An Endpoint Estimate for the Commutators of Singular Integral Operators with Rough Kernels

Guoen Hu1 · Xiangxing Tao2

Received: 22 November 2020 / Accepted: 16 June 2021 / Published online: 13 August 2021
© The Author(s), under exclusive licence to Springer Nature B.V. 2021

Abstract

Let $\Omega$ be homogeneous of degree zero and have mean value zero on the unit sphere $S^{d-1}$, $T_\Omega$ be the homogeneous singular integral operator with kernel $\frac{\Omega(x)}{|x|^d}$ and $T_\Omega, b$ be the commutator of $T_\Omega$ with symbol $b$. In this paper, we prove that if $\Omega \in L(\log L)^2(S^{d-1})$, then for $b \in \text{BMO}(\mathbb{R}^d)$, $T_\Omega, b$ satisfies an endpoint estimate of $L \log L$ type.

Keywords Rough singular integral operator · Commutator · Weak type endpoint estimate

Mathematics Subject Classification (2010) 42B20

1 Introduction

In this paper, we will work on $\mathbb{R}^d$, $d \geq 2$. Let $T$ be a linear operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ and $b \in L^1_{\text{loc}}(\mathbb{R}^d)$. The commutator of $T$ with symbol $b$, is defined by

$$T_b f(x) = b(x) T f(x) - T(b f)(x).$$

A celebrated result of Coifman, Rochberg and Weiss [5] states that if $T$ is a Calderón-Zygmund operator, then $T_b$ is bounded on $L^p(\mathbb{R}^d)$ for every $p \in (1, \infty)$ and also a converse result in terms of the Riesz transforms. Pérez [18] considered the weak type endpoint estimate for the commutator of Calderón-Zygmund operator, and proved the following result.
Theorem 1.1 Let $T$ be a Calderón-Zygmund operator and $b \in \text{BMO}(\mathbb{R}^d)$. Then for any $\lambda > 0$,

$$|\{x \in \mathbb{R}^d : |T_b f(x)| > \lambda\}| \lesssim \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right)dx,$$

where and in the following, $\Phi(t) = t \log(e + t)$.

Let $\Omega$ be homogeneous of degree zero, integrable and have mean value zero on the unit sphere $S^{d-1}$. Define the singular integral operator $T_\Omega$ by

$$T_\Omega f(x) = \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(y')}{|y'|^d} f(x - y)dy,$$

where and in the following, $y' = y/|y|$ for $y \in \mathbb{R}^d$. This operator was introduced by Calderón and Zygmund [2], and has been proved to be bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, under various assumptions on the homogeneous function $\Omega$. For instance, Calderón and Zygmund [3] proved that if $\Omega \in L \log L(S^{d-1})$, then $T_\Omega$ is bounded on $L^p(\mathbb{R}^d)$ for $p \in (1, \infty)$. Ricci and Weiss [20] improved the result of Calderón-Zygmund, and showed that $\Omega \in H^1(S^{d-1})$ guarantees the $L^p(\mathbb{R}^d)$ boundedness for $p \in (1, \infty)$. Seeger [21] showed that $\Omega \in L \log L(S^{d-1})$ is a sufficient condition such that $T_\Omega$ is bounded from $L^1(\mathbb{R}^d)$ to $L^{1, \infty}(\mathbb{R}^d)$. For other works about the mapping properties of $T_\Omega$, we refer to the papers [4, 7, 8, 14, 20] and the references therein.

We now consider the commutator of $T_\Omega$ with symbol in BMO$(\mathbb{R}^d)$. Let $p \in [1, \infty)$ and $w$ be a nonnegative, locally integrable function on $\mathbb{R}^d$. We say that $w \in A_p(\mathbb{R}^d)$ if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx\right) \left(\frac{1}{|Q|} \int_Q w^{1 - p'}(x)dx\right)^{1 - p'} < \infty, \quad p \in (1, \infty),$$

the supremum is taken over all cubes in $\mathbb{R}^d$, $p' = p/(p - 1)$, and $w \in A_1(\mathbb{R}^d)$ if

$$\text{ess sup}_{x \in \mathbb{R}^d} \frac{Mw(x)}{w(x)} < \infty,$$

see [9, Chapter 9] for the properties of $A_p(\mathbb{R}^d)$. By the result of Duandikoetxea and Rubio de Francia [8] (see also [7]), we know that if $\Omega \in L^q(S^{d-1})$ for some $q \in (1, \infty)$, then for $p \in (q', \infty)$ and $w \in A_{p/q'}(\mathbb{R}^d)$

$$\|T_\Omega f\|_{L^p(\mathbb{R}^d, w)} \lesssim_{d, p, w} \|f\|_{L^p(\mathbb{R}^d, w)}.$$

This, together with [1, Theorem 2.13], tells us that if $\Omega \in L^q(S^{d-1})$ for $q \in (1, \infty)$, then for $b \in \text{BMO}(\mathbb{R}^d)$,

$$\|T_\Omega, b f\|_{L^p(\mathbb{R}^d, w)} \lesssim_{d, p, w} \|b\|_{\text{BMO}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d, w)}, \quad p \in (q', \infty), \quad w \in A_{p/q'}(\mathbb{R}^d).$$

Hu [10] proved that if $\Omega \in L(\log L)^2(S^{d-1})$, then $T_\Omega, b$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$, see also [11] for the $L^p(\mathbb{R}^d)$ boundedness of $T_\Omega, b$ when $\Omega$ satisfies another minimum size condition.

The weak type endpoint estimates of $T_\Omega, b$ are of interest. By Theorem 1.1, we know that if $\Omega \in \text{Lip}_\alpha(S^{d-1})$ with $\alpha \in (0, 1]$ and $b \in \text{BMO}(\mathbb{R}^d)$, then for any $\lambda > 0$,

$$|\{x \in \mathbb{R}^d : |T_\Omega, b f(x)| > \lambda\}| \lesssim \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right)dx.$$

Recently, Lan, Tao and Hu [15] established the weak type endpoint estimates for $T_\Omega, b$ when $\Omega$ satisfies only size condition. They proved that
Theorem 1.2 Let $\Omega$ be homogeneous of degree zero and have mean value zero on $S^{d-1}$, $b \in \text{BMO}(\mathbb{R}^d)$. Suppose that $\Omega \in L^q(S^{d-1})$ for some $q \in (1, \infty]$, then for any $\lambda > 0$ and weight $w$ such that $w^\varphi \in A_1(\mathbb{R}^d)$,

$$w \left( \{ x \in \mathbb{R}^d : |T_{\Omega, b} f(x)| > \lambda \} \right) \lesssim_{d, w} \int_{\mathbb{R}^d} \Phi \left( \frac{D |f(x)|}{\lambda} \right) w(x) dx,$$

with $D = \|\Omega\|_{L^q(S^{d-1})} \|b\|_{\text{BMO}(\mathbb{R}^d)}$.

The purpose of this paper is to give a weak type endpoint estimate of $T_{\Omega, b}$ when $\Omega$ satisfies certain minimum size condition. For a function $\Omega$ on $S^{d-1}$ and $\kappa \geq 0$, we say that $\Omega \in L(\log L)^{\kappa}(S^{d-1})$, if

$$\| \Omega \|_{L(\log L)^{\kappa}(S^{d-1})} := \int_{S^{d-1}} |\Omega(x')| \log(e + |\Omega(x')|) dx' < \infty.$$

Our main result can be stated as follows.

Theorem 1.3 Let $\Omega$ be homogeneous of degree zero, have mean value zero on $S^{d-1}$ and $\Omega \in L(\log L)^{2}(S^{d-1})$, $T_{\Omega}$ be the operator defined by (1.1). Then for $b \in \text{BMO}(\mathbb{R}^d)$ and $\lambda > 0$,

$$\{ x \in \mathbb{R}^d : |T_{\Omega, b} f(x)| > \lambda \} \lesssim \int_{\mathbb{R}^d} \Phi \left( \frac{|f(x)|}{\lambda} \right) dx.$$ (1.3)

Remark 1.4 For $r \in (1, \infty)$, let $\mathcal{M}_{r, T_{\Omega}}$ be the maximal operator defined by

$$\mathcal{M}_{r, T_{\Omega}} f(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |T_{\Omega}(f 1_Q)(\xi)|^r d\xi \right)^{1/r},$$ (1.4)

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ containing $x$. This operator was introduced by Lerner [14], who proved that for any $r \in (1, \infty),

$$\| \mathcal{M}_{r, T_{\Omega}} f \|_{L^{1, \infty}(\mathbb{R}^d)} \lesssim r \| \Omega \|_{L^\infty(S^{d-1})} \| f \|_{L^1(\mathbb{R}^d)},$$ (1.5)

see [14, Lemma 3.3]. The crucial estimate in the proof of Theorem 1.2 is

$$\| \mathcal{M}_{r, T_{\Omega}} f \|_{L^{1, \infty}(\mathbb{R}^d)} \lesssim r \| f \|_{L^1(\mathbb{R}^d)},$$ (1.6)

when $\Omega \in L^q(S^{d-1})$ for some $q > 1$. However, the estimate (1.6) does not hold and the argument used in [15] does not apply when $\Omega \in L(\log L)^{2}(S^{d-1})$. In fact, as in the proof of Theorem 1.2 in [15], the estimate (1.6) implies the $L^p(\mathbb{R}^d, w)$ boundedness of $T_{\Omega}$ for large $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^d)$ for some $s \geq 1$, which is impossible when $\Omega \in L(\log L)^{2}(S^{d-1})$ (see [16, Theorem 1]). To prove Theorem 1.3, we will employ some ideas and estimates of Ding and Lai [6] (see also Seeger [21]). However, the estimate for $T_{\Omega, b} h$ is much more complicated and more refined than the estimate of $T_{\Omega} h$ in [6, 21], here $h$ is the bad part in the Calderón-Zygmund decomposition of function $f$. Some computations of Luxmberg norms, interpolation between Orlicz spaces, an observation of Hytönen and Pérez [12] and the interpolation with changes of measures, are involved in the estimate $T_{\Omega, b} h$; see Lemma 2.1, Lemma 4.1 and Lemma 5.2 for details.

Remark 1.5 Let $\widetilde{T}_{\Omega}$ be the operator defined by

$$\widetilde{T}_{\Omega} f(x) = \text{p.v.} \int_{\mathbb{R}^d} \Omega(x - y) K(x, y) f(y) dy.$$ (1.7)
Suppose that $\tilde{T}_\Omega$ is bounded on $L^2(\mathbb{R}^d)$. For $b \in \text{BMO}(\mathbb{R}^d)$, define the commutator of $\tilde{T}_\Omega$ by
\[
\tilde{T}_\Omega, b f (x) = b(x)\tilde{T}_\Omega f (x) - \tilde{T}_\Omega (bf) (x)
\]initially for $f \in \mathcal{S}(\mathbb{R}^d)$. Mimicking the proof of Theorem 1.3, we can prove the following result.

