Liénard–Wiechert solution revisited

Nikolai V. Mitskievich∗†

Abstract

A self-sufficient consideration of the Liénard–Wiechert solution is given including its heuristic deduction, which involves a future light cone (thus with lightlike propagation of information from an arbitrarily moving pointlike charge), and physical interpretation of this field via application of three distinct reference frames (of an inertial observer, then a non-inertial one retardedly co-moving with the charged source, and finally, co-moving with the electromagnetic field). In the last frame the magnetic part of the Liénard–Wiechert field identically (though not asymptotically) vanishes in all spacetime together with the Poynting vector. In the second frame, the properties of energy redistribution and radiation are discussed. The dynamically caused propagation velocity of the Liénard–Wiechert electromagnetic field in a vacuum at any final distance from the source is less than that of light.

Key words: Light cone. Propagation of information. Propagation of electromagnetic field. Classification of electromagnetic fields. Transforming away the magnetic field. Reference frame co-moving with the electromagnetic field.

∗Physics Department, CUCEI, University of Guadalajara, Guadalajara, Jalisco, Mexico.
†Postal address: Apartado Postal 1-2011, C.P. 44100, Guadalajara, Jalisco, México. E-mail: mitskievich03@yahoo.com.mx
1 Introduction

1.1 Preliminaries

In this paper we consider Maxwellian electromagnetic fields in the flat Minkowski spacetime with the metric tensor $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, thus taking Cartesian coordinates (algebraic relations used or deduced here, frequently remain unaltered also in the framework of general relativity). Greek indices are four- and Latin, three-dimensional. However we more frequently use as three-dimensional quantities four-dimensional vectors (tensors) orthogonal to the timelike unit vector describing the reference frame (the monad). Different frames may be simultaneously applied (the test-object property which is essential in treatment of reference frames in non-quantum theory). In general, we do not mutually relate reference frames and systems of coordinates. A comma (,) followed by an index is used to denote partial differentiation with respect to the corresponding coordinate. We also use natural units in which the velocity of light in a vacuum is $c = 1$. Round brackets mean symmetrization and square brackets, antisymmetrization in the indices contained in them (the so-called Bach brackets). In concrete calculations no approximations are assumed.

This material is essentially the final chapter of my unpublished one-semester course “Relativistic Physics” given to undergraduate (Licenciatura) students at the Physics Department of the University of Guadalajara during the last seven years. The students first have to attend another course on tensor calculus which also includes the formalism of Cartan forms with some applications in physics. The subsection 4.3 was included in my course only in the last semester.

1.2 Preview of the paper

More than one hundred years ago, A. Liénard [1] and E. Wiechert [2] discovered an exact solution of Maxwell’s equations describing electromagnetic field of a pointlike electric charge in an arbitrary motion. A frequently used treatment of this solution can be found in [3], and its more general deduction, including the use of an arbitrary mixture of retarded and advanced potentials, in [1]. In section 2 we consider a simple and direct deduction of the Liénard–Wiechert (below abbreviated as LW) solution with the use of the light cone concept which involves a supposition of lightlike propagation of
information from this pointlike source. Some important general properties of the LW solution are discussed in section 3. Here a general classification of electromagnetic fields is outlined, and it is found that the LW field belongs to the pure electric type, thus its magnetic part can be transformed away when one passes to certain non-inertial reference frames. It is well known that in a vacuum electromagnetic waves propagate with the fundamental velocity $c$ ($= 1$). However, as it is shown in section 4, a mixture of non-radiative and radiative electromagnetic fields has another propagation velocity ($< 1$). For this reason, when we speak above and in sections 2 and 5 about ‘propagation of information,’ we do not speak strictly about propagation of electromagnetic field in the general sense. In subsection 4.3 the general method of finding reference frames co-moving with electromagnetic fields is formulated (mostly for the case of pure subtypes of electric or magnetic types of fields via transformation away of the magnetic or electric field, respectively; however also in the impure subtypes, though there it is impossible to transform away one or — asymptotically — both fields $\mathbf{E}$ and $\mathbf{B}$, one always may make these fields mutually parallel, thus transforming away the Poynting vector in the respective frame). In frames co-moving with the electromagnetic field, the Poynting vector automatically vanishes. This method is then applied to the LW field. Relative motion of different reference frames is considered in subsection 4.4 first in general and then for the LW solution. In section 5 some results obtained in the paper are discussed. In two appendices, A and B, a short review of the Ehlers–Zel’manov covariant theory of reference frames (its algebraic part) is given together with applications to the description of electric and magnetic fields.

2 A systematic deduction of the LW solution

Let us consider a pointlike charge $Q$ in a motion along a worldline $L$ parametrically described as

$$\mathbf{r}' = \mathbf{r}'(t'), \text{ equivalently, } x'^\mu = x'^\mu(t'),$$

(2.1)

$t' = x'^0, i = 1, 2, 3$. We shall determine at an arbitrary, but fixed spacetime point $P$ with coordinates $x^\mu$ (not on $L$), the electromagnetic field created by the charge $Q$ being at another point $P'$ on $L$; the coordinates are chosen to be Cartesian. It is obvious that the electromagnetic field created by a pointlike charge should have a singularity on $L$, this is why we exclude here
the case of coincidence of the points $P$ and $P'$. Note that the coordinates of $P$ represent four independent scalar variables $x^\mu$, and those of $P'$ merely are scalar functions of some parameter (this may be $s'$, but we shall use the retarded time $t'$) along the worldline $L$, $x^\mu(t')$ (there are three equations, the fourth being simply an identity, $x^0 = t'$). To mutually relate the spacetime points $P$ and $P'$, we use a hypothesis that the information about position and state of motion of the charge propagates with the fundamental velocity (that of light) in an accordance with the relativistic causality law. If the point $P$ and worldline $L$ are given, the point $P'$ can be determined as that of intersection of the past light cone with a vertex at $P$ and the line $L$ (this simultaneously means that $P$ is on the future light cone with a vertex at $P'$). This constructive definition is important in the subsequent calculations, but fortunately the concrete relation between the position of $P$ and the corresponding retarded time $t'$ at $P'$ turns out to be of no importance. Thus $t'$ is a function of all four coordinates of $P$ — we write it as $t'(x)$; we shall easily calculate the explicit form of derivatives of $t'$ with respect to the coordinates $x^\mu$ without an explicit knowledge of $t'(x)$.

We take the Minkowski metric as $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ (in fact, this is the definition of Cartesian coordinates), thus the tangent vector to $L$, $u^\mu = dx^\mu/ds'$ (the four-velocity of the charge) taken at the retarded point $P'$, is timelike and unitary ($u' \cdot u' \equiv u^\mu u'_\mu = 1$), its timelike property being manifested by the relation $ds'^2 > 0$ along $L$. Locally, $u'$ determines the direction of growth of the proper time $s'$, being simultaneously the projector onto the (retarded) physical time direction of the (retarded) reference frame (retardedly) co-moving with the charge. Another projector, now a tensor, can be constructed as (A.2), here

$$b_{\mu\nu} = g_{\mu\nu} - u'_\mu u'_\nu. \quad (2.2)$$

It is (a) symmetric ($b_{\mu\nu} = b_{\nu\mu}$), (b) orthogonal to $u'$, thus realizing projection onto the subspace $\perp u'$ (the physical three-space of the just mentioned inertial reference frame at $P'$); (c) it possesses the property of idempotent ($b^\lambda_\mu b^\lambda_\nu = b^\lambda\nu$ with $\det b^\lambda_\mu = 0$), and (d) plays the rôle of the three-dimensional metric in the mentioned subspace, with the signature $(0, -, -, -)$ (zero is inserted in the four-dimensional sense). Thus $g^\lambda_\lambda = 3^\lambda_\lambda = 4$ and $b^\lambda_\lambda = 3$ give dimensionalities of the space-time and subspace under consideration.

