Medium corrections to the CP-violating parameter in leptogenesis

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In two recent papers, arXiv:0909.1559 and arXiv:0911.4122, it has been demonstrated that one can obtain quantum corrected Boltzmann kinetic equations for leptogenesis using a top-down approach based on the Schwinger–Keldysh/Kadanoff–Baym formalism. These “Boltzmann-like” equations are similar to the ones obtained in the conventional bottom-up approach but differ in important details. In particular there is a discrepancy between the CP-violating parameter obtained in the first-principle derivation and in the framework of thermal field theory. Here we demonstrate that the two approaches can be reconciled if causal n-point functions are used in the thermal field theory approach. The new result for the medium correction to the CP-violating parameter is qualitatively different from the conventional one. The analogy to a toy model considered earlier enables us to write down consistent quantum corrected Boltzmann equations for thermal leptogenesis in the SM+3νR which include quantum statistical terms and medium corrected expressions for the CP-violating parameter.

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I. INTRODUCTION

To calculate the baryon asymmetry generated during the epoch of leptogenesis [1] in the standard model extended by three right-handed neutrinos (SM+3νR) and its extensions one usually uses standard Boltzmann kinetic equations. The collision terms (and in particular the CP-violating parameters) in this equations are computed in vacuum in the in-out formalism [2, 3] and do not take into account effects induced by the hot medium of the early universe. Such effects can be consistently taken into account in a top-down approach based on the Schwinger–Keldysh/Kadanoff–Baym formalism. In [4, 5] we have applied it to a simple toy model of leptogenesis and derived a new (quantum corrected) form of the Boltzmann equations, which includes quantum statistical factors and takes the medium effects into account. We have found that the medium corrections to the CP-violating parameter depend only linearly on the one-particle distribution functions (see also [6–8]). In the analysis based on finite temperature field theory for the phenomenological scenario of thermal leptogenesis [2, 3, 9] and for GUT baryogenesis [10] the medium corrections to the CP-violating parameter depend quadratically on the distribution functions.

This discrepancy has been noted in the context of leptogenesis in [4] for the vertex contribution to the CP-violating parameter and later in [5] for the self-energy contribution. Here, we use a finite temperature equivalent of the Cutkosky cutting rules [11–14] to derive thermal corrections to the expression for the imaginary part of the three-point vertex function and the self-energy loop and to calculate the corresponding medium-corrected CP-violating parameters. We show that the discrepancy is due to an ambiguity in the real-time (RTF) formulation of thermal quantum field theory and disappears if one considers retarded or advanced n-point functions. In the framework of the toy model this has been demonstrated recently in [15]. Together with the new form of the Boltzmann equation derived in [4, 5] this puts us in the position to write down quantum corrected Boltzmann equations for the phenomenological scenario of thermal leptogenesis which consistently include the medium corrected CP-violating parameter and quantum statistical terms.

In section II, we introduce our notations for the CP-violating parameters and the thermal field theory formalism. Then, in section III we review the conventional calculation of the thermal corrections, and in section IV we demonstrate how to reconcile them with the recent results from nonequilibrium field theory.

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Finally, in section V we present the quantum corrected Boltzmann equations taking medium corrections into account.

II. CP-VIOLATING PARAMETER AND THERMAL FIELD THEORY

In the phenomenological scenario of thermal leptogenesis as well as in the toy model, the matter-antimatter asymmetry is generated by the decay of a heavy species. In both cases the CP violation in this decay is caused by interference between the tree level and the one-loop diagrams, see fig. 1. In the phenomenological scenario $\psi_i = N_i$ are heavy Majorana neutrinos which decay via Yukawa interactions $\mathcal{L} = h_{\alpha\ell}N_i \bar{\ell}\phi + \text{h.c.}$ into leptons $\alpha = \ell$ and Higgs $\beta = \phi$ or their anti-particles. In the toy model $\psi_i$ is a heavy real scalar particle which decays via Yukawa interactions $\mathcal{L} = -\frac{g_i}{2}\psi_i \bar{b}b + \text{h.c.}$ into two light scalars $\alpha = \beta = b$ or the conjugate $\bar{b}$. The CP-violating parameter $\epsilon_i$ for the decay of $\psi_i$ is defined as

$$\epsilon_i = \epsilon_i^V + \epsilon_i^S = \frac{\Gamma_{\psi_i \to \alpha \beta} - \Gamma_{\psi_i \to \bar{\alpha} \bar{\beta}}}{\Gamma_{\psi_i \to \alpha \beta} + \Gamma_{\psi_i \to \bar{\alpha} \bar{\beta}}} \tag{1}$$

where $\Gamma_{N_i \to \ell, \phi}$ includes a sum over flavour indices and loop-internal Majorana neutrino generations in the case of the phenomenological scenario: $\Gamma_{N_i \to \ell, \phi} = \sum_{\alpha, \beta} \Gamma_{N_i \to \ell, \phi}$ (we do not consider flavor effects here). If the tree level and one-loop contributions are written as $\lambda_0 A_0$ and $\lambda_1 A_1$, respectively, where all coupling constants are absorbed in $\lambda_0(1)$, the CP-violating parameter becomes at lowest order:

