Quantum MIMO n-Systems and Conditions for Stability

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Abstract

In this paper we present some conditions for the (strong) stabilizability of an n-D Quantum MIMO system P(X). It contains two parts. The first part is to introduce the n-D Quantum MIMO systems where the coefficients vary in the algebra of Q-meromorphic functions. Then we introduce some conditions for the stabilizability of these systems. The second part is to show that this Quantum system has the n-D system as its quantum limit and the results for the SISO, SIMO, MISO, MIMO are obtained again as special cases.

Introduction

In modern communication technic, multiple-input and multiple-output, or MIMO, is the use of multiple antennas at both the receiver and transmitter to improve communication performance. It is one of several forms of smart antenna (SA), and the state of the art of SA technology [1], [5].

MIMO technology has attracted attention, since it offers significant increases in data throughput and link range without additional bandwidth or transmit power. It achieves this by higher spectral efficiency (more bits per second Hertz of bandwidth) and link reliability or diversity (reduced fading). Because of these properties, MIMO is a current direction of international wireless research [4].

The goal in this work is to study the conditions for stability of n-D systems. A dynamical system can be interpreted as a vector field on $\mathbb{R}^n$ presented by an ordinary differential equation. In dynamical systems we
deal with trajectories. Since some systems are very complicated, the understanding of trajectories will be with difficulties and so sometimes in studying the system, the notion of stability has to be introduced.

In this work we would like to study dynamical systems on the Quantized spaces. The notion of functional quantization is introduced in [1].

The paper is organized as follows: First we introduce the notion of trajectories and vector fields on quantized spaces and obtain some facts about them which makes us ready to define dynamics of these systems. Then we see what can be meant by stability of such systems and study the conditions of stable systems. In the end we notice that the quantum limit of the q-system is the dynamical system on the classical space. We end the paper by presenting some examples of such systems and find their stability conditions and also their quantum limit.

1. Quantum Trajectories

Before going further lets recall the definition of functional quantization and quantized spaces from[1].

Let $S_1$ and $S_2$ be sets. Let $x \in S_1$. Assume that $A$ is a unital $C$-algebra of complex valued functions on $S_1$. Suppose that $\mathcal{B}$ is the $C$-vector space of the algebra of complex valued functions on $S_1 \times S_2$. Let a composition law $\star$ makes $\mathcal{B}$ into an associative, unital, not necessary commutative $A$-algebra. Denote the restriction of an element $f \in \mathcal{B}$ to $\{x\} \times S_2$ by $f_x$ and let

$$B = \{f_x | f \in \mathcal{B}\}$$

This can be considered as the subalgebra of complex valued functions on $S_2$. Also let define $\delta_x : A \rightarrow C$ defined by $\delta_x(g) = g(x)$ be the character of $A$.

With these notations $\mathcal{B}$ is called a $(x, S_1, A)$ functional quantization of $B$ and the homomorphism

$$\phi : \mathcal{B} \rightarrow B$$

defined by $\phi(f) = f_x$ with the property $\phi(gf) = \delta_x(g)\phi(f)$ for all $f \in \mathcal{B}$ and $g \in A$, is called the quantization map.

By a quantum space we mean the functional quantization of the coordinate algebra of $R^n$ as follows:
For the coordinates $x$ and $y$ in $R^2$, let $*$ be defined as $y * x = qx * y$ with $q \in D - \{0\}$.

Now let $D = \{ q \in C | |q| \leq 1 \}$ be the unit disc in $C$ and $A_1(q)$ be the $C$-algebra of all absolutely convergent power series $\sum_{i=0}^{\infty} a_i q^i$ in $D$ with coefficients in $C$. Also let $A_0(q)$ be the $C$-algebra of all absolutely convergent power series $\sum_{i>\infty} c_i q^i$ in $D - \{0\}$ with coefficients in $C$. The $(1, D - \{0\}, A_0(q))$ functional quantization of $M$: the $C$-algebra of all absolutely convergent power series $\sum_{i>\infty} \alpha_{ij} t^1_1 t^2_2$ on $R - \{0\} \times R - \{0\}$ with coefficients in $C$, is called the quantum 2-space and we denote it by $Q^2$. This is a unital non commutative associative $A_0(q)$-algebra. Elements of this algebra has a representation of the form $\sum_{i>\infty} a_{ij} x^i y^j$, where $a_{ij}$s are in $A_0(q)$.

