Sigma_f) \leq 23$.

Then $\pi_1(X \setminus \Delta)$ is trivial.

**Proof of Claim 2.** By Lemma 4.3, the assumption $MW_f = (0)$ implies that the cohomology classes $[O]$ and $[C_{v,i}] (v \in R_f, i = 1, \ldots, r_v)$ of the irreducible components of $\Gamma_f$ span $NS_X$. The relations among these generators are generated by

$$\sum_{i=1}^{r_v} m_{v,i} C_{v,i} = \sum_{i=1}^{r_{v'}} m_{v',i} C_{v',i} \quad (v, v' \in R_f).$$

Therefore the condition (Z1) implies that the cohomology classes of the irreducible components of $\Delta$ constitute a subset of a $\mathbb{Z}$-basis of $NS_X$. Hence $\mathbb{Z}[\Delta]$ is primitive in $H^2(X; \mathbb{Z})$. In particular, $\pi_1(X \setminus \Delta)$ is a perfect group by Lemma 4.6. On the other hand, the condition (Z1) implies that there exists a point $v_0 \in \mathbb{P}^1$ such that every fiber of the restriction

$$f|_{X \setminus (\Delta \cup f^{-1}(v_0))} : X \setminus (\Delta \cup f^{-1}(v_0)) \rightarrow \mathbb{P}^1 \setminus \{v_0\}$$

of $f$ has a reduced irreducible component. Then, by Nori’s lemma [13, Lemma 1.5 (C)], if $U$ is a non-empty connected classically open subset of $\mathbb{P}^1 \setminus \{v_0\}$, then the inclusion of $f^{-1}(U) \setminus (f^{-1}(U) \cap \Delta)$ into $X \setminus (\Delta \cup f^{-1}(v_0))$ induces a surjection on the fundamental groups. The inclusion of $X \setminus (\Delta \cup f^{-1}(v_0))$ into $X \setminus \Delta$ also induces a surjection on the fundamental groups. We shall show that there exists a small open disk $U$ on $\mathbb{P}^1 \setminus \{v_0\}$ such that

$$G_U := \pi_1(f^{-1}(U) \setminus (f^{-1}(U) \cap \Delta))$$

is abelian. When (Z2·a) occurs, we take a small open disk disjoint from $R_f$ as $U$. Then $G_U$ is abelian, because of $f^{-1}(U) \cap \Delta = \emptyset$. Suppose that (Z2·b) occurs. We can take $v_0$ from $\mathbb{P}^1 \setminus \{v_1\}$, because $F_1$ has no irreducible components of multiplicity.
\[ \geq 2. \] We choose as \( U \) a small open disk with the center \( v_1 \). There is a contraction from \( f^{-1}(U) \setminus (f^{-1}(U) \cap \Delta) \) to \( F_1 \setminus (F_1 \cap \Delta) \). Because \( \pi_1(F_1 \setminus (F_1 \cap \Delta)) \) is abelian, so is \( G_U \). Suppose that \( (Z2-c) \) occurs. By Lemma 2.2, there exists a singular fiber \( F_2 := f^{-1}(v_2) \) of type I, II, III or IV. Because \( F_2 \) has no irreducible components of multiplicity \( \geq 2 \), we can choose \( v_0 \) from \( \mathbb{P}^1 \setminus \{v_2\} \). If \( F_2 \) is of type I or II, then \( F_2 \cap \Delta \) consists of a nonsingular point of \( F_2 \), and \( \pi_1(F_2 \setminus (F_2 \cap \Delta)) \) is abelian. Hence \( G_U \) is also abelian. If \( F_2 \) is of type III or IV, then \( F_2 \cap \Delta \) cannot coincide with the whole fiber \( F_2 \). Hence Lemma 4.8 implies that \( G_U \) is abelian. Therefore we see that \( \pi_1(X \setminus \Delta) \) is abelian. Being both perfect and abelian, \( \pi_1(X \setminus \Delta) \) is trivial.

Now we proceed to the proof of Claim 1. We list up all \( \Sigma \) satisfying the condition \((N2)\) and \( \text{rank}(L(\Sigma)) = 18 \). It consists of 297 elements. Among them, 199 elements can be the type \( \Sigma_f \) of singular fibers of some extremal elliptic \( K3 \) surface \( f : X \to \mathbb{P}^1 \) with \( MW_f = 0 \). For these configurations, \( \pi_1(X \setminus \Delta) \) is trivial by Claim 2. The remaining 98 configurations are listed in the second column of Table 1 below. Each of them is a sub-configuration of \( \Gamma_f \) satisfying the conditions \((Z1)\) and \((Z2)\), where \( f : X \to \mathbb{P}^1 \) is the extremal elliptic \( K3 \) surface with \( MW_f = 0 \) whose number in Table 2 is given in the third column of Table 1. The fourth and fifth columns of Table 1 indicate \( \Sigma_f \) and \( eu(\Sigma_f) \), respectively. In the case nos. 20, 28, 39, 41 and 85 in Table 1, we can choose the embedding of \( \Delta \) into \( \Gamma_f \) in such a way that \((Z2-b)\) holds. In the case nos. 30, 37, 57 and 63 in Table 1, we can choose the embedding of \( \Delta \) into \( \Gamma_f \) in such a way that \((Z2-a)\) holds. By Claim 2 again, \( \pi_1(X \setminus \Delta) \) is trivial for these 98 configurations \( \Delta \).

