Advances in sequential measurement and control of open quantum systems

Stefano Gherardini,1,2,3 Andrea Smirne,4 Matthias M. Müller,1,2 and Filippo Caruso1,2

1Department of Physics and Astronomy, University of Florence, via G. Sansone 1, I-50019 Sesto Fiorentino, Italy.
2LENS and QSTAR, via N. Carrara 1, I-50019 Sesto Fiorentino, Italy.
3CNR-INO, Largo E. Fermi 6, I-50125 Firenze, Italy.
4Institute of Theoretical Physics, and IQST, Universität Ulm, Albert-Einstein-Allee 11, 89069 Ulm, Germany.

Novel concepts, perspectives and challenges in measuring and controlling an open quantum system via sequential schemes are shown. In particular, we discuss how protocols, relying both on repeated quantum measurements and dynamical decoupling control pulses, can allow to: (i) Confine and protect quantum dynamics from decoherence in accordance with the Zeno physics. (ii) Analytically predict the probability that a quantum system is transferred into a target quantum state by means of stochastic sequential measurements. (iii) Optimally reconstruct the spectral density of environmental noise sources by orthogonalizing in the frequency domain the filter functions driving the designed quantum-sensor. (iv) Characterize the relation between the non-Markovianity of an open quantum system and the multi-time statistics outcomes obtained by its monitoring, beyond the hypothesis of time-translational invariance. The achievement of these tasks will enhance our capability to observe and manipulate open quantum systems, thus bringing advances to quantum science and technologies.

Let us consider a non-isolated quantum mechanical system $S$, defined within the finite-dimensional Hilbert space $\mathcal{H}_S$. Its dynamics is governed by a time-dependent Hamiltonian of the form $H(t) = H_0 + H_{\text{cont}}(t)$, where $H_0$ is the Hamiltonian of the system without taking into account any interaction with the external environment $E$. Instead, $H_{\text{cont}}(t) = \lambda(t)\sigma_S$ denotes a coherent control term where $\sigma_S$ is an arbitrary operator acting on $S$. $H_{\text{cont}}(t)$ depends on the function $\lambda(t)$ that is properly modulated so as to fulfill the control tasks required by the user.

In case $S$ is a closed system, $H_0$ is a time-independent Hamiltonian term, and the only ways to interact with the system are given by performing control terms and measurements, usually on a portion of the system. Conversely, $S$ has to be considered an open quantum system when is in contact with other external systems $E$. Usually, the effects of such interactions affect only $S$ and they can be easily modeled by adding in the Hamiltonian $H_0$ a non-deterministic term proportional to a stochastic field $E(t)$, to be effectively seen as an environmental noise contribution. Under this hypothesis, the evolution of the system is described by a stochastic quantum dynamics; in this regard, results stemming from the statistical field theory [2] have been recently derived [3,4]. Moreover, the noise can involve not only $S$ but also the control pulse and the measurement device. Though the effects of such noise sources lead to systematic errors that can be selectively identified and attenuated, they need to be properly modeled so as to avoid a substantial loss of efficiency and accuracy. Finally, one can adopt the standard description of open quantum systems, whereby the system is physically coupled to a structured non-equilibrium environment modeling its surroundings. In this case, the global dynamics is governed by an Hamiltonian $H(t)$ including also the term $H_{\text{int}}$ describing the interaction between $S$ and $E$. Then, the effects of $E$ on the dynamics of $S$ are obtained just by applying the partial trace of the density matrix of the composite system $S + E$ over the degrees of freedom of the environment. In case $H_{\text{int}}$ is fully known and described by deterministic coupling terms, the dynamics of $S$ is deterministic as well; conversely, by including in $H_{\text{int}}$ also the action of stochastic fields, one can recover the stochastic dynamics like the one in [2].

