A plethora of Type IIA embeddings for $d = 5$ minimal supergravity

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Abstract

We construct multiple embeddings of all solutions of $d = 5$ minimal (un)gauged supergravity into massive Type IIA supergravity. The internal spaces and warpings of such embeddings are the same as those of the $\mathcal{N} = 1$ supersymmetric (Mink$_5$) AdS$_5$ vacua, with the slight modification that the U(1) R-symmetry direction becomes fibered over the external space by the $d = 5$ gauge field. In addition the fluxes are appropriately modified. There are many distinct types of the aforementioned internal spaces and as such many different embeddings of the $d = 5$ supergravity. As examples of our setup we provide new solutions dual to six-dimensional, $\mathcal{N} = (1, 0)$ SCFTs compactified on the product of a constant curvature Riemann surface and a spindle. We also provide a multitude of massive Type IIA embeddings for rotating, asymptotically AdS$_5$ black hole solutions.
1 Introduction

Supersymmetric solutions of supergravity theories play a key role in the study of string theory. Supersymmetric compactifications provide a setting for obtaining realistic models of particle physics, while a microscopic derivation of the black hole entropy in string theory is best understood for supersymmetric black holes. Supersymmetric solutions have also supported the development of the gauge/gravity duality.

Supersymmetry is, in part, technically a simplifying assumption in the construction of solutions. Still, especially in the absence of a high degree of supersymmetry or other symmetries, the construction of solutions of ten- or eleven-dimensional supergravity theories is a challenging task which calls for advanced mathematical tools, especially in the field of geometry.

Complementary to the construction of solutions directly in ten or eleven dimensions, is the uplift of solutions of lower-dimensional supergravity theories. The latter is feasible due to the existence of consistent truncations of the infinite Kaluza–Klein tower of higher-dimensional compactifications.
to a finite set of modes, so that a solution of the lower-dimensional equations of motion is also a solution of the ten- or eleven-dimensional ones. Examples include consistent truncations on spheres down to maximal gauged supergravities [1–7], Sasaki–Einstein manifolds [8–11], weak-$G_2$ holonomy manifolds [8] and tri-Sasakian manifolds [12], SU(2)-structure [13, 14] and SU(3)-structure [15] manifolds, as well as spaces including brane singularities [16]. Recently, a framework based on exceptional generalised geometry and exceptional field theory has emerged, that allows for a systematic treatment of consistent truncations [17–21]. Despite these successes the exceptional field theory framework for consistent truncations has only been fully worked out for reductions to maximal and half-maximal gauged supergravities.\(^1\)

In the present work we construct a universal consistent truncation of massive Type IIA supergravity on a five-dimensional Riemannian manifold $M_5$, to minimal (un)gauged supergravity in five dimensions. The manifold $M_5$ can be any of the class of manifolds that constitute the internal space of five-dimensional, $\mathcal{N} = 1$ supersymmetric Minkowski ($\text{Mink}_5$) or anti-de Sitter ($\text{AdS}_5$) solutions of massive Type IIA supergravity [24].\(^2\) We apply the technical methodology of that work, the bi-spinor formalism in conjunction with $G$-structures, to the construction of the consistent truncation Ansatz.\(^3\)

Five-dimensional minimal supergravity is a rich theory [29], and our work paves the way for the uplift of many interesting solutions that reside in it (e.g. [30]), in a multitude of ways, and their subsequent study in Type IIA supergravity. We consider the uplift of two classes of solutions as examples of our consistent truncation. The first includes the solution of [30] describing a black hole with two independent angular momenta and a single magnetic charge. The second is the near-horizon of a black string which has a spindle horizon, first studied in [31] where it was uplifted to Type IIB supergravity on a Sasaki–Einstein manifold. Later work has generalised the spindle solutions to different dimensions $\geq 4$ and different embeddings in string/M-theory, [32–47].

The rest of the paper is organised as follows:

In section 2 we lay down the groundwork to embed $d = 5$ minimal (un)gauged supergravity into massive Type IIA supergravity. We discuss the ($\text{Mink}_5$) $\text{AdS}_5$ vacua of massive Type IIA supergravity with generalised structures in section 2.1, which reviews and slightly generalises (allowing for $\text{Mink}_5$) the results of [24]. An important part of this section for our later generalisation is appreciating that $\text{AdS}_5$ vacua support both null and time-like Killing vectors (in the sense of [48]), with the latter yielding information pertinent to our ultimate aim more readily. We review the

\(^1\)That being said, in [22] (which generalises the earlier half-maximal work of [23]) there is a generic prescription to obtain a consistent truncation preserving any amount of supersymmetry.

\(^2\)The corresponding truncation of Type IIB supergravity was carried out in [25], (see also [26]). Our work includes the embeddings of [27].

\(^3\)See also [28].
known class of AdS$_5$ vacua in section 2.2, writing them in a convenient form for our later purposes. In section 2.3 we derive a new class of Mink$_5$ vacua, relevant for the ungauged limit of the $d = 5$ supergravity.

In section 3 we derive an embedding of $d = 5$ minimal (un)gauged supergravity into massive Type IIA supergravity under the assumption that the ten-dimensional solution decomposes as a warped product with the five-dimensional U(1) gauge field appearing as a connection in the ten-dimensional metic. We further assume that the ten-dimensional bosonic fields depend on $d = 5$ minimal (un)gauged supergravity only through its bosonic fields — so we can obtain an embedding that does not depend on external supersymmetry. To derive the embedding we make use of the same language of generalised structures used to derive the (Mink$_5$) AdS$_5$ vacua, a major benefit being that there is no need to make an ansatz for the flux, which is uniquely fixed by our previous assumptions. We show that the internal space of the ten-dimensional solutions is a mild generalisation of that of (Mink$_5$) AdS$_5$ vacua and provide simple replacement rules to map a ten-dimensional vacuum solution to an embedding of a generic solution of $d = 5$ minimal (un)gauged supergravity. Given a solution to $d = 5$ supergravity, there are as many embeddings as there are ten-dimensional vacua, i.e. many. This section is supplemented by appendix C where we prove that for any of these embeddings ten-dimensional supersymmetry is preserved whenever five-dimensional supersymmetry holds and that one has a solution to the ten-dimensional equations of motion regardless.

In section 4 we uplift two classes of solutions to massive Type IIA supergravity. The seed AdS$_5$ solutions were constructed in [49] and consist of a constant curvature Riemann surface present in the internal space as well as an O8–D8 stack, D6-brane and D4-brane sources, localized and partially localized. The solutions are characterised by a cubic polynomial with different global completions depending on the choice of four parameters. The solutions have the natural interpretation of being the holographic duals of the four-dimensional superconformal field theories (SCFTs) arising in the IR limit of placing a six-dimensional, $\mathcal{N} = (1,0)$ theory on the Riemann surface. Using the seed solutions we show how to uplift two classes of solutions of $d = 5$ minimal gauged supergravity to the massive Type IIA one. The first class of solutions is the Gutowski–Reall black hole solutions with equal angular momenta parameters [30]; one could also use our formulae to uplift the CCLP solution [50] which has two independent angular momenta. The second class are the AdS$_3 \times \mathbb{WCP}^1_{n,\pm}$ spindle solutions [31], which also include in a particular limit the AdS$_3 \times \Sigma_{g>1}$ solutions.

The work is supplemented by technical appendices referred to in the main text.

2 Generalised structures and vacua

Before proceeding with the embedding of $d = 5$ minimal (un)gauged supergravity into massive Type IIA supergravity, it will be useful to review some features of the $\mathcal{N} = 1$ supersymmetric AdS$_5$
vacua of the latter. These were originally classified in [24], with their local form significantly refined in [49]. Something that will be particularly useful going forward is how these vacua arise from the necessary and sufficient conditions for supersymmetry phrased in terms of generalised structures in $d = 10$ [48].

### 2.1 Bi-spinor equations

The fundamental objects appearing in the classification of [48] are bi-linears of the two Majorana–Weyl supersymmetry parameters of Type II supergravity $\epsilon_{1,2}$, namely

$$
K^{(10)} \equiv \frac{1}{64}(\tau_1 \Gamma_M \epsilon_1 + \tau_2 \Gamma_M \epsilon_2) dx^M, \quad \tilde{K}^{(10)} \equiv \frac{1}{64}(\tau_1 \Gamma_M \epsilon_1 - \tau_2 \Gamma_M \epsilon_2) dx^M, \quad \Psi^{(10)} \equiv \epsilon_1 \otimes \epsilon_2.
$$

(2.1)

Necessary conditions for supersymmetry are given in terms of these and the dilaton, NSNS 3-form and RR polyform, respectively ($\Phi, H, F$),

$$
d_H(e^{-\Phi} \Psi^{(10)}) = -(\iota_{K^{(10)}} + \tilde{K}^{(10)} \wedge) F, \quad \nabla_{(M} K_{N)}^{(10)} = 0, \quad d\tilde{K}^{(10)} = \iota_{K^{(10)}} H,
$$

(2.2a) (2.2b)

where $d_H \equiv d - H \wedge$. These conditions imply that

$$
\mathcal{L}_{K^{(10)}} \Psi^{(10)} = \mathcal{L}_{K^{(10)}} \Phi = 0,
$$

(2.3)

and further, when the Bianchi identities for the fluxes are assumed, namely

$$
dH = 0, \quad dH F = 0,
$$

(2.4)

that $\mathcal{L}_{K^{(10)}} H = \mathcal{L}_{K^{(10)}} F = 0$. Thus, $(K^{(10)})^M \partial_M$ is an isometry of any supersymmetric solution, under which $\epsilon_{1,2}$ are singlets. The conditions (2.2a)-(2.2b) are not by themselves sufficient for supersymmetry generically; for that one must also solve some so called pairing constraints — however, for the cases we are interested in they are actually implied so we shall not quote them here.

An AdS$_5$ vacuum solution of massive Type IIA supergravity must have bosonic fields decomposing as

$$
ds_{10}^2 = e^{2A} ds^2(\text{AdS}_5) + ds^2(\text{M}_5), \quad F = f_+ + e^{5A} \text{vol}(\text{AdS}_5) \wedge \ast (f_+),
$$

(2.5)

where $(e^{2A}, f_+)$ have support on $\text{M}_5$ only and likewise for the NSNS 3-form and dilaton, while we assume AdS$_5$ has inverse radius $m$.\footnote{Note: By definition $F = \sum_{k=0}^{\frac{5}{2}} F_{2k}$, $f_+ = F_0 + f_2 + f_4$ is the magnetic part of this RR polyform, while $\lambda(C_k) = (-1)^{\frac{3}{2}} C_k$ for a $k$-form $C_k$.} When such vacua are supersymmetric they can be extracted
from (2.2a)-(2.2b) by decomposing the $d = 10$ Killing spinors as

$$
\epsilon_1 = \frac{1}{\sqrt{2}} \left( \frac{1}{i} \right) \otimes \zeta \otimes \chi_1 + \text{m.c.}, \quad \epsilon_2 = \frac{1}{\sqrt{2}} \left( \frac{1}{-i} \right) \otimes \zeta \otimes \chi_2 + \text{m.c.,}
$$

(2.6)

where $\chi_{1,2}$ are Dirac spinors on the internal space, $\zeta$ are Killing spinors on AdS$_5$ obeying

$$
\nabla_\mu \zeta = \frac{m}{2} \gamma_\mu \zeta,
$$

(2.7)

and here and elsewhere m.c. stands for Majorana conjugate. Our conventions for gamma matrices can be found in appendix A. Defining bi-spinors on AdS$_5$ and M$_5$ as

$$
\phi^1 = \zeta \otimes \bar{\zeta}, \quad \phi^2 = \zeta \otimes \bar{\zeta}^c,
$$

$$
\psi^1 = \chi_1 \otimes \chi_2^\dagger, \quad \psi^2 = \chi_1 \otimes \chi_2^\dagger c,
$$

(2.8)

where $\phi^2$ only has non-trivial 2- and 3-form contributions, one finds that the $d = 10$ bi-linears decompose as

$$
K^{(10)} = \frac{1}{16} (q_k - f \xi), \quad \tilde{K}^{(10)} = \frac{1}{16} (q_k - f \tilde{\xi}), \quad q_\pm = \frac{e^A}{2} (|\chi_1|^2 \pm |\chi_2|^2),
$$

$$
\Psi^{(10)} = \left( i \phi^1 \text{Im} \psi^1_+ + e^{2A} \phi^1_\dagger \wedge \text{Im} \psi^1_+ \right) + e^A \phi^1_\dagger \wedge \text{Im} \psi^1_+ + e^{2A} \left( \phi^1_2 \wedge \text{Re} \psi^1_+ + \text{Im} (\phi^1_2 \wedge \psi^2_+) \right) + e^{3A} \left( i \phi^1_3 \wedge \text{Re} \psi^1_+ + \text{Re} (\phi^1_3 \wedge \psi^1_-) \right) + ie^{4A} \phi^1_4 \wedge \text{Im} \psi^1_+,
$$

(2.9)

where we introduced the following real function $f$ and real 1-forms $(k, \xi, \tilde{\xi})$:

$$
f \equiv -iA \phi^1_0, \quad k \equiv 4\phi^1_1, \quad \xi \equiv \frac{1}{2} (\chi_1^t \gamma_2 \chi_1 - \chi_2^t \gamma_2 \chi_2) e^2, \quad \tilde{\xi} \equiv \frac{1}{2} (\chi_1^t \gamma_2 \chi_1 + \chi_2^t \gamma_2 \chi_2) e^2.
$$

(2.10)

It is a simple application of Fierz identities to establish that (2.7) implies the following equations\(^5\)

$$
d\phi^1_- = m(\text{deg}) \phi^1_+, \quad d\phi^1_+ = 0, \quad d\phi^2_2 = 3m\phi^2_3, \quad d\phi^2_3 = 0, \quad \nabla_\nu (\phi^1_\nu) = 0,
$$

(2.11)

so that in particular $f$ is constant and $k^\mu \partial_\mu$ is a Killing vector. One can show in general\(^6\) that $i_k k = -f^2$ and it follows from (2.11) (given identities in appendix B) that

$$
\mathcal{L}_{k} \phi^1 = 0, \quad \mathcal{L}_{k} \phi^2 = 3imf \phi^2,
$$

(2.12)

so the nature of $k^\mu \partial_\mu$, null/time-like, singlet/charged is intimately related to the value of $f$. There are of course two types of supercharges that AdS$_5$ preserves: Poincaré supercharges $\zeta_P$ and conformal supercharges $\zeta_C$.\(^7\) We can choose to align $\zeta$ along any (non-zero) linear combination of

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\(^5\)Here (deg) indicates that the form degree appears here, i.e. (deg) $C_k = kC_k$.

\(^6\)I.e. this follows from the generic properties of a Lorentzian bi-linear in $d = 5$.

