Solution properties of convex sweeping processes with velocity constraints

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\textbf{ABSTRACT}

Some properties of solutions of convex sweeping processes with velocity constraints are studied in this paper. Namely, the solution stability with respect to the initial value, the boundedness, the closedness, and the convexity of the solution set are discussed in detail. In addition, an outer estimate for the solution set is given here for the first time and two open questions are raised for further research. Our investigations complement the preceding ones on the solution existence and the solution uniqueness of convex sweeping processes with velocity constraints.

\textbf{1. Introduction}

Sweeping processes with velocity constraints, which are nontrivial generalizations of certain evolution variational inequalities, were studied firstly by Siddiqi and Manchada [1]. Since these models have various applications in mechanics, physics, and engineering (see [2, p. 8] and [3, Section 6.4]), several forms of such processes have been considered in the literature; see [2,4–9].

In this paper, we study the form of sweeping processes with velocity constraints proposed by Adly, Haddad and Thibault [2], which is defined as follows. Suppose that $\mathcal{H}$ is a real Hilbert space, $T$ a positive real number, and $C : [0, T] \rightarrow \mathcal{H}$ a set-valued map having nonempty closed convex values. Let $A_0, A_1 : \mathcal{H} \rightarrow \mathcal{H}$ be positive semidefinite, bounded, symmetric linear operators and $f : [0, T] \rightarrow \mathcal{H}$ be a continuous mapping. Consider the following differential inclusion, which is called sweeping process with velocity constraint:

$$\begin{cases}
A_1 \dot{u}(t) + A_0 u(t) - f(t) \in - \mathcal{N}_C(t)(\dot{u}(t)) & \text{a.e. } t \in [0, T], \\
u(0) = u_0,
\end{cases}$$

where $\mathcal{N}_C(t)(\dot{u}(t))$ is the normal cone to $C(t)$ at $\dot{u}(t)$ in the sense of convex analysis. An \textit{absolutely continuous} function $u : [0, T] \rightarrow \mathcal{H}$ is said to be a \textit{solution} of (P) if it satisfies the conditions stated in the formulation of problem (P). Since the Hilbert space $\mathcal{H}$ has the Radon-Nikodým property, the Fréchet derivative $\dot{u}(t)$ of $u$ exists for almost every $t \in [0, T]$ (see Section 2 for some relevant references).

For sweeping processes with velocity in a moving bounded convex set in separable Hilbert spaces, Adly et al. [2] have established sufficient conditions for the solution existence and the solution uniqueness. Later, Adly and Le [4] have generalized the solution existence result of [2] to the case where the
moving set can be unbounded and the operator \( A_1 \) is semicoercive. An application to non-regular electrical circuits was given in [2,4]. In a subsequent paper, by weakening the continuity condition of the moving constraint set, Vilches and Nguyen [9, Section 5] have obtained a refinement of the corresponding result of [4]. For implicit sweeping processes of a general type, Jourani and Vilches [8] have proved the solution existence and uniqueness by using the concept of quasistatic evolution variational inequalities from [10]. Considering history-dependent implicit sweeping processes, Migórski et al. [11] have extended the results in [10] including the existence, uniqueness, and stability of the solution. Later, Adly and Sofonea [12] and Nacry and Sofonea [13] have also provided the unique solvability of a class of time-dependent inclusions with history-dependent operators. The results are then applied to obtain the existence and uniqueness of solution to several sweeping processes with the history-dependent operators deeply influencing the right-hand side.

Relaxing the convexity of the constraint sets, Bounkhel [7] has obtained the solution existence and the solution uniqueness for (P), provided that \( N_{C(t)}(\dot{u}(t)) \) is replaced by the proximal normal cone \( N^P_{C(t)}(\dot{u}(t)) \), \( A_0 \equiv 0, A_1 \) is the identity operator, and \( C(t) \) are uniformly prox-regular for all \( t \in [0, T] \). Recently, by using a result of Yen [14] on the solution sensitivity of parametric variational inequalities, Adly et al. [6] have investigated the problem (P) in the case where the set-valued mapping \( t \mapsto C(t), t \in [0, T], \) is locally Lipschitz-like. The authors have also established several solution existence results for the case, where \( C(t) \) is a finite union of disjoint convex sets. More comments and remarks on the solution existence and solution uniqueness of (P) in both convex and nonconvex cases can be found in [6].

Our aim is to study some fundamental properties of the solutions of (P). When the problem has a unique solution, it is of interest to study the continuity of the solution with respect to the initial value \( u_0 \). We prove that if the sufficient conditions for the solution existence and uniqueness either in [2] or in [6] are satisfied, then the solution is Lipschitz continuous on the initial value. Then, we show that the solution set is bounded if some assumptions used in [2,4,6] are fulfilled. The solution set is not always closed in the space of continuous vector-valued functions. However, it is a closed subset in an appropriate space. Two sets of sufficient conditions for the convexity of the solution set are obtained.

Interestingly, a sharp outer estimate for the solution set can be established. It is worthy to stress that the just-mentioned properties of the solutions of (P) are investigated here for the first time. To the best of our knowledge, analogous results are not available in the literature.

The paper is organized as follows. Some preliminaries, including a lemma on a relation between strong convergence of sequence of functions in \( L^1([0, T], \mathcal{H}) \) and its pointwise convergence, are presented in Section 2. The solution sensitivity with respect to the initial value is addressed in Section 3. Three theorems on the boundedness of the solution set are proved in Section 4. We establish in Section 5 the closedness of the solution set of (P) in the Sobolev space \( W^{1,1}([0, T], \mathcal{H}) \). Section 6 is devoted to the convexity of the solution set, an outer estimate for the set, and two interesting open questions. The obtained results are summarized in the last section.

2. Preliminaries

Throughout this paper, let \( \mathcal{H} \) be a real Hilbert space equipped with the norm \( \| \cdot \| \) and the scalar product \( \langle \cdot, \cdot \rangle \). The open ball (resp., closed ball) in \( \mathcal{H} \) with center \( x \) and radius \( r > 0 \) is denoted by \( B(x, r) \) (resp., \( \overline{B}(x, r) \)). The distance from \( x \) to \( \Omega \) is \( d(x, \Omega) := \inf_{y \in \Omega} \| x - y \| \). The projection of a point \( x \in \mathcal{H} \) onto \( \Omega \) is defined by \( P_{\Omega}(x) = \{ y \in \Omega \mid d(x, \Omega) = \| x - y \| \} \). The Hausdorff distance between nonempty subsets \( \Omega_1, \Omega_2 \) of \( \mathcal{H} \) is defined by the formula

\[
d_{\mathcal{H}}(\Omega_1, \Omega_2) = \max \left\{ \sup_{x \in \Omega_1} d(x, \Omega_2), \sup_{y \in \Omega_2} d(y, \Omega_1) \right\}.
\]

For a convex set \( \Omega \subset \mathcal{H} \), the normal cone to \( \Omega \) at \( x \in \mathcal{H} \) in the sense of convex analysis is \( N_\Omega(x) := \{ x^* \in \mathcal{H} \mid \langle x^*, y - x \rangle \leq 0, \forall y \in \Omega \} \) if \( x \in \Omega \) and \( \emptyset \) if \( x \notin \Omega \).
By \( \mathbb{N} \) we denote the set of positive integers. The notation \([a, b]\) (resp., \((a, b)\)) stands for a closed interval (resp., an open interval) in the real line \( \mathbb{R} \). The Banach space of continuous functions defined on \([a, b]\) with values in \( \mathcal{H} \) is denoted by \( C^0([a, b], \mathcal{H}) \). Here, \( \|x\|_{C^0} = \max_{t \in [a, b]} \|x(t)\| \).

**Definition 2.1:** A function \( x : [a, b] \to \mathcal{H} \) is said to be absolutely continuous on \([a, b]\) if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( \sum_{k=1}^n \|x(b_k) - x(a_k)\| < \varepsilon \) for any finite system of pairwise disjoint subintervals \((a_k, b_k) \subseteq [a, b]\) of total length \( \sum_{k=1}^n (b_k - a_k) \) less than \( \delta \).

Any absolutely continuous function \( u : [0, T] \to \mathcal{H} \) is Fréchet differentiable almost everywhere on \([0, T]\) with respect to the Lebesgue measure of the segment; see, for example, [15, Corollary 13 of Chapter 3, Theorem 2 on p. 107, and Section 6 of Chapter VII] or [16, Corollary 5.12 and Theorem 5.21].

**Definition 2.2** (See [17, Definition 1.40] and [18, Definition 3.1]): One says that a set-valued mapping \( K : \Lambda \to \mathcal{H} \), where \( \Lambda \) is a metric space, is Lipschitz-like around a point \((\lambda, \bar{x})\) in its graph, which is the set \( \{(\lambda, x) \in \Lambda \times \mathcal{H} | x \in K(\lambda)\} \), if there exist a neighborhood \( V \) of \( \bar{x} \), a neighborhood \( W \) of \( \lambda \) and a constant \( \kappa > 0 \) such that

\[
K(\lambda) \cap W \subset K(\lambda') + \kappa d(\lambda, \lambda')\overline{B}(0, 1), \quad \forall \lambda, \lambda' \in V.
\]

**Definition 2.3:** A linear operator \( A : \mathcal{H} \to \mathcal{H} \) is coercive if there exists a positive constant \( c \) such that

\[
\langle Ax, x \rangle \geq c \|x\|^2 \quad \forall x \in \mathcal{H}.
\]

If there is \( c > 0 \) such that (1) holds, then \( c \|x\|^2 \leq \langle Ax, x \rangle \leq \|A\| \|x\|^2 \) for all \( x \in \mathcal{H} \). Thus, we must have \( c \leq \|A\| \), provided that \( \mathcal{H} \neq \{0\} \). Let \( A \) be bounded and coercive. Set

\[
\bar{c} = \sup \left\{ c \in \mathbb{R}_+ \mid \langle Ax, x \rangle \geq c \|x\|^2 \quad \forall x \in \mathcal{H} \right\}.
\]

By the definition of supremum, there exists a sequence \( \{c_k\} \subset \mathbb{R}_+ \) satisfying the inequality \( \langle Ax, x \rangle \geq c_k \|x\|^2 \) for all \( x \in \mathcal{H} \) and \( c_k \to \bar{c} \) as \( k \to \infty \). Hence, one has \( \langle Ax, x \rangle \geq \bar{c} \|x\|^2 \) for all \( x \in \mathcal{H} \).