Let us introduce a vector connecting the four-points (events) $P'$ and $P$,

$$R^\mu = x^\mu - x'^\mu(t'). \quad (2.3)$$
Of course, this is not a vector under more general transformations than the Lorentz ones (like the Euclidean ‘radius vector’ is a vector only in Cartesian systems). Since $R^\mu$ lies on the light cone,

$$R^\mu R_\mu = 0,$$

(2.4)

this vector is null. Its projection onto $u'$ is denoted as $D$, and onto the retarded three-space, as $D^\mu$:

$$D := u'^\mu R_\mu \equiv u' \cdot R, \quad D^\mu = R'^\nu b^\nu_\mu = R^\mu - D u'^\mu, \quad D \perp u'.$$

(2.5)

Due to (2.2), $\Rightarrow \delta^\mu_\nu = b^\mu_\nu + u'^\mu u'_\nu$, and the null property (2.4),

$$D^\mu D_\mu = -D^2, \quad D = \sqrt{-D^\mu D_\mu},$$

(2.6)

thus we call $D^\mu$ the ‘retarded spatially projected vector between $P'$ and $P$.’ Similarly, $D$ is interpreted as the retarded three-dimensional distance between $P'$ and $P$. Recall also that

$$u'^\mu = \frac{dx'^\mu}{ds'} = \frac{dx'^0}{ds'} \frac{dx'^\mu}{dx'^0} = u'^0 \cdot (1, v'^i).$$

(2.7)

Now we are ready to calculate all necessary derivatives (of $t', R^\mu, D, u'^\mu$, and more) with respect to $x^\alpha$. The first step is to write

$$R^\mu_\cdot,\alpha = \frac{\partial x'^\mu}{\partial x^\alpha} - \frac{\partial x'^0}{\partial x^\alpha} = \delta^\mu_\alpha - \frac{dx'^\mu}{ds'} \frac{dx'^0}{dt'} \partial t',$$

that is,

$$R^\mu_\cdot,\alpha = \delta^\mu_\alpha - \frac{u'^\mu}{u'^0} t',\alpha.$$  

(2.8)

Differentiation of (2.4) yields

$$R_\mu R^\mu_\cdot,\alpha \equiv \frac{1}{2} (R_\mu R^\mu)_\cdot,\alpha = 0,$$

thus

$$t',\alpha = \frac{u'^0 R_\alpha}{D},$$

(2.9)

and its substitution into (2.8) yields

$$R^\mu_\cdot,\alpha = \delta^\mu_\alpha - \frac{u'^\mu R_\alpha}{D}.$$  

(2.10)
Now,
\[
 u^{\mu,\alpha} = \frac{du^{\mu}}{dt'} \tau'^{,\alpha} = \frac{du^{\mu}}{ds'} \tau'^{,\alpha} = \frac{a^{\mu}}{D} R_{\alpha}
\]
(similar derivatives of all primed objects are proportional to \( R \) with the differentiation subindex), where
\[
a^{\mu} = \frac{du^{\mu}}{ds'}
\]
is the acceleration four-vector (at \( P' \)) obviously possessing the property of four-orthogonality to \( u' \):
\[
u^{\mu} a^{\mu} \equiv 0.
\]
This use of the acceleration four-vector is more economic than of the respective three-vector, though their mutual relation is somewhat indirect; the reader, beginning with (A.8), may easily reconstruct the corresponding formulae and apply them to interpretation of the results and to make a comparison with the treatment of LW problem in [3]. The final step in this part of calculations is to differentiate \( D \):
\[
 D_{,\alpha} = (u' \cdot R)_{,\alpha} = u'_{\mu,\alpha} R^{\mu} + u'_{\mu} R^{\mu,\alpha} = u'_{\alpha} - \frac{R_{\alpha}}{D} (1 - a' \cdot R)
\]
where, of course, \( a' \cdot R := a'_{\mu} R^{\mu} \equiv a' \cdot D \). Let us also take into account that
\[
 R'_{\nu,\nu} = 3 \quad \text{and} \quad a^{\mu}_{\nu,\nu} = \frac{da^{\mu}}{ds'} R_{\nu}
\]
[see a comment to (2.11)].

The second, and last, preparatory part of our calculations is to write down Maxwell’s equations. Outside the sources, their four-dimensional form is
\[
 F^{\mu\nu,\nu} = 0
\]
where
\[
 F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}
\]
is the field tensor written in terms of the four-potential \( A_{\mu} \), thus \( F^{\mu\nu,\nu} = \square A^{\mu} + (A'_{\nu})^{\mu} = 0 \), the d’Alembertian operator being \( \square = \Delta - \partial^2 / \partial t^2 \). The \( A'_{\nu,\nu} \)-term can be eliminated if we use the Lorenz condition\(^1\)
\[
 A'^{\nu} = 0
\]
\(^1\)This condition is due not to H.A. Lorentz as admits the majority of physicists, but to L.V. Lorenz (born in Elsinore, Denmark, in 1829), see the footnote related to formula (5.1.47) in [3], p. 321.
which only fixes global gauge of the four-potential without any other restrictions. The alternative form of Maxwell’s equations should then include the Lorenz condition, thus in the form of a system

\[ \Box A^\mu = 0 \quad \text{and} \quad A^\nu,_{\nu} = 0. \]  

(2.19)

The well-known Coulomb potential in a vacuum in electrostatics can be written as  

\[ A^\mu = \frac{Q}{r} \delta^\mu_0 \]  

for a pointlike charge \( Q \) located at the spatial origin. One notices that the four-velocity of the charge at rest is \( u^\mu = u^\mu = \delta^\mu_0 \). This potential exactly satisfies both equations of (2.19) when \( r \neq 0 \). We shall now show that a simple generalization of the Coulomb potential is also an exact solution of Maxwell’s equations, and this is precisely that of Liénard–Wiechert.

