Special Geometry and Automorphic Forms

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Abstract

We consider the special geometry of the vector multiplet moduli space in compactifications of the heterotic string on $K3 \times T^2$ or the type IIA string on $K3$-fibered Calabi-Yau threefolds. In particular, we construct a modified dilaton that is invariant under $SO(2, n; \mathbb{Z})$ $T$-duality transformations at the non-perturbative level and regular everywhere on the moduli space. The invariant dilaton, together with a set of other coordinates that transform covariantly under $SO(2, n; \mathbb{Z})$, parameterize the moduli space. The construction involves a meromorphic automorphic function of $SO(2, n; \mathbb{Z})$, that also depends on the invariant dilaton. In the weak coupling limit, the divisor of this automorphic form is an integer linear combination of the rational quadratic divisors where the gauge symmetry is enhanced classically. We also show how the non-perturbative prepotential can be expressed in terms of meromorphic automorphic forms, by expanding a $T$-duality invariant quantity both in terms of the standard special coordinates and in terms of the invariant dilaton and the covariant coordinates.

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1 Introduction

Theories with \( N = 2 \) extended supersymmetry in \( d = 4 \) space-time dimensions have proven to be a remarkably fertile area of study, exhibiting a rich variety of physical phenomena, while at the same time being sufficiently constrained to be amenable to a quantitative analysis. This has been used fruitfully in quantum field theory with rigid supersymmetry as well as in supergravity and also in \( N = 2 \) string theory compactifications to \( d = 4 \).

In this paper, we will consider the theories that arise when, for example, the heterotic string is compactified to four dimensions on \( K3 \times T^2 \). At the perturbative level, such a model is invariant under \( T \)-duality transformations that form a group isomorphic to \( SO(2, n; \mathbb{Z}) \) for some \( n \) \([4]\). This symmetry is seemingly explicitly broken by the non-perturbative corrections, though. The classically invariant dilaton will transform in a non-trivial way under \( T \)-duality in the exact theory. Furthermore, the exact holomorphic prepotential \( F \), that describes the vector multiplet moduli space, is a transcendental function (in so called special coordinates) with polylogarithm singularities and complicated transformation properties under \( SO(2, n; \mathbb{Z}) \).

It has been known for some time that at the perturbative level in the heterotic string, one can introduce an invariant dilaton \([2]\), which however is not a special coordinate. This construction involves a certain automorphic form of \( SO(2, n; \mathbb{Z}) \). Furthermore, by taking a suitable number of derivatives of the perturbative contribution to \( F \) one obtains a meromorphic automorphic form of \( SO(2, n; \mathbb{Z}) \) \([2, 3, 4, 5, 6]\). In this paper, we will generalize these ideas to the non-perturbative level by constructing a distinguished set of coordinates, that are not of the special coordinates type, but transform in a simple way under \( T \)-duality. More precisely, we construct a \( T \)-duality invariant dilaton and a set of coordinates that transform covariantly under \( SO(2, n; \mathbb{Z}) \). In a sense, by going to this frame we compensate for the quantum corrections. To get an invariant dilaton which is regular everywhere on the moduli space, and thus can be used as an expansion parameter, we introduce a meromorphic automorphic function of \( SO(2, n; \mathbb{Z}) \), that also depends on the invariant dilaton. In the limit where the invariant dilaton is large, i.e. the weak coupling limit, the divisor of this automorphic function is an integer linear combination of so called rational quadratic divisors \([7]\). These are the divisors where the heterotic string classically acquires an enhanced non-abelian gauge symmetry and the perturbative description of the type II string breaks down. The invariant dilaton and the covariant coordinates may be used to express the non-perturbative prepotential in terms of meromorphic automorphic forms of \( SO(2, n; \mathbb{Z}) \). The idea is to expand some \( T \)-duality invariant quantity both in terms of the standard special coordinates and in terms of the invariant dilaton and the covariant coordinates. Such formulas were given for a particular two-parameter model in \([8]\), but without the geometric interpretation discussed in this paper.

We see several possible applications of the geometric structure discussed in this paper. It is clearly related to the problem of understanding the heterotic string at a non-perturbative level. Using the type IIA picture, we get a relationship between the number of rational curves on \( K3 \)-fibered Calabi-Yau threefolds and automorphic forms that would be interesting to study further. On the type IIB side, the Picard-Fuchs equations for the
periods of the mirror manifold of a $K3$-fibered Calabi-Yau threefold $[9, 10]$ should simplify in the covariant coordinates. This is probably related to the problem of finding a criterion characterizing the mirror manifolds of $K3$-fibered Calabi-Yau threefolds. More speculatively, it would be nice if these ideas generalize to $K3$-fibered Calabi-Yau spaces of higher dimension and their mirrors.

The meromorphic automorphic function that arises in the construction of the perturbatively invariant dilaton has remarkable mathematical properties. Its divisor carries information about the spectrum of massless particles. Furthermore, it can be written as an infinite product, where the exponents are the Fourier coefficients of an $PSL(2; \mathbb{Z})$ modular form $[6, 7, 11]$. (The latter form is essentially the elliptic genus of the $K3$ non-linear sigma model with an $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$ vector bundle that defines the heterotic compactification). Most importantly, this form is the denominator formula for a generalized Kac-Moody algebra that has been conjectured to be a broken gauge symmetry algebra in string theory $[8]$. The gauge bosons of this algebra are the BPS saturated states, which naturally form an algebra $[12]$. It would be very interesting to give a non-perturbative generalization of these statements. In the type II picture, it would be desirable to relate the properties of the automorphic forms that arise in this construction to the data specifying the $K3$-fibered Calabi-Yau threefold for example as a subvariety of a toric variety.

This paper is organized as follows: In the next section, we give a quick review of theories with $N = 2$ supersymmetry in $d = 4$ space-time dimensions, with particular emphasis on special geometry of the type that arises in heterotic compactifications on $K3 \times T^2$ and type IIA compactifications on $K3$-fibered Calabi-Yau threefolds. In section three we discuss these two types of compactifications in more detail. In section four, we discuss the singularity structure of the vector multiplet moduli space. In section five, which is the main part of the paper, we introduce the covariant coordinates. These are used in section six to show how the prepotential can be expressed in terms of automorphic forms. Finally, in the appendix, we discuss some cases with a small number of moduli more explicitly.

2 Review of $N = 2$ in $d = 4$

At low energies, theories with $N = 2$ supersymmetry in $d = 4$ space-time dimensions are described by some effective Lagrangian governing the dynamics of the massless multiplets. The massless spectrum consists of:

1. Exactly one supergravity multiplet (in the case of theories with local supersymmetry), consisting of a spin 2 graviton, two spin 3/2 gravitini and a spin 1 graviphoton.

2. $N_V$ vector multiplets, each consisting of a spin 1 gauge field, two spin 1/2 gaugini, and two spin 0 fields. These fields transform in the adjoint of the gauge group $G$ of the theory.

3. $N_H$ hyper multiplets, each consisting of two spin 1/2 quarks and four spin 0 fields. These fields transform in some representation $R \oplus \bar{R}$ of $G$. 
A most important point is that the superpotential for the scalar fields in the vector and hyper multiplets has flat directions, so we have a moduli space $\mathcal{M}$ of inequivalent vacua. Furthermore, the couplings between vector and hyper multiplets of this Lagrangian are constrained to be a supersymmetric extension of ‘minimal coupling’ [13]. In particular, there are no couplings between neutral hyper multiplets and vector multiplets. It follows that, at least locally, the moduli space factorizes as

$$\mathcal{M} \cong \mathcal{M}_V \times \mathcal{M}_H,$$

where the two factors $\mathcal{M}_V$ and $\mathcal{M}_H$ are parameterized by the vacuum expectation values of the scalars in the vector and hyper multiplets respectively. The hyper multiplet moduli space $\mathcal{M}_H$ has the structure of a quaternionic manifold [14]. In this paper, we will focus on the vector multiplet moduli space $\mathcal{M}_V$, which is subject to the constraints of special geometry [15, 13, 16]. The implications of these constraints will be reviewed below.

A universal modulus in string theory is the dilaton-axion $\theta/2\pi + i4\pi/g^2$, where $g$ and $\theta$ are the coupling constant and the theta angle respectively. This modulus organizes the perturbative and non-perturbative quantum corrections. It is therefore important to know to which type of multiplet it belongs. We can get $N = 2$ local supersymmetry in $d = 4$ space-time dimensions by compactifying

1. A string theory with $N = 2$ supersymmetry in $d = 10$ dimensions, i.e. the type IIA or type IIB string, on a manifold of $SU(3)$ holonomy, i.e. an arbitrary Calabi-Yau threefold. The dilaton-axion is then part of a hyper multiplet [17].

