Quaternionic 1-Factorizations and Complete Sets of Rainbow Spanning Trees

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Abstract
A 1-factorization \( F \) of a complete graph \( K_{2n} \) is said to be \( G \)-regular, or regular under \( G \), if \( G \) is an automorphism group of \( F \) acting sharply transitively on the vertex-set. The problem of determining which groups can realize such a situation dates back to a result by Hartman and Rosa (Eur J Comb 6:45–48, 1985) on cyclic groups and it is still open when \( n \) is even, although several classes of groups were tested in the recent past. It has been recently proved, see Rinaldi (Australas J Comb 80(2):178–196, 2021) and Mazzuoccolo et al. (Discret Math 342(4):1006–1016, 2019), that a \( G \)-regular 1-factorization, together with a complete set of rainbow spanning trees, exists for each group \( G \) of order \( 2n \), \( n \) odd. The existence for each even \( n > 2 \) was proved when either \( G \) is cyclic and \( n \) is not a power of 2, or when \( G \) is a dihedral group. Explicit constructions were given in all these cases. In this paper we extend this result and give explicit constructions when \( n > 2 \) is even and \( G \) is either abelian but not cyclic, dicyclic, or a non cyclic 2-group with a cyclic subgroup of index 2.

Keywords
Regular 1-factorizations · Complete graph · Sharply transitive permutation groups · Starter · Rainbow spanning trees

Mathematics Subject Classification 05C70 · 05C15 · 05C05 · 05C51

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1 Introduction

It is well known that the number of non-isomorphic 1-factorizations of $K_{2n}$, the complete graph on $2n$ vertices, goes to infinity with the positive integer $n$, [10]. Therefore, attempts to achieve classifications can be facilitated by imposing additional conditions either on the 1-factorization or on its automorphism group. For example, a precise description of the 1-factorization and of its automorphism group was given when the group is assumed to act multiply transitively on the vertex set, [11].

A few years ago, the following question was addressed:

**Question** Let $G$ be a group of order $2n$. Does there exist a 1-factorization of $K_{2n}$ admitting $G$ as an automorphism group acting sharply transitively on the vertex-set of $K_{2n}$?

A 1-factorization of $K_{2n}$ satisfying the above condition is said to be regular under $G$ or $G$-regular.

This question is a restricted version of problem n.4 in the list of [29]. Although the word “sharply” does not appear there, but the two versions are equivalent for abelian groups, since every transitive abelian permutation group is sharply transitive. When $n$ is odd the problem simplifies somewhat: $G$ must be the semi-direct product of $Z_2$ with its normal complement and $G$ always realizes a 1-factorization of $K_{2n}$, upon which it acts sharply transitively on vertices, see [3, Remark 1]. When $n$ is even, the complete answer is still unknown.

If $G$ is a cyclic group, then Hartman and Rosa proved in [16] that the answer is negative when $n$ is a power of 2 greater than 2, while it is affirmative for all other values of $n$. In the recent past, an affirmative answer was given for several other classes of groups, see for example [3, 4, 7, 26] which respectively consider the class of abelian, dihedral, dicyclic and other nilpotent groups. In [6, 24] a positive answer was found for the class of 2-groups with an elementary abelian Frattini subgroup and for some non-solvable groups, respectively. Also, nonexistence results were achieved by assuming the existence of a fixed 1-factor, [20, 26]. Further results were obtained when the number of fixed 1-factors is as large as possible, [3], or when the 1-factors satisfy some additional conditions, [5]. Recently, we focused our attention on the existence of $G$-regular 1-factorizations of $K_{2n}$ which possess a complete set of rainbow spanning trees, [22, 27].

We recall that a rainbow spanning tree is a spanning tree sharing exactly one edge with each 1-factor of the given 1-factorization. In other words, a 1-factorization of $K_{2n}$ corresponds to a proper edge coloring of $K_{2n}$ with precisely $2n - 1$ colors: each color appears exactly $n$ times and corresponds to a 1-factor. Therefore, a spanning tree is rainbow if its edges have distinct colors. In this case, it is also said to be orthogonal to the 1-factorization.

A set of rainbow spanning trees is said to be a complete set if the trees form a partition of the edge set of $K_{2n}$. It is easy to prove that a complete set cannot exist in $K_4$, so we restrict our discussion to complete sets in $K_{2n}$ with $n \geq 3$. Also, since each rainbow spanning tree has $2n - 1$ edges, $n$ is the number of disjoint trees in a complete set.
Finally, we recall that if $T$ is any subgraph of $K_{2n}$ with exactly $2n - 1$ edges, then $T$ is a spanning tree if and only if $T$ is a spanning connected graph, see for instance [30, 6, p.68].

In [27] it is proved that, regardless of the isomorphism type of $G$, a $G$-regular 1-factorization of $K_{2n}$, together with a rainbow spanning tree whose orbit under a subgroup of $G$ gives rise to a complete set, exists if and only if $n \geq 3$ is an odd number. The problem of determining for which groups $G$ a $G$-regular 1-factorization, together with a complete set of rainbow spanning trees, exists remains largely open when the order of $G$ is twice an even number. There are some exceptions where the matter has been resolved. In [22] a complete set of rainbow spanning trees was explicitly constructed in the family of cyclic regular 1-factorizations of [16] for each $n \geq 3$, except when $n = 2^s$, $s \geq 2$. In [27] an explicit construction was given for the class of dihedral groups of order twice an even number.

Both dihedral and cyclic groups possess a cyclic subgroup of index 2 which plays an important role in the constructions. In the present paper, we feel it is rather natural to extend the analysis to other classes of groups with the same property. More precisely, we consider both dicyclic groups and abelian groups with a cyclic subgroup of index 2. We obviously exclude the family of cyclic 2-groups, since a regular 1-factorization does not exist in these cases, [16]. We also consider all the non-cyclic 2-groups admitting a cyclic subgroup of index 2.

The state of art can be resumed in the following Theorem.

**Theorem 1** Let $G$ be a group of order $2n$, $n > 2$ even. A $G$-regular 1-factorization together with a complete set of rainbow spanning trees exists whenever $G$ is one of the following: a dihedral group; a dicyclic group; an abelian group admitting a cyclic subgroup of index 2 and different from a cyclic 2-group; a non-cyclic 2-group admitting a cyclic subgroup of index 2.

The dihedral case and the cyclic case were already considered in [22, 27], respectively. In the following Sects. 2, 3, 4, we will show explicit constructions which will prove the existence in all the other cases of Theorem 1.

We recall that the finite non-abelian 2-groups (of order $\geq 8$) admitting a cyclic subgroup of index 2 are known. Satz 14.9 in [18] divides them into four isomorphism types: (1), (2), (3), (4). Groups of type (1) are the dihedral groups already considered in [27], while, each group $G$ of type (2), (3) or (4) is considered in this paper.

In this paper we will make use of the regular 1-factorizations already obtained in [4] and we refer to [7] for the abelian case. The 1-factorizations constructed in [4] were referred to as *quaternionic 1-factorizations*, since a type (2) group is a quaternionic one. This inspired the title of the present paper.

Our main interest fits in the general problem of characterizing $G$-regular 1-factorizations satisfying additional properties.

For the sake of completeness, we recall that the problem of determining whether every given 1-factorization of a complete graph possesses a complete set of rainbow spanning trees dates back to a conjecture of Brualdi and Hollingsworth, [8], and to a conjecture of Constantine when the trees are asked to be pairwise isomorphic as uncolored ones, [13]. A recent asymptotic result settles both these conjectures for all
sufficiently large \( n \), [15]. Nevertheless, the solution for each given \( n \) remains nontrivial even if one is allowed to choose the 1-factorization.

Most of the papers about these conjectures treat the general case by methods of extremal graph theory/probabilistic methods which can be applied for every 1-factorization of \( K_{2n} \). The best known results hold for large \( n \) and mainly give lower-bounds on the number of rainbow spanning trees. Together with [15] we recall some other important papers in this direction: [1, 14, 17, 21, 23, 25]. The Brualdi-Hollingsworth conjecture was extended also in [19], by stating that edges of every properly colored \( K_n \) (not necessarily colored by a 1-factorization) can be partitioned into rainbow spanning trees. Results are, for example, contained in [2, 12, 23], and for large \( n \), the results of [25] improved the best known bounds for the three conjectures in [8, 13, 19].

Some examples of 1-factorizations of \( K_{2n} \) satisfying the above conjectures without imposing conditions on \( n \) are also available. Constantine himself proved the existence of a suitable 1-factorization satisfying his conjecture for the case \( 2n \) a power of 2 or five times a power of two, [13].

Also, a first family of 1-factorizations for which the conjecture of Brualdi and Hollingsworth can be verified for each \( n \geq 3 \) was recently shown in [9].

1.1 Preliminaries

We refer to the monograph [30] for the general notions on graphs and 1-factorizations that will not be explicitly defined here. Let \( G \) be a group of even order \( 2n \). We use for \( G \) a multiplicative notation and denote by \( 1_G \) its identity, we also use 1 if the group \( G \) is clear from the context. Let us denote by \( V \) and \( E \) the set of vertices and edges of \( K_{2n} \), respectively. We identify the vertices of \( K_{2n} \) with the group-elements of \( G \). We shall denote by \([x, y]\) the edge with vertices \( x \) and \( y \). Following [7] we always consider \( G \) in its right regular permutation representation. In other words, each group-element \( g \in G \) is identified with the permutation \( V \to V, x \mapsto xg \). This action of \( G \) on \( V \) induces actions on the subsets of \( V \) and on sets of such subsets. Hence if \( g \in G \) is an arbitrary group-element and \( S \) is any subset of \( V \) then we write \( S \cdot g = \{xg : x \in S\} \). In particular, if \( S = [x, y] \) is an edge, then \([x, y] \cdot g = [xg, yg]\). Furthermore, if \( U \) is a collection of subsets of \( V \), then we write \( U \cdot g = \{S \cdot g : S \in U\} \). In particular, if \( U \) is a collection of edges of \( K_{2n} \) then \( U \cdot g = \{[xg, yg] : [x, y] \in U\} \). The \( G \)-orbit of an edge \([x, y]\) has either length \( 2n \) or \( n \) and we speak of a long orbit or a short orbit, respectively, and we call \([x, y]\) a long edge or a short edge, respectively. If \([x, y]\) is a short edge, then there is a non-trivial group element \( g \) so that \([xg, yg] = [x, y]\). Such a \( g \) is unique (\( g = x^{-1}y \)) and is an involution; we call this \( g \) the involution associated with the short edge \([x, y]\). Obviously, the element \( yx^{-1} \) is an involution as well.

