Investigation of a nonlinear three-dimensional diffusion model of a porous medium

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Abstract. We study a general three-dimensional nonlinear diffusion model of porous medium with non-stationary source or absorption. For the model, admitting the widest group Lie of the transformations we found some invariant solutions of rank 1. These include the following solutions: the generalizations of "a layered circular pie", "a layered plane pie", a generalization of "a layered spiral pie", "a layered conical pie", a solution with a radial distribution of the pressure, "a layered spherical pie". The obtained results can be used to study the processes associated with an underground fluid or gas flow, with water filtration, with the engineering surveys in the construction of the buildings, and also with shale oil and gas production.

1. Introduction

There are a number of physical applications in which the model of porous medium is used by a natural way. First of all, this model is used to describe the processes associated with an underground fluid or gas flow, with water filtration, with the engineering surveys in the construction of the buildings, and also with shale oil and gas production. The study of fluid and gas motion in porous media within the framework of classical models does not always adequately describe real processes. This is due to the fact that these models do not take into account the presence of non-stationary source or absorption (see, for example, [1 – 5] and its references). For a more adequate description of the real processes, it is necessary to obtain and study new more complex three-dimensional models with non-stationary source or absorption.

Many mathematical models of physics and continuum mechanics are formulated in the form of linear and quasi-linear differential equations. Mathematical model is a description of the real scheme by mathematical language. The symmetry analysis of the equations of the models of physics and mechanics of continuous media is one of the most effective ways to obtain quantitative and qualitative characteristics of the physical processes. The main task of the symmetry analysis of differential equations is to study the set of the solutions of these equations. All algorithms of the symmetry analysis are the preparation for achieving this purpose. The modern concept of the symmetry analysis is understood as the fullest using of the group of transformations admitted by the equations of model primarily to obtain and research the exact solutions. Exact solutions allow us to describe the specific physical processes. Exact solutions can be used as test solutions in numerical calculations, which perform in the studies of the real processes. Exact solutions allow us to assess the degree of adequacy of a given mathematical model to the real physical processes, after carrying out experiments.
corresponding to these decisions, and estimating the deviations that arise.

In [6] with a help of the algorithm proposed in [7, 8] we will solve the problem of the group classification of the equation of general three-dimensional nonlinear diffusion model of porous medium with non-stationary source or absorption, which is described by an equation:

$$p_t = \Delta \Phi(p) - f(t)p,$$

(1)

where $p = p(t, x)$ is a pressure; $t$ is a time, $x = (x, y, z) \in \mathbb{R}^3$, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$; $\Phi(p)$ defines a nonlinearity of the process; $f(t)$ defines nonstationary nature of the source or absorption. $\Phi(p)$ and $f(t)$ are any functions, which are determined empirically. A case $f(t) < 0$ corresponds to the presence of a source. A case $f(t) > 0$ corresponds to the presence of an absorption. The functions $\Phi(p)$ and $f(t)$ satisfy to the condition

$$\Phi'(p)f'(t) \neq 0.$$  

(2)

This condition means that the process is nonlinear and the source or absorption is nonstationary.

In contrast to the classical algorithm presented in [9, 10], this algorithm, firstly, avoids significant analytical difficulties associated with the analysis of the classification equations arising in the application of the algorithm from [9, 10]; secondly, it significantly reduces the number of calculations. This algorithm was successfully used in [11 – 18] for group classification of various equations of mechanics and mathematical physics.

In [6] we found all invariant solutions of rank 0 of the equation (1) under condition (2) for the having the greatest symmetry basic model

$$f = -\alpha \left(\ln g'(t)\right)' > 0, \quad \alpha \left(\ln g'(t)\right)' \neq 0, \quad \Phi = \begin{cases} \frac{\alpha+1}{p}, & \alpha > -1 \\ \alpha, & \alpha = -1 \\ \ln p, & \alpha < -1 \end{cases}$$

(3)

The function $g(t)$ is expressed via the function $f(t)$, by the formula

$$g(t) = a_1 \int \exp\left(-\frac{1}{\alpha} f(t)dt\right) dt + a_2,$$

where $a_1$ and $a_2$ are real constants.

