ELLIPTIC ANALOGUE OF IRREGULAR PRIME NUMBERS FOR THE $p^n$-DIVISION FIELDS OF THE CURVES $y^2 = x^3 - (s^4 + t^2)x$

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Abstract. A prime number $p$ is said to be irregular if it divides the class number of the $p$-th cyclotomic field $\mathbb{Q}(\zeta_p) = \mathbb{Q}(\mathbb{G}_m[p])$. In this paper, we study its elliptic analogue for the division fields of an elliptic curve. More precisely, for a prime number $p \geq 5$ and a positive integer $n$, we study the $p$-divisibility of the class number of the $p^n$-division field $\mathbb{Q}(E[p^n])$ of an elliptic curve $E$ of the form $y^2 = x^3 - (s^4 + t^2)x$. In particular, we construct a certain infinite subfamily consisting of curves with novel properties that they are of Mordell-Weil rank 1 and the class numbers of their $p^n$-division fields are divisible by $p^{2n}$. Moreover, we can prove that these division fields are not isomorphic to each other. In our construction, we use recent results obtained by the first author.

1. Introduction and main results

1.1. Background. For every integer $N$ and an abelian group $A$, we denote the subgroup of $A$ consisting of elements whose orders divide $N$ by $A[N]$.

Let $N$ be an integer, $K$ be a field, $\overline{K}$ be a fixed algebraic closure of $K$, and $\mathfrak{A}$ be a commutative group scheme of finite type (e.g. an elliptic curve $E$) defined over $K$. For simplicity, suppose that $K$ is perfect. Then, we have a natural Galois representation $\rho_{K,\mathfrak{A},n} : \text{Gal}(\overline{K}/K) \to \text{Aut}_K(\mathfrak{A}(\overline{K})[N])$.

We define the $N$-division field $K(\mathfrak{A}[N])$ of $\mathfrak{A}$ over $K$ by the fixed field $K(\mathfrak{A}[N]) := K^{\text{Ker}(\rho_{K,\mathfrak{A},n})}$.

One of the most classical examples of the $N$-division field is a cyclotomic field $\mathbb{Q}(\zeta_N)$, which corresponds to the case where $K = \mathbb{Q}$ and $\mathfrak{A}$ is the multiplicative group scheme $\mathbb{G}_m$. In this setting, a prime number $p$ is called irregular if the class number of $\mathbb{Q}(\zeta_p) = \mathbb{Q}(\mathbb{G}_m[p])$ is divisible by $p$ because Kummer’s approach to Fermat’s Last Theorem breaks down for irregular prime numbers [40]. It has been an open question over 150 years whether there exist infinitely many regular (i.e., not irregular) prime numbers. On the other hand, it has been well-known that there exist infinitely many irregular prime numbers. We refer the reader to Carlitz’s elegant proof [3], which has a qualitative refinement [26]. Moreover, Ernvall [15, 16] generalized the infinitude of prime numbers $p$ dividing the class numbers of $\mathbb{Q}(\mathbb{G}_m[p])$ to the infinitude of prime numbers $p$ (prime to a fixed $N$) dividing the class numbers of $\mathbb{Q}(\mathbb{G}_m[Np])$ by means of generalized Bernoulli numbers. These results lead us naturally to the following questions on elliptic analogue of irregular prime numbers.

Question 1.1. Let $E$ be an elliptic curve defined over $\mathbb{Q}$. Then, how often the class numbers of the $p$-division fields $\mathbb{Q}(E[p])$ are divisible by $p$ when $p$ runs over the prime numbers?
Recent results by Ray and Weston \cite{RayWeston} Theorems 5.1 and 5.2 give partial answers to this question. On the other hand, since there are infinitely many non-isomorphic elliptic curves over $\mathbb{Q}$, it is also natural to ask the following variant of Question \ref{Q1.3}.

**Question 1.2.** Let $p$ be a prime number. Then, how often the class numbers of the $p$-division fields $\mathbb{Q}(E[p])$ are divisible by $p$ when $E$ runs over the elliptic curves over $\mathbb{Q}$?

To Question 1.2 a recent conditional result by Ray and Weston \cite{RayWeston} Theorem 4.4] in the same paper gives an effective lower bound for the density $\delta(E_p)$ of elliptic curves $E$ for which $\text{Hom}_{\text{Gal}}(\mathbb{Q}(E[p])/\mathbb{Q})\left(\text{Cl}(\mathbb{Q}(E[p])), E[p]\right) = 0$ for each odd prime number $p$. Here and after, $\text{Cl}(K)$ denotes the ideal class group of a number field $K$. There are also unconditional results for small prime numbers $p$. For $p = 2$, we have some qualitative results to Question 1.2 e.g. each of \cite{2, 11, 21}. A similar result can be deduced by combining \cite{2} Theorem 1.4 and \cite{3} Theorem 1.1. In fact, the last result \cite{3} Theorem 1.1] is generalized for $p = 3$ and $5$ in \cite{4} and \cite{5} respectively, hence it might be possible to obtain similar $p$-divisibility results of the class numbers also for these small $p$ along this line unconditionally.

Both of the Questions \ref{Q1.1} and \ref{Q1.2} ask the stochastic aspects of the $p$-divisibility of the class number of $\mathbb{Q}(E[p])$. On the other hand, there are several recent results on the deterministic aspects of the growth of the $p$-part of $\text{Cl}(\mathbb{Q}(E[p^n]))$ along $n \to \infty$. For example, the preceding works \cite{20} Theorem 1.1], \cite{33} Theorem 1.1], and \cite{34} Theorem 1.1] give sufficient conditions on triplet $(E, p, n)$ so that the class number of $\mathbb{Q}(E[p^n])$ is divisible by a power of $p$. More precisely, all of their main results have the following form

$$\text{ord}_p(\#\text{Cl}(\mathbb{Q}(E[p^n]))) \geq 2n \text{ rank } E(\mathbb{Q}) - (\text{local contributions}).$$

In fact, these general bounds can imply the $p$-divisibility of the class number of $\mathbb{Q}(E[p])$ only if $\text{rank } E(\mathbb{Q}) \geq 2$ because of the above local contributions. Hence, if we are interested in elliptic curves $E$ such that $\text{rank } E(\mathbb{Q}) \leq 1$, we need a different approach. Moreover, it is fair to say that the condition $\text{rank } E(\mathbb{Q}) \geq 2$ is too restrictive at least hypothetically because the conjecture of Goldfeld \cite{19} and Katz-Sarnak \cite{23} §5] predicts that a half of elliptic curves over $\mathbb{Q}$ have rank 0 and the other have rank 1. Thus, it is natural to focus on an infinite family of elliptic curves $E$ such that $\text{rank } E(\mathbb{Q}) \leq 1$ especially for $p \geq 7$.

In this paper, for any arbitrarily given prime power $p^n$ with $p \geq 5$, we construct an explicit infinite family consisting of elliptic curves $E$ such that $\text{rank } E(\mathbb{Q}) = 1$ and the class numbers of $\mathbb{Q}(E[p^n])$ are divisible by $p^{2n}$.