It is easy to show that a 1-factor of \( K_{2n} \) which is fixed by \( G \) necessarily coincides with a short \( G \)-orbit of edges.

If \( e \) is an edge, respectively if \( S \) is a set of edges, we will denote by \( Orb_G(e) \), respectively by \( Orb_G(S) \), the orbit of \( e \), respectively of the set \( S \), under the action of \( G \).
If $H$ is a subgroup of $G$ then a system of distinct representatives for the left cosets of $H$ in $G$ will be called a left transversal for $H$ in $G$.

If $[x, y]$ is an edge in $K_{2n}$ we define

$$
\partial([x, y]) = \begin{cases} 
\{xy^{-1}, yx^{-1}\} & \text{if } [x, y] \text{ is long} \\
\{xy^{-1}\} & \text{if } [x, y] \text{ is short}
\end{cases}
$$

$$
\phi([x, y]) = \begin{cases} 
\{x, y\} & \text{if } [x, y] \text{ is long} \\
\{x\} & \text{if } [x, y] \text{ is short}
\end{cases}
$$

Roughly speaking, we also say that the edge $[x, y]$ has difference set $\partial([x, y])$, or that $\{xy^{-1}, yx^{-1}\}$ are the differences of $[x, y]$.

It is clear that the set of all the edges sharing a common difference form a unique $G$-orbit.

If $S$ is a set of edges of $K_{2n}$ we define

$$
\partial S = \bigcup_{e \in S} \partial(e) \quad \phi(S) = \bigcup_{e \in S} \phi(e)
$$

where, in either case, the union may contain repeated elements and so, in general, will return a multiset.

In [7, Definition 2.1] a starter in a group $G$ of even order is a set $\Sigma = \{S_1, \ldots, S_k\}$ of subsets of $E$ together with associated subgroups $H_1, \ldots, H_k$ which satisfy the following conditions:

(i) $\partial S_1 \cup \cdots \cup \partial S_k = G\{1_G\}$;
(ii) for $i = 1, \ldots, k$, the set $\phi(S_i)$ is a left transversal for $H_i$ in $G$;
(iii) for $i = 1, \ldots, k$, $H_i$ must contain the involutions associated with any short edge in $S_i$.

We note that $G\{1_G\}$ is a set, so that $\partial S_1 \cup \cdots \cup \partial S_k$ is a list of distinct elements, the edges of $S_1 \cup \cdots \cup S_k$ are all distinct and lie in distinct $G$-orbits. Hence it also follows that $S_i$ can have no edges in common with $S_j$ for $i \neq j$. Moreover, each $\phi(S_i)$ is a set and therefore the edges of $S_i$ are vertex disjoint.

It is proved in [7], that the existence of a starter in a finite group $G$ of order $2n$ is equivalent to the existence of a $G$-regular 1-factorization of $K_{2n}$. Property (i) in the previous definition ensures that every edge of $K_{2n}$ will occur in exactly one $G$-orbit of an edge from $S_1 \cup \ldots \cup S_k$. Properties (ii) and (iii) ensure the union of the $H_i$-orbits of edges from $S_i$ will form a 1-factor. Namely, for each index $i$, we form a 1-factor $F_i = \bigcup_{e \in S_i} Orb_{H_i}(e)$, whose stabilizer in $G$ is the subgroup $H_i$; the $G$-orbit $Orb_G(F_i) = \{F_i^1, \ldots, F_i^{t_i}\}$, which has length $t_i = |G : H_i|$ (the index of $H_i$ in $G$), is then included in the 1-factorization.

Observe also that the existence of a 1-factor, say $F_1$, which is fixed by $G$ is equivalent to the existence in $\Sigma$ of a set $S_1 = \{e\}$, where $e$ is a short edge. Moreover, $\phi(S_i)$ and $\partial S_i$ both contain $t_i$ elements and $t_i$ is equal to the number of short edges in $S_i$ plus twice
the number of long edges in \( S_i \). It is also true that the unique 1-factor which contains a chosen edge \( e \) with differences in \( \partial S_i \) is one of the 1-factors in \( \{ F_i^1, \ldots, F_i^{|\Sigma|} \} \).

For the purpose of the following Lemma 1, suppose \( n > 2 \) to be even and \( G \) to contain a cyclic subgroup \( H \) of index 2. Let \( j \) be the unique involution in \( H \) and let \( \{ h_1, \ldots, h_{\frac{n}{2}} \} \) be a set of distinct representatives for the cosets of \( \{ 1, j \} \) in \( H \). Suppose \( \Sigma = \{ S_1, \ldots, S_r \} \) to be a starter in \( G \) with associated subgroups \( H_1, \ldots, H_r \), and such that \( S_1 = \{ e \} \), with \( \partial e = \{ j \} \). Let \( \mathcal{F} \) be the \( G \)-regular 1-factorization equivalent to \( \Sigma \).

In the following Lemma 1 we describe a subgraph \( R \) of \( K_{2n} \) which leads to the construction of a complete set of spanning trees orthogonal to \( \mathcal{F} \).

**Lemma 1** Let \( R = R_2 \cup \cdots \cup R_r \) be a subgraph of \( K_{2n} \) such that:

1. For each \( i \in \{ 2, \ldots, r \} \), the set \( R_i \) contains \( t_i = [G : H_i] \) edges: one for each 1-factor of the set \( \{ F_i^1, \ldots, F_i^{h_i} \} \), and the set of distinct elements of \( \partial R_i \) coincides with \( \partial S_i \).
2. If \( l \) is a long edge of \( R_i \), \( i \in \{ 2, \ldots, r \} \), then there is exactly one edge \( l' \in R_i \) such that \( \partial l = \partial l' \) and \( l' \notin Orb_H(l) \). Additionally, if \( l \) is a short edge of \( R_i \), \( i \in \{ 2, \ldots, r \} \), then it is the unique edge of \( R_i \) with difference set \( \partial l \).
3. There exist two distinct edges \( e_1 \) and \( e_2 \) of the fixed 1-factor \( F_1 \) such that \( Orb_H(e_1) \cap Orb_H(e_2) = \emptyset \) and both \( R \cup \{ e_1 \} \) and \( R \cup \{ e_2 \} \) are spanning connected graphs.

Let \( T_1 = R \cup \{ e_1 \} \) and \( T_2 = Rj \cup \{ e_2 \} \).

The set \( \mathcal{T} = \{ T_1 h_1, \ldots, T_1 h_{\frac{n}{2}} \} \cup \{ T_2 h_1, \ldots, T_2 h_{\frac{n}{2}} \} \) is a complete set of rainbow spanning trees.

**Proof** Conditions 1 and 3 assure that both \( T_1 = R \cup \{ e_1 \} \) and \( R \cup \{ e_2 \} \) are spanning connected graphs with \( 2n - 1 \) edges belonging to distinct 1-factors, therefore they are spanning rainbow trees. Since \( \left( R \cup \{ e_2 \} \right) j = Rj \cup \{ e_2 \} = T_2 \), therefore \( T_2 \) is a rainbow spanning tree as well. We also conclude that each graph in \( \mathcal{T} \) is a rainbow spanning tree. We now prove that \( \mathcal{T} \) is a partition of the edge-set of \( K_{2n} \).

Let \( f \) be an edge of \( K_{2n} \). We have three possibilities: either \( f \) is long, or \( f \) is short with \( \partial f = \{ j_1 \}, j_1 \neq j \), or \( f \) is short and \( \partial f = \{ j \} \). In all these cases we prove that \( f \) belongs to a unique spanning tree of \( \mathcal{T} \).

Suppose \( f \) is a long edge, then there exists a unique \( S_i \in \Sigma \setminus \{ S_1 \} \) such that \( \partial f \in \partial S_i \) and \( f \) is an edge of \( F_i^1 \cup \cdots \cup F_i^{h_i} \). Conditions 1 and 2 assure the existence of \( l, l' \in R_i \) such that \( \partial f = \partial l = \partial l' \), with \( l' \notin Orb_H(l), f \notin Orb_G(l) = Orb_G(l') \). Let \( g_1, g_2 \in G \) be the unique elements such that \( f = lg_1 = l'g_2 \). Since \( g_1, g_2^{-1} \notin H \), just one of the two elements \( g_1 \) or \( g_2 \) is in \( H \) and then there is a unique graph of the set \( \{ Rh \mid h \in H \} \) containing \( f \). Therefore \( f \) belongs to a unique tree of \( \mathcal{T} \).

Now suppose \( f \) is short and \( \partial f = \{ j_1 \}, j_1 \neq j \). Let \( l \) be the unique edge of \( R \) with \( \partial l = \partial f \). Since \( j_1 \notin H \), all the \( n \) edges of \( K_{2n} \) with difference set \( \{ j_1 \} \) are in \( Orb_H(l) \). We conclude that a unique tree of \( \mathcal{T} \) contains \( f \).

