Multiplicity of chaotic attractors in a model of lasers with variable feedback delay

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Abstract. We demonstrate coexistence of chaotic attractors induced by periodic modulation of the delay time in the feedback circuit which controls pumping rate in a laser. Depending on the $n$-number of pulses in the delay interval, the dynamics of the pulse regime is described by the dynamics of the $(2n+2)$-dimensional map. It is possible to switch chaotic regimes of different structures by a single impulse perturbation of pumping in a suitable phase of the oscillations.

1. Introduction

Delayed feedback (FB) is widely used to control the dynamics of nonlinear optical and electro-optical systems [1]. Recently, modifications of standard FB schemes have been actively studied, in particular, high-frequency periodic modulation of the delay time in the FB scheme was discussed to stabilize the state of unstable equilibrium [2,3]. On the other hand, the variation of the delay time with a frequency comparable with the relaxation one can be used to obtain oscillating regimes with given characteristics. This may be of interest for methods of optical processing and information coding, optical vibrometry and in other applications, see, for example, [5].

In this paper, we show the coexistence of irregular pulse regimes caused by periodic modulation of the delay time in the FB circuit. Such regimes can be classified as slowly oscillating with inter-spikes intervals longer than the delay time or fast oscillating with inter-spikes intervals shorter than the delay.

Of particular importance can be the task of purposeful switching of coexisting attractors. Previously, quick switching periodic cycles by external impulse was suggested for a loss-driven CO\textsubscript{2} laser [6]. With a suitable choice of the switching pulse characteristics the system may be found on stable periodic cycles, as well as on unstable cycles. It was found that there is an optimal time of application of impact, which corresponds to the minimum duration of the transition process to the target cycle. The theoretical explanation was given on the base of the 3-dimensional non-autonomous model [7].

In the present paper, we study coexisting chaotic attractors with basins that may have fractal boundaries and are embedded in the infinite-dimensional phase space of a system with retarded argument. Thus, the possibility of a single-cycle transition is not obvious. Nevertheless, we note the conditions that can ensure successful switching by a quick change of the system
position within the phase space. To this end we derive asymptotically the finite-dimensional maps responsible for the dynamics of spikes. With an impulse-perturbation of the pumping rate in a suitable phase of the oscillation, it is possible to achieve switching chaotic regimes of different structures.

2. Model of laser dynamics

Relaxation oscillations in a semiconductor laser with an optoelectronic FB can be studied on the base of the model proposed in [8], which we supplements by periodic modulation of the delay time,

\[
\frac{du}{dt} = vu(y - 1), \quad \frac{dy}{dt} = q + \gamma u(g(t)) - y - yu, \tag{1}
\]

where \( q \) is the pumping rate; \( v \) is the ratio of the rate of decay photons in the cavity to the rate of relaxation of populations; \( \gamma \) is the feedback factor (FB level); \( t \) is current time normalized at the time of relaxation of the population inversion. The optoelectronic FB is represented by the term \( \gamma u(g(t)) \), where \( \gamma \) is the feedback coefficient, the delayed argument has the form \( g(t) = t - (\tau_0 + B \cos \Phi(t)) \), the modulator phase \( \Phi(t) = \omega t + \varphi \), \( \tau_0 \) is the constant time of the radiation transformation in the FB circuit, \( B \) and \( \omega \) is the amplitude and frequency of the delay modulation, respectively, \( \varphi \) is the initial phase of the modulating signal.

Here we will consider positive FB with \( \gamma > 0 \). The delay modulation amplitude and frequency should be limited by the inequalities which ensures positive values of the delay \( \tau(t) > 0 \) and positive derivation \( g'(t) > 0 \) (ensures pulsed structure of the solution). Further we study the system with the assumptions

\[
B < \tau_0, \quad B\omega < 1. \tag{2}
\]

