The smoothness of the stationary measure

Italo Cipriano

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Abstract

We study the smoothness of the stationary measure with respect to smooth perturbations of the iterated function scheme and the weight functions that define it. Our main theorems relate the smoothness of the perturbation of the iterated function scheme and the weight functions; to the smoothness of the perturbation of the stationary measure. The results depend on the smoothness of: the iterated function scheme and the weights functions; and the space on which the stationary measure acts as a linear operator. As a consequence we also obtain the smoothness of the Hausdorff dimension of the limit set and of the Hausdorff dimension of the stationary measure.

1 Introduction

An IFS (iterated function scheme) with constant weight functions has a unique stationary measure associated sometimes also called self-similar measure. Self-similar measures were originally defined in [9]. The most studied features of IFSs (iterated function schemes) are their fractal properties, like the Hausdorff dimension of its limit set [6, 9] and the Hausdorff dimension of its stationary probability measure [8, 13, 14, 19, 29]. In this paper we are concerned with analytic properties of conformal IFSs, mostly motivated by [15, 21, 23]. A particularly natural special case is that of a finite family of contractions on the unit interval. For definiteness, let us consider the following setting:

Definition 1.1. Assume that $\epsilon > 0$ small, $\beta, \epsilon > 0$, $k, l, m \in \mathbb{N} \setminus \{1\}$, $r \in \mathbb{N}$ and call the interval $(-\epsilon, \epsilon) \subset \mathbb{R}$ by $I_\epsilon$. Then

i. let $T^{(\lambda)} = \{T_i^{(\lambda)}\}_{i=1}^k$ with $\lambda \in I_\epsilon$ be a family of $C^{m+\beta}$ contractions on $[0,1]$. Assume that we can expand for $\lambda \in I_\epsilon$,

$$T_i^{(\lambda)} = T_i + \lambda T_{i,1} + \cdots + \lambda^{m-1}T_{i,m-1} + o(\lambda^{m-1}),$$

where $T_i, T_{i,j} \in C^{m+\beta}([0,1], [0,1])$, $\|dT_i\|_{\mathbb{C}} < 1$, $dT_i = dT_j$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m-1\}$; and

ii. let $G^{(\theta)} = \{g_i^{(\theta)}\}_{i=1}^k$ with $\theta \in I_\epsilon$ be a family of $C^{l+\epsilon}([0,1], \mathbb{R}^+)$ positive weight functions on $[0,1]$ satisfying the following two conditions:

$$\sum_{i=1}^k g_i^{(\theta)} = 1$$

(1)
\[
\sum_{i=1}^{k} \|g_i^{(\theta)}\|_{C^0} \text{Lip} \left( T_i^{(\lambda)} \right) < 1 \text{ for all } \lambda, \theta \in I_e
\]

where

\[
g_i^{(\theta)} = g_i + \theta g_{i,1} + \cdots + \theta^r g_{i,r} + o(\theta^r)
\]

\(g_i, g_{i,j} \in C^{l+r}(\{0,1\}, \mathbb{R}^+)\) for \(i \in \{1, \ldots, k\}\) and \(j \in \{1, \ldots, r\}\).

In this case the stationary measure \(\mu = \mu_{\lambda,\theta}\) is the unique probability measure on \([0,1]\) that satisfies

\[
\int f(x) d\mu(x) = \sum_{i=1}^{k} \int g_i^{(\theta)}(x) f(T_i^{(\lambda)} x) d\mu(x)
\]

for any continuous function \(f : [0,1] \to \mathbb{R}\).

The existence of such a measure is well known and discussed in Subsection 2.1. There is an equivalent definition of stationary measure which is perhaps somewhat more intuitive and particularly useful for simulations that is given by the following rather well known lemma.

**Lemma 1.2.** For any \(x_0 \in [0,1]\) we can write \(\mu\) as the weak star limit of finitely supported probability measures, indeed

\[
\mu = \lim_{n \to +\infty} \sum_{i \in \{1,\ldots,k\}^n} g_i^{(\theta)}(x_0) \delta_{T_i^{(\lambda)}(x_0)},
\]

where for each of the \(k^n\) strings \(\underline{i} = (i_1, \ldots, i_n)\) we write (for \(n \in \mathbb{N}\)):

\[
T_{\underline{i}}^{(\lambda)} := T_{i_1}^{(\lambda)} \circ \cdots \circ T_{i_n}^{(\lambda)} : \mathbb{R} \to \mathbb{R};
\]

\[
g^{(\theta)}_{\underline{i}}(x_0) := g_{i_1}^{(\theta)} \left( T_{i_1}^{(\lambda)} \cdots T_{i_n}^{(\lambda)}(x_0) \right) \cdots g_{i_{n-1}}^{(\theta)} \left( T_{i_n}^{(\lambda)}(x_0) \right) \cdot g_{i_n}^{(\theta)}(x_0); \text{ and}
\]

\(\delta_{T_{\underline{i}}^{(\lambda)}(x_0)}\) denotes the Dirac measure supported on \(T_{\underline{i}}^{(\lambda)}(x_0)\).

Our first main result is about the differentiability of the dependence of this measure.

**Theorem 1.3.** Assume \(\delta \in (0,1), k, l, m, s \in \mathbb{N} \setminus \{1\}\) and \(r \in \mathbb{N}\), then:

i. Given \(\theta \in I_e\), the measure \(\mu_{\lambda,\theta}\) has a \(C^{(1,m,s^{-1})}_{\min}\) dependence on \(\lambda \in I_e\) as an element of \(C^{s+\delta}([0,1], \mathbb{R})^*\).

ii. Given \(\lambda \in I_e\), the measure \(\mu_{\lambda,\theta}\) has a \(C^r\) dependence on \(\theta \in I_e\) as an element of \(C^1([0,1], \mathbb{R})^*\).

**Remark 1.4.** In Theorem 1.3, when we study the dependence of the measure \(\mu = \mu_{\lambda,\theta}\) on \(\lambda\), it is essential to consider the measure \(\mu\) as an element of \(C^{s+\delta}([0,1], \mathbb{R})^*\) for \(s \in \mathbb{N} \setminus \{1\}\), i.e. we identify \(\mu\) with the functional \(\mathcal{M} : C^{s+\delta}([0,1], \mathbb{R}) \to \mathbb{R}\) defined by \(C^{s+\delta}([0,1], \mathbb{R}) \ni w \mapsto \int_0^1 w(\bar{x}) d\mu(\bar{x}) \in \mathbb{R}\).

We have the following simple corollary from Theorem 1.3.
Corollary 1.5. Let \( w : [0, 1] \rightarrow \mathbb{R} \) be a \( C^\infty \) function. Given \( \theta \in \mathcal{I}_\epsilon \), the function \((-\epsilon, \epsilon) \ni \lambda \mapsto \int wd\mu_{\lambda, \theta} \in \mathbb{R} \) is \( C^{\min(l, m) - 1} \).

The next corollary applies under the hypothesis that the weight functions are \( C^\infty \). In particular, this is true in the special case of constant weight functions.

Corollary 1.6. Suppose that the family \( \mathcal{G}(\theta) = \{ g_i(\theta) \}_{i=1}^k \) of weights satisfies \( g_i(\theta) \in C^\infty([0, 1], \mathbb{R}^+) \) for every \( i \in \{1, \ldots, k\} \). Let \( w : [0, 1] \rightarrow \mathbb{R} \) be a \( C^\infty \) function. Given \( \theta \in \mathcal{I}_\epsilon \), the function \((-\epsilon, \epsilon) \ni \lambda \mapsto \int wd\mu_{\lambda, \theta} \in \mathbb{R} \) is \( C^{m-1} \).

Our second result is on the differentiability of the Hausdorff dimension of the limit set \( K_\lambda \) of \( T_\lambda \).

Theorem 1.7. Let \( T \) be an IFS as in Definition 1.1 such that the sets \( T_i(\lambda)[0, 1] \) are pairwise disjoint for \( i \in \{1, \ldots, k\} \). Then the dependence \((-\epsilon, \epsilon) \ni \lambda \mapsto HD(K_\lambda) \) of the Hausdorff dimension of the limit set of \( T_\lambda \), is \( C^{m-2} \).

Our last result is on the differentiability of the Hausdorff dimension of the stationary measure.

Theorem 1.8. Let \( \delta \in (0, 1) \), \( k, l, m, s \in \mathbb{N} \setminus \{1\} \) and \( r \in \mathbb{N} \). Consider \( T_\lambda = \{ T_i^{(\lambda)} \}_{i=1}^k \) and \( \mathcal{G}(\theta) = \{ g_i^{(\theta)} \}_{i=1}^k \) be as in Definition 1.1 with the property that the sets \( T_i^{(\lambda)}[0, 1] \) are pairwise disjoint for \( i \in \{1, \ldots, k\} \). If there exists \( \rho > 0 \) such that
\[
\min_{i} \inf_{\lambda} \inf_{x} |dT_i^{(\lambda)}(x)| > \rho, \tag{4}
\]
then
i. given \( \theta \in \mathcal{I}_\epsilon \), the dependence \((-\epsilon, \epsilon) \ni \lambda \mapsto HD(\mu_{\lambda, \theta}) \) of the Hausdorff dimension of the measure \( \mu_{\lambda, \theta} \), is \( C^{\min(l-1, m-2)} \); and

ii. given \( \lambda \in \mathcal{I}_\epsilon \), the dependence \((-\epsilon, \epsilon) \ni \theta \mapsto HD(\mu_{\lambda, \theta}) \) of the Hausdorff dimension of the measure \( \mu_{\lambda, \theta} \), is \( C^r \).

Our results use basic facts of IFS and are closely related to [26], see Subsection 4.4. However, our proof relies on a result of composition of operators in [5] and structural stability, whereas the proof in [26] uses Proposition 2.3 in [26] and [25].

The structure of the paper is the following: In Section 2 we explain the background, in particular, we define and justify the existence and unicity of the stationary measures, we define the Hausdorff dimension of a limit set and the Hausdorff dimension of a stationary measure, we also state Bowen’s formula and the volume lemma. In Section 3 we prove our main results. Finally, in Section 4 we exhibit some examples of application of our results.

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2 Background

We introduce iterated functions schemes, limit sets, stationary measures, projection maps, some basic results on thermodynamic formalism and the Hausdorff dimension of sets and measures.

2.1 Stationary measures

We are only concerned with the study of stationary measures for IFSs, i.e. for a finite family of contractions with respect to the Lipschitz norm on a complete metric space. To make this precise, consider two complete metric spaces \((\mathcal{M}, d)\) and \((\mathcal{N}, \tilde{d})\).

Define the Lipschitz semi norm \(\text{Lip}(A) := \sup_{x \neq y} \frac{\tilde{d}(A(x), A(y))}{d(x, y)}\).

Definition 2.1 (Iteration function scheme). An IFS is a finite family of contractions with respect to \(\text{Lip}\), i.e. a family of maps \(\mathcal{T} = \{T_i\}_{i=1}^n\) where \(T_i : \mathcal{M} \to \mathcal{M}\) and \(\max_{i=1,\ldots,n} \text{Lip}(T_i) < 1\).

Given a finite family of contractions, an interesting class of sets to study are those invariant under the contractions. The next lemma says that in the case of IFSs there exists a unique such set.

Lemma 2.2. If \(\mathcal{T} = \{T_i\}_{i=1}^n\) be an IFS, then there exists a unique closed bounded set \(\mathcal{K} \subset \mathcal{M}\) such that \(\mathcal{K} = \bigcup_{i=1}^n T_i \mathcal{K}\).

We call \(\mathcal{K}\) the limit set of \(\mathcal{T}\).

A proof of this can be found in [9]. A basic example to keep in mind is the case of \((\mathcal{M}, d) = ([0, 1], | |)\), for the unit interval \([0, 1]\) and the absolute value \(| |\) on \(\mathbb{R}\), and \(T_1(x) = \frac{x}{3}, T_2(x) = \frac{x}{3} + \frac{2}{3}\). The limit set in this example is the famous middle third Cantor set.

An important property of an IFS is the open set condition that was introduced in [17] to compute the Hausdorff dimension of the limit set.

Definition 2.3 (Open set condition). An IFS \(\mathcal{T} = \{T_i\}_{i=1}^n\) is said to satisfy the open set condition if there is a non-empty open set \(\mathcal{V} \subset \mathcal{M}\) such that

\[\bigcup_{i=1}^n T_i(\mathcal{V}) \subset \mathcal{V} \text{ and } T_i(\mathcal{V}) \cap T_j(\mathcal{V}) = \emptyset \text{ for } i \neq j.\]  \hspace{1cm} (5)

We say that \(\mathcal{T}\) satisfies the open set condition for the open set being \(\mathcal{V}\), if \(\mathcal{V} \subset \mathcal{M}\) such that (5).
Associated to an IFS \( \mathcal{T} = \{T_i\}_{i=1}^n \), we can consider a family of weight functions \( \mathcal{G} = \{g_i\}_{i=1}^n \), \( g_i : \mathcal{M} \to (0, 1) \) such that

\[
\sum_{i=1}^n g_i \equiv 1 \quad \text{and} \quad \sum_{i=1}^n \|g_i\|_{\text{Lip}}(T_i) < 1,
\]

where \( \|g\| = \sup\{g(x) : x \in \mathcal{M}\} \).

