Path Integral Equivalence between Super D-string 
and IIB Superstring

Ichiro Oda

Edogawa University, 474 Komaki, Nagareyama City, Chiba 270-01, JAPAN

Abstract

We show that the super D-string action is exactly equivalent to the IIB Green-Schwarz superstring action with some "theta term" in terms of the path integral. Since the "theta term" imposes the Gauss law constraint on the physical state but contributes to neither the mass operator nor the constraints associated with the kappa symmetry and the reparametrization, this exact equivalence implies that the impossibility to disentangle the first and second class fermionic constraints covariantly in the super D-string action is generally inherited from the IIB Green-Schwarz superstring action except specific gauge choices which make the ground state massive, such as the static gauge. Moreover, it is shown that if the electric field is quantized to be integers, the super D-string action can be transformed to the IIB Green-Schwarz superstring action with $SL(2,\mathbb{Z})$ covariant tension.

---

1 E-mail address: ioda@edogawa-u.ac.jp
1 Introduction

The recent discovery of an exact conformal field theory description of Type II D p-branes makes it possible to understand various non-perturbative properties of superstring theory [1]. In particular, it is remarkable that the Type IIB superstring in ten dimensions has an infinite family of soliton and bound state strings permuted by $SL(2, Z)$ S duality group [4].

More recently, several groups have presented supersymmetric D-brane actions with local kappa symmetry [3, 4, 5], to which a covariant quantization has been performed by adopting the so-called static gauges $X^m = \sigma^m$ for the bosonic world volume reparametrization invariance and the covariant gauge $\theta_1 = 0$ for the fermionic kappa symmetry [4]. In particular, since there is the $SL(2, Z)$ dual symmetry between IIB superstring and super D-string as mentioned above, this means that a consistent covariant gauge fixing of the fundamental IIB superstring has been successfully achieved.

Soon after this success of the covariant quantization, a natural question has been presented in the form that ”Does it mean that the previous attempt to covariantly quantize the Green-Schwarz string [6] missed the point, or something else happened? ” [7] From the recent works [7, 8], by now it seems to be generally accepted that the key ingredients for the covariant quantization of the kappa symmetry can be summarized to the following two points. One is that the consistent covariant gauge fixing of the kappa symmetry requires strictly massive ground state. For instance, it is easy to see that the static gauge $X^1 = \sigma$ for the world sheet reparametrization symmetry in the case of super D-string [4] makes the ground state massive. This statement appears quite plausible from our past knowledges because the origin of a difficulty of the covariant quantization exists in only the massless sector of superstring.

The other is that the space-time supersymmetry must be extended, i.e., $N > 1$, but this is only a necessary, not sufficient condition. To understand this importance, it is valuable to recall why we have not be able to quantize the Green-Schwarz superstring [6] in a Lorentz covariant way. The problem lies in the fact that in case of $N = 1$ supersymmetry it is impossible to disentangle covariantly 8 first class constraints generating the kappa symmetry and 8 second class constraints since the minimum off-shell dimension of covariant spinor (Majorana-Weyl spinor) in ten dimensions is equal to 16. However, provided that there is $N = 2$ supersymmetry we have a possibility of combining a pair of 8 first class constraints into 16 dimensional Lorentz covariant spinor representation, whose situation precisely occurs in the procedure performed in [4].

The purpose of this paper is two-fold. One of them is to show the exact equivalence between the super D-string action and the IIB Green-Schwarz superstring action with some ”theta term” in the framework of the path integral. It is valuable to point out that this action has been recently derived by the authors [9] within the framework of the canonical formalism where they have also showed that canonical transformation to the type IIB theory with dynamical tension is constructed to establish the $SL(2, Z)$ covariance. By contrast, in this paper, we would like to show the following thing. Namely, the ”theta term” imposes the Gauss law constraint on the physical state but does not contribute to the mass operator and the fermionic constraints, so this equivalence explicitly proves that the impossibility
to disentangle the first and second class fermionic constraints covariantly in the super D-string action is generally inherited from the IIB Green-Schwarz superstring action except specific gauge choices which make the ground state massive, such as the static gauge. The other purpose is to show that if the electric field is quantized to be integers, the super D-string action can be transformed to the IIB Green-Schwarz superstring action with $SL(2, \mathbb{Z})$ covariant tension without appealing to any semiclassical approximation.

