Abstract. We prove Beurling’s theorem for the full group $SL(2, \mathbb{R})$. This is the master theorem in the quantitative uncertainty principle as all the other theorems of this genre follow from it.

1. Introduction

Our starting point is the following theorem of Hörmander (\cite{Hor}):

**Theorem 1.1.** (Hörmander 1991) Let $f \in L^1(\mathbb{R})$. If

$$
\int_\mathbb{R} \int_\mathbb{R} |f(x)\bar{f}(y)|e^{ixy} dx dy < \infty
$$

where $\bar{f}(\lambda) = \int_\mathbb{R} f(t)e^{-i\lambda t} dt$, then $f = 0$ almost everywhere.

Hörmander attributes this theorem to A. Beurling. We will follow his practice and call theorem 1.1 as Beurling’s theorem. This theorem is an instance of the quantitative uncertainty principle (QUP) (see \cite{Hor}). We recall that the uncertainty principle is the paradigm in harmonic analysis which says that a function and its Fourier transform cannot both be very rapidly decreasing at infinity. Some other well known QUP’s like Cowling-Price theorem, Gelfand-Shilov theorem, Morgan’s theorem and Hardy’s theorem (see section 4 for the precise statements) become corollaries of this theorem. Thus Beurling’s theorem can be regarded as the “Master Theorem” in the context of the uncertainty principle.

In recent years mathematicians have taken up the QUP problems on semisimple Lie groups and on Riemannian symmetric spaces and proved versions of Hardy’s, Cowling-Price, Gelfand-Shilov and Morgan’s theorems (see \cite{C, G, H, S, H1, H2} and the references therein). However, Beurling’s theorem has not yet been proved for any semisimple Lie group. The aim of this article is to prove Beurling’s theorem for the full group $SL(2, \mathbb{R})$. The other four theorems mentioned above will follow from it.

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Our principal tool is the Abel transform. This is a well-known device for handling $K$-biinvariant functions on $G$. In this article we extend its scope to bi $K$-finite functions and use this in a crucial way to obtain our result (see Theorem 3.1) where no $K$-finiteness restriction is imposed.

It may be noted that this is the first instance when the discrete series representations appear in the hypothesis in a theorem of uncertainty. See Remark 4.1 for further details.

The plan of the article is as follows. In the next section we record some preliminary material and set up the necessary notation. In section 3 we give the statement and proof of our main result. In section 4 we indicate how the other QUP’s viz Hardy’s theorem, Morgan’s theorem etc. follow from our main result.

2. Notation and Preliminaries

The letter $C$ will denote a positive constant, not necessarily the same at each occurrence. We will mainly use the notation of [1] with a few exceptions which we will mention here. For ready reference we will also quote from [1] the things which we will frequently use. Let $G = SL(2, \mathbb{R})$. Let $k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ and $n_\xi = \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix}$.

Then $K = \{k_\theta | \theta \in [0, 2\pi)\}$, $A = \{a_t | t \in \mathbb{R}\}$ and $N = \{n_\xi | \xi \in \mathbb{R}\}$ are three particular subgroups of $G$ of which $K$ is a maximal compact subgroup $SO(2)$ of $G$. It is clear from the above that both $A$ and $N$ can be identified with $\mathbb{R}$. Let $G = KAN$ be an Iwasawa decomposition of $G$ and for $x \in G$, let $x = k_\theta a_t n_\xi$ be its corresponding decomposition. We will write $H(x)$ for $t$ and $K(x)$ for $k_\theta$. Clearly $H$ is left $K$-invariant and right $N$-invariant. The Haar measure $dx$ of $G$ splits according to this decomposition as $dx = e^{2t} dk dt dn$ where $dk = dk_\theta = \frac{d\theta}{2\pi}$ is the normalised Haar measure of $K$ and $dn = dn_\xi = d\xi$ as well as $da = da_t = dt$ are both Lebesgue measures on $\mathbb{R}$.

We also recall that $G$ has Cartan decomposition $G = K \overline{A} K$, $x = k_1 a_t k_2$ where $k_1, k_2 \in K, t \geq 0$. The Haar measure of $G$ splits according to this decomposition as $dx = dk_1 \sinh 2t dt dk_2$. Let $\sigma(x) = \sigma(k_1 a_t k_2) = |t|$. In fact $\sigma(x) = d(xK, o)$, where $o = eK$ is the ‘origin’ of the symmetric space $G/K$ and $d$ is the distance function on $G/K$. 
Let $\hat{K} = \{ e_n | n \in \mathbb{Z} \}$ be the set of continuous characters of $K$, where $e_n(k) = e^{i n \theta}$. Instead of $e_n$, by abuse of language, we will call the integers $n$ as $K$-types. A complex valued function $f$ on $G$ is said to be of left (respectively right) $K$-type $n$ if $f(kx) = e_n(k)f(x)$ (respectively $f(xk) = e_n(k)f(x)$) for all $k \in K$ and $x \in G$. A function is of type $(m,n)$ if its left $K$-type is $m$ and right $K$-type is $n$. For a suitable function $f$, the $(m,n)$-th isotypical component of $f$ is denoted by $f_{m,n}$ and this is given by:

$$\int_K \int_K e_m(k_1) e_n(k_2) f(k_1 x k_2) dk_1 dk_2 = \int_K \int_K e^{-i m \theta} e^{-i n \theta} f(k_0 x k_0) dk_0 dk_0.$$  

(2.1)

It can be verified that $f_{m,n}$ is itself a function of type $(m,n)$ and $f_{m,n} \equiv 0$ when $m$ and $n$ are of opposite parity. The function $f$ can be decomposed as $f = \sum_{m,n \in \mathbb{Z}} f_{m,n}$. In fact when $f \in C^\infty(G)$ this is an absolutely convergent series in the $C^\infty$-topology. When $f \in L^p(G)$, $p \in [1, \infty)$, the equality is in the sense of distribution.

