Cellular Automata with Symmetric Local Rules

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Abstract. The cellular automata with local permutation invariance are considered. We show that in the two-state case the set of such automata coincides with the generalized Game of Life family. We count the number of equivalence classes of the rules under consideration with respect to permutations of states. This reduced number of rules can be efficiently generated in many practical cases by our C program. Since a cellular automaton is a combination of a local rule and a lattice, we consider also maximally symmetric two-dimensional lattices. In addition, we present the results of compatibility analysis of several rules from the Life family.

1 Introduction

The number of possible local rules for $q$-state cellular automaton defined on a $(k+1)$-cell neighborhood is double exponential of $k$, namely, $q^{4k+1}$. It is natural to restrict our attention to special classes of local rules. S. Wolfram showed [1] that even simplest 2-state 3-cell automata, which he terms elementary cellular automata (there are $2^{2^3} = 256$ possible rules for these automata), demonstrate the unexpectedly complex behavior.

We consider here a class of symmetric local rules defined on a $(k+1)$-cell neighborhood. Here by a ‘symmetry’ we mean a symmetry with respect to all permutations of $k$ cells (points or vertices) surrounding $(k+1)$th cell, which time evolution the local rule determines. The reasons to distinguish such rules are

- The number of possible symmetric rules is the single exponential of polynomial of $k$ (see formula (2) below). For example, for $k = 8$, $q = 2$ (the case of the Conway’s game of Life) the number of symmetric rules is $262144 \approx 2.6 \times 10^5$, whereas the number of all possible rules $\approx 1.3 \times 10^{154}$.
- The symmetry of the neighborhood under permutations is in a certain sense a discrete analog of general local diffeomorphism invariance which is believed must hold for any fundamental physical theory based on continuum spacetime.\(^1\)

\(^1\) The symmetric group of any finite or infinite set $M$ is often denoted by $\text{Sym}(M)$. If $M$ is a manifold, then a diffeomorphism of $M$ is nothing but a special — continuous and differentiable — permutation from $\text{Sym}(M)$.
This class of rules contains such widely known automata as *Conway’s Life*. In fact, as we show below, any symmetric rule is a natural generalization of the *Life* rule.

## 2 Symmetric Local Rules and Generalized Life

### 2.1 Symmetric Rules

We interpret a \((k + 1)\)-cell neighborhood of a cellular automaton as a \(k\)-star graph, i.e., rooted tree of height 1 with \(k\) leaves. We call this the \(k\)-valent neighborhood. We adopt the convention that the leaves are indexed by the numbers 1, 2, \ldots, \(k\) and the root is numbered by \(k + 1\). For example, the trivalent neighborhood looks like this

\[
\begin{align*}
\bullet & \quad x_3 \\
\bullet & \quad x_4 \\
\bullet & \quad x_1 \\
\bullet & \quad x_2
\end{align*}
\]

A *local rule* is a function specifying one time step evolution of the state of root

\[
x'_{k+1} = f(x_1, \ldots, x_k, x_{k+1}).
\]

We consider the set \(R_{S_k}\) of local rules symmetric with respect to the group \(S_k\) of all permutations of leaves, i.e., variables \(x_1, \ldots, x_k\). We will consider also the subset \(R_{S_{k+1}} \subset R_{S_k}\) of rules symmetric with respect to permutations of all \(k + 1\) points of the neighborhood. For brevity we shall use the terms \(k\)-symmetry and \((k + 1)\)-symmetry, respectively.

