Nonholonomic Ricci Flows: 
III. Curve Flows and Solitonic Hierarchies

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Abstract

The geometric constructions are elaborated on (semi) Riemannian manifolds and vector bundles provided with nonintegrable distributions defining nonlinear connection structures induced canonically by metric tensors. Such spaces are called nonholonomic manifolds and described by two equivalent linear connections also induced in unique forms by a metric tensor (the Levi Civita and the canonical distinguished connection, d–connection). The lifts of geometric objects on tangent bundles are performed for certain classes of d–connections and frame transforms when the Riemann tensor is parametrized by constant matrix coefficients. For such configurations, the flows of non–stretching curves and corresponding bi–Hamilton and solitonic hierarchies encode information about Ricci flow evolution, Einstein spaces and exact solutions in gravity and geometric mechanics. The applied methods were elaborated formally in Finsler geometry and allows us to develop the formalism for generalized Riemann–Finsler and Lagrange spaces. Nevertheless, all geometric constructions can be equivalently re–defined for the Levi Civita connections and holonomic frames on (semi) Riemannian manifolds.

Keywords: Ricci flow, curve flow, (semi) Riemannian spaces, Finsler and Lagrange geometry, nonholonomic manifold, nonlinear connection, bi–Hamiltonian, solitonic equations.

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1 Introduction

Both the theory of Ricci flows and the theory of integrable partial differential equations have deep links to the geometry of curves and surfaces, generalized Riemann–Finsler spaces and geometric analysis:

Originally, the Ricci flow theory has addressed geometrical and topological issues, and certain applications in physics, for Riemannian manifolds [1, 2, 3, 4, 5] (we cite here some reviews on Hamilton–Perelman theory [6, 7, 8, 9]). In parallel, it was found that various classes of solitonic equations (for instance, the sine–Gordon, SG, and modified Korteweg–de Vries, mKdV, equations) and along with their hierarchies of symmetries, conservation laws and associated recursion operators can be encoded into the geometry of flows of non–stretching curves in Riemannian symmetric spaces and related Lie algebras and Klein spaces [10, 11, 12, 13, 14, 15, 16, 17], see also reviews and new results in Refs. [18, 19].

In [20], it was proven that a more general class of (pseudo) Riemannian spaces can be encoded into bi–Hamilton structures and related solitonic hierarchies. The key construction was to deform nonholonomically the frame and linear connection structures in order to get constant matrix curvature coefficients, with respect to certain classes of nonholonomic frames. Such frames are adapted to a nonlinear connection (N–connection) structure induced by some generic off–diagonal metric coefficients. We also concluded that having generated the corresponding solitonic hierarchies for the so–called canonical distinguished connection (d–connection), we can re–define equivalently the geometric objects, conservation laws and basic equations and solutions in terms of the usual Levi Civita connection.

The formalism applied in [20], based on the geometry of moving nonholonomic frames with associated N–connection structure, was originally developed in Finsler–Lagrange geometry and generalizations [21, 22, 23]. Our idea [24] was to apply it to usual (semi) Riemannian spaces, or to Riemann–Cartan ones (with nontrivial torsion), prescribing certain nonholonomic distributions arising naturally if we constrain partially some degrees of freedom. For such systems, there are defined certain classes of preferred frames and symmetries for the gravitational and matter field interactions. Following this approach, it was possible to construct various classes of exact solutions in Einstein and string gravity modelling Finsler like locally anisotropic structure, possessing noncommutative symmetries and defining generically off–diagonal metrics and nonlinear interactions of pp–waves, two and three dimensional gravitational solitons and spinor fields [24, 26, 27].

Nevertheless, if realistic theories of gravitational and gauge field interactions and/or generalized Finsler models of geometric mechanics are intro-
duced into consideration, the solitonic encoding of metric, connection and frame structures is possible for certain effective generalized Lagrange spaces. In this case, we model the geometric constructions on couples of symmetric Riemannian spaces provided with nonholonomic distributions. The work [25] concluded an approach when different theories of gravity and geometric mechanics are treated in a unified geometric way as nonholonomic (semi) Riemannian manifolds, or vector bundles. It was also proven that the data for geometric objects and fundamental physical equations (their symmetries and conservation laws) can be encoded into bi–Hamilton structures and correspondingly derived solitonic hierarchies.

Integrable and nonintegrable (i.e. holonomic and nonholonomic / anholonomic) flows of geometric structures were also recently considered in a series of works on nonholonomic Ricci flows [28, 29] and applications in physics [30, 31, 32, 33]. Some important results of those works were the proofs that constrained Ricci flows of (semi) Riemannian metrics result in Finsler like metrics and generalizations and, inversely, Finsler–Lagrange type geometrical objects can be described equivalently by flows on Riemann (in general, Riemann–Cartan) spaces.

The goal of this paper, the third one in a series defined together with [28, 29], is to prove that solitonic hierarchies can be generated by any (semi) Riemannian metric $g_{ij}$ on a manifold $V$ of dimension $\text{dim} V = n \geq 2$ if the geometrical objects are lifted in the total space of the tangent bundle $TV$, or of a vector bundle $\mathcal{E} = (M, \pi, E)$, $\text{dim} E = m \geq n$, by defining such frame transforms when constant matrix curvatures are defined canonically with respect to certain classes of preferred systems of reference. We shall also define the criteria when families of bi–Hamilton structures and solitonic hierarchies encode (in general, nonholonomic) Ricci flow evolutions of geometric objects and/ or exact solutions of gravitational field equations.

The paper is organized as follows:

In section 2 we outline the geometry of nonholonomic manifolds and vector/ tangent bundles provided with nonlinear connection structure. We emphasize the possibility to define fundamental geometric objects induced by a (semi) Riemannian metric on the base space when the Riemannian curvature tensor has constant coefficients with respect to a preferred nonholonomic basis. We also present some results on evolution equations of nonholonomic Ricci flows and exact solutions in gravity.

In section 3 we consider Ricci flow families of curve flows on nonholonomic vector bundles. We sketch an approach to classification of such spaces defined by conventional horizontal and vertical symmetric (semi) Riemannian subspaces and provided with nonholonomic distributions defined by the nonlinear connection structure. It is constructed a corresponding family of non-
holonomic Klein spaces for which the bi–Hamiltonian operators are defined by canonical distinguished connections, adapted to the nonlinear connection structure, for which the distinguished curvature coefficients are constant.

Section 4 is devoted to the formalism of distinguished bi–Hamiltonian operators and vector soliton equations for arbitrary (semi) Riemannian spaces. Then we consider the properties of cosympletic and symplectic operators adapted to the nonlinear connection structure. We define the basic equations for nonholonomic curve flows and parametrize their possible Ricci flows.

Section 5 is devoted to formulation of the Main Result: a proof that for any nonholonomic Ricci flow system, one can be defined a natural family of $N$–adapted bi–Hamiltonian flow hierarchies inducing anholonomic solitonic configurations. There are constructed in explicit form the solitonic hierarchies corresponding to the bi–Hamiltonian anholonomic curve flows. Finally, there are speculated the conditions when from solitonic hierarchies we can extract solutions of the Ricci flow and/or field equations.

We summarize and discuss the results in section 6. For convenience, we outline the necessary definitions and formulas from the geometry of nonholonomic manifolds in Appendix A. Then, in Appendix B, we consider the geometry of $N$–anholonomic Klein spaces. A proof of the Main Theorem is sketched in Appendix C.

Notation remarks:

There are considered two types of flows of geometrical objects on manifolds of necessary smooth class, induced by 1) non–stretching curve flows $\gamma(\tau, l)$, defined by real parameters $\tau$ and $l$, and 2) Ricci flows of metrics $g_{ij}(\chi)$, parametrized by a real $\chi$. The non–stretching flows of a curve are constrained by the condition $g_{ij}(\gamma_\tau, \gamma_1) = 1$, which under Ricci flows transforms into a family of such conditions, $g_{ij}(\gamma_\tau, \gamma_1, \chi) = 1$. For Ricci flows, we get evolutions of families of non–stretching curves parametrized by hypersurfaces $\gamma(\tau, l, \chi) \equiv \chi(\gamma(\tau, l))$. It is convenient to use in parallel two types of notations for the geometric objects subjected to both curve and Ricci flows: by emphasized all dependencies on parameters $\tau, l$ and $\chi$ or by introducing “up/lower” labels like $\chi \gamma = \gamma(\ldots, \chi)$ or $\chi A = A(\ldots, \chi)$.

We shall also write ”boldface” symbols for geometric objects and spaces adapted to a noholonomic/ nonlinear connection structure, like $\mathbf{V}, \mathbf{E}, \ldots$ and write $V, E, \ldots$ if the nonholonomic structure became trivial, i.e. integrable/holonomic. In order to investigate the properties of curve and Ricci flow evolution equations it is convenient to use both abstract/global denotations and coefficient formulas with respect to coordinate or nonholonomic bases.
A nonholonomic distribution with associated nonlinear connection structure splits the manifolds into conventional horizontal \((h)\) and vertical \((v)\) subspaces. The geometric objects, for instance, a vector \(X\) can be written in abstract form as \(X = (hX, vX) = (\hat{h}X, \hat{v}X)\), or in coefficient forms as \(X^\alpha = (X^i, X^a) = (\hat{h}X^i, \hat{v}X^a)\), where \(X^\alpha e_\alpha = X^i e_i + X^a e_a = X^i \partial_i + X^a \partial_a\) can be equivalently decomposed with respect to a general nonholonomic frame \(e_\alpha = (e_i, e_a)\) or coordinate frame \(\partial_\alpha = (\partial_i, \partial_a)\) for local \(h\)- and \(v\)-coordinates \(u = (x, y)\), or \(u^\alpha = (x^i, y^a)\) when indices will be underlined if it is necessary to emphasize certain decompositions are defined for coordinate bases. The \(h\)-indices \(i, j, k, ... = 1, 2, ... n\) will be used for base/ nonholonomic vector objects and the \(v\)-indices \(a, b, c... = n + 1, n + 2, ... n + m\) will be used for fiber/ holonomic vector objects. Greek indices of type \(\alpha, \beta, ...\) will be used as cumulative ones.

Finally, we note that we shall omit labels, indices and parametric/ coordinate dependencies for some formulas if it does not result in ambiguities.

## 2 Nonholonomic Lifts and Ricci Flows

In this section, we prove that for any family of (semi) Riemannian metrics \(g_{ij}(\chi)\) on a manifold \(V\), parametrized by a real parameter \(\chi\), it is possible to define lifts to the tangent bundle \(TV\) provided with canonical nonlinear connection (in brief, \(N\)-connection), Sasaki type metrics and (linear) canonical distinguished connection (\(d\)-connection) structures. We also outline some important formulas for nonholonomic Ricci flow evolution equations of geometric structures. The reader is recommended to consult Refs. [24, 26, 21, 22, 28, 29] and Appendix A on details on \(N\)-connection geometry and recent developments in modern gravity and Ricci flow theory.

### 2.1 \(N\)-connections induced by families of Riemannian metrics

Let \(\mathcal{E} = (E, \pi, F, M)\) be a (smooth) vector bundle over base manifold \(M\), \(\dim M = n\) and \(\dim E = (n+m)\), for \(n \geq 2\), and \(m \geq n\) being the dimension of the typical fiber \(F\). It is defined a surjective submersion \(\pi : E \to M\). In any point \(u \in E\), the total space \(E\) splits into "horizontal", \(M_u\), and "vertical", \(F_u\), subspaces. We denote the local coordinates in the form \(u = (x, y)\), or \(u^\alpha = (x^i, y^a)\), with horizontal indices \(i, j, k, ... = 1, 2, ... n\) and vertical
indices \(a, b, c, \ldots = n + 1, n + 2, \ldots, n + m\). The summation rule on the same “up” and “low” indices will be applied.

The base manifold \(M\) is provided with a family of (semi) Riemannian metrics, nondegenerate second rank tensors, 

\[
\mathbf{h}(\chi) = g_{ij}(x, \chi) dx^i \otimes dx^j, \text{ for } 0 \leq \chi \leq \chi_0 \in \mathbb{R}.
\]

We also introduce a family of vertical metrics \(v\mathbf{g}(\chi) = g_{\alpha\beta}(x, \chi) dy^\alpha \otimes dy^\beta\) by completing the matrices \(g_{ij}(x, \chi)\) diagonally with \(\pm 1\) till any nondegenerate second rank tensor \(g_{\alpha\beta}(x, \chi)\) if \(m > n\) and then subjecting to any frame transforms. This way, we define certain families of metrics \(\mathbf{g}(\chi) = [h\mathbf{g}(\chi), v\mathbf{g}(\chi)]\) (we shall also use the notation \(g_{\alpha\beta}(\chi) = [g_{ij}(\chi), g_{\alpha\beta}(\chi)]\) on \(E\). Considering frame (vielbein) transforms, 

\[
g_{\alpha\beta}(x, y, \chi) = e^\alpha_\alpha(x, y, \chi) e^\beta_\beta(x, y, \chi) g_{\alpha\beta}(x, \chi), \tag{1}
\]

where \(g_{\alpha\beta}(x, \chi)\) is written in equivalent form \(g_{\alpha\beta}(x, \chi)\), we can deform the metric structures, \(g_{\alpha\beta} \rightarrow g_{\alpha\beta} = [g_{ij}, g_{\alpha\beta}]\) (we shall omit dependencies on coordinates and parameters if it does not result in ambiguities). The coefficients \(e^\alpha_\alpha(x, y, \chi)\) will be defined below (see formula (6)) from the condition of generating curvature tensors with constant coefficients with respect to certain preferred systems of reference.

For any \(g_{ab}(\chi)\) from the set \(g_{\alpha\beta}(\chi)\), we can construct a family of effective generation functions

\[
\mathcal{L}(x, y, \chi) = g_{ab}(x, y, \chi) y^a y^b
\]

inducing families of vertical metrics

\[
\tilde{g}_{ab}(x, y, \chi) = \frac{1}{2} \frac{\partial^2 \mathcal{L}(x, y, \chi)}{\partial y^a \partial y^b} \tag{2}
\]

which is ”weakly” regular if \(\det |\tilde{g}_{ab}| \neq 0\).

By straightforward computations, we prove \(^3\).

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\(^1\)In a particular case, we have a tangent bundle \(E = TM\), when \(n = m\); for such bundles both type of indices run the same values but it is convenient to distinguish the horizontal and vertical ones by using different groups of small Latin indices. Here one should be noted that on \(TM\) we are able to contract the vertical indices with the corresponding horizontal ones, and inversely, but not on a general nonholonomic manifold \(V\), or \(E\).

\(^2\)in physical literature, one uses the term (pseudo) Riemannian/Euclidean space

\(^3\)see Refs. \([21, 22]\) for details of a similar proof; here we note that in our case, in general, \(e^\alpha_\alpha \neq \delta^\alpha_\alpha\).
**Theorem 2.1** The family of Lagrangians $L(\chi) = \sqrt{|L(\chi)|}$, where $y^i = \frac{dx^i}{d\tau}$ for paths $x^i(\tau)$ on $M$, depending on parameter $\tau$, with weakly regular metrics induces a family of Euler–Lagrange equations on $T M$,

$$\frac{d}{d\tau} \left( \frac{\partial L(\chi)}{\partial y^i} \right) - \frac{\partial L(\chi)}{\partial x^i} = 0,$$

which are equivalent to a corresponding family of “nonlinear” geodesic equations

$$\frac{d^2 x^i}{d\tau^2} + 2\tilde{G}^i(x^k, \frac{dx^j}{d\tau}, \chi) = 0$$

defining paths of a canonical semispray $S(\chi) = y^i \frac{\partial}{\partial x^i} - 2\tilde{G}^i(x, y, \chi) \frac{\partial}{\partial y^i}$, where

$$2\tilde{G}^i(x, y, \chi) = \frac{1}{2} \tilde{g}^{ij}(\chi) \left( \frac{\partial^2 L(\chi)}{\partial y^i \partial x^k} y^k - \frac{\partial L(\chi)}{\partial x^i} \right)$$

with $\tilde{g}^{ij}(\chi)$ being inverse to $\tilde{g}^{ij}$.

The Theorem 2.1 states the possibility to geometrize the regular Lagrange mechanics by geometric objects on nonholonomic spaces and inversely:

**Conclusion 2.1** For any family of (semi) Riemannian metrics $g_{ij}(x, \chi)$ on $M$, we can associate canonically certain families of effective regular Lagrange mechanical systems on $T M$ with the Euler–Lagrange equations transformed into corresponding families of nonlinear (semispray) geodesic equations.

**Theorem 2.2** Any family of (semi) Riemannian metrics $g_{ij}(x, \chi)$ on $M$ induces a corresponding family of canonical $N$–connection structures on $T M$.

**Proof.** We sketch a proof by defining the coefficients of $N$–connection, see (A.1),

$$\tilde{N}^i_{\ j}(x, y, \chi) = \frac{\partial \tilde{G}^i(x, y, \chi)}{\partial y^j} \quad (3)$$

where

$$\tilde{G}^i(\chi) = \frac{1}{4} \tilde{g}^{ij} \left( \frac{\partial^2 L}{\partial y^i \partial x^j} y^k - \frac{\partial L}{\partial x^j} \right) = \frac{1}{4} \tilde{g}^{ij}(\chi) g_{jk}(\chi) \gamma^h_{il}(\chi) y^l y^m, \quad (4)$$

$$\gamma^i_{\ lm}(\chi) = \frac{1}{2} \tilde{g}^{ih}(\chi) \left[ \partial_m g_{lh}(\chi) + \partial_l g_{mh}(\chi) - \partial_h g_{lm}(\chi) \right], \quad \partial_h = \partial/\partial x^h,$$

with $g_{ab}(\chi)$ and $\tilde{g}_{ab}(\chi)$ defined respectively by formulas (1) and (2), $\square$.

