Covers of multiplicative groups of algebraically closed fields of arbitrary characteristic

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Abstract

We show that algebraic analogues of universal group covers, surjective group homomorphisms from a \(\mathbb{Q}\)-vector space to \(F^\times\) with ‘standard kernel’, are determined up to isomorphism of the algebraic structure by the characteristic and transcendence degree of \(F\) and, in positive characteristic, the restriction of the cover to finite fields. This extends the main result of ‘Covers of the Multiplicative Group of an Algebraically Closed Field of Characteristic Zero’ (Zilber, JLMS 2007), and our proof fills a hole in the proof given there.

1. Introduction

This paper was conceived as an extension of the main results of [10] to fields of positive characteristic. But in the course of proving the main result, a gap in the proof of the main technical theorem, Theorem 2 (in the case of \(n > 1\) fields), was detected. So the aim of this paper has become two-fold (to fix the proof in the characteristic 0 case and to extend it to all characteristics). This goal has now been achieved.

The reader will see that we had to correct the formulation of the theorem of [10]. Theorem 3 now requires that the fields \(L_1, \ldots, L_n\) are from an independent system, in the same sense as in [8, Section 4], and in accordance with Shelah’s theory of excellence. Indeed, the necessity of this condition has stressed again the amazingly tight interaction of field-theoretic algebra and very abstract model theory.

A simple but instructive case of Theorem 3 is the following statement.

Let \(L_1\) and \(L_2\) be linearly disjoint algebraically closed subfields of a common field of characteristic 0 and \(L_1L_2\) their composite. Then the multiplicative group \((L_1L_2)^\times\) of the composite is of the form \(A \times (L_1^\times \cdot L_2^\times)\), for some locally free abelian group \(A\). Surprisingly, even this was apparently unknown.

In characteristic \(p\) the statement is true with \(A\) a locally free \(\mathbb{Z}/[1/p]\)-module written multiplicatively.

Here locally free, also known as \(\aleph_1\)-free, means that any finite rank subgroup (submodule) is free as an abelian group (module). Note that this definition does not agree with the definition of ‘locally free’ in general group theory.

Our main technical proposition, Proposition 1, exhibits a construction that produces fields \(K\) with the multiplicative group of the form \(A \times D\), where \(A\) is locally free and \(D\) possesses \(n\)-roots of elements, for any \(n\). This construction is suggested by Shelah’s notion of independent system and plays a crucial role in proving the uniqueness of universal covers of the multiplicative group of an algebraically closed field.

Received 7 August 2008; revised 18 June 2010; published online 26 February 2011.

2000 Mathematics Subject Classification 12F10 (primary), 03C60, 12L12 (secondary).

This research was supported by EPSRC grant EP/P500397/1 and by the Marie Curie project MRTN-CT-2004-512234 (MODNET).
2. Statement of results and outline of proof

The main theorem of [10] is the following.

**Theorem 1.** For each cardinal \( \kappa \) there is up to isomorphism a unique 2-sorted structure \( \langle \langle V; + \rangle; \langle F; +, \ast \rangle; \text{ex} : V \to F \rangle \) with \( V \) a divisible torsion-free abelian group and \( F \) an algebraically closed field of transcendence degree \( \kappa \) such that

\[
0 \longrightarrow \mathbb{Z} \longrightarrow V \xrightarrow{\text{ex}} F^\times \longrightarrow 1 \tag{2.1}
\]

is an exact sequence of groups.

In positive characteristic the statement must be modified.

**Theorem 2.** Given a choice of structure \( \mathfrak{C}_0 := \langle \langle \mathbb{Q}; + \rangle; \mathbb{F}_p \text{alg}; \text{ex}_0 : \mathbb{Q} \to \mu \rangle \), where \( \mu = (\mathbb{F}_p \text{alg})^\times \), such that

\[
0 \longrightarrow \mathbb{Z} \langle \frac{1}{p} \rangle \longrightarrow \mathbb{Q} \xrightarrow{\text{ex}_0} \mu \longrightarrow 1 \tag{2.2}
\]

is an exact sequence of groups, for each cardinal \( \kappa \) there is up to isomorphism a unique 2-sorted structure \( \mathfrak{C} := \langle \langle V; + \rangle; \langle F; +, \ast \rangle; \text{ex} : V \to F \rangle \) extending \( \mathfrak{C}_0 \) with \( V \) a divisible torsion-free abelian group and \( F \) an algebraically closed field of characteristic \( p \) and transcendence degree \( \kappa \) such that

\[
0 \longrightarrow \mathbb{Z} \langle \frac{1}{p} \rangle \longrightarrow V \xrightarrow{\text{ex}} F^\times \longrightarrow 1 \tag{2.3}
\]

is an exact sequence of groups.

Theorems 1 and 2 are proved by showing quasi-minimal excellence [9] of the class of models of an appropriate \( L_{\omega_1, \omega} \)-sentence, expressing that we have such a sequence and, in positive characteristic, that \( \text{ex} \) is as specified on \( \mathbb{Q} \cdot \ker(\text{ex}) \).

For reference, we give a quick outline of the main stages in the proof now.

The characteristic \( p \) is zero or prime, and \( \mathfrak{C} \) is an arbitrary fixed algebraically closed field of characteristic \( p \).

We use a version of Shelah’s notion of an independent system.

**Definition 1.** We say algebraically closed subfields \( L_1, \ldots, L_n \) of \( \mathfrak{C} \) are from an independent system if and only if there exist an algebraically independent set \( B \subseteq \mathfrak{C} \) and subsets \( \overline{B}_i \subseteq B \) such that \( B = \bigcup_i \overline{B}_i \) and \( L_i = \text{acl}^{\mathfrak{C}}(\overline{B}_i) \).

In the case \( n = 2 \), this condition reduces to saying that \( L_1 \) is linearly disjoint from \( L_2 \) over \( L_1 \cap L_2 \).

**Definition 2.** If \( \overline{c} \in \mathfrak{C}^\times \) is a \( k \)-tuple, a division system below \( \overline{c} \) consists of a system of roots \( (\overline{c}^{1/n})_{n \in \mathbb{N}} \) such that \( \overline{c}^1 = \overline{c} \) and \( (\overline{c}^{1/nm})^n = \overline{c}^{1/m} \). For a rational \( q = m/n \), we define \( c^q_i := (c^{1/n}_i)^m \). For an \( l \times k \) rational matrix \( M = (q_{i,j})_{i,j} \in \text{Mat}_{l,k}(\mathbb{Q}) \), we define \( \overline{c}^M \) to be the \( l \)-tuple \( (\Pi_j c^q_j)_{i,j} \), and define \( \overline{c}^Q := (\overline{c}^M)_{M \in \text{Mat}_{1,k}(\mathbb{Q})} \subseteq \mathfrak{C}^\times \).
If \( K \leq \mathcal{C} \), we say that division systems below \( \overline{c} \) are finitely determined over \( K \) if and only if there exists \( m \in \mathbb{N} \) such that if \( (\overline{c}_1^{1/n})_n \) and \( (\overline{c}_2^{1/n})_n \) are division systems below \( \overline{c} \) with \( \overline{c}_1^{1/m} = \overline{c}_2^{1/m} \), then, for all \( n \in \mathbb{N} \), we have that \( \overline{c}_1^{1/n} \) and \( \overline{c}_2^{1/n} \) have the same field type over \( K \).

