Large diffeomorphisms in (2+1)-quantum gravity on the torus

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Abstract

The issue of how to deal with the modular transformations – large diffeomorphisms – in (2+1)-quantum gravity on the torus is discussed. I study the Chern-Simons/connection representation and show that the behavior of the modular transformations on the reduced configuration space is so bad that it is possible to rule out all finite dimensional unitary representations of the modular group on the Hilbert space of $L^2$-functions on the reduced configuration space. Furthermore, by assuming piecewise continuity for a dense subset of the vectors in any Hilbert space based on the space of complex valued functions on the reduced configuration space, it is shown that finite dimensional representations are excluded no matter what inner-product we define in this vector space. A brief discussion of the loop- and ADM-representations is also included.

PACS numbers: 04.60.-m, 04.60.Ds, 04.60.Kz

1 Introduction

During the last six years there has been an increasing interest in the study of (2+1)-dimensional quantum gravity, and specifically ”(2+1)-gravity on the torus”. However, although one now can say that gravity has been successfully quantized in (2+1)-dimensions with the use of several different formulations as well as methods, there are still not many physically interesting questions that have been answered definitely. (Maybe because there are no definite answers.) Examples of such interesting questions are; 1) Is area or length

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quantized in (2+1)-quantum gravity? 2) What happens to the classical spacetime singularity upon quantization? 3) What is a complete set of observables in quantum gravity, and what is the Hilbert space upon which these observables act irreducibly?

Furthermore, although most workers in the field seem to agree on how to handle the local symmetries – in principle – there still seems to be quite a lot of disagreement of how to handle the global symmetries of the theory. Some authors claim that the quantum theory must be invariant under those symmetries as well \[1\], \[2\], \[3\], \[4\] while others believe it is enough to have the wavefunctions transforming under a finite-dimensional unitary representation of the symmetry group \[5\], \[6\]. A third alternative is given in \[7\] \[8\] where one basically claims that the global symmetries should almost not restrict the quantum theory at all; it is enough if the wavefunctions transform under any unitary representation of the symmetry group. A fourth group of papers do not explicitly discuss how to deal with the global symmetries \[9\], \[10\], \[11\], \[12\], \[13\], \[14\].

Now, to better understand why different researchers in the field have such a different opinion about the treatment of the global symmetries in quantum gravity, we may study the conventional treatment of global symmetries in standard quantum mechanics. Then we need to distinguish between the two cases: 1) we have a global symmetry that commutes with all the observables in the theory, 2) or the global symmetry does not commute with all observables.

In the first case, we get a so called super selection rule. That is, if we assume that our Hilbert space may be written as a direct sum (integral) of subspaces left invariant by the symmetry, there is no way we can distinguish between states that only differ by the superposition coefficients between these subspaces. Thus, the conclusion must be that every subspace defines a separate quantum representation, and since they do not communicate there is no reason to treat them together; we might as well study each sector separately. Often, these subspaces are of the same dimension – infinite dimensional – and can be considered isomorphic.

One of the simplest example of such a situation is the parity symmetry in the Hilbert space of \(L^2(\mathbb{R})\)-functions. Suppose that all our observables are invariant under parity. Then there is no way to distinguish between the wavefunctions \(\Psi_1(x) := \Psi_s(x) + \Psi_a(x)\) and \(\Psi_2(x) := \Psi_s(x) - \Psi_a(x)\). Here \(\Psi_s(x) = \Psi_s(-x)\) and \(\Psi_a(x) = -\Psi_a(-x)\). The wavefunctions \(\Psi_s(x)\) and \(\Psi_a(x)\) transform under different unitary irreducible representations and they do not communicate. Thus, the even and odd sectors define two isomorphic quantum theories.

A similar situation appears in \(SU(N)\) Yang-Mills theory \[15\]. Here the global symmetries are the large \(SU(N)\) gauge transformations. Since the physical observables in this theory normally are defined in terms of gauge invariant functions of the electric and magnetic field, the observables are automatically invariant under the large gauge transformations as well. We get super selection rules. By assuming the gauge transformations to fall off to the identity-transformation at spatial infinity, one may show that the group of large transformations modulo the small ones is isomorphic to the group of integers. Thus, the only irreducible representations are the one-dimensional ones and they are given by \(D(g) = \exp(i\theta)\) where \(g\) is the generator of the group and \(\theta\) is a parameter labeling the different representations. Hence, we see that we get an infinite number of irreducible representations and since they do not communicate, there is no way to distinguish between states that only differ in the superposition between the representations; we could study
one sector at the time.

Normally, in these cases, one implicitly assumes that every unitary representation can be written as a direct sum (integral) of unitary irreducible representations of the symmetry group. This is not always the case. There is a classification of group representations into type I, II and III \[16\], and it is only for type I representations that this decomposition always can be performed. For type II and III it may happen that it is impossible to write a given representation as a direct sum (integral) of irreducible representations, or, if a decomposition exists, it may not be unique. The relevant group of large diffeomorphisms for quantum gravity on the torus, $\text{PSL}(2, \mathbb{Z})$, allows representations of type II and III, and there is therefore no reason to believe that the Hilbert space should be totally decomposable into irreducible subspaces.

In case 2 – where the global symmetry does not commute with all observables – we get a similar situation as for case 1. Here, however, since the observables do not commute with the symmetry, we can distinguish between wavefunctions that only differ by a symmetry transformation. Thus, the situation will be that our Hilbert space may be decomposable into a direct sum (integral) of irreducible subspaces, but since the sectors now do communicate we cannot study each sector separately; we need all related sectors. However, note here as well that if our representation is not of type I, there is no guarantee that the Hilbert space is decomposable into irreducible pieces.

