Near-optimal multiple testing in Bayesian linear models with finite-sample FDR control

Taejoo Ahn† Licong Lin† Song Mei‡

July 25, 2023

Abstract

In high dimensional variable selection problems, statisticians often seek to design multiple testing procedures that control the False Discovery Rate (FDR), while concurrently identifying a greater number of relevant variables. Model-X methods, such as Knockoffs and conditional randomization tests, achieve the primary goal of finite-sample FDR control, assuming a known distribution of covariates. However, whether these methods can also achieve the secondary goal of maximizing discoveries remains uncertain. In fact, designing procedures to discover more relevant variables with finite-sample FDR control is a largely open question, even within the arguably simplest linear models.

In this paper, we develop near-optimal multiple testing procedures for high dimensional Bayesian linear models with isotropic covariates. We introduce Model-X procedures that provably control the frequentist FDR from finite samples, even when the model is misspecified, and conjecturally achieve near-optimal power when the data follow the Bayesian linear model. Our proposed procedure, PoEdCe, incorporates three key ingredients: Posterior Expectation, distilled Conditional randomization test (dCRT), and the Benjamini-Hochberg procedure with e-values (eBH). The optimality conjecture of PoEdCe is based on a heuristic calculation of its asymptotic true positive proportion (TPP) and false discovery proportion (FDP), which is supported by methods from statistical physics as well as extensive numerical simulations. Our result establishes the Bayesian linear model as a benchmark for comparing the power of various multiple testing procedures.

1 Introduction

High dimensional variable selection problems arise pervasively in a broad range of scientific domains, including genetics, healthcare, economics, and political science. These problems are often framed by statisticians within a multiple testing context, where the objectives are twofold: to control the false discovery rate (FDR) and to identify as many relevant variables as possible. The first goal of FDR control is usually much more crucial and expected to be achieved under much weaker model assumptions compared to the goal of variable discovery. To uncover more relevant variables, statisticians typically employ strong model assumptions that incorporate prior knowledge of the scientific domain. However, FDR control, often having more significant consequences or risks in scientific applications, is desired even if the model and the prior are misspecified. The contrasting model assumptions needed for these two objectives pose a significant challenge in high-dimensional variable selection tasks.

Considerable previous work has focused on controlling the frequentist FDR in variable selection problems. Among these, Model-X methods such as Knockoffs and conditional randomization tests [BC15, CFJL18, LKJR22], have proven successful in controlling FDR from finite samples under mild model assumptions. These methods, more specifically, presume a known covariate distribution but allow any correlation between the response and the covariates, employing resampling techniques to convert any base statistics into test statistics that control finite-sample FDR. Recent work has also studied the power of Model-X procedures

†Department of Statistics, University of California, Berkeley. Email: taejoo.ahn@berkeley.edu
†Department of Statistics, University of California, Berkeley. Email: liconglin@berkeley.edu
‡Department of Statistics and EECS, University of California, Berkeley. Email: songmei@berkeley.edu
Code for our experiments is available at https://github.com/taejoo-ahn/FDR_Bayes_figures.
when combined with LASSO-based statistics [WBC17, WSB+20, LR19, WJ22]. However, how to leverage Model-X methods to establish procedures with optimal power in specific models remains largely unanswered.

An alternative line of research has sought to derive optimal FDR control procedures using the Bayesian approach. The central quantity in these Bayesian methods is the local false discovery rate (local fdr) [ETST01, Efr05], which is the posterior probability of the null hypothesis being applicable. Prior work [MPRR04, MPR06, SC07, XCML11] has demonstrated that truncating local fdrs results in the most powerful procedure among those controlling Bayesian FDR. However, the Bayesian FDR control of these methods relies heavily on the assumption that the model and the prior are correctly specified. These procedures could potentially lose FDR control whenever the prior is incorrect or the model is misSpecified.

