Multiplicity and concentration of nontrivial solutions for the generalized extensible beam equations

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\begin{abstract}
In this paper, we study a class of generalized extensible beam equations with a super-linear nonlinearity
\begin{equation}
\begin{aligned}
\Delta^2 u - M \left( \| \nabla u \|_{L^2}^2 \right) \Delta u + \lambda V(x) u &= f(x, u) \quad \text{in } \mathbb{R}^N, \\
u &\in H^2(\mathbb{R}^N),
\end{aligned}
\end{equation}
where $N \geq 3$, $M(t) = at^\delta + b$ with $a, \delta > 0$ and $b \in \mathbb{R}$, $\lambda > 0$ is a parameter, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. Unlike most other papers on this problem, we allow the constant $b$ to be nonpositive, which has the physical significance. Under some suitable assumptions on $V(x)$ and $f(x, u)$, when $a$ is small and $\lambda$ is large enough, we prove the existence of two nontrivial solutions $u_{a,\lambda}^{(1)}$ and $u_{a,\lambda}^{(2)}$, one of which will blow up as the nonlocal term vanishes. Moreover, $u_{a,\lambda}^{(1)} \to u_\infty^{(1)}$ and $u_{a,\lambda}^{(2)} \to u_\infty^{(2)}$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \to \infty$, where $u_\infty^{(1)} \neq u_\infty^{(2)} \in H^0_0(\Omega)$ are two nontrivial solutions of Dirichlet BVPs on the bounded domain $\Omega$. It is worth noting that the regularity of weak solutions $u_\infty^{(i)} (i = 1, 2)$ here is explored. Finally, the nonexistence of nontrivial solutions is also obtained for $a$ large enough.

\textbf{Keywords:} Extensible beam equations, Nontrivial solution, Multiplicity, Concentration, Nonexistence.
\end{abstract}

1 Introduction

Consider the nonlinear generalized extensible beam equations in the form:
\begin{equation}
\begin{aligned}
\Delta^2 u - M \left( \| \nabla u \|_{L^2}^2 \right) \Delta u + \lambda V(x) u &= f(x, u) \quad \text{in } \mathbb{R}^N, \\
u &\in H^2(\mathbb{R}^N),
\end{aligned}
\end{equation}
where $N \geq 3$, $\Delta^2 u = \Delta(\Delta u)$, $M(t) = at^\delta + b$ with $a, \delta > 0$ and $b \in \mathbb{R}$, $\lambda > 0$ is a parameter, and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. We assume that the potential $V(x)$ satisfies the following assumptions:

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\end{itemize}
(V1) \(V \in C(\mathbb{R}^N)\) and \(V(x) \geq 0\) for all \(x \in \mathbb{R}^N\);

(V2) there exists \(c_0 > 0\) such that the set \(\{V < c_0\} := \{x \in \mathbb{R}^N \mid V(x) < c_0\}\) has finite positive Lebesgue measure for \(N \geq 4\) and

\[
|\{V < c_0\}| < S^{-2}_\infty \left(1 + \frac{A_0^2}{2}\right)^{-1} \quad \text{for} \quad N = 3,
\]

where \(|\cdot|\) is the Lebesgue measure, \(S_\infty\) is the best Sobolev constant for the imbedding of \(H^2(\mathbb{R}^N)\) in \(L^\infty(\mathbb{R}^N)\) for \(N = 3\), and \(A_0\) is defined in (1.6) below;

(V3) \(\Omega = \text{int}\{x \in \mathbb{R}^N : V(x) = 0\}\) is nonempty and has smooth boundary with \(\overline{\Omega} = \{x \in \mathbb{R}^N : V(x) = 0\}\).

The hypotheses (V1) – (V3), suggested by Bartsch et. al. \[3\], imply that \(\lambda V(x)\) represents a potential well whose depth is controlled by \(\lambda\). If \(\lambda\) is sufficiently large, then \(\lambda V(x)\) is known as the steep potential well. About its applications, we refer the reader to \[14, 22, 23, 24, 25, 26, 34, 35\] and references therein.

Eq. (E) arises in an interesting physical context. In 1950, Woinowsky and Krieger \[30\] introduced the following extensible beam equation:

\[
\rho u_{tt} + EI u_{xxxx} - \left(\frac{Eh}{2T} \int_0^L |u_x|^2 \, dx + P_0\right) u_{xx} = 0, \quad (1.1)
\]

where \(L\) is the length of the beam in the rest position, \(E\) is the Young modulus of the material, \(I\) is the cross-sectional moment of inertia, \(\rho\) is the mass density, \(P_0\) is the tension in the rest position and \(h\) is the cross-sectional area. This model is used to describe the transverse deflection \(u(x, t)\) of an extensible beam of natural length \(L\) whose ends are held a fixed distance apart. Such problems are often referred to as being nonlocal because of the presence of the term \(\left(\int_0^L |u_x|^2 \, dx\right) u_{xx}\), which indicates the change in the tension of the beam due to its extensibility.

The qualitative and stable analysis of solutions for Eq. (1.1) can be traced back to the 1970s, for instance in the papers by Ball \[2\], Dickey \[9\] and Medeiros \[20\].

As a simplification of the von Karman plate equation, Berger \[4\] proposed the plate model describing large deflection of plate as follows

\[
u_{tt} + \Delta^2 u - \left(\int_\Omega |\nabla u|^2 \, dx + Q_0\right) \Delta u = f(u, u_t, x), \quad (1.2)
\]

where \(\Omega \subset \mathbb{R}^N (N = 1, 2)\) is a bounded domain with a sufficiently smooth boundary, the parameter \(Q_0\) is in-plane forces applied to the plate (\(Q_0 > 0\) represents outward pulling forces and \(Q_0 < 0\) means inward extrusion forces) and the function \(f\) represents transverse loads which may depend on the displacement \(u\) and the velocity \(u_t\). Apparently, when \(N = 1\) and \(f \equiv 0\) in Eq. (1.2), the corresponding equation becomes the extensible beam equation (1.1).

Owing to its importance, the various properties of solutions for Eq. (1.2) have been treated by many researchers; see for example, \[8, 19, 21, 33, 36\]. More precisely, Patecheu \[21\] investigated the existence and decay property of global solutions to the Cauchy problem of Eq. (1.2) with \(f(u, u_t, x) \equiv f(u_t)\) in the abstract form. Yang \[33\] studied the global existence, stability and the longtime dynamics of solutions to the initial boundary value problem (IBVP) of an
admits one or two nontrivial solutions, respectively. When \( f(x, u, x, u) = g(u_t) + h(u) + k(x) \),

In the last two decades, the stationary form of Eq. (1.2), of the form similar to Eq. (E), has begun to attract attention, especially on the existence and multiplicity of nontrivial solutions, but the relevant results are rare. We refer the reader to [7, 11, 15, 18, 27, 28, 31, 32] and references therein. To be precise, Ma [18] studied the existence of nontrivial solutions for biharmonic equations without nonlocal term; see, for example, [13, 17, 25, 29, 34]. Specifically, Sun et al. [25] investigated the following biharmonic multiplicity of nontrivial solutions for biharmonic equations without nonlocal term, via Lions’ second concentration-compactness principle.

On the other hand, steep potential well has been applied to the study of the existence and multiplicity of nontrivial solutions for biharmonic equations without nonlocal term; see, for example, [13, 17, 25, 29, 34]. Specifically, Sun et al. [25] investigated the following biharmonic equations with \( p \)-Laplacian and steep potential well

\[
\begin{cases}
\Delta^2 u - M \left( \| \nabla u \|_{L^2}^2 \right) \Delta u + u = \lambda f(u) + |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\
u \in H^2(\mathbb{R}^N),
\end{cases}
\]

where \( M : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) are continuous increasing functions, \( f \in C(\mathbb{R}, \mathbb{R}) \), \( 2^* = \frac{2N}{N-4} \) with \( N \geq 5 \) and \( \lambda > 0 \) is a parameter. By using the minimax theorem and the truncation technique, the existence of nontrivial solutions of Eq. (1.4) is proved for \( \lambda \) sufficiently large. Later, Liang and Zhang [15] obtained the existence and multiplicity of nontrivial solutions for Eq. (1.3) via Lions’ second concentration-compactness principle.

We focus our attention on the multiplicity and concentration of nontrivial solutions for Eq. (E) with a superlinear nonlinearity, one of which will blow up as the nonlocal term vanishes; (III) we would like to explore the phenomenon of concentrations of two different nontrivial solutions as \( \lambda \rightarrow \infty \), which seems to be less involved in extensible beam equations.

It is noteworthy that in analysis, we have to face some challenges. First, since the constant \( b \leq 0 \) is allowed, how to construct an appropriate norm of the working space such that this norm is associated with the norm \( \| \nabla u \|_{L^2} = \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{1/2} \) is crucial. Second, having considered the fact that the norms \( \| \nabla u \|_{L^2} \) and \( \| u \|_{H^2} = \left( \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + u^2) dx \right)^{1/2} \) are not equivalent,
how to verify that the energy functional of Eq. (E) is bounded below and coercive in $H^2(\mathbb{R}^N)$ is critical. Third, we note that $\Delta u|_{\partial \Omega} = 0$ is not included in the space $H^1_0(\Omega) \cap H^2(\Omega)$. In view of this, about the concentration of nontrivial solutions, how to prove the functions of convergence satisfy the second boundary condition $\Delta u|_{\partial \Omega} = 0$ in Navier boundary conditions is the key.

In order to overcome these difficulties, in this paper some new inequalities are established and new research techniques are introduced. In addition, the regularity of weak solutions for Navier BVPs to generalized extensible beam equations is discussed. By so doing, we obtain the existence of two nontrivial solutions for Eq. (E) by the minimax theory and the nonexistence of nontrivial solutions. Furthermore, we successfully figure out the concentrations of two different nontrivial solutions for Eq. (E) as $\lambda \to \infty$.

Before stating our results, we shall first introduce some notations. Denote the best Sobolev constant for the imbedding $H^2(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ ($2 \leq r < +\infty$) by $S_r$ for $N = 4$. Let $A_0 > 0$ be a Gagliardo-Nirenberg constant satisfying the following Gagliardo-Nirenberg inequality

$$
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \leq A_0^2 \left( \int_{\mathbb{R}^N} |\Delta u|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^N} u^2 \, dx \right)^{1/2}.
$$

(1.6)

Set

$$
\beta_N := \begin{cases} 
(1 + \frac{A_0^2}{2}) \left( 1 + \frac{16}{N} |\{a < c_0\}|^{4/N} \right) & \text{for } N = 3, 4, \\
(1 + \frac{A_0^2}{2}) \left( 1 + B_N^2 |\{V < c_0\}|^{4/N} \right) & \text{for } N \geq 4,
\end{cases}
$$

and

$$
\Theta_{2,N} := \begin{cases} 
\left( 1 + \frac{A_0^2}{2} \right)^{-1} - S_\infty^2 |\{V < c_0\}|^{-1} & \text{for } N = 3, \\
S_2^{-2} \left( 1 + \frac{A_0^2}{2} \right) & \text{for } N = 4, \\
1 + \frac{A_0^2}{2} & \text{for } N > 4.
\end{cases}
$$

We now summarize our main results as follows.