Thus, we may now understand one reason for disagreement about the treatment of global diffeomorphisms in quantum gravity; if one believes that all physical observables should be invariant under all diffeomorphisms, we get case 1, and if one only requires the physical observables to be invariant under the small diffeomorphisms, we get case 2. Consequently, the situation seems rather clear in both cases whenever the representation of the symmetry group is of type I. In these cases one could just follow the standard textbook treatment.

Now, what are the arguments to support these two different viewpoints? It seems that if we take seriously the fact that all diffeomorphisms – large or small – are pure gauge and we can therefore never physically distinguish between systems related by diffeomorphisms, we are inevitably led to the conclusion that all our physical observables must be invariant under all diffeomorphisms.

The other viewpoint – that observables do not have to commute with the large diffeomorphisms or gauge transformations – follows directly from the conventional Dirac-definition of observables in constrained systems; a physical observable is a function on the phase space that commutes with all first class constraints.

As I see it, there are huge problems with both these viewpoints and we cannot categorically say that we always should treat all diffeomorphisms in a unified way. For instance, by always demanding invariance under all diffeomorphisms, we would have to require wavefunctions for an asymptotically flat spacetime to transform trivially under diffeomorphisms that tend to Poincare transformations at infinity. Most physicists would probably agree that this is a too restricted quantum theory. The problem with the other viewpoint

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\[2\] Amongst the separable locally compact groups which are known to have only type I representations are the compact groups, the commutative groups, the connected semi-simple Lie-groups and the connected real algebraic groups. However there are many examples of connected solvable Lie groups which have non-type I representations. It is known that the regular representation of a discrete group is of type II whenever the subgroup generated by the finite conjugate classes has infinite index \[16\].
is that we now allow objects to be called physical observables although we in principle never can set up a measuring-apparatus to measure them. Normally, in standard quantum mechanics, we say that not all self adjoint operators can be considered as physical observables; a physical observable needs to be in principle measurable. Take for instance the quantum theory of identical particles. We have a global symmetry; permutations of the particles. According to the Dirac-definition of a physical observable, there is nothing wrong with observables that do not commute with the permutation operator. However, we know that if our particles are truly indistinguishable, we can never set up a measuring device to measure any quantity that is not invariant under permutations of the particles. We define a physical observable to be a self adjoint operator that commutes with the permutation operator [17].

Is there any way out of this dilemma? Yes, I believe that if we stop trying to quantize gravity by applying standard quantization methods to the entire spacetime and instead start studying quantum gravity for bounded regions of spacetime, this problem will be naturally solved. That is, when we are trying to use standard arguments from conventional quantum mechanics in a closed quantum mechanical system; quantum gravity for the the entire spacetime, we are really applying our quantization procedures to a situation outside the domain where our quantization rules applies. Quantum mechanics – in its current form – is constructed to be applied to physical systems consisting of a classical observer together with the quantum system. If we try to quantize closed systems without classical observers, we have actually no real guidance from conventional quantum mechanics and we are therefore free to choose a definition of what we mean by physical observables. At the end, experiments will tell.

However, if we instead study bounded regions in spacetime, and quantize only the interior of the region while keeping the outside classical, we could probably carry over most arguments and procedures from standard quantum mechanics.

In short, I believe the situation would be the following: all diffeomorphisms that leave the boundary invariant (diffeomorphisms that goes to the identity transformation at the boundary) should be considered pure gauge and should therefore define superselection rules for the wavefunctions. However, for diffeomorphisms that do not leave the boundary invariant, we just get ordinary symmetries and there is no a priori reason to restrict the quantum theory to any specific representation. This other group of diffeomorphisms – diffeomorphisms that do not go to the identity at the boundary – will therefore be treated analogous to Galilean transformations in classical Newtonian mechanics and Poincare transformations in special relativity.

Thus my opinion is that conventional quantum mechanics does not really give us any instructions of how to handle quantum mechanics for closed systems without classical observers. Or, it may even be that it does not make sense to talk about closed quantum systems in the absence of classical observers. And, if we still want to study such quantum theories, we are free to choose a definition of a physical observable ourselves, and therefore it makes sense to study both approaches: 1) observables must commute with all unphysical transformations, 2) observables just need to commute with small transformations.

In this paper, I choose to study the first approach. I require all observables to be invariant under the full group of diffeomorphisms on the torus, and therefore – according to standard arguments for superselection rules in quantum mechanics [17] – I should only
study the one-dimensional representations. (Just to be safe I will keep the representation slightly more general; finite dimensional, unitary, and irreducible.) I will show that in the connection representation of quantum gravity on the torus, the quantum theory that follows from this approach becomes completely trivial; the Hilbert space is at most one-dimensional if we require the wavefunctions to both be piecewise continuous somewhere on the reduced configuration space and also to transform under a finite-dimensional representation of $\text{PSL}(2, \mathbb{Z})$. I also give a brief discussion of the case of the loop-representation and the reduced ADM-phase space quantization. There, the irreducible sectors seems to be non-trivial.

2 The Chern-Simons/connection approach

In the Chern-Simons/connection approach to Einstein gravity, the starting point is the Chern-Simons action

$$S_{CS} = \int d^3 x \epsilon^{\alpha\beta\delta} \text{Tr} \left( A_\alpha \partial_\beta A_\delta + \frac{1}{3} A_\alpha A_\beta A_\delta \right)$$

or the Hilbert-Palatini action

$$S_{HP} = \int d^3 x \epsilon^{\alpha\beta\delta} e^I_\alpha R_{\beta\delta I} (\omega)$$