So, \( \bar{c} \in (0, \|A\|] \). For a bounded coercive linear operator \( A : \mathcal{H} \to \mathcal{H} \), the constant \( \bar{c} \) defined by (2) is called the modulus of coercivity of \( A \).

We now recall the definition of Bochner integral.

**Definition 2.4** (See [15, pp. 44–45]): Let \((\Omega, \Sigma, \mu)\) be a finite measurable space and \( X \) be a Banach space. A \( \mu \)-measurable function \( f : \Omega \to X \) is called Bochner integrable if there is a sequence of simple functions \( \{f_k\} \) such that \( \lim_{k \to \infty} \int_\Omega \|f_k(\omega) - f(\omega)\|_X \, d\mu = 0 \). In this case, \( \int_E f(\omega) \, d\mu \) is defined for each \( E \in \Sigma \) by \( \int_E f(\omega) \, d\mu = \lim_{k \to \infty} \int_E f_k(\omega) \, d\mu \), where \( \int_E f_k(\omega) \, d\mu \) is defined in an obvious way.

As noted in [15, p. 45], the limit in Definition 2.4 exists and is independent of the defining sequence \( \{f_k\} \). According to [15, Theorem 2, p. 45], a \( \mu \)-measurable function \( f : \Omega \to X \) is Bochner integrable if and only if \( \int_\Omega \|f(\omega)\|_X \, d\mu < \infty \).

If \( u : [0, T] \to \mathcal{H} \) is an absolutely continuous function, then the function \( \dot{u}(\cdot) \) is Bochner integrable on \([0, T]\) (see the proof of [15, Theorem 2, p. 107] for detailed explanations).

For every \( p \in [1, \infty) \), the Bochner space \( L^p(\Omega, X) \) consists of all \( \mu \)-measurable functions \( f : \Omega \to X \) satisfying

\[
\|f\|_p = \left( \int_\Omega \|f(\omega)\|_X^p \, d\mu \right)^{1/p} < \infty
\]

(see, e.g. [15, pp. 49–50]). The space \( L^p(\Omega, X) \) for any \( 1 \leq p < \infty \) is a Banach space and the set of simple functions is dense in \( L^p(\Omega, X) \) (see, e.g. [15, p. 97]).
The following lemma gives a relation between strong convergence of sequence of functions in $L^1([0, T], \mathcal{H})$ and its pointwise convergence.

**Lemma 2.1:** Let $\{x_n\}$ be a sequence in $L^1([0, T], \mathcal{H})$ and let $x \in L^1([0, T], \mathcal{H})$ be such that $x_n$ converges strongly to $x$ in $L^1([0, T], \mathcal{H})$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}(t)$ converges to $x(t)$ almost everywhere on $[0, T]$.

**Proof:** Since $\{x_n\}$ is a strongly convergent sequence, it is a Cauchy sequence. Hence, for every positive integer $k$, we can find a positive integer $n_k$ such that

$$
\|x_m - x_q\|_{L^1} \leq \frac{1}{2^k} \quad (\forall m \geq n_k, \forall q \geq n_k).
$$

Without loss of generality we may assume that $n_{k_1} < n_{k_2}$ whenever $k_1 < k_2$. Clearly, the above choice of $\{n_k\}$ implies that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ having the property

$$
\|x_{n_{k+1}} - x_{n_k}\|_{L^1} \leq \frac{1}{2^k} \quad \forall k \geq 1. \tag{3}
$$

Define

$$
y_m(t) = \sum_{k=1}^{m} \|x_{n_{k+1}}(t) - x_{n_k}(t)\|. \tag{4}
$$

For all $t \in [0, T]$, by (4) and (3), we have

$$
|y_m(t)| = \sum_{k=1}^{m} \|x_{n_{k+1}}(t) - x_{n_k}(t)\| \leq \sum_{k=1}^{m} \frac{1}{2^k} \leq 1.
$$

Thus, $|y_m(t)| \leq 1$ for every $t \in [0, T]$. Since $x_n \in L^1([0, T], \mathcal{H})$ is measurable for all $n \in \mathbb{N}$, the function $y_m : [0, T] \to \mathbb{R}$ is also measurable for all $m \in \mathbb{N}$. As $\{y_m\}$ is an increasing sequence of real-valued functions, by the monotone convergence theorem [19, Theorem 4.1] one can assert that $y_m(t)$ converges to a function $y(t)$ almost everywhere on $[0, T]$. Since $|y(t)| \leq 1$ for all $t \in [0, T]$, we see that $y \in L^1([0, T], \mathbb{R})$. On the other hand, for $i > j \geq 2$, we have

$$
\|x_{n_i}(t) - x_{n_j}(t)\| \leq \|x_{n_i}(t) - x_{n_{i-1}}(t)\| + \ldots + \|x_{n_{j+1}}(t) - x_{n_j}(t)\| \leq y(t) - y_{n_{j-1}}(t). \tag{5}
$$

It follows that, for almost every $t \in [0, T]$, $\{x_{n_k}(t)\}$ is a Cauchy sequence in $\mathcal{H}$ and it converges to a finite limit, say, $\tilde{x}(t)$. From (5), letting $i$ tend to infinity, we obtain

$$
\|\tilde{x}(t) - x_{n_j}(t)\| \leq y(t) - y_{n_{j-1}}(t) \leq y(t)
$$

for almost every $t \in [0, T]$ and for any $j \geq 2$. Hence, one has $\tilde{x} \in L^1([0, T], \mathcal{H})$. Since $\|x_{n_k}(t) - \tilde{x}(t)\|^2 \to 0$ and $\|x_{n_k}(t) - \tilde{x}(t)\| \leq y(t)$ almost everywhere on $[0, T]$, using the dominated convergence theorem [15, Theorem 3, p. 45], we can deduce that $\|x_{n_k} - \tilde{x}\|_1 \to 0$. Since $x_n$ converges strongly to $x$ in $L^1([0, T], \mathcal{H})$ and $L^1([0, T], \mathcal{H})$ is a subspace of $L^1([0, T], \mathcal{H})$, $x_n$ converges strongly to $x$ in $L^1([0, T], \mathcal{H})$. By the uniqueness of the limit, we have $\tilde{x} = x$. Therefore, we have shown that $x_{n_k}(t)$ converges to $x(t)$ almost everywhere on $[0, T]$.

The proof is complete. 

**Remark 2.1:** In the formulation of Lemma 2.1, one can replace $L^1([0, T], \mathcal{H})$ by any Bochner space $L^p(\Omega, X)$ with $1 \leq p < \infty$. The proof remains the same, provided that one writes $L^p(\Omega, X)$ instead of $L^1([0, T], \mathcal{H})$ and $L^p([0, T], \mathbb{R})$ instead of $L^1([0, T], \mathbb{R})$. 

Proposition 2.1 (See [21, Theorem 1.4.35]):

$\forall x \in \Omega$, there exists a function $y(x)$ such that for almost every $t \in \Omega$, $x$ is absolutely continuous, differentiable almost everywhere and $x(t)$ has compact support in $\Omega$.

The weak derivative of $f \in L^p(\Omega, X)$ is uniquely defined up to a set of measure zero (see [22, Proposition 23.18]).

Definition 2.5: Let $f \in L^p(\Omega, X)$, where $p \in [1, \infty)$, a function $\tilde{f} \in L^1_{\text{loc}}(\Omega, X)$ is said to be a weak derivative of $f$ if

$$\int_{\Omega} \tilde{g}(\tau)f(\tau) \, d\tau = -\int_{\Omega} g(\tau)\tilde{f}(\tau) \, d\tau,$$

for all $g \in C^\infty_0(\Omega)$, where $C^\infty_0(\Omega)$ the space of all real-valued functions that are infinitely differentiable and have compact support in $\Omega$.

From the above definition, we see that if a sequence $(f_k)$ converges strongly to $f$ in $W^{1,p}(\Omega, X)$, then $f_k$ (resp., $\tilde{f}_k$) converges strongly to $f$ (resp., $\tilde{f}$) in $L^p(\Omega, X)$. It is well known [21, Proposition 1.4.34] that $W^{1,p}(\Omega, X)$ is a Banach space for all $p \in [1, +\infty)$.

