On the NP-Hardness of Approximating Ordering Constraint Satisfaction Problems

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Abstract

We show improved NP-hardness of approximating Ordering Constraint Satisfaction Problems (OCSPs). For the two most well-studied OCSPs, Maximum Acyclic Subgraph and Maximum Betweenness, we prove inapproximability of $14/15 + \varepsilon$ and $1/2 + \varepsilon$.

An OCSP is said to be approximation resistant if it is hard to approximate better than taking a uniformly random ordering. We prove that the Maximum Non-Betweenness Problem is approximation resistant and that there are width-$m$ approximation-resistant OCSPs accepting only a fraction $1/(m/2)!$ of assignments. These results provide the first examples of approximation-resistant OCSPs subject only to $P \neq NP$.

Contents

1 Introduction................................................. 2
  1.1 Results................................................. 3
  1.2 Proof Overview........................................... 3

2 Preliminaries............................................... 5
  2.1 Ordering Constraint Satisfaction Problems.................. 5
  2.2 Label Cover and Inapproximability.......................... 6
  2.3 Primer on Real Analysis.................................. 6

3 A General Hardness Result................................ 8
  3.1 Dictatorship Test........................................ 8
  3.2 Reduction from Label Cover................................. 9

4 Applications of the General Result......................... 10
  4.1 Hardness of Maximum Betweenness.......................... 10
  4.2 Hardness of Maximum Non-Betweenness...................... 11
  4.3 Hardness of Maximum Acyclic Subgraph...................... 12
  4.4 Hardness of Maximum 2t-Same Order......................... 13

5 Analysis of the Reduction.................................. 14
  5.1 Bucketing............................................... 14
  5.2 Soundness of the Dictatorship Test........................ 16
  5.3 Soundness of the Reduction................................ 19
1 Introduction

We study the NP-hardness of approximating a rich class of optimization problems known as the Ordering Constraint Satisfaction Problems (OCSPs). An instance of an OCSP is described by a set of variables $X$ and a set of local ordering constraints $C$. Each constraint specifies a set of variables and a set of permitted permutations of these variables. The objective is to find a permutation of $X$ that maximizes the fraction of constraints satisfied by the induced local permutations.

A simple example of an OCSP is the Maximum Acyclic Subgraph (MAS) where one is given a directed graph $G = (V, A)$ with the task of finding an acyclic subgraph of $G$ with the maximum number of edges. Phrased as an OCSP, $V$ is the set of variables and each edge $u \rightarrow v$ is a constraint “$u \prec v$” dictating that $u$ should precede $v$. The maximum fraction of constraints simultaneously satisfiable by an ordering of $V$ is then exactly the normalized size of the largest acyclic subgraph, also called the value of the instance. Since the constraints in an MAS instance are on two variables, it is an OCSP of width two. Another example of an OCSP is the Maximum Betweenness (Max BTW) problem \cite{GJ79}. In this width-three OCSP, a constraint on a triplet of variables $(x, y, z)$ is satisfied by the local ordering $x \prec z \prec y$ and its reverse, $y \prec z \prec x$; in other words, $z$ has to be between $x$ and $y$, giving rise to the name for the problem.

Determining the value of a MAS instance is already NP-hard and one turns to approximation algorithms. An algorithm is called a $c$-approximation if, when applied to an instance $I$, it is guaranteed to produce an ordering satisfying at least a fraction $c \cdot \text{val}(I)$ of the constraints. Every OCSP admits a naive approximation algorithm which picks an ordering of $X$ uniformly at random without even looking at the constraints. For MAS, this algorithm yields a $1/2$-approximation in expectation as each constraint is satisfied with probability $1/2$ for a random ordering. Surprisingly, there is evidence that this mindless procedure achieves the best approximation ratio possible in polynomial time: assuming Khot’s Unique Games Conjecture (UGC) \cite{Kho02}, MAS is hard to approximate within $1/2 + \varepsilon$ for every $\varepsilon > 0$ \cite{GMR08, GHM11}. An OCSP is called approximation resistant if it exhibits this behavior, i.e., if it is NP-hard to improve on the guarantee of the random-ordering algorithm by $\varepsilon$ for every $\varepsilon > 0$. In fact, the results of \cite{GHM11} are much more general: assuming the UGC, they prove that every OCSP of bounded width is approximation resistant.

In many cases – such as for Vertex Cover, Max Cut, and as we just mentioned, for all OCSPs – the UGC allows us to prove optimal NP-hard inapproximability results which are not known without the conjecture. For instance, the problems MAS and MAX BTW were to date only known to be NP-hard to approximate within $65/66 + \varepsilon$ \cite{New01} and $47/48 + \varepsilon$ \cite{CS98}, which comes far from matching the random assignment thresholds of $1/2$ and $1/3$, respectively. In fact, while the UGC implies that all OCSPs are approximation resistant, there were no results proving NP-hard approximation resistance of an OCSP prior to this work. In contrast, there is a significant body of work on NP-hard approximation resistance of classical Constraint Satisfaction Problems (CSPs) \cite{Has01, ST00, EH08, Cha13}. Furthermore, the UGC is still very much open and recent algorithmic advances have given rise to subexponential algorithms for Unique Games \cite{ABS10, BBH12} putting the conjecture in question. Several recent works have also been aimed at bypassing the UGC for natural problems by providing comparable results without assuming the conjecture \cite{GRSW12, Cha13}.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{ MAS_instance.png}
\caption{An MAS instance with value $5/6$.}
\end{figure}
1.1 Results

In this work we obtain improved NP-hardness of approximating various OCSPs. While a complete characterization such as in the UG regime still eludes us, our results improve the knowledge of what we believe are four important flavors of OCSPs; see Table 1 for a summary of the present state of affairs.

We address the two most studied OCSPs: MAS and Max BTW. For MAS, we show a factor \((14/15 + \varepsilon)-\)inapproximability improving the factor from 65/66 + \(\varepsilon\) [New01]. For Max BTW, we show a factor \((1/2 + \varepsilon)-\)inapproximability improving from 47/48 + \(\varepsilon\) [CS98].

**Theorem 1.1.** For every \(\varepsilon > 0\), it is NP-hard to distinguish between MAS instances with value at least \(15/18 - \varepsilon\) from instances with value at most \(14/18 + \varepsilon\).

**Theorem 1.2.** For every \(\varepsilon > 0\), it is NP-hard to distinguish between Max BTW instances with value at least \(1 - \varepsilon\) from instances with value at most \(1/2 + \varepsilon\).

The above two results are inferior to what is known assuming the UGC and in particular do not prove approximation resistance. We introduce the Maximum Non-Betweenness (Max NBTW) problem which accepts the complement of the predicate in Max BTW. This predicate accepts four of the six permutations on three elements and thus a random ordering satisfies two thirds of the constraints in expectation. We show that this is optimal up to smaller-order terms.

**Theorem 1.3.** For every \(\varepsilon > 0\), it is NP-hard to distinguish between Max NBTW instances with value at least \(1 - \varepsilon\) from instances with value at most \(2/3 + \varepsilon\).

Finally, we address the approximability of a generic width-\(m\) OCSP as a function of the width \(m\). In the CSP world, the generic version is called \(m\)-CSP and we call the ordering version \(m\)-OCSP. We devise a simple predicate, “2\(t\)-Same Order” (2\(t\)-SO), on \(m = 2t\) variables that is satisfied only if the first \(t\) elements are relatively ordered exactly as the last \(t\) elements. A random ordering satisfies only a fraction \(1/t\) of the constraints and we prove that this is essentially optimal, implying a \((1/\lfloor m/2 \rfloor! + \varepsilon)-\)factor inapproximability of \(m\)-OCSP.

**Theorem 1.4.** For every \(\varepsilon > 0\) and integer \(m \geq 2\), it is NP-hard to distinguish \(m\)-OCSP instances with value at least \(1 - \varepsilon\) from value at most \(1/\lfloor m/2 \rfloor! + \varepsilon\).

1.2 Proof Overview

With the exception of MAS, our results follow a route which is by now standard in inapproximability: starting from the optimization problem Label Cover (LC), we give a reduction to the
problem at hand using a dictatorship-test gadget, also known as a long-code test. We describe this reduction in the context of Max NBTW to highlight the new techniques in this paper.

The reduction produces an instance $I$ of Max NBTW from an instance $L$ of LC such that $\text{val}(I) > 1 - \eta$ if $\text{val}(L) = 1$ while $\text{val}(I) < 2/3 + \eta$ if $\text{val}(L) \leq \delta$. By the PCP Theorem and the Parallel Repetition Theorem [AS98, ALM+98, Raz98], it is NP-hard to distinguish between $\text{val}(L) = 1$ and $\text{val}(L) \leq \delta$ for every constant $\delta > 0$ and thus we obtain the result in Theorem 1.3. The core component in this paradigm is the design of a dictatorship test: a Max NBTW instance on $[q]^L \cup [q]^R$, for integers $q$ and label sets $L$ and $R$. Let $\pi$ be a map $R \rightarrow L$. Each constraint is a tuple $(x, y, z)$ where $x \in [q]^L$, while $y, z \in [q]^R$. The distribution of tuples is obtained as follows.

First, pick $x$ and $y$ uniformly at random from $[q]^L$, and $[q]^R$. Set $z_j = y_j + x_{\pi(j)} \mod q$. Finally, add noise by independently replacing each coordinate $x_i$, $y_j$ and $z_j$ with a uniformly random element from $[q]$ with probability $\gamma$.