Repeated quantum measurements.- Let us assume to monitor the dynamics of $S$ within the range of time $[0, t_{\text{fin}}]$, which is defined by $m$ distinct time instants $t_{\text{fin}} = t_m > t_{m-1} > \ldots > t_1 > t_0 = 0$, not necessarily equidistributed in time. Protocols allowing it are given by a sequence of quantum measurements, locally performed on $S$ and in correspondence of the time instants $t_k$, $k = 1, \ldots, m$, of the observables $O_k \equiv O_{S,k} \otimes I_S$, where $O_{S,k} \equiv F_{0_k}$ and $I$ denotes the identity operator. Specifically, $\{\theta_k\}$ is the set of the possible measurement outcomes, while $\{F_\theta\}$ denotes the set of positive Hermitian semi-definite operators on $\mathcal{H}_S$ satisfying the relation $\sum_\theta F_\theta = I_S$. Given the system density matrix $\rho_{S,k}$ at time $t_k$, the probability that the outcome $\theta$ associated with the measurement operator $F_\theta$ occurs is returned by the trace $\text{Tr}[\rho_{S,k} F_\theta]$, while the post-measurement state of $S$ equals to $\hat{\rho}_{S,k} = (M_\theta \rho_{S,k} M_\theta^\dagger) / \text{Tr}[M_\theta \rho_{S,k} M_\theta^\dagger]$, where $M_\theta$ fulfills the identity $F_\theta = M_\theta^\dagger M_\theta$.

Coherent pulsed-control couplings.- Coherent control pulses are an essential tool to efficiently perform quantum sensing, i.e., to control a quantum system or quantum phenomena with the aim to perform a measurement of a physical quantity [4,5]. Here, before showing techniques for noise quantum sensing, we briefly introduce the formalism of the open-loop (i.e., in absence of feedback coming from the recording of
measurements outcomes) control strategy given by the application of a sequence of \( \pi\)–pulses. The latter, usually called in quantum computing with the acronym of Dynamical Decoupling (DD) [8], is based on applying short and strong control pulses that invert the phase of the quantum system \( \mathcal{S} \) - usually a qubit - used as a sensor. In the noise sensing context, the qubit.sensor is placed in interaction with an external stochastic field \( E(t) \) with the aim to infer its fluctuation profile. For this purpose, the qubit is prepared in the ground state \( |0\rangle \) and a \( \pi/2 \)-pulse is first performed so as to transfer the system in the superposition state \((|0\rangle + |1\rangle)/\sqrt{2}\), where \( |1\rangle \) is the corresponding excited state. Only at this point the sequence of \( \pi \)-pulse is applied to \( \mathcal{S} \), which thus acquires a phase \( \phi(t) \), providing us (if measured) information about the fluctuating field. In fact, at the end of the DD sequence at time \( t_{\text{fin}} \), the state of the qubit-sensor is \( |\psi_{\text{fin}}\rangle = \left[ |0\rangle + e^{-i\phi(t_{\text{fin}})}|1\rangle \right]/\sqrt{2} \), where \( \phi(t_{\text{fin}}) \equiv \int_{t_{\text{fin}}}^{t_{\text{fin}}} y(t')E(t')dt' \) and \( y(t) \in \{-1,1\} \) is the control pulse modulation function that switches sign whenever a \( \pi \)-pulse occurs. Finally, a second \( \pi/2 \)-pulse brings the qubit-sensor into the final state \( |\psi_{\text{fin}}\rangle = \left[ (e^{i\phi(t_{\text{fin}})} + e^{-i\phi(t_{\text{fin}})})|0\rangle + (e^{i\phi(t_{\text{fin}})} - e^{-i\phi(t_{\text{fin}})})|1\rangle \right]/2 \) and the probability \( p_{0}(t_{\text{fin}}) \equiv \langle |\psi_{\text{fin}}\rangle |0\rangle^2 \) that the qubit-sensor is in the state \( |0\rangle \) at time \( t_{\text{fin}} \) is measured.

In the following, we will show three approaches of measurement and control theory: The first two, based on the physics of Zeno phenomena [3, 5] and noise-sensing spectroscopy [8], rely on applying repeatedly repeated quantum measurements and DD sequences by using models with fluctuating parameters entering in the dynamics of \( \mathcal{S} \). Instead, the third one, focused on the characterization of memory effects [11] within the system dynamics through its monitoring, will adopt the formalism of quantum maps.

