Modal logic with the difference modality of topological $T_0$-spaces

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Abstract

The aim of the paper is to study the topological modal logic of $T_0$ spaces, with the difference modality (for $T_n$, where $n \geq 1$ the corresponding logics were known). We consider propositional modal logic with two modal operators $\square$ and $[\neq]$. $\square$ is interpreted as an interior operator and $[\neq]$ corresponds to the inequality relation. We introduce the logic $S4DT_0$ and show that $S4DT_0$ is the logic of all $T_0$ spaces and has the finite model property.

Keywords: Kripke semantics, finite model property, completeness, topological semantics

Introduction

In this paper we study the topological semantics of modal logics. Several interpretations of the modal box as an operator over a topological space are possible. Namely diamond-as-closure-operator and diamond-as-derivation-operator have been pioneering in the semantics of modal logic as far back as in 1944, in the celebrated paper of McKinsey and Tarski (cf. [6]). They showed that $S4$ is the logic of all topological spaces and the logic of any metric dense-in-itself space is $S4$. This remarkable result also demonstrates a relative weakness of the interior operator to distinguish between interesting topological properties.

The second interpretation gives more expressive power. $T_0$ and $T_D$ separation axioms become expressible (cf. [2], [3]); the real line can be distinguished from the rational line (cf. [12]). It also has its limitations (for example, it is still impossible to distinguish $\mathbb{R}^2$ from $\mathbb{R}^3$).

We can increase the expressive power by adding extra modalities (cf. [10], [11]). For example, connectedness is expressible in modal logic with the interior

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and the universal modality (cf. \[18\]) and $T_1$ separation axiom becomes expressible in modal logic with the interior and the difference modality (cf. \[19\]).

In this paper we add the difference modality (or modality of inequality) $\not=\,$, interpreted as "true everywhere except here". The expressive power of this language in topological spaces has been studied by Gabelaia in \[10\], where he presented an axiom that defines $T_0$ spaces.

The first section contains basic information, definitions and results from the theory of modal logics and general topology.

In the last two sections we formulate completeness of $S_4DT_0$ logic with respect to $T_0$ spaces respectively and the finite modal property.

1. Language, axioms and logic

In this paper, we study propositional modal logics with two modal operators, $\Box$ and $\not=\,$. A formula is defined as follows:

$$\phi ::= p \mid \bot \mid \phi \rightarrow \phi \mid \Box \phi \mid \not=\phi$$

The classic logic operators $\lor$, $\land$, $\neg$, $\top$, $\equiv$ are expressed in terms of $\rightarrow$ and $\bot$ in the standard way. The dual modal operators $\Diamond$ and $\langle\not=\,\rangle$ are defined as usual:

$$\Diamond \phi = \neg \Box \neg \phi, \langle\not=\,\rangle \phi = \neg [\not=\,] \neg \phi$$

respectively. We denote $[\not=\,] \phi \wedge \phi$ by $[\forall] \phi$.

The set of all bimodal formulas is called the bimodal language and is denoted by $\mathcal{ML}_2$.

**Definition 1.1.** A (normal bimodal) logic is a set of formulas $L \subseteq \mathcal{ML}_2$ such that:

1. $L$ contains all the classical tautologies.
2. $L$ contains the modal axioms of normality:

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q),\quad [\not=\,](p \rightarrow q) \rightarrow ([\not=\,]p \rightarrow [\not=\,]q).$$

3. $L$ is closed with respect to the following inference rules:

$$\frac{\phi \rightarrow \psi, \phi}{\psi} \quad (\text{MP}),$$
$$\frac{\phi}{\Box \phi} \quad (\rightarrow \Box),$$
$$\frac{\phi}{\phi} \quad ([\not=\,] \rightarrow [\not=\,]),$$
$$\frac{\phi}{[\psi/p] \phi} \quad (\text{Sub}).$$

Let $L$ be a logic and $\Gamma$ be a set of formulas. The minimal logic containing $L \cup \Gamma$ is denoted by $L + \Gamma$. We also write $L + \psi$ instead of $L + \{\psi\}$.

In this paper we will use the following axioms:

$$(T) \quad \Box p \rightarrow p,$$
$$(4) \quad \Box p \rightarrow \boxdot \Box p,$$
$$(D) \quad [\forall] p \rightarrow \Box p,$$
$$(B) \quad p \rightarrow [\not=\,](\not=\,p),$$
$$(4D) \quad [\forall] p \rightarrow [\not=\,][\not=\,] p.$$
\[(AT_0) \quad (p \land [\neg] \neg p \land (\neg q \land [\neg] \neg q)) \rightarrow (\Box q \lor (\neg q \land [\neg] \neg p)).\]
We introduce the notation for the following logics:

\[S4 = K_1 + T_\Box + 4\Box\]
\[S4D = K_2 + T_\Box + 4\Box + D_\Box + B_\Box + 4D\]
\[S4DT_0 = S4D + AT_0\]

2. Topological semantics

**Definition 2.1.** A topological space is a pair \(X = (X, \Omega)\) where \(X\) is a nonempty set (the domain of the space) and \(\Omega\) is a set of subsets of \(X\) satisfying the following properties:

1. The empty set \(\emptyset\) and \(X\) itself belong to \(\Omega\).
2. The union of any collection of sets from \(\Omega\) is contained in \(\Omega\).
3. The intersection of any finite collection of sets from \(\Omega\) is also contained in \(\Omega\).

The elements of \(\Omega\) are called open sets and \(\Omega\) is called a topology on \(X\). If \((X, \Omega)\) is a topological space and \(x\) is a point in \(X\), a neighbourhood of \(x\) is an open set \(U\) containing \(x\). A closed set is a set whose complement is an open set.