We deduce quasi-minimal excellence from the following theorem, the analogue of [10, Theorem 2].

**Theorem 3.** Let \( n \geq 1 \) and let \( L_1, \ldots, L_n \) be algebraically closed subfields of \( \mathcal{C} \) from an independent system. Let \( (\overline{a}, \overline{b}) \in \mathcal{C}^\times \) be multiplicatively independent over the product \( \Pi_i L_i^\times \).

Then division systems below \( \overline{b} \) are finitely determined over \( L_1 L_2 \ldots L_n(\overline{\beta}) \).

Theorem 3 will in turn follow by Kummer theory from the following proposition describing the structure of the multiplicative groups of finitely generated perfect extensions of composites of algebraically closed fields from an independent system.

**Proposition 1.** Let \( \mathcal{C} \) be an algebraically closed field and let \( L_1, \ldots, L_n \leq \mathcal{C} \) be algebraically closed subfields from an independent system, with \( n \geq 1 \). Let \( K \) be the perfect closure of a finitely generated extension \( L_1 \ldots L_n(\overline{\beta}) \leq \mathcal{C} \) of \( L_1 \ldots L_n \).

Then \( K^\times / \Pi_i L_i^\times \) is a locally free \( R_p \)-module.

Although Proposition 1 will suffice along with some results from [10] to prove Theorem 2, we state here a natural extension.

**Proposition 2.** In each of the following situations, \( (K^{\text{per}})^\times / H \) is a locally free \( R_p \)-module, where \( K^{\text{per}} \) is the perfect closure of \( K \):

- (i) \( K \) is a finitely generated extension of the prime field and \( H \) is the torsion group of \( K^\times \);
- (ii) \( K \) is a finitely generated extension of the field generated by the group \( \mu \) of all roots of unity and \( H = \mu \);
- (iii) \( K \) is a finitely generated extension of the composite \( L_1 \ldots L_n \) of algebraically closed fields from an independent system and \( H = \Pi_i L_i^\times \).

In the first two cases, and in the third if \( K \) is countable or \( n = 1 \), \( (K^{\text{per}})^\times / H \) is free.

**Remark 1.** Theorem 2 of [10] claims the statement of Theorem 3 for arbitrary finite-dimensional algebraically closed fields \( L_i \), with no independence assumption. The proof given there was flawed, but we have no counter-example to this statement; it would be interesting to determine whether it is true.

3. **Torsion-free \( R_p \)-modules**

**Definition 3.** (i) For \( p \) a positive prime, let \( R_p \) be the subring \( \mathbb{Z}[1/p] \) of \( \mathbb{Q} \).

(ii) For \( p = 0 \), let \( R_p \) be the ring \( \mathbb{Z} \).

To prove Theorem 3, we will need to work with the multiplicative groups of perfect (that is, definably closed) subfields of \( \mathcal{C} \). These have the natural structure of \( R_p \)-modules. If \( p = 0 \),
an $R_p$-module is just an abelian group, and for $p > 0$ they can be treated analogously; we therefore borrow definitions and developments from the theory of abelian groups.

In this section, $M$ will be a torsion-free $R_p$-module written additively.

Here, and throughout the paper, we use tuple notation. A tuple is a sequence $\overline{a} = (a_i)_{i \in \lambda}$. All tuples will be finite, that is, $\lambda \in \omega$, unless otherwise specified. We write (slightly abusively) $\overline{a} \in A$ to mean that $\overline{a}$ is a finite tuple such that $a_i \in A$ for all $i$. Unary functions lift to tuples co-ordinatetwise; for example, if $f : A \to B$ is a function, and $\overline{a} \in A$, then $f(\overline{a}) = (f(a_1), \ldots, f(a_n)) \in B$.

The ring $R_p$ is a principal ideal domain with fraction field $\mathbb{Q}$, so we have the usual definitions.

**Definition 4.** (i) The span $\langle A \rangle \leq M$ of $A \subseteq M$ is the $R_p$-submodule generated by $A$.

(ii) $\overline{b}$ is independent over $A \leq M$ if and only if

$$\forall \overline{a} \in R_p \ (\sum_i n_i b_i \in A \implies \overline{a} = \overline{0}).$$

$B \subseteq M$ is independent over $A$ if and only if every finite tuple $\overline{b} \in B$ is independent over $A$.

(iii) The rank $r(A)$ of $A \leq M$ is the cardinality of any maximal independent $B \subseteq A$. This is well defined.

(iv) $M$ is free of rank $\kappa$ if and only if it is isomorphic to the direct sum of $\kappa$ copies of $R_p$, and equivalently if it is the span of an independent set (called a basis of $M$) of cardinality $\kappa$.

(v) $M$ is locally free if and only if any finite rank submodule is free.

(vi) $M$ embeds in its divisible hull $\text{divHull}(M) := M \otimes_{R_p} \mathbb{Q}$, a $\mathbb{Q}$-vector space, and $A \leq M$ embeds in the subspace $\text{divHull}(A) := A \otimes_{R_p} \mathbb{Q}$ of $M \otimes_{R_p} \mathbb{Q}$, and the embeddings commute. $R_p$-independence agrees with $\mathbb{Q}$-independence in the divisible hull, and $r(A)$ is the vector space dimension of $\text{divHull}(A)$.

Our aim is to show that certain $R_p$-modules are locally free. To this end we develop the notions of purity and simplicity.

**Definition 5.** (i) The pure hull of a submodule $A \leq M$ is $\text{pureHull}_M(A) := \{ x \in M | \exists n \in R_p \setminus \{0\}, nx \in A \}$.

(ii) A submodule $A \leq M$ is pure in $M$ if and only if $\text{pureHull}_M(A) = A$.

(iii) A tuple $\overline{a} \in M$ is simple in $M$ if and only if $\overline{a}$ is independent and $\langle \overline{a} \rangle$ is pure in $M$. If $A \leq M$ is a pure submodule, then $\overline{a} \in M$ is simple in $M$ mod $A$ if and only if $\overline{a} / A$ is simple in the torsion-free $R_p$-module $M / A$.

**Remark 2.** For $A \leq M$, the quotient $R_p$-module $M / A$ is torsion-free if and only if $A$ is pure in $M$.

**Remark 3.** In the next section, we will be considering quotients of multiplicative groups of perfect fields by divisible subgroups containing the torsion. It follows from Remark 2 that such quotients are torsion-free $R_p$-modules.