**Theorem 1.1** Suppose that $N \geq 3, \delta \geq \frac{2}{N-2}, b > -2A_0^{-2} \beta_N^{-1}$ and conditions (V1) – (V3) hold. In addition, we assume that the function $f$ satisfies the followings:

\begin{itemize}
  \item [(F1)] $f(x,s)$ is a continuous function on $\mathbb{R}^N \times \mathbb{R}$;
  \item [(F2)] there exists a constant $0 < d_0 < \alpha$ such that
\end{itemize}

$$
pF(x,s) - f(x,s)s \leq d_0 s^2 \quad \text{for all } x \in \mathbb{R}^N \text{ and } s \in \mathbb{R},
$$

where

$$
\alpha = \begin{cases} 
\frac{1}{2} \delta \Theta_{2,N}^{-2} (2 + bA_0^2 \beta_N) & \text{if } -2A_0^{-2} \beta_N^{-1} < b < 0, \\
\frac{\delta \Theta_{2,N}^{-2}}{\Theta_{2,N}^{-2}} & \text{if } b \geq 0,
\end{cases}
$$

and $F(x,s) = \int_0^s f(x,t) \, dt$;

\begin{itemize}
  \item [(F3)] for each $\epsilon \in \left( 0, \frac{1}{2} (2 + bA_0^2 \beta_N) \Theta_{2,N}^{-2} \right)$, there exist constants $2 < p < \frac{2N}{N-2}$ and $C_{1,\epsilon}, C_{2,\epsilon} > 0$ satisfying $C_{1,\epsilon} > \frac{2N+2-p}{6p} C_{2,\epsilon}$ such that for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$,
\end{itemize}

$$
C_{2,\epsilon} s^{p-1} - \gamma s \leq f(x,s) \leq \epsilon s + C_{1,\epsilon} s^{p-1}
$$

for some constant $\gamma$ independent on $\epsilon$. 

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Then there exists constants $\Lambda_1, a_* > 0$ such that for every $\lambda \geq \Lambda_1$ and $0 < a < a_*$, Eq. (E) admits at least two nontrivial solutions $u^{(1)}_{a,\lambda}$ and $u^{(2)}_{a,\lambda}$ satisfying $J_{a,\lambda}(u^{(2)}_{a,\lambda}) < 0 < J_{a,\lambda}(u^{(1)}_{a,\lambda})$. In particular, $u^{(2)}_{a,\lambda}$ is a ground state solution of Eq. (E). Furthermore, when $\delta > \frac{2}{N-2}$, for every $\lambda \geq \Lambda_1$ there holds

$$J_{a,\lambda}(u^{(2)}_{a,\lambda}) \to -\infty \quad \text{and} \quad \|u^{(2)}_{a,\lambda}\|_\lambda \to \infty \quad \text{as} \quad a \to 0,$$

where $J_{a,\lambda}$ is the energy functional of Eq. (E) and $\|\cdot\|_\lambda$ is defined as (2.1).

**Theorem 1.2** Suppose that $N \geq 3, \delta \geq \frac{2}{N-2}, b > -2A_0^{-2}\beta_N^{-1}$ and conditions (V1) – (V2) hold. In addition, we assume that the function $f$ is a continuous function on $\mathbb{R}^N \times \mathbb{R}$ satisfying:

$$(F3)' \quad \text{for each} \quad \epsilon \in \left(0, b\beta_N^2 \left\{\{V < c_0\}\right\}^{-2/N}\right), \quad \text{there exists constants} \quad 2 < p < \frac{2N}{N-2} \quad \text{and} \quad C_{1,\epsilon} > 0 \quad \text{such that for all} \quad x \in \mathbb{R}^N \quad \text{and} \quad s \in \mathbb{R},$$

$$f(x, s) \leq \epsilon s + C_{1,\epsilon}s^{p-1}.$$

Then there exists $a^* > 0$ such that for every $a > a^*$, Eq. (K$_{a,\lambda}$) does not admit any nontrivial solution for all $\lambda > bc_0^{-1}\beta_N^{-2} \left\{\{V < c_0\}\right\}^{-2/N}$.

**Theorem 1.3** Assume that $N \geq 5$. Let $u^{(1)}_{a,\lambda}$ and $u^{(2)}_{a,\lambda}$ be the solutions obtained by Theorem 1.1. Then $u^{(1)}_{a,\lambda} \to u^{(1)}_\infty$ and $u^{(2)}_{a,\lambda} \to u^{(2)}_\infty$ in $H^2(\mathbb{R}^N)$ as $\lambda \to \infty$, where $u^{(1)}_\infty \neq u^{(2)}_\infty \in H^2_0(\Omega)$ are two nontrivial solutions of the following Dirichlet BVPs:

$$\begin{cases} \Delta^2 u - M \left(\int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x, u) & \text{in} \ \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & \text{on} \ \partial \Omega. \end{cases} \quad (K_\infty)$$

The remainder of this paper is organized as follows. After presenting some preliminary results in section 2, we prove Theorem 1.1 in section 3, and demonstrate proof of Theorem 1.2 in Sections 4. Sections 5 is dedicated to the proof of Theorem 1.3.

## 2 Preliminaries

Let

$$X = \left\{ u \in H^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} (|\Delta u|^2 + V(x) u^2) \, dx < \infty \right\}$$

be equipped with the inner product and norm

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + V(x) uv) \, dx, \quad \|u\| = \langle u, u \rangle^{1/2}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$\langle u, v \rangle_\lambda = \int_{\mathbb{R}^N} (\Delta u \Delta v + \lambda V(x) uv) \, dx, \quad \|u\|_\lambda = \langle u, u \rangle_\lambda^{1/2}. \quad (2.1)$$

It is clear that $\|u\| \leq \|u\|_\lambda$ for $\lambda \geq 1$. Now we set $X_\lambda = (X, \|u\|_\lambda).$
By the Young and Gagliardo-Nirenberg inequalities, there exists a sharp constant \( A_0 > 0 \) such that

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \leq \frac{A_0^2}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + u^2) \, dx. \tag{2.2}
\]

This shows that

\[
\int_{\mathbb{R}^N} (|\Delta u|^2 + u^2) \, dx \leq \| u \|_{H^2}^2 \leq \left( 1 + \frac{A_0^2}{2} \right) \int_{\mathbb{R}^N} (|\Delta u|^2 + u^2) \, dx. \tag{2.3}
\]

For \( N = 3, 4 \), applying condition (V1) and the Hölder, Young and Gagliardo-Nirenberg inequalities, there exists a sharp constant \( A_N > 0 \) such that

\[
\int_{\mathbb{R}^N} u^2 \, dx \leq \frac{1}{c_0} \int_{\{ V \geq c_0 \}} V(x) u^2 \, dx + \left( |\{ V < c_0 \}| \int_{\mathbb{R}^N} |u|^4 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{c_0} \int_{\mathbb{R}^N} V(x) u^2 \, dx + \frac{N A_N^{16/N}}{8} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{8 - N}{8} \int_{\mathbb{R}^N} u^2 \, dx,
\]

which shows that

\[
\int_{\mathbb{R}^N} u^2 \, dx \leq \frac{8}{N c_0} \int_{\mathbb{R}^N} V(x) u^2 \, dx + \overline{A}_N^{16/N} \int_{\{ a < c_0 \}} u^2 \, dx \tag{2.4}
\]

It follows from (2.3) and (2.4) that

\[
\| u \|_{H^2}^2 \leq \left( 1 + \frac{A_0^2}{2} \right) \max \left\{ 1 + \overline{A}_N^{16/N} |\{ a < c_0 \}|^{4/N}, \frac{8}{N c_0} \right\} \| u \|^2. \tag{2.5}
\]

Similarly, we also obtain that

\[
\| u \|_{H^2}^2 \leq \left( 1 + \frac{B_0^2}{2} \right) \left( 1 + \overline{B}_N^{16/N} |\{ a < c_0 \}|^{4/N} \right) \| u \|_{\lambda}^2 \tag{2.6}
\]

for \( \lambda \geq 8 N^{-1} c_0^{-1} \left( 1 + \overline{A}_N^{16/N} |\{ a < c_0 \}|^{4/N} \right) \). For \( N > 4 \), by conditions (V1) – (V2), Hölder and Gagliardo-Nirenberg inequalities, there exists a sharp constant \( \overline{B}_N > 0 \) such that

\[
\int_{\mathbb{R}^N} u^2 \, dx = \int_{\{ V \geq c_0 \}} u^2 \, dx + \int_{\{ V < c_0 \}} u^2 \, dx
\]

\[
\leq \frac{1}{c_0} \int_{\mathbb{R}^N} V(x) u^2 \, dx + \overline{B}_N^2 \int_{\{ V < c_0 \}} |\{ V < c_0 \}|^{4/N} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx.
\]

Combining the above inequality with (2.3) yields

\[
\| u \|_{H^2}^2 \leq \left( 1 + \frac{A_0^2}{2} \right) \max \left\{ 1 + \overline{B}_N^2 |\{ V < c_0 \}|^{4/N}, \frac{1}{c_0} \right\} \| u \|^2. \tag{2.7}
\]

Similarly, we also have

\[
\| u \|_{H^2}^2 \leq \left( 1 + \frac{A_0^2}{2} \right) \left( 1 + \overline{B}_N^2 |\{ V < c_0 \}|^{4/N} \right) \| u \|_{\lambda}^2 \tag{2.8}
\]
Since the imbedding $H^\beta \hookrightarrow G^\alpha$ and Gagliardo-Nirenberg inequalities again, it follows that for any $\lambda > c_0^{-1}\left(1 + \mathcal{E}_N^2 \|\{V < c_0\}\|^{4/N}\right)$. Set

$$\alpha_N := \begin{cases} 
\left(1 + \frac{A_0^2}{2}\right) \max \left\{ \frac{1}{N} \mathcal{A}_{16/N} \|\{a < c_0\}\|^{4/N}, \frac{8}{N c_0} \right\} & \text{for } N = 3, 4, \\
\left(1 + \frac{A_0^2}{2}\right) \max \left\{ 1 + \mathcal{B}_N^2 \|\{V < c_0\}\|^{4/N}, \frac{1}{c_0} \right\} & \text{for } N \geq 5.
\end{cases}$$

Thus, it follows from (2.5) and (2.7) that

$$\|u\|_{L^2}^2 \leq \alpha_N \|u\|_\lambda^2,$$

which implies that the imbedding $X \hookrightarrow H^2(\mathbb{R}^N)$ is continuous. If we set

$$\Lambda_N := \begin{cases} 
8N^{-1}c_0^{-1}\left(1 + \mathcal{A}_{16/N}^{16/N} \|\{a < c_0\}\|^{4/N}\right) & \text{for } N = 3, 4, \\
c_0^{-1}\left(1 + \mathcal{B}_N^2 \|\{V < c_0\}\|^{4/N}\right) & \text{for } N \geq 5,
\end{cases}$$

then we have

$$\|u\|_{L^2}^2 \leq \beta_N \|u\|_\lambda^2$$

for $\lambda \geq \Lambda_N$, (2.10)

where $\beta_N$ is defined as (1.6). Furthermore, by (2.2), (2.3) and (2.10) one has

