Abelian Zero Modes in Odd Dimensions

Gerald V. Dunne
Department of Physics, University of Connecticut, Storrs, CT 06269

Hyunsoo Min
Department of Physics, University of Seoul, Seoul 130-743, Korea

We show that the Loss-Yau zero modes of the 3d abelian Dirac operator may be interpreted in a simple manner in terms of a stereographic projection from a 4d Dirac operator with a constant field strength of definite helicity. This is an alternative to the conventional viewpoint involving Hopf maps from $S^3$ to $S^2$. Furthermore, our construction generalizes in a straightforward way to any odd dimension. The number of zero modes is related to the Chern-Simons number in a nonlinear manner.

The behavior of quantized charged fermions in ultra-strong magnetic fields has applications in atomic physics [1], particle and condensed matter physics [2], and astrophysics [3]. Key properties determining stability are the existence of zero modes for the 3 dimensional Dirac operator, and the associated magnetic helicity. In studying the stability of atoms in magnetic fields, Loss and Yau [1] found the surprising result that the abelian Dirac operator in 3 dimensions, $D_3 \equiv i\gamma_\mu (\partial/\partial x^\mu - iA_\mu)$, can have exact zero modes

$$i\gamma_\mu \left( \frac{\partial}{\partial x^\mu} - iA_\mu \right) \psi^{(0)} = 0 \quad (1)$$

for smooth, localized magnetic fields $\vec{B} = \vec{\nabla} \times \vec{A}$. In even dimensions there is a well-known relation between zero modes and the topology of gauge fields [4], but in odd dimensions, where the relevant index theorem is due to Callias [5], the situation is somewhat different, as the index is determined by the topology of the coupling to a Higgs field. In this Brief Report we present a simple new interpretation of the Loss-Yau zero-mode-supporting abelian gauge fields, and show that this construction generalizes to all odd dimensions.

Loss and Yau’s simplest example [1] is the gauge field

$$\vec{A}_{LY} = \frac{3}{1 + \vec{x}^2} \hat{N}, \quad \hat{N} = \frac{1}{1 + \vec{x}^2} \left( \begin{array}{c} 2x_1x_3 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{array} \right), \quad \vec{B}_{LY} = \frac{12}{(1 + \vec{x}^2)^2} \hat{N}, \quad (2)$$

for which the zero mode is

$$\psi_{LY}^{(0)} = \frac{4}{(1 + \vec{x}^2)^{3/2}} (1 + i\gamma_\mu x_\mu) \left( \begin{array}{c} 1 \\ 0 \end{array} \right). \quad (3)$$

For this field, the Chern-Simons number, or magnetic helicity, is

$$N_{LY}^{CS} = \frac{1}{16\pi^2} \int d^3x \vec{A} \cdot \vec{B} = \frac{9}{16} . \quad (4)$$

The associated magnetic field is plotted in Figure 1 showing the localized but non-trivial structure of the field. This basic example may be extended [1,6] to fields with multiple zero modes:

$$\vec{A}_{LY} = \frac{(2k + 3)}{1 + \vec{x}^2} \hat{N}, \quad \vec{B}_{LY} = \frac{4(2k + 3)}{(1 + \vec{x}^2)^2} \hat{N}, \quad N_{LY}^{CS} = \frac{(2k + 3)^2}{16} . \quad (5)$$

Here $k \geq 0$ is an integer, and this field has $(k+1)$ zero modes that can be expressed in terms of 3d spherical harmonics [1,6].

These fields have since been discussed in terms of Hopf maps [6,7], which are maps $\chi: S^3 \rightarrow S^2$ such that the magnetic field is

$$\vec{B}_H = \frac{2}{i} \frac{\vec{\nabla}_\chi \times \vec{\nabla}\bar{\chi}}{(1 + \chi\bar{\chi})^2} . \quad (6)$$

*Electronic address: dunne@phys.uconn.edu
†Electronic address: hsmin@dirac.uos.ac.kr
FIG. 1: Plot of the magnetic field vector $\vec{B}_{LY}$ in (2). Note that the magnitude is highly localized around the origin, while the direction winds in a non-trivial manner.

