CONVERGENCE OF INVARIANT MEASURES FOR SINGULAR
STOCHASTIC DIFFUSION EQUATIONS

IOANA CIOTIR
Department of Mathematics, Faculty of Economics and Business Administration,
“Al. I. Cuza” University, Bd. Carol no. 9–11, Iași, Romania

JONAS M. TÖLLE
Institut für Mathematik, Technische Universität Berlin (MA 7-5), Straße des 17.
Juni 136, 10623 Berlin, Germany

Abstract. It is proved that the solutions to the singular stochastic 
\( p \)-Laplace equation, \( p \in (1,2) \) and the solutions to the stochastic fast diffusion equation 
with nonlinearity parameter \( r \in (0,1) \) on a bounded open domain \( \Lambda \subset \mathbb{R}^d \) with Dirichlet boundary conditions are continuous in mean, uniformly in time, with 
respect to the parameters \( p \) and \( r \) respectively (in the Hilbert spaces \( L^2(\Lambda), H^{-1}(\Lambda) \) respectively). The highly singular limit case \( p = 1 \) is treated with the 
help of stochastic evolution variational inequalities, where \( \mathbb{P} \)-a.s. convergence, 
uniformly in time, is established.

It is shown that the associated unique invariant measures of the ergodic 
semigroups converge in the weak sense (of probability measures).

1. Introduction

Let \( \Lambda \subset \mathbb{R}^d \) be a bounded open domain with Lipschitz boundary \( \partial \Lambda \). Let 
\( \{ W(t) \}_{t \geq 0} \) be a \( U \)-valued cylindrical Wiener process on some filtered probability 
space \( (\Omega, \mathcal{F}, \{ \mathcal{F}(t) \}_{t \geq 0}, \mathbb{P}) \), where \( U \) is a separable Hilbert space.

We are interested in the following two (families of) stochastic diffusion equations, the stochastic \( p \)-Laplacian equation, \( p \in (1, \infty) \), \( B \in L_2(U, L^2(\Lambda)) \),

\[
(PL_p) \quad \left\{ \begin{array}{ll}
    dX_p(t) = \text{div} \left[ |\nabla X_p(t)|^{p-2} \nabla X_p(t) \right] dt + B \, dW(t) & \text{in } (0,T) \times \Lambda, \\
    X_p(t) = 0 & \text{on } (0,T) \times \partial \Lambda, \\
    X_p(0) = x \in L^2(\Lambda) & \text{in } \Lambda.
\end{array} \right.
\]

E-mail addresses: ioana.ciotir@feaa.uaic.ro, toelle@math.tu-berlin.de.

2000 Mathematics Subject Classification. 60H15; 35K67, 37L40, 49J45.

Key words and phrases. Stochastic evolution equation, stochastic diffusion equation, \( p \)-Laplace 
equation, 1-Laplace equation, total variation flow, fast diffusion equation, ergodic semigroup, 
unique invariant measure, variational convergence.

Financial support of the DFG Collaborative Research Center 701 (SFB 701) Spectral Structures 
and Topological Methods in Mathematics (Bielefeld) and the DFG Research Group 718 
(Forschergruppe 718) Analysis and Stochastics in Complex Physical Systems (Berlin-Leipzig) is 
gratefully acknowledged.

Both authors would like to thank Viorel Barbu and Michael Röckner for helpful comments. The 
authors are grateful for the remarks of two referees which helped in improving the paper.

1
The deterministic $p$-Laplace equation arises from geometry, quasi-regular mappings, fluid dynamics and plasma physics, see [19, 20]. In [27], $(\text{PL}_p)$ with $B \equiv 0$ is suggested as a model of motion of non-Newtonian fluids. See [28] for the stochastic equation.

We are also interested in the stochastic fast diffusion equation $r \in (0, \infty), B \in L^2(U, H^{-1}(\Lambda))$,

\begin{align*}
(\text{FD}_r) \quad \left\{ \begin{array}{ll}
dY_r(t) & = \Delta \left( |Y_r(t)|^{r-1}Y_r(t) \right) \ dt + B \ dW(t), & \text{in } (0, T) \times \Lambda, \\
y_r(t) & = 0, & \text{on } (0, T) \times \partial \Lambda, \\
y_r(0) & = y \in H^{-1}(\Lambda), & \text{in } \Lambda,
\end{array} \right.
\end{align*}

which models diffusion in plasma physics, curvature flows and self-organized criticality in sandpile models, see e.g. [12, 14, 36, 41] and the references therein.

The above equations considered are called singular for $p \in (1, 2), r \in (0, 1)$ and degenerate for $p \in (2, \infty), r \in (1, \infty)$ (porous medium equation). In this paper, we shall investigate the former case.

For $p = 1$, equation $(\text{PL}_1)$ can be heuristically written as a stochastic evolution inclusion, $B \in L^2(U, H^{-1}(\Lambda))$,

\begin{align*}
(\text{PL}_1) \quad \left\{ \begin{array}{ll}
dX_1(t) & \in \text{div } [\text{Sgn}(\nabla X_1(t))] \ dt + B \ dW(t) \quad \text{in } (0, T) \times \Lambda, \\
x_1(t) & = 0 \quad \text{on } (0, T) \times \partial \Lambda, \\
x_1(0) & = x \in L^2(\Lambda) \quad \text{in } \Lambda,
\end{array} \right.
\end{align*}

where $\text{Sgn} : \mathbb{R}^d \rightarrow 2^{\mathbb{R}^d}$ is defined by

$$
\text{Sgn}(u) := \begin{cases} 
\frac{u}{|u|}, & \text{if } u \in \mathbb{R}^d \setminus \{0\}, \\
\{v \in \mathbb{R}^d \mid |v| \leq 1\}, & \text{if } u = 0.
\end{cases}
$$

A precise characterization of the 1-Laplace operator can be found in [2, 3, 37]. A typical 2-dimensional example for the so-called total variation flow can be found in image restoration, see [1, 3, 6] and the references therein.

We shall, however, take use of the stochastic evolution variational inequality formulation as in [11].

We are particularly interested in continuity of the solutions in the parameters $p$ and $r$, especially for the case $p \rightarrow 1$. Stochastic Trotter-type results in this direction have been obtained by the first named author in [15, 16, 17]. However, for the case $p \rightarrow 1$, we shall need the theory of Mosco convergence of convex functionals as in [4], since no strong characterization of the limit is available (which could be treated by Yosida-approximation methods). For $B = 0$ (i.e., the deterministic equation), the convergence of solutions to the evolution problem $(\text{PL}_p)$ was proved in [23, 40]. See also [39, Ch. 8.3].

With the help of a uniqueness result for invariant measures of the equations considered, obtained by Liu and the second named author [29], we prove tightness and the weak convergence (weak continuity) of invariant measures associated to the ergodic semigroups of the equations $(\text{PL}_p)$ and $(\text{FD}_r)$. See [9, 10, 18, 22] for other result in this direction.

**Organization of the paper.** In Section 2 we prove that the solutions to the basic examples are continuous in the parameters $p$ and $r$ resp.

In Section 3 the result of Section 2 is combined with the uniqueness of invariant measures proved in [29] in order to obtain the weak continuity of invariant measures in the parameters $p$ and $r$ resp.

In Section 4 we prove a convergence result for the stochastic $p$-Laplace equation as $p \rightarrow 1$, using another notion of a solution. For the limit $p = 1$, however,
uniqueness of the invariant measure is an open question. The matter is further investigated in [22].

The Appendix collects some well-known results on Mosco (variational) convergence and Mosco convergence in $L^p$-spaces, needed for the proof in Section 4.

2. CONVERGENCE OF SOLUTIONS

Compare with [16, Theorem 2].

**Theorem 2.1.** Let $\{p_n\} \subset \left(1 \vee \frac{2d}{d+2}, 2\right]$, $n \in \mathbb{N}$, $p_0 \in \left(1 \vee \frac{2d}{d+1}, 2\right]$ such that $p_n \to p_0$. Let $X_n := X_{p_n}$, $n \in \mathbb{N}$, $X_0 := X_{p_0}$ be the solutions to $(\text{PL}_{p_n})$, $n \in \mathbb{N}$, $(\text{PL}_{p_0})$ resp. Then for $x \in L^2(\Lambda)$,

$$\lim_{n} \mathbb{E} \left[ \sup_{t \in [0,T]} \|X_n(t) - X_0(t)\|_{L^2(\Lambda)}^2 \right] = 0.$$

**Proof.** For $p \in (1, \infty)$, define $a_p : \mathbb{R}^d \to \mathbb{R}^d$ by $a_p(x) := |x|^{p-2}x$. Furthermore, let $A_p : W^{1,p}_0(\Lambda) \to (W^{1,p}_0(\Lambda))^*$ be defined by $A_p(y) := -\text{div}[a_p(\nabla y)]$, where $y \in W^{1,p}_0(\Lambda)$. To be more specific,

$$(W^{1,p}_0(\Lambda), \langle A_p(y), z \rangle_{W^{1,p}} = \int_{\Lambda} \langle a_p(\nabla y), \nabla z \rangle \, d\xi, \quad \forall z \in W^{1,p}_0(\Lambda)).$$

We first consider the following approximating equations for $(\text{PL}_{p_n})$

$$\begin{cases}
    dX_n^\varepsilon(t) + A_p^\varepsilon(X_n^\varepsilon(t)) \, dt = B \, dW(t) \\
    X_n^\varepsilon(0) = x 
\end{cases}$$

(2.1)

where for any $u \in L^2(\Lambda)$,

$$A_p^\varepsilon(u) = -(1 - \varepsilon \Delta)^{-1} \text{div} \left[a_p^\varepsilon(\nabla (1 - \varepsilon \Delta)^{-1} u)\right]$$

and $a_p^\varepsilon$ is the Yosida approximation of $a_p$ i.e., for any $r \in \mathbb{R}^d$,

$$a_p^\varepsilon(r) = \frac{1}{\varepsilon} \left(1 - (1 + \varepsilon a_p)^{-1}(r)\right).$$

In particular, for $u, v \in L^2(\Lambda)$,

$$\langle A_p^\varepsilon(u), v \rangle_{L^2(\Lambda)} = \int_{\Lambda} \langle a_p^\varepsilon(\nabla R_c u), \nabla R_c (v) \rangle \, d\xi,$$

where $R_c := (1 - \varepsilon \Delta)^{-1}$ is the resolvent of the Dirichlet Laplacian.