\(^7\)If one parametrises AdS$_5$ as $ds^2(\text{AdS}_5) = e^{2mr}(dx^\alpha)^2 + dr^2$ for $\alpha = 0, \ldots, 3$, then in the obvious frame this suggests, these are $\zeta_P = e^{mr} \zeta^0_+$ and $\zeta_C = (e^{-mr} + me^{4mr}x^2 \gamma_2^0) \zeta^0_-$ where $\zeta^0_\pm$ are constant spinors obeying $\gamma_5 \zeta^0_\pm = \pm \zeta^0_\pm$. 

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these without changing anything physical about the AdS vacua, however taking without loss of
generality
\[ \zeta = \zeta_P + i\zeta_C \quad \Rightarrow \quad f = 2\text{Re}(\bar{\zeta}_P\zeta_C), \]  
so \( f \) is only non-zero if we align \( \zeta \) along both such charges and we can in fact extract information
more easily from (2.9) by making this choice. To see this one can consider for instance (2.2b): plugging (2.9) into this one finds it requires
\[ e^{-2A}q_+ = c, \quad mq_- = 0, \quad dq_- = 0, \quad f\nabla_{(a}\xi_{b)} = 0, \quad f(d\tilde{\xi} - \iota_\xi H) = 0, \]  
where \( c > 0 \) is a constant, so (2.2b) imply that \( \xi^a\partial_a \) is a Killing vector and fixes the part of
\( H \) parallel to it but only when \( f \neq 0 \). Of course as nothing physical should depend on how \( \zeta \) is parameterised, and hence the value of \( f \), \( \xi^a\partial_a \) should always be Killing — indeed [24], which implicitly assumes \( f = 0 \), show this explicitly, albeit with a less direct computation. Likewise given (2.12) the first of (2.3) imposes
\[ f(\mathcal{L}_\xi \psi^1) = f(\mathcal{L}_\xi \psi^2 - 3imc\psi^2) = 0, \]  
making clear that \( \xi^a\partial_a \) is the U(1) R-symmetry one expects an \( \mathcal{N} = 1 \) supersymmetric AdS\(_5\) solution to support. Finally, plugging (2.9) and the second of (2.5) into (2.2a), we find further differential conditions, under the assumption that we solve \( mq_- = 0 \) as \( q_- = 0 \) (necessary for AdS\(_5\)). In summary, supersymmetric AdS\(_5\) vacua must satisfy
\[ e^{-2A}q_+ = c, \quad \nabla_{(a}\xi_{b)} = 0, \quad d\tilde{\xi} = \iota_\xi H, \]  
\[ d_H(e^{2A-\Phi}\psi^2_+) = 0, \quad d_H(e^{3A-\Phi}\psi^2_-) - 3me^{2A-\Phi}\psi^2_+ = 0, \]  
\[ d_H(e^{3A-\Phi}\text{Re}\psi^1_+) = 0, \quad d_H(e^{A-\Phi}\text{Im}\psi^1_-) = 0, \]  
\[ d_H(e^{A-\Phi}\text{Re}\psi^1_+) + 2me^{A-\Phi}\text{Im}\psi^1_- = 0, \]  
\[ d_H(e^{4A-\Phi}\text{Im}\psi^1_+) - 4me^{3A-\Phi}\text{Re}\psi^1_- - C e^{5A} \ast \lambda(f_+) = 0, \]  
\[ d_H(e^{-\Phi}\text{Im}\psi^1_+) + \frac{1}{4}(\iota_\xi + \tilde{\xi} \wedge)f_+ = 0, \quad d_H(e^{5A-\Phi}\text{Im}\psi^1_-) + \frac{1}{4}e^{5A}(\iota_\xi + \tilde{\xi} \wedge) \ast \lambda(f_-) = 0. \]  
These conditions are necessary and sufficient for AdS\(_5\) vacua. The conditions (2.16f) when extracted are multiplied by \( f \), however since they are implied by the rest of the conditions we present, irrespective of the value of \( f \) we can remove the \( f \) multiplicative factor. They will be important for the embedding of the \( d = 5 \) minimal supergravity. Note that when one fixes \( m = 0 \) we also have conditions for Mink\(_5\) vacua, though not completely general ones which do not demand \( q_- = 0 \) — however this constraint is necessary if one wishes to allow for purely RR sources.
2.2 AdS$_5$ vacua

Following [24] one solves the bi-spinor constraints of the previous section by first decomposing the internal spinors in a common basis in terms of a single spinor $\chi$ with norm $||\chi||^2 = e^A c$. This leads to the bi-spinors
\[
\chi \otimes \chi^\dagger = \frac{e^A c}{4} (1 + v) \wedge e^{-ij_2}, \quad \chi \otimes \chi'^\dagger = \frac{e^A c}{4} (1 + v) \wedge \omega_2,
\]
where $(v, j_2, \omega_2)$ span an SU(2)-structure on $M_5$. Consistency with (2.14) and the 0-form part of the second of (2.16b) restricts this decomposition to
\[
\chi_1 = \chi, \quad \chi_2 = a \chi + \frac{b}{2} \bar{w} \chi, \quad a = a_1 + ia_2, \quad a_1^2 + a_2^2 + b^2 = 1,
\]
for $w$ a holomorphic 1-form such that $||w||^2 = 2$, $\iota_v w = 0$ and $w \chi = 0$. Defining a second 1-form as $z = -\frac{i}{2} \iota_w \omega_2$, the SU(2)-structure forms then decompose as
\[
\dot{j}_2 = \frac{i}{2} (w \wedge \bar{w} + z \wedge \bar{z}), \quad \omega_2 = w \wedge z,
\]
with $\{v, \text{Re} w, \text{Im} w, \text{Re} z, \text{Im} z\}$ giving a vielbein on $M_5$. The internal bi-linears of the previous section then decompose in terms of this vielbein as
\[
\xi = e^A c b (v - \text{Re}(aw)), \quad \tilde{\xi} = e^A c (b \text{Re}(aw) + (1 - b^2)v),
\]
\[
\psi^1_+ = \frac{e^A c}{4} \overline{\alpha} e^{-ij_2 + \frac{b}{2} \bar{v} \wedge w}, \quad \psi^1_- = \frac{e^A c}{4} (\overline{\alpha} v + bw) \wedge e^{-ij_2},
\]
\[
\psi^2_+ = \frac{e^A c}{4} (aw - bv) \wedge z \wedge e^{-ij_2}, \quad \psi^2_- = -\frac{e^A c}{4} （b z \wedge e^{-ij_2 - \frac{b}{2} \bar{v} \wedge w}
\]
which span an identity-structure.\(^9\) The condition that $\xi$ is Killing allows us to parameterise it as
\[
\xi = \frac{||\xi||^2}{3c} D\psi, \quad D\psi \equiv d\psi + V, \quad ||\xi|| = be^A c,
\]
with $\partial_\psi$ a Killing vector and $V$ a 1-form with support on the directions of $M_5$ that are not $\psi$. We then have that $M_5$ decomposes as a $U(1)$ fibration over a four-dimensional base as
\[
ds^2(M_5) = \frac{||\xi||^2}{9c^2} D\psi^2 + ds^2(M_4),
\]
\(^8\)With respect to [24] we have a sign change in $\omega_2$. This is due to a difference in phase in the internal intertwiner defining Majorana conjugation (see appendix A). The choice we make here ensures that (2.20b)-(2.20c) takes the same form as [24] (up to the $e^A c$ factor explained in the next footnote), which is what actually matters if we wish to use their results.

\(^9\)In [24] $c = 1$, which one can choose to fix without loss of generality. The $e^A$ factor has been extracted appearing instead in (2.16a)-(2.16e).
with $M_4$ independent of $\psi$ (at least locally).

Introducing coordinates $(s, u, x_1, x_2)$ on $M_4$, the problem of finding supersymmetric AdS$_5$ solutions can be recast in terms of two functions ($D_u, D_s$) depending on $(s, u, x_1, x_2)$, subject to partial differential equations [49].

The metric for a general supersymmetric AdS$_5$ solution is

$$ds_{10}^2 = e^{2A} \left[ ds^2(\text{AdS}_5) + e^{2\varphi} \left( dx_1^2 + dx_2^2 \right) + \frac{1}{3} e^{-6\lambda} ds_3^2 \right], \quad (2.23a)$$

$$ds_3^2 = -\frac{4}{\partial_s D_s} D\psi^2 - \partial_s \tilde{D}_s ds^2 - 2\partial_u D_s duds - \partial_u D_u du^2, \quad (2.23b)$$

where

$$D\psi = d\psi - \frac{1}{2m} \ast_2 d_2 D_s$$

and $\tilde{D}_s \equiv D_s - \frac{3}{2} \ln s$. The Hodge star operator $\ast_2^{10}$ and the exterior derivative $d_2$ are taken over the $(x_1, x_2)$ plane.

The functions appearing in the metric are given in terms of $(D_u, D_s)$ as follows:

$$e^{-6\lambda} = \frac{1}{8m^2 s} \det(h), \quad e^{4A} = -\frac{\partial_s D_s}{3 \det(h)}, \quad e^{2\varphi} = \frac{1}{24m^2} \det(h) e^{D_s}, \quad (2.25)$$

with

$$\det(g) = \partial_u D_u \partial_s \tilde{D}_s - (\partial_u D_s)^2,$$

$$\det(h) = \partial_u D_u \partial_s D_s - (\partial_u D_s)^2.$$

The dilaton can be expressed as

$$e^{2\Phi} = e^{6A} e^{-6\lambda}. \quad (2.27)$$

The NSNS field strength $H$ is given by

$$H = \frac{1}{3c} d \left[ \tilde{\xi} \wedge D\psi + \frac{c}{8m^2 \sqrt{2s}} \partial_u (e^{D_s}) dx_1 \wedge dx_2 \right] + \frac{1}{36m^2 \sqrt{2s}} du \wedge d \ast_2 d_2 D_s - \frac{e^{D_s}}{12cm} \det(g) \tilde{\xi} \wedge dx_1 \wedge dx_2,$$

$$\tilde{\xi} = \sum_{i=1,2} \frac{c}{6m \det(g) \sqrt{2s}} \left( \frac{3}{2s} \partial_u D_s ds + \det(h) du \right) \quad (2.28)$$

$^{10}$The convention for its action is $\ast_2 dx_1 = dx_2$ and $\ast_2 dx_2 = -dx_1$. 

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The RR field strengths read
\[ F_0 = 36\sqrt{2sm^2} \partial_u (\partial_s D_u - \partial_u D_s) / \partial_s D_s, \]  
(2.29)

\[ F_2 = \frac{1}{3c} F_0 \tilde{\xi} \wedge D\psi - d \left( \ast_2 d_2 D_u + 2m \frac{\partial_u D_u}{\partial_s D_s} D\psi \right) + \left( \Delta_2 D_u - \partial_u (e^{D_s} s \det(g)) \right) dx_1 \wedge dx_2 + \ast_2 d_2 (\partial_u D_s - \partial_s D_u) \wedge ds, \]  
(2.30)

\[ F_4 = \frac{1}{3c} F_2 \wedge \tilde{\xi} \wedge D\psi - \frac{1}{36m} d \left( \sqrt{2s} e^{D_s} \det(h) dx_1 \wedge dx_2 \wedge D\psi \right) + \frac{1}{18m\sqrt{2s}} ds \wedge d(\ast_2 d_2 D_s) \wedge D\psi + \frac{\partial_u D_s}{18m\sqrt{2s}\partial_s D_s} \left[ du \wedge \left( d(\ast_2 d_2 D_s) + \frac{1}{2} e^{D_s} \det(h) dx_1 \wedge dx_2 \right) + \frac{3}{2} d(\partial_u (e^{D_s})) \wedge dx_1 \wedge dx_2 \right] \wedge D\psi, \]  
(2.31)

where \( \Delta_2 \) is the Laplace operator \( \Delta_2 = \partial_{x_1}^2 + \partial_{x_2}^2 \).

The Bianchi identity of the Romans mass \( F_0 \) sets it to a constant. The Bianchi identity of the NSNS field strength, \( dH = 0 \), yields an equation for \( D_s \):

\[ \Delta_2 D_s = \partial_s (s \det(g) e^{D_s}) + \frac{1}{24m^2\sqrt{2s}} F_0 \partial_s e^{D_s}, \]  
(2.32)

which is actually also required for supersymmetry to hold, so there can be no NSNS sources. Given the above, it follows that the Bianchi identity of \( F_2, dF_2 - F_0 H = 0 \), is equivalent to

\[ \Delta_2 (\partial_u D_u) = \partial_u^2 (s \det(g) e^{D_s}) + \frac{1}{36m^2\sqrt{2s}} F_0 s \partial_s (\det(h) e^{D_s}). \]  
(2.33)

The Bianchi identity of \( F_4 \) is automatically satisfied.

A general class of solutions contained within this framework are the AdS\(_7\) holographic duals of six-dimensional, \( \mathcal{N} = (1,0) \) theories studied in [51], compactified on a Riemann surface, giving rise to four-dimensional, \( \mathcal{N} = 1 \) SCFTs and their anti-de Sitter duals. Examples of AdS\(_5\) solutions arising from compactifications on a Riemann surface with genus \( g > 1 \) were studied in [24, 52]. The extension to punctured Riemann surfaces was studied in [49]. Another class of solutions which may be embedded in the above classification, with vanishing Romans mass, are the abelian and non-abelian T-duals of the Sasaki–Einstein solutions, though the details of the explicit embedding have not been worked out fully. Indeed consistent truncations on the non-abelian T-duals of \( S^5, T^{1,1} \) and \( Y^{p,q} \) to \( d = 5 \) minimal gauged supergravity were constructed in [27] recently and can be seen as a particular choice of background of the general formalism we present in the following sections.
2.3 Mink\textsubscript{5} vacua

In this section we shall present a sub-class of possible Mink\textsubscript{5} vacua, namely those consistent with the spinor ansatz (2.18) with \( b \neq 0 \) that can be used to embed \( d = 5 \) minimal ungauged supergravity into massive Type IIA supergravity. To our knowledge these do not appear anywhere else in the literature.

