Two-sided estimates of minimum-error
distinguishability of mixed quantum states via
generalized Holevo-Curlander bounds

Jon Tyson
Jefferson Lab, Harvard University

November 27, 2008
Journal ref: J. Math. Phys. 50, 032106 (2009)

Abstract
We prove a concise factor-of-two estimate for the failure-rate of optimally dis-
tinguishing an arbitrary ensemble of mixed quantum states, generalizing work
of Holevo [Theor. Probab. Appl. 23, 411 (1978)] and Curlander [Ph.D. The-
sis, MIT, 1979]. A modification of the minimal principle of Concha and Poor
[Proceedings of the 6th International Conference on Quantum Communication,
Measurement, and Computing (Rinton, Princeton, NJ, 2003)] is used to derive
a sub-optimal measurement which has an error rate within a factor of two of
the optimal by construction. This measurement is quadratically weighted, and
has appeared as the first iterate of a sequence of measurements proposed by
Ježek, Reháček, and Fiurášek [Phys. Rev. A 65, 060301]. Unlike the so-called
“pretty good” measurement, it coincides with Holevo’s asymptotically-optimal
measurement in the case of non-equiprobable pure states. A quadratically-
weighted version of the measurement bound by Barnum and Knill [J. Math.
Phys. 43, 2097 (2002)] is proven. Bounds on the distinguishability of syn-
dromes in the sense of Schumacher and Westmoreland [Phys. Rev. A 56,
131 (1997)] appear as a corollary. An appendix relates our bounds to the
trace-Jensen inequality.

*jonetyson@X.Y.Z, where X=post, Y=Harvard, Z=edu
I Introduction

The minimum-error quantum distinguishability problem is of obvious practical importance in the design of optical detectors [1] and of fundamental importance in subject of quantum information [2–4] and quantum computation [5–10]:

If an unknown state $\rho_k$ is randomly chosen from a known ensemble of quantum states, what is the chance that the value of $k$ will be discovered by an optimal measurement?

Although various necessary and sufficient conditions for optimal measurements have been derived [11,17] (see also [18]), and a number of numerical algorithms for computing optimal measurements have been implemented [18,22], it is unlikely that an explicit general solution is forthcoming. A number of works give interesting general upper and/or lower bounds on quantum distinguishability [2,23–29].

The theory of optimal measurements has been generalized in several directions, including to Belavkin and Maslov’s theory of wave discrimination [30] and to the theory of optimal quantum channel reversals, in the sense of average entanglement fidelity [24,31–35]. Success rates of optimal measurements have recently been expressed in terms of the conditional min-entropy of bipartite classical-quantum states in Theorem 1 of [36]. In particular, the problem of finding the conditional min-entropy of an arbitrary bipartite quantum state generalizes the optimal distinguishability problem.

IA Results

Theorem 9 of section IV combines ideas of Holevo [37], Curlander [38], and Concha & Poor [39,41] to give mathematically concise upper and lower distinguishability bounds for arbitrary ensembles of mixed quantum states. Employing an approximate minimal principle, a suboptimal measurement is derived which has a failure rate within a factor of two of the optimal by construction. This measurement is observed to be the first iteration in the sequence of measurements of Ježek, Řeháček, and Fiurášek [20]. In the case of pure states this measurement reduces to Holevo’s asymptotically-optimal measurement [37], which is the quadratically-weighted Belavkin square root measurement [14,15,42].

Theorem 10 of section V combines ideas of Curlander and Holevo to give somewhat tighter distinguishability bounds. Furthermore, a quadratically-weighted version of Barnum and Knill’s measurement bounds [24] are obtained, as are bounds on distinguishability of syndromes in the sense of Schumacher and Westmoreland.

To motivate our considerations in the case of mixed states, section II revisits Holevo and Curlander’s pure-state bounds.

Future directions appear in the final section. The appendix relates our bounds to the trace-Jensen inequality.
II Holevo-Curlander pure state distinguishability bounds

Before attempting to distinguish elements of mixed-state ensembles, it is instructive to revisit a pure-state bound used by Holevo in his proof of the asymptotic optimality Theorem (Theorem \[\text{6}\] below).

**Theorem 1 (Holevo \[\text{37}\])** The ensemble $E_m = \{(\psi_k, p_k)\}_{k=1}^{m}$ of linearly-independent pure states has the following minimum-error distinguishability bound:

$$P_{\text{fail}}^{\text{optimal}} \leq 2\gamma_{\text{Holevo}},$$

where

$$\gamma_{\text{Holevo}} = 1 - \text{Tr} \left( \sum_k p_k^2 |\psi_k\rangle \langle \psi_k| \right),$$

and $P_{\text{fail}}^{\text{optimal}}$ is the failure rate of the optimal measurement.

A two-sided version of Holevo’s bound (1)

$$P_{\text{fail}}^{\text{optimal}} \in [\gamma_{\text{Holevo}}, 2\gamma_{\text{Holevo}}],$$

was proved using different techniques by Curlander \[\text{38}\] under the additional assumption of equiprobability ($p_k = 1/m$). Equation (3) follows without Curlander’s restriction by the following trivial modification of Holevo’s argument.