Such a Hopf map can also be viewed as a map $\chi : \mathbb{R}^3 \to \mathbb{R}^2$, and the simplest example

$$\chi = \frac{(x_1 + ix_2)}{2x_3 - i(1 - x^2)} ; \quad \vec{B}_H = \frac{16}{(1 + x^2)^2} \hat{N} = \frac{4}{3} \vec{B}_{LY} ; \quad \mathcal{N}_H^{CS} = 1$$

(7)

gives a magnetic field proportional to the Loss-Yau field in (2). Geometrically, $\vec{B}_H$ is tangent to the closed curves in $\mathbb{R}^3$ given by $\chi =$ constant. Erdős and Solovej [7] gave an elegant interpretation of these zero-mode-supporting gauge fields in terms of pull-backs (to $\mathbb{R}^3$) of 2 dimensional magnetic fields, and the 3 dimensional zero modes were related to the Aharonov-Casher zero modes in 2 dimensions [8]. Further results have been found in [6, 9], and these gauge fields have also been understood in terms of projections of non-abelian fields [10].

However, a number of questions remain. The fundamental mismatch of the coefficient [the factor $4/3$ in (7)] does not have an elegant interpretation in the Hopf map language. In one picture, one introduces an additional "background" field with a correcting coefficient [6]; and in another picture [7], one includes a magnetic monopole of a particular strength at the centre of the $S^2$ to adjust the strength of the area form.

In this short note we present another characterization of these 3d abelian zero-mode-supporting gauge fields, in terms of four dimensional gauge fields of fixed helicity. This construction is extremely simple, and furthermore it generalizes naturally to other odd dimensions.

Our basic example [the analogue of (2)] is expressed for arbitrary odd dimension by a stereographic projection from $\mathbb{R}^{2n} \supset S^{2n-1} \to \mathbb{R}^{2n-1}$. We define coordinates $x_\mu (\mu = 1, 2, \ldots, 2n - 1)$ on $\mathbb{R}^{2n-1}$, and coordinates $y_a (a = 1, \ldots, 2n)$ on $\mathbb{R}^{2n}$. Consider a 2n-dimensional gauge field corresponding to a constant field strength, and such that the field has fixed helicity:

$$A_a = -\frac{F}{2} (y_2, -y_1, y_4, -y_3, \ldots, y_{2n-2}, -y_{2n-3}, -y_{2n}, y_{2n-1})$$

$$\equiv -\frac{F}{2} J_{ab} y_b$$

(8)

where the antisymmetric matrix $J = \text{diag}(i\sigma_2, \ldots, i\sigma_2, -i\sigma_2)$. [The sign-flip in the last diagonal entry is a parity convention chosen to agree with the choice of Loss-Yau.] This gauge field is in Fock-Schwinger gauge: $y_a A_a = 0$. Now restrict to $S^{2n-1}$ by imposing the condition $y^2 = 1$, and stereographically project from $S^{2n-1}$ to $\mathbb{R}^{2n-1}$ via:

$$y_\mu = \frac{2x_\mu}{1 + x^2} , \quad y_{2n} = \frac{1 - x^2}{1 + x^2} .$$

(9)
The projected $(2n - 1)$-dimensional gauge field $A_\mu$ is

$$A_\mu = \frac{\partial y_a}{\partial x_\mu} A_a$$  \hspace{1cm} (10)

One finds by a simple computation

$$A_i = 2\mathcal{F}\left( -\frac{J_{ij}x_j + x_i x_{2n-1}}{1 + x^2} \right), \quad i = 1, 2 \ldots 2n - 2$$

$$A_{2n-1} = \mathcal{F}\left( \frac{1 - x^2 + 2x^2_{2n-1}}{1 + x^2} \right)$$  \hspace{1cm} (11)

When $n = 2$ (i.e., a 3 dimensional gauge field $A_\mu$) this reproduces precisely the form of the original Loss-Yau gauge field [1] in [2], although the coefficient $\mathcal{F}$ is not yet determined.