We shall use the following strategy ($\mathbb{P}$-a.s.)

$$\|X_n(t) - X_0(t)\|_{L^2(\Lambda)}^2 \leq 3 \|X_n(t) - X_n^\varepsilon(t)\|_{L^2(\Lambda)}^2 + 3 \|X_0^\varepsilon(t) - X_0(t)\|_{L^2(\Lambda)}^2 + 3 \|X_n^\varepsilon(t) - X_0^\varepsilon(t)\|_{L^2(\Lambda)}^2$$

$$=: I_1(n, \varepsilon) + I_2(n, \varepsilon) + I_3(\varepsilon).$$

uniformly in $t \in [0,T]$.

At this point we need to prove the following lemma. We introduce the notation $r_p^\varepsilon(r) := (1 + \varepsilon a_p)^{-1}(r)$.

**Lemma 2.2.** Under our assumptions, if we let $X_n^\varepsilon$ be the solution to (2.1) and $	ilde{X}_p^\varepsilon := (1 - \varepsilon \Delta)^{-1} X_n^\varepsilon$, we have that

$$\mathbb{E} \int_0^t \int_{\Lambda} \left| r_p^\varepsilon \left(\nabla \tilde{X}_p^\varepsilon(s)\right) \right|^p \, d\xi \, ds \leq C_t \left(\|x\|_{L^2(\Lambda)}^2 + \|B\|_{H^2_S}\right),$$

(2.2)
for all $t \in [0, T]$.

**Proof.** We know by the definition of $a_p$ that
\[
\langle a_p (r), r \rangle \geq |r|^p.
\]

On the other hand we have by Itô’s formula, applied to the function $u \mapsto \|u\|_{L^2(\Lambda)}^2$, that
\begin{equation}
\mathbb{E} \left| X^\varepsilon_p (t) \right|^2 + 2\mathbb{E} \int_0^t \left\langle a^\varepsilon_p \left( \nabla \hat{X}^\varepsilon_p (s) \right), \nabla \hat{X}^\varepsilon_p (s) \right\rangle \, ds
\leq C_t \left( \|x\|_{L^2(\Lambda)}^2 + \|B\|_{HS}^2 \right).
\end{equation}

By the definition of the Yosida approximation we have that
\[
a^\varepsilon_p (r) = a_p (r^\varepsilon_p (r))
\]
and
\[
\langle a^\varepsilon_p (r), r \rangle = \langle a^p (r^\varepsilon_p (r)), r^\varepsilon_p (r) \rangle + \frac{1}{\varepsilon} |r - r^\varepsilon_p (r)|^2.
\]

We rewrite as follows
\[
\mathbb{E} \int_0^t \int_\Lambda \left\langle a^\varepsilon_p \left( \nabla \hat{X}^\varepsilon_p (s) \right), \nabla \hat{X}^\varepsilon_p (s) \right\rangle \, d\xi \, ds
\geq \mathbb{E} \int_0^t \int_\Lambda \left\langle a^\varepsilon_p \left( r^\varepsilon_p \left( \nabla \hat{X}^\varepsilon_p (s) \right) \right), r^\varepsilon_p \left( \nabla \hat{X}^\varepsilon_p (s) \right) \right\rangle \, d\xi \, ds
\geq \mathbb{E} \int_0^t \int_\Lambda \left| r^\varepsilon_p \left( \nabla \hat{X}^\varepsilon_p (s) \right) \right|^p \, d\xi \, ds.
\]

Plugging into (2.3) proves (2.2). \qed

We shall prove now that $\mathbb{P}$-a.s.
\[
\lim_{\varepsilon \to 0} \sup_{t \in [0, T]} \| X^\varepsilon_p (t) - X^\varepsilon_p (t) \|_{L^2(\Lambda)}^2 = 0, \quad \text{uniformly in } p \in \left( \frac{2d}{d+2}, 2 \right).
\]

We set $\hat{X}^\varepsilon_p = (1 - \varepsilon \Delta)^{-1} X^\varepsilon_p$ and $\hat{X}^\lambda_p = (1 - \lambda \Delta)^{-1} X^\lambda_p$. Then by (2.1), we have that
\[
\frac{1}{2} \left\| X^\varepsilon_p (t) - X^\lambda_p (t) \right\|_{L^2(\Lambda)}^2 + \int_0^t \int_\Lambda \left\langle a^\varepsilon_p \left( \nabla \hat{X}^\varepsilon_p (s) \right) - a^\lambda_p \left( \nabla \hat{X}^\lambda_p (s) \right), \nabla \hat{X}^\varepsilon_p (s) - \nabla \hat{X}^\lambda_p (s) \right\rangle \, d\xi \, ds = 0 \quad \mathbb{P}\text{-a.s.}
\]

Setting $\nabla \hat{X}^\varepsilon_p (s) = u^\varepsilon$ and $\nabla \hat{X}^\lambda_p (s) = u^\lambda$ and using
\[
a^\varepsilon_p (u) \in a_p \left( (1 + \varepsilon a_p)^{-1} (u) \right),
\]
we get by the monotonicity of $a_p$ that
\[
\langle a^\varepsilon_p (u^\varepsilon) - a^\lambda_p (u^\lambda), u^\varepsilon - u^\lambda \rangle
\geq \langle a^\varepsilon_p (u^\varepsilon) - a^\lambda_p (u^\lambda), \varepsilon a^\varepsilon_p (u^\varepsilon) - \lambda a^\lambda_p (u^\lambda) \rangle.
\]

This leads to
\begin{equation}
\frac{1}{2} \left\| X^\varepsilon_p (t) - X^\lambda_p (t) \right\|_{L^2(\Lambda)}^2
\leq \int_0^t \int_\Lambda \left( \varepsilon |a^\varepsilon_p \left( \nabla \hat{X}^\varepsilon_p (s) \right)|^2 + \lambda |a^\lambda_p \left( \nabla \hat{X}^\lambda_p (s) \right)|^2 \right) \, d\xi \, ds \quad \mathbb{P}\text{-a.s.}
\end{equation}

We can now prove that $\mathbb{P}$-a.s.
for some $C_t$ independent of $p$ and $\epsilon$.
Using Jensen’s inequality (for $t \mapsto t^{p/(2p-2)}$) and taking into account that $|a_p(r)| \leq \rho |r|^{p-1}$, we obtain

\[
\int_0^t \int_{\Lambda} \left| a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right) \right|^2 d\xi ds \leq C_t
\]

(2.5)

where $A$ and that follows from $r \leq p_1$.

Using Jensen’s inequality (for $t \mapsto t^{p/(2p-2)}$) and taking into account that $|a_p(r)| \leq \rho |r|^{p-1}$, we obtain

\[
\int_0^t \int_{\Lambda} \left| a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right) \right|^2 d\xi ds
\]

\[
\leq (t |\Lambda|)^{1-((2p-2)/p)} \left( \int_0^t \int_{\Lambda} \left| a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right) \right|^p d\xi ds \right)^{(2p-2)/p}
\]

(2.6)

where $|\Lambda| = \int_{\Lambda} d\xi$.

Now by Lemma 2.2 we have (2.5) for a constant $C_t$ independent of $p$ and $\epsilon$, and passing to the limit for $\epsilon, \lambda \to 0$ in (2.4) we get that $P$-a.s.

\[
\lim_{\epsilon \to 0, t \in [0, T]} \sup_{\epsilon} \|X_p^\epsilon (t) - X_p^\epsilon (t)\|_{L^2(\Lambda)} = 0, \quad \text{uniformly in } p \in \left( 1 + \frac{2d}{d+2}, \frac{2d}{d+2} \right).
\]

As a consequence, $I_1(n, \epsilon)$ and $I_3(\epsilon)$ tend to zero as $\epsilon \downarrow 0$, uniformly in $n$. For $I_2(n, \epsilon)$, using the monotonicity of $a_p^\epsilon$, we have

\[
\frac{1}{2} \|X_p^\epsilon (t) - X_p^\epsilon (t)\|_{L^2(\Lambda)}^2
\]

+ \int_0^t \int_{\Lambda} \left( a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right) - a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right), \nabla \tilde{X}_p^\epsilon (s) - \nabla \tilde{X}_p^\epsilon (s) \right) d\xi ds \leq 0.

