Approximations of the Reproducing Kernel Hilbert Space (RKHS) Embedding Method over Manifolds

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Abstract—The reproducing kernel Hilbert space (RKHS) embedding method is a recently introduced estimation approach that seeks to identify the unknown or uncertain function in the governing equations of a nonlinear set of ordinary differential equations (ODEs). While the original state estimate evolves in Euclidean space, the function estimate is constructed in an infinite dimensional RKHS that must be approximated in practice. When a finite dimensional approximation is constructed using a basis defined in terms of shifted kernel functions centered at the observations along a trajectory, the RKHS embedding method can be understood as a data-driven approach. This paper derives sufficient conditions that ensure that approximations of the unknown function converge in a Sobolev norm over a submanifold that supports the dynamics. Moreover, the rate of convergence for the finite dimensional approximations is derived in terms of the fill distance of the samples in the embedded manifold. A numerical simulation of an example problem is carried out to illustrate the qualitative nature of convergence results derived in the paper.

I. INTRODUCTION

Data-driven modeling of uncertain or unknown nonlinear dynamic systems has been a topic of great interest over the past few years. This field synthesizes methods from dynamical system theory, estimation and control theory, approximation theory, and operator theory and uses observations of states or observables to estimate quantities associated with an unknown dynamic system [1]–[3]. The collection of algorithms that can, in some sense, be viewed as data-dependent methods is vast. Specific examples include the following: the collection of studies on the extended dynamic mode decomposition (EDMD) algorithm and its variants that are based on Koopman theory [4]–[10]; adaptive basis methods for online adaptive estimation [11]–[13]; fuzzy control methods based on neural networks [14], [15]; and strategies from distribution-free learning theory and nonlinear regression [16].

Recently the authors have introduced a novel approach, the RKHS embedding method in [17]–[20], for the estimation of uncertain systems. This method likewise can be viewed as a type of data-dependent algorithm when bases of approximation are selected along the trajectory of an unknown system. The RKHS embedding method generalizes estimators used in conventional adaptive estimation over finite dimensional state spaces. The approach essentially lifts the learning law of the estimation scheme to an infinite dimensional RKHS $H$ of real-valued functions defined over the state space.

The unknown function $f(\cdot)$ that characterizes the uncertainty in the ordinary differential equations (ODEs) of dynamical system of is assumed to be an element of the RKHS $H$. The resulting overall estimator is thereby defined for both the states and the unknown function, and it defines an evolution in $\mathbb{R}^d \times H$. Since the evaluation functional $\mathcal{E}_x$ is linear and bounded in the RKHS $H$, the unknown nonlinearity defined by $x \mapsto f(x)$ in the original ODE can be expressed as $\mathcal{E}_x f$ in the RKHS embedding formulation, which is essentially a linear operator acting on the function $f \in H$. In this way, the nonlinearity in the original ODEs is avoided. The trade-off is that one has to conduct analysis in the infinite dimensional spaces, which is usually (much) more complicated.

Those familiar with the general philosophy of Koopman theory will recognize a qualitative similarity between the advantages of the RKHS embedding method and approaches based on Koopman theory. Both seek to replace a nonlinear system with one that is linear, and both must address the issues regarding convergence of approximations in coordinate realizations. One significant difference might be that the RKHS method defines an estimator in continuous time via an infinite dimensional distributed parameter system (DPS). As discussed in detail in [21], it is possible to further relate the two methods by showing that particular entries of the operator matrix that defines the DPS in RKHS embedding can be identified with some types of Koopman operators.

In a way that is analogous to the conventional adaptive estimation in finite dimensional spaces, the convergence of the RKHS embedded estimator can be guaranteed with the satisfaction of a condition of persistence of excitation (PE). The notion of the PE condition for the RKHS embedding method has been introduced and studied in the authors’ previous work [22], [23]. Given a subset $\Omega \subseteq \mathbb{R}^d$, a necessary condition for $\Omega$ to be PE by a positive orbit $\Gamma^+(x_0)$ starting at $x_0$ is that all the neighborhoods of points in $\Omega$ are “visited infinitely often” by the trajectory. This means that $\Omega$ must be a subset of the $\omega$-limit set, in which every point is the limit of a subsequence of points extracted from the trajectory [23]. A convergence result in [23] states that over the indexing set $\Omega$ that is persistently excited, the estimate of the unknown function converges to the actual function [22]. However, the RKHS estimate generated by the DPS lies in the infinite dimensional RKHS. In order to obtain estimates that can be computed in practice, a finite dimensional approximation of the RKHS embedded equations has to be implemented. This paper studies the approximation of the adaptive RKHS embedded estimator.

Most of time, a discussion about the convergence of

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approximations in an RKHS can be transformed into a discussion about the operator \( I - P_{\Omega_n}, \) where \( P_{\Omega_n} : H_X \rightarrow H_{\Omega_n} \) denotes the orthogonal projection onto the finite dimensional approximant subspace \( H_{\Omega_n}. \) Additional insight, or sometimes a finer analysis, can be obtained by interpreting this projection error in other well-known spaces. It is known that many commonly used RKHS are either embedded in or equivalent to some Sobolev space \( W^{r,2}(\mathbb{R}^d). \) The fact that the family of Sobolev spaces provides a refined characterization of the smoothness of functions is useful for estimating the approximation error. In this paper, we first review carefully the relationship between some types of RKHS and Sobolev spaces. Then the error equations for some type of RKHS embedded estimator are recast in Sobolev spaces to facilitate the error analysis. Using the recently introduced results on the Sobolev error bounds for interpolation operator \([24]–[26]\), the rate of convergence is derived in terms of the fill distance. More recent results on the Sobolev error bounds for \( C(X) \) are used to estimate the error decays like \( \sqrt{\mathcal{R}(x,x)} \leq \tilde{k} < \infty \) for all \( x \in \Omega \). In fact, by the Cauchy-Schwarz inequality we have

\[
|\mathcal{E}_x(f)| = |(f, \hat{\mathcal{R}}_K)_{H_X}| \leq \|f\|_{H_X} \|\hat{\mathcal{R}}_K\|_{H_X} = \|f\|_{H_X} \sqrt{\mathcal{R}(x,x)}.
\]

