Edge theories for polarized quantum Hall states

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ABSTRACT

Starting from recently proposed bosonic mean field theories for fully and partially polarized quantum Hall states, we construct corresponding effective low energy theories for the edge modes. The requirements of gauge symmetry and invariance under global O(3) spin rotations, broken only by a Zeeman coupling, imply boundary conditions that allow for edge spin waves. In the generic case, these modes are chiral, and the spin stiffness differs from that in the bulk. For the case of a fully polarized $\nu = 1$ state, our results agree with previous Hartree-Fock calculations.

1. INTRODUCTION

The study of edge modes in the quantum Hall (QH) system is motivated by their importance for the charge transport, and also because they are believed to give important information about the bulk state.

A class of models which have been quite successful in describing the bulk QH states, are the Chern-Simons Ginzburg-Landau (CSGL) effective field theories. The simplest form of such a theory describes spinless QH systems at Laughlin filling fractions $\nu = 1/m$, but generalizations can also describe more complicated filling fractions, multilayer systems and effects of spin. Starting from a CSGL theory, it is possible to construct effective, one-dimensional chiral boson models describing the low energy physics at the edge. These models, both for the simple single-component spinless case and more general multicomponent cases, have been analyzed by Wen and others.

The QH states are incompressible, i.e. there is a finite energy gap to density fluctuations in the bulk. Because of the magnetic field, the ground state is usually at least partially polarized, and the spectrum also includes ferromagnetic spin waves. Generically, one would expect the Zeeman gap to be very large, but material effects can renormalize the $g$-factor to small enough values for the gap to be comparable to typical Coulomb energies. Thus it is interesting to study polarization effects, both in bulk and at the edge.

The perhaps most striking polarization effect is that, under certain conditions, the lowest charged excitation in a $\nu = 1$ state is a topological soliton, a so-called skyrmion, as originally predicted by Sondhi et al. Concerning the ground states, there is strong experimental evidence that at low electron densities fractional QH ground states at certain fractions such as 2/5 and 4/3 are not fully polarized. This has been established by studying the transport properties at a fixed filling fraction as a function of the applied magnetic field $B$ (either by tilting the field or by changing the electron density).

There have also been many studies of spin polarization effects at the edge of QH-states, and in particular Karlhede et al., using Hartree-Fock techniques and an effective sigma-model, predicted that for smooth enough potential the edge would reconstruct to a spin-textured state. Similar results were obtained by Oaknin et al. for quantum dots and by Leinaas and Viefers, who solved a full CSGL model of fully polarized QH states in the presence of an infinite edge.

Several papers have studied the interplay between density and spin fluctuations at the edge and derived the pertinent dispersion relations. In a very recent TDHF-study of the $\nu = 1$ state Karlhede, Lejnell and Sondhi find that in addition to the edge magnetoplasmons and bulk spin waves, there are also edge spin waves, which generically are chiral.

With this background, it is clearly interesting to construct a class of effective theories that describe the low energy physics both in the bulk and at the edge, and which are general enough to describe polarization effects around both fully and partially polarized ground states.
This is the aim of the present paper, where we start from a recently proposed effective CSGL model describing fully and partially polarized states. In addition to a two-component bosonic field $\phi$ and two Chern-Simons (CS) gauge fields, which together describe the electrons, this model also contains a $\sigma$-model field $\hat{m}$, describing the spin polarization direction. In the special case of full polarization, the model only involves a single scalar field $\phi$ and one CS field.

The main result of our analysis is an effective action consisting of a bulk part that is derived from the CSGL model by integrating out the high energy modes, and an edge part which is essentially a chiral boson Lagrangian of the type originally proposed by Wen, but coupled to the bulk in such a way that not only the total charge current, but also the total spin current is conserved (up to an explicit breaking due to the Zeeman effect). We analyze the small fluctuation spectrum of this model and find, in addition to the expected edge magnetoplasmons, also (gapped) edge spin waves which generically are chiral and, for soft edge potentials, propagate in the same direction as the edge magnetoplasmons.

The paper is organized as follows. In the next section we give the effective bulk Lagrangians together with a short physical motivation for their general form. The actual derivation from CSGL theories, which is rather technical, is referred to Appendix A. In the following two sections we construct the corresponding edge theories and show how the presence of edge currents and charge densities implies a boundary condition on the sigma model field describing the spin. Section 5 contains an analysis of the excitation spectrum of our model with emphasis on the edge spin waves. Here we also compare with the work of Kariyada et al. and Milovanović. We summarize our results in section 6, and an alternative way to derive the effective low-energy model for partially polarized states, using a dual CSGL formulation, is given in Appendix B.

2. CHERN-SIMONS BULK LAGRANGIANS

Consider a two-dimensional electron gas subject to a magnetic field of constant magnitude and direction, $\vec{B} = \nabla \times \vec{A}_{bg}$. In addition we probe the system with a weak external electromagnetic potential $\delta A^\mu$, so the total potential is $A^\mu = A^\mu_{bg} + \delta A^\mu$. In the following we shall include the electron charge in the gauge potentials, i.e. to get back to physical units one should make the replacement $A^\mu \rightarrow -e A^\mu$, where $e > 0$. In particular this means that electron densities and charge densities are both positive, and since the physical background magnetic field is taken in the positive $z$-direction, the background field $B_\perp \equiv e^i \partial_i A_{j,bg}$ will be negative. We also put $\hbar = c = 1$. Here and in the following repeated indices are summed over, but there is no distinction between upper and lower indices, and all signs are written explicitly (in particular, note that the potential energy is $-A_0$).

At the special filling fractions corresponding to QH plateaux, the electrons form incompressible quantum liquids, characterized by quantized Hall conductivities, $\sigma_H$. The electromagnetic response of such states is described by effective actions of the CS type; in the simplest case (corresponding to spinless electrons in the principal Laughlin states) the effective low-energy Lagrangian reads

$$\mathcal{L}_{el} = -\frac{l}{4\pi} \epsilon^{\mu\nu\sigma} A_\mu \partial_\nu A_\sigma \equiv -\frac{l}{4\pi} \delta A \quad ,$$

(2.1)

where $l^{-1}$ is an odd integer. The corresponding current is given by

$$J_\mu = \frac{\delta S}{\delta A_\mu} = -\frac{l}{2\pi} \epsilon_{\mu\nu\sigma} \partial^\nu A_\sigma \quad ,$$

(2.2)

with the correct Hall conductivity, $\sigma_H = l/2\pi$.

We shall now consider systems where the Zeeman gap is small so that spin waves should be included in the effective low energy theories; the Lagrangians will thus depend on a dynamical $\sigma$-model field $\hat{m}$, describing the spin polarization direction, in addition to the electromagnetic field $A$. In this section we give some simple arguments to motivate the form of the Lagrangians and then simply state the final results. The actual derivation from CSGL-theories is outlined in Appendix A.