Holevo restricted attention to orthonormal von Neumann measurement bases $\{e_k\}$ with phases chosen so that

$$\langle e_k, \psi_k \rangle \geq 0.$$  

Instead of minimizing the probability of failure

$$P_{\text{fail}} (\{e_k\}) = \sum_{k=1}^{m} p_k \left( 1 - |\langle e_k, \psi_k \rangle|^2 \right),$$

Holevo considered the tractable approximation

$$C_{\text{Holevo}} (\{e_k\}) = \sum_{k=1}^{m} p_k \|\psi_k - e_k\|^2,$$

which is equation 8 of \[\text{37}\]. Since the phase condition (4) implies that

$$1 - |\langle \psi_k, e_k \rangle|^2 = \frac{\|e_k - \psi_k\|^2}{2} (1 + \langle \psi_k, e_k \rangle) \in [1/2, 1] \times \|e_k - \psi_k\|^2,$$

\[\text{1}\]Rank-1 projective measurements are optimal for distinguishing linearly-independent pure states.\[1, 14, 43, 44\]
one has
\[ P_{\text{fail}} \left( \{e_k\} \right) \in [1/2, 1] \times C_{\text{Holevo}} \left( \{e_k\} \right), \tag{8} \]
where we use the notation \([a, b] \times c = [\alpha c, \beta c]\). The bound \([3]\) follows from minimization of \(C_{\text{Holevo}}\). The minimizer is the (usually sub-optimal) measurement basis\(^2\)
\[ e_k^{\text{Holevo}} = \left( \sum p_k^2 |\psi_k\rangle \langle \psi_k| \right)^{-1/2} p_k |\psi_k\rangle. \tag{9} \]

### III Definitions, background, and notation:

In this section we collect the technical definitions and mathematical background needed for the rest of the paper. Throughout we shall consider an ensemble
\[ \mathcal{E}_m = \{(\rho_k, p_k)\}_{k=1, \ldots, m} \tag{10} \]
of quantum states \(\rho_k\) on a Hilbert space \(\mathcal{H}\) with a-priori probabilities \(p_k\), with \(\sum p_k = 1\) and \(\text{Tr} \rho_k = 1\). One may take \(m = \infty\) without changing our results. For the special case of pure states, \(\rho_k\) will be denoted by \(\rho_k = |\psi_k\rangle \langle \psi_k|\).

**Definition 2** The ensemble \(\mathcal{E}_m\) is **equiprobable** if \(p_k = 1/m\) for all \(k\). The subspace \(\text{Span} (\mathcal{E}_m)\) is the **span** of the ensemble \(\mathcal{E}_m\), i.e. the span of the ranges of the \(p_k\rho_k\). A **positive-operator valued measure (POVM)** (see, for example, p. 74 of \([1]\)) for distinguishing \(\mathcal{E}_m\) is a collection of positive semidefinite operators \(\{M_k\}_{k=1, \ldots, m}\) such that \(\sum M_k = \mathbb{1}_{\mathcal{H}}\) or \(\mathbb{1}_{\text{Span} (\mathcal{E}_m)}\). The probability that the value \(i\) is detected when the POVM is applied to the state \(\rho_j\) is given by \(p_{ij} = \text{Tr} M_i \rho_j\). In particular, the **success rate** for the POVM to distinguish the ensemble \(\mathcal{E}_m\) is given by
\[ P_{\text{succ}} (M_k) = \sum_{k=1}^m p_k \text{Tr} (\rho_k M_k) = 1 - P_{\text{fail}}. \tag{11} \]

The **optimal success rate** is
\[ P_{\text{optimal}} = \sup_{\text{POVMs} \{M_k\}} P_{\text{succ}} (M_k) = 1 - P_{\text{fail}}^{\text{optimal}}. \tag{12} \]

A common POVM is

**Definition 3** The Belavkin-Hausladen-Wootters “pretty good measurement” (PGM)\(^3\) is given by
\[ M_k = \left( \sum_{\ell=1}^m p_{\ell} \rho_{\ell} \right)^{-1/2} p_k \rho_k \left( \sum_{\ell=1}^m p_{\ell} \rho_{\ell} \right)^{-1/2}, \tag{13} \]

\(^2\)The final section of Holevo’s paper contains minor algebra errors. A corrected version of the minimizer \([9]\) appears in \([16]\), which also removes Holevo’s assumption of linear-independence. (This generalization may also be accomplished simply by using Naimark’s theorem, as advocated by Kebo \([16]\).) Holevo’s measurement \([9]\) belonged to the previously-considered class of Belavkin weighted square root measurements. \([14, 15]\), also called “weighted least-squares measurements” \([45]\) and “generalized ‘pretty good’ measurements” \([44]\).

\(^3\)The PGM for non-equiprobable pure states appeared in 1975 as an optimal measurement under conditions of equality along the diagonal of the Graham matrix \([14, 15]\), and reappeared in 1993 as an approximately-optimal measurement \([47, 48]\).
where one defines

$$A^{-1/2} = \sum_{\lambda_j > 0} \lambda_j^{-1/2} |\psi_j\rangle \langle \psi_j|,$$  \hspace{1cm} (14)

for a spectral decomposition $A = \sum \lambda_j |\psi_j\rangle \langle \psi_j|.$

Numerical evidence \cite{20, 21} suggests that the following sequence of measurements converges to the optimal measurement:

**Definition 4** The Ježek-Řeháček-Fiurášek iterative measurements $\{M_k^{(n)}\}_{k=1,\ldots,m}, n \in \mathbb{Z}^+,$ are recursively defined by \cite{20, 21}

$$M_k^{(0)} = \mathbb{I}/m \text{ for } m < \infty, \ M_k^{(0)} = \mathbb{I} \text{ for } m = \infty \hspace{1cm} (15)$$

$$M_k^{(n)} = \left(\sum_{\ell=1}^{m} p_\ell^2 \rho_\ell M_k^{(n-1)} \rho_\ell^{1/2}\right)^{-1/2}$$

$$+ p_k^2 \rho_k M_k^{(n-1)} \rho_k \left(\sum_{\ell=1}^{m} p_\ell^2 \rho_\ell M_k^{(n-1)} \rho_\ell\right)^{-1/2} \hspace{1cm} (16)$$

If $E_m$ is a pure-state ensemble then Holevo’s measurement \cite{37} is given by

$$M_k = |e_k^{\text{Holevo}}\rangle \langle e_k^{\text{Holevo}}|,$$  \hspace{1cm} (17)

where $e_k^{\text{Holevo}}$ is given by (9). In particular, Holevo’s measurement is the pure-state version of the first Ježek-Řeháček-Fiurášek iterate.

