Almost-commuting matrices are almost jointly diagonalizable

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Abstract

We study the relation between approximate joint diagonalization of self-adjoint matrices and the norm of their commutator, and show that almost commuting self-adjoint matrices are almost jointly diagonalizable by a unitary matrix.

1 Introduction

The study of almost commuting matrices has been of interest in theoretical mathematics and physics communities, with the main question: are almost commuting matrices close to matrices that exactly commute? This question was answered positively for self-adjoint (Hermitian) matrices by Lin [14], and studied for additional different cases and settings [1, 18, 19, 11, 9, 15, 10].

In this paper, we study the relation between commutativity and joint diagonalizability of matrices: while it is well-known that commuting matrices are jointly diagonalizable, to the best of our knowledge, no results exist for almost-commuting matrices. Our result is that almost commuting self-adjoint matrices are almost jointly diagonalizable by a unitary matrix, and vice versa, in a sense that will be explained later.
Besides theoretical interest, this result has practical applications given the recent use of simultaneous approximate diagonalization of matrices in signal processing [7, 5, 6], machine learning [8], and computer graphics [13]. In particular, Kovnatsky et al. [13] used joint diagonalizability of Laplacian matrices as a criterion of similarity between 3D shapes (isometric shapes have jointly diagonalizable Laplacians). Since the joint diagonalization procedure is computationally expensive, the easily computable norm of the commutator can be used instead; our result justifies this use.

2 Background

Let \(A, B\) be two \(n \times n\) complex matrices. We denote by

\[
\|A\|_F = \left(\sum_{ij} |a_{ij}|^2\right)^{1/2} = \left(\text{tr}(A^*A)\right)^{1/2} ;
\]

\[
\|A\|_2 = \max_{x \in \mathbb{R}^n : \|x\|_2 = 1} \|Ax\|_2 = (\lambda_{\text{max}}(A^*A))^{1/2},
\]

the Frobenius and the operator norm (induced by the Euclidean vector norm) of \(A\), respectively. Here \(A^*\) is the adjoint (conjugate transpose) of \(A\).

We say that \(A, B\) are jointly diagonalizable if there exists a unitary matrix \(U\) such that \(U^*AU = \Lambda_A\) and \(U^*BU = \Lambda_B\) are diagonal. In general, two matrices \(A, B\) are not necessarily jointly diagonalizable, however, we can approximately diagonalize them by minimizing

\[
\min_U J(A, B, U) \text{ s.t. } U^*U = I,
\]

where

\[
J(A, B, U) = \text{off}(U^*AU) + \text{off}(U^*BU),
\]

and \(\text{off}(A) = \sum_{i \neq j} |a_{ij}|^2\) is the sum of the squared absolute values of the off-diagonal elements. In the following, we denote \(J(A, B) = \min_{U^*U=I} J(A, B, U)\). Numerically, this optimization problem can be solved by a Jacobi-type iteration, referred to as the JADE algorithm [4, 6].

Furthermore, we say that \(A\) and \(B\) commute if \(AB = BA\), and call \([A, B] = AB - BA\) their commutator. It is well-known that commuting self-adjoint matrices are jointly diagonalizable [12], which can be expressed as

\[
\|[A, B]\|_F = 0 \text{ iff } J(A, B) = 0.
\]

We are interested in extending this relation for the case \(\|[A, B]\|_F > 0\) (respectively, \(J(A, B) > 0\)).
The main result of this paper is that if \( \| [A, B] \|_F \) is sufficiently small, then \( J(A, B) \) is also small, and vice versa, i.e., almost commuting matrices are almost jointly diagonalizable. We can state this as the following

\textbf{Theorem 2.1 (main theorem).} There exist functions \( \epsilon_1(x), \epsilon_2(x) \) satisfying \( \lim_{x \to 0} \epsilon_i(x) = 0 \), \( i = 1, 2 \), such that for any two self-adjoint \( n \times n \) matrices \( A, B \) with \( \| A \|_F = \| B \|_F = 1 \),

\[ \epsilon_1(\| [A, B] \|_F) \leq J(A, B) \leq n \epsilon_2(\| [A, B] \|_F). \]

The lower bound is discussed in Section 3. We show that this bound is independent of \( n \) and is tight. The upper bound is discussed in Section 4. Besides showing the existence of the bounds, we also state them explicitly.