The generalization is simply

\[ A^\mu = \frac{Qu^\mu}{D}. \]  

(2.20)

The proof that this is the exact solution is quite short for the Lorenz condition:

\[ A^\nu,_{\nu} = \frac{Q}{D} \left( u^\nu,_{\nu} - \frac{u^\nu D,_{\nu}}{D} \right) = \frac{Q}{D^2} \left[ a' \cdot R - 1 + \frac{R,_{\nu}u^\nu}{D}(1 - a' \cdot R) \right] = 0, \]

and for the d’Alembert equation [the first in (2.19)], a little tedious. First, we calculate

\[ A_{\mu,\nu} = \frac{Q}{D^2} \left[ a'_\mu R,_{\nu} - u'_\mu \left( u^\nu - R^\nu \frac{1 - a' \cdot R}{D} \right) \right]. \]  

(2.21)

Turning now to the rest of (2.19), we see that it is necessary to consider \( \Box A_\mu = -A_{\mu,\nu}^{\nu,\nu} \), taking into account (2.15) and the already known derivatives of \( u', a', R^\alpha \), and \( D \). The reader can verify after performing differentiation that for \( D \neq 0 \) all terms identically cancel:

\[ \left\{ \frac{Q}{D^2} \left[ a'_\mu R^\nu - u'_\mu \left( u^\nu - R^\nu \frac{1 - a' \cdot R}{D} \right) \right] \right\}_{\nu,\nu} = 0. \]

This completes the proof.
Since we shall need the full expression of $F_{\mu\nu}$ in the subsequent calculations, let us now antisymmetrize the expression (2.21) (the first term in round brackets is immediately cancelled):

$$F_{\mu\nu} = \frac{Q}{D^2} \left[ R_\mu \left( a'_\nu + u'_\nu \frac{1 - a' \cdot R}{D} \right) - R_\nu \left( a'_\mu + u'_\mu \frac{1 - a' \cdot R}{D} \right) \right].$$

(2.22)

This is a specific type of skew-symmetric tensor sometimes called simple bivector since it represents an antisymmetrization of only two vectors, $R_\mu$ and $U_\mu = \frac{Q}{D^2} \left( a'^\mu + u'^\mu \frac{1 - a' \cdot R}{D} \right)$:

$$F_{\mu\nu} = R_\mu U_\nu - U_\mu R_\nu$$

(2.23)

which can be written as a 2-form $F = R \wedge U$, $R = R_\mu dx^\mu$ and $U = U_\mu dx^\mu$.

3 General properties of the LW field

First it is worth mentioning the obvious fact that the Coulomb field is a special case of the LW solution: one simply has to consider a pointlike charge at rest, that is $u'_\mu = \delta^\mu_0$ for any $P'$, thus $a'^\mu = 0$. This is the reason why the LW solution has to be interpreted as the electromagnetic field of an arbitrarily moving pointlike charge (of course, the Gauss theorem is here also applicable, for example, in an inertial frame instantaneously co-moving with the central charge at $P'$).

3.1 Classification of electromagnetic fields and its application to the LW solution

The classification of electromagnetic fields is based on existence of only two invariants built with the field tensor $F_{\mu\nu}$, while all other invariants are merely algebraic functions of these two (if not vanish identically). The first invariant is $I_1 = F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$, and the second, $I_2 = F^*_{\mu\nu} F^{\mu\nu} = 4E \cdot B$, cf. (B.2) and (B.3); the definition of $I_2$ contains dual conjugation of $F_{\mu\nu}$,

$$F^*_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}, \quad F^*_{\mu\nu} := -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}. \quad (3.1)$$

Here $\epsilon_{\mu\nu\alpha\beta}$ is the completely skew-symmetric object (not exactly a tensor) with $\epsilon_{0123} = +1$, known as the Levi-Civitá symbol. In fact, only the squared
$I_2$ is really invariant, and $I_2$ itself is a pseudo-invariant which acquires the factor $J/|J|$ by a general transformation of coordinates, $J$ being the Jacobian of the transformation, thus the concrete sign of $I_2$ does not matter. In terms of $I_1$ the invariant classification suggests three types of fields: $I_1 < 0$ is the electric type (the electric field dominates), $I_1 > 0$ gives the magnetic type, and to $I_1 = 0$ corresponds the null type. On the pseudo-invariant $I_2$ the further working out in detail of the classification is based: the additional subtypes are impure ($I_2 \neq 0$) and pure ($I_2 = 0$). It is important that the pure electric case permits (at least, locally, if one considers only inertial frames) to completely eliminate the magnetic field, and similarly, the pure magnetic field permits to completely eliminate the electric field, while the pure null electromagnetic field in a vacuum permits to find a coordinate system (reference frame) in which the electric and magnetic field intensities would take any desired finite (nonzero and non-infinite) and equal values, but, of course, the field will continue to pertain to the same pure null type (in this case, both fields $\mathbf{E}$ and $\mathbf{B}$ will be ever equal in their absolute values and mutually orthogonal, as can be seen from the structure of both invariants). This last property is closely related to the Doppler effect (not only in the sense of the frequency, but — and more profoundly — also of the field intensity), in particular, a complete elimination of the pure null type field is ‘possible’ only asymptotically (in less rich-in-content terms, this means ‘impossible’), since there cannot exist any reference frame moving with the speed of light with respect to an arbitrary permissible reference frame. The impure electric, magnetic, and null types obviously do not permit such manipulations with the three-dimensional parts $\mathbf{E}$ and $\mathbf{B}$ of the electromagnetic field (in the impure electric and magnetic cases it is impossible to transform away the counterparts of these respective fields).

Let us now apply this classification to the LW electromagnetic field. Since

$$I_2 = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \equiv 0$$

for any simple bivector, even with arbitrary $R$ and $U$, the field is pure. Then it is pure electric since

$$I_1 = -\frac{2Q^2}{D^3} < 0$$

(remarkably, the structure of $I_1$ is exactly Coulombian). This means that at any point of the spacetime (any finite value of the distance $D$, i.e. not asymptotically) it is possible to transform away the magnetic part of the field; moreover, it is possible to find such a global reference frame in which only electric part of the field will be present. This possibility can be globally
realized for any concrete choice of the motion of the pointlike charge. In these specific reference frames which are in general non-inertial, but naturally admissible in special relativity (like those to which we are accustomed in non-relativistic physics, the area much more restricted than special relativity), the Poynting vector of the LW field will vanish globally. This fact will be discussed in more concrete details below. Its physical meaning is that at any finite point of the spacetime the electromagnetic LW field propagates with sub-luminal velocity.

4 Propagation of the LW electromagnetic field

4.1 Viewpoint of an inertial observer

This is the least interesting case of the reference frame application to LW solution while the approach reduces to use of a monad adapted to Cartesian coordinates. Let the inertial observer at \( P \) measure electric and magnetic fields \( E \) and \( B \) as well as electromagnetic energy density \( w \) and Poynting vector \( S \) which are two of the three decomposition parts of (B.7) (we shall not consider the stress tensor) with respect to this observer’s monad \( \tau^\mu = \delta^\mu_0 \) (the observer is at rest with respect to the Cartesian coordinates) and to the corresponding orthogonal projector \( b^\mu_\nu = \delta^\mu_\nu - \delta^\mu_0 \delta^\nu_0 \leftrightarrow \delta^\mu_i \delta^\nu_j \Rightarrow (0, \delta^i_j) \) [see (2.2)],

\[
W \equiv T_{em0}^0 = \frac{1}{4\pi} \left( \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - F_{0\alpha} F^{0\alpha} \right) = \frac{1}{8\pi} (E^2 + B^2), \tag{4.1}
\]

