A HIGHER-DIMENSIONAL CHEVALLEY RESTRICTION THEOREM FOR ORTHOGONAL GROUPS

LEI SONG, XIAOPENG XIA, AND JINXING XU

Abstract. We prove a higher-dimensional Chevalley restriction theorem for orthogonal groups, which was conjectured by Chen and Ngô for reductive groups. In characteristic $p > 2$, we also prove a weaker statement. In characteristic 0, the theorem implies that the categorical quotient of a commuting scheme by the diagonal adjoint action of the group is integral and normal. As applications, we deduce some trace identities and a certain multiplicative property of the Pfaffian over an arbitrary commutative algebra.

1. Introduction

Let $G$ be a reductive group over an algebraically closed field $\mathbb{K}$ with Lie algebra $\mathfrak{g}$. For an integer $d \geq 2$, let $\mathcal{C}_{\mathfrak{g}}^d \subset \mathfrak{g}^d$ be the commuting scheme, which is defined as the scheme-theoretic fiber of the commutator map over the zero

$$\mathfrak{g}^d \to \prod_{i < j} \mathfrak{g}, \quad (x_1, \cdots, x_d) \mapsto \prod_{i < j} [x_i, x_j].$$

Its underlying variety (the reduced induced closed subscheme) $\mathcal{C}_{\mathfrak{g},\text{red}}^d$ is called the commuting variety. As a set, $\mathcal{C}_{\mathfrak{g},\text{red}}^d$ consists of $d$-tuples $(x_1, \cdots, x_d) \in \mathfrak{g}^d$ such that $[x_i, x_j] = 0$, for all $1 \leq i, j \leq d$. It is a long-standing open question whether or not $\mathcal{C}_{\mathfrak{g}}^d$ is reduced, that is $\mathcal{C}_{\mathfrak{g}}^d = \mathcal{C}_{\mathfrak{g},\text{red}}^d$. When the characteristic of $\mathbb{K}$ (char $\mathbb{K}$ for short) is zero, Charbonnel [3] recently claims a proof for $\mathcal{C}_{\mathfrak{g}}^2 = \mathcal{C}_{\mathfrak{g},\text{red}}^2$. Although there is no adequate evidence to expect that $\mathcal{C}_{\mathfrak{g}}^d$ is reduced for general $d$, one can study the categorical quotient $\mathcal{C}_{\mathfrak{g}}^d//G$ and ask the same question. Here $G$ acts on $\mathfrak{g}^d$ by the diagonal adjoint action, and the action leaves $\mathcal{C}_{\mathfrak{g}}^d$ stable.

Let $T$ be a maximal torus of $G$ and $t$ be the Lie algebra of $T$. Then the Weyl group $W := N_G(T)/T$ acts on $t^d$ diagonally. The embedding $t^d \hookrightarrow \mathfrak{g}^d$ factors through $\mathcal{C}_{\mathfrak{g}}^d$ and induces the natural morphism

$$\Phi : t^d//W \to \mathcal{C}_{\mathfrak{g},\text{red}}^d//G.$$

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In studying the Hitchin morphism from the moduli stack of principal $G$-Higgs bundles on a proper smooth variety $X$ of dimension $d \geq 2$, Chen and Ngô [4] are led to

**Conjecture 1.1** (Chen-Ngô). The morphism $\Phi : t^d/W \to \mathfrak{g}^d/G$ is an isomorphism.

When $d = 1$ and $\text{char } \mathbb{K} = 0$, Conjecture 1.1 is simply the classical Chevalley restriction theorem. Since in the context of Higgs bundles, $d$ is the dimension of the underlying variety $X$, we view the conjecture as a higher-dimensional analogue of Chevalley restriction theorem. Note when $d = 2$ and $\text{char } \mathbb{K} = 0$, this conjecture is a special case (degree zero part) of a more general conjecture proposed by Berest et al. [1].

If $\text{char } \mathbb{K} = 0$, Conjecture 1.1 is known to hold for $G = \text{GL}_n(\mathbb{K})$ (Vaccarino [15], Domokos [6], and later Chen-Ngô [4] independently; see also Gan-Ginzburg [7] for case $d = 2$) and for $G = \text{Sp}_n(\mathbb{K})$ (Chen-Ngô [5]). A weaker version $t^d/W \sim \mathfrak{g}^d,_{\text{red}}/G$ is proved by Hunziker [9] if $G$ is of type $A, B, C, D$ or $G_2$.

If $\text{char } \mathbb{K} > 0$, Conjecture 1.1 is largely open. However, Vaccarino [15] proved the weaker version $t^d/W \sim \mathfrak{g}^d,_{\text{red}}/G$ for $G = \text{GL}_n(\mathbb{K})$.

The main purpose of the article is to prove Conjecture 1.1 for orthogonal groups in case $\text{char } \mathbb{K} = 0$ and to prove a weaker version of this conjecture in case $\text{char } \mathbb{K} > 2$. To be more precise, our main result is the following (see Theorems 3.4, 3.5, 4.3 and 4.4):

**Theorem 1.2.** Suppose $n \geq 2$, $d \geq 1$, and $G$ is an orthogonal group $O_n(\mathbb{K})$ or a special orthogonal group $SO_n(\mathbb{K})$.

1. If $\text{char } \mathbb{K} = 0$, then $\Phi : t^d/W \to \mathfrak{g}^d/G$ is an isomorphism.
2. If $\text{char } \mathbb{K} > 2$, then $\Phi : t^d/W \to \mathfrak{g}^d,_{\text{red}}/G$ is an isomorphism.

Our proof can also treat in a uniform way the case $G = \text{Sp}_n(\mathbb{K})$ ($n$ even), which is due to Chen-Ngô [5] if char $\mathbb{K} = 0$.

Since $W$ is finite and $t^d$ is an affine space, the theorem implies

**Corollary 1.3.** If char $\mathbb{K} = 0$ and $G = O_n(\mathbb{K})$ or $SO_n(\mathbb{K})$, the quotient $\mathfrak{g}^d/G$ is integral (i.e. reduced and irreducible) and normal. □

We will divide the proof into two parts, according to the type of the Weyl group $W$. In the first part, $W$ is of type $B$, and this includes cases $G = O_n(\mathbb{K})$, $SO_n(\mathbb{K})$ ($n$ odd), and $Sp_n(\mathbb{K})$ ($n$ even). Based on the results in the first part (Theorems 3.4 and 3.5), we prove Theorem 4.3 and Theorem 4.4 in the second one, which corresponds to the $W$ of type $D$ ($G = SO_n(\mathbb{K})$, $n$ even).

Next let us explain the strategy of our proof of Theorem 3.4. Let $\mathbb{K}[\mathfrak{g}^d]$ (resp. $\mathbb{K}[t^d]$) be the coordinate ring of $\mathfrak{g}^d$ (resp. $t^d$). In order to show the restriction homomorphism $\Phi : \mathbb{K}[\mathfrak{g}^d]^G \to \mathbb{K}[t^d]^W$ is an isomorphism, it
suffices to construct a $K$-linearly spanning set of $\mathbb{K}[c_d^d]^G$, which are mapped bijectively to a $K$-linear basis of $\mathbb{K}[t^d]^W$.

The results of Procesi [13] give a set of generators of $\mathbb{K}[M_n(\mathbb{K})]^d]^G$, which induces a set of generators of $\mathbb{K}[c_d^d]^G$ under the surjective homomorphism $\mathbb{K}[M_n(\mathbb{K})]^d]^G \to \mathbb{K}[c_d^d]^G$. We shall encode these generators in the determinant of a universal matrix. In (3.0.1), we define the formal power series
\[
F_g = \det(I_n + \sum_{(i_1, \ldots, i_d) \in S} X(1)^{i_1} X(2)^{i_2} \cdots X(d)^{i_d} T_{i_1 \cdots i_d}),
\]
with $F_g \in \mathbb{K}[c_d^d]^G[[T_{i_1 \cdots i_d} | (i_1, \ldots, i_d) \in S]]$, where the index set $S$ is \[
\{(i_1, \ldots, i_d) | i_1, \ldots, i_d \in \mathbb{Z}_{\geq 0}, (i_1, \ldots, i_d) \neq (0, \ldots, 0), i_1 + \cdots + i_d \text{ is even}\},
\]
and $X(1), \ldots, X(d)$ are “generic” commuting skew symmetric $n \times n$ matrices over $\mathbb{K}[c_d^d]$. Based on Procesi’s results, we can show that a $K$-linearly spanning set of $\mathbb{K}[c_d^d]^G$ is given by the coefficients of $F_g$, and in turn by the coefficients of $\sqrt{F_g}$, the unique square root of $F_g$ with constant coefficient 1.

Under the restriction homomorphism $\Phi : \mathbb{K}[c_d^d]^G \to \mathbb{K}[t^d]^W$, we deduce that
\[
\Phi(\sqrt{F_g}) = \sqrt{F_1}.
\]
See (3.0.1), (3.0.2) for the definition of $F_1$ and $\sqrt{T_1}$. Now explicit computations show $\deg \sqrt{T_1} \leq \lfloor \frac{d}{2} \rfloor$, that is, $\sqrt{T_1}$ can be written as
\[
\sqrt{T_1} = 1 + \sum_{w \in M([\lfloor \frac{d}{2} \rfloor])} c_w w,
\]
where $c_w \in \mathbb{K}[t^d]^W$ and $M([\lfloor \frac{d}{2} \rfloor])$ is the set of non-empty monomials in variables $\{T_{i_1 \cdots i_d} | (i_1, \ldots, i_d) \in S\}$ whose degree is less than or equal to $\lfloor \frac{d}{2} \rfloor$. We can also show that the coefficients of $\sqrt{T_1}$ form a $K$-linear basis of $\mathbb{K}[t^d]^W$. So to finish, all we need to show is $\deg \sqrt{F_g} \leq \lfloor \frac{n}{2} \rfloor$. This turns out to be the technical heart of this paper.

To prove $\deg \sqrt{F_g} \leq \lfloor \frac{n}{2} \rfloor$, we explicitly construct a power series $N$ with constant coefficient 1 such that $N^2 = F_g$. By the uniqueness of square roots with constant coefficient 1, we see $N = \sqrt{F_g}$. So the degree bound follows from the explicit construction of $N$.

If $G = S_p_n(\mathbb{K})$, our construction of $N$ is essentially the same as that presented in Chen-Ngô [5]. However, if $G = O_n(\mathbb{K})$, the construction of $N$ is much more involved and poses significant challenges. We provide the full details in Section 5.2.

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2. Notations and preliminaries

In this section we fix some notations and record some useful lemmas that will be used frequently in the subsequent sections.

Throughout the paper, $\mathbb{K}$ is an algebraically closed field of characteristic not equal to 2. All rings are commutative, unless otherwise specified.

Given a ring $R$, we denote by $M_n(R)$ the set of $n \times n$ matrices over $R$, and for a matrix $M$, we denote by $M^t$ its transpose, and by $M(i,j)$ the $(i,j)$-entry of $M$. We denote the coordinate ring of an affine $\mathbb{K}$-scheme $X$ by $\mathbb{K}[X]$. In particular, if $V$ is a $\mathbb{K}$-linear space, $\mathbb{K}[V]$ means the $\mathbb{K}$-algebra of polynomial functions on $V$. If a group $G$ acts $\mathbb{K}$-linearly on $V$, then $G$ acts naturally on $\mathbb{K}[V]$ by $g \cdot f(v) = f(g^{-1} \cdot v)$, for $g \in G$, $f \in \mathbb{K}[V]$, $v \in V$. The $\mathbb{K}$-algebra of $G$-invariant polynomials on $V$ is denoted by $\mathbb{K}[V]^G$. For $n \in \mathbb{Z}$, we denote by $[n]$ the maximal integer less than or equal to $n$.

Here are some lemmas on square roots of formal power series.