Remark 4.10. The graph \( \Gamma(A_{19}) \) (resp. \( \Gamma(D_{19}) \)) can be embedded into \( \Gamma_f \) in such a way that \((Z1)\) and \((Z2)\) are satisfied, where \( f : X \to \mathbb{P}^1 \) is the extremal elliptic \( K3 \) surfaces whose number in Table 2 is 312 (resp. 320). Therefore, if \( \Gamma(\Delta) \) is embedded in \( \Gamma(A_{19}) \) or \( \Gamma(D_{19}) \), then \( \Gamma(\Delta) \) can be embedded in \( \Gamma_f \) in such a way that \((Z1)\) and \((Z2)\) are satisfied.
| No | \( \Delta \) | No | \( \Sigma f \) | \( e_n(\Sigma f) \) |
|---|---|---|---|---|
| 1 | \( A_2 + A_3 + 2 A_4 + A_5 \) | 19 | \( A_2 + 2 A_3 + A_4 + A_6 \) | 23 |
| 2 | \( A_1 + A_2 + A_3 + 2 A_6 \) | 23 | \( A_1 + A_2 + A_4 + A_5 + A_6 \) | 23 |
| 3 | \( 2 A_1 + A_6 + 2 A_6 \) | 23 | \( A_1 + A_2 + A_4 + A_5 + A_6 \) | 23 |
| 4 | \( 2 A_2 + 2 A_4 + A_6 \) | 23 | \( A_1 + A_2 + A_4 + A_5 + A_6 \) | 23 |
| 5 | \( A_1 + A_5 + 2 A_6 \) | 40 | \( A_1 + A_4 + A_6 + A_7 \) | 22 |
| 6 | \( A_1 + 2 A_7 \) | 52 | \( A_4 + A_6 + A_8 \) | 21 |
| 7 | \( A_1 + A_2 + 2 A_4 + A_7 \) | 23 | \( A_1 + A_2 + A_4 + A_5 + A_6 \) | 23 |
| 8 | \( A_3 + 2 A_4 + A_7 \) | 24 | \( A_3 + A_4 + A_5 + A_6 \) | 22 |
| 9 | \( A_2 + 2 A_4 + A_8 \) | 36 | \( A_2 + A_4 + A_5 + A_7 \) | 22 |
| 10 | \( 2 A_3 + A_4 + A_8 \) | 46 | \( A_1 + A_2 + A_4 + A_5 + A_8 \) | 23 |
| 11 | \( A_3 + A_7 + A_8 \) | 53 | \( A_1 + A_2 + A_7 + A_8 \) | 22 |
| 12 | \( A_1 + 2 A_2 + A_4 + A_9 \) | 46 | \( A_1 + A_2 + A_3 + A_4 + A_8 \) | 23 |
| 13 | \( A_2 + A_3 + A_4 + A_9 \) | 71 | \( 2 A_2 + A_4 + A_{10} \) | 22 |
| 14 | \( A_3 + A_4 + A_{11} \) | 93 | \( A_2 + A_4 + A_{12} \) | 21 |
| 15 | \( A_7 + A_{11} \) | 312 | \( A_{10} + E_8 \) | 21 |
| 16 | \( 2 A_3 + A_{12} \) | 93 | \( A_2 + A_4 + A_{12} \) | 21 |
| 17 | \( A_3 + A_{15} \) | 312 | \( A_{10} + E_8 \) | 21 |
| 18 | \( A_2 + 2 A_6 + D_4 \) | 99 | \( A_2 + A_3 + A_{13} \) | 21 |
| 19 | \( 2 A_4 + A_6 + D_4 \) | 18 | \( A_1 + A_3 + 2 A_4 + A_6 \) | 23 |
| 20 | \( 2 A_4 + A_5 + A_6 + D_4 \) | 20 | \( A_1 + 2 A_2 + A_3 + A_4 + A_6 \) | 24 |
| 21 | \( A_2 + A_4 + A_8 + D_4 \) | 44 | \( 2 A_1 + 2 A_4 + A_8 \) | 23 |
| 22 | \( A_6 + A_8 + D_4 \) | 50 | \( 2 A_1 + A_2 + A_6 + A_6 \) | 23 |
| 23 | \( 2 A_2 + A_{10} + D_4 \) | 72 | \( 2 A_1 + A_2 + A_4 + A_{10} \) | 23 |
| 24 | \( A_4 + A_{10} + D_4 \) | 72 | \( 2 A_1 + A_2 + A_4 + A_{10} \) | 23 |
| 25 | \( A_2 + A_{12} + D_4 \) | 90 | \( 2 A_1 + 2 A_2 + A_{12} \) | 23 |
| 26 | \( A_{14} + D_4 \) | 320 | \( D_{10} + E_8 \) | 22 |
| 27 | \( 2 A_2 + A_4 + 2 D_5 \) | 210 | \( 2 A_2 + D_{14} \) | 22 |
| 28 | \( A_1 + 2 A_2 + 2 A_4 + D_5 \) | 157 | \( A_1 + A_2 + 2 A_4 + D_7 \) | 24 |
| 29 | \( A_2 + A_3 + 2 A_4 + D_5 \) | 46 | \( A_1 + A_2 + A_3 + A_4 + A_8 \) | 23 |
| 30 | \( A_2 + A_6 + 2 D_5 \) | 193 | \( A_2 + A_6 + D_{10} \) | 22 |
| 31 | \( A_3 + A_4 + A_6 + D_5 \) | 18 | \( A_1 + A_3 + 2 A_4 + A_6 \) | 23 |
| 32 | \( A_2 + A_4 + A_7 + D_5 \) | 72 | \( 2 A_1 + A_2 + A_4 + A_{10} \) | 23 |
| 33 | \( A_6 + A_7 + D_5 \) | 50 | \( 2 A_1 + A_2 + A_6 + A_8 \) | 23 |
| 34 | \( A_6 + A_9 + D_5 \) | 50 | \( 2 A_1 + A_2 + A_6 + A_8 \) | 23 |
| 35 | \( A_2 + A_3 + A_8 + D_5 \) | 69 | \( A_1 + 2 A_2 + A_3 + A_{10} \) | 23 |
| 36 | \( A_2 + A_{11} + D_5 \) | 90 | \( 2 A_1 + 2 A_2 + A_{12} \) | 23 |
| 37 | \( A_4 + 2 D_7 \) | 213 | \( A_4 + D_{14} \) | 21 |
| 38 | \( A_3 + 2 A_4 + D_7 \) | 44 | \( 2 A_1 + 2 A_4 + A_8 \) | 23 |
| 39 | \( 2 A_2 + A_3 + A_4 + D_7 \) | 20 | \( A_1 + 2 A_2 + A_3 + A_4 + A_6 \) | 24 |
| 40 | \( A_2 + A_4 + A_5 + D_7 \) | 23 | \( A_1 + A_2 + A_4 + A_5 + A_6 \) | 23 |
| 41 | \( A_1 + 2 A_2 + A_6 + D_7 \) | 14 | \( 2 A_1 + 2 A_2 + 2 A_6 \) | 24 |
Table 1. List of embedding of $\Delta$ in $\Gamma_f$

