ON THE $U_p$ OPERATOR IN CHARACTERISTIC $p$

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ABSTRACT. For a perfect field $\kappa$ of characteristic $p > 0$, a positive integer $N$ not divisible by $p$, and an arbitrary subgroup $\Gamma$ of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$, we prove (with mild additional hypotheses when $p \leq 3$) that the $U$-operator on the space $M_k(\mathcal{P}_1/\kappa)$ of (Katz) modular forms for $\Gamma$ over $\kappa$ induces a surjection $U : M_k(\mathcal{P}_1/\kappa) \twoheadrightarrow M_k'(\mathcal{P}_1/\kappa)$ for all $k \geq p + 2$, where $k = (k - k_0)/p + k_0$ with $2 \leq k_0 \leq p + 1$ the unique integer congruent to $k$ modulo $p$. When $\kappa = \mathbb{F}_p$, $p \geq 5$, $N \neq 2, 3$, and $\Gamma$ is the subgroup of upper-triangular or upper-triangular unipotent matrices, this recovers a recent result of Dewar [Dew12].

1. Introduction

Fix a prime $p$, an integer $N > 0$ with $p \nmid N$, and a subgroup $\Gamma$ of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Let $\tilde{\Gamma}$ be the preimage in $\text{SL}_2(\mathbb{Z})$ of $\Gamma := \Gamma \cap \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$, and write $\tilde{M}_k(\tilde{\Gamma})$ for the space of weight $k$ mod $p$ modular forms for $\tilde{\Gamma}$ (in the sense of Serre [Ser73, §1.2]). When $N = 1$, a classical result of Serre [Ser73, §2.2, Théorème 6] asserts that the $U_p$ operator is a contraction: for $k \geq p + 2$, the map $U_p : \tilde{M}_k(\tilde{\Gamma}(1)) \to \tilde{M}_k(\tilde{\Gamma}(1))$ factors through the subspace $\tilde{M}_{k'}(\tilde{\Gamma}(1))$ for some $k' < k$ satisfying $pk' \leq k + p^2 - 1$. In fact, Serre’s result may be generalized and significantly sharpened:

Theorem 1.1. Let $\kappa$ be a perfect field of characteristic $p$ and denote by $M_k(\mathcal{P}_1/\kappa)$ the space of weight $k$ Katz modular forms for $\Gamma$ over $\kappa$ (see §3). Let $k_0$ be the unique integer between 2 and $p + 1$ congruent to $k$ modulo $p$, and if $p \leq 3$, assume that $N > 4$ and that $\Gamma_0$ is a subgroup of the upper-triangular unipotent matrices. Then for $k \geq p + 2$, the $U$-operator (see §3) acting on $M_k(\mathcal{P}_1/\kappa)$ induces a surjection $U : M_k(\mathcal{P}_1/\kappa) \twoheadrightarrow M_{k'}(\mathcal{P}_1/\kappa)$, for $k' := (k - k_0)/p + k_0$.

When $\tilde{\Gamma} = \Gamma_*(N)$ for $* = 0, 1$ and $\kappa = \mathbb{F}_p$, the endomorphism $U$ coincides with the usual Atkin $U_p$ operator $U_p$ (see Corollary 3.3). In particular, if $p \geq 5$, so $\tilde{M}_k(\tilde{\Gamma}) \simeq M_k(\mathcal{P}_1/\mathbb{F}_p)$ (by Theorems 1.7.1, and 1.8.1–1.8.2 of [Kat73]) and $N \neq 2, 3$, Theorem 1.1 is due to Dewar [Dew12]. Both Serre’s original result and Dewar’s refinement of it rely on a delicate analysis of the interplay between the operators $U_p, V_p$, and $\theta$ acting on mod $p$ modular forms. In the present note, we take an algebro-geometric perspective, and show how Theorem 1.1 follows immediately from a (trivial extension of a) general theorem of Tango [Tan72] on the behavior of vector bundles under the Frobenius map. In this optic, the contractivity of $U_p$ in characteristic $p$ is simply an instance of the “Dwork Principle” of analytic continuation along Frobenius. In particular, we use neither the $\theta$-operator, nor the notion of “filtration” of a mod $p$ modular form.