\[
S^i = T_{em0}^0 = -\frac{1}{4\pi} F_{i\alpha} F^{0\alpha} = \frac{1}{4\pi} (E \times B)^i, \tag{4.2}
\]

cf. (B.8). Here

\[
E^i = \frac{Q}{D^3} \left[ u'_\alpha \left( R^i - R_0 v'^i \right) (1 - a' \cdot R) + D \left( a'_\alpha R^i - R_0 a'^i \right) \right] \tag{4.3}
\]

and \( B^i = \ast (dt \wedge R \wedge U) \equiv (n \times E)^i \) where (for the inertial frame) \( n = R/R_0 \) and the electromagnetic field 2-form \( F = R \wedge U \) where \( R \) and \( U \) are 1-forms built of the respective covectors found in (2.23); see also the definitions (B.5) and (B.3). Taking into account (A.8) and relations \( D = u'_\alpha (R_0 - R^i v'^i) \) and \( R^i (R^i - R_0 v'^i) = R_0 (R_0 - R^i v'^i) \), it is easy to verify that (4.3) coincides with
the expression given by Landau and Lifshitz ([3], (63.8)) — in our notations,

\[ E^i = \frac{Q}{(R_0 - R^i v^j)^3} \left\{ \frac{1}{u'_0^2} \left( R^i - R_0 v^i \right) + [R \times ((R - R_0 v') \times v')] \right\}. \quad (4.4) \]

However, since the Poynting vector expression is nonlinear in characteristics of the electromagnetic field (due to multiplication of electric and magnetic vectors), we prefer our consideration given in the next subsection to that which splits (4.4) in two parts one of which should describe the outgoing radiation; this reasoning works only asymptotically, and the expression (4.3) is in this case more transparent than (4.4) due to the factor \( D \) in the corresponding term in square brackets in (4.3).

Finally, it is worth mentioning that the differential characteristics of any inertial frame (its acceleration, rotation, and rate-of-strain tensor), including the frame considered above, identically vanish, thus of course simplifying the considerations given in [3], though at the cost of omission of some important details.

### 4.2 The retarded reference frame co-moving with the charge

The retarded reference frame at \( P \) co-moving with the charge at \( P' \) is determined by the monad \( \tau^\mu = u'^\mu \). Thus for electric and magnetic fields we have

\[ E = \frac{Q}{D^2} \left[ (1 - a' \cdot R) n - D a' \right], \quad B = \frac{Q}{D} a' \times n. \quad (4.5) \]

However it is more direct to use projections considered in appendix [3] which result in the non-inertial reference frame where the Poynting vector is

\[ S^\mu := T^\nu u^\lambda b^\mu_\nu, \]

and projections have to be applied to \( F_{\lambda\alpha} F^{\nu\alpha} \) using (2.23) and the relation \( R_\mu U^\mu = Q/D^2 \) obvious from \( U^\mu \) given just before that expression for \( F_{\mu\nu} \). Then

\[ F_{\lambda\alpha} F^{\nu\alpha} u^\lambda b^\mu_\nu = \frac{Q^2}{D^4} \left[ D D^\mu a' a' - D a'^\mu - \frac{D^\mu}{D} a' \cdot D (1 - a' \cdot D) \right]. \]

In order to find a more concise form of the last expression, let us introduce the unit radial vector \( n \) perpendicular to the monad:

\[ n^\sigma := \frac{D^\sigma}{D}, \quad n \cdot n = -1. \quad (4.6) \]
Then

\[
S^\mu = -\frac{Q^2}{4\pi D^2} \left[ n^\mu \left( a' \cdot a' + (n \cdot a')^2 \right) - \frac{1}{D} \left( a'^\mu + n^\mu (n \cdot a') \right) \right].
\]

This expression however takes more transparent form if we also use the projector onto the two-dimensional surface simultaneously orthogonal to both \( u' \) and \( n \). This will be a spherical surface of radius \( D \) not in a hyperplane perpendicular to \( u' \), but on the future light cone with its vertex at \( P' \) (a sphere corresponding to the retarded time in analogy with determination of the LW field). Thus we introduce the projector

\[
c^{\sigma \tau} := b^{\sigma \tau} + n^{\sigma} n^{\tau} \equiv \eta^{\sigma \tau} - u'^{\sigma} u'^{\tau} + n^{\sigma} n^{\tau},
\]

\[
c^{\sigma \tau} c_{\rho \tau} = c^{\rho \sigma}, \quad c^{\sigma \tau} n_{\sigma} = 0, \quad c^{\sigma}_{\sigma} = 2,
\]

and relations similar to \( a^\epsilon \equiv b^{\epsilon \sigma} a_{\sigma} \) should be also taken into account. This new projection tensor plays the rôle of metric tensor on the two-dimensional sphere with the signature \((0,0,-,-)\) involving two zeros, one with respect to direction of the proper time from the viewpoint of all four dimensions, and the second, in the sense of the radial direction \((n)\) which corresponds to the sphere. Finally, the Poynting vector takes the form

\[
S^\mu = \frac{Q^2}{4\pi} \left( \frac{1}{D^3} c^{\mu \tau} a'^{\tau} - \frac{1}{D^2} n^\mu c^{\sigma \tau} a'^{\sigma} a'^{\tau} \right).
\]

This remarkably simple expression of the LW energy flux suggests the following two conclusions. First, the part proportional to \( 1/D^3 \) and linear in the retarded four-acceleration \( a' \), is perpendicular to the radial direction \( n \) (i.e., it is restricted to the corresponding two-sphere on the future light cone with its vertex at \( P' \)). Thus it describes a redistribution of energy at the fixed retarded distance \( D \) from the field source. The integral redistribution flux becomes smaller with more distant location of the observer and asymptotically \((D \to \infty)\) tends to zero due to multiplication of \((4.8)\) by the two-dimensional surface element of the sphere \((\sim D^2)\), while the integration is performed only in the sense of angular coordinates on the sphere. Of course, the very surface (if taken not on the light cone), as well as the reference frame’s three-space, is non-holonom since in general this frame possesses rotation

\[
\omega = *(u' \wedge du') = *(a' \wedge u' \wedge n) = a' \times n,
\]

(4.9)
see (A.9), (A.5), (2.11), and the final remarks in appendix A. Second, the part proportional to $1/D^2$ has positive radial direction (take into account that it gives exactly this contribution since four-dimensional square of the spacelike vector $a'$ is negative due to the space-time signature). Thus it describes an energy flux from the charge to spatial infinity. Moreover, all this part of energy really goes to infinity without being accumulated or rarefied at any values of $D$. Hence this term really describes radiation of energy by the accelerated charge. The non-holonomicity remark is here also relevant, and in this situation one has to take certain caution; this is why we mentioned a roundabout approach involving the light cone which always exists and represents a real hypersurface, though its normal vector is null, thus at the same time it is on the light cone itself. This problem goes beyond the bounds of our paper, and we only mention here that it was successfully treated in last few decades in general relativity. After all, we are living and working in the rotating reference frame of our planet, therefore our three-dimensional physical space certainly is non-holonomic, but this does not prevent us to do physics and to apply it quite well.