Since we consider systems in a strong magnetic field, we expect the electrons to be (at least partially) polarized. Thus, the ground state is ferromagnetic, and the dynamics of the spin field $\hat{m}$ is described by an $O(3)$ $\sigma$-model. It would, however, be too simplistic just to add a $\sigma$-model Lagrangian to (2.1), since there is a non-trivial connection between spin and charge due to the strong magnetic field. This can be understood in many ways, e.g. by calculating the Berry phase corresponding to adiabatic motion of a single electron in a (static) polarized background of the other electrons. Assuming that the spin of the electron remains aligned with that of the background at all times, the Berry phase can be expressed as an Aharonov-Bohm phase due to a topological vector potential $\tilde{a}$, related to the spin mean field $\hat{m}$ via
\[ \hat{\alpha}^\mu = \chi^\dagger_\mu i \partial^\mu \chi_\mu \]  
(2.3)

where the spinor \( \chi_\mu \) is defined by

\[ \hat{\sigma}^i = \chi^\dagger_\mu \sigma^i \chi_\mu \]  
(2.4)

up to a gauge transformation. For fully polarized (fp) states, the electron cannot distinguish between the real electromagnetic potential \( A \) and the topological potential \( \hat{\sigma} \), so they can appear only in the combination \( A + \hat{\sigma} \). In this case the effective Lagrangian is given by

\[ \mathcal{L}_{eff}^{fp} = -\frac{\mu}{4\pi} (A + \hat{\sigma}) d(A + \hat{\sigma}) - \frac{\kappa \hat{\rho}}{4} (\hat{\nabla}^\mu \hat{\mu})^2 \]  
(2.5)

with the corresponding current

\[ J_\mu = -\frac{1}{2\pi} \epsilon_{\mu \nu \rho} \partial^\nu (A^\rho + \hat{\sigma}) \]  
(2.6)

where we denoted the ground state density by \( \hat{\rho} \), i.e. \( \hat{\rho} = \langle J_0 \rangle \). (In this context, we can replace \( J_0 \) by \( \hat{\rho} \) since the difference is given by higher derivative terms, as discussed in Appendix A.) It is not obvious that the Lagrangian \( (2.5) \) describes a \( \sigma \)-model, but in fact it does. The CS-part contains a term \( \sim B \cdot \hat{\sigma} \) which is nothing but the kinetic part of the \( \sigma \)-model, and the Zeeman term is included as a spin-dependent part of the scalar potential i.e. \( A^0 \rightarrow A^0 - (\mu / 2) \hat{B} \cdot \hat{\sigma} \). For further details we again refer to Appendix A. The Lagrangian \( (2.5) \) has also been discussed by Baez et al. [3].

Partially polarized (pp) ground states are characterized by two conserved charges, being the total number of electrons with spin pointing along and opposite to the mean field, \( \hat{m} \), respectively. Assuming that these charges are separately conserved amounts to neglecting the effects of spin-flip transitions. A spin wave corresponds to fluctuations in the direction of \( \hat{m} \), while keeping the local polarization \( \hat{P} \) fixed at its mean field value, \( \hat{\rho} \), and \( \hat{\sigma} \) with a shift in the potential \( \hat{P} \) (see eq. (A.15)). Corresponding to the two conserved charges, there are now two gauge symmetries: The first is related to electromagnetism, described by the potential \( \hat{A} \equiv A_1 \), and the second is related to the Berry phase, i.e. the potential \( \hat{\sigma} \equiv A_2 \). Here, we have introduced the convenient notation \( \hat{A}_\alpha, \alpha = 1, 2 \) for the two gauge fields. In this case the effective Lagrangian reads

\[ \mathcal{L}_{eff}^{pp} = -\frac{1}{2\pi} \epsilon_{\alpha \beta} \hat{A}_\alpha d\hat{A}_\beta - \frac{\kappa \hat{\rho} \hat{P}}{4} (\hat{\nabla} \hat{m})^2 \]  
(2.7)

where the form of the matrix \( \hat{I} \) is given in Appendix A. Again, the kinetic term for the \( \sigma \)-model is hidden in the CS-term, and the Zeeman term is included by a shift in the potential \( \hat{A}^0_2 \). The two conserved currents corresponding to \( (2.7) \) are

\[ J_\mu^\alpha = -\frac{1}{\pi} \epsilon_{\alpha \beta \kappa} \partial^\mu \hat{A}_\kappa \]  
(2.8)

In addition to the gauge symmetries of the fully- and partially polarized models, there is, in the absence of a Zeeman interaction, a global O(3) symmetry corresponding to rotations of the spin field \( \hat{m} \). In the following we shall assume that, even in the presence of an edge, the only explicit breaking of this symmetry is due to the Zeeman term.

The effective low-energy Lagrangians \( (2.3) \) and \( (2.7) \) are the starting point for the next section, where we construct the corresponding edge Lagrangians.

### 3. Edge Theories

In principle one should be able to derive the edge theory directly from the microscopic physics, and for the case of a spin polarized \( \nu = 1 \) state this was done by Stone [4]. For general filling fractions and polarizations, a possible route would be to start from a CSGL formulation and use mean field theory to find an explicit ground state solution in the presence of an edge potential. Then one would try to extract an effective edge Lagrangian. A step in this direction was taken by Leinaas and Viefers [18] who (numerically) found solutions to the CSGL mean field equations, showing the existence of edge spin textures, and further studied the gapless edge excitations (edge magnetoplasmons) of the system. In this paper we shall not take this approach, but instead, following Wen and others [12, 19] construct the edge theories using general arguments based on symmetry, incompressibility and topological order. In the spinless case, these properties imply the existence of massless chiral degrees of freedom at the edge of the quantum Hall droplet. These edge modes are described by chiral boson theories, where the charges reflect the topological order in the bulk, and the velocities of the different modes are non-universal parameters depending on the edge potentials.
An important aspect of the edge theories is that they are needed to cancel the gauge non-invariant terms that appear in CS-theories in the presence of boundaries. (With a common abuse of terminology we shall refer to these terms as “anomalies”.) To be specific, we shall use a straight edge geometry, where the bulk is in the upper half-plane, and the edge is at \( y = 0 \). This is made explicit by writing the bulk action as \( S = \int d^3r \, \theta(y) L_{eff} \), where \( \theta(y) \) is the step function and \( L_{eff} \) is given by (2.3) or (2.7). Under the gauge-transformation

\[
A_\alpha \rightarrow A_\alpha - d \epsilon_\alpha ,
\]

we get the anomalies

\[
\delta S_f^p = \frac{1}{4 \pi} \int d^3r \, \delta(y)(E^x + \tilde{e}^x) \epsilon_1 \quad (3.2)
\]

\[
\delta S^{pp} = \frac{1}{2 \pi} \epsilon_{\alpha \beta} \int d^3r \, \delta(y) E_{\alpha}^x \epsilon_{\beta} \quad (3.3)
\]

where \( E^x, \tilde{e} \) and \( \tilde{E}_\alpha \) are the electric fields corresponding to the potentials \( A, \tilde{a} \) and \( A_\alpha \) respectively, \( E_x = -\epsilon_{ab} \partial_a A_b \) (the indices \( a, b \) denote the edge coordinates \((0, x)\)).