**Remark:** $M_k^{(0)}$ is also a POVM for $m < \infty.$ Note that since the recursion formula (16) is invariant under rescalings $M_k^{(n-1)} \rightarrow \lambda \times M_k^{(n-1)},$ the cases $m < \infty$ and $m = \infty$ are essentially the same for $n > 0.$

Holevo studied measurements which were asymptotically optimal in the following precise sense:

**Definition 5** A measurement procedure $G$ is a mapping from ensembles to corresponding POVMs. It is asymptotically optimal \cite{37} for distinguishing pure-state ensembles if for fixed $p_1, \ldots, p_m$ one has

$$\frac{P_{\text{fail}}(E_m) (E_m)}{P_{\text{fail}}^\text{opt} (E_m)} \rightarrow 1$$  \hspace{1cm} (18)

as the states $\psi_k$ of $E_m$ approach an orthonormal set.

Holevo showed that

\footnote{It is presumably intractable to produce a closed-form measurement process $G$ for which $P_{\text{fail}}^G(E_m) (E_m) / P_{\text{fail}}^\text{opt} (E_m) \rightarrow 1$ as the $\psi_k$ and $p_k$ are arbitrarily varied in such a way that $P_{\text{fail}}^\text{opt} (E_m) \rightarrow 0.$ Otherwise, one could recover the optimal measurement for a fixed ensemble $E_m$ on $\mathcal{H}$ by taking the $\lambda \rightarrow 1^-$ limit of the ensemble $E_{m+1}' \equiv \{\psi_k, (1-\lambda) p_k\} \cup \{\phi, \lambda\}$ on a dilation $\mathcal{H}' \supset \mathcal{H},$ with $\phi \perp \mathcal{H}.$}
Theorem 6 (Holevo’s asymptotic-optimality Theorem (1977) [37]) Holevo’s measurement is asymptotically optimal. Furthermore, for fixed \( \{p_k\} \) one has

\[
\frac{2\gamma_{Holevo}(E_m)}{P_{\text{fail}}(E_m)} \to 1
\]

(19)
as \( \langle \psi_i, \psi_j \rangle \to \delta_{ij} \), where \( \gamma_{Holevo} \) is given by (2).

A converse was proven in [42].

The following norms will be used:

Definition 7 Let \( \mathcal{H} \) and \( \mathcal{K} \) be Hilbert spaces, and let \( A : \mathcal{H} \to \mathcal{K} \) be a bounded linear operator. The **absolute value** is \( |A| = \sqrt{A^\dagger A} \). The **trace norm** is \( \|A\|_1 = \text{Tr} |A| \). The **Frobenius norm** is \( \|A\|_2 = \sqrt{\text{Tr} A^\dagger A} \). The **operator norm** is given by

\[
\|A\| = \sup_{0 \neq \psi \in \mathcal{H}} \frac{\|A\psi\|}{\|\psi\|}.
\]

(20)

A is an **isometry** if \( A^\dagger A = 1 \).

It will be assumed that the reader is familiar with the following properties of the trace-norm, which may be found in [49]:

1. \( |\text{Tr } A| \leq \|A\|_1 = \|A^\dagger\|_1 \)
2. \( \|WA\|_1 \leq \|W\| \times \|A\|_1 \)
3. If \( \dim \mathcal{K} \geq \dim \mathcal{H} \) then

\[
\sup_{\text{isometries } U : \mathcal{H} \to \mathcal{K}} \text{Re} \left( \text{Tr } A^\dagger A \right) = \|A\|_1 ,
\]

(21)

where \( U \) is a maximizer iff

\[
U|_{\text{Ran}(A^\dagger A)} = A \left( A^\dagger A \right)^{-1/2}.
\]

(22)

**Note:** Property 3 is a simple consequence of the singular-value decomposition.
IV  Mixed-state distinguishability bounds using Holevo’s method

The first step in constructing a mixed-state version of the argument of section III is to construct a mixed-state version of the underlying estimate (7):

**Lemma 8** Let \( \rho \) be a density matrix on \( \mathcal{H} \) and let \( E : \mathcal{H} \rightarrow \mathcal{H} \) be an operator with \( \|E\| \leq 1 \). Then

\[
1 - \text{Tr} \left( E^\dagger E \rho \right) \in [1, 2] \times \left( 1 - \|E\rho\|_1 \right)
\]

**Proof.** The lower bound follows from the properties of the trace-norm:

\[
1 - \text{Tr} \left( E^\dagger E \rho \right) \geq 1 - \|E^\dagger E \rho\|_1 \geq 1 - \|E\| \times \|E\rho\|_1 \geq 1 - \|E\rho\|_1
\]

To prove the upper bound, define the pre-inner product on the bounded operators on \( \mathcal{H} \) by

\[
\langle E, F \rangle_\rho = \text{Tr} \left( E^\dagger F \rho \right).
\]