In the retarded co-moving reference frame of the pointlike charge the LW electromagnetic energy flux has no other constituent parts. Since the problem does not take into account the sources of acceleration of the charge (the lack of a strict auto-consistency of the problem), the energy flux does not result here in any change of the state of motion of charge. One may say that there is implicitly some kind of engine which prescribes the exact world line of the charged particle (the LW problem does not involve any information about the particle’s mass and energy), thus this “engine” automatically “takes into account” the particle’s energy loss due to radiation (which at finite distances is not lightlike, see below). Other details follow from the further consideration of a new reference frame in which the magnetic field of the LW solution simply vanishes.

### 4.3 LW solution in the reference frame co-moving with electromagnetic field, but not with the charge

In a reference frame which is co-moving with electromagnetic field, the Poynting vector should vanish. This can occur for two alternative reasons (to be realized in this frame): either electric and magnetic vectors are mutually parallel (this is the impure classification subcase), or one of them is equal to
zero (the pure subcase). The first case was considered by Wheeler \cite{7} toward other ends. The second case pertains naturally to the LW field since this is a pure electric one (thus Wheeler’s approach is not applicable, and the magnetic part can be transformed away \textit{via} a proper choice of the reference frame). In fact, this possibility is scarcely encountered in literature (I even don’t know any references), and it would be interesting to investigate it in more detail. We shall see that this task is much simpler than one could expect.

Remember the general form of the LW field tensor, \( F_{\mu\nu} = R_{\mu} U_{\nu} - U_{\mu} R_{\nu} \). Let us (algebraically) regauge the vector \( U \rightarrow V = U + kR \) where \( k \) is a scalar function. This does not change the field tensor, \( F_{\mu\nu} = R_{\mu} V_{\nu} - V_{\mu} R_{\nu} \). \( (4.10) \)

Applying now the 1-form definition of the magnetic vector in a \( \tau \)-frame \( (B.3) \) and taking the monad as \( \tau = NV \) where the scalar normalization factor is \( N = (V \cdot V)^{-1/2} \), we obviously come to \( B = 0 \) in this frame. The problem is thus reduced to a proper choice of \( k \) such that \( V \) will be a suitable real timelike vector with \( V \cdot V > 0 \). This method should work in our case (for a pure magnetic field, a similar technique can be applied, though requiring automatic representation of \( *F \) as a simple bivector).

We see that

\[
V^\mu = \frac{Q}{D^2} \left( a'^\mu + \frac{1 - a' \cdot R}{D} u'^\mu + kR^\mu \right),
\]

\( (4.11) \)

thus it was natural to include before \( k \) the scalar coefficient \( Q/D^2 \). Then

\[
V \cdot V = \left( \frac{Q}{D^2} \right)^2 \left[ a' \cdot a' + \frac{(1 - a' \cdot R)^2}{D^2} + 2k \right].
\]

\( (4.12) \)

In fact, \( k \) still remains arbitrary. Let it be

\[
k = \frac{1}{2} \left[ \frac{1}{D^2} - a' \cdot a' - \frac{(1 - a' \cdot R)^2}{D^2} \right]
\]

\( (4.13) \)

(the first term in the square brackets, \( 1/D^2 \), got its denominator to fit the dimensional considerations). Finally,

\[
V \cdot V = \left( \frac{Q}{D^3} \right)^2 > 0
\]

\( (4.14) \)
\[ \hat{\tau}^\mu = Da^\mu + (1 - a' \cdot R) u^\mu + \frac{1}{2D} \left[ 1 - D^2 a' \cdot a' - (1 - a' \cdot R)^2 \right] R^\mu \quad (4.15) \]

(it is clear that \( \hat{\tau} \cdot \hat{\tau} = +1 \)). By its definition, the monad \( \hat{\tau} \) describes the reference frame co-moving with the LW electromagnetic field: in this frame the Poynting vector of the field vanishes, and the electromagnetic energy flux ceases to exist due to the absence of magnetic part \( \hat{\mathbf{B}} \) of the field in this frame (applicable at any finite distance \( D \), not asymptotically). Really, (4.10) now can be rewritten as

\[ F_{\mu\nu} = \frac{Q}{D^3} (R_{\mu} \hat{\tau}_{\nu} - \hat{\tau}_{\mu} R_{\nu}) , \]

thus the expression of \( \hat{\mathbf{B}} \) (4.3) contains \( \hat{\tau} \wedge R \wedge \hat{\tau} \equiv 0 \).

Let us now calculate the electric vector \( \hat{\mathbf{E}} \) in the frame \( \hat{\tau} \). A combination of (4.15), (4.11), and (4.10) gives

\[ F = R \wedge V = \frac{Q}{D^3} R \wedge \hat{\tau} , \quad (4.16) \]

see also (B.1). Then the expression (B.2) yields

\[ \hat{\mathbf{E}} = * (\hat{\tau} \wedge * F) = \frac{Q}{D^2} * [\hat{\tau} \wedge * (R \wedge \hat{\tau})] = \frac{Q}{D^2} \hat{\mathbf{n}} \quad (4.17) \]

which is, up to an understandable reinterpretation of notations, exactly the form known as the Coulomb field vector. Here \( \hat{\mathbf{n}}^\mu = \hat{\mathbf{D}}^\mu / D (\perp \hat{\tau}) \) where \( R^\mu u_\mu' =: D \equiv \hat{\mathbf{D}} := R^\mu \hat{\tau}_\mu \) and \( \hat{\mathbf{D}} = \hat{\mathbf{b}}^\mu R_{\mu} \) with \( \hat{\mathbf{b}}^\mu = \delta^\mu_{\nu} - \hat{\tau}^\mu \hat{\tau}_\nu \), hence

\[ \hat{\mathbf{D}}^\mu = -D^2 a^\mu - D (1 - a' \cdot R) u^\mu + \frac{1}{2} \left[ 1 + D^2 a' \cdot a' + (1 - a' \cdot R)^2 \right] R^\mu , \quad (4.18) \]

\( \hat{\mathbf{D}}^\mu \neq D^\mu \); note that \( \hat{\mathbf{D}}^\mu \hat{\mathbf{D}}_\mu = -D^2 \), as this was the case for \( D^\mu \) in (2.16). It is clear that \( \hat{\mathbf{D}}^\mu + D \hat{\tau}^\mu = R^\mu \).

