THE STABLE MAPPING CLASS GROUP OF SIMPLY CONNECTED
4-MANIFOLDS

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ABSTRACT. We consider mapping class groups $\Gamma(M) = \pi_0\text{Diff}(M \text{ fix } \partial M)$ of smooth compact simply connected oriented 4–manifolds $M$ bounded by a collection of 3–spheres. We show that if $M$ contains $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ as a connected summand then all Dehn twists around 3–spheres are trivial, and furthermore, $\Gamma(M)$ is independent of the number of boundary components. By repackaging classical results in surgery and handlebody theory from Wall, Kreck and Quinn, we show that the natural homomorphism from the mapping class group to the group of automorphisms of the intersection form becomes an isomorphism after stabilization with respect to connected sum with $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. We next consider the 3+1 dimensional cobordism 2–category $\mathcal{C}$ of 3–spheres, 4–manifolds (as above) and enriched with isotopy classes of diffeomorphisms as 2–morphisms. We identify the homotopy type of the classifying space of this category as the Hermitian algebraic $K$-theory of the integers. We also comment on versions of these results for simply connected spin 4–manifolds. Finally, we observe that a related 4–manifold operad detects infinite loop spaces.

1. Introduction

In this paper we shall be interested in the mapping class groups

$$\Gamma(M) := \pi_0\text{Diff}(M \text{ fix } \partial M)$$

of smooth compact simply connected oriented 4-manifolds $M$ bounded by any number of ordinary 3–spheres. Our strategy is to compare the mapping class group to the group $\text{Aut}(Q_M)$ of automorphisms of the intersection form $Q_M$ of $M$, which is an object with a more algebraic character and which has a far clearer structure. This problem is analogous to the well-studied problem in surface theory of understanding the Torelli group. For a surface of genus $g$, the Torelli group $I_g$ is the kernel of the natural map $\Gamma_g \to Sp_{2g,\mathbb{Z}}$ which sends an isotopy class to its induced automorphism of the intersection form (which is a symplectic form for surfaces). The Torelli group for surfaces contains quite a lot of rich structure, even in the stable setting of infinite genus. In contrast we show that the situation is as different as possible for simply connected 4-manifolds. After stabilizing by taking connected sums with $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, the stable mapping class group becomes isomorphic to the stable automorphism group, and thus the stable Torelli group of simply connected 4-manifolds vanishes.

We next turn towards cobordism categories of simply connected 4-manifolds. This is motivated by Witten’s and Morava’s ideas about “topological quantum gravity” (e.g. [Wit91], [Mor01], [Mor04]), and also by results in dimension 2 (such as the proof of the stable Mumford conjecture) which have origins tracing back to Tillmann’s analysis of the analogous category in dimension 2 [Til97], [Til99]. We find that the homotopy type

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of the classifying space of the cobordism 2–category of \( (3\text{-spheres, simply connected 4-manifolds, and isotopy classes of diffeomorphisms}) \) is equivalent as infinite loop spaces to the Hermitian algebraic \( K \)-theory of the integers. We also give a variant of this analysis for the restriction to spin 4-manifolds.

Finally, we observe that the operad constructed from mapping class groups of simply connected 4–manifolds fits into Tillmann’s framework \([Til00]\). Thus the 4–manifold operad also detects infinite loop spaces.

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2. Overview and statement of results

Given a simply connected 4-manifold \( M \), we let
\[
\Gamma(M) := \pi_0 \text{Diff}(M \text{ fix } \partial M)
\]
denote the mapping class group of isotopy classes of orientation-preserving diffeomorphisms which are the identity on \( \partial M \). The intersection pairing \( Q_M : H_2(M; \mathbb{Z}) \otimes H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z} \) (thought of as a symmetric bilinear form) determines a group \( \text{Aut}(Q_M) \subset GL(H_2(M; \mathbb{Z})) \) of automorphisms which preserve the intersection pairing. There is a homomorphism \( \Gamma(M) \rightarrow \text{Aut}(Q_M) \) induced by sending a diffeomorphism to the induced automorphism on \( H_2 \), and the kernel of this map is defined to be the 4-manifold Torelli group \( I(M) \).

Let us briefly recall some known facts about this map for surfaces and 4-manifolds. For surfaces, as soon as the genus is larger than 1 the components of the diffeomorphism group are contractible so \( \text{Diff}(F_g) \rightarrow \Gamma_g \) is a homotopy equivalence. The rational cohomology of \( \Gamma_g \) (in a stable range proportional to the genus) is a polynomial algebra \( \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \) on the Miller-Morita-Mumford classes \( \kappa_i \). The odd half of these classes pull-back from the integral symplectic group via the map \( \Gamma_g \rightarrow Sp_{2g}(\mathbb{Z}) \), but the even half do not. Even at the cohomological level the two groups are fairly different.

However, the situation is markedly simpler for topological 4-manifolds.

Theorem 2.1 \([Qui86]\). For a simply connected topological 4-manifold \( M \),
\[
\pi_0 \text{Homeo}(M) \cong \text{Aut}(Q_M).
\]

Of course, once we set foot in the land of smooth manifolds, the situation becomes somewhat more interesting, for Ruberman \([Rub99]\) has constructed examples of smooth 4-manifolds for which the homomorphism \( \pi_0 \text{Diff}(M) = \Gamma(M) \rightarrow \text{Aut}(Q_M) \) has non-finitely generated kernel! One detects and distinguishes elements in the kernel using a gauge theoretic invariant. Note that in a rough sense gauge theory tends to only
detect properties which are *unstable* with respect to connected sum. For instance, the Donaldson polynomial vanishes after a single connected sum with $S^2 \times S^2$ \cite[Theorem 1.3.4, p.26]{DK90}. Furthermore, if we allow ourselves to start taking connected sums with $S^2 \times S^2$ then Wall’s stable classification theorem \cite[Theorem 1.3.4, p.26]{Val64b} tells us that the stable diffeomorphism type is determined entirely by the intersection form. Motivated by these examples of how the unusual phenomena of smoothness in dimension 4 tend to go away in stabilization, there is hope that we may understand $\Gamma(M)$ stably.

In the world of surfaces stabilization is a familiar concept—one stabilizes by sequentially gluing on tori to let the genus go to infinity. There are a multitude of results which illustrate the utility of considering this stabilization. Tillmann’s theorem \cite{Til97} that the classifying space of the stable mapping class group is an infinite loop space after applying Quillen’s plus construction is one such example. This and many other results stem from Harer’s homological stability theorem \cite{Har85}, which is considered to be one of the high-points of surface theory. It says that increasing the genus and number of boundary components induces isomorphisms on the homology of the mapping class groups in a stable range of degrees that depends only on the genus. (The stability range was later improved by Ivanov \cite{Iva89}.)

For 3–manifolds, Hatcher and Wahl \cite{HW05} have proven that the homology of the mapping class group modulo all Dehn twists around embedded 2–spheres has a similar homological stability property. (Note that in dimension 2 Dehn twists are known by the Dehn-Lickorish Theorem \cite{Deh87, Lic62} to generate the mapping class group.) In dimension 4 however there is no known theorem analogous to Harer’s stability. Nevertheless, we shall find that there is still appreciable utility in studying 4-manifold mapping class groups under stabilization.