Let $\mathfrak{a}$ be the Lie algebra of $A$. Let $\mathfrak{a}^\ast$ be the real dual of $\mathfrak{a}$ and $\mathfrak{a}_\mathbb{C}^\ast$ be the complexification of $\mathfrak{a}^\ast$. Then $\mathfrak{a}^\ast$ and $\mathfrak{a}_\mathbb{C}^\ast$ can be identified with $\mathbb{R}$ and $\mathbb{C}$ respectively via $\rho$, the half-sum of the positive roots, i.e. $\rho = 1$ under this identification. Let $M$ be $\{ \pm I \}$, where $I$ is the $2 \times 2$ identity matrix. The unitary dual of $M$ is $\hat{M} = \{ \sigma_+, \sigma_- \}$ of which $\sigma_+$ is the trivial representation of $M$ and $\sigma_-$ is the only nontrivial unitary irreducible representation of $M$. Let $\mathbb{Z}^{\sigma_+}$ (respectively $\mathbb{Z}^{\sigma_-}$) be the set of even (respectively odd) integers.

For $\sigma \in \hat{M}$ and $\lambda \in \mathfrak{a}_\mathbb{C}^\ast = \mathbb{C}$, let $(\pi_{\sigma,\lambda}, H_\sigma)$ be the principal series representation of $G$ where $H_\sigma$ is the subspace of $L^2(K)$ generated by the orthonormal set $\{ e_n | n \in \mathbb{Z}^\sigma \}$. The representation $\pi_{\sigma,\lambda}$ is normalized so that it is unitary if and only if $\lambda \in i \mathfrak{a}^\ast = i \mathbb{R}$. In fact (\cite[4.1]{P}):  

$$\langle \pi_{\sigma,\lambda}(x) e_n \rangle(k) = e^{-(\lambda+1)H(x^{-1}k^{-1})} e_{-n}(K(x^{-1}k^{-1})).$$  

(2.2)

For every $k \in \mathbb{Z}^\ast$, the set of nonzero integers, there is a discrete series representation $\pi_k$ which occurs as a subrepresentation of $\pi_{\sigma,|k|}$ so that $k \in \mathbb{Z} \setminus \mathbb{Z}^\sigma$. For $m, n \in \mathbb{Z}^\sigma$ and $k \in \mathbb{Z} \setminus \mathbb{Z}^\sigma$, let $\Phi_{\sigma,\lambda}^{m,n}(x) = \langle \pi_{\sigma,\lambda}(x) e_m, e_n \rangle$ and $\Psi_{k}^{m,n}(x) = \langle \pi_k(x) e_m^k, e_n^k \rangle_k$, be the matrix coefficients of the principal series and discrete series representations respectively, where $\{ e_n^k \}$ are the renormalised basis and $\langle \ , \ \rangle_k$ is the renormalised inner product of $\pi_k$ (see \cite[p. 20]{P}). In particular $\Phi_{\sigma_+,\lambda}^{0,0}$ is clearly the elementary spherical
function, which we also denote by \( \phi_\lambda \). For details of the parametrization of the representations \( \pi_{\sigma, \lambda} \) and \( \pi_k \) and their realizations we refer to [1].

For a function \( f \in L^1(G) \), let \( \hat{f}(\sigma, \lambda) \) and \( \hat{f}(k) \) denote its (operator valued) principal and discrete Fourier transforms at \( \pi_{\sigma, \lambda} \) and \( \pi_k \) respectively. Precisely:

\[
\hat{f}(\sigma, \lambda) = \int_G f(x) \pi_{\sigma, \lambda}(x^{-1}) dx \quad \text{and} \quad \hat{f}(k) = \int_G f(x) \pi_k(x^{-1}) dx.
\]

The \((m, n)\)-th matrix entries of \( \hat{f}(\sigma, \lambda) \) and \( \hat{f}(k) \) are denoted by \( \hat{f}(\sigma, \lambda)_{m,n} \) and \( \hat{f}(k)_{m,n} \) respectively. Thus

\[
\hat{f}(\sigma, \lambda)_{m,n} = \langle \hat{f}(\sigma, \lambda) e_m, e_n \rangle = \int_G f(x) \Phi_{\sigma, \lambda}^{m,n}(x^{-1}) dx \quad \text{and} \quad \hat{f}(k)_{m,n} = \int_G f(x) \Psi_k^{m,n}(x^{-1}) dx.
\]

As

\[
\int_G f(x) \Phi_{\sigma, \lambda}^{m,n}(x^{-1}) dx = \int_{m,n} f_{m,n}(x) \Phi_{\sigma, \lambda}^{m,n}(x^{-1}) dx,
\]

clearly, \( \hat{f}(\sigma, \lambda)_{m,n} = \hat{f}_{m,n}(\sigma, \lambda) \). Similarly \( \hat{f}(k)_{m,n} = \hat{f}_{m,n}(k) \). Henceforth we will not distinguish between \( \hat{f}(\sigma, \lambda)_{m,n} \) (respectively \( \hat{f}(k)_{m,n} \)) and \( \hat{f}_{m,n}(\sigma, \lambda) \) (respectively \( \hat{f}_{m,n}(k) \)). Notice that integers \( m, n \) of the same parity uniquely determine a \( \sigma \in \hat{M} \) by \( m, n \in \mathbb{Z}_\sigma \). Therefore we may sometimes omit the obvious \( \sigma \) and write \( \Phi_{\lambda}^{m,n} \) for \( \Phi_{\sigma, \lambda}^{m,n} \) and \( \hat{f}_{m,n}(\lambda) \) for \( \hat{f}_{m,n}(\sigma, \lambda) \).

