ON THE STRONG HOMOTOPY LIE-RINEHART ALGEBRA OF A
FOLIATION

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Abstract. It is well known that a foliation $\mathcal{F}$ of a smooth manifold $M$ gives rise to a
rich cohomological theory, its characteristic (i.e., leafwise) cohomology. Characteristic coho-
mologies of $\mathcal{F}$ may be interpreted, to some extent, as functions on the space $P$ of integral
manifolds (of any dimension) of the characteristic distribution $C$ of $\mathcal{F}$. Similarly, characteris-
tic cohomologies with local coefficients in the normal bundle $TM/C$ of $\mathcal{F}$ may be interpreted
as vector fields on $P$. In particular, they possess a (graded) Lie bracket and act on charac-
teristic cohomology $H$. In this paper, I discuss how both the Lie bracket and the action on
$H$ come from a strong homotopy structure at the level of cochains. Finally, I show that such
a strong homotopy structure is canonical up to canonical isomorphisms.

1. Introduction

The space of leaves of a foliation is not necessarily a smooth manifold. However, there exists
a formal, cohomological way of defining a differential calculus on it. Namely, a foliation is a
special instance of a diffiety. A diffiety (or a $D$-scheme, in the algebraic geometry language)
is a geometric object formalizing the concept of partial differential equation (PDE). Basically,
it is a (possibly infinite dimensional) manifold $M$ with an involutive distribution $C$ (in the
case when $M$ is finite dimensional, $(M,C)$ is the same as a foliation of $M$). It emerges in
the geometric theory of PDEs as the infinite prolongation of a given system of differential
equations [2]. Solutions (initial data, etc.) of a system of PDEs with $n$ independent variables,
correspond bijectively to $n$-dimensional ($(n-1)$-dimensional, etc.) integral submanifolds of the
corresponding diffiety. Vinogradov developed a theory, which is known as secondary calculus
[28, 29, 30], formalizing in cohomological terms the idea of a differential calculus on the space
of solutions of a given system of PDEs, or, which is roughly the same, the space of integral
manifolds of a given diffiety $(M,C)$. In other words, secondary calculus provides substitutes
for vector fields, differential forms, differential operators, etc., on a (generically) very singular
space where these objects cannot be defined in the usual (smooth) way.

Namely, let $(\mathfrak{X},\mathfrak{j})$ be the differential graded (DG) commutative algebra of differential forms
on $M$ longitudinal along $C$ (i.e., the exterior algebra of the module of sections of the quotient
bundle $T^*M/C^\perp$, see Section 6 for details) and $\mathfrak{X}$ the module of vector fields transversal to
$C$ (i.e., the module of sections of the quotient bundle $TM/C$, see again Section 6 for details).
The $\mathfrak{X}$-module, $\mathfrak{X} \otimes \mathfrak{X}$ possesses a differential $\mathfrak{j}$ which makes it a DG module over $(\mathfrak{X},\mathfrak{j})$.
Now, secondary functions are just cohomologies of $(\mathfrak{X},\mathfrak{j})$, and secondary vector fields are
cohomologies of $(\mathfrak{X} \otimes \mathfrak{X},\mathfrak{j})$. Similarly, secondary differential forms, etc., are characteristic (i.e.,
longitudinal along $C$) cohomologies of $(M,C)$ (with local coefficients in transversal differential
forms, etc.). All constructions of standard calculus on manifolds (Lie bracket of vector fields,
action of vector fields on functions, exterior differential, insertion of vector fields in differential forms, Lie derivative of differential forms along vector fields, etc.) have a secondary analogue, i.e., a formal, cohomological analogue within secondary calculus. In the trivial case when $\dim C = 0$, secondary calculus reduces to standard calculus on the manifold $M$ (see the first part of [31] for a compact review of secondary Cartan calculus).

Since secondary constructions are algebraic structures in cohomology, it is natural to wonder whether they come from algebraic structures “up to homotopy” at the level of cochains.

The present paper is the first in a series aiming at exploring the following

**Conjecture 1.** All secondary constructions come from suitable homotopy structures at the level of (characteristic) cochains.

A few instances motivating Conjecture 1 are scattered throughout the literature. Namely, Barnich, Fulp, Lada, and Stasheff [1] proved that a (secondary) Poisson bracket on the space of histories of a field theory (which is nothing but the space of solutions of the trivial PDE $0 = 0$, whose underlying diffiety is a “free” one, i.e., an infinite jet space) comes from a (non-canonical) $L_\infty$-structure at the level of horizontal forms. Similarly, Oh and Park [23] showed that the Poisson bracket on characteristic cohomologies of the degeneracy distribution of a presymplectic form comes from an $L_\infty$-structure on longitudinal forms. Finally, C. Rogers [24] showed that $L_\infty$ algebras naturally appear in multisymplectic geometry. More precisely, he proved that Hamiltonian forms in multisymplectic geometry build up an $L_\infty$-algebra (see also [34] for a generalization of the results of Rogers to field theories with non-holonomic constraints). In fact, Rogers’ $L_\infty$-algebra induces the standard Lie algebra of conservation laws in the characteristic cohomology of the covariant phase space of a multisymplectic field theory (see [31]). In its turn, such a Lie algebra can be understood as a secondary analogue of the Lie algebra of first integrals in Hamiltonian mechanics.

In this paper, I show that the Lie-Rinehart algebra of secondary vector fields comes from a strong homotopy (SH) Lie-Rinehart algebra structure on the corresponding cochains, i.e., transversal vector field valued longitudinal forms. To keep things simpler, I assume $M$ to be finite dimensional. In fact, all the proofs are basically algebraic and immediately generalize to the infinite dimensional case.

I have to mention here that three papers already appeared containing results closely related to results in this paper. Firstly, in [11] Huebschmann shows that higher homotopies naturally emerge in the theory of characteristic cohomologies of foliations. Specifically, he proposes a definition of “homotopy Lie-Rinehart algebra”, which he calls quasi-Lie-Rinehart algebra, and proves (among numerous other things) that a (split) Lie subalgebroid in a Lie algebroid gives rise to a quasi-Lie-Rinehart algebra. In fact, the homotopy Lie-Rinehart algebra presented in this paper coincides with Huebschmann’s quasi-Lie-Rinehart algebra in the case of the Lie subalgebroid defined by a foliation. Indeed, a quasi-Lie Rinehart algebra is a special type of SH Lie-Rinehart algebra, but this is not explicitly stated by Huebschmann in his paper. In the subsequent sections, I discuss the precise relation between quasi-Lie-Rinehart algebras and SH Lie-Rinehart algebras, and clarify the novelty of the present paper with respect to [11] (see Remark 13 of Section 3 and last paragraph of Section 6). Secondly, very recently, Chen, Stiénon, and Xu [5] showed that the Lie bracket in the cohomology of a Lie subalgebroid with values in the quotient module comes from a homotopy Leibniz algebra at the level of cochains (see the end of Section 3 for a comparison between their results and results in this paper).
Thirdly, when I was preparing a revised version of my manuscript arXiv:1204.2467v1, there appeared on the arXiv itself the paper [14] by Ji. Ji proves that a (split) Lie subalgebroid in a Lie algebroid gives rise to an $L_\infty$-algebra. In fact, again in the case of the Lie subalgebroid defined by a foliation, Ji’s $L_\infty$-algebra can be obtained by the SH Lie-Rinehart algebra of this paper forgetting about the anchors (see the first appendix for the relation between Ji’s construction and the construction in this paper).

The paper is organized as follows. It is divided in three parts. The first one contains algebraic foundations and it consists of three sections. In Section 2, I recall the definitions of (and fix the conventions about) SH algebras (including their morphisms), SH modules and SH Lie-Rinehart algebras (which, to my knowledge, have been defined for the first time by Kjeseth in [15]). In Section 3, I present in details the DG algebra approach to SH Lie-Rinehart algebras which is dual to the coalgebra approach of Kjeseth [15] (computational details, are postponed to Appendix A). The algebra approach is, in my opinion, more suitable for the aims of this paper. Indeed, the existence of the SH Lie-Rinehart algebra of a foliation is an immediate consequence of the existence of the exterior differential in the algebra of differential forms on the underlying manifold (see Section 8).

In Section 4, I use the DG algebra approach to discuss morphisms of SH Lie-Rinehart algebras, over the same DG algebra. The second part of the paper contains the geometric applications and it consists of five sections. Section 5 reviews fundamentals of the Frölicher-Nijenhuis calculus on form-valued vector fields (more often named vector-valued differential forms [7]). The SH Lie-Rinehart algebra of a foliation has a nice description in terms of Frölicher-Nijenhuis calculus. In Section 6, I briefly review the characteristic cohomology of a smooth foliation, and state the theorem about the occurrence of a SH Lie-Rinehart algebra in the theory of foliations. Section 7 contains more preliminaries on geometric structures over a foliated manifold. In Section 8, I present the SH Lie-Rinehart algebra of a foliation and describe it in terms of Frölicher-Nijenhuis calculus, thus answering a question posed by Huebschmann after a remark by Michor (see Remark 4.16 of [11]).

In Section 9, I remark that the SH Lie-Rinehart algebra of a foliation is independent of the complementary distribution appearing in the definition, up to isomorphisms, and describe a canonical isomorphism between the SH Lie-Rinehart algebras determined by different complementary distributions. In Section 10, as a further example of the emergence of SH structures in secondary calculus, I consider the integral foliation of the degeneracy distribution of a presymplectic form and prove that there exists a canonical morphism from the SH algebra of Oh and Park to the SH Lie-Rinehart algebra of the foliation.

The third part of the paper contains the appendixes. The first appendix contains some computational details omitted in Sections 3 and 4. In the second appendix, I show that the higher brackets in a SH Lie-Rinehart algebra are actually derived brackets, according to the construction of T. Voronov [33]. In the third appendix, I briefly present an alternative derivation of the SH Lie-Rinehart algebra of a foliation which does not apply to the general case of a Lie subalgebroid. Finally, in the last appendix, I present an alternative formulas for the binary operations in the SH Lie-Rinehart algebra of a foliation which could be useful for some purposes.

1.1. Conventions and notations. I will adopt the following notations and conventions throughout the paper. Let $k_1, \ldots, k_\ell$ be positive integers. I denote by $S_{k_1, \ldots, k_\ell}$ the set of
strong homotopies, i.e., permutations $\sigma$ of \{1, \ldots, k_1 + \cdots + k_\ell\} such that
\[\sigma(k_1 + \cdots + k_{i-1} + 1) < \cdots < \sigma(k_1 + \cdots + k_{i-1} + k_i), \quad i = 1, \ldots, \ell.\]

If $S$ is a set, I denote
\[S^{\times k} := S \times \cdots \times S, \quad \text{k times}\]
and the element $(s, \ldots, s) \in S^{\times k}$ of the diagonal will be simply denoted by $s^k$, $s \in S$.

The degree of a homogeneous element $v$ in a graded vector space will be denoted by $\bar{v}$. However, when it appears in the exponent of a sign ($-$), I will always omit the overbar, and write, for instance, $(-)^v$ instead of $(-)^\bar{v}$.

Every vector space will be over a field $K$ of zero characteristic, which will actually be $\mathbb{R}$ in Part 2 (and Appendixes C and D).

If $V = \bigoplus V^i$ is a graded vector space, I denote by $V[1] = \bigoplus V^i[1]$ (resp., $V[-1] = \bigoplus V^i[-1]$) its suspension (resp., de-suspension), i.e., the graded vector space defined by putting $V[1]^i = V^{i+1}$ (resp., $V[-1]^i = V^{i-1}$). Let $V_1, \ldots, V_n$ be graded vector spaces,
\[v = (v_1, \ldots, v_n) \in V_1 \times \cdots \times V_n,\]
and $\sigma$ a permutation of \{1, \ldots, $n$\}. I denote by $\alpha(\sigma, v)$ (resp., $\chi(\sigma, v)$) the sign implicitly defined by
\[v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)} = \alpha(\sigma, v) v_1 \odot \cdots \odot v_n\]
(resp., $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \chi(\sigma, v) v_1 \wedge \cdots \wedge v_n$)
where $\odot$ (resp., $\wedge$) is the graded symmetric (resp., graded skew-symmetric) product in the symmetric (resp., exterior) algebra of $V_1 \oplus \cdots \oplus V_n$.

Let $V,W$ be graded vector spaces, $\Phi : V^{\times k} \longrightarrow W$ a graded, multilinear map, $v = (v_1, \ldots, v_k) \in V^{\times k}$, and $\sigma \in S_k$. I call $\alpha(\sigma, v)\Phi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$ a Koszul signed permutation of $(v_1, \ldots, v_k)$ (in $\Phi(v_1, \ldots, v_k)$).

Now, let $M$ be a smooth manifold. I denote by $C^\infty(M)$ the real algebra of smooth functions on $M$, by $\mathfrak{X}(M)$ the Lie-Rinehart algebra of vector fields on $M$, and by $\Lambda(M)$ the DG algebra of differential forms on $M$. Elements in $\mathfrak{X}(M)$ are always understood as derivations of $C^\infty(M)$. Homogeneous elements in $\Lambda(M)$ are always understood as $C^\infty(M)$-valued, skew-symmetric, multilinear maps on $\mathfrak{X}(M)$. I simply denote by $\omega_1 \omega_2$ (instead of $\omega_1 \wedge \omega_2$) the (wedge) product of differential forms $\omega_1, \omega_2$. I denote by $d : \Lambda(M) \longrightarrow \Lambda(M)$ the exterior differential. Every tensor product will be over $C^\infty(M)$, if not explicitly stated otherwise, and will be simply denoted by $\otimes$. Finally, I adopt the Einstein summation convention.

Part 1. Algebraic Foundations

2. Strong Homotopy Structures

Let $(V, \delta)$ be a complex of vector spaces and $\mathfrak{A}$ be any kind of algebraic structure (associative algebra, Lie algebra, module, etc.). Roughly speaking, a homotopy $\mathfrak{A}$-structure on $(V, \delta)$ is an algebraic structure on $V$ which is of the kind $\mathfrak{A}$ only up to $\delta$-homotopies, and a strong homotopy (SH) $\mathfrak{A}$-structure is a homotopy structure possessing a full system of (coherent) higher homotopies. In this paper, I will basically deal with three kinds of SH structures, namely SH Lie algebras (also named $L_\infty$-algebras), SH Lie modules (also named $L_\infty$-modules), and
SH Lie-Rinehart algebras (that, actually, encompass the latter). For them I provide detailed definitions below.

Let $L$ be a graded vector space, and let $\mathcal{L} = \{[\cdot, \ldots, \cdot], \ k \in \mathbb{N}\}$ be a family of $k$-ary, multilinear, homogeneous of degree $2 - k$ operations

$$[\cdot, \ldots, \cdot]_k : L^{\times k} \to L, \quad k \in \mathbb{N}.$$ 

If the $[\cdot, \ldots, \cdot]_k$’s are graded skew-symmetric, then the $k$-th Jacobiator of $\mathcal{L}$ is, by definition, the multilinear, homogeneous of degree $3 - k$ map

$$J^k : L^{\times k} \to L,$$

defined by

$$J^k(v_1, \ldots, v_k) := \sum_{i+j=k} (-)^{ij} \sum_{\sigma \in S_i,j} \chi(\sigma, v) [[v_{\sigma(1)}, \ldots, v_{\sigma(i)}], v_{\sigma(i+1)}, \ldots, v_{\sigma(i+j)}],$$

$v = (v_1, \ldots, v_k) \in L^{\times k}$. I will often omit the subscript $k$ in $[\cdot, \ldots, \cdot]$ when it is clear from the context, and I will do the same for other $k$-ary operations in the paper without further comments.

**Definition 2.** An $L_\infty$-algebra is a pair $(L, \mathcal{L})$, where $L$ is a graded vector space, and $\mathcal{L} = \{[\cdot, \ldots, \cdot], \ k \in \mathbb{N}\}$ is a family of $k$-ary, multilinear, homogeneous of degree $2 - k$ operations

$$[\cdot, \ldots, \cdot]_k : L^{\times k} \to L, \quad k \in \mathbb{N},$$
such that

1. $[\cdot, \ldots, \cdot]_k$ is graded skew-symmetric, and
2. the $k$-th Jacobiator of $\mathcal{L}$ vanishes identically,

for all $k \in \mathbb{N}$, (in particular, $(L, [\cdot]_1)$ is a complex).

Notice that if $L$ is concentrated in degree 0, then an $L_\infty$-algebra structure on $L$ is simply a Lie algebra structure for degree reasons. Similarly, if $[\cdot, \ldots, \cdot]_k = 0$ for all $k > 2$, then $(L, \mathcal{L})$ is a DG Lie algebra.

Now, let $(L, \mathcal{L}), \mathcal{L} = \{[\cdot, \ldots, \cdot], \ k \in \mathbb{N}\}$, be an $L_\infty$-algebra, $M$ a graded vector space, and let $\mathcal{M} = \{[\cdot, \ldots, \cdot], \ k \in \mathbb{N}\}$ be a family of $k$-ary, multilinear, homogeneous of degree $2 - k$ operations,

$$[\cdot, \ldots, \cdot]_k : L^{\times(k-1)} \times M \to M, \quad k \in \mathbb{N}.$$ 

If the $[\cdot, \ldots, \cdot]_k$’s are graded skew-symmetric in the first $k - 1$ entries, then the $k$-th Jacobiator of $\mathcal{M}$ is, by definition, the multilinear, homogeneous of degree $3 - k$, map

$$J^k : L^{\times(k-1)} \times M \to M,$$

defined by

$$J^k(v_1, \ldots, v_{k-1}|m) := \sum_{i+j=k} (-)^{ij} \sum_{\sigma \in S_i,j} \chi(\sigma, b) [[b_{\sigma(1)}, \ldots, b_{\sigma(i)}], b_{\sigma(i+1)}, \ldots, b_{\sigma(i+j)}],$$

$b = (v_1, \ldots, v_{k-1}, m) \in L^{\times(k-1)} \times M$, where the $[\cdot, \ldots, \cdot]^{\oplus}$’s are new operations

$$[\cdot, \ldots, \cdot]^{\oplus}_k : (L \oplus M)^{\times k} \to L \oplus M, \quad k \in \mathbb{N},$$
defined by extending the $[\cdot, \ldots, \cdot]_k$’s and the $[\cdot, \ldots, \cdot]^{\oplus}$’s by multilinearity, skew-symmetry, and the condition that the result is zero if more than one entry is from $M$. 
Definition 3. An $L_\infty$-module over the $L_\infty$-algebra $(L, \mathcal{L})$, $\mathcal{L} = \{[\cdot, \cdot, \cdot, \cdot]_k, k \in \mathbb{N}\}$, is a pair $(M, \mathcal{M})$, where $M$ is a graded vector space, and $\mathcal{M} = \{[\cdot, \cdot, \cdot, \cdot]_k, k \in \mathbb{N}\}$ is a family of $k$-ary, multilinear, homogeneous of degree $1$ operations,

$$[\cdot, \cdot, \cdot, \cdot]_k : L^{(k-1)} \times M \rightarrow M, \quad k \in \mathbb{N},$$

such that

1. $[\cdot, \cdot, \cdot, \cdot]_k$ is graded skew-symmetric in the first $k-1$ entries, and
2. the $k$-th Jacobiator of $\mathcal{M}$ vanishes identically, for all $k \in \mathbb{N}$ (in particular, $(M, [\cdot, \cdot]_1)$ is a complex).

