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Near-Heisenberg-Limited Atomic Clocks in the Presence of Decoherence

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The ultimate stability of atomic clocks is limited by the quantum noise of the atoms. To reduce this noise it has been suggested to use entangled atomic ensembles with reduced atomic noise. Potentially this can push the stability all the way to the limit allowed by the Heisenberg uncertainty relation, which is denoted the Heisenberg limit. In practice, however, entangled states are often more prone to decoherence, which may prevent reaching this performance. Here we present an adaptive measurement protocol that in the presence of a realistic source of decoherence enables us to get near-Heisenberg-limited stability of atomic clocks using entangled atoms. The protocol may thus realize the full potential of entanglement for quantum metrology despite the detrimental influence of decoherence.

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Atomic clocks provide some of the most accurate time measurements in physics. One of the main limitations to the stability of atomic clocks is the quantum noise of the atoms, which leads to the standard quantum limit where the stability scales as $1/\sqrt{N}$ with $N$ being the number of atoms [1, 2]. To overcome this noise it has been suggested to use entangled states with reduced atomic noise [3–7]. Ultimately this may lead to a stability at the Heisenberg limit where the resolution scales as $1/N$, and recently the first proof of principle experiments have demonstrated these concepts experimentally [8–13]. In practice, however, entangled states are often more prone to decoherence, and to fully assess the advantage it is essential to study the performance in the presence of decoherence [14]. In Ref. [5] it was proven that entanglement can be used to improve the long-term stability of atomic clocks in the presence of the dominant practical source of decoherence, but the improvement identified was rather limited. Here we show that it is possible to obtain a large improvement in the stability of the clock by combining entanglement with an adaptive measurement protocol (inspired by Refs. [15, 16]). With our adaptive measurement protocol the entangled states are not more sensitive to the decoherence than disentangled states [cf. Fig. 1(b)]. As a consequence the long-term stability of the atomic clock can be improved almost to the Heisenberg limit even in the presence of decoherence.

Many atomic clocks are operated by locking a local oscillator (LO) to an atomic transition via a feedback loop. The feedback is typically based on a measurement of the LO frequency offset $\delta \omega$ compared to the atomic transition through Ramsey spectroscopy [17]. Here the atoms are first prepared in one of the two clock states by, e.g., a laser pulse. During the Ramsey sequence the atoms interact with the LO field. This interaction consists of three parts: first the atoms are subject to a near-resonant $\pi/2$ pulse from the LO followed by the Ramsey time $T$ of free evolution, and finally another near-resonant $\pi/2$ pulse is applied to the atoms. During the free evolution the LO acquires a phase $\delta \phi = \delta \omega T$ relative to the atoms. Due to the last $\pi/2$ pulses this phase can be measured as a population difference between the two clock levels. $\delta \omega$ can thus be estimated from the measurement and used for a feedback that steers the frequency of the LO to the atomic frequency. The stability of the clock will improve with $T$ since a longer $T$ improves the relative sensitivity of the frequency measurement. For current atomic fountain clocks, $T$ is limited by gravity and can hardly be varied [18]. Here on the other hand we consider trapped particles, where $T$ can be increased until it is limited by the decoherence in the system [19–21]. The long-term stability thus depends on the nature of the decoherence.

To take decoherence into account Ref. [14] considered single atom dephasing. For this model Ref. [14] showed that entanglement cannot improve the stability of atomic clocks considerably (although an improvement is possible