### 4.4 Relative three-velocities of reference frames

Let us now simultaneously consider three distinct reference frames and denote them as A, B, and C. Between such frames there can be established quite a few algebraic relations having a clear and important physical meaning, and it is interesting that these relations hold equally in general and special
relativity. One defines the relative three-velocity of frame B with respect to frame A (and measured in A) as a (co)vector $v_{BA}$ perpendicular to the monad $\tau_A$. According to (A.6),

$$\tau_B = (\tau_A + v_{BA})(\tau_A \cdot \tau_B)$$

and

$$v_{BA}^\mu = \frac{\tau_B^\nu b_{A\nu}^\mu}{\tau_A \cdot \tau_B}$$

(4.19)

(here the relation $\tau_B^\mu - \tau_A^\mu(\tau_A \cdot \tau_B) \equiv \tau_{AB}^\mu b_{A\nu}^\mu$ was used); hence,

$$\tau_A \cdot \tau_B = \frac{1}{\sqrt{1 + v_{BA} \cdot v_{BA}}} \equiv \frac{1}{\sqrt{1 - v_{BA} \cdot v_{BA}}} = \frac{1}{\sqrt{1 - v_{AB}^2}}.$$  (4.20)

It is clear that similar relations exist for any pair of reference frames whatever when the respective monads are introduced. We see that there is a symmetry for squared three-velocities between any pair of frames, in particular, $v_{BA}^2 = v_{AB}^2$. Since these three-velocities are described as four-vectors perpendicular to the respective monads (of the frames corresponding to the frame subindex of $\tau$ and of $b$), they belong to different (local) three-spatial sections of spacetime and in general cannot be directly compared by measurements ones with others without further projections onto alternative subspaces. The inevitability of such a situation is quite obvious. Even in the generally used special-relativistic composition-of-velocities formula for globally inertial frames in motion along “same spatial direction,” this is in fact also the case which is tacitly assumed, but frequently not properly understood. Its strict formulation when these velocities are not mutually “parallel,” is however more laborious.

Another useful step in our calculations is to apply same procedure as in (4.19), but taken with respect to the frames C and A, then to C and B, and further applying it to the free $\tau_B$, thus $\tau_C = (\tau_A + v_{CA})(\tau_A \cdot \tau_C) = (\tau_B + v_{CB})(\tau_B \cdot \tau_C) = [(\tau_A + v_{BA})(\tau_B \cdot \tau_B) + v_{CB}](\tau_B \cdot \tau_C)$. When this expression is multiplied by $b_{A\nu}$ under a contraction with the lower (component) index of this factor, we come to

$$v_{CA}^\nu = \left[ v_{BA}^\nu (\tau_A \cdot \tau_B) + v_{CB}^{\mu b_{A\nu}} \right] \frac{\tau_B \cdot \tau_C}{\tau_A \cdot \tau_C}.$$  (4.21)

In fact, this is the local velocities composition formula $A \rightarrow B \rightarrow C$ for general (not only inertial) frames in both relativities, special as well as general one. Here, of course, one has to take into account the relation (4.20). In this paper we do not consider further details of the usual composition formula.
Other relations which are worth being mentioned, are the following ones: those with projections onto the alternative monads,

\[ \mathbf{v}_{BA}^\nu b_{BP}^\mu = - (\tau_A \cdot \tau_B) \mathbf{v}_{AB}^\mu \quad \text{and} \quad \mathbf{v}_{AB}^\nu b_{AP}^\mu = - (\tau_A \cdot \tau_B) \mathbf{v}_{BA}^\mu; \]  

(4.22)

further, due to (4.19) and (4.22),

\[ \mathbf{v}_{AB} \cdot \mathbf{v}_{BA} = - (\tau_A \cdot \tau_B) \mathbf{v}_{BA}^2 = (\tau_A \cdot \mathbf{v}_{AB})^2 / \mathbf{v}_{AB}^2 \]  

(4.23)

(here the obvious symmetry \( \tau_A \cdot \mathbf{v}_{AB} = \tau_B \cdot \mathbf{v}_{BA} \) was taken into account); finally,

\[ \mathbf{v}_{AB} = - (\tau_A \cdot \tau_B) \mathbf{v}_{BA} + (\mathbf{v}_{AB} \cdot \tau_A) \tau_A \]  

(4.24)

(decomposition with respect to the frame A). Note that \( \mathbf{v}_{AB}^2 := \mathbf{v}_{AB} \cdot \mathbf{v}_{AB} > 0 \).

Let us globally (at any \( \mathbf{P} \)) denote in the LW problem the reference frame of inertial observer as \( A \), \( \tau^\mu_A = \delta^\mu_0 \), the retarded frame co-moving with the charge as \( B \), \( \tau^\mu_B = u' \), and the frame co-moving with the field and introduced in subsection 4.3, as \( C \) (\( \tau^\mu_C = \hat{\tau}^\mu \)). Then, on the one hand,

\[ (\tau_B \cdot \tau_C) = (u' \cdot \hat{\tau}) = 1 - \frac{1}{2} \left[ D^2 a' \cdot a' + (a' \cdot R)^2 \right]. \]  

(4.25)

On the other hand,

\[ \mathbf{v}_{CB} = \frac{\hat{\tau}}{(u' \cdot \hat{\tau})} - u'. \]  

(4.26)

Rotation of the frame \( C \) takes the (not quite easily deducible) form

\[ \hat{\omega} = - \frac{1 - D(\hat{a}' \cdot R)}{1 - a' \cdot R} \hat{a}' \times \hat{n} + D \hat{a}' \times \hat{n} \]  

(4.27)

where 1-form \( \hat{a}' = (da'_\mu / ds') dx^\mu \) describes the retarded third proper-time derivative of position of the charge in its motion along the worldline \( L \). It is worth giving some hints for the deduction of (4.27): The exterior product of any odd-rank forms \( \alpha \) and \( \beta \) is skew-symmetric, thus \( \alpha \wedge \alpha \equiv 0 \). The vector product (A.5) is applicable to a pair of arbitrary vectors, thus it automatically projects each of them onto the three-dimensional subspace orthogonal to the monad. One now has to apply the definition of rotation (A.9) to the monad \( \hat{\tau} \). Some simplifications follow immediately. Then to complete the simplification one has to take into account a relation following from the form (not directly from the general definition) of \( \hat{\tau} \) (4.15) and \( \mathbf{D} \) (4.18):

\[ \mathbf{D}_{\mu} = D \hat{\tau}_{\mu} - 2 D^2 a'_\mu - 2 D(1 - a' \cdot R) u'_\mu + \left[ D^2 (a' \cdot a') + (1 - a' \cdot R)^2 \right] R_\mu \]  

(at each subsequent step only very few terms survive). The final result is (4.27) which should be compared with (4.9).
5 Concluding remarks

We tried to give in this paper a self-sufficient consideration of the LW solution, from its heuristic deduction to an analysis of important properties of the obtained field. One of these properties is that of field’s motion with respect to a given reference frame. In fact, one can relate this motion to the monad describing the frame in which the electromagnetic field does not propagate (its Poynting vector, the electromagnetic energy flux density, vanishes in this frame). It is possible to find such a frame in all cases with the exception of pure null electromagnetic fields: in this latter case both electromagnetic invariants are equal to zero, consequently there remains only an asymptotic possibility to transform away the field’s motion, but then it is transformed away always together with the field itself (this is precisely the asymptotic limit of the Doppler effect). This asymptotic situation does not belong to any admissible reference frame or system of coordinates since such a frame (or, of one wishes, a system) is a degenerate one and thus excluded from consideration (whose region of application is an open one, and the ‘boundary’ is excluded from it, though we can approach it as ‘near’ as we wish, making the non-zero field as weak as we choose it to become). In this pure null case (the definition see in section 3.1) the field by itself exercises lightlike (null) motion, that with the velocity of light. But then there cannot exist a co-moving (with this field) reference frame since its four-velocity should coincide with the monad of the co-moving frame, and the monad vector is timelike by its definition. (More physical reasons are related to the fact that the continuous swarm of observers forming, together with their measuring equipment, a reference frame, and thus being co-moving with it, should always possess non-zero rest masses, though, of course, these masses have to be infinitesimal ones to guarantee the test property of a classical frame of reference. The non-zero rest mass means a timelike worldline of the corresponding object, thus the lightlike motion of any reference frame is physically impossible.) In all other cases concerning electromagnetic fields’ types a co-moving frame is easily realizable (in this paper we discussed the pure electric and pure magnetic types, and all impure subcases should be dealt with according to the method used by Rainich and Wheeler, see [7]).