where $A_\alpha$ is an $\text{ISO}(1, 2)$ valued connection, $\text{Tr}$ is a non-degenerate invariant bilinear form on the Lie-algebra of $\text{ISO}(1, 2)$, $e^I_\alpha$ is the triad field, and $\omega^I_\alpha$ is an $\text{SO}(1, 2)$ connection. These two actions can be shown to be equivalent provided $A_\alpha = e^I_\alpha P_I + \omega^I_\alpha J_I$, where $P_I$ are the generators of three-dimensional translations and $J_I$ are the generators of $\text{so}(1, 2)$. See [11] for details. To quantize this formulation, one may proceed as follows: 1) Do a (2+1)-decomposition to the Hamiltonian formulation. 2) Identify the phase space variables, and the first class constraints. 3) Solve the constraints and gauge fix the local symmetries. 4) Identify the remaining physical degrees of freedom and the Hamiltonian. 5) Quantize. I will not discuss these steps in detail here. See e.g. [1], [2], [13], [7] for details. In short, one notices that the constraints tell us that the reduced configuration space consists of gauge equivalent classes of flat $\text{so}(1, 2)$ connections, and since a flat connection is completely determined – up to gauge transformations – by its holonomies around the non-contractible loops on the hypersurface, one may parameterize the reduced configuration space by two commuting $\text{SO}(1, 2)$ element. They have to commute since they are suppose to give a representation of the fundamental group of the torus, which is $\pi_1(T^2) \sim \mathbb{Z} \oplus \mathbb{Z}$. This naturally split up the reduced configuration space into three sectors labelled by the kind of vector the $\text{SO}(1, 2)$ elements stabilize; time-like, null or space-like. Only the space-like sector correspond to conventional geometrodynamics, and if the intention is to compare with e.g the ADM-approach, one chooses this sector. In studying this sector, I choose to use the choice made by Carlip [1] for the representation of $\pi_1(T^2)$:

$$A_1 := \begin{pmatrix} \cosh(\lambda) & \sinh(\lambda) & 0 \\ \sinh(\lambda) & \cosh(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 := \begin{pmatrix} \cosh(\mu) & \sinh(\mu) & 0 \\ \sinh(\mu) & \cosh(\mu) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with the identification $(\lambda, \mu) \sim -(\lambda, \mu)$ that follows from a small gauge transformation. Thus the resulting reduced theory is described by two pair of canonically conjugate variables and a vanishing Hamiltonian, i.e the theory is completely trivial. In the notation
used by Carlip (1), we have \( \{ \mu, a \} = \frac{1}{2} \) and \( \{ \lambda, b \} = -\frac{1}{2} \) and all other Poisson brackets vanish. Here, \( a \) and \( b \) are the phase space momenta. The configuration space \( \mathcal{C} \) is topologically the punctured plane \( \mathbb{R}^2 - (0,0) \) where points reflected through the origin are identified. Now, in the transition to the reduced phase space, we have taken care of all the small \( SO(1,2) \) transformations as well as all the small diffeomorphisms. However, we still have the large \( O(1,2) \) transformations as well as the large diffeomorphisms to deal with. These large transformations are symmetries of the classical theory, and should therefore presumably also be symmetries of the quantum theory. In this case, however, it is enough to have the wavefunctions transforming under a unitary representation of this remaining symmetry group. The large or global diffeomorphisms on the torus are normally called modular transformations. Their action on the configuration space is given by

\[
S: \lambda \rightarrow \mu, \quad \mu \rightarrow -\lambda \tag{4}
\]
\[
T: \lambda \rightarrow \lambda, \quad \mu \rightarrow \mu + \lambda \tag{5}
\]

What is left of the large \( O(1,2) \) transformations are the internal time-reversal and parity transformations. However, as is easily seen, they have trivial action on the configuration space, meaning that we do not have to bother about them. Thus, we only have the modular transformations left. By representing a point in the configuration space as \( v := \left( \begin{array}{c} \mu \\ \lambda \end{array} \right) \), we may represent the \( S \) and \( T \) transformations as

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6}
\]
\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{7}
\]

which are easily recognized as the generators of the group \( SL(2, \mathbb{Z}) \). However, since we should identify two points that are related via a reflection through the origin \( v \sim -v \), it follows that we have to identify two \( SL(2, \mathbb{Z}) \) elements that differ only by the sign. Thus, effectively we are dealing with the group \(^3 PSL(2, \mathbb{Z}) \simeq SL(2, \mathbb{Z})/\{1, -1\} \). Using this representation of the generators of the group, two important relations follow immediately:

\[
S^2 = (S \cdot T)^3 = -1 \sim 1. \tag{8}
\]

Now it is time to study some finite dimensional irreducible representations of our symmetry group \( PSL(2, \mathbb{Z}) \) in a Hilbert space. I will work with two different Hilbert spaces:

Case I: \( \mathcal{H} = L^2(\mathcal{C}, d\mu d\lambda) \) and every vector in \( \mathcal{H} \) is required to transform under a \( N \)-dimensional unitary representation of \( PSL(2, \mathbb{Z}) \) according to

\[
\Psi(v) = \sum_{l=1}^{N} \sum_{m=1}^{\text{Dim} \mathcal{H}/N} C_{lm} B_{lm}(v) \tag{9}
\]

\(^3\)Since \( PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{1, -1\} \), an element in \( PSL(2, \mathbb{Z}) \) is really the equivalence class \( \{G, -G\}, G \in SL(2, \mathbb{Z}) \). However, since our configuration space is such that \( v = -v \), we may just choose one representative from the class to represent the \( PSL(2, \mathbb{Z}) \) element.
\[ \Psi(G \cdot v) = \sum_{l=1}^{N} \sum_{n=1}^{N} \sum_{m=1}^{\text{Dim}\mathcal{H}/N} C_{lm}^{n} D_{l}^{N}(G)_{ln} B_{nm}(v) \]  