The families of $N$–adapted partial derivative and differential operators, see Appendix for more general formulas (A.2) and (A.3), are defined by
the N–connection coefficients (3) and may be denoted respectively \( \tilde{e}_e(\chi) = (\tilde{e}_i(\chi), e_a) \) and \( \tilde{\nu}^\mu(\chi) = (e^i, \tilde{e}_a(\chi)) \).

For any metric structure \( g \) on a manifold, there is the unique metric compatible and torsionless Levi Civita connection \( \nabla \) for which \( \nabla T^\alpha = 0 \) and \( \nabla g = 0 \). This connection is not a d–connection because it does not preserve under parallelism the N–connection splitting (A.1). One has to consider less constrained cases, admitting nonzero torsion coefficients, when a d–connection is constructed canonically for a d–metric structure. A simple minimal metric compatible extension of \( \nabla \) is that of canonical d–connection \( \hat{\nabla} \), with \( T^i_{jk} = 0 \) and \( T^a_{bc} = 0 \) but, in general, nonzero \( T^a_{ji} \) and \( T^a_{bi} \), see (A.9). The coefficient formulas for such connections are given in Appendix, see (A.15) and related discussion. It should be noted that on tangent bundle \( TM \) it is possible to define the torsionless canonical d–connection (A.16) which is completely similar to the Levi Civita connection. For families of metrics \( g(\chi) \), we get certain families of connections \( \nabla(\chi) \) and \( \hat{\nabla}(\chi) \).

**Theorem 2.3** Any family of (semi) Riemannian metrics \( g_{ij}(x, \chi) \) on \( M \) induces a parametrized by \( \chi \) family of nonholonomic (semi) Riemannian structures on \( TM \).

**Proof.** The family \( g_{ij}(x, \chi) \) on \( M \) induces a family of canonical d–metric structures on \( TM \),

\[
\tilde{g}(\chi) = \tilde{g}_{ij}(x, y, \chi) \ e^i \otimes e^j + \tilde{g}_{ij}(x, y, \chi) \ \tilde{e}^i(\chi) \otimes \tilde{e}^j(\chi),
\]

where \( \tilde{e}^i(\chi) \) are elongated as in (A.3), but with \( \tilde{N}^i_j(\chi) \) from (3). Then, we note that there are canonical d–connections on \( TM \) induced by \( g_{ij}(x, \chi) \) : we can construct them in explicit form by introducing \( \tilde{g}_{ij}(\chi) \) and \( g_{ab}(\chi) \) in formulas (A.16), in order to compute the coefficients \( \tilde{\Gamma}^\alpha_{\beta\gamma}(\chi) = (\tilde{\cal{L}}^i_{jk}(\chi), \tilde{C}^a_{bc}(\chi)) \).

The corresponding curvature curvature tensor

\[
\tilde{R}^\alpha_{\beta\gamma\tau}(\chi) = \{ \tilde{R}_{hjk}(\chi), \tilde{P}_{jka}(\chi), \tilde{S}_{bcd}(\chi) \}
\]

can be computed by introducing respectively the values \( \tilde{g}_{ij}(\chi), \tilde{N}^i_j(\chi) \) and \( \tilde{e}_k(\chi) \) into (A.16), defining \( \tilde{\Gamma}^\alpha_{\beta\gamma}(\chi) = (\tilde{\cal{L}}^i_{jk}(\chi), \tilde{C}^a_{bc}(\chi)) \) and then into formulas (A.21). Here one should be noted that the constructions on \( TM \) depend on arbitrary vielbein coefficients \( e^\alpha_\beta(x, y, \chi) \) in (11). We can restrict such sets of coefficients in order to generate various particular classes of (semi) Riemannian geometries on \( TM \), for instance, in order to generate symmetric Riemannian spaces with constant curvature, see Refs. [36, 37, 38].

8
Corollary 2.1 There are lifts of a family of (semi) Riemannian metric $g_{ij}(x, \chi)$ on $M$, $\dim M = n$, generating a corresponding family Riemannian structures on $TM$ with the curvature coefficients of the canonical d-connections coinciding (with respect to $N$-adapted bases) to those for the families of Riemannian space of constant curvature of dimension $n+n$.

Proof. For a given set $g_{ij}(x, \chi)$ on $M$, in (1), we chose such coefficients $e_\alpha^a(x, y, \chi) = \{e_a^\alpha(x, y, \chi)\}$ that

$$g_{ab}(x, y, \chi) = e_\alpha^a(x, y, \chi) e_\beta^b(x, y, \chi) g_{ab}(x, \chi)$$

results in (2) of type

$$\tilde{g}_{ef}(\chi) = \frac{1}{2} \frac{\partial^2 L(\chi)}{\partial y^e \partial y^f} = \frac{1}{2} \frac{\partial^2 (e_a^\alpha e_b^\beta y^a y^b)}{\partial y^e \partial y^f} g_{ab}(x, \chi) = \tilde{g}_{ef}(\chi),$$

where $\tilde{g}_{ab}(\chi)$ are metrics of symmetric Riemannian space (of constant curvature). Considering a prescribed set $\tilde{g}_{ab}(\chi)$, we have to integrate two times on $y^e$ in order to find any solution for $e_\alpha^a(\chi)$ defining a frame structure in the vertical subspace. The next step is to construct the d–metric $\tilde{g}_{ab}(\chi) = [\tilde{g}_{ij}(\chi), \tilde{g}_{ab}(\chi)]$ of type (4), in our case, with respect to a nonholonomic base elongated by $\tilde{N}_{ij}(\chi)$, generated by $\tilde{g}_{ij}(x, \chi)$ and $\tilde{g}_{ef} = \tilde{g}_{ab}(\chi)$, like in (3) and (4). This defines a constant curvature Riemannian space of dimension $n+n$. The coefficients of the canonical d–connection, which in this case coincide with those for the Levi Civita connection, and the coefficients of the Riemannian curvature can be computed respectively by introducing $\tilde{g}_{ef} = \tilde{g}_{ab}(\chi)$ in formulas (A.16) and (A.21). Finally, we note that the induced symmetric Riemannian spaces contain additional geometric structures like the $N$–connection and anholonomy coefficients $W^\gamma_{\alpha\beta}(\chi)$, see (A.3).

There are various possibilities to generate on $TM$ nonholonomic Riemannian structures from a given set $g_{ij}(x, \chi)$ on $M$. They result in different geometrical and physical models.

Remark 2.1 We can simplify substantially the geometric constructions if instead of families of constant coefficients $\tilde{g}_{ef}(\chi)$, we consider only one set of constant coefficients $\tilde{g}_{ef} = \tilde{g}_{ef}(\chi_0)$. This is possible even $g_{ab}(x, \chi)$ have quite general dependencies on $\chi$ but supposing that we can define such $e_\alpha^a(x, y, \chi)$ when (6) can be solved for a fixed right side. For simplicity, in our further considerations we shall fix any set $\tilde{g}_{ef}$ and parametrize the dependencies on $\chi$ for $g_{ij}$ and $e_\alpha^a$. 

9
In this work, we emphasize the possibility of generating spaces with constant curvature because for such symmetric spaces it was elaborated a bi–Hamiltonian approach and corresponding solitonic hierarchies \cite{15, 16, 18, 19}. For general Riemannian and/or Finsler–Lagrange spaces it is not possible to get constant coefficient curvature coefficients for the Levi Civita connection but for the corresponding lifts to the canonical d–connection there are constructions generating constant curvature coefficients \cite{20, 25}. This will allow us to construct the corresponding solitonic hierarchies from which, by imposing the corresponding constraints, it will be possible to extract the information for very general classes of metrics.

**Example 2.1** The simplest example when a Riemannian structure with constant matrix curvature coefficients is generated on $T M$ is given by a $d$–metric induced by $\tilde{g}_{ij} = \delta_{ij}$, i.e.

$$\tilde{g}[E] = \delta_{ij} e^i \otimes e^j + \delta_{ij} \tilde{e}^i \otimes \tilde{e}^j,$$

with $\tilde{e}^i$ defined by $\tilde{N}^i_j$ in their turn induced by a given set $g_{ij}(x)$ on $M$. For families of geometric objects, we consider

$$\tilde{g}[E](\chi) = \delta_{ij} e^i \otimes e^j + \delta_{ij} \tilde{e}^i(\chi) \otimes \tilde{e}^j(\chi),$$

when $\tilde{N}^i_j(\chi)$ are defined by a given set $\tilde{g}_{ij}(x, \chi)$.

For more general nonholonomic configurations on $T M$, we can consider families of metrics of type

$$g(\chi) = g_{ij}(x, y, \chi) e^i \otimes e^j + g_{ab}(x, y, \chi) e^a(\chi) \otimes e^b(\chi),$$

where

$$g_{ij}(x, y, \chi) = \eta_{ij}(x, y, \chi) \tilde{g}_{ij}(x, y, \chi) \text{ and } g_{ab}(x, y, \chi) = \eta_{ab}(x, y, \chi) \tilde{g}_{ab}(x, y, \chi)$$

and $e^a(\chi)$ are elongated by $N^i_j(x, y, \chi) = \eta^i_j(x, y, \chi) \tilde{N}^i_j(x, y, \chi)$. We note that in the formulas defining the coefficients of the metrics \cite{8} one does not consider summation on repeating indices which are not ”cross” one, i.e. $\eta_{ij} \tilde{g}_{ij}$ means a simple product $\eta_{ij} \times \tilde{g}_{ij}$ between deformation function $\eta_{ij}$ and metric coefficient $\tilde{g}_{ij}$, for any fixed values $i, j, ...$ or $a, b, ...$ It is possible to write down the metrics \cite{8} in ”generic off–diagonal forms”, see \cite{A.11} and \cite{A.12}, for any fixed value of $\chi$. 

10
2.2 Nonholonomic Ricci flows and Einstein spaces

In the theory of Ricci flows, the families of metrics \( \mathcal{R} \) must satisfy certain evolution equations on parameter \( \chi \). For normalized (holonomic) Ricci flows \([3, 6, 8, 9]\), with respect to a coordinate base \( \partial_\alpha = \partial/\partial u^\alpha \), the evolution equations are postulated in the form

\[
\frac{\partial}{\partial \chi} g_{\alpha\beta} = -2 \hat{R}_{\alpha\beta} + \frac{2}{3} g_{\alpha\beta},
\]

(9)

where the normalizing factor \( r = \int \omega \cdot dV / dV \) is introduced in order to preserve the volume \( V \) and the metric coefficients \( g_{\alpha\beta} \) are parametrized in the form (A.12), and \( \hat{R}_{\alpha\beta} \) is the Ricci tensor for the Levi Civita connection \( \nabla \). In Refs. [28, 29, 33] we discuss in details the \( N \)-anholonomic Ricci flows and prove that the nonholonomic version of (9) can be proven by a \( N \)-adapted calculus from the Perelman’s functionals,

\[
\frac{\partial}{\partial \chi} g_{ij} = -2 \left[ \hat{R}_{ij} - \lambda g_{ij} \right] - \frac{\partial}{\partial \chi} \left( g_{ab}N^a_iN^b_j \right),
\]

(10)

\[
\frac{\partial}{\partial \chi} g_{ab} = -2 \left( \hat{R}_{ab} - \lambda g_{ab} \right),
\]

(11)

\[
\hat{R}_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta,
\]

(12)

where \( \lambda = r/5 \). We note that the equations (12) constrain nonholonomic Ricci flows to result in symmetric metrics and that we wrote them with respect to \( N \)-adapted frames. A simple class of solutions can be constructed for the families of \( N \)-anholonomic Einstein spaces when

\[
\hat{R}_{ij} - \lambda g_{ij} = 0, \quad \hat{R}_{ab} - \lambda g_{ab} = 0
\]

(13)

and

\[
\frac{\partial}{\partial \chi} g_{ij} = -g_{ab} \frac{\partial}{\partial \chi} (N^a_iN^b_j), \quad \frac{\partial}{\partial \chi} g_{ab} = 0.
\]

(14)

Such equations define some effective Einstein metrics subjected to Ricci flows under evolution of the \( N \)-anholonomic structure \( N^a_i \) correlated with the evolutions of \( h \)-metric \( g_{ij} \) but \( g_{ab} \) stated for a fixed value of \( \chi_0 \).

For our further considerations, we need the results of two Corollaries (see Refs. [6, 28, 29] for detailed proofs and discussions both for holonomic and nonholonomic manifolds):

**Corollary 2.2** The evolution, for all \( \chi \in [0, \chi_0] \), of preferred frames on a \( N \)-anholonomic manifold

\[
e_a(\chi) = e_\alpha(\chi, u) \partial_\alpha
\]

11
is defined by the coefficients
\[
\begin{align*}
\mathbf{e}^a_\alpha(x, u) &= \begin{bmatrix} e^i_\alpha(x, u) & N^b_i(x, u) & e^a_b(x, u) \\
0 & e^a_\alpha(x, u) & 0
\end{bmatrix}, \\
\mathbf{e}_a^\alpha(x, u) &= \begin{bmatrix} e^i_\alpha(x, u) \\
0
\end{bmatrix}
\end{align*}
\]
with
\[
\begin{align*}
g_{ij}(x) &= e^i_\alpha(x, u) e^j_\beta(x, u) \eta_{\alpha\beta}, \quad \text{and} \quad g_{ab}(x) = e^a_\alpha(x, u) e^b_\beta(x, u) \eta_{\alpha\beta},
\end{align*}
\]
where \(\eta_{\alpha\beta} = \text{diag}[\pm 1, \ldots, \pm 1]\) and \(\eta_{ab} = \text{diag}[\pm 1, \ldots, \pm 1]\) establish the signature of \(g_{\alpha\beta}^0(u)\), is given by equations
\[
\frac{\partial}{\partial \chi} e^\alpha_a = g^{\alpha\beta} \hat{R}^\beta_{\alpha\gamma} e^\gamma_a.
\]
(15)

For simplicity, we omit formulas for h- and v–decomposition of (15).

**Corollary 2.3** The scalar curvature (A.20)
\[
\hat{R} = g^{\alpha\beta} \hat{R}_{\alpha\beta} = g^{ij} \hat{R}_{ij} + h^{ab} \hat{S}_{ab} = \bar{R} + \bar{S}
\]
for the canonical d–connection on \(TM\) satisfies the evolution equations
\[
\frac{\partial \hat{R}}{\partial \chi} = \hat{D}_i \hat{D}^i \hat{R} + 2 \hat{R}_{ij} \hat{R}^{ij} \quad \text{and} \quad \frac{\partial \bar{S}}{\partial \chi} = \hat{D}_a \hat{D}^a \bar{S} + 2 \hat{S}_{ab} \hat{S}^{ab}. \quad (16)
\]

**Proof.** It is similar to that for the Levi Civita connection because on \(TM\) the coefficients of canonical d–connection with respect to N-adapted frames are the same as those for the Levi Civita but decomposed into h– and v–components. We note that on \(TM\) the Ricci d–tensor \(\hat{R}_{\alpha\beta}\) is symmetric which does not hold true for a general nonholonomic manifold or vector bundle, see formulas (A.19). The evolution equations (16) consist a particular case of more general formulas for Ricci flows on N–anholonomic manifolds proved in Theorem 4.1 of Ref. [28] (on \(TM\) the distorsion tensor transforming the \(\nabla\) into \(\hat{D}\) is zero). \(\square\)

Finally, in this section, we note that a number of geometric ideas and methods applied in this section were considered in the approaches to the geometry of nonholonomic spaces and generalized Finsler–Lagrange geometry elaborated by the schools of G. Vranceanu and R. Miron and by A. Bejancu in Romania [39, 40, 21, 22, 23, 34]. We emphasize that this way it
is possible to construct geometric models with metric compatible linear connections which is important for elaborating standard approaches in modern (non)commutative gravity and string theory [26, 24]. For Finsler spaces with nontrivial nonmetricity, for instance, for those defined by the the Berwald and Chern connections, see details in [41], the physical theories with local anisotropy are not imbedded into the class of standard models. It is also a more cumbersome task to elaborate a theory of Ricci flows of noncompatible metrics and connection structures.

3 N–Adapted Curve Flows in Vector Bundles

We formulate the geometry of curve flows adapted to the nonlinear connection structures constructed by certain classes of canonical lifts from the base space and nonholonomic frame deformations resulting into constant curvature coefficients for the canonical $\nabla$–connection. The case of tangent bundles will be emphasized as a special one when both $\nabla$ and $\hat{\nabla}$ can be torsionless.

3.1 Non–stretching and N–adapted curve flows

Let us consider a vector bundle $E = (E, \pi, F, M)$, $\dim E = n + m$ (in a particular case, $E = TM$, when $m = n$) provided with a d–metric $g = [g, h]$ (A.10) and N–connection $N^a_i$ (A.1) structures. A non–stretching curve $\gamma(\tau, l)$ on $E$, where $\tau$ is a parameter and $l$ is the arclength of the curve on $E$, is defined with such evolution d–vector $Y = \gamma_\tau$ and tangent d–vector $X = \gamma_l$ that

$$ \quad \text{g}(X, X) = 1. $$ (17)

The curve $\gamma(\tau, l)$ swept out a two–dimensional surface in $T_{\gamma(\tau, l)}V \subset TV$. If the geometric objects evolve as Ricci flows on parameter $\chi$, we get an additional parameter for the geometric objects like connections and metrics and the non–stretching condition (17) transforms into $\text{g}(X, X, \chi) = 1$ for a family of d–metrics $\chi \text{g} \equiv \text{g}(\chi) = [g(\chi), h(\chi)]$ and $N^a_i(\chi)$ which can be satisfied by certain families of curves, $\gamma(\tau, l, \chi) \equiv \chi \gamma(\tau, l)$ (briefly, we shall write only $\chi \gamma = \gamma(\chi)$) and related curve evolution and tangent vectors, $X(\chi) = \chi X = \chi \gamma_1$ and $Y(\chi) = \chi X = \chi \gamma_1$.