1.2. Main results. Let $s, t \in \mathbb{Z}$ be integers such that $(s, t) \neq (0, 0)$ and $E_{s,t}$ be an elliptic curve defined by $y^2 = x^3 - (s^4 + t^2)x$. The first main result of this paper is the following:

**Theorem 1.3.** Let $p$ be a prime number and $n$ be an integer. Suppose that $p \geq 5$. Then, there exist infinitely many prime numbers of the form $l = s^4 + t^2$ with $s, t \in \mathbb{Z}$ such that $\text{rank } E_{s,t}(\mathbb{Q}) = 1$ and the class number of $\mathbb{Q}(E_{s,t}[p^n])$ is divisible by $p^{2n}$. Moreover, these number fields $\mathbb{Q}(E_{s,t}[p^n])$ and $\mathbb{Q}(E_{s',t'}[p^n])$ are isomorphic to each other if and only if the prime number $s^4 + t^2$ coincides with the other $s'^4 + t'^2$.

The proof is based on the following theorem, which is the second main result of this paper.

**Theorem 1.4.** Let $p$ be a prime number and $n$ be an integer. Suppose that $p \geq 5$, $\text{gcd}(s, t) = 1$, $st \equiv 0 \bmod p^{n+1}$, and $s^4 + t^2$ is fourth-power-free and not a square. Then, there exists a non-zero $\text{Gal}(\mathbb{Q}(E_{s,t}[p])/\mathbb{Q})$-equivariant homomorphism

$$\text{Cl}(\mathbb{Q}(E_{s,t}[p^n])) \to E_{s,t}[p^n].$$

Moreover, the class number of $\mathbb{Q}(E_{s,t}[p^n])$ is divisible by $p^{2n}$.
The above results show the divisibility of the order of the ideal class group \( \text{Cl}(\mathbb{Q}(E_{s,t}[p^n])) \). Since it admits a natural Galois action, it is also natural to study the Galois module structure of \( \text{Cl}(\mathbb{Q}(E_{s,t}[p^n])) \). For \( n = 1 \), we give the following theorem, which is our third main result.

**Theorem 1.5.** Let \( p \) be a prime number. Suppose that \( p \geq 5 \), \( t = \tau^2 \) (\( \tau \in \mathbb{Z} \)), \( s^4 + \tau^4 \) is fourth-power-free, \( st \equiv 0 \mod p^2 \), and \( E[p] \) is an irreducible \( \mathbb{F}_p[\text{Gal}(\mathbb{Q}(E_{s,t}[p])/\mathbb{Q})] \)-module. Then, there exists a surjective \( \text{Gal}(\mathbb{Q}(E_{s,t}[p])/\mathbb{Q}) \)-equivariant homomorphism

\[
\left( \frac{\text{Cl}(\mathbb{Q}(E_{s,t}[p]))}{p \text{Cl}(\mathbb{Q}(E_{s,t}[p]))} \right)_{ss} \rightarrow E_{s,t}[p^2].
\]

In particular, the class number of \( \mathbb{Q}(E_{s,t}[p]) \) is divisible by \( p^4 \).

Let us explain the outline of this paper. In §2, we review the previous work of the first author [12], which gives a lower bound for the number of Galois equivariant morphisms of \( \text{Cl}(\mathbb{Q}(E[p^n])) \) to \( E[p^n] \) under certain technical conditions on an elliptic curve \( E \). After that, in §3, we verify these conditions for our elliptic curves \( E_{s,t} \), which completes the proof of Theorem 1.4. A key point is that a recent work of [39] ensures that an obvious rational point \( P_{s,t} := (-s^2, st) \) on \( E_{s,t} \) does not fall into a subgroup \( pE_{s,t}(\mathbb{Q}) \) of \( E_{s,t}(\mathbb{Q}) \), which ensures the most important necessary condition in [12]. In §4, we deduce Theorem 1.3 from Theorem 1.4. This will be done by a collaboration of two monumental works in number theory; one is the Néron-Ogg-Shafarevich criterion [35] for good reduction of elliptic curves and the other is the Friedlander-Iwaniec’s theorem [17] for prime numbers of the form \( s^4 + t^2 \) with integers \( s, t \). Finally, in §5, we give a proof of the fact that \( \text{rank}(E(\mathbb{Q})) = 1 \) if \( s^4 + t^2 \) is a prime number congruent to 9 mod 16. This implies that for any arbitrarily given prime power \( p^n \) with \( p \geq 5 \), we can construct an explicit infinite family consisting of elliptic curves \( E \) such that \( \text{rank}(E(\mathbb{Q})) = 1 \), their \( p^n \)-division fields \( \mathbb{Q}(E[p^n]) \) are non-isomorphic to each other, and the class numbers of \( \mathbb{Q}(E[p^n]) \) are divisible by \( p^{2n} \). Although there are several preceding works (e.g. [20][33][34] mentioned above and [29][30]) on the \( p \)-divisibility of the class numbers of \( \mathbb{Q}(E[p^n]) \), as far as the authors know, there is no literature including an infinite family of elliptic curves having the above properties.

2. A SUFFICIENT CONDITION FOR \( p \)-DIVISIBILITY OF THE CLASS NUMBER

Let \( i \) be a non-negative integer. For every separable extension \( L/K \) of fields and a \( \text{Gal}(L/K) \)-module \( M \), we abbreviate the Galois cohomology group \( H^i(\text{Gal}(L/K), M) \) by \( H^i(L/K, M) \). Moreover, if \( L \) is a separable closure of \( K \), then we abbreviate \( H^i(L/K, M) \) by \( H^i(K, M) \). For general properties of Galois cohomology groups, we refer the reader to [28].

In what follows, we assume that \( K \) is a finite extension of \( \mathbb{Q} \).

Let \( \mathfrak{A} \) be a commutative group scheme of finite type defined over \( K \), and \( N \) be an integer. We are interested in the ideal class group \( \text{Cl}(K(\mathfrak{A}[N])) \). By the class field theory (see e.g. [22][28]), we have a canonical isomorphism \( \text{Cl}(K(\mathfrak{A}[N])) \simeq \text{Gal}(K(\mathfrak{A}[N])^\text{ur}/K(\mathfrak{A}[N])) \), where \( K(\mathfrak{A}[N])^\text{ur} \) denotes the maximal unramified abelian extension of \( K(\mathfrak{A}[N]) \). Therefore, in order to obtain information of \( \text{Cl}(K(\mathfrak{A}[N])) \), it is natural to study the subgroup of \( \text{Hom}(\text{Gal}(K(\mathfrak{A}[N])/K), A) \) consisting of unramified homomorphisms for several (or all if possible) finite abelian groups \( A \). Moreover, since the group \( \text{Cl}(K(\mathfrak{A}[N])) \) admits a natural action of the Galois group \( \text{Gal}(K(\mathfrak{A}[N])/K) \), it is more natural to study the subgroup

\[
\text{Hom}_{\text{Gal}(K(\mathfrak{A}[N])/K)}(\text{Gal}(K(\mathfrak{A}[N])/K), M) \simeq H^1(K(\mathfrak{A}[N]), M)^{\text{Gal}(K(\mathfrak{A}[N])/K)}
\]
consisting of unramified elements for several (or all if possible) finite $\text{Gal}(K(\mathfrak{A}[N])/K)$-modules $M$. In particular, it is interesting, at least as a first step, to carry out the above plan for the $\text{Gal}(K(\mathfrak{A}[N])/K)$-module $M = \mathfrak{A}[N]$.