I. STOCHASTIC QUANTUM ZENO PHYSICS

The main purpose to apply sequences of quantum measurements based on the quantum Zeno physics [11] is to force the dynamics of \( \mathcal{S} \) to be confined within the Hilbert subspace defined by the measurement observable. Since their introduction, the standard observation protocols, given by sequences of repeated projective measurements, have been applied to closed quantum systems by assuming that between each measurement the system freely evolves with unitary dynamics for a constant small time interval \( \tau \). According to the measurement postulate of quantum mechanics, a projective measurement leads to an instantaneous collapse of the system wave-function \( |\psi_{\mathcal{S}}\rangle \) onto the measurement basis depending on the value assumed by \( |\psi_{\mathcal{S}}\rangle \) before the measurement. Thus, sequential projective measurements imply a time series of resets of the system dynamics. More formally, the measurement observables \( \mathcal{O}_{\mathcal{S},k} \) are all equal for any \( k \) to the projector (Hermitian, idempotent operator, in general with rank greater than 1) \( \Pi \) that defines the confinement Hilbert subspace, i.e., \( \mathcal{H}^{\Pi}_{\mathcal{S}} = \Pi \mathcal{H}_{\mathcal{S}} \), so that the dynamics of the system is described exclusively by the projected (or Zeno) Hamiltonian \( \Pi H_{\Pi} \). Let us also observe that the application of the same projector along a sequence of consecutive measurements denotes the presence of correlations, induced by the surroundings, which can be seen also as control terms.

Recently, the Probability Density Functions (PDF) \( p(\tau) \) and \( p(E(t)) \) respectively of the time intervals \( \tau \) between measurements or the stochastic field \( E(t) \) have been taken into account and the acronym Stochastic Quantum Zeno Dynamics (SQZD) introduced [3, 5]. Here, the peculiarity is that the interaction model with \( E \) is given by fluctuation profiles of one or more parameters entering in the dynamics of \( \mathcal{S} \). It has been proved that the presence of such a noise is detrimental for the confinement of quantum system dynamics by using sequence of quantum measurements in the Zeno regime; however, on the other side it was observed the increasing of the system capability to explore a larger number of configurations within \( \mathcal{H}^{\Pi}_{\mathcal{S}} \). In a controlled setup this viewpoint could make one consider the noise, i.e., the presence of an external environment, as a resource [12, 16]. In this regard, a first result is given by the analytical expression of the probability distribution that \( \mathcal{S} \) belongs to the confinement subspace \( \mathcal{H}^{\Pi}_{\mathcal{S}} \) after a large enough number \( m \) of sequential measurements. In particular, let us define the survival probability \( \mathcal{P}_{\text{fin}} \equiv \text{Prob}(\rho_{\mathcal{S},t_{\text{fin}}} \in \mathcal{H}^{\Pi}_{\mathcal{S}}) \) to find the system in the confinement subspace. In case the time intervals \( \tau_j \) between measurements are independent and identically distributed random variables, \( \mathcal{P}_{\text{fin}} \) is equal to

\[
\mathcal{P}_{\text{fin}} = \prod_{j=1}^{m} \text{Tr} \left[ \Pi U_{j-1:j} \Pi \rho_{\mathcal{S},t_{j}} \Pi U_{j-1:j}^\dagger \right],
\]

where \( U_{j-1:j} \equiv \hat{T} \exp \left( -(i/\hbar) \int_{t_{j-1}}^{t_{j}} H(t) dt \right) \), with \( \hat{T} \) the time ordering operator. Being able to take values from an ensemble of configurations, the density matrix of \( \mathcal{S} \) at time \( t_{\text{fin}} \) and the survival probability \( \mathcal{P}_{\text{fin}} \) are random quantities. Thus, our prediction power is constrained by the ability to compute the most probable value \( \mathcal{P}^* \) of the survival probability after a single realization of the sequence of measurements. In this regard, by using the Large Deviation (LD) theory, it has been derived also the analytical expression of \( \mathcal{P}^* \) for a large enough value of \( m \) [3], i.e.,