From [2] it follows that for \( \lambda \in \mathbb{C}, \sigma \in \hat{M} \) and \( m, n \in \mathbb{Z}_\sigma \),

\[
\Phi_{\sigma, \lambda}^{m,n}(x) = \int_K e^{-(\lambda+1)H(x^{-1}k^{-1})} e^{m(K(x^{-1}k^{-1}))} e^{n(k^{-1})} dk.
\]

Hence, \( |\Phi_{\lambda}^{m,n}(x)| \leq \int_K e^{-(\Re \lambda+1)H(x^{-1}k^{-1})} dk = \Phi_{\Re \lambda}^{0,0} = \phi_{\Re \lambda} \), where \( \Re \lambda \) stands for the real part of \( \lambda \). It is well known ([5]) that \( |\phi_\lambda(x)| \leq 1 \) for \( x \in G \) and \( \lambda \in \mathbb{C} \) with \( |\Re \lambda| \leq 1 \). Combining this with the following two estimates ([3] proposition 4.6.4 and [1] 3.2):

\[
|\phi_\lambda(x)| \leq e^{|\Re \lambda|\sigma(x)} \Xi(x) \quad \text{for} \quad \lambda \in \mathbb{C} \text{ and } \Xi(x) \leq C(1 + \sigma(x)) e^{-\sigma(x)}
\]

where \( \Xi = \Phi_{0,0}^{0,0} = \phi_0 \), we get \( |\Phi_{\lambda}^{m,n}(x)| \leq C e^{|\Re \lambda|\sigma(x)} (1 + \sigma(x)) e^{-\sigma(x)} \leq C e^{|\Re \lambda|\sigma(x)}, \forall \lambda \in \mathbb{C} \).

It follows from the estimates above and Morera’s theorem that \( \hat{f}_{m,n}(\lambda) \) is a holomorphic function in the Helgason-Johnson strip, \( \{ \lambda \in \mathbb{C} \mid |\Re \lambda| < 1 \} \). In particular the restriction of \( \hat{f}_{m,n} \) to the imaginary axis is a (complex valued) real analytic function and hence its zeros form a set of Lebesgue measure zero.
We conclude this section with a brief discussion of Plancherel measure (see [7]). This measure on the unitary principal series representations (parametrized by $i\mathbb{R}$) is

$$d\mu(\sigma, \lambda) = \mu(\sigma, \lambda)d\lambda$$

(2.5) \(\mu(\sigma_+, i\xi) = (\frac{\xi}{2\pi}) \tanh(\frac{\xi}{2\pi})\) and \(\mu(\sigma_-, i\xi) = (\frac{\xi}{2\pi}) \coth(\frac{\xi}{2\pi})\) for \(\xi \in \mathbb{R}\).

Here again we may omit \(\sigma\) and write \(\mu(\lambda)\) for \(\mu(\sigma, \lambda)\), when there is no confusion about \(\sigma \in \hat{M}\). The Plancherel measure on the discrete series is given by \(\mu(\pi_k) = \frac{|k|}{2\pi}\), for \(k \in \mathbb{Z}^*\).

3. Statement and Proof of the main theorem

In the proofs of the theorems, Lemmas etc. we will use Fubini’s theorem freely without explicitly mentioning it every time.

**Theorem 3.1.** Let \(f \in L^2(G)\). If

$$\int_G \int_{i\mathbb{R}} |f(x)||\hat{f}(\sigma, \lambda)||2\phi_{|\lambda|}(x)dxd\mu(\sigma, \lambda) < \infty,$$

for all \(\sigma \in \hat{M}\) and

$$\sum_{k \in \mathbb{Z}^*} \frac{|k|}{2\pi} \int_G |f(x)||\hat{f}(k)||2\phi_{|\lambda|}(x)dx < \infty,$$

then \(f = 0\) almost everywhere. Here \(\| \cdot \|_2\) is the Hilbert-Schmidt norm.

As \(\phi_{|\lambda|}(x) \leq e^{\lambda|\sigma(x)|}\Xi(x)\) (see [2.4]), we have the following immediate corollary:

**Corollary 3.2.** Let \(f \in L^2(G)\). Suppose

$$\int_G \int_{i\mathbb{R}} |f(x)||\hat{f}(\sigma, \lambda)||2e^{\lambda|\sigma(x)|}\Xi(x)dxd\mu(\sigma, \lambda) < \infty.$$

for all \(\sigma \in \hat{M}\) and

$$\sum_{k \in \mathbb{Z}^*} \frac{|k|}{2\pi} \int_G |f(x)||\hat{f}(k)||2e^{k|\sigma(x)|}\Xi(x)dx < \infty.$$

Then \(f = 0\) almost everywhere.

**Remark 3.3.** Note that a naive analogue of Beurling’s theorem would use the weight \(e^{\lambda|\sigma(x)|}\). Instead here we use \(\phi_{|\lambda|}(x)\) which has less decay (since \(\phi_{|\lambda|}(x) \leq e^{\lambda|\sigma(x)|}\Xi(x)\) and \(\Xi(x) \leq 1\)) and hence obtain
a stronger result. We also feel that the formulation of the theorem is natural as $\phi_\lambda(x)$ plays the role of $e^{i\lambda x}$, at least for the $K$-biinvariant functions on $G$.