For class B lasers including semiconductor lasers, some solid-state lasers, CO\(_2\) gas lasers typical values \( v \sim 10^3 \) are large while the other parameters are of the order of unit. Considering \( v \gg 1 \) as a large parameter, an asymptotic approach for spiking can be done. The method of investigation is following. The phase space of system (1) is the direct product of the Banach space \( C[\tau, 0] \) of continues functions by the number line \( R^1 \), i.e., the values of the functions from \( C[\tau, 0] \) and the value \( y(0) \in R^1 \) are given as initial conditions. In this space, we shall distinguish a (fairly wide) set \( S(\xi) \) dependent on the vector parameter \( \xi \) and consider the solutions with initial conditions from this set. It is possible to construct uniform asymptotic approximations of all such solutions and show that after a certain time these solutions again fall within \( S(\xi) \). Thus, the operator of the shifting along the trajectories, which makes a function from \( S \) corresponds to a function also from \( S \), is naturally determined. The properties of this operator are mainly determined by the finite-dimensional map \( \xi = f(\xi) \). To a fixed point of the map there corresponds the fixed point of the operator, and to the later point there corresponds a periodic solution of the same stability in the original system. Examples of the investigation of similar systems were given in our previous works [9].

Solutions in form of spikes can be classified as slowly oscillating (SO) solutions with inter-spikes intervals longer than the delay time, fast oscillating (FO\(n\)) solutions with inter-spikes intervals shorter than the delay, mixed slowly and fast (MSF) oscillating solutions. For each type of solutions we will find asymptotically (at \( v \rightarrow \infty \)) the maps responsible for the dynamics of spikes.

3. Slowly oscillating solution

For SO solutions any interval between spikes greater than the time delay. Fix a time point when the radiation spike begins as the starting point, so that radiation intensity \( u(0) = 1 \) whereas before that, in the delay interval, the radiation intensity is of noise level.
The set of initial conditions can be defined as follows,
\[ S(c, \varphi) = \{ y(0) = c, \quad \Phi(0) = \varphi, \quad u(s) = \psi(s), \quad s \in [-\tau_0 - B, 0] \} \] (3)
where \( c \in (1, q] \), \( \varphi \in [0, 2\pi] \), and the function \( \psi(s) \in S_0 \) is chosen from the set \( S_0 \) of the functions with the properties,
\[ S_0 = \{ \psi(s) : 0 \leq \psi(s) \ll 1, \quad \psi(0) = 1, \quad \int_0^0 \psi(s) ds \leq v^{-1/2} \} \].

Note, the parameter \( c > 1 \) that provides \( \dot{u}(0) > 0 \). The form of the functions \( \psi(s) \) is not specified, hence, the set \( S(c, \varphi) \) is wide enough.

Now we integrate the system step-by-step dividing the evolution path into segments \([t_i, t_{i+1}]\), where the asymptotic estimates (under \( v \to \infty \)) for functions \( u(t - \tau) \) and \( u(t) \) are known. The solution values at the end of the interval are used as the initial conditions for the next interval. In this way we obtain an analytical estimate of the SO solution up to the time \( t = t_2 \), when the next radiation pulse begins.

The main result is that the problem of further constructing the solution for \( t > t_2 \) has returned to the original problem with the initial conditions \( u(t_2 + s) = \tilde{\psi}(s) \in S_0, \quad s \in [-\tau_0 + B, 0] \) and \( y(t_2) = \tilde{c} + o(1), \quad \Phi(t_2) = \tilde{\varphi}, \) which belong to set \( S \) given by Eq.(3) with replacing \( (c, \varphi) \) by \( (\tilde{c}, \tilde{\varphi}) \), namely,
\[
\tilde{c} = q + (c - p - q)e^{-T} + \frac{\gamma p}{g'(\tilde{t}_0)} e^{-(T - \tilde{t}_0)}, \\
\tilde{\varphi} = \omega T + \varphi, \quad \text{mod} \ 2\pi, \quad (4)
\]
where \( \tilde{t}_0(\varphi) \) is the positive root of the equation \( \tau_0 + B \cos(\omega \tilde{t}_0 + \varphi) = \tilde{t}_0; \) \( p(c) \) is the positive root of the equation \( c - p = ce^{-p} \); and \( T(c, \varphi) \) is the root of the equation
\[ (q - 1)T + (c - p - q)(1 - e^{-T}) + \gamma \frac{p}{g'(\tilde{t}_0)} (1 - e^{-(T - \tilde{t}_0)}) = 0. \] (5)

Note, the value of \( p \) characterizes the pulse energy and the value of \( T \) characterizes the interpulse interval. Integrating the system, we have supposed that inequality
\[ a(\tilde{t}_0) < 0 \quad (6) \]
where \( a(t) = (q - 1)t + (c - p - q)(1 - e^{-t}) \) is fulfilled in order to get the \( T > \tau_0 + B \).

