Quasi-normal modes of superfluid neutron stars

L. Gualtieri\textsuperscript{1}, E. M. Kantor\textsuperscript{2}, M. E. Gusakov\textsuperscript{2,3}, A. I. Chugunov\textsuperscript{2} \\
\textsuperscript{1}Dipartimento di Fisica, “Sapienza” Università di Roma & Sezione INFN Roma1, Piazzale Aldo Moro 5, 00185, Roma, Italy  
\textsuperscript{2}Ioffe Physical-Technical Institute of the Russian Academy of Sciences, Polytekhnicheskaya 26, 194021 Saint-Petersburg, Russia and  
\textsuperscript{3}Saint-Petersburg State Polytechnical University, Polytekhnicheskaya 29, 195251 Saint-Petersburg, Russia

We study non-radial oscillations of neutron stars with superfluid baryons, in a general relativistic framework, including finite temperature effects. Using a perturbative approach, we derive the equations describing stellar oscillations, which we solve by numerical integration, employing different models of nucleon superfluidity, and determining frequencies and gravitational damping times of the quasi-normal modes. As expected by previous results, we find two classes of modes, associated to superfluid and non-superfluid degrees of freedom, respectively. We study the temperature dependence of the modes, finding that at specific values of the temperature, the frequencies of the two classes of quasi-normal modes show avoided crossings, and their damping times become comparable. We also show that, when the temperature is not close to the avoided crossings, the frequencies of the modes can be accurately computed by neglecting the coupling between normal and superfluid degrees of freedom. Our results have potential implications on the gravitational wave emission from neutron stars.

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I. INTRODUCTION

When a neutron star (NS) is excited by an external or internal event – such as a glitch, a close interaction with an orbital companion, or the gravitational collapse from which it is born – it can be set into non-radial oscillations, emitting gravitational waves (GWs) at the frequencies of its quasi-normal modes (QNMs). Such oscillations are damped, due to GW emission and to dissipative processes. In some cases – for instance, in presence of rotation – unstable modes can also be present in the spectrum; these modes do not require a specific excitation mechanism, since small fluctuations in this case grow exponentially.

The QNMs of a NS carry invaluable information on the state and composition of matter at the extreme densities and pressures prevailing in its core, which are still poorly understood (see, e.g., \cite{1} and references therein). Detection of the gravitational emission from a non-radially oscillating NS (by second- or, more likely, third-generation gravitational interferometers \cite{2,3}) would allow us to measure the frequencies and damping times of the NS QNMs, extracting information on the behaviour of matter in the stellar core \cite{4}. In addition, NS oscillations are probably associated to a wide variety of interesting astrophysical phenomena, such as quasi-periodic oscillations of the electromagnetic radiation observed in giant flares of soft gamma repeaters \cite{5,6}.

It is then not surprising that in the last decades, a huge effort has been done, on the theoretical side, to model – in a general relativistic framework – NS oscillations, taking into account all relevant features of the matter composing the NS. For many years, however, most studies neglected an important feature of NS matter: baryon superfluidity.

Theoretical studies (see, e.g., reviews \cite{7,8}) show that baryon matter in NS cores becomes superfluid at $T \lesssim 10^8 - 10^{10}$ K. This is also suggested by astrophysical observations. For instance, it is difficult to explain the phenomenon of pulsar glitches without invoking baryon superfluidity \cite{9}. Recent observations of the real-time cooling of the NS in Cassiopeia A supernova remnant \cite{10} can also be explained by NS models with superfluid baryons in the core \cite{11,12}.

Non-radial oscillations of relativistic stars have been studied since the late sixties (e.g., \cite{13,14}), but the pioneer works neglected superfluidity. Oscillations of superfluid stars, in which different components of the fluid can have different velocities, were first studied in a Newtonian framework \cite{15,16}, and, more recently, in general relativity \cite{17,18}. Most of these papers assumed vanishing temperature. However, this approximation is not justified: although after the first minutes of life, the temperature $T$ of a NS is much lower than the Fermi energy of neutrons [which allows one to use a zero-temperature equation of state (EoS)], it can be comparable to the critical temperature $T_{ci}$ at which baryon species $i$ becomes superfluid. Therefore, the temperature $T$ determines the fraction of paired baryons as well as the size of the superfluid region in the core and thus affects the dynamical properties of a NS. As discussed in \cite{19,20}, the assumption $T = 0$ can lead to qualitatively incorrect results.

Non-radial oscillations of superfluid NSs in general relativity, taking into account finite temperature effects, have been first studied in \cite{21,22}, in the so-called “decoupled limit”, in which the (small) coupling between non-superfluid and superfluid degrees of freedom is neglected (see also \cite{23}). In this paper we do not neglect this coupling. We derive the fully coupled equations describing non-radial oscillations of non-rotating, superfluid NSs, generalizing the equations of Lindblom & Detweiler \cite{24,25} to the case of superfluid nuclear matter. We also perform numerical integrations of these
II. SUPERFLUID FINITE-TEMPERATURE HYDRODYNAMICS

The equations of superfluid relativistic finite-temperature hydrodynamics were reviewed in many papers, see, e.g., Refs. [42, 43, 53, 10]. Here, following Ref. [42], we assume that the matter of NS cores consists of neutrons \((n)\), protons \((p)\) and electrons \((e)\), i.e., \(npe\)-matter; and that when the temperature is small enough (see below), neutrons are superfluid, and protons and electrons are superconducting.

Superfluidity affects the dynamical properties of a fluid, leading to a possibility of co-existence, without dissipation, of several independent motions with different velocities [17]. In particular, superfluid \(npe\)-matter is described by the three four-velocities: \(u^\mu, v^\mu_{(n)},\) and \(v^\mu_{(p)}\), where \(u^\mu\) is the velocity of the normal (non-superfluid) liquid component (electrons and Bogoliubov excitations of neutrons and protons) and \(v^\mu_{(i)}\) is the “superfluid” velocity of particle species \(i = n, p\) (the velocity of superfluid condensate of species \(i\)). In what follows, instead of \(v^\mu_{(i)}\), we will use the four-vector \(w^\mu_{(i)} = \mu_i (v^\mu_{(i)} - u^\mu)\), where \(\mu_i\) is the relativistic chemical potential for particle species \(i\).

The existence of two additional velocities in superfluid \(npe\)-matter modifies the expressions for neutron and proton conserved current densities, which become

\[
\begin{align*}
\gamma_{(i)}^\mu & = n_i u^\mu + Y_{ik} w^\mu_{(k)} \\
\gamma_{(e)}^\mu & = n_e u^\mu.
\end{align*}
\]  

(1)

(c.f. with the ordinary expression \(\gamma_{(i)}^\mu = n_i u^\mu\)). At the same time, the electron current density remains unaffected by superfluidity,

\[
\gamma_{(e)}^\mu = n_e u^\mu.
\]  

(2)

Here and below, the indexes \((i, k, l)\) refer to particle species; in particular, indexes \(i, k\) refer to nucleons \((i = n, p)\), \(l\) refers to nucleons and electrons \((l = n, p, e)\); \(n_l\) is the number density of the particle specie \(l\). Greek letters \((\mu, \nu, \ldots)\) refer to spacetime indexes, and the index \(j\) refers to purely spatial indexes. Unless otherwise stated, a summation is assumed over repeated indexes. We use geometrized units in which \(G = c = 1\).

In Eq. (1), \(Y_{ik}\) is the symmetric relativistic entrainment matrix, which is a generalization of the so-called Andreek-Bashkin matrix \(\rho_{ik}\) [48, 51] to the relativistic case [52, 53]. It was first introduced in Ref. [43] and accurately calculated in Refs. [52, 54]. In the non-relativistic limit this matrix is related to \(\rho_{ik}\) by the condition

\[
\rho_{ik} = m_i m_k Y_{ik},
\]  

(3)

where \(m_i\) is the bare nucleon mass, and there is no sum over the indexes \(i, k\). In the case of a one-component superfluid liquid, \(\rho_{ik}\) reduces to the so-called superfluid density \(\rho_s\) (see, e.g., [17]). The (symmetric) matrix \(Y_{ik}\) generally depends on the Landau parameters \(F_{ik}^1\) of asymmetric nuclear matter and on the temperature \(T\) [52]. In beta-equilibrium \(Y_{ik}\) can be expressed as a function only depending on the energy density \(\rho\) (or the baryon number density \(n_b = n_n + n_p\)) and the combinations \(T/T_{cn}\) and \(T/T_{cp}\), where \(T_{cn}(\rho)\) and \(T_{cp}(\rho)\) are the density-dependent neutron and proton critical temperatures, respectively. If, for example, \(T > T_{cn}\), then all neutrons are normal and the corresponding matrix elements \(Y_{nk} = Y_{kn}\) vanish.

In what follows we will be interested in low-frequency oscillations of a NS (\(p-\) and \(f-\)modes) which are well below the electron and proton plasma frequencies, and therefore preserve quasi-neutrality, \(n_e = n_p\). For a non-rotating non-magnetized NS this condition, together with continuity equations for electrons and protons, implies that \(\gamma_{(p)}^\mu = \gamma_{(e)}^\mu\) or, in view of Eqs. (1) and (2), that

\[
Y_{pk} w^\mu_{(k)} = 0,
\]  

(4)

that is, \(w^\mu_{(p)}\) and \(w^\mu_{(n)}\) are interrelated.