**Theorem 1.6** Let $\Omega$ be homogeneous of degree zero, have mean value zero on $S^{d-1}$ and $\Omega \in L(\log L)^2(S^{d-1})$, $\tilde{T}_\Omega$ be the operator defined by (1.7) and $\tilde{T}_\Omega, b$ be the commutator defined by (1.8). Suppose that $T_\Omega$ and $T_{\Omega, b}$ are bounded on $L^2(\mathbb{R}^d)$, $K$ satisfies the size condition that
\[
|K(x, y)| \lesssim \frac{1}{|x - y|^d},
\]
and the regularity that for some $\delta \in (0, 1]$,
\[
|K(x_1, y) - K(x_2, y)| \lesssim \frac{|x_1 - x_2|^\delta}{|x_1 - y|^{d+\delta}}, \quad |x_1 - y| \geq 2|x_1 - x_2|,
\]
\[
|K(x, y_1) - K(x, y_2)| \lesssim \frac{|y_1 - y_2|^\delta}{|x - y_1|^{d+\delta}}, \quad |x - y_1| \geq 2|y_1 - y_2|.
\]
Then for $b \in \text{BMO}(\mathbb{R}^d)$ and $\lambda > 0$,
\[
|\{x \in \mathbb{R}^d : |\tilde{T}_\Omega, b f (x)| > \lambda\}| \lesssim \int_{\mathbb{R}^d} \Phi \left( \frac{|f(x)|}{\lambda} \right) dx.
\]

As it was pointed out in [6], Theorem 1.6 is more general than Theorem 1.3.

This paper is organized as follows. In Section 2, we outline some known facts about Orlicz spaces, and give a lemma concerning the interpolation between Orlicz spaces. In Section 3, we reduce the proof of Theorem 1.3 to the proof of two key estimates (3.5) and (3.6). In Section 4 and Section 5, we prove (3.5) and (3.6) respectively.

Throughout this paper, $C$ always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant $C$ such that $A \leq CB$. Specially, we use $A \lesssim_{d, p} B$ to denote that there exists a positive constant $C$ depending only on $d, p$ such that $A \leq CB$. Constant with subscript such as $C_1$, does not change in different occurrences. For any set $E \subset \mathbb{R}^d$, $\chi_E$ denotes its characteristic function. For a cube $Q \subset \mathbb{R}^d$ and $\lambda \in (0, \infty)$, we use $\ell(Q)$ to denote the side length of $Q$, and $\lambda Q$ to denote the cube with the same center as $Q$ and whose side length is $\lambda$ times that of $Q$. For a local function $b$ and a cube $Q$, $\langle b \rangle_Q$ denotes the mean value of $b$ on $Q$.

## 2 Preliminary Results on Orlicz Spaces

In this section, we list some known facts about Orlicz spaces. These facts can be found in [19]. Let $\Psi : [0, \infty) \rightarrow [0, \infty)$ be Young function, namely, $\Psi$ is convex and continuous on $[0, \infty)$, $\Psi(0) = 0$ and $\lim_{t \to \infty} \Psi(t) = \infty$. We always assume that $\Psi$ satisfies a doubling condition, that is, $\Psi(2t) \leq C \Psi(t)$ for any $t \in (0, \infty)$. A Young function $\Psi$ is called an $N$-function, if $\Psi(t) = 0$ only in $t = 0$, and
\[
\lim_{t \to 0} \frac{\Psi(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{\Psi(t)}{t} = \infty.
\]
Let $\Psi$ be a Young function, and $Q \subset \mathbb{R}^d$ be a cube. Define the space $L^\Psi(Q)$ as

$$L^\Psi(Q) = \{ f : f \text{ is measurable on } Q, \| f \|_{L^\Psi(Q)} < \infty \},$$

with $\| \cdot \|_{L^\Psi(Q)}$ the Luxemburg norm defined by

$$\| f \|_{L^\Psi(Q)} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Psi \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$ 

Then we have

$$\frac{1}{|Q|} \int_Q \Psi(|f(x)|) dx \leq 1 \iff \| f \|_{L^\Psi(Q)} \leq 1,$$

see [19, p. 54]. Also, we have that and

$$\| f \|_{L^\Psi(Q)} \leq \inf \left\{ \lambda + \frac{\lambda}{|Q|} \int_Q \Psi \left( \frac{|f(x)|}{\lambda} \right) dx : \lambda > 0 \right\} \leq 2\| f \|_{L^\Psi(Q)},$$

see [19, p. 69].

Let $\Psi$ be a Young function. We define its complementary function $\Psi^*$ on $[0, \infty)$ by

$$\Psi^*(t) = \sup\{st - \Psi(s) : s \geq 0\}.$$ 

Then $\Psi^*$ is also a Young function. We have that

$$t_1 t_2 \leq \Psi(t_1) + \Psi^*(t_2), \quad t_1, t_2 \in [0, \infty),$$

and consequently, the generalized Hölder inequality

$$\frac{1}{|Q|} \int_Q |f(x)h(x)| dx \leq \| f \|_{L^\Psi(Q)} \| h \|_{L^{\Psi^*}(Q)}$$

holds for $f \in L^\Psi(Q)$ and $h \in L^{\Psi^*}(Q)$. see [19, p. 6]. Also, we have

$$C \| f \|_{L^\Psi(Q)} \leq \sup_{\| h \|_{L^{\Psi^*}(Q)} \leq 1} \frac{1}{|Q|} \int_Q f(x)h(x) dx \leq \| f \|_{L^\Psi(Q)},$$

see inequality (18) in [19, p.62]. When the functions $\Psi$ and $\Psi^*$ are $N$-functions, the inequality

$$t \leq \Psi^{-1}(t)(\Psi^*)^{-1}(t) \leq 2t,$$

holds true for all $t > 0$, where $\Psi^{-1}(t)$ is the inverse of $\Psi(t)$ (see [19, p.13] for details).

Now let $p \in (1, \infty)$ and $\alpha \in \mathbb{R}$, set $\Phi_{p, \alpha}(t) = t^p \log^\alpha(e + t).$ Note that $\Phi_{p, p}(t) = (\Phi(t))^p.$ As it is well known, for $p \in (1, \infty)$ and $\alpha \in [0, \infty)$, the complementary function of $\Phi_{p, \alpha}$ is

$$\Phi^*_{p, \alpha}(t) \approx t^p \log^{-\alpha/(p-1)}(e + t),$$

see [17]. Usually, we denote $\| f \|_{L^{\Phi_{p, \alpha}}(Q)}$ as $\| f \|_{L^p(\log L)^\alpha, Q}.$ Observe that when $p \in (1, \infty), \Phi_{p, \alpha}(t)$ satisfies the doubling condition.

As it is well known, for $\Phi(t) = t \log(e + t)$, we have that $\Phi^*(t) \approx e^t - 1. \quad 1$. For a cube $Q \subset \mathbb{R}^d$, we also define $\| f \|_{\text{exp}L, Q}$ by

$$\| f \|_{\text{exp}L, Q} = \inf \left\{ t > 0 : \frac{1}{|Q|} \int_Q \Phi^* \left( \frac{|f(y)|}{t} \right) dy \leq 1 \right\}.$$ 

Let $b \in \text{BMO}(\mathbb{R}^d)$. The John-Nirenberg inequality tells us that for any $Q \subset \mathbb{R}^d$,

$$\| b - \langle b \rangle_Q \|_{\text{exp}L, Q} \leq \| b \|_{\text{BMO}(\mathbb{R}^d)}.$$ 

 Springer
This, together with the generalization of Hölder’s inequality, shows that
\[ \frac{1}{|Q|} \int_Q |b(x) - \langle b \rangle_Q| |h(x)| dx \lesssim \|h\|_{L^{1+\log L}, \bar{Q}} \|b\|_{\text{BMO}(\mathbb{R}^d)}. \] (2.3)

The following lemma will be used in the proof of Theorem 1.3.

**Lemma 2.1** Let \( Q \subset \mathbb{R}^d \) be a cube, \( p \in (1, \infty) \), \( \alpha \in [0, \infty) \) and \( C_1 \in (0, 1] \). Suppose that
\[ \frac{1}{|Q|} \int_Q |f(y)| dy \leq C_1, \quad \|f\|_{L^p(\log L)^{-\alpha}}, Q \leq 1. \]

Then for \( q \in (1, p), r \in (0, 1) \) such that \( 1/q = r + (1-r)/p \), and \( \varepsilon \in (0, r) \),
\[ \left( \frac{1}{|Q|} \int_Q |f(y)| q dy \right)^{\frac{1}{q}} \lesssim C_1^\varepsilon. \]

**Proof** At first, we claim that for \( q_1 \in (1, p) \),
\[ \left( \frac{1}{|Q|} \int_Q |h(y)| q_1 dy \right)^{\frac{1}{q_1}} \lesssim \|h\|_{L^p(\log L)^{-\alpha}}, Q. \] (2.4)
To prove this, we assume that \( \|h\|_{L^p(\log L)^{-\alpha}}, Q = 1 \), which means that
\[ \frac{1}{|Q|} \int_Q \Phi_{p,-\alpha}(|h(x)|) dx \leq 1. \]
Observe that when \( t \in [1, \infty) \), \( t^{\alpha p - p} \lesssim \log^{-\alpha}(e + t) \). Therefore,
\[ \frac{1}{|Q|} \int_Q |h(y)| q_1 dy \leq 1 + \frac{1}{|Q|} \int_{\{y \in Q : |h(y)| \geq 1\}} |h(y)| q_1 dy \]
\[ \lesssim 1 + \frac{1}{|Q|} \int_{\{y \in Q : |h(y)| \geq 1\}} \Phi_{p,-\alpha}(|h(y)|) dy \lesssim 1. \]
This verifies (2.4). For fixed \( q \in (1, \infty) \) and \( \varepsilon \in (0, r) \), we choose \( q_1 \in (1, p) \) such that \( 1/q = \varepsilon + (1-\varepsilon)/q_1 \). It then follows from (2.4) that
\[ \left( \int_Q |f(y)| q dy \right)^{\frac{1}{q}} \leq \left( \int_Q |f(y)| dy \right)^{\varepsilon} \left( \int_Q |f(y)| q_1 dy \right)^{\frac{1}{q_1}} \lesssim C_1^\varepsilon |Q|^{1/q}, \]
and then completes the proof of Lemma 2.1. \( \square \)

### 3 Proof of Theorem 1.3

In this section, we will start to prove Theorem 1.3. In particular, we reduces its proof to two estimates (3.5) and (3.6), which will be proved in Section 4 and Section 5 respectively.

