Robust Instance-Optimal Recovery of Sparse Signals at Unknown Noise Levels

Hendrik Bernd Petersen * Peter Jung †

Abstract

We consider the problem of sparse signal recovery from noisy measurements. Many of frequently used recovery methods rely on some sort of tuning depending on either noise or signal parameters. If no estimates for either of them are available, the noisy recovery problem is significantly harder. The square root LASSO and the least absolute deviation LASSO are known to be noise-blind, in the sense that the tuning parameter can be chosen independent on the noise and the signal. We generalize those recovery methods to the rLASSO and give a recovery guarantee once the tuning parameter is above a threshold. Moreover we analyze the effect of mistuning on a theoretic level and prove the optimality of our recovery guarantee. Further, for Gaussian matrices we give a refined analysis of the threshold of the tuning parameter and proof a new relation of the tuning parameter on the dimensions. Indeed, for a certain amount of measurements the tuning parameter becomes independent on the sparsity. Finally, we verify that the least absolute deviation LASSO can be used with random walk matrices of uniformly at random chosen left regular bipartite graphs.

Introduction

We consider the problem of sparse signal recovery from noisy measurements. Classical recovery methods require additional information about either the noise or the signal. The basis pursuit denoising needs to be tuned in the order of the noise power [1, Theorem 4.22], the $\ell_1$-norm constrained least residual needs to be tuned in the order of the $\ell_1$-norm of the signal [2, Theorem 11.1] and $\ell_1$-norm penalized least squares (LASSO) allegedly needs to be tuned depending on the noise power [2, Theorem 11.1]. If no prior information about the signal and noise are available, these methods fail or, in their sub-optimally tuned versions, yield a sub-optimal recovery guarantee [3]. Thus, it is desirable to find other noise-blind recovery methods. A commonly used approach is cross validation which is often computationally more expensive and theoretical guarantees are not fully understood, see exemplary [4] for further discussion. If the signal is non-negative, the non-negative least squares [5] and non-negative least absolute deviation [6] are tuning free methods that achieve almost as good robustness bounds as the optimally tuned basis pursuit denoising and $\ell_1$-norm constrained least residual. Without the non-negativity assumption this problem is harder.

The square-root LASSO, introduced in [7], is an alteration of the LASSO, where the square of the $\ell_2$-norm is removed. The square-root LASSO is known for being a noise-blind recovery method. Indeed, in [7] it has been shown that the tuning parameter can be chosen independent on the noise power. Further, the square root LASSO has been studied in [8][9][10][11][12][13][14].

The least absolute deviation LASSO is an alteration of the square root LASSO, where the $\ell_2$-norm of the data fidelity term is replaced by an $\ell_1$-norm. The least absolute deviation LASSO has also been studied frequently [15][16][17][18][19][20][21][22][23][24][25][26][27]. Under the assumption that the measurement matrix extended by the identity of the measurement domain has a null space property, it was proven that the least absolute deviation LASSO can recover sparse signals exactly even in the presence of sparse noise [18], see also [19][20][21][22][23][26][27].

Our Contribution

We will introduce the notion of a stable and robust decoder and generalize the square root LASSO and the least absolute deviation LASSO to the “$p$th-root LASSO “ (rLASSO). Under the assumption of a robust null space property we generalize the recovery guarantee of the square root LASSO by proving that robust recovery is possible if the tuning parameter is larger than a threshold. Further, this threshold is a smooth function of the parameters of the robust null space property. We will then discuss the effect of the tuning parameter on the recovery guarantee in a larger theoretical detail. In particular, we prove that the error bound does not
degnerate when the tuning parameter is chosen too large. On the other hand we prove that if the tuning parameter is chosen smaller than the threshold of our recovery guarantee, recovery has to fail for at least one sparse signal. This yields that the recovery guarantee is optimal in a certain sense and can not be improved. In the second part of our work we focus on the estimation of the tuning parameter. For Gaussian matrices we use Gordon’s escape through the mesh \[28\] to estimate the tuning parameter from the phase transition and refine the dependence of the tuning parameter on the dimensions. This dependence coincides with the general established rule, to choose the tuning parameter in the order of the square root of the sparsity, only if sufficient measurements are present, but if the number of measurements is close to the optimal number of measurements, a different rule for the tuning parameter is better suitable. Further, we will establish that rLASSO can be used with random walk matrices of uniformly at random chosen left regular bipartite graphs and will prove that the tuning parameter can be chosen in the order of a constant, independent on all dimensions. Lastly, we will verify our theoretical results by short numerical tests.

1 Preliminaries

Given a set \(C \subseteq \mathbb{R}^N\) and a function \(f : C \rightarrow \mathbb{R}\) we denote the set of minimizers of \(f\) on \(C\) by

\[
\arg\min_{x \in C} f(x) := \left\{ x \in C : f(x) = \inf_{x' \in C} f(x') \right\}.
\]

For \(q \in [1, \infty)\) and \(x \in \mathbb{R}^N\) we denote the \(\ell_q\)-norm by \(\|x\|_q := \left( \sum_{n=1}^N |x_n|^q \right)^{\frac{1}{q}}\) and the \(\ell_\infty\)-norm by \(\|x\|_\infty := \sup_{n \in [N]} |x_n|\). If \(q = \infty\), we use the notation \(\frac{1}{q} := q^{-1} = 0\). For a number \(N \in \mathbb{N}\) we denote \([N] := \{1, \ldots, N\}\).

For a set \(T \subseteq [N]\) we denote the number of elements in \(T\) by \(#(T)\). We also write \(#(T) \leq S\) to mean \(T \subseteq [N]\) and \(#(T) \leq S\). For \(T \subseteq [N]\) we denote the projection onto the subspace \(\{z \in \mathbb{R}^N : \text{supp}(z) \subseteq T\}\) by \(\cdot|_T\). For \(x \in \mathbb{R}^N\) it is given by

\[
x|_T := \left\{ \begin{array}{ll}
(x|_T)_n = x_n & \text{if } n \in T, \\
(x|_T)_n = 0 & \text{if } n \notin T
\end{array} \right\}.
\]

We call \(A \in \mathbb{R}^{M \times N}\) a measurement matrix and any map \(Q : \mathbb{R}^M \rightarrow \mathbb{R}^N\) a decoder. A vector \(x \in \mathbb{R}^N\) is called signal, and any \(y \in \mathbb{R}^M\) is called measurement. Given all these we denote \(e := y - Ax\) the noise. A signal \(x\) is called \(S\)-sparse if \(\|x\|_0 := #(\text{supp}(x)) \leq S\). The set of \(S\)-sparse signals is denoted by \(\Sigma_S := \{z \in \mathbb{R}^N : \|z\|_0 \leq S\}\). Given some \(q \in [1, \infty]\) and \(S \in [N]\) the compressibility of a signal \(x\) is measured by

\[
d_q(x, \Sigma_S) = \inf_{z \in \Sigma_S} \|x - z\|_q
\]

and quantizes how close a signal is to being \(S\)-sparse. Motivated by \[28\] we would like to get decoders that are able to bound the estimation error by a linear function in the uncertainties \(d_1(x, \Sigma_S)\) and \(\|e\|\), for some norm \(\|\cdot\|\) on \(\mathbb{R}^M\).

Definition 1.1. Let \(S \in [N], q \in [1, \infty]\) and \(\|\cdot\|\) be any norm on \(\mathbb{R}^M\). Let \(A \in \mathbb{R}^{M \times N}\) and \(Q : \mathbb{R}^M \rightarrow \mathbb{R}^N\). If there exist \(C, D \in [0, \infty)\) such that

\[
\|Q(y) - x\|_q \leq CS^{\frac{q}{2} - 1}d_1(x, \Sigma_S) + D\|y - Ax\| \quad \text{for all } x \in \mathbb{R}^N, y \in \mathbb{R}^M
\]

holds true, then we say \(Q\) is an \(\ell_q\)-stable robust decoder of order \(S\) with respect to \(\|\cdot\|\) for \(A\) with constants with constants \(C\) and \(D\). We shorten this to \(\ell_q\)-SRD of order \(S\) wrt \(\|\cdot\|\) for \(A\) with constants \(C\) and \(D\) and omit parts of it in case they are not of importance or clear from context.

To find such decoders the measurement matrix needs to obey certain properties. We consider a robust null space property introduced in \[1\] Definition 4.21).

Definition 1.2 (\[1\] Definition 4.21). Let \(S \in [N], q \in [1, \infty]\) and \(\|\cdot\|\) be any norm on \(\mathbb{R}^M\) and \(A \in \mathbb{R}^{M \times N}\). If there exist \(\rho \in [0, 1]\) and \(\tau \in [0, \infty)\) such that

\[
\|v|_T\|_q \leq \rho S^{\frac{q}{2} - 1}\|v|_{\tau}\|_1 + \tau \|Av\| \quad \text{for all } \#(T) \leq S \text{ and } v \in \mathbb{R}^N
\]

holds true, then we say \(A\) has the \(\ell_q\)-robust null space property of order \(S\) with respect to \(\|\cdot\|\) with constants \(\rho\) and \(\tau\). We shorten this to \(\ell_q\)-RNSP of order \(S\) wrt \(\|\cdot\|\) with constants \(\rho\) and \(\tau\) and omit parts of it in case they are not of importance or clear from context. \(\rho\) is called stableness constant and \(\tau\) is called robustness constant.
It is well known that certain decoders (basis pursuit denoising, \(\ell_1\)-norm constrained least residual, LASSO) obey robust recovery guarantees if the measurement matrix obeys an RNSP [11 Theorem 4.22][2 Theorem 11.1][2 Theorem 11.1]. However, these fail to define an SRD for \(A\), since they rely on some form of a priori knowledge to achieve these bounds. Only under the additional assumption of a quotient property, the basis pursuit defines an SRD for \(A\), [3].

We introduce the \(p\)th root LASSO (rLASSO) as follows: Let \(|\cdot|\) be any norm on \(\mathbb{R}^M\) and \(\lambda \in [0, \infty)\). Then rLASSO with input \(y \in \mathbb{R}^M\) and \(A \in \mathbb{R}^{M \times N}\) is the optimization problem

\[
\underset{z \in \mathbb{R}^n}{\text{argmin}} \|z\|_1 + \lambda \|y - Az\|.
\]

(rLASSO\(_\lambda\))

rLASSO has been studied in the case \(|\cdot| = \|\cdot\|_2\) under the name square root LASSO [7][8][9][10][11][12][13][14] and in the case \(|\cdot| = \|\cdot\|_1\) under the name least absolute deviation LASSO [15][16][17][18][19][20][21][22][23][24][25][26][27]. The first recovery guarantee of the square root LASSO has been presented in [7]. To obtain recovery guarantees the authors assumed a compatibility condition (or sometimes restricted eigenvalue condition) and other minor conditions. Note that by Proposition 5.1 the compatibility condition is equivalent to the corresponding RNSP, and thus our work is a generalization of [7]. Relations of the compatibility condition to other conditions for LASSO can be found in [30].

One of the first recovery guarantee of the least absolute deviation LASSO was presented in [15]. Closest to our work and results is [27], although it considers a certain structured sparsity model. In particular, under the assumption of a 2-level robust null space property (definition see [15]) they prove that the least absolute deviation LASSO can recover sparse signals exact, even in the presence of sparse noise. There are numerous works considering such stronger requirements that yield exact recovery in the presence of sparse noise [10][20][21][22][23][20][27]. We will not consider such stronger requirements, but only use the weakest requirements possible, i.e. the RNSP.

2 Theoretic Results for rLASSO

2.1 Recovery Guarantee for rLASSO

The main statement here is that robust recovery independent on the noise power is possible, as long as \(\lambda\) is above a threshold. This threshold is a smooth function in the constants \(\tau\) and \(S\) of the RNSP.

**Theorem 2.1 (Recovery with rLASSO).** Let \(A \in \mathbb{R}^{M \times N}\) have \(\ell_q\)-RNSP of order \(S\) wrt \(|\cdot|\) with constants \(\rho\) and \(\tau\). Let

\[
\lambda > \tau S^{1 - \frac{1}{q}}\quad \text{and set} \quad \rho' = \left\{ \begin{array}{ll}
\max \left\{ \frac{\rho}{2\tau}, S^{1 - \frac{1}{q}} \left(1 + \sqrt{\frac{2}{\tau}S^{\frac{1}{q}} - 1}\right) - 1 \right\} & \text{if } q \in (1, \infty] \\
\max \left\{ \frac{\rho}{2\tau}, \frac{\rho}{\tau} - 1 \right\} & \text{if } q = 1
\end{array} \right.
\]

(1)

Then, \(\rho' \in [\rho, 1)\) and for all \(x \in \mathbb{R}^N\) and \(y \in \mathbb{R}^M\) any minimizer \(x^#\) of

\[
\min_{z \in \mathbb{R}^n} \|z\|_1 + \lambda \|y - Az\|
\]

obeys

\[
\|x^# - x\|_q \leq \left\{ \begin{array}{ll}
2 (1 + \rho')^2 S^{\frac{1}{q}} \left[ d_1(x, \Sigma_S) + \frac{3 + \rho'}{1 - \rho'} \tau \left(1 + \frac{\rho'}{1 - \rho'} S^{\frac{1}{q}} - 1\right) \|y - Ax\| \right] & \text{if } q \in (1, \infty) \\
2 (1 + \rho')^2 d_1(x, \Sigma_S) + \frac{2}{1 - \rho'} \left[ \left(1 + \frac{\rho'}{1 - \rho'} \lambda \|y - Ax\| \right] \right. & \text{if } q = 1
\end{array} \right.
\]

(2)

In particular, \(\rho' = \rho\) happens if and only if

\[
\lambda \geq \left\{ \begin{array}{ll}
\frac{3 + \rho'}{1 - \rho'} \tau S^{1 - \frac{1}{q}} \quad \text{if } q \in (1, \infty] \\
\frac{2}{1 - \rho'} \tau \quad \text{if } q = 1
\end{array} \right.
\]

(3)

The result is proven in Subsection 6.1. Note that \(S\) from the threshold is not \(||x||_0\), but the order of the RNSP of the matrix \(A\) and exact recovery in the absence of noise is possible for all \(x\) with \(||x||_0 \leq S\). The case \(q = 1\) is obviously interesting, since the sparsity disappears from the condition for \(\lambda\). If \(\lambda\) obeys (3) with equality, then the bound (2) is the same error bound as the so far best known result for the optimally tuned basis pursuit denoising and \(\ell_1\)-norm constrained least residual [11 Theorem 4.22].

At this point we have three open problems to address. The first problem is: The error bound of the recovery guarantee scales with \(\lambda\). What happens when \(\lambda\) converges to infinity? This problem will be studied in Subsection...
2.2 The second problem is: Is there a recovery guarantee for \( \lambda \leq \tau S^{1/4} \)? This problem is related to the optimality of the recovery guarantee and will be answered in Subsection 2.3. The third problem is: Given a choice on \( \lambda \), how do we determine whether or not the threshold \( \lambda > \tau S^{1/4} \) is fulfilled? This problem will be answered in Section 3. In particular, we will study the threshold explicitly for Gaussian matrices in Subsection 3.1.

Using the threshold (1) or (3) with \( q = 2 \) suggests to choose \( \lambda \approx \sqrt{S} \). A similar argument has been used in [27, Section 5.1.4]. However, this ignores the dependence of \( \tau \) on the dimensions \( M, N, S \). We will give a more detailed analysis by estimating \( \tau \) from the phase transition inequality with Gordon's escape through the mesh [28]. From this it will follow that \( \lambda \approx \sqrt{S} \) is only valid if the number of measurements is suboptimal and close to the optimal number of measurements the tuning parameter scales differently. For the exact results we refer to Theorem 3.2 and the discussion afterwards. Before we proceed with these problems, we formulate one result.

**Corollary 2.2.** Let \( A \in \mathbb{R}^{M \times N} \) have \( \ell_q \)-RNSP of order \( S \) wrt \( || \cdot || \) with constants \( \rho \) and \( \tau \). Then, with \( rLASSO \) there exists an \( \ell_q \)-SRD of order \( S \) wrt \( || \cdot || \) for \( A \) with constants \( C = 2 \frac{1+\rho^2}{1-\rho} \) and \( D = 2 \frac{1+\rho}{1-\rho} \). In particular, if \( q = 1 \), we get the improved constants \( C = 2 \frac{1+\rho^2}{1-\rho} \) and \( D = 2 \frac{1+\rho}{1-\rho} \).

The result is proven in Subsection 6.1.