**Lemma 2.1.** Suppose $R = \bigoplus_{i=0}^{\infty} R_i$ is a graded $\mathbb{K}$-algebra. Let $f_1, f_2 \in R$ and let $c_j \in R_0$ be the degree zero part of $f_j$, $j = 1, 2$. If $f_1^2 = f_2^2$, $c_1 = c_2$, and if $c_1$ is not a zero-divisor in $R$, then $f_1 = f_2$.

**Proof.** Put $f_1 = c_1 + g_1$ and $f_2 = c_2 + g_2$. Since $f_1^2 = f_2^2$ and $c_1 = c_2$, we see

$$(c_1 + c_2 + g_1 + g_2)(g_1 - g_2) = 0.$$

Since $c_1 + c_2 = 2c_1$ is not a zero-divisor in $R$, it is easy to see $c_1 + c_2 + g_1 + g_2$ is not a zero-divisor in $R$. From this, we deduce $g_1 = g_2$ and hence $f_1 = f_2$. $\square$

**Lemma 2.2.** Let $R$ be a $\mathbb{K}$-algebra, and $R[[t_1, t_2, \cdots]]$ be the formal power series ring with countable variables.

1. Suppose $f_1, f_2 \in R[[t_1, t_2, \cdots]]$, and let $c_j = f_j(0,0,\cdots) \in R$ be the constant coefficient of $f_j$. If $f_1^2 = f_2^2$, $c_1 = c_2$, and if $c_1$ is not a zero-divisor in $R$, then $f_1 = f_2$.

2. Suppose $g \in R[[t_1, t_2, \cdots]]$ has constant coefficient 1, then there exists a unique $f \in R[[t_1, t_2, \cdots]]$ whose constant coefficient is 1 and which satisfies $f^2 = g$. Moreover, the $\mathbb{K}$-subalgebra of $R$ generated by the coefficients of $g$ coincides with that generated by the coefficients of $f$.

**Proof.** (1) The proof is the same as that for the previous lemma.
(2) The uniqueness of $f$ follows from (1). In order to show its existence, let $\mathcal{T}$ be the set of all nonempty monomials in the variables \{t_1, t_2, \cdots \}. For $w_1$, $w_2 \in \mathcal{T}$, define $w_1 \leq w_2$ if there exists $w \in \{1\} \cup \mathcal{T}$ such that $w_2 = w_1w$. This gives a partial order on $\mathcal{T}$. Write $g = 1 + \sum_{w \in \mathcal{T}} a_w$, with $a_w \in R$. For each $w \in \mathcal{T}$, we will define by induction $b_w \in R$, so that $f := 1 + \sum_{w \in \mathcal{T}} b_w$ satisfies $f^2 = g$.

Suppose $w \in \mathcal{T}$ and suppose for any $v \in \mathcal{T}$ with $v < w$, the element $b_v \in R$ has been defined. Let $\mathcal{T}_w := \{(v_1, v_2) \in \mathcal{T}^2 \mid v_1 < w, \ v_2 < w, v_1v_2 = w\}$. Then define

$$b_w := \frac{1}{2}(a_w - \sum_{(v_1, v_2) \in \mathcal{T}_w} b_{v_1}b_{v_2}).$$

By direct computations, we can see that $f := 1 + \sum_{w \in \mathcal{T}} b_w$ satisfies $f^2 = g$. Moreover, it follows from the explicit expressions that the $\mathbb{K}$-subalgebras $\mathbb{K}[a_w | w \in \mathcal{T}]$ and $\mathbb{K}[b_w | w \in \mathcal{T}]$ of $R$ coincide.

**Lemma 2.3.** Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded $\mathbb{K}$-algebra, with $R_0 = \mathbb{K}$. Suppose $f, g \in R[[y, z_0, z_1, \cdots]]$ satisfy $f^2 = y^{2n}g$ and $g = z_0^{2m} + g_1 + g_2 + \cdots$, for \(g_i \in R_i[[y, z_0, z_1, \cdots]]\), with $n, m \geq 0$, then there exists $\tilde{f} \in R[[y, z_0, z_1, \cdots]]$ satisfying $\tilde{f} = y^n\tilde{g}$. Moreover, if $f^2 = y^{2n}$, then $f = \pm y^n$.

**Proof.** Let $f = f_0 + f_1 + \cdots$, with $f_i \in R_i[[y, z_0, z_1, \cdots]]$. By $f^2 = y^{2n}g$ we get $f_0^2 = y^{2n}z_0^m$ in $R_0[[y, z_0, z_1, \cdots]] = \mathbb{K}[[y, z_0, z_1, \cdots]]$. This implies $f_0 = \pm y^n z_0^m$. Assume without loss of generality that $f_0 = y^n z_0^m$. Now suppose $i \geq 1$ and suppose for each $1 \leq j < i$, there exists $\tilde{f}_j \in R_j[[y, z_0, z_1, \cdots]]$ such that $f_j = y^n\tilde{f}_j$. By comparing the $R_i[[y, z_0, z_1, \cdots]]$ part of $f^2$ and $y^{2n}g$, we get $\sum_{j=0}^{i} f_j f_{i-j} = y^{2n}g_i$. Since $\sum_{j=0}^{i} f_j f_{i-j} = 2f_0 f_i + \sum_{j=1}^{i-1} f_j f_{i-j} = 2y^n z_0^m f_i + y^n \sum_{j=1}^{i-1} f_j f_{i-j}$, we obtain $2y^n z_0^m f_i = y^{2n}(g_i - \sum_{j=1}^{i-1} f_j f_{i-j})$. As neither $z_0$ nor $y$ is a zero-divisor in $R[[y, z_0, z_1, \cdots]]$, this implies $2z_0^m f_i = y^n(g_i - \sum_{j=1}^{i-1} f_j f_{i-j})$, and then $f_i = y^n \tilde{f}_i$ for some $\tilde{f}_i \in R_i[[y, z_0, z_1, \cdots]]$. By induction on $i$ we see that for each $i \geq 1$, there exists $\tilde{f}_i \in R_i[[y, z_0, z_1, \cdots]]$ satisfying $f_i = y^n \tilde{f}_i$. Since $f \in R[[y, z_0, z_1, \cdots]]$, the formal sum $\tilde{f} := z_0^m + \sum_{i=1}^{\infty} \tilde{f}_i$ can be viewed as an element of $R[[y, z_0, z_1, \cdots]]$ and it satisfies $f = y^n \tilde{f}$.

If $f^2 = y^{2n}$, then $\tilde{f}^2 = 1$, and this implies $\tilde{f} \in R_0 = \mathbb{K}$. So $\tilde{f} = \pm 1$ and $f = \pm y^n$. \hfill \qed

3. Main theorems: type $B$ case

Suppose $n \geq 2$, $d \geq 1$ are positive integers. Let

$$O_n(\mathbb{K}) := \{ A \in M_n(\mathbb{K}) \mid AA^t = I_n \}$$
be the orthogonal group, and if \( n \) is even, let

\[
Sp_n(\mathbb{K}) := \{ A \in M_n(\mathbb{K}) \mid A^t J A = I_n \}
\]

be the symplectic group, where \( J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \).

Throughout this section, \( G \) is one of the following groups:

\[
G = \begin{cases} 
O_n(\mathbb{K}), & \text{if } G \text{ is } O_n(\mathbb{K}) \text{ or } SO_n(\mathbb{K}); \\
Sp_n(\mathbb{K}), & \text{if } G \text{ is } Sp_n(\mathbb{K}), \\
SO_n(\mathbb{K}), & \text{if } G \text{ is } SO_n(\mathbb{K}),
\end{cases}
\]

where we fix the realizations of the Lie algebras as matrices:

\[
so_n = \{ A \in M_n(\mathbb{K}) \mid A + A^t = 0 \}, \\
sp_n = \{ A \in M_n(\mathbb{K}) \mid A^t J + J A = 0 \}.
\]

The Lie algebra \( t \) of \( T \) is a Cartan subalgebra of \( g \). Via the diagonal adjoint representation, the group \( G \) acts on \( g^d \) and this induces an action of \( W \) on \( t^d \). For \( 1 \leq k \leq d \), \( 1 \leq i, j \leq n \), let \( x(k)_{ij} \) be the polynomial function of \( g^d \) whose value at a point \((A_1, \ldots, A_d) \in g^d \) is the \((i, j)\)-entry of the matrix \( A_k \in M_n(\mathbb{K}) \).

Over the ring \( \mathbb{K}[g^d] \), consider the “generic” \( n \times n \) matrices \( X(1), X(2), \ldots, X(d) \), such that the \((i, j)\)-entry of \( X(k) \) is \( x(k)_{ij} \). Let \( I \) be the ideal of \( \mathbb{K}[g^d] \) generated by all of the entries of the matrices \( [X(k), X(l)] := X(k)X(l) - X(l)X(k), 1 \leq k, l \leq d \), and define the quotient ring

\[
\mathbb{K}[\mathfrak{e}^d_0] := \mathbb{K}[g^d]/I.
\]

This ring can be viewed as the coordinate ring of the commuting scheme \( \mathfrak{e}^d_0 \). Since \( I \) is obviously a homogeneous ideal of the polynomial ring \( \mathbb{K}[g^d] \), the quotient ring \( \mathbb{K}[\mathfrak{e}^d_0] = \bigoplus_{i=0}^\infty \mathbb{K}[\mathfrak{e}^d_0]_i \) is a graded \( \mathbb{K} \)-algebra. Moreover, the degree zero part \( \mathbb{K}[\mathfrak{e}^d_0]_0 = \mathbb{K} \), and the degree one part \( \mathbb{K}[\mathfrak{e}^d_0]_1 \) is \( \mathbb{K} \)-linearly spanned by \( x(k)_{ij} \), \( 1 \leq k \leq d, 1 \leq i, j \leq n \). From now on, we view \( X(k) \) \((1 \leq k \leq d)\) as matrices over \( \mathbb{K}[\mathfrak{e}^d_0] \). Note these matrices are mutually commutative by the definition of \( I \), and \((X(1), \ldots, X(d)) \in \mathfrak{e}^d_0(\mathbb{K}[\mathfrak{e}^d_0])\) can also be viewed as the tautological \( \mathbb{K}[\mathfrak{e}^d_0] \)-valued point of the commuting scheme \( \mathfrak{e}^d_0 \).

Since obviously \( I \) is \( G \)-invariant, we have the induced action of \( G \) on \( \mathbb{K}[\mathfrak{e}^d_0] \). Moreover, the action preserves the degrees on \( \mathbb{K}[\mathfrak{e}^d_0] \), so that the invariant subring \( \mathbb{K}[\mathfrak{e}^d_0]^G \) is still a graded \( \mathbb{K} \)-algebra whose degree zero part is equal...
to $\mathbb{K}$. Any polynomial function on $\mathfrak{g}^d$ restricts to a polynomial function on $t^d$ through the inclusion $t^d \subset \mathfrak{g}^d$, and the restriction homomorphism $\mathbb{K}[\mathfrak{g}^d] \to \mathbb{K}[t^d]$ factors through $\mathbb{K}[\mathfrak{c}_0^d]$. This induces the following restriction homomorphism between the invariant rings:

$$\Phi : \mathbb{K}[\mathfrak{c}_0^d]^G \to \mathbb{K}[t^d]^W.$$ 

For $1 \leq k \leq d$, let $Y(k)$ be the image of $X(k)$ under the restriction homomorphism $M_n(\mathbb{K}[\mathfrak{c}_0^d]) \to M_n(\mathbb{K}[t^d])$. Then $Y(1), \ldots, Y(d)$ are commuting $n \times n$ matrices over $\mathbb{K}[t^d]$. We define the following formal power series:

$$F_0 := \det(N + \sum_{(i_1, \ldots, i_d) \in S} X(1)^{i_1} X(2)^{i_2} \cdots X(d)^{i_d} T_{i_1, \ldots, i_d}),$$

$$F_1 := \det(N + \sum_{(i_1, \ldots, i_d) \in S} Y(1)^{i_1} Y(2)^{i_2} \cdots Y(d)^{i_d} T_{i_1, \ldots, i_d}),$$

(3.0.1)

with $F_0 \in \mathbb{K}[\mathfrak{c}_0^d][T_{i_1, \ldots, i_d} | (i_1, \ldots, i_d) \in S]$ and $F_1 \in \mathbb{K}[t^d][T_{i_1, \ldots, i_d} | (i_1, \ldots, i_d) \in S]$. Here and throughout this paper, $S$ denotes the index set

$$\{(i_1, \ldots, i_d) | i_1, \ldots, i_d \in \mathbb{Z}_{\geq 0}, (i_1, \ldots, i_d) \neq (0, \ldots, 0), i_1 + \cdots + i_d \text{ is even}\}.$$ 

Since determinants are invariant under conjugations, we see in fact $F_0 \in \mathbb{K}[\mathfrak{c}_0^d]^G[T_{i_1, \ldots, i_d} | (i_1, \ldots, i_d) \in S]$ and $F_1 \in \mathbb{K}[t^d]^W[T_{i_1, \ldots, i_d} | (i_1, \ldots, i_d) \in S]$. Moreover, under the restriction homomorphism, we have $\Phi(F_0) = F_1$.

