Rank and border rank of real ternary cubics

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Abstract
In this work we give the classification of real ternary cubic forms with respect to rank and border rank up to $SL(3)$-action and we examine the differences with the complex case.

1 Introduction
In this paper we solve the classification’s problem of real ternary cubic forms with respect to rank and border rank. This problem is related to the representation of symmetric tensors with real coefficients and it is relevant in applications as Electrical Engineering (such as Antenna Array Processing), in Algebraic Statistic, in Computer Science, in Data Analysis and in other scientific areas. This topic is a particular case of the more general “Waring decomposition” of a polynomial as a sum of powers that goes back to J.J. Sylvester, A. Clebsch, A. Palatini, A. Terracini and many other mathematicians of XIX and XX centuries. The problem was to decompose a homogeneous polynomial of degree $d$ as a sum of $d$-th powers of linear forms.

The decomposition of forms as a sum of linear powers over $\mathbb{R}$ is actually not so developed as its complex counterpart because over $\mathbb{R}$ there can be more than one generic rank.

Only recently, Comon and Ottaviani (5) conjectured a list of typical ranks for binary real forms over the reals and G. Blekherman was able to prove this conjecture (2).

Let $V$ be a vector space of dimension $n+1$ over the field $\mathbb{R}$, and let $S^d V$ be the space of symmetric tensors of order $d$, and let $f \in S^d V$.

The definition of rank for a homogeneous polynomial of degree $d$, $f \in \mathbb{R}[x_0, .., x_n]$ is:

**Definition 1.1.** The (Waring or symmetric) real rank of $f$, denoted by $r_kR(f)$, is the minimum integer $r$ such that

$$f = \sum_{i=1}^{r} \lambda_i l_i^d$$

where $l_i$ are linear forms and $\lambda_i \in \mathbb{R}$.
If the field is $\mathbb{R}$, the coefficients $\lambda_i$ can be reduced to be only $\pm 1$, while in the complex case, we can impose $\lambda_i = 1$ for all $i$.

The (symmetric) real border rank of a polynomial $P$, denoted by $rk_{\mathbb{R}}(P)$, is defined in terms of limit:

**Definition 1.2.** The (symmetric) real border rank of a homogeneous polynomial $P$, denoted by $rk_{\mathbb{R}}(P)$, is the smallest positive integer $r$ such that there exists a sequence of polynomials $P_\epsilon$, each of real rank $r$, such that

$$P = \lim_{\epsilon \to 0} P_\epsilon.$$  \hfill (2)

We may define in the same way $rk_{\mathbb{C}}(f)$ and $rk_{\mathbb{C}}(f)$ for a homogeneous polynomial $f$ (see [11]).

We remark that, for a general tensor $T$, the rank depends on the field but for any field $rk(T) \geq rk(T)$.

We get for real polynomial $f$, $rk_{\mathbb{R}}(f) \geq rk_{\mathbb{C}}(f)$, $rk_{\mathbb{R}}(f) \geq rk_{\mathbb{C}}(f)$ and inequality can be strict between real and complex rank as the following example

$$f = 2x^3 - 6xy^2 = (x + iy)^3 + (x - iy)^3 = (\sqrt[3]{4}x)^3 - (x + y)^3 - (x - y)^3$$

shows.

In the above example,

$$rk_{\mathbb{C}}(f) = rk_{\mathbb{C}}(f) = 2$$

while

$$rk_{\mathbb{R}}(f) = rk_{\mathbb{R}}(2x^3 - 6xy^2) = 3.$$  

Sylvester gave a method to compute the symmetric rank of a symmetric tensor in $\mathbb{P}(S^dV)$ when $\dim(V)=2$ and Comas and Seiguer implemented this method by giving a complete classification over the complex numbers ([3]).

Over $\mathbb{R}$ there are algorithms for computing rank of a general real binary form of degree $d = 4$ and $d = 5$ (see [3], [4], [2] for the decomposition of real forms). The research of an explicit decomposition algorithm of a (symmetric) tensor is an open problem and much of the paper of Landsberg-Teitler ([12]) is devoted to such important area.

## 2 Statement of main result

The aim of this paper is to compute explicitly the (real) rank for real ternary cubic forms and to show that the maximal real rank five is obtained in three cases, that is, the union of a conic and a tangent line (see [12]), the new two cases of the cubic that factor as the union of a conic and an external line and the imaginary triangle (see §6.3 and §6.4).

We compare our Table 1 with the Table 2 obtained in [12] where the ranks and border ranks of plane cubic curves are computed over $\mathbb{C}$.

Over $\mathbb{R}$ we have obtained that the difference between the rank and the border
rank is 5-3=2 in the case of real conic plus tangent line and 5-4=1 in the case of real conic plus external line and the imaginary triangle.