The coefficient $\mathcal{F}$ is fixed by an explicit construction of the zero mode, directly from the zero mode equation (11). Straightforward Dirac algebra manipulations [1, 14] show that the gauge field in (1) can be expressed in terms of the zero mode $\psi(0)$

$$\psi(0) = \frac{1}{1 + x^2} (1, \ldots, 1, 0, \ldots, 0)^T$$  \hspace{1cm} (13)

Then the resulting gauge field constructed from (12) in $(2n - 1)$ dimensions is

$$A_i = 2(2n - 1) \left( -\frac{J_{ij}x_j + x_i x_{2n-1}}{1 + x^2} \right), \quad i = 1, 2 \ldots 2n - 2$$

$$A_{2n-1} = (2n - 1) \left( 1 - x^2 + 2x^2_{2n-1} \right)$$  \hspace{1cm} (14)

which is precisely the same as (11), but now the overall coefficient has been fixed to be $\mathcal{F} = 2n - 1$. When $n = 2$ (corresponding to 3 dimensions) this reproduces the original Loss-Yau gauge field in [2]. Note that the zero mode is normalizable in all odd dimensions $d = 2n - 1 \geq 3$.

The degeneracy of the abelian zero modes can be deduced by group theoretic arguments for spinors in arbitrary dimensions. Define $2^{n-1} \times 2^{n-1}$ Dirac matrices $\gamma_\mu$ ($\mu = 1, \ldots, (2n - 1)$) for $\mathbb{R}^{2n-1}$, and $2^n \times 2^n$ Dirac matrices $\Gamma_a$ ($a = 1, \ldots, 2n$) for $\mathbb{R}^{2n}$. These can be related as

$$\Gamma_\mu = \begin{pmatrix} 0 & i\gamma_\mu \\ -i\gamma_\mu & 0 \end{pmatrix}; \quad \Gamma_{2n} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \Gamma_{2n+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (16)

Then the spin matrices in $\mathbb{R}^{2n}$ may be block-decomposed as

$$\Sigma_{ab} \equiv \frac{1}{4i} [\Gamma_a, \Gamma_b] = \Sigma^+_{ab} - \Sigma^-_{ab}$$  \hspace{1cm} (17)

where $\Sigma^\pm_{\mu\nu} = \sigma_{\mu\nu} \equiv \frac{1}{4i} [\gamma_\mu, \gamma_\nu]$; and $\Sigma^\pm_{\mu,2n} = \pm i \gamma_\mu$. We also define the $2n$ dimensional angular momentum generators

$$L_{ab} \equiv -i \left( y_a \frac{\partial }{\partial y_b} - y_b \frac{\partial }{\partial y_a} \right)$$  \hspace{1cm} (18)

Then a canonical result of stereographic projection of the free Dirac equation from $S^{2n-1}$ (defined by $y_a y_a = 1$) to $\mathbb{R}^{2n-1}$ is that

$$\left( \frac{1 + x^2}{2} \right)^{2n-1} i \gamma_\mu \frac{\partial }{\partial x_\mu} = \left( \frac{1 + x^2}{2} \right)^{n-3/2} V \left[ \Sigma^+_{ab} L_{ab} + \left( n - \frac{1}{2} \right) 1 \right] \left( \frac{1 + x^2}{2} \right)^{n-3/2}$$  \hspace{1cm} (19)
where $V \equiv \frac{1}{\sqrt{2}} (1 + i \gamma_\mu x_\mu)$.

