A Markov product for tail dependence functions

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Abstract

We introduce a Markov product structure for multivariate tail dependence functions, building upon the well-known Markov product for copulas. We investigate algebraic and monotonicity properties of this new product as well as its role in describing the tail behaviour of the Markov product of copulas. For the bivariate case, we show additional smoothing properties and derive a characterization of idempotents together with the limiting behaviour of n-fold iterations. Finally, we establish a one-to-one correspondence between bivariate tail dependence functions and a class of positive, substochastic operators. These operators are contractions both on $L^1(\mathbb{R}_+)$ and $L^\infty(\mathbb{R}_+)$ and constitute a natural generalization of Markov operators.

Keywords: Copula, Tail dependence, Markov product, Markov operator, Substochastic operator

2010 MSC: 37A30, 60E05, 62H05

1. Introduction

In many applications, there is a need to quantify the dependence between different random variables. Examples range from finance to hydrology, where the dependence can have a global, for instance a linear or a functional, or a local character. In the following, we are interested in a certain type of local dependence, the tail dependence, which describes the extremal behaviour between multiple random variables. A natural application are the joint losses of multiple stocks in a portfolio. The lower tail dependence function

$$\Lambda((w_1, w_2); X, Y) := \lim_{s \to 0} \mathbb{P}(X \leq F_X^{-1}(sw_1) \mid Y \leq F_Y^{-1}(sw_2)) = \lim_{s \to 0} \frac{C_{XY}(sw_1, sw_2)}{s},$$

of two continuous random variables $X$ and $Y$ allows a scale-free characterization of the joint behaviour in the extremes, in this case the jointly occurring extreme losses. Properties and applications of the tail dependence functions can be found in Joe (2015), while estimators and their statistical properties have been established in Schmidt and Stadtmüller (2006).

This paper treats the tail properties of a certain class of $d$-variate copulas, namely the ones constructed via the (generalized) Markov product. For the set of 2-copulas, denoted by $\mathcal{C}_2$, the Markov product $\ast$ has become an important tool in the modelling and description of dependencies. First introduced by Darsow et al. (1992) to model transition probabilities in the context of Markov processes through a rephrasing of the Chapman-Kolmogorov-equations in terms of consistency conditions imposed on a family of copulas, it also plays an essential role in the theory of complete dependence (see Siburg and Stoimenov (2008) and Frütschig (2011)) and the study of extremal elements (see Darsow et al. (1992)). An extensive overview over the properties and applications of the Markov product can be found in Durante and Sempi (2015). Some results of the tail behaviour of similar constructions have been achieved in the context of vine-copulas by Joe et al. (2010) and more recently

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\textsuperscript{1}Supported by the German Academic Scholarship Foundation.
by Jaworski (2015). To facilitate the study of the extremal behaviour of the Markov product on $C_2$, we introduce a generalized version of the Markov product on the set of all tail dependence functions $M_2$ and link both under appropriate regularity conditions. One of the most important properties of the Markov product $M_2$ is a monotonicity property, which results in an overall dependence reduction and, in general, does not hold for 2-copulas. Using this monotonicity, we treat iterates of the Markov product and derive additional smoothing properties akin to those presented in Trutschnig (2013a).

Finally, we connect the set of all bivariate tail dependence functions equipped with $\ast$ to a certain class of substochastic operators and their composition which generalize the well-known Markov operators. The paper is structured as follows. Section 2 presents the necessary notation and some preliminaries. Section 3 introduces the Markov product for tail dependence functions and establishes a link to the Markov product of copulas. Section 4 discusses the monotonicity of the Markov product, while Section 5 employs these results to derive the behaviour of iterates, idempotents and averages. Finally, Section 6 links $(M, \ast)$ to a class of linear operators $(T, \circ)$.

2. Notation and preliminaries

A $d$-copula is a $d$-variate distribution function on $[0, 1]^d$ with uniform margins. We denote $\mathbb{R}_+ := [0, \infty)$.

**Definition 2.1.** For a $d$-copula $C$, the lower tail dependence function $\Lambda(\cdot ; C) : \mathbb{R}_+^d \to \mathbb{R}_+$ is defined as

$$\Lambda(w) := \Lambda(w ; C) := \lim_{s \downarrow 0} \frac{C(sw)}{s},$$

provided that the limit exists for all $w$ in $\mathbb{R}_+^d$.

Let $C_d$ and $M_d$ denote the set of $d$-copulas and the set of $d$-variate tail dependence functions, respectively. We refer to the lower Frchet-Hoeffding-bound by $C^-$, the upper Frchet-Hoeffding-bound by $C^+$ and the product copula by $\Pi$. Many properties of the copula $C$ immediately transfer to the tail dependence function $\Lambda$ (see, Propositions 4 and 6 in Jaworski (2006)):

**Proposition 2.2.** A function $\Lambda : \mathbb{R}_+^d \to \mathbb{R}_+$ is the tail dependence function of a copula $C$ if and only if

1. $\Lambda$ is bounded from below by 0 and from above by $\Lambda^+ := \Lambda(\cdot ; C^+)$.
2. $\Lambda$ is $d$-increasing, i.e. the $\Lambda$-volume of any rectangle in $\mathbb{R}_+^d$ is nonnegative.
3. $\Lambda$ is homogeneous of order 1, i.e. $\Lambda(sw) = s\Lambda(w)$ for any $s \in \mathbb{R}_+$ and $w \in \mathbb{R}_+^d$.

Furthermore, for any tail dependence function $\Lambda$, we have

a. $\Lambda$ is Lipschitz continuous with constant $1$.

b. $\Lambda$ is concave.

**Remark 2.3.** Due to the positive homogeneity of $\Lambda$, in the bivariate case, we will often only consider $\Lambda$ on the unit simplex $S^1 := \{ w \in \mathbb{R}_+^2 \ | \ w = (t, 1 - t) \text{ with } t \geq 0 \}$ and identify $S^1$ with $[0, 1]$ using

$$\Lambda|_{[0,1]}(t) := \Lambda(t, 1 - t). \tag{1}$$

Whenever the meaning is clear, we will write $\Lambda$ instead of $\Lambda|_{[0,1]}$.

We investigate a generalized version of the Markov product introduced by Darsow et al. (1992), which was discussed in Jaworski (2013) in the context of vine-copulas.
Definition 2.4. Let $C_1, \ldots, C_d$ be 2-copulas and let $C$ be a $d$-copula. Then, the $(d+1)$-copula

$$
\phi_{u,C} (C_1, \ldots, C_d) (v_1, \ldots, v_d) := \int_0^u C (\partial_1 C_1(t, v_1), \ldots, \partial_1 C_d (t, v_d)) \, dt
$$

is called the $C$-lifting of the copulas $C_1, \ldots, C_d$. Furthermore, we define the $d$-copula

$$
\phi_C (C_1, \ldots, C_d) (v_1, \ldots, v_d) := \int_0^1 C (\partial_1 C_1(t, v_1), \ldots, \partial_1 C_d (t, v_d)) \, dt = \phi_{1,C} (C_1, \ldots, C_d) (v_1, \ldots, v_d)
$$

to be the generalized Markov product of $C_1, \ldots, C_d$ induced by $C$.

Note that for $d = 2$, the previously defined generalized Markov product

$$
\phi_C (C_1, C_2) (v_1, v_2) = \int_0^1 C (\partial_1 C_1(t, v_1), \partial_1 C_2 (t, v_2)) \, dt
$$

maps $C_2 \times C_2$ onto $C_2$ and is closely related to the traditional Markov product of 2-copulas via

$$
C_1 \ast C_2 (v_1, v_2) = \phi_{\Pi} (C_1^T, C_2) (v_1, v_2),
$$

where $C_1^T (v_1, v_2) := C_1 (v_2, v_1)$.

3. A Markov product for tail dependence functions

Similar to this construction of higher dimensional copulas from bivariate copulas, we introduce an operation for bivariate tail dependence functions.

Definition 3.1. Let $\Lambda_1, \ldots, \Lambda_d \in M_2$ and $C \in C_d$. We call

$$
\phi_{w_0,C} (\Lambda_1, \ldots, \Lambda_d) (w_1, \ldots, w_d) := \int_0^{w_0} C (\partial_1 \Lambda_1 (t, w_1), \ldots, \partial_1 \Lambda_d (t, w_d)) \, dt
$$

the $C$-lifting of the tail dependence functions $\Lambda_1, \ldots, \Lambda_d$. Similarly, the generalized Markov product of $\Lambda_1, \ldots, \Lambda_d$ induced by $C$ is defined by

$$
\phi_C (\Lambda_1, \ldots, \Lambda_d) (w_1, \ldots, w_d) := \int_0^\infty C (\partial_1 \Lambda_1 ((t, w_1)), \ldots, \partial_1 \Lambda_d ((t, w_d))) \, dt.
$$

First, we verify that the $C$-lifting and the generalized Markov product do in fact generate new tail dependence functions.