It is possible to show that when \( m = 0 \) (2.16a)-(2.16d) can be solved in terms of local coordinates \((\psi, x_1, x_2, s, u)\) and the vielbein

\[
v = e^A \left( \frac{b^2}{3} D\psi + e^{-5A+\Phi}(a_1 du - a_2 e^{2A+k} ds) \right), \quad z = -ie^{-4A+\Phi}b^{-1}(dx_1 + i dx_2),
\]

\[
w = b^{-1} \left( -a \left( \frac{e^{A}b^2}{3} D\psi - ie^{-4A+\Phi}(a_2 du + e^{2A+k} a_1 ds) \right) + e^{-4A+\Phi}b^2(du + ie^{2A+k} ds) \right),
\]

where \( e^k \) is a function of \( s \) only, and parametrises diffeomorphism invariance in this direction and

\[
a = e^{-\Phi}(c_0 p^{-\frac{5}{4}}(q - l^2)^{\frac{3}{2}} + ip^{-\frac{3}{4}}(q - l^2)^{\frac{1}{2}}), \quad b = e^{-\Phi}p^{-\frac{5}{4}}(q - l^2)^{\frac{3}{4}}, \quad e^A = p^{\frac{1}{4}}(q - l^2)^{-\frac{1}{4}},
\]

where \( c_0 \) is a constant and the constraint \( |a|^2 + b^2 = 1 \) fixes \( e^{-\Phi} \). Here \( p \) has support on \((s, x_1, x_2)\) and \((q, l)\) on \((u, x_1, x_2)\). The connection appearing in \( D\psi \) is fixed such that

\[
dV = -3du \wedge *2d2l + 3\partial_u l dx_1 \wedge dx_2 - 3c_0 e^k ds \wedge *2d2p,
\]

where for consistency with \( d^2 V = 0 \) we should have

\[
c_0 \Delta_2 p = 0, \quad (\partial_u^2 + \Delta_2)l = 0,
\]

where \( \Delta_2 \) is again flat space Laplacian on \((x_1, x_2)\). What remains non-trivial in (2.16a)-(2.16d) fixes the NSNS flux; we find

\[
H = \frac{1}{3c} \left( D\psi \wedge d\xi + d(\xi \wedge V) \right) + e^k ds \wedge du \wedge *2d2p.
\]

What remains to solve is (2.16e)-(2.16f), which simply define the RR fluxes. We shall quote them along with our summary of the class.

In summary, we find a class of Mink\textsubscript{5} vacua with NSNS sector of the form

\[
ds_{10}^2 = \sqrt{p} \left[ \frac{1}{\sqrt{\xi_1}} ds^2(\text{Mink}_5) + \sqrt{\Xi_1} \left( \frac{1}{\Xi_2} \left( \frac{1}{9q} D\psi^2 + Du^2 \right) + e^{2k}p \frac{q}{q} ds^2 + dx_1^2 + dx_2^2 \right) \right],
\]

\[
H = \frac{1}{3c} d(\xi \wedge D\psi) + e^k ds \wedge du \wedge *2d2p, \quad e^{-\Phi} = p^{-\frac{3}{4}}\Xi_1^{\frac{1}{4}} \sqrt{q^{-2}}.
\]
where we define
\[
\xi_1 \equiv (q - l^2), \quad \xi_2 \equiv 1 + \frac{p}{q_0} \xi_1, \quad \xi \equiv \frac{cp}{q} (c_0 \xi_1 du - le^k ds), \quad Du \equiv du + c_0 \frac{pl}{q} e^k ds. \tag{2.40}
\]

These backgrounds support several RR fluxes, which can be compactly expressed in terms of
\[
B = \frac{1}{3c} \tilde{\xi} \wedge D\psi - e^k u ds \wedge \star_2 d^2 p, \tag{2.41}
\]

which away from NSNS sources is such that \(dB = H\); we find
\[
F_0 = \frac{1}{p} \left( c_0 \partial_u l - e^{-k} q \partial_s (p^{-1}) \right),
\]
\[
F_2 = F_0 B - \frac{1}{3} d \left( \frac{l}{p} D\psi \right) - du \wedge \star_2 d_2 (qp^{-1}) + p^{-1} \partial_u \left( q + c_0^2 p \xi_1 \right) dx_1 \wedge dx_2 + e^k ds \wedge (F_0 u \star_2 d_2 p - c_0 \star_2 d_2 l),
\]
\[
F_4 = B \wedge F_2 - \frac{1}{2} B \wedge BF_0 - \frac{c_0}{3} d (\xi_1 D\psi) \wedge dx_1 \wedge dx_2 + \frac{1}{9} e^k ds \wedge dV \wedge D\psi
\]
\[
- \frac{1}{3} e^k ds \wedge \left[ d \left( ulp^{-1} D\psi \right) + 3 u du \wedge \star_2 d_2 (qp^{-1}) \right] \wedge \star_2 d_2 p. \tag{2.42}
\]

Away from the loci of sources the Bianchi identities of the RR and NSNS fluxes demand \(F_0\) is constant and that the following partial differential equations are solved,
\[
\Delta_2 p = 0, \quad (\partial_u^2 + \Delta_2) l = 0, \quad \Delta_2 (qp^{-1}) + \partial_u^2 (q + c_0^2 p \xi_1) = 0 \tag{2.43}
\]

which define solutions in this class. The second of these is implied by \(d^2 V = 0\), while the first is also implied by this when \(c_0 \neq 0\). We remind the reader that \(\partial_u p = \partial_u q = \partial_u l = 0\), so arranging for \(F_0 = \) constant and solving the final partial differential equation leads to branching classes of solutions, the most obvious are those for which either of the \(s\) or \(u\) directions become an isometry direction.

To our knowledge this is the first time this class of Mink_5 solutions has appeared in the literature, a detailed analysis of the solutions it contains is outside the scope of this work. We note however that constructing compact solutions is not particularly difficult: The simplest non-trivial solution is probably given by fixing
\[
c_0 = l = 0, \quad q = 1, \quad p = e^{-k} = h_8^{-1}, \tag{2.44}
\]
for \(h_8 = h_8(x)\). This reduces the class to the solution of formal D8-branes along Mink_5 \(\times T^4\) with \(h_8\) locally a linear function and \(\partial_x h_8 = F_0\). Locally this makes \(x\) span a semi-infinite interval bounded at one end by an O8–D8 system — globally however one can glue such local patches together with D8-branes in the fashion of [53] (see section 4.1 therein) thereby bounding \(x\) between two O8–D8 singularities with additional D8-branes along the interior.
3 Embedding of $d = 5$ minimal (un)gauged supergravity into massive Type IIA supergravity

In this section we will embed $d = 5$ minimal (un)gauged supergravity into massive Type IIA supergravity. The action of the bosonic part of this theory, in mostly positive metric conventions, is

$$S = \int \left[ (R^{(5)} + 12m^2) \star_5 1 - \frac{1}{6} \mathcal{F} \wedge \star_5 \mathcal{F} - \frac{1}{27} \mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F} \right],$$

where $\mathcal{F} = d\mathcal{A}$. The equations of motion following from the action are

$$R^{(5)}_{\mu\nu} = -4m^2 g^{(5)}_{\mu\nu} + \frac{1}{6} \mathcal{F}_{\mu\rho} \mathcal{F}_\nu^\rho - \frac{1}{36} g^{(5)}_{\mu\nu} \mathcal{F}_{\rho\sigma} \mathcal{F}^{\rho\sigma},$$

$$d \star_5 \mathcal{F} + \frac{1}{3} \mathcal{F} \wedge \mathcal{F} = 0,$$

while the preservation of supersymmetry requires the vanishing of the gravitino variation which implies

$$\left[ \nabla_\mu + \frac{m}{2} A_\mu - \frac{m}{2} \gamma_\mu + \frac{1}{24} \mathcal{F}_{\rho\sigma} \left( \gamma_\mu^{\rho\sigma} - 4 \delta_\mu^{\rho} \gamma_\sigma \right) \right] \zeta = 0.$$  

Notice that when $A = 0$ this reduces to the Killing spinor equation of AdS$_5$, and the equations of motion reduce to $R_{\mu\nu} = -4m^2 g_{\mu\nu}$ making AdS$_5$ the vacuum of this theory, at least for $m \neq 0$. Solutions of minimal gauged supergravity were classified in [29], and in the ungauged limit $m = 0$ in [54].

One can embed $d = 5$ minimal supergravity into $d = 10$ by again taking the spinor ansatz (2.6), with $\zeta$ now taken to obey (3.3). The $d = 10$ bi-linears decompose in the same fashion as they do in (2.9) for AdS$_5$ vacua, albeit now for generalised $d = 5$ bi-spinors $\phi^{1,2}$. As the external spinor now obeys (3.3), clearly (2.11) are no longer valid, indeed one can show these are modified to

$$\nabla_{(\mu} k_{\nu)} = 0,$$

$$d\phi_-^1 = m(\text{deg}) \phi_+^1 + \frac{2i}{3(\text{deg}!)} \phi_+^1 \wedge \mathcal{F} - \frac{1}{12} t_k \star_5 \mathcal{F}, \quad d\phi_+^1 = -\frac{i}{12} t_k \mathcal{F},$$

$$(d + im A \wedge) \phi_-^2 = 3m \phi_+^2, \quad (d + im A \wedge) \phi_-^3 = \frac{i}{3} \mathcal{F} \wedge \phi_+^2.$$  

We again define $(f, k)$ as in (2.10) (note that $f$ is no longer necessarily constant), it then follows from the differential bi-spinor relations that

$$\mathcal{L}_k \phi^1 = 0, \quad \mathcal{L}_k \phi^2 = im(3f - t_k A) \phi^2, \quad d\mathcal{F} = 0 \Rightarrow \mathcal{L}_k \mathcal{F} = 0,$$

which makes $k^\mu \partial_\mu$ a Killing vector under which $\zeta$ is charged, as it was for AdS$_5$. It then follows again that $\xi^a \partial_a$ must be a Killing vector for $\nabla_{(M} K^{(10)}_{N)} = 0$ to hold.
We seek an embedding of $d = 5$ minimal supergravity that, like the Type IIB and M-theory examples [25], does not ultimately require supersymmetry to hold. As such the $d = 10$ bosonic fields of massive Type IIA supergravity should only depend on those of the $d = 5$ theory, and not $\phi^{1,2}$ which require a Killing spinor to define. We shall assume that, like for AdS$_5$, $\xi$ does not vanish so we have a U(1) isometry which can be fibred over the $d = 5$ supergravity directions by $\mathcal{A}$. We thus take the ten-dimensional metric to be

$$ds_{10}^2 = e^{2A}g^{(5)}_{\mu\nu}dx^\mu dx^\nu + \frac{\xi^2}{||\xi||^2} + ds^2(M_4), \quad \frac{\xi}{||\xi||} = \frac{||\xi||}{3c}\mathcal{D}\psi, \quad \mathcal{D}\psi \equiv d\psi + V - \mathcal{A} , \quad (3.6)$$

where $e^{2A}$ and the dilaton have support on $M_4$ alone. Consequently we have

$$16(K^{10})^M\partial_M = e^{-2A}q_+(k^\mu\partial_\mu + \iota_k\mathcal{A}\partial_\psi) - 3e^{-2A}q_+f\partial_\psi, \quad \xi^a\partial_a = 3c\partial_\psi. \quad (3.7)$$

In addition to imposing that $\xi^a\partial_a$ is a Killing vector (2.2b) demands, for the NSNS 3-form to be independent of $\phi^{1,2}$, that

$$e^{-2A}q_+ = c, \quad q_- = 0, \quad H = H_3 + \frac{1}{3c}(\mathcal{D}\psi \wedge d\bar{\xi} + \bar{\xi} \wedge \mathcal{F}), \quad (3.8)$$

for $c > 0$ a constant and where $H_3$ is orthogonal to both $\mathcal{D}\psi$ and the external directions — notice that $d\mathcal{F} = 0$ implies that $dH$ is independent of external data. The first of (2.3), given (3.5), then furnishes us with information about the charge of the internal bi-spinors under $\partial_\psi$, namely

$$\mathcal{L}_\xi\psi^1 = 0, \quad \mathcal{L}_\xi\psi^2 = 3imc\psi^2, \quad (3.9)$$

meaning that one can locally take the only functional dependence of $\psi$ in these bi-spinors to be an $e^{im\psi}$ factor in $\psi^2$. To proceed we demand that the RR fluxes can close on the Bianchi identity and equation of motion of $\mathcal{F}$ which means it can only depend on external data through $(\mathcal{F}, \star_5\mathcal{F}, \mathcal{D}\psi)$ restricting its form to

$$F = f_+ + e^{2A}\mathcal{F} \wedge g_+ - e^{3A}\star_5\mathcal{F} \wedge \star\lambda(g_+) + e^{5A}\text{vol}_5 \wedge \star\lambda(f_+), \quad (3.10)$$

11The precise numerical factor multiplying $\mathcal{A}$ can be fixed in several ways, perhaps the quickest is consistency with the fact that $(K^{(10)})^M\partial_M$ should be a Killing vector under the assumption that $\partial_\psi$ is itself Killing and given that $k^\mu\partial_\mu$ is Killing.

12The $\mathcal{D}\psi$ terms are contained implicitly in $(g_+, f_+)$. One might think of including all the combinations one can construct out of $(\mathcal{A}, \mathcal{F})$, utilising hodge duals and wedge products, however there are no necessary external conditions which these close on. For instance $\mathcal{F} \wedge \mathcal{F}$ at first sight may appear reasonable to include, but the self-duality constraint $F$ must obey means this must come with $\star_5(\mathcal{F} \wedge \mathcal{F})$, which need obey no special identity under $d$. 

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where \((f_+, g_+)\) are defined on \((\mathcal{D}\psi, \mathcal{M}_4)\) and are to be determined. We are now ready to reduce \((2.2a)\) to conditions on the internal space. To deal with the fact that \(\mathcal{D}\psi\) also contains the external potential \(\mathcal{A}\), one can decompose all the objects defined on \(\mathcal{M}_5\) into their parts defined along the base and fibre directions, i.e.

\[
\psi^1 = \psi^{1B} + \mathcal{D}\psi \wedge \psi^{1F}, \quad \psi^2 = e^{i\nu\psi}(\psi^{2B} + \mathcal{D}\psi \wedge \psi^{2F}),
\]

and so on. One then again substitutes for \(\Psi^{(10)}\) in \((2.2a)\), this time making use of \((3.4b)-(3.4c)\) and attempts to factor out the external data. Since we want to embed all solutions of \(d = 5\) minimal supergravity in a common framework, there are no identities we can assume that the wedge products of the external fields and bi-spinors obey, i.e. we must take \((\mathcal{F} \wedge \phi^{1})_n\) to be independent of \(\phi^{1}_{n+2}\) and so on. However there are important identities that the internal bi-spinors obey, namely

\[
(t\xi + \tilde{\xi} \wedge)\psi^{1,2}_+ = e^{A}\psi^{1,2}_-, \quad (t\xi + \tilde{\xi} \wedge)\psi^{1,2}_- = 0,
\]

which can in turn be decomposed into conditions on base and fiber directions as in \((3.11)\). Putting this all together, after a lengthy computation, we find that \((2.2a)\) reduces to a number of conditions on the internal space we can most succinctly express as

\[
e^{2A}g_+ = -\frac{4}{3c}e^{-\Phi}\text{Im}\psi^1_+,
\]

\[
d_H(e^{2A-\Phi}\psi^2_+)\bigg|_{\mathcal{A}=0} = 0, \quad d_H(e^{3A-\Phi}\psi^2_-) - 3me^{2A-\Phi}\psi^2_+\bigg|_{\mathcal{A}=0} = 0,
\]

\[
d_H(e^{3A-\Phi}\text{Re}\psi^1_+)\bigg|_{\mathcal{A}=0} = 0, \quad d_H(e^{A-\Phi}\text{Im}\psi^1_+\bigg|_{\mathcal{A}=0} = 0,
\]

\[
d_H(e^{2A-\Phi}\text{Re}\psi^1_-) + 2me^{A-\Phi}\text{Im}\psi^1_-\bigg|_{\mathcal{A}=0} = 0,
\]

\[
d_H(e^{4A-\Phi}\text{Im}\psi^1_+) - 4me^{3A-\Phi}\text{Re}\psi^1_- - \frac{c}{4}e^{5A} \star \lambda(f_+)\bigg|_{\mathcal{A}=0} = 0,
\]

\[
d_H(-\Phi\text{Im}\psi^1_+) + \frac{1}{4}(t\xi + \tilde{\xi} \wedge)f_+\bigg|_{\mathcal{A}=0} = 0, \quad d_H(e^{5A-\Phi}\text{Im}\psi^1_-) + \frac{1}{4}e^{5A}(t\xi + \tilde{\xi} \wedge) \star \lambda(f_+)\bigg|_{\mathcal{A}=0} = 0,
\]

where we prove these conditions are indeed necessary and sufficient for supersymmetry in appendix C.1. We have now reproduced \((2.16a)-(2.16f)\) and therefore the internal space of the embedding of \(d = 5\) minimal gauged supergravity is mostly the same as it is for the AdS\(_5\) vacua, with the only

