Feynman’s proper time approach to QED

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Abstract

The genesis of Feynman’s original approach to QED is reviewed. The main ideas of his original presentation at the Pocono Conference are discussed and compared with the ones involved in his action-at-distance formulation of classical electrodynamics. The role of the de Sitter group in Feynman’s visualization of space-time processes is pointed out.

“I had used it to formulate quantum electrodynamics. I invented it to do that. It was in fact the mathematical formulation that I expressed at the Pocono Conference—that was this crazy language. Dates don’t mean anything. It was published in 1951, but it had all been invented by 1948. I called it the operator calculus.” R.P. Feynman

1 Introduction

We devote this article to review Feynman’s ideas since we think that there is an incomplete knowledge and appreciation of them in quantum field theory (QFT). The reason is that most people have only paid attention to Feynman’s papers of 1949 and learned the derivation of Feynman’s rules for QED from Dyson’s paper of 1948. So some important ideas involved in the papers of 1950 and 1951—in which Feynman exposed his own way—were practically forgotten. Fortunately some recent works of Schweber and Mehra have rescued them. Here we present a brief account of these ideas to prepare the discussion about a hidden de Sitter invariance in QED which can be taken as the starting point to reformulate it.
2 The Wheeler-Feynman theory

The genesis of Feynman’s approach to QED has its origin in his celebrated paper with Wheeler [11]. There they considered an action-at-distance theory of Fokker’s type. Their purpose was to formulate a classical theory free from the well-known divergences due to self-interaction, so they originally assumed that the charges do not act on themselves. The idea of an action-at-distance theory was pursued again by Feynman in his article “A Relativistic Cut-off for Classical Electrodynamics” [12] but this time self-interaction was allowed. He considered an action of the form

\[ S = \sum_n m_n \int ds_n + \frac{1}{2} \sum_{nm} e_n e_m \int f(s_{nm}) dx_n^\mu dx_m^\mu, \]  

where \( s_{nm}^2 = (x_n^\mu - x_m^\mu)(x_n^{\mu\prime} - x_m^{\mu\prime}) \), and \( x_n^\mu, s_n, m_n \), and \( e_n \) are the coordinates, proper time, mass, and charge of the \( n \)-particle respectively. \( f \) is an invariant function of \( s_{nm}^2 \), which behaves like the Dirac delta, \( \delta \), for large distances but has a cut-off for small ones. This was the trick he used in this case to avoid the divergences of the self-interaction. The variation of the action (1) leads to the Lorentz force-law for the particles

\[ m_n \frac{d^2 x_n^\mu}{ds_n^2} = e_n \frac{dx_n^\mu}{ds_n} \sum_m F_{\mu\nu}^m(x_n), \]  

where \( F_{\mu\nu}^m(x) = \partial^\mu A_\nu^m - \partial^\nu A_\mu^m \), is the field caused by the \( m \)-particle corresponding to the potential

\[ A_\nu^m(x_n) = e_m \int f(s_{nm}^2) dx_n^\mu. \]  

As \( \delta(s_{nm}^2) \) is a Green function of d’Alembertian operator with the boundary condition corresponding to the half-advanced plus half-retarded solution, \( A_\nu^m \) satisfies Maxwell equations at long distances of the sources.

Concluding his 1948 article [12] Feynman described another important result. He noticed that the explicit covariant formulation of classical electrodynamics allowed him to introduce the notion of antiparticles. In fact form Eq. (2) it follows that the proper time reversal is equivalent to conjugate the sign of the charge. Following this idea, suggested to him by Wheeler in 1941, Feynman described pair production in external fields.

3 The Pocono Conference

At the Pocono Conference (1948) Feynman gave a dissertation about an “Alternative Formulation of Quantum Electrodynamics.” The notes of Feynman’s talk taken by Wheeler were recently published by Schweber [7, 8]. These important works of Schweber clarify the genesis of Feynman’s ideas, which is not so explicit in the most widely known published material (the two papers of 1949 [2, 3] and the Lectures on QED given by Feynman at Caltech in 1953 [13]), and only appeared in the appendices of his later publications [5, 6].

In his dissertation Feynman introduced the following parametrization of the Dirac equation in an external electromagnetic field

\[ -i \frac{\partial \Psi(x, s)}{\partial s} = \gamma^\mu (i \partial_\mu - eA_\mu) \Psi(x, s), \]  

where \( A_\mu \) is the electromagnetic potential. However it is important to remark that he introduced the fifth parameter in a purely formal way and did not
Hence the Fourier transform of the retarded Green function \( G_\pi(4) \) is a Schroedinger equation in the invariant parameter \( s \) which labels the evolution of states out of the mass-shell. The mass-shell condition is satisfied by stationary states of mass \( m \), \( \Psi(x^\mu, s) = \psi_m(x^\mu)e^{ims} \), where \( \psi_m(x^\mu) \) are solutions of the Dirac equation for a system with mass \( m \), properly attributed any physical meaning to it. He denoted it as \( w \), but we prefer to call it \( s \) because, in the classical limit, it can be identified with the proper time \( \tau \). 