2. A string theory with $N = 1$ supersymmetry in $d = 10$ dimensions, i.e. the $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$ heterotic string or the type I string, on a manifold of $SU(2)$ holonomy, i.e. $K3 \times T^2$. In the heterotic case, the dilaton-axion is part of a vector multiplet[4], whereas it is a combination of vector and hyper multiplet moduli in the type I case [18].

There are also more general possibilities such as, for instance, M-theory compactified on $CY_3 \times S^1$, or F-theory compactified on $CY_3 \times T^2$. The former vacua are dual to type IIA theory on the same Calabi-Yau threefold while the latter is dual to the heterotic string on $K3 \times T^2$. Note however, that while in the case of M-theory there is no condition on the manifold, F-theory requires that the Calabi-Yau is an elliptic fibration. In this paper, we will restrict our study to the special geometry that arises in compactifications of the heterotic string on $K3 \times T^2$ or the type IIA string on $K3$-fibered Calabi-Yau threefolds. In many cases, these models are dual to each other [13, 20, 21]. One can also use mirror symmetry to relate them to compactifications of the type IIB string on mirror manifolds of $K3$-fibered Calabi-Yau threefolds [22]. The geometry of $\mathcal{M}_V$ is encoded in a holomorphic function $F$ called the prepotential. In the type II case, $F$ can be calculated exactly at string tree level by going to a weak coupling limit. For type IIA, $\mathcal{M}_V$ can be identified with the complexified Kähler cone of the Calabi-Yau threefold $X$. In the large radius limit, $F$ is essentially the triple intersection form on $H^{1,1}(X; \mathbb{Z})$ [23], but world-sheet instantons,

\[ \text{More precisely, the dilaton-axion belongs to a vector-tensor multiplet, which on-shell is dual to a vector multiplet [2].} \]
i.e. rational curves on $X$, give corrections at finite radius \[24\]. In principle these instanton corrections can be computed directly on the type IIA side \[24\]. However, the calculation is more tractable by going to the mirror manifold $\tilde{X}$ and the corresponding type IIB theory. For type IIB, $\mathcal{M}_{V}$ can be identified with the complex structure moduli space of the Calabi-Yau threefold on which the theory is compactified. $F$ can then be determined exactly at world-sheet tree level by going to a large radius limit. The computation amounts to determining the periods of the holomorphic, nowhere vanishing section of the canonical bundle of $\tilde{X}$. The prepotential is expressed in terms of these periods. By using the mirror map, relating the periods to the so called special coordinates, we obtain $F$ for the type IIA theory. In the heterotic case, $F$ is classically given by the metric on the Narain moduli space of the two-torus, but it receives perturbative (one-loop only) and non-perturbative quantum corrections controlled by the dilaton-axion. The non-perturbative corrections can be attributed to space-time instantons.

### 2.1 Review of Special Geometry

The constraints of special geometry amount to the following \[26\]: $\mathcal{M}_{V}$ is a Kähler manifold of restricted type\[5\], i.e. the cohomology class of the Kähler form is an even element of $H^2(\mathcal{M}_{V}; \mathbb{Z})$. This determines a line bundle $\mathcal{L}$ over $\mathcal{M}_{V}$, the first Chern class of which equals the Kähler class of $\mathcal{M}_{V}$. Furthermore, there exists a flat $Sp(4+2n; \mathbb{R})$ holomorphic vector bundle $\mathcal{H}$ over $\mathcal{M}_{V}$ with a compatible hermitian metric where $n = \dim_{\mathbb{C}} \mathcal{M}_{V} - 1$, and a holomorphic section $\Pi$ of $\mathcal{L} \otimes \mathcal{H}$ such that the Kähler potential $K$ of $\mathcal{M}_{V}$ can be written as

\[
K = - \log i \left( \bar{\Pi}' J \Pi \right),
\]

where $J$ is the $Sp(4+2n)$ invariant metric. In a more general setting further conditions are needed to define special geometry \[27\]. We work in a particular set of homogeneous special coordinates $X^I$, which are functions defined on $\mathcal{M}_{V}$, for which the section $\Pi$ can be written in the form

\[
\Pi' = \left( X^0, X^1, \ldots, X^{n+1}; F_0, F_1, \ldots, F_{n+1} \right),
\]

where $F_I = \partial_I F = \frac{\partial F}{\partial X^I}$, and the prepotential $F$ is a function of the $X^I$ of degree two in $X^I$. In this basis, the $Sp(4+2n)$ invariant metric is

\[
J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\]

where $I$ is the $(2+n) \times (2+n)$ unit matrix. The homogeneous special coordinates arise naturally in supergravity: The $X^\alpha$ for $\alpha = 1, \ldots, n+1$ are the complex scalars of the vector multiplets, and $X^0$ is a non-dynamical complex scalar corresponding to the graviphoton. As coordinates on $\mathcal{M}_{V}$ we can then take the inhomogeneous special coordinates $X^\alpha/X^0$ for $\alpha = 1, \ldots, n+1$.

In general, $\mathcal{M}_{V}$ has singularities along certain divisors where additional fields become massless. If we encircle such a singularity, the section $\Pi$ will undergo a monodromy

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5 Also called a Hodge manifold.
6 Indices $I, J, \ldots$ always take the values $0, 1, \ldots, n + 1$. 

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transformation of the form $\Pi \to M \Pi$, where $M \in Sp(4+2n; \mathbb{Z})$, i.e. $M^t J M = J$. These transformations generate the monodromy group of the theory. Note that the Kähler potential $K$ is invariant under such a transformation.

3 The cases with $S$- and $T$-duality

We will be interested in the particular cases corresponding to the compactifications of

1. The $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$ heterotic string on $K3 \times T^2$.

2. The type IIA string on a $K3$-fibered Calabi-Yau threefold.

3. The type IIB string on the mirror manifold of a $K3$-fibered Calabi-Yau threefold.

These compactifications are conjectured to be related by heterotic/type IIA duality and mirror symmetry between type IIA and type IIB.

The special geometry of the corresponding vector multiplet moduli space $\mathcal{M}_V$ can be characterized as follows: We work with a particular set of homogeneous special coordinates $X^I$. The corresponding inhomogeneous special coordinates, that parameterize $\mathcal{M}_V$, are denoted $S = X^1/X^0$ and $T^I = X^I/X^0$. Following standard practice, it is convenient to introduce the period vector $\hat{\Pi}$, which is related to $\Pi$ by an $Sp(4+2n; \mathbb{Z})$ transformation, as

$$\hat{\Pi}^t = \left( \hat{X}^0, \hat{X}^1, \hat{X}^2, \ldots, \hat{X}^{n+1}; \hat{F}_0, \hat{F}_1, \hat{F}_2, \ldots, \hat{F}_{n+1} \right) = \left( X^0, F_1, X^2, \ldots, X^{n+1}; F_0, -X^1, F_2, \ldots, F_{n+1} \right).$$

We also define the symmetric tensor $\eta_{IJ}$ by $\eta_{01} = -\frac{1}{2}$, $\eta_{00} = \eta_{11} = \eta_{0i} = \eta_{ii} = 0$, and $\eta_{ij} = \text{diag}(-1,1,\ldots,1)$. The extra condition that we impose on the theory is that the monodromy group contains a subgroup generated by the elements of a group isomorphic to $\mathbb{Z} \times SO(2n; \mathbb{Z})$, where we refer to the two factors as the $S$- and $T$-duality groups respectively. The $S$-duality group acts as

$$\hat{X}^I \to \hat{X}^I, \quad \hat{F}_I \to +\hat{F}_I + 2b\eta_{IJ}\hat{X}^J,$$  

where $b \in \mathbb{Z}$. An element $U$ of the $T$-duality group, defined as the group of matrices $U^I_J$ such that $(U^t)^K_{KL}U^L_J = \eta_{IJ}$, acts as

$$\hat{X}^I \to U^I_J \hat{X}^J, \quad \hat{F}_I \to (U^{t-1})^I_J \left( \hat{F}_J + (\Lambda_U)_{JK}\hat{X}^K \right),$$

where $(\Lambda_U)_{JK}$ is some symmetric, integer-valued matrix. We will discuss the form of the $\Lambda_U$-matrices later. Finally, we require $F_1$ to be regular as $X^1/X^0 \to i\infty$ with $X^0$ and the $X^I$ held fixed.

Indices $i,j,\ldots$ always take the values $2,\ldots,n+1$. 

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The general solution to the conditions on the prepotential following from the S-duality transformation (3.2) is

\[ F = \frac{X^1}{X^0}X^i\eta_{ij}X^j + f, \]  

(3.4)

where the function \( f \) is of degree two in \( X^I \), invariant under \( X^1 \to X^0 + X^1 \), and regular in the \( X^1/X^0 \to i\infty \) limit. This means that \( f \) can be expanded in a Fourier series as

\[ f = \sum_{k=0}^{\infty} q^k f_k, \]  

(3.5)

where \( q = e^{2\pi i S} \) and the functions \( f_k \) are independent of \( X^1 \) and of degree two in \( X^0 \) and the \( X^i \). The T-duality transformations impose some constraints on these functions, which we will analyze in detail later.