Theorem 3.2. Suppose $C \in C_d$ and $\Lambda_1, \ldots, \Lambda_d \in M_2$. Then $\phi_{w_0,C} (\Lambda_1, \ldots, \Lambda_d)$ and $\phi_C (\Lambda_1, \ldots, \Lambda_d)$ are $(d+1)$-variate and $d$-variate tail dependence functions, respectively.

Proof. First, the tail dependence functions $\Lambda_\ell$, for $\ell = 1, \ldots, d$, are positive, monotone in each component, Lipschitz continuous and thus have partial derivatives almost everywhere. Moreover, the partial
derivatives are bounded on $[0,1]$. Therefore, we have

$$\phi_{w_0,C}(A_1, \ldots, A_d)(w_1, \ldots, w_d) \leq \int_0^\infty C^+(\partial_1 A_1(t, w_1), \ldots, \partial_1 A_d(t, w_d)) \, dt$$

which establishes the existence of the integral. The second inequality is due to $\Lambda$ being increasing in each component and bounded above by $\Lambda^+$. Thus, we can define

$$\phi(w) := \int_0^{w_0} C(\partial_1 A_1(t, w_1), \ldots, \partial_1 A_d(t, w_d)) \, dt .$$

It remains to verify properties 1-3 of Proposition 2.2 which characterizes tail dependence functions. For the first property, note that due to all copulas being bounded from above by $C^+$ and as tail dependence functions have bounded partial derivatives between 0 and 1, it holds

$$0 \leq \int_0^{w_0} C(\partial_1 A_1(t, w_1), \ldots, \partial_1 A(t, w_d)) \, dt$$

The $(d + 1)$-increasing property of $\phi$ needs to be verified on every rectangle $R = \bigotimes_{\ell=0}^d [x_\ell^1, x_\ell^2]$ with $x_\ell^1 \leq x_\ell^2$. Then, with $N(z) := |\{\ell \in \{1, \ldots, d\} \mid z_\ell = x_\ell^1\}|$, the $\phi$-volume of $R$ equals

$$\sum_{z \in \bigotimes_{\ell=0}^d \{x_\ell^1, x_\ell^2\}} (-1)^{N(z)} \phi(z) = \sum_{\tilde{z} \in \bigotimes_{\ell=0}^d \{x_\ell^1, x_\ell^2\}} (-1)^{N(\tilde{z})} (\phi(x_0^2, \tilde{z}) - \phi(x_0^1, \tilde{z}))$$

where the last inequality holds due to $\partial_1 A_\ell(t, x_\ell^1) \leq \partial_1 A_\ell(t, x_\ell^2)$. Lastly, the positive homogeneity can be established via a change of variables and the positive homogeneity of order 0 of the partial
derivatives of $\Lambda$,

$$
\phi(sw) = \int_0^{sw_0} \frac{C(\partial_1 \Lambda_1(t, sw_1), \ldots, \partial_1 \Lambda_d(t, sw_d))}{dt}
= \int_0^{sw_0} \frac{C(\partial_1 \Lambda_1(s/t, w_1), \ldots, \partial_1 \Lambda_d(s/t, w_d))}{dt}
= s \int_0^{w_0} \frac{C(\partial_1 \Lambda_1(z, w_1), \ldots, \partial_1 \Lambda_d(z, w_d))}{dz} = s \phi(w).
$$

By Proposition 2.2, we can thus find a copula $C^*$ with $\Lambda^* (w; C^*) = \phi(w)$ for all $w \in \mathbb{R}^{d+1}$. The proof that $\phi_C (\Lambda_1, \ldots, \Lambda_d)$ is a $d$-variate tail dependence function works analogously. \qed

Remark 3.3. Note that the first part of the proof only requires that all $\Lambda_\ell$ are 2-increasing functions bounded by $\Lambda^+$. Furthermore, $\phi$ is positive homogeneous of order one if and only if $\partial_1 \Lambda ((t, w_\ell); C_\ell)$ is homogeneous of order zero for all $\ell = 1, \ldots, d$.

The next proposition compiles basic algebraic properties of $\phi_C (\Lambda_1, \ldots, \Lambda_d)$ and $\phi_{w_0, C} (\Lambda_1, \ldots, \Lambda_d)$.

Proposition 3.4. Suppose $\Lambda_1, \ldots, \Lambda_d$ are bivariate tail dependence functions and $C$ is a d-copula. Then

1. $\Lambda^+ = \Lambda (\cdot; C^+)$ is the unit element in the sense that if $\Lambda_\ell = \Lambda^+$, then
   \[\phi_{w_0, C} (\Lambda_1, \ldots, \Lambda_d) (w_1, \ldots, w_d) = \phi_{\min \{w_0, w_\ell\}, \tilde{C}_\ell} (\Lambda_1, \ldots, \Lambda_{\ell-1}, \Lambda_{\ell+1}, \ldots, \Lambda_d) (\tilde{w}_\ell),\]
   where $\tilde{w}_\ell := (w_1, \ldots, w_{\ell-1}, w_{\ell+1}, \ldots, w_d)$ and $\tilde{C}_\ell := C(u_1, \ldots, u_{\ell-1}, 1, u_{\ell+1}, \ldots, u_d)$.

2. $\Lambda^0 := \Lambda (\cdot; \Pi)$ is the null element in the sense that if $\Lambda_\ell = \Lambda^0$, then
   \[\phi_{w_0, C} (\Lambda_1, \ldots, \Lambda_d) = \Lambda ((w_0, \ldots, w_d); \Pi^{d+1}) = 0.\]

3. If $C$ is convex resp. concave in the $\ell$-th component, then $\phi_{w_0, C} (\cdot)$ is convex resp. concave in the $\ell$-th component.

4. For every permutation $\pi$ on $\{1, \ldots, d\}$, we have
   \[\phi_C (\Lambda_1, \ldots, \Lambda_d) (w_{\pi(1)}, \ldots, w_{\pi(d)}) = \phi_{C^\pi} (\Lambda_{\pi(1)}, \ldots, \Lambda_{\pi(d)}) (w_1, \ldots, w_d),\]
   where $C^\pi (u_1, \ldots, u_d) := C(u_{\pi(1)}, \ldots, u_{\pi(d)}).

5. Let $C_n \in C_d$ with $C_n \to C$ pointwise. Then
   \[\phi_{w_0, C_n} (\Lambda_1, \ldots, \Lambda_d) \to \phi_{w_0, C} (\Lambda_1, \ldots, \Lambda_d)\]
   pointwise.

6. If $C \leq D$ pointwise, then
   \[\phi_{w_0, C} (\Lambda_1, \ldots, \Lambda_d) \leq \phi_{w_0, D} (\Lambda_1, \ldots, \Lambda_d).\]

Remark 3.5. The term “unit-element” stems from the bivariate case, where $\phi_C : M_2 \times M_2 \to M_2$ constitutes a genuine product, which fulfills

\[\phi_C (\Lambda^+, \Lambda) (w_1, w_2) = \Lambda(w_1, w_2).\]
In analogy to the binary product $\ast$,

**Proof.** 1. Without loss of generality, we consider $\ell = 1$. As $\partial_1 \Lambda ((t, w_1) ; C^+) = 1_{[0, w_1]}(t)$, we have

$$
\phi_{w_0, C}(\Lambda_1, \ldots, \Lambda_d)(w_1, \ldots, w_d) = \min_{\{w_0, w_1\}} \int_0^{\min\{w_0, w_1\}} C(1, \partial_1 \Lambda_2(t, w_2), \ldots, \partial_1 \Lambda_d(t, w_d)) \, dt
$$

$$
= \int_0^{\min\{w_0, w_1\}} C_1(\partial_1 \Lambda_2(t, w_2), \ldots, \partial_1 \Lambda_d(t, w_d)) \, dt
$$

$$
= \phi_{\min\{w_0, w_1\}, C_1}(\Lambda_2, \ldots, \Lambda_d)(w_2, \ldots, w_d).
$$

2. The second result is obvious since $\Lambda (\cdot ; \Pi) \equiv 0$ and $C(0, u) = 0$.

3. The third result follows immediately from the pointwise inequality of $C$.

4. A direct calculation yields

$$
\phi_C(\Lambda_1, \ldots, \Lambda_d)(w_{\pi(1)}, \ldots, w_{\pi(d)}) = \int_0^\infty C(\partial_1 \Lambda_1(t, w_{\pi(1)}), \ldots, \partial_1 \Lambda_d(t, w_{\pi(d)})) \, dt
$$

$$
= \int_0^\infty C_{\pi}(\partial_1 \Lambda_{\pi(1)}(t, w_1), \ldots, \partial_1 \Lambda_{\pi(d)}(t, w_d)) \, dt
$$

$$
= \phi_{C, \pi}(\Lambda_{\pi(1)}, \ldots, \Lambda_{\pi(d)})(w_1, \ldots, w_d).
$$

5. A combination of $C(\partial_1 \Lambda_1(t, w_1), \ldots, \partial_1 \Lambda_d(t, w_d)) \leq \partial_1 \Lambda_1(t, w_1)$ and the dominated convergence theorem yields the desired result.

