A covariant approach to general field space metric in multi-field inflation

Jinn-Ouk Gong\textsuperscript{*} and Takahiro Tanaka\textsuperscript{†}

\textsuperscript{*} Theory Division, CERN
CH-1211 Genève 23, Switzerland

\textsuperscript{†} Yukawa Institute for Theoretical Physics, Kyoto University
Kyoto 606-8502, Japan

January 17, 2012

Abstract

We present a covariant formalism for general multi-field system which enables us to obtain higher order action of cosmological perturbations easily and systematically. The effects of the field space geometry, described by the Riemann curvature tensor of the field space, are naturally incorporated. We explicitly calculate up to the cubic order action which is necessary to estimate non-Gaussianity and present those geometric terms which have not yet known before.
1 Introduction

Inflation \(^{1}\) is currently the leading candidate to lay down the necessary initial conditions for the successful hot big bang evolution of the universe \(^{2}\). The most recent observations from the cosmic microwave background (CMB) are consistent with the predictions of the inflationary paradigm \(^{3}\): the universe is homogeneous and isotropic with vanishing spatial curvature, and the primordial scalar perturbation is dominantly adiabatic and follows almost perfect Gaussian statistics with a nearly scale invariant power spectrum. Thus, any small deviation from these predictions would provide crucial information for us to distinguish different models of inflation. Especially, the non-linearities in the primordial perturbation have received an extensive interest nowadays in the light of upcoming precise cosmological observations. For example, while the current bound on the non-linear parameter \(f_{\text{NL}}\) \(^{4}\) is constrained to be \(|f_{\text{NL}}| \lesssim \mathcal{O}(100)\) from the Wilkinson Microwave Anisotropy Probe observation on the CMB \(^{3}\), the Planck satellite can probe with better precision to detect \(|f_{\text{NL}}| = \mathcal{O}(5)\) \(^{5}\). The sensitivity may be even further improved from the observations on large scale structure \(^{6}\).

The absence of the relevant scalar field which can support inflation in the standard model (SM) of particle physics\(^{3}\) demands that inflation be described in the context of the theories beyond the SM. Typically there are plenty of scalar fields which can contribute to the inflationary dynamics \(^{10}\). Further, in multi-field system we can obtain interesting observational signatures which deviate from the predictions of the single field models of inflation and can be detected in near future, such as isocurvature perturbation \(^{11}\) or non-Gaussianity \(^{12}\). Thus we have both theoretical and phenomenological motivations to develop a complete formulation of general multi-field inflation.

An important point in multi-field system is that in the field space which generally has non-trivial field space metric, the scalar fields play the role of the coordinate. Naturally, as we do in general relativity, it is preferable to formulate the dynamics in the field space in the coordinate independent manner. That is, we need a covariant formulation of multi-field inflation which allows us to describe the inflationary dynamics with arbitrary field space. However, most studies on multi-field inflation, especially regarding non-linear perturbations, are based on trivial field space \(^{13}\) or non-covariant description \(^{14, 15, 16}\). The existing studies with covariant approach to general field space metric are mostly on linear perturbation theory \(^{17}\).

In this note, we develop a fully covariant formulation of non-linear perturbations in general multi-field inflation. Along with the covariance for general field space, it allows us to obtain arbitrary higher order action of cosmological perturbations easily and systematically. We consider the matter Lagrangian which is a generic function of the field space metric \(G_{IJ}\) with \(I\) and \(J\) being generic field space indices, kinetic function \(\partial^\mu \phi^I \partial_\mu \phi^J\) and the fields \(^{18}\). This form includes not only the matter Lagrangian with the standard canonical kinetic term but also more generic ones motivated from high energy theories, such as the Dirac-Born-Infeld (DBI) type \(^{19}\).

This note is outlined as follows. In Section 2 we set up the geodesic equation to describe the field fluctuation around the background trajectory. In Section 3 we consider pure matter Lagrangian and present a covariant formulation to describe the field fluctuations up to arbitrary perturbations.
order. The extension to include gravity follows in Section 4 and we explicitly compute the perturbed action up to cubic order. We also discuss the genuine multi-field effects briefly. We conclude in Section 5. Technical details to compare with the previously known non-covariant description are presented in the Appendix.

2 Issue of mapping

To begin with, first let us consider how to describe the physical field fluctuation $\delta \phi^I$ in the field space in a covariant manner. We can think of the background field trajectory parametrized by a single parameter, usually taken as the cosmic time $t$: $\phi^I_0 = \phi^I_0(t)$. The real physical field in a fixed gauge $\phi^I$ incorporates quantum fluctuations $\delta \phi^I$ around this background trajectory. However, the fluctuations $\delta \phi^I$ are coordinate dependent, and hence they are not covariant. These two points, $\phi^I_0(t)$ and $\phi^I$, can be connected by a unique geodesic with respect to the field space metric $G_{IJ}$ as long as their separation is sufficiently small. This geodesic can be specified by the initial point $\phi^I_0$ and its initial velocity, which we denote by $Q^I$. This situation is depicted in Figure 1. Hence, the issue here is the "mapping" beyond linear order between the finite displacement $\delta \phi^I \equiv \phi^I - \phi^I_0$ and a vector $Q^I$ living in the tangent space at $\phi^I_0$. Let us parametrize the geodesic trajectory in the field space by $\lambda$, which runs from 0 to $\epsilon > 0$: $\lambda = 0$ and $\lambda = \epsilon$ correspond to $\phi^I_0$ and $\phi^I$, respectively. Here $\epsilon$ is a parameter introduced to count the order of perturbation just for a bookkeeping purpose, and hence it is set to unity at the end of calculation.

![Figure 1: A schematic figure showing a physical field $\phi^I$ in the field space around the background trajectory $\phi^I_0(t)$. The geodesic connecting $\phi^I$ and $\phi^I_0$ is parametrized $\lambda$, which runs from 0 to $\epsilon$.](image)
Denoting the covariant differentiation in \( \lambda \)-direction by \( D_\lambda \equiv D/d\lambda \), the geodesic equation for \( \phi^I(\lambda) \) is written as
\[
D_\lambda^2 \phi^I = \frac{d^2 \phi^I}{d\lambda^2} + \Gamma^I_{JK} \frac{d\phi^J}{d\lambda} \frac{d\phi^K}{d\lambda} = 0,
\]
and the initial conditions are
\[
\phi^I |_{\lambda=0} = \phi_0^I, \quad \quad \quad \quad (2)
\]
\[
D_\lambda \phi^I |_{\lambda=0} = \left. \frac{d\phi^I}{d\lambda} \right|_{\lambda=0} = Q^I. \quad \quad \quad \quad (3)
\]
Now, we expand \( \phi^I(\lambda = \epsilon) \) as a power series with respect to \( \epsilon \) from \( \lambda = 0 \) as
\[
\phi^I(\lambda = \epsilon) = \phi^I |_{\lambda=0} + \left. \frac{d\phi^I}{d\lambda} \right|_{\lambda=0} \epsilon + \frac{1}{2!} \left. \frac{d^2 \phi^I}{d\lambda^2} \right|_{\lambda=0} \epsilon^2 + \frac{1}{3!} \left. \frac{d^3 \phi^I}{d\lambda^3} \right|_{\lambda=0} \epsilon^3 + \cdots. \quad \quad \quad \quad (4)
\]
Note that the derivatives with respect to \( \lambda \) here are not covariant ones. Thus, we can trade quadratic and higher derivatives with single derivatives by means of the geodesic equation (1). Namely, we can replace a quadratic derivative with
\[
\frac{d^2 \phi^I}{d\lambda^2} = -\Gamma^I_{JK} \frac{d\phi^J}{d\lambda} \frac{d\phi^K}{d\lambda}, \quad \quad \quad \quad (5)
\]
and the third order derivative with
\[
\frac{d^3 \phi^I}{d\lambda^3} = \left( \Gamma^I_{LM} \Gamma^M_{JK} - \Gamma^I_{JMKL} \right) \frac{d\phi^L}{d\lambda} \frac{d\phi^J}{d\lambda} \frac{d\phi^K}{d\lambda}, \quad \quad \quad \quad (6)
\]
and so on. Thus, we can write (4) as
\[
\phi^I(\lambda = \epsilon) = \phi_0^I + Q^I \epsilon - \frac{1}{2} \Gamma^I_{JK} Q^J Q^K \epsilon^2 + \frac{1}{6} \left( \Gamma^I_{LM} \Gamma^M_{JK} - \Gamma^I_{JMKL} \right) Q^J Q^K Q^L \epsilon^3 + \cdots, \quad \quad \quad \quad (7)
\]
which we can continue up to arbitrary non-linear order. In the end, setting \( \epsilon = 1 \), we obtain
\[
\phi^I - \phi_0^I \equiv \delta \phi^I = Q^I - \frac{1}{2} \Gamma^I_{JK} Q^J Q^K + \frac{1}{6} \left( \Gamma^I_{LM} \Gamma^M_{JK} - \Gamma^I_{JMKL} \right) Q^J Q^K Q^L + \cdots. \quad \quad \quad \quad (8)
\]
If we truncate (8) at linear order, we can identify \( \delta \phi^I \) and \( Q^I \). Then, we do not have to pay attention to the difference between them. However, when we consider non-linear perturbations, we have to distinguish them clearly. Only when we write the equations in terms of \( Q^I \), they can be expressed in a covariant manner.