If the above constant \( \tilde{k} \) exists, then we have

\[
\|f\|_{C(X)} = \sup_{x \in \Omega} |\mathcal{E}_x(f)| \leq \|f\|_{H_X} \sup_{x \in \Omega} \sqrt{\mathcal{R}(x,x)} \leq \tilde{k} \|f\|_{H_X},
\]

which implies \( H_X \hookrightarrow C(X) \). In all the following discussions, we assume such constant \( \tilde{k} \) always exists.

In some cases it is useful to consider the space of restrictions \( R_{\Omega}H_X \). Clearly, the restriction operator \( R_{\Omega} : H_X \rightarrow R_{\Omega}H_X \) is linear and onto. It follows from Eq. [1] that \( R_{\Omega}V_{\Omega} = \{0\} \). In fact, one can show that \( R_{\Omega} \) is a bijection over the subspace \( H_{\Omega}. \) To see why this is true, suppose \( f_1, f_2 \in H_{\Omega} \) are different functions. If \( R_{\Omega}f_1 - R_{\Omega}f_2 = R_{\Omega}(f_1 - f_2) = 0, \) then \( f_1 - f_2 \in V_{\Omega} \) because \( (f_1 - f_2, \hat{\mathcal{R}}_K)_{H_X} = 0 \) for all \( x \in \Omega. \) But \( f_1 - f_2 \in H_{\Omega} \) because \( H_{\Omega} \) is a subspace, so \( f_1 - f_2 = 0, \) which contradicts the assumption. With \( R_{\Omega}|_{H_{\Omega}} \) being a bijection, the inverse operator \( (R_{\Omega}|_{H_{\Omega}})^{-1} : R_{\Omega}H_X \rightarrow H_{\Omega} \) is well defined. We denote the inverse as the extension operator \( E_{\Omega} := (R_{\Omega}|_{H_{\Omega}})^{-1}. \) It follows that

\[
P_{\Omega} = E_{\Omega}R_{\Omega}: H_X \rightarrow H_{\Omega}.
\] (2)

By defining the inner product in \( R_{\Omega}H_X \) as \( \langle f, g \rangle_{R_{\Omega}H_X} := \langle E_{\Omega}f, E_{\Omega}g \rangle_{H_X}, \) we can show that \( R_{\Omega}H_X \) is an RKHS. The associated reproducing kernel is shown to be the restriction of the reproducing kernel \( \hat{\mathcal{R}}(\cdot, \cdot) \) over \( \Omega \times \Omega, \) that is, \( R_{\Omega}\hat{\mathcal{R}} := \hat{\mathcal{R}}|_{\Omega \times \Omega} \)[24].

**B. RKHS embedded estimator**

We assume the governing equation of a partially unknown dynamic system can be written as follows

\[
\dot{x}(t) = Ax(t) + Bf(x(t)),
\] (3)

where \( A \in \mathbb{R}^{d \times d} \) is a Hurwitz matrix, \( B \in \mathbb{R}^{d \times 1} \), and \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is the unknown nonlinear function. As discussed more fully in [22], this equation suffices for the proofs that follow, and more general governing equations can be treated as discussed in this reference. The governing equations of the RKHS embedded estimator contain one equation for the state estimates and another for the function estimates,

\[
\dot{x}(t) = A\hat{x}(t) + B\mathcal{E}_{x(t)}\hat{f}(t),
\]

\[
\dot{\hat{f}}(t) = \gamma(B\mathcal{E}_{x(t)})^*P(x(t) - \hat{x}(t)),
\] (4)
where \( \hat{f}(t) \in H_X \) is the time-varying function estimate in the RKHS, \( \mathcal{E}_x(t) \) denotes the evaluation operator, and \( P \) is the solution to the Lyapunov equation associated to the Hurwitz matrix \( A \). That is, \( P \) satisfies \( A^T P + P A = -Q \) for some positive definite \( Q \in \mathbb{R}^{d \times d} \). Assuming \( f \in H_X \), we can replace the nonlinear term \( f(x(t)) \) with \( \mathcal{E}_x(t) f \) in Eq. 5. Denote the estimation errors by \( \hat{x}(t) = x(t) - \bar{x}(t) \) and \( \tilde{f}(t) = f - \hat{f}(t) \). The estimation error equations can be written as follows,
\[
\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + BE_x(t)\hat{f}(t), \\
\dot{\tilde{f}}(t) &= -\gamma(BE_x(t))^*P\tilde{x}(t),
\end{align*}
\]
which resembles its counterpart in conventional adaptive estimation, but evolves in infinite dimensional space \( \mathbb{R}^d \times H_X \). It has been proven that the equilibrium the system expressed in Eq. \( \mathcal{E} \) at the origin is uniformly asymptotically stable (UAS) if the PE condition is satisfied. See [22], [23] for discussion about the PE condition and the stability of the error dynamics.

