Theoretical proposal for dual transformation between the Josephson effect and quantum phase slip in single junction systems and nanowires

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A method was devised to construct a generalized dual field theory in the quantum field theory. As a simple example using this method, we examined the duality between coherent quantum phase slip and the Josephson effect in single junction systems and nanowires. The this method was proved to be reliable within the Villain approximation.

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Introduction

The dual transformation has been known to be a useful tool in various physical systems. In particular, in quantum field theory and statistical mechanics, many studied cases incorporating duality are known. Similarly, electric circuits arranged in series and parallel within classical electrical engineering exhibit similar adding laws, which are satisfied by interchanging the role of resistance and conductance, inductance and capacitance, current and voltage. This rule is known as the duality principle of electrical circuits. Similarly, electric circuits arranged in series and parallel within classical electrical engineering exhibit similar adding laws, which are satisfied by interchanging the role of resistance and conductance, inductance and capacitance, current and voltage. This rule is known as the duality principle of electrical circuits. In recent years, numerous experiments and theoretical discussions have been conducted on the potential of quantum phase-slip system using nanowires. The existence of quantum phase-slip system is confirmed by both experiments and theoretical calculations. In this paper, we introduce two Hamiltonians, which are dual to each other, and propose a general theory to construct a dual system by applying the dual condition between current and voltage in an electric circuit. This method was named the dual Hamiltonian (DH) method. By using this method, the Hamiltonians of the QPS system and the JJ system are proved to be equivalent to each other by dual transformation and also prove to be an exact dual system. The remainder of this paper is organized as follows. In the next section, the DH method is applied to build a quantum LC circuit as a simple example. In section 2, as an application of the preceding section, the relationship between the QPS system and the JJ system in a single junction is introduced. In section 3, self-duality in various quantum junction circuits is briefly proved. In the section 4, superconductors and superinsulators are discussed from the standpoint of quantum phase transition. In section 5, duality is examined for the partition function of a single junction that incorporates quantum effects using path integration. In section 6, the derivation of the anisotropic XY (AXY) model and dual anisotropic XY (DAXY) model are described in the classical 1 + 1 dimensional system equivalent to the JJ and QPSJ in a nanowire, which is a quantum one-dimensional system. In section 7, the duality between the AXY model and DAXY model is proved by the Villain approximation. In the last section, the summary, discussion, and conclusions are presented.

I. DH METHOD IN THE QUANTUM LC CIRCUIT

In this section, examples of quantum LC circuits are presented as the simplest application of the DH method. First, the Hamiltonian of a normal quantum LC circuit is introduced as follows:

$$H_{LC} = \frac{Q^2(t)}{2C} + \frac{\Phi^2(t)}{2L},$$

(1)

where $C$ and $L$ are the capacitance and inductance of a quantum LC circuit, respectively. The commutation relationship between the electric charge $Q(x)$ and magnetic flux $\Phi(x)$ is described as follows:

$$[\Phi(t), Q(t)] = i\hbar.$$  

(2)

From the Hamiltonian of Eq.(1), the equations of motion are given by:

$$I(t) = \frac{dQ(t)}{dt} = i\frac{\hbar}{L}[H_{LC}, Q(t)] = -\frac{\Phi(t)}{L},$$

$$V(t) = \frac{d\Phi(t)}{dt} = i\frac{\hbar}{C}[H_{LC}, \Phi(t)] = -\frac{Q(t)}{C},$$

(3)

where $I(t)$ and $V(t)$ are the current and voltage of an LC circuit, respectively. The dual Hamiltonian $\tilde{H}_{LC}$ is then introduced for the quantum LC circuit, assuming the following:

$$\tilde{H}_{LC} = \frac{\tilde{Q}^2(t)}{2C} + \frac{\tilde{\Phi}^2(t)}{2L},$$

(4)

where $\tilde{C}$, $\tilde{L}$, $\tilde{Q}(t)$ and $\tilde{\Phi}(t)$ are dual capacitance, dual inductance, dual electric charge, and dual magnetic flux,
respectively. The commutation relationship between the dual charge $\tilde{Q}(t)$ and dual flux $\tilde{\Phi}(t)$ is:

$$\left[\tilde{\Phi}(t), \tilde{Q}(t)\right] = i\hbar. \quad (5)$$

From the dual Hamiltonian of Eq. (3), the equations of motion are given by:

$$\tilde{I}(t) \equiv \frac{d\tilde{Q}(t)}{dt} = \frac{i}{\hbar} \left[\tilde{H}_{LC}, \tilde{Q}(t)\right] = -\frac{\tilde{\Phi}(t)}{L}, \quad (6)$$

$$\tilde{V}(t) \equiv \frac{d\tilde{\Phi}(t)}{dt} = \frac{i}{\hbar} \left[\tilde{H}_{LC}, \tilde{\Phi}(t)\right] = \frac{\tilde{Q}(t)}{C}. \quad (6)$$

where, $\tilde{I}(t)$ and $\tilde{V}(t)$ denoted by the tilde, are the dual current and dual voltage in the dual quantum LC circuit, respectively. As the first step of the dual Hamiltonian method, two dual conditions between equation Eq. (3) and the dual equations of Eq (6) are assumed as follows:

$$V(t) \equiv \tilde{I}(t), \quad I(t) \equiv \tilde{V}(t). \quad (7)$$

The two conditions of Eq. (7) are called dual conditions. The next step of the dual Hamiltonian method is to derive a relational expression for the canonically conjugate operators that act on each other according to the duality condition (7). According to this, the following two relational expressions between charge and flux in dual systems are derived as shown below:

$$\Phi(t) \equiv -\tilde{Q}(t), \quad \tilde{\Phi}(t) \equiv Q(t). \quad (8)$$

The last step of the dual Hamiltonian method is to derive a relational expression between the constants according to the duality condition of Eq. (7). According to this, as shown below, two relational expressions between the electrostatic capacitance and the inductance within the dual systems are derived:

$$C \equiv L, \quad L \equiv -C. \quad (9)$$

In this section, the conditions under which the dual Hamiltonian of a quantum LC circuit, which is a trivial self-dual system, were established. In particular, the duality condition of (7) is very important, because it becomes an index for defining an exact dual system.