### 3 General matter Lagrangian

Now, let us consider the general effective matter Lagrangian \( P \), which is a function of the field space metric \( G_{IJ} \), kinetic function \( X^{IJ} = -g^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J / 2 \) and \( \phi^I \), i.e.
\[
P = P(G_{IJ}, X^{IJ}, \phi^I). \quad \quad \quad \quad (9)
\]
Note that we do not restrict the kinetic function to a function of $X = G_{IJ} X^{IJ}$, i.e. all the indices contracted with the metric. This is because we may have a term like $G_{IK} G_{JL} X^{IJ} X^{KL}$, as is typical in the multi-field DBI inflation. Also we do not consider higher derivative terms such as $\Box \phi^I$, which usually leads to ghost except for some special combinations such as Galileon [20]. In this section, we treat the spacetime metric $g_{\mu\nu}$ as a given background, but it is not necessarily spatially homogeneous. Inclusion of the metric perturbation will be discussed in the succeeding section.

We consider that the fields $\phi^I$ contained in $P$ are all functions of $\lambda$, a parameter along the geodesic in the field space. Then, $P$ as a whole is a function of $\lambda$ and is a scalar with respect to the field space indices. We expand $P$ in terms of the parameter $\lambda$, and set $\lambda$ to $\epsilon$, to obtain

$$P = P|_{\lambda=0} + D_\lambda P|_{\lambda=0} \epsilon + \frac{1}{2!} D_\lambda^2 P|_{\lambda=0} \epsilon^2 + \frac{1}{3!} D_\lambda^3 P|_{\lambda=0} \epsilon^3 + \cdots ,$$  

(10)

where we have used the fact that an ordinary derivative of a field space scalar is identical to a covariant one. First let us consider the linear variation, $D_\lambda P$. At this stage, we find

$$D_\lambda P = \frac{\partial P}{\partial X^{IJ}} D_\lambda X^{IJ} + \frac{\partial P}{\partial \phi^I} D_\lambda \phi^I ,$$

(11)

without any subtle issue. Here we have used $D_\lambda G_{IJ} = 0$ that follows from the definition of the covariant differentiation, and we have also assumed that a derivative of $P$ with respect to $X^{IJ}$ is automatically symmetrized, i.e.

$$\frac{\partial P}{\partial X^{IJ}} \rightarrow \frac{1}{2} \left( \frac{\partial P}{\partial X^{IJ}} + \frac{\partial P}{\partial X^{JI}} \right) \equiv P_{(IJ)} .$$

(12)

However, from the quadratic variation, we find that our notation becomes a little uncomfortable. Explicitly, we can write

$$D_\lambda^2 P = D_\lambda^2 X^{IJ} P_{(IJ)} + D_\lambda X^{IJ} (D_\lambda P_{(IJ)}) + D_\lambda^2 \phi^I P_{(I)} + D_\lambda \phi^I (D_\lambda P_{(I)}),$$

(13)

where the third term vanishes due to the geodesic equation of $\phi^I$. The difficulty is in the second and the last terms: how to write the covariant derivatives of the derivative of $P$? In fact, we can easily come to know that the differentiation of $P$ with respect to $X^{IJ}$ should be understood as an ordinary one because $X^{IJ}$ is not a coordinate in the field space but a tensor living in the tangent space. On the other hand, the differentiation with respect to $\phi^I$ should be understood as a covariant one because $\phi^I$ is a coordinate of the field space: any differentiation in the field space necessarily incorporates parallel transport.

While the above considerations are legitimate, it is very uncomfortable to have covariant and ordinary differentiations mixed. Moreover, covariant and ordinary differentiations do not commute. We have $P_{(IJ)K}$ coming from the second term of (13) and $P_{K(IJ)}$ coming from the last term, but they are not the same. Explicitly, $P_{(IJ)K} = P_{(KIJ)} - \Gamma^K_{LK} P_{(IL)} - \Gamma^K_{JK} P_{(IL)}$. Therefore if we rewrite one expression with the other, the result contains the Christoffel symbols and is not manifestly covariant.

To avoid this mess, we consider an alternative description. We assume that $P$ depends on $\phi^I$ only through field space tensors such as $f^a_{J_1 \cdots J_n} (\phi^I)$, where the subscript $a$ is introduced to discriminate different kinds of such tensors. One most important example is the potential
\( V(\phi^I) \), which is a field space scalar. Here we are assuming that there is no spacetime derivatives of fields in \( f_a^{J_1 \ldots J_{n_a}}(\phi^I) \). With this, first let us consider a single derivative. From
\[
P = P \left[ G_{IJ}, X^{IJ}, f_a^{J_1 \ldots J_{n_a}}(\phi^I) \right],
\]
a single derivative with respect to \( \lambda \) is easily calculated as
\[
D_\lambda P = D_\lambda G_{IJ} \frac{\partial P}{\partial G_{IJ}} + D_\lambda X^{IJ} \frac{\partial P}{\partial X^{IJ}} + \sum_a D_\lambda f_a^{J_1 \ldots J_{n_a}} \frac{\partial P}{\partial f_a^{J_1 \ldots J_{n_a}}},
\]
where we have defined \( P_{\{J_1 \ldots J_{n_a}\}a} \equiv \partial P / \partial f_a^{J_1 \ldots J_{n_a}} \).

