σ(e⁺e⁻ → hadrons) and the Heavy Quark Masses

M. Steinhauser

II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee, D-22761 Hamburg, Germany

The precise data for the total cross section σ(e⁺e⁻ → hadrons) from the charm threshold region, when combined with the evaluation of moments with three loop accuracy, lead to a direct determination of the short distance MS charm quark mass

\[ m_c(m_c) = 1.304(27) \]

Applying the same approach to the bottom quark we obtain

\[ m_b(m_b) = 4.191(51) \text{ GeV} \]

A complementary method for the determination of \( m_b \) is based on the analysis of the \( \Upsilon(1S) \) system, which is confronted with a next-to-next-to-next-to-leading order calculation of the corresponding energy level. This leads to

\[ m_b(m_b) = 4.346(70) \text{ GeV} \]

1. Introduction

During the past years new and more precise data for \( \sigma(e^+e^- \rightarrow \text{hadrons}) \) have become available in the low energy region between 2 and 10 GeV. At the same time increasingly precise calculations have been performed in the framework of perturbative QCD (pQCD), both for the cross section as a function of the center-of-mass energy \( \sqrt{s} \), including quark mass effects, and for its moments which allow for a precise determination of the quark mass. A fresh look at the evaluation of the charm quark mass with the help of sum rules is thus an obvious task. We will concentrate on low moments as suggested by the ITEP group long ago [1]. This is a natural route to determine directly a short distance mass, say \( m_c(m_c) \), in the \( \overline{\text{MS}} \) scheme [1, 2].

The same method can also be applied to the bottom system leading directly to \( m_b(m_b) \).

A complementary method for the determination of \( m_b \) is based on the analysis of the \( \Upsilon(1S) \) system. Recently a major improvement on the theoretical side has been achieved by the evaluation of the next-to-next-to-next-to-leading order (N^3LO) corrections to the energy level. The disadvantage of this method is due to the large non-perturbative effects.

2. Charm and bottom quark mass from low-order moment sum rules

A very elegant method for the determination of the charm quark mass is based on the direct comparison of theoretical and experimental moments of the charm quark contribution to the photon polarization function. In the limit of small momentum the latter can be cast into the form [3]

\[ \Pi_c(q^2) = \frac{Q_c^2}{16m_c^2} \sum_{n \geq 0} \bar{C}_n z^n, \]

with \( z = q^2/(4m_c^2) \) where \( m_c = m_c(\mu) \) is the \( \overline{\text{MS}} \) charm quark mass at the scale \( \mu \). The perturbative series for the coefficients \( \bar{C}_n \) up to \( n = 8 \) is known analytically [3] up to order \( \alpha_s^2 \). We define the moments through

\[ M_n^{\text{exp}} = \int \frac{ds}{s^{n+1}} R_c(s) \]

\[ = M_n^{\text{th}} = \frac{9}{4} Q_c^2 \left( \frac{1}{4m_c^2} \right)^n \bar{C}_n, \]

which leads to the following formula for the charm quark mass

\[ m_c(\mu) = \frac{1}{2} \left( \frac{\bar{C}_n(\ln m_c)}{M_n^{\text{exp}}} \right)^{1/(2n)}. \]

The analysis of the experimental moments with \( n = 1, \ldots, 4 \) leads to the results displayed in Fig. [3].
The moment with \( n = 1 \) is evidently least sensitive to non-perturbative contributions from condensates, to the Coulombic higher order effects, the variation of \( \mu \) and the parametric \( \alpha_s \) dependence. Hence

\[
m_c(m_c) = 1.304(27) \text{ GeV}, \tag{4}
\]
is adopted as the final result \([1]\). In principle the same analysis can be performed using the pole mass scheme for the quarks. However, the final results are quite unstable in contrast to the \( \overline{\text{MS}} \) scheme.

The same approach is also applicable to the determination of \( m_b \). Again a significant improvement of the stability of the prediction after inclusion of the NNLO terms is observed. As final result one finds

\[
m_b(m_b) = 4.191(51) \text{ GeV}, \tag{5}
\]

3. The \( \Upsilon(1S) \) system and the bottom quark mass

In contrast to the previous section one has to deal with a non-relativistic system of a bound state of a heavy quark-antiquark pair which is governed by a complicated multiscale dynamics. In the nonrelativistic regime, where the heavy-quark velocity \( v \) is of the order of the strong-coupling constant \( \alpha_s \), the Coulomb effects are crucial and have to be taken into account to all orders in \( \alpha_s \). This makes the use of the effective theory mandatory. This approach allows us to separate the scales and to implement the expansion in \( v \) at the level of the Lagrangian.

