INFINITE ARMS BANDIT: OPTIMALITY VIA CONFIDENCE BOUNDS

BY HOCK PENG CHAN* AND SHOURI HU

National University of Singapore

The infinite arms bandit problem was initiated by Berry et al. (1997). They derived a regret lower bound of all solutions for Bernoulli rewards, and proposed various bandit strategies based on success runs, but which do not achieve this bound. We propose here a confidence bound target (CBT) algorithm that achieves extensions of their regret lower bound for general reward distributions and distribution priors. The algorithm does not require information on the reward distributions, for each arm we require only the mean and standard deviation of its rewards to compute a confidence bound. We play the arm with the smallest confidence bound provided it is smaller than a target mean. If the confidence bounds are all larger, then we play a new arm. We show how the target mean can be computed from the prior so that the smallest asymptotic regret, among all infinite arms bandit algorithms, is achieved. We also show that in the absence of information on the prior, the target mean can be determined empirically, and that the regret achieved is comparable to the smallest regret. Numerical studies show that CBT is versatile and outperforms its competitors.

1. Introduction. Berry, Chen, Zame, Heath and Shepp (1997) initiated the infinite arms bandit problem on Bernoulli rewards. They showed in the case of uniform prior on the mean of an arm, a $\sqrt{2n}$ regret lower bound for $n$ rewards, and provided algorithms based on success runs that achieve no more than $2\sqrt{n}$ regret. Bonald and Proutière (2013) extended these to a two-target stopping-time algorithm that can get arbitrary close to Berry et al.’s lower bound. Wang, Audibert and Munos (2008) considered the infinite arms bandit problem with bounded rewards and general priors. Vermorel and Mohri (2005) proposed a POKER algorithm for general reward distributions and priors.

The confidence bound method is arguably the most influential approach for the (fixed arm-size) multi-armed bandit problem over the past thirty years. Lai and Robbins (1985) derived the smallest asymptotic regret that a multi-armed bandit algorithm can achieve. Lai (1987) showed that by constructing an upper confidence bound (UCB) for each arm, playing the arm

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with the largest UCB, this smallest regret is achieved in exponential families. The UCB approach was subsequently extended to unknown time-horizons and other parametric families in Agrawal (1995), Auer, Cesa-Bianchi and Fischer (2002), Burnetas and Katehakis (1996), Cappé, Garivier, Maillard, Munos and Stoltz (2013) and Kaufmann, Cappé and Garivier (2012), and it has been shown to perform well in practice, achieving optimality beyond exponential families. Chan (2018) modified the subsampling approach of Baransi, Maillard and Mannor (2014) and showed that optimality is achieved in exponential families, despite not applying parametric information in the selection of arms. The method can be considered to be applying confidence bounds that are computed empirically from subsample information, which substitutes for the missing parametric information. Good performances and optimality has also been achieved by Bayesian approaches to the multi-armed bandit problem, see Berry and Fridstedt (1985), Gittins (1989) and Thompson (1933) for early groundwork on the Bayesian approach, and Korda, Kaufmann and Munos (2013) for more recent advances.

In this paper we show how the confidence bounds method can be extended to the infinite arms bandit problem, with differences to adjust for the infinite arms that are available, in particular the specification of a target mean that we desire from our best arm, and a modification of the bounds for quicker rejection of weak arms. For each arm a lower confidence bound of its mean is computed, using only information on the sample mean and standard deviation of its rewards. We are interested in the minimization of rewards ("penalty" is probably a more apt term than "rewards"), hence the consideration of lower rather than upper confidence bounds. We play an arm as long as its confidence bound is below the target mean. If it is above, then a new arm is played in the next trial.

We start by deriving the smallest possible regret that any infinite arms bandit algorithm can achieve, as the number of rewards goes to infinity. This is followed by showing how to choose the target mean so that the confidence bound target (CBT) algorithm described above achieves this regret. The optimal target mean depends only on the distribution prior of the arm means and not on the reward distributions. In the absence of information on the distribution prior, we show how to adapt via empirical determination of the target mean. Numerical studies on Bernoulli rewards and a URL dataset show that CBT achieves smaller regret compared to its competitors.

The layout of the paper is as follows. In Section 2 we review a number of infinite arms bandit algorithms and describe CBT. In Section 3 we motivate why a particular choice of the target mean leads to the smallest regret and state the optimality results. In Section 4 we provide an empirical version of
CBT to tackle unknown priors. In Section 5 we perform numerical studies. In Sections 6 and 7 we prove the optimality of CBT.

2. Methodology. Let \( X_{k1}, X_{k2}, \ldots \) be i.i.d. non-negative rewards from an arm \( \Pi_k, 1 \leq k < \infty \), with mean \( \mu_k \). Let \( \mu_1, \mu_2, \ldots \) be i.i.d. with prior density \( g \) on \((0, \infty)\). Let \( F_\mu \) denote the reward distribution of an arm with mean \( \mu \), and let \( E_\mu (P_\mu) \) denote expectation (probability) with respect to \( X \sim F_\mu \). Let \( E_g(\cdot) = \int_0^\infty E_\mu(\cdot)g(\mu)d\mu \) and similarly for \( P_g \).

A bandit algorithm is required to select one of the arms to be played at each trial, with the choice informed from past outcomes. We measure the effectiveness of a bandit algorithm by its regret

\[
R_n = E\left( \sum_{k=1}^K n_k \mu_k \right),
\]

where \( K \) is the total number of arms played, \( n_k \) the number of rewards from \( \Pi_k \) and \( n = \sum_{k=1}^K n_k \).

Berry et al. (1997) showed that if \( F_\mu \) is Bernoulli and \( g \) uniform on \((0, 1)\), then a regret lower bound

\[
(2.1) \quad \liminf_{n \to \infty} \frac{R_n}{\sqrt{n}} \geq \sqrt{2}
\]

is unavoidable. They proposed a number of bandit strategies that we describe below. It should be clarified that in our notation success refers to observing a reward of 0, and failure refers to observing a reward of 1.

1. \( f \)-failure strategy. The same arm is played until \( f \) failures are encountered. When this happens, we switch to a new arm. We do not go back to a previously played arm, that is the strategy is non-recalling.

2. \( s \)-run strategy. We restrict ourselves to no more than \( s \) arms, following the 1-failure strategy in each, until a success run of length \( s \) is observed in an arm. When this happens, we play the arm for the remaining trials. If no success runs of length \( s \) is observed in all \( s \) arms, the arm with the highest proportion of successes is played for the remaining trials.

3. Non-recalling \( s \)-run strategy. We follow the 1-failure strategy until an arm produces a success run of length \( s \). When this happens, we play this arm for the remaining trials. If no arm produces a success run of length \( s \), the 1-failure strategy is used for all \( n \) trials.

4. \( m \)-learning strategy. We follow the 1-failure strategy for the first \( m \) trials, with the arm at trial \( m \) played until it yields a failure. Thereafter we play, for the remaining trials, the arm with the highest proportion of successes.
Berry et al. (1997) showed that $R_n \sim n/ (\log n)$ for the $f$-failure strategy for any $f \geq 1$, whereas for the $\sqrt{n}$-run strategy, the $\log n \sqrt{n}$-learning strategy and the non-recalling $\sqrt{n}$-run strategy,

$$\limsup_{n \to \infty} \frac{R_n}{\sqrt{n}} \leq 2.$$

Bonald and Proutière (2013) proposed a two-target algorithm that gets arbitrarily close to the lower bound in (2.1). Let $\lfloor \cdot \rfloor$ denote the greatest integer function. The target values are $s_1 = \lfloor \frac{\sqrt{2}}{f} \rfloor$ and $s_f = \lfloor \sqrt{\frac{2}{f}} \rfloor$, where $f \geq 2$ is user-defined. An arm is discarded if it fails to achieve the target of $s_1$ successes before the first failure, or $s_f$ successes before $f$ failures. If both targets are met, then we accept the arm and play it for the remaining trials. Bonald and Proutière (2013) showed that for a uniform prior, the two-target algorithm satisfies

$$\limsup_{n \to \infty} \frac{R_n}{\sqrt{n}} \leq \sqrt{2} + \frac{1}{f \sqrt{2}},$$

and we get arbitrarily close to the lower bound of Berry et al. by selecting $f$ large.

Wang, Audibert and Munos (2008) proposed a UCB-F algorithm for rewards taking values in $[0, 1]$. They showed that if $P_g(\mu_k \leq \mu) = O(\mu^\beta)$ for some $\beta > 0$,

then under suitable regularity conditions, $R_n = O(n^{\frac{\beta}{\beta+1}} \log n)$. In UCB-F an order $n^{\frac{\beta}{\beta+1}}$ arms are chosen, and confidence bounds are computed on these arms to determine which arm to play. There are additional constants in these confidence bounds, compared to the usual confidence bounds for finite arms, that inflate the bounds. Though there is more distribution flexibility in Wang et al., when restricted to the Bernoulli setting of Berry et al. and Bonald and Proutière, its numerical performances are considerably weaker. Carpentier and Valko (2014) also considered distributions on $[0, 1]$, but their interest in maximizing the selection of a good arm is different from the aims here and in the papers described above.