Now the differentiations of \( P \) are all ordinary ones, and those of \( f_a^{J_1 \ldots J_{n_a}} \) are all covariant ones. In this way, we can straightforwardly write up to cubic order expansion of the general matter Lagrangian \( P \) with respect to \( \lambda \) as
\[
P = P|_{\lambda=0} + P_{(IJ)} \delta X^{IJ} + P_a \delta f_a + \frac{1}{2!} P_{(IJ)(KL)} \delta X^{IJ} \delta X^{KL} + P_{\{IJ\}a} \delta X^{IJ} \delta f_a + \frac{1}{2!} P_{ab} \delta f_a \delta f_b \\
+ \frac{1}{3!} P_{(IJ)(KL)(MN)} \delta X^{IJ} \delta X^{KL} \delta X^{MN} + \frac{1}{2!} P_{(IJ)(KL)} \delta X^{IJ} \delta X^{KL} \delta f_a \\
+ \frac{1}{3!} P_{(IJ)ab} \delta X^{IJ} \delta f_a \delta f_b + \frac{1}{3!} P_{abc} \delta f_a \delta f_b \delta f_c + \cdots ,
\]
where we have assumed that the field space tensors \( f_a^{J_1 \ldots J_{n_a}} \) are all scalars for simplicity, and introduced the following notations
\[
P_a \equiv \frac{\partial P}{\partial f_a},
\]
\[
\delta X^{IJ} \equiv \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} D_\lambda^n X^{IJ}|_{\lambda=0},
\]
\[
\delta f_a \equiv \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} D_\lambda^n f_a|_{\lambda=0}.
\]

With the aid of the geodesic equation (1), it is trivial to find that the derivatives of \( f_a \) are given by
\[
D_\lambda f_a|_{\lambda=0} = f_a : I Q^I ,
\]
\[
D_\lambda^2 f_a|_{\lambda=0} = f_a : IJ Q^I Q^J ,
\]
\[
D_\lambda^3 f_a|_{\lambda=0} = f_a : IJK Q^I Q^J Q^K .
\]
Obtaining the derivatives of \( X^{IJ} \) with respect to \( \lambda \) needs some manipulation. First, we should understand that \( \partial_\mu \phi^I \) is a vector living in the tangent space. Hence, the covariant differentiation of \( \partial_\mu \phi^I \) is given by
\[
D_\lambda \partial_\mu \phi^I = \partial_\mu \frac{d\phi^I}{d\lambda} + \Gamma^I_{JK} \partial_\mu \phi^J \frac{d\phi^K}{d\lambda} \equiv D_\mu \frac{d\phi^I}{d\lambda}.
\]
When we recursively act the covariant differentiation \( D_\lambda \), we need the commutator between \( D_\mu \) and \( D_\lambda \). The necessary commutation relation can be derived in the same manner as in the derivation of the geodesic deviation equation, e.g. for an arbitrary vector \( V^I \),

\[
[D_\lambda, D_\mu]V^I = R^I_{\ JKL}V^J \frac{d\phi^K}{d\lambda} \partial_\mu \phi^L .
\]  

(24)

Then, we obtain

\[
D_\lambda X^{IJ}|_{\lambda=0} = -g^{\mu\nu} D_\mu Q^{(I} \partial_\nu \phi_{0}^{J)} ,
\]  

(25)

\[
D_\lambda^2 X^{IJ}|_{\lambda=0} = -g^{\mu\nu} \left[ R^{(I}_{\ KLM} \partial_\mu \phi_{0}^{J)} \partial_\nu \phi_{0}^{M} Q^K Q^L + D_\mu Q^{I} D_\nu Q^{J} \right] ,
\]  

(26)

\[
D_\lambda^3 X^{IJ}|_{\lambda=0} = -g^{\mu\nu} \left[ R^{(I}_{\ KLM;N} \partial_\mu \phi_{0}^{J)} \partial_\nu \phi_{0}^{M} Q^N Q^K Q^L + R^{(I}_{\ KLM} \partial_\mu \phi_{0}^{J)} Q^K Q^L D_\nu Q^{M} 
\right.
\]
\[
+ \left. 3R^{(I}_{\ KLM} D_\nu Q^{J)} \partial_\mu \phi_{0}^{M} Q^K Q^L \right] ,
\]  

(27)

where parentheses over the indices denote symmetrization. Here we have written down explicitly how the inverse metric \( g^{\mu\nu} \) is contained in the expressions for the later convenience when we consider metric perturbations.

4 Gravity

4.1 General arguments

Until now, we have only considered matter Lagrangian and treated the metric as a given background. But to describe real physics we must take into account the dynamics of gravitational degrees of freedom: additional 4 scalar, 4 vector, and 2 tensor degrees of freedom. Here, scalar, vector and tensor are those with respect to the three dimensional isometry. However, not all of them are physical. The fictitious gauge degrees of freedom can be removed by imposing appropriate gauge conditions. Here in this note we choose the flat gauge as we will explain immediately below, neglecting the vector and tensor degrees of freedom. Their contributions, especially those of tensor perturbations, to the higher order correlation functions of the curvature perturbation enter only through loop corrections, which are highly suppressed.

At the beginning, we have \( n + 4 \) scalar variables: \( n \) from \( n \) scalar field components, 4 from the metric. Since there are 1 temporal and 1 spatial gauge transformations in the scalar sector, we can eliminate 2 of them. In the flat gauge, we impose the conditions that the perturbations of three dimensional spatial metric on each time slice vanish. The remaining metric degrees of freedom are perturbations of the lapse function and the shift vector. We denote them by \( \xi^\alpha \) symbolically. Further, by solving 2 constraint equations, we can also remove the remaining two degrees of freedom \( \xi^\alpha \), so that after all \( n \) degrees of freedom are left. Namely, we can write all the metric degrees of freedom solely in terms of the field fluctuations \( \delta \phi^I \).

First let us formally expand the metric fluctuations \( \delta \xi^\alpha \) in \( \epsilon \) as

\[
\xi^\alpha (\lambda = \epsilon) = \xi^\alpha_0 + \xi^\alpha_1 \epsilon + \xi^\alpha_2 \epsilon^2 + \cdots .
\]  

(28)

The constraint equations are simply given by the variation of the action with respect to \( \xi^\alpha \),

\[
\frac{\delta S}{\delta \xi^\alpha} = 0 .
\]  

(29)
When we expand the action with respect to $\xi_\alpha^{(n)}$, the $n$-th order term in $\xi^\alpha$, we find

\begin{equation}
S = S|_{\xi_\alpha^{(n)}=0} + \frac{\delta S}{\delta \xi_\alpha^{(n)}}|_{\xi_\alpha^{(n)}=0} \xi_\alpha^{(n)} + \frac{1}{2} \frac{\delta^2 S}{\delta \xi_\alpha^{(n)} \delta \xi_\beta^{(n)}}|_{\xi_\alpha^{(n)}=0} \xi_\alpha^{(n)} \xi_\beta^{(n)} + \cdots. \tag{30}
\end{equation}

Then, writing (29) as

\begin{equation}
\left. \frac{\delta S}{\delta \xi_\alpha} \right|_{\xi_\mu^{(n)}=0} = - \left. \frac{\delta^2 S}{\delta \xi_\beta \delta \xi_\alpha} \right|_{\xi_\mu^{(n)}=0} \xi_\beta^{(n)} - \frac{1}{2} \left. \frac{\delta^3 S}{\delta \xi_\gamma \delta \xi_\alpha \delta \xi_\beta} \right|_{\xi_\mu^{(n)}=0} \xi_\gamma^{(n)} \xi_\alpha^{(n)} \xi_\beta^{(n)} + \cdots = O(\epsilon^n), \tag{31}
\end{equation}

we find that both the second and the third terms on the right hand side of (30) are $O(\epsilon^{2n})$. Hence, when we want to know the action to, say, the cubic order in $\epsilon$, the second and higher order of $\xi^\alpha$ are not necessary. To obtain the linear order of $\xi^\alpha$, we only need to solve the constraint equations (29) expanded up to linear order in $\epsilon$,

\begin{equation}
\left. \left( \frac{\delta S}{\delta \xi_\alpha} \right) \right|_{(1)} = 0. \tag{32}
\end{equation}

Plugging the solution for $\xi_\alpha^{(1)}$ of the above constraint equations back into the action, we obtain the action written in terms of the field perturbation $Q^I$.