| no | $\Delta$ | No | $\Sigma_f$ | $e_\Delta(\Sigma_f)$ |
|----|----------|----|-----------|------------------|
| 42 | $2A_2 + A_7 + D_7$ | 90 | $2A_1 + 2A_2 + A_{12}$ | 23 |
| 43 | $A_4 + A_7 + D_7$ | 44 | $2A_1 + 2A_4 + A_8$ | 23 |
| 44 | $A_1 + A_2 + A_8 + D_7$ | 50 | $2A_1 + A_2 + A_6 + A_8$ | 23 |
| 45 | $A_3 + A_8 + D_7$ | 44 | $2A_1 + 2A_4 + A_8$ | 23 |
| 46 | $A_{11} + D_7$ | 320 | $D_{10} + E_8$ | 22 |
| 47 | $A_2 + A_4 + D_5 + D_7$ | 200 | $A_2 + A_5 + D_{11}$ | 22 |
| 48 | $A_6 + D_5 + D_7$ | 186 | $A_9 + D_9$ | 21 |
| 49 | $A_2 + 2A_4 + D_8$ | 66 | $A_2 + A_7 + A_9$ | 21 |
| 50 | $A_1 + A_6 + D_8$ | 23 | $A_1 + A_2 + A_4 + A_5 + A_6$ | 23 |
| 51 | $A_2 + A_8 + D_8$ | 50 | $2A_1 + A_2 + A_6 + A_8$ | 23 |
| 52 | $A_{10} + D_8$ | 320 | $D_{10} + E_8$ | 22 |
| 53 | $A_1 + 2A_4 + D_9$ | 44 | $2A_1 + 2A_4 + A_8$ | 23 |
| 54 | $A_2 + A_3 + A_4 + D_9$ | 46 | $A_1 + A_2 + A_3 + A_4 + A_8$ | 23 |
| 55 | $A_3 + A_6 + D_9$ | 76 | $2A_1 + A_6 + A_{10}$ | 22 |
| 56 | $A_2 + A_7 + D_9$ | 50 | $2A_1 + A_2 + A_6 + A_8$ | 23 |
| 57 | $2A_2 + D_5 + D_9$ | 210 | $2A_2 + D_{14}$ | 22 |
| 58 | $A_2 + D_7 + D_9$ | 186 | $A_9 + D_9$ | 21 |
| 59 | $2A_2 + A_4 + D_{10}$ | 72 | $2A_1 + A_2 + A_4 + A_{10}$ | 23 |
| 60 | $A_3 + A_4 + D_{11}$ | 44 | $2A_1 + 2A_1 + A_8$ | 23 |
| 61 | $A_2 + D_{11}$ | 320 | $D_{10} + E_8$ | 22 |
| 62 | $A_2 + D_5 + D_{11}$ | 186 | $A_9 + D_9$ | 21 |
| 63 | $D_7 + D_{11}$ | 218 | $D_{18}$ | 20 |
| 64 | $A_2 + A_4 + D_{12}$ | 72 | $2A_1 + A_2 + A_4 + A_{10}$ | 23 |
| 65 | $A_6 + D_{12}$ | 320 | $D_{10} + E_8$ | 22 |
| 66 | $A_1 + 2A_2 + D_{13}$ | 90 | $2A_1 + 2A_2 + A_{12}$ | 23 |
| 67 | $A_2 + A_3 + D_{13}$ | 72 | $2A_1 + A_2 + A_4 + A_{10}$ | 23 |
| 68 | $A_3 + D_{15}$ | 320 | $D_{10} + E_8$ | 22 |
| 69 | $A_2 + D_{16}$ | 320 | $D_{10} + E_8$ | 22 |
| 70 | $2A_1 + A_4 + 2E_6$ | 303 | $A_1 + A_4 + A_5 + E_8$ | 23 |
| 71 | $2A_1 + A_2 + 2A_4 + E_6$ | 23 | $A_1 + A_2 + A_4 + A_5 + A_6$ | 23 |
| 72 | $A_2 + 2A_3 + A_4 + E_6$ | 46 | $A_1 + A_2 + A_3 + A_4 + A_8$ | 23 |
| 73 | $2A_6 + E_6$ | 37 | $A_1 + 2A_2 + A_6 + A_7$ | 23 |
| 74 | $2A_3 + A_6 + E_6$ | 41 | $A_5 + A_6 + A_7$ | 21 |
| 75 | $A_9 + A_3 + A_7 + E_6$ | 37 | $A_1 + 2A_2 + A_6 + A_7$ | 23 |
| 76 | $2A_4 + D_4 + E_6$ | 182 | $A_4 + A_5 + D_9$ | 22 |
| 77 | $A_2 + A_6 + D_4 + E_6$ | 183 | $A_1 + A_2 + A_6 + D_9$ | 23 |
| 78 | $A_8 + D_4 + E_6$ | 186 | $A_9 + D_9$ | 21 |
| 79 | $A_1 + D_5 + 2E_6$ | 320 | $D_{10} + E_8$ | 22 |
| 80 | $A_2 + 2D_5 + E_6$ | 320 | $D_{10} + E_8$ | 22 |
| 81 | $A_1 + A_2 + A_4 + D_5 + E_6$ | 193 | $A_2 + A_6 + D_{10}$ | 22 |
| 82 | $A_2 + A_3 + D_7 + E_6$ | 200 | $A_2 + A_5 + D_{11}$ | 22 |
Table 1. List of embedding of $\Delta$ in $\Gamma_f$

| no | \(\Delta\)   | No  | \(\Sigma_f\)     | \(eu(\Sigma_f)\) |
|----|----------------|-----|------------------|-------------------|
| 83 | \(A_5 + D_7 + E_6\) | 320 | \(D_{10} + E_8\) | 22                |
| 84 | \(A_2 + D_{10} + E_6\) | 193 | \(A_2 + A_6 + D_{10}\) | 22                |
| 85 | \(A_1 + A_2 + 2 A_4 + E_7\) | 17 | \(2 A_1 + A_2 + 2 A_4 + A_6\) | 24                |
| 86 | \(A_3 + 2 A_4 + E_7\) | 18 | \(A_1 + A_3 + 2 A_4 + A_6\) | 23                |
| 87 | \(2 A_2 + D_7 + E_7\) | 210 | \(2 A_2 + D_{14}\) | 22                |
| 88 | \(A_2 + 2 A_4 + E_8\) | 36 | \(A_2 + A_4 + A_5 + A_7\) | 22                |
| 89 | \(2 A_1 + 2 A_2 + A_4 + E_8\) | 30 | \(2 A_2 + A_3 + A_4 + A_7\) | 23                |
| 90 | \(2 A_3 + A_4 + E_8\) | 24 | \(A_3 + A_4 + A_5 + A_6\) | 22                |
| 91 | \(A_3 + A_7 + E_8\) | 46 | \(A_1 + A_2 + A_3 + A_4 + A_8\) | 23                |
| 92 | \(A_2 + A_4 + D_4 + E_8\) | 182 | \(A_4 + A_5 + D_9\) | 22                |
| 93 | \(A_6 + D_4 + E_8\) | 186 | \(A_9 + D_9\) | 21                |
| 94 | \(A_1 + 2 A_2 + D_5 + E_8\) | 210 | \(2 A_2 + D_{14}\) | 22                |
| 95 | \(A_2 + A_3 + D_5 + E_8\) | 198 | \(2 A_2 + A_3 + D_{14}\) | 23                |
| 96 | \(A_3 + D_7 + E_8\) | 213 | \(A_4 + D_{14}\) | 21                |
| 97 | \(A_2 + D_8 + E_8\) | 210 | \(2 A_2 + D_{14}\) | 22                |
| 98 | \(2 A_1 + A_2 + E_6 + E_8\) | 320 | \(D_{10} + E_8\) | 22                |
Table 2. List of extremal elliptic $K3$ surfaces

| No | $\Sigma$ | $MW$ | $a$ | $b$ | $c$ |
|----|---------|------|----|----|----|
| 1  | $6 A_3$ | $\mathbb{Z}/(4) \times \mathbb{Z}/(4)$ | 4  | 0  | 4  |
| 2  | $2 A_1 + 4 A_4$ | $\mathbb{Z}/(5)$ | 10 | 0  | 10 |
| 3  | $2 A_2 + 2 A_3 + 2 A_4$ | (0) | 60 | 0  | 60 |
| 4  | $3 A_1 + 3 A_5$ | $\mathbb{Z}/(2) \times \mathbb{Z}/(6)$ | 2  | 0  | 6  |
| 5  | $4 A_2 + 2 A_5$ | $\mathbb{Z}/(3) \times \mathbb{Z}/(3)$ | 6  | 0  | 6  |
| 6  | $A_3 + 3 A_5$ | $\mathbb{Z}/(6)$ | 4  | 0  | 6  |
| 7  | $2 A_1 + 2 A_3 + 2 A_5$ | $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 12 | 0  | 12 |
| 8  | $A_1 + 2 A_2 + A_3 + 2 A_5$ | $\mathbb{Z}/(6)$ | 6  | 0  | 12 |
| 9  | $2 A_1 + 2 A_5$ | (0) | 30 | 0  | 30 |
| 10 | $2 A_2 + A_4 + 2 A_5$ | $\mathbb{Z}/(3)$ | 6  | 0  | 30 |
| 11 | $A_1 + A_3 + A_4 + 2 A_5$ | $\mathbb{Z}/(2)$ | 12 | 0  | 30 |
| 12 | $A_1 + A_2 + 2 A_3 + A_4 + A_5$ | $\mathbb{Z}/(2)$ | 24 | 12 | 36 |
| 13 | $3 A_6$ | $\mathbb{Z}/(7)$ | 2  | 1  | 4  |
| 14 | $2 A_1 + 2 A_2 + 2 A_6$ | (0) | 42 | 0  | 42 |
| 15 | $2 A_3 + 2 A_6$ | (0) | 28 | 0  | 28 |
| 16 | $A_2 + A_4 + 2 A_6$ | (0) | 28 | 7  | 28 |
| 17 | $2 A_1 + A_2 + 2 A_4 + A_6$ | (0) | 50 | 20 | 50 |
| 18 | $A_1 + A_3 + 2 A_4 + A_6$ | (0) | 10 | 0  | 140|
| 19 | $A_2 + 2 A_3 + A_4 + A_6$ | (0) | 24 | 12 | 76 |
| 20 | $A_1 + 2 A_2 + A_3 + A_4 + A_6$ | (0) | 30 | 0  | 84 |
| 21 | $2 A_1 + 2 A_5 + A_6$ | $\mathbb{Z}/(2)$ | 12 | 6  | 24 |
| 22 | $A_1 + 2 A_3 + A_5 + A_6$ | $\mathbb{Z}/(2)$ | 4  | 0  | 84 |
| 23 | $A_1 + A_2 + A_4 + A_5 + A_6$ | (0) | 30 | 0  | 42 |
| 24 | $A_3 + A_4 + A_5 + A_6$ | (0) | 18 | 6  | 72 |
| 25 | $4 A_1 + 2 A_7$ | $\mathbb{Z}/(2) \times \mathbb{Z}/(4)$ | 4  | 0  | 4  |
| 26 | $2 A_2 + 2 A_7$ | (0) | 24 | 0  | 24 |
| 27 | $A_1 + A_3 + 2 A_7$ | $\mathbb{Z}/(8)$ | 2  | 0  | 4  |
| 28 | $2 A_1 + 3 A_3 + A_7$ | $\mathbb{Z}/(2) \times \mathbb{Z}/(4)$ | 4  | 0  | 8  |
| 29 | $A_2 + 3 A_3 + A_7$ | $\mathbb{Z}/(4)$ | 4  | 0  | 24 |
| 30 | $2 A_2 + A_3 + A_4 + A_7$ | (0) | 12 | 0  | 120|
| 31 | $2 A_1 + A_2 + A_3 + A_4 + A_7$ | $\mathbb{Z}/(2)$ | 20 | 0  | 24 |
| 32 | $A_1 + 2 A_5 + A_7$ | $\mathbb{Z}/(2)$ | 6  | 0  | 24 |
| 33 | $3 A_1 + A_3 + A_5 + A_7$ | $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 8  | 0  | 12 |
Table 2. List of extremal elliptic $K3$ surfaces