2.1. Confidence bound target. We construct a confidence bound for each arm and play an arm as long as its confidence bound is under a target mean. Let $b_n \to \infty$ and $c_n \to \infty$ increase slowly with $n$. In particular we require $b_n$ and $c_n$ to be sub-polynomial in $n$. In our numerical studies, we select $b_n = c_n = \log(\log n)$. 


For an arm $\Pi_k$ that has been played $t$ times, we compute its confidence bound by

$$L_{kt} = \max \left( \frac{\bar{X}_{kt}}{b_n}, \bar{X}_{kt} - c_n \frac{\hat{\sigma}_{kt}}{\sqrt{t}} \right),$$

where $\bar{X}_{kt} = \frac{1}{t} \sum_{u=1}^t x_{ku}$ and $\hat{\sigma}_{kt}^2 = \frac{1}{t} \sum_{u=1}^t (x_{ku} - \bar{X}_{kt})^2$.

Let $\mu_* > 0$ be a target mean. We shall discuss in Section 3 how $\mu_*$ should be selected to achieve optimality. It suffices to mention here that it is small for large $n$, more specifically it decreases at a polynomial rate with $n$. The algorithm is non-recalling, an arm is played until its confidence bound goes above $\mu_*$, and it is not played after that.

Confidence Bound Target (CBT)

For $k = 1, 2, \ldots$; Select $n_k$ rewards from $\Pi_k$, where

$$n_k = \inf \{ t \geq 1 : L_{kt} > \mu_* \} \land \left( n - \sum_{\ell=1}^{k-1} n_\ell \right).$$

The number of arms played is $K = \min \{ k : \sum_{\ell=1}^k n_\ell = n \}$.

There are essentially three types of arms that we need to take care of, and that explains the design of the confidence bound $L_{kt}$. The first type are arms with means $\mu_k$ significantly larger than $\mu_*$. For these arms, we would like to reject them as quickly as we can. The condition that an arm be rejected when $\bar{X}_{kt}/b_n$ exceeds $\mu_*$ is key to the achievement of this goal.

The second type of arms are those with means $\mu_k$ larger than $\mu_*$, but not by as much as those in the first type. For these arms, we are unlikely to reject them quickly, as it may be hard to determine whether $\mu_k$ is larger than or less than $\mu_*$ based on a small sample. Rejecting $\Pi_k$ when $\bar{X}_{kt} - c_n \hat{\sigma}_{kt}/\sqrt{t}$ exceeds $\mu_*$ ensures that $\Pi_k$ is rejected only when it is statistically significant that $\mu_k$ is larger than $\mu_*$. Though there may be large number of rewards from this group of arms, their contributions to the regret are small because their means are small.

The third group of arms are those with means $\mu_k$ smaller than $\mu_*$. For this group of arms, the best strategy (when $\mu_*$ is chosen correctly) is to play them for the remaining trials. Selecting $b_n \to \infty$ and $c_n \to \infty$ in (2.2) ensures that the probabilities of rejecting these arms are small.
3. Optimality. In Lemma 1 below we motivate the choice of $\mu_*$. Let $\lambda = \int_0^\infty E_\mu(X|X > 0)g(\mu)d\mu$ be the (finite) mean of the first non-zero reward from a random arm. The value $\lambda$ represents the cost of experimenting with a new arm. We consider $E_\mu(X|X > 0)$ instead of $\mu$ because we are able to reject an arm only when there is a non-zero reward. For Bernoulli rewards, $\lambda = 1$. Let $p(\mu) = P_\mu(\mu_1 \leq \mu)$ and $v(\mu) = E_\mu(\mu - \mu_1)^+$, with $\mu_1 \sim g$.

Consider an idealized algorithm which plays $\Pi_k$ until a non-zero reward is observed, and $\mu_k$ is revealed when that happens. If $\mu_k > \mu_*$, then $\Pi_k$ is rejected and a new arm is played next. If $\mu_k \leq \mu_*$, then we end the experimental stage and play $\Pi_k$ for the remaining trials. Assuming that the experimental stage uses $o(n)$ trials and $\mu_*$ is small, the regret of this algorithm is asymptotically $r(\mu_*)$, where

\begin{equation}
(3.1)\quad r(\mu) = \frac{\lambda}{p(\mu)} + nE_\mu(\mu_1|\mu_1 \leq \mu).
\end{equation}

The first term in the expansion of $r(\mu)$ approximates $E_\mu(\sum_{k=1}^K n_k \mu_k)$ whereas the second term approximates $E(nK\mu K)$.

**Lemma 1.** If $\mu_*$ is such that $v(\mu_*) = \lambda n^{-1}$, then

$$\min_{0 \leq \mu \leq 1} r(\mu) = r(\mu_*) = n\mu_*.$$

**Proof.** Since $E_\mu(\mu - \mu_1|\mu_1 \leq \mu) = v(\mu)/p(\mu)$, it follows from (3.1) that

\begin{equation}
(3.2)\quad r(\mu) = \frac{\lambda}{p(\mu)} + n\mu - \frac{nv(\mu)}{p(\mu)}.
\end{equation}

Since $\frac{d}{d\mu} v(\mu) = p(\mu)$ and $\frac{d}{d\mu} p(\mu) = g(\mu)$, it follows that

$$\frac{d}{d\mu} r(\mu) = \frac{g(\mu) - \lambda}{p(\mu)},$$

and Lemma 1 follows from solving $\frac{d}{d\mu} r(\mu) = 0$. □

Consider:

(A1) There exists $\alpha > 0$ and $\beta > 0$ such that $g(\mu) \sim \alpha \mu^{\beta-1}$ as $\mu \to 0$.

Under (A1), $p(\mu) = \int_0^\mu g(x)dx \sim \frac{\alpha}{\beta} \mu^\beta$ and $v(\mu) = \int_0^\mu p(x)dx \sim \frac{\alpha}{\beta(\beta+1)} \mu^{\beta+1}$.

Hence $v(\mu_*) = \lambda n^{-1}$ implies that

\begin{equation}
(3.3)\quad \mu_* \sim Cn^{-\frac{1}{\beta+1}}, \quad \text{where} \quad C = \left(\frac{\lambda\beta(\beta+1)}{\alpha}\right)^{\frac{1}{\beta+1}}.
\end{equation}

In Theorem 1 below we also assume that:
(A2) There exists $a_1 > 0$ such that $P_\mu(X > 0) \geq a_1 \min(\mu, 1)$ for all $\mu$.

This assumption is to avoid the situation of playing a bad arm a large number of times because the rewards are mostly zeros but may be very big when non-zeros.

**Theorem 1.** Assume (A1) and (A2). For any infinite arms bandit algorithm, its regret satisfies

\[(3.4) \quad R_n \geq [1 + o(1)]n\mu_* \sim Cn^{\frac{\theta}{\beta+1}}.\]

**Example 1.** Consider $X \sim \text{Bernoulli}(\mu)$. Assumption (A2) holds with $a_1 = 1$. If $g$ is uniform on $(0,1)$, then (A1) holds with $\alpha = \beta = 1$. Since $\lambda = 1$, by (3.3), $\mu_* \sim (2/n)^{1/2}$. Theorem 1 says that $R_n \geq [1 + o(1)]\sqrt{2n}$, agreeing with Theorem 3 of Berry et al. (1997).

Wang, Audibert and Munos (2008) proposed an algorithm achieving, under (A1), regret of $n^{\frac{\theta}{\beta+1}}$ plus additional log $n$ terms. In Theorem 2 we shall show that CBT achieves the lower bound in (3.4) without the log $n$ terms and with the same constant, and hence that (3.4) is sharp. Before that we state conditions on discrete rewards under (B1) and continuous rewards under (B2) for which Theorem 2 holds. Let $M_\mu(\theta) = E_\mu e^{\theta X}$.

(B1) The rewards are non-negative integer-valued. For $0 < \delta \leq 1$, there exists $\theta_\delta > 0$ such that for $\mu > 0$,

\[(3.5) \quad M_\mu(\theta_\delta) \leq e^{(1+\delta)\theta_\delta \mu},\]
\[(3.6) \quad M_\mu(-\theta_\delta) \leq e^{-(1-\delta)\theta_\delta \mu}.
\]

In addition,

\[(3.7) \quad P_\mu(X > 0) \leq a_2 \mu \text{ for some } a_2 > 0,\]
\[(3.8) \quad E_\mu X^4 = O(\mu) \text{ as } \mu \to 0.\]

(B2) The rewards are continuous random variables satisfying

\[(3.9) \quad \sup_\mu P_\mu(X \leq \gamma \mu) \to 0 \text{ as } \gamma \to 0.\]

For $0 < \delta \leq 1$, there exists $\tau_\delta > 0$ such that for $0 < \theta \mu \leq \tau_\delta$,

\[(3.10) \quad M_\mu(\theta) \leq e^{(1+\delta)\theta \mu},\]
\[(3.11) \quad M_\mu(-\theta) \leq e^{-(1-\delta)\theta \mu}.
\]
In addition (3.8) holds and for each $t \geq 1$, there exists $\nu_t > 0$ such that

\begin{equation}
\sup_{\mu \leq \nu_t} P_\mu(\hat{\sigma}_t^2 \leq \gamma \mu^2) \to 0 \text{ as } \gamma \to 0,
\end{equation}

where $\hat{\sigma}_t^2 = t^{-1} \sum_{u=1}^t (X_u - \bar{X}_t)^2$.