Since

\[
\frac{1}{2} \|X_p^\epsilon (t) - X_p^\epsilon (t)\|_{L^2(\Lambda)}^2
\]

\[
\leq \int_0^t \int_{\Lambda} \left( (1 - \rho \Delta)^{-1} \text{div} \left( a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right) - a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right) \right) \right) [X_p^\epsilon (s) - X_p^\epsilon (s)] d\xi ds
\]

\[
\leq \left( \int_0^t \int_{\Lambda} \left( (1 - \rho \Delta)^{-1} \text{div} a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right) - (1 - \rho \Delta)^{-1} \text{div} a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right) \right)^2 d\xi ds \right)^{1/2}
\]

\[
\times \left( \int_0^t \int_{\Lambda} [X_p^\epsilon (s) - X_p^\epsilon (s)]^2 d\xi ds \right)^{1/2}
\]

We only need to prove that

\[
\left( \int_0^t \int_{\Lambda} \left( (1 - \rho \Delta)^{-1} \text{div} a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right) - (1 - \rho \Delta)^{-1} \text{div} a_p^\epsilon \left( \nabla \tilde{X}_p^\epsilon (s) \right) \right)^2 d\xi ds \right)^{1/2}
\]

\[
\to 0
\]

and that follows from

\[
A_p^\epsilon (u) \to A_p^\epsilon (u), \quad \text{strongly in } L^2((0, T) \times \Lambda),
\]

(2.7)

where $A_p^\epsilon (u) = (1 - \rho \Delta)^{-1} \text{div} a_p^\epsilon (u)$ (as in (2.4)).

Indeed, we obtain (2.7) by the following arguments:
Theorem 2.3. Let \( p \) constants. Inequality and Grönwall’s lemma. We refer to [38] for the convergence of the resolvent in \( L^2((0, T) \times \Lambda) \), \#n\( \) for \( \) in \( L^2((0, T) \times \Lambda) \).

That means
\[
\text{div} a_{p_n}^\varepsilon (u) \to \text{div} a_{p_0}^\varepsilon (u), \text{ weakly in } L^2((0, T) \times \Lambda)
\]
and this leads to
\[
(1 - \varepsilon \Delta)^{-1} \text{div} a_{p_n}^\varepsilon (u) \to (1 - \varepsilon \Delta)^{-1} \text{div} a_{p_0}^\varepsilon (u), \text{ strongly in } L^2((0, T) \times \Lambda),
\]
which is (2.7).

We have proved that
\[
\lim_{n} \sup_{t \in [0, T]} \| X_n(t) - X_0(t) \|_{L^2(\Lambda)} = 0 \quad \text{P-a.s.}
\]

The convergence
\[
\lim_{n} \mathbb{E} \left[ \sup_{t \in [0, T]} \| X_n(t) - X_0(t) \|_{L^2(\Lambda)}^2 \right] = 0
\]
is established by Lebesgue’s dominated convergence theorem and [28, Eq. (1.3)], where the constant can be controlled uniformly in \( p \) by Itô’s formula, Poincaré inequality and Grönwall’s lemma. We refer to [33] for the \( p \)-dependence of Poincaré constants. \( \square \)

Theorem 2.3. Let \( \{ r_n \} \subseteq \left( 0 \lor \frac{2}{n+2} , 1 \right) \), \( n \in \mathbb{N} \), \( r_0 \in \left( 0 \lor \frac{4}{n+2} , 1 \right) \) such that \( r_n \to r_0 \). Let \( Y_n := Y_{r_n} \), \( n \in \mathbb{N} \), \( Y_0 := Y_{r_0} \) be the solutions to (FD\( r_n \)), \( n \in \mathbb{N} \), (FD\( r_0 \)) resp. Then for \( y \in H^{-1}(\Lambda) \),

\[
\lim_{n} \mathbb{E} \left[ \sup_{t \in [0, T]} \| Y_n(t) - Y_0(t) \|_{H^{-1}(\Lambda)}^2 \right] = 0.
\]

Proof. We need to show that
\[
\lim_{n} \mathbb{E} \left[ \sup_{t \in [0, T]} \| Y_n(t) - Y_0(t) \|_{H^{-1}(\Lambda)}^2 \right] = 0.
\]

Using the same approximation as in [10] consider

\[
\| Y_n(t) - Y_0(t) \|_{H^{-1}(\Lambda)} \leq \| Y_n(t) - Y_0^\varepsilon(t) \|_{H^{-1}(\Lambda)} + \| Y_0^\varepsilon(t) - Y_0^\varepsilon(t) \|_{H^{-1}(\Lambda)} + \| Y_0^\varepsilon(t) - Y_0(t) \|_{H^{-1}(\Lambda)}
\]
\[
= I_1 + I_2 + I_3.
\]

For \( I_1 \) and \( I_2 \), we have the convergence uniformly in \( r_n \) for \( r_n > 1/2 \), arguing as in [10], Proposition 2.6 and using at the end Jensen’s inequality for \( L^2(\Lambda) \subset L^2(r_n, \Lambda) \).

For \( I_2 \) note that the pointwise convergence of \( \Psi_{r_n}(x) = |x|^{r_n - 1} x \) to \( \Psi_{r_0}(x) = |x|^{r_0 - 1} x \) imply the convergence of the resolvent in \( \mathbb{R} \) and then we get the result arguing as in [15]. \( \square \)
3. Convergence of invariant measures

In this section, we shall present a result on convergence of invariant measures associated to equations (PL\(_p\)), (FD\(_r\)) respectively.

Let \(\{X^p_t(t)\}_{t \geq 0}\) be the variational solution associated to equation (PL\(_p\)) starting at \(x \in L^2(\Lambda)\). Similarly, let \(\{Y^p_t(t)\}_{t \geq 0}\) be the variational solution associated to equation (FD\(_r\)) starting at \(y \in H^{-1}(\Lambda)\).

Let
\[
P^n_t F(x) := \mathbb{E}[F(X^p_t(t))], \quad F \in C_b(L^2(\Lambda)), \quad t \geq 0,
\]
be the semigroup associated to equation (PL\(_p\)).

Let
\[
Q^n_t G(y) := \mathbb{E}[G(Y^p_t(t))], \quad G \in C_b(H^{-1}(\Lambda)), \quad t \geq 0,
\]
be the semigroup associated to equation (FD\(_r\)).

Recently, Liu and the second named author obtained the following result:

**Proposition 3.1.** Suppose that \(p \in \left(1 \vee \frac{2d}{2d+1}, 2\right], \quad r \in \left(0 \vee \frac{d-2}{d+2}, 1\right]\). Then \(\{P^n_t\}\) and \(\{Q^n_t\}\) are ergodic and admit unique invariant measures \(\mu_p, \nu_r\) respectively. It holds that \(\mu_p\) is supported by \(W^{1,p}_0(\Lambda)\) and \(\nu_r\) is supported by \(L^{r+1}(\Lambda)\).

\[
\int_{L^2(\Lambda)} \|x\|^{p_1}_1 \mu_p(dx) < +\infty,
\]
and
\[
\int_{H^{-1}(\Lambda)} \|y\|^{r+1}_1 \nu_r(dy) < +\infty.
\]

**Proof.** See [29] Propositions 3.2 and 3.4. \(\square\)

**Theorem 3.2.** (i) Let \(\{p_n\} \subset \left(1 \vee \frac{2d}{2d+1}, 2\right], \quad n \in \mathbb{N}, \quad p_0 \in \left(1 \vee \frac{2d}{2d+1}, 2\right]\) such that \(p_n \to p_0\). Set \(P^n_t := P^{p_n}_t, \quad P^0_t := P^{p_0}_t\).

Then the unique invariant measures \(\mu_n, \quad n \in \mathbb{N}, \quad \mu_0\) resp. associated to \(\{P^n_t\}, \quad n \in \mathbb{N}, \quad \{P^0_t\}\) converge in the weak sense, i.e.

\[
\lim_n \int_{L^2(\Lambda)} F(x) \mu_n(dx) = \int_{L^2(\Lambda)} F(x) \mu_0(dx) \quad \forall F \in C_b(L^2(\Lambda)).
\]

(ii) Let \(\{r_n\} \subset \left(0 \vee \frac{d-2}{d+2}, 1\right], \quad n \in \mathbb{N}, \quad r_0 \in \left(0 \vee \frac{d-2}{d+2}, 1\right]\) such that \(r_n \to r_0\). Set \(Q^n_t := Q^{r_n}_t, \quad Q^0_t := Q^{r_0}_t\).

Then the unique invariant measures \(\nu_n, \quad n \in \mathbb{N}, \quad \nu_0\) resp. associated to \(\{Q^n_t\}, \quad n \in \mathbb{N}, \quad \{Q^0_t\}\) converge in the weak sense, i.e.

\[
\lim_n \int_{L^2(\Lambda)} F(x) \nu_n(dx) = \int_{L^2(\Lambda)} F(x) \nu_0(dx) \quad \forall F \in C_b(L^2(\Lambda)).
\]

**Proof.** Let us prove (i) first. By Proposition 3.1 we see that \(\{P^n_t\}, \quad n \in \mathbb{N}, \quad \{P^0_t\}\) admit unique invariant measures \(\mu_n, \quad n \in \mathbb{N}, \quad \mu_0\) resp. Let \(p_1 := \inf_n p_n\). By the convergence \(p_n \to p_0, \quad p_1 \in \left(1 \vee \frac{2d}{2d+1}, 2\right]\) and the embedding \(W^{1,p_1}_0(\Lambda) \subset L^2(\Lambda)\) is compact.

