TOWARDS A COMBINATORIAL INTERSECTION COHOMOLOGY FOR FANS

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Abstract. The real intersection cohomology $IH^\bullet(X_\Delta)$ of a toric variety $X_\Delta$ is described in a purely combinatorial way using methods of elementary commutative algebra only. We define, for arbitrary fans, the notion of a “minimal extension sheaf” $E^\bullet$ on the fan $\Delta$ as an axiomatic characterization of the equivariant intersection cohomology sheaf. This provides a purely algebraic interpretation of the $f$- and $g$-vector of an arbitrary polytope or fan under a natural vanishing condition. — The results presented in this note originate from joint work with G.Barthel, J.-P.Brasselet and L.Kaup (see [3]).

1. Minimal Extension Sheaves

We introduce the notion of a minimal extension sheaf on a fan and study some elementary properties of such sheaves.

Let $\Delta$ be a fan in a real vector space $V$ of dimension $n$. We endow $\Delta$ with the fan-topology, having the subfans $\Lambda$ of $\Delta$ as the non-empty open subsets. In particular, affine subfans, i.e. fans $\langle \sigma \rangle$ consisting of $\sigma$ and its faces, are open; for simplicity we usually write $\sigma$ instead of $\langle \sigma \rangle$. A sheaf $\mathcal{F}$ of real vector spaces on $\Delta$ is determined by the collection of vector spaces $F_\sigma := \mathcal{F}(\sigma)$ for each $\sigma \in \Delta$ together with the restriction homomorphisms $F_\sigma \to F_\tau$ for $\tau \preceq \sigma$. An important example is the sheaf $\mathcal{A}^\bullet$ with $\mathcal{A}^\bullet(\sigma) := A_\sigma^\bullet := S^\bullet(V_\sigma^*)$, where $V_\sigma := \text{span}(\sigma)$, and the natural restriction homomorphisms. Its sections over a subfan $\Lambda$ are the piecewise polynomial functions on the support $|\Lambda|$ of $\Lambda$. 
**Warning:** We use a topologically motivated grading for $S^\bullet(V^\ast)$: Linear polynomials are of degree 2, etc.

We set $A^\bullet := S^\bullet(V^\ast)$. For a graded $A^\bullet$-module $F^\bullet$, let $\overline{F^\bullet} := R^\bullet \otimes_{A^\bullet} F^\bullet = F^\bullet/mF^\bullet$ denote the residue class vector space modulo $m := A^{>0}$, where $R^\bullet := A^\bullet/m$.

**Definition:** A sheaf $E^\bullet$ of graded $A^\bullet$-modules on the fan $\Delta$ is called a minimal extension sheaf (of $R^\bullet$) if it satisfies the following conditions:

(N) **Normalization:** One has $E^\bullet_o \cong A^\bullet_o = R^\bullet$ for the zero cone $o$.

(PF) **Pointwise Freeness:** For each cone $\sigma \in \Delta$, the module $E^\bullet_\sigma$ is free over $A^\bullet_\sigma$.

(LME) **Local Minimal Extension mod** $m$: For each cone $\sigma \in \Delta \setminus \{0\}$, the restriction mapping $\varphi_\sigma: E^\bullet_\sigma \to E^\bullet_{\partial \sigma}$ induces an isomorphism $\overline{\varphi_\sigma}: \overline{E^\bullet_\sigma} \cong \overline{E^\bullet_{\partial \sigma}}$ of graded real vector spaces.

Condition (LME) implies that $E^\bullet$ is minimal in the set of all flabby sheaves of graded $A^\bullet$-modules satisfying conditions (N) and (PF), whence the name “minimal extension sheaf”. Moreover, $E^\bullet$ vanishes in odd degrees. For a cone $\sigma \in \Delta$ and for a subfan $\Lambda \preceq \Delta$, the $A^\bullet$-modules $E^\bullet_\sigma$ and $E^\bullet_\Lambda$ are finitely generated.

If $\Delta$ is a rational fan for some lattice $N \subset V$ of maximal rank, then there is an associated toric variety $X_\Delta$ with the action of an algebraic torus $T \cong (\mathbb{C}^*)^n$. Let $IH^\bullet_T(X_\Delta)$ denote the equivariant intersection cohomology of $X_\Delta$ with real coefficients. In [1], the following theorem was proved; it has been the starting point to investigate minimal extension sheaves.