By the definition of the $\text{Gal}(K(\mathfrak{A}[N])/K)$-module $\mathfrak{A}[N]$, we have an exact sequence of $\text{Gal}(\overline{K}/K)$-modules

$$0 \rightarrow \mathfrak{A}[N] \rightarrow \mathfrak{A}(\overline{K}) \xrightarrow{N} \mathfrak{A}(\overline{K}).$$

In what follows, we assume that the last map $N : \mathfrak{A}(\overline{K}) \rightarrow \mathfrak{A}(\overline{K})$ is surjective. Then, the above exact sequence yields a short exact sequence of the Galois cohomology groups, the so-called Kummer sequence:

$$0 \rightarrow \frac{\mathfrak{A}(K)}{N\mathfrak{A}(K)} \xrightarrow{\kappa_{\mathfrak{A},N}} H^1(K, \mathfrak{A}[N]) \rightarrow H^1(K, \mathfrak{A}[N]) \rightarrow 0.$$

On the other hand, the following inflation-restriction exact sequence is one of the most elementary tools in the theory of Galois cohomology groups:

$$1 \rightarrow H^1(K(\mathfrak{A}[N])/K, \mathfrak{A}[N]) \xrightarrow{\text{inf}} H^1(K, \mathfrak{A}[N]) \xrightarrow{\text{res}} H^1(K(\mathfrak{A}[N]), \mathfrak{A}[N])^{\text{Gal}(K(\mathfrak{A}[N])/K)} \rightarrow \cdots$$

Therefore, we obtain a natural composite homomorphism induced by the Kummer map $\kappa$ and the restriction map

$$\frac{\mathfrak{A}(K)}{N\mathfrak{A}(K)} \xrightarrow{\kappa_{\mathfrak{A},N}} H^1(K, \mathfrak{A}[N]) \rightarrow \text{Hom}_{\text{Gal}(K(\mathfrak{A}[N])/K)}(\text{Gal}(\overline{K}/K(\mathfrak{A}[N])), \mathfrak{A}[N]).$$

This suggests that we can construct non-zero elements in the right-most side by using non-zero elements in the left-most side. In this view point, it is clear that $H^1(K(\mathfrak{A}[N])/K, \mathfrak{A}[N])$ plays a role as the obstruction for the transformation of the non-zero elements in $\mathfrak{A}(K)/N\mathfrak{A}(K)$ to non-zero homomorphisms.

On the other hand, since we are interested in unramified homomorphisms of $\text{Gal}(\overline{K}/K(\mathfrak{A}[N]))$ to $\mathfrak{A}[N]$, we want to check that which elements in $\mathfrak{A}(K)/N\mathfrak{A}(K)$ correspond to unramified homomorphisms. This motivates us to define

$$\mathfrak{A}(K)_{\text{ur},N} = \text{Ker} \left( \mathfrak{A}(K) \rightarrow \prod_v \frac{\mathfrak{A}(K_{v}^{\text{ur}})}{N\mathfrak{A}(K_{v}^{\text{ur}})} \right).$$

Along this line, in a forthcoming paper [12], the first author studied some conditions under which $\text{res}(\kappa(\mathfrak{A}(K)_{\text{ur},N})) \neq 0$ for a fixed elliptic curve $\mathfrak{A} = E$ defined over $K = \mathbb{Q}$ and a fixed prime power $N = p^n$. The results are summarized as follows:

**Theorem 2.1** ([12]). *Let $E$ be an elliptic curve defined over $\mathbb{Q}$, $p$ be a prime number, and $n$ be a positive integer. Suppose that the following conditions hold.*

1. $E$ has good reduction at $p$.
2. $p \geq 5$ and the denominator of $j(E)$ is $p$-th power free.
3. $H^1(\mathbb{Q}(E[p^n])/\mathbb{Q}, E[p^n]) = 0$.

*Then, we have $E(\mathbb{Q})_{\text{ur},p^n} = E(\mathbb{Q}) \cap p^nE(\mathbb{Q}_p)$. Moreover, if we denote the length of $E(\mathbb{Q})_{\text{ur},p^n}/p^nE(\mathbb{Q})$ as a $\mathbb{Z}_p$-module by $r_{\text{ur},p^n}$. Then, the following inequality holds.*

$$\text{length}_{\mathbb{Z}_p} \text{Hom}_{\text{Gal}(K(E[p^n])/\mathbb{Q})}(\text{Cl}(\mathbb{Q}(E[p^n])), E[p^n]) \geq r_{\text{ur},p^n}.$$
Remark 2.4. (1) In [24], the triples $(\mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})])$ whose domain is the semi-simplification of $\mathbb{Q}(E[p])/\mathbb{Q})$ for which the first condition holds for $\mathbb{Q}(E[p])/\mathbb{Q})$, and $\mathbb{Q}(E[p])/\mathbb{Q})$ as an $\mathbb{F}_p[\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})]$-module. Then, there exists a $\mathbb{F}_p[\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})]$-equivariant surjective homomorphism

$$
\left( \frac{\mathrm{Cl}(\mathbb{Q}(E[p]))}{p \cdot \mathrm{Cl}(\mathbb{Q}(E[p]))} \right)^{\text{ss}} \to \mathbb{Q}(E[p])^{\text{ur,p}},
$$

whose domain is the semi-simplification of $\mathbb{Q}(E[p])/\mathbb{Q})/p \cdot \mathrm{Cl}(\mathbb{Q}(E[p]))$ as an $\mathbb{F}_p[\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})]$-module.

In application, the following corollary of Theorem 2.1 is valuable.

Corollary 2.3 ([24], see also [29, Lemma 2.10]). Assume the same conditions as Theorem 2.1.

1. Suppose that $E$ has a Weierstrass equation $y^2 = x^3 + ax + b$ over $\mathbb{Z}_p$ and there exists a rational point $P \in \mathbb{Q}(\mathbb{Q}) \setminus p^n E(\mathbb{Q})$ and a divisor $f_0$ of $\#(\mathbb{F}_p)$ prime to $p$ such that $x(f_0 P) \not\in \mathbb{Z}_p$ and $x(f_0 P)/y(f_0 P) \in p^n \mathbb{Z}_p$. Then, the class number of $\mathbb{Q}(E[p^n])$ is divisible by $p$.

2. Suppose that $E[p]$ is an irreducible $\mathbb{F}_p[\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})]$-module. Then, the class number of $\mathbb{Q}(E[p^n])$ is divisible by $p^{2\text{ur,p}^n}$.

Remark 2.4. (1) In [24], the triples $(E, p, n)$ for which the third condition

$$H^1(\mathbb{Q}(E[p^n])/\mathbb{Q}, E[p^n]) = 0$$

of Theorem 2.1 fails are classified in terms of isogeny on $E$ of degree $p$. In particular, the third condition holds for all $(E, p, n)$ such that $p = 13$. Moreover it holds for all $(E, 11, n)$ (resp. $(E, 7, n)$, $(E, 5, n)$) whenever $E$ is not 121c1 nor 121c2 in Cremona’s table [9] (resp. $E(\mathbb{Q})[7] = 0$, $E[5]$ is an irreducible $\mathbb{F}_5[\mathrm{Gal}(\mathbb{Q}/\mathbb{Q})]$-module and the quadratic twist of $E$ by $D = 5$ has no rational points of order 5).