\section{Lower bound}

\textbf{Theorem 3.1 (lower bound).} Let \( A, B \) be self-adjoint matrices such that \( \| A \|_F = \| B \|_F = 1 \). Then,

\[ \frac{1}{4} \| [A, B] \|_F^2 \leq J(A, B). \]

\textit{Proof.} Let us denote the minimizer \( V = \arg\min_{U} J(A, B, U) \), and decompose

\[ V^* AV = D_A + X; \]
\[ V^* BV = D_B + Y, \]

Here \( D_A, D_B \) are diagonal matrices, and \( X = V^* AV - D_A, Y = V^* BV - D_B \) have zeroes on their diagonal. This implies that \( J(A, B) = \| X \|_F^2 + \| Y \|_F^2 \). Since \( \| V^* AV \|_F^2 = \| D_A \|_F^2 + \| X \|_F^2 \) and \( \| V^* BV \|_F^2 = \| D_B \|_F^2 + \| Y \|_F^2 \), and using the invariance of the Frobenius norm to a unitary transformation, we get

\[ \| D_A \|_F^2 = \| V^* AV \|_F^2 - \| X \|_F^2 = \| A \|_F^2 - \| X \|_F^2 \leq \| A \|_F^2 \leq 1; \]

in the same way, we establish that \( \| D_B \|_F^2 \leq 1 \).

Rewriting (1) as \( A = VD_A V^* + VXV^* \) and \( B = VD_B V^* + VYV^* \), we get

\[ AB = VD_A V^* VD_B V^* + VD_A V^* VXV^* + VXV^* VD_B V^* + VXV^* VYV^* \]
\[ = VD_A D_B V^* + VD_A YV^* + VX D_B V^* + VX Y V^* \]
and
\[ AB = VD_BD_AV^* + VD_B XV^* + VYD_AV^* + VYXV^*. \]

Thus, we can express
\[ [A, B] = AB - BA = V([D_A, D_B] + [D_A, Y] + [X, D_B] + [X, Y])V^*; \]
since \( D_A, D_B \) are diagonal, \([D_A, D_B] = 0\), and we have
\[ [A, B] = V([D_A, Y] + [X, D_B] + [X, Y])V^* = V([D_A + X, Y] + [X, D_B])V^*, \]
and finally, by the triangle inequality and the invariance of \( \| \cdot \| \) with respect to unitary transformations
\[ \| [A, B] \|_F \leq \| [D_A + X, Y] \|_F + \| [X, D_B] \|_F. \]

Next, we use the bound of of Böttcher and Wenzel\footnote{This bound was conjectured by Böttcher and Wenzel \cite{BottcherWenzel2} for real square matrices, and proved later for different settings in \cite{BottcherWenzel1,BottcherWenzel3,BottcherWenzel16,BottcherWenzel17}.} \[ \| [A, B] \|_F^2 \leq 2 \| A \|_F \| B \|_F \]
together with (1) and (2) to get
\[ \| [D_A + X, Y] \|_F \leq \sqrt{2} \| D_A + X \|_F \| Y \|_F = \sqrt{2} \| A \|_F \| Y \|_F \leq \sqrt{2} \| Y \|_F; \]
\[ \| [X, D_B] \|_F \leq \sqrt{2} \| D_B \|_F \| X \|_F \leq \sqrt{2} \| X \|_F. \]

This implies
\[ \| [A, B] \|_F \leq \| [D_A + X, Y] \|_F + \| [X, D_B] \|_F \leq \sqrt{2} (\| X \|_F + \| Y \|_F) \]
\[ \leq \sqrt{2} (2 \| X \|_F^2 + 2 \| Y \|_F^2)^{1/2} = 2 J^{1/2}(A, B), \]
which proves the theorem. \( \Box \)

**Remark 3.2.** The bound is tight, which can be seen by considering the 2 \( \times \) 2 matrices
\[ A_2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -0.5 & -0.5 + \epsilon \\ -0.5 + \epsilon & 0.5 \end{pmatrix}, \]
\[ B_2 := \hat{B}_2/\| B_2 \| \text{ for } \epsilon \to 0. \] This example extends to any dimension \( n > 2 \) by defining \( n \times n \) matrices
\[ A_n = \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_n = \begin{pmatrix} B_2 & 0 \\ 0 & 0 \end{pmatrix}. \]
Figure 1: Visualization of the bounds for 100 real symmetric $n \times n$ matrices drawn uniformly on the unit sphere for different values of $n$. Lower bound is shown in black line.

