Von Neumann and Luders postulates and quantum information theory

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Abstract

This note is devoted to some foundational aspects of quantum mechanics (QM) related to quantum information (QI) theory, especially quantum teleportation and “one way quantum computing.” We emphasize the role of the projection postulate (determining post-measurement states) in QI and the difference between its Lüders and von Neumann versions. These projection postulates differ crucially in the case of observables with degenerate spectra. Such observables play the fundamental role in operations with entangled states: any measurement on one subsystem is represented by an observable with degenerate spectrum in the Hilbert space of a composite system. If von Neumann was right and Lüders was wrong the canonical schemes of quantum teleportation and “one way quantum computing” would not work. Surprisingly, we found that, in fact, von Neumann’s description of measurements via refinement implies (under natural assumptions) Lüders projection postulate. It seems that this important observation was missed during last 70 years. This result closed the problem of the proper use of the projection postulate in quantum information theory. One can proceed with Lüders postulate (as people in quantum information really do).

1 Introduction

Although the QI project approached the stage of technological (at least experimental) realizations, research on foundational problems related to quantum information processing\footnote{See, e.g., recent book of G. Jaeger \cite{Jaeger} and paper of M. Asano, M. Ohya, and Y. Tanaka \cite{Asano}.} did not become less important. Moreover, many problems in foundations of QM which were considered as of pure theoretical (or even philosophical) value nowadays play an important role in (expensive) technological projects. Thus such problems could not be simply ignored. Development of QI also induces new approaches which foundational basis should be carefully analyzed. Among such novel approaches I would like to mention quantum teleportation and “one way
quantum computing”, see, e.g., [3]–[5] – an exciting alternative to the
conventional scheme of quantum computing.

In a recent series of papers [6]–[9] the author paid attention on crucial
difference of consequences of von Neumann [10] and Lüders [11] projec-
tion postulates for QI, staring with EPR-argument [12]. These postulates
coincide for observables with nondegenerate spectra, but they differ in the
case of degenerate spectra. We remark that the latter case is the most
important for quantum information theory, since measurement on one of
systems in a pair of entangled systems is represented by an operator with
degenerate spectrum.

While Lüders [11] projection postulate is fine for QI, the appeal to
von Neumann postulate induces serious problems [13]. In the first case mea-
surement on a subsystem produces a pure state for another subsystem
and it is good for quantum teleportation and computing. However, in
the second case even starting with a pure state for a composite system,
one obtains in general a statistical mixture. Moreover, by von Neumann
the formalism of QM does not predict the post measurement state in the
case of degenerate spectrum. Thus even mentioned statistical mixture is
unknown. In [10] it was emphasized that measurement of observables
represented by operators with degenerate spectra are ambiguous. This
problem can be solved (due to von Neumann) only via refinement mea-
surements. One should find an observable, say $B$, represented by an
operator $\hat{B}$ with nondegenerate spectrum which commutes with the original
operator $\hat{A}$ with degenerate spectrum. Then results of $A$-measurement
are obtained as $A = f(B)$, where $f$ is the function coupling the operators:
$A = f(\hat{B})$. Since $B$ can be chosen in various ways, one can select various
measurement procedures for $A$-measurement. It is crucial for foundations
of QI that for composite systems refinement of measurement on one of
subsystems can be approached only via combined measurement on both
subsystems. If it is really the case and von Neumann was right, then
foundations of QI should be carefully reconsidered, since a number of im-
portant procedures in QI processing is based on Lüders postulate. First
of all we mention quantum teleportation. It were impossible to teleport
an unknown quantum state in von Neumann’s framework, see [10]. Alice
evidently uses Lüders postulate to be sure that her measurement would
produce the corresponding pure state for Bob (then Bob needs only to
perform a local unitary evolution to get the proper state).

The situation in quantum computing is not completely clear. It seems
that the post-measurement state does not play any role in the conven-
tional scheme of quantum computation: unitary evolution and, finally, measure-
ment of a proper observable. It seems that only probabilities of results are
important. Probabilities are calculated in the same way both in Lüders
and von Neumann’s approach. The situation is completely different in the
case of so called “one way quantum computing”, see, e.g., [3]–[5]. This
scheme (based on measurements, instead of unitary evolution) depends
crucially on the possibility to use Lüders postulate. It would not work if
von Neumann was right and Lüders was wrong.