**Theorem 2.** Assume (A1), (A2) and either (B1) or (B2). For CBT with $\mu_*$ satisfying (3.3),

\begin{equation}
R_n \sim n\mu_* \text{ as } n \to \infty.
\end{equation}

**Example 2.** If $X \sim \text{Bernoulli}(\mu)$ under $P_\mu$, then

$$M_\mu(\theta) = 1 - \mu + \mu e^\theta \leq \exp[\mu(e^\theta - 1)],$$

and (3.5), (3.6) follow from selecting $\theta_\delta > 0$ such that

\begin{equation}
e^{\theta_\delta} - 1 \leq \theta_\delta(1 + \delta) \text{ and } e^{-\theta_\delta} - 1 \leq -\theta_\delta(1 - \delta).
\end{equation}

We check that (3.7) holds with $a_2 = 1$, and that $E_\mu X^4 = \mu$, hence (3.8) holds as well.

**Example 3.** If $X \sim \text{Poisson}(\mu)$ under $P_\mu$, then

$$M_\mu(\theta) = \exp[\mu(e^\theta - 1)],$$

and (3.5), (3.6) again follow from (3.14). Since $P_\mu(X > 0) = 1 - e^{-\mu}$, (A2) holds with $a_1 = 1 - e^{-1}$, and (3.7) holds with $a_2 = 1$. We also check that

$$E_\mu X^4 = \sum_{k=1}^{\infty} \frac{k^4 \mu^k e^{-\mu}}{k!} = \mu e^{-\mu} + e^{-\mu}O\left(\sum_{k=2}^{\infty} \mu^k\right),$$

and (3.8) holds.

**Example 4.** Let $Y$ be a continuous non-negative random variable with mean 1, with $Ee^{\tau_0 Y} < \infty$ for some $\tau_0 > 0$. Let $X$ be distributed as $\mu Y$ under $P_\mu$. Assumption (A2) holds with $a_1 = 1$. We check that

$$\sup_{\mu} P_\mu(X \leq \gamma \mu) = P(Y \leq \gamma) \to 0 \text{ as } \gamma \to 0,$$

and (3.9) holds. Let $0 < \delta \leq 1$. Since $\lim_{\tau \to 0} \tau^{-1} \log Ee^{\tau Y} = EY = 1$, there exists $\tau_\delta > 0$ such that for $0 < \tau \leq \tau_\delta$,

\begin{equation}
Ee^{\tau Y} \leq e^{(1+\delta)\tau} \text{ and } Ee^{-\tau Y} \leq e^{-(1-\delta)\tau}.
\end{equation}
Since $M_\mu(\theta) = E_\mu e^{\theta X} = E e^{\theta \mu Y}$ and $M_\mu(-\theta) = E e^{-\theta \mu Y}$, we can conclude (3.10) and (3.11) from (3.15) with $\tau = \theta \mu$. We conclude (3.8) from $E_\mu X^4 = \mu^4 EY^4$, and (3.12), for arbitrary $\nu_t > 0$, from

$$P_\mu(\hat{\sigma}_t^2 \leq \gamma \mu^2) = P_\mu(\hat{\sigma}_{tY}^2 \leq \gamma) \to 0 \text{ as } \gamma \to 0,$$

where $\hat{\sigma}_{tY}^2 = t^{-1} \sum_{u=1}^t (Y_u - \bar{Y}_t)^2$, for i.i.d. $Y$ and $Y_u$.

4. Empirical CBT for unknown priors. The optimal implementation of CBT, in particular the computation of the best target mean $\mu_*$, assumes knowledge of the behaviour of $g(\mu)$ for $\mu$ near 0. For $g$ unknown we rely on Theorem 2 to motivate the empirical implementation of CBT.

What is striking about (3.13) is that it relates the optimal $\mu_*$ with $\frac{\mu}{n}$, and moreover this relation does not depend on either the prior $g$, or the reward distributions. We suggest therefore, in an empirical implementation of CBT, to replace $\mu_*$ by

$$\hat{\mu}_* = \frac{S_t}{n},$$

(4.1)

where $S_t$ is the sum of the $t$ total rewards that have been played on all arms.

In the beginning with $t$ small, $\hat{\mu}_*$ underestimates $\mu_*$ but that is not a problem since this will only encourage exploration, which is the right strategy at the beginning. Over time $\hat{\mu}_*$ will get closer to the desired $\mu_*$, and empirical CBT will behave like CBT in deciding whether to play an arm further. Unlike CBT however empirical CBT is recalling as it decides from among all arms which to play further, not just the current arm.

**Empirical CBT**

Let $t$ be the total number of rewards, with $t_k$ of them from $\Pi_k$, $1 \leq k \leq K_t$, where $K_t$ is the total number of arms played.

For $t = 0$, play $\Pi_1$. Hence $K_1 = 1$ and $t_1 = 1$.

For $t = 1, \ldots, n - 1$:

1. Let $\hat{\mu}_*$ be computed as in (4.1). If $\min_{1 \leq k \leq K_t} L_{k|t} \leq \hat{\mu}_*$, then play the arm minimizing $L_{k|t}$.

2. If $\min_{1 \leq k \leq K_t} L_{k|t} > \hat{\mu}_*$, then play a new arm, that is $\Pi_{K_t+1}$.

Unlike CBT, empirical CBT does not achieve the smallest regret. This is because when a good arm (that is an arm with mean below optimal target) appears early, we are not sure if this is due to good fortune or that the prior is disposed towards arms with small means, so we experiment with more arms before we are certain and play the good arm for the remaining trials.
Similarly when no good arm appears after some time, we may conclude that the prior is disposed towards arms with large means, and play an arm with mean above the optimal target for the remaining trials, even though it is advantageous to experiment further.

In (4.2) below we provide the regret of empirical CBT. Let \( \Gamma(u) = \int_0^\infty x^{u-1}e^{-x}dx \). In the Appendix we show that under (A1), (A2), \( E_gX^2 < \infty \) and either (B1) or (B2), as \( n \to \infty \),

\[
R_n \sim I_\beta n \mu_*,
\]

where \( I_\beta = \left( \frac{1}{\beta+1} \right)^{\frac{1}{\beta+1}} (2 - \frac{1}{(\beta+1)^2}) \Gamma(2 - \frac{1}{\beta+1}) \).

The constant \( I_\beta \) is the inflation of the regret due to applying empirical CBT for \( g \) unknown. It increases from 1 (at \( \beta = 0 \)) to 2 (at \( \beta = \infty \)), so the worst-case inflation is not more than 2. The increase is quite slow so for reasonable values of \( \beta \) it is closer to 1 than 2. For example \( I_1 = 1.10 \), \( I_2 = 1.17 \), \( I_3 = 1.24 \) and \( I_{10} = 1.53 \). The predictions from (4.2), that the inflation of the regret increases with \( \beta \), and that it is not more than 25% for \( \beta = 1, 2 \) and 3, are validated by our simulations in Section 5.

5. Numerical studies. We study here arms with Bernoulli rewards as well as a URL dataset with unknown reward distributions. In our simulations 10,000 datasets are generated for each entry, and standard errors are after the ± sign. In both CBT and empirical CBT, we let \( b_n = c_n = \log(\log n) \).

Even though optimal CBT performs better than empirical CBT in Example 5, optimal CBT assumes knowledge of the prior to find \( \mu_* \), which differs with the cases. On the other hand the same algorithm is used for all cases when applying empirical CBT, and in fact the same algorithm is also used on the URL dataset in Example 6, with no knowledge of the reward distributions.

**Example 5.** We consider Bernoulli rewards for the following priors:

1. \( g(\mu) = 1 \), which satisfies (A1) with \( \alpha = \beta = 1 \),
2. \( g(\mu) = \frac{\pi}{2} \sin(\pi \mu) \), which satisfies (A1) with \( \alpha = \frac{\pi^2}{2} \) and \( \beta = 2 \),
3. \( g(\mu) = 1 - \cos(\pi \mu) \), which satisfies (A1) with \( \alpha = \frac{\pi^2}{2} \) and \( \beta = 3 \).

