Gaussian limits for multidimensional random sequential packing at saturation (extended version)

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Abstract

Consider the random sequential packing model with infinite input and in any dimension. When the input consists of non-zero volume convex solids we show that the total number of solids accepted over cubes of volume $\lambda$ is asymptotically normal as $\lambda \to \infty$. We provide a rate of approximation to the normal and show that the finite dimensional distributions of the packing measures converge to those of a mean zero generalized Gaussian field. The method of proof involves showing that the collection of accepted solids satisfies the weak spatial dependence condition known as stabilization.

1 Main results

Given $d \in \mathbb{N}$ and $\lambda \geq 1$, let $U_{1,\lambda}, U_{2,\lambda}, \ldots$ be a sequence of independent random $d$-vectors uniformly distributed on the cube $Q_\lambda := [0, \lambda^{1/d})^d$. Let $S$ be a fixed bounded

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closed convex set in $\mathbb{R}^d$ with non-empty interior (i.e., a ‘solid’) with centroid at the origin $0$ of $\mathbb{R}^d$ (for example, the unit ball), and for $i \in \mathbb{N}$, let $S_{i,\lambda}$ be the translate of $S$ with centroid at $U_{i,\lambda}$. So $S_{\lambda} := (S_{i,\lambda})_{i \geq 1}$ is an infinite sequence of solids arriving at uniform random positions in $Q_\lambda$ (the centroids lie in $Q_\lambda$ but the solids themselves need not lie wholly inside $Q_\lambda$).

Let the first solid $S_{1,\lambda}$ be packed, and recursively for $i = 2, 3, \ldots$, let the $i$-th solid $S_{i,\lambda}$ be packed if it does not overlap any solid in $\{S_{1,\lambda}, \ldots, S_{i-1,\lambda}\}$ which has already been packed. If not packed, the $i$-th solid is discarded; we sometimes use accepted as a synonym for ‘packed’. This process, known as random sequential adsorption (RSA) with infinite input, is irreversible and terminates when it is not possible to accept additional solids. At termination, we say that the sequence of solids $S_{\lambda}$ jams $Q_\lambda$ or saturates $Q_\lambda$. The jamming number $N_{\lambda} := N_{\lambda}(S_{\lambda})$ denotes the number of solids accepted in $Q_\lambda$ at termination. We use the words ‘jamming’ and ‘saturation’ interchangeably in this paper.

Jamming numbers $N_{\lambda}$ arise naturally in the physical, chemical, and biological sciences. They are considered in the description of the irreversible deposition of colloidal particles on a substrate (see the survey [1] and the special volume [20]), hard core interactions (see the survey [7]; also [25]), adsorption modelling (see [3] and the survey [24]) and also in the modelling of communication and reservation protocols (see [4, 5]).

The extensive body of experimental results related to the large scale behavior of packing numbers stands in sharp contrast with the limited collection of rigorous mathematical results, especially in $d \geq 2$. The main obstacle to a rigorous mathematical treatment of the packing process is that the short range interactions of arriving particles create long range spatial dependence, thus turning $N_{\lambda}$ into a sum of spatially correlated random variables.

In the case where $d = 1$ and $S = [0, 1]$, a famous result of Rényi [21] shows that jamming limit, defined as $\lim_{\lambda \to \infty} \lambda^{-1} \mathbb{E} N_{\lambda}$, exists as an integral which evaluates to roughly 0.748; also in this case, Mackenzie [10] shows that $\lim_{\lambda \to \infty} \lambda^{-1} \text{Var} N_{\lambda}$ exists as an integral which evaluates to roughly 0.03815. Dvoretzky and Robbins [6] show that the jamming numbers $N_{\lambda}$ are asymptotically normal as $\lambda \to \infty$, but their techniques do not address the case $d > 1$.

Since the above results were established in the 1960s, progress in extending them
rigorously to higher dimensions has been slow until recently. Penrose [11] establishes the existence of a jamming limit for any \( d \geq 1 \) and any choice of \( S \), and also [12] obtains a CLT for a related model (monolayer ballistic deposition with a rolling mechanism) but comments in [12] that ‘Except in the case \( d = 1 \) ... a CLT for infinite-input continuum RSA remains elusive.’

In the present work we show for any \( d \) and \( S \) that \( \lambda^{-1} \text{Var} N_\lambda \) converges to a positive limit and that \( N_\lambda \) satisfies a central limit theorem, i.e., the fluctuations of the random variable \( N_\lambda \) are indeed Gaussian in the large \( \lambda \) limit. This puts the recent experimental results and Monte Carlo simulations of Quintanilla and Torquato [22] and Torquato (ch. 11.4 of [25]) on rigorous footing. We also provide a bound on the rate of convergence to the normal, and on the rate of convergence of \( \lambda^{-1} \mathbb{E} N_\lambda \) to the jamming limit.

Throughout \( \mathcal{N}(0, 1) \) denotes a mean zero normal random variable with variance one.

**Theorem 1.1** Let \( S_\lambda \) be as above and put \( N_\lambda := N_\lambda(S_\lambda) \). There are constants \( \mu := \mu(S, d) \in (0, \infty) \) and \( \sigma^2 := \sigma^2(S, d) \in (0, \infty) \) such that as \( \lambda \to \infty \) we have

\[
|\lambda^{-1} \mathbb{E} N_\lambda - \mu| = O(\lambda^{-1/d})
\]  

and \( \lambda^{-1} \text{Var} N_\lambda \to \sigma^2 \) with

\[
\sup_{t \in \mathbb{R}} \left| P \left[ \frac{N_\lambda - \mathbb{E} N_\lambda}{(\text{Var} N_\lambda)^{1/2}} \leq t \right] - P[\mathcal{N}(0, 1) \leq t] \right| = O((\log \lambda)^{3d} \lambda^{-1/2}).
\]  

The process of accepted solids in \( Q_\lambda \) induces a natural random point measure \( \nu_\lambda \) on \([0, 1]^d\) given by

\[
\nu_\lambda := \sum_{i=1}^{\infty} \delta_{\lambda^{-1/d} U_{i,\lambda}}^1_{\{S_{i,\lambda} \text{ is accepted}\}}
\]

where \( \delta_x \) stands for the unit point mass at \( x \). It also induces a natural random volume measure \( \nu'_\lambda \) on \( \mathbb{R}^d \), normalized to have the same total measure as \( \nu_\lambda \), defined for all Borel \( A \subseteq \mathbb{R}^d \) by

\[
\nu'_\lambda(A) := \frac{\lambda}{|S|} \left| A \cap \left( \bigcup_{i \geq 1} [\lambda^{-1/d} S_{i,\lambda} : \{S_{i,\lambda} \text{ is accepted}\}] \right) \right|
\]
where $|\cdot|$ denotes Lebesgue measure and $\lambda^{-1/d}A := \{\lambda^{-1/d}x : x \in A\}$. The measure $\nu'_\lambda$ is not necessarily supported by $Q_1$ due to boundary effects, but for $\lambda > 1$ it is supported by $Q_1^+ := [-1, 2]^d$ (a fattened version of $Q_1$).

Let $\tilde{\nu}_\lambda := \nu_\lambda - \mathbb{E}[\nu_\lambda]$ and $\tilde{\nu}'_\lambda := \nu'_\lambda - \mathbb{E}[\nu'_\lambda]$. Let $\mathcal{R}(Q_1^+)$ denote the class of bounded, almost everywhere continuous functions on $Q_1^+$. For $f \in \mathcal{R}(Q_1^+)$ and $\mu$ a signed measure on $\mathbb{R}^d$ with finite total mass, let $\langle f, \mu \rangle := \int_{\mathbb{R}} f \, d\mu$. The following theorem provides the limit theory (law of large numbers and central limit theorems) for the integrals of test functions $f \in \mathcal{R}(Q_1^+)$ against the random point measure $\nu_\lambda$ and the random volume measure $\nu'_\lambda$ induced by the packing process. In particular, it shows that the finite dimensional distributions of the centered packing point measures $(\tilde{\nu}_\lambda)_\lambda$ converge to those of a certain mean zero generalized Gaussian field, namely white noise on $Q_1$ with variance $\sigma^2$ per unit volume, and likewise for the centered packing volume measures $(\tilde{\nu}'_\lambda)_\lambda$.

**Theorem 1.2** Let $\mu$ and $\sigma^2$ be as in Theorem 1.1. Then for any $f, g$ in $\mathcal{R}(Q_1^+)$,

$$
\lim_{\lambda \to \infty} \lambda^{-1} \mathbb{E}[\langle f, \nu_\lambda \rangle] = \mu \int_{[0,1]^d} f(x) \, dx
$$

and

$$
\lim_{\lambda \to \infty} \lambda^{-1} \text{Cov}(\langle f, \nu_\lambda \rangle, \langle g, \nu_\lambda \rangle) = \sigma^2 \int_{[0,1]^d} f(x)g(x) \, dx.
$$

Also, the finite-dimensional distributions of the random field $(\lambda^{-1/2}\langle f, \tilde{\nu}_\lambda \rangle, f \in \mathcal{R}(Q_1^+))$ converge as $\lambda \to \infty$ to those of a mean zero generalized Gaussian field with covariance kernel

$$(f, g) \mapsto \sigma^2 \int_{[0,1]^d} f(x)g(x) \, dx, \quad f, g \in \mathcal{R}(Q_1^+).$$

Moreover, the same conclusions hold with $\nu_\lambda$ and $\tilde{\nu}_\lambda$ replaced by $\nu'_\lambda$ and $\tilde{\nu}'_\lambda$ respectively.

**Remarks.**

1. **Finite input.** Let $\tau \in (0, \infty)$ and let $\lceil x \rceil$ denote the smallest integer greater than or equal to $x$. Inputting only the first $\lceil \lambda \tau \rceil$ solids of the sequence $S_{1,\lambda}, S_{2,\lambda}, ..., S_{\lceil \lambda \tau \rceil,\lambda}$ yields RSA packing of the cube $Q_\lambda$ with finite input. The finite-input packing number, i.e., the total number of solids accepted from $S_{1,\lambda}, S_{2,\lambda}, ..., S_{\lceil \lambda \tau \rceil,\lambda}$, is asymptotically
normal as \( \lambda \to \infty \) with \( \tau \) fixed. This is proved in [17], and extended in [2] to the case where the spatial coordinates come from a non-homogeneous point process. Packing measures induced by RSA packing with finite input have finite dimensional distributions converging to those of a mean zero generalized Gaussian field with a covariance structure depending upon the underlying density of points [2].

2. **Stabilization.** One might expect that the restriction of the packing measure \( \nu_\lambda \) or \( \nu'_\lambda \) to a localized region of space depends only on incoming particles with ‘nearby’ spatial locations, in some well-defined sense. This local dependency property is denoted *stabilization*; when the region of spatial dependency has a diameter with an exponentially decaying tail, it is called *exponential stabilization*. These notions are spelt out in general terms in Section 2. Theorem 2.1 provides a general spatial limit theory for exponentially stabilizing measures; this is an infinite-input analog to known results [2, 13, 14, 15] for the finite-input setting, and is of independent interest.

A form of stabilization for infinite input RSA was proved in [11], but without any tail bounds. *Exponential stabilization* in the infinite input setting is perhaps not surprising, but it has been challenging to rigorously establish this key localization feature. In Section 3, we show that infinite-input packing measures stabilize exponentially, so that the general results of Section 2 are applicable to these measures.

3. **Related models** in the literature (see e.g. [17]) include cooperative sequential adsorption, RSA with solids of random size or shape, ballistic deposition with a rolling mechanism, and spatial birth-growth models. For all of these models, limit theorems in the finite-input setting are discussed in [12]. It seems likely that these can be extended to the infinite-input setting using the methods of this paper, although we do not discuss any of them in detail. Nor do we consider non-homogeneous point processes as input.