By Lemma 2.2, there exists a unique $\sqrt{F_0} \in \mathbb{K}[\mathfrak{c}_0^d]^G[T_{i_1, \ldots, i_d} | (i_1, \ldots, i_d) \in S]$ with constant coefficient 1 and satisfying $\sqrt{F_0} = F_0$. Similarly, we denote $\sqrt{F_1} \in \mathbb{K}[t^d]^W[T_{i_1, \ldots, i_d} | (i_1, \ldots, i_d) \in S]$ as the unique square root of $F_1$ with constant coefficient 1. Define $\mathcal{M}$ to be the set of non-empty monomials in the variables $\{T_{i_1, \ldots, i_d} | (i_1, \ldots, i_d) \in S\}$. We expand these two formal power series as follows:

$$\sqrt{F_0} = 1 + \sum_{w \in \mathcal{M}} d_w w, \quad \sqrt{F_1} = 1 + \sum_{w \in \mathcal{M}} c_w w,$$

(3.0.2)

with $d_w \in \mathbb{K}[\mathfrak{c}_0^d]^G$, $c_w \in \mathbb{K}[t^d]^W$.

For a commutative ring $R$, and for a nonzero formal power series $f \in R[[T_{i_1, \ldots, i_d} | (i_1, \ldots, i_d) \in S]]$, we write $f = a_0 + \sum_{w \in \mathcal{M}} a_w w$ with $a_0, a_w \in R$.

Define the degree $\deg f \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ of $f$ by

$$\deg f := \begin{cases} 0, & \text{if } f = a_0; \\ \sup\{\deg w | w \in \mathcal{M}, a_w \neq 0\}, & \text{if } f \neq a_0. \end{cases}$$

We say $f$ is of finite degree if $\deg f < +\infty$.

Let $\bigoplus_{S} \mathbb{K}$ be the direct sum of $\mathbb{K}$ indexed by $S$. Note that for an element $t = (t_{i_1, \ldots, i_d}) \in \bigoplus_{S} \mathbb{K}$, the components $t_{i_1, \ldots, i_d} = 0$ for all but finitely many $(i_1, \ldots, i_d) \in S$. Then if a formal power series $f \in R[[T_{i_1, \ldots, i_d} | (i_1, \ldots, i_d) \in S]]$ is of finite degree, we get a well-defined element $f(t) \in R$ by evaluating
Lemma by the coefficients of \( F \)

Proof. Let \( f \) is the zero element in \( \bigoplus_{S} \mathbb{K} \).

Let \( G \) act on \( M_n(\mathbb{K})^d \) via simultaneous conjugation: \( g \cdot (A_1, \cdots, A_d) = (gA_1g^{-1}, \cdots, gA_dg^{-1}) \), for \( g \in G \), \( (A_1, \cdots, A_d) \in M_n(\mathbb{K})^d \). Then the inclusion \( g^d \subset M_n(\mathbb{K})^d \) induces a restriction homomorphism \( \varphi : \mathbb{K}[M_n(\mathbb{K})^d]^G \to \mathbb{K}[\mathfrak{q}^d]^G \), and we have the following diagram:

\[
\mathbb{K}[M_n(\mathbb{K})^d]^G \xrightarrow{\varphi} \mathbb{K}[\mathfrak{q}^d]^G \xrightarrow{\Phi} \mathbb{K}[t^d]^W.
\]

**Proposition 3.1.** The image of \( \varphi \) is \( \mathbb{K} \)-linearly spanned by the coefficients \( \{1\} \cup \{d_w | w \in M\} \) of \( \sqrt{T_g} \).

**Proof.** Let \( S' := \{(i_1, \cdots, i_d) | i_1, \cdots, i_d \in \mathbb{Z}_{\geq 0}, i_1 + \cdots + i_d \text{ odd}\} \), and define the following \( n \times n \) matrix

\[
A := I_n + \sum_{(i_1, \cdots, i_d) \in S \cup S'} X(1)^{i_1} \cdots X(d)^{i_d} T_{i_1 \cdots i_d}.
\]

Let \( F' := \det A \), a formal power series of the variables \( \{T_{i_1 \cdots i_d} | (i_1, \cdots, i_d) \in S \cup S'\} \). By Procesi [13] (char. 0 case) and Zubkov [16] (char. \( p > 2 \) case), \( \text{Im } \varphi \) is generated as a \( \mathbb{K} \)-algebra by the coefficients of \( \text{tr } A \). From this or directly from Lopatin [11, Corollary 2], we deduce that \( \text{Im } \varphi \) is generated by the coefficients of \( F' \), and hence by the coefficients of \( (F')^2 \) according to Lemma 2.2 (2). Note if \( G = O_n(\mathbb{K}) \) or \( SO_n(\mathbb{K}) \), the matrix \( X(1)^{i_1} \cdots X(d)^{i_d} \) is symmetric for \((i_1, \cdots, i_d) \in S \) and skew-symmetric for \((i_1, \cdots, i_d) \in S' \), so in these cases

\[
(F')^2 = \det(AA') = \det[(I_n + \sum_{(i_1, \cdots, i_d) \in S} X(1)^{i_1} \cdots X(d)^{i_d} T_{i_1 \cdots i_d})^2 - \left( \sum_{(i_1, \cdots, i_d) \in S'} X(1)^{i_1} \cdots X(d)^{i_d} T_{i_1 \cdots i_d} \right)^2].
\]

Similarly if \( G = Sp_n(\mathbb{K}) \), we have

\[
(F')^2 = \det(A(JAJ)^t) = \det[(I_n + \sum_{(i_1, \cdots, i_d) \in S} X(1)^{i_1} \cdots X(d)^{i_d} T_{i_1 \cdots i_d})^2 - \left( \sum_{(i_1, \cdots, i_d) \in S'} X(1)^{i_1} \cdots X(d)^{i_d} T_{i_1 \cdots i_d} \right)^2].
\]

In either case, each coefficient of \( (F')^2 \) is a \( \mathbb{K} \)-linear combination of the coefficients of \( F_g = \det(I_n + \sum_{(i_1, \cdots, i_d) \in S} X(1)^{i_1} \cdots X(d)^{i_d} T_{i_1 \cdots i_d}) \). Then \( \text{Im } \varphi \) is generated by the coefficients of \( F_g \), or equivalently, by the evaluations \( F_g(\ell) \) (\( \ell \in \bigoplus_{S} \mathbb{K} \)). By the multiplicative property of determinants, for any
$\ell_1, \ell_2 \in \bigoplus_S \mathbb{K}$, there exists $\ell_3 \in \bigoplus_S \mathbb{K}$, such that

\begin{equation}
F_\theta(\ell_1)F_\theta(\ell_2) = F_\theta(\ell_3).
\end{equation}

So $\text{Im } \varphi$ is $\mathbb{K}$-linearly spanned by $F_\theta(\ell) \ (\ell \in \bigoplus_S \mathbb{K})$.

By Lemma 2.2 (2), the $\mathbb{K}$-subalgebra of $\mathbb{K}[\mathfrak{c}_d^G]$ generated by the coefficients of $\sqrt{F_\theta}$ coincides with that generated by the coefficients of $F_\theta$, and hence is equal to $\text{Im } \varphi$. Moreover, the equality (3.1.1) implies that $(\sqrt{F_\theta}(\ell_1)\sqrt{F_\theta}(\ell_2))^2 = (\sqrt{F_\theta}(\ell_3))^2$. By applying Lemma 2.1 to the graded $\mathbb{K}$-algebra $\mathbb{K}[\mathfrak{c}_d^G]$, we obtain $\sqrt{F_\theta}(\ell_1)\sqrt{F_\theta}(\ell_2) = \sqrt{F_\theta}(\ell_3)$. So finally $\text{Im } \varphi$ is $\mathbb{K}$-linearly spanned by the coefficients of $\sqrt{F_\theta}$.

The following two lemmas are important, but technical in nature, so their proofs will be postponed to Section 5. Here and throughout this paper, $\mathcal{M}(\frac{n}{2}) := \{w \in \mathcal{M} | \text{deg } w \leq \lfloor \frac{n}{2} \rfloor \}$ is the subset of $\mathcal{M}$ consisting of monomials with degree less than or equal to $\lfloor \frac{n}{2} \rfloor$.

**Lemma 3.2.** deg $\sqrt{F_\theta} \leq \lfloor \frac{n}{2} \rfloor$, and $\{1\} \cup \{c_w | w \in \mathcal{M}(\lfloor \frac{n}{2} \rfloor)\}$ is a $\mathbb{K}$-linear basis of $\mathbb{K}[t]^W$.

**Lemma 3.3.** deg $\sqrt{F_\theta} \leq \lfloor \frac{n}{2} \rfloor$.

Assuming these lemmas, we can now prove one of our main theorems.

**Theorem 3.4.** If char $\mathbb{K} = 0$, then the restriction homomorphism $\Phi : \mathbb{K}[\mathfrak{c}_d^G] \rightarrow \mathbb{K}[t]^W$ is an isomorphism of $\mathbb{K}$-algebras.

**Proof.** By Lemma 3.2 and 3.3, the expansions (3.0.2) reduce to the following:

\begin{equation}
\sqrt{F_\theta} = 1 + \sum_{w \in \mathcal{M}(\lfloor \frac{n}{2} \rfloor)} d_w w, \quad \sqrt{F_\theta} = 1 + \sum_{w \in \mathcal{M}(\lfloor \frac{n}{2} \rfloor)} c_w w.
\end{equation}

Since $\Phi(\sqrt{F_\theta})^2 = \Phi(F_\theta) = F_\theta(\sqrt{F_\theta})$, and both the constant coefficients of $\Phi(\sqrt{F_\theta})$, $\sqrt{F_\theta}$ are $1$, we deduce from Lemma 2.2 that $\Phi(\sqrt{F_\theta}) = \sqrt{F_\theta}$. So

\begin{equation}
\Phi(d_w) = c_w, \quad \text{for all } w \in \mathcal{M}(\lfloor \frac{n}{2} \rfloor).
\end{equation}

Since char $\mathbb{K} = 0$, the reductive group $G$ is linearly reductive. Then $\varphi : \mathbb{K}[M_n(\mathbb{K})^d] \rightarrow \mathbb{K}[\mathfrak{c}_d^G]$ is surjective, because it is induced from the surjective restriction homomorphism $\mathbb{K}[M_n(\mathbb{K})^d] \rightarrow \mathbb{K}[\mathfrak{c}_d^G]$. By Proposition 3.1 and (3.4.1), $\mathbb{K}[\mathfrak{c}_d^G]$ is $\mathbb{K}$-linearly spanned by $\{1\} \cup \{d_w | w \in \mathcal{M}(\lfloor \frac{n}{2} \rfloor)\}$. Then by Lemma 3.2 and (3.4.2), the homomorphism $\Phi$ maps a $\mathbb{K}$-linearly spanning set bijectively to a $\mathbb{K}$-linear basis, hence it is an isomorphism.