We work with families of N–adapted bases (A.2) and (A.3) and the connection 1–forms $\Gamma^a_\beta(\chi) = \Gamma^a_\beta(\chi) e^\gamma(\chi)$ (equivalently, $\chi \Gamma^a_\beta = \chi \Gamma^a_\beta$) with the coefficients $\Gamma^a_{\beta\gamma}(\chi) = \chi \Gamma^a_{\beta\gamma}$, for the canonical d–connection operator $D(\chi) \equiv \chi D$ (A.15) acting in the form

$$ D_Xe_\alpha = (X_i \Gamma^i_\alpha) e_\gamma \quad \text{and} \quad D_Y e_\alpha = (Y_j \Gamma^j_\alpha) e_\gamma, $$ (18)
where "\( \rfloor \)" denotes the interior product and the indices are lowered and raised respectively by the d–metric \( g_{\alpha\beta} = [g_{ij}, h_{ab}] \) and its inverse \( g^{\alpha\beta} = [g^{ij}, h^{ab}] \).

We note that \( D_{X(\gamma)} = \chi X^\alpha \chi D_\alpha \) is the covariant derivation operator along curve \( \gamma(\tau, 1, \chi) \). It is convenient to orient the N–adapted frames to be parallel respectively to curves \( \chi \gamma \)

\[
e^1 \div hX, \text{ for } i = 1, \text{ and } \hat{e}^i, \text{ where } h g(hX, e^i) = 0,\]

\[
e^{a+1} \div vX, \text{ for } a = n + 1, \text{ and } \hat{e}^a, \text{ where } v g(vX, e^a) = 0,
\]

for \( i = 2, 3, \ldots n \) and \( \hat{a} = n + 2, n + 3, \ldots, n + m \). For such frames, the covariant derivatives of each "normal" d–vectors \( \chi e^\hat{a} \) result into the d–vectors adapted to \( \chi \gamma \),

\[
\chi D_X \chi e^\hat{i} = -\rho^\hat{i}(u, \chi) \chi X \text{ and } \chi D_{hX} h \chi X = \rho^\hat{i}(u, \chi) \chi e^\hat{i},\]

\[
\chi D_X \chi e^\hat{a} = -\rho^\hat{a}(u, \chi) \chi X \text{ and } \chi D_{vX} v \chi X = \rho^\hat{a}(u, \chi) \chi e^\hat{a},
\]

which holds for certain classes of functions \( \chi \rho^\hat{i} = \rho^\hat{i}(u, \chi) \) and \( \chi \rho^\hat{a} = \rho^\hat{a}(u, \chi) \).

The formulas (13) and (20) are distinguished into \( h^- \) and \( v^- \)–components for \( \chi X = hX(\chi) + vX(\chi) \) and \( \chi D = (hD(\chi), vD(\chi)) \) for \( \chi D = \{ \chi \Gamma^\gamma_{\alpha\beta} \} \), where \( hD(\chi) = \{ \chi L^i_{jk}, \chi L^\alpha_{wk} \} \) and \( vD(\chi) = \{ \chi C^\alpha_{\gamma j}, \chi C^\alpha_{\beta k} \} \).

Along any curve \( \gamma(\chi) \), we can move differential forms in a parallel N–adapted form. For instance, \( \Gamma^\alpha_{\chi} \div X(\chi) | \chi \Gamma^\alpha_{\beta} \) which for families of d–objects is to be written \( \chi \Gamma^\alpha_{\chi} \div X(\chi) | \chi \Gamma^\alpha_{\beta} \). The algebraic characterization of such spaces, can be obtained if we perform a frame transform preserving the decomposition (A.1) to an orthonormalized basis \( e_a \), when

\[
e_a \to A_a^{\alpha'}(u) e_{\alpha'}, \quad (\chi e_a \to A_a^{\alpha'}(u, \chi) \chi e_{\alpha'}),\]

called orthonormal d–basis (family of d–bases). In this case, the coefficients of the d–metric (A.10) transform into the Euclidean ones, \( g_{\alpha'\beta'} = \delta_{\alpha'\beta'} \), (we can define such frame transform (6) when the the same constant coefficients for d–metric are generated for all values of parameter \( \chi \); for such configurations, we do not emphasize the labels/dependencies on Ricci flow parameter which are present in d–connection operators and N–connection coefficients). In distinguished form, we obtain families of two skew matrices

\[
\chi \Gamma_{hX}^{\alpha'\beta'} = hX(\chi) | \chi \Gamma^{\alpha'\beta'} = 2 e_{hX}^{\alpha'\beta'}, \chi \rho^{\alpha'\beta'}
\]

and

\[
\chi \Gamma_{vX}^{\alpha'\beta'} = vX(\chi) | \chi \Gamma^{\alpha'\beta'} = 2 e_{vX}^{\alpha'\beta'}, \chi \rho^{\alpha'\beta'}
\]

\(^{4}\)For simplicity, we shall omit the Ricci flow parameter if it does not result in ambiguities.
where
\[ e_{hX} = g(hX, e^i) = [1, 0, \ldots, 0] \text{ and } e_{vX} = h(vX, e^i) = [1, 0, \ldots, 0], \]
we omitted the Ricci flow label performing the constructions according Remark 2.1 and
\[
\chi_{X}^{\gamma} = \left[ \begin{array}{ccc} 0 & \chi_{\rho}^{i} & \chi_{\mu}^{j} \\ -\chi_{\rho}^{i} & 0_{[n]} \end{array} \right] \text{ and } \Gamma_{vX^\alpha \beta}^{\gamma} = \left[ \begin{array}{ccc} 0 & \chi_{\rho}^{i} & \chi_{\mu}^{j} \\ -\chi_{\rho}^{i} & 0_{[n]} \end{array} \right]
\]
with \(0_{[n]}\) and \(0_{[m]}\) being respectively \((n - 1) \times (n - 1)\) and \((m - 1) \times (m - 1)\) matrices. The above presented row–matrices and skew–matrices show that locally an total space of a vector bundle of dimension \(n + m\), with respect to distinguished orthonormalized frames are characterized algebraically by couples of unit vectors in \(\mathbb{R}^n\) and \(\mathbb{R}^m\) preserved respectively by the \(SO(n - 1)\) and \(SO(m - 1)\) rotation subgroups of the local \(N\)-adapted frame structure group \(SO(n) \oplus SO(m)\). The connection matrices \(\chi_{X}^{\gamma} \Gamma^{\gamma}_{hX^\alpha \beta} \) and \(\chi_{X}^{\gamma} \Gamma^{\gamma}_{vX^\alpha \beta} \) belong to the orthogonal complements of the corresponding Lie subalgebras and algebras, \(\mathfrak{so}(n - 1) \subset \mathfrak{so}(n)\) and \(\mathfrak{so}(m - 1) \subset \mathfrak{so}(m)\).

The torsion (A.8) and curvature (A.17) tensors can be in orthonormalized component form with respect to (19) mapped into a distinguished ortho-normalized dual frame (21),
\[
T^\alpha = D_X e_\gamma^\alpha - D_Y e_\gamma^\alpha + e_\gamma^\beta \Gamma^\gamma_{X^\alpha \beta} - e_\gamma^\beta \Gamma^\gamma_{Y^\alpha \beta}
\]
and
\[
R_{\beta^\gamma}^{}(X, Y) = D_Y \Gamma^\gamma_{X^\alpha \beta} - D_X \Gamma^\gamma_{Y^\alpha \beta} + \Gamma^\gamma_{Y^\alpha \beta} \Gamma^\gamma_{X^\alpha \beta} - \Gamma^\gamma_{X^\alpha \beta} \Gamma^\gamma_{Y^\alpha \beta},
\]
where \(e_\gamma^\alpha = g(Y, e_\alpha^\gamma)\) and \(\Gamma^\gamma_{Y^\alpha \beta} = Y | Y^\gamma \Gamma^\gamma_{X^\alpha \beta} = g(e_\alpha^\gamma, D_Y e_\gamma^\beta)\) define respectively the \(N\)-adapted orthonormalized frame row–matrix and the canonical \(d\)-connection skew–matrix in the flow directs, and \(R_{\beta^\gamma}^{}(X, Y) = g(e_\alpha^\gamma, [D_X, D_Y] e_\gamma^\beta)\) is the curvature matrix. Both torsion and curvature components can be distinguished in \(h\)- and \(v\)-components like (A.9) and (A.18), by considering \(N\)-adapted decompositions of type
\[
g = [g, h], e_\beta = (e_\beta^\gamma, e_\gamma^\beta), e_\alpha^\gamma = (e_\alpha^\gamma, e_\gamma^\alpha), X = hX + vX, D = (hD, vD).
\]
Finally, we note that the matrices for torsion (22) and curvature (23) can be computed for any families, parametrized by \(\chi\), metric compatible linear connection like the Levi Civita and the canonical \(d\)-connection. For our purposes, in this work, we are interested to define such a frame of reference with respect to which the curvature tensor has constant coefficients and the torsion tensor vanishes.
3.2 N–anholonomic bundles with constant matrix curvature

For vanishing N–connection torsion and constant matrix curvature of the canonical d–connection, we get couples of holonomic Riemannian manifolds and the equations (22) and (23) directly encode couples of bi–Hamiltonian structures, see details in Refs. [20, 25, 16, 18, 19]. A well known class of Riemannian manifolds for which the frame curvature matrix constant consists of the symmetric spaces $M = G/H$ for compact semisimple Lie groups $G \supset H$. A complete classification and summary of main results on such spaces are given in Refs. [36, 37, 38].

We suppose that the base manifold is a symmetric space $M = hG/SO(n)$ with the isotropy subgroup $hH = SO(n) \supset O(n)$ and the typical fiber space to be a symmetric space $F = vG/SO(m)$ with the isotropy subgroup $vH = SO(m) \supset O(m)$. This means that $hG = SO(n + 1)$ and $vG = SO(m + 1)$ which is enough for a study of real holonomic and nonholonomic manifolds and geometric mechanics models.

Our aim is to solder in a canonic way (like in the N–connection geometry) the horizontal and vertical symmetric Riemannian spaces of dimension $n$ and $m$ with a (total) symmetric Riemannian space $V$ of dimension $n + m$, when $V = G/SO(n + m)$ with the isotropy group $H = SO(n + m) \supset O(n + m)$ and $G = SO(n + m + 1)$. We note that for the just mentioned horizontal, vertical and total symmetric Riemannian spaces one exists natural settings to Klein geometry. For instance, the metric tensor $hg = \{\hat{g}_{ij}\}$ on a symmetric Riemannian space $M$ is defined by the Cartan–Killing inner product $<\cdot, \cdot>_{hG}$ restricted to the Lie algebra quotient spaces $h\mathfrak{p} = hg/hh$, with $T_xhH \simeq hh$, where $hg = hh \oplus hp$ is stated such that there is an involutive automorphism of $hG$ under $hh$ is fixed, i.e. $[hh, hp] \subseteq hh$ and $[hp, hp] \subseteq hh$. In a similar form, we can define the group spaces and related inner products and Lie algebras,

$$
\text{for } vg = \{\hat{g}_{ab}\} <\cdot, \cdot>_v, T_yvG \simeq vg, vp = vg/vh, \text{ with } T_yvh \simeq vh, vg = vh \oplus vp, \text{where } [vh, vp] \subseteq vp, [vp, vp] \subseteq vh; \tag{24}
$$

$$
\text{for } g = \{g_{\alpha\beta}\} <\cdot, \cdot>_g, T_{(x,y)}G \simeq g, p = g/h, \text{ with } T_{(x,y)}h \simeq h, g = h \oplus p, \text{where } [h, p] \subseteq p, [p, p] \subseteq h.
$$

We parametrize the metric structure with constant coefficients on $V =$

---

5It is necessary to consider $hG = SU(n)$ and $vG = SU(m)$ for the geometric models with spinor and gauge fields.
\[ G/\text{SO}(n+m) \] in the form
\[ \hat{g} = \hat{g}_{\alpha\beta} du^\alpha \otimes du^\beta, \]
where \( u^\alpha \) are local coordinates and
\[
\hat{g}_{\alpha\beta} = \begin{bmatrix}
\hat{g}_{ij} + \hat{N}^a_i \hat{N}^b_j h_{ab} & \hat{N}^a_i \hat{h}_{ae} \\
\hat{N}^b_j \hat{h}_{be} & \hat{h}_{ab}
\end{bmatrix}
\]
(25)
where trivial, constant, N–connection coefficients are computed \( \hat{N}^e_i = \hat{h}^{eb} \hat{g}_{jb} \) for any given sets \( \hat{h}^{eb} \) and \( \hat{g}_{jb} \), i.e. from the inverse metrics coefficients defined respectively on \( \hat{G} = \text{SO}(n+1) \) and by off–blocks \((n \times n)–\) and \((m \times m)–\) terms of the metric \( \hat{g}_{\alpha\beta} \). As a result, we define an equivalent d–metric structure of type (A.10)
\[
\hat{g} = \hat{g}_{ij} e^i \otimes e^j + \hat{h}_{ab} \hat{e}^a \otimes \hat{e}^b,
\]
(26)
defining a trivial \((n+m)–\)splitting \( \hat{g} = \hat{g} \oplus \hat{h} \) because all nonholonomy coefficients \( \hat{W}^\gamma_{\alpha\beta} \) and N–connection curvature coefficients \( \hat{\Omega}^a_{ij} \) are zero. In more general form, we can consider any covariant coordinate transforms of (26) preserving the \((n+m)–\)splitting resulting in any \( \hat{W}^\gamma_{\alpha\beta} = 0 \) (A.5) and \( \hat{\Omega}^a_{ij} = 0 \) (A.4). It should be noted that even such trivial parametrizations define algebraic classifications of symmetric Riemannian spaces of dimension \( n+m \) with constant matrix curvature admitting splitting (by certain algebraic constraints) into symmetric Riemannian subspaces of dimension \( n \) and \( m \), also both with constant matrix curvature and introducing the concept of N–anholonomic Riemannian space of type \( \hat{V} = [hG = \text{SO}(n+1), vG = \text{SO}(m+1), \hat{N}^e_i] \). One can be considered that such trivially N–anholonomic group spaces have possess a Lie d–algebra symmetry \( \text{so}_{\hat{N}}(n+m) \cong \text{so}(n) \oplus \text{so}(m) \).

The simplest generalization on a vector bundle \( \hat{E} \) is to consider nonholonomic distributions on \( V = G/\text{SO}(n+m) \) defined locally by families of N–connection coefficients \( N^a_i(x,y,\chi) \) with nonvanishing \( \hat{W}^\gamma_{\alpha\beta}(\chi) \) and \( \hat{\Omega}^a_{ij}(\chi) \) but with constant d–metric coefficients when
\[
g(\chi) = \hat{g}_{ij} e^i \otimes e^j + \hat{h}_{ab} \chi e^a \otimes \chi e^b,
\]
(27)
\[
e^i = dx^i, \quad \chi e^a = dy^a + N^a_i(x,y,\chi) dx^i.
\]
This family of metric is very similar to (7) but with the coefficients \( \hat{g}_{ij} \) and \( \hat{h}_{ab} \) induced by the corresponding Lie d–algebra structure \( \text{so}_{\hat{N}}(n+m) \). Such spaces transform into families of N–anholonomic Riemann–Cartan manifolds \( \hat{V}_N = [hG = \text{SO}(n+1), vG = \text{SO}(m+1), N^e_i] \) with nontrivial N–connection curvature and induced d–torsion coefficients of the canonical d–connection (see formulas (A.9) computed for constant d–metric coefficients.
and the canonical d–connection coefficients in (A.15)). One has zero curvature for the canonical d–connection (in general, such spaces are curved ones with generic off–diagonal metric (27) and nonzero curvature tensor for the Levi Civita connection). This allows us to classify the N–anholonomic manifolds (and vector bundles) as having the same group and algebraic structures of couples of symmetric Riemannian spaces of dimension n and m but nonholonomically soldered to the symmetric Riemannian space of dimension n + m. With respect to N–adapted orthonormal bases (21), with distinguished h– and v–subspaces, we obtain the same inner products and group and Lie algebra spaces as in (24).

The classification of N–anholonomic vector bundles is almost similar to that for symmetric Riemannian spaces if we consider that n = m and try to model tangent bundles of such spaces, provided with N–connection structure. For instance, we can take a (semi) Riemannian structure with the N–connection induced by a absolute energy structure like in (3) and with the canonical d–connection structure (A.15), for $\tilde{\gamma}_{ef} = \hat{\gamma}_{ab}$, like in (6). A straightforward computation of the canonical d–connection coefficients, and of d–curvatures for $\tilde{\gamma}_{ij}$ and $\tilde{\Omega}_{i j}$ proves that the nonholonomic Riemannian manifold $(M = SO(n + 1)/SO(n), \mathcal{L})$ possess constant both zero canonical d–connection curvature and torsion but with induced nontrivial N–connection curvature $\tilde{\Omega}_{i j}$. Such spaces, being tangent to symmetric Riemannian spaces, are classified similarly to the Riemannian ones with constant matrix curvature, see (24) for $n = m$ but provided with a nonholonomic structure induced by generating function $\mathcal{L}$. We can introduce Ricci flows on parameter $\chi$ when for certain systems of coordinates the metric coefficients are constant but satisfy the evolution equations (13) and (14).

4 Basic Equations for N–anholonomic Curve Flows

Introducing N–adapted orthonormalized bases, for N–anholonomic spaces of dimension $n + n$, with constant curvatures of the canonical d–connection, we can derive bi–Hamiltonian and vector soliton structures similarly to [19, 18, 16]. In symbolic, abstract index form, the constructions for nonholonomic vector bundles are similar to those for the Riemannian symmetric–spaces soldered to Klein geometry. We have to distinguish the horizontal and ver-
tical components of geometric objects and related equations. The previous bi–Hamiltonian and solitonic constructions were for an extrinsic approach soldering the Riemannian symmetric–space geometry to the Klein geometry [38]. For the N–anholonomic spaces of dimension $n + n$, with constant d–curvatures, similar constructions hold true but we have to adapt them to the N–connection structure, see Appendix [B].