We introduce two four-vectors which will be useful later,

\[
X^\mu = \frac{Y_{nk} w^\mu_{(k)}}{n_b},
\]  

(5)

which depends on the superfluid degrees of freedom, and

\[
U_{(b)}^\mu = u^\mu + X^\mu,
\]  

(6)

which we call “baryon four-velocity” (strictly speaking, \(U_{(b)}^\mu\) is not a four-velocity, since \(U_{(b)}^\mu U_{(b)}_{\mu} = -1\) only in the linearized theory; see the footnote 4 in Ref. [42] for more details). As follows from Eqs. (1), (2) and (4), the baryon current density \(\gamma_{(b)}^\mu = \gamma_{(p)}^\mu + \gamma_{(n)}^\mu\) is related to \(U_{(b)}^\mu\) by the standard equation

\[
\gamma_{(b)}^\mu = n_b U_{(b)}^\mu,
\]  

(7)

while \(\gamma_{(e)}^\mu\) equals

\[
\gamma_{(e)}^\mu = n_e \left[U_{(b)}^\mu - X^\mu\right].
\]  

(8)
Together with the quasi-neutrality condition \((n_e = n_p)\) and Eq. \((3)\), the equations of superfluid hydrodynamics are (e.g., Ref. \([12]\)):

1. Continuity equations for baryons \((b)\) and electrons \((e)\),
   \[
   j^\mu_{(b); \mu} = 0, \quad j^\mu_{(e); \mu} = 0. \tag{9}
   \]

2. Energy-momentum conservation
   \[
   T^\mu_{\; \nu} = 0, \tag{11}
   \]

   where the stress-energy tensor for the superfluid described above is:
   \[
   T^\mu_{\; \nu} = (P + \rho) u^\mu u^\nu + Pg^\mu_{\; \nu} + Y_{ik} \left( w^{\mu}_{(i)} w^{\nu}_{(k)} + \mu_i w^{\mu}_{(k)} u^{\nu} + \mu_k w^{\nu}_{(i)} u^{\mu} \right). \tag{12}
   \]

3. Potentiality condition for superfluid motion of neutrons
   \[
   \partial_\nu [w^{(n)}_{(\mu)} + \mu_n u_{\nu}] = \partial_\mu [w^{(n)}_{(\nu)} + \mu_n u_{\nu}]. \tag{13}
   \]

4. The second law of thermodynamics
   \[
   d\rho = T\, dS + \mu_n \, dn_l + \frac{Y_{ik}}{2} d \left( w^{(n)}_{(i)} w^{(n)}_{(k)} \right). \tag{14}
   \]

In formulas \((9)-(14)\) \(g^{\mu\nu}\) is the metric tensor; \(\partial_\mu = \partial/(\partial x^\mu)\); \(\rho\), \(S\), and \(\mu_e\) are the energy density, entropy density, and relativistic electron chemical potential, respectively. Finally, \(P\) is the pressure given by
   \[
   P = -\rho + \mu_n n_l + TS. \tag{15}
   \]

The equations of superfluid hydrodynamics described above should be supplemented by two additional conditions on the four-vectors \(w^\mu\) and \(w^{\mu}_{(n)}\):
   \[
   u_\mu u^\mu = -1, \tag{16}
   \]
   \[
   u_\mu w^{\mu}_{(n)} = 0, \tag{17}
   \]

i.e., the normalization condition for the four-velocity \(u^\mu\), and the requirement that in the comoving frame, the four-vector \(w^{\mu}_{(n)}\) is purely spatial. Then we have, using Eqs. \((1), (12), (16)\) and \((17)\), \(n_l = -u_\mu j^\mu_{(l)} \) \((l = n, p, e)\) and \(\rho = u_\mu u_\nu T^{\mu\nu}\).

### III. Perturbations of Neutron Stars with a Superfluid Phase

The theory of relativistic stellar perturbations has been developed, e.g., in Refs. \([13, 21]\). It allows one to describe the oscillations of a relativistic star (such as a NS), and in particular to determine the QNMs of the star. We have generalized this theory, originally developed to describe non-superfluid matter, to include a superfluid phase, which is described within the approach discussed in Sec. \([11]\). We here present the derivation of this generalization. Our starting point is the Lindblom & Detweiler (LD) formulation of the relativistic theory of stellar perturbations for non-superfluid stars \([10, 21]\). We follow the notation and conventions of Ref. \([20]\).

#### A. The stationary, spherically symmetric background

We describe stellar oscillations as linear perturbations of a stationary, spherically symmetric background, i.e., we expand
\[
\begin{align*}
g_{\mu\nu} &= g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \\
u^\mu &= u^{(0)}\mu + \delta u^\mu, \\
U^{(0)}_{(b)} &= U^{(b)}_{(0)} + \delta U^{(b)}_{(0)}. \tag{18}
\end{align*}
\]

The Schwarzschild background metric \(g^{(0)}_{\mu\nu}\), in the coordinates \(x^\mu = (t, r, \theta, \phi)\), can be written as
\[
(ds^{(0)})^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{19}
\]
where \(\nu(r), \lambda(r)\) are solutions to Eqs. \((21)\) [see below]. The background normal four-velocity is \(u^{(0)}\mu = (e^{-\nu/2}, 0, 0, 0)\).

In a non-rotating and non-magnetized star, the unperturbed velocities of normal and superfluid liquid components coincide, that is (see, e.g., \([42, 43, 45]\))
\[
u^{(0)}_{(n)} = w^{(0)}_{(p)} = X^{(0)} = 0. \tag{20}
\]

Hence \(U^{(0)}_{(b)} = u^{(0)}_{(b)} = (e^{-\nu/2}, 0, 0, 0)\) [see Eq. \((1)\)], while the background stress-energy tensor has exactly the same form as for a perfect fluid in the absence of superfluidity:
   \[
   T^{\mu}_{\; \nu} = (\rho^{(0)} + P^{(0)}) u^\mu u^\nu + P^{(0)} g^{\mu}_{\; \nu}. \tag{18}
   \]

In other words, the stationary configuration, i.e., the structure equations and the background spacetime metric, are not affected by superfluidity (see Ref. \([42]\) for a detailed discussion of this issue).

Therefore, the hydrostatic structure of a superfluid non-rotating NS is determined by the solution of standard Tolman-Oppenheimer-Volkov (TOV) equations [hereafter, a prime denotes differentiation with respect to the radial coordinate \(r\) and the notation \(Z^{(0)}\) denotes the value of some generic quantity \(Z\) (e.g., \(P, \rho\) etc.) in the unperturbed star),
\[
\begin{align*}
\nu' &= \frac{2e^\lambda}{r^3}(m + 4\pi P^{(0)} r^3), \\
m' &= 4\pi P^{(0)} r^2, \\
P^{(0)'} &= -\frac{1}{2}(\rho^{(0)} + P^{(0)}) \nu', \tag{21}
\end{align*}
\]
where $m(r) = r(1 - e^{-\lambda})/2$ is the gravitational mass inside the radius $r$. If the equation of state (EoS), providing a relation between $P(0)$ and $\rho(0)$, is known, the TOV equations allow to compute the gravitational mass $M$ and the circumferential radius $R$ of the NS.

In general, the EoS has the form $P = P(n_b, x_s, x_t)$, where $x_s$ is the entropy per baryon, and $x_t = m/n_b$ are the chemical fractions of different species. Notice that since the stellar temperature is much smaller than the chemical potentials, thermal effects do not affect the EoS (but they are relevant for the transition to the superfluid phase). Similarly, superfluidity of baryons in NS interiors does not significantly affect the EoS, because the superfluid energy gaps are negligible in comparison with the chemical potentials. Moreover, as discussed in Section [II] we consider npe-matter assuming charge quasi-neutrality ($n_p = n_e$). Therefore, the fluid can be described by a two-parameter EoS, say $P = P(n_b, x_e)$.

**B. Linear perturbations and harmonic expansion**

We shall only consider perturbations with polar parity, with harmonic index $l \geq 2$. The perturbations of the spacetime metric are expanded in tensor spherical harmonics, and Fourier transformed, as follows:

$$
\delta g_{\mu\nu}dx^\mu dx^\nu = \left[ e^{\lambda}H_0^{lm} dt^2 + 2i\omega r H_1^{lm} dt dr \\
+ e^{\lambda}H_2^{lm} dr^2 + r^2 K^{lm}(d\theta^2 \\
+ \sin^2 \theta d\phi^2) \right] Y^{lm}\epsilon^{i\omega t},
$$

(22)

where $Y^{lm}(\theta, \phi)$ are the usual spherical harmonics (not to be confused with the entrainment matrix $Y_{lk}$), while the functions $H_0^{lm}$, $H_1^{lm}$, $H_2^{lm}$, $K^{lm}$ depend on $r$ only.