The basic strategy of our proof of theorem 3.1 is to reduce the theorem to the Euclidean situation by using the Abel transform. We shall therefore begin with a short discussion on the Abel transform. For $f \in L^1(G)$ of type $(m, n)$, we define the Abel transform

$$Af(t) = e^t \int_N f(a_t n) dn.$$ 

Therefore

$$|Af(t)| \leq e^t \int_N |f(a_t n)| dn = A|f|(t)$$

(3.5)

Since $|f|$ is integrable and $K$-biinvariant we have $Af \in L^1(\mathbb{R})$. Furthermore $Af$ is an even function of $t$ (vide [4]). Therefore $Af \in L^1(\mathbb{R})$. We need the following lemma:

**Lemma 3.4.** Let $\sigma \in \hat{M}$ and let $f \in L^1(G)_{m,n}$ for some $m, n \in \mathbb{Z}^\sigma$. Then $\widehat{f}_{m,n}(\sigma, \lambda) = \widetilde{A}f(-i\lambda)$ for $\lambda \in i\mathbb{R}$. Here $\widetilde{A}f(\nu) = \int_{\mathbb{R}} Af(t)e^{-i\nu t} dt$.

**Proof.** For reason mentioned in section 2 we will omit $\sigma$ and write $\Phi_{\lambda}^{m,n}$ for $\Phi_{\sigma,\lambda}^{m,n}$ and $\widehat{f}(\lambda)$ for $\widehat{f}_{m,n}(\sigma, \lambda)$.

From (2.3) we have for $\lambda \in i\mathbb{R}$,

$$\widehat{f}_{m,n}(\lambda) = \int_G f(x) \int_K e^{-(\lambda+1)H(xk^{-1})} e_{-m}(K(xk^{-1})) e_n(k^{-1}) dk dx$$

$$= \int_K \int_G f(x) e^{-(\lambda+1)H(xk^{-1})} e_{-m}(K(xk^{-1})) e_n(k^{-1}) dx dk.$$

Substituting $k^{-1}yk$ for $x$, we get,

$$\widehat{f}_{m,n}(\lambda) = \int_K \int_G f(k^{-1}yk)e^{-(\lambda+1)H(k^{-1}y)} e_{-m}(K(k^{-1}y)) e_n(k^{-1}) dy dk$$

as the Haar measure of $G$ is invariant under this substitution.

Since $f(k^{-1}yk) = f(y)e_{m}(k^{-1}) e_{-m}(K(k^{-1}y)) = e_{m}(k)e_{-m}(K(y))$ and $H(k^{-1}y) = H(y)$, we have

$$\widehat{f}_{m,n}(\lambda) = \int_G f(y)e^{-(\lambda+1)H(y)} e_{m}(K(y)) dy.$$
Using the Iwasawa decomposition $G = KAN$ and the identification of $A$ and $\mathbb{R}$, we obtain,

$$
\hat{f}_{m,n}(\lambda) = \int_K \int_{AN} f(katn) e^{-\langle \lambda + 1 \rangle t} e^{-m(k)} dk e^{2t} dt dn
$$

$$= \int_{\mathbb{R}} \int_{AN} f(atn) e^{-\langle \lambda + 1 \rangle t} e^{2t} dt dn
$$

$$= \int_{\mathbb{R}} e^t \int_{AN} f(atn) dne^{-\lambda t} dt
$$

$$= \int_{\mathbb{R}} A(f)(t)e^{-i(-i\lambda)t} dt.
$$

Thus $\hat{f}_{m,n}(\lambda) = \tilde{A}f(-i\lambda)$. □

Note that the lemma above is valid, for any $\lambda \in \mathbb{C}$ for which both sides of the equality are well-defined.

Looking back at theorem 1.1 we see that it can be rewritten as: For $g \in L^1(\mathbb{R})$, if

$$
(3.6) \quad \int_{\mathbb{R}} M(g)(\lambda) |g(\lambda)| d\lambda < \infty,
$$

where $M(g)(\lambda) = \int_{\mathbb{R}} |g(x)| e^{\langle \lambda \rangle |x|} dx$ then $g = 0$ almost everywhere.

With this preparation we are now ready to prove theorem 3.1

Proof. We shall divide the proof in a few steps for convenience. Before proving Step 1, let us note that if $\hat{f}(\sigma, \cdot) \equiv 0$ on $i\mathbb{R}$, for all $\sigma \in \hat{M}$, then the Fourier transform of $f$ (hence of $f_{m,n}$) is supported on the discrete series representations. In this case we can directly go to Step 4.

**Step 1:** Let us fix a $\sigma \in \hat{M}$ such that $\hat{f}(\sigma, \cdot) \neq 0$ on $i\mathbb{R}$. (If there is no such $\sigma$ then we go to Step 4 as mentioned above.) In this step we shall show that $f \in L^1(\mathbb{R})$ and for any $m, n \in \mathbb{Z}^\sigma$, $\hat{f}(\sigma, \lambda)_{m,n}$ can be defined for any $\lambda \in \mathbb{C}$. We will use the following asymptotic behaviour of $\phi_\lambda$ (see [5, p. 447]):

$$
(3.7) \quad \lim_{t \to \infty} e^{(-|\lambda|+1)t} \phi_\lambda(a_t) = c(|\lambda|) \quad \text{for} \quad \lambda \neq 0
$$

where $c(\lambda) = \frac{\Gamma(\lambda/2)}{\Gamma(1+\lambda/2)}$ ([1, p. 24]) is the Harish-Chandra c-function for $G$.

Let $B \subset G$ be a large compact set containing the identity. Since $f \in L^2(G)$, $f$ is a locally integrable function on $G$ and hence $\int_B |f(x)| dx < \infty$ and $\int_B |f(x)| |\Phi^{m,n}_{\sigma,\lambda}(x)| dx < \infty$.