### 3.1 The heterotic picture

For the case of a compactification of the \( E_8 \times E_8 \) or \( Spin(32)/\mathbb{Z}_2 \) heterotic string on \( K3 \times T^2 \) the interpretation of the above structure is as follows: The modulus \( S \) is the dilaton-axion \( \theta/2\pi + i4\pi/g^2 \), and the \( T^i \) parameterize the moduli space of the torus and a flat \( U(1)^{n-2} \) bundle over it, i.e. \( T = T^2 - T^3 \) and \( U = T^2 + T^3 \) are the complex structure and complexified Kähler moduli respectively and \( V^a = T^a \) for \( a = 4, \ldots, n + 1 \) are the Wilson line moduli [28]. The moduli of the \( K3 \) surface and the \( E_8 \times E_8 \) or \( Spin(32)/\mathbb{Z}_2 \) bundle parameterize the hyper multiplet moduli space \( \mathcal{M}_H \). The dimension of this space may jump at particular points in \( \mathcal{M}_V \) where additional hyper multiplets become massless. By ‘Higgsing’, i.e. giving vacuum expectation values to such hyper multiplets, we may give masses to some vector multiplets and thus change the number \( n \) in the interval \( 0 \leq n \leq 18 \). Classically the vector multiplet moduli space \( \mathcal{M}_V \) is a direct product:

\[
\mathcal{M}_V^{\text{classical}} \cong \mathcal{M}_V^{\text{dilaton}} \times \mathcal{M}_V^{\text{Narain}} \\
\cong (\mathbb{Z} \backslash SU(1,1)/U(1)) \times (SO(\Gamma^{2,n})/SO(2) \times SO(n)),
\]

(3.6)

where \( \Gamma^{2,n} \) is the Narain lattice of the indicated signature. Spaces that locally are of the form (3.6) are in fact the only special geometries that can be written as direct products [29]. The prepotential of such a space is precisely given by the first term in (3.4). Quantum mechanically, the geometry receives perturbative and non-perturbative corrections that are encoded in the second term in (3.4). In the expansion (3.5), the \( q^k f_k \) term is the one-loop contribution in the \( k \)-instanton sector. Note that \( q \) is precisely the exponential of the instanton action. In particular, the \( f_0 \) term is thus the perturbative contribution. At the semi-classical level, where \( f_k = 0 \) for \( k > 0 \), the theory is in fact invariant under continuous real shifts of \( S \) and this symmetry is explicitly broken down to the discrete Peccei-Quinn subgroup (3.2) by instantons just as expected. We expect the \( SO(\Gamma^{2,n}) \) T-duality group to survive in the exact theory. Indeed, it can be interpreted as a discrete gauge symmetry and should therefore not be explicitly broken by any anomalies [30].
3.2 The type II picture

Next we turn to the case of compactification of the type IIA string on a Calabi-Yau threefold $X$. The hyper multiplet moduli space $\mathcal{M}_H$ is then $4(h^{2,1}(X) + 1)$ real-dimensional and is parameterized by the complex structure of $X$, the moduli arising from compactifications of the Ramond-Ramond three-form, and the dilaton-axion. The vector multiplet moduli space $\mathcal{M}_V$ is $h^{1,1}(X)$ complex-dimensional and can be identified with the complexified Kähler cone of $X$. Using the isomorphism $H^2(X; \mathbb{Z}) \cong H_4(X; \mathbb{Z})$ we can associate divisors $D_1$ and $D_i$ in the latter group to the moduli $T^1 = S$ and the $T^i$. The prepotential $F$ can then be written in the form

$$F = -\frac{1}{3!}(D_\alpha \cdot D_\beta \cdot D_\gamma)T^\alpha T^\beta T^\gamma + \frac{i\zeta(3)}{16\pi^3} \chi(X) + \frac{1}{(2\pi i)^3} \sum_R n(R) \text{Li}_3(\exp 2\pi i R^\alpha T^\alpha),$$  \hspace{1cm} (3.7)

where $D_\alpha \cdot D_\beta \cdot D_\gamma$ are the triple intersection numbers of $X$, $\chi(X)$ is the Euler characteristic of $X$, and $n(R)$ is the number of rational curves of multi degree $R = (R_1, \ldots, R_{n+1})$ on $X$. The mirror hypothesis states the equivalence of the compactifications of the type IIA string on $X$ and the type IIB string on the mirror manifold $\tilde{X}$ of $X$. The data entering in (3.7) can in many cases be effectively computed by considering the complex structure moduli space of $\tilde{X}$. Comparing (3.7) with (3.4) and (3.5), we see that the divisor $D_1$ squares to zero, and that $C \cdot D_1 \geq 0$ for all curves $C$ in $X$. The existence of such a nef divisor of numerical $D$-dimension equal to one implies that $X$ is a fiber bundle over $\mathbb{P}^1$ with the generic fiber being a K3-surface [31]. Thus a necessary condition for the compactification of the type IIA theory on $X$ to be dual to a compactification of the heterotic string on $K3 \times T^2$ is that $X$ is K3-fibered [3, 21]. A point on the $\mathbb{P}^1$ base times the generic K3 fiber equals the divisor $D_1$ when considered as an element of $H_4(X; \mathbb{Z})$. If we neglect contributions to $H_4(X; \mathbb{Z})$ from reducible ‘bad’ fibers, the remaining generators are given by the fundamental class of the $\mathbb{P}^1$ base times algebraic two-cycles in the generic K3 fiber that are invariant under the monodromy around closed curves on the base. The lattice of such two-cycles equals the monodromy invariant part of the Picard lattice $\text{Pic}(D_1)$ of the generic K3 fiber, and has rank $n$ in the interval $1 \leq n \leq \rho(D_1)$. Here $\rho(D_1)$ is the Picard number of the generic K3 fiber, and $1 \leq \rho(D_1) \leq 20$. The modulus $S$ is the complexified Kähler form on the $\mathbb{P}^1$ base. Shifting the $B$-field of the complexified Kähler form on the $\mathbb{P}^1$ base by an element of $H^2(\mathbb{P}^1; \mathbb{Z}) \cong \mathbb{Z}$ is physically trivial, so $S$-duality should be valid in the exact theory. Furthermore, the moduli $T^i$ parameterize the moduli space of the complexified Kähler form on the generic K3 fiber. This space is isomorphic to the second factor in (3.6), where $\Gamma^{2,n}$ now is the monodromy invariant part of the ‘quantum Picard lattice’, i.e. the direct sum of $\text{Pic}(D_1)$ and the hyperbolic plane $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Again we expect the $T$-duality group to be a subgroup of the monodromy group in the exact theory. We have thus found completely analogous structures for the two different string theory realizations of these special geometries.
4 The singularity structure

We now turn to the singularities of the vector multiplet moduli space $\mathcal{M}_V$. The subspace of $\mathcal{M}_V$ at $q = 0$, i.e. in the $S \to i\infty$ limit, takes the form of the second factor in (3.6), where $\Gamma^{2,n}$ is either the Narain lattice or the monodromy invariant part of the quantum Picard lattice. This space can be regarded as the Grassmannian of space-like two-planes in $\Gamma^{2,n} \otimes \mathbb{R}$ modulo automorphisms of $\Gamma^{2,n}$. Let $\alpha \in \Gamma^{2,n}$ be a root, i.e. $\alpha \cdot \alpha = -2$. We define the rational quadratic divisor associated to $\alpha$ to be the subspace of the Grassmannian where $\alpha$ is in the orthogonal complement of the space-like two-plane $[7]$. In the heterotic picture, this means that we have a purely left-moving vector with length squared equal to two in the Narain lattice, i.e. we are at an enhanced symmetry point where the unbroken gauge group acquires an $SU(2)$ factor and some number $N_f$ of hyper multiplet doublets of $SU(2)$ become massless. Larger non-abelian groups arise at the intersections of rational quadratic divisors. In the type IIA picture, we use the fact that a root $\alpha$ of the Picard lattice corresponds to a rational curve, i.e. an embedded $\mathbb{P}^1$. On the associated rational quadratic divisor, $\alpha$ is orthogonal to the space-like two-plane spanned by the complexified Kähler form of the generic $K3$ fiber, i.e. the volume of and $B$-flux through this $\mathbb{P}^1$ vanish. The perturbative analysis of the theory thus breaks down, and we expect an $SU(2)$ factor in the gauge group together with a number $N_f$ of massless hyper multiplet doublets $[32, 33]$.