(2) For the convenience of the reader, we give a sketch of the proof of Theorem 2.1. As we have already seen, the third condition is equivalent to say that the restriction map

$$H^1(\mathbb{Q}, E[p^n]/\mathbb{Q})) \cong H^1(\mathbb{Q}(E[p^n]), E[p^n]^{\text{Gal}(\mathbb{Q}(E[p^n])/\mathbb{Q})})$$

$$\cong \text{Hom}_{\text{Gal}(\mathbb{Q}(E[p^n])/\mathbb{Q})}(\mathbb{Q}(\mathbb{Q}(E[p^n]), E[p^n]))$$

is injective. The first condition ensures that the image of the set $E(\mathbb{Q})_{\text{ur,n}}$ by the mod $p^n$ Kummer map $\kappa : E(\mathbb{Q}) \to H^1(\mathbb{Q}, E[p^n])$ consists of 1-cocycle classes unramified at $p$. Finally, the second condition ensures that every 1-cocycle classes in $H^1(\mathbb{Q}, E[p^n])$ is unramified at every place $v \neq p$.

(3) Theorem 2.2 means that if one decomposes the semi-simplification of the $\mathbb{F}_p[\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})]$-module $\mathrm{Cl}(\mathbb{Q}(E[p]))/p \cdot \mathrm{Cl}(\mathbb{Q}(E[p]))$ into its irreducible components, then the module $E[p]$ appears at least $r_{\text{ur,1}}$-times. Therefore, we can obtain an explicit lower bound for the multiplicity of this special component contained in our object $\mathrm{Cl}(\mathbb{Q}(E[p]))/p \cdot \mathrm{Cl}(\mathbb{Q}(E[p]))$ as the name of Theorem 2.2 suggests.

(4) Corollary 2.3(1) gives a useful condition to prove the $p$-divisibility of the class number of $\mathbb{Q}(E[p^n])$ if one knows a specific rational point on $E$. On the other hand, Corollary 2.3(2) gives a useful condition to prove that the class number of $\mathbb{Q}(E[p^n])$ is divisible by a higher power of $p$ if $p$ one knows a better lower bound for $r_{\text{ur,p}^n}$.

Now, let us explain what the above theorems give for the proof of Theorem 1.4. Since we assume that $st \equiv 0 \pmod{p}$ and $s^4 + t^2$ is square-free, we see that $s^4 + t^2$ is prime to $p$. Hence, the first condition holds for $(E, p, n) = (E_{s,t}, p, 1)$ because $E_{s,t}$ has good reduction.

2 More precisely, every 1-cocycle is unramified at the infinite place because $p$ is odd.
outside \(2(s^4 + t^2)\). Moreover, the second condition holds for our elliptic curves \(E_{s,t}\) because \(j(E_{s,t}) = 2^6 \cdot 3^3\). Finally, we can verify that \(H^1(\mathbb{Q}, E_{s,t}[p^n]) = 0\) by checking the following facts.

(1) For \(p = 11\), \(E_{s,t} \not\cong 121c1, 121c2\) in Cremona’s table [9].
(2) For \(p = 7\), \(E_{s,t}(\mathbb{Q})[7] = 0\) by [36, Ch. X, Proposition 6].
(3) For \(p = 5\), \(E_{s,t}[5]\) is an irreducible \(\mathbb{F}_p[\text{Gal}(\mathbb{Q}(E_{s,t}[5])/\mathbb{Q})]\)-module by [13, Theorem 7], and the quadratic twist of \(E_{s,t}\) by \(D = 5\) is given by \(y^2 = x^3 - 25(s^4 + t^2)x\) and has no rational 5-torsion point by [36, Ch. X, Proposition 6].

Thus, thanks to Corollary 2.3(1), it is sufficient for the proof of Theorem 1.4 to check the followings.

(1) If \(s^4 + t^2\) is fourth-power-free and not a square, then an obvious rational point \(P_{s,t} := (-s^2, st) \in E_{s,t}(\mathbb{Q})\) does not lie in the subgroup \(pE_{s,t}(\mathbb{Q})\).
(2) If \(p\) divides exactly one of \(s\) and \(t\) and \(p^{n+1}\) divides \(st\), then \(x(2P_{s,t}) \notin p^{n+1}\mathbb{Z}_p\) and \(x(2P_{s,t})/y(2P_{s,t}) \in p^{n+1}\mathbb{Z}_p\). Here, note that since the group \(E_{s,t}(\mathbb{F}_p)\) of the modulo \(p\) rational points contains a 2-torsion point \((0,0)\), the order \(\#E_{s,t}(\mathbb{F}_p)\) is even.

In the next section, we verify these conditions.

3. Proof of Theorems 1.4 and 1.5

3.1. Primitivity of the obvious rational point. The goal of this subsection is to prove the following:

**Theorem 3.1** (cf. [14, Theorem 10.1], [18, Theorem 1.5(1)]). Let \(E_{s,t}\) be an elliptic curve defined by \(y^2 = x^3 - (s^4 + t^2)x\) with \(s, t \in \mathbb{Z}_{\geq 1}\) and \(P_{s,t} := (-s^2, st)\) be a rational point on \(E_{s,t}\). Suppose that \(s^4 + t^2\) is forth-power free and not a square. Then, the rational point \(P_{s,t}\) can be extended to a minimal system of generators of \(E_{s,t}(\mathbb{Q})\).

**Remark 3.2.** If \(s^4 + t^2\) is a prime number congruent to 9 mod 16, then \(E_{s,t}(\mathbb{Q})\) is generated by \((0,0)\) and \(P_{s,t}\). For the proof, see §5.

The proof of Theorem 3.1 is based on some inequalities on height of rational points on \(E_{s,t}\). Before the proof, we recall these inequalities with some terminologies on height.

First, we recall two kinds of heights of a rational point \(P\) on an elliptic curve \(E\).

Suppose that \(E\) is defined by \(y^2 = x^3 + ax + b\) with \(a, b \in \mathbb{Z}\). Let \(n, d \in \mathbb{Z}\) such that \(x(P) = n/d\) and \(\gcd(n, d) = 1\). Then, the *absolute logarithmic height* \(h(n/d)\) of the rational number \(n/d\) is defined by

\[
h(n/d) := \sum_{v : \text{place of } \mathbb{Q}} \log^+ |n/d|_v = \log \max \{|n|, |d|\},
\]

where \(\log^+(\alpha) := \log \max \{1, \alpha\}\), and the *naive* (or *Weil*) height \(h(P)\) of the rational point \(P\) is defined by

\[
h(P) := h(x(P)) = h(n/d).
\]

The naive height of a given point is easy to calculate, but it is non-canonical in the sense that it depends on a fixed defining equation of \(E\).

On the other hand, it is known [36, Proposition 9.1] that the limit

\[
\hat{h}(P) = \lim_{\mathrm{deg}(x) \to \infty} \frac{h(kP)}{k^2} = \frac{1}{2} \lim_{n \to \infty} \frac{\hat{h}(2^n P)}{4^n}
\]

exists and defines a positive definite quadratic form on the \(\mathbb{R}\)-vector space \(\mathbb{R} \otimes E(\mathbb{Q})\). We call the function \(\hat{h} : E(\mathbb{Q}) \to \mathbb{R}\) as the *canonical* (or *Néron-Tate*) height on \(E\).
Remark 3.3. Here, we follow the definition of the canonical height in [36, Ch. VIII.6] and [37]. It should be remarked that several authors use the other normalization of the canonical height. For example, in [10]-[14]-[18], the canonical height is defined as the twice of \( \hat{h} \). For such a discordance of the “canonical” heights (and of the local heights), the contents of [10, §4] is worth reading.