We also find the real decomposition of normal forms in each case under the action of $SL(3)$. Our result was obtained by looking at the singularity of the Hessian of each normal forms.

The main result is the following table that shows the rank and border rank of ternary cubics on $\mathbb{R}$, up to $SL(3)$-action.

**Theorem 2.1.** Ranks and border ranks of real cubics, up to $SL(3)$, are as in Table 1.

| Description                  | normal form                      | $r_k^R$ | $r_k^G$ | Hessian up to scalar               |
|------------------------------|----------------------------------|---------|---------|------------------------------------|
| 1)triple line                | $x^3$                            | 1       | 1       |                                    |
| 2)imaginary concurrent lines | $x(x^2 + y^2)$                    | 2       | 2       |                                    |
| 3)real concurrent lines      | $x(x^2 - y^2)$                    | 3       | 3       |                                    |
| 4)double line+line           | $x^2y$                           | 3       | 2       |                                    |
| 5)imaginary conic+line       | $(x^2 + y^2 + z^2)x$              | 4       | 4       | $-x(-3x^2 + y^2 + z^2)$ real conic+ext. line |
| 6)real conic+external line   | $(x^2 + y^2 - z^2)z$              | 5       | 4       | $-z(x^2 + y^2 + 3z^2)$ im. conic+line |
| 7)real conic+secant line     | $(x^2 + y^2 - z^2)y$              | 4       | 4       | $y(x^2 - 3y^2 - z^2)$ real conic+sec. line |
| 8)real conic+tangent line    | $(x^2 + y^2 - z^2)(y - z)$        | 5       | 3       | $(y - z)^3$ triple line            |
| 9)real Fermat (Hesse pencil for $\lambda = 0$) | $x^3 + y^3 + z^3$ | 3       | 3       | $xyz$ real triangle                |
| 10)imaginary Fermat (Hesse pencil for $\lambda = 1$) | $x^3 + y^3 + z^3 + 6xyz$ | 4       | 3       | $x^3 + y^3 + z^3 - 3xyz$ imaginary triangle |
| 11)Hesse pencil for $\lambda \neq -\frac{1}{2}, 0, 1$ | $x^3 + y^3 + z^3 + 6\lambda xyz$ | 4       | 4       | $-\lambda(x^3 + y^3 + z^3)$ +$(1 + 2\lambda^3)xyz$ Hesse pencil |
| 12)imaginary triangle (Hesse pencil for $\lambda = -\frac{1}{2}$) | $x^3 + y^3 + z^3 - 3xyz$ | 5       | 4       | $x^3 + y^3 + z^3 - 3xyz$ imaginary triangle |
| 13)cusp                      | $y^2z - x^3$                      | 4       | 3       | $xyz$ double line +line            |
| 14)nodal cubic               | $x^3 + y^3 + 6xyz$                | 4       | 4       | $xyz + x^3 + y^3$ nodal cubic      |
| 15)cubica punctata           | $y^2z - x^3 + x^2z$               | 4       | 4       | $3xy^2 - x^2z - y^2z$ cubica punctata |
| 16)real triangle             | $xyz$                            | 4       | 4       | $xyz$ real triangle                |

**Table 1:** Ranks and border ranks of ternary cubics on $\mathbb{R}$

**Proof.** The proof of theorem 2.1 will be divided in the following sections where
each case is settled. □

It follows from theorem 2.1 that 4 is the only typical rank for real ternary cubics. This has been proved also in [1].

The following theorem and Table 2 are in [12], [11], and appeared first in [15]. We modified the canonical form of orbit 9) of the Table 2) in order to have uniform notation with the real case and we correct a small misprint that appeared in errata corrigè in Landsberg web page.

**Theorem 2.2.** (L.-T. [12]) The ranks and border ranks of cubic curves over \( \mathbb{C} \) are as in Table 2 (§96 in [15]).