From Table 1, we see that among the algorithms in Berry et al. (1997) for the uniform prior, the best performing is the non-recalling \( \sqrt{n} \)-run algorithm. The two-target algorithm does better with \( f = 3 \) at smaller \( n \) and \( f = 6 \) at larger \( n \). CBT is the best performer uniformly over \( n \), and empirical CBT is also competitive against the two-target algorithm with \( f \) fixed. We ran
Algorithm | Regret
---|---
CBT $\mu_* = \sqrt{2/n}$ | $n=100$ | $n=1000$ | $n=10,000$ | $n=100,000$
empirical | 14.6±0.1 | 51.5±0.3 | 162±1 | 504±3
Berry et al. 1-failure | $n=100$ | 15.6±0.1 | 54.0±0.3 | 172±1 | 531±3
$\sqrt{n}$-run (non-recall) | $n=1000$ | 19.1±0.2 | 74.7±0.7 | 260±3 | 844±9
$\log n\sqrt{n}$-learning | $n=10,000$ | 15.4±0.1 | 57.7±0.4 | 193±1 | 618±4
Two-target | $n=100,000$ | 18.7±0.1 | 84.4±0.6 | 311±3 | 1060±9

Table 1

The regrets for Bernoulli rewards with uniform prior.

Prior $g(\mu)$ | Algorithm | Regret
---|---|---
$\frac{\pi}{2} \sin(\pi \mu)$ | CBT | $n=100$ | $n=1000$ | $n=10,000$ | $n=100,000$
emp. CBT | 24.9±0.1 | 124.8±0.5 | 575±3 | 2567±12
$n^{\frac{1}{n+1}}$-run | 25.6±0.1 | 132.3±0.6 | 604±2 | 2816±11
L. Bound $C n^{\frac{1}{n+1}}$ | 28.1±0.1 | 172.5±0.9 | 903±5 | 4434±28

Table 2

The regrets for Bernoulli rewards with non-uniform priors.

Example 6. We consider a URL dataset studied in Vermorel and Mohri (2005),
where a POKER algorithm for dealing with large number of arms is proposed. We reproduce part of their Table 1 in our Table 3, together with new simulations on empirical CBT. The dataset consists of the retrieval latency of 760 university home-pages, in milliseconds, with a sample size of more than 1300 for each home-page. The dataset can be downloaded from “sourceforge.net/projects/bandit”.

In our simulations, the rewards for each home-page are randomly permuted in each run. We see from Table 3 that POKER does better than empirical CBT at $n = 130$, whereas for $n = 1300$ empirical CBT does better. The other algorithms are uniformly worse than both POKER and empirical CBT.

The algorithm $\epsilon$-first refers to exploring with the first $\epsilon n$ rewards, with random selection of the arms to be played. This is followed by pure exploitation for the remaining $(1 - \epsilon) n$ rewards, on the “best” arm (with the largest sample mean). The algorithm $\epsilon$-greedy refers to selecting, in each play, a random arm with probability $\epsilon$, and the best arm with the remaining $1 - \epsilon$ probability. The algorithm $\epsilon$-decreasing is like $\epsilon$-greedy except that in the $t$th play, we select a random arm with probability $\min(1, \frac{\epsilon}{t})$, and the best arm otherwise. Both $\epsilon$-greedy and $\epsilon$-decreasing are disadvantaged by not making use of information on the total number of rewards. Vermorel and Mohri also ran simulations on more complicated strategies like LeastTaken, SoftMax, Exp3, GaussMatch and IntEstim, with average regret ranging from 287–447 for $n = 130$ and 189–599 for $n = 1300$.

6. Proofs of Theorems 1 and 2. We prove Theorems 1 and 2 in Sections 6.1 and 6.2 respectively. Preliminary lemmas used in their proofs are proved in Section 7.

6.1. Proof of Theorem 1. Let the infinite arms bandit problem be labelled as Problem A, and let $R_A$ be the smallest regret for this problem. We shall now describe two related problems, Problems B and C.

Problem B is like Problem A except that when we observe the first non-zero reward from $\Pi_k$, its mean $\mu_k$ is revealed. In Problem B, the best solution involves an initial experimental phase of $M$ rewards in which we play $K$ arms, each until its first non-zero reward. This is followed by an exploitation phase in which we play the best arm for the remaining $n - M$ trials. For continuous rewards $M = K$. Let $\mu_b (= \mu_{\text{best}}) = \min_{1 \leq k \leq K} \mu_k$. Let $R_B$ be the smallest regret for Problem B. Since all solutions of Problem A are solutions of Problem B, $R_A \geq R_B$.

In Problem C like in Problem B, the mean $\mu_k$ of $\Pi_k$ is revealed upon the observation of its first non-zero reward. The difference is that instead
of playing the best arm for an additional $n - M$ trials, we play it for $n$ additional trials, for a total of $n + M$ trials. Let $R_C$ be the smallest regret of Problem C, the expected value of $\sum_{k=1}^{K} n_k \mu_k$ with $\sum_{k=1}^{K} n_k = n + M$.

We can extend the best solution of Problem B to a solution of Problem C by simply playing the best arm a further $M$ times. Hence

$$R_A + E(M \mu_b) \geq R_B + E(M \mu_b) \geq R_C.$$

Theorem 1 follows from Lemmas 2 and 3 below. We shall prove the more technical Lemma 3 in Section 7.

**Lemma 2.** $R_C = n \mu_*$. 

**Proof.** Consider $k$ arms played so far in the experimental phase, with the best arm $\Pi_j$ having mean $\mu_j = \min_{1 \leq \ell \leq k} \mu_\ell$. We want to choose between trying out a new arm and exploiting arm $j$ for $n$ plays.

The cost of trying out a new arm is $\lambda$. The gain is $n E_g(\mu_j - \mu) = n v(\mu_j)$. Hence to minimize regret, we should try out a new arm if and only if $v(\mu_j) > \lambda n - 1$, or equivalently $\mu_j > \mu_*$, where $v(\mu_*) = \lambda n - 1$. Since we need on the average $1/p(\mu_*)$ arms before achieving $\mu_j \leq \mu_*$,

$$R_C = \frac{\lambda}{p(\mu_*)} + n E_g(\mu|\mu \leq \mu_*) = r(\mu_*),$$

see (3.1), and Lemma 2 follows from Lemma 1. □

**Lemma 3.** $E(M \mu_b) = o(n^{\beta/\beta+1})$.

6.2. **Proof of Theorem 2.** We preface the proof of Theorem 2 with the following supporting lemmas. Consider $X_1, X_2, \ldots$ i.i.d. $F_\mu$. Let $S_t = \sum_{u=1}^{t} X_t$, $\bar{X}_t = \frac{S_t}{t}$ and $\sigma^2_t = t^{-1} \sum_{u=1}^{t} (X_u - \bar{X}_t)^2$. Let

$$T_1 = \inf \{ t : S_t > b_n t \mu_* \},$$

$$T_2 = \inf \{ t : S_t > t \mu_* + c_n \sigma_t \sqrt{t} \},$$

with $b_n \to \infty$, $c_n \to \infty$ and $b_n + c_n = o(n^\delta)$ for all $\delta > 0$. Let $d_n = n^{-\omega}$ for some $0 < \omega < \frac{1}{\beta+1}$. Let $\mu_* \sim p n^{-\frac{\delta}{\beta+1}}$ for some $\rho > 0$. In Theorem 2 we require Lemmas 4–7 for $\rho = C$ only. The generality is required for the calculations behind (4.2). Let $a \wedge b = \min(a, b)$.

**Lemma 4.** As $n \to \infty$,

$$\sup_{\mu \geq d_n} [\min(\mu, 1) E_\mu T_1] = O(1),$$

$$\lim_{n \to \infty} E_g(T_1 \mu I_{\{\mu \geq d_n\}}) = \lambda.$$
LEMMA 5. Let $\epsilon > 0$. As $n \to \infty$,
\begin{align}
(6.6) \quad & \sup_{(1+\epsilon)\mu_\star \leq \mu \leq d_n} [\mu E_\mu(T_2 \wedge n)] = O(c_3^3 + \log n), \\
(6.7) \quad & E_g[(T_2 \wedge n) \mu 1_{\{(1+\epsilon)\mu_\star \leq \mu \leq d_n\}}] \to 0.
\end{align}

LEMMA 6. Let $0 < \epsilon < 1$. As $n \to \infty$, $\sup_{\mu \leq (1-\epsilon)\mu_\star} P_\mu(T_1 < \infty) \to 0$.

LEMMA 7. Let $0 < \epsilon < 1$. As $n \to \infty$, $\sup_{\mu \leq (1-\epsilon)\mu_\star} P_\mu(T_2 < \infty) \to 0$.