**II. DH METHOD BETWEEN THE JJ AND QPSJ IN A SINGLE JUNCTION**

In this section, according to Hamiltonian of QPSJ introduced by Mooij et al. [1] which is already known prior research, using the method introduced in the previous section, we investigate the duality between JJ and QPSJ for case of single junction. [3, 4] First, the Hamiltonian $H$ of the single JJ and the Hamiltonian $\tilde{H}$ of the single QPSJ are shown as follows:

$$H(\theta, N) = E_cN^2 + E_J(1 - \cos \theta), \quad (10)$$

$$\tilde{H}\left(\tilde{\theta}, \tilde{N}\right) = E_c\tilde{N}^2 + E_J\left(1 - \cos \tilde{\theta}\right). \quad (11)$$

In Eq. (10), $E_c \equiv (2e)^2/2C$ is charging energy per Cooper pair, therefore, $E_J \equiv \Phi_0L/2\pi$ is the Josephson energy, $L_c$ and $\Phi_0 \equiv h/2e$ are the critical current and the magnetic flux-quantum, respectively, and $N$ and $\theta$ are the number of the Cooper pair and the phase of the Cooper pair, respectively. In Eq. (11), $E_L \equiv \Phi_0^2/2L$ is the inductive energy per magnetic flux quantum, $E_S \equiv 2eV_c/2\pi$ is the QPS amplitude, $V_c$ is the critical voltage, $\tilde{N}$ and $\tilde{\theta}$ are the number of magnetic flux-quantum and the phase of magnetic flux-quantum in QPS junction respectively. The commutation relations by Hamiltonian $H$ and $\tilde{H}$ canonical conjugate variables are described as follows:

$$\left[\theta(t), N(t')\right] = i\hbar, \quad \left[\tilde{\theta}(t), \tilde{N}(t')\right] = i\hbar. \quad (12)$$

From the equation of motion for each Hamiltonian, we derived the Josephson’s equation for two sets is derived. One set are the usual Josephson’s equations as follows:

$$V = \frac{\hbar}{2e} \frac{\partial \theta}{\partial t} = \frac{1}{2e} \frac{\partial H}{\partial N} = \frac{2N}{2e} E_c, \quad (13)$$

$$I = 2e \frac{\partial^2 N}{\partial t \partial \theta} = -\frac{2e}{h} \frac{\partial H}{\partial \theta} = 2\pi E_J \sin \theta, \quad (13)$$

where $I$ and $V$ are the current and voltage in the JJ, respectively. The other set are the dual Josephson’s equations in the QPSJ as follows [3, 4]:

$$\tilde{V} = \frac{\hbar}{\Phi_0} \frac{\partial \tilde{\theta}}{\partial t} = \frac{1}{\Phi_0} \frac{\partial \tilde{H}}{\partial \tilde{N}} = \frac{2\tilde{N}}{2e} E_L, \quad (14)$$

$$\tilde{I} = -\Phi_0 \frac{\partial \tilde{N}}{\partial \tilde{\theta}} = \frac{\Phi_0}{h} \frac{\partial \tilde{H}}{\partial \tilde{\theta}} = \frac{2\pi}{2e} E_S \sin \tilde{\theta}. \quad (14)$$

When the condition of Eq. (7) is imposed between Eq. (13) and Eq. (14), the following two relational expressions between phase and number of particles between dual systems are derived as shown below. One of them is the relationship between the phase $\theta(t)$ of the Cooper pair and the number $N(t)$ of the magnetic flux-quantum, and the other is the relationship between the phase $\tilde{\theta}(t)$ of the magnetic flux-quantum and the number $\tilde{N}(t)$ of the Cooper pair, as follows:

$$\theta(t) = \sin^{-1}\left[-2\pi \tilde{N}(t)\right], \quad \tilde{\theta}(t) = \sin^{-1}\left[2\pi N(t)\right]. \quad (15)$$

If it is recognized that the relationships described in Eq. (13) and Eq. (14) are satisfied, the relationship between the QPS amplitude and charging energy per single-charge, and the relationship between Josephson energy and inductive energy per magnetic flux-quantum, are as follows:

$$E_S = \frac{1}{2\pi^2} E_c, \quad E_J = \frac{1}{2\pi} E_L. \quad (16)$$

Furthermore, inductance and capacitance are related to the critical current and the critical voltage, respectively, as follows:

$$L = \frac{\Phi_0}{2\pi I_c}, \quad C = \frac{2\pi}{2eV_c}. \quad (17)$$
The linear approximation of Eq. (15) is a well-known relationship between the phase and the number of particles, as shown in the following equations:

\[
\theta(t) = -2\pi \tilde{N}(t) = -2\phi \frac{\Phi_0}{\Phi_0},
\]

\[
\bar{\theta}(t) = 2\pi N(t) = 2\pi \frac{Q}{2e}.
\] (18)

To compare with the existing theoretical formula, calculating the kinetic inductance \( L_{\text{kin}} \) and the kinematic capacitance \( C_{\text{kin}} \) defined by \( L_{\text{kin}}^{-1} = \Phi_0^2 H / \Phi_0^2 \) and \( C_{\text{kin}}^{-1} = \Phi_0^2 H / \Phi_0^2 \), respectively, according to Eq. (15) yields the following equations:

\[
L_{\text{kin}} = \left[ 1 - (2\pi \Phi_0)^2 \right] \frac{\Phi_0}{2\pi I_c \cos \theta},
\]

\[
C_{\text{kin}} = \left[ 1 - (2\pi q/2e)^2 \right] \frac{2e}{2\pi V_c \cos \theta}.
\] (19)

To summarize the results of this section, by accepting the results of Eq. (13) to Eq. (17) obtained under the double condition of Eq. (4), the Hamiltonian in Eq. (4) and the Eq. (17), it was found that its duality was completely guaranteed. Among the results, the Eq. (13) is particularly important as it becomes the starting point as a relational expression for creating a self-dual system in the next section. The phase and number of particles between the dual systems of Eq. (4) are nonlinear, and its linear approximation Eq. (18) is consistent with the generally well known relationship of Mooij et al. in Eq. (17). We note that between the QPS amplitude in QPS and the charging energy in JJ, between the Josephson energy in JJ and the induced energy in QPS are connected by 1/2\( \pi^2 \) times relation. This relation is important, but it is not mentioned in the paper by Mooij et al. [11]. The kinetic inductance and kinematic capacitance of Eq. (18) have the difference with \(-2\pi \Phi_0^2 / \Phi_0 I_c \cos \theta \) and \(-2\pi q^2 / 2e V_c \cos \theta \), respectively, from the calculation result of Mooij et al. [11].