1. **Non-superfluid phase**

The perturbations of the fluid four-velocity are expanded in tensor spherical harmonics, and Fourier transformed, as:

$$
\delta u^\mu = \left[ \delta u^0 e^{-(\nu + \lambda)/2}r^{l-1}W^{lm}Y^{lm}, \\
-i\omega e^{-\nu/2}r^{l-1}V^{lm}Y^{lm}, \\
-i\omega e^{-\nu/2}r^{l-1}\sin^{-2}\theta V^{lm}_{\theta}Y^{lm} \right] \epsilon^{i\omega t},
$$

(23)

where the functions $W^{lm}$ and $V^{lm}$ depend on $r$ only. In addition, if we denote a generic (scalar) fluid quantity (such as $P$, $\rho$, $n_b$, etc.) as $Z$, its Eulerian perturbation $\delta Z$ is decomposed as

$$
\delta Z = \delta Z^{lm} Y^{lm}\epsilon^{i\omega t}.
$$

(24)

The Lagrangian perturbation $\Delta Z$ (i.e., the perturbation of a given fluid element) is related to $\delta Z$ by

$$
\Delta Z = \delta Z + \xi^i Z_{;i},
$$

(25)

where $\xi^i$ is the Lagrangian displacement of the fluid element, related to the four-velocity perturbation by $\delta u^i = u^0 \xi^i + i\omega e^{-\nu/2}Z^i j$ (recall that the index $j$ denotes space components, i.e. $j = 1, 2, 3$). We note that, in the linear approximation, $u^0 Z_{\mu} = i\omega e^{-\nu/2}(\delta Z + \xi^i Z_{;i})$. The Lagrangian perturbation of the quantity $Z$ is expanded as $\Delta Z = \Delta Z^{lm} Y^{lm}\epsilon^{i\omega t}$. From Eq. (23), we can see that $\xi^i = e^{-\nu/2}W^{lm}Y^{lm}/r$. Therefore

$$
\Delta Z^{lm} = \delta Z^{lm} + Z^{(0)} e^{-\nu/2/r} W^{lm}.
$$

(26)

2. **Superfluid phase**

As discussed in Sec. [II] in a superfluid phase we introduce the four-vector $U^{(b)}_\mu = u^0 + X^\mu$, which we call "baryon four-velocity", satisfying (at first order in the perturbations) $U^{(b)}_\mu = \mu(0) = -\omega/2 \cdot 0, 0, 0, 0$ and its perturbation $\delta U^{(b)}_\mu$ has the same expansion as $\delta u^\mu$,

$$
\delta U^{(b)}_\mu = \left[ \delta U^{(0)}_\mu e^{-\nu/2}r^{l-1}W^{lm}Y^{lm}, \\
-i\omega e^{-\nu/2}r^{l-1}V^{lm}Y^{lm}, \\
-i\omega e^{-\nu/2}r^{l-1}\sin^{-2}\theta V^{lm}_{\theta}Y^{lm} \right] \epsilon^{i\omega t},
$$

(27)

in terms of the perturbation functions $W^{lm}(r)$ and $V^{lm}(r)$.

In a superfluid phase, we define Lagrangian perturbations in terms of the baryon four-velocity; therefore, if $Z$ is a generic fluid quantity, $U^{(b)}_\mu Z_{\mu} = i\omega e^{-\nu/2}\Delta Z$.

**C. Pulsation energy**

The mechanical energy stored in a QNM with frequency $\omega = \sigma + i\tau_{GW}$ can be evaluated, as discussed in Ref. [42], in terms of the eigenfunctions of the mode and can be split into two terms:

$$
E_{\text{mech}} = E_{\text{mech} (b)} + E_{\text{mech} (sfl)},
$$

(28)

where (see Eqs. (72) and (73) of Ref. [42])

$$
E_{\text{mech} (b)}(t) = e^{-2t/\tau_{GW}} \frac{\sigma^2}{2} \int_0^R \left[ P^{(0)} + \rho^{(0)} \right] e^{(\lambda - \nu)/2} r^{2l} \left[ |W^{lm}(r)|^2 + l(l+1) |V^{lm}(r)|^2 \right] dr,
$$

(29)

$$
E_{\text{mech} (sfl)}(t) = e^{-2t/\tau_{GW}} \frac{\sigma^2}{2} \int_0^R \left[ P^{(0)} + \rho^{(0)} \right] e^{(\lambda - \nu)/2} r^{2l} y \left[ |W^{lm}_{(sfl)}(r)|^2 + l(l+1) |V^{lm}_{(sfl)}(r)|^2 \right] dr,
$$

(30)

and

$$
y = \frac{n_b Y_{pp}}{\mu_n (Y_{nn} Y_{pp} - Y_{np}^2)} - 1.
$$

(31)
In Eq. (30) the perturbation functions $W^{lm}_{(sl)}$ and $V^{lm}_{(sl)}$ represent the harmonic components of the four-vector $X^\mu$ defined in Eq. (31):

$$X^\mu = \left(0, i\omega e^{-i(\nu+3)/2}r^{-1}W^{lm}_{(sl)}\right)^{\nu} \gamma^{lm},$$

$$-i\omega e^{-i(\nu+3)/2}r^{-2}V^{lm}_{(sl)}\gamma^{lm},$$

$$-i\omega e^{-i(\nu+3)/2}r^{-2} \sin^{-\theta} \theta V^{lm}_{(sl)}\gamma^{lm} \right) \delta_{\nu}w(t). \quad (32)$$

Vanishing of $X^0$ follows from Eqs. (31), (5), (17), and (20) (see also Eq. (42) of [12]). We note that Eq. (6) implies

$$W^{lm}_{(sl)} = W^{lm}_{(b)} - W^{lm},$$

$$V^{lm}_{(sl)} = V^{lm}_{(b)} - V^{lm}. \quad (33)$$

The four-vector $X^\mu$ – and thus its harmonic components $W^{lm}_{(sl)}$, $V^{lm}_{(sl)}$ – is associated to superfluid degrees of freedom. If $X^\mu = 0$ then superfluid degrees of freedom are not excited and superfluid and normal liquid components move with the same velocity (the so-called “co-moving” oscillations, similar to those of a non-superfluid matter) \(^1\). In this case only the first term, $E_{\text{mech}}(b)$, survives in Eq. (28).

D. The Lindblom & Detweiler equations (non-superfluid matter)

We here briefly discuss the derivation of the LD equations in the case of a NS composed of non-superfluid matter.

If we substitute the expansions (22), (23), and (24) into the linearized Einstein’s equations

$$\delta G_{\mu\nu} = 8\pi \delta T_{\mu\nu}, \quad (34)$$

where

$$\delta T_{\mu\nu} = (\delta \rho + \delta P) u^0_{\mu} u^0_{\nu} + (\rho^0 + P^0)(u^0_{\mu}) u^0_{\nu} + u^0_{\nu} \delta u_{\mu} + P^0 \delta g_{\mu\nu} + \delta P g^0_{\mu\nu}, \quad (35)$$

we get a set of equations for the seven perturbation functions $H^{lm}_{0}$, $H^{lm}_{1}$, $K^{lm}$, $W^{lm}$, $V^{lm}$, $\delta P^{lm}$ ($H^{lm}_{2}$) does not appear explicitly in the equations because Einstein’s equations imply $H^{lm}_{2} = H^{lm}_{0}$ for $l \geq 2$). In the LD formulation (21), there are four first-order differential equations [equations (8)–(11) of Ref. (20) and two algebraic relations [equations (5) and (6) of Ref. (20)]. In addition, $\delta P^{lm}$ and $\delta M^{lm}$ are related by the EoS, since

$$\Delta P^{lm} = c_s^2 \Delta \rho^{lm} \quad (36)$$

\(^1\) Here and in what follows by “superfluid” we, by definition, understand degrees of freedom associated with the vector $X^\mu$, whose spatial components depend on the difference between the normal and superfluid velocities (see Sec. II). Correspondingly, “normal” we imply degrees of freedom associated with the baryon four-velocity $U^{b}_{\mu}$. We remark that this is only a convention, even though in Sec. V C we will justify this choice a posteriori.

This is due to the fact that we consider non-dissipative hydrodynamics, therefore in the oscillation timescale $\Delta x_{e} = 0$, and the term $\partial P/\partial x_{e}\Delta x_{e}$ in Eq. (36) vanishes. The property $\Delta x_{e} = 0$ can be shown, for instance, as follows. Continuity equation for baryons $\left(n_{b}u^{\mu}_{b,\mu} = 0\right)$ can be written as

$$u^{\mu}_{n,\mu} + n_{b} u^{\nu}_{b,\mu} = i\omega e^{-i\nu/2} n_{b} + n_{b} u^{\nu}_{b,\mu} = 0. \quad (38)$$

Since continuity equation for electrons $\left(n_{e}u^{\mu}_{\text{e},\mu} = 0\right)$ also holds, we have that $i\omega e^{-i\nu/2} \Delta n_{b} + n_{e} u^{\nu}_{\text{e},\mu} = 0$, which, compared with (38), yields $\Delta n_{b}/n_{b} = \Delta n_{e}/n_{e}$, and then $\Delta x_{e} = 0$. We remark that Eq. (36) is equivalent to

$$\Delta P^{lm} = \frac{\gamma P^{(0)}}{n_{b}} \Delta n_{b}^{lm}, \quad (39)$$

where

$$\gamma = \frac{n_{b}^{(0)}}{P^{(0)}} \left(\frac{\partial P}{\partial x_{e}}\right)^{(0)} \quad (40)$$

Eq. (36) allows one to reduce the number of perturbation functions to six:

$$H^{lm}_{0}, H^{lm}_{1}, K^{lm}, W^{lm}, V^{lm}, X^{lm}, \quad (41)$$

where we have defined

$$X^{lm} \equiv - e^{-i\nu/2} \Delta P^{lm} = - e^{-i\nu/2} \delta P^{lm} - P^{(0)} \frac{e^{-i\nu/2}}{r} W^{lm} \quad (42)$$

(not to be confused with the four-vector $X^{\mu}$ introduced in the previous Section). Therefore, in the non-superfluid case the LD equations are fully determined, because they are six (differential or algebraic) equations for six perturbation functions.