PROOF OF THEOREM 2. The number of times $\Pi_k$ is played is $n_k$, and it is distributed as $T_1 \wedge T_2 \wedge (n - \sum_{k=1}^{j-1} n_k)$. Let $0 < \epsilon < 1$. We can express
\begin{equation}
R_n - n\mu_\star = z_1 + z_2 + z_3,
\end{equation}
where $z_i = E[\sum_{k: \mu_k \in D_i} n_k(\mu_k - \mu_\star)]$, with
\begin{align}
D_1 &= [(1+\epsilon)\mu_\star, \infty), \quad D_2 = ((1-\epsilon)\mu_\star, (1+\epsilon)\mu_\star) \text{ and } D_3 = (0, (1-\epsilon)\mu_\star].
\end{align}
By (6.5) and (6.6),
\begin{equation}
z_1 \leq E_g(n_1\mu_1 1_{\{\mu_1 \geq (1+\epsilon)\mu_\star\}})EK \leq [\lambda + o(1)]EK.
\end{equation}
By Lemmas 6 and 7,
\begin{equation}
q_n := \sup_{\mu \leq (1-\epsilon)\mu_\star} [P_\mu(T_1 < \infty) + P_\mu(T_2 < \infty)] \to 0,
\end{equation}
and therefore
\begin{equation}
EK \leq \frac{1}{(1-q_n)p((1-\epsilon)\mu_\star)} \sim \frac{1}{(1-\epsilon)^{\beta} p(\mu_\star)}.
\end{equation}
It is easy to see that
\begin{equation}
z_2 \leq n\epsilon\mu_\star.
\end{equation}
Let $j = \inf\{k : \mu_k \leq (1+\epsilon)\mu_\star\}$. Let $M = \sum_{i=1}^{j-1} n_i$. By (6.4) and (6.6),
\begin{equation}
EM \leq \frac{1}{p((1+\epsilon)\mu_\star)} E_g(n_1|\mu_1 > (1+\epsilon)\mu_\star)
= O(n^{3+\beta}(c_3^3 + \log n) \int_{(1+\epsilon)\mu_\star}^{\infty} \frac{g(\mu)}{\min(\mu, 1)} d\mu
= O(n^{3+\beta}(c_3^3 + \log n) \max(n^{\frac{3-\beta}{2}}, 1) = o(n).
\end{equation}
The density of $\mu_j$, conditioned on $\mu_j \leq (1 + \epsilon)\mu_*$, is $\frac{g(\mu)}{p((1+\epsilon)\mu_*)}$. If $\mu_j \leq (1 - \epsilon)\mu_*$, the probability is at least $1 - q_n$ that $n_j \geq n - M$. Therefore by (6.12),

\begin{equation}
(6.13) \quad z_3 \leq (n - EM)(1 - q_n)\frac{\int_0^{(1-\epsilon)\mu_*} g(\mu)(\mu-\mu_*)d\mu}{p((1+\epsilon)\mu_*)} \leq n\left[1 - \frac{1}{(1+\epsilon)^\theta} + o(1)\right]\frac{[\frac{\text{TV}(\mu_*) + \epsilon^2\mu_2}{p(\mu_*)}]^\beta}{p(\mu_*)}.
\end{equation}

By (6.8)–(6.13),

\[ R_n - n\mu_* \leq [1 + o(1)]\left(\frac{\lambda - nv(\mu_*)}{p(\mu_*)} + A\epsilon n^{\frac{\beta}{1+\beta}}\right), \]

with $A \rightarrow 0$ as $\epsilon \rightarrow 0$. Since $nv(\mu_*) \sim \lambda$, we can conclude Theorem 2 by letting $\epsilon \rightarrow 0$. \(\Box\)

7. Proofs of supporting lemmas. In Section 7.1 we prove Lemma 3. In Sections 7.2 and 7.3 we prove Lemmas 4–7 for discrete and continuous rewards respectively.

7.1. Proof of Lemma 3. Express $E(M\mu_b) = \sum_{i=1}^5 E(M\mu_b 1_{D_i})$, where

\begin{align*}
D_1 &= \{\mu_b \leq \frac{\mu_*}{\log n}\},
D_2 &= \{\mu_b > \frac{\mu_*}{\log n}, K > n\mu_*(\log n)^{\beta+2}\},
D_3 &= \{\frac{\mu_*}{\log n} < \mu_b \leq \mu_*(\log n)^{\beta+3}, K \leq n\mu_*(\log n)^{\beta+2}\},
D_4 &= \{\mu_b > \mu_*(\log n)^{\beta+3}, K \leq n\mu_*(\log n)^{\beta+2}, M > \frac{n}{2}\},
D_5 &= \{\mu_b > \mu_*(\log n)^{\beta+3}, K \leq n\mu_*(\log n)^{\beta+2}, M \leq \frac{n}{2}\}.
\end{align*}

It suffices to show that for all $i$,

\begin{equation}
(7.1) \quad E(M\mu_b 1_{D_i}) = o(n^{\frac{\beta}{1+\beta}}).
\end{equation}

Since $\frac{M\mu_*}{\log n} \leq \frac{n\mu_*}{\log n} = o(n^{\frac{\beta}{1+\beta}})$, (7.1) holds for $i = 1$. Let $\hat{\mu}_b = \min_{k \leq \hat{K}} \mu_k$, where $\hat{K} = \lfloor n\mu_*(\log n)^{\beta+2}\rfloor$. We have

\begin{equation}
(7.2) \quad E(M\mu_b 1_{D_2}) = O(n)P(D_2) = O(n)P(\hat{\mu}_b > \frac{\mu_*}{\log n}).
\end{equation}

Substituting

\[ P(\hat{\mu}_b > \frac{\mu_*}{\log n}) = [1 - p(\frac{\mu_*}{\log n})]^\hat{K} = \exp\{-[1 + o(1)]\hat{K} \frac{\mu_*}{p(\mu_*)} \beta\} = O(n^{-1}) \]

into (7.2) shows (7.1) for $i = 2$.\]
Let \( M_j \) be the number of plays to first non-zero reward of \( \Pi_j \). Since 

\[
E_\mu M_j \leq \mathcal{K} E_\mu 1_{(\mu_1 > \frac{\mu_\star}{\log n})} \mu_\star (\log n)^{\beta + 3}
\]

(7.3) follows from (A2) that

\[
E(M_\mu 1_{D_4}) \leq \mathcal{K} \left( \int_{\frac{\mu_\star}{\log n}}^\infty \frac{g(\mu)}{a_1 \min(\mu, 1)} d\mu \right) \mu_\star (\log n)^{\beta + 3}.
\]

Substituting

\[
\int_{\frac{\mu_\star}{\log n}}^1 \frac{g(\mu)}{\mu} d\mu = \left\{ \begin{array}{ll} O(1) & \text{if } \beta \geq 1, \\
O(\left( \frac{\mu_\star}{\log n} \right)^{\beta - 1}) & \text{if } \beta < 1,
\end{array} \right.
\]

into (7.3) shows (7.1) for \( i = 3 \).

If \( \mu_j > \mu_\star (\log n)^{\beta + 3} \), then by (A2), \( M_j \) is bounded above by a geometric random variable with mean \( \nu^{-1} \), where \( \nu = a_1 \mu_\star (\log n)^{\beta + 3} \). Hence for \( 0 < \theta < \log \left( \frac{1}{1-\nu} \right) \),

\[
E(e^{\theta M_j} 1_{(\mu_j > \mu_\star (\log n)^{\beta + 3})}) \leq \sum_{i=1}^{\infty} e^{\theta i} \nu (1 - \nu)^{i-1} = \frac{\nu e^\theta}{1 - e^{\theta (1 - \nu)}},
\]

implying that

(7.4) 

\[
E(e^{\theta M_j} 1_{D_4}) \leq \left( \frac{\nu e^\theta}{1 - e^{\theta (1 - \nu)}} \right) \mathcal{K}.
\]

Consider \( e^{\theta} = 1 + \frac{\nu}{2} \), and check that \( e^{\theta (1 - \nu)} \leq 1 - \frac{\nu}{2} \). It follows from (7.4) and Markov’s inequality that

\[
P(D_4) \leq e^{-\frac{\nu}{2} \left( \frac{\nu e^\theta}{1 - e^{\theta (1 - \nu)}} \right) \mathcal{K}} = 2^{\mathcal{K} e^{\theta (\mathcal{K} - \frac{\nu}{2})}} = \exp[\mathcal{K} \log 2 + (1 + o(1)) \frac{\nu}{2} (\mathcal{K} - \frac{\nu}{2})] = O(n^{-1}).
\]

Since \( M \leq n \), (7.1) holds for \( i = 4 \).

Finally under the event \( D_5 \), for \( n \) large,

\[
(n - M) v(\mu_b) > \frac{n}{2} v(\mu_\star (\log n)^{\beta + 3}) > \lambda.
\]

The optimal solution of Problem B would require us to experiment further since the cost of experimentation \( \lambda \) is less than the gain. In other words, \( D_5 \) is an event of zero probability, and therefore (7.1) holds for \( i = 5 \). \( \square \)
7.2. Proofs of Lemmas 4–7 for discrete rewards. In the case of discrete rewards, one difficulty is that for \( \mu_k \) small, there are potentially multiple plays on \( \Pi_k \) before a non-zero reward is observed. Assumption (A2) is helpful in ensuring that the mean of this non-zero reward is not too large.