Proposition 2.1 (See [21, Theorem 1.4.35]): Let $p \in [1, \infty)$ and $x \in L^p(\Omega, X)$. The following conditions are equivalent

(a) $x \in W^{1,p}(\Omega, X)$.
(b) $x$ is absolutely continuous, differentiable almost everywhere and $\dot{x} \in L^p(\Omega, X)$.
(c) there exists a function $y \in L^p(\Omega, X)$ such that for almost every $t_0, t \in \Omega$, one has

$$x(t) = x(t_0) + \int_{t_0}^{t} y(\tau) \, d\tau.$$

Remark 2.2: For $\Omega = (0, T)$, if $x : \Omega \to \mathcal{H}$ is an absolutely continuous function, then it is a simple matter to prove that the limits $\lim_{t \to \infty} x(t)$ and $\lim_{t \to -\infty} x(t)$ exist. So, setting $x(0) = \lim_{t \to 0^+} x(t)$ and $x(T) = \lim_{t \to T^-} x(t)$ gives an absolutely continuous function defined on $[0, T]$. Therefore, by Proposition 2.1 one can identify the Sobolev space $W^{1,1}(\Omega, X)$, where $\Omega = (0, T)$, with the space of absolutely continuous functions $u : [0, T] \to \mathcal{H}$ equipped with the norm

$$\|u\|_{W^{1,1}} = \int_0^T \|u(t)\| \, d\tau + \int_0^T \|\dot{u}(t)\| \, d\tau.$$

We use this identification and write $W^{1,1}([0, T], \mathcal{H})$ for $W^{1,1}((0, T), \mathcal{H})$. 
Throughout this paper, $A_0, A_1 : \mathcal{H} \to \mathcal{H}$ are positive semi-definite, bounded symmetric linear operators and $f : [0, T] \to \mathcal{H}$ is a continuous mapping. We denote by $\text{Sol}(P, u_0)$ the solution set of $(P)$ with the initial value $u_0$. Before investigating the solution properties for problem $(P)$, we present some assumptions that were used in preceding works [2,4,6].

**Assumption (H1):** The constraint sets $C(t), t \in [0, T]$, are nonempty, closed, and convex.

**Assumption (H1a):** The constraint sets $C(t), t \in [0, T]$, are nonempty and convex.

**Assumption (H2a):** The set-valued mapping $C$ is continuous in the Hausdorff distance sense, i.e. there exists a continuous function $g : [0, T] \to \mathbb{R}$ such that

$$d_H(C(s), C(t)) \leq |g(s) - g(t)| \quad \forall \ s, t \in [0, T]. \quad (7)$$

**Assumption (H2b):** $C$ is Lipschitz-like around every point in its graph.

**Assumption (H3a):** The constraint set $C(0)$ is bounded.

**Assumption (H3b):** There exist positive constants $c_1, c_2$ such that

$$\langle A_1 x, x \rangle \geq c_1 \|x\|^2 - c_2, \quad \forall \ x \in C(0).$$

**Assumption (H3c):** There exist positive constants $c_1, c_2$ such that

$$\langle A_1 x, x \rangle \geq c_1 \|x\|^2 - c_2, \quad \forall \ t \in [0, T], \quad \forall \ x \in C(t).$$

First, we recall some solution existence and solution uniqueness results of $(P)$.

**Theorem 2.1 (See [2, Theorems 5.1]):** Suppose that $\mathcal{H}$ is separable. If (H1), (H2a), and (H3a) are satisfied, then $(P)$ has at least one Lipschitz solution.

**Theorem 2.2 (See [4, Theorem 1]):** Suppose that $\mathcal{H}$ is separable. If the Assumptions (H1), (H2a), and (H3b) are satisfied, then $(P)$ has at least one Lipschitz solution.

**Theorem 2.3 (See [6, Theorem 3.3]):** Let $A_0 = 0, A_1 : \mathcal{H} \to \mathcal{H}$ be coercive. If the assumptions (H1) and (H2b) are satisfied, then $(P)$ has a unique solution $u$, which is a Lipschitz function. Moreover, the unique solution is a continuously differentiable function.

In [2,6], some conditions for the solution uniqueness of $(P)$, which require the coerciveness of either $A_0$ or $A_1$, have been given.

**Theorem 2.4 (See [2, Theorem 5.2]):** If $A_0$ is coercive and $C(t)$ is nonempty and convex for every $t \in [0, T]$, then $(P)$ has at most one solution.

**Theorem 2.5 (See [6, Theorem 3.4]):** If $A_1$ is coercive and $C(t)$ is nonempty and convex for every $t \in [0, T]$, then $(P)$ has at most one solution.

For a detailed discussion on the above assumptions and results, we refer to [6].
3. Solution stability with respect to the initial value

In this section, we investigate the solution sensitivity of (P) with respect to the initial value when the solution is unique. The following theorem takes account of the case where the operator \( A_0 \) is coercive.

**Theorem 3.1:** If the Assumption (H1a) is satisfied, \( \text{Sol}(P,u_0) \) is nonempty for every \( u_0 \in C(0) \), and \( A_0 \) is coercive with the modulus of coercivity \( \alpha_0 \), then the mapping \( \varphi : C(0) \to C^0([0,T],\mathcal{H}) \), \( u_0 \mapsto u(u_0,\cdot) \), where \( u(u_0,\cdot) \) denotes the unique solution of (P), is Lipschitz continuous with the modulus

\[
\sqrt{\frac{\|A_0\|}{\alpha_0}}.
\]

**Proof:** Let \( x_0, y_0 \in C(0) \) be given arbitrarily. Then, by our assumptions and Theorem 2.4, the sweeping process (P) has a unique solution \( x(\cdot) \) with the initial value \( x_0 \) (resp., a unique solution \( y(\cdot) \) with the initial value \( y_0 \)). Since \( C(t) \) is convex, the inclusion

\[
A_1 \dot{u}(t) + A_0 u(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}(t))
\]

in the formulation of (P) can be rewritten equivalently as

\[
\langle A_1 \dot{u}(t) + A_0 u(t) - f(t), \dot{u}(t) - z \rangle \leq 0 \quad \forall \ z \in C(t).
\]

As \( \mathcal{N}_{C(t)}(\dot{u}(t)) = \emptyset \) if \( \dot{u}(t) \notin C(t) \), the fulfillment of (8) for almost every \( t \in [0,T] \) implies that \( \dot{u}(t) \in C(t) \) for almost every \( t \in [0,T] \). Hence, the inclusions \( \dot{x}(t) \in C(t) \) and \( \dot{y}(t) \in C(t) \) hold for almost every \( t \in [0,T] \). So, we have

\[
\begin{cases}
\langle A_1 \dot{x}(t) + A_0 x(t) - f(t), \dot{x}(t) - \dot{y}(t) \rangle \leq 0, \\
\langle -A_1 \dot{y}(t) - A_0 y(t) + f(t), \dot{x}(t) - \dot{y}(t) \rangle \leq 0
\end{cases}
\]

for almost every \( t \in [0,T] \). Adding the inequalities in (9) side by side yields

\[
\langle A_1 (\dot{x}(t) - \dot{y}(t)), \dot{x}(t) - \dot{y}(t) \rangle + \langle A_0 (x(t) - y(t)), \dot{x}(t) - \dot{y}(t) \rangle \leq 0 \quad \text{a.e. } t \in [0,T].
\]

Since \( A_1 \) is positive semi-definite, this implies that

\[
\langle A_0 (x(t) - y(t)), \dot{x}(t) - \dot{y}(t) \rangle \leq 0 \quad \text{a.e. } t \in [0,T].
\]

Taking the Lebesgue integral on both sides of the inequality in (10) and applying [23, Remarks 11.23(c)], we obtain

\[
\int_0^t \langle A_0 (x(\tau) - y(\tau)), \dot{x}(\tau) - \dot{y}(\tau) \rangle \, d\tau \leq 0 \quad (\forall \ t \in [0,T]).
\]

As \( \frac{d}{dt} \langle A_0 (x(\tau) - y(\tau)), x(\tau) - y(\tau) \rangle = 2\langle A_0 (x(\tau) - y(\tau)), \dot{x}(\tau) - \dot{y}(\tau) \rangle \) at every point \( \tau \) where both derivatives \( \dot{x}(\tau), \dot{y}(\tau) \) exist, by [24, Theorem 6, p. 340] one has

\[
\int_0^t \langle A_0 (x(\tau) - y(\tau)), \dot{x}(\tau) - \dot{y}(\tau) \rangle \, d\tau = \frac{1}{2} \left[ \langle A_0 (x(t) - y(t)), x(t) - y(t) \rangle - \langle A_0 (x(0) - y(0)), x(0) - y(0) \rangle \right].
\]

Then, from (11) it follows that

\[
\langle A_0 (x(t) - y(t)), x(t) - y(t) \rangle - \langle A_0 (x(0) - y(0)), x(0) - y(0) \rangle \leq 0.
\]

Hence, by the coerciveness of \( A_0 \), we get

\[
\alpha_0 \|x(t) - y(t)\|^2 \leq \langle A_0 (x(t) - y(t)), x(t) - y(t) \rangle \leq \langle A_0 (x_0 - y_0), x_0 - y_0 \rangle \leq \|A_0\| \|x_0 - y_0\|^2.
\]
Therefore, \(\|x(t) - y(t)\| \leq \sqrt{\frac{1}{\alpha_0}} \|x_0 - y_0\|\) for all \(t \in [0, T]\). So, the inequality
\[
\|x - y\|_{C^0} \leq \sqrt{\frac{\|A_0\|}{\alpha_0}} \|x_0 - y_0\|
\]
holds for any \(x_0, y_0 \in C(0)\). We have thus proved that the mapping \(\varphi\) is Lipschitz continuous on \(C(0)\) with the modulus \(\sqrt{\frac{\|A_0\|}{\alpha_0}}\).

According to Theorem 2.5, the nonemptiness and convexity of \(C(t)\) together with the coerciveness of \(A_1\) can also guarantee the solution uniqueness for (P) if such a solution exists. A natural question arises: Could we get a similar result as the one in Theorem 3.1 for the case under consideration? The next theorem gives a complete answer to this question.

**Theorem 3.2:** If the Assumption (H1a) is fulfilled, \(\text{Sol}(P, u_0)\) is nonempty for every \(u_0 \in C(0)\), and \(A_1\) is coercive with the modulus of coercivity \(\alpha_1\), then the mapping \(\varphi: C(0) \to C^0([0, T], \mathcal{H}), u_0 \mapsto (u_0, \cdot)\), where \(u(u_0, \cdot)\) denotes the unique solution of (P), is Lipschitz continuous with the modulus \(\sqrt{\frac{T\|A_0\|}{2\alpha_1}} + 1\).