**Definition 2.2.** The interior of a set \(A\) in a topological space \(X\) is the greatest (with respect to inclusion) open set in \(X\) contained in \(A\), i.e., an open set that contains any other open subset of \(A\). It is denoted by \(\text{Int} A\).

**Definition 2.3.** The closure of a set \(A\) is the smallest closed set containing \(A\). It is denoted \(\text{Cl} A\).

**Definition 2.4.** Let \(X\) be a topological space, then \(X\) is an Alexandroff space if arbitrary intersections of open sets are open.

**Definition 2.5.** A topological space \(X\) is a \(T_0\)-space if for every pair of distinct points of \(X\), at least one of them has a neighborhood not containing the other.

**Definition 2.6.** A topological model on a topological space \(X := (X, \Omega)\) is a pair \((X, V)\), where \(V : PV \rightarrow P(X)\) (the set of all subsets), i.e., a function that assigns each propositional variable to a set \(V(p) \subseteq X\) and is called a valuation. The truth of a formula \(\phi\) at a point \(x\) of a topological model \(\mathcal{M} = (X, V)\) (notation: \(\mathcal{M}, x \models \phi\)) is defined by induction:

\[\mathcal{M}, x \models p \iff x \in V(p),\]
\[\mathcal{M}, x \not\models \bot,\]
\[\mathcal{M}, x \not\models \phi \rightarrow \psi \iff \mathcal{M}, x \not\models \phi \text{ or } \mathcal{M}, x \models \psi,\]
\[\mathcal{M}, x \models \Box \phi \iff \exists U \in \Omega(x \in U \text{ and } \forall y \in U(\mathcal{M}, y \models \phi)),\]
\[\mathcal{M}, x \models [\neg] \phi \iff \forall y \neq x(\mathcal{M}, y \not\models \phi).\]

**Definition 2.7.** Let \(\mathcal{M} = (X, \Omega, V)\) be a topological model and \(\phi\) be a formula. We say that the formula \(\phi\) is true in the model \(\mathcal{M}\) (notation: \(\mathcal{M} \models \phi\)), if it is true at all points of the space, i.e.
\[ M \models \phi \iff \forall x \in X, M, x \models \phi. \]

**Definition 2.8.** Let \( X = (X, \Omega) \) be a topological space, \( \mathcal{C} \) be a class of spaces and \( \phi \) be a formula. We say that a formula is valid in \( X \) (notation: \( X \models \phi \)) if it is true in every model on this topological space, i.e.

\[ X \models \phi \iff \forall V (X, V \models \phi). \]

We say that the formula \( \phi \) is valid in \( \mathcal{C} \) if it is valid in every space in \( \mathcal{C} \).

**Definition 2.9.** The logic of a class of topological spaces \( \mathcal{C} \) (denoted by \( \text{Log}(\mathcal{C}) \)) is the set of all formulas of the language \( \mathcal{ML}^2 \) that are valid in all spaces of the class \( \mathcal{C} \).

**Theorem 2.1.** (c.f. [14]). Let \( \mathcal{C} \) be a class of topological spaces. Then \( \text{Log}(\mathcal{C}) \) is a modal logic.

**Lemma 2.1.** Let \( M = (X, \Omega, V) \) be a topological model. Denote \( V(\phi) = \{ x \in X \mid M, x \models \phi \} \), where \( \phi \) is an arbitrary formula. Then we have:

1. \( V(\phi \lor \psi) = V(\phi) \cup V(\psi) \)
2. \( V(\phi \land \psi) = V(\phi) \cap V(\psi) \)
3. \( V(\neg \phi) = X - V(\phi) \)
4. \( V(\phi \rightarrow \psi) = (X - V(\phi)) \cup V(\psi) \)
5. \( M \models \phi \rightarrow \psi \iff V(\phi) \subseteq V(\psi) \)
6. \( V(\top) = X \)
7. \( V(\Box \phi) = \text{Int}(V(\phi)) \)
8. \( V(\Diamond \phi) = \text{Cl}(V(\phi)) \)

**Proof.** Since the first 6 points are obvious, so we prove only last 2 points.

8. \( x \in V(\Box \phi) \)
   \( \iff M, x \models \Box \phi \)
   \( \iff \exists U \in \Omega (x \in U \land \forall y \in U (M, y \models \phi)) \)
   \( \iff \exists U \in \Omega (x \in U \land \forall y \in U \models V(\phi)) \)
   \( \iff \exists U \in \Omega (x \in U \land U \subseteq V(\phi)) \)
   \( \iff x \in \text{Int}(V(\phi)) \)
9. \( x \in V(\Diamond \phi) \)
   \( \iff M, x \models \Diamond \phi \)
   \( \iff M, x \models \neg \Box \neg \phi \)
   \( \iff x \in X - \text{Int}(X - \text{Int}(\phi)) \)
   \( \iff x \in \text{Cl}(V(\phi)) \)

**Lemma 2.2.** Let \( X = (X, \Omega) \) be a topological space then \( X \models AT_0 \) iff \( X \) is a \( T_0 \) space.