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\(^{13}\)These follow from the necessary \(d = 10\) condition \((t_K^{(10)} + \tilde{K}^{(10)} \wedge)\Psi^{(10)} = 0\) given \((3.8)\) and the identities involving \(\phi^{1,2}\) in appendix B.
modifications happening in the fluxes and U(1) fiber in terms of $(\mathcal{A}, \mathcal{F})$. In particular, away from the loci of sources, we necessarily have that

$$dH\bigg|_{\mathcal{A}=0} = 0, \quad d_HF\bigg|_{\mathcal{A}=0} = 0,$$

as the left-hand side of these expressions reduce to their AdS$_5$ vacua values.\footnote{Strictly speaking the form of vol$_5$ depends on the specific solution of $d = 5$ minimal supergravity, but the parts of $d_HF$ parallel and orthogonal to this vanish independently, so this subtlety is immaterial.} It is then a simple matter to show that the NSNS Bianchi identity is implied by $d\mathcal{F} = 0$ and with a little more effort that

$$d_HF = d_HF\bigg|_{\mathcal{A}=0}$$

is implied by (3.13a)-(3.13f), the identities (3.12) and the Bianchi identity and equation of motion of $\mathcal{F}$. Therefore the Bianchi identities of the fluxes are implied by the AdS$_5$ result. We prove in appendix C.2 that all the equations of motion of Type IIA supergravity are implied by what we present in this section, irrespective of whether the solution on the external space is supersymmetric or not.

In summary the embedding of $d = 5$ minimal supergravity into massive Type IIA supergravity is given by

$$ds^2_{10} = e^{2\mathcal{A}}g^{(5)}_{\mu\nu}dx^\mu dx^\nu + \frac{||\xi||^2}{9c^2}D\psi^2 + ds^2(M_4), \quad D\psi \equiv d\psi + V - \mathcal{A}$$

$$H = H_3 + \frac{1}{3c}(D\psi \wedge d\tilde{\xi} + \tilde{\xi} \wedge \mathcal{F}),$$

$$F = f_+ + e^{5\mathcal{A}}\text{vol}_5 \wedge \star\lambda(f_+) - \frac{4}{3c}\left[\mathcal{F} \wedge (e^{-\Phi}\text{Im}\psi_1^-) - \star_5\mathcal{F} \wedge (e^{\mathcal{A}-\Phi}\text{Im}\psi_1^-)\right].$$

(3.16)

More specifically $d = 5$ minimal gauged supergravity can be embedded into massive Type IIA supergravity by making the following substitutions\footnote{Note that $D\psi \rightarrow D\psi = D\psi - \mathcal{A}$ must be substituted before evaluating $dD\psi$ where it appears.} in the AdS$_5$ vacua of (2.23a)-(2.31)

$$ds^2(\text{AdS}_5) \rightarrow g^{(5)}_{\mu\nu}dx^\mu dx^\nu, \quad \text{vol(AdS}_5) \rightarrow \text{vol}_5,$$

$$D\psi \rightarrow D\psi = D\psi - \mathcal{A}, \quad F_4 \rightarrow F_4 - \frac{1}{3\sqrt{2}s}\left(-\frac{1}{3}\mathcal{F} \wedge D\psi + \star_5\mathcal{F}\right) \wedge ds.$$  

(3.17)
The ungauged limit, $m = 0$, on the other hand can be embedded into massive Type IIA supergravity by making the following substitutions in the Mink$_5$ vacua of (2.39)- (2.42)

$$ds^2(\text{Mink}_5) \rightarrow g^{(5)}_{\mu\nu} dx^\mu dx^\nu, \quad \text{vol(\text{Mink}_5)} \rightarrow \text{vol}_5,$$

$$D\psi \rightarrow D\psi = D\psi - A, \quad F_4 \rightarrow F_4 + \frac{e^k}{3} \left( -\frac{1}{3} F \wedge D\psi + *_5 F \right) \wedge ds. \quad (3.18)$$

## 4 Uplift examples

In this section we will use the consistent truncation constructed in the previous section to uplift two families of solutions of $d = 5$ minimal gauged supergravity to massive Type IIA supergravity. As seed AdS$_5$ solution on which to perform the uplift we take the infinite family of solutions constructed in [49] (BPT). Another interesting class of solutions which we could use as our seed solutions are the ones found in [24]. The field theory duals of these solutions, along with the interpolating flow between the AdS$_7$ and AdS$_5$ vacua have recently been studied in [55]; see also [56]. One could then use these solutions to construct supergravity solutions dual to the compactification of a six-dimensional, $\mathcal{N} = (1, 0)$ theory down to, for example, a two-dimensional SCFT, in a two stage process. This should, in theory, allow for greater control in understanding the two-dimensional SCFT through this chain of reductions rather than performing a direct reduction on a four-dimensional space. It would thus be interesting to further compactify the quiver theories studied there, the holographic duals of which will be accessible by using our truncation. We begin by reviewing the seed AdS$_5$ solutions, before rewriting them in a form consistent with the uplift formula presented in section 3. We then review the two classes of solutions of $d = 5$ minimal gauged supergravity that we will uplift, before studying some basic properties of the uplifted solutions.

### 4.1 Reduction on solutions of BPT

In this section we will review the solutions found in [49] before constructing solutions using the truncation discussed above.$^{16}$ The solutions of [49] have the broad interpretation of placing a six-dimensional, $\mathcal{N} = (1, 0)$ theory on a constant curvature Riemann surface of genus $g$, though different completions allow for different interesting physics. The metric takes the form

$$ds^2_{10} = e^{2A} \left[ ds^2(\text{AdS}_5) - \frac{p'(z)}{9z^2} ds^2(X_5) \right], \quad (4.1)$$

$$ds^2(X_5) = ds^2(\Sigma_g) + \frac{3zdz^2}{p(z)} + \frac{9z^3}{3p(z) - zp'(z)} \left[ \frac{k}{1 - k^3} dk^2 + \frac{4(1 - k^3)p(z)}{3(3p(z) - zp'(z)(1 - k^3))} D\psi^2 \right], \quad (4.2)$$

$^{16}$We will set $c = 1$ and $m = 1$ in this section to avoid cluttering the equations.
where \( p(z) \) is the cubic function
\[
p(z) = (z - z_0)(\kappa(z^2 + z_0z + z_0^2) - 3\ell z_1^2), \quad \Rightarrow \quad p'(z) = 3(\kappa z^2 - \ell z_1^2),
\]
and
\[
D\psi \equiv d\psi - A_g, \quad dA_g = \text{vol}(\Sigma_g).
\]
The parameters \( \kappa, z_0, z_1 \) and \( \ell \) are all real, with \( \kappa = 0, \pm 1 \) and \( \ell \) constrained to be \( \ell = \pm 1 \). Without loss of generality one can restrict to \( z_1 > 0 \). The warp factor is fixed to be
\[
e^{4A} = \frac{z(3p(z) - zp'(z)(1 - k^3))}{-p'(z)k}.
\]
The metric has the correct signature provided
\[
zp(z) \geq 0, \quad -p'(z) \geq 0, \quad 0 \leq k \leq 1.
\]
Since the solution is invariant under the simultaneous reflection \( z \to -z, \; z_0 \to -z_0 \) we can further restrict to \( z \geq 0 \). The dilaton is given by
\[
e^{4\Phi} = \frac{1}{F_0^3} - \frac{(3p(z) - zp'(z)(1 - k^3))^3}{-p'(z)(3p(z) - zp'(z))^2z^3k^5},
\]
while the fluxes may be succinctly written in terms of the potentials
\[
B = -\frac{2}{3} \frac{z^2p'(z)}{3p(z) - zp'(z)} dk \wedge D\psi - \frac{kpp'(z) - zp''(z)}{9} \text{vol}(\Sigma_g),
\]
\[
C_1 = \frac{2F_0}{3} \frac{kz^2p'(z)(1 - k^3)}{3p(z) - zp'(z)(1 - k^3)} D\psi,
\]
\[
C_3 = \frac{2F_0}{9} k^2 \left[ \frac{p'(z) - zp''(z)}{3p(z) - zp'(z)(1 - k^3)} p(z) + \frac{zp''(z)}{6} \right] D\psi \wedge \text{vol}(\Sigma_g),
\]
where
\[
H = dB, \quad F_2 = dC_1 + F_0 B, \quad F_4 = dC_3 + B \wedge F_2 - \frac{1}{2} F_0 B \wedge B.
\]
The above local solution has many different global completions depending on how the space is ended, see [49]. Different degenerations of the space lead to the inclusion of different brane sources and thus different physics. We will review some of the possible degenerations briefly but refer to [49] for further details. All the solutions we will present here allow for any of the global completions studied in [49] however given the plethora of solutions we will focus on a single example for exposition.

There are various points where either the metric is singular or the circle parametrised by the coordinate \( \psi \) shrinks. First consider where the \( S^1 \) shrinks at either \( p(z) = 0 \) or \( k = 1 \). For \( z_1 \) a
single root of $p(z)$ the circle shrinks smoothly provided $\psi$ has period $2\pi$. Similarly at $k = 1$ the circle shrinks smoothly if $\psi$ has period $2\pi$. Despite the two limits being separately smooth, the double limit is singular and corresponds to the presence of D6-branes with worldvolume $\text{AdS}_5 \times \Sigma_g$. The metric is singular at the three points $k = 0$, $z = 0$ and $p'(z) = 0$. At $k = 0$ and away from the two other degenerations the metric degenerates due to the presence of a stack of D8-branes on top of an O8-plane. There are smeared D4-branes located at $z = 0$ and $p'(z) = 0$. For the former the D4-branes are smeared along the Riemann surface, while for the latter the D4-branes are smeared along both the Riemann surface and the $S^2$. Note that $p'(z_1) = 0$ provided $\kappa = \ell \neq 0$.

We may now fix the range of the coordinates. From the above we see that the coordinate ranges of $\psi$ and $k$ are uniquely fixed: $\psi$ has $2\pi$ period while $k \in [0, 1]$. For the $z$ coordinate there are a larger number of options to take.

For $\kappa = 0$ or $\kappa = -\ell$, $p(z)$ only admits one real root at $z_0$, moreover $p'(z)$ has no real roots. Positivity of $p'(z)$ implies $\kappa = -\ell = -1$ and the $z$ coordinate is fixed between $z \in [0, z_0]$. There are smeared D4-branes at $z = 0$ and a shrinking circle at $z = z_0$.

Instead for $\kappa = \ell$, $p'(z)$ has roots at $z = \pm z_1$ and $p(z)$ can have three real roots. Writing

$$p(z) = \kappa(z - z_0)(z - z_-)(z - z_+),$$

where the roots satisfy

$$z_0 + z_- + z_+ = 0, \quad 3z_1^2 = -z_0 z_- - z_- z_+ - z_+ z_0.$$  \hspace{1cm} (4.13)

We take $z_0$ to be real without loss of generality and then the other two roots are real if $z_0^2 \leq 4z_1^2$. When the roots $z_\pm$ are complex the positivity conditions for the metric to be well-defined require

$$z \in \begin{cases} [0, z_1] \quad \text{for} \quad \kappa = 1, \quad z_0 < -2z_1, \\ [z_1, z_0] \quad \text{for} \quad \kappa = -1, \quad z_0 > 2z_1. \end{cases}$$  \hspace{1cm} (4.14)

For all three roots being real one finds that at least one root is always negative and at least one is always positive. The ranges are then

$$z \in \begin{cases} [0, z_0] \quad \text{for} \quad \kappa = 1, \quad 0 < z_0 \leq z_1, \\ [z_1, z_0] \quad \text{for} \quad \kappa = -1, \quad z_1 < z_0 \leq 2z_1. \end{cases}$$  \hspace{1cm} (4.15)

Having given the broad outline of the solutions we are able to present the truncation on this family of solutions. Following the truncation ansatz derived in section 3 the metric is

$$ds_{10}^2 = e^{2A} \left[ g^{(5)}_{\mu\nu} dx^\mu dx^\nu - \frac{p'(z)}{9z^2} \left( ds^2(\Sigma_g) + \frac{3zd^2}{p(z)} \right) + \frac{9z^3}{3p(z) - zp'(z)} \left\{ \frac{k}{1 - k^3} dk^2 + \frac{4(1 - k^3)p(z)}{3(p(z) - zp'(z)(1 - k^3))} (D\psi - A)^2 \right\} \right].$$  \hspace{1cm} (4.16)
To compute the modification of the fluxes we must put them into the form used in (2.28) and (2.29)-(2.31). First we should identify the 1-form $\tilde{\xi}$ given in (2.28),

$$\tilde{\xi} = kdz + \frac{z(3p(z) + zp'(z))}{3p(z) - zp'(z)}dk.$$  \hspace{1cm} (4.17)

From the replacement rule in (3.16) it follows that the NSNS 3-form is

$$H = d\left(-\frac{k p'(z) - zp''(z)}{9} \right)_{\text{vol}(\Sigma_g)} + 2 \frac{z^2p'(z)}{3p(z) - zp'(z)}dk \wedge \text{vol}(\Sigma_g) + \frac{1}{3} \left( D\psi \wedge d\tilde{\xi} + \tilde{\xi} \wedge F \right),$$  \hspace{1cm} (4.18)

where $D\psi = d\psi - A_g - \mathcal{A}$. The potential may be written as

$$B = \frac{1}{3} \left( \tilde{\xi} \wedge D\psi + \frac{k(3z^2\kappa - p'(z))}{3z} \right) \text{vol}(\Sigma_g),$$  \hspace{1cm} (4.19)

where the gauging is done through $D\psi$ term and reproduces (4.18). The 2-form $F_2$ becomes

$$F_2 = \frac{1}{3} F_0 \left( \tilde{\xi} \wedge D\psi + \frac{k(3z^2\kappa - p'(z))}{3z} \right) \text{vol}(\Sigma_g) - d \left( F_0 k z (3p(z) + (1 - k^3)z p'(z))_{\text{vol}(\Sigma_g)} \right),$$  \hspace{1cm} (4.20)

where we may identify the first bracketed part as $F_0B$ using the gauge given in (4.19). Finally the 4-form flux is given by

$$F_4 = \frac{1}{3} F_2 \wedge \tilde{\xi} \wedge D\psi + \frac{F_0 k z}{18} d(kz) \wedge D\psi \wedge \text{vol}(\Sigma_g) - \frac{F_0}{36} d \left( k^2 p'(z) D\psi \right) \wedge \text{vol}(\Sigma_g) +$$

$$+ \frac{F_0 k z}{3} d(kz) \wedge \left( *_5 F - \frac{1}{3} F \wedge D\psi \right) - F_0 \left( 3z^2 + 4p'(z) \right) \left( k^2 z dk + (k^3 - 2) dz \right) \wedge \text{vol}(\Sigma_g) \wedge D\psi.$$  \hspace{1cm} (4.21)

### 4.2 Solutions of $d = 5$ minimal gauged supergravity

Let us now give some explicit supersymmetric solutions to $d = 5$ minimal supergravity which may be uplifted to massive Type IIA supergravity using the results of the previous section.