2.2 Asymptotic Analysis of rLASSO for \( \lambda \to \infty \)

Heuristically, if \( \lambda \) goes to infinity, the second summand of \( rLASSO \) becomes more dominant and we expect that the minimizer needs to be closer to a minimizer of \( \min_{z \in \mathbb{R}^N} ||y - Az|| \). \( rLASSO \) then basically only minimizes \( ||z|| \) under the restriction that \( ||y - Az|| \) is almost minimal. In this section we will prove that indeed for \( \lambda \) going to infinity the minimizers of \( rLASSO \) move closer to the minimizers of

\[
\min_{z \in \text{argmin}_{z' \in \mathbb{R}^N} ||Ax' - y||} ||z||_1. \tag{BPImp}
\]

Note that if \( y \in \text{Ran}(A) \), this problem is the basis pursuit and \( rLASSO \) can be used to approximate a minimizer of basis pursuit. Further, we get a verifiable condition if a minimizer of \( rLASSO \) is also an optimizer of \( \text{BPImp} \).

**Theorem 2.3.** Let \( || \cdot || \) be any norm on \( \mathbb{R}^M \), \( A \in \mathbb{R}^{M \times N} \) and \( y \in \mathbb{R}^M \). For every \( \lambda \in [0, \infty) \) let \( x^\lambda \) be any minimizer of

\[
\min_{z \in \mathbb{R}^N} ||z||_1 + \lambda ||y - Az||.
\]

Then we have the following results:

1. We have two stopping criteria:
   - \( x^\lambda \) is a minimizer of \( \text{BPImp} \) if and only if \( ||x^\lambda||_1 = \inf_{z \in \argmin_{z' \in \mathbb{R}^N} ||Ax' - y||} ||z||_1 \), and
   - \( x^\lambda \) is a minimizer of \( \text{BPTImp} \) if and only if \( ||Ax^\lambda - y|| = \inf_{z' \in \mathbb{R}^N} ||Az' - y|| \).

2. We have the convergence of the stopping criteria
   \[
   ||x^\lambda||_1 \xrightarrow[\lambda \to \infty]{} \inf_{z \in \argmin_{z' \in \mathbb{R}^N} ||Az' - y||} ||z||_1 \quad \text{as} \, \lambda \to \infty \quad \text{and}
   \]
   \[
   ||Ax^\lambda - y|| \xrightarrow[\lambda \to \infty]{} \inf_{z \in \mathbb{R}^N} ||Az' - y|| \quad \text{as} \, \lambda \to \infty \quad \text{with distance bounded by} \, \lambda^{-1} \left( \inf_{z \in \mathbb{R}^N} ||Az' - y|| \right). \tag{5}
   \]

3. The sequence \( x^\lambda \) converges to the set of minimizers of \( \text{BPImp} \), meaning that
   \[
   \lim_{\lambda \to \infty} \inf_{\text{minimizer of} \, \text{BPImp}} ||x^\lambda - z||_2 = 0.
   \]

4. Let \( \lim_{\lambda \to \infty} \lambda^\lambda = \infty \) and consider the sequence \( (x^\lambda)_{\lambda \in \mathbb{N}} \). If this sequence converges or the minimizer of \( \text{BPImp} \) is unique, the sequence converges to a minimizer of \( \text{BPImp} \). In particular, the sequence always has a subsequence converging to a minimizer of \( \text{BPImp} \).

5. If \( y \in \text{Ran}(A) \), we have the convergence of optimal values
   \[
   \lim_{\lambda \to \infty} ||x^\lambda||_1 + \lambda ||Ax^\lambda - y|| = \inf_{z \in \mathbb{R}^N} ||z||_1.
   \]
The result is proven in Subsection 6.2. Surprisingly the convergence for $\lambda \to \infty$ occurs at a finite value, whenever $A$ is surjective, since then the operator $A^T$ is bounded below.

**Proposition 2.4 (Finite Convergence).** Let $||\cdot||$ be a norm on $\mathbb{R}^M$ with dual norm $||\cdot||_*$ := sup$_{||w||\leq 1} ||(w, \cdot)||$. Let $A \in \mathbb{R}^{M \times N}$ be surjective and $\lambda > \lambda_{\infty} := \sup_{0 \neq w \in \mathbb{R}^M} ||w||_{\infty} / ||A^T w||_\infty$. Then, $\lambda_{\infty} < \infty$. Further, for $y \in \mathbb{R}^M$ any minimizer $x^*$ of $\min_{x \in \mathbb{R}^N} ||y - Ax||$ is also a minimizer of $\min_{z \in \mathbb{R}^N} A z = y ||z||$.

The result is proven in Subsection 6.2. If $||\cdot|| = ||\cdot||_p$, the dual norm is $||\cdot||_* = ||(1 - 1/p)^{-1} \cdot ||$ and we can calculate an upper bound for $\lambda_{\infty}$ in polynomial time. Let $A \in \mathbb{R}^{M \times N}$ be surjective and have the singular value decomposition $A = U \Sigma V^T$ with $U \in \mathbb{R}^{M \times M}$, $\Sigma \in \mathbb{R}^{M \times M}$ and $V \in \mathbb{R}^{N \times N}$. Then $\Sigma$ is invertible and $A^T \Sigma := U \Sigma^{-1} V^T \in \mathbb{R}^{M \times N}$ is the Moore-Penrose inverse of $A^T$ and obeys $A^T \Sigma A^T = \text{Id}_{\text{mat}} \in \mathbb{R}^{M \times M}$. It follows that

$$
\lambda_{\infty} = \sup_{0 \neq w \in \mathbb{R}^M} ||(1 - 1/p)^{-1} \cdot ||_{\infty} = \sup_{0 \neq w \in \mathbb{R}^M} ||A^T w||_{\infty} / ||(1 - 1/p)^{-1} \cdot ||_{\infty} = \sup_{0 \neq w \in \text{ran}(A^T)} ||A^T v||_{\infty} / ||v||_{\infty} \leq M^{1 - (1 - 1/p)} \sup_{0 \neq v \in \mathbb{R}^N} ||A^T v||_{\infty} / ||v||_{\infty} = M^\frac{1}{p} ||A^T||_{\infty \to \infty}.
$$

This value is computable in polynomial time, since the norm is the maximum absolute row sum of $A^T$. Other possible bounds involve the smallest, non-zero singular value.

**Large Tuning Parameters for rLASSO and the Quotient Property**

By statement (3) of Theorem 2.3 the minimizers of rLASSO converge to the minimizers of $\text{BPTImp}$ for $\lambda \to \infty$. This suggests that the error bound of Theorem 2.1 is not tight for large $\lambda$ and it should be able to replace it by a constant independent on $\lambda$. Indeed, this is possible if $A$ obeys a quotient property.

**Definition 2.5.** Let $q \in [1, \infty]$ and $||\cdot||$ be any norm on $\mathbb{R}^M$ and $A \in \mathbb{R}^{M \times N}$. If there exists $d \in [0, \infty)$ such that

$$
\text{for all } e \in \mathbb{R}^M \text{ there exists } v \in \mathbb{R}^N \text{ such that } A v = e \text{ and } ||v||_q \leq d ||e||
$$

holds true, then we say $A$ has $\ell_q$-quotient property with constant $d$ relative to $||\cdot||$.

In [3] it was shown that the additional assumption of the quotient property yields that the basis pursuit obeys robust recovery guarantees. Further, it was shown that Gaussian matrices obey a good quotient property with high probability. In [1] Chapter 11.2 the techniques were adapted to account for the RNSP instead of the restricted isometry property. In particular, from [1] Lemma 11.15 and 11.16 we can deduce directly the following result.

**Proposition 2.6 ([1], Chapter 11.2).** Under the additional assumption that $A \in \mathbb{R}^{M \times N}$ has $\ell_q$-quotient property with constant $d$ relative to $||\cdot||$ and that [3] holds true, the error bound [2] of Theorem 2.1 can be replaced by

$$
||x^* - x||_q \leq \left\{ \begin{array}{ll}
\frac{2(1 + \rho^2)}{1 - \rho} S^{1/2} \frac{d_1(x, \Sigma S)}{\sqrt{1 - \rho}} + \frac{(1 + \rho)(3 + \rho)}{1 - \rho^2} S^{1/2} \frac{d + \tau}{d} ||y - Ax||_1 & \text{if } q \in (1, \infty] \\
\frac{3 + \rho}{1 - \rho} d_1(x, \Sigma S) + \frac{3 + \rho}{1 - \rho} ||y - Ax||_1 & \text{if } q = 1
\end{array} \right.,
$$

which is independent on the possibly large $\lambda$.

A sketch for the proof can be found in Subsection 6.2. The strength of rLASSO is not that it achieves the error bound from Proposition 2.6 whenever $A$ suffices a quotient property, but that it achieves stable and robust recovery even if $A$ has a bad quotient property constant. Especially if the number of measurements $M$ increases, the quotient property is harder to fulfill and the constant $d$ gets worse. Thus, the error bound of Proposition 2.6 gets worse, just as the error bound of the minimizer of the basis pursuit [31] Figure 1]. Opposed to that, the stableness constant $\rho$ and robustness constant $\tau$ of the RNSP get better when the number of measurements increases. If we get a good estimate on these parameters, Theorem 2.1 with

$$
\lambda = \left\{ \begin{array}{ll}
\frac{3 + \rho}{1 - \rho} \tau S^{1 - 1/\rho} & \text{if } q \in (1, \infty] \\
\frac{3 + \rho}{1 - \rho} \tau S^{1 - 1/\rho} & \text{if } q = 1
\end{array} \right.
$$

gives an error bound that gets better with increasing number of measurements $M$! This effect will also be verified numerically in Subsection 4.3.
2.3 Equivalent Conditions for Successful Recovery with rLASSO

In this subsection we consider the second problem from Subsection 2.1 i.e. what happens if \( \lambda \) goes to the threshold \( \tau S \frac{1}{2} \) from Theorem 2.1. For this we need to introduce some new null space properties. In Definition 1.2 we have introduced a robust null space property, but before this property was introduced in the way we have it here, there have been different notions of null space properties. We will give a brief history of null space properties, but for a general overview about null space properties we refer the reader to [1, Section 4].

Definition 2.7. Let \( S \in [N] \), \( q \in [1, \infty] \) and \( \| \cdot \| \) be any norm on \( \mathbb{R}^M \) and \( A \in \mathbb{R}^{M \times N} \).

1. If
   \[
   \| v_T \|_q < S^{\frac{q}{2}} - 1 \| v_{|\tau} \|_1 \quad \text{for all } \#(T) \leq S \text{ and } v \in \ker(A) \setminus \{0\}
   \]
   holds true, then we say \( A \) has \( \ell_q \)-null space property of order \( S \). We shorten this to \( \ell_q \)-NSP of order \( S \).

2. If there exists a \( \tau \in [0, \infty) \) such that
   \[
   \| v_T \|_q < S^{\frac{q}{2}} - 1 \| v_{|\tau} \|_1 + \tau \| Av \| \quad \text{for all } \#(T) \leq S \text{ and } v \in \mathbb{R}^N \setminus \{0\}
   \]
   holds true, then we say \( A \) has \( \ell_q \)-only robust null space property of order \( S \) with respect to \( \| \cdot \| \) with constant \( \tau \). We shorten this to \( \ell_q \)-ORNSP of order \( S \) wrt \( \| \cdot \| \) with constant \( \tau \).

We omit parts of these definitions in case they are not of importance or clear from context.

The first null space property used was the NSP for noiseless recovery of sparse signals using the basis pursuit. One of the first uses was [32], although the term null space property was not used. To account for compressibility one considered the stableness constant. In [29] the stable null space property was used for the first time and also the term null space property appeared for the first time. Lastly the robustness constant was added in addition to the stableness constant to account for additive noise. The result is the RNSP defined in Definition 1.2. See for instance [1, Section 4.3]. However, if we only add the robustness to the NSP, we obtain the ORNSP, which only accounts for noise in the measurements but not for compressibility. To the best of the knowledge of the authors this property has not been used before. By Lemma 6.7 and Lemma 6.8 all null space properties introduced are equivalent to each other.

Theorem 2.8 (Equivalent Condition for Stable and Robust Decodability). Let \( S \in [N] \) and \( \| \cdot \| \) be any norm on \( \mathbb{R}^M \). Let \( A \in \mathbb{R}^{M \times N} \) and \( \lambda \in [0, \infty) \). Then, the following are equivalent:

1. \( A \) has \( \ell_1 \)-ORNSP of order \( S \) wrt \( \| \cdot \| \) with constant \( \lambda \).

2. Any decoder \( Q : \mathbb{R}^M \rightarrow \mathbb{R}^N \) such that \( Q(y) \in \argmin_{z \in \mathbb{R}^N} \| z \|_1 + \lambda \| Az - y \| \) for all \( y \in \mathbb{R}^M \) is an \( \ell_1 \)-SRD of order \( S \) wrt \( \| \cdot \| \) for \( A \).

3. For all \( x \in \Sigma_S \) we have \( \{ x \} = \argmin_{z \in \mathbb{R}^N} \| z \|_1 + \lambda \| Az - Ax \| \).

The result is proven in Subsection 6.3. None of the statements is equivalent to \( A \) having \( \ell_1 \)-RNSP of order \( S \) with robustness constant \( \lambda \), since for such a matrix we can only guarantee the \( S \)-RNSP with constant \( \tau > \lambda \), see Lemma 6.8. Further, the equivalence fails to hold if only one possible decoder mapping to solutions of rLASSO is an SRD of order \( S \), but considering all possible decoders allows to draw the result. To find the exact parameters \( \lambda \), which give a recovery guarantee, it thus remains to characterize all constants of the ORNSP. We thus, introduce the NSP shape constant.

Definition 2.9. Let \( S \in [N] \), \( q \in [1, \infty] \) and \( \| \cdot \| \) be a norm on \( \mathbb{R}^M \). Let \( A \in \mathbb{R}^{M \times N} \) have \( \ell_q \)-NSP of order \( S \).

The constant
\[
\tau^0_q := \sup_{\#(T) \leq S} \sup_{v \in \mathbb{R}^N \setminus \ker(A)} \frac{\| v_T \|_q - S^{\frac{q}{2}} - 1 \| v_{|\tau} \|_1}{\| Av \|}
\]
is called NSP shape constant and the function
\[
\rho_q(\tau) := \max \left\{ 0, \sup_{\#(T) \leq S} \sup_{v \in \mathbb{R}^N : v_T \neq 0} \frac{\| v_T \|_q - \tau \| Av \|}{S^{\frac{q}{2}} - 1 \| v_{|\tau} \|_1} \right\} \quad \text{for all } \tau \in \left[ \tau^0_q, \infty \right).
\]
is called NSP shape function.
We say that a random variable is an $N(\mu, \sigma^2)$ random variable, if it is normal distributed with expectation $\mu$ and variance $\sigma^2$. In this section we calculate a threshold for $rLASSO$ to actually recover a signal, we thus have to prove that there are methods to calculate a tuning parameter such that the minimizers of $rLASSO$ are the minimizers of the basis pursuit and come back to the original problem, namely what happens if $\lambda \in (\tau_1^0, \infty)$? Note that this remark only states the optimality of the threshold but not optimality of the bounds of Theorem 2.1.

### 3 Estimating the Tuning Parameter

In this section we consider the remaining problem from Subsection 2.1. Namely, how to determine whether or not the threshold $\lambda > \tau S^{1-\frac{1}{q}}$ from Theorem 2.1 holds true. By Corollary 2.10 it would be sufficient to calculate the NSP shape constant $\tau^0_q$. However, in [33] it was proven that given the order $S$ of a stable null space property, calculating the smallest stableness constant of the stable null space property is NP-hard in general. We have to accept that calculating the NSP shape constant $\tau^0_q$ might also be NP-hard in general. If we want to use $rLASSO$ to actually recover a signal, we thus have to prove that there are methods to calculate a threshold $\lambda$ that obeys a bound for a recovery guarantee in a polynomial time. In Proposition 2.3 we have proven that if $A$ is surjective, there exists a tuning parameter such that the minimizers of $rLASSO$ are the minimizers of the basis pursuit and thus this tuning parameter has a recovery guarantee. Furthermore, it is computable in polynomial time. Hence given the order $S$ of a null space property, there are methods that calculate an upper bound on the NSP shape constant in polynomial time. We thus hope that there are methods that calculate better bounds in polynomial time, although we have no such method yet. The problem becomes easier, when we consider random matrices, since we can use the following idea: For certain random matrices the phase transition inequality is an intrinsic function in the variable $\tau$ and possibly $\rho$. Solving for $\tau$, we get a function that maps the dimensions $M, N, S$ and the constant $\rho$ to a tuning parameter $\lambda$ that obeys [33].
function in the variable \( \tau \) and thus, \( \tau \) can be estimated from the other constants. In view of the threshold \( \tau_0^2 \) of Theorem 2.1 we want to estimate the smallest possible robustness constant, i.e. we want to estimate the NSP shape constant \( \tau_0^2 \). We adapt the proof of \([33]\) Theorem 11 suitably to be able to estimate the best possible \( \tau \). Indeed, the threshold will calculate depends on the following constant.

**Definition 3.1.** Let \( M \in [N] \) and the entries of \( g \in \mathbb{R}^M \) be independent \( \mathcal{N} (0, M^{-1}) \) random variables. Then \( E_M := \mathbb{E} \| g \|_2 \) is called Gordon’s constant.