The relation between $\delta P^{lm}$ and $\delta \rho^{lm}$ (or between $\delta P^{lm}$ and $\delta n_{b}^{lm}$) only affects one of the LD equations, namely

$$W^{lm}_{1} = - \frac{l + 1}{r} W^{lm}_{0} + r e^{\lambda/2} \frac{\gamma P^{(0)}}{n_{b}} X^{lm} \frac{e^{-i\nu/2}}{r} \left[H^{lm}_{0} + K^{lm} \right]. \quad (43)$$

To show how this occurs let us note that Eq. (43) can be derived from the continuity equation for baryons, which in our perturbative scheme can be written in the form [35]. Substitution of the perturbative expansion for the four-velocity (23) into Eq. (43) yields

$$W^{lm}_{1} = - \frac{l + 1}{r} W^{lm}_{0} + r e^{\lambda/2} \left[\frac{\Delta n_{b}^{lm}}{n_{b}^{(0)}} \right] \left[\frac{\Delta n_{b}^{lm}}{n_{b}^{(0)}} \right]

$$

$$- \frac{l + 1}{r} \frac{V^{lm}_{0}}{V^{lm}_{0}} + \frac{1}{2} \frac{H^{lm}_{0}}{H^{lm}_{0}} + K^{lm} \right]. \quad (44)$$

Then, using Eq. (43) together with the definition (12), we obtain Eq. (44), that is, Eq. (10) in the article by Detweiler & Lindblom (26).
E. Perturbation equations for a superfluid star

A remarkable property of the formulation of Refs. [42, 44] is that in a superfluid phase the stress-energy tensor perturbation has formally the same form as in a non-superfluid phase [see Eq. (43)], with \( \delta u^\mu \) replaced by \( \delta U^{(b)}_\mu \):

\[
\delta T_{\mu\nu} = (\delta \rho + \delta P) U^{(b)}_{(\mu} U^{(b)}_{\nu)} + \left( \rho^{(b)} + P^{(b)} \right) U^{(b)}_{(\mu} \delta U^{(b)}_{\nu)} + P^{(b)} \delta g_{\mu\nu} + \delta P g^{(b)}_{\mu\nu}.
\]

Therefore, the perturbation equations have formally the same expressions as the LD equations, with \( W^{lm} \) and \( V^{lm} \) [the variables of the expansion (23)] replaced, respectively, by \( W^{lm}_{(b)} \) and \( V^{lm}_{(b)} \) [the variables of the expansion (27)]. The only exception is equation (10) of Ref. [20] [i.e., Eq. (43) of this paper], which was derived by making use of the relation (39), not valid in superfluid matter (see below).

As we have noted in Sec. III D, Eq. (43) follows from the perturbative expansion of the baryon continuity equation and from Eq. (39), which relates \( \Delta P^{lm} \) and \( \Delta n^{lm}_e \). We shall now determine how these equations are modified in the superfluid case.

As discussed in Sec. III the baryon current density \( j^\mu_{(b)} = n_b U^\mu_{(b)} \) and the electron current density \( j^\mu_e = n_e \delta n^\mu_e \) satisfy the continuity equations \( \partial \delta n^\mu_e = 0 \). This implies

\[
\left[ n_b U^\mu_{(b)} \right]_{;\mu} = 0,
\]

\[
\left[ n_e U^\mu_{(b)} \right]_{;\mu} = (n_e X^\mu)_{;\mu}.
\]

Following [42], we define

\[
\delta n_e = \frac{e^{-\nu/2}}{i\omega} (n_e X^\mu)_{;\mu}.
\]

The continuity equation for baryons (40) has the same form as in the non-superfluid case [with \( u^\mu \) replaced by \( U^\mu_{(b)} \)], and its perturbative expansion yields [see Eq. (14)]

\[
W^{lm}_e = -\frac{l + 1}{r} W^{lm}_{(b)} + re^{\lambda/2} \left[ -\frac{\Delta n^{lm}_e}{n_b^{(0)}} \right] - \frac{l(l + 1)}{r^2} V^{lm}_{(b)} + \frac{1}{2} H^{lm}_0 + K^{lm}.
\]

The continuity equation for electrons (47), instead, is different from that in the non-superfluid case, and gives

\[
\frac{\Delta n_e}{n^{(0)}_{e}} = \frac{ie^{-\nu/2}}{\omega} \delta n^{\mu}_{(b)} m^{\mu}_{;\mu} = \frac{\Delta n_{b}}{n^{(0)}_{b}}.
\]

This implies that (expanding \( \delta n^{\mu}_{(b)} = \delta n^{\mu}_{(b)} + \delta n^{\mu}_{(sfl)} \))

\[
n_e^{(0)} \Delta x^{lm}_e = \Delta n_e^{lm} - x^{(0)}_e \Delta n_b^{lm} = \delta n^{lm}_{(sfl)}.
\]

In the superfluid case, then, \( \Delta x^{lm}_e \) does not vanish, and Eq. (39) is replaced by

\[
\Delta P^{lm}_{(b)} = \frac{\gamma P^{(0)}_{(b)}}{n_b} \Delta n_b^{lm} + \left[ \frac{\partial P^{(0)}}{\partial x_e} \right]^{(0)}_{(b)} \frac{n_b^{(0)}}{n_b} \delta n^{lm}_{(sfl)}.
\]

Therefore

\[
-\Delta n^{lm}_b = -\frac{\Delta P^{lm}_{(b)}}{\gamma P^{(0)}_{(b)}} + \frac{1}{\gamma P^{(0)}_{(b)}} \left[ \frac{\partial P^{(0)}}{\partial x_e} \right]^{(0)}_{(b)} \delta n^{lm}_{(sfl)}.
\]

\[
e^{-\nu/2} \gamma P^{(0)}_e \lambda^{lm},
\]

where we have defined

\[
\lambda^{lm} = X^{lm} + e^{\nu/2} \left( \frac{\partial P^{(0)}}{\partial n_e} \right)^{0}_{(b)} \delta n^{lm}_{(sfl)}.
\]

Eq. (43) is then replaced, in the superfluid case, by

\[
W^{lm}_{(b)} = -\frac{l + 1}{r} W^{lm}_{(b)} + re^{\lambda/2} \left[ e^{-\nu/2} \gamma P^{(0)}_e \lambda^{lm} - \frac{l(l + 1)}{r^2} V^{lm}_{(b)} + \frac{1}{2} H^{lm}_0 + K^{lm} \right].
\]

The new set of equations depends on a new perturbation quantity, \( \delta n^{lm}_{(sfl)} \), which, as shown in Ref. [42], can be expressed in terms of the redshifted chemical potential imbalance \( \delta \mu^{\infty} \), defined as:

\[
\delta \mu^{\infty} = e^{-\nu/2} \delta \mu \equiv e^{-\nu/2} (\mu_n - \mu_p - \mu_e).
\]

Note that \( \delta \mu^{\infty} \) is a first order quantity, since it vanishes on the background. As discussed in Refs. [41, 42, 44], the chemical potential imbalance is related to the space components of the four-vector \( X^\mu = U^\mu_{(b)} - u^\mu \) by

\[
X^\mu = \frac{\mu_n}{\mu_p - \mu_e} \frac{\partial_j (\delta \mu^{\infty})}{\partial_j},
\]

where \( y \) is given by Eq. (31). Expanding \( \delta \mu^{\infty} = \delta \mu^{\text{norm}} \delta \mu^{\text{e\ell}} \), we have [cf. Eqs. (92) and (100) of Ref. [42]]

\[
\delta n^{lm}_{(sfl)} = \frac{e^{-\nu/2}}{n_b} \left( \frac{\partial \delta \mu^{\text{norm}}}{\partial n_e} \right)^{0}_{(b)} \left( \delta \mu^{\text{e\ell}} - \delta \mu^{\text{norm}} \right),
\]

where

\[
\delta \mu^{\text{norm}} = e^{-\nu/2} n_b \left( \frac{\partial \delta \mu^{\text{norm}}}{\partial n_e} \right)^{0}_{(b)} \beta^{lm}_{1}.
\]

Note 2: This expansion is slightly different from that in Ref. [42]; the quantity \( \delta \mu^{\text{e\ell}} \) appearing in Ref. [42] reads, with our conventions, \( v^{\text{e\ell}} \delta \mu^{\text{e\ell}} \).
If the star has a three-layer structure, neutrons at
the LD equations are modified as follows:

\[ \beta_1^{lm} = K^{lm} + \frac{1}{2} H_0^{lm} - \frac{e^{-\nu/2}}{r} \left( W^{lm}_r + \frac{l+1}{r} W^{lm}_p \right) \]

\[ - \frac{l(l+1)}{r^2} V^{lm}_p, \]  \hspace{1cm} (60)