| No | $\Sigma$ | $MW$ | $a$ | $b$ | $c$ |
|----|----------|------|-----|-----|-----|
| 34 | $A_1 + A_2 + A_3 + A_5 + A_7$ | $\mathbb{Z}/(2)$ | 12 | 0 | 24 |
| 35 | $2A_1 + A_4 + A_5 + A_7$ | $\mathbb{Z}/(2)$ | 2 | 0 | 120 |
| 36 | $A_2 + A_4 + A_5 + A_7$ | (0) | 6 | 0 | 120 |
|    |          |      | 24 | 0 | 30 |
| 37 | $A_1 + 2A_2 + A_6 + A_7$ | (0) | 24 | 0 | 42 |
| 38 | $2A_1 + A_3 + A_6 + A_7$ | $\mathbb{Z}/(2)$ | 12 | 4 | 20 |
| 39 | $A_2 + A_3 + A_6 + A_7$ | (0) | 4 | 0 | 168 |
| 40 | $A_1 + A_4 + A_6 + A_7$ | (0) | 2 | 0 | 280 |
|    |          |      | 18 | 4 | 32 |
| 41 | $A_5 + A_6 + A_7$ | (0) | 16 | 4 | 22 |
| 42 | $2A_1 + 2A_8$ | (0) | 18 | 0 | 18 |
|    |          |      | $\mathbb{Z}/(3)$ | 4 | 2 | 10 |
| 43 | $A_1 + 3A_2 + A_3 + A_8$ | $\mathbb{Z}/(3)$ | 12 | 0 | 18 |
| 44 | $2A_1 + 2A_4 + A_8$ | (0) | 20 | 10 | 50 |
| 45 | $3A_2 + A_4 + A_8$ | $\mathbb{Z}/(3)$ | 12 | 3 | 12 |
| 46 | $A_1 + A_2 + A_3 + A_4 + A_8$ | (0) | 6 | 0 | 180 |
| 47 | $A_1 + 2A_2 + A_5 + A_8$ | $\mathbb{Z}/(3)$ | 6 | 0 | 18 |
| 48 | $A_2 + A_3 + A_5 + A_8$ | $\mathbb{Z}/(3)$ | 4 | 0 | 18 |
| 49 | $A_1 + A_4 + A_5 + A_8$ | (0) | 18 | 0 | 30 |
| 50 | $2A_1 + A_2 + A_6 + A_8$ | (0) | 18 | 0 | 42 |
| 51 | $A_1 + A_3 + A_6 + A_8$ | (0) | 10 | 4 | 52 |
| 52 | $A_4 + A_6 + A_8$ | (0) | 18 | 9 | 22 |
| 53 | $A_1 + A_2 + A_7 + A_8$ | (0) | 18 | 0 | 24 |
| 54 | $2A_9$ | (0) | 10 | 0 | 10 |
|    |          |      | $\mathbb{Z}/(5)$ | 2 | 0 | 2 |
| 55 | $A_1 + A_2 + 2A_3 + A_9$ | $\mathbb{Z}/(2)$ | 4 | 0 | 60 |
| 56 | $2A_1 + 2A_2 + A_3 + A_9$ | $\mathbb{Z}/(2)$ | 6 | 0 | 60 |
| 57 | $A_1 + 2A_4 + A_9$ | $\mathbb{Z}/(5)$ | 2 | 0 | 10 |
| 58 | $3A_1 + A_2 + A_4 + A_9$ | $\mathbb{Z}/(2)$ | 20 | 10 | 20 |
| 59 | $2A_1 + A_3 + A_4 + A_9$ | $\mathbb{Z}/(2)$ | 10 | 0 | 20 |
| 60 | $2A_1 + A_2 + A_5 + A_9$ | $\mathbb{Z}/(2)$ | 12 | 6 | 18 |
| 61 | $A_1 + A_3 + A_5 + A_9$ | $\mathbb{Z}/(2)$ | 10 | 0 | 12 |
| 62 | $A_4 + A_5 + A_9$ | (0) | 10 | 0 | 30 |
|    |          |      | $\mathbb{Z}/(2)$ | 10 | 5 | 10 |
| 63 | $3A_1 + A_6 + A_9$ | $\mathbb{Z}/(2)$ | 4 | 2 | 36 |
| 64 | $A_1 + A_2 + A_6 + A_9$ | (0) | 10 | 0 | 42 |
Table 2. List of extremal elliptic $K3$ surfaces