There is an isomorphism between the real space $\mathfrak{so}(n)$ and the Lie algebra of $n \times n$ skew–symmetric matrices. This allows us to establish an isomorphism between $\mathfrak{h}_p \simeq \mathbb{R}^n$ and the tangent spaces $T_xM = \mathfrak{so}(n+1)/\mathfrak{so}(n)$ of the Riemannian manifold $M = SO(n+1)/SO(n)$ as described by the following canonical decomposition

$$\mathfrak{h}_g = \mathfrak{so}(n+1) \supset \mathfrak{h}_p \in \begin{bmatrix} 0 & \mathfrak{h}_p \\ -\mathfrak{h}_p^T & \mathfrak{h}_0 \end{bmatrix} \text{ for } \mathfrak{h}0 \in \mathfrak{h}_h = \mathfrak{so}(n)$$

with $\mathfrak{h}_p = \{p^i\} \in \mathbb{R}^n$ being the h–component of the d–vector $p = (p^i, p^a)$ and $\mathfrak{h}_p^T$ mean the transposition of the row $\mathfrak{h}_p$. The Cartan–Killing inner product on $\mathfrak{h}_g$ is stated following the rule

$$\mathfrak{h}_p \cdot \mathfrak{h}_p = \left\langle \begin{bmatrix} 0 & \mathfrak{h}_p \\ -\mathfrak{h}_p^T & \mathfrak{h}_0 \end{bmatrix}, \begin{bmatrix} 0 & \mathfrak{h}_p \\ -\mathfrak{h}_p^T & \mathfrak{h}_0 \end{bmatrix} \right\rangle \div \frac{1}{2} \text{tr} \left\{ \begin{bmatrix} 0 & \mathfrak{h}_p \\ -\mathfrak{h}_p^T & \mathfrak{h}_0 \end{bmatrix}^T \begin{bmatrix} 0 & \mathfrak{h}_p \\ -\mathfrak{h}_p^T & \mathfrak{h}_0 \end{bmatrix} \right\},$$

where $\text{tr}$ denotes the trace of the corresponding product of matrices. This product identifies canonically $\mathfrak{h}_p \simeq \mathbb{R}^n$ with its dual $\mathfrak{h}_p^* \simeq \mathbb{R}^n$. In a similar form, we can consider

$$\mathfrak{v}_g = \mathfrak{so}(m+1) \supset \mathfrak{v}_p \in \begin{bmatrix} 0 & \mathfrak{v}_p \\ -\mathfrak{v}_p^T & \mathfrak{v}_0 \end{bmatrix} \text{ for } \mathfrak{v}0 \in \mathfrak{v}_h = \mathfrak{so}(m)$$

with $\mathfrak{v}_p = \{p^a\} \in \mathbb{R}^m$ being the v–component of the d–vector $p = (p^i, p^a)$ and define the Cartan–Killing inner product $\mathfrak{v}_p \cdot \mathfrak{v}_p \div \frac{1}{2} \text{tr}\{\ldots\}$. In general, in the tangent bundle of a N–anholonomic manifold, we can consider the Cartan–Killing N–adapted inner product $\mathfrak{p} \cdot \mathfrak{p} = \mathfrak{h}_p \cdot \mathfrak{h}_p + \mathfrak{v}_p \cdot \mathfrak{v}_p$.

Following the introduced Cartan–Killing parametrizations, we analyze the flow $\gamma(\tau,1)$ of a non–stretching curve in $V_N = G/SO(n) \oplus SO(m)$. Let us introduce a family of coframes $e \in T^*_\gamma V_N \otimes (\mathfrak{h}_p \oplus \mathfrak{v}_p)$, which is a N–adapted $(SO(n) \oplus SO(m))$–parallel basis along $\gamma$, and its associated canonical d–connection 1–form $\chi \in T^*_\gamma V_N(\chi) \otimes (\mathfrak{so}(n) \oplus \mathfrak{so}(m))$. Such d–objects are respectively parametrized:

$$e_x = e_{hX} + e_{vX},$$

19
for
\[ e_h X = \gamma_h X \left[ \begin{array}{c} 0 \\ -(1, \vec{0})^T \end{array} \right] \left( \begin{array}{c} 1, \vec{0} \\ h0 \end{array} \right) \]
and
\[ e_v X = \gamma_v X \left[ \begin{array}{c} 0 \\ -(1, \vec{0})^T \end{array} \right] \left( \begin{array}{c} 1, \vec{0} \\ v0 \end{array} \right), \]
where we write \((1, \vec{0}) \in \mathbb{R}^n, \vec{0} \in \mathbb{R}^{n-1}\) and \((1, \vec{0}) \in \mathbb{R}^m, \vec{0} \in \mathbb{R}^{m-1};\)
\[ \chi \Gamma = [ \chi \Gamma_h X, \chi \Gamma_v X ]; \]
for
\[ \chi \Gamma_h X = \chi \gamma_h X \left[ \begin{array}{c} 0 \\ -(0, \vec{0})^T \end{array} \right] \left( \begin{array}{c} 0, \vec{0} \\ \chi L \end{array} \right) \in \mathfrak{so}(n+1), \]
where
\[ \chi L = \left[ \begin{array}{c} 0 \\ -\chi \nabla \vec{v} \\ h0 \end{array} \right] \in \mathfrak{so}(n), \ \chi \nabla \vec{v} \in \mathbb{R}^{n-1}, \ h0 \in \mathfrak{so}(n-1), \]
and
\[ \chi \Gamma_v X = \chi \gamma_v X \left[ \begin{array}{c} 0 \\ -(0, \vec{0})^T \end{array} \right] \left( \begin{array}{c} 0, \vec{0} \\ \chi C \end{array} \right) \in \mathfrak{so}(m+1), \]
where
\[ \chi C = \left[ \begin{array}{c} 0 \\ -\chi \nabla \vec{v} \\ v0 \end{array} \right] \in \mathfrak{so}(m), \ \chi \nabla \vec{v} \in \mathbb{R}^{m-1}, \ v0 \in \mathfrak{so}(m-1). \]

The above parametrizations are fixed in order to preserve the \(SO(n)\) and \(SO(m)\) rotation gauge freedoms on the \(N\)–adapted coframe and canonical \(d\)–connection 1–form, distinguished in \(h\)- and \(v\)–components.

There are defined decompositions of horizontal \(SO(n+1)/ SO(n)\) matrices like
\[ h_p \equiv \left[ \begin{array}{cc} 0 & h_p \end{array} \right] \left[ \begin{array}{cc} 0 & (h_p, \vec{0})^T \\ -(h_p, \vec{0})^T & h0 \end{array} \right] \]
\[ + \left[ \begin{array}{cc} 0 \\ -(0, h_p \nabla)^T \\ h0 \end{array} \right] \left( \begin{array}{c} 0 \\ (0, h_p \nabla)^T \\ h0 \end{array} \right), \]
into tangential and normal parts relative to \(e_h X\) via corresponding decompositions of \(h\)–vectors \(h_p = (h_p, h_p \nabla) \in \mathbb{R}^n\) relative to \((1, \vec{0})\), when \(h_p\)
is identified with \( h\mathbf{p}_C \) and \( h\mathbf{p}_\perp \) is identified with \( h\mathbf{p}_\perp = h\mathbf{p}_{C\perp} \). In a similar form, it is possible to decompose vertical \( SO(m+1)/SO(m) \) matrices,

\[
v\mathbf{p} \ni \begin{bmatrix} 0 & v\mathbf{p} \\ -v\mathbf{p}^T & v\mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & (v\mathbf{p}_\parallel, \overrightarrow{0}) \\ -\left((v\mathbf{p}_\parallel, \overrightarrow{0})^T v\mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 \\ -\left((0, v\mathbf{p}_\perp)^T v\mathbf{0} \end{bmatrix},
\]

into tangential and normal parts relative to \( \mathbf{e}_v \mathbf{X} \) via corresponding decompositions of \( h\)-vectors

\[
v\mathbf{p} \ni \begin{bmatrix} 0 & (v\mathbf{p}_\parallel, v\mathbf{\overrightarrow{p}}_\perp) \\ -\left((v\mathbf{p}_\parallel, v\mathbf{\overrightarrow{p}}_\perp)^T v\mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 & (0, v\mathbf{p}_\perp)^T v\mathbf{0} \end{bmatrix},
\]

of \( h\mathbf{p}_\perp \) and \( h\mathbf{p}_\perp \) is identified with \( v\mathbf{p}_\perp \) and \( v\mathbf{p}_\perp \) is identified with \( v\mathbf{p}_\perp = v\mathbf{p}_{C\perp} \).

The canonical \( d\)-connection induces matrices decomposed with respect to the flow direction. In the \( h\)-direction, we parametrize

\[
\mathbf{e}_{h\mathbf{Y}} = \gamma_r[h\mathbf{e}] = \begin{bmatrix} 0 \\ -\left((h\mathbf{e}_\parallel, h\overrightarrow{\mathbf{e}}_\perp)^T h\mathbf{0} \end{bmatrix},
\]

when \( \mathbf{e}_{h\mathbf{Y}} \in h\mathbf{p}, (h\mathbf{e}_\parallel, h\overrightarrow{\mathbf{e}}_\perp) \in \mathbb{R}^n \) and \( h\overrightarrow{\mathbf{e}}_\perp \in \mathbb{R}^{n-1} \),

\[
\chi_{\Gamma_{h\mathbf{Y}}} = \chi_{\gamma_r[h\mathbf{e}]} \chi_{\mathbf{L}} = \begin{bmatrix} 0 & (0, \overrightarrow{0}) \\ -\left((0, \overrightarrow{0})^T h\mathbf{\omega}_r(\chi) \end{bmatrix} \in \mathfrak{so}(n+1),
\]

where

\[
h\mathbf{\omega}_r(\chi) = \begin{bmatrix} 0 & \overrightarrow{\mathbf{w}}(\chi) \\ -\overrightarrow{\mathbf{w}}^T(\chi) h\Theta(\chi) \end{bmatrix} \in \mathfrak{so}(n), \overrightarrow{\mathbf{w}}(\chi) \in \mathbb{R}^{n-1}, h\Theta(\chi) \in \mathfrak{so}(n-1).
\]

In the \( v\)-direction, we parametrize

\[
\mathbf{e}_{v\mathbf{Y}} = \chi_{\gamma_r[v\mathbf{e}]} = \begin{bmatrix} 0 & (v\mathbf{e}_\parallel, v\overrightarrow{\mathbf{e}}_\perp) \\ -\left((v\mathbf{e}_\parallel, v\overrightarrow{\mathbf{e}}_\perp)^T v\mathbf{0} \end{bmatrix},
\]

when \( \mathbf{e}_{v\mathbf{Y}} \in v\mathbf{p}, (v\mathbf{e}_\parallel, v\overrightarrow{\mathbf{e}}_\perp) \in \mathbb{R}^m \) and \( v\overrightarrow{\mathbf{e}}_\perp \in \mathbb{R}^{m-1} \),

\[
\chi_{\Gamma_{v\mathbf{Y}}} = \chi_{\gamma_r[v\mathbf{e}]} \chi_{\mathbf{C}(\chi)} = \begin{bmatrix} 0 & (0, \overrightarrow{0}) \\ -\left((0, \overrightarrow{0})^T v\mathbf{\omega}_r(\chi) \end{bmatrix} \in \mathfrak{so}(m+1),
\]

where

\[
v\mathbf{\omega}_r(\chi) = \begin{bmatrix} 0 & \overrightarrow{\mathbf{w}}(\chi) \\ -\overrightarrow{\mathbf{w}}^T(\chi) v\Theta(\chi) \end{bmatrix} \in \mathfrak{so}(m), \overrightarrow{\mathbf{w}}(\chi) \in \mathbb{R}^{m-1}, v\Theta(\chi) \in \mathfrak{so}(m-1).
\]
The components \( h^e \parallel (\chi) \) and \( h^e \perp (\chi) \) correspond to the decomposition

\[
\chi^e h^Y = h^g (\chi^e h^Y, \chi^e h^X) + (\chi^e h^Y) h^e(\chi)
\]

into tangential and normal parts relative to \( \chi^e h^X \). In a similar form, one considers \( v^e \parallel (\chi)(\chi) \) and \( v^e \perp (\chi) \) corresponding to the decomposition

\[
\chi^e v^Y = v^g (\chi^e v^Y, \chi^e v^X) + (\chi^e v^Y) v^e(\chi).
\]

Using the above stated matrix parametrizations, we get

\[
[ \chi^e h^X, \chi^e h^Y ] = - \begin{bmatrix} 0 & 0 \\ 0 & h^e(\chi) \end{bmatrix} \in so(n+1), \tag{29}
\]

for \( h^e(\chi) = \begin{bmatrix} 0 \\ -(h^e \perp (\chi))^T h^0 \end{bmatrix} \in so(n) \);

\[
[ \chi^e h^X, \chi^e h^Y ] = - \begin{bmatrix} 0 \\ -(0, \chi^e \perp) \end{bmatrix} \in hp; \tag{30}
\]

\[
[ \chi^e h^X, \chi^e h^Y ] = \begin{bmatrix} 0 \\ \chi \left( -v^e \cdot h^e \perp, h^e \parallel v^e \right) \end{bmatrix} \in hp; \tag{31}
\]

and

\[
[ \chi^e v^X, \chi^e v^Y ] = - \begin{bmatrix} 0 & 0 \\ 0 & v^e \perp(\chi) \end{bmatrix} \in so(m+1), \tag{29}
\]

for \( v^e \perp(\chi) = \begin{bmatrix} 0 \\ -(v^e \perp)^T(\chi) \end{bmatrix} \in so(m) \);

\[
[ \chi^e v^X, \chi^e v^Y ] = - \begin{bmatrix} 0 \\ -(0, \chi^e \perp) \end{bmatrix} \in vp; \tag{30}
\]

\[
[ \chi^e v^X, \chi^e v^Y ] = \begin{bmatrix} 0 \\ \chi \left( -v^e \cdot v^e \perp, v^e \parallel v^e \right) \end{bmatrix} \in vp. \tag{31}
\]

We can use formulas (29) and (30) in order to write the structure equations (22) and (23) in terms of N–adapted curve flow operators soldered to the geometry Klein N–anholonomic spaces using the relations (B.1). One obtains respectively the \( G \)–invariant N–adapted torsion and curvature generated by the canonical d–connection,

\[
T(\gamma, \gamma) = (D^X \gamma - D^Y \gamma) e = D^X e_Y - D^Y e_X + [\gamma^X, e_Y] - [\gamma^Y, e_X] \tag{31}
\]
and

$$R(\gamma_\tau, \gamma_\eta)e = [D_X, D_Y]e = D_X\Gamma_Y - D_Y\Gamma_X + [\Gamma_X, \Gamma_Y]$$  \hspace{1cm} (32)$$

where $e_X \equiv \gamma_\tau|e$, $e_Y \equiv \gamma_\eta|e$, $\Gamma_X \equiv \gamma_\tau|\Gamma$ and $\Gamma_Y \equiv \gamma_\eta|\Gamma$. The formulas (31) and (32) are equivalent, respectively, to (A.19) and (A.18). In general, $T(\gamma_\tau, \gamma_\eta) \neq 0$ and $R(\gamma_\tau, \gamma_\eta)e$ can not be defined to have constant matrix coefficients with respect to a $N$–adapted basis. For $N$–anholonomic spaces with dimensions $n = m$, we have $\varepsilon T(\gamma_\tau, \gamma_\eta) = 0$ and $\varepsilon R(\gamma_\tau, \gamma_\eta)e$ defined by constant, or vanishing, $d$–curvature coefficients (see discussions related to formulas (8.21) and (8.16)). For such cases, we can consider the $h$– and $v$–components of (31) and (32) in a similar manner as for symmetric Riemannian spaces but with the canonical $d$–connection instead of the Levi Civita one. One obtains, respectively,

$$0 = (D_h X \gamma_\tau - D_h Y \gamma_\eta)|he$$
$$= D_h X e_h Y - D_h Y e_h X + [L_h X, e_h Y] - [L_h Y, e_h X];$$

and

$$hR(\gamma_\tau, \gamma_\eta)he = [D_h X, D_h Y]he = D_h X L_h Y - D_h Y L_h X + [L_h X, L_h Y].$$

$$vR(\gamma_\tau, \gamma_\eta)ve = [D_v X, D_v Y]ve = D_v X C_v Y - D_v Y C_v X + [C_v X, C_v Y].$$

Following the $N$–adapted curve flow parametrizations (29) and (30), the equations (33) and (34) are written

$$0 = D_h X e_h || v + \overrightarrow{v} \cdot h \overrightarrow{e}_{\perp}, \hspace{0.5cm} 0 = D_v X e_v || v + \overrightarrow{v} \cdot v \overrightarrow{e}_{\perp};$$

and

$$D_h X \overrightarrow{\omega} - D_h Y \overrightarrow{\nu} + \overrightarrow{\nu} | \ h \Theta = h \overrightarrow{e}_{\perp}, \hspace{0.5cm} D_v X \overrightarrow{\omega} - D_v Y \overrightarrow{\nu} + \overrightarrow{\nu} | \ h \Theta = v \overrightarrow{e}_{\perp};$$

and

$$D_h X h \Theta - \overrightarrow{\nu} \otimes \overrightarrow{\omega} + \overrightarrow{\omega} \otimes \overrightarrow{\nu} = 0, \hspace{0.5cm} D_v X v \Theta - \overrightarrow{\nu} \otimes \overrightarrow{\omega} + \overrightarrow{\omega} \otimes \overrightarrow{\nu} = 0.$$  \hspace{1cm} (36)$$

The tensor and interior products, for instance, for the $h$–components, are defined in the form: $\otimes$ denotes the outer product of pairs of vectors ($1 \times n$ row matrices), producing $n \times n$ matrices $\overrightarrow{A \otimes B} = \overrightarrow{A^T B}$, and $|$ denotes multiplication of $n \times n$ matrices on vectors ($1 \times n$ row matrices); one holds the properties $\overrightarrow{A} | (\overrightarrow{B \otimes C}) = (\overrightarrow{A \cdot B}) \overrightarrow{C}$ which is the transpose of the standard matrix product on column vectors, and $(\overrightarrow{B \otimes C}) \overrightarrow{A} = (\overrightarrow{C \cdot A}) \overrightarrow{B}$. 