This test instance has canonical assignments that satisfy almost all the constraints. These are obtained by picking an arbitrary $j \in [R]$, and partitioning the variables into $q$ sets $S_0, \ldots, S_q-1$ where $S_t = \{x \in [q]^R | x_{\pi(j)} = t\} \cup \{y \in [q]^L | y_j = t\}$. If a constraint $(x, y, z)$ is so that $x \in S_t$, $y \in S_u$ then $z \notin S_v$ for any $v \in \{t+1, \ldots, u-1\}$ except with probability $O(\gamma)$. This is because $(a + b) \mod q$ is never strictly between $a$ and $b$. Further, the probability that any two of $x$, $y$, and $z$ fall in the same set $S_i$ is simply the probability that any two of $x_{\pi(j)}$, $y_j$, and $z_j$ are assigned the same value, which is at most $O(1/q)$. Thus, ordering the variables such that $S_0 < S_1 < \ldots < S_q-1$ with an arbitrary ordering of the variables within a set satisfies a fraction $1 - O(1/q) - O(\gamma)$ constraints.

The proof of Theorem 1.3 requires a partial converse of the above: every ordering that satisfies more than a fraction $2/3 + \varepsilon$ of the constraints is more-or-less an ordering that depends on a few coordinates $j$ as above. This proof involves three steps. First, we show that there is a $\Gamma = \Gamma(q, \gamma, \beta)$ such that every ordering $O$ of $[q]^L$ or $[q]^R$ can be broken into $\Gamma$ sets $S_0, \ldots, S_{\Gamma-1}$ such that one achieves expected value at least $\text{val}(O) - \beta$ for arbitrarily small $\beta$ by ordering the sets $S_0 < \ldots < S_{\Gamma-1}$ and within each set ordering elements randomly. The proof of this “bucketing” uses hypercontractivity of noised functions from a finite domain. We note that a related bucketing argument is used in proving inapproximability of OCSPs assuming the UGC [GMR08, GHM+11]. Their bucketing argument is tied to the use of the UGC, where $|L| = |R|$ for the corresponding dictatorship test, and does not extend to our setting. In particular, our approach yields $\Gamma \gg q$ while they crucially require $\Gamma \ll q$ in their work. We believe that our bucketing argument is more general and a useful primitive.

Then, similarly to [GMR08, GHM+11], the bucketing argument allows an OCSP to be analyzed as if it were a CSP on a finite domain, enabling us to use powerful techniques developed in this setting. In particular, we show that unless $\text{val}(L) > \delta$, the distribution of constraints $(x, y, z)$ can be regarded as obtained by sampling $x$ independently up to an error $\eta$ in the payoff; in other words, $x$ is “decoupled” from $(y, z)$. We note that the marginal distribution of the tuple $(y, z)$ is already symmetric with respect to swaps: $P(y = y, z = z) = P(y = z, z = y)$. In order to prove approximation resistance, we combine three of these dictatorship tests: the $j$th variant has $x$ as the $j$th component of the 3-tuple. We show that the combined instance is symmetric with respect to every swap up to an error $O(\eta)$ unless $\text{val}(L) > \delta$. This implies that the instance has value at most $2/3 + O(\eta)$ hence proving approximation resistance of Max NBTW.

For Max BTW and Max 2t-SO, we do not require the final symmetrization and instead use a dictatorship test based on a different distribution. Finally, the reduction to MAS is a simple gadget reduction from Max NBTW. For hardness results of width-two predicates, such gadget
reductions presently dominate the scene of classical CSPs and also define the state of affairs for MAS. As an example, the best-known \( \text{NP} \)-hard approximation hardness of \( 16/17 + \varepsilon \) for \text{MAX CUT} is via a gadget reduction from \text{MAX 3-Lin-2} [Hás01, TSSW00]. The previously best approximation hardness of MAS was also via a gadget reduction from \text{MAX 3-Lin-2} [New01], although with the significantly smaller gap \( 65/66 + \varepsilon \). By reducing from a problem more similar to MAS, namely \text{MAX NBTW}, we improve to the approximation hardness to \( 14/15 + \varepsilon \). The gadget in question is quite simple and we have in fact already seen it in Section 1.

**Organization.** Section 2 sets up the notation used in the rest of the article. Section 3 gives a general hardness result based on a test distribution which is subsequently used in Section 4 to derive our main results. The proof of the soundness of the general hardness reduction is largely given in Section 5.

## 2 Preliminaries

We denote by \([n]\) the integer interval \([0, \ldots, n - 1]\). Given a tuple of reals \( \mathbf{x} \in \mathbb{R}^m \), we write \( \sigma(x) \in S_m \) for the natural-order permutation on \( \{1, \ldots, m\} \) induced by \( \mathbf{x} \). For a distribution \( \mathcal{D} \) over \( \Omega_1 \times \cdots \times \Omega_m \), we use \( \mathcal{D}_{\leq t} \) and \( \mathcal{D}_{> t} \) to denote the projection to coordinates up to \( t \) and the remaining, respectively.

### 2.1 Ordering Constraint Satisfaction Problems.

We are concerned with predicates \( \mathcal{P} : S_m \rightarrow [0,1] \) on the symmetric group \( S_m \). Such a predicate specifies a width-\( m \) \text{OCSP} written as \text{OCSP}(\mathcal{P}). An instance \( \mathcal{I} \) of \text{OCSP}(\mathcal{P}) problem is a tuple \((\mathcal{X}, \mathcal{C})\) where \( \mathcal{X} \) is the set of variables and \( \mathcal{C} \) is a distribution over ordered \( m \)-tuples of \( \mathcal{X} \) referred to as the constraints.

An assignment to \( \mathcal{I} \) is an injective map \( \mathcal{O} : \mathcal{X} \rightarrow \mathbb{Z} \) called an ordering of \( \mathcal{X} \). For a tuple \( \mathbf{c} = (v_1, \ldots, v_m) \), \( \mathcal{O}_c \) denotes the tuple \( (\mathcal{O}(v_1), \ldots, \mathcal{O}(v_m)) \). An ordering is said to satisfy the constraint \( c \) when \( \mathcal{P}(\sigma(\mathcal{O}_c)) = 1 \). The value of an ordering is the probability that a random constraint \( c \leftarrow \mathcal{C} \) is satisfied by \( \mathcal{O} \) and the value \( \text{val}(\mathcal{I}) \) of an instance is the maximum value of any ordering. Thus,

\[
\text{val}(\mathcal{I}) = \max_{\mathcal{O} : \mathcal{X} \rightarrow \mathbb{Z}} \text{val}(\mathcal{O}; \mathcal{I}) = \max_{\mathcal{O} : \mathcal{X} \rightarrow \mathbb{Z}} \mathbb{E} \left[ \mathcal{P}(\mathcal{O}_c) \right].
\]

We extend the definition of value to include orderings that are not strictly injective as follows. Extend the predicate to \( \mathcal{P} : \mathbb{Z}^m \rightarrow [0,1] \) by setting \( \mathcal{P}(a_1, \ldots, a_m) = \mathbb{E}_\sigma [\mathcal{P}(\sigma)] \) where \( \sigma \) is drawn uniformly at random over all permutations in \( S_m \) such that \( \sigma_i < \sigma_j \) whenever \( a_i < a_j \). Note that the value of an instance is unchanged by this extension as there is always a complete ordering that attains the value of a non-injective map.

We define the predicates and problems studied in this work. MAS is exactly OCSP(\{(1,2)\}). The betweenness predicate BTW is the set \{(1,3,2), (3,1,2)\} and NBTW is \( S_3 \setminus \text{BTW} \). We define MAX BTW as \text{OCSP}(\text{BTW}) and MAX NBTW as \text{OCSP}(\text{NBTW}). Finally, introduce \( 2t \)-SO as the subset of \( S_{2t} \) such that the induced ordering on the first \( t \) elements coincides with the ordering of the last \( t \) elements, i.e.

\[
2t \text{-SO} \overset{\text{def}}{=} \{ \pi \in S_{2t} | \sigma(\pi(1), \ldots, \pi(r)) = \sigma(\pi(r + 1), \ldots, \pi(2t)) \}.
\]
This predicate has \((20)/\varepsilon\) satisfying orderings and will be used in proving the inapproximability of the generic \(m\)-OCSP with constraints of width at most \(m\).

### 2.2 Label Cover and Inapproximability.

The problem LC is a common starting point of strong inapproximability results. An LC instance \(\mathcal{L} = (U, V, E, L, R, \Pi)\) consists of a bipartite graph \((U \cup V, E)\) associating with every edge \(u, v\) a projection \(\pi_{uv} : R \rightarrow L\) with the goal of labeling the vertices, \(\lambda : U \cup V \rightarrow L \cup R\), to maximize the fraction of projections for which \(\pi_{uv}(\lambda(v)) = \lambda(u)\). The following well-known hardness result follows from the PCP Theorem [ALM\textsuperscript{+}98] and the Parallel Repetition Theorem [Raz\textsuperscript{98}].

**Theorem 2.1.** For every \(\varepsilon > 0\), there exists fixed label sets \(L \text{ and } R\) such that it is \(\text{NP}\)-hard to distinguish LC instances of value one from instances of value at most \(\varepsilon\).

### 2.3 Primer on Real Analysis

We refer to a finite domain \(\Omega\) along with a distribution \(\mu\) as a probability space. Given a probability space \((\Omega, \mu)\), the \(n\)th tensor power is \((\Omega^n, \mu^\otimes n)\) where \(\mu^\otimes n(\omega_1, \ldots, \omega_n) = \mu(\omega_1) \cdots \mu(\omega_n)\). The \(\ell_p\) norm of \(f : \Omega \rightarrow \mathbb{R}\) w.r.t. \(\mu\) is denoted by \(\|f\|_{\mu, p}\) and is defined as \(\mathbb{E}_{\mathbf{x} \sim \mu} [\|f(\mathbf{x})\|^p]^{1/p}\) for real \(p \geq 1\) and \(\max_{x \in \text{supp}(\mu)} f(x)\) for \(p = \infty\). When clear from the context, we omit the distribution \(\mu\). The following noise operator and its properties play a pivotal role in our analysis.