**Definition 2.4 (Stationary measure).** Let \( \mathcal{T} = \{T_i\}_{i=1}^n \) be an IFS with weight functions \( \mathcal{G} = \{g_i\}_{i=1}^n \) and let \( \mathcal{P}(\mathcal{M}) \) be the set of Borel regular probability measures having bounded support. A stationary measure \( \mu \in \mathcal{P}(\mathcal{M}) \) is a fixed point for the operator \( \mathcal{S} = \mathcal{S}_{\mathcal{T}, \mathcal{G}} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M}) \) defined by

\[
\mathcal{S}(\nu)(f) := \sum_{i=1}^n \int g_i(x)f(T_i(x))d\nu(x),
\]

where \( \nu \in \mathcal{P}(\mathcal{M}) \) and \( f : \mathcal{M} \to \mathbb{R} \) is a continuous compactly supported function.

**Remark 2.5.** Two direct but important facts from the definition of stationary measure are the following:

i. A stationary measure for \( (\mathcal{T}, \mathcal{G}) \) is supported on the limit set of \( \mathcal{T} \) (a proof is given in [9], Section 4.4).

ii. A probability measure \( \mu \in \mathcal{P}(\mathcal{M}) \) is a fixed point of \( \mathcal{S} \) if and only if

\[
\mathcal{S}(\mu)(f) = \int f(x)d\mu(x)
\]

for every continuous compactly supported function \( f : \mathcal{M} \to \mathbb{R} \).

We have the following well known theorem:

**Theorem 2.6.** Suppose that \( \mathcal{M} \) is a compact metric space. An IFS \( \mathcal{T} \) with weight functions \( \mathcal{G} \) satisfying (6) and (7) has a unique stationary measure.

A proof of this theorem can be found in [9] for constant weight functions, using the contractive mapping principle. A small modification of the same argument can be applied here. Recall also that the existence of a stationary measure is a classic result [7] (Lemma 1.2).

**Proof.** The space \( \mathcal{P}(\mathcal{M}) \) can be equipped with the Kantorovich-Rubinshtein norm [1]

\[
\|\mu\| = \sup \left\{ \int fd\mu : f : \mathcal{M} \to \mathbb{R}, \text{Lip}(f) \leq 1 \right\}.
\]

The operator \( \mathcal{S} \) is a contraction on the space \( (\mathcal{P}(\mathcal{M}), \|\|) \). Indeed, for \( \mu, \nu \in \mathcal{P}(\mathcal{M}) \) and a function \( f : \mathcal{M} \to \mathbb{R} \), we have that

\[
\mathcal{S}(\mu)(f) - \mathcal{S}(\nu)(f) = \int \sum_{i=1}^n g_i(x)f(T_i(x))(d\mu - d\nu)(x). \quad (8)
\]
If \( g : \mathcal{M} \to (0, 1) \) and \( T : \mathcal{M} \to \mathcal{M} \) with \( \text{Lip}(T) < \infty \), then
\[
\sup \left\{ \int f(T(x))g(x)d\mu(x) : f : \mathcal{M} \to \mathbb{R}, \text{Lip}(f) \leq 1 \right\}
\leq \text{Lip}(T)\|g\| \sup \left\{ \int fd\mu(x) : f : \mathcal{M} \to \mathbb{R}, \text{Lip}(f) \leq 1 \right\}.
\]

From Equation (8) and the last observation we conclude that
\[
\|\mathcal{S}(\mu) - \mathcal{S}(\nu)\| \leq \left( \sum_{i=1}^{n} \text{Lip}(T_i)\|g_i\| \right) \|\mu - \nu\| = L \|\mu - \nu\|,
\]
where \( L = \sum_{i=1}^{n} \text{Lip}(T_i)\|g_i\| < 1 \) by hypothesis, and thus \( \mathcal{S} \) is a contraction. On the other hand, \( \mathcal{P}(\mathcal{M}) \) with the metric \( \|\|\| \) is a complete metric space. It follows that \( \mathcal{S} \) has a unique fixed point on \( \mathcal{P}(\mathcal{M}) \) by the contraction mapping principle. \( \square \)

**Remark 2.7.** A complete proof of the fact that \( \mathcal{P}(\mathcal{M}) \) with the metric \( \|\|\| \) is a complete metric space can be found in [10], Chapter 8, §4, where it is proved that \( (\mathcal{P}(\mathcal{M}), \|\|\|) \) is a compact metric space. A more general result can be found in [11], Theorem 4.2. On the other hand, it is also possible to prove the completeness of \( \mathcal{P}(\mathcal{M}) \) with the metric \( \|\|\| \) by using similar arguments than in [18].

### 2.2 Projection map and thermodynamic formalism

To introduce our setting we need to define the metric space
\[ \mathcal{X} := \left\{ x = (x_n)_{n=0}^{\infty} : x_n \in \{1, \ldots, k\}, n \in \mathbb{N}_0 \right\} = \{1, \ldots, k\}^{\mathbb{N}_0} \]
with the metric
\[ d(x, y) := \sum_{n=0}^{\infty} \frac{1 - \delta_{\{x_n\}}(y_n)}{2^n}. \]

We consider \( \mathcal{X} \) with the action of the shift \( \sigma : \mathcal{X} \to \mathcal{X} \), defined by \( (\sigma(x))_n = x_{n+1} \) for \( n \in \mathbb{N} \), where \( x = (x_n)_{n=0}^{\infty} \in \mathcal{X} \). The space \( \mathcal{X} \) with the shift action is called a shift space.

**Definition 2.8 (Projection map).** Let \( \mathcal{T} = \{T_i\}_{i=1}^{n} \) be an IFS on the unit interval. We define the projection map \( \pi : \mathcal{X} \to [0, 1] \) by
\[ \pi(x) = \pi_{\mathcal{T}}(x) := \lim_{n \to \infty} T_{x_0} \circ T_{x_1} \circ \cdots \circ T_{x_n}(0), \]
where \( x = (x_i)_{i=0}^{\infty} \).

We recall some results on thermodynamic formalism and in particular we define the pressure function, Gibbs measures and the transfer operator. They will be useful in the proofs of the main theorems. We begin with the definition of the space of \( \alpha \)-Hölder functions.
**Definition 2.9.** Given $0 < \alpha < 1$, let $\mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ denote the Banach space of $\alpha$-Hölder continuous functions (or simply $\alpha$-Hölder functions) $f : \mathcal{X} \to \mathbb{R}$ with norm

$$\|f\| := \max\{\|f\|_\alpha, K\|f\|_\infty\},$$

where

$$\|f\|_\alpha := \sup_{x \neq y} \left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \right\} \quad \text{and} \quad \|f\|_\infty := \sup_x |f(x)|$$

and $K > 0$ is a constant.

We now define the pressure function.

**Definition 2.10.** Let $P : \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathbb{R}$ denote the pressure defined by

$$P(\varphi) = \lim_{n \to +\infty} \frac{1}{n} \log \left( \sum_{\sigma^k x = x} \exp \left( \sum_{k=0}^{n-1} \varphi(\sigma^k x) \right) \right)$$

where $\varphi \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$.

The basic properties can be found in [3], [20], for example. The following result gives an alternative definition of the pressure.

**Lemma 2.11** (Variational principle). We can write

$$P(\varphi) = \sup \left\{ h(\nu) + \int \varphi d\nu : \nu \text{ is } \sigma \text{ invariant probability measure} \right\},$$

where $h(\nu)$ is the measure theoretic entropy with respect to $\nu$. Moreover, there is a unique $\sigma$ invariant probability measure $\mu_{\varphi}$ on $\mathcal{B}_\mathcal{X}$ which satisfies $P(\varphi) = h(\mu_{\varphi}) + \int \varphi d\mu_{\varphi}$.

This leads to the following definition.

**Definition 2.12.** The measure $\mu_{\varphi}$ is called the Gibbs measure (or equilibrium state) for $\varphi \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$.

The basic properties of the pressure function we need are the following.

**Lemma 2.13.** The function $P : \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathbb{R}$ is analytic. Moreover, the first and second derivatives are given by:

i. \(\frac{dP(\varphi + t\psi)}{dt}\big|_{t=0} = \int \psi d\mu_{\varphi}\); and

ii. \(\frac{\partial^2 P(\varphi + t\psi + t\xi)}{\partial \psi \partial \xi}\big|_{(0,0)} = \sigma_{\mu_{\varphi}}^2(\varphi, \xi)\) where $\sigma_{\mu_{\varphi}}^2(\psi, \xi)$ is the variance of $\mu_{\varphi}$ and $\psi, \xi \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$.

This result can be found in [24] or [20]. For a proof including the details see [30], Propositions 6.12 and 6.13 in Section 6.6.

Now we proceed to the definition of the Transfer operator.

**Definition 2.14** (Transfer operator). Let $\mathcal{T} = \{T_i\}_{i=1}^n$ be an IFS on the unit interval with weight functions $\mathcal{G} = \{g_i\}_{i=1}^n$ and let $\psi \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ be a Hölder function. We define the transfer operator $\mathcal{L}_{\psi} : \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ by

$$\mathcal{L}_{\psi} w(x) = \sum_{\sigma y = x} e^{\psi(y)} w(y) \quad \text{where } w \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}).$$
2.3 Hausdorff Dimension

The notion of Hausdorff dimension allows to measure Borel sets in $\mathbb{R}^n$ associating to them a real number. This number is particularly useful to study fractal geometry, however in many cases hard to calculate. A complete discussion of the Hausdorff dimension of a set can be found in [6].

**Definition 2.15.** Let $\mathcal{E}$ be a Borel set in $\mathbb{R}^n$. The Hausdorff dimension $HD(\mathcal{E})$ of $\mathcal{E}$ is defined by

$$HD(\mathcal{E}) := \inf\{\alpha > 0 : H_\alpha(\mathcal{E})\} = 0,$$

where

$$H_\alpha(\mathcal{E}) := \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(B_i)^\alpha : \{B_i\} \text{ covers } \mathcal{E} \text{ and diam}(B_i) \leq \varepsilon \right\}.$$

In this paper we are concerned with the Hausdorff dimension of the limit set $\mathcal{K}$ of an IFS $\mathcal{T}$. In this case, Bowen [4] introduced a method relating the Hausdorff dimension $s$ of $\mathcal{K}$ with the solution of the equation $P(s\Phi) = 0$, where $P$ is the pressure function (Definition 2.10) and $\Phi$ is an appropriate function that depends on $\mathcal{T}$. Some memorable references for applications of this approach are [22], [23], [16], [15].

The definition of Hausdorff dimension of a measure that we will use was introduced by Young in [29].

**Definition 2.16 (Hausdorff dimension of $\mu$).** Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ with bounded support. The Hausdorff dimension $HD(\mu)$ of $\mu$ is defined by

$$HD(\mu) := \inf\{HD(\mathcal{E}) : \mu(\mathbb{R}^d \setminus \mathcal{E}) = 0\}.$$

Young proved the following theorem.

**Theorem 2.17 ([29]).** Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ with bounded support. If

$$\lim_{\delta \to 0^+} \frac{\log \mu(B_\delta(x))}{\log \delta} = \alpha \text{ for } \mu - a.e. x \in \text{supp}(\mu), \tag{9}$$

then $HD(\mu) = \alpha$.

A Borel probability measure on $\mathbb{R}^d$ satisfying the condition (9) is called dimensional exact measure. In this paper we will study the Hausdorff dimension of the stationary measure of an IFS which satisfy the Open Set Condition. In this case, the stationary measure is well known to be a dimensional exact measure [19]. Moreover, we have the following theorem.

**Theorem 2.18 (Volume Lemma).** Let $\mu$ be the stationary measure of an IFS on the unit interval which satisfy the Open Set Condition. Let $\nu$ be the unique probability measure on the shift space $\mathcal{X}$ such that $\pi_*\nu := \nu \circ \pi^{-1} = \mu$, where $\pi : \mathcal{X} \to [0, 1]$ is the projection map. If

$$\chi_{\nu} := -\int \log |dT_{x_0}(\pi(x))| d\nu(x) < \infty,$$
then
\[ HD(\mu) = \frac{h_\mu(\sigma)}{\chi_\nu}. \]

This theorem is from [19]. Earlier versions of it under stronger conditions can be found in [8, 13].