The first example that we will consider is a local solution, which possesses a number of different global completions. Our focus will be on two different global completions of this local solution. The first is a spindle while the second is constant curvature hyperbolic space. The local solution giving rise to both of these solutions is\(^{17}\)

$$g_{\mu\nu} dx^\mu dx^\nu = P(y)^{1/3} \left( ds^2(\text{AdS}_3) + \frac{1}{4q(y)} dy^2 + \frac{q(y)}{P(y)} d\phi^2 \right),$$

$$\mathcal{A} = \frac{3y}{y - a} d\phi, \quad P(y) = (y - a)^3, \quad q(y) = P(y) - y^2.$$  \hspace{1cm} (4.22)

\(^{17}\)This solution was first considered as a spindle in [34] in U(1)\(^3\) gauged supergravity with the solution here obtained by taking the minimal limit ($A^1 = A^2 = A^3 = \mathcal{A}$). The resultant solution is then a coordinate transformation (and rescaling owing to different conventions for the action) away from the solution in [31].
**Spindle solution**  Let us first consider the spindle, we will try to be as brief as possible since many of the details are by now well studied. For the space to be compact we require that the polynomial \( q(y) \) has three real roots. It follows that this requires

\[
-\frac{4}{27} \leq a \leq 0. \tag{4.23}
\]

Note that the end-points of the interval are special. For \( a = 0 \) the space is actually AdS\(_5\) written with an AdS\(_3\) slicing, while for \( a = -\frac{4}{27} \) \( q(y) \) has a double root at \( y = \frac{8}{27} \). We will come back to this latter special point later.

We should then fix \( a \) to be strictly within this domain which implies that there are three single roots. It follows that two roots are necessarily positive while the third is necessarily negative. Let us denote the roots as \( y^-, y^+, y^* \), with \( y^- < 0 < y^+ < y^* \). Then we must bound the \( y \) coordinate as \( y \in [y^-, y^+] \). At either end-point the space develops a conical deficit angle \( 2\pi(1 - n_+^{-1}) \) giving rise to the orbifold \( \Sigma = \mathbb{CP}^1_{[n_-, n_+]} \). The Euler characteristic of the space and magnetic charge of the solution are

\[
\chi(\Sigma) = \frac{1}{n_+} + \frac{1}{n_-}, \quad Q = \frac{1}{2\pi} \int_\Sigma F = \frac{1}{n_-} - \frac{1}{n_+}, \tag{4.24}
\]

and thus exhibits an anti-twist, see [40]. Given the form of the roots we must take \( n_+ > n_- > 0 \) and additionally require them to be relatively prime. In terms of the orbifold weights the period \( \Delta \phi \), parameter \( a \) and roots take the form:\(^{18}\)

\[
\frac{\Delta \phi}{2\pi} = \frac{n_+^2 + n_+ n_- + n_-^2}{3n_+ n_- (n_+ + n_-)}, \quad a = -\frac{(n_+ - n_-)^2 (2n_+ + n_-)^2 (2n_- + n_+)^2}{27(n_+^2 + 2n_+ n_- + n_-^2)^3}, \tag{4.25}
\]

\[
y_+ = \frac{(n_+ - n_-)^3 (2n_+ + n_-)^3}{27(n_+^2 + 2n_+ n_- + n_-^2)^3}, \quad y_- = -\frac{(n_+ - n_-)^3 (2n_- + n_+)^3}{27(n_+^2 + 2n_+ n_- + n_-^2)^3}. \tag{4.26}
\]

**Hyperbolic space**  We noted earlier that \( a = -\frac{4}{27} \) is a special point where the function \( q(y) \) develops a non-trivial double root. As we will show, by taking a certain scaling limit to this point we obtain the metric on a constant curvature hyperbolic disc. To wit, set \( a = -\frac{4}{27} \) and define

\[
y = \frac{8}{27} + \epsilon Y, \quad \phi = \frac{4\chi}{9\epsilon}. \tag{4.27}
\]

Expanding the metric around \( \epsilon = 0 \) we find

\[
g^{(5)}_{\mu\nu} dx^\mu dx^\nu = \frac{4}{9} \left[ ds^2(\text{AdS}_3) + \frac{3}{4} \left( \frac{dY^2}{Y^2} + Y^2 d\chi^2 \right) \right], \tag{4.28}
\]

\(^{18}\)To work this out it is simplest to solve for the roots \( y_\pm \) and \( a \) in terms of the third root and then to solve the period constraint.
which is the direct product of AdS$_3$ and $\mathbb{H}^2$. The gauge field works similarly, though one must add in a pure gauge term

$$A \rightarrow A - \frac{8}{9\epsilon} d\chi \xrightarrow{\epsilon \rightarrow 0} Y d\chi.$$  \hspace{1cm} (4.29)

The normalisation of the metric and gauge field implies

$$2(1 - g) = \chi(\mathbb{H}^2) = \frac{1}{4\pi} \int_{\mathbb{H}^2} R \text{vol}(\mathbb{H}^2) = -\frac{1}{2\pi} \text{Vol}(\mathbb{H}^2) = -\frac{1}{2\pi} \int_{\mathbb{H}^2} F.$$  \hspace{1cm} (4.30)

We therefore see that the magnetic charge perfectly cancels the Euler characteristic and supersymmetry is preserved via a topological twist.

**Gutowski–Reall black hole**  The second class of solution that we may uplift is the Gutowski–Reall black hole solution [30]. This is an asymptotically AdS$_5$ rotating black hole of $d = 5$ minimal gauged supergravity,\(^\text{19}\)

$$g^{(5)}_{\mu \nu} dx^\mu dx^\nu = -\frac{u}{\Lambda} dt^2 + \frac{dr^2}{u} + \frac{r^2}{4} \left( L_1^2 + L_2^2 + \Lambda (L_3 - \Omega dt)^2 \right),$$

$$A = -3 \left( \left( 1 - \frac{r_0^2}{r^2} - \frac{m^2 r_0^4}{2 r^2} \right) dt + \epsilon \frac{m r_0^4}{4 r^2} L_3 \right), \quad dL_i = \frac{1}{2} \epsilon_{ijk} L_j \wedge L_k,$$

$$u = \left( 1 - \frac{r_0^2}{r^2} \right)^2 \left( 1 + m^2(r^2 + 2 r_0^2) \right), \quad \Lambda = 1 + m^2 \left( \frac{r_0^6}{r^8} - \frac{r_0^8}{4 r^6} \right),$$

$$\Omega = 2 m \epsilon \frac{r_0^4}{\Lambda} \left[ \frac{3}{2} + r_0^2 m^2 \right] \frac{r_0^4}{r^4} - \left( \frac{1}{2} + \frac{m^2 r_0^2}{4} \right) \frac{r_0^6}{r^6}, \quad \epsilon^2 = 1.$$  \hspace{1cm} (4.31)

We may now simply insert the solutions outlined here into the uplift worked out in the previous section 4.1 to obtain new solutions of massive Type IIA supergravity. It is interesting to understand the form of the solutions that we obtain. Recall that the seed solution on which we performed the truncation can be interpreted as the holographic duals of the IR limit of wrapping one of the six-dimensional, $\mathcal{N} = (1, 0)$ theories studied in [57] on a constant curvature Riemann surface. For the case of the solutions of $d = 5$ minimal gauged supergravity on a spindle and hyperbolic space we may interpret the uplifted solutions as the holographic duals of two-dimensional SCFTs obtained by compactifying the six-dimensional, $\mathcal{N} = (1, 0)$ theory on the four-manifold consisting of the direct product of the seed Riemann surface with either a spindle or two-dimensional hyperbolic space, see for example [42–45] for similar setups in different theories. A similar interpretation for the uplift of the Gutowski–Reall solution is somewhat more subtle. It would be interesting to understand the thermodynamics of the black hole in this uplift and to identify the microstates of the black hole.

\(^{19}\)To embed this into the class of solutions in section 4.1 one must fix $m = 1$ below.
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A  Conventions

In this appendix we spell out our conventions. First off, we shall use Roman and Greek letters (the latter reserved for the external space) to indicate curved indices and underline them to indicate flat indices. We follow the conventions of [48] with the hodge dual defined as

$$\star e^{M_1...M_k} = \frac{1}{(d-k)!} \epsilon^{M_{k+1}...M_{d-k}} e^{M_{k+1}...M_{d-k}},$$

(A.1)

The self-duality relation for the $d=10$ polyform flux is

$$F = \star \lambda(F),$$

(A.2)

where $\lambda(C_k) = (-1)^{\lfloor k/2 \rfloor} C_k$, for a $k$-form $C_k$. We define the matrix

$$\hat{\gamma} = \eta \gamma^{1...d} = \eta(-1)^t \gamma^{1...d}$$

(A.3)

in all dimensions and signatures, where $t$ is the number of time-like directions, such that

$$\eta^2 = (-1)^t (-1)^{\lfloor d/2 \rfloor}. $$

(A.4)

In odd dimensions we shall fix

$$\hat{\gamma} = I$$

(A.5)

while in even dimensions $\hat{\gamma}$ is the chirality matrix. We assume, as [48] does, that in ten dimensions

$$\eta^{(10)} = 1.$$ 

(A.6)

Through the Clifford map we have the following action on forms

$$\hat{\gamma} C_k = \eta \star \lambda(C_k), \quad C_k \hat{\gamma} = \eta(-)^{\lfloor d/2 \rfloor} \lambda(\star C_k).$$

(A.7)
In $d = 10$ we also define
\[ \tau = (\Gamma_0 \epsilon) = \epsilon^\dagger \Gamma^0, \] (A.8)
and make use of some shorthand notation
\[ C_M = t_{dx^M} C, \quad C^2 = \sum_k \frac{1}{k!} (C_k)_{M_1...M_k} (C_k)^{M_1...M_k}, \]
\[ C^2_{MN} = \sum_k \frac{1}{(k-1)!} (C_k)_{MM_1...M_{k-1}} (C_k)^{M_1...M_{k-1}}. \] (A.9)

Note also
\[ \sqrt{|g|} \frac{1}{k!} (C_k)_{M_1...M_k} (C_k)^{M_1...M_k} = \star C_k \wedge C_k. \] (A.10)

We shall be interested in a split of the gamma matrices into $10 = 5 + 5$, as such we shall parameterise them as
\[ \Gamma_\mu = \sigma_3 \otimes \gamma_\mu \otimes \mathbb{I}, \quad \Gamma_a = \sigma_1 \otimes \mathbb{I} \otimes \gamma_a, \] (A.11)
where we have split the $d = 10$ index $M = (\mu, a)$ for $\mu$ an index on a $d = 5$ Lorentzian space and $a$ an index on a $d = 5$ Euclidean space. We work in conventions where $\gamma_{12345} = 1$, as such
\[ \hat{\Gamma} = \sigma_2 \otimes \mathbb{I} \otimes \mathbb{I}. \] (A.12)

The intertwiner defining $d = 10$ Majorana conjugation (m.c.) as $\epsilon^c = B^{(10)} \epsilon^*$ then decomposes in terms of corresponding intertwiners $\hat{B}$ on the external and $B$ on the internal space as
\[ B^{(10)} = \sigma_1 \otimes \hat{B} \otimes B, \] (A.13)
where
\[ \bar{B}^{-1} \gamma^*_\mu \bar{B} = -\gamma^*_\mu, \quad \bar{B} \bar{B}^* = -\mathbb{I}, \quad \bar{B}^\dagger = \bar{B}. \] (A.14)
and
\[ B^{-1} \gamma^*_a B = \gamma^*_a, \quad B B^* = -\mathbb{I}, \quad B^\dagger = B, \] (A.15)

Finally we define the spin covariant derivative as
\[ \nabla_M = \partial_M + \frac{1}{4} \omega_M^{PQ} \Gamma_{PQ}, \quad de^M + \omega^M_N \wedge e^N = 0, \] (A.16)
and spinorial Lie derivative as
\[ \mathcal{L}_K \epsilon = K^M \nabla_M \epsilon + \frac{1}{4} \nabla_M K_N \Gamma^{MN} \epsilon. \] (A.17)
B Some details of $d = 5$ Lorentzian bi-linears

In this appendix we provide some details of the $d = 5$ Lorentzian bi-linears we refer to in the main text.

In terms of a generic Dirac spinor in $d = 5$ Lorentzian space, $\zeta$, we define the following bi-spinors

$$\phi^1 = \zeta \otimes \zeta, \quad \phi^2 = \zeta \otimes \zeta^c,$$

for which (given (A.14)) all of $\phi^1$ but only $(\phi^2)_{2,3}$ are non-trivial and

$$\phi^{1,2} = i \star_5 \lambda (\phi^{1,2}).$$

This suggests defining

$$if = \bar{\zeta} \zeta, \quad k_\mu = \bar{\zeta} \gamma_\mu \zeta, \quad X_{\mu\nu} = \bar{\zeta} \gamma_{\mu\nu} \zeta, \quad Y_{\mu\nu} = \bar{\zeta}^c \gamma_{\mu\nu} \zeta,$$

so that

$$\phi^1 = \frac{1}{4}(1 + i \star_5 \lambda)(if + k - X) = \frac{f}{4} \left( i + \frac{k}{j} \right) \wedge e^{i \frac{\zeta}{j}}, \quad \phi^2 = -\frac{1}{4}(1 + i \star_5 \lambda)Y = \frac{f}{4} \left( 1 - i \frac{k}{j} \right) \wedge \frac{Y}{j},$$

where $(f, k, X)$ are real and $Y$ complex and we do not attempt to refine things further as $f$ is not necessarily non-vanishing. Note also that

$$k \zeta = if \zeta, \quad X \zeta = -2if \zeta, \quad \tau_k k = -f^2, \quad \tau_k \phi^{1,2} = if \phi^{1,2}, \quad k \wedge \phi^{1,2} = if \phi^{1,2}$$

and that

$$Y \wedge X = 0, \quad X \wedge X = \frac{1}{2} Y \wedge Y, \quad \tau_k X = \tau_k Y = 0.$$

Note that we have not used (3.3) to derive any of these conditions, they are completely general.