If both $L$ and $M$ are concentrated in degree 0, then an $L_\infty$-module structure on $M$ over $L$ is simply a Lie module structure over the Lie algebra $L$. Similarly, if $[\cdot, \cdot, \cdot, \cdot]_k = 0$ and $[\cdot, \cdot, \cdot]_k = 0$ for all $k > 2$, then $(M, \mathcal{M})$ is a DG Lie module over the DG Lie algebra $L$.

The sign conventions in Definitions 2 and 3 are the same as in [20, 19]. However, in this paper, I will mainly use a different sign convention [33]. Namely, I will deal with what are often called $L_\infty[1]$-algebras and $L_\infty[1]$-modules, whose definitions I recall now.

Let $L$ be a graded vector space, and let $\mathcal{L} = \{[\cdot, \cdot, \cdot]_k, k \in \mathbb{N}\}$ be a family of $k$-ary, multilinear, homogeneous of degree 1 operations

$$\{\cdot, \cdot, \cdot\}_k : L^k \rightarrow L, \quad k \in \mathbb{N}.$$ 

If the $\{\cdot, \cdot, \cdot\}_k$’s are graded symmetric, then the $k$-th Jacobiator of $\mathcal{L}$ is, by definition, the multilinear, homogeneous of degree 2 map

$$J^k : L^k \rightarrow L,$$

defined by

$$J^k(v_1, \ldots, v_k) := \sum_{i+j=k} \sum_{\sigma \in S_{i,j}} \alpha(\sigma, \nu) \{v_{\sigma(1)}, \ldots, v_{\sigma(i)}, v_{\sigma(i+1)}, \ldots, v_{\sigma(i+j)}\},$$

$\nu = (v_1, \ldots, v_k) \in L^k$.

Definition 4. An $L_\infty[1]$-algebra is a pair $(L, \mathcal{L})$, where $L$ is a graded vector space, and $\mathcal{L} = \{[\cdot, \cdot, \cdot]_k, k \in \mathbb{N}\}$ is a family of $k$-ary, multilinear, homogeneous of degree 1 operations,

$$\{\cdot, \cdot, \cdot\}_k : L^k \rightarrow L, \quad k \in \mathbb{N},$$

such that

1. $\{\cdot, \cdot, \cdot\}_k$ is graded symmetric,
2. the $k$-th Jacobiator of $\mathcal{L}$ vanishes identically, for all $k \in \mathbb{N}$.

There is a one-to-one correspondence between $L_\infty$-algebra structures $\{[\cdot, \cdot, \cdot]_k, k \in \mathbb{N}\}$ in a graded vector space $L$, and $L_\infty[1]$-algebra structures $\{\cdot, \cdot, \cdot\}_k$, $k \in \mathbb{N}$ in $L[1]$, given by

$$\{v_1, \ldots, v_k\} = (-)^{(k-1)}v_1 + (k-2)v_2 + \cdots + v_k - [v_1, \ldots, v_k], \quad v_1, \ldots, v_k \in L, \quad k \in \mathbb{N}.$$ 

$L_\infty[1]$-algebras build up a category whose morphisms are defined as follows.
Let \((L, \{\cdot, \cdot, \cdot \}_{k}, \ k \in \mathbb{N})\) and \((L', \{\cdot, \cdot, \cdot \}'_{k}, \ k \in \mathbb{N})\) be \(L_{\infty}[1]\)-algebras, and let \(f = \{f_{k}, \ k \in \mathbb{N}\}\) be a family of \(k\)-ary, multilinear, homogeneous of degree 0 maps
\[
f_{k} : L^{\times k} \rightarrow L', \quad k \in \mathbb{N}.
\]
If the \(f_{k}\)'s are graded symmetric, define multilinear, homogeneous of degree 1 maps
\[
K^{k}_f : L^{\times k} \rightarrow L',
\]
by putting
\[
K^{k}_f(v_1, \ldots, v_k)
\]
\[
:= \sum_{i+j=k} \sum_{\sigma \in S_{i,j}} \alpha(\sigma, v) f_{i+j+1}(\{v_{\sigma(1)}, \ldots, v_{\sigma(i)}\}, v_{\sigma(i+1)}, \ldots, v_{\sigma(i+j)})
\]
\[
- \sum_{k} \sum_{k_1 + \cdots + k_l = k} \sum_{\sigma \in S_{k_1, \ldots, k_l}} \alpha(\sigma, v) f_{l} (v_{\sigma(1)}, \ldots, v_{\sigma(k_1)}), \ldots, f_{l} (v_{\sigma(k-k_l+1)}, \ldots, v_{\sigma(k)}),'
\]
where \(v = (v_1, \ldots, v_k) \in L^{\times k}\), where \(S_{k_1, \ldots, k_l} = \{i \mid 1 \leq i \leq k\}\) is the set of \((k_1, \ldots, k_l)\)-unshuffles such that
\[
\sigma(k_1 + \cdots + k_{l-1} + 1) < \sigma(k_1 + \cdots + k_{l-1} + k_l + 1) \quad \text{whenever} \quad k_l = k_{l+1}.
\]

**Definition 5.** A morphism \(f : L \rightarrow L'\) of the \(L_{\infty}[1]\)-algebras \((L, \{\cdot, \cdot, \cdot \}_{k}, \ k \in \mathbb{N})\) and \((L', \{\cdot, \cdot, \cdot \}'_{k}, \ k \in \mathbb{N})\) is a family \(f = \{f_{k}, \ k \in \mathbb{N}\}\) of \(k\)-ary, multilinear, homogeneous of degree 0 maps,
\[
f_{k} : L^{\times k} \rightarrow L', \quad k \in \mathbb{N},
\]
such that
\begin{enumerate}
\item \(f_{k}\) is graded symmetric, and
\item \(K^{k}_f\) vanishes identically,
\end{enumerate}
for all \(k \in \mathbb{N}\).

An identity morphism \(\mathbb{I} : L \rightarrow L\) is defined by \(\mathbb{I} := \{\mathbb{I}_k, \ k \in \mathbb{N}\}\), where \(\mathbb{I}_1 : L \rightarrow L\) is the identity map, and \(\mathbb{I}_k : L^{\times k} \rightarrow L\) is the zero map for \(k > 1\).

If \(f : L \rightarrow L'\) and \(g : L' \rightarrow L''\) are morphisms of \(L_{\infty}[1]\)-algebras, the composition \(g \circ f : L \rightarrow L''\) is defined as \(g \circ f := \{g \circ f\}_k, \ k \in \mathbb{N}\), where
\[
(g \circ f)_k(v_1, \ldots, v_k)
\]
\[
:= \sum_{1 \leq k_1 + \cdots + k_l = k} \sum_{\sigma \in S_{k_1, \ldots, k_l}} \alpha(\sigma, v) g_{l} (f_{k_1} (v_{\sigma(1)}), \ldots, v_{\sigma(k_1)}), \ldots, f_{k_l} (v_{\sigma(k-k_l+1)}, \ldots, v_{\sigma(k)}))
\]
for all \(v = (v_1, \ldots, v_k) \in L^{\times k}, \ k \in \mathbb{N}\). \(g \circ f\) is a morphism as well.

Now, let \((L, \mathcal{Z}), \mathcal{Z} = \{\cdot, \cdot, \cdot \}_{k}, \ k \in \mathbb{N}\), be an \(L_{\infty}[1]\)-algebra, \(M\) a graded vector space, and let \(\mathcal{M} = \{\cdot, \cdot, \cdot \}_{k}, \ k \in \mathbb{N}\) be a family of \(k\)-ary, multilinear, homogeneous of degree 1 operations,
\[
\{\cdot, \cdot, \cdot \}_{k} : L^{\times (k-1)} \times M \rightarrow M, \quad k \in \mathbb{N}.
\]
If the \{ \cdot, \ldots, \cdot \}_k \text{'s are graded symmetric in the first } k - 1 \text{ entries, then the } k\text{-th Jacobiator of } \mathcal{M} \text{ is, by definition, the multilinear, homogeneous of degree 2, map }

\[ J^k : L^{(k-1)} \times M \rightarrow M, \]

defined by

\[ J^k(v_1, \ldots, v_{k-1}|m) := \sum_{i+j=k} \sum_{\sigma \in S_{i,j}} \alpha(\sigma, b) \{ b_{\sigma(1)}, \ldots, b_{\sigma(i)} \}^\oplus \{ b_{\sigma(i+1)}, \ldots, b_{\sigma(i+j)} \}^\oplus, \quad (1) \]

\[ b = (v_1, \ldots, v_{k-1}, m) \in L^{(k-1)} \times M, \text{ where the } \{ \cdot, \ldots, \cdot \}^\oplus \text{'s are new operations } \]

\[ \{ \cdot, \ldots, \cdot \}^\oplus : (L \oplus M)^k \rightarrow L \oplus M, \quad k \in \mathbb{N}, \]
defined by extending the \{ \cdot, \ldots, \cdot \}_k \text{'s and the } \{ \cdot, \ldots, \cdot \} \text{'s by multilinearity, symmetry, and the condition that the result is zero if more than one entry is from } M. \]

**Definition 6.** An \( L_\infty[1] \)-module over the \( L_\infty[1] \)-algebra \( (L, \mathcal{L}) \), \( \mathcal{L} = \{ \cdot, \ldots, \cdot \}_k, \quad k \in \mathbb{N} \}, \) is a pair \( (M, \mathcal{M}) \), where \( M \) is a graded vector space, and \( \mathcal{M} = \{ \cdot, \ldots, \cdot \}^\oplus, \quad k \in \mathbb{N} \) is a family of \( k \)-ary, multilinear, homogeneous of degree 1, operations,

\[ \{ \cdot, \ldots, \cdot \}^\oplus : L^{(k-1)} \times M \rightarrow M, \quad k \in \mathbb{N}, \]
such that

1. \{ \cdot, \ldots, \cdot \}^\oplus_k \text{ is graded symmetric in the first } k - 1 \text{ entries, and}
2. the \( k \)-th Jacobiator of \( \mathcal{M} \) vanishes identically, for all \( k \in \mathbb{N} \).

Finally, define SH Lie-Rinehart algebras. I will use the same sign convention as in the definition of \( L_\infty[1] \)-algebras (and \( L_\infty[1] \)-modules). For simplicity, I call the resulting objects \( LR_\infty[1] \)-algebras. To my knowledge, (a version of) this definition has been proposed for the first time by Kjeseth in [15]. Recall that a Lie-Rinehart algebra is a (purely algebraic) generalization of a Lie algebroid.

**Definition 7.** An \( LR_\infty[1] \)-algebra is a pair \( (A, \mathcal{Q}) \), where \( A \) is an associative, graded commutative, unital algebra, and \( (\mathcal{Q}, \mathcal{D}) \) is an \( L_\infty[1] \)-algebra, \( \mathcal{D} = \{ \cdot, \ldots, \cdot \}_k, \quad k \in \mathbb{N} \}. \) Moreover,

- \( \mathcal{Q} \) possesses the structure of an \( A \)-module,
- \( A \) possesses the structure \( \mathcal{M} = \{ \cdot, \ldots, \cdot \}_k, \quad k \in \mathbb{N} \) of an \( L_\infty[1] \)-module over \( (\mathcal{Q}, \mathcal{D}) \),

such that,

- \( \{ \cdot, \ldots, \cdot \}_k : \mathcal{Q}^{(k-1)} \times A \rightarrow A \) is a derivation in the last entry and is \( A \)-multilinear in the first \( k - 1 \) entries;
- Formula

\[ \{ q_1, \ldots, q_{k-1}, a q_k \} = \{ q_1, \ldots, q_{k-1} \} a \cdot q_k + (-)^{a(q_1 + \cdots + q_{k-1} + 1)} a \cdot \{ q_1, \ldots, q_{k-1}, q_k \}, \quad (2) \]

holds for all \( q_1, \ldots, q_k \in \mathcal{Q}, \quad a \in A, \quad k \in \mathbb{N} \) (in particular, \( (\mathcal{Q}, \{ \cdot \}_1) \) is a DG module over \( (A, \{ \cdot \}_1) \)).

The map \( \{ \cdot, \ldots, \cdot \}_k : \mathcal{Q}^{(k-1)} \times A \rightarrow A \) is called the \( k \)-th anchor, \( k \in \mathbb{N} \).
Note that the brackets \{ ·, ·, · \}_k in the above definition are only \(K\)-linear, in general. Formula (2) is a higher generalization of the standard identity fulfilled by the anchor in a Lie-Rinehart algebra.

If \(Q\) is concentrated in degree \(-1\), and \(A\) is concentrated in degree 0, \((A, Q[-1])\) is a Lie-Rinehart algebra.

In the smooth setting, i.e., when \(A\) is the algebra of smooth functions on a smooth manifold \(M\) (in particular \(A\) is concentrated in degree 0), and \(Q[-1]\) is the \(A\)-module of sections of a graded bundle \(E\) over \(M\), then \(E\) is sometimes called an \(L_\infty\)-algebroid [25, 26, 4, 3].

Remark 8. In [11], Huebschmann proposes a definition of a homotopy version of a Lie-Rinehart algebra, called a quasi Lie-Rinehart algebra. Although he mentions the earlier work [15] of Kjeseth, he doesn’t discuss the relation between quasi Lie-Rinehart algebras and Kjeseth’s homotopy Lie-Rinehart pairs. For instance, he doesn’t state explicitly that a quasi Lie-Rinehart algebra is, in particular, an \(L_\infty\)-algebra. Actually, this is an immediate consequence of the description of \(LR_\infty[1]\)-algebras in terms of their Chevalley-Eilenberg algebras (also known as Maurer-Cartan algebras) discussed in the next section.

3. Homotopy Lie-Rinehart Algebras and Multi-Differential Algebras

In this section, I discuss a DG algebraic approach to \(LR_\infty[1]\)-algebras, which is especially suited for the aim of this paper, where the main \(LR_\infty[1]\)-algebra comes from a DG algebra of differential forms. Propositions in this section are known to specialists but, to my knowledge, explicit formulas and detailed proofs are not available. I include some of them here and others in Appendix A.

Notice that the approach in terms of DG algebras (as opposed to the one in terms of coalgebras) has the slight disadvantage of necessitating extra finiteness conditions: a certain module has to possess a nice biduality property.

Let \(A\) be an associative, commutative, unital algebra over a field \(K\) of zero characteristic, and \(Q\) an \(A\)-module. It is well known that a Lie-Rinehart algebra structure on \((A, Q)\) determines a homological derivation \(D\) in the graded algebra \(\text{Alt}_A(Q, A)\) of alternating, \(A\)-valued, \(A\)-multilinear maps on \(Q\). The DG algebra \((\text{Alt}_A(Q, A), D)\) is the Chevalley-Eilenberg algebra of \(Q\). On the other hand, if \(Q\) is projective and finitely generated, then \(\text{Alt}_A(Q, A)\) is isomorphic to \(\Lambda^\bullet A Q^*\), the exterior algebra of the dual module, and a homological derivation in it determines a Lie-Rinehart algebra structure on \((A, Q)\).

Similarly, let \(A\) be a commutative, unital \(K\)-algebra, and \(Q\) a graded \(A\)-module. An \(LR_\infty[1]\)-algebra structure on \((A, Q)\) determines a formal homological derivation \(D\) in the graded algebra \(\text{Sym}_A(Q, A)\) of graded, graded symmetric, \(A\)-valued, \(A\)-multilinear maps on \(Q\) (see below). In [15], Kjeseth describes \(D\) in coalgebraic terms and call \((\text{Sym}_A(Q, A), D)\) the homotopy Rinehart complex of \(Q\). On the other hand, if \(Q\) is projective and finitely generated, then \(\text{Sym}_A(Q, A) \simeq S^\bullet A Q^*\), the graded symmetric algebra of the dual module, and a formal homological derivation in it determines an \(LR_\infty[1]\)-algebra structure on \(Q\). This is shown below. Instead of using the language of formal derivations, I prefer to use the language of multi-differential algebra structures (named multi-algebras in [11]), which makes manifest the role of higher homotopies.

Definition 9. A multi-differential algebra is a pair \((A, \mathcal{D})\), where \(A = \bigoplus_{r,s} A^{r,s}\) is a bigraded algebra, understood as a graded algebra with respect to the total degree \(r + s\) (now on,
named simply the degree), and $D = \{ d_k, k \in \mathbb{N}_0 \}$ is a family of graded derivations
\[ d_k : A \rightarrow A, \]
of bi-degree $(k, -k+1)$ (in particular $d_k$ is homogeneous of degree 1), such that the derivations
\[ E_k := \sum_{i+j=k} [d_i, d_j] : A \rightarrow A, \quad k \in \mathbb{N} \]
vanish for all $k \in \mathbb{N}$ (in particular, $(A, d_0)$ is a DG algebra).

Huebschmann \cite{11} introduces multi-differential algebras, under the name of multi-algebras, but then he concentrates on the case $d_k = 0$ for $k > 2$. Indeed, a multi-differential algebra with $d_k = 0$ for $k > 2$ is naturally associated with a Lie subalgebroid in a Lie algebroid. However, the general case is relevant as well. For instance, multi-differential algebras are at the basis of the BFV-BRST formalism \cite{10, 27} (see also \cite{10}).

**Remark 10.** If for any homogeneous element $\omega \in A$, $d_k \omega = 0$ for $k \gg 1$, one can consider the derivation $D := \sum_k d_k$ (otherwise $D$ is just a formal derivation). Condition $E_k = 0$ for all $k$ is then equivalent to $D^2 = 0$.