In analogy to the binary product $\ast$ on $C_2 \times C_2$ induced by $\Pi$, we introduce $\ast$ on $M_2 \times M_2$ via

$$(\Lambda_1 \ast \Lambda_2)(w_1, w_2) := \phi_{\Pi}(\Lambda_1^T, \Lambda_2)(w_1, w_2) = \int_0^\infty \partial_2 \Lambda_1(w_1, t) \partial_1 \Lambda_2(t, w_2) \, dt.
$$

Its properties closely resemble those of the Markov product on $C_2 \times C_2$. In particular, $\Lambda^+$ and $\Lambda^0$ are the unit and null element of $\ast$, respectively, and $\ast$ is associative as well as skew-symmetric, i.e.

$$(\Lambda_1 \ast \Lambda_2)^T = \Lambda_2^T \ast \Lambda_1^T.
$$

With these basic algebraic properties, we will develop two conditions under which the Markov product commutes with the tail dependence function, i.e.

$$
\Lambda(w; C_1 \ast C_2) = \Lambda(\cdot ; C_1) \ast \Lambda(\cdot ; C_2)(w).
$$

The first approach utilizes the Lipschitz continuity of $C$ and follows an idea from Jaworski (2013). Theorem 7 therein derives the tail behaviour of the $C$-lifting

$$
\Lambda((w_0, \ldots, w_d); \phi_{\cdot, C}(C_1, \ldots, C_d)) = \phi_{w_0, C}(\Lambda(\cdot ; C_1), \ldots, \Lambda(\cdot ; C_d))(w_0, \ldots, w_d)
$$

under a Sobolev-type condition imposed on $C_1, \ldots, C_d$.

**Theorem 3.6.** Suppose that $C$ is a $d$-copula and that $C_1, \ldots, C_d$ are 2-copulas with existing bivariate tail dependence functions, which fulfill the Sobolev-type condition

$$
\lim_{s \downarrow 0} \int_0^\infty \left| \partial_1 C_i(st, sw) 1_{[0, \frac{1}{2}]}(t) - \partial_1 \Lambda((t, w); C_i) \right| \, dt = 0
$$

(2)
for all \( w \in \mathbb{R}_+ \) and all \( i = 1, \ldots, d \). Then,
\[
\phi_C (\Lambda (\cdot ; C_1), \ldots, \Lambda (\cdot.; C_d)) (w) = \Lambda (w ; \phi_C (C_1, \ldots, C_d))
\]
for all \( w \in \mathbb{R}_+^d \), or, equivalently,
\[
\begin{array}{c}
\phi_C (\Lambda (\cdot ; C_1), \ldots, \Lambda (\cdot.; C_d)) (w) \\
\Leftrightarrow
\end{array}
\]

Proof. The Lipschitz continuity and groundedness of \( C \) yield
\[
\frac{\partial^1 C_i}{\partial t} (s \tau, s w_1) \ |_{[0, \frac{1}{s}]} (\tau) - \frac{\partial^1 \Lambda (\cdot.; C_i)}{\partial t} (s \tau, s w_i) \ |_{[0, \frac{1}{s}]} (\tau) 
\]
for all \( i = 1, \ldots, d \).

The next approach does not utilize the Lipschitz continuity of the copula \( C \) and yields a different condition in terms of the convergence of the partial derivatives.

Theorem 3.7. Suppose \( C \) is a \( d \)-copula and that the \( 2 \)-copulas \( C_1, \ldots, C_d \) as well as their generalized Markov product have a tail dependence function. If for all \( w \in \mathbb{R}_+ \) and almost all \( t \in \mathbb{R}_+ \)
\[
\lim_{s \searrow 0} \frac{\partial^1 C_i}{\partial t} (s t, s w) = \frac{\partial^1 \Lambda (\cdot.; C_i)}{\partial t} (t, w) 
\]
for all \( i = 1, \ldots, d \),
then
\[
\phi_C (\Lambda (\cdot; C_1), \ldots, \Lambda (\cdot.; C_d)) (w) \leq \Lambda (w ; \phi_C (C_1, \ldots, C_d)) .
\]
Additionally, if there exists an \( \ell \in \{1, \ldots, d\} \) such that
\[
\frac{\partial^1 C_k}{\partial t} (s \tau, s w_k) \ |_{[0, \frac{1}{s}]} (\tau) \leq g_{w_k} (\tau)
\]
for all \( w_k \in [0,1] \) and some family \( (g_{w})_{w \in [0,1]} \) of integrable functions, it holds
\[
\phi_C (\Lambda (\cdot; C_1), \ldots, \Lambda (\cdot.; C_d)) (w) = \Lambda (w ; \phi_C (C_1, \ldots, C_d)) .
\]

Proof. By the definition of the tail dependence function and an application of Fatou’s lemma for
positive measurable functions, it holds that

\[
\Lambda (w ; \phi_C (C_1, \ldots, C_d)) = \lim_{s \to 0} \frac{1}{s} \int_0^s C (\partial_i C_1(t, sw_1), \ldots, \partial_i C_d(t, sw_d)) \, dt
\]

For all \( w \in \mathbb{R}_+ \), we have that for \( \tau \leq 1/s \)

\[
C (\partial_i C_1(\tau, sw_1), \ldots, \partial_i C_d(\tau, sw_d)) \leq C^+ (\partial_i C_1(\tau, sw_1), \ldots, \partial_i C_d(\tau, sw_d))
\leq \partial_t C_\ell (\tau, sw_\ell) \leq g_{sw} (\tau)
\]

The desired result follows from the dominated convergence theorem.

**Remark 3.8.** Assume that in addition to the almost everywhere pointwise convergence of the partial derivatives required in Theorem 3.7, the tail dependence functions of \( C_i \) are strict, i.e. \( \lim_{t \to \infty} \Lambda ((t, w) ; C_i) = w \) for all \( w \in \mathbb{R}_+ \). Then an application of Scheff’s Lemma (see, Novinger (1972)) yields

\[
\lim_{s \to 0} \int_0^\infty \left| \partial_i C_\ell (st, sw) \right| \left| \partial_t \Lambda ((t, w) ; C_i) \right| \, dt = 0
\]

for all \( i = 1, \ldots, d \), which implies

\[
\phi_C (\Lambda (\cdot ; C_1), \ldots, \Lambda (\cdot ; C_d)) (w) = \Lambda (w ; \phi_C (C_1, \ldots, C_d))
\]

due to Theorem 3.6

The lower bound behaviour stated in Theorem 3.7 is generally the best result possible, as can be seen from the following example.

**Example.** Consider the lower Frchet-Hoeffding bound \( C^- \), which is symmetric and left invertible, i.e., \((C^-)^T \ast C^- = C^+ \). Then an application of Theorem 5.5.3 in Durante and Sempi (2015) yields

\[
\phi_C (C^-, C^-) = C^- \ast C^- = (C^-)^T \ast C^- = C^+ .
\]

Hence for \( w = (w_1, w_2) \) in \( \mathbb{R}_+^2 \),

\[
\phi_C (\Lambda (\cdot ; C^-), \Lambda (\cdot ; C^-)) (w) = 0 \leq \min \{ w_1, w_2 \} = \Lambda (w ; C^+) = \Lambda (w ; \phi_C (C^-, C^-)),
\]

which is strict for every \( w \) in \((0, \infty)^2\).

Let us now study some examples to investigate the behaviour of the Markov product on \( \mathcal{M}_2 \) for different 2-copulas \( C \).
dependence functions $\Lambda_1$, $\Lambda_2$. Then

**Example.** Let $\phi^+(\Lambda_1, \Lambda_2)$ bound $\Lambda$. The influence of the choice of $C$ on the product is depicted in Figure 1. For the two tail dependence functions $\Lambda_1(t, 1-t) = \min\left\{ \frac{2t}{3}, 1-t \right\}$ and $\Lambda_2(t, 1-t) = \min\left\{ \frac{t}{2}, \frac{1-3t}{4} \right\}$ are depicted in black, the upper bound $\Lambda^+$ in grey.