**Theorem.** Let $\Delta$ be a rational fan.

i) The assignment $\mathcal{H}^\bullet_T: \Lambda \mapsto IH^\bullet_T(X_\Lambda)$ defines a sheaf on the fan space $\Delta$; it is a minimal extension sheaf.

ii) If $E^\bullet$ is a minimal extension sheaf on $\Delta$ and $\sigma$ a $k$-dimensional cone, then for the local intersection cohomology $\mathcal{H}^\bullet_x$ of $X_\Delta$ in a point $x$ belonging to the orbit corresponding to $\sigma$, we have: $\mathcal{H}^\bullet_x \cong \overline{E^\bullet_\sigma}$.

iii) For a complete fan $\Delta$ or an affine fan $\Delta = \langle \sigma \rangle$ with a cone $\sigma$ of dimension $n$, one has $IH^\bullet(X_\Delta) \cong \overline{E^\bullet_\Delta}$. 

The vanishing axiom for local intersection cohomology together with statement ii) in the above theorem yields that, in the case of a rational fan, the following vanishing condition is satisfied:

**Vanishing Condition** \( V(\sigma) \): For a cone \( \sigma \) and a minimal extension sheaf \( E^\bullet \) on a fan \( (\sigma) \), we have \( E^q_\sigma = 0 \) for \( q \geq \dim \sigma \).

On every fan \( \Delta \) there exists a minimal extension sheaf \( E^\bullet \). Furthermore, for any two such sheaves \( E^\bullet, F^\bullet \) on \( \Delta \), each isomorphism \( E^\bullet \cong F^\bullet \) extends (non-canonically) to an isomorphism \( E^\bullet \cong F^\bullet \) of graded \( \Delta A^\bullet \)-modules, which is unique in the case of a simplicial fan. Simplicial fans are easily characterized in terms of minimal extension sheaves: The sheaf \( \Delta A^\bullet \) is a minimal extension sheaf if and only if the fan \( \Delta \) is simplicial.

2. Combinatorial equivariant perverse sheaves

We propose a definition for (combinatorially) “perverse” sheaves, here called semisimple sheaves.

**Definition:** A (combinatorially) semi-simple sheaf \( F^\bullet \) on a fan space \( \Delta \) is a flabby sheaf of graded \( A^\bullet \)-modules such that, for each cone \( \sigma \in \Delta \), the \( A^\bullet_\sigma \)-module \( F^\bullet_\sigma \) is finitely generated and free.

For each cone \( \tau \in \Delta \), we construct inductively a “simple” sheaf \( \tau E^\bullet \) on \( \Delta \) as follows: For \( \sigma \in \Delta^{\leq \dim \tau} := \{\sigma \in \Delta; \dim \sigma \leq \dim \tau\} \) we set

\[
\tau E_\sigma^\bullet := \tau E^\bullet(\sigma) := \begin{cases} A^\bullet_\tau & \text{if } \sigma = \tau, \\ 0 & \text{otherwise.} \end{cases}
\]

Now, if \( \tau E^\bullet \) has been defined on \( \Delta^{\leq m} \) for some \( m \geq \dim \tau \), then for each \( \sigma \in \Delta^{m+1} \), we set \( \tau E_\sigma^\bullet := A^\bullet_\sigma \otimes_R \tau E^\bullet_{\partial \sigma} \) with the restriction map \( \tau E_\sigma^\bullet \to \tau E^\bullet_{\partial \sigma} \) being induced by some homogeneous \( R \)-linear section \( s : \tau E^\bullet_{\partial \sigma} \to \tau E^\bullet_{\partial \sigma} \) of the residue class map \( \tau E^\bullet_{\partial \sigma} \to \tau E^\bullet_{\partial \sigma} \).

**Decomposition Theorem:** Every semi-simple sheaf \( F^\bullet \) on \( \Delta \) is isomorphic to a finite direct sum \( F^\bullet \cong \bigoplus_i \tau_i E^\bullet [-\ell_i n_i] \) of shifted simple sheaves with uniquely determined cones \( \tau_i \in \Delta \), natural numbers \( n_i \geq 1 \) and integers \( \ell_i \in \mathbb{Z} \).