### 4.2 Explicit calculations

Now let us move onto more explicit computations. We consider a general matter Lagrangian which describes multi-field system minimally coupled to Einstein gravity in the Arnowitt-Deser-Misner form [21],

\begin{equation}
S = \int d^4x \sqrt{\gamma} \left\{ \frac{m^2_{\text{Pl}}}{2} R^{(3)} + \frac{1}{N^2} \left( E^i_j E_j^i - E^2 \right) \right\} + P, \tag{33}
\end{equation}

where $R^{(3)}$ is the 3-curvature scalar constructed from the spatial metric $\gamma_{ij}$, and

\begin{equation}
E_{ij} \equiv \frac{1}{2} \left( \dot{\gamma}_{ij} - N_{ij} - N_{ji} \right), \tag{34}
\end{equation}

with a vertical bar denoting a covariant differentiation with respect to $\gamma_{ij}$. The gauge we choose is, as advertised, the so-called flat gauge, in which the spatial metric $\gamma_{ij}$ is unperturbed, i.e.

\begin{equation}
\gamma_{ij} = a^2 \delta_{ij}, \tag{35}
\end{equation}

which completely fixes both spatial slicing and temporal threading beyond linear level [22], as long as one neglects the vector and tensor perturbations. In this gauge, we separate the action into the gravity and matter sectors,

\begin{equation}
S = S^{(G)} + S^{(M)}, \tag{36}
\end{equation}
with
\[
S^{(G)} = \int d^4x \frac{a^3m^2_{Pl}}{2N} (E_i^j E^j_i - E^2)
\]
\[
= \int d^4x \frac{a^3m^2_{Pl}}{N} \left[ -3H^2 + 2HN^i_{,i} + \frac{1}{4} \left( N_{i,j}N^{i,j} + N_{i,j}N^{j,i} - 2N_{i,i}N^{j,j} \right) \right],
\]
(37)
\[
S^{(M)} = \int d^4x a^3NP.
\]
(38)

We choose the background values of the metric variables, which we associate with a subscript \((0)\), as \(N_{(0)} = 1\) and \(N^i_{(0)} = 0\), corresponding to the Friedmann-Lemaître-Robertson-Walker model written using the cosmological time coordinate. As we have explained above, to obtain the cubic order action, we only need to keep the linear order for the metric perturbations \(\xi^\alpha\). In the action (36), therefore we set
\[
N = 1 + N_{(1)} \epsilon, \quad (39)
\]
\[
N^i = N^i_{(1)} \epsilon. \quad (40)
\]

### 4.2.1 Action expansion including metric perturbations

It is straightforward to write down the gravity part of the action. All we need to do is just to plug the expansions (39) and (40) into the gravity action (37). To the cubic order, we have
\[
S^{(G)} = \int d^4x a^3m^2_{Pl} \left[ 1 - N_{(1)} \epsilon + N_{(1)}^2 \epsilon^2 - N_{(1)}^3 \epsilon^3 \right]
\]
\[
\times \left[ -3H^2 + 2HN^i_{(1),i} \epsilon + \frac{1}{4} \left( N^i_{(1),j}N^{i,j}_{(1)} + N^i_{(1),j}N^{j,i}_{(1)} - 2N^i_{(1),i}N^j_{(1),j} \right) \right]^2. \quad (41)
\]

The expansion of the matter Lagrangian is a little more non-trivial. However, by assumption, our matter Lagrangian contains the spacetime metric only through \(X^{IJ}\). Therefore, all we have to do is just to replace the expression for \(\delta X^{IJ}\) to the one that explicitly includes the expansion with respect to metric perturbations. As the inverse metric \(g^{\mu\nu}\) is given by
\[
g^{\mu\nu} \partial_\mu \partial_\nu = -\frac{1}{N^2} (\partial_t - N^j \partial_j)^2 + \gamma^{ij} \partial_i \partial_j, \quad (42)
\]
(25), (26) and (27) are more explicitly written down as
\[
D_\lambda X^{IJ}|_{\lambda=0} = \frac{1}{N^2} \tilde{D}_t Q^I(\tilde{\phi}_0^J), \quad (43)
\]
\[
D^2_\lambda X^{IJ}|_{\lambda=0} = \frac{1}{N^2} \left[ R^{(I}_{KLM} \tilde{\phi}_0^J) \tilde{\phi}_0^M Q^K Q^L + \tilde{D}_t Q^I \tilde{D}_t Q^J \right] - \gamma^{ij} \partial_i \partial_j Q^I Q^J, \quad (44)
\]
\[
D^3_\lambda X^{IJ}|_{\lambda=0} = \frac{1}{N^2} \left[ R^{(I}_{KLM} \tilde{\phi}_0^J) \tilde{\phi}_0^M Q^K Q^L + R^{(I}_{KLM} \tilde{\phi}_0^J) Q^K Q^L \tilde{D}_t Q^M \right.
\]
\[
+ 3R^{(I}_{KLM} \tilde{\phi}_0^J) Q^K Q^L \tilde{D}_t Q^J], \quad (45)
\]
where we have defined
\[
\tilde{D}_t \equiv D_t - N^j \partial_j. \quad (46)
\]
More explicitly expanding the perturbation of $X^{IJ}$ in terms of $\epsilon$, we obtain

$$X^{IJ} = X_0^{IJ} + X_1^{IJ} \epsilon + X_2^{IJ} \epsilon^2 + X_3^{IJ} \epsilon^3 + \cdots,$$  \hspace{1cm} (47)

with

$$X_1^{IJ} = - N_{(1)} \phi_0^{I} \phi_0^{J} + D_t Q^{(I} \phi_0^{J)},$$  \hspace{1cm} (48)

$$X_2^{IJ} = \frac{3}{2} N_{(1)} \phi_0^{I} \phi_0^{J} - 2 N_{(1)} D_t Q^{(I} \phi_0^{J)} - N_{(1)} \phi_0^{I} \phi_0^{J} + \frac{1}{2} \left[ D_t Q^{(I} D_t Q^{J)} - \gamma^{ij} \partial_i Q^{(I} \partial_j Q^{J)} + R^{(I}_{KLM} \phi_0^{J)^*} Q^K Q^L \right],$$  \hspace{1cm} (49)

$$X_3^{IJ} = - 2 N_{(1)} \phi_0^{I} \phi_0^{J} + 3 N_{(1)} D_t Q^{(I} \phi_0^{J)} - N_{(1)} \phi_0^{I} \phi_0^{J} + \frac{1}{2} \left[ 3 R^{(I}_{KLM} D_t Q^{J)} \phi_0^{*} Q^K Q^L + R^{(I}_{KLM} \phi_0^{J)^*} D_t Q^M Q^K Q^L + R^{(I}_{KLM} N \phi_0^{J)^*} Q^M Q^K Q^L \right].$$  \hspace{1cm} (50)