A natural idea for circumventing the restrictions of both frequentist and Bayesian methods involves developing a procedure that integrates these two approaches. More specifically, one might use local fdrs as base statistics and wrap them using Model-X methodologies. Moreover, the Bayesian linear model, which is applicable to a broad range of scientific problems, presents itself as arguably the most suitable model for exploring such an idea. This prompts us to pose the following question:

**Is there a procedure that controls FDR from finite samples and achieves near-optimal power under well-specified Bayesian linear models?**

In this paper, we investigate this question and suggest an affirmative answer through the introduction of two procedures, PoPCE (pronounced as “pop-see”) and PoEdCE (pronounced as “pod-see”). These procedures utilize local fdr and posterior expectation as the base statistics, apply the conditional randomization test (CRT) [CFJL18] to compute the p-value of each hypothesis, and then implement the eBH procedure [WR22] on the obtained p-values. We show that PoPCE and PoEdCE always control finite-sample FDR and conjecturally achieve near-optimal power under Bayesian linear models with a known prior distribution. The conjectured result is supported by methods from statistical physics as well as extensive numerical simulations. This result establishes the Bayesian linear model as a benchmark for comparing the power of various multiple testing procedures.

### 1.1 Model setup

Suppose we observe independent and identically distributed (i.i.d.) samples \(\{(\mathbf{x}_i, y_i)\}_{i \in [n]} \subseteq \mathbb{R}^d \times \mathcal{Y}\) from the joint distribution \(P \in \mathcal{P}(\mathbb{R}^d \times \mathcal{Y})\) over variables \(\mathbf{X} = (X_1, X_2, \ldots, X_d)\) and \(Y\). Throughout the paper, we represent the response vector as \(Y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n\) and the covariate matrix as \(\mathbf{X} = (x_1, \ldots, x_n)^T = (x_{1j}, \ldots, x_{nj}) \in \mathbb{R}^{n \times d}\), where \(\{\mathbf{x}_i\}_{i \in [n]} \subseteq \mathbb{R}^d\) and \(\{x_{ij}\}_{j \in [d]} \subseteq \mathbb{R}^n\). In the case when the response \(Y\) only depends on a small subset of the covariates, our goal is to identify this subset from samples. This problem can be cast into the multiple testing framework. Namely, a variable \(X_j\) is said to be “null” if and only if under the joint distribution \(P \in \mathcal{P}(\mathbb{R}^d \times \mathcal{Y})\), \(Y\) is independent of \(X_j\) when conditioned on the other variables \(\mathbf{X}_{-j} = \{X_1, \ldots, X_{d}\} \setminus \{X_j\}\). The subset of null variables is denoted by \(\mathcal{H}_0 = \mathcal{H}_0(P) \subseteq [d]\). Conversely, a variable \(X_j\) is called “nonnull” if \(j \notin \mathcal{H}_0\), with the subset ofnonnull variables denoted by \(S = S(P) = [d] \setminus \mathcal{H}_0(P)\).

A test statistics \(\mathbf{T} = (T_j)_{j \in [d]}\) is a vector of possibly random functions \(T_j : \Omega \rightarrow \{0, 1\}\) where \(\Omega = (\mathbb{R}^d \times \mathcal{Y})^n\). We say that the \(j\)-th null hypothesis is rejected if and only if \(T_j(D) = 1\). Given the dataset \(\mathcal{D} = (\mathbf{X}, \mathbf{Y})\) and the joint distribution \(P \in \mathcal{P}(\mathbb{R}^d \times \mathcal{Y})\), the number of discoveries \(\mathcal{R}\), the number of false discoveries \(\mathcal{F}_D\), and the number of true discoveries \(\mathcal{TD}\) of the test statistics \(\mathbf{T}\) are respectively defined as:

\[
\begin{align*}
\mathcal{R}(\mathbf{T}; \mathcal{D}) &\equiv \#\{j \in [d] : T_j(D) = 1\}, \\
\mathcal{F}_D(\mathbf{T}; \mathcal{D}, P) &\equiv \#\{j \in \mathcal{H}_0(P) : T_j(D) = 1\}, \\
\mathcal{TD}(\mathbf{T}; \mathcal{D}, P) &\equiv \#\{j \in S(P) : T_j(D) = 1\}.
\end{align*}
\]

Furthermore, we define the false discovery proportion (FDP) and the true positive proportion (TPP) of the test statistics \(\mathbf{T}\) as

\[
\begin{align*}
\text{FDP}(\mathbf{T}; \mathcal{D}, P) &\equiv \frac{\mathcal{F}_D(\mathbf{T}; \mathcal{D}, P)}{\mathcal{R}(\mathbf{T}; \mathcal{D}) \lor 1}, \\
\text{TPP}(\mathbf{T}; \mathcal{D}, P) &\equiv \frac{\mathcal{TD}(\mathbf{T}; \mathcal{D}, P)}{|S(P)| \lor 1}.
\end{align*}
\]
In the above expressions, we explicitly detail the dependence on $D$ and $P$ for clarity. We will abbreviate as $\text{FDP}(T) = \text{FDP}(T; D, P)$ when the dataset $D$ and the distribution $P$ are unambiguous from the context (similar abbreviations apply to TPP, R, FD, and TD).