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \leq \frac{1}{2} A_0^2 \beta_N \|u\|_\lambda^2$$

for $\lambda \geq \Lambda_N$. (2.11)

Since the imbedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ is continuous, by (2.6), for any $r \in [2, +\infty)$ we have

$$\int_{\mathbb{R}^3} |u|^r \, dx \leq \|u\|_{L^\infty}^{r-2} \int_{\mathbb{R}^3} u^2 \, dx \leq S_{(r-2)}^{(r-2)} \left(1 + \frac{A_0^2}{2}\right)^{r/2} \left(1 + \mathcal{A}_3^{16/3} \|\{a < c_0\}\|^{4/3}\right)^{r/2} \|u\|_\lambda^r$$

for $\lambda \geq \frac{8}{3c_0} \left(1 + \mathcal{A}_3^{16/3} \|\{a < c_0\}\|^{4/3}\right)$. Moreover, using the fact that the imbedding $H^2(\mathbb{R}^4) \hookrightarrow L^r(\mathbb{R}^4)$ ($2 \leq r < +\infty$) is continuous and (2.3), for any $r \in [2, +\infty)$ one has

$$\int_{\mathbb{R}^4} |u|^r \, dx \leq S_r^{(r-r)} \left(1 + \frac{A_0^2}{2}\right)^{r/2} \left(1 + \mathcal{A}_4^4 \|\{a < c_0\}\|^{4/3}\right)^{r/2} \|u\|_\lambda^r$$

for $\lambda \geq 2c_0^{-1}\left(1 + \mathcal{A}_4^4 \|\{a < c_0\}\|^{4/3}\right)$, where $S_r$ is the best Sobolev constant for the imbedding of $H^2(\mathbb{R}^4)$ in $L^r(\mathbb{R}^4)$ ($2 \leq r < +\infty$). Finally, for $N > 4$, from conditions $(V1) - (V2)$, (2.8) and Hölder and Gagliardo-Nirenberg inequalities again, it follows that for any $r \in [2, \frac{2N}{N-4})$,

$$\int_{\mathbb{R}^N} |u|^r \, dx \leq C_0^{N(r-2)/4} \left(\int_{\mathbb{R}^N} |u|^2 \, dx\right)^{[2N-r(N-4)]/8} \left(\int_{\mathbb{R}^N} |\Delta u|^2 \, dx\right)^{N(r-2)/8}$$

for $\lambda \geq \frac{1 + C_0^2 \|\{V < c_0\}\|^{4/N}}{c_0}$.

(2.14)
It is easily seen that Eq. (2.14) is variational and its solutions are critical points of the functional defined in $X_\lambda$ by

$$J_{a,\lambda}(u) = \frac{1}{2} \|u\|_{X_\lambda}^2 + \frac{a}{2} \|\nabla u\|_{L^2}^{2(1+\delta)} + \frac{b}{2} \|\nabla u\|_{L^2}^2 - \int_{\mathbb{R}^N} F(x, u) \, dx.$$  

It is not difficult to prove that the functional $J_{a,\lambda}$ is of class $C^1$ in $X_\lambda$, and that

$$\langle J_{a,\lambda}'(u), v \rangle = \int_{\mathbb{R}^N} [\Delta u \cdot \Delta v + \lambda V(x) uv] \, dx + a \|\nabla u\|_{L^2}^{2(1+\delta)} + b \|\nabla u\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x, u) \, vdx.$$  

Furthermore, we have the following results.

**Lemma 2.1** Suppose that $N \geq 3$ and $\delta \geq \frac{2}{N-2}$. In addition, we assume that conditions (V1) – (V2), (F1) and (F3) hold. Then the energy functional $J_{a,\lambda}$ is bounded below and coercive on $X_\lambda$ for all $a > 0$ and

$$\lambda \geq \Lambda_0 := \begin{cases} \max \left\{ \Lambda_N, \frac{2c_0}{a} \right\} & \text{if } \delta > \frac{2}{N-2}, \\ \max \left\{ \Lambda_N, \frac{2c_0}{a} + \frac{4C_{1,4}}{c_0 p} \left( \frac{2C_{1,4}(1+\delta)}{a p S^{2N/(N-2)}} \right)^{\frac{N-2}{2N}} \right\} & \text{if } \delta = \frac{2}{N-2}. \end{cases}$$

Furthermore, for all $a > 0$ and $\lambda \geq \Lambda_0$, there exists a constant $R_a > 0$ such that

$$J_{a,\lambda}(u) \geq 0 \text{ for all } u \in X_\lambda \text{ with } \|u\|_\lambda \geq R_a.$$

**Proof.** Let $u \in X_\lambda$. Note that for any $2 \leq r \leq 2^* := \frac{2N}{N-2}$, there holds

$$\int_{\mathbb{R}^N} |u|^r \, dx \leq \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{2^*-r}{2^*-2}} \left( \int_{\mathbb{R}^N} |u|^{r} \, dx \right)^{\frac{r-2}{2^*-2}} \leq \left( \frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda V(x) u^2 \, dx + S^{-2} \|\{V < c_0\}\|^\frac{1}{p} \|\nabla u\|_{L^2}^2 \right)^{\frac{2^*-r}{2^*-2}} \left( S^{-1} \|\nabla u\|_{L^2} \right)^{\frac{N(r-2)}{2}}.$$
where we have used the Hölder and Sobolev inequalities and \( \overline{S} \) is the best Sobolev constant for the imbedding of \( D^{1,2}(R^N) \) in \( L^{2^*}(R^N) \). We now divide the proof into two separate cases:

**Case A** : \( \int_{R^N} \lambda V(x) u^2 dx \geq \lambda c_0 \left( \frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)} \right) \left( \overline{S}^{-1} \| \nabla u \|_{L^2} \right)^{2^*} \). It follows from condition (F3) and (2.19) that

\[
J_{a,\lambda}(u) \geq \frac{1}{2} \| u \|_\lambda^2 + \frac{a}{2(1 + \delta)} \| \nabla u \|_{L^2}^{2(1+\delta)} + \frac{b}{2} \| \nabla u \|_{L^2}^2 - \frac{\epsilon}{2} \int_{R^N} u^2 dx - \frac{C_{1,\epsilon}}{p} \int_{R^N} |u|^p dx
\]

\[
\geq \frac{1}{4} \| u \|_\lambda^2 + \frac{a}{2(1 + \delta)} \| \nabla u \|_{L^2}^{2(1+\delta)} + \frac{1}{2} \left( b - \epsilon \overline{S}^{-2} |\{ V < c_0 \}| \right) \| \nabla u \|_{L^2}^2
\]

\[
- \frac{C_{1,\epsilon}}{pS^p} \{ \{ V < c_0 \} \}^{\frac{2N-p(N-2)}{2N}} \| \nabla u \|_{L^2}^p.
\]

Since \( \delta \geq \frac{2}{N-2} \), we have \( 1 + \delta > \frac{2}{N-2} > 1 \). Then there exists a constant \( D_a \) such that

\[
D_a = \min_{t \geq 0} \left[ \frac{at^{1+\delta}}{2(1 + \delta)} + \frac{t}{2} \left( b - \epsilon \left( \frac{|\{ V < c_0 \}|}{\overline{S}^2} \right)^\frac{2}{N} - \frac{C_{1,\epsilon} t^\frac{2}{N}}{pS^p} \right) \{ V < c_0 \}^{\frac{2N-p(N-2)}{2N}} \right]
\]

\[
< 0,
\]

and \( D_a \to -\infty \) as \( a \to 0 \). Using this, together with the above inequality leads to

\[
J_{a,\lambda}(u) \geq \frac{1}{4} \| u \|_\lambda^2 + D_a \geq D_a,
\]

which implies that \( J_{a,\lambda}(u) \) is bounded below and coercive on \( X_\lambda \) for all \( a > 0 \) and \( \lambda > \max \{ \lambda_0, \frac{2 \epsilon}{C_0} \} \).

**Case B** : \( \int_{R^N} \lambda V(x) u^2 dx < \lambda c_0 \left( \frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)} \right) \left( \overline{S}^{-1} \| \nabla u \|_{L^2} \right)^{2^*} \). By virtue of (2.19) one has

\[
\int_{R^N} |u|^p dx
\]

\[
\leq \left( \frac{1}{\lambda c_0} \int_{R^N} \lambda V(x) u^2 dx + \frac{|\{ V < c_0 \}|^2}{\overline{S}^2} \| \nabla u \|_{L^2}^2 \right)^{\frac{2^*-p}{2^*-2}} \left( \overline{S}^{-1} \| \nabla u \|_{L^2} \right)^{\frac{N(p-2)}{2N}}
\]

\[
\leq \overline{S}^{-2^*} \left( \frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)} \right)^{\frac{2N-p(N-2)}{(p-2)(N-2)}} \| \nabla u \|_{L^2}^{2^*} + \frac{|\{ V < c_0 \}|^{\frac{2N-p(N-2)}{2N}}}{S^p} \| \nabla u \|_{L^2}^p.
\]

Using this, together with condition (F3), gives

\[
J_{a,\lambda}(u) \geq \frac{1}{2} \| u \|_\lambda^2 + \frac{a}{2(1 + \delta)} \| \nabla u \|_{L^2}^{2(1+\delta)} + \frac{b}{2} \| \nabla u \|_{L^2}^2
\]

\[
- \frac{\epsilon}{2} \int_{R^N} u^2 dx - \frac{C_{1,\epsilon}}{p} \int_{R^N} |u|^p dx
\]

\[
\geq \frac{1}{4} \| u \|_\lambda^2 + \frac{a}{2(1 + \delta)} \| \nabla u \|_{L^2}^{2(1+\delta)} + \frac{1}{2} \left( b - \epsilon \overline{S}^{-2} |\{ V < c_0 \}| \right) \| \nabla u \|_{L^2}^2
\]

\[
- \frac{C_{1,\epsilon}}{pS^p} \left( \frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)} \right)^{\frac{2N-p(N-2)}{(p-2)(N-2)}} \| \nabla u \|_{L^2}^{2^*} - \frac{C_{1,\epsilon}}{pS^p} \{ \{ V < c_0 \} \}^{\frac{2N-p(N-2)}{2N}} \| \nabla u \|_{L^2}^p.
\]
If \( \delta = \frac{2}{N-2} \), then for
\[
\lambda > \frac{2\epsilon}{c_0} + \frac{4C_{1,\epsilon}}{c_0p} \left[ \frac{2C_{1,\epsilon} (1 + \delta)}{apS^{\frac{2}{pN}}} \right]^{\frac{(p-2)(N-2)}{2N-p(N-2)}},
\]
there exists a constant \( D_a < D_a < 0 \) such that
\[
J_{a,\lambda}(u) \geq \frac{1}{4} \| u \|_\lambda^2 + \frac{a}{2(1+\delta)} \| \nabla u \|_{L^2}^{2(1+\delta)} + \frac{1}{2} \left( b - \epsilon S^{-2} \left| \{ V < c_0 \} \right|^{\frac{2}{N}} \right) \| \nabla u \|_{L^2}^2
- \frac{C_{1,\epsilon}}{pS^{\frac{2}{p}}} \left( \frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)} \right)^{\frac{2N-p(N-2)}{p(2-N)(N-2)}} \| \nabla u \|_{L^2}^2 \geq \frac{1}{4} \| u \|_\lambda^2 + D_a \geq D_a.
\]