The above expression can be explicitly calculated for some choices of $C$ (see, Figure 2):

1. If $C = \Pi$, then $\phi_\Pi(\Lambda_1, \Lambda_2)(w_1, w_2) = \Lambda_2(\beta w_1, \alpha w_2)$.

2. If $C = C^-$, we have

$$\phi_{C^-}(\Lambda_1, \Lambda_2)(w_1, w_2) = \Lambda_2(p^* \wedge \beta w_1, w_2) + \Lambda(\alpha p^*, \beta w_1; C^+) - p^* ,$$

since the monotonicity of $\partial_1 \Lambda_2$ yields the existence of a $p^* = p^*(\alpha, w_2) \geq 0$ with

$$\partial_1 \Lambda_2(t, w_2) + \alpha - 1 \geq 0 \text{ for all } t \leq p^* \text{ and } \partial_1 \Lambda_2(t, w_2) + \alpha - 1 \leq 0 \text{ for all } t > p^* .$$

3. By a similar argument, for $C = C^+$, it holds

$$\phi_{C^+}(\Lambda_1, \Lambda_2)(w_1, w_2) = \Lambda(\alpha p^*, \beta w_1; C^+) + \Lambda_2\left(\frac{\beta}{\alpha} w_1, w_2\right) - \Lambda_2\left(p^* \wedge \frac{\beta}{\alpha} w_1, w_2\right) ,$$

where $p^* = p^*(1 - \alpha, w_2)$.

![Figure 1: Plots of the product $\phi_C(\Lambda_1, \Lambda_2)(t, 1-t)$ for different choices of $C$ (red line) following Remark 2.3. The tail dependence functions $\Lambda_1(t, 1-t) = \min\left\{ \frac{2t}{3}, 1-t \right\}$ and $\Lambda_2(t, 1-t) = \min\left\{ \frac{t}{2}, \frac{1-3t}{4} \right\}$ are depicted in black, the upper bound $\Lambda^+$ in grey.](image-url)
4. Monotonicity of the Markov product

Figures 1 and 2 already suggest a monotonicity of the Markov product whenever $C$ fulfils a negative dependence property. We will treat this property in more detail in this section.

**Theorem 4.1.** Let $\Lambda_1, \ldots, \Lambda_d \in M_2$ and $C \in C_d$ be negatively dependent, i.e. $C \leq \Pi$. Then, for $k = 1, \ldots, d$ and $w \in \mathbb{R}^d_+$,

$$
\phi_C(\Lambda_1, \ldots, \Lambda_d)(w) \leq \phi_\Pi(\Lambda_1, \ldots, \Lambda_d)(w) \leq \min_{m=1, \ldots, d; m \neq k} \Lambda_k(w_m, w_k).
$$

This result decidedly contrasts with the behaviour of the Markov product for 2-copulas, where for example

$$
C^- \leq C^+ = C^- \ast C^-.
$$

Theorem 4.1 is incorrect without the assumption that $C \leq \Pi$, as can be seen in Figure 2(c). We will give two different proofs of Theorem 4.1; the second proof is deferred to Section 6 since it uses the theory of substochastic operators developed there.

**Proof.** Due to $\Lambda_1 \leq \Lambda^+$, we have

$$
\int_0^t \partial_1 \Lambda_1(t, w_1) \, ds \leq \int_0^t \partial_1 \Lambda^+(t, w_1) \, ds
$$

for all $w_1, t \in [0, \infty)$. Hardy’s Lemma (see, *Bennett and Sharpley (1988)*) yields for any positive decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ that

$$
\int_0^\infty \partial_1 \Lambda_1(s, w_1) f(s) \, ds \leq \int_0^\infty \partial_1 \Lambda^+(s, w_1) f(s) \, ds = \int_0^{w_1} f(s) \, ds.
$$

Thus, for all tail dependence functions $\Lambda_2, \ldots, \Lambda_d$ and any $w \in \mathbb{R}^d_+$

$$
\phi_\Pi(\Lambda_1, \ldots, \Lambda_d)(w) = \int_0^\infty \partial_1 \Lambda_1(s, w_1) \partial_1 \Lambda_2(s, w_2) \cdots \partial_1 \Lambda_d(s, w_d) \, ds
$$

$$
\leq \int_0^{w_1} \partial_1 \Lambda_2(s, w_2) \cdots \partial_1 \Lambda_d(s, w_d) \, ds
$$

$$
= \phi_{w_1, \Pi}(\Lambda_2, \ldots, \Lambda_d)(w_2, \ldots, w_d).
$$

An application of 4. in Proposition 3.4 yields the desired result. \qed
Corollary 4.2. Let $C$ be an idempotent 2-copula, i.e. $C * C = C$. Then for all $w \in \mathbb{R}^2$,

$$\Lambda (w ; C * C) \geq \Lambda (\cdot ; C) * \Lambda (\cdot ; C) (w).$$

Proof. Theorem 4.1 in combination with $C * C = C$ immediately yields

$$\Lambda (w ; C * C) = \Lambda (w ; C) \geq \Lambda (\cdot ; C) * \Lambda (\cdot ; C) (w).$$

Theorem 4.1 can be strengthened for bivariate tail dependence functions at zero and at one. Due to the concavity of tail dependence functions and Remark 2.3, an application of Theorem 4.1 yields

$$(\Lambda_1 * \Lambda_2)'(0) = \lim_{s \downarrow 0} \frac{(\Lambda_1 * \Lambda_2)(s)}{s} \leq \lim_{s \downarrow 0} \min_{s \downarrow 0} \{\Lambda_1(s), \Lambda_2(s)\} \leq \min \{\Lambda_1'(0), \Lambda_2'(0)\},$$

despite of Figure 3 suggesting a much stronger result.

Proposition 4.3. For $\Lambda_1, \Lambda_2 \in \mathcal{M}_2$, it holds

$$(\Lambda_1 * \Lambda_2)'(0) = \Lambda_1'(0) \cdot \Lambda_2'(0) \text{ and } (\Lambda_1 * \Lambda_2)'(1) = -\Lambda_1'(1) \cdot \Lambda_2'(1).$$

Furthermore, for any negatively dependent $C \in C_2$, i.e. $C \leq \Pi$,

$$-\Lambda_1'(1) \cdot \Lambda_2'(1) \leq (\Lambda_1 * C \Lambda_2)'(t) \leq \Lambda_1'(0) \cdot \Lambda_2'(0)$$

for all $t \in [0, 1]$.

Proof. The positive homogeneity of a tail dependence function $\Lambda$ leads to

$$\partial_1 \Lambda(x, y) = \Lambda|_{[0, 1]} \left( \frac{x}{x + y} \right) + \frac{y}{x + y} \Lambda'|_{[0, 1]} \left( \frac{x}{x + y} \right),$$

so that, for any $y > 0$, we obtain

$$\partial_1 \Lambda(0, y) = \Lambda'|_{[0, 1]} (0).$$
An application of the product rule for the Stieltjes integral yields

\[
(A_1 * A_2)'(0) = \partial_1 (A_1 * A_2)(0, y) = \partial_x \int_0^\infty \partial_2 A_1(x, t) \partial_1 A_2(t, y) \, dt \bigg|_{x=0}
\]

\[
= \int_0^\infty \partial_1 A_2(t, y) \, \partial_1 A_1(x, dt) \bigg|_{x=0}
\]

\[
= \partial_1 A_2(\infty, y) \partial_1 A_1(x, \infty) - \partial_1 A_2(0, y) \partial_1 A_1(x, 0) - \int_0^\infty \partial_1 A_1(x, t) \, \partial_1 A_2(dt, y) \bigg|_{x=0}
\]

\[
= -\int_0^\infty \partial_1 A_1(0, t) \, \partial_1 A_2(dt, y) = -A_1'(0) \int_0^\infty \partial_1 A_2(dt, y)
\]

\[
= -A_1'(0) [\partial_1 A_2(\infty, y) - \partial_1 A_2(0, y)] = A_1'(0) \cdot A_2'(0) ,
\]

where the third equality can be shown analogously to Lemma 3.1 of [Darsow et al. 1992]. The second claim can be derived by observing that \((A_1 * A_2)'(1) = (A_1^2 * A_2^2)'(0)\). Finally, the last assertion stems from the fact that \(A_1 * A_2\) is concave and thus has a monotone derivative.

This factorization of \(A_1 * A_2|_{[0,1]}\) is only valid in 0 and 1 and does not generally hold for \(s \in (0,1)\), i.e. \((A_1 * A_2)'(s) \neq A_1'(s) \cdot A_2'(s)\), see for example Figure 3. Nevertheless, a general smoothing property concerning the Markov product can be derived, which is reminiscent of [Trutschnig 2013a].

**Theorem 4.4.** Suppose \(A_1, A_2 \in \mathcal{M}_2\). Then \(A_1 * A_2|_{[0,1]}\) is differentiable if \(A_1|_{[0,1]}\) or \(A_2|_{[0,1]}\) is differentiable.