Now, let $A$ be an associative, graded commutative, unital algebra, and $Q$ a graded $A$-module. Let $\text{Sym}^r(Q, A)$ be the graded $A$-module of graded, graded symmetric, $A$-multilinear maps with $r$ entries. A homogeneous element $\omega \in \text{Sym}^r(Q, A)$ is a homogeneous, graded symmetric, $K$-multilinear map
\[ \omega : Q^r \rightarrow A, \]
such that
\[ \omega(aq_1, q_2, \ldots, q_r) = (-)^{\omega} \omega(q_1, aq_2, \ldots, q_r), \quad a \in A, \quad q_1, \ldots, q_r \in Q. \]
In particular, $\text{Sym}^1(Q, A) = A$ and $\text{Sym}^1(Q, A) = \text{Hom}_A(Q, A)$ (I will also denote it by $Q^*$). Consider also $\text{Sym}^r(Q, A) = \bigoplus_{s \geq 0} \text{Sym}^s(Q, A)$. It is a bi-graded $A$-module in an obvious way. Moreover, $\text{Sym}^r(Q, A)$ is a graded, associative, graded commutative, unital algebra: for $\omega \in \text{Sym}^r(Q, A)$, $\omega' \in \text{Sym}^s(Q, A)$, $q_1, \ldots, q_{r+s} \in Q$,
\[ (\omega \omega')(q_1, \ldots, q_{r+s}) = \sum_{\sigma \in S_{r+s}} (-)^{\sigma} \omega'(q_{\sigma(1)} + \cdots + q_{\sigma(r)}) \omega(q_{\sigma(1)}, \ldots, q_{\sigma(r)}) \omega'(q_{\sigma(r+1)}, \ldots, q_{\sigma(r+s)}). \]
(3)
Since $\omega \omega' \in \text{Sym}^{r+s}(Q, A)$, the algebra $\text{Sym}^r(Q, A)$ is actually bigraded. Namely, if $\omega \in \text{Sym}^r(Q, A)$ is homogeneous, then its bidegree is defined as $(r, \omega - r)$. I denote by $\text{Sym}^r(Q_A, A)^s$ the subspace of elements in $\text{Sym}^r(Q, A)$ of bidegree $(r, s)$.

**Remark 11.** Suppose $Q = A \otimes A_0 Q[1]$, where $A_0$ is the zeroth homogeneous component of $A$ and $Q$ is a projective and finitely generated $A_0$-module. Then $\text{Sym}^r_Q(A, Q) \cong A \otimes A_0 \Lambda^*_{A_0} Q^*$ as an $A$-module. There is a pre-existing, graded algebra structure on $A \otimes A_0 \Lambda^*_{A_0} Q^*$ given by the exterior product
\[ (a \otimes \omega) \wedge (b \otimes \xi) := (-)^{\omega} ab \otimes \omega \wedge \xi, \quad a, b \in A, \quad \omega, \xi \in Q^*. \]
The isomorphism $\text{Sym}^r(Q, A) \cong A \otimes A_0 \Lambda^*_{A_0} Q^*$ can be chosen so that it identifies the algebra structures in $A \otimes A_0 \Lambda^*_{A_0} Q^*$ and $\text{Sym}^r(Q, A)$. In order to do that, one should identify $a \otimes \omega \in \text{Sym}^r(Q, A)$ with $a \otimes \omega$.
\[ A \otimes_A \Lambda^* \Lambda \circ Q^* \text{ with the unique element } \Omega \text{ in } \operatorname{Sym}^*_A(Q, A) \text{ such that} \]

\[ \Omega(q_1, \ldots, q_r) = (-)^r(r-1)/2 a \omega(q_1, \ldots, q_r) \]

for all \( q_1, \ldots, q_r \in Q \).

**Theorem 12.** Let \( A \) be an associative, graded commutative, unital algebra and \( Q \) a projective and finitely generated \( A \)-module. An \( LR_{\infty}[1] \)-algebra structure on \((A, Q)\) is equivalent to a multi-differential algebra structure \( \{d_k, k \in \mathbb{N}_0\} \) on \( \operatorname{Sym}_A(Q, A) \).

**Remark 13.** Huebschmann \cite{Hue} basically defines a quasi Lie-Rinehart algebra as the datum of an \( A \)-module \( Q \) and a multi-differential algebra structure \( \{d_k, k \in \mathbb{N}_0\} \) on \( \operatorname{Sym}_A(Q, A) \) such that \( d_k = 0 \) for \( k > 2 \). The derivations \( d_0, d_1, d_2 \) induce unary, binary and tertiary brackets in \( Q \), and unary and binary anchors. However, in \cite{Hue} Huebschmann does not spell out explicitly all identities determined, among the former operations, by the identities among the \( d_k \)'s. The proof of Theorem \cite{Hue} fills this out. In particular, it shows explicitly that a quasi Lie-Rinehart algebra is a special case of an \( LR_{\infty}[1] \)-algebra, thus relating definitions by Huebschmann and Kjeseth. Notice also that, very recently, Huebschmann himself \cite{Hue} proposed an alternative proof of Theorem \cite{Hue} using the language of cocommutative coalgebras and twisting cochains.

**Remark 14.** It is well known that the datum of a Lie-Rinehart algebra structure on a module \( Q \) is also equivalent to the datum of a suitable Poisson (resp., Schouten) algebra structure on \( S^*Q \) (resp., \( \Lambda^*Q \)) (see, for instance, \cite{Hue2}, for details). Similarly, the datum of an \( LR_{\infty}[1] \)-algebra structure on \( Q \) (over a commutative DG algebra \( A \)) is also equivalent to the datum of a suitable homotopy, Schouten (resp., Poisson) algebra structure on \( \Lambda^*_A Q[-1] \) (resp., \( S^*_A Q[-1] \)) (see \cite{Hue} for details). The description of \( LR_{\infty}[1] \)-algebra structures in terms of multi-differential algebras, however, looks the most convenient for the purposes of this paper (see Section \ref{section} for details).

**Proof of Theorem \cite{Hue}** Here is a sketch. I postpone the (computational) details to Appendix \ref{appendix}. Let \( (A, Q) \) possess the structure of an \( LR_{\infty}[1] \)-algebra. Denote brackets and anchors as usual. Since \( Q \) is projective and finitely generated, \( \operatorname{Sym}_A(Q, A) \simeq S_A^* Q^* \), which is generated by \( A \) and \( Q^* \). Define \( d_k : \operatorname{Sym}_A(Q, A) \rightarrow \operatorname{Sym}_A(Q, A) \) on generators in the following way. For \( a \in A \), put

\[ (d_k a)(q_1, \ldots, q_k) := (-)^n(q_1 + \cdots + q_n)\{q_1, \ldots, q_k|a\}, \quad q_1, \ldots, q_k \in Q, \quad k \geq 0. \]

Obviously \( d_k a \in \operatorname{Sym}_A^k(Q, A) \). Similarly, for \( \omega \in Q^* \), put

\[ (d_k \omega)(q_1, \ldots, q_{k+1}) := \sum_{i=1}^{k+1} (-)^i \{q_1, \ldots, \hat{q_i}, \ldots, q_{k+1}|\omega(q_i)\} + (-)^n \omega(\{q_1, \ldots, q_{k+1}\}), \]

where \( \hat{\cdot} \) denotes omission, and \( k \geq 0 \). It is easy to show that \( d_k \omega \in \operatorname{Sym}_A^{k+1}(Q, A) \), in particular, it is \( A \)-multilinear (see Lemma \ref{lemma} in Appendix \ref{appendix}).

Now define \( d_k \) by (uniquely) extending to \( \operatorname{Sym}_A(Q, A) \) as a derivation. This is possible, indeed, for \( a, \omega \) as above,

\[ d_k(a \omega) = (d_k a) \omega + (-)^n a(d_k \omega), \]
(see Lemma 37 in Appendix A). Notice that $d_k$ satisfies the following higher Chevalley-Eilenberg formula:

$$(d_k \omega)(q_1, \ldots, q_{r+k}) := \sum_{\sigma \in S_{k,r}} (-)^{\omega(q_\sigma(1)+\cdots+q_\sigma(k))} \alpha(q, q_\sigma(1), \ldots, q_\sigma(k) \mid \omega(q_\sigma(k), \ldots, q_\sigma(k+r)))$$

and

$$- \sum_{\tau \in S_{k+1,r-1}} (-)^{\omega} \alpha(q, q_\tau(1), \ldots, q_\tau(k+1) \mid q_\tau(k+1), \ldots, q_\tau(k+r)), \quad \omega \in \text{Sym}_A^s(Q, A)$$

$q_1, \ldots, q_{r+k} \in Q$ (see Lemma 38 in Appendix A).

It remains to prove that $E_k := \sum_{\ell+m=k}[d_{\ell}, d_m] = 0$. It is a degree 2 derivation of $\text{Sym}_A^s(Q, A)$. To show that it vanishes, it is enough to prove that it vanishes on $A$ and $Q^*$. Now, for $a \in A, \omega \in Q^*$ and $q_1, \ldots, q_k \in Q$

$$(E_k \omega)(q_1, \ldots, q_k) = (-)^{a(q_1+\cdots+q_k)} J^{k+1}(q_1, \ldots, q_k \mid a) = 0,$$

and

$$(E_k \omega)(q_1, \ldots, q_{k+1}) = \sum_{i=1}^{k+1} (-)^{\chi} J^{k+1}(q_1, \ldots, q_i, \ldots, q_{k+1} \mid \omega(q_i)) - \omega(J^{k+1}(q_1, \ldots, q_{k+1})),\quad \chi = \bar{\omega} \sum_{j \neq i} q_j + \bar{q}_i \sum_{j > i} q_j,$$

(see Lemma 39 in Appendix A).

Conversely, let $\{d_k, k \in \mathbb{N}_0\}$ be a family of derivations of $\text{Sym}_A^s(Q, A)$ such that $d_k$ maps $\text{Sym}_A^s(Q, A)^s$ to $\text{Sym}_A^{s-k+1}(Q, A)^{s-k+1}$. For all $a \in A$, and $q_1, \ldots, q_k \in Q$, put

$$\{q_1, \ldots, q_{k-1} \mid a\}_k := (-)^{a(q_1+\cdots+q_{k-1})} (d_{k-1}a)(q_1, \ldots, q_{k-1}) \in A$$

and let $\{q_1, \ldots, q_k\}_k \in Q$ be implicitly defined by

$$\omega(\{q_1, \ldots, q_k\}) := (-)^{\omega} \sum_{i=1}^{k} (-)^{a(q_1+\cdots+q_{i-1})} d_{k-1}(\omega(q_i))(q_1, \ldots, \hat{q}_i, \ldots, q_k)$$

$$- (-)^{\omega}(d_{k-1} \omega)(q_1, \ldots, q_k),$$

where $\omega \in Q^*$. Computations in Appendix A (but the other way round) show that i) $\{q_1, \ldots, q_{k-1} \mid a\}$ is symmetric and $A$-linear in the first entries and a graded derivation in the last one, ii) $\{q_1, \ldots, q_k\}$ is symmetric and satisfies the Lie-Rinehart property \( II \) iii) $J^1(\cdot, \ldots, \cdot) = 0$ and $J^k(\cdot, \ldots, \cdot) = 0$, iff $E_{k-1} := \sum_{\ell+m=k-1}[d_{\ell}, d_m] = 0$, $k \in \mathbb{N}$. \( \square \)

In the smooth setting, the multi-differential algebra $(\text{Sym}_A^s(Q, A), \{d_k, k \in \mathbb{N}_0\})$ determined be an $LR_{\infty}[1]$-algebra $Q$ is sometimes called the Chevalley-Eilenberg algebra of $Q$.

**Corollary 15.** Any degree 1, homological derivation $D$ of $\text{Sym}_A(Q, A)$ determines an $LR_{\infty}[1]$-algebra structure on $(A, Q)$. 
Proof. It is enough to define $d_k$ as the composition

$$\text{Sym}^r_A(Q,A)^s \xrightarrow{D} \text{Sym}_A(Q,A) \xrightarrow{} \text{Sym}^{r+k}_A(Q,A)^{s-k+1}$$

where the second arrow is the projection. Then $(\text{Sym}_A(Q,A),\{d_k, k \in \mathbb{N}_0\})$ is a multi-differential algebra and $(A,Q)$ gets the structure of an $LR_{\infty}[1]$-algebra. □

4. Morphisms of Homotopy Lie-Rinehart Algebras

Let $A$ be an associative, graded commutative, unital algebra. From now on, I will only consider $LR_{\infty}[1]$-algebras $(A,Q)$, with the extra finiteness condition that $Q$ is a projective and finitely generated $A$-module, without further comments. I will also denote by $D_Q = \sum_k d_k$ the formal derivation of $\text{Sym}_A(Q,A)$ encoding brackets and anchors in $(A,Q)$. Finally, I occasionally denote by $p : \text{Sym}_A(Q,A) \rightarrow A$ the projection.

The equivalent description of $LR_{\infty}[1]$-algebras in terms of multi-differential algebras suggests a simple definition of morphisms between $LR_{\infty}[1]$-algebras $(A,P)$ and $(A,Q)$.

**Definition 16.** A morphism $\phi : (A,P) \rightarrow (A,Q)$ of $LR_{\infty}[1]$-algebras is a (degree 0) morphism of graded, unital algebras $\psi : \text{Sym}_A(Q,A) \rightarrow \text{Sym}_A(P,A)$ such that

1. $\psi$ is a morphism of multi-differential algebras, i.e., formally, $\psi \circ D_Q = D_P \circ \psi$ (which is to be understood component-wise);
2. $p \circ \psi = p$.

I now re-express it in terms of brackets and anchors. To my knowledge, Formula (8) is presented here for the first time. As a morphism of DG algebras, $\psi$ is completely determined by its restrictions to $A$ and $Q^*$. Moreover, composing with the projections $\text{Sym}_A(P,A) \rightarrow \text{Sym}_A^k(P,A)$, one get degree 0 maps

$$\psi_k : A \rightarrow \text{Sym}_A^k(P,A), \quad \Psi_k : Q^* \rightarrow \text{Sym}_A^k(P,A), \quad k \geq 0,$$

determining $\psi$ in an obvious way. Notice that, by definition, $\psi_0 = \text{id}_A$ and $\Psi_0 = 0$. The maps $\psi_k$ and $\Psi_k$ are not $A$-linear in general. They determine degree 0 maps

$$\phi_k : P^\times k \times A \rightarrow A, \quad \Phi_k : P^\times k \rightarrow Q, \quad k \geq 1,$$

as follows. Let $p_1, \ldots, p_k \in P$. Put

$$\phi_k(p_1,\ldots,p_k) := (-)^{a(p_1+\cdots+p_k)}\psi_k(a)(p_1,\ldots,p_k).$$

Notice that $\phi_k$ is $A$-linear and graded symmetric in the $p$’s. On the other hand, let $\Phi_k$ be defined (inductively on $k$) by the implicit formula:

$$\omega(\Phi_k(p_1,\ldots,p_k)) = \Psi_k(\omega)(p_1,\ldots,p_k)$$

$$- \sum_{i+j=k} \sum_{\sigma \in S_{i,j}} \alpha(\sigma,p)\psi_{\sigma}(\omega(\Phi_\sigma(p_{\sigma(1)},\ldots,p_{\sigma(i)}))(p_{\sigma(i+1)},\ldots,p_{\sigma(i+j)})), \quad \omega \in Q^*.$$
\[ \omega(\Phi_2(p_1, p_2)) = \Psi_2(\omega)(p_1, p_2) \\
- \psi_1(\Psi_1(\omega)(p_1))(p_2) + \frac{p_1 p_2}{2} , \]

where \( a \leftrightarrow b \) denotes (Koszul signed) transposition of \( a, b \), and

\[ \omega(\Phi_3(p_1, p_2, p_3)) = \Psi_3(\omega)(p_1, p_2, p_3) \\
- \psi_1(\Psi_2(\omega)(p_1, p_2))(p_3) + \frac{p_1 p_2 p_3}{6} , \]

where \( a \circ b \circ c \) denotes (Koszul signed) cyclic permutations of \( a, b, c \). Notice that, before one could even write \( \Phi \), one should prove that the (lower) \( \Phi \)'s are well defined, specifically, that the right hand side \( R_k(\omega) \) of \( \Phi \) is \( A \)-linear in \( \omega \). I do this in Appendix [A](#) (see Lemma 40 therein).

My next aim is to express condition \( \psi \circ D_Q = D_P \circ \psi \) in terms of the \( \phi \)'s and the \( \Phi \)'s. Notice that for any morphism of graded algebras \( \psi : \text{Sym}_A(\mathcal{Q}, A) \rightarrow \text{Sym}_A(\mathcal{P}, A) \), \([\psi, D] := \psi \circ D_Q - D_P \circ \psi \) is a formal derivation along \( \psi \). In particular, \([\psi, D] = 0 \) iff it vanishes on \( A \) and \( \mathcal{Q} \). Let \( \omega \in \text{Sym}_A(\mathcal{P}, A) \). In the following I will denote by \( \omega_k \) its projection onto \( \text{Sym}_A^k(\mathcal{P}, A) \), \( k \geq 0 \).

**Lemma 17.** Let \( \omega \in \text{Sym}_A^k(\mathcal{Q}, A) \). Then

\[
\psi(\omega)_k(p_1, \ldots, p_k) = \sum_{l_0 + \cdots + l_r = k} \sum_{\ell_0 \leq \cdots \leq \ell_r} (-1)^{\chi}(\sigma; \alpha)(p_1, \ldots, p_k) |\omega(\Phi_{1,\sigma}(p_1), \ldots, \Phi_{r,\sigma}(p_r))|,
\]

for all \( p = (p_1, \ldots, p_k) \in \mathcal{P}^k \), where \( \chi := \omega(p_1) + \cdots + p_k \),

\[
\Phi_{i,\sigma}(p) := \Phi_{i,\sigma}(p_{\sigma(0) + \cdots + \ell_i + 1}, \ldots, p_{\sigma(0) + \cdots + \ell_i + 1}), \quad i > 0,
\]

and \( T_{\ell_0, \ell_1, \ldots, \ell_r} \) is the set of permutations of \( \{1, \ldots, k\} \) such that \( i) \sigma(\ell_0 + \ell_1 + \cdots + \ell_i + 1) < \cdots < \sigma(\ell_0 + \ell_1 + \cdots + \ell_i + 1) \), and \( ii) \sigma(\ell_0 + \ell_1 + \cdots + \ell_i + 1) < \sigma(\ell_0 + \ell_1 + \cdots + \ell_i + \ell_{i+1} + 1) \) whenever \( \ell_i = \ell_{i+1}, i > 0 \).

**Proof.** see Appendix [A](#) \( \square \)

Now I want to characterize morphisms of \( LR_\infty[1] \)-algebras. If \( a \in A \), then \( \psi(D_Qa) = D_P \psi(a) \) means that

\[
\psi(D_Qa)_k(p_1, \ldots, p_k) = (D_P \psi(a))_k(p_1, \ldots, p_k), \quad p_1, \ldots, p_k \in \mathcal{P}, \quad (5)
\]

for all \( k \). Since both hand sides of \( (5) \) are graded \( K \)-multilinear, and graded symmetric, they coincide if and only if they coincide on equal, even arguments \( p_1 = \cdots = p_k = p \) (in the following, and especially in Appendix [A](#)) to prove that two multilinear, graded symmetric maps are equal, I will often use the trick of evaluating them on equal, even arguments, without further
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Comments). Now, \( \psi(D_Qa)_k = \sum_m \psi(d_m a)_k \). In its turn, in view of Lemma 17, \( \psi(d_m a)_k \) is given by

\[
\psi(d_m a)_k(p^k) = \sum_{\ell_0 + \cdots + \ell_m = k} C(\ell_0, \ell_1, \ldots, \ell_m) \phi_{\ell_0}(p^{\ell_0}|\Phi_{\ell_1}(p^{\ell_1}), \ldots, \Phi_{\ell_m}(p^{\ell_m})|a),
\]

where \( C(\ell_0, \ell_1, \ldots, \ell_m) \) is the cardinality of \( T_{\ell_0}\ell_1 \cdots \ell_m \). On the other hand, \( (D_P\psi(a))_k = \sum_m (d_m \psi(a))_k \) and a short computation shows that

\[
(d_m \psi(a)_k(p^k) = (k_m(p^m|\phi_{k-m}(p^{k-m}|a)) - (m+1)_m \phi_{k-m}((p^m), p^{k-m-1}|a).
\]

I conclude that

\[
\sum_m \sum_{\ell_0 + \cdots + \ell_m = k, \ell_1 \leq \cdots \leq \ell_m} C(\ell_0, \ell_1, \ldots, \ell_m) \phi_{\ell_0}(p^{\ell_0}|\Phi_{\ell_1}(p^{\ell_1}), \ldots, \Phi_{\ell_m}(p^{\ell_m})|a)
\]

\[
= \sum_m (k_m(p^m|\phi_{k-m}(p^{k-m}|a)) - \sum_m (m+1)_m \phi_{k-m}((p^m), p^{k-m-1}|a),
\]

for all \( k \).