The following inequalities give bounds for the difference of these two kinds of heights.

**Theorem 3.4** ([37, Theorem 1.1], [14, Proposition 8.1]). Let \( E \) be an elliptic curve defined by \( y^2 = x^3 + ax + b \) with \( a, b \in \mathbb{Z} \), \( \Delta(E) \) be its discriminant, \( j(E) \) be its \( j \)-invariant, and \( P \neq \infty \) be a rational point on \( E \). Then, the following inequalities hold:

\[
-\frac{h(j(E))}{8} - \frac{h(\Delta(E))}{12} - 0.973 \leq \hat{h}(P) - \frac{1}{2}h(P) \leq \frac{h(j(E))}{12} + \frac{h(\Delta(E))}{12} + 1.07.
\]

In particular, if \( a = -(s^4 + t^2) \) with \( s, t \in \mathbb{Z} \) and \( b = 0 \), then we have the upper bound for the canonical height of \( P \) as follows:

\[
\hat{h}(P) \leq \frac{1}{2}h(P) + \frac{1}{4}\log(s^4 + t^2) + 2.03781.
\]

Therefore, if we obtain an explicit lower bound for the canonical height, then we can obtain an explicit lower bound for the naive height.

**Theorem 3.5** ([39, Theorem 1.2]). Let \( E_a \) be an elliptic curve defined by \( y^2 = x^3 + ax \) with a forth-power free integer \( a \) and \( Q \in E_a(\mathbb{Q}) \) be a non-torsion point. Then, we have the following estimate:

\[
\hat{h}(Q) > \frac{1}{16}\log|a| + \begin{cases} 
\frac{1}{4}\log 2 & \text{if } a > 0 \text{ and } a \equiv 1, 5, 7, 9, 13, 15 \mod 16 \\
\frac{1}{4}\log 2 & \text{if } a > 0 \text{ and either } a \equiv 20, 36 \mod 64, \\
or a \equiv 2, 3, 6, 8, 10, 11, 12, 14 \mod 16 \\
\frac{1}{8}\log 2 & \text{if } a > 0 \text{ and } a \equiv 4, 52 \mod 64 \\
\frac{1}{16}\log 2 & \text{if } a < 0 \text{ and } a \equiv 1, 5, 7, 9, 13, 15 \mod 16 \\
\frac{1}{16}\log 2 & \text{if } a < 0 \text{ and either } a \equiv 20, 36 \mod 64 \\
or a \equiv 2, 3, 6, 8, 10, 11, 12, 14 \mod 16 \\
\frac{1}{16}\log 2 & \text{if } a < 0 \text{ and } a \equiv 4, 52 \mod 64
\end{cases}
\]

Now, we can prove Theorem 3.1.

**Proof of Theorem 3.1.** Since we assume that \( s^4 + t^2 \) is not a square, [39, Ch. X, Proposition 6.1(a)] shows that

\[ E_{s,t}(\mathbb{Q})_{\text{tors}} = \langle (0, 0) \rangle \simeq \mathbb{Z}/2\mathbb{Z}. \]

Thus, it is sufficient to prove that if there exists a rational point \( Q \in E_{s,t}(\mathbb{Q}) \), a positive integer \( n \), and \( \delta \in \{0, 1\} \) such that

\[ P_{s,t} = nQ + \delta(0, 0), \]

then \( n = 1 \). Here, note that \( Q \neq (0, 0), \infty \) because \( s \neq 0 \).

First, we prove that \( n \neq 2 \). Indeed, if \( n = 2 \) and \( \delta = 0 \), then the duplication formula implies that

\[ -s^2 = x(2Q) = \left( \frac{x(Q)^2 + s^4 + t^2}{2y(Q)} \right)^2, \]

where \( x(x^2 + s^4 + t^2)^2 \) is not a square as \( x^2 + s^4 + t^2 \neq 0 \). If \( x(2Q) \) is rational, then \( x(Q)^2 + s^4 + t^2 \) is rational. However, \( x(Q) \) is not rational because \( Q \neq (0, 0), \infty \). Therefore, \( x(2Q) \) is not rational, which contradicts the assumption. Hence, \( n \neq 2 \).
which is impossible for \( s \neq 0 \). On the other hand, if \( n = 2 \) and \( \delta = 1 \), then the addition formula implies that
\[
-s^2 = x(2Q + (0, 0)) = -\frac{s^4 + t^2}{x(2Q)}, \quad \text{i.e.,} \quad \frac{s^4 + t^2}{s^2} = x(2Q) = \left(\frac{x(Q)^2 + s^4 + t^2}{2y(Q)}\right)^2,
\]
which is impossible for non-square \( s^4 + t^2 \).

Next, since
\[
\frac{\hat{h}(P_{s,t})}{\hat{h}(Q)} = n^2,
\]
it is sufficient to prove that the left hand side is smaller than 9. Suppose that \( s^4 + t^2 \) is odd. Then, since \( s^4 + t^2 \equiv 1 \mod 8 \), Theorems 3.4 and 3.5 imply the desired bound
\[
\frac{\hat{h}(P_{s,t})}{\hat{h}(Q)} \leq \frac{\log(s^4 + t^2) + \frac{1}{2}\log(s^2) + 2.03781}{\frac{9}{16}\log(s^4 + t^2) + \frac{9}{16}\log 2} < 9,
\]
where the second inequality follows from
\[
\frac{1}{2}\log(s^4 + t^2) + 2.03781 \leq \frac{9}{16}\log(s^4 + t^2) + 3.50905, \quad \text{i.e.,} \quad s^4 + t^2 > e^{-1.47124\times16}.
\]

Suppose that \( s^4 + t^2 \) is even. If \( s^4 + t^2 = 2 \), then we can check the assertion e.g. by using MAGMA command \textbf{Generators}. If \( s^4 + t^2 \neq 2 \) and even, then since we assume that \( s^4 + t^2 \) is not a square, we have \( s^4 + t^2 \geq 6 \). Moreover, since we assume that \( s^4 + t^2 \) is divisible by 2, we see that \( s^4 + t^2 \equiv 2 \mod 4 \) or \( s^4 + t^2 \equiv 4, 20 \mod 32 \). Therefore, Theorem 3.5 implies the desired bound
\[
\frac{\hat{h}(P_{s,t})}{\hat{h}(Q)} \leq \frac{\log(s^4 + t^2) + \frac{1}{2}\log(s^2) + 2.03781}{\frac{9}{16}\log(s^4 + t^2) + \frac{9}{16}\log 2} < 9
\]
where the second inequality follows from
\[
\frac{1}{2}\log(s^4 + t^2) + 2.03781 \leq \frac{9}{16}\log(s^4 + t^2) + 1.94947, \quad \text{i.e.,} \quad s^4 + t^2 > e^{0.08834\times16} \approx 4.1.
\]

This completes the proof of Theorem 3.1.

\[\square\]

3.2. Unramifiedness at \( p \). The goal of this subsection is to prove the following:

\textbf{Proposition 3.6.} Let \( E_{s,t} \) be an elliptic curve defined by \( y^2 = x^3 - (s^4 + t^2)x \) with \( s, t \in \mathbb{Z} \), \( p \) be an odd prime number, and \( n \) be a positive integer. Suppose that

\begin{enumerate}
\item \( p \) divides exactly one of \( s \) and \( t \).
\item \( p^{n+1} \) divides \( st \).
\end{enumerate}

Then, \( 2P_{s,t} \in p^nE_{s,t}(\mathbb{Q}_p) \).