The particular stabilization we focus on in the present paper is that of repeatedly taking connected sums with $\mathbb{C}P^2 \# \mathbb{C}P^2$. (Here $\mathbb{C}P^2$ denotes the complex projective plane with the opposite orientation.) Though perhaps less familiar than using $S^2 \times S^2$, this stabilization makes sense for the following two reasons. (i) There is a diffeomorphism $(S^2 \times S^2) \# \overline{\mathbb{C}P^2} \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$, so our stabilization process automatically implicitly contains the more familiar stabilization with respect to $S^2 \times S^2$. (ii) Intersection forms (which are integral unimodular forms) are either *even* or *odd*, and *definite* or *indefinite*. Since connected sum of manifolds corresponds to direct sum of intersection forms, the odd indefinite quadrant of the classification is the only one which cannot be exited by taking connected sums, and a form in this quadrant is isomorphic to $n(1) \oplus m(-1)$ by the classical Hasse-Minkowski classification. The intersection form of $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is $(1) \oplus (-1)$, so our stabilization process puts us immediately into the land of odd indefinite forms and sends the numbers of $(1)$ and $(-1)$ summands both to infinity.

To stabilize the mapping class group one must have a way to extend an isotopy class $[\phi]$ across a connected sum. In general this is impossible since one needs a fixed disc in which to perform the cutting and pasting, and a chosen representative $\phi$ need not fix a disc anywhere. Of course, one can always choose a representative which does fix a disc, but then the isotopy class resulting from extending this representative may depend on
the choice. Instead we shall use manifolds with boundary and stabilize by gluing along
the boundary. Let $M$ be a 4-manifold bounded by some number of ordinary 3–spheres,
and let $X$ denote $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ with the interiors of two discs removed. We may glue $X$
along a selected boundary component of $M$ to obtain a new manifold denoted $MX$, and
then iterate by gluing along the remaining boundary of $X$, as in Figure 1. Extension
by the identity on $X$ determines a system of maps

$$\Gamma(M) \to \Gamma(MX) \to \Gamma(MX^2) \to \cdots$$

and the \textit{stable mapping class group} of $M$, written $\Gamma_\infty(M)$, is defined to be the colimit
of this sequence.

On the intersection form, gluing $M$ to $N$ along a boundary 3–sphere (or even just
a homology 3–sphere) induces an inclusion $\text{Aut}(Q_M) \hookrightarrow \text{Aut}(Q_{MN})$ coming from the
block addition of intersection forms, $Q_M \mapsto Q_{MN} = Q_M \oplus Q_N$. We thus define the \textit{stable automorphism group} of $M$,

$$\text{Aut}_\infty(M) := \colim_{n \to \infty} \{ \text{Aut}(Q_M) \hookrightarrow \text{Aut}(Q_{nM}) \hookrightarrow \cdots \}.$$ 

Our main result is the following.

**Theorem 2.2.** The stable groups $\Gamma_\infty(M)$ and $\text{Aut}_\infty(Q_M)$ do not depend on the choice
of the initial manifold $M$ within the class of smooth compact oriented simply connected
4-manifolds bounded by a collection of ordinary 3–spheres, and in particular, they are
independent of the number of boundary components of $M$. Furthermore, $\Gamma_\infty \cong \text{Aut}_\infty \cong O_{\infty,\infty}(\mathbb{Z})$, with the first isomorphism induced by the natural map $\Gamma \to \text{Aut}$ and the
second coming from choosing a basis.

(Saeki [Sae06] has independently proved a more general version of this theorem by
similar methods.) Here $O_{\infty,\infty}(\mathbb{Z})$ is the group of automorphisms of the quadratic form
$\infty(1) \oplus \infty(-1)$ on $\mathbb{Z}^{\infty}$. This group is closely related to the Hermitian $K$-theory of the
integers. See the discussion immediately after Theorem 2.6.

**Corollary 2.3.** The 4-manifolds stable Torelli group $I_\infty$ is zero.
Along the way we find it necessary to analyze the elements of the mapping class group represented by Dehn twists around 3–spheres embedded with trivial normal bundle. In section 3 we define these elements and show that the set of twists around boundary 3–spheres generates the kernel of the surjective homomorphism induced by filling the boundary components with discs. Furthermore, we find that a $\mathbb{C}P^2$ summand will kill all Dehn twists.

**Theorem 2.4.** Let $M$ be a smooth compact 4-manifold (nontrivial fundamental group and nonempty boundary are allowed) of the form $N \# \mathbb{C}P^2$ (or $N \# \overline{\mathbb{C}P^2}$). Then any Dehn twist on $M$ is isotopic to the identity.

**Corollary 2.5.** Let $M$ be a simply connected oriented closed 4-manifold and let $M'$ be the result of removing the interiors of $n$ disjoint discs in $M$; let $K$ denote the kernel of the surjective map $\Gamma(M') \to \Gamma(M)$.

\(\begin{align*}
(i) & \ K \text{ is generated by Dehn twists around the boundary spheres of } M'. \\
(ii) & \text{If } M \text{ is of the form } N \# \mathbb{C}P^2 \text{ (or } N \# \overline{\mathbb{C}P^2} \text{) then } K = 0. \\
(iii) & \text{If } M \text{ is spin then } K \text{ is either } (\mathbb{Z}/2)^{n-1} \text{ or } (\mathbb{Z}/2)^n.
\end{align*}\)

The above corollary may be viewed as a very strong form of stability with respect to increasing the number of boundary components for mapping class groups of 4-manifolds. It holds at the level of groups and it requires only a single stabilization step, whereas for surfaces Harer-Ivanov stability says that the analogue of the above map is merely a homology isomorphism and only in a stable range depending on the genus. The proofs of 2.4 and 2.5 are entirely elementary, whereas the proof Harer-Ivanov stability requires the machinery designed to deal with curve-complexes.

Theorem 2.2 is partly a repackaging of classical theorems in 4-manifold topology due to Wall, Kreck, and Quinn. Wall [Wal64a] proved that $\Gamma(M) \to \text{Aut}(Q_M)$ is surjective when $M$ is indefinite and contains $S^2 \times S^2$ as a connected summand. On the other hand, Kreck [Kre79] proved that the map is always injective, once one descends from isotopy to pseudo-isotopy. Finally, Quinn [Qui86] proved that pseudo-isotopy implies isotopy in a stable sense. Together these results yield a lifting of the automorphism group of the intersection form of $M$ into the stabilized mapping class group of $M$ and in the colimit this becomes the inverse to $\Gamma_{\infty} \to \text{Aut}_{\infty}$. This material is covered in more detail in section 4.

Our results above have a close connection to what Morava [Mor01] calls a *theory of topological gravity* in 4-dimensions. Generalizing Witten’s theory [Wit91] in 2–dimensions, Morava defines such a theory to be a representation of a topological 2–category $\mathcal{G}$ where objects are 3–manifolds, morphisms are 4–dimensional cobordisms and 2–morphisms are diffeomorphisms of the cobordisms. There are many possible variations in the definition of this category. The symmetric monoidal product given by disjoint union implies that the classifying space of the cobordism category is an infinite loop space, and a representation induces an infinite loop map into some variant of a $K$–theory infinite loop space. Thus a theory of topological gravity produces an element in some version of the $K$–theory of $BG$. As a rough first step towards constructing or classifying topological
gravity theories one would thus like to understand the homotopy type of $BG$ (or at least some version of its $K$-theory).

The recent work of Galatius, Madsen, Tillmann and Weiss \cite{GMTW06} determines the homotopy types of many versions of the category $G$ in terms of more accessible spaces: the zeroth spaces of certain Thom spectra. Given their result, one might wonder why it should be necessary to study the homotopy type with any other methods. However, their argument only applies when $G$ is maximal in the sense that it is built using all manifolds of appropriate dimensions; their theorem does not determine the homotopy type of sub-categories which are obtained by restricting the objects or morphisms.

Morava \cite{Mor04} has indicated that one such subcategory out of reach to the GMTW theorem is potentially interesting; he argues based on an analogy with the Virasoro algebra that symmetries of the Tate cohomology $t^*_{SU(2)} kO$ should play a role in the representation theory of the cobordism category of smooth spin 4–manifolds bounded by ordinary 3–spheres.