**Definition 2.2.** Let \((\Omega, \mu)\) be a probability space and \(f : \Omega^n \rightarrow \mathbb{R}\) be a function on the \(n\)th tensor power. For a parameter \(\rho \in [0, 1]\), the noise operator \(T_\rho\) takes \(f\) to \(T_\rho f : \Omega \rightarrow \mathbb{R}\) defined by

\[
T_\rho f(x) = \mathbb{E} [f(y)|x],
\]

where the \(i\)th coordinate of \(y\) is chosen as \(y_i = x_i\) with probability \(\rho\) and otherwise as an independent new sample.

The noise operator preserves the mass \(\mathbb{E} [f]\) of a function while spreading it in the space. The quantitative bound on this is referred to as hypercontractivity.

**Theorem 2.3** ([Wol\textsuperscript{07}; Theorem 3.16, 3.17 of [Mos\textsuperscript{10}]].) Let \((\Omega, \mu)\) be a probability space in which the minimum nonzero probability of any atom is \(\alpha < 1/2\). Then, for every \(q > 2\) and every \(f : \Omega^n \rightarrow \mathbb{R}\),

\[
\|T_\rho(q) f\|_q \leq \|f\|_2,
\]

where for \(\alpha < 1/2\) we set \(A = \frac{1-\alpha}{\alpha}; 1/q' = 1 - 1/q\); and \(\rho(q, \alpha) = \left(\frac{A^{1/q} - A^{-1/q}}{A^{1/q'} - A^{-1/q'}}\right)^{1/2}\). For \(\alpha = 1/2\), we set \(\rho(q) = (q - 1)^{-1/2}\).

For a fixed probability space, the above theorem says that for every \(\gamma > 0\), there is a \(q > 2\) such that \(\|T_{1-\gamma} f\|_q \leq \|f\|_2\). For our application, we need the easy corollary that the reverse direction also holds: for every \(\gamma > 0\), there exists a \(q > 2\) such that hypercontractivity to the \(\ell_2\)-norm holds.

**Lemma 2.4.** Let \((\Omega, \mu)\) be a probability space in which the minimum nonzero probability of any atom is \(\alpha \leq 1/2\). Then, for every \(f : \Omega^n \rightarrow \mathbb{R}\), small enough \(\gamma > 0\),

\[
\|T_{1-\gamma} f\|_{2+\delta} \leq \|f\|_2
\]

for any \(0 < \delta \leq \delta(\gamma, \alpha) = 2^{\log((1-\gamma)^{-2})-1} / \log(A)\) with \(A = \frac{1-\alpha}{\alpha} > 1\). Further, \(\delta(\gamma, 1/2) = \gamma(2-\gamma)(1-\gamma)^{-2}\).
Proof. The estimate for \(\delta(\gamma, 1/2)\) follows immediately from the above theorem assuming \(\gamma < 1/2\). In the case when \(\alpha < 1/2\), solving \(\rho^2 \overset{\text{def}}{=} (1 - \gamma)^2 = (A^{1/q} - A^{-1/q})(A^{1-1/q} - A^{1/q-1})^{-1}\) for \(q\) gives, for \(\gamma < 1 - A^{-1/2}\),
\[
\delta = q - 2 = \frac{2 \log(A)}{\log(1 + \rho^2 A)} - 2 \geq \frac{2 \log((1 - \gamma)^2) - 1}{\log(A)}.
\]
\[\square\]

Efron-Stein Decompositions. Our proofs make use of the Efron-Stein decomposition which has useful properties akin to Fourier decompositions.

Definition 2.5. Let \(f : \Omega^{(1)} \times \cdots \times \Omega^{(n)} \to \mathbb{R}\) and \(\mu\) a measure on \(\prod \Omega^{(t)}\). The Efron-Stein decomposition of \(f\) with respect to \(\mu\) is defined as \(\{f_S\}_{S \subseteq [n]}\) where \(f_S(x) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \mathbb{E}[f(x') \mid x'_T = x_T]\).

Lemma 2.6 (Efron and Stein [ES81], and Mossel [Mos10]). Assuming \(\{\Omega^{(t)}\}_{t}\) are independent, the Efron-Stein decomposition \(\{f_S\}_S\) of \(f : \prod \Omega^{(t)} \to \mathbb{R}\) satisfies:
- \(f_S(x)\) depends only on \(x_S\),
- for any \(S, T \subseteq [m]\), and \(x_T \in \prod_{t \in T} \Omega^{(t)}\) such that \(S \setminus T \neq \emptyset\), \(\mathbb{E}[f_S(x') \mid x'_T = x_T] = 0\).

We use the standard notion of influence and noisy influence as in previous work.

Definition 2.7. Let \(f : \Omega^n \to \mathbb{R}\) be a function on a probability space. The influence of the \(1 \leq i \leq n\) coordinate is
\[\text{Inf}_i(f) = \mathbb{E}[\text{Var}_{\Omega_i}(f)]\,.
\]
Additionally, given a noise parameter \(\gamma\), the noisy influence of the \(i^{th}\) coordinate is
\[\text{Inf}_i^{(1-\gamma)}(f) = \mathbb{E}[\text{Var}_{\Omega_i}(T_{1-\gamma}f)]\,.
\]

The following bounds on the noisy influence are handy for the analysis.

Lemma 2.8. For every \(\gamma > 0\), natural numbers \(i\) and \(n\) such that \(1 \leq i \leq n\), and every \(f : \Omega^n \to \mathbb{R}\),
\[\text{Inf}_i^{(1-\gamma)}(f) \leq \text{Var}(f).
\]

Lemma 2.9. For every \(\gamma > 0\), and every \(f : \Omega^n \to \mathbb{R}\),
\[\sum_i \text{Inf}_i^{(1-\gamma)}(f) \leq \frac{\text{Var}(f)}{\gamma}.
\]

We introduce the notion of cross influence between functions which is a notion implicitly prevalent in modern LC reductions, either for noised or for analytically similar degree-bounded functions:
\[\text{CrInf}_\pi^{(1-\gamma)}(f, g) \overset{\text{def}}{=} \sum_{(i,j) \in \pi} \text{Inf}_i^{(1-\gamma)}(f) \text{Inf}_j^{(1-\gamma)}(g).
\]

We note that our definition differs somewhat from the cross-influence, denoted ‘XInf’, used by Samorodnitsky and Trevisan [ST09].
3 A General Hardness Result

In this section, we prove a general inapproximability result for OCSPs which, similar to results for classic CSPs, permit us to deduce hardness of approximation based on the existence of certain simple distributions. The proof is via a scheme of reductions from LC to OCSPs. For an \(m\)-width predicate \(P\), we instantiate this scheme with a distribution \(D\) over \(Q_1^t \times Q_2^{m-t}\); for some parameters \(t\), \(Q_1\), and \(Q_2\); to obtain a reduction to OCSP(\(P\)) instances. Straightforward applications of this result using appropriate distributions yields Theorems 1.2 to 1.4.

The reduction itself is composed of pieces known as dictatorship test which is described in the next section. Section 3.2 uses this test to construct the overall reduction and also contains the properties of this reduction. Throughout this section, we assume \(P\) is the \(m\)-width predicate of interest and that \(D\) is the distribution of the appropriate signature.

3.1 Dictatorship Test

The dictatorship test uses a distribution parametrized by \(\gamma\), and \(\pi\) and is denoted by \(T_{\pi}(\gamma)(D)\); its definition follows.

**Procedure 3.1 (Test Distribution).**

**Parameters:**
- distribution \(D\) over \(Q_1^t \times \cdots \times Q_1^t \times Q_2 \times \cdots \times Q_2^{m-t}\);
- noise probability, \(\gamma > 0\);
- projection map \(\pi : R \rightarrow L\);

**Output:** Distribution \(T_{\pi}(\gamma)(D)\) over
\[
(x^{(1)}, \ldots, x^{(t)}, y^{(t+1)}, \ldots, y^{(m)}) \in (Q_1^t \times \cdots \times Q_1^t) \times (Q_2^R \times \cdots \times Q_2^R).
\]

1. pick a random \(|L| \times t\) matrix \(X\) over \(Q_1\) by letting each row be a sample from \(D_{\leq t}\), independently.

2. pick a random \(|R| \times (m-t)\) matrix \(Y \equiv (y^{(t+1)}, \ldots, y^{(m)})\) over \(Q_2\) by letting the \(i^{th}\) row be a sample from \(D_{\geq t}\) conditioned on \(D_{\leq t} = X_{\pi(i)} = (x_{\pi(i)}^{(1)}, \ldots, x_{\pi(i)}^{(t)})\).

3. for each entry of \(X\) (resp. \(Y\)) independently, replace it with a sample from \(Q_1\) (resp. \(Q_2\)) with probability \(\gamma\).

4. output \((X, Y)\).

Recall our convention from Section 2.1 of extending \(P\) to a predicate \(P : \mathbb{Z}^m \rightarrow [0,1]\). For a pair of functions \((f, g)\), let \((f, g) \circ (X, Y)\) denote the tuple \((f(x^{(1)}), \ldots, f(x^{(t)}), g(y^{(t+1)}), \ldots, g(y^{(m)}))\).