Finally suppose \( f \) to be a short edge with \( \partial f = \{ j \} \), i.e., \( f \in F_1 \). Condition 3 implies that \( F_1 \) contains the \( n \) distinct edges \( \{ e_1 h_i, e_2 h_i \mid i = 1, \ldots, \frac{n}{2} \} \). Therefore, a unique tree of \( \mathcal{T} \) contains \( f \). \( \square \)
For the rest of the paper, two long edges $l, l'$ satisfying condition 2 of the previous Lemma 1 will be said to be $H$-paired.

2 Dicyclic Groups and Complete Sets of Rainbow Spanning Trees

A dicyclic group $G$ of order $2n = 4s$, $s \geq 2$, can be presented as follows [28, p.189]:

$$G = \langle a, b : a^{2s} = 1, b^2 = a^s, b^{-1}ab = a^{-1} \rangle.$$

We have $G = \{1, a, \ldots, a^{2s-1}, b, ba, \ldots, ba^{2s-1}\}$ and the relations $a^rb = ba^{-r}$, $ba^r(ba^s)^{-1} = a^{r-s}$, $ba^r = ba^{r+s}$, $(ba^r)^2 = a^s$ hold for $r, t = 0, 1, \ldots, (2s-1)$. Furthermore $a^s$ is the unique involution in $G$. In particular, if $s = 2^m - 1$, then $G$ is a generalized quaternion group of order $2^{m+1}$.

We give explicit constructions which allow to prove the following Proposition.

**Proposition 1** Let $G$ be a dicyclic group of order $2n \geq 6$. There exists a $G$-regular 1-factorization of $K_{2n}$ together with a complete set of rainbow spanning trees.

**Proof** We consider the $G$-regular 1-factorization of $K_{2n}$ constructed in [4]. The description is given in terms of starters according to whether $s$ is even or odd. A starter in the case $s$ even is the set:

$$\Sigma = \{S\} \cup \left\{S_{2i+1} : 0 \leq i \leq \frac{s-2}{2}\right\} \cup \left\{S_j^* : 0 \leq j \leq s-1, j \neq \frac{s}{2}\right\} \cup \{S_s\},$$

with:

$$S = \left\{[a^t, a^{-t}] : t = 1, \ldots, \frac{s}{2} - 1\right\} \cup \{[1, ba^\frac{s}{2}]\};$$

$$S_{2i+1} = \left\{[1, a^{2i+1}]\right\}, \quad 0 \leq i \leq \frac{s-2}{2};$$

$$S_j^* = \left\{[1, ba^j]\right\}, \quad 0 \leq j \leq s-1, j \neq \frac{s}{2};$$

$$S_s = \{[1, a^s]\}.$$

Take the subgroups:

$$\langle b \rangle = \{1, b, a^s, ba^s\} \text{ and } \langle b, a^2 \rangle = \{1, a^2, a^4, \ldots, a^{2n-2}, b, ba^2, \ldots, ba^{2n-2}\}.$$

We have:

$$\partial S = \{a^{2t}, a^{-2t} : t = 1, \ldots, \frac{s}{2} - 1\} \cup \{ba^\frac{s}{2}, ba^{-\frac{s}{2}}\} \text{ and } \phi(S) \text{ is a left transversal for } \langle b \rangle,$$

$$\partial S_{2i+1} = \{a^{2i+1}, a^{-2i-1}\} \text{ and } \phi(S_{2i+1}) = \{1, a^{2i+1}\} \text{ is a left transversal for the subgroup } \langle b, a^2 \rangle,$$

$$\partial S_j^* = \{ba^j, ba^{j+s}\} \text{ and } \phi(S_j^*) = \{1, ba^j\} \text{ is a left transversal for the cyclic subgroup } \langle a \rangle.$$
$\partial S_s = \{a^s\}$ and $\phi(S_s) = \{1\}$.

With the starter above, we construct the following 1-factors:

$$F = \text{Orb}_b(S) = \left\{ [1, ba^2], [b, a^2], [a^s, ba^{s+2}], [ba^s, a^{s+2}], [a^t, a^{-t}] \right\}. $$

$[ba^{-t}, ba^t], [a^{s+t}, a^{s-t}], [ba^{-t}, ba^{s+t}] : t = 1, \ldots, \frac{s}{2} - 1 \}$.

$F_{2i+1} = \text{Orb}_{(b,a^2)}(S_{2i+1}) = \left\{ [a^{2k}, a^{2i+1+2k}], [ba^{2k}, ba^{2k-2i-1}] : k = 0, \ldots, s-1 \right\}$ with $0 \leq i < \frac{s-2}{2}$.

$F_j^* = \text{Orb}_{(a)}(S_j^*) = \left\{ [a^k, ba^{j+k}] : k = 0, \ldots, 2s-1 \right\}$ with $0 \leq j \leq s-1, j \neq \frac{s}{2}$.

$F_s = \text{Orb}_G([1, a^s])$.

These 1-factors give rise to the 1-factorization. Namely:

The 1-factor $F$ is fixed by $\langle b \rangle$ and its $G$-orbit yields the 1-factors:

$$F, Fa, Fa^2, \ldots, Fa^{s-1}$$

These 1-factors cover all long edges with difference set in $\partial S$.

For each $0 \leq i \leq \frac{s-2}{2}$, the 1-factor $F_{2i+1}$ is fixed by $\langle b, a^2 \rangle$ and its $G$-orbit yields the 1-factors:

$$F_{2i+1}, F_{2i+1}a$$

These 1-factors cover all edges with difference set in $\partial S_{2i+1}$.

For each $0 \leq j \leq s-1, j \neq \frac{s}{2}$, the 1-factor $F_j^*$ is fixed by $\langle a \rangle$ and its $G$-orbit yields the 1-factors:

$$F_j^*, F_j^*b$$

These 1-factors cover all edges with difference set in $\partial S_j^*$.

Finally $F_s$ is a fixed 1-factor which contains all edges with difference set $\{a^s\}$.

A starter in the case $s$ odd is the set:

$$\Sigma = \{S \} \cup \left\{ S_i^* : 0 \leq i \leq s-1, i \neq \frac{s-1}{2} \right\} \cup \{S_s\},$$

with:

$$S = \left\{ [a^t, a^{s-t-1}], [ba^t, ba^{s-t-2}] : 0 \leq t \leq \frac{s-3}{2} \right\} \cup \{[a^{\frac{s-1}{2}}, ba^{s-1}]\};$$

$$S_i^* = \{[1, ba^i] \}, 0 \leq i \leq s-1, i \neq \frac{s-1}{2};$$

$$S_s = \{[1, a^s] \}$$
We have:
\[ \partial S = \{a^j : 1 \leq j \leq 2s - 1, j \neq s\} \cup \{ba^{s^t-1}, ba^{s+2s-t+1}\} \] and \( \phi(S) \) is a left transversal for \( \langle a^s \rangle \).
\[ \partial S_i^* = \{ba^i, ba^{i+s}\} \text{ and } \phi(S_i^*) \text{ is a left transversal for the subgroup } \langle a \rangle. \]
\[ \partial S_s = \{a^s\} \text{ and } \phi(S_s) = \{1\}. \]

With the starter above, we construct the following 1-factors:

\[ F = Orb_{\langle a^s \rangle}(S) = \left\{ \left[a^t, a^{s-t-1}\right], \left[a^{t+s}, a^{2s-t-1}\right], \left[ba^t, ba^{s-t-2}\right], \left[ba^{t+s}, ba^{2s-t-2}\right], \left[a^{s-1}, ba^{s-1}\right], \left[a^{s+t-1}, ba^{2s-1}\right] : 0 \leq t \leq \frac{s-3}{2} \right\} \]

\[ F_i^* = Orb_{\langle a \rangle}(S_i^*) = \{ [a^r, ba^{i+r}] : r = 0, \ldots, 2s-1 \} \]

with \( 0 \leq i < s-1, i \neq \frac{s-1}{2} \).

\[ F_s = Orb_G([1, a^s]). \]

These 1-factors give rise to the 1-factorization. Namely:
The 1-factor \( F \) is fixed by \( \langle a^s \rangle \) and its \( G \)-orbit yields the 1-factors:

\[ F, Fa, Fa^2, \ldots, Fa^{s-1}, Fb, Fba, Fba^2, \ldots, Fba^s-1 \]

These 1-factors cover all long edges with difference set in \( \partial S \).

For each \( 0 \leq i \leq s-1, i \neq \frac{s-1}{2} \), the 1-factor \( F_i^* \) is fixed by \( \langle a \rangle \) and its \( G \)-orbit yields the 1-factors:

\[ F_i^*, F_i^* b \]

These 1-factors cover all edges with difference set in \( \partial S_i^* \).

Finally \( F_s \) is a fixed 1-factor which contains all edges with difference set \( \{a^s\} \).

Using the method explained in the previous Lemma 1, we are now able to construct a complete set of rainbow spanning trees in both of these two cases.

**Construction of a complete set in the case \( s \) even.**

Let \( s \equiv 2 \pmod{4} \).

Suppose \( s \geq 6 \). Consider the forest induced by the following set \( T \) of edges:

\[ T = \left\{ [1, ba^2], [1, ba^{s+2t}], [1, a^{2t}], [b, ba^{s+2t}] : t = 1, \ldots, \frac{s}{2} - 1 \right\}. \]

We have \( [1, ba^2] \in F, [1, ba^{s+2t}] \in F a^{s^t}, \) we also have: \( [1, a^{2t}] \in Fa^t, [b, ba^{s+2t}] \in Fa^{s^t+t}. \) In fact: \( [1, ba^{s+2t}] = [a^{s+2t}, ba^s]a^2, [1, a^{2t}] = [a^{-t}, a^t]a^t, [b, ba^{s+2t}] = [ba^{s+(s-t)}a^2, ba^{s-(s-t)}a^{s^t+t}. \) Moreover, for each \( t = 1, \ldots, \frac{s}{2} - 1 \), the two edges \( [1, a^{2t}] \) and \( [b, ba^{s+2t}] \) are \( \langle a \rangle \)-paired, as well as the two edges \( [1, ba^2], [1, ba^{s+2t}] \). For each \( i = 0, \ldots, \frac{s-2}{2} \), let \( T_{2i+1} = \{ [1, a^{2i+1}], [b, ba^{2i+1}] \}. \)
We have \( [1, a^{2i+1}] \in F_{2i+1} \) and \( [b, ba^{2i+1}] \in F_{2i+1} \), in fact \( [b, ba^{-2i-1}] \in F_{2i+1} \) and then \( [ba^{2i+1}, b] \in F_{2i+1} \) as \( F_{2i+1} a^{2i+1} = F_{2i+1} a^2 = F_{2i+1} \). Moreover, the two edges of \( T_{2i+1} \) are \( (a) \)-paired.