$$\epsilon_i = \frac{|\lambda_0 A_0 + \lambda_1 A_1|^2 - |\lambda_0^* A_0 + \lambda_1^* A_1|^2}{|\lambda_0 A_0 + \lambda_1 A_1|^2 + |\lambda_0^* A_0 + \lambda_1^* A_1|^2} \simeq -2 \text{Im} \left\{ \lambda_0^* \lambda_1 \right\} \text{Im} \left\{ A_0^* A_1 \right\} \frac{1}{|\lambda_0|^2 |A_0|^2} \tag{2}$$

where the sum over lepton flavour and Majorana neutrino generation indices is again implicit. In the case of thermal leptogenesis this leads to

$$\epsilon_i = -2 \sum_{j \neq i} \frac{\text{Im} \left\{ (h^\dagger h)^{ij} \right\} \text{Im} \left\{ A_0^* A_1 \right\}}{2 q \cdot k}, \quad i = 1, 2, 3, \tag{3}$$

where $q$ and $k$ denote the four-momenta of the Majorana neutrino and the lepton respectively, see fig. 3, and for the toy model to

$$\epsilon_i = -2 |g_j|^2 \text{Im} \left\{ \frac{g_i g_j^*}{g_i^* g_j} \right\} \text{Im} \left\{ A_0^* A_1 \right\}, \quad i \neq j, \quad i = 1, 2. \tag{4}$$

This means that one needs to compute the imaginary (absorptive) part of the vertex and the self-energy loop contributions $\text{Im} \left\{ A_0^* A_1 \right\}$. In vacuum this can be done conveniently with help of the Cutkosky cutting rules [16–18]. In thermal quantum field theory these can be generalized in order to take into account interactions of internal lines in the loops with the background medium [11, 13, 14]. In the real-time formalism of thermal quantum field theory two types of fields, termed type-1 and type-2 fields, are introduced in order to avoid pathological singularities [19]. Vertices can be of either type, differing only by a relative minus sign.
We denote them by \(g^1 = -ig\) and \(g^2 = +ig\) for a generic coupling \(g\). The propagators connecting the different types of vertices can be considered as components of a \(2 \times 2\) propagator matrix:

\[
G_\alpha(p) = \begin{pmatrix} G_{11}^\alpha(p) & G_{12}^\alpha(p) \\ G_{21}^\alpha(p) & G_{22}^\alpha(p) \end{pmatrix} = \begin{pmatrix} \Delta_\alpha(p) & e^{-\beta p_\mu/2} \Delta_\alpha^+(p) \\ e^{\beta p_\mu/2} \Delta_\alpha^-(p) & \Delta_\alpha^+(p) \end{pmatrix}.
\]

For a scalar particle \(b\) the components are

\[
\Delta_b(p) = D_b(p), \quad \Delta_b^+(p) = D_b^+(p).
\]

For a fermion \(f\) the components are

\[
\Delta_f(p) = (\gamma \cdot p + m_f)D_f(p), \quad \Delta_f^+(p) = (\gamma \cdot p + m_f)D_f^+(p).
\]

For brevity we have defined

\[
D_\alpha(p) = \frac{i}{p^2 - m_\alpha^2 + i\epsilon} - 2\pi \xi_\alpha f^{\alpha,eq}(p)\delta(p^2 - m_\alpha^2),
\]

\[
D_\alpha^+(p) = 2\pi \Theta(\pm p_0) - \xi_\alpha f^{\alpha,eq}(p)\delta(p^2 - m_\alpha^2),
\]

where \(\xi_\alpha = +1\) for fermions and \(\xi_\alpha = -1\) for bosons. Here, we denote by \(f^{b,eq}(p)\) and \(f^{f,eq}(p)\) the equilibrium distribution function for bosons and fermions, respectively, given by

\[
f^{\alpha,eq}(p) = \left[\exp \left(\beta |p_\mu U^\mu|\right) + \xi_\alpha \right]^{-1}.
\]

They are functions of the Lorentz invariant product \(p_\mu U^\mu\) of the particles’ four-momentum and the four-velocity \(U\) of the plasma in a general frame. In the rest-frame of the plasma, \(U = (1, 0, 0, 0)\), we obtain the standard form which depends on \(p_0\). In the following we assume that it is sufficient to replace the different propagators in our toy model and the phenomenological theory by their thermal field theory equivalents given in eqn. (5). This approach has been followed in previous works for the baryogenesis and leptogenesis scenarios [2, 3, 9, 10]. We ignore further thermal effects, such as thermal corrections to the masses and wave function renormalization here.

Denoting vertices attached to external lines by \(x_j\) and those attached to internal lines only by \(z_j\) we can formally denote an amputated \(n\)-point graph by \(F(x_1, \ldots, x_n; z_j)\). Here we assume that \(F\) is given in momentum space, writing the position space coordinates in order to identify the individual vertices. The contribution of this graph to the amplitude is \(-i F(x_1, \ldots, x_n; z_j)\).

Physical amplitudes involve a sum over possible combinations of types of internal vertices:

\[
F(x_1, \ldots, x_n; z_j) = \sum_{\text{type } z_j} F(x_1, \ldots, x_n; z_j).
\]

For external vertices of fixed type it has been shown [11, 12] that this sum is equivalent to a sum over all possible “circlings” of the internal vertices:\(^3\)

\[
F(x_1, \ldots, x_n; z_j) = \sum_{\text{circling } z_j} F_\xi(x_1, \ldots, x_n; z_j).
\]

\(F_\geq\) and \(F_\leq\) with “cirled” vertices represent graphs computed using the set of rules, given in fig. 2. These differ for the computation of \(F_\geq\) and \(F_\leq\) by interchange of the \(\Delta^+\) and \(\Delta^-\) propagators. In

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1 At the end of the calculation of \(\text{Im} \{ A_0^4 A_4 \}\) we set \(g = 1\), since the physical coupling constants have been factored out into \(\lambda_0\) and \(\lambda_1\).