To prove Theorem 1.3, we will employ the well known micro-local decomposition introduced by Seeger [21], see [6, Section 2] for its variant. For \( s > 3 \), let \( \mathcal{E}^s = \{e^s_{\nu} \}_{\nu \in \Lambda_s} \) be a collection of unit vectors on \( \mathbb{S}^{d-1} \) such that
(a) \( |e^s_{\nu} - e^s_{\nu'}| > 2^{-s} \gamma^{-4} \) when \( \nu \neq \nu' \);
(b) for each \( \theta \in \mathbb{S}^{d-1} \), there exists an \( e^s_{\nu} \) such that \( |e^s_{\nu} - \theta| \leq 2^{-s} \gamma^{-4} \),

\( \square \) Springer
where $\gamma \in (0, 1)$ is a constant. The set $\mathcal{E}^s$ can be constructed as in [6, Section 2]. Observe that $\text{card}(\mathcal{E}^s) \lesssim 2^{s\gamma(d-1)}$. Let $\zeta$ be a smooth, nonnegative, radial function, such that $\text{supp} \zeta \subset B(0, 1)$ and $\zeta(r) = 1$ for $|r| \leq 1/2$. Set

$$\widehat{\Gamma}_s^\zeta(\xi) = \zeta \left(2^{s\gamma} \left(\frac{\xi}{|\xi|} - e_\nu^s\right)\right)$$

and

$$\Gamma_s^\zeta(\xi) = \widehat{\Gamma}_s^\zeta(\xi) \left(\sum_{\nu \in \Lambda_s} \widehat{\Gamma}_s^\zeta(\xi)\right)^{-1}.$$ 

It is easy to verify that $\Gamma_s^\zeta$ is homogeneous of degree zero, and for all $s$,

$$\sum_{\nu \in \Lambda_s} \Gamma_s^\zeta(\xi) = 1, \xi \in S^{d-1}.$$ 

Let $\psi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \psi \leq 1$, $\text{supp} \psi \subset [-4, 4]$ and $\psi(t) \equiv 1$ when $t \in [-2, 2]$. Define the multiplier operator $G_\psi^s$ by

$$\widehat{G_\psi^s f}(\xi) = \psi \left(2^{s\gamma} \langle \xi/|\xi|, e_\nu^s\rangle\right) \hat{f}(\xi),$$

where and in the following, for a suitable function $f$, $\hat{f}$ denotes the Fourier transform of $f$. Take a smooth radial nonnegative function $\phi$ on $\mathbb{R}^d$ such that $\text{supp} \phi \subset \{x: \frac{1}{2} \leq |x| \leq 2\}$ and $\sum_j \phi_j(x) = 1$ for all $x \in \mathbb{R}^d \setminus \{0\}$, where $\phi_j(x) = \phi(2^{-j}x)$.

Recall that, $\mathcal{D}$, the standard dyadic grid in $\mathbb{R}^d$ consists of all cubes of the form

$$2^{-k}([0, 1)^d + l), k \in \mathbb{Z}, l \in \mathbb{Z}^d.$$ 

For $j \in \mathbb{Z}$, let $\mathcal{D}_j = \{Q \in \mathcal{D}: \ell(Q) = 2^j\}$.

*Proof of Theorem 1.3.* By homogeneity, it suffices to prove (1.3) for the case of $\lambda = 1$. Applying the Calderón-Zygmund decomposition to $\Phi(|f|)$ at level 1, we can obtain a collection of non-overlapping closed dyadic cubes $S = \{Q\}$, such that $\|f\|_{L^{\infty}(\mathbb{R}^d \setminus \cup_{Q \in S} Q)} \lesssim 1$, and

$$\int_Q \Phi(|f(x)|)dx \lesssim |Q|, \sum_{Q \in S} |Q| \lesssim \int_{\mathbb{R}^d} \Phi(|f(x)|)dx.$$ 

Let $E = \cup_{Q \in S} 2^{200}Q$, it is obvious that $|E| \lesssim \int_{\mathbb{R}^d} \Phi(|f(x)|)dx$. Set

$$g(x) = f(x)\chi_{\mathbb{R}^d \setminus \cup_{Q \in S} Q}(x) + \sum_{Q \in S} (f)_Q \chi_Q(x),$$

and

$$h(x) = \sum_{Q \in S} h_Q(x), \text{ with } h_Q(x) = (f(x) - (f)_Q) \chi_Q(x).$$

It is easy to verify that for each cube $Q \in S$,

$$\|h_Q\|_{L^{\infty} \log L, Q} \lesssim 1.$$ 

By $L^2(\mathbb{R}^d)$ boundedness of $T_{\Omega, b}$, we have that

$$\{x \in \mathbb{R}^d: |T_{\Omega, b} g(x) > 1/2| \} \lesssim \int_{\mathbb{R}^d} |f(x)|dx.$$ 

(3.1)

Let

$$E_0 = \{x' \in S^{d-1}: |\Omega(x')| \leq 1\}$$

and

$$E_i = \{x' \in S^{d-1}: 2^{i-1} < |\Omega(x')| \leq 2^i\} (i \in \mathbb{N}).$$
Denote
\[ \Omega_0(x') = \Omega(x') \chi_{E_0}(x'), \quad \Omega_i(x') = \Omega(x') \chi_{E_i}(x') \quad (i \in \mathbb{N}). \]

Set \( K_j(x) = \frac{\Omega(x')}{|k|^d} \phi_j(x), \) \( K^i_j(x) = \frac{\Omega_i(x')}{|k|^d} \phi_j(x), \) \( K^{i,s}_j(x) = \frac{\Omega_i(x')}{|k|^d} \phi_j(x) \Gamma^s_v(x'), \) \( T_j \) be the convolution operators with kernel \( K_j, \) and
\[ T^i_j u(x) = K^i_j * u(x), \quad T^{i,s}_j u(x) = K^{i,s}_j * u(x). \]

Observe that for each fixed \( s, \) \( T^i_j u(x) = \sum_v T^{i,s}_j u(x). \) It is obvious that \( \text{supp} T_j h_Q \subset 2^{100} Q \) when \( Q \in \mathcal{S}_{j-s} \) with \( j \in \mathbb{Z} \) and \( s < 100. \) Set \( \mathcal{S}_j = \mathcal{D}_j \cap \mathcal{S}. \) For \( x \in \mathbb{R}^d \setminus E, \) we can decompose \( T_{\Omega, b} h \) as
\[ T_{\Omega, b} h(x) = \sum_Q (b - \langle b \rangle Q) T_{\Omega} h_Q(x) - T_{\Omega} \left( \sum_Q (b - \langle b \rangle Q) h_Q \right)(x). \]

Recall that \( T_{\Omega} \) is bounded from \( L^1(\mathbb{R}^d) \) to \( L^{1,\infty}(\mathbb{R}^d). \) An application of (2.3) tells us that
\[ \left\| \left\{ x \in \mathbb{R}^d : \left| T_{\Omega} \left( \sum_Q (b - \langle b \rangle Q) h_Q \right)(x) \right| > \frac{1}{4} \right\} \right\| \lesssim \sum_Q \| (b - \langle b \rangle Q) h_Q \|_{L^1(\mathbb{R}^d)} \lesssim \sum_Q |Q| \| h_Q \|_{L^{1,\infty}(\mathbb{R}^d)} \lesssim \int_{\mathbb{R}^d} \Phi(|f(x)|) dx. \quad (3.2) \]

With estimates (3.1) and (3.2) in hand, it suffices to prove that
\[ \left\| \left\{ x \in \mathbb{R}^d \setminus E : \sum_{s \geq 100} \sum_{j \in \mathbb{Z}} \sum_Q \left( b(x) - \langle b \rangle Q \right) T_j h_Q(x) \right| > \frac{1}{4} \right\| \lesssim \| f \|_{L^1(\mathbb{R}^d)}. \quad (3.3) \]

To prove (3.3), let
\[ U_1 h(x) = \sum_{i=0}^{\infty} \sum_{100 \leq s \leq N_0 i} \sum_{j \in \mathbb{Z}} \sum_Q \left( b(x) - \langle b \rangle Q \right) T^i_j h_Q(x), \]
\[ U_2 h(x) = \sum_{i=0}^{\infty} \sum_{s > N_0 i} \sum_{j \in \mathbb{Z}} \sum_Q \sum_{v} G^s_v \left[ \left( b - \langle b \rangle Q \right) T^{i,s}_j h_Q \right](x), \]
and
\[ U_3 h(x) = \sum_{i=0}^{\infty} \sum_{s > N_0 i} \sum_{j \in \mathbb{Z}} \sum_Q \sum_{v} \left[ \left( b(x) - \langle b \rangle Q \right) T^i_j h_Q(x) \right. \]
\[ - \sum_{v} G^s_v \left[ \left( b - \langle b \rangle Q \right) T^{i,s}_j h_Q \right](x) \].
where and in the following, \( N_0 \in \mathbb{N} \) is a constant which will be chosen in the estimate for \( U_2 \) and \( U_3 \), see (5.8) in Section 5. For \( x \in \mathbb{R}^d \setminus E \), we write

\[
\sum_{s \geq 100} \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{S}_{j-s}} (b(x) - \langle b \rangle_Q) T_j h_Q (x) = U_1 h(x) + U_2 h(x) + U_3 h(x).
\]

To estimate term \( U_1 \), we claim that for each cube \( Q \in \mathcal{S}_{j-s} \),

\[
\|(b - \langle b \rangle_Q) T_j h_Q \|_{L^1(\mathbb{R}^d)} \lesssim \left( 2^{-i} + (i + s) \| \Omega_i \|_{L^1(\mathcal{S}^{d-1})} \right) \| h_Q \|_{L^1(\mathbb{R}^d)}, \tag{3.4}
\]

To see this, let \( x_Q \) be the center of \( Q \). It is easy to see that \( \text{supp} \ T_j h_Q \subset B(x_Q, \ 10d2^j) \), and \( \| (b) - \langle b \rangle_Q \| \lesssim s \). Observing that for each \( y \in Q \) and \( \lambda > 0 \),

\[
\int_{B_Q} \frac{|\Omega_i(x - y)|}{\lambda} \log \left( e + \frac{|\Omega_i(x - y)|}{\lambda} \right) dx \lesssim 2^j \int_{\mathcal{S}^{d-1}} \frac{|\Omega_i(\theta)|}{\lambda} \log \left( e + \frac{|\Omega_i(\theta)|}{\lambda} \right) d\theta,
\]

we thus get that for \( y \in Q \),

\[
\| \Omega_i(\cdot - y) \|_{L^\infty \log L, B_Q} \lesssim \inf \left\{ \lambda > 0 : \| \Omega_i \|_{L^1(\mathcal{S}^{d-1})} \log \left( e + \frac{|\Omega_i(\cdot)|}{\lambda} \right) \leq 1 \right\}
\]

\[
\lesssim \| \Omega_i \|_{L^\infty(\mathcal{S}^{d-1})} + \| \Omega_i \|_{L^1(\mathcal{S}^{d-1})} \log(e + \| \Omega_i \|_{L^\infty(\mathcal{S}^{d-1})})
\]

\[
\lesssim 2^{-i} + i \| \Omega_i \|_{L^1(\mathcal{S}^{d-1})}.
\]

It then follows from inequality (2.3) that for each \( y \in Q \),

\[
\int_{B_Q} \left| K_j^i(x - y) \right| |b(x) - \langle b \rangle_Q| dx \leq 2^{-jd} \int_{B_Q} |\Omega_i(x - y)| |b(x) - \langle b \rangle_Q| dx + 2^{-jd} \int_{B_Q} |\Omega_i(x - y)| dx |(b) - \langle b \rangle_Q| \lesssim 2^{-i} + (i + s) \| \Omega_i \|_{L^1(\mathcal{S}^{d-1})}.
\]