To my surprise, recently I found that, in fact, von Neumann’s descrip-
tion of measurements via refinement implies (under natural assumptions)
Lüders projection postulate. It seems that this important observation was
missed during last 70 years. This result closed the problem of the proper

2By using an observable represented by an operator with nondegenerate spectrum commuting with operator with degenerate spectrum representing the original observable.
use of the projection postulate in quantum information theory. One can proceed with Lüders postulate (as people in quantum information really do).

2 Von Neumann’s and Lüders’ postulates for pure states

2.1 Nondegenerate (discrete) spectrum

 Everywhere below $\mathcal{H}$ denotes complex Hilbert space. Let $\psi \in \mathcal{H}$ be a pure state, i.e., $\|\psi\|^2 = 1$. We remark that any pure state induces density operator:

$$\rho_{\psi} = \psi \otimes \psi = \hat{P}_{\psi}$$

where $\hat{P}_{\psi}$ denotes the orthogonal projector on the vector $\psi$. This operator describes an ensemble of identically prepared systems each of them in the same state $\psi$.

For an observable $A$ represented by the operator $\hat{A}$ with nondegenerate spectrum von Neumann’s and Lüders projection postulates coincide. For simplicity we restrict our considerations to operators with purely discrete spectra. In this case spectrum consists of eigenvalues $\alpha_k$ of $\hat{A} : \hat{A}e_k = \alpha_k e_k$. Nondegeneracy of spectrum means that subspaces consisting of eigenvectors corresponding to different eigenvalues are one dimensional.

PP: Let $A$ be an observable described by the self-adjoint operator $\hat{A}$ having purely discrete nondegenerate spectrum. Measurement of observable $A$ on a system in the (pure) quantum state $\psi$ producing the result $A = \alpha_k$ induces transition from the state $\psi$ into the corresponding eigenvector $e_k$ of the operator $\hat{A}$.

If we select only systems with the fixed measurement result $A = \alpha_k$, then we obtain an ensemble described by the density operator $\hat{q}_{\psi} = e_k \otimes e_k$. Any system in this ensemble is in the same state $e_k$. If we do not perform selections, we obtain an ensemble described by the density operator

$$\hat{q}_{\psi} = \sum_k |\langle \psi, e_k \rangle|^2 \hat{P}_{e_k} = \sum_k \langle \hat{\rho}_{\psi} e_k, e_k \rangle \hat{P}_{e_k} = \sum_k \hat{P}_{e_k} \hat{\rho}_{\psi} \hat{P}_{e_k},$$

where $\hat{P}_{e_k}$ is projector on the eigenvector $e_k$.

2.2 Degenerate (discrete) spectrum: Lüders viewpoint

Lüders generalized this postulate to the case of operators having degenerate spectra. Let us consider spectral decomposition for a self-adjoint operator $\hat{A}$ having purely discrete spectrum:

$$\hat{A} = \sum_i \alpha_i \hat{P}_i,$$

where $\alpha_i \in \mathbb{R}$ are different eigenvalues of $\hat{A}$ (so $\alpha_i \neq \alpha_j$) and $\hat{P}_i, i = 1, 2, \ldots$, is projector onto subspace $\mathcal{H}_i$ of eigenvectors corresponding to $\alpha_i$. 

By Lüders’ postulate after measurement of an observable $A$ represented by the operator $\hat{A}$ that gives the result $\alpha_i$ the initial pure state $\psi$ is transformed again into a pure state, namely,

$$\psi_i = \frac{\hat{P}_i \psi}{\|\hat{P}_i \psi\|}.$$ 

Thus for corresponding density operator we have

$$\hat{Q}_i = \psi_i \otimes \psi_i = \frac{\hat{P}_i \psi \otimes \hat{P}_i \psi}{\|\hat{P}_i \psi\|^2} = \frac{\hat{P}_i \rho \hat{P}_i}{\|\hat{P}_i \psi\|^2}.$$ 

If one does not make selections corresponding to values $\alpha_i$ the final post-measurement state is given by

$$\hat{q}_\psi = \sum_i p_i \hat{Q}_i, \quad p_i = \|\hat{P}_i \psi\|^2, \quad (1)$$

or simply

$$\hat{q}_\psi = \sum_i \hat{q}_i, \quad \hat{q}_i = \hat{P}_i \rho \hat{P}_i. \quad (2)$$

This is the statistical mixture of pure states $\psi_i$. Thus by Lüders there is no essential difference between measurements of observables with degenerate and nondegenerate spectra.