**Lemma 3.1.** Suppose that $A, B, C$ are $R_p$-modules and $B$ is an extension of $A$ by $C$:

$$A \xrightarrow{\phi} B \xrightarrow{\phi} C.$$ (3.1)

(i) If $A$ and $C$ are free, then $B$ is free.

(ii) If $A$ and $C$ are locally free, then $B$ is locally free.
Proof. (i) Say \((\phi(b_i))_{i \in I}\) is a basis for \(C\). Then \((b_i)_{i \in I}\) are independent, and \(B = A \oplus \langle(b_i)_{i \in I}\rangle\). So \(B\) is the direct sum of free modules, hence is free.

(ii) Let \(B'\) be a finite rank submodule of \(B\). Then we have the exact sequence

\[
A \cap B' \rightarrow B' \rightarrow \phi(B').
\]

But \(A \cap B'\) and \(\phi(B')\) are both finite rank and hence free; so \(B'\) is free by (i).

The following facts are standard results on modules over principal ideal domains.

**Fact 3.2.** Any finitely generated torsion-free \(R_p\)-module is free.

**Fact 3.3.** Any submodule of a free torsion-free \(R_p\)-module is free.

**Lemma 3.4.** A torsion-free \(R_p\)-module \(M\) is locally free if and only if, for every finite independent \(a \in M\), the pure hull of \(\langle a \rangle\) in \(M\) is free.

Proof. The forward direction is immediate from the definition of local freeness. For the converse, suppose \(A \leq M\) is finite rank. Let \(a \in A\) be a maximal independent set. Then \(A\) is contained in the pure hull of \(\langle a \rangle\), which is free by assumption. So \(A\) is free by Fact 3.3.

The next two lemmas reduce the condition of purity of a finitely generated submodule to an easily checked condition on the divisibility of points.

**Lemma 3.5.** A finitely generated submodule \(A \leq M\) is pure in \(M\) if and only if every \(a \in A\) which is simple in \(A\) is simple in \(M\).

Proof. The forward implication is clear. Conversely, suppose that \(A\) is not pure in \(M\). Say \(a \in M \setminus A\), and \(ma = a \in A\) for some \(m \in R_p \setminus \{0\}\). By Fact 3.2, \(A\) is free, so the pure hull of \(a\) in \(A\) is free of rank 1, say generated by \(a'\). Then \(a'\) is simple in \(A\) but not in \(M\).

**Lemma 3.6.** An element \(a \in M\) is not simple in \(M\) if and only if \(l\alpha = a\) for some \(\alpha \in M\) and some prime \(l \neq p\).

Proof. Suppose that \(a\) is not simple in \(M\). Then \(m\beta = na\) for some \(\beta \in M \setminus \langle a \rangle\) and some \(m, n \in R_p\). Multiplying on both sides of the equation by a power of \(p\), we can take \(m, n \in \mathbb{Z}\), and by changing \(\beta\) we can then take \(m \notin p\mathbb{Z}\). We may assume \(gcd(m, n) = 1\). So there exist \(s, t \in \mathbb{Z}\) such that \(sm + tn = 1\). Then \(m(t\beta + sa) = a\). We complete the proof by taking \(l\) to be a prime divisor of \(m\).

We will have to deal with the delicate question of when a quotient of a locally free torsion-free \(R_p\)-module \(M\) by a pure submodule \(B\) is locally free, and more generally when, for a finite tuple \(\tau \in M\) independent over \(B\), we have that the pure hull of \(\tau/B\) in \(M/B\) is free. Note that if \(B\) is finitely generated (equivalently, finite rank) then \(M/B\) is locally free, but that the quotient by an infinite rank submodule need not be locally free.
The following lemma shows that if, in a certain sense, all the ‘extra divisibility’ of \( \pi \) introduced by quotiening by \( B \) is explained by a finite rank portion of \( B \), then the pure hull of \( \pi/B \) is indeed free.

For \( D \) an \( R_p \)-module and \( m \in R_p \), we say that \( d \in D \) is \('m\)-divisible in \( D' \) if and only if \( \exists d' \in D, md' = d \).

**Lemma 3.7.** Let \( M \) be a locally free torsion-free \( R_p \)-module.
Suppose that \( A \leq B \leq M \), that \( B \) is pure in \( M \) and that \( A \) is finitely generated.
Let \( \pi \in M \) be independent over \( B \).
Suppose that it holds for all \( c \in \langle \pi \rangle \) and all \( m \in R_p \) that if \( c/B \) is \('m\)-divisible in \( M/B \), then already \( c/A \) is \('m\)-divisible in \( M/A \).

Then the pure hull of \( \langle \pi/B \rangle \) is free.

**Proof.** Say \( A = \langle \pi \rangle \). By local freeness, the pure hull of \( \langle \pi \rangle \) is free, say freely generated by \( \pi \).

**Claim 1.** \( \langle \pi/B \rangle \) is the pure hull of \( \langle \pi/B \rangle \) in \( M/B \).

**Proof of Claim.** (i) \( \langle \pi/B \rangle \subseteq \text{pureHull}_{M/B}(\langle \pi/B \rangle) \): Indeed let \( c/B \in \langle \pi/B \rangle \). Without loss of generality, \( e \in \langle \pi \rangle \). Then \( e \) is in the pure hull of \( \langle \pi \rangle \), so say \( s \cdot e = a + c \), where \( a \in \langle \pi \rangle \) and \( c \in \langle \pi \rangle \). But \( A \subseteq B \), so \( s \cdot c/B = c/B \in \langle \pi/B \rangle \).

(ii) \( \langle \pi/B \rangle \) is pure in \( M/B \): Indeed suppose \( m \cdot \alpha/B = c/B \), where \( e \in \langle \pi \rangle \) and \( \alpha \in M \). As above, let \( c \in \langle \pi \rangle \) and \( s \in \mathbb{Z} \) be such that \( s \cdot c/B = c/B \). Then \( sm \cdot \alpha/B = c/B \), so by the assumption, for some \( \alpha'/M \), we have \( sm \cdot \alpha'/A = c/A \). Hence, \( sm \cdot \alpha' = c + a \) say, so \( \alpha' \in \langle \pi \rangle \).

But \( sm \cdot (\alpha - \alpha') \in B \), and so, by the purity of \( B \), we have \( \alpha/B = \alpha'/B \). So \( \alpha/B \in \langle \pi/B \rangle \), as required.

Now \( \langle \pi/B \rangle \) is finitely generated, and so by Fact 3.2 is free. Hence, the result follows from the claim.

**4. Proof of Proposition 1**

**Notation.** For subfields \( F, F' \) of an algebraically closed field \( \mathcal{C} \), we write \( F \vee F' \) for the perfect closure of the compositum in \( \mathcal{C} \) of \( F \) and \( F' \) (so in model-theoretic terms, \( F \vee F' = \text{dcl}(F \cup F') \)), and we write \( F \vee \pi \) for \( F \vee F' \), where \( F' \) is the subfield of \( \mathcal{C} \) generated over the prime field by \( \{a_1, \ldots, a_n\} \).