The dynamics of a nonrelativistic quark-antiquark pair is characterized by four different regions, the hard region, the soft region, the potential region, and the ultrasoft region. Nonrelativistic QCD (NRQCD) \([8]\) is obtained by integrating out the hard modes. Subsequently integrating out the soft modes and the potential gluons results in the effective theory of potential NRQCD (pNRQCD) \([8]\), which contains potential heavy quarks and ultrasoft gluons, ghosts, and light quarks as active particles. The effect of the modes that have been integrated out is twofold: higher-dimensional operators appear in the effective Hamiltonian, corresponding to an expansion in \( v \), and the Wilson coefficients of the operators in the effective Hamiltonian acquire corrections, which are series in \( \alpha_s \). In \([5,8]\) the ingredients for the N\(^3\)LO Hamiltonian have been completed using an efficient combination of the effective theory formalism and the threshold expansion \([8]\).

Once the Hamiltonian is available the only task is to solve the corresponding Schrödinger equation up to third order using the usual time-independent perturbation theory. A new feature which appears for the first time at N\(^3\)LO are the retardation effects which arise from the chromoelectric dipole interaction of the heavy quarkonium with a virtual ultrasoft gluon.

The third order correction to the lowest energy level was calculated in \([8,10]\) and will be used in this work in order to determine the bottom quark mass. The application to the top quark system can be found in \([11]\). Recently the N\(^3\)LO Hamiltonian has been used to compute the corrections of order \( \alpha_s^4 \ln \alpha_s \) to the wave function \([12]\).

The perturbative expansion of the energy level with quantum number \( n \) looks as follows

\[
E_{n}^\text{pert} = E_{n}^C + \delta E_{n}^{(1)} + \delta E_{n}^{(2)} + \delta E_{n}^{(3)} + \ldots, \tag{6}
\]

where the \( O(\alpha_s^3) \) correction, \( \delta E_{n}^{(3)} \), arises from the following sources:

(i) matrix elements of the N\(^3\)LO operators of the effective Hamiltonian between Coulomb wave functions;

(ii) higher iterations of the NLO and NNLO operators of the effective Hamiltonian in time-independent perturbation theory;

(iii) matrix elements of the N\(^3\)LO instantaneous operators generated by the emission and absorption of ultrasoft gluons; and

(iv) the retarded ultrasoft contribution.

The analytical result for \( \delta E_{n}^{(3)} \) can be found in \([13]\). In numerical form it reads (adopting the choice \( \mu_s = C_F \alpha_s(m_s) m_q \)) for the bottom and top system

\[
\delta E_{1}^{(3)} = \alpha_s^3(m_s) E_{1}^C \left[ \begin{array}{c} 70.590 |_{\text{bottom}} \\ 56.732 |_{\text{top}} \end{array} \right].
\]
where we have separated the contributions arising from $\alpha_3 \text{ and } \beta_0^3$. The only unknown ingredient in the result for $\delta E_1^{(3)}$ is the three-loop $\overline{\text{MS}}$ coefficient $a_3$ of the corrections to the static potential. Up to now there are only estimates based on Padé approximation [13] which is used in our analysis. However, the final result only changes marginally even for a rather large deviations of $a_3$ from its Padé estimate.

The starting point for the determination of the bottom quark mass is its relation to the mass of the $\Upsilon(1S)$ resonance

$$M_{\Upsilon(1S)} = 2m_b + E_1^{\text{pert}} + \delta^{\text{n.p.}} E_1,$$

with $M_{\Upsilon(1S)} = 9.46030(26)$ GeV. Here $\delta^{\text{n.p.}} E_1$ is the nonperturbative correction to the ground state energy. The dominant contribution is associated with the gluon condensate and gives

$$\delta^{\text{n.p.}} E_1 = 60 \pm 30 \text{ MeV}.$$  

Combining Eq. (8) with the result for the perturbative ground state energy up to $O(m_b^2 \alpha_s^2)$ one obtains the bottom quark mass as a function the renormalization scale of the strong coupling constant normalization, $\mu$, which is plotted in Fig. 2(a). For the numerical evaluation we extract $\alpha_s^{(2)}(m_b)$ with $m_b = 4.83$ GeV from its value at $M_Z$ using four-loop $\beta$-function accompanied with three-loop matching $\alpha_s^{(3)}(m_b)$ is used as starting point in order to evaluate $\alpha_s^{(4)}(\mu)$ at $N^k\text{LO}$ with the help of the $k$-loop $\beta$ function.

We use the package RunDec [14] to perform the running and matching of $\alpha_s$. 