Let \(\theta > 0\). Set
\[
K_\theta := \left\{x \in L^2(\Lambda) \mid \|x\|^{p_1}_{1,p_1} \leq \theta^{-1} + |\Lambda|\right\}.
\]
Clearly, \(K_\theta\) is compact in \(L^2(\Lambda)\). Now by (5.1),
\[
\mu_n(K_\theta) = \mu_n \left\{\|x\|^{p_1}_{1,p_1} \leq \theta^{-1} + |\Lambda|\right\} \leq \theta \int_{L^2(\Lambda)} \|x\|^{p_1}_{1,p_1} \mu_n(dx) \leq \theta \|B\|_{HS}^2.
\]
Hence the family of measures \( \{\mu_n\}_{n \in \mathbb{N}} \) is tight and has a weak accumulation point \( \tilde{\mu} \), i.e. \( \mu_n \to \tilde{\mu} \) weakly. By the Krylov–Bogoliubov theorem, for \( F \in C_b(L^2(\Lambda)) \),

\[
\int_{L^2(\Lambda)} F(x) \mu_n(dx) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T P_t^{\mu_n} F(x) dt
\]

\[
= \lim_{T \to +\infty} \frac{1}{T} \int_0^T (P_t^{\mu_n} F(x) - P_t^{\mu_0} F(x)) dt
\]

\[
+ \lim_{T \to +\infty} \frac{1}{T} \int_0^T P_t^{\mu_0} F(x) dt
\]

\[
= \varepsilon_k + \int_{L^2(\Lambda)} F(x) \mu_0(dx)
\]

By Theorem 2.1 and dominated convergence, \( \varepsilon_k \to 0 \) as \( k \to +\infty \) and hence

\[
\int_{L^2(\Lambda)} F(x) \tilde{\mu}(dx) = \int_{L^2(\Lambda)} F(x) \mu_0(dx).
\]

As a consequence, for the whole sequence, \( \mu_n \to \mu_0 \) weakly.

The proof for (ii) can be carried out by similar arguments. \( \square \)

4. The case \( p = 1 \)

For \( p = 1 \), the situation is more complicated. We would like to find a convex functional \( \Phi^1 \) such that the stochastic 1-Laplace equation

\[
(P_{L1}) \begin{cases}
\; dX_1(t) = \text{div} \left( \frac{\nabla X_1(t)}{|\nabla X_1(t)|} \right) dt + B \, dW(t) & \text{in } (0,T) \times \Lambda, \\
\; X_1(t) = 0 & \text{on } (0,T) \times \partial \Lambda, \\
\; X_1(0) = x & \text{in } \Lambda,
\end{cases}
\]

can be written as

\[
(4.1) \begin{cases}
\; dX_1(t) \in -\partial \Phi^1(X_1(t)) dt + B \, dW(t) & \text{in } (0,T), \\
\; X_1(0) = x,
\end{cases}
\]

where \( \partial \Phi^1 \) is the subdifferential of \( \Phi^1 \).

We shall need the spaces \( BV(\Lambda) \) and \( BV(\mathbb{R}^d) \). For \( f \in L^1_{\text{loc}}(\Lambda) \), define the total variation

\[
\|Df\|(\Lambda) = \sup \left\{ \int_\Lambda f \text{ div } \psi \, d\xi \ : \ \psi \in C^\infty_c(\Lambda; \mathbb{R}^d), \ |\psi| \leq 1 \right\}
\]

\( BV(\Lambda) \) is defined to be equal to \( \{f \in L^1(\Lambda) \mid \|Df\|(\Lambda) < \infty\} \). Denote the \( d \)-1-dimensional Hausdorff measure on \( \partial \Lambda \) by \( H^{d-1} \). For \( f \in BV(\Lambda) \) there is an element \( f^\Lambda \in L^1(\partial \Lambda, dH^{d-1}) \) called the trace such that

\[
\int_\Lambda f \text{ div } \psi \, d\xi = -\int_\Lambda \langle \psi, df \rangle + \int_{\partial \Lambda} \langle \psi, \nu \rangle f^\Lambda \, dH^{d-1} \ \forall \psi \in C^1(\Lambda; \mathbb{R}^d),
\]

where \([df]\) denotes the distributional gradient of \( f \) on \( \Lambda \) (which is a \( \mathbb{R}^d \)-valued Radon measure here) and \( \nu \) denotes the outer unit normal on \( \partial \Lambda \). \( BV(\mathbb{R}^d) \) is defined similarly by setting \( \Lambda = \mathbb{R}^d \). Define also \( \|Df\|(\mathbb{R}^d) \) in the above manner. Note that for \( f \in BV(\Lambda) \) (extended by zero outside \( \Lambda \)) it holds that \( f \in BV(\mathbb{R}^d) \) and that

\[
(4.2) \quad \|Df\|(\mathbb{R}^d) = \|Df\|(\Lambda) + \int_{\partial \Lambda} |f^\Lambda| \, dH^{d-1},
\]

cf. [I] Theorem 3.87.
Remark 4.1. By Ambrosio et al. [1] Corollary 3.49, if \( d \in \{1, 2\} \), then
\[
W^{1, 1}_0(\Lambda) \subset BV(\Lambda) \subset L^2(\Lambda)
\]
continuously. If \( d = 1 \), then
\[
BV(\Lambda) \subset L^2(\Lambda)
\]
compactly.

For further results in spaces of functions of bounded variation, we refer to [1] Ch. 3.

We shall return to equation (4.1). Recall that the subdifferential \( \partial \Phi^1 \) in \( L^2(\Lambda) \) is defined by \( \eta \in \partial \Phi^1(x) \) iff
\[
\Phi^1(x) - \Phi^1(y) \leq \int_\Lambda \eta(x - y) \, d\xi, \quad \forall y \in \text{dom} \Phi^1.
\]

One possible choice for \( \Phi^1 \) is the (homogeneous) energy
\[
\tilde{\Phi}(u) := \begin{cases} f(x)|\nabla u| \, d\xi, & \text{if } u \in W^{1, 1}_0(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus W^{1, 1}_0(\Lambda). \end{cases}
\]
In this case, if \( u \in W^{1, 1}_0(\Lambda) \), and if \( U := -\text{div}(\text{sgn}(\nabla u)) \subset L^2(\Lambda) \), then we have that \( u \in \text{dom} \tilde{\Phi} \) and \( U = \partial \tilde{\Phi}(u) \).

However, \( \tilde{\Phi} \) fails to be lower semi-continuous in \( L^2(\Lambda) \) which is a necessary ingredient for the theory. Therefore, it is convenient to consider its relaxed functional in \( L^2(\Lambda) \), which is equal to
\[
\Phi^1(u) := \begin{cases} \|Du\|_2(\mathbb{R}^d), & \text{if } u \in BV(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus BV(\Lambda), \end{cases}
\]
see equation (4.2) above. \( \Phi^1 \) is proper, convex and lower semi-continuous in \( L^2(\Lambda) \) and an extension of \( \tilde{\Phi} \) in the sense that \( \text{dom} \Phi^1 \supset \text{dom} \tilde{\Phi} \) and \( \Phi^1 \equiv \tilde{\Phi} \). Compare with [3] [21] [57] [40].

Following the approach of Barbu, Da Prato and Röckner [1], we shall give the definition of a solution for equations (PL_p), \( p \in [1, 2] \).

**Definition 4.2.** Set \( V_p := W^{1, p}_0(\Lambda), \, p \in (1, 2) \), \( V_1 := BV(\Lambda) \). Let \( \Phi^1 \) be defined as above. For \( p \in (1, 2) \), let
\[
\Phi^p(x) := \begin{cases} \frac{1}{p} \int_\Lambda |\nabla x|^p \, d\xi, & \text{if } u \in W^{1, p}_0(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus W^{1, p}_0(\Lambda). \end{cases}
\]
A stochastic process \( X = X^x \) with \( \mathbb{P} \)-a.s. continuous sample paths in \( H := L^2(\Lambda) \) is said to be a solution to equation (PL_p), \( p \in [1, 2] \) if
\[
X \in C_W([0, T]; H) \cap L^p((0, T) \times \Omega, V_p), \quad X(0) = x \in H
\]
and
\[
\frac{1}{2} \|X(t) - Y(t)\|_{L^2(\Lambda)}^2 + \int_0^t (\Phi^p(X(s)) - \Phi^p(Y(s))) \, ds \leq \frac{1}{2} \|x - Y(0)\|_{L^2(\Lambda)}^2 + \int_0^t (G(s), X(s) - Y(s))_{L^2(\Lambda)} \, ds, \quad t \in [0, T],
\]
for all \( G \in L^2_\mathbb{F}(0, T; H) \) and \( Y \in C_W([0, T]; H) \cap L^p((0, T) \times \Omega; V_p) \) satisfying the equation
\[
dY(t) + G(t) \, dt = B \, dW(t), \quad t \in [0, T].
\]
Suppose for a while that $1 < p < 2$, $d = 1, 2$. Arguing as in [31] Example 4.1.9, Theorem 4.2.4, we can easily prove existence and uniqueness of the solution $X_p$ for equation (PL$_p$), in the usual (strong) variational sense, as in Pardoux, Krylov, Rozovski [29,30]. We shall refer to Prévôt, Röckner [31, Definition 4.2.1]. By Itô’s formula, we see that $X_p$ is also a solution in the sense of the definition above.