4. **Rates of convergence.** Even in \( d = 1 \), the rate given by Theorem 1.1 is new. Quintanilla and Torquato [22] use Monte Carlo simulations to predict convergence of the distribution function for \( N_\lambda \) to that of a normal, but they do not obtain rates. Penrose and Yukich [19] obtain rates of approximation to the normal for RSA packing with finite (Poisson) input.

5. **Numerical values.** We do not provide any new analytical methods for computing numerical values of \( \mu \) and \( \sigma^2 \) when \( d \geq 1 \).
6. **Jamming variability.** A significant amount of work is needed (see Section 4) to show that the limiting variance $\sigma^2$ in Theorems 1.1 and 1.2 is non-zero, and we prove this using the following notions.

Given $L > 0$, we shall say that a point set $\eta \subset \mathbb{R}^d \setminus [0, L]^d$ is **admissible** if the translates of $S$ centered at the points of $\eta$ are non-overlapping. Given such an $\eta$, let $N([0, L]^d|\eta)$ denote the (random) number of solids from the sequence $S_{L,t}$ which are packed in $[0, L]^d$ given the **pre-packed configuration** $\eta$. In other words, $N([0, L]^d|\eta)$ arises as the number of solids packed in $[0, L]^d$ in the course of the usual infinite input packing process subject to the additional rule that an incoming solid is discarded should it overlap any solid centered at a point of $\eta$. Say that the convex body $S$ has **jamming variability** if there exists a $L > 0$ such that $\inf_{\eta} \text{Var} N([0, L]^d|\eta) > 0$ with the infimum taken over admissible point sets $\eta \subset \mathbb{R}^d \setminus [0, L]^d$.

In Proposition 4.1 we shall show that each bounded convex body $S \subset \mathbb{R}^d$ with non-empty interior has jamming variability.

7. We let $d_S$ stand for the diameter of $S$. In our proofs, we shall assume that $2d_S < 1$. This assumption entails no loss of generality, since once we have proved Theorems 1.1 and 1.2 under this assumption, the results follow for general $S$ by obvious scaling arguments.

## 2 Terminology, auxiliary results

Let $\mathbb{R}_+ := [0, \infty)$. Given a point $(x_1, \ldots, x_d, t) = (x, t) \in \mathbb{R}^d \times \mathbb{R}_+$, the first $d$ coordinates of the point will be interpreted as spatial components with the $(d + 1)$-st regarded as a time mark. Let us say a point set $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$ is **temporally locally finite** (or TLF for short) if $\mathcal{X} \cap (\mathbb{R}^d \times [0, t])$ is finite for all $t > 0$. Loosely speaking, $\mathcal{X}$ is TLF if it is finite in the spatial directions and locally finite in the time direction.

In this section we adapt the general results and terminology from [2, 14, 15, 19] on limit theory for stabilizing spatial measures defined in terms of finite point sets in $\mathbb{R}^d$, to to the setting of spatial measures defined in terms of TLF point sets in $\mathbb{R}^d \times \mathbb{R}_+$ (typically obtained as Poisson processes). In subsequent sections, we show that these general results can be applied to obtain the limit theorems for RSA described in Section 1.

For $x \in \mathbb{R}^d$ and $r > 0$, let $B_r(x)$ denote the Euclidean ball centered at $x$ of
radius $r$. We abbreviate $B_r(0)$ by $B_r$. Given $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, $a > 0$ and $y \in \mathbb{R}^d$, we let $y + a\mathcal{X} := \{(y + ax, t) : (x, t) \in \mathcal{X}\}$; in other words, scalar multiplication and translation on $\mathbb{R}^d \times \mathbb{R}_+$ act only on the spatial components. For $A \subset \mathbb{R}^d$ we write $y + aA$ for $\{y + ax : x \in A\}$; also, we write $\partial A$ for the boundary of $A$, and write $A_+$ for $A \times \mathbb{R}_+$. For nonempty subsets $A, A'$ of $\mathbb{R}^d$, write $y + aA$ for $\{y + ax : x \in A\}$; also, we write $\partial A$ for the boundary of $A$, and write $A_+$ for $A \times \mathbb{R}_+$. For nonempty subsets $A, A'$ of $\mathbb{R}^d$, write $D_2(A, A') := \inf \{|x - y| : x \in A, y \in A'\}$.

Let $\xi(\mathcal{X}, A)$ be an $\mathbb{R}_+$-valued function defined for all pairs $(\mathcal{X}, A)$, where $\mathcal{X}$ is a TLF subset of $\mathbb{R}^d \times \mathbb{R}_+$ and $A$ is a Borel subset of $\mathbb{R}^d$. Throughout this section we make the following assumptions on $\xi$:

1. $\xi(\cdot, A)$ is measurable for each Borel $A$,

2. $\xi(\mathcal{X}, \cdot)$ is a finite measure on $\mathbb{R}^d$ for each TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$,

3. $\xi$ is translation invariant, that is $\xi(i + \mathcal{X}, i + A) = \xi(\mathcal{X}, A)$ for all $i \in \mathbb{Z}^d$, all TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, and all Borel $A \subset \mathbb{R}^d$,

4. $\xi$ is uniformly locally bounded (or just bounded for short) in the sense that there is a finite constant $||\xi||_\infty$ such that for all TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$ we have

$$\xi(\mathcal{X}, [0, 1]^d) \leq ||\xi||_\infty. \tag{2.1}$$

5. $\xi$ is locally supported, i.e. there exists a constant $\rho$ such that $\xi(\mathcal{X}, A) = 0$ whenever $D_2(\mathcal{X}, A) > \rho$.

Note that if $\xi(\mathcal{X}, \cdot)$ is a point measure supported by the points of $\mathcal{X}$, then $\xi$ is locally supported (in fact, in this case we can set $\rho = 0$).

For all $\lambda > 0$, let $\mathcal{P}_\lambda$ denote a homogeneous Poisson point process in $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\lambda dx \times ds$, with $dx$ denoting Lebesgue measure on $\mathbb{R}^d$ and $ds$ Lebesgue measure on $\mathbb{R}_+$. We put $\mathcal{P} := \mathcal{P}_1$.

Thermodynamic limits and central limit theorems for functionals in geometric probability are often proved by showing that the functionals satisfy a type of local spatial dependence known as stabilization [2, 13, 14, 15, 17, 18, 23] and that will be our goal here as well. First, we adapt the definitions in [2, 13, 14] to the context of measures defined in terms of TLF point sets in $\mathbb{R}^d$. Recall that $Q_\lambda$ denotes the cube $[0, \lambda^{1/d})^d$. 

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Definition 2.1 We say $\xi$ is homogeneously stabilizing if there exists an a.s. finite random variable $R'$ (a radius of homogeneous stabilization for $\xi$) such that for all TLF $\mathcal{X} \subset (\mathbb{R}^d \setminus B_{R'})_+$ we have

$$\xi((\mathcal{P} \cap (B_{R'})_+) \cup \mathcal{X}, Q_1) = \xi(\mathcal{P} \cap (B_{R'})_+, Q_1).$$

(2.2)

We say $\xi$ is exponentially stabilizing if (i) it is homogeneously stabilizing and $R'$ can be chosen so that $\limsup_{L \to \infty} L^{-1} \log P[R' > L] < 0$, and (ii) for all $\lambda \geq 1$ and all $i \in \mathbb{Z}^d$, there exists a random variable $R := R^\xi(i, \lambda)$ (a radius of stabilization for $\xi$ at $i$ with respect to $\mathcal{P}$ in $(Q_\lambda)_+$) such that for all TLF $\mathcal{X} \subset [Q_\lambda \setminus B_{R(i)}]_+$, and all Borel $A \subseteq Q_1$, we have

$$\xi((\mathcal{P} \cap [B_{R(i)} \cap Q_\lambda]_+) \cup \mathcal{X}, i + A) = \xi(\mathcal{P} \cap [B_{R(i)} \cap Q_\lambda]_+, i + A)$$

(2.3)

and moreover the tail probability $\tau(L)$ defined for $L > 0$ by

$$\tau(L) := \sup_{\lambda \geq 1, i \in \mathbb{Z}^d} P[R^\xi(i, \lambda) > L]$$

(2.4)

satisfies

$$\limsup_{L \to \infty} L^{-1} \log \tau(L) < 0.$$

Loosely speaking, $R := R^\xi(i, \lambda)$ is a radius of stabilization if the $\xi$-measure on $i + Q_1$ is unaffected by changes to the Poisson points outside $B_{R(i)}$ (but inside $Q_\lambda$). When $\xi$ is homogeneously stabilizing, the limit

$$\xi(\mathcal{P}, i + Q_1) := \lim_{r \to \infty} \xi(\mathcal{P} \cap (B_r(i))_+, i + Q_1)$$

exists almost surely for all $i \in \mathbb{Z}^d$. The random variables $(\xi(\mathcal{P}, i + Q_1), i \in \mathbb{Z}^d)$ form a stationary random field.

Given $\xi$, for all $\lambda > 0$, all TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, and all Borel $A \subset \mathbb{R}^d$ we let $\xi_\lambda(\mathcal{X}, A) := \xi(\lambda^{1/d} \mathcal{X}, \lambda^{1/d} A)$. Define the random measure $\mu^\xi_\lambda$ on $\mathbb{R}^d$ by

$$\mu^\xi_\lambda(\cdot) := \xi_\lambda(\mathcal{P}_\lambda \cap Q_1, \cdot)$$

(2.5)

and the centered version $\mu^\xi_\lambda := \mu^\xi_\lambda - \mathbb{E}[\mu^\xi_\lambda]$. By the assumed locally supported property of $\xi$, $\mu_\lambda$ is supported by the fattened cube $Q^+_1 := [-1, 2)^d$ for large enough $\lambda$. 

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If $\xi$ is stabilizing, define $\mu(\xi) := \mathbb{E}[\xi(P,Q_1)]$ and if $\xi$ is exponentially stabilizing, define

$$
\sigma^2(\xi) := \sum_{i \in \mathbb{Z}^d} \operatorname{Cov}[\xi(P,Q_1), \xi(P,i+Q_1)],
$$

where the sum can be shown to converge absolutely by exponential stabilization and (2.1). The following general theorem provides laws of large numbers and normal approximation results for $\langle f, \mu_{\lambda}^{\xi} \rangle$, suitably scaled and centered, for $f \in \mathcal{R}(Q_1^+)$. This set of results for measures determined by TLF point sets is similar to previously known results for measures determined by finite point sets (Theorem 2.1 of [18], Theorem 2.1 of [2], Theorem 2.3 of [2], and Corollary 2.4 of [19]).

**Theorem 2.1** Suppose that $\xi$ is exponentially stabilizing. Then as $\lambda \to \infty$, for $f$ and $g$ in $\mathcal{R}(Q_1^+)$ we have

$$
\lim_{\lambda \to \infty} \lambda^{-1} \mathbb{E}[\langle f, \mu_{\lambda}^{\xi} \rangle] = \mu(\xi) \int_{[0,1]^d} f(x) dx
$$

and

$$
\lim_{\lambda \to \infty} \lambda^{-1} \operatorname{Cov}[\langle f, \mu_{\lambda}^{\xi} \rangle, \langle g, \mu_{\lambda}^{\xi} \rangle] = \sigma^2(\xi) \int_{[0,1]^d} f(x) g(x) dx.
$$

Also,

$$
|\lambda^{-1} \mathbb{E}[\mu_{\lambda}^{\xi}(Q_1^+)] - \mu(\xi)| = O(\lambda^{-1/d}).
$$

Moreover, if $\sigma^2(\xi) > 0$ then

$$
\sup_{t \in \mathbb{R}} \left| P \left[ \frac{\mu_{\lambda}^{\xi}(Q_1^+) - \mathbb{E}[\mu_{\lambda}^{\xi}(Q_1^+)]}{\sqrt{\operatorname{Var}[\mu_{\lambda}^{\xi}(Q_1^+)]/\lambda}} \leq t \right] - P[\mathcal{N}(0,1) \leq t] \right| = O((\log \lambda)^{3d} \lambda^{-1/2})
$$

and the finite-dimensional distributions of the random field $(\lambda^{-1/2}\langle f, \mu_{\lambda}^{\xi} \rangle, f \in \mathcal{R}(Q_1^+))$ converge as $\lambda \to \infty$ to those of a mean zero generalized Gaussian field with covariance kernel

$$
(f,g) \mapsto \sigma^2(\xi) \int_{[0,1]^d} f(x) g(x) dx, \quad f,g \in \mathcal{R}(Q_1^+).
$$

We shall use Theorem 2.1 to prove the results on RSA described in Section 1. It seems likely that Theorem 2.1 can also be applied to obtain similar results for the related models listed in Remark 3 of Section 1. For some of these, certain
generalizations of Theorem 2.1 may be needed; for example, in some cases one may need to allow for the Poisson points to carry independent identically distributed random marks, and in others the boundedness condition \((2.1)\) may need to be relaxed to a moments condition. It seems likely that little change to the proof of Theorem 2.1 will be needed to cover these generalizations.