The constant originates from Gordon’s escape through the mesh theorem \([11, \text{Theorem 9.21}]\), which we will use in the proof of the result. By \([11, \text{Proposition 8.1(b)}]\) Gordon’s constant obeys

\[
E_M := \mathbb{E} \| g \|_2 \leq E_M \leq 1.
\]

For high \( M \) it is thus feasible to estimate \( E_M \approx 1 \), and \( E_M \) should be considered as a constant. The following result is basically \([33]\) Theorem 11, however their result uses estimates that calculate a suboptimal \( \tau \). Since we want to estimate \( \tau_0^2 \), we need to optimize with respect to \( \tau \).

**Theorem 3.2.** Let \( \rho \in (0, 1) \), \( \eta \in (0, 1) \) and the entries of \( A \in \mathbb{R}^{M \times N} \) be independent \( \mathcal{N} (0, M^{-1}) \) random variables. If

\[
\tau \geq \left( E_M - \sqrt{1 + (1 + \rho^{-1})^2} \left( \sqrt{\frac{2 \ln \left( e \frac{N}{S} \right)}{M}} + \sqrt{\frac{S}{M}} - \sqrt{\frac{2}{M} \ln (\eta^{-1})} \right) \right)^{-1} > 0,
\]

then, with probability of at least \( 1 - \eta \), \( A \) has \( \ell_2\)-RNSP of order \( S \) wrt \( \| \cdot \|_2 \) with constants \( \rho \) and \( \tau \). In this case, setting \( \lambda := \frac{3 + \rho}{(1 + \rho)^2} \sqrt{S} \) yields that for all \( x \in \mathbb{R}^N \) and \( y \in \mathbb{R}^M \) any minimizer \( x^\# \) of

\[
\min_{z \in \mathbb{R}^n} \| z \|_1 + \lambda \| y - Az \|_2
\]

obeys

\[
\| x^\# - x \|_2 \leq \frac{2(1 + \rho)^2}{1 - \rho} S^{-\frac{1}{2}} d_1 (x, \Sigma_S) + \frac{3 + \rho}{1 - \rho} \| y - Ax \|_2.
\]

The result is proven in Subsection 7.1. Combining (7) and \( \lambda = \frac{3 + \rho}{(1 + \rho)^2} \sqrt{S} \) yields

\[
\lambda \geq \frac{3 + \rho}{(1 + \rho)^2} \left( E_M - \sqrt{1 + (1 + \rho^{-1})^2} \left( \sqrt{\frac{2 \ln \left( e \frac{N}{S} \right)}{M}} + \sqrt{\frac{S}{M}} - \sqrt{\frac{2}{M} \ln (\eta^{-1})} \right) \right)^{-1} \sqrt{S}.
\]

This allows us to directly estimate \( \lambda \) from \( M, N, S \) and any choice of \( \eta, \rho \), such that the right hand side of (8) is positive. In practice such an estimation is often infeasible due to suboptimally chosen bounds in some inequalities of the proof. Thus, the value of this inequality lies not in the direct relation of the tuning parameter to the dimensions \( M, N, S \), but in the order of this relation.

In particular, (7) yields that the robustness constant depends on the order of the term

\[
\text{SecOrd} := E_M \sqrt{M} - \sqrt{1 + (1 + \rho^{-1})^2} \left( \sqrt{2 \ln \left( e \frac{N}{S} \right)} + \sqrt{S} \right) - \sqrt{2 \ln (\eta^{-1})}.
\]

If for instance for some \( \alpha > 0 \) one of the three equalities

\[
\text{SecOrd} = \begin{cases} \alpha & \alpha S^\frac{1}{2} \\ \alpha M^\frac{1}{2} \end{cases}
\]

holds true, then, with probability of at least \( 1 - \eta \), \( A \) has \( \ell_2\)-RNSP of order \( S \) wrt \( \| \cdot \|_2 \) with constants \( \rho \) and

\[
\tau = \sqrt{M} \text{SecOrd} = \begin{cases} \alpha^{-1} M^\frac{1}{2} & \alpha^{-1} \left( \frac{M}{S} \right)^\frac{1}{2} \end{cases}
\]

respectively. In this cases we should choose the tuning parameter

\[
\lambda = \frac{3 + \rho}{(1 + \rho)^2} \tau S^{1 - \frac{1}{2}} = \frac{3 + \rho}{(1 + \rho)^2} \alpha^{-1} \left( \frac{\sqrt{MS}}{\sqrt{S}} \right).
\]
respectively. Only in the third case this coincides with the simple rule to choose \( \lambda \approx \sqrt{S} \). However, this case is also the case with the most measurements. This suggests that \( \lambda \approx \sqrt{MS} \) is a better choice and requires less measurements. In between we have that \( \lambda \approx \sqrt{M} \) is also a viable choice, that has the advantage that it does not require knowledge about the possible unknown \( S \). In Subsection \[33\] we will verify in a short numeric experiment that a choice of tuning parameter independent on \( S \) is indeed possible.

In any case it should be noted that the choice of the tuning parameter is more complicated than \( \lambda \approx \sqrt{S} \) rather given by the relation in \([5]\). It is rather impressive that, if the measurements are optimal in the sense that \( E_M \sqrt{M} \) is in the same order as \( \sqrt{1 + (1 + \rho^{-1})^2} \left( \sqrt{2S \ln \left( e \frac{N}{S} \right)} + \sqrt{S} \right) + \sqrt{2 \ln(\eta^{-1})} \) with the same leading constant, the tuning parameter depends heavily on the second order term, i.e. the difference

\[
\text{SecOrd} = E_M \sqrt{M} - \sqrt{1 + (1 + \rho^{-1})^2} \left( \sqrt{2S \ln \left( e \frac{N}{S} \right)} + \sqrt{S} \right) - \sqrt{2 \ln(\eta^{-1})}.
\]

It is remarkable that contrary to other compressed sensing results the constant \( C \) from the phase transition inequality \( M \geq CS \ln \left( e \frac{N}{S} \right) \), is of importance. Not only this, also the second order term \( \sqrt{M} - \sqrt{\frac{1}{\rho} \sqrt{S \ln \left( e \frac{N}{S} \right)}} \) is of importance as well.

In view of \([Corollary 2.10]\) it is desirable to estimate \( \tau_2^0 \) and thus \( \tau_1^0 \leq \tau_2^0 \sqrt{S} \) as good as possible to determine the exact set of tuning parameters that yield an \( \ell_2 \)-SRD for \( A \). If the entries of \( A \in \mathbb{R}^{M \times N} \) are independent \( N(0, M^{-1}) \) random variables, then \( A \) can have \( \ell_2 \)-NSP of order \( S \). Restricted to this event, the NSP shape constant \( \tau_2^0 \) is well defined and a random variable. The following result states two things. At first it bounds the probability that \( \tau_2^0 \) is bounded above by a constant. Secondly given a sufficiently large tuning parameter, it bounds the probability that the sufficient condition for recovery with rLASSO is fulfilled.

**Proposition 3.3.** Let the entries of \( A \in \mathbb{R}^{M \times N} \) be independent \( N(0, M^{-1}) \) random variables. If

\[
\lambda > \sqrt{S} \left( E_M - \sqrt{S} \left( \sqrt{2 \frac{S}{M} \ln \left( e \frac{N}{S} \right)} + \sqrt{S} \right) \right)^{-1} > 0,
\]

then, with probability of at least

\[
1 - \exp \left( -\frac{1}{2} \left( E_M - \sqrt{S} \left( \sqrt{2 \frac{S}{M} \ln \left( e \frac{N}{S} \right)} + \sqrt{S} \right)^2 \right) \right) \in (0, 1),
\]

\( A \) has \( \ell_2 \)-NSP of order \( S \) and the NSP shape constant obeys \( \lambda > \tau_2^0 \sqrt{S} \). In this case any decoder \( Q : \mathbb{R}^M \to \mathbb{R}^N \) with \( Q(y) \in \text{argmin} \{ \|z\|_1 + \lambda \|Az - y\|_2 \} \) for all \( y \in \mathbb{R}^M \) is an \( \ell_2 \)-SRD of order \( S \) wrt \( \|\cdot\|_2 \) for \( A \).

The proof can be found in Subsection \[7.1\].

### 3.2 Random Walk Matrices of Uniformly Distributed \( D \)-Left Regular Bipartite Graphs

In view of \([Theorem 2.1]\) it is desirable to find matrices, which obey the \( \ell_1 \)-RNSP, since \( S \) does not appear in the threshold for \( \lambda \). To generate such matrices we introduce left regular bipartite graphs. Although we will not present a detailed analysis as in Subsection \[3.1\], we can still deduce a result for the least absolute deviation LASSO.

**Definition 3.4.** Let \( A \in \{0, 1\}^{M \times N} \) and \( D \in [M] \). For \( T \subseteq N \) we set \( \text{Row}(T) := \bigcup_{n \in T} \{ m \in [M] : A_{m,n} = 1 \} \). If \( \#(\text{Row}(\{n\})) = D \) for all \( n \in [N] \), then \( D^{-1}A \) is called a random walk matrix of a \( D \)-left regular bipartite graph.

Uniformly at random chosen \( D \)-left regular bipartite graph are similar to Gaussian matrices. In particular, if \( M \geq CS \log \left( e \frac{N}{S} \right) \) they obey with high probability the lossless expansion property, which is the \( \ell_1 \) counterpart to the restricted isometry property and yields an \( \ell_1 \)-RNSP. For more details see Subsection \[7.2\].

**Theorem 3.5 (Left Regular Bipartite Graph \( \Rightarrow \lambda = 3 \).** Let \( \theta \in (0, \frac{1}{2}) \) and \( D := \left\lfloor \frac{2}{\theta} \ln \left( e \frac{N}{2S} \right) \right\rfloor \). Let \( A \in \{0, D^{-1}\}^{M \times N} \) be a uniformly at random chosen \( D \)-left regular bipartite graph. If

\[
M \geq \frac{4}{\theta} \exp \left( \frac{2}{\theta} \right) S \ln \left( e \frac{N}{2S} \right),
\]
then, with probability of at least $1 - \frac{1}{\ell^2}$, the matrix $A$ has $\ell_1$-RNSP of order $S$ w.r.t $\|\cdot\|_1$ with constants $\rho = \frac{2\theta}{1 - \theta}$ and $\tau = \frac{1}{1 - 4\theta}$. In this case, setting $\lambda := \frac{2}{1 - \rho} \tau = \frac{2}{1 - 2\theta} \in (2, 3)$ yields that for all $x \in \mathbb{R}^N$ and $z \in \mathbb{R}^M$ any minimizer $x^\#$ of

$$\min_{z \in \mathbb{R}^N} \|z\|_1 + \|y - Az\|_1$$

obeys

$$\|x^\# - x\|_1 \leq 2 \frac{1 - 2\theta}{1 - 6\theta} S \|x\|_1 + \frac{4}{1 - 6\theta} \|y - Ax\|_1.$$ 

The result is proven in Subsection 7.2. In Subsection 4.4 we will verify that the upper bound $\lambda := 3$ is a viable choice for rLASSO with uniformly at random chosen $D$-left regular bipartite graph. Note that rLASSO defines an $\ell_1$-SRD of order $S$ w.r.t $\|\cdot\|_1$ in this case and can be used without prior information about the noise $e$, the signal $x$ or the order $S$ of the RNSP.

4 Numerical Experiments

For $q \in [1, \infty]$ and $N \in \mathbb{N}$ we denote the unit sphere in the $\ell_q$ norm by $S^{N-1}_q := \{z \in \mathbb{R}^N : \|z\|_q = 1\}$. Given $q \in \{1, 2\}$ we will compare the estimation error of the the minimizers of

$$\min_{z \in \mathbb{R}^N} \|z\|_1 + \lambda \|Az - y\|_q$$

against the estimation errors of the following well known minimizers

$$\min_{x \in \mathbb{R}^N, Az = y} \|z\|_1,$$ (BP)

$$\min_{x \in \mathbb{R}^N, \|Az - y\|_q \leq \varepsilon} \|z\|_1,$$ with $\varepsilon := \|e\|_q$ (BPDN)

$$\min_{x \in \mathbb{R}^N, \|Az - y\|_q \leq \tau} \|Az - y\|_q$$ with $\tau := \|x\|_1.$ (CLR)

We will estimate some minimizer of these problems using the CVX toolbox of Matlab. For the BPDN and the CLR we use the optimal tuning $\varepsilon = \|e\|_q$ and $\tau = \|x\|_1$ respectively. These two represent a best case benchmark using the unknown prior information $\|e\|_q$ and $\|x\|_1$. As a worst case benchmark we use the BP, as it requires no prior information about the noise or the signal. Tuning the BPDN too high often leads to worse estimation errors than tuning it too low [37, Figure 1]. Thus, it is reasonable to choose the BP as the best worst case benchmark. Given $M, N, S, D, SNR \in \mathbb{N}$ the following experiment will be repeated multiple times:

Experiment 1. For each $k \in [100]$ and given SNR do the following:

1. If $q = 2$ draw each component of $A_k \in \mathbb{R}^{M \times N}$ independent as $N(0, M^{-1})$ random variable. If $q = 1$ draw $A_k \in \{0, D^{-1}\}^{M \times N}$ as a uniformly at random drawn $D$-left regular bipartite graph.
2. Draw the signal $x_k$ uniformly at random from $\Sigma_S \cap S^{N-1}_q$.
3. Draw the noise $e_k$ uniformly at random from $\|A_k x_k\|_q \mathcal{N}(0, 1)$.
4. Define the observation $y_k := A_k x_k + e_k$.
5. For each optimization problem estimate a minimizer by $x^k_\#$ and collect the relative estimation errors $\frac{\|x_k - x^k_\#\|_q}{\|x_k\|_q}$ and the noise powers $\|e_k\|_q$.

Calculate the mean normalized $\ell_q$-estimation error and the mean normalized $\ell_q$-estimation error per noise power, i.e.

$$\frac{\|x - x^\#\|_q}{\|x\|_q} := \frac{1}{100} \sum_{k \in [100]} \frac{\|x_k - x^\#_k\|_q}{\|x_k\|_q}$$ and $$\frac{\|x - x^\#\|_q}{\|x\|_q \|e\|_q} := \frac{1}{100} \sum_{k \in [100]} \frac{\|x_k - x^\#_k\|_q}{\|x_k\|_q \|e_k\|_q}.$$ 

where the left hand sides are understood as an assigned symbol.

Note that in this experiment we have that the relative and absolute $\ell_q$ estimation errors coincide, i.e. $\frac{\|x_k - x^\#_k\|_q}{\|x_k\|_q} = \|x_k - x^\#_k\|_q$, since $x_k$ is normalized.
For $S = 256$, the BPDN only performs as a best case benchmark for $\lambda$. Against our expectation, the BPDN only performs as a best case benchmark for $\lambda$. As expected, the CLR performs as a best case benchmark and the BPDN performs like a worst case benchmark. All estimation errors grow rapidly, and we expect that the recovery fails. Thus, we discarded plots for $\lambda$.

We fix the parameters $N$, $\lambda$, and $\tau$ as proposed in Theorem 3.2 and in the argumentation around (7). Thus, we set $\lambda$ exactly like the basis pursuit. The tuning methods even better estimation errors than the optimally tuned BPDN. However, rLASSO does not require any prior information about the noise or the signal. We deduce that rLASSO is noise-blind.

We investigate the dependence of the tuning parameter $\lambda$ on the dimensions $N$, $M$, $S$, for Gaussian matrices as proposed in Theorem 3.2 and in the argumentation around (7). Thus, we set $q = 2$. We fix $N = 1024$, $M = 256$, $S = 32$, and vary the signal-to-noise-ratio $SNR = \sqrt{M}/\sqrt{\lambda}$ in Experiment 1. The results are plotted in Figure 1(a). For $\lambda \leq 8$ the recovery with rLASSO seems to fail. For $\lambda > 8$, the recovery with rLASSO succeeds, but the estimation errors of BP and rLASSO are the same, suggesting that they return a similar minimizer, as proposed in Theorem 2.3. For $\lambda \to \infty$, the estimation error does not diverge to infinity and is capped by the estimation error from the basis pursuit, which is a consequence of the quotient property. This coincides with the results of Proposition 2.6. The recovery succeeds roughly for $\lambda \geq 8$, suggesting that $\tau_1^0 = 8$ and thus $\tau_2^0 = \tau_1^0 = \sqrt{2}$. The optimal estimation error is achieved at $\lambda = 10.5$.