Equation (3) is a Schroedinger equation in the invariant parameter \( s \) which enabled one to derive the corresponding mass-shell Green function \( G_m(x, x') \), i.e. 

\[
\gamma^\mu(\pi_\mu - eA_\mu) - m | \psi_m(x) = 0.
\]

The key idea of Feynman [4] was that by Fourier transforming in \( s \) any solution \( \Psi(x, s) \) of Eq. (3) a solution \( \psi_m(x) \) of Eq. (8) can be obtained, namely 

\[
\psi_m(x) = \int_{-\infty}^{+\infty} \Psi(x, s)e^{-ims}ds.
\]

Hence the Fourier transform of the retarded Green function \( G(x, x', s) \) of Eq. (3) \( (\pi_\mu = i\partial_\mu - eA_\mu) \), 

\[
\left( \gamma^\mu\pi_\mu - i\frac{\partial}{\partial s} \right) G(x, x', s) = \delta(x, x')\delta(s),
\]

enables one to derive the corresponding mass-shell Green function \( G_m(x, x') \), i.e. 

\[
(\gamma^\mu\pi_\mu - m) G_m(x, x') = \delta(x, x').
\]

That is [as \( G(x, x', s) = 0 \) for \( s \leq 0 \)] 

\[
G_m(x, x') = \int_0^{+\infty} G(x, x', s)e^{-ims}ds.
\]

Taking into account that the off-shell retarded Green function is \( G(x, x', s) = -i\Theta(s) \langle x | e^{i\gamma^\mu\pi_\mu s} | x' \rangle \) and using the formal identity \( i/(a + i\epsilon) = \int_0^\infty \exp[i\epsilon(a + i\epsilon)]ds \) for \( a = \gamma^\mu\pi_\mu - m \), one immediately sees that such retarded boundary condition for \( G(x, x', s) \) naturally leads to the Feynman \( i\epsilon \) prescription for avoiding the poles in the on-shell Green function: 

\[
G_m(x, x') = \left< x \left| \frac{1}{\gamma^\mu\pi_\mu - m + i\epsilon} \right| x' \right>. \tag{10}
\]

This formal trick allowed Feynman to discuss external field problems of QED keeping it up at a first-quantized level.

In order to discuss the radiative processes Feynman wrote down a closed expression for the off-shell amplitude of a system of spin-half charges 

\[
\Psi(x_1, \ldots x_n, s_1, \ldots s_n) = \exp \left\{-i \sum_n \int_0^s \gamma^\mu_n(s'_n)\pi^\mu_n(s'_n)ds'_n \right. \\
+ \sum_{nm} e_n e_m \frac{1}{2} \int_0^s \int_0^s \gamma^\mu_n(s'_n)\gamma^\mu_m(s'_m)\delta_+(|x'_n(s'_n) - x'_m(s'_m)|^2)ds'_nds'_m \left| (x'_n, \ldots 0, \ldots 0) \right). \tag{11}
\]

Equation (11), after an integration in the parameters \( s_n \), as in (3), allowed him to describe any process of QED in which only virtual photons are present. From that one he deduced the corresponding diagrammatic expansion of QED.
He explicitly showed how to derive the vertex diagram to order $e^2$ and lowest order in the external field and the self-energy terms to order $e^2$.

Feynman proposed Eq. (11) in 1948 at the Pocono Conference, but he did not have a formal derivation of it, whose justification was given later in his paper of 1951. He was led for the intuition gained in his previous works in which via the path integral method he computed the transition amplitude between processes in which the initial and final states correspond to the radiation oscillators in the vacuum. In such a case he could eliminate the radiation oscillators in the action of scalar electrodynamics and obtain an effective action similar to (11) but with the Green function $\delta_+(s_{nm}^2)$ instead of $f(s_{nm}^2)$. At that time the path integration for spinning particles was unknown. So to deal with the spin-half case Feynman had to invent an alternative formalism, his operator calculus. The idea is that the order in which the operators act is determined by the order of some associated parameters, which in the case of expression (11) are the evolution parameters $s_n$. The exponential in Eq. (11) is the evolution operator for a system of interacting charges. In the first sum we recognize the evolution operator of the external field problem for each charge. The second bilinear expression resembles the one which appears in the action-at-distance theory. When writing Eq. (11) Feynman just associated the proper time velocities $\frac{dx_\mu}{ds}$ of his effective action for the scalar case with the Dirac matrices $\gamma^\mu$.