If \( \delta > \frac{2}{N-2} \), then for \( \lambda > \frac{2\epsilon}{c_0} \), there exists a constant \( \tilde{D}_a < 0 \) such that
\[
J_{a,\lambda}(u) \geq \frac{1}{4} \| u \|_\lambda^2 + \frac{a}{2(1+\delta)} \| \nabla u \|_{L^2}^{2(1+\delta)} + \frac{1}{2} \left( b - \epsilon S^{-2} \left| \{ V < c_0 \} \right|^{\frac{2}{N}} \right) \| \nabla u \|_{L^2}^2
- \frac{C_{1,\epsilon}}{pS^{\frac{2}{p}}} \left( \frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)} \right)^{\frac{2N-p(N-2)}{p(2-N)(N-2)}} \| \nabla u \|_{L^2}^2 \geq \tilde{D}_a.
\]

This indicates that \( J_{a,\lambda} \) is bounded below and coercive on \( X_\lambda \) for all \( a > 0 \) and \( \lambda \geq \Lambda_0 \). Furthermore, for all \( a > 0 \) and \( \lambda \geq \Lambda_0 \), it is clear that there exists a constant \( R_a > 0 \) such that
\[
J_{a,\lambda}(u) \geq 0 \text{ for all } u \in X_\lambda \text{ with } \| u \|_\lambda \geq R_a.
\]
Consequently, the proof is complete.

Next, we give a useful theorem, which is the variant version of the mountain pass theorem. It can help us to find a so-called Cerami type \((PS)\) sequence.

**Lemma 2.2 ([10], Mountain Pass Theorem).** Let \( E \) be a real Banach space with its dual space \( E^* \), and suppose that \( I \in C^1(E,R) \) satisfies
\[
\max\{I(0),I(e)\} \leq \mu < \eta \leq \inf_{\| u \| = \rho} I(u),
\]
for some \( \mu < \eta, \rho > 0 \) and \( e \in E \) with \( \| e \| > \rho \). Let \( c \geq \eta \) be characterized by
\[
c = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I(\gamma(\tau)),
\]
where \( \Gamma = \{ \gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e \} \) is the set of continuous paths joining 0 and \( e \), then there exists a sequence \( \{ u_n \} \subset E \) such that
\[
I(u_n) \to c \geq \eta \quad \text{and} \quad (1 + \| u_n \|) \| I'(u_n) \|_{E^*} \to 0 \quad \text{as} \ n \to \infty.
\]
In what follows, we give two lemmas which ensure that the functional \( J_{a,\lambda} \) has the mountain pass geometry.

**Lemma 2.3** Suppose that \( b > -2A_0^{-2}\beta_N^{-1} \). In addition, assume that conditions (V1)–(V2), (F1) and (F3) hold. Then there exists \( \rho > 0 \) such that for every \( a > 0 \) and \( \lambda > \Lambda_N \),

\[
\inf \{ J_{a,\lambda}(u) : u \in X_\lambda \text{ with } \|u\| = \rho \} > \eta
\]

for some \( \eta > 0 \).

**Proof.** By (2.11) and the condition (F3), for all \( u \in X_\lambda \) one has

\[
J_{a,\lambda}(u) \geq \frac{1}{2} \|u\|_\lambda^2 + \frac{a}{2} \|\nabla u\|_{L^2}^{2(1+\delta)} + \frac{b}{2} \|\nabla u\|_{L^2}^2 - \frac{\epsilon}{2} \int_{\mathbb{R}^N} u^2 \, dx - \frac{C_{1,\epsilon}}{p} \int_{\mathbb{R}^N} |u|^p \, dx
\]

\[
\geq \begin{cases} \\
\frac{1}{2} \left( 1 - \epsilon \Theta_{2,N}^2 \right) \|u\|_\lambda^2 - \frac{C_{1,\epsilon} \Theta_p^{p,N}}{p} \|u\|_\lambda^p & \text{if } b \geq 0, \\
\frac{1}{2} \left( 1 + \frac{bA_0^2}{2} \beta_N - \epsilon \Theta_{2,N}^2 \right) \|u\|_\lambda^2 - \frac{C_{1,\epsilon} \Theta_p^{p,N}}{p} \|u\|_\lambda^p & \text{if } -2A_0^{-2}\beta_N^{-1} < b < 0.
\end{cases}
\]

Let

\[
g(t) = \frac{1}{2} \left( 1 - \epsilon \Theta_{2,N}^2 \right) t^2 - \frac{C_{1,\epsilon} \Theta_p^{p,N}}{p} t^p \text{ for } t \geq 0.
\]

A direct calculation shows that

\[
\max_{t \geq 0} g(t) = g(\bar{t}) = \frac{(p-2)}{2p} \left( 1 - \epsilon \Theta_{2,N}^2 \right)^{p/(p-2)} \left( C_{1,\epsilon} \Theta_p^{p,N} \right)^{-2/(p-2)},
\]

where

\[
\bar{t} = \left[ \frac{(1 - \epsilon \Theta_{2,N}^2)}{C_{1,\epsilon} \Theta_p^{p,N}} \right]^{1/(p-2)}.
\]

This shows that when \( b \geq 0 \), for every \( u \in X_\lambda \) with \( \|u\|_\lambda = \bar{t} \) we have

\[
J_{a,\lambda}(u) \geq g(\bar{t}) > 0.
\]

Choosing \( \rho = \bar{t} \) and

\[
\eta = \frac{(p-2)}{2p} \left( 1 - \epsilon \Theta_{2,N}^2 \right)^{p/(p-2)} \left( C_{1,\epsilon} \Theta_p^{p,N} \right)^{-2/(p-2)} > 0,
\]

it is easy to see that the result holds. Similarly, when \( -2A_0^{-2}\beta_N^{-1} < b < 0 \), for every \( u \in X_\lambda \) with

\[
\|u\|_\lambda = \tilde{t} = \left[ \frac{1 + \frac{bA_0^2}{2} \beta_N - \epsilon \Theta_{2,N}^2}{C_{1,\epsilon} \Theta_p^{p,N}} \right]^{1/(p-2)},
\]

we can take \( \rho = \tilde{t} \) and

\[
\eta = \frac{(p-2)}{2p} \left( 1 + \frac{bA_0^2}{2} \beta_N - \epsilon \Theta_{2,N}^2 \right)^{p/(p-2)} \left( C_{1,\epsilon} \Theta_p^{p,N} \right)^{-2/(p-2)}
\]
such that the result holds. This completes the proof. \(\square\)

Define

\[
\Pi_\lambda = \sup_{u \in X_\lambda \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |u|^p \, dx}{\|u\|_\lambda}.
\]  
(2.20)

It follows from (2.16) that

\[
\Pi_\lambda \leq \Theta_{p,N} \text{ for } \lambda \geq \Lambda_N.
\]  
(2.21)

Furthermore, by Appendix A there exist \(\Lambda_1 \geq \Lambda_N\) and \(\phi_\lambda \in X_\lambda \setminus \{0\}\) such that

\[
\Pi_\lambda = \left(\frac{\int_{\mathbb{R}^N} |\phi_\lambda|^p \, dx}{\|\phi_\lambda\|_\lambda}\right)^{1/p} > 0 \text{ for every } \lambda \geq \Lambda_1,
\]  
(2.22)

and there exists a constant \(\Pi_\infty > 0\) independent on \(\lambda\) such that

\[
\Pi_\lambda \searrow \Pi_\infty \text{ as } \lambda \to \infty.
\]  
(2.23)

Setting

\[
a_\ast := \frac{2^{2+\delta} C_{2,\epsilon} \Pi_\infty^p (1+\delta) (p-2)}{\delta p A_0^{2(1+\delta)} \beta_N^{1+\delta}} \left[ \frac{C_{2,\epsilon} \Pi_\infty^p (2\delta + 2 - p)}{\delta p \left(1 + \frac{b_2 A_0^2}{2} \beta_N + \gamma \Theta_{2,N}^2\right)} \right]^{\frac{2\delta + 2 - p}{p-2}}.
\]

**Lemma 2.4** Assume that \(b \in R\), conditions (V1) – (V3), (F1) and (F3) hold. Let \(\rho > 0\) be as in Lemma 2.3 Then for every \(\lambda \geq \Lambda_1\) and \(0 < a < a_\ast\), there exists \(e \in X_\lambda\) satisfying

\[
\|e\|_\lambda > \rho \text{ and } \|e\|_\lambda \to \infty \text{ as } a \to 0
\]

such that

\[
J_{a,\lambda}(e) < 0 \text{ and } J_{a,\lambda}(e) \to -\infty \text{ as } a \to 0.
\]

**Proof.** Let \(\phi_\lambda \in X_\lambda \setminus \{0\}\) be as in (2.22) and let

\[
I(t) = I_{a,\lambda}(t \phi_\lambda)
\]

\[
= \frac{t^2}{2} \|\phi_\lambda\|_\lambda^2 + \frac{\alpha t^{2(1+\delta)}}{2(1+\delta)} \|\nabla \phi_\lambda\|_{L^2}^{2(1+\delta)} + \frac{\beta t^2}{2} \|\nabla \phi_\lambda\|_{L^2}^2
\]

\[
+ \frac{\delta p}{2} \int_{\mathbb{R}^N} \phi_\lambda \, dx - \frac{C_{2,\epsilon} \Pi_\infty^p}{p} \int_{\mathbb{R}^N} |\phi_\lambda|^p \, dx \text{ for } t > 0.
\]

Then it follows from (2.11) and (2.16) that

\[
I(t) \leq \frac{A_0^{2(1+\delta)} \beta_N^{1+\delta} \|\phi_\lambda\|_\lambda^{2(1+\delta)} t^2}{2^{2+\delta} (1+\delta)} \cdot \left[ \frac{\alpha t^{2+\delta} (1+\delta)^2 + \frac{b_2 A_0^2}{2} \beta_N + \gamma \Theta_{2,N}^2}{A_0^{2(1+\delta)} \beta_N^{1+\delta} \|\phi_\lambda\|_\lambda^{2(1+\delta)}} - \frac{2^{3+\delta} (1+\delta) C_{2,\epsilon} \Pi_\infty^p}{p A_0^{2(1+\delta)} \beta_N^{1+\delta} \|\phi_\lambda\|_\lambda^{2(1+\delta)} - p^{2-2}} \right].
\]