In particular, among the solutions found there are solutions that we called "a layered circular pie", "a layered spiral pie", "a layered plane pie" and "a layered spherical pie". The solution "a layered circular pie" describes a motion of the liquid or gas in a porous medium, for which a pressure is the same at each fixed moment of a time at all points of each circle from the family of concentric circles. The solution "a layered spiral pie" describes a motion of the liquid or gas in a porous medium, for which a pressure is the same at each fixed moment of a time at all points of each logarithmic spiral, from the obtained family of logarithmic spirals. The solution "a layered spherical pie" describes a motion of the liquid or gas in a porous medium, for which a pressure is the same at each fixed moment of a time at all points of each sphere, from the family of concentric spheres. A set of the solutions "a layered circular pie", "a layered spiral pie" and "a layered spherical pie" contains the solutions describing a distribution of the pressure in a porous medium after a point blast or a point hydraulic shock. Also this set contains the solutions describing a stratified with respect to the pressure a motion of liquid or gas in a porous medium, with a very high pressure at infinity in a presence of a very strong absorption at a point. The solution "a layered plane pie" describes a motion of the liquid or gas in a porous medium, for which a pressure is the same at each fixed moment of a time at all points of each plane, from the family of parallel planes. A set of the solutions "a layered plane pie" contains the
solutions describing a motion of the liquid or gas in a porous medium with a very high pressure near a fixed plane in a presence of a very strong absorption at infinity. Also this set contains the solutions describing a motion of the liquid or gas in a porous medium with a very high pressure at infinity in a presence of a very strong absorption on a fixed plane.

The main group $G_9$ of the model (1) – (3) is generated by the operators

$$\mathbf{X} = \left( X_1, X_2, X_3 \right) = \partial_x, \quad \mathbf{Z} = \left( Z_1, Z_2, Z_3 \right) = x \times \partial_x$$

$$R_2 = R_2(\alpha) = x \cdot \partial_x + 2 \alpha p \partial_p, \quad Y_1 = Y_1(\alpha) = \frac{1}{(\ln g(t))} \partial_t - \alpha \left( \frac{1}{(\ln g(t))} \right)' p \partial_p$$

$$Y_2 = Y_2(\alpha) = \frac{1}{g'(t)} \partial_t - \alpha \left( \frac{1}{g'(t)} \right)' p \partial_p$$

2. Exact solutions

A model (1) with functions $\Phi(p) = p^2, \quad f(t) = 0$, which does not take into account the source or absorption, is a classical model which is the most often used in calculating the pressure of a liquid or gas in a porous medium [1–5]. The model (3), taking into account the presence of a source or absorption, with $\alpha = 1$ is a generalization of this classical model. Below, exact solutions will be obtained mainly for this generalization.

In [6] we obtained the optimal systems of the $n$-parametric ($n = 1, 2, ..., 5$) subgroups of the group $G_9$ and universal invariants of these subgroups. The invariant solutions of rank 1 of the equation (1) under condition (2) are the invariant $H$-solutions where $H$ is one of the 31 subgroups of the optimal system of the 3-parametric subgroups and 11 subgroups of the optimal system of the 4-parametric subgroups.

In all of the subsequent formulas the values $c_1$ and $c_2$ are arbitrary real constants, $r = \sqrt{x^2 + y^2}$, $\varphi = \arctan \frac{y}{x}$.

* Invariant $\left< Z_3, \quad R_2, \quad Y_2 \right>$-solution for $\alpha = 1$ has a form

$$p = z^2 g'(t) \sqrt{U(\xi)}, \quad \xi = \frac{r}{z}$$

(4)

The factor equation is reduced to the Gauss hypergeometric equation. As a result, the function $Q(\xi)$ is defined by the formula

$$U(\xi) = c_1 \left( 1 - \frac{6}{7} (\xi^2 + 1) + \frac{3}{35} (\xi^2 + 1)^2 \right) +$$

$$+ c_2 \left( 105 F \left( 3, \frac{3}{2}; \frac{3}{2}; \frac{1}{\xi^2 + 1} \right) \sqrt{\xi^2 + 1} - \frac{\pi}{128} \left( 35 - 10 (\xi^2 + 1) + (\xi^2 + 1)^2 \right) \right)$$

(5)
where  \( F\left(3, -\frac{3}{2}; \frac{3}{2}; \frac{1}{\xi^2+1}\right) \) is the hypergeometric series, which converges for all \( \xi \neq 0 \). The solution (4), (5) describes the motion of a liquid or gas in a porous medium for which a pressure is the same at each fixed moment in time on each plane \( z = \text{const} \) at all points of each circle from the family of concentric circles \( r = \text{const} \). This solution is a generalization of the solution "a layered circular pie".