### III. SIMPLE PROOF OF SELF-DUALITY IN VARIOUS QUANTUM JUNCTION CIRCUITS

In this section, a simple proof of duality in the single JJ system is presented. First, the half angle version of Eq. (17) is introduced as follows:

\[
\theta(t) = \sin^{-1}(-\pi \tilde{N}(t)), \quad \bar{\theta}(t) = \sin^{-1}(-\pi N(t)).
\] (20)

Eq. (20) is in agreement with Eq. (13) and Eq. (18) within the range of the linear approximation. Substituting Eq. (13) into the second term of Eq. (11) and Eq. (11), their Hamiltonians are described as follows:

\[
H\left(\tilde{N}, N\right) = E_c N^2 + 2\pi E_J \tilde{N}^2,
\] (21)

\[
\bar{H}\left(N, \tilde{N}\right) = E_L N^2 + 2\pi E_S \tilde{N}^2.
\] (22)

It is trivial that these two Hamiltonians are equal by applying the relation of Eq. (16) to Eq. (21) and Eq. (22). It is also trivial that Eq. (21) is quite equivalent to the quantum LC circuit discussed in Section 1. Further, when the linear relational expression of Eq. (18) is used for the second terms of Eq. (21) and Eq. (22), the equations are expressed as follows:

\[
H\left(\tilde{N}, \theta\right) = E_c N^2 + \frac{1}{2} E_J \bar{\theta}^2,
\] (23)

\[
\bar{H}\left(\tilde{N}, \theta\right) = E_L N^2 + \frac{1}{2} E_S \bar{\theta}^2.
\] (24)

Regarding the second term of Eq. (23) and Eq. (24), it is obvious that this is a Gaussian approximation of the cosine term of the second term of Eq. (10) and Eq. (11). Next, substituting Eq. (13) into the first terms of Eq. (10) and Eq. (11), their Hamiltonians can be expressed as follows:

\[
H\left(\theta, \theta\right) = \frac{E_c}{2\pi^2} \left(1 - \cos \theta\right) + E_J \left(1 - \cos \theta\right),
\] (25)

\[
\bar{H}\left(\theta, \theta\right) = \frac{E_L}{2\pi^2} \left(1 - \cos \theta\right) + E_S \left(1 - \cos \theta\right),
\] (26)

As with the relation of Eq. (21) and Eq. (22), it is trivial that these two Hamiltonians are equal by substituting Eq. (20) into Eq. (25) and Eq. (26). Further, when the linear relational expression of Eq. (18) is substituted in the first terms of Eq. (21) and the Eq. (22), the equations can be rewritten as follows:

\[
H\left(\theta, N\right) = \frac{E_c}{2\pi^2} \left[1 - \cos (2\pi N)\right] + E_J \left(1 - \cos \theta\right),
\] (27)

\[
\bar{H}\left(\theta, N\right) = \frac{E_L}{2\pi^2} \left[1 - \cos (2\pi N)\right] + E_S \left(1 - \cos \theta\right).
\] (28)

By imposing the conditions of Eq. (17) and Eq. (13) on Eq. (27) and Eq. (28), these two Hamiltonians are equal, i.e., self-dual. A Hamiltonian with two cosine terms, competing with each other, similar to Eq. (25) to Eq. (26) is a new form which has not been known until now. Such a system is a system in which both JJ and QPSJ which are in a coherent state compete with each other, and the circuit in which JJ and QPSJ are connected in series is called a JJ - QPSJ competitive circuit. FIG.1 (a) shows the quantum LC circuit represented by the Hamiltonian of Eq. (21) and Eq. (22) or Eq. (23) and Eq. (24). FIG.1 (b) and (c) show a single JJ represented by the Hamiltonian of Eq. (21) and a single QPSJ represented by the Hamiltonian of the Eq. (11), respectively. FIG.1 (d) shows the JJ - QPSJ competitive circuit represented by the Hamiltonian of Eq. (27) and Eq. (28).
IV. SUPERCONDUCTOR- INSULATOR TRANSITIONS

In this section, superconductor-insulator transitions are discussed from the viewpoint of quantum phase transitions. Quantum resistance was derived using the following two methods with the Josephson’s equations of Eq. (13) and the dual Josephson’s equations of Eq. (14) and Eq. (15). One of the methods uses the ratio between the fluctuation of the number of Cooper pairs and the fluctuation of the number of magnetic flux-quantums. The resistance can thus be derived as follows:

\[ R = \frac{V}{I} = \frac{R_Q}{2\pi} \frac{d\theta}{dN} = \frac{R_Q E_c N}{2\pi^2 E_J N}, \]

where \( R_Q \equiv \hbar/(2e)^2 \approx 6.4 \, [k\Omega] \) is the universal critical sheet resistance. The other method uses the ratio between the fluctuation of the phase of the Cooper pair and the fluctuation of the phase of the magnetic flux-quantum. The quantum conductance can thus be derived as follows:

\[ G = \frac{\tilde{V}}{I} = \frac{1}{2\pi} G_Q \frac{d\theta}{dN} = G_Q \frac{1}{2\pi^2} \frac{E_{L, J} \tilde{N}}{E_S N}, \]