3 Proofs

The main goal of this part is to prove Theorem 1.3, from it, we will deduce the other results. We have divided this section into six subsections. In the first, we study composition of functions, we settle some of our notation and we show some results on composition operators required in our proof. The results in this subsection follow from [5]. In the second, we study the projection map, in particular we prove a useful result for the smoothness of projection map. Indeed, we prove that
\[ \mathcal{I}_\epsilon \ni \lambda \mapsto \pi(\lambda) \in C^{n-1}(\mathcal{X}, \mathbb{R}) \]
where (\mathcal{X}, \sigma) is a subshift of finite type. In the third we use some basic thermodynamic formalism results that we apply in the following subsections. In the fourth, we prove Theorem 1.3. In the fifth, we prove Theorem 1.7. Finally, in the sixth, we prove Theorem 1.8.

3.1 First requirement: composition of functions

We will use results on composition of functions which are related to those in [5]. For the first part of the proofs, we do not really need to work with the full composition operator, whose definition depends on further smoothing conditions of its domain, but with a simpler map whose definition only depends on the space \( C^\alpha(\mathcal{X}, \mathbb{R}) \).

Definition 3.1. Given a function \( v : [0, 1] \to \mathbb{R} \), we define the map
\[ v_* : C^n(\mathcal{X}, [0, 1]) \to C^n(\mathcal{X}, \mathbb{R}) \]
\[ f \mapsto v_*(f) := v \circ f. \]

Most of the results in this section deal with the regularity of the map \( v_* \). In order to state them precisely, we need to introduce the spaces of functions \( C^{n+\delta}([0, 1], \mathbb{R}) \), for \( 0 < \delta < 1 \) and \( n > 0 \), which correspond to the classic spaces of \( n \)-times continuously differentiable functions with the \( n \)-th derivatives are \( \delta \)-Hölder. We define these spaces rigorously.

Definition 3.2. For each \( i > 0 \), we denote the \( i \)-th derivative of \( v : [0, 1] \to \mathbb{R} \), when it exists, by \( d^i v \) (where \( d^0 v = v \)).

Given \( n > 0 \) and \( 0 < \delta < 1 \), the space \( C^{n+\delta}([0, 1], \mathbb{R}) \) is defined to be the space of functions \( v : [0, 1] \to \mathbb{R} \) such that \( v \) is \( n \)-times differentiable and
\[ \|v\|_{C^n} := \sup_{\tilde{x} \in [0,1]} |v(\tilde{x})| < \infty, \]
\[ \|v\|_{C^{n+\delta}} := \max_{i \in \{0, \ldots, n\}} \|d^i v\|_{C^\delta} < \infty. \]
and
\[ \|d^nv\|_{C^\delta} := \sup_{\tilde{x} \neq \tilde{y}} \frac{|d^n v(\tilde{x}) - d^n v(\tilde{y})|}{|\tilde{x} - \tilde{y}|^\delta} < \infty. \]

We endowed it with the norm
\[ \|v\|_{C^{n+\delta}} = \sup(\|d^n v\|_{C^\delta}, \|v\|_{C^n}). \]

This is a Banach space and in the case \( n \in \mathbb{N} \) we have that
\[ \|v\|_{C^{n+\delta}} = \sup(\|v\|_{C^n}, \|dv\|_{C^{n-1+\delta}}). \]

**Remark 3.3.** Given an integer \( n > 0 \), any function \( v \in C^{n+1}([0,1], \mathbb{R}) \) has \( i \)-th Lipschitz derivative for \( i = 0, 1, \ldots, n \), i.e.
\[ \text{Lip}(d^i v) := \sup_{\tilde{x} \neq \tilde{y}} \frac{|d^i v(\tilde{x}) - d^i v(\tilde{y})|}{|\tilde{x} - \tilde{y}|} < \infty, \]
for \( i \in \{0, \ldots, n\} \).

This implies that \( C^n([0,1], \mathbb{R}) \subset C^{m+\delta}([0,1], \mathbb{R}) \), for every \( 0 \leq \delta \leq 1 \) and \( m < n \), because Lipschitz functions are automatically \( \delta \)-Hölder for \( 0 < \delta \leq 1 \).

The following result is analogous to the proof of Proposition 6.2, part ii.2) in [5].

**Lemma 3.4.** If \( v \in C^{1+\delta}([0,1], \mathbb{R}) \), then the map \( v_* \) is \( C^0 \).

**Proof.** We can choose arbitrarily \( f_1, f_2 \in C^\alpha(\mathcal{X}, [0,1]) \) and \( x, y \in \mathcal{X} \). We can then consider a path \( \gamma_1 : [0,1] \to [0,1] \) joining \( f_1(x) \) and \( f_1(y) \) defined by \( \gamma_1(t) = (1 - t)f_1(x) + tf_1(y) \) and a path \( \gamma_2 : [0,1] \to [0,1] \) joining \( f_2(x) \) and \( f_2(y) \), defined by \( \gamma_2(t) = (1 - t)f_2(x) + tf_2(y) \). We then have the following inequalities
\[
\begin{align*}
|v(f_1(x)) - v(f_2(x)) - v(f_1(y)) + v(f_2(y))| &\leq \int_0^1 |dv(\gamma_1(t))\frac{d\gamma_1}{dt}(t) - dv(\gamma_2(t))\frac{d\gamma_2}{dt}(t)|dt \\
&\leq \int_0^1 |(dv(\gamma_1(t)) - dv(\gamma_2(t)))\frac{d\gamma_1}{dt}(t)|dt + \int_0^1 |dv(\gamma_2(t))(\frac{d\gamma_1}{dt}(t) - \frac{d\gamma_2}{dt}(t))|dt \\
&\leq \|v\|_{C^{1+\delta}}(f_2(x) - f_1(x)) + |f_2(y) - f_1(y)|^\delta |f_1(x) - f_1(y)| + \|v\|_{C^1}|f_1(x) - f_2(x) - f_1(y) + f_2(y)|.
\end{align*}
\]

In particular, dividing both sides of the inequality by \( d(x, y)^\alpha \) and taking the supremum over the set \( \{x, y : x, y \in \mathcal{X}, x \neq y\} \), we obtain
\[
\|v_*(f_1) - v_*(f_2)\|_\alpha = \sup_{x \neq y} \frac{|v \circ f_1 - v \circ f_2(x) - (v \circ f_1 - v \circ f_2)(y)|}{d(x, y)^\alpha} \\
\leq 2^\delta \|v\|_{C^{1+\delta}}\|f_2 - f_1\|_{C^\alpha}^\delta \|f_1\|_\alpha + \|v\|_{C^1}\|f_1 - f_2\|_\alpha. \quad (10)
\]

The result follows. \( \square \)
The next lemma is similar to the proof of Proposition 6.7 in [5]. In preparation, we need to introduce some definitions of differentiable operators.

Let $\mathcal{E}, \mathcal{F}$ be Banach spaces with norms $\|\cdot\|_\mathcal{E}$ and $\|\cdot\|_\mathcal{F}$, respectively. We denote the space of bounded linear functions from $\mathcal{E}$ to $\mathcal{F}$ by $L(\mathcal{E}, \mathcal{F})$. Let $U \subset \mathcal{E}$ be an open set. We recall that a function $f : U \to \mathcal{F}$ is Fréchet differentiable at $u \in U$ if we can find a bounded linear function $df(u)$ such that

$$
\lim_{\epsilon \to 0} \frac{\|f(u + \epsilon h) - f(u) - \epsilon df(u)h\|_\mathcal{F}}{\epsilon} = 0
$$

for every $h \in \mathcal{E}$ and uniformly with respect to $h \in B_1(0) := \{y \in \mathcal{E} : \|y\|_\mathcal{E} < 1\}$. We say that $f$ is differentiable in $U$ if $f$ is differentiable at every point $u \in U$. We say that $f$ is of class $C^1$ if it is differentiable and the mapping $df : U \to L(\mathcal{E}, \mathcal{F})$, $u \mapsto df(u)$ is continuous for the topology induced by the norm. Inductively, we define $d^nf$ to be the differential of $d^{n-1}f$ and we say that a function $f$ is $C^n$ ($n$ times continuously differentiable) if $df : U \to L(\mathcal{E}, \mathcal{F})$ is $(n-1)$ times continuously differentiable.

**Lemma 3.5.** If $v \in C^{2+\delta}([0, 1], \mathbb{R})$, then $v_*$ is $C^1$ and for all $f, h \in C^\alpha(X, [0, 1])$ the derivative of $v_*$ is given by $(dv_*)(f)(h) = (dv)_*(f) \cdot h$.

**Proof.** If $v \in C^{2+\delta}([0, 1], \mathbb{R})$, then it has a $C^{2+\delta}$ extension to an open neighbourhood of $[0, 1]$, i.e. $v \in C^{2+\delta}((-\epsilon_1, 1 + \epsilon_1), \mathbb{R})$ for some $\epsilon_1 > 0$. This induces an extension of $v_*$ to $C^\alpha(X, (-\epsilon_1, 1 + \epsilon_1))$. Let $f \in C^\alpha(X, [0, 1])$ and $h \in C^\alpha(X, \mathbb{R})$.

To complete the proof we will need two simple inequalities: choose $0 < \epsilon_2 < 1$ sufficiently small such that $\max_{t \in [0, 1]}\|f + t\epsilon_2h\|_\infty < 1 + \epsilon_1$, then

$$
\int_0^1 \|dv \circ (f + t\epsilon_2h) - dv \circ f\|_\infty dt \leq \|h\|\|v\|_{c^2+1}\epsilon_2^\delta
$$

and

$$
\|dv \circ (f + t\epsilon_2h) - dv \circ f\|_\alpha \leq 2^\delta\|v\|_{c^{2+\delta}}\|\epsilon_2h\|_\infty^\delta\|f\|_\alpha + \|v\|_{c^2}\|\epsilon_2h\|_\alpha.
$$

To prove (11), we use that for every $t \in [0, 1]$ and $x \in X$

\[
\frac{|dv \circ (f(x) + t\epsilon_2h(x)) - dv \circ f(x)|}{\epsilon_2} = \frac{|d^2v(f(x)) \cdot te_2h(x) + o(te_2h(x))|}{\epsilon_2} \leq \|d^2v(f(x))|.|h(x)| + |h(x)|o(\epsilon_2)\frac{\epsilon_2}{\epsilon_2} \leq \|h\|\|v\|_{c^2+1}.
\]

To prove (12) we notice that by definition $dv \circ (f + t\epsilon_2h) - dv \circ f = (dv)_*(f + t\epsilon_2h) - (dv)_*f$ and use inequality (10) with $dv$ instead of $v$, $f + t\epsilon_2h$ instead of $f_1$ and $f$ instead of $f_2$. 

11
Fix $0 < \epsilon_2 < 1$ sufficiently small for equation (11) to hold, then
\[
\frac{1}{\epsilon_2} \|v_*(f + \epsilon_2 h) - v_*(f) - \epsilon_2 (dv)_*(f) \cdot h\|_\alpha
\]
\[
= \frac{1}{\epsilon_2} \|v \circ (f + \epsilon_2 h) - v \circ f - \epsilon_2 (dv \circ f) \cdot h\|_\alpha
\]
\[
= \|\int_0^1 \left[ (dv \circ (f + t\epsilon_2 h) - dv \circ f) \cdot h dt \right]\|_\alpha
\]
\[
\leq \|h\|_\infty \int_0^1 \|dv \circ (f + t\epsilon_2 h) - dv \circ f\|_\alpha dt
\]
\[
+ \|h\|_\alpha \int_0^1 \|dv \circ (f + t\epsilon_2 h) - dv \circ f\|_\infty dt
\]
\[
\leq (2^\delta \|v\|_{C^{2+\delta}} \|\epsilon_2 h\|_{\infty} \|f\|_\alpha + \|v\|_{C^2} \|\epsilon_2 h\|_\alpha) + \|h\| (\|v\|_{C^2} + 1) \epsilon_2^\delta
\]
\[
\leq (4\|v\|_{C^{2+\delta}} \max\{\|f\|_{\alpha, 1} + 1\}) \|h\| \epsilon_2^\delta,
\]
which proves the second part of the lemma. We used inequalities (11) and (12) in the penultimate inequality.

Now that we have the formula for the derivative of $v_* :$
\[
d(v_*)(f)(h) = (dv)_*(f) \cdot h
\]
for all $f, h \in C^\alpha(\mathcal{X}, [0, 1])$, we can prove that $v_*$ is $C^1$. For this, it is enough to show that $d(v_*)$ is continuous. From (13) we can see that $d(v_*)$ corresponds to $(dv)_*$, followed by the continuous linear map
\[
\mathcal{L} : C^\alpha(\mathcal{X}, L(\mathbb{R}, \mathbb{R})) \to L(C^\alpha(\mathcal{X}, [0, 1]), C^\alpha(\mathcal{X}, \mathbb{R})),
\]
\[
\xi \mapsto [\mathcal{L}(\xi) : h \mapsto \xi \cdot h].
\]
Thus we have that $d(v_*) = \mathcal{L} \circ (dv)_*$ is continuous, since $(dv)_*$ is continuous by Lemma 3.4.