**Proof.** \((\Rightarrow)\) We prove by contradiction. Assume \( X \models AT_0 \) and let there be points \( x \neq y \) such that \( \forall U \in \Omega, x \in U \iff y \in U \). Define a valuation \( V \) such that \( V(p) = \{ x \} \) and \( V(q) = \{ y \} \). Then \( X, V, x \models p \land [\neg p \land (\neg q) \land \neg q] \) and \( X, V, x \models \Box \neg q \lor (\neg q \land \Box \neg p) \). This contradicts the fact that \( X \models AT_0 \).
Assume $X$ is a $T_0$ space. Let $X, V, x \models p \land \neg p \land \langle \neq \rangle (q \land \neg q)$. Then there is a point $y$, such that $V(q) = \{y\}$. Further, at least one of the points $x$ and $y$ is contained in a neighborhood that does not contain the other. That means $X, V, x \models \Box \neg q$ or $X, V, y \models \Box \neg p$ which proves our assertion.

Definition 2.10. A logic $L$ is complete with respect to a class of topological spaces $C$ if $Log(C) = L$.

Theorem 2.2. (cf. [14]) The logic $S4D$ is complete with respect to all topological spaces.

3. Kripke semantics.

Definition 3.1. A Kripke frame is a tuple $\langle W, R_1, \ldots, R_n \rangle$, where $W \neq \emptyset$ is a set, and $R_i$ (for $i = 1, \ldots, n$) is a binary relation on $W$. Elements of $W$ are called points or worlds, and $R_i$ for $i = 1, \ldots, n$ is an accessibility relation.

In this article we will deal with Kripke frames with two binary relations. The first relation will be denoted by $R$, the second by $R_D$.

Definition 3.2. A valuation on a Kripke frame $F = (W, R_1, R_2, \ldots, R_n)$ is a function $V : PV \rightarrow 2^W$. A Kripke model is a pair $M = (F, V)$. Then we inductively define the notion of a formula $\phi$ being true in $M$ at a point $x$ as follows:

$$
M, x \models p \iff x \in V(p), \text{ for } p \in PV
$$

$$
M, x \not\models \bot
$$

$$
M, x \models \phi \rightarrow \psi \iff M, x \not\models \phi \text{ or } M, x \models \psi
$$

$$
M, x \models \Box_i \phi \iff \forall y (x R_i y \Rightarrow M, y \models \phi)
$$

For a subset $U \subseteq W$ $M, U \models \phi$ denotes that for any $x \in U$ ($M, x \models \phi$). We say that a formula $\phi$ is valid in a model $M$ (notation: $M \models \phi$), if $\forall x \in W$ ($M, x \models \phi$).

We also say that a formula $\phi$ is valid on a frame $F$ (notation: $F \models \phi$) if it is valid in all models of the frame $F$ and a formula is valid in a class of frames if it is valid in every frame from this class.

Definition 3.3. The logic of a class of frames $C$ (in notation $Log(C)$) is the set of formulas that are valid in all frames from $C$. For a single frame $F$, $Log(F)$ stands for $Log(\{F\})$.

Definition 3.4. A logic $L$ is called Kripke complete if there exists a class of frames $C$, such that $L = Log(C)$.

Definition 3.5. Let $L$ be a modal logic. A frame $F$ is called an $L$-frame if $L \subseteq Log(F)$.

Theorem 3.1. (c.f. [14]). Let $F$ be a Kripke frame. Then $Log(F) = \{ \phi \mid F \models \phi \}$ is a modal logic.
Let us introduce some notations. Let \( W \) be an arbitrary nonempty set, \( B \subseteq W \); \( R, R' \subseteq W \times W \) are relations on \( W \).

\[
R|_B = R \cap (B \times B);
Id_W = \{(x, x) | x \in W\};
R^+ = R \cup Id_W \text{ (reflexive closure)};
R \circ R' = \{(x, z) | \exists y (xRy \land yR'z)\};
R^n = Id_W;
R^{n+1} = R^n \circ R;
R^* = \bigcup_{n=0}^{\infty} R^n \text{(reflexive and transitive closure)}.
\]

Let \( F = (W, R_1, ..., R_n) \) be a frame, and let \( x \in W \). \( R_i(x) = \{y | xR_iy\} \), \( R_i^{-1}(x) = \{y | yR_ix\} \). Let \( U \subseteq W \), then \( R_i(U) = \bigcup_{x \in U} R_i(x) \), \( R_i^{-1}(U) = \bigcup_{x \in U} R_i^{-1}(x) \).

**Definition 3.6.** Let \( F = (W, R_1, ..., R_n) \) be a Kripke frame and \( S^* \) be the transitive and reflexive closure of the relation \( S = (\bigcup_{i=0}^{n} R_i) \). For \( x \in W \), \( W^x = \{y | xS^*y\} \) (the set of all points reachable from the point \( x \) by relation \( S^* \)). Frame \( F^x = (W^x, R_1|_{W^x}, ..., R_n|_{W^x}) \) is called a generated subframe (cone).

**Lemma 3.1.** Let \( F = (W, R_1, R_2, ..., R_n) \) be a Kripke frame and \( \mathcal{C} \) be a class of Kripke frames, then

1. \( \operatorname{Log}(F) = \bigcap_{x \in W} \operatorname{Log}(F^x) = \operatorname{Log}(\{F^x | x \in W\}) \).
2. \( \operatorname{Log}(\mathcal{C}) = \operatorname{Log}(\{F^x | F \in \mathcal{C}, x \in F\}) \)

Let \( F = (W, R) \) be an \( \text{S4} \)-frame, then the set of subsets \( T = \{U \subseteq W | R(U) \subseteq U\} \) defines a topology on \( W \). Topological space \( (W, T) \) is denoted by \( \text{Top}(F) \). This topology is Alexandroff (see [14]) (since \( R(x) \) is the minimal open neighborhood of \( x \) for any \( x \)).