A direct calculation shows that there exists

\[
t_{a,\lambda} := \left(\frac{2^{2+\delta} C_{2,\epsilon} \Pi_\infty^p (1+\delta) (p-2)}{a \delta p A_0^{2(1+\delta)} \beta_N^{1+\delta}}\right)^{1/(2\delta + 2 - p)} \|\phi_\lambda\|_\lambda^{-1} > 0
\]

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such that for every $0 < a < a_*$,

$$
\begin{align*}
\alpha t^{2\delta}_{a,\lambda} + 2 (1 + \delta) \left( b + \gamma \Theta^2_{2,N} \right) & \quad - \quad \frac{4 (1 + \delta) C_{2,\epsilon} \Pi_p}{p A_0^2 \beta_N \| \phi \|^{2\delta}_\lambda} \frac{E_{\alpha\lambda}^{p-2}}{p A_0^2 \beta_N \| \phi \|^{2(1+\delta)-p}_\lambda} \\
& = \frac{2^{2+\delta} (1 + \delta) \| \phi \|^{-2\delta}_\lambda}{A_0^{1+\delta} \beta_N} \cdot \left[ 1 + \frac{b A_0^2}{2} \beta_N + \gamma \Theta^2_{2,N} \right] - \frac{C_{2,\epsilon} \Pi_p}{\delta p a A_0^{2(1+\delta)}} \left( 2^{2+\delta} C_{2,\epsilon} \Pi_p (1 + \delta) (p - 2) \right) \left( \frac{2^{2+\delta} C_{2,\epsilon} \Pi_p (1 + \delta) (p - 2)}{\delta p A_0^{2(1+\delta)}} \right)^{\frac{p}{p-2}} \\
& < 0,
\end{align*}
$$

this implies that

$$
I (t, \alpha) = I_{a,\lambda} (t, \alpha \phi) < 0 \text{ for } 0 < a < a_*
$$

and

$$
I_{a,\lambda} (t, \alpha \phi) \to -\infty \text{ as } a \to 0.
$$

Choosing $e = t_{a,\lambda} \phi$. Clearly,

$$
\| e \| = \| t_{a,\lambda} \phi \| = \left[ \frac{2^{2+\delta} C_{2,\epsilon} \Pi_p (1 + \delta) (p - 2)}{a \delta p A_0^{2(1+\delta)}} \frac{1}{\beta_N^{1+\delta}} \right]^{\frac{1}{2(\delta+2-p)}} \to \infty \text{ as } a \to 0.
$$

Note that for $0 < a < a_*$, by (2.21) and (2.23), there holds

$$
\left[ \frac{2^{2+\delta} C_{2,\epsilon} \Pi_p (1 + \delta) (p - 2)}{a \delta p A_0^{2(1+\delta)}} \frac{1}{\beta_N^{1+\delta}} \right]^{\frac{1}{2(\delta+2-p)}} > \left[ \frac{\delta p \left( 1 + \frac{b A_0^2}{2} \beta_N + \gamma \Theta^2_{2,N} \right)}{C_{2,\epsilon} \Theta^p \Pi_p \beta_N^{2(\delta+2-p)}} \right]^{\frac{1}{p-2}},
$$

by using (2.21). Using this, together with condition (F3), leads to

$$
\| e \| > \rho := \left\{ \begin{array}{ll}
\left( \frac{1 - e \Theta^2_{2,N}}{\epsilon_1 \Theta^p \Pi_p} \right)^{\frac{1}{(p-2)}} & \text{if } b \geq 0, \\
\left( \frac{1 - \frac{b A_0^2}{2} \beta_N - \epsilon \Theta^2_{2,N}}{\epsilon_1 \Theta^p \Pi_p} \right)^{\frac{1}{p-2}} & \text{if } -2 A_0^{-2} \beta_N^{-1} < b < 0
\end{array} \right.,
$$

where $\rho > 0$ is as in Lemma 2.3. Moreover, by condition (F3), there holds $J_{a,\lambda} (e) \leq I_{a,\lambda} (e) < 0$ for $0 < a < a_*$. Consequently, the lemma is proved. 

Define

$$
c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{0 \leq t \leq 1} J_{a,\lambda} (\gamma (t))
$$

and

$$
c_0 (\Omega) = \inf_{\gamma \in \Gamma_\lambda (\Omega)} \max_{0 \leq t \leq 1} J_{a,\lambda} |H^2_0 (\Omega) \gamma (t)),
$$

where $J_{a,\lambda} |H^2_0 (\Omega)$ is a restriction of $J_{a,\lambda}$ on $H^2_0 (\Omega)$,

$$
\Gamma_\lambda = \{ \gamma \in C([0, 1], X_\lambda) : \gamma (0) = 0, \gamma (1) = e \}$$
and
\[
\Gamma_\lambda(\Omega) = \{ \gamma \in C([0, 1], H_0^2(\Omega)) : \gamma(0) = 0, \gamma(1) = e \}.
\]

Note that for \( u \in H_0^2(\Omega) \),
\[
J_{a,\lambda}|_{H_0^2(\Omega)}(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx + \frac{a}{2(1 + \delta)} \left( \int_\Omega |\nabla u|^2 \, dx \right)^{2(1+\delta)} + \frac{b}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega F(x, u) \, dx
\]
and \( c_0(\Omega) \) independent of \( \lambda \). Moreover, if conditions (F1) and (F3) hold, then by the proofs of Lemmas 2.3 and 2.4 we can conclude that \( J_{a,\lambda}|_{H_0^2(\Omega)} \) satisfies the mountain pass hypothesis as in Theorem 2.2.

Since \( H_0^2(\Omega) \subset X_\lambda \) for all \( \lambda > 0 \), one can see that \( 0 < \eta \leq c_\lambda \leq c_0(\Omega) \) for all \( \lambda \geq \Lambda_N \). Take \( D_0 > c_0(\Omega) \). Then we have
\[
0 < \eta \leq c_\lambda \leq c_0(\Omega) < D_0 \text{ for all } \lambda \geq \Lambda_N.
\]

By Lemmas 2.3, 2.4 and Theorem 2.2 we obtain that for each \( \lambda \geq \Lambda_N \), there exists a sequence \( \{u_n\} \subset X_\lambda \) such that
\[
J_{a,\lambda}(u_n) \to c_\lambda > 0 \quad \text{and} \quad (1 + \|u_n\|_\lambda)\|J'_{a,\lambda}(u_n)\|_{X_\lambda^{-1}} \to 0 \quad \text{as } n \to \infty.
\]

3 Proof of Theorem 1.1

Recall that a \( C^1 \)-functional \( J_{a,\lambda} \) satisfies Cerami condition at level \( c \) ((\( C\))\(_c\)-condition for short) if any sequence \( \{u_n\} \subset X_\lambda \) satisfying
\[
J_{a,\lambda}(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|_\lambda)\|J'_{a,\lambda}(u_n)\|_{X_\lambda^{-1}} \to 0,
\]
has a convergent subsequence, and such sequence is called a \( (C)\)\(_c\)-sequence.

Lemma 3.1 Assume that \( N \geq 1, \delta > 0 \) and \( b > -2A_0^{-2}\beta_N^{-1} \). In addition, assume that conditions (V1) – (V3), (F1) and (F3) hold. Then \( \{u_n\} \) is bounded in \( X_\lambda \) for each \( \lambda \geq \Lambda_0 \), where \( \{u_n\} \) is a \( (C)\)\(_c\)-sequence.

Proof. Following the argument of Lemma 2.1 we can conclude that the \( (C)\)\(_c\)-sequence \( \{u_n\} \) is bounded in \( X_\lambda \) for each \( \lambda \geq \Lambda_0 \).

Proposition 3.2 Assume that \( b > -2A_0^{-2}\beta_N^{-1} \). In addition, we assume that conditions (V1) – (V3) and (F1) – (F3) hold. Then for each \( D > 0 \), there exists \( \Lambda_1 := \Lambda_1(D) \geq \Lambda_0 > \Lambda_N \) such that \( J_{a,\lambda} \) satisfies the \( (C)\)\(_c\)-condition in \( X_\lambda \) for all \( c < D \) and \( \lambda > \Lambda_1 \).

Proof. Let \( \{u_n\} \) be a \( (C)\)\(_c\)-sequence with \( c < D \). By Lemma 3.1 \( \{u_n\} \) is bounded in \( X_\lambda \) and there exists \( D_0 > 0 \) such that \( \|u_n\|_\lambda \leq D_0 \). Then there exist a subsequence \( \{u_n\} \) and \( u_0 \) in \( X_\lambda \) such that
\[
\begin{align*}
 u_n &\to u_0 \text{ weakly in } X_\lambda, \\
u_n &\to u_0 \text{ strongly in } L_{loc}^r(\mathbb{R}^N), \text{ for } 2 \leq r < 2^*_s, \\
u_n &\to u_0 \text{ a.e. in } \mathbb{R}^N.
\end{align*}
\]
Moreover, using \((2.11)\) and \((2.16)\) implies that the imbedding \(X_\lambda \hookrightarrow W^{1,2}(\mathbb{R}^N)\) is continuous, which shows that

\[ u_n \rightharpoonup u_0 \text{ weakly in } W^{1,2}(\mathbb{R}^N). \]

Similar to the proof of Lemma 4.4 in [12], one can easily obtain that

\[ \nabla u_n(x) \rightarrow \nabla u_0(x) \text{ a.e. in } \mathbb{R}^N. \]

Thus, it follows from Brezis-Lieb lemma [6] that

\[
\int_{\mathbb{R}^N} |\nabla (u_n - u_0)|^2 \, dx = \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx + o(1).
\]

(3.1)

Now we prove that \(u_n \rightarrow u_0\) strongly in \(X_\lambda\). Let \(v_n = u_n - u_0\). Then \(v_n \rightharpoonup 0\) in \(X_\lambda\). By the condition (V2), we have

\[
\int_{\mathbb{R}^N} v_n^2 \, dx = \int_{\{V \geq c_0\}} v_n^2 \, dx + \int_{\{V < c_0\}} v_n^2 \, dx \leq \frac{1}{\lambda c_0} \|v_n\|_2^2 + o(1).
\]

(3.2)

Using (3.2), together with the Hölder and Sobolev inequalities, for any \(\lambda > \Lambda_N\), we check the following estimation:

Case (i) \(N = 3\):

\[
\int_{\mathbb{R}^N} |v_n|^r \, dx \leq \|v_n\|_{L^\infty}^{r-2} \int_{\mathbb{R}^N} v_n^2 \, dx \leq \frac{S^{r-2}_{\infty}}{\lambda c_0} \|v_n\|_{H^2} \|v_n\|_\lambda^2 + o(1)
\]

\[
\leq \frac{S^{r-2}_{\infty}}{\lambda c_0} \left[ \left( 1 + \frac{A_0^2}{2} \right)^{-1} - S^2_{\infty} \{V < c_0\} \right] \|v_n\|_\lambda^r + o(1).
\]