As we shall see shortly, the thermodynamic limits \((2.6)\) and \((2.8)\) do not require exponential decay of the stabilization radius for \(\xi\), but in fact hold under weaker decay conditions. We expect that \((2.7)\) also holds under weaker decay conditions on the stabilization radius, and also that the boundedness condition \((2.1)\) can be relaxed to a moments condition in Theorem 2.1, but for simplicity we shall assume throughout that \(\xi\) is exponentially stabilizing and satisfies \((2.1)\). Also, if we restrict attention to \(f\) supported by \(Q_1\), we do not need the condition that \(\xi\) be locally supported.

The rest of this section is devoted to proving Theorem 2.1. We shall use the following notation. Given \(f \in \mathcal{R}(Q_1^+)\), we extend \(f\) to the whole of \(\mathbb{R}^d\) by setting \(f(x) = 0\) for \(x \in \mathbb{R}^d \setminus Q_1^+\). Given TLF \(\mathcal{X} \subset (\mathbb{R}^d)_+\), and \(\lambda > 0\), write \(\langle f, \xi_\lambda(\mathcal{X}) \rangle\) for \(\int_{\mathbb{R}^d} f(x) \xi_\lambda(\mathcal{X}, dx)\) (the integral of \(f\) with respect to the measure \(\xi_\lambda(\mathcal{X}, \cdot)\)). For \(j \in \lambda^{-1/d} \mathbb{Z}^d\), let \(f_{\lambda,j} : \mathbb{R}^d \to \mathbb{R}\) be given by \(f_{\lambda,j}(x) = f(x)\) for \(x \in j + Q_{1/\lambda}\), and \(f_{\lambda,j}(x) = 0\) otherwise. Then

\[
\langle f, \mu_\lambda^\xi \rangle = \sum_{j \in \lambda^{-1/d} \mathbb{Z}^d} \langle f_{\lambda,j}, \mu_\lambda^\xi \rangle. \tag{2.10}
\]

Also, let

\[
\overline{f}(\lambda, j) := \sup\{f(x) : x \in j + Q_{1/\lambda}\}; \quad \underline{f}(\lambda, j) := \inf\{f(x) : x \in j + Q_{1/\lambda}\}.
\]

For \(x \in \mathbb{R}^d\) let \(i_\lambda(x)\) be the choice of \(i \in \lambda^{-1/d} \mathbb{Z}^d\) such that \(x \in i + Q_{1/\lambda}\).

**Proof of (2.6).** Let \(f \in \mathcal{R}(Q_1^+)\). Then by (2.10), we have

\[
\lambda^{-1} \mathbb{E}[\langle f, \mu_\lambda^\xi \rangle] = \lambda^{-1} \sum_{j \in \lambda^{-1/d} \mathbb{Z}^d} \mathbb{E}\left[\langle f_{\lambda,j}, \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+) \rangle\right]
= \int_{\mathbb{R}^d} \mathbb{E}\left[\langle f_{\lambda,i_\lambda(x)}, \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+) \rangle\right] dx. \tag{2.11}
\]
For $x \in \mathbb{R}^d \setminus \partial Q_1$, with $f$ continuous at $x$, we assert that as $\lambda \to \infty$,
\[
\mathbb{E} \left[ \left( f_{\lambda,i_{\lambda}(x)}, \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+) \right) \right] \to \mu(\xi) f(x) 1_{Q_1}(x). \tag{2.12}
\]
This clearly holds for $x \in \mathbb{R}^d \setminus [0,1]^d$, since both sides are zero for large $\lambda$, by the locally supported property of $\xi$. To see (2.12) for $x \in (0,1)^d$, observe that the left side has the upper bound
\[
\mathbb{E} \left[ \left( f_{\lambda,i_{\lambda}(x)}, \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+) \right) \right] \leq \overline{f}(\lambda, i_{\lambda}(x)) \mathbb{E} \left[ \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+, i_{\lambda}(x) + Q_{1/\lambda}) \right]
\]
\[
= \overline{f}(\lambda, i_{\lambda}(x)) \mathbb{E} \left[ \xi(\mathcal{P} \cap (Q_{1})_+, i_1(\lambda^{1/d}x) + Q_1) \right]. \tag{2.13}
\]
and has a similar lower bound with $f(\lambda, i_{\lambda}(x))$ instead of $\overline{f}(\lambda, i_{\lambda}(x))$. If $f$ is continuous at $x$, then both $f(\lambda, i_{\lambda}(x))$ and $\overline{f}(\lambda, i_{\lambda}(x))$ tend to $f(x)$, so to prove (2.12) it suffices to show the expectation in the last line of (2.13) converges to $\mu(\xi)$. By translation invariance, this expectation equals
\[
\mathbb{E} \left[ \xi(\mathcal{P} \cap (-i_1(\lambda^{1/d}x) + Q_{1/\lambda})_+, Q_1) \right].
\]
For $x$ in the interior of $Q_1$, the set $-i_1(\lambda^{1/d}x) + Q_{1/\lambda}$ has limit set $\mathbb{R}^d$ as $\lambda \to \infty$, i.e. for any $r < \infty$ the ball $B_r$ is contained in $-i_1(\lambda^{1/d}x) + Q_{1/\lambda}$ for large enough $\lambda$. Hence by stabilization,
\[
\xi(\mathcal{P} \cap (-i_1(\lambda^{1/d}x) + Q_{1/\lambda}), Q_1) \overset{\text{a.s.}}{\longrightarrow} \xi(\mathcal{P}, Q_1) \tag{2.14}
\]
and by (2.1), the corresponding expectations converge. This demonstrates (2.12).

The integrand in (2.11) is dominated by a constant for $x \in Q^+_1$, and is zero for $x \notin Q^+_1$. So by (2.12) and dominated convergence applied to (2.11), we obtain (2.6).

\[\square\]

Proof of (2.8). For this proof, set $f(x) \equiv 1$ on $Q^+_1$. We need to bound the error term in (2.6) for this choice of $f$, which we do by using (2.11) again. For $x \in \mathbb{R}^d$, let $X(x, \lambda)$ be the integrand in (2.11), i.e. set $X(x, \lambda) := \langle f_{\lambda,i_{\lambda}(x)}, \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+) \rangle$ with our current choice of $f$; also set $Y(x, \lambda) := \xi(\mathcal{P} \cap (-i_1(\lambda^{1/d}x) + Q_{1/\lambda}), Q_1)$. If $x \in (0,1 - \lambda^{-1/d})^d$, then
\[
\mathbb{E} \left[ X(x, \lambda) \right] = \mathbb{E} \left[ \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+, i_{\lambda}(x) + Q_{1/\lambda}) \right]
\]
\[
= \mathbb{E} \left[ \xi(\mathcal{P} \cap (Q_{1/\lambda})_+, i_1(\lambda^{1/d}x) + Q_1) \right]
\]
\[
= \mathbb{E} \left[ \xi(\mathcal{P} \cap (-i_1(\lambda^{1/d}x) + Q_{1/\lambda}), Q_1) \right] = \mathbb{E} \left[ Y(x, \lambda) \right]. \tag{2.15}
\]
Abbreviating the Euclidean distance $D_2(\{y\}, A)$ by $D_2(y, A)$ is we have

$$D_2(0, \partial(-i_1(\lambda^{1/d}x) + Q_\lambda)) = D_2(i_1(\lambda^{1/d}x), \partial Q_\lambda) \geq D_2(\lambda^{1/d}x, \partial Q_\lambda) - \sqrt{d} = \lambda^{1/d}D_2(x, \partial Q_1) - \sqrt{d}. $$

Hence the ball $B_{\lambda^{1/d}D_2(x, \partial Q_1) - \sqrt{d}}$ is contained in the box $-i_1(\lambda^{1/d}x) + Q_\lambda$, so with $R'$ denoting the radius of homogeneous stabilization of $\xi$,

$$Y(x, \lambda)1\{R' < \lambda^{1/d}D_2(x, \partial Q_1) - \sqrt{d}\} = \xi(\mathcal{P}, Q_1)1\{R' < \lambda^{1/d}D_2(x, \partial Q_1) - \sqrt{d}\}. $$

(2.16)

Set $\mu := \mu(\xi) = E[\xi(\mathcal{P}, Q_1)]$. By (2.15), (2.16) and (2.1), we have for $x \in (0, 1 - \lambda^{-1/d})d$ that

$$|E[X(x, \lambda)] - \mu| = |E[Y(x, \lambda)] - \mu| = |E[(Y(x, \lambda) - \xi(\mathcal{P}, Q_1))1\{R' \geq \lambda^{1/d}D_2(x, \partial Q_1) - \sqrt{d}\}]| \leq 2\|\xi\|_{\infty}P[R' > \lambda^{1/d}D_2(x, \partial Q_1) - \sqrt{d}]$$

and so by exponential stabilization, there is a constant $K > 0$ such that

$$|E[X(x, \lambda)] - \mu| \leq K \exp(-\lambda^{1/d}D_2(x, \partial Q_1)/K). $$

(2.17)

Also by (2.1), for suitable $K$ the same bound (2.17) for holds trivially for $x \in Q_1 \setminus (0, 1 - \lambda^{-1/d})d$, and hence (2.17) holds for all $x \in Q_1$. By (2.17), it is straightforward to deduce that

$$\int_{Q_1} |E[X(x, \lambda)] - \mu|dx = O(\lambda^{-1/d}). $$

(2.18)

Also, for $x \in \mathbb{R}^d \setminus Q_1$ with $D_2(x, \partial Q_1) > \lambda^{-1/d}$ we have $E[X(x, \lambda)] = 0$, and $X(x, \lambda)$ is uniformly bounded by (2.1), so that

$$\int_{\mathbb{R}^d \setminus Q_1} |E[X(x, \lambda)]|dx = O(\lambda^{-1/d}).$$

Combining this with (2.18) and using (2.11) gives us (2.8). 

Proof of (2.7). Let $f \in \mathcal{R}(Q_1^+)$ and assume $f$ is nonnegative. By linearity, it suffices to prove (2.7) in the case where $f$ is nonnegative and $f \equiv g$, so we now
assume this. First, we assert that there is a constant $K$, independent of $\lambda$, such that for all $\lambda \geq 1$ and all $i \in \lambda^{-1/d} \mathbb{Z}^d$, $z \in \mathbb{Z}^d$, we have
\[
|\text{Cov}[(f_{\lambda,i}, \xi_\lambda(\mathcal{P} \cap (Q_1)_+)), (f_{\lambda,i+\lambda^{-1/d} z}, \xi_\lambda(\mathcal{P} \cap (Q_1)_+))]| \leq K \exp(-|z|/K). \tag{2.19}
\]
This can be proved by arguments similar to those in, e.g., the proof of Lemma 4.1 in [2] or that of Lemma 4.2 in [15]. By (2.10), we have
\[
\lambda^{-1} \text{Var}[(f, \mu_\lambda^x)] = \lambda^{-1} \sum_{i,j \in \lambda^{-1/d} \mathbb{Z}^d} \text{Cov}[(f_{\lambda,i}, \xi_\lambda(\mathcal{P} \cap (Q_1)_+)), (f_{\lambda,j}, \xi_\lambda(\mathcal{P} \cap (Q_1)_+)])
\]
\[
= \int_{\mathbb{R}^d} dx \sum_{z \in \mathbb{Z}^d} \text{Cov}[(f_{\lambda,i_{\lambda}(x)}, \xi(\mathcal{P} \cap (Q_1)_+)), (f_{\lambda,i_{\lambda}(x) + \lambda^{-1/d} z}, \xi(\mathcal{P} \cap (Q_1)_+)])
\]
\[
\tag{2.20}
\]
where the inner sum converges absolutely by (2.19) and is zero for $x \notin Q^+_1$.