Consider the interpretation of the language \( \mathcal{ML}_2 \) in topological spaces with a binary relation of the form \((\mathbb{X}, R)\), where \( \Box \) is interpreted in the same way as in topological semantics, and \( \{\neq\} \) using \( R \) as in Kripke semantics.

If the reflexive closure of the binary relation \( R \) is the universal relation (i.e., \( R \cup Id_W = W \times W \)), then the relation \( R \) can be characterized by the set of all irreflexive points, which we call selected points.

The following lemma is well-known (cf. [1], [4])

**Lemma 3.2.** (see [2], [3]) Let \( F = (W, R, R_D) \) be a Kripke frame, then

1. \( \vdash B_D \iff \forall x, y \in W (xR_Dy \Rightarrow yR_Dx) \iff R_D \text{ is symmetric}; \)
2. \( \vdash 4_D \iff R^2_D \subseteq R_D \cup Id_W \iff R_D \cup Id_D \text{ is transitive}; \)
3. \( \vdash 4_\Box \iff \forall x, y, z \in W (xR_Dy \land yR_Dz \Rightarrow xR_Dz) \iff R \text{ is transitive}; \)
4. \( \vdash 1_\Box \iff \forall x \in W \ xRx \Rightarrow R \text{ is reflexive}; \)
5. \( \vdash D_\Box \iff \exists R \subseteq R_D \cup Id_W. \)

Now let \( F = (W, R, R_D) \) be an \( \text{S4D} \)-cone, so \( R_D \cup Id_W = W \times W \). We define a space with selected points \( \text{Top}_D(F) = (\text{Top}(F), A) \), where \( A = \{v | \neg \forall v \neg R_Dv\} \). Note that we may consider topological space \( \mathbb{X} \) as \((\mathbb{X}, \mathbb{X})\), where the domain \( \mathbb{X} \) is the set of selected points, in other words all the points are selected.
Lemma 3.3. Let \((F,V)\) be a model on \(F\), where \(F = (W,R,R_D)\) is an S4D-cone, then
\[
F,V,x \models \phi \iff \text{Top}_D(F),V,x \models \phi,
\]
for any \(x \in W\) and for any formula \(\phi\).

Proof. The standard proof is carried out by the induction on the length of the formula. \(\square\)

Corollary 3.1. Let \(F = (W,R,R_D)\) be an S4D-cone, then
\[
\text{Log}(F) = \text{Log}(\text{Top}_D(F)).
\]

Lemma 3.4. Let \(F = (W,R,R_D)\) be an S4D-cone, then:
\[
F \models AT_0 \iff \forall x,y \in W (x \neq y \land xRy \land yRx \implies xR_Dx \lor yR_Dy)
\]

Proof. Suppose there are two points \(x, y\) such that they both are irreflexive with respect to the second relation (\(R_D\)-irreflexive) and mutually reachable by the first relation. We define a model \(M = (F,V)\) by defining valuation as follows:
\[
V(p) = \{x\}, V(q) = \{y\}.\]
Then
\[
M,x \models p \land [\neq] \neg p \land (\neq)(q \land [\neq] \neg q) \quad \text{and} \quad M,x \not\models \Box q \lor (\neq)(q \land \Box \neg p).
\]
Conversely, suppose \(M,x \models p \land [\neq] \neg p \land (\neq)(q \land [\neq] \neg q)\). Then \(V(p) = \{x\}\) and there is a point \(y\) such that \(V(q) = \{y\}\). These points are \(R_D\)-irreflexive. By assumption, they are not mutually accessible by the first relation. We consider 2 cases:
1. \(\neg xRy\). Then
\[
M,y \models \Box \neg q \implies M,x \models \Box \neg q \lor (\neq)(q \land \Box \neg p).
\]
2. \(\neg yRx\). Then
\[
M,y \models q \land \Box \neg p \implies M,x \models (\neq)(q \land \Box \neg p) \implies M,x \models \Box \neg q \lor (\neq)(q \land \Box \neg p).
\]

\(\square\)

4. \(p\)-morphism

For two topological spaces \(\mathcal{X}\) and \(\mathcal{Y}\) a map \(f : \mathcal{X} \to \mathcal{Y}\) is said to be \textit{continuous} if for every open subset \(U \subset \mathcal{Y}\), the inverse image \(f^{-1}(U) \subset \mathcal{X}\) is open in \(\mathcal{X}\). Map \(f\) is said to be \textit{open} if for every open set \(U\) in \(\mathcal{X}\), \(f(U)\) is open in \(\mathcal{Y}\). We call \(f\) \textit{interior} if it is both open and continuous.

Definition 4.1. A map between topological spaces \(f : \mathcal{X} \to \mathcal{Y}\) is called \(p\)-morphism if it is surjective and interior (notation: \(f : \mathcal{X} \xrightarrow{p} \mathcal{Y}\)).
Definition 4.2. A map between topological spaces with selected points $\mathcal{X} = (X, A_X)$ and $\mathcal{Y} = (Y, A_Y)$ is called a p-morphism if it is a p-morphism of topological spaces $f: X \to Y$, and

$$A_Y = \{y \mid \exists x \in A_X (f^{-1}(y) = \{x\})\}$$

Lemma 4.1. (cf. [4]) For a given map $f: X \to Y$ the following statements are equivalent:
1. $f$ is interior;
2. $f^{-1}(\text{Int } Z) = \text{Int } f^{-1}(Z)$ for any $Z \subseteq Y$;
3. $f^{-1}(\text{Cl } Z) = \text{Cl } f^{-1}(Z)$ for any $Z \subseteq Y$.