### 2.3 Degenerate (discrete) spectrum: von Neumann’s viewpoint

Von Neumann had the completely different viewpoint on the post-measurement state [10]. Even for a pure state $\psi$ the post-measurement state (for measurement with selection with respect to a fixed result $A = \alpha_k$) will not be a pure state again. If $\hat{A}$ has degenerate (discrete) spectrum, then according to von Neumann [10]

*A measurement of an observable $A$ giving the value $A = \alpha_i$ does not induce projection of $\psi$ on the subspace $H_i$.*

The result will not be the fixed pure state, in particular, not Lüders’ state $\psi_i$. Moreover, the post-measurement state, say $\hat{q}_\psi$, is not determined by the formalism of QM! Only a subsequent measurement of an observable $D$ such that $A = f(D)$ and $\hat{D}$ is an operator with nondegenerate spectrum (“refinement measurement”) will determine the final state.

Following von Neumann, we choose in each $H_i$ an orthonormal basis $\{e_{in}\}$. Let us take sequence of real numbers $\{\gamma_{in}\}$ such that all numbers are distinct. We define the corresponding self-adjoint operator $\hat{D}$ having eigenvectors $\{e_{in}\}$ and eigenvalues $\{\gamma_{in}\}$:

$$\hat{D} = \sum_i \sum_n \gamma_{in} \hat{P}_{e_{in}}.$$ 

A measurement of the observable $D$ represented by the operator $\hat{D}$ can be considered as measurement of the observable $A$, because $A = f(D)$, where $f$ is some function such that $f(\gamma_{in}) = \alpha_i$. The $D$-measurement (without post-measurement selection with respect to eigenvalues) produces the statistical mixture

$$\hat{O}_{D,\psi} = \sum_i \sum_n |\langle \psi, e_{in} \rangle|^2 \hat{P}_{e_{in}}. \quad (3)$$
By selection for the value \( \alpha_i \) of \( A \) (its measurements realized via measurements of a refinement observable \( D \)) we obtain the statistical mixture described by normalization of the operator

\[
\hat{O}_{i,D} = \sum_n |\langle \psi, e_n \rangle|^2 \hat{P}_{e,n}.
\] (4)

Von Neumann emphasized that the mathematical formalism of QM could not describe the post-measurement state for measurements (without refinement) of degenerate observables. He did not discuss directly properties of such a state, he described them only indirectly via refinement measurements. We would like to proceed by considering this (“hidden”) state under assumption that it can be described by a density operator, say \( \hat{g}_\psi \).

We formalize a list of properties of this hidden (post-measurement) state which can be extracted from von Neumann’s considerations on refinement measurements. Finally, we prove, see Theorem 1, that \( \hat{g}_\psi \) should coincide with the post-measurement state postulated by Lüders, (2).

Consider the \( A \)-measurement without refinement. By von Neumann, for each quantum system \( s \) in the initial pure state \( \psi \), the \( A \)-measurement with the \( \alpha_i \)-selection transforms the \( \psi \) in one of states \( \phi = \phi(s) \) belonging to the eigensubspace \( H_i \). Unlike Lüders’ approach, it implies that, instead of one fixed state, namely, \( \psi_i \in H_i \), such an experiment produces a probability distribution of states on the unit sphere of the subspace \( H_i \).

