Stein factors for variance-gamma approximation in the Wasserstein and Kolmogorov distances

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Abstract

We obtain new bounds for the solution of the variance-gamma (VG) Stein equation that are of the correct form for approximations in terms of the Wasserstein and Kolmogorov metrics. These bounds hold for all parameters values of the four parameter VG class. As an application we obtain explicit Wasserstein and Kolmogorov distance error bounds in a six moment theorem for VG approximation of double Wiener-Itô integrals.

Keywords: Stein’s method; variance-gamma approximation; Stein factors; Wasserstein distance; Kolmogorov distance

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1 Introduction

The variance-gamma (VG) distribution with parameters $r > 0, \theta \in \mathbb{R}, \sigma > 0, \mu \in \mathbb{R}$ has probability density function

$$p(x) = \frac{1}{\sigma \sqrt{\pi} \Gamma(\frac{r}{2})} e^{\frac{\theta}{\sigma^2}(x-\mu)} \left( \frac{|x-\mu|}{2\sqrt{\theta^2 + \sigma^2}} \right)^{\frac{r-1}{2}} K_{\frac{r-1}{2}} \left( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} |x-\mu| \right),$$

with support $\mathbb{R}$. In the limit $\sigma \to 0$ the support becomes the region $(\mu, \infty)$ if $\theta > 0$, and is $(-\infty, \mu)$ if $\theta < 0$. Here $K_\nu(x)$ is a modified Bessel function of the second kind, defined in Appendix [A]. For a random variable $Z$ with density (1.1), we write $Z \sim \text{VG}(r, \theta, \sigma, \mu)$. Different parametrisations are given in [19] and the book [37], in which they refer to the distribution as the generalized Laplace distribution.

The VG distribution is widely used in financial modelling [43, 44]; an overview of this and other applications are given in [37]. The VG distribution also has a rich distributional theory (see Chapter 4 of [37] and [24]), and the class contains several classical distributions as special or limiting cases, such as the normal, gamma, Laplace, product of zero mean normals and difference of gammas (see Proposition 1.2 of [24] for a list of further cases).

Stein’s method [61] is a powerful and widely used approach for deriving quantitative limit theorems in probability. Originally developed for normal approximation, it has been

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or the product of two mean zero normal random variables. Distributions can be represented as the difference of two centered gamma random variables. Further VG approximations are given in [1, 5, 6, 7], in which the limiting theorem of [51] for normal approximation; see also [3] for a similar result for gamma. This result complements and in some senses generalises the celebrated optimal fourth moment VG approximation of double Wiener-Itô integrals. Recently, with the aid of results from this paper (Theorem 3.1 and Corollary 3.3), [4] have achieved a six moment for the VG approximation of double Wiener-Itô integrals. Recently, with the aid of results from [23, 24], they were able to obtain “six moment” theorems for the VG approximation of double Wiener-Itô integrals. The Malliavin-Stein method [47, 50] for VG approximation was developed by [21], and to the VG distribution by [23, 24], with subsequent technical advances made by [30].

Stein’s method was extended to several other distributions such as the Poisson [12], exponential [11, 54], gamma [33, 41] and Laplace [57]; for an overview see [39]. Stein’s method was extended to several other distributions such as the Poisson [12], exponential [11, 54], gamma [33, 41] and Laplace [57]; for an overview see [39]. Stein’s method was extended to several other distributions such as the Poisson [12], exponential [11, 54], gamma [33, 41] and Laplace [57]; for an overview see [39]. Stein’s method was extended to several other distributions such as the Poisson [12], exponential [11, 54], gamma [33, 41] and Laplace [57]; for an overview see [39].

The starting point of Stein’s method for VG approximation is the Stein equation [24]

\[ L_{r, \theta, \sigma, \mu} f(x) := \sigma^2 (x - \mu) f''(x) + (\sigma^2 r + 2 \theta (x - \mu)) f'(x) + (r \theta - (x - \mu)) f(x) = \tilde{h}(x), \quad (1.2) \]

where \( \tilde{h}(x) = h(x) - \mathbb{E} h(Z) \) for \( h : \mathbb{R} \to \mathbb{R} \) and \( Z \sim \text{VG}(r, \theta, \sigma, \mu) \), and is such that \( \mathbb{E} |h(Z)| < \infty \). Here \( L_{r, \theta, \sigma, \mu} \) is the VG Stein operator. Along with the Stein equations of [55] and [57], this was one of the first second order Stein equations in the literature. Let us now set \( \mu = 0 \); we recover the general case using the translation relation that if \( Z \sim \text{VG}(r, \theta, \sigma, \mu) \) then \( Z - \mu \sim \text{VG}(r, \theta, \sigma, 0) \). The solution to (1.2) is (see [24, Lemma 3.3] here we have used a different parametrisation for the VG distribution)

\[
\begin{align*}
 f_h(x) &= - \frac{e^{-\theta x/\sigma^2}}{\sigma^2 x^\nu} I_\nu \left( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} x \right) \int_0^x e^{\theta t/\sigma^2} |t|^\nu I_\nu \left( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} t \right) \tilde{h}(t) \, dt \\
 &\quad - \frac{e^{-\theta x/\sigma^2}}{\sigma^2 x^\nu} I_\nu \left( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} x \right) \int_x^\infty e^{\theta t/\sigma^2} |t|^\nu K_\nu \left( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} t \right) \tilde{h}(t) \, dt, \quad (1.3)
\end{align*}
\]

where \( \nu = \frac{r - 1}{2} \) and \( I_\nu(x) \) is a modified Bessel function of the first kind, defined in Appendix A. If \( h \) is bounded, then \( f_h(x) \) and \( f'_h(x) \) are bounded for all \( x \in \mathbb{R} \), and (1.3) is the unique bounded solution when \( r \geq 1 \), and the unique solution with bounded first derivative for \( r > 0 \) (see [23, Lemma 3.14]).

One may approximate a random variable of interest \( W \) by a VG random variable \( Z \sim \text{VG}(r, \theta, \sigma, 0) \) by evaluating both sides of (1.2) at \( W \), taking expectations and then taking the supremum of both sides over a class of functions \( \mathcal{H} \) to arrive at

\[
\sup_{h \in \mathcal{H}} |\mathbb{E} h(W) - \mathbb{E} h(Z)| = \sup_{h \in \mathcal{H}} \mathbb{E} |L_{r, \theta, \sigma, \mu} f_h(W)|.
\]

This is important because many standard probability metrics have a representation of the form \( \sup_{h \in \mathcal{H}} |\mathbb{E} h(W) - \mathbb{E} h(Z)| \). In particular, taking

\[
\begin{align*}
\mathcal{H}_K &= \{ \mathbf{1}(\cdot \leq z) \mid z \in \mathbb{R} \}, \\
\mathcal{H}_W &= \{ h : \mathbb{R} \to \mathbb{R} \mid h \text{ is Lipschitz, } \|h\| \leq 1 \}, \\
\mathcal{H}_{BW} &= \{ h : \mathbb{R} \to \mathbb{R} \mid h \text{ is Lipschitz, } \|h\| \leq 1 \text{ and } \|h'\| \leq 1 \}
\end{align*}
\]
yields the Kolmogorov, Wasserstein and bounded Wasserstein distances. We shall denote the Kolmogorov and Wasserstein distances by $d_K$ and $d_W$, respectively. Here, and throughout the paper, $\| \cdot \| := \| \cdot \|_\infty$ denotes the usual supremum norm of a real-valued function on $\mathbb{R}$.

In order for the above procedure to be effective, it is crucial to have suitable bounds on the solution (1.3), which are often referred to in the literature as Stein factors. This is technically demanding, due to the presence of modified Bessel functions in the solution together with the singularity at the origin in the Stein equation (1.2). The first bounds in the literature [24] (given for the case $\theta = 0$) resulted from a brute force approach that involved the writing of three papers on modified Bessel functions [25, 26, 27] and long calculations (given in Section 3.3 and Appendix D of the thesis [23]). A significant advance was later made by [17]. Their iterative approach reduced the problem of bounding derivatives of arbitrary order to bounding just the solution and its first derivative. Consequently, they were able to obtain bounds for derivatives of any order for the whole class of VG distributions. However, the dependence of the bounds of [17] on the test function $h$ meant that they were only suitable for approximation in metrics that are weaker than the Wasserstein and Kolmogorov metrics. A technical advance was made in the recent work [30] in which the iterative technique of [17] and new inequalities for integrals of modified Bessel functions [28] were used to obtain bounds suitable for Wasserstein and Kolmogorov distance error bounds in the case $\theta = 0$.

In this paper, we complement the work of [30] by obtaining analogous bounds for the whole class of VG distributions. Our results have been made possible due to a very recent work on inequalities for integrals of modified Bessel functions [32]. The bounds we establish have followed from a series of contributions to the problem of bounding derivatives of solutions of Stein equations together with technical results for modified Bessel functions spanning several papers, and the overall task of establishing such bounds for VG approximation has arguably been more demanding than for any other distribution for which this step of Stein’s method has been achieved. Like a number of other papers in the literature, for example [8, 9, 10, 14, 15, 16, 17, 38, 40, 42, 58], the main focus of this paper is to obtain new Stein factors, although we do present a simple application to complement the work of [21] on the Malliavin-Stein method for VG approximation. Here our work fixes a technical issue and provides explicit constants for their quantitative limit theorems, with Corollary 4.2 giving a quantitative sixth moment theorem for the VG approximation of double Wiener-Itô integrals, with explicit bounds in the Wasserstein and Kolmogorov distances. We give a demonstration of these general bounds by using them to obtain bounds on the rate of convergence, with respect to the Wasserstein and Kolmogorov distances, in one of the main results of the recent work [7, Theorem 2.4], which concerns the generalized Rosenblatt process at extreme critical exponent. Here the limiting distribution is a VG distribution with $\theta \neq 0$, highlighting the importance of our generalisation of the Stein factor bounds of [30] to the general $\theta \in \mathbb{R}$ case.

The rest of this paper is organised as follows. In Section 2 we present some basic properties of VG distributions that will be used in the paper. In Section 3 we obtain our new bounds for the solution of the VG Stein equation. We also provide a connection between Kolmogorov and Wasserstein distances between a general distribution and a
VG distribution. Our application to the Malliavin-Stein method for VG approximation is given in Section 4. Proofs of some technical results are given in Section 5. Appendix A lists some relevant basic properties and inequalities for modified Bessel functions. Appendix B provides a list of uniform bounds for expressions involving integrals of modified Bessel functions that we use in obtaining our bounds for the solution of the VG Stein equation.

2 The class of variance-gamma distributions

In this section, we present some basic properties of the class of variance-gamma (VG) distributions that will be useful in the remainder of the paper; for further properties, see [24] and Chapter 4 of [37].

The modified Bessel function in the probability density function (1.1) makes it difficult to parse on first inspection. We can gain some understanding from the following limiting forms. Applying the limiting form (A.52) to (1.1) gives that,

\[ p(x) \sim \frac{1}{2\pi(\theta^2 + \sigma^2)^{\frac{3}{2}}} |x|^{\frac{r-3}{4}} \exp \left( \frac{\theta^2 + \sigma^2}{2\sigma^2} |x - \mu| \right), \quad |x| \to \infty, \]

which is valid for all \( r > 0, \theta \in \mathbb{R}, \sigma > 0 \) and \( \mu \in \mathbb{R} \). Similarly, this time using the limiting form (A.51), we have that (see [23])

\[ p(x) \sim \begin{cases} \frac{1}{2\sigma\sqrt{\pi}(1 + \theta^2/\sigma^2)^{\frac{r-1}{2}}} \frac{\Gamma\left(\frac{r-1}{2}\right)}{\Gamma\left(\frac{r}{2}\right)}, & x \to \mu, \quad r > 1, \\ \frac{\pi}{\sigma} \log |x - \mu|, & x \to \mu, \quad r = 1, \\ \frac{1}{(2\sigma)^r\sqrt{\pi}} \frac{\Gamma\left(\frac{1-r}{2}\right)}{\Gamma\left(\frac{r}{2}\right)} |x - \mu|^{r-1}, & x \to \mu, \quad 0 < r < 1. \end{cases} \tag{2.4} \]

We see that the density has a singularity at \( x = \mu \) if \( r \leq 1 \). In fact, for all parameter values, the VG(\( r, \theta, \sigma, \mu \)) distribution is unimodal. The following properties of the mode \( M \) can be found in [31]. For \( 0 < r \leq 2 \), \( \theta \in \mathbb{R}, \sigma > 0 \), or \( r > 0, \theta = 0, \sigma > 0 \) the mode is equal to \( \mu \). Suppose now that \( r > 2, \theta \in \mathbb{R}, \sigma > 0 \). Then \( M = \mu + \text{sgn}(\theta) \cdot x^* \), where \( x^* \) is the unique positive solution of the equation

\[ K_{\frac{r-3}{2}} \left( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} x \right) = \frac{\theta}{\sqrt{\theta^2 + \sigma^2}} K_{\frac{r-1}{2}} \left( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} x \right). \tag{2.5} \]

For \( r > 2, \theta > 0, \sigma > 0 \), we have the two-sided inequality

\[ \theta(r-3)_+ < M - \mu < \theta(r-2), \tag{2.6} \]

with the inequality reversed for \( \theta < 0 \). Here \( x_+ = \max\{0, x\} \).

The following result is new and needed in the proof of Proposition 3.7. Both Proposition 2.1 and Proposition 3.7 are proved in Section 5.
Proposition 2.1. \( r > 2, \theta \in \mathbb{R}, \sigma > 0, \mu \in \mathbb{R} \) the VG\((r, \theta, \sigma, \mu)\) density \((1.1)\) can be bounded above for all \(x \in \mathbb{R}\) by
\[
p(x) \leq \frac{\Gamma\left(\frac{r-1}{2}\right)}{2\sigma\sqrt{\pi\Gamma\left(\frac{r}{2}\right)}} \left(\frac{\sigma^2}{\theta^2 + \sigma^2}\right)^{\frac{r-1}{2}} e^{\frac{\sigma^2}{\theta^2 + \sigma^2}(r-2)}.
\] (2.7)

For \( r > 3, \theta \neq 0, \sigma > 0, \mu \in \mathbb{R} \), the following bound improves on (2.7) for all \(x \in \mathbb{R}\):
\[
p(x) < \frac{1}{\sigma\sqrt{\pi\Gamma\left(\frac{r}{2}\right)}} e^{\frac{\sigma^2}{\theta^2 + \sigma^2}(r-2)} \left(\frac{\theta(r-3)}{2\theta^2 + \sigma^2}\right)^{\frac{r-1}{2}} K_{r-1} \left(\frac{\theta\sqrt{\theta^2 + \sigma^2}}{\sigma^2}(r-3)\right).
\] (2.8)

Remark 2.2. The proof of inequality (2.8) in Proposition 2.1 makes use of the two-sided inequality (2.6). An alternative lower bound for the mode of the VG\((r, \theta, \sigma, \mu)\) distribution is given in [31, Corollary 2.6], which improves on the lower bound of (2.6) for \( r > 4 \). Applying this inequality in the proof would lead to a more accurate bound than (2.8) for \( r > 4 \), but the resulting bound would be more complicated.

The mean and variance of \( Z \sim \text{VG}(r, \theta, \sigma, \mu) \) are given by (see [37])
\[
\mathbb{E}Z = \mu + r\theta, \quad \text{Var}(Z) = r(\sigma^2 + 2\theta^2).
\]

An application of the Cauchy-Schwarz inequality then yields
\[
\mathbb{E}|Z - \mu| \leq \sqrt{r(\sigma^2 + 2\theta^2) + r^2\theta^2}.
\] (2.9)

Let us write VG\(_c\)(\(r, \theta, \sigma\)) for VG\((r, \theta, \sigma, -r\theta)\), the VG distribution with zero mean. (The notation VG\(_c\)(\(r, \theta, \sigma\)) was introduced by [21], and whilst when introducing the notation they mention that this denotes the VG\((r, \theta, \sigma, 0)\) distribution, it is quite clear when studying their paper that they instead meant VG\((r, \theta, \sigma, -r\theta)\).) The following formulas for the cumulants of \( Y \sim \text{VG}_c(r, \theta, \sigma) \) [21, Lemma 3.6] will be used in the proof of Corollary 4.2
\[
\kappa_2(Y) = r(\sigma^2 + 2\theta^2), \quad \kappa_3(Y) = 2r\theta(3\sigma^2 + 4\theta^2), \quad \kappa_4(Y) = 6r(\sigma^4 + 8\sigma^2\theta^2 + 8\theta^4), \quad \kappa_5(Y) = 24r\theta(5\sigma^4 + 20\sigma^2\theta^2 + 16\theta^4), \quad \kappa_6(Y) = 120r(\sigma^2 + 2\theta^2)(\sigma^4 + 16\sigma^2\theta^2 + 16\theta^4).
\]

3 Bounds for the solution of the Stein equation

In this section, we establish new bounds for the solution \((1.3)\) of the VG\((r, \theta, \sigma, \mu)\) Stein equation \((1.2)\) that have the correct dependence on the test function \(h\) for the purposes of using Stein’s method to derive Wasserstein and Kolmogorov distance error bounds for VG approximation. Our bounds are valid for the entire parameter space \( r > 0, \theta \in \mathbb{R}, \sigma > 0 \) and \( \mu \in \mathbb{R} \).