(10)

where \( B_{lm}(v) \) is an orthogonal basis in \( \mathcal{H} \), \( G \in PSL(2, \mathbb{Z}) \), and \( D_{l}^{N}(G)_{ln} \) is an element of an \( N \)-dimensional unitary irreducible matrix-representation of \( PSL(2, \mathbb{Z}) \). Note that if \( \mathcal{H} \) is infinite dimensional we may choose a continuous label \( m \) for the "basis" and have to replace the summation over \( m \) by an integral. The inner-product in \( \mathcal{H} \) is:

\[ \langle \Psi_1 | \Psi_2 \rangle := \int_{C} d\mu d\lambda \bar{\Psi}_1 \Psi_2 \]

Case II: \( \tilde{\mathcal{H}}_N = \sum_{s} \mathcal{H}_s = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_N \), where each \( \mathcal{H}_s = L^2(C, d\mu d\lambda) \), and I require the vectors in \( \tilde{\mathcal{H}}_N \) to transform under an \( N \)-dimensional unitary representation of \( PSL(2, \mathbb{Z}) \) according to:

\[ \Psi(v) := \begin{pmatrix} \Psi_1(v) \\ \Psi_2(v) \\ \vdots \\ \Psi_N(v) \end{pmatrix} \quad \Psi_s(v) \in \mathcal{H}_s \]  

(11)

\[ \Psi(G \cdot v) = D_{l}^{N}(G) \cdot \Psi(v) \]  

(12)

where \( G \in PSL(2, \mathbb{Z}) \) and \( D_{l}^{N}(G) \) is an element of an \( N \)-dimensional unitary irreducible matrix-representation of \( PSL(2, \mathbb{Z}) \). The inner-product is here given by:

\[ \langle \Psi_1 | \Psi_2 \rangle := \int_{C} d\mu d\lambda \bar{\Psi}_1 \cdot \Psi_2 \]

Now, if we manage to show that \( \tilde{\mathcal{H}}_N \) is empty, we have automatically also shown that \( \mathcal{H} \) is empty, and it is thus sufficient to study only case II to prove the non-existence of the finite dimensional representations. To see that this is indeed the case, suppose that \( \tilde{\mathcal{H}}_N \) is empty while \( \mathcal{H} \) is not. This means that there exist a set of orthonormal basis vectors in \( \mathcal{H} \) such that:

\[ B_{lm}(G \cdot v) = \sum_{n=1}^{N} D_{l}^{N}(G)_{ln} B_{nm}(v) \]  

Now, construct the following vector in \( \tilde{\mathcal{H}}_N \):

\[ \Psi_B(v) := \sum_{m=1}^{\text{Dim}\mathcal{H}/N} \begin{pmatrix} f_{1}^{m} B_{1m}(v) \\ f_{2}^{m} B_{2m}(v) \\ \vdots \\ f_{N}^{m} B_{Nm}(v) \end{pmatrix} \]  

(13)

where the \( f_{s}^{m} \) are arbitrary Fourier-coefficients. As is easily checked, this vector transforms correctly to belong to \( \tilde{\mathcal{H}}_N \) and its norm is:

\[ \langle \Psi_B | \Psi_B \rangle = \sum_{s=1}^{N} \sum_{m=1}^{\text{Dim}\mathcal{H}/N} \bar{f}_{s}^{m} f_{s}^{m} \]  

(14)

and thus, by properly choosing the \( f_{s}^{m} \)'s we have an element in \( \tilde{\mathcal{H}}_N \), which is a contradiction. Therefore we can conclude that if \( \tilde{\mathcal{H}}_N \) is empty, it follows that no basis \( B_{lm}(v) \) in \( \mathcal{H} \) with the correct transformation property exists and hence, \( \mathcal{H} \) is empty as well.

\[ ^4 \text{I gratefully thank Don Marolf for suggesting this proof.} \]
To see what kinds of restrictions we get on the finite dimensional irreducible representations, we will need some results concerning the orbits of the modular transformation on our configuration space. Basically, I intend to show that all the rational lines \( \frac{\mu}{\lambda} \) is rational – are equivalent under the modular transformations, and that every point on a rational line is left invariant by an abelian invariant subgroup of \( PSL(2, \mathbb{Z}) \) – different subgroups for different rational lines. Moreover, I will show that it is possible to transform two different points on different rational lines arbitrarily close to different points on the same rational line.

These three results are enough to put extremely hard restrictions on the admissible representations as well as wavefunctions. Since every point on a rational line is left invariant by the action of an element – the generator of the abelian subgroup that stabilizes the line – of \( PSL(2, \mathbb{Z}) \), we have to require the wavefunctions to be eigenvectors with eigenvalue one to the representation of the element under which the line is left invariant. That is, we have to require our representation of \( PSL(2, \mathbb{Z}) \) to be such that the representation of the group elements that have fixpoints all have the eigenvalue one in its spectrum.

And since there are an infinite number of such elements in \( PSL(2, \mathbb{Z}) \) this seems to be a severe restriction on what representations that are allowed. Furthermore, even if such non-trivial representations may be found we still have to require the wavefunctions to be continuously extendable outside the rational lines. Actually, in the finite dimensional case it is enough to require the wavefunctions to be piecewise continuous somewhere to rule out all representations besides the trivial one, and for that representation we can only allow the constant wavefunction.