**Proof.** First, the derivative of \(A_1 * A_2|_{[0,1]}\) can be rewritten as

\[
(A_1 * A_2)'(s) = \partial_1 (A_1 * A_2)(s, 1 - s) - \partial_2 (A_1 * A_2)(s, 1 - s) .
\]

We will treat both terms on the right-hand side separately. First,

\[
\partial_1 (A_1 * A_2)(s, 1 - s) = \partial_1 \int_0^\infty \partial_2 A_1(s, t) \partial_1 A_2(t, 1 - s) \, dt
\]

\[
= \int_0^\infty \partial_1 A_2(t, 1 - s) \, \partial_1 A_1(s, dt) ,
\]

where w.l.o.g. \(\partial_1 A_1(s, t)\) exists for all \(s \in [0,1]\) and is increasing in \(t\), otherwise switch the roles of \(A_1\) and \(A_2\) due to symmetry. Analogously,

\[
\partial_2 (A_1 * A_2)(s, 1 - s) = \int_0^\infty \partial_1 A_2(t, 1 - s) \, \partial_1 A_1(s, dt)
\]

\[
= -\int_0^\infty \partial_2 A_2(t, 1 - s) \, \partial_2 A_1(s, dt) .
\]

While the inverses with respect to the Markov product for 2-copulas can be used to analyse complete dependence and extremal points of \(C_2\), the reduction property impedes an analogy for tail dependence functions.
Theorem 4.5. Suppose \( \Lambda \in \mathcal{M}_2 \).

1. If \( \Lambda \) is left-invertible, i.e. there exists a bivariate tail dependence function \( \xi \) such that \( \xi * \Lambda(w) = \Lambda(w ; C^+) \), then \( \Lambda(w) = \Lambda(w ; C^+) \).

2. If \( \partial_1 \Lambda(w_1, w_2) \in \{0, 1\} \) for almost all \( w_2 \in \mathbb{R}_+ \), then \( \Lambda(w_1, w_2) = \Lambda(w_1, aw_2 ; C^+) \) for some \( a \in [0, 1] \).

Proof. 1. If \( \Lambda \) is left-invertible with left-inverse \( \xi \), then

\[ \Lambda(w ; C^+) = \xi * \Lambda(w) \leq \Lambda(w) \leq \Lambda(w ; C^+) . \]

2. Assuming \( \Lambda \) is a tail dependence function with \( \partial_1 \Lambda(w_1, w_2) \in \{0, 1\} \) for almost all \( w_2 \in \mathbb{R}_+ \), then there exists a function \( \alpha : [0, \infty) \to [0, 1] \) such that

\[ \partial_1 \Lambda(w_1, w_2) = \mathbb{1}_{[0, \alpha(w_2)]}(w_1) . \]

The positive homogeneity of \( \Lambda \) implies that \( \partial_1 \Lambda \) is positive homogeneous of order 0, i.e. constant along rays. Thus, for all \( s > 0 \) this leads to

\[ \mathbb{1}_{[0, \alpha(sw_2)]}(w_1) = \mathbb{1}_{[0, \alpha(sw_2)]}(sw_1) = \partial_1 \Lambda(sw_1, sw_2) = \partial_1 \Lambda(w_1, w_2) = \mathbb{1}_{[0, \alpha(w_2)]}(w_1) . \]

Consequently, \( \alpha(sw_2) = \alpha(w_2) = \alpha \).

Lastly, we derive a monotonicity property of the Markov product with respect to the pointwise order of tail dependence functions.

Corollary 4.6. For \( \Lambda_1, \Lambda_2 \in \mathcal{M}_2 \), the following are equivalent:

1. \( \Lambda_1(w) \leq \Lambda_2(w) \) for all \( w \in \mathbb{R}^d_+ \).
2. \( (\Lambda_1 * \Lambda)(w) \leq (\Lambda_2 * \Lambda)(w) \) for all \( w \in \mathbb{R}^d_+ \) and \( \Lambda \in \mathcal{M}_2 \).

Proof. The implication 2 \( \Rightarrow \) 1 follows immediately from the choice \( \Lambda = \Lambda^+ \). Conversely, assuming \( \Lambda_1(w) \leq \Lambda_2(w) \) for all \( w \in \mathbb{R}^d_+ \), we have

\[ \int_0^{w_2} \partial_2 \Lambda_1(w_1, t) \, dt = \Lambda_1(w) \leq \Lambda_2(w) = \int_0^{w_2} \partial_2 \Lambda_2(w_1, t) \, dt . \]

Since \( \partial_1 \Lambda(\cdot, w_2) \) is non-negative and decreasing for any tail dependence function \( \Lambda \in \mathcal{M}_2 \), Proposition 2.3.6 in [Bennett and Sharples, 1988] yields

\[ (\Lambda_1 * \Lambda)(w) = \int_0^\infty \partial_2 \Lambda_1(w_1, t) \partial_1 \Lambda(t, w_2) \, dt \leq \int_0^\infty \partial_2 \Lambda_2(w_1, t) \partial_1 \Lambda(t, w_2) \, dt = (\Lambda_2 * \Lambda)(w) . \]

5. Iterates of the Markov product

In the context of 2-copulas, the concepts of iterates, idempotents, and Cesro sums of the Markov product are widely investigated, see, for example, [Darsow and Olsen, 2010] or [Trutschnig, 2013a]. To investigate these concepts in the setting of tail dependence functions, we define the \( n \)-th iterate of the Markov product for 2-copulas and tail dependence functions as

\[ C^{*n} := C * \ldots * C \text{ and } \Lambda^{*n} := \Lambda * \ldots * \Lambda , \]
respectively. Trutschnig (2013b) showed the existence of the Cesaro sums
\[ \hat{C} := \lim_{n \to \infty} \frac{1}{n} \sum_{\ell=1}^{n} C^{*\ell} \]
for general 2-copulas \( C \) and treated their limit behaviour using ergodic theory. Here, we will study the asymptotic behaviour of \( \Lambda^{*n} \) and extend the results to an averaging of the Markov product. First, we will develop an understanding using two simple examples.

**Example.** Consider a copula \( C \) such that
\[ \partial_1 \Lambda (w ; C) = \mathbb{I}_{[0, \alpha w_2]}(w_1) \quad \text{with} \quad \alpha \in [0, 1) . \]
A simple calculation yields
\[ \Lambda (\cdot ; C)^{*2} (w) = \int_0^\infty \partial_2 \Lambda ((w_1, t) ; C) \mathbb{I}_{[0, \alpha w_2]}(t) \, dt = \Lambda ((w_1, \alpha w_2) ; C) \]
and iteratively
\[ \Lambda (\cdot ; C)^{*n} (w) = \Lambda ((w_1, \alpha^{n-1} w_2) ; C) \to \begin{cases} 0 & \text{for } \alpha \in [0, 1) \smallbreak \min \{w_1, w_2\} & \text{for } \alpha = 1 \end{cases} . \]
Thus, in this example, the limiting behaviour of \( \Lambda^{*n} \) is either given by \( \Lambda^0 \) or \( \Lambda^+ \).

The next example treats a class of tail dependence functions, which will be utilized to dominate arbitrary tail dependence functions and ultimately characterize idempotents.

**Example.** For \( 0 \leq p \leq \frac{1}{2} \) define the function
\[ \Lambda (s) := \begin{cases} s & \text{for } 0 \leq s \leq p \smallbreak p & \text{for } p \leq s \leq 1 - p \smallbreak 1 - s & \text{for } 1 - p \leq s \leq 1 \end{cases} . \quad (3) \]
Following Remark 2.3, \( \Lambda \) can be extended to a tail dependence function on \( \mathbb{R}_+^2 \). A straightforward calculation with \( q := \frac{1-p}{p} \) yields the recurrence equation
\[ \Lambda^{*(n+1)} (w_1, w_2) = (1-p) \Lambda^{*n} \left( \frac{1}{q} w_1, w_2 \right) + p \Lambda^{*n} (qw_1, w_2) . \]
It can be solved in two steps. First, it holds
\[ \Lambda^{*(n+1)} (w_1, w_2) = \sum_{\ell=0}^{n} a_\ell^n \Lambda \left( q^{n-2\ell} w_1, w_2 \right) \]
with \( a_\ell^n \in \mathbb{R}_+ \) such that
\[ a_0^0 = 1 , \quad a_0^{n+1} = p^n \quad \text{and} \quad a_\ell^{n+1} = (1-p)a_{\ell-1}^n + pa_\ell^n \quad \text{for } 1 \leq \ell \leq n . \]
The general solution of multivariate recurrences of this type was derived by Neuwirth (2001) and Mansour and Shattuck (2013) and is given by
\[ a_\ell^n = \binom{n}{\ell} (1-p)^\ell p^{n-\ell} \quad \text{for } 0 \leq \ell \leq n . \]
Due to Theorem 5.1. Let previous examples dominate any tail dependence function, we arrive at the following result. Using the monotonicity property of the Markov product from Corollary 4.6 and the fact that the second part converges to zero as the asymptotic behaviour of $\Lambda$ for $n \to \infty$. Due to the iterated Markov product being symmetric and due to the monotonicity of $\Lambda$, we arrive at the solution $\Lambda^{(2k+1)}(1/2,1/2) = p^{2k} \sum_{\ell=0}^{2k} \binom{2k}{\ell} \Lambda(q^{2k-\ell}, q^\ell) = p^{2k} \sum_{\ell=0}^{2k} \binom{2k}{\ell} (q^{2k-\ell} + q^\ell) \Lambda \left( \frac{q^{2k-\ell}}{q^{2k-\ell} + q^\ell} \right) \leq p^{2k} \sum_{\ell=0}^{k} \binom{2k}{\ell} q^{\ell} - p^{2k+1} q^k \binom{2k}{k} \leq \sum_{\ell=0}^{k} \binom{2k}{\ell} (1-p)^\ell p^{2k-\ell} - \binom{2k}{k} p^{k+1}(1-p)^k$, where the inequality is due to the definition of $\Lambda(s)$ and equality holds in case of $p = 1/2$. While the second part converges to zero as $n \to \infty$, the first part is a truncated binomial sum and by the weak law of large numbers, we have