From the theorem in section 3 we then obtain the following consequence:
Corollary 1: Let \( \pi: \hat{\Delta} \to \Delta \) be a refinement map of fans with minimal extension sheaves \( \hat{E}^\bullet \) resp. \( E^\bullet \). Then the direct image sheaf \( \pi_*(\hat{E}^\bullet) \) is a semisimple sheaf, in particular there is a decomposition \( \pi_*(\hat{E}^\bullet) \cong E^\bullet \oplus \bigoplus \tau_i E^\bullet[-\ell_i]^{n_i} \) with cones \( \tau_i \in \Delta^{\geq 2} \) and positive integers \( \ell_i, n_i \).

Corollary 2: For a simplicial refinement \( \hat{\Delta} \) of \( \Delta \), let \( \hat{A}^\bullet \) be the sheaf of \( \hat{\Delta} \)-piecewise polynomial functions on \( \Delta \). Then a minimal extension sheaf \( \Delta E^\bullet \) on \( \Delta \) can be realized as a subsheaf of \( \Delta \hat{A}^\bullet \).

3. Cellular Cech Cohomology of Minimal Extension Sheave

In this section, we investigate under which assumptions the module of global sections \( \Delta E^\bullet := E^\bullet(\Delta) \) of a minimal extension sheaf \( E^\bullet \) is a free \( \hat{A}^\bullet \)-module.

Definition: A fan \( \Delta \) is called quasi-convex if for a minimal extension sheaf \( E^\bullet \) on \( \Delta \), the \( \hat{A}^\bullet \)-module \( E^\bullet(\Delta) \) is free.

According to Proposition 6.1 in [1], a rational fan \( \Delta \) is quasi-convex if and only if the intersection cohomology of the associated toric variety \( X_\Delta \) vanishes in odd degrees. — The main tool to be used in the sequel is the “complex of cellular cochains with coefficients in \( E^\bullet \)”: To a sheaf \( F \) of real vector spaces on the fan \( \Delta \), we associate its “cellular cochain complex” \( C^\bullet(\Delta, F) \). The cochain module in degree \( k \) is \( \bigoplus_{\dim \sigma = n-k} F(\sigma) \), the coboundary operator \( \delta^k: C^k(\Delta, F) \to C^{k+1}(\Delta, F) \) is defined with respect to fixed orientations as in the usual Čech cohomology. For \( F = E^\bullet \), the above complex is – up to a rearrangement of the indices – a “minimal complex” in the sense of Bernstein and Lunts [4].

We also have to consider relative cellular cochain complexes with respect to the boundary subfan \( \partial \Delta \) of a purely \( n \)-dimensional fan \( \Delta \), supported by the topological boundary of \( \Delta \).

Definition: If the fan \( \Delta \) is purely \( n \)-dimensional, then for a sheaf \( F \) of real vector spaces on \( \Delta \) we set

\[
C^\bullet(\Delta, \partial \Delta; F) := C^\bullet(\Delta; F)/C^\bullet(\partial \Delta; F),
\]

where \( C^\bullet(\partial \Delta; F) \subset C^\bullet(\Delta; F) \) is the subcomplex of cochains supported in \( \partial \Delta \).
We also need the augmented complex
\[ \tilde{C}^\bullet(\Delta, \partial \Delta; \mathcal{F}) : 0 \rightarrow \mathcal{F}(\Delta) \rightarrow C^0(\Delta, \partial \Delta; \mathcal{F}) \rightarrow \ldots \rightarrow C^n(\Delta, \partial \Delta; \mathcal{F}) \rightarrow 0 \]
and its cohomology groups \( \tilde{H}^q(\Delta, \partial \Delta; \mathcal{F}) := H^q(\tilde{C}^\bullet(\Delta, \partial \Delta; \mathcal{F})) \).

**Theorem:** For a purely \( n \)-dimensional fan \( \Delta \) and a minimal extension sheaf \( \mathcal{E}^\bullet \) on \( \Delta \), the following statements are equivalent:

i) We have \( \tilde{H}^\bullet(\Delta, \partial \Delta; \mathcal{E}^\bullet) = 0 \).

ii) The \( A^\bullet \)-module \( E_\Delta^\bullet := \mathcal{E}^\bullet(\Delta) \) of global sections is free.

iii) The support \( |\partial \Delta| \) of the boundary subfan is a real homology manifold.