- Invariant \( H \)-solution for
  \[
  H \in \left\{ \lambda R_2 + Z_3, X_1, X_2 \right\}, \quad \left\{ R_2, X_1, X_2 \right\}, \quad \left\{ R_2, Z_3, X_1, X_2 \right\} \quad (\lambda = \text{const} \neq 0)
  \]
  is defined by the formula
  \[
  p = \left( z^2 g'(t) \right)^2 \left( \frac{2(\alpha+1)(2\alpha-3)}{\alpha^2} g(t) \right)^{-\alpha}, \quad \alpha \neq -1
  \]
  The solution (6) has the structure of "a layered plane pie". For \( \alpha < 0 \) a pressure \( p \to \infty \) when \( z \to 0 \) and \( p \to 0 \) when \( z \to \infty \). This solution describes vertically stratified with respect to the pressure a flow of the liquid or gas in a porous medium, with a very high pressure near a plane \( z = 0 \) in a presence of a very strong absorption at infinity. For \( \alpha > 0 \) a pressure \( p \to 0 \) when \( z \to 0 \) and \( p \to \infty \) when \( z \to \infty \). This solution describes vertically stratified with respect to the pressure a flow of the liquid or gas in a porous medium, with a very high pressure at infinity in a presence of a very strong absorption on a plane \( z = 0 \).

- Invariant \( \left\{ \lambda R_2 + Z_3, X_3, Y_2 \right\} \)-solution for \( \alpha = 1 \), \( \lambda = \text{const} \) has a form
  \[
  p = r^2 g'(t) \sqrt{U(\xi)}, \quad \xi = r \exp(-\lambda \phi)
  \]
  The function \( U(\xi) \) is defined by the formula
  \[
  U(\xi) = \xi^{\frac{4}{\lambda^2+1}} \left( c_1 \sin \frac{4 \ln \xi}{\lambda^2+1} + c_2 \cos \frac{4 \ln \xi}{\lambda^2+1} \right)
  \]
  The solution (7), (8) describes the motion of a liquid or gas in a porous medium for which at each fixed moment in time a value \( r^{-2} p(t, x) \) is the same at all points of each logarithmic spiral \( r \exp(\lambda \phi) = c = \text{const} \). This solution is a generalization of the solution "a layered spiral pie". For \( |\lambda| > 1 \) a pressure \( p \to \infty \) when \( r \to \infty \) and \( p \to 0 \) when \( r \to 0 \). This solution describes a flow of the liquid or gas in a porous medium with a very high pressure at infinity in a presence of a very strong absorption in the point \( x = y = 0 \). For \( |\lambda| < 1 \) a pressure \( p \to 0 \) when \( r \to \infty \) and \( p \to \infty \) when \( r \to 0 \). This solution describes a flow of the liquid or gas in a porous medium, with a very high pressure in the point \( x = y = 0 \) with a presence of a very strong absorption at infinity.

- Invariant \( \left\{ Z_3, X_3 + Y_1, Y_2 \right\} \)-solution for \( \alpha = 1 \) is defined by the formula
  \[
  p = g'(t) \exp(-z) \left[ c_1 J_0(r) + c_2 N_0(r) \right]
  \]
  where \( J_0(r) \) is Bessel function of the first kind, \( N_0(r) \) is Bessel function of the second kind. This solution describes the motion of a liquid or gas in a porous medium for which at each fixed moment
in time a pressure is the same at each fixed moment in time on each plane $z = \text{const}$ at all points of each circle from the family of concentric circles $r = \text{const}$. This solution is a generalization of the solution "a layered circular pie".

- Invariant $\left\{Z_3, 2R_2 + Y_1, Y_2\right\}$-solution for $\alpha = 1$ is defined by the formula

$$ p = g'(t) \sqrt{c_2 + c_1 \ln \left|\frac{r^2 - z^2}{r^2 + z^2}\right|} \left(|x| \neq 0\right) \quad (10) $$

The solution (10) describes the motion of a liquid or gas in a porous medium for which at each fixed moment in time a pressure is the same at each fixed moment in time at all points of each cone $z^2 = c \left(x^2 + y^2\right)$, $c = \text{const}$. This solution is conically stratified with respect to pressure. We will be called this solution "a layered conical pie".

Let $c_2 = 0$, $c_1 < 0$. Then a pressure is zero at the plane $z = 0$ and axis $Oz$, that are the degenerate cones. A pressure on the cone $z^2 = \left(x^2 + y^2\right)$ is infinitely high. The solution (9) describes the flow of a liquid or gas when there are a very strong source at the cone $z^2 = c \left(x^2 + y^2\right)$ and a very strong absorption at the plane $z = 0$ and axis $Oz$. Physically, this means a conical hydraulic shock for fracturing layer in the production of shale oil or gas.