| No  | $\Sigma$ | $MW$ | $a$ | $b$ | $c$ |
|-----|----------|------|-----|-----|-----|
| 65  | $A_3 + A_6 + A_9$ | (0)  | 2   | 0   | 140 |
| 66  | $A_2 + A_7 + A_9$ | (0)  | 10  | 0   | 24  |
| 67  | $A_1 + A_8 + A_9$ | (0)  | 10  | 0   | 18  |
| 68  | $A_2 + 2A_3 + A_{10}$ | (0) | 24  | 12  | 28  |
| 69  | $A_1 + 2A_2 + A_3 + A_{10}$ | (0) | 12  | 0   | 66  |
| 70  | $2A_4 + A_{10}$ | (0)  | 10  | 5   | 30  |
| 71  | $2A_2 + A_4 + A_{10}$ | (0) | 6   | 3   | 84  |
| 72  | $2A_1 + A_2 + A_4 + A_{10}$ | (0) | 2   | 0   | 330 |
| 73  | $A_1 + A_3 + A_4 + A_{10}$ | (0) | 20  | 0   | 22  |
| 74  | $A_1 + A_2 + A_5 + A_{10}$ | (0) | 6   | 0   | 66  |
| 75  | $A_3 + A_5 + A_{10}$ | (0)  | 4   | 0   | 66  |
| 76  | $2A_1 + A_6 + A_{10}$ | (0)  | 12  | 2   | 26  |
| 77  | $A_2 + A_6 + A_{10}$ | (0)  | 4   | 1   | 58  |
| 78  | $A_1 + A_7 + A_{10}$ | (0)  | 2   | 0   | 88  |
| 79  | $A_8 + A_{10}$ | (0)  | 10  | 1   | 10  |
| 80  | $A_1 + 3A_2 + A_{11}$ | $\mathbb{Z}/(3)$ | 6   | 0   | 12  |
| 81  | $3A_1 + 2A_2 + A_{11}$ | $\mathbb{Z}/(6)$ | 2   | 0   | 12  |
| 82  | $A_1 + 2A_3 + A_{11}$ | $\mathbb{Z}/(4)$ | 4   | 0   | 6   |
| 83  | $2A_2 + A_3 + A_{11}$ | $\mathbb{Z}/(3)$ | 4   | 0   | 12  |
| 84  | $2A_1 + A_2 + A_3 + A_{11}$ | $\mathbb{Z}/(6)$ | 4   | 2   | 4   |
| 85  | $3A_1 + A_4 + A_{11}$ | $\mathbb{Z}/(2)$ | 6   | 0   | 20  |
| 86  | $A_1 + A_2 + A_4 + A_{11}$ | (0)  | 12  | 0   | 30  |
| 87  | $2A_1 + A_5 + A_{11}$ | $\mathbb{Z}/(2)$ | 6   | 0   | 12  |
| 88  | $A_2 + A_5 + A_{11}$ | $\mathbb{Z}/(6)$ | 2   | 0   | 4   |
| 89  | $A_1 + A_6 + A_{11}$ | (0)  | 4   | 0   | 42  |
| 90  | $2A_1 + 2A_2 + A_{12}$ | (0)  | 12  | 6   | 42  |
| 91  | $A_1 + A_2 + A_3 + A_{12}$ | (0) | 6   | 0   | 52  |
Table 2. List of extremal elliptic $K3$ surfaces

| No | $\Sigma$ | $MW$ | $a$ | $b$ | $c$ |
|----|-------|------|----|----|----|
| 92 | $2A_1 + A_4 + A_{12}$ | (0) | 2 | 0 | 130 |
|    |       |      | 18 | 8  | 18  |
| 93 | $A_2 + A_4 + A_{12}$  | (0) | 6 | 3  | 34  |
| 94 | $A_1 + A_5 + A_{12}$  | (0) | 10| 2  | 16  |
| 95 | $A_6 + A_{12}$        | (0) | 2 | 1  | 46  |
| 96 | $A_1 + 2A_2 + A_{13}$ | (0) | 6 | 0  | 42  |
|    |       | $\mathbb{Z}/(2)$ | 6 | 3  | 12  |
| 97 | $3A_1 + A_2 + A_{13}$ | $\mathbb{Z}/(2)$ | 2 | 0  | 42  |
| 98 | $2A_1 + A_3 + A_{13}$ | $\mathbb{Z}/(2)$ | 6 | 2  | 10  |
| 99 | $A_2 + A_3 + A_{13}$  | (0) | 4 | 0  | 42  |
| 100| $A_1 + A_4 + A_{13}$  | (0) | 2 | 0  | 70  |
|    |       | $\mathbb{Z}/(2)$ | 8 | 2  | 18  |
| 101| $A_5 + A_{13}$        | (0) | 4 | 2  | 22  |
| 102| $2A_2 + A_{14}$       | $\mathbb{Z}/(3)$ | 4 | 1  | 4   |
| 103| $2A_1 + A_2 + A_{14}$ | (0) | 12| 6  | 18  |
|    |       | $\mathbb{Z}/(3)$ | 2 | 0  | 10  |
| 104| $A_1 + A_3 + A_{14}$  | (0) | 10| 0  | 12  |
| 105| $A_4 + A_{14}$        | (0) | 10| 5  | 10  |
| 106| $3A_1 + A_{15}$       | $\mathbb{Z}/(4)$ | 2 | 0  | 4   |
| 107| $A_1 + A_2 + A_{15}$  | (0) | 10| 2  | 10  |
|    |       | $\mathbb{Z}/(2)$ | 4 | 0  | 6   |
| 108| $A_3 + A_{15}$        | $\mathbb{Z}/(4)$ | 2 | 0  | 2   |
| 109| $2A_1 + A_{16}$       | (0) | 2 | 0  | 34  |
|    |       | $\mathbb{Z}/(3)$ | 4 | 2  | 18  |
| 110| $A_2 + A_{16}$        | (0) | 6 | 3  | 10  |
| 111| $A_1 + A_{17}$        | (0) | 4 | 2  | 10  |
|    |       | $\mathbb{Z}/(3)$ | 2 | 0  | 2   |
| 112| $A_{18}$              | (0) | 2 | 1  | 10  |
| 113| $2A_4 + 2D_5$         | (0) | 20| 0  | 20  |
| 114| $A_3 + 2A_5 + D_5$    | $\mathbb{Z}/(2)$ | 12| 0  | 12  |
| 115| $2A_4 + A_5 + D_5$    | (0) | 20| 0  | 30  |
| 116| $A_1 + A_3 + A_4 + A_5 + D_5$ | $\mathbb{Z}/(2)$ | 12| 0  | 20  |
| 117| $A_1 + 2A_6 + D_5$    | (0) | 14| 0  | 28  |
| 118| $2A_2 + A_3 + A_6 + D_5$ | (0) | 12| 0  | 84  |
| 119| $A_1 + A_2 + A_4 + A_6 + D_5$ | (0) | 20| 0  | 42  |
Table 2. List of extremal elliptic K3 surfaces