23
Here we note that similar formulas hold for the v–components but, for instance, we have to change, correspondingly, $n \to m$ and $\vec{A} \to \vec{A}$.

The variables $e_\parallel$ and $\Theta$, written in h– and v–components, can be expressed corresponding in terms of variables $\vec{v}$, $\vec{w}$, $h \vec{e}_\perp$ and $\vec{v}$, $\vec{w}$, $v \vec{e}_\perp$ (see respectively the first two equations in (35) and the last two equations in (36)),

$$h e_\parallel = -D_{hX}^{-1}(\vec{v} \cdot h \vec{e}_\perp), \quad ve_\parallel = -D_{vX}^{-1}(\vec{v} \cdot v \vec{e}_\perp),$$

and

$$h \Theta = D_{hX}^{-1}(\vec{v} \otimes \vec{w} - \vec{w} \otimes \vec{v}), \quad v \Theta = D_{vX}^{-1}(\vec{v} \otimes \vec{w} - \vec{w} \otimes \vec{v}).$$

Substituting these values, correspondingly, in the last two equations in (35) and in the first two equations in (36), we express

$$\vec{w} = -D_{hX}h \vec{e}_\perp - D_{hX}^{-1}(\vec{v} \cdot h \vec{e}_\perp) \vec{w}, \quad \vec{w} = -D_{vX}v \vec{e}_\perp - D_{vX}^{-1}(\vec{v} \cdot v \vec{e}_\perp) \vec{w},$$

contained in the h– and v–flow equations respectively on $\vec{v}$ and $\vec{w}$, considered as scalar components when $D_{hY} \vec{v} = \vec{v}_\tau$ and $D_{hY} \vec{w} = \vec{w}_\tau$,

$$\vec{v}_\tau = D_{hX} \vec{w} - \vec{v} \mid D_{hX}^{-1}(\vec{v} \otimes \vec{w} - \vec{w} \otimes \vec{v}) - \vec{R}h \vec{e}_\perp, \quad \vec{w}_\tau = D_{vX} \vec{w} - \vec{v} \mid D_{vX}^{-1}(\vec{v} \otimes \vec{w} - \vec{w} \otimes \vec{v}) - \vec{S}v \vec{e}_\perp,$$

where the scalar curvatures of the canonical d–connection, $\vec{R}$ and $\vec{S}$ are defined by formulas (A.20). For symmetric Riemannian spaces like $SO(n + 1)/SO(n) \simeq S^n$, the value $\vec{R}$ is just the scalar curvature $\chi = 1$, see [19]. On tangent bundles, it is possible that $\vec{R}$ and $\vec{S}$ are certain zero or nonzero constants with the h–part equivalent to the base scalar curvature.

For Ricci flows of geometric objects, the curve flow evolution equations (37) contain additional dependencies on parameter $\chi$,

$$x \vec{v}_\tau = x D_{hX} \vec{w}(\chi) - \vec{R}(\chi) h \vec{e}_\perp(\chi) - x \vec{v} \mid x D_{hX}^{-1}(x \vec{v} \otimes \vec{w}(\chi) - \vec{w}(\chi) \otimes x \vec{v}),$$

$$x \vec{w}_\tau = x D_{vX} \vec{w} - \vec{S}(\chi)v \vec{e}_\perp(\chi), - x \vec{w} \mid x D_{vX}^{-1}(x \vec{w} \otimes \vec{w}(\chi) - \vec{w}(\chi) \otimes x \vec{w}(\chi))$$

where the scalar curvatures evolve following formulas (16) for Ricci flows.

The above presented considerations consist the proof of

24
Lemma 4.1 There are canonical lifts of Ricci flows from a (semi) Riemannian manifold $M$ to $T M$ when certain families of constant curvature matrix coefficients for the canonical d–connections define families of $N$–adapted Hamiltonian sympletic operators,

$$h \mathcal{J}(\chi) = \chi D_{hX} + \chi D_{hX}^{-1} (\chi \mathcal{V}) \chi \mathcal{V},$$

$$v \mathcal{J}(\chi) = \chi D_{vX} + \chi D_{vX}^{-1} (\chi \mathcal{V}) \chi \mathcal{V},$$

and cosympletic operators

$$h \mathcal{H}(\chi) = \chi D_{hX} + \chi \mathcal{V} \chi D_{hX}^{-1} (\chi \mathcal{V}) \chi \mathcal{V},$$

$$v \mathcal{H}(\chi) = \chi D_{vX} + \chi \mathcal{V} \chi D_{vX}^{-1} (\chi \mathcal{V}) \chi \mathcal{V},$$

where, for instance, $\mathcal{V} \mathcal{W} = \mathcal{V} \otimes \mathcal{W} - \mathcal{W} \otimes \mathcal{V}$.

For any fixed value $\chi = \chi_0$, the formulas for this Lemma transform into similar ones from Ref. [20]. The properties of operators (38) and (39) are defined by

Theorem 4.1 The Ricci flows of d–operators $^x\mathcal{J} = (h \mathcal{J}(\chi), v \mathcal{J}(\chi))$ and $^x\mathcal{H} = (h \mathcal{H}(\chi), v \mathcal{H}(\chi))$ are defined respectively by $(O(n - 1), O(m - 1))$–invariant Hamiltonian sympletic and cosympletic d–operators with respect to the corresponding Hamiltonian d–variables $(^x\mathcal{V}, ^x\mathcal{W})$. Such d–operators defines the Hamiltonian form for the curve and Ricci flows equations on $N$–anholonomic tangent bundles with constant d–connection curvature: the $h$–flows are given by

$$^x\mathcal{V} \tau = h \mathcal{H} (^x\mathcal{V}, \chi) - \mathcal{R}(\chi) \ h \mathcal{E}(\chi),$$

$$^x\mathcal{W} = h \mathcal{J} (^x\mathcal{E}(\chi), \chi),$$

$$\frac{\partial \mathcal{R}}{\partial \chi} = \mathcal{D}_a \mathcal{D}^a \mathcal{R} + 2 \mathcal{R}_{ij} \mathcal{R}^{ij};$$

the $v$–flows are given by

$$^x\mathcal{V} \tau = v \mathcal{H} (^x\mathcal{V}(\chi), \chi) - \mathcal{S}(\chi) \ v \mathcal{E}(\chi),$$

$$^x\mathcal{W} = v \mathcal{J} (^x\mathcal{E}(\chi), \chi),$$

$$\frac{\partial \mathcal{S}}{\partial \chi} = \mathcal{D}_a \mathcal{D}^a \mathcal{S} + 2 \mathcal{S}_{ab} \mathcal{S}^{ab};$$
where the so-called Ricci flows of hereditary recursion \( d \)-operator has the respective \( h \)- and \( v \)-components

\[
hR(\chi) = hH(\chi) \circ hJ(\chi) \quad \text{and} \quad vR(\chi) = vH(\chi) \circ vJ(\chi)
\]
and \( \hat{D}_i\hat{D}^i \hat{R} = \hat{D}_a\hat{D}^a \hat{S} = 0 \) and \( \hat{R}_{ij}\hat{R}^{ij} = \text{const}, \hat{S}_{ab}\hat{S}^{ab} = \text{const} \) for lifts to constant curvature matrices.

**Proof.** One follows from the Lemma 4.1, Corollaries 2.2 and 2.3 and (37). In a detailed form, for holonomic structures, it is given in Ref. [16] and discussed in [19]. □

Finally, we note that for any fixed value \( \chi_0 \) we get a Theorem from [20], on curve flows in symmetric Riemannian spaces, which has generalizations for curve flows on generalized Lagrange and Finsler spaces [25].

5 Curve Flows and Solitonic Hierarchies for Ricci Flows

The final aim of this paper is to prove that for any nonholonomic Ricci flow system we can define naturally a family of \( N \)-adapted bi-Hamiltonian flow hierarchies inducing anholonomic solitonic configurations.

5.1 Formulation of the main theorem

Following a usual solitonic techniques generalized for \( N \)-anholonomic spaces, see details in Ref. [20] [25] [18] [19], the recursion \( h \)-operators from (42),

\[
hR(\chi) = \chi D_{hX} \left( \chi D_{hX} + \chi D_{hX}^{-1} \chi \bigwedge \chi \bigwedge \chi \right)
\]

\[
\quad + \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi
\]

\[
= \chi D_{hX}^2 + \chi D_{hX}^2 \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi
\]

generate a family of horizontal hierarchies of commuting Hamiltonian vector fields \( h e^{(k)}(\chi) \) starting from \( h e^{(0)}(\chi) = \chi \bigwedge \chi \) given by the infinitesimal generator of \( l \)-translations in terms of arclength \( l \) along the curve (we use a boldface \( l \) in order to emphasized that the curve is on a \( N \)-anholonomic manifold when the geometric objects are subjected to Ricci flows). A family of vertical hierarchies of commuting vector fields \( v e^{(k)}(\chi) \) starting from \( v e^{(0)}(\chi) = \chi \bigwedge \chi \) is generated by the recursion \( v \)-operators

\[
vR(\chi) = \chi D_{vX} \left( \chi D_{vX} + \chi D_{vX}^{-1} \chi \bigwedge \chi \bigwedge \chi \right)
\]

\[
\quad + \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi
\]

\[
= \chi D_{vX}^2 + \chi D_{vX}^2 \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi
\]

\[
\quad - \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi
\]

\[
= \chi D_{vX}^2 + \chi D_{vX}^2 \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi
\]

\[
\quad - \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi \bigwedge \chi
\]
There are related hierarchies, generated by adjoint operators \( \mathcal{R}^* \), of involutive Hamiltonian \( h \)-covector fields \( \bar{\omega}^{(k)} = \delta (h H^{(k)}) / \delta \bar{v} \) in terms of Hamiltonians \( h H = h H^{(k)}(\bar{v}, \bar{v}_1, \bar{v}_{21}, ...) \) starting from \( \bar{\omega}^{(0)} = \bar{v}, h H^{(0)} = \frac{1}{2} |\bar{v}|^2 \) and of involutive Hamiltonian \( v \)-covector fields \( \bar{\pi}^{(k)} = \delta (v H^{(k)}) / \delta \bar{v} \) in terms of Hamiltonians \( v H = v H^{(k)}(\bar{v}, \bar{v}_1, \bar{v}_{21}, ...) \) starting from \( \bar{\pi}^{(0)} = \bar{v}, v H^{(0)} = \frac{1}{2} |\bar{v}|^2 \).

The relations between different type families of hierarchies are established correspondingly by formulas

\[
\begin{align*}
\hat{h} \bar{e}^{(k)}_{\perp}(\chi) &= h \mathcal{H} (\bar{\omega}^{(k)}, \bar{\omega}^{(k+1)}, \chi) = h \mathcal{J} (\hat{h} \bar{e}^{(k)}_{\perp}, \chi) \\
v \bar{e}^{(k)}_{\perp}(\chi) &= v \mathcal{H} (\bar{\omega}^{(k)}, \bar{\omega}^{(k+1)}, \chi) = v \mathcal{J} (v \bar{e}^{(k)}_{\perp}, \chi),
\end{align*}
\]

where \( k = 0, 1, 2, ... \). All hierarchies (horizontal, vertical and their adjoint ones) have a typical mKdV scaling symmetry, for instance, \( l \rightarrow \lambda l \) and \( \chi \bar{v} \rightarrow \lambda^{-1} \chi \bar{v} \) under which the values \( \hat{h} \bar{e}^{(k)}_{\perp}(\chi) \) and \( h H^{(k)}(\chi) \) have scaling weight \( 2 + 2k \), while \( \bar{\omega}^{(k)}(\chi) \) has scaling weight \( 1 + 2k \). Following the above presented considerations, we prove

**Corollary 5.1** There are Ricci flow families of \( N \)-adapted hierarchies of distinguished horizontal and vertical commuting bi–Hamiltonian flows, correspondingly, on \( \bar{v} \) and \( \bar{v} \) associated to the recursion \( d \)-operator \([42]\) given by families of \( O(n-1) \oplus O(m-1) \)–invariant \( d \)-vector \( h \)-evolution equations,

\[
\begin{align*}
\chi \bar{v}_\tau &= h \bar{e}^{(k+1)}_{\perp}(\chi) - \bar{R}(\chi) \ h \bar{e}^{(k)}_{\perp}(\chi) = h \mathcal{H} \left( \delta \left( h H^{(k+1)}(\chi) \right) / \delta \chi \bar{v} \right) \\
&= (h \mathcal{J}(\chi))^{-1} \left( \delta \left( h H^{(k+1)}(\chi) \right) / \delta \chi \bar{v} \right),
\end{align*}
\]

with families of horizontal Hamiltonians

\[
h H^{(k+1)}(\chi) = h H^{(k+1)}(\chi) - \bar{R}(\chi) \ h H^{(k)}(\chi),
\]

and \( v \)-evolution equations

\[
\begin{align*}
\chi \bar{v}_\tau &= v \bar{e}^{(k+1)}_{\perp}(\chi) - \bar{S}(\chi) \ v \bar{e}^{(k)}_{\perp}(\chi) = v \mathcal{H} \left( \delta \left( v H^{(k+1)}(\chi) \right) / \delta \chi \bar{v} \right) \\
&= (v \mathcal{J}(\chi))^{-1} \left( \delta \left( v H^{(k+1)}(\chi) \right) / \delta \chi \bar{v} \right),
\end{align*}
\]

and with families of vertical Hamiltonians

\[
v H^{(k+1)}(\chi) = v H^{(k+1)}(\chi) - \bar{S}(\chi) \ v H^{(k)}(\chi),
\]

27
for \( k = 0, 1, 2, \ldots \). The Ricci flows of \( d \)-operators \( \mathcal{H}(\chi) \) and \( \mathcal{J}(\chi) \) are \( N \)-adapted and mutually compatible from which we can construct a family of alternative (explicit) Hamilton \( d \)-operators

\[
^a\mathcal{H}(\chi) = \chi \mathcal{H} \circ \mathcal{J}(\chi) \circ \chi \mathcal{H} = \chi \mathcal{R} \circ \mathcal{H}(\chi). 
\]

The Main Result of this work is formulated in the form:

**Theorem 5.1** For any vector/tangent bundle with Ricci flows of \( d \)-metric structures, one can be defined a family of hierarchies of bi-Hamiltonian \( N \)-adapted flows of curves \( \chi(\tau, l) = h(\tau, l) + v(\tau, l) \) described by families of geometric nonholonomic map equations:

The 0 flows are defined as convective (travelling wave) maps

\[
\chi_{\gamma \tau} = \chi_{\gamma l}, \text{ distinguished as } \quad (45)
\]

\( (h\gamma)_\tau(\chi) = (h\gamma)_h X(\chi) \quad \text{and} \quad (v\gamma)_\tau(\chi) = (v\gamma)_v X(\chi). \)

There are families of \(+1\) flows defined as Ricci flows of non-stretching mKdV maps

\[
\chi_{\gamma \tau} = \left[ \chi_{D_h^2 X} + \frac{3}{2} \chi_{D_h X} (h\gamma)_h X(\chi) \right] (h\gamma)_h X(\chi), \quad (46)
\]

\[
\chi_{\gamma \tau} = \left[ \chi_{D_v^2 X} + \frac{3}{2} \chi_{D_v X} (v\gamma)_v X(\chi) \right] (v\gamma)_v X(\chi),
\]

and the families of \(+2,\ldots\) flows as higher order analogs.

There are also families of \(-1\) flows defined by the kernels of recursion operators \((43)\) and \((44)\) inducing non-stretching maps

\[
\chi_{D_h X} (h\gamma)_h X(\chi) = 0 \quad \text{and} \quad \chi_{D_v X} (v\gamma)_v X(\chi) = 0. \quad (47)
\]

Proof is outlined in Appendix C.

### 5.2 Nonholonomic mKdV and SG hierarchies

Let us consider some explicit constructions when families of solitonic hierarchies are derived following the conditions of Theorem 5.1.