The conserved currents associated with the gauge symmetries have both a bulk and an edge contribution, \( i.e. \)

\[
J^\mu_\alpha = \theta(y) J^\alpha_\mu + \delta(y) J^a_\alpha \delta_{\mu a} \quad (fp) \tag{3.9}
\]

\[
J^{tot, \alpha} = \theta(y) J^\alpha_\mu + \delta(y) j^a_\alpha \delta_{\mu a} \quad (pp) \tag{3.9}
\]

where the bulk currents are given by (2.6) and (2.8), respectively, and the edge currents by

\[
j^a = \frac{\partial L^{gi}_{ed}}{\partial A^a} + \frac{1}{4 \pi} \epsilon_{ab} \Delta_b \phi \quad (fp) \tag{3.10}
\]

\[
j^a_\alpha = \frac{\partial L^{gi}_{ed}}{\partial A^a_\alpha} + \frac{1}{2 \pi} \epsilon_{\alpha \beta} \epsilon_{ab} \Delta_b \phi_{\beta} \quad (pp) \tag{3.11}
\]
Some care is needed in deriving these expressions from the action (3.7). There are contributions both from the bulk and the edge actions, and only the sum of these is gauge invariant. Note that although the total currents (3.3) are conserved, the edge parts (3.10) and (3.11) are not, but satisfy

\[
\partial_0 j^a = -\frac{i}{2\pi} (E^1 + \dot{e}^1) = -J^y \quad (fp)
\]

\[
\partial_0 j_a^x = -\frac{1}{\pi} l_{\alpha \beta} E_\beta^1 = -J^y_a \quad (pp),
\]

where the last identities follow from the explicit expressions (2.6) and (2.8) for the bulk currents.

At this point it is illuminating to recall the well understood relation between anomaly cancellation and charge conservation. As stressed by Stone\[18\], the current flow from the bulk into the edge means that the edge currents are not conserved. This is easily understood by considering the region \([x - \epsilon, x + \epsilon]\) of the edge, where charge conservation is expressed as

\[
\partial_0 \int_{x-\epsilon}^{x+\epsilon} dx' j^0(x') = -j^x(x + \epsilon) + j^x(x - \epsilon) - \int_{x-\epsilon}^{x+\epsilon} dx' J^y(x') \quad .
\]

By taking the limit \(\epsilon \rightarrow 0\), we obtain (3.12). Below, we shall use the corresponding relation for the spin current to derive boundary conditions on the spin field \(\hat{n}\).

The form of the gauge invariant part of the edge action, \(\mathcal{L}_{ed}^g(\Delta^a \phi)\), is not determined by the anomaly structure. In fact, not even the number of modes is fixed, and in the above discussion we implicitly assumed a minimal field content on the edge (this might well change in real systems with flat edge potentials where edge reconstruction is expected). We shall take the following Lagrangians,

\[
\mathcal{L}_{ed}^g = \frac{1}{4\pi} \left( \Delta x_\phi \Delta_\phi - v(\Delta_x \phi)^2 \right) + (j^a) \Delta_a \phi \quad (fp)
\]

\[
\mathcal{L}_{ed}^g = \frac{1}{4\pi} \left( l_{\alpha \beta} \Delta_\phi \Delta_\phi - v_{\alpha \beta} \Delta_x \phi \Delta_\phi \phi_\beta + (j^a) \Delta_\phi \phi_\alpha \right) \quad (pp),
\]

where the quadratic parts are identical to the ones proposed by Wen and others\[2\], but where we also added explicit terms corresponding to constant (i.e. space and time independent) ground state charge- and current densities, for which we shall use the notation \(\rho_{ed} = \langle j^0 \rangle\) and \(J_{ed} = \langle j^x \rangle\) etc.. Note that \(\rho_{ed}\) and \(J_{ed}\) are parameters in the model that will depend on the microscopic details of the edge, as discussed below. Also note that the terms containing \((j^a)\) do not contribute to the equation of motion for the field \(\phi\), and in particular that the mean field ground state for any edge potential is simply \(\phi = 0\). The velocity \(v\), in the (fp) case, and the velocity matrix \(v\), in the (pp) case, are also parameters of the model. We will use a sign convention where \(v > 0\). In the (pp) case, the velocity matrix \(v\) can in general be diagonalized simultaneously with \(\Delta_x \phi\). In our case it is physically quite reasonable to assume that these two matrices are simply proportional to each other, \(v = e v\), meaning that the edge velocity of a charged excitation does not depend on its spin. Although not necessary, we shall for simplicity restrict our discussion to this special case. Note that the spin on the edge is given by the boundary value of the bulk field, so there is no independent spin degree of freedom on the edge.

The edge charge and edge current for the (fp) case can now be obtained from (3.10) and (3.12):

\[
\begin{align*}
j^0 &= \rho_{ed} + \frac{i}{2\pi} \Delta x \phi \quad (fp) \\
j^x &= J_{ed} - v \frac{1}{2\pi} \Delta x \phi \quad ,
\end{align*}
\]

and for the (pp) case (3.11) and (3.15) give

\[
\begin{align*}
\begin{align*}
\end{align*}
\end{align*}
\]

Note that although the edge theories defined by (3.8) and (3.14) - (3.15) look identical to the spinless case when expressed in the covariant derivatives (3.5) and (3.6), there are in fact many spin dependent terms, both via \(\bar{a}\) and via the Zeeman term included in \(A^0\).
We end this section with the following observation. Since the edge is along the $x$-axis, it is natural to fix the axial gauge $\tilde{a}_y = 0$. For this choice, the bulk charge per unit length in the $x$-direction due to spin texturing can be expressed as,

$$\rho^{\text{top}}(x) = \int_0^\infty dy (\rho - \tilde{\rho}) = -\frac{l}{2\pi} \int_0^\infty dy \tilde{b} = -\frac{l}{2\pi} \tilde{a}_x(x,0) \quad (fp) . \quad (3.20)$$

If the topological charge is accumulated along the edge (a concrete example is given in the next section), it is natural to define the total edge charge density by

$$\tilde{j}_0^0 \equiv \rho^{\text{top}}(x) + j^0 = \rho_{\text{ed}} + \frac{l}{2\pi} \partial_x \phi \quad (fp) \quad (3.21)$$

(with the gauge choice $A_0(y = 0) = 0$). In particular, an $x$-independent $\phi$ simply corresponds to a redistribution of charge in the edge region. The corresponding expressions for the partially polarized case are

$$\tilde{j}_0^0 \equiv \rho^{\text{top}}(x) + j^0_1 = \rho_{\text{ed},1} + \frac{l}{\pi} l_{1\beta} \partial_x \phi_\beta \quad (pp) \quad (3.22)$$

$$\tilde{j}_0^0_{\text{tot},s} \equiv \rho^{\text{top}}(x) + j^0_2 = \rho_{\text{ed},2} + \frac{l}{\pi} l_{2\beta} \partial_x \phi_\beta \quad (pp) . \quad (3.23)$$

where $\rho^{\text{top}}_s(x)$ is the “spin charge density” defined by $\int dy (\rho P - \tilde{\rho} \tilde{P})$.

4. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

In the following we shall treat the fully and partially polarized cases in parallel. In the partially polarized case, the edge current $j^0$ without subscript will always refer to the spin current $j^0_2$, and similarly, $\rho_{\text{ed}}$ refers to $\rho_{\text{ed},2}$ and $\tilde{j}_\mu^{\text{tot}}$ to $\tilde{j}_\mu^{\text{tot},2}$.