where \( G_Q \equiv R_Q^{-1} \) is the universal critical sheet conductance. In Eq. (29) when the conditions \( \Delta \theta >> \Delta N \), or \( E_c >> E_J \) and \( N >> \tilde{N} \), are met, the equation represents an insulator state. In particular, when \( R \), it represents a superinsulator state. The reverse case occurs when the conditions of \( \Delta \theta << \Delta N \) or, \( E_c << E_J \) and \( N << \tilde{N} \), are met, the equation represents a conductor state. In particular, when \( R_0 \), it represents a superconductor state. In the special case of \( \Delta \theta \approx 2\pi \Delta N \), or \( E_c \approx 2\pi^2 E_J \) and \( N \approx \tilde{N} \), the equation represents a critical state. In Eq. (30), when the conditions: \( \Delta \theta >> \Delta \tilde{N} \), or \( E_L >> E_S \) and \( N >> N \), are met, the equation represents a conductor state. In particular, when \( G \), it represents a superconductor state. The reverse case occurs when the conditions of \( \Delta \theta << \Delta \tilde{N} \), or \( E_L << E_S \) and \( N << N \), are met, in which case the equation represents an insulator state. In particular, when \( G_0 \), it represents a superinsulator state.

V. PARTITION FUNCTION OF THE JJ AND QPSJ IN SINGLE JUNCTION

Up to the preceding section, the JJ and QPSJ have been dealt with in a single junction for Hamiltonian form at the level of classical mechanics. In this section, the partition function of a single junction is investigated, incorporating the quantum effect by path integral and its duality is considered. The partition function of a single JJ system is expressed using Eq. (11) as follows:

\[ Z = \int D\tilde{N}D\theta \exp \left( \frac{\beta}{\hbar} \int_0^\beta \frac{d\theta}{\beta} \left[ 2\pi \left( \frac{\partial \theta}{\partial \beta} \right) - N E_c N^2 - E_J (1 - \cos \theta) \right] \right), \]

where, \( \int D\tilde{N} = \prod \int_{-\infty}^{\infty} d\tilde{N}(\tau) \), \( \int D\theta = \prod \int_0^\beta d\theta/\beta, \beta \equiv (k_B T)^{-1} \), and \( \tau \equiv \beta \hbar \) is the imaginary time. In the same manner, the partition function of a single QPSJ system is expressed using Eq. (11) as follows:

\[ \tilde{Z} = \int D\tilde{N}D\tilde{\theta} \exp \left( \frac{\beta}{\hbar} \int_0^\beta \frac{d\tilde{\theta}}{\beta} \left[ 2\pi \left( \frac{\partial \tilde{\theta}}{\partial \beta} \right) - E_{L, J} \tilde{N}^2 - E_S (1 - \cos \tilde{\theta}) \right] \right). \]

First, to make computation using path integration of Eq. (31) and Eq. (32) convenient, imaginary time is changed from the continuous value to the discrete value, the differential operator is changed to the difference operator, and the integral \( \int_0^\beta d\tau \) is changed to the sum \( \sum_{\tau=1}^{M_{\tau}} \). These partition functions can then be expressed as follows:

\[ Z = \int D\tilde{N}D\theta \exp \sum_{\tau=1}^{M_{\tau}} \left[ iN \nabla_\tau \theta - E_J' N^2 - E_J' (1 - \cos \theta) \right], \]

\[ \tilde{Z} = \int D\tilde{N}D\theta \exp \sum_{\tau=1}^{M_{\tau}} \left[ i\tilde{N} \nabla_\tau \tilde{\theta} - E_J' \tilde{N}^2 - E_J' (1 - \cos \tilde{\theta}) \right], \]

where \( E_J', E_J', E_{L, J}' \) and \( E_S' \) are the dimensionless energy defined by \( \Delta \tau E_c/h, \Delta \tau E_J/h, \Delta \tau E_L/h \) and \( \Delta \tau E_S/h \), respectively. In addition, \( \Delta \tau \equiv \tau_{\max}/M_{\tau}, \tau_{\max}, \Delta \tau, \tau_{\max}, \nabla_\tau (\theta) \equiv \theta (\tau) - \theta (\tau - \Delta \tau) \) are the minimum imaginary time interval, the maximum imaginary time, the division number and the difference operator in imaginary time, respectively. When Eq. (33) and Eq. (34) are integrated with respect to \( N(\tau) \) and \( \tilde{N}(\tau) \), respectively, the following equations are obtained:

\[ Z = \int D\theta \exp \sum_{\tau=1}^{M_{\tau}} \left[ -\frac{1}{2} E_J' \nabla_\tau \theta^2 - E_J' (1 - \cos \theta) \right], \]

\[ \tilde{Z} = \int D\tilde{\theta} \exp \sum_{\tau=1}^{M_{\tau}} \left[ -\frac{1}{2} E_S' \nabla_\tau \tilde{\theta}^2 - E_S' (1 - \cos \tilde{\theta}) \right], \]

where \( E_J' \) and \( E_S' \) represent the dimensionless energy of the imaginary time component in the JJ and QPSJ, respectively, and are defined by the following equations:

\[ E_J' = \frac{1}{2E_c'}. \]
\[ E_S^0 \equiv \frac{1}{2E_L'} \]  \hspace{1cm} (38)

On the contrary, when Eq. (33) and Eq. (34) are integrated with respect to \( \theta(\tau) \) and \( \dot{\theta}(\tau) \), respectively, the following equations are obtained:

\[ Z = \int D\theta \exp \sum_{\tau=1}^{M_\tau} \left[ -E_J' - E_S'N^2 + \ln I_{0}(E_J') \right], \]  \hspace{1cm} (39)

\[ \tilde{Z} = \int \tilde{D}\tilde{\theta} \exp \sum_{\tau=1}^{M_\tau} \left[ -E_S' - E_L'\tilde{N}\tilde{N}^2 + \ln I_{\tilde{0}}(E_S') \right], \]  \hspace{1cm} (40)

where \( I_{0}(E_J') \) and \( I_{\tilde{0}}(E_S') \) represent modified Bessel functions of order \( \alpha(\tau) = -\nabla N(\tau) \) and order \( \tilde{\alpha}(\tau) = \tilde{\nabla}\tilde{N}(\tau) \) respectively. When the Villain approximation is introduced into the modified Bessel functions of Eq. (33) and Eq. (34), the following equations are obtained:

\[ Z = \int D\theta \exp \sum_{\tau=1}^{M_\tau} \left[ -E_J' - E_S'N^2 + \ln I_{0}(E_J') - \frac{1}{2(E_J')_v} \left( \nabla_{\tau}N \right)^2 \right], \]  \hspace{1cm} (41)

\[ \tilde{Z} = \int \tilde{D}\tilde{\theta} \exp \sum_{\tau=1}^{M_\tau} \left[ -E_S' - E_L'\tilde{N}\tilde{N}^2 + \ln I_{\tilde{0}}(E_S') - \frac{1}{2(E_S')_v} \left( \nabla_{\tau}\tilde{N} \right)^2 \right], \]  \hspace{1cm} (42)

where \( (E_J')_v \) and \( (E_S')_v \) are Villain’s parameters and are defined as follows:

\[ (E_J')_v = \frac{1}{2} \ln \frac{1}{I_{0}(E_J')/I_{0}(E_J'}), \]  \hspace{1cm} (43)

\[ (E_S')_v = \frac{1}{2} \ln \frac{1}{I_{\tilde{0}}(E_S')/I_{\tilde{0}}(E_S')}, \]  \hspace{1cm} (44)

When the conversion formula between the number of particles and its dual phase in Eq. (13) is substituted in Eq. (41) and Eq. (42), the following equations are obtained:

\[ Z \equiv \int D\theta \exp \sum_{\tau=1}^{M_\tau} \left[ -E_J' + \ln \left( \frac{1}{2\pi} \cos \frac{\theta}{2} \right) + \ln I_{0}(E_J') - \cos^2 \left( \frac{\theta}{2} \right)/(8\pi^2(E_J')_v) \left( \nabla_{\tau}\theta \right)^2 - \frac{E_J'}{2\pi} \left( 1 - \cos \theta \right) \right], \]  \hspace{1cm} (45)

\[ \tilde{Z} \equiv \int \tilde{D}\tilde{\theta} \exp \sum_{\tau=1}^{M_\tau} \left[ -E_S' + \ln \left( \frac{1}{2\pi} \cos \frac{\theta}{2} \right) + \ln I_{\tilde{0}}(E_S') - \cos^2 \left( \frac{\theta}{2} \right)/(8\pi^2(E_S')_v) \left( \nabla_{\tau}\tilde{\theta} \right)^2 - \frac{E_S'}{2\pi} \left( 1 - \cos \theta \right) \right]. \]  \hspace{1cm} (46)

By comparing Eq. (45) with Eq. (46), and by considering that \( (E_J')_v \approx E_J' \) and \( \cos^2(\theta/2) \approx 1 \) are established respectively at the limit of large \( E_J' \) and the limit of small \( \theta \), it can be understood that the relational expressions \( E_S = E_c/2\pi^2 \) and \( E_L = 2\pi^2E_J \), introduced in Eq. (11) are established. Similarly, by comparing Eq. (40) with Eq. (33), and by considering that \( (E_S')_v \approx E_S' \) and \( \cos^2(\theta/2) \approx 1 \) are established respectively at the limit of large \( E_S' \) and the limit of small \( \theta \), it can be understood that the relational expressions \( E_J = E_L'/2\pi^2 \) and \( E_L = 2\pi^2E_S \) introduced in Eq. (11) are established. From the above results, at least at the level of the Villain approximation, the partition functions of Eq. (31) and Eq. (32) are proved to be in a dual relationship with each other.

VI. PARTITION FUNCTION OF THE JJ AND QPSJ IN A ONE-DIMENSIONAL NANOWIRE

In the previous sections, the duality for the JJ and QPSJ was examined in a single junction. In this section, this is extended to consider the dual model for the JJ and QPSJ in a nanowire, which is a one-dimensional system. The Hamiltonians obtained by extending Eq. (10) and Eq. (11) into a one-dimensional nanowire are as follows:

\[ H(\theta, N) = E_c \sum_{x=1}^{M_x} \left\{ N(x, \tau)^2 + E_J \left[ 1 - \cos \nabla_\tau \theta(x, \tau) \right] \right\} , \]  \hspace{1cm} (47)

\[ \tilde{H}(\tilde{\theta}, \tilde{N}) = E_L \sum_{x=1}^{M_x} \left\{ \tilde{N}(x, \tau)^2 + E_S \left[ 1 - \cos \tilde{\nabla}_\tau \tilde{\theta}(x, \tau) \right] \right\} , \]  \hspace{1cm} (48)

where \( x, a, L, M_x \equiv L/a \) and \( \nabla_\tau \theta(x, \tau) \equiv \theta(x-a, \tau) \) are the space variable, the lattice spacing, the length of the one-dimensional nanowire, the division number of the space and difference operator of the space, respectively. The partition functions of Eq. (47) and Eq. (48) can be expressed as follows:

\[ Z = \int D\theta \exp \sum_{x=1}^{M_x} \left[ iN(\nabla_\tau \theta - E_J'N^2 - E_J' \left( 1 - \cos \nabla_\tau \theta \right) \right] \]  \hspace{1cm} (49)

\[ \tilde{Z} = \int \tilde{D}\tilde{\theta} \exp \sum_{x=1}^{M_x} \left[ i\tilde{N}(\tilde{\nabla}_\tau \tilde{\theta} - E_S'\tilde{N}^2 - E_S' \left( 1 - \cos \tilde{\nabla}_\tau \tilde{\theta} \right) \right] \]  \hspace{1cm} (50)

where, \( Z \) and \( \tilde{Z} \) represent the partition function of the JJ and the QPSJ, respectively, in the one-dimensional nanowire. When Eq. (14) and Eq. (15) are integrated with respect to \( N(x, \tau) \) and \( \tilde{N}(x, \tau) \), respectively, the following equations are obtained:

\[ Z = \int D\theta \exp \sum_{x, \tau} \left[ -\frac{1}{2} E_J' \left( \nabla_\tau \theta \right)^2 - E_J' \left( 1 - \cos \nabla_\tau \theta \right) \right] , \]  \hspace{1cm} (51)