Substituting this expression into Eq. (59), we find

\[ \beta_1^{lm} = -\frac{e^{-\nu/2}}{\gamma P^{(0)}} \chi^{lm} \]  \hspace{1cm} (61)

and from Eqs. (52), (53), (54) we finally obtain

\[ \chi^{lm} = \left( 1 - \frac{1}{\gamma_2} \right) \left( \chi^{lm} + \gamma_3 n_b^{(0)} \delta\mu^{lm} \right), \]  \hspace{1cm} (62)

where

\[ \gamma_2 \equiv \left( \frac{\partial P}{\partial n_e} \right)^{0}_{n_b} \left( \frac{\partial \delta\mu}{\partial n_e} \right)^{0}_{n_b}, \quad \gamma_3 \equiv \left( \frac{\partial P}{\partial n_e} \right)^{0}_{n_b} \left( \frac{\partial \delta\mu}{\partial n_e} \right)^{(0)}_{n_b}. \]  \hspace{1cm} (63)

The perturbation functions \( H^{lm}_0, H^{lm}_1, K^{lm}, W^{lm}_p, V^{lm}_p, \) \( X^{lm} \) are then coupled by Eqs. (55) to the new quantity \( \delta\mu^{lm} \), describing the superfluid degrees of freedom.

The equation for the perturbation function \( \delta\mu^{lm} \) follows from the energy-momentum conservation \( (1) \) and the potentiality condition \( (1) \), and was obtained in Ref. 13. In the notations of this article, it can be written as

\[ \delta\mu^{lm} = \left[ h' \frac{h'}{h} + \frac{2(l+1)}{r} \right] \delta\mu^{lm}, \]

\[ \left[ (1 - e^{\nu/2}) \frac{l(l+1)}{r^2} + \frac{l}{r} \left( \frac{h'}{h} - \frac{\nu}{2} \right) \right] \delta\mu^{lm}, \]

\[ + \frac{e^{\lambda-\nu/2}}{h^2 B} \left[ \frac{\partial \delta\mu}{\partial n_e} \right]^{(0)}_{n_b} \left( \frac{\gamma_2}{\gamma_3} \right) \chi^{lm}, \]  \hspace{1cm} (64)

where

\[ B = \left( \frac{\partial \delta\mu}{\partial n_e} \right)^{(0)}_{n_b} ; \quad h = \frac{e^{\nu/2}}{\mu_n^{(0)} n_b^{(0)} y}. \]  \hspace{1cm} (65)

We can conclude that, in the case of superfluid matter, the LD equations are modified as follows:

- the functions \( W^{lm}(r) \) and \( V^{lm}(r) \) are replaced with \( W^{lm}_p(r) \) and \( V^{lm}_p(r) \), respectively;

- a new perturbation function \( \delta\mu^{lm} \), satisfying Eq. (64), is introduced;

- Eq. (10) of Ref. 20 is replaced by our Eq. (65).

Note that the quantity \( \chi^{lm} \) (which is related to \( \delta\mu^{lm} \) by Eq. (21)) only enters in this equation; the other LD equations [Eqs. (5), (6), (8), (9), (11) of 20] depend on \( \chi^{lm} = -e^{\nu/2} \Delta p^{lm} \), and have the same form (with the replacement \( W^{lm} \rightarrow W^{lm}_p \) and \( V^{lm} \rightarrow V^{lm}_p \)) as in the non-superfluid case.

The full set of the perturbation equations is summarized in the Appendix.

### F. Boundary conditions

We look for solutions of the perturbation equations describing a star oscillating in its QNMs. We here discuss the boundary conditions corresponding to such solutions.

The boundary conditions depend on the structure of the superfluid phase of neutrons. Microscopic calculations predict the so-called bell-shaped profile of critical temperature (see Sec. 11), which has a maximum at a certain value of the density, and decreases at larger and lower densities (see, e.g., Figs. 1 and 2). As a result, depending on the parameters of the neutron critical temperature profile, stellar model, and stellar temperature we have two possibilities: two-layer stars or three-layer stars.

In the case of a two-layer star we have a superfluid internal layer [where \( T(r) < T_{cn}(r) \)] and a non-superfluid external layer (see Fig. 1) where the dashed region is superfluid at a redshifted temperature \( T^\infty = 4 \times 10^8 \text{ K} \). When the maximum of \( T_{cn} \) corresponds to a density which is lower than the central density of the star, configurations with three layers are possible. In this case we have non-superfluid internal and external layers and neutron superfluidity in between (see the dashed region in Fig. 2). We denote the inner and outer radii of the neutron superfluid phase by \( r_i \) and \( r_f \), respectively.

We note that when \( T_{cn}(r) = T(r) \) then \( h = 0 \), as expected from Eqs. (61) and (65), since \( Y_{ni} \rightarrow 0 \) in this limit. Therefore, Eq. (64) implies

\[ \left( \delta\mu^{lm} + \frac{1}{r} \delta\mu^{lm} \right)_{T_{cn}(r) = T(r)} = \left[ e^{\lambda-\nu/2} \frac{\omega^2}{h^2 B} \left( \frac{\partial \delta\mu}{\partial n_e} \right)^{(0)}_{n_b} \right]_{T_{cn}(r) = T(r)} \]  \hspace{1cm} (66)

1. Inner boundary conditions

To impose boundary conditions at the stellar center we have to consider an asymptotic expansion of the perturbation equations at \( r \rightarrow 0 \). We expand the perturbation functions as \( X^{lm}(r) = X^{lm}(0) + \frac{1}{r^2} X^{lm}(0) + \ldots \), \( H^{lm}_1(r) = H^{lm}_1(0) + \frac{1}{r^2} H^{lm}_1(0) + \ldots \), etc. and the background quantities as \( P^{(0)} = P_0 + \frac{1}{r^2} P_2 + \ldots \), etc., and replace these expressions in the perturbation equations.

- If the star has a three-layer structure, neutrons at its center are non-superfluid, and the perturbations at the center are described by the LD equations

---

3 For simplicity we do not account for superfluidity of neutrons in the crust, which could lead to additional layers; we also assume constant redshifted stellar temperature.
We find at the lowest order \[ 20 \]:
\[
X^{lm}(0) = (\rho_0 + P_0) e^{-\nu_0/2} \left\{ \frac{4\pi}{3} (\rho_0 + 3P_0) - \omega^2 \frac{e^{-\nu_0}}{l} \right\} W^{lm}(0) + \frac{1}{2} K^{lm}(0),
\]
\[
H_1^{lm}(0) = \frac{2lK^{lm}(0) + 16\rho_0 (\rho_0 + P_0) W^{lm}(0)}{l(l+1)}. \tag{67}
\]
This lowest order is sufficient to solve the LD equations and find the QNMs with good accuracy: we do not need to include second order terms in the expansion.

Imposing the boundary conditions \(67\) we have, for each value of \(\omega\), two independent solutions of the LD equations.

We integrate the LD equations up to \(r = r_1\), where we require \(W^{lm}_{(b)}(r_+ = W^{lm}(r_-))\), continuity of \(H_1^{lm}\), \(K^{lm}\), \(X^{lm}\), and impose Eq. \(66\) which allows us to determine \(\delta\mu^{lm}\) up to an arbitrary constant. Therefore, we have three independent solutions of Eqs. \(A.1) - A.8\) satisfying the boundary conditions.

If the star has a two-layer structure, we have to consider at \(r \rightarrow 0\) the asymptotic expansion of the full set of equations \(A.1) - A.8\), in which the quantities \(W^{lm}_{(b)}\), \(H_1^{lm}\), \(K^{lm}\), \(X^{lm}\) are coupled with the superfluid degree of freedom \(\delta\mu^{lm}\). We find that the relations \(67\) remain unchanged [provided that one makes a replacement \(W^{lm}(0) \rightarrow W^{lm}_{(b)}(0)\)], while the expansion of the equation \(66\) at \(r \rightarrow 0\), yields a new boundary condition:
\[
\delta\mu^{lm}(0) = \frac{1}{2l + 3} \left\{ \frac{8\pi \rho_0}{3} (l + 2) \delta\mu^{lm}(0) - \frac{h_2}{h_0} \delta\mu^{lm}(0) \right\} + \frac{e^{-\nu_0/2} \omega^2}{h_0 b_0} \left\{ \delta\mu^{lm}(0) + \frac{\nu_2}{n b_0 \gamma_3 0} X^{lm}(0) \right\}. \tag{68}
\]
We note that since the differential equation for \(\delta\mu^{lm}\) is of the second order, in this case we need to include the second order term \(\delta\mu^{lm}(0)\) in the expansion.

Imposing the boundary conditions \(67\) and \(68\) we have, for each value of \(\omega\), three independent solutions of the perturbation equations \(A.1) - A.8\).