2 In [2] and elsewhere resummed propagators have been used in this place to prevent the appearance of singularities. Since we are mainly interested in the structure of the thermal corrections we stick to the free thermal propagators here.

3 The historic origin of this formula was that the external fields where considered to be all of type 1 (physical).
\(F_\geq(x_1, \ldots, x_n; z_j)\) we explicitly denote circling of a vertex \(\alpha\) as \(F_\geq(x_1, \ldots, \alpha, \ldots, x_n; z_j)\). Note that the two ways of defining \(F\) in terms of \(F_>\) and \(F_<\) in eqn. (9) are in agreement only if the Kubo–Martin–Schwinger (KMS) boundary condition,

\[
\Delta^-_\alpha(p) = -\xi_\alpha e^{-\beta p \cdot U} \Delta^+_\alpha(p),
\]

is satisfied. This is the case in thermal equilibrium.

\[
\begin{align*}
\times & = -ig \\
\bigcirc & = +ig \\
\bigcirc & = \Delta(p), \\
\bigcirc & = \Delta^-(p), \\
\bigcirc & = \Delta^+(p)
\end{align*}
\]

FIG. 2: Circling rules for a generic theory used for the computation of \(F_>\) in momentum space. The rules for the computation of \(F_<\) differ by interchange of \(\Delta^+(p)\) and \(\Delta^-(p)\). The \(\Delta^\pm\) propagators connecting circled and uncircled vertices may be interpreted as cut propagators. In vacuum they correspond to the cut propagators in the Cutkosky rules.

From \(F\) we can then compute \(\text{Im}\{A_0^+, A_1\}\) as

\[
\text{Im}\{A_0^+, A_1\} = -\text{Im}\left\{\frac{i^{-1} F}{g_1g_2g_3}\right\}.
\]

where \(g_1, g_2\) and \(g_3\) stand for the generic couplings associated with the three vertices in the one-loop diagrams at Fig. 2.

III. PHYSICAL AND GHOST FIELDS

In this section we briefly review the conventional calculation of the CP-violating parameter in real-time thermal field theory. However, we use a notation that is helpful to understand the ambiguities emerging there, and that can be more easily compared to the results from non-equilibrium field theory.

An obvious problem with the real-time formulation for the computation of \(n\)-point functions is that there are in general \(2^n\) such functions which differ in the types of the external vertices. Historically the correct function was considered to be the one with all external vertices of type-1 (physical). In this case eqn. (9) leads to the following formula for the imaginary part of a graph’s contribution to the amplitude:

\[
\text{Im} \left\{i^{-1} F(1, \ldots, 1; z_j)\right\} = \frac{1}{2} \sum_{\text{circling } (x_i), z_j} F_\geq(x_1, \ldots, x_n; z_j),
\]

where the sum includes all possible circlings of the internal vertices \(z_j\) but only those circlings of external vertices \(x_i\) which include both, circled and uncircled vertices (indicated by the brackets around \(x_i\)).

The six diagrams contributing to the imaginary part of the three-point vertex function according to eqn. (12) are shown in fig. 4. The circlings contributing to the self-energy part are shown in fig. 5.

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4 In the phenomenological scenario the Feynman rules for Majorana neutrinos include spinors, charge conjugation and projection operators which we assume to be included in \(F_\geq\).
FIG. 3: Momentum flow in the vertex and the self-energy loop.

FIG. 4: Circlings contributing to \( \text{Im} \{ i^{-1} F(1, 1, 1) \} \) for the vertex loop. At one-loop level the circlings can be interpreted as cuts, as indicated, by the lines separating circled from uncircled regions \[11\]. The contributions from diagrams involving cuts through the \( x_2-x_3 \) line are suppressed relative to the others in the hierarchical limit.

FIG. 5: Circlings contributing to \( \text{Im} \{ i^{-1} F(1, 1; z) \} \) for the self-energy loop. The graphs (b) and (c) vanish since \( \psi_i \) and \( \psi_j \) cannot be on-shell simultaneously. Note that we consider only the diagrams with \( \psi_i \) in the external and \( \psi_j \) in the internal line \((i \neq j)\) because these are the only ones which contribute to \( \epsilon_i \).

The contributions which correspond to cuts through the \( \psi_j \) line are suppressed in the hierarchical limit. If they are neglected the application of this circling formula leads for the toy model to the result

\[
\epsilon_{i}^{V,\text{th}} = -\frac{1}{8\pi} \frac{|g_j|^2}{M_i^2} \text{Im} \left( \frac{g_i g_j^*}{g_i^* g_j} \right) \int \frac{d\Omega}{4\pi} \frac{1 + f_{E_1}^{\bar{b},eq} + f_{E_2}^{\bar{b},eq} + 2 f_{E_1}^{\bar{b},eq} f_{E_2}^{\bar{b},eq}}{M_j^2/M_i^2 + \frac{1}{2} (1 + \cos \theta_i)} + \ldots, \tag{13}
\]

for the vertex contribution and

\[
\epsilon_{i}^{S,\text{th}} = -\frac{|g_j|^2}{16\pi} \text{Im} \left( \frac{g_i g_j^*}{g_i^* g_j} \right) \frac{1}{M_j^2 - M_i^2} \int \frac{d\Omega}{4\pi} \{ 1 + f_{E_1}^{\bar{b},eq} + f_{E_2}^{\bar{b},eq} + 2 f_{E_1}^{\bar{b},eq} f_{E_2}^{\bar{b},eq} \} \tag{14}
\]

for the self-energy contribution. The distribution functions are to be evaluated for the energies \( E_1 \) and \( E_2 \).
given by\(^5\)

\[
E_{1,2} = \frac{1}{2} [E_q^{\psi_1} \mp |q| (\sin \theta_1 \cos \varphi_l \cos \delta' + \sin \delta')],
\]