2 The IIB string with $SL(2, \mathbb{Z})$ covariant tension

In this section, we consider the super D-string action in the flat space-time geometry, from which we would like to derive the fundamental IIB superstring action with the $SL(2, \mathbb{Z})$ covariant tension given by Schwarz formula [2]. This derivation is a straightforward generalization to the supersymmetric D-string of the bosonic D-string performed in the reference [10] but it is worthwhile to expose the full detail of it since we will make use of a similar technique in proving the exact equivalence between the super D-string and the IIB Green-Schwarz superstring with some "theta term" in the next section.

The $\kappa$-symmetric super D-string action in the flat background geometry is given by [4]

$$S = -n \int d^2 \sigma \left[ e^{-\phi} \left\{ \sqrt{-\det(G_{\mu\nu} + F_{\mu\nu})} + \epsilon^{\mu\nu} \Omega_{\mu\nu}(\tau_1) \right\} + \frac{1}{2} \epsilon^{\mu\nu} \chi F_{\mu\nu} \right],$$

(1)

where

$$\det(G_{\mu\nu} + F_{\mu\nu}) = \det G_{\mu\nu} + (F_{01})^2, \quad G_{\mu\nu} = \Pi^m \Pi^\mu_{\nu} \eta_{mn},$$

$$\Pi^m_{\mu} = \partial_\mu X^m - \theta^A \Gamma^m \partial_\mu \theta^A, \quad F_{01} = F_{01} - \epsilon^{\mu\nu} \Omega_{\mu\nu}(\tau_3),$$

$$\Omega_{\mu\nu}(\tau_1) = \left\{ -\frac{1}{2} \partial^A \Gamma_m \tau_1 \partial_\mu \theta^A (\partial_\nu X^m - \frac{1}{2} \partial^A \Gamma^m \partial_\nu \theta^A) \right\} - (\mu \leftrightarrow \nu),$$

$$\Omega_{\mu\nu}(\tau_3) = \left\{ -\frac{1}{2} \partial^A \Gamma_m \tau_3 \partial_\mu \theta^A (\partial_\nu X^m - \frac{1}{2} \partial^A \Gamma^m \partial_\nu \theta^A) \right\} - (\mu \leftrightarrow \nu).$$

(2)

Here $\mu, \nu, \cdots = 0, 1$ are the world sheet indices, $m, n, \cdots = 0, 1, \cdots, 9$ ten-dimensional space-time ones, and $A = 1, 2$ is the two-dimensional spinor index. Throughout this paper, we assume that the space-time metric takes the flat Minkowskian form defined as $\eta_{mn} = \text{diag}(- + \cdots +)$. Finally note that we confine ourselves to be only a constant dilaton $\phi$ and a constant axion $\chi$, and set the antisymmetric tensor fields to be zero.

Now we are ready to show how this super D-string action becomes a fundamental superstring action with the $SL(2, \mathbb{Z})$ covariant tension by using the path integral. The equivalence in the case of the bosonic string has already been shown in the paper [11, 10]. We shall follow the strategy found by de Alwis and Sato [11] since their method does not rely on any approximation. Incidentally, it is necessary to use a saddle point approximation if we want to apply this method to super D p-branes with $p > 1$ because of the nonlinear feature of the p-brane actions.
The major difference between super D-branes and super F-branes is the presence of $U(1)$
gauge field in the former. Hence in order to show the path integral equivalence between two
actions it is enough to concentrate on the $U(1)$ gauge sector in the super D-branes. Following
de Alwis and Sato [10], let us define the theory in terms of the first-order Hamiltonian form
of the path integral. The canonical conjugate momenta $\pi_\mu$ corresponding to the gauge field
$A_\mu$ are given by