We write \( \mu \) for the multiplicative group of all roots of unity.

We make use of some notions from valuation theory. We consider a place of a field \( \pi : K \to k \) to be a partially defined ring homomorphism such that the domain of definition \( \mathcal{O}_\pi := \text{dom}(\pi) \) is a valuation ring. If \( k \subseteq K \), then we write \( \pi : K \to k \) to indicate that \( \pi \) is the identity on \( k \), in other words, that the field embedding of \( k \) in \( K \) is a section of \( \pi \). Such a \( \pi \) is sometimes called a specialization of \( K \) to \( k \).

We make use of the Newton–Puiseux theorem, or rather the following generalization to arbitrary characteristic.

**Fact 4.1** (Rayner [7], cited in [3]). Let \( L \) be an algebraically closed field of characteristic \( p \geq 0 \). Let \( L((t^Q)) \) be the field of generalized formal power series in \( t \) with coefficients in \( L \) and
rational exponents, and let \( L\{\{t\}\} \subseteq L((t^2)) \) be the subfield consisting of those power series with support \( S \subseteq \mathbb{Q} \) satisfying the following condition:

(i) there exists \( m \in \mathbb{Z} \setminus \{0\} \) such that \( mS \subseteq R_p \).

Then \( L\{\{t\}\} \) is an algebraically closed field.

**Lemma 4.2.** Let \( L \) be an algebraically closed subfield of an algebraically closed field \( \mathcal{C} \); suppose that \( L \) contains algebraically closed subfields \( k_i \) for \( i \in \{1, \ldots, n\} \); let \( \lambda \in \mathcal{C} \) be transcendental over \( L \); let \( K := \text{acl}(L(\lambda)) \supseteq L \); and let \( k'_i := \text{acl}(k_i(\lambda)) \). Further, let \( k_0 \leq L \) be a perfect subfield and let \( k'_0 := k_0 \).

Then, for any place \( \pi : K \rightarrow L \) such that \( \pi(\lambda) \subseteq \bigcap_{i>0} k_i \),

\[
\pi \left( \bigvee_{i \geq 0} k'_i \right) = \bigvee_{i \geq 0} k_i.
\]

**Proof.** Since replacing \( \lambda \) with \( \lambda - \pi(\lambda) \) does not alter \( K \) or \( k'_i \), and \( \lambda - \pi(\lambda) \) is also transcendental over \( L \), we may assume that \( \pi(\lambda) = 0 \).

Let \( L\{\{\lambda\}\} \) be the field of generalized Puiseux series, as defined in Fact 4.1. Let \( \pi' : L\{\{\lambda\}\} \rightarrow L \) be the standard power series residu map.

Here \( \pi' \) agrees with \( \pi \) on \( L(\lambda) \), and so, by the Conjugation Theorem [1, 3.2.15], we may embed \( K \) into \( L\{\{\lambda\}\} \) over \( L(\lambda) \) in such a way that \( \pi' \) agrees with \( \pi' \).

Now, for \( i > 0 \), the subfield \( k_i \{\{\lambda\}\} \subseteq L\{\{\lambda\}\} \) of power series with coefficients from \( k_i \) is algebraically closed and contains \( k_i(\lambda) \), and so contains \( k'_i \). Similarly \( k'_0 = k_0 \leq k_0 \{\{\lambda\}\} \).

Now

\[
\pi \left( \bigvee_{i \geq 0} k'_i \right) \leq \pi' \left( \bigvee_{i \geq 0} (k_i \{\{\lambda\}\}) \right) \leq \pi' \left( \left( \bigvee_{i \geq 0} k_i \right) \{\{\lambda\}\} \right) = \bigvee_{i \geq 0} k_i.
\]

**Lemma 4.3.** Suppose \( L_1, \ldots, L_n \subseteq \mathcal{C} \) are algebraically closed subfields from an independent system, witnessed by an independent set \( B = B_1 \cup \ldots \cup B_n \) as in Definition 1. Let \( B^0 \subseteq B \) and define \( B^0_i := B_i \cap B^0 \) and \( L^0_i := \text{acl}(B^0_i) \). Let \( C \subseteq \text{acl}(B^0) \).

Then there exists a place \( \pi : \text{acl}(B) \rightarrow \text{acl}(B^0) \) such that \( \pi(L_i) = L^0_i \) and \( \pi(\bigvee L_i \cup C) = \bigvee L^0_i \cup C \).

Furthermore, for any finite tuple \( \pi \in \text{acl}(B)^\times \), \( \pi \) can be chosen such that \( \pi(\pi) \in \text{acl}(B^0)^\times \).

**Proof.** Let the possibly infinite tuple \( \bar{b} = (b_\alpha)_{\alpha < \lambda} \) enumerate \( B \setminus B^0 \).

For \( \beta \leq \lambda \), define \( B^\beta := B^0 \cup \{b_\alpha | \alpha < \beta\} \); \( L^\beta := \text{acl}(B^\beta) \); \( B_\beta^i := B_i \cap B^\beta \); \( L_\beta^i := \text{acl}(B_\beta^i) \); and \( K^\beta := \bigvee L_\beta^i \cup C \).

Let \( f_{i,j}(\bar{b}) \in L^0[\bar{b}] \) be the non-zero coefficients of a minimal polynomial in \( L^0[\bar{b}][X] \) for \( c_i \) over \( L^0(\bar{b}) \). Let \( \pi = (a_\alpha)_{\alpha < \lambda} \in \text{acl}(\emptyset) \) be such that \( f_{i,j}(\pi) \neq 0 \) for all \( i, j \).

We define, by transfinite recursion on \( \beta \leq \lambda \), places \( \pi^\beta : L^\beta \rightarrow L^0 L^0 \), such that \( \pi^\beta(b_\alpha) = a_\alpha \) for \( \alpha < \beta \), and \( \pi^\beta(L^\beta_i) = L^0_i \) and \( \pi^\beta(K^\beta) = K^\lambda \), and \( \pi^\gamma \upharpoonright L^\gamma = \pi^\gamma \) for \( \gamma < \beta \).
Define $\pi^0 := \text{id}_{L^0}$, and take unions at limit ordinals. If $\beta = \gamma + 1$ is a successor ordinal, by Lemma 4.2 if $\pi^\gamma + 1 : L^\gamma + 1 \to L^\gamma$ is a place such that $\pi^\gamma + 1(b_\gamma) = a_\gamma$, then $\pi^\gamma + 1(K^\gamma + 1) = K^\gamma$; clearly we also have $\pi^\gamma + 1(L^\gamma_i + 1) = L^\gamma_i$. So $\pi^\gamma + 1 := \pi^\gamma \circ \pi^\gamma + 1$ is as required.