Let us introduce the trajectory shift operator \( \Pi(c, \varphi, \psi(s)) = (y(t), \Phi(t), u(s + t)) \), \( s \in [-\tau_0 - B, 0] \), which associates (by means of solutions) any element \( (c, \varphi) \) from the set \( S \) with another element \( (\tilde{c}, \tilde{\varphi}) \) from the same set \( S \). Evolution of the operator and, in turn, evolution of slowly oscillating pulsated solution is determined by the iterations of the two-dimensional map (5). In particular, to the stable fixed point \((c_0, \varphi_0)\) there corresponds stable periodic slowly oscillating pulsations of the period \( T_0 = T(c_0, \varphi_0) \) in the original system and \( T_0 > \tau_0 + B \). If map (5) has another attractor and inequality (6) is fulfilled for each iteration of the map, then the attractor corresponds to a pulsed solution of SO structure in system (1). To bifurcations of the mapping attractor there correspond bifurcations of pulsated regime. Inequality (6) provides SO structure of such solutions and, in this way, restricts the domain in parameters space and in phase space where SO solutions can realized.

By computing map (5) with \( q = 1.5, \quad \gamma = 0.3, \quad \tau_0 = 0.3 \) and \( \omega = 29.8 \) we find that at sufficiently small modulation amplitudes, \( 0 < B < 0.012 \), there is a stable mapping point which corresponds to a stable cycle with a period longer than the delay. At \( 0.012 < B < 0.04 \), cycles
and chaotic attractors (quasiperiodic and chaotic SO oscillations) are observed. At $B > 0.04$ that corresponds to a violation of the condition $Bw < 1$, a transition takes place to oscillations a with interpulse intervals longer and shorter than the delay time.

Thus, there are sufficiently wide domains of modulation parameters at which chaotic spikes of special SO-structure may be observed.

4. Fast oscillating solutions

Inequality (6) determines the parameters with which the intervals between neighboring pulses are greater than the delay. Violation of this condition leads to occurrence of $n \geq 1$ pulses on any interval of delay length. We call such regimes fast oscillating (FO) modes, since the time intervals between pulses are shorter than the delay time in the FB circuit.

Consider FO1 regimes with one pulse on the delay interval. The set of initial conditions for Eqs. (1) can be given as follows,

$$S(c, \varphi, \xi, p_1) = \{ y(0) = c, \Phi(0) = \varphi, u(s) = \psi_1(s), s \in [-\tau_0 - B, 0] \},$$

where $c \in (1, q], \varphi \in [0, 2\pi], \psi_1(s) \in S_1(\xi, p_1)$, and

$$S_1(\xi, p_1) = \left\{ \psi_1(s) \in C_{[-\tau_0 - B, 0]} : \psi_1(0) = 1, \int_{-\tau + \xi + \delta_1}^{-\tau + \xi} \psi_1(s) ds = p_1, \int_{-\tau}^{-\tau + \xi} \psi_1(s) ds + \int_{-\tau + \xi}^{0} \psi_1(s) ds < v^{-1/2} \right\}.$$  

The parameter $\xi$ determines the moment of pulse onset on the delay interval so that the value $T = (\tau_0 + B \cos \varphi - \xi) > 0$ is the interval between pulses, the parameter $p_1 > 0$ is the energy of the pulse in the delay interval. The $\psi_1(s)$ values are asymptotically small in the intervals between pulses, the pulse shape is not specified, for example, a pulse can be square (non-smooth). It is only necessary that the pulse width $\delta_1$ should be sufficiently short, $\delta_1 \to 0$, with $v \to \infty$. Note also, the condition $c > 1$ ensures the positive derivative $u'(0) > 0$ at the initial moment, which we determine at the moment of a new pulse onset. Thus the set $S(c, \varphi, \xi, p_1)$, depending on four parameters, is rather wide.