4.2 Noise Blindness

We will verify that the choice of $\lambda$ is indeed independent on the noise power. For this we consider the Gaussian case $q = 2$. We fix $N = 1024$, $M = 256$, $S = 32$, and vary the signal-to-noise-ratio $SNR = \sqrt{M}/\sqrt{\lambda}$ in Experiment 1. The resulting errors are plotted in Figure 1(b). Note that due to a quotient property and the optimal tuning, all decoders achieve robust recovery guarantees. In particular, since $\|x_k\|_2 = 1$, Theorem 3.2 and corresponding results yield that for each decoder the quantity

$$\frac{\|x - x_k\|_q}{\|x\|_q} = \frac{1}{100} \sum_{k \in [100]} \frac{\|x_k - x_k^0\|_q}{\|x_k\|_q} \|e_k\|_q = \frac{1}{100} \sum_{k \in [100]} \frac{\|x_k - x_k^0\|_q}{\|e_k\|_q}$$

should be bounded by some constant $D$ independent on the signal-to-noise-ratio $SNR$. Indeed, we can see that this quantity stays constant and even the relative proportions between all decoders stay constant. Remarkably, rLASSO with the tuning $\lambda = 0.65\sqrt{M}$ achieves the same estimation errors as the optimally tuned CLR and even better estimation errors than the optimally tuned BPDN. However, rLASSO does not require any prior information about the noise or the signal. We deduce that rLASSO is noise-blind.

4.3 Gaussian Matrices

We investigate the dependence of the tuning parameter $\lambda$ on the dimensions $M$, $N$, $S$ for Gaussian matrices as proposed in Theorem 3.2 and in the argumentation around (7). Thus, we set $q = 2$. We fix $N = 1024$, $M = 256$, $SNR = 100$, and vary $S \in [128]$ in Experiment 1. The results are plotted in Figure 2(a). For $S \geq 60$ all estimation errors grow rapidly, and we expect that the recovery fails. Thus, we discarded plots for $S > 60$. As expected, the CLR performs as a best case benchmark and the BP performs like a worst case benchmark. Against our expectation, the BPDN only performs as a best case benchmark for $S \leq 15$. The authors have no explanation for that yet. For $S \geq 30$ the tuning $\lambda = 2.5\sqrt{S}$ and for $S \geq 45$ the tuning $\lambda = 2\sqrt{S}$ perform exactly like the basis pursuit. The tuning methods $\lambda = 2^{-\frac{1}{2}}\sqrt{M}$ and $\lambda = 0.65\sqrt{M}$ do not suffer from this problem. In particular, they perform nearly as good as the best case benchmark CLR for $S \geq 20$ and $S \geq 15$ respectively. However, the tuning $\lambda = 0.65\sqrt{M}$ performs slightly better than the tuning $\lambda = 2^{-\frac{1}{2}}\sqrt{M}$ for $S \leq 40$.
while the opposite is true for $S \geq 40$. We deduce that the tuning parameter might be chosen independent of $S = \|x\|_0$, and still achieve fine error bounds that might even outperform tuning with $\lambda \propto \sqrt{S}$. Since for $S \leq 10$ all tuning methods of rLASSO perform sub-optimal, we deduce that the optimal tuning parameter depends non-trivially on $S$. As an alternative experiment we fix $N = 1024, S = 32, SNR = 100$ and vary $M = 5[100]$ in [Experiment 1]. The results are plotted in Figure (2b). For $M \leq 270$ the tuning $\lambda = 2.5\sqrt{S}$ performs exactly like the BPDN and the tuning $\lambda = 2\sqrt{S}$ has a similar problem for $M \leq 220$. Also if $M \geq 400$, the tuning $\lambda = 2\sqrt{S}$ gets suboptimal. The tunings $\lambda = 2^{-\frac{1}{2}}\sqrt{M}$ and $\lambda = 0.65\sqrt{M}$ do not share these same problems. In particular, the tuning parameter $\lambda = 0.65\sqrt{M}$ seems to be indistinguishable from the optimal benchmark. We deduce that $\lambda \propto \sqrt{M}$ might reflect the behavior of the tuning parameter better than $\lambda \propto \sqrt{S}$. If the number of measurements increases, the gap between rLASSO and BP increases. This is due to the fact that the quotient property gets harder to fulfill and thus the constant of the quotient property gets worse. Thus, rLASSO works better than the basis pursuit with a quotient property, whenever the number of measurements is not optimal, as it was proposed in the argumentation following [Proposition 2.6]...

4.4 Random Walk Matrices of Uniformly Distributed $D$-Left Regular Bipartite Graphs

We will verify that the choice of tuning parameters from [Theorem 3.5] is viable and thus set $q = 1$. We fix $N = 1024, M = 256, D = 10, SNR = 100$ and vary $S \in [128]$ in [Experiment 1]. The results are plotted in Figure (3). Similar to Gaussian matrices the estimation errors grow rapidly for $S \geq 60$ and we expect that the recovery fails. For $S \geq 35$ the tuning $\lambda = 2$ fails, for $S \geq 55$ the tuning $\lambda = 3$ fails. For smaller $S$, smaller tuning parameters perform better. This suggests that the optimal tuning parameter is depending on at least the dimensions $S$, even though the requirement $\lambda > \tau S^{1-\frac{1}{4}} = \tau$ suggests otherwise. We deduce that recovery with rLASSO with a constant tuning parameter is possible, but the optimal tuning is depending at least on...
some parameters. A more detailed analysis, as it was done for Gaussian matrices in Proposition 3.2 is required to understand the optimal tuning parameter better.

5 Proofs of Section 1 Preliminaries

The RNSP and the compatibility condition are equivalent.

Proposition 5.1. Let $S \in [N], A \in \mathbb{R}^{M \times N}$ and $\# (T) \leq S$. Then, the following statements are equivalent

1. There exists $\rho \in (0, 1)$ and $\tau \in (0, \infty)$ such that
   $$\|v\|_{T} \leq \rho \|v\|_{T^c} + \tau \|Av\|_{2}$$
   for all $v \in \mathbb{R}^{N}$ and $v_{T} \neq 0$ holds.

2. There exists $L \in (1, \infty)$ such that for the set
   $$\Delta_{L,T} := \{v \in \mathbb{R}^{N} : \|v\|_{T} \leq L \|v\|_{1} \text{ and } \|v\|_{T^c} \neq 0\}$$
   the condition
   $$\inf_{v \in \Delta_{L,T}} \frac{S_{L,T}^{\|Av\|_{2}^{2}}}{\|v\|_{T}^{2}} > 0$$
   holds true.

The constants may change and this change may depend on the dimensions.

Proof. (1)⇒(2): Pick any $L \in (1, \rho^{-1})$, which is a non empty interval since $\rho < 1$. Let $v \in \Delta_{L,T}$ be arbitrary. The assumption yields

$$\|v\|_{T} \leq \rho \|v\|_{T^c} + \tau \|Av\|_{2} \leq \rho L \|v\|_{T^c} + \tau \|Av\|_{2}$$

and by algebraic manipulation $\frac{1-\rho L}{\tau} \leq \frac{\|Av\|_{2}}{\|v\|_{T^c}}$. By the choice of $L$ we get

$$\inf_{v \in \Delta_{L,T}} \frac{S_{L,T}^{\|Av\|_{2}^{2}}}{\|v\|_{T}^{2}} \geq S \left(1 - \frac{\rho L}{\tau}\right)^{2} > 0. \quad (10)$$

(2)⇒(1): Set $\rho := L^{-1}$ and $\tau := S_{L,T}^{\frac{\|Av\|_{2}^{2}}{\|v\|_{T}^{2}}}$. Then, $\rho \in (0, 1)$ and $\tau > 0$. At first let $v \notin \Delta_{L,T}$. If $\|v\|_{T} = 0$ the bound to prove holds trivially. So let $\|v\|_{T} \neq 0$. Hence, we have for any $\tau' > 0$

$$\|v\|_{T} < L^{-1} \|v\|_{T^c} \leq \rho \|v\|_{T^c} + \tau' \|Av\|_{2},$$

Now on the other hand assume that $v \in \Delta_{L,T}$. Then we have for any $\rho' > 0$

$$\|v\|_{T} = \left(\frac{\|v\|_{T^c}}{S_{L,T}^{\|Av\|_{2}}}\right)^{\frac{1}{2}} S_{L,T}^{\frac{\|Av\|_{2}^{2}}{\|v\|_{T}^{2}}} \leq \left(\inf_{v \in \Delta_{L,T}} \frac{S_{L,T}^{\|Av\|_{2}^{2}}}{\|v\|_{T}^{2}}\right)^{-\frac{1}{2}} S_{L,T}^{\frac{\|Av\|_{2}^{2}}{\|v\|_{T}^{2}}} \|Av\|_{2} = \tau \|Av\|_{2} \leq \rho' \|v\|_{T^c} + \tau' \|Av\|_{2}.$$ 

This finishes the proof for all $v \in \mathbb{R}^{N}$ if $\rho'$ and $\tau'$ are the particular $\rho$ and $\tau$ above.

6 Proofs of Section 2 Theoretic Results for rLASSO

6.1 Proofs of Subsection 2.1 Recovery Guarantees for rLASSO

The following statement is well known for $p, q \in [1, \infty)$ as Stechkin bound, see for instance [1] Proposition 2.3. We note that it holds even for $p, q \in [1, \infty]$.

Lemma 6.1 ( [1] Proposition 2.3 ). Let $p, q \in [1, \infty], S \in [N]$ and $v \in \mathbb{R}^{N}$ with $q \geq p$. Then,

$$d_{q}(v, \Sigma_{S}) \leq S^{\frac{1}{q'}} \|v\|_{p}.$$ 

Proof. For $q, p \in [1, \infty]$ the statement is [1] Proposition 2.3. Now let $p, q \in [1, \infty]$ with $q \geq p$ and $v \in \mathbb{R}^{N}$. Let $T$ be the set of the $S$ indices with largest absolute value of $v$. Then $\inf_{x \in \Sigma_{S}} \|v - z\|_{q} = \|v - v_{T}\|_{q}$. The proof now follows from a limit argument and the fact that $\lim_{r \to \infty} ||v'||_{r} = ||v'||_{\infty}$ holds true for all $v' \in \mathbb{R}^{N}$, which is a consequence of $||v||_{\infty} \leq ||v||_{r} \leq N^{\frac{1}{r}} ||v||_{\infty}$. 

\[ \boxdot \]
To prove recovery guarantees we require a lemma, which is proven in [1, Theorem 4.25] and [1, Theorem 4.20]. With the improved Stechkin bound the proof of [1, Theorem 4.25] also holds for \( q = \infty \).

**Lemma 6.2 (1, Theorem 4.25) & (1, Theorem 4.20).** Let \( A \in \mathbb{R}^{M \times N} \) have \( \ell_q \)-RNSP of order \( S \) wrt \( \| \cdot \| \) with constants \( \rho \) and \( \tau \). Then, for all \( x', z \in \mathbb{R}^N \) it holds that

\[
\| z - x' \|_q \leq \begin{cases} \frac{(1 + \rho)^2}{1 - \rho} S^{1 - \frac{q}{2}} \left( \| z \|_1 - \| x' \|_1 + 2d_1 (x', \Sigma_S) \right) + \frac{3 + \rho}{1 - \rho} \tau \| Ax' - Az \| & \text{if } q \in (1, \infty) \\
\frac{3 + \rho}{1 - \rho} \tau S^{1 - \frac{q}{2}} & \text{if } q = 1 \end{cases}
\]

We prove an auxiliary statement that looks similar to Theorem 2.1.

**Theorem 6.3 (Weak bound on \( \lambda \)).** Let \( A \in \mathbb{R}^{M \times N} \) have \( \ell_q \)-RNSP of order \( S \) wrt \( \| \cdot \| \) with constants \( \rho \) and \( \tau \). Let

\[
\lambda \geq \begin{cases} \frac{3 + \rho}{1 - \rho} \tau S^{1 - \frac{q}{2}} & \text{if } q \in (1, \infty) \\
\frac{2}{1 - \rho} d_1 (x, \Sigma_S) + \left( \frac{3 + \rho}{1 - \rho} \tau \right) \| y - Ax \| & \text{if } q = 1 \end{cases}
\]

Then, for all \( x \in \mathbb{R}^N \) and \( y \in \mathbb{R}^M \) any minimizer \( x^\# \) of

\[
\min_{x \in \mathbb{R}^N} \| x \|_1 + \lambda \| y - Ax \|
\]

obeys

\[
\| x^\# - x \|_q \leq \begin{cases} \frac{2 (1 + \rho)^2}{1 - \rho} S^{1 - \frac{q}{2}} - d_1 (x, \Sigma_S) + \frac{3 + \rho}{1 - \rho} \tau \| y - Ax \| & \text{if } q \in (1, \infty) \\
\frac{3 + \rho}{1 - \rho} \tau S^{1 - \frac{q}{2}} - d_1 (x, \Sigma_S) + \left( \frac{3 + \rho}{1 - \rho} \tau \right) \| y - Ax \| & \text{if } q = 1 \end{cases}
\]

**Proof.** We apply Lemma 6.2 for \( q \in (1, \infty) \) with \( x' := x \) and \( z := x^\# \) and obtain

\[
\| x^\# - x \|_q \leq \begin{cases} \frac{(1 + \rho)^2}{1 - \rho} S^{1 - \frac{q}{2}} - d_1 (x, \Sigma_S) + \frac{3 + \rho}{1 - \rho} \tau \| y - Ax \| & \text{if } q \in (1, \infty) \\
\frac{2 (1 + \rho)^2}{1 - \rho} S^{1 - \frac{q}{2}} - d_1 (x, \Sigma_S) + \left( \frac{3 + \rho}{1 - \rho} \tau \right) \| y - Ax \| & \text{if } q = 1 \end{cases}
\]

Since \( x \) is feasible and \( x^\# \) is a minimizer, we get

\[
\| x^\# - x \|_q \leq \frac{2 (1 + \rho)^2}{1 - \rho} S^{1 - \frac{q}{2}} - d_1 (x, \Sigma_S) + \frac{3 + \rho}{1 - \rho} \tau \| y - Ax \| + \left( \frac{3 + \rho}{1 - \rho} \tau \right) \| y - Ax \|.
\]

This proves the case \( q \in (1, \infty) \). For the other case we apply Lemma 6.2 for \( q = 1 \) with \( x' := x \) and \( z := x^\# \) and obtain

\[
\| x^\# - x \|_1 \leq \frac{1 + \rho}{1 - \rho} \| x^\# \|_1 - \| x \|_1 + 2d_1 (x, \Sigma_S) + \frac{2}{1 - \rho} \| y - Ax \|
\]

\[
\leq \frac{1 + \rho}{1 - \rho} \| x^\# \|_1 - \| x \|_1 + 2d_1 (x, \Sigma_S) + \frac{2}{1 - \rho} \| y - Ax \| + \frac{2}{1 - \rho} \| y - Ax \|.
\]

\[
= 2 \frac{1 + \rho}{1 - \rho} d_1 (x, \Sigma_S) + \frac{1 + \rho}{1 - \rho} \| y - Ax \| + \frac{1 + \rho}{1 - \rho} \left( \| x^\# \|_1 + \frac{2}{1 + \rho} \| y - Ax \| \right)
\]

\[
\leq 2 \frac{1 + \rho}{1 - \rho} d_1 (x, \Sigma_S) + \frac{1 + \rho}{1 - \rho} \| x \|_1 + \frac{2}{1 - \rho} \| y - Ax \| + \frac{1 + \rho}{1 - \rho} \left( \| x^\# \|_1 + \frac{2}{1 + \rho} \| y - Ax \| \right).
\]
Since $\mathbf{x}$ is feasible and $x^\#$ is a minimizer, we obtain

$$
\|x^\# - x\|_1 \leq 2\frac{1 + \rho}{1 - \rho} d_1 (x, \Sigma_S) + \frac{1 + \rho}{1 - \rho} \|x\|_1 + \frac{2}{1 - \rho} \tau \|y - A x\| + \frac{1 + \rho}{1 - \rho} \| y - A x \|_1,
$$

$$
= 2\frac{1 + \rho}{1 - \rho} d_1 (x, \Sigma_S) + \left( \frac{2}{1 - \rho} \tau + \frac{1 + \rho}{1 - \rho} \lambda \right) \|y - A x\|.
$$

This proves the case $q = 1$.

It turns out that the bounds on $\lambda$ of Theorem 6.3 are not tight. We can even use smaller parameters in a trade-off with possibly worse bounds for the estimation error. The reason for this is that the functions $\rho \mapsto \frac{1 + \rho}{1 - \rho}$ and $\rho \mapsto \frac{1}{\rho}$ are monotonically decreasing on $[0, 1)$. Thus, if we artificially assume that $A$ has a worse stableness constant $\rho'$ the bound on $\lambda$ will get loosed and we can deduce Theorem 2.1 from Theorem 6.3.