The next corollary follows by induction.

**Corollary 3.6.** If $v \in C^{n+\delta}([0, 1], \mathbb{R})$ for some integer $n \in \mathbb{N}$, and thus $v_*$ is $C^{n-1}$, as required.

**Proof.** The case $n = 1$ is covered by Lemma 3.4. If the result holds for $n$ and $v \in C^{n+1+\delta}([0, 1], \mathbb{R})$, then $(dv)_*$ is $C^{n-1}$ by the inductive hypothesis. We can use the same argument as in the last lines of the proof of Lemma 3.5 to obtain that $d(v_*) = \mathcal{L} \circ (dv)_*$, where $\mathcal{L}$ is a continuous linear map, then $d(v_*)$ is $C^{n-1}$. Therefore, by definition, $v_*$ is $C^n$, which concludes the proof.

A simple argument based in the previous corollary gives the following result that we use to prove the smoothness of the stationary probability measure.

**Corollary 3.7.** Suppose that we have a family of maps $\{v_i \in C^{n+\delta}([0, 1], \mathbb{R}) : i \in \{1, \ldots, k\}\}$ for some integer $n \in \mathbb{N}$, and consider the map $F : C^\alpha(\mathcal{X}, [0, 1]) \to C^\alpha(\mathcal{X}, \mathbb{R})$, defined\(^1\) by $F(\Pi)(x) := v_{x_0}(\Pi(\sigma x))$, where $\Pi \in C^\alpha(\mathcal{X}, [0, 1])$ and $x \in \mathcal{X}$. Then $F$ is $C^{n-1}$. Moreover, for all $f, h \in C^\alpha(\mathcal{X}, [0, 1])$ the derivative of $F$ is given by
\[
d(F)(f)(h)(x) = (dv_{x_0})(f(\sigma x)) \cdot h(\sigma x) \text{ for } x \in \mathcal{X}.
\]

\(^1\)The notation $v_{x_0}(\Pi(\sigma x))$ denotes $v_i(\Pi(\sigma x))$ if $x_0 = i$. 

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12
Proof. The map \( l_1 : \mathcal{C}^\alpha(\mathcal{X}, [0, 1]) \to [\mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})]^k \), defined by
\[
l_1(\Pi(x)) := [v_1(\Pi(x)), \ldots, v_k(\Pi(x))] \in [\mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})]^k
\]
is \( C^{n-1} \) by Lemma 3.5, and the map \( l_2 : [\mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})]^k \to \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \), defined by
\[
l_2([f_1(x), \ldots, f_k(x)]) = f_{x_0}(\sigma x)
\]
is linear and continuous. It follows that the map \( F = l_2 \circ l_1 \) is \( C^{n-1} \).

To prove the formula for the derivative of \( F \) we can use the chain rule and the fact that \( l_2 \) is linear to deduce that \( dF = l_2 \circ dl_1 \) and \( dl_1 = [d(v_1)_*, \ldots, d(v_k)_*] \). This together with the formula for \( d(v_i)_* \) for \( i \in \{1, \ldots, k\} \) in Lemma 3.5 concludes the proof. \( \square \)

To prove the smoothness of the Hausdorff dimension of the support of the stationary measure we additionally need the following results, whose proofs are analogous to the proofs in [5] combined with simple arguments similar to the used in this section.

**Definition 3.8.** Given \( n > 0 \) and \( 0 < \delta < 1 \), we define the composition operator by
\[
\text{Comp} : \mathcal{C}^{n+\delta}([0, 1], \mathbb{R}) \times \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})
\]
\[
(v, f) \mapsto \text{Comp}(v, f) := v \circ f.
\]

**Proposition 3.9.** Given \( n \in \mathbb{N} \) and \( 0 < \delta < 1 \), the composition operator \( \text{Comp} : \mathcal{C}^{n+\delta}([0, 1], \mathbb{R}) \times \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \) is \( C^{n-1} \).

This leads to the following corollaries.

**Corollary 3.10.** The map \([\mathcal{C}^{n+\delta}([0, 1], \mathbb{R})]^k \times \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \ni ([v_1, \ldots, v_k], f) \mapsto v_{x_0} \circ f(x) \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})\) is \( C^{n-1} \).

**Corollary 3.11.** Let \( n \in \mathbb{N} \), \( 0 < \delta < 1 \), \( \epsilon > 0 \) and suppose that we have for each \( \lambda \in \mathcal{I}_\epsilon \) a family of maps \( \{v_i(\lambda) \in \mathcal{C}^{n+\delta}([0, 1], \mathbb{R}) : i \in \{1, \ldots, k\}\} \) and a map \( f(\lambda) \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \). If the map \( \mathcal{I}_\epsilon \ni \lambda \mapsto [v_1(\lambda), \ldots, v_k(\lambda)] \in [\mathcal{C}^{n+\delta}([0, 1], \mathbb{R})]^k \) is \( C^{n_1} \) for some \( n_1 > 0 \), and the map \( \mathcal{I}_\epsilon \ni \lambda \mapsto f(\lambda) \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \) is \( C^{n_2} \) for some \( n_2 > 0 \), then the map \( \mathcal{I}_\epsilon \ni \lambda \mapsto v_{x_0}^{(\lambda)} \circ f(\lambda)(x) \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \) is \( C^{\min(n_1, n_2, n-1)} \).

### 3.2 Second requirement: projection map
We will introduce a projection map \( \pi(\lambda) : \mathcal{X} \to [0, 1] \) for \( \lambda \in \mathcal{I}_\epsilon \) that will be essential to study the differentiability of the stationary measure.

**Definition 3.12.** For each \( \lambda \in \mathcal{I}_\epsilon \) we define the projection map \( \pi(\lambda) : \mathcal{X} \to [0, 1] \) by
\[
\pi(\lambda)(x) := \lim_{n \to \infty} T_{x_0}^{(\lambda)} \circ T_{x_1}^{(\lambda)} \circ \cdots \circ T_{x_n}^{(\lambda)}(0),
\]
where \( x = (x_i)_{i=0}^\infty \).
The following result is easily seen.

**Lemma 3.13.** There exists $\alpha > 0$ such that each individual map $\pi^{(\lambda)} : \mathcal{X} \to [0, 1]$ is $\alpha$-Hölder continuous.

**Proof.** Define $a := \max_{i \in \{1, \ldots, k\}} \sup_{\lambda \in \mathcal{I}} \{\|dT_i^{(\lambda)}\|_{C^0}\} < 1$ and $\alpha := -\frac{\log(a)}{\log(2)}$. Suppose that $x, y \in \mathcal{X}$ and chose $n = n(x, y)$ such that $x_i = y_i$ for $i \leq n$ and $x_{n+1} \neq y_{n+1}$, then

$$|\pi^{(\lambda)}(x) - \pi^{(\lambda)}(y)| \leq a^n = 1^{2^\alpha n} \leq d(x, y)^\alpha.$$  

This completes the proof. \(\square\)

To make further use of the functional analytic approach it helps to choose a specific Banach space of Hölder continuous functions.

**Remark 3.14.** We are now at liberty to choose values of $\alpha$ and $K$ which are most convenient for us in definition of Hölder norm on $\mathcal{X}$ (i.e., Definition 2.9). Denote $\theta_0 := \|dT_1^{(0)}\|_{C^0}$ and then fix a choice of $\theta_0 < \theta < 1$. We can then choose $0 < \alpha < 1$ sufficiently small such that $2^\alpha \theta_0 < \frac{\theta + \theta_0}{2}$. Finally, let us choose $K > 0$ sufficiently large such that

$$\text{Lip}(dT_1)\|\pi^{(0)}\|_\alpha \frac{2^\alpha}{K} < \theta - \theta_0$$

where $\text{Lip}(dT_1)$ is the Lipschitz constant of the derivative of the contraction $T_1$.

We may now prove the main proposition in this section.

**Proposition 3.15.** Provided $\alpha > 0$ is chosen sufficiently small, the map $\mathcal{I}_\epsilon \ni \lambda \mapsto \pi^{(\lambda)} \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{m-1}$.

**Proof.** For each $\lambda \in (-\epsilon, \epsilon)$ we let $R^{(\lambda)} : C^\alpha(\mathcal{X}, \mathbb{R}) \to C^\alpha(\mathcal{X}, \mathbb{R})$ be defined by

$$(R^{(\lambda)}(\Pi))(x) := T^{(\lambda)}_{x_0}(\Pi(\sigma x)),$$

and we construct the map $F : \mathcal{I}_\epsilon \times C^\alpha(\mathcal{X}, \mathbb{R}) \to C^\alpha(\mathcal{X}, \mathbb{R})$ defined by $F(\lambda, \Pi) = \left(I - R^{(\lambda)}\right)(\Pi)$, where $\Pi \in C^\alpha(\mathcal{X}, \mathbb{R})$. As usual $D_2F(0, \pi^{(0)})$ denotes the partial derivative of $F$ with respect to the second coordinate and evaluated in $(0, \pi^{(0)})$, i.e. for $F(0, \cdot) : C^\alpha(\mathcal{X}, \mathbb{R}) \to C^\alpha(\mathcal{X}, \mathbb{R})$ defined by $F(0, \cdot)(\Pi) = F(0, \Pi)$, we define $D_2F(0, \pi^{(0)}) := dF(0, \cdot)(\pi^{(0)})$.

We begin with some preliminary observations.

i. First observe that $\pi^{(\lambda)}$ is a fixed point, i.e., $R^{(\lambda)} \pi^{(\lambda)} = \pi^{(\lambda)}$.

ii. We next observe that the family of maps $(-\epsilon, \epsilon) \times C^\alpha(\mathcal{X}, \mathbb{R}) \ni (\lambda, \Pi) \mapsto R^{(\lambda)}(\Pi) \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{m-1}$. Clearly it is $C^{m-1}$ in $\lambda$, whilst it is $C^{m-1}$ in $\Pi$ by Corollary 3.7.
iii. $D_2 F(0, \pi^{(0)})$ is a linear homeomorphism of $C^\alpha(X, \mathbb{R})$ onto $C^\alpha(X, \mathbb{R})$. Moreover, we will prove that $(I - D_2 (R^{(0)} \pi^{(0)}))$ is invertible. We call

$$\mathcal{R}^{(0)} := D_2 (R^{(0)} \pi^{(0)}).$$

On $\Pi \in C^\alpha(X, \mathbb{R})$, $\mathcal{R}^{(0)}$ is given by

$$\mathcal{R}^{(0)}(\Pi)(x) = dT_{x_0}^{(0)} \left( \pi^{(0)}(\sigma x) \right) \cdot \Pi(\sigma x), \quad x \in X,$$

and this is clear using Corollary 3.7. Since each $T_i$ is a contraction it is easy to see that $\mathcal{R}^{(0)} : C^0(X, \mathbb{R}) \to C^0(X, \mathbb{R})$ satisfies $\|\mathcal{R}^{(0)}\|_\infty < 1$, i.e. $\mathcal{R}^{(0)}$ is a contraction on $C^0$. Using Remark 3.14 we will prove that $\mathcal{R}^{(0)}$ is also a contraction on $C^\alpha(X, \mathbb{R})$. For this, assume $\|\Pi\| \leq 1$ (and thus, in particular, $\|\Pi\|_\alpha \leq 1$ and $\|\Pi\|_\infty \leq 1/K$). We can then use the triangle inequality to bound

$$\|\mathcal{R}^{(0)}(\Pi)(x) - \mathcal{R}^{(0)}(\Pi)(y)\| \leq \|dT_{x_0}^{(0)}\|_\alpha \|\Pi(\sigma x) - \Pi(\sigma y)\| \leq \|dT_{x_0}^{(0)}\|_\alpha \|\Pi(\sigma x)\|_\alpha \leq \|\Pi\|_\alpha < 1.$$

For this, assume $\|\Pi\|_\alpha \leq 1$ (and thus, in particular, $\|\Pi\|_\infty \leq 1/K$). We can then use the triangle inequality to bound

$$\|\mathcal{R}^{(0)}(\Pi)(x) - \mathcal{R}^{(0)}(\Pi)(y)\| \leq \|dT_{x_0}^{(0)}\|_\alpha \|\Pi(\sigma x) - \Pi(\sigma y)\| \leq \|dT_{x_0}^{(0)}\|_\alpha \|\Pi(\sigma x)\|_\alpha \leq \|\Pi\|_\alpha < 1.$$

To end the proof we will use the implicit function theorem for Banach spaces (see for example [28]). The map $F$ is $C^{m-1}$ in a neighbourhood of $(0, \pi^{(0)})$ of $I_x \times C^\alpha(X, \mathbb{R})$ and since $\max \{\|\mathcal{R}^{(0)}\|_\infty, \|\mathcal{R}^{(0)}\|_\alpha\} < 1$ we see that $D_2 F(0, \pi^{(0)}) = I - \mathcal{R}^{(0)}$ is invertible. Thus the hypotheses of the implicit function theorem are satisfied and the result follows.

\[\Box\]

Example 3.16. If $T_0(x) = \lambda x$, $T_1(x) = \lambda x + t$ and $X = \{0, 1\}^{\mathbb{N}_0}$, then we can explicitly write the map $\pi : X \to \mathbb{R}$ as an infinite series:

$$\pi \left( (x_n)_{n=0}^\infty \right) = t \sum_{n=0}^\infty \lambda^n x_n.$$

3.3 Third requirement: thermodynamic formalism

We can deduce by classical techniques and an argument based in composition of operators the differentiability of a Gibbs measure that we will relate with the stationary measure using the projection maps. Also, we relate the Hausdorff dimension
with the zero of $t \mapsto P(-t\Phi)$ by Bowen’s method for some appropriate function $\Phi$. This will be used to deduce the differentiability of the Hausdorff dimension.