Here, $W(t)$ is a cylindrical Wiener process on $L^2(\Lambda)$ of the form

$$W(t) = \sum_{n=1}^{\infty} \gamma_n(t)e_n, \quad t \geq 0,$$

where $\{\gamma_n\}$ is a sequence of mutually independent real Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and $\{e_n\}$ is an orthonormal basis of $L^2(\Lambda)$. We shall make further specifications. $BB^*$ is assumed to be a linear, continuous, non-negative, symmetric operator on $L^2(\Lambda)$ with eigenbasis $\{e_n\}$ and corresponding sequence of eigenvalues $\{\lambda_n\}$. Let $(-\Delta, \text{dom}(-\Delta))$ be the Dirichlet Laplacian in $L^2(\Lambda)$, in particular, $\text{dom}(-\Delta) = H^2(\Lambda) \cap H^1_0(\Lambda)$. Assume for simplicity that $\{e_n\}$ is an eigenbasis of $-\Delta$ with corresponding sequence of eigenvalues $\{\mu_n\}$. We shall assume that

$$\sum_{n=1}^{\infty} \lambda_n^{1+\kappa} \mu_n < \infty$$

for some $\kappa > 0$. For the situation considered in this paper, it is enough to set $Q := (-\Delta)^{-1-\delta}$ with $\delta > \frac{1}{2} + \kappa$ for $d = 1$ and $\delta > 1 + \kappa$ for $d = 2$.

Regarding equation (PL$_1$), well-posedness of the problem as well as existence and uniqueness of the solution were proved by Barbu, Da Prato and Röckner in [31].

**Remark 4.3.** Note that in [31], the space $BV_0(\Lambda)$ is introduced, consisting of $BV(\Lambda)$-functions with zero trace. They claim, however, that the energy

$$\Psi(u) := \begin{cases} \|Du\|_1(\Lambda), & \text{if } u \in BV_0(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus BV_0(\Lambda). \end{cases}$$

is lower semi-continuous which is not the case. Consider, for example, a sequence $u_n$ of trace zero Lipschitz functions on $\Lambda$ with $\|Du_n\|_1(\Lambda) = 1$ converging in $L^2(\Lambda)$ to $\mathbb{1}_\Lambda$. Then

$$\lim_{n} \Psi(u_n) = 1 < +\infty = \Psi(\mathbb{1}_\Lambda).$$

Fortunately, all results of [31] remain true, if one replaces $\Psi$ (denoted by $\Phi$ in their paper) by $\Phi^1$. We do not repeat the steps taken in the proof of [31] here, but note that for their existence and uniqueness result relies on an approximation $\{\Psi^\varepsilon\}$ of $\Psi$ which “does not see” the trace-term in (4.2), i.e. maps $L^2(\Lambda)$ functions on a joint subspace of $BV_0(\Lambda)$ and dom($\Phi^1$). In fact, $\{\Psi^\varepsilon\}$ is defined similarly to (4.1).

Other results of stochastic evolution variational inequalities can be found in [8,13,32,33,51].

We are now able to formulate the main result of this section.

**Theorem 4.4.** Let $d \in \{1, 2\}$. The sequence of solutions $\{X_p\}_p$ to equations (PL$_p$) is convergent for $p \to 1$ to the solution $X_1$ of equation (PL$_1$), strongly in $L^2(\Lambda)$, uniformly on $[0, T]$, $\mathbb{P}$-a.s., i.e.,

$$\lim_{p \to 1} \sup_{t \in [0, T]} \|X_p(t) - X_1(t)\|_{L^2(\Lambda)} = 0, \quad \mathbb{P} \text{-a.s.}$$

There is some evidence that the following conjecture is true, see [21,22,25].
Conjecture 4.5. Let \( d \in \{1, 2\} \). Then the semigroup

\[
P^t_F(x) := \mathbb{E}[F(X_1(t, x))], \quad F \in C_b(L^2(\Lambda)),
\]

admits a unique invariant measure \( \mu_1 \).

Theorem 4.6. Let \( d = 1 \). Suppose that Conjecture 4.5 is true. Let \( X_p = X_p(t, x) \) be the solution to equation \((PL_p)\), \( p \in \{1, 2\} \). Let \( \{p_n\} \subset (1, 2] \) such that \( \lim_n p_n = 1 \). Let

\[
P^t_p F(x) := \mathbb{E}[F(X_1^p(t))], \quad \varphi \in C_b(L^2(\Lambda)),
\]

be the semigroup associated to equation \((PL_p)\). Let \( \mu_{p_n}, n \in \mathbb{N}, \mu_1 \) be the associated unique invariant measures on \( L^2(\Lambda) \). Then

\[
\mu_{p_n} \to \mu_1 \quad \text{in the weak sense.}
\]

Proof. Note that by Remark 4.1, the embedding \( BV(\Lambda) \subset L^2(\Lambda) \) is compact. The proof is similar to that of Theorem 3.2, \( W^1_0(\Lambda) \) therein replaced by \( BV(\Lambda) \). \( \square \)

Proof of Theorem 4.4. For each \( \varepsilon > 0 \), let \( R_\varepsilon := (1 - \varepsilon \Delta)^{-1} \) be the resolvent of the Dirichlet Laplace operator \((-\Delta, \text{dom}(\Delta))\), where \( \text{dom}(\Delta) = H^1_0(\Lambda) \cap H^2(\Lambda) \).

For \( p \in \{1, 2\}, \varepsilon > 0 \), let

\[
\Phi^p_\varepsilon(u) := \int_\Lambda j^p_\varepsilon(\nabla R_\varepsilon u) d\xi, \quad u \in L^2(\Lambda).
\]

Lemma 4.7. Let \( \{p_n\} \subset (1, 2] \) such that \( \lim_n p_n = 1 \). Let \( \varepsilon > 0 \). Then for \( u \in L^2(\Lambda) \), we have that

\[
\lim_n \Phi^{p_n}_\varepsilon(u_n) = \Phi^1_\varepsilon(u).
\]

Furthermore, if \( u_n \rightharpoonup u \) converges weakly in \( L^2(\Lambda) \), we have that

\[
\lim_n \Phi^{p_n}_\varepsilon(u_n) \geq \Phi^1_\varepsilon(u).
\]

Also, each \( \Phi^p_\varepsilon, p \in \{1, 2\}, \varepsilon > 0 \), is continuous w.r.t. the weak topology of \( L^2(\Lambda) \).

Proof. Since \( R_\varepsilon \) maps to \( \text{dom}(\Delta) \subset H^1_0(\Lambda) \), it is clear that \( \nabla R_\varepsilon u \in L^2(\Lambda; \mathbb{R}^d) \) and hence \( \Phi^p_\varepsilon \) follows from (A.2).

Let \( u_n \in L^2(\Lambda), n \in \mathbb{N}, u \in L^2(\Lambda) \), such that \( u_n \rightharpoonup u \) weakly in \( L^2(\Lambda) \). If we can prove that \( \nabla R_\varepsilon u_n \rightharpoonup \nabla R_\varepsilon u \) weakly in \( L^2(\Lambda; \mathbb{R}^d) \), we can apply (A.3) and (4.7) follows. Indeed, we even have that \( \nabla R_\varepsilon u_n \to \nabla R_\varepsilon u \) strongly in \( L^2(\Lambda; \mathbb{R}^d) \).

The last part follows by repeating the compactness argument above and the strong \( L^2(\Lambda; \mathbb{R}^2) \)-continuity of the \( \Phi^p_\varepsilon \)'s. \( \square \)

We first consider the following approximating equations for \((PL_p)\)

\[
\begin{cases}
  dX^p_\varepsilon(t) + A^p_\varepsilon(X^p_\varepsilon(t)) dt = B dW(t) \\
  X^p_\varepsilon(0) = x
\end{cases}
\]

where for any \( u \in L^2(\Lambda) \),

\[
A^p_\varepsilon(u) = - (1 - \varepsilon \Delta)^{-1} \text{div} \left[ a^p_\varepsilon \left( \nabla (1 - \varepsilon \Delta)^{-1} u \right) \right]
\]

and \( a^p_\varepsilon \) is the Yosida approximation of \( a_p \) i.e., for any \( r \in \mathbb{R}^d \),

\[
a^p_\varepsilon(r) = \frac{1}{\varepsilon} \left( 1 - (1 + \varepsilon a_p)^{-1}(r) \right).
\]

In particular, for \( u, v \in L^2(\Lambda) \),

\[
(A^p_\varepsilon(u), v)_{L^2(\Lambda)} = \int_\Lambda \langle a^p_\varepsilon(\nabla R_\varepsilon u), \nabla R_\varepsilon(v) \rangle d\xi.
\]
We shall consider a similar approximation for equation (PL₁)

\( \begin{cases} dX_1^\varepsilon (t) + A^\varepsilon \left( X_1^\varepsilon \right) dt = B dW (t) \\ X_1^\varepsilon (0) = x \end{cases} \) (4.10)

where for any \( u \in L^2(\Lambda) \),

\( A^\varepsilon (u) = -(1 - \varepsilon \Delta)^{-1} \text{div} \left( \beta^\varepsilon \left( \nabla (1 - \varepsilon \Delta)^{-1} u \right) \right) \).

with

\[ \beta^\varepsilon (r) = \begin{cases} r, & \text{if } |r| \leq \varepsilon, \\ \frac{r}{|r|}, & \text{if } |r| > \varepsilon. \end{cases} \]

In particular, for \( u,v \in L^2(\Lambda) \),

\[ (A^\varepsilon (u), v)_{L^2(\Lambda)} = \int_\Lambda \langle \beta^\varepsilon (\nabla R_\varepsilon u), \nabla R_\varepsilon (v) \rangle d\xi. \]