**Proof of Theorem 2.1** Note that the RNSP is being preserved under increases in the stableness constant $\rho$. Hence, $\rho' \in [\rho, 1)$ yields that $A$ has the $\ell_q$-RNSP of order $S$ wrt $\| \cdot \|$ with constants $\rho'$ and $\tau$ and the error bound of Theorem 6.3 follows from $\rho' \mapsto \frac{1}{\rho'}$. It is thus sufficient to prove $\rho \in [0, 1)$ and the “in particular part”. At first let $q \in (1, \infty]$. Note that the function $f: [0, 1] \to [1, 3]$ mapping $t$ to $f(t) := \frac{3 + \rho}{1 + \rho} t S^{\frac{1}{2} - 1}$ is by differentiation strictly monotonically decreasing. Hence it is invertible and its inverse is also strictly monotonically decreasing and given by $g : [1, 3] \to [0, 1]$ mapping $r$ to $g(r) := \frac{r}{3} (1 + \sqrt{8r + 1}) - 1$. Now $\lambda \geq \frac{3 + \rho}{1 + \rho} t S^{\frac{1}{2} - 1}$ is equivalent to $\frac{1}{\tau} S^{\frac{1}{2} - 1} \leq f(\rho)$ and by the strict monotonicity we get the logical statement

$$
\lambda \geq \frac{3 + \rho}{1 + \rho} r S^{\frac{1}{2} - 1} \Leftrightarrow g\left(\frac{\lambda}{\tau}\right) \leq \rho.
$$

(11)

The definition of $\rho'$ can be rewritten as $\rho' = \max \left\{ \rho, g\left(\frac{1}{\tau} S^{\frac{1}{2} - 1}\right) \right\}$. By (11) we get the “in particular part”, namely that $\lambda \geq \frac{3 + \rho}{1 + \rho} r S^{\frac{1}{2} - 1}$ is equivalent to $\rho' \geq \rho$. In order to prove $\rho' \in [\rho, 1)$ we distinguish two cases. If $\lambda \geq \frac{3 + \rho}{1 + \rho} r S^{\frac{1}{2} - 1}$, then (11) yields that $\rho' = \max \left\{ \rho, g\left(\frac{1}{\tau} S^{\frac{1}{2} - 1}\right) \right\} = \rho$, and thus $\rho' \in [\rho, 1)$. If $\rho \leq \frac{3 + \rho}{1 + \rho} r S^{\frac{1}{2} - 1}$, then (11) yields that $\rho' = \max \left\{ \rho, g\left(\frac{1}{\tau} S^{\frac{1}{2} - 1}\right) \right\} = g\left(\frac{1}{\tau} S^{\frac{1}{2} - 1}\right) > \rho$. Now by assumption $\lambda \leq r S^{\frac{1}{2} - 1}$, which is equivalent to $\frac{1}{\tau} S^{\frac{1}{2} - 1} > 1$. By the strict monotonicity of $g$ we have $\rho' = g\left(\frac{1}{\tau} S^{\frac{1}{2} - 1}\right) < g(1) = 1$ and thus $\rho' \in [\rho, 1)$, i.e., the case $q = 1$ works similarly by choosing $f(t) := \frac{2}{1 + \tau}$, since then $g(r) := \frac{r}{\tau} - 1$.

From Theorem 6.3 we can deduce that rLASSO defines an SRD.

**Proof of Corollary 2.2** Let $q > 1$ and set $\lambda := \frac{3 + \rho}{1 + \rho} t S^{\frac{1}{2} - 1}$. For $y$ we set $Q(y)$ as any minimizer of $\ell_q$-RNSP with input $y$ and $A$. This defines a mapping $Q : \mathbb{R}^M \to \mathbb{R}^N$. By Theorem 6.3 with $q \in (1, \infty]$ we have for all $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$ that

$$
\|Q(y) - x\|_q \leq 2\frac{(1 + \rho)^2}{1 - \rho} S^{\frac{1}{2} - 1} d_1 (x, \Sigma_S) + \left( 3 + \rho + \frac{1 + \rho}{1 - \rho} S^{\frac{1}{2} - 1} \lambda \right) \|y - A x\| + \frac{3 + \rho}{1 - \rho} \|y - A x\|_1.
$$

Thus, $Q$ is an $\ell_q$-SRD of order $S$ wrt $\| \cdot \|$ for $A$ with constants $C, D$. For the “in particular part” we repeat the same steps with $\lambda = \frac{3 + \rho}{1 + \rho} t$ and use the bound of Theorem 6.3 with $q = 1$ instead.

### 6.2 Proofs of Subsection 2.2 Asymptotic Analysis for rLASSO

To prove the claimed convergence, we prove an auxiliary statement.

**Lemma 6.4.** Let $\| \cdot \|$ be any norm on $\mathbb{R}^M$, $A \in \mathbb{R}^{M \times N}$ and $y \in \mathbb{R}^M$. For every $\lambda \in [0, \infty)$ let $x^\lambda$ be any minimizer of

$$
\min_{x \in \mathbb{R}^N} \|x\| + \lambda \|y - Ax\|.
$$

Then, $\{x^\lambda\}$ has a minimizer $x^\text{BPImp}$ and the following statements hold true:

(a) The function $\lambda \mapsto \|x^\lambda\|_1$ is monotonically increasing.
(b) The function $\lambda \mapsto \|A\lambda - y\|$ is monotonically decreasing.

(c) The estimators are bounded $\|x^\lambda\|_1 \leq \|x^{BPImp}\|_1$.

(d) The residuals are bounded

$$\|A\lambda - y\| \leq \|A x^{BPImp} - y\| + \lambda^{-1} (\|x^{BPImp}\|_1 - \|x^\lambda\|_1). \tag{12}$$

(e) If $\|x^\lambda\|_1 \geq \|x^{BPImp}\|_1$, then $x^\lambda$ is a minimizer of $\{BPImp\}$.

(f) If $\|A\lambda - y\| \leq \|A x^{BPImp} - y\|$, then $x^\lambda$ is a minimizer of $\{BPImp\}$.

Proof. At first we will prove that $\{BPImp\}$ indeed has an optimizer. Let $(z'_n)_{n \in N}$ be a sequence such that $\lim_{n \to \infty} \|Az'_n - y\| = \inf_{z' \in \mathbb{R}^N} \|Az' - y\|$. Let $z''_n$ be the orthogonal projection of $z'_n$ onto $\ker (A)^\perp$. Since $A$ is injective on the finite-dimensional space $\ker (A)^\perp$, it is also bounded below on this, i.e. there exists some $C > 0$ such that $\|Az''_n\| \geq C \|z''_n\|_2$ for all $n \in [N]$. Hence,

$$\|z''_n\|_2 \leq C^{-1} \|Az''_n\| \leq C^{-1} (\|Az''_n - y\| + \|y\|).$$

This together with the convergence of $\|Az'_n - y\|$ yields that $\|z''_n\|_2$ is bounded and thus $(z''_n)_{n \in N}$ contains some convergent subsequence $(z'''_n)_{n \in N}$ converging to some $z'''_n$. It follows that

$$\inf_{z' \in \mathbb{R}^N} \|Az' - y\| = \lim_{n \to \infty} \|Az'_n - y\| = \lim_{n \to \infty} \|Az''_n - y\| = \inf_{z' \in \mathbb{R}^N} \|Az' - y\| = \|Az''_n - y\|.$$

Thus, the problem $\min_{z' \in \mathbb{R}^N} \|Az' - y\|$ has a minimizer. Since the objective function is continuous and the set of feasible vectors is closed, $\arg\min \|Az' - y\|$ is closed and non-empty. Now let $(z_n)_{n \in N}$ be a sequence such that $z_n \in \arg\min \|Az' - y\|$ and $\lim_{n \to \infty} \|z_n\|_1 = \inf_{z' \in \mathbb{R}^N} \|Az' - y\| \|z\|_1$, which is finite since there are feasible points. The sequence $(z_n)_{n \in N}$ is thus bounded and contains a convergent subsequence $(z'_n)_{n \in N}$ converging to some $x^{BPImp}$, which lies in $\arg\min \|Az' - y\|$ due to closedness. Hence, $x^{BPImp}$ is feasible for $\{BPImp\}$ and

$$\|x^{BPImp}\|_1 = \lim_{n \to \infty} \|x'_n\|_1 = \lim_{n \to \infty} \|z_n\|_1 = \inf_{z' \in \mathbb{R}^N} \|Az' - y\| \|z\|_1.$$

Thus, $\{BPImp\}$ has some minimizer $x^{BPImp}$. We now prove the remaining statements.

(b): Let $\lambda > \lambda$. We use the optimality of $x^\lambda$ and $x^{\lambda'}$ to obtain

$$\|x^\lambda\|_1 + \lambda \|A x^\lambda - y\| \leq \|x^{\lambda'}\|_1 + \lambda' \|A x^{\lambda'} - y\| = \|x^{\lambda'}\|_1 + \lambda' \|A x^{\lambda'} - y\| - (\lambda' - \lambda) \|A x^{\lambda'} - y\|$$

$$\leq \|x^\lambda\|_1 + \lambda' \|A x^{\lambda'} - y\| - (\lambda' - \lambda) \|A x^{\lambda'} - y\|.$$

Since $\lambda' - \lambda > 0$, it follows that $\|A x^{\lambda'} - y\| \leq \|A x^\lambda - y\|$. (a): Let $\lambda > \lambda$. We use the optimality of $x^\lambda$ and statement (b) to get

$$\|x^\lambda\|_1 = \|x^\lambda\|_1 + \lambda \|A x^\lambda - y\| - \lambda \|A x^\lambda - y\| \leq \|x^{\lambda'}\|_1 + \lambda \|A x^{\lambda'} - y\| - \lambda \|A x^{\lambda'} - y\| \leq \|x^{\lambda'}\|_1 \tag{2}.$$

(c): We use the optimality of $x^\lambda$ and the feasibility of $x^{BPImp}$ to obtain

$$\|x^\lambda\|_1 = \|x^\lambda\|_1 + \lambda \|A x^\lambda - y\| - \lambda \|A x^\lambda - y\| \leq \|x^{BPImp}\|_1 + \lambda \|A x^{BPImp} - y\| - \lambda \|A x^{BPImp} - y\| \leq \|x^{BPImp}\|_1.$$

(d): We use the optimality of $x^\lambda$ to obtain

$$\|A x^\lambda - y\| = \lambda^{-1} (\|x^\lambda\|_1 + \lambda \|A x^\lambda - y\|) - \lambda^{-1} \|x^\lambda\|_1$$

$$\leq \lambda^{-1} (\|x^{BPImp}\|_1 + \lambda \|A x^{BPImp} - y\|) - \lambda^{-1} \|x^\lambda\|_1$$

$$= \|A x^{BPImp} - y\| + \lambda^{-1} (\|x^{BPImp}\|_1 - \|x^\lambda\|_1).$$

(f): Note that the assumption means that $x^\lambda$ is a feasible point of $\{BPImp\}$ and hence $\|A x^\lambda - y\| = \|A x^{BPImp} - y\|$. This and the optimality of $x^\lambda$ yields

$$\|x^\lambda\|_1 = \|x^\lambda\|_1 + \lambda \|A x^\lambda - y\| - \lambda \|A x^{BPImp} - y\| \leq \|x^{BPImp}\|_1.$$
and $x^\lambda$ is a minimizer of $\{BP \text{ Imp}\}$.

(e): We use statement (d) to get

$$\|Ax^\lambda - y\| \leq \|Ax^{BP \text{ Imp}} - y\| + \lambda^{-1} (\|x^{BP \text{ Imp}}\|_1 - \|x^\lambda\|_1) \leq \|Ax^{BP \text{ Imp}} - y\|.$$ 

By statement (f) we obtain that $x^\lambda$ is a minimizer of $\{BP \text{ Imp}\}$. 

This lemma allows us to deduce the results of Theorem 2.3

**Proof of Theorem 2.3** Let $x^{BP \text{ Imp}}$ be any minimizer of $\{BP \text{ Imp}\}$, which exists by Lemma 6.4

(1): This statement follows from statements (e) and (f) of Lemma 6.4.

(2): By the feasibility of $x^{BP \text{ Imp}}$ and statement (d) of Lemma 6.4 we have

$$0 \leq \|Ax^\lambda - y\| - \|Ax^{BP \text{ Imp}} - y\| \leq \lambda^{-1} (\|x^{BP \text{ Imp}}\|_1 - \|x^\lambda\|_1).$$

This yields the convergence and the bound on the distance in [5]. The monotonicity of the convergence follows from statement (b) of Lemma 6.4. Now let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence with $\lim_{n \to \infty} \lambda_n = \infty$. By statement (c) of Lemma 6.4 the sequence $(x^{\lambda_n})_{n \in \mathbb{N}}$ is bounded. Hence, there exists a subsequence $(\lambda'_n)_{n \in \mathbb{N}}$ such that $(x^{\lambda'_n})_{n \in \mathbb{N}}$ converges to some $x$. [5] yields that $x$ is a feasible point for $\{BP \text{ Imp}\}$. By optimality of $x^{BP \text{ Imp}}$ we get

$$\lim_{n \to \infty} \|x^{\lambda'_n}\|_1 = \|x\|_1 \geq \|x^{BP \text{ Imp}}\|_1$$

and by statement (c) of Lemma 6.4 we also get $\|x\|_1 = \lim_{n \to \infty} \|x^{\lambda_n}\|_1 \leq \|x^{BP \text{ Imp}}\|_1$. Combining these two inequalities yields

$$\lim_{n \to \infty} \|x^{\lambda_n}\|_1 = \lim_{n \to \infty} \|x^{\lambda'_n}\|_1 = \|x^{BP \text{ Imp}}\|_1 = \inf_{z \in \text{argmin} \|Ax^\lambda - y\|} \|z\|_1.$$ 

Doing this for all possible sequences with $\lim_{n \to \infty} \lambda_n = \infty$ we obtain the convergence

$$\lim_{\lambda \to \infty} \|x^\lambda\|_1 = \inf_{z \in \text{argmin} \|Ax^\lambda - y\|} \|z\|_1.$$ 

The monotonicity of convergence follows from statement (a) of Lemma 6.4

(3): Towards a contradiction assume that $\inf_{z \text{ minimizer of } x^\lambda} \|x^\lambda - z\|_2$ does not converge to zero. Then, there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ and $\epsilon > 0$ with $\lim_{n \to \infty} \lambda_n = \infty$ and

$$\inf_{z \text{ minimizer of } x^\lambda} \|x^\lambda_n - z\|_2 > \epsilon \text{ for all } n \in \mathbb{N}. $$

By statement (c) of Lemma 6.4 the sequence $x^{\lambda_n}$ is bounded. Thus, there exists a subsequence $(\lambda'_n)_{n \in \mathbb{N}}$ such that $x^{\lambda'_n}$ converges to some $x$ and $\lim_{n \to \infty} \lambda'_n = \infty$. [5] yields that $x$ is feasible for $\{BP \text{ Imp}\}$. [3] gives that $x$ is a minimizer of $\{BP \text{ Imp}\}$. Hence, we have

$$\epsilon < \lim_{n \to \infty} \inf_{z \text{ minimizer of } x^\lambda} \|x^\lambda - z\|_2 \leq \lim_{n \to \infty} \|x^{\lambda'_n} - x\|_2 = 0.$$ 

This is a contradiction to the assumption and thus proves statement (3).

(4): We note that the set of minimizer of $\{BP \text{ Imp}\}$ is closed, since the set of feasible points is closed and the objective function continuous. Since the objective function is a norm in a finite-dimensional space, the set of minimizers of $\{BP \text{ Imp}\}$ is bounded and hence compact. By a continuity/compactness argument $\inf_{z \text{ minimizer of } x^\lambda} \|x^\lambda - z\|_2$ is attained for some $x^{BP \text{ Imp}}$ that is a minimizer of $\{BP \text{ Imp}\}$. Statement (3) yields that $\lim_{n \to \infty} \|x^{\lambda_n} - x^{BP \text{ Imp}}\|_2 = 0$. If $x = \lim_{n \to \infty} x^{\lambda_n}$, it follows that

$$\lim_{n \to \infty} \|x - x^{BP \text{ Imp}}\|_2 \leq \lim_{n \to \infty} \|x - x^{\lambda_n}\|_2 + \|x^{\lambda_n} - x^{BP \text{ Imp}}\|_2 = 0 + 0.$$ 

By the closedness of the set of minimizers $x$ is also a minimizer of $\{BP \text{ Imp}\}$, The existence of such a convergent sequence follows from the boundedness ensured by statement (c) of Lemma 6.4

(5): By statement (d) of Lemma 6.4 we have

$$0 \leq \lambda \|Ax^\lambda - y\| \leq \lambda \|Ax^{BP \text{ Imp}} - y\| + (\|x^{BP \text{ Imp}}\|_1 - \|x^\lambda\|_1) \leq \|x^{BP \text{ Imp}}\|_1 - \|x^\lambda\|_1,$$
where the last inequality holds since $x^{BPlmp}$ is feasible and $y \in \text{Ran}(A)$. (3) yields that the right hand side and thus $\lambda \|Ax^\lambda - y\|$ converges to zero. Using this and (1) again gives

$$\lim_{\lambda \to \infty} \|x^\lambda\|_1 + \lambda \|Ax^\lambda - y\| = \inf_{z \in \text{Ran}(Ax^\lambda - y)} \|z\|_1 + 0.$$  

Since $y \in \text{Ran}(A)$, this yields the last statement.