Practical implementations of the RKHS embedded estimator require the construction of approximations in finite dimensional subspaces \( \mathbb{R}^d \times H_{\Omega_n} \). We denote the states of the finite-dimensional estimator with \( (\hat{x}_n(t), \tilde{f}_n(t)) \). The approximation \( \tilde{f}_n(t) \) lies in the approximant space \( H_{\Omega_n} \) defined as
\[
H_{\Omega_n} = \text{span}\{\mathcal{R}(\cdot, x) : x \in \Omega_n\} \subseteq H_X
\]
for some finite collections of distinct centers \( \Omega_n = \{\xi_i\}_{i=1}^n \). In this paper we explore the case when the centers are taken from the positive orbit \( \Gamma^+(x_0) = \bigcup_{t \geq 0} x(t) \) of the uncertain system. This choice makes the approach a data-driven method. The evolution equations of the finite dimensional estimator are written as
\[
\begin{align*}
\dot{\hat{x}}_n(t) &= A\hat{x}_n(t) + BE_x(t)\hat{f}_n(t), \\
\dot{\tilde{f}}_n(t) &= -\gamma(BE_x(t))^*P\tilde{x}_n(t),
\end{align*}
\]
and the difference \( \tilde{f}_n(t) = f - \hat{f}_n(t) \) the total error of the function estimate. It is the summation of two parts, the “infinite dimensional” error \( \tilde{f}(t) = f - \hat{f}(t) \) and the finite dimensional approximation error \( \hat{f}(t) = f - \hat{f}_n(t) \). The corresponding notations \( \tilde{x}(t) \) and \( \hat{x}(t) \) are defined accordingly for the state estimate. From Eq. 3, 4 and 5 we can derive the dynamical equations as follows that characterize the evolution of the approximation error,
\[
\begin{align*}
\dot{\hat{x}}_n(t) &= A\hat{x}_n(t) + BE_x(t)\hat{f}_n(t), \\
\dot{\tilde{f}}_n(t) &= -\gamma(BE_x(t))^*P\tilde{x}_n(t) + \gamma(I - P_{\Omega_n})(BE_x(t))^*P\tilde{x}(t).
\end{align*}
\]

\[ C. \text{} Sobolev Spaces \]

For a subset \( \Omega \subseteq X = \mathbb{R}^d \) and a positive integer \( s \), the Sobolev space \( W^{s,2}(\Omega) \) is the collection of functions in \( L^2(\Omega) \) that have weak derivatives of all orders less than or equal to \( s \) that are also in \( L^2(\Omega) \). The space \( W^{s,2}(\Omega) \) is equipped with the norm
\[
\|f\|_{W^{s,2}(\Omega)}^2 = \sum_{0 \leq |\alpha| \leq s} \int_{\Omega} |\partial^\alpha f|_2^2 \, d\mu,
\]
where \( \alpha = (\alpha_1, ..., \alpha_d) \) denotes the multi-indices of derivative with \( \sum_{i=1}^d \alpha_i = |\alpha| \). The integral is taken with respect to the Lebesgue measure. The Sobolev spaces for non-integer \( s \) are defined in terms of interpolation space theory. Similar definitions hold for Sobolev spaces \( W^{s,2}(\Omega) \) over a manifold \( \Omega \), with the derivatives replaced by covariant intrinsic derivatives on \( \Omega \) and \( \mu \) the volume measure of the manifold. See [28] for details.

As discussed above, we must consider the restrictions of functions to subsets. One particularly important case needed in this paper is when \( \Omega \subseteq X \) is a \( k \)-dimensional smooth compact embedded manifold. By trace theorem (Proposition 2 in [24]), if \( \tau > (d-k)/2 \), then we have
\[
R_{\Omega} W^{\tau,2}(X) \approx W^{\tau-(d-k)/2,2}(\Omega)
\]
with equivalent norms. The important implication in the above equivalence is that an amount of “smoothness” is lost due to the restriction operation onto the low-dimensional smooth submanifold, which will affect the convergence rates in this paper.

Many Sobolev spaces are also reproducing kernel Hilbert spaces. Let \( \mathcal{R} : X \times X \rightarrow \mathbb{R} \) be a reproducing kernel and \( \mathcal{R} := \mathcal{F}(\mathcal{R}) \) be the Fourier transform of \( \mathcal{R} \). Suppose \( \mathcal{R} \) has algebraic decay as follows
\[
\mathcal{R}(\xi) \sim (1 + \|\xi\|^2)^{-\tau}, \quad \tau > d/2.
\]
Then it has been shown in [24], [27] that the RKHS \( H_X \) induced by \( \mathcal{R} \) itself is a Sobolev space with smoothness index \( \tau \). Specifically, we have
\[
H_X \approx W^{\tau,2}(X)
\]
with equivalent norms. Following the arguments of Theorem 5 in [24] the equivalence displayed in Eq. 11 can be extended to the space of restrictions. When \( \Omega \subseteq X \) is the submanifold defined above, then we have
\[
R_{\Omega} H_X \approx R_{\Omega} W^{\tau,2}(X) \approx W^{\tau-(d-k)/2,2}(\Omega).
\]

III. MAIN RESULTS

A. Restriction of Evolution on Manifolds

Ultimately we are interested in restricting the evolution equations governed by the DPS over \( X \) to a subset \( \Omega \subseteq X \). This section shows that
\[
\frac{d}{dt} (R_{\Omega} h(t)) = R_{\Omega} \left( \frac{d}{dt} h(t) \right) = R_{\Omega} \dot{h}(t),
\]
where the derivative is understood in the strong sense. We begin with a lemma that shows that \( R_{\Omega} h(t) = \frac{d}{dt} (R_{\Omega} h(t)) \) where \( \frac{d}{dt} \) in the proof is temporarily understood as the weak derivative with respect to time.

