The thermodynamic quantities of a black hole with an $f(R)$ global monopole

Jingyun Man      Hongbo Cheng*
Department of Physics, East China University of Science and Technology, Shanghai 200237, China
The Shanghai Key Laboratory of Astrophysics, Shanghai 200234, China

Abstract

The thermodynamic quantities such as the local temperature, heat capacity, off-shell free energy and the stability of a black hole involving a global monopole within or outside the $f(R)$ gravity are examined. We compare the two classes of results to show the influence from the generalization of the general relativity. It is found that the $f(R)$ theory will modify the thermodynamic properties of black holes, but the shapes of curves for thermodynamic quantities with respect to the horizon are similar to the results within the frame of general relativity. In both cases there will exist a small black hole which will decay and a large stable black hole in the case that the temperatures are higher than their own critical temperature.

PACS number(s): 04.70Bw, 14.80.Hv
Keywords: Black hole, Global monopole, $f(R)$ gravity

*E-mail address: hbcheng@ecust.edu.cn
I. Introduction

Recently more contributions were paid to the thermodynamics of various kinds of black holes. More than thirty years ago, Bekenstein pointed out that the entropy of a black hole is proportional to its surface area [1-3]. Hawking also discussed the particle creation around a black hole to show that the black hole has a thermal radiation with the temperature subject to its surface gravity [4]. The issues about phase transitions of black holes in the frame of semiclassical gravity were listed in Ref. [5]. Further the thermodynamic properties of modified Schwarzschild black holes have been investigated [6, 7]. The thermodynamic behaviours including phase transition in Born-Infeld-anti-de Sitter black holes were explored by means of various ways [8, 9]. The phase transition of the quantum-corrected Schwarzschild black hole was discussed, which fosters the research on the quantum-mechanical aspects of thermodynamic behaviours [10].

In the process of the vacuum phase transition in the early Universe the topological defects such as domain walls, cosmic strings and monopoles were generated from the breakdown of local or global gauge symmetries [11, 12]. Among these topological defects, a global monopole as a spherical symmetric topological defect occurred in the phase transition of a system composed by a self-coupling triplet of scalar field whose original global O(3) symmetry is spontaneously broken to U(1). It has been shown that the metric outside a monopole has a deficit solid angle [13]. We researched on the strong gravitational lensing for a massive source with a global monopole and find that the deficit angle associated with the monopole affect the lensing properties [14]. H. A. Buchdahl proposed a modified gravity theory named as $f(R)$ gravity to explain the accelerated expansion of the Universe instead of adding unknown forms of dark energy or dark matter [15-18]. The metric outside a gravitational object involving a global monopole in the context of $f(R)$ gravity theory has been studied [19]. Further the classical motion of a massive test particle around the gravitational source with an $f(R)$ global monopole is probed [20]. We also examine the gravitational lensing for the same object in the strong field limit [21].

Here we plan to investigate the thermodynamics of a static and spherically symmetric black hole swallowing a global monopole or an $f(R)$ global monopole. It is significant to understand the $f(R)$ theory in a new direction. We wish to find the influences from the modified gravity on the thermal properties of black holes. First of all we introduce a black hole containing a global monopole in the context of $f(R)$ gravity theory and certainly the black hole metric will recover to be one of the metrics of black hole with a deficit solid angle. We show the dependence of the horizon on the mass and parameters describing the monopole and the modification from $f(R)$ theory. We exhibit the thermodynamic characteristics due to the parameters of the black hole. The thermodynamic stability will also be checked. We are going to discuss the results in the end.

II. The thermodynamics of black holes involving an $f(R)$ global monopole

We adopt the line element,
\[ ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \] (1)

which is static and spherical. In the \( f(R) \) gravity theory, the action introduced is,

\[ I = \frac{1}{2\kappa} \int d^4x \sqrt{-g}f(R) + I_m \] (2)

where \( f(R) \) is an analytical function of Ricci scalar \( R \) and \( \kappa = 8\pi G \), \( G \) is the Newton constant, \( g \) is the determinant of metric and \( I_m \) is the action of matter fields which can be denoted as,

\[ I_m = \int d^4x \sqrt{-g}\mathcal{L} \] (3)