As was emphasized by Schweber, Feynman was harshly criticized by the audience. Bohr misunderstood his pictorial use of particle trajectories prohibited by the uncertainty principle. Dirac did not receive a satisfactory answer to his question about the unitarity of the transformation (11). This was the reason why he felt somewhat discouraged and decided to publish his result in the way he did in his two celebrated papers of 1949.

## 4 The papers of 1949

In 1949 Feynman published two papers just presenting the rules of QED by means of some plausible intuitive arguments. He avoided the use of path integration or his operator calculus, both techniques weird at that time. It is clear that he decided to conceal his thoughts in order to be better understood. He didn’t reveal them until he wrote his papers of 1950 and 1951.

Some time before the papers of 1949 appeared in the Physical Review, Dyson published a paper about the equivalence of Feynman’s rules with the Tomonaga and Schwinger formulation. Dyson’s paper had a double effect. On the one hand it legitimized the use of Feynman diagrams in the community, which quickly became more and more popular as an easier calculation device. But on the other hand, it made people forget the ideas presented by Feynman at Pocono in 1948, which was the original way in which Feynman had derived the rules of QED. These ideas are very important because they represent an alternative formulation of QED to the one given by Tomonaga, Schwinger, and Dyson in the framework of QFT. Moreover, Dyson’s paper does not imply that both theories are strictly equivalent. It only implies that both theories have in common the same diagrammatic expansion of the $S$ matrix on the mass shell.

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1See the comments in footnote 19 of Feynman’s 1951 paper.

2In the papers of 1949 Feynman avoided the use of path integrals for evaluating the Green functions, and only used the Stueckelberg interpretation for antiparticles on the mass shell to determine their boundary conditions.
5 The de Sitter invariance of QED

The purpose of this section is to show that the Pocono formulation is richer than QFT.

In effect by multiplying on the left by $\gamma^5$ we can rewrite Eq. (4) as a five-dimensional massless Dirac equation ($A = 0, 1, 2, 3, 5$)

$$\Gamma^A (i\partial_A - e A_\mu) \Psi = M \Psi, \quad M = 0,$$

where $A_\mu = (A_\mu, A_5 = 0)$ and $\Gamma^\mu = \gamma^5 \gamma^\mu$ and $\Gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$ satisfy the Dirac algebra

$$\Gamma^A \Gamma^B + \Gamma^B \Gamma^A = 2g^{AB},$$

(13)

corresponding to a five-dimensional manifold of coordinates $x^A = (x^\mu, x^5)$ endowed with a super-Minkowskian metric $g^{AB} = \text{diag}(+, -, -, -, -)$. We have identified $x^5$ with $s$, due to as $M = 0$, the five-line element

$$dS^2 = g^{AB} dx_A dx_B = g^{\mu\nu} dx_\mu dx_\nu - (dx^5)^2$$

(14)

vanishes. In other words, the super-particles (off-shell particles) go at the speed of light ($dS = 0$)

$$\frac{dx^\mu}{ds} \frac{dx_\mu}{dx^5} = 1,$$

(15)

in analogous sense that the standard particles go at the speed of light ($ds = 0$)

$$\frac{dx_\mu}{dt} \frac{dx^\mu}{x_5} = 1,$$

when $m = 0$.

The evolution of any operator $A$ in the Heisenberg picture is given by

$$\frac{dA}{ds} = -i[H, A],$$

(17)

which is the proper time derivative originally proposed by Beck [17] in 1942. In this formalism, the coordinate time $x^0$ has been elevated to the status of an operator canonically conjugated to the energy $p^0$. Their commutation relation and the standard canonical commutation relation for the three-position and momentum can be summarized in the covariant commutation relation

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3Note however that $x^5$ is arbitrary in principle, and the identification of $x^5$ with $s$ is only valid in the equations of motions whose solutions lies on the super-light cone.

4The connection of this formalism with classical theories of spinning particles is discussed in Ref. [18].

5During the last fifty years the Feynman parametrization and the Beck proper time derivative were rediscovered or discussed by many authors (for a list of references see Refs. [18, 19]).
\[ [x^\mu, p^\nu] = -i\eta^{\mu\nu}. \]  

(18)

Using (17) for \( A = x^\mu \) and (18) we obtain the covariant generalization of Breit’s formula

\[ \frac{dx^\mu}{ds} = \gamma^\mu, \]  

(19)

which justifies the association made by Feynman in writing Eq. (11).