In this subsection we consider an IFS $T^{(\lambda)} = \{ T_i^{(\lambda)} \}_{i=1}^n$ for $\lambda \in I_\epsilon := (-\epsilon, \epsilon)$ and the family $G^{(\theta)}$ of weights $G^{(\theta)} = \{ g_i^{(\theta)} \}_{i=1}^k$ for $\theta \in I_\epsilon$. We associate a Hölder continuous function $\psi^{(\lambda,\theta)} \in C^\alpha(X, \mathbb{R})$ defined by
\[
\psi^{(\lambda,\theta)}(x) := \log \left( g_i^{(\theta)}(\pi^{(\lambda)}(\sigma x)) \right).
\]

**Remark 3.17.** We see from the definition of $L_{\psi^{(\lambda,\theta)}}$ and the property that $\sum_{i=1}^k g_i^{(\theta)} = 1$ that $L_{\psi^{(\lambda,\theta)}}1 = 1$, i.e., $L_{\psi^{(\lambda,\theta)}}$ preserves the constant functions.

We next recall the following classical result.

**Theorem 3.18 (Ruelle Operator Theorem).** There exists a maximal positive simple isolated eigenvalue 1. Moreover,

i. there is a positive eigenvector $w_{\psi^{(\lambda,\theta)}}$, i.e., $L_{\psi^{(\lambda,\theta)}} w_{\psi^{(\lambda,\theta)}} = w_{\psi^{(\lambda,\theta)}}$;

ii. the equilibrium state $\nu_{\psi^{(\lambda,\theta)}}$ is a fixed point for the dual operator, i.e.,
\[
L^*_{\psi^{(\lambda,\theta)}} \nu_{\psi^{(\lambda,\theta)}} = \nu_{\psi^{(\lambda,\theta)}}
\]
thus $\int f d\nu_{\psi^{(\lambda,\theta)}} = \int (L_{\psi^{(\lambda,\theta)}} f) d\nu_{\psi^{(\lambda,\theta)}}$ for every continuous $f : X \to \mathbb{R}$.

**Proof.** The spectral properties of the operator follow from the general results of Ruelle for transfer operators with any Hölder continuous function [3], [24]. In this particular case the fact that the maximal eigenvalue is 1 and the corresponding eigen-distribution is the equilibrium state follows from the property that $L_{\psi^{(\lambda,\theta)}}1 = 1$ and [27], [12].

### 3.4 Proof of Theorem 1.3

We need to relate the Gibbs measure to the stationary measure $\mu_{\lambda,\theta}$, recall its definition in (3). The strategy of the proof of Theorem 1.3 consists of the following steps:

i. We construct a probability measure $\nu_{\lambda,\theta}$ on the Borel sets of $X := \{1, \ldots, k\}^\mathbb{N}$ such that for $w \in C^{s+\delta}([0, 1], \mathbb{R})$ we have
\[
\int_X w \circ \pi^{(\lambda)}(x) d\nu_{\lambda,\theta}(x) = \int_0^1 w(\tilde{x}) d\mu_{\lambda,\theta}(\tilde{x}), \tag{14}
\]
where $\pi^{(\lambda)} \in C^\alpha(X, [0, 1])$ for $\lambda \in I_\epsilon$. The probability measure $\nu_{\lambda,\theta}$ corresponds to the Gibbs measure of an explicitly constructed Hölder potential that depends on both $T^{(\lambda)}$ and $G^{(\theta)}$.

ii. We prove that $C^\alpha(X, \mathbb{R}) \ni \Pi \mapsto w \circ \Pi \in C^\alpha(X, \mathbb{R})$ is $C^{s-1}$. To achieve this, we use an argument of composition of operators (following de la Llave and Obaya) which requires $w \in C^{s+\delta}([0, 1], \mathbb{R})$. 

16
iii. A similar argument is used to show that $I_\epsilon \ni \lambda \mapsto \pi^{(\lambda)} \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{m-1}$.

In order to apply the result in this case we need to use that $T^{(\lambda)}$ is a family of $C^{m+\beta}$ functions. We use an argument based on the implicit function theorem that requires the family $T^{(\lambda)}$ to be contractions.

iv. We use a classical result about regularity of Gibbs measures to prove that $I_\epsilon \ni \lambda \mapsto \nu_{\lambda, \theta} \in C^\alpha(\mathcal{X}, \mathbb{R})^*$ is $C^{l-1}$.

v. As a consequence of the previous parts, we have that the map $I_\epsilon \ni \lambda \mapsto (\nu_{\lambda, \theta}, w \circ \pi^{(\lambda)}) \in C^\alpha(\mathcal{X}, \mathbb{R})^* \times C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{\min(l, m, \alpha)-1}$. On the other hand, the map $C^\alpha(\mathcal{X}, \mathbb{R})^* \times C^\alpha(\mathcal{X}, \mathbb{R}) \ni (\nu_{\lambda, \theta}, w \circ \pi^{(\lambda)}) \mapsto \nu_{\lambda, \theta}(w \circ \pi^{(\lambda)}) = \int_X w \circ \pi^{(\lambda)}(x) d\nu_{\lambda, \theta}(x) \in \mathbb{R}$ is $C^\infty$. This, together with equation (14) concludes the proof.

Now we can show the following result.

**Lemma 3.19.** Consider the family $G^{(\theta)}$ of weights $g^{(\theta)}_j$ for $j = 1, \cdots, k$ and $-\epsilon < \theta < \epsilon$. Then the stationary measure for $T^{(\lambda)}$ and $G^{(\theta)}$ is the image of the eigen-distribution $\nu_{\psi^{(\lambda)}}$ for $\psi^{(\lambda)}$, i.e., $(\pi^{(\lambda)})_* \nu_{\psi^{(\lambda)}} = \mu_{\lambda, \theta}$.

**Proof.** By the uniqueness of the stationary measure, it is enough for us to check that

$$\int f(\tilde{x}) d\left( (\pi^{(\lambda)})_* \nu_{\psi^{(\lambda)}} \right)(\tilde{x}) = \sum_{i=1}^k \int g^{(\theta)}_i(\tilde{x}) f(T, \tilde{x}) d\left( (\pi^{(\lambda)})_* \nu_{\psi^{(\lambda)}} \right)(\tilde{x})$$

holds for any continuous $f : [0, 1] \to \mathbb{R}$ and $\tilde{x} \in [0, 1]$. A straightforward manipulation yields

$$\sum_{i=1}^k \int g^{(\lambda)}_i(\tilde{x}) f(T, \tilde{x}) d\left( (\pi^{(\lambda)})_* \nu_{\psi^{(\lambda)}} \right)(\tilde{x}) = \int \left( \sum_{y \in \sigma^{-1}x} e^{\psi^{(\lambda)}(y)} f(\pi^{(\lambda)} y) \right) d\nu_{\psi^{(\lambda)}}(x)$$

$$= \int \mathcal{L}^{\psi^{(\lambda)}}(f \circ \pi^{(\lambda)})(x) d\nu_{\psi^{(\lambda)}}(x)$$

$$= \int f \circ \pi^{(\lambda)}(x) d\nu_{\psi^{(\lambda)}}(x)$$

$$= \int f(\tilde{x}) d\left( (\pi^{(\lambda)})_* \nu_{\psi^{(\lambda)}} \right)(\tilde{x})$$

for every continuous function $f : [0, 1] \to \mathbb{R}$, where we have used that $\mathcal{L}^{\psi^{(\lambda)}}(\nu_{\psi^{(\lambda)}}) = \nu_{\psi^{(\lambda)}}.$

**Lemma 3.20.** For fixed $\theta \in I_\epsilon$, the map $I_\epsilon \ni \lambda \mapsto \psi^{(\lambda, \theta)} \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{\min(l, m)-1}$.

**Proof.** Consider $\theta \in I_\epsilon$ fixed. By Corollary 3.7 we have that $C^\alpha(\mathcal{X}, \mathbb{R}) \ni \Pi \mapsto g^{(\theta)}_x(\Pi(\sigma x)) \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{l-1}$ and by Proposition 3.15 the map $I_\epsilon \ni \lambda \mapsto \pi^{(\lambda)} \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{m-1}$, then the map $I_\epsilon \ni \lambda \mapsto g^{(\theta)}_x(\pi^{(\lambda)}(\sigma x)) \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{\min(m, l)-1}$. This proves that the map $I_\epsilon \ni \lambda \mapsto \psi^{(\lambda, \theta)}(x) = \log \left( g^{(\theta)}_x(\pi^{(\lambda)}(\sigma x)) \right) \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{\min(m, l)-1}$, which concludes the proof.

**Lemma 3.21.** For fixed $\lambda \in I_\epsilon$, the map $I_\epsilon \ni \theta \mapsto \psi^{(\lambda, \theta)} \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{r}$.  

17
Proof. From the hypothesis on the family $G^{(\theta)}$ and the definition of $\psi^{(\lambda,\theta)}$

$$\psi^{(\lambda,\theta)}(x) = \log \left( g_{x_0}^{(\theta)}(\pi^{(\lambda)}(\sigma x)) \right)$$

$$= \log \left( g_{x_0}(\pi^{(\lambda)}(\sigma x)) + \theta g_{x_0,1}(\pi^{(\lambda)}(\sigma x)) + \cdots + \theta^r g_{x_0,r}(\pi^{(\lambda)}(\sigma x)) + o(\theta^r) \right)$$

$$=: t(\theta)$$

where $t(\theta) = t(0) + dt(0)\theta + \frac{1}{2!} d^2t(0)\theta^2 + \cdots + o(\theta^r)$, and where $d^2t(0) \in C^a(\mathcal{X}, \mathbb{R})$ is given by

$$d^2t(0)(x) = \frac{p_i \left[ g_{x_0}(\pi^{(\lambda)}(\sigma x)), g_{x_0,1}(\pi^{(\lambda)}(\sigma x)), \ldots, g_{x_0,r}(\pi^{(\lambda)}(\sigma x)) \right]}{g_{x_0}(\pi^{(\lambda)}(\sigma x))^i},$$

where $p_i (i \in \{0, \ldots, r\})$ are polynomials.

Using standard analytic perturbation theory (cf. [24]) and the previous corollary we have the following.

**Corollary 3.22.**

i. For fixed $\theta \in \mathcal{I}_e$, the map $(-\epsilon, \epsilon) \ni \lambda \mapsto \nu^{(\lambda,\theta)} \in C^a(\mathcal{X}, \mathbb{R})^*$ is $C^{\min(l,m)-1}$.

ii. For fixed $\lambda \in \mathcal{I}_e$, the map $(-\epsilon, \epsilon) \ni \theta \mapsto \nu^{(\lambda,\theta)} \in C^a(\mathcal{X}, \mathbb{R})^*$ is $C^r$.

In particular, this implies the following.

**Corollary 3.23.** Given a Hölder continuous function $f \in C^a(\mathcal{X}, \mathbb{R})$.

i. For any fixed $\theta \in (-\epsilon, \epsilon)$, the map $(-\epsilon, \epsilon) \ni \lambda \mapsto \int f d\nu^{(\lambda,\theta)} \in \mathbb{R}$ is $C^{\min(l,m)-1}$.

ii. For any fixed $\lambda \in (-\epsilon, \epsilon)$, the map $(-\epsilon, \epsilon) \ni \theta \mapsto \int f d\nu^{(\lambda,\theta)} \in \mathbb{R}$ is $C^r$.

We now turn to the proof of Theorem 1.3.