Similarly, let \( \omega \in Q^* \). Then \( \psi(D_Q\omega) = D_P\psi(\omega) \) means that \( \psi(D_Q\omega)_k = (D_P\psi(\omega))_k \) for all \( k \), which can be compactly rewritten in the form

\[
\sum_{m+r=k} (m+r)_m \phi_r(p^r|\omega(K^m_{\Phi}(p^m))) = 0,
\]

where

\[
K^m_{\Phi}(p^n) = \sum_m \sum_{\ell_1 + \cdots + \ell_m = n, \ell_1 \leq \cdots \leq \ell_m} C(\ell_1, \ldots, \ell_m) \Phi_{\ell_1}(p^{\ell_1}), \ldots, \Phi_{\ell_m}(p^{\ell_m}) - \sum_{m+r=n} (m+r)_m \Phi_{r+1}(\{p^m\}, p^r),
\]

and \( C(\ell_1, \ldots, \ell_m) \) is the cardinality of \( S^{<}_{\ell_1 \cdots \ell_m} \) (see Lemma 33 in Appendix A). From Formula (7) it is easy to see, by induction on \( r \), that \( K^m_{\Phi} \) vanishes for all \( r \). I have thus proved the following

**Theorem 18.** A \( (\text{degree } 0) \) morphism of graded, unital algebras \( \psi : \text{Sym}_A(Q,A) \rightarrow \text{Sym}_A(P,A) \) such that \( p \circ \psi = p \) is a morphism of \( LR_\infty[1]-\text{algebras} \phi : (A,P) \rightarrow (A,Q) \) iff

(1) \( \phi = \{\phi_k, \ k \in \mathbb{N}\} \) is a morphism of \( L\infty[1]-\text{algebras} \)

(2) Formula

\[
\sum_m \sum_{\ell_0 + \cdots + \ell_m = k, \sigma \in T_{\ell_0}|\cdots|\ell_m} (-1)^{p_{\sigma(1)} + \cdots + p_{\sigma(\ell)}} \alpha(\sigma,p) \phi_{\ell_0}(p_{\sigma(1)}, \ldots, p_{\sigma(\ell)}|\Phi_{1,\sigma}(p), \ldots, \Phi_{m,\sigma}(p)|a)
\]

\[
= \sum_{\ell+m=k} \sum_{\sigma \in S_{\ell+m}} \alpha(\sigma,p) \phi_{m+1}(\{p_{\sigma(1)}, \ldots, p_{\sigma(\ell)}\}, p_{\sigma(\ell+1)}, \ldots, p_{\sigma(m+1)}|a)
\]

holds for all \( p = (p_1, \ldots, p_k) \in P^{\times k}, a \in A, \text{ and } k \in \mathbb{N}_0. \)
Formula (5) could be hardly guessed without the description of $LR_{\infty}[1]$-algebras in terms of multi-differential algebras.

\section{Part 2. Geometric Applications}

\subsection{5. Form-Valued Vector Fields}

Let $M$ be a smooth manifold. A \textit{form-valued vector field} on $M$ is an element of the (graded) $\Lambda(M)\otimes X(M)$-module obtained from $X(M)$ by extension of scalars, i.e., $\Lambda(M)\otimes X(M)$. Notice that a form-valued vector field $Z$ on $M$ may be understood as a derivation of the algebra $C^\infty(M)$ with values in the $\Lambda(M)$-module $\Lambda(M)$. It may be understood also as a $X(M)$-valued, skew-symmetric, multilinear map on $X(M)$. In the following I will take both points of view. Form-valued vector fields inherit a rich calculus which I call \textit{Fr"ohlicher-Nijenhuis calculus}. In this section, I briefly review it, referring to \cite{7} and \cite{22} for details.

\begin{theorem}[see, for instance, \cite{22}]
Let $Z \in \Lambda(M)\otimes X(M)$. There exist unique graded derivations $i_Z, L_Z : \Lambda(M) \rightarrow \Lambda(M)$ (called the insertion of $Z$, and the Lie derivative along $Z$, respectively) such that
\begin{enumerate}
  \item $i_Z$ is $C^\infty(M)$-linear and $i_Z df = Z(f)$ for all $f \in C^\infty(M)$,
  \item $L_Z$ commutes (in the graded sense) with $d$ and $L_Z f = Z(f)$ for all $f \in C^\infty(M)$.
\end{enumerate}

Conversely, let $\Delta : \Lambda(M) \rightarrow \Lambda(M)$ be a graded derivation. There exist unique form-valued vector fields $Z, Y$ such that
\begin{align*}
\Delta &= i_Z + L_Y.
\end{align*}
\end{theorem}

Notice that, if $Z \in \Lambda^k(M)\otimes X(M)$ then $i_Z$ (resp., $L_Z$) is homogeneous of degree $k-1$ (resp., $k$). Conversely if $\Delta = i_Z + L_Y : \Lambda(M) \rightarrow \Lambda(M)$ is a homogeneous derivation of degree $\ell$, then $Z \in \Lambda^{\ell+1}(M)\otimes X(M)$ and $Y \in \Lambda^{\ell}(M)\otimes X(M)$. For $Z \in \Lambda(M)\otimes X(M)$, abusing the notation, I also denote by $i_Z$ the $C^\infty(M)$-linear map
\begin{align*}
i_Z \otimes \id : \Lambda(M)\otimes X(M) \rightarrow \Lambda(M)\otimes X(M).
\end{align*}

Derivations of $\Lambda(M)$ of the form $i_Z$ (resp., $L_Z$) form a Lie subalgebra. Namely, let $Z_1, Z_2 \in \Lambda(M)\otimes X(M)$. Then there exists a unique $Z \in \Lambda(M)\otimes X(M)$ such that
\begin{align*}
[i_{Z_1}, i_{Z_2}] = i_Z \quad \text{(resp., } [L_{Z_1}, L_{Z_2}] = L_Z).\end{align*}

Z is denoted by $[Z_1, Z_2]_{nr}$ (resp., $[Z_1, Z_2]$) and called the \textit{Nijenhuis-Richardson} (resp., \textit{Fr"ohlicher-Nijenhuis}) \textit{bracket} of $Z_1$ and $Z_2$. The brackets $[\cdot, \cdot]_{nr}$ (resp., $[\cdot, \cdot]$) give $(\Lambda(M)\otimes X(M))[1]$ (resp., $\Lambda(M)\otimes X(M)$) the structure of a graded Lie algebra. In particular, $[\cdot, \cdot]_{nr}$ (resp., $[\cdot, \cdot]$) satisfies a suitable graded Jacobi identity.
Theorem 20. Let $\omega \in \Lambda(M)$, and $X, Y, Z \in \Lambda(M) \otimes \mathfrak{x}(M)$ be homogeneous elements. The following formulas hold
\[
i_{\omega Z} = \omega i_Z, \\
L_{\omega Z} = \omega L_Z + (-)^{\omega + Z}d\omega i_Z, \\
[i_{\omega}, d] = L_Z, \\
\omega X = L_{\omega X} - (-)^{\omega X}i_{[\omega, X]}, \\
[Z, Y]_{\text{nr}} = i_{\omega X}Y - (-)^{\omega Y}(X-1)_{\text{Y}}i_{X}Z, \\
[\omega Z, Y]_{\text{nr}} = \omega[Z, \text{Y}]_{\text{nr}} - (-)^{\omega Z-1}(\omega X-1)_{\text{Y}}(i_{X}\omega)Z, \\
\omega X = \omega[Z, Y] - (-)^{\omega X}Y(L_{\omega X})Z + (-)^{\omega X}d\omega i_{Z}Y, \\
iX[Z, Y] = [i_{X}Z, Y] + (-)^{(X-1)Z}[Z, i_{Y}Z] + (-)^{Z}i_{Y}[X, Z]Y - (-)^{(X-1)}i_{Z}Y[i_{X}, Y]Z, \\
\quad \text{and} \\
[X, [Z, Y]_{\text{nr}}] \\
= [[X, Z], Y]_{\text{nr}} + (-)^{(X-1)}([Z, [X, Y]_{\text{nr}}] - [i_{Z}X, Y]) + (-)^{(X+Z-1)}Y(i_{X}Z, Y). \tag{14}
\]

Below, I will often use formulas in the above theorem, sometimes without any comment.

Remark 21. If $(C^\infty(M), L)$ is the Lie-Rinehart algebra of sections of a Lie algebroid over $M$, then there is an analogue of the Frölicher-Nijenhuis calculus on $\Lambda^*L^* \otimes L$. In particular, elements $Z \in \Lambda^*L^* \otimes L$ determine derivations $i_Z, L_Z$ of $\Lambda^*L^*$, satisfying analogues of formulas (9)-(14) (see, for instance, [19]).

6. Homological Algebra of Foliations

Let $M$ be a smooth manifold and $\mathcal{C}$ an involutive $n$-dimensional distribution on it. Let $\mathcal{F}$ be the integral foliation of $C$. I will denote by $\mathcal{C}X$ the submodule of $\mathfrak{X}(M)$ made of vector fields in $C$. Moreover, following A. Vinogradov (and his school) [28] [29] [30] [2], I will denote by $CA^1 := \mathcal{C}X^* \subset \Lambda^1(M)$ the annihilator of $\mathcal{C}X$. Put
\[
\mathfrak{X} := \mathfrak{X}(M)/\mathcal{C}X, \\
\mathfrak{B} := \Lambda^1(M)/\Lambda^1(M).
\]

Then $CA^1 \simeq \mathfrak{X}$ and $\mathfrak{B}^1 \simeq \mathcal{C}X^*$. In view of the Fröbenius theorem, there always exist coordinates $\ldots, x^1, \ldots, u^\alpha, \ldots, i = 1, \ldots, n, \alpha = 1, \ldots, \dim M - n$, adapted to $C$, i.e., such that $\mathcal{C}X$ is locally spanned by $\ldots, \partial_i := \partial/\partial x^i, \ldots$ and $CA^1$ is locally spanned by $\ldots, du^\alpha, \ldots$. Consider the Chevalley-Eilenberg DG algebra $(\mathfrak{X}, \partial)$ associated to the Lie algebroid $\mathcal{C}X$, i.e., $(\mathfrak{X}, \partial)$ is the exterior algebra of $\mathfrak{X}$ and
\[
\partial(\lambda)(X_1, \ldots, X_{k+1}) \\
= \sum_i (-)^{i+1}X_i(\lambda(\ldots, \hat{X}_i, \ldots)) + \sum_{i<j} (-)^{i+j}\lambda([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots)
\]
where $\lambda \in \mathfrak{X}$ is understood as a $C^\infty(M)$-valued, $k$-multilinear, skew-symmetric map on $\mathcal{C}X$ and $X_1, \ldots, X_{k+1} \in C\mathcal{C}X$. The DG algebra $(\mathfrak{X}, \partial)$ is the quotient of $(\Lambda(M), d)$ over the
differentially closed ideal generated by $C \Lambda^1$. In particular, it is generated by degree 0, and $\overline{d}$-exact degree 1 elements.

In the following, I write $\omega \mapsto \overline{\omega}$ the projection $\Lambda(M) \to \overline{\Lambda}$.

The Lie algebroid $C \mathfrak{X}$ acts on $\overline{\mathfrak{X}}$ via the Bott connection. Namely, write $X \mapsto \overline{X}$ the projection $X(M) \to \overline{\mathfrak{X}}$. Then

$$X \cdot Y := [X, Y] \in \overline{\mathfrak{X}}, \quad X \in C \mathfrak{X}, \quad Y \in \mathfrak{X}(M).$$

Accordingly, there is a differential $(\overline{\Lambda}, \overline{d})$-module $(\overline{\Lambda} \otimes \overline{\mathfrak{X}}, \overline{d})$ whose differential is given by the usual Chevalley-Eilenberg formula:

$$(\overline{d}Z)(X_1, \ldots, X_{k+1}) = \sum_i (-)^{i+1} X_i \cdot Z(\ldots, \hat{X}_i, \ldots) + \sum_{i<j} (-)^{i+j} Z([X_i, X_j], \ldots, \hat{X}_i, \ldots),$$

where $Z \in \overline{\Lambda}^k \otimes \overline{\mathfrak{X}}$ is understood as a $\overline{\mathfrak{X}}$-valued, $k$-multilinear, skew-symmetric map on $C \mathfrak{X}$, and $X_1, \ldots, X_{k+1} \in C \mathfrak{X}$. The tensor product

$$\Lambda(M) \otimes \mathfrak{X}(M) \to \overline{\Lambda} \otimes \overline{\mathfrak{X}}.$$

of projections $\Lambda(M) \to \overline{\Lambda}$ and $\mathfrak{X}(M) \to \overline{\mathfrak{X}}$ will be written $Z \mapsto \overline{Z}$.

**Remark 22.** The $\overline{\mathfrak{d}}$ differentials in $\overline{\mathfrak{X}}$ and $\overline{\mathfrak{X}}$ can be uniquely extended to the whole tensor algebra

$$\bigoplus_{i,j} \overline{\Lambda}^i \otimes \overline{\mathfrak{X}}^j \otimes (C \Lambda^1)^{\otimes j},$$

by requiring Leibniz rules with respect to tensor products and contractions.

The zeroth cohomology $H^0(\overline{\mathfrak{X}}, \overline{\mathfrak{d}})$ is made of functions on $M$ which are constant along the leaves of $C$ and, therefore, elements in it are naturally interpreted as functions on the “space of leaves”. In secondary calculus, one is also concerned with lower dimensional integral submanifolds of $C$. In this respect, it is natural to understand the whole

$$C^\infty\mathfrak{F} := H(\overline{\mathfrak{X}}, \overline{\mathfrak{d}})$$

as algebra of functions over the “space of integral manifolds”. Notice that $C^\infty\mathfrak{F}$ is nothing but the leaf-wise (de Rham) cohomology of $\mathfrak{F}$. The zeroth cohomology $H^0(\overline{\Lambda} \otimes \overline{\mathfrak{X}}, \overline{\mathfrak{d}})$ is made of vector fields on $M$ preserving $C$ (modulo vector fields in $C$) and, therefore, elements in it are naturally interpreted as vector fields on the “space of leaves”. Just as above, it is natural to understand the whole

$$\mathfrak{X}\mathfrak{F} := H(\overline{\Lambda} \otimes \overline{\mathfrak{X}}, \overline{\mathfrak{d}})$$

as Lie algebra of vector fields over the “space of integral manifolds”. The geometric interpretation of cohomologies of $\overline{\mathfrak{d}}$ is very fruitful and far reaching [30, 31]. The following theorem supports this interpretation.

Denote by $[\theta]$ the cohomology class in either $C^\infty\mathfrak{F}$ or $\mathfrak{X}\mathfrak{F}$ of a cocycle $\theta$ in either $\overline{\Lambda}$ or $\overline{\Lambda} \otimes \overline{\mathfrak{X}}$, respectively.

**Theorem 23.** The pair $(C^\infty\mathfrak{F}, \mathfrak{X}\mathfrak{F})$ possesses a canonical structure of graded Lie-Rinehart algebra. Namely,
Theorem 24. The pair what they call a Lie pair up to a sign (due to the chosen sign conventions), and properties 4, 5 imply properties 4, 5 of Lie subalgebroid authors of [5] prove that such a Lie algebra comes from a SH Leibniz algebra structure on L from an L algebra is a genuine L algebra.

Moreover, structures in 1, 2, 3 induce structures in 1, 2, 3 of Theorem 23 in cohomology, In Section 8, I show that the graded Lie-Rinehart algebra of Theorem 23 actually comes from an L/L-connection extending the canonical L/L-connection in L/L, and \([\lambda \cdot Z] := [X, fY] = (X \cdot f) \cdot Y + (-)^X f \cdot [X, Y],\)

holds for all \(X, Y \in \mathfrak{X}_\mathbb{F}, f \in C^\infty_\mathbb{F}.\)

In Section 8 I show that the graded Lie-Rinehart algebra of Theorem 23 actually comes from an \(LR_\infty[1]\)-algebra structure on \((\mathbb{X}, \mathbb{X} \otimes \mathbb{X}[1])\) (see also [11, 14]), according to the following

Theorem 24. The pair \((\mathbb{X}, \mathbb{X} \otimes \mathbb{X}[1])\) possesses a structure of \(LR_\infty[1]\)-algebra. Namely,

1. \(\mathbb{X} \otimes \mathbb{X}[1]\) has an \(L_\infty[1]\)-algebra structure, denoted \(L = \{\cdot, \cdot, \cdot\}, k \in \mathbb{N}\), such that \(\{\cdot\}_1 = \mathbb{X}\).

2. \(\mathbb{X}\) has an \(L_\infty[1]\)-module structure over \(\mathbb{X} \otimes \mathbb{X}[1], L\), denoted \(\{\cdot, \cdot, \cdot\}_k, k \in \mathbb{N}\), such that \(\{\cdot\}_1 = \mathbb{X}\).

3. \(\mathbb{X} \otimes \mathbb{X}[1]\) has a graded \(\mathbb{X}\)-module structure.

4. \(\{\cdot, \cdot, \cdot\}_k : (\mathbb{X} \otimes \mathbb{X}[1])^{k-1} \times \mathbb{X} \rightarrow \mathbb{X}\) is a derivation in the last argument and it is \(\mathbb{X}\)-multilinear in the first \(k\) entries, \(k \in \mathbb{N}\).

5. the formula

\[\{Z_1, \ldots, Z_{k-1}, \lambda Z_k\} = \{Z_1, \ldots, Z_{k-1}\}_k \lambda Z_k + (-)^{\lambda(Z_1+\cdots+Z_{k-1})+1} \lambda Z_1, \ldots, Z_{k-1}, Z_k\]

holds for all \(Z_1, \ldots, Z_{k-1}, Z_k \in \mathbb{X} \otimes \mathbb{X}, \lambda \in \mathbb{X},\) and \(k \in \mathbb{N}\).