By Bessel’s inequality

\[
\text{Tr} \left( E^\dagger E \rho \right) = \|E\rho\|_\rho^2 \geq \sup_{U \text{ unitary}} \left| \langle U, E \rangle_\rho \right|^2 = \sup \frac{\left| \text{Tr} U^\dagger E \rho \right|^2}{(\text{Tr} \rho)^2} = \|E\rho\|_1^2.
\]

Subtracting both sides from 1,

\[
1 - \text{Tr} \left( E^\dagger E \rho \right) \leq (1 + \|E\rho\|_1) (1 - \|E\rho\|_1) \leq 2 (1 - \|E\rho\|_1).
\]

To find the measurement properly analogous to (9), one simply needs to minimize the cost function arising from (23):

**Theorem 9** Let \( M_k = E_k^\dagger E_k \) be a POVM on \( \text{Span} (\mathcal{E}_m) \) minimizing the approximate cost function

\[
C \left( \{E_k\} \right) = \sum_k p_k \left( 1 - \|E_k \rho_k\|_1 \right).
\]

Then \( M_k \) is the first Ježek-Řeháček-Fiurášek iterate

\[
M_k = \left( \sum p_k^2 \rho_k^2 \right)^{-1/2^+} \rho_k \left( \sum p_k^2 \rho_k^2 \right)^{-1/2^+},
\]

and

\[
\Gamma \leq P_{\text{fail}}^{\text{optimal}} \leq P_{\text{fail}} (\{M_k\}) \leq 2 \Gamma,
\]

where

\[
\Gamma = \Gamma (\mathcal{E}_m) = \min_{\{E_k\}} C \left( \{E_k\} \right) = 1 - \text{Tr} \sqrt{\sum_{k=1}^m p_k^2 \rho_k^2} \in [0, 1).
\]

7
Remark: The approximate cost function \((24)\) is a somewhat-disguised modification of the minimal principle of Concha and Poor \([39–41]\), which was reverse-engineered to reproduce the \textit{ad-hoc} mixed-state PGM. The measurement \((25)\) is an example of a mixed-state Belavkin weighted measurement. (See section 2.2 of \([30]\).) A discussion of the relative merits of various weightings, including the cubic weighting of \([50, 51]\), may be found in \([42]\).

Proof. By lemma \(P_{\text{fail}} \left( \left\{ E_k^\dagger E_k \right\} \right) \in [1, 2] \times C (\{ E_k \})\) for all POVMs \(M_k = E_k^\dagger E_k\).

Hence all that is required to get a factor-of-two estimate of \(P_{\text{fail}}^{\text{opt}} (\mathcal{E}_m)\) is to minimize \(C\) subject to the constraint \(\sum E_k^\dagger E_k = 1\) \(\|\text{Span}(\mathcal{E}_m)\|\). Note that the replacement \(E_k \rightarrow W_k E_k\) for unitary \(W_k\) does not alter \(C (\{ E_k \})\) or the quantities \(E_k^\dagger E_k\). Hence the polar decomposition allows imposition of the additional constraint \(E_k \rho_k \geq 0\), giving the expression

\[
C (\{ E_k \}) = \tilde{C} (U) = 1 - \text{Tr} V^\dagger U,
\]

where \(U, V : \text{Span} (\mathcal{E}_m) \rightarrow \text{Span} (\mathcal{E}_m) \otimes \mathbb{C}^m\) are defined by

\[
U \psi = \sum_{k=1}^m |E_k \psi\rangle \otimes |k\rangle_{\mathbb{C}^m}
\]

\[
V \psi = \sum_{k=1}^m |p_k \rho_k \psi\rangle \otimes |k\rangle_{\mathbb{C}^n}.
\]

Here \(|k\rangle_{\mathbb{C}^m}\) is the standard basis of \(\mathbb{C}^m\). Note that \(U\) is an isometry iff \(M_k = E_k^\dagger E_k\) is a POVM on \(\text{Span} (\mathcal{E}_m)\). By equation \(21\)

\[
\min_{\text{isometries } U} \tilde{C} (U) = 1 - \|V\|_1 = 1 - \text{Tr} \left( \sum_{k=1}^m p_k^2 \rho_k^2 \right),
\]

with minimizer

\[
U = V (V^\dagger V)^{-1/2}.
\]

This gives

\[
E_k^{\text{min}} = \langle k |_{\mathbb{C}^m} U = p_k \rho_k \left( \sum_{\ell=1}^m p_\ell^2 \rho_\ell \right)^{-1/2}.
\]

Since \(E_k^{\text{min}} \rho_k \geq 0\), the theorem follows. \(\blacksquare\)
V  Generalization of Curlander’s upper bound

The upper bound of (26) may be sharpened by combining Holevo’s measurement (9) with Curlander’s argument of Ref. [38]:

Theorem 10  The optimal failure rate for distinguishing the arbitrary mixed-state ensemble $\mathcal{E}_m = \{(\rho_k, p_k)\}_{k=1,\ldots,m}$ satisfies

$$\Gamma \leq P_{\text{fail}}^{\text{opt}} \leq P_{\text{HJRF}}^{\text{fail}} \leq \Gamma (2 - \Gamma) \leq 2\Gamma,$$

(28)

where $P_{\text{HJRF}}^{\text{fail}}$ is the failure rate of the measurement (25) and $\Gamma = \Gamma (\mathcal{E}_m)$ is given by (27). Furthermore,

$$P_{\text{fail}}^{\text{opt}} \leq P_{\text{HJRF}}^{\text{fail}} \leq (1 + P_{\text{succ}}^{\text{opt}}) P_{\text{fail}}^{\text{opt}},$$

(29)

Note: Curlander proved (28) in the special case of equiprobable pure states. [38] Barnum and Knill have already shown that the bound (29) holds for the mixed-state “pretty good” measurement [24]. Note that the RHS of (29) never exceeds 1, so the bound is always meaningful.