**Proof:** For any \(x_0, y_0 \in C(0)\), the assumptions made and Theorem 2.5 assure that (P) has a unique solution \(x(\cdot)\) (resp., \(y(\cdot)\)) with the initial value \(x_0\) (resp., \(y_0\)). Then, arguing similarly as in the proof of Theorem 3.1, we have
\[
\langle A_1 \dot{x}(t) + A_0 x(t) - f(t), \dot{x}(t) - \dot{y}(t) \rangle \leq 0
\]
and
\[
\langle A_1 \dot{y}(t) + A_0 y(t) - f(t), \dot{y}(t) - \dot{x}(t) \rangle \leq 0
\]
for almost every \(t \in [0, T]\). Adding the last inequalities side by side, one obtains
\[
\langle A_1 (\dot{x}(t) - \dot{y}(t)), \dot{x}(t) - \dot{y}(t) \rangle + \langle A_0 (x(t) - y(t)), \dot{x}(t) - \dot{y}(t) \rangle \leq 0
\]
for almost every \(t \in [0, T]\). Combining the coerciveness of \(A_0\) with (13) yields
\[
\alpha_1 \|\dot{x}(t) - \dot{y}(t)\|^2 \leq -\langle A_0 (x(t) - y(t)), \dot{x}(t) - \dot{y}(t) \rangle \quad \text{a.e. } t \in [0, T].
\]

Since the function \(t \mapsto -\langle A_0 (x(t) - y(t)), \dot{x}(t) - \dot{y}(t) \rangle\) is integrable (in the Lebesgue sense), from (14), we can deduce that the function \(t \mapsto \alpha_1 \|\dot{x}(t) - \dot{y}(t)\|^2\) is also integrable. Integrating both sides of the inequality in (14), we obtain
\[
\int_0^t \alpha_1 \|\dot{x}(\tau) - \dot{y}(\tau)\|^2 \, d\tau \leq -\int_0^t \langle A_0 (x(\tau) - y(\tau)), \dot{x}(\tau) - \dot{y}(\tau) \rangle \, d\tau.
\]

At every point \(\tau\) where both derivatives \(\dot{x}(\tau), \dot{y}(\tau)\) exist, we have
\[
\frac{d}{d\tau} \langle A_0 (x(\tau) - y(\tau)), x(\tau) - y(\tau) \rangle = 2 \langle A_0 (x(\tau) - y(\tau)), \dot{x}(\tau) - \dot{y}(\tau) \rangle.
\]

Hence, as noted in the preceding proof, by [24, Theorem 6, p. 340] we have (12). Consequently, from (15), it follows that
\[
\int_0^t \alpha_1 \|\dot{x}(\tau) - \dot{y}(\tau)\|^2 \, d\tau \leq -\frac{1}{2} \left[ \langle A_0 (x(t) - y(t)), x(t) - y(t) \rangle - \langle A_0 (x(0) - y(0)), x(0) - y(0) \rangle \right].
\]
Since $A_0$ is positive semidefinite, the latter implies
\[
\int_0^t \alpha_1 \| \dot{x}(\tau) - \dot{y}(\tau) \|^2 \, d\tau \leq \frac{1}{2} \langle A_0 (x(0) - y(0)), x(0) - y(0) \rangle \leq \frac{\|A_0\|}{2} \|x_0 - y_0\|^2.
\]
So, we have
\[
\int_0^t \| \dot{x}(\tau) - \dot{y}(\tau) \|^2 \, d\tau \leq \frac{\|A_0\|}{2\alpha_1} \|x_0 - y_0\|^2.
\] (16)

In addition, for each $t \in [0, T]$ one has
\[
\|x(t) - y(t)\| = \left\| \left( x_0 + \int_0^t \dot{x}(\tau) \, d\tau \right) - \left( y_0 + \int_0^t \dot{y}(\tau) \, d\tau \right) \right\|
\leq \|x_0 - y_0\| + \int_0^t \| \dot{x}(\tau) - \dot{y}(\tau) \| \, d\tau.
\] (17)

The inequality shows that the function $t \mapsto \| \dot{x}(t) - \dot{y}(t) \|$ belongs to the space $L^2([0, T], \mathbb{R})$. Therefore, setting $\beta(t) = 1$ for $t \in [0, T]$ and using the Hölder’s inequality (see [19, Theorem 4.6] and [24, p. 385]) for functions from $L^2([0, T], \mathbb{R})$, we have
\[
\int_0^t (\beta(\tau) \| \dot{x}(\tau) - \dot{y}(\tau) \|) \, d\tau \leq \left( \int_0^t \beta(\tau)^2 \, d\tau \right)^{\frac{1}{2}} \left( \int_0^t \| \dot{x}(\tau) - \dot{y}(\tau) \|^2 \, d\tau \right)^{\frac{1}{2}}.
\]

Then, combining this with (16) yields
\[
\int_0^t \| \dot{x}(\tau) - \dot{y}(\tau) \| \, d\tau \leq \sqrt{t} \sqrt{\frac{\|A_0\|}{2\alpha_1} \|x_0 - y_0\|} \leq \sqrt{T} \sqrt{\frac{\|A_0\|}{2\alpha_1} \|x_0 - y_0\|}
\]
for every $t \in [0, T]$. Hence, thanks to (17), we get
\[
\|x(t) - y(t)\| \leq \|x_0 - y_0\| + \sqrt{T} \frac{\|A_0\|}{2\alpha_1} \|x_0 - y_0\| = \left( \sqrt{T} \frac{\|A_0\|}{2\alpha_1} + 1 \right) \|x_0 - y_0\|
\]
for all $t \in [0, T]$. This implies that the mapping $\varphi$ defined in the statement of the theorem is Lipschitz continuous on $C(0)$ with the modulus $\sqrt{\frac{T\|A_0\|}{2\alpha_1} + 1}$. $\blacksquare$

### 4. Boundedness of the solution set

Noting that the Sobolev space $W^{1,1}([0, T], \mathcal{H})$ is the space of all absolutely continuous functions with its derivative in $L^1([0, T], \mathcal{H})$ (see Proposition 2.1), we can view the solution set of $(P)$ as a subset of $W^{1,1}([0, T], \mathcal{H})$. Of course, at the same time, it is a subset of $C^0([0, T], \mathcal{H})$.

If $(P)$ has a unique solution then, under suitable conditions, we have established the solution sensitivity with respect to the initial value. When the solution uniqueness is not guaranteed, the solution set of $(P)$ may be unbounded. Let us consider an example.

**Example 4.1:** Let $\mathcal{H} = \mathbb{R}^2$, $A_0 = A_1 = \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, $u_0 = (0, 0)$, $f(t) = (0, t)$, and $C(t) = \mathbb{R} \times \{0\}$ for all $t \in [0, T]$. For every $\lambda \in \mathbb{R}$, we define a function by setting $u^{(\lambda)}(t) = (\lambda t, 0)$ for all $t \in [0, T]$. Clearly,
Lemma 4.1 (See [25, Lemma 4.2.1]): generalization of Gronwall’s inequalities.

\[ A_1 \dot{u}^{(\lambda)}(t) + A_0 u^{(\lambda)}(t) - f(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{u}_1^{(\lambda)}(t) \\ \dot{u}_2^{(\lambda)}(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1^{(\lambda)}(t) \\ u_2^{(\lambda)}(t) \end{pmatrix} - (0) = (0) \] .

Since \( \mathcal{N}_{C(t)}(\dot{u}^{(\lambda)}(t)) = [0] \times \mathbb{R} \), this yields \( A_1 \dot{u}^{(\lambda)}(t) + A_0 u^{(\lambda)}(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}^{(\lambda)}(t)) \) for all \( t \in [0, T] \). Thus, for any \( \lambda \in \mathbb{R} \), \( u^{(\lambda)} \) is a solution of (P). As \( \|u^{(\lambda)}\|_{C^0} = |\lambda|T \), the solutions of (P) form an unbounded subset of \( C^0([0, T], \mathcal{H}) \).

Our aim in this section is to establish some sets of conditions ensuring that the solution set of (P) is bounded.

**Theorem 4.1:** If \( C(t) \) is nonempty for all \( t \in [0, T] \) and the Assumptions (H2a), (H3a) are satisfied then, for any \( u_0 \in C(0) \), the solution set Sol(P, \( u_0 \)) is bounded in both spaces \( C^0([0, T], \mathcal{H}) \) and \( W^{1,1}([0, T], \mathcal{H}) \).