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1Tango’s paper, which appeared the year prior to Serre’s [Ser73], is perhaps not as well-known as it should be.
2. Tango’s Theorem

Fix a perfect field $\kappa$ of characteristic $p$, and write $\sigma : \kappa \to \kappa$ for the $p$-power Frobenius automorphism of $\kappa$. Let $X$ be a smooth, proper, and geometrically connected curve over $\kappa$ of genus $g$. Attached to $X$ is its Tango number:

\[(2.1) \quad n(X) := \max \left\{ \sum_{x \in X(\kappa)} \left[ \frac{\text{ord}_x(df)}{p} \right] : f \in \kappa(X) \setminus \kappa(X)^p \right\},\]

where $\kappa(X)$ is the function field of $X_{\kappa}$. As in Lemma 10 and Proposition 14 of [Tan72], it is easy to see that $n(X)$ is well-defined and is an integer satisfying $-1 \leq n(X) \leq [(2g - 2)/p]$, with the lower bound an equality if and only if $g = 0$.

**Proposition 2.1 (Tango).** Let $S \neq X$ be a reduced closed subscheme of $X$ with corresponding ideal sheaf $\mathcal{I}_S \subseteq \mathcal{O}_X$, and let $\mathcal{L}$ be a line bundle on $X$. If $\deg \mathcal{L} > n(X)$ then the natural $\sigma$-linear map

\[(2.2) \quad F^* : H^1(X, \mathcal{L}^{-1} \otimes \mathcal{I}_S) \to H^1(X, \mathcal{L}^{-p} \otimes \mathcal{I}_S)\]

induced by pullback by the absolute Frobenius of $X$ is injective, and the natural $\sigma^{-1}$-linear “trace map”

\[(2.3) \quad F_* : H^0(X, \Omega^1_{X/\kappa}(S) \otimes \mathcal{L}^p) \to H^0(X, \Omega^1_{X/\kappa}(S) \otimes \mathcal{L})\]

given by the Cartier operator ([Car57], [Ser58, §10]) is surjective.

**Proof.** First note that the formation of (2.2) and (2.3) is compatible, via $\sigma$- (respectively $\sigma^{-1}$-) linear extension, with any scalar extension $\kappa \to \kappa'$ to a perfect field $\kappa'$; we may therefore assume that $\kappa$ is algebraically closed. As the two assertions are dual\footnote{Note that $\kappa$-linear duality interchanges $\sigma$-linear maps with $\sigma^{-1}$-linear ones.} by Serre duality [Ser58, §10, Proposition 9], it suffices to prove the injectivity of (2.2). The case $S = \emptyset$ is Tango’s Theorem\footnote{Strictly speaking, Tango requires $g > 0$; however, by tracing through Tango’s argument—or by direct calculation—one sees easily that the result holds when $g = 0$ as well.} [Tan72, Theorem 15]. We reduce the general case to this one as follows: using that $\deg(\mathcal{L}) > 0$ and that $\mathcal{O}_X/\mathcal{I}_S^p$ is a skyscraper sheaf for all $j > 0$, one finds a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & H^0(X, \mathcal{O}_X/\mathcal{I}_S) & \to & H^1(X, \mathcal{L}^{-1} \otimes \mathcal{I}_S) & \to & H^1(X, \mathcal{L}^{-1}) & \to & 0 \\
& & F^* \downarrow & & F^* \downarrow & & F^* \downarrow & & \\
0 & \to & H^0(X, \mathcal{O}_X/\mathcal{I}_S^p) & \to & H^1(X, \mathcal{L}^{-p} \otimes \mathcal{I}_S^p) & \to & H^1(X, \mathcal{L}^{-p}) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(X, \mathcal{O}_X/\mathcal{I}_S) & \to & H^1(X, \mathcal{L}^{-p} \otimes \mathcal{I}_S) & \to & H^1(X, \mathcal{L}^{-p}) & \to & 0 \\
\end{array}
\]