4 Upper bound

Theorem 4.1. There exists a function $\hat{\epsilon}(\delta)$ satisfying $\lim_{\delta \to 0} \hat{\epsilon}(\delta) = 0$ with the following property: If $A, B$ are two self-adjoint $n \times n$ matrices satisfying $\|A\|_2, \|B\|_2 \leq 1$, and $\|[A,B]\|_2 \leq \delta$, then

$$J(A, B) \leq n\hat{\epsilon}(\delta)$$

In the proof of Theorem 4.1, we will use the following two auxiliary results. The first result is Huaxin Lin’s theorem, asserting that almost commuting matrices are close to commuting matrices:

Theorem 4.2 (Lin 1995). There exists a function $\epsilon(\delta)$ satisfying $\lim_{\delta \to 0} \epsilon(\delta) = 0$ with the following property: If $A, B$ are two self-adjoint $n \times n$ matrices satisfying $\|A\|_2, \|B\|_2 \leq 1$, and $\|[A,B]\|_2 \leq \delta$, then there exists a pair $A', B'$ of commuting matrices satisfying $\|A - A'\|_2 \leq \epsilon(\delta)$ and $\|B - B'\|_2 \leq \epsilon(\delta)$.

For a proof for the complex Hermitian case, we refer the reader to [14, 19]. The first proof for the real case of symmetric matrices was given by Loring
and Sørensen [15]. The second result is the following property of the function $J$:

**Lemma 4.3.** Let $A, B, C, D$ be self-adjoint $n \times n$ matrices, and let $U$ denote a $n \times n$ unitary matrix. Then,

$$|J(A, B, U) - J(C, D, U)| \leq \|A + C\|_F \|A - C\|_F + \|B + D\|_F \|B - D\|_F.$$

**Proof.** For notational convenience, let us define $J(A, U) = \text{off}(U^*AU)$, such that $J(A, B, U) = J(A, U) + J(B, U)$. We can also express

$$J(A, U) = \text{tr}((M \circ (U^*AU))\circ(U^*AU)),$$

where $M$ is a matrix with elements $m_{ij} = 1 - \delta_{ij}$ and $\circ$ denotes the Hadamard (element-wise) matrix product. Using the relation $\text{tr}((X + Y)^*(X - Y)) = \text{tr}(X^*X - Y^*Y)$, we have

$$|J(A, U) - J(C, U)| =$$

$$= |\text{tr}((M \circ (U^*AU))\circ(U^*AU)) - \text{tr}((M \circ (U^*CU))\circ(U^*CU))|$$

$$= |\text{tr}((M \circ (U^*AU) + M \circ (U^*CU))\circ(U^*AU) - M \circ (U^*CU))|$$

$$= |\text{tr}((M \circ (U^*(A + C)U))\circ(U^*(A - C)U))|.$$

Employing the Cauchy-Schwartz inequality $|\text{tr}(X^*Y)| \leq \|X\|_F \|Y\|_F$, we get

$$|J(A, U) - J(C, U)| = |\text{tr}((M \circ (U^*(A + C)U))\circ(U^*(A - C)U))|$$

$$\leq \|M \circ (U^*(A + C)U)\|_F \|M \circ (U^*(A - C)U)\|_F$$

$$\leq \|U^*(A + C)U\|_F \|U^*(A - C)U\|_F$$

$$= \|A + C\|_F \|A - C\|_F.$$

By the same argument, $|J(B, U) - J(D, U)| \leq \|B + D\|_F \|B - D\|_F$. Finally,

$$|J(A, B, U) - J(C, D, U)| = |J(A, U) + J(C, U) - J(B, U) - J(D, U)|$$

$$\leq |J(A, U) - J(C, U)| + |J(B, U) - J(D, U)|$$

$$\leq \|A + C\|_F \|A - C\|_F + \|B + D\|_F \|B - D\|_F,$$

which completes the proof of the lemma. \qed

We now state the proof of our upper bound:
Proof of Theorem 4.1. Let \( \| [A, B] \|_F \leq \delta \) which implies \( \| [A, B] \|_2 \leq \delta \), and \( \| A \|_2 \leq \| A \|_F \leq 1, \| B \|_2 \leq \| B \|_F \leq 1 \). By Lin’s theorem, there are commuting matrices \( A', B' \) such that \( \| A - A' \|_F \leq \sqrt{n} \| A - A' \|_2 \leq \sqrt{n} \epsilon(\delta) \).