This, via duality argument, verifies (3.4). Now we obtain from (3.4) that

\[
\| U_1 h \|_{L^1(\mathbb{R}^d)} \lesssim \sum_{i=0}^{\infty} \sum_{100 \leq s \leq N_0i} \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{S}_{j-s}} \| (b - \langle b \rangle_Q) T_j h_Q \|_{L^1(\mathbb{R}^d)}
\]

\[
\lesssim \sum_{i=0}^{\infty} \sum_{100 \leq s \leq N_0i} \left( 2^{-i} + (i + s) \| \Omega_i \|_{L^1(\mathcal{S}^{d-1})} \right) \int_{\mathbb{R}^d} |f(x)| dx
\]

\[
\lesssim \left( 1 + \| \Omega \|_{L(\log L)^2(\mathcal{S}^{d-1})} \right) \int_{\mathbb{R}^d} |f(x)| dx.
\]

Therefore,

\[
\left\{ \left| \int_{\mathbb{R}^d \setminus \{ x \in \mathbb{R}^d : |U_1 h(x)| > 1/12 \} } \right| \lesssim \int_{\mathbb{R}^d} |f(x)| dx.
\]

The proof of (3.3) is now reduced to proving that

\[
\left\{ \left| \int_{\mathbb{R}^d \setminus \{ x \in \mathbb{R}^d : |U_2 h(x)| > 1/12 \} } \right| \lesssim \int_{\mathbb{R}^d} |f(x)| dx, \tag{3.5}
\]

and

\[
\left\{ \left| \int_{\mathbb{R}^d \setminus \{ x \in \mathbb{R}^d : |U_3 h(x)| > 1/12 \} } \right| \lesssim \int_{\mathbb{R}^d} |f(x)| dx. \tag{3.6}
\]
The proofs of these two inequalities are long and complicated, and will be given in Sections 4 and 5 respectively.

4 Proof of Inequality (3.5)

Let \( \Omega \) be homogeneous of degree zero and \( \Omega \in L^\infty(S^{d-1}) \). For each \( j \in \mathbb{Z} \) and \( \nu \in \Lambda_j \), define operator \( T_{j,\nu}^s \) by

\[
T_{j,\nu}^s f(x) = K_{j,\nu}^s \ast f(x),
\]

where \( K_{j,\nu}^s(x) = \Omega(x')|x|^{-d}\phi_j(x)\Gamma_\nu^s(x') \). Let \( S \) be a collection of dyadic cubes with disjoint interiors. For \( m \in \mathbb{Z} \), let \( S_m = S \cap \mathcal{D}_m \). Then for each \( \nu \) and \( s \geq 3 \),

\[
\left\| \sum_j \sum_{Q \in \Omega_{j-s}} T_{j,\nu}^s h_Q \right\|^2_{L^2(\mathbb{R}^d)} \lesssim 2^{-2ys(d-1)}\|\Omega\|^2_{L^\infty(S^{d-1})} \sum_j \sum_{Q \in \Omega_{j-s}} \|h_Q\|_{L^1(\mathbb{R}^d)},
\]

where \( \Omega_{j-s} \subset S_{j-s} \), each \( h_Q \) is supported on cube \( Q \in \Omega_{j-s} \) and \( \|h_Q\|_{L^1(\mathbb{R}^d)} \leq |Q| \). This fact was proved in [6, p.1658] (also [21, p. 99]) and plays an important role in the weak type endpoint estimate for \( T_\Omega \).

To prove inequality (3.5), we need the following key lemma which can be considered as a refined version of the estimate (4.2).

Lemma 4.1 Let \( \Omega \) be homogeneous of degree zero and \( \Omega \in L^\infty(S^{d-1}) \), \( S \) be a collection of dyadic cubes with disjoint interiors. For each cube \( Q \in S \), let \( h_Q \) be an integrable function supported in \( Q \) satisfying \( \|h_Q\|_{L^1(\mathbb{R}^d)} \leq |Q| \). Then for \( b \in BMO(\mathbb{R}^d) \) and \( s \geq 100 \),

\[
\left\| \sum_j \sum_{Q \in \Omega_{j-s}} \sum_{\nu} G_\nu^s \left( (b - \langle b \rangle_Q)T_{j,\nu}^s h_Q \right) \right\|^2_{L^2(\mathbb{R}^d)} \lesssim \|\Omega\|^2_{L^\infty(S^{d-1})} 2^{-s\gamma/2} \sum_{Q \in S} \|h_Q\|_{L^1(\mathbb{R}^d)}.
\]

Proof For \( f \in L^2(\mathbb{R}^d) \), it follows from Cauchy-Schwarz inequality that

\[
\left| \sum_j \sum_{Q \in \Omega_{j-s}} \sum_{\nu} \int_{\mathbb{R}^d} G_\nu^s \left( (b - \langle b \rangle_Q)T_{j,\nu}^s h_Q \right) (x) f(x) dx \right|
\]

\[
= \int_{\mathbb{R}^d} \sum_{\nu} G_\nu^s f(x) \sum_j \sum_{Q \in \Omega_{j-s}} (b(x) - \langle b \rangle_Q)T_{j,\nu}^s h_Q(x) dx
\]

\[
\leq \left( \sum_{\nu} |G_\nu^s f|^2 \right)^{\frac{1}{2}} \left( \sum_{\nu} \left\| \sum_j \sum_{Q \in \Omega_{j-s}} (b - \langle b \rangle_Q)T_{j,\nu}^s h_Q \right\|^2_{L^2(\mathbb{R}^d)} \right)^{\frac{1}{2}}.
\]

Plancherel’s theorem, via the estimate

\[
\sup_{\xi \neq 0} \sum_{\nu} |\psi(2^{s\gamma} \langle e_\nu^s, \xi/|\xi| \rangle)|^2 \lesssim 2^{s\gamma(d-2)}
\]
(see [6, inequality (3.1)], implies that
\[
\left\| \left( \sum_{\nu} \left| G_{\nu} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^d)}^2 = \sum_{\nu} \int_{\mathbb{R}^d} \left| \psi \left( 2^\nu \langle \xi / |\xi|, e_\nu^* \rangle \right) \right|^2 \left| \hat{f}(\xi) \right|^2 d\xi \\
\lesssim 2^{2\nu(d-2)} \left\| f \right\|_{L^2(\mathbb{R}^d)}^2.
\] (4.3)
Recall that \( \text{card}(E^s) \lesssim 2^{2s(d-1)} \). It suffices to prove that for each fixed \( \nu \in \Lambda_s \),
\[
\left\| \sum_j \sum_{Q \in S_{j-1}} (b - \langle b \rangle_Q) T_{j\nu}^s h_Q \right\|_{L^2(\mathbb{R}^d)}^2 \lesssim 2^{-s(2d-5)} \left\| \Omega \right\|_{L^\infty(\mathbb{R}^d)}^2 \left\| \sum_{Q \in S} h_Q \right\|_{L^1(\mathbb{R}^d)}.
\] (4.4)
By homogeneity, we may assume that \( \left\| \Omega \right\|_{L^\infty(\mathbb{R}^d)} = \left\| b \right\|_{\text{BMO}(\mathbb{R}^d)} = 1 \).

We now prove (4.4). Write
\[
\left\| \sum_j \sum_{Q \in S_{j-1}} (b - \langle b \rangle_Q) T_{j\nu}^s h_Q \right\|_{L^2(\mathbb{R}^d)}^2 \\
= \sum_j \sum_{Q \in S_{j-1}} \sum_{I \in S_{j-1}} \int_{\mathbb{R}^d} h_Q(x) T_{j\nu}^s ((b - \langle b \rangle_Q)(b - \langle b \rangle_I) T_{j\nu}^s h_I)(x) dx \\
+ 2 \sum_j \sum_{Q \in S_{j-1}} \sum_{i < j} \sum_{I \in S_{i-1}} \sum_{I \cap (x + R_{j\nu}^s) \neq \emptyset} \int_{\mathbb{R}^d} h_Q(x) T_{j\nu}^s ((b - \langle b \rangle_Q)(b - \langle b \rangle_I) T_{j\nu}^s h_I)(x) dx.
\] (4.5)
For each fixed \( j, \nu \) and \( s \), let
\[
\widetilde{R}_{j\nu} = \{ x \in \mathbb{R}^d : |\langle y, e_\nu^* \rangle| \leq 2^{j+2}, |y - \langle y, e_\nu^* \rangle e_\nu^*| \leq 2^{2j+2-s}, \}
\]
and
\[
R_{j\nu} = \widetilde{R}_{j\nu} + \check{R}_{j\nu}.
\]
As it was pointed out by Seeger [21, p. 99] (see also Ding and Lai [6, p. 1659]), when \( i \leq j \), we have that
\[
\sum_{I \in S_{i-1}} T_{i\nu}^s ((b - \langle b \rangle_Q)(b - \langle b \rangle_I) T_{i\nu}^s h_I)(x) \\
= \sum_{I \in S_{i-1}, I \cap (x + R_{j\nu}^s) \neq \emptyset} \int_{\mathbb{R}^d} K_{i\nu}^s (x - y)(b(y) - \langle b \rangle_Q)(b(y) - \langle b \rangle_I) T_{i\nu}^s h_I(y) dy.
\]
Observe that
\[
|x + 2R_{j\nu}^s| \lesssim 2^{j-\gamma s(d-1)}.
\]
For each fixed \( Q \in S_{j-s} \) and \( x \in Q \), we can find a cube \( R_{j,s}^x \) centered at \( x \), such that \( Q \subset R_{j,s}^x \), \( |R_{j,s}^x| \approx 2^{jd} \), and
\[
\bigcup_{i \leq j, I \in S_{i-1}, I \cap (x + R_{j\nu}^s) \neq \emptyset} I \subset x + 2R_{j\nu}^s \subset R_{j,s}^x.
\]
For each fixed \( i \leq j, I \in S_{i-1} \), let \( I^s = 2^{i+4}dI \). Then \( |\langle b \rangle_I - \langle b \rangle_{I^s}| \lesssim s \). Observe that for each \( r \in [1, \infty) \),
\[
\| b - \langle b \rangle_{I^s} \|_{L^{r'}(I^s)} \lesssim 2^{id/r'}, \| K_{i\nu}^s \|_{L^r(\mathbb{R}^d)} \lesssim 2^{-\gamma s(d-1)/r} 2^{-id/r'},
\]
Recall that \( \text{supp} K^s_{I_0} \subset \{ x : |x| \leq 2^{j+2} \} \). Thus for each \( I \in S_{I_0} \), \( \text{supp} T^s_{I_0} h_I \subset I^s \). A trivial computation involving Hölder’s inequality gives us that

\[
\begin{align*}
&\sum_{i \leq j} \sum_{I \in S_{I_0}, \text{ } |I| \leq 2^{j+2}} |b - \langle b \rangle \| T^s_{I_0} h_I \|_2 \|_2 \\
\lesssim & \sum_{i \leq j} \sum_{I \in S_{I_0}, \text{ } |I| \leq 2^{j+2}} (s \| T^s_{I_0} h_I \|_2 + \| (b - \langle b \rangle \| T^s_{I_0} h_I \|_2) \\
\lesssim & s 2^{-y s (d-1)} \sum_{i \leq j} \sum_{I \in S_{I_0}, \text{ } |I| \leq 2^{j+2}} \| h_I \|_2 \\
&+ \sum_{i \leq j} \sum_{I \in S_{I_0}, \text{ } |I| \leq 2^{j+2}} \| b - \langle b \rangle \|_2 \| L^s(I^s) \| K^s_{I_0} \| L^s(I^s) \| h_I \|_2 \\
\lesssim & 2^{-ys(d-1)/r} 2^{jd}.
\end{align*}
\]