NOTE ADDED TO ARXIV VERSION: It was not realized at the time of publication that the lower bound of (28) admits a generalization using the theory of matrix monotonicity [54]. Furthermore, this generalization is a minor variation of a similar bound of [55].

Proof. First restrict consideration to pure-state ensembles $\mathcal{E}_m = \{(|\psi_k\rangle, p_k)\}_{k=1,\ldots,m}$. By the convexity of $x \mapsto x^2$ and Jensen’s inequality, 6

$$P_{\text{HJRF}}^{\text{succ}} = \sum_k p_k \langle \psi_k | \epsilon_k^{\text{Holevo}} \rangle^2 \geq \left( \sum_k p_k \langle \epsilon_k^{\text{Holevo}} | \psi_k \rangle \right)^2 = \left( \sum_k p_k |\langle \psi_k | \psi_k \rangle| \right)^{-1/2} \left( \sum_{k,\ell} p_{k,\ell}^2 \langle \psi_k | \psi_{\ell} \rangle \right)^{-1/2} P_{\text{HJRF}}^{\text{fail}} \leq \left( \sum_k p_k |\langle \psi_k | \psi_k \rangle| \right)^{-1/2} \left( \sum_{k,\ell} p_{k,\ell}^2 \langle \psi_k | \psi_{\ell} \rangle \right)^{-1/2} P_{\text{HJRF}}^{\text{fail}} \leq (1 - \Gamma)^2
$$

(30)

so that

$$P_{\text{HJRF}}^{\text{fail}} \leq 1 - (1 - \Gamma)^2 = \Gamma (2 - \Gamma).$$

The left-most inequality of (28) was already proved in Theorem 9.

In the more general case of mixed states, take spectral decompositions $\rho_k = \sum_\ell \mu_{k,\ell} |\psi_{k,\ell}\rangle \langle \psi_{k,\ell}|$ and consider the pure-state ensemble

$$\mathcal{E}_m^* = \{(|\psi_{k,\ell}\rangle, p_k \mu_{k,\ell})\}.$$

(31)

5 An erratum or comment will be sent to JMP to this effect.
6 The author’s argument, which is similar to Curlander’s, was originally motivated by that used to prove Lemma 2 of [25].
Note that any measurement \( \{M_{k\ell}\} \) for \( \mathcal{E}_m^* \) may be converted into a measurement \( M_k = \sum_{\ell} M_{k\ell} \) for \( \mathcal{E}_m \), which trivially satisfies

\[
P_{\text{fail}} (\{M_{k\ell}\}) \geq P_{\text{fail}} (\{M_k\}).
\]

In particular, \( \mathcal{E}_m^* \) is less distinguishable than \( \mathcal{E}_m \), and the measurement (25) is less successful at distinguishing it. Then using (26) and the pure-state case,

\[
\Gamma (\mathcal{E}_m) \leq P_{\text{opt}} (\mathcal{E}_m) \leq P_{\text{HJRF}} (\mathcal{E}_m) \leq P_{\text{HJRF}} (\mathcal{E}_m^*) \leq \Gamma (\mathcal{E}_m^*). \tag{32}
\]

Note that last inequality used the identity \( \Gamma (\mathcal{E}_m^*) = \Gamma (\mathcal{E}_m) \).

Note that \( \Gamma \in [0,1) \) by (27). Because \( \gamma \mapsto \gamma (2 - \gamma) \) is monotonic increasing on \( \gamma \in [0,1) \), the chain of inequalities (29) follows by plugging in the left-hand-side of the first inequality of (28) into the right-hand-side of the third.

**Remark:** In Schumacher and Westmoreland’s classic paper [3], the elements of \( \mathcal{E}_m \) appear as “codewords,” with “syndromes” given by elements of \( \mathcal{E}_m^* \). Schumacher and Westmoreland assert that measurements of \( \mathcal{E}_m^* \) are “not really more difficult” than measurements of \( \mathcal{E}_m \). It is now easy to quantify this assertion:

**Corollary 11** Let \( \mathcal{E}_m^* \) be the ensemble (31) of eigenvectors of the elements of \( \mathcal{E}_m \).

Then

\[
P_{\text{fail}}^{\text{opt}} (\mathcal{E}_m) \leq P_{\text{fail}}^{\text{opt}} (\mathcal{E}_m^*) \leq (1 + P_{\text{succ}}^{\text{opt}} (\mathcal{E}_m)) P_{\text{fail}}^{\text{opt}} (\mathcal{E}_m).
\]

**Proof.** Simply replace the quantity \( P_{\text{HJRF}}^{\text{fail}} (\mathcal{E}_m) \) by \( P_{\text{fail}}^{\text{opt}} (\mathcal{E}_m^*) \) in the chain of inequalities (32), and continue as in the proof of (28). ■

### VI Reflections on the quadratic weighting

As we have seen, the quadratic weighting gives rise to some particularly simple bounds for distinguishability of quantum states. For comparison, substituting the linearly-weighted “pretty good” measurement (13) into equation (30) gives the upper bound

\[
P_{\text{fail}}^{\text{PGM}} \leq 1 - \text{Tr} \left( \left( \sum_{\ell} p_{k\ell} |\psi_{k\ell}\rangle \langle \psi_{k\ell}| \right)^{-1/2} \left( \sum_{\ell} p_{\ell}^{3/2} |\psi_{\ell}\rangle \langle \psi_{\ell}| \right) \right).
\]

The relative simplicity of the quadratic bound (28) is not surprising. As shown by the author in [42], Holevo’s pure-state measurement (9) has the following conceptual and practical advantages over the ad-hoc “pretty good” measurement:

1. Holevo’s asymptotic-optimality property uniquely specifies Holevo’s measurement among the class of Belavkin weighted measurements.