Now let $\pi := \pi^\gamma$. By the condition on $\pi$, we have $\pi(e_i) \in L^0$. □

**Lemma 4.4.** Let $K \supseteq L$ be algebraically closed fields and let $\pi : K \to L$ be a place. Let $k_0 \leq K$ be a perfect subfield such that $\pi k_0 \leq k_0$. Let $k_1 \geq k_0$ be a finite extension.

Then there exists a finite extension $k' \geq k_1$ such that $\pi k' \leq k'$.

**Proof.** We may assume that $k_1/k_0$ is Galois.

For $i \geq 1$, define $k_{i+1} := k_i(\pi k_i)$.

A finite extension of a perfect field is perfect, so each $k_i$, and hence each $\pi k_i$, is perfect.

Normality of a finite field extension implies $\left[ K : k_i \right] = \left[ K : k_{i+1} \right]$; normality of the corresponding extension of residue fields; it follows inductively that, for all $i \geq 0$, the extensions $k_{i+1}/k_i$ and $\pi k_{i+1}/\pi k_i$ are Galois.

Now $k_{i+2}$ is generated over $k_{i+1}$ by $\pi k_{i+1}$, and $\pi k_i \leq k_{i+1}$, so $[k_{i+2} : k_{i+1}] \leq [\pi k_{i+1} : \pi k_i]$.

Also, $[\pi k_{i+1} : \pi k_i] \leq [k_{i+1} : k_i]$. Hence, after some $n$, the degrees reach their minimum level, say

$$d = [\pi k_{n+2} : \pi k_{n+1}] = [k_{n+2} : k_{n+1}] = [\pi k_{n+1} : \pi k_n] = [k_{n+1} : k_n].$$

By the fundamental inequality of valuation theory [1, 3.3.4],

(I) any $\pi \in \text{Gal}(k_{n+1}/k_n)$ preserves $\mathcal{O}_\pi \cap k_{n+1};$

(II) any $\pi \in \text{Gal}(k_{n+2}/k_{n+1})$ preserves $\mathcal{O}_\pi \cap k_{n+2}.$

Now $\pi k_{n+1} = (\pi k_n)(\pi \beta)$, say, for some $\beta \in k_{n+1}$. Let $\beta = \beta_1, \beta_2, \ldots, \beta_n$ be the $k_n$-conjugates of $\beta$. By (I), $\beta_i \in \mathcal{O}_\pi$ for all $i$. Applying $\pi$ to the minimum polynomial $\Pi_i(x - \beta_i)$, we see that $s = d$ and the $(\pi k_n)$-conjugates of $\pi \beta$ are precisely $(\pi \beta)_i$.

Now suppose for a contradiction that $\pi \in \text{Gal}(k_{n+2}/k_{n+1}) \setminus \{\text{id}\}$. We have $k_{n+2} = k_{n+1}(\pi \beta)$, so $\sigma(\pi \beta) = \pi \beta_i$ some $i > 1$.

Now $\beta - \pi \beta \notin m_\pi \cap k_{n+1}$, but $\sigma(\beta - \pi \beta) = \beta - \pi \beta \in \mathcal{O}_\pi \cap k_{n+1}$. This contradicts (II).

Thus, $d = 1$, and so $\pi k_n \leq k_n$. □

**Fact 4.5** [6, Proposition 1]. Let $E \supseteq F$ be a finitely generated regular extension. Then $E^x/F^x$ is free as an abelian group.

This fact slightly extends the second statement of [10, Lemma 2.1]. The proof involves considering the Weil divisors of a normal projective variety over $F$ with function field $E$.

We translate this result to our context of perfect fields and $R_p$-modules.

**Corollary 4.6.** Let $E^\per$ be the perfect closure of a finitely generated regular extension $E$ of a perfect field $F$. Then $E^\per x/F^x$ is free as an $R_p$-module.

**Proof.** This is immediate from Fact 4.5, on noting that if $(e_i/F^x)_{i<\kappa}$ is a basis for $E^x/F^x$ as an abelian group, then $(e_i/F^x)_{i<\kappa}$ is a basis for $E^\per x/F^x$ as an $R_p$-module. □

**Proposition 1.** Let $\mathcal{C}$ be an algebraically closed field and let $L_1, \ldots, L_n \leq \mathcal{C}$ be algebraically closed subfields from an independent system, with $n \geq 1$. Let $\beta \in \mathcal{C}$ be an arbitrary finite tuple and let $K := L_1 \vee \ldots \vee L_n \vee \beta \leq \mathcal{C}$.  

Then $K^x / \Pi_{i} L_{i}^x$ is a locally free $R_p$-module.

**Proof.** The $n = 1$ case of the proposition follows from Corollary 4.6; we proceed to prove the proposition by induction on $n$.

Let $B, B_1$ be as in Definition 1.

Let $L := L_1$, let $P := \bigvee_{i>1} L_i$ and let $H := \Pi_{i>1} L_i^x \leq P^x$.

We first show that we may reduce to the case where $\overline{\beta}$ is algebraic over $P \vee L = \bigvee_i L_i$.

Indeed, the relative algebraic closure of $P \vee L$ in $P \vee L \vee \overline{\beta}$, is an algebraic subextension of the finitely generated extension $(P \vee L)(\overline{\beta})$ of $P \vee L$ and so is a finite extension $P \vee L \vee \overline{\beta}$, say, where $\overline{\beta} \in \text{acl}^\kappa(P \vee L)$.

By Corollary 4.6, $(P \vee L \vee \overline{\beta})^x / (P \vee L \vee \overline{\beta})^x$ is free. So, by Lemma 3.1, we need only show that $(P \vee L \vee \overline{\beta})^x / H L^x$ is locally free.

So we suppose that $\overline{\beta} \in \text{acl}^\kappa(P \vee L)$.

We claim further that we may assume $B$ to be finite. Indeed, suppose $B^0 \subseteq_{\text{fin}} B$ is such that $\overline{\beta} \in \text{acl}^\kappa(B^0)$. Let $B^0 := B_1 \cap B^0$, and define $L^0 := \text{acl}^\kappa(B^0)$ and $K_0 := \bigvee L^0 \vee \overline{\beta} \subseteq K$.

Note that $\Pi_{i} L_{i}^x \cap K_0^x = \Pi_{i} L_{i}^0$. Indeed, if $x = \Pi_{i} a_i \in \Pi_{i} L_{i}^x \cap K_0^x$, then, by Lemma 4.3, there exists a place $\pi_i : K \rightarrow K^0$ such that $\pi_i(a_i) \in L_i^x$, and so $x = \pi_i(x) = \Pi_{i} \pi_i(a_i) \in \Pi_{i} L_{i}^0$.

