Vortex distribution in the Lowest Landau Level

Amandine Aftalion, Xavier Blanc

Laboratoire Jacques-Louis Lions, UMR-CNRS 7598, Université Paris 6, 175 rue du Chevaleret, 75013 Paris, France.

Francis Nier
IRMAR, UMR-CNRS 6625, Université Rennes 1, 35042 Rennes Cedex, France.

(Dated: November 7, 2018)

We study the vortex distribution of the wave functions minimizing the Gross Pitaevskii energy for a fast rotating condensate in the Lowest Landau Level (LLL): we prove that the minimizer cannot have a finite number of zeroes thus the lattice is infinite, but not uniform. This uses the explicit expression of the projector onto the LLL. We also show that any slow varying envelope function can be approximated in the LLL by distorting the lattice. This is used in particular to approximate the inverted parabola and understand the role of “invisible” vortices: the distortion of the lattice is very small in the Thomas Fermi region but quite large outside, where the “invisible” vortices lie.

PACS numbers: 03.75.Hh, 05.30.Jp, 67.40.Db, 74.25.Qt

The fast rotating regime for a Bose Einstein condensate in a harmonic trap, observed experimentally in [1, 2, 3], displays analogies with type II superconductors behaviors and Quantum Hall Physics. However, some different features have emerged and are of interest, in particular due to the existence of a potential trapping the atoms.

A quantum fluid described by a macroscopic wave function rotates through the nucleation of quantized vortices [4]. For a condensate confined in a harmonic potential rotation, observed experimentally in [1, 2, 3], displays analogies with type II superconductors behaviors and Quantum Hall Physics. However, some different features have emerged and are of interest, in particular due to the existence of a potential trapping the atoms. The visible vortices arrange themselves in a triangular Abrikosov lattice. The system is strongly confined along the axis of rotation, and it is customary to restrict to a two dimensional analysis in the $x-y$ plane. We will call $z = x + iy$. The Hamiltonian is similar to that for a charged particle in a magnetic field: for rotational angular velocities just below the transverse trap frequency, the wave function of the condensate can be described using only components in the Lowest Landau Level (LLL):

$$
\Psi(z) = \Phi_0 \prod_{i=1}^{N} (z - z_i) e^{-|z|^2/2}
$$

(1)

where $\Phi_0$ is a normalization factor and the $z_i$ are the location of the vortices. In rescaled units, the reduced energy in the LLL is [5, 6, 7, 10]

$$
E_{LLL}(\Psi) = \int \left[(1 - \Omega)|\Psi|^2 + \frac{G}{2} |\Psi|^4\right] d^2 r
$$

(2)

under $\int d^2 r |\Psi|^2 = 1$, where $\Omega$ is the rotational velocity, the transverse trap frequency is scaled to 1, and $G$ models the interaction term: $G = Ng/(d\sqrt{2\pi})$, where $g$ is the two body interaction strength and $d$ is the characteristic size of the harmonic oscillator in the direction of the rotation.

In the absence of a confining potential, the problem is reduced to the one studied by Abrikosov [8] for a type II superconductor and the minimizer is a wave function with a uniform triangular lattice [9]; its modulus vanishes once in each cell and is periodic over the lattice. The presence of the confining potential is at the origin of a slow varying density profile, which can be described as the mean of the modulus of the wave function on many cells. Ho [10] predicted that for a uniform lattice, the smoothed density profile is a gaussian. Various contributions [6, 7, 10] then pointed out that the energy can be lowered if this smoothed density distribution is an inverted parabola rather than a gaussian. This type of density profile can be achieved either by taking wave functions with a uniform lattice but with components outside the LLL [10], or by remaining in the LLL and distorting the lattice. The study of the distortion has been the focus of recent papers [10, 11] and raises the issue of the optimal vortex distribution. In the LLL description, there are two kinds of vortices: the “visible vortices”, which lie in the region where the wave function is significant (for instance inside the Thomas Fermi region in the case of the inverted parabola), and the “invisible vortices” which are in the region where the modulus of the wave function is small. The visible vortices form a regular triangular lattice, while the invisible ones seem to have a strong distorted shape, whose distribution is essential to recreate the inverted parabola profile inside the LLL approximation. These latter are not within reach of experimental evidence, but can be computed numerically [10, 11]. An important theoretical question is the distribution of these invisible vortices, their number, or an estimate of how many of them are necessary to approximate the inverted parabola properly inside the LLL.
Our main result is to prove that in order to minimize the energy in the LLL, there is a need for an infinite number of vortices. The main tool that we use is an explicit expression of the projector onto the LLL. This projector also allows us to approximate any slow varying density profile by LLL wave functions.