in which the lower vertical arrows are induced by the inclusion of ideal sheaves $\mathcal{I}^p_S \subseteq \mathcal{I}_S$. Using that $\kappa = \kappa_p$ and identifying $H^0(X, \mathcal{O}_X/\mathcal{I}_S)$ with $\kappa^S$, the left vertical composite is easily seen to coincide with the map $\oplus_S \sigma : \kappa^S \to \kappa^S$ which is $\sigma$ on each factor; it is therefore injective. As the right vertical composite map is injective by Tango’s Theorem, an easy diagram chase finishes the proof. \(\blacksquare\)
3. Modular forms mod \( p \) as differentials on the Igusa curve

In order to apply Tango’s Theorem to prove Theorem 1.1, we must recall Katz’s geometric definition of mod \( p \) modular forms, and Serre’s interpretation of them as certain meromorphic differentials on the Igusa curve.

Let us write \( R_{\Gamma} := (\mathbb{Z}[\zeta_N])^{\det(\Gamma)} \), and for any \( R_{\Gamma} \)-algebra \( A \) denote by \( \mathcal{P}_{\Gamma}/A \) the moduli problem \((\Gamma(N)/\Gamma)_{\text{R}_{\Gamma}-\text{can}} \otimes_{\mathbb{R}_\delta} A\) on \((\text{Ell} / A)\) (see §3.1, §7.1, 9.4.2, and 10.4.2 of [KM85]) and by \( M_k(\mathcal{P}_{\Gamma}/A) \) the space of weight \( k \) Katz modular forms for \( \mathcal{P}_{\Gamma}/A \) (e.g. [Ulm90, §6]) that are holomorphic at \( \infty \) in the sense of [Kat73, §1.2]. Equivalently, \( M_k(\mathcal{P}_{\Gamma}/A) \) is the \( A \)-submodule of level \( N \), weight \( k \) modular forms in the sense of [DR73, VII.3.6] that are invariant under the natural action of \( \Gamma_0 \). Viewing \( C \) as an \( R_{\Gamma} \)-algebra via \( \zeta_N \mapsto \exp(2\pi i / N) \), we remark that \( M_k(\mathcal{P}_{\Gamma}/C) \) is the “classical” space of weight \( k \) modular forms for \( \Gamma \) over \( C \) defined via the transcendental theory [DR73, VII.4].

Now fix a ring homomorphism \( R_{\Gamma} \to \kappa \) with \( \kappa \) a perfect field of characteristic \( p \). From here until the end of the section we will assume that \( \mathcal{P}_{\Gamma}/\kappa \) is representable and that \(-1\) acts without fixed points on the space of cusp-labels for \( \Gamma \) (see [KM85, §10.6] and c.f. [KM85, 10.13.7–8]). We will later explain how to relax these hypotheses to those of Theorem 1.1. We write \( Y_{\Gamma} \) (respectively \( X_{\Gamma} \)) for the associated (compactified) moduli scheme; by [KM85, 10.13.12], one knows that \( X_{\Gamma} \) is a proper, smooth, and geometrically connected curve over \( \kappa \). Writing \( \rho : \mathcal{E} \to Y_{\Gamma} \) for the universal elliptic curve, our hypothesis that \(-1\) acts without fixed points ensures that the line bundle \( \omega_{\Gamma} := \rho_*(\Omega^1_{\mathcal{E}/Y_{\Gamma}}) \) on \( Y_{\Gamma} \) admits a canonical extension, again denoted \( \omega_{\Gamma} \), to a line bundle on \( X_{\Gamma} \) [KM85, 10.13.4, 10.13.7]. By definition, \( M_k(\mathcal{P}_{\Gamma}/\kappa) = H^0(X_{\Gamma}, \omega_{\Gamma}^k) \).