Another property is also related to propagation, however not of the field but of the information about its sources, thus this property belongs to the deduction of the LW field. This is a rare case when we encounter in a classical physical context the concept of information usually alien to it. And here
information propagates with the velocity of light in a vacuum.

Appendices

A Description of reference frames

In this paper we use notations and definitions from [8], see also references therein. A reference frame is understood as the splitting of general four-dimensional physical quantities into parts referred to observer’s local time direction and the corresponding local three-dimensional physical subspace orthogonal to it, however the latter (or both parts) are written as four-dimensional tensor quantities (of naturally determined ranks) being orthogonal (or also, if we would wish to emphasize this geometrically, parallel) to observer’s time direction. This direction is expressed via the unit vector (or covector, the distinction should be understandable from the context, frequently mathematical) $\tau$, the monad, tangent to the observer’s world line, thus interpreted as the observer’s four-velocity at the event (four-dimensional point) where is located the quantity (object) under consideration. Thus we speak about a continuous swarm of observers, a congruence of their world lines without singularities (the lines do not intersect, and through any event goes one and only one such line). The monad and the metric tensor at each event are necessary and sufficient for a complete description of a reference frame. Of course, this presence of a swarm of observers, with all their equipment necessary for measuring of all physical quantities at any event, should not disturb both usual physical fields and (in general relativity) the spacetime geometry (the gravitational field). Here we consider such arbitrary reference frames only in the framework of special relativity, thus the simplest choice of coordinates is Cartesian which we use in this paper. In our treatment reference frames are generally not related to systems of coordinates, and in one and the same system of coordinates any choice of a reference frame (or different choices simultaneously) may be used.

To split spacetime tensors into their above-mentioned parts, two typical projectors are used. A projector is an idempotent, which means that its repeated action automatically reduces to a single action of it, and it differs from the metric tensor possessing a similar (just mentioned) property by the fact that an application of a projector leads to certain partial loss of information. If we describe a projector as a $4 \times 4$ matrix (really, a rank two
tensor), its determinant should be equal to zero. In more concrete terms, the matrix rank of a projector should be equal to one when we speak about a projector onto a single direction (here, $\tau$), or three when we perform a projection onto the local three-dimensional physical space orthogonal to $\tau$. Thus in the first case we can use the projector

$$\pi^\mu_\nu = \tau^\mu \tau_\nu$$  \hspace{1cm} (A.1)

and in the second case, 

$$b^\mu_\nu = g^\mu_\nu - \tau^\mu \tau_\nu,$$  \hspace{1cm} (A.2)

hence

$$\pi^\mu_\lambda \pi^\lambda_\nu = \pi^\mu_\nu, \quad b^\mu_\lambda b^\lambda_\nu = b^\mu_\nu, \quad b^\mu_\nu \pi^\nu_\lambda = 0, \quad b^\mu_\nu \tau^\nu = 0.$$  \hspace{1cm} (A.3)

However in the first case we frequently use a mere interior multiplication (that is, with a contraction) by $\tau$ since this leads to a four-dimensionally well defined quantity. It is also clear that $b^\mu_\nu + \pi^\mu_\nu = g^\mu_\nu$. It is worth being repeated that the matrices corresponding to (A.1) and (A.2) are respectively of ranks one and three.

Traditionally, in the literature one usually finds an implicit identification of a four-dimensional Cartesian system of coordinates and the corresponding (“co-moving”) reference frame. This does not pose any ambiguities, only if different reference frames are not considered simultaneously on the background of same system of coordinates, or a non-inertial reference frame is involved. However it is better to take into account that this traditional approach represents a tacit admission that the monad coincides with the unit (timelike) vector along the $t$-axis and any orthonormal transformation is accompanied with a corresponding change of the monad. There is also a widespread prejudice that non-inertial frames cannot be used in or they contradict to the special theory of relativity, but this is nothing more than a prejudice. In this paper we consider such frames of non-inertial observers in two concrete cases, and the monad approach works perfectly in description of physical situation in these non-inertial frames. We also use another projector (of rank-two matrix, that is, realizing projection onto a two-dimensional subspace) when it simplifies description of the situation, and there should exist a naturally determined spatial direction which enables this description.

It is convenient, in the sense of both calculations and adequate work of physical intuition, to use the vector symbolics of scalar and vector products denoted as $\bullet$ and $\times$. In fact, these operations are coincident with those
of the three-dimensional vector algebra, though the objects to which they are applied are four-dimensional vectors restricted to the three-dimensional subspace orthogonal to the monad (not always to the global subspace corresponding in particular to an inertial frame, but, in rotating frames, changing to the more general local non-holonomic case: see in the end of this appendix comments related to the three-dimensional subspaces then having such a local meaning only). These products are defined as

\[ p \cdot q := -b_{\mu\nu} p^\mu q^\nu \equiv \ast[(\tau \wedge p) \wedge \ast(\tau \wedge q)] \quad (A.4) \]

and

\[ p \times q = \ast(p \wedge \tau \wedge q). \quad (A.5) \]

We use here the Cartan exterior forms notations such as the wedge product \( \wedge \), the Hodge star operation \( \ast \) (the dual conjugation of a \( p \)-form, not necessarily of a 2-form = skew-symmetric rank-two tensor), and, later, the exterior differential \( d \), see for details and references [8].

In Cartesian coordinates, due to the spacetime signature \((+, -, -, -)\), the monad of the frame co-moving with these coordinates is \( \tau^\mu = \delta_0^\mu \), \( \tau_j = \delta_0^j \). Thus \( (A.5) \) becomes \((p \times q)^i = \epsilon_{ijk} p^j q^k \). The (co)vectors lying in the three-dimensional subspace of a reference frame are usually written as four-dimensional ones, but in some important cases we put them in boldface printing (as \( E \) and \( B \) for electric and magnetic vectors). Then \( E^2 \equiv E \cdot E = -b_{\mu\nu} E^\mu E^\nu \), etc.