Here the Lagrangian for global monopole is [13],

\[ \mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - \frac{1}{4}\lambda(\phi^a \phi^a - \eta^2)^2 \] (4)

where \( \lambda \) and \( \eta \) are parameters. The ansatz for the triplet of field configuration showing a monopole is \( \phi^a = \eta h(r)\frac{x^a}{r} \) with \( x^a x^a = r^2 \), where \( a = 1, 2, 3 \). \( h(r) \) is a dimensionless function to be determined by the equation of motion. This model has a global O(3) symmetry, which is spontaneously broken to U(1). The field equation reads,

\[ F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = \kappa T_{\mu\nu} \] (5)

with \( F(R) = \frac{df(R)}{dR} \) and \( T_{\mu\nu} \) is the minimally coupled energy-momentum tensor. The field equation (5) was solved under the weak field approximation that assumes the components of metric tensor like \( A(r) = 1 + a(r) \) and \( B(r) = 1 + b(r) \) with \( |a(r)| \) and \( |b(r)| \) being smaller than unity [19]. Here the modified theory of gravity corresponds to a small correction on the general relativity like \( F(R(r)) = 1 + \psi(r) \) with \( \psi(r) \ll 1 \). It is clear that \( F(R) = 1 \) is equivalent to the conventional general relativity. Further the modification can be taken as the simplest analytical function of the radial coordinate \( \psi(r) = \psi_0 r \). In this case the factor \( \psi_0 \) reflects the deviation of standard general relativity. The external metric of the black hole with a global monopole is found finally [19, 20],

\[ A = B^{-1} = 1 - 8\pi G\eta^2 - \frac{2GM}{r} - \psi_0 r \] (6)

where \( M \) is the mass parameter. It should be pointed out that the parameter \( \eta \) is of the order \( 10^{16} \text{GeV} \) for a typical grand unified theory, which means \( 8\pi G\eta^2 \approx 10^{-5} \). If we choose \( \psi_0 = 0 \) excluding the modification from \( f(R) \) theory, the metric (6) will recover to be the result by M. Barriola et.al. [13] like,

\[ ds^2 = (1 - 8\pi G\eta^2 - \frac{2GM}{r})dt^2 - \frac{dr^2}{1 - 8\pi G\eta^2 - \frac{2GM}{r}} - r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \] (7)

The function \( A(r) \) is plotted in Figure 1 for comparison of two metrics for a global monopole and \( f(R) \) monopole black hole. According to the behavior of \( f(R) \) curve, it is clear that the black
hole has two horizons, one is the inner horizon which is supposed to be the event horizon \( r_H \), the other one is the outer horizon. The event horizons of two metrics seem to be the same from Figure 1, because the given parameters \( 8\pi G\eta^2 \) is very small, but the analytic expression of \( r_H \) are quite distinct. Solving the equation \( A(r_H) = 0 \), the event horizon of metric (6) is located at,

\[
r_H = \frac{(1 - 8\pi G\eta^2) - \sqrt{(1 - 8\pi G\eta^2)^2 - 8\psi_0 GM}}{2\psi_0}.
\]  

(8)

It also gives the relation between the mass parameter \( GM \) and the event horizon \( r_H \),

\[
GM = \frac{1}{2} r_H (1 - 8\pi G\eta^2 - \psi_0 r_H)
\]  

(9)

for \( f(R) \) monopole metric. It is seen from Figure 2 that there is a maximum \( GM_0 = \frac{(1 - 8\pi G\eta^2)^2}{8\psi_0} \) at \( r_{H_0} = \frac{1 - 8\pi G\eta^2}{2\psi_0} \) where the inner and outer horizon meets. The Hawking temperature can be obtained,

\[
T_{HM} = T_{HM}(\eta, \psi_0)
\]

\[
= \frac{1}{4\pi} (-g^{tt}g^{rr}g'_{tt})|_{r=r_H}
\]

\[
= \frac{1}{4\pi} \left( 1 - 8\pi G\eta^2 - 2\psi_0 \right)
\]  

(10)

where the prime stands for the derivative with respect to the radial coordinate \( r \). The local temperature is given by [22],

\[
T_{loc}^M = \frac{T_{HM}}{\sqrt{A(r)}}
\]

\[
= \frac{1}{4\pi} \left( 1 - 8\pi G\eta^2 \right) \frac{r}{\psi_0 r_H^2 - (1 - 8\pi G\eta^2)r_H + (1 - 8\pi G\eta^2)r - \psi_0 r^2}.
\]  

(11)

Having omitted the influence from the modified gravity, we obtain the local temperature as,

\[
T_{loc} = \frac{1}{4\pi r_H} \sqrt{\frac{(1 - 8\pi G\eta^2)r}{r - r_H}}.
\]  