\[
\mathcal{P}^* = \exp \left( m \int_{\tau,\eta(t)} p(\tau) p(\eta(t)) \ln (q(\tau, \eta(t))) d\tau d\eta(t) \right),
\]

where \( q(\tau, \eta(t)) \) is a functional identifying the probability that \( \mathcal{S} \) belongs to \( \mathcal{H}^{\Pi}_{\mathcal{S}} \) at time \( t \) after the application of a
projective measurement interspersed by the previous one of a time interval $\tau$. In the limit of small $\tau$'s, $q(\tau, \eta(t))$ admits the second-order expansion $q(\tau, \eta(t)) \approx 1 - \eta(t)^2 \tau^2$.

In particular,

$$\eta(t) = \Delta_{\eta, t} H_\Pi(t)$$

(3)

is the standard deviation of $H_\Pi(t)$ with respect to the system density matrix within the confinement subspace given by $H^\Pi_H \equiv U^\Pi_H \Pi_{t-\tau} U^{\Pi H}_{t-\tau}$, with $U^\Pi_H \equiv \hat{T} \exp \left(-i/\hbar \int_{t-\tau}^t \Pi(t) \Pi dt\right)$. Moreover, in Eq. (2) $p(\eta(t))$ is an artificial PDF obeying the relation

$$\int p(\eta(t)) |\eta(t)|^2 d\eta(t) = \frac{1}{t_{\text{fin}}^t} \int_{t_{\text{fin}}}^t \Delta^2_{\eta, t} H_\Pi(t) dt,$$

i.e., $\int p(\eta(t)) |\eta(t)|^2 d\eta(t) = \frac{1}{t_{\text{fin}}^t} \int_{t_{\text{fin}}}^t \eta(t)^2 dt$, that fixes on average the leakage dynamics of $S$ outside the confinement subspace $H^\Pi_H$.

SQZD is a special class of dynamics induced by protocols based on sequential measurements. In the more general case, the measurement observables within the sequence are no longer equal to a single projector, and also the presence of coherent control terms in the Hamiltonian $H(t)$ can be taken into account. Such protocols are expected to provide the proper tool to explore the whole Hilbert space of a quantum system by engineering the occurrence of the measurement operators in specific time instants, so as to move the system population from one portion of the Hilbert space to another. This question is still challenging, because it requires to properly modulate the control pulse $\lambda(t)$ so that the probability distribution of $t_{\text{fin}}$ is peaked in correspondence of a target value chosen by the user. As final remark, let us observe also that coherent dynamical couplings with an auxiliary system have been studied as tools playing the role of a measurement. In other words the back-action induced by a quantum measurement has been reproduced by using a coherent pulse, and an equivalence between sequences of repeated measurements and pulsed control couplings have been established. However, although such an approximation revealed to experimentally work quite well in peculiar dynamical conditions, as shown e.g. in [17], one has always to keep in mind that, even when the measurement outcomes are not recorded, sequential measurements can lead to dissipative dynamics, thus implying loss of quantum coherence. On the other side, pulsed control couplings are not always able to reproduce the measurement back-action; e.g., as shown in [3], the ideal confinement of quantum dynamics is ensured only by sequence of projective measurements.

II. NOISE-ROBUST QUANTUM SENSING

The prediction power of the results shown in the previous section is ensured by knowing (also partially) the fluctuation profiles of the parameters entering in the dynamics of $S$. In this regard, noise sensing (or spectroscopy) aims to determine the spectral density of the noise originated by the interaction between the quantum system $S$ used as a probe and its external environment $E$. Let us simply take a qubit as sensing device to detect the presence of stochastic (time-varying) magnetic fields $E(t)$. Thus, assuming that the qubit-sensor is coherently manipulated, the application of different and optimized sequences of control pulses allows to enhance the sensor sensitivity in probing the target field. In this regard, the DD control strategy has been successfully applied to noise sensing [21]. Its main peculiarity is that the decay rate (or decoherence function) $\chi(t)$ of the qubit-sensor due the presence of $E$ is related to the probability $p_{00}(t_{\text{fin}})$ via the equation $\chi(t) = -\ln(1 - 2p_{00}(t_{\text{fin}}))$ (see e.g. [3]), and is simply given by the overlap in the frequency domain between the environmental spectral density function $S(\omega)$ and the filter function $F(\omega)$ of the DD sequence driving the qubit. More specifically,