We begin by stating two bounds from [17] (see inequalities (3.31) and (3.32) from that reference) that are the only bounds in the current literature that are of a suitable form for deriving Kolmogorov distance bounds for the whole class of VG distributions; no bounds in the literature are suitable for the purposes of obtaining Wasserstein distance
bounds for the entire VG class. We will use these bounds in our proof of Theorem 3.1. For bounded and measurable \( h: \mathbb{R} \to \mathbb{R} \),

\[
\| f \| \leq \frac{\| \tilde{h} \|}{\sqrt{\theta^2 + \sigma^2}} \left( \frac{2}{r} + A_{r, \theta, \sigma} \right), 
\]  
(3.10)

\[
\| f' \| \leq \frac{\| \tilde{h} \|}{\sigma^2} \left( \frac{2}{r} + A_{r, \theta, \sigma} \right),
\]  
(3.11)

where

\[
A_{r, \theta, \sigma} = \begin{cases} 
\frac{2\sqrt{\pi}}{\sqrt{2r-1}} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{r-1}{2}}, & r \geq 2, \\
12\Gamma \left( \frac{r}{2} \right) \left( 1 + \frac{\theta^2}{\sigma^2} \right), & 0 < r < 2.
\end{cases}
\]

Our presentation of inequalities (3.10) and (3.11) differs a little from that given in [17]. This is discussed in Remark 3.4.

Let us now state our main result. The theorem extends Theorem 3.1 of [30], which was given for the \( \theta = 0 \) case, to cover the entire class of VG distributions.

**Theorem 3.1.** Let \( f \) denote the solution (1.3) of the VG\((r, \theta, \sigma, \mu)\) Stein equation (1.2).

1. Suppose \( h: \mathbb{R} \to \mathbb{R} \) is bounded and measurable. Then

\[
\| (x - \mu)f(x) \| \leq \left( 1 + \frac{6}{r} + B_{r, \theta, \sigma} \right) \| \tilde{h} \|,
\]  
(3.12)

\[
\| (x - \mu)f'(x) \| \leq \frac{2\sqrt{\theta^2 + \sigma^2}}{\sigma^2} \left( 1 + \frac{6}{r} + B_{r, \theta, \sigma} \right) \| \tilde{h} \|,
\]  
(3.13)

\[
\| (x - \mu)f''(x) \| \leq \frac{1}{\sigma^2} \left\{ 5 + 2r A_{r, \theta, \sigma} + \left( \frac{4\theta^2}{\sigma^2} \right) \left[ 1 + \frac{6}{r} + B_{r, \theta, \sigma} \right] \right\} \| \tilde{h} \|,
\]  
(3.14)

where

\[
B_{r, \theta, \sigma} = \begin{cases} 
\sqrt{\frac{\pi(r-1)}{2}} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{r-1}{2}}, & r \geq 2, \\
2, & 0 < r < 2.
\end{cases}
\]

2. Suppose now that \( h: \mathbb{R} \to \mathbb{R} \) is Lipschitz. Then

\[
\| f \| \leq \left\{ 4 + \frac{2\sqrt{2}}{\sqrt{r}} + \sqrt{2\pi(r+1)} |\theta| \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{r-1}{2}} \right\} \| h' \|,
\]  
(3.15)

\[
\| f' \| \leq \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} C_{r, \theta, \sigma} \| h' \|,
\]  
(3.16)

\[
\| f'' \| \leq \frac{1}{\sigma^2} \left( \frac{2}{r + 1} + A_{r+1, \theta, \sigma} \right) \left\{ 1 + \left( 2 + \frac{\theta^2}{\sigma^2} \right) C_{r, \theta, \sigma} \right\} \| h' \|,
\]  
(3.17)

where

\[
C_{r, \theta, \sigma} = 6 + \frac{2\sqrt{2}}{\sqrt{r}} + 2\sqrt{2\pi(r+1)} |\theta| \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{r-1}{2}} + 2\left( \sqrt{2r + r} \right) A_{r, \theta, \sigma}.
\]  
(3.18)
We also have that
\[\| (x - \mu) f'(x) \| \leq \left( 1 + \frac{6}{r + 1} + B_{r+1, \theta, \sigma} \right) \left\{ 1 + \left( 2 + \frac{\theta^2}{\sigma^2} \right) C_{r, \theta, \sigma} \right\} \| h' \|,\] (3.19)
\[\| (x - \mu) f''(x) \| \leq \frac{2\sqrt{\theta^2 + \sigma^2}}{\sigma^2} \left( 1 + \frac{6}{r + 1} + B_{r+1, \theta, \sigma} \right) \left\{ 1 + \left( 2 + \frac{\theta^2}{\sigma^2} \right) C_{r, \theta, \sigma} \right\} \| h' \|,\] (3.20)
\[\| (x - \mu) f^{(3)}(x) \| \leq \frac{1}{\sigma^4} \left\{ 5 + 2(r + 1) A_{r+1, \theta, \sigma} + \left( 5 + \frac{4\theta^2}{\sigma^2} \right) \left( 1 + \frac{6}{r + 1} + B_{r+1, \theta, \sigma} \right) \right\} \times \left\{ 1 + \left( 2 + \frac{\theta^2}{\sigma^2} \right) C_{r, \theta, \sigma} \right\} \| h' \|.\] (3.21)

Remark 3.2. In the light of the singularity in the Stein equation (1.2) at \( x = \mu \), the factor of \( (x - \mu) \) in the left-hand sides of the estimates (3.12)–(3.14) and (3.19)–(3.21) is quite natural.

The bounds in Theorem 3.1 can be used together with the iterative technique of [17] to obtain bounds on higher order derivatives of the solution of the VG Stein equation. These bounds will necessarily involve higher order derivatives of the test function \( h \); see Proposition 3.6. An example is given in the following corollary, which improves on a bound for \( \| f^{(3)} \| \) given on page 18 of [17] by only depending on \( \| h'' \| \) and \( \| h' \| \) (the bound of [17] also had a term involving \( \| \tilde{h} \| \)).

Corollary 3.3. Let \( f \) denote the solution (1.3) of the VG(\( r, \theta, \sigma, \mu \)) Stein equation (1.2). Let \( h : \mathbb{R} \to \mathbb{R} \) be such that its first derivative \( h' \) is bounded and Lipschitz. Then
\[\| f^{(3)} \| \leq \frac{1}{\sigma^4} \left( 2 + A_{r+2, \theta, \sigma} \right) \left\{ 1 + \left( 2 + \frac{\theta^2}{\sigma^2} \right) C_{r+1, \theta, \sigma} \right\} \left\{ \| h'' \| + \sqrt{\theta^2 + \sigma^2} C_{r, \theta, \sigma} + |\theta| \left( 1 + \left( 2 + \frac{\theta^2}{\sigma^2} \right) C_{r, \theta, \sigma} \right) \right\} \| h' \|,\] where \( A_{r, \theta, \sigma} \) and \( C_{r, \theta, \sigma} \) are defined as in Theorem 3.1.

Proof of Theorem 3.1. In order to simplify the calculations, we make the following change of parameters
\[\nu = \frac{r - 1}{2}, \quad \alpha = \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2}, \quad \beta = \frac{\theta}{\sigma^2}.\] (3.22)
We also set \( \mu = 0 \); the bounds for the general case follow from a simple translation. With these parameters, the solution (1.3) can be written as
\[f(x) = -\frac{e^{-\beta x} K_{\nu}(\alpha |x|)}{\sigma^2 |x|^{\nu}} \int_0^x e^{\beta t} |t|^{\nu} I_{\nu}(\alpha |t|) \tilde{h}(t) \, dt - \frac{e^{-\beta x} I_{\nu}(\alpha |x|)}{\sigma^2 |x|^{\nu}} \int_x^\infty e^{\beta t} |t|^{\nu} K_{\nu}(\alpha |t|) \tilde{h}(t) \, dt\] (3.23)
By the triangle inequality,

\[
-\frac{e^{-\beta x}K_\nu(\alpha|x|)}{\sigma^2|x|^\nu} \int_0^x e^{\beta t} |t|^\nu I_\nu(\alpha|t|) \tilde{h}(t) \, dt + \frac{e^{-\beta x} I_\nu(\alpha|x|)}{\sigma^2|x|^\nu} \int_{-\infty}^x e^{\beta t} |t|^\nu K_\nu(\alpha|t|) \tilde{h}(t) \, dt.
\]  

(3.24)

We have equality between the different representations of the solution (3.23) and (3.24) because, letting \( Z \sim VG(r, \theta, \sigma, 0) \), we have that

\[
\int_{-\infty}^x e^{\beta t} |t|^\nu K_\nu(\alpha|t|) \tilde{h}(t) \, dt - \left( - \int_x^\infty e^{\beta t} |t|^\nu K_\nu(\alpha|t|) \tilde{h}(t) \, dt \right)
\]

\[
= \int_{-\infty}^\infty e^{\beta t} |t|^\nu K_\nu(\alpha|t|) \tilde{h}(t) \, dt = \frac{1}{\sigma^\nu \Gamma(\frac{\nu}{2})} \left( \frac{1}{2 \sqrt{\theta^2 + \sigma^2}} \right)^{\frac{r-1}{2}} \mathbb{E}[\tilde{h}(Z)] = 0,
\]

where in the penultimate step we recalled the change of parameters (3.22) to observe that \( e^{\beta t} |t|^\nu K_\nu(\alpha|t|) \) is the VG(r, \theta, \sigma, 0) density up to the normalising constant. This equality is useful because it means that to obtain uniform bounds for all \( x \in \mathbb{R} \) it is sufficient to bound the solution and its derivatives in the region \( x \geq 0 \), provided we consider both the cases of negative and positive \( \beta \). We shall therefore proceed by deriving bounds for \( x \geq 0 \), which must also hold for \( x \leq 0 \), and thus for all \( x \in \mathbb{R} \).

Suppose first that \( h : \mathbb{R} \to \mathbb{R} \) is Lipschitz. We begin by proving the bound for \( \|f\| \), which will be used in the derivation of some of the other bounds. The mean value theorem gives that \( |\tilde{h}(x)| \leq \|h'|(|x| + \mathbb{E}[Z]) \), where \( Z \sim VG(r, \theta, \sigma, 0) \). We recall from (2.9) that \( \mathbb{E}[Z] \leq \sqrt{r(\sigma^2 + 2\theta^2) + r^2\theta^2} \). Using these two inequalities, together with the integral inequalities (B.70), (B.71), (B.74) and (B.75), gives that, for \( x \geq 0 \),

\[
|f(x)| \leq \frac{\|h'\|}{\sigma^2} \left\{ \frac{e^{-\beta x} K_\nu(\alpha|x|)}{x^\nu} \int_0^x e^{\beta t} (t + \mathbb{E}[Z]) t^\nu I_\nu(\alpha t) \, dt + \frac{e^{-\beta x} I_\nu(\alpha|x|)}{x^\nu} \int_x^\infty e^{\beta t} (t + \mathbb{E}[Z]) t^\nu K_\nu(\alpha t) \, dt \right\}
\]

\[
\leq \frac{\|h'\|}{\sigma^2} \left\{ \frac{2\sigma^2}{\sqrt{\theta^2 + \sigma^2}} \mathbb{E}[Z] + \left( \frac{\sigma^4}{\theta^2 + \sigma^2} + \sqrt{2\pi} \theta \sigma \sqrt{r + 1} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{r-1}{2}} \right)^{\frac{r-1}{2}} + \frac{\sigma^2 A_{r,\theta,\sigma}}{\sqrt{\theta^2 + \sigma^2}} \mathbb{E}[Z] \right\}
\]

\[
\leq \frac{\|h'\|}{\sigma^2} \left\{ \frac{2\sigma^2}{\sqrt{\theta^2 + \sigma^2}} \sqrt{r(\sigma^2 + 2\theta^2) + r^2\theta^2} + \frac{\sigma^4}{\theta^2 + \sigma^2} + \sqrt{2\pi} \theta \sigma \sqrt{r + 1} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{r-1}{2}} + \frac{\sigma^2 A_{r,\theta,\sigma}}{\sqrt{\theta^2 + \sigma^2}} \sqrt{r(\sigma^2 + 2\theta^2) + r^2\theta^2} \right\}.
\]

(3.25)

By the triangle inequality,

\[
\frac{\sqrt{r(\sigma^2 + 2\theta^2) + r^2\theta^2}}{\sqrt{\theta^2 + \sigma^2}} < \frac{\sqrt{2r(\sigma^2 + \theta^2) + r|\theta|}}{\sqrt{\theta^2 + \sigma^2}} < \sqrt{2r + r},
\]

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and we also have that $\frac{x^4}{\theta + \sigma} \leq \sigma^2$. Therefore the bound in (3.25) simplifies to

$$|f(x)| \leq \left\{ 1 + \frac{2}{r} \left( \sqrt{2r} + r \right) + 1 + \sqrt{2\pi(r + 1)} \frac{\theta}{\sigma} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{1}{\sigma^2}} + \left( \sqrt{2r} + r \right) A_{r, \theta, \sigma} \right\} \|h'||$$

$$= \left\{ 4 + \frac{2\sqrt{2}}{\sqrt{r}} + \sqrt{2\pi(r + 1)} \frac{\theta}{\sigma} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{1}{\sigma^2}} + \left( \sqrt{2r} + r \right) A_{r, \theta, \sigma} \right\} \|h'||.$$

It was sufficient to deal with the case $x \geq 0$, and so we have proved inequality (3.15).

We now prove the bound for $\|f'||$. For $x \geq 0$, the first derivative of $f$ is given by

$$f'(x) = -\frac{1}{\sigma^2} \left[ \frac{d}{dx} \left( \frac{e^{-\beta x} K_{\nu}(\alpha x)}{x^\nu} \right) \right] \int_x^\infty e^{\beta t} t^{\nu} I_{\nu}(\alpha t) \tilde{h}(t) dt - \frac{1}{\sigma^2} \left[ \frac{d}{dx} \left( \frac{e^{-\beta x} I_{\nu}(\alpha x)}{x^\nu} \right) \right] \int_x^\infty e^{\beta t} t^{\nu} K_{\nu}(\alpha t) \tilde{h}(t) dt. \quad (3.26)$$

Therefore, using inequalities (B.85), (B.86), (B.87) and (B.90), and inequality (2.9), to bound $E|Z|$, we have that, for $x \geq 0$,

$$|f'(x)| = \frac{\|h'||}{\sigma^2} \left\{ \left| \frac{d}{dx} \left( \frac{e^{-\beta x} K_{\nu}(\alpha x)}{x^\nu} \right) \right| \int_x^\infty e^{\beta t} (t + E|Z|) t^{\nu} I_{\nu}(\alpha t) dt + \left| \frac{d}{dx} \left( \frac{e^{-\beta x} I_{\nu}(\alpha x)}{x^\nu} \right) \right| \int_x^\infty e^{\beta t} (t + E|Z|) t^{\nu} K_{\nu}(\alpha t) dt \right\} \leq \frac{\|h'||}{\sigma^2} \left\{ 2\sqrt{\theta^2 + \sigma^2} + \frac{2}{r} \sqrt{r(\sigma^2 + 2\theta^2) + r^2\theta^2} + \frac{2\sigma^2}{\sqrt{\theta^2 + \sigma^2}} \right.$$  

$$+ 2\sqrt{2\pi\theta} \sqrt{r + 1} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{1}{2}} + 2A_{r, \theta, \sigma} \sqrt{r(\sigma^2 + 2\theta^2) + r^2\theta^2} \right\}. \quad (3.27)$$

We bound the upper bound (3.27) similarly to how we bounded (3.25) in bounding $\|f\|$ to obtain the simpler bound

$$|f'(x)| \leq \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} \left\{ 2 + \frac{2}{r} (\sqrt{2r} + r) + 2 + 2 \sqrt{2\pi(r + 1)} \frac{\theta}{\sigma} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{1}{2}} + 2(\sqrt{2r} + r) A_{r, \theta, \sigma} \right\} \|h'||$$

$$= \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} C_{r, \theta, \sigma} \|h'||.$$

Again, it suffices to consider $x \geq 0$, so we have proved inequality (3.16).