**Proof:** Let \( u_0 \in C(0) \) be given arbitrarily. If Sol(P, \( u_0 \)) is empty, then it is bounded. Suppose that Sol(P, \( u_0 \)) \( \neq \emptyset \) and \( u \) is an element from Sol(P, \( u_0 \)). As \( C(0) \) is bounded, we can find \( \rho_0 > 0 \) such that \( C(0) \subset \rho_0 \overline{\mathbb{R}}(0, 1) \). Let \( g : [0, T] \to \mathbb{R} \) be a continuous function satisfying (7). Thus, for all \( t \in [0, T] \) one has \( C(t) \subset C(0) + |g(0) - g(t)| \). So, \( C(t) \subset \rho \overline{\mathbb{R}}(0, 1) \) for all \( t \in [0, T] \), where \( \rho := \rho_0 + \max\{|g(0) - g(s)| : s \in [0, T]\} \). Since \( \dot{u}(t) \in C(t) \) for almost every \( t \in [0, T] \), one has \( \|\dot{u}(t)\| \leq \rho \) for almost every \( t \in [0, T] \). For any \( t \in [0, T] \), we define two sets \( \Omega_1(t) = \{ s \in [0, t] \mid \|\dot{u}(s)\| \leq \rho \} \) and \( \Omega_2(t) = \{ s \in [0, t] \mid \|\dot{u}(s)\| > \rho \} \). Then, the sets \( \Omega_1(t) \) and \( \Omega_2(t) \) are measurable, and \( \mu(\Omega_2(t)) = 0 \) with \( \mu \) being the Lebesgue measure on \( \mathbb{R} \). So, by [6, Remark 3.4(c)] and [15, Theorem 4, p. 46], we have

\[
\|u(t)\| = \|u_0\| + \int_0^t \|\dot{u}(\tau)\| d\tau = \|u_0\| + \int_{\Omega_1(t)} \|\dot{u}(\tau)\| d\tau + \int_{\Omega_2(t)} \|\dot{u}(\tau)\| d\tau \\
\leq \|u_0\| + \int_{\Omega_1(t)} \|\dot{u}(\tau)\| d\tau + \int_{\Omega_2(t)} \|\dot{u}(\tau)\| d\tau \\
\leq \|u_0\| + \rho \mu(\Omega_1(t)) \\
\leq \|u_0\| + \rho T.
\]

Thus, \( \|u\|_{C^0} \leq \|u_0\| + \rho T \). This establishes the boundedness of Sol(P, \( u_0 \)) in \( C^0([0, T], \mathcal{H}) \). Since \( \|u(t)\| \leq \|u_0\| + \rho T \) for all \( t \in [0, T] \), \( \|\dot{u}(t)\| \leq \rho \) for a.e. \( t \in [0, T] \), and \( u \in \text{Sol}(P, u_0) \) was chosen arbitrarily, by (6), we can assert that Sol(P, \( u_0 \)) is a bounded subset of the Sobolev space \( W^{1,1}([0, T], \mathcal{H}) \).

To deal with the case where the sets \( C(t), t \in [0, T] \), can be unbounded, we will need the following generalization of Gronwall’s inequalities.

**Lemma 4.1 (See [25, Lemma 4.2.1]):** Let \( f \) be a Lebesgue integrable, real-valued function defined on \( [0, T] \). If

\[
f(t) \leq a + b \int_0^t f(\tau) d\tau \quad \text{a.e. } t \in [0, T]
\]

for some constants \( a, b \) with \( b \neq 0 \), then \( \int_0^t f(\tau) d\tau \leq \frac{a}{b} \left( \exp(bt) - 1 \right) \) for all \( t \in [0, T] \).

**Theorem 4.2:** If the Assumptions (H1a), (H2a) and (H3b) are satisfied then, for any \( u_0 \in C(0) \), the solution set Sol(P, \( u_0 \)) is bounded in both spaces \( C^0([0, T], \mathcal{H}) \) and \( W^{1,1}([0, T], \mathcal{H}) \).
Proof: Given any $u_0 \in C(0)$. If $\text{Sol}(P, u_0)$ is empty, then it is bounded. Suppose that $\text{Sol}(P, u_0)$ is nonempty. Take any $u \in \text{Sol}(P, u_0)$ and let $\varepsilon > 0$ be given arbitrarily. Since $C(t)$ is nonempty, for any $t \in [0, T]$ there exists $z_t \in C(t)$ satisfying

$$
\|u_0 - z_t\| < d(u_0, C(t)) + \varepsilon.
$$

By (H2a), we have

$$
\|z_t\| - \|u_0\| \leq \|u_0 - z_t\| < d(u_0, C(t)) + \varepsilon \leq d_P(C(0), C(t)) + \varepsilon \leq |g(0) - g(t)| + \varepsilon.
$$

Then, setting $\beta := \|u_0\| + \max_{\tau \in [0, T]} |g(0) - g(\tau)| + \varepsilon$, we get $\|z_t\| < \beta$. So, for every $t \in [0, T]$ one can find some $z_t \in C(t)$ such that $\|z_t\| < \beta$. As $u \in \text{Sol}(P, u_0)$, by (H1a), one has for almost every $t \in [0, T]$ that

$$
\langle A_1 \dot{u}(t) + A_0 u(t) - f(t), \dot{u}(t) - z_t \rangle \leq 0 \quad \forall z \in C(t).
$$

Substituting $z = z_t$ into the above inequality yields

$$
\langle A_1 \dot{u}(t) + A_0 u(t) - f(t), \dot{u}(t) - z_t \rangle \leq 0
$$

for almost every $t \in [0, T]$. Thus,

$$
\langle A_1 \dot{u}(t), \dot{u}(t) \rangle - \langle A_1 \dot{u}(t), z_t \rangle + \langle A_0 u(t) - f(t), \dot{u}(t) \rangle - \langle A_0 u(t) - f(t), z_t \rangle \leq 0.
$$

(19)

Using the assumptions (H2a), (H3b), and [6, Remark 3.2], we can find positive constants $\hat{c}_1, \hat{c}_2$ such that $\langle A_1 x, x \rangle \geq \hat{c}_1 \|x\|^2 - \hat{c}_2$ for all $t \in [0, T]$ and $x \in C(t)$. Then, (19) implies that

$$
\hat{c}_1 \|\dot{u}(t)\|^2 - \hat{c}_2 - \langle A_1 \dot{u}(t), z_t \rangle + \langle A_0 u(t) - f(t), \dot{u}(t) \rangle - \langle A_0 u(t) - f(t), z_t \rangle \leq 0
$$

for a.e. $t \in [0, T]$. So, one has

$$
\hat{c}_1 \|\dot{u}(t)\|^2 - \hat{c}_2 - \beta \|A_1\| \|\dot{u}(t)\| - (\|A_0\| \|u(t)\| + \|f\|_{C^0}) \|\dot{u}(t)\| - \beta (\|A_0\| \|u(t)\| + \|f\|_{C^0}) \leq 0
$$

for a.e. $t \in [0, T]$. For each $t \in [0, T]$, setting $a_1(t) = \beta \|A_1\| + \|A_0\| \|u(t)\| + \|f\|_{C^0}$ and

$$
a_2(t) = \beta (\|A_0\| \|u(t)\| + \|f\|_{C^0}) + \hat{c}_2,
$$

we get

$$
\hat{c}_1 \|\dot{u}(t)\|^2 - a_1(t) \|\dot{u}(t)\| - a_2(t) \leq 0 \quad \text{a.e. } t \in [0, T].
$$

(20)

As one has $\hat{c}_1 > 0$ and $a_2(t) > 0$ for every $t \in [0, T]$, the quadratic polynomial $q(x) := \hat{c}_1 x^2 - a_1(t) x - a_2(t)$ has two roots with different signs. Hence, (20) holds if and only if

$$
\|\dot{u}(t)\| \leq \frac{a_1(t) + \sqrt{a_1(t)^2 - 4 \hat{c}_1 a_2(t)}}{2 \hat{c}_1} \quad \text{a.e. } t \in [0, T].
$$

Since $\sqrt{a_1(t)^2 - 4 \hat{c}_1 a_2(t)} \leq a_1(t)$, this yields $\|\dot{u}(t)\| \leq \frac{a_1(t)}{\hat{c}_1}$ for a.e. $t \in [0, T]$. Therefore,

$$
\|\dot{u}(t)\| \leq \frac{\beta \|A_1\| + \|A_0\| \|u(t)\| + \|f\|_{C^0}}{\hat{c}_1}
$$

for a.e. $t \in [0, T]$. Then one has

$$
\|\dot{u}(t)\| \leq \gamma (1 + \|u(t)\|) \quad \text{a.e. } t \in [0, T],
$$

(21)
where $\gamma := \max\{\frac{\beta}{\|A_1\| + \|f\|_{C_0}}, \frac{\|A_0\|}{c_1}\}$. Since
\[
\|u(t)\| = \|u_0 + \int_0^t \dot{u}(\tau) \, d\tau\| \leq \|u_0\| + \int_0^t \|\dot{u}(\tau)\| \, d\tau
\]  
(22)
(see [6, Remark 3.4(c)] and [15, Theorem 4(ii), p. 46]), from (21) it follows that
\[
\|\dot{u}(t)\| \leq \gamma (1 + \|u_0\|) + \gamma \int_0^t \|\dot{u}(\tau)\| \, d\tau \quad \text{a.e. } t \in [0, T].
\]
So, applying Lemma 4.1 for $f(t) := \|\dot{u}(t)\|$, $a := \gamma (1 + \|u_0\|)$, and $b := \gamma$ gives
\[
\int_0^t \|\dot{u}(\tau)\| \, d\tau \leq (1 + \|u_0\|)(\exp(\gamma t) - 1) \leq (1 + \|u_0\|)(\exp(\gamma T) - 1) \quad \forall \ t \in [0, T].
\]
Combining this with (22) yields
\[
\|u(t)\| \leq \|u_0\| + (1 + \|u_0\|)(\exp(\gamma T) - 1) \quad \forall \ t \in [0, T].
\]  
(23)
It follows that $\|u\|_{C_0} \leq \|u_0\| + (1 + \|u_0\|)(\exp(\gamma T) - 1)$. So, $\text{Sol}(P, u_0)$ is a bounded subset of $C^0([0, T], \mathcal{H})$. Finally, using the estimates (21), (23), and formula (6), we can find a constant $\rho > 0$ such that $\|u\|_{W^{1,1}} \leq \rho$ for any $u \in \text{Sol}(P, u_0)$. The proof is complete. 