Variation of the total action (3.7) with respect to the field $\phi$ (using (3.8), (3.14) and (3.15)) gives the following equations of motion,

$$\partial_x (\Delta_\phi - \nu \Delta_x) \phi = 0 \quad (fp) \quad (4.1)$$

$$l_{\alpha\beta} \partial_x (\Delta_\phi - \nu \Delta_x) \phi_\beta = 0 \quad (pp) \quad .$$

Variation with respect to the bulk field $\hat{m}$, taking the constraint $\hat{m}^2 = 1$ properly into account, gives two contributions. First the usual ferromagnetic spin wave equation,

$$\partial_0 \hat{m} - \kappa \hat{m} \times \nabla^2 \hat{m} - \mu_e \hat{B} \times \hat{m} = 0 \quad , \quad (4.2)$$

but also an edge part \sim $\delta(y)$ which is not a dynamical equation, but provides a boundary condition for the spin field $\hat{m}$. An equivalent and physically more instructive way to obtain this boundary condition is to require the spin current to be conserved at the edge:

The total spin current is obtained from the full action $S^{\text{tot}}$ using Noether’s theorem and the global symmetry transformation (3.11),

$$\tilde{j}_\mu^{\text{tot}} = \frac{1}{2} \tilde{m}_\mu J_\mu^{\text{tot}} - \frac{\kappa \tilde{P}}{2} \hat{m} \times \partial_\mu \hat{m} (1 - \delta_{\mu 0}) \theta(y) \quad . \quad (4.3)$$

where the current $J_\mu^{\text{tot}}$ is given by (3.9). (To derive (4.3) it is convenient to take variations with respect to the spinor $\chi_{\hat{m}}$ and use the relation (2.3).)

As expected, the spin current has a convective part, corresponding to the spin of the moving charges, and a ferromagnetic part, corresponding to spin waves for fixed charge current. The convective part, being proportional to the total current, has both a bulk and an edge contribution, while the ferromagnetic component only exists in the bulk. In writing (3.7) we assumed that the only explicit breaking of the $O(3)$ spin rotation symmetry comes from the Zeeman term. In particular, we ignore possible spin-orbit effects, both at the boundary and in the bulk. The divergence of the spin current is thus given only by the Zeeman term in $A^0$ (recall $J_0^{\text{tot},A^0} = J_0^{\text{tot}} (-\frac{1}{2} \mu_e \hat{B} \cdot \hat{m} + \delta A^0)$) and reads

$$\partial_\mu \tilde{j}_\mu^{\text{tot}} = \frac{J_0^{\text{tot}}}{2} \mu_e \hat{B} \times \hat{m} \quad . \quad (4.4)$$
The boundary condition now follows by using the same arguments as for the gauge currents (or equivalently by inserting (4.3) into (4.4) and identifying the terms $\sim \delta(y)$). The expression corresponding to (3.12) becomes

$$\partial_a(j^a\frac{1}{2}\hat{\rho}n) + \vec{J}_y = \frac{1}{2}\mu_e\vec{B} \times \hat{m}.$$  (4.5)

We now see that the anomaly cancellation conditions (3.12) imply that the $\hat{m}$-component of this equation is identically satisfied, and the equations for the remaining two components give the boundary condition

$$j^a\partial_a\hat{m} - \vec{\rho} \vec{A}(\hat{m} \times \partial_y \hat{m}) - \mu_e j^0 \vec{B} \times \hat{m} = 0$$  (4.6)

that relates the $\hat{m}$ field and its derivatives to the chiral edge current. Had it not been for the anomaly cancellation that eliminated one of the equations in (4.5), the equations of motion would in general only have trivial solutions since there are only two degrees of freedom in the $\hat{m}$-field. The boundary condition (4.6) is one of the main results of this paper.

5. EDGE MODELS AND EDGE MODES

The boundary condition (4.6) involves the edge charge and current densities, and when studying small fluctuations, these can simply be put equal to their ground state values $\rho_{ed}$ and $j_{ed}$. To relate these parameters to the edge potential, we shall use a simple model, where the edge is defined as a region of size $\Delta y$ where there is an electric field in the $y$-direction. To be able to compare our results with those of Karlhede et al.\textsuperscript{13}, we shall follow their construction and divide the external electrostatic potential into one piece corresponding to a background charge $A^0_{bg}(y)$, which completely cancels the charge of the electrons in the ground state, and an additional piece $A^0$ that can either tend to contract or expand the edge, according to the direction of the corresponding electric field. Note that for the “ideal edge” where $A^0 = 0$, there is no net electric field and thus no drift velocity and no edge drift current in the ground state. For the non-ideal edge, there is an electric field, $E_y = \partial_y A^0$ and a corresponding drift velocity $v_d = E_y/B_\perp$ in the ground state. If $v_d$ is constant over the edge, corresponding to a constant electric field, the edge charge and edge currents in the ground state are related by $j_{ed} = v_d \rho_{ed}$. Note that $v_d$, which depends on the details of the edge, can have any sign and is not to be confused with the chiral velocity $v$, which characterizes the excitations and is always positive.

If we now further assume that the electron density in the edge region does not differ much from the bulk value $\tilde{\rho}$ (it must of course go to zero at the very edge), we have

$$\rho_{ed} = \Delta y \tilde{\rho}$$  (5.1)

$$j_{ed} = v_d \rho_{ed} = \frac{\Delta A^0}{B_\perp} = -\frac{\nu}{2\pi} \Delta A^0,$$

where $\Delta A^0 = A^0(\Delta y) - A^0(0)$ is the total potential difference over the edge, and $\nu$ is the filling fraction.

Note that in this simple model, the edge current is related to the (at least in principle) measurable quantity $\Delta A^0$. On the other hand, there is presumably no unambiguous way to define the width $\Delta y$, so $\rho_{ed}$ will remain a free parameter in the model.

In order to study the dynamics of the low energy spin and edge excitations, we fix the gauge by expressing the spinor $\chi_{\hat{m}}$ in terms of a small complex number, $z$,

$$\chi_{\hat{m}} = \left( \begin{array}{c} \frac{z}{1 - \frac{1}{2}|z|^2} \\ \frac{1}{2} \end{array} \right)$$  (5.2)

and keep only terms up to $O(z^2)$. The sigma model fields are then given by

$$\hat{m} = \left[ (z + \bar{z}), i(z - \bar{z}), 2|z|^2 - 1 \right]$$

$$(\nabla \hat{m})^2 = 4 (\nabla \bar{z}) (\nabla z)$$

$$\tilde{a}_\mu = \frac{i}{2} (\bar{z} \partial_\mu z - z \partial_\mu \bar{z})$$  (5.3)

Extracting the $\sigma$-model kinetic energy $\sim \tilde{a}^0$ and the Zeeman term included in $A^0$ and $A^2_\perp$ from (2.3) and (2.7), and using the expressions (5.3), we get
\[
\mathcal{L} = \bar{\rho} \bar{P} \left[ i \bar{\varepsilon} \partial_0 z + \kappa \bar{\varepsilon} \nabla^2 z + \mu_e B \left( \frac{1}{2} - \bar{\varepsilon} \right) \right],
\]

(5.4)

where \( \bar{P} = 1 \) and \( B = |\vec{B}| \) for the fully polarized case. Substituting the ground state values for the edge charge and current density into (5.4) gives a boundary condition for \( z \),

\[
(\rho_{ed} \bar{\varepsilon} \partial_0 + j_{ed} \bar{\varepsilon} \partial_x - \rho_{ed} \mu_e B) z + \kappa \bar{\rho} \partial_y z = 0 .
\]

(5.5)

(Here we might worry that the relevant edge charge would be \( \langle j^0_{tot} \rangle \), from (3.21) rather than \( \rho_{ed} \), but for the ansatz (5.2), the difference is \( \sim |A|^2 k \), i.e. of higher order in the small fluctuation amplitude \( A \).)

There will be several classes of low-lying modes: bulk spin waves, edge density waves (which we shall refer to as edge magnetoplasmons - EMP) and edge spin waves (ESW).