So the $R_p$-module $M(B^0) := K_0^x / \Pi_{i} L_{i}^0$ is isomorphic to $K_0^x / \Pi_{i} L_{i}^0$. By the existence of $\pi_i$, we have that $M(B^0)$ is pure in $M := K^x / \Pi_{i} L_{i}^x$. So, assuming the current lemma for finite $B$, we have that $M$ is the union of the locally free pure submodules $M(B^0)$ as $B^0$ ranges through the finite subsets of $B$ for which $\overline{\beta} \in \text{acl}^\kappa(B^0)$, and so $M$ is locally free as required.

So we assume that $B$ is finite.

We aim to apply Lemma 3.4. So let $\overline{\beta} \in P \vee L \vee \overline{\beta}$ be multiplicatively independent over $H L^x$; we want to show that the pure hull of $(\overline{\beta})_{H L^x}$ in $(P \vee L \vee \overline{\beta})^x / H L^x$ is free.

Let $(c_i)_{i}$ enumerate $\overline{\beta}$.

**Claim 2.** There exist a finitely generated extension $k$ of $P$ and a place $\pi : \text{acl}^\kappa(LP) \rightarrow L$ such that:

(i) $k \vee L \geq P \vee L \vee \overline{\beta}$;

(ii) $\forall_i c_i \in k$;

(iii) $L = \text{acl}^\kappa(k \cap L)$;

(iv) $\pi(k) = k \cap L$;

(v) $\pi(c_i) \in L^x$.

**Proof.** By Lemma 4.3 with $B^0 := B_1$ and $C := B_1$, there exists a place $\pi : \text{acl}^\kappa(LP) \rightarrow L$ such that $\pi(P \vee B_1) = L$ and $\pi(c_i) \in L^x$.

By Lemma 4.3, there exists a finite extension $k$ of $P \vee B_1 \vee \overline{\beta}$ such that $\pi(k) \leq k$.

Then $k$ and $\pi$ are as required.

**Claim 3.** If $b \in k^x$ is simple in $k^x \mod (k^x \cap H L^x)$, then $b$ is simple in $(k \vee L)^x \mod H L^x$.

Furthermore, identifying $k^x / (k \cap H L^x)$ with the submodule $k^x / H L^x$ of $(k \vee L)^x / H L^x$, we have that, for any $\overline{\pi} \in k^x$, if $(\overline{\pi}) / H L^x$ is pure in $k^x / H L^x$, then it is pure in $(k \vee L)^x / H L^x$. 


Proof of Claim 3. Suppose that $b$ is not simple in $(k \lor L)^\times \mod HL^\times$. By Lemma 3.6 and the fact that $HL^\times$ is divisible in $(k \lor L)^\times$, we have $\alpha^q = b$ for some $\alpha \in (k \lor L)^\times \setminus k^\times$ and some prime $q \neq p$.

Now $k(\alpha)$ is a degree $q$ cyclic extension of $k$, so this is a Galois extension, $\text{Gal}(k(\alpha)/k) \cong \mathbb{Z}/q\mathbb{Z}$, and $k(\alpha)$ is perfect.

Let $F_0 := k \cap L$ and $F_1 := k(\alpha) \cap L$. Let $F_2 \leq L$ be a finite extension of $F_1$ such that $\alpha \in k \lor F_2$ and $F_2$ is Galois over $F_0$. Note that $F_2 \cap k = F_0$ and $F_2 \cap k(\alpha) = F_1$.

By [5, VI Theorem 1.12], $k \lor F_2$ is Galois over $k$ and restriction to $F_2$ gives an isomorphism of finite groups

$$\leftarrow F_2: \text{Gal}(k \lor F_2/k) \rightarrow \text{Gal}(F_2/F_0),$$

and $\text{Gal}(F_2/F_1)$ is the image under $\leftarrow F_2$ of the normal subgroup $\text{Gal}(k \lor F_2/k(\alpha))$ of $\text{Gal}(k \lor F_2/k)$.

So $F_1$ is Galois over $F_0$ and

$$\text{Gal}(F_1/F_0) \cong \frac{\text{Gal}(F_2/F_0)}{\text{Gal}(F_2/F_1)} \cong \frac{\text{Gal}(k \lor F_2/k)}{\text{Gal}(k \lor F_2/k(\alpha))} \cong \text{Gal}(k(\alpha)/k) \cong \mathbb{Z}/q\mathbb{Z}.$$

By [5, VI Theorem 1.12] again, $\text{Gal}(kF_1/k) \cong \text{Gal}(F_1/F_0) \cong \text{Gal}(k(\alpha)/k)$. So $k \lor F_1 = kF_1 = k(\alpha)$, and we have the following lattice diamond:

Since the torsion group $\mu$ is contained in $(k \lor L)^\times$, by [5, VI 6.2] $F_1 = (k \lor L)(\gamma)$ for some $\gamma$ such that $\gamma^q \in k \lor L$.

Now $k(\alpha) = k \lor F_1 = k(\gamma)$, so say $\gamma = \sum_{i<q} c_i \alpha^i$, with $c_i \in k$. Let $\sigma \in \text{Gal}(k(\alpha)/k)$ restrict non-trivially to $F_1$. Say $\sigma(\alpha) = \zeta \alpha$ and $\sigma(\gamma) = \zeta^i \gamma$, where $(l, q) = 1$, and $\zeta$ is a primitive $q$th root of unity. So

$$\sum_{i<q} c_i \zeta^i \alpha^i = \zeta^i \gamma = \sigma(\gamma) = \sum_{i<q} c_i \zeta^i \alpha^i.$$

Since $(\alpha^i)$ is a basis for the $k$-vector space $k(\alpha)$, we have $\gamma = c_0 \alpha^l$.

Now say $sl + tq = 1$. Then $\gamma^q = c_0^q \alpha^{tq}$. So letting $d := c_0^{-tq} b \in k$, we have

$$d^q \gamma^q = \alpha^q = b.$$

But $\gamma^q = (\gamma^q)^q \in (k \lor L)^\times$, so, by Lemma 3.6, $b$ is not simple in $k^\times \mod (k^\times \lor HL^\times)$.

This completes the proof of the first statement in Claim 3. The ‘Furthermore’ part follows by Lemma 3.5.

We aim to apply Lemma 3.7. Let $N := k^\times \lor H \lor$, which by the inductive hypothesis is a torsion-free locally free $R_p$-module; let $D := (k^\times \lor HL^\times)/H$, which is a pure submodule of $N$ and let $A := \langle \pi(b)/H \rangle \subseteq D$.

Claim 4. Let $b/H \in \langle \pi(b)/H \rangle$ and let $m \in R_p$.

If $b/H$ has an $m$th root modulo $D$ in $N$, then $b/H$ has an $m$th root modulo $A$ in $N$.\hfill \Box
Proof of Claim 4. Say $\gamma_H(\alpha_H)^m = b_H$, where $\alpha \in k^\times$, and $\lambda \in H L^\times$. Since $H$ is divisible, we may suppose that $\lambda \in L^\times$, where $b \in (\bar{b})$, and $\lambda \alpha^m = b$.