$$\lim_{k \to \infty} \max_{w_1 + w_2 = 1} \Lambda^{(2k+1)}(w_1, w_2) = \lim_{k \to \infty} \Lambda^{(2k+1)} \left( \frac{1}{2} \right) \leq \lim_{k \to \infty} \sum_{\ell=0}^{k} \binom{2k}{\ell} (1-p)^\ell p^{2k-\ell} - \binom{2k}{k} p^{k+1}(1-p)^k \leq \begin{cases} 0 & \text{for } p < \frac{1}{2} \\ \frac{1}{2} & \text{for } p = \frac{1}{2} \end{cases}.$$  

Due to $0 \leq \Lambda^{(2k+1)}$, the above inequality is in fact an equality.

Using the monotonicity property of the Markov product from Corollary 4.9 and the fact that the previous examples dominate any tail dependence function, we arrive at the following result.

**Theorem 5.1.** Let $\Lambda$ be a bivariate tail dependence function. Then

$$\lim_{n \to \infty} \Lambda^n(w) = \begin{cases} \Lambda(w; C^+) & \text{for } \Lambda = \Lambda(\cdot; C^+) \\ \Lambda(w; \Pi) & \text{for } \Lambda \neq \Lambda(\cdot; C^+) \end{cases}$$
and the Cesro sum equals

\[ \Lambda^*(w) := \lim_{n \to \infty} \frac{1}{n} \sum_{\ell=1}^{n} \Lambda^{*\ell}(w) = \lim_{n \to \infty} \Lambda^{*n}(w). \]

This result gives another indication that the Markov product has smoothing properties, as tail independence corresponds to Fréchet-differentiability of \( C \) in zero.

**Proof.** If \( \Lambda = \Lambda^+ \), the result is immediate. Thus, consider a tail dependence function \( \Lambda \) with \( \Lambda \neq \Lambda^+ \). Define

\[ p := \max_{t \in [0,1]} \Lambda(t) < \frac{1}{2}. \]

and set

\[ \Lambda_p(s) := \begin{cases} 
  s, & 0 \leq s \leq p \\
  p, & p \leq s \leq 1-p \\
  1-s, & 1-p \leq s \leq 1 
\end{cases} \]

Thus, \( \Lambda_p \) dominates \( \Lambda \), i.e. \( \Lambda \leq \Lambda_p \), and Corollary 4.6 yields by induction

\[ \Lambda^{*n}(w) = \Lambda^{*(n-1)} * \Lambda(w) \leq \Lambda^{*(n-1)} * \Lambda_p(w) \leq \Lambda^{*n}_p(w) \to 0 \]

for any \( p < \frac{1}{2} \). For the second statement, we only need to verify that the partial Cesro sums are decreasing. Applying the monotonicity of \( * \) yields

\[
\frac{1}{n} \sum_{\ell=1}^{n} \Lambda^{*\ell} - \frac{1}{n+1} \sum_{\ell=1}^{n+1} \Lambda^{*\ell} = \left( \frac{1}{n} - \frac{1}{n+1} \right) \sum_{\ell=1}^{n} \Lambda^{*\ell} - \frac{1}{n+1} \Lambda^{*(n+1)} \\
= \frac{1}{n(n+1)} \sum_{\ell=1}^{n} \Lambda^{*\ell} - \frac{1}{n+1} \Lambda^{*(n+1)} \\
\geq \frac{1}{n(n+1)} \sum_{\ell=1}^{n} \Lambda^{*n} - \frac{1}{n+1} \Lambda^{*(n+1)} \\
= \frac{\Lambda^{*n} - \Lambda^{*(n+1)}}{n+1} \geq 0.
\]

The limit of a mean of concave functions is again concave and bounded and thus a bivariate tail dependence function. Moreover, Dini’s theorem implies that the monotone convergence of continuous functions on a compact set to a continuous function must be uniform, i.e.

\[
\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{\ell=1}^{n} \Lambda^{*\ell} - \Lambda^{*} \right\|_\infty = 0.
\]

This theorem has two immediate corollaries, one in regard to idempotent tail dependence functions, and the other to the connection to the tail behaviour of the generalized Markov product.

**Corollary 5.2.** A bivariate tail dependence function \( \Lambda \in M_2 \) is idempotent, i.e. \( \Lambda * \Lambda = \Lambda \), if and only if \( \Lambda = \Lambda^+ \) or \( \Lambda = \Lambda^0 \).

**Proof.** If \( \Lambda \) is idempotent, we have

\[
\Lambda = \lim_{n \to \infty} \Lambda^{*n}(w) = \begin{cases} 
  \Lambda\left( w; C^+ \right) & \text{for } \Lambda = \Lambda\left( \cdot; C^+ \right) \\
  \Lambda\left( w; \Pi \right) & \text{for } \Lambda \neq \Lambda\left( \cdot; C^+ \right) 
\end{cases}.
\]

Finally, we link the previous results to the tail behaviour of iterates and idempotents of 2-copulas.
Corollary 5.3. Suppose $C$ is a twice continuously differentiable 2-copula on $(0,1)^2$ with a strict tail dependence function. If we define $\hat{C}$ as the Cesaro sum of $C$ with respect to $\ast$, then

$$\Lambda\left(w; \hat{C}\right) = \lim_{n \to \infty} \frac{1}{n} \sum_{\ell=1}^{n} \Lambda\left(w; C\ast\ell\right) = \begin{cases} \Lambda\left(w; C^{+}\right) & \text{for } \Lambda\left(\cdot; C\right) = \Lambda\left(\cdot; C^{+}\right) \\ \Lambda\left(w; \Pi\right) & \text{for } \Lambda\left(\cdot; C\right) \neq \Lambda\left(\cdot; C^{+}\right). \end{cases}$$

Proof. Consider the tail dependence function of $\hat{C}$, i.e.

$$\Lambda\left(w; \hat{C}\right) = \lim_{s \searrow 0} \frac{\hat{C}(su)}{s} = \lim_{s \searrow 0} \frac{1}{n} \sum_{\ell=1}^{n} \frac{C^{*}\ell(su)}{s} =: \lim_{s \searrow 0} \lim_{n \to \infty} f(n,s).$$

Note that $f(n,s)$ converges pointwise for fixed $s$ as $n \to \infty$ as well as pointwise for fixed $n$ as $s \searrow 0$. Moreover, Theorem 2 in Trutschnig (2013b) implies the uniform convergence of $\lim_{n \to \infty} f(n,s)$. Thus, the iterated limit above can be interchanged and it holds

$$\Lambda\left(w; \hat{C}\right) = \lim_{s \searrow 0} \lim_{n \to \infty} \frac{1}{n} \sum_{\ell=1}^{n} \frac{C^{*}\ell(su)}{s} = \lim_{n \to \infty} \lim_{s \searrow 0} \frac{1}{n} \sum_{\ell=1}^{n} \Lambda\left(u; C^{*}\ell\right).$$

The last equality stems from an inductive argument utilizing

$$\Lambda\left(w; C \ast D\right) = \Lambda\left(\cdot; C\right) \ast \Lambda\left(\cdot; D\right)(w)$$

for all twice differentiable 2-copulas $C$ and $D$. The result follows from observing that $C \ast D$ is again twice differentiable if $C$ and $D$ are twice differentiable, and strict if both tail dependence functions are strict. 