$$\chi(t) = \frac{1}{2} \int_0^t \int_0^t y(t')y(t'')g(t'-t'')dt'dt'',$$

(5)

where $g(t'-t'') \equiv \langle E(0)E(t'-t'') \rangle$ given by the integral $\int_{-\infty}^\infty \int_{-\infty}^\infty \rho(E(0),E(t'-t''))E(0)E(t'-t'')dE(0)dE(t'-t'')$ is the autocorrelation function of the fluctuating field $E(t)$, with $\rho(E(0),E(t'-t''))$ the joint PDF of $E(t)$ in the two time instants $t = 0$ and $t = t' - t''$. Eq. (4) is valid if we assume that the mean value of $E(t)$ is equal to zero, i.e., $\langle E(t) \rangle = 0$, and $E(t)$ is a stationary process so that $g(t', t'') = g(t' - t'')$. Then, being $S(\omega)$ and $F(\omega)$ defined respectively as the Fourier transform of the autocorrelation function $g(t)$ and the pulse modulation function $y(t)$, one has that

$$\chi(t) = \int_{-\infty}^\infty S(\omega)F(\omega)d\omega,$$

(6)

where the filter function $F(\omega) \equiv \frac{1}{\pi}|Y(\omega)|^2$. As a result, from the measurement of $p_{00}$ at the end of the protocol, one can obtain the corresponding value of the decoherence function at time $t_{\text{fin}}$, i.e., $\chi(t_{\text{fin}})$.

As second step, different filter functions $F(\omega)$ can be designed by engineering the pulse modulation function, with the aim to reconstruct $S(\omega)$ in a range $\omega \in [0, \omega_c]$ for a given cut-off frequency $\omega_c$. To this end, let us consider a set of $N$ filter functions $F_n(\omega)$, $n = 1, \ldots, N$, generated by equidistant $\pi$-pulse sequences with a different number of pulses placed in correspondence of the zeros of $\cos[\omega_n(\omega)\frac{n-1}{N}]$. As given by the Filter Orthogonalization (FO) protocol, introduced in [3] for noise-robust quantum sensing, we quantify the overlap between the $N$ filter functions $F_n(\omega)$ in the frequency domain, by computing the $N \times N$ symmetric matrix $A$ with matrix elements

$$A_{nl} \equiv \int_0^{\omega_c} F_n(\omega)F_l(\omega)d\omega.$$

(7)
An accurate estimate of $S(\omega)$ is then obtained in case of no overlaps between the $F_n(\omega)$’s, i.e., if the filter functions are orthogonal and they span a $N$-dimensional space. Otherwise, we can orthogonalize the matrix $A$ by using the transformation $VAV^\dagger = \text{diag}(\chi_1, \ldots, \chi_N)$, where $V$ is an orthogonal matrix and $\chi_n$ are the eigenvalues of $A$. In this way, we will determine a transformed (“rotated”) version of the filter functions $F_n(\omega)$, i.e.,

$$\tilde{F}_n(\omega) = \frac{1}{\sqrt{\chi_n}} \sum_{i=1}^{N} V_{ni} F_i(\omega), \quad n = 1, \ldots, N, \quad (8)$$

that are all orthogonal functions. The procedure is concluded by expanding $S(\omega)$ in the transformed orthogonal basis, so that also the $\chi(t_{fin})$’s are accordingly modified in the transformed coefficients

$$\tilde{\chi}_n \equiv \int_0^\infty S(\omega) \tilde{F}_n(\omega) = \frac{1}{\sqrt{\chi_n}} \sum_{i=1}^{N} \chi_i(t_{fin}) V_{ni}, \quad (9)$$

and the estimate of $S(\omega)$, i.e., $\tilde{S}(\omega)$, is simply given by the expansion