The massless spectrum on the rational quadratic divisors determines the singularities of the functions $f_k$ in (3.3). This is most easily understood by comparing the heterotic picture with the corresponding $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory with $N_f$ massless hyper multiplet doublets. The one-loop beta function is proportional to $4 - N_f$, so asymptotic freedom and non-trivial infrared dynamics gives the restriction $N_f = 0, 1, 2, 3$. The low-energy effective gauge couplings, given by the matrix of second-derivatives of the prepotential, diverge as we approach the divisor where $SU(2)$ is restored. In particular, the perturbative contribution has a logarithmic divergence proportional to $4 - N_f$. The contribution from the $k$-instanton sector has a pole of order $(4 - N_f)k$ $[34]$.

These results determine the singularities of the functions $f_k$ in the string theory prepotential as we approach a rational quadratic divisor $D(X) = 0$ where a $U(1)$ factor gets enhanced to $SU(2)$ and $N_f$ hyper multiplet doublets become massless $[35]$. The function $f_0$ has a trilogarithm singularity such that

$$\partial_i \partial_j F = \frac{1}{2\pi i} (4 - N_f)(\Omega_D)_{IJ} \log D(X) + \text{regular}$$

for some integer-valued symmetric matrix $(\Omega_D)_{IJ}$, the precise form of which depends on which $U(1)$ factor gets enhanced to $SU(2)$. The $f_k$ for $k > 0$ behave like

$$f_k \sim (D(X))^{2-k(4-N_f)}.$$ 

The matrices $\Lambda_U$ for $U \in SO(2, n; \mathbb{Z})$ that appear in the $T$-duality transformations (3.3) are also constrained by the massless spectrum on the rational quadratic divisors. To analyze these constraints, it is convenient to regard the heterotic monodromy group as the fundamental group $\pi_1(\mathcal{M}_V)$ of the vector multiplet moduli space $\mathcal{M}_V$. We have already mentioned the fact that the classical vector multiplet moduli space $\mathcal{M}_V^{\text{classical}}$...
is of the form \((3.6)\). It follows that \(\pi_1(\mathcal{M}_V^{\text{classical}}) \cong \mathbb{Z} \times SO(\Gamma^{2,n})\). To get the true vector multiplet moduli space \(\mathcal{M}_V\), we have to remove the singularity locus given by the rational quadratic divisors before modding out by \(SO(\Gamma^{2,n})\), and this change affects the fundamental group \(\pi_1(\mathcal{M}_V)\). This group can still be regarded as being generated by the elements of \(SO(\Gamma^{2,n})\), but these generators no longer fulfill the relations of \(SO(\Gamma^{2,n})\). The reason is that a sequence of transformations that would be equivalent to the identity element of \(SO(\Gamma^{2,n})\) and thus of \(\pi_1(\mathcal{M}_V^{\text{classical}})\) might amount to encircling a rational quadratic divisor \(D(X) = 0\) and thus gives a non-trivial element of \(\pi_1(\mathcal{M}_V)\). Indeed, under such a transformation the prepotential \(F\) would be shifted by \(\frac{1}{2}(4 - N_f)\hat{X}^I(\Omega_D)_{IJ}\hat{X}^J\), where \((\Omega_D)_{IJ}\) is the matrix in \((4.1)\). The induced transformation of the period vector \(\hat{\Pi}\) is

\[
\hat{X}^I \rightarrow \hat{X}^I \\
\hat{F}_I \rightarrow \hat{F}_I + (4 - N_f)(\Omega_D)_{IJ}\hat{X}^J.
\]

The requirements that the \(T\)-duality transformations \((3.3)\) form a group isomorphic to \(\pi_1(\mathcal{M}_V)\) and that encircling a rational quadratic divisor induces the above transformation gives a set of equations relating linear combinations of the \(\Lambda_U\)-matrices to the \(\Omega_D\) matrices. The general solution to this system can be written in the form

\[
\Lambda_U = \Lambda_U^{\text{part}} + \Lambda_U^{\text{hom}},
\]

where \(\Lambda_U^{\text{part}}\) is some particular solution and \(\Lambda_U^{\text{hom}}\) is the general solution to the corresponding homogeneous equations, i.e. the equations with all the \(\Omega\)-matrices equal to zero. The latter equations read \((d^{(1)}\Lambda^{\text{hom}})_{U_1,U_2} = 0\) for all \(U_1, U_2 \in SO(2, n; \mathbb{Z})\), where

\[
(d^{(1)}\Lambda^{\text{hom}})_{U_1,U_2} = \Lambda^{\text{hom}}_{U_1U_2} - \Lambda^{\text{hom}}_{U_2U_1} - U_2^t \Lambda^{\text{hom}} U_1.
\]

The interpretation is that in the absence of the singularities on the rational quadratic divisors described by the \(\Omega_D\)-matrices, the monodromy group is isomorphic to \(SO(2, n; \mathbb{Z})\), and performing first a \(U_2\) transformation and then a \(U_1\) transformation is then equivalent to performing a \(U_1U_2\) transformation. Adding a physically trivial term \(\hat{X}^IM_{IJ}\hat{X}^J\) to the prepotential, where \(M_{IJ}\) is an arbitrary symmetric matrix, would change the \(\Lambda_U\)-matrices by \((d^{(0)}M)_U\), where

\[
(d^{(0)}M)_U = U^tMU - M.
\]

Noting that \((d^{(1)}d^{(0)}M)_{U_1,U_2} = 0\) for all \(U_1, U_2 \in SO(2, n; \mathbb{Z})\), we see that we are really interested in the cohomology group

\[
H^1 = \text{Ker}(d^{(1)})/\text{Im}(d^{(0)}).
\]

This group is trivial for \(SO(2,1; \mathbb{Z})\) and \(SO(2,2; \mathbb{Z})\) \((4)\), and we believe this to be true in the general case of \(SO(2, n; \mathbb{Z})\) as well. The matrices \(\Lambda_U\) for \(U \in SO(2, n; \mathbb{Z})\) would then be uniquely determined by the spectrum of massless particles on the various rational quadratic divisors.
5 The invariant dilaton and the covariant coordinates

The functions $f_k$ appearing in (3.3) are constrained by the $T$-duality transformations (3.3). These constraints can be worked out explicitly, but they are fairly complicated, due to quantum corrections to the transformation laws of the homogeneous special coordinates $X^I$. Our aim in this section is to find a new set of homogeneous coordinates $Y^I$, that are not special coordinates, but which have the virtue that they transform in a simpler way under $SO(2, n; \mathbb{Z})$. If we then express some quantity which is invariant under $SO(2, n; \mathbb{Z})$ in these new coordinates, we get a set of functions related to the $f_k$ but with simpler transformation properties.

Our strategy is now as follows: We will try to construct a set of quantities $\hat{Y}^I$ that are invariant under the $S$-duality transformations (3.2), transform as

$$\hat{Y}^I \rightarrow U^I_J \hat{Y}^J$$

under the $T$-duality transformations (3.3), and fulfill

$$\hat{Y}^I \eta_{IJ} \hat{Y}^J = 0$$

identically on $\mathcal{M}_V$. Furthermore, we will need an additional quantity, the invariant dilaton $S^{\text{inv}}$, that transforms as

$$S^{\text{inv}} \rightarrow S^{\text{inv}} + b$$

under $S$-duality and is invariant under $T$-duality. The motivation is that these properties characterize the $\hat{X}^I$ and the perturbatively invariant dilaton in the semi-classical case, i.e. when $f = f_0$. The new coordinates $Y^I$ are then defined as

$$Y^0 = \hat{Y}^0$$
$$Y^1 = \hat{Y}^0 S^{\text{inv}}$$
$$Y^i = \hat{Y}^i.$$  

(5.4)

We can of course also work with the inhomogeneous covariant coordinates $S^{\text{inv}}$ and $(T^{\text{cov}})^i = Y^i / Y^0$, which parameterize $\mathcal{M}_V$.

We begin by regarding the $\hat{X}^I$ as homogeneous coordinates on $\mathcal{M}_V$. We introduce the notation $\partial_I = \frac{\partial}{\partial \hat{X}^I}$. The Jacobian matrix and its inverse for the transformation $X^I \rightarrow \hat{X}^I$ are

$$\frac{\partial \hat{X}^I}{\partial X^J} = \delta^I_J + \delta^I_1 B_J, \quad \frac{\partial X^I}{\partial \hat{X}^J} = \delta_J^I - \delta_1^I \frac{B_J}{1 - B_1}.$$  

(5.5)

Here, $B_J = 2\eta_{JK} \hat{X}^K / \hat{X}^0 + \partial_J \partial f + \delta_1^0 \frac{\partial f}{\hat{X}^0}$. Hence, the change of coordinates from $X^I$ to $\hat{X}^I$ is singular whenever the Jacobian determinant

$$\det \left( \partial_I \hat{X}^J \right) = \partial_1^2 f$$

vanishes but is regular generically. It follows from the expansion (3.5) of $f$ that an irreducible component of the singular locus is given by the divisor $q = 0$, i.e. $S = i\infty$. Indeed, the $\hat{X}^I$ fulfill the relation $\hat{X}^I \eta_{IJ} \hat{X}^J = 0$ at $q = 0$ and are thus not independent.
there. In a sense, there is a physical singularity in the ‘weak coupling limit’ \( q = 0 \), and encircling this divisor produces exactly the \( S \)-duality monodromy transformation (3.2). Although a prepotential does not exist in the semi-classical limit (essentially because the transformation \( X \to \hat{X} \) is singular in this limit) a prepotential generically exists. In fact, \( \hat{F} = \frac{1}{2} \hat{X}^I \hat{F}_I \) has the property \( \hat{F}_I = \partial_I \hat{F} \).