Then, the acceptance probability on \(T_{\pi}(\gamma)(D)\) for a pair of functions \((f, g)\) where \(f : Q_1^t \rightarrow \mathbb{Z}\) and \(g : Q_2^R \rightarrow \mathbb{Z}\) is:

\[
\text{Acc}_{f,g}(T_{\pi}(\gamma)(D)) \equiv \mathbb{E}_{(X,Y) \leftarrow T_{\pi}(\gamma)(D)}[P((f, g) \circ (X, Y))].
\] (1)
This definition is motivated by the overall reduction described in the next section. The distribution is designed so that functions \((f, g)\) that are dictated by a single coordinate have a high acceptance probability, justifying the name of the test.

**Lemma 3.2.** Let \(g : Q_2^R \to \mathbb{Z}\) and \(f : Q_1^I \to \mathbb{Z}\) be defined by \(g(y) = y_i\) and \(f(x) = x_{\pi(i)}\) for some \(i \in R\). Then, \(\text{Acc}_{f,g}(T_{\pi}^{(\gamma)}(\mathcal{D})) \geq E_{x \sim \mathcal{D}}[P(x)] - \gamma m\).

**Proof.** The vector \((f(x^{(1)}), \ldots, f(x^{(t)}), g(y^{(t+1)}), \ldots, g(y^{(m)})\)) simply equals the \(\pi(i)\)th row of \(X\) followed by the \(i\)th row of \(Y\). With probability \((1 - \gamma)^m \geq 1 - \gamma m\) this is a sample from \(\mathcal{D}\) and is hence accepted by \(\mathcal{P}\) with probability at least \(E_{x \sim \mathcal{D}}[P(x)] - \gamma m\). \(\square\)

We prove a partial converse of the above: unless \(f\) and \(g\) have influential coordinates \(i\) and \(j\) such that \(\pi(j) = i\), the distribution \(\mathcal{D}\) can be replaced by a product of two distributions with a negligible loss in the acceptance probability. We define this product distribution below and postpone the analysis to Section 5.2 in order to complete the description of the reduction.

**Definition 3.3.** Given the base distribution \(\mathcal{D}\), the decoupled distribution \(\mathcal{D}^\perp\) is obtained by taking independent samples \(x\) from \(\mathcal{D}_{\leq t}\) and \(y\) from \(\mathcal{D}_{>t}\).

### 3.2 Reduction from Label Cover

**Procedure 3.4 (Reduction \(R_{\mathcal{D},\gamma}^{(\mathcal{P})}\)).**

**Parameters:** distribution \(\mathcal{D}\) over \(Q_1^I \times Q_2^{m-t}\) and noise parameter \(\gamma > 0\).

**Input:** a Label Cover instance \(\mathcal{L} = (U, V, E, L, R, \Pi)\).

**Output:** a weighted OCSP(\(P\)) instance \(\mathcal{I} = (\mathcal{X}, \mathcal{C})\) where \(\mathcal{X} = (U \times Q_1^I) \cup (V \times Q_2^P)\). The distribution of constraints in \(\mathcal{C}\) is obtained by sampling a random edge \(e = (u, v) \in E\) with projection \(\pi_e\) and then \((X, Y)\) from \(T_{\pi_e}^{(\gamma)}(\mathcal{D})\); the constraint is the predicate \(\mathcal{P}\) applied on the tuple \(((u, x^{(1)}), \ldots, (u, x^{(t)}), (v, y^{(t+1)}), \ldots, (v, y^{(m)})\)).

An assignment to \(\mathcal{I}\) is seen as a collection of functions, \(\{f_u\}_{u \in U} \cup \{g_v\}_{v \in V}\), where \(f_u : Q_1^I \to \mathbb{Z}\) and \(g_v : Q_2^R \to \mathbb{Z}\). The value of an assignment is:

\[
E_{e=(u,v) \in E, (X,Y) \in T_{\pi_e}^{(\gamma)}(\mathcal{D})} \mathcal{P}((f_u, g_v) \circ (X, Y)) = E_{e=(u,v) \in E} [\text{Acc}_{f_u, g_v}(T_{\pi_e}^{(\gamma)}(\mathcal{D}))].
\]

**Lemma 3.5.** If \(\lambda\) is a labeling of \(\mathcal{L}\) satisfying a fraction \(c\) of its constraints, then the ordering assignment \(f_u(x) = x_{\lambda(u)}\), \(g_v(y) = y_{\lambda(v)}\) satisfies at least a fraction \(c \cdot (E_{x \sim \mathcal{D}}[\mathcal{P}(\sigma(x))]) - \gamma m\) of the constraints of \(R_{\mathcal{D},\gamma}^{(\mathcal{P})}(\mathcal{L})\). In particular, there is an ordering of \(R_{\mathcal{D},\gamma}^{(\mathcal{P})}(\mathcal{L})\) attaining a value \(\text{val}(\mathcal{L}) \cdot (E_{x \sim \mathcal{D}}[\mathcal{P}(\sigma(x))]) - \gamma m\) that is oblivious to the distribution \(\mathcal{D}\).

On the other hand, we also extend the decoupling property of the dictatorship test to the instance output if \(\text{val}(\mathcal{L})\) is small. This is the technical core of the paper and is proved in Section 5.
Theorem 3.6. Suppose that $D$ over $Q_1^t \times Q_2^{m-t}$ satisfies the following properties:

- $D$ has uniform marginals.
- For every $i > t$, $D_i$ is independent of $D_{\leq t}$.

Then, for every $\varepsilon > 0$ and $\gamma > 0$ there exists $\varepsilon_{LC} > 0$ such that if $\text{val}(L) \leq \varepsilon_{LC}$ then for every assignment $A = \{f_u\}_{u \in U} \cup \{g_v\}_{v \in V}$ to $I$ it holds that

$$\text{val}(A; R^{(P)}_{D,\gamma}(L)) \leq \text{val}(A; R^{(P)}_{D_{\perp \gamma}}(L)) + \varepsilon.$$ 

In particular, $\text{val}(R^{(P)}_{D,\gamma}(L)) \leq \text{val}(R^{(P)}_{D_{\perp \gamma}}(L)) + \varepsilon$.

4 Applications of the General Result

In this section, we prove the inapproximability of $\text{Max BTW}$, $\text{Max NBTW}$, and $\text{Max 2t-SO}$, using the general hardness result of Section 3. We also prove the hardness of $\text{MAS}$ using a gadget reduction from $\text{Max NBTW}$.

4.1 Hardness of Maximum Betweenness

For an integer $q$, define the distribution $D$ over $\{-1, q\} \times [q] \times [q]$ by picking $x_1 \sim \{-1, q\}$, $y_2 \sim [q]$, and setting $y_3 = y_2 + 1 \mod q$ if $x_1 = q$ and $y_2 - 1$ otherwise. This distribution has the following properties which can be readily verified.

Proposition 4.1. Let $(x_1, y_2, y_3) \sim D$. Then the following holds:

1. $D$ has uniform marginals.
2. The marginals $y_2$ and $y_3$ are independent of $x$.
3. $(y_2, y_3)$ has the same distribution as $(y_3, y_2)$.
4. $E_{x,y_2,y_3 \sim D}[\text{BTW}(x_1, y_2, y_3)] \geq 1 - 1/q$.

Let $D_{\perp}$ be the decoupled distribution of $D$ which draws the first coordinate independently of the remaining and $\gamma > 0$ a noise parameter. Given a LC instance $L$ and consider applying Reduction 3.4 to $L$ with test distributions $D$ and $D_{\perp}$, obtaining $\text{Max BTW}$ instances $I = R_{D,\gamma}(L)$ and $I_{\perp} = R_{D_{\perp},\gamma}(L)$.

Lemma 4.2 (Completeness). If $\text{val}(L) = 1$ then $\text{val}(I) \geq 1 - 1/q - 3\gamma$.

Proof. This is an immediate corollary of Lemma 3.5 and Proposition 4.1. \qed

Lemma 4.3 (Soundness). For every $\varepsilon > 0$, $\gamma > 0$, $q$, there is an $\varepsilon_{LC} > 0$ such that if $\text{val}(L) \leq \varepsilon_{LC}$ then $\text{val}(I) \leq 1/2 + \varepsilon$. 

10
Proof. We note that Proposition 4.1 asserts that $D$ satisfies the conditions of Theorem 3.6 and it suffices to show $\text{val}(I^\perp) \leq 1/2$. Let $\{f_u : \{0, 1\}^L \to \mathbb{Z}\}_{u \in U}, \{g_v : [q]^R \to \mathbb{Z}\}_{v \in V}$ be an arbitrary assignment to $I^\perp$. Fix an LC edge $\{u, v\}$ with projection $\pi$ and consider the mean value of constraints produced for this edge by the construction:

$$E_{x^{(1)}, y^{(2)}, y^{(3)} \leftarrow I^\perp_{\pi} (D^\perp)} \left[ \text{BTW}(f_u(x^{(1)}), g_v(y^{(2)}), g_v(y^{(3)})) \right].$$

(2)

As noted in Proposition 4.1, $(y^{(2)}, y^{(3)})$ has the same distribution as $(y^{(3)}, y^{(2)})$ when drawn from $D$. Consequently, when drawing arguments from the decoupled test distribution, the probability of a specific outcome $(x^{(1)}, y^{(2)}, x^{(3)})$ equals the probability of $(x^{(1)}, y^{(3)}, x^{(2)})$. For strict orderings, at most one of the two can satisfy the predicate BTW. Thus, the expression in (2), and in effect $\text{val}(I^\perp)$, is bounded by $1/2$. □

Theorem 1.2 is now an immediate corollary of Lemmas 4.2 and 4.3, taking $q = \lceil 2/\varepsilon \rceil$ and $\gamma = \varepsilon/6$.