We next define the usual frequentist FDR and TPR [BH95], which is the expectation of FDP and TPP over the observed samples following the distribution $\{(x_i, y_i)\}_{i \in [n]} \sim \text{i.i.d.} P$, and over the randomness of the test statistics $T$

\[
\text{FDR}(T, P) \equiv \mathbb{E}_{D \sim P, T} \left[ \frac{\text{FD}(T; D, P)}{\mathbb{R}(T; D) \lor 1} \right], \quad \text{TPR}(T, P) \equiv \mathbb{E}_{D \sim P, T} \left[ \frac{\text{TD}(T; D, P)}{|S(P)| \lor 1} \right].
\]

A test statistics $T$ is said to control the frequentist $\text{FDR}$ at level $\alpha$ over a collection of distributions $\mathcal{M}$, if $\text{FDR}(T; P) \leq \alpha$ for any distribution $P \in \mathcal{M}$. Additionally, statisticians often consider Bayesian variants of FDR and TPR. Given a family of distributions $\{P_Z(\beta_0)\}_{\beta_0 \in \mathcal{B}} \subseteq \mathcal{P}(\mathbb{R}^d \times \mathcal{Y})$ parameterized by $\beta_0 \in \mathcal{B}$, and given a prior $\beta_0 \sim \Pi \in \mathcal{P}(\mathcal{B})$, we can define the Bayesian false discovery rate (BFDR) and the Bayesian true positive rate (BTPR) [MPRR04, MPR06, SC07, Sto07] of the test statistics $T$ as

\[
\text{BFDR}(T, \Pi) \equiv \mathbb{E}_{\beta_0 \sim \Pi} \left\{ \mathbb{E}_{D \sim P_{Z(\beta_0)}} \left[ \frac{\text{FD}(T; D, P_{Z(\beta_0)})}{\mathbb{R}(T; D) \lor 1} \right] \right\},
\]

\[
\text{BTPR}(T, \Pi) \equiv \mathbb{E}_{\beta_0 \sim \Pi} \left\{ \mathbb{E}_{D \sim P_{Z(\beta_0)}} \left[ \frac{\text{TD}(T; D, P_{Z(\beta_0)})}{|S(P_{Z(\beta_0)})| \lor 1} \right] \right\}.
\]

In the literature, people have also considered the marginal false discovery rate (mFDR) and marginal true positive rate (mTPR) defined as (with the convention that $0/0 = 0$)

\[
\text{mFDR}(T, \Pi) \equiv \frac{\mathbb{E}_{\beta_0 \sim \Pi} \{\mathbb{E}_{D \sim P_{Z(\beta_0)}} \text{FD}(T; D, P_{Z(\beta_0)})\}}{\mathbb{E}_{\beta_0 \sim \Pi} \{\mathbb{E}_{D \sim P_{Z(\beta_0)}} \mathbb{R}(T; D)\}},
\]

\[
\text{mTPR}(T, \Pi) \equiv \frac{\mathbb{E}_{\beta_0 \sim \Pi} \{\mathbb{E}_{D \sim P_{Z(\beta_0)}} \text{TD}(T; D, P_{Z(\beta_0)})\}}{\mathbb{E}_{\beta_0 \sim \Pi} \{S(P_{Z(\beta_0)})\}}.
\]

In high dimensional statistical models, due to the concentration phenomenon, different average notions of false discovery proportion (and true positive proportion) often approximate each other asymptotically. In simpler terms, in high dimensions, we often observe $\text{FDP} \approx \text{FDR} \approx \text{BFDR} \approx \text{mFDR}$ and $\text{TPP} \approx \text{TPR} \approx \text{BTPR} \approx \text{mTPR}$ [GW02].