| Description            | Normal form | \( r_k^C \) | \( r_k^H \) | Hessian                        | Correspond. Real Cases |
|------------------------|-------------|-------------|-------------|--------------------------------|------------------------|
| 1) triple line         | \( x^3 \)   | 1           | 1           | 1                              |                        |
| 2) three concurrent line | \( xy(x + y) \) | 2           | 2           | 2, 3                           |                        |
| 3) double line + line  | \( x^2y \)  | 3           | 2           | 4                              |                        |
| 4) conic + secant line | \( x(x^2 + yz) \) | 4           | 4           | conic + secant line | 5, 6, 7                |
| 5) conic + tangent line| \( y(x^2 + yz) \) | 5           | 3           | triple line                    | 8                      |
| 6) irreducible Fermat  | \( y^2z - x^3 - z^3 \) | 3           | 3           | triangle                       | 9, 10                  |
| 7) nodal               | \( y^2z - x^3 - x^2z \) | 4           | 4           | nodal                          | 14, 15                 |
| 8) cusp                | \( y^2z - x^3 \) | 4           | 3           | double line + line            | 13                     |
| 9) smooth for \( \lambda \neq -\frac{1}{2}, 0, 1 \) | \( x^3 + y^3 + z^3 + 6\lambda xyz \) | 4           | 4           | smooth                         | 11                     |
| 10) triangle           | \( xyz \)   | 4           | 4           | triangle                       | 12, 16                 |

Table 2: Ranks and border ranks of plane cubic curves on \( \mathbb{C} \)

3 Apolarity

Let \( V \) be a real vector space of dimension \( n + 1 \) and \( V^\vee \) its dual space.

Following [7] a homogeneous form \( \Phi \in S^k(V) \) is apolar to a homogeneous form \( F \in S^d(V^\vee) \) if \( \Phi \) belong to the kernel of following map:

\[
ap_k^F : S^k(V) \rightarrow S^{d-k}(V^\vee)
\]
given by

\[
\Phi \mapsto P_{\Phi}(F)
\]

where \( P_{\Phi}(F) = \Phi \cdot F \) is the contraction operator.

This map is called the apolarity map. The kernel of the apolarity map is the linear space (denoted by \( AP_k \) in [8]) of apolar to \( F \) of degree \( k \).

**Definition 3.1.** (cfr. [10]) The matrix \( \text{Cat}(F) \) of the above linear map \( ap_k^F \) with respect to two basis of monomials of \( S^k V \) and \( S^{d-k}(V^\vee) \) is called the \( k \)-th catalecticant matrix of the homogeneous form \( F \) and if \( n = 2k \) the determinant of this
matrix is the catalecticant matrix of \( F \).

In particular, for \( \dim V = 2 \), that is for homogeneous polynomials over \( \mathbb{P}^1 \), after identification \( \wedge^2 V = \mathbb{R} V \) can be identified with its dual space \( V' \) and if \( f = (ax + by)^d \in S^d V \) and \( g = (cx + dy)^d \in S^d V \) the contraction operator of \( f \) and \( g \) will be \((ad - bc)^d \) up to scalar and this extends by linearity to every \( f, g \in S^d V \).

Let \( f \) be a nonsingular cubic in the projective plane \( \mathbb{P}^2(\mathbb{R}) \), defined by a homogeneous cubic equation \( f(x, y, z) = 0 \).

To find the degenerate polar conic of \( f \), let us write the equation of the polar conic \( P_Y(f) \) with respect to the point \( Y = (x_0, y_0, z_0) \) as:

\[
P_Y(f) = x_0 \frac{\partial f}{\partial x} + y_0 \frac{\partial f}{\partial y} + z_0 \frac{\partial f}{\partial z}.
\]

This polar conic splits as a pair of lines if

\[
H(f) = \begin{vmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\
\frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\
\frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2}
\end{vmatrix} = 0.
\]

So there are infinitely many points \( Y \) such that the polar conics degenerate in two lines and the locus of such points is given by the above equation that is a cubic curve called the Hessian of \( f \).

**Definition 3.2.** The Hessian curve \( H(f) \) of a plane cubic curve \( f = 0 \) is the plane cubic curve defined by the equation \( H(f) = 0 \), where \( H(f) \) is the determinant of the matrix of the second partial derivatives of \( f \).

The inflection points of \( f = 0 \) are the nine points of \( \{f = 0\} \cap \{H(f) = 0\} \).

The Hessian of a cubic polynomial \( F \) is a covariant of \( F(\mathbb{R}) \).

## 4 Table of \( \text{SL}(2) \)-orbits of real binary cubics

Consider the catalecticant matrix of the cubic binary form \( f = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 \) that can be written as

\[
\begin{pmatrix}
A & B & C \\
B & C & D
\end{pmatrix}
\]

and the discriminant of \( f \)

\[
\Delta(f) = \begin{vmatrix}
A & C \\
B & D
\end{vmatrix}^2 - 4 \begin{vmatrix}
B & C \\
C & D
\end{vmatrix} \begin{vmatrix}
A & B \\
B & C
\end{vmatrix}.
\]

Over \( \mathbb{R} \), the discriminant \( \Delta \) is positive or negative corresponding to \( f \) having one real root or three real distinct roots, respectively.