$$\tilde{S}(\omega) = \sum_{n=1}^{N} \tilde{\chi}_n \tilde{F}_n. \quad (10)$$

### III. PROBING NON-MARKOVIANITY

In this section, we are interested into analyzing the Non-Markovianity (NM) [22, 23] of the open quantum system $S$ in reference to the multi-time statistics obtained by locally measuring $S$ [10]. Indeed, in case our knowledge of the system-environment interaction is a-priori unknown (or partially known), our capability to evaluate the NM of the system dynamics is simply given by the outcomes from a sequence of measurements and so we monitor the changes of the system state due to the presence of $E$. In the quantum mechanical context, this is a challenging issue, since it requires to understand which is the role and the effects on the system dynamics of the measurement back-action in probing the NM of $S$.

The dynamics of the composite system $S + E$ strictly depends on the interaction Hamiltonian $H_{int}$ which is fixed but unknown. $S + E$ is initialized in the product state $\rho_0 = \rho_{S,0} \otimes \rho_{E,0}$ and the formalism of quantum dynamics maps [23] is adopted to describe the dynamics of $S$. Thus, we define the family of Completely Positive, Trace-Preserving (CPTP) quantum maps $\{\Phi(t, t_0)\}$ with $\Phi(t, t_0) : L(H_S) \to L(H_S)$, where $L(H_S)$ denotes the set of density operators (non-negative operators with unit trace) defined on $H_S$. The peculiarity of our analysis is to take into account the value of the dynamical maps of $S$ only at the discrete time instants $t_k$, $k = 1, \ldots, m$, i.e., $\Phi(t_k, t_0) = \Phi_k$, so that $\rho_{S,k} = \Phi_k[\rho_{S,0}]$, because we want to focus our attention to the statistics from the measurement outcomes at those separate times. Moreover, given the initial and final time instants $t_0$ and $t_{fin}$, the value of $m$ is chosen by using the relation $m = (t_{fin} - t_0)/\delta t$, where $\delta t$ denotes the time interval below which it is no longer possible to evaluate the divisibility of the system dynamics [22], i.e., the existence of a CPTP quantum map $E(t_2, t_1)$ such that $\Phi(t_2, t_0) = E(t_2, t_1)\Phi(t_1, t_0)$ – also called composition law – for arbitrary $t_2 \geq t_1 \geq 0$.

For this reason, the Transfer Tensors (TT) method introduced in [24] is adopted. It relies on the transformations $\Phi_k = \sum_{j=0}^{k-1} T_{k,j} \Phi_j$, i.e., $T_{k,0} = \Phi_k - \sum_{j=1}^{k-1} T_{k,j} \Phi_j$, where $T_{k,j}$ are the so-called transfer tensors that allow for the propagation of the state of $S$ at time $t_j$ into the corresponding state at $t_k$. The state propagation law is, thus, $\rho_{S,k} = \sum_{j=0}^{k-1} T_{k,j} \rho_{S,j}$. Notice that, by construction, the set $\{T\}$ of TT does not depend on the initial reduced density matrix $\rho_{S,0}$, being fully determined by the dynamical maps of $S$, and allows for a recursive expansion of the $\Phi_k$’s as a function of $\Phi_0$. For the sake of clarity, we show the first terms of such a recursive expansion:

$$\Phi_0 = I_S[\rho_{S,0}], \quad \Phi_1 = T_{1,0}\Phi_0 = T_{1,0}[\rho_{S,0}], \quad \Phi_2 = T_{2,0}\Phi_0 + T_{2,1}\Phi_1 = (T_{2,0} + T_{2,1}T_{1,0})[\rho_{S,0}], \quad \text{with} \quad I_S \text{ identity channel of } S.$$ 

Such structure internally contains information about the memory effects within the dynamics of $S$ due to the interaction with $E$, and the latter are described by the conditional dynamics that $S$ is propagated to the state at time $t_k$ given that the system was in the corresponding states at times $t_{k-1}, t_{k-2}, \ldots, t_0$. With this formalism the conditional dynamics of $S$ are evaluated by dividing $m$ times the evolution of the system, and one can show that the TT are directly related to the notion of NM of the quantum system dynamics. The latter has been identified with the property of CP-divisibility [22], meaning that $E(t_2, t_1)$ is a completely positive quantum map that fulfills the composition law for every $t_1$ and $t_2$. Such property can be shown [11] to hold if the $T_{k,k-1}$’s are the only non-vanishing TT and, in addition, are completely positive.