Note that \( \beta^\varepsilon \) is the Yosida approximation of the sign function, i.e., for any \( r \in \mathbb{R}^d \),

\[ \beta^\varepsilon (r) = \frac{1}{\varepsilon} \left( 1 - (1 + \varepsilon \text{ sgn})^{-1} (r) \right). \]

In particular, \( \beta^\varepsilon = \nabla j^\varepsilon \), where \( j^\varepsilon \) is the convex function defined by

\[ j^\varepsilon (r) = \begin{cases} \frac{|r|^2}{2\varepsilon}, & \text{if } |r| \leq \varepsilon, \\ \frac{|r| - \varepsilon}{2}, & \text{if } |r| > \varepsilon. \end{cases} \]

We shall use the following strategy to prove the main result

\[ \|X_p (t) - X_1 (t)\|_{L^2(\Lambda)} \leq \|X_p (t) - X_1^\varepsilon (t)\|_{L^2(\Lambda)} + \|X_1^\varepsilon (t) - X_1^\varepsilon (0)\|_{L^2(\Lambda)} + \|X_1^\varepsilon (t) - X_1 (t)\|_{L^2(\Lambda)} \]

\[ \mathbb{P}\text{-a.s. and uniformly in } t \in [0, T]. \]

**Step I**

We note that, taking Remark 4.3 into account, the result of [11, equation (4.8)] remains valid in our case. Hence,

\[ \lim_{\varepsilon \to 0} \sup_{t \in [0, T]} \|X_1^\varepsilon (t) - X_1 (t)\|_{L^2(\Lambda)} = 0, \quad \mathbb{P}\text{-a.s.} \]

**Step II**

Note that we have proved above (proof of Theorem 2.1) that

\[ \lim_{\varepsilon \to 0} \sup_{t \in [0, T]} \|X_p (t) - X_1^\varepsilon (t)\|_{L^2(\Lambda)} = 0, \quad \mathbb{P}\text{-a.s. uniformly in } p \in (1, 2). \]

**Step III**

In order to complete the proof we still need to show that for all \( \varepsilon > 0 \) fixed we have

\[ \lim_{p \to 1} \sup_{t \in [0, T]} \|X_p^\varepsilon (t) - X_1^\varepsilon (t)\|_{L^2(\Lambda)} = 0, \quad \mathbb{P}\text{-a.s.} \]

To this aim, we consider the definition of the solution for equations

\[ \begin{cases} dX_p^\varepsilon (t) + A_p^\varepsilon \left( X_p^\varepsilon \right) dt = B dW (t) \\ X_p^\varepsilon (0) = x \end{cases} \]
as
\[ \frac{1}{2} \| X^\varepsilon_p(t) - Y(t) \|^2_{L^2(\Lambda)} + \int_0^t (\Phi^p_\varepsilon(X^\varepsilon_p(s)) - \Phi^p(\varepsilon(Y(s))) \, ds \]
\[ \leq \frac{1}{2} \| x - Y(0) \|^2_{L^2(\Lambda)} + \int_0^t (G(s), X^\varepsilon_p(s) - Y(s))_{L^2(\Lambda)} \, ds, \]
for all \( t \in [0, T] \), \( \mathbb{P} \)-a.s.

We take \( Y = X^\varepsilon_1 \), the solution of equation
\[ \begin{cases} 
  dX^\varepsilon_1(t) + A^\varepsilon(X^\varepsilon_1) \, dt = B \, dW(t) \\
  X^\varepsilon_1(0) = x. 
\end{cases} \]
and using the definition of the subdifferential we get that
\[(4.11) \quad \frac{1}{2} \| X^\varepsilon_p(t) - X^\varepsilon_1(t) \|^2_{L^2(\Lambda)} + \int_0^t (\Phi^p_\varepsilon(X^\varepsilon_p(s)) - \Phi^p_\varepsilon(X^\varepsilon_1(s)) + \Phi^1_\varepsilon(X^\varepsilon_1(s)) - \Phi^1_\varepsilon(X^\varepsilon_p(s))) \, ds \leq \frac{1}{2} \| x - X^\varepsilon_1(0) \|^2_{L^2(\Lambda)} = 0, \]
for \( t \in [0, T] \) and \( \mathbb{P} \)-a.s. By estimate (2.3), we can extract a subsequence \( \{p_n\} \) with \( \lim_{n} p_n = 1 \) such that for \( X^\varepsilon := X^\varepsilon_{p_n} \) we have that for \( dt \)-a.a. \( t \in [0, T] \), \( X^\varepsilon(t) \rightharpoonup Z^\varepsilon(t) \) weakly in \( L^2(\Lambda) \) \( \mathbb{P} \)-a.s. for some \( dt \otimes \mathbb{P} \)-measurable \( Z^\varepsilon \) that satisfies
\[ \sup_{t \in [0, T]} \| Z^\varepsilon(t) \|_{L^2(\Lambda)} \leq \lim_{n} \sup_{t \in [0, T]} \| X_n(t) \|_{L^2(\Lambda)} \quad \mathbb{P} \text{-a.s.} \]

We shall need following lemma. Set \( \Phi^\varepsilon := \Phi^p_{\varepsilon}. \)

**Lemma 4.8.**

\( \Phi^\varepsilon(X^\varepsilon_t(\cdot)) - \Phi^\varepsilon(X^\varepsilon_1(\cdot)) + \Phi^1_\varepsilon(X^\varepsilon_1(\cdot)) - \Phi^1_\varepsilon(X^\varepsilon_t(\cdot)) \)

is \( \mathbb{P} \)-a.s. bounded above by a function in \( L^\infty(0, T) \).

**Proof.** Set \( u := X^\varepsilon_n(\cdot), \ v := X^\varepsilon_1(\cdot) \). Recall that in our notation, \( R_\varepsilon := (1 - \varepsilon \Delta)^{-1} \).

Let us treat the term \( \Phi^1_\varepsilon(u) - \Phi^1_\varepsilon(v) \) first. By the definition of the subgradient it is bounded by \( (\nabla \Phi^1_\varepsilon(u), u - v)_{L^2(\Lambda)} \). But this term is equal to
\[ \int_\Lambda \langle \beta^\varepsilon(\nabla R_\varepsilon(u)), \nabla R_\varepsilon(u - v) \rangle \, d\xi. \]
Since \( |\beta^\varepsilon| \leq 1 \), we get that the latter is bounded by \( \| \nabla R_\varepsilon(u - v) \|_{L^2(\Lambda; \mathbb{R}^d)} \). By the proof of Lemma 4.7 \( \nabla R_\varepsilon \) is a bounded operator from \( L^2(\Lambda) \) to \( L^2(\Lambda; \mathbb{R}^d) \).

We get that
\[ \Phi^1_\varepsilon(X^\varepsilon_t(\cdot)) - \Phi^1_\varepsilon(X^\varepsilon_1(\cdot)) \leq C \sup_n \| X^\varepsilon_n(\cdot) \|_{L^2(\Lambda)} + C \| X^\varepsilon_1(\cdot) \|_{L^2(\Lambda)} \]
which is \( \mathbb{P} \)-a.s. in \( L^\infty(0, T) \) again by estimate (2.3).

We continue with the term \( \Phi^\varepsilon_p(u) - \Phi^\varepsilon_p(v) \). By the definition of the subgradient it is bounded by \( (\nabla \Phi^\varepsilon_p(v), u - v)_{L^2(\Lambda)} \), which is equal to
\[ \int_\Lambda \langle a^\varepsilon_p(\nabla R_\varepsilon(v)), \nabla R_\varepsilon(v - u) \rangle \, d\xi. \]
Noticing that \( r^\varepsilon \) is a contraction on \( \mathbb{R}^d \), we can use a similar estimate as in (2.4) to get that the latter is bounded by
\[ C + C \| \nabla R_\varepsilon(u) \|_{L^2(\Lambda; \mathbb{R}^d)} \| \nabla R_\varepsilon(v - u) \|_{L^2(\Lambda; \mathbb{R}^d)}. \]
Arguing as above, we see that this term is bounded by
\[ C + C \sup_n \|X_n^ε(\cdot)\|_{L^2(\mathcal{A})} \|X_1^ε(\cdot)\|_{L^2(\mathcal{A})} + C \|X_1^ε(\cdot)\|^2_{L^2(\mathcal{A})}, \]
which is \(\mathbb{P}\)-a.s. in \(L^\infty(0,T)\) by estimate (2.33).

We take the limit superior in (4.11) and continue investigating
\[
\lim_n \int_0^t [\Phi^ε_n(X_n^ε(t)) - \Phi^ε_n(X_n^ε(0)) + \Phi_1^ε(X_n^ε(s)) - \Phi_1^ε(X_1^ε(s))] \, ds.
\]
By Lemma A.3, we can apply (reverse) Fatou’s lemma such that it is sufficient to prove that
\[
\lim_n \left[ \Phi^ε_n(X_1^ε(s)) - \Phi^ε_n(X_n^ε(s)) + \Phi_1^ε(X_n^ε(s)) - \Phi_1^ε(X_1^ε(s)) \right] \leq 0
\]
\(\mathbb{P}\)-a.s. and for \(ds\)-a.e. \(s \in [0,T]\). At this point, we apply Lemma A.7 and get that
\[
\lim_n \left[ \Phi^ε_n(X_1^ε(s)) - \Phi^ε_n(X_n^ε(s)) + \Phi_1^ε(X_n^ε(s)) - \Phi_1^ε(X_1^ε(s)) \right]
\leq \lim_n \Phi^ε_n(X_1^ε(s)) - \lim_n \Phi^ε_n(X_n^ε(s)) + \lim_n \Phi_1^ε(X_n^ε(s)) - \Phi_1^ε(X_1^ε(s))
\leq \Phi_1^ε(X_1^ε(s)) - \Phi_1^ε(Z^ε(s)) + \Phi_1^ε(Z^ε(s)) - \Phi_1^ε(X_1^ε(s)) = 0,
\]
\(\mathbb{P}\)-a.s. and for \(ds\)-a.e. \(s \in [0,T]\).