Set \( T' = T \cup \bigcup_{i=0}^{s-2} T_{2i+1} \). The graph \( T' \) is a rainbow tree which is given by the union of a star at \( 1 \) and a star at \( b \) which are connected through the edge \( [1, ba^z] \). Moreover \( T' \) covers all the vertices of \( K_{4s} \) except for those in the set:

\[
\{a^{s+i} : 0 \leq i \leq s-1\} \cup \left\{ba^{s-2t} : 0 \leq t \leq \frac{s}{2} - 1\right\} \cup \left\{ba^{s+2j+1} : 0 \leq j \leq \frac{s-2}{2}, j \neq \frac{s-2}{4}\right\}
\]

Consider the star at \( a^s \) induced by the set:

\[
U_1 = \left\{[a^s, ba^{s-2}], [a^s, ba^{s+2j+1}] : 0 \leq t \leq \frac{s-2}{2}, 0 \leq j \leq \frac{s-2}{2}, j \neq \frac{s-2}{4}\right\},
\]

together with the star at \( ba^{s+1} \) induced by:

\[
U_2 = \left\{[ba^{s+1}, a^{2s-2j}] : 1 \leq j \leq \frac{s-2}{2}, j \neq \frac{s-2}{4}\right\},
\]

and the star at \( ba^{2s-1} \) induced by:

\[
U_3 = \left\{[ba^{2s-1}, a^{s+2t-1}] : 1 \leq t \leq \frac{s-2}{2}\right\}.
\]

Now let:

\[
T'' = U_1 \cup U_2 \cup U_3 \cup \{[ba^s, a^{2s-1}], [ba^z+1, a^{s+z+1}]\}.
\]

The graph \( T'' \) is a tree, \( T' \) and \( T'' \) are disconnected and all together cover all the vertices of \( K_{4s} \). Moreover, you can partition \( T'' \) into the following pairs of edges:

\[
T_0^* = \{[a^s, ba^s], [ba^z+1, a^{s+z+1}]\}, \quad T_1^* = \{[a^s, ba^{s+1}], [ba^s, a^{2s-1}]\},
\]

\[
T_{2j+1}^* = \{[a^s, ba^{s+2j+1}], [ba^{s+1}, a^{2s-2j}]\} \text{ with } 1 \leq j \leq \frac{s-2}{2}, j \neq \frac{s-2}{4},
\]

\[
T_{s-2t}^* = \{[ba^{2s-1}, a^{s+2t-1}], [a^s, ba^{s-2t}]\} \text{ with } 1 \leq t \leq \frac{s-2}{2}.
\]

The edges of \( T_0^* \) belong to \( F_0^* \) and \( F_0^* b \), respectively. In fact: \( [1, b] \in F_0^* \), \( F_0^* \) is fixed by \( (a) \) and then: \( [a^s, ba^s] \in F_0^* \). Also, \( [ba^z+1, a^{s+z+1}] = [ba^z+1, b^2 a^z+1] = [a^{-z-1} b, ba^{-z-1} b] = [a^{-z-1}, ba^{-z-1}] b \in F_0^* b \). Moreover, they are \( (a) \)-paired.
The edges of $T_1^*$ belong to $F_1^*$ and $F_1^{*b}$, respectively. In fact: $[1, ba] \in F_1^*$, $F_1^*$ is fixed by $\langle a \rangle$ and then: $[a^s, ba^{s+1}] \in F_1^*$.

For each $1 \leq j \leq \frac{s-2}{2}$, $j \neq \frac{s-2}{4}$, the edges of $T_j^*$ belong to $F_j^*$ and $F_j^{*b}$, respectively. In fact: $[1, ba^{2j+1}] \in F_{2j+1}^*$, $F_{2j+1}$ is fixed by $\langle a \rangle$ and then: $[a^s, ba^{s+2j+1}] \in F_{2j+1}^*$. Also, $[a^{-s-1}, ba^{-s-1+2j+1}] \in F_{2j+1}$ and $[a^{-s-1}b, ba^{-s-1+2j+1}] = [ba^{s+1}, b^2a^{s-2j}] = [ba^{s+1}, a^{2s-2j}]$. Moreover, they are $\langle a \rangle$-paired.

When $1 \leq t \leq \frac{s-2}{2}$, the edges of $T_{s-2t}^*$ belong to $F_{s-2t}^*$ and $F_{s-2t}^{*b}$, respectively. In fact: $[1, ba^{s-2t}] \in F_{s-2t}^*$, $F_{s-2t}$ is fixed by $\langle a \rangle$ and then: $[1, ba^{s-2t}]a^{s+2t-1} = [a^{s+2t-1}, ba^{2s-1}] \in F_{s-2t}$. Also, $[1, ba^{s-2t}]b \in F_{s-2t}$ and $[b, ba^{s-2t}] = [b, b^2a^{s-2t}] = [a, a^2t]$. Therefore $[b, a^2t] \in F_{s-2t}$ and $[ba^{s-2t}, a^2t] \in F_{s-2t}^*$, which with $F_{s-2t}ba^{s-2t} = F_{s-2t}^*a^{s+2t}b = F_{s-2t}^*b$. Moreover, they are $\langle a \rangle$-paired.

Therefore, the graph $R = T' \cup T''$ satisfies conditions (1) and (2) of Lemma 1. Let now $e_1 = [1, a^s] \in F_s$ and $e_2 = [b, ba^s] \in F_s$, they are in distinct orbits under $\langle a \rangle$ and both connect $T'$ and $T''$ in such a way that $R \cup \{e_1\}$ and $R \cup \{e_2\}$ satisfy condition (3) of Lemma 1. We conclude that $T = \{T_1a^j : 0 \leq i \leq s-1\} \cup \{T_2a^j : 0 \leq i \leq s-1\}$ with $T_1 = R \cup \{e_1\}$ and $T_2 = Ra^s \cup \{e_2\}$ is a complete set of rainbow spanning trees.

If $s = 2$, the dicyclic group is the quaternion group $Q_8$ and it is easy to observe that $R = T' \cup T''$ with $T' = \{[1, ba], [1, ba^3], [1, a], [b, ba], [ba, a^3]\}$ and $T'' = \{[a^2, ba^2]\}$ is rainbow and satisfies (1) and (2) of Lemma 1 and the above construction can be repeated with $e_1 = [1, a^2]$ and $e_2 = [b, ba^2]$.

For the readers’ convenience, in Fig. 1 we picture $R \cup \{e_1\}$ and $Ra^s \cup \{e_2\}$ when $s = 2$ and we point out $e_1$ and $e_2$ with a different color.

In Fig. 2 we show $R \cup \{e_1\}$ when $s = 6$, in particular we picture the sets $T', T''$ and the edge $e_1$ assigning a color to each of them. Let $s \equiv 0 \pmod{4}$.
With a slight modification of the construction above, we construct a complete set of rainbow spanning trees. Namely, take $T' = T \cup \left( \bigcup_{i=0}^{\frac{s-2}{2}} T_{2i+1} \right)$ exactly as above and recall that $T'$ is a rainbow tree. It is the union of a star at 1 together with a star at $b$ connected through the edge $[1, ba^{s+\frac{r}{2}}]$. Let

$$U_1 = \left\{ [a^s, ba^{s-2j}], [a^s, ba^{s+2j+1}] : 0 \leq t \leq \frac{s-2}{2}, t \neq \frac{s}{4}, 0 \leq j \leq \frac{s-2}{2} \right\},$$

$$U_2 = \left\{ ba^{s+1}, a^{2s-2j} \right\} : 1 \leq j \leq \frac{s-2}{2},$$

$$U_3 = \left\{ ba^{2s-1}, a^{s+2r-1} \right\} : 1 \leq t \leq \frac{s-2}{2}, t \neq \frac{s}{4}.$$ 

$T'' = U_1 \cup U_2 \cup U_3 \cup \{[ba^s, a^{2s-1}]\}.$

It is easy to observe that $T''$ and $T' \cup \{[ba^{\frac{s}{2}-1}, a^{s+\frac{r}{2}-1}]\}$ are trees, they are vertex disjoint and all together cover all the vertices of $K_{4s}$. Moreover, you can partition $T'' \cup \{[ba^{\frac{s}{2}-1}, a^{s+\frac{r}{2}-1}]\}$ into the following pairs of $\langle a\rangle$-paired edges:

$$T^*_0 = \{[a^s, ba^s], [ba^{s-1}, a^{s+\frac{r}{2}-1}]\}, T^*_1 = \{[a^s, ba^{s+1}], [ba^s, a^{2s-1}]\},$$

$$T_{2j+1}^* = \{[a^s, ba^{s+2j+1}], [ba^{s+1}, a^{2s-2j}]\} \text{ with } 1 \leq j \leq \frac{s-2}{2},$$

$$T_{s-2t}^* = \{[ba^{2s-1}, a^{s+2r-1}], [a^s, ba^{s-2t}]\} \text{ with } 1 \leq t \leq \frac{s-2}{2}, t \neq \frac{s}{4}.$$ 

Proceeding as above, we can conclude that $R = T' \cup \{[ba^{\frac{s}{2}-1}, a^{s+\frac{r}{2}-1}]\} \cup T''$ satisfies conditions (1) and (2) of Lemma 1. Taking $e_1 = [1, a^s] \in F_s$ and $e_2 = [b, ba^s] \in F_r$ the set $T = \{T_i a^i : 0 \leq i \leq s-1\} \cup \{T_2 a^i : 0 \leq i \leq s-1\}$, with $T_1 = R \cup \{e_1\}$ and $T_2 = Ra^s \cup \{e_2\}$, is a complete set of rainbow spanning trees.