6. Substochastic operators

We previously saw the close resemblance of the set of 2-copulas endowed with the Markov-product and the set of bivariate tail dependence function endowed with $\ast$. In case of the set of 2-copulas, Olsen et al. (1996) derived an isomorphy to integral-preserving linear operators. Along those lines, we will subsequently draw a connection between a certain class of linear operators and bivariate tail dependence functions. For this we define the underlying space

$$L^{1}(\mathbb{R}_{+}) + L^{\infty}(\mathbb{R}_{+}) := \left\{ f + g \mid f \in L^{1}(\mathbb{R}_{+}) \text{ and } g \in L^{\infty}(\mathbb{R}_{+}) \right\}$$

and both $L^{1}(\mathbb{R}_{+})$ and $L^{\infty}(\mathbb{R}_{+})$ are subsets of $L^{1}(\mathbb{R}_{+}) + L^{\infty}(\mathbb{R}_{+})$.

Definition 6.1. A linear operator $T : L^{1}(\mathbb{R}_{+}) + L^{\infty}(\mathbb{R}_{+}) \to L^{1}(\mathbb{R}_{+}) + L^{\infty}(\mathbb{R}_{+})$ is called doubly substochastic if

1. $T$ is positive, i.e. $Tf \geq 0$ whenever $f \geq 0$.
2. $T(L^{1}(\mathbb{R}_{+})) \subset L^{1}(\mathbb{R}_{+})$ and $T(L^{\infty}(\mathbb{R}_{+})) \subset L^{\infty}(\mathbb{R}_{+})$. 

17
3. \( T \) is a contraction on \( L^1(\mathbb{R}_+) \) and \( L^\infty(\mathbb{R}_+) \), respectively, i.e. \( \|Tf\|_1 \leq \|f\|_1 \) and \( \|Tg\|_\infty \leq \|g\|_\infty \) for all \( f \in L^1(\mathbb{R}_+) \) and \( g \in L^\infty(\mathbb{R}_+) \).

\( T \) is called equivariant if
\[
T(f \circ \sigma) = (Tf) \circ \sigma
\]
holds for all dilations \( \sigma(x) := \frac{x}{s} \) with \( s > 0 \).

Substochastic operators can be seen as a generalization of Markov operators, in the same way as doubly substochastic matrices generalize doubly stochastic matrices. A complete introduction can be found in Bennett and Sharpley (1988). In the following, we will establish a one-to-one correspondence between substochastic operators and subdistribution functions (see, Theorem 6.6). While many of the proofs work similarly to the case of compact spaces in Olsen et al. (1996), some care is needed due to the underlying non-finiteness of the measure space \( \mathbb{R}_+ \).

**Definition 6.2.** A function \( F : \mathbb{R}_+^d \to \mathbb{R}_+ \) is called a subdistribution function if it is positive, \( d \)-increasing and bounded by \( \Lambda^+ \).

**Remark 6.3.** Note that the class of \( d \)-variate tail dependence functions equals the positive homogeneous subdistribution functions.

**Lemma 6.4.** Let \( T \) be a doubly substochastic operator. Then
\[
F_T(x, y) := \int_0^x T \mathbb{I}_{[0,y]}(s) \, ds
\]
is a bivariate subdistribution function. If \( T \) is additionally equivariant, then \( F_T \) is a bivariate tail dependence function, i.e. \( F_T(\cdot) = \Lambda(\cdot; C) \) for some \( C \in \mathcal{C}_2 \).

**Proof.** We will check the properties 1 - 3 of Proposition 2.2:

1. Because \( 0 \leq F_T \) is immediate for positive \( T \), we only need to show that \( F_T \) is bounded from above by \( \Lambda(\cdot; C^+) \):
\[
\int_0^x T \mathbb{I}_{[0,y]}(s) \, ds \leq \begin{cases} \int_0^\infty T \mathbb{I}_{[0,y]}(s) \, ds \leq \int_0^\infty \mathbb{I}_{[0,y]}(s) \, ds = y \\
\int_0^x T \mathbb{I}_{\mathbb{R}_+}(s) \, ds = x \end{cases} .
\]

2. Let \( R = [x_1, x_2] \times [y_1, y_2] \) with \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \). Then the linearity of \( T \) yields
\[
V_{F_T}(R) = \int_{x_1}^{x_2} T \mathbb{I}_{[y_1,y_2]}(s) \, ds \geq 0 .
\]

Hence, \( F_T \) is a bivariate subdistribution function. Finally, the positive homogeneity of \( F_T \) follows from
\[
F_T(sx, sy) = \int_0^{sx} T \mathbb{I}_{[0,y]}(t) \, dt = \int_0^{sx} T \mathbb{I}_{[0,y]} \left( \frac{t}{s} \right) \, dt = \int_0^x T \mathbb{I}_{[0,y]} \left( \frac{z}{s} \right) s \, dz = sF_T(x, y)
\]
for any \( s \geq 0 \). Thus, \( F_T \) is a positive homogeneous, bounded and 2-increasing function, and the claim follows from Proposition 2.2. \( \square \)
Lemma 6.5. Let $F$ be a bivariate subdistribution function. Then

$$T_F : L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+) \to L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+)$$

$$T_F f(x) = \partial_x \int_0^\infty \partial_t F(x,t) f(t) \, dt$$

defines a doubly substochastic operator. Moreover, if $F$ is a bivariate tail dependence function, then $T_F$ is equivariant.

Proof. As $\partial_t F(x,t)$ is increasing in $x$, we have that for $|f| \pm f \geq 0$

$$\int_0^\infty \partial_t F(x,t) (|f| \pm f) (t) \, dt$$

is again an increasing function in $x$ and its derivative with respect to the first component exists. Thus, representing $f$ as a linear combination of $|f| + f$ and $|f| - f$ implies that $T_F f$ exists. Let us now verify properties [1, 3] of Definition 6.1.

1. Let $f$ be positive. As $\partial_t F(x_2,t) - \partial_t F(x_1,t) \geq 0$ for $x_1 \leq x_2$, we have that

$$\int_0^\infty \partial_t F(x_2,t) f(t) \, dt - \int_0^\infty \partial_t F(x_1,t) f(t) \, dt \geq 0$$

and hence $T_F f \geq 0$.

2. To prove 2 and 3, we first consider $f \in L^\infty(\mathbb{R}^+)$ and note that

$$g(x) := \int_0^\infty \partial_t F(x,t) f(t) \, dt$$

is Lipschitz continuous with Lipschitz constant $L = \|f\|_\infty$. Because for $x_1 \leq x_2$, we have

$$|g(x_2) - g(x_1)| = \left| \int_0^\infty (\partial_t F(x_2,t) - \partial_t F(x_1,t)) f(t) \, dt \right|$$

$$\leq \|f\|_\infty \int_0^\infty |\partial_t F(x_2,t) - \partial_t F(x_1,t)| \, dt$$

$$= \|f\|_\infty \int_0^\infty (\partial_t F(x_2,t) - \partial_t F(x_1,t)) \, dt$$

$$= \|f\|_\infty \lim_{t \to \infty} |F(x_2,t) - F(x_1,t)|^R_0 \leq \|f\|_\infty |x_2 - x_1|,$$

where the second equality is due to $\partial_t F(x,t)$ being increasing, as $F$ is 2-increasing. Thus, $T_F$ is a contraction on $L^\infty(\mathbb{R}^+)$. Now let $f$ be in $L^1(\mathbb{R}^+)$. Combining the linearity and positivity of $T_F$ leads to

$$|Tf| = |T(f^+ - f^-)| \leq T f^+ + T f^- = T |f|.$$
Thus, without loss of generality, let \( f \) be positive. Then using the absolute continuity of \( g \), we have
\[
\int_0^\infty T_F f(x) \, dx = \int_0^\infty \partial_x \int_0^\infty \partial_t F(x, t) f(t) \, dt \, dx = \lim_{R \to \infty} \int_0^R \partial_x \int_0^\infty \partial_t F(x, t) f(t) \, dt \, dx
\]
\[
= \lim_{R \to \infty} \int_0^\infty \partial_t F(R, t) f(t) \, dt \leq \lim_{R \to \infty} \int_0^\infty f(t) \, dt = \int_0^\infty f(t) \, dt
\]
and \( T_F \) is a contraction on \( L^1(\mathbb{R}_+) \).

Combining the previous two results, one sees that \( T \) is an operator from \( L^1(\mathbb{R}_+) + L^\infty(\mathbb{R}_+) \) onto \( L^1(\mathbb{R}_+) + L^\infty(\mathbb{R}_+) \) and therefore doubly substochastic. If \( F \) is also positive homogeneous, then for any \( s > 0 \)
\[
T_F \mathbb{1}_{[0, sy]}(x) = \partial_x \int_0^\infty \partial_t F(x, t) \mathbb{1}_{[0, sy]}(t) \, dt = \partial_x \int_0^\infty \partial_t F(x, t) \mathbb{1}_{[0, y]} \left( \frac{t}{s} \right) \, dt
\]
\[
= \partial_x \int_0^\infty \partial_2 F(x, sz) \mathbb{1}_{[0, y]}(z) s \, dz
\]
\[
= \partial_x \int_0^\infty \partial_2 F \left( \frac{x}{s}, z \right) \mathbb{1}_{[0, y]}(z) \, dz = T_F \mathbb{1}_{[0, y]} \left( \frac{x}{s} \right).
\]

As a consequence of Lemma 6.4 and 6.5 we obtain the main result establishing the correspondence between subdistribution functions and substochastic operators.