**Proof of Lemma 4.** Recall that

\[
T_1 = \inf \{ t : S_t > b_n t \mu_* \},
\]

and that \( d_n = n^{-\omega} \) for some \( 0 < \omega < \frac{1}{2(\beta + 1)} \). We want to show that for \( \mu_* \sim \rho n^{-\frac{1}{\beta+1}} \),

\[
\sup_{\mu \geq d_n} [\min(\mu, 1) E_\mu T_1] = O(1),
\]

(7.5)

\[
\lim_{n \to \infty} E_g(T_1 \mu 1_{\{\mu \geq d_n\}}) = \lambda.
\]

(7.6)

Since \( X \) is integer-valued, it follows from Markov’s inequality that

\[
P_\mu(S_t \leq b_n t \mu_*) \leq \left[ e^{\theta b_n \mu_* M_\mu(-\theta)} \right]^t \leq \{ e^{\theta b_n \mu_* [P_\mu(X = 0) + e^{-\theta}]} \}^t.
\]

(7.7)

Consider \( \theta = 2 \omega \log n \). By (A2) and (7.7), uniformly over \( \mu \geq d_n \),

\[
E_\mu T_1 = 1 + \sum_{t=1}^{\infty} P_\mu(T_1 > t)
\]

\[
\leq 1 + \sum_{t=1}^{\infty} P_\mu(S_t \leq b_n t \mu_*)
\]

\[
\leq \{ 1 - e^{\theta b_n \mu_* [P_\mu(X = 0) + e^{-\theta}]} \}^{-1}
\]

\[
= \{ 1 - [1 + o(d_n)] [P_\mu(X = 0) + d_n^2] \}^{-1}
\]

\[
= [P_\mu(X > 0) + o(d_n)]^{-1} \sim [P_\mu(X > 0)]^{-1}.
\]

We conclude (7.5) from (7.8) and (A2). By (7.8),

\[
E_g(T_1 \mu 1_{\{\mu \geq d_n\}}) \leq [1 + o(1)] \int_{d_n}^{\infty} E_\mu(X|X > 0) g(\mu) d\mu \to \lambda,
\]

and we can conclude (7.6) from \( E_\mu T_1 \geq [P_\mu(X > 0)]^{-1} \). \( \square \)

**Proof of Lemma 5.** Recall that \( T_2 = \inf \{ t : S_t > t \mu_* + c_n \tilde{\sigma}_t \sqrt{t} \} \) and let \( \epsilon > 0 \). We want to show that uniformly over \( (1 + \epsilon) \mu_* \leq \mu \leq d_n \),

\[
\mu E_\mu(T_2 \wedge n) = O(c_n^3 + \log n),
\]

(7.9)

\[
E_g(T_2 \wedge n) \mu 1_{\{(1+\epsilon) \mu_* \leq \mu \leq d_n\}} \to 0.
\]

(7.10)
We shall first show that there exists $\kappa > 0$ such that as $n \to \infty$,

\begin{equation}
\mu \sum_{t=1}^{n} P\left(\hat{\sigma}_t^2 \geq \kappa \mu \right) = O(\log n),
\end{equation}

uniformly over $\mu \leq d_n$. Indeed by (3.8), there exists $\kappa > 0$ such that $E\mu X^2 \leq \frac{\kappa \mu}{2}$ for $\mu \leq d_n$ and $n$ large, and therefore by (3.8) again and Chebyshev’s inequality,

\begin{equation}
P\mu\left(\hat{\sigma}_t^2 \geq \kappa \mu \right) \leq P\left(\sum_{u=1}^{t} X_u^2 \geq t\kappa \mu \right) \leq e^{t\var_{\mu}(X^2)/(t\kappa \mu \mu)} = O((t\mu)^{-1}),
\end{equation}

and (7.11) follows.

By (7.11), uniformly over $(1 + \epsilon)\mu_* \leq \mu \leq d_n$,

\begin{equation}
E\mu(T_2 \land n) = 1 + \sum_{t=1}^{n-1} P\mu(T > t)
\leq 1 + \sum_{t=1}^{n-1} P\mu(S_t \leq t\mu_*) + c'_{n}\sqrt{\mu t} + O(\mu^{-1} \log n),
\end{equation}

where $c'_{n} = c_n \sqrt{\kappa}$.

Uniformly over $t \geq c_n^3 \mu^{-1}$, $\mu t / (c'_n \sqrt{\mu t}) \to \infty$ and therefore by (3.6), for $\mu \geq (1 + \epsilon)\mu_*$ and $0 < \delta < \frac{1}{2}$ to be further specified, for $n$ large,

\begin{equation}
P\mu(S_t \leq t\mu_* + c'_n \sqrt{\mu t}) \leq e^{\theta_5 t(\mu_* + \delta \mu)} M_t^t(-\theta_5) \leq e^{\eta \theta_5 (\mu_* - (1 - 2\delta)\mu)} \leq e^{-\eta \theta_5 \mu},
\end{equation}

where $\eta = 1 - 2\delta - \frac{1}{1+\epsilon} > 0$ (with $\delta$ chosen small enough). Since

\begin{equation}
c_n^3 \mu^{-1} + \sum_{t \geq c_n^3 \mu^{-1}} e^{-\eta \theta_5 \mu} = O(c_n^3 \mu^{-1}),
\end{equation}

substituting (7.13) into (7.12) gives us (7.9). By (7.9),

\begin{equation}
E_g[(T_2 \land n)\mu 1_{\{1+\epsilon\mu_* \leq \mu \leq d_n\}}] = P\mu(\mu \leq d_n) O(c_n^3 + \log n) = d_n^3 O(c_n^3 + \log n),
\end{equation}

and (7.10) holds as well. \(\square\)

**Proof of Lemma 6.** We want to show that

\begin{equation}
P\mu(S_t > t\mu_* \mu_\epsilon \text{ for some } t \geq 1) \to 0,
\end{equation}

for $(1 + \epsilon)\mu_* \leq \mu \leq d_n$, uniformly over $\mu$. Indeed by (3.8), there exists $\kappa > 0$ such that $E\mu X^2 \leq \frac{\kappa \mu}{2}$ for $\mu \leq d_n$ and $n$ large, and therefore by (3.8) again and Chebyshev’s inequality,

\begin{equation}
P\mu\left(\hat{\sigma}_t^2 \geq \kappa \mu \right) \leq P\mu\left(\sum_{u=1}^{t} X_u^2 \geq t\kappa \mu \right) \leq e^{t\var_{\mu}(X^2)/(t\kappa \mu \mu)} = O((t\mu)^{-1}),
\end{equation}

and (7.11) follows.

By (7.11), uniformly over $(1 + \epsilon)\mu_* \leq \mu \leq d_n$,

\begin{equation}
E\mu(T_2 \land n) = 1 + \sum_{t=1}^{n-1} P\mu(T > t)
\leq 1 + \sum_{t=1}^{n-1} P\mu(S_t \leq t\mu_*) + c'_{n}\sqrt{\mu t} + O(\mu^{-1} \log n),
\end{equation}

where $c'_{n} = c_n \sqrt{\kappa}$.

Uniformly over $t \geq c_n^3 \mu^{-1}$, $\mu t / (c'_n \sqrt{\mu t}) \to \infty$ and therefore by (3.6), for $\mu \geq (1 + \epsilon)\mu_*$ and $0 < \delta < \frac{1}{2}$ to be further specified, for $n$ large,

\begin{equation}
P\mu(S_t \leq t\mu_* + c'_n \sqrt{\mu t}) \leq e^{\theta_5 t(\mu_* + \delta \mu)} M_t^t(-\theta_5) \leq e^{\eta \theta_5 (\mu_* - (1 - 2\delta)\mu)} \leq e^{-\eta \theta_5 \mu},
\end{equation}

where $\eta = 1 - 2\delta - \frac{1}{1+\epsilon} > 0$ (with $\delta$ chosen small enough). Since

\begin{equation}
c_n^3 \mu^{-1} + \sum_{t \geq c_n^3 \mu^{-1}} e^{-\eta \theta_5 \mu} = O(c_n^3 \mu^{-1}),
\end{equation}

substituting (7.13) into (7.12) gives us (7.9). By (7.9),

\begin{equation}
E_g[(T_2 \land n)\mu 1_{\{1+\epsilon\mu_* \leq \mu \leq d_n\}}] = P\mu(\mu \leq d_n) O(c_n^3 + \log n) = d_n^3 O(c_n^3 + \log n),
\end{equation}

and (7.10) holds as well. \(\square\)
uniformly over \( \mu \leq (1 - \epsilon)\mu_* \).