To bound $\|f''\|$, we use the iterative technique of [17]. A detailed account of the technique is given on pages 4–5 of [17], and it was noted in that work that the technique is applicable to the VG Stein equation (1.2) [see their Assumption 2.1 and Remark 2.2].

We differentiate both sides of the VG($r, \theta, \sigma, 0$) Stein equation (1.2) and rearrange to obtain

$$\sigma^2 x f^{(3)}(x) + (\sigma^2 (r + 1) + 2\theta x) f''(x) + ((r + 1)\theta - x) f'(x) = h'(x) + f(x) + \theta f'(x). \quad (3.28)$$
We recognise (3.28) as the VG$(r+1, \theta, \sigma, 0)$ Stein equation, applied to $f'$, with test function $h'(x) + f(x) + \theta f'(x)$. Indeed, (3.28) can be written compactly as $L_{r+1, \theta, \sigma, 0} f'(x) = h'(x) + f(x) + \theta f'(x)$, where $L_{r+1, \theta, \sigma, 0}$ is the VG$(r+1, \theta, \sigma, 0)$ Stein operator. Here, the test function $h'(x) + f(x) + \theta f'(x)$ has mean zero with respect to the random variable $Y \sim$ VG$(r + 1, \theta, \sigma, 0)$. We will make use of this property when we later apply inequality (3.11). As $h$ is Lipschitz, it follows from inequalities (3.15) and (3.16) that $E|h'(Y) + f(Y) + \theta f'(Y)| < \infty$. In particular, because (3.28) is the VG$(r+1, \theta, \sigma, 0)$ Stein equation applied to $f'$, it follows that $E[L_{r+1, \theta, \sigma, 0} f'(Y)] = 0$, and so $E[h'(Y) + f(Y) + \theta f'(Y)] = 0$. An application of inequality (3.11), with $r$ replaced by $r + 1$ and test function $h'(x) + f(x) + \theta f'(x)$, now gives that

$$
\|f''\| = \frac{1}{\sigma^2} \left( \frac{2}{r + 1} + A_{r+1, \theta, \sigma} \right) \|h'(x) + f(x) + \theta f'(x)\| 
\leq \frac{1}{\sigma^2} \left( \frac{2}{r + 1} + A_{r+1, \theta, \sigma} \right) (\|h'\| + \|f\| + |\theta||\|f'\|).
$$

(3.29)

To obtain the bound (3.17) for $\|f''\|$, we use (3.15) and (3.16) to bound $\|f\|$ and $\|f'\|$, respectively, and simplify to obtain

$$
\|h'\| + \|f\| + |\theta||\|f'\| \leq \left\{ 1 + \left( 1 + \frac{|\theta|\sqrt{\theta^2 + \sigma^2}}{\sigma^2} \right) C_{r, \theta, \sigma} \right\} \|h'\| 
\leq \left\{ 1 + \left( 2 + \frac{\theta^2}{\sigma^2} \right) C_{r, \theta, \sigma} \right\} \|h'\|,
$$

(3.30)

where we used the inequality $\frac{|\theta|\sqrt{\theta^2 + \sigma^2}}{\sigma^2} < 1 + \frac{\theta^2}{\sigma^2}$, since $|\theta| < \sqrt{\theta^2 + \sigma^2}$. Combining inequalities (3.29) and (3.30) gives us the bound (3.17), as required.

Suppose now that $h : \mathbb{R} \to \mathbb{R}$ is bounded and measurable. We now prove the bounds (3.12)–(3.14). Using the integral inequalities (B.76) and (B.78), we obtain, for $x \geq 0$,

$$
|xf(x)| = \left| \frac{e^{-\beta x} K_v(\alpha x)}{\sigma^2 x^{\nu-1}} \int_0^x e^{\beta t} t^{\nu} I_\nu(\alpha t) \tilde{h}(t) \, dt + \frac{e^{-\beta x} I_\nu(\alpha x)}{\sigma^2 x^{\nu-1}} \int_x^\infty e^{\beta t} t^{\nu} K_\nu(\alpha t) \tilde{h}(t) \, dt \right|
\leq \frac{\|\tilde{h}\|}{\sigma^2} \left\{ \frac{e^{-\beta x} K_v(\alpha x)}{x^{\nu-1}} \int_0^x e^{\beta t} t^{\nu} I_\nu(\alpha t) \, dt + \frac{e^{-\beta x} I_\nu(\alpha x)}{x^{\nu-1}} \int_x^\infty e^{\beta t} t^{\nu} K_\nu(\alpha t) \, dt \right\}
\leq \frac{\|\tilde{h}\|}{\sigma^2} \left\{ \frac{(1 + \frac{6}{\sigma^2} + \sigma^2 B_{r, \theta, \sigma}}{r} \right\} = \|\tilde{h}\| \left( 1 + \frac{6}{\sigma^2} + B_{r, \theta, \sigma} \right).
$$

Using the formula (3.26) for the first derivative of $f$, followed by an application of the integral inequalities (B.88) and (B.89), gives that for $x \geq 0$,

$$
|xf'(x)| = \left| \frac{x}{\sigma^2} \left[ \frac{d}{dx} \left( \frac{e^{-\beta x} K_v(\alpha x)}{x^{\nu}} \right) \right] \int_0^x e^{\beta t} t^{\nu} I_\nu(\alpha t) \tilde{h}(t) \, dt + \frac{x}{\sigma^2} \left[ \frac{d}{dx} \left( \frac{e^{-\beta x} I_\nu(\alpha x)}{x^{\nu}} \right) \right] \int_x^\infty e^{\beta t} t^{\nu} K_\nu(\alpha t) \tilde{h}(t) \, dt \right|
$$

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and (3.13). We now obtain the bound for \( \|VG(x) \| \).

Again, it was sufficient to deal with the \( x \geq 0 \) case, and so we have proved (3.12) and (3.13). We now obtain the bound for \( \|xf''(x)\| \), and we begin by rearranging the VG\((r, \theta, \sigma, 0)\) Stein equation and using the triangle inequality to obtain that, for \( x \in \mathbb{R} \),

\[
|xf''(x)| = \frac{1}{\sigma^2} \left| \tilde{h}(x) - (\sigma^2 r + 2\theta x) f'(x) - (r \theta - x) f(x) \right|
\leq \frac{1}{\sigma^2} \|\tilde{h}\| + r \|f'\| + \frac{2|\theta|}{\sigma^2} \|xf'(x)\| + \frac{r|\theta|}{\sigma^2} \|f\| + \frac{1}{\sigma^2} \|xf(x)\|.
\]

We now use (3.11) to bound \( |f'| \), (3.13) to bound \( |xf'(x)| \), (3.10) to bound \( \|f\| \), and (3.12) to bound \( \|xf(x)\| \), which gives us the bound

\[
\|xf''(x)\| \leq \left\{ \begin{array}{l}
\frac{1}{\sigma^2} + r \cdot \frac{2}{\sigma^2} \left( \frac{2}{r} + A_{r, \theta, \sigma} \right) + \frac{2|\theta|}{\sigma^2} \cdot \frac{2\sqrt{\theta^2 + \sigma^2}}{\sigma^2} \left( 1 + \frac{6}{r} + B_{r, \theta, \sigma} \right) \\
\frac{r|\theta|}{\sigma^2} \cdot \frac{1}{\sqrt{\theta^2 + \sigma^2}} \left( \frac{2}{r} + A_{r, \theta, \sigma} \right) + \frac{1}{\sigma^2} \cdot \left( 1 + \frac{6}{r} + B_{r, \theta, \sigma} \right) \\
+ \frac{r}{\sigma^2} \left( \frac{2}{r} + A_{r, \theta, \sigma} \right) + \frac{1}{\sigma^2} \cdot \left( 1 + \frac{6}{r} + B_{r, \theta, \sigma} \right) \end{array} \right\} \|\tilde{h}\|
\]

\[
\leq \left\{ \begin{array}{l}
\frac{1}{\sigma^2} + r \cdot \frac{2}{\sigma^2} \left( \frac{2}{r} + A_{r, \theta, \sigma} \right) + \frac{4(\theta^2 + \sigma^2)}{\sigma^4} \left( 1 + \frac{6}{r} + B_{r, \theta, \sigma} \right) \\
+ \frac{r}{\sigma^2} \left( \frac{2}{r} + A_{r, \theta, \sigma} \right) + \frac{1}{\sigma^2} \cdot \left( 1 + \frac{6}{r} + B_{r, \theta, \sigma} \right) \end{array} \right\} \|\tilde{h}\|
\]

\[
= \frac{1}{\sigma^2} \left\{ 5 + 2rA_{r, \theta, \sigma} + \left( 5 + \frac{4\theta^2}{\sigma^2} \right) \left( 1 + \frac{6}{r} + B_{r, \theta, \sigma} \right) \right\} \|\tilde{h}\|.
\]

Finally, we bound \( \|x f'(x)\| \), \( \|xf''(x)\| \) and \( \|xf^{(3)}(x)\| \) for Lipschitz \( h \). We do so through a similar application of the iterative technique of [17] to the one we used to establish inequality (3.17). The setting is the same in that (3.28) is the VG\((r + 1, \theta, \sigma, 0)\) Stein equation, applied to \( f' \), with the test function \( h'(x) + f(x) + \theta f'(x) \) having zero mean with respect to the VG\((r + 1, \theta, \sigma, 0)\) measure. We apply inequalities (3.12), (3.13) and (3.14), respectively, with \( r \) replaced by \( r + 1 \) and test function \( h'(x) + f(x) + \theta f'(x) \), to obtain the bounds

\[
\|xf'(x)\| \leq \left( 1 + \frac{6}{r + 1} + B_{r + 1, \theta, \sigma} \right) \|h'(x) + f(x) + \theta f'(x)\|
\leq \left( 1 + \frac{6}{r + 1} + B_{r + 1, \theta, \sigma} \right) (\|h'\| + \|f\| + \|\theta\|\|f'\|),
\]

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\[\|xf''(x)\| \leq \frac{2\sqrt{\theta^2 + \sigma^2}}{\sigma^2} \left( 1 + \frac{6}{r+1} + B_{r+1,\theta,\sigma} \right) \left( \|h''\| + \|f\| + |\theta||f'| \right),\]

\[\|xf^{(3)}(x)\| \leq \frac{1}{\sigma^2} \left\{ 5 + 2(r+1)A_{r+1,\theta,\sigma} + \left( 5 + \frac{4\theta^2}{\sigma^2} \right) \left( 1 + \frac{6}{r+1} + B_{r+1,\theta,\sigma} \right) \right\} \times \left( \|h''\| + \|f\| + |\theta||f'| \right).\]

Using inequality (3.30) to bound \(\|h''\| + \|f\| + |\theta||f'| \) then yields the bounds (3.19)–(3.21). This completes the proof. \(\square\)

**Proof of Corollary 3.3.** As in the proof of Theorem 3.1, we set \(\mu = 0\). We obtain the bound by using a similar implementation of the iterative technique of [17] to those used in the proof of Theorem 3.1. Recall that (3.28) is the VG\((r + 1, \theta, \sigma, 0)\) Stein equation, applied to \(f'\), with the test function \(h'(x) + f(x) + \theta f'(x)\) having zero mean with respect to the VG\((r + 1, \theta, \sigma, 0)\) measure. By applying inequality (3.17) with \(r\) replaced by \(r + 1\) and test function \(h'(x) + f(x) + \theta f'(x)\), we obtain the bound

\[\|f^{(3)}\| \leq \frac{1}{\sigma^2} \left\{ \frac{2}{r+2} + A_{r+2,\theta,\sigma} \right\} \left\{ 1 + \left( 2 + \frac{\theta^2}{\sigma^2} \right) C_{r+1,\theta,\sigma} \right\} \|h''(x) + f'(x) + \theta f''(x)\| \]

\[\leq \frac{1}{\sigma^2} \left\{ \frac{2}{r+2} + A_{r+2,\theta,\sigma} \right\} \left\{ 1 + \left( 2 + \frac{\theta^2}{\sigma^2} \right) C_{r+1,\theta,\sigma} \right\} \left( \|h''\| + \|f'\| + |\theta||f''| \right).\]

Bounding \(\|f'\|\) and \(\|f''\|\) using inequalities (3.16) and (3.17), respectively, then yields the desired bound on \(\|f^{(3)}\|\). \(\square\)

**Remark 3.4.** Recall the change of parameters (3.22). The following bound (3.31) was given on p. 24 of [17], and the bound (3.32) is a slight improvement on one given on p. 24 of [17]:

\[\|f\| \leq \frac{\|\tilde{h}\|}{\sigma^2\alpha} \left( \frac{2}{2\nu + 1} + M_{\nu,\gamma} \right),\]  

(3.31)

\[\|f'\| \leq \frac{\|\tilde{h}\|}{\hat{\sigma}} \left( \frac{2}{2\nu + 1} + M_{\nu,\gamma} \right),\]  

(3.32)

where \(M_{\nu,\gamma}\) is defined in (3.70). The improvement in inequality (3.32) for \(\|f'\|\) comes from using the integral inequality (3.90), which improves on the analogous integral inequality that was used by [17] in deriving their bound for \(\|f'\|\). The bounds for \(\|f\|\) and \(\|f'\|\) of [17] were translated into the VG\((r, \theta, \sigma, \mu)\) parametrisation on p. 17 of [17] at the cost of two typos. By using the improved integral inequality (3.90), we have been able to fix one of the typos made by [17] (this concerns the factor \(\frac{2}{r}\) in the bound (3.11)). We correct the other typo by correctly bounding \(M_{\nu,\gamma} < A_{r,\theta,\sigma}\) (see (3.80)), which leads to different, corrected, bounds to those stated by [17]. In obtaining the inequality \(M_{\nu,\gamma} < A_{r,\theta,\sigma}\) in Appendix B we also obtained a slight simplification on the presentation given in [17] by using the upper bound in (3.81) to bound a ratio of gamma functions by a power function.

It is a natural question to ask whether bounds of the form \(\|f''\| \leq M_{r,\theta,\sigma}\|\tilde{h}\|\) and \(\|f^{(3)}\| \leq M_{r,\theta,\sigma}\|h''\|\), where \(M_{r,\theta,\sigma} > 0\) is a constant not involving \(x\), could be obtained
that hold for all bounded and measurable \( h : \mathbb{R} \to \mathbb{R} \), and all Lipschitz \( h : \mathbb{R} \to \mathbb{R} \), respectively. We will show that this is not possible through the following two propositions, which are proved in Section 5. Analogous results for the \( \theta = 0 \) case are given in [30]; our propositions show that no such bounds are attainable for any possible choice of parameter values in the four parameter VG class. We also refer the reader to [20] for similar results concerning solutions of Stein equations for a wide class of distributions.

**Proposition 3.5.** Denote by \( f_z \) the solution to the \( \text{VG}(r, \theta, \sigma, \mu) \) Stein equation (1.2) with test function \( h(x) = 1(x \leq z) \). Then \( f'_z(x) \) is discontinuous at \( x = \mu \).

**Proposition 3.6.** Let \( f \) denote the solution to the \( \text{VG}(r, \theta, \sigma, \mu) \) Stein equation with Lipschitz test function \( h : \mathbb{R} \to \mathbb{R} \). Then there does not exist a positive constant \( M_{r, \theta, \sigma} \) such that the bound \( \| f^{(3)} \| \leq M_{r, \theta, \sigma} \| h' \| \) holds for all Lipschitz \( h : \mathbb{R} \to \mathbb{R} \).

We end this section by stating the following proposition, which relates the Kolmogorov and Wasserstein distances between a general distribution and a VG distribution. The proof of Proposition 3.7 is postponed to Section 5. This is a useful result, because, for continuous target distributions, it is typically easier to obtain Wasserstein distance bounds via Stein’s method than Kolmogorov distance bounds. This is indeed the case in our application to the Malliavin-Stein method for VG approximation in Section 4.