(12)

The local temperatures for a global monopole and an \( f(R) \) global monopole are shown respectively in the Figure 3. There is a minimal local temperature \( T_c \), which means if the local temperature is below \( T_c \), no black hole exists. When the temperatures for the two kinds of black holes are high enough, there will both exist two black holes, one is small and the other is large. From the calculation of the extrema of local temperature, \( \frac{\partial T_{loc}}{\partial r_H} = 0 \), the minimal local temperature can be obtained as,

\[
T_{c}^M = \frac{\sqrt{T}}{\pi} \frac{\psi_0}{(1 - 8\pi G\eta^2)^{\frac{3}{4}} - (1 - 8\pi G\eta^2 - 2\psi_0 r)^{\frac{3}{4}}}. 
\]  

(13)
The modifying factor $\psi_0$ from $f(R)$ leads the minimum of local temperature lower than the one for global monopole black hole,

$$T_c = \frac{3\sqrt{3}}{8\pi r} \sqrt{1 - 8\pi G\eta^2} \quad (14)$$

for $8\pi G\eta^2 = 10^{-5}$, $r = 10$ and $\psi_0 = 0.02$, then $T_c^M = 0.01836$, $T_c = 0.02067$. The Figure 4 shows that the critical temperature is a decreasing function of the modifying factor $\psi_0$ denoting the influence from $f(R)$ gravity here.

According to Bekenstein’s opinion, the entropy is proportional to the area of event horizon and denoted as [1-3],

$$S = \frac{A_0}{4} = \pi r_H^2 \quad (15)$$

leading $dS = 2\pi r_H dr_H$.

From the first law of thermodynamics $dE_{loc} = T_{loc} dS$, the thermodynamical local energy can be derived as,

$$E^M_{loc} = E_0 + \int_{S_0}^S T^M_{loc} dS$$
$$= r\sqrt{(1 - 8\pi G\eta^2) - \psi_0 r - \sqrt{r(r - r_H)}} \sqrt{(1 - 8\pi G\eta^2) - \psi_0 (r + r_H)} \quad (16)$$

Here $S_0$ represents that $M = 0$. For simplicity $E_0 = 0$. The thermodynamic local energy in the case not belonging to $f(R)$ theory becomes,

$$E_{loc} = \sqrt{(1 - 8\pi G\eta^2)r} (\sqrt{r} - \sqrt{r - r_H}). \quad (17)$$

For the purpose of checking the stability of the black holes, we should discuss their heat capacity,

$$C^M = \left( \frac{\partial E^M_{loc}}{\partial T^M_{loc}} \right)_r$$
$$= 2\pi (r - r_H)[(1 - 8\pi G\eta^2) - 2\psi_0 r_H][(1 - 8\pi G\eta^2) - \psi_0 (r + r_H)]$$
$$\times[(1 - 8\pi G\eta^2)\left(\frac{r_H^2}{r_H^2} - 3\right)\psi_0 + (1 - 8\pi G\eta^2)\frac{2r_H^2 - 2r}{2r_H^2} + 2\psi_0^2 r_H]^{-1}. \quad (18)$$

Similarly we choose the modifying factor $\psi_0 = 0$, the heat capacity of black holes leads,

$$C = \left( \frac{\partial E_{loc}}{\partial T_{loc}} \right)_r$$
$$= 4\pi \frac{r_H^2 (r - r_H)}{3r_H - 2r} \quad (19)$$
According to Eq. (18) and Eq. (19), we compare the heat capacities of these two kinds of black holes in the Figure 5. The shapes of the curves of heat capacities are similar, but can be recognized explicitly. The expression of heat capacity for a Schwarzschild black hole with a global monopole is exactly the same as the one for original Schwarzschild black hole. In Figure 5, the $f(R)$ curve shifts from the conventional one. In every case the relatively smaller horizon leads the heat capacity to be negative and the positive capacity is due to the larger horizon, which meaning that the larger black holes are stable. It should be emphasized that only huge black holes can survive for long time within the frame of $f(R)$ theory. When $r_H > \frac{1-8\pi G \eta^2 - [(1-8\pi G \eta^2)(2\psi_0 r - (1-8\pi G \eta^2))^2]^{3/2}}{2\psi_0}$, the $f(R)$ heat capacity is positive and a large black hole is stable. A small black hole appears unstable for $0 < r_H < \frac{1-8\pi G \eta^2 - [(1-8\pi G \eta^2)(2\psi_0 r - (1-8\pi G \eta^2))^2]^{3/2}}{2\psi_0}$ since the heat capacity is negative. A large Schwarzschild black hole with a monopole is stable when $r_H > 2\frac{\psi}{3}$, while it is unstable on $0 < r_H < 2\frac{\psi}{3}$.