**Projection onto the LLL and infinite number of zeroes** We define a small parameter \( \varepsilon = \sqrt{1 - \Omega} \) and make the change of variables \( \psi(z) = \sqrt{\varepsilon} \Psi(\sqrt{\varepsilon} z) \), so that the condensate is of size of order 1 and the lattice spacing is expected to be of order \( \sqrt{\varepsilon} \). The energy gets rescaled as \( \mathcal{E}_{LLL}(\Psi) = \varepsilon E_{LLL}(\psi) \) where

\[
E_{LLL}(\psi) = \int \left[ |z|^2 |\psi|^2 + \frac{G}{2} |\psi|^4 \right] d^2 r.
\]

Moreover, \( \psi \) belongs to the LLL so that \( f(z) = \psi(z)e^{i|z|^2/2\varepsilon} \) is a holomorphic function and thus belongs to the so called Fock-Bargman space

\[
\mathcal{F} = \{ f \text{ is holomorphic} , \int |f|^2 e^{-|z|^2/\varepsilon} d^2 r < \infty \}.
\]

Let us point out that such a function \( f \) is not only determined by its zeroes and normalization factor as in [1], but also by a globally defined phase, which is a holomorphic function. The space \( \mathcal{F} \) is a Hilbert space endowed with the scalar product \( \langle f, g \rangle = \int f(z)\overline{g(z)} e^{-|z|^2/\varepsilon} d^2 r \). The point of considering this space is that the projection of a general function \( g(z, \bar{z}) \) onto \( \mathcal{F} \) is explicit, and called the Szego projector [12 13]:

\[
\Pi(g) = \frac{1}{\pi \varepsilon} \int e^{-|z|^2/\varepsilon} e^{-\frac{|z'|^2}{\varepsilon}} g(z', \bar{z'}) d^2 r'.
\]

If \( g \) is a holomorphic function, then an integration by part yields \( \Pi(g) = g \).

If one considers the minimization of \( E_{LLL}(\psi) \) without the holomorphic constraint on \( f \), then the minimization process yields that \( |z|^2 + G|\psi|^2 - \mu = 0 \), where \( \mu \) is the chemical potential due to the constraint \( \int |\psi|^2 = 1 \), so that \( |\psi| \) is the inverted parabola

\[
|\psi|^2(z) = \left( 1 - \frac{2G}{\pi R^2} \right) \mathbf{1}_{|z| \leq R}, R = \sqrt{\mu} = \left( \frac{2G}{\pi} \right)^{1/4}.
\]

The restriction to the LLL prevents from achieving this specific inverted parabola since \( \psi e^{i|z|^2/2\varepsilon} \) cannot be a holomorphic function. The advantage of the explicit formulation of the projector \( \Pi \) is that it allows us to derive an equation satisfied by \( \psi \) or rather \( f \) when minimizing the energy in the LLL. A proper distribution of zeroes can approximate an inverted parabola profile but is going to modify the radius \( R \) by a coefficient \( b^{1/4} \) coming from the contribution of the vortex lattice to the energy.

**Theorem 1** If \( f \in \mathcal{F} \) minimizes

\[
E(f) = \int \left[ |z|^2 |f|^2 e^{-|z|^2/\varepsilon} + \frac{G}{2} |f|^4 e^{-|z|^2/\varepsilon} \right] d^2 r
\]

under \( \int |f|^2 e^{-|z|^2/\varepsilon} d^2 r = 1 \), then \( f \) is a solution of the following equation

\[
\Pi \left( (|z|^2 + G|f|^2 e^{-|z|^2/\varepsilon} - \mu) f \right) = 0
\]

where \( \mu \) is the chemical potential coming from the mass constraint.