**Definition 1.1.** Let $\psi : A_0(q) \to R$ be the $C$-algebra homomorphism defined by $\psi(q) = 1$. Let $I \subseteq R$ be an open interval. Any $\psi$- homomorphism $\alpha : Q^2 \to C^\infty(I)$ is called a quantum trajectory on the quantum 2-space $Q^2$. We call it in short a Q-trajectory. In other words a Q-trajectory is a $C$-linear map $\alpha$ satisfying $\alpha(q^i x^j y^k) = \psi(q)^i \alpha(x)^j \alpha(y)^k$.

**Remark.** From the above property we see that $\alpha$ is completely determined by its values $x$ and $y$, i.e by $\alpha(x)$ and $\alpha(y)$.

**Definition 2.1.** For the point $P = (p_1, p_2) \in R^2$ we say that $\alpha$ passes through $P$ at $t = 0$ if for each $f \in Q^2$, $\alpha(f)(0) = f(P)$. In particular if $\alpha$ passes through $P$ at $t = 0$, then $\alpha(x)(0) = p_1$ and $\alpha(y)(0) = p_2$. And so we can use the familiar notation $\alpha(0) = (\alpha(x)(0), \alpha(y)(0)) = (p_1, p_2) = P$.

**Definition 3.1.** If we write $\alpha$ as $\alpha(t) = (\alpha(x)(t), \alpha(y)(t))$, then the velocity vector for $\alpha$ at $t$ is defined by

$$\dot{\alpha}(t) = \left( \frac{d}{dt} \alpha(x)(t), \frac{d}{dt} \alpha(y)(t) \right)$$

**Definition 4.1.** Let $f \in Q^2$ and let $\alpha : Q^2 \to C^\infty(I)$ be a Q-trajectory. The rate of change of $f$ in $\alpha$ direction $\dot{\alpha} f$ is defined by

$$\dot{\alpha} f(t) = \frac{d}{dt} \alpha(f)(t)$$
Lemma 5.1. $\dot{\alpha}(t)$ can be considered as a derivation of $Q^2$, i.e. as a $C$-linear map $\dot{\alpha} : Q^2 \to C^\infty(I)$ with the property that for each $f$ and $g$ in $Q^2$:
$$\dot{\alpha}(fg)(t) = \dot{\alpha}(f)(t)\dot{\alpha}(g)(t) + \alpha(f)(t)\dot{\alpha}(g)(t)$$

Definition 6.1. Any $A_0(q)$-homomorphism $X : Q^2 \to Q^2$ is called a vector field on $Q^2$. A Q-trajectory $\alpha$ is the integral curve for $X$ if
$$aoX(t) = \frac{d}{dt}\alpha(t)$$

Lemma 7.1. For vector fields $X$ and $Y$ their composition is a vector field and so if we define their bracket by
$$[X, Y] = XoY - YoX$$
then the set of vector fields with this bracket is a Lie algebra. Furthermore the Liebniz rule satisfies, i.e. for vector fields $X, Y$ and $Z$
$$[X, YZ] = [X, Y]Z + Y[X, Z]$$
Where we define the product of two vector fields by
$$XY(f) = X(f).Y(f)$$
for all $f \in Q^2$.

Proposition 8.1. For each vector field $X : Q^2 \to Q^2$ and each point $P = (p_1, p_2) \in R^2$, there exists a Q-trajectory $\alpha : Q^2 \to C^\infty(I)$ passing through $P$ and satisfies
$$\alpha X(0) = \dot{\alpha}(0)$$