### 4.2 Hardness of Maximum Non-Betweenness

For an implicit parameter $q$, define a distribution $D$ over $[q]^3$ by picking $x_1, x_2 \sim [q]$ and setting $x_3 = x_1 + x_2 \mod q$.

**Proposition 4.4.** The distribution $D$ satisfies the following:

1. $D$ is pairwise independent with uniform marginals,
2. and $E_{x_1, x_2, x_3 \sim D} [\text{NBTW}(x_1, x_2, x_3)] \geq 1 - 3/q$.

A straightforward application of the general inapproximability with $t = 1$ shows that $x_1$ is decoupled from $x_2$ and $x_3$ unless $\text{val}(L)$ is large. Further, pairwise independence implies that the decoupled distribution is simply the uniform distribution over $[q]^3$. However, this does not suffice to prove approximation resistance and in fact the value could be greater than $2/3$. To see this, note that if $\{f_u\}_{u \in U}, \{g_v\}_{v \in V}$ is an ordering of the instance from the reduction, then the first coordinate of every constraint is a variable of the form $f_u(\cdot)$ while the rest are $g_v(\cdot)$. Thus, ordering the $f_u(\cdot)$ variables in the middle and randomly ordering $g_v(\cdot)$ on both sides satisfies a fraction $3/4$ of the constraints.

To remedy this and prove approximation resistance, we permute $D$ by swapping the last coordinate with each of the remaining coordinates and overlay the instances obtained by the reduction obtained from these respective distributions. More specifically, for $1 \leq j \leq 3$, define $D_j$ as the distribution over $[q]^3$ obtained by first sampling from $D$ and then swapping the $j$th and third coordinate, i.e., the $j$th coordinate is the sum of the other two which are picked independently at random. Similarly, define NBTW$_j$ as the ordering predicate which is true if the $j$th argument does not lie between the other two, e.g., NBTW$_3$ = NBTW.

As in the previous section, take a LC instance $L$ and consider applying Reduction 3.4 to $L$ with the distributions $D_j$, and write $I_j = R_{D_j, \gamma}^{\text{NBTW}_j}(L)$. Similarly write $I^\perp_j = R_{D^\perp_j, \gamma}^{\text{NBTW}_j}(L)$ for the corresponding decoupled instances.

As the distributions $D_j$ are over the same domain $[q]^3$, the instances $I_1, I_2, I_3$ are over the same variables. We define a new instance $I$ over the same variables as the “sum” $\frac{1}{3} \sum_{j \in [3]} I_j$, defined by taking all constraints in $I_1, I_2, I_3$ with multiplicities and normalizing their weights by $1/3$. 

11
Lemma 4.5 (Completeness). If \( \text{val}(L) = 1 \) then \( \text{val}(I) \geq 1 - 3/q - 3\gamma \).

Proof. This is an immediate corollary of Lemma 3.5 and Proposition 4.4.

Lemma 4.6 (Soundness). For every \( \varepsilon > 0, \gamma > 0, q \), there is an \( \varepsilon_{LC} > 0 \) such that if \( \text{val}(L) \leq \varepsilon_{LC} \) then \( \text{val}(I) \leq 2/3 + \varepsilon \).

Proof. Again our goal is to use Theorem 3.6 and we start by bounding \( \text{val}(I^\perp) \). To do this, note that the decoupled distributions \( D_j \) are in fact the uniform distribution over \( [q]^3 \) and in particular do not depend on \( j \). This means that the distributions of variables which NBTW \( j \) is applied to in \( I_j^\perp \) is independent of \( j \), e.g., if \( I_j^\perp \) contains the constraint NBTW \( (z_1, z_2, z_3) \) with weight \( w \) then \( I_j^\perp \) contains the constraint NBTW \( (z_1, z_2, z_3) \) with the same weight. In other words, \( I_j^\perp \) can be thought of as having constraints of the form \( E_j[NBTW_j(z_1, z_2, z_3)] \). It is readily verified that \( E_j[NBTW_j(a, b, c)] \leq 2/3 \) for every \( a, b, c \).

Getting back to the main task – bounding \( \text{val}(I) \) – fix an arbitrary assignment \( A = \{ f_v : [q]^L \}_{v \in V} \cup \{ g_v : [q]^R \}_{v \in V} \) of \( I \). By Theorem 3.6, \( \text{val}(A; I_j) \leq \text{val}(A; I_j^\perp) + \varepsilon \) for \( j \in \{3\} \). It follows that \( \text{val}(A; I) \leq \text{val}(A; I^\perp) + \varepsilon \) and therefore, since \( A \) was arbitrary, it holds that \( \text{val}(I) \leq \text{val}(I^\perp) + \varepsilon \leq 2/3 + \varepsilon \), as desired.

4.3 Hardness of Maximum Acyclic Subgraph

The inapproximability of MAS is from a simple gadget reduction from MAX NBTW. We claim the following properties of the directed graph shown in Section 1, defined formally as follows.

Definition 4.7. Define the MAS gadget \( H \) as the directed graph \( H = (V, A) \) where \( V = \{x, y, z, a, b\} \) and \( A \) consists of the walk

\[
b \rightarrow x \rightarrow a \rightarrow z \rightarrow b \rightarrow y \rightarrow a.
\]

Lemma 4.8. Consider an ordering \( O \) of \( x, y, z \). Then,

1. if \( \text{NBTW}(O(x), O(y), O(z)) = 1 \), then \( \max_{O'} \text{val}(O'; H) = 5/6 \) where the max is over all extensions \( O' : V \rightarrow \mathbb{Z} \) of \( O \) to \( V \).

2. if \( \text{NBTW}(O(x), O(y), O(z)) = 0 \), then \( \max_{O'} \text{val}(O'; H) = 4/6 \) where the max is over all extensions of \( O \) to \( V \).

Proof. To find the value of the gadget \( H \), we individually consider the optimal placement of \( a \) and \( b \) relative \( x, y, z \). There are three edges in which the respective variables appear: \( a \) appears in \( (x, a), (y, a) \) and \( (a, z) \); while \( b \) appears in \( (z, b), (b, x) \), and \( (b, y) \).

From this, we gather that two out of the three respective constraints can always be satisfied by placing \( a \) after \( x, y, z \) and similarly placing \( b \) before \( x, y, z \). We also see that all three constraints involving \( a \) can be satisfied if and only if \( z \) comes after both \( x \) and \( y \). Similarly, satisfying all three constraints involving \( b \) is possible if and only if \( z \) comes before both \( x \) and \( y \). From this, one concludes that if \( \text{NBTW}(x, y, z) = 1 \), i.e., if \( z \) comes first or last, then we can satisfy five out of the six constraints, whereas if \( z \) is the middle element of \( O \), we can satisfy only four out of the six constraints.

The proof of Theorem 1.1 is now a routine application of the MAS gadget.
Proof of Theorem 1.1. Given an instance \(\mathcal{I}\) of \(\text{Max NBTW}\), construct a directed graph \(G\) by replacing each constraint \(\text{NBTW}(x, y, z)\) of \(\mathcal{I}\) with a MAS gadget \(H\), identifying \(x, y, z\) with the vertices \(x, y, z\) of \(H\) and using two new vertices \(a, b\) for each constraint of \(\mathcal{I}\).

By Lemma 4.8, it follows that \(\text{val}(G) = \frac{5}{6}\text{val}(\mathcal{I}) + \frac{1}{6}(1 - \text{val}(\mathcal{I}))\). By Theorem 1.3, it is \(\text{NP-hard to distinguish between val}(\mathcal{I}) \geq 1 - \varepsilon\) and \(\text{val}(\mathcal{I}) \leq 2/3 + \varepsilon\) for every \(\varepsilon > 0\), implying that it is \(\text{NP-hard to distinguish val}(G) \geq 5/6 - \varepsilon/6\) from \(\text{val}(G) \leq 7/9 + \varepsilon/6\), providing a hardness gap of \(\frac{7}{9} + \varepsilon' = 14/15 + \varepsilon'\).

\[\square\]

4.4 Hardness of Maximum 2t-Same Order

We establish the hardness of \(\text{Max 2t-SO}\), Theorem 1.4, via the approximation resistance of the relatively sparse predicate \(2t\)-SO. The proof is similar to the that of \(\text{Max BTW}\) (see Section 4.1).

Let \(q_1 < q_2\) be integer parameters and define the base distribution \(D\) over \([q_1]^t \times [q_2]^t\) as follows: draw \(x_1, \ldots, x_t\) uniformly at random from \([q_1]\), draw \(z\) uniformly at random from \([q_2]\), and for \(1 \leq j \leq t\) set \(y_j = x_j + z \mod q_2\). The distribution of \((x_1, \ldots, x_t, y_1, \ldots, y_t)\) defines \(D\). For a permutation \(\sigma \in S_t\), let \(1_\sigma(\cdot)\) be the ordering predicate which is \(1\) on \(\sigma\) and \(0\) on all other inputs.

Proposition 4.9. \(D\) satisfies the following properties.

1. \(D\) has uniform marginals.
2. For every \(i > t\), \(D_i\) is independent of \(D_{\leq t}\).
3. For every \(\sigma \in S_t\), \(E[1_\sigma(x_1, \ldots, x_t)] = E[1_\sigma(y_{t+1}, \ldots, y_m)] = 1/t!\)
4. \(E_{x_1, \ldots, x_t, y_{t+1}, \ldots, y_m \sim D}[2t\cdot\text{SO}(x_1, \ldots, x_t, y_{t+1}, \ldots, y_m)] \geq 1 - \frac{t^2}{2q_1} - \frac{q_1}{q_2}\).