Since \( A', B' \) commute, they are jointly diagonalizable, implying that \( J(A', B') = 0 \), and that there exists a common diagonalizing matrix \( W' = \arg\min_{U} U^{*} U = I \) such that \( J(A', B', U) = 0 \). Applying Lemma 4.3, we get

\[
J(A, B) = J(A, B) - J(A', B') \leq J(A, B, W') - J(A', B', W')
\]

\[
\leq \| A + A' \|_F \| A - A' \|_F + \| B + B' \|_F \| B - B' \|_F
\]

\[
\leq \| A + A' \|_F \| A - A' \|_F + \| B + B' \|_F \| B - B' \|_F
\]

\[
\leq (2 + \sqrt{n} \epsilon(\delta)) \| A - A' \|_F + (2 + \sqrt{n} \epsilon(\delta)) \| B - B' \|_F
\]

\[
\leq 2(2 + \sqrt{n} \epsilon(\delta)) \sqrt{n} \epsilon(\delta).
\]

Now \( 2(2 + \sqrt{n} \epsilon(\delta)) \sqrt{n} \epsilon(\delta) \leq 2n(2/\sqrt{n} + \epsilon(\delta)) \epsilon(\delta) \leq 2n(\sqrt{2} + \epsilon(\delta)) \epsilon(\delta) = n \hat{\epsilon}(\delta) \) where we defined \( \hat{\epsilon}(\delta) = 2(\sqrt{2} + \epsilon(\delta)) \epsilon(\delta) \), satisfying \( \lim_{\delta \to 0} \hat{\epsilon}(\delta) = 0 \) which finishes the proof of the theorem.

Remark 4.4. The drawback of our Theorem 4.1 is that it does not provide an explicit bound on \( J(A, B) \) in terms of \( \| A, B \|_F \), but rather proves asymptotic behavior allowing to conclude that if two matrices almost commute, they are also almost jointly diagonalizable. In order to obtain an explicit bound, one can resort to different, more ‘constructive’ alternatives to Lin’s theorem:

1. Hastings [11] showed that \( \epsilon(\delta) = E(\delta^{-1}) \delta^{1/5} \), where \( E(x) \) is a function independent on \( n \) that grows slower than any power of \( x \), without, however, specifying the function \( E \) explicitly.

2. There are different results [18, 10, 9], which, under the assumptions of Theorem 4.1, allow to calculate positive constants \( c > 0, \frac{1}{2} \leq p, q \leq 1 \) such that if \( \| [A, B] \|_F \leq \delta \), then

\[
\| A - A' \|_F, \| B - B' \|_F \leq cn^{p} \delta^{q}.
\]

By means of the arguments used for the proof of Theorem 4.1, together with the Böttcher-Wenzel bound \( \delta \leq \sqrt{2} \) [2] and the fact that \( \frac{2}{n^p} \leq \sqrt{2} \) for
\[ \frac{1}{2} \leq p \leq 1, n \geq 2, \] this leads to the bound

\[
J(A, B) \leq 2n^{2p}(\frac{2}{n^p} + c\delta^q)c\delta^q
\leq 2n^{2p}(\sqrt{2} + c\sqrt{2})c\delta^q
\leq Cn^{2p}\|\lbrack A, B\rbrack\|_F^q
\]

with \( C = 2\sqrt{2}(c + 1)c \). For example, Pearcy and Shields \cite{18} obtained
\( c = \frac{1}{\sqrt{2}}, p = \frac{1}{4}, q = \frac{1}{2} \), Glebsky \cite{10} \( c = 12, p = \frac{5}{12}, q = \frac{1}{6} \), and Filonov and Kachkovskiy \cite{9} \( c = 2, p = \frac{3}{8}, q = \frac{1}{4} \).

Remark 4.5. We observed that none of the upper bounds derived by these theorems lead to realistic values which are useful for numerical computations, so we do not discuss these results here in detail, and we leave this subject for further research.

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Peary and Shields use the operator norm \( \| \cdot \|_2 \) in the derivation of their bound, so the relation \( \| \cdot \|_F \leq \sqrt{n} \| \cdot \|_2 \) has to be taken into account.

In \cite{9, 10} instead of the Frobenius norm the authors use the normalized Frobenius norm \( \| \cdot \|_F = \frac{1}{\sqrt{n}} \| \cdot \|_F \), so the assumptions and the assertion have to be adjusted accordingly.
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