C Proving sufficiency of the embedding

In appendix C.1 we prove that the embedding of $d = 5$ minimal supergravity, presented in the main text, indeed preserves $d = 10$ supersymmetry, when the background on the external space preserves $d = 5$ supersymmetry. Furthermore, in appendix C.2, we prove that the embedding gives a solution to the $d = 10$ equations of motion, even when the solution of the $d = 5$ supergravity is not supersymmetric. For clarity’s sake, let us stress that through out this appendix we are using the term AdS$_5$ vacua loosely. We include also the limit where the inverse radius $m = 0$, i.e. the Mink$_5$ vacua limit, where it is ungauged supergravity that is being embedded in ten dimensions.
C.1 Sufficiency for supersymmetry

In this appendix we prove that our embedding of $d = 5$ minimal gauged supergravity into massive Type IIA supergravity preserves supersymmetry in $d = 10$ provided that it preserves supersymmetry in $d = 5$. We find it easiest to do this in terms of the necessary spinorial conditions.

A solution of Type IIA supergravity preserves supersymmetry if it supports two Majorana–Weyl Killing spinors such that the gravitino and dilatino variations, respectively
\[\delta\psi^1_M = \left(\nabla^{(10)}_M - \frac{1}{4} H_M\right)\epsilon_1 + \frac{e^\Phi}{16} F\Gamma_M\epsilon_2,\]
\[\delta\psi^2_M = \left(\nabla^{(10)}_M + \frac{1}{4} H_M\right)\epsilon_2 + \frac{e^\Phi}{16} \lambda(F)\Gamma_M\epsilon_1,\]
\[\delta\lambda^1 = \left(- \frac{1}{2} H + d\Phi\right)\epsilon_1 + \frac{e^\Phi}{16} \Gamma^M F\Gamma_M\epsilon_2,\]
\[\delta\lambda^2 = \left(\frac{1}{2} H + d\Phi\right)\epsilon_2 + \frac{e^\Phi}{16} \Gamma^M \lambda(F)\Gamma_M\epsilon_1,\]

all vanish. For the case at hand the fields $(F, H, \Phi)$ and metric are defined in (3.16), the flat space gamma matrices in the preceding appendix and the Killing spinors are as in (2.6) with $\zeta$ obeying (3.3).

Let us begin by considering the vanishing of the gravitino variations, which requires us to decompose the covariant derivative and curved space gamma matrices on a space of the form
\[ds_{10}^2 = e^{2A}g^{(5)}_{\mu\nu}dx^\mu dx^\nu + ds^2(M_4) + e^{2C}D\psi^2, \quad D\psi \equiv d\psi + V - A.\] (C.2)

One can show that the spin covariant derivative on this space decomposes as
\[\nabla^{(10)}_{\mu} = \nabla_{\mu} - A_{\mu}(\nabla_{\psi} - \partial_{\psi}) + \frac{1}{2}(\Gamma_{\mu} + A_{\mu}\Gamma_\psi)\partial A + \frac{1}{4}\Gamma_\psi F_{\mu},\]
\[\nabla^{(10)}_i = \nabla_i + V_i(\nabla_{\psi} - \partial_{\psi}) - \frac{1}{4}\Gamma_\psi (dV)_i,\]
\[\nabla^{(10)}_{\psi} = \nabla_{\psi} + \frac{1}{4}e^{2C}F, \quad \nabla_{\psi} = \partial_{\psi} + \frac{1}{2}\Gamma_\psi \partial C - e^{2C}\frac{1}{4}dV\] (C.3)
where we have further split the internal index as $a = (\psi, i)$. The gamma matrices likewise decompose as
\[\Gamma_{\mu} = e^A\sigma_3 \otimes \gamma_\mu \otimes \mathbb{1} - A_{\mu}\sigma_1 \otimes \mathbb{1} \otimes \gamma_\psi, \quad \Gamma_i = \sigma_1 \otimes \mathbb{1} \otimes (\gamma_i + V_i\gamma_\psi), \quad \gamma_\psi = e^C\gamma_\psi.\] (C.4)

To proceed we observe that (A.2) and (A.7) together imply that
\[\hat{\Gamma}F = F, \quad \hat{\Gamma}\lambda(F) = -\lambda(F),\] (C.5)
allowing us to simplify the RR flux terms in the gravitino variations a little, for instance

\[ F\Gamma_M\epsilon_2 = (1 + \hat{\Gamma})\left(f_+ - \frac{4}{3c} e^{-\Phi} F \wedge \text{Im}\psi_1^1\right)\Gamma_M\epsilon_2 = 2\left(f_+ - \frac{4}{3c} e^{-\Phi} F \wedge \text{Im}\psi_1^1\right)\Gamma_M\epsilon_2. \]  

(C.6)

We also find it helpful to bring the external gravitino condition to the form

\[ \left(\nabla_\mu + \frac{i}{2} m A_\mu\right) \zeta = \left(\frac{m}{2} \gamma_\mu \zeta - \frac{3i}{24} F\gamma_\mu + \frac{i}{24} F\gamma_\mu F\right)\zeta, \]  

(C.7)

and write the \( d = 10 \) spinors as

\[ \epsilon_1 = \theta_+ \otimes \left[ \zeta \otimes \chi_1 - i\zeta^c \otimes \chi_1^i \right], \quad \epsilon_2 = \theta_- \otimes \left[ \zeta \otimes \chi_2 + i\zeta^c \otimes \chi_2^i \right], \]  

(C.8)

with \( \theta_{\pm} \) short hand for the auxiliary vectors appearing in (2.6). To make progress the important thing to appreciate is that when we fix \( \mathcal{A} = 0 \), the supersymmetry variations (C.1) reduce to those of AdS\(_5\) vacua once the external spinors have been factored out, so

\[ \delta\psi_1^1\bigg|_{\mathcal{A}=0} = \delta\psi_2^2\bigg|_{\mathcal{A}=0} = 0, \]  

(C.9)

or equivalently

\[ (me^{-A} + i\partial A)\chi_1 + \frac{e^\Phi}{4} f_+ \chi_2 = 0, \quad (me^{-A} - i\partial A)\chi_2 + \frac{e^\Phi}{4} \lambda(f_+)\chi_1 = 0, \]

\[ \left(\nabla_\psi - \frac{1}{4} H_2\right)\chi_1 - i\frac{e^\Phi}{8} f_+ \gamma_\psi \chi_2 = 0, \quad \left(\nabla_\psi + \frac{1}{4} H_2\right)\chi_2 + i\frac{e^\Phi}{8} \lambda(f_+)\gamma_\psi \chi_1 = 0, \]

\[ \left(\nabla_i - V_i \partial_\psi - \frac{1}{4} \gamma_\psi (V_i - (H_2)i) - \frac{1}{4} (H_3)i\right)\chi_1 - i\frac{e^\Phi}{8} f_+ \gamma_\psi \chi_2 = 0, \]

\[ \left(\nabla_i - V_i \partial_\psi - \frac{1}{4} \gamma_\psi (V_i + (H_2)i) + \frac{1}{4} (H_3)i\right)\chi_2 - i\frac{e^\Phi}{8} \lambda(f_+)\gamma_\psi \chi_1 = 0, \]  

(C.10)

hold by definition as the internal space (modulo the \( \mathcal{A} \) dependence in the fibre) is that of the AdS\(_5\) vacua. Our task then is to show that \( \delta\psi_1^{1,2} = \delta\psi_2^{1,2}\bigg|_{\mathcal{A}=0} \) when (C.7) is assumed to hold for a non-trivial \( \zeta \). To this end the following identities are useful

\[ \partial_\psi \chi_{1,2} = \frac{i}{2} m \chi_{1,2}, \quad \gamma_\psi = \frac{e^C}{||\xi||} \xi, \quad \text{Im}\lambda(\psi_+^1)\xi \chi_1 = \frac{i}{2} ||\xi||^2 \chi_2, \quad \text{Im}\psi_+^1 \xi \chi_2 = \frac{i}{2} ||\xi||^2 \chi_1. \]  

(C.11)

These and the rest of the identities we use can be easily proved with a concrete representative spinor \( \chi \) that gives rise to the internal bi-spinors \( \psi^{1,2} \) as in section 2.2. Using these it is now easy to show that

\[ \delta\psi_1^1 - \left(\delta\psi_1^1\bigg|_{\mathcal{A}=0}\right) = \frac{e^{C-2A}}{4} \left[e^C - \frac{||\xi||}{3c}\right] \theta_+ \otimes \mathcal{F}\zeta \otimes \chi_1 + \text{m.c.}, \]

\[ \delta\psi_2^2 - \left(\delta\psi_2^2\bigg|_{\mathcal{A}=0}\right) = \frac{e^{C-2A}}{4} \left[e^C - \frac{||\xi||}{3c}\right] \theta_- \otimes \mathcal{F}\zeta \otimes \chi_2 + \text{m.c.}, \]  

(C.12)
which is zero on our classes of solutions by definition. Likewise, using the identities

\[ \text{Im} \psi_1^1 \chi_2 = - \frac{i}{2} e^A c \chi_1, \quad \text{Im} \lambda(\psi_1^1) \chi_1 = \frac{i}{2} e^A c \chi_2, \quad \xi \chi_1 = e^A c (\chi_1 - \bar{\chi}_2), \quad \bar{\xi} \chi_1 = e^A c \bar{\chi}_2, \quad (C.13) \]

we find

\[
\begin{align*}
\delta \psi_1^1 + A_\mu \delta \psi_1^1 - \left( \delta \psi_1^1 \bigg|_{A=0} \right) &= A_\mu \theta_+ \otimes \zeta \otimes \left[ \partial_\mu \chi_1 - \frac{i}{2} m \chi_1 \right] \\
&\quad + \frac{ie^{-A}}{4||\xi||} \left[ e^C - \frac{||\xi||}{3c} \right] \theta_+ \otimes F_\mu \zeta \otimes (\bar{\xi} \chi_1 - e^A c \chi_1) + \text{m.c.}, \\
\delta \psi_2^2 + A_\mu \delta \psi_2^2 - \left( \delta \psi_2^2 \bigg|_{A=0} \right) &= A_\mu \theta_- \otimes \zeta \otimes \left[ \partial_\mu \chi_2 - \frac{i}{2} m \chi_2 \right] \\
&\quad + \frac{ie^{-A}}{4||\xi||} \left[ e^C - \frac{||\xi||}{3c} \right] \theta_- \otimes F_\mu \zeta \otimes (\bar{\xi} \chi_2 - e^A c \chi_2) + \text{m.c.}, \quad (C.14)
\end{align*}
\]

where every term in square brackets is necessarily zero. Finally, one can show that

\[
\begin{align*}
\delta \psi_1^1 - V_i \delta \psi_1^1 - \left( \delta \psi_1^1 \bigg|_{A=0} \right) &= \frac{e^{-2A}}{12c} \theta_+ \otimes F_\zeta \otimes \left[ -\bar{\xi} \chi_1 + 2i \text{Im} \psi_1^1 \gamma_i \chi_2 \right] + \text{m.c.}, \\
\delta \psi_2^2 - V_i \delta \psi_2^2 - \left( \delta \psi_2^2 \bigg|_{A=0} \right) &= \frac{e^{-2A}}{12c} \theta_- \otimes F_\zeta \otimes \left[ -\bar{\xi} \chi_2 + 2i \text{Im} \lambda(\psi_1^1) \gamma_i \chi_1 \right] + \text{m.c.}, \quad (C.15)
\end{align*}
\]

where the right-hand side vanishes via the identities

\[ \text{Im} \psi_1^1 \gamma_i \chi_2 = - \frac{i}{2} \bar{\xi} \chi_1, \quad \text{Im} \lambda(\psi_1^1) \gamma_i \chi_1 = \frac{i}{2} \bar{\xi} \chi_2. \quad (C.16) \]

This exhausts all the directions of the gravitino variations. Moving now onto the dilatino variations, we again have by definition that

\[ \delta \lambda^1 \bigg|_{A=0} = \delta \lambda^2 \bigg|_{A=0} = 0 \quad (C.17) \]

holds because the internal space is fixed so as to match that of the AdS_5 vacua. It is simple to show that

\[
\begin{align*}
\delta \lambda^1 - \left( \delta \lambda^1 \bigg|_{A=0} \right) &= \frac{e^{-2A}}{6c} \theta_- \otimes F_\zeta \otimes \left[ i \bar{\xi} \chi_1 - (\gamma^a \text{Im} \psi_1^1 \gamma_a + \text{Im} \psi_1^1) \chi_2 \right] + \text{m.c.}, \quad (C.18) \\
\delta \lambda^2 - \left( \delta \lambda^2 \bigg|_{A=0} \right) &= \frac{e^{-2A}}{6c} \theta_+ \otimes F_\zeta \otimes \left[ i \bar{\xi} \chi_2 + (\gamma^a \text{Im} \lambda(\psi_1^1) \gamma_a + \text{Im} \lambda(\psi_1^1)) \chi_1 \right] + \text{m.c.}, \quad (C.19)
\end{align*}
\]

which vanish if the quantities in square brackets sum to zero, which indeed turns out to be the case for the bi-linears and spinors defined as in section 2.2. This completes our proof that the embedding of any supersymmetric solution of \( d = 5 \) minimal supergravity into massive Type IIA supergravity preserves \( d = 10 \) supersymmetry.
C.2 Sufficiency for equations of motion

In this appendix we prove that the embedding of any solution of \( d = 5 \) minimal supergravity gives rise to a solution of the Type IIA supergravity equations of motion, irrespective of whether external supersymmetry is assumed to hold or not.