Moreover, structures in 1, 2, 3 induce structures in 1, 2, 3 of Theorem 23 in cohomology, up to a sign (due to the chosen sign conventions), and properties 4, 5 imply properties 4, 5 of Theorem 23 respectively.

When I was finalizing the first version of this paper, Chen, Stiénon, and Xu published an e-print [5] who were present similar results in a much wider context. Namely, they consider what they call a Lie pair, i.e., a Lie algebroid \(L\) (generalizing \(TM\) in this paper) with a Lie subalgebroid \(L_1\) (generalizing \(C\) in this paper). There is a differential in \(L^\bullet L_1 \otimes L/L_1\) (generalizing the differential \(\overline{d}\) in \(\mathbb{X} \otimes \mathbb{X}\)) and a Lie algebra structure on cohomology. The authors of [5] prove that such a Lie algebra comes from a SH Leibniz algebra structure on \(L^\bullet L_1 \otimes L/L_1\) defined by means of: 1) a splitting of the inclusion of modules \(L_1 \subset L\) and 2) an \(L\)-connection extending the canonical \(L_1\)-connection in \(L/L_1\). Finally, such an SH Leibniz algebra is a genuine \(L_\infty[1]\)-algebra if the inclusion \(L_1 \subset L\) is split via another Lie subalgebroid. They prove similar results for a general \(L_1\)-module. Obviously, their framework encompasses...
mine. However, their results do not encompass mine for many reasons: i) the SH structure \( \mathcal{D} \) on \( \Lambda \otimes \Pi \) described in this paper is always a true \( L_{\infty}[1] \)-algebra, and not just a SH Leibniz algebra; ii) I define \( \mathcal{D} \) by just a splitting of the inclusion \( C \Lambda \subset \Pi(M) \), and then prove that it is independent of the splitting up to isomorphisms; iii) the SH structure \( \mathcal{D} \) possesses only one higher homotopy (a third level one); iv) when the splitting is made via another involutive distribution, \( \mathcal{D} \) actually becomes a DG Lie algebra (the higher homotopy vanishes) up to a sign (due to the chosen sign conventions); v) I also discuss the question: where does the structure of a Lie-Rinehart algebra on \( (C_{\mathcal{F}}^{\infty}, \mathfrak{x}_F) \) comes from? And not only the questions: where does the Lie algebra (resp., Lie module) structure on \( \mathfrak{x}_F \) (resp., \( C_{\mathcal{F}}^{\infty} \)) come from?

I have to mention also that, while I was preparing a revised version of this paper (already published as e-print arXiv:1204.2467v1), Ji published an e-print \[14\] were he basically presents, among other things, (part of) my results, in the already mentioned wider context of \[3\]. Ji’s main aim is to discuss deformations of Lie pairs (in the sense of \[3\]). Notice that, i) he defines only the \( L_{\infty}[1] \)-algebra structure determined by a Lie pair, and not the \( LR_{\infty}[1] \)-algebra structure; ii) he does this via T. Voronov’s derived bracket formalism \[32\], however (it is not hard to see that) his \( L_{\infty}[1] \)-algebra coincides with the one of Section \[5\] (see the first appendix); iii) he does not discuss the dependence on the choice of the above mentioned splitting. Notice also that the methods of this paper, including the use of the Frölicher-Nijenhuis calculus (see Remark \[21\]), can be straightforwardly generalized to the case of a Lie pair more general than an involutive distribution. However, I prefer to stay on the latter case. Indeed, it is the relevant one for applications in secondary calculus which is the ultimate goal of the paper.

Finally, I stress again that the existence of a quasi Lie-Rinehart algebra associated to a Lie pair had been already proved by Huebschmann in 2005 \[11\]. The quasi Lie-Rinehart algebra of Huebschmann coincides with the \( LR_{\infty}[1] \)-algebra in Section \[8\]. However, I decided to present again its derivation in this paper for various reasons. From an algebraic point of view, for the reasons already discussed in Remarks \[8\] \[13\]. From a geometric point of view, because 1) the presentation in terms of Frölicher-Nijenhuis calculus is somewhat more explicit and easy to work with, 2) I complement it with a proof of canonicity (see Section \[9\]), 3) I relate it to the work of Oh and Park \[23\] (see Section \[10\]).

### 7. Adding a Complementary Distribution to an Involutive Distribution

The exact sequence

\[ 0 \longrightarrow C \mathfrak{x} \longrightarrow \mathfrak{x}(M) \longrightarrow \Pi \longrightarrow 0 \]

splits. The datum of a splitting is equivalent to the datum of a distribution \( V \) complementary to \( C \). From now on, fix such a distribution. I will always identify \( \mathfrak{x} \) (resp., \( \Pi \)) with the corresponding submodule in \( \mathfrak{x}(M) \) (resp., \( \Lambda^1(M) \)) determined by \( V \). The triple \( (C_{\infty}(M), C \mathfrak{x}, \Pi) \) is actually a special instance of Huebschmann’s Lie-Rinehart triple \[11\].

Let \( \mathcal{L}(M) := \bigoplus_k C \Lambda k \subset \Lambda(M) \) be the \( C_{\infty}(M) \)-subalgebra generated by \( C \Lambda^1 \), \( C \Lambda^k := \Lambda^k C \Lambda^1 \). Then

\[ \Lambda(M) \simeq \Lambda^* \Lambda^1(M) \simeq \Lambda^* (\Pi^* \oplus C \Lambda^1) \simeq \Pi \otimes C \Lambda. \quad (15) \]

In the following, I will identify \( \Lambda(M) \) and \( \Pi \otimes C \Lambda \). The transversal distribution \( V \simeq TM/C \) is locally spanned by vector fields \( \ldots, V_\alpha, \ldots \) of the form \( V_\alpha := \partial/\partial u^\alpha + \sum \partial_i u^\alpha, \alpha = 1, \ldots, \dim M - n, \) for some local functions \( \ldots, \, V_i^\alpha, \ldots \), and its annihilator \( V^\perp \simeq C^* \) is locally spanned by differential forms \( \ldots, \, dC x^\alpha := dx^\alpha - V_\alpha^i du^i, \ldots \).
Denote by $P^C, P^V \in \Lambda^1(M) \otimes \mathfrak{X}(M)$ the projectors onto $C, V$, and by $d^C, d^V$ Lie derivatives of differential forms along them, respectively (this is consistent with our previous notations). The form valued vector field $P^C$ belongs to $\overline{\Lambda}^1 \otimes C \mathfrak{X}$, and it is locally given by
$$P^C = d^C x^i \otimes \partial_i.$$ Similarly, $P^V \in C \Lambda^1 \otimes \mathfrak{X}$, and it is locally given by
$$P^V = d\alpha \otimes V_{\alpha}.$$ By definition, $P^C + P^V = 1$, the identity in $\Lambda^1(M) \otimes \mathfrak{X}(M)$, so that $d^C + d^V = d$. The curvature of the splitting $V$ is, by definition,
$$R := \frac{1}{2}[P^C, P^C] \in \Lambda^2(M) \otimes \mathfrak{X}(M).$$ It is easy to see that $R \in C \Lambda^2 \otimes \mathfrak{X}$. Moreover, $P^C, P^V$ and $R$ generate a Lie subalgebra of $(\Lambda(M) \otimes \mathfrak{X}(M), [[\cdot, \cdot]])$ with relations summarized in the following table:

| $[\cdot, \cdot]$ | $P^C$ | $P^V$ | $R$ |
|------------------|-------|-------|-----|
| $P^C$           | $2R$  | $-2R$ | 0   |
| $P^V$           | $-2R$ | $2R$  | 0   |
| $R$             | 0     | 0     | 0   |

(16)
Relations $[P^C, R] = [P^V, R] = 0$ are the Bianchi identities. Table (16) implies that $d^C, d^V, i_R$ and $L_R$ generate a Lie algebra of derivations of $\Lambda(M)$:

| $[\cdot, \cdot]$ | $d^C$ | $d^V$ | $i_R$ | $L_R$ |
|------------------|-------|-------|-------|-------|
| $d^C$           | $2L_R$| $-2L_R$| $L_R$ | 0     |
| $d^V$           | $-2L_R$| $2L_R$ | 0     | 0     |
| $i_R$           | $L_R$ | 0     | 0     | 0     |
| $L_R$           | 0     | 0     | 0     | 0     |

(17)
Lemma 25. Let $\lambda \in \overline{\Lambda}$. Then
$$\overline{d}\lambda = d^C \lambda - i_R \lambda.$$ Proof. Both $\overline{d}$ and $d^C - i_R$ are $\Lambda(M)$-valued derivations of $\overline{\Lambda}$. They coincide provided they coincide on functions and $\overline{d}$-exact elements in $\overline{\Lambda}$. For $f \in C^\infty(M)$,
$$(d^C - i_R)f = d^C f = \overline{d} f.$$ Moreover,
$$(d^C - i_R)\overline{d} f = ((d^C)^2 - i_R d^C)f = L_R f - [i_R, d^C]f = 0 = \overline{d} d f.$$ □

Corollary 26. Let $\lambda \in \overline{\Lambda}$. Then
$$\overline{d}\lambda = \overline{d^C}\lambda.$$ Proof. It immediately follows from the above lemma and the obvious fact that $i_R \lambda = 0$. □

Lemma 27. Let $Z \in \overline{\Lambda} \otimes \overline{\mathfrak{X}}$. Then
$$\overline{d} Z = [P^C, Z] - [R, Z]_{nr}.$$
Proof. Interpret both \(d\) and \(\delta := [P^C, \cdot] - [R, \cdot]_{nr}\) as operators from \(\Lambda \otimes \overline{\Lambda}\) to \(\Lambda \otimes \Lambda\). If \(\Delta = d\) or \(\delta\), then
\[
\Delta(\lambda Z) = (d\lambda)Z + (-)^{\lambda}\Delta(Z).
\]
Therefore, \(d\) and \(\delta\) coincide provided they coincide on zero degree elements. Thus, let \(Y \in \mathfrak{X}(M)\) and \(X \in C\mathfrak{X}\). Then \(\overline{Y} = [Y, P^V]_{nr}\) and
\[
i_X \overline{dY} = (d\overline{Y})(X)
= [X, \overline{Y}] = -[[Y, X], P^V]_{nr}
= -[[\overline{Y}, [X, P^V]_{nr}], + [X, [\overline{Y}, P^V]]_{nr}
= i_X[P^C, \overline{Y}]
= i_X([P^C, \overline{Y}] - [R, \overline{Y}]_{nr})
= i_X\delta \overline{Y},
\]
where I used Formula [14]. It follows from arbitrariness of \(Y\) and \(X\) that \(d = \delta\). \(\Box\)

Corollary 28. Let \(Z \in \overline{\Lambda} \otimes \overline{\Lambda}\). Then
\[
\overline{d}Z = [P^C, Z].
\]
Proof. It immediately follows from the above lemma and the fact that \([R, Z]_{nr} = 0\). \(\Box\)

Notice that, a priori, \(\overline{d}\) is defined on longitudinal differential forms only. However, I need to extend it to the whole \(\Lambda(M)\). This can be done in two ways, both exploiting the transversal distribution \(V\). Namely, consider the derivation \(d^C - i_R : \Lambda(M) \to \Lambda(M)\). In view of Lemma 25 it extends \(\overline{d}\). Alternatively, identify \(\Lambda(M)\) and \(\overline{\Lambda} \otimes CA\) and extend \(\overline{d}\) to it as in Remark 22. It is easy to see that, actually, these two extensions coincide. Indeed, \(CA\) is generated by differential 1-forms of the kind \(P^V f, f \in C^\infty(M)\). Let \(Y \in \overline{\mathfrak{X}}\), then
\[
i_Y \overline{d} P^V f = i_Y P^V f - \overline{d} i_Y P^V f = i_{[P^C, Y]} P^V f - d^C i_Y P^V f = i_Y (d^C - i_R) P^V f.
\]
It follows from the arbitrariness of \(f\) and \(Y\), that \(\overline{d} = d^C - i_R\).

8. The Homotopy Lie-Rinehart Algebra of a Foliation

In the following put \(A := \overline{\mathfrak{X}}\) and \(Q := \overline{\Lambda} \otimes \overline{\Lambda}[1]\). According to Remark 11, one can identify \(\text{Sym}_A(\mathfrak{Q}, A)\) with \(\overline{\Lambda} \otimes CA = \Lambda(M)\) in such a way that the product [3] identifies with the exterior product of differential forms. In particular \(\text{Sym}_A^*(\mathfrak{Q}, A)^*\) identifies with \(\overline{\Lambda}^* \otimes CA^*\). In the following, I will assume this identification. For \(\omega \in \overline{\Lambda} \otimes CA^k = \text{Sym}_A^k(\mathfrak{Q}, A)\), I denote by
\[
\langle \omega | Z_1, \ldots, Z_k \rangle \in A
\]
its action on elements \(Z_1, \ldots, Z_k \in \mathfrak{Q}\), so not to cause confusion with the action of differential forms on vector fields. It is easy to see that
\[
\langle \omega | Z_1, \ldots, Z_k \rangle = (-)^k i_{Z_1} \cdots i_{Z_k} \omega, \quad \text{where} \ \chi = r + \tilde{\omega} \left( \frac{r(r-1)}{2} + \sum_{i=1}^{k} Z_i \right).
\]
In view of Corollary 15, the existence of the de Rham differential \(d : \text{Sym}_A(\mathfrak{Q}, A) \to \text{Sym}_A(\mathfrak{Q}, A)\) by itself implies the existence of an \(LR_{\infty}[1]\)-algebra structure on \(Q\). Denote

\[
\langle \omega | Z_1, \ldots, Z_k \rangle = (-)^k \alpha_1 \cdots \alpha_k \tilde{\omega} \left( \frac{r(r-1)}{2} + \sum_{i=1}^{k} \tilde{Z}_i \right).
\]
anchors and brackets as usual. I want to describe them. First of all, notice that $d$ decomposes as

$$d = d_0 + d_1 + d_2$$

where

$$d_0 = d\bar{a} = d^C - i_R$$
$$d_1 = d^V + 2i_R$$
$$d_2 = -i_R,$$

$d_k$ is of bidegree $(k, -k+1)$, $k = 0, 1, 2$, and $R$ is the curvature of the splitting $V$ (see previous section). Moreover, from $d^2 = 0$ it follows that $\sum_{i+j=k}[d_i, d_j] = 0$, $k = 0, \ldots, 4$, i.e.,

$$[d_0, d_0] = [d_0, d_1] = [d_1, d_2] = [d_2, d_2] = 0,$$

and

$$[d_1, d_1] = -2[d_0, d_2] = 2L_R,$$

where I also used Table (17). The pair $(\Lambda(\mathcal{M}); \{d_0, d_1, d_2\})$ is the Maurer-Cartan algebra of a foliation first discussed by Huebschmann in [11]. In Remark 4.16 of [11], Huebschmann mentions that “P. Michor pointed out [...] a possible connection of the notion of quasi-Lie-Rinehart bracket with that of Frölicher-Nijenhuis bracket”. Such a connection exists indeed, as shown by the following theorem, which provides a description of the $LR_\infty[1]$-algebra structure, determined by the de Rham differential of $M$ via Corollary 15, in terms of Frölicher-Nijenhuis calculus.

**Theorem 29.** The pair $(A, \mathcal{Q})$ possesses the structure of an $LR_\infty[1]$-algebra, such that

$$\{\|\lambda\|\} = \bar{d}\lambda$$
$$\{Z|\lambda\} = -(-)^Z LZ\lambda + i_{[R,Z]_{nr}}\lambda$$
$$\{Z_1, Z_2|\lambda\} = -i_{[R,Z_1]_{nr}, Z_2]_{nr}}\lambda$$
$$\{Z_1, \ldots, Z_k-1|\lambda\} = 0 \quad \text{for } k > 3$$

and

$$\{Z\} = \bar{d}Z$$
$$\{Z_1, Z_2\} = -(-)^{Z_1[Z_1, Z_2]} + [R, Z_1]_{nr}, Z_2]_{nr}$$
$$\{Z_1, Z_2, Z_3\} = -[[R, Z_1]_{nr}, Z_2]_{nr}, Z_3]_{nr}$$
$$\{Z_1, \ldots, Z_k\} = 0 \quad \text{for } k > 3$$

for all $\lambda \in A$, $Z, Z_i \in \mathcal{Q}$, $i = 1, 2, 3, \ldots, k$.

**Proof.** First of all notice that the right hand sides of all identities in the statement of the theorem are $\mathbb{R}$-multilinear and graded symmetric in the $Z_i$’s. Therefore, I can apply the
standard trick and prove the identities just for $Z_i = Z$ even. Compute the anchors: let $\lambda \in A$, 

$$\{ Z | \lambda \} = (d_1 \lambda)(Z)$$

$$= -i_Z (d^V + 2i_R) \lambda$$

$$= -[i_Z, d^V + 2i_R] \lambda$$

$$= -(L_Z + i_{[PC,Z]} + 2i_{[Z,R_{nr}]} \lambda)$$

$$= -(L_Z - i_{[R,Z]_{nr}} + i_{Z} \lambda)$$

$$= -(L_Z - i_{[R,Z]_{nr}}) \lambda,$$

where I used Lemma 27 and the fact the $i_Z \lambda = 0$ for all $Z \in Q$ and $\lambda \in A$.

Similarly,

$$\{ Z, Z | \lambda \} = (d_2 \lambda)(Z,Z)$$

$$= -i_Z i_Z i_R \lambda$$

$$= -i_Z i_Z i_R \lambda$$

$$= -i_{[R,Z]_{nr}} \lambda.$$

Since $d_k = 0$ for $k > 2$, higher anchors vanish.