| No | $\Sigma$ | $MW$ | $a$ | $b$ | $c$ |
|----|--------|------|----|----|----|
| 120 | $A_2 + A_5 + A_6 + D_5$ | (0) | 6 | 0 | 84 |
| 121 | $A_1 + A_7 + 2D_5$ | $\mathbb{Z}/(4)$ | 12 | 0 | 42 |
| 122 | $A_1 + A_3 + A_7 + D_5$ | $\mathbb{Z}/(4)$ | 6 | 0 | 8 |
| 123 | $2A_1 + A_4 + A_7 + D_5$ | $\mathbb{Z}/(2)$ | 8 | 0 | 20 |
| 124 | $A_8 + 2D_5$ | (0) | 8 | 4 | 20 |
| 125 | $A_1 + A_4 + A_8 + D_5$ | (0) | 2 | 0 | 180 |
| 126 | $A_5 + A_8 + D_5$ | (0) | 12 | 0 | 18 |
| 127 | $2A_2 + A_9 + D_5$ | (0) | 6 | 0 | 60 |
| 128 | $2A_1 + A_2 + A_9 + D_5$ | $\mathbb{Z}/(4)$ | 2 | 0 | 60 |
| 129 | $A_1 + A_3 + A_9 + D_5$ | $\mathbb{Z}/(2)$ | 8 | 4 | 12 |
| 130 | $A_4 + A_9 + D_5$ | (0) | 10 | 0 | 20 |
| 131 | $A_1 + A_2 + A_{10} + D_5$ | (0) | 14 | 4 | 20 |
| 132 | $2A_1 + A_{11} + D_5$ | $\mathbb{Z}/(4)$ | 2 | 0 | 6 |
| 133 | $A_2 + A_{11} + D_5$ | $\mathbb{Z}/(2)$ | 6 | 0 | 6 |
| 134 | $A_1 + A_{12} + D_5$ | (0) | 2 | 0 | 52 |
| 135 | $A_{13} + D_5$ | (0) | 6 | 2 | 10 |
| 136 | $3D_6$ | $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 2 | 0 | 2 |
| 137 | $2A_3 + 2D_6$ | $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 4 | 0 | 4 |
| 138 | $2A_2 + 2A_4 + D_6$ | (0) | 30 | 0 | 30 |
| 139 | $2A_1 + 2A_5 + D_6$ | $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 6 | 0 | 6 |
| 140 | $A_1 + 2A_3 + A_5 + D_6$ | $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 4 | 0 | 12 |
| 141 | $A_3 + A_4 + A_5 + D_6$ | $\mathbb{Z}/(2)$ | 4 | 0 | 30 |
| 142 | $2A_6 + D_6$ | (0) | 14 | 0 | 14 |
| 143 | $A_2 + A_4 + A_6 + D_6$ | (0) | 6 | 0 | 70 |
| 144 | $A_1 + 2A_2 + A_7 + D_6$ | $\mathbb{Z}/(2)$ | 6 | 0 | 24 |
| 145 | $A_2 + A_3 + A_7 + D_6$ | $\mathbb{Z}/(2)$ | 4 | 0 | 24 |
| 146 | $A_1 + A_4 + A_7 + D_6$ | $\mathbb{Z}/(2)$ | 6 | 2 | 14 |
| 147 | $A_4 + A_8 + D_6$ | (0) | 4 | 2 | 46 |
| 148 | $A_1 + A_2 + A_9 + D_6$ | $\mathbb{Z}/(2)$ | 6 | 0 | 10 |
| 149 | $A_3 + A_9 + D_6$ | $\mathbb{Z}/(2)$ | 4 | 2 | 16 |
| 150 | $A_2 + A_{10} + D_6$ | (0) | 6 | 0 | 22 |
| 151 | $A_1 + A_{11} + D_6$ | $\mathbb{Z}/(2)$ | 4 | 0 | 6 |
Table 2. List of extremal elliptic $K3$ surfaces

| No  | $\Sigma$               | $MW$                | $a$ | $b$ | $c$ |
|-----|------------------------|---------------------|-----|-----|-----|
| 152 | $A_{12} + D_6$         | (0)                 | 4   | 2   | 14  |
| 153 | $A_2 + A_5 + D_5 + D_6$| $\mathbb{Z}/(2)$   | 6   | 0   | 12  |
| 154 | $A_7 + D_5 + D_6$      | $\mathbb{Z}/(2)$   | 4   | 0   | 8   |
| 155 | $2A_2 + 2D_7$          | (0)                 | 12  | 0   | 12  |
| 156 | $A_2 + 3A_3 + D_7$     | $\mathbb{Z}/(4)$   | 8   | 4   | 8   |
| 157 | $A_1 + A_2 + 2A_4 + D_7$| (0)             | 10  | 0   | 60  |
| 158 | $A_2 + A_3 + A_6 + D_7$| (0)               | 8   | 4   | 44  |
| 159 | $A_1 + A_4 + A_6 + D_7$| (0)             | 4   | 0   | 70  |
| 160 | $A_5 + A_6 + D_7$      | (0)                 | 2   | 0   | 84  |
| 161 | $2A_1 + A_2 + A_7 + D_7$| $\mathbb{Z}/(2)$ | 4   | 0   | 24  |
| 162 | $A_1 + A_3 + A_7 + D_7$| $\mathbb{Z}/(4)$   | 2   | 0   | 8   |
| 163 | $2A_1 + A_9 + D_7$     | $\mathbb{Z}/(2)$   | 4   | 0   | 10  |
| 164 | $A_2 + A_9 + D_7$      | (0)                 | 2   | 0   | 60  |
| 165 | $A_1 + A_{10} + D_7$   | (0)                 | 4   | 0   | 22  |
| 166 | $A_{11} + D_7$         | $\mathbb{Z}/(4)$   | 2   | 1   | 2   |
| 167 | $A_1 + A_5 + A_5 + D_7$| $\mathbb{Z}/(2)$   | 4   | 0   | 12  |
| 168 | $A_5 + D_6 + D_7$      | $\mathbb{Z}/(2)$   | 2   | 0   | 12  |
| 169 | $2A_1 + 2D_8$          | $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 2 | 0 | 2  |
| 170 | $2A_2 + 2A_3 + D_8$    | $\mathbb{Z}/(2)$   | 12  | 0   | 12  |
| 171 | $2A_5 + D_8$           | $\mathbb{Z}/(2)$   | 6   | 0   | 6   |
| 172 | $2A_1 + A_5 + A_5 + D_8$| $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 2 | 0 | 12  |
| 173 | $A_1 + A_4 + A_5 + D_8$| $\mathbb{Z}/(2)$   | 2   | 0   | 30  |
| 174 | $2A_2 + A_6 + D_8$     | (0)                 | 12  | 6   | 24  |
| 175 | $A_1 + A_2 + A_7 + D_8$| $\mathbb{Z}/(2)$   | 2   | 0   | 24  |
| 176 | $A_1 + A_9 + D_8$      | $\mathbb{Z}/(2)$   | 2   | 0   | 10  |
| 177 | $2D_5 + D_8$           | $\mathbb{Z}/(2)$   | 4   | 0   | 4   |
| 178 | $A_1 + A_3 + D_6 + D_8$| $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 2 | 0 | 4  |
| 179 | $2D_9$                 | (0)                 | 4   | 0   | 4   |
| 180 | $A_1 + 2A_2 + A_4 + D_9$| (0)             | 12  | 0   | 30  |
| 181 | $A_1 + A_3 + A_5 + D_9$| $\mathbb{Z}/(2)$   | 4   | 0   | 12  |
| 182 | $A_4 + A_5 + D_9$      | (0)                 | 4   | 0   | 30  |
| 183 | $A_1 + A_2 + A_6 + D_9$| (0)             | 4   | 0   | 42  |
| 184 | $2A_1 + A_7 + D_9$     | $\mathbb{Z}/(2)$   | 4   | 0   | 8   |
| 185 | $A_1 + A_8 + D_9$      | (0)                 | 4   | 0   | 18  |
| 186 | $A_9 + D_9$            | (0)                 | 4   | 0   | 10  |
| 187 | $A_4 + D_5 + D_9$      | (0)                 | 4   | 0   | 20  |
Table 2. List of extremal elliptic $K3$ surfaces