In Fig. 3 we show $R \cup \{e_1\}$ when $s = 4$, in particular we picture the sets $T' \cup \{[ba^{\frac{s}{2}-1}, a^{s+\frac{r}{2}-1}]\}, T''$ and the edge $e_1$ assigning a color to each of them.
Construction of a complete set in the case $s$ odd.
Consider the forest $T'$ induced by the following set of edges:

$$
\left\{ [1, a^{2t}], [b, ba^{2s-2t}], [1, a^{2t-1}], [b, ba^{2s-2t+1}] : 1 \leq t \leq \frac{s-1}{2} \right\} \cup
\cup \{[a^{\frac{s+1}{2}}, ba^s], [a^s, ba^{\frac{s-1}{2}}]\}.
$$

Observe that $T'$ is rainbow as it contains exactly one edge for each 1-factor of the set $\{Fa^i, Fba^i : 1 \leq i \leq s-1\}$.

More precisely: $[1, a^{2t}] \in Fa^{\frac{s+1}{2}+t}$, $[b, ba^{2s-2t}] \in Fba^{\frac{s-1}{2}-t}$, $[1, a^{2t-1}] \in Fba^{\frac{s+1}{2}+t}$, $[b, ba^{2s-2t+1}] \in Fa^{\frac{s+1}{2}+t}$, $1 \leq t \leq \frac{s-1}{2}$, and also $[a^{\frac{s+1}{2}}, ba^s] \in Fa$, $[a^s, ba^{\frac{s-1}{2}}] \in Fba^{s-1}$.

In fact:

$$
[1, a^{2t}] = [a^{\frac{s+1}{2}-t}, a^{s-\frac{s-1}{2}+t-1}]a^{\frac{s}{2}+t} \in Fa^{\frac{s-1}{2}+t} = Fa^{\frac{s+1}{2}+t},
[b, ba^{2s-2t}] = [1, a^{2t}]b \in Fa^{\frac{s+1}{2}+t}b = Fa^{s-\frac{s+1}{2}+t}b = Fba^{\frac{s-1}{2}-t},
[1, a^{2t-1}] = [ba^{\frac{s-1}{2}+t}, ba^{\frac{s-1}{2}-1}]ba^{s-\frac{s-3}{2}+t} \in Fba^{s+\frac{s-3}{2}+t} = Fba^{s-\frac{s+1}{2}+t},
[b, ba^{2s-2t+1}] = [1, a^{2t-1}]b \in Fba^{\frac{s+1}{2}+t}b = Fa^{s-\frac{s-3}{2}-t} = Fa^{\frac{s-1}{2}+t},
[a^{\frac{s+1}{2}}, ba^s] = [a^{\frac{s-1}{2}}, ba^{s-1}]a \in Fa.
$$

Moreover, you can partition $T'$ into the following pairs of edges:

$$
\{[1, a^{2t}], [b, ba^{2s-2t}]\}, \{[1, a^{2t-1}], [b, ba^{2s-2t+1}]\}, \quad 1 \leq t \leq \frac{s-1}{2}
$$

and

$$
\{[a^{\frac{s+1}{2}}, ba^s], [a^s, ba^{\frac{s-1}{2}}]\}.
$$

It is easy to observe that two edges in the same pair are $(a)$-paired.
Consider the forest $T''$ induced by the following set of edges:
\[\{[1, ba^i], [ba^s, a^{2s-i}] : 1 \leq i \leq s-1, \ i \neq \frac{s-1}{2}\} \cup \]
\[\cup[[a^{s+\frac{s-1}{2}}, ba^{\frac{s-1}{2}}], [a^{s+\frac{s+1}{2}}, ba^{s+\frac{s+1}{2}}]].\]
Observe that: $[a^{s+\frac{s+1}{2}}, ba^{s+\frac{s+1}{2}}] \in F_0^s$ and $[a^{s+\frac{s-1}{2}}, ba^{s-\frac{s-1}{2}}] \in F_0^s b$. In fact: the first edge is contained in $\text{Orb}(a)([1, b])$, while $[a^{s+\frac{s-1}{2}}, ba^{\frac{s-1}{2}}]$ is contained in $\text{Orb}(a)([a^s, b])$ with $[a^s, b] = [1, b]b \in F_0^s b$. Moreover, the two edges are $\langle a \rangle$-paired. For each $i$, with $1 \leq i \leq s - 1$, $i \neq \frac{s-1}{2}$, we obviously have $[1, ba^i] \in F_i^s$ and $[ba^s, a^{2s-i}] \in F_i^s b$ and these two edges are $\langle a \rangle$-paired.

The graph $T' \cup T''$ covers all the vertices of $K_{4s}$ and it is formed by two connected components. Namely: a first component is given by a star at 1 connected to a star at $ba^s$ through the edge $[a^{s+\frac{s+1}{2}}, ba^{s}]$, plus the two edges $[ba^{s+\frac{s-1}{2}}, a^s], [ba^{\frac{s+1}{2}}, a^{s+\frac{s-1}{2}}]$. A second component is given by a star at $b$ plus the edge $[ba^s, a^{2s-i}] \in F_i^s b$.

Moreover $R = T' \cup T''$ satisfies conditions (1) and (2) of Lemma 1.

Taking $e_1 = [a^{s+\frac{s+1}{2}}, a^{s+\frac{s-1}{2}}] \in F_s$ and $e_2 = [ba^{s+\frac{s+1}{2}}, ba^{s+\frac{s-1}{2}}] \in F_s$, the set $T = \{T_1a^i : 0 \leq i \leq s - 1\} \cup \{T_2a^i : 0 \leq i \leq s - 1\}$, with $T_1 = R \cup \{e_1\}$ and $T_2 = Ra^s \cup \{e_2\}$, is a complete set of rainbow spanning trees.

In Fig. 4 we show $R \cup \{e_1\}$ when $s = 5$, in particular we picture the sets $T', T''$ and the edge $e_1$ assigning a color to each of them. \hfill \Box

### 3 Abelian Groups with a Cyclic Subgroup of Index 2 and Complete Sets of Rainbow Spanning Trees

We give an explicit construction which allows to prove the following Proposition.

**Proposition 2** Let $G$ be an abelian non-cyclic group $G$ of order $2n$, $n \geq 4$ even, possessing a cyclic subgroup $H$ of index 2. There exists a $G$-regular 1-factorization of $K_{2n}$ together with a complete set of rainbow spanning trees.

**Proof** It is well-known that $G$ is the direct product of the subgroup $H$ by a cyclic group, say $K$, of order 2. Namely, we set $G = KH$ with $K = \langle b \rangle$ and $H = \langle a \rangle$ and $G$ has three involutions: $b, a^\frac{n}{2}, ba^\frac{n}{2}$.

\[ \text{Springer} \]
A starter in $G$ can be constructed as follows, see [7]:

If $n > 4$ let

$$\Sigma = \{S, S'\} \cup \left\{S_i : 1 \leq i \leq \frac{n}{2} - 1, i \neq \frac{n}{4}\right\} \cup \{S^*, S^*_1, S^*_2, S^*\},$$

if $n = 4$ let $\Sigma = \{S, S'\} \cup \{S^*_1, S^*_2, S^*\}$, with:

$$S = \{[a^i, a^{-i+1}] : 1 \leq i \leq \frac{n}{4}\}; \quad S' = \{[a^i, a^{n-i}] : 1 \leq i < \frac{n}{4}\} \cup \{(1, ba^{\frac{n}{4}})\};$$

$$S_i = \{(1, ba^i)\}, 1 \leq i \leq \frac{n}{2} - 1, i \neq \frac{n}{4}; \quad S^*_1 = \{(1, b)\}; \quad S^*_2 = \{(1, ba^{\frac{n}{2}})\};$$

$$S^* = \{(1, a^{\frac{n}{2}})\}.$$  

We have:

$$\partial S = \{a^{2i-1} : 1 \leq i \leq \frac{n}{2}\}; \quad \partial S' = \{a^{n-2r} : 1 \leq r < \frac{n}{2}\} \cup \{ba^{\frac{3n}{4}}, ba^{\frac{n}{4}}\};$$

and both $\phi(S)$ and $\phi(S')$ are a left transversal for the subgroup $I = \{1, b, a^{\frac{n}{2}}, ba^{\frac{n}{4}}\}$.  

$\partial S_i = \{ba^i, ba^{n-i}\}$ and $\phi(S_i)$ is left transversal for $(a)$.  

Finally, we have $\partial S^*_1 = \{b\}, \partial S^*_2 = \{ba^{\frac{n}{2}}\}, \partial S^* = \{a^{\frac{n}{2}}\}$. Moreover, $\phi(S^*_1) = \phi(S^*_2) = \phi(S^*) = \{1\}$.  

With the starter above, we construct the following 1-factors:

$$F_S = Orb_I(S)$$

whose $G$-orbit gives the 1-factors: $F_S, F_{Sa}, \ldots, F_{Sa^{\frac{n}{2}-1}}$.  

$$F'_S = Orb_I(S')$$

whose $G$-orbit gives the 1-factors: $F'_S, F'_Sa, \ldots, F'_Sa^{\frac{n}{2}-1}$.  

$$F_i = Orb_{(a)}(S_i)$$

whose $G$-orbit gives the 1-factors: $F_i, F_i b$, for each $i$ with $1 \leq i \leq \frac{n}{2} - 1, i \neq \frac{n}{4}$.  