Note that given the relation between \( f \) and \( \psi \), \( E(f) \) and \( E_{LLL}(\psi) \) are identical. Equation (8) comes from the fact that for any \( g \) in \( \mathcal{F} \) with \( \langle f, g \rangle = 0 \), if \( f \) minimizes \( E \), then we have

\[
\int \left[ |z|^2 |\bar{f}f|^2 e^{-|z|^2/\varepsilon} + \frac{G}{2} |\bar{f}f|^4 e^{-|z|^2/\varepsilon} \right] d^2 r = 0
\]

and we use the scalar product in \( \mathcal{F} \) and the definition of the projector to conclude.

The equation for the minimizer allows us to derive that this minimizer cannot be a polynomial:

**Theorem 2** If \( f \in \mathcal{F} \) minimizes \( E \), then \( f \) has an infinite number of zeroes.

We are going to argue by contradiction and assume that \( f \) is a polynomial.

1. The proof first requires another formulation of (5). The projector \( \Pi \) has many properties [12 17]: in particular, one can check, using an integration by part in the expression of \( \Pi \), that \( \Pi \left( |z|^2 f \right) = ze\partial_z f + \varepsilon f \). As for the middle term in the equation, one can compute that if \( f \) is a polynomial, \( \Pi \left( e^{-\frac{|z|^2}{\varepsilon}} |f|^2 \right) = \Pi \left( e^{-\frac{|z|^2}{\varepsilon}} |f|^2 \right) \Pi \left( e^{-\frac{|z|^2}{\varepsilon}} f \right) \)

\[
= \Pi \left( \Pi \left( e^{-\frac{|z|^2}{\varepsilon}} f \right) \right) \Pi \left( e^{-\frac{|z|^2}{\varepsilon}} f \right) = \tilde{f} \Pi \left( e^{-\frac{|z|^2}{\varepsilon}} f \right).
\]

A simple change of variable yields \( \Pi \left( e^{-\frac{|z|^2}{\varepsilon}} f \right) = \frac{1}{2} \Pi \left( \tilde{f} \right) \left( \frac{\tilde{z}}{\tilde{r}} \right) = \frac{1}{2} \tilde{f} \left( \frac{\tilde{z}}{\tilde{r}} \right) \). Thus, we find the following simplification of (5):

\[
ze\partial_z f + \frac{G}{2} \tilde{f} \left( \frac{\tilde{z}}{\tilde{r}} \right) - (\mu - \varepsilon) f = 0.
\]

2. Now we assume that \( f \) is polynomial of degree \( n \) and a solution of (10). We are going to show that there is a contradiction due to the term of highest degree in the equation. Indeed, if \( f \) is a polynomial of degree \( n \), then \( \langle \varepsilon \partial_z f | \tilde{f} \rangle = \langle \tilde{f} \rangle \langle f^2 \rangle / 2 \) is of degree \( 2n - k \). But (10) implies that \( \tilde{f} \langle \varepsilon \partial_z f | \tilde{f} \rangle = \langle \tilde{f} \rangle \langle f^2 \rangle / 2 \) is of degree \( n \), hence \( f \) must be equal to \( cz^n \). This is indeed a solution of (10) if \( nz + G|c|^2 \pi e^n (2n)! / (2^{2n+1} n!) - \mu + \varepsilon = 0 \). Using that \( \int |f|^2 e^{-|z|^2/\varepsilon} = 1 \), we find that \( |c|^2 \pi e^{n+1} n! = 1 \). The Stirling formula provides the existence of a constant \( c_0 \)
such that $n \epsilon + \alpha_0 G/(2 \pi \epsilon \sqrt{n}) \leq \mu$. For the minimizer, $\mu$ is of the same order as the energy, thus of order 1, so that if $\epsilon$ is too small, no $n$ can satisfy this last identity hence the minimizer is not a polynomial. A similar argument can be used to check that, if $f$ is more generally a holomorphic function in $\mathcal{F}$, then it cannot have a finite number of zeroes. The detailed proof will be given in [1].