(15)

where \(\theta_1\) and \(\varphi_l\) are elements of the solid angle \(\Omega_l\) and the angle \(\delta'\) is given in the limit of massless products by \(\sin \delta' = (|p| - |k|)/|q|\). The dots in eqn. (13) represent further terms in \(f^{\psi_1,eq}\) which are neglected. Equivalently these results can be derived directly using only the 11 components of the propagators, because the vertex and the self-energy loop do not include internal vertices. Very similar results are known for the phenomenological scenario which can be obtained using the propagators in eqn. (6) for fermions for the Majorana neutrinos and leptons in the loops and eqn. (5) for the Higgs bosons \([2, 9]\). For the dependence on the distribution functions one obtains then the quadratic form

\[
1 - f_{E_1}^{\psi,eq} + f_{E_2}^{\psi,eq} - 2 f_{E_1}^{\psi,eq} f_{E_2}^{\psi,eq},
\]

(16)

The results obtained from non-equilibrium field theory in \([4, 5]\) differ from eqns. (13) and (14). The non-equilibrium results feature a different dependence on the distribution functions,

\[
1 + f_{E_1}^{\psi} f_{E_2}^{\psi} + 2 f_{E_1}^{\psi} f_{E_2}^{\psi} \quad \rightarrow \quad 1 + f_{E_1}^{\psi} + f_{E_2}^{\psi}.
\]

(17)

Note that the top-down results are valid even if \(f^b\) is not an equilibrium distribution \((f^b \simeq f^b\) must hold, however) and that the dependence is linear in the distribution function in contrast to eqns. (13), (14). The latter property contradicts the result derived from thermal quantum field theory.

In the phenomenological model, an analogous replacement leads to a particularly important discrepancy. Indeed, eqn. (16) would imply a cancellation of the leading effects since \(f_{E_1}^{\psi,eq} - f_{E_2}^{\psi,eq} = 2 f_{E_1}^{\psi,eq} f_{E_2}^{\psi,eq}\). The remaining effect is, in this case, entirely due to the fact that different energies enter the distribution functions of leptons and Higgs particles in eqn. (16). Since \(E_1 - E_2 \sim |q|\), this effect vanishes when the velocity of the Majorana neutrino in the medium rest-frame, \(|q|/E_{N1}\), becomes small. Therefore it is important to check whether a replacement of the form of eqn. (17) does also occur in the phenomenological scenario. This will be investigated in the next section.

IV. CAUSAL N-POINT FUNCTIONS

We will now see how the finite temperature field theory approach can be reconciled with the results derived from non-equilibrium quantum field theory. In \([21–23]\) it was shown that the combination

\[
F^{(\alpha)}_{R/I}(x_1, \ldots, x_n; z_j) = \sum_{\alpha \neq \alpha} \text{circling } z_i, z_j F_{\varphi}(x_1, \ldots, x_\alpha, \ldots, x_n; z_j),
\]

(18)

referred to as the retarded (advanced) product, has the distinguishing property that the time component \((x_\alpha)_0\) is singled out as being the largest (smallest). This becomes clear when we consider the so-called largest (smallest) time equation

\[
F_\varphi(x_1, \ldots, x_\alpha, \ldots, x_n) + F_\varphi(x_1, \ldots, x_\alpha, \ldots, x_n) = 0, \quad \text{if } (x_\alpha)_0 \text{ largest/smallest},
\]

(19)

which implies pairwise cancellation of the terms in eqn. (18) if any external vertex \(x_i\) with \(i \neq \alpha\) has the largest (smallest) time component. It has been realized that such causal products appear in Boltzmann equations in different cases, see for example \([22, 24]\). Furthermore, it has been shown that the causal

\[\text{Note that if in eqn. (14) the term quadratic in the distribution functions was absent, then by redefining the integration variable } \varphi_l \text{ we could write the energies } E_1 \text{ and } E_2 \text{ in the form } E_{1,2} = \frac{1}{2} [E_q^{\psi_1} + |q| (\sin \theta_1 \cos \varphi_l \cos \delta' + \cos \varphi_l \sin \delta')), \text{ which was used in } [20].\]
products agree with the results of the calculation in imaginary-time formalism analytically continued to real energies, at least in a few examples including the self-energy loop and the three-point vertex. The imaginary part of the causal product was shown in [22] to obey

\[
\text{Im} \{ i^{-1} F^{(a)}_{R/A} (x_1, \ldots, x_m, x_n; z_j) \} =
\]

\[
\pm \frac{1}{2} \sum_{\text{not all circling } x_i} \sum_{\text{circling } z_j} \text{Im} \{ i^{-1} F_\sigma (x_1, \ldots, x_m, x_n, z_j; z_i) - i^{-1} F_\sigma (x_1, \ldots, x_m, x_n; z_j) \},
\]

(20)

where “not all” means that not all \( x_j \) should be circled at the same time and the imaginary part is taken of the causal product in momentum space. Here, the vertex \( x_m \) with largest or smallest time is always circled.