The three-dimensional velocity \( \mathbf{v} \) (described as a four-vector \( \perp \tau \)) of a pointlike particle from the viewpoint of reference frame corresponding to the monad \( \tau \), is determined via the splitting of its four-velocity \( u^\mu = dx^\mu / ds \),

\[ u = (\tau \cdot u)(\tau + \mathbf{v}), \text{ or equivalently } v^\mu = b_{\nu\alpha} \frac{dx^\nu}{\tau_\alpha dx^\alpha} \quad (A.6) \]

where \( \tau_\alpha dx^\alpha / ds = (1-v^2)^{-1/2} \); cf. also \([1, 20]\) and the corresponding remarks. This is, of course, an exclusion in the general method of projecting vector and tensor quantities. Another exclusion is the relation between the four-dimensional acceleration and its usual three-dimensional counterpart which is applied in making an easier comparison with the Landau–Lifshitz treatment of the LW field \([3]\). It is now convenient to write the corresponding relations in the (local) three-dimensional subspace notations. The relativistic acceleration four-vector then is

\[ a^\mu = \frac{du^\mu}{ds'} = \frac{1}{\sqrt{1 - v'^2}} \left[ \frac{d}{dt'} \left( \frac{1}{\sqrt{1 - v'^2}} \right) (1, \mathbf{v'}) + \frac{1}{1 - v^2} (0, \dot{\mathbf{v}}') \right], \]
and the orthogonality of $a'$ and $u'$,

$$u' \cdot a' = \frac{d}{dt'} \left( \frac{1}{\sqrt{1 - v'^2}} \right) - \frac{1}{(1 - v'^2)^{3/2}} v' \cdot \dot{v}' = 0,$$

(A.7)

finally yields a simpler relation between the four- and three-acceleration

$$a^\mu = \frac{v' \cdot \dot{v}'}{(1 - v'^2)^2} (1, v') + \frac{1}{1 - v'^2} (0, \dot{v}').$$

(A.8)

Rotation of a reference frame is defined as

$$\omega = \star (\tau \wedge d\tau) \equiv 2 \star (\tau \wedge A), \quad A = \frac{1}{2} A_{\mu\nu} dx^\mu \wedge dx^\nu,$$

(A.9)

while in Cartesian coordinates and with $\tau$ describing a non-inertial frame, $A$ (not the electromagnetic four-potential 1-form, but the rotation 2-form) is the skew term in the natural decomposition of gradient of the monad,

$$\tau_{\mu,\nu} = \tau_{\nu} G_{\mu} + A_{\mu\nu} + D_{\nu\mu}, \quad A_{\mu\nu} = A_{[\mu\nu]}, \quad D_{\mu\nu} = D_{(\mu\nu)},$$

(A.10)

$G$ being acceleration of the reference frame and $D$, the frame’s symmetric rate-of-strain tensor; $G$, $A$, and $D$ belong to the above-mentioned three-dimensional (local) subspace. Of course, all these quantities become equal to zero in any inertial frame globally. When $A \neq 0$ (equivalent to $\omega \neq 0$), the three-dimensional subspace orthogonal to $\tau$ is non-holonom, that is, there only exists an overall distribution of elements of the corresponding (now non-holonom) hypersurface, but these elements do not fit together to form a global spatial hypersurface in the proper (holonom) sense, see [8], the fact well known in geometry of congruences (here we are dealing with the $\tau$-congruence).

B Electromagnetic fields in arbitrary reference frames

Let us now apply the definitions given in appendix A to the electromagnetic field and related quantities. The field tensor $F_{\alpha\beta}$ which also can be written as a 2-form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu,$$

(B.1)
splits into two four-dimensional vectors, electric
\[ \mathbf{E}_\mu = F_{\mu
u} \tau^\nu \quad \iff \quad \mathbf{E} = *(\tau \wedge *F) \quad \text{(B.2)} \]
and magnetic
\[ \mathbf{B}_\mu = -F_{\mu
u} \tau^\nu \quad \iff \quad \mathbf{B} = *(\tau \wedge F), \quad \text{(B.3)} \]
both \( \perp \tau \), see also (3.1); 2-form \( F := \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \). This splitting follows from an observation that the Lorentz force can be expressed as
\[ (\mathbf{E} + \mathbf{v} \times \mathbf{B})_\alpha = F_{\mu\nu} (\tau^\nu + v^\nu) b^\mu_\alpha. \quad \text{(B.4)} \]

In Cartesian coordinates (and with the corresponding inertial monad) we have the same relations as for usual contravariant three-vectors:
\[ \mathbf{E}^i = F_{i0} = -F^{i0}, \quad \mathbf{B}^i = \frac{1}{2} \epsilon_{ijk} F_{jk} = \frac{1}{2} \epsilon_{ijk} F^{jk}, \quad \text{(B.5)} \]
thus
\[ F_{ij} = F^{ij} = -\epsilon_{ijk} \mathbf{B}^k. \quad \text{(B.6)} \]

The electromagnetic stress-energy tensor is
\[ T^\nu_{\ em \mu} = \frac{1}{4\pi} \left( \frac{1}{4} F_{\kappa\lambda} F^{\kappa\lambda} \delta_\mu - F_{\mu\lambda} F^{\nu\lambda} \right) \quad \text{(B.7)} \]
(in Gaussian units). Its deduction is most simple when one considers Maxwell’s equations in tensor form in a vacuum and without sources. Its (single) contraction with arbitrary monad includes the Poynting vector in that frame,
\[ T^\nu_{\ em \mu} \tau_\nu = \frac{1}{8\pi} \left[ (\mathbf{E}^2 + \mathbf{B}^2) \tau_\mu + 2(\mathbf{E} \times \mathbf{B})_\mu \right], \quad \text{(B.8)} \]
and the squared expression is
\[ T^\nu_{\ em \mu} T^\mu_{\ em \xi} \tau_\nu \tau_\xi = \frac{1}{(8\pi)^2} \left[ (\mathbf{E}^2 + \mathbf{B}^2)^2 - 4(\mathbf{E} \times \mathbf{B})^2 \right] \]
\[ \equiv \frac{1}{(8\pi)^2} \left[ (\mathbf{B}^2 - \mathbf{E}^2)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2 \right] = \frac{1}{(16\pi)^2} \left( I_1^2 + I_2^2 \right) \quad \text{(B.9)} \]
(it is interesting that this expression is not only a scalar under transformations of coordinates, but it is also independent of the choice of reference frame: the right-hand side does not involve any mention of the monad at all). For the LW field [due to (3.2)] this takes a very concise form,
\[ T^\nu_{\ em \mu} T^\mu_{\ em \xi} \tau_\nu \tau_\xi = \left( \frac{Q^2}{8\pi D^4} \right)^2. \quad \text{(B.10)} \]
References

[1] A. Liénard (1898) L’Éclairage électrique 16, 5, 53, 106.

[2] E. Wiechert (1900) Archives Néerlandaises 5, 549.

[3] L.D. Landau and E.M. Lifshitz (1973) Field Theory (Theoretical Physics, Vol. II), 6th edition (Moscow: Nauka). In Russian.\(^2\)

[4] J.L. Synge (1965) Relativity: The Special Theory (Amsterdam: North-Holland).

[5] N.V. Mitskievich (1989) Ann. Phys. (Leipzig) 46, 425. In German.

[6] R. Penrose and W. Rindler (1986) Spinors and space-time, Vol. 1 (Cambridge: CUP)

[7] J.A. Wheeler (1962) Geometrodynamics (New York: Academic Press).

[8] N.V. Mitskievich (1996) Relativistic Physics in Arbitrary Reference Frames. Book preprint Arxiv [gr-qc/9606051]

---

\(^2\)This edition was not translated into English, though it is the best and most complete version of all Russian editions of this book. We use some notations accepted in it, but our deduction of the LW solution and its treatment are different from those given by Landau and Lifshitz.