Let $\mathbb{K}[\mathfrak{c}_d^G, \text{red}] := \mathbb{K}[\mathfrak{c}_d^G]/\sqrt{(0)}$ be the quotient of $\mathbb{K}[\mathfrak{c}_d^G]$ by its nilpotent radical. This is the coordinate ring of the commuting variety $\mathfrak{c}_d^G, \text{red}$. As $\mathbb{K}[t]^W$ is reduced, $\Phi : \mathbb{K}[\mathfrak{c}_d^G] \rightarrow \mathbb{K}[t]^W$ factors through $\mathbb{K}[\mathfrak{c}_d^{G, \text{red}}]^G$. 

- \[  \]
Theorem 3.5. If char \( \mathbb{K} = p > 2 \), then the restriction homomorphism \( \Phi : \mathbb{K}[\mathfrak{c}_d]^{G} \to \mathbb{K}[d^i]^{W} \) is an isomorphism of \( \mathbb{K} \)-algebras.

Proof. Let \( \pi : \mathbb{K}[\mathfrak{c}_d]^{G} \to \mathbb{K}[\mathfrak{c}_d]^{G}_{\text{red}} \) be the homomorphism induced by the natural quotient homomorphism \( \mathbb{K}[\mathfrak{c}_d] \to \mathbb{K}[\mathfrak{c}_d]^{G}_{\text{red}} \). Consider the composition of homomorphisms:

\[
\mathbb{K}[M_n(\mathbb{K})]^{G} \xrightarrow{\rho} \mathbb{K}[\mathfrak{c}_d]^{G}_{\text{red}} \xrightarrow{\pi} \mathbb{K}[\mathfrak{c}_d]^{G}_{\text{red}}.
\]

By the same arguments as above, we can see that when restricted on the subspace \( \text{Im} (\pi \circ \varphi) \), \( \Phi \) induces an isomorphism \( \text{Im} (\pi \circ \varphi) \cong \mathbb{K}[d]^{W} \) of \( \mathbb{K} \)-linear spaces. In particular, \( \Phi : \mathbb{K}[\mathfrak{c}_d]^{G}_{\text{red}} \to \mathbb{K}[d]^{W} \) is surjective.

For any \( f \in \mathbb{K}[\mathfrak{c}_d]^{G}_{\text{red}} \), it follows from Mumford-Fogarty-Kirwan [12, Lemma A.1.2] that there exists \( m \geq 1 \), such that \( f^m \in \text{Im} (\pi \circ \varphi) \). If \( \Phi(f) = 0 \), then \( \Phi(f^m) = 0. \) Since \( f^m \in \text{Im} (\pi \circ \varphi) \) and \( \Phi |_{\text{Im} (\pi \circ \varphi)} \) is an isomorphism, we obtain \( f^m = 0. \) This implies \( f = 0 \) as the ring \( \mathbb{K}[\mathfrak{c}_d]^{G}_{\text{red}} \) is reduced. So we obtain \( \Phi \) is injective. This in turn implies that \( \Phi : \mathbb{K}[\mathfrak{c}_d]^{G} \to \mathbb{K}[d]^{W} \) is an isomorphism.

\( \square \)

4. Main theorems: Type D case

In this section, keeping the same notations as in Section 3, we assume furthermore that \( n \geq 2 \) is even and \( G = O_n(\mathbb{K}) \). Let \( G' := SO_n(\mathbb{K}) \), and define the corresponding Weyl group by \( W' := N_{G'}(T)/T \). Note both \( G' \subset G \) and \( W' \subset W = N_G(T)/T \) are subgroups of index two. We have the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{K}[\mathfrak{c}_d]^{G'} & \xrightarrow{\Phi} & \mathbb{K}[d]^{W'} \\
\uparrow & & \uparrow \\
\mathbb{K}[\mathfrak{c}_d]^{G} & \xrightarrow{\Phi} & \mathbb{K}[d]^{W}
\end{array}
\]

Take \( w_0 \in W \) which generates the quotient group \( W/W' \cong \mathbb{Z}/2\mathbb{Z} \). Note \( W/W' \) acts naturally on \( \mathbb{K}[d]^{W'} \) and we have the eigen-subspace decomposition

\[
\mathbb{K}[d]^{W'} = \mathbb{K}[d]^{W'}_{(0)} \oplus \mathbb{K}[d]^{W'}_{(1)},
\]

where \( \mathbb{K}[d]^{W'}_{(0)} = \mathbb{K}[d]^{W} \) is the invariant part and

\[
\mathbb{K}[d]^{W'}_{(1)} = \{ v \in \mathbb{K}[d]^{W'} | w_0 v = -v \}.
\]

Similarly, the action of \( G/G' \cong \mathbb{Z}/2\mathbb{Z} \) on \( \mathbb{K}[\mathfrak{c}_d]^{G'} \) induces the eigen-subspace decomposition \( \mathbb{K}[\mathfrak{c}_d]^{G'} = \mathbb{K}[\mathfrak{c}_d]^{G'}_{(0)} \oplus \mathbb{K}[\mathfrak{c}_d]^{G'}_{(1)} \), with \( \mathbb{K}[\mathfrak{c}_d]^{G'}_{(0)} = \mathbb{K}[\mathfrak{c}_d]^{G} \) and \( \mathbb{K}[\mathfrak{c}_d]^{G'}_{(1)} = \{ v \in \mathbb{K}[\mathfrak{c}_d]^{G'} | g_0 v = -v \} \), where \( g_0 \) is a generator of \( G/G' \). Clearly the restriction homomorphism \( \Phi \) preserves these decompositions:

\( \Phi(\mathbb{K}[\mathfrak{c}_d]^{G'}_{(i)}) \subset \mathbb{K}[d]^{W'}_{(i)}, \ i = 0, 1. \)
Let \( S' := \{(i_1, \ldots, i_d)|i_1, \ldots, i_d \in \mathbb{Z}_{>0}, i_1 + \cdots + i_d \text{ odd}\} \), and \( M'(\frac{n}{2}) \) be the set of non-empty monomials in the variables \( \{T_{i_1 \cdots i_d}|(i_1, \ldots, i_d) \in S'\} \) whose degree is less than or equal to \( \frac{n}{2} \). Define the following formal power series:

\[
H_0 := \text{Pf}(\sum_{(i_1, \ldots, i_d) \in S'} X(1)^{i_1}X(2)^{i_2} \cdots X(d)^{i_d}T_{i_1 \cdots i_d}),
\]

\[
H_t := \text{Pf}(\sum_{(i_1, \ldots, i_d) \in S'} Y(1)^{i_1}Y(2)^{i_2} \cdots Y(d)^{i_d}T_{i_1 \cdots i_d}),
\]

with \( H_0 \in \mathbb{K}[\mathcal{C}^d_{\theta(1)}][[T_{i_1 \cdots i_d}|(i_1, \ldots, i_d) \in S']] \), \( H_t \in \mathbb{K}[\mathcal{C}^d_{\theta(1)}][[T_{i_1 \cdots i_d}|(i_1, \ldots, i_d) \in S']] \). Here Pf(\( A \)) means the Pfaffian of a skew symmetric matrix \( A \). Note for any \( P \in G = O_n(\mathbb{K}) \),

\[
P \cdot H_0 = \text{Pf}(\sum_{(i_1, \ldots, i_d) \in S'} P^{-1}X(1)^{i_1} \cdots X(d)^{i_d}PT_{i_1 \cdots i_d}) = \det P \cdot H_0,
\]

so \( H_0 \in \mathbb{K}[\mathcal{C}^d_{\theta(1)}][[T_{i_1 \cdots i_d}|(i_1, \ldots, i_d) \in S]] \). In a similar way, we can see that \( H_t \in \mathbb{K}[\mathcal{C}^d_{\theta(1)}][[T_{i_1 \cdots i_d}|(i_1, \ldots, i_d) \in S]] \). By constructions, \( \deg H_0 \leq \frac{n}{2} \) and \( \deg H_t \leq \frac{n}{2} \), so we can write

\[
(4.0.1) \quad H_0 = \sum_{w \in M'(\frac{n}{2})} d'_w w, \quad H_t = \sum_{w \in M'(\frac{n}{2})} c'_w w,
\]

with \( d'_w \in \mathbb{K}[\mathcal{C}^d_{\theta(1)}], \quad c'_w \in \mathbb{K}[\mathcal{C}^d_{\theta(1)}][W'] \).

We postpone the proof of the next lemma to Section 5.

**Lemma 4.1.** The set \( \{c'_w|w \in M'(\frac{n}{2})\} \) is a \( \mathbb{K} \)-linear basis of \( \mathbb{K}[\mathcal{C}^d_{\theta(1)}][W'] \).

Now consider the composition of homomorphisms:

\[
\mathbb{K}[M_n(\mathbb{K})^d]^G \xrightarrow{\varphi} \mathbb{K}[\mathcal{C}^d_{\theta(1)}]^G - \pi_1 \rightarrow \mathbb{K}[\mathcal{C}^d_{\theta(1)}]^G,
\]

where \( \varphi \) is the restriction homomorphism and \( \pi_1 \) is the projection under the decomposition \( \mathbb{K}[\mathcal{C}^d_{\theta(1)}]^G = \mathbb{K}[\mathcal{C}^d_{\theta(1)}]^G \oplus \mathbb{K}[\mathcal{C}^d_{\theta(1)}]^G \).

**Proposition 4.2.** The image of \( \varphi \) is \( \mathbb{K} \)-linearly spanned by \( \{d'_w|w \in M'(\frac{n}{2})\} \).

**Proof.** Note that \( \text{Im} \varphi = \text{Im} \varphi \cap \mathbb{K}[\mathcal{C}^d_{\theta(1)}][W'] \oplus \text{Im} \varphi \cap \mathbb{K}[\mathcal{C}^d_{\theta(1)}][W'], \) since \( \varphi \) is \( G/G' \)-equivariant. We can deduce from Lopatin [11, Corollary 2] that \( \text{Im} \varphi \) is generated as a \( \mathbb{K} \)-algebra by \( d'_w \) \((w \in M'(\frac{n}{2}))\) and \( \text{Im} \varphi \cap \mathbb{K}[\mathcal{C}^d_{\theta(1)}][W'] \). Note also \( \mathbb{K}[\mathcal{C}^d_{\theta(1)}][W] \subset \mathbb{K}[\mathcal{C}^d_{\theta(1)}][W] \) for \( i, j \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \). It follows that \( \text{Im} \varphi \cap \mathbb{K}[\mathcal{C}^d_{\theta(1)}][W] \) is \( \mathbb{K} \)-linearly spanned by

\[
\{a \cdot d'_w|a \in \text{Im} \varphi \cap \mathbb{K}[\mathcal{C}^d_{\theta(1)}][W], \ w \in M'(\frac{n}{2})\}.
\]

By the proof of Proposition 3.1, \( \text{Im} \varphi \cap \mathbb{K}[\mathcal{C}^d_{\theta(1)}][W] = \varphi(\mathbb{K}[M_n(\mathbb{K})^d]^G) \) is \( \mathbb{K} \)-linearly spanned by the evaluations \( F_{\theta(2)}^I \) \((I \in \bigoplus S)\). On the other hand,
it is easy to see the $\mathbb{K}$-linear subspace spanned by the coefficients $\{d'_{w} | w \in \mathcal{M}'(\frac{n}{2})\}$ coincides with the $\mathbb{K}$-linear subspace spanned by the evaluations $H_{g}(t_{(2)}) \cdot f_{S'}(t_{2}) \in \mathcal{S}$. It follows then that $\text{Im} \varphi \cap \mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1})$ is $\mathbb{K}$-linearly spanned by $\{F_{g}(t_{1}) \cdot H_{g}(t_{2})| t_{1} \in \mathcal{S}, t_{2} \in \mathcal{S}'\}$.