Fix $x \in (0, 1)^d$ and $z \in \mathbb{Z}^d$, with $f$ continuous at $x$. Then we have the upper bound
\[
\mathbb{E}[(f_{\lambda,i_{\lambda}(x)}, \xi_\lambda(\mathcal{P} \cap (Q_1)_+)) (f_{\lambda,i_{\lambda}(x) + \lambda^{-1/d} z}, \xi_\lambda(\mathcal{P} \cap (Q_1)_+)]) \leq \mathcal{F}(i_{\lambda}(x), \lambda) \mathcal{F}(i_{\lambda}(x) + \lambda^{-1/d} z, \lambda)
\]
\[
\times \mathbb{E}[\xi_\lambda(\mathcal{P} \cap (Q_1)_+, i_{\lambda}(x) + Q_{1/\lambda}) \xi_\lambda(\mathcal{P} \cap (Q_1)_+, i_{\lambda}(x) + \lambda^{-1/d} z + Q_{1/\lambda})] \tag{2.21}
\]
and a similar lower bound with $\mathcal{F}(i_{\lambda}(x), \lambda) \mathcal{F}(i_{\lambda}(x) + \lambda^{-1/d} z, \lambda)$ replaced by $\mathcal{F}(i_{\lambda}(x), \lambda) \mathcal{F}(i_{\lambda}(x) + \lambda^{-1/d} z, \lambda)$. Note that both $\mathcal{F}(i_{\lambda}(x), \lambda) \mathcal{F}(i_{\lambda}(x) + \lambda^{-1/d} z, \lambda)$ and $\mathcal{F}(i_{\lambda}(x), \lambda) \mathcal{F}(i_{\lambda}(x) + \lambda^{-1/d} z, \lambda)$ converge as $\lambda \to \infty$ to $f^2(x)$.

By scaling and translation invariance of $\xi$, we have
\[
\mathbb{E}[\xi_\lambda(\mathcal{P} \cap (Q_1)_+, i_{\lambda}(x) + Q_{1/\lambda}) \xi_\lambda(\mathcal{P} \cap (Q_1)_+, i_{\lambda}(x) + \lambda^{-1/d} z + Q_{1/\lambda})] = \mathbb{E}[\xi(\mathcal{P} \cap (Q_1)_+, i_{1}(\lambda^{1/d} x) + Q_1) \xi(\mathcal{P} \cap (Q_1)_+, i_{1}(\lambda^{1/d} x) + z + Q_1)]
\]
\[
= \mathbb{E}[\xi(\mathcal{P} \cap (-i_1(\lambda^{1/d} x) + Q_1)_+, Q_1) \xi(\mathcal{P} \cap (-i_1(\lambda^{1/d} x) + Q_1)_+, z + Q_1)].
\]

By a similar argument to (2.14), as $\lambda \to \infty$ we have
\[
\xi(\mathcal{P} \cap (-i_1(\lambda^{1/d} x) + Q_1)_+, Q_1) \xi(\mathcal{P} \cap (-i_1(\lambda^{1/d} x) + Q_1)_+, z + Q_1) \xrightarrow{a.s.} \xi(\mathcal{P}, Q_1) \xi(\mathcal{P}, z + Q_1)
\]
and since $\xi$ is bounded (2.1), the expectations converge. Hence, by (2.21) and the similar lower bound,
\[
\mathbb{E}[(f_{\lambda,i_{\lambda}(x)}, \xi_\lambda(\mathcal{P} \cap (Q_1)_+)) (f_{\lambda,i_{\lambda}(x) + z}, \xi_\lambda(\mathcal{P} \cap (Q_1)_+)]) \rightarrow f^2(x) \mathbb{E}[\xi(\mathcal{P}, Q_1) \xi(\mathcal{P}, z + Q_1)]. \tag{2.22}
\]
Also, $E[(f_{\lambda,i}(x), \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+))]$ converges to $f(x)\mu(\xi)$ by (2.12), and a similar argument yields

$$E[(f_{\lambda,i}(x)+\lambda^{-1/d}z, \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+))] \to f(x)\mu(\xi).$$

Combining these with (2.22), we obtain that for $x \in (0,1)^d$ with $f$ continuous at $x$,

$$\lim_{\lambda \to \infty} \text{Cov}[(f_{\lambda,i}(x), \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+)), (f_{\lambda,i}(x)+z, \xi_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+))] = f^2(x)\text{Cov}[\xi(\mathcal{P}, Q_1), \xi(\mathcal{P}, z+Q_1)1_{Q_1}(x)].$$ \tag{2.23}

Also, (2.23) holds for $x \in \mathbb{R}^d \setminus [0,1]^d$ as well, since both sides are zero for large $\lambda$. By (2.19), (2.23) and the dominated convergence theorem, applied to the last line of (2.20), we obtain

$$\lim_{\lambda \to \infty} \lambda^{-1}\text{Var}[(f, \mu_\lambda^\xi)] = \sigma^2(\xi) \int_{[0,1]^d} f^2(x)dx.$$

In other words, we have demonstrated (2.7) in the case where $f \equiv g$ and $f$ is nonnegative. Extending (2.7) to the general case is then a routine application of linearity. \hfill \Box

**Proof of (2.9) and the rest of Theorem 2.1.** Suppose $\sigma^2(\xi) > 0$ and take $f \in \mathcal{R}(Q_1^\uparrow)$ with $\int_{Q_1} f^2(x)dx > 0$. We prove asymptotic normality for $\langle f, \mu_\lambda^\xi \rangle$, with a rate of convergence. To do this we adapt the proof of Corollary 2.4 of [19] (Corollary 2.1 in the electronically available version of [19]), to the setting of functionals of TLF point sets in $\mathbb{R}^d$. The proof of Corollary 2.4 of [19] involves applying Stein’s method to a graph whose vertices are sub-cubes of the unit cube with edge length proportional to $(\log \lambda)^{-1/d}$ and with edges between sub-cubes whenever the distance between sub-cubes is within twice the common cube edge length. We make the following trivial modifications to the proof of Corollary 2.4 of [19].

Let $\lambda$ be fixed and large. Subdivide $Q_1^\uparrow$ into $V(\lambda) := 3^d \rho^{-d}_\lambda$ sub-cubes $C_i^\lambda$ of volume $\rho^{-d}_\lambda$, where $\rho_\lambda := \alpha \log \lambda$ for some suitably large $\alpha$, as in section four of [19]. For all $1 \leq i \leq V(\lambda)$, put

$$\xi_i^\lambda := \int_{C_i^\lambda} f(x)\xi_\lambda(\mathcal{P}_\lambda \cap Q_1, dx) = \int_{\lambda^{1/d} C_i^\lambda} f(y)\xi(\mathcal{P} \cap Q_\lambda, dy).$$

Then

$$\langle f, \mu_\lambda^\xi \rangle = \sum_{i=1}^{V(\lambda)} \xi_i^\lambda.$$
Note that $\xi_1^{\lambda}$ is the analog of $\sum_{j=1}^{\infty} |\xi_{ij}|$ of Lemma 4.3 of [19] and furthermore, by the boundedness (2.1) of $\xi$, for $q = 3$ there exists $K := K(q; f) < \infty$ such that $\|\xi_1^{\lambda}\|_q \leq K\rho_\lambda^d$.

Consider for all $1 \leq i \leq V(\lambda)$ the events
$$E_i := \bigcap_{j \in \mathbb{Z}^d: (j + Q_1)^\lambda C^\lambda_i \neq \emptyset} \{ R^\xi(j, \lambda) \leq \rho_\lambda \},$$
where $R^\xi(j, \lambda)$ is the radius of stabilization of $\xi$ at $j \in \mathbb{Z}^d$. Let
$$E_\lambda := \bigcap_{i=1}^{V(\lambda)} E_i,$$
and note that $P[E_\lambda^c] \leq \lambda \tau(\rho_\lambda)$, where $\tau$ is as in Definition 2.1.

Next, define the analog of $T'_\lambda$ in [19] by
$$\mu^\xi_\lambda := \sum_{i=1}^{V(\lambda)} \xi_1^{\lambda} 1_{E_i}$$
and note that $\xi_1^{\lambda} 1_{E_i}$ and $\xi_1^{\lambda} 1_{E_j}$ are independent whenever $D_2(C^\lambda_i, C^\lambda_j) > 2\lambda^{-1/d} \rho_\lambda$.

For $1 \leq i \leq V(\lambda)$, define
$$S_i := (\text{Var}[\mu^\xi_\lambda])^{-1/2} \xi_1^{\lambda} 1_{E_i}$$
and put
$$S := \sum_{i=1}^{V(\lambda)} (S_i - \mathbb{E} S_i).$$

As in [19] we define a dependency graph $G_\lambda := (\mathcal{V}_\lambda, \mathcal{E}_\lambda)$ for $\{S_i\}_{i=1}^{V(\lambda)}$. The set $\mathcal{V}_\lambda$ consists of the sub-cubes $C^\lambda_1, \ldots, C^\lambda_{V(\lambda)}$ and edges $(C^\lambda_i, C^\lambda_j)$ belong to $\mathcal{E}_\lambda$ if $D_2(C^\lambda_i, C^\lambda_j) \leq 2\lambda^{-1/d} \rho_\lambda$. Next, in parallel with the proof of Corollary 2.4 of [19], we notice that:

(i) $V(\lambda) := |\mathcal{V}_\lambda| = 3^d \lambda \rho_\lambda^{-d}$,
(ii) the maximal degree $D_\lambda$ of $G_\lambda$ satisfies $D_\lambda \leq 5^d$,
(iii) for all $1 \leq i \leq V(\lambda)$ we have $\|S_i\|_3 \leq K(\text{Var}[\mu^\xi_\lambda])^{-1/2} \rho_\lambda^d$,
(iv) $\text{Var}[\mu^\xi_\lambda] = O(\rho_\lambda^d \lambda)$,
and
(v) $|\text{Var}[\langle f, \mu^\xi_\lambda \rangle] - \text{Var}[\langle f, \mu^\xi_\lambda \rangle]| \leq K \lambda^{-2}$. 

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As in [19], we may use Stein’s method to deduce a normal approximation result for $S$ and then applying the estimates (iv) and (v) and following [19] verbatim we can turn this into a normal approximation result for $\langle f, \mu^\xi \rangle$, i.e., in this way we obtain the desired rate (2.9) when $f \equiv 1$.

The normal approximation result for $\langle f, \mu^\xi \rangle$, together with (2.7), implies that $\lambda^{-1/2} \langle f, \mu^\xi \rangle$ converges in distribution to a mean zero normal random variable with variance $\sigma^2(\xi) \int_{[0,1]^d} f^2(x)dx$. Given this, the convergence of the finite dimensional distributions in Theorem 2.1 is a standard application of the Cramér-Wold device. This completes the proof of Theorem 2.1.