We now define

\[
\tilde{S} = \frac{1}{2(n+2)} \eta^{IJ} \partial_I \hat{F}_J
\]  

(5.7)

and

\[
\hat{Y}^I = \frac{1}{2} \left( \hat{X}^I - \frac{\hat{X}^K \eta_{KL} \hat{X}^L}{\partial_M \hat{S} \eta^{MN} \partial_N \hat{S}} \eta^{IJ} \partial_J \hat{S} \right).
\]  

(5.8)

It follows from (3.2) and (3.3) that \( \tilde{S} \) transforms under \( S \)- and \( T \)-duality as

\[
\tilde{S} \to \tilde{S} + b
\]  

(5.9)

and

\[
\tilde{S} \to \tilde{S} + \frac{1}{2(n+2)} \eta^{IJ} (\Lambda_U)_{IJ},
\]  

(5.10)

whereas the \( \hat{Y}^I \) are invariant under \( S \)-duality, transform as (5.1) under \( T \)-duality and satisfy the constraint (5.2). (Here we have used that \( \hat{X}^I \partial_I \hat{S} = X^I \partial_I S = 0 \). Expressing \( \tilde{S} \) in terms of \( S \) and \( f \) we get

\[
\tilde{S} = S + \frac{1}{2(n+2)} \left( \Delta f - \frac{\partial_i \partial_j f \eta^{ij} \partial_j f}{\partial_i f} - \frac{4 \partial_i f}{X^0 \partial_i f} \right).
\]  

(5.11)

Note that \( \tilde{S} \) will have singularities on the rational quadratic divisors. A direct calculation gives

\[
\tilde{S} = S + \frac{1}{2(n+2)} \left( \Delta f_0 - \frac{4}{2\pi i} \right) + \mathcal{O}(q),
\]  

(5.12)

so \( \tilde{S} \) is regular in the semi-classical limit and generalizes the perturbatively pseudo-invariant dilaton that has been discussed in the literature [2]. Furthermore,

\[
\partial_I \tilde{S} = -\frac{2}{(2\pi i)^2 q f_1} \left( \eta_{IJ} \hat{X}^J + \mathcal{O}(q) \right)
\]  

(5.13)

and

\[
- \frac{\hat{X}^K \eta_{KL} \hat{X}^L}{\partial_M \hat{S} \eta^{MN} \partial_N \hat{S}} = \left( \frac{(2\pi i)^2 q f_1}{2} \right)^2 + \mathcal{O}(q^3),
\]  

(5.14)

so if we define the branch of the square root function in (5.8) by

\[
\sqrt{\frac{\hat{X}^K \eta_{KL} \hat{X}^L}{\partial_M \hat{S} \eta^{MN} \partial_N \hat{S}}} = \frac{(2\pi i)^2 q f_1}{2} + \mathcal{O}(q^2)
\]  

(5.15)

we get

\[
\hat{Y}^I = \hat{X}^I + \mathcal{O}(q).
\]  

(5.16)
The invariant dilaton can now be written as

\[ S_{\text{inv}} = \tilde{S} + \frac{1}{2(n + 2)} L, \]  

where the function \( L \) is of degree zero and is determined up to an additive constant by the requirements that \( S_{\text{inv}} \) should have no singularities on \( \mathcal{M}_V \), transform correctly under \( S \)-duality and be invariant under \( T \)-duality. \( L \) should thus be invariant under \( S \)-duality and transform as

\[ L \to L - \eta^{IJ} (\Lambda_U)_{IJ} \]  

under \( T \)-duality. We see that

\[ \Psi = \exp (2\pi i L) \]  

is invariant under \( T \)-duality because of the integrality of the \( \Lambda_U \) matrices. Introducing \( q^{\text{inv}} = e^{2\pi i S_{\text{inv}}} \), we can expand \( \Psi \) as

\[ \Psi = \sum_{k=0}^{\infty} (q^{\text{inv}})^k \Psi_k, \]  

where the functions \( \Psi_k \) are independent of \( Y^1 \) and of degree zero in \( Y^0 \) and the \( Y^i \).

6 Automorphic properties of the prepotential

In this section, we will use the invariant dilaton and the covariant coordinates \( \hat{Y}^I \) to express the functions \( f_k \), that enter in the expansion (3.5) of the prepotential, in terms of another set of functions with simple transformation properties under \( T \)-duality.

We will begin by reviewing the definition of an automorphic form of \( SO(\Gamma^{2,n}; \mathbb{Z}) \), see e.g. [7]. Define the domain \( P \subset \Gamma^{2,n} \otimes \mathbb{C} \) through

\[ P = \{ \hat{Y}^I \mid \hat{Y}^I \eta_{IJ} \hat{Y}^J = 0, \hat{Y}^I \eta_{IJ} \overline{\hat{Y}^J} < 0 \}. \]  

The \( T \)-duality group \( SO(\Gamma^{2,n}) \cong SO(2, n; \mathbb{Z}) \) has a natural action on \( P \). An automorphic form of weight \( w \) is now defined as a homogeneous function on \( P \) of degree \( -w \) that is invariant under \( SO(2, n; \mathbb{Z}) \). Alternatively, we can construct a model of the Hermitian symmetric space of \( SO(2, n) \)

\[ H \cong \frac{SO(2, n)}{SO(2) \times SO(n)}, \]  

as \( H \cong P/C^* \), where \( \lambda \in C^* \) acts as \( \hat{Y}^I \mapsto \lambda \hat{Y}^I \). \( P \) is then a principal \( C^* \)-bundle over \( H \) and an automorphic form of weight \( w \) is an \( SO(2, n; \mathbb{Z}) \) invariant holomorphic section of \( P^w \).

We have in fact already introduced a set of automorphic forms of weight zero, or automorphic functions, namely the \( \Psi_k \) of (5.20) if we regard them as functions defined on \( P \) rather than as functions of the \( Y^I \) by using the relations

\[ \hat{Y}^0 = Y^0 \]
\( \hat{Y}^1 = \frac{Y^i \eta_{ij} Y^j}{Y^0} \)
\( \hat{Y}^i = Y^i. \) (6.3)

We can now easily construct more examples of automorphic forms by expressing an arbitrary \( S - \) and \( T - \) duality invariant quantity \( A \) of degree \(-w\) in terms of the covariant coordinates \( Y^I \). Invariance of \( A \) under \( S - \) duality means that it can be expanded as
\[
A = \sum_{k=0}^{\infty} (q^\text{inv})^k A_k
\]
where the functions \( A_k \) are independent of \( Y^1 \) and of degree \(-w\) in \( Y^0 \) and the \( Y^i \). (We have assumed that \( A \) is regular in the \( S \to \infty \) limit.) Invariance of \( A \) under \( T - \) duality is now tantamount to the \( A_k \) being automorphic forms of weight \( w \) when regarded as functions on \( P \) via (6.3).

A simple example of such an invariant is \( \Upsilon = \hat{X}^I \eta_{IJ} \hat{X}^J = -X^0 \partial_1 f \), which is of degree 2. It can be expanded in terms of the functions \( f_k \) in (3.5) as
\[
\Upsilon = -2\pi i \sum_{k=1}^{\infty} k q^k f_k.
\] (6.5)
and in terms of the invariant dilaton and functions \( h_k \) of the covariant coordinates as
\[
\Upsilon = -2\pi i \sum_{k=1}^{\infty} k (q^\text{inv})^k h_k,
\] (6.6)
Comparing the two expansions (6.6) and (6.5) of \( \Upsilon \) and using the expression (5.17) with (5.7) for the invariant dilaton and (5.8) for the covariant coordinates, we can express the \( f_k \) in terms of the \( h_k \) and vice versa.