Finally, we have the three fixed 1-factors:

$$F^*_1 = Orb_G([1, b]), F^*_2 = Orb_G([1, ba^{\frac{n}{2}}]), F^* = Orb_G([1, a^{\frac{n}{2}}]).$$

Consider the graph $U_1$ induced by the following set of edges:

$$\{(1, a^{2i-1}), [ba^{\frac{n}{4}}, ba^{\frac{n}{4}+2i-1}] : 1 \leq i \leq \frac{n}{4}\}$$

The $\frac{n}{2}$ edges of $U_1$ belong to the $\frac{n}{2}$ distinct 1-factors $F_S, F_{Sa}, \ldots, F_{Sa^{\frac{n}{2}-1}}$. In fact:

$$[1, a^{2i-1}] = [a^i, a^{-i+1}]a^{-i-1} \in F_{Sa^{i-1}}$$

and $[ba^{\frac{n}{4}}, ba^{\frac{n}{4}+2i-1}] = [1, a^{2i-1}]ba^{\frac{n}{4}} \in F_{Sa^{\frac{n}{4}+i-1}}$. Moreover, for every $i$, the two edges $[1, a^{2i-1}], [ba^{\frac{n}{4}}, ba^{\frac{n}{4}+2i-1}]$ are $\langle a \rangle$-paired.  

Consider the graph $U_2$ induced by the following set of edges:

$$\left\{[1, a^\frac{n}{2}-2i], [ba^{\frac{n}{4}}, ba^{\frac{n}{4}+2i-2}] : 1 \leq i < \frac{n}{4} \right\} \cup \left\{[1, ba^{\frac{n}{2}}], [ba^{\frac{n}{4}}, a^{\frac{n}{2}}]\right\}$$

The $\frac{n}{2}$ edges of $U_2$ belong to the $\frac{n}{2}$ distinct 1-factors $F'_S, F'_Sa, \ldots, F'_Sa^{\frac{n}{2}-1}$. In fact: $[1, ba^{\frac{n}{4}}] \in F'_S$ and $[1, ba^{\frac{n}{4}}]ba^{\frac{n}{4}} = [ba^{\frac{n}{4}}, a^{\frac{n}{2}}] \in F'_Sa^{\frac{n}{2}}$. For every $i$, we have:

$$[1, a^{\frac{n}{2}-2i}] = [a^i, a^{\frac{n}{2}-i}]a^{-i} \in F'_Sa^{\frac{n}{2}-i}$$

and $[ba^{\frac{n}{4}}, ba^{\frac{n}{4}+2i-2}] = [1, a^{\frac{n}{2}-2i}]ba^{\frac{n}{4}} \in F'_{Sa^{\frac{n}{2}-1}}$.  

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Observe that $U_2 = \{[1, ba], [ba, a^2]\}$ whenever $n = 4$.

If $n > 4$, consider the graph $U_3$ induced by the following set of edges:

$\{[a^{n+1}, ba^3i], [b, a^{n-1}] \cup \{1, ba^i\}, [ba^{3n}, a^{3n+i}] : 1 \leq i \leq n - 2\}$

The $n - 4$ edges of $U_3$ belong to the $n - 4$ distinct 1-factors $F_i, F_i b, 1 \leq i \leq n - 4$, $i \neq \frac{n}{4}$. In fact:

$[1, ba^i] \in F_i$ and $[ba^{3n}, a^{3n+i}] \in F_i b, 1 \leq i \leq n - 1$. Moreover, for each fixed $i$, these two edges are $(a)$-paired.

$[a^{n+1}, ba^{3n+i+1}] \in F_1$ and $[ba^3, a^{3n+i}] \in F_1 b, 1 \leq i \leq n - 2$. Moreover, for each fixed $i$, these two edges are $(a)$-paired as well as the two edges $[a^{n+2}, ba^{3n+1}] \in F_{n-1}$ and $[b, a^{n-1}] \in F_{n-1} b$.

Finally, if $n > 4$, let $U_4$ be the graph induced by the following two short edges: $[a^{3n}, b a^3] \in F_1^*$, $[ba^3, a^{n+1}] \in F_2^*$, while if $n = 4$ let $U_4$ be the graph induced by the following two short edges: $[a^3, ba^3] \in F_1^*$, $[b, ba^3] \in F_2^*$.

If $n > 4$, let $R = U_1 \cup U_2 \cup U_3 \cup U_4$, while let $R = U_1 \cup U_2 \cup U_4$ whenever $n = 4$. As above observed, the graph $R$ satisfies conditions (1) and (2) of Lemma 1. Moreover, the graph $R$ has two connected components. In fact, if $n = 4$ the two connected components are clearly indicated in Fig. 5. If $n > 4$, a component is given by the star at $ba^{3n}$ containing all the vertices $\{a^{3n+i}, 0 \leq i \leq \frac{n}{4} - 1\}$; the other component is obtained as follows: a star at 1 containing all the vertices $\{a^i : 1 \leq i \leq \frac{n}{2} - 1\} \cup \{ba^i : 1 \leq i \leq \frac{n}{4}\}$ plus the edge $[b, a^{2n-1}]$ with $b$ of degree 1 and the edge $[a^{3n+2}, ba^{3n+1}]$ with $ba^{3n+1}$ of degree 1; a star at $a^{2n+1}$ containing all the vertices $\{ba^{3n+i+1} : 2 \leq i \leq \frac{n}{4} - 1\} \cup \{ba\}$ and which has just the vertex $ba$ in common with the star at 1; a star at $ba^{n}$ containing the vertex $a^n$ together with all the vertices $\{ba^{n+i} : 1 \leq i \leq \frac{n}{2} - 1\}$. This star has the unique vertex $ba^{n}$ in common with the star at 1 and no vertex in common with the star at $a^{2n+1}$. Finally, we have the edges $[ba^{n-2i}, a^{n-i}], 1 \leq i \leq \frac{n}{4} - 2$, which are connected to the star $ba^{n}$ and the vertices $a^{3n-i}$ have degree 1. Therefore, this component is a tree.

If $n = 4$, let $e_1 = [b, ba^2] \in F^*$ and $e_2 = [a, a^3] \in F^*$, while if $n > 4$, let $e_1 = [a^{3n}, a^n] \in F^*$ and $e_2 = [ba^{3n}, ba^n] \in F^*$. These two edges are distinct orbits under $(a)$ and the graphs $T_1 = R \cup \{e_1\}$ and $T_2 = R \cup \{e_2\}$ satisfy condition (3) of Lemma 1. Therefore, the set $T = \{T_1 a^i : 0 \leq i \leq s - 1\} \cup \{T_2 a^i : 0 \leq i \leq s - 1\}$ is a complete set of rainbow spanning trees.

In Figs. 5 and 6 we show $R \cup \{e_1\}$. In particular, we picture the two connected components of $R$ and the edge $e_1$ assigning a color to each of them.

\[\square\]
4 2-Groups with a Cyclic Subgroup of Index 2 and Complete Sets of Rainbow Spanning Trees

Apart from the abelian groups, the dihedral groups and the generalized quaternion groups, for which we refer to [22, 27] and to the previous sections, respectively, there are two more isomorphism types of groups of order $2n = 2^{m+1}$ with a cyclic subgroup of index 2, see Satz 14.9 in [18]. In particular, these have $n \geq 8$ and they can be presented as follows, [18, p.91]:

(i) \[ G = \langle a, b : a^n = b^2 = 1, bab = a^{\frac{n}{2} - 1} \rangle \] (semidihedral group)

(ii) \[ G = \langle a, b : a^n = b^2 = 1, bab = a^{\frac{n}{2} + 1} \rangle \]

For each case, we exhibit a starter, a 1-factorization and a complete set of rainbow spanning trees, thus proving the following Proposition.

**Proposition 3** Let $G$ be a 2-group of type (i) or (ii). There exists a $G$-regular 1-factorization of $K_{2n}$ together with a complete set of rainbow spanning trees.
Proof  We divide the proof into two parts, according to whether $G$ is of type (i) or (ii).

Type (i).

Let $0 \leq r \leq n - 1$. Observe that $a' b = ba^{-r}$ whenever $r$ is even, while $a' b = ba^{\frac{n}{2} - r}$ whenever $r$ is odd. Moreover, $G$ contains exactly $\frac{n}{2} + 1$ involutions: $a^2$ and $ba^r$, with $r$ even and $0 \leq r \leq n - 2$.

A starter can be constructed as follows, see \[4\]:

\[
\Sigma = \{S\} \cup \left\{ S_{2r+1} : 0 \leq t \leq \frac{n}{4} - 1 \right\} \cup \left\{ S_{2s} : 0 \leq s \leq \frac{n}{2} - 1 \right\} \cup \left\{ S'_{2r+1} : 0 \leq r \leq \frac{n}{4} - 1, r \neq \frac{n}{8} \right\} \cup \{S^*\},
\]

with:

\[
S = \left\{ [a', a^{-t}] : 1 \leq t \leq \frac{n}{4} - 1 \right\} \cup \{[1, ba^{\frac{n}{2}+1}]\};
\]

\[
S_{2r+1} = \{[1, a^{2r+1}]\}, 0 \leq t \leq \frac{n}{4} - 1;
\]

\[
S_{2s} = \{[1, ba^{2s}]\}, 0 \leq s \leq \frac{n}{2} - 1;
\]

\[
S'_{2r+1} = \{[1, ba^{2r+1}]\}, 0 \leq r \leq \frac{n}{4} - 1, r \neq \frac{n}{8};
\]

\[
S^* = \{[1, a^{\frac{n}{2}}]\}.
\]

We have:

\[
\partial S = \{a^{2t}, a^{-2t} : 1 \leq t \leq \frac{n}{4} - 1 \} \cup \{ba^{\frac{n}{2}+1}, ba^{-\frac{n}{2}+1}\} \text{ and } \phi(S) \text{ is a left transversal for the subgroup } \langle ba \rangle = \{1, ba, a^{\frac{n}{2}}, ba^{\frac{n}{2}+1}\}.
\]

For each $t, 0 \leq t \leq \frac{n}{4} - 1$, we have: $\partial S_{2r+1} = \{a^{2r+1}, a^{-2t-1}\}$ and $\phi(S_{2r+1})$ is a left transversal for the subgroup $\langle a^{2}, b \rangle = \{a^{2m}, ba^{2m} : 1 \leq m \leq \frac{n}{4}\}$.