**Proposition 3.7.** Let \( Z \sim \text{VG}(r, \theta, \sigma, \mu) \), where \( r > 0 \), \( \theta \in \mathbb{R} \), \( \sigma > 0 \) and \( \mu \in \mathbb{R} \). Let \( p_{r, \sigma, \theta}(x) \) denote the density (1.1) with \( \mu = 0 \). Then, for any random variable \( W \):

(i) If \( r > 1 \),

\[
d_K(W, Z) \leq D_{r, \sigma, \theta} \sqrt{d_W(W, Z)},
\]

where \( D_{r, \sigma, \theta} = \sup_{x \in \mathbb{R}} \sqrt{2p_{r, \sigma, \theta}(x)} \). When \( 1 < r \leq 2 \) we have

\[
D_{r, \sigma, \theta} = \sqrt{\frac{\Gamma\left(\frac{r-1}{2}\right)}{\sigma \sqrt{\pi \Gamma\left(\frac{1}{2}\right)}} \left(\frac{\sigma^2}{\theta^2 + \sigma^2}\right)^{\frac{r-1}{2}}},
\]

and when \( r > 2 \) we have

\[
D_{r, \sigma, \theta} = \sqrt{2p_{r, \sigma, \theta}(x^* \text{sgn}(\theta))} \leq \left\{ \frac{\Gamma\left(\frac{r-1}{2}\right)}{\sigma \sqrt{\pi \Gamma\left(\frac{1}{2}\right)}} \left(\frac{\sigma^2}{\theta^2 + \sigma^2}\right)^{\frac{r-1}{2}} e^{\frac{\sigma^2}{\sigma^2(r-2)}} \right\}^{\frac{1}{2}},
\]

where \( x^* \) is the unique positive solution of (2.5). In the case \( r > 3 \), \( \theta \neq 0 \), a more accurate bound on \( D_{r, \theta, \sigma} \) can be obtained by bounding \( p_{r, \sigma, \theta}(x^* \text{sgn}(\theta)) \) using inequality (2.8).

(ii) Let \( r = 1 \). Suppose that \( \frac{\sigma^2 + \sigma^2}{\sigma^2} d_W(W, Z) < 0.755 \). Then

\[
d_K(W, Z) \leq \left\{ 5 + \log \left(\frac{6}{\pi}\right) + \log \left(\frac{\sigma^3}{(\theta^2 + \sigma^2)d_W(W, Z)}\right) \right\} \sqrt{\frac{d_W(W, Z)}{6\pi \sigma}}.
\]

(iii) If \( 0 < r < 1 \),

\[
d_K(W, Z) \leq 2 \left(\frac{\Gamma\left(\frac{1-x}{2}\right)}{\sqrt{\pi 2^{x-1} \Gamma\left(\frac{x}{2}\right)}}\right)^{\frac{r+1}{r+1}} \left(\sigma^{-1} d_W(W, Z)\right)^{\frac{r}{r+1}}.
\]
Remark 3.8. (i) The assumption that \( \frac{\theta^2 + \sigma^2}{\sigma^2} d_W(W, Z) < 0.755 \) is quite mild. Indeed, if \( \theta = 0 \), then with \( d_W(W, Z)/\sigma = 0.755 \) we see that the upper bound in (3.33) is equal to 1.186, and thus uninformative. It is possible to increase the range of validity of inequality (3.33) (that is increase the numerical constant that \( \frac{\theta^2 + \sigma^2}{\sigma^2} d_W(W, Z) \) is bounded above by beyond 0.755) at the expense of larger numerical constants in the upper bound. This can be done by making a minor modification to derivation of inequality (3.33) by applying the more general part (ii) of Lemma 5.1 with \( c > 3 \), rather than part (iii) of that lemma with \( c = 3 \). We proceeded as we did to simplify the statement and proof of part (ii) of Proposition 3.7.

(ii) An analogue of Proposition 3.7 was given by [30, Proposition 4.1] for the \( \theta = 0 \) case. Our bounds for the general \( \theta \in \mathbb{R} \) case take the same functional form in terms of dependence on \( d_W(W, Z) \) as those of [30]. In fact, the bound (3.34), which does not involve \( \theta \), is exactly the same as that of [30] for \( 0 < r < 1 \) in the \( \theta = 0 \) case. In general, we expect our inequalities to yield suboptimal order Kolmogorov distance bounds. Indeed, an example has been given in the \( \theta = 0 \) case in which bounds for each of the cases \( r > 1 \), \( r = 1 \) and \( 0 < r < 1 \) are seen to suboptimal; see Remark 5.2 of [30].

4 Application to the Malliavin-Stein method for variance-gamma approximation

In this section, we obtain explicit constants in some of the main results of the paper [21] (see Theorem 4.1 and Corollary 4.2 below), which extended the Malliavin-Stein method to the VG distribution. In doing so, we fix a technical issue in that the Wasserstein distance bounds stated in [21] had only been proven in the weaker bounded Wasserstein distance. This is because at the time of [21] the only available bounds for the solution of the VG Stein equation [23, 24] had a dependence on the test function \( h \) that meant that this was the best that could be attained. We also give an illustrative example of the applicability of the general bound in Corollary 4.2 by obtaining bounds on the rate of convergence in a recent result of [7] concerning the generalized Rosenblatt process at extreme critical exponent.

We first introduce some notation; see the book [50] for further details. We write \( \mathbb{D}^{p,q} \) to denote the Banach space of all functions in \( L^q(\gamma) \), where \( \gamma \) is the standard Gaussian measure, whose Malliavin derivatives up to order \( p \) belong to \( L^q(\gamma) \). The class of infinitely many times Malliavin differentiable random variables is denoted by \( \mathbb{D}_\infty \). For a random variable \( F \in \mathbb{D}_\infty \), we iteratively define the gamma operators \( \Gamma_j \) by \( \Gamma_1(F) = F \) and, for \( j \geq 2 \),

\[
\Gamma_j(F) = \langle DF, -DL^{-1}\Gamma_{j-1}(F) \rangle_{S}.
\]

Here \( S \) is a real separable Hilbert space, \( D \) is the Malliavin derivative, and \( L^{-1} \) is the pseudo-inverse of the infinitesimal generator of the Ornstein-Uhlenbeck semi-group. Let \( S^{\odot 2} \) denote the second symmetric tensor product of \( S \). For \( f \in S^{\odot 2} \), the double Wiener-Itô integral is denoted by \( I_2(f) \) (see [50, Definition 2.7.1]). Some of the most important properties of multiple Wiener-Itô integrals are given in Section 2.7 of [50]. Double Wiener-Itô integrals also have several attractive properties and representations that are not shared
by higher order multiple Wiener-Itô integrals; see [50, Section 2.7.4]. Recall that we write \( \text{VG}_c(r, \theta, \sigma) \) for \( \text{VG}(r, \theta, \sigma, -r\theta) \).

**Theorem 4.1.** Let \( F \in \mathbb{D}^{3,8} \) and suppose that \( \Gamma_3(F) \) is square-integrable and \( EF = 0 \). Then, for \( Z \sim \text{VG}_c(r, \theta, \sigma) \),

\[
d_{W}(F, Z) \leq C_1 \left( \mathbb{E}[\sigma^2(F^2+r\theta)+2\theta\Gamma_2(F)-\Gamma_3(F)] \right)^{1/2} + C_2 \mathbb{E}[r(\sigma^2+2\theta^2)-\mathbb{E}[\Gamma_2(F)]], \tag{4.35}
\]

where

\[
C_1 = \frac{1}{\sigma^2} \left( \frac{2}{r+1} + A_{r+1, \theta, \sigma} \right) \left( 1 + \left( 2 + \frac{\theta^2}{\sigma^2} \right) C_{r, \theta, \sigma} \right), \quad C_2 = \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} C_{r, \theta, \sigma},
\]

with \( C_{r, \theta, \sigma} \) defined as in (5.18).

**Proof.** It was shown in the proof of Theorem 4.1 of [21] that, for functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) that are twice differentiable with bounded first and second derivative,

\[
\left| \mathbb{E}[\sigma^2 F f''(F) + \sigma^2 r f'(F) - F f(F)] \right| = \left| \mathbb{E} \left[ f''(F)(\sigma^2(F + r\theta) + 2\theta\Gamma_2(F) - \Gamma_3(F)) + f'(F)(r\sigma^2 + 2r\theta^2 - \mathbb{E}[\Gamma_2(F)]) \right] \right|
\]

\[
\leq ||f''|| \mathbb{E}[\sigma^2(F + r\theta) + 2\theta\Gamma_2(F) - \Gamma_3(F)] + ||f'|| \mathbb{E}[r(\sigma^2 + 2\theta^2) - \mathbb{E}[\Gamma_2(F)]]
\]

\[
\leq ||f''|| \left( \mathbb{E}[(\sigma^2(F + r\theta) + 2\theta\Gamma_2(F) - \Gamma_3(F))^2] \right)^{1/2} + ||f'|| \mathbb{E}[r(\sigma^2 + 2\theta^2) - \mathbb{E}[\Gamma_2(F)]],
\]

where a justification of the application of the Cauchy-Schwarz inequality in the final step is given in [21]. We know from Theorem 3.1 that, for \( h \in \mathcal{H}_W \), the solution \( f \) of the \( \text{VG}_c(r, \theta, \sigma) \) Stein equation satisfies the conditions of being twice differentiable with bounded first and second derivatives. We can bound \( ||f''|| \) and \( ||f'|| \) using the estimates (3.17) and (3.16) of Theorem 3.1 (with \( \|h'\| = 1 \)), which gives (4.35). \[ \square \]

**Corollary 4.2.** Consider the sequence \( (F_n = I_2(f_n) : n \geq 1) \) with \( f_n \in \mathbb{D}^{\sigma^2}, n \geq 1 \). Let \( Z \sim \text{VG}_c(r, \theta, \sigma) \) and write \( \tilde{\kappa}_i(F_n) := \kappa_i(F_n) - \kappa_i(Z), i = 2, 3, 4, 5, 6 \). Then

\[
d_{W}(F_n, Z) \leq C_1 \left( \frac{1}{120} \kappa_6(F_n) - \frac{\theta}{6} \kappa_5(F_n) + \frac{1}{3}(2\theta^2 - \sigma^2)\kappa_4(F_n) + (2 - r)\theta \sigma \kappa_3(F_n)
\]

\[
+ \frac{1}{4}(\kappa_3(F_n))^2 - 2\theta \kappa_2(F_n)\kappa_3(F_n) + (\sigma^4 + 4\theta^2\sigma^2)\kappa_2(F_n)
\]

\[
+ 4\theta^2(\kappa_2(F_n))^2 + r^2\theta^2\sigma^4 \right)^{1/2} + C_2 \mathbb{E}[r(\sigma^2 + 2\theta^2) - \kappa_2(F_n)], \tag{4.36}
\]

\[
\leq C_1 \left( \frac{1}{\sqrt{120}} \sqrt{\kappa_6(F_n)} + \frac{\sqrt{\theta}}{\sqrt{6}} \sqrt{\kappa_5(F_n)} + \frac{\sqrt{2\theta^2 - \sigma^2}}{\sqrt{3}} \sqrt{\kappa_4(F_n)}
\]

\[
+ \sigma \sqrt{(2 - r)\theta} \sqrt{\kappa_3(F_n)} + \frac{1}{2} \sqrt{\kappa_3^2(F_n)} + \sigma \sqrt{\sigma^2 + 4\theta^2} \sqrt{\kappa_2(F_n)}
\]

\[
+ 2\sqrt{|(2 - r)\theta|} \sqrt{\kappa_2(F_n)} + 2\theta \sqrt{\kappa_2(F_n)} \kappa_3(F_n) - \kappa_2(Z) \kappa_3(Z) \right) + C_2 \mathbb{E}[\tilde{\kappa}_2(F_n)], \tag{4.37}
\]
Proof. It is well-known that $\mathbb{E}[\Gamma_2(F_n)] = \kappa_2(F_n)$ (see [48]), and the equality

$$\mathbb{E}[(\sigma^2(F_n + r\theta) + 2\theta\Gamma_2(F_n) - \Gamma_3(F_n))^2]$$

$$= \frac{1}{120} \kappa_6(F_n) - \frac{\theta}{6} \kappa_5(F_n) + \frac{1}{3} (2\theta^2 - \sigma^2) \kappa_4(F_n) + (2 - r)\theta \sigma^2 \kappa_3(F_n)$$

$$+ \frac{1}{4} (\kappa_3(F_n))^2 - 2\theta \kappa_2(F_n) \kappa_3(F_n) + (\sigma^4 + 4r\theta^2 \sigma^2) \kappa_2(F_n) + 4\theta^2 (\kappa_2(F_n))^2 + r^2 \theta^2 \sigma^4$$

$$=: G_{r,\theta,\sigma}(F_n)$$

was shown in the proof of Theorem 5.8 of [21]. Substituting these formulas into (4.36) gives us (4.37).

A simple calculation using the VG$_c(r,\theta,\sigma)$ cumulant formulas given at the end of Section 2 gives that

$$G_{r,\theta,\sigma}(F_n) = \frac{1}{120} \tilde{\kappa}_6(F_n) - \frac{\theta}{6} \tilde{\kappa}_5(F_n) + \frac{1}{3} (2\theta^2 - \sigma^2) \tilde{\kappa}_4(F_n) + (2 - r)\theta \sigma^2 \tilde{\kappa}_3(F_n)$$

$$- 2\theta (\kappa_2(F_n) \kappa_3(F_n) - \kappa_2(Z) \kappa_3(Z)) + (\sigma^4 + 4r\theta^2 \sigma^2) \tilde{\kappa}_2(F_n)$$

$$+ 4\theta^2 (\tilde{\kappa}_2(F_n))^2.$$

Plugging this formula into the upper bound (4.36), using that $\kappa_2(Z) = r(\sigma^2 + 2\theta^2)$, and then using the triangle inequality gives us (4.37). \qed

Remark 4.3. We expect that, for any $r > 0$, our bound on $d_K(F_n, Z)$ will be of sub-optimal order. It is not possible to easily adapt the proof of Theorem 4.1 to obtain Kolmogorov distance bounds with the same rate of convergence as the Wasserstein distance bounds (4.35) and (4.36). This is because the first derivative of the solution $f_z$ of the VG$_c(r,\theta,\sigma)$ Stein equation with test function $h_z(x) = 1(x \leq z)$ has a discontinuity (see Proposition 3.3). This is in contrast to the case of normal approximation, for which bounds on the first derivative of the solution of the normal Stein equation suffice, and optimal order Kolmogorov distance bounds have been obtained [21].

Remark 4.4. Consider the smooth Wasserstein distance $d_{\mathcal{H}_2}(F, G)$ between the distributions of two random elements $F$ and $G$, defined by

$$d_{\mathcal{H}_2}(F, G) := \sup_{h \in \mathcal{H}_2} |\mathbb{E}h(F) - \mathbb{E}h(G)|,$$

where $\mathcal{H}_2 = \{h : \mathbb{R} \to \mathbb{R} | h' \text{ is Lipschitz}, \|h'\| \leq 1, \|h''\| \leq 1\}$ (see [2, 18]). Note that $d_{\mathcal{H}_2}(F, G) \leq d_W(F, G)$ for any random elements $F$ and $G$ such that $d_W(F, G)$ is well-defined. Let $F_n$ and $Z$ be defined as in Corollary 4.2. In addition, define

$$M(F_n) = \max\{||\tilde{\kappa}_i(F_n)|| : i = 2, 3, 4, 5, 6\}.$$
Recently, \cite{4} have obtained the following rather beautiful VG approximation with optimal rate of convergence: There exist constants $K_1, K_2 > 0$ only depending on $r, \theta$ and $\sigma$ such that

$$K_1 \mathcal{M}(F_n) \leq d_{\mathcal{H}_2}(F_n, Z) \leq K_2 \mathcal{M}(F_n).$$

(4.38)

The upper bound in (4.38) improves the bound of Corollary 4.2 by removing the square root factor. This improvement comes at the expense of being given with respect to the weaker $d_{\mathcal{H}_2}$ metric. As part of their proof, \cite{4} utilised bounds from Theorem 3.1 and Corollary 3.3. In the light of Proposition 3.6, it seems that a quite different approach to the one used by \cite{4} would be needed to achieve a bound of the form $d_W(F_n, Z) \leq K \mathcal{M}(F_n)$, assuming such a result holds.

A number of special and limiting cases of VG distributions are given in Proposition 1.2 of \cite{24}, and Corollary 4.2 can be specialised to these cases. We note two illustrative examples.