In this way we get
\[
h_1 = \exp \left( -\frac{2\pi i}{2(n+2)} \Delta f_0 - \frac{4}{2\pi i} + L_0 \right) f_1
\]
\[
h_2 = \frac{n}{n+2} \exp \left( -\frac{4\pi i}{2(n+2)} \Delta f_0 - \frac{4}{2\pi i} + L_0 \right) \left[ f_2 - \frac{2\pi i}{4n} f_1 \Delta f_1 - \frac{2\pi i}{4n} f_1 f_1 L_1
\right.
\]
\[
+ \frac{2\pi i}{16n} (2-n) \partial_i f_1 \eta^{ij} \partial_j f_2 + \frac{(2\pi i)^2}{16n} f_1 (\partial_i f_1 - \frac{2\pi i}{2(n+2)} f_1 \partial_i \Delta f_0) \eta^{ij} \partial_j L_0
\]
\[
+ \frac{(2\pi i)^2}{16n} f_1 \partial_i f_1 \eta^{ij} \partial_j \Delta f_0 - \frac{(2\pi i)^3}{64n(n+2)} f_1^2 \partial_i \Delta f_0 \eta^{ij} \partial_j \Delta f_0 \right]
\] (6.7)
where the \( L_k \)'s are defined through
\[
L = \sum_{k=0}^{\infty} L_k q^k.
\] (6.8)

The expressions for the \( h_k \) become increasingly complex for larger \( k \), but the general structure is that \( h_k \) equals \( \exp \left( -\frac{2\pi i k}{2(n+2)} (\eta^{ij} \partial_i \partial_j f_0 - \frac{1}{2\pi i} + L_0) \right) \) times a differential polynomial in the \( f_{k'} \) and \( L_{k'} \) of degree two and charge \( k \) if \( f_{k'} \) and \( L_{k'} \) are assigned charge \( k' \).
A heuristic argument goes as follows: First we observe that at any given order in the $q$ expansion of (5.6), only $f_1$ appears in the denominator. Hence generically $h_k$ should on the face of it be singular when $f_1 = 0$. However, $\Upsilon = -X^0 \partial_1 f$, and is regular at the zeros of $f_1$ so generically (assuming there is no great conspiracy) this should be true also term by term in the expansion. Furthermore, $f$ can be expanded (by comparing with the type II expression) in trilogarithms, and the first derivative of a trilogarithm is non-singular (except at infinity); in particular nothing special should happen when $f_1 = 0$. Hence, we conclude that terms which are singular when $f_1 = 0$ are absent.

Since $\hat{\partial}_I \cdots \hat{\partial}_k \hat{F}$ transforms as a tensor it can be used together with $\hat{X}^I$ to build further invariants. These can also be expanded in terms of automorphic forms in the same way as above. These are not independent, though, but can be expressed in terms of the $h_k$ above and certain automorphic combinations of their derivatives. The simplest such combination is the Schwarzian derivative

$$h \triangle h - \frac{n+2}{4} \eta^{ij} \partial_i h \partial_j h,$$

which is a meromorphic automorphic form of weight $-2$ whenever $h$ is.

The above construction goes through unaltered even if we set $L_k = 0$ for $k > 0$. The only difference is that $S^{inv}$ will have singularities somewhere on $\mathcal{M}_V$, but we can still construct meromorphic automorphic forms $h_k$ in terms of $\Psi_0$ and the $f_k$. We can go further and also set $L_0 = 0$. The $h_k$ will then transform with the phase $\exp \left( -\frac{2\pi i k}{2(n+2)} \eta^{IJ} (\Lambda_U)_{I,J} \right)$, but products of the form $h_{k_1} \cdots h_{k_m}$ such that $k_1 + \cdots + k_m = 2(n+2)$ will be true meromorphic automorphic forms due to the integrality of the $\Lambda_U$ matrices. Following the path outlined above we thus get explicit formulas relating the functions $f_k$ to automorphic forms.

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A  The low-dimensional cases

Our discussion so far applies to $SO(2, n; \mathbb{Z})$ for arbitrary $n$. In this appendix, we will make some comments on the cases when $n = 1, 2, 3$. Here we have a more explicit knowledge of the space of automorphic forms, which allows for quite explicit calculations to be performed.
A.1 \[ n = 1 \]

Here we use the isomorphism \( SO(2, 1; \mathbb{Z}) \cong PSL(2; \mathbb{Z}) \) given by

\[
PSL(2; \mathbb{Z}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto U = \begin{pmatrix} d^2 & c^2 & 2cd \\ b^2 & a^2 & 2ab \\ bd & ac & ad + bc \end{pmatrix} \in SO(2, 1; \mathbb{Z}). \tag{A.1}
\]

The action of the T-duality transformation \( \hat{X}^I \to U^I J \hat{X}^J \) on the inhomogeneous special coordinate \( T = X^2/X^0 = \hat{X}^2/\hat{X}^0 \) is then

\[ T \to \frac{aT + b}{cT + d}, \tag{A.2} \]

i.e. the standard \( PSL(2, \mathbb{Z}) \) action on the upper halfplane. Classically, the vector multiplet moduli space \( \mathcal{M}_V \) is thus given by the dilaton factor times the \( PSL(2, \mathbb{Z}) \) fundamental domain. The only rational quadratic divisor is \( T = i \), at which point a \( U(1) \) factor gets enhanced to \( SU(2) \) without any additional hyper multiplet doublets becoming massless.

Writing an automorphic form \( \Theta \) of weight \( w \) in the form

\[ \Theta = (X^0)^{-w} \theta(T), \tag{A.3} \]

we see that automorphy of \( \Theta \) is equivalent to \( \theta \) being an \( PSL(2, \mathbb{Z}) \) modular form of weight \( 2w \), i.e.

\[ \theta \left( \frac{aT + b}{cT + d} \right) = (cT + d)^{2w} \theta_k(T). \tag{A.4} \]

(Note the different conventions for the weights of \( SO(2, 1; \mathbb{Z}) \) automorphic forms and \( PSL(2, \mathbb{Z}) \) modular forms.)

The graded ring of holomorphic automorphic forms is freely generated by the Eisenstein series \( E_4(T) \) and \( E_6(T) \) of weights 4 and 6 respectively. They have simple zeros at \( T = \rho = \exp(\pi i/3) \) and \( T = i \) respectively. We will also have use for the weight 0 form \( j(T) = 1728E_4^3(T)/(E_4^2(T) - E_6^2(T)) \), which is a one-to-one mapping from the fundamental domain to the Riemann sphere such that \( j(T) \) has a simple pole at the cusp \( T = i\infty \) and a triple zero at \( T = \rho \) and \( j(T) - 1728 \) has a double zero at \( T = i \). The function \( L_0 \) should have logarithmic singularities at the rational quadratic divisor \( T = i \) and also at the cusp \( T = i\infty \); it is thus proportional to \( \log(j(T) - 1728) \).

A model of this kind can be obtained by compactifying the \( E_8 \times E_8 \) heterotic string on \( K3 \times T^2 \), restricting to the subspace where the complexified Kähler modulus \( U \) of the \( T^2 \) equals its complex structure modulus \( T \) so that an extra \( SU(2) \) group arises and breaking all non-abelian gauge symmetry by embedding 10 instantons in each \( E_8 \) factor and 4 instantons in the \( SU(2) \) factor. This model is dual to the type IIA string compactified on a degree 12 hypersurface in the weighted projective space \( \mathbb{P}^4_{1,1,2,2,6} \) with Hodge numbers \( h^{1,1} = 2 \) and \( h^{2,1} = 128 \)\footnote{This pair of dual models has been studied extensively \cite{13, 14, 15}.}. A particular feature of this Calabi-Yau space is the existence of a certain symmetry between the \( S \) and \( T \) moduli \cite{37, 38} which, together with the singularity structure described above, uniquely determines the special geometry \cite{11}.\footnote{This pair of dual models has been studied extensively \cite{13, 14, 15}.}
A.2 \( n = 2 \)

Here we use that \( SO(2, 2; \mathbb{Z}) \) is generated by two commuting \( PSL(2; \mathbb{Z}) \) subgroups with the standard action on the inhomogeneous special coordinates

\[
T = \frac{X^2 - X^3}{X^0}, \quad U = \frac{X^2 + X^3}{X^0},
\]

and a \( \mathbb{Z}_2 \) transformation that exchanges them. Classically, \( \mathcal{M}_V \) is given by the dilaton factor, parametrized as usual by \( S = X^1/X^0 \), times the product of two \( PSL(2; \mathbb{Z}) \) fundamental domains modulo the operation that exchanges them. The only rational quadratic divisor is \( T - U = 0 \), where an \( SU(2) \) factor arises without any extra hyper multiplets. (The points \( T = U = i \) and \( T = U = \rho \) are special in that the divisor intersects its images under \( SO(2, 2; \mathbb{Z}) \); at these points we instead get an \( SU(2) \times SU(2) \) or \( SU(3) \) factor respectively.)