In order to explore the phase transition among the black holes, we should derive their off-shell free energy. The off-shell free energy can be defined as,

$$F_{off}^M = E_{loc}^M - TS$$

where $E_{loc}^M$ is thermodynamic local energy from Eq. (16), $S$ is the entropy of the black hole from Eq. (15) and $T$ is an arbitrary temperature. The off-shell free energy of black holes containing a global monopole governed by $f(R)$ gravity theory can be calculated as,

$$F_{off}^M = r \sqrt{(1 - 8\pi G \eta^2) - \psi_0 r} - \sqrt{r(r - r_H)} \sqrt{(1 - 8\pi G \eta^2) - \psi_0 (r + r_H)} - \pi r_H^2 T.$$  \hspace{1cm} (21)

If we are not going to consider the deviation from $f(R)$ theory, the Eq. (21) will become the expression of the off-shell free energy for the black holes with only a deficit solid angle like,

$$F_{off} = \sqrt{(1 - 8\pi G \eta^2)r(\sqrt{T} - \sqrt{r - r_H})} - \pi r_H^2 T.$$ \hspace{1cm} (22)

In Figure 6 the behaviour of the off-shell free energy of black hole involving a global monopole proposed by Barriola and Vilenkin [13] is shown as a function of the horizon under several temperatures. Having considered extrema of off-shell free energy, $(\frac{\partial F_{off}^M}{\partial r_H})_r = 0$ or $(\frac{\partial F_{off}^M}{\partial r_H})_r = 0$, we give out the critical temperatures which has already been mentioned above. When the temperature is lower than the critical value, no black hole will appear. For the temperature above the critical one the large black holes are stable but the smaller ones are unstable. The dependence of the free energy of black holes with an $f(R)$ global monopole on the horizon is plotted in the Figure 7 when the temperature is chosen to be several values around their own critical temperature. The shapes of the off-shell free energy are similar to those in the Figure 6. When the temperature is lower than critical value, there is no real root from the equation $(\frac{\partial F_{off}^M}{\partial r_H})_r = 0$ or $(\frac{\partial F_{off}^M}{\partial r_H})_r = 0$, therefore no black hole will appear. There are two equal roots $r_{H1} = r_{H2} = \frac{1-8\pi G \eta^2 - [(1-8\pi G \eta^2)(2\psi_0 r - (1-8\pi G \eta^2))^2]^{3/2}}{2\psi_0}$ at critical temperature. When $T > T_c$, two physically meaningful roots exist. An unstable small black hole appears at event horizon $r_H = r_{H1}$, and a stable large black hole appears at $r_H = r_{H2}$. 
III. Discussion

We search for the thermodynamic quantities of the black holes with a global monopole in the context of $f(R)$ gravity theory. We also obtain the thermodynamic quantities excluding the modifications from $f(R)$ theory. We compare the two classes of results to show how the generalized gravity corrects the original thermodynamic quantities such as local temperature, heat capacity, off-shell free energy. It should be pointed out that the $f(R)$-modifications on these quantities are manifest. It is interesting that the shapes of these quantities of black holes controlled by two different gravity theories are similar although the quantities’ expressions are more different from each other. In both cases the small black holes are unstable and the large ones are stable when the temperature is higher than the critical temperature and no black hole can exist as the temperature is sufficiently low. The critical temperature has something to do with the modified factor from $f(R)$ theory. Here we open a new window to explore the $f(R)$ theory revising the Einstein Gravity.

In this paper, in what we pay more attention to is the thermodynamic behaviours of a black hole with an $f(R)$ global monopole, as the stability of the black hole is dependent on them. There are some methods proposed for probing a gravitational source which has a solid deficit angle subject to $f(R)$ global monopole. But one necessary condition is that the black hole has to be stable enough to be observed. Through calculation of the heat capacity, only when the event horizon is larger than \[ \frac{1-8\pi G\eta^2 - [(1-8\pi G\eta^2)(2\psi_0 r-(1-8\pi G\eta^2))^2]^{3/2}}{2\psi_0}, \] the black hole of an $f(R)$ global monopole can be stable. It is the condition for a large and more stable black hole given by extrema of off-shell free energy as well. Moreover, a stable black hole also requires the local temperature higher than the critical temperature $T_c$.