The Type IIA supergravity equations of motion and Bianchi identities, away from the loci of any possible sources, take the form

\[
d_H F = 0, \quad dH = 0, \quad \mathcal{H} \equiv d(e^{-2\Phi} \star_{10} H) - \frac{1}{2} (F, F)_8 = 0,
\]

\[
\mathcal{D} \equiv 2R^{(10)} - H^2 - 8e^\Phi (\nabla^{(10)})^2 e^{-\Phi} = 0, \quad \mathcal{E}_{AB} \equiv R_{AB}^{(10)} + 2\nabla_A^{(10)} \nabla_B^{(10)} \Phi - \frac{1}{2} H_{AB}^2 - \frac{e^\Phi}{4} (F)^2_{AB} = 0.
\]

(C.20)

We already establish that the first two of these hold in the main text, as

\[
dH = dH\bigg|_{A=0} = 0, \quad d_H F = d_H F\bigg|_{A=0} = 0,
\]

(C.21)

with the first equality following from the Bianchi identity and equation of motion of the \( F \) and the second because what is left is equal to the AdS_5 vacua result, which we know vanishes. For the equation of motion of the NSNS flux it should be clear from the form of \((F, H)\) (see (3.16)) that it decomposes into parts parallel to vol_5 which are implied because they hold for AdS_5 vacua so we will not quote explicitly, and the parts defined by the following

\[
e^{-2\Phi} \star_{10} H = \frac{e^{A-2\Phi}}{3c} \star \tilde{\xi} \wedge \star_5 \mathcal{F} + \ldots,
\]

(C.22)

\[
\frac{1}{2} (F, F)_8 = \frac{8e^{-2\Phi}}{9c^2} (\mathcal{F} \wedge \text{Im} \psi^1_+ + \mathcal{F} \wedge \text{Im} \psi^1_-) + \frac{2e^{A-\Phi}}{3c} \left[ (f_+ \star_5 \mathcal{F} \wedge \text{Im} \psi^1_+) + (\star_5 \mathcal{F} \wedge \text{Im} \psi^1_+, f_+) \right]
\]

\[
- \frac{2e^{-\Phi}}{3c} \left[ (f_+ \mathcal{F} \wedge \text{Im} \psi^1_+) + (\mathcal{F} \wedge \text{Im} \psi^1_+, f_+) \right]
\]

\[
- \frac{8e^{A-2\Phi}}{9c^2} \left[ (\mathcal{F} \wedge \text{Im} \psi^1_+ \star_5 \mathcal{F} \wedge \text{Im} \psi^1_-) + (\star_5 \mathcal{F} \wedge \text{Im} \psi^1_+, \mathcal{F} \wedge \text{Im} \psi^1_+) \right] + \ldots
\]

(C.23)
To show this is implied some identities are necessary; first off with the properties of the pairing one can establish that

\[
(F \wedge \text{Im}\psi_+^1, F \wedge \text{Im}\psi_+^1)_8 = -\frac{1}{8} e^A c F \wedge F \wedge \star \bar{\xi},
\]

(C.24)

\[
(f_+, F \wedge \text{Im}\psi_+^1)_8 + (F \wedge \text{Im}\psi_+^1, f_+)_8 = 0,
\]

\[
(f_+, *_5 F \wedge \text{Im}\psi_-^1)_8 + (*_5 F \wedge \text{Im}\psi_-^1, f_+)_8 = *_5 F \wedge (\text{Im}\psi_+^1, *\lambda(f_+) + \frac{1}{ce^A}(\tau_5 + \bar{\xi}) \wedge f_+) + 5,
\]

\[
= \frac{4}{c} *_5 F \wedge (\text{Im}\psi_+^1, e^{-5A}(d_H(e^{4A-\Phi}\text{Im}\psi_+^1) - 4me^{3A-\Phi}\text{Re}\psi_+^1) - e^{-A}d_H(e^{-\Phi}\text{Im}\psi_+^1))_5,
\]

\[
= \frac{16}{c} e^{-4-\Phi} *_5 F \wedge (\text{Im}\psi_+^1, dA \wedge \text{Im}\psi_+^1)_5 = 2e^{-\Phi} *_5 F \wedge dA \wedge \star \bar{\xi},
\]

(C.25)

\[
(F \wedge \text{Im}\psi_+^1, *_5 F \wedge \text{Im}\psi_-^1)_8 + (*_5 F \wedge \text{Im}\psi_-^1, F \wedge \text{Im}\psi_+^1)_8 = -\frac{1}{8} ||\xi|| F \wedge *_5 F \wedge *_4 \bar{\xi},
\]

(C.26)

where we have used the non-trivial fact that

\[
e^{4A}
\left[ d_H(e^{-\Phi}\text{Im}\psi_+^1) + \frac{1}{e^A}(\tau_5 + \bar{\xi}) f_+ \right] - \left[ d_H(e^{4A-\Phi}\text{Im}\psi_+^1) - 4me^{3A-\Phi}\text{Re}\psi_+^1 - \frac{c}{e^A} e^{5A} *\lambda(f_+) \right] = 0
\]

(C.27)

holds in general, the \(A\) dependence cancelling between the two terms and we also use that \((\text{Im}\psi_+^1, \text{Re}\psi_-^1)_5 = 0\). To deal with \(d \star \bar{\xi}\) we use an identity that must hold given that a time-like Killing vector can be defined on AdS\(_5\) (which leads to a time-like Killing vector in \(d = 10\))

\[
d(e^{-2\Phi} *_{10} \bar{K}^{(10)}) \bigg|_{A=0} = 0 \quad \Rightarrow \quad d(e^{5A-2\Phi} \star \bar{\xi}) \bigg|_{A=0} = 0
\]

(C.28)

\[
\Rightarrow \quad d(e^{5A-2\Phi} \star \bar{\xi}) = -e^{5A-2\Phi} \frac{||\xi||}{3c} F \wedge \star_4 \bar{\xi}
\]

(C.29)

(see [58] where we have corrected an obvious typo, given the conditions that lead to (D.3) there). With this it is now possible to establish that (3.13b)-(3.13f) and the external flux equation of motion imply

\[
\frac{1}{2} (F, F) = \frac{1}{3c} d(e^{A-2\Phi} \star \bar{\xi} \wedge *_5 F) + ... = d(e^{-2\Phi} \star H),
\]

(C.30)

so the NSNS flux equation of motion is implied. We now pause to make an observation: in [58] it is proven that supersymmetry plus the equations of motions and Bianchi identities of the fluxes imply the remaining equations of motion when \(K^{(10)}\) is assumed to be time-like. We further note that it is possible to show that \(K^{(10)}\) is time-like or null if and only if the external Killing vector \(k_\mu \partial_\mu\) is like-wise time-like or null. However as (3.16) does not depend on any of the external bi-linears (including \(k\)), and for \(A = 0\) we know all the \(d = 10\) equations of motion hold, then in general the \(d = 10\) equations of motion must be closing on the equations of motion of \(d = 5\) minimal gauged
supergravity. As such if these are implied for a time-like supersymmetric solution of this theory, they should be implied for all solutions of the theory. If one is willing to trust that argument, we have already shown what we set out to, however we are aware that the reader may be unsatisfied with this, for this reason we will now also complete the proof in a more direct way.

To proceed we must solve Einsteins equations and the dilaton equation of motion. We will achieve this in a similar fashion to how we established that supersymmetry holds, i.e. by assuming that the equations of motion of $d=5$ minimal gauged supergravity (3.2a)-(3.2b) hold we shall show that the equations of motion for generic $A$ reduce to those of the AdS$_5$ vacua for which $A = 0$. To this end we need to decompose several of the objects appearing here in terms of a metric of the form

$$ds^{2}_{10} = e^{2A}g^{(5)}_{\mu\nu}dx^\mu dx^\nu + ds^{2}(M_4) + e^{2C}D\psi^2. \quad (C.31)$$

Again decomposing the internal directions as $a = (\psi, i)$, the Ricci tensor and scalar decomposes in coordinate frame as

$$R^{(10)} = R^{0} + \frac{e^{-4A}}{18}(e^{2A} - 9e^{2C})F^2, \quad R^{(10)}_{\psi\psi} = R_{\psi\psi}^{0} + \frac{e^{-4(A-C)}}{2}F^2, \quad R^{(10)}_{ij} = R_{ij}^{0} - V_{i}V_{j}R^{(10)}_{\psi\psi} + 2V_{(i}R^{(10)}_{j)\psi},$$

$$R^{(10)}_{\mu\nu} = A_{\mu}R^{(10)}_{\psi\psi} + \frac{e^{-2(A-C)}}{2}\nabla^\alpha F_{\alpha\mu}, \quad R^{(10)}_{\psi\psi} = R^{0}_{\psi\psi} + V_{i}R_{\psi\psi}^{(10)}, \quad R^{(10)}_{\mu\nu} = -A_{\mu}R_{\psi\psi}^{(10)} + \frac{e^{-2(A-C)}}{2}V_{i}\nabla^\alpha F_{\alpha\mu},$$

$$R^{(10)}_{ij} = R_{ij}^{0} - \frac{1}{18}g^{(5)}_{ij}F^2 + e^{-2(A-C)}A_{(\mu}\nabla^\alpha F_{\nu)\alpha} + A_{\mu}A_{\nu}R_{\psi\psi}^{(10)} + \frac{1}{2}(\frac{1}{3} - e^{-2(A-C)})F^{2}_{\mu\nu}, \quad (C.32)$$

where the superscript 0 means these terms are independent of $A$ and so they take the form they do for AdS$_5$ vacua, we have made use of the equations of motion of $d=5$ supergravity so we have $g^{(5)}_{\mu\nu}$ rather than $R^{(5)}_{\mu\nu}$ appear and so on, specifically

$$R_{\psi\psi}^{0} = \frac{e^{4C}}{2}(dV)^2 - e^{-(5A+C)}\nabla_{i}(e^{5A}\nabla^{i}(e^{C})),$$

$$R_{\psi\psi}^{0} = -\frac{e^{-(5A+C)}}{2}d^k(e^{5A+3C}dV)_{ki},$$

$$R_{\mu\nu}^{0} = -(4m^2 + \frac{1}{5}e^{-3A-C}\nabla_{i}(e^{C}\nabla^i(e^{5A}))g^{(5)}_{\mu\nu},$$

$$R_{ij}^{0} = R_{ij} - (5\nabla_{i}\nabla_{j}A + \nabla_{i}A\nabla_{j}A + \nabla_{i}\nabla_{j}C + \nabla_{i}A\nabla_{j}C) - \frac{e^{2C}}{2}dV_{ij}^2. \quad (C.33)$$
One can also show that the dilaton terms decompose as
\[
e^\Phi (\nabla^{(10)})^2 (e^{-\Phi}) = e^\Phi -(5A+C) \nabla_i (e^{5A+C} \nabla^i (e^{-\Phi})), \quad \nabla^{(10)} \psi \nabla^{(10)} \Phi = e^{2C} \nabla_i \Phi \nabla^i C,
\]
\[
\nabla_i^{(10)} \nabla_j^{(10)} \Phi = -V_i V_j \nabla^{(10)} \psi \nabla^{(10)} \Phi + 2V_i \nabla_j^{(10)} \nabla^{(10)} \psi + \nabla_i \nabla_j \Phi, \quad \nabla^{(10)} \nabla^{(10)} \Phi = -A_\mu \nabla_\psi \nabla^{(10)} \Phi,
\]
\[
\nabla_i^{(10)} \nabla_\psi^{(10)} \Phi = \frac{1}{2} e^{2C} \nabla^k \Phi (dV)_{ki} + \nabla_\psi^{(10)} \nabla^{(10)} \Phi, \quad \nabla_\mu^{(10)} \nabla_\psi^{(10)} \Phi = -A_\mu \nabla_\psi^{(10)} \Phi,
\]
\[
\nabla_\mu^{(10)} \nabla_\nu^{(10)} \Phi = e^{2A} g^{(5)}_{\mu \nu} \nabla^i \psi \Phi + A_\mu A_\nu \nabla_\psi^{(10)} \nabla^{(10)} \Phi.
\]
(C.34)

To show that Einstein’s equations are implied we need several identities, we shall quote them as they become relevant but they can all be derived from the bi-linears in section 2.2. First we consider \( \mathcal{E}_{\psi \psi} \), for this one can show that
\[
(t_\xi \psi^1)_x^2 = \frac{1}{16} \| \xi \|^2 (e^{2A} c \pm \| \xi \|^2) \quad \Rightarrow 
\frac{e^{-\Phi}}{4} (F)^2_{\psi \psi} = \frac{e^{-\Phi}}{4} (F)^2_{\psi \psi} \bigg|_{A=0} + \frac{e^{-4A} \| \xi \|^2}{16c^4} F^2,
\]
(C.35)

It then follows that
\[
\mathcal{E}_{\psi \psi} = \mathcal{E}^0_{\psi \psi} + \frac{e^{-4A}}{162} \left[ e^{4C} - \left( \frac{\| \xi \|}{3c} \right)^4 \right] F^2,
\]
(C.36)
the first term vanishing because it is precisely as it is for AdS5 and the second because \( 3ce^C = \| \xi \| \) for our background. Next we consider \( \mathcal{E}_{\mu \nu} \), here we have
\[
H_\mu = -A_\mu H_2 - \frac{1}{3c} \hat{\xi} \wedge F_\mu, \quad F_\mu = -A_\mu F_\psi - \frac{4}{3c} e^{-\Phi} \left[ F_\mu \wedge \text{Im} \psi^1_+ - e^A (\star_5 F)_\mu \wedge \text{Im} \psi^1_+ \right],
\]
(C.37)
where one needs to bear in mind that, in addition to itself, \( (\star_5 F)_\mu \) can contract with the \( F \) term within \( F_\psi \). Through the identities
\[
\sum_k \frac{1}{k!} (t_\xi \text{Im} \psi^1_+)_{a_1 \ldots a_k} (\text{Im} \psi^1_-)_{a_1 \ldots a_k} = \frac{1}{8} \| \xi \|^2 e^A c, \quad (\text{Im} \psi^1_+)^2 = (\text{Im} \psi^1_-)^2 = \frac{e^{2A} c^2}{8},
\]
(C.38)
one establishes that
\[
H^2_{\mu \nu} = A_\mu A_\nu (H)^2_{\psi \psi} + \frac{e^{-2A} \| \xi \|^2}{9c^2} F^2_{\mu \nu},
\]
(C.39)
\[
e^\Phi (F)^2_{\mu \nu} = e^\Phi (F)^2_{\mu \nu} \bigg|_{A=0} + A_\mu A_\nu e^\Phi (F)^2_{\psi \psi} - \frac{e^{-2A} \| \xi \|^2}{27c^2} A_{(\mu} \epsilon_{\nu) a_1 \ldots a_4} F^{a_1 a_2} F^{a_3 a_4} + \frac{2}{9} (2F^2_{\mu \nu} - g^{(5)}_{\mu \nu} F^2)
\]
and so after substituting for \( e^C \) we find
\[
\mathcal{E}_{\mu \nu} = \mathcal{E}^0_{\mu \nu} + A_\mu A_\nu \mathcal{E}_{\psi \psi} + \frac{e^{-2A} \| \xi \|^2}{9c^2} A_{(\mu} \delta F_{\nu)} + \frac{1}{18} \left[ 1 - \frac{1}{e^{2A} c^2} (\| \xi \|^2 + \| \xi \|^2) \right] F^2_{\mu \nu},
\]
(C.40)
where $||\xi||^2 + ||\bar{\xi}||^2 = e^{2A}c^2$ holds in general so the term in square brackets is zero, we already established $\mathcal{E}_{\psi\psi} = 0$ and $\mathcal{E}^0_{\mu\nu}$ takes exactly the same form as it does for AdS$_5$ so also vanishes. The final term is simply the equation of motion of $F$ written in component form, i.e.

$$
\delta F_{\nu} = \nabla^\mu F_{\mu\nu} + \frac{1}{12} \epsilon_{\alpha_1...\alpha_4} F^{\alpha_1\alpha_2} F^{\alpha_3\alpha_4},
$$

(C.41)

clearly this should vanish by definition. At this point we have provided a detailed proof of sufficiency for a solution when external supersymmetry holds, this follows from the integrability arguments of [58, 59].