The main result is that in time $t_2 = T + o(1)$ we find the system in the state analogous to the initial one with replacing $(c, \xi, p_1, \varphi)$ by $(\tilde{c}, \tilde{\xi}, \tilde{p}_1, \tilde{\varphi})$, where

$$\begin{align*}
\tilde{c} &= q + (c - p - q)e^{-T} + \frac{\gamma p_1}{g'(\tilde{\xi})} e^{\tilde{\xi} - T}, \\
\tilde{p}_1 &= p, \\
\tilde{\xi} &= \tau_0 + B \cos \varphi - T, \\
\tilde{\varphi} &= \omega T + \varphi, \mod 2\pi,
\end{align*}$$

with $\tilde{\xi} = \tilde{\xi}(\xi, \varphi)$ is the root of the equation $\tilde{\xi} - B \cos(\omega \tilde{\xi} + \varphi) = \xi - B \cos \varphi$, $p = p(c)$ is the root of the equation $c - p = c \exp(-p)$ and $T = T(c, \varphi, p_1, \xi)$ being positive root of the equation

$$(q - 1)T + (c - p - q)(1 - e^{-T}) + \frac{\gamma p_1}{g'(\tilde{\xi})}(1 - e^{\tilde{\xi} - T}) = 0.$$  

The obtained 4D-map describes the dynamics of FO1-solutions if the inequalities

$$a(\tilde{\xi}) < 0, \quad T(c, \varphi, p_1, \xi) < \tau_0 + B$$

(9)
are valid for each iteration of the map. Conditions (9) provide the reproduction of FO\(^1\)-structure of pulsed solution.

Numerical simulations show that the conditions can be valid for map’s attractors only in the case of \(\gamma > 0\) (positive feedback). With modulation amplitude \(B\) increases, chaotic attractor can form, hence, one can expect chaotic spiking of special FO\(^1\)-structure in the original infinite-dimensional delayed system.

Inequalities (9) restrict the domain in the parameter space of the system, where solutions of this type can be realized. It appears that the parameter domain for FO\(^1\)-solution may intersect with the domain of SO-solutions, hence the multiplicity of different periodic spike regimes takes place.

It is possible to construct analogously \((2n + 2)\)-dimensional map responsible for FO\(^n\) solutions representing \(n > 1\) spikes within the delay interval and find the parameter domain where such solutions coexist. Below we numerically demonstrate the coexistence of chaotic spike regimes in the case of the modulated delay.

5. Numerical simulation

In order to verify the above theoretical conclusions we provide numerical simulations of the original delayed system with \(v = 10^3\) and \(q = 1.5\) that are typical for a semiconductor laser. Note, \(v\) takes sufficiently large value. Other parameters were chosen so that both maps (5) and (8) have the chaotic attractor: FB level \(\gamma = 0.3\), the delay \(\tau_0 = 0.6\), modulation amplitude of the delay \(B = 0.032\), modulation frequency \(\omega = 29.814\) was chosen comparable with natural relaxation frequency. Preliminary calculation of maps is really necessary, because the parameter regions where various solutions are realized are relatively narrow, and also depend on the initial conditions.

In accordance with set (3) we choose the initial function \(u(s)\) in the form \(u(s) = 0\), \(s \in [-(\tau_0 + B), 0)\), \(u(0) = 1\). The initial value of the modulation phase is \(\Phi(0) = 3.6\). The initial value of the inversion \(y(0) = 1.25\) was chosen so that map (3) has the SO chaotic attractor. At \(y(0) = 1.15\) map (6) has the FO\(^1\) chaotic attractor and at \(y(0) = 1.08\) there is FO\(^2\) chaotic attractor. In order to specify structure of irregular spiking, we show the dependence of inter-

![Figure 1](image-url)  
**Figure 1.** Coexistence of SO (1), FO\(^1\) (2) and FO\(^2\) (3) spiking which are obtained under different initial conditions for the delayed system (1).