To simplify the transformations, I choose my configuration space to be defined as \( \lambda \geq 0, \mu \in \mathbb{R} \) with the identification \( (\lambda, \mu) = (0, \mu) \sim (0, -\mu) \) and define the new coordinates
\[
y := \frac{\mu}{\lambda}, \quad x := \lambda.
\]

In these coordinates, the modular transformations become
\[
S : y \rightarrow -\frac{1}{y}, \quad x \rightarrow x y \quad (16)
\]
\[
T : y \rightarrow y + 1, \quad x \rightarrow x \quad (17)
\]

and \( v = \left( \begin{array}{c} x \\ y \\ x \end{array} \right) \). Throughout the rest of the paper, I will use three different notations to denote a point in \( C \): \( v, (\cdot, \cdot)_{xy} \) or \( (\cdot, \cdot)_{\lambda\mu} \). Consider now the transformation \( V := T^2 \cdot S \) where \( T^2 := T \cdot T \). Under multiple applications of \( V \) on a generic starting point \( v = v_0 \), we get \( v_n = V^n \cdot v_0 \):

\[
\left( \begin{array}{c} x_n \\ y_n \\ x_n \end{array} \right) = \left( \begin{array}{cc} 2 & -1 \\ 1 & 0 \end{array} \right)^n \left( \begin{array}{c} x_0 \\ y_0 \\ x_0 \end{array} \right) = \left( \begin{array}{cc} (n+1) & -n \\ n & -(n-1) \end{array} \right) \cdot \left( \begin{array}{c} x_0 \\ y_0 \\ x_0 \end{array} \right) = \left( \begin{array}{c} x_0(1+n) - n \\ x_0(n y_0 + (1 - n)) \end{array} \right). \quad (18)
\]

We can directly read off \( x_n = x_0(n y_0 + (1 - n)) \) and \( y_n = \frac{y_0(1+n) - n}{n y_0 + (1 - n)} \). Thus for \( n \gg 1 \) and \( |n(y_0 - 1)| \gg 1 \) we get \( y_n \sim 1 + \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2(y_0 - 1)}\right) \), and we see that we can map a
generic starting point $x_0, y_0$ arbitrarily close to the line $y = 1$. For rational $y$ it is actually possible to prove an even stronger statement:

**Theorem:** All rational lines $y = \frac{p}{q}$, $p, q \in \mathbb{Z}$ are equivalent under $PSL(2, \mathbb{Z})$ transformations.

**Proof:** Consider a point on the line $y = \frac{m}{n}$, $m, n \in \mathbb{Z}$ and $g.c.d(m, n) = 1$. Here, $g.c.d(\cdot, \cdot)$ denotes the greatest common divisor of the entries. First of all, we may note that all points on rational lines are of the above form.

To see that the line $y = \frac{m}{n}$ always can be mapped into the line $y = 0$, consider

$$\begin{pmatrix} 0 \\ x' \end{pmatrix} = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} \cdot \begin{pmatrix} x \\ \frac{m}{n} \end{pmatrix} = \frac{x}{n} \begin{pmatrix} mN_1 + nN_2 \\ mN_3 + nN_4 \end{pmatrix} \tag{19}$$

where $N_1, N_2, N_3, N_4 \in \mathbb{Z}$ and $N_1N_4 - N_2N_3 = 1$. Thus we see that $N_1 = pn$ and $N_2 = -pm$, $p \in \mathbb{Z}$. Then, we just need to make sure that there exist integers $N_3$ and $N_4$ such that $pnN_4 + pmN_3 = 1$. However, as is well known \[25\], this equation has a solution iff $g.c.d(pn, pm) = 1 \iff p = \pm 1$, and hence $x' = \pm \frac{x}{n}$. Thus, we get the result

$$(x, \frac{m}{n}) \xrightarrow{SL(2, \mathbb{Z})} (\frac{x}{n}, 0) \tag{20}$$

for $x \in \mathbb{R}$ and $m, n \in \mathbb{Z}$ and $g.c.d(m, n) = 1$. This proves the theorem.

Thus, we now know that all points on rational lines are equivalent under the modular transformations. Furthermore, for irrational lines, we have seen that it is possible to transform us arbitrarily close to the line $y = 1$ by repeatedly applying modular transformations. With this result at hand it is rather obvious that there does not exist a fundamental region for the $PSL(2, \mathbb{Z})$ transformations in the configuration space. Remember that the conventional definition of a fundamental region \[19\] says that the interior of the fundamental region should contain one and only one representative of points equivalent under the symmetry transformation. (If one includes the total boundary of the fundamental region, one has to allow points on the boundary to be equivalent.) In our case, it seems rather natural to require the fundamental region to contain one and only one rational line. However, there are no such connected open regions. As soon as we try to go outside the rational line by appending a tiny area to the rational line, we are automatically including segments from an infinite number of rational lines and these segments can all be mapped to the original rational line, meaning that there does not exist any fundamental region in the conventional sense.

A direct consequence of the absence of a fundamental region is that no wavefunctions that transform under a finite dimensional unitary representation of $PSL(2, \mathbb{Z})$ will be normalizable w.r.t. the inner-product that is defined over the entire configuration space. That is, $\hat{H}_N$ is empty and consequently so is $\hat{H}$! To see this, consider the inner-product in $\hat{H}_N$: $\langle \Psi_1 | \Psi_2 \rangle := \int d\mu d\lambda \Psi^*_1 \cdot \Psi_2$, and split up the configuration space into an infinite number of regions equivalent under $PSL(2, \mathbb{Z})$ transformations: $C = \bigcup_{k \in \mathbb{Z}} W_k$ where $W_k$ is defined as: $x > 0$ and $k \leq y < k + 1$. Since $W_k$ is mapped into $W_{k+1}$ by a $T$ transformation, we see that all $W_k$ regions are equivalent under $T$ transformations. But, since our wavefunction just transform under a finite dimensional unitary matrix-
representation of $PSL(2, \mathbb{Z})$, we directly see that the inner-product becomes:

$$< \Psi_1 | \Psi_2 > = \int_C d\mu d\lambda \Psi^*_1 \cdot \Psi_2 = \infty \times \int_{W_0} d\mu d\lambda \Psi^*_1 \cdot \Psi_2 \quad (21)$$

Consequently, the inner-product diverges for all $\Psi$ and $\mathcal{H}_N$ is thus empty. Note also that the result would have been the same for any finite dimensional representation, reducible or irreducible. If the representation is infinite dimensional, however, one needs to make sure that the infinite scalar product $\Psi^*_1 \cdot \Psi_2$ in the redefinition of the inner-product always converges. This result – the non-existence of finite dimensional $SL(2, \mathbb{Z})$-invariant closed subspaces of $L^2(\mathbb{R}^2, dx dy)$ – has recently been proven for $\mathcal{H}$ with the use of representation theory for $SL(2, \mathbb{R})$ in $L^2(\mathbb{R}^2, dx dy)$ [20].