Now compute the brackets. Let $\omega \in Q^\ast = \overline{A} \otimes CA^1$. Then

$$(-)^\omega(\omega | \{ Z \}) = \{ \omega | Z \} - \langle d_1 \omega | Z \rangle$$

$$= -di_Z \omega + i_Z (d \omega)$$

$$= -[d, i_Z] \omega$$

$$= -i_{\partial} \omega$$

$$= (-)^\omega(\omega | d \omega),$$

where I used (18). Similarly,

$$(-)^\omega(\omega | \{ Z, Z \}) = 2\{ Z, \omega | Z \} - \langle d_1 \omega | Z, Z \rangle$$

$$= 2\{ \omega | (d^V + 2i_R) \omega \} - \langle d_1 \omega | Z, Z \rangle$$

$$= 2(L_Z - i_{[R,Z]_{nr}}) i_Z \omega - [i_Z^2, (d^V + 2i_R)] \omega$$

$$= 2(L_Z - i_{[R,Z]_{nr}}) i_Z \omega - [i_Z, (d^V + 2i_R)] i_Z \omega$$

$$- i_Z [i_Z, (d^V + 2i_R)] \omega$$

$$= 2(L_Z - i_{[R,Z]_{nr}}) i_Z \omega - (L_Z - i_{[R,Z]_{nr}} + i_{Z} \lambda) i_Z \omega$$

$$- i_Z (L_Z - i_{[R,Z]_{nr}} + i_{Z} \lambda) i_Z \omega$$

$$= [L_Z - i_{[R,Z]_{nr}}] i_Z \omega$$

$$= i_{[Z,Z] - [R,Z]_{nr}, Z]_{nr}} \omega$$

$$= (-)^\omega(\omega - [Z, Z] + [R, Z]_{nr}, Z]_{nr}).$$
Finally, 
\[ (-)^{\omega} (\omega|[Z, Z]) = 3\{Z, Z|[\omega|Z]\} = (d_2 \omega|Z, Z) \]
\[ = -3i^2\Lambda i[Z, Z]|_{\text{nr}} iZ\omega + iZ^2iR\omega \]
\[ = -3i^2\Lambda i[Z, Z]|_{\text{nr}} iZ\omega + iZ^2iR_i\omega \]
\[ = -3i^2\Lambda i[Z, Z]|_{\text{nr}} iZ\omega - iZ^2i[Z, iR_i]\omega - iZi[R, iZ]\omega \]
\[ = -3i^2\Lambda i[Z, Z]|_{\text{nr}} iZ\omega - iZ^2i[Z, iR_i]\omega - iZi[R, iZ]\omega \]
\[ = -3i^2\Lambda i[Z, Z]|_{\text{nr}} iZ\omega - iZ^2i[Z, iR_i]\omega - iZi[R, iZ]\omega \]
\[ = -i(Z, iR_i[Z, Z]|_{\text{nr}})iZ\omega + iZi[R, Z]|_{\text{nr}} iZ\omega \]
\[ = -i(Z, iR_i[Z, Z]|_{\text{nr}})iZ\omega + iZi[R, Z]|_{\text{nr}} iZ\omega \]
\[ = i(Z, iR_i[Z, Z]|_{\text{nr}})iZ\omega + iZi[R, Z]|_{\text{nr}} iZ\omega \]
\[ = (\omega|[Z, Z]) \] 
\[ \omega_i(Z, iR_i[Z, Z]|_{\text{nr}})iZ\omega + iZi[R, Z]|_{\text{nr}} iZ\omega \]
\[ = (\omega|[Z, Z]) \]

Similarly as above, higher brackets vanish. \( \square \)

9. Change of Splitting

A priori the \( LR_\infty[1] \)-algebra described in the previous section depends on the choice of the complementary distribution \( V \). In fact, it does not, up to isomorphisms, as an immediate consequence of its derivation from the (multi-)differential algebra \( \Lambda(M, d) \). Namely, let \( V' \) be a different complementary distribution. Denote by \( \Lambda(M, d) \) the (multi-)differential algebra described in the previous section depends on the choice of the \( \Lambda(M, d) \). Let \( \Lambda(M, d) \) be locally given by vector fields \( \alpha = \alpha^i \partial_i \) and \( \Lambda(M, d) \) be locally given by vector fields \( \alpha = \alpha^i \partial_i \). The algebras \( \Lambda(M, d) \) and \( \Lambda(M, d) \) both identify with \( \Lambda(M, d) \) and, in view of Definition 16, \( d \) induces isomorphic \( LR_\infty[1] \)-algebra structures on \( (\Lambda, Q) \) and \( (\Lambda, Q') \) (up to the obvious identifications \( \Lambda \otimes C \Lambda \) and \( \Lambda \otimes C \Lambda \) all the \( \Lambda \)'s appearing below in this section are due to this). Let \( \psi : \Lambda \otimes C \Lambda \rightarrow \Lambda \otimes C \Lambda \) be the composition of isomorphisms

\[ \Lambda \otimes C \Lambda \rightarrow \Lambda(M, d) \rightarrow \Lambda \otimes C \Lambda. \]

Now, I describe the isomorphism \( Q' \rightarrow Q \). I will use the same notations as in Section 4. Let \( \Lambda^C = \Lambda \otimes C \Lambda \) be the projector on \( C \) determined by \( V' \).

If \( V' \) is locally spanned by vector fields \( \partial_i \partial_i, \partial_i \partial_i, \ldots, \partial_i \partial_i \), then \( \Lambda^C = \partial_i \partial_i \). Put \( \Delta := \Lambda^C - \Lambda^C \subset C \Lambda \otimes C \Lambda \). Locally

\[ \Delta = \Delta_i^j \partial_i \partial_i, \quad \Delta_i^j := \Lambda^C - \Lambda^C \]

Proposition 30. The maps \( \psi_k : \Lambda \rightarrow \Lambda \) and \( \Lambda \) determined by \( \psi \) are given by \( i^k \Delta \) and \( i^{k-1} \Delta \) respectively.

Proof. The simplest proof is in local coordinates. Thus, let \( \lambda \in \Lambda \) be locally given by

\[ \lambda = \lambda_{i_1 \ldots i_q} \frac{\partial}{\partial x^{i_1}} \ldots \frac{\partial}{\partial x^{i_q}} = \lambda_{i_1 \ldots i_q} \left( \frac{\partial}{\partial x^{i_1}} + \Delta_i^{i_1} \partial_i \right) \ldots \left( \frac{\partial}{\partial x^{i_q}} + \Delta_i^{i_q} \partial_i \right). \]

Its component in \( \Lambda \otimes C \Lambda \) is

\[ \psi_k(\lambda) = (-)^{kq} k!(q)^k \Delta_i^{i_1} \ldots \Delta_i^{i_k} \lambda_{i_1 \ldots i_q} \partial_i \partial_i \ldots \partial_i \partial_i \partial_i \partial_i \partial_i = i^k \Delta \lambda. \]
Similarly, let \( \omega \in \Lambda \otimes CA^1 \) be locally given by
\[
\omega = \omega_\alpha \otimes du^\alpha, \quad \omega_\alpha \in \Lambda.
\]
Then
\[
\Psi_k(\omega) = \psi_{k-1}(\omega_\alpha) \otimes du^\alpha = i^k_\Delta - 1 \omega_\alpha \otimes du^\alpha = i^k_\Delta - 1 \omega.
\]
\( \square \)

Now I will describe the maps \( \phi_k : (Q')^{\times k} \times \Lambda \to \Lambda \) and \( \Phi_k : (Q')^{\times k} \to Q \). For \( Z' \in Q' \), it is convenient to put
\[
\Delta Z' := -i_\Delta Z'.
\]
Now, let \( Z'_1, \ldots, Z'_k \in Q' \) and \( \lambda \in \Lambda \). Clearly,
\[
\phi_k(Z'_1, \ldots, Z'_k|\lambda) = (-)^{\chi(Z'_1+\cdots+Z'_k)} i_\Delta \Psi_k(\lambda' | Z'_1, \ldots, Z'_k) = (-)^{\chi} i_\Delta \left( i_{Z'_1} \cdots i_{Z'_k} i^k_\Delta \lambda \right)
\]
with
\[
\chi = r + \lambda \left( r(r-1) + \sum_{i=1}^k Z'_i \right).
\]

**Proposition 31.** \( \Phi_1(Z') = i_\Delta(Z') \) and
\[
\Phi_k(Z'_1, \ldots, Z'_k) = i_\Delta \sum_{\lambda \in S_k} \alpha(\sigma, Z') i_{Z'_{\sigma(1)}} i_{\Delta Z'_{\sigma(2)}} \cdots i_{\Delta Z'_{\sigma(k)}} \Delta Z'_{\sigma(k)} (19)
\]
for all \( k > 1 \) and \( Z', Z'_1, \ldots, Z'_k \in Q' \).

**Proof.** Let \( \omega \in Q^* \) and \( Z' \) be an even element of \( Q' \). Then
\[
i_{\Phi_k(Z^k)} \omega = -\langle \omega | \Phi_k(Z^k) \rangle
\]
\[
= -i_\Delta \langle \Psi_k(\omega) | Z^k \rangle + i_\Delta \sum_{m_1+m_2=k} \left( \begin{array}{c} k \\ m_1 \end{array} \right) \langle \psi_{m_1} \langle \omega | \Phi_{m_2}(Z^{m_2}) \rangle | Z^{m_1} \rangle
\]
\[
= -(-)^{k} i_\Delta \left( i_{Z^k} i^k_\Delta - 1 \omega \right) + i_\Delta \sum_{m_1+m_2=k} \left( \begin{array}{c} k \\ m_1 \end{array} \right) (-)^{m_1} i_{Z^k} i^k_{\Delta Z^{m_2}} \Phi_{m_2}(Z^{m_2}) \omega
\]
\[
= \cdots
\]
\[
= i_\Delta \sum_{s=1}^k (-)^{k+s} \sum_{m_1+\cdots+m_s=k} \left( \begin{array}{c} k \\ m_1, \ldots, m_s \end{array} \right) i_{Z^{m_1}} i^1_{\Delta Z^{m_2}} \cdots i^1_{\Delta Z^{m_s-1}} \Phi_{m_2}(Z^{m_2}) \cdots i^1_{\Delta Z^1} i^1_{\Delta Z} \omega. \quad (20)
\]

In particular, for \( k = 1 \), one gets
\[
i_{\Phi_1(Z')} \omega = i_\Delta(i_{Z'} \omega) = i_\Delta(Z') \omega \Rightarrow \Phi_1(Z') = i_\Delta(Z'),
\]
which is the base of induction. Notice that, by linearity, \( \Phi_k(Z^{nk}) \) is known provided \( i_{\Phi_k}(Z^{nk}) \omega \) is known for all \( \omega \in C^1 \). But in this case \( i_{\Delta} \omega = 0 \), and (21) reduces to

\[
\begin{align*}
    i_{\Phi_k}(Z^{nk}) \omega &= \frac{1}{k!} \prod_{s=1}^{k} (-)^{k+s} \sum_{m_1, \ldots, m_{k-1} = k-1} m_1 \prod_{s=1}^{k} \sum_{m_{s+1}, \ldots, m_{s+k} = k-1} m_{s+1} \frac{(-)^{k-s+1}}{m_1 \cdots m_{s+k} > 0} \left( \frac{(-)^{m_1+\cdots+m_{s+k} = k-1}}{m_1 \cdots m_{s+k} > 0} \right) \left( i_{Z_1} \Delta \cdots i_{Z_s} \Delta \cdots \right) \iota_{Z^{nk}} \omega \\
    &= \frac{1}{k!} \prod_{s=1}^{k} (-)^{k+s} \sum_{m_1, \ldots, m_{k-1} = k-1} m_1 \prod_{s=1}^{k} \sum_{m_{s+1}, \ldots, m_{s+k} = k-1} m_{s+1} \frac{(-)^{m_1+\cdots+m_{s+k} = k-1}}{m_1 \cdots m_{s+k} > 0} \left( i_{Z_1} \Delta \cdots i_{Z_s} \Delta \cdots \right) \iota_{Z^{nk}} \omega.
\end{align*}
\]

For \( k > 1 \) one gets

\[
\begin{align*}
    i_{\Phi_k}(Z^{nk}) \omega &= \frac{1}{k!} \prod_{s=1}^{k} (-)^{k+s} \sum_{m_1, \ldots, m_{k-1} = k-1} m_1 \prod_{s=1}^{k} \sum_{m_{s+1}, \ldots, m_{s+k} = k-1} m_{s+1} \frac{(-)^{m_1+\cdots+m_{s+k} = k-1}}{m_1 \cdots m_{s+k} > 0} \left( i_{Z_1} \Delta \cdots i_{Z_s} \Delta \cdots \right) \iota_{Z^{nk}} \omega \\
    &= \frac{1}{k!} \prod_{s=1}^{k} (-)^{k+s} \sum_{m_1, \ldots, m_{k-1} = k-1} m_1 \prod_{s=1}^{k} \sum_{m_{s+1}, \ldots, m_{s+k} = k-1} m_{s+1} \frac{(-)^{m_1+\cdots+m_{s+k} = k-1}}{m_1 \cdots m_{s+k} > 0} \left( i_{Z_1} \Delta \cdots i_{Z_s} \Delta \cdots \right) \iota_{Z^{nk}} \omega.
\end{align*}
\]

Now let \( Z' \) be locally given by \( Z' = Z^\alpha \otimes V'_\alpha \), \( Z^\alpha \in \mathcal{X} \), so that \( \Delta Z \) is locally given by \( \Delta Z = W^\beta_\delta du^\beta \otimes V'_\alpha \), where \( W^\beta_\delta := \Delta^\beta_\delta (\partial_j) Z^\alpha \). Similarly, let \( \Phi_k(Z^{nf}) \) be locally given by \( \Phi_k(Z^{nf}) = \Phi^n_\alpha \otimes V'_\alpha \), \( \Phi^n_\alpha \in \mathcal{X} \). It follows from (22) that

\[
\begin{align*}
    \Phi_k(Z^{nk}) &= k! \Phi_k \iota_{Z_1} \iota_{Z_2} \cdots \iota_{Z_{k-1}} \iota_{Z_1} \Delta \cdots \iota_{Z_k} \Delta \cdots W^\alpha_\beta W^\alpha_\gamma \cdots W^\alpha_{k-1} \Delta W^\alpha_0 = k! \iota_{Z_1} \iota_{Z_2} \cdots \iota_{Z_{k-1}} \iota_{Z_1} \Delta \cdots \iota_{Z_k} \Delta \cdots W^\alpha_\beta W^\alpha_\gamma \cdots W^\alpha_{k-1} \Delta W^\alpha_0
\end{align*}
\]

so that

\[
\Phi_k(Z^{nk}) = k! \iota_{Z_1} \iota_{Z_2} \cdots \iota_{Z_{k-1}} \iota_{Z_1} \Delta \cdots \iota_{Z_k} \Delta \cdots W^\alpha_\beta W^\alpha_\gamma \cdots W^\alpha_{k-1} \Delta W^\alpha_0
\]

\[\square\]

10. On the Homotopy Lie-Rinehart Algebra of a Foliation

The key idea behind secondary calculus \[29,30\] is to interpret characteristic cohomologies of an involutive distribution as geometric structures on the space \( P \) of integral manifolds. In Section 6 I provided two examples of this, namely \( C_\mathcal{F} := H(X,d) \) and \( \mathcal{X}_\mathcal{F} := H(X \otimes \mathcal{F}, \mathcal{F}) \). Within secondary calculus, they are interpreted as functions and vector fields on \( P \), respectively. As I have already remarked, Theorem 23 supports this interpretation. I will now discuss some supporting facts. Recall that a \textit{presymplectic} manifold \( (M, \Omega) \) is a smooth manifold together with a constant rank, closed 2-form \( \Omega \). Typical examples of presymplectic manifolds come from symplectic geometry. Namely, let \( M \) be a submanifold in a symplectic manifold \( (N, \omega) \). Then, \( M \) together with the restricted 2-form \( \omega|_M \), is (almost everywhere, locally) a presymplectic manifold. Thus, let \( (M, \Omega) \) be a presymplectic manifold, let \( C \) be its characteristic (involutive) distribution, i.e. a vector field \( X \) is in \( C \mathcal{X} \) if \( i_X \Omega = 0 \), and \( \mathcal{F} \) its integral foliation. The two forms \( \Omega \) is naturally interpreted as if it were a genuine symplectic structure on the space \( P \) of leaves of \( C \) (for instance, when \( P \) is a manifold and the projection \( \pi : M \rightarrow P \) is a submersion, then \( \Omega := \pi^* \Omega_0 \) for a unique symplectic form on \( P \)). This statement can be given the more precise formulation of Theorem 32 below. Before stating it,
I give some definitions. First of all notice that, by definition, $\Omega \in C\Lambda^2$. Moreover, it follows from $d\Omega = 0$, that

$$d\Omega = d_2\Omega + d_3\Omega = 0.$$ 

Now, as above, chose a distribution $V$ which is complementary to $C$. There is a unique bivector $P \in \Lambda^2\bar{\Lambda}$, “inverting $\omega$ on $\bar{\Lambda}$”. Clearly, $\bar{\Omega}P = 0$. However, as discussed in [23], $P$ is Poisson iff $R = 0$, i.e., $V$ is involutive as well. Nonetheless, it defines an isomorphism

$$\sharp : \bar{\Omega} \otimes CA^1 \longrightarrow \bar{\Omega} \otimes \bar{\Omega}$$

of $\bar{\Omega}$-modules in an obvious way. For $\omega_1, \omega_2 \in \bar{\Omega} \otimes CA^1$, put

$$\langle \omega_1 | \omega_2 \rangle_{\Omega} := \langle \omega_1 | \sharp (\omega_2) \rangle.$$ 

**Theorem 32.**

1. Cohomology $C^\infty_{\Omega}$ possesses a canonical structure of graded Poisson algebra $\{ \cdot , \cdot \}$ given by

$$\{ [\lambda] , [\lambda'] \} := \{ (d_1\lambda)(d_1\lambda') \} \in \bar{\Omega}_{\bar{\Omega}} , \quad [\lambda] , [\lambda'] \in C^\infty_{\Omega} , \quad \lambda , \lambda' \in \bar{\Omega} .$$

The brackets $\{ \cdot \cdot \cdot \}$ is independent of the choice of $V$.

2. There is a canonical morphism of graded Lie algebras $X : (C^\infty_{\Omega} , \{ \cdot , \cdot \} ) \rightarrow (\bar{\Omega}_{\bar{\Omega}} , [\cdot , \cdot ])$ given by

$$X : C^\infty_{\Omega} \ni [\lambda] \longmapsto [d_1 \lambda] \in \bar{\Omega}_{\bar{\Omega}} , \quad [\lambda] \in C^\infty_{\Omega} , \quad \lambda \in \bar{\Omega} .$$

The morphism $X_{\bar{\Omega}}$ is independent of the choice of $V$.

The graded Poisson algebra of Theorem 32 actually comes from an $L_{\infty}[1]$-algebra structure $\mathcal{P}$ in $\bar{\Omega}$, and the morphism $X$ comes from a morphism of $L_{\infty}[1]$-algebras $(\bar{\Omega} , \mathcal{P}) \rightarrow (Q , \mathcal{Q})$ (here, $\mathcal{Q}$ is the canonical $L_{\infty}[1]$-algebra structure on $Q = \bar{\Omega} \otimes \bar{\Omega}[1]$), according to the following

**Theorem 33.**

1. The vector space $\bar{\Omega}[1]$ possesses a structure of $L_{\infty}[1]$-algebra $\mathcal{L} = \{ \cdot \cdot \cdot \}_{k} \in \mathbb{N}$ such that $\{ \cdot \}_{1}^{op} = \mathcal{Q}$.

2. There is a morphism of $L_{\infty}[1]$-algebras $X : (\bar{\Omega} , \mathcal{P}) \rightarrow (Q , \mathcal{Q})$.

Moreover, the structure in [1] (resp., the morphism in [2]), induce the structure in [1] (resp., the morphism in [2]) of Theorem 32 in cohomology, up to a sign (due to the chosen sign conventions).