Approximation of a slowly varying profile by the LLL

The Abrikosov solution

The Abrikosov problem [8] consists in minimizing the ratio $\langle |u|^4 \rangle / \langle |u|^2 \rangle^2$ over periodic functions, where $\langle \cdot \rangle$ denotes the average value over a cell, for functions $u$ obtained as limits of LLL functions. The minimum is achieved for $u = u_\epsilon(z, e^{2i\pi/3})$ where [14]

$$u_\epsilon(z, \tau) = \epsilon^{-|z|^2/2\epsilon} f_\epsilon(z, \tau), \quad f_\epsilon(z, \tau) = \epsilon^2/2\epsilon \Theta(\sqrt{\pi \epsilon/\tau}, \tau)$$

and for any complex number $\tau = \tau_R + i \tau_I$,

$$\Theta(v, \tau) = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} (-1)^n e^{i \pi n (n+1)^2/2} e^{2(n+1) \pi i \tau}.$$  \hspace{1cm} (11)

The $\Theta$ function has the following properties

$$\Theta(v + k + l \tau, \tau) = (-1)^{k+l} e^{-2i\pi l \tau} e^{-i \pi k \tau} \Theta(v, \tau)$$  \hspace{1cm} (13)

so that $|u_\epsilon(z, \tau)|$ is periodic over the lattice $\sqrt{\pi \epsilon/\tau} \oplus \sqrt{\pi \epsilon/\tau} \mathbb{Z}$, and vanishes at each point of the lattice. Without loss of generality, one can restrict $\tau$ to vary in $|\tau| \geq 1$, $-1/2 \leq \tau_R < 1/2$: this is equivalent to require that the smallest period for $\Theta$ is 1 and along the $x$ axis (see [15]) and any lattice in the plane, can be obtained from one of these by similarity. For any $\tau$, $f_\epsilon$ given by [11] is a solution of

$$\Pi(|\epsilon f_\epsilon|^2 e^{-|z|^2/\epsilon} f_\epsilon) = \lambda_\epsilon f_\epsilon,$$

with $\lambda_\epsilon = \langle |u_\epsilon|^2 \rangle b(\tau)$,  \hspace{1cm} (14)

and

$$b(\tau) = \langle |u_\epsilon|^4 \rangle / \langle |u_\epsilon|^2 \rangle^2 = \sum_{k,l \in \mathbb{Z}} e^{-\pi |k \tau - l \tau|^2/\tau_I}.$$  \hspace{1cm} (15)

This expression can be obtained using arguments in [10]. The minimal value of $b(\tau) \sim 1.16$ is achieved for $\tau = e^{2i\pi/3}$, that is for the triangular lattice [3]; in [8], it is argued that one can restrict to $\tau_R = -1/2$, and vary $\tau_I$ in $(1/2, \sqrt{3}/2)$. Accepting this restriction, they compute the variations of $b$ which depends on a single parameter and is indeed minimal for the triangular lattice. In [17], we prove that this restriction is rigorous using the description of these lattices by varying $\tau$ for $|\tau| = 1$ and $\tau_R \in (-1/2, 0)$.

If one compares [14] and [5], one notices that they only differ by the term $\Pi(|z|^2 f) = \epsilon z \partial_z \bar{f} + \epsilon f$, which is negligible on the lattice size, but plays a role on the shape of the density profile.

The role of the confining potential

A natural candidate to approximate any slow varying profile $\alpha(z, \bar{z})$ is to take $\alpha(z, \bar{z}) u_\epsilon(z, \tau)$, where $u_\epsilon$ is the periodic function defined in [11]. Of course, such a function is not in the LLL, but can be well approximated in the LLL by $f^\alpha e^{-|z|^2/2\epsilon}$ where $f^\alpha = \Pi(\alpha f_\epsilon)$. $\Pi$ is the projector onto the LLL and $f_\epsilon$ comes from [10]. Estimating the energy of $f^\alpha$ yields $E(\alpha) - \int |z|^2 |\alpha|^2 \langle |u_\epsilon|^2 \rangle + \frac{G(b(\tau))}{2} |\alpha|^4 \langle |u_\epsilon(z, \bar{z})|^2 \rangle d^2r \sim C \epsilon^{-1/4}$. This computation uses calculus on $\Pi$, and that $u_\epsilon$ and $\alpha$ do not vary on the same scale, hence the integrals can be decoupled. The contribution of $u_\epsilon$ to the energy is through the coefficient $b(\tau)$, which is minimum for $\tau = e^{2i\pi/3}$.