The \( h \)-flow and \( v \)-flow equations resulting from \((47)\) are

\[
\chi_{\gamma \tau} = -\overline{R}(\chi) \ h e_\perp(\chi) \quad \text{and} \quad \chi_{\gamma \tau} = -\overline{S}(\chi) \ v e_\perp(\chi), \quad (48)
\]

when, respectively,

\[
0 = \chi_{\gamma v} = -\chi_{D_h X} h e_\perp(\chi) + h e_\parallel(\chi) \ \chi_{\gamma v}, \quad \chi_{D_h X} h e_\parallel(\chi) = h e_\perp(\chi) \cdot \chi_{\gamma v}
\]

\[
0 = \chi_{\gamma h} = -\chi_{D_v X} v e_\perp(\chi) + v e_\parallel(\chi) \ \chi_{\gamma h}, \quad \chi_{D_v X} v e_\parallel(\chi) = v e_\perp(\chi) \cdot \chi_{\gamma h}
\]

\[
0 = \chi_{\gamma} = -\chi_{D_h v} h e_\perp(\chi) + h e_\parallel(\chi) \ \chi_{\gamma}, \quad \chi_{D_h v} h e_\parallel(\chi) = h e_\perp(\chi) \cdot \chi_{\gamma}
\]

\[
0 = \chi_{\gamma} = -\chi_{D_v h} v e_\perp(\chi) + v e_\parallel(\chi) \ \chi_{\gamma}, \quad \chi_{D_v h} v e_\parallel(\chi) = v e_\perp(\chi) \cdot \chi_{\gamma}
\]
and

\[ 0 = \chi \tilde{\alpha} = -\chi D_{v}X v e_{\perp} (\chi) + v e_{\parallel} (\chi) \chi \tilde{\alpha}, \quad \chi D_{v}X v e_{\parallel} (\chi) = v e_{\perp} (\chi) \cdot \chi \tilde{\alpha}. \]

The d–flow equations possess horizontal and vertical conservation laws

\[ \chi D_{h}X ((he_{\parallel} (\chi))^2 + |h e_{\perp} (\chi)|^2) = 0, \]

for \((he_{\parallel} (\chi))^2 + |h e_{\perp} (\chi)|^2 =< he_{\gamma} (\chi), he_{\tau} (\chi) >_{h_{0}} = |(h \gamma)_{\tau} (\chi)|^2_{h_{0}}, \text{ and} \]

\[ \chi D_{v}Y ((ve_{\parallel} (\chi))^2 + |v e_{\perp} (\chi)|^2) = 0, \]

for \((ve_{\parallel} (\chi))^2 + |v e_{\perp} (\chi)|^2 =< ve_{\gamma} (\chi), ve_{\tau} (\chi) >_{v_{0}} = |(v \gamma)_{\tau} (\chi)|^2_{v_{0}}. \] This corresponds to

\[ \chi D_{h}X (h \gamma)_{\tau} (\chi)|^2_{h_{0}} = 0 \quad \text{and} \quad \chi D_{v}X (v \gamma)_{\tau} (\chi)|^2_{v_{0}} = 0. \]

In general, such laws are more sophisticated than those on (semi) Riemannian spaces because of nonholonomic constraints resulting in non–symmetric Ricci tensors and different types of identities. But for the geometries modelled for dimensions \(n = m\) with canonical d–connections, we get similar h– and v–components of the conservation law equations as on symmetric Riemannian spaces.

It is possible to rescale conformally the variable \(\tau\) and obtain \(|(h \gamma)_{\tau} (\chi)|^2_{h_{0}} = 1\) and \((v \gamma)_{\tau} (\chi)|^2_{v_{0}} = 1\), i.e.

\[(he_{\parallel} (\chi))^2 + |h e_{\perp} (\chi)|^2 = 1 \quad \text{and} \quad (ve_{\parallel} (\chi))^2 + |v e_{\perp} (\chi)|^2 = 1.\]

We can express \(he_{\parallel} (\chi)\) and \(h e_{\perp} (\chi)\) in terms of \(\chi \tilde{\alpha}\) and its derivatives and, similarly, we can express \(ve_{\parallel} (\chi)\) and \(v e_{\perp} (\chi)\) in terms of \(\chi \tilde{\alpha}\) and its derivatives, which follows from \(\chi \tilde{\alpha}\). The N–adapted wave map equations describing the -1 flows reduce to a system of two independent nonlocal evolution equations for the h– and v–components, parametrized by \(\chi\),

\[ \chi \tilde{\alpha}_{\tau} = -\chi D_{h}X^{-1} \left( \sqrt{\tilde{R}^2 (\chi) - |\chi \tilde{\alpha}_{\tau}|^2 \chi \tilde{\alpha}} \right), \]

\[ \chi \tilde{\alpha}_{\tau} = -\chi D_{v}X^{-1} \left( \sqrt{\tilde{S}^2 (\chi) - |\chi \tilde{\alpha}_{\tau}|^2 \chi \tilde{\alpha}} \right). \]

For N–anholonomic spaces of constant scalar d–curvatures, we can rescale the equations on \(\tau\) to the case where the terms \(\tilde{R}^2 (\chi)\) and \(\tilde{S}^2 (\chi)\) are constant,

\[ ^{8}\text{We note that the problem of formulating conservation laws on N–anholonomic spaces (in particular, on nonholonomic vector bundles) in analyzed in Ref. \cite{24}.} \]
and the evolution equations transform into a system of hyperbolic d–vector equations,

\[ \frac{\partial}{\partial \tau} \mathbf{v} = -\text{sgn}(v) \frac{3}{2} |v|^2 \mathbf{v}, \quad (49) \]

\[ \frac{\partial}{\partial \tau} \mathbf{v} = -\text{sgn}(v) \frac{3}{2} |v|^2 \mathbf{v}, \]

where \( \partial_{\mathbf{h}} \) and \( \partial_{\mathbf{v}} \) are usual partial derivatives on direction \( \mathbf{h} = \mathbf{v} + \mathbf{v} \) with \( \mathbf{v} \neq 0 \) and \( \mathbf{v} \) considered as scalar functions for the covariant derivatives \( \partial_{\mathbf{h}} \) and \( \partial_{\mathbf{v}} \) defined by the canonical d–connection. It also follows that \( h \mathbf{e} \) and \( v \mathbf{e} \) obey corresponding vector sine–Gordon (SG) equations

\[ \left( \sqrt{1 - |h \mathbf{e}|^2} \right)^{-1} \partial_{\mathbf{h}}(h \mathbf{e}) = -h \mathbf{e}, \quad (50) \]

\[ \left( \sqrt{1 - |v \mathbf{e}|^2} \right)^{-1} \partial_{\mathbf{v}}(v \mathbf{e}) = -v \mathbf{e}. \quad (51) \]

The above presented formulas and Corollary 5.1 imply

**Conclusion 5.1** The Ricci flow families of recursion d–operators \( \mathbf{R} = (h \mathbf{R}(\chi), h \mathbf{R}(\chi)) \) (42), see (43) and (44), generate two hierarchies of vector mKdV symmetries: the first one is horizontal,

\[ \mathbf{v}^{(0)} = h \mathbf{R}(\mathbf{v}_h, \chi) = h \mathbf{v}_h + \frac{3}{2} |\mathbf{v}_h|^2 \mathbf{v}_h, \]

\[ \mathbf{v}^{(1)} = h \mathbf{R}(\mathbf{v}_h, \chi) = h \mathbf{v}_h + \frac{3}{2} |\mathbf{v}_h|^2 \mathbf{v}_h, \]

\[ \mathbf{v}^{(2)} = h \mathbf{R}(\mathbf{v}_h, \chi) = h \mathbf{v}_h + \frac{3}{2} |\mathbf{v}_h|^2 \mathbf{v}_h, \]

with all such terms commuting with the -1 flow

\[ \left( \sqrt{1 - |h \mathbf{e}|^2} \right)^{-1} = h \mathbf{e}, \quad (53) \]

associated to the vector SG equation (50); the second one is vertical,

\[ \mathbf{v}^{(0)} = v \mathbf{R}(\mathbf{v}_v, \chi) = v \mathbf{v}_v + \frac{3}{2} |\mathbf{v}_v|^2 \mathbf{v}_v, \]

\[ \mathbf{v}^{(1)} = v \mathbf{R}(\mathbf{v}_v, \chi) = v \mathbf{v}_v + \frac{3}{2} |\mathbf{v}_v|^2 \mathbf{v}_v, \]

\[ \mathbf{v}^{(2)} = v \mathbf{R}(\mathbf{v}_v, \chi) = v \mathbf{v}_v + \frac{3}{2} |\mathbf{v}_v|^2 \mathbf{v}_v, \]

\[ \left( \sqrt{1 - |v \mathbf{e}|^2} \right)^{-1} = v \mathbf{e}, \quad (54) \]
with all such terms commuting with the -1 flow

\[(\chi \overrightarrow{v})^{-1} = v \overrightarrow{e}_{\perp}(\chi)\]  

(55)

associated to the vector SG equation \([57]\).

In its turn, using the above Conclusion, we derive that the family of adjoint d–operators \(\mathcal{R}^* (\chi) = \chi \mathcal{J} \circ \mathcal{H} (\chi)\) generates corresponding families of horizontal hierarchies of Hamiltonians (for simplicity, here we omit labels/dependences on \(\chi\)),

\[
hH^{(0)} = \frac{1}{2} |\overrightarrow{v}|^2, \quad hH^{(1)} = -\frac{1}{2} |\overrightarrow{v}_{h1}|^2 + \frac{1}{8} |\overrightarrow{v}|^4, \quad (56)
\]

and of vertical hierarchies of Hamiltonians

\[
vH^{(0)} = \frac{1}{2} |\overleftarrow{v}|^2, \quad vH^{(1)} = -\frac{1}{2} |\overleftarrow{v}_{v1}|^2 + \frac{1}{8} |\overleftarrow{v}|^4, \quad (57)
\]

all of which are conserved densities for respective horizontal and vertical -1 flows and determining higher conservation laws for the corresponding hyperbolic equations (50) and (51).

The above presented horizontal equations (50), (52), (53) and (56) and of vertical equations (51), (54), (55) and (57) have similar mKdV scaling symmetries but on different parameters \(\lambda_h\) and \(\lambda_v\) because, in general, there are two independent values of scalar curvatures \(\overrightarrow{R}\) and \(\overleftarrow{S}\), see (A.20). The horizontal scaling symmetries are \(h1 \rightarrow \lambda_h h1, \overrightarrow{v} \rightarrow (\lambda_h)^{-1} \overrightarrow{v}\) and \(\tau \rightarrow (\lambda_h)^{1+2k}\), for \(k = -1, 0, 1, 2, \ldots\) For the vertical scaling symmetries, one has \(v1 \rightarrow \lambda_v v1, \overleftarrow{v} \rightarrow (\lambda_v)^{-1} \overleftarrow{v}\) and \(\tau \rightarrow (\lambda_v)^{1+2k}\), for \(k = -1, 0, 1, 2, \ldots\).

Finally, we discuss how exact solutions for the Einstein equations

\[
\chi \hat{\mathcal{R}}_{\alpha\beta} = \chi \hat{\lambda} g_{\alpha\beta}
\]  

(58)

can be extracted from Ricci flows of solitonic hierarchies (such spaces with nonhomogeneous effective constants \(\chi \hat{\lambda}(u)\) were examined in details in Refs. \([24, 26]\), for exact solutions in gravity, and \([29, 30, 31, 32]\), for exact solutions and applications in physics of the nonholonomic Ricci flow theory).

**Corollary 5.2** Ricci flows of nonhomogeneous Einstein spaces, of signature \((- + \ldots +, - + \ldots +)\) defined by (58) are can be extracted from solitonic hierarchies satisfying the conditions of Theorem 5.1 by certain classes of constraints for

\[
\overrightarrow{R}(\chi) = (n - 1) \chi \hat{\lambda} = \overleftarrow{S}(\chi) = (m - 1) \chi \hat{\lambda}
\]  

(59)
solving the equations

\[
\frac{\partial \hat{\lambda}(\chi)}{\partial \chi} - \left[ \hat{\lambda}(\chi) \right]^2 = D_i \hat{D}^i \hat{\lambda}(\chi) = \hat{D}^a \hat{D}_a \hat{\lambda}(\chi) \quad (60)
\]

and evolution of N–adapted frames stated by formulas

\[
\frac{\partial e^\alpha_a}{\partial \chi} = \hat{\lambda}(\chi)e^\alpha_a. \quad (61)
\]

Proof. The equations (60) follow from equations (40) and (41), see also (16), when hold true (59). The equation (61) is a consequence of (15). Similar proofs can be provided for different signatures of metrics. □

The vacuum equations in Einstein gravity can be parametrized as solitonic hierarchies with constant on \(\chi\) N–anholonomic frames, see (61). The corresponding SG hierarchies are defined as solutions of the equations (49) and (50) and (51), with constant scalar curvature and frame coefficients.

6 Conclusion

In summary, we have considered a geometric formalism of encoding general (semi) Riemannian metrics and their lifts to tangent bundles into families of nonholonomic hierarchies of bi–Hamiltonian structures and related solitonic equations derived for curve flows on tangent spaces. Towards this ends, we have applied a programme of study that is based on prior works on nonholonomic Ricci [28, 29] and curve flows [20, 25]. The premise of this methodology is that one can derive solitonic hierarchies for non–stretching curve flows on constant curvature Riemannian spaces [14, 15, 16, 19]. The validity of this approach was substantiated by the encoding into solitonic hierarchies of arbitrary (semi) Riemannian and Finsler–Lagrange metrics [20, 25].

Our analysis was completed by explicit constructions related to solitonic encoding of Ricci flow equations [11, 2, 3, 14, 5]. First of all, we note that it was not possible to perform such constructions working only with the Levi Civita connection because, in general, the curvature tensor for this connection can not be parametrized by constant coefficients. The idea was to re–define equivalently the geometric objects and basic Ricci flow and field equations for other classes of linear connections generated on vector/ tangent bundles and/or (semi) Riemannian manifolds enabled with certain classes of preferred frames with associated nonlinear connection (N–connection) structure. In particular, we elaborated such lifts of (semi) Riemannian metrics to the tangent bundle when in a canonical form there are defined metric structures, a class of N–connections and distinguished (d) connections when with
respect to certain classes of N–adapted frames it is possible to get constant
curvature coefficients and zero torsion. Such geometric models and their lifts
to different classes of fibred spaces are related to effective Lagrangians and,
inversely, any regular geometric mechanics can be encoded into geometric
objects on nonholonomic Riemannian manifolds/ tangent bundles.

Secondly, we considered Ricci flows of geometric objects and fundamental
evolution/field equations. We found that a nonholonomically constrained
flow of (semi) Riemannian metrics can result in generalized Finsler–Lagrange
configurations which motivates both application of such Finsler geometry
methods for usual Riemannian spaces if moving frames are introduced into
consideration and a deep study of nonholonomic structures with associated
N–connections soldering couples of Klein spaces and/or constant curvature
Riemannian spaces.

Third, we argued that the geometry Riemann and Finsler–Lagrange spa-
aces can be encoded into bi–Hamilton structures and nonholonomic solitonic
equations and anticipated that such curve flow – solitonic hierarchies can
be constructed in a similar manner for exact solutions of Einstein–Yang–
Mills–Dirac equations, derived following the anholonomic frame method, in
noncommutative generalizations of gravity and geometry and possible quan-
tum models based on nonholonomic Lagrange–Fedosov manifolds. In spite
of the fact that there are a number of conceptual and technical difficulties
(such as the physical meaning of the general N–connections, additionally to
the preferred frame systems and nonlinear generalizations of the usual lin-
ear connections, explicit relations of the Ricci flows to renormalization group
flows in quantum gravity models, cumbersome geometric analysis calculus...)
the outcome of such approach is almost obvious that we can encode nonlin-
ear fundamental field/evolution equations in terms of corresponding vector
solitonic equations, their hierarchies and conservation laws.

Finally, we tried not only to encode some geometric information about
metrics, connections and frames into solitons but also formulated certain
criteria when from such solitonic equations we can extract vacuum gravita-
tional spaces or certain more general solutions for the Ricci flow equations
and related Einstein spaces or Euler–Lagrange equations in geometric me-
chanics. For more general classes of solutions and extensions to quantum
gravity, noncommutative geometry, these are desirable purposes for further
investigations.

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A The Geometry of N–anholonomic Vector Bundles

We denote by $\pi^{\top}: TE \rightarrow TM$ the differential of map $\pi : E \rightarrow M$ defined by fiber preserving morphisms of the tangent bundles $TE$ and $TM$. The kernel of $\pi^{\top}$ is just the vertical subspace $vE$ with a related inclusion mapping $i : vE \rightarrow TE$.

**Definition A.1** A nonlinear connection (N–connection) $\mathbf{N}$ on a vector bundle $\mathcal{E}$ is defined by the splitting on the left of an exact sequence

$$0 \rightarrow vE \xrightarrow{i} TE \xrightarrow{} TE/vE \rightarrow 0,$$

i.e. by a morphism of submanifolds $\mathbf{N} : TE \rightarrow vE$ such that $\mathbf{N}i$ is the unity in $vE$.

In an equivalent form, we can say that a N–connection is defined by a Whitney sum of conventional horizontal (h) subspace, ($hE$), and vertical (v) subspace, ($vE$),

$$TE = hE \oplus vE. \quad (A.1)$$

This sum defines a nonholonomic (equivalently, anholonomic, or nonitegrable) distribution of horizontal and vertical subspaces on $TE$. Locally, a N–connection is defined by its coefficients $N^a_i(u)$,

$$\mathbf{N} = N^a_i(u)dx^i \otimes \frac{\partial}{\partial y^a}.\quad (A.2)$$

The well known class of linear connections consists on a particular subclass with the coefficients being linear on $y^a$, i.e., $N^a_i(u) = \Gamma^a_{bj}(x)y^b$.\footnote{We use “boldface” symbols if it is necessary to emphasize that any space and/or geometrical objects are provided/adapted to a N–connection structure, or with the coefficients computed with respect to N–adapted frames.}

**Remark A.1** A bundle space, or a a manifold, is called nonholonomic if it provided with a nonholonomic distribution (see historical details and summary of results in [34]). In particular case, when the nonholonomic distribution is of type (A.1), such spaces are called N–anholonomic [24].