The table over \( \mathbb{R} \) is the following (see Table 3):

**Proposition 4.1.** Cases 1,2,3,4 of Table 1 are settled.
Table 3: Ranks of binary cubics over \( \mathbb{R} \)

| Description | normal form | rk | rk catalec | \( \Delta \) |
|-------------|-------------|----|------------|-----|
| 1) \( \mathbb{P}^3 \setminus \text{Tan}(C_3)_+ \) | \( x^3 \) | 1 | 1 | |
| 2) \( \mathbb{P}^3 \setminus \text{Tan}(C_3)_+ \) | \( x^3 + y^3 \) | 2 | 2 | > 0 |
| 3) \( \mathbb{P}^3 \setminus \text{Tan}(C_3)_- \) | \( x(x^2 - y^2) \) | 3 | 2 | < 0 |
| 4) \( \text{Tan}(C_3) \setminus C_3 \) | \( xy \) | 3 | 2 | 0 |

5 Real De Paolis algorithm

There is an algorithm, which dates back to De Paolis, that gives an useful method to find a decomposition of a general plane cubic curve as a sum of at most 4 cubes when the first cube \( l_0 \) is given and \( l_0 \) must be such that \( l_0 \cap H(f) \) is given by three real points. Let \( C_3 \) a real cubic curve defined by a cubic polynomial \( F \in \mathbb{S}_3^3 \mathbb{R}^3 \).

It is known that (classic theorem):

**Theorem 5.1.** (\[9\] Book 3, Cap.III) A nonsingular real cubic has exactly three real inflection points and these points are collinear.

**Proposition 5.2.** Let \( F \) be a real ternary cubic such that \( l_0 \cap H(F) \) consists of three distinct points \( P_1, P_2, P_3 \). Then there are defined three real lines \( l_i \ni P_i \) and three real scalar \( c_i \) (\( i=1,2,3 \)) such that \( F = \sum_{i=0}^{3} c_i l_i^3 \).

The algorithm to find this decomposition is described in the proof.

**Proof.** The singular point of a real singular conic is always real.

The algorithm is:

**INPUT** \( F \) a real plane cubic and \( l_0 \) satisfying the assumptions.

\( l_0 \) a line such that \( l_0 \cap H(F) \) consist of three distinct points.

**COMPUTE** \( l_0 \cap H(F) = \{P_1, P_2, P_3\} \).

**COMPUTE** \( Q_1 \) the singular points of the polar conic \( P_{P_i}(F) \) for \( i=1,2 \).

**COMPUTE** \( l_1 = \langle P_1, Q_2 \rangle, l_2 = \langle P_2, Q_1 \rangle, l_3 = \langle P_3, Q_1 \rangle \).

**SOLVE** the linear system \( F = \sum_{i=0}^{3} c_i l_i^3 \).

**OUTPUT** lines \( l_1, l_2, l_3 \) and numbers \( c_i \in \mathbb{R}, i=0,1,2,3 \), such that \( F = \sum_{i=0}^{3} c_i l_i^3 \), indeed

\[
\begin{align*}
P_{P_1}(F) &= c_2 l_2^3 + c_3 l_3^3 \quad \text{hence} \quad Q_1 \in l_2, l_3 \\
P_{P_2}(F) &= c_1 l_1^3 + c_3 l_3^3 \quad \text{hence} \quad Q_2 \in l_1, l_3 \\
P_{P_3}(F) &= c_1 l_1^3 + c_2 l_2^3 \quad \text{hence} \quad Q_3 \in l_1, l_2
\end{align*}
\]

moreover the points \( Q_i \) are real, because singular points of a real singular conic are always real, hence the lines \( l_i \) are real.

This algorithm tells us that for real plane cubics there is only one typical rank which is 4.

\( \Box \)
5.1 Hesse pencil

Every smooth plane cubic is projectively equivalent to a member of the Hesse pencil

\[ F_\lambda = x^3 + y^3 + z^3 + 6\lambda xyz = 0. \]  

(4)

The nine base points of the pencil are the flexes of every smooth member of the family. Three base points are reals, namely \((1, -1, 0), (1, 0, -1), (0, 1, -1)\), moreover there are three pairs of conjugate base points. The Hessian of each member of the Hesse pencil is still a member in the Hesse pencil. There are four singular members of the Hesse pencil, for \(\lambda = \infty\) (real triangle) and for \(\lambda = -\frac{1}{2}, -\frac{1}{2}, -\frac{\tau^2}{2}\) (imaginary triangle composed by a real line and a pair of complex conjugate lines), where \(\tau\) is a primitive cube root of unity. Working on real numbers, we will consider just the value \(\lambda = -\frac{1}{2}\).