This result is of course nothing new and especially strange; we have the same situation for all theories having symmetries that cut up the configuration space into an infinite number of non-zero area pieces. For example, one may think of wavefunctions on $\mathcal{R}$ that transform under a finite dimensional representation of the $a$-translation group; translations with a fix $a$. However, in this case we have the natural option of redefining our inner-product to be integrated only over the fundamental region: $0 \leq x < a$.

In our case, since there is no fundamental region, there is no natural redefinition of the inner-product. One could take the drastic step to define the inner-product to be integrated only over a rational line. However, by doing so, one is also drastically changing the Hilbert space; a rational line is a region of measure zero in the full configuration space. Perhaps a better choice is to define the inner-product over a finite area – w.r.t the Euclidean metric on $\mathcal{C}$ – region in $\mathcal{C}$. I will, however, not make such a choice here. I just assume that one can find a nice, well defined inner-product for wavefunctions transforming under a finite dimensional unitary representation, and make appropriate changes in the previous definition of the inner-product.

Now, returning to the study of the orbits of the modular transformations on $\mathcal{C}$, I intend to show that every point on a rational line is a fixpoint under the action of one element of $PSL(2, \mathbb{Z})$. A generic point on a rational line is given by

$$v_{p,q} := \left( \begin{array}{c} \frac{p}{q} \\ x \\ \frac{q}{x} \end{array} \right) \quad (22)$$

where $p, q \in \mathbb{Z}$, $x \in \mathcal{R}$. This point is left invariant by the $SL(2, \mathbb{Z})$ element

$$G_{p,q} := \left( \begin{array}{cc} 1 - p q & p^2 \\ -q^2 & 1 + p q \end{array} \right). \quad (23)$$

Actually, it is left invariant by all elements of the form $G_{\alpha p, \alpha q}$ for $\alpha^2 \in \mathbb{Z}$. However, these elements are generated by $G_{p,q}$ as $G_{\alpha p, \alpha q} = G_{p,q}^\alpha$ and are thus not important here.

The next result I will need is the fact that two points on different rational lines may be mapped arbitrarily close to different points on the same rational line. Consider points of the form

$$v_{(\lambda, \alpha, p, q, k)} := \lambda \alpha \left( \frac{p}{q}(1 - \frac{1}{pqk(\frac{1}{\alpha} - 1)}) \right) \quad (24)$$
where \( p, q, k \in \mathbb{Z} \) and \( \lambda, \alpha \in \mathbb{R} \). This line \( 0 < \lambda \alpha < \infty \) has the property that it approaches the line \( y = \frac{p}{q} \) as \( k \to \infty \). By using a transformation of the form \( G_{p,q}^k - G_{p,q} \) raised to the power \( k \) – this point is mapped into \( v(G_{p,q}, G_{p,q}, k) = G_{p,q}^k \cdot v(\lambda, \alpha, p, q, k) \):

\[
\lambda \left( \frac{p}{q} \frac{1 - \frac{\alpha}{p} (\frac{1}{\alpha} - 1)}{1} \right) = \left( 1 - k^p q^2 \quad k^p q^2 \quad 1 + k^p q \right) \cdot \lambda \alpha \left( \frac{p}{q} \frac{1 - \frac{1}{p} (\frac{1}{\alpha} - 1)}{1} \right)
\]

which in the limit \( k \to \infty \) says \( (\lambda, \frac{p}{q}) \sim (\lambda, \frac{\alpha}{q}) \). Or in words; the points \( \lambda \) and \( \alpha \) on the line \( y = \frac{p}{q} \) is approximately related by an \( PSL(2, \mathbb{Z}) \) transformation for all \( \lambda, \alpha \in \mathbb{R} \). Note, however, that this relation never is exact. One can actually easily prove that no two points on the same rational line are equivalent under a \( PSL(2, \mathbb{Z}) \) transformation. For continuous wavefunctions, this approximate relation is, however, good enough.

Now, what does all this mean for a wavefunction transforming under a finite dimensional representation of \( PSL(2, \mathbb{Z}) \)? First of all, since every point on a rational line is a fixpoint for an element of \( PSL(2, \mathbb{Z}) \), we have to require:

\[
\Psi(v_{p.q}) = \Psi(G_{p.q} \cdot v_{p.q}) = D^N(G_{p.q}) \cdot \Psi(v_{p.q}) \tag{26}
\]

Hence, the wavefunction is either zero on the line \( y = \frac{p}{q} \) or an eigenvector with eigenvalue one to \( D^N(G_{p.q}) \). However, if the wavefunction is identically zero on one rational line it follows that it must be identically zero on all rational lines. (Remember that all rational lines are equivalent under \( PSL(2, \mathbb{Z}) \) transformations.) Thus, non-trivial piecewise continuous wavefunctions must be eigenvectors with eigenvalue one to \( D^N(G_{p.q}) \). And this means that we can only allow representations where \( D^N(G_{p.q}) \) has the eigenvalue one in its spectrum for all \( p \) and \( q \)! Such representations exists. One example is the trivial representation; all elements in \( PSL(2, \mathbb{Z}) \) are represented by the identity matrix. It may, however, be impossible to find a faithful finite dimensional unitary representation that allows this.