2. Holevo’s measurement categorically outperforms the PGM for ensembles of two pure states.

---

\(^7\)It is of course assumed that the \( a\text{-priori} \) probabilities \( p_k \) are not all the same, or Holevo’s pure-state measurement and the PGM would be identical.
3. The optimality conditions for Holevo’s measurement are particularly simple.

The previous sections provide more examples of this theme:

4. The quadratically-weighted mixed-state measurement gives particularly simple pure- and mixed-state distinguishability bounds.

5. The approximate cost function \( P_{\text{fail}}(M_k) \) for the quadratic measurement is within a factor of two of the function \( P_{\text{fail}}(M_k) \). (The corresponding cost functions for the pure and mixed-states PGMs [39–41, 45] admit no such comparison.)

VII Conclusion and Future Directions

As we have seen, mathematically concise (and reasonably tight) bounds on the distinguishability of mixed quantum states may be obtained by combining the ideas of Holevo, Curlander, and Concha & Poor. In the above we have not explained the connection between these ideas and the iterative algorithm of Ježek, Řeháček, and Fiurášek, other than to recognize that a natural generalization of Holevo’s argument gives the first iterate of Ježek et al’s measurements.

A proper setting to explore such questions is in the theory of approximate quantum channel reversals, which Barnum and Knill [24] have already investigated using a generalization of the “pretty good” measurement. We will consider an abstract form of JRF iteration, study its convergence properties, and construct bounds on channel reversibility and relative min-entropy in future work [52, 56]. We will also attempt to reconsider Holevo’s notion of asymptotic optimality in this setting.

Acknowledgements: I would like to thank Aram Harrow, Julio Concha, V. P. Belavkin, and Vincent Poor for pointing out useful references, Julio Concha and Andrew Kebo for providing copies of their theses, and William Wootters for providing a copy of Hausladen’s thesis, and Stephanie Wehner for a valuable discussion. I would also like to thank Arthur Jaffe, Chris King, and Peter Shor for their encouragement and the editors and anonymous referees for their useful comments and suggestions.

Appendix A: An application of the trace-Jensen inequality

The following theorem makes it transparent that \( 1 - \text{Tr} \sqrt{\sum p_k^2 \rho_k^2} \geq 0 \), giving some insight into the bounds (26) and (28):

**Theorem 12** Let \( f : [0, \infty) \to \mathbb{R} \) be concave with \( f(0) = 0 \), and consider positive semidefinite operators \( A_k \) on a Hilbert space \( \mathcal{H} \). Then

\[
\text{Tr} f \left( \sum_{k=1}^{N} A_k \right) \leq \text{Tr} \sum_{k=1}^{N} f(A_k),
\]
where $f(A)$ is defined using the functional calculus [49]. (In particular, $f(A) = \sum f(\lambda_i) |\psi_i\rangle \langle \psi_i|$.)

**Proof.** The case $N = 2$ is sufficient. By the trace-Jensen inequality [53]

$$\text{Tr} f(A) = \text{Tr} f\left(E_1^\dagger (A + B) E_1 + E_2^\dagger 0 E_2\right) \geq \text{Tr} \left(E_1^\dagger f(A + B) E_1 + E_2^\dagger f(0) E_2\right) = \text{Tr} \left(A^{1/2}(A + B)^{-1/2+} f(A + B)(A + B)^{-1/2+} A^{1/2}\right) \quad (A1)$$

where

$$E_1^\dagger = A^{1/2}(A + B)^{-1/2+}$$

$$E_2 = \sqrt{1 - E_1^\dagger E_1}.$$

Similarly,

$$\text{Tr} f(B) \geq \text{Tr} \left(B^{1/2}(A + B)^{-1/2+} f(A + B)(A + B)^{-1/2+} B^{1/2}\right). \quad (A2)$$

The conclusion follows by adding (A1) and (A2) and applying the cyclicity of the trace. ■

**Note added in proof:** The lower bound $\Gamma(\xi) \leq P_{\text{fail}}(M_{\text{opt}}^k)$ of Theorem 10 admits a simple generalization proved using matrix monotonicity:

$$1 - \text{Tr} \left[\left(\sum_{k=1}^m p_k^s \rho_k^s\right)^{1/s}\right] \leq P_{\text{fail}}(M_{\text{opt}}^k),$$

for any $s \in [1, \infty)$. This is addressed in a short note which has been submitted to this journal. [54]

**References**

[1] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York 1976).

[2] P. Hausladen, R. Josza, B. Schumacher, M. Westmoreland, and W. K. Wootters, “Classical information capacity of a quantum channel,” Phys Rev A **54**, 1869 (1996).

[3] B. Schumacher and M. D. Westmoreland, “Sending classical information via noisy quantum channels,” Phys Rev A **56**, 131 (1997).