Lemma 4.2. Let $\mathcal{X} = (X, A_X)$ and $\mathcal{Y} = (Y, A_Y)$ be topological spaces with selected points and $f: X \to Y$ be a p-morphism. Let $V_Y$ be a valuation on topological space $Y$ and $V_X(p) = f^{-1}(V_Y(p))$ for all $p \in PV$. Then for any formula $\phi$ the following holds:

$$\forall x \in X, (\mathcal{X}, V_X, x \models \phi \iff \mathcal{Y}, V_Y, f(x) \models \phi).$$

Proof. The statement of the lemma can be rewritten as follows:

$$V_X(\phi) = f^{-1}(V_Y(\phi))$$

The proof proceeds in a straightforward way by induction on the length of the formula. Let us consider cases when $\phi = \Box \psi$ and $\phi = [\neq] \psi$. The other cases are trivial.

Suppose that $\phi = \Box \psi$. First, we use the assertion of the lemma 2.1, then the previous lemma.

$$f^{-1}(V_Y(\Box \psi)) = f^{-1}(\text{Int } (V_Y(\psi))) = \text{Int } (f^{-1}(V_Y(\psi))) \overset{\text{H}}{=} \text{Int } (V_X(\psi)) = V_X(\Box \psi).$$

Now we turn to $[\neq]$. Suppose $\mathcal{Y}, V_Y, f(x) \models [\neq] \psi$. If $f(x) \in A_Y$, then

$$\forall z \neq f(x) (\mathcal{Y}, V_Y, z \models \psi) \overset{\text{H}}{\Rightarrow} \forall y \neq x (\mathcal{X}, V_X, y \models \psi) \Rightarrow \mathcal{X}, V_X, x \models [\neq] \psi.$$ If $f(x) \notin A_Y$, then

$$\forall z (\mathcal{Y}, V_Y, z \models \psi) \overset{\text{H}}{\Rightarrow} \forall y (\mathcal{X}, V_X, y \models \psi) \Rightarrow \mathcal{X}, V_X, x \models [\neq] \psi.$$ Suppose $\mathcal{X}, V_X, x \models [\neq] \psi$. There are 3 cases:
1. $x \in A_X \& f(x) \in A_Y$
2. $x \notin A_X \& f(x) \notin A_Y$
3. $x \in A_X \& f(x) \notin A_Y$

The first two cases are obvious, so we only consider the last case. By the definition of p-morphism, there is another point $x'$, such that $f(x) = f(x')$. Then

$$\mathcal{X}, V_X, x \models [\neq] \psi \Rightarrow \mathcal{X}, V_X, x' \models [\neq] \psi \overset{\text{H}}{\Rightarrow} \mathcal{Y}, V_Y, f(x) \models [\neq] \psi \Rightarrow \mathcal{X}, V_X, x \models [\neq] \psi.$$ It follows that,

$$\forall z (\mathcal{X}, V_X, z \models \psi) \overset{\text{f-surjective}}{\Rightarrow} \forall y (\mathcal{Y}, V_Y, y \models \psi) \Rightarrow \mathcal{Y}, V_Y, f(x) \models [\neq] \psi.$$
5. Canonical frames and Kripke completeness

The axioms $T□, A□, B\neg D, \neg A\neg D$ are Sahlqvist formulas, so by the Sahlqvist theorem we obtain the canonicity and Kripke completeness for logic $S4D$ (see [1]). To prove the Kripke completeness of logic $S4DT_0$, we use the canonical model construction.

**Definition 5.1.** The canonical frame for $L$ is $F_L = (W_L, R_1, L, \ldots, R_n, L)$, where $W_L$ is the set of all maximal consistent theories over $L$ (see [13]) and $xR_Ly$ if for every $\square A \in x$ we have $A \in y$.

**Definition 5.2.** The canonical model for $L$ is a model $M_L$ on the frame $F_L$ with a valuation function $V_L$ such that $V_L(p) = \{x | p \in x\}$.

**Theorem 5.1. (Canonical Model Theorem, cf. [1], [4])** For the modal logic $L$ and its canonical model $M_L = (W_L, R_1, L, \ldots, R_n, L, V_L)$, it is true that,

1. $\forall \phi \forall x \in W \quad i. \ M_L, x \models \phi \iff \phi \in x$
2. $\forall \phi \forall x \in W \quad ii. \ M_L \models \phi \iff \phi \in L$

**Lemma 5.1.** $S4DT_0$ logic is Kripke complete.

**Proof.** We take the canonical model $M_L = (F_L, V_L)$ of logic $L = S4DT_0$. By the Sahlqvist theorem $F$ is an $S4D$-frame (see [1] and [11]). Consider a cone $M = M_L^z$ and assume there exist different points $z$ and $y$ such that they are $R_D$-irreflexive ($\neg zR_Dz$ and $\neg yR_Dy$) and are mutually reachable by the first relation (i.e. $zRy \land yRz$). Note that, by definition of the canonical model $(\neg zR_Dz \iff \exists \phi([\neq] \phi \in z \land \phi \notin z))$. But on the other hand $\phi$ is true at all the other points of the cone. Hence, $\neg \phi$ is false everywhere, except for the point $z$. Similarly, it can be shown that there exists a formula $\psi$ that is true only at $y$. Hence,

$$M, z \models \neg \phi \land [\neq] \phi \land (\neq) (\psi \land [\neq] \neg \psi)$$

On the other hand,

$$M, z \not\models \boxdot \neg \psi \lor (\neq) (\psi \land \boxdot \phi).$$

As a result, assuming the opposite, we get $M, z \not\models AT_0$, which contradicts to the previous theorem.