As $\det A \cdot Pf(B) = Pf(ABA)$ for any $n \times n$ symmetric matrix $A$ and skew symmetric matrix $B$, we can see for any $t_{1} \in \mathcal{S}, t_{2} \in \mathcal{S}'$, there exists $t_{3} \in \mathcal{S}'$, such that $F_{g}(t_{1}) \cdot H_{g}(t_{2}) = H_{g}(t_{3})$. So $\text{Im} \pi_{1} \circ \varphi = \text{Im} \varphi \cap \mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1})$ is $\mathbb{K}$-linearly spanned by $\{H_{g}(t_{1})| t_{1} \in \mathcal{S}\}$, and hence by the coefficients $\{d'_{w} | w \in \mathcal{M}'(\frac{n}{2})\}$. □

**Theorem 4.3.** If $\text{char} \mathbb{K} = 0$, then the restriction homomorphism $\Phi : \mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1}) \sim \mathbb{K}[\mathcal{G}]$ is an isomorphism of $\mathbb{K}$-algebras.

**Proof.** By Theorem 3.4, $\Phi$ induces an isomorphism $\mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1}) \sim \mathbb{K}[\mathcal{G}]$. Since $\text{char} \mathbb{K} = 0$, the special orthogonal group $\mathcal{G}$ is linearly reductive, and then $\varphi : \mathbb{K}[M_{n}(\mathbb{K})]/(w_{1}) \sim \mathbb{K}[\mathcal{C}(\mathcal{G})]$ is surjective as it is induced from the surjection $\mathbb{K}[M_{n}(\mathbb{K})]/(w_{1}) \rightarrow \mathbb{K}[\mathcal{C}(\mathcal{G})]$ by Proposition 4.2. $\mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1})$ is $\mathbb{K}$-linearly spanned by $\{d'_{w} | w \in \mathcal{M}'(\frac{n}{2})\}$. Note $\Phi(H_{g}) = H_{1}$, so $\Phi(d'_{w}) = c_{w}$, for any $w \in \mathcal{M}'(\frac{n}{2})$. By Lemma 4.1, the set $\{c_{w} | w \in \mathcal{M}'(\frac{n}{2})\}$ is a $\mathbb{K}$-linear basis of $\mathbb{K}[\mathcal{G}]$. It follows that $\Phi$ maps the $\mathbb{K}$-linearly spanning set $\{d'_{w} | w \in \mathcal{M}'(\frac{n}{2})\}$ of $\mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1})$ bijectively to the $\mathbb{K}$-linear basis $\{c_{w} | w \in \mathcal{M}'(\frac{n}{2})\}$ of $\mathbb{K}[\mathcal{G}]$, and hence $\Phi$ induces a $\mathbb{K}$-linear isomorphism $\mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1}) \sim \mathbb{K}[\mathcal{G}]$. Combing with the isomorphism $\mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1}) \sim \mathbb{K}[\mathcal{G}]$, we finish the proof. □

**Theorem 4.4.** If $\text{char} \mathbb{K} = p > 2$, then the restriction homomorphism

$$
\Phi : \mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1}) \sim \mathbb{K}[\mathcal{G}]
$$

is an isomorphism of $\mathbb{K}$-algebras.

**Proof.** By Theorem 3.5, $\Phi$ induces an isomorphism $\mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1}) \sim \mathbb{K}[\mathcal{G}]$. So it suffices to show $\mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1}) \sim \mathbb{K}[\mathcal{G}]$ is an isomorphism.

Let $V$ be the image of the following composition of homomorphisms:

$$
\mathbb{K}[M_{n}(\mathbb{K})]/(w_{1}) \rightarrow \mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1}) \rightarrow \mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1}) \rightarrow \mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1})
$$

By Proposition 4.2, $V$ is $\mathbb{K}$-linearly spanned by $\{d'_{w} | w \in \mathcal{M}'(\frac{n}{2})\}$. The same arguments as above show that the restriction $\Phi|_{V} : V \rightarrow \mathbb{K}[\mathcal{G}]$ is an isomorphism. In particular, $\Phi : \mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1}) \sim \mathbb{K}[\mathcal{G}]$ is surjective.

Since $\mathbb{K}[M_{n}(\mathbb{K})] \rightarrow \mathbb{K}[\mathcal{C}(\mathcal{G})]$ is surjective, it follows from Mumford-Fogarty-Kirwan [12, Lemma A.1.2] that for any $f \in \mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1})$, a positive power of $f$ is in the image of $\mathbb{K}[M_{n}(\mathbb{K})]/(w_{1})$. So for any $f \in \mathbb{K}[\mathcal{C}(\mathcal{G})]/(w_{1})$, there
exists \( m \geq 1 \), such that \( f^m \in K[\mathfrak{g}_{\text{red}}^{(0)}] \) (\( m \) even) or \( f^m \in V \) (\( m \) odd). If \( \Phi(f) = 0 \), then \( \Phi(f^m) = 0 \). Since either \( \Phi : K[\mathfrak{g}_{\text{red}}^{G'}] \to K[\mathfrak{t}^{W'}] \) or \( \Phi|_V : V \to K[\mathfrak{t}^{W'}] \) is injective, we obtain \( f^m = 0 \), and then \( f = 0 \) by the reducedness of \( K[\mathfrak{g}_{\text{red}}^{G'}] \). This shows \( \Phi : K[\mathfrak{g}_{\text{red}}^{G'}] \to K[\mathfrak{t}^{W'}] \) is injective, and we have finished the proof. \( \square \)

5. Proofs of some lemmas

The section is the most technical part of the paper. The reader is advised to skip the section at the first reading. We keep the same notations as in Section 3, 4, and let \( m = \lceil \frac{n}{2} \rceil \).

5.1. Proofs of Lemma 3.2 and 4.1. Since for any two Cartan subalgebras \( t_1, t_2 \) of \( \mathfrak{g} \), there exists an element \( h \in G \), such that \( Ad(h)t_1 = t_2 \) and since \( Ad(h)F_0 = F_0 \), Lemma 3.2 and 4.1 hold for \( t_1 \) if and only if they hold for \( t_2 \). As a result, we will choose a specific Cartan subalgebra \( t \) for the proofs.

If \( n \) is even and \( G = Sp_n(K) \), let

\[
 t = \{ \text{diag}(x_1, \ldots, x_m, -x_1, \ldots, -x_m) \mid x_i \in K, 1 \leq i \leq m \}. 
\]

If \( G = O_n(K) \) or \( SO_n(K) \), let

\[
 t = \{ \text{SK}(x_1, \ldots, x_m) \mid x_i \in K, 1 \leq i \leq m \},
\]

where \( \text{SK}(x_1, \ldots, x_m) \) is the \( n \times n \) skew symmetric matrix:

\[
\begin{pmatrix}
 0 & \sqrt{-1}x_1 & & \\
 -\sqrt{-1}x_1 & 0 & \sqrt{-1}x_2 & \\
 & \cdots & \ddots & \sqrt{-1}x_2 & \\
 & & \sqrt{-1}x_2 & 0 \\
\end{pmatrix}
\]

(5.0.1)

In other words, for \( 1 \leq i, j \leq n \), the \((i, j)\)-entry of \( \text{SK}(x_1, \ldots, x_m) \) is

\[
\begin{cases}
 \sqrt{-1}x_p & \text{if } (i, j) = (2p - 1, 2p) \text{ and } 1 \leq p \leq m, \\
 -\sqrt{-1}x_p & \text{if } (i, j) = (2p, 2p - 1) \text{ and } 1 \leq p \leq m, \\
 0 & \text{otherwise.}
\end{cases}
\]

Proof of Lemma 3.2. Let \( V = K^m \), we identify \( V \) and \( t \) by:

\[
\begin{cases}
 (x_1, \ldots, x_m) \mapsto \text{diag}(x_1, \ldots, x_m, -x_1, \ldots, -x_m), & \text{if } n \text{ even}, G = Sp_n(K); \\
 (x_1, \ldots, x_m) \mapsto \text{SK}(x_1, \ldots, x_m), & \text{if } G = O_n(K) \text{ or } SO_n(K).
\end{cases}
\]

Under this identification, the Weyl group \( W \) acts on \( V \) by permuting the coordinates \( x_1, \ldots, x_m \) and sign changing \( x_i \mapsto -x_i \). Let \( x_{ij} \) be the linear function on \( V^d \) whose value at a point \((v_1, \ldots, v_d)\) is the \( i \)-th component of
\(v_j\). A direct computation shows that under the identification of \(V\) and \(t\), we have \(F_i = N_i^2\), where

\[
N_i = \prod_{k=1}^{m} (1 + \sum_{(i_1, \ldots, i_d) \in S} x_{k_1}^{i_1} x_{k_2}^{i_2} \cdots x_{k_d}^{i_d} T_{i_1 \cdots i_d}).
\]

The constant coefficient of \(N_i\) is 1, so by the uniqueness of square root (Lemma 2.2), \(N_i = \sqrt{F_i}\), and hence \(\deg \sqrt{F_i} = \deg N_i \leq m = \lfloor \frac{n}{2} \rfloor\).

Next we will show that the coefficients of \(N_i\) form a \(K\)-linear basis of \(K[V^d]^W\). If \(\text{char } K = 0\), this is a direct consequence of Hunziker [9, Lemma 2.2]. In the following we shall adapt Hunziker’s proof slightly so that it works for all the char \(K \neq 2\) cases.

We denote by \(\Lambda\) the following set of nonzero \(m \times d\) matrices:

\[
\Lambda := \{ \lambda = \begin{pmatrix} \lambda_{i1} & \cdots & \lambda_{id} \\ \vdots & \ddots & \vdots \\ \lambda_{m1} & \cdots & \lambda_{md} \end{pmatrix} | \lambda_{ij} \in \mathbb{Z}_{\geq 0}, \lambda \neq 0 \}.
\]

Let \(\Lambda_{\text{even}} := \{ \lambda \in \Lambda | \sum_{j=1}^{d} \lambda_{ij} \text{ is even, for } 1 \leq i \leq m \}\).

The symmetric group \(S_m\) acts on \(\Lambda\) by permuting the rows. Let \(\Lambda_{\text{even}}^+ \subset \Lambda_{\text{even}}\) be the subset of all \(\lambda \in \Lambda_{\text{even}}\) such that \(\lambda_1 \geq \cdots \geq \lambda_m\) with respect to the lexicographic order on the rows, where \(\lambda_i\) is the \(i\)-th row of \(\lambda\). Then for \(\lambda \in \Lambda_{\text{even}}\), the orbit \(S_m \cdot \lambda \subset \Lambda_{\text{even}}\) contains a unique element in \(\Lambda_{\text{even}}^+\).