Any N–connection $\mathbf{N} = \{N^a_i(u)\}$ may be characterized by a N–adapted frame (vielbein) structure $e_\nu = (e_i, e_a)$, where

$$e_i = \frac{\partial}{\partial x^i} - N^a_i(u)\frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a}, \quad (A.2)$$
and the dual frame (coframe) structure \( e^a = (e^i, e^a) \), where
\[
e^i = dx^i \quad \text{and} \quad e^a = dy^a + N^a_i (u) dx^i.
\] (A.3)

For any N–connection, we can introduce its N–connection curvature
\[
\Omega = \frac{1}{2} \Omega^a_{ij} \, d^i \wedge d^j \otimes \partial_a,
\]
with the coefficients defined as the Neijenheuse tensor,
\[
\Omega^a_{ij} = e^i_a N_i^a - e^j_a N_i^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N^b_i \frac{\partial N_j^a}{\partial y^b} - N^b_j \frac{\partial N_i^a}{\partial y^b}.
\] (A.4)

The vielbeins (A.3) satisfy the nonholonomy (equivalently, anholonomy) relations
\[
[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma
\] (A.5)
with (antisymmetric) nontrivial anholonomy coefficients \( W^b_{ia} = \partial_a N_i^b \) and \( W^a_{ji} = \Omega^a_{ij} \).

The geometric objects can be defined in a form adapted to a N–connection structure, following decompositions being invariant under parallel transports preserving the splitting (A.1). In this case we call them to be distinguished (by the connection structure), i.e. d–objects. For instance, a vector field \( X \in TV \) is expressed
\[
X = (hX, vX), \quad \text{or} \quad X = X^\alpha e_\alpha = X^i e_i + X^a e_a,
\]
where \( hX = X^i e_i \) and \( vX = X^a e_a \) state, respectively, the adapted to the N–connection structure horizontal (h) and vertical (v) components of the vector (which following Refs. [21, 22] is called a distinguished vector, in brief, d–vector). In a similar fashion, the geometric objects on \( V \), for instance, tensors, spinors, connections, ... are called respectively d–tensors, d–spinors, d–connections if they are adapted to the N–connection splitting (A.1).

**Definition A.2** A distinguished connection (in brief, d–connection) \( D = (hD, vD) \) is a linear connection preserving under parallel transports the nonholonomic decomposition (A.1).

The N–adapted components \( \Gamma^\gamma_{\alpha\beta} \) of a d–connection \( D_\alpha = (e_\alpha] D) \) are defined by the equations
\[
D_\alpha e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma, \quad \text{or} \quad \Gamma^\gamma_{\alpha\beta} (u) = (D_\alpha e_\beta)] e^\gamma.
\] (A.6)
The $N$–adapted splitting into $h$– and $v$–covariant derivatives is stated by

$$
hD = \{ D_k = (L^i_{jk}, L^a_{bk}) \}, \quad vD = \{ D_c = (C^i_{jk}, C^a_{bc}) \},
$$

where, by definition, $L^i_{jk} = (D_k e^j)e^i$, $L^a_{bk} = (D_k e_b)e^i$, $C^i_{jk} = (D_c e_j)e^i$, $C^a_{bc} = (D_c e_b)e^a$. The components $\Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jk}, C^a_{bc})$ completely define a $d$–connection $D$ on $E$.

The simplest way to perform $N$–adapted computations is to use differential forms. For instance, starting with the $d$–connection 1–form,

$$\Gamma_{\alpha\beta} = \frac{\partial N^a_{\alpha\beta}}{\partial y^a}, \quad (A.7)$$

with the coefficients defined with respect to $N$–elongated frames ($A.3$) and ($A.2$), the torsion of a $d$–connection,

$$T^\alpha = de^\alpha + \Gamma^\alpha_{\beta\gamma} e^\beta, \quad (A.8)$$

is characterized by (N–adapted) $d$–torsion components,

$$T^i_{jk} = L^i_{jk} - L^j_{ki}, \quad T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji}, \quad (A.9)$$

For $d$–connection structures on $TM$, we have to identify indices in the form $i \leftrightarrow a, j \leftrightarrow b, ...$ and the components of $N$– and $d$–connections, for instance, $N^a_i \leftrightarrow N^i_a$ and $L^i_{jk} \leftrightarrow L^a_{bk}, C^i_{ja} \leftrightarrow C^b_{ca} \leftrightarrow C^i_{jk}$.

**Definition A.3** A distinguished metric (in brief, $d$–metric) on a vector bundle $E$ is a usual second rank metric tensor $g = g^{\alpha\beta}N^a h$, equivalently,

$$g = g_{ij}(x, y) \ e^i \otimes e^j + h_{ab}(x, y) \ e^a \otimes e^b, \quad (A.10)$$

adapted to the $N$–connection decomposition ($A.1$).

With respect to a coordinate basis, the metric $g$ ($A.10$) can be written in the form

$$g = g_{\alpha\beta}(u) \ du^\alpha \otimes du^\beta \quad (A.11)$$

where

$$g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N^i_j N^j_i h_{ab} & N_c^i g_{ae} \\ N_c^i g_{be} & g_{ab} \end{bmatrix}. \quad (A.12)$$

From the class of arbitrary $d$–connections $D$ on $V$, one distinguishes those which are metric compatible (metrical) satisfying the condition

$$Dg = 0 \quad (A.13)$$
including all h- and v-projections \( D_j g_{kl} = 0, D_ag_{kl} = 0, D_j h_{ab} = 0, D_a h_{bc} = 0 \). For d–metric structures on \( V \simeq TM \), with \( g_{ij} = h_{ab} \), the conditions of vanishing "nonmetricity" \( (A.13) \) transform into

\[
h D(g) = 0 \text{ and } v D(h) = 0,
\]

\( (A.14) \)
i.e. \( D_j g_{kl} = 0 \) and \( D_a g_{kl} = 0 \).

There are two types of preferred linear connections uniquely determined by a generic off–diagonal metric structure with \( n+m \) splitting, see \( g = g \oplus Nh \):

1. The Levi Civita connection \( \nabla = \{ \Gamma_{\beta \gamma}^\alpha \} \) is by definition torsionless, \( T = 0 \), and satisfies the metric compatibility condition, \( \nabla g = 0 \).

2. The canonical d–connection \( \hat{\Gamma}_{\alpha \beta}^\gamma = (\hat{L}_{jk}^i, \hat{L}_{bk}^a, \hat{C}_{jc}^i, \hat{C}_{bc}^a) \) is also metric compatible, i.e. \( \hat{D} g = 0 \), but the torsion vanishes only on h– and v–subspaces, i.e. \( \hat{T}_{jk}^i = 0 \) and \( \hat{T}_{bc}^a = 0 \), for certain nontrivial values of \( \hat{T}_{ja}^i, \hat{T}_{bi}^a, \hat{T}_{ji}^a \). For simplicity, we omit hats on symbols and write, for simplicity, \( L_{jk}^i \) instead of \( \hat{L}_{jk}^i \), \( T_{jk}^i \) instead of \( \hat{T}_{jk}^i \) and so on...but preserve the general symbols \( \hat{D} \) and \( \hat{\Gamma}_{\alpha \beta}^\gamma \).

With respect to \( N \)-adapted frames \( (A.2) \) and \( (A.3) \), we can verify that the requested properties for \( \hat{D} \) on \( E \) are satisfied if

\[
L_{jk}^i = \frac{1}{2} g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \quad (A.15)
\]

\[
L_{bk}^a = e_b (N_k^a) + \frac{1}{2} h^{ac} (e_a h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d),
\]

\[
C_{jc}^i = \frac{1}{2} g^{jk} e_c g_{jk}, \quad C_{bc}^a = \frac{1}{2} h^{ad} (e_a h_{bd} + e_c h_{cd} - e_d h_{bc}).
\]

For \( E = TM \), the canonical d–connection \( \bar{D} = (h \bar{D}, v \bar{D}) \) can be defined in torsionless form\(^{10}\) with the coefficients \( \Gamma_{\beta \gamma}^\alpha = (L_{jk}^i, L_{bc}^a) \),

\[
L_{jk}^i = \frac{1}{2} g^{ih}(e_k g_{jh} + e_j g_{kh} - e_h g_{jk}), \quad (A.16)
\]

\[
C_{bc}^a = \frac{1}{2} h^{ae} (e_c h_{be} + e_b h_{ce} - e_e h_{bc}).
\]

The curvature of a d–connection \( D \),

\[
\mathcal{R}^\alpha_{\beta \gamma} = \bar{D} \Gamma^\alpha_{\beta \gamma} = d \Gamma^\alpha_{\beta \gamma} - \Gamma^\alpha_{\beta \gamma} \wedge \Gamma^\alpha_{\gamma \beta}, \quad (A.17)
\]

\(^{10}\) i.e. it has the same coefficients as the Levi Civita connection with respect to \( N \)-elongated bases \( (A.2) \) and \( (A.3) \).
splits into six types of N–adapted components with respect to (A.2) and (A.3),
\[ R^a_{\beta\gamma\delta} = \left( R^i_{hjk}, R^a_{bjk}, P^i_{hja}, P^c_{bjc}, S^i_{jbe}, S^a_{bcd} \right), \]

\[
\begin{align*}
R^i_{hjk} &= e_k L^i_{hj} - e_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{ha} \Omega^a_{kj}, \\
R^a_{bjk} &= e_k L^a_{bj} - e_j L^a_{bk} + L^c_{bj} L^a_{ck} - L^c_{bk} L^a_{cj} - C^a_{bc} \Omega^c_{kj}, \\
P^i_{jka} &= e_a L^i_{jk} - D_k C^i_{ja} + C^i_{jb} T^b_{ka}, \\
P^c_{bjc} &= e_a L^c_{bk} - D_k C^c_{ba} + C^c_{bd} T^c_{ka}, \\
S^i_{jbc} &= e_a C^i_{jb} - e_b C^i_{jc} + C^h_{jc} C^i_{hc} - C^c_{jc} C^i_{hb}, \\
S^a_{bcd} &= e_a C^a_{bc} - e_c C^a_{bd} + C^e_{bc} C^a_{ed} - C^e_{bd} C^a_{ec}.
\end{align*}
\]

Contracting respectively the components, \( R_{\alpha\beta} \div R^r_{\alpha\beta r} \), one computes the h–v–components of the Ricci d–tensor (there are four N–adapted components)
\[
R_{ij} \div R^k_{ijk}, \quad R_{ia} \div -P^k_{ika}, \quad R_{ai} \div P^b_{aib}, \quad S_{ab} \div S^c_{abc}.
\]

The scalar curvature is defined by contracting the Ricci d–tensor with the inverse metric \( g^{\alpha\beta} \),
\[
\overrightarrow{R} \div g^{\alpha\beta} R_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} S_{ab} = \overrightarrow{R} + \overrightarrow{S}.
\]

If \( E = TM \), there are only three classes of d–curvatures,
\[
\begin{align*}
R^i_{hjk} &= e_k L^i_{hj} - e_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{ha} \Omega^a_{kj}, \\
P^i_{jka} &= e_a L^i_{jk} - D_k C^i_{ja} + C^i_{jb} T^b_{ka}, \\
S^a_{bcd} &= e_a C^a_{bc} - e_c C^a_{bd} + C^e_{bc} C^a_{ed} - C^e_{bd} C^a_{ec},
\end{align*}
\]

where all indices \( a, b, ..., i, j, ... \) run the same values and, for instance, \( C^e_{bc} \rightarrow C^i_{jk}, ... \)

**B  N–anholonomic Klein Spaces**

There are Ricci flow families of two Hamiltonian variables given by the principal normals \( h^\nu \) and \( v^\nu \), respectively, in the horizontal and vertical subspace, defined by the canonical d–connections \( \chi D = (h \chi D, v \chi D) \), see formulas (19) and (20),
\[
h^\nu(\chi) \div \chi D_h x h X = \nu^i(\chi) \chi e_i \text{ and } v^\nu(\chi) \div \chi D_v x v X = \nu^\beta(\chi) \varepsilon^\beta.
\]

This normal d–vectors \( v(\chi) = (h^\nu(\chi), v^\nu(\chi)) \), with components of type
\[
\nu^\alpha(\chi) = (\nu^i(\chi), \nu^p(\chi)) = (\nu^1(\chi), \nu^2(\chi), \nu^{n+1}(\chi), \nu^{n+2}(\chi)),
\]

38
are oriented in the tangent directions of curves $\gamma$. There is also the principal normal $d$-vectors $\varpi(\gamma) = (h \varpi(\gamma), v \varpi(\gamma))$ with components of type $\varpi^a(\gamma) = (\varpi^1(\gamma), \varpi^a(\gamma)) = (\varpi^1(\gamma), \varpi^a(\gamma), \varpi^{a+1}(\gamma), \varpi^a(\gamma))$ in the flow directions, with

$$h \varpi(\gamma) = \chi D_h Y h X = \varpi^i(\gamma) \chi e_i, \ v \varpi(\gamma) = \chi D_v Y v X = \varpi^a(\gamma) e_a,$$

representing a Hamiltonian $d$-covector field. We can consider that the normal part of the flow $d$-vector

$$h_\perp(\gamma) = Y_\perp(\gamma) = h^i(\gamma) \chi e_i + h^a(\gamma) e_a$$

represents a Hamiltonian $d$-vector field. For such configurations, we can consider parallel N-adapted frames $\chi e_{a'} = (\chi e_{i'}, e_{a'})$ when the $h$-variables $\nu^i(\gamma), \varpi^a(\gamma), h^a(\gamma)$ are respectively encoded in the top row of the horizontal canonical $d$-connection matrices $\chi \Gamma_{hX} j' j''$ and $\chi \Gamma_{hY} j' j''$ and in the row matrix $(\chi e_{Y_\perp}) = (\chi e_{Y'}, \chi e_\chi X')$ where $\chi g|| = \chi g(hY, hX)$ is the tangential $h$-part of the flow $d$-vector. A similar encoding holds for $v$-variables $\nu^i(\gamma), \varpi^a(\gamma), h^a(\gamma)$ in the top row of the vertical canonical $d$-connection matrices $\chi \Gamma_{vX} a' b'$ and $\chi \Gamma_{vY} a' b'$ and in the row matrix $(\chi e_{Y'}) = (\chi e_{Y'}, \chi e_{X'})$ where $\chi h || = \chi h(vY, vX)$ is the tangential $v$-part of the flow $d$-vector. In a compact form of notations, we shall write $\nu^a(\gamma)$ and $\varpi^a(\gamma)$ where the primed small Greek indices $a', \beta', \ldots$ will denote both N-adapted and then orthonormalized components of geometric objects (like $d$-vectors, $d$-covectors, $d$-tensors, $d$-groups, $d$-algebras, $d$-matrices) admitting further decompositions into $h$- and $v$-components defined as non-integrable distributions of such objects.

With respect to N-adapted orthonormalized frames, the geometry of N-anholonomic manifolds is defined algebraically, on their tangent bundles, by couples of horizontal and vertical Klein geometries considered in [38] and for bi-Hamiltonian soliton constructions in [18]. The N-connection structure induces a N-anholonomic Klein space stated by two left-invariant $hG$- and $vG$-valued Maurer–Cartan form on the Lie d-group $G = (hG, vG)$ is identified with the zero-curvature canonical $d$-connection 1-form $G \Gamma = \{ G \Gamma_{a'} \}$, where

$$G \Gamma_{a'} = \chi \Gamma_{a' b'} e^b = hG L^{j' k'} e^{j'} + vG C^{j' k'}. $$

For trivial N-connection structure in vector bundles with the base and typical fiber spaces being symmetric Riemannian spaces, we can consider that $hG L^{j' k'}$ and $vG C^{j' k'}$ are the coefficients of the Cartan connections $hG L$ and $vG C$, respectively for the $hG$ and $vG$, both with vanishing curvatures, i.e. with

$$d G \Gamma + \frac{1}{2} [G \Gamma, G \Gamma] = 0$$

39
and $h$– and $v$–components, $d^hG\mathbf{L} + \frac{1}{2} [h^G\mathbf{L}, h^G\mathbf{L}] = 0$ and $d^vG\mathbf{C} + \frac{1}{2} [v^G\mathbf{C}, v^G\mathbf{C}] = 0$, where $d$ denotes the total derivatives on the $d$–group manifold $\mathbb{G} = h\mathbb{G} \oplus v\mathbb{G}$ or their restrictions on $h\mathbb{G}$ or $v\mathbb{G}$. We can consider that $\mathbb{G} \Gamma$ defines the so–called Cartan d–connection for nonintegrable $N$–connection structures, see details and supersymmetric/ noncommutative developments in [26, 24].

Through the Lie $d$–algebra decompositions $\mathfrak{g} = h\mathfrak{g} \oplus v\mathfrak{g}$, for the horizontal splitting: $h\mathfrak{g} = \mathfrak{so}(n) \oplus h\mathfrak{p}$, when $[h\mathfrak{p}, h\mathfrak{p}] \subset \mathfrak{so}(n)$ and $[\mathfrak{so}(n), h\mathfrak{p}] \subset h\mathfrak{p}$; for the vertical splitting $v\mathfrak{g} = \mathfrak{so}(m) \oplus v\mathfrak{p}$, when $[v\mathfrak{p}, v\mathfrak{p}] \subset \mathfrak{so}(m)$ and $[\mathfrak{so}(m), v\mathfrak{p}] \subset v\mathfrak{p}$, the Cartan d–connection determines an $N$–anholonomic Riemannian structure on the nonholonomic bundle $\hat{\mathcal{E}} = [h\mathbb{G} = SO(n + 1), v\mathbb{G} = SO(m + 1), N^e]$. For $n = m$, and canonical $d$–objects ($N$–connection, $d$–metric, $d$–connection, ...) derived from [27], or any $N$–anholonomic space with constant $d$–curvatures, the Cartan $d$–connection transform just in the canonical $d$–connection (A.10). It is possible to consider a quotient space with distinguished structure group $\mathbb{V}_N = \mathbb{G}/SO(n) \oplus SO(m)$ regarding $\mathbb{G}$ as a principal ($SO(n) \oplus SO(m)$)–bundle over $\hat{\mathcal{E}}$, which is a $N$–anholonomic bundle. In this case, we can always fix a local section of this bundle and pull–back $\mathbb{G} \Gamma$ to give a $(h\mathfrak{g} \oplus v\mathfrak{g})$–valued 1–form $\theta \Gamma$ in a point $u \in \hat{\mathcal{E}}$. Any change of local sections define $SO(n) \oplus SO(m)$ gauge transforms of the canonical $d$–connection $\theta \Gamma$, all preserving the nonholonomic decomposition (A.1).