In sections $5$ and $6$ of this paper we study the homotopy type of two simplified variants of Morava’s category. Let $C$ denote the cobordism 2–category where the objects of $C$ are unions of 3–spheres, the morphisms are disjoint unions of simply connected “tree-like” 4–manifolds — meaning that each component has precisely one outgoing boundary sphere; this is imposed so that compositions stay within the simply connected realm — and the 2–morphisms are isotopy classes of diffeomorphisms.

We will construct a map from $\Omega B C$ into the Hermitian algebraic $K$–theory space $\mathbb{Z}^2 \times BO_{\infty,\infty}(\mathbb{Z})^+$; it is induced by the natural 2–functor from $C$ to the 2–category $K$ constructed similarly, with 2–morphisms now being isometries (with respect to the intersection forms) of $H_2$. We define these categories more carefully in section $5$. These categories are strict symmetric monoidal under disjoint union, and hence their classifying spaces are infinite loop spaces. Our main result here is:

**Theorem 2.6.** There is a homology equivalence $\mathbb{Z}^2 \times BO_{\infty,\infty}(\mathbb{Z}) \to \Omega B C$, and hence a homotopy equivalence of $\Omega^\infty$–spaces

$$\mathbb{Z}^2 \times BO_{\infty,\infty}(\mathbb{Z})^+ \simeq \Omega B C.$$  

Here “$+$” denotes Quillen’s plus construction with respect to the commutator subgroup of $\pi_1 BO_{\infty,\infty}(\mathbb{Z}) \cong O_{\infty,\infty}(\mathbb{Z})$, which is perfect by \cite{Vas70} (or see \cite{HO89} 5.4.6, p.246). The plus construction preserves (generalized) (co)homology and kills a selected perfect subgroup of $\pi_1$; in this case it kills the commutator subgroup and thus abelianizes the fundamental group. The space $\mathbb{Z}^2 \times BO_{\infty,\infty}(\mathbb{Z})^+$ is precisely the Hermitian algebraic $K$–theory of the integers, it is known to be an infinite loop space (see for example \cite{Lod76}). Closely related to this, the Hermitian algebraic $K$–theory of $\mathbb{Z}[1/2]$ has recently been studied by Berrick and Karoubi in \cite{BK05}; they compute the rational and 2–primary

\footnote{Note that one must be careful whether one works with automorphisms of quadratic or symmetric bilinear forms; these groups are slightly different when 2 is not invertible in the ring. However, it can be shown that the associated $K$–theories are rationally the same.}
homotopy groups. The space $BO_{\infty, \infty}(\mathbb{Z})^+$ is rationally equivalent to $BO$, and it is equivalent to $BO_{\infty, \infty}(\mathbb{Z}[1/2])^+$ away from the prime 2.

For the second variant of Morava’s category, (slightly closer to the original), let $C_{\text{spin}} \subset C$ denote the sub-2–category with only spin 4–manifolds (since they are simply connected, this is equivalent to using only 4–manifolds with even intersection form). This spin sub-2–category is described in more detail in section 7.

**Theorem 2.7.** There is a map $\mathbb{Z}^2 \times B\text{Aut}(\infty H \oplus \infty (-E_8)) \to \Omega BC_{\text{spin}}$ which is a homology equivalence away from the prime 2, and hence there is an $\Omega^\infty$–map

\[
\mathbb{Z}^2 \times B\text{Aut}(\infty H \oplus \infty (-E_8))^+ \to \Omega BC_{\text{spin}}
\]

which is a homotopy equivalence away from the prime 2.

Here $E_8$ is the rank 8 irreducible form and $H$ is the standard rank 2 hyperbolic form

\[
H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

Note that the commutator subgroup of $\text{Aut}(\infty H \oplus \infty (-E_8))$ is perfect—this follows from the argument in [HO89] given for $O_{\infty, \infty}(\mathbb{Z})$.

The proofs of Theorems 2.6 and 2.7 are based on Tillmann’s generalized group completion theorem [Til97], together with the isomorphisms of Theorem 2.2 and Corollary 2.5.

From the perspective of physics most of the interesting mathematics is related to the representation theory of the identity component of the diffeomorphism group rather than the group of components. Unfortunately our isotopy variants of the Morava categories lose sight of this aspect entirely. However, there is also an interest in homotopy QFTs, or flat QFTs, which are essentially representations of isotopy 2–categories such as ours. See for example [Tur99], [BTW03].

The are operads closely related to the cobordism 2–categories we define. As an application of our analysis of mapping class groups in dimension 4 we observe that Tillmann’s higher genus surface operad [Til00] has an a 4–manifold analogue. Tillmann’s argument applies to this operad as well, so that the 4–manifold mapping class group operad detects infinite loop spaces.

**Organization of the paper.** The remainder of the paper is organized as follows. In section 3 we establish some properties of Dehn twists from which we deduce Theorem 2.4. In section 4 we review the results of Kreck, Wall and Quinn and then combine them to give the proof of the the stable isomorphism theorem, Theorem 2.2. The remainder of the paper is concerned with the cobordism 2–categories of 4–manifolds. In section 5 we construct of the category $C$ and the map into $K$–theory in detail, and in section 6 study this map and prove Theorem 2.6. Discussion of the spin case and the proof of Theorem 2.7 is contained in section 7. In section 8 we discuss the 4–manifold mapping class group operad.
3. Dehn twists

The purpose of this section is to prove Theorem 2.4, which gives a sufficient condition for when the Dehn twist around a 3–sphere is actually isotopic (fixing the boundary) to the identity.

By a Dehn twist we shall mean the element of the mapping class group constructed as follows. The data required to construct a Dehn twist is an embedding $S^3 \hookrightarrow M$ with trivial normal and a loop $\alpha \in \Omega \text{Diff}(S^3)$. We think of the loop as parametrized by the interval $(-1, 1)$. Let $V$ be a tubular neighborhood of the embedded sphere. One constructs a diffeomorphism $\phi : M \to M$ by defining it to be the identity outside of $V$, and on $V \cong S^3 \times (-1, 1)$ one sets $\phi(z, t) = \alpha_t(z)$. One easily sees that the isotopy class of $\phi$ depends only on the homotopy class of $\alpha$ and the isotopy class of the embedding.

Let $M$ be a simply connected closed oriented 4–manifold, and let $M'$ be the result of cutting out $n$ disjoint discs in $M$, so $M'$ is bounded by $n$ 3–spheres. Our goal is to study the map $\Gamma(M') \to \Gamma(M)$.

**Proposition 3.1.** The homomorphism $\Gamma(M') \to \Gamma(M)$ is surjective with kernel a quotient of $(\mathbb{Z}/2)^n$ generated by Dehn twists around the boundary components.

The proof will follow from a bit of elementary differential topology. There is a fibration

$$\text{Diff}(M' \text{ fix } \partial M') \to \text{Diff}(M) \to \text{Emb} \left( \bigcap_{i=1}^n D^4, M \right)$$

where Emb is the space of embeddings (which extend to diffeomorphisms of $M$). For a single disc, linearization at the center of the disc yields a homotopy equivalence between $\text{Emb}(D^4, M)$ and the frame bundle of $M$; see for instance [Iva02, Theorem 2.6.C]. Similarly, when there is more than one disc then there is a homotopy equivalence

$$\text{Emb} \left( \bigcap_{i=1}^n D^4, M \right) \simeq FC_n(M)$$

where $FC_n(M)$ is the framed configuration space, consisting of configurations of $n$ distinct ordered points in $M$ equipped with (oriented) framings. When $M$ is connected $FC_n(M)$ is also connected. Forgetting the framings gives a map to the usual configuration space $C_n(M)$ of $n$ ordered points which fits into a fibration

$$SO(4)^n \to FC_n(M) \to C_n(M).$$

**Lemma 3.2.** If $M$ is closed and simply connected then $\pi_1 FC_n(M)$ is a quotient of $(\mathbb{Z}/2)^n$, with the generators corresponding to rotations of each of the framings.