Figure 1. $m_c(m_c)$ for $n = 1, 2, 3$ and $4$. For each value of $n$ the results from left to right correspond to the inclusion of terms of order $\alpha_s^0$, $\alpha_s^1$ and $\alpha_s^2$ in the coefficients $\bar{C}_n$. Note, that for $n = 3$ and $n = 4$ the errors can not be determined with the help of Eq. (3) in those cases where only the two-loop corrections of order $\alpha_s$ are included into the coefficients $\bar{C}_n$ as the equation cannot be solved for $m_c$.
The uncertainties in the obtained value of $\overline{m}_b(\mu^*)$ have been investigated in detail in [10]. Finally, the prediction for the $\overline{\text{MS}}$ bottom quark mass $\overline{m}_b(\mu^*)$ is used to account for these corrections in a most accurate way. We suggest the following procedure to take into account these corrections in a most accurate way. The idea is that from the one-parametric family of $\overline{m}_b(\mu)$ we can choose a representative corresponding to some scale $\mu^*$ in such a way that $\overline{m}_b(\mu^*) = m_b$.

For a given fixed-order value of the pole mass Eq. (10) can be solved for $\mu^*$. In particular, for a pole mass to $N^k$LO we use the $k$-loop relation between the $\overline{\text{MS}}$ and pole mass in Eq. (10). Afterwards $\overline{m}_b(\mu^*)$ can be computed from $\overline{m}_b(\mu^*)$ solving the renormalization group (RG) equation. The advantage of this approach is obvious: we use the finite order relation between $\overline{\text{MS}}$ and pole mass at the scale where they are perturbatively close while the large difference between $\overline{m}_b(\mu^*)$ and $m_b$ is completely covered by the RG evolution which can be computed with very high accuracy as the corresponding anomalous dimension is known to four-loop approximation [17, 18]. The only restriction on the method could be connected to the value of $\mu^*$. It should be large enough to allow for a reliable use of the RG equation which in practice indeed is the case [10].

In Fig. 2(b) our result for $\overline{m}_b(m_b)$ is plotted at NLO, NNLO and N$^3$LO as a function of the normalization scale $\mu$ which is used to obtain the pole mass $m_b$ (cf. Fig. 2(a)). It is remarkable that close to $\mu = 2.7$ GeV, which is consistent with the physically motivated soft scale $\mu_s \approx 2$ GeV, both the second and the third order corrections vanish. This fact is a rather strong indication of the convergence of the series for $\overline{m}_b(m_b)$.

From Fig. 2(a) we see that the dependence on the renormalization scale becomes very strong below $\mu \sim 2$ GeV which indicates that the perturbative corrections are not under control. However, even above this scale the perturbative series for the pole mass shows no sign of convergence. This means that one can assign a numerical value to the pole mass only in a specified order of perturbation theory.

On the contrary, it is widely believed that the $\overline{\text{MS}}$ mass $\overline{m}_b(\mu)$ at the scale $\mu = \overline{m}_b(\mu)$ is a short-distance object which has much better perturbative properties. Thus, it seems to be reasonable to convert our result for the pole mass into $\overline{m}_b(\overline{m}_b)$. The relation between $m_b$ and $\overline{m}_b(\overline{m}_b)$ is known up to three-loop approximation [15, 16] and shows sizable perturbative corrections. For this reason we suggest the following procedure to take into account these corrections in a most accurate way. The idea is that from the one-parametric family of $\overline{m}_b(\mu)$ we can choose a representative corresponding to some scale $\mu^*$ in such a way that

$$\overline{m}_b(\mu^*) = m_b.$$  

The uncertainties in the obtained value of $\overline{m}_b(\overline{m}_b)$ have been investigated in detail in [10].
mass reads
\[ m_c(m_b) = 4.346 \pm 0.070 \text{ GeV}. \] (11)

4. Conclusions

For the comparison of the results discussed in this contribution with the literature we refer to [1, 11]. It is, however, interesting to compare the value for the charm quark mass with a very recent result obtained in a lattice calculation [19]. Their final result in quenched approximation, \( m_c(m_c) = 1.301(34) \text{ GeV}, \) is impressively close to ours (cf. Eq. (10)) with comparable errors. In order to estimate the uncertainty induced by the quenched approximation we performed a “perturbative quenching” by setting the number of active flavours to zero in the calculation of [4]. This leads to a small uncertainty of \( \approx 30 \text{ MeV}, \) which agrees with the estimate of [19].

Although formally slightly beyond 1\( \sigma \) the \( \overline{\text{MS}} \) quark mass obtained from the low-moment sum rule approach is in very good agreement with the one from the \( \Upsilon(1S) \) system. One needs to have in mind that both the experimental data and the theoretical calculations are completely different. Whereas no further improvement in the \( \Upsilon(1S) \) method can be expected there is significant improvement possible in the approach discussed in Section 2. In particular, after reducing the experimental error of \( R(s) \) in the charm and bottom threshold region and the one of the leptonic widths of the narrow resonances to roughly 2\% a reduction of the uncertainty in the charm and bottom quark mass to 15 MeV and 30 MeV, respectively, can be expected. More details can be found in Ref. [12].

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