We apply De Paolis algorithm to the Hesse pencil (see Figure 4 for the general pencil and Figure 2 and 3 for the pencil with two components and one component respectively):

\[ F_\lambda = x^3 + y^3 + z^3 + 6\lambda xyz = 0 \]  

(5)

with the condition of non singularity \(1 + 8\lambda^3 \neq 0\).

\[ S = \lambda - \lambda^4 = \lambda(1 - \lambda)(\lambda^2 + \lambda + 1) \]
Figure 2: two components $\lambda > -\frac{1}{2}$

Figure 3: one component $\lambda < -\frac{1}{2}$

Figure 4: Hesse pencil
(16) is, up to scalar, the invariant of degree four of plane cubics. The other invariant derived from the invariant $S$ is an invariant of the sixth order in the coefficients, that for the canonical form is

$$T = 1 - 20\lambda^3 - 8\lambda^6$$

up to scalar, so for $S = 0$ the curve is an equianharmonic cubic and it is in the orbit of the Fermat cubic, only if $\lambda = 0$, or $\lambda = 1$ on the real numbers. The discriminant for the Hesse pencil (4) is (see [14] pag. 189 where is denoted by $R$)

$$\Delta = T^2 + 64S^3 = (1 + 8\lambda^3)^3.$$ (6)

Proposition 5.3. Let $F$ a smooth real cubic curve with $S = 0$. The discriminant $\Delta = T^2 + 64S^3$ is different from zero for $\lambda \neq \frac{-1}{2}$ and if $T \geq 0$ then $rk_R(F) = 3$ and this is the case 9) of Table 1 and if $T \leq 0$ then $rk_R(F) = 4$ and this is the case 10) of Table 1.

Proof. If $S = 0$ then the cubic is a sum of three independent linear powers. There are two cases:

1. all the linear forms are real
2. one is real and two complex conjugate

The second case is given by the cubic form

$$f = x^3 + (y + iz)^3 + (y - iz)^3 = x^3 + 2y^3 - 6yz^2 = x^3 - (y + z)^3 - (y - z)^3 + 4y^3$$

with $rk_R(f) = 4$ and $rk_C(f) = 3$. □

Proposition 5.4. The Waring decomposition of Hesse pencil for $\lambda \neq \frac{-1}{2}, 0$ is

$$F_\lambda = c_0(x+y+z)^3 + c_1((1+\lambda)x-\lambda y-\lambda z)^3 + c_2(-\lambda x+(1+\lambda)y-\lambda z)^3 + c_3(-\lambda x-\lambda y+(1+\lambda)z)^3$$

where $c_i$ are described in the proof. For $\lambda = 0$ we have case 9) of Table 1.

Proof. In this proof we apply De Paolis algorithm. The Hessian of (4) is again of this form, that is a curve of the pencil: we have the equation

$$H(F_\lambda) = -\lambda(x^2 + y^2 + z^2) + (1 + 2\lambda^2)xyz = 0$$

and we can choose three real collinear flexes as $P_1 = (0, 1, -1), P_2 = (1, 0, -1), P_3 = (1, -1, 0)$. This flexes belong to the line

$$l_0 = x + y + z = 0.$$ 

Compute the equation of the polar conics $P_{i}(F_\lambda)$ for $i=1,2,3$:

$$P_{P_1}(F_\lambda) = 3y^2 + 6\lambda xz - 3z^2 - 6\lambda xy = 0$$

$$P_{P_2}(F_\lambda) = 3x^2 + 6\lambda yz - 3z^2 - 6\lambda xy = 0$$
so we get three singular points

\[ Q_1 = (1, \lambda, \lambda) \]
\[ Q_2 = (\lambda, 1, \lambda) \]
\[ Q_3 = (\lambda, \lambda, 1) \]

Solving the linear system

\[ F_{\lambda} = \sum_{i=0}^{3} c_i \lambda^i \]

we get the value of the coefficients \( c_i \).

The solution of this system is:

\[ c_0 = \frac{\lambda(\lambda^2 + \lambda + 1)}{(2\lambda + 1)^2} \]

and

\[ c_1 = c_2 = c_3 = \frac{1}{(2\lambda + 1)^2} \]

In conclusion we settle the cases 9,10,11 of Table 1.

6 Union of conic and non tangent line

6.1 Union of imaginary conic and line

The cubic has the equation \( F = (x^2 + y^2 + z^2)x \) and its Hessian is \( H(F) = (9x^2 - y^2 - 3z^2)(8x) \).

In this case

\[ F_z = 2zx = \frac{1}{2}(z + x)^2 - \frac{1}{2}(z - x)^2 \]

hence

\[ F = \frac{1}{6}[(z + x)^3 - (z - x)^3] + \phi(x, y) \]

where \( \phi(x, y) = x(x^2 + y^2) \).