pulse intervals \(T_{i+1}\) on \(T_i\) that correlates with a projection of the maps obtained. In Fig.2 coexistence of three chaotic attractors of different structure is demonstrated. One can see that all points of the map corresponding to SO spikes are located in domain \(T_i > \tau_0\). In domain \(\tau_0/2 < T_i < \tau_0\) the map corresponds to FO\(^1\) chaotic attractor, and in domain \(\tau_0/3 < T_i < \tau_0/3\) the map corresponds to FO\(^2\) chaotic attractor with two pulses on any delay interval.
5.1. Switching to the desired attractor

Here we study the effects of an additional short-time square-shape control signal in the pumping circuit. In Eqs. (1) we set

\[ q(t) = q + \rho, \quad t \in [t_x, t_x + \Theta], \]

where \( \rho \) is the force amplitude, \( t_x \) is the moment of the signal application, \( \Theta \) is its duration of the order or shorter than radiation pulse width, \( \eta = \rho \Theta \) is the pulse energy which assumed to be not an asymptotically small value.

In order to switch on the target attractor one has to move the phase trajectory into its attractive basin. The phase space of the delayed system (1) is infinite-dimensional. Therefore, we can not fully describe the attractive basins of coexisting solutions. But, using asymptotic conditions (6) and (9) for existence of the solution of each type, it is possible to propose perturbations guaranteeing access to the target attractor.

In order to move from an SO solution to a FO solution, it is necessary to violate condition (6). To do this, we must apply a perturbation of the corresponding energy \( \eta \) at the time \( t_x \) in the interval less than \( \tau_0 \), following the radiation pulse, as shown in Fig. 2a.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Switching induced by pump-perturbation signal: a) from irregular SO spiking to FO\(^1\) spiking, b) from irregular FO\(^1\) spiking to SO spiking, c) from irregular FO\(^1\) spiking to FO\(^2\) spiking. The moment of application of the control signal is marked with a black triangle.}
\end{figure}

In order to move from an FO\(^1\)-solution to a SO-solution, it is necessary to violate condition (9), namely to get \( T_i > \tau_0/2 \). To do this, we must apply a perturbation of the corresponding energy \( \eta \) at the time \( t_x \) synchronized with the radiation pulse, as shown in Fig. 2b. At last, if the perturbation restricts condition (9), namely to get \( T_i < \tau_0/3 \), FO\(^1\)-solution can switch to FO\(^2\)-solution, as shown in Fig. 2c.

6. Conclusion

In conclusion, we have analytically described pulsed solutions of various structure in the laser diode with variable delayed FB. Doing so, the problem on dynamics of the original infinite-dimensional system has been reduced to the problem on the dynamics of nonlinear finite-
dimensional maps. The advantages of the proposed asymptotic method are as follows: 1) the initial conditions (attractive basin) are determined for the desired regime; 2) the domain of parameters can be found for the desired regime; 3) the dependencies of characteristics (amplitude and inter-spike interval) can be obtained from the control parameters; 4) bifurcations of spikes can be followed.

A hierarchy of spiking regimes has been proposed on the base of pattern complexity, i.e. on the number $n$ of the pulses within the delay interval. We distinguish between slowly oscillating solutions determined by the 2D-map, fast oscillating solutions determined by the $(2n+2)$D-maps and mixed solutions determined by the 2D-map. Since the dynamics of such a map completely determines pulse dynamics, including chaotic pulsing, we expect that the dimension of the corresponding attractor in the infinite-dimensional phase space would be respectively limited. With increasing amplitude of modulation of the delay time, we observed intermittency of slowly and fast oscillating solutions and merge of attractors. Such a scenario leading to annihilation one of the coexisting states can be proposed for controlled monostability of a chaotic attractor.

The obtained maps can be used for finding the parameters of the system at which chaotic pulsing with special characteristics are realized. In the case of negative FB, we demonstrate a chaotic spikes following in strictly alternating intervals of less than and more than the delay time. In the case of positive FB, we demonstrate possible coexistence of slowly and fast oscillating solutions (multistability of spiking) at the same parameters. These results are promising for developing methods for dynamic control, in particular, for the fast switching of attractors by a suitable external shock.

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