It can be proved that at a semiclassical level \[ \text{(20)} \]

It means that, according to the Stueckelberg \[ \text{(15)} \] interpretation, super-particles and super-antiparticles states have positive and negative “norms” respectively. This is the root of the indefinite character of the “inner product.” Frequently, this fact is considered as an anomaly of the theory, but it is a consequence of having a canonical description with particles and antiparticles on the same footing \[ \text{(19)} \].

We have appealed to the Stueckelberg interpretation for antiparticles at the classical level. Let us see that it naturally arises in the formalism according to the more familiar notion based on charge conjugation. The operation that conjugates the charge in Eq. (4) is \[ \text{(21)} \]

where \( c = \gamma^5 K \) is the standard charge conjugation operator. The remarkable point is that this operation coincides with the \( s \)-time reversal operation in the Wigner sense

\[ C = S. \]  

(22)

The identity \[ \text{(22)} \] is the quantum analogue of the Feynman observation quoted in Sec. 2. That is, charge conjugation in the Lorentz force law is equivalent to an inversion of the sign of \( \frac{dx^0}{ds} \).

The comparison between Pocono formula \[ \text{(11)} \] and the classical Fokker’s action \[ \text{(1)} \] clearly shows that the classical limit of the Pocono theory is an action-at-distance formulation of electrodynamics with the Feynman boundary conditions for the on-shell Green function of the d’Alembertian

\[ \partial^\mu \partial_\mu \delta_+ \left[ (x_\mu - x'_\mu)^2 \right] = \delta(x, x'). \]  

(23)

Let us show that such boundary conditions correspond to the retarded ones in a five-dimensional electromagnetic theory

\[ \partial^A \partial_A A^B = j^B, \quad \partial_A A^A = 0. \]  

(24)

The off-shell Green function \( D(x, x', s) \) of the wave operator \( \partial^A \partial_A \),

\[ \left[ \partial^\mu \partial_\mu - \frac{\partial^2}{\partial s^2} \right] D(x, x', s) = \delta(x, x')\delta(s), \]  

(25)

is related to the spin 1/2 Green function by

\[ \text{Note that we can rewrite in action (I-1) } dx^\mu_n = \frac{ds_n}{\delta s_n} \text{ and identify } \frac{ds_n}{\delta s_n} \text{ with } \gamma_\mu \text{ and } m_n \text{ with the mass operator } \gamma^{\mu\nu} \pi_{\nu}. \]

\[ \text{Five-dimensional electromagnetism is also discussed by Saad, Horwitz, and Arshansky} \text{ and Shnerb and Horwitz.} \]
\[ G(x, x', s) = - \left[ \gamma^\mu i \partial_\mu + i \frac{\partial}{\partial s} \right] D(x, x', s). \]  

(26)

Integrating Eq. (26) between \( s = 0 \) and \( s = \infty \) and assuming the retarded boundary conditions \([D(x, x', \infty) = D(x, x', 0) = 0]\) we have

\[ G_{m=0}(x, x') = -\gamma^\mu i \partial_\mu D_{m=0}(x, x'), \]

(27)

where

\[ D_{m=0}(x, x') = \int_0^{+\infty} D(x, x', s) ds. \]  

(28)

As we know from (10) that \( G_{m=0}(x, x') \) satisfies Feynman’s boundary conditions, \( D_{m=0}(x, x') \) also satisfies it. But multiplying (27) by \( \gamma^\mu i \partial_\mu \) and using (8) we see that \( D_{m=0}(x, x') = \delta_+ \left[ (x_\mu - x'_\mu)^2 \right] \).

Notice also that (28) is the analogue of (9). The exponential factor does not appear in this case because the photon mass is zero.

6 Final remarks

Once a line of thought is installed in any field of science it has a very high possibility of success—greater than any other better alternative point of view—due to the great number of thinkers playing with the leading ideas. In the case of QED, most physicists have learnt Feynman’s rules from Dyson works, and probably they did not know very much about Feynman’s original ideas. Of course, in part this because Feynman himself concealed part of his thoughts in his papers of 1949. It is clear that Feynman did not find a framework to justify all his intuitions and finally he plunged into an orthodox point of view (albeit not completely). Remind that when Dyson told him about his paper \[ 9 \], Feynman had no qualms about giving him a free hand to publish his ideas before Feynman had published them himself. Feynman just said: “Well, that’s great! Finally I am respectable.”

In this work we have presented an intent to rescue the original ideas since from Feynman’s view we can go beyond QFT. As we have shown, while QFT rests on Poincaré invariance the Pocono formulation hides the de Sitter invariance of QED.

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\[ ^8 \text{A notable exception is Nambu \[ 23 \], who based on Feynman’s 1949 papers and an earlier work of Fock \[ 24 \] also realized of the importance of proper time in QED.} \]

\[ ^9 \text{See Mehra’s book \[ 1 \], Chap. 13.} \]
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