3 Stabilization of infinite input packing functionals

In this section, we show that the random packing measures $\nu_\lambda$ and $\nu'_\lambda$ described in Section 1 can each be expressed in terms of a suitably defined measure-valued functional $\xi$ of TLF point sets in $\mathbb{R}^d \times \mathbb{R}_+$, of the general type considered in Section 2, applied to a Poisson point process in space-time. Then we show that in both cases the appropriate choice of $\xi$ satisfies the exponential stabilization condition described in Definition 2.1, so that Theorem 2.1 is applicable to this choice of $\xi$. We defer to the next section the proof that in both cases the appropriate choice of $\xi$ satisfies $\sigma^2(\xi) > 0$.

Let us say that two points $(x, t)$ and $(y, u)$ in $\mathbb{R}^d \times \mathbb{R}_+$ are adjacent if $(x + S) \cap (y + S) \neq \emptyset$. Given TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, let us first list the points of $\mathcal{X}$ in order of increasing time-marks using the lexicographic ordering on $\mathbb{R}^d$ as a tie-breaker in the case of any pairs of points of $\mathcal{X}$ with equal time-marks. Then consider the points of $\mathcal{X}$ in the order of the list; let the first point in the list be accepted, and let each subsequent point be accepted if it is not adjacent to any previously accepted point of $\mathcal{X}$; otherwise let it be rejected. We call this the usual rule for packing points of $\mathcal{X}$, since it corresponds to the packing rule of Section 1 with the input ordering determined by time-marks. Let $\mathcal{A}(\mathcal{X})$ denote the subset of $\mathcal{X}$ consisting of all accepted points when the points of $\mathcal{X}$ are packed according to the usual rule.

We consider two specific measure-valued functionals $\xi^*$ and $\xi'$ on TLF point sets in $\mathbb{R}^d \times \mathbb{R}_+$, of the general type considered in Section 2, which are defined as
follows. For any TLF point set $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$ and bounded Borel $A \subset \mathbb{R}^d$, recall that $A_+ := A \times \mathbb{R}_+$. Let $\xi^*(\mathcal{X}, A)$ be the number of points of $A(\mathcal{X})$ which lie in $A_+$, and with $|\cdot|$ denoting Lebesgue measure, let

$$\xi'(\mathcal{X}, A) := |S|^{-1} \left| \bigcup_{(x,t) \in \mathcal{A}(\mathcal{X})} (x + S) \right|.$$ 

Then $\xi^*$ and $\xi'$ are clearly translation invariant, and are bounded (i.e., satisfy (2.1)), since only a bounded number of solids can be packed in any fixed bounded cube.

Recall that $\mathcal{P}_\lambda$ denotes a homogeneous Poisson point process of intensity $\lambda$ on $\mathbb{R}^d \times \mathbb{R}_+$, and $\mathcal{P} = \mathcal{P}_1$. Assume $\mathcal{P}_\lambda$ is obtained from $\mathcal{P}$ by $\mathcal{P}_\lambda := \lambda^{-1/d} \mathcal{P}$. For all $\lambda > 0$, recall the definition of $\xi_\lambda$ in Section 2, and define the random measures

$$\mu^{\xi^*}_\lambda(\cdot) := \xi^*_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+, \cdot) \quad \text{and} \quad \mu^{\xi'}_\lambda(\cdot) := \xi'_\lambda(\mathcal{P}_\lambda \cap (Q_1)_+, \cdot).$$

Let $N^{\xi^*}_\lambda$ denote the total mass of $\mu^{\xi^*}_\lambda$, i.e.

$$N^{\xi^*}_\lambda := \mu^{\xi^*}_\lambda(\mathcal{P}_\lambda \cap [0, 1]^d_+ \times [0, 1]^d).$$

Then $\mu^{\xi^*}_\lambda$ and $\mu^{\xi'}_\lambda$ are the random packing point measure and the random packing volume measure, respectively, corresponding to the random sequential adsorption process obtained by taking the spatial locations of the points of $\mathcal{P} \cap Q_\lambda$, in order of increasing time-mark, as the input sequence. Since these spatial locations are independent and uniformly distributed on $Q_\lambda$, we have the distributional equalities

$$\mu^{\xi^*}_\lambda \overset{D}{=} \nu_\lambda, \quad \mu^{\xi'}_\lambda \overset{D}{=} \nu'_\lambda, \quad \text{and} \quad N^{\xi^*}_\lambda \overset{D}{=} N_\lambda,$$  \hspace{1cm} (3.1)

where the measures $\nu_\lambda$ and $\nu'_\lambda$ are given in (1.3) and the jamming number $N_\lambda$ is also given in Section 1.

We show in Lemmas 3.4 and 3.6 below that both $\xi^*$ and $\xi'$ are exponentially stabilizing, and therefore we can apply Theorem 2.1 to either of these choices of $\xi$. To proceed with the proof of exponential stabilization, consider a partition of $\mathbb{R}^d$ into translates of the unit cube $C := Q_1 = [0, 1)^d$. It is convenient to index these translates as $C_i, \ i := (i_1, \ldots, i_d) \in \mathbb{Z}^d$, with $C_i := (i_1, \ldots, i_d) + C$. We shall write $C_i^+ := \bigcup_{j \in \mathbb{Z}^d, \|i-j\|_\infty \leq 1} C_j$, that is to say $C_i^+$ is the union of $C_i$ and its neighboring cubes. We also consider the moat $\Delta C_i := C_i^+ \setminus C_i$. 

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We need further terminology. Given TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, and given $A \subset \mathbb{R}^d$, we say that $\mathcal{X}$ fully packs the region $A$ if every point in $A_+$ is adjacent to at least one point of $\mathcal{A}(\mathcal{X})$. For $t > 0$, we say $\mathcal{X}$ fully packs $A$ by time $t$ if $\mathcal{X} \cap (\mathbb{R}^d \times [0, t])$ fully packs $A$. Given $B \subseteq \mathbb{R}^d$, we say that a finite point configuration $\mathcal{X} \subset (B \cap C_i^+)_{+}$ is maximal or strongly saturates the cube $C_i$ in $B$ if for each TLF external configuration $\mathcal{Y} \subset (B \setminus C_i^+)_+$, $\mathcal{X} \cup \mathcal{Y}$ fully packs the region $B \cap C_i$ (the existence of maximal configurations is guaranteed by Lemmas 3.1 and 3.3 below).

We shall be interested in strong saturation of $C_i$ in $B$ when $B = \mathbb{R}^d$ or when $B = Q_\lambda$. The reason for our interest is this: If we knew that there was a constant $\tau < \infty$ such that $\mathcal{P} \cap (C_0^+ \times [0, \tau])$ strongly saturated $C_0$ in $\mathbb{R}^d$ a.s., then points in $\mathcal{P}$ with time marks exceeding $\tau$ would have no bearing on the packing status of points in $\mathcal{P} \cap (C_0)_+$. Thus, to check stabilization of $\xi$ at $0$ it would be enough to replace $\mathcal{P}$ by the Poisson point process $\mathcal{P} \cap (\mathbb{R}^d \times [0, \tau])$, and follow the stabilization arguments for packing with finite Poisson input (section four of [17]). While clearly no such constant $\tau$ exists, we shall show in Lemma 3.3 that a finite random $\tau$ exists.

We say that $\mathcal{X}$ locally strongly saturates $C_i$ if for each $\eta \subseteq \mathcal{X} \cap (\Delta C_i)_+$, the point set $(\mathcal{X} \cap (C_i)_+) \cup \eta$ fully packs $C_i$. The following lemma shows that local strong saturation implies strong saturation.

**Lemma 3.1** Suppose $\mathcal{X} \subset (C_i^+)_{+}$ is TLF and locally strongly saturates $C_i$. Then for any $B \subseteq \mathbb{R}^d$ with $C_i \subseteq B$, $\mathcal{X} \cap B$ strongly saturates $C_i$ in $B$.

*Proof.* Let $\mathcal{Y} \subset (B \setminus C_i^+)_+$ be TLF. Let $\eta := \mathcal{A}((\mathcal{X} \cap B_{+}) \cup \mathcal{Y}) \cap (\Delta C_i)_+$. We claim that

$$\mathcal{A}((\mathcal{X} \cap B_{+}) \cup \mathcal{Y}) \cap (C_i^+)_{+} = \mathcal{A}((\mathcal{X} \cap (C_i)_+) \cup \eta).$$

(3.2)

Indeed, considering each point of $(\mathcal{X} \cap (C_i)_+) \cup \eta$ in the usual temporal order, we see that the decision on whether to accept is the same for these points whether we are applying the usual packing rule to $(\mathcal{X} \cap B_{+}) \cup \mathcal{Y}$ or to $(\mathcal{X} \cap (C_i)_+) \cup \eta$.

Since we assume $\mathcal{X}$ locally strongly saturates $C_i$, $(\mathcal{X} \cap (C_i)_+) \cup \eta$ fully packs $C_i$, and so by (3.2), $(\mathcal{X} \cap B_{+}) \cup \mathcal{Y}$ fully packs $C_i$. \hfill $\Box$

We will use one more auxiliary lemma.

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Lemma 3.2 With probability 1, \( P \) has the property that for any \( \eta \subseteq P \cap (\Delta C_0)_+ \), there exists \( T < \infty \) such that the point set \((P \cap (C_0)_+) \cup \eta\) fully packs \( C_0 \) by time \( T \).

Proof. Suppose that for each rational hypercube \( Q \) contained in \( C_0 \), \( P \cap Q_+ \neq \emptyset \); this event has probability 1.

Take \( \eta \subset P \cap (\Delta C_0)_+ \). Let \( A := A((P \cap (C_0)_+) \cup \eta) \). Clearly \( A \) is finite. Let \( V \) be the set of \( x \in C_0 \) such that \((x, 0)\) does not lie adjacent to any point of \( A \). Then \( V \) is open in \( C_0 \) (because we assume \( S \) is closed) and if it is non-empty, it contains a rational cube contained in \( C_0 \) so that \( V_+ \) contains a point of \( P \cap (C_0)_+ \). But then this point should have been accepted so there is a contradiction. Hence \( V \) is empty and since \( A \) is finite this shows that \( C_0 \) is fully packed within a finite time. \( \square \)

For \( i \in \mathbb{Z}^d \), let \( T_i := T_i(P) \) denote the time till local strong saturation, defined to be the smallest \( t \in [0, \infty] \) such that \( C_i \) is locally strongly saturated by the point set \((P \cap (C_i)_+) \cap (\mathbb{R}^d \times [0, t]) \) (and set \( T_i = \infty \) if no such \( t \) exists). Clearly, \( T_i, i \in \mathbb{Z}^d \), are identically distributed random variables depending only on \( P \cap (C_i)_+ \). In particular, \((T_i, i \in \mathbb{Z}^d)\) forms a 2-dependent random field, meaning that \( T_i \) is independent of \((T_j, \|j - i\|_\infty > 2) \) for each \( i \in \mathbb{Z}^d \). We can now prove the key result that \( T_0 \) is almost surely finite.

Lemma 3.3 It is the case that \( P[T_0 = \infty] = 0 \).

Proof. Suppose that \( T_0 = \infty \). Then for each positive integer \( \tau \) there exists \( \eta_\tau \subseteq P \cap (\Delta C_0)_+ \) such that \((P \cap (C_0)_+) \cup \eta_\tau \) does not fully pack \( C_0 \) by time \( \tau \).

Assume \( P \cap (\Delta C_0)_+ \) is locally finite (this happens almost surely). Then \( P \cap (\Delta C_0 \times [0, 1]) \) is finite so that we can take a subsequence \( \tau' \to \infty \) of \( \tau \) along which \( \eta_{\tau'} \cap (\Delta C_0 \times [0, 1]) \) is the same for all \( \tau' \). Then we can take a further subsequence \( \tau'' \) of \( \tau' \) along which \( \eta_{\tau''} \cap (\Delta C_0 \times [0, 2]) \) is the same for all \( \tau'' \). Repeating this procedure and using Cantor’s diagonal argument, we can find a subsequence \( \tau_n \) tending to infinity, and a limit set \( \eta \subseteq (\Delta C_0 \times \mathbb{R}_+) \), such that for all \( k \), it is the case that

\[
\eta_{\tau_n} \cap (\Delta C_0 \times [0, k]) = \eta \cap (\Delta C_0 \times [0, k])
\]

(3.3) for all but finitely many \( n \).
Let $k > 0$, and choose $n$ to be large enough so that $\tau_n \geq k$ and such that (3.3) holds. Then the point set $(\mathcal{P} \cap (C_0)_+) \cup \eta_n$ does not yet fully pack $C_0$ by time $\tau_n$, and therefore $(\mathcal{P} \cap (C_0)_+) \cup \eta$ does not yet fully pack $C_0$ by time $k$.