The convergence to $\{BPlmp\}$ occurs at a finite value, but to prove this we need to introduce subdifferentials.

**Definition 6.5.** Let $C \subseteq \mathbb{R}^M$ be a convex set and $f : C \to \mathbb{R}$ be a convex function and $w \in C$. The set

$$\partial f|_w := \{ g \in \mathbb{R}^M : \langle g, w' - w \rangle \leq f(w') - f(w) \text{ for all } w' \in C \}$$

is called the subdifferential of $f$ at $w$. Any vector $g \in \partial f|_w$ is called subgradient of $f$ at $w$.

We only require simple statements about subdifferentials, namely: $x^\#$ is a minimizer of $\min_{x \in C} f$ if and only if $0 \in \partial f|_{x^\#}$ [38, Section 5.2]. If $f$ and $g$ are convex with common domain $C$, then $\partial (f + g)|_w = \partial f|_w + \partial g|_w$ [38, Section 5.1]. And lastly the concatenation of a convex function with an affine transformation obeys $\partial f(A \cdot y)|_x = A^T \partial f|_{Ax \cdot y}$ [38, Section 5.1]. For more information about subdifferentials we refer to [38].

The subdifferential of an arbitrary norm is not always a unique vector, but it has a nice characterization by its dual norm.

**Lemma 6.6.** Let $\| \cdot \|$ be a norm on $\mathbb{R}^M$ with dual norm $\| \cdot \|_* := \sup_{\|w\| \leq 1} \langle \cdot, w \rangle$. Then,

$$\partial \| \cdot \|_w = \{ g \in \mathbb{R}^M : \langle g, w \rangle = \|w\| \text{ and } \|g\|_* \leq 1 \}$$

and in particular, if $w \neq 0$, we have

$$\partial \| \cdot \|_w = \{ g \in \mathbb{R}^M : \langle g, w \rangle = \|w\| \text{ and } \|g\|_* = 1 \}.$$  

**Proof.** At first let $g \in \{ g' \in \mathbb{R}^M : \langle g', w \rangle = \|w\| \text{ and } \|g'\|_* \leq 1 \}$. Then, for any $w' \in \mathbb{R}^M$ we have

$$\langle g, w' - w \rangle = \langle g, w' \rangle - \langle g, w \rangle \leq \|g\|_* \|w'\| - \|w\| \leq \|w'\| - \|w\|$$

and thus $g$ is a subgradient. Now let $g \in \partial \| \cdot \|_w$. If we apply the definition of a subgradient once for $w' := 0 \in \mathbb{R}^M$ and once for $w' := 2w$, we get

$$\langle g, w \rangle = \langle g, 2w - w \rangle \leq 2 \|w\| - \|w\| = \|w\|$$

and $\langle g, -w \rangle = \langle g, 0 - w \rangle \leq \|0\| - \|w\| = - \|w\|$.  

Hence, $\langle g, w \rangle = \|w\|$. By a continuity/compactness argument, there exists a $w' \in \mathbb{R}^M$ such that $\|w'\| \leq 1$ and $\|g\|_* = \langle g, w' \rangle$. If we apply the definition of a subgradient for $w'$, it follows that

$$\|g\|_* = \langle g, w' - w \rangle + \langle g, w \rangle \leq \|w'\| - \|w\| + \langle g, w \rangle = \|w'\| \leq 1.$$  

Thus, we obtain the second inclusion. For the “in particular part” note that if $w \neq 0$, we additionally get $1 = \langle g, w \rangle \|w\|^{-1} \leq \|g\|_*$, which proves the last statement.

Now, we can prove the result about convergence to BPlmp for a finite value $\lambda$.

**Proof of Proposition 2.4.** Since $A$ is surjective, $A^T : \mathbb{R}^M \to \text{Ran}(A^T)$ is bijective and there exists an inverse mapping $B : \text{Ran}(A^T) \to \mathbb{R}^M$. Since the spaces are finite-dimensional, the operator norm of $B$ is finite and further given by

$$\|B\|_\infty \leq \sup_{0 \neq v \in \text{Ran}(A^T)} \frac{\|Bv\|_\infty}{\|v\|_\infty} = \sup_{0 \neq w \in \mathbb{R}^M} \frac{\|BA^Tw\|_*}{\|A^Tw\|_\infty} = \sup_{0 \neq w \in \mathbb{R}^M} \frac{\|w\|_*}{\|A^Tw\|_\infty} = \lambda_{\infty}.$$  

Towards a contradiction assume that $Ax^\lambda \neq y$. By the affine transformation concatenation formula and Lemma 6.6 we have

$$\partial (\lambda \|A \cdot - y\|)_{x^\lambda} = \lambda A^T \partial \| \cdot \|_{Ax^\lambda - y} = \{ \lambda A^T g : g \in \mathbb{R}^M \text{ and } \langle g, Ax^\lambda - y \rangle = \|Ax^\lambda - y\| \text{ and } \|g\|_* = 1 \}.$$  

Let $g' = \lambda A^T g \in \partial (\lambda \|A \cdot - y\|)|_{x^\lambda}$ be any subgradient. By the bound on $\lambda$ it follows that

$$\|g'\|_\infty = \lambda \|A^T g\|_\infty \geq \|g\|_* \lambda \|B\|_{\infty}^{-1} > \|g\|_* = 1,$$
i.e. no vector in \( \partial (\lambda \|A \cdot -y\|) \) lies in the \( \ell_\infty \) unit ball. By Lemma 6.6 on the other hand \( \partial \|\cdot\|_{1,\infty} \) is a subset of the \( \ell_\infty \) unit ball and thus

\[
\partial (\lambda \|A \cdot -y\|) \cap (\partial \|\cdot\|_{1,\infty}) = \emptyset.
\]

(13)

Since \( x^\lambda \) is an optimizer, the optimality criterion of convex optimization yields that \( 0 \in \partial (\|\|_{1} + \lambda \|A \cdot -y\|) \in \partial \|\cdot\|_{1,\infty} \), which is a contradiction to (13). Hence, the original assumption was wrong and we have \( \|Ax^\lambda - y\| = 0 \). By statement (1) of Theorem 2.3 \( x^\lambda \) is a minimizer of (BPLLmp).

We only give a sketch for a proof of Proposition 2.6, since we would need to cite too many results for a complete proof.

Sketch for a proof of Proposition 2.6. The proof is a consequence of [1, Lemma 11.15] and [1, Lemma 11.16]. In order to use (11) Lemma 11.15, we need to consider (11) Definition 11.2 and (11) Definition 11.4. We set \( c := S \) and \( s := 1 \). If \( q \neq 1 \), we apply (11) Lemma 11.16 to obtain the simultaneous \((\ell_q, \ell_1)\)-quotient property relative to \( \|\cdot\| \) with constant \( D = (1 + \rho)S^{\frac{1}{q} - 1} + \tau \) and \( D^* = d \). If \( q = 1 \), the simultaneous \((\ell_q, \ell_1)\)-quotient property relative to \( \|\cdot\| \) is directly fulfilled with constant \( D = d \) and \( D^* = d \). In both cases, this yields the second requirement of (11) Lemma 11.15. Let \( \Delta (y) \in \text{argin} \|z\|_1 + \lambda \|Ax - y\| \) for all \( y \in \mathbb{R}^N \). If (3) holds true, then \( q' = \rho \) and (11) Theorem 2.1 yields that \((A, \Delta)\) is mixed \((\ell_q, \ell_1)\)-instance optimal of order \( S \) with constant 

\[
C = \begin{cases} 
2(1+\rho)^2 & \text{if } q \in (1, \infty) \\frac{2+\rho}{1+\rho} & \text{if } q = 1
\end{cases}
\]

Thus, the other requirement of (11) Lemma 11.15 is fulfilled, which then yields the claim.

6.3 Proofs of Subsection 2.3 Equivalent Conditions for Successful Recovery with rLASSO

In order to prove Theorem 2.8 and Corollary 2.10 we need to prove that certain null space properties are equivalent. This is in general straightforward, but we also need to prove that certain constants can be preserved and this is highly nontrivial. The corresponding statements will also give a construction formula for missing constants, which we require for the proofs of Theorem 2.8 and Corollary 2.10. However, these constructions require calculating \( \binom{N}{S} \) bounds for all subsets of \( N \) with \( S \) elements. Hence the constants can not be calculated in polynomial time using these results. See also [33, Section IV].

Lemma 6.7. (NSP and ORNSP). Let \( A \in \mathbb{R}^{M \times N}, S \in [N], q \in [1, \infty] \) and \( \|\cdot\| \) be a norm on \( \mathbb{R}^M \). Then we have the equivalence:

1. If \( A \) has \( \ell_q \)-ORNSP of order \( S \) wrt \( \|\cdot\| \) with constant \( \tau \), then \( A \) has \( \ell_q \)-NSP of order \( S \).
2. If \( A \) has \( \ell_q \)-NSP of order \( S \), then \( \tau_q \) \in (0, \infty) \) and for every \( \tau' > \tau_q \) \( A \) has \( \ell_q \)-ORNSP of order \( S \) wrt \( \|\cdot\| \) with constant \( \tau' \).

Proof. Statement (1): Let \( v \in \ker(A) \setminus \{0\} \) and \( \#(T) \leq S \). By the ORNSP we have

\[
\|v\|_T < S^\frac{1}{q} - 1 \|v\|_T \| + \tau \|Av\| = S^\frac{1}{q} - 1 \|v\|_T \|.
\]

Statement (2): Let \( T \subseteq [N] \) be an arbitrary set with \( \#(T) \leq S \). We set

\[
\tau_T := \sup_{v \in \mathbb{R}^N \setminus \ker(A)} \frac{\|v\|_T - S^\frac{1}{q} - 1 \|v\|_T \|}{\|Av\|} = \sup_{v \in \mathbb{R}^N \setminus \ker(A), \|v\|_T = 1} \frac{\|v\|_T - S^\frac{1}{q} - 1 \|v\|_T \|}{\|Av\|}.
\]

If \( \ker(A) = \mathbb{R}^N \), then \( A \) is the zero matrix, which does not have \( \ell_q \)-NSP of order \( S \geq 1 \). Thus, we have \( \mathbb{R}^N \setminus \ker(A) \neq \{0\} \) and hence \( \tau_T > -\infty \). For now assume \( \tau_T < \infty \) for all \( \#(T) \leq S \) and note that \( \tau_q = \sup_{\#(T) \leq S} \tau_T \). Since this supremum is being taken over finitely many elements, we have \( \tau_q < \infty \). Since \( A \) has the \( \ell_q \)-NSP with \( S \geq 1 \), any standard unit vector \( e_n^{\prime} \) can not be an element of \( \ker(A) \). For \( n \in T \), it follows that \( \tau_T \geq \frac{1}{\|Ae_n^{\prime}\|} > 0 \). Thus, we have \( \tau_q \in (0, \infty) \). Now let \( \tau' > \tau_q \) be arbitrary. We get for all \( v \notin \ker(A) \)

\[
\|v\|_T \|\leq \tau_q \|Av\| + S^\frac{1}{q} - 1 \|v\|_T \| \leq \tau_T \|Av\| + S^\frac{1}{q} - 1 \|v\|_T \| \leq \tau_q \|Av\| + S^\frac{1}{q} - 1 \|v\|_T \| < \tau' \|Av\| + S^\frac{1}{q} - 1 \|v\|_T \|.
\]
For all \( v \in \ker(A) \setminus \{0\} \) we get by the NSP anyway
\[
\| v_T \|_q < S^{\frac{1}{q}-1}\| v_{T^c} \|_1 = \tau' \| A v \| + S^{\frac{1}{q}-1}\| v_{T^c} \|_1 .
\]
So \( A \) has \( \ell_q \)-ORNSP of order \( S \) wrt \( \| \cdot \| \) with the claimed constant. It remains to prove that \( \tau_T < \infty \). Recall that \( \mathbb{R}^N \setminus \ker(A) \neq \{0\} \). Suppose there exists a sequence of vectors \( (v_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^N \setminus \ker(A) \) such that \( \| v_k \|_2 = 1 \) and
\[
\frac{\| v_k \|_{T^c} - S^{\frac{1}{q}-1}\| v_k \|_{T^c} \|_1}{\| A v_k \|} \rightarrow \infty.
\]
Since the sequence \( (v_k)_{k \in \mathbb{N}} \) is bounded it contains a convergent subsequence \( (v'_{k})_{k \in \mathbb{N}} \). Let \( v := \lim_{k \rightarrow \infty} v'_k \). Since the sequence \( v_k \) is bounded we have \( \| v'_{k} \|_{T^c} - S^{\frac{1}{q}-1}\| v'_{k} \|_{T^c} \|_1 \leq \| v'_{k} \|_q \leq R \) for some \( R > 0 \). Thus we get
\[
\frac{\| v'_{k} \|_{T^c} - S^{\frac{1}{q}-1}\| v'_{k} \|_{T^c} \|_1}{\| A v'_{k} \|} \leq \frac{R}{\| A v'_{k} \|}.
\]
The left hand side goes to infinity for \( k \rightarrow \infty \), thus the denominator on the right hand side needs to go to zero. Hence, we obtain that \( v \in \ker(A) \). Since also \( \| v \|_2 = \lim_{k \rightarrow \infty}\| v'_k \|_2 = 1 \neq 0 \) we have \( v \in \ker(A) \setminus \{0\} \). We can use the NSP to obtain \( \| v_T \|_q - S^{\frac{1}{q}-1}\| v_T \|_{T^c} \|_1 < 0 \). By continuity there exists a \( k_0 \) such that for all \( k \geq k_0 \) we also have the strict inequality
\[
\| v'_{k} \|_{T^c} - S^{\frac{1}{q}-1}\| v'_{k} \|_{T^c} \|_1 < 0,
\]
but this is a contradiction to
\[
\lim_{k \rightarrow \infty} \frac{\| v'_{k} \|_{T^c} - S^{\frac{1}{q}-1}\| v'_{k} \|_{T^c} \|_1}{\| A v'_{k} \|} = \lim_{k \rightarrow \infty} \frac{\| v_k \|_{T^c} - S^{\frac{1}{q}-1}\| v_k \|_{T^c} \|_1}{\| A v_k \|} = 0.
\]
So it follows that \( \tau_T < \infty \). \( \Box \)

As we will see in Corollary 6.9 we can not improve this result to any \( \tau' < \tau_T^0 \).

**Lemma 6.8 (Equivalence of ORNSP and RNSP).** Let \( A \in \mathbb{R}^{m \times N} \), \( S \in [N] \), \( q \in [1,\infty) \) and \( \| \cdot \| \) be a norm on \( \mathbb{R}^M \). Then we have the equivalence:

1. If \( A \) has \( \ell_q \)-RNSP of order \( S \) wrt \( \| \cdot \| \) with constants \( \rho \) and \( \tau \), then for every \( \tau' > \tau \) \( A \) has \( \ell_q \)-ORNSP of order \( S \) wrt \( \| \cdot \| \) with constant \( \tau' \).
2. If \( A \) has \( \ell_q \)-ORNSP of order \( S \) wrt \( \| \cdot \| \) with constant \( \tau \), then \( \rho_q(\tau) \in [0,1] \) and \( A \) has \( \ell_q \)-RNSP of order \( S \) wrt \( \| \cdot \| \) with constants \( \rho_q(\tau) \) and \( \tau \).

**Proof.** Statement (1): Let \( \tau' > \tau \). Let \( v \notin \ker(A) \) and \( \#(T) \leq S \). By the RNSP we have
\[
\| v_T \|_q \leq \rho S^{\frac{1}{q}-1}\| v_{T^c} \|_1 + \tau \| A v \| \leq S^{\frac{1}{q}-1}\| v_{T^c} \|_1 + \tau \| A v \| \leq S^{\frac{1}{q}-1}\| v_{T^c} \|_1 + \tau' \| A v \| .
\]
Now let \( v \in \ker(A) \setminus \{0\} \). Then either \( v_T \neq 0 \) or \( v_{T^c} \neq 0 \). Suppose that \( v_T \neq 0 \). Then, the RNSP yields that \( 0 < \| v_T \|_q \leq \rho S^{\frac{1}{q}-1}\| v_{T^c} \|_1 + \tau \| A v \| = \rho S^{\frac{1}{q}-1}\| v_{T^c} \|_1 \). Thus, in both cases \( v_{T^c} \neq 0 \). Using this, \( \rho < 1 \) and the RNSP once more yields
\[
\| v_T \|_q \leq \rho S^{\frac{1}{q}-1}\| v_{T^c} \|_1 + \tau \| A v \| < S^{\frac{1}{q}-1}\| v_{T^c} \|_1 + \tau \| A v \| = S^{\frac{1}{q}-1}\| v_{T^c} \|_1 + \tau' \| A v \| .
\]
It follows that \( A \) has \( \ell_q \)-ORNSP of order \( S \) wrt \( \| \cdot \| \) with constant \( \tau' \).