**Bulk spin waves**

As expected, there are bulk spin wave solutions of eq.(4.2) of the form

\[
z = e^{i(kx-\omega t)} \cos(k_y y + \alpha)
\]

(5.6)

The dispersion relation,

\[
\omega = \mu_e B + \kappa k^2 + \kappa \lambda^2
\]

(5.7)

follows immediately from (5.4), and the phase shift \( \alpha \) is determined from the boundary condition (5.5)

\[
\rho_{ed} (\omega - \mu_e B) - j_{ed} k \cos \alpha - \kappa \bar{\rho} \bar{k} \sin \alpha = 0
\]

(5.8)

We now turn to the more interesting case of the edge modes.

**Edge magnetoplasmons**

These are the well known chiral edge modes discussed by e.g. Wen. In our model they appear as solutions where \( z = 0 \), so only the \( \phi \) field is excited. In the fully polarized case, there is one such mode with a linear dispersion \( \omega = -vk \) where \( k < 0 \). For a model with the general form (3.15), there are in general two EMP eigenmodes with different velocities \( v_\alpha \). However, as already mentioned, in the special case \( v = v_\parallel \) they both have the same velocity \( v \).

By analyzing the boundary condition (5.5) it is not hard to see that, at least in our approximation, there is no solution that simultaneously excites both the \( \phi \) and the \( z \) fields, i.e. there is no mixing between the EMP and the ESW.

**Edge spin waves**

In order to study the dispersion of edge spin waves, we now make the ansatz

\[
z = A e^{i(kx-\omega t)} e^{-\lambda y}
\]

(5.9)

with \( \lambda > 0 \). (5.5) gives

\[
\rho_{ed} (\omega - \mu_e B) - j_{ed} k - \bar{\rho} \bar{k} \lambda = 0
\]

(5.10)

The equation of motion of \( \bar{z} \) corresponding to (5.4) gives

\[
\omega = \mu_e B + \kappa k^2 - \kappa \lambda^2
\]

(5.11)

The dispersion relation is found by eliminating the parameter \( \lambda \) from (5.10) and (5.11).

\[
\lambda = \frac{\bar{\rho} \bar{P}}{2 \rho_{ed}} \left[ -1 + \sqrt{1 + \left( \frac{2 \rho_{ed}}{\bar{\rho} \bar{P}} \right)^2 \left( \frac{j_{ed}}{\kappa \rho_{ed}} k + k^2 \right)^2} \right]
\]

(5.12)

\[
= \frac{j_{ed}}{\bar{\rho} \lambda} k + \frac{\rho_{ed}}{\bar{\rho} P} \left( 1 - \left( \frac{j_{ed}}{\kappa \bar{\rho} \bar{P}} \right)^2 \right) k^2 + O(k^3)
\]
In the generic case where \( j_{ed} \neq 0 \), we can substitute the model dependent expression for \( j_{ed} \) from (5.1) to get

\[
\lambda = \frac{\nu \Delta A^0}{2 \pi \kappa \rho} k + O(k^2) ,
\]

and the corresponding dispersion relation

\[
\omega = \mu_e B + \kappa \left( 1 - \left( \frac{\nu \Delta A^0}{2 \pi \kappa \rho} \right)^2 \right) k^2 ,
\]

is that of an ESW with a spin stiffness differing from the bulk value \( \kappa \). Note that since \( \lambda \) must be positive, the sign of \( k \) has to be the same as that of \( \Delta A^0 \). In particular this means that for a potential that tends to smoothen the edge, which with our sign conventions corresponds to \( \Delta A^0 < 0 \), we have \( k < 0 \), i.e. the ESWs propagate in the same direction as the EMPs. This qualitative result was also found by Karlhede et al. for the special case of a fully polarized \( \nu = 1 \) state. Also, and perhaps more surprising, if we take \( P = \nu = 1 \) together with the \( \kappa \) value pertinent to Coulomb interaction, (5.13) exactly coincides with the result of [14].

For the special case of \( \Delta A^0 = 0 \), corresponding to zero drift current in the ground state, we get

\[
\lambda = \frac{\rho_{ed} k^2}{P \rho}
\]

and the corresponding dispersion relation

\[
\omega = \mu_e B + \kappa k^2 .
\]

This mode is not chiral, but exists for both signs of \( k \), and the spin stiffness is not renormalized; corrections to the bulk spin wave dispersion are \( O(k^4) \). Both these qualitative features were found in the \( \nu = 1 \) case in [13]. Since in this case \( \lambda \) depends explicitly on \( \rho_{ed} \), we cannot relate the actual value of \( \lambda \) to that in [14]. It is interesting to note, however, that in our approach the physical condition for having a chiral ESW mode is rather transparent: The convective part of the spin current (4.3) has a chiral edge component as soon as there is a charge drift current in the ground state. Via the boundary condition (4.6) this also gives chirality to the ESWs. When there is no current in the ground state, the ESW is non-chiral.

Here we should mention that our results differ from those of a previous sigma model analysis of the fully polarized case by Milovanovic et al. who e.g. finds a term \( \sim k \) in the dispersion relation for the ESW. (The differences can be traced back to our requirement that the gauge anomaly related to the Berry gauge field \( \tilde{a}_\mu \) should cancel between bulk and edge.)

We end this section with a discussion of a possible generalization of the edge Lagrangian. A basic assumption in constructing the edge theories was that \( \hat{m} \) is continuous at the edge, so that the \( \hat{m} \)-field in the boundary theory (that enters \( \mathcal{L}_{ed} \) via \( \tilde{a}_\mu \)) is just the boundary value of the bulk \( \hat{m} \) field, i.e. the spin on the boundary does not constitute an independent degree of freedom. Even with this assumption, however, the action \( S^{tot} \) in (3.7) is not the most general one. We already mentioned the possibility of extra chiral modes, corresponding to a (non-textured) reconstruction of the edge, but there is also the possibility of having an extra \( \sigma \)-model term in the Hamiltonian corresponding to an edge spin stiffness,

\[
\mathcal{H}_{\sigma,ed} = \int dx \frac{k^2}{4} j^0 (\partial_x \hat{m}(x, 0))^2 .
\]

Adding this piece to the previous expression (3.9) for the spin current and proceeding as before, we find that the boundary conditions (4.8) are modified into

\[
 j^0 \partial_x \hat{m} - \bar{P} \kappa (\hat{m} \times \partial_y \hat{m}) - j^0 \kappa' (\hat{m} \times \partial_y^2 \hat{m}) - j^0 \mu_e \vec{B} \times \hat{m} = 0 .
\]

This gives a correction to the term \( \sim k^2 \) in the expression for \( \lambda \), (5.12); in the special case \( \kappa' = \kappa \), the \( k^2 \) term is cancelled. In the generic case where \( j_{ed} \neq 0 \), both the penetration depth, \( \lambda^{-1} \), and the ESW dispersion relation, are unchanged to leading order, while for the non-chiral case, \( j_{ed} = 0 \), \( \lambda \) depends sensitively on \( \kappa' \). This further emphasizes our previous comment that the non-chiral case can only be qualitatively described within our model.
6. SUMMARY AND OUTLOOK

Since the derivation of the combined effective theory for the bulk and edge is rather technical, we want to summarize the main steps in order to highlight the rather simple and compelling logic.

First the bulk theories are derived from the CSGL-Lagrangians by integrating out the hard density and polarization modes (for details, see Appendix A). The resulting effective actions depend on the background electromagnetic field and a single dynamical spin field, \( \hat{m} \). We make a derivative expansion and neglect terms with more than two derivatives of \( \hat{m} \). Zeeman terms are included as spin dependent scalar potentials. For the partially polarized case there are two gauge invariances, one related to electromagnetism and another related to the choice of spin basis. In the case of full polarization, these gauge invariances coincide.