Proof. The first three properties are immediate from the construction and recalling the extension of predicates to non-unique values. For the last property, note that \(2t\cdot\text{SO}(x_1, \ldots, x_t, y_{t+1}, \ldots, y_m) = 1\) if \(x_1, \ldots, x_t\) are distinct and \(z < q_2 - q_1\). The former event occurs with probability at least \(1 - \frac{t^2}{2q_1}\) and the latter independently with probability at least \(1 - q_1/q_2\); a union bound implies the claim. \(\square\)

As in the proof of Theorem 1.2, take a LC instance \(\mathcal{L}\) and let \(\mathcal{I} = R_{D, \gamma}^{2t\cdot\text{SO}}(\mathcal{L})\) and \(\mathcal{I}^\perp = R_{D, \gamma}^{2t\cdot\text{SO}}(\mathcal{L})\) be the instances produced by Reduction 3.4 using the base distribution \(D\) and the decoupled version \(D^\perp\) and some noise parameter \(\gamma > 0\).

The following lemma is an immediate corollary of Lemma 3.5 and Proposition 4.9, Item 4.

Lemma 4.10 (Completeness). If \(\text{val}(\mathcal{L}) = 1\) then \(\text{val}(\mathcal{I}) \geq 1 - \frac{t^2}{2q_1} - \frac{q_1}{q_2} - 3\gamma\).

For the soundness, we have the following.

Lemma 4.11 (Soundness). For every \(\varepsilon > 0, \gamma > 0\), and \(1 \leq q_1 \leq q_2\), there is an \(\varepsilon_{LC} > 0\) such that if \(\text{val}(\mathcal{L}) \leq \varepsilon_{LC}\) then \(\text{val}(\mathcal{I}) \leq \frac{1}{t!} + \varepsilon\).

Proof. As in the proof of Lemma 4.3, it suffices to prove \(\text{val}(\mathcal{I}^\perp) \leq 1/t!\). Let \(\{f_u : [q_1]^L \to \mathbb{Z}\}_{u \in U},\)
\(\{g_v : [q_2]^R \to \mathbb{Z}\}_{v \in V}\) be an arbitrary assignment to \(\mathcal{I}^\perp\). Fix an arbitrary edge \(\{u, v\}\) of \(\mathcal{L}\) with
projection \(\pi\). The value of constraints corresponding to \(\{u, v\}\) satisfied by the assignment is

\[
\mathbb{E}_{(X,Y) \in T_c^{(\gamma)(D^\perp)}} \left[ 2t^{\perp}\text{-SO}(f_u(x^{(1)}), \ldots, f_u(x^{(t)}), g_v(y^{(t+1)}), \ldots, g_v(y^{(m)})) \right] \\
= \sum_{\sigma \in S_t} \mathbb{E}_{(X,Y) \in T_c^{(\gamma)(D^\perp)}} \left[ 1_{\sigma} \left\{ f_u(x^{(1)}), \ldots, f_u(x^{(t)}) \right\} 1_{\sigma} \left\{ g_v(y^{(t+1)}), \ldots, g_v(y^{(m)}) \right\} \right] \\
= \sum_{\sigma \in S_t} \mathbb{E}_{(X,Y) \in T_c^{(\gamma)(D^\perp)}} \left[ 1_{\sigma} \left\{ f_u(x^{(1)}), \ldots, f_u(x^{(t)}) \right\} \right] \mathbb{E}_{(X,Y) \in T_c^{(\gamma)(D^\perp)}} \left[ 1_{\sigma} \left\{ g_v(y^{(t+1)}), \ldots, g_v(y^{(m)}) \right\} \right] \\
= 1/t!,
\]

where the penultimate step uses the independence of \(X\) and \(Y\) in the decoupled distribution, and the final step Item 3 of Proposition 4.9.

Theorem 1.4 is an immediate corollary of Lemmas 4.10 and 4.11, taking \(q_1 = \lceil 2t^2/\varepsilon \rceil\), \(q_2 = \lceil 3q_1/\varepsilon \rceil\) and \(\gamma = \varepsilon/9\).

5 Analysis of the Reduction

In this section we prove Theorem 3.6 which bounds the value of the instance generated by the reduction in terms of the decoupled distribution. Throughout, we fix an LC instance \(\mathcal{L}\), a predicate \(\mathcal{P}\), an OCSP instance \(\mathcal{I}\) obtained by the procedure \(R_{\mathcal{D}, \gamma}^{(\mathcal{P})}\) for a distribution \(\mathcal{D}\) and noise-parameter \(\gamma\), and finally an assignment \(A = \{f_u\}_{u \in U} \cup \{g_v\}_{v \in V}\).

The proof involves three major steps. First, we show that the assignment functions, which are \(Z\)-valued, can be approximated by functions on finite domains via bucketing (see Section 5.1). This approximation makes the analyzed instance value susceptible to tools developed in the context of finite-domain CSPs [Wen12, Cha13] which are used in Section 5.2 to prove the decoupling property of the dictatorship test. Finally, this decoupling is extended to the reduction hence bounding the value of \(\mathcal{I}\) (see Section 5.3).

5.1 Bucketing

For an integer \(\Gamma\), we approximate the function \(f_u : Q_1^L \rightarrow \mathbb{Z}\) by partitioning the domain into \(\Gamma\) pieces. Set \(q_1 = \lceil |Q_1| \rceil\) and partition the set \(Q_1^L\) into sets \(B_{i}^{(f_u)}\), \(B_{i+1}^{(f_u)}\) of size \(q_1^L/\Gamma\) such that if \(x \in B_{i}^{(f_u)}\) and \(y \in B_{j}^{(f_u)}\) for some \(i < j\) then \(f(x) < f(y)\). Note that this is possible as long as the parameter \(\Gamma\) divides \(q_1^L\) which will be the case. Let \(F_u : Q_1^L \rightarrow [\Gamma]\) specify the mapping of points to the bucket containing it, and \(F_u^{(a)} : Q_1^L \rightarrow \{0, 1\}\) the indicator of points assigned to \(B_{i}^{(f_u)}\). Partition \(g_v : Q_2^R \rightarrow \mathbb{Z}\) similarly into buckets \(\{B_{i}^{(g_v)}\}\) obtaining \(G_v : Q_2^R \rightarrow [\Gamma]\) and \(G_v^{(a)} : Q_2^R \rightarrow \{0, 1\}\).

Now we show that the acceptance probability of the dictatorship test – see (1) in Section 3 – applied to an edge \(e = (u, v)\) of the LC instance \(\mathcal{L}\) can be approximated by a bucketed version. Fix an edge \(e = (u, v)\) and put \(f = f_u, \ g = g_v\). As before, we denote a query tuple, \((x^{(1)}), \ldots, x^{(t)}, y^{(t+1)}, \ldots, y^{(m)})\) concisely as \((X, Y)\) and the tuple of assignments by a pair of functions \((f, g), (f(x^{(1)}), \ldots, f(x^{(t)}), g(y^{(t+1)}), \ldots, g(y^{(m)}))\) as \((f, g)\circ(X, Y)\). Define the bucketed payoff
function with respect to \( f \) and \( g \), \( \varphi^{(f,g)} : [\Gamma]^m \to [0,1] \) as:
\[
\varphi^{(f,g)}(a_1, \ldots, a_m) = \mathbb{E}_{x^{(i)} \sim B^{(f)}_i, i \leq t, y^{(j)} \sim B^{(g)}_j, i < j} [P((f,g) \circ (X,Y))]
\]
and the \text{bucketed} acceptance probability,
\[
\text{BAcc}^{f,g}(T^{(\gamma)}(D)) = \mathbb{E}_{x \sim (X,Y) \sim T^{(\gamma)}(D)} [\varphi^{(f,g)}((F,G) \circ (X,Y))].
\]

In other words, bucketing corresponds to generating a tuple \( a = (f,g) \circ (X,Y) \) and replacing each coordinate \( a_i \) with a random value from the bucket \( a_i \) fell in. We show that above is close to the true acceptance probability \( \text{Acc}^{f,g}(T^{(\gamma)}(D)) \).

\textbf{Theorem 5.1.} For every predicate \( P \), every distribution \( D \) with uniform marginals, every pair of orderings \( f : Q_1^L \to Z \) and \( g : Q_2^R \to Z \), every \( \gamma > 0 \), projection \( \pi : R \to L \), and every \( \Gamma \),
\[
|\text{Acc}^{f,g}(T^{(\gamma)}(D)) - \text{BAcc}^{f,g}(T^{(\gamma)}(D))| \leq m^2\Gamma^{-\delta},
\]
for some \( \delta = \delta(\gamma, Q) > 0 \) with \( Q = \max\{|Q_1|, |Q_2|\} \).

To prove this, we show that \( f \) and \( g \) have few overlapping pairs of buckets and that the probability of hitting any particular pair is small. Let \( R_a^{(f)} \) be the smallest interval in \( Z \) containing \( B_a^{(f)} \); and similarly \( R_a^{(g)} \).

\textbf{Lemma 5.2 (Few Buckets Overlap).} For every integer \( \Gamma \) there are at most \( 2\Gamma \) choices of pairs \((a,b) \in [\Gamma] \times [\Gamma]\) such that \( R_a^{(f)} \cap R_b^{(g)} \neq \emptyset \).