**Proof.** Since $M$ is 4–dimensional, removal of a finite number of points in $M$ preserves simple-connectedness and connectedness. Induction on $k$ with the Fadell–Neuwirth fibrations $M - (k \text{ points}) \to C_{k+1}(M) \to C_k(M)$ shows that the unframed configuration spaces are all simply connected. The result then follows from the homotopy exact sequence of the fibration [I].
From the fibration (1) there is an exact sequence of homotopy groups,
\[
\pi_1 \text{Diff}(M) \xrightarrow{\alpha} \pi_1 FC_n(M) \xrightarrow{\delta} \Gamma(M') \rightarrow \Gamma(M) \rightarrow 0.
\]
Note that a rotation of a framing in \(\pi_1 FC_n(M)\) is sent by \(\delta\) to the Dehn twist around the corresponding boundary sphere of \(M'\) in \(\Gamma(M')\), and Proposition 3.1 follows. From this we also see that,

**Lemma 3.3.** The map \(\Gamma(M') \rightarrow \Gamma(M)\) is an isomorphism if and only if \(\alpha\) is surjective.

Fixing distinct points \(p_1, \ldots p_n \in M\), the diagram of fibrations
\[
\begin{array}{cccccc}
\text{Diff} \left( M \text{ fix } \bigcup p_i \right) & \rightarrow & \text{Diff}(M) & \xrightarrow{\beta} & C_n(M) \\
\downarrow & & \downarrow & & \downarrow \\
SO(4)^n & \rightarrow & FC_n(M) & \rightarrow & C_n(M)
\end{array}
\]
induces a homomorphism of long exact sequences,
\[
\begin{array}{cccccc}
\pi_2 C_n(M) & \rightarrow & \pi_1 \text{Diff} \left( M \text{ fix } \bigcup p_i \right) & \rightarrow & \pi_1 \text{Diff}(M) & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{\beta} & & \downarrow{\alpha} \\
\pi_2 C_n(M) & \rightarrow & \pi_1 SO(4)^n & \rightarrow & \pi_1 FC_n(M) & \rightarrow & 0
\end{array}
\]
from which we see that,

**Lemma 3.4.** \(\alpha\) is surjective if and only if \(\beta\) is.

*Proof.* One direction is immediate and the other follows from the Five Lemma. \(\square\)

**Lemma 3.5.** \(\Gamma(\mathbb{C}P^2 - \{2 \text{ discs}\}) \cong \Gamma(\mathbb{C}P^2)\).

*Proof.* By Lemma 3.4 it suffices to show that \(\beta\) is surjective. We do this by constructing \(S^1\)–actions on \(\mathbb{C}P^2\) which hit each of the generators of \(\mathbb{Z}/2 \times \mathbb{Z}/2 \rightarrow \pi_1 FC_2(\mathbb{C}P^2)\). Let \(p_1 = [0,0,1] \in \mathbb{C}P^2\) and \(p_2 = [1,0,0] \in \mathbb{C}P^2\), and consider the \(S^1\)–action defined by \(\lambda \cdot [x,y,z] = [x,y,\lambda z]\). This action fixes both \(p_1\) and \(p_2\), and hence there is a representation of \(S^1\) on the tangent space at each of these two points. The complex 1–dimensional representations of \(S^1\) are labelled by the integers and one can easily identify the representations on the tangent spaces of the fixed points by choosing local coordinates: the representation on \(T_{p_1} \mathbb{C}P^2\) is \((-1) \oplus (-1)\), and at \(p_2\) the representation is \((0) \oplus (1)\). These correspond to compositions of group homomorphisms
\[
S^1 \xrightarrow{p_1} U(1) \times U(1) \xrightarrow{1} U(2) \xrightarrow{i} SO(4).
\]
The induced maps on fundamental groups are
\[
\mathbb{Z} \xrightarrow{p_1} \mathbb{Z} \times \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/2
\]
with $\iota_*$ being addition and $\rho_*$ determined by the representation of $S^1$. That is,
\[
\rho_1 : n \mapsto (-n, -n), \quad \rho_2 : n \mapsto (0, n).
\]
Thus $(0, 1)$ is in the image of
\[
\beta : \pi_1 \Diff \left( \mathbb{C}P^2 \text{ fix } \bigcup p_i \right) \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \twoheadrightarrow \pi_1 \Diff_2(\mathbb{C}P^2).
\]
By letting $S^1$ act instead on the first coordinate of $\mathbb{C}P^2$, one sees that $(1, 0)$ is also in the image of $\beta$. □

As a consequence we have that a Dehn twist around either of the boundary components in $\mathbb{C}P^2 - \{2 \text{ discs} \}$ is isotopic (keeping the boundary fixed) to the identity.

The spin case in Theorem 2.5 is handled by the following lemma.

**Lemma 3.6.** Suppose $M$ is a smooth closed oriented simply connected 4–manifold which is spin, and $M'$ is the result of removing the interiors of $n$ disjoint discs. There is a homomorphism $\Gamma(M') \to H^1(M', \partial M'; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{n-1}$ whose restriction to the kernel $K$ of $\Gamma(M') \to \Gamma(M)$ is surjective.

**Proof.** Let $s$ be the unique spin structure on $M$. Spin structures on $M'$ (relative to the boundary) are an affine space over $H^1(M', \partial M'; \mathbb{Z}/2)$, and there is a natural choice of basepoint given by the restriction of $s$ to $M'$. Now, define a homomorphism $\Gamma(M') \to H^1(M', \partial M'; \mathbb{Z}/2)$ by
\[
\varphi \mapsto \varphi^*(s) - s \in H^1(M', \partial M'; \mathbb{Z}/2);
\]
this measures the extent to which $\varphi$ fails to preserve the spin structure $s$. by Proposition 3.1 $K$ is generated by Dehn twists around the boundary spheres, and these twists act transitively on the set of spin structures on $(M', \partial M')$. To see this, think of the relative 1–skeleton of $M'$ as a union of arcs joining one fixed boundary sphere with each of the other boundary spheres as in Figure 2. Spin structures on $(M', \partial M')$ correspond to labelings of the arcs in the relative 1–skeleton by elements of $\mathbb{Z}/2$. If $\varphi$ is a Dehn twist around a boundary sphere $S$ then $\varphi^*$ reverses the label of each arc having an endpoint on $S$. Hence the homomorphism defined above is surjective. □

**Proof of Theorem 2.4.** Recall the setup: $M$ is an arbitrary 4-manifold containing $\mathbb{C}P^2$ as a connected summand. Let $S \hookrightarrow M$ be a 3–sphere embedded in $M$ with trivial normal bundle and let $\alpha$ denote the Dehn twist around $S$. We cut $M$ along $S$, producing a manifold $M'$, and we now regard $\alpha$ as Dehn twist around one of the new boundary components. Up to diffeomorphism, we may assume that the boundary component around which $\alpha$ twists lies on the $\mathbb{C}P^2$ summand, as in figure 3, so $\alpha$ is in the image of the composition
\[
\Gamma(\mathbb{C}P^2 - \{2 \text{ discs} \}) \to \Gamma(M') \to \Gamma(M).
\]
Lemma 3.5 tells us that the element of $\Gamma(\mathbb{C}P^2 - \{2 \text{ discs} \})$ which maps to $\alpha$ is the zero element, so $\alpha$ is zero in $\Gamma(M)$. The same argument holds with $\mathbb{C}P^2$ replaced by $\overline{\mathbb{C}P^2}$ throughout. □
Proof of Corollary 2.5. We have $M'$ obtained from $M$ by removing a collection of discs. The map $\Gamma(M') \to \Gamma(M)$ given by gluing the discs back in is surjective with kernel $K$. By Proposition 3.1 $K$ is generated entirely of Dehn twists around boundary 3–spheres. If $M$ contains $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ as a connected summand then all Dehn twists are isotopic to the identity by Theorem 2.4, so $K = 0$. If $M$ is spin then by Lemma 3.6 $K$ is isomorphic to either $(\mathbb{Z}/2)^n$ or $(\mathbb{Z}/2)^{n-1}$ since it is a quotient of $(\mathbb{Z}/2)^n$. □

In the spin case of Corollary 2.5 it appears to be a difficult problem to decide in general precisely how many powers factors of $\mathbb{Z}/2$ there are in $K$. As easy case is connected sums of $S^2 \times S^2$.