Now we observe that for the $2n$ dimensional gauge field $A_\mu$ defined in [11] and the $(2n-1)$-dimensional gauge field $A_\mu$ defined in [15], this projection property of the free Dirac equation is maintained once the gauge field is included:

$$
\left( \frac{1 + \vec{x}^2}{2} \right)^{2n-1} \gamma_\mu \left( \frac{\partial}{\partial x_\mu} - i A_\mu \right) = \left( \frac{1 + \vec{x}^2}{2} \right)^{n-3/2} \nabla^T \left[ \Sigma^+_{ab} \mathcal{L}_{ab} + \left( n - \frac{1}{2} \right) \right] V \left( \frac{1 + \vec{x}^2}{2} \right)^{n-3/2}
$$

(20)

where $\mathcal{L}_{ab} \equiv L_{ab} + \left( y^a A_b - y^b A_a \right)$. Thus, the zero-mode equation on $\mathbb{R}^{2n-1}$ can be lifted to a zero-mode equation on $S^{2n-1}$, where the solutions can be written in terms of the spinor spherical harmonics in $\mathbb{R}^{2n}$.

To illustrate this explicitly we consider the $n = 2$ case. The 4-dimensional gauge field may be written as $A_\mu = -\bar{\mathcal{F}}/2\eta^a_{ab} y^b$, where $\eta^a_{ab}$ is the 3rd isospin component of the standard 4-dim. 't Hooft tensor [12, 13]. The 4-component spinor zero mode $\psi_{(0)}$ in $\mathbb{R}^4$ may be written in terms of a 2-component spinor $u$ of definite (we choose positive) helicity:

$$
\psi_{(0)} = \begin{pmatrix} u \\ 0 \end{pmatrix}
$$

(21)

Then using (20) the zero mode equation (11) becomes an algebraic equation

$$
\left[ \Sigma^+_{ab} \mathcal{L}_{ab} + 3/2 \right] u = \left( 4 \vec{S} \cdot \vec{L} + 3/2 - \frac{\mathcal{F}}{2} \sigma_3 \right) u = 0
$$

(22)

where $u = ((1 + \vec{x}^2)/2)^{l/2} V \phi$, and where $\vec{S}$ and $\vec{L}$ are angular momentum operators, of spin 1/2 and $l (= \text{half integer})$, respectively. We define the total angular momentum $\vec{J} = \vec{S} + \vec{L}$, and use the spinor spherical harmonics [16]

$$
u^{(\pm)} = \frac{1}{\sqrt{2l + 1}} \begin{pmatrix} \pm \sqrt{l + 1/2 \pm M} Y^k_{m,M-1/2} \\ \sqrt{l + 1/2 \mp M} Y^k_{m,M+1/2} \end{pmatrix}
$$

(23)

with $j = l \pm 1/2$, $-j \leq M \leq j$, and $-l \leq m \leq l$. Then

$$
4 \vec{S} \cdot \vec{L} \nu^{(\pm)} = \begin{cases} 2l & \text{if } \mathcal{F} = 4l + 3 \\ -2l - 2 & \text{if } \mathcal{F} = 4l + 3 \\ \end{cases}
$$

(24)

Thus a zero mode can only occur when $j = M = l + 1/2$ and

$$
2l + 3/2 - \frac{\mathcal{F}}{2} = 0 \quad \rightarrow \quad \mathcal{F} = 4l + 3
$$

(25)

for arbitrary value of $-l \leq m \leq l$. In this case $\nu^{(\pm)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} Y^l_{m,l}$. Recalling that $l = k/2$ is a half-integer, we arrive at the general Loss-Yau case [3]. Furthermore, the degeneracy is simply given by $2l + 1 = k + 1$, as found by Loss-Yau in [1]. This construction makes it clear why the zero mode degeneracy factor is linear in the integer $k$, while the Chern-Simons number is quadratic in $k$. An analogous construction is clearly possible in higher dimensions using generalized spinor spherical harmonics, but we do not present the details here.

To conclude, we have given a simple new interpretation of the Loss-Yau abelian zero-mode-supporting gauge fields in three dimensions, and have extended the construction to obtain new zero-mode-supporting abelian gauge fields in other odd dimensions. An interesting outstanding problem is the possibility of including an interaction with a scalar field, which might shed light on the possible relation of these fields to the Callias index theorem [2].

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[17] Such gauge field projections have been studied extensively for projections to $R^4$ [11, 12, 13]; here we consider analogous projections to odd dimensional spaces.