**Lemma 1:** Let \( h \in C^1([0, T], H_\Omega) \). Then,
\[
R_{\Omega} \dot{h}(t) = \frac{d}{dt} (R_{\Omega} h(t))
\]
with \( \frac{d}{dt} \) the weak derivative in time.

**Proof:** A mapping \( h : [0, T) \to H_\Omega \) is weakly differentiable if there is a distribution \( v = \dot{h} : [0, T) \to H_\Omega \) such that

\[
\int_0^T (h(\zeta), \dot{\phi}(\zeta)w)_{H_\Omega} d\zeta = -\int_0^T (v, \phi(\zeta)w)_{H_\Omega} d\zeta,
\]

(13)

for all \( w \in H_\Omega \) and \( \phi \in C_0^\infty[0, T] \) [29]. From Section I we know that the projection operator can be expressed as \( P_\Omega = E_0 R_\Omega \). Note that for \( f \in H_\Omega \) the projection operator acts as the identity, so the left hand side of Eq. (14) can be written as

\[
\int_0^T (P_\Omega h(\zeta), P_\Omega \dot{\phi}(\zeta)w)_{H_\Omega} d\zeta
\]

\[
= \int_0^T (E_0 R_\Omega h(\zeta), E_0 R_\Omega \dot{\phi}(\zeta)w)_{H_\Omega} d\zeta,
\]

\[
= \int_0^T (R_\Omega h(\zeta), R_\Omega \dot{\phi}(\zeta)w)_{H_\Omega} d\zeta.
\]

(14)

The right hand side of Equation (14) can similarly be expressed as

\[
\int_0^T (v(\zeta), \phi(\zeta)w)_{H_\Omega} d\zeta,
\]

\[
= \int_0^T (E_0 R_\Omega v(\zeta), E_0 R_\Omega \phi(\zeta)w)_{H_\Omega} d\zeta,
\]

\[
= \int_0^T (R_\Omega v(\zeta), \phi(\zeta)R_\Omega w)_{H_\Omega} d\zeta.
\]

(15)

Combining Equations (14) and (15) we have

\[
\int_0^T (R_\Omega h(\zeta), \dot{\phi}(\zeta)R_\Omega w)_{H_\Omega} d\zeta
\]

\[
= -\int_0^T (R_\Omega v(\zeta), \phi(\zeta)R_\Omega w)_{H_\Omega} d\zeta
\]

(16)

for all \( \phi \in C_0^\infty[0, T] \) and \( w \in H_\Omega \). Since the operator \( R_\Omega \) is onto \( R_\Omega H_\Omega \), the conclusion of the lemma follows. \( \blacksquare \)

Since \( R_\Omega \) is a bounded linear operator, we also know that

\[
\frac{\| (R_\Omega f)(t + \Delta) - (R_\Omega f)(t) \|_{H_\Omega}}{\Delta} \leq \frac{\| f(t + \Delta) - f(t) \|_{H_\Omega}}{\Delta} \to \frac{\| f \|_{H_\Omega}}{\Delta}
\]

which implies that the map \( t \mapsto (R_\Omega f)(t) \) is strongly differentiable. This means that \( R_\Omega \dot{f} = \frac{d}{dt} (R_\Omega f) \) where now both derivatives are interpreted in the strong sense.

With the conclusion from Lemma 1, we are able to derive the evolution of the finite dimensional approximation restricted to the manifold \( \Omega \). Apply the restriction operator \( R_\Omega \) to both sides of Eq. (8) and the error equation with respect to the restriction \( R_\Omega f_n(t) \) is

\[
\frac{d}{dt} (R_\Omega f_n(t)) = R_\Omega \dot{f}_n(t)
\]

\[
= -\gamma R_\Omega P_{\Omega_n}(B\mathcal{E}_{x(t)})^* P\tilde{x}_n(t)
\]

\[
+ \gamma R_\Omega (I - P_{\Omega_n})(B\mathcal{E}_{x(t)})^* P\tilde{x}(t).
\]

(17)

**B. Error Estimates of Sobolev Type**

In this section we derive the primary theoretical result of this paper that is summarized in Theorem 2 and Corollary 2. First we review some results of an analysis in [30], [31] about Sobolev error bounds for scattered data interpolation. Let \( t \) and \( \mu \) be the orders of two Sobolev spaces of functions defined over a smooth manifold \( \Omega \). Clearly, given \( t > \mu \), it follows that \( W^{t,2}(\Omega) \subset W^{\mu,2}(\Omega) \). Suppose the function \( u \) has a set of zero points \( \Omega_0 \subset \Omega \) (i.e. \( u|_{\Omega_0} \equiv 0 \)) which are distributed densely enough in \( \Omega \). This theorem characterizes the relationship between \( \| u \|_{W^{t,2}(\Omega)} \) and \( \| u \|_{W^{\mu,2}(\Omega)} \) in the term of the fill distance of zeros \( \Omega_0 \), which is defined below.

**Definition 1 (Fill Distance [24]):** For a finite set of discrete points \( \Omega_n = \{ \xi_i \}_{i=1}^n \) in a metric space \( \Omega \), the fill distance \( h_{\Omega_0, \Omega} \) of \( \Omega_n \) with respect to \( \Omega \) is defined as

\[
h_{\Omega_0, \Omega} := \inf \min_{x \in \Omega, \xi \in \Omega_n} d(x, \xi),
\]

where \( d(\cdot, \cdot) \) is the intrinsic metric in \( \Omega \).