### 4.2.2 Linear order action

First we consider the first order terms, where we can extract the background equations of motion. Collecting the results that we have obtained in the preceding sections, the first order action becomes

$$S_1 = \int d^4x a^3 \left[ \left( 3m_{\text{Pl}}^2 H^2 + P_0 - P_{(IJ)} \phi_0^{I} \phi_0^{J} \right) N_{(1)} + P_{(IJ)} D_t Q^{I} \phi_0^{J} + P_a f_{a;I} Q^I \right],$$  \hspace{1cm} (51)

where by $P_0$ we denote the matter Lagrangian with the background quantities substituted. What we can immediately see is that we can derive two equations of motion by varying with respect to the lapse perturbation $N_{(1)}$ and the field fluctuation $Q^I$. Taking a variation of (51) with respect to $N_{(1)}$, we obtain

$$H^2 = \frac{1}{3m_{\text{Pl}}^2} \left( P_{(IJ)} \phi_0^{I} \phi_0^{J} - P_0 \right),$$  \hspace{1cm} (52)

which is the background Friedmann equation. We can also immediately obtain the equation of the background field $\phi_0^I$ as

$$\frac{1}{a^3} D_t \left( a^3 P_{(IJ)} \phi_0^{J} \right) = P_a f_{a;I},$$  \hspace{1cm} (53)

or more explicitly,

$$\left( P_{(IJ)} + P_{(IK)(JL)} \phi_0^{K} \phi_0^{L} \right) D_t \phi_0^{I} + \left( 3H P_{(IJ)} + P_{(IJK)} a f_{a;K} \phi_0^{K} \right) \phi_0^{J} - P_a f_{a;I} = 0.$$  \hspace{1cm} (54)
4.2.3 Quadratic order action

Again after straightforward manipulations, we find

\[
S_2 = \int d^4 x a^3 \left\{ \frac{1}{2} \left[ P_{(IJ)} \left( R^{I} K_L M \phi^I_0 \phi^M_0 Q^K Q^L + D_t Q^I D_t Q^J - \gamma^{ij} \partial_i Q^I \partial_j Q^J \right) + Pa f_{a;I} Q^I Q^J \right. \\
\left. + P_{(IJ)(KL)} D_t Q^I \phi^I_0 \phi^K_0 Q^L + 2P_{(IJ)} a D_t Q^I \phi^I_0 f_{a;K} Q^K + P_{ab} f_{a;I} f_{b;J} Q^I Q^J \right] \\
+ N_{(1)} \left[-P_{(IJ)} D_t Q^I \phi^I_0 + P_a f_{a;I} Q^I - \left( P_{(IJ)(KL)} D_t Q^K \phi^I_0 + P_{(IJ)} a f_{a;K} Q^K \right) \phi^I_0 \phi^J_0 \right] \\
+ \frac{N_{(1)}^2}{2} \left(-6m_{Pl}^2 H^2 + P_{(IJ)} \phi^K_0 \phi^I_0 + P_{(IJ)(KL)} \phi^K_0 \phi^L_0 \phi^I_0 \phi^0_0 - 2m_{Pl}^2 H N_{(1)} N_{(1),i} \right) \\
- P_{(IJ)} N_{(1)i} \partial_i Q^I \phi^I_0 + \frac{m_{Pl}^2}{4} \left( N_{(1)i} N_{(1)}^{j,i} + N_{(1)i} N_{(1)}^{j,i} - 2N_{(1),i} N_{(1),j} \right) \right\}. \tag{55}
\]

From the second order action (55), we can derive the linear order metric perturbations, \(N_{(1)}\) and \(N_{(1)i}\), which we have not specified yet. Varying the quadratic action (55) with respect to \(N_{(1)i}\) and \(N_{(1)}\), we obtain the constraint equations which is easily solved as

\[
N_{(1)} = \frac{1}{2m_{Pl}^2 H} P_{(IJ)} Q^I \phi^I_0 = N_I Q^I, \tag{56}
\]

\[
-2m_{Pl}^2 H \frac{\Delta}{a^2} \chi = N_{(1)} \left(P_{(IJ)} \phi^K_0 \phi^I_0 - 2P_0 - P_{(IJ)(KL)} \phi^K_0 \phi^I_0 \phi^L_0 \phi^J_0 \right) \\
+ \left(P_{(IJ)} + P_{(IJ)(KL)} \phi^K_0 \phi^L_0 \phi^I_0 \phi^0_0 \right) D_t Q^I \phi^I_0 + \left(-P_a f_{a;I} + P_{(JK)} a f_{a;I} \phi^K_0 \phi^L_0 \phi^I_0 \phi^0_0 \right) Q^I, \tag{57}
\]

where we have set \(N_{(1)i} = \partial_i \chi\), which is allowed when we consider only scalar perturbations.

4.2.4 Cubic order action

Now we turn to the third order action. First, we present the contributions coming from the gravity sector \(S^{(G)}\). We can easily collect the third order terms to obtain

\[
S^{(G)}_3 = \int d^4 x a^3 \left\{ 3m_{Pl}^2 H^2 N_{(1)}^3 + 2m_{Pl}^2 H \frac{\Delta}{a^2} \chi N_{(1)}^2 - \frac{m_{Pl}^2}{2a^4} \left[ \chi ;^i j \chi ;^i j - (\Delta \chi)^2 \right] N_{(1)} \left\} \right. \tag{58}
\]

Next we consider the contributions from matter sector, \(S^{(M)}\). Using the notation introduced in (53), after some arrangement we find

\[
S^{(M)}_3 = \int d^4 x a^3 \left\{ (g_1)_{IJ} Q^I Q^J Q^K + (g_2)_{IJ} D_t Q^I Q^J Q^K + (g_3)_{IJ} D_t Q^I D_t Q^J Q^K \\
+ (g_4)_{IJ} D_t Q^I D_t Q^J D_t Q^K + (g_5)_{IJ} \partial_t Q^I \partial_t Q^J N_{(1)} + (g_6)_{IJ} \partial_t Q^I \partial_t Q^J N_{(1)} \\
+ (g_7)_{IJ} \partial_t Q^I \partial_t Q^J \partial_t Q^K + (g_8)_{IJ} \partial_t Q^I \partial_t Q^J \partial_t Q^K \right\}, \tag{59}
\]

\]
where
\[(g_1)_{IJK} = \frac{1}{6} \left( P_{,LM} R^L_{,IJN,K} \dot{\phi}_0^M \dot{\phi}_0^N + P_a f_{a;IJK} + 3 P_{,LM} a R^L_{,IJN} \dot{\phi}_0^M \dot{\phi}_0^N f_{a;K} \right. \]
\[+ 3 P_{,ab} f_{a;I} f_{b;K} + P_{abc} f_{a;I} f_{b;L} f_{c;K} \right) \]
\[+ \frac{1}{2} N_{K} \left[ - P_{,LM} R^L_{,IJN} \dot{\phi}_0^M \dot{\phi}_0^N + P_a f_{a;IJ} + P_{ab} f_{a;IJ} \dot{\phi}_0^B \dot{\phi}_0^C \right. \]
\[- \phi_0^L \dot{\phi}_0^M \left( P_{,LM}(AB) R^A_{,IJC} \dot{\phi}_0^B \dot{\phi}_0^C + P_{,LM} a f_{a;IJ} + P_{,LM} a f_{a;IJ} \right) \]
\[+ \frac{1}{2} N_{JK} \left( P_{,AB} f_{a;I} \dot{\phi}_0^A \dot{\phi}_0^B + P_{,AB}(CD) a f_{a;I} \dot{\phi}_0^A \dot{\phi}_0^B \dot{\phi}_0^C \dot{\phi}_0^D \right) \]
\[- N_{IJK} \left( \frac{1}{2} P_{,AB} \dot{\phi}_0^A \dot{\phi}_0^B + P_{,AB}(CD) \dot{\phi}_0^A \dot{\phi}_0^B \dot{\phi}_0^C \dot{\phi}_0^D + \frac{1}{6} P_{,AB}(CD)(EF) \dot{\phi}_0^A \dot{\phi}_0^B \dot{\phi}_0^C \dot{\phi}_0^D \right), \]
\[\tag{60} \]