Since $(\mathcal{P} \cap (C_0)_+) \cup \eta$ does not yet fully pack $C_0$ by time $k$ for any $k$, we are in the complement of the event described in Lemma 3.2. Thus by that result, the event $\{T_0 = \infty\}$ is contained in an event of probability zero, which completes the proof of Lemma 3.3.

Using Lemma 3.3, we can now prove that $\xi^*$ and $\xi'$, defined at the start of this section, satisfy the first part of exponential stabilization (exponential decay of the tail of $R'$).

**Lemma 3.4** There exists a positive constant $K_1$ such that for either $\xi = \xi^*$ or $\xi = \xi'$, there is a stabilization radius $R'$ as described in Definition 2.1, satisfying

$$P[R' > L] \leq K_1 \exp(-L/K_1), \quad \forall L > 0.$$ 

**Proof.** Let $\delta_1 > 0$ be a number falling below the critical probability $p_c$ for site percolation on $\mathbb{Z}^d$ with neighborhood relation $i = (i_1, \ldots, i_d) \sim j = (j_1, \ldots, j_d)$ if and only if $\|i - j\|_\infty \leq 1$, see Grimmett [8].

We will apply a domination by product measures result of [9], more precisely Theorem 0.0 in [9]. This tells us that, for a family of $\{0, 1\}$-valued random variables indexed by lattice vertices, if we are able to show that for each given site the probability of seeing 1 there conditioned on the configuration outside a fixed size neighborhood of the site exceeds certain large enough $p$, then this random field dominates a product measure with positive density $q$ which can be made arbitrarily close to 1 by appropriate choice of $p$. By this result, with $\delta_1$ as chosen above we can find $\delta_2 > 0$ such that any 2-dependent random field $(Y_i, i \in \mathbb{Z}^d)$ with $Y_i$ taking values in $\{0, 1\}$ and $P[Y_i = 1] \geq 1 - \delta_2$ for each $i$, this random field dominates the product measure with density $1 - \delta_1$.

Using Lemma 3.3, take $T^* > 0$ such that $P[T_0 > T^*] < \delta_2$. Then by the conclusion of the preceding paragraph, $\mathcal{P}$ can be coupled on a common probability space with an i.i.d. $\{0, 1\}$-valued random field $\pi_i, i \in \mathbb{Z}^d$, so that, for all $i \in \mathbb{Z}^d$,

- $P[\pi_i = 1] \geq 1 - \delta_1$, 

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- we have $T_i < T^*$ whenever $\pi_i = 1$.

Let us say that the cube $C_i$ is $T^*$-saturated if $T_i \leq T^*$. By Lemma 3.1, if $C_i$ is $T^*$-saturated then for any $B \subseteq \mathbb{R}^d$ with $C_i \subseteq B$, $\mathcal{P} \cap ([C_i^+ \cap B] \times [0, T^*])$ strongly saturates $C_i$ in $B$.

We declare a point $(x, t) \in \mathcal{P} \cap (C_i)_+$ to be causally relevant if either

- $\pi_i = 0$,
- or $\pi_i = 1$ and $t \leq T^*$.

Otherwise the point $x \in \mathcal{P} \cap (C_i)_+$ is declared causally irrelevant.

We now argue as follows, directly adapting the oriented percolation based technique introduced in section four of [17]. We convert the collection of points $\mathcal{P}$ (in $\mathbb{R}^d \times \mathbb{R}_+$) into a directed graph by providing a directed connection from $(y, s)$ to $(x, t)$ whenever $|y - x| \leq 2d_S$ and $s < t$ and, moreover, both $(x, t)$ and $(y, s)$ are causally relevant. By the causal cluster $\text{Cl}[(x, t); \mathcal{P}]$ of $(x, t) \in \mathcal{P}$ we understand the set of all causally relevant points $(y, s)$ of $\mathcal{P}$ such that there is a directed path from $(y, s)$ to $(x, t)$ (referred to as a causal chain for $(x, t)$ in the sequel). Necessarily the points in the causal cluster for $(x, t)$ have time mark at most $t$.

For each $(x, t) \in \mathcal{P}$ we define the causal cube cluster of $(x, t)$ in $\mathbb{R}^d$ by

$$
\text{Cl}[(x, t); \mathcal{P}] := \bigcup \{C_j^+ : (C_j)_+ \cap \text{Cl}[(x, t); \mathcal{P}] \neq \emptyset\}
$$

and for each $i \in \mathbb{Z}^d$ we define its causal cube cluster as the union of clusters given by

$$
\text{Cl}[i; \mathcal{P}] := \bigcup_{(x, t) \in \mathcal{P} \cap (C_i^+)_+} \text{Cl}[(x, t); \mathcal{P}]. \quad (3.4)
$$

The significance of causal cube clusters is as follows. First, we assert that the packing status of a given point $(x, t)$ is unaffected by changes to $\mathcal{P}$ outside $\text{Cl}[(x, t); \mathcal{P}]$. Indeed, viewing the directed connections as potential direct interactions between overlapping solids in the course of the sequential packing process, we can repeat the corresponding argument from Lemma 4.1 in [17], adding the extra observation that causally irrelevant points will not be accepted regardless of the outside packing configuration and hence do not have to be taken into account. Similarly, the packing status of the totality of points falling within distance $d_S$ of the
cube \( C \) can only be affected by the status of points falling in the causal cube cluster \( \bar{C}[i; \mathcal{P}] \). Consequently, we see that for either \( \xi = \xi^* \) or \( \xi = \xi' \), we can define a radius of stabilization by

\[
R' := \text{diam}(\bar{C}[0; \mathcal{P}]).
\]

We need to show that \( R' \) is almost surely finite with an exponentially decaying tail. Given \( L > 0 \), let \( E_1(L) \) be the event that there is a ‘path of zeros’ from some site \( i \in \{-1, 0, 1\}^d \) to the complement of \( B_{L/2-\sqrt{d}} \) in the Bernoulli random field \( (\pi_i, i \in \mathbb{Z}^d) \). More formally, \( E_1(L) \) is the event that there exists there is a sequence \( i_0, i_1, i_2, \ldots, i_n \), such that (a) \( i_0 \in \{-1, 0, 1\}^d \), and (b) \( i_n \not\in B_{L/2-2\sqrt{d}} \), and (c) for \( j = 1, \ldots, n \), \( i_j \in \mathbb{Z}^d \) and \( ||i_j - i_{j-1}||_{\infty} = 1 \) and \( \pi_{i_j} = 0 \).

For \( i \in \mathbb{Z}^d \), let \( E_2(L, i) \) be the event that there exists \( (x, t) \in \mathcal{P} \cap (C_i)_+ \), such that \( t \leq T^* \) and there exists a causal chain for \( (x, t) \) which starts at some point of \( \mathcal{P} \setminus (B_{L-2\sqrt{d}})_+ \). Define the event

\[
E_2(L) := \bigcup \{E_2(L, i) : i \in \mathbb{Z}^d, C_i \cap B_{L/2} \neq \emptyset \}.
\]

Then we assert that the event \( \{R' > L\} \) is contained in \( E_1(L) \cup E_2(L) \). Indeed, if \( E_2(L) \) does not occur, then for any causal chain for any \( (x, t) \in \mathcal{P} \cap (C_0^+)_+ \) starting outside \( (B_{L-2\sqrt{d}})_+ \), all points in the causal chain of \( (x, t) \) lying inside \( (B_{L/2})_+ \) must have time-coordinate greater than \( T^* \); if also \( E_1(L) \) does not occur, at least one of these points must lie in a cube which is \( T^* \)-saturated, and therefore be causally irrelevant, so in fact there is no causal chain for any \( (x, t) \in \mathcal{P} \cap (C_0^+)_+ \) starting outside \( (B_{L-2\sqrt{d}})_+ \). Hence, \( \bar{C}[0, \mathcal{P}] \subseteq B_L \), so that \( R' \leq L \).

By the choice of \( \delta_1 \) and by the exponential decay of the cluster size in the subcritical percolation regime (see e.g. Sections 5.2 and 6.3 in Grimmett [8]), we have exponential decay of \( P[E_1(L)] \). That is, there is a constant \( K_2 \) such that \( P[E_1(L)] \leq K_2 \exp(-L/K_2) \) for all \( L \).

Since \( T^* \) is fixed, we can use the methods of [17] for finite (Poisson) input packing, in particular the argument leading to Lemma 4.2 in [17], to see that there is a constant \( K_3 \) such that \( P[E_2(L, i)] \leq K_3 \exp(-L/K_3) \) for all \( i \in \mathbb{Z}^d \cap B_{L/2} \). Since the number of such \( i \) is only \( O(L^d) \), we see that \( P[E_2(L)] \) also decays exponentially in \( L \), and hence so does \( P[E_1(L)] + P[E_2(L)] \). Since the event \( \{R' > L\} \) is contained in \( E_1(L) \cup E_2(L) \), the lemma is proved. \( \square \)
To finish checking that $\xi^*$ and $\xi'$ satisfy the conditions for Theorem 2.1, we consider strong saturation, not only of unit cubes but of cubes of slightly less than unit size. Let $Q_\xi^+$ denote the cube $[-\zeta^{1/d},2\zeta^{1/d})^d$, i.e., the cube of side $3\zeta^{1/d}$ concentric with $Q_\zeta$. Let us say that $Q_\zeta$ is locally strongly saturated by a finite point set $X \subset (Q_\zeta^+ \cup \eta$ fully packs $Q_\zeta$.

**Lemma 3.5** Given $\delta > 0$, there exist constants $\varepsilon > 0$ and $t_0 < \infty$ such that for all $\zeta \in [1-\varepsilon,1]$, 

$$P[\mathcal{P} \cap (Q_\zeta^+ \times [0,t_0]) \text{ locally strongly saturates } Q_\zeta] > 1 - \delta. \quad (3.6)$$

*Proof.* By Lemma 3.3, we can choose $t_0$ such that $\mathcal{P} \cap (Q_1^+ \times [0,t_0])$ locally strongly saturates $Q_1$, with probability at least $1 - \delta/2$. Having chosen $t_0$ in this way, we can then choose $\varepsilon$, with $2d_S < (1-\varepsilon)^{1/d}$, so that for any $\zeta \in [1-\varepsilon,1]$, 

$$P[\mathcal{P} \cap ((Q_1 \setminus Q_\zeta) \times [0,t_0]) \neq \emptyset] < \delta/2.$$ 

For $\zeta < 1$ with $2d_S < \zeta^{1/d}$, if $\mathcal{P} \cap (Q_1^+ \times [0,t_0])$ strongly saturates $Q_1$, and $\mathcal{P} \cap ((Q_1 \setminus Q_\zeta) \times [0,t_0])$ is empty, then $\mathcal{P} \cap (Q_\zeta^+ \times [0,t_0])$ strongly saturates $Q_\zeta$. Hence, the preceding probability estimates complete the proof. \hfill $\Box$

**Lemma 3.6** There exists a positive constant $K_4$ such that for either $\xi = \xi^*$ or $\xi = \xi'$, there is a family of stabilization radii $R(i,\lambda) = R^\xi(i,\lambda)$, defined for $\lambda \geq 1$ and $i \in \mathbb{Z}^d$ as described in Definition 2.1, which satisfy

$$\sup_{\lambda \geq 1, i \in \mathbb{Z}^d} P[R(i,\lambda) > L] \leq K_4 \exp(-L/K_4). \quad (3.7)$$