We postulate

\[ \text{DO For any value } \alpha_i \text{ such that } \hat{P}_i \psi \neq 0, \text{ the post-measurement probability distribution on } H_i \text{ can be described by a density operator, say } \hat{G}_i. \]

Here \( \hat{G}_i : H_i \to H_i \) is such that \( \hat{G}_i \geq 0 \) and \( \text{Tr} \hat{G}_i = 1 \). Consider now the corresponding density operator \( \hat{G}_i \) in \( H \). Its restriction on \( H_i \) coincides with \( \hat{G}_i \). In particular this implies its property:

\[ \hat{G}_i(H_i) \subset H_i. \] (5)

We remark that \( \hat{G}_i \) is determined by \( \psi \), so \( \hat{G}_i \equiv \hat{G}_{i,\psi} \).

We would like to present the list of other properties of \( \hat{G}_i \) induced by von Neumann’s considerations on refinement. Since, for each refinement measurement \( D \), the operators \( \hat{A} \) and \( \hat{D} \) commute, the measurement of \( A \) with refinement can be performed in two ways. First we perform the \( D \)-measurement and then we get \( A \) as \( A = f(D) \). However, we also can first perform the \( A \)-measurement, obtain the post-measurement state described by the density operator \( \hat{G}_i \), then measure \( D \) and, finally, we again find \( A = f(D) \).

Take an arbitrary \( \phi \in H_i \) and consider a refinement measurement \( D \) such that \( \phi \) is an eigenvector of \( \hat{D} \). Thus \( \hat{D} \phi = \gamma_\phi \phi \). Then for the cases – [direct measurement of \( D \)] and [first \( A \) and then \( D \)] – we get probabilities which are coupled in a simple way. In the first case (by Born’s rule)

\[ P(D = \gamma_\phi | \hat{D}_\psi) = | \langle \psi, \phi \rangle |^2. \] (6)

In the second case, after the \( A \)-measurement, we obtain the state \( \hat{G}_i \) with probability

\[ P(A = \alpha_i | \hat{P}_i \psi) = \| \hat{P}_i \psi \|^2. \]

\(^3\)For him this state was a kind of hidden variable. It might even be that he had in mind that it “does not exist at all”, i.e., it could not be described by a density operator.
Performing the $D$-measurement for the state $\tilde{G}_i$ we get the value $\gamma_\phi$ with probability:

$$P(D = \gamma_\phi | \tilde{G}_i) = \text{Tr} \tilde{G}_i \tilde{P}_\phi.$$

(7)

By (classical) Bayes’ rule

$$P(D = \gamma_\phi | \tilde{P}_\psi) = P(A = \alpha_i | \tilde{P}_\psi) P(D = \gamma_\phi | \tilde{G}_i).$$

(8)

Finally, we obtain

$$P(D = \gamma_\phi | \tilde{G}_i) = \text{Tr} \tilde{G}_i \tilde{P}_\phi = \frac{|\langle \psi, \phi \rangle|^2}{\|\tilde{P}_i \psi\|^2}.$$  (9)

Thus

$$\text{Tr} \tilde{G}_i \tilde{P}_\phi = \frac{|\langle \psi, \phi \rangle|^2}{\|\tilde{P}_i \psi\|^2}.$$  (10)

This is one of the basic features of the post-measurement state $\tilde{G}_i$ (for the $A$-measurement with the $\alpha_i$-selection, but without any refinement).

Another basic equality we obtain in the following way. Take an arbitrary $\phi' \in H_i^{\perp}$, and consider a measurement of the observable described by the orthogonal projector $\tilde{P}_\psi$ under the state $\tilde{G}_i$. Since the later describes a probability distribution concentrated on $H_i$, we have:

$$P(P_{\phi'} = 1 | \tilde{G}_i) = 0.$$  (11)

Thus

$$\text{Tr} \tilde{G}_i \tilde{P}_{\phi'} = 0.$$  (12)

This is the second basic feature of the post-measurement state. Our aim is to show that (10) and (12) imply that, in fact,

$$\tilde{G}_i = \tilde{P}_i \tilde{\rho}_\psi \tilde{P}_i / \|\tilde{P}_i \psi\|^2 \equiv \tilde{P}_i \psi \otimes \tilde{P}_i \psi / \|\tilde{P}_i \psi\|^2.$$  (13)

i.e., to derive Lüders postulate which is a theorem in our approach.