\textbf{Proof.} By the duplication formula, we have
\[
x(2P_{s,t}) = \left(\frac{2s^4 + t^2}{2st}\right)^2.
\]
Moreover, the assumptions implies that
\[
v_p(x(2P_{s,t})) = -2v_p(st) \leq -2(n + 1) < 0,
\]
hence
\[
v_p(y(2P_{s,t})) = \frac{1}{2}v_p(x(2P_{s,t})^3 - (s^4 + t^2)x(P_{2s,t})) = \frac{3}{2}v_p(x(2P_{s,t})) = -3v_p(st) < 0.
\]
In particular, we have $2P_{s,t} \in \text{Ker}(E_{s,t}(\mathbb{Q}_p) \to E_{s,t}(\mathbb{F}_p))$ and
$$v_p \left( \frac{x(2P_{s,t})}{y(2P_{s,t})} \right) = v_p(st) \geq n + 1.$$ 

On the other hand, [36, Ch. IV, Theorem 6.4] gives an isomorphism
$$z : \text{Ker}(E_{s,t}(\mathbb{Q}_p) \to E_{s,t}(\mathbb{F}_p)) \cong p\mathbb{Z}_p; (x, y) \mapsto \frac{x}{y}.$$ 

In particular, we see that $2P_{s,t} \in p^nE_{s,t}(\mathbb{Q}_p)$. Since $p$ is odd, we obtain the assertion $P_{s,t} \in p^nE_{s,t}(\mathbb{Q}_p)$. This completes the proof. 

\[\square\]

3.3. Proof of Theorem [1.4]

Proof. As we have seen in the last of §2, it is sufficient to check the following two conditions.

1. If $s^4 + t^2$ is fourth-power-free and not a square, then an obvious rational point $P_{s,t} := (-s^2, st) \in E_{s,t}(\mathbb{Q})$ does not lie in the subgroup $pE_{s,t}(\mathbb{Q})$.

2. If $p$ divides exactly one of $s$ and $t$ and $p^{n+1}$ divides $st$, then $x(2P_{s,t}) \not\equiv p^{n+1}\mathbb{Z}_p$ and $x(2P_{s,t})/y(2P_{s,t}) \in p^{n+1}\mathbb{Z}_p$. Here, note that since the group $E_{s,t}(\mathbb{F}_p)$ of the modulo $p$ rational points contains a 2-torsion point $(0, 0)$, the order $\#E_{s,t}(\mathbb{F}_p)$ is even.

Each of them follows immediately from Theorem [3.1] and Proposition [3.6] respectively. 

\[\square\]

3.4. Proof of Theorem [1.5]

Proof of Theorem [1.3] Suppose that $t$ is a square, say $t = \tau^2$ for some $\tau \in \mathbb{Z}$. Then, from the symmetry of $s$ and $\tau$, we can find pairs of rational points $P_{s,\tau^2}, P_{\tau,s^2} \in E_{s,\tau^2}(\mathbb{Q})$. Moreover, Proposition [3.6] shows that if $p$ divides exactly one of $s$ and $\tau$, $p^{n+1}$ divides $s\tau$, and $E_{s,\tau^2}(\mathbb{Q}_p)[p] = 0$, then $P_{s,\tau^2}, P_{\tau,s^2} \in p^nE_{s,\tau^2}(\mathbb{Q})$. On the other hand, Fujita and Terai [18, Theorem 1.5(1)] proved that these two rational points can be extended to a system of generators of $E_{s,\tau^2}(\mathbb{Q})$ whenever $s^4 + \tau^4$ is fourth-power-free. As a consequence, the same argument in the proof of Theorem [1.4] with a replacement of Theorem [2.1] to Theorem [2.2] implies the assertion. 

\[\square\]

4. Proof of Theorem [1.3]

Let $l \geq 5$ be a prime number, $E = E^{(0)}$ be an elliptic curve defined by $y^2 = x^3 - lx$, $E$ be the Néron model of $E$ over $\mathbb{Z}_l$ (see s.g. [38 Ch. IV]), $\mathcal{E}$ be the special fiber of $E$, $E^{(0)}$ be the identity component of $E$, and $c(\mathcal{E}) = [\mathcal{E} : E^{(0)}]$. Then, since the minimal discriminant of $E$ is $2^6 \cdot l^3$ and we assume that $l \geq 5$, the table of types of elliptic curves in [38, p. 365] shows that $E$ is of Type III, hence $c(E/\mathbb{Q}_l) = 2$. Therefore, (the proof of) the Néron-Ogg-Shafarevich criterion [35, Theorem 1] implies that for every prime number $p \neq 2, l$, the Galois extension $\mathbb{Q}(E[p])/\mathbb{Q}$ is ramified at $v = 2, l \ (p, \infty)$ and unramified outside $2, l, p$ and $\infty$.

Proof of Theorem [1.3] In view of the above argument, it is sufficient to confirm that there exists infinitely many prime numbers $l$ of the form $l = s^4 + t^2$ such that $s \equiv 0 \mod p^{n+1}$ or $t \equiv 0 \mod p^{n+1}$. This is a consequence of a refinement of [17, Theorem 1] (see [17, p. 947]). This completes the proof of Theorem [1.3]. 

\[\square\]

In what follows, we review the outline of the proof of [35, Theorem 1] for the convenience of the reader.
4.1. The mod $p$ Néron-Ogg-Shafarevich criterion for abelian varieties à la Serre-Tate \cite{35}. The Néron-Ogg-Shafarevich criterion for abelian varieties (defined over a finite extension of the field $\mathbb{Q}_v$ of $v$-adic numbers) is usually stated as follows:

**Theorem 4.1 (\cite{35} Theorem 1).** Let $K$ be a finite extension of $\mathbb{Q}_v$ and $A$ be an abelian variety defined over $K$. Then, the following conditions are equivalent to each other:

1. The abelian variety $A$ has good reduction.
2. For every integer $N$ prime to $v$, the module $A[N]$ of $N$-torsion points in $A(K)$ is an unramified $\text{Gal}(K/K)$-module.
3. There exist infinitely many integers $N$ prime to $v$ such that $A[N]$ is an unramified $\text{Gal}(K/K)$-module.
4. For every/some prime number $p$ different from $v$, the $p$-adic Tate module $T_p(A) := \lim_{\rightarrow n} A[p^n]$ of $A$ is an unramified $\text{Gal}(K/K)$-module.

However, the proof of Serre and Tate in \cite{35} actually shows a finer statement. Let $A$ be the Néron model of $A$ (\cite{27}) over the ring $\mathcal{O}_K$ of integers in $K$, $\tilde{A}$ be the special fiber of $A$, $\tilde{A}^0$ be the identity component of $A$, and $c(\tilde{A}) = [\tilde{A} : \tilde{A}^0]$. Then, a refinement of Theorem 1\textsuperscript{.1} can be stated as follows:

**Theorem 4.2.** The following condition is equivalent to each of the conditions in Theorem 1\textsuperscript{.1}:

5. There exists an integer $N$ prime to $v$ and $c(\tilde{A})$ such that $A[N]$ is an unramified $\text{Gal}(K/K)$-module.