FIG. 1 (color online). (a) The atomic state just before the measurement of $J_z$ for (A) uncorrelated atoms, (B) moderately squeezed atoms, and (C) highly squeezed atoms. (b) Stability as a function of the Ramsey time ($\gamma T$) for $N = 10^5$. $\blacksquare$, $\blacktriangle$ is the conventional protocol of Ref. [5] for optimal squeezing (uncorrelated) atoms while $\bullet$, $\blacktriangle$ is the adaptive protocol for optimal squeezing (uncorrelated) atoms. The adaptive protocol allows for $\gamma T \sim 0.3$ while the conventional protocol only allows for $\gamma T \sim 0.1$ (see the Supplemental Material [32]).
for non-Markovian noise [22]). A more realistic model of
the decoherence was described in Ref. [5] where the pri-
mary noise source is the frequency fluctuations of the
LO [23]. In this work a small improvement in the long-
term stability, scaling as \( \sim N^{1/6} \), was identified for
entangled atoms. Here we use the same decoherence model
and disregard any decoherence of atoms, to show that
entanglement and adaptive measurements may improve
the performance and give near-Heisenberg-limited atomic
clocks. Although the assumption of negligible atomic
decoherence may be hard to fulfill for the highly entangled
states considered here, our results highlight that there is no
fundamental obstacle to reaching the Heisenberg limit.

Another approach to increase the stability is to increase
the frequency fluctuations of the
LO [23]. In this work a small improvement in the long-
term noise source is the frequency fluctuations of the
LO [23]. In particular Ref. [26] increases
through a
measurement protocol highly related to ours. However that
work considers a scenario where the clock is limited by
technical noise so that a direct comparison with our results
is not possible. Which protocol is advantageous is thus an
open question beyond the scope of this Letter.

We consider an ensemble of \( N \) two-level atoms, which
we model as a collection of spin-1/2 particles with total
angular momentum \( \vec{J} \). The angular momentum operators
\( \vec{J}_{x,y,z} \) give the projections of \( \vec{J} \) on the \( x, y, \) and \( z \) axis.
The atoms are initially pumped to have a mean spin along
the \( z \) axis, \( \langle \vec{J}_z \rangle = 0 \). After the Ramsey sequence the Heisenberg evolution of \( \vec{J}_z \) is
\( \vec{J}_i(\delta \phi) = \hat{J}_z, \quad \hat{J}_y(\delta \phi) = \sin(\delta \phi)\hat{J}_x - \cos(\delta \phi)\hat{J}_z, \quad \text{and} \quad \hat{J}_z(\delta \phi) = \cos(\delta \phi)\hat{J}_y + \sin(\delta \phi)\hat{J}_z. \) At the end of the
Ramsey sequence \( \hat{J}_z \) is measured and used to estimate
\( \delta \phi \). The \( \hat{J}_i \) term in \( \hat{J}_z \) results in the so-called projection noise in the phase estimate \( \sim \Delta \hat{J}_z/\langle \hat{J}_z \rangle \). For uncorrelated
atoms \( \Delta \hat{J}_y \Delta \hat{J}_x = \langle \hat{J}_z \rangle/2 = N/4 \) and the projection noise
causes the stability of the clock to scale as \( \sim 1/\sqrt{N} \). For a spin squeezed state the variance of \( \hat{J}_z \) is reduced to
obtain a better phase estimate. Such a spin squeezed state is depicted in Fig. 1(a), which shows how the spin squeezed state looks like a “flat banana” on the Bloch sphere. The more we squeeze, the longer and more narrow the banana is and significant extra noise is added to the mean spin direction. For a phase estimate based on a direct measurement of \( \hat{J}_z \) this gives an additional noise term \( \sim \delta \phi \Delta \hat{J}_z/\langle \hat{J}_z \rangle \). This extra noise limited the performance in Ref. [5] if strongly squeezed states were used. We avoid this problem by using an adaptive scheme with weak measurements to make a rough estimate of \( \delta \phi \) and then rotate the spins of the atoms such that the mean spin is almost along the \( y \) axis. The flat banana depicted in Fig. 1(a) will then lie in the \( xy \) plane and this will decrease the noise from \( \Delta \hat{J}_z \) in subsequent measurements (see Fig. 2). Having eliminated the noise from \( \Delta \hat{J}_z \) we can allow strong squeezing in \( \Delta \hat{J}_x \) and obtain near-Heisenberg-limited stability.