\[ \tilde{Z} = \int \tilde{D}\tilde{\theta} \exp \sum_{x, \tau} \left[ -\frac{1}{2} E_S' \left( \tilde{\nabla}_\tau \tilde{\theta} \right)^2 - E_S' \left( 1 - \cos \tilde{\nabla}_\tau \tilde{\theta} \right) \right] , \]  \hspace{1cm} (52)

where, \( \sum_{x, \tau} \equiv \sum_{x=1}^{M_x} \sum_{\tau=1}^{M_\tau} \) and the first terms of Eq. (51) and Eq. (52) are expressed in a quadratic form for the imaginary time difference of each phase, but the second terms
are expressed in a cosine form for the spatial difference of each phase, but these second terms are expressed in a cosine form for the spatial difference of each phase. Here, in consideration of the periodicity of the lattice space, the cosine form is also introduced for the first terms of the Eq.(51) and Eq.(52), as follows:

\[ Z - \int D\theta \exp \sum_{x,\tau} \left[ -E^{0}_0 \left( 1 - \cos \nabla_x \theta \right) - E^{0}_j \left( 1 - \cos \nabla_x \theta \right) \right], \]  

(53)

\[ \tilde{Z} = \int D\bar{\theta} \exp \sum_{x,\tau} \left[ -E^{0}_S \left( 1 - \cos \nabla_x \bar{\theta} \right) - E^{0}_S \left( 1 - \cos \nabla_x \bar{\theta} \right) \right], \]  

(54)

where, \( Z_{AXY} \) and \( Z_{DAXY} \) represent the partition function of the anisotropic \( XY' \) (AXY) model and the dual anisotropic \( XY' \) (DAXY) model, respectively, in 1+1 dimensions. That is, the \( AXY \) model in the 1+1 dimension of Eq.(53) is equivalent to the JJ model in the one-dimensional nanowire of Eq.(49), and the DAXY model in the 1+1 dimension of Eq.(54) is equivalent to the QPSJ model in the one-dimensional nanowire of Eq.(46). These relationships, which are known as one-dimensional quantum models, are equivalent to 1+1 dimensional classical \( XY \) models [2-4]. In Eq.(53) and Eq.(54), to make handling convenient, the partition functions \( Z'_{AXY} \) and \( Z'_{DAXY} \), are defined, the constant term is removed and a pure cosine exponent remains as follows:

\[ Z'_{AXY} \equiv \exp \left[ -E^{0}_0 \left( 1 - \cos \nabla_x \theta \right) \right] M_x M_j \]  

(55)

\[ Z'_{DAXY} \equiv \exp \left[ -E^{0}_S \left( 1 - \cos \nabla_x \bar{\theta} \right) \right] M_x M_j \]  

(56)

\[ Z'_{AXY} \text{ and } Z'_{DAXY} \] are the starting points for discussing dual transformation by the Villain approximation in the next section.

VII. DUALITY BETWEEN THE AXY MODEL AND DAXY MODEL BY VILLAIN APPROXIMATION

The Villain approximation is first applied to \( Z'_{AXY} \) and \( Z'_{DAXY} \), introduced in the previous section, as follows:

\[ Z_{QV} = R_{QV} \int D\theta \sum_{(\omega)} \sum_{x,\tau} \left[ -E^{0}_0 \left( \nabla_x \theta - 2\pi \omega \right) \right] \frac{1}{2} \left( \nabla_x \theta - 2\pi \omega \right) \]  

(57)

\[ \tilde{Z}_{QV} = R_{\tilde{Q}V} \int D\bar{\theta} \sum_{(\omega)} \sum_{x,\tau} \left[ -E^{0}_S \left( \nabla_x \bar{\theta} - 2\pi \omega \right) \right] \frac{1}{2} \left( \nabla_x \bar{\theta} - 2\pi \omega \right) \]  

(58)

where \( Z_{QV} \) and \( \tilde{Z}_{QV} \) are Villain approximations of the partition functions \( Z'_{AXY} \) and \( Z'_{DAXY} \) respectively,
\[ Z_{QDV} = C_{QV} \sum_{\{l\}} e^{-\beta \tilde{V} \int \int D\tilde{B}_n \exp \sum_{x,\tau} \left[ -\frac{\beta \tilde{V} (x,\tau)}{2 (E_n^{(x,\tau)})} + \frac{\beta^2 \tilde{V}^2 (x,\tau)}{4 (E_n^{(x,\tau)})^2} \right] } \]

Integrating over the continuous field variable \( B \) and \( \tilde{B} \) of Eq. (77) and Eq. (78) yields the following equations:

\[ Z_{QV} = R_{QV} \sum_{\{l\}} e^{-\beta \tilde{V} \int \int D\tilde{B}_n \exp \sum_{x,\tau} \left[ -2 \tilde{V} (E_n^{(x,\tau)}) \tilde{E}_n^{(x,\tau)} - 2 \tilde{V} (E_n^{(x,\tau)}) \tilde{E}_n^{(x,\tau)} \right] } \]

\[ Z_{QV} = R_{QV} \sum_{\{l\}} e^{-\beta \tilde{V} \int \int D\tilde{B}_n \exp \sum_{x,\tau} \left[ -2 \tilde{V} (E_n^{(x,\tau)}) \tilde{E}_n^{(x,\tau)} - 2 \tilde{V} (E_n^{(x,\tau)}) \tilde{E}_n^{(x,\tau)} \right] } \]

The Kronecker deltas, when rewritten in the integral form, allow the equations to be written as follows:

\[ Z_{QV} = R_{QV} \sum_{\{l\}} e^{-\beta \tilde{V} \int \int D\tilde{B}_n \exp \sum_{x,\tau} \left[ -2 \tilde{V} (E_n^{(x,\tau)}) \tilde{E}_n^{(x,\tau)} - 2 \tilde{V} (E_n^{(x,\tau)}) \tilde{E}_n^{(x,\tau)} \right] } \]