We can now compute the imaginary part of the advanced product \( \text{Im} \{ i^{-1} F^{(1)}_A (x_1, x_2, x_3) \} \) for the three-point vertex with smallest time component \( (x_1)_0 \) of the decaying particle. The relevant circlings are shown in fig. 6. As before the contributions fig. 6(b) and (c) are suppressed due to the cut through the \( \psi_j \) propagator line.

![Fig. 6: Circlings contributing to \( \text{Im} \{ i^{-1} F^{(1)}_A (x_1, x_2, x_3) \} \) for the vertex loop. The advanced three-point function involves a difference of \( F_\sigma \) and \( F_\sigma \) contributions which differ by the replacement \( \Delta^\pm \leftrightarrow \Delta^\mp \) in the circling rules. Since the finite temperature contributions (terms proportional to \( f^{(a)} \)) to the latter are the same, all contributions quadratic in the distribution functions cancel.](image)

We compute the remaining contribution from fig. 6(a):

\[
\text{Im} \{ i^{-1} F^{(1)}_A (x_1, x_2, x_3) \} = \frac{1}{2} \text{Im} \{ i^{-1} F_\sigma (x_1, x_2, x_3) - i^{-1} F_\sigma (x_1, x_2, x_3) \}
\]

\[
= \frac{1}{2} \text{Im} \left\{ i^{-1} \int \frac{d^4 l}{(2\pi)^4} \left[ (+i g_1)(-i g_2)(-i g_1)D_{\psi j}^{+} (q + l) D_{\psi j}^{+} (q + l - k) D_{\psi j}^{+} (l) - (+i g_1)(-i g_2)(-i g_3) D_{\psi j}^{+} (q + l) D_{\psi j}^{+} (q + l) D_{\psi j}^{+} (l) \right] S \right\},
\]

(21)

where we take \( F_\sigma \) to include the spinors and charge conjugation operators \( C \) as well as projection operators \( P_R, P_L \) associated with the vertices (for the Majorana neutrino interactions). This leads to the trace part denoted by \( S \) [2, 25, 26]. In the massless lepton and Higgs limit:

\[
S = \sum_{\text{spms}} \left[ \bar{u}_t(k) P_L u_N(q) \right]^* \left[ \bar{u}_t(k) P_L (\gamma \cdot (q + l - k) + M_j) C^{-1} P_L (\gamma \cdot l) P_R C u_N(q) \right]
\]

\[
= \text{Tr} \left[ (\gamma \cdot q - M_i)(\gamma \cdot k) M_j (\gamma \cdot l) P_R \right]
\]

\[
= -2 M_i M_j k \cdot l
\]

(22)

and \( S = 1 \) for the toy model.

It turns out that the pole of the \( \psi_j \) propagator does not lie in the loop integration region, so we can drop the \( i\epsilon \) prescription. We then get (the upper and lower signs correspond to the toy model and phenomenological scenario respectively)
\[ \text{Im} \left\{ i^{-1} F_A^{(1)}(x_1, x_2, x_3) \right\} = -\frac{1}{2} \text{Im} \int \frac{d^4l}{(2\pi)^2} \delta((q + l)^2 - m_\beta^2) \delta(l^2 - m_\alpha^2) \frac{i}{(q + l - k)^2 - M_j^2} \times \]
\[ \left\{ \left[ \Theta(-(q_0 + l_0))\Theta(l_0) - \Theta(q_0 + l_0)\Theta(-l_0) \right] \right. \]
\[ \pm \left[ \Theta(-(q_0 + l_0)) - \Theta(q_0 + l_0) \right] f_{\alpha, eq}^i + \]
\[ + \left[ \Theta(l_0) - \Theta(-l_0) \right] f_{\beta, eq}^i \bigg\} \bigg\} \bigg\} \bigg\} \bigg\} \bigg\} S, \quad (23) \]

which becomes
\[ \text{Im} \left\{ i^{-1} F_A^{(1)}(x_1, x_2, x_3) \right\} = \]
\[ = -\frac{1}{2} \int \frac{d^4l}{(2\pi)^2} \delta((q + l)^2 - m_\beta^2) \delta(l^2 - m_\alpha^2) \frac{1}{(q + l - k)^2 - M_j^2} \left\{ 1 \pm f_{\alpha, eq}^i + f_{\beta, eq}^i \right\} S. \quad (24) \]

Performing the integration over \( d[l] \, dl_0 \), this indeed leads to the result for the CP-violating parameter obtained in the top-down approach with correct dependence on the distribution functions eqn. (17). For the phenomenological scenario we obtain in the limit of massless lepton and Higgs:
\[ \epsilon_i^{V,\text{th}} = \frac{1}{16\pi} \sum_{j \neq i} \text{Im} \left\{ \left( \frac{h^+ h}{{(h^+ h)}_{ii}} \right) \right\} \frac{M_j}{M_i} \int \frac{d\Omega_i}{4\pi} \frac{1 - \cos \theta_i}{M_j^2/M_i^2 + \frac{1}{2}(1 + \cos \theta_i)} \left\{ 1 - f^{t, eq}_i + f^{t, eq}_i \right\} \quad (25) \]

where \( E_{1,2} \) are given by eqn. (15). In the zero temperature limit this reduces to the well-known result
\[ \epsilon_i^{V,\text{vac}} = -\frac{1}{8\pi} \sum_{j \neq i} \text{Im} \left\{ \left( \frac{h^+ h}{{(h^+ h)}_{ii}} \right) \right\} f \left( \frac{M_j^2}{M_i^2} \right), \quad (26) \]

with
\[ f(x) = \sqrt{x} \left[ 1 - (1 + x) \ln \left( \frac{1 + x}{x} \right) \right]. \]