[4] A. S. Holevo, “The capacity of the quantum channel with general signal states,” IEEE Trans. Inf. Theory **44**, 269 (1998).
[5] L. Ip, “Shor’s algorithm is optimal,” http://lawrenceip.com/papers/hpsdpababstract.html (2003).

[6] D. Bacon, A. M. Childs, and W. van Dam, “Optimal measurements for the dihedral hidden subgroup problem,” Chicago J. of Theoret. Comput. Sci. 2006, (2006); e-print arXiv: quant-ph/0501044.

[7] D. Bacon, A. M. Childs, and W. van Dam, “From optimal measurement to efficient quantum algorithms for the hidden subgroup problem over semidirect product groups,” Proceedings of the 46th IEEE Symp. Foundations of Computer Science, (IEEE, Los Alamitos, CA, 2005), pp. 469-478 (2005).

[8] A. M. Childs and W. van Dam, “Quantum algorithm for a generalized hidden shift problem,” Proceedings of the 18th ACM-SIAM Symp. Discrete Algorithms, (Society for Industrial and Applied Mathematics, Philadelphia, PA, 2007), pp. 1225-1234; e-print arXiv:quant-ph/0507190.

[9] C. Moore and A. Russell, “For Distinguishing Hidden Subgroups, the Pretty Good Measurement is as Good as it Gets,” Quantum Inform. Compu. 7, 752 (2007); e-print arXiv:quant-ph/0501177.

[10] D. Bacon and T. Decker, “The optimal single-copy measurement for the hidden-subgroup problem,” Phys. Rev. A 77, 032335 (2008); e-print arXiv:0706.4478.

[11] H. P. Yuen, R. S. Kennedy, and M. Lax, “Optimum testing of multiple hypotheses in quantum detection theory,” IEEE Trans. Inf. Theory, IT-21, 125 (1975).

[12] A. S. Holevo, “Statistical Decision Theory for Quantum Systems,” J. Multivariate Anal. 3, 337 (1973).

[13] A. S. Holevo, “Remarks on optimal measurements,” Problemy Peredachi Informatsii, 10 no. 4, 51-55; English Translation: Problems of Information Transmission 10, no.4, 317-320 (1974).

[14] V. P. Belavkin, “Optimal distinction of non-orthogonal quantum signals,” Radio Eng. Electron. Phys., 20, 39 (1975).

[15] V. P. Belavkin, “Optimal multiple quantum statistical hypothesis testing,” Stochastics 1, 315 (1975). Note: Inequality 4.3 in the statement of Theorem 5 is backwards.

[16] V. P. Belavkin and A. G. Vantsian, “On sufficient conditions of optimality of quantum signal processing,” Radio Eng. Electron. Phys. 19, 1391 (1974).

[17] S. M. Barnett and S. Croke, “On the conditions for discrimination between quantum states with minimum error,” J. Phys. A: Math. Theor. 42 062001 (2009); e-print arXiv:0810.1919.
[18] Y. C. Eldar, A. Megretski, and G. C. Verghese, “Designing Optimal Quantum Detectors Via Semidefinite Programming,” IEEE Trans. Inf. Theory, 49, 1007 (2003).

[19] C. W. Helstrom, “Bayes-Cost reduction algorithm in quantum hypothesis testing,” IEEE Trans. Inf. Theory IT-28, 359 (1982). Note: Ref. [20] asserts that the presented algorithm does not always converge to an optimal measurement.

[20] M. Ježek, J. Řeháček, and J. Fiurášek, “Finding optimal strategies for minimum-error quantum state discrimination,” Phys. Rev. A 65, 060301 (2002); quant-ph/0201109.

[21] Z. Hradil, J. Řeháček, J. Fiurášek, and M. Ježek, “Maximum-Likelihood Methods in Quantum Mechanics,” Lecture Notes in Physics 649, pp. 59-112 (2004).

[22] J. Tyson, “Estimates of non-optimality of quantum measurements and a simple iterative method for computing optimal measurements,” e-print arXiv:0902.0395.

[23] P. Hayden, D. Leung, and G. Smith, “Multiparty data hiding of quantum information,” Phys Rev A 71, 062339 (2005).

[24] H. Barnum and E. Knill, “Reversing quantum dynamics with near-optimal quantum and classical fidelity,” J. Math. Phys. 43, 2097 (2002); e-print arXiv:quant-ph/0004088.

[25] A. Montanaro, “On the distinguishability of random quantum states,” Commun. Math. Phys. 273, 619 (2007).

[26] D. Qiu, “Minimum-error discrimination between mixed quantum states,” Phys Rev A 77, 012328 (2008).

[27] A. Montanaro, “A lower bound on the probability of error in quantum state discrimination,” Proc. IEEE Information Theory Workshop 2008, pp. 378-380; e-print arXiv:0711.2012.

[28] M. Hayashi, A. Kawachi, and H. Kobayashi, “Quantum measurements for Hidden Subgroup Problems with Optimal Sample Complexity,” Quantum Inform. and Compu. 8, 0345 (2008); e-print arXiv:quant-ph/0604174.

[29] D. Qiu and L. Li, “Bounds on the minimum-error discrimination between mixed quantum states,” eprint arXiv: 0812.2378.

[30] V. P. Belavkin and V. Maslov, “Design of Optimal Dynamic Analyzer: Mathematical Aspects of Wave Pattern Recognition” In Mathematical Aspects of Computer Engineering, edited by V. Maslov, pp. 146-237 (Mir, Moscow 1987); e-print arXiv:quant-ph/0412031 Note: The first two equations on page 40 should be $F_i = H_i (L^o)^{-1/2}$ and $M_i^o = (L^o)^{+1/2} D_i (L^o)^{+1/2}$. 