Applying $\pi$, we obtain (recalling that $\pi(b_i) \in L^\times$ and that $\pi$ fixes $L \ni \lambda$)

$$\lambda = \pi(\lambda) = \pi(b)/\pi(\alpha)^m.$$ 

So

$$\pi(b) \left( \frac{\alpha}{\pi(\alpha)} \right)^m = b.$$ 

But $\pi(b) \in (\pi(b))$ and $\pi(\alpha) \in \pi(k) \subseteq k$, so this shows that $b_H$ has an $m$th root modulo $A$ in $N$.

This concludes the proof of Claim 4. 

It follows from Lemma 3.7 and Claim 4 that the pure hull of $(\bar{b}/H L^\times)$ in $(k^\times/H L^\times)$ is free; by Claim 3, the pure hull in $(P \cap \bar{b})/H L^\times = ((k \cap \bar{b})^\times)/H L^\times$ is also free.

Applying Lemma 3.4, this completes the proof of Proposition 1. 

PROPOSITION 2. In each of the following situations, $K H^{\times} / H$ is a locally free $R_p$-module:

(i) $K$ is a finitely generated extension of the prime field and $H$ is the torsion group of $K^\times$;

(ii) $K$ is a finitely generated extension of the field generated by the group $\mu$ of all roots of unity and $H = \mu$;

(iii) $K$ is a finitely generated extension of the composite $L_1 \ldots L_n$ of algebraically closed fields from an independent system and $H = \Pi_i L_i^\times$.

In the first two cases, and in the third if $K$ is countable or $n = 1$, $(K H^{\times}) / H$ is free.

Proof. In characteristic 0, the first case is the first part of the statement of [10, Lemma 2.1], and the second case is [10, Lemma 2.14(i)].

In characteristic $p > 0$, both the first and second case follow from Corollary 4.6 with $F$ being $K \cap F_p^{\text{alg}}$.

In all characteristics the third case is precisely Proposition 1. Freeness in the countable case follows from Pontyragin’s theorem [2, 19.1], and in the $n = 1$ case from Fact 4.5. 

5. Proof of Theorem 3

Theorem 3 will follow from Proposition 1 by Kummer theory, our use of which is packaged in the following lemma.

LEMMA 5.1. Let $K$ be a perfect field containing the roots of unity $\mu$ and let $F \supseteq K$ be algebraically closed. Let $\bar{a} \in K^\times$ such that $\bar{a}/\mu$ is simple in $K^\times/\mu$. Let $n \in \mathbb{N}$. Then all choices of $\bar{a} \in F^\times$, such that $\bar{a}^n = \bar{a}$, have the same field type over $K$.

Proof. Let $\bar{a}$ be such. Say $n = p^t m$ where $(m, p) = 1$. Since the field type of $\bar{a}$ is determined by that of $\bar{a}^p$, it suffices to consider the case where $t = 0$. By Kummer theory [5, VI § 8],

$$\text{Gal}(K(\bar{a})/K) \cong \text{Hom} \left( \frac{(\bar{a})^\times}{(\bar{a})^\times \cap (K^\times)^n}, \frac{\mathbb{Z}}{n \mathbb{Z}} \right) \cong \frac{(\bar{a})^\times}{(\bar{a})^\times \cap (K^\times)^n},$$

where $(K^\times)^n$ is the $n$-powers subgroup of $K^\times$.

By simplicity, $\langle \bar{a} \rangle \cap (K^\times)^n = \langle \bar{a}^n \rangle$. So $\text{Gal}(K(\bar{a})/K) \cong \left( \frac{\mathbb{Z}}{n \mathbb{Z}} \right)^{\bar{a}}$. 

\[]
THEOREM 3. Let \( n \geq 1 \) and let \( L_1, \ldots, L_n \) be algebraically closed subfields of \( \mathcal{C} \) from an independent system. Let \( (\bar{a}, \bar{b}) \in \mathcal{C}^\times \) be multiplicatively independent over the product \( \Pi_i L_i^\times \).

Let \( (\bar{a}^{1/n})_{n \in \mathbb{N}} \) be a division system below \( \bar{a} \).

Then division systems below \( \bar{b} \) are finitely determined over \( L_1L_2 \ldots L_n(\bar{a}^3) \).

Proof. Let \( \tau := \bar{a}^3 \).

Let \( K := \bigvee L_i \vee \tau \). Let \( \Gamma_1 \) be the pure hull of \( \tau / \Pi_i L_i^\times \) in \( K^\times / \Pi_i L_i^\times \) and let \( \Gamma \) be the pure hull of \( \tau / \Pi \).

Since \( \tau \) is multiplicatively independent over \( \Pi_i L_i^\times \) and \( \mu \subseteq \Pi_i L_i^\times \subseteq K^\times \), the \( R_p \)-modules \( \Gamma_1 \) and \( \Gamma \) are isomorphic; by Proposition 1 and Lemma 3.4, \( \Gamma_1 \), and hence \( \Gamma \), is free.

Now let \( m \) be such that \( \Gamma^m \leq (\tau / \mu) \). Suppose that \( (\bar{b}_1^1/n)_{n \in \mathbb{N}} \) and \( (\bar{b}_2^1/n)_{n \in \mathbb{N}} \) are division systems below \( \bar{b} \) such that \( \bar{b}_1^1/n = \bar{b}_2^1/n \), and let \( \bar{k} \in \mathbb{N} \); we claim that \( \bar{b}_1^{1/k} \) and \( \bar{b}_2^{1/k} \) have the same field type over \( \bigvee L_i \vee \tau^Q \). Since this holds for all \( l \), we have that \( \bar{b}_1^{1/k} \) and \( \bar{b}_2^{1/k} \) have the same field type over \( \bigvee L_i \vee \tau^Q \), as required. \( \square \)

6. Proof of Theorems 1 and 2

We now deduce Theorems 1 and 2 by proving quasi-minimal excellence of an appropriate class of structures corresponding to the exact sequences of (2.1) and (2.3). We use [4] as our reference for the theory of quasi-minimal excellent classes.

The following is a corrected and abbreviated version of the argument in [10, Section 3].

Let \( L \) be the one-sorted language \( \langle +, (\mu_q)_{q \in \mathbb{Q}}, \omega, \langle W_f \rangle_{f \in \mathbb{Z}[X_1, \ldots, X_n], n \in \mathbb{N}}, E \rangle \). If \( p > 0 \), fix a map \( \text{ex}_0 \) as in (2.2). Let \( \Sigma \) be the \( L_{\omega, \omega}(L) \)-sentence expressing that for a model \( V \):

(I) \( \langle V; +, (\mu_q)_{q \in \mathbb{Q}} \rangle \) is a \( \mathbb{Q} \)-vector space (we write \( qx \) for \( \mu_q(x) \));

(II) \( E \) is an equivalence relation on \( V \);

(III) \( V/E \) can be identified with the multiplicative group \( F^\times \) of a characteristic \( p \) algebraically closed field \( (F; +, \cdot) \), such that \( (x + y)/E = x/E \cdot y/E \), and, for each \( n \in \mathbb{N} \) and each polynomial \( f \in \mathbb{Z}[X_1, \ldots, X_n] \), we have \( V \models W_f(x_1, \ldots, x_n) \) if and only if \( f(x_1/E, \ldots, x_n/E) = 0 \);

(IV) \( x \in E \) if and only if \( \forall z \in E \) \( y/z \in E \);

(V) (if \( p > 0 \)) for each tuple of rationals \( q \in \mathbb{Q} \), the field types of \( \langle q_1 \omega/E, \ldots, q_n \omega/E \rangle \) and \( \langle \text{ex}_0(q_1), \ldots, \text{ex}_0(q_n) \rangle \) are equal.

