Inverse transient thermoelastic problem with heat source in an annular disc.

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Abstract. This article deals with formulating an inverse problem of heat conduction to analyse thermoelastic stress in a thick annular disc due to internal heat generation within the solid. The integral transformation techniques, Marchi-Zgrablich transform and finite Marchi-Fasulo integral transform, are employed in order to solve governing transient heat conduction equation, under stated thermal boundary conditions. The outcomes obtained from analytical solution are represented in the form of infinite series consisting Bessel’s functions. Numerical verification is done for a particular case of a thick annular disc made of aluminum metal and results are depicted graphically.

Keywords – Inverse Thermoeelastic problem, Marchi-Zgrablich, Marchi –Fasulo transform.

1. Introduction

Thermoelasticity is the domain of science that deals with the study of temperature distribution, thermal stresses and strain developed in a material body as an effect of application of thermo-mechanical load. Thermoelasticity describes a broad range of phenomena and hence it is of paramount importance in the stress analysis. In extensive engineering applications, mainly in the field of design of a structural element, thermoelastic behaviour of a material plays a primary and decisive role. Due to change in temperature, thermal stresses occur in the material body, which in turn results in failure of a structural element. Thermal stresses, being one of the most important factor that affect the life of the material body, its analysis is very important in plenty of engineering applications. As a consequence, the demand for the study of thermal behaviour of the solids increases.

With the progress of science and technology, thermoelasticity becomes more and more attractive subject for researchers. A lot of investigators are interested to deal with the thermoelastic behaviour of the annular discs constituting the foundation of containers for hot gases, furnaces and many more applications. In addition, the inverse problems of thermoelasticity are of keen interest in view of its relevance in aerospace engineering.

A thorough literature review shows that, though many studies have reported the direct thermoelastic problems, very little research has been done in the area of inverse thermoelastic problems of thick solid bodies. Marchi E, Zgrablich G [1] highlighted the study of vibrations considering hollow circular membrane having elastic supports. The heat conduction in sector of
hollow cylinder with radiation is discussed by Marchi E. and Fasulo A. [2]. An inverse unsteady thermoelastic problem of a transversely isotropic body. Thermal stress analysis of a direct thermoelastic problem of a thick annular disc with radiation conditions is elaborated by Khobragade N. [4]. A thin annular disc has been analyzed by Khobragade K. et al. [5] to study an inverse transient thermoelastic problem. Analysis of thermal stresses in a thick annular disc is done by Kulkarni and Deshmukh [6]. An axisymmetric inverse steady state problem of thermoelastic deformation of a finite length hollow cylinder is discussed by Deshmukh and Wankhede [7]. Recently, influence of internal heat generation on thermal stresses is determined by Shinde A. et al. [8] by considering transient thermoelastic problem of a thick circular plate.

The purpose of this paper is to analyze the impact of internal heat source on thermoelastic behaviour of a thick annular disc. Marchi-Zgrablich and finite Marchi-Fasulo integral transforms are employed to tackle the expression of heat conduction. The expressions are derived for different field parameters such as temperature distribution, displacement and thermal stresses. To demonstrate the analytical solution, numerical computations and graphic plots are done for thick annular disc of aluminum material. A special case of interest has also been presented.

2. Mathematical formulation
Consider a thick annular disc having thickness $2h$, defined by $a \leq r \leq b$, $-h \leq z \leq h$, which is initially at zero temperature. For times $t > 0$, heat is generated within the disc at the rate $g(r, z, t)$ and dissipated by convection from its fixed circular edges $r = a$ and $r = b$ into a surrounding at zero temperature. On the lower surface $z = -h$, homogeneous boundary condition of the third kind is maintained. With given initial condition and stated boundary conditions, the unknown temperature $G(r, t)$ on upper surface $z = h$ and the thermal stresses need to be determined.