For a rational fan \( \Delta \), the above conditions are equivalent to

iv) For the toric variety \( X_\Delta \) associated to \( \Delta \), we have \( IH^{\text{odd}}(X_\Delta) = 0 \).

Since complete fans are quasi-convex, the previous results provide a proof of a conjecture of Bernstein and Lunts (see [4], 15.9).

**Corollary:** For a complete fan \( \Delta \), the minimal complex of Bernstein and Lunts is exact.

Furthermore, for a quasi-convex fan \( \Delta \) and a minimal extension sheaf \( \mathcal{E}^\bullet \) on \( \Delta \), even the \( A^\bullet \)-submodule \( E^\bullet_{(\Delta, \partial \Delta)} \) of \( E_\Delta^\bullet \) consisting of the global sections vanishing on the boundary subfan \( \partial \Delta \) is a free \( A^\bullet \)-submodule.

**4. Poincaré Polynomials and Poincaré duality**

For a quasi-convex fan \( \Delta \) and a minimal extension sheaf \( \mathcal{E}^\bullet \) on \( \Delta \), we want to discuss the Poincaré polynomials related to \( \Delta \) and the pair \( (\Delta, \partial \Delta) \).

**Definition:** The Poincaré polynomial of \( \Delta \) is the polynomial \( P_\Delta(t) := \sum_{q \geq 0} \dim E^q_\Delta \cdot t^q \).

The relative Poincaré polynomial \( P_{(\Delta, \partial \Delta)}(t) \) is defined in an analogous manner.

The relation between a global Poincaré polynomial \( P_\Delta \) and its local Poincaré polynomials \( P_\sigma \) for \( \sigma \in \Delta \) is rather explicit:

**Local-to-Global Formula:** If \( \Delta \) is a quasi-convex fan of dimension \( n \), we have
\[
P_\Delta(t) = \sum_{\sigma \in \Delta \setminus \partial \Delta} (t^2 - 1)^{n - \dim \sigma} P_\sigma(t) \quad \text{and} \quad P_{(\Delta, \partial \Delta)}(t) = \sum_{\sigma \in \Delta} (t^2 - 1)^{n - \dim \sigma} P_\sigma(t) .
\]
The proof of the above formulæ depends on the fact that $C^*(\Delta, \partial \Delta; E^*)$ resp. $C^*(\Delta; E^*)$ are resolutions of $E^*_\Delta$ resp. $E^*_\Delta$. Hence, the Poincaré series of $E^*_\Delta$ resp. $E^*_\Delta$ equals the alternating sum of the Poincaré series of the cochain modules $C^i(\ldots)$. Finally, we use the fact that $E^*_\Delta \cong A^* \otimes_R E^*_\Delta$, since $E^*_\Delta$ is free, and similarly for $E^*_{\Delta, \partial \Delta}$, while $E^*_\sigma \cong A^*_\sigma \otimes_R E^*_\sigma$.

Using an induction argument one proves:

**Corollary:** Let $\Delta$ be a quasi-convex fan.

i) The relative Poincaré polynomial $P_{\Delta, \partial \Delta}$ is monic of degree $2n$.

ii) The absolute Poincaré polynomial $P_\Delta$ is of degree $2n$ iff $\Delta$ is complete; otherwise, it is of strictly smaller degree.

iii) For a non-zero cone $\sigma$, the local Poincaré polynomial $P_\sigma$ is of degree at most $2 \dim \sigma - 2$.

Of course, statement ii) is a rather weak vanishing estimate; in fact we expect the much stronger vanishing condition $V(\sigma)$ to hold.

In order to have a recursive algorithm for the computation of global Poincaré polynomials, we relate the local Poincaré polynomial $P_\sigma$ to the global one of some fan $\Lambda_\sigma$ in a vector space of lower dimension: The fan $\Lambda_\sigma$ “lives” in the quotient vector space $V_\sigma/L$, where $L$ is a line in $V$ passing through the relative interior of $\sigma$. For the projection $\pi: V_\sigma \to V_\sigma/L$, we pose $\Lambda_\sigma := \{ \pi(\tau); \tau \prec \sigma \}$. The homeomorphism $\pi|_{\partial \sigma}: |\partial \sigma| \to V_\sigma/L$ induces an isomorphism of the fans $\partial \sigma$ and $\Lambda_\sigma$. Using the truncation operator $\tau_{<j}(\sum a_q t^q) = \sum_{q \leq r} a_q t^q$ we can now formulate the next step:

**Local Recursion Formula:** Let $\sigma$ be a non-zero cone.

i) If $\sigma$ is simplicial, then we have $P_\sigma \equiv 1$.

ii) If the condition $V(\sigma)$ of section 1 is satisfied, then we have $P_\sigma(t) = \tau_{<\dim \sigma}((1 - t^2)P_{\Lambda_\sigma}(t))$.