This paper will focus on the Bayesian linear model with isotropic Gaussian covariates. The same model was previously investigated in [WBC17, WSB+20] to analyze the efficacy of the Knockoff approach when applied to LASSO-based statistics. In the Bayesian linear model, the response $Y \in \mathbb{R}$ and the covariates $X \in \mathbb{R}^d$ form a linear relationship $Y = (X, \beta_0) + \varepsilon$, where the covariates $X$, the parameters $\beta_0 = (\beta_{0,j})_{j \in [d]} \in \mathbb{R}^d$, and the random noise $\varepsilon \in \mathbb{R}$ are mutually independent. We assume that $X \sim \mathcal{N}(0, (1/n)I_d)$, and assume that the prior of $\beta_0$ has i.i.d. coordinates $(\beta_{0,j})_{j \in [d]} \sim \text{i.i.d.} \Pi$, where $\Pi \in \mathcal{P}(\mathbb{R})$ is a distribution supported on the real line. We further assume that $\Pi$ has a point mass at 0, and the weight of this point mass gives $\pi_0 \in (0, 1)$, which is roughly the proportion of parameters $\{\beta_{0,j}\}_{j \in [d]}$ that equal 0. It should be noted that in linear models, under certain mild conditions (such as one cannot perfectly predict any $X_j$ from knowledge of $X_{-j}$, [CFJL18, Proposition 2.2]), the $j$-th null hypothesis $H_{0j} : Y \perp X_j | X_{-j}, \beta_0$ can be demonstrated to be equivalent to the hypothesis $H_{0j} : \beta_{0,j} = 0$. A more comprehensive description of the Bayesian linear model is provided in Assumption 3.

### 1.2 Frequentist optimality in the Model-X setting and Bayesian optimality

We now focus on the Model-X setting [CFJL18, LKJR22], in which $P|_X$, the marginal distribution of the covariates of $P \in \mathcal{P}(\mathbb{R}^d \times \mathcal{Y})$, is known to the statistician. For a distribution over covariates $P_X \in \mathcal{P}(\mathbb{R}^d)$, we let

\[
\mathcal{M}(P_X) \equiv \{P \in \mathcal{P}(\mathbb{R}^d \times \mathcal{Y}) : P|_X = P_X\}
\]

represent the ensemble of probability distributions where the marginal distribution of the covariates aligns with $P_X$. Notice that our goal is to propose test statistics that control the frequentist FDR from finite

3
samples and ensure asymptotic optimality given the model is correctly specified. This gives rise to the following optimality notion.

**Definition 1** (Optimal test with finite-sample FDR control). Let \( \{ P_{Z|\beta_0} \}_{\beta_0 \in \mathcal{B}} \subseteq \mathcal{P}(\mathbb{R}^d \times \mathcal{Y}) \) be a family of distributions and let \( \Pi \subseteq \mathcal{P}(\mathcal{B}) \) be a prior. Assume that \( \{ P_{Z|\beta_0} \}_{\beta_0 \in \mathcal{B}} \subseteq \mathcal{M}(P_X) \) for a distribution \( P_X \in \mathcal{P}(\mathbb{R}^d) \). For some scalars \( \alpha \in (0, 1) \) and \( \varepsilon > 0 \), we say a test statistics \( T_* : \Omega \to \{0, 1\}^d \) is \((\alpha, \mathcal{M}(P_X), \Pi, \varepsilon)\)-optimal with frequentist FDR control, if its frequentist FDR is controlled at level \( \alpha \) over the collection of distributions \( \mathcal{M}(P_X) \),

\[
\sup_{P \in \mathcal{M}(P_X)} \text{FDR}(T_*, P) \leq \alpha,
\]

and its \( mTPR \) with prior \( \Pi \) is nearly maximized across all tests that control frequentist FDR at level \( \alpha \),

\[
mTPR(T_*, \Pi) \geq \sup_{T : \Omega \to \{0, 1\}^d} \left\{ \sup_{P \in \mathcal{M}(P_X)} \text{FDR}(T, P) \leq \alpha \right\} - \varepsilon.
\]

In Definition 1, an alternative to employing \( mTPR \) as the power measure is to use \( BTPR \) instead. As we have mentioned, \( BTPR \) and \( mTPR \) will be asymptotically equal in many high dimensional models, so it will not make a big difference in choosing either as the power measure. For the purposes of this paper, we specifically choose \( mTPR \) as the power measure due to its ability to facilitate a more straightforward theoretical framework.

In addition to the aforementioned, there are two other crucial optimality concepts, namely, the optimal test with \( \text{BFDR} \) control and the optimal test with \( mFDR \) control.