\[\text{Theorem 4.3: If the assumptions (H1a), (H2b) and (H3c) are satisfied then, for any } u_0 \in C(0), \text{ the solution set } \text{Sol}(P, u_0) \text{ is bounded in both spaces } C^0([0, T], \mathcal{H}) \text{ and } W^{1,1}([0, T], \mathcal{H}).\]

\[\text{Proof: For each } t \in [0, T], \text{ pick a point } x_t \in C(t). \text{ As } C \text{ is Lipschitz-like around } (t, x_t), \text{ there exist an open neighborhood } V_t \text{ of } t \text{ in the induced topology of } [0, T] \subset \mathbb{R}, \text{ a neighborhood } W_t \text{ of } x_t \text{ in } \mathcal{H}, \text{ and a constant } \kappa_t > 0 \text{ such that}
\]
\[
C(t') \cap W_t \subset C(t'') + \kappa_t|t' - t''|\mathbb{B}(0, 1) \quad \forall t', \ t'' \in V_t.
\]  
(24)
Since $[0, T] = \bigcup_{t \in [0, T]} V_t$, the compactness of $[0, T]$ implies the existence of $t_1, \ldots, t_k$ in $[0, T]$ such that $[0, T] = \bigcup_{i=1}^k V_{t_i}$. For each $i \in \{1, \ldots, k\}$, we have $x_{t_i} \in W_{t_i}$. So, thanks to (24), for every $t \in V_{t_i}$ we can find $z_{t_i}^{(i)} \in C(t)$ and $\xi_{t_i}^{(i)} \in \mathbb{B}(0, 1)$ satisfying $x_{t_i} = z_{t_i}^{(i)} + \kappa_{t_i}|t - t_i|$$\xi_{t_i}^{(i)}$. Then,
\[
\|z_{t_i}^{(i)}\| \leq \|x_{t_i}\| + \kappa_{t_i}|t - t_i| \leq \|x_{t_i}\| + \kappa_{t_i}T.
\]  
(25)
Setting $\beta = \max\{\|x_{t_i}\| + \kappa_{t_i}T \mid i \in \{1, \ldots, k\}\}$, we have $\beta > 0$. For each $t \in [0, T]$, there is some $i \in \{1, \ldots, k\}$ such that $t \in V_{t_i}$ and, by (25), the element $z_{t_i}^{(i)} \in C(t)$ satisfies the estimate $\|z_{t_i}^{(i)}\| \leq \beta$. Therefore, for every $t \in [0, T]$, there exists at least one point of the form $z_{t_i}^{(i)}$ such that $z_{t_i}^{(i)} \in C(t)$ and $\|z_{t_i}^{(i)}\| \leq \beta$.

Let $u_0 \in C(0)$ be given arbitrarily. Since Sol$(P, u_0)$ bounded if it is empty, it suffices to consider the case $\text{Sol}(P, u_0) \neq \emptyset$. Take any $u \in \text{Sol}(P, u_0)$. By (H1a), we deduce for almost every $t \in [0, T]$ that
\[
\langle A_1\dot{z}(t) + A_0z(t) - f(t), \dot{z}(t) - z(t) \rangle \leq 0 \quad \forall \ z \in C(t).
\]
Substituting $z = z_{t_i}^{(i)}$ into the last inequality yields
\[
\langle A_1\dot{z}(t) + A_0z(t) - f(t), \dot{z}(t) - z_{t_i}^{(i)} \rangle \leq 0
\]
for almost every $t \in [0, T]$. Using the assumption (H3c) and repeating the final part of the proof of Theorem 4.1 (starting from inequality (19)), we can show that the solution set Sol$(P, u_0)$ is bounded in both spaces $C^0([0, T], \mathcal{H})$ and $W^{1,1}([0, T], \mathcal{H})$. 

\[\blacksquare\]
Remark 4.1: The boundedness of Sol(P, u₀) in Theorem 4.3 is also valid if instead of the assumption (H2b) one requires that C is inner semicontinuous at every point in its graph, i.e. for every (t, x) ∈ [0, T] × ℋ with x ∈ C(t), if U ⊂ ℋ is an open set containing x, then there exists a neighborhood V of t in [0, T] such that C(t') ∩ U ≠ ∅ for all t' ∈ V. Indeed, for each t ∈ [0, T], select a point x_t ∈ C(t). The inner semicontinuity of C at (t, x_t) assures that there is an open neighborhood V_t of t in the induced topology of [0, T] such that C(t') ∩ V_t ≠ ∅ for every t' ∈ V_t. By the compactness of [0, T], from the open covering {V_t}_{t∈[0,T]} of the segment we can extract a finite subcover V_{t₁},..., V_{t_k}. So, for each t ∈ [0, T], there exists an index i ∈ {1,...,k} such that t ∈ V_{t_i}. Since C(t) ∩ B(x_{t_i}, 1) ≠ ∅, there is a vector z_{t_i} ∈ C(t) ∩ B(x_{t_i}, 1). Then one has ||z_{t_i}|| ≤ β, where β := max{||x_i|| + 1 | i ∈ {1,...,k}}. Consequently, for each t ∈ [0, T], there exists at least one point of the form z_{t_i} such that z_{t_i} ∈ C(t) and ||z_{t_i}|| ≤ β. Then, as noted above, the usage of (H3c) and the repetition of the final part of the proof of Theorem 4.1 yield the desired assertion.

Remark 4.2: If a set-valued mapping is Lipschitz-like around a point in its graph then it is inner semicontinuous at that point (see, e.g. [?, Proposition 3.1]). On the other hand, there exist locally Lipschitz-like mappings which are not continuous in the Hausdorff distance sense (see [6, Example 3.1] and the discussion therein). Clearly, if the mapping C : [0, T] ⇒ ℋ is continuous in the Hausdorff distance sense, then it is inner semicontinuous at every point in its graph. The just cited example of [6] shows that the converse is not true in general.

Remark 4.3: The continuity in the Hausdorff distance sense of C(⋅) together with the Assumption (H3b) implies (H3c) (see [6, Remark 3.2]). However, a similar implication may not hold under the inner semicontinuity of C(⋅) at every point in its graph or even under the Lipschitz-likeness of C(⋅) around every point in its graph.

5. Closedness of the solution set

First, let us show that the closedness of Sol(P, u₀) may not be available even for very simple problems in finite dimensions.

Proposition 5.1: The solution set of (P) may not be closed in C⁰([0, T], ℋ).

Proof: We will prove the proposition by constructing a suitable example. Let ℋ = ℝ, A₀ = 0, A₁ = 0, u₀ = 0, f(t) ≡ 0, and C(t) = ℝ for all t ∈ [0, T]. Then, an absolutely continuous function u : [0, T] → ℝ is a solution of (P) if and only if

\[
\begin{array}{l}
0 \in \mathcal{N}_{C(t)}(u(t)) \quad \text{a.e. } t \in [0, T], \\
u(0) = 0.
\end{array}
\]

Since C(t) = ℝ for all t ∈ [0, T], \(\mathcal{N}_{C(t)}(u(t)) = \{0\}\) for any t where u(t) exists. So, any absolutely continuous function u : [0, T] → ℝ with u(0) = 0 is a solution of (P). For k ∈ ℕ, let

\[x_k(t) = \begin{cases} 
t^2 \sin \left(\frac{1}{t^2}\right) & \text{if } t \in \left[0, \frac{1}{k}\right], \\
\frac{t}{k} \sin(k^2) & \text{if } t \in \left[\frac{1}{k}, \frac{1}{k}\right],
\end{cases}\]

and

\[x(t) = \begin{cases} 
t^2 \sin \left(\frac{1}{t^2}\right) & \text{if } t \in (0, T], \\
0 & \text{if } t = 0.
\end{cases}\]
Clearly, \( x_k(\cdot) \) is a Lipschitz function for each \( k \in \mathbb{N} \). Since \( x_k(0) = 0 \), \( x_k(\cdot) \) is a solution of (P) for every \( k \in \mathbb{N} \). In addition, for any \( k \in \mathbb{N} \), we have

\[
\sup_{t \in [0,T]} |x(t) - x_k(t)| = \sup_{0 < t \leq \frac{1}{k}} \left| t^2 \sin \left( \frac{1}{t^2} \right) - \frac{t}{k} \sin(k^2) \right| \\
\leq \sup_{0 < t \leq \frac{1}{k}} \left| t^2 \sin \left( \frac{1}{t^2} \right) \right| + \sup_{0 < t \leq \frac{1}{k}} \left| \frac{t}{k} \sin(k^2) \right| \\
\leq t^2 + \frac{t}{k} \\
= \frac{2}{k^2}.
\]

Therefore, \( x_k \) strongly converges to \( x \) in \( C^0([0, T], \mathbb{R}) \) as \( k \to \infty \). However, since \( x(\cdot) \) is not of bounded variation (see [24, Problem 2, p. 331]), it is not absolutely continuous. Hence, \( x \) is not a solution of (P). We have thus shown that \( \text{Sol}(P, u_0) \) is non-closed in \( C^0([0, T], \mathcal{H}) \).

Next, we will prove that the solution set of (P) is closed if it is regarded as a subset of an appropriate space. More precisely, the following theorem confirms that the Sobolev space \( W^{1,1}([0, T], \mathcal{H}) \) is such a space. (This result can be explained by the well-known fact that the norm of \( W^{1,1}([0, T], \mathcal{H}) \) is finer than the one of \( C^0([0, T], \mathcal{H}) \).)

**Theorem 5.1:** If the Assumption (H1) is satisfied then, for any \( u_0 \in C(0) \), the solution set \( \text{Sol}(P, u_0) \) is closed in \( W^{1,1}([0, T], \mathcal{H}) \).