Let \( I_{\Gamma} \) be the Igusa curve of level \( p \) over \( X_{\Gamma} \); by definition, \( I_{\Gamma} \) is the compactified moduli scheme associated to the simultaneous problem \([\mathcal{P}_{\Gamma}/\kappa, [\mathcal{E} / \kappa]]\) on \((\text{Ell} / \kappa)\) [KM85, §12]. By [KM85, 12.7.2], the Igusa curve is proper, smooth, and geometrically connected, and the natural map \( \pi : I_{\Gamma} \to X_{\Gamma} \), is finite étale and Galois with group \((\mathbb{Z}/p\mathbb{Z})^\times\) outside the supersingular points, and totally ramified over every supersingular point. We define \( \omega := \pi^*\omega_{\Gamma} \), and recall [KM85, 12.8.2–3] that there is a canonical section \( q \in H^0(I_{\Gamma}, \omega) \) which has \( q \)-expansion equal to \( 1 \), vanishes to order \( 1 \) at each supersingular point, and on which \( \delta \in (\mathbb{Z}/p\mathbb{Z})^\times \) acts (via its action on \( I_{\Gamma} \)) through \( \chi^{-1} \), for \( \chi : (\mathbb{Z}/p\mathbb{Z})^\times = \mathbb{F}_p^\times \to \mathbb{F}_p^\times \) the mod \( p \) Teichmüller character. The following is a straightforward generalization of a theorem of Serre; see [KM85, §12.8] and c.f. Propositions 5.7–5.10 of [Gro90].

**Proposition 3.1.** Fix an integer \( k \geq 2 \) and let \( k_0 \leq k \) be any integer with \( 2 \leq k_0 \leq p + 1 \). The map \( f \mapsto f/a^{k_0-2} \) induces an natural isomorphism of \( \kappa \)-vector spaces

\[
M_k(\mathcal{P}_{\Gamma}/\kappa) \simeq H^0(I_{\Gamma}, \Omega^1_{I_{\Gamma}/\kappa}(\text{cusps} + \delta_{k_0} : \text{ss})) \otimes \omega^{k-k_0}(\kappa^{k_0-2}),
\]

where \( \delta_{k_0} = 1 \) when \( k_0 = p + 1 \) and is zero otherwise; here, ss, cusps are the reduced supersingular and cuspidal divisors, respectively.

**Proof.** The proof is a straightforward adaptation of Propositions 5.7–5.10 of [Gro90]; for the convenience of the reader, we sketch the argument. Thanks to [KM85, 10.13.11], the Kodaira-Spencer map [KM85, 10.13.10] provides an isomorphism of line bundles \( \omega_{I_{\Gamma}}^2 \simeq \Omega^1_{X_{\Gamma}/\kappa}(\text{cusps}) \) on \( X_{\Gamma} \) which, after pullback along \( \pi \), gives an isomorphism

\[
\omega^2 \simeq \Omega^1_{I_{\Gamma}/\kappa}(-p - 2)\text{ss} \oplus \text{cusps}
\]

of line bundles on \( I_{\Gamma} \) as \( \pi \) is étale outside ss and totally (tamely) ramified at each supersingular point.