There are involutive automorphisms $h\sigma = \pm 1$ and $v\sigma = \pm 1$, respectively, of $h\mathfrak{g}$ and $v\mathfrak{g}$, defined that $\mathfrak{so}(n)$ (or $\mathfrak{so}(m)$) is eigenspace $h\sigma = +1$ (or $v\sigma = +1$) and $h\mathfrak{p}$ (or $v\mathfrak{p}$) is eigenspace $h\sigma = -1$ (or $v\sigma = -1$). Taking into account the existing eigenspaces, when the symmetric parts $\Gamma = \frac{1}{2} (\theta \Gamma + \theta (\theta \Gamma))$, with respective $h$– and $v$–splitting, $\mathbf{L} = \frac{1}{2} (h^G\mathbf{L} + h\sigma (h^G\mathbf{L}))$ and $\mathbf{C} = \frac{1}{2} (v^G\mathbf{C} + h\sigma (v^G\mathbf{C}))$, defines a $(\mathfrak{so}(n) \oplus \mathfrak{so}(m))$–valued $d$–connection 1–form, we can construct $N$–adapted decompositions.

The antisymmetric part $\mathbf{e} = \frac{1}{2} (\theta \mathcal{E} - \theta (\theta \mathcal{E}))$, with $h$– and $v$–splitting, $h\mathbf{e} = \frac{1}{2} (h^G\mathbf{L} - h\sigma (h^G\mathbf{L}))$ and $v\mathbf{e} = \frac{1}{2} (v^G\mathbf{C} - h\sigma (v^G\mathbf{C}))$, defines a $(h\mathfrak{p} \oplus v\mathfrak{p})$–valued $N$–adapted coframe for the Cartan–Killing inner product $\cdot \cdot \cdot >_p$ on $T_u\mathbb{G} = h\mathfrak{g} \oplus v\mathfrak{g}$ restricted to $T_u\mathbb{V}_N \simeq \mathfrak{p}$. This inner product, distinguished into $h$– and $v$–components, provides a $d$–metric structure of type $\mathfrak{g} = [g, h]$ (A.10), where $g = \langle he \otimes he \rangle >_h$ and $h = \langle ve \otimes ve \rangle >_v$ on $\mathbb{V}_N = \mathbb{G}/SO(n) \oplus SO(m)$.

We generate a $\mathbb{G}(= h\mathbb{G} \oplus v\mathbb{G})$–invariant $d$–derivatives $\mathcal{D}$ whose restriction to the tangent space $T\mathbb{V}_N$ for any $N$–anholonomic curve flow $\gamma(\tau, l, \chi)$ in $\mathbb{V}_N = \mathbb{G}/SO(n) \oplus SO(m)$ is defined via

$$\chi \mathcal{D}_x \chi \mathfrak{e} = [\chi \mathfrak{e}, \gamma_1] \chi \psi$$ and $$\chi \mathcal{D}_y \chi \mathfrak{e} = [\chi \mathfrak{e}, \gamma_2] \chi \psi,$$

admitting further $h$– and $v$–decompositions. The derivatives $\chi \mathcal{D}_x$ and $\chi \mathcal{D}_y$
in (B.1) are equivalent to those considered in (18) and obey the Cartan structure equations (22) and (23). For the canonical d–connections, a large class of N–anholonomic spaces of dimension \( n = m \), the d–torsions are zero and the d–curvatures are with constant coefficients.

Let \( \chi e^{\alpha'} = (e^{\alpha'}, \chi e^{\alpha'}) \) be a family of N–adapted orthonormalized coframes identified with the \( (h \mathfrak{p} \oplus v \mathfrak{p}) \)–valued coframe \( e \) in a fixed orthonormal basis for \( \mathfrak{p} = h \mathfrak{p} \oplus v \mathfrak{p} \subset h \mathfrak{g} \oplus v \mathfrak{g} \). Considering the kernel/ cokernel of Lie algebra multiplications in the h– and v–subspaces, respectively, \([e_hX, \cdot]_{h \mathfrak{g}}\) and \([e_vX, \cdot]_{v \mathfrak{g}}\), we can decompose the coframes into parallel and perpendicular parts with respect to \( e_X \). We write

\[
\chi e = (e_C = he_C + ve_C, e_{C\perp} = he_{C\perp} + ve_{C\perp}),
\]

for \( \mathfrak{p} = (h \mathfrak{p} \oplus v \mathfrak{p}) \)–valued mutually orthogonal d–vectors \( e_C \) and \( e_{C\perp} \), when there are satisfied the conditions \([e_X, e_C]_{h \mathfrak{g}} = 0 \) but \([e_X, e_{C\perp}]_{h \mathfrak{g}} \neq 0 \); such conditions can be stated in h– and v–component form, respectively, \([he_X, he_C]_{h \mathfrak{g}} = 0, [he_X, he_{C\perp}]_{h \mathfrak{g}} \neq 0 \) and \([ve_X, ve_C]_{v \mathfrak{g}} = 0, [ve_X, ve_{C\perp}]_{v \mathfrak{g}} \neq 0 \). One holds also the algebraic decompositions

\[
T_u V_N \simeq \mathfrak{p} = h \mathfrak{p} \oplus v \mathfrak{p} = h \mathfrak{g} \oplus v \mathfrak{g} / \mathfrak{so}(n) \oplus \mathfrak{so}(m),
\]

\[
p = p_C \subset p_{C\perp} = (h \mathfrak{p}_C \oplus v \mathfrak{p}_C) \oplus (h \mathfrak{p}_{C\perp} \oplus v \mathfrak{p}_{C\perp}),
\]

with \( \mathfrak{p}_\parallel \subset \mathfrak{p}_C \) and \( \mathfrak{p}_{C\perp} \subset \mathfrak{p}_\perp \), where \([\mathfrak{p}_\parallel, \mathfrak{p}_C] = 0, < \mathfrak{p}_{C\perp}, \mathfrak{p}_C >> 0 \), but \([\mathfrak{p}_\parallel, \mathfrak{p}_{C\perp}] \neq 0 \) (i.e. \( \mathfrak{p}_C \) is the centralizer of \( e_X \) in \( \mathfrak{p} = h \mathfrak{p} \oplus v \mathfrak{p} \subset h \mathfrak{g} \oplus v \mathfrak{g} \)). Using the canonical d–connection derivative \( D_X \) of a d–covector perpendicular (or parallel) to \( e_X \), we get a new d–vector which is parallel (or perpendicular) to \( e_X \), i.e. \( D_X e_C \in \mathfrak{p}_{C\perp} \) (or \( D_X e_{C\perp} \in \mathfrak{p}_C \)); in h– and v–components such formulas are written \( D_{hX}he_C \in h \mathfrak{p}_{C\perp} \) (or \( D_{hX}he_{C\perp} \in h \mathfrak{p}_C \)) and \( D_{vX}ve_C \in v \mathfrak{p}_{C\perp} \) (or \( D_{vX}ve_{C\perp} \in v \mathfrak{p}_C \)). All such d–algebraic relations can be written in N–anholonomic manifolds and canonical d–connection settings, for instance, using certain relations of type

\[
\chi D_X(\chi e^{\alpha'})_C = \chi v^{\alpha'}_{\beta'}(\chi e^{\beta'})_{C\perp} \quad \text{and} \quad \chi D_X(\chi e^{\alpha'})_{C\perp} = -\chi v^{\alpha'}_{\beta'}(\chi e^{\beta'})_C,
\]

for some antisymmetric d–tensors \( \chi v^{\alpha'\beta'} = -\chi v^{\beta'\alpha'} \). We get a N–adapted \( (\mathfrak{so}(n) \oplus \mathfrak{so}(m)) \)-parallel frame defining a generalization of the concept of Riemannian parallel frame on N–adapted manifolds whenever \( \mathfrak{p}_C \) is larger than \( \mathfrak{p}_\parallel \). Substituting \( \chi e^{\alpha'} = (e^{\alpha'}, \chi e^{\alpha'}) \) into the last formulas and considering
h- and v–components, we define $SO(n)$–parallel and $SO(m)$–parallel frames (for simplicity we omit these formulas when the Greek small letter indices are split into Latin small letter h- and v–indices).

The final conclusion of this section is that the Cartan structure equations on hypersurfaces swept out by nonholonomic curve flows on N–anholonomic spaces with constant matrix curvature for the canonical d–connection geometrically encode two $O(n−1)$– and $O(m−1)$–invariant, respectively, horizontal and vertical bi–Hamiltonian operators. This holds true if the distinguished by N–connection freedom of the d–group action $SO(n) \oplus SO(m)$ on $\chi e$ and $\chi \Gamma$ is used to fix them to be a N–adapted parallel coframe and its associated canonical d–connection 1–form is related to the canonical covariant derivative on N–anholonomic manifolds.

C Proof of the Main Theorem

We provide a proof of Theorem 5.1 for the horizontal Ricci and curve flows (similar results were published in [25] and [13], respectively, for Lagrange–Finsler and symmetric Riemannian spaces). The vertical constructions are similar but with respective changing of h– variables / objects into v–variables / objects.

One obtains a vector mKdV equation up to a convective term, which can be absorbed by redefinition of coordinates, defining the +1 flow for $h \overrightarrow{e}_\perp (\chi) = \chi \overrightarrow{v}_1$,

$$\chi \overrightarrow{v}_\tau = \chi \overrightarrow{v}_3 + \frac{3}{2} \overrightarrow{v}(\chi)^2 - \overrightarrow{R}(\chi) \chi \overrightarrow{v}_1,$$

when the +(k+1) flow gives a vector mKdV equation of higher order $3+2k$ on $\overrightarrow{v}$ and there is a 0 h–flow $\overrightarrow{v}_\tau = \overrightarrow{v}_1$ arising from $h \overrightarrow{e}_\perp = 0$ and $h \overrightarrow{e}_\parallel = 1$ belonging outside the hierarchy generated by $h\mathcal{R}(\chi)$. Such flows correspond to N–adapted horizontal motions of the curve $\chi \gamma(\tau, 1) = h \gamma(\tau, 1) + v \gamma(\tau, 1, \chi)$, given by

$$(h \gamma)_\tau (\chi) = f ((h \gamma)_{hX}(\chi), \chi D_{hX} (h \gamma)_{hX}(\chi), \chi D_{hX}^2 (h \gamma)_{hX}(\chi), ...)$$

subject to the non–stretching condition $|(h \gamma)_{hX}(\chi)|_{hg} = 1$, when the equation of motion is to be derived from the identifications

$$(h \gamma)_\tau (\chi) \longleftrightarrow e_{hX}(\chi), \chi D_{hX} (h \gamma)_{hX}(\chi) \longleftrightarrow \chi D_{hX} \chi e_{hX} = [ \chi L_{hX}, \chi e_{hX}]$$

and so on, which maps the constructions from the tangent space of the curve.
to the space $hp$. For such identifications, we have

$$\left[ \chi_{L_h X}, \chi_{e_h X} \right] = -\begin{bmatrix} 0 & (0, \chi_{\bar{v}}) \\ -\left(0, \chi_{\bar{v}} \right)^T & h0 \end{bmatrix} \in hp,$$

$$\left[ \chi_{L_h X}, [ \chi_{L_h X}, \chi_{e_h X} ] \right] = -\begin{bmatrix} 0 & \left( |\chi_{\bar{v}}|^2, 0 \right) \\ -\left( |\chi_{\bar{v}}|^2, 0 \right)^T & h0 \end{bmatrix}$$

and so on, see similar calculus in [29]. At the next step, stating for the $+1$\h–flow

$$h\bar{e}_\perp(\chi) = \chi_{\bar{v}} \perp \chi_{\bar{v}} \parallel (\chi) = -\chi_{D_h X} (\chi_{\bar{v}} \cdot \chi_{\bar{v}}) = -\frac{1}{2} |\bar{v}(\chi)|^2,$$

we compute

$$\chi_{e_h Y} = \begin{bmatrix} 0 \\ -\left(he_\parallel, h\bar{e}_\perp \right)^T (\chi) \end{bmatrix} \in \begin{bmatrix} h0 \\ \chi_{\bar{v}} \end{bmatrix}$$

$$= -\frac{1}{2} |\bar{v}(\chi)|^2 \begin{bmatrix} 0 \\ \left(1, 0 \right)^T \\ -\left(0, 0 \right)^T \\ h0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -\left(0, \chi_{\bar{v}} \parallel \chi_{\bar{v}} \right)^T \end{bmatrix} = -\chi_{D_h X} [ \chi_{L_h X}, \chi_{e_h X}] + \frac{1}{2} \left[ \chi_{L_h X}, [ \chi_{L_h X}, \chi_{e_h X}] \right]$$

Following above presented identifications related to the first and second terms, when

$$|\bar{v}(\chi)|^2 = \left[ \chi_{L_h X}, \chi_{e_h X} \right], [ \chi_{L_h X}, \chi_{e_h X}] > hp$$

$$\iff h\gamma \left( \chi_{D_h X} (h\gamma)_h X (\chi), \chi_{D_h X} (h\gamma)_h X (\chi) \right)$$

$$= \chi_{D_h X} (h\gamma)_h X (\chi)$$

we can identify $\chi_{D_h X} [ \chi_{L_h X}, \chi_{e_h X}]$ to $\chi_{D_h X}^2 (h\gamma)_h X (\chi)$ and write

$$-\chi_{e_h Y} \iff \chi_{D_h X}^2 (h\gamma)_h X (\chi) + \frac{3}{2} \left| \chi_{D_h X} (h\gamma)_h X (\chi) \right|^2 h\gamma \gamma_h X (\chi)$$

which is just the first equation [40] in the Theorem [5, 1] defining a family of non–stretching mKdV map h–equations induced by the h–part of the family of canonical d–connections.
Using the adjoint representation \( \text{ad} (\cdot) \) acting in the Lie algebra \( h\mathfrak{g} = h\mathfrak{p} \oplus \mathfrak{so}(n) \), with
\[
\text{ad}([\chi L_{hX}, \chi e_{hX}]) \chi e_{hX} = \begin{bmatrix}
0 \\
-\langle 0, \overrightarrow{v}(\chi) \rangle T
\end{bmatrix} \in \mathfrak{so}(n + 1),
\]
where
\[
\overrightarrow{v}(\chi) = -\begin{bmatrix}
0 \\
-\overrightarrow{v}(\chi) h0
\end{bmatrix} \in \mathfrak{so}(n),
\]
and the derived (applying \( \text{ad}([\chi L_{hX}, \chi e_{hX}]) \) again)
\[
\text{ad}([\chi L_{hX}, \chi e_{hX}])^2 \chi e_{hX} = -|\overrightarrow{v}(\chi)|^2 \begin{bmatrix}
0 \\
-\langle 1, \overrightarrow{v}(\chi) \rangle T
\end{bmatrix} \in \mathfrak{so}(n + 1),
\]
the equation (46) can be represented in alternative form
\[
-(h\gamma)_\tau (\chi) = \chi D^2_{hX} (h\gamma)_{hX} (\chi) - \frac{3}{2} \overrightarrow{R}^{-1}(\chi) \text{ad} (\chi D_{hX} (h\gamma)_{hX} (\chi)) \overrightarrow{R}^{-1}(\chi),
\]
which is more convenient for analysis of higher order flows on \( \overrightarrow{v}(\chi) \) subjected to higher-order geometric partial differential equations. Here we note that the 0 flow one \( \overrightarrow{v}(\chi) \) correspond to just convective (linear travelling h–wave but subjected to certain nonholonomic constraints) map equations (45).

Now we consider -1 flows contained in the family of h–hierarchies derived from the property that \( h \overrightarrow{e}_\perp(\chi) \) is annihilated by the h–operator \( hJ(\chi) \) and mapped into \( h\mathfrak{R}(h \overrightarrow{e}_\perp(\chi)) = 0 \). This states that \( h\mathfrak{J}(h \overrightarrow{e}_\perp(\chi)) = \chi \overrightarrow{e} = 0 \). Such properties together with (28) and equations (37) imply \( \chi \text{L}_\tau = 0 \) and hence \( h\text{D}_\tau e_{hX}(\chi) = [\chi \text{L}_\tau, \chi e_{hX}] = 0 \) for \( h\text{D}_\tau(\chi) = h\text{D}_\tau(\chi) + [\chi \text{L}_\tau, \cdot] \).

We obtain the equation of motion for the h–component of curve, \( h\gamma(\tau, l, \chi) \), following the correspondences
\[
\text{D}_{hY} \longleftrightarrow h\text{D}_\tau(\chi) \text{ and } h\gamma(\chi) \longleftrightarrow \chi e_{hX},
\]
which is just the first equation in (47).

Finally, we note that the formulas for the v–components, stated by Theorem 5.1 can be derived in a similar form by respective substitution in the the above proof of the h–operators and h–variables into v–ones, for instance, \( h\gamma \rightarrow v\gamma, h \overrightarrow{e}_\perp \rightarrow v \overrightarrow{e}_\perp, \overrightarrow{v} \rightarrow \overrightarrow{v}, \overrightarrow{e} \rightarrow \overrightarrow{e}, h\text{D}_{hX} \rightarrow \text{D}_{vX}, h\text{D}_{hY} \rightarrow \text{D}_{vY}, h\mathfrak{R} \rightarrow \mathfrak{S}, h\mathfrak{D} \rightarrow v\mathfrak{D}, h\mathfrak{R} \rightarrow v\mathfrak{R}, h\mathfrak{J} \rightarrow v\mathfrak{J} \), ...where, for simplicity, we omit parametric dependencies on \( \chi \).
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