In the case which is of the most interest to this paper, the set \( \Omega \) is a compact smooth Riemannian submanifold in \( \mathbb{R}^d \), and the discrete set \( \Omega_n \) is the set of interpolation points in the manifold. With this definition in mind, the following theorem states the relationship between \( \| u \|_{W^{t,2}(\Omega)} \) and \( \| u \|_{W^{\mu,2}(\Omega)} \).

**Theorem 1:** Let \( \Omega \subseteq \mathbb{R}^d \) be a smooth \( k \)-dimensional manifold, \( t \in \mathbb{R} \) with \( t > k/2 \), \( \mu \in \mathbb{N}_0 \) with \( 0 \leq \mu \leq \lfloor t \rfloor - 1 \). Then there is a constant \( h_{\Omega_0} \) such that if the fill distance \( h_{\Omega_0, \Omega} \leq h_{\Omega_0} \) and \( u \in W^{t,2}(\Omega) \) satisfies \( u|_{\Omega_0} \equiv 0 \), then

\[
\| u \|_{W^{\mu,2}(\Omega)} \lesssim h_{\Omega_0, \Omega}^{t-\mu} \| u \|_{W^{t,2}(\Omega)}.
\]

We denote the interpolation operator over \( \Omega_n \) by \( I_{\Omega_n} \). For a function \( f \in W^{t,2}(\Omega) \), the interpolation error \( (I - I_{\Omega_n}) f \) by definition has zeros over \( \Omega_n \). A corollary is introduced in [24] to characterize the decaying rate of interpolation error in the native space (i.e. RKHS).

**Corollary 1:** Let \( \Omega \subseteq X := \mathbb{R}^d \) be a \( k \)-dimensional smooth manifold, and let the native space \( H_X \) be continuously embedded in a Sobolev space \( W^{t,2}(X) \) with \( t > d/2 \), so that \( \| f \|_{W^{t,2}(X)} \lesssim \| f \|_{H_X} \). Define \( s = t - (d-k)/2 \) and let \( 0 \leq \mu \leq \lfloor s \rfloor - 1 \). Then there is a constant \( h_{\Omega_0} \) such that if \( h_{\Omega_0, \Omega} \leq h_{\Omega_0} \), then for all \( f \in R_{\Omega}(H_X) \) we have

\[
\| (I - I_{\Omega_n}) f \|_{W^{\mu,2}(\Omega)} \lesssim h_{\Omega_0, \Omega}^{h_{\Omega_0, \Omega}} \| f \|_{R_{\Omega}(H_X)}.
\]

With this error bound in mind for interpolation and projection, we now turn to the main result of this paper.

**Theorem 2:** Suppose that \( \Omega \) is a compact, connected, \( k \)-dimensional, regularly embedded Riemannian submanifold of \( X := \mathbb{R}^d \), \( R_{\Omega}(H_X) \to W^{t,2}(\Omega) \) for some \( s > k/2 \), and the orbit of the unknown system \( \Gamma^x(\epsilon_0) \subseteq \Omega \). Then there exist two constants \( a, b > 0 \) such that for all \( t \in [0, T] \),

\[
\| \tilde{x}(t) \|^2 + \| R_{\Omega} f_n(t) \|^2_{W^{t,2}(\Omega)}
\]

\[
\leq e^{bt} \left( \| (I - I_{\Omega_n}) R_{\Omega} f_0 \|^2_{W^{t,2}(\Omega)}
\]

\[
+ a \int_0^t \| (I - I_{\Omega_n}) R_{\Omega}(\mathcal{E}_{x(t)}) \|^2_{W^{t,2}(\Omega)} d\zeta \right).
\]
Here $\Pi_{\Omega_n} : \mathbf{R}_{\Omega}H_X \rightarrow \mathbf{R}_{\Omega}H_{\Omega_n}$ denotes the projection operator defined over the space of restriction $\mathbf{R}_{\Omega}H_X$ onto the space $\mathbf{R}_{\Omega}H_{\Omega_n}$.

**Proof:** The proof of the inequality below uses the fact that $\mathbf{R}_{\Omega}(\frac{d}{dt}h(t)) = \frac{d}{dt}(\mathbf{R}_{\Omega}h(t))$ where the time derivative is understood in the strong sense, i.e. $\mathbf{R}_{\Omega}h(t) = \frac{d}{dt}(\mathbf{R}_{\Omega}h(t))$.

By Eq. 17 and 17 we have

$$
\frac{d}{dt}\left(\|\bar{x}_n(t)\|^2 + \|\mathbf{R}_{\Omega}\bar{f}_n(t)\|^2_{W^{s,2}(\Omega)}\right) = (\dot{\bar{x}}_n(t), \bar{x}_n(t)) + 2\left(\mathbf{R}_{\Omega}\bar{f}_n(t), \frac{d}{dt}(\mathbf{R}_{\Omega}\bar{f}_n(t))\right)_{W^{s,2}(\Omega)}
$$

In several of the steps that follow, we use conclusions that result from the assumptions that certain spaces are continuously embedded in others. By choosing certain types of reproducing kernels, it can be guaranteed that the injection $j : \mathbf{R}_{\Omega}(H_X) \rightarrow W^{s,2}(\Omega)$ is such that we have the continuous embedding

$$
\mathbf{R}_{\Omega}(H_X) \hookrightarrow W^{s,2}(\Omega)
$$

for some $s > k/2$. By the Sobolev embedding theorem the injection $i : W^{s,2}(\Omega) \rightarrow C(\Omega)$ is also continuous, that is

$$
W^{s,2}(\Omega) \hookrightarrow C(\Omega).
$$

This implies that

$$
|\mathcal{E}_{s,x}(f)| := |f(x)| \leq \|i\|_C \|\bar{x}_n(t)\|_{W^{s,2}(\Omega)},
$$

and it follows that each evaluation functional $\mathcal{E}_{s,x} : W^{s,2}(\Omega) \rightarrow \mathbb{R}$ is uniformly bounded by $\|i\|$. By assumption we have that the forward orbit $\Gamma^+(x_0) \subseteq \Omega$, we conclude that term 1 can be bounded by the expression

$$
|\left(B\mathcal{E}_{x}(t)\mathbf{R}_{\Omega}\bar{f}_n(t), \bar{x}_n(t)\right)| \leq \|B\|\|i\|\|\bar{x}_n(t)\|\|\mathbf{R}_{\Omega}\bar{f}_n(t)\|_{W^{s,2}(\Omega)}. \tag{18}
$$