The same computation can be performed for the self-energy loop. The possible circlings are shown in fig. 7:

![Circlings](attachment:fig7.png)

**FIG. 7**: Circlings contributing to \( \text{Im} \left\{ i^{-1} F_A^{(1)}(x_1, x_2, z) \right\} \) for the self-energy loop. Graph (b) vanishes since \( \psi_i \) and \( \psi_j \) cannot be on-shell simultaneously.

\[ \text{Im} \left\{ i^{-1} F_A^{(1)}(x_1, x_2, z) \right\} = \frac{1}{2} \text{Im} \left\{ i^{-1} F_{\alpha} (x_1, x_2, x_3) - i^{-1} F_{\beta} (x_1, x_2, x_3) \right\} \]
\[ = -\frac{1}{2} \text{Im} \left\{ \int \frac{d^4l}{(2\pi)^2} \left[ (+ig_\alpha)(-ig_\beta)(-ig_3) D_\beta^+(q + l) D_{\alpha}(q) D_{\alpha}^+(q) - \right. \right. \]
\[ \left. \left. - (+ig_\alpha)(-ig_\beta)(-ig_3) D_\beta^+(q + l) D_{\alpha}(q) D_{\alpha}^+(q) \right] \bigg\} S, \quad (27) \]
where (the result for) $S$ coincides with eqn. (22) in the phenomenological scenario, while $S = 1/2!$ includes an additional symmetrization factor in the toy model. This becomes

$$\text{Im}\left\{ i^{-1} F_A^{(1)}(x_1, x_2; z) \right\} = \frac{1}{2} \int \frac{d^4q}{(2\pi)^2} \delta((q + l)^2 - m_3^2) \delta(l^2 - m_0^2) \frac{1}{q^2 - M_j^2} \left\{ 1 \pm f_1^{\alpha,eq} + f_3^{\beta,eq} \right\} S. \quad (28)$$

This corresponds to the result for the self-energy contribution in the hierarchical limit in the top-down approach [5] if the equilibrium distribution functions are replaced with non-equilibrium ones. In the zero temperature limit this leads to the correct vacuum result. Thus, we have shown that (within the toy model) the CP-violating parameter $\epsilon^{th}$ obtained with help of thermal quantum field theory coincides with the one obtained in the top-down approach (in the approximately symmetric case) when one uses causal products instead of the conventional ones which assume type-1 external vertices. Furthermore, by comparing with the top-down result, we find that the thermal field theory result can be generalized to a (symmetric) non-equilibrium configuration for the toy model by the canonical replacement of the equilibrium distribution functions with the non-equilibrium ones: $f^{eq} \rightarrow f$.

For the phenomenological scenario we obtain in the limit of massless lepton and Higgs for the self-energy contribution (including a factor of 2, because the two components of the lepton doublet can propagate in the self-energy loop for a given transition)

$$\epsilon^{S,th}_i = -\frac{1}{8\pi} \sum_{j \neq i} \text{Im}\left\{ \frac{(h_i^h)^2}{(h_i^h)^2} \right\} \frac{M_i M_j}{M_i^2 - M_j^2} \int \frac{d\Omega_l}{4\pi} (1 - \cos \theta_l) \left\{ 1 - f_{E_1}^{\ell,eq} + f_{E_2}^{\phi,eq} \right\}, \quad (29)$$

where $E_{1,2}$ are again given by eqn. (15). In the zero temperature limit this reduces to the standard result

$$\epsilon^{S,vac}_i = -\frac{1}{8\pi} \sum_{j \neq i} \text{Im}\left\{ \frac{(h_i^h)^2}{(h_i^h)^2} \right\} \frac{M_i M_j}{M_i^2 - M_j^2}.$$

The complete CP-violating parameter is given by

$$\epsilon^{th}_i = \epsilon^{V,th}_i + \epsilon^{S,th}_i, \quad (30)$$

where the vertex and self-energy contributions are given by eqns. (25) and (29) respectively. Therefore the overall dependence on the distribution functions (vertex and self-energy contribution) is given by

$$1 - f_{E_1}^{\ell,eq} + f_{E_2}^{\phi,eq}.$$

In contrast to previous findings eqn. (16), this does not vanish in the limit when the Majorana neutrino decays at rest assuming massless $\ell$ and $\phi$. Therefore, it is qualitatively different from the conventional result. The new expression can lead to a significant enhancement of the CP-violating parameter, see fig. 8. Similar formulas can be derived for processes such as $\phi \rightarrow N_1 \ell$ in the standard model (which can become relevant at higher temperatures) or for similar MSSM processes involving sneutrinos and sleptons. The size of the medium corrections depends primarily on the statistics of the particles in the loop, see fig. 9.