![Figure 1. Geometry of the problem.](image)

The differential equation governing the temperature function $T(r, z, t)$ is given by

$$\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] + \frac{g(r, z, t)}{k} = \frac{1}{a} \frac{\partial T}{\partial t} \quad (2.1)$$

Subject to initial condition,

$$T(r, z, t) = 0; \quad \text{at} \ t = 0 \quad (2.2)$$

and the boundary conditions

$$T(r, z, t) + h_1 \frac{\partial T(r, z, t)}{\partial r} = 0 \quad \text{at} \ r = a \quad (2.3)$$

$$T(r, z, t) + h_2 \frac{\partial T(r, z, t)}{\partial r} = 0 \quad \text{at} \ r = b \quad (2.4)$$

$$T(r, z, t) + h_3 \frac{\partial T(r, z, t)}{\partial z} = 0 \quad \text{at} \ z = -h \quad ; a \leq r \leq b \quad (2.5)$$

$T(r, z, t)) + h_4 \frac{\partial T(r, z, t)}{\partial z} = G(r, t) \quad (unknown) \ \text{at} \ z = h; \quad a \leq r \leq b \quad (2.6)$

the interior condition,
The displacements, $u_r$ and $u_z$, in radial and axial directions respectively, are expressed in terms of the Goodier’s thermoelastic displacement potential $\phi$ and Love’s function $\psi$ as

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 \phi}{\partial r \partial z}$$
$$u_z = \frac{\partial \phi}{\partial z} + 2(1 - \nu)\nabla^2 L - \frac{\partial^2 \phi}{\partial z^2}$$

$\phi$ should satisfy the governing equation

$$\nabla^2 \phi = KT = \left(1 + \frac{\nu}{1 - \nu}\right)\alpha T$$

where, the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$

vis Poisson’s ratio and $\alpha$ is linear coefficient of thermal expansion of the material of the disc.

Also, $L$ must satisfy the equation $\nabla^2 \nabla^2 L = 0$

The components of the stress are

$$\sigma_{rr} = 2G \left\{ \frac{(\partial^2 \phi)}{\partial r^2} - KT \right\} + 2(1 - \nu)\nabla^2 L - \frac{(\partial^2 \phi)}{\partial z^2}$$
$$\sigma_{zz} = 2G \left\{ \frac{(\partial^2 \phi)}{\partial z^2} - KT \right\} + 2(1 - \nu)\nabla^2 L - \frac{(\partial^2 \phi)}{\partial z^2}$$

where $G$ is shear modulus.

On the traction free surfaces of the thick annular disc, the boundary conditions are

$$\sigma_{rz} = \sigma_{zr} = 0 \text{ at } z = \pm h$$

This system of equations (2.1) to (2.16) constitutes the mathematical formulation of the thermoelastic problem for displacement and associated thermal stresses developed within the thick annular disc as an effect of temperature distribution.

### 3. The integral transforms required to find analytical solution

#### 3.1. Finite Marchi-Zgrablich integral transform

The finite Marchi-Zgrablich integral transform of $f(r)$ of order $p$ is defined as

$$f^*_p(m) = \int_{r=a}^{r=b} f(r) S_p(h_1, h_2, \mu m) dr$$

where

$$C_m = \frac{b^2}{2} \left\{ S_p^2(h_1, h_2, \mu m) - S_{p-1}(h_1, h_2, \mu m) S_{p+1}(h_1, h_2, \mu m) \right\}$$

and the property of this transform is

$$\int_a^b r \left\{ \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{p^2}{r^2} f \right\} S_p(h_1, h_2, \mu m) dr$$

$$= \frac{b}{h_2} S_p(h_1, h_2, \mu m) \left\{ f + h_2 \frac{\partial f}{\partial r} \right\}_{r=b} - \frac{a}{h_1} S_p(h_1, h_2, \mu m) \left\{ f + h_1 \frac{\partial f}{\partial r} \right\}_{r=a} - \mu m^2 f^*_p(m)$$

where $h_1$, $h_2$, $\mu$ are constants in the boundary conditions

$$f(r) + h_1 f'(r)|_{r=a} = 0$$
$$f(r) + h_2 f'(r)|_{r=b} = 0$$
\( f^*(m) \) is the transform of \( f(r) \) with respect to nucleus \( S_p(h_1, h_2, \mu_m r) \) and weight function \( r \). The eigen values \( \mu_m \) are the positive roots of the characteristic equation