Suppose now that we have found such a representation and we want to realize it in the space of piecewise continuous complex valued \( N \)-dimensional vector-functions on \( C \). Now, since we require the functions to transform as \( \Psi(G \cdot v) = D^N(G) \cdot \Psi(v) \) eq. (27) says

\[
\Psi(G_{p,q}^k \cdot v(\lambda, \alpha, p, q, k)) = \left( D^N(G_{p,q}) \right)^k \cdot \Psi(v(\lambda, \alpha, p, q, k)) \tag{27}
\]

which in the limit \( k \to \infty \) becomes

\[
\Psi(\lambda, \frac{p}{q})_{xy} = \Psi(\lambda, \frac{\alpha}{q})_{xy} \tag{28}
\]

To see this, try to map a point on the line \( y = 0 \) to a different point on the same line: \( \begin{pmatrix} 0 \\ x' \end{pmatrix} = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} N_2 x \\ N_4 x \end{pmatrix} \), \( x' \neq x \). We get \( N_2 = 0 \) and \( N_4 \neq 1 \). But the only element of \( SL(2, \mathbb{Z}) \) satisfying this is \( G = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \) which maps \( x \to -x \) on the line \( y = 0 \). and since these points are identified in our configuration space, we see that no two point on the line \( y = 0 \) are related by a \( PSL(2, \mathbb{Z}) \) transformation. Furthermore, since every point on a rational line has a unique image on the line \( y = 0 \), the result applies to all rational lines.
where I have used that $\Psi(x, \frac{p}{q})_{xy}$ is stable under $D^N(G_{p,q})$, and that entries in unitary matrices have modulus $\leq 1$. This makes the limit well defined.

Thus, since eq. (28) is true for all $\lambda, \alpha \in \mathbb{R}$, we may conclude that the wavefunction is constant on every rational line, and hence it is constant everywhere in $\mathbb{C}$.

This completes the proof of the nonexistence of nontrivial piecewise continuous wave-functions transforming under a unitary finite dimensional representation of $PSL(2, \mathbb{Z})$.

Before leaving this representation, I want to comment on possible ways of fixing the problem of using finite dimensional representations. If we want to construct our Hilbert space from the vector space of piecewise continuous complex valued functions on $\mathbb{C}$, there are no good solutions. That is, since already this underlying vector space does not allow finite dimensional irreducible representations, there is no way to fix our problem by making a clever choice of an inner-product. However, if we are prepared to relax the smoothness of the functions to only requiring continuity along the rational lines, it may be possible to use finite dimensional representations. That is, we will get wavefunctions that are nowhere continuous in a two-dimensional neighborhood of a point in $\mathbb{C}$. Our inner-product could be defined as the integral over one rational line. This situation will very much resemble the case for the loop-representation, discussed below. However, one should be aware of that this construction is very different from the conventional Hilbert space constructions in standard quantum mechanics. Normally, although one sometimes restricts the integration region to a fundamental region, there is no problem to extend the wavefunctions smoothly outside this fundamental region to the entire configuration space. This will not be possible here. Also, with such a Hilbert space, it is not clear how to find a quantum representation of e.g. the classical Poisson algebra for the phase space variables.

Another, perhaps more promising way to fix the problem is to choose a different polarization of the phase space. Since we know that there is a one-to-one mapping between the ADM-phase space and half the connection phase space, and there exist a fundamental region in the ADM phase space, there must exist a fundamental region in the connection phase space $[1]$. I have only shown that this fundamental region ”has a non-trivial projection” down to the configuration space. Thus, by choosing a different polarization it may be possible to use a conventional Hilbert space to find a non-trivial quantum theory even for these sectors $[7]$. However, such a quantum representation is strictly speaking not a connection representation. Results of such a quantization will be reported elsewhere.

3 Loop-representation

There is a closely related formulation of quantum gravity called the loop-representation $[7], [13]$. In that formulation, one starts from the same Hamiltonian formulation as for the Chern-Simons/connection formulation, but instead of quantizing the classical Poisson
algebra of the reduced phase space variables, one uses a classical $T$-algebra of holonomy-like variables $T^0, T^1$ to coordinatize the phase space. By using these variables which are manifestly gauge invariant, one has automatically taken care of the local gauge invariance. The local diffeomorphism invariance is then handled by restricting the theory to flat connections. In the quantization of this algebra, one defines a quantum representation in the loop-space of the hypersurface. And, basically due to the fact that one uses flat connections, these $T$-operators will only be able to distinguish between homotopically inequivalent loops. On the torus, every homotopy class of loops is uniquely labelled by two integers; the winding numbers around the two noncontractible independent loops on the torus. Thus, all information in a function on loop-space as seen by the $T$-operators are given by these two integers. Effectively, this means that the Hilbert space will be based on the vector space of complex valued functions on the two-dimensional lattice $\mathbb{Z}^2$. Actually, there is one additional degeneracy; due to classical relations among the $T$-variables, the quantum representation is only well defined on wavefunctions satisfying $\Psi(n_1, n_2) = \Psi(-n_1, -n_2)$. Effectively this means that we should identify two points on the lattice that are related via a reflection in the origin: $(n_1, n_2) \sim -(n_1, n_2)$. What is left of the global transformations is again the modular transformations of the torus; they are given by \[ S: \quad (n_1, n_2) \to (n_2, -n_1) \quad (29) \]
\[ T: \quad (n_1, n_2) \to (n_1, n_1 + n_2) \quad (30) \]
Thus, basically we are now dealing with the lattice formulation of the configuration space for the Chern-Simons/connection representation.

One could now try to go through the same analysis of the irreducible sectors of this quantum theory as was done for the connection representation. In this case, however, we do not find such strong statements about the finite dimensional irreducible representations. Basically, this comes about because the underlying space is discrete; we cannot use continuity argument anymore. We still have the problem of finding representations where all $D^N(G_{p,q})$ has the eigenvalue one in its spectrum, but once this is done, there should not be any problem of finding non-trivial wavefunctions transforming under this representation. As an example of allowed representations we have the trivial one-dimensional representation.