We bound term 2 by

$$
\left(\mathbf{R}_{\Omega}(I - \Pi_{\Omega_n})\mathcal{E}_{x}(t)B^TP\tilde{x}(t), \mathbf{R}_{\Omega}\bar{f}_n(t)\right)_{W^{s,2}(\Omega)} \leq \|\mathbf{R}_{\Omega}(I - \Pi_{\Omega_n})\mathcal{E}_{x}(t)B^TP\tilde{x}(t)\|_{W^{s,2}(\Omega)}
$$

$$
\|\mathbf{R}_{\Omega}\bar{f}_n(t)\|_{W^{s,2}(\Omega)} \leq \|B\|\|P\|\|\tilde{x}(t)\|\|I - \Pi_{\Omega_n}\|\|\mathbf{R}_{\Omega}\tilde{z}(t)\|_{W^{s,2}(\Omega)}
$$

$$
\|\mathbf{R}_{\Omega}\bar{f}_n(t)\|_{W^{s,2}(\Omega)}. \tag{19}
$$

We next consider term 3, which satisfies the inequality

$$
\left(\mathbf{R}_{\Omega}\mathbf{P}_{\Omega_n}\mathcal{E}_{x}(t)B^TP\tilde{x}(t), \mathbf{R}_{\Omega}\bar{f}_n(t)\right)_{W^{s,2}(\Omega)} \leq \|\mathbf{R}_{\Omega}\mathbf{P}_{\Omega_n}\mathcal{E}_{x}(t)B^TP\tilde{x}(t)\|_{W^{s,2}(\Omega)}
$$

$$\times \|\mathbf{R}_{\Omega}\bar{f}_n(t)\|_{W^{s,2}(\Omega)} \times \|\mathbf{R}_{\Omega}\bar{f}_n(t)\|_{W^{s,2}(\Omega)}.
$$

Combining all the terms above, we obtain

$$
\frac{d}{dt}\left(\|\bar{x}_n(t)\|^2 + \|\mathbf{R}_{\Omega}\bar{f}_n(t)\|^2_{W^{s,2}(\Omega)}\right)
$$

$$\leq \gamma\|B\|^2\|P\|^2\|\tilde{x}(t)\|^2\|I - \Pi_{\Omega_n}\|\mathbf{R}_{\Omega}\bar{f}_n(t)\|_{W^{s,2}(\Omega)}^2
$$

$$+ (2\|A\| + \|i\|\|B\|)^2\|\bar{x}_n(t)\|^2
$$

$$+ (\gamma \bar{k}^2\|P\|^2\|\mathbf{R}_{\Omega}\bar{f}_n(t)\|_{W^{s,2}(\Omega)})^2
$$

$$+ (2\gamma + \|i\|\|P\|)^2\|\bar{x}_n(t)\|^2
$$

Let the constants $a, b$ be defined as follows.

$$
a := \gamma\|B\|^2\|P\|^2 \sup_{\zeta \in [0,T]} \|\tilde{x}(\zeta)\|^2,
$$

$$b := \max\{2\gamma + \|i\|\|P\|, 2\|A\| + \|i\|\|B\| + \gamma \bar{k}^2\|P\|^2\|\mathbf{R}_{\Omega}\bar{f}_n(t)\|_{W^{s,2}(\Omega)}\}.
$$

When we integrate the inequality above, it follows that

$$
\|\bar{x}_n(t)\|^2 + \|\mathbf{R}_{\Omega}\bar{f}_n(t)\|^2_{W^{s,2}(\Omega)}
$$

$$\leq \|\bar{x}_n(0)\|^2 + \|\mathbf{R}_{\Omega}\bar{f}_n(0)\|^2_{W^{s,2}(\Omega)}
$$

$$+ \int_0^t a\|(I - \Pi_{\Omega_n})\mathbf{R}_{\Omega}\mathbf{R}_{\Omega_n}(\zeta)\|_{W^{s,2}(\Omega)}d\zeta
$$

$$+ \int_0^t (\|\tilde{x}(t)\|^2 + \|\mathbf{R}_{\Omega}\bar{f}_n(\zeta)\|^2)_{W^{s,2}(\Omega)}d\zeta.
$$

But since $\hat{f}_0 \in H_{\Omega}$, we know that $\mathbf{P}_{\Omega}\hat{f}_0 = \mathbf{E}_{\Omega}\mathbf{R}_{\Omega}\hat{f}_0 = \hat{f}_0$, and $\mathbf{R}_{\Omega}\Pi_{\Omega_n}\hat{f}_0 = \Pi_{\Omega_n}\mathbf{R}_{\Omega}\hat{f}_0$. Thus

$$
\|\mathbf{R}_{\Omega}\hat{f}_0(0)\|_{W^{s,2}(\Omega)} = \|(I - \Pi_{\Omega_n})\mathbf{R}_{\Omega}\hat{f}_0\|_{W^{s,2}(\Omega)}.
$$

Combining the above inequalities yields

$$
\|\bar{x}_n(t)\|^2 + \|\mathbf{R}_{\Omega}\bar{f}_n(t)\|^2_{W^{s,2}(\Omega)}
$$

$$\leq e^{bt}\|(I - \Pi_{\Omega_n})\mathbf{R}_{\Omega}\hat{f}_0\|_{W^{s,2}(\Omega)}^2
$$

$$+ ae^{bt}\int_0^t \|(I - \Pi_{\Omega_n})\mathbf{R}_{\Omega}\tilde{z}(\zeta)\|^2_{W^{s,2}(\Omega)}d\zeta.
$$

The next corollary combines the results of Theorem 2 and Corollary 2 to obtain the error rates in terms of the fill distance of samples in the manifold.