Proposition 3.7. If $M$ is a connected sum of copies of $S^2 \times S^2$ and $M'$ is obtained from $M$ by deleting the interiors of $n$ discs then $\ker(\Gamma(M') \to \Gamma(M)) = K \cong \mathbb{Z}/2^{n-1}$.

Proof. First consider the homomorphism

$$\Gamma(S^2 \times S^2 - \{\text{a disc}\}) \to \Gamma(S^2 \times S^2)$$

This is actually an isomorphism. The idea of the proof is the same as for Lemma 3.5; we look for a circle action on $S^2 \times S^2$ which fixes a point $p$ and rotates the tangent
space $T_p$ through the nontrivial element of $\pi_1 SO(4)$. Rotation about the polar axis on the first sphere, with $p = (\text{north pole, north pole})$ does the job.

From the isomorphism (4) it follows that if $M$ is a connected sum of copies of $S^2 \times S^2$ then gluing in a disc on $M'$ kills one factor of $\mathbb{Z}/2$ in $K$ as long as there is at least one remaining boundary component because the Dehn twist around any boundary sphere is isotopic to the product of the twists around each of the other boundary spheres. The conclusion now follows by induction on $n$. 

A connected sum of copies of $S^2 \times S^2$ is essentially the only case which can be handled directly by circle actions in light of the classification of locally smooth circle actions on closed simply connected 4–manifolds given in [Fin77]. If such an action exists then the manifold must be a connected sum of copies of $S^2 \times S^2$, $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, or a homotopy 4–sphere.

4. The stable groups $\text{Aut}_\infty$ and $\Gamma_\infty$

In this section we recall the theorems of Wall, Kreck, and Quinn and show how they combine to give a lifting of $\text{Aut}(Q_M)$ into $\Gamma(MX^k)$ which stabilizes to an inverse of $\Gamma_\infty(M) \to \text{Aut}_\infty(Q_M)$.

In the statements of the following three theorems $M$ shall be a simply connected compact oriented smooth 4–manifold, possibly bounding some number of homology 3–spheres.

**Theorem 4.1.** [Wal64b] If $M$ is of the form $N \# (S^2 \times S^2)$ with $Q_N$ either indefinite or of rank $\leq 8$ then $\Gamma(M) \to \text{Aut}(Q_M)$ is surjective.

Two diffeomorphisms $\varphi_0, \varphi_1$ of $M$ are said to be pseudo-isotopic if they are the restrictions to $0 \times M$ and $1 \times M$ respectively of a diffeomorphism $\Phi$ of $I \times M$. Pseudo-isotopy is in general a coarser equivalence relation than isotopy, but is finer than homotopy. Let $P(M)$ denote the group of pseudo-isotopy classes of diffeomorphisms of $M$; this group is a quotient of $\Gamma(M)$, and the morphism from $\Gamma(M)$to $\text{Aut}(M)$ descends to $P(M)$.

**Theorem 4.2.** [Kre79] $P(M) \to \text{Aut}(Q_M)$ is always injective, provided that the pseudo-isotopies are not required to fix the boundary point-wise.

Thus $P(M) \cong \text{Aut}(Q_M)$ whenever $M$ satisfies the conditions of Wall’s theorem and one is lead to ask what the relationship between $\Gamma(M)$ and $P(M)$ is. Quinn has given a good answer to this question; he proves that in dimension 4 pseudo-isotopy implies isotopy-after-stabilization.

**Theorem 4.3.** [Qui86] If $\varphi \in \text{Diff}(M)$ is pseudo-isotopic to the identity then for $k$ large enough its extension (by identity on $S^2 \times S^2$) to an element of $\text{Diff}(M \# k(S^2 \times S^2))$ is isotopic to the identity. (All isotopies and pseudo-isotopies fix the boundary of $M$ pointwise.)
Remark 4.4. For our purposes we require only this weak version of Quinn’s theorem—the stronger form actually gives existence of an isotopy not merely between the $t = 1$ end of the pseudo-isotopy and the identity diffeomorphism on $M$, but between the pseudo-isotopy (regarded as an element of $\text{Diff}(M \# k(S^2 \times S^2) \times I, \partial M \times I)$) and the identity on $M \# k(S^2 \times S^2) \times I$.

Note that the value of $k$ may depend on $\varphi$; it could potentially be unbounded as $\varphi$ ranges over all pseudo-isotopy classes. However, in all known examples $k = 1$ suffices.

Quinn’s theorem applies even when $M$ has boundary and all diffeomorphisms and (pseudo)-isotopies are taken to fix the boundary point-wise, as does Wall’s surjectivity result; however, the injectivity result of Kreck is no longer valid if one requires the boundary to be fixed because there can be a kernel consisting of twists around boundary components which induce isomorphism on homology. This limitation is precisely what necessitates our analysis of Dehn twists in the previous section since our stabilization process requires that diffeomorphisms fix the boundary point-wise in order to have a well-defined extension.

We now consider stabilization of $\Gamma_M$ and $\text{Aut}(Q_M)$ by gluing on copies of $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} - \{2 \text{ discs}\}$.

Lemma 4.5. Suppose $M$ is a simply connected oriented smooth compact 4–manifold bounded by a collection of 3–spheres. Then each automorphism of $Q_M$ is induced (as an element of $\text{Aut}_\infty(Q_M)$) by a unique element of $\Gamma_\infty(M)$.

Proof. Let $\alpha \in \text{Aut}(Q_M)$. Even if $\alpha$ is not induced by a diffeomorphism, Theorem 4.1 implies that for $\ell \geq 2$ its image in $\text{Aut}(Q_M^\ell)$ is induced by a diffeomorphism. Any two diffeomorphisms representing $\alpha$ are pseudo-isotopic after closing off the boundary with discs by Theorem 4.2. Furthermore, by Theorem 4.3 these two representatives are actually isotopic after extending to $M^\ell$, for some $k \geq \ell$ large enough, and then closing off the boundary with discs. But the extensions are isotopic even before closing off the boundary by Corollary 2.6. □

This next lemma follows immediately from the previous.

Lemma 4.6. There is a unique inclusion $\text{Aut}(Q_M) \hookrightarrow \Gamma_\infty(M)$ such that the following diagram commutes,

\[ \begin{array}{ccc} & & \Gamma_\infty(M) \\ & \searrow & \downarrow \ \\ \text{Aut}(Q_M) & \hookrightarrow & \text{Aut}_\infty(Q_M). \end{array} \]

Proof of Theorem 2.2. We produce an inverse to the map $\Gamma_\infty(M) \rightarrow \text{Aut}_\infty(Q_M)$ by exhibiting compatible maps on each of the terms in the directed system which defines $\text{Aut}_\infty(Q_M)$. The liftings of Lemma 4.6 serve this purpose. We need only check that
the diagrams

\[
\begin{array}{ccc}
\text{Aut}(Q_{MX^k}) & \longrightarrow & \Gamma_\infty(M) \\
\downarrow & & \downarrow \\
\text{Aut}(Q_{MX^{k+1}}) & \hookrightarrow & \Gamma_\infty(M)
\end{array}
\]

all commute, but this follows immediately from Lemmas 4.5 and 4.6. Hence we obtain a homomorphism \(\text{Aut}_\infty(Q_M) \rightarrow \Gamma_\infty(M)\) which is the desired inverse by construction.