| No  | $\Sigma$ | $MW$          | $a$ | $b$ | $c$ |
|-----|----------|---------------|-----|-----|-----|
| 188 | $2A_1 + 2A_3 + D_{10}$ | $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 4   | 0   | 4   |
| 189 | $2A_4 + D_{10}$            | (0)            | 10  | 0   | 10  |
| 190 | $A_1 + A_3 + A_4 + D_{10}$ | $\mathbb{Z}/(2)$ | 2   | 0   | 20  |
| 191 | $3A_1 + A_5 + D_{10}$      | $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ | 4   | 2   | 4   |
| 192 | $A_3 + A_5 + D_{10}$       | $\mathbb{Z}/(2)$ | 2   | 0   | 12  |
| 193 | $A_2 + A_6 + D_{10}$       | (0)            | 2   | 0   | 42  |
| 194 | $A_8 + D_{10}$             | (0)            | 2   | 0   | 18  |
| 195 | $A_1 + A_2 + D_5 + D_{10}$ | $\mathbb{Z}/(2)$ | 4   | 0   | 6   |
| 196 | $A_2 + D_6 + D_{10}$       | $\mathbb{Z}/(2)$ | 2   | 0   | 6   |
| 197 | $A_1 + D_7 + D_{10}$       | $\mathbb{Z}/(2)$ | 2   | 0   | 4   |
| 198 | $2A_2 + A_3 + D_{11}$      | (0)            | 12  | 0   | 12  |
| 199 | $A_1 + A_2 + A_4 + D_{11}$ | (0)            | 6   | 0   | 20  |
| 200 | $A_2 + A_5 + D_{11}$       | (0)            | 6   | 0   | 12  |
| 201 | $A_1 + A_6 + D_{11}$       | (0)            | 6   | 2   | 10  |
| 202 | $2A_1 + 2A_2 + D_{12}$     | $\mathbb{Z}/(2)$ | 6   | 0   | 6   |
| 203 | $A_1 + A_2 + A_3 + D_{12}$ | $\mathbb{Z}/(2)$ | 4   | 0   | 6   |
| 204 | $2A_1 + A_4 + D_{12}$      | $\mathbb{Z}/(2)$ | 4   | 2   | 6   |
| 205 | $A_1 + D_5 + D_{12}$       | $\mathbb{Z}/(2)$ | 2   | 0   | 4   |
| 206 | $D_6 + D_{12}$             | $\mathbb{Z}/(2)$ | 2   | 0   | 2   |
| 207 | $A_1 + A_4 + D_{13}$       | (0)            | 2   | 0   | 20  |
| 208 | $A_5 + D_{14}$             | (0)            | 2   | 0   | 12  |
| 209 | $D_5 + D_{13}$             | (0)            | 4   | 0   | 4   |
| 210 | $2A_2 + D_{14}$            | (0)            | 6   | 0   | 6   |
| 211 | $2A_1 + A_2 + D_{14}$      | $\mathbb{Z}/(2)$ | 2   | 0   | 6   |
| 212 | $A_1 + A_3 + D_{14}$       | $\mathbb{Z}/(2)$ | 2   | 0   | 4   |
| 213 | $A_4 + D_{14}$             | (0)            | 4   | 2   | 6   |
| 214 | $A_1 + A_2 + D_{15}$       | (0)            | 4   | 0   | 6   |
| 215 | $2A_1 + D_{16}$            | $\mathbb{Z}/(2)$ | 2   | 0   | 2   |
| 216 | $A_2 + D_{16}$             | $\mathbb{Z}/(2)$ | 2   | 1   | 2   |
| 217 | $A_1 + D_{17}$             | (0)            | 2   | 0   | 4   |
| 218 | $D_{18}$                   | (0)            | 2   | 0   | 2   |
| 219 | $3E_6$                     | $\mathbb{Z}/(3)$ | 2   | 1   | 2   |
| 220 | $2A_3 + 2E_6$              | (0)            | 12  | 0   | 12  |
| 221 | $A_1 + A_3 + 2A_4 + E_6$   | (0)            | 20  | 0   | 30  |
| 222 | $A_1 + A_5 + 2E_6$         | $\mathbb{Z}/(3)$ | 2   | 0   | 6   |
| 223 | $A_2 + 2A_5 + E_6$         | $\mathbb{Z}/(3)$ | 6   | 0   | 6   |
Table 2. List of extremal elliptic $K3$ surfaces

| No | $\Sigma$ | $MW$ | $a$ | $b$ | $c$ |
|----|----------|------|-----|-----|-----|
| 224 | $2A_2 + A_3 + A_5 + E_6$ | $\mathbb{Z}/(3)$ | 6 | 0 | 12 |
| 225 | $A_3 + A_4 + A_5 + E_6$ | (0) | 12 | 0 | 30 |
| 226 | $A_6 + 2E_6$ | (0) | 6 | 3 | 12 |
| 227 | $A_1 + A_2 + A_3 + A_6 + E_6$ | (0) | 6 | 0 | 84 |
| 228 | $A_1 + A_4 + A_6 + E_6$ | (0) | 20 | 10 | 26 |
| 229 | $A_2 + A_4 + A_6 + E_6$ | (0) | 18 | 3 | 18 |
| 230 | $A_1 + A_5 + A_6 + E_6$ | (0) | 6 | 0 | 42 |
| 231 | $A_1 + A_4 + A_7 + E_6$ | (0) | 2 | 0 | 120 |
| 232 | $A_5 + A_7 + E_6$ | (0) | 6 | 0 | 24 |
| 233 | $2A_2 + A_8 + E_6$ | $\mathbb{Z}/(3)$ | 6 | 3 | 6 |
| 234 | $2A_1 + A_2 + A_8 + E_6$ | $\mathbb{Z}/(3)$ | 2 | 0 | 18 |
| 235 | $A_1 + A_3 + A_8 + E_6$ | (0) | 12 | 0 | 18 |
| 236 | $A_4 + A_8 + E_6$ | (0) | 12 | 3 | 12 |
| 237 | $A_1 + A_2 + A_9 + E_6$ | (0) | 12 | 6 | 18 |
| 238 | $A_3 + A_9 + E_6$ | (0) | 10 | 0 | 12 |
| 239 | $2A_1 + A_{10} + E_6$ | (0) | 2 | 0 | 66 |
| 240 | $A_2 + A_{10} + E_6$ | (0) | 6 | 3 | 18 |
| 241 | $A_1 + A_{11} + E_6$ | (0) | 6 | 0 | 12 |
| 242 | $A_{12} + E_6$ | (0) | 4 | 1 | 10 |
| 243 | $A_3 + A_4 + D_5 + E_6$ | (0) | 12 | 0 | 20 |
| 244 | $A_1 + A_6 + D_5 + E_6$ | (0) | 2 | 0 | 84 |
| 245 | $A_7 + D_5 + E_6$ | (0) | 8 | 0 | 12 |
| 246 | $D_6 + 2E_6$ | (0) | 6 | 0 | 6 |
| 247 | $A_2 + A_4 + D_6 + E_6$ | (0) | 6 | 0 | 30 |
| 248 | $A_6 + D_6 + E_6$ | (0) | 4 | 2 | 22 |
| 249 | $A_1 + A_4 + D_7 + E_6$ | (0) | 4 | 0 | 30 |
| 250 | $D_5 + D_7 + E_6$ | (0) | 4 | 0 | 12 |
| 251 | $A_4 + D_8 + E_6$ | (0) | 8 | 2 | 8 |
| 252 | $A_1 + A_2 + D_9 + E_6$ | (0) | 6 | 0 | 12 |
| 253 | $A_3 + D_9 + E_6$ | (0) | 4 | 0 | 12 |
| 254 | $A_1 + D_{11} + E_6$ | (0) | 2 | 0 | 12 |
| 255 | $D_{12} + E_6$ | (0) | 4 | 2 | 4 |
| 256 | $2A_2 + 2E_7$ | (0) | 6 | 0 | 6 |
| 257 | $A_1 + A_3 + 2E_7$ | $\mathbb{Z}/(2)$ | 2 | 0 | 4 |
Table 2. List of extremal elliptic $K3$ surfaces