(4.6)

Now we claim that for \( p \in (1, \infty) \),

\[
\begin{align*}
&\sum_{i \leq j} \sum_{I \in S_{I_0}, \text{ } |I| \leq 2^{j+2}} \| (b - \langle b \rangle \| T^s_{I_0} h_I \|_p \|_p \\
\lesssim & s.
\end{align*}
\]

(4.7)

To prove this, let \( f \subset R^s_{j,s} \) with \( \| f \|_{L^p(\log L)^s} = 1 \), namely,

\[
\int_{R^s_{j,s}} \Phi_p, p'(|f(z)|)dz \leq |R^s_{j,s}|.
\]

Let \( M \) be the Hardy-Littlewood maximal operator. A straightforward computation involving inequality (2.1) leads to that for \( I \in S_{I_0} \) and \( y \in I \),

\[
\begin{align*}
|K^s_{I_0} \ast (|b - \langle b \rangle \| f \|)(y) & \lesssim s |K^s_{I_0} \ast |f \| (y) + |K^s_{I_0} \ast (|b - \langle b \rangle \| f \|)(y) \\
& \lesssim |K^s_{I_0} \ast \exp \left( \frac{|b - \langle b \rangle \| f \|}{C |b||_{\text{BMO}(\mathbb{R}^d)}} \right)(y) + s |K^s_{I_0} \ast (\Phi(|f|))(y) \\
& \lesssim 1 + s \inf_{z \in I} M(\Phi(f))(z).
\end{align*}
\]

Recall that \( \text{supp} f \subset R^s_{j,s} \), we then have that

\[
\int_{R^s_{j,s}} M(\Phi(f))(y)dy \lesssim 2^{jd/p} \| M(\Phi(|f|)) \|_{L^p(\mathbb{R}^d)} \lesssim 2^{jd/p} \Phi(|f|) \|_{L^p(\mathbb{R}^d)} \lesssim 2^{jd}.
\]
Therefore,

\[
\sum_{i \leq j} \sum_{I \in S_{j,i}} \int_{R_{j,i}^s} |f(y)(b(y) - \langle b \rangle I)T_{iv}^s h_I(y)| dy \\
\leq \sum_{i \leq j} \sum_{I \in S_{j,i}} \|h_I\|_{L^1(\mathbb{R}^d)} \|K_{iv}^x \ast (|b - \langle b \rangle I|f)\|_{L^1(\mathbb{R}^d)} \\
\lesssim s \sum_{i \leq j} \sum_{I \in S_{j,i}} \|h_I\|_{L^1(\mathbb{R}^d)} + \sum_{i \leq j} \sum_{I \in S_{j,i}} |I| \inf_{z \in I} M(\Phi(f))(z) \\
\lesssim s 2^{-ys(d-1)} 2^{jd} + s \int_{R_{j,i}^s} M(\Phi(f))(y) dy \lesssim s 2^{jd}.
\]

This, via inequality (2.2), leads to (4.7).

Inequalities (4.6) and (4.7), via Lemma 2.1, state that for each fixed \( \varepsilon \in (0, 1) \), we can choose \( q \in (1, 2) \) which is close to 1 sufficiently, such that

\[
\left\| \sum_{i \leq j} \sum_{I \in S_{j,i}} |(b - \langle b \rangle I)T_{iv}^s h_I| \right\|_{L^q(R_{j,i}^s)} \lesssim 2^{jd/q} 2^{-2\varepsilon ys(d-1)}.
\]

Let \( j \in \mathbb{Z}, Q \in S_{j-\varepsilon} \) and \( x \in Q \). Another application of Hölder’s inequality yields

\[
\sum_{i \leq j} \sum_{I \in S_{j,i}} \left| \int_{\mathbb{R}^d} K_{jv}^x (x - y)(b(y) - \langle b \rangle Q)(b(y) - \langle b \rangle I)T_{iv}^s h_I(y) dy \right| \\
\lesssim 2^{-jd} \left( \int_{R_{j,i}^s} |b(y) - \langle b \rangle Q|^{q'} dy \right)^{1/q'} \left\| \sum_{i \leq j} \sum_{I \in S_{j,i}} |(b - \langle b \rangle I)T_{iv}^s h_I| \right\|_{L^q(R_{j,i}^s)} \\
\lesssim s 2^{-2\varepsilon ys(d-1)}.
\]

since \( |(b) Q - \langle b \rangle R_{j,i}^s| \lesssim s \). This, in turn, implies that

\[
\left\| \sum_{j} \sum_{Q \in S_{j-\varepsilon}} (b - \langle b \rangle Q)T_{jv}^s h_Q \right\|_{L^2(\mathbb{R}^d)} \lesssim s 2^{-2\varepsilon ys(d-1)} \sum_{Q} \|h_Q\|_{L^1(\mathbb{R}^d)}.
\]

We choose \( \varepsilon \in (0, 1) \) such that \( 2\varepsilon(d - 1) = 2d - 7/3 \). The last estimate, along with (4.5), establishes (4.4) and then completes the proof of Lemma 4.1.
Proof of the inequality (3.5) It follows from Lemma 4.1 that

\[
\begin{aligned}
|\{x \in \mathbb{R}^d \setminus E : |U_2 h(x)| > \frac{1}{12}\}| & \leq \|U_2 h\|^2_{L^2(\mathbb{R}^d)} \\
& \leq \left( \sum_{i=0}^{\infty} \sum_{s > N_0 i} \left\| \sum_j \sum_{Q \in S_{j-s}} \sum_v G^j_v \left[ (b - \langle b \rangle_Q) T^j_i h_Q \right] \right\|_{L^2(\mathbb{R}^d)} \right)^2 \\
& \lesssim \left( \sum_{i \geq 0}^{2^i} \sum_{s > N_0 i} 2^{-s^2/4} \left( \sum_Q \|h_Q\|_{L^1(\mathbb{R}^d)} \right)^{\frac{1}{2}} \right)^2 \lesssim \int_{\mathbb{R}^d} |f(x)| dx,
\end{aligned}
\]

if we choose \(N_0 \in \mathbb{N}\) and \(\gamma \in (0, 1)\) such that \(N_0 \gamma > 16\). This proves (3.5).

\[\square\]

5 Proof of Inequality (3.6)

To prove (3.6), we will employ some lemmas.

Lemma 5.1 Let \(m\) be a complex-valued bounded function on \(\mathbb{R}^d \setminus \{0\}\) such that

\[|\hat{m}(\xi)| \leq A|\xi|^{-|\alpha|}\]

for all multi indices \(\alpha\) with \(|\alpha| \leq [d/2] + 1\), where and in the following, \([d/2]\) denote the integer part of \(d/2\). Let \(T_m\) be the multiplier operator defined by

\[\overline{T_m f}(\xi) = m(\xi) \hat{f}(\xi)\]

Then for \(w \in A_2(\mathbb{R}^d)\), \(T_m\) is bounded on \(L^2(\mathbb{R}^d, w)\) with bound \(C_{d, |w|A_2}(\|m\|_{L^\infty(\mathbb{R}^d)} + A)\), and is bounded from \(L^1(\mathbb{R}^d)\) to \(L^{1, \infty}(\mathbb{R}^d)\) with bound \(C_d(\|m\|_{L^\infty(\mathbb{R}^d)} + A)\).

The boundedness of \(T_m\) on \(L^2(\mathbb{R}^d, w)\) with \(w \in A_2(\mathbb{R}^d)\) and from \(L^1(\mathbb{R}^d, w)\) to \(L^{1, \infty}(\mathbb{R}^d, w)\) with \(w \in A_1(\mathbb{R}^d)\) was proved by Kurtz and Wheeden [13]. Repeating the proof of Theorem 1 in [13], we can verify the bound of \(T_m\) on \(L^2(\mathbb{R}^d, w)\) (\(w \in A_2(\mathbb{R}^d)\)) is less than \(C_{d, |w|A_2}(\|m\|_{L^\infty(\mathbb{R}^d)} + A)\), while the bound from \(L^1(\mathbb{R}^d)\) to \(L^{1, \infty}(\mathbb{R}^d)\) is less than \(C_d(\|m\|_{L^\infty(\mathbb{R}^d)} + A)\).

Let \(\eta \in C^\infty_0(\mathbb{R}^d)\) be a radial function such that \(\text{supp } \eta \subset \{|\xi| \leq 2\}, 0 \leq \eta \leq 1\) and \(\eta(\xi) = 1\) when \(|\xi| \leq 1\). Define \(\varphi_k(\xi) = \eta(2^k \xi) - \eta(2^{k+1} \xi)\), then \(\text{supp } \varphi_k \subset \{2^{-k-1} \leq |\xi| \leq 2^{-k+1}\}\). Define multipliers operators \(V_k\) and \(W_k\) by

\[\overline{V_k f}(\xi) = \eta(2^k \xi) \hat{f}(\xi), \quad \overline{W_k f}(\xi) = \varphi_k(\xi) \hat{f}(\xi),\]

respectively. Observe that for any \(m \in \mathbb{Z}\), \(I = V_m + \sum_{k < m} W_k\).

Lemma 5.2 Let \(b \in BMO(\mathbb{R}^d)\). Under the same hypothesis and notations as in Lemma 4.1, we have that for \(m \in \mathbb{Z}\) and \(s \geq 100\),

\[
\left\| \sum_j \sum_v G^j_v b T^j_i H_{j-s} \right\|_{L^2(\mathbb{R}^d)}^2 \lesssim \|\Omega\|_{L^\infty(S^{d-1})}^2 2^{-s^2/2} \sum_Q \|h_Q\|_{L^1(\mathbb{R}^d)},
\]
where and in the following, $G^s_{\nu, b}$ is the commutator of $G^s_{\nu}$ with $b$, and for $j \in \mathbb{Z}$, $H_j(x) = \sum_{Q \in S_j} h_Q(x)$.