We now claim that the stabilized automorphism group \(\text{Aut}_\infty(Q_M)\) is isomorphic (non-canonically) to the split-signature orthogonal group \(O_{\infty,\infty}(\mathbb{Z}) = \text{colim}_k O_{k,k}(\mathbb{Z})\), which is clearly independent of the initial manifold \(M\). By the Hasse-Minkowski classification of odd indefinite unimodular forms, \(Q_M \oplus (1) \oplus (-1) \cong a(1) \oplus b(-1)\) for some natural numbers \(a, b\). Thus

\[
\text{Aut}_\infty(Q_M) = \text{Aut}_\infty(Q_M \oplus (1) \oplus (-1)) \\
\cong \text{colim}_{k \rightarrow \infty} O_{a+k,b+k}(\mathbb{Z}) \\
\cong O_{\infty,\infty}(\mathbb{Z}).
\]

Note that passing from the first to the second line requires choosing a basis of vectors of length 1, so the resulting identification is non-canonical. In passing to the third line we have used the fact that \(\text{colim}_k O_{k,k}(\mathbb{Z}) \cong \text{colim}_k O_{k,k+n}(\mathbb{Z})\); this can be proved by comparing both sides to \(\text{colim}_j,k O_{j,k}(\mathbb{Z})\). The two embeddings \(\mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}\) given by \(k \mapsto (k, k)\) and \(k \mapsto (k, k+n)\) are both co-final subsystems. \(\square\)

In applying Tillmann's group completion theorem to identify the homotopy type of the cobordism categories described in the next section we will need an additional result about this group which we record now. Let \(L\) be a simply connected smooth compact oriented 4–manifold with outgoing boundary (a union of 3–spheres) compatible with the incoming boundary of \(M\) so that \(LM\) is a well defined composition.

**Lemma 4.7.** The inclusion \(i : \text{Aut}_\infty(Q_M) \hookrightarrow \text{Aut}_\infty(Q_{LM}) = \text{Aut}_\infty(Q_L \oplus Q_M)\) is an integral homology equivalence.

**Proof.** The Hasse-Minkowski classification implies that

\[
Q_{LMX^2} \cong (Q_L \oplus Q_X) \oplus (Q_M \oplus Q_X) \\
\cong (\ell_1(1) \oplus \ell_2(-1)) \oplus (m_1(1) \oplus m_2(-1)),
\]

so it suffices to show that

\[
\text{colim}_{k \rightarrow \infty} O_{k,k+n}(\mathbb{Z}) \hookrightarrow \text{colim}_{k \rightarrow \infty} O_{k+\ell_1,k+n+\ell_2}(\mathbb{Z})
\]

is a homology isomorphism.

The group \(O_{k+\ell_1,k+n+\ell_2}\) contains \(\Sigma_{k+\ell_1} \times \Sigma_{k+n+\ell_2}\) as permutations of basis vectors. Thus for some \(k'\) large enough there exists a basis permutation which conjugates the image of the stabilization embedding \(O_{k+k',k+n+\ell_2}(\mathbb{Z}) \hookrightarrow O_{k'+\ell_1,k'+n+\ell_2}(\mathbb{Z})\) into the image of the embedding \(O_{k'+\ell_1,k'+n+\ell_2}(\mathbb{Z}) \hookrightarrow O_{k'+\ell_1,k'+n+\ell_2}(\mathbb{Z})\). Since conjugation induces an isomorphism
on group homology, and each class in $H_*O_{\infty,\infty}(\mathbb{Z})$ comes from $H_*O_{k,k+n}(\mathbb{Z})$ for some $k$ large enough, this proves surjectivity. Injectivity follows from a similar argument. Note that this argument can be easily modified to work also in the case of even indefinite forms.

Since $\text{Aut}_\infty(Q_M) \cong \Gamma_\infty(M)$, we also have:

**Corollary 4.8.** $\Gamma_\infty(M) \to \Gamma_\infty(LM)$ is also an integral homology equivalence.

Note that the analogue of Corollary 4.8 for surfaces holds by virtue of Harer stability. In dimension 4 we are able to replace the need for homological stability with general properties of the homology of linear groups.

### 5. Definitions of the categories of 4–manifolds

In this section we construct the 2–category $\mathcal{C}$ and the map into $K$-theory which comes from a 2–functor into a closely related category $\mathcal{K}$. The definitions we use are natural extensions of the definitions found in [Til97] and [Til99]. In particular, our 2–category $\mathcal{C}$ is constructed precisely along the lines of the surface cobordism 2–category in the second reference above.

Both $\mathcal{C}$ and $\mathcal{K}$ have the same underlying ordinary category (i.e. the same objects and morphisms) which we denote by $\mathcal{C}_0$; conceptually, this category should be thought of as the cobordism category of (unions of) 3–spheres and simply connected oriented 4–manifolds. However, one must be careful in defining the morphisms so that composition is well-defined and the result is a small category. The 2–morphisms of $\mathcal{C}$ and $\mathcal{K}$ will be isotopy classes of diffeomorphisms and isomorphisms of the intersection form, respectively. Let us proceed in detail.

**Objects of $\mathcal{C}_0$:** The objects are the non-negative integers, with $n$ thought of as representing a disjoint union of $n$ copies of $S^3$.

**Morphisms of $\mathcal{C}_0$:** Let $\mathcal{A}(n)$ denote a set of manifolds containing precisely one representative from each diffeomorphism class of compact oriented connected simply connected 4–manifolds bounded by $n + 1$ ordered 3–spheres. We equip each boundary sphere with a collar and we consider the first $n$ boundary components as *in-going* and the final boundary component as *out-going*.

We now allow these *atomic* manifolds to freely generate the morphism sets via finite sequences of the three operations of:

1. Gluing the out-going boundary of one morphism to the in-going boundary of another using the ordering of the boundary components and the collars.
2. Taking disjoint unions.
3. Renumbering the in-going and out-going boundary components.
The morphism set \( C_0(m, n) \) consists of all such composites with \( m \) incoming and \( n \) outgoing boundary components respectively, together with an identity morphism when \( m = n \). Composition of morphisms is given by gluing the out-going end of one to the in-going end of the other. Given \( M \in C_0(a, b) \) and \( N \in C_0(c, d) \), the ordering of the in-going boundary components of a disjoint union \( M \sqcup N \in C_0(a+c, b+d) \) is determined by taking first the in-going boundary components of \( M \) followed by those of \( N \), and likewise for the out-going boundary ordering.

**Remark 5.1.** We have imposed the requirement that each component has precisely one outgoing boundary sphere to ensure that all compositions remain simply connected.

It may at first seem strange that the atomic manifolds are allowed to freely generate the morphisms, since clearly a given cobordism \( M^4 \) can be written as a composition of smaller pieces in many different ways and one would want the different decompositions of \( M \) to all represent the same cobordism. However, we follow an aspect of the philosophy of 2-categories; rather than trying to equate all of the different decompositions of \( M \) into atomic manifolds, we use the 2-morphisms to encode the property that the different decompositions are all isomorphic.