**Theorem 6.6.** Let \( F \) be a bivariate subdistribution function and \( T \) a substochastic operator, and define \( \Phi(T) := F_T \) and \( \Psi(F) := T_F \). Then \( \Phi \circ \Psi \) and \( \Psi \circ \Phi \) define identities on their respective spaces. Furthermore, \( F \) is positive homogeneous if and only if \( T_F \) is equivariant.

**Proof.** First, let \( F \) be a subdistribution function. Then using the Lipschitz continuity of \( F \)
\[
\Phi \circ \Psi(F)(x, y) = \int_0^x \Psi(F) \mathbb{1}_{[0, y]}(s) \, ds = \int_0^x \partial_x \int_0^\infty \partial_t F(s, t) \mathbb{1}_{[0, y]}(t) \, dt \, ds
\]
\[
= \int_0^x \partial_x \int_0^y \partial_t F(s, t) \, dt \, ds = \int_0^x \partial_x F(s, y) \, ds = F(x, y).
\]

Conversely, let \( T \) be a substochastic operator and \( f(t) = \mathbb{1}_{[0, y]}(t) \). Then the absolute continuity
\[
\Psi \circ \Phi(T)f(x) = \partial_x \int_0^\infty \partial_t \Phi(T)(x, t) f(t) \, dt = \partial_x \int_0^\infty \int_0^x T \mathbb{1}_{[0, t]}(s) \, ds f(t) \, dt
\]
\[
= \partial_x \int_0^y \partial_t \int_0^x T \mathbb{1}_{[0, t]}(s) \, ds \, dt = \partial_x \int_0^y T \mathbb{1}_{[0, x]}(s) \, ds = T \mathbb{1}_{[0, y]}(x).
\]

Thus \( \Psi \circ \Phi(T) \) and \( T \) are substochastic operators which agree on \([0, y]\). Following the argument in Lemma 2.2 of Olsen et al. [1996] yields the assertion. Finally, Lemma 6.4 and 6.5 yield the equivalence between the positive homogeneity of \( F \) and the equivariance of \( T \).
The correspondence between substochastic operators and subdistribution functions is a structure-preserving isomorphism translating $*$ into $\circ$ and vice versa.

**Theorem 6.7.** Let $F$ and $G$ be subdistribution functions. Then

$$T_{F\ast G} = T_F \circ T_G.$$  

**Proof.** In view of Theorem 6.6, it suffices to prove that

$$\Phi \left( T_F \circ T_G \right) (w) = \Phi \left( T_{F\ast G} \right) (w) = (F \ast G)(w)$$

for all $w \in \mathbb{R}_+^2$. To do so, we use the Lipschitz continuity to obtain

$$\Phi \left( T_F \circ T_G \right) (w) = \int_0^{w_1} \int_0^{w_2} \partial_s F(s, t) T_G \mathbb{1}_{[0, w_2]}(t) \, dt \, ds$$

$$= \int_0^{w_1} \int_0^{w_2} \partial_s F(w_1, t) T_G \mathbb{1}_{[0, w_2]}(t) \, dt$$

$$= \int_0^{w_1} \int_0^{w_2} \partial_s F(w_1, t) \partial_t \int_0^t \partial_s G(t, s) \mathbb{1}_{[0, w_2]}(s) \, ds \, dt$$

$$= \int_0^{w_1} \int_0^{w_2} \partial_s F(w_1, t) \partial_t G(t, w_2) \, dt = F \ast G(w) .$$

The Banach space adjoint of a substochastic operator $T_F$ corresponds to the doubly substochastic operator associated with the transpose $F^T$ of $F$ where $F^T(x, y) := F(y, x)$.

**Proposition 6.8.** Let $f \in L^1(\mathbb{R}_+) + L^\infty(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. Then

$$\int_0^\infty T_F f(x) g(x) \, dx = \int_0^\infty f(x) T_{F^T} g(x) \, dx .$$

**Proof.** Let $f \in L^1(\mathbb{R}_+) + L^\infty(\mathbb{R}_+)$ and $g$ in the dual space

$$\left( L^1(\mathbb{R}_+) + L^\infty(\mathbb{R}_+) \right)' = L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) .$$

As the space of compactly supported and smooth functions is dense in $L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, we only need to show the desired result for $g \in C^\infty_c(\mathbb{R}_+)$. Then an identical calculation to Lemma 2.4 from Olsen et al. (1996) yields the result, except that for the partial integration, we additionally require $g(\infty) = 0$ and $\partial_t F(0, t) = 0$, which holds due to $F(0, t) \equiv 0$.

Using this connection between the adjoint of $T$ and the transpose of $F$, we can establish a relation between strict subdistribution functions and Markov operators.

**Definition 6.9.** Let $F$ be a bivariate subdistribution function. Then we call $F$ strict if

$$\lim_{t \to \infty} F(w_1, t) = w_1 \quad \text{and} \quad \lim_{t \to \infty} F(t, w_2) = w_2$$

for all $w$ in $\mathbb{R}_+^2$. 

21
Definition 6.10. Let $T$ be a doubly substochastic operator. $T$ is called a doubly stochastic operator or Markov operator if

$$T \mathbb{1}_{\mathbb{R}_+} = \mathbb{1}_{\mathbb{R}_+} \quad \text{and} \quad \int_0^\infty Tf(x) \, dx = \int_0^\infty f(x) \, dx$$

for all $f$ in $L^1(\mathbb{R}_+)$. 

Proposition 6.11. Let $F$ be a bivariate subdistribution function. Then $F$ is strict if and only if $T_F$ and $T_{FT}$ are Markov operators.

Proof. First, let $F$ be strict. Then,

$$T_F \mathbb{1}_{\mathbb{R}_+}(x) = \partial_x \int_0^\infty \partial_t F(x,t) \mathbb{1}_{\mathbb{R}_+}(t) \, dt = \partial_x \int_0^\infty \partial_t F(x,t) \, dt$$

$$= \partial_x (F(x,\infty) - F(x,0)) = \partial_x x = \mathbb{1}_{\mathbb{R}_+}(x)$$

for all $x \in \mathbb{R}_+$. Now let $f$ be in $L^1(\mathbb{R}_+)$, then it holds

$$\int_0^\infty T_f(x) \, dx = \int_0^\infty \partial_x \int_0^\infty \partial_t F(x,t) f(t) \, dt \, dx = \int_0^\infty \partial_t F(\infty,t) f(t) \, dt = \int_0^\infty f(t) \, dt$$

using the strictness of $F$ and absolute continuity. The claims for $T_{FT}$ can be proven analogously by exploiting the strictness in the second component of $F$. Conversely, if $T_F$ and $T_{FT}$ are doubly stochastic, then

$$\lim_{t \to \infty} F(t,w_2) = \lim_{t \to \infty} \int_0^t T_F \mathbb{1}_{[0,w_2]}(s) \, ds = \int_0^\infty T_F \mathbb{1}_{[0,w_2]}(s) \, ds = \int_0^\infty \mathbb{1}_{[0,w_2]}(s) \, ds = w_2$$

and, analogously, for $\lim_{t \to \infty} F(w_1,t) = w_1$. 

Finally, we present an alternative proof of Theorem 4.1 using the theory of substochastic operators.

Proof of Theorem 4.1. For every substochastic operator $T$ and every $t \in [0,\infty)$, it holds

$$\int_0^t (Tf)^*(s) \, ds \leq \int_0^t f^*(s) \, ds$$

or, in short, $Tf \preceq f$, where $f^*$ denotes the decreasing rearrangement of $f$ (see, Chapter 1 in Bennett and Sharpley (1988)). Thus

$$\partial_1 \Lambda_2(w_1,w_2) \succ T_{\Lambda^T} \partial_1 \Lambda_2(\cdot,w_2)(w_1)$$

$$= \partial_1 \int_0^\infty \partial_2 \Lambda_1(w_1,s) \partial_1 \Lambda_2(s,w_2) \, ds$$

$$= \partial_1 (\Lambda_1 \ast \Lambda_2)(w_1,w_2)$$

together with the monotonicity of the tail dependence function yields

$$(\Lambda_1 \ast \Lambda_2)(w_1,w_2) \leq \Lambda_2(w_1,w_2).$$

□
Acknowledgements

We are grateful to Piotr Jaworski for interesting discussions. The second author gratefully acknowledges financial support from the German Academic Scholarship Foundation.

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