Obviously the total numbers of \(k\)- and \((k + 1)\)-symmetric rules are, respectively,

\[
\begin{align*}
N_{S_k}^q &= q^{(k+q-1)q-1} \\
N_{S_{k+1}}^q &= q^{(k+q)q-1}
\end{align*}
\]

### 2.2 Life Family

The “Life family” is a set of 2-dimensional, binary cellular automata similar to *Conway’s Life* [2], which rule is defined on 9-cell \((3 \times 3)\) Moore neighborhood and is described as follows. A cell is “born” if it has exactly 3 alive neighbors, “survives” if it has 2 or 3 such neighbors, and dies otherwise. This rule is symbolized in terms of the “birth”/“survival” lists as B3/S23. Another examples of automata from this family are *HighLife* (the rule B36/S23), and *Day&Night* (the rule B3678/S34678). The site [3] contains collection of more than twenty rules from the Life family with Java applet to run these rules and descriptions of their behavior.
Generalizing this type of local rules, we define a \( k \)-valent Life rule as a binary rule on a \( k \)-valent neighborhood, described by two arbitrary subsets of the set \( \{0, 1, \ldots, k\} \). These subsets \( B, S \subseteq \{0, 1, \ldots, k\} \) contain conditions for the \( x_k \rightarrow x'_k \) transitions of the forms \( 0 \rightarrow 1 \) and \( 1 \rightarrow 1 \), respectively. Since the number of subsets of any finite set \( A \) is \( 2^{|A|} \), the number of rules defined by two sets \( B \) and \( S \) is equal to \( 2^{k+1} \times 2^{k+1} = 2^{2k+2} \), which in turn is equal to \( \mathbb{2} \) evaluated at \( q = 2 \). On the other hand, different pairs \( B/S \) define different rules.

Thus, we have the obvious

**Proposition.** For any \( k \) the set of \( k \)-symmetric binary rules coincides with the set of \( k \)-valent Life rules.

This proposition implies, in particular, that one can always express any symmetric binary rule in terms of “birth”/“survival” lists.

### 2.3 Equivalence with Respect to Permutations of States

Exploiting the symmetry with respect to renaming of \( q \) states of cellular automata allows us to reduce the number of rules to consider. Namely, it suffices to consider only orbits (equivalence classes) of the rules under \( q! \) permutations forming the group \( S_q \). For counting orbits of a finite group \( G \) acting on a set \( R \) (rules, in our context) there is the formula called Burnside’s lemma. This lemma states (see, e.g., [4]) that the number of orbits, denoted \( |R/G| \), is equal to the average number of points \( R^g \subset R \) fixed by elements \( g \in G \):

\[
|R/G| = \frac{1}{|G|} \sum_{g \in G} |R^g|.
\]  

Thus, the problem is reduced to finding the sets of fixed points.

Since we are mainly interested here in the binary automata, we shall consider further the case \( q = 2 \) only. For this case the combinatorics is rather simple.

Specializing (2) and (3) for \( q = 2 \) we have the numbers of binary \( k \)- and \((k + 1)\)-symmetric rules, respectively,

\[
N_{S_k} = 2^{2k+2},
\]

\[
N_{S_{k+1}} = 2^{k+2}.
\]  

The group \( S_2 \) contains two elements \( e = (0)(1) \) and \( c = (01) \) (in cyclic notation). After S.Wolfram, we shall call the permutation \( c \in S_2 \) “black-white” (shortly BW) transformation.

Since the number of fixed points for \( e \) is either \( \mathbb{6} \) or \( \mathbb{1} \), all we need is to count the number of fixed points for the BW transformation in both \( k \)- and \((k + 1)\)-symmetry cases. The rules in both these cases can be represented, respectively, by the bit strings

\[
\alpha_1 \alpha_2 \cdots \alpha_{2k+2}
\]

and

\[
\alpha_1 \alpha_2 \cdots \alpha_{k+2}
\]

in accordance with the tables
One can see from these tables that the BW transformation acts similarly on both strings $\text{(7)}$ and $\text{(8)}$, namely,

$$\alpha_1 \alpha_2 \cdots \alpha_n \rightarrow \overline{\alpha_n} \overline{\alpha_{n-1}} \cdots \overline{\alpha_2} \overline{\alpha_1},$$

where bar means complementary transformation of bit, i.e., $\overline{\alpha} = \alpha + 1 \mod 2$, and $n = 2k + 2$ or $n = k + 2$.