*Proof.* First let us restrict attention to $\lambda$ with $\lambda^{1/d} \in \mathbb{N}$. Adapting notation from the preceding proof, for $(x,t) \in \mathcal{P} \cap (Q_\lambda)_+$ we let $\text{Cl}[(x,t);\mathcal{P} \cap (Q_\lambda)_+]$ denote the set of all causally relevant points $(y,s)$ of $\mathcal{P} \cap (Q_\lambda)_+$ such that there is a directed path from $(y,s)$ to $(x,t)$, with all points in the path lying inside $(Q_\lambda)_+$. Then define the causal cube cluster in $Q_\lambda$ for $(x,t)$ by

$$\text{Cl}[(x,t);\mathcal{P} \cap (Q_\lambda)_+] := \bigcup C_j^+ \cap Q_\lambda : (C_j)_+ \cap \text{Cl}[(x,t);\mathcal{P} \cap (Q_\lambda)_+] \neq \emptyset$$

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and for \( i \in \mathbb{Z}^d \) by

\[
\bar{C}\text{l}[i; \mathcal{P} \cap (Q_\lambda)_+] = \bigcup_{(x,t) \in \mathcal{P} \cap (Q_\lambda \cap C_i)_+} \bar{C}\text{l}[(x,t); \mathcal{P} \cap (Q_\lambda)_+].
\]

Define

\[
R(i, \lambda) := \text{diam}(\bar{C}\text{l}[i; \mathcal{P} \cap (Q_\lambda)_+]), \quad \lambda^{1/d} \in \mathbb{N}. \tag{3.8}
\]

Then for \( i \in \mathbb{Z}^d \), the packing statuses of points of \( \mathcal{P} \cap (C_i^+ \cap Q_\lambda)_+ \) are unaffected by changes to \( \mathcal{P} \cap (Q_\lambda)_+ \) in the region \( (Q_\lambda \setminus B_{R(i,\lambda)}(i))_+ \), by the same argument as in the preceding proof. Here we are using the fact that \( \lambda^{1/d} \in \mathbb{Z} \), and that if \( C_i \subset Q_\lambda \) is \( T^* \)-saturated then \( C_i \) is strongly saturated in \( Q_\lambda \) by \( \mathcal{P} \cap (Q_\lambda \times [0,T^*]) \) (Lemma 3.1). Thus, \( R(i, \lambda) \) serves as a radius of stabilization in the sense of Definition 2.1 (for either \( \xi^* \) or \( \xi' \)). Moreover, \( \bar{C}\text{l}[i; \mathcal{P} \cap (Q_\lambda)_+] \subseteq \bar{C}\text{l}[i; \mathcal{P}] \), and so with \( K_1 \) as in the the preceding proof we have \( P[R(i, \lambda) > L] \leq K_1 \exp(-L/K_1) \), uniformly over \( i, \lambda \) with \( \lambda^{1/d} \in \mathbb{N} \).

Now suppose \( \lambda^{1/d} \notin \mathbb{N} \). In this case, instead of dividing \( Q_\lambda \) into cubes of side 1, some of which would not fit exactly, we divide \( Q_\lambda \) into cubes of side slightly less than 1, which do fit exactly, and repeat the above argument.

More precisely, we modify the proof of Lemma 3.4. With \( \delta_2 \) as in that proof, we use Lemma 3.5 to choose constants \( \varepsilon > 0 \) and \( T^* < \infty \) (with \( \max(2d_S, 1/2) < (1 - \varepsilon)^{1/d} \)) in such a way that for any \( \zeta \in [1 - \varepsilon, 1] \) we have

\[
P[\mathcal{P} \cap (Q_\zeta^+ \times [0,T^*])] \text{ locally strongly saturates } Q_\zeta > 1 - \delta_2.
\]

With \( \varepsilon \) thus fixed, for all large enough \( \lambda \) we can choose \( \zeta = \zeta(\lambda) \in [1 - \varepsilon, 1] \) in such a way that \( \lambda^{1/d}/\zeta^{1/d} \) is an integer. Partitioning \( \mathbb{R}^d \) into cubes \( C_i' \) of volume \( \zeta \), we can then follow the argument already given for the case \( \lambda^{1/d} \in \mathbb{N} \), using the fact that the each of the unit cubes \( i + Q_1 \), for which we need to check conditions in Theorem 2.1, is contained in the union of at most \( 2^d \) cubes in the partition \( \{C_j'\} \).

\[ \square \]

### 4 Jamming variability, variance asymptotics

At the end of this section, we complete the proofs of Theorems 1.1 and 1.2. First, we need to show that the limiting variance \( \sigma^2(S,d) \) is non-zero for all \( d \) and all
$S$. This is achieved by Proposition 4.1 and Lemma 4.1 below. The first of these results establishes that any convex $S \subseteq \mathbb{R}^d$ with nonempty interior satisfies jamming variability (as defined in remark 6, Section 1), and the second establishes that this is sufficient to guarantee that $\sigma^2(S, d) > 0$. Recall from (3.1) that we can work just as well with $\mathcal{N}_\nu^\ast$ as with $\mathcal{N}_\lambda$.

**Proposition 4.1** The convex body $S$ has jamming variability.

**Proof.** Given $S$, for all $x \in \mathbb{R}^d$ define

$$\|x\| := \sup \{a \geq 0 : (x + aS) \cap aS = \emptyset\}.$$  

It is straightforward to verify that $\| \cdot \|$ is a norm on $\mathbb{R}^d$, using the convexity of $S$ to verify the triangle inequality. For nonempty $A \subset \mathbb{R}^d$, and $x \in \mathbb{R}^d$, write $D(x, A)$ for $\inf\{\|x - y\| : y \in A\}$. By our earlier assumption that $2dS < 1$ we have $\|x\| < \|x\|_\infty$ for all $x \in \mathbb{R}^d$.

For $L \subset \mathbb{R}^d$, we shall say $L$ is **packed** if $\|x - y\| \geq 1$ for all $x \in L$, $y \in L$, and that $L$ is **maximally packed** if it is packed and

$$D(w, L) < 1, \quad \forall w \in \mathbb{R}^d.$$

(4.1)

We shall say $L$ is a **periodic set** if for all $x \in L$ and $z \in \mathbb{Z}^d$ we have $x + z \in L$.

Let $L$ be a maximally packed periodic subset of $\mathbb{R}^d$ (it is not hard to see that such an $L$ exists). Then the function $x \mapsto D(x, L)$ is a continuous function on $\mathbb{R}^d$ that is periodic (i.e., $D(x, L) = D(x + z, L)$ for all $x \in \mathbb{R}^d, z \in \mathbb{Z}^d$). Hence the range of this function is the continuous image of the compact torus $\mathbb{R}^d/\mathbb{Z}^d$, and so is compact. Hence by (4.1) we have

$$\beta := \sup\{D(w, L) : w \in \mathbb{R}^d\} < 1.$$  

Then for $x \in \mathbb{R}^d$ and $\alpha > 0$, by scaling

$$D(x, \alpha L) = \alpha D(\alpha^{-1}x, L) \leq \alpha \beta.$$  

(4.2)

Choose $\delta > 0$ such that $\beta(1 + 6\delta) < 1 - 2\delta$. For $i = 1, 2$, let $L_i := (1 + 3i\delta)L$. By (4.2) and the choice of $\delta$ we have for all $x \in \mathbb{R}^d$ and $i = 1, 2$ that

$$D(x, L_i) < 1 - 2\delta.$$  

(4.3)
Let $c_1$ denote the number of points of $L$ in $[0, 1]^d$. Denote by $\text{Box}(L)$ the hypercube $[-L/2, L/2]^d$. For $i = 1, 2$, let $n_i(L)$ denote the number of points of $L_i$ in $\text{Box}(L - 4)$. Then as $L \to \infty$, for $i = 1, 2$ we have
\[
n_i(L) \sim c_1(1 + 3\delta_i)^{-d}L^d. \tag{4.4}
\]
Let $n_3(L)$ denote the maximum integer $m$ such that there exists a packed subset of $\text{Box}(L) \setminus \text{Box}(L - 6)$ with $m$ elements. Then there is a finite constant $c_2$ such that for all $L \geq 6$ we have
\[
n_3(L) \leq c_2L^{d-1}. \tag{4.5}
\]
By (4.4) and (4.5), we can choose $L_0$ such that for $L \geq L_0$ we have
\[
n_3(L) < n_1(L) - n_2(L). \tag{4.6}
\]
For $x \in \mathbb{R}^d$ and $r > 0$, set $\tilde{B}_r(x) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$ (a ball of radius $r$ using the norm $\| \cdot \|$). For bounded $A \subset \mathbb{R}^d$, let $T(A)$ denote the time of the first Poisson arrival in $A$, i.e set
\[
T(A) := \inf\{t : \mathcal{P} \cap (A \times \{t\}) \neq \emptyset\},
\]
with the convention that the infimum of the empty set is $\infty$. Fix $L \geq L_0$, and for $i = 1, 2$ define the event $E_i$ by
\[
E_i := \left\{ \max\{T(\tilde{B}_\delta(x)) : x \in L_i \cap \text{Box}(L - 4)\} < T\left(\text{Box}(L) \setminus \bigcup_{x \in L_i \cap \text{Box}(L - 4)} \tilde{B}_\delta(x)\right) \right\}.
\]
Let $i = 1$ or $i = 2$. If $y, y'$ are distinct points of $L_i$ then $\|y - y'\| \geq 1 + 3\delta$. Hence, if also $w \in \tilde{B}_\delta(y)$ and $w' \in \tilde{B}_\delta(y')$, then $\|w - w'\| \geq 1 + \delta$ by the triangle inequality. Moreover, for $x \in \mathbb{R}^d$, by (4.3) and the triangle inequality we can find $y = y(x) \in L_i$ such that $\|x - w\| \leq 1 - \delta$ for all $w \in \tilde{B}_\delta(y)$. Hence, if $E_i$ occurs then the set of accepted points (i.e., centroids of accepted shapes) of the infinite input packing process on $\text{Box}(L)$ induced by $\mathcal{P}$ with arbitrary external pre-packed configuration $\eta$ in $\mathbb{R}^d \setminus \text{Box}(L)$, includes one point from each $\tilde{B}_\delta(x), x \in L_i \cap \text{Box}(L - 4)$, and also contains no other points from $\text{Box}(L - 6)$.

Thus for any pre-packed configuration $\eta$ in $\mathbb{R}^d \setminus \text{Box}(L)$, if $E_1$ occurs the number of accepted points in $\text{Box}(L)$ is at least $n_1(L)$, and if $E_2$ occurs the number of accepted points is at most $n_2(L) + n_3(L)$. Also, the probabilities $P[E_1]$ and $P[E_2]$
are strictly positive and do not depend on \( \eta \). By (4.6), it follows that there is a constant \( \varepsilon > 0 \), independent of \( \eta \), such that \( \text{Var}[N^\varepsilon_{\lambda}(\text{Box}(L))|\eta] \geq \varepsilon \). Thus we have established the required jamming variability. \( \square \)

**Lemma 4.1** It is the case that \( \lim_{\lambda \to \infty} \lambda^{-1} \text{Var}[N^\varepsilon_{\lambda}] > 0 \).

**Proof.** By Proposition 4.1, there exists \( L > 0 \) such that \( \inf_{\eta} \text{Var}[N[[0,L]^d|\eta] > 0 \), where the infimum is over all admissible \( \eta \subset \mathbb{R}^d \setminus [0,L]^d \). We consider \( \lambda \) with \( \lambda^{1/d}/(L+4) \in \mathbb{N} \). We subdivide the cube \( Q_\lambda \) into \( n(\lambda) := \lambda/(L+4)^d \) equal-sized sub-cubes \( \tilde{C}_{1,\lambda}, \tilde{C}_{2,\lambda}, \ldots, \tilde{C}_{n(\lambda),\lambda} \) arising as translates of \( \text{Box}(L+4) \) centered at \( x_{1,\lambda}, \ldots, x_{n(\lambda),\lambda} \) respectively. For \( 1 \leq i \leq n(\lambda) \), let \( \tilde{C}_{i,\lambda}^- \) be the translate of \( \text{Box}(L) \) centered at \( x_{i,\lambda} \), and let \( M_{i,\lambda} \) be the translate of \( \text{Box}(L+2) \setminus \text{Box}(L) \) centered at \( x_{i,\lambda} \) (a ‘moat’ around \( \tilde{C}_{i,\lambda}^- \)).