**Example 4.5.** The VG$_c(r, 0, \sigma/\sqrt{r})$ distribution converges to the $N(0, \sigma^2)$ distribution as $r \to \infty$. It is readily seen that $\lim_{r \to \infty} A_{r,0,\sigma/\sqrt{r}} = 0$ and $\lim_{r \to \infty} C_{r,0,\sigma/\sqrt{r}} = 6$. In this limit, we have that $C_1 = 2(1 + 2 \cdot 6) = 26$. Let $F_n = I_2(f_n)$ and suppose that $\mathbb{E}[F_n^2] = \sigma^2$. Then, with $Z \sim N(0, \sigma^2)$, we obtain from (4.37) the bound

$$d_W(F_n, Z) \leq 26 \left( \frac{1}{\sqrt{120}} \sqrt{\kappa_6(F_n)} + \frac{\sigma}{\sqrt{3}} \sqrt{\kappa_4(F_n)} + \frac{1}{2} |\kappa_3(F_n)| \right).$$

As expected, given its derivation from a general theorem for VG approximation, this result is weaker than the quantitative Gaussian fourth moment theorem of \cite{47}. It is worth noting that, for $F_n = I_2(f_n)$ and $Z \sim N(0, 1)$, we have that, for $k \geq 3$, $|\mathbb{E}[F_n^k] - \mathbb{E}[Z^k]| \leq c_k \sqrt{\mathbb{E}[F_n]} - 3$, where $c_k > 0$ is an explicit constant depending only on $k$ (see \cite{47}). Therefore, for $k = 3, 6$, $\kappa_k(F_n) \leq c_k \sqrt{\mathbb{E}[F_n]} - 3$ (for some $c_k > 0$), which is consistent with the famous condition of \cite{52} that convergence in distribution of a sequence of random variables, with zero mean and unit variance, living in a Wiener chaos of fixed order to the standard Gaussian distribution occurs if and only if the sequence of fourth moments converges to that of a $N(0, 1)$ random variable.

**Example 4.6.** The VG$_c(2, 0, \sigma)$ distribution corresponds to the Laplace$(0, \sigma)$ distribution with density $p(x) = \frac{1}{2\sigma} e^{-|x|/\sigma}$, $x \in \mathbb{R}$. We have that $A_{2,0,\sigma} = \frac{2\sigma}{\sqrt{3}}$ and $C_{2,0,\sigma} = 8 + \frac{16\sigma}{\sqrt{3}}$. In this case, we have $C_1 = \frac{134.978 \ldots}{\sigma^2} < \frac{135}{\sigma^2}$. Let $F_n = I_2(f_n)$ be such that $\mathbb{E}[F_n^2] = 2\sigma^2$. Then, with $Z \sim \text{Laplace}(0, \sigma)$, we obtain from (4.37) the bound

$$d_W(F_n, Z) \leq \frac{135}{\sigma^2} \left( \frac{1}{\sqrt{120}} \sqrt{\kappa_6(F_n)} + \frac{\sigma}{\sqrt{3}} \sqrt{\kappa_4(F_n)} + \frac{1}{2} |\kappa_3(F_n)| \right).$$

We end this section by demonstrating how Corollary 4.2 can be used to obtain bounds on the rate of convergence in a recent result of \cite{7}.

**Example 4.7** (The generalized Rosenblatt process at extreme critical exponent). Consider the Rosenblatt process $Z_{\gamma_1,\gamma_2}(t)$, introduced by \cite{43} as the double Wiener-Itô integral

$$Z_{\gamma_1,\gamma_2}(t) = \int_{\mathbb{R}^2} \left( \int_0^t (s - x_1)^{\gamma_1}(s - x_2)^{\gamma_2} \, ds \right) dB_{x_1} dB_{x_2},$$

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where the prime \( t \) indicates exclusion of the diagonals \( x_1 = x_2 \) in the stochastic integral, \( B_x \) is standard Brownian motion and \( \gamma_i \in (−1, −\frac{1}{2}) \), \( i = 1, 2 \), and \( \gamma_1 + \gamma_2 > −\frac{3}{2} \). The Rosenblatt process [62] is the special case \( Z_{\gamma}(t) = Z_{\gamma,\gamma}(t) \), \( −\frac{4}{3} < \gamma < −\frac{1}{2} \). By a change of variables and using the scale invariant property of Brownian motion it can be shown that

\[
Z_{\gamma_1,\gamma_2}(t) = t^{2+\gamma_1+\gamma_2} \int_{\mathbb{R}^2} \left( \int_0^1 (s-x_1)^{\gamma_1}(s-x_2)^{\gamma_2} \, ds \right) dB_{x_1} dB_{x_2},
\]

and so \( Z_{\gamma_1,\gamma_2}(t) \equiv t^{2+\gamma_1+\gamma_2} Z_{\gamma_1,\gamma_2}(1) \). From now on, for simplicity, we will work with the random variable \( Z_{\gamma_1,\gamma_2}(1) \); results for the general \( t > 0 \) case can be inferred from a rescaling. For \( \rho \in (0,1) \), define the random variable \( Y_{\rho} \) by

\[
Y_{\rho} = \frac{a_{\rho}}{\sqrt{2}}(X_1 - 1) - \frac{b_{\rho}}{\sqrt{2}}(X_2 - 1),
\]

where \( X_1 \) and \( X_2 \) are independent \( \chi^2_{(1)} \) random variables and

\[
a_{\rho} = \frac{(2\sqrt{\rho})^{-1} + (\rho + 1)^{-1}}{\sqrt{(2\rho)^{-1} + 2(\rho + 1)^{-2}}}, \quad b_{\rho} = \frac{(2\sqrt{\rho})^{-1} - (\rho + 1)^{-1}}{\sqrt{(2\rho)^{-1} + 2(\rho + 1)^{-2}}}.
\]

We follow [4] and suppose for simplicity that \( \gamma_1 \geq \gamma_2 \) and that \( \gamma_2 = (\gamma_1 + \frac{1}{2})/\rho - \frac{1}{2} \).

It was recently shown by [4] that, as \( \gamma_1 \to -\frac{1}{2} \),

\[
d_{W_2}(Z_{\gamma_1,\gamma_2}(1), Y_{\rho}) \leq C_{\rho} \sqrt{−\gamma_1 − \frac{1}{2}}, \quad (4.39)
\]

where \( C_\rho > 0 \) is a constant depending solely on \( \rho \) and \( d_{W_2} \) is the Wasserstein-2 distance. (Note that if \( \gamma_1 \to −\frac{1}{2} \), then automatically \( \gamma_2 \to −\frac{1}{2} \).) Working with respect to the weaker \( d_{H_2} \) metric, [4] have very recently obtained a faster rate of convergence: as \( \gamma_1 \to −\frac{1}{2} \),

\[
d_{H_2}(Z_{\gamma_1,\gamma_2}(1), Y_{\rho}) \leq C_{\rho} \left| −\gamma_1 − \frac{1}{2} \right| \quad (4.40)
\]

With these results, [4] and [4] have given bounds on the rate of convergence in a recent limit theorem of [4, Theorem 2.4]. To obtain the bound \( (4.39) \), [4] showed that, for any \( m \geq 2 \), as \( \gamma_1 \to −\frac{1}{2} \),

\[
\kappa_m(Z_{\gamma_1,\gamma_2}(1)) = \kappa_m(Y_{\rho}) + O \left( −\gamma − \frac{1}{2} \right), \quad (4.40)
\]

and inserted this asymptotic relation into a general Wasserstein-2 distance bound (expressed in terms of cumulants) in which the limit distribution can be represented as linear combinations of centered chi-square random variables. (The statement of the asymptotic relation \( (4.40) \) in [4] is only given for \( m \geq 3 \), but on inspecting their proof it can be seen that the asymptotic relation is also valid in the case \( m = 2 \).) Similarly, [4], substituted \( (4.40) \) into the upper bound of \( (4.38) \). As the Wasserstein-2 metric is stronger than the Wasserstein metric, it is immediate that, as \( \gamma_1 \to −\frac{1}{2} \),

\[
d_W(Z_{\gamma_1,\gamma_2}(1), Y_{\rho}) \leq C_{\rho} \sqrt{−\gamma_1 − \frac{1}{2}}, \quad (4.41)
\]
We now show that we can apply bound (4.37) of Corollary 4.2 to obtain an alternative proof of (4.41). This is a weaker result than that of [1], but the example is useful in demonstrating the applicability of Corollary 4.2. We also apply Proposition 3.7 to obtain a bound on the rate of convergence in the Kolmogorov distance, which is a new result.

We first recognise $Y_\rho$ as a VG random variable. Let $\Gamma(r, \lambda)$ denote a gamma random variable with density $p(x) = \frac{1}{\Gamma(r)} x^{r-1} e^{-\lambda x}, x > 0$. Then, Proposition 1.2 of [24] tells us that if $G_1 \sim \Gamma(r, \lambda_1)$ and $G_2 \sim \Gamma(r, \lambda_2)$ are independent, then $G_1 - G_2 \sim \text{VG}(r, (2\lambda_1)^{-1} - (2\lambda_2)^{-1}, (\lambda_1\lambda_2)^{-1/2}, 0)$. Observe that $a_\rho > 0$, $b_\rho > 0$ and that, since $\chi^2(1) = \Gamma(\frac{1}{2}, \frac{1}{2})$, we have that $\frac{a_{\sqrt{2}}}{{\sqrt{2}}} X_1 \sim \Gamma(\frac{1}{2}, \frac{1}{2a_{\rho}})$ and $\frac{b_{\sqrt{2}}}{{\sqrt{2}}} X_2 \sim \Gamma(\frac{1}{2}, \frac{1}{2b_{\rho}})$. Therefore, $Y_\rho \sim \text{VG}_c(1, \frac{a_{\sqrt{2}}}{\sqrt{2}}, \frac{b_{\sqrt{2}}}{\sqrt{2}})$. As $Z_{\gamma_1, \gamma_2}(1)$ is a double Wiener-Itô integral (and hence also satisfies $\mathbb{E}[Z_{\gamma_1, \gamma_2}(1)] = 0$), we may apply bound (4.37) of Corollary 4.2 together with the asymptotic relation (4.40) to obtain (4.41). By part (ii) of Proposition 3.7 (note that here $r = 1$) we can then obtain that, as $\gamma_1 \to \frac{1}{2}$,

$$d_K(Z_{\gamma_1, \gamma_2}(1), Y_\rho) \leq C'_\rho \left( -\gamma_1 - \frac{1}{2} \right)^{\frac{r}{2}} \log \left( \frac{1}{-\gamma_1 - \frac{1}{2}} \right),$$

where $C'_\rho > 0$ depends only on $\rho$.

5 Further proofs

Proof of Proposition 3.7. We first prove inequality (2.7). We prove the result for the case $\theta \neq 0$ and then treat the case $\theta = 0$. Let $r > 2$ and $\sigma > 0$. We prove the result for $\mu = 0$; the extension to general $\mu \in \mathbb{R}$ is obvious. We also fix $\theta > 0$; the case $\theta < 0$ is very similar because, for $Z \sim \text{VG}(r, \theta, \sigma, 0)$, we have that $-Z \sim \text{VG}(r, -\theta, \sigma, 0)$. Recall that, for $x > 0$, the VG$(r, \theta, \sigma, 0)$ density is given by

$$p(x) = \frac{1}{\sigma \sqrt{\pi} \Gamma(\frac{r}{2}) \left( \frac{\sigma^2}{2(\theta^2 + \sigma^2)} \right)^{\frac{r}{2}}} u(x)v(x),$$

where

$$u(x) = e^{\frac{\theta^2}{2} x}, \quad v(x) = \left( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma} x \right)^{\frac{r-1}{2}} K_{\frac{r-1}{2}} \left( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} x \right).$$

It suffices to consider $x > 0$, because the mode of the VG$(r, \theta, \sigma, 0)$ distribution is strictly positive for $\theta > 0$. By the upper bound in (2.6), we know that $M < \theta(r-2)$. Now, $u(x)$ is a strictly increasing function of $x$ on $(0, \infty)$, and $v(x)$ is a strictly decreasing function of $x$ on $(0, \infty)$ (see (A.54)). Therefore, for all $x > 0$,

$$p(x) \leq \frac{1}{\sigma \sqrt{\pi} \Gamma(\frac{r}{2}) \left( \frac{\sigma^2}{2(\theta^2 + \sigma^2)} \right)^{\frac{r}{2}}} u(\theta(r-2)) \lim_{x \to 0} v(x),$$

where $\lim_{x \to 0} v(x)$ can be calculated using the limiting form (A.51). As it suffices to consider $x > 0$ and $\theta > 0$, the proof of inequality (2.7) is complete for the case $\theta \neq 0$. To
extend the range of validity of inequality (2.7) to \( \theta \in \mathbb{R} \), we use that in the \( \theta = 0 \) case the mode of the VG\( (r, \theta, \sigma, 0) \) distribution is 0. Letting \( x \to 0 \) in the VG density using (A.51) gives that, for \( r > 2, \theta = 0, \sigma > 0 \),

\[
p(x) \leq \lim_{x \to 0} p(x) = \frac{\Gamma\left(\frac{r-1}{2}\right)}{2\sigma\sqrt{\pi}\Gamma\left(\frac{r}{2}\right)},
\]

which verifies that inequality (2.7) is also valid for \( \theta = 0 \).

We now set \( z = 0 \). On evaluating \( u(\theta(r-2))v(\theta(r-3)) \), we obtain the upper bound in (2.8). As it sufficed to consider \( x > 0 \) and \( \theta > 0 \), this completes the proof of inequality (2.8).

Finally, the assertion that inequality (2.7) is less accurate than (2.8) for \( r > 3, \theta \neq 0 \) follows because \( v(x) \) is a strictly decreasing function of \( x \) on \((0, \infty)\).

\( \square \)

**Proof of Proposition 2.3.** To simplify the notation we set \( \mu = 0 \); the general case follows from a simple translation. To further simplify the notation, we shall work with the change of parameters (3.22) and set \( \alpha = 1 \), so that \( |\beta| < 1 \), with the general \( \alpha > 0 \) case following from rescaling. With this change of parameters, the solution of the VG\( (r, \theta, \sigma, 0) \) Stein equation with test function \( h_z(x) = 1(x \leq z) \) is given by

\[
f_z(x) = -\frac{e^{-\beta x}K_\nu(|x|)}{\sigma^2|x|^\nu} \int_0^x e^{\beta t}|t|^\nu I_\nu(|t|)[1(t \leq z) - \mathbb{P}(Z \leq z)] \, dt - \frac{e^{-\beta x}I_\nu(|x|)}{\sigma^2|x|^\nu} \int_x^\infty e^{\beta t}|t|^\nu K_\nu(|t|)[1(t \leq z) - \mathbb{P}(Z \leq z)] \, dt.
\] (5.42)

We now set \( z = 0 \). Differentiating (5.42) using the formulas (A.55) and (A.56) gives us

\[
f'_z(x) = \frac{e^{-\beta x}}{\sigma^2} \left( \beta \frac{K_\nu(|x|)}{|x|^\nu} + \frac{K_{\nu+1}(|x|)}{|x|^\nu} \text{sgn}(x) \right) \int_0^x e^{\beta t}|t|^\nu I_\nu(|t|)[1(t \leq 0) - \mathbb{P}(Z \leq 0)] \, dt
+ \frac{e^{-\beta x}}{\sigma^2} \left( \beta \frac{I_\nu(|x|)}{|x|^\nu} - \frac{I_{\nu+1}(|x|)}{|x|^\nu} \text{sgn}(x) \right) \int_x^\infty e^{\beta t}|t|^\nu K_\nu(|t|)[1(t \leq 0) - \mathbb{P}(Z \leq 0)] \, dt.
\]