Final step. Going back to
\[
\|X_p(t) - X_1(t)\|_{L^2(\mathcal{A})} \leq \|X_p(t) - X_p^ε(t)\|_{L^2(\mathcal{A})} + \|X_p^ε(t) - X_1^ε(t)\|_{L^2(\mathcal{A})} + \|X_1^ε(t) - X_1(t)\|_{L^2(\mathcal{A})}
\]
\(\mathbb{P}\)-a.s. and uniformly in \(t \in [0,T]\), we can complete the proof using Steps I–III as follows. Let \(\delta > 0\). Pick \(\varepsilon_0 > 0\), such that the first and the third term are less than \(\delta/3\). Having fixed \(\varepsilon_0\) in such a way, we can pick \(p\) such that the second term is less than \(\delta/3\). \(\square\)

**Appendix A. Some results on variational convergence**

Let \(H\) be a separable Hilbert space. For a proper, convex functional \(\Phi : H \to (-\infty, +\infty]\), the Legendre transform \(\Phi^*\) is defined by
\[
\Phi^*(y) := \sup_{x \in H} \langle x, y \rangle_H - \Phi(x), \quad y \in H.
\]
For two functionals \(F,G : H \to (-\infty, +\infty]\) the infimal convolution \(F\#G\) is defined by
\[
(F\#G)(y) := \inf_{x \in H} \langle F(x) + G(y - x) \rangle, \quad y \in H.
\]
For a proper, convex, l.s.c. functional \(\Phi : H \to (-\infty, +\infty]\), for each \(\varepsilon > 0\), define the Moreau-Yosida regularization
\[
\Phi_\varepsilon := \Phi \# \frac{1}{2\varepsilon} \|\cdot\|_H^2.
\]
\(\Phi_\varepsilon\) is a continuous convex function. Also, \(\lim_{\varepsilon \searrow 0} \Phi_\varepsilon = \Phi\) pointwise.

It holds that
\[
(\Phi_\varepsilon)^* = \Phi^* + \frac{\varepsilon}{2} \|\cdot\|_H^2.
\]
see e.g. [7] §2.2 and [4] Ch. 3.

Recall following definition.
Definition A.1 (Mosco convergence). Let $\Phi^n : H \to (-\infty, +\infty]$, $n \in \mathbb{N}$, $\Phi : H \to (-\infty, +\infty]$ be proper, convex, l.s.c. functionals. We say that $\Phi^n \xrightarrow{M} \Phi$ in the Mosco sense if

\begin{align*}
(\text{M1}) & \quad \forall x \in H \forall x_n \in H, n \in \mathbb{N}, x_n \rightharpoonup x \text{ weakly in } H : \lim_{n} \Phi^n(x_n) \geq \Phi(x). \\
(\text{M2}) & \quad \forall y \in H \exists y_n \in H, n \in \mathbb{N}, y_n \to y \text{ strongly in } H : \lim_{n} \Phi^n(y_n) \leq \Phi(y).
\end{align*}

We shall need following theorem.

Theorem A.2. Let $\Phi^n : H \to (-\infty, +\infty]$, $n \in \mathbb{N}$, $\Phi : H \to (-\infty, +\infty]$ be proper, convex, l.s.c. functionals. Then the following conditions are equivalent.

(i) $\Phi^n \xrightarrow{M} \Phi$.
(ii) $(\Phi^n)^* \xrightarrow{M} \Phi^*$.
(iii) $\forall \varepsilon > 0$, $\forall x \in H : \lim_n \Phi^n(x) = \Phi_\varepsilon(x)$.

Proof. See [4, Theorems 3.18 and 3.26].

Corollary A.3. Suppose that $\Phi^n \xrightarrow{M} \Phi$. Then for each $\varepsilon > 0$, $\Phi^n \xrightarrow{M} \Phi_\varepsilon$, too.

Proof. Suppose that $\Phi^n \xrightarrow{M} \Phi$. By Theorem A.2, $(\Phi^n)^* \xrightarrow{M} \Phi^*$, too.

If we can prove for each $\varepsilon > 0$ that $(\Phi^n)^* \xrightarrow{M} (\Phi_\varepsilon)^*$, we are done by Theorem A.2 (M2) in Definition A.1 follows easily, using equation (A.1) and (M1) for convex, l.s.c. functionals. Then the following conditions are equivalent.

\begin{align*}
(\text{A.1}) & \quad \forall \varepsilon > 0, \forall x \in H : \lim_n \Phi^n(x) = \Phi_\varepsilon(x).
\end{align*}

Proof. Straightforward from [35, Theorem 14.60].

A.1. The $L^p$-case. Let $p \in [1, 2]$. We define $j^p : \mathbb{R}^d \to \mathbb{R}$ by $j^p(x) := \frac{1}{p} |x|^p$. Obviously, if $p > 1$, each $j^p$ is a convex $C^1$-function. For $\varepsilon > 0$, let

\[ j^p_\varepsilon(x) := \inf_{y \in \mathbb{R}^d} \left[ j^p(y) + \frac{1}{2\varepsilon} |x - y|^2 \right] \]

be its regularization. For $u \in L^2(\Lambda; \mathbb{R}^d)$, set

\[ \Psi^p(u) := \int_\Lambda j^p(u) \, d\xi. \]

$\Psi^p$ is a continuous convex functional on $L^2(\Lambda; \mathbb{R}^d)$ for each $p \in [1, 2]$.

Lemma A.4. For $\varepsilon > 0$, let $\Psi^p_\varepsilon$ be the Moreau-Yosida regularization of $\Psi^p$ in $L^2(\Lambda; \mathbb{R}^d)$. Then

\[ \Psi^p_\varepsilon(v) = \int_\Lambda j^p_\varepsilon(v) \, d\xi \quad \forall v \in L^2(\Lambda; \mathbb{R}^d). \]

Proof. Straightforward from [35, Theorem 14.60].

We would like to prove a convergence result, which shall be useful later. See the appendix for the terminology. Compare also with [35].

Lemma A.5. Let $(p_n) \subset [1, 2]$, $p_0 \in [1, 2]$ such that $\lim_n p_n = p_0$. Then

$\Psi^{p_n} \xrightarrow{M} \Psi^{p_0}$ in the Mosco sense in $L^2(\Lambda; \mathbb{R}^d)$. 

Proof. Let us prove (M1) in Definition [A.1] first. Let \( u_n \in L^2(\Lambda; \mathbb{R}^d) \), \( n \in \mathbb{N} \), \( u \in L^2(\Lambda; \mathbb{R}^d) \) such that \( u_n \rightharpoonup u \) weakly in \( L^2(\Lambda; \mathbb{R}^d) \). W.l.o.g. \( \lim_n \Psi^{p_n}(u_n) < +\infty \). 

Extract a subsequence (also denoted by \( \{u_n\} \)) such that
\[
\lim_n \Psi^{p_n}(u_n) = \lim_n \Psi^{p_n}(u_n).
\]

Let \( v \in L^\infty(\Lambda; \mathbb{R}^d) \). Clearly,
\[
\lim_n \int_\Lambda \langle u_n, v \rangle \, d\xi = \int_\Lambda \langle u, v \rangle \, d\xi.
\]

Also, by Hölder’s inequality,
\[
\frac{1}{p_n} \left| \int_\Lambda \langle u_n, v \rangle \, d\xi \right|^{p_n} \leq \Psi^{p_n}(u_n) \times \left\{ \begin{array}{ll}
|\Lambda|^{p_n-1} \|v\|^{p_n}_{L^\infty(\Lambda; \mathbb{R}^d)}, & \text{if } p_0 = 1, \\
\left( \int_\Lambda |v|^{p_n/(p_0-1)} \, d\xi \right)^{p_n-1}, & \text{if } p_0 > 1,
\end{array} \right.
\]

(here \( |\Lambda| = \int_\Lambda d\xi \)). Upon taking the limit \( n \to \infty \), we get that
\[
\frac{1}{p_0} \left| \int_\Lambda \langle u, v \rangle \, d\xi \right|^{p_0} \leq \lim_n \Psi^{p_0}(u_n) \times \left\{ \begin{array}{ll}
\|v\|^{p_0}_{L^\infty(\Lambda; \mathbb{R}^d)}, & \text{if } p_0 = 1, \\
\left( \int_\Lambda |v|^{p_0/(p_0-1)} \, d\xi \right)^{p_0-1}, & \text{if } p_0 > 1.
\end{array} \right.
\]

Taking the supremum over all \( v \in L^\infty(\Lambda; \mathbb{R}^d) \) with \( \|v\|^{p_0/(p_0-1)}_{L^\infty(\Lambda; \mathbb{R}^d)} \leq 1 \) and using the l.s.c. property of the supremum, we get that
\[
\Psi^{p_0}(u) = \frac{1}{p_0} \int_\Lambda |u|^{p_0} \, d\xi \leq \lim_n \Psi^{p_0}(u_n).
\]

Since the same argument works for any subsequence of \( \{u_n\} \), we have proved (M1).