Since \( a \in H^0(I_{\Gamma}, \omega) \) has \( p \)-zeroes at the supersingular points, via (3.2) any global section \( f \) of \( \omega^2_{I_{\Gamma}} \) induces a global section \( \pi^*f/\omega^{k_0-2} \) of \( \Omega^1_{I_{\Gamma}/\kappa}(\text{cusps} + \delta_{k_0} : \text{ss}) \otimes \omega^{k-k_0} \) on which \((\mathbb{Z}/p\mathbb{Z})^\times \) acts through

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\( ^4 \)Here, we follow the notation of [KM85, §9.4]: By definition \( \mathbb{Z}[\zeta_N] \) is the finite free \( \mathbb{Z} \)-algebra \( \mathbb{Z}[X]/\Phi_N(X) \), where \( \Phi_N \) is the \( N \)-th cyclotomic polynomial and \( \zeta_N \) corresponds to \( X \), equipped with its natural Galois action of \((\mathbb{Z}/N\mathbb{Z})^\times \).
\( \chi^{k_0-2} \); thus the map (3.1) is well-defined. Since the \( q \)-expansion of \( a \) is 1 and \( I_r \) is geometrically connected, the \( q \)-expansion principle then shows that (3.1) is injective. To prove surjectivity, observe that by (3.2), a global section of \( \Omega_{1_r}^{1/k} \) (cusps + \( \delta_{k_0} \cdot ss \)) \( \otimes \omega^{k-k_0} \) gives a meromorphic section \( h \) of \( \omega^{k-k_0+1} \) satisfying \( \text{ord}_x(h) \geq -(p-1) \) at each supersingular point \( x \), with equality possible only when \( k_0 = p + 1 \). If \( h \) lies in the \( (k_0-2) \)-eigenspace of the action of \( (\mathbb{Z}/p\mathbb{Z})^\times \), then \( f := a^{k_0-2}h \) descends to a meromorphic section of \( \omega^k \) over \( X_r \) satisfying

\[
(p-1) \text{ord}_x(f) = \text{ord}_x(h) + k_0 - 2 \geq k_0 - p - 1
\]

at each supersingular point \( x \in X_r(\bar{\kappa}) \), with equality possible only when \( k_0 = p + 1 \). Since the left side is a multiple of \( p-1 \) and \( k_0 \geq 2 \), we must have \( \text{ord}_x(f) \geq 0 \) in all cases, and \( f \) is a global (holomorphic) section of \( \omega^k \) over \( X_r \) with \( \pi^*f/a^{k_0-2} = h \).

Using Proposition 3.1, the Cartier operator \( F_\tau \) on meromorphic differentials induces, by “transport of structure”, a \( \sigma^{-1} \)-linear endomorphism \( U : M_k(\mathcal{M}_{\Gamma}/\kappa) \to M_k(\mathcal{M}_{\Gamma}/\kappa) \). If \( G \) is any group of automorphisms of \( X(\Gamma) \), then the action of \( G \) commutes with \( F_\tau \) (ultimately because the \( p \)-power map in characteristic \( p \) commutes with all ring homomorphisms), and we likewise obtain a \( \sigma^{-1} \)-linear endomorphism \( U \) of \( M_k(\mathcal{M}_{\Gamma}/\kappa)^G \). This allows us to define \( U \) even when \( \mathcal{M}_{\Gamma}/\kappa \) is not representable as follows. Choose a prime \( \ell > 3N \), and let \( \Gamma' \) be the unique subgroup of \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) projecting to the trivial subgroup of \( \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \) at \( \Gamma \). Then for any perfect field \( \kappa' \) of characteristic \( p \) admitting a map from \( R_{\Gamma'} \), the moduli problem \( \mathcal{M}_{\Gamma'}/\kappa' \) is representable, there is a natural action of \( G := \text{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \) on \( M_k(\mathcal{M}_{\Gamma'}/\kappa') \), and one has \( M_k(\mathcal{M}_{\Gamma'}/\kappa') = M_k(\mathcal{M}_{\Gamma'/\kappa'})^G \) (c.f. [DR73, VII.3.3] and [Kat73, §1.2]). Since \( M_k(\mathcal{M}_{\Gamma}/\kappa) \otimes_{\kappa} \kappa' \simeq M_k(\mathcal{M}_{\Gamma'/\kappa'}) \), we obtain the desired endomorphism \( U \) of \( M_k(\mathcal{M}_{\Gamma}/\kappa) \) by descent, and it is straightforward to check that it is independent of our initial choices of \( \ell \) and \( \kappa' \).