**Lemma.** The post-measurement density operator $\tilde{G}_i$ maps $H$ into $H_i$.

**Proof.** By (15) it is sufficient to show that $\tilde{G}_i (H_i^{\perp}) \subset H_i$. By (12) we obtain

$$< \tilde{G}_i \phi', \phi' > = 0$$  (14)

for any $\phi' \in H_i^{\perp}$. This immediately implies that $< \tilde{G}_i \phi'_1, \phi'_2 > = 0$ for any pair of vectors from $H_i^{\perp}$. The latter implies that $\tilde{G}_i \phi' \in H_i$ for any $\phi' \in H_i^{\perp}$.

Consider now the $A$-measurement without refinement and selection. The post-measurement state $\tilde{g}_\psi$ can be represented as

$$\tilde{g}_\psi = \sum_m p_m \tilde{G}_m, \quad p_m \|\tilde{P}_m \psi\|^2,$$  (15)

**Proposition 1.** For any pure state $\psi$ and self-adjoint operator $\tilde{A}$ (with purely discrete degenerate) spectrum the post-measurement state (in the absence of refinement measurement) can be represented as

$$\tilde{g}_\psi = \sum_m \tilde{g}_m,$$  (16)

where $\tilde{g}_m : H \rightarrow H_m, \tilde{g}_m \geq 0$, and, for any $\phi \in H_m,$

$$< \tilde{g}_m \phi, \phi > = |< \psi, \phi >|^2.$$  (17)
3 Derivation of Luders’ postulate from von Neumann’s postulate

**Theorem.** Let \( \hat{\varrho} \equiv \hat{\varrho}_\psi \) be a density operator described by Proposition 1. Then

\[
\hat{g}_m = \hat{P}_m \psi \otimes \hat{P}_m \psi. \tag{18}
\]

**Proof.** Let \( \{e_{mk}\} \) be an orthonormal basis in \( H_m \) and let \( u \in H \). We represent it as \( u = u_m + u_m^\perp \), where \( u_m \in H_m \) and \( u_m^\perp \in H_m^\perp \). Then

\[
< \hat{g}_m u, u > = < \hat{g}_m u_m, u_m > + < \hat{g}_m u_m^\perp, u_m > + < \hat{g}_m u_m^\perp, u_m^\perp > + < \hat{g}_m u_m^\perp, u_m^\perp >.
\]

The second and last terms equals to zero, since \( \hat{g}_m : H \rightarrow H_m \). To show that the third term also equals to zero, we should use self-adjointness of \( \hat{g}_m \). Thus

\[
< \hat{g}_m u, u > = \sum_{k,k'} < u, e_{ku} > < e_{mk'}, u > < \hat{g}_m e_{ku}, e_{mk'} >.
\]

For each \( e_{mn} \), we have \( < \hat{g}_m e_{mn}, e_{mn} > = | < \psi, e_{mn} > |^2 \). Thus the diagonal elements of the matrix of operator \( \hat{g}_m \) coincide with diagonal elements of operator \( \hat{P}_m \psi \otimes \hat{P}_m \psi \). Take now another basis in \( H_m \) which is constructed in the following way. We fix two indexes, say \( n \) and \( j \), and choose two new basis vectors:

\[
f_{mn} = (e_{mn} + e_{mj})/\sqrt{2}, \quad f_{mj} = (e_{mn} - e_{mj})/\sqrt{2}.
\]

Then we have \( < \hat{g}_m f_{mn}, f_{mn} > = | < \psi, f_{mn} > |^2 \), or

\[
< \hat{g}_m e_{mn}, e_{mn} > + < \hat{g}_m e_{mj}, e_{mj} > + < \hat{g}_m e_{mn}, e_{mj} > + < \hat{g}_m e_{mj}, e_{mn} >
\]

\[
= | < \psi, e_{mn} > |^2 + | < \psi, e_{mj} > |^2 + < \psi, e_{mn} > < e_{mj}, \psi > + < \psi, e_{mj} > < e_{mn}, \psi >.
\]

Thus

\[
< \hat{g}_m e_{mn}, e_{mj} > + < \hat{g}_m e_{mn}, e_{mj} >
\]

\[
= < \psi, e_{mn} > < e_{mj}, \psi > + < \psi, e_{mn} > < e_{mj}, \psi >.
\]