For each $s, 0 \leq s \leq \frac{n}{2} - 1$, we have: $\partial S_{2s} = \{ba^{2s}\}$ and $\phi(S_{2s}) = \{1\}$.

For each $r, 0 \leq r \leq \frac{n}{4} - 1, r \neq \frac{n}{8}$, we have: $\partial S'_{2r+1} = \{ba^{2r+1}, ba^{\frac{n}{2}+2r+1}\}$ and $\phi(S'_{2r+1})$ is a left transversal for the subgroup $\langle a \rangle$.

Finally, we have $\partial S^* = \{a^{\frac{n}{2}}\}$ and $\phi(S^*) = \{1\}$.

With the starter above, we construct the following 1-factors:

\[
F = Orb_{\langle ba \rangle} (S) \text{ whose } G \text{-orbit gives the 1-factors: } F, Fa, \ldots, Fa^{\frac{n}{2}-1}.
\]

For each $t, 0 \leq t \leq \frac{n}{4} - 1$, we obtain the 1-factors $F_{2t+1}$ and $F_{2t+1}a$, with $F_{2r+1} = \text{Orb}_{\langle a^2, b \rangle \{1, a^{2r+1}\}}$.

For each $s, 0 \leq s \leq \frac{n}{2} - 1$, we have the fixed 1-factor $F_{2s} = \text{Orb}_{\langle a \rangle} ([1, ba^{2s}])$.

For each $r, 0 \leq r \leq \frac{n}{4} - 1, r \neq \frac{n}{8}$, we obtain the 1-factors $F'_{2r+1}$ and $F'_{2r+1}b$, with $F'_{2r+1} = \text{Orb}_{\langle a \rangle} ([1, ba^{2r+1}])$.

We also have the fixed 1-factor $F^* = \text{Orb}_{\langle a \rangle} ([1, a^{\frac{n}{2}}])$.

Consider the graph $U_1$ induced by the following set of edges:

\[
\left\{ [1, ba^{\frac{n}{2}+1}], [b, a^{\frac{n}{2}-1}] \cup \{[1, a^{2t}], [b, ba^{\frac{n}{2}+2t}] : 1 \leq t \leq \frac{n}{4} - 1 \right\}
\]
The \( \frac{n}{2} \) edges of \( U_1 \) belong to the \( \frac{n}{2} \) distinct 1-factors \( F, Fa, \ldots, Fa^{n-1} \). In fact, observe that \( [1, ba^{n+1}] \in F \) and \( [b, ba^{n+1}] = [1, ba^{n+1}] 
 b \in Fb = Fba^{n+1}a^{n-1} = Fa^{n-1}. \) Moreover, these two edges are \((a)\)-paired.

Observe also that \( [1, a^{2t}] = [a^{-t}, a^t]a^t \in Fa^t \) and \( [b, ba^{n+2t}] = [1, a^{-2(n+t)}] \in F a^{-(n+t)}b = Fba^{n+t} = Fba^{n+2t} = Fba^{n+2t} = Fa^{n+2t} \). Moreover, for each \( t \), \( 1 \leq t \leq \frac{n}{4} - 1 \), the two edges \([1, a^{2t}] \) and \([b, ba^{n+2t}] \) are \((a)\)-paired.

Let \( U_2 \) be the graph induced by the following set of edges:

\[
\{ [1, a^{2t+1}], [ba, ba^{n-2t-2}] : 0 \leq t \leq \frac{n}{4} - 1 \}
\]

The \( \frac{n}{2} \) edges of \( U_2 \) belong to the \( \frac{n}{2} \) distinct 1-factors \( F_{2r+1}, F_{2r+1}a \), with \( 0 \leq t \leq \frac{n}{4} - 1 \). In fact, it is \( [1, a^{2t+1}] \in F_{2r+1} \) and \( [1, a^{2t+1}]ba = [ba, ba^{n-2t-2}] \in F_{2r+1}ba = F_{2r+1}a. \) Moreover, for each \( t \) with \( 0 \leq t \leq \frac{n}{4} - 1 \), these two edges are \((a)\)-paired.

Let \( U_3 \) be the graph induced by the following set of edges:

\[
\{ [a^{n+2s-2}, ba^{n-2s-2}] \cup [a^{n+1}, ba^{2t+1}] : 1 \leq t \leq \frac{n}{2} - 1 \}
\]

The \( \frac{n}{2} \) edges of \( U_3 \) are short and they belong to the \( \frac{n}{2} \) distinct fixed 1-factors \( F_{2s} \), \( 0 \leq s \leq \frac{n}{2} - 1 \). If \( 1 \leq t \leq \frac{n}{4} - 1 \), we have \( [a^{n+1}, ba^{2t+1}] \in F_{2s+2t} = F_{2s} \) with \( \frac{n}{4} + 1 \leq s \leq \frac{n}{2} - 1 \). If \( \frac{n}{4} \leq s \leq \frac{n}{2} - 1 \), we have \( [a^{n+1}, ba^{2t+1}] \in F_{2s+2t} = F_{2s} \) with \( 0 \leq s \leq \frac{n}{4} - 1 \). Finally \( \partial(a^{n+2s-2}, ba^{n-2s-2}) = ba^n \) and \( [a^{n+2s-2}, ba^{n-2s-2}] \in F_{2s}. \)

If \( n = 8 \) let \( U_4 \) be the graph induced by the following pair of edges:

\[
\{ [ba^7, a^6], [a^7, ba^4] \}
\]

While, if \( n > 8 \), let \( U_4 \) be the graph induced by the following set of edges:

\[
\{ [ba^{n-1}, a^{n-2r-2}], [a^{n+2r+3}, ba^{4r+4}] : 0 \leq r \leq \frac{n}{4} - 2, r \neq \frac{n}{8} \} \cup
\]

\[
\{ [ba^{n-1}, a^{2g}], [a^{n+2g+3}, ba^{2g+2}] \}
\]

If \( n = 8 \), we have: \( [ba^7, a^6] \in F_1' \) and \( [a^7, ba^4] = [ba^7, a^6]a^6b \in F_1'b \) since \( F_1' \) is fixed by \((a)\). Moreover, these two edges are \((a)\)-paired.

If \( n > 8 \), we have: \( [ba^{n-1}, a^{n-2r-2}] \in F_{2r+1}' \) and \( [a^{n+2r+3}, ba^{4r+4}] = [ba^{n-1}, a^{n-2r-2}]a^{2r-2}b \in F_{2r+1}'b \) since \( F_{2r+1}' \) is fixed by \((a)\). We also have: \( [ba^{n-1}, a^{2g}] \in F_{2g-1}' \) and \( [a^{n+2g+3}, ba^{2g+2}] = [ba^{n-1}, a^{2g}]a^{2g-2}b \in F_{2g-1}'b \) since \( F_{2g-1}' \) is fixed by \((a)\). Moreover, for each fixed \( r \), with \( 0 \leq r \leq \frac{n}{4} - 2, r \neq \frac{n}{8} \), the two edges \([ba^{n-1}, a^{n-2r-2}], [a^{n+2r+3}, ba^{4r+4}] \) are \((a)\)-paired as well as the two edges \([ba^{n-1}, a^{2g}], [a^{n+2g+3}, ba^{2g+2}] \).
The graph $R = U_1 \cup U_2 \cup U_3 \cup U_4$ satisfies conditions (1) and (2) of Lemma 1. If $n = 8$, the graph $R$ has two connected components: one is given by the three edges: $[ba, ba^2], [ba, ba^4], [ba^4, a^7]$ and the other by the remaining ones. Let $e_1 = [a^5, a^7] \in E^*$ and $e_2 = [b, ba^4] \in E^*$, these two edges are in distinct orbits under $\langle a \rangle$ and the graphs $T_1 = R \cup \{e_1\}$ and $T_2 = R \cup \{e_2\}$ satisfy condition (3) of Lemma 1. Therefore, the set $T = \{T_1a^i : 0 \leq i \leq 3\} \cup \{T_2a^j : 0 \leq i \leq 3\}$ is a complete set of rainbow spanning trees.

If $n > 8$, the graph $R$ has two connected components, both without cycles. More precisely, one component, say $R'$, is given by a star at $ba$ together with the edges of the set $\{(ba^\frac{n}{2} - 2, a^\frac{n}{2} - 2)\} \cup \{(ba^{4t+4}, a^{\frac{n}{2}+2t+3}) : t = 0, \ldots, \frac{n}{8} - 1\}$. The other component, say $R''$, is given by four stars: a star at 1, a star at $b$, a star at $a^{\frac{n}{2}+1}$ and a star at $ba^{n-1}$. The stars at 1 and at $b$ are connected through the unique common vertex $a^{\frac{n}{2}-1}$. Their union is connected to the star at $a^{\frac{n}{2}+1}$ through the unique common vertex $ba^{\frac{n}{2}+1}$. Finally, the union of these three stars is connected to the star at $ba^{n-1}$ through the unique common edge $[a^{\frac{n}{2}+1}, ba^{n-1}]$. Both these connected components have no cycles. Let $e_1 = [a^{\frac{n}{2}+\frac{2}{3}-2}, a^{\frac{n}{2}-2}] \in E^*$ and $e_2 = [b, ba^{\frac{n}{2}}] \in E^*$. Observe that $a^{\frac{n}{2}+\frac{2}{3}-2}$ is a vertex of $R'$ while $a^{\frac{n}{2}-2}$ is a vertex of $R''$, in the same manner $b$ is a vertex of $R''$ while $ba^{\frac{n}{2}}$ is a vertex of $R'$. Moreover, $e_1$ and $e_2$ are in distinct orbits under $\langle a \rangle$ and then $T_1 = R \cup \{e_1\}$ and $T_2 = R \cup \{e_2\}$ satisfy condition (3) of Lemma 1. Now, the set $T = \{T_1a^i : 0 \leq i \leq \frac{n}{2} - 1\} \cup \{T_2a^j : 0 \leq i \leq \frac{n}{2} - 1\}$ is a complete set of rainbow spanning trees.