**Corollary 2:** Suppose that $\Omega$ is a compact, connected, regularly embedded $k$-dimensional submanifold of $X := \mathbb{R}^d$, the kernel $k$ is selected so that $H_X \rightarrow W^{s,k}(\mathcal{X})$ for $\tau > d/2$, define $s := \tau - (d-k)/2$, and let $s \in [k/2, \lceil s \rceil - 1]$. Then we have

$$
\|\bar{x}_n(t)\|^2 + \|\mathbf{R}_{\Omega}\bar{f}_n(t)\|^2_{W^{s,2}(\Omega)}
$$

$$\leq \left(\|\mathbf{R}_{\Omega}\hat{f}_0\|^2_{\Omega_{\Omega_n}(H_X)} + a\bar{k}^2t\right)e^{bt}\|\tilde{h}_{\Omega_{\Omega_n}}^{2(s-\mu)}.
$$
Proof: We first observe that under the stated hypotheses the native space $R_{\Omega} H_X \hookrightarrow W^{\mu,2}(\Omega)$. For any two positive $r_1 \geq r_2 > 0$ the associated Sobolev spaces are a continuous scale of spaces with $W^{r_1,2}(\Omega) \hookrightarrow W^{r_2,2}(\Omega)$. This implies that $W^{s,2}(\Omega) \hookrightarrow W^{\mu,2}(\Omega)$ since $s \geq \mu$. Also, the trace theorem yields
\[
\|f\|_{W^{s,2}(\Omega)} \lesssim \|f\|_{W^{\mu,2}(\Omega)} = \|E_{\Omega} R_{\Omega} f\|_{W^{s,2}(\Omega)} \lesssim \|E_{\Omega} f\|_{W^{s,2}(\Omega)} \lesssim \|f\|_{R_{\Omega} H_X},
\]
where the constants in the above string of inequalities depend on $\|E_{\Omega}\|$, $\|R_{\Omega}\|$, and the norm of the embedding of $H_X$ into $W^{s,2}(X)$. Combining these results yields $R_{\Omega} H_X \hookrightarrow W^{\mu,2}(\Omega)$ with $\mu \geq k/2$. We can now apply the results of Theorem 2 for the choice $s = \mu$ and write
\[
\|\bar{x}_n(t)\|^2 + \|R_{\Omega} \bar{f}_n(t)\|^2_{W^{s,2}(\Omega)} \\
\leq e^{bs} \|(I - \Pi_{\Omega}) \bar{x}_n(t)\|^2_{W^{s,2}(\Omega)} + a e^{bs} \|(I - \Pi_{\Omega}) R_{\Omega} \bar{x}_n(t)\|^2_{W^{s,2}(\Omega)},
\]
where $b$ is chosen such that $e^{bs} R_{\Omega} H_X \hookrightarrow W^{s,2}(\Omega)$, then $\sup_{t \in [0,T]} \|R_{\Omega} \bar{x}_n(t)\|_{H_X} \leq \bar{k}$ for each $t \in [0,T]$ by Theorem 11 of [24]. The bound now follows since
\[
\sup_{t \in [0,T]} \|R_{\Omega} \bar{x}_n(t)\|_{H_X} = \sup_{t \in [0,T]} \|\bar{x}_n(t)\|_{H_X} \leq \bar{k}.
\]

IV. NUMERICAL SIMULATION

Corollary 2 gives a rate of convergence for finite dimensional approximations of the RKHS embedded estimator. The rate of convergence depends on the density of the sample set $\Omega_n$ in the manifold $\Omega$, which is characterized by the fill distance $h_{\Omega_n, \Omega}$. In this section, this rate of convergence is illustrated qualitatively using numerical simulations. Following the formulation of Eq. 3 the governing equations of the unknown system are selected to be
\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
x_1^2(t) \\
0
\end{bmatrix},
\]
where $B = [1,0]^T$ and $f(x_1,x_2) = x_1^2$. Here we assume the linear coefficient matrix $A_0$ is known. By adding and subtracting a selected Hurwitz term $A x(t)$, the governing equations of the original system can be written as
\[
\dot{x}(t) = A x(t) + (A_0 - A) x(t) + B E_{\Omega} x(t) f.
\]
Since $A_0$ and $A$ are known, the term $(A_0 - A) x(t)$ can be canceled in the error equation. The governing equations of the finite dimensional RKHS embedded estimator are chosen as
\[
\begin{align*}
\dot{\hat{x}}_n(t) &= A \hat{x}_n(t) + (A_0 - A) x(t) + B \hat{E}_{\Omega} x(t) P_{\Omega} \hat{f}_n(t), \\
\dot{\hat{f}}_n(t) &= \gamma P_{\Omega} (B \hat{E}_{\Omega} x(t))^* P_{\Omega} x(t) - \hat{x}_n(t).
\end{align*}
\]
This choice yields the error equations that have the form studied in this paper. The phase portraits of the original system in Eq. 21 are shown in Fig. 1. A first integral of the unknown system is
\[
\Phi(x) := x_2 + x_1^2 - 0.5 e^{2x_2} = c. 
\]