However, the results obtained by evaluating a system dynamical behaviour are not usually the same of those obtained by observing/measuring the system. Nevertheless, the TT method is the most appropriate to characterize the NM of $S$ if we also assume to monitor the system via a sequence of repeated quantum measurements. Accordingly, let us introduce the measurement multi-time statistics that is defined by the joint probability distributions to get the measurement outcomes $\theta_1, \theta_2, \ldots, \theta_k$ respectively at the time instants $t_1, t_2, \ldots, t_k$. By using the measurement super-operator $R_{\theta_k} [\rho] \equiv M_{\theta_k} \rho M_{\theta_k}^\dagger$ and the global unitary dynamics $U_{\theta_k} [\rho] = U_{\theta_k} \rho U_{\theta_k}^\dagger$, the $k$-th joint probability distribution is equal to

$$q_k = \text{Tr} \left[ (R_{\theta_k} \otimes I_E) U_{\theta_{k-1}; t_k} \cdots (R_{\theta_1} \otimes I_E) U_{\theta_0; t_1} [\rho_0] \right]. \quad (11)$$

Notice that the previous definition involves the global
on the Markov property for quantum multi-time statistics. Measuring the open quantum system affects not only its current state, but also its future evolution as a consequence of the change in the correlations between $S$ and $E$ due to the measurement itself. Thus, we will use again the notion of conditional dynamics to take into account also the random occurrence of the measurement operators during the dynamics, since the system admits a different dynamics for each set of measurement outcomes, which corresponds to a specific system trajectory. Such dynamics are described by stochastic CTPT maps, denoted as $\Phi_{k}^{\theta}$, that depend on the time instants $\mathcal{I} \equiv (t_1, \ldots, t_k)$ as well as on the measurement outcomes $\theta \equiv (\theta_1, \ldots, \theta_{k-1})$. Thus, as proved in [10], a stochastic version of the TT transformation can be introduced with the aim to link together the maps $\Phi_{k}^{\theta}$'s at time instants $t_j$, $j = 1, \ldots, k$. Explicitly, one has

$$\Phi_{k}^{\theta} = \sum_{j=0}^{k-1} T_{k,j}^{\theta} R_{\theta_j} \Phi_{j}^{\theta_j},$$

(12)

where also the new set $\{ \mathcal{T} \}$ of TT depends on the instants and outcomes from the sequence of quantum measurements. As a result, via the conditional TT $T_{k,j}$ we are able to derive the stochastic dynamical maps $\Phi_{k}$ and, hence, determine the non-Markovianity of the conditional evolution. In this regard, let us observe that, being the non-Markovian behaviour of $S$ modified by the sequence of measurements, in general the $T_{k,j}$'s are not equal to the conditional TT $T_{k,j}$'s. In particular, if all the $T_{k,j}$'s vanish except for the one-step tensors $T_{k,k-1}$, $k = 1, \ldots, m$, then $T_{k,k-1}$ (that has to be always CP) equals to the one-step propagators $A_{t_{k-1}:t_k}$ defined by the equation $\rho_{S,k} = A_{t_{k-1}:t_k} \rho_{S,k-1}$. This implies that the $k$-th joint probability distribution $q_k$ can be expressed in terms of the initial reduced state and the one-step propagators, which corresponds to state that the Quantum Regression Theorem (QRT) [1] holds also in case the dynamics of the system is interspersed by the super-operators $R_{\theta}$. The QRT represents the counterpart of the Markov property for quantum multi-time statistics [27] and can be then used to introduce a proper notion of quantum Markovianity if also sequential measurements on $S$ are taken into account.

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