The operation of the clock consists of repeating the
clock cycle illustrated in Fig. 2. The total cycle duration
\( T_{c} \) will be larger than the period of free evolution due to the
time spent on preparation and measurement of the atoms,
and this dead time introduces Dick noise to the stability
[28]. To focus on the atomic noise we assume that the dead
time is negligible (\( T_{c} \sim T \)) so that we can ignore the Dick
noise. (Alternatively some clock based measurements can
also be constructed which are immune to Dick noise
[29–31]). This assumption is further discussed in the
Supplemental Material [32]. We discretize time in the
number of clock cycles \( (k) \) such that at time \( t_k = kT \)
the frequency correction \( \Delta \omega(t_k) = \Delta \phi(t_k)/T \) is
applied to the LO where \( \alpha \) sets the strength of the feedback
loop and \( \delta \phi(t_k) \) is the estimate of the accumulated phase
\( \delta \phi(t_k) \) between time \( t_{k-1} \) and \( t_k \). The frequency offset of the
LO at time \( t_k \) is then \( \delta \omega(t_k) = \delta \omega_0(t_k) + \sum_{l=1}^{k-1} \Delta \omega(t_l) \),
where \( \delta \omega_0(t_k) \) is the frequency fluctuation of the unlocked
LO. The mean frequency offset after running for a period
\( \tau = lT \) \( (l \gg 1) \) is (see the Supplemental Material [32])
\[
\langle \delta \omega(\tau) \rangle = \frac{1}{T} \sum_{i=1}^{l} \delta \phi(t_k) - \delta \phi(\tau) \rangle \end{equation}
resulting in the long-term stability of the atomic clock:
\[
\sigma_{\gamma}(\tau) = \langle (\delta \omega(\tau)/\omega)^2 \rangle^{1/2} = \sqrt{\frac{1}{\tau \omega^2} \left( \frac{1}{l} \sum_{i=1}^{l} \langle (\delta \phi(t_k) - \delta \phi(\tau))^2 \rangle \right)^{1/2}}.
\]

We initially assume that the phase offset of the unlocked
LO \( \delta \phi_0 \) is due to frequency fluctuations in the LO with a
white noise spectrum. Later we will also consider the case
where the fluctuations have a \( 1/f \) spectrum. For white
noise we have \( \langle \delta \phi_0^2 \rangle = \gamma T \end{equation} \( (\delta \phi_0 = 0) \) where \( \gamma \) is a
parameter characterizing the fluctuations. In the limit $\alpha \ll 1$, the phases are uncorrelated (see the Supp. Mat. [32]) such that $\sigma_\gamma(\tau) = \sqrt{\gamma/\pi} \omega^{-\gamma((\delta \phi_0 - \delta \phi_{\epsilon})^2)/\gamma \tau}^{1/2}$. This expression shows that for fixed $\gamma$ and $\tau$ the stability of the clock only depends on how precisely we can estimate $\delta \phi_0$.

We now describe our adaptive measurements in detail. Our weak measurements are based on the strategy developed and demonstrated in Refs. [10,11,33,34] where a light field dispersively interacts with the spin and is subsequently measured. This is described by a Hamiltonian $H_{\text{int}} = -\chi_j \hat{J}_x \hat{X}_1$ where $\chi_j$ is the interaction strength and $\hat{X}_1$ is the canonical position operator of the light [35–37].

The measurement results in a rotation around $\hat{J}_1$ described by the rotation matrix $R_3(\hat{J}_1)$, where $\hat{J}_1 = \Omega_j \hat{X}_1$. $\Omega_j = \Omega_1 \mu_1$ is the measurement strength and $\mu_1$ is the measurement time. The canonical momentum operators of the light before ($\hat{P}_1$) and after ($\hat{P}_1'$) the interaction are then related by $\hat{P}_1' = \hat{P}_1 - \Omega_j \hat{J}_1$. $\hat{P}_1'$ is measured using homodyne detection [38] and the phase is estimated as $\delta \phi_i = \beta_{\epsilon} \hat{P}_1' / \Omega_j$, where $\beta_{\epsilon}$ is found from minimizing $\langle (\delta \phi_0 - \delta \phi_{\epsilon})^2 \rangle$. Based on the phase estimate we rotate the spin of the atoms around $\hat{J}_1$ in order to compensate for the extra noise added ($\Delta \hat{J}_y$) by the spin squeezing. This is described by a rotation matrix $R(\delta \phi_{\epsilon})$. The process can be iterated such that after $n - 1$ weak measurements the Heisenberg evolution of the original operators ($\hat{J}_1, \hat{J}_2, \hat{J}_3$) is