For the proof, we consider a minimal extension sheaf $G^*$ on the fan $\Lambda := \Lambda_\sigma$. Let $B^*$ be the polynomial algebra on $V_\sigma/L$, considered as subalgebra of $A^*_\sigma \cong B^*[T]$. Then we have an $B^*$-module isomorphism $G^*_\Lambda \cong E^*_{\partial \sigma}$, such that $\overline{E^*}_{\partial \sigma} \cong \overline{G^*_\Lambda}/T\overline{G^*_\Lambda}$; Here $\overline{G^*_\Lambda}$ is the residue class module of the $B^*$-module $G^*_\Lambda$ and $T$ acts on it via the
isomorphism $G'^\bullet \cong E_{\partial\sigma}',$ the latter module living over $A^\bullet_{\sigma}.$ The action of $T$ on $G'^\bullet$ coincides with the multiplication with the piecewise linear strictly convex function $\psi := T \circ (\pi|_{\partial\sigma})^{-1} \in \mathcal{A}^2(\Lambda)$. Now we use the following combinatorial version of the Hard Lefschetz Theorem:

**Combinatorial Hard Lefschetz Theorem:** In the same notations as in the proof of the Local Recursion Formula we set $m := \dim(V_{\sigma}/L) = \dim \sigma - 1.$ If the condition $V(\sigma)$ is satisfied, then $\mu: G'^\bullet_{\Lambda} \rightarrow G'^\bullet_{\Lambda}[2], f \mapsto \psi f$ induces a map $\mu^{2q} : \overline{G}_{\Lambda}^{2q} \rightarrow \overline{G}_{\Lambda}^{2q+2},$ which is injective for $2q \leq m - 1$ and surjective for $2q \geq m - 1.$

The surjectivity is nothing but a reformulation of the vanishing condition $V(\sigma),$ whence the injectivity is obtained via Poincaré duality for the real vector space $\overline{G}_{\Lambda},$ the map $\overline{\mu}$ being selfadjoint with respect to the Poincaré duality pairing:

Such a Poincaré duality on a quasi-convex fan $\Delta$ is obtained as follows: By a stepwise procedure, one constructs an internal (non-canonical) intersection product $\mathcal{E}^\bullet \times \mathcal{E}^\bullet \rightarrow \mathcal{E}^\bullet.$ Then one composes the induced product on the level of global sections with an evaluation map $E^\bullet_{(\Delta,\partial\Delta)} \rightarrow A^\bullet[-2n],$ which is homogeneous of degree 0 and unique up to a non-zero real factor. That construction uses the above corollary and the freeness of $E^\bullet_{(\Delta,\partial\Delta)}.$ The pairing thus obtained induces a pairing on the level of residue class vector spaces.

**Poincaré Duality Theorem:** For every quasi-convex fan $\Delta$, the pairings

$$E^\bullet_{\Delta} \times E^\bullet_{(\Delta,\partial\Delta)} \rightarrow E^\bullet_{(\Delta,\partial\Delta)} \rightarrow A^\bullet[-2n]$$

and

$$\overline{E}^\bullet_{\Delta} \times \overline{E}^\bullet_{(\Delta,\partial\Delta)} \rightarrow \overline{E}^\bullet_{(\Delta,\partial\Delta)} \rightarrow \mathbb{R}^\bullet[-2n].$$

are dual pairings of free $A^\bullet$-modules resp. of $\mathbb{R}$-vector spaces.

We end this section with a numerical version of Poincaré duality:

**Corollary:** For a quasi-convex fan $\Delta$, the global Poincaré polynomials $P_{\Delta}$ and $P_{(\Delta,\partial\Delta)}$ are related by the identity

$$P_{(\Delta,\partial\Delta)}(t) = t^{2n}P_{\Delta}(t^{-1}).$$

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