When restricting the bulk theory to a half-plane, there are gauge non-invariant terms in the action, so we add a non-gauge invariant edge action such that the combined theory is invariant under both gauge transformations. This implies that both the charge current and the \( \hat{m} \)-component of the spin current are conserved. The gauge invariant part of the edge action is not fixed by invariance arguments, and we choose a minimal model, (3.14) and (3.15), which describes the edge as a chiral Luttinger liquid.

Assuming that the global O(3) spin symmetry is violated only by the Zeeman term, we derive a boundary condition for the spin field \( \hat{m} \). Together with the equations of motion for \( \hat{m} \) in the bulk and \( \phi \) on the boundary, this boundary condition defines the combined bulk - edge theory. It is important that in this model the spins on the boundary do not constitute any independent degrees of freedom.

Finally, we did a small fluctuation analysis of the full model and found a spectrum in agreement with the \( \nu = 1 \) Hartree-Fock results in Ref.\[14\]. It is encouraging that our results, which were derived in a very general context and based mainly on symmetry considerations, give the same results as a specific, much more controlled calculation, both qualitatively (number of modes and their chirality) and to some extent also quantitatively (the exact expression for the penetration depth in the chiral case). This makes it plausible that our effective action really does describe the charge and spin-dynamics of quantum Hall edges, at non-integer filling fractions and/or partial polarization.

A natural extension of the present work would be to study textured edges. To do this one would have to keep the potential terms that we neglected since they included higher derivatives. Such terms would affect the boundary condition, but in principle it is straightforward to write it down together with the corresponding Hamiltonian. Since, for certain choices of the parameters, the ESW equation (5.14) can become soft for large \( k \), it is not inconceivable that texturing the edge will lower the energy. By expanding the action around such a static solution, one could then go on to study the interplay between the expected gapless spin waves and the EMP modes.

A potentially interesting application of the theory developed in this paper, would be to study the interplay between EMP modes and spin waves in a narrow strip. By expanding our Lagrangian to higher order in small fluctuations, one generates interaction vertices between the EMP:s and the spin waves and one could then use perturbation theory to calculate an effective spin-mediated interaction between the EMP:s at the two edges.

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APPENDIX A: FROM CS LAGRANGIANS TO EFFECTIVE \( \sigma \)-MODELS

In this Appendix we shall outline the derivation of the effective Lagrangians (2.5) and (2.7). We consider three different cases. The first, idealized case of spinless (sl) electrons is relevant for systems with a very large Zeeman gap and is included for reference. We then turn to the fully polarized (fp) and then to the partially polarized (pp) states. To make the presentation self-contained, we reproduce some relevant definitions and equations from section 2.

We shall describe all three systems with the following Lagrangian of the CSGL type,

\[ \mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_{CS}(a_\alpha^\dagger) - \mathcal{H}_\sigma. \]  \hspace{1cm} (A.1)

In the partially polarized case, \( \phi \) is a complex bosonic spinor describing the electrons, in the fully polarized and spinless models it is just a single complex field, and \( a_\alpha^\dagger \) are auxiliary CS fields. The sigma model term \( \mathcal{H}_\sigma \) is absent in the spinless case. In all cases we have

\[ \mathcal{L}_\phi = \phi^\dagger iD_0 \phi - \frac{1}{2m_e} |\vec{D}\phi|^2 - V(\rho), \]  \hspace{1cm} (A.2)

where \( m_e \) is the electron mass, and \( \rho = \phi^\dagger \phi \) is the density. The exact form of the potential \( V(\rho) \) varies for the different cases, but is of no importance for the following discussion. For the spinless and the fully polarized cases the covariant derivative and the CS Lagrangian are given by
gauge-symmetry is fundamental, the second charge is only conserved under certain physical assumptions, namely corresponding to O(3) spin rotations. There it is also stressed that while the charge corresponding to having another conserved charge corresponding to the local polarization density, as discussed in some detail in

Netic spin waves, and in the presence of a symmetry breaking Zeeman term they get a Zeeman gap \( \mu_c |\vec{B}| \).

For the partially polarized states the mean field ground state is given by

\[ \tilde{\rho} = \langle \rho \rangle = \frac{l}{2\pi} b_1 = -\frac{l}{2\pi} B_\perp \]  

with filling fraction \( \nu = l \). For the fully polarized case, the same condition holds, and the O(3) symmetry is spontaneously broken so that the \( \tilde{m} \) field points in a fixed direction \( \tilde{m}_0 \). The corresponding Goldstone bosons are ferromagnetic spin waves, and in the presence of a symmetry breaking Zeeman term they get a Zeeman gap \( \mu_c |\vec{B}| \).

For the partially polarized states the mean field ground state is given by

\[ iD^\mu = i\partial^\mu + a_1^\mu + A^\mu \]  \hspace{1cm} (sl)  
\[ iD^\mu = i\partial^\mu + a_1^\mu + A^\mu + \tilde{a}^\mu \]  \hspace{1cm} (fp)  
\[ \mathcal{L}_{CS} = -\frac{1}{4\pi} \epsilon_{\mu\nu\sigma} a_1^\mu \partial^\nu a_1^\sigma \equiv -\frac{1}{4\pi} a_1^\mu da_1 \]  \hspace{1cm} (A.5)

Please recall that the three-vector notation is only for notational convenience; the metric is Euclidian, \( i.e. D^\mu = D_\mu \), and all signs are written explicitly. \( l^{-1} \) is an odd integer, and the topological vector potential \( \tilde{a}^\mu \) is defined by

\[ \tilde{a}^\mu = \chi_{\tilde{m}} i\partial^\mu \chi_{\tilde{m}} \]  \hspace{1cm} (A.6)

with the spinor \( \chi_{\tilde{m}} \) defined by

\[ \hat{m}^i = \chi_{\tilde{m}} \sigma^i \chi_{\tilde{m}} \]  \hspace{1cm} (A.7)

up to a gauge transformation. As discussed in [4], the potential \( \tilde{a}^\mu \) represents the Berry phases picked up by the spins on the electrons, and the gauge freedom corresponds to the freedom of picking the local spin basis. The spinless case differs from the fully polarized only in that the gauge field \( \tilde{a}^\mu \) is absent. The vector field \( \hat{m} \), which is present in addition to the field \( \phi \) and the CS fields \( a_1^\mu \), describes the direction of the spin polarization. The corresponding expressions in the partially polarized case are

\[ iD^\mu = i\partial^\mu + a_1^\mu + a_2^\mu \hat{\sigma} \cdot \hat{m} + A^\mu - \frac{1}{2} (\hat{m} \times \partial^\mu \hat{m}) \cdot \tilde{\sigma} \]  \hspace{1cm} (A.8)
\[ \mathcal{L}_{CS} = -\frac{1}{2\pi} l_\alpha \epsilon_{\mu\nu\sigma} a_2^\nu \partial_\sigma a_1^\mu \equiv -\frac{1}{2\pi} l_\alpha a_\alpha da_\beta \]  \hspace{1cm} (A.9)

where the elements of the symmetric matrix \( l^{-1} \) are integers, with diagonal elements being both either even or odd.