The stability of the system depends on the initial condition $(x_1(0), x_2(0))$. When the initial condition $x(0)$ is such that the constant $c < 0$, the system is stable and the positive orbit $\Gamma^+(x(0))$ itself is the invariant manifold $\Omega := \{ x \in \mathbb{R}^2 : \Phi(x) = c \}$. The manifold $\Omega$ is a smooth, one dimensional, regularly embedded submanifold in the phase space $\mathbb{R}^2$. In this example, we choose the trajectory for of which the constant $c = -0.1$ in Eq. 22. The samples $\Omega_n = \{ \xi_i \}_{i=1}^N$ are taken uniformly along the manifold with respect to the intrinsic metric of the manifold $\Omega$. Although in practice, such sampling procedure cannot be accomplished without knowing the manifold a priori, it is not difficult to picture that as $t \to \infty$, the set $\{ x(t_i) \}_{i=1}^N$ gradually fills the manifold $\Omega$. The samples are used to construct the approximant RKHS.
The Sobolev-Mate\'n kernel $\mathcal{R}_\nu$ is used to induce the RKHS. The subscript $\nu$ denotes the order of the kernel. If $\nu > d/2$, then all the functions in the RKHS $H_X$ induced by $\mathcal{R}_\nu$ over $X = \mathbb{R}^d$ also belong to every Sobolev space $W^{\tau,2}(\mathbb{R}^d)$ where $\tau > 2\nu - d/2$ [24]. The general expression of $\mathcal{R}_\nu$ is defined using a Bessel function, but when $\nu = p + 1/2$, $p \in \mathbb{N}$ the kernel has the following closed-form expressions

$$\mathcal{R}_{3/2}(x, y) = \left(1 + \frac{\sqrt{3}r}{l}\right) \exp\left(-\frac{\sqrt{3}r}{l}\right),$$

$$\mathcal{R}_{5/2}(x, y) = \left(1 + \frac{\sqrt{5}r}{l} + \frac{5r^2}{3l^2}\right) \exp\left(-\frac{\sqrt{5}r}{l}\right),$$

where $r = \|x - y\|$, and $l$ is the scaling factor of length [32].

Fig. 2 shows the contour of the estimation error $|f(x) - \hat{f}_n(x)|$ in $\mathbb{R}^d$ using when $N = 100$ and $\nu = 5/2$. The result is as expected. The estimate of error in the unknown function is close to zero along the manifold $\Omega = \{x \in \mathbb{R}^2 : \Phi(x) = -0.1\}$. The rate of convergence with respect to the number of samples $N$ is shown in Fig. 3. Note that the manifold $\Omega$ is a closed curve, and the samples are taken uniformly in metric. As a result, the fill distance $h_{\Omega_n,\Omega} \sim N^{-1}$. With this in mind, by Corollary 2 we have the following relationship

$$\|\mathbf{R} \mathcal{R}_\nu(\hat{f}(t) - \hat{f}_n(t))\|_{W^{\mu,2}(\Omega)} \sim N^{-(s-\mu)}.$$

In this example, the set $\Omega$ is PE, so $\hat{f}(t) \rightarrow f$ over $\Omega$. The RKHS $H_X \hookrightarrow W^{\tau,2}(\mathbb{R}^2)$ where $\tau < 2\nu - 1$. Thus the space of restrictions $\mathbf{R}_\nu H_X \hookrightarrow W^{\tau-0.5,2}(\Omega)$, and $s \leq \tau - 0.5 < 2\nu - 1.5$. On the other hand, we must have $\mu \in [k/2, [s] - 1]$ so that $W^{s,2}(\Omega) \hookrightarrow W^{\mu,2}(\Omega) \hookrightarrow C(\Omega)$. In this way, we have

$$\|f - \hat{f}_n(t)\|_{C(\Omega)} = \sup_{x \in \Omega} |f(x) - (\hat{f}_n(t))(x)| \sim N^{-(s-\mu)}.$$

From the analysis above, we obtain the rates of convergence for the $C(\Omega)$-norm. When $\nu = 3/2$, the order $s = \mu \geq 1$. When $\nu = 5/2$, the order $s = \mu \geq 2$. Taking the logarithms for both sides of the equation above, the values calculated above are the worst case of slope bounds in Fig. 3. In both figures, the actual error curves are below the slope bounds, which validates the conclusions in Corollary 2. One assumption for the Theorem 1 to hold is that $h_{\Omega_n,\Omega}$ must be smaller than a threshold. This assumption may explain the flat error curve when $N \leq 30$.

V. CONCLUSIONS

The RKHS embedding method constructs estimates of the unknown or uncertain functions that appear in types of ODEs in an infinite dimensional RKHS. This paper considers the practical problem of formulating finite dimensional approximations for this technique. The convergence of approximations is proven, and the rates of convergence are derived. By selecting the reproducing kernel that has algebraic decaying Fourier transform, the induced RKHS is embedded in or equivalent to a standard Sobolev space. The error equation of approximation is recast in the Sobolev space, and bounds on the error of interpolation in Sobolev spaces are applied to analyse the error of approximation. When the trajectory of the unknown system concentrates in a compact, regularly embedded submanifold of the state space, the rate of convergence for finite dimensional approximation is derived in terms of the fill distance of the samples. It is shown that as the samples becomes increasingly dense in the submanifold, the approximation error decays accordingly.

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