To each \(\lambda \in \Lambda\) corresponds to the monomials \(x^\lambda := \prod_{k=1}^{m} x_{k1}^{\lambda_{k1}} x_{k2}^{\lambda_{k2}} \cdots x_{kd}^{\lambda_{kd}}\) and \(T_\lambda := \prod_{k=1}^{m} T_{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kd}}\). Here by convention \(T_{0, \ldots, 0} = 1\). For \(\lambda \in \Lambda_{\text{even}}^+\) we put

\[
a_\lambda := \sum_{\mu \in S_m \cdot \lambda} x^\mu.
\]

Using these notations, we can rewrite \(N_i\) as

\[
N_i = 1 + \sum_{\lambda \in \Lambda_{\text{even}}^+} a_\lambda T_\lambda.
\]

It is direct to see \(a_\lambda \in K[V^d]^W\), and \(a_\lambda\) (\(\lambda \in \Lambda_{\text{even}}^+\)) are linearly independent. Suppose \(f \in K[V^d]^W\) is nonzero and write \(f = b_0 + \sum_{\lambda \in \Lambda} b_\lambda x^\lambda\), with \(b_0, b_\lambda \in K\). From \(w f = f\) for all sign changes \(w\) we deduce \(\lambda \in \Lambda_{\text{even}}\) if \(b_\lambda \neq 0\). Then \(\sum_{\lambda \in \Lambda_{\text{even}}} b_\lambda x^w \lambda = \sum_{\lambda \in \Lambda_{\text{even}}} b_\lambda x^\lambda\), for \(w \in S_m\). This implies \(b_w \lambda = b_\lambda\), for any \(w \in S_m\). So \(f\) is a linear combination of \(1\) and \(a_\lambda\), \(\lambda \in \Lambda_{\text{even}}^+\). Finally \(\{1\} \cup \{a_\lambda | \lambda \in \Lambda_{\text{even}}^+\}\), i.e., the coefficients of \(N_i = \sqrt{F_i}\), are a \(K\)-linear basis of \(K[V^d]^W\).

Proof of Lemma 4.1. Keep notations as above. Define

\[
\Lambda_{\text{odd}} := \{ \lambda \in \Lambda | \sum_{j=1}^{d} \lambda_{ij} \text{ is odd , for } 1 \leq i \leq m \},
\]

\(\Lambda_{\text{odd}}\) is a \(K\)-linear basis of \(K[V^d]^W\).
and let $\Lambda_{odd}^+ \subset \Lambda_{odd}$ be the subset of all $\lambda \in \Lambda_{odd}$ such that $\lambda_1 \geq \cdots \geq \lambda_m$ with respect to the lexicographic order on the rows, where $\lambda_i$ is the $i$-th row of $\lambda$. Then for $\lambda \in \Lambda_{odd}$, the orbit $S_m \cdot \lambda \subset \Lambda_{odd}$ contains a unique element in $\Lambda_{odd}^+$. For $\lambda \in \Lambda_{odd}^+$ we put

$$a_\lambda = \sum_{\mu \in S_m \cdot \lambda} x^\mu.$$  

Then under the identification of $V$ and $t$, a direct computation shows that

$$H_t = (\sqrt{-1})^m \sum_{\lambda \in \Lambda_{odd}^+} a_\lambda T_\lambda.$$  

The Weyl group $W$ of $G$ acts on $V$ by permuting the coordinates $x_1, \cdots, x_d$ and sign changing $\tau_i : x_i \mapsto -x_i$. The Weyl group $W'$ of $G'$ is then the index two subgroup of $W$, generated by the permutations and $\tau_i \circ \tau_j$, $1 \leq i, j \leq d$. It follows that

$$a_\lambda \in \mathbb{K}[V^d]_{(1)}^W,$$

for all $\lambda \in \Lambda_{odd}^+$, where $\mathbb{K}[V^d]_{(1)}^W$ is the $(-1)$-eigen subspace of $\mathbb{K}[V^d]_{(1)}^W$ under the action of $W/W'$.

Moreover, it is easy to see $a_\lambda$ ($\lambda \in \Lambda_{odd}^+$) are linearly independent. Similar to the proof of Lemma 3.2, we can see $\mathbb{K}[V^d]_{(1)}^{W'}$ is linearly spanned by $a_\lambda$ ($\lambda \in \Lambda_{odd}^+$). So the coefficients of $H_t$ form a basis of $\mathbb{K}[V^d]_{(1)}^{W'}$. Under the identification $V = t$, this means $\{c'_d \mid w \in \mathcal{M}'(\frac{d}{2})\}$ is a $\mathbb{K}$-linear basis of $\mathbb{K}[t^d]_{(1)}^W$, as asserted.

\[\square\]

5.2. Proof of Lemma 3.3. If $n$ is even and $G = Sp_n(\mathbb{K})$, the following matrix

$$A := J + J \sum_{(i_1, \cdots, i_d) \in S} X(1)^i_1 \cdots X(d)^i_d T_{i_1 \cdots i_d}$$

is skew-symmetric. According to Chen-Ngô [5], let

$$N := Pf(A) Pf(J)^{-1} \in \mathbb{K}[[T_{i_1 \cdots i_d} \mid (i_1, \cdots, i_d) \in S]].$$

Then the constant coefficient of $N$ is 1 and

$$N^2 = \det(I_n + \sum_{(i_1, \cdots, i_d) \in S} X(1)^i_1 \cdots X(d)^i_d T_{i_1 \cdots i_d}) = F_g.$$  

By Lemma 2.2, we get $N = \sqrt{F_g}$. Since $\deg N \leq \frac{d}{2}$ by its explicit construction, we see $\deg \sqrt{F_g} \leq \frac{d}{2}$. This gives a proof of Lemma 3.3 in the $G = Sp_n(\mathbb{K})$ case.

In the remaining of this subsection, we assume $G$ is one of the following orthogonal groups.

$$G = \begin{cases} O_n(\mathbb{K}), \\ SO_n(\mathbb{K}), \quad n \text{ odd.} \end{cases}$$
We will need the following lemma, see e.g. [10], [14].

**Lemma 5.1.** Let $R$ be a ring, and $X_{ij} \in M_j(R)$, for $1 \leq i, j \leq k$. Let $\tilde{X} \in M_{k^2}(R)$ be the following block matrix

\[
\begin{pmatrix}
X_{11} & X_{12} & \cdots & X_{1k} \\
X_{21} & X_{22} & \cdots & X_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
X_{k1} & X_{k2} & \cdots & X_{kk}
\end{pmatrix}
\]

If $X_{ij}$ commute pairwise, then

\[\det \tilde{X} = \det(\sum_{\sigma \in S_k} \text{sgn } \sigma)X_{1\sigma(1)}X_{2\sigma(2)}\cdots X_{k\sigma(k)}).\]

Suppose $d \geq 1$ and consider the following $(2d+3)\times(2d+3)$ skew symmetric matrix $T$ over the polynomial ring $\mathbb{K}[t_{ij}|1 \leq i < j \leq 2d+3]$:

\[
T(i,j) = \begin{cases} 
  t_{ij}, & 1 \leq i < j \leq 2d+3; \\
  -t_{ji}, & 1 \leq j < i \leq 2d+3; \\
  0, & 1 \leq i = j \leq 2d+3.
\end{cases}
\]

For any $k \geq 1$ and any $1 \leq i_1, \ldots, i_k \leq 2d+3$, let $T(\hat{i}_1, \ldots, \hat{i}_k)$ be the matrix obtained from $T$ by deleting the $\hat{i}_j$-th row and the $\hat{i}_j$-th column for all $1 \leq j \leq k$, and let $h_{i_1,\ldots,i_k} = \text{Pf } T(\hat{i}_1, \ldots, \hat{i}_k)$ be the corresponding Pfaffian. Define the $2d \times 2d$ matrix $A$ over $\mathbb{K}[t_{ij}|1 \leq i < j \leq 2d+3]$ by $A(i,j) := h_{i,j,2d+2}^2 + h_{i,2d+1,2d+2}^2 + h_{j,2d+1,2d+2}^2$, for $1 \leq i, j \leq 2d$.

**Lemma 5.2.** The following system of equations for the variables $t_{ij}$ ($1 \leq i < j \leq 2d+3$) has a solution in $\mathbb{K}$:

\[
\begin{cases} 
  h_{2d+2} = 0; \\
  h_{2d+3} \det A \neq 0.
\end{cases}
\]

**Proof.** It suffices to show that in the polynomial ring $\mathbb{K}[t_{ij}|1 \leq i < j \leq 2d+3]$, the polynomial $h_{2d+3} \det A$ is not contained in the ideal $\sqrt{(h_{2d+2})}$. Since the Pfaffians $h_{2d+2}$ and $h_{2d+3}$ are coprime irreducible polynomials (cf. Goodman-Wallach [8, Lemma B.2.10]), we only need to prove $\det A \not\in (h_{2d+2})$. We will proceed by induction on $d$.

If $d = 1$, by direct computations, we have

\[h_{2d+2} = h_4 = t_{12}t_{35} - t_{13}t_{25} + t_{15}t_{23},\]

and

\[
\det A = 4h_{3,4}^2 h_{2,3,4}^2 - (h_{1,2,4}^2 + h_{1,3,4}^2 + h_{2,3,4}^2)^2 \\
= -(t_{35}^2 + (t_{25} + t_{15})^2)(t_{35}^2 + (t_{25} - t_{15})^2).
\]

Since $h_4$ is irreducible, we can directly verify that $\det A \not\in (h_4)$. 
Suppose $d \geq 2$ and the statement holds for $d - 1$. Suppose to the contrary that $\det A \in (h_{2d+2})$, so there exists $g \in \mathbb{K}[t_{ij}]|1 \leq i < j \leq 2d + 3|$ such that

(5.2.1) \hspace{2cm} \det A = g h_{2d+2}.

On the one hand the degree of $\det A$ with respect to the variable $t_{12}$ is at most $4d - 4$, and the coefficient of $t_{12}^{4d-4}$ is given by

$$
\det \left( \begin{array}{cc}
2h^2_{1,2d+1,2d+2} & h^2_{1,2d+1,2d+2} + h^2_{2,2d+1,2d+2} + h^2_{2,2d+1,2d+2} \\
h^2_{1,2d+1,2d+2} + h^2_{2,2d+1,2d+2} + h^2_{2,2d+1,2d+2} & h^2_{2,2d+1,2d+2} + h^2_{2,2d+1,2d+2} + h^2_{2,2d+1,2d+2}
\end{array} \right) \cdot \det A',
$$

where $A'$ is the $(2d-2) \times (2d-2)$ matrix over $\mathbb{K}[t_{ij}]|3 \leq i < j \leq 2d + 3|$ with

$$
A'(i, j) = h^2_{1,2,i+2,j+2,2d+2} + h^2_{2,1,2,i+2,2d+2} + h^2_{2,2,1,2d+2,2d+2}.
$$

On the other hand, the degree of $h_{2d+2}$ with respect to $t_{12}$ is 1, and the coefficient of $t_{12}$ is $h_{1,2,2d+2}$. By comparing the coefficient of $t_{12}^{4d-4}$, we obtain from (5.2.1) the following equality:

$$
\det \left( \begin{array}{cc}
2h^2_{1,2d+1,2d+2} & h^2_{1,2d+1,2d+2} + h^2_{2,2d+1,2d+2} + h^2_{2,2d+1,2d+2} \\
h^2_{1,2d+1,2d+2} + h^2_{2,2d+1,2d+2} + h^2_{2,2d+1,2d+2} & h^2_{2,2d+1,2d+2} + h^2_{2,2d+1,2d+2} + h^2_{2,2d+1,2d+2}
\end{array} \right) \cdot \det A' = g_1 h_{1,2,2d+2},
$$

where $g_1 \in \mathbb{K}[t_{ij}]|1 \leq i < j \leq 2d + 3|$. A further simplification yields that

$$(h^2_{1,2d+1,2d+2} - h^2_{2,2d+1,2d+2})^2 \det A' = g_2 h_{1,2,2d+2},$$

for some $g_2 \in \mathbb{K}[t_{ij}]|1 \leq i < j \leq 2d + 3|$. Applying the induction hypothesis to the skew symmetric matrix $T(\hat{1}, \hat{2})$, we obtain

$$
\det A' \notin (h_{1,2,2d+2}).
$$

Then $(h^2_{1,2d+1,2d+2} - h^2_{2,2d+1,2d+2})^2 \in (h_{1,2,2d+2})$, since $h_{1,2,2d+2}$ is irreducible. Note the variable $t_{3,2d+1}$ is absent from $(h^2_{1,2d+1,2d+2} - h^2_{2,2d+1,2d+2})^2$, and the degree of $h_{1,2,2d+2}$ with respect to $t_{3,2d+1}$ is 1. This obviously contradicts the relation $(h^2_{1,2d+1,2d+2} - h^2_{2,2d+1,2d+2})^2 \in (h_{1,2,2d+2})$. So $\det A \notin (h_{2d+2})$, as desired.