(3.3)

Case (ii) \(N = 4\):

\[
\int_{\mathbb{R}^N} |v_n|^r \, dx \leq \left( \int_{\mathbb{R}^N} v_n^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^N} v_n^{2(r-1)} \, dx \right)^{1/2}
\]

\[
\leq \left( \frac{1}{\lambda c_0} \|v_n\|_\lambda^2 + o(1) \right)^{1/2} \frac{S^{r-(r-1)}_{2(r-1)}}{\sqrt{\lambda c_0}} \left( 1 + \frac{A_0^2}{2} \right)^{(r-1)/2} \|v_n\|_{\lambda^{r-1}}
\]

\[
= \frac{S^{r-(r-1)}_{2(r-1)}}{\sqrt{\lambda c_0}} \left( 1 + \frac{A_0^2}{2} \right)^{(r-1)/2} \|v_n\|_\lambda^r + o(1).
\]

(3.4)

Case (iii) \(N > 4\):

\[
\int_{\mathbb{R}^N} |v_n|^r \, dx \leq \left( \int_{\mathbb{R}^N} v_n^2 \, dx \right)^{\frac{2-\sigma}{2}} \left( \int_{\mathbb{R}^N} v_n^{2\sigma} \, dx \right)^{\frac{\sigma-2}{2}}
\]

\[
\leq C_{0,\sigma}^{\frac{2(r-2)}{2-\sigma}} \left( \frac{1}{\lambda c_0} \right)^{\frac{2-\sigma}{2-\sigma}} \|v_n\|_\lambda^r + o(1).
\]

(3.5)

Set

\[
\Psi_r := \begin{cases}
\frac{S^{r-2}_{\infty}}{\lambda c_0} \left[ \left( 1 + \frac{A_0^2}{2} \right)^{-1} - S^2_{\infty} \{V < c_0\} \right]^{-1/2} & \text{if } N = 3, \\
\frac{S^{r-(r-1)}_{2(r-1)}}{\sqrt{\lambda c_0}} \left( 1 + \frac{A_0^2}{2} \right)^{(r-1)/2} & \text{if } N = 4, \\
C_0^{-\frac{r-2}{2-\sigma}} \left( \frac{1}{\lambda c_0} \right)^{(2-\sigma)/(2-\sigma)} & \text{if } N > 4.
\end{cases}
\]
Clearly, \( \Psi_r \to 0 \) as \( \lambda \to \infty \). The inequalities (3.3) – (3.5) indicate that
\[
\int_{\mathbb{R}^N} |v_n|^r \, dx \leq \Psi_r \|v_n\|_\lambda^r + o(1). \tag{3.6}
\]

Following the argument of \[22\], it is easy to verify that
\[
\int_{\mathbb{R}^N} F(x, v_n) \, dx = \int_{\mathbb{R}^N} F(x, u_n) \, dx - \int_{\mathbb{R}^N} F(x, u_0) \, dx + o(1) \tag{3.7}
\]
and
\[
\sup_{\|h\|_\lambda = 1} \int_{\mathbb{R}^N} [f(x, v_n) - f(x, u_n) + f(x, u_0)] h(x) \, dx = o(1).
\]

Thus, using (3.1), (3.7) and Brezis-Lieb Lemma [6], we deduce that
\[
J_{a,\lambda} (u_n) - J_{a,\lambda} (u_0) = \frac{1}{2} \|v_n\|_\lambda^2 + \frac{a}{2(1+\delta)} \left( \|\nabla u_n\|_{L^2}^{2(1+\delta)} - \|\nabla u_0\|_{L^2}^{2(1+\delta)} \right)
+ \frac{b}{2} \|\nabla v_n\|_{L^2}^2 - \int_{\mathbb{R}^N} F(x, v_n) \, dx + o(1). \tag{3.8}
\]

Moreover, it follows from the boundedness of the sequence \( \{u_n\} \) in \( X_\lambda \) and (2.11) that there exists a constant \( A > 0 \) such that
\[
\|\nabla u_n\|_{L^2}^2 \to A \text{ as } n \to \infty.
\]

It indicates that for any \( \varphi \in C_0^\infty (\mathbb{R}^N) \), there holds
\[
o(1) = \left< J'_{a,\lambda} (u_n), \varphi \right>
= \int_{\mathbb{R}^N} \Delta u_n \Delta \varphi \, dx + \int_{\mathbb{R}^N} \lambda V(x) u_n \varphi \, dx + a \|\nabla u_n\|_{L^2}^{2\delta} \int_{\mathbb{R}^N} \nabla u_n \nabla \varphi \, dx
+ b \left( \frac{1}{2} \|\nabla u_n\|_{L^2}^2 - \int_{\mathbb{R}^N} F(x, v_n) \, dx \right)
\rightarrow \int_{\mathbb{R}^N} \Delta u_0 \Delta \varphi \, dx + \int_{\mathbb{R}^N} \lambda V(x) u_0 \varphi \, dx + a A^\delta \int_{\mathbb{R}^N} \nabla u_0 \nabla \varphi \, dx
+ b \left( \frac{1}{2} \|\nabla u_0\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x, u_0) u_0 \, dx \right) \text{ as } n \to \infty,
\]
which shows that
\[
\|u_0\|_\lambda^2 + (a A^\delta + b) \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx - \int_{\mathbb{R}^N} f(x, u_0) u_0 \, dx = o(1).
\]

Note that
\[
o(1) = \left< J'_{a,\lambda} (u_n), u_n \right>
= \|u_n\|_\lambda^2 + a \|\nabla u_n\|_{L^2}^{2(1+\delta)} + b \|\nabla u_n\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x, u_n) u_n \, dx.
\]
Combining the above two equalities gives

\[ o(1) = \|u_n\|_\lambda^2 + a \|\nabla u_n\|_{L^2}^{2(1+\delta)} + b \|\nabla u_n\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x, u_n) u_n dx \]

\[ - \|u_0\|_\lambda^2 - (A\delta + b) \int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \int_{\mathbb{R}^N} f(x, u_0) u_0 dx \]

\[ = \|v_n\|_\lambda^2 + a \|\nabla u_n\|_{L^2}^{2(1+\delta)} - a \|\nabla u_n\|_{L^2}^{2\delta} \|\nabla u_0\|_{L^2}^2 \]

\[ + b \|\nabla v_n\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x, v_n) v_n dx + o(1) \]

\[ = \|v_n\|_\lambda^2 + a \|\nabla u_n\|_{L^2}^{2\delta} \|\nabla v_n\|_{L^2}^2 + b \|\nabla v_n\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x, v_n) v_n dx + o(1). \]  

(3.9)

By Lemma 2.1 there exists a constant \( K < 0 \) such that

\[ J_{a,\lambda}(u_0) \geq K. \]  

(3.10)

Thus, in virtue of condition \((F2)\) and \((3.8) - (3.10)\) one has

\[ D - K \geq c - J_{a,\lambda}(u_0) \]

\[ \geq J_{a,\lambda}(u_n) - J_{a,\lambda}(u_0) + o(1) \]

\[ \geq \frac{1}{2} \|v_n\|_\lambda^2 + \frac{a}{2(1+\delta)} \left( \|\nabla u_n\|_{L^2}^{2(1+\delta)} - \|\nabla u_0\|_{L^2}^{2(1+\delta)} \right) \]

\[ + \frac{b}{2} \|\nabla v_n\|_{L^2}^2 - \int_{\mathbb{R}^N} F(x, v_n) dx + o(1) \]

\[ \geq \frac{\delta}{2(1+\delta)} \|v_n\|_\lambda^2 + \frac{b\delta}{2(1+\delta)} \|\nabla v_n\|_{L^2}^2 - \frac{d_0}{2(1+\delta)} \int_{\mathbb{R}^N} v_n^2 dx \]

\[ + \frac{a}{2(1+\delta)} \left( \|\nabla u_n\|_{L^2}^{2\delta} - \|\nabla u_0\|_{L^2}^{2\delta} \right) \|\nabla u_0\|_{L^2}^2 + o(1) \]

\[ \geq \frac{\delta - d_0 \Theta_{2,N}^2}{2(1+\delta)} \|v_n\|_\lambda^2 + \frac{b\delta}{2(1+\delta)} \|\nabla v_n\|_{L^2}^2 \]

\[ + \frac{a}{2(1+\delta)} \left( \|\nabla u_n\|_{L^2}^{2\delta} - \|\nabla u_0\|_{L^2}^{2\delta} \right) \|\nabla u_0\|_{L^2}^2 + o(1), \]

which implies that there exists a constant \( \hat{D} = \hat{D}(a, D) > 0 \) such that

\[ \|v_n\|_\lambda^2 \leq \hat{D} + o(1) \text{ for every } \lambda > \Lambda_N. \]  

(3.11)

It follows from the condition \((F3), (3.6)\) and \((3.11)\) that

\[ o(1) = \|v_n\|_\lambda^2 + a \|\nabla u_n\|_{L^2}^{2\delta} \|\nabla v_n\|_{L^2}^2 + b \|\nabla v_n\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x, v_n) v_n dx \]

\[ \geq \|v_n\|_\lambda^2 + b \|\nabla v_n\|_{L^2}^2 - \epsilon \int_{\mathbb{R}^N} v_n^2 dx - C_{1,\epsilon} \int_{\mathbb{R}^N} |v_n|^p dx \]

\[ \geq \begin{cases} \|v_n\|_\lambda^2 - \epsilon \Psi_1^2 \hat{D} - C_{1,\epsilon} \Psi_{p}^{p/2} & \text{if } b \geq 0, \\ \frac{1}{2} \left( 2 + b \lambda_0^2 \beta_N \right) \|v_n\|_\lambda^2 - \epsilon \Psi_1^2 \hat{D} - C_{1,\epsilon} \Psi_{p}^{p/2} & \text{if } -2\lambda_0^{-2} \beta_N^{-1} < b < 0, \end{cases} \]
Lemma 3.4 Suppose that \( \lambda > \Lambda \)

Similar to the proof of Theorem 3.3, there exist a subsequence \( \{X_u\} \) nonzero critical point \( v_{a,\lambda}^{(1)} \in X_\lambda \) such that \( J_{a,\lambda}(v_{a,\lambda}^{(1)}) = c_\lambda > 0 \).

Proof. By virtue of Theorem 2.2, Lemmas 2.3 and 2.4, for every \( \{\} \) exists a sequence \( \{\} \).

By Lemma 3.1, one has \( \{\} \).

This completes the proof.

Theorem 3.3 Assume that \( N \geq 3, \delta \geq \frac{2}{N-2} \) and \( b > -2A_0^{-2} \beta_N^{-1} \). In addition, we assume that conditions (V1) – (V3) and (F1) – (F3) are satisfied. Then for each \( 0 < a < a_* \) and \( \lambda > \Lambda_1 \), \( J_{a,\lambda} \) has a nonzero critical point \( v_{a,\lambda}^{(1)} \in X_\lambda \) such that \( J_{a,\lambda}(v_{a,\lambda}^{(1)}) = c_\lambda > 0 \).