- Invariant $\left\{R_2 + Y_1, X_3, Y_2\right\}$-solution for $\alpha = 1$ has a form

$$ p = xg'(t) \sqrt{U(\xi)}, \quad \xi = \frac{y}{x}, \quad x^2 + y^2 \neq 0 \quad (11) $$

The function $U(\xi)$ is defined by the formula

$$ U(\xi) = \left(\xi^2 - 2\right) \left\{c_1 \left(4\text{arctg}\sqrt{\xi^2 + 1} - \ln \left|\frac{1 - \sqrt{\xi^2 + 1}}{1 + \sqrt{\xi^2 + 1}}\right| - \frac{\sqrt{\xi^2 + 1}}{\xi^2}\right) + c_2\right\} \quad (12) $$

The solution (11), (12) describes the motion of a liquid or gas in a porous medium for which at each fixed moment in time a value $xp(t, x)$ is the same at all points of each straight line $y = cx$, $c = \text{const}$. Physically, this means a radial distribution of the pressure.

- Invariant $\left\{Z_1, Z_2, Z_3, Y_1\right\}$-solution for $\alpha = 1$ is defined by the formula

$$ p = g'(t) \sqrt{[\xi] \left[c_1J_1(4\sqrt{[\xi]}) + c_2N_1(4\sqrt{[\xi]})\right]} \quad (13) $$

where $J_1(\eta)$ is Bessel function of the first kind, $N_1(\eta)$ is Bessel function of the second kind. This solution has the structure of "a layered spherical pie".

- Invariant $\left\{Z_1, Z_2, Z_3, R_2 + 2Y_1\right\}$-solution for $\alpha = 1$ has a form

$$ p = g'(t) \sqrt{U(\xi)}, \quad \xi = \frac{|x|}{g(t)} \quad (14) $$

where the function $U(x)$ is a solution of the integral equations:

\[ \text{integral equations} \]
\[ U(\xi) = c_1 + c_2 \ln \xi + \frac{1}{2} \int_0^\xi \left( 3 \ln \frac{\xi}{\eta} - 1 \right) \eta^2 \sqrt{U(\eta)} \, d\eta \]  

(15)

where \( \xi_0 = \frac{|x_0|}{\sqrt{g(t_0)}} \), and \( t_0 \geq 0, \ |x_0| > 0 \) are arbitrary real constants. This solution has a structure of "a layered spherical pie".

The solution (14), (15) can be used for describing the motion of a liquid or gas in a porous medium for which at an initial time \( t = t_0 \) at a fixed point \( x = x_0 \) a pressure and its radial derivative are given by the formulas

\[ p(t_0, x_0) = p_0 > 0, \quad \frac{\partial p}{\partial |x|}(t_0, x_0) = p_1. \]  

(16)

where \( p_0 \) and \( p_1 \) are arbitrary real constants. In this case the constants \( c_1, \ c_2 \) are determined by the formulas:

\[ c_1 = p_0^2 \left( g'(t_0) \right)^2 + c_2 \left( \frac{1}{2} \ln g(t_0) - \ln |x_0| \right) \]
\[ c_2 = p_0 |x_0| \left( g'(t_0) \right)^2 \left( 2 p_1 + \frac{1}{2} |x_0|^2 \frac{g'(t_0)}{g^3(t_0)} \right) \]  

(17)

We established that in the neighborhood of the point \( (t_0, x_0) \) there is a unique solution of the equation (1) for the model (3) satisfying to the conditions (16), for which the value \( (g'(t))^{-1} p(t, x) \) is constant along each trajectory \( |x| = c \sqrt{g(t)} \ (c = \text{const}) \). These solutions are determined by the formulas (14), (15), (17). The physical meaning of this solution is as follows:

Let the pressure is determined by the law (14). If at a point \( x = x_0 \) at the time \( t_0 \), we have measured a pressure and its radial derivative, then we can uniquely determine the pressure arising in the neighborhood of the sphere \( |x| = |x_0| \) in the neighborhood of the time \( t_0 \).

3. Conclusion and discussion

For the model (3), admitting the widest group Lie of the transformations we found explicitly the following invariant solutions of rank 1: (4) and (9) – the generalizations of "a layered circular pie", (6) – "a layered plane pie", (7) – a generalization of "a layered spiral pie", (10) – "a layered conical pie", (11) – a solution with a radial distribution of the pressure, (13) – "a layered spherical pie". The solution (14) is "a layered spherical pie". A finding of this solution is reduced to the solving of the nonlinear integral equation, given by the formula (15). The presence of the arbitrary constants in the integral equation (15) allowed us to study the motion of a liquid or gas in a porous medium, for which at the initial instant of time in a fixed point a pressure and its radial derivative are given. We established the existence and uniqueness of the solution of this boundary value problem and indicated the physical meaning of this solution.

The obtained results can be used to study the processes associated with an underground fluid or gas flow, with water filtration, with the engineering surveys in the construction of the buildings, and also with shale oil and gas production.
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