6. Finite model property

A formula is **satisfiable in a frame $F$ (in a class of frames $C$)** if it is true at a point of some model over $F$ (over some frame in $C$).

**Definition 6.1.** Logic $L$ has the **finite model property** if $L = L(C)$ for some class of finite frames $C$.  

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Logic $L = L(C)$ has the finite model property if and only if each satisfiable in $C$ formula is satisfiable in a finite $L$-frame.

**Definition 6.2.** Let us consider a frame $F = (W, R_1, R_2)$ and an equivalence relation $\sim$ on $W$. A frame $F/\sim = (W/\sim, R_1/\sim, R_2/\sim)$ is said to be the minimal filtration of $F$ through $\sim$, if for $U_1, U_2 \in W/\sim$ and $i = 1, 2$

$$U_1 R_i/\sim U_2 \iff \exists u \in U_1 \exists v \in U_2 u R_i v$$

**Definition 6.3.** Let $M$ be a model and $\phi$ be a formula. We define the equivalence $\sim_{\phi}$ induced by the formula $\phi$ on the points of $M$ as follows: $u \sim_{\phi} v$ iff every subformula of $\phi$ is simultaneously true or false in $u$ and in $v$.

We say that an equivalence $\sim$ agrees with formula $\phi$ in a model if $\sim \subseteq \sim_{\phi}$.

**Lemma 6.1.** (cf. [9]) If a formula $\phi$ is satisfiable in a model $M$ over a frame $F$ and an equivalence $\sim$ agrees with formulas $\phi$, then $\phi$ is satisfiable in $F/\sim$.

A partition of a set $W$ is a family of disjoint subsets of $W$ whose union coincides with $W$. If $A$ and $B$ are partitions of a set $W$ and each element of $A$ is a subset of one element from $B$, then we say $A$ is a refinement of $B$. We denote by $\sim_A$ the equivalence relation whose set of classes coincides with $A$:

$$A = W/\sim_A.$$

We write $F_A$ and $R_A$ instead of $F/\sim_A$ and $R/\sim_A$.

**Theorem 6.1.** $S4DT_0$ has the finite model property.

**Proof.** Let $F = (W, R, R_D)$ be an $S4DT_0$-cone, and a formula $\phi$ is satisfiable in it. We will show that there is a finite $S4DT_0$-frame in which $\phi$ is satisfiable. First we construct the minimal filtration of $M = (F, V) (\exists x \in W (M, x \models \phi))$ via $\sim_{\phi}$ and denote the resulting model as $M' = (F', V')$, where $F' = (W', R', R_D')$. The points of $F'$ we will call equivalence classes or classes. Since all the different points of $W$ see each other by the second relation, then each $R_D'$-irreflexive class consists of a single $R_D'$-irreflexive point.

For validity of formulas $T\square, D\square, B_D$ and $4_D$ in the resulting frame [see [14], page 48], $T_0$ is preserved by minimal filtration, since if there are $R_D'$-irreflexive classes mutually reachable by the first relation we will get a contradiction with the theorem 4.5. The resulting frame may not be $R'$-transitive (Fig.1). To satisfy $4_D$ axiom we can consider transitive closure of the minimal filtration. But if we do this, $T\square, D\square, B_D$ and $4_D$ axioms will still be valid, but $T_0$ may become false. Indeed, consider Fig.1.
If we take the transitive closure of $R'$ (denote as $R''$) we may have two mutually reachable by $R'' R'_D$-irreflexive classes.

Note that there is a finite number of equivalence classes in $M'$, hence there is a finite number of $R'$-paths (i.e., finite sequences $x_0 x_1 \ldots x_n$ such that $x_i R' x_{i+1}$ for any $i < n$) from one $R'_D$-irreflexive class to another, satisfying the following conditions

- no classes are repeated,
- there are no $R'_D$-irreflexive classes except the beginning and the end of the path.

Let $L_1, L_2, \ldots, L_m$ are all such paths. $A_{n_1}, A_{n_2}, \ldots, A_{n_n}$, are all the $R'_D$-reflexive classes that appear in $L_1$. We consider all this classes ascending their numbering. Assume that $A_{n_i}$ is visible from class $A_{n_j}$ and sees class $A_{n_k}$ in $L_1$. We devide points of class $A_{n_i}$ into four parts.

1. Points of class $A_{n_i}$ that are visible from the class $A_{n_j}$ and see the class $A_{n_k}$ at the same time will be denoted by $N$.
2. Points of class $A_{n_i}$ that are visible from the class $A_{n_j}$ and don’t see the class $A_{n_k}$ will be denoted by $N_1$.
3. Points of class $A_{n_i}$ that are not visible from the class $A_{n_j}$ and see the class $A_{n_k}$ are denoted by $N_2$.
4. The last class is $A_{n_i} \setminus (N_1 \cup N_2 \cup N)$. 
We do this division for all the \( n' \) classes. Then we do the same procedure for \( L_2, L_3, \ldots, L_m \). In result we get divisions for all \( R_D \)-reflexive classes \( A_1, A_2, \ldots, A_l \), that appear in the above paths. Then we take the intersection of all divisions of \( A_1 \) and replace \( A_1 \) to its subclasses generated by intersection. Then we do the same replacement for all \( A_2, \ldots, A_l \) and get the frame \( F'' = (W'', R'', R''_D) \).

Then we take the transitive closure of \( R'' \) (denote \( G = (W'', S, R''_D) \), where \( S \) is the transitive closure of \( R'' \)).