**Proof:** Let \( u_0 \in C(0) \) be given. Suppose that \( \{u_k\} \subset \text{Sol}(P, u_0) \) is a sequence converging strongly in \( W^{1,1}([0, T], \mathcal{H}) \) to \( u \) as \( k \to \infty \). Then, \( u \) is an absolutely continuous function. To prove that \( u \) satisfies the initial condition in (P), we can argue as follows. Since the norm in \( W^{1,1}([0, T], \mathcal{H}) \) is given by (6), we have

\[
\lim_{k \to \infty} \int_0^T \|u_k(\tau) - u(\tau)\| \, d\tau = 0 \tag{26}
\]

and

\[
\lim_{k \to \infty} \int_0^T \|\dot{u}_k(\tau) - \dot{u}(\tau)\| \, d\tau = 0. \tag{27}
\]

Note that \( u_k(t) = u_k(0) + \int_0^t \dot{u}_k(s) \, ds \) and \( u(t) = u(0) + \int_0^t \dot{u}(s) \, ds \) for every \( t \in [0, T] \) and for all \( k \in \mathbb{N} \) (see [6, Remark 3.4(c)]). Hence, from (26), (27), and [15, Theorem 4, p. 46], it follows that

\[
0 = \lim_{k \to \infty} \int_0^T \|u_k(\tau) - u(\tau)\| \, d\tau \\
= \lim_{k \to \infty} \left[ \int_0^T \|u_k(0) - u(0)\| + \int_0^T (\dot{u}_k(s) - \dot{u}(s)) \, ds \right] \, d\tau \\
\geq \liminf_{k \to \infty} \left[ \int_0^T \left( \|u_k(0) - u(0)\| - \int_0^T (\dot{u}_k(s) - \dot{u}(s)) \, ds \right) \right] \, d\tau \\
\geq \liminf_{k \to \infty} \left[ \int_0^T \left( \|u_k(0) - u(0)\| - \int_0^T \|\dot{u}_k(s) - \dot{u}(s)\| \, ds \right) \right] \, d\tau
\]
\[
\begin{aligned}
\liminf_{k \to \infty} & \left[ T\|u_0 - u(0)\| - T \int_0^T \|\dot{u}_k(s) - \dot{u}(s)\| \, ds \right] \\
= & \ T\|u_0 - u(0)\|
\end{aligned}
\]

So, \( u(0) = u_0 \).

It remains to prove that \( u \) satisfies the differential inclusion in (P).

Setting \( C = \{ \varphi \in L^1([0, T], \mathcal{H}) \mid \varphi(t) \in C(t) \text{a.e. } t \in [0, T] \} \), we will prove that \( C \) is closed in \( L^1([0, T], \mathcal{H}) \). Let \( \{ \varphi_m \} \subset D \) be a sequence converging strongly in \( L^1([0, T], \mathcal{H}) \) to a function \( \psi \). Thanks to Lemma 2.1, we can find a subsequence \( \{ \varphi_{m_j} \} \) of \( \{ \varphi_m \} \) such that \( \varphi_{m_j}(t) \) converges to \( \psi(t) \) for almost every \( t \in [0, T] \). Since \( \varphi_{m_j}(t) \in C(t) \) a.e. \( t \in [0, T] \) and \( C(t) \) is closed, we have \( \psi(t) \in C(t) \) a.e. \( t \in [0, T] \). Hence, one has \( \psi \in C \). This shows that \( C \) is closed in \( L^1([0, T], \mathcal{H}) \).

Since \( \{ u_k \} \subset \text{Sol}(P, u_0) \), we have \( \dot{u}_k \in C \) for all \( k \in \mathbb{N} \). From (27), it follows that \( \dot{u} \in C \). So, \( \dot{u}(t) \in C(t) \) for almost every \( t \in [0, T] \). As \( C(t) \) is convex for all \( t \in [0, T] \), the inclusion \( A_1 \dot{u}_k(t) + A_0 u_k(t) - f(t) \in -N_{C(t)}(\dot{u}_k(t)) \) is equivalent to

\[
\langle A_1 \dot{u}_k(t) + A_0 u_k(t) - f(t), \dot{u}_k(t) - z \rangle \leq 0 \quad \forall \, z \in C(t). \tag{28}
\]

For each \( k \in \mathbb{N} \), (28) holds for almost every \( t \in [0, T] \). Thus, there exists a subset \( D_k \subset [0, T] \) having zero Lebesgue measure that (28) holds for every \( t \in [0, T] \setminus D_k \). Putting \( D = \bigcup_{k \in \mathbb{N}} D_k \), we see that \( D \) is a subset of zero Lebesgue measure and (28) holds for all \( k \in \mathbb{N} \) and for every \( t \in [0, T] \setminus D \). For each \( t \) from \([0, T] \setminus D\), passing the inequality in (28) to the limit yields

\[
\langle A_1 \dot{u}(t) + A_0 u(t) - f(t), \dot{u}(t) - z \rangle \leq 0 \quad \forall \, z \in C(t).
\]

Thus, for almost every \( t \in [0, T] \), one has \( A_1 \dot{u}(t) + A_0 u(t) - f(t) \in -N_{C(t)}(\dot{u}(t)) \).

We have thus proved that \( u \in \text{Sol}(P, u_0) \) and, therefore, established the desired closedness of \( \text{Sol}(P, u_0) \) in \( W^{1,1}([0, T], \mathcal{H}) \). \( \blacksquare \)

6. Convexity of the solution set

As the normal cone in the sense of convex analysis to a convex set can be presented in a variational way, sweeping processes and variational inequalities are closely related. So, the convexity of the solution set of a sweeping process may have some connections with that property of the solution set of a variational inequality.

**Theorem 6.1:** If the assumption (H1) is fulfilled and \( A_0 = 0 \), then \( \text{Sol}(P, u_0) \) is convex for every \( u_0 \in C(0) \).

**Proof:** Let \( u_0 \in C(0) \) be taken arbitrarily. It suffices to consider the case where \( \text{Sol}(P, u_0) \) is nonempty. Under the Assumption (H1) and the condition \( A_0 = 0 \), an absolutely continuous function \( u \) belongs to \( \text{Sol}(P, u_0) \) if and only if \( u(0) = u_0 \) and

\[
\langle A_1 \dot{u}(t) - f(t), y - \dot{u}(t) \rangle \geq 0 \quad \forall \, y \in C(t)
\]

for a.e. \( t \in [0, T] \). The latter means that \( z(t) := \dot{u}(t) \) is a solution of the variational inequality

\[
\langle F(z, t), y - z \rangle \geq 0 \quad \forall \, y \in C(t)
\]

for a.e. \( t \in [0, T] \), where \( F(z, t) := A_1 z - f(t) \). By the assumed positive semidefiniteness of \( A_1 \), one has

\[
(F(z', t) - F(z, t), z' - z) = \langle A_1(z' - z), z' - z \rangle \geq 0
\]

for every \( z, z' \in \mathcal{H} \). Hence, \( F(\cdot, t) : \mathcal{H} \to \mathcal{H} \) is a monotone operator. Moreover, since the linear operator \( A_1 \) is bounded, \( F(\cdot, t) \) is continuous. Therefore, applying Minty’s lemma [26, Lemma 1.5] for
the monotone variational inequality (29), we can assert that the solution set of (29) is closed
convex for every \( t \in [0, T] \). Consequently, if \( u, v \) are two elements of \( \text{Sol}(P, u_0) \) and \( \lambda \in (0, 1) \) is
given arbitrarily, \((1 - \lambda)\dot{u}(t) + \lambda \dot{v}(t)\) is a solution of (29) for almost every \( t \in [0, T] \). Since \( t \mapsto (1 - \lambda)\dot{u}(t) + \lambda \dot{v}(t) \) is Bochner integrable (see [21, Proposition 1.4.17]), the formula
\[
w(t) := u_0 + \int_0^t [(1 - \lambda)\dot{u}(\tau) + \lambda \dot{v}(\tau)] \, d\tau
\]
defines an absolutely continuous function. Clearly, \( w(0) = u_0 \). In addition, we have
\[
\dot{w}(t) = (1 - \lambda)\dot{u}(t) + \lambda \dot{v}(t)
\]
for a.e. \( t \in [0, T] \) (see, e.g. [6, Remark 3.4(d)]). So, \( w(t) \) is a solution of (29) for a.e. \( t \in [0, T] \). This implies that
\[
A_1 \dot{w}(t) + A_0 w(t) - f(t) = -NC_C(t)(\dot{w}(t)) \quad \text{a.e. } t \in [0, T].
\]

Hence, \( w \in \text{Sol}(P, u_0) \). The convexity of \( \text{Sol}(P, u_0) \) has been proved. \( \blacksquare \)

The kernel of the operator \( A_0 : \mathcal{H} \to \mathcal{H} \) plays an important role in the forthcoming results. Recall that \( \ker A_0 := \{ x \in \mathcal{H} \mid A_0 x = 0 \} \). Note that the quadratic form \( \varphi(y) := \langle A_0 y, y \rangle \) is Fréchet differentiable on \( \mathcal{H} \) because \( A_0 \) is bounded (see, e.g. [27, Proposition 2.1]). Since \( \langle A_0 y, y \rangle \geq 0 \) for all \( y \in \mathcal{H} \), a vector \( x \in \mathcal{H} \) satisfies the equality \( \langle A_0 x, x \rangle = 0 \) if and only if \( x \) solves the optimization problem
\[
\min \{ \varphi(y) \mid y \in \mathcal{H} \}.
\]
If \( x \) is a solution of the latter, then by the Fermat rule one has \( \nabla \varphi(x) = 0 \), i.e. \( A_0 x = 0 \). Conversely, if \( A_0 x = 0 \) then \( \varphi(x) = 0 \). Therefore, we have
\[
\{ x \in \mathcal{H} \mid \langle A_0 x, x \rangle = 0 \} = \ker A_0.
\]

Under a mild assumption, using one solution \( u \) of \((P)\), we can construct a closed convex set \( \mathcal{K} \) in \( W^{1,1}([0, T], \mathcal{H}) \), such that the solution set \( \text{Sol}(P, u_0) \) is contained in \( u + \mathcal{K} \). Thus, the closed convex set \( u + \mathcal{K} \) is an outer estimate for \( \text{Sol}(P, u_0) \). The estimate is sharp, because in some cases it holds as an equality (see Theorem 6.3).