**Proof** For each fixed $f \in L^2(\mathbb{R}^d)$, we have by Cauchy-Schwarz inequality that

$$\left| \sum_{\nu} \sum_{j} \int_{\mathbb{R}^d} G^s_{\nu, b} T^s_{j\nu} H_{j-s}(x) f(x) dx \right|$$

$$= \left| \sum_{\nu} \sum_{j} \int_{\mathbb{R}^d} G^s_{\nu, b} f(x) T^s_{j\nu} H_{j-s}(x) dx \right|$$

$$\leq \left( \sum_{\nu} \left| G^s_{\nu, b} f \right|^2 \right)^{1/2} \left\| L^2(\mathbb{R}^d) \left( \sum_{\nu} \left| \sum_{j} T^s_{j\nu} H_{j-s} \right|^2 \right) \right\|^{1/2}_{L^2(\mathbb{R}^d)}.$$ 

It follows from (4.2) that

$$\left( \sum_{\nu} \left| \sum_{j} T^s_{j\nu} H_{j-s} \right|^2 \right)^{1/2} \left\| L^2(\mathbb{R}^d) \right\|^{2}_{L^2(\mathbb{R}^d)} \leq 2^{-(s\gamma)(d-1)} \Omega \left\| L^2(\mathbb{R}^d) \right\| \sum_{Q} h_Q \left\| L^1(\mathbb{R}^d) \right\|.$$ 

On the other hand, we have by Cauchy-Schwarz inequality that

$$\left| \sum_{\nu} \sum_{j} \int_{\mathbb{R}^d} G^s_{\nu, b} V_m T^s_{j\nu} H_{j-s}(x) f(x) dx \right|$$

$$\leq \left( \sum_{\nu} \left| G^s_{\nu, b} f \right|^2 \right)^{1/2} \left\| L^2(\mathbb{R}^d) \left( \sum_{\nu} \left| V_m \sum_{j} T^s_{j\nu} H_{j-s} \right|^2 \right) \right\|^{1/2}_{L^2(\mathbb{R}^d)}$$

$$\leq \left( \sum_{\nu} \left| G^s_{\nu, b} f \right|^2 \right)^{1/2} \left\| L^2(\mathbb{R}^d) \right\|^2_{L^2(\mathbb{R}^d)} \left( \sum_{\nu} \left| \sum_{j} T^s_{j\nu} H_{j-s} \right|^2 \right)^{1/2}_{L^2(\mathbb{R}^d)},$$

where in the last inequality, we have invoked Plancherel’s theorem and the fact that

$$\| V_m f \|_{L^2(\mathbb{R}^d)} = \| \overline{V_m f} \|_{L^2(\mathbb{R}^d)} = \| \eta (2^m \cdot) \hat{f} \|_{L^2(\mathbb{R}^d)} \leq \| f \|_{L^2(\mathbb{R}^d)}.$$ 

If we can prove that

$$\left( \sum_{\nu} \left| G^s_{\nu, b} f \right|^2 \right)^{1/2} \left\| L^2(\mathbb{R}^d) \right\|^2_{L^2(\mathbb{R}^d)} \leq 2^{s\gamma(d-1)-s\gamma/2} \| f \|_{L^2(\mathbb{R}^d)},$$

the inequalities (5.1) and (5.2) then follow from duality directly.

To prove (5.3), we will employ an observation of Coifman, Rochberg and Weiss (see [5, pp. 620-621]), which shows that certain weighted $L^p(\mathbb{R}^d)$ estimates for linear operators imply the $L^p(\mathbb{R}^d)$ estimates for the corresponding commutators, see also [12, Section 7]. We present the details here mainly to make the bound clearer. We can verify that

$$|\partial_{\xi}^\alpha \psi (2^{s\gamma} (\xi / |\xi|))| \leq 2^{s\gamma (1 + |\alpha|)} |\xi|^{-|\alpha|}, \ |\alpha| \leq \left\lfloor \frac{d}{2} \right\rfloor + 1$$

(5.4)
for all \( v \in \Lambda_s \). Let \( w \in A_2(\mathbb{R}^d) \) such that \( w^{1+\epsilon} \in A_2(\mathbb{R}^d) \) for \( \epsilon = 2d + 6 \). We then have by Lemma 5.1 that
\[
\sum_{v \in \Lambda_s} \| G_v^s f \|_{L^2(\mathbb{R}^d, w^{1+\epsilon})}^2 \lesssim d, [w^{1+\epsilon}]_{A_2} 2^{2^s \gamma (d-1) - \frac{2^{2^s \gamma (d-1) + 1}}{2} + 1} \| f \|_{L^2(\mathbb{R}^d, w^{1+\epsilon})}^2 \lesssim d, [w^{1+\epsilon}]_{A_2} 2^{2^s \gamma (d-1) + 1} \| f \|_{L^2(\mathbb{R}^d, w^{1+\epsilon})}^2. \tag{5.5}
\]
Note that \( f \to (\sum_v |G_v^s f|^2)^{1/2} \) is sublinear. Applying interpolation theorem of Stein and Weiss [22], we deduce from (4.3) and (5.5) that
\[
\left\| \left( \sum_v |G_v^s f|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^d, w^{1+\epsilon})} \lesssim d, [w^{1+\epsilon}]_{A_2} 2^{2^s \gamma (d-1) + \frac{2^{2^s \gamma (d-1) + 1}}{2} + 1} \| f \|_{L^2(\mathbb{R}^d, w^{1+\epsilon})} \lesssim d, [w^{1+\epsilon}]_{A_2} 2^{2^s \gamma (d-1) + \frac{2^{2^s \gamma (d-1) + 1}}{2} + 1} \| f \|_{L^2(\mathbb{R}^d, w^{1+\epsilon})}. \tag{5.6}
\]
with \( t = \frac{1}{1+\epsilon} \). Now let \( b \in \text{BMO}(\mathbb{R}^d) \). [12, Lemma 7.3] tells us that there exists a constant \( c_d \) such that
\[
[e^{(1+\epsilon)2\text{Re} \theta b}]_{A_2} \leq \left( \frac{C}{2(1+\epsilon)} \right) b \| b \|_{\text{BMO}(\mathbb{R}^d)}. \]
For \( z \in \mathbb{C} \), let
\[
G_v^z_{\nu, s, b} f = e^{zb} G_v^s (e^{-zb} f).
\]
It is obvious that
\[
\| G_v^z_{\nu, s, b} f \|_{L^2(\mathbb{R}^d)} = \| G_v^s (e^{-zb} f) \|_{L^2(\mathbb{R}^d, e^{2z b})}.
\]
As in [12, inequality (7.7)], we choose \( \rho = \frac{e_d}{4(1+\epsilon)\| b \|_{\text{BMO}(\mathbb{R}^d)}} \), have that
\[
\left\| G_v^s f \right\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{2\pi \rho^2} \int_{|z| = \rho} \left\| G_v^z f \right\|_{L^2(\mathbb{R}^d)} |dz| \leq \frac{1}{2\pi \rho^2} \left( \int_{|z| = \rho} \| G_v^z f \|_{L^2(\mathbb{R}^d)}^2 |dz| \right) \frac{1}{2}.
\]
It now follows from (5.6) (with \( w = e^{2z b} \)) that,
\[
\sum_v \left\| G_v^s f \right\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{4 \pi \rho^3} \int_{|z| = \rho} \sum_v \| G_v^s (e^{-zb} f) \|_{L^2(\mathbb{R}^d, e^{2z b})}^2 |dz| \lesssim 2^{2^s \gamma (d-\frac{3}{2})} \| f \|_{L^2(\mathbb{R}^d)}^2.
\]
This leads to (5.3) and then completes the proof of Lemma 5.2. \( \square \)

**Lemma 5.3** For fixed \( k, s, j, v \), let \( K^s_{k, j, v}(x, y) \) be the kernel of the operator \( (I - G_v^s)W_k T_{j}^s \), namely,
\[
(I - G_v^s)W_k T_{j}^s h(x) = \int_{\mathbb{R}^d} K^s_{k, j, v}(x, y) h(y) dy.
\]
Under the hypothesis of Theorem 1.3, we have that for each \( x, y \in \mathbb{R}^d \) and \( N_1 \in \mathbb{N} \),
\[
| K^s_{k, j, v}(x, y) | \lesssim N_1 2^{\gamma (N_1 + 2N_2)} 2^{-j + k} \| \Omega \|_{L^\infty(S^{d-1})} \times \int_{S^{d-1}} |\Gamma^s_v(\theta)| \int_{2^{j-k}}^{2^{j+1}} \frac{1}{(1 + 2^{-2k}|x - y - r\theta|^2)^N} dr d\theta,
\]
where and in the following, \( N = \lfloor d/2 \rfloor + 1 \).
Proof We follow the argument in [6, Section 4.2]. Let
\[ L_{k, s, \nu}(\xi) = (1 - \psi(2^{2^j} \langle \xi / |\xi|, e_{2^j}^s \rangle)) \varphi_k(\xi), \]
and write
\[ K_{k, j, \nu}^s(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x - z, \xi)} L_{k, s, \nu}(\xi) d\xi K_{j, \nu}^s(z - y) dz \]
\[ = \frac{1}{(2\pi)^d} \int_{S^{d-1}} \Omega(\theta) \Gamma_{\nu}^s(\theta) \left\{ \int_{\mathbb{R}^d} \int_0^\infty e^{i(x - y - r\theta, \xi)} \phi(2^{-j} r) r^{-1} d\phi L_{k, s, \nu}(\xi) d\xi \right\} d\theta. \]
Recall that supp \( \phi \subset \{1/2 \leq |z| \leq 2\}. Integrating by parts with \( r \), we deduce that
\[ \int_0^\infty e^{i(x - y - r\theta, \xi)} \phi(2^{-j} r) r^{-1} d\phi = \int_0^\infty e^{i(x - y - r\theta, \xi)} (i \langle \theta, \xi \rangle)^{-N_1} (\phi(2^{-j} r) r^{-1})^{(N_1)} d\phi. \]
On the other hand, integrating by parts with \( \xi \) leads to that
\[ \int_{\mathbb{R}^d} e^{i(x - y - r\theta, \xi)} (i \langle \theta, \xi \rangle)^{-N_1} L_{k, s, \nu}(\xi) d\xi \]
\[ = \int_{\mathbb{R}^d} e^{i(x - y - r\theta, \xi)} \frac{(I - 2^{-2k} \Delta_{\xi})^N}{(1 + 2^{-2k}|x - y - r\theta|^2)^N} \left( L_{k, s, \nu}(\xi)(i \langle \theta, \xi \rangle)^{-N_1} \right) d\xi. \]
Therefore,
\[ K_{k, j, \nu}^s(x, y) = \frac{1}{(2\pi)^d} \int_{S^{d-1}} \Omega(\theta) \Gamma_{\nu}^s(\theta) \int_{\mathbb{R}^d} e^{i(x - y - r\theta, \xi)} \int_0^\infty (\phi(2^{-j} r) r^{-1})^{(N_1)} d\phi L_{k, s, \nu}(\xi)(i \langle \theta, \xi \rangle)^{-N_1} d\xi d\theta. \]
Since
\[ |(I - 2^{-2k} \Delta_{\xi})^N (i \langle \theta, \xi \rangle)^{-N_1} L_{k, s, \nu}(\xi)| \lesssim_{N_1} 2^{(s\gamma + k)N_1 + 2s\gamma N}, \]
see [6, (4.12)], we obtain that
\[ |K_{k, j, \nu}^s(x, y)| \lesssim_{N_1} 2^{(s\gamma + k)N_1 + 2s\gamma N} \int_{S^{d-1}} |\Omega(\theta)\Gamma_{\nu}^s(\theta)| \]
\[ \times \int_{2^{-k-1} \leq |\theta| \leq 2^{-k+1}} \int_0^\infty (\phi(2^{-j} r) r^{-1})^{(N_1)} dr \frac{1}{(1 + 2^{-2k}|x - y - r\theta|^2)^N} d\xi d\theta \]
\[ \lesssim_{N_1} 2^{(s\gamma + k)N_1 + 2s\gamma N} 2^{-j(N_1 + 1) - 2kd} \|\Omega\|_{L^\infty(S^{d-1})} \]
\[ \times \int_{S^{d-1}} |\Gamma_{\nu}^s(\theta)| \int_{2^{-j-1}}^{2^{j+1}} \frac{1}{(1 + 2^{-2k}|x - y - r\theta|^2)^N} dr d\theta. \]
This leads to our desired conclusion.
Lemma 5.4 For \( j, m \in \mathbb{Z}, \nu \in \Delta_s \) and \( s \in \mathbb{N} \) with \( s \geq 100 \), let \( F^s_{j,v,m}(x, y) \) be the kernel of the operator \( V^s_{j,v} \). Let \( Q \) be a cube with \( \ell(Q) = 2^{j-s} \). Then for any \( x \in \mathbb{R}^d, y, y_0 \in Q \),

\[
|F^s_{j,v,m}(x, y) - F^s_{j,v,m}(x, y_0)| \lesssim 2^{-s-m-md} \| \Omega \|_{L^\infty(S^{d-1})} \int_{S^{d-1}} \Gamma_v^s(\theta) \int_{2^{j-1}}^{2^{j+1}} \left[ \int_0^1 \frac{N2^{-m}|x - (ty + (1 - t)y_0) - r\theta| dt}{(1 + 2^{-2m}|x - (ty + (1 - t)y_0) - r\theta|^2)^{N+1}} + \frac{1}{(1 + 2^{-2m}|x - y - r\theta|^2)^N} \right] \phi_j(r) dr d\theta.
\]