2. Outer boundary conditions

At the outer boundary \((r = r_f)\) the oscillation equations imply \(W^{lm}_{(b)}(r_{f-}) = W^{lm}(r_{f+})\), and continuity of \(H_1^{lm}\), \(K^{lm}\), \(X^{lm}\). These conditions coincide with the corresponding boundary conditions at \(r = r_1\). The situation with the boundary condition for the quantity \(\delta\mu^{lm}\) is more subtle. If the superfluid phase does not extend up to the crust, one has to impose Eq. \(66\) at \(r = r_f\). If, instead, the superfluid phase extends up to the crust, a boundary condition on \(\delta\mu^{lm}\) has to be imposed at the crust-core interface \((r_f = R_{cc})\), where Eq. \(66\) does not apply, because \(h(r = R_{cc}) \neq 0\). In that case the appropriate boundary condition (see Ref. \[42\]) follows from the requirement of the absence of particle transfer (baryons and electrons) through the interface, that implies continuity of the radial velocity \(\delta u^r\) through the crust-core interface; this, combined with the condition \(W^{lm}_{(b)}(R_{cc-}) = W^{lm}(R_{cc+})\), yields \(X^r = 0\) at \(r = R_{cc}\).

Using Eq. \(66\) the latter condition can be rewritten as
\[
\left( \frac{\delta\mu^{lm}}{r} + \gamma^{lm} \right)_{r=R_{cc}} = 0. \tag{69}
\]

The condition at the outer boundary of the superfluid phase [either \(66\) or \(69\)] reduces the number of independent solutions to two. We then integrate the standard LD equations, in terms of \(W^{lm}\), \(V^{lm}\), etc., up to the NS surface, where we impose the vanishing of the Lagrangian pressure perturbation, \(X^{lm}(R) = 0\). After that only one solution meeting all the boundary conditions inside the star survives. Outside the star we solve the Zerilli equation with two boundary conditions at the stellar surface \(\text{S}\).

Finally, at infinity, we impose the vanishing of the ingoing gravitational radiation. This condition is satisfied by a discrete set of (complex) frequencies \(\omega\): the QNMs of the star.

IV. STELLAR MODELS

Microphysics input and equilibrium stellar models adopted in the present paper are essentially the same as in Ref. \[42\]. We briefly describe them here in order to make our presentation more self-contained.

As mentioned in Sec. 11, we consider the simplest \(npe\)-matter composition of NS core. We adopt the Akmal-
Pandharipande-Ravenhall EoS \[56\] parametrized in Ref. \[57\] in the core and the equation of state \[58\] in the crust. All numerical results presented here are obtained for a NS with mass \(M = 1.4M_\odot\). The circumferential radius for such star is \(R = 12.2\,\text{km}\), the central density is \(\rho_c = 9.26 \times 10^{14}\,\text{g}\,\text{cm}^{-3}\). We set the crust-core interface at \(\rho_{cc} = 2 \times 10^{14}\,\text{g}\,\text{cm}^{-3}\), at the distance \(R_{cc} = 10.9\,\text{km}\) from the centre.

We consider an isothermal temperature profile, i.e., we assume that the redshifted temperature \(T^\infty = e^{\nu/2} T\) is uniform over the core of the star. We also assume triplet pairing of neutrons and singlet pairing of protons in the NS core. The neutron superfluidity in the stellar crust is ignored; this assumption should not noticeably affect global oscillations of NSs.

Following Ref. \[42\], we consider two models of nucleon superfluidity, which we denote by “A” and “B”, as representatives of a two-layer and a three-layer structure for the superfluid NS, respectively.

In model A the redshifted proton critical temperature is constant over the core, \(T_{cp}^\infty = T_{cp} e^{\nu/2} = 5 \times 10^8\,\text{K}\); the redshifted neutron critical temperature \(T_{cn}^\infty = T_{cn} e^{\nu/2}\) increases with the energy density \(\rho\) and reaches the maximum value \(T_{cn,\max}^\infty = 6 \times 10^8\,\text{K}\) at the stellar centre \((r=0)\). A similar model of neutron superfluidity (with the maximum of \(T_{cn}^\infty(\rho)\) at the stellar centre) has been recently considered in Ref. \[52\] and agrees with the results of some microscopic calculations \[60\].

In model B both critical temperatures \(T_{cn}^\infty\) and \(T_{cp}^\infty\) are density-dependent and, depending on the value of the temperature, the superfluid NS can have two or three layers. A similar model of neutron superfluidity has been recently used to explain observations of the cooling NS in Cassiopeia A supernova remnant \[16\], and agrees with the results of microscopic calculations (see, e.g., Refs. \[11\], \[12\]).

Models A and B are shown in Figs. \[4\] and \[2\]. These figures coincide with, respectively, Figs. 1 and 2 of Ref. \[42\]. The functions \(T_{ci}(\rho)\) are shown in the left panels of both figures; the right panels show the dependence \(T_{ci}^\infty(r)\) \((i = n\) and \(p)\). As the redshifted temperature \(T^\infty\) decreases, the size of the superfluid region [given by the condition \(T^\infty < T_{cn}^\infty(r)\)] increases or remains unaffected.

For illustration, we shaded in Figs. \[4\] and \[2\] the superfluid region corresponding to \(T^\infty = 4 \times 10^8\,\text{K}\). One can see that in model B there can be three-layer configurations of a star with no neutron superfluidity in the centre and in the outer region but with superfluid intermediate region, or, for lower temperatures, two-layer configurations. In contrast, in model A only two-layer configurations are possible.

This can also be seen looking at the profiles of the function \(h(r)\) defined in Eq. \[65\], which vanishes in the non-superfluid region, and is non-vanishing in the superfluid region. In Fig. \[3\] we show \(h(r)\) for models A, B and for different values of the temperature. We can see that model A yields two-layer configurations, while in model B we have two-layer configurations for \(T^\infty \lesssim 2 \times 10^8\,\text{K}\), and three-layer configurations for \(T^\infty \gtrsim 2 \times 10^8\,\text{K}\). For \(T^\infty \geq 6 \times 10^8\,\text{K}\) (model A) or \(T^\infty \geq 5 \times 10^8\,\text{K}\) (model B), the superfluid region disappears.

As we already emphasized in Sec. \[11\] the entrainment matrix \(Y_{ik}\) depends on the critical temperature profiles \(T^\infty_{ci}(\rho)\), and on the value of the stellar temperature \(T^\infty\) as well. We have computed \(Y_{ik}\) for the models A and B following the same procedure as in Refs. \[42\], \[51\], \[53\].
V. RESULTS

Here we describe the results of our numerical integrations of the perturbative equations derived in Sec. III E to find the QNMs of superfluid NSs.

A. General structure of the QNM spectrum

As first noted by Epstein [22] and Lindblom and Mendell [23], when a superfluid phase is present, there are two classes of QNMs. The first class is formed by “normal” (or “ordinary”) modes, which correspond (with small deviations) to modes of a non-superfluid NS. These modes, then, follow the standard classification (particularly, for \( l \geq 2 \)) in a fundamental mode (the \( f \)-mode), and a set of pressure modes (the \( p_i \)-modes) [61]. The second class of modes is associated to the new degrees of freedom due to the relative motion of the fluids; they are called “superfluid” modes. Notice, that superfluidity crucially affects the buoyancy modes, i.e., \( g \)-modes [62, 63], which cannot be classified neither as “normal” modes, nor as “superfluid” modes.

There is no standard notation for the superfluid modes. Some papers, such as [23, 24, 38], consider the superfluid modes as belonging to an unique class, and denote them as \( s_i \), or \( \beta_i \), etc. Other works [28, 51] instead, follow the suggestion of [37], where it was argued that a sort of doubling of the degrees of freedom occurs in a superfluid star, so there are two modes – one ordinary and one superfluid – for each \( f \)- or \( p \)-mode of a non-superfluid star. Therefore, there are the \( f^o \)- and the \( f^s \)-modes, the \( p_i^o \)- and the \( p_i^s \)-modes. However, as noted in [28], this labeling is a pure convention, also because the number of nodes in the radial velocity eigenfunction does not always match with the order of the mode (in addition, it is not possible to define a single radial velocity eigenfunction in the superfluid case). We then choose to treat the superfluid modes as part of an unique class, and denote them as \( s_i^f \) (\( i = 0, 1, \ldots \)) (we do not call them \( s_i \)) to avoid confusion with purely gravitational modes with axial parity [64], which are also called \( s_i \).

B. Comparison with the results of Refs. [41, 42]

In Refs. [41, 42] the spectrum of non-radial oscillations for superfluid NSs was computed in the so called “decoupled limit”, in which equations governing the superfluid modes are completely decoupled from those governing the normal (“ordinary”) modes. As shown in Ref. [44] this approximation is very well justified, because the dimensionless parameter (the coupling constant) \( s = [\nu_n \partial P(n_b, n_e)/\partial n_e]/[\nu_b \partial P(n_b, n_e)/\partial n_b] \), that couples superfluid and normal degrees of freedom is actually small for realistic EoSs, \( s \sim 0.01 - 0.05 \).