We have that, for all \( \nu > -\frac{1}{2} \) and \(-1 < \beta < 1\),

\[
\lim_{x \to 0} \left[ \frac{e^{-\beta x}K_\nu(|x|)}{|x|^\nu} \int_0^x e^{\beta t}|t|^\nu I_\nu(|t|)[1(t \leq 0) - \mathbb{P}(Z \leq 0)] \, dt \right] = 0,
\]

\[
\lim_{x \to 0} \left[ \frac{e^{-\beta x}I_{\nu+1}(|x|)}{|x|^\nu} \int_x^\infty e^{\beta t}|t|^\nu K_\nu(|t|)[1(t \leq 0) - \mathbb{P}(Z \leq 0)] \, dt \right] = 0.
\]
Here the first limit is readily seen to be equal to 0 through an application of the limiting forms (A.50) and (A.51), whilst the second limit can be seen to be equal to 0 through an application of (A.50) and by identifying \( e^{\beta t} |t|^\nu K_\nu(|t|) \) as a constant multiple of the \( V_G(r, \theta, \sigma, 0) \) density, which means that the integral must be bounded for all \( x \in \mathbb{R} \). The term
\[
J(x) := \frac{\beta e^{-\beta x} I_\nu(|x|)}{\sigma^2 |x|^\nu} \int_x^\infty e^{\beta t} |t|^\nu K_\nu(|t|) [1(t \leq 0) - \mathbb{P}(Z \leq 0)] \, dt
\]
is the product of two functions that are continuous at \( x = 0 \),
\[
u(x) = \frac{\beta e^{-\beta x} I_\nu(|x|)}{\sigma^2 |x|^\nu}, \quad v(x) = \int_x^\infty e^{\beta t} |t|^\nu K_\nu(|t|) [1(t \leq 0) - \mathbb{P}(Z \leq 0)] \, dt,
\]
meaning that \( J(0-) = J(0+) \). Therefore
\[
f'_b(0+) = -\mathbb{P}(Z \leq 0) \lim_{x \downarrow 0} \left[ \frac{e^{-\beta x} K_{\nu+1}(x)}{\sigma^2 x^\nu} \int_0^x e^{\beta t} t^\nu I_\nu(t) \, dt \right] + J(0+),
\]
\[
f'_b(0-) = -(1 - \mathbb{P}(Z \leq 0)) \lim_{x \downarrow 0} \left[ \frac{e^{-\beta x} K_{\nu+1}(-x)}{\sigma^2 (-x)^\nu} \int_0^x e^{\beta t} t^\nu I_\nu(t) \, dt \right] + J(0-)
\]
\[= (1 - \mathbb{P}(Z \leq 0)) \lim_{x \downarrow 0} \left[ \frac{e^{-\beta x} K_{\nu+1}(-x)}{\sigma^2 (-x)^\nu} \int_0^{-x} e^{-\beta u} u^\nu I_u(u) \, du \right] + J(0+).
\]
The above limits can be calculated using (A.50) and (A.51), which gives \( f'_b(0+) = -\frac{1}{\sigma^2(2\nu+1)} \mathbb{P}(Z \leq 0) + J(0+) \) and \( f'_b(0-) = \frac{1}{\sigma^2(2\nu+1)} (1 - \mathbb{P}(Z \leq 0)) + J(0+) \), thus proving the assertion. \( \square \)

**Proof of Proposition 3.6.** Again, we set \( \mu = 0 \). To simplify the expressions, we shall also work with a rescaling of the solution \( g(x) := \sigma^2 f(x) \), which will remove a multiplicative constant of \( \frac{1}{\sigma^2} \) from the calculations. The analogous approach to the proof of Proposition 3.5 would be to find a Lipschitz test function \( h \) for which \( g'' \) has a discontinuity. This would be quite a tedious undertaking, and instead we choose a highly oscillating test function and perform an asymptotic analysis. Let \( h(x) = \frac{\sin(ax)}{a} \in H_W \). If a general bound of the form \( \|g^{(3)}\| \leq M_{r, \theta, \sigma} \|h'\| \) was available, then we would be able to find a constant \( N_{r, \theta, \sigma} > 0 \), that does not involve \( a \), such that \( \|g^{(3)}\| \leq N_{r, \theta, \sigma} \). We will show that such a bound is not possible by showing that, with the choice of test function \( h(x) = \frac{\sin(ax)}{a} \), the third derivative \( g^{(3)}(x) \) blows up if we let \( a \gg 1 \) and choose \( x \) such that \( ax \ll 1 \ll a^2 x \). This means that it is not possible to obtain such a bound for \( \|f^{(3)}\| \), which will prove the proposition. Before beginning this analysis, it is worth noting that \( h''(x) = -a \sin(ax) \) blows up if \( a \gg 1 \) and \( x \) is chosen such that \( ax \ll 1 \ll a^2 x \), which can be seen from the expansion \( \sin(t) = t + O(t^3) \), \( t \to 0 \). It is therefore still possible that a general bound of the form \( \|g^{(3)}\| \leq M_{r, \theta, \sigma, 0} \|h\| + M_{r, \theta, \sigma, 1} \|h'\| + M_{r, \theta, \sigma, 2} \|h''\| \) can be obtained. Indeed, such a bound has been obtained; see Section 3.1.7 of [17].

Let \( x > 0 \). We first obtain a formula for \( g^{(3)}(x) \). We have already obtained a formula for \( g'(x) \) (see (3.26)), and differentiating this formula and then simplifying using the differentiation formulas (A.50) and (A.51) followed by an application of the Wronskian.
formula $I_{\nu}(x)K_{\nu+1}(x) + I_{\nu+1}(x)K_{\nu}(x) = \frac{1}{x}$ gives

$$g''(x) = \frac{\bar{h}(x)}{x} - \left[ \frac{d^2}{dx^2} \left( \frac{e^{-\beta x} K_{\nu}(x)}{x^{\nu}} \right) \right] \int_0^x e^{\beta t\nu} I_{\nu}(t) \bar{h}(t) \, dt$$

$$- \left[ \frac{d^2}{dx^2} \left( \frac{e^{-\beta x} I_{\nu}(x)}{x^{\nu}} \right) \right] \int_x^\infty e^{\beta t\nu} K_{\nu}(t) \bar{h}(t) \, dt.$$  

Differentiating again gives

$$g'''(x) = \frac{\bar{h}'(x)}{x} - \frac{\bar{h}(x)}{x^2} - \left[ \frac{d^3}{dx^3} \left( \frac{e^{-\beta x} K_{\nu}(x)}{x^{\nu}} \right) \right] \int_0^x e^{\beta t\nu} I_{\nu}(t) \bar{h}(t) \, dt + R_1$$

$$+ \bar{h}(x) \left\{ - x^{\nu} I_{\nu}(x) \frac{d^2}{dx^2} \left( \frac{e^{-\beta x} K_{\nu}(x)}{x^{\nu}} \right) + x^{\nu} K_{\nu}(x) \frac{d^2}{dx^2} \left( \frac{e^{-\beta x} I_{\nu}(x)}{x^{\nu}} \right) \right\}$$

$$= \frac{\bar{h}'(x)}{x} - \left( \frac{2\nu + 2}{x^2} + \frac{2\beta}{x} \right) \bar{h}(x) - \left[ \frac{d^3}{dx^3} \left( \frac{e^{-\beta x} K_{\nu}(x)}{x^{\nu}} \right) \right] \int_x^\infty e^{\beta t\nu} K_{\nu}(t) \bar{h}(t) \, dt$$

$$+ R_1,$$  

(5.43)

where

$$R_1 = - \left[ \frac{d^3}{dx^3} \left( \frac{e^{-\beta x} I_{\nu}(x)}{x^{\nu}} \right) \right] \int_x^\infty e^{\beta t\nu} K_{\nu}(t) \bar{h}(t) \, dt.$$

Here, in simplifying to obtain the formula (5.43) we used the differentiation formulas (A.60) and (A.61) followed by an application of the Wronskian formula. We can bound $R_1$ using inequalities (A.67) and (B.75) to obtain that, for all $\nu > -\frac{1}{2}$, $-1 < \beta < 1$ and $x > 0$,

$$|R_1| \leq \|\bar{h}\| \left[ \frac{d^3}{dx^3} \left( \frac{e^{-\beta x} I_{\nu}(x)}{x^{\nu}} \right) \right] \int_x^\infty e^{\beta t\nu} K_{\nu}(t) \, dt$$

$$\leq 8\|\bar{h}\| \frac{e^{-\beta x} I_{\nu}(x)}{x^{\nu}} \int_x^\infty e^{\beta t\nu} K_{\nu}(t) \, dt \leq 8M_{\nu,\beta}\|\bar{h}\|,$$

where $M_{\nu,\beta}$ is defined in (B.79). We have that $\|\bar{h}\| \leq 2\|h\| = \frac{2}{a}$, and so the term $R_1$ does not blow up in the limit $a \to \infty$.

An application of integration by parts to (5.43) gives that

$$g'''(x) = \frac{\bar{h}'(x)}{x} + \left[ \frac{d^3}{dx^3} \left( \frac{e^{-\beta x} K_{\nu}(x)}{x^{\nu}} \right) \right] \int_0^x h'(u) \int_0^u e^{\beta t\nu} I_{\nu}(t) \, dt \, du + R_1 + R_2,$$

where

$$R_2 = -\bar{h}(x) \left\{ \frac{2\nu + 2}{x^2} + \frac{2\beta}{x} + \left[ \frac{d^3}{dx^3} \left( \frac{e^{-\beta x} K_{\nu}(x)}{x^{\nu}} \right) \right] \int_0^x e^{\beta t\nu} I_{\nu}(t) \, dt \right\}$$

$$= -\bar{h}(x) T_{\nu,\beta}(x).$$

We will show that, for all $\nu > -\frac{1}{2}$ and $-1 < \beta < 1$, there exists a constant $C_{\nu,\beta} > 0$, that does not involve $x$, such that $T_{\nu,\beta}(x) \leq C_{\nu,\beta}$ for all $x > 0$. For this purpose, it will be
sufficient to examine the function $T_{\nu,\beta}(x)$ in the limits $x \downarrow 0$ and $x \to \infty$. We have that $T_{\nu,\beta}(x) \to 0$ as $x \to \infty$. This can be shown by using the differentiation formula (A.62) followed by an application of the limiting form (A.52) and the following limiting form (see (29)). For $\nu > -\frac{1}{2}$, $-1 < \beta < 1$, we have that, as $x \to \infty$,

$$
\int_0^x e^{\beta t' I_{\nu}(t)} \, dt \sim \frac{1}{\sqrt{2\pi(1+\beta)}} x^{\nu-\frac{1}{2}} e^{(1+\beta)x}.
$$

(5.44)

In addition, by applying the differentiation formula (A.62) and then the limiting forms (A.50) and (A.51) together with the expansion $e^{-\beta x} = 1 - \beta x + O(x^2)$, as $x \to 0$, we obtain that, for $\nu > -\frac{1}{2}$, $-1 < \beta < 1$, as $x \downarrow 0$,

$$
\left[ \frac{d^3}{dx^3} \left( \frac{e^{-\beta x} K_{\nu}(x)}{x^\nu} \right) \right] \int_0^x e^{\beta t' I_{\nu}(t)} \, dt
\begin{align*}
&= -e^{-\beta x} \left\{ \left( 3\beta + \beta^3 + \frac{2\nu+1}{x} \right) \frac{K_{\nu}(x)}{x^\nu} \\
&\quad + \left( 1 + 3\beta^2 + \frac{3\beta(2\nu+1)}{x} + \frac{(2\nu+1)(2\nu+2)}{x^2} \right) \frac{K_{\nu+1}(x)}{x^\nu} \right\} \int_0^x e^{\beta t' I_{\nu}(t)} \, dt \\
&= - \left\{ (1 - \beta x) \left( \frac{(2\nu+1)(2\nu+2)}{x^2} + \frac{3\beta(2\nu+1)}{x} \right) \frac{2\nu\Gamma(\nu+1)}{x^{2\nu+1}} + O(x^{-2\nu-1}) \right\} \times \\
&\quad \times \int_0^x \left( \frac{t^{2\nu}}{2\nu\Gamma(\nu+1)} + \frac{\beta t^{2\nu+1}}{2\nu\Gamma(\nu+1)} + O(t^{2\nu+2}) \right) \, dt \\
&= - \left( \frac{(2\nu+1)(2\nu+2)}{x^{2\nu+3}} - \beta(2\nu+1)(2\nu-1) \right) \left( \frac{x^{2\nu+1}}{2\nu+1} + \frac{\beta x^{2\nu+2}}{2\nu+2} \right) + O(1) \\
&= -\frac{2\nu+2}{x^2} - \frac{2\beta}{x} + O(1).
\end{align*}
$$

One needs to argue carefully that the remainder term in the curly brackets in the second equality is $O(x^{-2\nu-1})$. For $\nu > 0$, this is justified because we have the expansions $K_{\nu}(x) = 2^{\nu-1}\Gamma(\nu)x^{-\nu} + O(x^{-\nu+2})$ and $K_{\nu+1}(x) = 2^{\nu}\Gamma(\nu+1)x^{-\nu-1} + O(x^{-\nu+1})$, as $x \downarrow 0$ (see (A.51)). However, for $-\frac{1}{2} < \nu \leq 0$ the second term in the asymptotic expansion of $K_{\nu+1}(x)$ is larger than $O(x^{-\nu+1})$, as $x \downarrow 0$, so we need to work a little harder. For $-\frac{1}{2} < \nu \leq 0$, we use the identity (A.49) followed by the limiting form (A.51) to get that, as $x \downarrow 0$,

$$(2\nu+1) \frac{K_{\nu}(x)}{x^{\nu+1}} + (2\nu+1)(2\nu+2) \frac{K_{\nu+1}(x)}{x^{\nu+2}} = (2\nu+1) \frac{K_{\nu+2}(x)}{x^{\nu+1}}
\begin{align*}
&= (2\nu+1) \frac{2^{\nu+1}\Gamma(\nu+2)}{x^{2\nu+3}} + O(x^{-2\nu-1}) \\
&= (2\nu+1)(2\nu+2) \frac{2^{\nu}\Gamma(\nu+1)}{x^{2\nu+3}} + O(x^{-2\nu-1}),
\end{align*}
$$
as required. This argument is of course also valid for $\nu > 0$. Thus, we have show that $T_{\nu,\beta}(x)$ is bounded as $x \downarrow 0$, as well as in the limit $x \to \infty$, and so we have been able to shown that $R_2$ does not blow up when $a \to \infty$. 

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From the differentiation formula $g^{(3)}(x) = \frac{h'(x)}{x} - \frac{(2\nu + 1)(2\nu + 2)e^{-\beta x} K_{\nu+1}(x)}{x^{\nu+2}} \int_0^x h'(u) \int_0^u e^{\beta t} I_{\nu}(t) \, dt \, du + R_1 + R_2 + R_3,$

where

$$|R_3| = \left| e^{-\beta x} \left\{ \left( \beta + \beta^3 + \frac{2\nu + 1}{x} \right) \frac{K_{\nu}(x)}{x^{\nu}} + \left( 1 + \beta^2 + \frac{\beta(2\nu + 1)}{x} \right) \frac{K_{\nu+1}(x)}{x^{\nu}} \right\} \right| \times \int_0^x h'(u) \int_0^u e^{\beta t} I_{\nu}(t) \, dt \, du \right| \leq e^{-\beta x} \left\{ \left( \beta + \beta^3 + \frac{2\nu + 1}{x} \right) \frac{K_{\nu}(x)}{x^{\nu}} + \left( 1 + \beta^2 + \frac{\beta(2\nu + 1)}{x} \right) \frac{K_{\nu+1}(x)}{x^{\nu}} \right\} \times \int_0^x \int_0^u e^{\beta t} I_{\nu}(t) \, dt \, du,$$

and we used that $\|h'\| = 1$ in obtaining the inequality. We have that

$$\int_0^x \int_0^u e^{\beta t} I_{\nu}(t) \, dt \, du \sim \begin{cases} \frac{x^{2\nu+2}}{2^{2\nu + 1}(2\nu + 1)(2\nu + 2)\Gamma(\nu + 1)}, & x \downarrow 0, \\ \frac{1}{\sqrt{2\pi(1 + \beta)^2}} e^{-\frac{1}{2}x^{1+\beta}}, & x \to \infty. \end{cases} \tag{5.46}$$

Here the limiting form in the case $x \downarrow 0$ readily follows from an application of (A.50), whilst the limiting form for $x \to \infty$ results from an application of the limiting form (5.44) followed by a standard asymptotic analysis of the integral $\int_0^x u^{\nu-\frac{1}{2}} e^{(1+\beta)u} \, du$ in the limit $x \to \infty$. Using the limiting form (5.46) together with the limiting forms (A.51) and (A.52) for the modified Bessel function of the second kind proves that the upper bound (5.45) does not blow up in either the limits $x \downarrow 0$ or $x \to \infty$, and can thus be uniformly bounded for all $x \geq 0$. Therefore, the term $R_3$ does not explode when $a \to \infty$.

Finally, we analyse $g^{(3)}(x)$ in a neighbourhood of $x = 0$ when $a \to \infty$. This analysis proceeds almost exactly as this stage of the proof of Proposition 3.6 of [30], but we repeat the details for completeness. We have shown that, for all $x \geq 0$, $R_1$, $R_2$ and $R_3$ are $O(1)$ as $a \to \infty$. Therefore using the limiting forms (A.50) and (A.51) gives

$$g^{(3)}(x) = -\frac{\cos(ax)}{x} + \frac{(2\nu + 1)(2\nu + 2)}{x^{\nu+1}} \cdot \frac{2^{\nu}\Gamma(\nu + 1)}{x^{\nu+2}} \times \int_0^x \cos(au) \int_0^u t^{2\nu} \, dt \, du + O(1)$$

$$= -\frac{\cos(ax)}{x} + \frac{2\nu + 2}{x^{2\nu+3}} \int_0^x u^{2\nu+1} \cos(au) \, du + O(1), \quad x \downarrow 0.$$

As well as letting $x \downarrow 0$ and $a \to \infty$, we let $ax \downarrow 0$. Using the expansion $\cos(t) =
\[ 1 - \frac{1}{2} t^2 + O(t^4) \] as \( t \downarrow 0 \), we have, in this regime,

\[
g^{(3)}(x) = -\frac{1}{x} \left( 1 - \frac{a^2 x^2}{2} \right) + \frac{(\nu + 1) a^2 x}{2 \nu + 4} + O(1) = \frac{a^2 x}{2(\nu + 2)} + O(1).
\]

For \( x \) chosen such that \( ax \ll 1 \ll a^2 x \), we have that \( g^{(3)}(x) \) blows up, and this proves the proposition. \( \Box \)

We will need to following lemma for the proof of Proposition 3.7.