By post-composition with the \( \sigma \)-linear isomorphism

\[
M_k(\mathcal{M}_{\Gamma}/\kappa) \simeq M_k(\mathcal{M}_{\Gamma'}/\kappa')
\]

induced by the “exotic isomorphism” of moduli problems \( \mathcal{M}_{\Gamma}/\kappa \simeq \mathcal{M}_{\Gamma'}/\kappa' \) [KM85, 12.10.1] we obtain a \( \kappa \)-linear map \( U^\#: M_k(\mathcal{M}_{\Gamma}/\kappa) \to M_k(\mathcal{M}_{\Gamma'}/\kappa') \). When \( \mathcal{M}_{\Gamma} \) is defined over \( \mathbb{F}_p \), in the sense that \( R_{\Gamma} \) admits a (necessarily unique) surjection to \( \mathbb{F}_p \), one has canonically \( \mathcal{M}_{\Gamma}/\mathbb{F}_p = \mathcal{M}_{\Gamma}/\mathbb{F}_p \) as problems on \( (\mathbb{E}_l/\mathbb{F}_p) \), and \( U^\# \) is an endomorphism of \( M_k(\mathcal{M}_{\Gamma}/\mathbb{F}_p) \). The maps \( U \) and \( U^\# \) are natural generalizations of Atkin’s \( U_p \)-operator:

**Proposition 3.2.** Suppose that \( \mathcal{M}_{\Gamma}/\kappa \) is representable and let \( c \) be any cusp of \( X(\Gamma) \) defined over \( \kappa \). Then \( q^{1/e} \) is a uniformizing parameter at \( c \) for some divisor \( e \) of \( N \), and for any \( f \in M_k(\mathcal{M}_{\Gamma}/\kappa) \), the formal expansions of \( Uf \) at \( c \) and of \( U^\#f \) at \( c^{e^{-1}} \) are given by

\[
Uf = \sum_{n \geq 0} \sigma^{-1}(a_{np})q^{n/e} \quad \text{and} \quad U^\#f = \sum_{n \geq 0} a_{np}q^{n/e}
\]

respectively, where \( f = \sum_{n \geq 0} a_nq^{n/e} \).

**Proof.** Using the well-known local description of the Cartier operator on meromorphic differentials (e.g. [Ser58, §10, Proposition 8]), the result follows easily from the arguments of Propositions 2.8 and 5.7 of [Gro90]; see also (the proof of) [Gro90, Proposition 5.9].

**Corollary 3.3.** Suppose that \( \Gamma = \Gamma(N) \) for \( s = 0, 1 \). Then \( R_{\Gamma} = \mathbb{Z} \) and the resulting endomorphisms \( U \) and \( U^\# \) of \( M_k(\mathcal{M}_{\Gamma}/\mathbb{F}_p) \) coincide with the Atkin operator \( U_p \), whether or not \( \mathcal{M}_{\Gamma}/\mathbb{F}_p \) is representable.

**Proof.** That \( R_{\Gamma} = \mathbb{Z} \) is clear, as \( \text{det}(\Gamma) = (\mathbb{Z}/N\mathbb{Z})^\times \). By the discussion above, we may reduce to the representable case, and the result then follows from Proposition 3.2 and the \( q \)-expansion principle.