In order to test our code, we have computed the frequencies of the superfluid QNMs in the decoupled limit (\( s = 0 \)), and we have compared them with those obtained in Refs. [41, 42], using a completely different code. We have considered model A and three values of the temperature: \( T^\infty = 0 \) (cold star), \( T^\infty = 3.16 \times 10^7 \) K (such that the superfluid phase fills the whole core) and \( T^\infty = 6 \times 10^7 \) K (at which the superfluid phase does not fill the whole core). We find [see Tab. I] in which the frequency is shown in units of \( \nu = c/(2\pi R) \) a relative discrepancy \( \sim 10^{-4} \), which we think can be explained in terms of the different interpolation schemes which have been used.

C. The QNM spectrum of superfluid NSs

We have computed the frequencies and gravitational damping times of the first QNMs of the star, including the coupling between superfluid and non-superfluid degrees of freedom, for models A and B. As expected (see Sec. VA) we find two classes of QNMs: the normal \( f \)- and \( p \)-modes, and the superfluid modes.

\[ \begin{array}{cccc}
\hline
\text{\( T^\infty \)} & \text{\( T^\infty = 0 \) K} & \text{\( T^\infty = 3.16 \times 10^7 \) K} & \text{\( T^\infty = 6 \times 10^7 \) K} \\
\text{\( \nu_f \) (our code) \[42\]} & \text{\( 0.8309 \)} & \text{\( 0.8008 \)} & \text{\( 0.7088 \)} \\
\text{\( \nu_f \) (our code) \[42\]} & \text{\( 1.6137 \)} & \text{\( 1.5794 \)} & \text{\( 1.1220 \)} \\
\text{\( \nu_f \) (our code) \[42\]} & \text{\( 2.3166 \)} & \text{\( 2.2203 \)} & \text{\( 1.6364 \)} \\
\text{\( \nu_f \) (our code) \[42\]} & \text{\( 6.1370 \)} & \text{\( 5.7994 \)} & \text{\( 5.7090 \)} \\
\hline
\end{array} \]

TABLE I: Comparison between the frequencies of the first \( l = 2 \) superfluid modes (\( s_f^0, s_f^1, s_f^2 \)), computed in the decoupled case for model A, using the code developed for this work, and using the code employed in Ref. [42]. The frequencies are expressed in units of \( \nu = c/(2\pi R) \), as in Ref. [42].

1. Frequencies

In Fig. 3 we show the frequencies of the first \( l = 2 \) QNMs as functions of the redshifted temperature, for model A (upper panel) and B (lower panel). In each panel we can see the first two normal modes (thin horizontal lines), which are the same for models A and B and correspond to the frequencies of a non-superfluid star with the same mass and EoS: the \( f \)-mode (at frequency \( \nu_f = 1838 \) Hz) and the \( p_1 \) mode (at frequency \( \nu_{p_1} = 5935 \) Hz). These frequencies are generally not significantly affected by the presence of the superfluid phase. Thus, the dependence of the normal-like modes on the temperature is negligible.

Conversely, the superfluid modes strongly depend on the temperature, because it determines the structure of the superfluid phase. In general, the frequency of a superfluid mode decreases as the temperature increases, but in model B, at temperatures close to \( 2 \times 10^8 \) K, the behaviour is different. The reason is that when the temperature becomes larger than \( 2 \times 10^8 \) K, a phase transition occurs, due to the appearance of a non-superfluid region.
at the center of the star, and the structure of the superfluid NS changes from two-layers to three-layers (see Sec. IV). This transition is evident in the lower panel of Fig. 4. This behavior was also evident in the decoupled limit studied in [41, 42] (see, e.g., Fig. 6 of [42]).

Fig. 4 also shows the occurrence of avoided crossings: at particular values of the temperature (which we call resonance temperatures $T_{i}^{\infty}$) the frequencies of some normal and superfluid modes become very close, but the curves do not cross. A detail of the avoided crossing is shown in the inset in upper panel of Fig. 4 for model A. This phenomenon was expected, since it occurs in the case of radial pulsations [43, 44]. A similar phenomenon was also shown to occur, e.g., in Refs. [28, 38] (in non-rotating stars) and [65, 66] (for inertial modes of rotating stars), studied in the zero-temperature limit. In these cases, the frequencies of the modes were computed as functions of the entrainment parameter, and it was shown that those curves had avoided crossings.

Finally, Fig. 4 shows (thin solid lines) the frequencies of superfluid and normal modes calculated in the decoupled limit. It is clear that the frequencies of the QNMs in the coupled and decoupled limits are very similar for $T^{\infty} \neq T_{i}^{\infty}$. This is expected, since, as it was already noted in Sec. IV B, the coupling parameter $s$ is small for realistic EoSs [44]. The coupling is crucial to determine the avoided crossings but, far from the resonance temperatures $T_{i}^{\infty}$, the frequencies of the QNMs are barely affected by the coupling. We can conclude that the approximation of decoupled superfluid and normal modes works perfectly well for calculation of the QNMs of superfluid NSs.

2. Gravitational damping times

In Fig. 5 we show the gravitational damping times $\tau_{GW}$ of the lowest frequency QNMs, as functions of redshifted temperature, for model A (upper panel) and model B (lower panel). In principle, our approach allows us to compute $\tau_{GW}$ for all of the QNMs [4]. However, when the imaginary part of the mode is much smaller than the real part, numerical errors make it difficult to compute the damping times with good accuracy; this problem seems to be more severe for temperatures $\lesssim 5 \times 10^7$ K, and for damping times $\gtrsim 10^3 - 10^4$ s. Still, we think that the values shown in Fig. 5 provide a reliable estimate at least of the order of magnitude of the damping times, and of their dependence on the temperature.

We can see that at the resonance temperatures $T_{i}^{\infty}$, the curves of the damping times do cross, and the modes change their nature from normal to superfluid and vice versa. Fig. 5 also shows that, far from the resonance temperatures $T_{i}^{\infty}$, the superfluid modes have damping times $\gtrsim 10^2 - 10^3$ s, much larger than those of the normal modes ($\sim 0.1 - 1$ s). This result is consistent with calculations of Ref. [38] and the prediction of Ref. [41] (see also Ref. [44]) that the intensity of the gravitational radiation should be smaller, by a factor of $\sim 5^2 \approx 10^{-3}$, for

\[\nu_{C_{\min}} = 6 \times 10^8 \text{ K} \text{ for model A, } T_{C_{\min}}^{\infty} \approx 5.09 \times 10^8 \text{ K} \text{ for model B.}\]
because we have been able to compute $\tau_{GW}$ for these modes only. We note that, as the order of the mode increases, the gravitational damping time increases, while the viscous damping time decreases [12], therefore it is reasonable to expect that $\tau_{GW}$ becomes larger than $\tau_{b+s}$ for high-order superfluid modes. Moreover, even low-order modes will be damped mostly due to (shear) viscosity if the stellar temperature is sufficiently small.

QNM{s with shorter damping times are more efficient in emitting GW{s. Indeed, the GW flux can be estimated as $L_{GW} \approx 2E_{mech}/\tau_{GW}$ [13], where $E_{mech}$ is the mechanical pulsation energy stored in the mode, introduced in Sec. III C. Therefore, for generic values of the temperature the superfluid modes are not good sources of GW{s, because their damping times are large; but, at temperatures close to the resonance temperatures $T_{i}^{\infty}$, their damping times become comparable to those of the normal modes, and they can become much more efficient in emitting GW{s. We can expect, then, that at certain stages of NS thermal evolution, when a NS reaches one of the resonance temperatures, a new QNM — in principle detectable by GW observers — can appear in the GW spectrum.

3. Eigenfunctions

In Fig. 5 we show the velocity eigenfunctions for the $f$-mode (upper panel), the $p_1$-mode (middle panel) and the $sf_0$-mode (lower panel), for model A at $T_{i}^{\infty} = 6 \times 10^7$ K. We show the $(l = 2)$ quantities $W_{lm}^{(b)}$, $V_{lm}^{(b)}$, obtained expanding the radial and angular components, respectively, of the perturbation $\delta U_{lm}$ in spherical harmonics [27], and the quantities $W_{lm}^{(sf)}$, $V_{lm}^{(sf)}$, obtained expanding in the same way $X_{lm}^{(sfl)}$ [see Eq. (32)]. Note that the knowledge of these quantities allows one, using Eq. (43), to calculate also the functions $W_{lm}^{(i)}$ and $V_{lm}^{(i)}$, defined by the expansion of $\delta U_{lm}$ [28].