\[(g_2)_{IJK} = \frac{1}{6} \left( P_{,LM} R^L_{,JKI} + 3 P_{,IL} R^L_{,JKM} \right) \dot{\phi}_0^M + \frac{1}{2} P_{,IL} a \dot{\phi}_0^L f_{a;JK} + \frac{1}{2} P_{,IL}(AB) R^A_{,JKM} \dot{\phi}_0^B \dot{\phi}_0^C \dot{\phi}_0^M \]
\[+ \frac{1}{2} P_{,IL} a \dot{\phi}_0^L f_{a;JK} - N_{K} \left( P_{,IL} a \dot{\phi}_0^L f_{a;J} + P_{,IL}(MN) \dot{\phi}_0^L \dot{\phi}_0^M f_{a;J} \right) \]
\[+ N_{JK} \left( P_{,IL} \dot{\phi}_0^L + \frac{5}{2} P_{,IL}(MN) \dot{\phi}_0^L \dot{\phi}_0^M \dot{\phi}_0^N + \frac{1}{2} P_{,IL}(MN)(AB) \dot{\phi}_0^L \dot{\phi}_0^M \dot{\phi}_0^N \dot{\phi}_0^B \right), \]
\[\tag{61} \]

\[(g_3)_{IJK} = - \frac{1}{5} N_{K} \left[ P_{,IJ} + \left( P_{,IJ}(LM) + 3 P_{,ILJM} \right) \dot{\phi}_0^L \dot{\phi}_0^M + P_{,IL}(JM) \dot{\phi}_0^L \dot{\phi}_0^M \dot{\phi}_0^A \dot{\phi}_0^B \right] \]
\[+ \frac{1}{2} \left( P_{,IJa} + P_{,IL}(JM) a \dot{\phi}_0^L \dot{\phi}_0^M \right) f_{a;K}, \]
\[\tag{62} \]

\[(g_a)_{IJK} = \frac{1}{2} P_{,IL}(KL) \dot{\phi}_0^L + \frac{1}{6} P_{,IL}(JM)(KN) \dot{\phi}_0^M \dot{\phi}_0^N, \]
\[\tag{63} \]

\[(g_a)_{IJ} = N_{I} \left( P_{,JK} \dot{\phi}_0^K + P_{,JK}(LM) \dot{\phi}_0^K \dot{\phi}_0^M \right) - P_{,JK} a f_{a;I} \dot{\phi}_0^K, \]
\[\tag{64} \]

\[(g_b)_{IJK} = - P_{,IL} \dot{\phi}_0^L - P_{,IL}(JK) \dot{\phi}_0^L, \]
\[\tag{65} \]

\[(g_c)_{IJK} = \frac{1}{2} N_{I} \left( - P_{,JK} + P_{,JK}(LM) \dot{\phi}_0^L \dot{\phi}_0^M \right) - \frac{1}{2} P_{,JK} a f_{a;I}, \]
\[\tag{66} \]

\[(g_d)_{IJK} = - \frac{1}{2} P_{,IL}(JK) \dot{\phi}_0^L. \]
\[\tag{67} \]

### 4.3 Effects of field space geometry

As we have computed up to the cubic order action in the covariant form, now we can easily appreciate the effects of field space geometry. An elementary consideration comes from the second order action \[55\]. For this purpose, we may restrict ourselves to the simplest case of a canonical two-field model where the matter Lagrangian is given by \( P = G_{IJ} X^{IJ} - V \) with a constant field space curvature \( R^I_{,JKL} = K \left( \delta^I_{,K} G_{IJ} - \delta^I_{,L} G_{JK} \right) \), with \( K \) being a constant called Gaussian curvature: this form of the curvature tensor describes a two-dimensional surface with a constant curvature. Further, we can choose the basis in such a way that one is pointing along and the other is orthogonal to the field trajectory, so that we may interpret the former as the curvature mode \( \sigma \) and the latter the isocurvature mode \( s \) with \( G_{IJ} \) being diagonal, i.e. \( G_{\sigma s} = 0 \).
Then, the curvature term in (55) becomes
\[ S_2 \supset \int d^4x \frac{a^3}{2} R_{IKLJ} \dot{\phi}_0^K \dot{\phi}_0^L Q^I Q^J , \] (68)
with the field space index either \( \sigma \) or \( s \). By the symmetry of the Riemann curvature tensor, the only non-zero component is \( R_{\sigma s \sigma s} \). Further, by definition \( \dot{\phi}_0^s = 0 \), since the isocurvature mode remains always orthogonal to the trajectory. Thus, the only non-zero contribution is
\[ S_2 \supset \int d^4x \frac{a^3}{2} R_{\sigma s \sigma s} \dot{\phi}_0^\sigma \dot{\phi}_0^s Q^s Q^s = -\frac{K}{2} \int d^4x a^3 \dot{\phi}_0^2 (Q^s)^2 . \] (69)

Thus, we can immediately see that only the perturbation in the isocurvature mode is affected by the field space curvature, either enhanced \( (K < 0) \) or suppressed \( (K > 0) \) depending on the signature of the curvature, while that in the curvature mode remains intact. Such a constant, negative curvature can be realized, for example, for the motion of a D-brane in the internal anti de Sitter space.

To generate significantly large contribution of isocurvature perturbation at the epoch when the relevant scales cross the horizon during inflation, the mass squared in this direction should be suppressed compared with \( H^2 \). Otherwise, it decays exponentially. It is, however, hard to imagine that the mass squared is largely negative because the background trajectory will be unstable. The region with negative mass squared cannot extend indefinitely, and it should be surrounded by the regions where the mass squared is positive. To keep the trajectory along the region with the mass squared negative, it is difficult to avoid the tuning problem of the initial conditions for the background trajectory. Therefore, it would be natural to assume that the mass squared is non-negative at the early stage, and then the corresponding isocurvature perturbation is not amplified during its super-horizon evolution. At a later epoch, the mass squared in the initial isocurvature direction can become negative. However, if the mass squared is largely negative, the field rapidly rolls away from the initial isocurvature direction. Thus, the stage in which the mass squared is negative will not last long. Therefore it is difficult to selectively enhance the initial isocurvature perturbation during inflation.

It is, however, not impossible to enhance the isocurvature perturbation by incorporating the field space curvature as follows. The typical size of the mass squared induced by the field space curvature would be
\[ m_{eff}^2 \sim \dot{\phi}^2 R \sim \epsilon \beta H^2 , \] (70)
where \( \epsilon \) is the standard slow-roll parameter and \( \beta \equiv R m_{Pl}^2 \) is the ratio of the field space curvature to its typical value in the context of supergravity. Now, we consider the case of negative curvature. Then, the effective mass squared of the isocurvature perturbation may be negative even if the background trajectory keeps along the valley of the potential with the “bare” mass squared of the isocurvature perturbation positive. In this case, we can make the effective mass squared negative everywhere without fine tuning of the background trajectory. The magnitude of the magnification effect due to this effective negative mass squared is evaluated by the integral
\[ \exp \left( \int \frac{m_{eff}^2}{H^2} dN \right) \sim e^{\int \epsilon \beta dN} . \] (71)
Using the estimate \( \epsilon \sim 1/\Delta N \) valid for the standard slow-roll, where \( \Delta N \) is the e-folding number during inflation, the amplification effect is already marginally significant for \( \beta = 1 \). If
we have a negative $R$ with the magnitude being larger than $m_{\text{Pl}}^{-2}$, the curvature effect can easily give rise to large amplification of the isocurvature perturbation.