The fixed point condition

$$\alpha_1 \alpha_2 \cdots \alpha_{n-1} \alpha_n = \overline{\alpha_n} \overline{\alpha_{n-1}} \cdots \overline{\alpha_2} \overline{\alpha_1}$$

implies that the string is defined by a half of bits, i.e., by $k + 1$ or by $(k + 2)/2$ bits. In other words, the numbers of different bit strings satisfying condition $\text{(9)}$ are, respectively, $2^{k+1}$ and $2^{(k+3)/2}$ (for $k$ even). For the $(k + 1)$-symmetry case with odd $k = 2m + 1$ condition $\text{(9)}$ leads to the contradiction

$$\overline{\alpha}_{m+1} = \alpha_{m+1},$$

which means zero number of bit strings in this case.

Summarizing our calculations (recollecting that $G = S_2$ in formula $\text{(1)}, i.e., |G| = 2$), we have:

- in the case of $k$-symmetry
  - number of BW-symmetric rules
    $$N_{S_k, BW} = 2^{k+1},$$
  - number of non-equivalent rules
    $$N_{S_k / BW} = 2^{2k+1} + 2^k,$$

- in the case of $(k + 1)$-symmetry
  - number of BW-symmetric rules
    $$N_{S_{k+1, BW}} = \begin{cases} 2^{k/2+1} & \text{if } k = 2m, \\ 0 & \text{if } k = 2m + 1, \end{cases}$$
  - number of non-equivalent rules
    $$N_{S_{k+1 / BW}} = \begin{cases} 2^{k+1} + 2^{k/2} & \text{if } k = 2m, \\ 2^{k+1} & \text{if } k = 2m + 1. \end{cases}$$

For example, for trivalent rules the numbers are

$$N_{S_3, BW} = 16, \quad N_{S_3 / BW} = 136, \quad N_{S_3+1, BW} = 0, \quad N_{S_3+1 / BW} = 16.$$
3 Assembling Neighborhoods into Regular Lattices

$k$-valent neighborhoods can be gathered into a lattice in many ways. In fact, any $k$-regular graph may serve as a space for a cellular automaton. In applications a cellular automaton acts, as a rule, on a lattice embedded in a metric space, usually 2- or 3-dimensional Euclidean space. Any graph, being 1-dimensional simplicial complex, can be embedded into $\mathbb{E}^3$. When dealing with symmetric local rules it is natural to consider equidistant regular systems of cells. We consider here only two-dimensional case as more simple and suitable for visualization of the automaton behavior. Most symmetric 2D lattices correspond to the tilings by congruent regular polygons. The number of different types of such tilings is rather restricted (see, e.g., [5]). To denote a $k$-valent lattice composed of regular $p$-gons we use the Schlöfli symbol $\{p, k\}$.

3.1 2D Euclidean Metric

There are only three regular lattices in $\mathbb{E}^2$ (see Fig. 1).

Since real computers have finite memory, cellular automata can be simulated only on a finite lattice. Usually the universe of a cellular automaton is a rectangle instead of an infinite plane. There are different ways to handle the edges of the rectangle. One possible method is to fix states of the border cells. This breaks the symmetry of the lattice and is thus not interesting for us. Another way is to glue together the opposite edges of the rectangle.

The toroidal arrangement is a standard practice, but it would be interesting to study cellular automata on nonorientable surfaces also. There are 3 different identifications of opposite sides of a rectangle: the torus $T^2$, the Klein bottle $K^2$, and the projective plane $P^2$. All these spaces (the gluing does not affect their Euclidean metric) are shown in the figure below.
We need to check whether the regular lattices like in Fig. 1, i.e., with the Schlafli symbols \( \{p, k\} = \{6, 3\}, \{4, 4\}, \{3, 6\} \) can be embedded in \( T^2 \), \( K^2 \), or \( P^2 \). To do this we must solve the system of equations

\[
V - E + F = \chi(M), \quad pF = kV = 2E,
\]

where \( V, E, F \) are numbers of vertices, edges, and faces, respectively, \( \chi(M) \) is the Euler characteristic of a manifold \( M \). Since \( \chi(T^2) = \chi(K^2) = 0 \) and \( \chi(P^2) = 1 \), we see that the regular lattices are possible only in the torus and the Klein bottle and impossible in the projective plane, as well as in any other closed surfaces.