We claim that $\hat{f}(\sigma, \cdot)$ cannot be supported on a set of finite measure in $i\mathbb{R}$. Suppose $\hat{f}(\sigma, \cdot)$ is supported on a set of finite measure. Then for any $m, n \in \mathbb{Z}^\sigma$ $\hat{f}(\sigma, \lambda)_{m,n}$ is also supported on a set of finite measure. Now as $\hat{f}(\sigma, \cdot) \neq 0$, there exists $\lambda_0 \neq 0$ such that $\hat{f}(\sigma, \lambda_0) \neq 0$ and from (3.1)
\[ \int_G |f(x)| \phi(x) dx < \infty. \] Suppose \(|\lambda_0| = r > 0\). Using (3.7) we have \(\int_{G \setminus B} |f(x)| e^{(r-1)\sigma(x)} dx < \infty\). Again using the fact that \(f\) is locally integrable, we see that \(\int_B |f(x)| e^{(r-1)\sigma(x)} dx < \infty\), for \(r\) as above.

Together they imply, \(\int_G |f(x)| e^{(r-1)\sigma(x)} dx < \infty\). Using the Cartan decomposition \(G = K \mathbb{A}^+ K\), we can rewrite it as

\[
\int_K \int_0^\infty \int_K |f(k_1a_kk_2)| e^{(r-1)t} \sinh 2t dk_1 dt dk_2 < \infty.
\]

We have \(\hat{f}_{m,n}(\sigma, \lambda) = \int_G f(x) \Phi_{\lambda}^{m,n}(x^{-1}) dx\). Then

\[
|\int_G f(x) \Phi_{\lambda}^{m,n}(x^{-1}) dx| \leq \int_G |f(x)| |\Phi_{\lambda}^{m,n}(x^{-1})| dx
\]

\[
\leq \int_{G} |f(x)| e^{||\lambda|| \sigma(x)} \Xi(x) dx.
\]

\[
\leq \int_K \int_0^\infty \int_K |f(k_1a_kk_2)| e^{||\lambda|| - 1} \sinh 2t dk_1 dt dk_2.
\]

In the last two steps we have used the estimates of \(\Phi_{\lambda}^{m,n}(x)\) and \(\Xi(x)\) (see (2.4)) and the Cartan decomposition. Thus we have for \(0 < r' < r\) and for \(\lambda \in \mathbb{C}\) with \(|\Re \lambda| < r'\),

\[
|\int_G f(x) \Phi_{\lambda}^{m,n}(x^{-1}) dx| \leq \int_K \int_0^\infty \int_K |f(k_1a_kk_2)| e^{(r-1)t} e^{||\lambda|| - r'} e^{(r'-r)t} \sinh 2t dk_1 dt dk_2.
\]

Hence by (3.8), \(|\int_G f(x) \Phi_{\lambda}^{m,n}(x^{-1}) dx| < \infty\) for any \(\lambda \in \mathbb{C}\) with \(|\Re \lambda| < r'\).

By a standard use of Morera’s theorem it follows that \(\hat{f}_{m,n}\) is analytic in the open strip \(|\Re \lambda| < r'\) in \(a^*_C = \mathbb{C}\). This contradicts the assumption that \(\hat{f}(\sigma, \cdot)\) and hence \(\hat{f}_{m,n}(\sigma, \cdot)\) is supported on a set of finite measure. Thus our claim is established, i.e. \(\hat{f}(\sigma, \cdot)\) is supported on a set of infinite measure.

Now as \(\hat{f}(\sigma, \cdot)\) is supported on a set of infinite measure, from (3.1) and (3.7), it follows that for any large \(M > 0\) there exists \(\lambda \in i\mathbb{R}, |\lambda| > M\) such that \(\int_{G \setminus B} |f(x)| e^{(l|\lambda| - 1)\sigma(x)} dx < \infty\). This implies that \(\int_{G \setminus B} |f(x)| dx < \infty\). Since \(\int_B |f(x)| dx < \infty\) we immediately see that \(f \in L^1(G)\).

Now given any \(\lambda' \in \mathbb{C}\) with \(|\lambda'| = l\) say, we choose \(M = l\) in the above. Then

\[
\int_{G \setminus B} |f(x)| e^{(l|\lambda| - 1)\sigma(x)} dx < \infty \text{ for some } \lambda \text{ with } |\lambda| > l. \] 

Since (see section 2)

\[
|\Phi_{\sigma, \lambda'}^{m,n}(x)| \leq \phi_{\Re \lambda'}(x) \leq e^{||\Re \lambda'|| \sigma(x)} \Xi(x) \leq e^{(||\Re \lambda'|| - 1)\sigma(x)} (1 + \sigma(x)), \]

we see that \(\int_{G \setminus B} |f(x)| |\Phi_{\sigma, \lambda'}^{m,n}(x)| dx < \infty\). Combining this with the fact \(\int_B |f(x)| |\Phi_{\sigma, \lambda'}^{m,n}(x)| dx < \infty\), we have \(\int_G |f(x)| |\Phi_{\lambda, \lambda'}^{m,n}(x)| dx < \infty\). Thus we have established that the Fourier transform \(\hat{f}(\sigma, \lambda)_{m,n}\) of \(f\)
exists for every $\lambda \in \mathbb{C}$. Notice that we have actually established that the function $|f|$ has enough decay so that for any $m, n \in \mathbb{Z}^\sigma$, $|f|^-(\sigma, \lambda)_{m,n}$ exists for any $\lambda \in \mathbb{C}$.