Before closing this section, we should also mention the effects by curved background trajectories \[\text{[23]},\] which is another genuine phenomenon in multi-field system. A convenient way of describing perturbation around a curved trajectory is to introduce the decomposition into curvature and isocurvature modes, i.e. to construct a set of bases which is moving with the trajectory, with one of them pointing along and the others being orthogonal to the trajectory. An advantage of using such decomposition is its clear meaning throughout the evolution of perturbation. However, even when all components of the mass matrix are negligible small, in general, we have continuous mixing between curvature and isocurvature modes when the trajectory is curved.

We can consider an alternative to the decomposition of the curvature-isocurvature modes. It comes from the observation that in the presence of the potential the equation of the background trajectory is not the geodesic with respect to the field space metric. This makes it impossible to introduce such a convenient coordinate system that erases the Christoffel symbol along the trajectory, with one basis vector being identical to the direction of the background trajectory. Instead, we can introduce coordinates by parallelly transporting the basis vectors chosen at an arbitrary time as

$$D_t e^I_a = 0,$$

where $a$ represents the new tetrad frame indices. In this case Christoffel symbol does not vanish even on the background trajectory. Nevertheless, it looks more convenient to use such coordinates since the computation becomes more economical and intuitive. The covariant derivatives acting on the perturbation variable $Q^I$ all appear in the form of $D_t Q^I$. Using (72), we find that

$$e^a_I D_t Q^I = \partial_t Q^a,$$

where $Q^a \equiv e^a_I Q^I$. Namely, those derivatives become ordinary partial derivatives. If we use such coordinates, all the information about the linear evolution of perturbation is confined in the effective mass matrix projected onto this tetrad frame. Since the effective mass matrix is not diagonal in general, the calculation is not so straightforward. But still the description in this manner will help our intuitive understanding of the effect of curved trajectories in curved field space. It may deserve further study, which is beyond the scope of the present paper.

## 5 Conclusions

In this note, we have studied a covariant formulation of general multi-field inflation. Starting from the geodesic equation parametrized by $\lambda$ which connects a point on the background trajectory to the corresponding point with field perturbations, we have found the non-linear relation between the real physical field fluctuation $\delta \phi^I$ and the vector $Q^I$ living on the tangent space. Using this relation, we have expanded the general matter Lagrangian $P(G_{IJ}, X^{IJ}, \phi^I)$ in terms of $\lambda$ up to cubic order in $Q^I$. The resulting expression is fully covariant with the Riemann curvature tensor $R_{IJKL}$ describing the geometry of the field space.

Including gravity, we have chosen the flat gauge where metric perturbations are given by the solutions of the constraint equations in terms of $Q^I$. For an explicit calculation up to cubic order, which is necessary to find the leading contribution to the bispectrum of the curvature
perturbation, we need only the linear solutions of the metric perturbation which could be found from the second order action. With these solutions, we have explicitly computed the cubic order action in a fully covariant manner. Although we have presented up to cubic order action, our formulation can be straightforwardly extended to find arbitrary higher order action. We have also discussed briefly the genuine effects in multi-field inflation generated by the isocurvature perturbations.

Acknowledgement

JG thanks Ana Achucarro for important conversations. JG is grateful to the Yukawa Institute for Theoretical Physics at Kyoto University for hospitality during the long-term workshop “Gravity and Cosmology 2010 (GC2010)” (YITP-T-10-01) and the YKIS symposium “Cosmology – The Next Generation –” (YKIS2010), where this work was initiated, and the 20th Workshop on General Relativity and Gravitation in Japan (YITP-W-10-10) where this work was under progress. This work was supported in part by a Korean-CERN fellowship, the Japanese Society for Promotion of Science Grants N. 21244033, the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence”, and the Grant-in-Aid for Scientific Research on Innovative Areas (Ns. 21111006 and 22111507) from the MEXT.

A Comparison with non-covariant expression

Here we show a method to derive our new covariant expression from the previously known non-covariant expression [15], in order to clarify the equivalence between them. One trivial replacement is to change all the partial differentiations with respect to $\phi^I$ to the corresponding covariant ones. A non-trivial point is that the covariant expression contains the terms depending on the curvature of the field space, $R^A_{BCD}$.

To obtain the terms with curvature, we focus on the fact that $R^I_{ABJ}Q^A Q^B$ contains a term with second derivative of the metric contracted with $Q^A$ in the form $G_{KJ,AB} Q^A Q^B$,

$$R^I_{ABJ}Q^A Q^B \supset \frac{1}{2} G_{KJ,AB} Q^A Q^B.$$  \hspace{1cm} (A.1)

Such second derivatives of the field space metric arise in the non-covariant expression from the second or higher derivatives of $P$. In the covariant formulation, differentiations acting on the field space metric vanish by definition, while they do not in the non-covariant notation. Here, we recall that we are assuming that $P$ is a function of $X^I_J \equiv X^{IK} G_{KJ}$ and scalar functions $f_a(\phi^A)$. Indices among $X^I_J$ should be completely contracted in $P$, i.e. there is no other quantity having the field space indices in $P$. Using this fact, derivatives of $P$ with respect to $G_{KJ}$ can be related to those with respect to $X^{IK}$. Namely, we have

$$P_{,A} \supset \frac{\partial P}{\partial X^I_J} X^{IK} G_{KJ,A} = G^{LJ} P_{(IL)} X^{IK} G_{KJ,A}.$$  \hspace{1cm} (A.2)

Therefore, we find that $P_{,AB}$ contains a part of curvature contribution,

$$P_{,AB} \supset G^{LJ} P_{(IL)} X^{IK} G_{KJ,AB} \approx 2 P_{(L)} X^{LK} R^I_{ABK}.$$  \hspace{1cm} (A.3)
Here $A$ and $B$ indices are understood to be contracted with $Q^A$ and $Q^B$, and “$\approx$” means the equality that is valid focusing only on the term $G_{KJ,AB}Q^A Q^B$, neglecting the other terms in the curvature. In a similar way, we have

$$
P_{ABC} \supset G^{LJ} P_{(LI)} X^{1K} G_{KJ,ABC} + \left[ G^{LJ} P_{(LI),C} X^{1K} G_{KJ,AB} + \left( 2 \text{ permutations among } A, B, C \right) \right]
$$

$$
\approx 2 P_{(LI)} X^{LK} R^I_{ABK;C} + \left[ 2 P_{(LI);C} X^{LK} R^I_{ABK} + \left( 2 \text{ permutations among } A, B, C \right) \right],
$$

(A.4)

and

$$
P_{(IJ),AB} \supset \frac{\partial}{\partial X^{IJ}} \left( G^{NL} P_{(LM)} X^{MK} G_{KN,AB} \right)
$$

$$
= G^{NL} P_{(IJ)(LM)} X^{MK} G_{KN,AB} + \frac{1}{2} \left( G^{KL} P_{(IL)} G_{KJ,AB} + G^{KL} P_{(JI)} G_{KI,AB} \right)
$$

$$
\approx 2 P_{(IJ)(LM)} X^{MK} R^L_{ABK} + P_{(IL)} R^L_{ABJ} + P_{(JL)} R^L_{ABI}.
$$

(A.5)