\[
J_p(h_1, \mu a)Y_p(h_2, \mu b) - J_p(h_2, \mu b)Y_p(h_1, \mu a) = 0
\]

and the kernel function \( S_p(h_1, h_2, \mu_m r) \) can be defined as

\[
S_p(h_1, h_2, \mu_m r) = J_p(\mu_m r)[Y_p(h_1, \mu_m a) + Y_p(h_2, \mu_m b)] - Y_p(h_1, \mu_m a)[J_p(h_1, \mu_m a) + J_p(h_2, \mu_m b)]
\]

where, \( J_p(h_i, \mu r) = J_p(\mu r) \) and \( Y_p(h_i, \mu r) = Y_p(\mu r) \) for \( i = 1, 2 \)

where \( J_p(\mu_m r) \) is Bessel function of first kind and \( Y_p(\mu_m r) \) is Bessel function of second kind.

3.2 Finite Marchi-Fasulo integral transform:

Again, another integral transform, the finite Marchi-Fasulo integral transform of \( F(z) \), \(-h < z < h\) which responds to radiation type boundary conditions is

Integral transform: \( \bar{F}(n) = \int_{-h}^{h} F(z)P_n(z)dz \) (3.2.1)

Inversion formula: \( F(z) = \sum_{n=1}^{\infty} \frac{\bar{F}(n)}{\lambda_n} P_n(z) \) (3.2.2)

where,

\[
P_n(z) = Q_n \cos(a_n z) - W_n \sin(a_n z)
\]

\[
Q_n = a_n(\alpha_1 + \alpha_2) \cos(a_n h) + (\beta_1 - \beta_2) \sin(a_n h)
\]

\[
W_n = (\beta_1 + \beta_2) \cos(a_n h) + (\alpha_2 - \alpha_1) a_n \sin(a_n h)
\]

\[
\lambda_n = \int_{-h}^{h} P_n^2(z)dz = h[Q_n^2 + W_n^2] + \frac{\sin(2a_n h)}{2a_n} [Q_n^2 - W_n^2]
\]

The eigen values \( a_n \) are the solutions of the equation

\[
[a_1 a \cos(ah) + \beta_1 \sin(ah)] \times [\beta_2 a \cos(ah) + a_2 a \sin(ah)]
\]

\[
= [\alpha_2 a \cos(ah) - \beta_2 \sin(ah)] \times [\beta_1 a \cos(ah) - a_1 a \sin(ah)]
\]

The sum in (3.2.2) must be taken on \( n \) corresponding to the positive roots of the equation (3.2.4)

The integral transform (3.2.1) has the following property:

\[
\int_{-h}^{h} \frac{\partial^2 f(z)}{\partial z^2} P_n(z)dz = \frac{P_n(h)}{\alpha_1} \left[ \beta_1 f(z) + \alpha_1 \frac{\partial f(z)}{\partial z} \right]_{z=-h} - \frac{P_n(-h)}{\alpha_2} \left[ \beta_2 f(z) + \alpha_2 \frac{\partial f(z)}{\partial z} \right]_{z=-h} - a_n^2 F(n)
\]

4. The analytical solution in the integral transform domain

4.1 Solution of temperature Distribution

The transform defined by (3.1.1) for \( p = 0 \), when applied to the equations (2.1), (2.2), (2.5), (2.6), (2.7) and using (2.3) and (2.4), resp. gives,

\[
-\mu_m^2 T^* + \frac{d^2 T^*}{dz^2} + \frac{g^*}{\kappa} = \frac{1}{a} \frac{dT^*}{dt}
\]

(4.1.1)

\[
T^* = 0
\]

(4.1.2)

\[
T^* + h_3 \frac{dT^*}{dz} = 0 \quad \text{at} \quad z = -h
\]

(4.1.3)

\[
T^* + h_6 \frac{dT^*}{dz} = F^*(m, t) \quad \text{at} \quad z = h
\]