Proof The proof is essentially given in [6, Section 4.3]. For the sake of self-contained, we include a concise proof. By integrating by parts, we write

\[
F^s_{j,v,m}(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x - y - z, \xi)} \eta(2^m \xi) d\xi K^s_{j,v}(z) dz
\]

\[
= \frac{1}{(2\pi)^d} \Omega(\theta) \Gamma_v^s(\theta) \left\{ \int_0^\infty \int_{\mathbb{R}^d} e^{i(x - y - r\theta, \xi)} \times (I - 2^{-2m} \Delta \xi)^N \eta(2^m \xi) \frac{d\xi \phi_j(r) r^{-1} dr}{(1 + 2^{-2m}|x - y - r\theta|^2)^N} \right\} d\theta.
\]

Let

\[
D_1(x, y, y_0) = \frac{1}{(2\pi)^d} \Omega(\theta) \Gamma_v^s(\theta) \left\{ \int_0^\infty \int_{\mathbb{R}^d} (e^{i(y, \xi)} - e^{i(y_0, \xi)}) e^{i(x - y, \xi)} \times (I - 2^{-2m} \Delta \xi)^N \eta(2^m \xi) \frac{d\xi \phi_j(r) r^{-1} dr}{(1 + 2^{-2m}|x - y - r\theta|^2)^N} \right\} d\theta,
\]

and

\[
D_2(x, y, y_0) = \frac{1}{(2\pi)^d} \Omega(\theta) \Gamma_v^s(\theta) \left\{ \int_0^\infty \int_{\mathbb{R}^d} e^{i(x - y_0 - r\theta, \xi)} \times (I - 2^{-2m} \Delta \xi)^N \eta(2^m \xi) \frac{1}{\gamma(x, y, m)} - \frac{1}{\gamma(x, y_0, m)} \frac{d\xi \phi_j(r) r^{-1} dr}{(1 + 2^{-2m}|x - y - r\theta|^2)^N} \right\} d\theta,
\]

where \( \gamma(x, y, m) = (1 + 2^{-2m}|x - y - r\theta|^2)^N \). We then have that

\[
F^s_{j,v,m}(x, y) - F^s_{j,v,m}(x, y_0) = D_1(x, y, y_0) + D_2(x, y, y_0).
\]

By the facts that

\[
|e^{i(y, \xi)} - e^{i(y_0, \xi)}| \lesssim |y - y_0||\xi|
\]

and

\[
|(I - 2^{-2m} \Delta \xi)^N \eta(2^m \xi)| \lesssim \chi_{|\xi| \leq 2^{-m}}(\xi), \tag{5.7}
\]

it follows that

\[
|D_1(x, y, y_0)| \lesssim 2^{-s-m-md} \int_{S^{d-1}} |\Omega(\theta) \Gamma_v^s(\theta)| \int_0^\infty \frac{\phi_j(r) r^{-1} dr}{(1 + 2^{-2m}|x - y - r\theta|^2)^N} d\theta.
\]

\( \Box \) Springer
On the other hand, a trivial computation shows that

$$|\mathcal{Y}(x, y, m) - \mathcal{Y}(x, y_0, m)| = \left| \int_0^1 \langle (y - y_0, \nabla \mathcal{Y}(ty + (1 - t)y_0) \rangle \right|$$

$$\lesssim |y - y_0|^{2 - m} \int_0^1 \frac{N2^{-m}|x - (ty + (1 - t)y_0) - r\theta|}{(1 + 2^{-2m}|x - (ty + (1 - t)y_0) - r\theta|^2)^{N+1}} \, dt.$$ 

This, along with inequality (5.7), yields

$$|D_2(x, y, y_0)| \lesssim 2^{-j-m-md} \|\Omega\|_{L^{\infty}(S^{d-1})} \int_{S^{d-1}} \Gamma_k^\varepsilon(\theta) \int_{2^{j-1}}^{2^j} \int_0^1 \frac{N2^{-m}|x - (ty + (1 - t)y_0) - r\theta|}{(1 + 2^{-2m}|x - (ty + (1 - t)y_0) - r\theta|^2)^{N+1}} \, dt \, d\tau \, dr \, d\theta.$$ 

Combining estimates for $D_1(x, y, y_0)$ and $D_2(x, y, y_0)$ then finishes the proof of Lemma 5.4.

We are now ready to prove (3.6).

**Proof of the inequality (3.6)** We choose $\gamma \in (0, \frac{1}{6d})$, $N_0 \in \mathbb{N}$ such that

$$N_0 \gamma > 16, \quad \left[ \frac{1}{3} - \gamma (3d/2 + 1) \right] N_0 > \frac{3}{2}. \quad (5.8)$$

For $b \in \text{BMO}(\mathbb{R}^d)$ and $m = j - \lfloor \frac{d}{2} \rfloor$, write

$$(b(x) - \langle b \rangle_{\mathcal{Q}}) T_{j,\mathcal{Q}}^i h_{\mathcal{Q}}(x) = \sum_v G_v^i \left[ (b - \langle b \rangle_{\mathcal{Q}}) T_{j,\mathcal{Q}}^{i,v} h_{\mathcal{Q}} \right](x)$$

$$\quad = \sum_v (I - G_v^i) \left[ (b - \langle b \rangle_{\mathcal{Q}}) V_m T_{j,\mathcal{Q}}^{i,v} h_{\mathcal{Q}} \right](x)$$

$$\quad + \sum_v G_v^i b T_{j,\mathcal{Q}}^{i,v} h_{\mathcal{Q}}(x) - \sum_v G_v^i b (V_m T_{j,\mathcal{Q}}^{i,v} h_{\mathcal{Q}})(x)$$

$$\quad + (b(x) - \langle b \rangle_{\mathcal{Q}}) \sum_v (I - G_v^i) \left[ \sum_{k < m} W_k T_{j,\mathcal{Q}}^{i,v} h_{\mathcal{Q}} \right](x)$$

$$\quad =: U_{31,j}^{i,v} h_{\mathcal{Q}}(x) + U_{32,j}^{i,v} h_{\mathcal{Q}}(x) + U_{33,j}^{i,v} h_{\mathcal{Q}}(x) + U_{34,j}^{i,v} h_{\mathcal{Q}}(x).$$

As in the estimate for $U_2$, it follows from (5.1) in Lemma 5.2 that

$$\left\| \sum_{i=0}^\infty \sum_{s > N_0 i} \sum_{j \in \mathbb{Z}} \sum_{\mathcal{Q} \in S_{j-s}} U_{32,j}^{i,v} h_{\mathcal{Q}}(x) \right\| \geq \frac{1}{48}$$

$$\lesssim \left( \sum_{i=0}^\infty \sum_{s > N_0 i} \left\| \sum_{j \in \mathbb{Z}} \sum_{\mathcal{Q} \in S_{j-s}} G_v^i b T_{j,\mathcal{Q}}^{i,v} h_{\mathcal{Q}} \right\|_{L^2(\mathbb{R}^d)} \right)^2 \lesssim \int_{\mathbb{R}^d} |f(x)| \, dx, \quad (5.9)$$
and from (5.2) in Lemma 5.2 that
\[
\left\{ x \in \mathbb{R}^d \setminus E : \left| \sum_{i=0}^{\infty} \sum_{s > N_0 i} \sum_{j \in \mathbb{Z}} \sum_{Q \in S_{j-s}} U_{34}^{i,s,j} h_Q(x) \right| > \frac{1}{48} \right\} \]
\[
\lesssim \left( \sum_{i=0}^{\infty} \sum_{s > N_0 i} \left\| \sum_{j \in \mathbb{Z}} \sum_{Q \in S_{j-s}} \sum_{v} G_{v,ib} V_{m} T_{jv}^{i,s} h_Q \right\|_{L^2(\mathbb{R}^d \setminus E)} \right)^2 \lesssim \int_{\mathbb{R}^d} |f(x)| dx. \tag{5.10}
\]