Statement (2): Let \( T \subseteq [N] \) be an arbitrary set with \( \#(T) \leq S \). We set
\[
\rho_T := \sup_{v_T \neq 0, \| v \|_{T^c} = 1} \frac{\| v_T \|_q - \tau \| A v \|}{S^{\frac{1}{q}-1}\| v_{T^c} \|_1} = \sup_{v_T \neq 0, \| v \|_{T^c} = 1} \frac{\| v_{T^c} \|_1 - \tau \| A v \|}{S^{\frac{1}{q}-1}\| v_{T^c} \|_1} .
\]
By the ORNSP we have \( \rho_T \leq 1 \). For now assume that \( \rho_T < 1 \) for all \( \#(T) \leq S \) and note that \( \rho_q(\tau) = \max \{0, \sup_{\#(T) \leq S} \rho_T\} \). Since this supremum is being taken over finitely many elements we have \( \rho_q(\tau) \in [0,1) \). For all \( v \) such that \( v_{T^c} \neq 0 \) we get
\[
\| v_T \|_q = \frac{\| v_T \|_q - \tau \| A v \|}{S^{\frac{1}{q}-1}\| v_{T^c} \|_1} S^{\frac{1}{q}-1}\| v_{T^c} \|_1 + \tau \| A v \| \leq \rho_T S^{\frac{1}{q}-1}\| v_{T^c} \|_1 + \tau \| A v \| \leq \rho_q(\tau) S^{\frac{1}{q}-1}\| v_{T^c} \|_1 + \tau \| A v \| .
\]
For all \( v \neq 0 \) such that \( v|_{T^c} = 0 \) we get using the ORNSP
\[
\|v\|_q \leq S^\frac{1}{q} \|v|_{T^c}\|_1 + \tau \|Av\| = \rho_q(\tau) S^\frac{1}{q} \|v|_{T^c}\|_1 + \tau \|Av\|.
\]
So \( A \) has \( \ell_q \)-RNSP of order \( S \) wrt \( \|\cdot\| \) with the claimed stableness constant.
It remains to prove \( \rho_T < 1 \). If \( \{v\} \) such that \( v|_{T^c} \neq 0 \), then \( \rho_T = -\infty < 1 \). On the other hand assume \( \{v\} \) such that \( v|_{T^c} \neq 0 \). Suppose there exists a sequence of vectors \( \{v_k\}_{k \in \mathbb{N}} \) such that \( \|v_k\|_2 = 1 \), \( v_k|_{T^c} \neq 0 \) and
\[
\frac{\|v_k|_{T^c}\|_q - \tau \|Av_k\|}{S^\frac{1}{q} \|v_k|_{T^c}\|_1} \rightarrow 1.
\]
Since \( \{v_k\}_{k \in \mathbb{N}} \) is bounded, it contains a convergent subsequence \( \{v'_k\}_{k \in \mathbb{N}} \). Let \( v := \lim_{k \rightarrow \infty} v'_k \). There are now two cases both resulting in a contradiction. The first one is \( v|_{T^c} \neq 0 \). Then we have by the ORNSP
\[
1 = \lim_{k \rightarrow \infty} \frac{\|v'_k|_{T^c}\|_q - \tau \|Av'_k\|}{S^\frac{1}{q} \|v'_k|_{T^c}\|_1} = \frac{\|v|_{T^c}\|_q - \tau \|Av\|}{S^\frac{1}{q} \|v|_{T^c}\|_1} < 1
\]
which is a contradiction. The second case is \( v|_{T^c} = 0 \). Since \( \|v\|_2 = 1 \neq 0 \), the ORNSP yields that \( \|v|_{T^c}\|_q < S^\frac{1}{q} \|v|_{T^c}\|_1 + \tau \|Av\| = \tau \|Av\| \). By continuity there exists a \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \) we also have the strict inequality
\[
\|v'_k|_{T^c}\|_q - \tau \|Av'_k\| < 0,
\]
but this is a contradiction to
\[
\lim_{k \rightarrow \infty} \frac{\|v'_k|_{T^c}\|_q - \tau \|Av'_k\|}{S^\frac{1}{q} \|v'_k|_{T^c}\|_1} = 1.
\]
It follows that \( \rho_T < 1 \).

Recall that we want to prove Theorem 2.8 In order to do that we need to prove that \( A \) has \( \ell_q \)-ORNSP of order \( S \) with some constant \( \tau < \lambda \). This topological property is also a consequence of Lemma 6.7 as we will prove next.

**Corollary 6.9.** Let \( S \in \mathbb{N} \), \( q \in [1, \infty) \) and \( \|\cdot\| \) be a norm on \( \mathbb{R}^M \). Let \( A \in \mathbb{R}^{M \times N} \) have \( \ell_q \)-NSP of order \( S \). Then we have
\[
(\tau \in [0, \infty) : A \text{ has } \ell_q \text{-ORNSP of order } S \text{ wrt } \|\cdot\| \text{ with constant } \tau = (\tau^0_q, \infty),
\]
which is an open set.

**Proof.** Note that by Lemma 6.7 we obtain the inclusion \( \mathbb{C} \). By the definiton of \( \tau^0_q \) there exist \( T \), \( \{v_k\}_{k \in \mathbb{N}} \) be such that \( \|v_k\|_2 = 1 \) and
\[
\tau^0_q = \lim_{k \rightarrow \infty} \frac{\|v_k|_{T^c}\|_q - S^\frac{1}{q} \|v_k|_{T^c}\|_1}{\|Av_k\|}.
\]
Since \( \|v_k\|_2 = 1 \) there exists a subsequence \( \{v'_k\}_{k \in \mathbb{N}} \) that converges to some \( v \) with \( \|v\|_2 = 1 \). If \( v \in \ker(A) \), we set \( \epsilon := S^\frac{1}{q} \|v|_{T^c}\|_1 - \|v|_{T^c}\|_q \), which is strictly positive by the \( \ell_q \)-NSP. Since \( v \in \ker(A) \), \( \|Av'_k\| \) converges to zero. Hence \( \|v'_k|_{T^c}\|_q - S^\frac{1}{q} \|v'_k|_{T^c}\|_1 \) needs to converge to zero too. By continuity there exists a \( k_0 \) such that for all \( k \geq k_0 \) we have
\[
\|v'_k|_{T^c}\|_q - S^\frac{1}{q} \|v'_k|_{T^c}\|_1 \leq \frac{\epsilon}{2}.
\]
Hence
\[
\epsilon = S^\frac{1}{q} \|v|_{T^c}\|_1 - \|v|_{T^c}\|_q = \lim_{k \rightarrow \infty} \|v'_k|_{T^c}\|_q - S^\frac{1}{q} \|v'_k|_{T^c}\|_1 \leq \frac{\epsilon}{2},
\]
which is a contradiction. Thus we assume \( v \notin \ker(A) \). Then, we have
\[
\|v|_{T^c}\|_q = \frac{\|v|_{T^c}\|_q - S^\frac{1}{q} \|v|_{T^c}\|_1}{\|Av\|} \|Av\| + S^\frac{1}{q} \|v|_{T^c}\|_1 = \tau^0_q \|Av\| + S^\frac{1}{q} \|v|_{T^c}\|_1.
\]
In this case \( \tau^0_q \) is not an ORNSP constant, since we lack the strict inequality. Since we can increase ORNSP constants arbitrarily, it follows that no element from \( [0, \tau^0_q] \) is an ORNSP constant. This is the inclusion \( \subseteq \) and finishes the proof.
We can finally proof [Theorem 2.8 and Corollary 2.10]

Proof of [Theorem 2.8] (1)⇒(2): By Corollary 6.9 we have
\[ \lambda \in \{ \tau \in [0, \infty) : A \text{ has } \ell_1-\text{ORNSP of order } S \text{ wrt } \| \cdot \| \text{ with constant } \tau \} = (\tau_1^0, \infty) \]
and thus A has \( \ell_1-\text{ORNSP of order } S \text{ wrt } \| \cdot \| \text{ with constant } \tau' := \frac{\lambda}{\rho} \). By Lemma 6.8 A has \( \ell_1-\text{RNSP} \) of order S wrt \( \| \cdot \| \) with some stableness constant \( \rho \) and robustness constant \( \tau' \). By Theorem 2.1 the SRD property follows since \( \lambda > \tau' \) and \( q = 1 \).

(2)⇒(3): It is helpful to consider the set
\[ \text{Dec } (A) := \left\{ Q : \mathbb{R}^M \to \mathbb{R}^N \text{ such that } Q(y) = \arg \min_{z \in \mathbb{R}^N} \| z \|_1 + \lambda \| Az - y \| \text{ for all } y \in \mathbb{R}^M \right\}, \]
which by assumption only contains \( \ell_1-\text{SRD} \) of order S wrt \( \| \cdot \| \) for A. Now let \( x \) be \( S \)-sparse and set \( y := Ax \).
For any minimizer \( x^\# \) of \( rLASSO \) with input \( y \) choose one decoder \( Q_x \) in \( \text{Dec } (A) \) that maps \( y \) to \( x^\# \).
Since it is an \( \ell_1-\text{SRD} \) of order S wrt \( \| \cdot \| \) for A, there exists \( C_x, D_x \) such that
\[ \| Q_x (y) - x \|_1 \leq C_x d_1 (x, \Sigma_S) + D_x \| y - Ax \| = 0 + D_x \| Ax - Ax \| = 0 \]
holds true. It follows that \( x^\# = x \) and the minimizer of \( rLASSO \) with input \( y = Ax \) is unique and \( x \).

(3)⇒(1): Let \( v \in \mathbb{R}^M \setminus \{0\} \) and \# (T) ≤ S. Set \( y := A v_T \). By assumption we obtain that \( v_T \) is the minimizer of \( rLASSO \) with input \( y \). Since \( -v_{T^c} \) is feasible we have
\[ \| v_T \|_1 + \lambda \| y - A v_T \| \leq \| -v_{T^c} \|_1 + \lambda \| y - A (-v_{T^c}) \|. \]
After \( v \neq 0 \), we have
\[ v_T = v + v_{T^c} = v - v_{T^c} \neq -v_{T^c}. \]
By the assumption we also obtain that \( v_T \) is the unique minimizer of \( rLASSO \) with input \( y \). Thus, the inequality in (14) is a strict inequality and we get
\[ \| v_T \|_1 + \lambda \| y - A v_T \| < \| -v_{T^c} \|_1 + \lambda \| y - A (-v_{T^c}) \|. \]
Since we have set \( y = A v_T \), it follows that
\[ \| v_T \|_1 < \| -v_{T^c} \|_1 + \lambda \| A (v_T + v_{T^c}) \| = \| v_T \|_1 + \lambda \| Av \| \]
holds true. Doing this for all \( T \) with \# (T) ≤ S and all \( v \in \mathbb{R}^N \setminus \{0\} \) yields that A has the \( \ell_1-\text{ORNSP} \) of order S wrt \( \| \cdot \| \) with constant \( \tau = \lambda \).

Proof of [Corollary 2.10] By basic norm inequalities \( Q : \mathbb{R}^M \to \mathbb{R}^N \) is an \( \ell_1-\text{SRD} \) of order S wrt \( \| \cdot \| \) for A if and only if it is an \( \ell_1-\text{SRD} \) of order S wrt \( \| \cdot \| \) for A. By this and by Theorem 2.8 we obtain the equality in (6).
Now suppose that A has \( \ell_1-\text{NSP} \) of order S. Note that by a general norm inequality we have
\[ \| v_T \|_1 \leq S^{1 - \frac{q}{q'}}, \| v_{T^c} \|_q \text{ for all } \# (T) \leq S \text{ and } v \in \mathbb{R}^N. \]
It immediately follows that A has \( \ell_1-\text{NSP} \) of order S. In particular, using (15) on the definition of \( \tau_1^0 \) yields that \( \tau_1^0 \leq S^{1 - \frac{q}{q'}} \).
By Corollary 6.9 and by Theorem 2.8 \( (\tau_1^0, \infty) \) equals one and thus both sets of (6).

7 Proofs of Section 3 rLASSO is a Practical Usable Recovery Algorithm

7.1 Proofs of Subsection 3.1 Gaussian Measurements
In order to prove Theorem 3.1 we follow the proof of [14] Theorem 11 and adapt to account for a better robustness constant \( \tau \).

Definition 7.1. For \( S \in [N] \) and \( \rho \in [0, 1) \) the set
\[ T_{\rho, S} := \left\{ v \in \mathbb{R}^N \text{ such that } \exists \# (T) \leq S \text{ with } \| v_T \|_q \geq \rho S^{1 - \frac{q}{q'}} \| v_{T^c} \|_1 \right\} \]
is called robustness cone for \( S \) and \( \rho \).
The robustness cone can generally be interpreted as the set of vectors where the robustness summand of the RNSP inequality is required. We can use the robustness cone to get an estimate for the robustness constant.

Lemma 7.2. Let $q \in [1, \infty]$ and $\|\cdot\|$ be a norm on $\mathbb{R}^M$. Let $A \in \mathbb{R}^{M\times N}$, $S \in [N]$, $\rho \in (0, 1)$, and $\tau \in (0, \infty)$. If \( \inf_{v \in T^q_{\rho, S} \cap S_{\rho, l}} \|Av\| \geq \tau^{-1} > 0 \), then $A$ has $\ell_q$-RNSP of order $S$ wrt $\|\cdot\|$ with constants $\rho$ and $\tau$.

Proof. If on the one hand $v \in \mathbb{R}^N \setminus T^q_{\rho, S}$, then we have for all $# (T) \leq S$

\[
\|v\|_q < \rho S^{1 + \frac{\tau}{q}} \|v\|_1 + \tau \|Av\|.
\]

If on the other $v \in T^q_{\rho, S} \setminus \{0\}$, then $\frac{v}{\|v\|_q} \in T^q_{\rho, S} \cap S_{\rho, l}$ and we have

\[
\tau \|Av\| = \|v\|_q \left\| A \frac{v}{\|v\|_q} \right\| \geq \|v\|_q \left\| \inf_{v' \in T^q_{\rho, S} \cap B_{\rho, l}} \|Av'\| \right\| \geq \|v\|_q.
\]

For any $# (T) \leq S$ it follows that

\[
\|v\|_q \leq \rho S^{1 + \frac{\tau}{q}} \|v\|_1 + \tau \|Av\|.
\]

Consequently $A$ has $\ell_q$-RNSP of order $S$ wrt $\|\cdot\|$ with constants $\rho$ and $\tau$.

Interestingly all normalized (and rescaled) vectors of the robustness cone are a convex combination of sparse normalized vectors. The prove is given in \cite{34} Lemma 3(b)].

Lemma 7.3 (\cite{34} Lemma 3(b)]). Let $S \in [N]$ and $\rho \in (0, 1)$. Then

\[
T^2_{\rho, S} \cap S_{\rho, l}^{N-1} \subseteq \sqrt{1 + (1 + \rho^{-1})^2 \text{conv} (\Sigma^2_S)}.
\]

We introduce the Gaussian width.

Definition 7.4. Let $T \subseteq \mathbb{R}^N$ and let the entries of $g \in \mathbb{R}^N$ be independent $\mathcal{N} (0, 1)$ random variables. Then

\[
\ell (T) := \mathbb{E} \left\{ \sup_{v \in T} (g, v) \right\}
\]

is called Gaussian width of $T$.

Further, we need an estimate for the Gaussian width of $\text{conv} (\Sigma^2_S)$. A proof can be found in \cite{34} Lemma 4.

Lemma 7.5 (\cite{34} Lemma 4). Let $S \in [N]$. Then

\[
\ell (\text{conv} (\Sigma^2_S)) \leq \sqrt{2S \ln \left( \frac{e^N}{S} \right) + \sqrt{S}}.
\]

Lastly we introduce Gordon’s escape through the mesh theorem. It was originally proven in \cite{28}. A different proof can be found in \cite{1} Theorem 9.21].

Theorem 7.6 (\cite{1} Theorem 9.21). Let the entries of $A \in \mathbb{R}^{M\times N}$ be independent $\mathcal{N} (0, 1)$ random variables and $T \subseteq S_{\rho, l}^{N-1}$. Then, for any $t \in (0, \infty)$ we have

\[
P \left\{ \inf_{v \in T} \|Av\|_2 \leq E_M \sqrt{M} - \ell (T) - t \right\} \leq \exp \left( -\frac{t^2}{2} \right).
\]

With all these statements we can prove Theorem 3.2.