We can see that for the first superfluid mode $W_{lm}^{(i)} \gg W_{lm}^{(b)}$, $V_{lm}^{(i)} \gg V_{lm}^{(b)}$. This is a natural result, since the coupling parameter $s$ is small and superfluid oscillations almost do not excite baryon current (see Ref. [44]). This supports the interpretation [see Ref. [12] and the footnote in Sec. III C] of $W_{lm}^{(i)}$, $V_{lm}^{(i)}$ as describing non-superfluid degrees of freedom and $W_{lm}^{(sfl)}$, $V_{lm}^{(sfl)}$ as describing superfluid degrees of freedom. For the first pressure mode we obtain $W_{lm}^{(i)} \sim W_{lm}^{(sf)}$, $V_{lm}^{(i)} \sim V_{lm}^{(sf)}$, while for the fundamental mode $W_{lm}^{(sfl)} \ll W_{lm}^{(b)}$, $V_{lm}^{(sfl)} \ll V_{lm}^{(b)}$. The latter result is also expected and follows from the two facts [12]: (i) superfluid degrees of freedom (i.e., the quantities $W_{lm}^{(sfl)}$, $V_{lm}^{(sfl)}$) are excited by the gradient of the chemical potential imbalance $\delta \mu$ [see Eqs. (42) and (57)] and (ii) $f$-mode oscillations are almost incompressible (i.e., deviation from the beta-equilibrium in the course of $f$-mode oscillations is small), thus $\delta \mu$ is only weakly perturbed for $f$-modes.
From Fig. 6 one can see that the radial velocity eigenfunctions $W_{(b)}^{29}$, $W_{(adi)}^{29}$, $V_{(b)}^0$ and $V_{(adi)}^0$ have no nodes inside the star in the case of the $f$-mode, one node in the case of the $p_1$-mode. In the case of the $s_{fl0}$ mode, the eigenfunction $W_{(b)}^{s_{fl0}}$ has one node, but $W_{(adi)}^{s_{fl0}}$ (which is by far the largest) has no nodes.

The eigenfunctions $\delta \mu^{lm}(r)$ for the $l = 2$ $s_{fl0}$ and $s_{fl1}$ modes calculated for model A and $T^\infty = 6 \times 10^7$ K are shown in Fig. 7 in the coupled (dashed lines) and decoupled (thin solid lines) cases. The very good agreement between the two solutions demonstrates the accuracy of the decoupled limit.

4. Pulsation energy

We have computed the mechanical pulsation energy $E_{mech} = E_{mech (sfl)} + E_{mech (b)}$ stored in the QNMs, using Eqs. (29) and (30). In Table I we show the ratio $E_{mech (sfl)} / E_{mech (b)}$ for the $f$-mode, the $p_1$-mode and the $s_{fl0}$-mode, for models A and B, at different values of the redshifted temperature. We can see that when the star oscillates in a non-superfluid mode, $E_{mech (sfl)} \ll E_{mech (b)}$, i.e., most of the energy is stored in non-superfluid degrees of freedom (this is more evident for the $f$-mode than for the $p_1$-mode). When a NS oscillates in a superfluid mode, $E_{mech (sfl)} \gg E_{mech (b)}$, i.e., most of the energy is stored in superfluid degrees of freedom, while baryon currents are almost not excited.

VI. CONCLUSIONS

In this article we have derived the equations describing, in a general relativistic framework, non-radial oscillations of non-rotating NSs with a superfluid phase, including – for the first time – finite temperature effects. We have numerically solved these equations, finding the QNMs of the NS. We have employed two different models of nucleon superfluidity, as representatives of a two-layer and a three-layer structure, respectively; similar models are currently used in the literature to explain astrophysical observations [14, 17, 52, 57, 68].

We find (as expected from previous results) two classes
of modes: normal modes, corresponding (with very minor differences) to the fundamental and pressure modes of non-superfluid stars; and superfluid modes, directly associated to the superfluid degrees of freedom.

The frequencies of normal modes are almost independent of the NS temperature, but those of superfluid modes have a strong temperature dependence. The curves $\nu(T^\infty)$ of normal and superfluid modes show avoided crossings at specific resonance values $T^\infty_i$ of the temperature. Far from these values, the frequencies of the modes are accurately described by the decoupling approximation formulated in Ref. [44] and studied in Refs. [11, 22], where the coupling between superfluid and non-superfluid degrees of freedom was neglected; on the other hand, this coupling is important at temperatures close to the resonance temperatures $T^\infty_i$.

Our approach allows to directly compute the gravitational damping times of the QNMs. We find (consistently with the results of Refs. [11, 14]) that the gravitational damping times of superfluid modes are much larger ($\gtrsim 10^2 - 10^3$ s) than those of normal modes, but at the resonance temperatures they have a sharp decrease and become similar to those of normal modes.\footnote{Analogous behaviour was noted in Ref. [22] for the viscous damping times $\tau_{b+s}$ in the decoupled limit. Viscous damping times for normal modes are much larger than for superfluid modes, and decrease sharply near the resonance temperatures. We also note that for the lowest frequency modes $T_{GW} < \tau_{b+s}$ at $T^\infty > 3 \times 10^7$ K, therefore these modes are mainly damped by GW emission rather than by viscosity.} These results imply that, when a NS, during its cooling, reaches one of the resonance temperatures $T^\infty_i$, the superfluid modes become potentially efficient GW sources, and may appear in the GW spectrum.

The fact that the frequencies of the QNMs as functions of the temperature show avoided crossings confirms previous results for radial modes \cite{43,13}, and suggests that $r$-modes of rotating, superfluid NSs could have the same structure. This would have far-reaching consequences, since – as shown in Refs. \cite{69,70} – such hypothesis could allow to explain the puzzling observations of hot rapidly rotating NSs in low-mass X-ray binaries.

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### Appendix: Explicit form of the perturbation equations

Non-radial perturbations of stationary, spherically symmetric, superfluid stars are described in general relativity by four first-order differential equations for the quantities $H_1^{lm}$, $K_1^{lm}$, $W_1^{lm}$, $X_1^{lm}$ (for brevity of notation we here omit the superscript $(0)$ for background quantities):

\begin{equation}
H_1^{lm'} = -\frac{1}{r} \left[ (\ell + 1 + 2m)e^\lambda/r + 4\pi^2 e^\lambda (P - \rho) \right] H_1^{lm} + \frac{e^\lambda}{r} \left[ H_0^{lm} + K_0^{lm} - 16\pi (\rho + P) V_0^{lm} \right], \quad (A.1)
\end{equation}

\begin{equation}
K_1^{lm'} = \frac{1}{r} H_0^{lm} + \frac{\ell (\ell + 1)}{2r} H_1^{lm} - \left[ \frac{(\ell + 1) - \nu/2}{r} \right] K_1^{lm} - 8\pi (\rho + P) e^{\lambda/2}/r W_1^{lm}, \quad (A.2)
\end{equation}

\begin{equation}
W_1^{lm'} = -\frac{\ell + 1}{r} W_1^{lm} + \rho e^{\lambda/2} \left[ e^{-\nu/2}/\gamma P X_1^{lm} - \frac{\ell (\ell + 1)}{r^2} V_1^{lm} + \frac{1}{2} H_0^{lm} + K_1^{lm} \right], \quad (A.3)
\end{equation}

\begin{equation}
X_1^{lm'} = -\frac{\ell}{r} X_1^{lm} + (\rho + P)e^{\nu/2}/r \left\{ \left( \frac{1}{r} + \frac{\nu}{2} \right) H_0^{lm} + \frac{\ell - \nu^2}{2r^2} V_1^{lm} + \frac{2}{r} e^{\lambda/2}/\gamma P \left[ 4\pi (\rho + P)e^{\lambda/2} + \omega^2 e^{\lambda/2} \right] W_1^{lm} \right\}, \quad (A.4)
\end{equation}

and one second-order differential equation

\begin{equation}
\delta \mu^{lm''} = - \left[ \frac{h'}{h} - \frac{\lambda'}{2} + \frac{2(\ell + 1)}{r} \right] \delta \mu^{lm'}, \quad \text{with} \quad h'^2 = h \lambda'' + \frac{8}{r} \rho \mu'' \left( e^{\lambda/2} + \omega^2 e^{\lambda/2} \right).
\end{equation}
\[ -\left[ (1-e^\lambda) \frac{l(l+1)}{r^2} + \frac{l}{r} \left( \frac{h'}{h} - \frac{\lambda}{2} \right) \right] \delta \mu^{\ell m} + \frac{\nu e^{\nu/2}}{hE} \left( \delta \mu^{\ell m} + \frac{\gamma_2}{n_b \gamma_3} \chi^{\ell m} \right) \]  
\[ \text{(A.5)} \]

for \( \delta \mu^{\ell m} \). The quantities \( H_0^{\ell m}, \chi_0^{\ell m}, X^{\ell m} \) are given by the algebraic relations:

\[ 3m + \frac{(\ell-1)(\ell+2)}{2} r + 4\pi r^3 P \]  
\[ \text{(A.6)} \]

\[ H_0^{\ell m} - 8\pi r^3 e^{-\nu/2} X^{\ell m} \]

\[ \text{+} \left[ \frac{\ell(\ell+1)}{2} (m + 4\pi r^3 P) - \omega^2 r^3 e^{-\nu} X^{\ell m} \right] \]  
\[ \text{(A.7)} \]

\[ H_1^{\ell m} \]

\[ - \left[ \frac{(\ell-1)(\ell+2)}{2} r - \omega^2 r^3 e^{-\nu} \right] \]

\[ \text{(A.8)} \]

\[ \omega^2 (\rho + P) e^{-\nu/2} V_{\ell m}^{(b)} = X^{\ell m} + \frac{P'}{r} e^{(\nu-\lambda)/2} W_{\ell m}^{(b)} \]

\[ e^{-\nu/2} \left( \rho + P \right) H_0^{\ell m}, \]

\[ \chi^{\ell m} = \frac{1}{1 - \gamma_2} \left( X^{\ell m} + \frac{\gamma_3 n_b \delta \mu^{\ell m}}{} \right). \]

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