Then we proved that \( \text{Re} [ < \hat{g}_m e_{mn}, e_{mj} > ] = \text{Re} [ < \psi, e_{mn} > < e_{mj}, \psi > ] \). Let us now choose two new basis vectors

\[
\bar{f}_{mn} = (e_{mn} + ie_{mj})/\sqrt{2}, \quad \bar{f}_{mj} = (e_{mn} + ie_{mj})/\sqrt{2}.
\]

Then we have:

\[
< \hat{g}_m \bar{f}_{mn}, \bar{f}_{mn} > = < \hat{g}_m e_{mn}, e_{mn} > + < \hat{g}_m e_{mj}, e_{mj} >
\]

\[
+ i < \hat{g}_m e_{mj}, e_{mn} > - i < \hat{g}_m e_{mn}, e_{mj} >
\]

\[
= | < \psi, e_{mn} > |^2 + | < \psi, e_{mj} > |^2 + i < \psi, e_{mn} > < e_{mj}, \psi > - i < \psi, e_{mj} > < e_{mn}, \psi >.
\]

Thus:

\[
< \hat{g}_m e_{mn}, e_{mj} > + < \hat{g}_m e_{mn}, e_{mj} >
\]

\[
= < \psi, e_{mn} > < e_{mj}, \psi > + < \psi, e_{mn} > < e_{mj}, \psi >.
\]

Thus \( < \hat{g}_m e_{mn}, e_{mj} > = | < \psi, e_{mn} > < e_{mj}, \psi > |^2 \). We obtained the following representation for the quadratic form of the operator \( \hat{g}_m \)

\[
< \hat{g}_m u, u > = \sum_{k,k'} | < u, e_{mk} > < e_{mk'}, u > < \psi, e_{mk'}, e_{mk} > < \psi, e_{mk}, \psi > |^2.
\]

Hence \( \hat{g}_m = \hat{P}_m \psi \otimes \hat{P}_m \psi \).
Conclusion: The general scheme of measurement of observables with degenerate spectra provided by von Neumann [10] implies, in fact, the Lüders projection postulate. This postulate is a theorem (missed for 70 years) in von Neumann’s framework. Thus (in the canonical formalism of QM) the post-measurement state is always a pure state. This supports existing schemes of quantum teleportation and computing.

References

[1] G. Jaeger, Quantum information. An overview. Springer, Berlin, 2007.
[2] M. Asano, M. Ohya, and Y. Tanaka, Complete m-level teleportation based on Kossakowski-Ohya scheme. Proceedings of QBIC-2, Quantum Probability and White Noise Analysis, 24, 19-29 (2009).
[3] R. Rausendorf and J. Briegel, A one way quantum computer. Phys. Rev. Lett 86, 5188(2001).
[4] G. Vallone, E. Pomarico, F. De Martini, and P. Mataloni, One way quantum computation with two-photon multiqubit cluster state, arxiv:0807.3887.
[5] N. C. Menicucci, S. T. Flammia, O. Pfister, One way quantum computing in optical frequency comb, arxiv 0804.4468.
[6] A. Yu. Khrennikov, The role of von Neumann and Lüders postulates in the Einstein, Podolsky, and Rosen considerations: Comparing measurements with degenerate and nondegenerate spectra, J. Math. Phys., 49, N 5, art. no. 052102 (2008).
[7] A. Yu. Khrennikov, Analysis of the role of von Neumann’s projection postulate in the canonical scheme of quantum teleportation, J. Russian Laser Research, 29, N 3, 296-301 (2008).
[8] Khrennikov, Analysis of explicit and implicit assumptions in the theorems of J. Von Neumann and J. Bell. J. Russian Laser Research 28, 244 (2007).
[9] A. Yu. Khrennikov, EPR "Paradox", projection postulate, time synchronization “nonlocality”. Int. J. Quantum Information (IJQI), 7, N 1, 71 - 8 (2009),
[10] J. von Neumann, Matematische Grundlagen der Quantenmechanik (Springer, Berlin, 1932).
[11] G. Lüders, Ann. Phys., Lpz 8 322 (1951).
[12] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. 47, 777 (1935).