$$\pi_0 = 0, \quad \pi_1 = \frac{ne^{-\phi}F_{01}}{\sqrt{-\det(G_{\mu\nu} + F_{\mu\nu})}} - n\chi,$$

from which the Hamiltonian has the form

$$H = T_D\sqrt{-\det G_{\mu\nu} + \epsilon^{\mu\nu}\Omega_{\mu\nu}(\tau_D) - A_0}\partial_0\pi_1 + \partial_1(A_0\pi_1),$$

where $T_D$ and $\tau_D$ are defined as

$$T_D = \sqrt{(\pi_1 + n\chi)^2 + n^2e^{-2\phi}},$$

$$\tau_D = (\pi_1 + n\chi)\tau_3 + n e^{-\phi}\tau_1.$$ (5)

Then the partition function is defined by the first-order Hamiltonian form with respect to
only the gauge field as follows:

$$Z = \int D\pi_1 DA_0 DA_1 \exp i \int d^2\sigma (\pi_1\partial_0 A_1 - H)$$

$$= \int D\pi_1 DA_0 DA_1$$

$$\times \exp i \int d^2\sigma \left[-A_1\partial_0\pi_1 + A_0\partial_1\pi_1 - T_D\sqrt{-\det G_{\mu\nu} + \epsilon^{\mu\nu}\Omega_{\mu\nu}(\tau_D) - \partial_1(A_0\pi_1)}\right].$$ (6)

where we have canceled the gauge group volume against $\int D\pi_0$. Note that if we take the
boundary conditions for $A_0$ and/or $\pi_1$ such that the last surface term in the exponential
identically vanishes, then we can carry out the integrations over $A_\mu$, which gives us $\delta$
functions

$$Z = \int D\pi_1 \delta(\partial_0\pi_1)\delta(\partial_1\pi_1) \exp i \int d^2\sigma \left[-T_D\sqrt{-\det G_{\mu\nu} - \epsilon^{\mu\nu}\Omega_{\mu\nu}(\tau_D)}\right].$$ (7)

The existence of the $\delta$ functions reduces the integral over $\pi_1$ to the one over only its zero-
modes. If we require that one space component is compactified on a circle, these zero-modes
are quantized to be integers [12]. Consequently, the partition function becomes

$$Z = \sum_{m \in \mathbb{Z}} \exp i \int d^2\sigma \left[-t_D\sqrt{-\det G_{\mu\nu} - \epsilon^{\mu\nu}\Omega_{\mu\nu}(\eta_D)}\right],$$ (8)

where

$$t_D \equiv \sqrt{(m + n\chi)^2 + n^2e^{-2\phi}},$$

$$\eta_D \equiv (m + n\chi)\tau_3 + n e^{-\phi}\tau_1.$$ (9)
To adapt the action in the above partition function to the form of the Green-Schwarz superstring action [6], one needs to replace $\eta_D$ in the argument of $\Omega_{\mu\nu}$ with $\tau_3$ by performing the $SO(2)$ rotation $\theta^A = u^{AB}\tilde{\theta}^B$ where $u$ is an orthogonal matrix with constant elements. It is easy to carry out this procedure by selecting the orthogonal matrix, for example,

$$u = \frac{1}{b} \begin{pmatrix} m + n\chi + t_D & -ne^{-\phi} \\ ne^{-\phi} & m + n\chi + t_D \end{pmatrix},$$

with $b \equiv \sqrt{(m + n\chi + t_D)^2 + n^2 e^{-2\phi}}$. From the equation $u^T \eta_D u = t_D\tau_3$, we finally arrive at a desired form of the partition function

$$Z = \sum_{m \in \mathbb{Z}} \exp i \int d^2\sigma t_D \left( -\sqrt{-\det G_{\mu\nu}} - \epsilon^{\mu\nu} \Omega_{\mu\nu}(\tau_3) \right).$$

From this expression of the partition function, we can read off the action

$$S = -t_D \left( \sqrt{-\det G_{\mu\nu}} + \epsilon^{\mu\nu} \Omega_{\mu\nu}(\tau_3) \right),$$

which implies that the super D-string action is transformed to the Type IIB Green-Schwarz superstring action with the $SL(2,\mathbb{Z})$ covariant tension $t_D$. Note that we have obtained this result without making any approximation, which is a novel feature of string theory.