By (3.7) and Bonferroni’s inequality,

\[
\begin{align*}
(7.15) & \quad P_\mu(S_t > tb_n\mu_* \text{ for some } 1 \leq t \leq \frac{1}{\sqrt{b_n\mu_*}}) \\
& \leq P_\mu(X_t > 0 \text{ for some } 1 \leq t \leq \frac{1}{\sqrt{b_n\mu_*}}) \leq \frac{a_2\mu}{\sqrt{b_n\mu_*}} \to 0.
\end{align*}
\]

By (3.5) and a change-of-measure argument, for \( n \) large,

\[
(7.16) \quad P_\mu(S_t > tb_n\mu_* \text{ for some } t > \frac{1}{\sqrt{b_n\mu_*}}) \\
\leq \sup_{t > \frac{1}{\sqrt{b_n\mu_*}}} \left[ e^{-\theta_1 b_n\mu_* M_\mu(\theta_1)} \right]^t \leq e^{-\theta_1(b_n\mu_* - 2\mu)/(\mu_* \sqrt{b_n})} \to 0.
\]

More specifically the first inequality of (7.16) follows from

\[
P_\mu(T < \infty) = E^{\theta_1}_\mu(e^{-\theta_1 S_T} M^T_\mu(\theta_1) \mathbf{1}_{(T < \infty)})
\]

and Markov’s inequality, where \( T = \inf\{t > \frac{1}{\sqrt{b_n\mu_*}} : S_t > tb_n\mu_*\} \), and \( E^{\theta_1}_\mu \) is expectation under which \( X_t \) are i.i.d. with distribution function

\[
F^{\theta_1}_\mu(x) = \left[M_\mu(\theta_1)\right]^{-1} \int_0^x e^{\theta_1 y} dF_\mu(y).
\]

Combining (7.15) and (7.16) gives us (7.14). \( \square \)

**Proof of Lemma 7.** We want to show that

\[
(7.17) \quad P_\mu(S_t > t\mu_* + c_n\hat{\sigma}_t\sqrt{t} \text{ for some } t \geq 1) \to 0,
\]

uniformly over \( \mu \leq (1 - \epsilon)\mu_* \).

By (3.7) and Bonferroni’s inequality,

\[
(7.18) \quad P_\mu(S_t > t\mu_* + c_n\hat{\sigma}_t\sqrt{t} \text{ for some } t \leq \frac{1}{c_n\mu}) \\
\leq P_\mu(X_t > 0 \text{ for some } t \leq \frac{1}{c_n\mu}) \leq \frac{a_2}{c_n} \to 0,
\]

whereas

\[
(7.19) \quad P_\mu(S_t > t\mu_* + c_n\hat{\sigma}_t\sqrt{t} \text{ for some } t > \frac{1}{c_n\mu}) \leq (I) + (II),
\]

where

\[
(I) = P_\mu(S_t > t\mu_* + c_n(\mu t/2)^{\frac{3}{2}} \text{ for some } t > \frac{1}{c_n\mu}),
\]

\[
(II) = P_\mu(\hat{\sigma}_t^2 \leq \frac{\mu}{2} \text{ and } S_t \geq t\mu_* \text{ for some } t > \frac{1}{c_n\mu}).
\]

By (7.18) and (7.19), to show (7.17), it suffices to show that (I) \( \to 0 \) and (II) \( \to 0 \).
Let $0 < \delta \leq 1$ be such that $1 + \delta < (1 - \epsilon)^{-1}$. Hence $\mu \leq (1 - \epsilon)\mu_* \implies \mu_* \geq (1 + \delta)\mu$. It follows from (3.5) that for $n$ large,
\[
(\text{I}) \leq \sup_{t > \frac{1}{c_n\mu}} [e^{-\theta|t\mu_* + c_n(\mu/2)|}M_{\mu}^t(\theta)] \leq e^{-\theta\delta\mu_*(1+\delta)/(c_n\mu) - \theta\sqrt{c_n/2}} \rightarrow 0.
\]
Since $X_t$ is non-negative integer-valued, $S_t \geq t\mu_* \geq t\mu$, and this, together with $\hat{\sigma}_t^2 \leq \frac{\mu}{2}$ implies that $X_t^2 \geq \frac{\mu}{2}$. Hence by (3.5), for $n$ large,
\[
(\text{II}) \leq P_{\mu}(\hat{X}_t \geq \sqrt{\frac{\mu}{2}} \text{ for some } t > \frac{1}{c_n\mu}) \leq \sup_{t > \frac{1}{c_n\mu}} [e^{-\theta_1\sqrt{\frac{\mu}{2}}M_{\mu}(\theta_1)]^t \leq e^{-\theta_1[\sqrt{\mu/2-2\mu}]/(c_n\mu) \rightarrow 0}.
\]

7.3. Proofs of Lemmas 4–7 for continuous rewards. In the case of continuous rewards, the proofs are simpler due to rewards being non-zero, and we have $\lambda = E_g\mu$.

**Proof of Lemma 4.** To show (6.4) and (6.5), it suffices to show that
\[
E_\mu T_1 \rightarrow 1 \text{ uniformly over } \mu \geq d_n.
\]
Let $\theta > 0$ to be further specified. Since
\[
P_{\mu}(S_t \leq b_nt\mu_*) \leq e^{\theta b_n\mu_* - M_{\mu}(-\theta)]^t,}
M_{\mu}(-\theta) \leq P_{\mu}(X \leq \gamma\mu) + e^{-\gamma\theta\mu},
\]
it follows that
\[
E_\mu T_1 \leq 1 + \sum_{t=1}^{\infty} P_{\mu}(S_t \leq b_nt\mu_*) \leq \{1 - e^{\theta b_n\mu_* [P_{\mu}(X \leq \gamma\mu) + e^{-\gamma\theta\mu}]}\}^{-1}.
\]
Consider $\theta = n^{\omega}$ for some $\omega < \eta < \frac{1}{\beta+1}$ and $\gamma = \frac{1}{\log n}$. By (3.9), for $\mu \geq d_n,$
\[
e^{\theta b_n\mu_*} \rightarrow 1, \quad e^{-\gamma\theta\mu} \rightarrow 0, \quad P_{\mu}(X \leq \gamma\mu) \rightarrow 0,
\]
and (7.20) follows from (7.21). \(\square\)

**Proof of Lemma 5.** To show (6.6) and (6.7), we can proceed as in the proof of Lemma 5 for discrete rewards, applying (3.11) in place of (3.6), with any fixed $\theta > 0$ in place of $\theta_5$ in (7.13). \(\square\)
Proof of Lemma 6. It follows from (3.10) with \( \theta = \frac{\sqrt{\mu}}{\mu} \) that for \( n \) large,
\[
P_\mu(S_t > t\mu_\ast \text{ for some } t \geq 1) \leq \sup_{t \geq 1} [e^{-\theta t \mu_\ast} M_\mu(t)]^t \leq e^{-\theta (b_\ast \mu_\ast - 2\mu)} \to 0,
\]
and Lemma 6 thus holds. □

Proof of Lemma 7. By (3.12), for each \( u \geq 1 \) and \( \eta > 0 \), we can select \( \gamma > 0 \) such that for \( n \) large (so that \( \mu \) is small),
\[
\sum_{t=1}^{u} P_\mu(\tilde{\sigma}_t^2 \leq \gamma \mu^2) \leq \eta.
\]
Let \( c_n' = c_n \sqrt{\tau} \) and \( \theta = \frac{\sqrt{\mu}}{\mu} \). By (3.10) and (7.22),
\[
P_\mu(S_t > t\mu_\ast + c_n \tilde{\sigma}_t \sqrt{t} \text{ for some } 1 \leq t \leq u) \leq \eta + \sum_{t=1}^{u} P_\mu(S_t \geq c_n' \mu \sqrt{t}) \leq \eta + \sum_{t=1}^{u} e^{-\theta c_n' \mu \sqrt{t} M_\mu^t(\theta)} \leq \eta + \sum_{t=1}^{u} e^{-\tau_1(c_n' \sqrt{\tau} - 2t)} \to \eta.
\]

Let \( \delta > 0 \) be such that \((1 + \delta)(1 - \epsilon) < 1\). It follows from (3.10) for \( \theta = \frac{\sqrt{\mu}}{\mu} \) that for \( n \) large,
\[
P_\mu(S_t \geq t\mu_\ast \text{ for some } t > u) \leq \sup_{t > u} [e^{-\theta t \mu_\ast} M_\mu(t)]^t \leq e^{-u \theta (\mu_\ast - (1 + \delta)\mu)} \leq e^{-u \tau_1 [1 - (1 - \epsilon)^{-1} - (1 + \delta)]} \leq \eta
\]
for \( u \) large enough. Lemma 7 follows from (7.23) and (7.24) since \( \eta \) can be chosen arbitrarily small. □

APPENDIX A: DERIVATION OF (??)

In Section 6.1 we proved Theorem 1 by first considering a simpler Problem C. Likewise we shall derive (4.2) by applying empirical CBT on a simpler “Problem C”.