Using pseudo differential calculus, one can show [17], when $\epsilon$ is small, that $f^\alpha$ is very close to $\alpha u_\epsilon$: the error is at most like $\epsilon^{1/4}$ if $\alpha$ is not more singular than an inverted parabola. In particular, when $\alpha$ is an inverted parabola, this implies that in the Thomas Fermi region, the distribution of visible vortices is almost that of the triangular lattice since $\alpha u_\epsilon$ is a good approximation. Outside the support of the inverted parabola, where $f^\alpha$ is very small, one can check that the density of distribution of zeroes of $f_\epsilon$ decreases like $1/|z|$ for large $|z|$. Contrary to what was explained in [10].

The special shape of the inverted parabola comes out if one wants to approximate the equation of the minimizer of the energy: for any $\lambda$, we can prove that

$$\Pi\left(|z|^2 + G|f^\alpha|^2 e^{-|z|^2/\epsilon} - \lambda f^\alpha\right) + C \epsilon^{-1/4} \sim \Pi\left(|z|^2 + G b(\tau) \langle |u_\epsilon|^2 \rangle |\alpha|^2 - \lambda \alpha f_\epsilon\right)$$  \hspace{1cm} (16)

where $C$ only depends on bounds on $\alpha$. In other words, in the equation for $f^\alpha$, one can separate in the term $|f^\alpha|^2 e^{-|z|^2/\epsilon}$ the contributions due to the lattice and to the profile. The right hand side of (16) is zero if $\alpha$ is the inverted parabola

$$\alpha(z) = \frac{2}{\pi R_0^2 \langle |u_\epsilon|^2 \rangle} \left(1 - \frac{|z|^2}{R_0^2}\right), \quad R_0 = \left(\frac{2 G b(\tau)}{\pi}\right)^{1/4}$$

and $\lambda = R_0^2$, so that $f^\alpha$ is almost a solution of [3], up to an error in $\epsilon^{1/4}$.

Variations of the lattice

This approach can be used to study the variations in energy due to deformations of the lattice. The triangular lattice, corresponding to $\tau^1 = e^{2i\pi/3}$, is such that the Hessian of $b(\tau)$ is isotropic ($\sim 0.68 Id$). Two lattices close to each other can be described by two close complex numbers $\tau^1$ and $\tau^2$: the dif-
ference in energy between $E(f^\alpha(.,\tau^1))$ and $E(f^\alpha(.,\tau^2))$ is at leading order

$$G \frac{\partial^2 b}{\partial \tau^2_R} |\tau^1 - \tau^2|^2 \int |\alpha|^4 \langle |u_e|^2 \rangle^2 \, d^2 r \sim \frac{0.68 G}{3\pi R_0^6} |\tau^1 - \tau^2|^2$$

This computation justifies the approach which consists in decoupling the lattice contribution from the profile contribution in the energy $[18]$ but, given the definition of $f^\alpha$ using II, it relies on strong deformations of the lattice for points far away from the Thomas Fermi region. For a shear deformation for which $u_{ij}$ are the components of the deformation tensor, $\tau^2 - \tau^1 = i \sqrt{3} u_{xy}$. The elastic coefficient $C_2$ is defined by the fact that the difference in energy should be $4 C_2 u_{x'y}'$. This separation of scales allows to compute $C_2 \sim 0.68 G/(4 \pi R_0^5)$ (see also $[18, 19]$) and relate it in BEC to the same one computed for the Abrikosov solution.