Note also that here as well we get the problem that a wavefunction transforming under a finite dimensional representation will not be normalizable over the entire lattice. That is, if we define the inner-product to be
\[ \langle \Psi_1 | \Psi_2 \rangle := \sum_{n_1, n_2} \bar{\Psi}_1(n_1, n_2)\Psi_2(n_1, n_2) \quad (31) \]
we automatically exclude wavefunctions transforming under a finite dimensional representation of $PSL(2, \mathbb{Z})$. This can be understood from the fact that the two-dimensional lattice can be written as the infinite sum of all rational lines, and that all rational lines are equivalent under $PSL(2, \mathbb{Z})$ transformations. And if one tries to modify the inner-product by restricting the summation to only one rational line, the $T$-operators as defined in \[4\] will not be self adjoint. Thus, it is straightforward to find a non-trivial vector space of e.g modular invariant wavefunctions here, but we still do not have a good inner-product that makes these wavefunctions normalizable and makes the $T$-operators self adjoint.
4 The reduced ADM-phase space approach

In the ADM-approach to quantum gravity, one starts from the ADM-Hamiltonian formulation of gravity and reduces the phase space classically by solving all first class constraints and gauge fixes the local symmetries they generate [21], [1], [2], [4]. In the torus case, this can be done completely explicitly; the reduced configuration space is parametrized by the Teichmüller coordinates for the torus, and we get the Hamiltonian

\[ \hat{H} = \sqrt{\tau_2^2 \left( \frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right)} \]  (32)

where \( \tau_1 \) and \( \tau_2 \) are the coordinates on the Teichmüller space. What is left of the global symmetries are again the modular transformations, whose action on the configuration space is

\[ S: \quad \tau \rightarrow -\frac{1}{\tau} \]  (33)
\[ T: \quad \tau \rightarrow \tau + 1 \]  (34)

where \( \tau := \tau_1 + i\tau_2 \). These transformations on this space are well studied in the mathematics literature [19], [22] and it is known that we here get a fundamental region: \( M: |\tau| \geq 1 \) and \( |\tau_1| \leq \frac{1}{2} \). The Teichmüller space is naturally cut up into an infinite number of copies of this fundamental region. This means that we again has to choose integration region in our inner-product with care. Choosing an integration over the entire Teichmüller space automatically excludes all wavefunctions transforming under a finite dimensional representation of \( PSL(2, \mathbb{Z}) \). Therefore, since we here have a fundamental region, it seems more natural to define the inner-product over the moduli space \( M \). There also exist a natural metric, and hence a volume element, on the Teichmüller space; the Poincaré metric \( ds^2 = \tau_2^{-2}(d\tau_1^2 + d\tau_2^2) \) with volume element \( \sqrt{g}d\tau_1d\tau_2 = \tau_2^{-2}d\tau_1d\tau_2 \). The Hamiltonian given above is the square root of the Laplacian w.r.t this metric, and it may be shown that the classical solutions to this theory are given by geodesics w.r.t this metric [21]. Thus, it seems natural to choose the inner-product

\[ <\Psi_1|\Psi_2> := \int_M \frac{d\tau_1d\tau_2}{\tau_2^2} \bar{\Psi}_1 \Psi_2 \]  (35)

I do not know what can be said about the generic irreducible sector in this Hilbert space, it is, however, known that the trivial one-dimensional unitary representation – the space of moduli invariant wavefunctions – here is non-trivial. That is, this sector is known to infinite dimensional and mathematicians have numerically studied a basis of eigenfunctions to the Laplacian; the so called zero weight Maass-forms [22], [23].

5 Conclusions

In this paper, I have shown that the connection representation for (2+1)-quantum gravity on the torus is completely trivial if we require the wavefunctions to transform under a finite dimensional representation of \( PSL(2, \mathbb{Z}) \), and also require the wavefunctions to be piecewise continuous somewhere on the reduced configuration space. Thus if we strongly
believe that the wavefunctions must transform according to a one-dimensional representation of $\text{Diff}(T^2)/\text{Diff}(T^2)_0$ and want a nontrivial quantum theory, we are forced to accept a strangely looking Hilbert space; it will consist of wavefunctions that are nowhere continuous. And thus, since no one yet has shown how to find an commutator representation of some classical poisson-algebra in such a Hilbert space, we are yet far from having constructed a sensible nontrivial quantum theory in this sector. This immediately also implies problems for the loop-representation. According to [7], we probably need to relate the loop-representation to the connection representation via the loop-transform in order to be sure that our Hilbert space will be properly constructed as to contain the conventional geometrodynamical sector.

What is a bit puzzling here, is that we already have a nontrivial quantum theory in the trivial representation, for the torus case; the ADM-representation [11]. This seems to indicate that either the connection- and ADM-representation are not isomorphic/unitarily equivalent or that the ADM-representation is unitarily equivalent to a connection representation where our wavefunctions has to be strangely looking objects (nowhere continuous).

We should also remember that it may still be possible to find a nontrivial quantum theory with nice wavefunctions coming from the connection formulation if we use a different polarization of the phase space.

At last, note that there already exist a nontrivial quantum theory for (2+1)-quantum gravity on the torus in the connection representation if we are prepared to accept that the wavefunctions transform under an infinite dimensional, reducible representation of $PSL(2,\mathbb{Z})$ [7], [13], [20].

Acknowledgements
I thank Arlen Anderson, Abhay Ashtekar, Fernando Barbero, Ingemar Bengtsson, Domenico Giulini, Jorma Louko and Don Marolf for interesting discussions. I also thank Nigel Higson and Jean-Luc Brylinski for some information regarding representation theory for $SL(2,\mathbb{Z})$. This work was supported by the NFR contract F-PD 10070-300.
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