(4.1.4)

\[
T^* + \frac{dT^*}{dz} = F^*(m, t) \quad \text{at} \quad z = \xi
\]

(4.1.5)

Again, applying the transform defined in (3.2.1) to (4.1.1) gives,

\[
\frac{dT^*}{dt} + a(\mu_m^2 + a_n^2)T^* = a \left( \frac{P_n(\xi)}{h_5} F^*(m, t) + \frac{\xi}{\kappa} \right)
\]

(4.1.6)
Solving (4.1.6),
\[ T^* = e^{-\alpha(m^2 + a_n^2)t} \left\{ \int e^{\alpha(m^2 + a_n^2)t} \Psi dt + C \right\} \] (4.1.7)
where \( \Psi = \alpha \left(\frac{P_n(\xi)}{h_n} F^*(m, t) + \frac{\alpha^2}{k}\right) \)
Using initial condition, finding value of constant of integration \( C \), gives,
\[ T^* = e^{-\alpha(m^2 + a_n^2)t} \left\{ \int e^{\alpha(m^2 + a_n^2)t} \Psi dt - \left[ \int \Psi dt \right]_{t=0} \right\} \] (4.1.8)
Applying inverse transform defined in (3.2.2) to the equation (4.1.8),
\[ T^* = \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} e^{-\alpha(m^2 + a_n^2)t} \times \left\{ \int e^{\alpha(m^2 + a_n^2)t} \Psi dt - \left[ \int \Psi dt \right]_{t=0} \right\} \] (4.1.9)
Finally, applying inverse transform defined in (3.1.2) to the equation (4.1.9),
\[ T(r, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{P_n(z)}{\lambda_n} e^{-\alpha(m^2 + a_n^2)t} \times \left\{ \int e^{\alpha(m^2 + a_n^2)t} \Psi dt - \left[ \int \Psi dt \right]_{t=0} \right\} \times \frac{S_0(h_1, h_2, \mu_m r)}{c_m} \] (4.1.10)

4.2 Determination of displacement components
Assume the Love function \( L \), that satisfy equation (2.11) as
\[ L = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{S_0(h_1, h_2, \mu_m r)}{c_m} \frac{P_n(z)}{\lambda_n} \] (4.2.1)
Then from (2.10) and (4.1.10), the displacement potential \( \phi \)
\[ \phi(r, z) = K \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \frac{S_0(h_1, h_2, \mu_m r)}{c_m} \Omega(t) \] (4.2.2)
where \( \Omega(t) = \int e^{-\alpha(m^2 + a_n^2)t} \times \left\{ \int e^{\alpha(m^2 + a_n^2)t} \Psi dt - \left[ \int \Psi dt \right]_{t=0} \right\} dt \)
Now, to get \( u_r \) and \( u_z \), substitute (4.2.1) and (4.2.2) in equation (2.8) and (2.9), we get,
\[ u_r = K \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu_m S_0(h_1, h_2, \mu_m r)}{c_m} \left\{ \frac{P_n(z)}{\lambda_n} \Omega(t) - \frac{1}{K} \frac{P_n(z)}{\lambda_n} \right\} \] (4.2.3)
and,
\[ u_z = K \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{S_0(h_1, h_2, \mu_m r)}{c_m} \left\{ \frac{P_n(z)}{\lambda_n} \Omega(t) + \frac{(1-\nu)\mu_m S_0(h_1, h_2, \mu_m r)}{c_m} \right\} + \frac{\mu_m S_0(h_1, h_2, \mu_m r)}{c_m} \right\} \left\{ \frac{P_n(z)}{\lambda_n} \right\} \] (4.2.4)