For a detailed proof of the fact that any solution of the external theory gives rise to a solution in massive Type IIA supergravity we must show that the remaining components of Einstein’s equations and the dilaton equation of motion are implied. The first thing we need to consider is $E_{\bar{\psi}\psi}$ as this appears in later expressions. For this and the other $i$ dependent terms it is useful to define the following vielbein on $M_4$

$$
e_i = (\text{Re}z, \text{Im}z, e_\perp, \frac{1}{|\xi|} \bar{\xi}),
$$

(C.42)

then for a general $k$-form we can decompose

$$(C_k)_i = (C_k)_\psi V_i + c^i (t_{\bar{\epsilon}_a} C_k).$$

(C.43)

The important identity for the component at hand is

$$
\sum_k \frac{1}{k!} ((\text{Im}\psi^1_\pm)_a...ak (t_{\bar{\epsilon}_\bar{a}} \text{Im}\psi^1_\pm)^a...ak = 0,
$$

(C.44)

it is then simple to establish that

$$
\mathcal{E}_{\bar{\psi}\psi} = \mathcal{E}_{\bar{\psi}\psi}^0 + V_i \mathcal{E}_{\psi\psi},
$$

(C.45)

with each term in the sum necessarily zero. Next for $\mathcal{E}_{\mu\psi}$ we need only reuse the identities in (C.38), we find

$$
\mathcal{E}_{\mu\psi} = \mathcal{E}^0_{\mu\psi} - A_\mu \mathcal{E}_{\psi\psi} + \frac{e^{-2A}||\xi||^2}{18c^2} \delta F_\mu,
$$

(C.46)

with all terms in the sum again zero, at least when the external flux equations of motion is assumed to hold. For $\mathcal{E}_{\mu i}$ we can make use of (C.44) again and the identities

$$
\sum_k \frac{1}{k!} (t_{\bar{\epsilon}_a} \text{Im}\psi^1_+)_a...ak (\text{Im}\psi^1_-)^a...ak = 0, \quad \sum_k \frac{1}{k!} (t_{\bar{\epsilon}_a} \text{Im}\psi^1_-)_a...ak (\text{Im}\psi^1_+)^a...ak = \frac{e^{A_\psi}}{8} ||\xi|| \delta_{\bar{\xi}4}
$$

(C.47)

from which it follows that

$$
\mathcal{E}_{\mu i} = -A_\mu \mathcal{E}_{\bar{\psi}\psi} + \frac{e^{-2A}||\xi||^2}{18c^2} V_i \delta F_\mu,
$$

(C.48)
which is implied by the equation of motion of $F$ and the previous conditions. The final component of Einstein’s equations is $E_{ij}$, we have already quoted all but one of the required identities to tackle this, namely

$$\sum_k \frac{1}{k!} (t_{\epsilon \ell} \text{Im} \psi^{11})_{a_1 \ldots a_k} (t_{\epsilon \ell} \text{Im} \psi^{11})^{a_1 \ldots a_k} = \frac{1}{16} (e^{2A} c^2 \delta_{ij} = ||\xi||^2 \delta_{i4} \delta_{j4}),$$

we find this decomposes as

$$E_{ij} = E_{ij}^0 - V_i V_j E_{\psi\psi} + 2V_i (E_{j})_{\psi},$$

with each term in the sum again zero — this exhausts all components of Einstein’s equations. Fortuitously establishing that the dilaton equation of motion is implied is a much shorter computation, we find

$$D = D^0 + \frac{e^{-4A}}{9c^2} (e^{2A} c^2 - \xi^2) F^2 - e^{-4A} \frac{1}{9c^2} \tilde{c}^2 F^2$$

which is solved due to $||\xi||^2 + ||\tilde{\xi}||^2 = e^{2A} c^2$ and the fact that $D^0 = 0$ because $D^0$ is precisely equal to the $D$ of the AdS$_5$ vacua. Thus, the embedding of any solution of $d = 5$ minimal gauged supergravity, not merely the supersymmetric ones, into massive Type IIA supergravity always yields a solution of the Type IIA supergravity equations of motion.

References

[1] H. Nastase, D. Vaman and P. van Nieuwenhuizen, Consistent nonlinear K K reduction of 11-d supergravity on AdS(7) x S(4) and selfduality in odd dimensions, Phys. Lett. B 469 (1999) 96–102, [hep-th/9905075].

[2] H. Nastase, D. Vaman and P. van Nieuwenhuizen, Consistency of the AdS(7) x S(4) reduction and the origin of selfduality in odd dimensions, Nucl. Phys. B 581 (2000) 179–239, [hep-th/9911238].

[3] M. Cvetic, H. Lu, C. N. Pope, A. Sadrzadeh and T. A. Tran, Consistent SO(6) reduction of type IIB supergravity on S**5, Nucl. Phys. B 586 (2000) 275–286, [hep-th/0003103].

[4] A. Baguet, O. Hohm and H. Samtleben, Consistent Type IIB Reductions to Maximal 5D Supergravity, Phys. Rev. D 92 (2015) 065004, [1506.01385].

[5] A. Guarino and O. Varela, Consistent $\mathcal{N} = 8$ truncation of massive IIA on $S^6$, JHEP 12 (2015) 020, [1509.02526].

[6] B. de Wit and H. Nicolai, The Consistency of the $S^{**7}$ Truncation in D=11 Supergravity, Nucl. Phys. B 281 (1987) 211–240.
[7] O. Varela, Complete $D = 11$ embedding of SO(8) supergravity, Phys. Rev. D 97 (2018) 045010, [1512.04943].

[8] J. P. Gauntlett, S. Kim, O. Varela and D. Waldram, Consistent supersymmetric Kaluza-Klein truncations with massive modes, JHEP 04 (2009) 102, [0901.0676].

[9] D. Cassani, G. Dall’Agata and A. F. Faedo, Type IIB supergravity on squashed Sasaki-Einstein manifolds, JHEP 05 (2010) 094, [1003.4283].

[10] J. P. Gauntlett and O. Varela, Universal Kaluza-Klein reductions of type IIB to $N=4$ supergravity in five dimensions, JHEP 06 (2010) 081, [1003.5642].

[11] J. T. Liu, P. Szepietowski and Z. Zhao, Consistent massive truncations of IIB supergravity on Sasaki-Einstein manifolds, Phys. Rev. D 81 (2010) 124028, [1003.5374].

[12] D. Cassani and P. Koerber, Tri-Sasakian consistent reduction, JHEP 01 (2012) 086, [1110.5327].

[13] H. Triendl, Consistent truncations of M-theory for general SU(2) structures, JHEP 09 (2015) 068, [1505.05526].

[14] G. Larios and O. Varela, Minimal $D = 4$ $\mathcal{N} = 2$ supergravity from $D = 11$: An M-theory free lunch, JHEP 10 (2019) 251, [1907.11027].

[15] D. Cassani, P. Koerber and O. Varela, All homogeneous $N=2$ M-theory truncations with supersymmetric $AdS_4$ vacua, JHEP 11 (2012) 173, [1208.1262].

[16] A. Passias, A. Rota and A. Tomasiello, Universal consistent truncation for 6d/7d gauge/gravity duals, JHEP 10 (2015) 187, [1506.05462].

[17] E. Malek, H. Samtleben and V. Vall Camell, Supersymmetric $AdS_7$ and $AdS_6$ vacua and their minimal consistent truncations from exceptional field theory, Phys. Lett. B 786 (2018) 171–179, [1808.05597].

[18] E. Malek, H. Samtleben and V. Vall Camell, Supersymmetric $AdS_7$ and $AdS_6$ vacua and their consistent truncations with vector multiplets, JHEP 04 (2019) 088, [1901.11039].

[19] E. Malek and V. Vall Camell, Consistent truncations around half-maximal $AdS_5$ vacua of 11-dimensional supergravity, Class. Quant. Grav. 39 (2022) 075026, [2012.15601].

[20] G. Josse, E. Malek, M. Petrini and D. Waldram, The higher-dimensional origin of five-dimensional $\mathcal{N} = 2$ gauged supergravities, JHEP 06 (2022) 003, [2112.03931].
[21] M. Galli and E. Malek, *Consistent truncations to 3-dimensional supergravity*, JHEP 09 (2022) 014, [2206.03507].

[22] D. Cassani, G. Josse, M. Petrini and D. Waldram, *Systematics of consistent truncations from generalised geometry*, JHEP 11 (2019) 017, [1907.06730].

[23] E. Malek, *Half-Maximal Supersymmetry from Exceptional Field Theory*, Fortsch. Phys. 65 (2017) 1700061, [1707.00714].

[24] F. Apruzzi, M. Fazzi, A. Passias and A. Tomasiello, *Supersymmetric AdS5 solutions of massive IIA supergravity*, JHEP 06 (2015) 195, [1502.06620].

[25] J. P. Gauntlett and O. Varela, *Consistent Kaluza-Klein reductions for general supersymmetric AdS solutions*, Phys. Rev. D 76 (2007) 126007, [0707.2315].

[26] J. P. Gauntlett, E. O Colgain and O. Varela, *Properties of some conformal field theories with M-theory duals*, JHEP 02 (2007) 049, [hep-th/0611219].

[27] K. C. M. Cheung and R. Leung, *Type IIA embeddings of D = 5 minimal gauged supergravity via non-Abelian T-duality*, JHEP 06 (2022) 051, [2203.15114].

[28] D. Rosa and A. Tomasiello, *Pure spinor equations to lift gauged supergravity*, JHEP 01 (2014) 176, [1305.5255].

[29] J. P. Gauntlett and J. B. Gutowski, *All supersymmetric solutions of minimal gauged supergravity in five-dimensions*, Phys. Rev. D 68 (2003) 105009, [hep-th/0304064].

[30] J. B. Gutowski and H. S. Reall, *Supersymmetric AdS(5) black holes*, JHEP 02 (2004) 006, [hep-th/0401042].

[31] P. Ferrero, J. P. Gauntlett, J. M. Pérez Ipiña, D. Martelli and J. Sparks, *D3-Branes Wrapped on a Spindle*, Phys. Rev. Lett. 126 (2021) 111601, [2011.10579].

[32] P. Ferrero, J. P. Gauntlett, J. M. P. Ipiña, D. Martelli and J. Sparks, *Accelerating black holes and spinning spindles*, Phys. Rev. D 104 (2021) 046007, [2012.08530].

[33] S. M. Hosseini, K. Hristov and A. Zaffaroni, *Rotating multi-charge spindles and their microstates*, JHEP 07 (2021) 182, [2104.11249].

[34] A. Boido, J. M. P. Ipiña and J. Sparks, *Twisted D3-brane and M5-brane compactifications from multi-charge spindles*, JHEP 07 (2021) 222, [2104.13287].
[35] F. Faedo, S. Klemm and A. Viganò, *Supersymmetric black holes with spiky horizons*, JHEP 09 (2021) 102, [2105.02902].

[36] P. Ferrero, J. P. Gauntlett, D. Martelli and J. Sparks, *M5-branes wrapped on a spindle*, JHEP 11 (2021) 002, [2105.13344].

[37] D. Cassani, J. P. Gauntlett, D. Martelli and J. Sparks, *Thermodynamics of accelerating and supersymmetric AdS4 black holes*, Phys. Rev. D 104 (2021) 086005, [2106.05571].

[38] P. Ferrero, M. Inglese, D. Martelli and J. Sparks, *Multicharge accelerating black holes and spinning spindles*, Phys. Rev. D 105 (2022) 126001, [2109.14625].

[39] C. Couzens, K. Stemerdink and D. van de Heisteeg, *M2-branes on discs and multi-charged spindles*, JHEP 04 (2022) 107, [2110.00571].

[40] P. Ferrero, J. P. Gauntlett and J. Sparks, *Supersymmetric spindles*, JHEP 01 (2022) 102, [2112.01543].

[41] C. Couzens, *A tale of (M)2 twists*, JHEP 03 (2022) 078, [2112.04462].

[42] F. Faedo and D. Martelli, *D4-branes wrapped on a spindle*, JHEP 02 (2022) 101, [2111.13660].

[43] S. Giri, *Black holes with spindles at the horizon*, JHEP 06 (2022) 145, [2112.04431].

[44] K. C. M. Cheung, J. H. T. Fry, J. P. Gauntlett and J. Sparks, *M5-branes wrapped on four-dimensional orbifolds*, JHEP 08 (2022) 082, [2204.02990].

[45] M. Suh, *M5-branes and D4-branes wrapped on a direct product of spindle and Riemann surface*, 2207.00034.

[46] I. Arav, J. P. Gauntlett, M. M. Roberts and C. Rosen, *Leigh-Strassler compactified on a spindle*, JHEP 10 (2022) 067, [2207.06427].

[47] C. Couzens and K. Stemerdink, *Universal spindles: D2’s on Σ and M5’s on Σ × ℍ3*, 2207.06449.

[48] A. Tomasiello, *Generalized structures of ten-dimensional supersymmetric solutions*, JHEP 03 (2012) 073, [1109.2603].

[49] I. Bah, A. Passias and A. Tomasiello, *AdS5 compactifications with punctures in massive IIA supergravity*, JHEP 11 (2017) 050, [1704.07389].
[50] Z. W. Chong, M. Cvetic, H. Lu and C. N. Pope, *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, Phys. Rev. Lett. 95 (2005) 161301, [hep-th/0506029].

[51] F. Apruzzi, M. Fazzi, D. Rosa and A. Tomasiello, *All AdS$_7$ solutions of type II supergravity*, JHEP 04 (2014) 064, [1309.2949].

[52] F. Apruzzi, M. Fazzi, A. Passias, A. Rota and A. Tomasiello, *Six-Dimensional Superconformal Theories and their Compactifications from Type IIA Supergravity*, Phys. Rev. Lett. 115 (2015) 061601, [1502.06616].

[53] N. T. Macpherson, *Type II solutions on AdS$_3 \times S^3 \times S^3$ with large superconformal symmetry*, JHEP 05 (2019) 089, [1812.10172].

[54] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, *All supersymmetric solutions of minimal supergravity in five- dimensions*, Class. Quant. Grav. 20 (2003) 4587–4634, [hep-th/0209114].

[55] P. Merrikin, C. Nunez and R. Stuardo, *Compactification of 6d $\mathcal{N} = (1,0)$ quivers, 4d SCFTs and their holographic dual Massive IIA backgrounds*, 2210.02458.

[56] A. F. Faedo, C. Nunez and C. Rosen, *Consistent truncations of supergravity and $1/2$-BPS RG flows in 4d SCFTs*, JHEP 03 (2020) 080, [1912.13516].

[57] D. Gaiotto and A. Tomasiello, *Holography for (1,0) theories in six dimensions*, JHEP 12 (2014) 003, [1404.0711].

[58] A. Legramandi, L. Martucci and A. Tomasiello, *Timelike structures of ten-dimensional supersymmetry*, JHEP 04 (2019) 109, [1810.08625].

[59] S. Giusto, L. Martucci, M. Petrini and R. Russo, *6D microstate geometries from 10D structures*, Nucl. Phys. B 876 (2013) 509–555, [1306.1745].