Proof: Let $X(x) = f$ and $X(y) = g$. Set
$$\alpha(t) = (\alpha(x)(t), \alpha(y)(t)) = (p_1 + t\psi(f(p_1, p_2)), p_2 + t\psi(g(p_1, p_2)))$$
Then we have $\alpha(0) = (\alpha(x)(0), \alpha(y)(0)) = (p_1, p_2)$ and
$$\dot{\alpha}(x)(0) = \psi(f(p_1, p_2)) = \alpha(f)(0) = (\alpha X)(x)(0)$$
$$\dot{\alpha}(y)(0) = \psi(g(p_1, p_2)) = \alpha(g)(0) = (\alpha X)(y)(0)$$
Remark. The converse of the above proposition is also true. That is for any Q-trajectory \( \alpha : Q^2 \to C^\infty(I) \) if \( \alpha(0) = (\alpha(x)(0), \alpha(y)(0)) = (p_1, p_2) = P \), then there exists a vector field \( X : Q^2 \to Q^2 \) satisfying

\[
\begin{align*}
\alpha_0X(x)(0) &= \dot{\alpha}(x)(0) \\
\alpha_0X(y)(0) &= \dot{\alpha}(y)(0)
\end{align*}
\]

Proof: It is sufficient to define \( X(x) = f \) and \( X(y) = g \) where

\[
\begin{align*}
\psi(f(p_1, p_2)) &= \dot{\alpha}(x)(0) \\
\psi(g(p_1, p_2)) &= \dot{\alpha}(y)(0)
\end{align*}
\]

2. Quantum Systems

An autonomous (vector field) equation of the form

\[
P(X) : \alpha_0X(t) = \frac{d}{dt} \alpha(t)
\]

where \( \alpha \) is a Q-trajectory and \( X : Q^2 \to Q^2 \) is a vector field, is called an autonomous quantum (MIMO) system. A solution for this system is a Q-trajectory \( \alpha(t) \) such that \( \alpha_0X(t) = 0 \).

A nonautonomous quantum (MIMO) system is defined the same way except that any Q-trajectory \( \alpha \) satisfying the equation is a solution for the system.

Example 1.2. Let \( X : Q^2 \to Q^2 \) be the vector field defined by \( X(x) = y \) and \( X(y) = x \). Then an autonomous (time independent) solution for this system is the Q-trajectory \( \alpha \) defined by \( \alpha(t) = (0, 0) \) and a nonautonomous (time dependent) solution is the Q-trajectory

\[
\alpha(t) = (\alpha(x)(t), \alpha(y)(t)) = (ce^{-t}, -ce^{-t})
\]

Definition 2.2. The solution \( \alpha \) is called stable if for every \( \epsilon > 0 \) there exists a \( \delta(\epsilon) > 0 \) such that if \( \beta \) is another solution for the system and if

\[
|\alpha(t_0)(x) - \beta(t_0)(x)| < \delta, |\alpha(t_0)(y) - \beta(t_0)(y)| < \delta
\]
then
\[ |\alpha(t)(x) - \beta(t)(x)| < \epsilon, |\alpha(t)(y) - \beta(t)(y)| < \epsilon \]
for each \( t > t_0 \) and \( t_0 \in I \).

**Example 3.2.** Both the nonautonomous and autonomous solutions for example 1.2 are stable.

**Note.** We can generalize all the above results for the n-dimensional case.

### 3. Quantum Limit

In the quantum limit
\[ q \to 0 \]
the equations of motion and the evolution of the quantum system will have the classic meaning in terms of the classical Hamiltonian.

**References**

[1] Bruyn Lieven Le et al., "Canonical Systems and Noncommutative Geometry", *arxiv:math/0303304*.

[2] Milani V. et al, "Q-Analytic Functions on Quantum Spaces", *Journal of math. physics*, 35, No.9, (1994).

[3] Milani V. et al, "Geodesic Curves on Quantized Manifolds", *letters in math. physics*, 40, No.4, (1997).

[4] Pauraj A. et al., "Introduction to Space-Time Wireless Communications", Cambridge Univ. Press, 2003.

[5] Ying Jang Qian, "Conditions for Strong Stabilizabilities of n-dim Systems", *Multidimensional systems and Signal Processing*, Academic Press, no.9, 1998.

[6] Youla D. C. et al, "Single Loop Feed-Back Stabilization of Linear Multivariable Dynamical Plants", *Automatica*, Pergamon Press, 10, 1974.