Proof. By virtue of Theorem 2.2, Lemmas 2.3 and 2.4, for every \( \{\} \) exists a sequence \( \{\} \).

It follows from Lemmas 3.1, 3.4 and the Ekeland variational principle that there exists \( \{\} \).

Lemma 3.4 Suppose that \( N \geq 3, \delta \geq \frac{2}{N-2} \) and \( b > -2A_0^{-2} \beta_N^{-1} \). In addition, assume that conditions (V1) – (V3), (F1) and (F3) hold. Then for every \( 0 < a < a_* \) and \( \lambda > \Lambda_1 \) one has

\[
-\infty < \theta_a =: \inf \{J_{a,\lambda}(u) : u \in X_\lambda \text{ with } \rho < \|u\| < R_\alpha \} < \frac{\kappa}{2} < 0.
\]  

(3.12)

Proof. The proof directly follows from Lemmas 2.1 and 2.4.

Theorem 3.5 Suppose that \( N \geq 3, \delta \geq \frac{2}{N-2} \) and \( b > -2A_0^{-2} \beta_N^{-1} \). In addition, assume that conditions (V1) – (V3), (F1) – (F3) hold. Then for every \( 0 < a < a_* \) and \( \lambda > \Lambda_1 \), \( J_{a,\lambda} \) has a nonzero critical point \( u_{a,\lambda}^{(2)} \in X_\lambda \) such that

\[
J_{a,\lambda}(u_{a,\lambda}^{(2)}) = \theta_a < 0,
\]

where \( \hat{\theta}_a \) is as in (3.12). Furthermore, when \( \delta > \frac{2}{N-2} \), for every \( \lambda > \Lambda_1 \) there holds

\[
J_{a,\lambda}(u_{a,\lambda}^{(2)}) \to -\infty \text{ and } \|u_{a,\lambda}^{(2)}\|_\lambda \to \infty \text{ as } a \to 0,
\]

Proof. It follows from Lemmas 3.1, 3.4 and the Ekeland variational principle that there exists a minimizing bounded sequence \( \{u_n\} \subset X_\lambda \) with \( \rho < \|u_n\|_\lambda < R_\alpha \) such that

\[
J_{a,\lambda}(u_n) \to \theta_a \text{ and } J'_{a,\lambda}(u_n) \to 0 \text{ as } n \to \infty.
\]

Similar to the proof of Theorem 3.3, there exist a subsequence \( \{u_n\} \) and \( u_{a,\lambda}^{(2)} \in X_\lambda \) with \( \rho < \|u_{a,\lambda}^{(2)}\|_\lambda < R_\alpha \) such that \( u_n \to u_{a,\lambda}^{(2)} \) strongly in \( X_\lambda \), which implies that \( J'_{a,\lambda}(u_{a,\lambda}^{(2)}) = 0 \) and \( J_{a,\lambda}(u_{a,\lambda}^{(2)}) = \theta_a < 0 \). Furthermore, by Lemmas 2.1 and 2.4 we have

\[
J_{a,\lambda}(u_{a,\lambda}^{(2)}) \leq J_{a,\lambda}(u) \to -\infty \text{ as } a \to 0.
\]
It implies that 
\[ \| u_{a,\lambda}^{(2)} \|_\lambda \to \infty \text{ as } a \to 0, \]
Consequently, we complete the proof.

We are now ready to prove Theorem 1.1. By Theorems 3.3 and 3.5, for every \( 0 < a < a_\ast \) and \( \lambda > \Lambda_1 \), there exist two nontrivial solutions \( u_{a,\lambda}^{(1)} \) and \( u_{a,\lambda}^{(2)} \) of Eq. \((K_{a,\lambda})\) such that
\[ J_{a,\lambda}(u_{a,\lambda}^{(2)}) = \theta_a < \frac{\kappa}{2} < 0 < \eta < c_\lambda = J_{a,\lambda}(u_{a,\lambda}^{(1)}), \]
which implies that \( u_{a,\lambda}^{(1)} \neq u_{a,\lambda}^{(2)} \). Furthermore, when \( \delta > \frac{\sqrt{N}}{N-2} \), for every \( \lambda > \Lambda_1 \) there holds
\[ J_{a,\lambda}(u_{a,\lambda}^{(2)}) \to -\infty \text{ and } \| u_{a,\lambda}^{(2)} \|_\lambda \to \infty \text{ as } a \to 0. \]
Since \( J_{a,\lambda}(u) \geq 0 \) on \( \{ u \in X_\lambda \text{ with } \| u \|_\lambda \leq \rho \cup \| u \|_\lambda \geq R_\lambda \} \) by Lemmas 3.4 and 2.3, we conclude that \( u_{a,\lambda}^{(2)} \) is a ground state solution of Eq. \((K_{a,\lambda})\). This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. Let \( u_0 \) be a nontrivial solution of Eq. \((K_{a,\lambda})\). Then there holds
\[ \| u_0 \|_\lambda^2 + a \| \nabla u_0 \|_{L^2}^{2(1+\delta)} + b \| \nabla u_0 \|_{L^2}^2 - \int_{\mathbb{R}^N} f(x, u_0) u_0 \, dx = 0. \]
We now divide the proof into two separate cases:

Case A : \( \int_{\mathbb{R}^N} \lambda \nu(x) u_0^2 \, dx \geq \lambda c_0 \left( \frac{C_{1,\epsilon}}{\lambda c_0 - \epsilon} \right) \left( \frac{4}{(p-2)(N-2)} \right) \left( \frac{\sqrt{N}}{\lambda c_0 - \epsilon} \right)^{\frac{N}{2}} \| \nabla u_0 \|_{L^2}^2 \right)^{2p-2} \| \nabla u_0 \|_{L^2}^2 \]. It follows from the condition \((F3)\) and \((2.19)\) that
\[ 0 = \| u_0 \|_\lambda^2 + a \| \nabla u_0 \|_{L^2}^{2(1+\delta)} + b \| \nabla u_0 \|_{L^2}^2 - \int_{\mathbb{R}^N} f(x, u_0) u_0 \, dx \]
\[ \geq \| u_0 \|_\lambda^2 + a \| \nabla u_0 \|_{L^2}^{2(1+\delta)} + b \| \nabla u_0 \|_{L^2}^2 \]
\[ - \epsilon \left( \frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda \nu(x) u_0^2 \, dx + \frac{\sqrt{N}}{\lambda c_0 - \epsilon} \right)^\frac{2p}{N} \| \nabla u_0 \|_{L^2}^2 \]
\[ - \frac{C_{1,\epsilon}}{\sqrt{\lambda c_0 - \epsilon}} \left( \frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda \nu(x) u_0^2 \, dx + \frac{\sqrt{N}}{\lambda c_0 - \epsilon} \right)^\frac{2p}{N} \| \nabla u_0 \|_{L^2}^2 \]
\[ \geq \| u_0 \|_{L^2}^{2(1+\delta)} + \left( b - \epsilon \frac{\nu}{\lambda c_0} \right) \| \nabla u_0 \|_{L^2}^2 \]
\[ - \frac{C_{1,\epsilon}}{\sqrt{\lambda c_0 - \epsilon}} \| \nabla u_0 \|_{L^2}^2 \]
\[ \geq \| u_0 \|_{L^2}^{2(1+\delta)} + \left( b - \epsilon \frac{\nu}{\lambda c_0} \right) \| \nabla u_0 \|_{L^2}^2 - \frac{C_{1,\epsilon}}{\sqrt{\lambda c_0 - \epsilon}} \| \nabla u_0 \|_{L^2}^2 \]
\[ > 0. \]
provided that

\[
a > \frac{p - 2}{2\delta} \left[ \frac{2(\delta + 1) - p}{2\delta \left( b - \epsilon S^{-2} \left| \{ V < c_0 \} \right| \right)^{2/N}} \right] \left( C_{1,\epsilon} \right)^{2(\delta + 1) - p \over p - 2} \left( S^p \left| \{ V < c_0 \} \right| \frac{2N - p(N - 2)}{2N} \right)^{2\delta \over p - 2}.
\]

This is a contradiction.

Case B: \( \int_{\mathbb{R}^N} \lambda V(x) u_0^2 dx < \lambda c_0 \left[ \frac{C_{1,\epsilon}}{(\lambda c_0 - \epsilon)} \right]^{\frac{4}{(p - 2)(N - 2)}} \left( S^1 \left\| \nabla u_0 \right\|_{L^2} \right)^{2N - p(N - 2)} \). By virtue of \([2.19]\)

one has

\[
\int_{\mathbb{R}^N} |u_0|^p dx 
\leq \left( \frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda V(x) u_0^2 dx + \left\{ \left\| \nabla u_0 \right\|_{L^2} \right\} \left( S \right)^{-1} \left\| \nabla u_0 \right\|_{L^2} \right) \left( S \right)^{p - 2} \left( \lambda c_0 - \epsilon \right)^{- \frac{2N - p(N - 2)}{(p - 2)(N - 2)}} \left\| \nabla u_0 \right\|_{L^2}^2.
\]

Using this, together with condition \((F3)\), gives

\[
0 = \left\| u_0 \right\|_{L^2}^2 + a \left\| \nabla u_0 \right\|_{L^2}^2 + b \left\| \nabla u_0 \right\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x, u_0) u_0 dx 
\geq a \left\| \nabla u_0 \right\|_{L^2}^2 + \left( b - \epsilon S^{-2} \left\| \{ V < c_0 \} \right\| \right) \left\| \nabla u_0 \right\|_{L^2}^2 
- \frac{C_{1,\epsilon}}{S^{2\epsilon}} (\lambda c_0 - \epsilon)^{- \frac{2N - p(N - 2)}{(p - 2)(N - 2)}} \left\| \nabla u_0 \right\|_{L^2}^2 
- \frac{C_{1,\epsilon}}{S^{2\epsilon}} \left\| \{ V < c_0 \} \right\| \frac{2N - p(N - 2)}{2N} \left\| \nabla u_0 \right\|_{L^2}^p.
\]

If \( \delta = \frac{2}{N - 2} \), then for

\[
a > \frac{p - 2}{2\delta} \left[ \frac{2(\delta + 1) - p}{2\delta \left( b - \epsilon S^{-2} \left| \{ V < c_0 \} \right| \right)^{2/N}} \right] \left( C_{1,\epsilon} \right)^{2(\delta + 1) - p \over p - 2} \left( S^p \left| \{ V < c_0 \} \right| \frac{2N - p(N - 2)}{2N} \right)^{2\delta \over p - 2} 
+ \frac{C_{1,\epsilon}}{S^{2\epsilon}} \left( \lambda c_0 - \epsilon \right)^{- \frac{2N - p(N - 2)}{(p - 2)(N - 2)}} \left\| \nabla u_0 \right\|_{L^2}^2 
\]

there holds

\[
0 > a \left\| \nabla u_0 \right\|_{L^2}^2 + \left( b - \epsilon S^{-2} \left\| \{ V < c_0 \} \right\| \right) \left\| \nabla u_0 \right\|_{L^2}^2 
- \frac{C_{1,\epsilon}}{S^{2\epsilon}} \left( \lambda c_0 - \epsilon \right)^{- \frac{2N - p(N - 2)}{(p - 2)(N - 2)}} \left\| \nabla u_0 \right\|_{L^2}^2 
- \frac{C_{1,\epsilon}}{S^{2\epsilon}} \left\| \{ V < c_0 \} \right\| \frac{2N - p(N - 2)}{2N} \left\| \nabla u_0 \right\|_{L^2}^p > 0.
\]

This is a contradiction. If \( \delta > \frac{2}{N - 2} \), then we consider the following two cases:
In this section, we investigate the concentration for solutions and give the proof of Theorem 5 Concentration of solutions. We also get a contradiction. Therefore, there exists a constant \(a^*>0\) such that for every \(a>a^*\), Eq. \((K_{a,\lambda})\) does not admit any nontrivial solution for all \(\lambda > \Lambda_N\). This completes the proof of Theorem 1.2.