Writing an automorphic form \( \Theta \) of weight \( w \) as

\[
\Theta = (X^0)^2 \theta(T, U),
\]

we get that \( \theta \) is symmetric in \( T \) and \( U \) and has weight \( w \) with respect to both \( PSL(2; \mathbb{Z}) \) factors.

The graded ring of holomorphic automorphic forms is generated by \( E_4(T)E_4(U), \ E_6(T)E_6(U) \) and \( E_4(T)E_6(U) + E_4(U)E_6(T) \). We will also have use for \( j(T) - j(U) \), which has a simple zero on the rational quadratic divisor \( T - U = 0 \) and a simple pole at \( T = i\infty \) and \( U = i\infty \). In fact, \( L_0 \) is proportional to \( \log(j(T) - j(U)) \).

There exists three models of this type (two of which are isomorphic) which can be obtained by compactifying the \( E_8 \times E_8 \) heterotic string on \( K3 \times T^2 \) and embedding \( 12 - l \) and \( 12 + l \) instantons in the first and second \( E_8 \) factor respectively with \( l = 0, 1, 2 \) \cite{19, 33, 34}. It is then possible to break all the non-abelian gauge symmetries, leaving only a \( U(1)^4 \) gauge symmetry from the \( N_V = 3 \) vector multiplets and the graviphoton (inherited from the \( T^2 \) and the dilaton). These models are dual to type IIA compactifications on a particular kind of Calabi-Yau threefolds, elliptic fibrations over a so called Hirzebruch surface \( F_l \), where the latter is itself a \( \mathbb{P}^1 \) fibration over a base \( \mathbb{P}^1 \) \cite{41}. It has been shown that the cases \( l = 0 \) and \( l = 2 \) are isomorphic \cite{41, 42} and we will thus only consider \( l = 0, 1 \).

We will now study certain properties of the prepotential \( F \), in particular the contribution from the one-instanton correction, \( f_1 \). Recall, that \( F \) can be written as

\[
F = \frac{X^1}{X^0}X^i \eta_{ij} X^j + \sum_{k=0}^{\infty} q^k f_k.
\]

Along the lines of the discussion in section 6 one can show that \( f_k \) can be written in the form

\[
f_k = \exp \left( 2\pi i k \frac{1}{2(n + 2)} \Delta h_0 \right) h_k + \ldots.
\]

The functions \( h_k(X) \) are independent of \( X^1 \) and of degree two in \( X \) and transform (for \( k > 0 \)) as automorphic forms up to a phase in the sense that

\[
h_k(\tilde{X}) = \exp \left( -2\pi i k \frac{1}{2(n + 2)} \eta^{IJ} \Lambda_{IJ} \right) h_k(X).
\]
The ellipsis in the formula for $f_k$ denote a polynomial in $\triangle^m h_{k'}$ except $h_0$ and $\triangle h_0$.

Let us now consider the cases of interest. We introduce the inhomogeneous coordinates $S$, $T$, and $U$, which are the natural coordinates from the heterotic compactification, as well as
\[
q_S = q = \exp(2\pi i S), \quad q_T = \exp(2\pi i T), \quad q_U = \exp(2\pi i U).
\]
We can then rewrite (A.9) as
\[
F = STU + F_{\text{loop}}(T,U) + \sum_{k=1}^{\infty} q^k f_k(T,U)
\]
where
\[
F_{\text{loop}} = F_{\text{cubic}} + \frac{1}{(2\pi i)^3} \sum_{i,j} n(i,j,0) q_T^i q_U^{-j}
\]
and
\[
F_{\text{inst}} = f_1 = \frac{1}{(2\pi i)^3} \sum_{i,j=0}^{\infty} n(i,j,1) q_T^i q_U^{-j-1} q_S.
\]
Note that in order for the above expansion to be valid we take $\text{Im} U < \text{Im} T < \text{Im} S$.

We have here used the duality between the heterotic string on $K3 \times T^2$ and the dual type IIA Calabi-Yau manifold in order to express the 1-loop and 1-instanton corrected prepotential in terms of the number of rational curves $n(i,j,k)$ [8]. Although $n(i,j,k)$ does in general depend on the particular manifold, it turns out that for the two cases that we are studying, $l = 0, 1$, when restricted to the 1-loop corrections, they agree, i.e. $n(i,j,0)$ are the same for the two models. Furthermore, $F_{\text{cubic}}$ is given by
\[
F_{\text{cubic}} = \frac{1}{3} U^3 - \delta_{l1} \left( \frac{1}{2} U^2 T - \frac{1}{2} U T^2 \right)
\]
where $\delta_{l1}$ is the usual Kronecker delta. The cubic couplings in the two theories can however be made to agree if the dilaton in the $l = 1$ case is shifted by
\[
S \rightarrow S + \frac{1}{2} U - \frac{1}{2} T.
\]
Clearly, this shift will remove the mixed terms in $U$ and $T$ while leaving the rest of the 1-loop effect intact. However, the instanton corrections will pick up non-integer exponents of the type $(q_U q_T^{-1/2k})$ for the $k$-th order space-time instanton correction. Thus, although it seems to be a perfectly valid transformation from a perturbative point of view, we come to the conclusion that the shift of the dilaton is not allowed non-perturbatively. Hence, the two models are inequivalent.

By using the expression of $h_1$ in terms of $f_1$ and $f_0$,
\[
h_1 = \exp(-\pi i \partial_U \partial_T f_0) f_1
\]
it is indeed possible to write $h_1$ in terms of the generators of the graded ring of holomorphic forms;
\[
h_1 l = 0 = \frac{-2}{2\pi i (j(T) - j(U)) n^{12}(q_T) n^{12}(q_U)} \left( \frac{E_4(q_T)}{n^{12}(q_T)} \right) \left( \frac{E_6(q_T)}{n^{12}(q_U)} \right)
\]
\[9\]The $n(i,j,k)$ have been computed for the models in question using INSTANTON [8].
and

\[
h_1^{i=1} = \frac{1}{2\pi i} \frac{E_4(q_T)E_4(q_U)}{(j(T) - j(U))\eta_1^{12}(q_T)\eta_1^{12}(q_U)} \left\{ \left( \frac{E_6(q_T)}{\eta_1^{12}(q_T)} \right)^2 + \left( \frac{E_6(q_U)}{\eta_1^{12}(q_U)} \right)^2 \right\}
\]  

(A.17)

Note that these functions indeed transform with a phase under the \( SO(2, 2; \mathbb{Z}) \) transformation that exchanges \( T \) and \( U \).

### A.3 \( n = 3 \)

Here we use that \( SO(2, 2; \mathbb{Z}) \cong Sp(4; \mathbb{Z}) \). Assembling the inhomogeneous special coordinates

\[
T = \frac{X^2 - X^3}{X^0}, \quad U = \frac{X^2 + X^3}{X^0}, \quad V = \frac{X^4}{X^0}
\]

(A.18)

into \( \tau = \begin{pmatrix} T & V \\ V & U \end{pmatrix} \), the action of \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4; \mathbb{Z}) \) is

\[
\tau \rightarrow A\tau + B \\
C\tau + D
\]

(A.19)

which is the standard \( Sp(4; \mathbb{Z}) \) on the Siegel generalized upper halfplane. There are two different rational quadratic divisors \[\{13\}\]: In addition to \( T - U = 0 \) (which only gives an \( SU(2) \) factor) there is also \( V = 0 \) where we also get some model-dependent number of extra hyper multiplets \[\{14, 15\}\]. In the \( Sp(4) \) context, these divisors are known as the Humbert surfaces \( H_4 \) and \( H_1 \) respectively. Automorphic forms then correspond to Siegel modular forms for \( Sp(4; \mathbb{Z}) \). It follows from the Koechner boundedness principle that a nearly holomorphic automorphic form is in fact holomorphic. The ring of such forms is generated by the Eisenstein series \( E_4(T, U, V) \), \( E_6(T, U, V) \), \( E_{10}(T, U, V) \), and \( E_{12}(T, U, V) \) together with the cusp form \( \mathcal{C}_{35}(T, U, V) \) \[\{16\}\].

### References

[1] A. Giveon, M. Porrati and E. Rabinovici, “Target space duality in string theory”, \textit{Phys. Rep.} \textbf{244} (1994) 77–202, \texttt{hep-th/9401139}.

[2] B. de Wit, V. Kaplunovsky, J. Louis, and D. Lüst, “Perturbative couplings of vector multiplets in N=2 heterotic string vacua,” \textit{Nucl. Phys.} \textbf{B451} (1995) 53–95, \texttt{hep-th/9504006}.

[3] I. Antoniadis, S. Ferrara E. Gava, K. S. Narain, and T. R. Taylor, “Perturbative prepotential and monodromies in N=2 heterotic superstring,” \textit{Nucl. Phys.} \textbf{B447} (1995) 35–61, \texttt{hep-th/9504034}.