4.3 Determination of stress components
Substituting eq.(4.2.1) and (4.2.2) in equations (2.12)to (2.15), we have
\[ \sigma_{rr} = 2G \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ -\frac{K}{r} \frac{\mu_m S_0(h_1, h_2, \mu_m r)}{c_m} \frac{P_n(z)}{\lambda_n} + \frac{S_0(h_1, h_2, \mu_m r)}{c_m} \frac{P_n(z)}{\lambda_n} \Omega(t) \right\} \] (4.3.1)
\[ \sigma_{\theta\theta} = 2G \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ -K \left[ \frac{\mu_n^2 S_0^\prime(h_1, h_2, \mu_m r)}{C_m} + \frac{\mu_m S_0^\prime(h_1, h_2, \mu_m r)}{C_m} + \frac{S_0(h_1, h_2, \mu_m r)}{C_m} \right] \frac{\lambda_n}{\lambda_n} \right\} \Omega(t) + \mu_m \left[ \frac{\mu_m S_0^\prime(h_1, h_2, \mu_m r)}{C_m} + \frac{S_0(h_1, h_2, \mu_m r)}{C_m} \right] \frac{\lambda_n}{\lambda_n} \right\} \Omega(t) \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \r...
For sake of convenience, we set \( \beta = K = \frac{1+v}{1-v} \alpha \) and \( \gamma = 2G \)

**Figure 2.** Distribution of temperature \( T/\alpha \) versus radius \( r \).

Figure 2 represents distribution of temperature versus radius \( r \) for time \( t = 1,2,3,4,5 \) sec. It is observed that the temperature decreases uniformly from inner boundary \( (r = 1) \) to the outer boundary \( (r = 4) \). It attains peak at the inner circular boundary, become zero at \( r = 2.5 \) and then gradually decreases towards the outer circular boundary. As time increases, temperature also increases.

**Figure 3.** Radial displacement \( u_r/\beta \) versus radius \( r \).

The radial displacement is maximum near inner circular boundary, decreases in middle region and again rises towards outer circular boundary.

**Figure 4.** Axial displacement \( u_z/\beta \) versus radius \( r \).

From figure 4, the Axial displacement is maximum towards outer circular boundary, becomes zero near middle part at \( r = 2.5 \) and decreases towards inner circular curved boundary. It develops...
the compressive stresses within circular region $1 \leq r \leq 2.5$ and tensile stresses within an annular region where $2.5 \leq r \leq 4$.

![Figure 5](image)

**Figure 5.** Distribution of radial stress $\sigma_{rr}/\gamma$ versus radius $r$.

From figure 5, the radial stress function decreases as time increases. It is zero at inner circular boundary. At center, it takes steep decrease and then increases towards outer circular boundary.

![Figure 6](image)

**Figure 6.** Distribution of angular stress $\sigma_{\theta\theta}/\gamma$ versus radius $r$.

From figure 6, the angular stresses are inversely proportional to time. As time increases, angular stresses decreases. It vanishes at $r = 1.5$, take peak value near inner circular boundary. It develops compressive stresses in most region of annular disc.

![Figure 7](image)

**Figure 7.** Distribution of axial stress $\sigma_{zz}/\gamma$ versus radius $r$. 

From figure 7, the axial stresses $\sigma_{zz}$ shows varying nature. It develops tensile stresses near inner circular boundary, but in most of the region, in middle and outer circular boundary, it develops compressive stresses.

![Figure 8](image)

**Figure 8.** Distribution of resultant stress $\sigma_{zz}$ versus radius $r$.

From figure 8, the resultant stress function $\sigma_{zz}$ develops tensile stresses in annular region of disc. At center, it attains peak value, and attains equilibrium condition at both circular boundaries, i.e. at $r = 1$ and $r = 4$. It develops tensile stresses in the annular region of disc and as time increases, resultant stresses also increases.

6. Conclusion

The present paper focuses on the thermal stress analysis. The unknown temperature, displacement components and associated thermal stresses in thick annular disc under unsteady state conditions with radiation type boundary conditions are determined. The internal heat source has an essential and vital role in distribution of temperature and thermal stresses. The displacement and stress fields have both increasing and decreasing effects due to internal heat generation.

Any special Scenarios of thermoelastic behaviour of annular discs can be achieved by allocating particular values to the field parameters and functions in the derived expressions. The results presented here will be useful and applicable to many practical engineering applications where thermal environment plays a primary and decisive role.

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