**Step 2:** We have $f = \sum_{m,n\in\mathbb{Z}} f_{m,n}$ in the sense of distributions on $G$ (see section 2). Note that for each $m, n \in \mathbb{Z}$, $f_{m,n}$ is in $L^1(G) \cap L^2(G)$, since $f \in L^1(G) \cap L^2(G)$. Let us fix a $\sigma \in \mathbb{M}$ and take two arbitrary $m, n \in \mathbb{Z}^\sigma$. As $|f_{m,n}(x)| \leq \int_{K \times K} |f(k_1 x k_2)| dk_1 dk_2$, $\phi_{|\lambda|}(x)$ is $K$-bi-invariant and the Haar measure $dx$ is invariant under the transformation $x \mapsto k_1 x k_2$, we can substitute $f$ by $f_{m,n}$ in (3.1) and in (3.2). Also in (3.1) (respectively (3.2)) we can substitute $\|\hat{f}(\sigma, \lambda)\|_2$ by $|\hat{f}_{m,n}(\lambda)|$ (respectively $\|\hat{f}(k)\|_2$ by $|\hat{f}_{m,n}(k)|$) as $\|\hat{f}(\sigma, \lambda)\|_2^2 = \sum_{m,n} |\hat{f}_{m,n}(\sigma, \lambda)|^2$ (respectively $\|\hat{f}(k)\|_2^2 = \sum_{m,n} |\hat{f}_{m,n}(k)|^2$).

Thus we get

$$\int_G \int_{\mathbb{R}} |f_{m,n}(x)||\hat{f}_{m,n}(\sigma, \lambda)| \phi_{|\lambda|}(x) dxd\mu(\sigma, \lambda) < \infty,$$

(3.9)

$$\sum_{k \in \mathbb{Z}^*} |k| \int_G |f_{m,n}(x)||\hat{f}_{m,n}(k)| \phi_{|k|}(x) dx < \infty.$$  

Starting from (3.9) if we can show that $f_{m,n} = 0$ then we are done in view of the decomposition of $f$ in $f_{m,n}$. So, we can confine ourselves to the set of functions of type $(m, n)$ for some $m, n \in \mathbb{Z}$ of the same parity.

In order to avoid complicated notation we will simply write $\mathfrak{f}$ for $f_{m,n}$. Also by $\hat{\mathfrak{f}}(\lambda)$ (respectively $\hat{\mathfrak{f}}(k)$) we will mean $\hat{f}(\lambda)_{m,n}$ (respectively $\hat{f}(k)_{m,n}$). Notice that we have omitted $\sigma$ and have written $\hat{\mathfrak{f}}(\lambda)$ for $\hat{\mathfrak{f}}(\sigma, \lambda)$ as the $\sigma \in \mathbb{M}$ is fixed when $m, n$ are fixed by $m, n \in \mathbb{Z}^\sigma$. For the same reason we will write $\mu(\lambda)$ for $\mu(\sigma, \lambda)$. So we rewrite the inequalities (3.9) as:

$$\int_G \int_{\mathbb{R}} |\mathfrak{f}(x)||\hat{\mathfrak{f}}(\lambda)| \phi_{|\lambda|}(x) dxd\mu(\lambda) < \infty,$$

(3.10)

$$\sum_{k \in \mathbb{Z}^*} |k| \int_G |\mathfrak{f}(x)||\hat{\mathfrak{f}}(k)| \phi_{|k|}(x) dx < \infty,$$

where $\mathfrak{f} \in L^1(G) \cap L^2(G)$ and $\mathfrak{f}$ is of type $(m, n)$.

As $|\mathfrak{f}^-|(|\lambda|) = \int_G |\mathfrak{f}(x)||\hat{\mathfrak{f}}(\lambda)| d\mu(\lambda)$, the inequality above is equivalent to

$$\int_{\mathbb{R}} |\mathfrak{f}^-|(|\lambda|) |\hat{\mathfrak{f}}(\lambda)| d\mu(\lambda) = \int_{\mathbb{R}} |\mathfrak{f}^-|(|\lambda|) \hat{\mathfrak{f}}(i\lambda) |d\mu(i\lambda) < \infty.$$  

(Note that $|\mathfrak{f}^-|(|\lambda|)$ exists for $\lambda \in i\mathbb{R}$ for reasons mentioned in Step 1.)

Using lemma 3.4 we get

$$\int_{\mathbb{R}} \mathcal{A}|\mathfrak{f}^-|(-i|\lambda|) |\mathcal{A}\hat{\mathfrak{f}}(\lambda)| d\mu(i\lambda) < \infty.$$
Recall that \( f \) being a \( K \)-finite function of type \((m, n)\), \(|f|\) is \( K \)-bi-invariant and hence \( \mathcal{A}|f| \) is an even function. Therefore from above we have

\[
(3.11) \quad \int_\mathbb{R} \mathcal{A}(|f|)(i|\lambda|) |\hat{\mathcal{A}}f(\lambda)| d\mu(i\lambda) < \infty.
\]

Now \( \mathcal{A}(|f|)(i|\lambda|) = \int_\mathbb{R} \mathcal{A}|f|(t) e^{i|\lambda|t} dt \geq \int_0^\infty \mathcal{A}|f|(t) e^{i|\lambda|t} dt = \frac{1}{2} M(|f|)(\lambda) \), since \( |f| \) is an even function. Here \( M(\mathcal{A}|f|)(\lambda) \) is as defined in (3.5).

So from (3.11) we have

\[
\int_\mathbb{R} M(\mathcal{A}|f|)(\lambda) |\hat{\mathcal{A}}f(\lambda)| d\mu(i\lambda) < \infty.
\]

As \( |\mathcal{A}f(t)| \leq \mathcal{A}|f|(t) \) for all \( t \in \mathbb{R} \) (see (3.5)), we have \( M(\mathcal{A}f)(\lambda) \leq M(\mathcal{A}|f|)(\lambda) \) since \( e^{i|\lambda|t} \) is positive.