\textit{Proof.} Construct the bipartite intersection graph \( G_I = (U_I, V_I, E_I) \) where the vertex sets are disjoint copies of \( [\Gamma] \), and there is an edge between \( a \in U_I \) and \( b \in V_I \) iff \( R_a^{(f)} \cap R_b^{(g)} \neq \emptyset \). By construction of the buckets, the graph does not contain any pair of distinct edges \((u,v), (u',v')\) such that \( u < v \) and \( u' > v' \). Consequently, a vertex can have at most two neighbors with degree greater than one. Let \( A \) be the set of degree-one vertices. It follows that the maximum degree of the subgraph induced by \( A \) is at most two and contains at most \(|(U_I \cup V_I) \setminus A|\) edges. On the other hand, the number of edges incident to \( A \) is at most \(|A|\) implying a total of at most \(|U_I \cup V_I| = 2\Gamma\) intersections. \( \square \)

Next, we prove a bound on the probability that a fixed pair of the \( m \) queries fall in a fixed pair of buckets. For a distribution \( D \) over \( Q_1^L \times Q_2^R \), define \( D^{(\gamma)} \) as the distribution that samples from \( D \) and for each of the \(|L| + |R|\) coordinates independently with probability \( \gamma \) replaces it with a new sample from \( D \). The distribution \( D^{(\gamma)} \) is representative of the projection of \( T^{(\gamma)}(D) \) to two specific coordinates and we show that noise prevents the buckets from intersecting with good probability.

\textbf{Lemma 5.3.} Let \( D \) be a distribution over \( Q_1^L \times Q_2^R \) whose marginals are uniform in \( Q_1^L \) and \( Q_2^R \) and \( D^{(\gamma)} \) be as defined above. For every integer \( \Gamma \) and every pair of functions \( F : Q_1^L \to \{0,1\} \) and \( G : Q_2^R \to \{0,1\} \) such that \( \mathbb{E}[F(x)] = \mathbb{E}[G(y)] = 1/\Gamma \),
\[
\mathbb{E}_{(x,y) \in D^{(\gamma)}} [F(x)G(y)] \leq \Gamma^{-(1+\delta)}
\]
for some \( \delta = \delta(\gamma, Q) > 0 \) where \( Q = \min\{|Q_1|, |Q_2|\} \).
there are at most $2\Gamma$ possible independent samples of $D$ of the above lemma and Theorem 5.4. For every predicate $F$ and any noise rate $\gamma > 0$, projection $\pi : R \to L$, and bucketing parameter $\Gamma$, the following holds. For any functions $f : Q^L_1 \to \mathbb{Z}$, $g : Q^R_2 \to \mathbb{Z}$ with bucketing functions $F : Q^L_1 \to [\Gamma]$, $G : Q^R_2 \to [\Gamma],
$$\left| BA_{f,g,\pi}(T^{(\gamma)}_\pi(D)) - BA_{f,g,\pi}(T^{(\gamma)}_\pi(D^\perp)) \right| \leq \gamma^{-1/2} m^{1/2} 4^m \Gamma^m \sum_{a,b \in [\Gamma]} \text{CrInf}_{\pi}^{(1-\gamma)} \left( F^{(a)}, G^{(b)} \right) \right)^{1/2}.
$$

Recall that the decoupled version, $D^\perp$, of a base distribution $D$ is obtained by combining two independent samples of $D$, one for the first $t$ coordinates and the other for the remaining. A similar claim as above for the true acceptance probabilities of the dictator test is now a simple corollary of the above lemma and Theorem 5.1. This will be used later in extending the decoupling property to our general inapproximability reduction.

**Lemma 5.5.** For every predicate $\mathcal{P}$ and distribution $\mathcal{D}$ satisfying the conditions of Theorem 3.6, and any noise rate $\gamma > 0$, projection $\pi : R \to L$, and bucketing parameter $\Gamma$, the following holds. For any functions $f : Q^L_1 \to \mathbb{Z}$, $g : Q^R_2 \to \mathbb{Z}$ with bucketing functions $F : Q^L_1 \to [\Gamma]$, $G : Q^R_2 \to [\Gamma],
$$\left| Acc_{f,g}(T^{(\gamma)}_\pi(D)) - Acc_{f,g}(T^{(\gamma)}_\pi(D^\perp)) \right| \leq \gamma^{-1/2} m^{1/2} 4^m \Gamma^m \sum_{a,b \in [\Gamma]} \text{CrInf}_{\pi}^{(1-\gamma)} \left( F^{(a)}, G^{(b)} \right) \right)^{1/2} + 2\Gamma^{-\delta} m^2.
$$

\[5\]
The proof of the theorem is via the invariance principle and uses a few sophisticated but standard estimates developed in the works of Mossel [Mos10], Samorodnitsky and Trevisan [ST09], and Wenner [Wen12]. The first lemma essentially says that if a product of functions is influential, then at least one of the involved functions is influential. Applying the second lemma mostly involves introducing new notation where the notion of lifted functions is the most alien – a large-side table $g: [q_2]^R \to \mathbb{R}$ may equivalently be seen as the function $g': \Omega'^L \to \mathbb{R}$ where $\Omega' = [q_2]^d$ contains the values of all $d$ coordinates in $R$ projecting to the same coordinate in $L$. $g'$ is called the lifted analogue of $g$ with respect to the projection $\pi$ and the remark below essentially says that if the lifted analogue of $g$ is influential for a coordinate $i \in L$, then $g$ is influential in a coordinate projecting to $i$. The lemma will be used to – after massaging the expression – decoupling the small-side table from the lifted analogues of the large-side table as a function of their cross influence.

**Lemma 5.6** (Lemma 6.5, Mossel [Mos10]). Let $f_1, \ldots, f_t : \Omega^n \to [0,1]$ be arbitrary. Then for any $j$, $\text{Inf}_j \left( \prod_{r=1}^t f_r \right) \leq t \sum_{r=1}^t \text{Inf}_j (f_r)$.

**Remark 5.7** (Page 41, Wenner [Wen12]). Given a function $g : \Omega^n \to R$ and a projection $\pi : R \to L$ where $R = L \times [d]$ and $g' : (\Omega^n)^L \to \mathbb{R}$ is suitably defined. Then the influence of a coordinate $i \in L$ translates naturally to the sum of influences $j \in R$ projecting to $i$. Namely, we have $\text{Inf}_i (g^n) = \text{Inf}_{\pi^{-1}(i)} (g) \leq \sum_{j : \pi(j) = i} \text{Inf}_j (g)$. This follows from the expression of influences in decompositions of $g$ which equals $\sum_{T : i \in \pi(T)} \mathbb{E} \left[ g_T^2 \right]$ in the former two cases and $\sum_{T \cap \pi^{-1}(i)} \mathbb{E} \left[ g_T^2 \right]$ in the third.

**Theorem 5.8** (Theorem 3.21, Wenner [Wen12]). Consider functions $\{f(r) \in L^\infty(\Omega^n)\}_{r \in [m]}$ on a probability space $\mathcal{P} = (\prod_{i=1}^m \Omega_i, P, \mathcal{P})$, a set $M \subseteq [m]$, and a collection $\mathcal{C}$ of minimal sets $C \subseteq [m], C \not\subseteq M$ such that the spaces $\{\Omega_i\}_{i \in C}$ are dependent. Then,

$$\left| \mathbb{E} \left[ \prod_{r=1}^m f(r) \right] - \prod_{r \not\in M} \mathbb{E} \left[ f(r) \right] \mathbb{E} \left[ \prod_{r \in M} f(r) \right] \right| \leq 4^m \max_{C \in \mathcal{C}} \sqrt{\min_{r' \in \mathcal{C}} \text{TotInf} (f(r')} \sum_{l \in \mathcal{C} \setminus \{r'\}} \text{Inf}_l \left(f(r') \right) \prod_{r \not\in C} \left\| f(r) \right\|_\infty.$$ 

**Proof of Theorem 5.4**

Proof. We massage the expression $\text{BAcc}_{f,g}(T_{\pi}(\mathcal{D}))$ to a form suitable for applying Theorem 5.8. Recall that $F(a)$ denotes the indicator of “$F(x) = a”$ and similarly $G(a)$ of “$G(y) = a”$. Now, $\text{BAcc}_{f,g}(T_{\pi}(\mathcal{D}))$ equals

$$\sum_{a \in \Gamma^t, b \in \Gamma^{m-t}} \varphi(a, b) \mathbb{E}_{T_{\pi}(\mathcal{D})} \left[ \prod_{r=1}^t F(a_r)(x^{(r)}) \prod_{r=t+1}^m G(b_r)(y^{(r)}) \right],$$

in terms of these indicators. Consequently, $|\text{BAcc}_{f,g}(\mathcal{D}) - \text{BAcc}_{f,g}(\mathcal{D}^\perp)|$ may be bounded from above by

$$\sum_{a, b} \varphi(a, b) \mathbb{E}_{T_{\pi}(\mathcal{D})} \left[ \prod_{r=1}^t F(a_r)(x^{(r)}) \prod_{r=t+1}^m G(b_r)(y^{(r)}) \right] - \mathbb{E}_{T_{\pi}(\mathcal{D}^\perp)} \left[ \prod_{r=1}^t F(a_r)(x^{(r)}) \prod_{r=t+1}^m G(b_r)(y^{(r)}) \right].$$

(6)
We note that \( \phi \) is bounded by 1 in magnitude and proceed to bound the expression inside the summation. To this end, we must make a slight change of notation as discussed previously. The new notation may seem cumbersome; the high-level picture is that we group the first set of functions into a single function and redefine the latter to be functions on arguments indexed by \( L \) instead of \( R \).

Define \( m' = m - t + 1 \), \( \Omega_1 = [g_1]^t \), \( \Omega_2 = \ldots = \Omega_{m'} = [g_2]^{d} \) and let \( \bar{\gamma} \) denote \( 1 - \gamma \). Let \( \varsigma \) be a bijection \( L \times [d] \leftrightarrow R \) such that \( \pi(\varsigma(i, j')) = i \).