Acknowledgement

This work is supported by NSFC No. 10875043 and is partly supported by the Shanghai Research Foundation No. 07dz22020.
References

[1] J. D. Bekenstein, Lett. Nuovo Cim. 4(1972)737

[2] J. D. Bekenstein, Phys. Rev. D7(1973)2333

[3] J. D. Bekenstein, Phys. Rev. D9(1974)3292

[4] S. W. Hawking, Commun. Math. Phys. 43(1975)199

[5] G. J. Stephens, B. L. Hu, Int. J. Theor. Phys. 40(2001)2183

[6] W. Kim, E. J. Son, M. Yoon, JHEP 0804(2008)042

[7] R. G. Cai, L. M. Cao, N. Ohta, JHEP 1004(2010)082

[8] Y. S. Myung, Y. Kim, Y. Park, Phys. Rev. D78(2008)084002

[9] A. Lala, D. Roychowdhury, Phys. Rev. D86(2012)084027

[10] W. Kim, Y. Kim, arXiv: 1207.5318

[11] T. W. B. Kibble, J. Phys. A9(1976)1387

[12] A. Vilenkin, Phys. Rep. 121(1985)263

[13] M. Barriola, A. Vilenkin, Phys. Rev. Lett. 63(1989)341

[14] H. Cheng, J. Man, Class. Quantum Grav. 28(2011)015001

[15] H. A. Buchdahl, Non-linear Lagrangians and cosmological theory, MNRAS 150(1970)1

[16] S. Nojiri, S. D. Odintsov, Phys. Rev. D68(2003)125312

[17] S. M. Carroll, V. Duvvuri, M. Trodden, M. S. Turner, Phys. Rev. D70(2004)043528

[18] S. Fay, R. Tavakol, S. Tsujikawa, Phys. Rev. D74(2007)063509

[19] T. R. P. Carames, E. R. B. de Mello, M. E. X. Guimaraes, Int. J. Mod. Phys. A (2011)

[20] T. R. P. Carames, E. R. B. de Mello, M. E. X. Guimaraes, arXiv: 1111.1856

[21] J. Man, H. Cheng, arXiv: 1205.4857

[22] R. C. Tolman, Phys. Rev. 35(1930)904
Figure 1: The figure shows the function $A(r)$ of $f(R)$ monopole black hole (solid line) and Schwarzschild black hole with a global monopole (dot line). Here $8\pi G\eta^2 \approx 10^{-5}$, $\psi_0 = 0.02$ and $GM = 1$. 
Figure 2: The figure shows the mass parameter $GM$ for $f(R)$ monopole black hole (solid line) and global monopole black hole (dash line). Here $8\pi G\eta^2 \approx 10^{-5}$, $\psi_0 = 0.02$. 
Figure 3: The solid and dotted curves of the dependence of the local temperatures on the horizon with $8\pi G\eta^2 \approx 10^{-5}$, $r = 10$ and $\psi_0 = 0.02$ for the Schwarzschild black hole with an $f(R)$ global monopole or a global monopole respectively.
Figure 4: The dependence of the critical temperature for $8\pi G \eta^2 \approx 10^{-5}$ and $r = 10$ on the modifying factor $\psi_0$ from $f(R)$ gravity.
Figure 5: The solid and dotted curves correspond to the dependence of the heat capacities on the horizon with $8\pi G \eta^2 \approx 10^{-5}$, $r = 10$ and $\psi_0 = 0.02$ for the Schwarzschild black hole with an $f(R)$ global monopole or a global monopole respectively.
Figure 6: The dotted, solid and dashed curves correspond to the dependence of the off-shell free energy of the Schwarzschild black hole with a global monopole on the horizon with $8\pi G \eta^2 \approx 10^{-5}$ and $r = 10$ for $T = 0.018, 0.021, 0.03$ respectively.
Figure 7: The solid, dot and dashed curves correspond to the dependence of the off-shell free energy of the Schwarzschild black hole with an $f(R)$ global monopole on the horizon with $8\pi G\eta^2 \approx 10^{-5}$, $r = 10$ and $\psi_0 = 0.02$ for $T = 0.018, 0.021, 0.03$ respectively.