We call \( A \) parent class of \( a \) if \( a \subseteq A \), where \( A \in W' \) and \( a \in W'' \).

**Claim 6.1.** \( G = (W'', S, R''_D) \) is an SADT\(_0\) frame.

**Proof.** We check only \( AT_0 \) axiom. Assume opposite, i.e. there are mutually reachable by the first relation and irreflexive by the second relation two classes \( a \) and \( b \) in \( G \). Then, there must be a path \( a = x_1x_2\ldots x_n = b \) (or \( b = x_1x_2\ldots x_n = a \)), where \( x_iR''x_{i+1} \), for \( i = 1, \ldots, n - 1 \), (by the lemma 3.4) satisfying the following conditions:

- \( n > 2 \)
- no classes are repeated,
- there are no classes that have a point that sees a point from right neighbor class and is visible from a point of left neighbor class as in Fig.4.
Consider parent classes of \( a = x_1 \ldots x_n = b \). Since we didn’t devide classes irreflexive by the second relation, the parent classes of \( a = x_1, x_2 \ldots x_{n-1} x_n = b \) can be represented as \( a = x_1, X_2 \ldots X_{n-1} x_n = b \), where \( X_i \) is the parent class of \( x_i \).

Now we can delete all circles from (there can be many ways to do this, we do it in some way) \( a = x_1 X_2 \ldots X_{n-1} x_n = b \) and get a simple (a path without repeated classes) path from \( a \) to \( b \). As we don’t have any irreflexive by the second relation class between \( a \) and \( b \), the simple path will have the form \( a = x_1 X_{n_1} = X_2 \ldots, X_{n_{n'}}, x_n = b \) (denote this path as \( L \)), where \( 1, n_1, n_2, n_{n'}, n \) is subsequence of \( 1, 2, \ldots, n \).

We have already considered \( L \) and devided all classes of this path into 4 parts.

\[ a \]

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ N \]

\[ N_1 \]

\[ N_2 \]

\[ x_2 \]

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \text{Fig. 4} \]

It’s obvious that \( x_2 \subseteq N \) or \( x_2 \subseteq N_1 \) and all the points of \( N \) and \( N_1 \) are visible from \( a \). So, \( x_2 \) have a point that sees a point from right neighbor class and is visible from \( a \). This contradiction shows that, \( aSb \Rightarrow aRb \). Since \( F \) is an \( S4DT_0 \)-cone we finish the proof.

\[ \square \]

Then using the Lemma 1.1 we get the end of the proof.

\[ \square \]

7. Completeness of \( S4DT_0 \) with respect to \( T_0 \)-spaces

**Definition 7.1.** Let \( F = (W, R) \) be an \( S4 \)-frame, then the set \( C(x) = R(x) \cap R^{-1}(x) \) for some \( x \in W \) is called a cluster.

**Definition 7.2.** Let \( A_1, A_2, \ldots, A_n \) be sets of sets. We define \( \bigcup_{i=1}^{n} A_i = \{ a_1 \cup \cdots \cup a_n | a_i \in A_i \} \).
Theorem 7.1. The logic $\textbf{S4DT}_0$ is complete with respect to topological $T_0$-spaces.

Proof. By the lemma 3.4 each $R$-cluster in an $\textbf{S4DT}_0$-frame contains no more than one selected point. We know that the logic $\textbf{S4DT}_0$ has finite model property, in other words, there is a class $Q$ of finite $\textbf{S4DT}_0$ cones whose logic is $\textbf{S4DT}_0$. For each $\textbf{S4DT}_0$ cone $F \in Q$, we construct a $T_0$-space and a p-morphism from the space to $\text{Top}_m(F)$. Consider the following 3 cases:

I. Assume that the cone is a cluster without $R_D$-irreflexive points. As a domain of the space $X$, we take a countable set of points $X = \{x_1, \ldots, x_n, \ldots\}$. We define a topology on $X$ as $T = \{U_n \mid n \in \mathbb{N}\} \cup \{\emptyset\}$, where $U_n = \{x_m \mid m \geq n\}$. Let us verify that $X$ is indeed a topological space:
1. $X, \emptyset \in T$, because $U_1 = X$;
2. $\bigcup_{l \in I} U_k = U_{\min I}, \forall I \subseteq \mathbb{N}$;
3. $\bigcap_{k \in I} U_k = U_{\max I}, I$ is a finite subset of $\mathbb{N}$.

Let $W = \{w_1, w_2, \ldots, w_m\}$. We define the map as
\[ f(x_{mk+i}) = w_i, \]
where $m$ is the cardinality of $W$, $i \in \{1, \ldots, m\}$ and $k$ ranges over all natural numbers.

Function $f$ defined above is a p-morphism, since $f$ is surjective by the construction, the image of any open set in $T$ is either the empty set or $W$, the preimages of the only open sets $\emptyset$ and $W$ are either the empty set or $X$. The set of selected points of the cone is empty.

II. Let the cone be a cluster with one $R_D$-irreflexive point. As a domain of the space $X$, we take $X = \{x_1, x_2, \ldots, x_n, \ldots\} \cup \{\infty\}$. We define a topology on $X$ as $T = \{U'_n \mid n \in \mathbb{N}\} \cup \{\emptyset\}$, where $U'_n = \{x_m \mid m \geq n\} \cup \{\infty\}$. Let us verify that $X$ is indeed a topological space:
1. $X, \emptyset \in T$, because $U'_1 = X$;
2. $\bigcup_{l \in I} U'_k = U'_{\min I}, \forall I \subseteq \mathbb{N}$;
3. $\bigcap_{k \in I} U'_k = U'_{\max I}, I$ is a finite subset of $\mathbb{N}$.