A decomposition of \( F \) is

\[ F = \frac{1}{6}[(z + x)^3 - (z - x)^3] + \frac{1}{2}\left\{ \frac{1}{3}\sqrt{2}(\sqrt{2}x - y)^3 - \frac{1}{3}\sqrt{2}(-\sqrt{2}x - y)^3 \right\} \]

so

\[ \text{rk}_R(F) \leq 4 \]

and for \([12]\) \( \text{rk}_C(F) \geq 4 \), then \( \text{rk}_R(F) = 4 \).

The case 5) of Table 1 is settled.
6.2 Union of real conic and secant line

In this case the cubic is \( F = (x^2 + y^2 - z^2)y \) and the Hessian is \( H(F) = 8y(x^2 - 3y^2 - z^2) \).

In this case \( F_z = -2zy \)

and like the previous case we have the decomposition

\[
y(y^2 + xz) = \frac{1}{96}(4y(x + z)^3 + (4y - x - z)^3 - 2(2y + x - z)^3 - 2(2y - x + z)^3)
\]

so

\[ r_k(R(F)) \leq 4 \]

and for \( r_k(C(F)) \geq 4 \) then \( r_k(R(F)) = 4 \).

The case 7) of Table 1 is settled.

6.3 Union of a real conic and external line: the new case of \( r_k_R = 5 \)

In this case the cubic is \( F = (x^2 + y^2 - z^2)z \)

and the Hessian of \( F \) is

\[ H(F) = -8z(x^2 + y^2 + 3z^2), \]

that is the Hessian cubic curve \( H(F) = 0 \) is a imaginary conic plus a line.

We can write

\[ F = x^2z + z(y^2 - z^2) = x^2z + \varphi(y, z). \]

Now

\[ x^2z = \frac{1}{6}(x + z)^3 - (x - z)^3 - \frac{1}{3}z^3 \]

so

\[ x^2z + z(y^2 - z^2) = \frac{1}{6}(x + z)^3 - (x - z)^3 - \frac{4}{3}z^3 + zy^2 \]

and \( \varphi(y, z) = -\frac{4}{3}z^3 + zy^2 \).

But \( \varphi(y, z) \) is a binary cubic form with three real roots because

\[ zy^2 - \frac{4}{3}z^3 = z(y - \frac{2}{\sqrt{3}z})(y + \frac{2}{\sqrt{3}z}) \]

and then

\[ r_k[\varphi(y, z)] = 3. \]

Indeed we have

\[ y^2z = \frac{1}{6}(y + z)^3 - (y - z)^3 - 2z^3 \]

so

\[ zy^2 - \frac{4}{3}z^3 = \frac{1}{6}(y + z)^3 - (y - z)^3 - \frac{1}{3}z^3 - \frac{4}{3}z^3 \]
that is
\[ zy^2 - \frac{4}{3}z^3 = \frac{1}{6}(y + z)^3 - \frac{1}{6}(y - z)^3 - \frac{5}{3}z^3 \]
and finally the decomposition
\[ F = \frac{1}{6}\left((x + z)^3 - (x - z)^3\right) + \frac{1}{6}(y + z)^3 - \frac{1}{6}(y - z)^3 - \frac{5}{3}z^3 \]
so
\[ rk_R(F) \leq 5. \]
Now we have to prove that the rank of the above cubic can not be smaller than 5.
In fact, we have:

**Theorem 6.1.** The rank of the reducible cubic given by a real conic plus an external line is 5, that is
\[ rk_R(x^2 + y^2 - z^2)z = 5. \]

*Proof.* Suppose
\[ F = (x^2 + y^2 - z^2)z = l_1^3 + l_2^3 + l_3^3 + l_4^3 \]
where \( l_i, i = \{1, 2, 3, 4\} \) are linear real forms. Let \( Q = l_1 \cap l_2 \) be the point of intersection of the two lines \( l_1 \) and \( l_2 \). Then, up to scalar, the polar conic of \( F \) with respect to \( Q \)
\[ P_Q(F) = l_3^3 + l_4^3 \]
is necessarily singular.
Denote \( L \) the external line \( z = 0 \). Then the point \( Q \in H(F) \) and \( Q \in L = \{ z = 0 \} \), for the particular form of the Hessian, which in this case is
\[ H(F) = -8z(x^2 + y^2 + 3z^2) \]
a imaginary conic plus a line. Moreover, for the same argument, all the intersections \( l_i \cap l_j \in L = \{ z = 0 \} \) so there are two possibilities:
1) the line \( L = \{ z = 0 \} \) is one of the \( l_i, i = \{1, 2, 3, 4\} \) then
\[ \lambda F - z^3 \]
is a Fermat cubic curve for some \( \lambda \), or
2) the four lines are concurrent in \( \tilde{Q} \in L \) such that \( P_{\tilde{Q}}(F) \equiv 0 \).
The second case is impossible because \( F \) should be a cone with vertex in \( Q \). The first case also is impossible because then the Aronhold invariant \( S (\text{[8]}) \) should satisfy
\[ S(\lambda F - z^3) = 0. \]
This is an equation of degree four in \( \lambda \) with only root \( \lambda = 0 \). Indeed,
\[ S(\lambda F - z^3) = \lambda^4 \]
up to scalar, (see [13] where the classical Aronhold invariant \( S \) is expressed as a pfaffian).
The case 6) of Table 1 is settled. \( \Box \)
6.4 Imaginary triangle