Approximation by polynomials and modes An interesting issue, especially for the computations of modes, is to get an estimate of the degree of the polynomial which could approximate $f^\alpha$, since this function has an infinite number of zeroes. We can prove $[17]$ that as the degree of the polynomial gets large, the minimum of the energy for the problem restricted to polynomials (and the computation of modes) is a good approximation of the full problem. The convergence rates that we obtain are not satisfactory yet. We believe that a good degree should be $\kappa/\varepsilon$, with $\kappa > R_0^2$ and $R_0$ is the radius of the inverted parabola. Given that the volume of the cell is $\pi \varepsilon$, $\kappa = R_0^2$ would correspond to having only the visible vortices. Numerical simulations indicate that a sufficient number of invisible vortices is needed to recreate the inverted parabola profile $[11]$. There are two types of invisible vortices: those close to the boundary of the inverted parabola which contribute to the bulk modes and those sufficiently far away which produce single particle excitations as explained in $[11]$. An open issue is to understand the location of these latter invisible vortices; some simulations suggest that they lie on concentric circles, but then the density of these circles should be very low to match our predicted global vortex density far away which behaves like $1/|z|$. We have performed numerical simulations with $\Omega = 0.999$ and $G = 3$: this fixes the number of visible vortices to 30, and we vary the number of total vortices $N$. One needs at least $N = 52$ (that is 22 invisible vortices) to properly approximate the inverted parabola, the energy minimizer and the bulk modes. The distortion of the lattice is small at the edges but large at large distances. For $N$ too small, some modes do not appear (see Figure $[11]$), while for $N$ very large, one expects higher modes that $[11, 20]$ interpret as single particle modes.

Conclusion We have shown that for the minimizer of the Gross Pitaevskii energy in the LLL, the lattice of vortices is infinite, but not uniform. Any slow varying profile can be approximated in the LLL by distorting the lattice. This is proved using an explicit expression for the projection onto the LLL. Our results also give an insight on the elastic coefficient $C_2$ and the approximation of the minimizer and modes by polynomials.

Acknowledgements: We are very indebted to Jean Dalibard and Allan MacDonald for stimulating discussions. Part of them took place at the "Fondation des Treilles" in Tourtour which hosted a very fruitful interdisciplinary maths-physics meeting on these topics. We thank James Anglin, Sandy Fetter and Sandro Stringari for interesting comments.

[1] J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, Science 292, 476 (2001); C. Raman, J. R. Abo-Shaeer, J. M. Vogels, K. Xu, and W. Ketterle, Phys. Rev. Lett. 87, 210402 (2001).
[2] V. Schweikhard, I. Coddington, P. Engels, V. P. Mendoza, and E. A. Cornell, Phys. Rev. Lett. 92, 040404 (2004).
[3] I. Coddington, P. C. Haljan, P. Engels, V. Schweikhard, S. Tung, E. A. Cornell, Phys. Rev. A 70, 063607 (2004).
[4] R. J. Donnelly, Quantized Vortices in Helium II, (Cambridge, 1991), Chaps. 4 and 5.
[5] T. Y. Ho, Phys. Rev. Lett. 87, 060403 (2001).
[6] G. Watanabe, G. Baym and C. J. Pethick, Phys. Rev. Lett. 93, 190401 (2004).
[7] A. Aftalion, X. Blanc and J. Dalibard, Phys. Rev. A 71, 023611 (2005).
[8] A.A. Abrikosov, Zh. Eksp. Teor. Fiz. 32, 1442 (1957) [Sov. Phys.-JETP 5, 1174 (1957)].
[9] W. H. Klein, L. M. Roth and S. H. Autler, Phys. Rev. 133, A1226, (1964).
[10] N. R. Cooper, S. Komineas and N. Read, Phys. Rev. A 70, 033604 (2004).
[11] C.B. Hanna, A.J. Sup, and A.H. MacDonald, private communication.
[12] A. Martinez, Introduction to semiclassical and microlocal analysis New York: Springer-Verlag, 2002.
[13] G.B. Folland Harmonic analysis in phase space Princeton University Press, 1989.
[14] V.K. Tkachenko, Sov. Phys. JETP 22 1282 (1966); 23, 1049 (1966).
[15] K. Chandrasekharan *Elliptic functions* Springer 1985.

[16] O. Törnkvist, arXiv:hep-ph/9204235.

[17] A. Aftalion, X. Blanc and F. Nier, in preparation.

[18] E.B. Sonin, cond-mat/0411641.

[19] J. Sinova, C. B. Hanna, and A. H. MacDonald, Phys. Rev. Lett. 89, 030403 (2002).

[20] C.B. Hanna, A.J. Sup, and A.H. MacDonald, private communication.