Therefore

\[
(3.12) \quad \int_\mathbb{R} M(\mathcal{A}f)(\lambda) |\hat{\mathcal{A}}f(\lambda)| d\mu(i\lambda) < \infty.
\]

**Step 3** In this step we will first show that in (3.12) the Plancherel measure \( d\mu(i\lambda) \) can be substituted by the Lebesgue measure \( d\lambda \) and then conclude that \( \hat{f}(\lambda) = 0 \) for all \( \lambda \in i\mathbb{R} \).

We have \( \int_\mathbb{R} M(\mathcal{A}f)(\lambda) |\hat{\mathcal{A}}f(\lambda)| \mu(i\lambda) d\lambda < \infty \) since \( d\mu(i\lambda) = \mu(i\lambda) d\lambda \) where \( \mu(i\lambda) \) is as in (3.5). Therefore \( M(\mathcal{A}f)(\lambda) |\hat{\mathcal{A}}f(\lambda)| \mu(i\lambda) \) is finite for almost every \( \lambda \in \mathbb{R} \) with respect to the Lebesgue measure. Now \( \mu(i\lambda) \) and \( \hat{f}(i\lambda) = \hat{\mathcal{A}}f(\lambda) \) are real analytic functions for \( \lambda \in \mathbb{R} \). Hence \( M(\mathcal{A}f)(\lambda) \) is finite almost everywhere. Now by its very definition \( M(\mathcal{A}f)(\lambda) \) is even in \( \lambda \) and an increasing function of \(|\lambda|\). Consequently \( M(\mathcal{A}f)(\lambda) \) is finite everywhere and locally integrable, because for any \( R' > 0 \) we have \( \int_{|\lambda| \leq R} M(\mathcal{A}f)(\lambda) d\lambda \leq 2R' M(R') < \infty \). From (3.5) we see that there exists \( R > 0 \) such that \( \mu(i\lambda) \) is \( \geq \frac{1}{2} \) say for all \( \lambda \in \mathbb{R}, |\lambda| \geq R \) because \( \mu(i\lambda) \to \infty \) as \( |\lambda| \to \infty \). The Euclidean Riemann-Lebesgue lemma shows that \( \hat{\mathcal{A}}f(\lambda) \) is a bounded continuous function. Therefore \( \int_\mathbb{R} M(\mathcal{A}f)(\lambda) |\hat{\mathcal{A}}f(\lambda)| d\lambda \) is finite. This shows \( \mathcal{A}f \) satisfies the condition (3.5). Consequently \( \hat{\mathcal{A}}f(\lambda) = \hat{f}(i\lambda) = 0 \) for all \( \lambda \in \mathbb{R} \). That is \( \hat{f}(\lambda) = 0 \) for all \( \lambda \in i\mathbb{R} \). Hence the Fourier transform of \( f \) is supported on the discrete series.
Step 4 Let $D_{m,n} = \{ k \in \mathbb{Z}^* \mid \pi_k \text{ has } m \text{ and } n \text{ as } K \text{ types} \}$. That is $D_{m,n}$ is the set parametrizing the discrete series representations which admit the pair $(m, n)$ as $K$ types. Note that the cardinality of $D_{m,n}$ is finite. Since $f$ is of type $(m, n)$ and its Fourier transform is supported only on the discrete series, it must be a finite linear combination of matrix coefficients of discrete series representations parametrized by elements of $D_{m,n}$. That is $f = \sum_{k \in D_{m,n}} c_k \Psi_{k}^{m,n}$, where $\Psi_{k}^{m,n}$ is the $(m, n)$-th canonical matrix coefficient of $\pi_k$. Let $k_0 = \max\{|k| \mid k \in D_{m,n}\}$. Now (see [1, p. 70]) $\Psi_{k}^{m,n}(a_t)$ is asymptotic to $A e^{-(1+|k|)t}$ as $t \to \infty$ where $A$ is a nonzero constant depending on $k$. This shows that $|f(a_t)| \geq A' e^{-(1+k_0) t}$ for all $t > 0$ sufficiently large, where $A'$ is a positive constant depending on $f$. Now using Cartan decomposition and (3.7) we see that $\int_G |f(x)| \phi_{k_0}(x) dx = \int_0^{\infty} |f(a_t)| \phi_{k_0}(a_t) \sinh 2tdt$ is infinite, where $k_0$ is as above. Therefore $f$ cannot satisfy the second condition of the theorem unless it is zero almost everywhere.

Thus we have shown that $f(x) = f_{m,n}(x) = 0$ for almost all $x \in G$. As $(m, n)$ is arbitrary, in view of the decomposition $f = \sum_{m,n} f_{m,n}$ in the sense of distributions, it follows that $f(x) = 0$ for almost all $x \in G$. \qed

4. CONSEQUENCES

We have already mentioned that our theorem 3.1 implies the other QUP's. We now state these (see for instance [11, 9, 10, 8, 3, 12] for independent proofs of these theorems).

1. Hardy’s Theorem: Let $f : G \to \mathbb{C}$ be a complex valued measurable function and assume that,

\begin{align*}
(1) & \quad |f(x)| \leq Ce^{-\alpha \sigma(x)^2} \quad \text{for all } x \in G, \\
(2) & \quad \|\hat{f}(\sigma, \lambda)\|_2 \leq Ce^{-\beta|\lambda|^2} \quad \text{for all } \sigma \in \hat{M} \text{ and } \lambda \in i\mathbb{R},
\end{align*}

where $\alpha, \beta$ are positive constants. If $\alpha \beta > \frac{1}{4}$ then $f = 0$ almost everywhere.