The corresponding sigma model terms for the fully and partially polarized cases are both given by

\[ \mathcal{H}_\sigma = \frac{V_0}{2} (\partial_i \vec{S})^2 + \mu_c \vec{S} \cdot \vec{B} \]  \hspace{1cm} (A.10)

where \( -\mu_c = -e g/(2m) \) is the effective magnetic moment of the electron, \( V_0 \) is an interaction parameter, and the spin density \( \vec{S} \), is related to the unit vector field \( \hat{m} \), by \( \vec{S} = (1/2) \hat{m} \phi^\dagger \tilde{\sigma} \cdot \hat{m} \phi \).

The pertinent symmetries of the Lagrangian (A.1) are given by

\[ \chi \rightarrow e^{i\alpha(x)} \chi \]  \hspace{1cm} (A.11)
\[ a_1^\mu \rightarrow a_1^\mu + \partial^\mu \alpha(x) \]

\[ \chi \rightarrow e^{i\beta(x) |\tilde{\sigma}| \hat{m}(x)} \chi \]  \hspace{1cm} (A.12)
\[ a_2^\mu \rightarrow a_2^\mu + \partial^\mu \beta(x) \]

\[ \chi \rightarrow e^{i\hat{\tilde{L}} \cdot \tilde{m} \chi} \]  \hspace{1cm} (A.13)

where the \( 3 \times 3 \) matrix \( \tilde{L} \) is the angular momentum in the vector representation.

The gauge symmetry (A.11) connected to \( a_1 \) is related to electric charge conservation and is present in all three cases. The gauge symmetry (A.12) connected to \( a_2 \) is relevant only to the partially polarized case and is related to having another conserved charge corresponding to the local polarization density, as discussed in some detail in [4].

There it is also stressed that while the charge corresponding to \( a_1 \) is the usual electric charge, and the corresponding gauge-symmetry is fundamental, the second charge is only conserved under certain physical assumptions, namely neglecting spin-flip transitions. In this paper we simply assume that the partially polarized states can be described by the above theory with two gauge symmetries.

For theories with spin and in the absence of a Zeeman interaction, there is also a global symmetry (A.13), corresponding to O(3) spin rotations.

The mean field ground state in the spinless case is given by

\[ \tilde{\rho} = \langle \rho \rangle = \frac{l}{2\pi} b_1 = -\frac{l}{2\pi} B_\perp \]  \hspace{1cm} (A.14)
\[ \tilde{\rho} = -l_{11}B_1/\pi \]
\[ \tilde{P} = \langle \hat{n} \cdot \hat{m} \rangle = \cos \tilde{\alpha} = l_{12}/l_{11} \quad , \]

with filling fraction \( \nu = 2l_{11} \) and polarization \( \tilde{P} \).

We now assume that the \( \phi \)-factor is small enough for the spin waves to be considered as low-energy excitations, while there are large (cyclotron) gaps to density fluctuations. In the partially polarized case there are two density modes corresponding to independent fluctuations of spin up and spin down electrons, or equivalently, one density and one polarization mode. To get an effective low energy theory, the hard modes are integrated out. Technically one proceeds by making a phase-density decomposition of the fields (\( \phi \), polarization mode). To get an effective low energy theory, the hard modes are integrated out. Technically one proceeds corresponding to independent fluctuations of spin up and spin down electrons, or equivalently, one density and one polarization. To get an effective low energy theory, the hard modes are integrated out. Technically one proceeds.

Next we rewrite (A.20) in a way that is suitable for constructing the edge theory. Note that the Zeeman term in \( \mathcal{H}_\sigma \) is directly proportional to the density, while the spin stiffness term is proportional to the density squared. In the approximation we use, the density fluctuations are proportional to \( \tilde{b} \) and thus to second derivatives of the \( \tilde{m} \) field. Keeping terms only to quadratic order in derivatives of \( \tilde{m} \), we can thus make the replacement \( \rho \to \tilde{\rho} \) in the spin stiffness term and in the potential term (remember that \( V(\rho) = V(\tilde{\rho}) + O(\tilde{b}^2) \), but we have to keep \( \rho \) in the Zeeman term. Since a term proportional to the density is nothing but an electrostatic potential, we can incorporate the Zeeman term by shifting the potential \( A^0 \) as

\[ A^0 \to A^0 - \frac{1}{2} \mu_r \tilde{B} \cdot \hat{m} \quad . \]

Since gauge invariance will be a major guiding principle for constructing the edge theories, this way of including the Zeeman term is very advantageous, and throughout the paper we assume the shift (A.21) of \( A^0 \) if not stated otherwise. We shall also neglect the constant term \( V(\tilde{\rho}) \), so, to summarize, we have the following effective bulk Lagrangian,

\[ \mathcal{L}_{eff}^{fp} = -\frac{l}{4\pi} (A + \tilde{a})d(A + \tilde{a}) - \frac{k\tilde{\rho}}{4} (\nabla \tilde{m})^2 \quad . \]

It is straightforward to make a similar construction in the partially polarized case by noting that the gauge fields in (A.17) only occur in the combinations \( A + a_1 \) and \( \tilde{a} + a_2 \). Here we have two conserved currents corresponding to...
the two external potentials $A^\mu$ and $\tilde{a}^\mu$. Using the convenient notation $A_1 = A$ and $A_2 = \tilde{a}$ and performing the same
manipulations as in (A.18), gives, *mutatis mutandis*,

\[ J_\alpha^\mu = -\frac{1}{\pi} l_{\alpha\beta} \epsilon^{\mu\nu\sigma} d_\nu A_\sigma^\beta, \tag{A.23} \]

which can be integrated and combined with the $\sigma$-model action to give

\[ L_{\text{eff}}^{pp} = -\frac{1}{2\pi} l_{\alpha\beta} A_\alpha d A_\beta - \frac{\kappa \bar{\rho} \bar{P}}{4} (\nabla \hat{m})^2. \tag{A.24} \]

In this case the Zeeman term is included by a shift in the potential $A_0^2$. Note that in both cases, the topological vector potential $\tilde{a}^\mu$ is simply related to the ferromagnetic kinetic term by

\[ L_{\text{kin}} = \bar{\rho} \bar{P} \tilde{a}^0, \tag{A.25} \]

(see, e.g., 19), so using the mean field solutions (A.14) and (A.15), the term $\sim \tilde{a}^0 B$ in (A.22) and (A.24) provides the correct kinetic term for the sigma model field $\hat{m}$. To get (A.25), we again substituted $\rho \rightarrow \bar{\rho}$, but it is important to
note that the higher derivative term $\sim \partial^i \tilde{a}^i$ is very important when going beyond linear response. It is a so-called Hopf term, which is normalized such that skyrmions, which are topological soliton solutions to (A.22) and (A.24), get the correct (fractional) statistics. 20

**APPENDIX B: DUAL CS THEORY FOR PARTIALLY POLARIZED STATES**

Here we present the dual version of the CSGL model for partially polarized QH states and show how it can be used to give an alternative derivation of the low-energy effective bulk theory discussed in section 2. The advantage of this approach is that vortex currents are included in a very straightforward way. In this duality transformation, which is a direct generalization of the one for spinless electrons (as described in e.g. 21), one goes from the original CSGL model describing electrons in an external field, to a description where the quasiparticle excitations (vortices) are considered to be the fundamental particles.