Summarizing, for Euclidean metric there are only 3-valent hexagonal, 4-valent square and 6-valent triangular regular lattices in \( E^2 \), \( T^2 \), and \( K^2 \).

### 3.2 Hyperbolic Plane \( \mathbb{H}^2 \)

The hyperbolic (Lobachevsky) plane \( \mathbb{H}^2 \) allows infinitely many regular lattices. Poincaré proved that regular tilings \( \{p, k\} \) of \( \mathbb{H}^2 \) exist for any \( p, k \geq 3 \) satisfying \( \frac{1}{p} + \frac{1}{k} < \frac{1}{2} \).

For example, let us consider the octivalent Moore neighborhood used in the Life family and shown in the figure below.

![Octivalent regular lattice](image)

The Moore neighborhood can not form regular lattice in the Euclidean plane since the distances of surrounding cells from the center are different. But in the hyperbolic plane regular 8-valent lattices exist and there are infinitely many of them. The simplest one is shown (using the Poincaré disc model projection) in Fig. 2.

![Octivalent regular lattice \{3,8\} in \( \mathbb{H}^2 \)](image)
3.3 Sphere $S^2$

All regular lattices in the two-dimensional sphere correspond to the Platonic solids which are shown in Fig. 3.

![Fig. 3. All regular lattices in $S^2$](image)

In the sphere there are infinitely many other lattices, which are close to regular and may serve as spaces for 3-valent symmetric automata. They are called fullerenes.

The fullerenes were first discovered in carbon chemistry in 1985, and this discovery was rewarded with the 1996 Nobel Prize in Chemistry. A model of the first revealed fullerene — the carbon molecule $C_{60}$ — is displayed in Fig. 4 (the figure is borrowed from [6]). Later there were discovered other forms of large carbon molecules with structural properties of fullerenes (larger spherical fullerenes, carbon nanotubes, graphenes). Their unique properties promise they will have an important role in future technology, in particular, in nanotechnology engineering.

From a mathematical viewpoint, the structure of fullerene is a 3-valent convex polyhedron with pentagonal and hexagonal faces. In terms of graphs, fullerene can be defined as a 3-regular (3-valent) planar, or equivalently, embeddable in $M = S^2$, graph with all faces of size 5 or 6.

Let us generalize slightly this definition assuming that $M$ is not necessarily $S^2$, but may be closed surface of other type, orientable or nonorientable. Then the Euler–Poincaré equation together with the edge balance relations gives the
system of equations

\[ V - E + f_5 + f_6 = \chi(M), \quad 3V = 5f_5 + 6f_6 = 2E, \quad (15) \]

where \( f_5 \) and \( f_6 \) are numbers of pentagons and hexagons, respectively. The general solution of this system is

\[ f_5 = 6\chi(M), \quad (16) \]
\[ V = 2f_6 + 10\chi(M), \quad (17) \]
\[ E = 3f_6 + 15\chi(M). \quad (18) \]

We see that generalized fullerenes are possible only in the sphere \( S^2 \), in the projective plane \( \mathbb{P}^2 \), and in the torus \( \mathbb{T}^2 \) and Klein bottle \( K^2 \). Attempts to consider surfaces with greater genus lead due to (16) to senseless negative numbers of pentagons. Thus, we have for all generalized fullerenes:

- \( V = 2f_6 + 20, \quad E = 3f_6 + 30, \quad f_5 = 12, \) sphere \( S^2 \);
- \( V = 2f_6 + 10, \quad E = 3f_6 + 15, \quad f_5 = 6, \) projective plane \( \mathbb{P}^2 \);
- \( V = 2f_6, \quad E = 3f_6, \quad f_5 = 0, \) torus \( \mathbb{T}^2 \), Klein bottle \( K^2 \).