In the proof of Theorem 1\textsuperscript{.1} we used Theorem 1\textsuperscript{.2} for $N = p$.

The key lemmas for the proof of Theorem 4.2 are the following:

**Lemma 4.3 (cf. \cite{1,7,8,32}).** Let $k$ be a field and $G$ be a connected algebraic group scheme over $k$. Then, there exists a connected linear algebraic (i.e., affine) normal subgroup scheme $H$ of $G$ such that $G/H$ is an abelian (i.e., projective) variety.

**Lemma 4.4 (\cite{35} Lemma 1).** Let $k$ be a field, $\tilde{A}$ be a commutative algebraic group over $k$, and $\tilde{A}^0$ be the connected component of $\tilde{A}$. Suppose that $\tilde{A}^0$ is an extension of an abelian variety $B$ by an algebraic subgroup $H$ of $\tilde{A}$ and $H \cong T \times U$ for an algebraic torus $T$ and a unipotent algebraic group $U$. Let $c(\tilde{A}) := [\tilde{A} : \tilde{A}^0]$ be the index of $\tilde{A}^0$ in $\tilde{A}$. Suppose that $k$ is perfect and $N$ is an integer prime to $\text{char}(k)$. Then, the $\mathbb{Z}/N\mathbb{Z}$-module $\tilde{A}(\mathbb{Q})[N]$ is isomorphic to an extension of a cyclic group of order dividing $c(\tilde{A})$ by a free $\mathbb{Z}/N\mathbb{Z}$-module of rank $\dim T + 2\dim B$.

**Lemma 4.5 (\cite{35} Lemma 2).** Let $K$ be a finite extension of $\mathbb{Q}_v$, $\mathcal{O}$ be the ring of integers of $K$, $\mathfrak{m}$ be its maximal ideal, $k = \mathcal{O}/\mathfrak{m}$ be the residue field of $\mathcal{O}$, and $I$ be the inertia subgroup of the Galois group $G_K := \text{Gal}(\overline{K}/K)$. Let $A$ be an abelian variety defined over $K$ and $\tilde{A}$ be its Néron model. Let $N$ be an integer prime to $v$. Then the modulo $\mathfrak{m}$ reduction map defines an isomorphism $A[N][I] = A(\overline{K})[N][I] \cong A(\overline{\mathcal{O}})[N][I] \cong \tilde{A}(\overline{\mathbb{F}})[N]$.

\footnote{According to the introduction of \cite{32}, Lemma 1\textsuperscript{.3} was first announced by Chevalley in 1953.}
which is compatible with the action of the group $G_K/I \simeq \mathrm{Gal}(\overline{k}/k)$. Here, $\overline{O}$ denotes the ring of integers in $\overline{K}$.

The proof of $(1) \Rightarrow (2)$ is easy. Suppose that $A$ has good reduction. Then, Lemma 4.4 shows that the $\mathbb{Z}/N\mathbb{Z}$-modules $A(\overline{K})[N]$ and $\mathcal{A}(\overline{O})[N]$ are free of rank $2\dim A = 2\dim \mathcal{A}$. Hence, Lemma 4.5 implies an equality $A(\overline{K})[N] = A(\overline{K})[N]^T$, which means that $A(\overline{K})[N]$ is an unramified $\mathrm{Gal}(\overline{K}/K)$-module. This completes the proof of the implication $(1) \Rightarrow (2)$.

For the proof of $(5) \Rightarrow (1)$, suppose that there exists an integer $N$ prime to $v$ and $c(\mathcal{A})$ such that $A(\overline{K})[N]$ is an unramified $\mathrm{Gal}(\overline{K}/K)$-module. First, note that Lemma 4.3 implies that there exist an abelian variety $B$, an algebraic torus $T$, and an unipotent algebraic group $U$ such that $\mathcal{A}$ is an isomorphic to an extension of $B$ by $T \times U$ over the residue field $k$ of $K$. In particular, we have an inequality $\dim \mathcal{A} \geq \dim B + \dim T$. On the other hand, Lemma 4.5 shows an isomorphism $A(\overline{K})[N] \simeq A(\overline{O})[N]$ of $\mathbb{Z}/N\mathbb{Z}$-modules. Since we assume that $N$ is prime to $v = \text{char } k$ and $c(\mathcal{A})$, Lemma 4.4 shows that the $\mathbb{Z}/N\mathbb{Z}$-module $A(\overline{K})[N]$ is free of rank $2\dim A = \dim T + 2\dim B$. This implies that $\dim T = 0$, hence $\dim A = \dim B = \dim \mathcal{A}$. In particular, $\mathcal{A} \simeq B$ is an abelian variety, especially a proper scheme over $k$. Finally, a purely scheme theoretic lemma [35, Lemma 3] shows that the Néron model $A$ of $A$ is a proper scheme over the ring $O$ of integers in $K$, hence $A$ has good reduction [27]. This completes the proof of the implication $(5) \Rightarrow (1)$.

5. Upper bounds for the Mordell-Weil rank of $E_{s,t}$

By a standard descent method for 2-isogeny, we can prove the following, which itself should be well-known for experts. (See e.g. [36, Ch. X, Propositions 6.1 and 6.2] for similar results.)

**Theorem 5.1.** Let $l$ be a non-zero integer and $E^{(l)}$ be an elliptic curve defined by $y^2 = x^3 - lx$. Suppose that $l$ is a prime number. Then, the following inequality holds.

$$\text{rank } E^{(l)}(\mathbb{Q}) \leq \begin{cases} 0 & \text{if } l \equiv 3, 11, 13 \text{ mod } 16, \\ 1 & \text{if } l \equiv 2, 5, 7, 9, 15 \text{ mod } 16, \\ 2 & \text{if } l \equiv 1 \text{ mod } 16. \end{cases}$$

Since the elliptic curve $E_{s,t} : y^2 = x^3 - (s^4 + t^2)x$ in Theorem 3.1 has an obvious rational point $P_{s,t} = (-s^2, st)$ whose order is infinite, we obtain the following:

**Corollary 5.2.** Let $s, t$ be integers and $E_{s,t}$ be an elliptic curve defined by $y^2 = x^3 - (s^4 + t^2)x$. Suppose that $s^4 + t^2$ is a prime number such that $s$ is even and $t \equiv \pm 3 \text{ mod } 8$. Then, the equality $\text{rank } E_{s,t}(\mathbb{Q}) = 1$ holds.

**Remark 5.3.** The preceding works [20, 33, 34] obtained some lower bounds for the orders of the $p$-parts of the ideal class groups $\mathrm{Cl}(\mathbb{Q}(E[p^n]))$ of the $p^n$-division fields $\mathbb{Q}(E[p^n])$ of elliptic curves $E$. However, their general results fail to give any non-trivial bounds when $\text{rank } E(\mathbb{Q}) \leq 1$. Therefore, it is natural to ask whether we can obtain some lower bounds for the order of $\mathrm{Cl}(\mathbb{Q}[p])[p]$ for any elliptic curve $E$ such that $\text{rank } E(\mathbb{Q}) \leq 1$. We can give an affirmative answer to this question from Theorem 1.3 and Corollary 5.2. Indeed, we can obtain an infinite family of elliptic curves $E_{s,t}$ such that $\text{rank } E_{s,t}(\mathbb{Q}) = 1$ and $\mathrm{Cl}(\mathbb{Q}(E_{s,t}[p]))[p] \neq 0$. In fact, Theorem 1.3 ensures that the set of such number fields $\mathbb{Q}(E_{s,t}[p])$ contains infinitely isomorphism classes.
5.1. **Proof of Theorem 5.1** For the convenience of the reader, we give a proof of Theorem 5.1 along the descent method in [36 Ch. X].