There is another origin of $G_{KJ,AB}Q^A Q^B$. The field perturbation introduced in non-covariant formulation $\delta \phi^A$ is related to our $Q^A$ by (8),

$$
\delta \phi^A = Q^A - \frac{1}{2} \Gamma^A_{IJ} Q^I Q^J - \frac{1}{6} G^{AI} G_{IJ,KL} Q^J Q^K Q^L + \cdots,
$$

(A.6)

where we have abbreviated several terms at the cubic order, except for the term containing the combination $G_{KJ,AB}Q^A Q^B$. One may think that this cubic order contribution is higher order in action since the perturbed action starts with the second order of perturbation. However, the absence of linear terms in the perturbed action is achieved only after using the background equation of motion. The use of background equation of motion erases linear terms in respective formulations, but the meaning of linear terms varies in different formulations. Therefore, the perturbed actions in different formulation naturally differ by the terms proportional to the background equation of motion. This explains that we have to take into account the linear term in the non-canonical expression to obtain the correct curvature correction. Discriminating the quantities in the non-canonical formulation by associating an underbar, we have

$$
\bar{P}_{(I)} \supset P_{(IJ)} X^{IJ} \supset P_{(IJ)} \dot{\phi}^I \delta \phi^J
$$

$$
\supset - \frac{1}{6} P_{(IJ)} \dot{\phi}^I G^{JK} G_{KB,CD} \dot{Q}^B Q^C Q^D
$$

$$
\approx - \frac{1}{3} P_{(IJ)} \dot{\phi}^I R^I_{CD} \dot{Q}^B Q^C Q^D.
$$

(A.7)

Following the rules mentioned above, all the terms in the perturbed action with the field space curvature can be reproduced correctly.

References

[1] A. H. Guth, Phys. Rev. D 23, 347 (1981); A. D. Linde, Phys. Lett. B 108, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).
See e.g. A. R. Liddle and D. H. Lyth, “Cosmological inflation and large-scale structure,” *Cambridge, UK: Univ. Pr.* (2000) 400 p; V. Mukhanov, “Physical foundations of cosmology,” *Cambridge, UK: Univ. Pr.* (2005) 421 p; S. Weinberg, “Cosmology,” *Oxford, UK: Oxford Univ. Pr.* (2008) 593 p.

E. Komatsu *et al.* [WMAP Collaboration], [arXiv:1001.4538 [astro-ph.CO]].

E. Komatsu and D. N. Spergel, Phys. Rev. D 63, 063002 (2001) [arXiv:astro-ph/0005036].

[Planck Collaboration], [arXiv:astro-ph/0604069].

A. Cooray, Phys. Rev. Lett. 97, 261301 (2006) [arXiv:astro-ph/0610257]; N. Dalal, O. Dore, D. Huterer and A. Shirokov, Phys. Rev. D 77, 123514 (2008) [arXiv:0710.4560 [astro-ph]]; S. Matarrese and L. Verde, Astrophys. J. 677, L77 (2008) [arXiv:0801.4826 [astro-ph]]; D. Jeong and E. Komatsu, Astrophys. J. 703, 1230 (2009) [arXiv:0904.0497 [astro-ph.CO]]; R. Jimenez and L. Verde, Phys. Rev. D 80, 127302 (2009) [arXiv:0909.0403 [astro-ph.CO]].

F. L. Bezrukov and M. Shaposhnikov, Phys. Lett. B 659, 703 (2008) [arXiv:0710.3755 [hep-th]].

C. P. Burgess, H. M. Lee and M. Trott, JHEP 0909, 103 (2009) [arXiv:0902.4465 [hep-ph]]; C. P. Burgess, H. M. Lee and M. Trott, JHEP 1007, 007 (2010) [arXiv:1002.2730 [hep-ph]]; G. F. Giudice and H. M. Lee, Phys. Lett. B 694, 294 (2011) [arXiv:1010.1417 [hep-ph]].

R. N. Lerner and J. McDonald, JCAP 1004, 015 (2010) [arXiv:0912.5463 [hep-ph]]; S. Ferrara, R. Kallosh, A. Linde, A. Marrani and A. Van Proeyen, Phys. Rev. D 83, 025008 (2011) [arXiv:1008.2942 [hep-th]]; F. Bezrukov, A. Magnin, M. Shaposhnikov and S. Sibiryakov, JHEP 1101, 016 (2011) [arXiv:1008.5157 [hep-ph]].

For a review, see e.g. D. H. Lyth and A. Riotto, Phys. Rept. 314, 1 (1999) [arXiv:hep-ph/9807278].

See e.g. K. Y. Choi, J. O. Gong and D. Jeong, JCAP 0902, 032 (2009) [arXiv:0810.2299 [hep-ph]].

For a recent collection of reviews, see e.g. Class. Quant. Grav. 27, “Focus section on non-linear and non-Gaussian cosmological perturbations” (2010); Adv. Astron. 2010, “Testing the Gaussianity and Statistical Isotropy of the Universe” (2010).

D. Seery and J. E. Lidsey, JCAP 0509, 011 (2005) [arXiv:astro-ph/0506056].

D. Langlois, S. Renaux-Petel, D. A. Steer and T. Tanaka, Phys. Rev. Lett. 101, 061301 (2008) [arXiv:0804.3139 [hep-th]].

D. Langlois, S. Renaux-Petel, D. A. Steer and T. Tanaka, Phys. Rev. D 78, 063523 (2008) [arXiv:0806.0336 [hep-th]].

F. Arroja, S. Mizuno and K. Koyama, JCAP 0808, 015 (2008) [arXiv:0806.0619 [astro-ph]].
[17] M. Sasaki and T. Tanaka, Prog. Theor. Phys. 99, 763 (1998) [arXiv:gr-qc/9801017]; S. Groot Nibbelink and B. J. W. van Tent, Class. Quant. Grav. 19, 613 (2002) [arXiv:hep-ph/0107272]; J. O. Gong and E. D. Stewart, Phys. Lett. B 538, 213 (2002) [arXiv:astro-ph/0202098]; D. Langlois and S. Renaux-Petel, JCAP 0804, 017 (2008) [arXiv:0801.1085 [hep-th]].

[18] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. B 458, 209 (1999) [arXiv:hep-th/9904075]; J. Garriga and V. F. Mukhanov, Phys. Lett. B 458, 219 (1999) [arXiv:hep-th/9904176].

[19] E. Silverstein and D. Tong, Phys. Rev. D 70, 103505 (2004) [arXiv:hep-th/0310221]; M. Alishahiha, E. Silverstein and D. Tong, Phys. Rev. D 70, 123505 (2004) [arXiv:hep-th/0404084].

[20] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D 79, 064036 (2009) [arXiv:0811.2197 [hep-th]].

[21] R. L. Arnowitt, S. Deser and C. W. Misner, arXiv:gr-qc/0405109.

[22] H. Noh and J. c. Hwang, Phys. Rev. D 69, 104011 (2004) [arXiv:astro-ph/0305123].

[23] A. Achucarro, J. O. Gong, S. Hardeman, G. A. Palma and S. P. Patil, arXiv:1005.3848 [hep-th]; A. Achucarro, J. O. Gong, S. Hardeman, G. A. Palma and S. P. Patil, JCAP 1101, 030 (2011) [arXiv:1010.3693 [hep-ph]].