Introduce the distribution \( \Omega_1^L \times \ldots \times \Omega_{m'}^L \ni (w, z^{(2)}, \ldots, z^{(m')}) \sim R(\mu) \) which samples \((x^{(1)}), \ldots, x^{(t)}, y^{(t+1)}, \ldots, y^{(m)}) \) from \( T_{\bar{\gamma}}^{(r)}(D) \), setting \( w_{i, r} = x_i^{(r)} \) and \( z_i^{(r)} = \{y_{i-1}^{(r)}(i, j')\}_{j'=1}^{\gamma} \). Let \( W(w) \) be defined as \( H^{(r-t+1)}(z) = (T_{\bar{\gamma}}G^{(r)})(y) \) where \( y_{k(i, j')} = z_i^{(r)} \). With this new notation, the difference within the summation in (6) is

\[
\left| \mathbb{E}_{R(D)} \left[ W(w) \prod_{r=2}^{m'} H^{(r)}(z^{(r)}) \right] - \mathbb{E}_{R(D)} \left[ W(w) \prod_{r=2}^{m'} H^{(r)}(z^{(r)}) \right] \right|.
\] (7)

We note that \( R \) is a product distribution \( R = \mu \otimes \ldots \otimes \mu \) for some \( \mu \) and for any \( 2 \leq t \leq m' \), \( \Omega^{(r)} \) is independent of \( \Omega^{(t)} \). Choosing \( M = \{2, \ldots, m'\} \), minimal indices \( C \) of dependent sets in \( \mu \) not contained in \( M \) contains 1 and at least two elements from \( M \), i.e. \( C \in C \) implies \( 1, e, e' \in C \) for some \( e \neq e' \in \{2, \ldots, m'\} \).

Applying Theorem 5.8 and choosing \( r' \neq 1 \) bounds the difference (7) by

\[
4^m \sqrt{\max_{C \in \mathcal{C}} \max_{e \in C} \sum_{i} \text{Inf}_i(W) \prod_{t \in C \setminus \{1, e\}} \text{Inf}_i(H^{(t)}) \prod_{r \notin C} \|H^{(r)}\|_{\infty}}.
\] (8)

As we assumed the codomain of the studied functions \( \{G^{(r)}\}_r \) to be \([0, 1]\) the same holds for \( \{H^{(r)}\}_r \) and consequently the influences and infinity norms in (8) are upper-bounded by one on account of Lemma 2.8, yielding

\[
(7) \leq 4^m \left( \max_{e} \text{TotInf}(H^{(e)}) \cdot \max_{e} \sum_{i} \text{Inf}_i(W)\text{Inf}_i(H^{(e)}) \right)^{1/2}.
\] (9)

We recall that \( W = \prod_{t=1}^T T_{\gamma}f^{(r)} \) and hence by Lemma 5.6, \( \text{Inf}_i(W) \leq t \sum_{r=1}^t \text{Inf}_i(T_{\bar{\gamma}}f^{(r)}) = t \sum_{r=1}^t \text{Inf}_i^{(\bar{\gamma})}(f^{(r)}) \). Similarly, Remark 5.7 implies that \( \text{Inf}_j(H^{(e-t+1)}) \leq \sum_{j \in \pi^{-1}} \text{Inf}_j(T_{\bar{\gamma}}G^{(e)}) = \text{Inf}_j^{(\bar{\gamma})}(G^{(e)}) \). Returning to (9), we have the bound

\[
(7) \leq 4^m \left( t \max_{e} \text{TotInf}^{(\bar{\gamma})}(G^{(e)}) \max_{e} \sum_{i} \sum_{r=1}^t \text{Inf}_i^{(\bar{\gamma})}(f^{(r)}) \sum_{j \in \pi^{-1}(i)} \text{Inf}_j^{(\bar{\gamma})}(G^{(e)}) \right)^{1/2}.
\]

Bounding the total influence using Lemma 2.9 and identifying the inner sum as a cross influence, we establish the desired bound on the lemma difference

\[
(7) \leq 4^m \left( t \gamma^{-1} \sum_{r, e} \text{CrInf}_e^{(1-\gamma)}(f^{(r)}, G^{(e)}) \right)^{1/2} \leq \gamma^{-1/2} m^{1/2} 4^m \sum_{r, r'} \text{CrInf}_{r, r'}^{(1-\gamma)}(f^{(r)}, G^{(r')})^{1/2}.
\]
Finally, as we noted before, since \(0 < \varphi < 1\), (6) is at most
\[
\gamma^{-1/2}m^{1/2}4^m\Gamma^m \sum_{a,b \in [\Gamma]} \text{CrInf}_{\pi}^{(1-\gamma)}(F^{(a)}, G^{(b)})^{1/2},
\]
as there are at most \(\Gamma^m\) terms in the summation.

5.3 Soundness of the Reduction

With the soundness for the dictatorship test in place, proving the soundness of the reduction (Theorem 3.6) is a relatively standard task of constructing noisy-influence decoding strategies.

The proof follows immediately from the a more general estimate given in the following Lemma by taking \(\Gamma = (4m^2/\varepsilon)^{1/\delta}\) and then \(\varepsilon_{LC} = \left(\frac{\varepsilon^{3/2}}{m^{1/2}4^m\Gamma^{m+1}}\right)^2\).

Lemma 5.9. Given an LC instance \(\mathcal{L} = (U, V, E, L, R, \Pi)\) and a collection of functions, \(f_u : Q^L_1 \rightarrow \mathbb{Z}\) for \(u \in U\); \(g_v : Q^R_2 \rightarrow \mathbb{Z}\) for \(v \in V\), and \(\Gamma, \gamma, \delta\) as in this section,
\[
E_{u,v \sim E} \left[ (\text{Acc}_{f_u,g_v}(T_{\pi}^{(\gamma)}(D)) - \text{Acc}_{f_u,g_v}(T_{\pi}^{(\gamma)}(D^\perp))) \right] \leq \gamma^{-1.5}m^{1/2}4^m\Gamma^{m+1}\text{val}(\mathcal{L})^{1/2} + 2\Gamma^{-\delta}m^2.
\]

Proof. For a function \(f : Q^L_1 \rightarrow \mathbb{Z}\) define a distribution \(\Psi(f)\) over \(L\) as follows. First pick \(a \sim [\Gamma]\) uniformly, then pick \(l \in L\) with probability \(\gamma \cdot \text{Inf}_{\pi}^{(1-\gamma)}(F^{(a)}_u)\) and otherwise an arbitrary label. Note that by Lemma 2.9, \(\sum_{l \in L} \text{Inf}_{\pi}^{(1-\gamma)}(F^{(a)}_u) \leq 1/\gamma\) and so picking \(l \in L\) with the given probabilities is possible. Define \(\Psi(g)\) over \(R\) for \(g : Q^R_2 \rightarrow \mathbb{Z}\) similarly. Now define a labeling of \(\mathcal{L}\) by, for each \(u \in U\) (resp. \(v \in V\)), sampling a label from \(\Psi(f_u)\) (resp. \(\Psi(g_u)\)) independently.

For an edge \(e = (u, v) \in E\), the probability that \(e\) is satisfied by the labeling equals \(P(\pi_e(\Psi(f_u)) = \Psi(g_v))\), which can be lower bounded by
\[
\sum_{i,j : \pi_e(i)=j} \gamma^2 E_{u,v \sim E} \left[ \text{Inf}_{\pi}^{(1-\gamma)}(F^{(a)}_u)\text{Inf}_{\pi}^{(1-\gamma)}(G^{(b)}_v) \right] = (\gamma/\Gamma)^2 \sum_{a,b} \text{CrInf}_{\pi_e}^{(1-\gamma)}(F^{(a)}_u, G^{(b)}_v).
\]

Taking the expectation over all edges of \(\mathcal{L}\) we get that the fraction of satisfied constraints is
\[
(\gamma/\Gamma)^2 E_{e \in (u,v)} \left[ \sum_{a,b} \text{CrInf}_{\pi_e}^{(1-\gamma)}(F^{(a)}_u, G^{(b)}_v) \right] \leq \text{val}(\mathcal{L}),
\]
and by concavity of the \(\sqrt{\cdot}\) function, this implies that \(E_{e \in (u,v)} \left[ \sum_{a,b} \text{CrInf}_{\pi_e}^{(1-\gamma)}(F^{(a)}_u, G^{(b)}_v)^{1/2} \right] \leq \Gamma^{-1}\text{val}(\mathcal{L})^{1/2}\). Plugging this bound on the total cross influence into the soundness for the dictatorship test Lemma 5.5, we obtain
\[
E_{e \in (u,v)} \left[ (\text{Acc}_{f_u,g_v}(T_{\pi_e}^{(\gamma)}(D)) - \text{Acc}_{f_u,g_v}(T_{\pi_e}^{(\gamma)}(D^\perp))) \right] \leq \gamma^{-1.5}m^{1/2}4^m\Gamma^{m+1}\text{val}(\mathcal{L})^{1/2} + 2\Gamma^{-\delta}m^2,
\]
as desired. \(\square\)
Conclusion

We gave improved inapproximability for several important OCSPs. Our characterization is by no means complete and several interesting problems are still open. Closing the gap in the approximability of MAS is wide open and probably no easier than resolving the approximability for MAX CUT and other 2-CSPs. In particular, getting any factor close to 1/2 seems to require new ideas. MAX BTW has an approximation algorithm that satisfies half of the constraints if all the constraints can be simultaneously satisfied [CS98]. Thus improving our result to obtaining perfect completeness is particularly enticing. Finally, though the inability to fold long codes is a serious impedement, improving our general hardness result to only requiring that $D$ is pairwise independent is interesting especially in light of the analogous results for CSPs [AM09, Cha13].

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