This is the other new case such that the rank over \( \mathbb{R} \) is 5.
The proof is like the previous one: if the real rank is four we can choose 4 points
on the real side of the triangle.
Let
\[
F = x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 - xy - xz + y^2 - yz + z^2).
\]
Since the Hessian of \( F \) coincides with \( F \) itself, we may repeat the argument of
proof of theorem 6.1 concluding that there are two possibilities:
1) \( \lambda F - (x + y + z)^3 \)
is a Fermat cubic for some \( \lambda \).
2) the four lines are concurrent in \( \hat{Q} \in L = (x + y + z) = 0 \) such that \( P_{\hat{Q}}(F) \equiv 0 \).
The first case is excluded because the Aronhold invariant \( S \) for this pencil is
\[
S(\lambda(x^3 + y^3 + z^3 - 3xyz) - (x + y + z)^3) = \lambda^4.
\]
Also the second case is excluded because \( F \) is not a cone. Then

**Theorem 6.2.** The rank of the imaginary triangle is 5, that is
\[
\text{rk}_{\mathbb{R}}(x^3 + y^3 + z^3 - 3xyz) = 5.
\]
The case 12) is settled.

6.5 Nodal cubic

Every plane cubic curve with a real node is projectively equivalent to the cubic
\[
F = x^3 + y^3 - 3xyz = 0.
\]
This is a famous cubic curve called “Folium of Descartes”.
It has a double point in \((0,0,1)\) and there has a node with tangents \( x = 0 \) and \( y = 0 \). The Hessian is \((x^3 + y^3 + xyz)(-54)\). In this case
\[
F_x = 3x^2 - 3yz \\
F_y = 3y^2 - 3xz \\
F_z = -3xy
\]
Let
\[
\alpha F_x + \beta F_y + \gamma F_z = 0
\]
be the equation of the polar conic with respect to the point \((\alpha, \beta, \gamma)\). This is a
reducible conic if
\[
A = \begin{pmatrix}
3\alpha & -\gamma & -\beta \\
-\gamma & \beta & -\frac{\alpha}{2} \\
-\beta & -\frac{\alpha}{2} & 0
\end{pmatrix}
\]
We deduce that \( \det A = 0 \) if \((\alpha, \beta, \gamma) = (1,-1,0)\). So the pencil is
\[
3x^3 - 3yz - 3y^2 + 3xz
\]
that factors as
\[
3(x + y + z)(x - y).
\]
So we can write the polar of \( F \) at the point \((1,-1,0)\) as:
\[
F_x - F_y = 3\left((x + y + z)(x - y)\right) = 3\left(\frac{1}{4}(2x + z)^2 - (2y + z)^2\right)
\]
because of the identity
\[
ab = \frac{(a + b)^2}{4} - \frac{(a - b)^2}{4}.
\]
So we have, integrating with respect to \( x \) and \( y \),
\[
F = \frac{1}{8}\left(2x + z\right)^3 + \left(2y - z\right)^3 + \left\{\text{function of two variables}\right\}.
\]
To find this function let us write the equality
\[
x^3 + y^3 - 3xyz = \frac{1}{8}\left((2x + z)^3 + (2y + z)^3\right) - z\left(\frac{3}{2}x^2 + \frac{3}{4}xz + \frac{1}{4}z^2 + \frac{3}{2}y^2 + \frac{3}{4}yz + 3xy\right).
\]
Let \( g \) be this cubic form; it depends on two essential variables, namely \( z \) and \( h = x + y \).
We have
\[
g(z, x + y) = g(z, h) = z\left(\frac{z^2}{4} + \frac{3}{4}zh + \frac{3}{2}h^2\right)
\]
and the discriminant of the polynomial of degree 2 into the square bracket
\[
z^2 + 3zh + 6h^2
\]
is
\[
\Delta = 9 - 6 \cdot 4 < 0
\]
so that this quadratic polynomial has rank 2 and the cubic nodal form has rank 4.
Then
\[
g(z, h) = \frac{2}{3} \left(\frac{1 + \sqrt{5}}{4} - z + h\right)^3 - 2\left(\frac{1 + \sqrt{5}}{4} - z - h\right)^3
\]
and
\[
x^3 + y^3 - xyz = \frac{1}{8}\left((2x + z)^3 + (2y + z)^3\right) + 2\left(\frac{1 - \sqrt{5}}{4} - z + (x + y)\right)^3 - 2\left(\frac{1 + \sqrt{5}}{4} - z - (x + y)\right)^3.
\]
Then
\[
\text{rk}_R(F) \leq 4
\]
and again for [12] we conclude that \( \text{rk}_R = 4 \) because
\[
4 \geq \text{rk}_R(F) \geq \text{rk}_C(F) = 4.
\]
The case 14) of Table 1 is settled.
6.6 Cubica punctata