Let $\phi : E^{(l)} \to E^{(-4l)}$ be a rational map defined by $\phi(x, y) := (y^2/x^2, -y(x^2+l)/x^2)$. Then, it is a 2-isogeny whose kernel is generated by the point $(0, 0)$ of order 2. Let $\hat{\phi}$ be the dual isogeny of $\phi$. Then, we have an exact sequence with natural maps

$$0 \to E^{(-4l)}(Q)[\hat{\phi}] / \phi(E^{(l)}(Q)[2]) \to E^{(-4l)}(Q) / \phi(E^{(l)}(Q)) \xrightarrow{\hat{\phi}} E^{(l)}(Q) / 2E^{(l)}(Q) \to E^{(l)}(Q) / \phi(E^{(-4l)}(Q)) \to 0.$$ 

Since $E^{(-4l)}(Q)[\hat{\phi}] = \{\infty, (0, 0)\}$ and $\phi(E^{(l)}(Q)[2]) = \phi(\{\infty, (0, 0)\}) = \{\infty\}$, we have

$$\text{rank } E^{(l)}(Q) = \dim_{F_2}(E^{(l)}(Q)/2E^{(l)}(Q)) - \dim_{F_2} E^{(l)}(Q)[2]$$

$$= \dim_{F_2}(E^{(-4l)}(Q) / \phi(E^{(l)}(Q))) - \dim_{F_2}(E^{(-4l)}(Q)[\hat{\phi}] / \phi(E^{(l)}(Q)[2])) + \dim_{F_2}(E^{(l)}(Q)/\phi(E^{(-4l)}(Q))) - 1$$

$$\leq \dim_{F_2}\text{Sel}(Q, E^{(l)}[\phi]) + \dim_{F_2}\text{Sel}(Q, E^{(-4l)}[\hat{\phi}]) - 2.$$

Here, recall that the $\phi$-Selmer group $\text{Sel}(Q, E^{(l)}[\phi]) = S^{(\phi)}(E^{(l)}/Q)$ in [36] can be identified with the $F_2$-vector space consisting of the isomorphism classes of principal homogeneous spaces for $E^{(l)}$ which are trivialized by $\phi$ and have $Q_v$-rational points at every place $v$. Such classes are parametrized by $dQ^{2} \in Q^{x}/Q^{x2}$ and has a representative given by

$$D^{(l)}_d : dX^4 = dX^4 + 4dX^4.$$

Thus, we can check that

$$\{D^{(l)}_d : dX^4 = dX^4 + 4dX^4\} \subset \text{Sel}(Q, E^{(l)}[\phi]) \subset \begin{cases} \{D^{(l)}_d, D^{(l)}_d\} & \text{if } l \not\equiv \pm 1, 7 \text{ mod } 16, \\ \{D^{(l)}_d, D^{(l)}_d, D^{(l)}_d, D^{(l)}_d\} & \text{if } l \equiv \pm 1, 7 \text{ mod } 16. \end{cases}$$

In a similar manner, we can check that

$$\{D^{(-4l)}_d, D^{(-4l)}_d\} \subset \text{Sel}(Q, E^{(-4l)}[\hat{\phi}]) \subset \begin{cases} \{D^{(-4l)}_d, D^{(-4l)}_d\} & \text{if } l \equiv 1, 2, 5, 9 \text{ mod } 16, \\ \{D^{(-4l)}_d, D^{(-4l)}_d\} & \text{if } l \equiv 3, 7, 11, 13, 15 \text{ mod } 16. \end{cases}$$

As a conclusion, we obtain the desired bound.

**Remark 5.4.** In fact, we can determine both groups $\text{Sel}(Q, E^{(l)}[\phi])$ and $\text{Sel}(Q, E^{(-4l)}[\hat{\phi}])$ completely. For the former, we have the equality

$$\text{Sel}(Q, E^{(l)}[\phi]) = \{D^{(l)}_d, D^{(l)}_d, D^{(l)}_d, D^{(l)}_d\} \quad \text{if } l \equiv \pm 1, 7 \text{ mod } 16,$$

which is a consequence of the following facts.

- $[1 : 1 : \sqrt{2(l+1)}] \in D^{(l)}_2(Q_2)$ for $l \equiv 1 \text{ mod } 16$, $[(8-l)^{1/4} : 1 : 4] \in D^{(l)}_2(Q_2)$ for $l \equiv 7 \text{ mod } 16$, and $[(-l)^{1/4} : 1 : 0] \in D^{(l)}_2(Q_2)$ for $l \equiv 15 \text{ mod } 16$.
- $[1 : 0 : \sqrt{2}] \in D^{(l)}_2(Q_0)$ for $l \equiv \pm 1 \text{ mod } 8$.
- $[1 : 0 : \sqrt{2}] \in D^{(l)}_2(Q_{\mathbb{R}})$ for $l > 0$.

\[\text{Indeed, since } [X_0 : X_1 : X_2] = [0 : 1 : 2] \in D^{(l)}_1(Q), \text{ it is sufficient to check that } D^{(l)}_1, D^{(l)}_2, D^{(l)}_2 \not\in \text{Sel}(Q, E^{(l)}[\phi]), \text{ which follows from } D^{(l)}_1(\mathbb{R}) = D^{(l)}_2(\mathbb{R}) = \emptyset \text{ for every } l, \text{ and } D^{(l)}_2(Q_2) = \emptyset \text{ for } l \equiv \pm 1, 2, 7 \text{ mod } 16.\]

\[\text{Indeed, since } [0 : 1 : 4] \in D^{(-4l)}_2(Q), \text{ it is sufficient to check that } D^{(-4l)}_2(Q_2) = D^{(-4l)}_2(Q_2) = \emptyset \text{ for odd } l, \quad D^{(-4l)}_2(Q_2) = \emptyset \text{ for } l \equiv 3 \text{ mod } 4 \text{ and } l \equiv 13 \text{ mod } 16.\]
For the latter, we have the equality
\[ \text{Sel}(\mathbb{Q}, E([-4l])|\hat{\phi}) = \{D_{\pm 1}([-4l]), D_{\pm l}([-4l])\} \] for \( l \equiv 1, 2, 5, 9 \mod 16, \]
which is a consequence of the following facts.

- \([2 : 1 : 4] \in D_{-1}([-8l])\) for \( l = 2 \).
- \([0 : 1 : \sqrt{2}] \in D_{\pm 1}([-4l])\) for \( l \equiv 1 \mod 8 \) and \([2(l - 4)^{1/4} : 1 : 8] \in D_{-1}([-4l])\) for \( l \equiv 1 \mod 16. \)
- \([1 : 0 : \sqrt{-1}] \in D_{-1}([-4l])\) for \( l \equiv 1 \mod 4. \)
- \([0 : 1 : \sqrt{l}] \in D_{-1}([-4l])\) for \( l > 0. \)

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