**Definition 5.2.** Let \( \mathcal{C} \) be the (strict) 2–category with underlying category \( C_0 \), and with 2–morphisms given by isotopy classes (using isotopies constant on the boundary) of diffeomorphisms which respect the parametrizations and ordering of the boundary components. Thus if \( M \) and \( N \) are morphisms that are diffeomorphic (respecting the boundary data), then the 2–morphisms from \( M \) to \( N \) are the isotopy classes of diffeomorphisms from \( M \) to \( N \) which respect the boundary data; if \( M \) and \( N \) are not diffeomorphic then the set of 2–morphisms between them is empty. In the case where \( M \) is a morphism that is obtained from an identity morphism by renumbering the boundary we take the 2–morphisms from \( M \) to \( N \) to be empty unless \( N = M \), in which case there is only the identity 2–morphism. Horizontal composition of 2-morphisms is induced by gluing; vertical composition of 2–morphisms is induced from composition of diffeomorphisms.

**Definition 5.3.** The (strict) 2–category \( \mathcal{K} \) has underlying category \( C_0 \) and the 2–morphisms are now the isomorphisms of the intersection forms: a 2–morphism from \( M \) to \( N \) is an isomorphism \( H_2 M \xrightarrow{\sim} H_2 N \) which preserves the intersection form.

One may form simplicial categories \( B\mathcal{C} \) and \( B\mathcal{K} \) by replacing the morphism categories in \( \mathcal{C} \) and \( \mathcal{K} \) with their nerves. The nerve of a simplicial category is a bisimplicial set; the geometric realization of a simplicial category is defined to be the geometric realization of this bisimplicial nerve. We thus define the geometric realization \( B\mathcal{C} \) (\( B\mathcal{K} \)) of the 2–category \( \mathcal{C} \) (\( \mathcal{K} \) resp.) as the geometric realization of the associated simplicial category;

\[
B\mathcal{C} := B(B\mathcal{C}), \quad B\mathcal{K} := B(B\mathcal{K}).
\]

Disjoint union provides a strict symmetric monoidal product on each of \( \mathcal{C} \) and \( \mathcal{K} \) and hence an infinite loop structure on their geometric realizations (since both spaces are connected). There is an obvious natural 2–functor \( F : \mathcal{C} \to \mathcal{K} \); it is identity on objects and morphisms, and it sends the isotopy class of a diffeomorphism \( \phi : M \to N \) to the induced isomorphism of intersection forms \( \phi_* : (H_2 M, Q_M) \to (H_2 N, Q_N) \).
6. The proof of Theorem 2.6

In this section we prove Theorem 2.6 by identifying the homotopy type of $\Omega BK$ and showing that the 2–functor $\mathcal{C} \to \mathcal{K}$ is a homotopy equivalence after group completion. The proof is based on a group-completion argument, closely following Tillmann’s proof \cite{Til97} that $\mathbb{Z} \times B\Gamma_\infty^+$ is an infinite loop space. At the conceptual level, where Tillmann uses Harer–Ivanov stability of mapping class groups to obtain a homology fibration we substitute Lemma 4.7 together with Theorem 2.2.

Fix a morphism $X \in C_0(1, 1)$ diffeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$ with two discs removed, and consider the contravariant functor $X_\infty : C_0 \to (\text{Simplicial Sets})$ defined by the telescope construction:

$$X_\infty(n) := \text{hoColim}\{ BC(n, 1) \xrightarrow{X} BC(n, 1) \xrightarrow{X} \cdots \},$$

where $BC(n, 1)$ is the simplicial nerve of the morphism category $C(n, 1)$. A morphism $L : m \to n$ induces a map $X_\infty(n) \to X_\infty(m)$ by gluing on the left. Similarly we set

$$Y_\infty(n) := \text{hoColim}\{ BK(n, 1) \xrightarrow{X} BK(n, 1) \xrightarrow{X} \cdots \}.$$

Lemma 6.1. $Y_\infty(n) \simeq \mathbb{Z}^2 \times BO_{\infty, \infty}(\mathbb{Z})$

Proof. Swapping the order of the classifying space functor and the homotopy colimit functor expresses $Y_\infty(n)$ as the classifying space of the colimit of the hom-set groupoids $\mathcal{K}(n, 1)$. The connected components of the groupoid $\mathcal{K}(n, 1)$ correspond to the diffeomorphism classes of cobordisms $n \to 1$ and are thus indexed by the atomic manifolds $\mathcal{A}(n, 1)$. As we stabilize by gluing on copies of $X$, two objects in the groupoid $\mathcal{C}(n, 1)$ eventually become isomorphic if and only if they have the same rank and the same signature. Thus the connected components of the colimit groupoid

$$\text{colim}\{ \mathcal{K}(n, 1) \to \mathcal{K}(n, 1) \to \cdots \}$$

are $\mathbb{Z}^2$, the group completion of the additive monoid formed by the pairs $\{(\text{rank}, \text{signature})\} \subset \mathbb{N} \times \mathbb{Z}$. Hence the components of $Y_\infty(n)$ are in bijection with $\mathbb{Z}^2$ and one sees that each component of $Y_\infty(n)$ is the classifying space of a groupoid with underlying group $O_{\infty, \infty}(\mathbb{Z})$. \hfill $\square$

Lemma 6.2. $X_\infty(n) \simeq \mathbb{Z}^2 \times BO_{\infty, \infty}(\mathbb{Z})$ and the 2–functor $\mathcal{C} \to \mathcal{K}$ induces a homotopy equivalence $X_\infty(n) \simeq Y_\infty(n)$.

Proof. In $X_\infty$ swap the classifying space functor with the homotopy colimit functor, so

$$X_\infty(n) = B \text{colim}\{ \mathcal{C}(n, 1) \to \mathcal{C}(n, 1) \to \cdots \}.$$ 

The connected components of the groupoid $\mathcal{C}(n, 1)$ correspond to the diffeomorphism classes of cobordisms $n \to 1$ and are thus indexed by the atomic manifolds $\mathcal{A}(n, 1)$. As we stabilize by gluing on copies of $X$, two objects in the groupoid $\mathcal{C}(n, 1)$ eventually become isomorphic if and only if their underlying cobordisms eventually become diffeomorphic. By Wall’s stable diffeomorphism classification \cite{Wal64}, this happens if and only if the two objects have intersection forms which are stably isomorphic. Hence,
as in Lemma \[6.1\] the set of components of $X_\infty(n)$ is $\mathbb{Z}^2$, corresponding to the rank and signature. Each component of $X_\infty(n)$ is easily seen to be the classifying space of the stable 4–manifolds mapping class group, $B\Gamma_\infty$—here we use \[2.2\] to know that this group is independent of the initial manifold and hence all components are homotopy equivalent.

Now the 2–functor $C \to K$ clearly induces a bijection on components, and on each component it induces the natural map $B\Gamma_\infty \to B\text{Aut}_\infty$ which is a homotopy equivalence by Theorem \[2.2\]. □

The functor $X_\infty(\mathcal{Y}_\infty(n))$ is a $BC$–diagram ($BK$–diagram, respectively) in the language of \[Til97\]. That is to say, the simplicial set

$$\prod_n X_\infty(n)$$

is equipped with a unital and associative simplicial action of $BC$:

$$BC(n, m) \times X_\infty(m) \to X_\infty(n)$$

defined by composition on the left. One may thus form the simplicial Borel construction (a.k.a the bar construction)

$$(E_{BC}X_\infty)_k = BC(-, -) \times_N \cdots \times_N BC(-, -) \times_N X_\infty(-).$$

$k$ factors

The Borel construction commutes with the telescope, so

$$E_{BC}X_\infty = \text{hoColim}_X \{E_{BC}BC(-, 1)\}.$$ 

As observed by Tillmann, $E_{BC}BC(-, 1)$ is precisely the nerve of the comma category $(BC \downarrow 1)$ of objects in $BC$ over 1. This latter category is contractible because it contains the identity $1 \to 1$ as a terminal object, and hence $E_{BC}X_\infty$ is contractible as it is a homotopy colimit (over a contractible category) of contractible spaces.