Models of \( \Sigma \) correspond to exact sequences as in (2.1) and (2.3); given a model \( V \models \Sigma \), we write \( \text{ex}_0 : V \to F^\times \) for the quotient map. For \( p > 0 \), Axiom (V) implies that, for an appropriate choice of embedding \( \mathbb{F}_p^\text{alg} \leq F^\times \), we have that \( \text{ex}_0 \) extends \( \text{ex}_0 \).

Let \( \mathcal{C} \) be the class of models of \( \Sigma \).

We equip \( V \in \mathcal{C} \) with a closure operation \( \text{cl}(X) := \text{ex}^{-1}(\text{acl}(\text{ex}(X))) \); as in [10, Lemma 3.4], this satisfies \( [4, \text{Axiom 1}] \) as well as the exchange and countable closure properties.

In the following, a partial embedding is a partial map \( f \) which extends to an isomorphism \( \langle f \rangle : \langle \text{dom}(f) \rangle \to \langle \text{im}(f) \rangle \), where, for \( A \subseteq V \in \mathcal{C} \), we define \( \langle A \rangle = \langle A \rangle^V \) to be the substructure of \( V \) generated by \( A \). By our choice of language, \( \langle A \rangle \) is the \( \mathbb{Q} \)-vector space span of \( A \cup \{ \omega \} \).

Although it is not specified by the axioms in [4], the following property is in fact necessary for the categoricity theorems [4, Theorems 2.1 and 3.3].
Lemma 6.1. If $V_1, V_2 \in \mathcal{C}$, then the substructures generated by the empty set, $\langle \emptyset \rangle^{V_i} \subseteq V_i$, are isomorphic.

Proof. In positive characteristic, this is immediate from Axiom (V).
In characteristic 0, we argue as follows. The map

$$\theta : \mu_1 \longrightarrow \mu_2$$

$$\theta(\text{ex}_1(q\omega_1)) := \text{ex}_2(q\omega_2)$$

is a group isomorphism of the torsion groups. It follows (see [5, VI 3.1]) that $\theta$ is a partial field isomorphism; hence, $q\omega_1 \mapsto q\omega_2$ is an isomorphism $\langle \emptyset \rangle^{V_1} \rightarrow \langle \emptyset \rangle^{V_2}$, as required.

We proceed to verify [4, Axioms II and III].
The following lemma proves [4, Axiom II].

Lemma 6.2 ($\omega$-homogeneity over submodels and $\emptyset$). Let $V_1, V_2 \in \mathcal{C}$, let $G_i \subseteq V_i$ be closed substructures or the empty set and let $g : G_1 \rightarrow G_2$ be an isomorphism or the empty map.

(i) If $x_i \in V_i \setminus \text{cl}(G_i)$, then $g \cup \{(x_1, x_2)\}$ is a partial embedding.

(ii) If $\overline{a}_i \in V_i$ and $g' : G_i \overline{a}_i \rightarrow V_2$ is a partial embedding extending $g$, then, for any $b_1 \in \text{cl}(G_1 \overline{a}_1)$, there exists $b_2 \in V_2$ such that $g' \cup \{(b_1, b_2)\}$ is a partial embedding.

Proof. We have $\text{ex}_i : V_i \rightarrow F_i^{\times}$.
Condition (i) is clear.
For (ii), suppose first that $G_i$ is closed; so $\text{ex}_i(G_i) = F_i^{\times}$, where $F_i^{\times} \subseteq F_i$ is an algebraically closed subfield.
We may assume that $(\overline{a}_1 b_1)$ is linearly independent over $G_1$. By the $n = 1$ case of Theorem 3, division systems below $\text{ex}(b_1)$ are finitely determined over $\text{ex}_1((G_1, \overline{a}_1))$. The result follows.

The case remains that $G_i = \emptyset$. In characteristic 0, we refer to [10, 3.5(ii)] for this. In characteristic $p > 0$, the substructure of $G_i$ generated by $\emptyset$ is a closed subset of $V_i$, and so we return to the case above.

The following lemma proves [4, Axiom III] for $\mathcal{C}$: that axiom refers to closed partial embeddings, but in $\mathcal{C}$ any partial embedding is closed.

Lemma 6.3. Let $V_1, V_2 \in \mathcal{C}$, let $B \subseteq V_1$ be a cl-independent set, let $B_1, \ldots, B_n \subseteq B$ and let $C := \bigcup \text{cl}(B_i) \subseteq V_1$. Let $g : C \rightarrow V_2$ be a partial embedding. Let $\overline{a} \in \text{cl}(C)$. Then there exists a finite subset $C_0$ of $C$ such that if $g' : C_0 \overline{a} \rightarrow V_2$ is a partial embedding extending $g \upharpoonright C_0$, then $g \cup g' : C \overline{a} \rightarrow V_2$ is a partial embedding.

Proof. Letting $L_i := \text{ex}(\text{cl}(B_i))$, we have that $L_1, \ldots, L_n$ are from an independent system.
We may assume that $\overline{a}$ is $\mathbb{Q}$-linearly independent over $\Sigma_i \text{cl}(B_i)$. By Theorem 3, division systems below $\text{ex}(\overline{a})$ are finitely determined over $\bigvee_i L_i$. Let $m$ be as in the definition of finite determination; $\text{ex}(\overline{a}/m)$ is algebraic over $\bigvee_i L_i$, and so its field type is isolated by some field formula $\phi(\overline{a}, \overline{b}_1, \ldots, \overline{b}_n)$, where $\overline{b}_i \in L_i$. Letting $C_0 \subseteq C$ be a finite subset such that $\overline{b}_i \in \text{ex}(C_0)$ for all $i$, we see that $C_0$ is as required.

We have now shown that $\mathcal{C}$ is a quasi-minimal excellent class in the sense of [4]. By [4, Theorem 3.3], therefore, there is at most one structure in $\mathcal{C}$ of a given cl-dimension – in other words with the corresponding algebraically closed field having a given transcendence
degree. That there exists such a structure in each transcendence degree is clear. Translating straightforwardly from our one-sorted set-up to the two-sorted set-up of their statements, this concludes the proofs of Theorems 1 and 2.

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