In Problem C, the mean \( \mu_k \) of an arm is revealed when its first non-zero reward appears, and further experimentation is not necessary. For simplicity,
we assign a fixed average cost of $\lambda$ to each arm that we experiment with, so that $\hat{\mu}_k = \frac{k\lambda}{n}$ after $K$ arms. We stop experimenting after $K$ arms, where

$$K = \inf \{ k : \min_{1 \leq j \leq k} \mu_j \leq \hat{\mu}_k \}, \quad \hat{\mu}_k = \frac{k\lambda}{n}. \tag{A.1}$$

In the exploitation phase, we play the arm $\Pi_j$ with $\mu_j = \min_{1 \leq k \leq K} \mu_j$, a total of $n$ times. The regret $R'_C = E(K\lambda + n\mu_j)$, and we want to show that

$$R'_C \sim CI\beta n^\frac{\beta}{\beta + 1}, \tag{A.2}$$

where $C = (\frac{\lambda\beta(\beta+1)}{\alpha})^\frac{1}{\beta + 1}$ and $I_\beta = (\frac{1}{\beta + 1})^\frac{1}{\beta + 1} (2 - \frac{1}{(\beta + 1)^2}) \Gamma(2 - \frac{\beta}{\beta + 1})$.

Let

$$D^1_k = \{ \hat{\mu}_k - \frac{\Delta}{n} < \min_{1 \leq j \leq k-1} \mu_j \leq \hat{\mu}_k \}, \quad D^2_k = \{ \min_{1 \leq j \leq k-1} \mu_j > \hat{\mu}_k, \mu_k \leq \hat{\mu}_k \}.$$ 

We check that $D^1_k \cup D^2_k$ are disjoint, and that $D^1_k \cup D^2_k = \{ K = k \}$.

For $k \sim \frac{\rho}{\lambda} n^{\frac{\beta}{\beta + 1}}$ with $\rho > 0$,

$$P(D^1_k) = [1 - p(\hat{\mu}_k - \frac{\Delta}{n})]^{k-1} - [1 - p(\hat{\mu}_k)]^{k-1}$$

$$= \{1 - p(\hat{\mu}_k) + [1 + o(1)] \frac{\lambda}{n} g(\hat{\mu}_k)\}^{k-1} - [1 - p(\hat{\mu}_k)]^{k-1}$$

$$\sim \exp(-\frac{\alpha\rho^{\beta+1}}{\beta \lambda})\alpha\rho^\beta n^{-\frac{\beta}{\beta + 1}}.$$

Moreover

$$E(R'_C|D^1_k) \sim k\lambda + n(\frac{\Delta}{n}) \sim 2\rho n^{\frac{\beta}{\beta + 1}}.$$ 

Likewise,

$$P(D^2_k) = \{[1 - p(\hat{\mu}_k)]^{k-1}\} p(\hat{\mu}_k)$$

$$\sim \exp(-\frac{\alpha\rho^{\beta+1}}{\beta \lambda})(\frac{\alpha\rho^\beta}{\beta}) n^{-\frac{\beta}{\beta + 1}},$$

$$E(R'_C|D^2_k) = k\lambda + n E(\mu_1 \mu \leq \hat{\mu}_k)$$

$$= 2k\lambda - \frac{n \hat{\nu}(\hat{\mu}_k)}{p(\hat{\mu}_k)} \sim (2 - \frac{1}{\beta + 1}) \rho n^{\frac{\beta}{\beta + 1}}.$$ 

Combining (A.3)–(A.6) formally gives us

$$R'_C = \sum_{k=1}^\infty [E(R'_C|D^1_k) P(D^1_k) + E(R'_C|D^2_k) P(D^2_k)]$$

$$\sim \sum_{\rho} \exp(-\frac{\alpha\rho^{\beta+1}}{\beta \lambda})(\frac{\alpha\rho^\beta}{\beta})(2\beta + 2 - \frac{1}{\beta + 1}) \rho,$$
with \( \rho \) summed over \( \lambda n^{-\frac{\beta}{\beta+1}}Z^+ \). It follows from (A.7) and a change-of-
variables \( x = \frac{\alpha^\beta + 1}{\beta} \) that

\[
R'_C \sim n^{\frac{\beta}{\beta+1}} (2 - \frac{1}{(\beta+1)^2}) \int_0^\infty xe^{-x}(\frac{\beta}{\alpha})^{\frac{1}{\beta+1}}x^{-\frac{\beta}{\beta+1}}dx,
\]

and (A.2) indeed holds.

To show (4.2) rigorously, we need to apply Lemmas 6 and 7 to justify the assumption that an arm with mean below target \( \hat{\mu} \) is played for the remaining trials, and Lemma 8 below to justify an average cost of \( \lambda \) for each arm above the target.

**Lemma 8.** Let \( \epsilon > 0, k \sim \frac{\epsilon}{\rho} n^{\frac{\beta}{\beta+1}} \) and \( \mu \sim \rho n^{-\frac{1}{\beta+1}} \) for some \( \rho > 0 \). Let \( S_{jt} = \sum_{u=1}^t X_{j_u} \) and

(A.8) \[ t_j = \inf\{t : L_{jt} > \mu_\ast\} \land n. \]

Under (A1), (A2), \( E_gX^2 < \infty \) and either (B1) or ((B2), as \( n \to \infty \),

(A.9) \[ E(S_{1t_1}1_{(\mu_1 \geq (1+\epsilon)\mu_\ast)}) \to \lambda, \]
(A.10) \[ \text{Var}(S_{1t_1}1_{(\mu_1 \geq (1+\epsilon)\mu_\ast)}) = O(\epsilon_n^6). \]

In particular for any \( \delta > 0, \)

\[
P\left\{ \left| \sum_{j=1}^k S_{jt_j}1_{(\mu_j \geq (1+\epsilon)\mu_\ast)} - \rho n^{\frac{\beta}{\beta+1}} \right| \geq \delta n^{\frac{\beta}{\beta+1}} \right\} \to 0.
\]

**Proof.** By (A.8), \( t_1 \) is distributed as \( T_1 \land T_2 \land n \) with \( \mu_1 \sim g \). Since

\[
E(S_{1t_1}1_{(\mu_1 \geq (1+\epsilon)\mu_\ast)}) = E(t_1\mu_11_{(\mu_1 \geq (1+\epsilon)\mu_\ast)}),
\]

we can conclude (A.9) from (6.5), (6.7) and \( E_\mu t_1 \geq \frac{1+o(1)}{E_\mu(X > 0)} \). We next modify the arguments in (7.8) or (7.21) for continuous rewards], to show that for \( \mu \geq d_n, \)

(A.11) \[ E_\mu t_1^2 \leq 1 + \sum_{t=1}^{\infty} 2tP_\mu(t_1 > t) \leq 1 + \sum_{t=1}^{\infty} 2tP_\mu(S_{1t} \leq b_n t \mu_\ast) \leq [1 + o(1)]P_\mu(X > 0)^{-2} = O(\max(1,\mu^{-2})). \]
with (A2) applied to get the last relation.

By a similar modification of (7.12), uniformly over \((1 + \epsilon)\mu_* \leq \mu \leq d_n,

\begin{align}
(A.12) \quad E_{\mu} t_1^2 &= 1 + \sum_{t=1}^{n-1} 2t P_{\mu}(t_1 > t) \\
&\leq 1 + \sum_{t=1}^{n-1} 2t P_{\mu}(S_{1t} \leq t_{\mu*} + c_n' \sqrt{\mu t}) + O(\mu^{-1} \log n).
\end{align}

By (7.13),

\[
P_{\mu}(S_{1t} \leq t_{\mu*} + c_n' \sqrt{\mu t}) \leq e^{-\eta t \theta \mu}
\]

for some \(\eta > 0\), \(\theta > 0\) and \(c_n' = c_n \sqrt{\kappa}\) for some \(\kappa > 0\) uniformly over \(t \geq c_n^3 \mu^{-1}\), and hence by (A.12),

\[
\mu^2 E_{\mu} t_1^2 = O(c_n^6),
\]

uniformly over \((1 + \epsilon)\mu_* \leq \mu \leq d_n\). Hence by (3.8), (6.4), (6.6) and (A.11),

\[
\text{Var}(S_{1t_1} 1_{\{\mu_1 \geq (1+\epsilon)\mu_*\}}) \\
\leq 2[\text{Var}((S_{1t_1} - t_1 \mu_1) 1_{\{\mu_1 \geq (1+\epsilon)\mu_*\}} + \text{Var}(t_1 \mu_1 1_{\{\mu_1 \geq (1+\epsilon)\mu_*\}})] \\
\leq 2E[t_1 (\text{Var}_{\mu_1} X) 1_{\{\mu_1 \geq (1+\epsilon)\mu_*\}}) + O(c_n^6) \\
= O(E(\mu_1 t_1 1_{\{d_n \geq \mu_1 \geq (1+\epsilon)\mu_*\}}) + E_{g} X^2 + c_n^6),
\]

and (A.10) follows from (6.7). \(\square\)

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