We see that any fullerene in \( S^2 \) or in \( \mathbb{P}^2 \) contains exactly 12 or 6 pentagons, respectively, and arbitrary number of hexagons. In the case of torus or Klein bottle the fullerene structure degenerates into purely hexagonal lattice without pentagons considered already in subsection 3.1. Note that in carbon chemistry one-layer graphite sheets (and similar structures with few pentagons or heptagons added) are called graphenes.

4 Canonical Decomposition of Some Rules from Life Family

In this section we present the canonical decompositions of relations for three rules from the Life family: the standard Conway’s Life, HighLife and Day&Night.

Recall that canonical decomposition \[ \Xi \] of relation on a set of points is the representation of the relation as a combination of its projections onto subsets
of points. This decomposition is discrete analog of compatibility analysis of algebraic and differential equations. To apply our approach we interpret the local rule \( x'_9 = f(x_1, x_2, \ldots, x_9) \) as a relation on 10 points \( x_1, x_2, \ldots, x_9, x'_9 \).

The \textit{HighLife} is interesting since for it the \textit{replicator} — a self-reproducing pattern — is known explicitly. For \textit{Conway’s Life}, the existence of replicators is proved, but no example is known.

The name \textit{Day&Night} reflects the BW-symmetry of this rule. Perhaps, it is the first (conceivably found “by hand”) automaton from the Life family with this symmetry property. Note that there are exactly 512 such automata in the Life family, and they all are generated easily (for time \(<1\) sec) by the C program mentioned in [7].

It might be easier to grasp the relations, if they are written in the form of polynomials over the field \( \mathbb{F}_2 \). In the below formulas we use \textit{elementary symmetric polynomials} in variables \( x_1, \ldots, x_8 \). Recall that the \( k \)th (of degree \( k \)) elementary symmetric polynomial of \( n \) variables \( x_1, \ldots, x_n \) is defined by the formula:

\[
\sigma_k (x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1}x_{i_2} \cdots x_{i_k}.
\]

Hereafter we use the notations \( \sigma_k \equiv \sigma_k(x_1, \ldots, x_8), \sigma^j_k \equiv \sigma_k(x_1, \ldots, \sigma_i, \ldots, x_8) \), \( \sigma^i_k \equiv \sigma_k(x_1, \ldots, \sigma_i, \ldots, \sigma_j, \ldots, x_8) \). The indices \( i, j, k, l \) satisfy \( 1 \leq i < j < k < l \leq 8 \).

The polynomials representing \textit{Conway’s Life}, \textit{HighLife} and \textit{Day&Night} take the forms, respectively

\[
P_{\text{Conway’s Life}} = x'_9 + x_9 \{ \sigma_7 + \sigma_6 + \sigma_3 + \sigma_2 \} + \sigma_7 + \sigma_3, \quad (19)
\]
\[
P_{\text{HighLife}} = x'_9 + x_9 \{ \sigma_3 + \sigma_2 \} + \sigma_6 + \sigma_3, \quad (20)
\]
\[
P_{\text{Day&Night}} = x'_9 + x_9 \{ \sigma_7 + \sigma_6 + \sigma_5 + \sigma_4 \} + \sigma_8 + \sigma_7 + \sigma_6 + \sigma_3. \quad (21)
\]

Note that these polynomials have degrees 8, 6, 8 and numbers of terms 185, 169, 256, respectively, so application of the Gröbner basis technique to their analysis may take some time (about 1 hour on 1.8GHz AMD Athlon notebook with the Maple 9 Gröbner procedure). Our program computes the decompositions for time \(<1\) sec.

The decompositions are:

- \textit{Conway’s Life}:

\[
x'_9 \{ \sigma_3 + \sigma_2 + 1 \} + \sigma_7 + \sigma_3 = 0, \quad (22)
\]
\[
x'_9 \sigma_1^j + x_9 \{ \sigma_2^i + 1 \} + x'_9 \{ \sigma_2^i + 1 \} + x_9 \{ \sigma_2^j + \sigma_6^j + \sigma_4^j + \sigma_2^j \} = 0, \quad (23)
\]
\[
x'_9 \{ \sigma_3^j + \sigma_2^j + \sigma_1^j + 1 \} = 0, \quad (24)
\]
\[
x'_9 (x_9 + 1) \{ \sigma_3^{ij} + \sigma_2^{ij} + \sigma_1^{ij} + 1 \} = 0, \quad (25)
\]
\[
x'_9 x'_9 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 = 0. \quad (26)
\]
\[ x'_9 \{ \sigma_3 + \sigma_2 + 1 \} + \sigma_7 + \sigma_3 = 0, \] (27)

\[ x'_9 x_9 \{ \sigma_4^i + \sigma_1^i \} + x'_9 \{ \sigma_5^i + \sigma_2^i + 1 \} + x_9 \{ \sigma_7^i + \sigma_6^i + \sigma_3^i + \sigma_2^i \} = 0, \] (28)

\[ x'_9 \{ \sigma_7^i + \sigma_3^i + \sigma_2^i + \sigma_4^i + 1 \} = 0, \] (29)

\[ x'_9 x_9 \{ \sigma_3^{ij} + \sigma_2^{ij} + \sigma_1^{ij} + 1 \} + x'_9 \{ \sigma_6^{ij} + \sigma_5^{ij} + \sigma_4^{ij} + \sigma_3^{ij} + \sigma_2^{ij} + \sigma_1^{ij} + 1 \} = 0, \] (30)

\[ x'_9 x_9 x_i x_j x_k x_l = 0. \] (31)

\[ \begin{align*}
    x'_9 & \{ \sigma_7 + \sigma_6 + \sigma_5 + \sigma_4 + 1 \} + \sigma_8 + \sigma_7 + \sigma_6 + \sigma_3 = 0, \quad \text{(32)} \\
    x'_9 x_9 \{ \sigma_7^i + \sigma_6^i + \sigma_4^i + \sigma_3^i \} + x'_9 \{ \sigma_7^i + \sigma_6^i + \sigma_5^i + \sigma_2^i + 1 \} + x_9 \{ \sigma_7^i + \sigma_3^i \} + \sigma_6^i = 0, \quad \text{(33)} \\
    x'_9 \{ \sigma_7^{ij} + \sigma_6^{ij} + \sigma_4^{ij} + \sigma_3^{ij} + \sigma_2^{ij} + \sigma_1^{ij} + 1 \} + \sigma_6^{ij} = 0. \quad \text{(34)}
\end{align*} \]

Note that system (34) of prime relations can be combined into the reducible relation (for terminology like prime and reducible see [7])

\[ x'_9 \{ \sigma_3^i + \sigma_2^i + 1 \} + \sigma_6^i = 0, \]

which looks nicer (the polynomial representation of relations is somewhat artificial), but depends on larger set of variables.

We see that the decomposition for Day&Night differs essentially from the decompositions for Conway’s Life and HighLife having resembling structures.

5 Conclusions

We proved that the cellular automata from the Life family are nothing but binary automata with the local rules symmetric with respect to all permutations of the outer cells in the neighborhood.

Then we showed that the number of non-equivalent with respect to renaming of states \( k \)-valent symmetric binary local rules is equal to \( 2^{2k+1} + 2^k \). Considering the 3-valent case — the next step up after the 2-valent Wolfram’s elementary automata — we see that the total number of non-equivalent symmetric rules is only 136.

All interesting 3-valent 2-dimensional regular (or almost regular) lattices are:

- hexagonal lattice \( \{6, 3\} \) in the plane \( \mathbb{E}^2 \), in the torus \( \mathbb{T}^2 \) and in the Klein bottle \( \mathbb{K}^2 \)
- tetrahedron \( \{3, 3\} \), hexahedron (cube) \( \{4, 3\} \) and dodecahedron \( \{5, 3\} \) in the sphere \( \mathbb{S}^2 \)
– fullerenes in $S^2$ and in the projective plane $\mathbb{P}^2$.

Combining these lattices with 136 symmetric rules we obtain a class of cellular automata which looks like quite available for systematic study. Since these automata have more interesting geometry than the elementary ones we may expect more interesting behavior of them.

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