[4] I. Antoniadis, S. Ferrara, E. Gava, K. S. Narain, and T. R. Taylor, “Duality symmetries in N = 2 heterotic superstring,” \textit{Nucl. Phys. Proc. Suppl.} \textbf{45BC} (1996) 177–187, \texttt{hep-th/9510079}.
[5] V. Kaplunovsky, J. Louis and S. Theisen, “Aspects of duality in $N = 2$ string vacua”, Phys. Lett. B357 (1995) 71–75, hep-th/9506110.

[6] J. A. Harvey and G. Moore, “Algebras, BPS states, and strings,” Nucl. Phys. B463 (1996) 315–368, hep-th/9510182.

[7] R. E. Borcherds, “Automorphic forms on $O_{s+2}(R)$ and infinite products,” Invent. Math. 120 (1995) 161–213;
R. E. Borcherds, “Automorphic forms with singularities on Grassmannians”, alg-geom/9609022.

[8] M. Henningson and G. Moore, “Counting curves with modular forms,” Nucl. Phys. B472 (1996) 518–528, hep-th/9602154.

[9] A. Klemm, W. Lerche, and P. Mayr, “K3 fibrations and heterotic type II string duality,” Phys. Lett. B357 (1995) 313–322, hep-th/9506112.

[10] B. H. Lian and S.-T. Yau, “Mirror maps, modular relations and hypergeometric series 1”, hep-th/9507151;
B. H. Lian and S.-T. Yau, “Mirror maps, modular relations and hypergeometric series 2”, hep-th/9507153.

[11] M. Henningson and G. Moore, “Threshold corrections in K(3) x T(2) heterotic string compactifications..,” Nucl. Phys. B482 (1996) 187–212, hep-th/9608145.

[12] J. A. Harvey and G. Moore, “On the algebras of BPS states.,” hep-th/9609017.

[13] B. de Wit, P. G. Lauwers, and A. Van Proeyen, “Lagrangians of N=2 supergravity - matter systems,” Nucl. Phys. B255 (1985) 569.

[14] J. Bagger and E. Witten, “Matter couplings in $N = 2$ supergravity, Nucl. Phys. B222 (1983) 1–10.

[15] B. de Wit and A. Van Proeyen, “Potentials and symmetries of general gauged N=2 supergravity - Yang-Mills models,” Nucl. Phys. B245 (1984) 89.

[16] E. Cremmer, C. Kounnas, A. Van Proeyen, J.-P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, “Vector multiplets coupled to $N = 2$ supergravity: superhiggs effect, flat potentials and geometric structure”, Nucl. Phys. B250 (1985) 385.

[17] N. Seiberg, “Observations on the modulispace of superconformal field theories”, Nucl. Phys. B303 (1988) 286.

[18] C. Vafa and E. Witten, “Dual string pairs with N=1 and N=2 supersymmetry in four- dimensions,” hep-th/9507050.

[19] S. Kachru and C. Vafa, “Exact results for $N = 2$ compactifications of heterotic strings”, Nucl. Phys. B450 (1995) 69–89, hep-th/9505105.
[20] S. Ferrara, J. Harvey, A. Strominger and C. Vafa, “Second quantized mirror symmetry”, *Phys. Lett.* **B361** (1995) 59–65, hep-th/9505162.

[21] P. S. Aspinwall and J. Louis, “On the ubiquity of K3 fibrations in string duality,” *Phys. Lett.* **B369** (1996) 233–242, hep-th/9510234.

[22] S.-T. Yau ed., *Essays on mirror symmetry*, (International Press, Hong Kong, 1992); B. Greene and S.-T. Yau eds., *Mirror symmetry II*, (International Press, Hong Kong, 1997).

[23] A. Strominger and E. Witten, “New manifolds for superstring compactification”, *Comm. Math. Phys.* **101** (1985) 341.

[24] M. Dine, N. Seiberg, X. G. Wen and E. Witten, “Nonperturbative effects on the string world sheet”, *Nucl. Phys.* **278** (1986) 769; M. Dine, N. Seiberg, X. G. Wen and E. Witten, “Nonperturbative effects on the string world sheet 2”, *Nucl. Phys.* **289** (1987) 319.

[25] P. S. Aspinwall and D. R. Morrison, “Topological field theory and rational curves” *Commun. Math. Phys.* **151** (1993) 245-262, hep-th/9110048.

[26] A. Strominger, “Special geometry,” *Commun. Math. Phys.* **133** (1990) 163–180.

[27] B. Craps, F. Roose, W. Troost and A. Van Proeyen, “What is special Kähler geometry?,” hep-th/9703082.

[28] K. S. Narain, “New heterotic string theories in uncompactified dimensions < 10”, *Phys. Lett.* **169B** (1986) 41; K. S. Narain, M. H. Sarmadi and E. Witten, “A note on toroidal compactification of heterotic string theory”, *Nucl. Phys.* **B279** (1987) 369.

[29] S. Ferrara and A. Van Proeyen, “A theorem on N=2 special Kähler product manifolds,” *Class. Quant. Grav.* **6** (1989) L243.

[30] M. Dine, P. Huet and N. Seiberg, “Large and small radius in string theory”, *Nucl. Phys.* **322** (1989) 301.

[31] K. Oguiso, “On algebraic fiber space structures on a Calabi-Yau 3-fold”, *Int. J. Math.* **4** (1993) 439–465.

[32] E. Witten, “String theory dynamics in various dimensions”, *Nucl. Phys.* **B443** (1995) 85–126, hep-th/9503124.

[33] P. S. Aspinwall, “Enhanced gauge symmetries and K3 surfaces.” *Phys. Lett.* **B357** (1995) 329, hep-th/9507012; P. S. Aspinwall, “Enhanced gauge symmetries and Calabi-Yau threefolds.” *Phys.Lett.* **B371** (1996) 231-237. hep-th/9511171.
[34] N. Seiberg and E. Witten, “Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory”, *Nucl. Phys.* **B426** (1994) 19–52, hep-th/9407087;
N. Seiberg and E. Witten, “Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD”, *Nucl. Phys.* **B431** (1994) 484–550, hep-th/9408099.

[35] S. Kachru, A. Klemm, W. Lerche, P. Mayr, and C. Vafa, “Nonperturbative results on the point particle limit of N=2 heterotic string compactifications,” *Nucl. Phys.* **B459** (1996) 537–558, hep-th/9508155.

[36] A. Ceresole, R. D’Auria, S. Ferrara, and A. Van Proeyen, “Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity,” *Nucl. Phys.* **B444** (1995) 92–124, hep-th/9502072.

[37] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. R. Morrison “Mirror symmetry for two parameter models – I”, *Nucl. Phys.* **B416** (1994) 481, hep-th/9308083.

[38] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, “Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces” *Nucl. Phys.* **B433** (1995) 501, hep-th/9406055;
“Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces” *Commun. Math. Phys.* **167** (1995) 301, hep-th/9308122.

[39] G. Aldazabal, L. E. Ibanez, A. Font and F. Quevedo, “Chains of N=2, D=4 heterotic/type II duals”, *Nucl. Phys.* **B461** (1996) 85, hep-th/9510093.

[40] M. J. Duff, R. Minasian and Edward Witten, “Evidence for Heterotic/Heterotic Duality”, *Nucl. Phys.* **B465** (1996) 413, hep-th/9601036.

[41] D. R. Morrison and C. Vafa, “Compactifications of F-Theory on Calabi–Yau threefolds – I” *Nucl.Phys.* **B473** (1996) 74-92, hep-th/9602114;
D. R. Morrison and C. Vafa, “Compactifications of F-Theory on Calabi–Yau threefolds – II” *Nucl.Phys.* **B476** (1996) 437, hep-th/9603161.

[42] E. Witten, “Phase Transitions In M-Theory And F-Theory”, *Nucl. Phys.* **B471** (1996) 195, hep-th/9603150.

[43] V. A. Gritsenko and V. V. Nikulin, “The Igusa modular forms and ‘the simplest’ Lorentzian Kac-Moody algebras”, alg-geom/9603010.

[44] G. L. Cardoso, G. Curio, and D. Lüst, “Perturbative couplings and modular forms in $N = 2$ string models with a Wilson line,” hep-th/9608154.

[45] G. L. Cardoso, “Perturbative gravitational couplings and Siegel modular forms in $D = 4$, $N = 2$ string models,” hep-th/9612200.

[46] J.-I. Igusa, ”On Siegel modular forms of genus two.” *Amer. J. Math.* **84** (1962) 175-200;
J.-I. Igusa, "On Siegel modular forms of genus two (II).," *Amer. J. Math.* **86** (1964) 392-412;
J.-I. Igusa, "Modular forms and projective invariants.," *Amer. J. Math.* **89** (1967) 817-55;