Let $W = \{w_0, w_1, \ldots, w_m\}$, where $w_0$ is $R_D$-irreflexive point. We define the map as
\[ f(x_{mk+i}) = w_i, i = 1, 2, \ldots, m, \]
\[ f(\infty) = w_0. \]

where $m$ is the cardinality of $W \setminus \{m_0\}$ and $k$ ranges over all natural numbers.

Function $f$ defined above is a p-morphism, since $f$ is surjective by the construction, the image of any open set in $T$ is either an empty set or $W$, the preimages of the only open sets $\emptyset$ and $W$ are either an empty set or $X$ and $f^{-1}(w_0) = \{\infty\}$ is irreflexive singleton.

III. Let us consider a general case. Let frame $F = (W, R, R_D)$ be a cone. The preorder $R$ induces an equivalence relation on $F$:
\[ x \sim y \iff xRy \land yRx. \]
It identifies points belonging to the same cluster. Let $F' = (W', R') = (W, R)/\sim$ is called the skeleton of $F$. Let $\text{Top}(F') = (Y, T_Y)$.

Now we can construct the required topological space and define required p-morphism. Let us construct disjoint topological spaces for each cluster according to cases I and II i.e., $X_i = (X_i, T_i)$ and the p-morphisms $f_i$ in the case of clusters. The domain of the space is $X = \bigcup_{i \in I} X_i$, where $I$ is the set of all clusters, $X_i$ is the domain of the corresponding space. We define the topology as $T_X = \{\emptyset\} \cup \bigcup_{U \in T_Y} O_U$, where $O_U = \bigcup_{a \in U}(T_a \setminus \{\emptyset\})$.

Each element $V$ of $T_X$ can be represented as $V = \bigcup_{a \in U} U_a$, where $U$ is an open set in $\text{Top}(F')$ that corresponds to $V$ and $U_a$ is a nonempty open set from the topological space $X_a$, corresponding to the cluster $a$.

Claim 7.1. $T_X$ is a topology.

Proof. That $\emptyset, X \in T_X$ is obvious.

Suppose $I$ is the set of all clusters and $\{U_j : j \in J\} \subset T_X$. Each open set $U_j$, where $j \in J$ corresponds to $V_j$, which is an open set in $T_Y$. Then, $\bigcup_{j \in J} V_j = V$ is open. Then $\bigcup_{j \in J} U_j = \bigcup_{a_k \in V} \bigcup_{U_j \in J}(X_{a_k} \cap U_j)$, where $X_{a_k}$ is the domain of the space that corresponds to $a_k \in V$. For each $a_k \in V$ the set $\bigcup_{U_j \in J}(X_{a_k} \cap U_j)$ is a nonempty open set in $T_{a_k}$. In the end, from the definition of open sets in $T_X$, we conclude that $\bigcup_{j \in J} U_j$ is open in $T_X$.

Then, assume $U'$ and $U''$ are open in $T_X$ and each $U'$ and $U''$ corresponds to $V'$ and $V''$, which are open in $T_Y$. Then, $U' \cap U'' = \bigcup_{a_j \in V} ((U' \cap X_{a_j}) \cap (U'' \cap X_{a_j}))$ is open, where $V = V' \cap V''$ and $X_{a_j}$ is the domain of the space that corresponds to $a_j \in V$.

Claim 7.2. $X$ is a $T_0$ space.

Proof. Consider two points $x \neq y$. If their image under the map $f$ falls into the same cluster $a$, then there is a space $(X_a, T_a)$ corresponding to this cluster which contains these points. Since this space is a $T_0$ space, then there exists a $U_a \in T_a$ that contains only one of these points. Then we take $\bigcup_{y \neq a, k \in Y} X_y \cup U_a$.

In the case when the points $x$ and $y$ lie on different spaces $X_a = (X_a, T_a)$ and $X_b = (X_b, T_b)$ correspondingly, we consider points $c_a, c_b \in F'$ which correspond to $X_a$ and $X_b$. Since $R'$ is partial order, then $\neg (c_a R' c_b)$ or $\neg (c_b R' c_a)$. In the first case $\bigcup_{c \in R'(c_a)} X_c$ is an open set, that contains $x$ but doesn’t contain $y$. The second case is treated similarly.

We define $f$ as the union of the maps $f_i$.

Claim 7.3. $f : X \rightarrow \text{Top}_D(F)$.

Proof. Surjectivity of $f$ follows from surjectivity of each $f_i$.

Suppose $V$ is open in $\text{Top}_D(F)$. $V$ corresponds to an open set $U$ in $\text{Top}(F')$ then $f^{-1}(V) = \bigcup_{a \in U} X_a$ that is open in $X$. 

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Assume \( U \) is open in \( X \) and corresponds to an open set \( V \) in \( \text{Top}(F') \) (i.e. \( U = \bigcup_{a \in V} U'_a \), where \( U'_a \) is open in \( X \)). Then, \( f(U) = \bigcup_{a \in V} f(U'_a) = \bigcup_{a \in V} f_a(U'_a) \), where \( f_a(U'_a) \) is cluster. Then \( R'(V) \subseteq V \Rightarrow R(U) \subseteq U \), i.e., \( U \) is open in \( \text{Top}_D(F) \).

So, we have constructed a topological space for each cone in \( Q \), and then a corresponding p-morphism. Further, by the lemma 4.1 and by the theorems 2.1, 2.2 and 5.1 we obtain the assertion of the theorem.

8. References

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