**Proof of Theorem 1.3.** There are two parts.

i. From Corollary 3.6 we deduce that for $f \in C^{s+\delta}([0,1], \mathbb{R})$, the map $C^a(\mathcal{X}, \mathbb{R}) \ni \Pi \mapsto f \circ \Pi \in C^a(\mathcal{X}, \mathbb{R})$ is $C^{s-1}$ and we know from Proposition 3.15 that $\mathcal{I}_e \ni \lambda \mapsto \pi^{(\lambda)} \in C^a(\mathcal{X}, \mathbb{R})$ is $C^{m-1}$, then the map $\mathcal{I}_e \ni \lambda \mapsto f \circ \pi^{(\lambda)} \in C^a(\mathcal{X}, \mathbb{R})$ is $C^{\min(s,m)-1}$. Using Corollary 3.22 we have that $(-\epsilon, \epsilon) \ni \lambda \mapsto \nu^{(\lambda,\theta)} \in C^a(\mathcal{X}, \mathbb{R})^*$ is $C^{\min(l,m)-1}$, therefore the map $l_1 : \mathcal{I}_e \to C^a(\mathcal{X}, \mathbb{R}) \times C^a(\mathcal{X}, \mathbb{R})^*$, defined by $l_1(\lambda) = (f \circ \pi^{(\lambda)}), \nu^{(\lambda,\theta)}$ is $C^{\min(l,m,s)-1}$. We define the map $l_2 : C^a(\mathcal{X}, \mathbb{R}) \times C^a(\mathcal{X}, \mathbb{R})^* \to \mathbb{R}$ by $l_2(v, \nu) = \int v d\nu$ for $v \in C^a(\mathcal{X}, \mathbb{R})$ and $\nu \in C^a(\mathcal{X}, \mathbb{R})^*$. The map $l_2$ is $C^\infty$.

We consider the map $F := l_2 \circ l_1$, so $F(\lambda) = \int f \circ \pi^{(\lambda)}d\nu^{(\lambda,\theta)}$, which also concludes the proof of part 1.

ii. For $f \in C^1([0,1], \mathbb{R})$, $f \circ \pi^{(\lambda)} \in C^a(\mathcal{X}, \mathbb{R})$ and the map $l_3 : \mathcal{I}_e \to C^a(\mathcal{X}, \mathbb{R})^*$ defined by $l_3(\theta) = \nu^{(\lambda,\theta)}$ is $C^r$ by Corollary 3.22. We consider the map $G : \mathcal{I}_e \to \mathbb{R}$, defined by $G(\theta) = l_2(f \circ \pi^{(\lambda)}, l_3(\theta))$, where $l_2$ is defined in the part 1 of this proof. By Lemma 3.19 we have $G(\theta) = \int f d\mu_{\lambda,\theta}$ and $G$ is $C^r$ since $l_3$ is $C^r$ and $l_2$ is $C^\infty$. This finishes the proof.
3.5 Proof of Theorem 1.7

Consider an IFS $\mathcal{T}$ as in Definition 1.1 such that the sets $T_i^{(\lambda)}[0,1]$ are pairwise disjoint for $i \in \{1, \ldots, k\}$. Recall the definition of the projection map $\pi^{(\lambda)} : \mathcal{X} \to \mathbb{R}$ and the definition of the pressure $P$ (Definition 2.10). It is well known that the Hausdorff dimension of the limit set $K^{(\lambda)}$, that we call by $HD(K^{(\lambda)})$, corresponds to the unique $s \in [0,1]$ such that $P(s\psi^{(\lambda)}) = 0$, where $\psi^{(\lambda)} : \mathcal{X} \to \mathbb{R}$ is defined by $\psi^{(\lambda)}(x) := \log |dT^{(\lambda)}(\pi^{(\lambda)}(\sigma x))|$.

**Proposition 3.24.** Independently of $G^{(\theta)}$, there exists a unique $t = t_{\lambda} = \text{dim}_H(\text{supp } \mu_{\lambda,\theta})$ such that

$$P\left(-t\psi^{(\lambda)}(x)\right) = 0.$$ 

We are interested in the differentiability of the map $\mathcal{I}_e \ni \lambda \mapsto t_{\lambda} \in \mathbb{R}$. Using Corollary 3.11 we can prove the main proposition we need.

**Proposition 3.25.** The map $\mathcal{I}_e \ni \lambda \mapsto \psi^{(\lambda)}(x) \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{m-2}$.

We can now prove our second theorem.

**Proof of Theorem 1.7.** Since $P : C^\alpha(\mathcal{X}, \mathbb{R}) \to \mathbb{R}$ is real analytic it follows that $\mathcal{I}_e \ni \lambda \mapsto t_{\lambda} \in \mathbb{R}$ is $C^{m-2}$ and using Proposition 3.24 we conclude the proof of Theorem 1.7.

3.6 Proof of Theorem 1.8

We proceed to the proof of the theorem immediately, as we have already developed all the machinery necessary for the proof.

**Proof.** We consider the map $\psi^{(\lambda,\theta)}(x) := \log \left(g^{(\theta)}(\pi^{(\lambda)}(\sigma x))\right)$ defined in Subsection 3.3. The unique probability measure $\nu$ such that $\pi^{(\lambda)} \nu = \mu_{\lambda,\theta}$ is $\nu = \nu_{\psi^{(\lambda,\theta)}} = \nu_{\lambda,\theta}$ (see Lemma 3.19).

For what follows we choose $\lambda_0, \theta_0 \in \mathcal{I}_e$ fixed. By Lemma 3.20 the map

$$\mathcal{I}_e \ni \lambda \mapsto \psi^{(\lambda,\theta_0)} \in C^\alpha(\mathcal{X}, \mathbb{R})$$

is $C^{\min(l,m)-1}$. By Lemma 3.21 the map

$$\mathcal{I}_e \ni \theta \mapsto \psi^{(\lambda_0,\theta)} \in C^\alpha(\mathcal{X}, \mathbb{R})$$

is $C^r$. Since $P : C^\alpha(\mathcal{X}, \mathbb{R}) \to \mathbb{R}$ is real analytic by Lemma 2.13, it follows that the map

$$\mathcal{I}_e \ni \lambda \mapsto P\left(\psi^{(\lambda,\theta_0)}\right) \in \mathbb{R}$$

is $C^{\min(l,m)-1}$ and the map

$$\mathcal{I}_e \ni \theta \mapsto P\left(\psi^{(\lambda_0,\theta)}\right) \in \mathbb{R}$$

is $C^r$. On the other hand, by Corollary 3.22 we have that the map

$$\mathcal{I}_e \ni \lambda \mapsto \nu_{\lambda,\theta_0} \in C^\alpha(\mathcal{X}, \mathbb{R})^*$$
is $C_{\min(l,m)}^{-1}$ and the map

$$\mathcal{I}_\varepsilon \ni \theta \mapsto \nu_{\lambda_0, \theta} \in C^\alpha(\mathcal{X}, \mathbb{R})^*$$

is $C^r$.

We use now similar arguments to those in the proof of Theorem 1.3. By the previous paragraph the map $A : \mathcal{I}_\varepsilon \to C^\alpha(\mathcal{X}, \mathbb{R}) \times C^\alpha(\mathcal{X}, \mathbb{R})^*$ defined by $A(\lambda) = (\psi(\lambda, \theta_0), \nu_{\lambda, \theta})$ is $C_{\min(l,m)}^{-1}$ and the map $B : \mathcal{I}_\varepsilon \to C^\alpha(\mathcal{X}, \mathbb{R}) \times C^\alpha(\mathcal{X}, \mathbb{R})^*$ defined by $B(\theta) = (\psi(\lambda_0, \theta), \nu_{\lambda_0, \theta})$ is $C^r$. We define the map $l : C^\alpha(\mathcal{X}, \mathbb{R}) \times C^\alpha(\mathcal{X}, \mathbb{R})^* \to \mathbb{R}$ by $l(v, \eta) = \int v d\eta$. The map $l$ is $C^\infty$, therefore, the maps $F_A(\lambda) := l \circ A(\lambda) = \int \psi(\lambda_0, \theta) d
u_{\lambda_0, \theta}$ and $F_B(\theta) := l \circ B(\theta) = \int \psi(\lambda_0, \theta) d
u_{\lambda_0, \theta}$ are $C_{\min(l,m)}^{-1}$ and $C^r$, respectively.

Using the variational principle we have that $P \left( \psi(\lambda, \theta) \right) - \int \psi(\lambda_0, \theta) d
u_{\lambda, \theta} = h_{\nu_{\lambda, \theta}}(\sigma)$. Combining this with the results of the previous paragraphs we conclude that the map

$$\mathcal{I}_\varepsilon \ni \lambda \mapsto P \left( \psi(\lambda, \theta_0) \right) - F_A(\lambda) = h_{\nu_{\lambda_0, \theta_0}}(\sigma)$$

is $C_{\min(l,m)}^{-1}$ and the map

$$\mathcal{I}_\varepsilon \ni \theta \mapsto P \left( \psi(\lambda_0, \theta_0) \right) - F_B(\theta) = h_{\nu_{\lambda_0, \theta}}(\sigma)$$

is $C^r$.

We use now Proposition 3.25 with $\psi(\lambda)(x) := \log |dT_{\pi_0}^\lambda(\pi^\lambda(\sigma x))|$, so that the map $\mathcal{I}_\varepsilon \ni \lambda \mapsto \psi(\lambda)(x) \in C^\alpha(\mathcal{X}, \mathbb{R})$ is $C^{m-2}$. This implies that the map

$$\mathcal{I}_\varepsilon \ni \lambda \mapsto l(\psi(\lambda), \nu_{\lambda_0, \theta_0}) = \chi_{\nu_{\lambda_0, \theta_0}} \in \mathbb{R}$$

is $C_{\min(l-1,m-2)}$ and the map

$$\mathcal{I}_\varepsilon \ni \theta \mapsto l(\psi(\lambda_0), \nu_{\lambda_0, \theta}) = \chi_{\nu_{\lambda_0, \theta}}$$

is $C^r$.

Finally, by (4) we have the hypothesis of Theorem 2.18, so that $HD(\mu_{\lambda, \theta}) = \frac{h_{\nu_{\lambda_0, \theta}}}{\chi_{\nu_{\lambda_0, \theta}}}$. Combining this and the results of the previous two paragraphs we deduce that the map

$$\mathcal{I}_\varepsilon \ni \lambda \mapsto HD(\mu_{\lambda, \theta_0}) \in \mathbb{R}^+$$

is $C_{\min(l-1,m-2)}$ and the map

$$\mathcal{I}_\varepsilon \ni \theta \mapsto HD(\mu_{\lambda_0, \theta}) \in \mathbb{R}^+$$

is $C^r$. \qed

## 4 Examples

In this section we exhibit different examples of application of our main results.
4.1 A simple example

Let $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ be the affine maps $T_1(x) = \alpha x + \beta_1$ and $T_2(x) = \alpha x + \beta_2$ with $0 < \alpha < 1$. Let us consider the weights $p_1, p_2 > 0$ with $p_1 + p_2 = 1$. The unique stationary probability measure $\mu = \mu_{\alpha, \beta_1, \beta_2, p_1, p_2}$ in this case is given by the limit in the weak topology

$$\mu := \lim_{n \to +\infty} \sum_{i_1, \ldots, i_n \in \{1, 2\}} p_{i_1} \cdots p_{i_n} \delta_{T_1 \circ \cdots \circ T_n(0)}.$$

If we further assume for simplicity that $\alpha = 0.5$ and $\beta_1 = 0$, $\beta_2 = \alpha$ then the two images $T_1[0, 1] = [0, \alpha]$, $T_2[0, 1] = [\alpha, 1]$ partition the unit interval and $\mu$ will be supported on the unit interval. Finally, in this case it is simple to see that $\mu$ is then the Lebesgue measure if and only if $p_1 = p_2 = 0.5$.

We can consider the dependence of the stationary measure on the parameters $\alpha, \beta_j$ and $p_j$ ($j = 1, 2$) which form a two dimensional space. For any $C^{2+\delta}$ function $w : [0, 1] \to \mathbb{R}$ (with $0 < \delta \leq 1$) we then have that the map

$$(0, 1) \ni \alpha \mapsto \int w d\mu_{\alpha, p_1} \in \mathbb{R},$$

is $C^1$, and

$$(0, 1) \ni p_1 \mapsto \int w d\mu_{\alpha, p_1} \in \mathbb{R},$$

is $C^\infty$, where we write $\mu_{\alpha, p_1} = \mu_{\alpha, 0, p_1, 1-p_1}$. It is clear in this example that the Hausdorff dimension of the limit set and the Hausdorff dimension of the measure $\mu$ are both $C^\infty$.

4.2 A geometric example

We present in Example 4.2 a result on classical Schottky groups. Our machinery is however limited to the case of unique contraction that we define in what follows.