| No  | $\Sigma$                             | $MW$ | $a$ | $b$ | $c$ |
|-----|--------------------------------------|------|-----|-----|-----|
| 258 | $A_4 + 2 E_7$                        | (0)  | 4   | 2   | 6   |
| 259 | $A_1 + 2 A_3 + A_4 + E_7$            | $\mathbb{Z}/(2)$ | 4   | 0   | 20  |
| 260 | $2 A_2 + A_3 + A_4 + E_7$            | (0)  | 12  | 0   | 30  |
| 261 | $2 A_3 + A_5 + E_7$                  | $\mathbb{Z}/(2)$ | 4   | 0   | 12  |
| 262 | $A_1 + A_2 + A_4 + A_5 + E_7$        | $\mathbb{Z}/(2)$ | 6   | 0   | 12  |
| 263 | $2 A_1 + A_4 + A_5 + E_7$            | $\mathbb{Z}/(2)$ | 8   | 2   | 8   |
| 264 | $A_2 + A_4 + A_5 + E_7$              | (0)  | 6   | 0   | 30  |
| 265 | $A_1 + 2 A_3 + A_6 + E_7$            | (0)  | 6   | 0   | 42  |
| 266 | $A_2 + A_3 + A_6 + E_7$              | (0)  | 4   | 0   | 42  |
| 267 | $A_1 + A_4 + A_6 + E_7$              | (0)  | 2   | 0   | 70  |
|     |                                      |      |     |     | 8   | 2   | 18  |
| 268 | $A_5 + A_6 + E_7$                    | (0)  | 4   | 2   | 22  |
| 269 | $2 A_2 + A_7 + E_7$                  | (0)  | 6   | 0   | 24  |
| 270 | $2 A_1 + A_2 + A_7 + E_7$            | $\mathbb{Z}/(2)$ | 2   | 0   | 24  |
| 271 | $A_1 + A_3 + A_7 + E_7$              | $\mathbb{Z}/(2)$ | 4   | 0   | 8   |
| 272 | $A_4 + A_7 + E_7$                    | (0)  | 6   | 2   | 14  |
| 273 | $A_1 + A_2 + A_8 + E_7$              | (0)  | 6   | 0   | 18  |
| 274 | $A_3 + A_8 + E_7$                    | (0)  | 4   | 0   | 18  |
| 275 | $2 A_1 + A_9 + E_7$                  | $\mathbb{Z}/(2)$ | 2   | 0   | 10  |
| 276 | $A_2 + A_9 + E_7$                    | (0)  | 6   | 0   | 10  |
|     |                                      |      |     |     | $\mathbb{Z}/(2)$ | 4   | 1   | 4   |
| 277 | $A_1 + A_{10} + E_7$                 | (0)  | 2   | 0   | 22  |
|     |                                      |      |     |     | 6   | 2   | 8   |
| 278 | $A_{11} + E_7$                       | (0)  | 4   | 0   | 6   |
| 279 | $D_4 + 2 E_7$                        | $\mathbb{Z}/(2)$ | 2   | 0   | 2   |
| 280 | $A_2 + A_4 + D_5 + E_7$              | (0)  | 6   | 0   | 20  |
| 281 | $A_1 + A_5 + D_5 + E_7$              | $\mathbb{Z}/(2)$ | 2   | 0   | 12  |
| 282 | $A_6 + D_5 + E_7$                    | (0)  | 6   | 2   | 10  |
| 283 | $A_2 + A_3 + D_6 + E_7$              | $\mathbb{Z}/(2)$ | 4   | 0   | 6   |
| 284 | $A_5 + D_6 + E_7$                    | $\mathbb{Z}/(2)$ | 4   | 2   | 4   |
| 285 | $D_5 + D_6 + E_7$                    | $\mathbb{Z}/(2)$ | 2   | 0   | 4   |
| 286 | $A_1 + A_3 + D_7 + E_7$              | $\mathbb{Z}/(2)$ | 4   | 0   | 4   |
| 287 | $A_4 + D_7 + E_7$                    | (0)  | 2   | 0   | 20  |
| 288 | $A_1 + A_2 + D_8 + E_7$              | $\mathbb{Z}/(2)$ | 2   | 0   | 6   |
| 289 | $A_2 + D_8 + E_7$                    | (0)  | 4   | 0   | 6   |
| 290 | $A_1 + D_{10} + E_7$                 | $\mathbb{Z}/(2)$ | 2   | 0   | 2   |
Table 2. List of extremal elliptic $K3$ surfaces

| No  | $\Sigma$                  | MW | a  | b  | c  |
|-----|---------------------------|----|----|----|----|
| 291 | $D_{11} + E_7$            | (0)| 2  | 0  | 4  |
| 292 | $A_2 + A_3 + E_6 + E_7$   | (0)| 6  | 0  | 12 |
| 293 | $A_1 + A_4 + E_6 + E_7$   | (0)| 2  | 0  | 30 |
| 294 | $A_5 + E_6 + E_7$         | (0)| 6  | 0  | 6  |
| 295 | $D_5 + E_6 + E_7$         | (0)| 2  | 0  | 12 |
| 296 | $2A_1 + 2E_8$             | (0)| 2  | 0  | 2  |
| 297 | $A_2 + 2E_8$              | (0)| 2  | 1  | 2  |
| 298 | $2A_2 + 2A_4 + E_8$       | (0)| 12 | 0  | 12 |
| 299 | $2A_1 + 2A_4 + E_8$       | (0)| 10 | 0  | 10 |
| 300 | $A_1 + A_2 + A_3 + A_4 + E_8$ | (0)| 6  | 0  | 20 |
| 301 | $2A_5 + E_8$              | (0)| 6  | 0  | 6  |
| 302 | $A_2 + A_3 + A_5 + E_8$   | (0)| 6  | 0  | 12 |
| 303 | $A_1 + A_4 + A_5 + E_8$   | (0)| 2  | 0  | 30 |
| 304 | $2A_2 + A_6 + E_8$        | (0)| 6  | 3  | 12 |
| 305 | $2A_1 + A_2 + A_6 + E_8$  | (0)| 2  | 0  | 42 |
| 306 | $A_1 + A_3 + A_6 + E_8$   | (0)| 6  | 2  | 10 |
| 307 | $A_4 + A_6 + E_8$         | (0)| 2  | 1  | 18 |
| 308 | $A_1 + A_2 + A_7 + E_8$   | (0)| 2  | 0  | 24 |
| 309 | $2A_1 + A_8 + E_8$        | (0)| 2  | 0  | 18 |
| 310 | $A_2 + A_8 + E_8$         | (0)| 6  | 3  | 6  |
| 311 | $A_1 + A_9 + E_8$         | (0)| 2  | 0  | 10 |
| 312 | $A_{10} + E_8$            | (0)| 2  | 1  | 6  |
| 313 | $2D_5 + E_8$              | (0)| 4  | 0  | 4  |
| 314 | $A_1 + A_4 + D_5 + E_8$   | (0)| 2  | 0  | 20 |
| 315 | $A_5 + D_5 + E_8$         | (0)| 2  | 0  | 12 |
| 316 | $2A_2 + D_5 + E_8$        | (0)| 6  | 0  | 6  |
| 317 | $A_4 + D_6 + E_8$         | (0)| 4  | 2  | 6  |
| 318 | $A_1 + A_2 + D_7 + E_8$   | (0)| 4  | 0  | 6  |
| 319 | $A_1 + D_9 + E_8$         | (0)| 2  | 0  | 4  |
| 320 | $D_{10} + E_8$            | (0)| 2  | 0  | 2  |
| 321 | $A_1 + A_3 + E_9 + E_8$   | (0)| 2  | 0  | 12 |
| 322 | $A_4 + E_6 + E_8$         | (0)| 2  | 1  | 8  |
| 323 | $D_4 + E_6 + E_8$         | (0)| 4  | 2  | 4  |
| 324 | $A_1 + A_2 + E_7 + E_8$   | (0)| 2  | 0  | 6  |
| 325 | $A_3 + E_7 + E_8$         | (0)| 2  | 0  | 4  |
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