14
[31] A. S. Fletcher, “Channel-Adapted Quantum Error Correction,” Ph. D. Thesis MIT Cambridge, MA 2007; e-print arXiv: 0706.3400.

[32] A. S. Fletcher, P. W. Shor, and M. Z. Win, “Optimum quantum error recovery using semidefinite programming,” Phys Rev A 75, 012338 (2007); e-print arXiv:quant-ph/0606035.

[33] A. S. Fletcher, P. W. Shor, and M. Z. Win, “Channel-Adapted Quantum Error Correction for the Amplitude Damping Channel,” IEEE Trans. Inf. Theory, 54 5705-5718 (2008); e-print arXiv:0710.1052.

[34] A. S. Fletcher, P. W. Shor, and M. Z. Win, “Structured near-optimal channel-adapted quantum error correction,” Phys Rev A 77, 012320 (2008); e-print arXiv:0708.3658.

[35] S. Taghavi, R. L. Kosut, and D. A. Lidar, “Channel-Optimized Quantum Error Correction,” e-print arXiv:0810.2524 (2008).

[36] R. König, R. Renner, and C. Schaffner, “The operational meaning of min- and max-entropy,” e-print arXiv:0807.1338.

[37] A. S. Kholevo, “On asymptotically optimal hypothesis testing in quantum statistics,” Theor. Probab. Appl. 23 411 (1978). Note: The displayed equation between (8) and (9) should be \( \sum_j \pi_j \|\psi_j - e_j\|^2 = 2 \left(1 - \text{Re} \text{Tr} \left(U \Pi \Gamma^{1/2} \right)\right) \). The line just after equation (9) should read “where \( V^* = [\Pi \Gamma^{1/2} (\Pi \Gamma^{1/2})^{-1}] \). The final expression in the paper should read 2 \( \left(1 - \text{Tr} \left|\Gamma^{1/2} \Pi\right|\right) \).

[38] P. J. Curlander, “Quantum Limitations on Communication Systems” Ph. D. Thesis, MIT Cambridge, MA 1979.

[39] J. I. Concha, “Signal detection in multiaccess quantum channels,” Ph. D. Thesis, Princeton University, Princeton, NJ 2002.

[40] J. I. Concha and H. V. Poor, “An Optimality property of the square-root measurement for mixed states” in Proceedings of the 6th International Conference on quantum communication, measurement, and computing, (Rinton, Princeton, NJ, 2003), pp. 329-332.

[41] J. I. Concha and H. V. Poor, “Advances in quantum detection,” Chapter 7 of Communications, Information, and Network Security, Edited by V. K. Bhargava, H. V. Poor, V. Tarokh, and S. Yoon. (Kluwer Academic Publishers, Norwell Massachusetts 2003).

[42] J Tyson, “Error rates of Belavkin weighted quantum measurements and a converse to Holevo’s asymptotic optimality Theorem,” Physical Review A 79, 032343 (2009).
[43] R. S. Kennedy, “On the optimum quantum receiver for the M-ary linearly independent pure state problem,” MIT Research Laboratory of Electronics Quarterly Progress Report, Technical Report No. 110, pp. 142-146 (1973).

[44] C. Mochon, “Family of generalized ‘pretty good’ measurements and the minimal-error pure-state discrimination problems for which they are optimal,” Phys Rev A 73, 032328 (2006).

[45] Y. C. Eldar and G. D. Forney, “On quantum detection and the square-root measurement,” IEEE Trans. Inf. Theory 47, 858 (2001); e-print arXiv:quant-ph/0005132.

[46] A. K. Kebo, “Quantum detection and finite frames,” Ph.D dissertation University of Maryland, College Park, 2005.

[47] P. Hausladen, “On the Quantum Mechanical Channel Capacity as a Function of the Density Matrix,” B. A. Thesis, Williams College, Williamstown, Massachusetts 1993.

[48] P. Hausladen and W. K. Wootters, “A ‘pretty good’ measurement for distinguishing quantum states,” J Mod Optic 41, 2385 (1994).

[49] M. Reed and B. Simon, Methods of Modern Mathematical Physics I: Functional Analysis (Academic, New York, 1980).

[50] S. Wehner, “Cryptography in a quantum world,” PhD Thesis: University of Amsterdam (2008); e-print arXiv:0806.3483.

[51] M. A. Ballester, S. Wehner, and A. Winter, “State discrimination with post-measurement information,” IEEE Trans. Inf. Theory 54, 4183 (2008); e-print arXiv:quant-ph/0608014.

[52] J. Tyson, in preparation.

[53] F. Hansen and G. K. Pedersen, “Jensen’s operator inequality,” B. Lond. Math. Soc. 35, 553 (2003); e-print arXiv:math/0204049.

[54] J. Tyson, “Minimum-error quantum distinguishability bounds from matrix monotone functions: A comment on ‘Two-sided estimates of minimum-error distinguishability of mixed quantum states via generalized Holevo-Curlander bounds’,” to appear in J. Math. Phys. (2009).

[55] T. Ogawa and H. Nagaoka, “Strong converse to the quantum coding theorem,” IEEE Transactions on Information Theory 45, 2486-2489 (1999).

[56] J. Tyson, “Two-sided estimates of minimum-error distinguishability of mixed quantum states via generalized Holevo-Curlander bounds (and extensions to channel reversibility and maximum overlaps),” 4th Workshop on Theory of Quantum Computation, Communication and Cryptography (TQC2009), Waterloo, Ontario, Canada (May 13, 2009).