In Figs. 7 and 8 we show $R \cup \{e_1\}$ when either $n = 8$ or $n = 16$. In particular we picture the two connected components of $R$ and the edge $e_1$ assigning a color to each of them.

Type (ii).
Let $0 \leq r \leq n - 1$. Observe that $a^rb = ba^r$ whenever $r$ is even, while $a^rb = ba^{\frac{n}{2}+r}$ whenever $r$ is odd. Moreover, $G$ contains exactly 3 involutions: $a^\frac{n}{2}, b$ and $ba^{\frac{n}{2}}$.

Let $n > 8$. A starter can be constructed as follows, [4]:
$$
\Sigma = \{S\} \cup \{S_{2t+1} : 0 \leq t \leq \frac{n}{8} - 1\} \cup \{S_{2t+1} : 0 \leq t \leq \frac{n}{4} + \frac{n}{8} - 1\} \cup \{S_{2s} : 1 \leq s \leq \frac{n}{4} - 1, s \neq \frac{n}{8}\} \cup \{S_{1}, S_{2}, S^*\},
$$

with:

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Consider the tree with $0 \leq \partial S \leq \frac{n}{4} - 1$ for the subgroup $\langle \phi(a) \rangle$.

Finally, we have:

$F_1 = Orb_G([1, b])$, $F_2 = Orb_G([1, ba^{n/2}])$, $F^* = Orb_G([1, a^{n/2}])$.

Consider the tree $U_1$ induced by the following set of edges:

$$\{[b, a^{n/2}], [ba^{n/2}, a^{n/2}]\} \cup \{[1, a^{n/2-2s}], [ba^{n/2}, ba^{n/2-s} : 1 \leq s \leq \frac{n}{4} - 1\} \cup \{[1, a^{n/2-2t-1}], [ba^{n/2}, ba^{n/2-2t-1} : 0 \leq t \leq \frac{n}{4} - 1\}$$
The $n$ edges of $U_1$ belong to the $n$ distinct 1-factors $F, Fa, \ldots, Fa^{n-1}$. In fact: $\{b, a^{n}\} = \{ba^n, a^2 + a^n\}a^2 \in Fa^{n}$ and $\{a^n, b\}ba^2 \in Fa^{n+\frac{n}{4}}$. Moreover, these two edges are $(a)$-paired. Observe that: $[1, a^{n-2}] = [a^{n+s}, a^{n-s}a^{n-s} \in Fa^{n-s}$ and $[ba^n, ba^{\frac{n}{4} + \frac{n}{4} - 2s}] \in Fa^{\frac{n}{4} - s}ba^2$ which is either $Fa^{\frac{n}{4} + \frac{n}{4} - s}$ or $Fa^{\frac{n}{4} - s}$ according to whether $s$ is even or odd. Moreover, for each $s, 0 \leq s \leq n - 1$, the two edges $[1, a^{\frac{n}{4} - 2s}]$ and $[ba^n, ba^{\frac{n}{4} + \frac{n}{4} - 2s}]$ are $(a)$-paired. Finally, observe that $[1, a^{-2t-1}] = [a^{t}, a^{-t-1}] \in Fa^{-t}$ and

$$\{ba^n, ba^{n-2t-1}\} = [1, a^{n-2t-1}]ba^2 \in Fa^{-t}ba^2$$

which is either $Fa^{n-t}$ or $Fa^{\frac{n}{4} + \frac{n}{4} - t}$ according to whether $t$ is even or odd. For each $i, 0 \leq t \leq n - 1$, the two edges $[1, a^{\frac{n}{4} - 2t-1}]$ and $[ba^n, ba^{\frac{n}{4} - 2t-1}]$ are $(a)$-paired. Let $U_2$ be the union of the two stars induced by the following sets of edges:

$$\{[ba^n, a^{n+2i}], [ba^n, a^{n+2i}] : 1 \leq i \leq \frac{n}{8} - 1\}$$

The $\frac{n}{4} - 4$ edges of $U_2$ belong to the distinct 1-factors $F_{2i}, F_{2i}b, 1 \leq s \leq \frac{n}{4} - 1, s \neq \frac{n}{4}$. In fact, for each $1 \leq i \leq \frac{n}{8} - 1$, we have: $[ba^n, a^{n+2i}] \in F_{2i+1}, ba^{n+2i} \in F_{2i+1}b, n \in F_{2i}b, [ba^n, a^{n+2i}] \in F_{2i}, F_{2i}a^n, ba^{n+2i} \in F_{n}$. Moreover, for each $i$, the two edges $[ba^n, a^{n+2i}], [a^{n+2i}, ba^{n+2i+2}]$ are $(a)$-paired as well as the two edges $[ba^n, a^{n+2i+2}], [a^{n+2i}, ba^{n+2i+2}]$. Let $U_3$ be the union of the three stars induced by the following set of edges:

$$\{[a^n, ba^{n+2i}], [b, a^{n+2i}] \in F_{2i+1}, [ba^n, a^{n+2i+1}], [a^{n+2i}, ba^{n+2i+1}] : 0 \leq i \leq \frac{n}{8} - 1\}$$

The $\frac{n}{2}$ edges of $U_3$ belong to the distinct 1-factors $F_{2i+1}, F_{2i+1}b$, with $0 \leq t \leq \frac{n}{8} - 1$ and $\frac{n}{4} \leq t \leq \frac{n}{4} + \frac{n}{8} - 1$. In fact: $[a^n, ba^{n+2i+1}] \in F_{2i+1}, [b, a^{n+2i+1}] \in F_{2i+1}b$. Also, $[ba^n, a^{n+2i+1}] \in F_{2i+1}a^n, [a^{n+2i}, ba^{n+2i+1}] = [1, ba^{n+2i+1}]a^{n+2i+1}, and [a^{n+2i}, ba^{n+2i+1}] \in F_{n+2i}^2$. Moreover, for each $i$, the two edges $[a^n, ba^{n+2i+1}], [b, a^{n+2i+1}]$ are $(a)$-paired as well as the two edges $[ba^n, a^{n+2i+1}], [a^n, ba^{n+2i+1}]$. Finally, let $U_4$ be induced by the two edges $[a^{n+2i}, ba^{n+2i+1}] \in F_{2i}a^n$ and $[b, a^{n+2i}] \in F_{2i+1}b$ which are both short.

The graph $R = U_1 \cup U_2 \cup U_3 \cup U_4$ satisfies conditions (1) and (2) of Lemma 1. It has two connected components. One is given by the union of 5 stars: a star at $a$ and a star at $ba^n$ without common vertices and connected through the unique edge $[ba^n, a^n]$; a star at $ba^n$ with just the two vertices $ba^n, a^n$ in common with the previous two stars, a star at $b$ which is connected to the previous three stars through the unique edge $[b, a^n]$ and a star at $a^n$ connected to the previous four stars through the unique edge $[b, a^n]$. The other component of $R$ is a star at $a^{\frac{n}{2} + \frac{n}{4}}$. 

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Let $e_1 = [a^\frac{n}{2} + a^n, a^\frac{n}{2}] \in F^*$ and $e_2 = [ba^\frac{n}{2} + a^n, ba^\frac{n}{2}] \in F^*$, these two edges are in distinct orbits under $\langle a \rangle$ and the graphs $T_1 = R \cup \{e_1\}$ and $T_2 = R \cup \{e_2\}$ satisfy condition (3) of Lemma 1. Therefore, the set $T = \{T_1a^i : 0 \leq i \leq \frac{n}{2} - 1\} \cup \{T_2a^i : 0 \leq i \leq \frac{n}{2} - 1\}$ is a complete set of rainbow spanning trees.

If $n = 8$, a starter is given by:

$\Sigma = \{S, S_1, S_5, S_1^*, S_2^*, S^*\}$, with:

$S = \{[1, a^3], [a, a^2], [a^5, a^7], [a^6, ba^4]\}$, $S_1 = \{[1, ba]\}$, $S_5 = \{[1, ba^5]\}$, $S_1^* = \{[1, b]\}$, $S_2^* = \{[1, ba^4]\}$, $S^* = \{[1, a^4]\}$.

We have the 1-factors:

$F = Orb_{(p)}(S)$ whose $G$-orbit gives the 1-factors: $F, Fa, \ldots, Fa^7$.

$F_1 = Orb_{(a)}(S_1)$ and $F_5 = Orb_{(a)}(S_5)$, whose $G$-orbits give the 1-factors: $F_1, F_1b$, $F_5, F_5b$.

$F_1^* = Orb_G([1, b])$, $F_2^* = Orb_G([1, ba^4])$, $F^* = Orb_G([1, a^4])$.

Then, we repeat the same construction above with the graph $R = U_1 \cup U_3 \cup U_4$.

In Figs. 9 and 10 we show $R \cup \{e_1\}$. In particular we picture the two connected components of $R$ and the edge $e_1$ assigning a color to each of them.
Propositions 1, 2, 3, together with the results of [22] and [27], prove Theorem 1. □

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**Declarations**

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