We consider the ring $\mathbb{K}[\mathcal{C}_S^d]$, by adding a formal variable $T_0$.

**Lemma 5.3.** There exists $\tilde{N} \in \mathbb{K}[\mathcal{C}_S^d][[T_{i_1 \ldots i_d}|(i_1, \ldots, i_d) \in S]]$ satisfying:

$$
\tilde{N}^2 = \det (T_0^2 I_n + \sum_{(i_1, \ldots, i_d) \in S} X(1)^{i_1} X(2)^{i_2} \cdots X(d)^{i_d} T_{i_1 \ldots i_d}).
$$

**Proof.** For ease of notation, set $R = \mathbb{K}[\mathcal{C}_S^d]$. Note to begin with that

$$
T_0^2 I_n + \sum_{(i_1, \ldots, i_d) \in S} X(1)^{i_1} X(2)^{i_2} \cdots X(d)^{i_d} T_{i_1 \ldots i_d} = T_0^2 I_n + \sum_{j=1}^d X(j) F(j),
$$

where

$$
F(j) = \sum_{(0, \ldots, 0, i_j, \ldots, i_d) \in S, i_j \geq 1} X(j)^{i_j-1} X(j + 1)^{i_j+1} \cdots X(d)^{i_d} T_{0 \ldots 0 i_j \ldots i_d}.
$$
By Lemma 5.2, we can take a skew symmetric matrix \( T \in M_{2d+3}(\mathbb{K}) \) such that \( h_{2d+2} = 0 \) and \( h_{2d+3} \det A \neq 0 \). Here recall \( h_{i_1, \ldots, i_k} = \text{Pf} T(i_1, \ldots, i_k) \), and \( A \in M_{2d}(\mathbb{K}) \) whose \((i, j)\)-entry is \( a_{ij} := h_{i,j,2d+2}^2 + h_{i,2d+1,2d+2}^2 + h_{j,2d+1,2d+2}^2 \). By scaling \( T \) if necessary, we can assume \( h_{2d+3} = 1 \).

Over the polynomial ring \( \mathbb{K}[y, x_0, x_i | 1 \leq i \leq 2d] \), let \( v := (0, \ldots, 0, 1, yx_0) \) be the \( 2d + 3 \)-tuple. Define the \((2d + 3) \times (2d + 3)\) matrix \( B(T) \) by

\[
B(T) := \text{diag}(yx_1, yx_2, \cdots, yx_{2d}, \sum_{i=1}^{2d} yx_i, 0, 0) + T,
\]

and the \((2d + 4) \times (2d + 4)\) matrix \( M(T) \) by

\[
M(T) := \begin{pmatrix}
B(T) & v \\
v^t & 0
\end{pmatrix}.
\]

By Lemma 5.4 below,

\[
\det M(T) = (h_{2d+2} - h_{2d+3}yx_0)^2 + \sum_{i=1}^{2d} y^2 x_i^2 (h_{i,2d+1,2d+2} - h_{i,2d+1,2d+3}yx_0)^2 \\
+ \sum_{1 \leq i < j \leq 2d} y^2 x_i x_j [(h_{i,j,2d+2} - h_{i,j,2d+3}yx_0)^2 + (h_{i,2d+1,2d+2} - h_{i,2d+1,2d+3}yx_0)^2 \\
+ (h_{j,2d+1,2d+2} - h_{j,2d+1,2d+3}yx_0)^2] + y^4 g,
\]

where \( g \in \mathbb{K}[y, x_0, x_i | 1 \leq i \leq 2d] \) and \( g \in (x_1, \cdots, x_{2d}) \). Since \( h_{2d+2} = 0 \), \( h_{2d+3} = 1 \), and \( h_{i,j,2d+2}^2 + h_{i,2d+1,2d+2}^2 + h_{j,2d+1,2d+2}^2 = a_{ij} \), the above expression can be simplified as

\[
det M(T) = y^2(x_0^2 + \frac{1}{2} \sum_{i=1}^{2d} \sum_{j=1}^{2d} a_{ij} x_i x_j) + y^3 g_1,
\]

where \( g_1 \in \mathbb{K}[y, x_0, x_i | 1 \leq i \leq 2d] \) and \( g_1 \in (x_1, \cdots, x_{2d}) \). Since \( A = (a_{ij}) \in M_{2d}(\mathbb{K}) \) is symmetric with \( \det A \neq 0 \), there exists an invertible \( P \in M_{2d}(\mathbb{K}) \) such that if we let \((y_1, \cdots, y_{2d}) = (x_1, \cdots, x_{2d}) \cdot P \), then

\[
\frac{1}{2} \sum_{i=1}^{2d} \sum_{j=1}^{2d} a_{ij} x_i x_j = \sum_{i=1}^{d} y_{2i-1} y_{2i} = y_1 y_2 + y_3 y_4 + \cdots + y_{2d-1} y_{2d-1}.
\]

In the matrix \( M(T) \), we replace \( x_0 \) by \( T_0 I_n, y_{2i-1} \) by the skew symmetric matrix \( X(i), y_{2i} \) by \( F(i) \), \( 1 \leq i \leq d \), and any constant number \( a \in \mathbb{K} \) by \( aL_n \). In this way, we obtain a \((2d + 4)n \times (2d + 4)n\) matrix \( \tilde{M} \) over \( R[y, T_0][[T_{i_1}, \cdots, T_{i_d}](i_1, \cdots, i_d) \in S] \). Note \( \tilde{M} \) is skew symmetric. Define \( Q_y := \text{Pf} \tilde{M} \in R[y, T_0][[T_{i_1}, \cdots, T_{i_d}](i_1, \cdots, i_d) \in S] \). Then by (5.3.1) and Lemma 5.1,

\[
Q_y^2 = \det \tilde{M} = y^{2n} \det(T_0^2 I_n + \sum_{i=1}^{d} X(i) F(i) + y g_2),
\]
where $g_2$ is an $n \times n$ matrix over $R[y, T_0][[T_{i_1 \cdots i_d}^{1}] (i_1, \cdots, i_d) \in S]$. Note
$R = \oplus_{i=0}^{\infty} R_i$ is a graded $\mathbb{K}$-algebra with $R_0 = \mathbb{K}$, and the entries of $X(i)$ are
all in $R_1$. Since $g_1 \in (x_1, \cdots, x_{2d})$, we can see
\[
\det(T_0^2 I_n + \sum_{i=1}^{d} X(i) F(i) + y g_2) - T_0^{2n} \in \oplus_{i=1}^{\infty} R_i[y, T_0][[T_{i_1 \cdots i_d}^{1}] (i_1, \cdots, i_d) \in S]].
\]
Now in $R[[y, T_0, T_{i_1 \cdots i_d}^{1}] (i_1, \cdots, i_d) \in S]]$, we can apply Lemma 2.3 to the
equality (5.3.2) and obtain $\tilde{N}_y \in R[[y, T_0, T_{i_1 \cdots i_d}^{1}] (i_1, \cdots, i_d) \in S]]$ satisfying
$\tilde{Q}_y = y^n \tilde{N}_y$. Moreover, $\tilde{N}_y \in R[y, T_0][[T_{i_1 \cdots i_d}^{1}] (i_1, \cdots, i_d) \in S]]$ since $\tilde{Q}_y \in R[y, T_0][[T_{i_1 \cdots i_d}^{1}] (i_1, \cdots, i_d) \in S]].$ By (5.3.2), we see
\[
(5.3.3) \quad \tilde{N}_y^2 = \det(T_0^2 I_n + \sum_{i=1}^{d} X(i) F(i) + y g_2).
\]
Now let $\tilde{N} \in R[T_0][[T_{i_1 \cdots i_d}^{1}] (i_1, \cdots, i_d) \in S]]$ be the evaluation of $\tilde{N}_y$ at $y = 0$,
we deduce from (5.3.3) the desired equation:
\[
\tilde{N}^2 = \det(T_0^2 I_n + \sum_{i=1}^{d} X(i) F(i)) = \det(T_0^2 I_n + \sum_{(i_1, \cdots, i_d) \in \mathcal{S} S} X(1)^{i_1} X(2)^{i_2} \cdots X(d)^{i_d} T_{i_1 \cdots i_d}^{1}).
\]
\[\square\]
Now we begin to complete the proof of Lemma 3.3. Still let $R = \mathbb{K}[\mathfrak{g}^{\mathbb{C}}]$.
By Lemma 3.3, we can find $\tilde{N} \in R[T_0][[T_{i_1 \cdots i_d}^{1}] (i_1, \cdots, i_d) \in S]]$ satisfying:
\[
(5.3.4) \quad \tilde{N}^2 = \det(T_0^2 I_n + \sum_{(i_1, \cdots, i_d) \in \mathcal{S} S} X(1)^{i_1} X(2)^{i_2} \cdots X(d)^{i_d} T_{i_1 \cdots i_d}^{1}).
\]
Let $\tilde{N}(\emptyset) \in R[T_0]$ be the valuation of $\tilde{N}$ at $T_{i_1 \cdots i_d} = 0$, $(i_1, \cdots, i_d) \in S$.
Then $\tilde{N}(\emptyset)^2 = \det(T_0^2 I_n) = T_0^{2n}$. Recall that $R = \oplus_{i=0}^{\infty} R_i$ is a graded $\mathbb{K}$-
algebra with $R_0 = \mathbb{K}$. By Lemma 2.2, we get $\tilde{N}(\emptyset) = \pm T_0^n$. We assume
without loss of generality that $\tilde{N}(\emptyset) = T_0^n$.

Let $N \in R[T_{i_1 \cdots i_d}][[T_{i_1 \cdots i_d} (i_1, \cdots, i_d) \in S]]$ be the evaluation of $\tilde{N}$ at $T_0 = 1$.
Obviously the constant coefficient of $N$ is 1, and
\[
N^2 = \det(1 + \sum_{(i_1, \cdots, i_d) \in \mathcal{S} S} X(1)^{i_1} X(2)^{i_2} \cdots X(d)^{i_d} T_{i_1 \cdots i_d}^{1}) = F_{\mathfrak{g}}.
\]
By applying Lemma 2.2 to $R[[T_{i_1 \cdots i_d} (i_1, \cdots, i_d) \in S]]$, we get $\sqrt{F_{\mathfrak{g}}} = N$.

For any non-zero $\lambda \in \mathbb{K}^*$, let $\tilde{N}_\lambda \in R[T_0][[T_{i_1 \cdots i_d} (i_1, \cdots, i_d) \in S]]$ be the
image of $\tilde{N}$ under the automorphism of $R$-algebras:
\[
R[T_0][[T_{i_1 \cdots i_d} (i_1, \cdots, i_d) \in S]] \rightarrow R[T_0][[T_{i_1 \cdots i_d} (i_1, \cdots, i_d) \in S]]
\]
\[T_0 \mapsto \lambda T_0
\]
\[T_{i_1 \cdots i_d} \mapsto \lambda^2 T_{i_1 \cdots i_d}
\]
By (5.3.4), $\tilde{N}_\lambda^2 = \lambda^{2n} \tilde{N}^2 = (\lambda^n \tilde{N})^2$. Since $\tilde{N}(\Omega) = T_0^n$, we see $\tilde{N}_\lambda(\Omega) = \lambda^n \tilde{N}(\Omega) = \lambda^n T_0^n$ is not a zero-divisor in $R[T_0][[T_{1}, \ldots, i_d] \in S]$. Then $N_\lambda = \lambda^n \tilde{N}$ by Lemma 2.2. From this we deduce that, by requiring $\text{deg} \ T_0 = 1$ and $\text{deg} \ T_{1, \ldots, i_d} = 2$ for all $(i_1, \ldots, i_d) \in S$, $\tilde{N}$ is a degree $n$ weighted homogeneous formal power series with respect to the variables $\{T_0, T_{1, \ldots, i_d} | (i_1, \ldots, i_d) \in S\}$. Thus $\text{deg} \sqrt{F_{\theta}} = \text{deg} N \leq \lfloor \frac{n}{2} \rfloor$, and this completes the proof of Lemma 3.3.