Proof of Theorem 3.2. Note that the phase transition inequality (7) is equivalent to

\[
E_M \sqrt{M} - \sqrt{1 + (1 + \rho^{-1})^2 \left( \sqrt{2S \ln \left( \frac{e^N}{S} \right) + \sqrt{S}} \right) - \sqrt{2\ln (\eta^{-1})}} \geq \tau^{-1} \sqrt{M}.
\]

\[\tag{16}
\text{(16)}\]
We set \( T := T_{p, S}^2 \cap S_{\ell_2}^{N-1} \). To estimate the Gaussian width of \( T \) let the entries of \( g \in \mathbb{R}^N \) be independent \( \mathcal{N}(0, 1) \) random variables. By Lemma 7.3 and Lemma 7.5 we can estimate

\[
\ell(T) = \ell(T_{p, S}^2 \cap S_{\ell_2}^{N-1}) = \mathbb{E} \left[ \sup_{v \in T_{p, S}^2 \cap S_{\ell_2}^{N-1}} \langle g, v \rangle \right] \leq \mathbb{E} \left[ \sup_{v \in \sqrt{1 + (1 + \rho^{-1})^2 T_{\text{conv}}^2}} \langle g, v \rangle \right] = \sqrt{1 + (1 + \rho^{-1})^2} \mathbb{E} \left[ \sqrt{2\ln \left( N \frac{e}{S} \right) + \sqrt{S}} \right].
\]

(17)

Setting \( t := \sqrt{2 \ln \left( \eta^{-1} \right)} \in (0, \infty) \) and using (17) and (16) yields that

\[
E_M \sqrt{M} - \ell(T) - t = E_M \sqrt{M} - \ell(T) - \sqrt{2 \ln \left( \eta^{-1} \right)} \geq E_M \sqrt{M} - \sqrt{1 + (1 + \rho^{-1})^2} \left( \sqrt{2\ln \left( N \frac{e}{S} \right) + \sqrt{S}} \right) - \sqrt{2 \ln \left( \eta^{-1} \right)} \geq \tau^{-1} \sqrt{M}.
\]

Hence we have the logical statement

\[
\inf_{v \in T} \left\| M^{**} A v \right\|_2 > E_M \sqrt{M} - \ell(T) - t \Rightarrow \inf_{v \in T} \left\| A v \right\|_2 > \tau^{-1}.
\]

(18)

Since the entries of \( M^{**} A \) are independent \( \mathcal{N}(0, 1) \) random variables, Theorem 7.6 together with (13) yields that

\[
\mathbb{P} \left[ \inf_{v \in T} \left\| A v \right\|_2 > \tau^{-1} \right] \geq \mathbb{P} \left[ \inf_{v \in T} \left\| M^{**} A v \right\|_2 > \sqrt{E_M \sqrt{M} - \ell(T) - t} \right] \geq 1 - \exp \left( -\frac{t^2}{2} \right) = 1 - \eta.
\]

Hence, by Lemma 7.2 \( A \) has \( \ell_2 \)-RNSP of order \( S \) wrt \( \| \cdot \|_2 \) with constants \( \rho \) and \( \tau \) with probability of at least \( 1 - \eta \).

In order to estimate the NSP shape constant \( \tau_0^a \) we want to optimize this to account for the smallest possible \( \tau \). At first we choose a particular \( \eta \) and draw a temporary result to remove \( \eta \).

**Corollary 7.7.** Let the entries of \( A \in \mathbb{R}^{M \times N} \) be independent \( \mathcal{N}(0, M^{-1}) \) random variables. If

\[
\tau > \left( E_M - \sqrt{5} \left( \sqrt{2 M \ln \left( \frac{e}{S} \right)} + \sqrt{\frac{S}{M}} \right) \right)^{-1} > 0,
\]

(19)

then \( \rho(\tau) := \left( \sqrt{\frac{E_M - \tau^{-1}}{\sqrt{2 M \ln \left( \frac{e}{S} \right)} + \sqrt{\frac{S}{M}}}} - 1 - 1 \right) \in (0, 1) \), and for any \( \rho \in (\rho(\tau), 1) \) with probability of at least

\[
1 - \exp \left( -\frac{1}{2} \left( \sqrt{1 + (1 + \rho(\tau)^{-1})^2} - 1 - \sqrt{1 + (1 + \rho^{-1})^2} \right)^2 \left( \sqrt{2 M \ln \left( \frac{e}{S} \right)} + \sqrt{\frac{S}{M}} \right)^2 \right) \in (0, 1)
\]

\( A \) has \( \ell_2 \)-RNSP of order \( S \) wrt \( \| \cdot \|_2 \) with constants \( \rho \) and \( \tau \).

**Proof.** To prove that \( \rho(\tau) \) is well defined note that we have the logical statements

\[
\tau > \left( E_M - \sqrt{5} \left( \sqrt{2 M \ln \left( \frac{e}{S} \right)} + \sqrt{\frac{S}{M}} \right) \right)^{-1} \Rightarrow \frac{E_M - \tau^{-1}}{\sqrt{2 M \ln \left( \frac{e}{S} \right)} + \sqrt{\frac{S}{M}}} \in \left( \sqrt{5}, \infty \right)
\]

\[
\Rightarrow \rho(\tau) = \left( \sqrt{\frac{E_M - \tau^{-1}}{\sqrt{2 M \ln \left( \frac{e}{S} \right)} + \sqrt{\frac{S}{M}}}} - 1 - 1 \right)^{-1} \in (0, 1).
\]
We set 
\[ \eta := \exp \left( \frac{1}{2} \left( \sqrt{1 + \left(1 + \rho(\tau)^{-1}\right)^2} - \sqrt{1 + (1 + \rho^{-1})^2} \right)^2 \left( \sqrt{2 \frac{S}{M} \ln \left( \frac{e}{N} \right)} + \sqrt{\frac{S}{M}} \right)^2 M \right), \]
which obeys \( \eta \in (0, 1) \), since \( \rho > \rho(\tau) \). It follows that
\[
\sqrt{1 + (1 + \rho^{-1})^2} \left( \sqrt{2S \ln \left( \frac{e}{N} \right)} + \sqrt{S} \right) + 2\ln(\eta^{-1})
\]
\[ = \sqrt{1 + (1 + \rho^{-1})^2} \left( \sqrt{2S \ln \left( \frac{e}{N} \right)} + \sqrt{S} \right) \]
\[ + \left( \sqrt{1 + (1 + \rho(\tau)^{-1})^2} - \sqrt{1 + (1 + \rho^{-1})^2} \right) \left( \sqrt{2 \frac{S}{M} \ln \left( \frac{e}{N} \right)} + \sqrt{\frac{S}{M}} \right) \sqrt{M}
\]
\[ = \sqrt{1 + (1 + \rho(\tau)^{-1})^2} \left( \sqrt{2S \ln \left( \frac{e}{N} \right)} + \sqrt{S} \right). \]

If we plug in the definition of \( \rho(\tau) \) into this, we obtain
\[
\sqrt{1 + (1 + \rho^{-1})^2} \left( \sqrt{2S \ln \left( \frac{e}{N} \right)} + \sqrt{S} \right) + 2\ln(\eta^{-1}) = \frac{E_M - \tau^{-1}}{\sqrt{2 \frac{S}{M} \ln \left( \frac{e}{N} \right)} + \sqrt{\frac{S}{M}}} \left( \sqrt{2S \ln \left( \frac{e}{N} \right)} + \sqrt{S} \right)
\]
\[ = (E_M - \tau^{-1}) \sqrt{M}. \]
Together with \( \tau > 0 \) from (19) this yields that (7) holds true. The proof follows now from Theorem 3.2.

By a certain choice of \( \rho \) in this result and a limit argument we can deduce Proposition 3.3 from this.

**Proof of Proposition 3.3.** Given \( \alpha \in (0, 1) \) and \( \tau \in \left(\left( E_M - \sqrt{5} \left( \sqrt{2 \frac{S}{M} \ln \left( \frac{e}{N} \right)} + \sqrt{\frac{S}{M}} \right) \right)^{-1}, \lambda S^{-\frac{1}{2}} \right) \) we set
\[ \tilde{\rho}(\alpha, \tau) := \left( \sqrt{1 + (1 + \rho(\tau)^{-1})^2 + \alpha \sqrt{5}} \right)^{-1}, \]
with \( \rho(\tau) \) from Corollary 7.7. By Corollary 7.7 we have \( \rho(\tau) < 1 \), which we can use once for each bound to obtain that
\[ \tilde{\rho}(\alpha, \tau) < \left( \sqrt{1 + (1 + 1^{-1})^2 + \alpha \sqrt{5}} \right)^{-1} = 1 \quad \text{and} \]
\[ \tilde{\rho}(\alpha, \tau) > \left( \sqrt{1 + (1 + \rho(\tau)^{-1})^2 + \alpha \sqrt{5}} \right)^{-1} = \rho(\tau), \]
i.e. the necessary requirement \( \tilde{\rho}(\alpha, \tau) \in (\rho(\tau), 1) \). By the definition of \( \tilde{\rho}(\alpha, \tau) \), we further get that
\[ \sqrt{1 + (1 + \rho(\tau)^{-1})^2} = (1 - \alpha) \sqrt{1 + (1 + \rho(\tau)^{-1})^2 + \alpha \sqrt{5}}. \]

Using this and \( \sqrt{1 + (1 + \rho(\tau)^{-1})^2} = \frac{E_M - \tau^{-1}}{\sqrt{2 \frac{S}{M} \ln \left( \frac{e}{N} \right)} + \sqrt{\frac{S}{M}}} \) yields
\[ \left( \sqrt{1 + (1 + \rho(\tau)^{-1})^2} - \sqrt{1 + (1 + \tilde{\rho}(\alpha, \tau)^{-1})^2} \right)^2 \left( \sqrt{2 \frac{S}{M} \ln \left( \frac{e}{N} \right)} + \sqrt{\frac{S}{M}} \right)^2 \]
\[ = \alpha^2 \left( \sqrt{1 + (1 + \rho(\tau)^{-1})^2} - \sqrt{5} \right)^2 \left( \sqrt{2 \frac{S}{M} \ln \left( \frac{e}{N} \right)} + \sqrt{\frac{S}{M}} \right)^2 \]
\[ = \alpha^2 \left( E_M - \sqrt{5} \left( \sqrt{2 \frac{S}{M} \ln \left( \frac{e}{N} \right)} + \sqrt{\frac{S}{M}} \right) - \tau^{-1} \right)^2. \]
Using this and Corollary 7.7 we obtain for every \( \alpha \in (0,1) \) and every

\[
\tau \in \left( E_M - \sqrt{5} \left( \sqrt{\frac{2}{M} \ln \left( \frac{eN}{S} \right) + \frac{S}{M} } \right)^{-1}, \lambda S^{-\frac{1}{2}} \right)
\]

that

\[
P \left[ \text{A has } \ell_2\text{-RNSP of order } S \text{ wrt } \| \cdot \|_2 \text{ with constants } \tilde{\rho}(\alpha, \tau) \text{ and } \tau \right] \\
\geq 1 - \exp \left( - \frac{1}{2} \alpha^2 \left( E_M - \sqrt{5} \left( \sqrt{\frac{2}{M} \ln \left( \frac{eN}{S} \right) + \frac{S}{M} } - \tau^{-1} \right)^2 M \right) \right)
\]

holds true. It follows that

\[
P \left[ \text{A has } \ell_2\text{-NSP of order } S \text{ and } \lambda > \tau_2^0 S^{\frac{1}{2}} \right] \\
\geq P \left[ \text{A has } \ell_2\text{-NSP of order } S \text{ and } \tau_2^0 < \lambda S^{-\frac{1}{2}} \right] \\
\geq P \left[ \text{A has } \ell_2\text{-ORNSP of order } S \text{ wrt } \| \cdot \|_2 \text{ with constant } \lambda S^{-\frac{1}{2}} \right]
\]

and

\[
P \left[ \text{A has } \ell_2\text{-RNSP of order } S \text{ wrt } \| \cdot \|_2 \text{ with constants } \tilde{\rho}(\alpha, \tau) \text{ and } \tau \right] \\
\geq 1 - \exp \left( - \frac{1}{2} \alpha^2 \left( E_M - \sqrt{5} \left( \sqrt{\frac{2}{M} \ln \left( \frac{eN}{S} \right) + \frac{S}{M} } - \tau^{-1} \right)^2 M \right) \right).
\]

where

21 follows from Lemma 6.7 and Corollary 6.9
22 follows from Lemma 6.8 and \( \lambda S^{-\frac{1}{2}} > \tau \)
23 follows from (20).

The function in (23) is continuous in \((\alpha, \tau)\) and we can send \( \tau \to \lambda S^{-\frac{1}{2}} \) and \( \alpha \to 1 \) to obtain

\[
P \left[ \text{A has } \ell_2\text{-NSP of order } S \text{ and } \lambda > \tau_2^0 S^{\frac{1}{2}} \right]
\geq 1 - \exp \left( - \frac{1}{2} \left( E_M - \sqrt{5} \left( \sqrt{\frac{2}{M} \ln \left( \frac{eN}{S} \right) + \frac{S}{M} } - \sqrt{S} \lambda^{-1} \right)^2 M \right) \right).
\]

In this case Corollary 2.10 yields \( \lambda > \tau_2^0 \sqrt{S} \geq \tau_2^0 \) and \( \lambda \in (\tau_2^0, \infty) \) and thus also that \( Q \) is an \( \ell_2\text{-SRD of order } S \) wrt \( \| \cdot \|_2 \) for \( A \). Alternatively one could also use Corollary 6.9 to obtain that \( A \) has \( \ell_2\text{-ORNSP of order } S \) wrt \( \| \cdot \|_2 \) with constant \( \frac{1}{2} \left( \lambda S^{-\frac{1}{2}} + \tau_2^0 \right) \) and Theorem 2.1 to obtain that \( A \) has \( \ell_2\text{-RNSP of order } S \) wrt \( \| \cdot \|_2 \) with some stabilness constant and robustness constant \( \frac{1}{2} \left( \lambda S^{-\frac{1}{2}} + \tau_2^0 \right) \) and we can send \( \tau \to \lambda S^{-\frac{1}{2}} \) and \( \alpha \to 1 \) to obtain

Theorem 2.1 to get the claim.

\[ \square \]

7.2 Proofs of Subsection 3.2 Random Walk Matrices of Uniformly Distributed
\[ \text{D-Left Regular Bipartite Graphs} \]

**Definition 7.8.** Let \( S \in [N] \) and \( D^{-1} A \in \{0, D^{-1}\}^{M \times N} \) be a \( D\)-left regular bipartite graph. If additionally there exists a \( \theta \in [0,1) \) such that

\[ \#(\text{Row}(T)) \geq (1 - \theta) D \#(T) \text{ for all } \#(T) \leq S, \]

holds true, then \( D^{-1} A \) is called a random walk matrix of an \((S, D, \theta)\)-lossless expander.

If we draw a \( D\)-left regular bipartite graph uniformly at random, it will be a random walk matrix of an \((S, D, \theta)\)-lossless expander with high probability.

**Proposition 7.9 ( [1] Corollary 13.7 ).** Let \( S \in [N], \theta \in (0,1), D := \left[ \frac{2}{\theta} \ln \left( \frac{eN}{S} \right) \right] \) and \( A \in \{0, D^{-1}\}^{M \times N} \) be a uniformly at random chosen \( D\)-left regular bipartite graph. If

\[ M \geq \frac{2}{\theta} \exp \left( \frac{2}{\theta} \right) S \ln \left( \frac{eN}{S} \right), \]

then \( A \) is the random walk matrix of an \((S, D, \theta)\)-lossless expander with probability of at least \( 1 - \frac{\sqrt{e}}{eN} \).
This statement is proven in [1 Corollary 13.7]. Further, the random walk matrix of a \((2S, D, \theta)\)-lossless expander has \(\ell_1\)-RNSP of order \(S\).

Lemma 7.10 ([1 Theorem 13.1]). Let \(2S \in [N], D \in [M]\) and \(\theta \in [0, \frac{1}{4})\). Let \(A \in \{0, D^{-1}\}^{M \times N}\) be the random walk matrix of a \((2S, D, \theta)\)-lossless expander graph. Then, \(A\) has \(\ell_1\)-RNSP of order \(S\) wrt \(\|\cdot\|_1\) with constants \(\rho = \frac{20}{1 - 4\theta}\) and \(\tau = \frac{1}{1 - 4\theta}\).

Proof. Note that [1 Theorem 13.1] yields that the matrix \(DA \in \{0, 1\}^{M \times N}\) has \(\ell_1\)-RNSP of order \(S\) wrt \(\|\cdot\|_1\) with constants \(\rho = \frac{20}{1 - 4\theta}\) and \(\tau = \frac{1}{1 - 4\theta}\). Rescaling yields the statement.

We are now able to recover signals from measurements matrices chosen uniformly at random among all \(D\)-left regular bipartite graph.

Proof of Theorem 3.5. By Proposition 7.9 \(A\) is the random walk matrix of a \((2S, D, \theta)\)-lossless expander with the given probability. In this case \(A\) has by Lemma 7.10 \(\ell_1\)-RNSP of order \(S\) wrt \(\|\cdot\|_1\) with constants \(\rho = \frac{20}{1 - 4\theta}\) and \(\tau = \frac{1}{1 - 4\theta}\). Since \(\theta \in (0, \frac{1}{6})\), we have that \(\lambda = \frac{2}{1 + \rho} \tau = \frac{2}{1 - 20\theta} \in (2, 3)\). The statement now follows from Theorem 2.1.

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