We now estimate \( U_{34}^{i,s,j} h_Q \). For each cube \( Q \in S_{j-s} \), \( y \in Q \), \( 2^{j-1} \leq r \leq 2^{j+1} \), and \( \theta \in S^{d-1} \), denote by \( Q_{y+r\theta, k} \) the cube centered at \( y + r\theta \) and having side length \( 2^k \). We have that
\[
|b_Q - b_{Q_{y+r\theta, k}}| \lesssim |b_Q - b_{Q_{y+r\theta, j-s}}| + |b_{Q_{y+r\theta, k}} - b_{Q_{y+r\theta, j-s}}| \lesssim |k - j| + s.
\]
Recall that \( N = \lfloor d/2 \rfloor + 1 \). A trivial computation yields
\[
\int_{\mathbb{R}^d} \frac{|b(x) - b_{Q_{y+r\theta, k}}|}{(1 + 2^{-2k}|x - y - r\theta|^2)^N} dx \lesssim 2^{kd}.
\]
This, in turn, implies that
\[
\int_{\mathbb{R}^d} \frac{|b(x) - b_{Q_{y+r\theta, k}}|}{(1 + 2^{-2k}|x - y - r\theta|^2)^N} dx \lesssim \int_{\mathbb{R}^d} \frac{|b(x) - b_{Q_{y+r\theta, k}}|}{(1 + 2^{-2k}|x - y - r\theta|^2)^N} dx \lesssim 2^{kd}(|k - j| + s). \tag{5.11}
\]
From Lemma 5.3, we deduce that
\[
\| (b - b_{Q}) (I - G_{v}') W_{k} T_{jv}^{i,s} h_Q \|_{L^1(\mathbb{R}^d)} 
\lesssim 2^{(s+y+k)N_1 + 2sN} 2^{-j(N_1+1)} 2^{-kd} \| \Omega_i \|_{L^{\infty}(S^{d-1})} \int_{\mathbb{R}^d} \int_{S^{d-1}} |\Gamma_{v}'(\theta)| 
\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|b(x) - b_{Q_{y+r\theta, k}}|}{(1 + 2^{-2k}|x - y - r\theta|^2)^N} dx dr d\theta \| h_Q \|_{L^1(\mathbb{R}^d)} dy 
\lesssim 2^{-s} 2^{y(d-1)} 2^{-j+k} N_1 2^{s} N_2 \| \Omega_i \|_{L^{\infty}(S^{d-1})} (|k - j| + s) \| h_Q \|_{L^1(\mathbb{R}^d)},
\]
since \( \| \Gamma_{v}' \|_{L^{\infty}(S^{d-1})} \lesssim 2^{-y s (d-1)} \). Note that for \( s \in \mathbb{N} \), \( s \lesssim 2^{y s} \). Then
\[
\left\{ x \in \mathbb{R}^d \setminus E : \left| \sum_{i=0}^{\infty} \sum_{s > N_0 i} \sum_{j \in \mathbb{Z}} \sum_{Q \in S_{j-s}} U_{34}^{i,s,j} h_Q(x) \right| > \frac{1}{48} \right\} 
\lesssim \sum_{i=0}^{\infty} \sum_{s > N_0 i} \sum_{j \in \mathbb{Z}} \sum_{Q \in S_{j-s}} \| (b - b_{Q}) (I - G_{v}') W_{k} T_{jv}^{i,s} h_Q \|_{L^1(\mathbb{R}^d)} 
\lesssim \sum_{i=0}^{\infty} \sum_{s > N_0 i} 2^{s} N_1 2^{y} N_2 \sum_{j} \sum_{k < m} 2^{(-j+k)N_1} (|j - k| + s) \sum_{Q \in S_{j-s}} \| h_Q \|_{L^1(\mathbb{R}^d)} 
\lesssim \sum_{i=0}^{\infty} \sum_{s > N_0 i} 2^{s} N_1 2^{y} N_2 \sum_{j} \sum_{k < m} 2^{(-j+k)N_1} (|j - k| + s) \| h_Q \|_{L^1(\mathbb{R}^d)} \lesssim \int_{\mathbb{R}^d} |f(x)| dx. \tag{5.12}
\]
provided we choose \( N_1 \in \mathbb{N} \) in Lemma 5.3 such that
\[
N_0(\frac{N_1/2 - \gamma N_1 - 2N\gamma - \gamma}{1}) > 1. 
\tag{5.13}
\]

It remains to consider term \((I - G_i^m)(b - \langle b \rangle Q)V_m T_{j,v}^{i,s} h_Q(x)\). For each cube \(Q \in \mathcal{S}_{j-\frac{1}{3}}\), let \(y_Q\) be the center of \(Q\). Applying Lemma 5.4 and the vanishing moment of \(h_Q\), we get that for each \(Q \in \mathcal{S}_{j-\frac{1}{3}}\),
\[
\| (b - \langle b \rangle Q)V_m T_{j,v}^{i,s} h_Q \|_{L^1(\mathbb{R}^d)} \\
\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |F_{j,v,m}^x(x, y) - F_{j,v,m}^y(x, y_Q)||b(x) - \langle b \rangle_Q|dx |h_Q(y)|dy \\
\leq 2^{-s - m - md} \| \Omega_i \|_{L^\infty(\mathbb{R}^{d-1})} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} \Gamma_j^x(\theta) \int_0^\infty \\
\left\{ \int_{\mathbb{R}^d} |b(x) - \langle b \rangle_Q| \int_0^1 \frac{N^2 - m|x - (ty + (1 - t)y_Q) - r\theta|}{(1 + 2^{-2m}|x - (ty + (1 - t)y_Q) - r\theta|^2)^N + 1} dt dx \\
+ \int_{\mathbb{R}^d} \frac{|b(x) - \langle b \rangle_Q|}{(1 + 2^{-2m}|x - y - r\theta|^2)^N} dx \right\} \phi_j(r) dr d\theta |h_Q(y)|dy.
\]

As in inequality (5.11), we have that for each \(t \in [0, 1]\),
\[
\int_{\mathbb{R}^d} |b(x) - \langle b \rangle_Q| \frac{N^2 - m|x - (ty + (1 - t)y_Q) - r\theta|}{(1 + 2^{-2m}|x - (ty + (1 - t)y_Q) - r\theta|^2)^N + 1} dx \lesssim 2^{md} s.
\]

This, along with (5.11), gives us that
\[
\| (b - \langle b \rangle Q)V_m T_{j,v}^{i,s} h_Q \|_{L^1(\mathbb{R}^d)} \lesssim 2^{\frac{j}{2}} 2^{-s - r}(d-1) \| \Omega_i \|_{L^\infty(\mathbb{R}^{d-1})} \| h_Q \|_{L^1(\mathbb{R}^d)}.
\]

Let \(L_{v,m}^{i,s}(x) = \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{S}_{j-\frac{1}{3}}} (b - \langle b \rangle Q)V_m T_{j,v}^{i,s} h_Q(x)\). We then have
\[
\sum_{i=0}^\infty 2^{i/2} \sum_{s > N_0} 2^{j} \sum_{v} 2^{s\gamma(d-1)} 2^{r\gamma(\frac{d}{2} + 1)} \| L_{v,m}^{i,s} \|_{L^1(\mathbb{R}^d)} \\
\lesssim \sum_{i=0}^\infty 2^{\frac{j}{2}} 2^{-s - r} 2^{s\gamma(d-1)} 2^{r\gamma(\frac{d}{2} + 1)} \| h_Q \|_{L^1(\mathbb{R}^d)} \lesssim \| f \|_{L^1(\mathbb{R}^d)},
\]

since \(N_0\) and \(\gamma\) satisfies (5.8). It follows from the pigeonhole principle, inequality (5.4) and Lemma 5.1 that for some constant \(C_0\),
\[
\left\{ x \in \mathbb{R}^d \setminus E : \sum_{i=0}^\infty \sum_{s > N_0} \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{S}_{j-\frac{1}{3}}} U_{j, s, i}^{i, j} h_Q(x) > \frac{1}{48} \right\} \\
\lesssim \sum_{i=0}^\infty \sum_{s > N_0} \sum_{v} \left\{ x \in \mathbb{R}^d \setminus E : |(I - G_i^m) L_{v,m}^{i,s}(x)| > C_0 2^{-i/2} 2^{-s/6 - r\gamma(d-1)} \right\} \\
\lesssim \sum_{i=0}^\infty 2^{i/2} \sum_{s > N_0} 2^{j} \sum_{v} 2^{s\gamma(d-1)} 2^{r\gamma(\frac{d}{2} + 1)} \| L_{v,m}^{i,s} \|_{L^1(\mathbb{R}^d)} \lesssim \| f \|_{L^1(\mathbb{R}^d)}. \tag{5.14}
\]

Combining estimates (5.9), (5.10), (5.12) and (5.14) yields
\[
|\{ x \in \mathbb{R}^d \setminus E : |U_j h(x)| > \frac{1}{12} \}| \lesssim \| f \|_{L^1(\mathbb{R}^d)}.
\]

This completes the proof of (3.6).
Acknowledgements The authors would like to express their sincerely thanks to the referee for his/her valuable remarks and suggestions, which made this paper more readable. Also, The authors would like to thank Dr. Israel. P. Rivera-Ríos for helpful comments.

References

1. Alvarez, J., Babgy, R.J., Kurtz, D., Pérez, C.: Weighted estimates for commutators of linear operators. Studia Math. 104, 195–209 (1993)
2. Calderón, A.P., Zygmund, A.: On the existence of certain singular integrals. Acta Math. 88, 85–139 (1952)
3. Calderón, A.P., Zygmund, A.: On singular integrals. Amer. J. Math. 78, 289–309 (1956)
4. Christ, M., Rubio de Francia, J.L.: Weak type (1,1) bounds for rough operators, II. Invent Math. 93, 225–237 (1988)
5. Coifman, R.R., Rochberg, R., Weiss, G.: Factorization theorems for Hardy spaces in several variables. Ann. of Math. 103, 611–635 (1976)
6. Ding, Y., Lai, X.: Weak type (1,1) bounded criterion for singular integral with rough kernel and its applications. Trans. Amer. Math. Soc. 371, 1649–1675 (2019)
7. Duoandikoetxea, J.: Weighted norm inequalities for homogeneous singular integrals. Trans. Amer. Math. Soc. 336, 869–880 (1993)
8. Duoandikoetxea, J., Rubio de Francia, J.L.: Maximal and singular integrals via Fourier transform estimates. Invent Math. 84, 541–561 (1986)
9. Grafakos, L. Modern Fourier Analysis Graduate Texts in Mathematics, 3rd edn., vol. 250. Springer, New York (2014)
10. Hu, G.: $L^p$ boundedness for the commutator of a homogeneous singular integral. Studia Math. 154, 13–47 (2003)
11. Hu, G., Sun, Q., Wang, X.: $L^p$ bounds for commutators of convolution operators. Colloquium Math. 93, 11–20 (2002)
12. Hytönen, T., Pérez, C.: Sharp weighted bounds involving $A_1$. Anal. PDE. 6, 777–818 (2013)
13. Kurtz, D., Wheeden, R.L.: Results on weighted norm inequalities for multiplier. Trans. Amer. Math. Soc. 255, 343–362 (1979)
14. Lerner, A.K.: A weak type estimates for rough singular integrals. Rev. Mat. Iberoam. 35, 1583–1602 (2019)
15. Lan, J., Tao, X., Hu, G.: Weak type endpoint estimates for the commutators of rough singular integral operators. Math. Inequal. Appl. 23, 1179–1195 (2020)
16. Muckenhoupt, B., Wheeden, R.L.: Weighted norm inequalities for singular and fractional integrals. Trans. Amer. Math. Soc. 161, 249–258 (1971)
17. O’Neil, R.: Fractional integration in Orlicz spaces. Trans. Amer. Math. Soc. 115, 300–328 (1965)
18. Pérez, C.: Endpoint estimates for commutators of singular integral operators. J. Funct. Anal. 128, 163–185 (1995)
19. Rao, M., Ren, Z.: Theory of Orlicz spaces Monographs and Textbooks in Pure and Applied Mathematics, vol. 146. Marcel Dekker Inc., New York (1991)
20. Ricci, F., Weiss, G.: A characterization of $H^1(\mathbb{S}^{n-1})$. In: Wainger, S., Weiss, G. (eds.) Proc. Sympos. Pure Math. of Amer. Math. Soc., vol. 35 I, pp. 289–294 (1979)
21. Seeger, A.: Singular integral operators with rough convolution kernels. J. Amer. Math. Soc. 9, 95–105 (1996)
22. Stein, E.M., Weiss, G.: Interpolation of operators with change of measures. Trans. Amer. Math. Soc. 87, 159–172 (1958)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.