$$\begin{pmatrix}
\hat{J}_1 \\
\hat{J}_2 \\
\hat{J}_3
\end{pmatrix}_n = R_3(\delta \phi_{n-1}) R_3(\hat{J}_1_{n-1}) \cdots \hat{R}_3(\delta \phi_1)$$

$$\times R_3(\hat{J}_1_{n-1}),$$

The final measurement is assumed to be a projective measurement and the final phase estimate $\delta \Phi_f$ is thus $\delta \phi_n = \beta_{\epsilon} \hat{J}_3 / \Omega_j$. The factors of $\beta_i$ in the phase estimates are found by minimizing $\langle (\delta \phi_0 - \sum_{i=1}^{j} \delta \phi_i)^2 \rangle$ with respect to $\beta_i$ after each measurement. The final estimate of $\delta \phi_0$ at the end of the measurement sequence is $\delta \phi_{\epsilon} = \sum_{i=1}^{n} \delta \phi_i$, where $\delta \phi_{\epsilon}$ is the phase estimate after the $i$th measurement.

We will now show semianalytically that the measurement strategy in Eq. (3) allows for near-Heisenberg-limited stability. For simplicity we set all $\beta_i = 1$ in our analytical calculations. After $j$ weak measurements the difference between the true phase and the estimated phase $\delta \Phi_j$ is

$$\delta \Phi_j = \delta \phi_0 - \sum_{i=1}^{j} \delta \phi_i = \delta \phi_0 - \sum_{i=1}^{j-1} \delta \phi_i - \delta \phi_{\epsilon}.$$
We have numerically minimized $\sigma_\gamma(\tau)$ in the degree of squeezing, the number of weak measurements, the Ramsey time, and the strengths of the measurements. Figure 3(a) shows the result of the optimization for both the adaptive protocol and the conventional protocol with and without squeezing. The adaptive protocol gives a significant improvement compared to using uncorrelated atoms resulting in near-Heisenberg-limited stability. The numerical calculations also agree nicely with the analytical calculations (see the Supplemental Material [32]). As noted above the adaptive protocol allows for a longer Ramsey time than the conventional protocol, which gives an improvement of roughly a factor 1.6 for uncorrelated atoms.

So far we have assumed white noise in the LO. In practice the noise of the LO is however more likely to have a nontrivial spectrum like $1/f$ noise. We have therefore repeated the numerical optimization with $1/f$ noise in the LO for which $\langle \omega(f)\omega(f') \rangle = \delta(f + f')/\gamma^2/f$ and the results are shown in Fig. 3(b) (see the Supplemental Material [32]). The improvement obtained using the adaptive scheme with correlated atoms persists also for $1/f$ noise. Again near-Heisenberg-limited stability is obtained using the adaptive protocol. The longer Ramsey time of the adaptive scheme compared to projective measurements gives an improvement of roughly a factor 1.3 for uncorrelated atoms.

In conclusion we have developed an adaptive measurement protocol which allows operating atomic clocks near the Heisenberg limit using entangled spin squeezed ensembles of atoms. These results clearly demonstrate that entanglement can be an important resource for quantum metrology. Importantly our results are obtained under realistic assumptions where we account for the dominant source of noise in practice. We find that in this situation we can gain nearly the full potential of entanglement estimated without accounting for decoherence. Furthermore the adaptive protocol allows for a higher Ramsey time, which gives an improvement even for uncorrelated atoms.

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