Part 1 of Theorem 33 has been proved by Oh and Park in [23] (which motivates the notation for the brackets in $\mathcal{L}$). In the remaining part of this section, I prove Part 2. First, I recall the definition of $\mathcal{L}$ [23]. Let $R^\sharp$ be the tensor obtained by contracting one lower index of $R$ with one upper index of $P$. Interpret $R^\sharp$ as an $\text{End} CA^1$-valued derivation of $C^\infty(M)$: $R^\sharp \in \text{End} CA^1 \otimes C\bar{\Omega}$. If $\lambda \in \bar{\Omega}[1]$ I will consider

$$i_{R^\lambda} \lambda \in \bar{\Omega} \otimes \text{End} CA^1 \simeq \text{End}_\bar{\Omega}(\bar{\Omega} \otimes CA^1) .$$

Then

$$\{ \lambda_1 , \ldots , \lambda_k \}^{op}_{k} := \sum_{\sigma \in S_k} \alpha(\sigma , \lambda)(d_1 \lambda_{\sigma(1)}(i_{R^\lambda} \lambda_{\sigma(2)}(\cdots (i_{R^\lambda} \lambda_{\sigma(k-1)}))d_1 \lambda_{\sigma(k)})\} \Omega \quad (23)$$

For all $\lambda_1 , \ldots , \lambda_k \in \bar{\Omega}[1]$. 


Now, I define the morphism \( X : (\overline{\mathcal{A}}, \mathcal{P}) \rightarrow (\mathcal{Q}, \mathcal{Q}) \). It is a homotopy version of the standard morphism sending Hamiltonians to their Hamiltonian vector fields on a symplectic manifold and, to my knowledge, it is defined here for the first time. Define maps
\[
X_k : \overline{\mathcal{A}}[1]^{\times k} \rightarrow \mathcal{Q}
\]
via
\[
X_k(\lambda_1, \ldots, \lambda_k)(f) := \{\lambda_1, \ldots, \lambda_k, f\}^{\text{op}}_{k+1}
\]
It follows from (23) that \( X_k(\lambda_1, \ldots, \lambda_k) \in \overline{\mathcal{A}} \otimes \overline{\mathcal{A}} \) so that \( X_k \) is a well defined degree 0 map for all \( k \).

Lemma 34. Let \( \lambda \in \overline{\mathcal{A}}[1] \) be even. Put \( Z_k := X_k(\lambda^k) \). Then
\[
\{ Z_k | \lambda' \} = \{ \lambda^k, \lambda' \}^{\text{op}} + \sum_{i+j=k \atop i,j>0} (i+j) iZ_i iZ_j R \lambda'
\]
for all \( \lambda' \in \overline{\mathcal{A}} \).

Proof. Both hand sides of (24) are derivations in the argument \( \lambda' \). Therefore, it is enough to check (24) on generators of \( \overline{\mathcal{A}} \), i.e., for \( \lambda' = f \) and \( \overline{\mathcal{A}} f \) for \( f \in C^\infty(M) \). When \( \lambda' = f \), (24) is trivially true by definition of \( Z_k \). Now, it easily follows from \( \overline{dP} = 0 \) and the Bianchi identities that
\[
\{ Z_k | \overline{d f} \} = k! \langle d \overline{d f} | (iR \lambda)^{k-1}(d \lambda) \rangle + k! \langle d \lambda | (iR \lambda)^{k-1}(d \overline{d f}) \rangle
\]
On the other hand
\[
\{ \lambda^k, \overline{d f} \}^{\text{op}} = k! \langle d \overline{d f} | (iR \lambda)^{k-1}(d \lambda) \rangle + k! \langle d \lambda | (iR \lambda)^{k-1}(d \overline{d f}) \rangle
\]
\[
+ \sum_{r+s=k-2} \binom{r+s}{r} \langle d \lambda | (iR \lambda)^r \circ (iR \lambda)^s \circ (d \lambda) \rangle
\]
\[
\cdot \sum_{i+j=k \atop i,j>0} (i+j) iZ_i iZ_j R(f),
\]
where the final equality can be easily checked using, for instance, local coordinates. \( \square \)

Proposition 35. \( X := \{ X_k, \ k \in \mathbb{N} \} \) is a morphism of \( L^\infty[1] \)-algebras.

Proof. Let \( \lambda \in \overline{\mathcal{A}}[1] \) be an even element. I will prove that
\[
K^m_X(\lambda^m) = 0
\]
for all such \( \lambda \). Now, for \( f \in C^\infty(M) \),
\[
K^m_X(\lambda^m)(f) = \sum_{j+k=m} \binom{j+k}{j} X_{k+1}(\{ \lambda^j \}^{\text{op}}, \lambda^k)(f)
\]
\[
- \sum_{r=1}^m \sum_{k_1+\cdots+k_r = m \atop 0 < k_1 \leq \cdots \leq k_r} C(k_1, \ldots, k_r) X_{k_1}(\lambda^{k_1}), \ldots, X_{k_r}(\lambda^{k_r}) \}(f),
\]
where
\[
X_{k+1}(\{ \lambda^j \}^{\text{op}}, \lambda^k)(f) = \{ \{ \lambda^j \}^{\text{op}}, \lambda^k, f \}^{\text{op}}.
\]
Now, put $Z_k := X_k(\lambda^k)$ for all $k$, and compute

$$\{Z_{k_1}, \ldots, Z_{k_r}\}(f) = \{\{Z_{k_1}, \ldots, Z_{k_r}\}|f\}$$

$$= - \sum_{s+t=r, \sigma \in S_{s,t}} \{\{Z_{k_{\sigma(1)}}, \ldots, Z_{k_{\sigma(s)}}\}, Z_{k_{\sigma(s+1)}}, \ldots Z_{k_{\sigma(s+t)}}\}$$

$$= - \sum_{s+t=r, \sigma \in S_{s,t}}\{Z_{k_{\sigma(s+1)}}, \ldots Z_{k_{\sigma(s+t)}}|\{Z_{k_{\sigma(1)}}, \ldots, Z_{k_{\sigma(s)}}\}|f\},$$  \hspace{1cm} (25)

while the highest anchor vanishes on functions and does not contribute. For the same reason only summands with $s = 0, 1$ survive in (25) and one gets

$$\{Z_{k_1}, \ldots, Z_{k_r}\}(f) = - \{Z_{k_1}, \ldots, Z_{k_r}|\{f\}\} - \sum_{i=1}^{r} \{Z_{k_1}, \ldots, \hat{Z}_{k_i}, \ldots, Z_{k_r}|\{Z_{k_i}|f\}\},$$

which is non-zero only when $r = 1, 2, 3$. In view of Lemma 34

$$\{Z_m\}(f) = -\{Z_m|\partial f\} - \partial\{Z_m|f\}$$

$$= -\{\lambda^m, \{f\}^{\text{op}}\}^{\text{op}} - \{\{\lambda^m, f\}^{\text{op}}\}^{\text{op}} + \sum_{i+j=m, i,j>0}^{i+j} iZ_j iZ_k iR \partial f,$$

and

$$\{Z_j, Z_k\}(f) = -\{Z_j, Z_k|\{f\}\} - \{Z_j|\{Z_k|f\}\} + \frac{i^k}{4}$$

$$= -\frac{i}{2} iZ_j iZ_k iR \partial f - \{\lambda^j, \{\lambda^k, f\}^{\text{op}}\}^{\text{op}} + \sum_{r+s=j, r,s>0}^{r+s} \{iZ_j iZ_k iR \{Z_s|f\}\} + \frac{i^k}{4},$$

and

$$\{Z_j, Z_k, Z_{\ell}\}(f) = -\{Z_j, Z_k|\{Z_{\ell}|f\}\} + \frac{j_{\ell} k_{\ell}}{4} = -iZ_j iZ_k iR \{Z_{\ell}|f\} + \frac{j_{\ell} k_{\ell}}{4}.$$  

I conclude that

$$K^m_X(\lambda^m)(f) = \sum_{j+k=m} \{i^k\} \{\lambda^j, \{\lambda^k, f\}^{\text{op}}\} + \{\lambda^m, \{f\}^{\text{op}}\}^{\text{op}} + \{\lambda^m, f\}^{\text{op}} + \{\lambda^m, \{f\}^{\text{op}}\}^{\text{op}}$$

$$+ \sum_{j+k=m, j,k>0} \{j^k\} \{\lambda^j, \{\lambda^k, f\}^{\text{op}}\}^{\text{op}},$$

where the right hand side is the $(m+1)$st Jacobiator of $L$. □

Part 3. Appendixes

**Appendix A. Proofs of Theorems 12 and 18: Computational Details**

For completeness, in this appendix, I add the (mostly straightforward) computational details of the proofs of Theorems 12 and 18. To be clear, I organize these details in lemmas. I use the same notations as in Sections 3 and 4.
Lemma 36. Let \( \omega \in Q^* \). For \( q_1, \ldots, q_{k+1} \in Q \),

\[
(d_k \omega)(q_1, \ldots, q_{k+1}) := \sum_{i=1}^{k} (-)^i \{q_1, \ldots, \hat{q}_i, \ldots, q_{k+1} \mid \omega(q_i)\} + (-)^\omega(\{q_1, \ldots, q_{k+1}\}),
\]

where \( \chi := \bar{\omega}(\bar{q}_1 + \cdots + \bar{q}_k) + \bar{q}_i(\bar{q}_{i+1} + \cdots + \bar{q}_{k+1}) \), and a hat \( \hat{\cdot} \) denotes omission. Then \( d_k \omega \in \text{Sym}_A^{k+1}(Q, A) \).

Proof. Clearly, \( d_k \omega \) is \( K \)-multilinear and graded symmetric. Now, let \( q_2 = \cdots = q_{k+1} = q \) be even. Compute

\[
(d_k \omega)(aq_1, q^k) = (-)^{a+q_1} k \{aq_1, q^{k-1} \mid \omega(q)\} + \{q^k \mid \omega(aq_1)\} - (-)^\omega(\{aq_1, q^k\})
\]

\[
= (-)^{a+q_1+\omega} k \{q_1, q^{k-1} \mid \omega(q)\} + (-)^a \{q^k \mid \omega(q_1)\}
\]

\[
- (-)^{a+\omega+q_1} a \{q^k \mid a \omega(q_1)\}
\]

\[
= (-)^{a+q_1+\omega} k \{q_1, q^{k-1} \mid \omega(q)\} + (-)^a \{q^k \mid \omega(q_1)\}
\]

\[
- (-)^{a+\omega+q_1} a \{q^k \mid a \omega(q_1)\}
\]

\[
= (-)^{a+q_1} a(d_k \omega)(q_1, q^k).
\]

\[\square\]

Lemma 37. Let \( a \in A \) and \( \omega \in Q^* \). Then

\[
d_k(a \omega) = (d_k a) \omega + (-)^a a(d_k \omega).
\]

Proof. Let \( q \in Q \) be even. Then

\[
d_k(a \omega)(q^{k+1}) = (k+1) \{q^k \mid (a \omega)(q)\} - (-)^{a+\omega}(a \omega)(\{q^{k+1}\})
\]

\[
= (-)^a (k+1) a \{q^k \mid \omega(q)\} + (k+1) \{q^k \mid a \omega(q)\} - (-)^a \omega \omega(\{q^{k+1}\})
\]

\[
= (-)^a a(d_k \omega)(q^{k+1}) + (k+1)(d_k a)(q^k) \omega(q)
\]

\[
= ((d_k a) \omega + (-)^a a(d_k \omega))(q^{k+1}).
\]

\[\square\]

Lemma 38. The derivation \( d_k : \text{Sym}_A(Q, A) \rightarrow \text{Sym}_A(Q, A) \) is given by the Chevalley-Eilenberg formula:

\[
(d_k \omega)(q_1, \ldots, q_{r+k-1}) := \sum_{\sigma \in S_{k,r}} (-)^{q_\sigma(1)+\cdots+q_\sigma(k)} \alpha(\sigma, q) \{q_\sigma(1), \ldots, q_\sigma(k) \mid \omega(q_\sigma(k), \ldots, q_\sigma(k+r))\}
\]

\[
- \sum_{\tau \in S_{k+1,r-1}} (-)^{q_\tau} \alpha(\tau, q) \omega(\{q_\tau(1), \ldots, q_{\tau(k+1)}\}, q_{\tau(k+1)}, \ldots, q_{\tau(k+r)}),
\]

\( \omega \in \text{Sym}_A^{r+1}(Q, A) \), \( q_1, \ldots, q_{r+k} \in Q \).
Proof. Let $\omega \in \text{Sym}^r_A(Q, A)$ be of the form $\omega = \omega_1 \cdot \cdots \cdot \omega_r$, $\omega_1 \in Q^*$, and let $q \in Q$ be even. Then

\[
(d_k \omega)(q^{r+k}) = \sum_{i=1}^{r} (-)^i \omega(\omega_1(\cdots \omega_i(q) \cdots) \cdot d_k \omega_i(q^{r+k})
= \frac{(k+r)!}{(k+1)!} \sum_{i=1}^{r} (-)^i \omega(\omega_1(q) \cdots \omega_i(q) \cdots \omega_r(q) (d_k \omega_i(q^{k+1})
= \frac{(k+r)!}{(k+1)!} \sum_{i=1}^{r} (-)^i \omega(\omega_1(q) \cdots \omega_i(q) \cdots \omega_r(q) (k+1) \{q^k \omega_i(q)\} + (-)^i \omega_i(q^{k+1})
= \frac{(k+r)!}{k!} (q^k |\omega(q^r)\} - (-)^i \omega_i(q^{k+1}) \omega(q^{k+1}, q^{r-1})
= \frac{(k+r)!}{k!} (q^k |\omega(q^r)\} - (-)^i \omega_i(q^{k+1}) \omega(q^{k+1}, q^{r-1}),
\]

where $\chi = \omega_1 + \cdots + \omega_{i-1} + (\omega_i + 1)(\omega_{i+1} + \cdots + \omega_r)$. \(\square\)

Lemma 39. Let $a \in A$, $\omega \in Q^*$, and $q_1, \ldots, q_k \in Q$. Then

\[
(E_k a)(q_1, \ldots, q_k) = (-)^{a(q_1+\cdots+q_k-1)} j^{k}(q_1, \ldots, q_k-1 | a) = 0,
\]

and

\[
(E_k \omega)(q_1, \ldots, q_{k+1}) = \sum_{i=1}^{k} (-)^i j^{k+1}(q_1, \ldots, q_i, \ldots, q_{k+1} | \omega(q_i)\} - \omega(j^{k+1}(q_1, \ldots, q_k+1)),
\]

where

\[
\chi := \bar{\omega} \sum_{j \neq i} q_j + q_i \sum_{j > i} q_j.
\]

Proof. Let $q \in Q$ be even. Then

\[
(E_k a)(q^k) = \sum_{\ell+m=k} (d_l d_m a)(q^{\ell+m})
= \sum_{\ell+m=k} \binom{\ell+m}{\ell} q^\ell |(d_m a)(q^m)\} + \sum_{\ell+m=k} \binom{\ell+m}{\ell+1} (-)^a (d_m a)(\{q^{\ell+1}\}, q^{m-1})
= \sum_{\ell+m=k} \binom{\ell+m}{\ell} q^\ell |\{q^m a\}\} + \sum_{\ell+m=k} \binom{\ell+m}{\ell+1} (-)^a \{q^{\ell+1}\}, q^{m-1}|a\}
= j^{k+1}(q^{k} |a).
Similarly,

\[(E_k\omega)(q^{k+1}) = \sum_{\ell + m = k} (d_\ell d_m\omega)(q^{\ell + m + 1})\]

\[
= \sum_{\ell + m = k} \binom{\ell + m + 1}{\ell} (q^\ell d_m\omega(q^{m+1})) + \sum_{\ell + m = k} \binom{\ell + m}{\ell+1} (-)^m (d_m\omega(q^{\ell+1}), q^m) \]

\[
= \sum_{\ell + m = k} \left( \binom{\ell + m + 1}{\ell} (m + 1)\{q^\ell | \{q^m | \omega(q)\}\} - (-)^m \binom{\ell + m + 1}{\ell} \{q^\ell | \omega(q)\} \right) \]

\[
+ (-)^m \binom{\ell + m}{\ell+1} m \{q^{\ell+1}, q^{m-1} | \omega(q)\} \]

\[
- (-)^m \binom{\ell + m}{\ell+1} \omega(\{q^{\ell+1}, q^m\}) \]

\[
= -\omega(J^{k+1}(q^{k+1})) + (k + 1)J^{k+1}(q^k | \omega(q)). \]

\[\Box\]

Lemma 40. Let \(\omega \in Q^*\) with \(p_1, \ldots, p_\ell \in \mathcal{P}\). Then the expression \(R_\ell(\omega)\), inductively defined by

\[R_\ell(\omega)(p_1, \ldots, p_\ell) := \Psi_\ell(\omega)(p_1, \ldots, p_\ell) \]

\[
- \sum_{i+j = \ell} \sum_{\sigma \in S_{i,j}} \alpha(\sigma,p) \psi_j(R_i(\omega)(p_{\sigma(1)}, \ldots, p_{\sigma(i)}))(p_{\sigma(i+1)}, \ldots, p_{\sigma(i+j)})\]

is \(A\)-linear in its argument \(\omega\).

Proof. Let \(p_1 = \cdots = p_\ell = p\) with \(p\) even. I use induction on \(\ell\). \(\Psi_1\) is \(A\)-linear, indeed,

\[\Psi_1(a\omega) = \psi_0(a)\Psi_1(\omega) + \psi_1(a)\Psi_0(\omega) = a\Psi_1(\omega). \]

This provides the base of induction. Now compute

\[R_{\ell+1}(a\omega) = \Psi_{\ell+1}(a\omega)(p^{\ell+1}) - \sum_{i+j = \ell+1} \binom{i+j}{j} \psi_j(R_\ell(a\omega))(p^{\ell+1})\]

\[
= a\Psi_{\ell+1}(\omega)(p^{\ell+1}) + \sum_{i+j = \ell+1} \binom{i+j}{j} \psi_j(a)(p^j)\Psi_\ell(\omega)(p^j) - \sum_{i+j = \ell+1} \binom{i+j}{j} \psi_j(aR_\ell(\omega))(p^j) \]

\[
= aR_{\ell+1}(\omega) + \sum_{i+j = \ell+1} \binom{i+j}{j} \psi_j(a)(p^j)\Psi_\ell(\omega)(p^j) \]

\[
- \sum_{i+j+k = \ell+1} \binom{i+j+k}{j+k} \psi_k(a)(p^k)\psi_j(R_\ell(\omega))(p^j). \]
The last two summands cancel. Indeed, they are 
\[
\sum_{i+j=\ell+1}^{i=0} (i^2j^2) \psi_j(a)(p^j)\Psi_i(o)(p^i) - \sum_{i+j+k=\ell+1}^{i=0} (i^2j^2+k^2) \psi_k(a)(p^k)\Psi_j(o)(p^j)
\]
\[
= \sum_{i+k=\ell+1}^{i=0} (i+k^2) \psi_k(a)(p^k)\Psi_i(o)(p^i) - \sum_{i+j+k=\ell+1}^{i=0} (i+k^2) \psi_k(a)(p^k)R_i(o)
\]
\[
- \sum_{i+j+k=\ell+1}^{i=0} (i+k^2) \psi_k(a)(p^k)\Psi_j(o)(p^j)
\]
\[
= \sum_{i+k=\ell+1}^{i=0} (i+k^2) \psi_k(a)(p^k) \cdot [\Psi_i(o)(p^i) - R_i(o)] - \sum_{s+j=i}^{s+j=0} (s^2+j^2) \psi_j(R_s(o))(p^j)]
\]
\[
= 0. \quad \square
\]

Now, I present the proof of Lemma 17 (see Section 4 for the statement).