**Theorem 6.2:** Suppose that \((H1)\) is satisfied. For any \( u_0 \in C(0) \), if \( \text{Sol}(P, u_0) \) is nonempty and \( u \) is a selected solution of \((P)\), then
\[
\text{Sol}(P, u_0) \subset u + \mathcal{K},
\]
where
\[
\mathcal{K} := \{ y \in W^{1,1}([0, T], \mathcal{H}) \mid y(0) = 0, \ y(t) \in (C(t) - \dot{u}(t)) \cap \ker A_0 \ \text{a.e. } t \in [0, T] \}.
\]
is a closed convex set.

**Proof:** Select a solution \( u \) of \((P)\). Let \( v \in \text{Sol}(P, u_0) \) be chosen arbitrarily. Since \((H1)\) is fulfilled, we have
\[
\begin{align*}
\langle A_1 \dot{u}(t) + A_0 u(t) - f(t), \dot{u}(t) - z \rangle & \leq 0 \quad \forall z \in C(t), \\
\langle A_1 \dot{v}(t) + A_0 v(t) - f(t), \dot{v}(t) - z \rangle & \leq 0 \quad \forall z \in C(t)
\end{align*}
\]
for a.e. \( t \in [0, T] \). As \( \dot{u}(t) \) and \( \dot{v}(t) \) belong to \( C(t) \) for almost every \( t \in [0, T] \), the latter implies that
\[
\langle A_1 \dot{u}(t) + A_0 u(t) - f(t), \dot{u}(t) - \dot{v}(t) \rangle \leq 0
\]
and
\[
\langle A_1 \dot{v}(t) + A_0 v(t) - f(t), \dot{v}(t) - \dot{u}(t) \rangle \leq 0
\]
for a.e. \( t \in [0, T] \). From the last inequalities one gets
\[
\langle A_1 (\dot{u}(t) - \dot{v}(t)) + A_0 (u(t) - v(t)), \dot{u}(t) - \dot{v}(t) \rangle \leq 0
\]
for a.e. \( t \in [0, T] \). As \( A_1 \) is positive semidefinite, it follows that

\[
\langle A_0(u(t) - v(t)), \dot{u}(t) - \dot{v}(t) \rangle \leq 0
\]

for a.e. \( t \in [0, T] \). Integrating both sides of the last inequality and applying [23, Remarks 11.23(c)] yield

\[
\int_0^t \langle A_0(u(\tau) - v(\tau)), \dot{u}(\tau) - \dot{v}(\tau) \rangle d\tau \leq 0 \quad \forall \ t \in [0, T].
\]

As it has been noted in the proof of Theorem 3.1, this implies

\[
\langle A_0(u(t) - v(t)), u(t) - v(t) \rangle - \langle A_0(u(0) - v(0)), u(0) - v(0) \rangle \leq 0 \quad \forall \ t \in [0, T].
\]

Since \( u(0) = v(0) \), the latter means that \( \langle A_0(u(t) - v(t)), u(t) - v(t) \rangle \leq 0 \) for all \( t \in [0, T] \). So, by the positive semidefiniteness of \( A_0 \), we obtain

\[
\langle A_0(u(t) - v(t)), u(t) - v(t) \rangle = 0 \quad \forall \ t \in [0, T].
\]

Therefore, setting \( x(t) := v(t) - u(t) \), \( t \in [0, T] \), by (30) we have \( x(0) \in \ker A_0 \) for all \( t \in [0, T] \). It is clear that \( x(0) = v(0) - u(0) = 0 \) and \( \dot{x}(t) = \dot{v}(t) - \dot{u}(t) \in C(t) - \dot{u}(t) \) for a.e. \( t \in [0, T] \). Since \( x(\cdot) \) is an absolutely continuous function, from the condition \( A_0x(t) = 0 \) for all \( t \in [0, T] \), we deduce that \( A_0\dot{x}(t) = 0 \) for a.e. \( t \in [0, T] \). Hence, \( \dot{x} \in \mathcal{K} \). We have thus shown that (31) is valid. The convexity and closedness of \( \mathcal{K} \) can be easily verified by using the convexity and closedness of \( C(t) \) for all \( t \in [0, T] \). \( \blacksquare \)

In the next theorem, we investigate the convexity of the solution set in the case where \( A_0 \neq 0 \).

**Theorem 6.3:** Suppose that (H1) is satisfied, \( A_1 = 0 \), and \( f(t) \perp \ker A_0 \) (i.e. \( \langle f(t), x \rangle = 0 \) for every \( x \in \ker A_0 \) for all \( t \in [0, T] \)). Then, \( \text{Sol}(P, u_0) \) is convex for every \( u_0 \in C(0) \).

**Proof:** Let \( u_0 \in C(0) \) be given arbitrarily and \( u \) be a solution of (P). By Theorem 31, the inclusion (31), where the set \( \mathcal{K} \) is defined in (32), holds. Take any \( x \in \mathcal{K} \). Then, the function \( v \) defined by setting \( v(t) = u(t) + x(t), t \in [0, T] \), is a solution of (P). Indeed, for almost every \( t \in [0, T] \), one has

\[
\dot{v}(t) = \dot{u}(t) + \dot{x}(t) = \dot{u}(t) + (C(t) - \dot{u}(t)) = C(t).
\]

Note that \( v(0) = u(0) + x(0) = u_0 \). Since \( \dot{x}(t) \in \ker A_0 \) for a.e. \( t \in [0, T] \), \( x(0) = 0 \), and the linear operator \( A_0 \) is bounded, by [21, Proposition 1.4.22], we have

\[
A_0x(t) = A_0 \left( x(0) + \int_0^t \dot{x}(\tau) \, d\tau \right) = A_0 \int_0^t \dot{x}(\tau) \, d\tau = \int_0^t A_0\dot{x}(\tau) \, d\tau = 0 \quad (33)
\]

for all \( t \in [0, T] \). By \( \Omega \), we denote the set of all \( t \in [0, T] \) where the derivatives \( \dot{u}(t), \dot{x}(t) \) exist, \( \dot{x}(t) \in (C(t) - \dot{u}(t)) \cap \ker A_0 \), and \( A_0u(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}(t)) \). By our assumptions, \( \Omega \) is a subset of full measure of \([0, T]\). For any \( t \in \Omega \) and for any \( z \in C(t) \), by (33) we have

\[
\langle A_0v(t) - f(t), z - v(t) \rangle = \langle A_0(u(t) + x(t)) - f(t), z - (\dot{u}(t) + \dot{x}(t)) \rangle
\]

\[
= \langle A_0u(t) - f(t), z - (\dot{u}(t) + \dot{x}(t)) \rangle
\]

\[
= \langle A_0u(t) - f(t), z - \dot{u}(t) \rangle - \langle A_0u(t), \dot{x}(t) \rangle + \langle f(t), \dot{x}(t) \rangle
\]

\[
= \langle A_0u(t) - f(t), z - \dot{u}(t) \rangle - (u(t), A_0\dot{x}(t)) + \langle f(t), \dot{x}(t) \rangle.
\]
Since \( \dot{x}(t) \in \ker A_0 \) and \( f(t) \perp \ker A_0 \), it follows that \( \langle u(t), A_0 \dot{x}(t) \rangle = 0 \) and \( \langle f(t), \dot{x}(t) \rangle = 0 \). Therefore,

\[
\langle A_0 v(t) - f(t), z - \dot{v}(t) \rangle = \langle A_0 u(t) - f(t), z - \dot{u}(t) \rangle.
\]

(34)

As \( u \in \text{Sol}(P, u_0) \), the right hand side of (34) is nonnegative. Hence, from (34), we can deduce that \( \langle A_0 v(t) - f(t), z - \dot{v}(t) \rangle \geq 0 \). Since \( z \in C(t) \) can be chosen arbitrarily, we get

\[
\langle A_0 v(t) - f(t), z - \dot{v}(t) \rangle \geq 0 \quad \forall \ z \in C(t)
\]

for all \( t \in \Omega \). Equivalently, \( A_0 v(t) - f(t) \in -N_{C(t)}(\dot{v}(t)) \) for all \( t \in \Omega \). It follows that \( v \) is a solution of \( (P) \). So, we have proved that \( u + \mathcal{K} \subset \text{Sol}(P, u_0) \). Combining this with (31) yields \( \text{Sol}(P, u_0) = u + \mathcal{K} \). Hence, the desired convexity of \( \text{Sol}(P, u_0) \) follows from the convexity of the set \( u + \mathcal{K} \). \[\square\]

In connection with Theorems 6.1–6.3, we would like to raise the following open questions.

**Question 1.** We wonder if the assumptions \( A_1 = 0 \) and \( f(t) \perp \ker A_0 \) for all \( t \in [0, T] \) could be dropped in the formulation of Theorem 6.3? In other words, does estimate (31) hold as an equality just under the Assumption (H1)?

**Question 2.** Is there any example showing that, under the Assumption (H1), the solution set of \( (P) \) could be nonconvex?

### 7. Conclusions

For sweeping processes with convex velocity constraints, we have obtained several new results on the solution sensitivity with respect to the initial value, as well as the closedness, the boundedness, and the convexity of the solution set. In addition, an outer estimate for the solution set is also given. Hoping for further in-depth studies on the solution set, we have proposed two open questions.

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