Using the identity of Eq. (83) for Eq. (77) and Eq. (78), respectively, the equations become:

\[ Z_{QV} = R_{QV} \sum_{\{l\}} e^{-\beta \tilde{V} \int \int D\tilde{B}_n \exp \sum_{x,\tau} \left[ -2 \tilde{V} (E_n^{(x,\tau)}) \tilde{E}_n^{(x,\tau)} - 2 \tilde{V} (E_n^{(x,\tau)}) \tilde{E}_n^{(x,\tau)} \right] } \]

By using the inverse transform of the Villain approximation introduced in Eq. (73) and Eq. (78) on Eq. (74) and Eq. (75), respectively, the equations can be rewritten as follows:

\[ Z_{QV} = C_{QV} \sum_{\{l\}} e^{-\beta \tilde{V} \int \int D\tilde{B}_n \exp \sum_{x,\tau} \left[ \frac{\tilde{V} \theta - 2 \pi n_\theta \tilde{E}_n^{(x,\tau)}}{8 \pi^2 (E_n^{(x,\tau)})} + \frac{\tilde{V} \theta - 2 \pi n_\theta \tilde{E}_n^{(x,\tau)}}{8 \pi^2 (E_n^{(x,\tau)})} \right] } \]

When the relationships of Eq. (77) and Eq. (78) are used in Eq. (79) and Eq. (80), the following equation can be derived:

\[ Z_{AXV} \approx Z_{AXV} \approx C_{QV} \frac{R_{QV}}{R_{QV}} \int \int D\tilde{B}_n \exp \sum_{x,\tau} \left[ E_n^{(x,\tau)} \cos \tilde{\theta} + E_n^{(x,\tau)} \sin \tilde{\theta} \right] \]

\[ Z_{AXV} \approx Z_{QPS} \approx C_{QV} \frac{R_{QV}}{R_{QV}} \int \int D\tilde{B}_n \exp \sum_{x,\tau} \left[ E_n^{(x,\tau)} \cos \tilde{\theta} + E_n^{(x,\tau)} \sin \tilde{\theta} \right] \]

In Eq. (79) and Eq. (80), it is guaranteed that \( Z_{AXV} \) and \( Z_{QPS} \) are completely dual relationships under the following condition regarding normalization parameters:

\[ \frac{C_{QV}}{R_{QV}} \frac{C_{QDV}}{R_{QDV}} \approx 1. \]

VIII. GINZBURG-LANDAU THEORY AND KOSTERLITZ-THOULESS TRANSITION

In this section, starting from the two partition functions \( Z_{AXV} \) and \( Z_{QPS} \) of JJ and QPS from Eq. (53) and Eq. (54) which are dual to each other, we consider two types of Ginzburg-Landau theory (GL theory) and two types of Kosterlitz-Thouless transition (KT transition). For each of Eq. (55) and Eq. (56), we introduce two element unit vectors \( U_i = [\cos \theta, \sin \theta] \) and \( U_i = [\cos \theta, \sin \theta] (i = 1, 2) \) respectively as follows:

\[ Z_{AXV} \left( E_n^{(x,\tau)}, E_n^{(x,\tau)} \right) = \int \int D\tilde{B}_n \exp \left\{ \sum_{x,\tau} \sum_{i=1}^{2} U_i (x, \tau) R U_i (x, \tau) \right\} \]

\[ Z_{QPS} \left( E_n^{(x,\tau)}, E_n^{(x,\tau)} \right) = \int \int D\tilde{B}_n \exp \left\{ \sum_{x,\tau} \sum_{i=1}^{2} U_i (x, \tau) R U_i (x, \tau) \right\} \]

Where the lattice difference operators \( R \) and \( \tilde{R} \) are respectively defined as:

\[ R \equiv \int \frac{1}{2d} \left( \nabla \psi_n + \gamma \nabla \psi_n \right) \cdot \gamma \equiv \frac{E_n^{(x,\tau)}}{E_n^{(x,\tau)}} \]

\[ \tilde{R} \equiv \int \frac{1}{2d} \left( \nabla \psi_n + \gamma \nabla \psi_n \right) \cdot \gamma \equiv \frac{E_n^{(x,\tau)}}{E_n^{(x,\tau)}} \]

Were \( d \equiv 1 + \gamma \) and \( \tilde{d} \equiv 1 + \tilde{\gamma} \) are anisotropic dimensional constants of JJ and QPS, respectively. Moreover in Eq. (87), we introduce two sets of real two component fields \( u_i \) and \( \psi_i \) \((l = 1, 2)\) which satisfy the following identity:

\[ \int_{-\infty}^{\infty} d\psi_1 d\psi_2 (2\pi)^2 \exp \left\{ -\psi_i (u_i - U_i) \right\} = \int_{-\infty}^{\infty} d\psi_1 d\psi_2 (2\pi)^2 (u_i - U_i) = 1. \]
where we have used that the product of $\theta$ functional integrals is given as follows:

$$\Pi \int \frac{d\psi(x,\tau)}{2\pi} \exp \left\{ E_j^d dU(x,\tau) \right\} = \exp \sum_{x,\tau} \left\{ \ln I_0(\psi(x,\tau)) \right\},$$  

(88)

where $I_0(|\psi|) = |\sqrt{\psi^2 + \psi^2_x}|$ is the modified Bessel functions of integer $0th$ order. In Eq.(87), performing the integrals over $u_l$ fields, we obtain the partition function by the complex field $\psi \equiv \psi_1 + i\psi_2$ and $\psi^* \equiv \psi_1 - i\psi_2$.

$$Z_{AXY}'(E^0_j, E'_j) = \prod_{x,\tau} \left\{ \int_{-\infty}^{\infty} \frac{d\psi(x,\tau)}{4\pi E_j^d} \right\} \exp \left\{ -F'(\psi, \psi^*) \right\},$$  

(89)

$$F'(\psi, \psi^*) = \sum_{x,\tau} \left\{ \frac{1}{4E_j^d} |\psi(x,\tau)|^2 - \ln I_0 \left( \psi(x,\tau) \right) \right\},$$  