V. BOLTZMANN EQUATIONS

We can assume in addition that the structure of the Boltzmann equations for the phenomenological scenario is analogous to the one given in [4, 5] with appropriate quantum statistical factors for bosons and fermions respectively and appropriate symmetrization factors. This defines the full set of Boltzmann equations including medium corrections to the CP-violating parameter for the phenomenological scenario as derived above. With these modifications, the minimal network of quantum corrected Boltzmann equations for
where the Liouville operator is given by

\[
\langle \epsilon_1 \rangle / \epsilon_1^{vac}, N_1 \rightarrow \phi \ell
\]

\[
\langle \epsilon_1^{th} \rangle / \epsilon_1^{vac}, N_1 \rightarrow \phi \ell
\]

\[
\langle \epsilon_1^{th, conv} \rangle / \epsilon_1^{vac}, N_1 \rightarrow \phi \ell
\]

If the generated asymmetry is small, as we assume here, then \( f^\ell \approx f^\phi \) and \( f^\phi \approx f^\phi \). In this case the CP-violating contributions to the right-hand side of eqn. (31a) cancel out and we obtain

\[
C_{N_1 \leftrightarrow \ell \phi}[f^{N_1}, f^\ell, f^\phi](|k|) + C_{N_1 \leftrightarrow \ell \phi}[f^{N_1}, f^\ell, f^\phi](|k|)
\]

\[
\approx 1/2 \int d\Omega^o_\ell d\Omega^o_\ell (2\pi)^4 \delta(k - p - q) |M_0|^2_{N_1 \rightarrow \ell \phi} (p, q)
\]

\[
\times \left\{ \left[ |f^{N_1}_k| f^\ell_{|p|} f^\phi_{|q|} - f^\ell_{|k|} f^\phi_{|p|} f^{N_1}_{|q|} \right] + \left[ |f^{N_1}_k| f^\ell_{|p|} f^\phi_{|q|} - f^\ell_{|k|} f^\phi_{|p|} f^{N_1}_{|q|} \right] \right\},
\]

FIG. 8: Temperature dependence of the CP-violating parameter in the Majorana neutrino decay relative to its vacuum value. Shown are the thermal average \( \langle \epsilon_1^{th} \rangle / \epsilon_1^{vac} \) (solid red line) and the values for various momentum modes \( \langle \epsilon_1^{th} \rangle / \epsilon_1^{vac} \) (dotted red lines) corresponding to \( |q| = T, -1 \leq \sin(\delta) \leq +1 \). For comparison we also show the conventional results \( \langle \epsilon_1^{th, conv} \rangle / \epsilon_1^{vac} \) (dashed black line), where the leading effects cancel as described in the text. Equilibrium distribution functions for bosons and fermions with negligible chemical potentials are assumed. Note that the shown behavior can be modified if thermal masses are included, since the decay \( N_1 \rightarrow \ell \phi \) (and the conjugate process) becomes kinematically forbidden if the thermal Higgs mass becomes too large. At even higher temperature the process \( \phi \rightarrow N_1 \ell \) becomes relevant instead [2].

thermal leptogenesis with hierarchical Majorana neutrino masses \( M_1 \ll M_2, M_3 \) takes the form (in homogeneous and isotropic Friedman–Robertson–Walker space-time and not writing equations for the Higgs fields \( \phi, \bar{\phi} \) which are considered to be in thermal equilibrium):

\[
L[f^{N_1}_k]|(|k|) = C_{N_1 \leftrightarrow \ell \phi}[f^{N_1}, f^\ell, f^\phi](|k|) + C_{N_1 \leftrightarrow \ell \phi}[f^{N_1}, f^\ell, f^\phi](|k|),
\]

\[
L[f^\ell_0]|(|k|) = C_{\ell \phi \leftrightarrow N_1}[f^\ell_0, f^\phi, f^{N_1}](|k|),
\]

\[
L[f^\phi_0]|(|k|) = C_{\ell \phi \leftrightarrow N_1}[f^\ell_0, f^\phi, f^{N_1}](|k|),
\]

where the Liouville operator is given by

\[
L[f^\alpha]|(x, k) = k^0 \left( \frac{\partial}{\partial \ell} - |k| H \frac{\partial}{\partial |k|} \right) f^\alpha(|k|).
\]

If the generated asymmetry is small, as we assume here, then \( f^\ell \approx f^\ell \) and \( f^\phi \approx f^\phi \). In this case the CP-violating contributions to the right-hand side of eqn. (31a) cancel out and we obtain

\[
C_{N_1 \leftrightarrow \ell \phi}[f^{N_1}, f^\ell, f^\phi](|k|) + C_{N_1 \leftrightarrow \ell \phi}[f^{N_1}, f^\ell, f^\phi](|k|)
\]

\[
\approx 1/2 \int d\Omega^o_\ell d\Omega^o_\ell (2\pi)^4 \delta(k - p - q) |M_0|^2_{N_1 \rightarrow \ell \phi} (p, q)
\]

\[
\times \left\{ \left[ |f^{N_1}_k| f^\ell_{|p|} f^\phi_{|q|} - f^\ell_{|k|} f^\phi_{|p|} f^{N_1}_{|q|} \right] + \left[ |f^{N_1}_k| f^\ell_{|p|} f^\phi_{|q|} - f^\ell_{|k|} f^\phi_{|p|} f^{N_1}_{|q|} \right] \right\},
\]
where, as usual, the tree-level amplitude for the Majorana neutrino decay is given by $|\mathcal{M}_{10}|^2_{\phi \rightarrow t, \phi} (p, q) = |\mathcal{M}_{10}|^2_{N_1 \rightarrow \tilde{\phi} \pi} (p, q) = 2 \langle h^* h \rangle_{11} p \cdot q$. The collision terms for the (inverse) decay of the heavy particle into a $\ell \phi$ or a $\ell \tilde{\phi}$ pair explicitly contain the CP-violating parameter $\epsilon_1$ given in eqn. (30) but with the equilibrium distributions replaced by non-equilibrium ones $f^{eq} \rightarrow f$:

$$
C_{\ell \phi \rightarrow N_1} [f^\ell, f^{\phi}, f^{N_1}](|k|) = \frac{1}{2} \int d\Pi_0^\ell \, d\Pi_1^{\phi} \, (2\pi)^4 \delta(k + p - q) |\mathcal{M}_{10}|^2_{N_1 \rightarrow \ell \phi} (k, p)[1 + \epsilon_1(|q|)]
	imes \left\{1 - f^\ell_{|k|}[1 + f^{\phi}_{|p|} f^{N_1}_{|q|}] - f^\phi_{|k|} f^{\phi}_{|p|}[1 - f^{N_1}_{|q|}]\right\},
$$

(34a)

$$
C_{\ell \tilde{\phi} \rightarrow N_1} [f^\ell, f^{\tilde{\phi}}, f^{N_1}](|k|) = \frac{1}{2} \int d\Pi_0^\ell \, d\Pi_1^{\tilde{\phi}} \, (2\pi)^4 \delta(k + p - q) |\mathcal{M}_{10}|^2_{N_1 \rightarrow \ell \tilde{\phi}} (k, p)[1 - \epsilon_1(|q|)]
	imes \left\{1 - f^\ell_{|k|}[1 + f^{\tilde{\phi}}_{|p|} f^{N_1}_{|q|}] - f^{\tilde{\phi}}_{|k|} f^{\tilde{\phi}}_{|p|}[1 - f^{N_1}_{|q|}]\right\}.
$$

(34b)

Note that the network of Boltzmann equations (31) should be understood in the generalized sense: the transition amplitudes differ from the usual perturbative matrix elements and do not have their symmetry properties as was noted in [4, 5]. The structure of the collision terms (34) differs from the conventional one. In particular, we did not include the processes $\ell \phi \leftrightarrow \ell \phi$ explicitly, because the collision terms for the processes $\ell \phi \leftrightarrow N_1$ and $\ell \tilde{\phi} \leftrightarrow N_1$ do not suffer from the generation of an asymmetry in equilibrium. To obtain a consistent set of equations in the canonical bottom-up approach we would need to subtract the RIS part of the $S$-matrix element for the processes $\ell \phi \leftrightarrow \ell \phi$. Note, however, that it may be necessary to include the collision terms for $\ell \phi \leftrightarrow \ell \tilde{\phi}$ (derived in the top-down approach) in quantitative studies, because these can violate CP in general. Further scattering processes with top-quarks and gauge-bosons can also give relevant contributions. We note here that this result should be treated with care, because additional new effects could arise when the phenomenological scenario is investigated in the top-down approach. In addition, the applicability of the quasi-particle picture can not be tested in the framework of thermal field theory. In particular the results presented above will only apply in the hierarchical case [5]. The analysis of the resonant case requires the use of the Kadanoff-Baym formalism, which allows us to take into account the in-medium spectral properties of the mixing fields.

FIG. 9: Medium correction to the CP-violating parameters in the MSSM. The lines correspond to the thermal averages $\langle \epsilon_1^{th} \rangle / \epsilon_1^{vac}$, and the shaded regions illustrate the momentum dependence of $\epsilon_1^{th} / \epsilon_1^{vac}$ for $0.5 \leq |q|/T \leq 4$ and $\delta_1 = 0$. Note also that in the weighted sum of $N_1 \rightarrow \phi^\ell$ and $N_1 \rightarrow \phi^\ell$ processes the cancellation of the medium contributions, observed in the earlier publications, does not occur anymore.
VI. CONCLUSIONS

Inspired by a discrepancy between conventional results for the thermal corrections to the CP asymmetries in thermal leptogenesis and recent new results from non-equilibrium quantum field theory we have reconsidered the calculation of the CP-violating parameters based on thermal quantum field theory. We find that, if causal products are used in the computation of the $n$-point functions, the results of both approaches can be brought into agreement in the framework of a toy model. We conclude that causal $n$-point functions must be used in the derivation of the CP-violating parameter in the phenomenological scenario as well.

This leads to new expressions for the thermal corrections to the vertex and self-energy CP-violating parameters. In contrast to the conventional results the thermal corrections do not vanish in the limit when the Majorana neutrino decays at rest assuming massless decay products. Therefore, it is qualitatively different from the conventional result and might give significant contributions to the generated baryon asymmetry.

In the range from 0.1 to 10 of the dimensionless inverse temperature, thermal effects can enhance the CP-violating parameter by up to an order of magnitude. The asymmetry can be computed using the minimal set of Boltzmann equations for leptogenesis in SM+$3\nu_R$ presented here, which are analogous to the equations which have been derived earlier in the framework of the toy model. These take into account decays and inverse decays and include all quantum statistical factors in a way which guarantees that no asymmetry is generated in equilibrium. They can be applied in the case of non-degenerate Majorana neutrino masses. For a detailed phenomenological analysis it will be necessary to take into account further thermal effects such as thermal masses and resummed thermal propagators as well as additional CP-violating processes which exist in phenomenological scenarios.

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