**Lemma 5.1.** (i) Let \( 0 < \nu \leq \frac{1}{2} \). Then \( e^{x^\nu} K_\nu(x) \) is a decreasing function of \( x \) on \((0, \infty)\) and satisfies the inequality \( e^{x^\nu} K_\nu(x) \leq 2^{\nu - 1} \Gamma(\nu) \) for all \( x > 0 \).

(ii) Fix \( c \geq 2 \) and let \( x_*^c \) be the unique positive solution to \( e^x K_0(x) = -c \log(x) \). Then, for \( 0 < x < x_*^c \), we have that \( e^{x} K_0(x) < -c \log(x) \).

(iii) Suppose \( 0 < x < 0.629 \). Then \( e^{x} K_0(x) < -3 \log(x) \).

**Proof.** (i) From (A.54) and (A.64) we have that, for all \( x > 0 \),

\[
\frac{d}{dx} \left( e^{x^\nu} K_\nu(x) \right) = e^{x^\nu} \left( K_\nu(x) - K_{\nu - 1}(x) \right) \leq 0.
\]

and so \( e^{x^\nu} K_\nu(x) \) is a decreasing function of \( x \) on \((0, \infty)\). By (A.51) we have \( \lim_{x \downarrow 0} x^\nu K_\nu(x) = 2^{\nu - 1} \Gamma(\nu) \), and so we obtain the inequality.

(ii) From the differentiation formula (A.53) we have that, for all \( x > 0 \),

\[
\frac{d}{dx} \left( -c \log(x) - e^{x} K_0(x) \right) = -\frac{c}{x} - e^{x} (K_0(x) - K_1(x)) < 0,
\]

where the inequality follows because \( K_1(x) < \frac{c}{x} \) for all \( x > 0 \) (see [30, Lemma 6.1]). Thus, \( -c \log(x) - e^{x} K_0(x) \) is a decreasing function of \( x \) on \((0, \infty)\). A simple asymptotic analysis using (A.51) and (A.52) shows that \( \lim_{x \downarrow 0} (-c \log(x) - e^{x} K_0(x)) > 0 \) and \( \lim_{x \to \infty} (-c \log(x) - e^{x} K_0(x)) < 0 \). The assertion now follows.

(iii) One can use Mathematica to numerically check that \( x_*^3 = 0.62927 \ldots > 0.629 \). \( \Box \)

**Proof of Proposition 3.7.** As usual, we will set \( \mu = 0 \). In this proof, \( Z \) will denote a VG\((r, \theta, \sigma, 0)\) random variable. For a further simplification, we shall suppose that \( \theta \geq 0 \); the argument for \( \theta < 0 \) is a very similar because \( -Z \sim \text{VG}(r, -\theta, \sigma, 0) \).

(i) Let \( r > 1 \). Proposition 1.2 of [59] asserts that if the random variable \( Y \) has Lebesgue density bounded by \( C \), then for any random variable \( W \),

\[
d_K(W, Y) \leq \sqrt{2C d_W(W, Y)}.
\]

We know that the VG\((r, \theta, \sigma, 0)\) distribution is unimodal [31] and that for \( r > 1 \) the density is bounded. If \( 1 < r \leq 2 \) the density is bounded above by \( C = \frac{1}{2\sqrt{\pi}} (1 + \theta^2/\sigma^2)^{-\frac{r-1}{2}} \Gamma\left(\frac{r-1}{2}\right)/\Gamma\left(\frac{r}{2}\right) \) (see [24.4]), and for \( r > 2 \) we can use Proposition 2.1 to bound the density. This gives us the desired bounds.

(ii) We now let \( r = 1 \). We follow the approach used in the proof of Proposition 1.2 of [59] (see also the proof of Theorem 3.3 of [13]), but modify part of the argument because the VG\((1, \theta, \sigma, 0)\) density \( p(x) \) is unbounded as \( x \to 0 \). Let \( \epsilon > 0 \) be a constant. Let
\( h_z(x) = \mathbf{1}(x \leq z) \), and let \( h_{z,\epsilon}(x) \) be defined to be one for \( x \leq z + 2\epsilon \), zero for \( x > z \), and linear between. Then

\[
P(W \leq z) - P(Z \leq z) = \mathbb{E}h_{z}(W) - \mathbb{E}h_{z,\epsilon}(Z) + \mathbb{E}h_{z,\epsilon}(Z) - \mathbb{E}h_{z}(Z)
\]

\[
\leq \mathbb{E}h_{z,\epsilon}(W) - \mathbb{E}h_{z,\epsilon}(Z) + \frac{1}{2}P(z \leq Z \leq z + 2\epsilon)
\]

\[
\leq \frac{1}{2\epsilon}d_W(W, Z) + \frac{1}{2}P(z \leq Z \leq z + 2\epsilon)
\]

\[
\leq \frac{1}{2\epsilon}d_W(W, Z) + \mathbb{P}(0 \leq Z \leq \epsilon),
\]  

(5.47)

where the third inequality follows because the VG(1, \( \theta, \sigma, 0 \)) density is positively skewed about \( x = 0 \) (since \( \theta \geq 0 \)), and is a decreasing function of \( x \) on \((0, \infty)\) and an increasing function on \((-\infty, 0)\). Suppose \( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} < 0.629 \). Then we can apply part (iii) of Lemma 5.1 to get

\[
P(0 \leq Z \leq \epsilon) = \int_{0}^{\epsilon} \frac{1}{\pi \sigma} e^{\theta t/\sigma^2} K_0 \left( \frac{\sqrt{\theta^2 + \sigma^2} t}{\sigma^2} \right) dt \leq \frac{1}{\pi \sqrt{\theta^2 + \sigma^2}} \int_{0}^{\frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2}} y K_0(y) dy
\]

\[
\leq \frac{1}{\pi \sqrt{\theta^2 + \sigma^2}} \int_{0}^{\frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2}} -3 \log(y) dy = \frac{3\epsilon}{\pi \sigma} \left[ 1 + \log \left( \frac{\sigma^2}{\epsilon \sqrt{\theta^2 + \sigma^2}} \right) \right].
\]

Plugging into (5.47) gives that, for any \( z \in \mathbb{R} \),

\[
P(W \leq z) - P(Z \leq z) \leq \frac{1}{2\epsilon}d_W(W, Z) + \frac{3\epsilon}{\pi \sigma} \left[ 1 + \log \left( \frac{\sigma^2}{\epsilon \sqrt{\theta^2 + \sigma^2}} \right) \right].
\]

We choose \( \epsilon = \sqrt{\pi \sigma d_W(W, Z)/6} \), which, due to the assumption \( \frac{\theta^2+\sigma^2}{\sigma^2}d_W(W, Z) < 0.755 \), guarantees that \( \frac{\sqrt{\theta^2+\sigma^2}}{\sigma^2} < 0.629 \). We therefore obtain the upper bound

\[
P(W \leq z) - P(Z \leq z) \leq \left\{ \frac{\sqrt{6}}{2} + \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} \log \left( \frac{6\sigma^3}{\pi (\theta^2 + \sigma^2) d_W(W, Z)} \right) \right\} \sqrt{\frac{d_W(W, Z)}{\pi \sigma}}
\]

\[
= \left\{ \frac{5\sqrt{6}}{\pi} + \log \left( \frac{\sigma^3}{(\theta^2 + \sigma^2) d_W(W, Z)} \right) \right\} \sqrt{\frac{d_W(W, Z)}{6\pi \sigma}}.
\]

A lower bound can be obtained similarly, which is the negative of the upper bound. This proves inequality (3.33).

(iii) Let \( 0 < r < 1 \). In this regime, the VG(\( r, \theta, \sigma, 0 \)) density is unbounded as \( x \to 0 \), positively skewed about \( x = 0 \) (since \( \theta \geq 0 \)), and is a decreasing function of \( x \) on \((0, \infty)\) and an increasing function on \((-\infty, 0)\). We therefore proceed as we did in part (ii) by bounding \( \mathbb{P}(0 \leq Z \leq \epsilon) \) and then substituting into (5.47). Let \( \nu = \frac{r-1}{2} \), meaning that
We optimise by taking \( \epsilon \) for differentiation formulas (A.57)–(A.62).

In this appendix, we present some basic properties of modified Bessel functions that are needed in this paper. All formulas are given in [53], except for the inequalities and the differentiation formulas (A.57)–(A.62).

Substituting \( \nu \) where we made a change of variables and applied (A.48) in the second step, and used Lemma 5.1 in the third. Therefore, for any \( z \in \mathbb{R} \),

\[
P(W \leq z) - P(Z \leq z) \leq \frac{1}{2\epsilon} d_W(W, Z) + C_{\nu, \sigma} e^{2\nu+1},
\]

We optimise by taking \( \epsilon = \left( \frac{d_W(W, Z)}{2(2\nu+1)C_{\nu, \sigma}} \right)^{\frac{1}{2(2\nu+1)}} \), which yields the upper bound

\[
P(W \leq z) - P(Z \leq z) \leq 2 \left( 2(2\nu+1)C_{\nu, \sigma} \right)^{\frac{1}{2(2\nu+1)}} \left( d_W(W, Z) \right)^{\frac{2\nu+1}{2(2\nu+1)}}
\]

We can similarly obtain a lower bound, which is the negative of the upper bound. By substituting \( \nu = \frac{r-1}{2} \) we obtain (3.34), completing the proof.

## A Elementary properties of modified Bessel functions

In this appendix, we present some basic properties of modified Bessel functions that are needed in this paper. All formulas are given in [53], except for the inequalities and the differentiation formulas (A.57)–(A.62).

The modified Bessel functions of the first kind \( I_\nu(x) \) and second kind \( K_\nu(x) \) are defined, for \( \nu \in \mathbb{R} \) and \( x > 0 \), by

\[
I_\nu(x) = \sum_{k=0}^{\infty} \frac{\left( \frac{x}{2} \right)^{\nu+2k}}{\Gamma(\nu+k+1)k!} \quad \text{and} \quad K_\nu(x) = \int_0^{\infty} e^{-x \cosh(t)} \cosh(\nu t) \, dt.
\]

For \( x > 0 \), the modified Bessel functions \( I_\nu(x) \) and \( K_\nu(x) \) are strictly positive for \( \nu \geq -1 \) and all \( \nu \in \mathbb{R} \), respectively. The modified Bessel function \( K_\nu(x) \) satisfies the following identities, which hold for all \( \nu \in \mathbb{R} \) and \( x \in \mathbb{R} \),

\[
K_{-\nu}(x) = K_\nu(x), \quad \text{(A.48)}
\]

\[
K_{\nu+1}(x) = K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x). \quad \text{(A.49)}
\]
The modified Bessel functions satisfy the following asymptotics:

\[ I_\nu(x) = \frac{1}{\Gamma(\nu + 1)} \left( \frac{x}{2} \right)^\nu (1 + O(x^2)), \quad x \downarrow 0, \quad (A.50) \]

\[ K_\nu(x) = \begin{cases} 
2^{\nu-1}|\nu| x^{-\nu} (1 + O(x^\nu)), & x \downarrow 0, \nu \neq 0, \\
- \log x + O(1), & x \downarrow 0, \nu = 0, 
\end{cases} \quad (A.51) \]

\[ I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} (1 + O(x^{-1})), \quad x \to \infty, \]

\[ K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} (1 + O(x^{-1})), \quad x \to \infty. \quad (A.52) \]

Here, 0 < \( p_\nu \leq 2 \) for all \( \nu \neq 0 \). In particular, \( p_\nu = 2 \) for \( \nu > 1 \). We also have the following differentiation formulas:

\[ \frac{d}{dx} (K_0(x)) = -K_1(x), \quad (A.53) \]

\[ \frac{d}{dx} (x^\nu K_\nu(x)) = -x^\nu K_{\nu-1}(x), \quad (A.54) \]

\[ \frac{d}{dx} \left( \frac{I_\nu(x)}{x^\nu} \right) = \frac{I_{\nu+1}(x)}{x^\nu}, \quad (A.55) \]

\[ \frac{d}{dx} \left( \frac{K_\nu(x)}{x^\nu} \right) = -\frac{K_{\nu+1}(x)}{x^\nu}, \quad (A.56) \]

\[ \frac{d^2}{dx^2} \left( \frac{I_\nu(x)}{x^\nu} \right) = \frac{I_\nu(x)}{x^\nu} - \frac{(2\nu + 1)I_{\nu+1}(x)}{x^{\nu+1}}, \quad (A.57) \]

\[ \frac{d^2}{dx^2} \left( \frac{K_\nu(x)}{x^\nu} \right) = \frac{K_\nu(x)}{x^\nu} + \frac{(2\nu + 1)K_{\nu+1}(x)}{x^{\nu+1}}, \quad (A.58) \]

\[ \frac{d^3}{dx^3} \left( \frac{K_\nu(x)}{x^\nu} \right) = -\frac{(2\nu + 1)K_\nu(x)}{x^{\nu+1}} - \left( 1 + \frac{(2\nu + 1)(2\nu + 2)}{x^2} \right) \frac{K_{\nu+1}(x)}{x^\nu}, \quad (A.59) \]

where formulas (A.57)–(A.59) are obtained from short calculations that involve differentiating using the formulas (A.55) and (A.53) followed by an application of the identity (A.49) to simplify the expressions. Using the Leibniz differentiation formula together with formulas (A.55)–(A.59) gives

\[ \frac{d^2}{dx^2} \left( \frac{e^{-\beta x} I_\nu(x)}{x^\nu} \right) = e^{-\beta x} \left\{ (1 + \beta^2) \frac{I_\nu(x)}{x^\nu} - \left( 2\beta + \frac{2\nu + 1}{x} \right) \frac{I_{\nu+1}(x)}{x^\nu} \right\}, \quad (A.60) \]

\[ \frac{d^2}{dx^2} \left( \frac{e^{-\beta x} K_\nu(x)}{x^\nu} \right) = e^{-\beta x} \left\{ (1 + \beta^2) \frac{K_\nu(x)}{x^\nu} + \left( 2\beta + \frac{2\nu + 1}{x} \right) \frac{K_{\nu+1}(x)}{x^\nu} \right\}, \quad (A.61) \]

\[ \frac{d^3}{dx^3} \left( \frac{e^{-\beta x} K_\nu(x)}{x^\nu} \right) = -e^{-\beta x} \left\{ \left( 3\beta^2 + \frac{2\nu + 1}{x} \right) \frac{K_\nu(x)}{x^\nu} \right. \\
+ \left. \left( 1 + 3\beta^2 + \frac{3\beta(2\nu + 1)}{x} + \frac{2\nu + 1}{x^2} \right) \frac{K_{\nu+1}(x)}{x^\nu} \right\}. \quad (A.62) \]
For $x > 0$, the following inequalities hold:

\begin{align*}
I_\nu(x) &< I_{\nu-1}(x), \quad \nu \geq \frac{1}{2}, \quad (A.63) \\
K_\nu(x) &\leq K_{\nu-1}(x), \quad \nu \leq \frac{1}{2}, \quad (A.64) \\
K_\nu(x) &\geq K_{\nu-1}(x), \quad \nu \geq \frac{1}{2}. \quad (A.65)
\end{align*}

We have equality in (A.64) and (A.65) if and only if $\nu = \frac{1}{2}$. These two inequalities can be found in [35]. Inequality (A.63) is given in [33] and [36], extending a result of [60]. Also, we have the following inequality for products of modified Bessel functions, which is given in Corollary 1 of [26] and is a simple consequence of a monotonicity result of [56] concerning the product $K_\nu(x)I_\nu(x)$. For $x \geq 0$,

$$K_\nu(x)I_\nu(x) \leq \frac{1}{2\nu}, \quad \nu > 0. \quad (A.66)$$

Inequality (D.4) of [23] states that, for $x > 0$,

$$\frac{d^3}{dx^3} \left( \frac{e^{-\beta x}I_\nu(x)}{x^\nu} \right) < \frac{8e^{-\beta x}I_\nu(x)}{x^\nu}, \quad \nu > -\frac{1}{2}, -1 < \beta < 1. \quad (A.67)$$

We also have the following integral inequality, which is a special case of inequality (2.6) of [23]. For $x \geq 0$,

$$\int_0^x t^\nu I_\nu(t) \, dt \leq \frac{2(\nu + 1)}{2\nu + 1} x^\nu I_{\nu+1}(x), \quad \nu > -\frac{1}{2}. \quad (A.68)$$

### B Uniform bounds for expressions involving integrals of modified Bessel functions

In this appendix, we present bounds of [26, 28, 32] that we will use to bound the solution of the VG Stein equation. The bounds of [26, 28, 32] are stated for the case $\alpha = 1$, $-1 < \beta < 1$; the bounds we state in this appendix follow from a simple change of variables. Let $\nu$, $\alpha$ and $\beta$ be such that $\nu > -\frac{1}{2}$ and $|\beta| < \alpha$. Also, let $\gamma = \beta/\alpha$. We will translate the bounds of [26, 28, 32] into the VG($r, \theta, \sigma, \mu$) parametrisation using the change of parameters

$$\nu = \frac{r - 1}{2}, \quad \alpha = \sqrt{\beta^2 + \sigma^2}, \quad \beta = \frac{\theta}{\sigma^2}.$$

We first give the following bound, which is not available in the literature, but is easy to derive. Suppose $\nu > -\frac{1}{2}$ and $0 \leq \beta < \alpha$. Then, for $x \geq 0$,

$$\frac{e^{-\beta x}K_{\nu+1}(\alpha x)}{x^\nu} \int_0^x e^{\beta t} t^\nu I_\nu(\alpha t) \, dt \leq \frac{K_{\nu+1}(\alpha x)}{x^\nu} \int_0^x t^\nu I_\nu(\alpha t) \, dt \leq \frac{2(\nu + 1)}{\alpha(2\nu + 1)} K_{\nu+1}(\alpha x) I_{\nu+1}(\alpha x) \leq \frac{1}{\alpha(2\nu + 1)}. \quad (B.69)$$
where in the first step we used that $e^{\beta t}$ is an increasing function of $t$; in the second we used inequality (A.68); and in the third we used inequality (A.66).