2. Morgan’s Theorem (strong version): Let $f : G \to \mathbb{C}$ be measurable and assume that,

\begin{align*}
(1) & \quad |f(x)| \leq Ce^{-\alpha \sigma(x)^p} \quad \text{for all } x \in G, \\
(2) & \quad \|\hat{f}(\sigma, \lambda)\|_2 \leq e^{-\beta|\lambda|^q} \quad \text{for all } \sigma \in \hat{M} \text{ and } \lambda \in i\mathbb{R},
\end{align*}

where $\alpha, \beta$ are positive constants, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

If $(\alpha p)^{\frac{1}{p}} (\beta q)^{\frac{1}{q}} > 1$, then $f = 0$ almost everywhere.
3. Cowling-Price Theorem: Let $f : G \to \mathbb{C}$ be measurable and assume that for positive constants $\alpha$ and $\beta$ we have

1. $e_\alpha f \in L^p(G)$,

2. $e_\beta \|f(\sigma, \lambda)\|_2 \in L^q(i\mathbb{R}; d\mu(\sigma, \lambda))$,

where $1 \leq p, q \leq \infty$, $e_\alpha(x) = e^{\alpha \sigma^2(x)}$ and $e_\beta(\lambda) = e^{\beta|\lambda|^2}$.

If $\alpha \beta > \frac{1}{4}$, then $f = 0$ almost everywhere.

4. Gelfand-Shilov: Let $f \in L^2(G)$. Suppose $f$ satisfies

1. $\int_G |f(x)| e^{\frac{\alpha \sigma^2(x)}{p}} \, dx < \infty$,

2. $\int_{i\mathbb{R}} \|f(\sigma, \lambda)\|_2 e^{\frac{(\beta \lambda)^2}{q}} \, d\mu(\sigma, \lambda) < \infty$,

where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \beta \geq 1$. Then $f = 0$ almost everywhere.

The proof of the deduction of these theorems from theorem 3.1 is similar to that in the Euclidean case. We illustrate this in the case of Cowling-Price theorem. As in the proof of theorem 3.1 we can reduce the proof to the case when $f$ is of type $(m, n)$. We will write $\hat{f}(\lambda)$ for $f(\sigma, \lambda)$ and $\mu(\lambda)$ for $\mu(\sigma, \lambda)$ as $\sigma \in \hat{M}$ is uniquely determined by $m, n \in \mathbb{Z}^\sigma$.

Let $f$ be a function of type $(m, n)$ which satisfies the conditions (1) and (2) of the Cowling-Price theorem. Then we can choose $0 < \alpha' < \alpha$ (respectively $0 < \beta' < \beta$) such that $\alpha' \beta' > \frac{1}{4}$. We have $e_{\alpha'}|f| \in L^1(G)$ (respectively $e_{\beta'}|\hat{f}| \in L^1(i\mathbb{R}, d\mu(\lambda))$). We will show that

\[ (4.1) \int_G \int_{i\mathbb{R}} |f(x)|\|\hat{f}(\lambda)|e^{\sigma(x)|\lambda|} \, dx \, d\mu(\lambda) < \infty. \]

We take $\beta'' < \beta'$ such that $\alpha' \beta'' = \frac{1}{4}$. Then $e_{\beta''} \hat{f} \in L^1(i\mathbb{R}, d\mu(\lambda))$ and

\[ \tilde{f} = \int_G \int_{i\mathbb{R}} e_{\alpha'}(x)|f(x)|e_{\beta''}(\lambda)|\hat{f}(\lambda)|e^{-\alpha' \sigma(x)^2} e^{-\beta''|\lambda|^2} e^{\sigma(x)|\lambda|} \, dx \, d\mu(\lambda) \]

\[ = \int_G \int_{i\mathbb{R}} e_{\alpha'}(x)|f(x)|e_{\beta''}(\lambda)|\hat{f}(\lambda)|e^{-(\sqrt{\alpha'} \sigma(x)-\sqrt{\beta''}|\lambda|)^2} \, dx \, d\mu(\lambda). \]

Since $e^{-(\sqrt{\alpha'} \sigma(x)-\sqrt{\beta''}|\lambda|)^2} \leq 1$, $\tilde{f} < \infty$.

Furthermore the rapid decay of $f$ namely $e_{\alpha'} f(x) \in L^1(G)$ immediately shows that $f$ also satisfies the second condition of theorem 3.1. That is the pair $(f, \hat{f})$ satisfies the condition of theorem 3.1. Hence $f = 0$ almost everywhere.

In view of the inequality $\alpha \beta \sigma(x)|\lambda| \leq \frac{\alpha}{p} \sigma(x)^p + \frac{\beta}{q} |\lambda|^q$ conditions (1) and (2) of the Gelfand-Shilov theorem immediately imply the first condition of theorem 3.1. Again the rapid decay of $f$ given in
condition (1) of the Gelfand-Shilov theorem shows that \(f\) also satisfies the second condition of theorem 3.1.

Morgan’s theorem easily follows from Gelfand-Shilov theorem. Note that the special case \(p = q = 2\) in Morgan’s theorem gives Hardy’s theorem.

**Remark 4.1.** A careful reader will observe that this is the first time when in a theorem of uncertainty, discrete series representations appear in the hypothesis; compare with for instance [2, 9]. It is obvious that for an integrable cusp form \(\Psi\), its Fourier transform vanishes identically on the unitary principal series and hence \(\Psi\) trivially satisfies the first condition of Beurling’s theorem. The non-appearance of the discrete series in the other QUP’s mentioned in this section can be explained by noting that in all of them we put very rapid decay on the function \(f\) which forces every \(K\)-finite component of \(f\) to satisfy the second condition of the Beurling’s theorem (see (3.2) and (3.4)).

Our result also indicates that for a group having real rank greater than 1, the hypothesis of Beurling’s theorem will involve all non-minimal principal series and discrete series, in contrast with the other QUP’s.

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