Let

\[ f = y^2z - x^3 + x^2z \]

be the so-called “cubica punctata” with a double point in the origin with two complex tangent lines \( x^2 + y^2 = 0 \) = \((x + iy)(x - iy) = 0 \).

We have

\[ f_x = x(2z - 3x) = \frac{1}{4} \left\{ (-2x + 2z)^2 - (4x - 2z)^2 \right\} \]

so

\[ f = \frac{1}{4} \left\{ \frac{-1}{2} (-2x + 2z)^3 \right\} - \frac{1}{4} \left\{ \frac{4x - 2z)^3}{3} \right\} + \phi(y, z) \]

with

\[ \phi(y, z) = y^2z + \frac{1}{6}z^3 = z(y^2 + \frac{1}{6}z^2) \]

so \( \phi \) is a binary cubic form with only one real root and for \( \text{[5]} \) has rank 2.

We get

\[ \phi(y, z) = \frac{1}{12} \left( \sqrt{2}y + z \right)^3 - \frac{1}{12} \left( \sqrt{2}y - z \right)^3 \]

so

\[ f = \frac{1}{4} \left\{ \frac{-1}{2} (-2x + 2z)^3 \right\} - \frac{1}{4} \left\{ \frac{4x - 2z)^3}{3} \right\} + \frac{1}{12} \left( \sqrt{2}y + z \right)^3 - \frac{1}{12} \left( \sqrt{2}y - z \right)^3 \]

and finally

\[ \text{rk}_R(f) \leq 4. \]

Again

\[ 4 \geq \text{rk}_R(f) \geq \text{rk}_C(f) = 4 \]

so the case 15) of Table 1 is settled.

6.7 Union of real conic plus tangent line

Let’s see the case of the conic plus tangent line where the rank is five.

\[ F = (x^2 + yz)(y - z) = x^2(y - z) + yz(y - z) \]

We have

\[ 3y^2z - 3yz^2 = (z - y)^3 + y^3 - z^3 \]

that is \( y^2z - yz^2 \) is a sum of three cubes.

On the other hand

\[ 3x^2(y - z) = \frac{1}{2} \left\{ (x + y - z)^3 - (x - y + z)^3 - 2(y - z)^3 \right\} \]

so

\[ (x^2 + yz)(y - z) = x^2(y - z) + yz(y - z) = \frac{1}{2} \left\{ (x+y-z)^3-(x-y+z)^3-2(y-z)^3 \right\} + \frac{1}{3} \left\{ (z-y)^3+y^3-z^3 \right\} \]
but 
\[ \frac{1}{3}(z - y)^3 - (y - z)^3 = -\frac{4}{3}(y - z)^3 \]
so the cubic is a sum of 5 cubes and we have the decomposition
\[ (x^2 + yz)(y - z) = \frac{1}{2}(x + y - z)^3 - \frac{1}{2}(x - y + z)^3 - \frac{4}{3}(y - z)^3 + y^3 - z^3. \]
We get
\[ r_{k_R}(F) \leq 5 \]
and again \( r_{k_R}(F) \geq r_{k_C}(F) \geq 5 \).
The case 8) of Table 1 is settled.

The case 13) is also settled in [12] where the decomposition is given. We have
\[ y^2z - x^3 = \frac{1}{6} \left( (y + z)^3 + (y - z)^3 - 2x^3 \right) - x^3 \]
The last case of the triangle is settled in [12]. The decomposition is:
\[ xyz = \frac{1}{24} \left( (x + y + z)^3 - (-x + y + z)^3 - (x - y + z)^3 - (x + y - z)^3 \right). \]
The case 16) is settled.
This concludes the proof of Theorem 2.1.

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