Suppose now that $\nu > -\frac{1}{2}$ and $|\beta| < \alpha$. Then, the bounds of [26, 28, 32] that we will need are the following. For all $x \geq 0$,

$$
e^{-\beta x}K_{\nu}(\alpha x) \int_0^x e^{\beta t} t^{\nu+1} I_{\nu}(\alpha t) \, dt < \frac{1}{2\alpha^2(1-|\gamma|)} < \sigma^2, \quad (B.70)
$$

$$
e^{-\beta x}I_{\nu}(\alpha x) \int_x^{\infty} e^{\beta t} t^{\nu+1} K_{\nu}(\alpha t) \, dt < \frac{1}{\alpha^2}\left(1 + \frac{2\sqrt{\pi}|\gamma|\Gamma(\nu + \frac{3}{2})}{(1 - \gamma^2)^{\nu+\frac{3}{2}}\Gamma(\nu + 1)} \right)
\cdot \left(1 + \frac{\theta^2}{\sigma^2}\right)^{\frac{\nu-1}{2}} < \frac{\sigma^4}{\theta^2 + \sigma^2} + 2\sqrt{\pi} |\theta| \sqrt{r+1}\left(1 + \frac{\theta^2}{\sigma^2}\right)^{\frac{\nu-1}{2}}, \quad (B.71)
$$

$$
e^{-\beta x}K_{\nu+1}(\alpha x) \int_0^x e^{\beta t} t^{\nu+1} I_{\nu}(\alpha t) \, dt < \frac{1}{2\alpha^2(1-|\gamma|)} < \sigma^2, \quad (B.72)
$$

$$
e^{-\beta x}K_{\nu+1}(\alpha x) \int_0^x e^{\beta t} t^{\nu} I_{\nu}(\alpha t) \, dt \leq \frac{2}{\alpha(2\nu + 1)} = \frac{2\sigma^2}{r\sqrt{\theta^2 + \sigma^2}}, \quad (B.73)
$$

$$
e^{-\beta x}K_{\nu}(\alpha x) \int_0^x e^{\beta t} t^{\nu} I_{\nu}(\alpha t) \, dt \leq \frac{2}{\alpha(2\nu + 1)} = \frac{2\sigma^2}{r\sqrt{\theta^2 + \sigma^2}}, \quad (B.74)
$$

$$
e^{-\beta x}I_{\nu}(\alpha x) \int_x^{\infty} e^{\beta t} t^{\nu} K_{\nu}(\alpha t) \, dt \leq \frac{M_{\nu,\gamma}}{\alpha} < \frac{\sigma^2 A_{\nu,\theta,\sigma}}{\sqrt{\theta^2 + \sigma^2}}, \quad (B.75)
$$

$$
e^{-\beta x}K_{\nu+1}(\alpha x) \int_0^x e^{\beta t} t^{\nu-1} I_{\nu}(\alpha t) \, dt < \frac{2\nu + 7}{2\alpha^2(2\nu + 1)(1-|\gamma|)} < \sigma^2 \left(1 + \frac{6}{r}\right), \quad (B.76)
$$

$$
e^{-\beta x}K_{\nu+1}(\alpha x) \int_0^x e^{\beta t} t^{\nu-1} I_{\nu}(\alpha t) \, dt < \frac{2\nu + 7}{2\alpha^2(2\nu + 1)(1-|\gamma|)} < \sigma^2 \left(1 + \frac{6}{r}\right), \quad (B.77)
$$

$$
e^{-\beta x}I_{\nu}(\alpha x) \int_x^{\infty} e^{\beta t} t^{\nu-1} K_{\nu}(\alpha t) \, dt < \frac{N_{\nu,\gamma}}{\alpha^2} < \sigma^2 B_{r,\theta,\sigma}, \quad (B.78)
$$

where

$$
M_{\nu,\gamma} = \begin{cases}
\frac{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}{(1 - \gamma^2)^{\nu+\frac{3}{2}}\Gamma(\nu + 1)}, & \nu \geq \frac{1}{2}, \\
\frac{6\Gamma(\nu + \frac{1}{2})}{1 - |\gamma|}, & |\nu| < \frac{1}{2},
\end{cases} \quad (B.79)
$$

and

$$
N_{\nu,\gamma} = \begin{cases}
\frac{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}{(1 - \gamma^2)^{\nu+\frac{3}{2}}\Gamma(\nu)}, & \nu \geq \frac{1}{2}, \\
\frac{1}{1 - |\gamma|}, & |\nu| < \frac{1}{2},
\end{cases}
$$
and in the VG($r, \theta, \sigma, \mu$) parametrisation

$$A_{r,\theta,\sigma} = \begin{cases} \frac{2\sqrt{\pi}}{\sqrt{2r-1}} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{r}{2}}, & r \geq 2, \\ 12\Gamma\left(\frac{r}{2}\right) \left( 1 + \frac{\theta^2}{\sigma^2} \right), & 0 < r < 2, \end{cases}$$

and

$$B_{r,\theta,\sigma} = \begin{cases} \frac{\pi(r-1)}{2} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{r-1}{2}}, & r \geq 2, \\ \frac{6\Gamma\left(\frac{r}{2}\right)}{\sqrt{\theta^2 + \sigma^2} - |\theta|}, & 0 < r < 2, \end{cases}$$

which satisfy the inequalities

$$M_{\nu,\gamma} < A_{r,\theta,\sigma}, \quad N_{\nu,\gamma} < \alpha^2\sigma^2 B_{r,\theta,\sigma}. \quad \text{(B.80)}$$

Written in the VG($r, \theta, \sigma, \mu$) parametrisation, $M_{\nu,\gamma}$ reads

$$M_{\nu,\gamma} = \begin{cases} \frac{\sqrt{\pi} \Gamma\left(\frac{r}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)} \left( 1 + \frac{\theta^2}{\sigma^2} \right)^{\frac{r}{2}}, & r \geq 2, \\ \frac{6\Gamma\left(\frac{r}{2}\right)}{\sqrt{\theta^2 + \sigma^2} - |\theta|}, & 0 < r < 2, \end{cases}$$

The inequality $M_{\nu,\gamma} < A_{r,\theta,\sigma}$ then follows from an application of the inequality

$$\frac{\sqrt{\theta^2 + \sigma^2}}{\sqrt{\theta^2 + \sigma^2} - |\theta|} = \frac{\sqrt{\theta^2 + \sigma^2}(\sqrt{\theta^2 + \sigma^2} + |\theta|)}{\sigma^2} < \frac{2(\theta^2 + \sigma^2)}{\sigma^2} = 2 \left( 1 + \frac{\theta^2}{\sigma^2} \right)$$

and the upper bound in the two-sided inequality

$$\sqrt{\frac{2}{r}} < \frac{\Gamma\left(\frac{r}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)} < \sqrt{\frac{2}{r-\frac{1}{2}}}, \quad r > 1. \quad \text{(B.81)}$$

The double inequality \[ \text{(B.81)} \] is obtained by combining the inequalities $\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} > \left( x + \frac{1}{4} \right)^{-\frac{1}{2}}$ for $x > 0$ [34], and $\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} < \left( x + \frac{1}{4} \right)^{-\frac{1}{2}}$ for $x > -\frac{1}{4}$ [22]. The inequality $N_{\nu,\gamma} < \alpha^2\sigma^2 B_{r,\theta,\sigma}$ can be seen to hold similarly, although this time the lower bound in \[ \text{(B.81)} \] is used to bound the ratio of gamma functions, and we use that $\alpha^2\sigma^2 = 1 + \frac{\theta^2}{\sigma^2}$.

In obtaining inequality \[ \text{(B.70)} \] we calculated

$$\frac{1}{\alpha^2(1-|\gamma|)} = \frac{\sigma^4}{\theta^2 + \sigma^2} \frac{\sqrt{\theta^2 + \sigma^2}}{\sqrt{\theta^2 + \sigma^2} - |\theta|} = \frac{\sigma^4}{\theta^2 + \sigma^2} \frac{\sqrt{\theta^2 + \sigma^2}(\sqrt{\theta^2 + \sigma^2} + |\theta|)}{\sigma^2} \frac{2(\theta^2 + \sigma^2)}{\sigma^2} = \frac{2}{\sqrt{\theta^2 + \sigma^2}} \left( 1 + \frac{\theta^2}{\sigma^2} \right),$$

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and in obtaining inequality (B.71) we calculated
\[
\frac{1}{\alpha^2} \frac{|\gamma|\Gamma(\nu + \frac{3}{2})}{(1 - \gamma^2)^{\nu + \frac{3}{2}} \Gamma(\nu + 1)} = \frac{\sigma^4}{\theta^2 + \sigma^2} \frac{|\theta|}{\sqrt{\theta^2 + \sigma^2}} \left(1 - \frac{\theta^2}{\theta^2 + \sigma^2}\right)^{-\frac{1}{2}} \frac{\Gamma\left(\frac{\nu}{2} + 1\right)}{\Gamma\left(\frac{\nu}{2} + \frac{3}{2}\right)}
\]
\[
= \sigma^2 \left(1 + \frac{\theta^2}{\sigma^2}\right)^{-1} |\theta| \frac{1}{\sigma} \left(1 + \frac{\theta^2}{\sigma^2}\right)^{-\frac{1}{2}} \left(1 + \frac{\theta^2}{\sigma^2}\right)^{\frac{\nu}{2} + 1} \frac{\Gamma\left(\frac{\nu}{2} + 1\right)}{\Gamma\left(\frac{\nu}{2} + \frac{3}{2}\right)}
\]
\[= |\theta| \frac{\Gamma\left(\frac{\nu}{2} + 1\right)}{\Gamma\left(\frac{\nu}{2} + \frac{3}{2}\right)} \left(1 + \frac{\theta^2}{\sigma^2}\right)^{-\frac{1}{2}}.
\]
The final inequality then follows from bounding the ratio of gamma functions using the lower bound in (B.81). All other conversions from the parameters \(\nu, \alpha, \beta\) to \(r, \theta, \sigma\) are simple and we provide no further details.

In order to obtain our bounds for the solution of the VG Stein equation, we also need some additional bounds that are an easy consequence of some of the inequalities (B.70)–(B.78). To this end, we note two simple inequalities that follow from using the differentiation formulas (A.55) and (A.56) and the inequalities (A.63) and (A.65), followed by an application of our assumption \(|\beta| < \alpha\) to simplify the bound. For \(\nu > -\frac{1}{2}, |\beta| < \alpha\) and \(x > 0\),
\[
\left| \frac{d}{dx} \left( \frac{e^{-\beta x} I_{\nu}(\alpha x)}{x^\nu} \right) \right| = \frac{e^{-\beta x}}{x^\nu} |\alpha I_{\nu+1}(x) - \beta I_{\nu}(x)| < 2\alpha \frac{e^{-\beta x} I_{\nu}(\alpha x)}{x^\nu}, \tag{B.82}
\]
\[
\left| \frac{d}{dx} \left( \frac{e^{-\beta x} K_{\nu}(\alpha x)}{x^\nu} \right) \right| = \frac{e^{-\beta x}}{x^\nu} |\alpha K_{\nu+1}(x) + \beta K_{\nu}(x)| < 2\alpha \frac{e^{-\beta x} K_{\nu+1}(\alpha x)}{x^\nu}. \tag{B.83}
\]
If we restrict to \(-\alpha < \beta \leq 0\), then we can improve (B.83) to
\[
\left| \frac{d}{dx} \left( \frac{e^{-\beta x} K_{\nu}(\alpha x)}{x^\nu} \right) \right| \leq \alpha \frac{e^{-\beta x} K_{\nu+1}(\alpha x)}{x^\nu}. \tag{B.84}
\]
Combining inequalities (B.82) and (B.83) with certain bounds from the list (B.70)–(B.78) then yields the following uniform bounds (B.85)–(B.89). Suppose \(\nu > -\frac{1}{2}\) and \(|\beta| < \alpha\). Then, for any \(x \geq 0\),
\[
\left| \frac{d}{dx} \left( \frac{e^{-\beta x} I_{\nu}(\alpha x)}{x^\nu} \right) \right| \int_0^x e^{\beta t} t^{\nu+1} I_{\nu}(\alpha t) \, dt < \frac{1}{\alpha (1 - |\gamma|)} \]
\[= \sqrt{\theta^2 + \sigma^2} + |\theta| < 2\sqrt{\theta^2 + \sigma^2}, \tag{B.85}
\]
\[
\left| \frac{d}{dx} \left( \frac{e^{-\beta x} I_{\nu}(\alpha x)}{x^\nu} \right) \right| \int_x^\infty e^{\beta t} t^{\nu+1} K_{\nu}(\alpha t) \, dt < \frac{2}{\alpha} \left(1 + \frac{2\sqrt{\pi} |\gamma| \Gamma\left(\frac{\nu}{2} + \frac{3}{2}\right)}{(1 - \gamma^2)^{\nu + \frac{3}{2}} \Gamma(\nu + 1)} \right)
\]
\[< 2\sqrt{\theta^2 + \sigma^2} \left\{ \frac{\sigma^4}{\theta^2 + \sigma^2} + \sqrt{2\pi |\theta|} \sigma \frac{1}{\sqrt{r + 1}} \right\}
\]
\[= \frac{2\sigma^2}{\sqrt{\theta^2 + \sigma^2}} + 2\sqrt{2\pi |\theta|} \sigma \left(1 + \frac{\theta^2}{\sigma^2}\right)^{\frac{\nu}{2}}, \tag{B.86}
\]
\[
\left| \frac{d}{dx} \left( \frac{e^{-\beta x} I_{\nu}(\alpha x)}{x^\nu} \right) \right| \int_x^\infty e^{\beta t} t^{\nu} K_{\nu}(\alpha t) \, dt < 2M_{\gamma, \nu} < 2A_{r, \theta, \sigma}, \tag{B.87}
\]
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We also have the bound
\[
\left| \frac{d}{dx} \left( \frac{e^{-\beta x} K_\nu(x)}{x^\nu} \right) \right| \int_0^x e^{\beta t^\nu} I_\nu(\alpha t) dt \leq \frac{2}{2\nu + 1} = \frac{2}{r}.
\] (B.90)

For $-\alpha < \beta \leq 0$, this bound follows from combining inequalities (B.73) and (B.84). For $0 \leq \beta < \alpha$, we combine inequalities (B.69) and (B.83).

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