(90)

$$\psi(x,\tau) = R^\dagger \psi(x,\tau) = \sqrt{1 + \frac{1}{2d} (\nabla_x \psi + \gamma \nabla_x \psi^*) \psi(x,\tau)},$$  

(91)

In Eq.(90), since $\psi$ and $\psi^*$ can be regarded as the order parameter of superconductivity, the dimensionless energy $F'(\psi, \psi^*)$ can be Landau expansion of terms up to $|\psi|^4$ and $|\nabla_x \psi|^2$ as follows:

$$F_{GL}'(\psi, \psi^*) = \sum_{x,\tau} \left\{ \frac{1}{8d} \left[ |\nabla_x \psi|^2 + \gamma |\nabla_x \psi^*|^2 \right] + \frac{1}{4} \left( \frac{1}{E_j^d} - 1 \right) |\psi|^2 + \frac{1}{64} |\psi^4| \right\},$$  

(92)

$F_{GL}'$ is Ginzburg-Landau (GL) energy of superconductivity or Pitaevskii energy of Superfluid in $1+\gamma$ dimension at absolute zero. Similarly, when GL energy is calculated from Eq.(3), it becomes as follows:

$$\tilde{F}_{GL}'(\tilde{\psi}, \tilde{\psi}^*) = \sum_{x,\tau} \left\{ \frac{1}{8d} \left[ |\nabla_x \tilde{\psi}|^2 + \gamma |\nabla_x \tilde{\psi}^*|^2 \right] + \frac{1}{4} \left( \frac{1}{E_j^d} - 1 \right) |\psi|^2 + \frac{1}{64} |\psi^4| \right\},$$  

(93)

$\tilde{F}_{GL}'$ is Dual Ginzburg-Landau (DGL) energy of superinsulator $1+\gamma$ dimension at absolute zero. Here $\psi$ and $\psi^*$ can be regarded as order parameters of superinsulator. As opposed to being a condensate of $2e$ in which the order parameter of the superconductor is twice the elementary charge $e$, the order parameter of the superinsulator can be thought of as a condensate of $2\phi_0$ which is twice the quantum vortex $\phi_0 = h/(2e)$. From Eq.(92) and Eq.(93), the critical values $E^0_{S, \gamma}$ and $E^s_{\gamma}$ of mean field approximation of $E^0_j$ and $E^s_S$ are as follows:

$$E^0_{S, \gamma} = d^{-1} = \frac{1}{1 + \gamma},$$  

(94)

On the other hand, AXY model of $1+\gamma$ dimension and DAXY model of $1+\gamma$ dimension becomes a pseudo two-dimensional XY model under the condition of $E^0_j = E^0_j(\gamma = 1)$ and $E^0_S = E^s_S(\gamma = 1)$ respectively. Therefore, it is possible for the pseudo two-dimensional AXY model and the DAXY model to generate KT transition in the pseudo two-dimensional. In this case, the critical values $E^0_j$ and $E^s_S$ based on the KT transition of $E^j_{KT}$ and $E^s_{KT}$ are as follows:

$$E^j_{KT} = \frac{2}{\pi} \approx 0.637,$$

(96)

$$E^s_{KT} = \frac{2}{\pi} \approx 0.637,$$

(97)

Similarly, from Eq.(94) and Eq.(95), the critical values of the mean field approximation under the conditions of $\gamma = 1$ and $\gamma = 1$ are respectively as follows:

$$E^j_{MF}(\gamma = 1) = 0.5,$$

(98)

$$E^s_{MF}(\gamma = 1) = 0.5,$$

(99)

From Eq.(96) and Eq.(97), the difference between KT transition and mean field approximation is about $E^j_{KT} - E^j_{MF} \approx 0.137$. The same is true for $E^s_{KT} - E^s_{MF}$. In the $2+1$ dimensional JJ array of $[33, 34, 35, 36]$, a self-dual model is obtained (in the sense of electromagnetic duality) by adding kinetic terms to the vortices.

**IX. SUMMARY AND CONCLUSION**

This section contains the summarized conclusions from each section in this paper as follows: Section 1: Two dual Hamiltonians were introduced into a quantum LC circuit, known as the simplest quantum dual system, and the dual condition was applied between the current and the voltage of the electric circuit. Thus introducing a general theory and method for constructing a dual system named the DH (dual Hamiltonian) method. Section 2: The DH method was applied between the JJ and the QPSJ in a single junction, allowing the following to be derived: two relational expressions of particle number and phase between dual systems, QPS amplitude and charge per flux quantum. Furthermore, kinetic inductance and kinematic capacitance were derived in a nonlinear form. This result is an extension of the result of Mooij et al. [38, 39]. Section 3: The relation between the Hamiltonian and the equivalent circuit of each quantum circuit was clarified by applying a simple proof of self-duality in various quantum junction circuits. Section 4: Owing to the duality of the JJ and the QPSJ, the transition of superconductorsuperconductor could be explained by simple consideration. This indicated the
possibility that a QPSJ could be constructed from the junction of two superinsulators. Section 5: The JJ and QPSJ of a single junction were examined using the partition function and incorporating the quantum effect by path integration. By introducing the energy of the imaginary time component in the JJ and QPSJ, the duality between them was demonstrated and approximately established. Section 6: The partition function of AXY model and DAXY model was examined in 1 + 1 dimensions equivalent to the JJ and QPSJ in a one-dimensional nanowire. Section 7: Within the Villain approximation, it was confirmed that duality was accurately established. Overall results and conclusions: Section 8: Starting with the two partition functions of JJ and QPS, which are dual each other, we have determined two critical values of two types of GL theory and KT transition, respectively. The most important result of this paper is that by introducing two Hamiltonians that were dual with each other, the DH method was established, which is a general method for constructing an accurate dual system. The reliability of this method was proved accurately within the Villain approximation for the partition function of the 1 + 1 dimensional AXY model and DAXY model corresponding to the JJ and QPSJ in a one-dimensional nanowire system. It is believed that the DH method will prove to be a very effective method for future research into the QPS and superinsulator phenomena.

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