Consider the $L_\phi + L_{CS}$ part of eq.(A.1) ($H_\sigma$ does not enter the duality transformation and is omitted for simplicity), and decompose the matter field as follows (for notation, see 14),

\[ \phi = \sqrt{\rho} e^{i \vartheta} e^{i \alpha \cdot \mathbf{\hat{e}} (\theta)} \chi \hat{m}. \tag{B.1} \]

As in the text, we neglect gradient terms in the density $\rho$ and polarization $\alpha$, but we decompose the phase angles $\vartheta$ and $\theta$ into regular parts that are absorbed in the gauge fields $a_1$ and $a_2$, and singular parts $\eta_1$ and $\eta_2$ that describe vortices,

\[ \vartheta = \vartheta_{\text{reg}} + \eta_1 + \eta_2 \]
\[ \theta = \vartheta_{\text{reg}} - 2\eta_2 \tag{B.2} \]

The singular phase angles are defined by $\oint d\mathbf{r} \nabla \eta_i = 2\pi n$ for an infinitesimal contour enclosing the singularity. Doing this, it is a matter of algebra to derive

\[ L = \rho \left[ a_{01} + \partial_0 \eta_1 + \cos \alpha (a_{02} + \tilde{a}_0 + \partial_0 \eta_2) + A_0 \right] \]
\[ - \frac{\rho}{2m} \left[ (a_{11} + \partial_1 \eta_1 + \cos \alpha (a_{12} + \tilde{a}_1 + \partial_1 \eta_2) + A_1) \right] \]
\[ + \sin^2 \alpha (a_{12} + \tilde{a}_1 + \partial_1 \eta_2)^2 - V \]
\[ - \frac{1}{2\pi} l_{\alpha\beta} \epsilon^{\mu\nu\sigma} a^\alpha_\mu \partial_\nu a^\beta_\sigma. \tag{B.3} \]

Next, the space components of the CS fields are split into their longitudinal and transverse parts,

\[ a^2_\alpha = \partial_i \theta_\alpha + \varepsilon_{ij} \partial_j \phi_\alpha. \tag{B.4} \]

Integrating out $a^0_\alpha$ gives the CS constraints
\[ \rho = \frac{1}{\pi} l_{1\beta} b_\beta \]  
(B.5)
\[ \rho \cos \alpha = \frac{1}{\pi} l_{2\beta} b_\beta \]  
(B.6)

The quadratic terms in the Lagrangian are linearized by introducing Hubbard-Stratonovic fields \( J_\mu^\alpha \), such that
\[
L = \rho \left[ A_0 + \partial_\mu \eta_1 + \cos \alpha (a_0 + \partial_\mu \eta_2) \right] + \frac{m}{2\rho} \left( J_1^2 + J_2^2 \right) 
+ J_{11} \left[ A_1 + \epsilon_{ij} \partial_j \phi_1 + \partial_i \eta_1 + \partial_\mu \eta_2 + \cos \alpha (a_1 + \epsilon_{ij} \partial_j \phi_2 + \partial_i \theta_2 + \partial_\mu \eta_2) \right] 
+ J_{21} \sin \alpha \left[ \tilde{a}_1 + \epsilon_{ij} \partial_j \phi_2 + \partial_i \theta_2 + \partial_\mu \eta_2 \right] - V
- \frac{1}{\pi} \alpha \beta \phi_\alpha \nabla^2 \partial_0 \theta_\beta .
\]  
(B.7)

Integrating out the phase fields \( \theta_\alpha \) gives the constraints \( \partial_\mu J_\mu^C = \partial_\mu J_\mu^S = 0 \) where the two conserved currents are
\[
J_\mu^C = \left( \rho, \vec{J}_1 \right) \equiv \epsilon_{\mu\nu\sigma} \partial_\nu B_\sigma \quad \text{(B.8)}
\]
\[
J_\mu^S = \left( \rho \cos \alpha, \vec{J}_1 \cos \alpha + \vec{J}_2 \sin \alpha \right) \equiv \epsilon_{\mu\nu\sigma} \partial_\nu B_\sigma \quad \text{(B.9)}
\]

Here, we have explicitly solved the constraints by introducing dual Chern-Simons fields \( B_\alpha^\mu \).

Using this definition along with (B.4) - (B.6), the fields \( \phi_\alpha \) can be expressed in terms of the dual CS fields as
\[
\phi_\alpha = -\frac{\pi}{\nabla^2} l_{\alpha\beta}^{-1} \epsilon^{ij} \partial_i B_j^\beta .
\]  
(B.10)

Thus eliminating \( \phi_\alpha \) from \( L \) and introducing the vortex currents
\[
\hat{J}_\mu^\alpha = \frac{1}{2\pi} \epsilon^{\nu\sigma} \partial_\nu \partial_\sigma \eta_\alpha ,
\]  
(B.11)

(note that this expression is not zero because of the singularity in \( \eta_\alpha \)), one finds the final expression for the complete dual bulk action,
\[
L = \frac{\pi}{2} l_{\alpha\beta}^{-1} \epsilon^{\mu\nu\sigma} B_\mu^\alpha \partial_\nu B_\sigma + \epsilon^{\mu\nu\sigma} A_\mu^\alpha \partial_\nu B_\sigma + 2\pi B_\mu^\alpha \hat{J}_\mu^\alpha + F^2 - V(\rho)
\]  
(B.12)

where \( F^2 \) represents terms which are higher order in derivatives and will be neglected from now on.

The effective action is obtained by integrating out the dual CS fields in (B.12). To this end, we rewrite the gauge part of (B.12) as
\[
L = \frac{\pi}{2} l_{\alpha\beta}^{-1} \epsilon^{\mu\nu\sigma} B_\mu^\alpha \partial_\nu B_\sigma + B_\mu^\alpha \mathcal{J}_\mu^\alpha \equiv -\frac{1}{2} B_\mu^\alpha \left( G_{\alpha\beta}^{\mu\nu} \right)^{-1} B_\nu^\beta + B_\mu^\alpha \mathcal{J}_\mu^\alpha
\]  
(B.13)

where
\[
\mathcal{J}_\mu^\alpha = 2\pi \hat{J}_\mu^\alpha + \epsilon^{\mu\nu\sigma} \partial_\nu A_\sigma^\alpha
\]  
(B.14)

and
\[
\left( G_{\alpha\beta}^{\mu\nu} \right)^{-1} = \pi l_{\alpha\beta}^{-1} \epsilon^{\mu\nu\sigma} \partial_\sigma .
\]  
(B.15)

A convenient gauge to choose for the external field is \( \partial_\mu A_\mu^{(1)} = 0 \). In this gauge, the propagator is
\[
G_{\alpha\beta}^{\mu\nu} = -\frac{1}{\pi} l_{\alpha\beta}^{-1} \epsilon^{\mu\nu\sigma} \frac{\partial_\sigma}{\nabla^2} .
\]  
(B.16)
Upon integrating out the fields $B^\alpha_\mu$, the action reduces to

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \mathcal{J}^\alpha_\mu \mathcal{G}^{\mu \nu} \mathcal{J}^\beta_\nu .$$

(B.17)

Inserting for $\mathcal{J}^\alpha_\mu$ and $\mathcal{G}^{\mu \nu} \mathcal{J}^\beta_\nu$ from (B.14) and (B.16) and neglecting terms $\sim (\hat{j})^2$ finally leads to the effective bulk action, in the notation used in section 2,

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2\pi} l_{\alpha \beta} A_\alpha dA_\beta - l_{\alpha \beta} \left( A^\alpha \hat{j}^\beta + A^\beta \hat{j}^\alpha \right) .$$

(B.18)

We see that in the absence of vortex currents, this reduces to the gauge part of the action (2.7) found from the original CSGL model.

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