We are left to prove (M2) in Definition [A.1]. Let \( u \in L^2(\Lambda; \mathbb{R}^d) \). Clearly for a.e. \( \xi \in \Lambda \)
\[
\lim_n \frac{1}{p_n} |u(\xi)|^{p_n} = \frac{1}{p_0} |u(\xi)|^{p_0}.
\]

But for all \( p \in [1, 2] \),
\[
\frac{1}{p} |u|^p \leq 1_\Lambda + |u|^2 \in L^1(\Lambda).
\]

Hence an application of Lebesgue’s dominated convergence theorem yields
\[
\lim_n \Psi^{p_0}(u) = \Psi^{p_0}(u).
\]

(M2) is proved. \( \square \)

Theorem [A.2] Corollary [A.3] and Lemmas [A.4] [A.5] together give:

**Corollary A.6.** Let \( \{p_n\} \subset [1, 2] \) such that \( \lim_n p_n = 1 \). Let \( \varepsilon > 0 \). Then for \( u \in L^2(\Lambda; \mathbb{R}^d) \), we have that
\[
\tag{A.2}
\lim_n \int_\Lambda j^{p_n}_\varepsilon(u) \, d\xi = \int_\Lambda j^1_\varepsilon(u) \, d\xi.
\]

Furthermore, if \( u_n \rightharpoonup u \) converges weakly in \( L^2(\Lambda; \mathbb{R}^d) \), we have that
\[
\tag{A.3}
\lim_n \int_\Lambda j^{p_n}_\varepsilon(u_n) \, d\xi \geq \int_\Lambda j^1_\varepsilon(u) \, d\xi.
\]
References

[1] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Clarendon Press, Oxford University Press, 2000.

[2] F. Andreu, C. Ballester, V. Caselles, and J. M. Mazón, *The Dirichlet problem for the total variation flow*, J. Funct. Anal. **180** (2001), no. 2, 347–403.

[3] F. Andreu-Vaillo, V. Caselles, and J. M. Mazón, *Parabolic quasilinear equations minimizing linear growth functionals*, Birkhäuser, Basel, 2003.

[4] H. Attouch, *Variational convergence for functions and operators*, Pitman, Boston–London–Melbourne, 1984.

[5] H. Attouch and R. Cominetti, *L^p approximation of variational problems in L^1 and L^\infty*, Nonlinear Anal., Ser. A: Theory Methods **36** (1999), no. 3, 373–399.

[6] G. Aubert and P. Kornprobst, *Mathematical problems in image processing, partial differential equations and the calculus of variations*, 2nd ed., Applied mathematical sciences, vol. 147, Springer, Berlin-Heidelberg-New York, 2006.

[7] V. Barbu, *Analysis and control of nonlinear infinite dimensional systems*, Mathematics in science and engineering, vol. 190, Academic Press, Inc., 1993.

[8] V. Barbu and G. Da Prato, *The Neumann problem on unbounded domains of \(\mathbb{R}^d\) and stochastic variational inequalities*, Comm. Partial Differential Equations **30** (2005), no. 7–9, 1217–1248.

[9] V. Barbu and G. Da Prato, *Ergodicity for nonlinear stochastic equations in variational formulation*, Appl. Math. Optim. **53** (2006), no. 2, 121–139.

[10] V. Barbu and G. Da Prato, *Invariant measures and the Kolmogorov equation for the stochastic fast diffusion equation*, Stochastic Process. Appl. **120** (2010), no. 7, 1247–1266.

[11] V. Barbu, G. Da Prato, and M. Röckner, *Stochastic nonlinear diffusion equations with singular diffusivity*, SIAM J. Math. Anal. **41** (2009), no. 3, 1106–1120.

[12] V. Barbu, G. Da Prato, and M. Röckner, *Stochastic porous media equation and self-organized criticality*, Comm. Math. Phys. **285** (2009), no. 3, 901–923.

[13] A. Bensoussan and A. Rascanu, *Stochastic variational inequalities in infinite dimensional spaces*, Numer. Funct. Anal. Optim. **18** (1997), no. 1–2, 19–54.

[14] J. G. Berryman and C. J. Holland, *Stability of the separable solutions for fast diffusion*, Arch. Ration. Mech. Anal. **74** (1980), no. 4, 379–388.

[15] I. Ciotir, *A Trotter type result for the stochastic porous media equations*, Nonlinear Anal. **71** (2009), no. 11, 5606–5615.

[16] I. Ciotir, *A Trotter-type theorem for nonlinear stochastic equations in variational formulation and homogenization*, Differential and Integral Equations **24** (2011), no. 3–4, 371–388.

[17] I. Ciotir, *Convergence of solutions for the stochastic porous media equations and homogenization*, J. Evol. Equ. **11** (2011), no. 2, 339–370.

[18] G. Da Prato and J. Zabczyk, *Ergodicity for infinite dimensional systems*, London Mathematical Society lecture note series ; 229, Cambridge Univ. Press, 2003.

[19] E. Di Benedetto, *Degenerate parabolic equations*, Universitext, Springer, Berlin–Heidelberg–New York, 1993.

[20] J. I. Díaz, *Nonlinear partial differential equations and free boundaries: Elliptic equations*, Research Notes in Mathematics, vol. 106, Pitman Advanced Publ. Program, Boston, 1985.

[21] A. Es-Sarhir and M.-K. von Renesse, *Ergodicity of stochastic curve shortening flow in the plane*, to appear in SIAM J. Math. Anal. (2010), 14 pp., arXiv:1003.2074, http://www.mathematik.uni-muenchen.de/~renesse/Docs/esserhir_renesse.pdf

[22] B. Gess and J. M. Tölle, *Multi-valued, singular stochastic evolution inclusions*, Preprint (2011), 34 pp., http://arxiv.org/abs/1112.5672

[23] Y. Giga, Y. Kashima, and N. Yamazaki, *Local solvability of a constrained gradient system of total variation*, Abstr. Appl. Anal. **8** (2004), 651–682.

[24] B. Kawohl and F. Schuricht, *Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem*, Commun. Contemp. Math. **9** (2007), no. 4, 515–543.

[25] T. Komorowski, S. Peszat, and T. Szarek, *On ergodicity of some Markov processes*, Ann. Probab. **38** (2010), no. 4, 1401–1443.

[26] N. V. Krylov and B. L. Rozovskii, *Stochastic evolution equations*, Current problems in mathematics, vol. 14, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979, pp. 71–147.

[27] O. A. Ladyženskaja, *New equations for the description of the motions of viscous incompressible fluids, and global solvability for their boundary value problems. (Russian)*, Trudy Mat. Inst. Steklov **102** (1967), 85–104.

[28] W. Liu, *On the stochastic p-Laplace equation*, J. Math. Anal. Appl. **360** (2009), no. 2, 737–751.
[29] W. Liu and J. M. Tölle, Existence and uniqueness of invariant measures for stochastic evolution equations with weakly dissipative drifts, Elect. Comm. in Probab. 16 (2011), no. 40, 447–457.

[30] É. Pardoux, Équations aux dérivées partielles stochastiques de type monotone, Séminaire sur les Équations aux Dérivées Partielles (1974–1975), III, Exp. No. 2., Collège de France, Paris, 1975, p. 10.

[31] C. Prévôt and M. Röckner, A concise course on stochastic partial differential equations, Lecture Notes in Mathematics, vol. 1905, Springer-Verlag, Berlin–Heidelberg–New York, 2007.

[32] A. Răşcanu, Existence for a class of stochastic parabolic variational inequalities, Stochastics 5 (1981), no. 3, 201–239.

[33] A. Răşcanu, On some stochastic parabolic variational inequalities, Nonlinear Anal. 6 (1982), no. 1, 75–94.

[34] A. Răşcanu, Deterministic and stochastic differential equations in Hilbert spaces involving multivalued maximal monotone operators, Panamer. Math. J. 6 (1996), 83–119.

[35] R. T. Rockafellar and R. J.-B. Wets, Variational analysis, Grundlehren der mathematischen Wissenschaften, vol. 317, Springer-Verlag, Berlin–Heidelberg–New York, 1998.

[36] Ph. Rosenan, Fast and super fast diffusion processes, Phys. Rev. Lett. 74 (1995), no. 11, 7–14.

[37] F. Schuricht, An alternative derivation of the eigenvalue equation for the $1$-Laplace operator, Arch. Math. (Basel) 87 (2006), no. 6, 572–577.

[38] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. (4) 110 (1976), 353–372.

[39] J. M. Tölle, Variational convergence of nonlinear partial differential operators on varying Banach spaces, Ph.D. thesis, Universität Bielefeld, 2010, published online on BeSOn, Bielefeld University Library, URN (NBN): urn:nbn:de:hbz:361-16758.

[40] J. M. Tölle, Convergence of solutions to the $p$-Laplace evolution equation as $p$ goes to 1, Preprint (2011), 11 pp., http://arxiv.org/abs/1103.0229v2.

[41] J. L. Vázquez, Smoothing and decay estimates for nonlinear diffusion equations: equations of porous medium type, Oxford lecture series in mathematics and its applications, vol. 33, Oxford University Press, Oxford, 2006.