For each $n$ we have a pull-back diagram

$$\begin{array}{ccc}
X_\infty(n) & \to & E_{BC}X_\infty \\
\downarrow & & \downarrow \\
n \downarrow & & BC \\
\end{array}$$

and thus there is a map into the homotopy fibre:

$$\begin{equation}
X_\infty(n) \to \Omega BC.
\end{equation}$$

The left translation maps $L^o : X_\infty(n) \to X_\infty(m)$ are all integral homology equivalences by Theorem \[2.2\] and Lemma \[4.7\] so Tillmann’s generalized group completion theorem implies that the group completion map \[5\] is an integral homology equivalence.

Replacing $\mathcal{X}$ with $\mathcal{Y}$ and $C$ with $K$, we obtain a homology equivalence

$$\mathcal{Y}_\infty(n) \to \Omega BK.$$
The homology equivalence of Theorem 2.6 now follows from Lemma 6.2 and the homotopy equivalence then follows from the properties of the plus construction together with the Whitehead theorem for simple spaces.

We note here that the infinite loop space structure on $\Omega BK$ coming from the symmetric monoidal product in $K$ coincides with the usual infinite loop structure induced from direct sum.

7. The spin case

To obtain the proof of Theorem 2.7 we simply restrict everything in sight to even intersection forms and then check that the proof of Theorem 2.6 goes through, at least away from the prime 2.

Let us proceed in more detail. Thus $C_{\text{spin}}$ is the sub-2-category of $C$ with the same objects and containing only those morphisms which have even intersection forms; since all 4–manifolds here are simply connected this is exactly equivalent to admitting a spin structure, and such a structure is unique whenever it exists. The 2–morphisms are all diffeomorphisms whose source and target are even; since we are dealing with simply connected spin manifolds, any diffeomorphism automatically respects the spin structure. Similarly, $K_{\text{spin}}$ is the restriction of $K$ to even morphisms.

We would like to form spin analogues of the telescopes $X_\infty$ and $Y_\infty$, but $CP^2 \# \overline{CP^2}$ is no longer available in the spin setting since it has odd intersection form, so instead we stabilize using $S^2 \times S^2$. For spin mapping class groups the analogue of Corollary 4.8 is now only a homology isomorphism with $\mathbb{Z}[1/2]$ coefficients. Corollary 2.5 no longer provides an isomorphism, so in the proof of Lemma 4.5 there is an indeterminacy when passing from an element in the mapping class group of a closed manifold to an element in the mapping class group of that manifold minus some discs. However, by Lemma 3.1 this indeterminacy is purely 2–torsion. Letting $\Gamma_{\text{spin}}(M)$ and $\text{Aut}_{\text{spin}}(Q_M)$ denote the mapping class group and automorphism group stabilized with $S^2 \times S^2$, we therefore have,

**Theorem 7.1.** The map $\Gamma_{\text{spin}}(M) \to \text{Aut}_{\text{spin}}(Q_M)$ is surjective with kernel of exponent 2. More precisely, the kernel is either $(\mathbb{Z}/2)^n-1$ or $(\mathbb{Z}/2)^n$, where $n$ is the number of boundary spheres that $M$ has (one of these is used for stabilization).

Note that $\text{Aut}_{\text{spin}}(Q_M) \cong \text{Aut}(\mathbb{Z}(\mathbb{Z}/2) \oplus \infty)$ and one clearly sees that it is independent of the initial manifold $M$. This is not quite true for $\Gamma_{\text{spin}}(M)$, but by Corollary 2.5 it does not depend on $M$ after localization away from 2.

By the proofs of Lemmas 6.1 and 6.2

**Lemma 7.2.** There is a homotopy equivalence $Y_{\text{spin}}(n) \simeq \mathbb{Z}^2 \times \text{BAut}(\mathbb{Z}(\mathbb{Z}/2) \oplus \infty)$, and the 2–functor $C_{\text{spin}} \to K_{\text{spin}}$ induces a map $X_{\text{spin}}(n) \to Y_{\text{spin}}(n)$ which is a homotopy equivalence away from the prime 2.

The group completion argument in the proof of Theorem 2.7 now goes through exactly as for the non-spin version.
8. Another infinite loop space operad

In [Til00] Tillmann gives a general construction which takes as input a family of group(oid) normal extensions of symmetric groups equipped with appropriate wreath products. The output of the construction is an operad. When the extensions are (stably) homologically trivial Tillmann shows that spaces with an action of the resulting operad are infinite loop spaces.

At the time of publication of [Til00] mapping class groups of surfaces (and their variants) were the only known examples of families of extensions which are not already trivial at the group level and which produce an infinite loop operad. We observe now that mapping class groups of simply connected 4–manifolds also form such a family.

Consider the family of groupoid extensions

$$H_n \rightarrow G_n \rightarrow \Sigma_n$$

where $H_n = \mathcal{C}(n,1)$ (see Definition 5.2), and $G_n$ is the larger groupoid of isotopy classes of diffeomorphisms which preserve the parametrization of the boundary but are no longer required to preserve the ordering. The epimorphism to the symmetric group is given by sending an isotopy class to the permutation of the in-coming boundary components that it induces. Note that $\Sigma_n$ acts on $H_n$ by permuting the ordering of the in-coming boundary components.

There are associative wreath products

$$G_n \wr G_k \rightarrow G_{nk}$$

induced by gluing the out-going boundary component of each of $n$ manifolds of type $(k,1)$ to the in-coming boundary components of a manifold of type $(n,1)$. The operad $\mathcal{E}$ formed from the above family of extensions (6) is

$$\mathcal{E}_n := BG_n,$$

and the operad composition maps are induced by the wreath products. The component of $\mathcal{E}_1$ corresponding to the identity morphism in $\mathcal{C}(1,1)$ is a point; this gives a unit for the operad. There is also a product given by a 4–sphere with three discs removed, but this product is not strictly associative or unital. One could correct this with a quotient construction, but that is not necessary for us.

The extensions (6) are nontrivial. However, they become homologically trivial when stabilized (as with $\mathcal{X}_\infty(n)$ in the previous section) by gluing copies of $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} - \{2 \text{ discs}\}$ to the in-coming boundary component and extending isotopy classes by the identity. The resulting stabilized extensions

$$H_{\infty,n} \rightarrow G_{\infty,n} \rightarrow \Sigma_n$$

have $H_{\infty,n} \cong \pi_1 \mathcal{X}_\infty(n) \cong O_{\infty,\infty}(\mathbb{Z})$, and $G_{\infty,n}$ the obvious analogue where boundary components can be permuted. Since $H_{\infty,n} \cong H_{\infty,0}$ the action of $\Sigma_n$ on $H_{\infty,n}$, and hence on $H_*(H_{\infty,n})$, is trivial.

There is also a spin analogue of the above discussion, leading to an operad $\mathcal{E}^{\text{spin}}$. Tillmann’s argument [Til00] applies verbatim to these.
Theorem 8.1. Algebras over the operad $E$ (or $E^{\text{spin}}$) are canonically infinite loop spaces.

One should be able to adapt Wahl’s comparison [Wah04] to these operads to show that the infinite loop space structure detected on the stable 4–manifold (spin) mapping class group agrees with the usual infinite loop space structure on $BO_{\infty,\infty}(\mathbb{Z})^+$ (resp. $B\text{Aut}(\infty(−E_8) \oplus \infty H)$).

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