Using terminology from Section 3, let \( F_{i,\lambda} \) be the event that the point set \( \mathcal{P} \cap (M_{i,\lambda}) \) fully packs \( M_{i,\lambda} \) by time 1, and let \( G_{i,\lambda} \) be the event that \( \mathcal{P} \cap ((\tilde{C}_{i,\lambda} \setminus M_{i,\lambda}) \times [0,1]) \) is empty. Let \( E_{i,\lambda} := F_{i,\lambda} \cap G_{i,\lambda} \). Then \( p := P[E_{i,\lambda}] \) satisfies \( p > 0 \), and does not depend on \( i \) or \( \lambda \).

Observing that the events \( E_{i,\lambda}, 1 \leq i \leq n(\lambda) \), are independent (the cubes \( \tilde{C}_i \) are disjoint), denote the (random) set of indices for which \( E_{i,\lambda} \) occurs by \( I(\lambda) := \{i_1, \ldots, i_{K(\lambda)}\} \). Then \( \mathbb{E}[K(\lambda)] = pn(\lambda) \). Conditional on the event \( E_{i,\lambda} \), the packing process inside \( \tilde{C}_{i,\lambda}^- \) has a particularly simple form - before time 1 there are no points in \( \tilde{C}_{i,\lambda}^- \), and after that time the newly arriving solids centered in \( \tilde{C}_{i,\lambda}^- \) undergo the packing process according to the usual rules with the additional restriction that a solid overlapping another one packed in \( M_{i,\lambda} \) before time 1 is rejected. Note that for \( i \in I(\lambda) \), no new solids are accepted in \( M_{i,\lambda} \) after time 1 and, moreover, the acceptance times of solids accepted in \( M_{i,\lambda} \) before time 1 have no influence on the behavior of the packing process in \( \tilde{C}_{i,\lambda}^- \) after time 1; only their spatial locations matter. For a configuration \( \eta \) of accepted points (only spatial locations taken into account) in \( M_{i,\lambda} \), the process described above will be referred to as packing in \( \tilde{C}_{i,\lambda}^- \) in the presence of the pre-packed configuration \( \eta \).

Let \( \mathcal{M}_\lambda \) be the sigma-algebra generated by the points of \( \mathcal{P} \cap (Q_\lambda \times [0,1]) \), i.e. the Poisson arrivals up to time 1. Event \( E_{i,\lambda} \) is \( \mathcal{M}_\lambda \)-measurable, for each \( i \).

By the conditional variance formula we have

\[
\text{Var}[N^\varepsilon_{\lambda}] = \mathbb{E}[\text{Var}(N^\varepsilon_{\lambda}|\mathcal{M}_\lambda)] + \text{Var}[\mathbb{E}(N^\varepsilon_{\lambda}|\mathcal{M}_\lambda)]
\]
\[ \geq \mathbb{E} \left[ \text{Var} \left( N_{\lambda}^{\xi^*} \mid \mathcal{M}_\lambda \right) \right] \]

\[ = \mathbb{E} \text{Var} \left[ \sum_{i \in I(\lambda)} N_{\lambda}^{\xi^*} [\bar{C}_{i,\lambda}] + \left( N_{\lambda}^{\xi^*} - \sum_{i \in I(\lambda)} N_{\lambda}^{\xi^*} [\bar{C}_{i,\lambda}] \right) \mid \mathcal{M}_\lambda \right], \]

where we set \( N_{\lambda}^{\xi^*} [\bar{C}_{k,\lambda}] := \xi^{(P \cap Q_{\lambda}, \bar{C}_{k,\lambda})}, \) the number of solids packed in \( \bar{C}_{k,\lambda}. \) Conditionally on \( \mathcal{M}_\lambda, \) the packing processes after time 1 over different sub-cubes \( \bar{C}_{i,\lambda}, i \in I(\lambda), \) are independent of each other and of the packing process after time 1 in \( Q_{\lambda} \setminus \cup_{i \in I(\lambda)} \bar{C}_{i,\lambda}. \) Hence,

\[ \text{Var} \left[ N_{\lambda}^{\xi^*} \right] \geq \mathbb{E} \sum_{i \in I(\lambda)} \left[ \text{Var} [N_{\lambda}^{\xi^*} [\bar{C}_{i,\lambda}] \mid \mathcal{M}_\lambda] \right] \geq \mathbb{E} [K] \inf_\eta \text{Var} N[[0, L]^d \mid \eta], \]

where the infimum is taken over all admissible configurations \( \eta \) outside \([0, L]^d,\) and where \( N[[0, L]^d \mid \eta] \) stands for number of solids packed in \([0, L]^d\) in the presence of the pre-packed configuration \( \eta. \) By Proposition 4.1, this infimum is strictly positive, and Lemma 4.1 follows.

### Proof of Theorems 1.1 and 1.2.

Let \( \xi \) be \( \xi^* \) as defined in Section 3. Then Lemmas 3.4 and 3.6 show that \( \xi = \xi^* \) satisfies the exponential stabilization conditions in Theorem 2.1, so it satisfies the conclusions (2.6), (2.7) and (2.8) of that result. The conclusion (2.8) gives us (1.1) of Theorem 1.1. Also, by putting \( f \equiv g \equiv 1 \) on \( Q_{\lambda}^+ \) and using (2.7), we obtain the variance convergence \( \lambda^{-1} \text{Var} N_{\lambda} \to \sigma^2 \) asserted in Theorem 1.1. By Lemma 4.1, we may therefore deduce that \( \sigma^2 > 0. \) Hence we may apply the last part of Theorem 2.1 to obtain the rest of the conclusions in Theorem 1.2 as they pertain to \( \nu_{\lambda}; \) also the conclusion (2.9) of Theorem 2.1 gives us (1.2).

To get the same results for \( \nu', \) we argue similarly with \( \xi = \xi'. \) We need to check that the limiting means and variances are the same, i.e. \( \mu(\xi') = \mu(\xi^*) \) and \( \sigma^2(\xi') = \sigma^2(\xi^*). \) To see this, note that if \( f \equiv 1 \) on \( Q_{\lambda}^+ \), then \( \langle f, \mu_{\lambda}\rangle = \langle f, \mu_{\lambda}^{\xi^*}\rangle \) so application of (2.6) to this choice of \( f \) yields

\[ \mu(\xi') = \lim_{\lambda \to \infty} \lambda^{-1} \mathbb{E} [\langle f, \mu_{\lambda}\rangle] = \lim_{\lambda \to \infty} \lambda^{-1} \mathbb{E} [\langle f, \mu_{\lambda}^{\xi^*}\rangle] = \mu(\xi^*) \]

and a similar argument using (2.7) shows that \( \sigma^2(\xi') = \sigma^2(\xi^*). \)

\[ \Box \]
References

[1] Z. Adamczyk, B. Siwek, M. Zembala, P. Belouschek (1994), Kinetics of localized adsorption of colloid particles, *Adv. in Colloid and Interface Sci.* **48**, 151-280.

[2] Yu. Baryshnikov and J. E. Yukich (2005), Gaussian limits for random measures in geometric probability, *Annals Appl. Prob.*, **15**, 1A, 213-253, Electronically available via [http://www.lehigh.edu/~jey0/publications.html](http://www.lehigh.edu/~jey0/publications.html).

[3] M. C. Bartelt and V. Privman (1991), Kinetics of irreversible monolayer and multilayer sequential adsorption, *Internat. J. Mod. Phys. B*, **5**, 2883-2907.

[4] E. G. Coffman, L. Flatto, P. Jelenković (2000), Interval packing: the vacant interval distribution *Annals of Appl. Prob.*, **10**, 240-257.

[5] E. G. Coffman, L. Flatto, P. Jelenković, and B. Poonen (1998), Packing random intervals on-line, *Algorithmica*, **22**, 448-476.

[6] A. Dvoretzky and H. Robbins (1964), On the “parking” problem, *MTA Mat Kut. Int. Kőzl.*, (Publications of the Math. Res. Inst. of the Hungarian Academy of Sciences), **9**, 209-225.

[7] J. W. Evans (1993), Random and cooperative adsorption, *Reviews of Modern Physics*, **65**, 1281-1329.

[8] G. Grimmett (1999), Percolation, Grundlehren der mathematischen Wissenschaften 321, Springer.

[9] T.M. Liggett, R.H. Schonmann, A.M. Stacey (1997), Domination by product measures, *Ann. Probab.* **25**, 71-95.

[10] J. K. Mackenzie (1962) Sequential filling of a line by intervals placed at random and its application to linear adsorption, *J. Chem. Phys.*, **37**, 4, 723-728.

[11] M.D. Penrose (2001), Random parking, sequential adsorption, and the jamming limit, *Communications in Mathematical Physics* **218**, 153-176.
[12] M. D. Penrose (2001), Limit theorems for monolayer ballistic deposition in the continuum, *J. Stat. Phys.*, **105**, 561-583.

[13] M.D. Penrose (2005), Multivariate spatial central limit theorems with applications to percolation and spatial graphs, *Ann. Prob.*, **33**, 1945-1991.

[14] M.D. Penrose (2005), Laws of Large numbers for random measures in geometric probability. Preprint.

[15] M.D. Penrose (2005), Gaussian limits for random geometric measures. Preprint.

[16] M.D. Penrose and J.E. Yukich (2001), Central limit theorems for some graphs in computational geometry, *Ann. Appl. Probab.*, **11**, 1005-1041.

[17] M.D. Penrose and J.E. Yukich (2002), Limit theory for random sequential packing and deposition, *Ann. Appl. Probab.*, **12**, 272-301.

[18] M.D. Penrose and J.E. Yukich (2003), Weak laws of large numbers in geometric probability, *Ann. Appl. Probab.*, **13**, pp. 277-303.

[19] M.D. Penrose and J.E. Yukich (2005), Normal approximation in geometric probability, in Stein’s Method and Applications, Lecture Note Series, Institute for Mathematical Sciences, National University of Singapore, **5**, A. D. Barbour and Louis H. Y. Chen, Eds., 37-58. Electronically available from ArXiv as math.PR/0409088.

[20] V. Privman (2000), Adhesion of Submicron Particles on Solid Surfaces, A Special Issue of Colloids and Surfaces A, **165**, edited by V. Privman.

[21] A. Rényi (1958), On a one-dimensional random space-filling problem, *MTA Mat Kut. Int. Kől.*, (Publications of the Math. Res. Inst. of the Hungarian Academy of Sciences) **3**, 109-127.

[22] J. Quintanilla and S. Torquato (1997), Local volume fluctuations in random media, *J. Chem. Phys.*, **106**, 2741-2751.

[23] T. Schreiber and J. E. Yukich (2005), Large deviations for functionals of spatial point processes with applications to random packing and spatial graphs, *Stochastic Processes and Their Applications*, **115**, 1332-1356.
[24] J. Talbot, G. Tarjus, P. R. Van Tassel, and P. Viot (2000), From car parking to protein adsorption: an overview of sequential adsorption processes, *Colloids and Surfaces A*, **165**, 287-324.

[25] S. Torquato (2002), Random Heterogeneous Materials, Springer Interdisciplinary Applied Mathematics, Springer-Verlag, New York.

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