**Definition 4.1** (Unique contraction). Let $\Gamma \subset SL(2, \mathbb{C})$ be a classical Schottky group and suppose that $\Gamma$ is generated by the Möbius transformations $\{\gamma_i\}_{i=1}^k$. For each $i \in \{1, \ldots, k\}$ define $U_i := \{z \in \mathbb{C} : |d\gamma_i(z)| < 1\} \subset \mathbb{C}$ and call by $T_i$ the map $\gamma_i|_{C \setminus U_i} : \mathbb{C} \setminus U_i \to U_i$. We say that $\Gamma$ has a unique contraction if $dT_i = dT_j$ for every $i, j \in \{2, \ldots, k\}$.

**Example 4.2.** For $\lambda \in \mathcal{I}_e = (-\epsilon, \epsilon)$, let $\Gamma_0 \subset SL(2, \mathbb{C})$ be a classical Schottky group such that $\Gamma_0$ has a unique contraction and $\mathcal{I}_e \ni \lambda \mapsto \Gamma_\lambda \subset SL(2, \mathbb{C})$ is $C^m$. Let $\mu_\lambda$ be the conformal probability measure that satisfies

$$g^* \mu_\lambda = |dg| \mathcal{H}_\lambda \mu_\lambda,$$

where $\mathcal{H}_\lambda = HD(\Lambda_\lambda)$ is the Hausdorff dimension of the limit set $\Lambda_\lambda$ for $\Gamma_\lambda$. If $w : \mathbb{C} \to \mathbb{R}$ is a compactly supported $C^{s+\delta}$ function then the map

$$(\mathcal{I}_e \ni \lambda \mapsto \int f d\mu_\lambda)$$

is $C^{\min(m, s-1)}$. 

21
Proof of Example 4.2. Suppose that $\Gamma_\lambda$ is generated by some Möbius transformations $\{\gamma_i^\lambda\}_i$ and for each $\gamma \in \{1, \ldots, k\}$ define $U_i^\lambda := \{ z \in \mathbb{C} : |d\gamma_i^\lambda(z)| < 1 \} \subset \mathbb{C}$. For each $i, j \in \{1, \ldots, k\}$ call by $T_i^\lambda$ the map $\gamma_i^\lambda |_{\partial U_i^\lambda} : \mathbb{C} \setminus U_i^\lambda \to U_i^\lambda$ and define the map $T_j^\lambda : U_i \to U_j$ such that $T_j^\lambda = T_j^\lambda |_{U_i}$. We consider the shift space

$$
\Sigma := \{ x = (x_n)_{n=0}^\infty : x_n \in \{1, \ldots, k\}, x_n \neq x_{n+1}, n \in \mathbb{N} \} \subset \{1, \ldots, k\}^{\mathbb{N}_0}
$$

and define the projection map $\pi^\lambda : \Sigma \to \Lambda_\lambda \subset \mathbb{C}$, by $x \mapsto \lim_{n \to \infty} T_i^\lambda T_j^\lambda \cdots T_k^\lambda (z_0)$ where $z_0 \in \mathbb{C}$ is fixed and $\Lambda_\lambda := \{ \lim_{n \to \infty} T_i^\lambda T_j^\lambda \cdots T_k^\lambda (z_0) : x \in \Sigma \}$ is the limit set for $\Gamma_\lambda$. We notice that $\pi^\lambda \in C^\alpha(\Sigma, \mathbb{C})$ for some small $\alpha > 0$. The conformal probability measure $\mu_\lambda$ satisfies that $\mu_\lambda = \pi^\lambda_* \mu_\lambda$ for $L^\mu_\lambda = \mu_\lambda$ where

$$
L_\lambda w(x) = \sum_{y \in \sigma^{-1}x \Sigma} |dT_{y_0,x_0}(\pi^\lambda y)| d\pi^\lambda w(T_{y_0,x_0}(\pi^\lambda y)), w : \Sigma \to \mathbb{R}, x \in \Sigma.
$$

We know from [22] that the Hausdorff dimensions of the limit set for $\Gamma$ is a real analytic function on the deformation space of a Schottky group, then the map $I_x \equiv \lambda \mapsto \mathcal{H}_\lambda \in \mathbb{R}$ is $C^m$. On the other hand, the map $I_x \equiv \lambda \mapsto \pi^\lambda \in C^\alpha(\Sigma, \mathbb{C})$ is $C^m$ (we can use the same proof of Proposition 3.15, the main difference is that now when applying Corollary 3.7 we obtain $C^m$ and not $C^{m-1}$ as the maps $T_i^\lambda$ are $C^\infty$ and not just $C^{m+\delta}$). Then the map $I_x \equiv \lambda \mapsto \mathcal{H}_\lambda \log|dT_{y_0,x_0}(\pi^\lambda y)|d\mu_\lambda \in \mathbb{R}$ is $C^m$ and by perturbation theory so is the map $I_x \equiv \lambda \mapsto \mu_\lambda \in C^\alpha(\Sigma, \mathbb{R})^\ast$. Finally, we have that for $w : \mathbb{C} \to \mathbb{R}$ a compactly supported $C^\infty$ function $f \circ \pi^\lambda d\mu_\lambda = f w d\mu_\lambda$, and therefore the map $\lambda \mapsto \int w d\mu_\lambda$ is $C^m$ by an application of Corollary 3.6, which concludes the proof. 

\[\square\]

4.3 Some general examples

A careful look at Theorem 1.3 and to it proof allows to obtain similar results to the ones showed in the introduction under much weaker hypotheses. This is the propose of this subsection. We start by modifying Definition 1.1 and replacing it by:

Definition 4.3. Assume that $\delta, \epsilon \in (0, 1)$, $k, l, m, n, p \in \mathbb{N} \setminus \{1\}$, $q \in \mathbb{N}$ and let $\Lambda, \Theta$ be open intervals $\Lambda, \Theta \subset \mathbb{R}$.

i. Let

$$
T = T(\Lambda, k, l, m, \delta) := \{ (T_i^\lambda)_{i=1}^k : \lambda \in \Lambda \}
$$

be a family of contractions such that for $\lambda \in \Lambda$ and $i \in \{1, \ldots, k\}$:

$$
T_i^\lambda = \bar{T}_i(\lambda, \cdot),
$$

where

(a) $\bar{T}_i(\lambda, \cdot) \in C^{l+\delta}([0, 1], [0, 1])$,
(b) $\sup_{\lambda \in \Lambda} \| \frac{\partial}{\partial \lambda} \bar{T}_i(\lambda, \cdot) \|_{C^0} < 1$,
(c) $\bar{T}_i(\cdot, \cdot) \in C^m(\Lambda \times [0, 1], [0, 1])$, and
(d) $\frac{\partial}{\partial \lambda} \bar{T}_i(0, x) = \frac{\partial}{\partial \lambda} \bar{T}_j(0, x)$ for all $i, j$. 

22
On a family $\mathcal{T}$ for every $\lambda \in \Lambda$, we define the limit set $\mathcal{K}(\lambda)$ as the unique non-empty closed set $\mathcal{K} \subset [0, 1]$ such that
$$\mathcal{K} = \bigcup_{i=1}^{k} T_{i}^{(\lambda)} \mathcal{K}.$$  

We define $$(\mathcal{T}, G)$$, where
$$G = G(\Theta, k, n, p, \epsilon) := \left\{ \{ g_i^{(\theta)} \}_{i=1}^{k} : \theta \in \Theta \right\}$$
is a family of weight functions such that

(a) $$\sum_{i=1}^{k} \| g_i^{(\theta)} \|_{C^0} \text{Lip} \left( T_{i}^{(\lambda)} \right) < 1$$ for all $\lambda \in \Lambda, \theta \in \Theta$;

and

(b) for every $\theta \in \Theta, i \in \{1, \ldots, k\}$ :
$$g_i^{(\theta)} = \tilde{g}_i(\theta)$$
where for some $\beta \in (0, 1/2)$ we have

i. $\tilde{g}_i(\theta) \in C^{a+\rho}([0, 1], \mathbb{R}^+)$,
ii. $\tilde{g}_i(\cdot) \in C^q (\mathcal{I}, C^{a+\rho}([0, 1], \mathbb{R}^+)).$

If we do not consider the normalisation condition on the weight functions, we require a generalised definition of stationary measures. In order to deal with this we introduce the next definition.

**Definition 4.4.** Given the families $(\mathcal{T}, G)$, define $h_i^{(\lambda, \theta)} := (g_i^{(\theta)})^{s_{\lambda, \theta}}$, where $s_{\lambda, \theta} \in [0, 1]$ is unique solution of $P \left( s_{\lambda, \theta} \log \left( \frac{g_i^{(\theta)}(\pi_{\lambda}(\sigma x))}{s_{\lambda, \theta}} \right) \right) = 0$ and $P$ is the Pressure. A generalized stationary measure $\mu = \mu_{\lambda, \theta}$ is the unique probability measure on $[0, 1]$ that satisfies
$$\int f(x) d\mu(x) = \sum_{i=1}^{k} \int h_i^{(\lambda, \theta)}(x) f(T_{i}^{(\lambda)}(x)) d\mu(x),$$
for any continuous function $f : [0, 1] \to \mathbb{R}$.

Under the hypotheses of Definition 4.3, a step-by-step equal proof that the one given for Theorem 1.3 gives us the following result:

**Theorem 4.5.** Let fix $a \in \mathbb{N} \setminus \{1\} \cup \{ \infty \}$ and $\rho \in (0, 1)$. On $(\mathcal{T}, G)$, for the generalized stationary probability measure $\mu_{\lambda, \theta}$ with $\lambda \in \Lambda, \theta \in \Theta$, or in the case $\Lambda = \Theta$, for the generalized stationary probability measure $\mu_{\lambda, \lambda} = \mu_{\lambda}$ for $\lambda \in \Lambda$, we have:

i. For $\theta \in \Theta$ and $f \in C^{a+\rho}(\hat{\mathcal{K}}, \mathbb{R})$, where $\hat{\mathcal{K}} \supset \bigcup_{\lambda \in \Lambda} \mathcal{K}(\lambda)$, the map $F : \Lambda \to \mathbb{R}$ defined by
$$F(\lambda) = \int f d\mu_{\lambda, \theta}$$
belongs to $C^r(\Lambda, \mathbb{R})$ with $r = \min\{l - 1, m - 1, a - 1\}$.  


ii. For \( \lambda \in \Lambda \) and \( f \in C^1(\hat{\mathcal{K}}, \mathbb{R}) \), the map \( F : \Theta \to \mathbb{R} \) defined by

\[
F(\lambda) = \int f d\mu_{\lambda, \vartheta}
\]

belongs to \( C^q(\Theta, \mathbb{R}) \).

iii. For \( \Lambda = \Theta \) and \( f \in C^{a+\rho}(\hat{\mathcal{K}}, \mathbb{R}) \), the map \( F : \Lambda \to \mathbb{R} \) defined by

\[
F(\lambda) = \int f d\mu_{\lambda}
\]

belongs to \( C^r(\Lambda, \mathbb{R}) \) with \( r = \min\{l - 1, m - 1, a - 1, n - 1, q\} \).

An easy example of application of Theorem 4.5 that Theorem 1.3 fails is the case that \( x_0 \in [0, 1] \setminus \cup_{\lambda \in \Lambda} \mathcal{K}(\lambda) \) and \( f(x) = |x - x_0| \).

We end this subsection with two examples. In the first we can apply our theorem and it is possible to experimentally see the regularity of the map \( F(\lambda) \). In the second, the hypothesis on the smoothness of the contractions is not satisfied. In this case, experimentally the map \( F(\lambda) \) looks \( C^0 \) but not \( C^1 \), however we cannot prove it, as our method of composition of operator does not work. The first example is the following:

**Example 4.6.** Let us consider \( \Lambda = \Theta = [1/6, 1/3] \), \( x \in [0, 1] \), \( n \in \mathbb{N} \), \( \lambda \in \Lambda \),

\[
\phi(x, n) = x^{n+1} \sin(1/x) \in C^n(\mathbb{R}, \mathbb{R}) \setminus C^{n+1}(\mathbb{R}, \mathbb{R}),
\]

\[
T_1^{(\lambda)}(x) = \lambda x + \phi(\lambda - 0.25, 3) + 0.01,
\]

\[
T_2^{(\lambda)}(x) = \lambda x + \frac{2}{3} + \phi(\lambda - 0.25, 3),
\]

\[
g_1^{(\lambda)}(x) = \lambda \mathbf{1}_{[0,1/2]}(x) + (1 - \lambda) \mathbf{1}_{[1/2,1]}(x),
\]

\[
g_2^{(\lambda)}(x) = (1 - \lambda) \mathbf{1}_{[0,1/2]}(x) + (\lambda) \mathbf{1}_{[1/2,1]}(x), \quad \text{and}
\]

\[
f(x) = \begin{cases} 
-\lambda & \text{if } x \in [0, 1/2) \\
x^2 & \text{if } x \in [1/2, 1].
\end{cases}
\]

Then the map \( F : \Lambda \to \mathbb{R} \), defined by \( F(\lambda) = \int f(x) d\mu_{\lambda}(x) \), belongs to \( C^1(\Lambda, \mathbb{R}) \). Moreover, for any interval \( \Lambda' \subset [1/6, 1/4) \) or \( \Lambda' \subset (1/4, 1/3) \), we have that \( F|_{\Lambda' \in C^\infty(\Lambda', \mathbb{R})} \).