3 The equivalence between super D-string and IIB superstring

In what follows let us turn our attention to main purpose in this article, that is, to show the exact equivalence between the super D-string action and the IIB Green-Schwarz superstring action with some ”theta term”. One of the motivations behind this study is to clarify that both the super D-string action and the Green-Schwarz superstring action possess a similar structure with respect to the local symmetries, which would in turn clarify the issue of the covariant quantization of the kappa symmetry.

To this aim, one needs to sophisticate the machinery developed in the previous section to adjust to the present problem. In particular, one has to deal with not $\eta_D$ involving the constant $m$ but $\tau_D$ including the field $\pi_1$. Moreover, a careful treatment of the functional measures in the case at hand gives rise to an additional complication. Keeping these technical complications in mind, let us challenge the above-mentioned problem.

As in the previous section, let us start with the super D-string action (1), and then define the partition function as in (6). However, we should remark that the total partition function $Z_T$ is really defined as

$$Z_T = \int \mathcal{D}X^m \mathcal{D}\theta \mathcal{D}Y \ Z,$$
where $DY$ generically denotes the functional measures of ghosts, auxiliary fields e.t.c. Of course, we can also consider the first-order Hamiltonian form of the path integral with respect to $X^m$ and $\theta$, but for simplicity here the second-order Lagrangian form of the path integral is taken into account. In order to rewrite the super D-string action into the form of the Green-Schwarz superstring action, first let us make the field redefinitions as follows:

$$\tilde{X}^m = T_D^{1/2} X^m, \quad \tilde{\theta} = T_D^{1/2} U^{-1} \theta,$$

where the orthogonal matrix $U$ is given by

$$U = \frac{1}{B} \left( \begin{array}{cc} \pi_1 + n\chi + T_D & -ne^{-\phi} \\ ne^{-\phi} & \pi_1 + n\chi + T_D \end{array} \right),$$

with $B \equiv \sqrt{(\pi_1 + n\chi + T_D)^2 + n^2e^{-2\phi}}$. The point to note here is that the functional measures in $Z_T$ are invariant under the field redefinitions (14). This is because if we fix the local symmetries the number of independent degrees of freedoms associated with $X^m$ and $\theta$ is respectively eight and sixteen so that the jacobian factors depending on $T_D$ exactly cancel out between bosons and fermions, and $\det U = 1$.

We next move on to consider how various quantities in the partition function (6) change under (14). For instance, we have

$$\sqrt{-\det G_{\mu\nu}} = T_D^{-1} \sqrt{-\det \tilde{G}_{\mu\nu} + f_1(\partial_\mu \pi_1)},$$

$$\Omega_{\mu\nu}(\tau_D) = \tilde{\Omega}_{\mu\nu}(\tau_3) + f_2(\partial_\mu \pi_1),$$

where $\tilde{G}$ and $\tilde{\Omega}$ are expressed in terms of $\tilde{X}^m$ and $\tilde{\theta}$, and $f_1$ and $f_2$ are certain functions whose concrete expressions are irrelevant for the present arguments. Thus after the field redefinitions we obtain the partition function

$$Z = \int D\pi_1 DA_0 DA_1 \times \exp i \int d^2 \sigma \left[ -A_1 \partial_0 \pi_1 + A_0 \partial_1 \pi_1 - \sqrt{-\det G_{\mu\nu} - e^{\mu\nu} \tilde{\Omega}_{\mu\nu}(\tau_3) + f(\partial_\mu \pi_1)} \right]$$