5 Concentration of solutions

In this section, we investigate the concentration for solutions and give the proof of Theorem 1.3.
Proof of Theorem 1.3. Following the arguments of [3], we firstly choose a positive sequence \( \{\lambda_n\} \) such that \( \Lambda_1 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to \infty \) as \( n \to \infty \). Let \( u^{(i)}_n := u^{(i)}_{a,\lambda_n} \) with \( i = 1, 2 \) be the critical points of \( J_{a,\lambda_n} \) obtained in Theorem 1.1. Since

\[
J_{a,\lambda_n} \left( u^{(2)}_n \right) < \frac{\kappa}{2} < 0 < \eta < c_{\lambda_n} = J_{a,\lambda_n} \left( u^{(1)}_n \right) < D, \tag{5.1}
\]

by Lemma 2.1 one has

\[
\left\| u^{(i)}_n \right\|_{\lambda_n} \leq C_0, \tag{5.2}
\]

where the constant \( C_0 > 0 \) is independent of \( \lambda_n \). This implies that \( \left\| u^{(i)}_n \right\|_{\lambda_1} \leq C_0 \). Thus, there exist \( u^{(i)}_\infty \in X \) (\( i = 1, 2 \)) such that

\[
\begin{align*}
 u^{(i)}_n &\rightharpoonup u^{(i)}_\infty \text{ weakly in } X_{\lambda_1}, \\
u^{(i)}_n &\to u^{(i)}_\infty \text{ strongly in } L^r_{\text{loc}}(\mathbb{R}^N), \text{ for } 2 \leq r < 2_*, \\
u^{(i)}_n &\to u^{(i)}_\infty \text{ a.e. in } \mathbb{R}^N.
\end{align*}
\]

Following the proof of Proposition 3.2, we can conclude that

\[
u^{(i)}_n \to u^{(i)}_\infty \text{ strongly in } X_{\lambda_1}. \]

This shows that \( u^{(i)}_n \to u^{(i)}_\infty \) strongly in \( H^2(\mathbb{R}^N) \) by (2.10) and that

\[
\left\| \nabla u^{(i)}_n \right\|_{L^2}^2 \to \left\| \nabla u^{(i)}_\infty \right\|_{L^2}^2 \text{ as } n \to \infty \tag{5.3}
\]

by (2.11) and (3.1).

Using Fatou’s Lemma leads to

\[
\int_{\mathbb{R}^N} V(x) \left( u^{(i)}_\infty \right)^2 dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} V(x) \left( u^{(i)}_n \right)^2 dx \leq \liminf_{n \to \infty} \frac{\left\| u^{(i)}_n \right\|_{\lambda_n}^2}{\lambda_n} = 0,
\]

which implies that \( u^{(i)}_\infty(x) = 0 \) a.e. in \( \mathbb{R}^N \setminus \overline{\Omega} \). Moreover, fixing \( \phi \in C_0^\infty(\mathbb{R}^N \setminus \overline{\Omega}) \), we have

\[
\int_{\mathbb{R}^N \setminus \overline{\Omega}} \nabla u^{(i)}_\infty(x) \phi(x) dx = - \int_{\mathbb{R}^N \setminus \overline{\Omega}} u^{(i)}_\infty(x) \nabla \phi(x) dx = 0.
\]

This indicates that

\[
\nabla u^{(i)}_\infty(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \overline{\Omega}.
\]

Since \( \partial \Omega \) is smooth, \( u^{(i)}_\infty \in H^2(\mathbb{R}^N \setminus \overline{\Omega}) \) and \( \nabla u^{(i)}_\infty \in H^1(\mathbb{R}^N \setminus \overline{\Omega}) \), it follows from Trace Theorem that there are constants \( \overline{C}, \widetilde{C} > 0 \) such that

\[
\left\| u^{(i)}_\infty \right\|_{L^2(\partial \Omega)} \leq \overline{C} \left\| u^{(i)}_\infty \right\|_{H^2(\mathbb{R}^N \setminus \overline{\Omega})} = 0,
\]

and

\[
\left\| \nabla u^{(i)}_\infty \right\|_{L^2(\partial \Omega)} \leq \widetilde{C} \left\| \nabla u^{(i)}_\infty \right\|_{H^1(\mathbb{R}^N \setminus \overline{\Omega})} = 0.
\]

These show that \( u^{(i)}_\infty \in H^2_0(\Omega) \).
Since \( \langle J_{a,\lambda_n}'(u^{(i)}_n), \varphi \rangle = 0 \) for any \( \varphi \in C_0^\infty(\Omega) \), combining (5.3), it is not difficult to check that

\[
\int_\Omega \Delta u^{(i)}_\infty \Delta \varphi \, dx + \left[ a \left( \int_\Omega |\nabla u^{(i)}_\infty|^2 \, dx \right)^\delta + b \right] \int_\Omega \nabla u^{(i)}_\infty \cdot \nabla \varphi \, dx = \int_\Omega f(x, u^{(i)}_\infty) \varphi \, dx,
\]

that is, \( u^{(i)}_\infty \) are the weak solutions of the equation

\[
\Delta^2 u - M \left( \int_\Omega |\nabla u|^2 \, dx \right) \Delta u = f(x, u) \text{ in } \Omega,
\]

where \( M(t) = at^\delta + b \). Since \( u^{(i)}_n \to u^{(i)}_\infty \) strongly in \( X \), using (5.1) and the fact that \( \eta \) and \( \kappa \) are independent of \( \lambda_n \) gives

\[
\frac{1}{2} \int_\Omega |\Delta u^{(1)}_\infty|^2 \, dx + \frac{a}{2(1+\delta)} \left( \int_\Omega |\nabla u^{(1)}_\infty|^2 \, dx \right)^\delta + \frac{b}{2} \int_\Omega |\nabla u^{(1)}_\infty|^2 \, dx - \int_\Omega F(x, u^{(1)}_\infty) \, dx \geq \eta > 0
\]

and

\[
\frac{1}{2} \int_\Omega |\Delta u^{(2)}_\infty|^2 \, dx + \frac{a}{2(1+\delta)} \left( \int_\Omega |\nabla u^{(2)}_\infty|^2 \, dx \right)^\delta + \frac{b}{2} \int_\Omega |\nabla u^{(2)}_\infty|^2 \, dx - \int_\Omega F(x, u^{(2)}_\infty) \, dx \leq \frac{\kappa}{2} < 0.
\]

These imply that \( u^{(i)}_\infty \neq 0 \) \((i = 1, 2)\) and \( u^{(1)}_\infty \neq u^{(2)}_\infty \). Consequently, this complete the proof.

### 6 Appendix A

Consider the following biharmonic equations

\[
\begin{align*}
\Delta^2 u + \lambda V(x) u &= |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \\
u &\in H^2(\mathbb{R}^N),
\end{align*}
\]

(6.1)

where \( N \geq 3, 2 < p < \frac{2N}{N-2} \), \( \lambda > 0 \) is a parameter and \( V(x) \) satisfies conditions (V1) – (V3).

Eq. (6.1) is variational and its solutions are critical points of the functional defined in \( X_\lambda \) by

\[
\mathcal{J}_\lambda(u) = \frac{1}{2} \|u\|_\lambda^2 - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx,
\]

where \( \|u\|_\lambda \) is defined as (2.1). It is easily seen that the functional \( J_\lambda \) is of class \( C^1 \) in \( X_\lambda \), and that

\[
\langle \mathcal{J}'_\lambda(u), v \rangle = \int_{\mathbb{R}^N} [\Delta u \cdot \Delta v + \lambda V(x) uv] \, dx - \int_{\mathbb{R}^N} |u|^{p-2} uv \, dx.
\]

Define the Nehari manifold by

\[
\mathcal{N} = \{u \in X_\lambda \setminus \{0\} \mid \langle \mathcal{J}'_\lambda(u), u \rangle = 0\}.
\]

Clearly, \( J_\lambda(u) \) is bounded below and coercive on \( \mathcal{N} \), since \( p > 2 \). Following the standard argument, we conclude that there exists a positive constant \( \Lambda_1 \geq \Lambda_N \) such that Eq. (6.1)
admits a positive ground state solution $\phi_\lambda \in H^2(R^N)$ for every $\lambda \geq \Lambda_1$. Similar to the argument of [10, Theorem 22], we obtain that $\Pi_\lambda$ defined as (2.20) is achieved and

$$
\Pi_\lambda = \frac{\left(\int_{R^N} |\phi_\lambda|^p \, dx\right)^{1/p}}{\|\phi_\lambda\|_\lambda} > 0 \text{ for every } \lambda \geq \Lambda_1.
$$

Furthermore, similar to the proof of Theorem 1.3, it follows that $\phi_\lambda \to \phi_\infty$ in $H^2(R^N)$ and in $L^p(R^N)$ as $\lambda \to \infty$, where $0 \neq \phi_\infty \in H^1_0(\Omega) \cap H^2(\Omega)$ is the weak solution of biharmonic equations as follows

$$
\Delta^2 u = |u|^{p-2} u \text{ in } \Omega.
$$

This implies that

$$
\Pi_\lambda \to \Pi_\infty := \frac{\left(\int_\Omega |\phi_\infty|^p \, dx\right)^{1/p}}{\left(\int_\Omega |\Delta \phi_\infty|^2 \, dx\right)^{1/2}} > 0 \text{ as } \lambda \to \infty.
$$

Note that

$$
\Pi_\lambda = \sup_{u \in X \setminus \{0\}} \frac{\left(\int_{R^N} |u|^p \, dx\right)^{1/p}}{\|u\|_\lambda}
$$

is decreasing on $\lambda$. Hence, we have

$$
\Pi_\lambda \nrightarrow \Pi_\infty \text{ as } \lambda \to \infty.
$$

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