Lemma 5.4. Let $R$ be a $\mathbb{K}$-algebra. Suppose $n \geq 2$, and $T = (t_{ij}) \in M_{2n}(R)$ is a skew symmetric matrix. Over the polynomial ring $R[x_1, \ldots, x_{2n-3}]$, let

$$M := T + \text{diag}(x_1, \ldots, x_{2n-3}, 0, 0, 0)$$

be the sum of $T$ and a diagonal matrix. Suppose $t_{i,2n} = 0$, for $i = 1, \ldots, 2n - 3$. Then the determinant $\det M$ has the following expansion as a polynomial in $x_1, \ldots, x_{2n-3}$:

$$\det M = (h_{2n-2,2n} t_{2n-2,2n} - h_{2n-1,2n} t_{2n-1,2n})^2$$

$$+ \sum_{1 \leq i < j \leq 2n-3} x_i x_j (h_{i,j,2n-2,2n} t_{2n-2,2n} - h_{i,j,2n-1,2n} t_{2n-1,2n})^2$$

$$+ \sum_{k=2}^{n-2} f_{2k}.$$  

Here for even $m$ and for $1 \leq i_1 < i_2 < \cdots < i_m \leq 2n$, the symbol $h_{i_1, \ldots, i_m}$ means $\text{Pf} \ T(\hat{i}_1, \ldots, \hat{i}_m)$, the Pfaffian of the matrix obtained from $T$ by deleting the $i_j$-th row and the $i_j$-th column for all $1 \leq j \leq m$, and $f_{2k}$ means a degree $2k$ homogeneous polynomial in $x_1, \ldots, x_{2n-3}$.

Proof. By expanding $\det M$ along the diagonal, we obtain

$$\det M = \det T + \sum_{m=1}^{2n-3} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq 2n-3} \det T(\hat{i}_1, \ldots, \hat{i}_m) \ x_{i_1} x_{i_2} \cdots x_{i_m}.$$  

Note when $m$ is odd, $\det T(\hat{i}_1, \ldots, \hat{i}_m) = 0$ because $T(\hat{i}_1, \ldots, \hat{i}_m)$ is a skew symmetric matrix of odd order. So the above expansion reduces to

$$(5.4.1) \quad \det M = \det T + \sum_{k=1}^{2n-4} \sum_{i_1, \ldots, i_{2k}} \det T(\hat{i}_1, \ldots, \hat{i}_{2k}) \ x_{i_1} x_{i_2} \cdots x_{i_{2k}},$$  

where the sum $\sum_{i_1, \ldots, i_{2k}}$ is over

$$\{(i_1, \ldots, i_{2k}) | 1 \leq i_1 < i_2 < \cdots < i_{2k} \leq 2n - 3\}.$$  

For a $2m \times 2m$ skew symmetric matrix $A = (a_{ij})$, the Pfaffian $\text{Pf} \ A$ can be expanded with respect to the last column (cf. Cayley [2]) as follows:

$$\text{Pf} \ A = \sum_{i=1}^{2m-1} (-1)^{i+1} \text{Pf} \ A(\hat{i}, \hat{2m}) a_{i,2m}.$$
Applying this expansion to the skew symmetric matrix $T(\hat{t}_1, \cdots, \hat{t}_{2k})$, we obtain
\[
Pf \ T(\hat{t}_1, \cdots, \hat{t}_{2k}) = -t_{2n-2,2n}Pf \ T(\hat{t}_1, \cdots, \hat{t}_{2k}, 2n-2, 2n) + t_{2n-1,2n}Pf \ T(\hat{t}_1, \cdots, \hat{t}_{2k}, 2n-1, 2n).
\]
Combining this with (5.4.1), we get the desired expansion of $\det M$.  

6. Applications

In this section, we apply the restriction isomorphism to obtain some identities about polynomial functions on commuting skew symmetric matrices.

From now on, we suppose char $K = 0$, and let $G = O_n(K)$ be the orthogonal group, so that its Lie algebra $\mathfrak{g}$ is the space of $n \times n$ skew symmetric matrices. We also fix the Cartan subalgebra
\[
t = \{SK (x_1, \cdots, x_{\lfloor \frac{n}{2} \rfloor}) \mid x_i \in K, \ \text{for all} \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}
\]
as in Section 5.1, where recall $SK (x_1, \cdots, x_{\lfloor \frac{n}{2} \rfloor})$ is the $n \times n$ skew symmetric matrix defined in (5.0.1). Recall $(X(1), \cdots, X(d)) \in C^d_K(K[d])$ is the tautological $K[C^d_K]$-valued point of $C^d_K$. Under the restriction homomorphism $M_n(K[d]) \to M_n(K[t])$, the skew symmetric matrices $X(i)$ are mapped to $Y(i)$.

Let $R$ be a $K$-algebra.

**Corollary 6.1.** Suppose $n \geq 3$ is odd and let $X_1, \cdots, X_d \in M_n(R)$ be commuting skew symmetric matrices. For any $f \in K[x_1, \cdots, x_d]$, if $f(0, \cdots, 0) = 0$, then $\det f(X_1, \cdots, X_d) = 0$.

**Proof.** Clearly there exists a $K$-algebra homomorphism $\varphi : K[C^d_K] \to R$ such that $\varphi$ maps the matrix $X(i)$ to $X_i$, $i = 1, \cdots, d$. Then $f(X_1, \cdots, X_d) = \varphi(\det f(X(1), \cdots, X(d)))$. Since $\det f(X(1), \cdots, X(d)) \in K[C^d_K]^G$, we can apply the restriction homomorphism
\[
\Phi : K[C^d_K]^G \to K[t]^W
\]
to obtain
\[
\Phi(\det f(X(1), \cdots, X(d))) = \det f(Y(1), \cdots, Y(d)).
\]
By the definition of $Y(i)$ we see $\det f(Y(1), \cdots, Y(d)) = 0$. Since $\Phi$ is an isomorphism by Theorem 3.4, we obtain $\det f(X(1), \cdots, X(d)) = 0$ and hence $\det f(X_1, \cdots, X_d) = 0$.  

**Corollary 6.2.** Suppose $n \geq 2$ is even. Let $X_1, X_2, X_3 \in M_n(R)$ be commuting skew symmetric matrices. Then
\[
Pf(X_1X_2X_3) = (-1)^{\frac{n}{2}}Pf(X_1)Pf(X_2)Pf(X_3).
\]
Proof. The case $n = 2$ is trivial, so we assume $n \geq 4$ and take $d = 3$. Let $G' = SO_d(\mathbb{K})$ be the special orthogonal group, which has the same Lie algebra as $G = O_n(\mathbb{K})$. There exists a $\mathbb{K}$-algebra homomorphism $\varphi : \mathbb{K}[C^d_0] \to R$ such that $\varphi(X(i)) = X_i, \ i = 1, 2, 3$. Then

$$\text{Pf}(X_1X_2X_3) - (-1)^{\frac{m}{2}}\text{Pf}(X_1)\text{Pf}(X_2)\text{Pf}(X_3)$$

$$= \varphi(\text{Pf}(X_1X_2X_3(2)) - (-1)^{\frac{m}{2}}\text{Pf}(X_1)\text{Pf}(X_2)\text{Pf}(X_3)) .$$

Let $r = \text{Pf}(X_1X_2X_3(2)) - (-1)^{\frac{m}{2}}\text{Pf}(X_1)\text{Pf}(X_2)\text{Pf}(X_3)$, we see $r \in \mathbb{K}[C^d_0]G'$. Then we apply the restriction homomorphism $\Phi : \mathbb{K}[C^d_0]G' \to \mathbb{K}[t^d]W'$ to obtain

$$\Phi(r) = \text{Pf}(X_1X_2X_3(2)) - (-1)^{\frac{m}{2}}\text{Pf}(X_1)\text{Pf}(X_2)\text{Pf}(X_3).$$

By direct computations, we can see $\Phi(r) = 0$. Since $\Phi$ is an isomorphism by Theorem 4.3, we finally get $r = 0$. \hfill \Box

For a positive integer $m$, write $P_m$ for the set of partitions $\lambda = \lambda_1 \cup \cdots \cup \lambda_h$ of the set $\{1, \cdots, m\}$ into the disjoint union of non-empty subsets $\lambda_i$, and denote $h(\lambda) = h$ the number of parts of the partition $\lambda$.

Corollary 6.3. Suppose $n \geq 2$, $d \geq 1$. Let $m = \lfloor \frac{d}{2} \rfloor + 1$. Suppose $X_1, \cdots, X_d \in M_n(R)$ are commuting skew symmetric matrices. For $j = 1, \cdots, m$, let $Y_j = \prod_{i=1}^d X_i^{a_{ij}} \in M_n(R)$ be a monomial of $X_1, \cdots, X_d$, with $\sum_{i=1}^d a_{ij} > 0$ even. Then

$$\sum_{\lambda \in P_m} \left( -\frac{1}{2} \right)^{h(\lambda)} \prod_{i=1}^{h(\lambda)} \binom{|\lambda_i| - 1}{1} \cdot \text{tr} \prod_{s \in \lambda_i} Y_s = 0.$$

Proof. In a similar way as above, it suffices to verify the identity under the assumption $R = \mathbb{K}[t^d]$ and $X_i = Y(i), \ i = 1, \cdots, d$. Then the required trace identity is just a reformulation of Domokos [6, Proposition 2.3]. \hfill \Box

Example 6.4. As an illustration, we take $n = 4$, so $m = 3$. All of the partitions of $\{1, 2, 3\}$ are:

$$\{1, 2, 3\}, \ \{1, 2\} \cup \{3\}, \ \{1, 3\} \cup \{2\}, \ \{2, 3\} \cup \{1\}, \ \{1\} \cup \{2\} \cup \{3\}.$$

Then according to Corollary 6.3, for any $d \geq 1$ commuting skew symmetric $4 \times 4$ matrices $X_1, \cdots, X_d \in M_4(R)$, and for $Y_1, Y_2, Y_3$ which are monomials of $X_i$ of even degree, we have the following trace identity:

$$2\text{tr}(Y_1)\text{tr}(Y_2Y_3) + 2\text{tr}(Y_2)\text{tr}(Y_1Y_3) + 2\text{tr}(Y_3)\text{tr}(Y_1Y_2)$$

$$= 8\text{tr}(Y_1Y_2Y_3) + \text{tr}(Y_1)\text{tr}(Y_2)\text{tr}(Y_3).$$

To be more specific, let $Y_i = X_i^2, \ i = 1, 2, 3$, then this identity reduces to:

$$2\text{tr}(X_1^2)\text{tr}(X_2^2X_3^2) + 2\text{tr}(X_2^2)\text{tr}(X_1^2X_3^2) + 2\text{tr}(X_3^2)\text{tr}(X_1^2X_2^2)$$

$$= 8\text{tr}(X_1^2X_2^2X_3^2) + \text{tr}(X_1^2)\text{tr}(X_2^2)\text{tr}(X_3^2).$$
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School of Mathematics, SUN YAT-sen University, No. 135 XINGANG XI ROAD, GUANGZHOU, GUANGDONG 510275, P. R. CHINA
Email address: songlei3@mail.sysu.edu.cn

School of Mathematical Sciences, University of Science and Technology of China, No. 96 JINZhai ROAD, HEFEI, ANHUI 230026, P. R. CHINA
Email address: xpxia@mail.ustc.edu.cn

School of Mathematical Sciences, University of Science and Technology of China, No. 96 JINZhai ROAD, HEFEI, ANHUI 230026, P. R. CHINA
Email address: xujx02@ustc.edu.cn