$$= \int D\pi_1 DA_0 DA_1 \times \exp i \int d^2 \sigma \left[ \frac{1}{2} \pi_1 e^{\mu\nu} F_{\mu\nu} - \sqrt{-\det G_{\mu\nu} - e^{\mu\nu} \tilde{\Omega}_{\mu\nu}(\tau_3) + f(\partial_\mu \pi_1)} \right],$$

where we have rewritten the terms involving the gauge field in terms of the field strength at the second stage. The remaining problem is how to deal with the last term $f(\partial_\mu \pi_1)$. Since this term is independent of the gauge field $A_\mu$ we can absorb it into the first term $\frac{1}{2} \pi_1 e^{\mu\nu} F_{\mu\nu}$ by performing an appropriate field redefinition of the gauge field. Alternatively, if we allow to carry out the path integral over the gauge field as in the previous section, we have the $\delta$-function $\delta(\partial_\mu \pi_1)$ so that the term $f(\partial_\mu \pi_1)$ vanishes identically.
After all, we reach the partition function which is exactly equivalent to that of the super D-string

$$Z = \int D\pi_1 DA_0 DA_1 \exp i \int d^2\sigma \left[ \frac{1}{2} \pi_1 \epsilon^{\mu\nu} F_{\mu\nu} - \sqrt{-\det \tilde{G}_{\mu\nu}} - \epsilon^{\mu\nu} \tilde{\Omega}_{\mu\nu}(\tau_3) \right].$$  \hspace{1cm} (18)$$

Even if the partition function was originally defined in terms of the first-order Hamiltonian form with respect to the gauge field, we can now regard it as the second-order Lagrangian form of the path integral where \( \pi_1 \) must be viewed as an auxiliary field. From this viewpoint, the action is of the form

$$S = -\int d^2\sigma \left[ \sqrt{-\det \tilde{G}_{\mu\nu}} + \epsilon^{\mu\nu} \tilde{\Omega}_{\mu\nu}(\tau_3) - \frac{1}{2} \pi_1 \epsilon^{\mu\nu} F_{\mu\nu} \right].$$  \hspace{1cm} (19)$$

In this way we have derived the action (19) which is equivalent to the super-D string action even in the quantum level as well as the classical one if we regard \( \pi_1 \) as an auxiliary field. Note that the action (19) has the form of the IIB Green-Schwarz action with the unit tension in addition to the "theta term" \( \frac{1}{2} \pi_1 \epsilon^{\mu\nu} F_{\mu\nu} \). It is quite of interest to point out that the same action as (19) has been recently derived by using the canonical transformations [9], in which it is shown that the constraints in the two actions (1) and (19) have one to one correspondence and two theories are canonically equivalent.

Let us examine more closely what implication this "theta term" has. First of all, if \( \pi_1 \) is quantized to be integers as investigated in the previous section, the "theta term" becomes the conventional two-dimensional theta term. Naively, when we neglect this true theta term it is obvious that we obtain the action (12) with \( t_D = 1 \). Of course, this difference of the tension between two actions is inessential since we can change the overall value of the tension at will by the field redefinitions.

Next, the more important point with respect to the "theta term" in (19) is that this term leads to the nontrivial constraint on the physical state. To make the arguments clear let us consider the canonical formalism. The canonical conjugate momenta to the gauge field \( A_\mu \) are given by

$$\pi_0 = 0, \quad \pi_1 = \frac{\delta S}{\delta A_1}.$$  \hspace{1cm} (20)$$

Here we have not treated \( \pi_1 \) as the dynamical variable, but although we have done so we would obtain the same result through the use of the Dirac bracket [13]. The consistency condition of the primary constraint \( \pi_0 \approx 0 \) under time evolution gives rise to the Gauss law constraint \( \partial_1 \pi_1 \approx 0 \) as the secondary constraint. According to Dirac [13], the first-class Gauss law constraint must be imposed on the state as the physical state condition

$$\partial_1 \pi_1|_{phys} \geq \partial_1 (-i \frac{\delta}{\delta A_1})|_{phys} \geq 0.$$  \hspace{1cm} (21)$$

The Gauss law constraint of two-dimensional gauge theory requires that the physical states are of the form \( \psi_p(A) = \exp ip \int_C A \) [12]. Since the gauge sector is completely decoupled from
$X^m$ and $\theta$ sector in the action (19), the total physical states are a direct product of $\psi_\mu(A)$ and the physical states of the $X^m$ and $\theta$ sector. Therefore the existence of the "theta term" in (19) has no effect on the mass operator and the constraints’ system in the $X^m$ and $\theta$ sector. From this reasoning, it is obvious that the super D-string action and the IIB Green-Schwarz superstring action share the common phase structure except the $U(1)$ gauge sector.