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5Explicitly, this isomorphism sends \( f \in M_k(\mathcal{M}_{\Gamma}/\kappa) \) to the modular form \( f^* \) defined by \( f^*(E, \alpha) := (E^* \alpha^*) \)

6A sufficient condition for this to happen is that \( \text{det}(\Gamma) \) contain the residue class of \( p \) mod \( N \).
4. Proof of Theorem 1.1

We now prove Theorem 1.1. Fix $k$ and let $k_0$ and $k'$ be as in the statement of Theorem 1.1. First suppose that $\mathcal{P}_\Gamma \otimes_{\mathcal{R}_k} \kappa$ is representable and that $-1$ acts without fixed points on the cusp-labels of $\Gamma$. Using (3.2) and the fact that $a$ has simple zeroes along ss we compute (c.f. [KM85, 12.9.4])

$$\deg \omega = \frac{2g - 2}{p} + \frac{1}{p} \deg(\text{cusps}) \geq \left\lfloor \frac{2g - 2}{p} \right\rfloor \geq n(I_\Gamma)$$

where $g$ is the genus of $I_\Gamma$. Applying Proposition 2.1 with $X = I_\Gamma$, $S = \text{cusps} + \delta_{k_0} \cdot \text{ss}$, and $\mathcal{L} = \omega$, we conclude from (2.3) and the relation $k - k_0 = p(k' - k_0)$ that the Cartier operator

$$F_* : H^0(I_\Gamma, \Omega^1_{I_\Gamma/\kappa}(\text{cusps} + \delta_{k_0} \cdot \text{ss}) \otimes \omega^{k-k_0}) \longrightarrow H^0(I_\Gamma, \Omega^1_{I_\Gamma/\kappa}(\text{cusps} + \delta_{k_0} \cdot \text{ss}) \otimes \omega^{k' - k_0})$$

is surjective whenever $k - k_0 \geq p$. Passing to $\chi^{k_0-2}$-eigenspaces for $(\mathbb{Z}/p\mathbb{Z})^\times$ and appealing to Proposition 3.1 and Corollary 3.3 then completes the proof in this case.

Now when $p \leq 3$, the hypotheses $N > 4$ and $\Gamma \subseteq \Gamma_1(N)$ of Theorem 1.1 ensure that $\mathcal{P}_\Gamma \otimes_{\mathcal{R}_k} \kappa$ is representable (as it maps to the moduli problem $[\Gamma_1(N)]$, which is representable for $N \geq 4$ by [KM85, 10.9.6]) and that $-1$ acts without fixed points on the cusp-labels of $\Gamma$ [KM85, 10.7.4]. If $p \geq 5$, we may choose a prime $\ell > 3N$ with $\ell \not\equiv 0, \pm 1 \mod p$, so that $p \nmid | \text{SL}_2(\mathbb{Z}/\ell\mathbb{Z})|$. Then for $N' := N\ell$ and $\Gamma' := 1 \times \Gamma \subseteq \text{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \text{SL}_2(\mathbb{Z}/N\ell\mathbb{Z})$, we have (after passing to an appropriate extension $\kappa'$ of $\kappa$) that $\mathcal{P}_{\Gamma'} \otimes_{\mathcal{R}_{\kappa'}} \kappa'$ is representable with $-1$ acting freely on the cusp-labels of $\Gamma'$ [KM85, 10.7.1, 10.7.3]. We conclude that the $U$-operator induces a surjection of $\kappa[\text{SL}_2(\mathbb{Z}/\ell\mathbb{Z})]$-modules $M_k(\mathcal{P}_{\Gamma'}/\kappa') \twoheadrightarrow M_k(\mathcal{P}_{\Gamma}/\kappa')$. Our choice of $\ell$ ensures that the ring $\kappa[\text{SL}_2(\mathbb{Z}/\ell\mathbb{Z})]$ is semisimple, so passing to $\text{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$-invariants is exact. As the space of $\text{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$-invariant weight $k$ modular forms for $\Gamma'$ coincides with $M_k(\mathcal{P}_{\Gamma'}/\kappa')$ (c.f. the definition of $U$ in §3), passing to invariants and descending from $\kappa'$ to $\kappa$ then completes the proof of Theorem 1.1 in the general case.

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