The second example, where our results are not longer valid, is the following:

**Example 4.7.** Let us consider \( \Lambda = \Theta = [1/6, 1/3] \), \( x \in [0, 1] \), \( n \in \mathbb{N} \), \( \lambda \in \Lambda \),

\[
T_1^{(\lambda)}(x) = \lambda x + \phi(\lambda - 0.25, 1) + 0.01,
\]

\[
T_2^{(\lambda)}(x) = \lambda x + \frac{2}{3} + \phi(\lambda - 0.25, 1),
\]

\[
g_1^{(\lambda)}(x) = \lambda \mathbf{1}_{[0,1/2]}(x) + (1 - \lambda) \mathbf{1}_{[1/2,1]}(x),
\]

\[
g_2^{(\lambda)}(x) = (1 - \lambda) \mathbf{1}_{[0,1/2]}(x) + (\lambda) \mathbf{1}_{[1/2,1]}(x), \quad \text{and}
\]

\[
f(x) = \begin{cases} 
-\lambda & \text{if } x \in [0, 1/2) \\
x^2 & \text{if } x \in [1/2, 1].
\end{cases}
\]

Does the map \( F : \Lambda \to \mathbb{R} \), defined by \( F(\lambda) = \int f(x) d\mu_{\lambda}(x) \), belongs to \( C^0(\Lambda, \mathbb{R}) \)?
Figure 1: Graph of $F : \Lambda \rightarrow \mathbb{R}$ in Example 4.6

Figure 2: Graph of $F : \Lambda \rightarrow \mathbb{R}$ in Example 4.7
4.4 A comparison with previous results

In this section we compare our results with the main theorems in [26], Theorem 4.12 and 4.14 here. We start by introducing some definitions, as the setting of [26] is more general than our. As a consequence of Theorem 4.12 and 4.14 we obtain Corollary 4.15 that we compare with Corollary 4.16, a similar result whose proof follows entirely from Section 3.

**Definition 4.8** (Graph iterated function system). A GIFS (Graph iterated function system) is defined by a triplet \((G, (J_v), (T_e))\) satisfying the following conditions:

i. \(G = (V, E, i, t)\) is a finite directed multigraph which consists of vertices set \(V\), a directed edges sets \(E\) and two functions \(i, t : E \to V\). For each \(e \in E\), \(i(e)\) is called the initial vertex of \(e\) and \(t(e)\) is called the terminal vertex of \(e\). Assume that the graph \(G\) is strongly connected and aperiodic.

ii. For each \(v \in V\), a subset \(J_v \subset \mathbb{R}^D\) is compact and connected so that the interior of \(J_v\) is not empty. For every \(v, v' \in V\) with \(v \neq v'\) we have that \(J_v\) and \(J_{v'}\) are disjoint.

iii. For each \(v \in V\) we consider certain connected open sets \(O_v \subset J_v\), so that, for each \(e \in E\), the map \(T_e : O_{t(e)} \to O_{i(e)}\) is conformal \(C^{1+\beta}\)-diffeomorphism with \(\beta > 0\) and satisfies \(0 < \|T'_e(x)\| < 1\) for \(x \in O_{t(e)}\), and for every \(e, e' \in E\) with \(e \neq e'\), \(i(e) = i(e')\) we have that \(T_e J_{t(e)}\) and \(T_{e'} J_{t(e')}\) are disjoint.

**Remark 4.9.** We stated the definition of GIFS in [26]. A more general one can be found in [19].

We notice that IFSs are in particular GIFSs, as they can always be represented by a 1-vertex GIFS. Moreover, GIFSs may exhibit more general phenomena than IFSs [2].

For GIFSs there is a definition of limit set, similar to the one for IFSs in Lemma 2.2.

**Definition 4.10** (Limit set). Given a GIFS \((G, (J_v), (T_e))\) we define its limit set by the set \(K = \cup_{v \in V} K_v\), where for each \(v \in V\) the subset \(K_v \subset J_v\) is the unique non-empty compact set such that

\[
K_v = \cup_{e \in E : i(e) = v} T_e (K_{t(e)})
\]

We now introduce a condition on the regularity of the maps \((T_e)\) from [26].

**Definition 4.11** \(((G)_n\) condition). We say that a family of GIFSs \((G, (J_v), (T_e(\cdot, \cdot)))\) for \(\epsilon > 0\) small satisfies the \((G)_n\) condition if

i. there exists numbers \(\beta > 0\) and \(\beta(\epsilon) > 0\) such that \(T_e\) is \(C^{n+1+\beta}\),

ii. there exists functions \(T_{e,1}\) of class \(C^{n+\beta}\), ..., \(T_{e,n}\) of class \(C^{1+\beta}\), and \(\hat{T}_{e,n}(\epsilon, \cdot)\) of class \(C^{1+\beta(\epsilon)}\) defined on \(O_{t(e)}\) for each \(e \in E\) such that

\[
T_e(\epsilon, \cdot) = T_e + T_{e,1}(\epsilon) + \cdots + T_{e,n}(\epsilon) + \hat{T}_{e,n}(\epsilon, \cdot)\epsilon^n \quad on J_{t(e)},
\]

where \(|\hat{T}_{e,n}(\epsilon, \cdot)| \to 0\) and \(\frac{\partial}{\partial \epsilon} \hat{T}_{e,n}(\epsilon, \cdot) \to 0\) as \(\epsilon \to 0\).
For what follows, let us consider a family of GIFSs \((G, (J_v), (T_v(\epsilon, \cdot)))\) and respective limit sets \(K(\epsilon)\) for \(\epsilon > 0\) small. The main theorem in [26] is the following.

**Theorem 4.12** (Theorem 1.1 in [26]). Assume that the \((G)_n\) condition is satisfied. Then there exist numbers \(s_1, \ldots, s_n \in \mathbb{R}\) such that \(HD(K(\epsilon)) = HD(K) + s_1\epsilon + \cdots + s_n\epsilon^n + o(\epsilon^n)\) in \(\mathbb{R}\), where \(HD(K(\epsilon))\) corresponds to the Hausdorff dimension of \(K(\epsilon)\).

In order to state the second main theorem in [26], we need to introduce a definition and some notation.

**Definition 4.13** \((G)'_n\) condition. We say that a family of GIFSs \((G, (J_v), (T_v(\epsilon, \cdot)))\) satisfies the \((G)'_n\) condition if it satisfies the \((G)_n\) condition and the small order parts \(\tilde{T}_{e,n}(\epsilon, x)\) satisfy

\[
\limsup_{\epsilon \to 0} \max_{e \in E} \sup_{x, y \in O(e) : x \neq y} \frac{\| \frac{\partial}{\partial x} \tilde{T}_{e,n}(\epsilon, x) - \frac{\partial}{\partial x} \tilde{T}_{e,n}(\epsilon, y) \|}{|x - y|^\beta} < \infty.
\]

Let \(r \in (0, 1)\) be such that \(r > \|T'_e\|\) and \(r > \|T'_e(\epsilon, \cdot)\|\) for any \(e \in E\) and any \(\epsilon > 0\) small. Denote \(E^\infty := \{w = (w_k)_{k=0}^\infty \in \prod_{k=0}^\infty E : t(w_k) = i(w_{k+1})\) for all \(k \geq 0\}\) and define the shift \(\sigma : E^\infty \to E^\infty\). Let \(\pi : E^\infty \to \mathbb{R}^D\) be the projection of the GIFS \((G, (J_v), (T_v))\) defined by \(\pi(w) := \cap_{k=0}^\infty T(w_0) \cdots T(w_k) J(w_k)\) for \(w \in E^\infty\). We define the function \(\varphi(w) := \log \|T'_e(\pi(\sigma w))\|\). For each \(\epsilon > 0\), we denote by \(\pi(\epsilon, w)\) the projection of the GIFS \((G, (J_v), (T_v(\epsilon, \cdot)))\) and we denote by \(\varphi(\epsilon, w)\) the function \(\varphi(\epsilon, w) := \log \|\frac{\partial}{\partial w} T(w_0, \pi(\epsilon, \sigma w))\|\). Finally, we denote by \(\mu\) the Gibbs measure of \(HD(K)\varphi\) on \(E^\infty\) and by \(\mu(\epsilon, \cdot)\) the Gibbs measure of \(HD(K(\epsilon))\varphi(\epsilon, \cdot)\) on \(E^\infty\).

**Theorem 4.14** (Theorem 1.2. in [26]). Assume that the \((G)'_n\) condition is satisfied. Choose any \(\theta_1 \in (r^3, 1)\). Then there exists linear functionals \(\mu_1, \mu_2, \ldots, \mu_n \in F_{\theta_1}(E^\infty, \mathbb{R})\), and numbers \(H_1, H_2, \ldots, H_n \in \mathbb{R}\) such that for each \(f \in F_{\theta_1}(E^\infty, \mathbb{C})\)

\[
\mu(\epsilon, f) = \mu(f) + \mu_1(f)\epsilon + \cdots + \mu_n(f)\epsilon^n + o(\epsilon^n) \text{ in } \mathbb{R}
\]

\[
h(\mu(\epsilon, \cdot)) = h(\mu) + H_1\epsilon + \cdots + H_n\epsilon^n + o(\epsilon^n) \text{ in } \mathbb{R},
\]

where \(h(\mu(\epsilon, \cdot))\) denotes the measure-theoretic entropy of the Gibbs measure \(\mu(\epsilon, \cdot)\).

The main ingredients in the proofs of Theorem 4.12 and 4.14 are Proposition 2.3 in [26], and Theorem 2.1 and Theorem 2.4 in [25].

In the particular case that the GIFS is also an IFS, we are in conditions to compare our results with Theorem 4.12 and 4.14. We concluded that we can apply our methods to obtain similar results, indeed, we can do the following.

Consider an IFS \(\mathcal{T}\) as in Definition 4.3 such that the sets \(T_i(\lambda)[0, 1]\) are pairwise disjoint for \(i \in \{1, \ldots, k\}\) and such that \(m = l\). Using our results in Section 3, we can deduce the following result.

**Corollary 4.15.** i. The dependence \(\mathcal{I} \ni \lambda \mapsto HD(K(\lambda))\) of the Hausdorff dimension of the limit set is \(C^{m-2}\).
ii. For $\alpha \in (0, 1)$ small enough so that $2^\alpha \|dT_1\|_{C^0} < 1$ and $\pi^{(\lambda)} : \mathcal{X} \to \mathbb{R}$ is $\alpha$-Hölder, the Gibbs measure $\mu_\varphi$ of $\varphi = \text{HD}(\mathcal{K}(\lambda))\psi^{(\lambda)} \in C^\alpha(\mathcal{X}, \mathbb{R})$ and the measure theoretic entropy $h(\mu_\varphi)$ of $\mu_\varphi$ have both a $C^{m-2}$ dependence on $\lambda \in \mathcal{I}$, when we consider $\mu_\varphi$ as an operator on $C^\alpha(\mathcal{X}, \mathbb{R})^*$. 

In the same setting, using Theorem 4.12 and 4.14 above, instead of our results in Section 3, one can deduce a stronger result under slightly different conditions.

**Corollary 4.16.**

i. The dependence $\mathcal{I} \ni \lambda \mapsto \text{HD}(\mathcal{K}(\lambda))$ of the Hausdorff dimension of the limit set is $C^{m-1}$.

ii. The Gibbs measure $\mu_\varphi$ of $\varphi = \text{HD}(\mathcal{K}(\lambda))\psi^{(\lambda)} \in C^\alpha(\mathcal{X}, \mathbb{R})$ and the measure theoretic entropy $h(\mu_\varphi)$ of $\mu_\varphi$ have both a $C^{m-1}$ dependence on $\lambda \in \mathcal{I}$, when we consider $\mu_\varphi$ as an operator on $C^\alpha(\mathcal{X}, \mathbb{R})^*$, where $\alpha \in (r^\beta, 1)$ and $r \in (0, 1)$ depends on the rate of contraction of $T^{(\lambda)}$.

The difference in the necessary conditions of both corollaries is that in Corollary 4.15 the Gibbs measure $\mu_\varphi$ is an operator on $C^\alpha(\mathcal{X}, \mathbb{R})^*$, where $\alpha \in (r^\beta, 1)$ and $r \in (0, 1)$ depends on the rate of contraction of $T^{(\lambda)}$, whereas, in Corollary 4.16, it is necessary $\alpha \in (0, 1)$ small enough so that $2^\alpha \|dT_1\|_{C^0} < 1$ and $\pi^{(\lambda)} : \mathcal{X} \to \mathbb{R}$ is $\alpha$-Hölder.

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