4 Discussions

In this paper, we have pursued the possibility of reformulating the super D-string action in terms of the IIB Green-Schwarz superstring action. It has been shown that the super D-string action is exactly equivalent to the IIB Green-Schwarz superstring action with some "theta term". Because this "theta term" completely decouples from the space-time coordinates $X^m$ and the spinor fields $\theta$ its existence does modify neither the mass spectrum nor the common constraints’ structure so that the infamous problem, the impossibility of the covariant quantization of the kappa symmetry still remains in both the actions as long as we do not choose specific gauge conditions which make the ground state massive. We believe that we have shed some light in this paper on the relation between the super D-string and the fundamental Green-Schwarz superstring.

Finally we would like to point out two issues for future work. One issue is that in order to clarify the $SL(2,\mathbb{Z})$ duality in more detail we should remove the restrictions on the flat space-time, the vanishing antisymmetric tensor fields and the constant background of the dilaton and the axion. It seems to be interesting to apply the analyses done in this paper to the more general super D-string and F-string. The other is to understand how to realize the $SL(2,\mathbb{Z})$ duality in the IIB matrix model [14]. We hope that we will return to these issues in near future.

Acknowledgement

We are grateful to K. Kamimura, R. Kuriki, A. Sugamoto and M. Tonin for valuable discussions. This work was supported in part by Grant-Aid for Scientific Research from Ministry of Education, Science and Culture No.09740212.

References

[1] J. Polchinski, TASI Lectures on D-branes, hep-th/9611050.
[2] J.H. Schwarz, Phys.Lett.B360 (1995) 13; ERRATUM ibid. B364 (1995) 252.
[3] M. Cederwall, A. von Gussich, B.E.W. Nilsson, and A. Westerberg, Nucl.Phys.\textbf{B490} (1997) 163, hep-th/9610148; M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell, and A. Westerberg, ibid. \textbf{B490} (1997) 179, hep-th/9611159.

[4] J. Aganagic, C. Popescu, and J.H. Schwarz, Phys.Lett.\textbf{B393} (1997) 311, hep-th/9610249; Nucl.Phys.\textbf{B495} (1997) 99, hep-th/9612080; J. Aganagic, J. Park, C. Popescu, and J.H. Schwarz, Nucl.Phys.\textbf{B496} (1997) 215, hep-th/9702133.

[5] E. Bergshoeff and P.K. Townsend, Nucl.Phys.\textbf{B490} (1997) 145, hep-th/9611173.

[6] M.B. Green and J.H. Schwarz, Phys.Lett.\textbf{B136} (1984) 367.

[7] R. Kallosh, Phys.Rev.\textbf{D56} (1997) 3515, hep-th/9705056; hep-th/9709069.

[8] M. Hatsuda and K. Kamimura, hep-th/9705056.

[9] Y. Igarashi, K. Itoh, K. Kamimura, and R. Kuriki, hep-th/9801118.

[10] S.P. de Alwis and K. Sato, Phys.Rev.\textbf{D53} (1996) 7187, hep-th/9601167.

[11] C. Schmidhuber, Nucl.Phys.\textbf{B467} (1996) 146, hep-th/9601003.

[12] E. Witten, Nucl.Phys.\textbf{460} (1996) 335, hep-th/9510133.

[13] P.A.M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science, Yeshiva University, (1964).

[14] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl.Phys.\textbf{498} (1997) 467, hep-th/9612113.