Proof of Lemma 17 Let \( p_1 = \cdots = p_k = p \) be even. For \( r = 0 \) the result follows immediately from the definition of \( \phi_k \). Thus, let \( r > 0 \), and \( \omega = \sigma_1 \cdots \sigma_r \), with \( \sigma_i \in Q^* \). Then
\[
\psi(o)(p^k) = \psi(\sigma_1 \cdots \sigma_r)(p^k)
\]
\[
= \sum_{m_1+\cdots+m_r=k} \psi(\sigma_1)m_1 \cdots \psi(\sigma_r)m_r(p^k)
\]
\[
= \sum_{m_1+\cdots+m_r=k} (m_1^r) \psi(\sigma_1)m_1(p^m_1) \cdots \psi(\sigma_r)m_r(p^m_r).
\]

Now, it follows from (11) that, if \( \sigma \in Q^* \), then
\[
\psi(m)(p^m) = \Psi(m)(p^m) = \sum_{s+t=m} (s+t)^m \psi(s)(\Phi_s(p^s))(p^t) = \sum_{s+t=m} (s+t)^m \psi(\Phi_s(p^s))(p^t),
\]
so that
\[
\psi(o)(p^k) = \sum_{s+t+k=s+t+k=0} (s+t+k) \psi(\sigma_1)(\Phi_s(p^s))(\sigma_t)(p^t) = \sum_{s+t+k=s+t+k=0} (s+t+k) \psi(\sigma_1)(\Phi_s(p^s))(\sigma_t)(p^t)
\]
\[
= \sum_{t_0+\cdots+t_r=t_0+\cdots+t_r=0} C(t_0|t_1,\ldots,t_r)\phi(t_0)(p^0)\omega(\Phi_{t_1}(p^{t_1}),\ldots,\Phi_{t_r}(p^{t_r})),
\]
where \( C(t_0|t_1,\ldots,t_r) \) is the cardinality of \( T_{t_0|t_1,...,t_r} \). \( \square \)
Lemma 41. Let \( \omega \in Q^* \), and \( p \in P \) even. Then \( \psi(D_Q\omega)_k = (D_P\psi(\omega))_k \) iff
\[
\sum_{m+\ell=k} \binom{m+\ell}{m}\phi_\ell(p^\ell|\omega(K^m_\Phi(p^m))) = 0, \tag{26}
\]

Proof. Compute
\[
\psi(D_Q\omega)_k(p^k) = \sum_m \psi(d_m\omega)_k(p^k)
= \sum_m \sum_{\ell_0+\cdots+\ell_{m+1}=k} C(\ell_0|\ell_1,\ldots,\ell_{m+1})\phi_\ell_0(p^{\ell_0}|(d_m\omega)(\Phi_{\ell_1}(p^{\ell_1}),\ldots,\Phi_{\ell_{m+1}}(p^{\ell_{m+1}})))
\]
\[-\sum_m \sum_{\ell_0+\cdots+\ell_{m+1}=k} (-)^\omega C(\ell_0|\ell_1,\ldots,\ell_{m+1})\phi_\ell_0(p^{\ell_0}|\omega(\{\Phi_{\ell_1}(p^{\ell_1}),\ldots,\Phi_{\ell_{m+1}}(p^{\ell_{m+1}})\})). \tag{27}
\]
Moreover,
\[
(D_P\psi(\omega))_k(p^k)
= \sum_{m+n=k} d_n\Psi_n(\omega)(p^k)
= \sum_{m+n=k} \binom{k}{m} \{p^m|\Psi_n(\omega)(p^n)\} - \sum_{m+n=k} (-)^\omega \binom{k}{m+1} \Psi_n(\omega)(\{p^{m+1},p^{n-1}\})
= \sum_{m+n+r=k} \binom{m+n+r}{m,n} \{p^m|\phi_\ell(p^\ell|\omega(\Phi_r(p^r)))\} - \phi_\ell(\{p^m\},p^r|\omega(\Phi_r(p^r)))
\]
\[-(-)^\omega \phi_\ell(p^\ell|\omega(\Phi_r(\{p^m\},p^r)))). \tag{28}
\]
Now, (28) implies that the first summand in (27) equals the first two summands in (28). It follows that
\[
\sum_m \sum_{\ell_0+\cdots+\ell_{m+1}=k} C(\ell_0|\ell_1,\ldots,\ell_{m+1})\phi_\ell_0(p^{\ell_0}|\omega(\{\Phi_{\ell_1}(p^{\ell_1}),\ldots,\Phi_{\ell_{m+1}}(p^{\ell_{m+1}})\}))
\]
\[= \sum_{m+\ell+r=k} \binom{m+\ell+r}{m,r} \phi_\ell(p^\ell|\omega(\Phi_r(\{p^m\},p^r))), \tag{29}
\]
that can be compactly rewritten in the form (26).

\[\Box\]

Appendix B. Higher Lie-Rinehart Brackets as Derived Brackets

The Lie brackets in a Lie algebroid can be presented as derived brackets [18]. Similarly, the higher brackets in an \( L_\infty \)-algebra can be presented as higher derived brackets following T. Voronov [33] (see [8]). Marco Zambon suggested to me that a similar statement could hold
for the higher brackets in a SH Lie-Rinehart algebra. This is indeed the case as I briefly show in this appendix. First, I recall the formalism of [33].

**Definition 42.** Let $\mathcal{V}$ be a DG Lie algebra (over a field $K$ of zero characteristic). Consider the following data:

1. an Abelian sub-algebra $\mathcal{A} \subset \mathcal{V}$,
2. a $K$-linear projector $P : \mathcal{V} \rightarrow \mathcal{A}$,
3. a degree 1 element $D \in \mathcal{V}$.

The data $(\mathcal{A}, P, D)$ are called $V$-data in $\mathcal{V}$ if

1. $\ker P \subset \mathcal{V}$ is a Lie subalgebra,
2. $PD = 0$,
3. $D^2 = 0$.

A triple $(\mathcal{A}, P, D)$ of $V$-data determines, in particular, $K$-multilinear, graded-symmetric maps

$$\{\cdot, \cdot, \cdot\}_k^D : \mathcal{A}^k \rightarrow \mathcal{A}, \quad k \in \mathbb{N}$$

via

$$\{a_1, a_2, \ldots, a_k\}_k^D := P[[\cdots[[D, a_1], a_2], \cdots], a_k], \quad a_1, a_2, \ldots, a_k \in \mathcal{A}.$$

**Theorem 43** (Voronov [33]). The brackets $\{\cdot, \cdot, \cdot\}_k^D$, $k \in \mathbb{N}$, give $\mathcal{V}$ the structure of an $L_\infty[1]$-algebra.

Now, let $A$ be a graded algebra and $Q$ a projective and finitely generated $A$-module. As in Section 3, consider the graded algebra $\mathcal{A} := \text{Sym}_A(Q, A)$. Let $\text{Der}(\mathcal{A})$ be the graded Lie algebra of derivations of $\mathcal{A}$, and let $\mathcal{V} := \text{Der}(\mathcal{A}) \oplus \mathcal{A}$ be the graded Lie algebra of first order differential operators in $\mathcal{A}$. The brackets in $\mathcal{V}$ are

$$[(\Delta, \omega), (\nabla, \rho)] = ([(\Delta), \nabla], \Delta \rho - (-)^{\omega} \nabla \omega), \quad \Delta, \nabla \in \text{Der}(\mathcal{A}), \quad \omega, \rho \in \mathcal{A}.$$

Notice that $Q \oplus A$ embeds into $\mathcal{V}$. Namely, consider the linear map

$$i : Q \oplus A \ni (q, a) \mapsto (i_q, i_a) \in \mathcal{V},$$

where $i_a := -a$, and $i_q$ is defined by

$$i_q \omega := (-)^q \omega(q, \cdot, \cdot, \cdot), \quad \omega \in \mathcal{A}.$$

The signs are chosen to simplify formulas below. The image $\mathcal{A}$ of $i$ is clearly an Abelian subalgebra in $\mathcal{V}$. There is a canonical projection

$$P : \mathcal{V} \ni (\Delta, \omega) \mapsto (P_Q \Delta, P_A \omega) \in Q \oplus A,$$

where $P_A \omega := -p \omega, \ p : A \rightarrow A$ being the natural projection, and $P_Q$ is implicitly defined by the formula

$$\omega(P_Q \Delta) := (-)^{\Delta \omega} p \Delta \omega \in A, \quad \omega \in Q^*.$$

Notice that the expression $(-)^{\Delta \omega} p \Delta \omega$ is graded, $A$-linear in $\omega$ so that $P_Q \Delta$ is a well defined element in $Q$. Obviously, $P$ is a left inverse of $i$. Finally, it is easy to see that $\ker P \subset \mathcal{V}$ is a subalgebra as well, so that, $(\mathcal{A}, P, D)$ are $V$-data in $\mathcal{V}$ for all homological derivations $D \in \text{Der}(\mathcal{A})$. In particular, a homological derivation $D$ in $\mathcal{A}$ determines an $L_\infty[1]$-algebra structure $\mathcal{L}_D := \{\cdot, \cdot, \cdot\}_k^D, \ k \in \mathbb{N}$ in $A \oplus Q$ via Theorem 43. On the other hand, a homological derivation in $\mathcal{A}$ determines an $L_\infty[1]$-algebra structure $\mathcal{L} \oplus := \{\cdot, \cdot, \cdot\}_k^{\oplus}, \ k \in \mathbb{N}$ in $A \oplus Q$. 
also via Corollary 15 (see Definition 3 for details about the brackets \{·, ·, ·\}⊗ determined by the brackets in \mathcal{Q} and the anchors).

Proposition 44. Let \( D \in \text{Der}_A \) be a homological derivation. The \( \mathcal{L}_\infty[1] \)-algebra structures \( \mathcal{L}^D \) and \( \mathcal{L}^{\oplus} \) coincide.

Proof. First of all, since \([D,a] = Da\) for all \( a \in A \), it is obvious that \( \{v_1, \ldots, v_k\}_k^D \) vanishes whenever two entries are from \( A \). This means that \( \mathcal{L}^D \) is the same as an \( \mathcal{L}_\infty[1] \)-algebra structure on \( \mathcal{Q} \), and an \( \mathcal{L}_\infty[1] \)-module structure on \( A \). Now, define derivations \( d_k : A \to A \) as in the proof of Corollary 15, and, for \( a \in A \) and \( q \in \mathcal{Q} \) even, compute,

\[
\{ q^{k-1}, a \}^D = \mathbb{P}[\cdots [D, i_q] \cdots, i_q], i_a] = (-)^k p i_q^{k-1} Da = (d_{k-1} a)(q^{k-1}) = \{ q^{k-1}, a \} = \{ q^{k-1}, a \}^{\oplus}.
\]

Similarly, let \( \omega \in \mathcal{Q}^* \) and compute

\[
\omega(\{ q^k \}^D) = \omega(\mathbb{P}[\cdots [D, i_q] \cdots, i_q]) = (-)^k p i_q^{k-1} Da = (d_{k-1} \omega)(q^k) = \omega(\{ q^k \}^{\oplus}).
\]

Notice that when \( \mathcal{Q} \) is as in Section 8 and \( D \) is the de Rham differential in \( A = \Lambda(M) \), then the \( V \)-data \((\mathcal{A}, \mathbb{P}, D) \) in \( V \) are precisely those constructed by Ji [14] in the case of a foliation.

APPENDIX C. THE HOMOTOPY LIE-RINEHART ALGEBRA OF A FOLIATION VIA HOMOTOPY TRANSFER

After the publication on arXiv of a preliminary version of this paper, Florian Schätz suggested to me that the \( \mathcal{L}_\infty[1] \)-algebra of a foliation could be derived from the DG Lie algebra \( \text{Der}_\Lambda \) of derivations of \( \Lambda \) via homotopy transfer (see [12] about the homotopy transfer of Lie algebra structures). This is indeed the case as I briefly discuss in this appendix. I first recall the version of the homotopy transfer theorem I will refer to.

Theorem 45 (Homotopy Transfer Theorem). Let \((L, \Delta, [\cdot, \cdot])\) be a DG Lie algebra over a field of 0 characteristic, and

\[
h \bigcup (L, \Delta) \xrightarrow{p} (H, \delta)
\]

contraction data, i.e., i) \( p \) and \( j \) are cochain maps, ii) \( p \circ j = \text{id}_H \), and iii) \( \text{id}_L - j \circ p = \Delta \circ h + h \circ \delta \). Then there is a natural \( \mathcal{L}_\infty \)-algebra structure \( \mathcal{L} \) on \((H, \delta)\).
There exists an explicit description of brackets in \((H,\mathcal{L})\) in terms of the contraction data by means of trees [17], or inductive formulas (see for instance [21], where inductive formulas for the transfer of an associative algebra structure are provided).

I’m not presenting here this description (the interested reader may see [32], where I recall the necessary formulas from [21] and apply them to prove the existence of more SH structures associated to a foliation). Notice, however, that, in a similar way, one can transfer the structure of a DG Lie module along contraction data, and get an \(L_\infty\)-module.

Now, \(\text{Der}\Lambda\) possesses the canonical differential \(\Delta := [d, \cdot]\) and \((\text{Der}\Lambda, \Delta, [\cdot, \cdot])\) is a DG Lie algebra (sometimes referred to as the deformation complex of the Lie algebroid \(\mathcal{X}\) [6]). It is known that \((\text{Der}\Lambda, \Delta)\) is homotopy equivalent to \((\Lambda \otimes \mathcal{X}, d)\) [6]. However, to my knowledge, there was as yet no explicit description of contraction data. I provide it in the proof of the next

**Proposition 46.** A distribution \(V\) complementary to \(C\) determines contraction data

\[
\begin{array}{ccc}
\Lambda \otimes \mathcal{X} & \xrightarrow{p} & \Lambda \\
\Lambda \otimes \mathcal{X} & \xrightarrow{j} & \mathcal{X}
\end{array}
\]

\[
\begin{array}{ccc}
\Lambda \otimes \mathcal{X} & \xrightarrow{h} & \Lambda \\
\Lambda \otimes \mathcal{X} & \xrightarrow{j} & \mathcal{X}
\end{array}
\]

**Proof.** For \(Z \in \Lambda \otimes \mathcal{X}\), and \(\lambda \in \Lambda\) put

\[
j(Z)(\lambda) := \overline{LZ\lambda}.
\]

Then \(j(Z) \in \text{Der}\Lambda\). For \(D \in \text{Der}\Lambda\), put \(p(D) := \overline{D|_{C^\infty(M)}} \in \Lambda \otimes \mathcal{X}\). Finally, let \(h(D) \in \text{Der}\Lambda\) be defined on generators \(f, df\) by

\[
h(D)(f) := 0
\]

\[
h(D)(df) := (-)^{\text{deg}(D)}(D - jpD)f,
\]

It is easy to see, using, for instance, local coordinates, that \(h(D)\) is well defined, and \(j, p, h\) are actually contraction data.

As an immediate corollary of the Homotopy Transfer Theorem and the above proposition, there is an \(L_\infty[1]\)-algebra structure on \(\Lambda \otimes \mathcal{X}[1]\). It is easy to see that such \(L_\infty[1]\)-algebra actually coincides with the one described in Section 8. Notice that the \(L_\infty[1]\)-module structure on \(\Lambda\) can be obtained from homotopy transfer as well.

**Appendix D. Alternative Formulas for Binary Operations**

Let \((A, Q) = (\Lambda, \Lambda \otimes \mathcal{X}[1])\) denote the \(LR_\infty[1]\)-algebra of a foliation. In this appendix I present alternative formulas for the binary operations in \(Q\). This is useful for some purposes, e.g., proving the homotopy transfer and the derived bracket [14] origins of \(Q\).

**Proposition 47.** Let \(Z_1, Z_2 \in Q\). Then

\[
\{Z_1, Z_2\} = -(-)^{Z_1}[Z_1, Z_2].
\]

**Proof.** Let \(X \in \mathcal{X}\) and \(\lambda \in \Lambda\). Then \(i_X \lambda = 0\) and \(L_X \lambda = \lambda' + \lambda''\), with \(\lambda' \in \Lambda\) and \(\lambda'' \in \Lambda \otimes \mathcal{X}A^1\). Therefore, in view of Formula (12),

\[
[Z_1, Z_2] = [Z_1, Z_2] + Z' + Z''
\]
with \( Z' \in \mathfrak{X} \otimes \mathcal{C}\mathfrak{X}, \ Z'' \in \mathcal{C}\Lambda^1 \otimes \mathfrak{X} \otimes \mathfrak{X} \). It follows that
\[
Z' = i_{[Z_1,Z_2]} P_C \quad \text{and} \quad Z'' = i_{P^v}[Z_1,Z_2],
\]
so that
\[
[[Z_1,Z_2]] = [[Z_1,Z_2]] - Z' - Z'' = i_{[Z_1,Z_2]} \llbracket - i_{[Z_1,Z_2]} P_C - i_{P^v}[Z_1,Z_2],
\]
\[
= i_{[Z_1,Z_2]} P^v - i_{P^v}[Z_1,Z_2] = [[Z_1,Z_2]], P^v]_{nr}.
\]
Now, it follows from Formula (14) that
\[
[[Z_1,Z_2]], P^v]_{nr} = [[Z_1,Z_2]], P^v]_{nr} = -i_{P^v}[R,Z_1]_{nr},
\]
where I also used that \([Z,Z_1]_{nr} = 0\) for all \(Z,Z_1 \in \mathcal{Q}\).

**Proposition 48.** Let \( Z \in \mathcal{Q} \) and \( \lambda \in \mathfrak{X} \). Then
\[
\{Z|\lambda\} = -(-)^{Z_1} Z_2 \Lambda^1.
\]

**Proof.** In view of Formula (9)
\[
L_Z \lambda = \overline{L_Z \lambda} + \omega',
\]
with \( \omega' \in \mathfrak{X} \otimes \mathcal{C}\Lambda^1 \). It follows that
\[
\omega' = i_{P^v} L_Z \lambda = \llbracket L_Z | \lambda \rrbracket = -(-)^{Z} i_{P^v}[P_C, Z] \lambda
\]
\[
= (-)^{Z} i_{[P^v, Z]} \lambda = (-)^{Z} i_{[R,Z]}_{nr} \lambda = (-)^{Z} i_{[R,Z]}_{nr} \lambda.
\]

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