**Definition 2** (Optimal test with \( \text{BFDR} \) control). Let \( \{ P_{Z|\beta_0} \}_{\beta_0 \in \mathcal{B}} \subseteq \mathcal{P}(\mathbb{R}^d \times \mathcal{Y}) \) be a family of distributions and let \( \Pi \subseteq \mathcal{P}(\mathcal{B}) \) be a prior on \( \beta_0 \). Let \( \alpha, \varepsilon \in [0, 1] \) be two scalars. For a procedure \( T_* : \Omega \to \{0, 1\}^d \), we call it \((\alpha, \Pi, \varepsilon)\)-optimal with \( \text{BFDR} \) control, if its \( \text{BFDR} \) with prior \( \Pi \) is controlled at level \( \alpha \),

\[
\text{BFDR}(T_*, \Pi) \leq \alpha,
\]

and its \( mTPR \) is nearly maximized across all tests that control \( \text{BFDR} \) at level \( \alpha \),

\[
mTPR(T_*, \Pi) \geq \max_{T : \Omega \to \{0, 1\}^d} \left\{ mTPR(T, \Pi) : \text{BFDR}(T, \Pi) \leq \alpha \right\} - \varepsilon.
\]

**Definition 3** (Optimal test with \( mFDR \) control). Let \( \{ P_{Z|\beta_0} \}_{\beta_0 \in \mathcal{B}} \subseteq \mathcal{P}(\mathbb{R}^d \times \mathcal{Y}) \) be a family of distributions and let \( \Pi \subseteq \mathcal{P}(\mathcal{B}) \) be a prior on \( \beta_0 \). Let \( \alpha, \varepsilon \in [0, 1] \) be two scalars. For a procedure \( T_* : \Omega \to \{0, 1\}^d \), we say that it is \((\alpha, \Pi, \varepsilon)\)-optimal with \( mFDR \) control, if its \( mFDR \) with prior \( \Pi \) is controlled at level \( \alpha \),

\[
mFDR(T_*, \Pi) \leq \alpha,
\]

and its \( mTPR \) is nearly maximized across all tests that control \( mFDR \) at level \( \alpha \),

\[
mTPR(T_*, \Pi) \geq \max_{T : \Omega \to \{0, 1\}^d} \left\{ mTPR(T, \Pi) : mFDR(T, \Pi) \leq \alpha \right\} - \varepsilon.
\]

An essential connection exists between the optimal test with frequentist FDR control and the optimal test with \( \text{BFDR} \) control, as per Definition 1 and 2. The following lemma shows that the power of the optimal test with \( \text{BFDR} \) control gives an upper bound for the power of the optimal test with frequentist FDR control. The proof of this is directly evident from Definition 1 and 2.

**Lemma 1.** Let \( \{ P_{Z|\beta_0} \}_{\beta_0 \in \mathcal{B}} \subseteq \mathcal{P}(\mathbb{R}^d \times \mathcal{Y}) \) be a family of distributions and let \( \Pi \subseteq \mathcal{P}(\mathcal{B}) \) be a prior. Assume that \( \{ P_{Z|\beta_0} \}_{\beta_0 \in \mathcal{B}} \subseteq \mathcal{M}(P_X) \) for a distribution \( P_X \in \mathcal{P}(\mathbb{R}^d) \). Then the \( \text{BFDR} \) value of the \((\alpha, \mathcal{M}(P_X), \Pi, \varepsilon)\)-optimal test with frequentist FDR control is always less than or equal to the \( mTPR \) value of the \((\alpha, \Pi, \varepsilon)\)-optimal test with \( \text{BFDR} \) control.

**Proof.** Note that \( \text{BFDR}(T, \Pi) = \mathbb{E}_{\beta_0 \sim \Pi} \text{FDR}(T, P_{Z|\beta_0}) \leq \mathbb{E}_{\beta_0 \sim \Pi} |\alpha| = \alpha \) for any test statistics \( T \) with frequentist FDR below \( \alpha \) (c.f. Eq. (7)). That is, any test statistics with frequentist FDR below \( \alpha \) (c.f. Eq. (7)) also has \( \text{BFDR} \) below \( \alpha \) (c.f. Eq. (9)). Lemma 1 follows immediately. 

**Lemma 1** suggests that, in order to prove that a test statistics \( T \) is near-optimal with frequentist FDR control, it is sufficient to demonstrate two things: (1) \( T \) controls finite-sample FDR; (2) the \( mTPR \) of \( T \) is close to the \( mTPR \) of the optimal test with \( \text{BFDR} \) control. In this paper, we will follow this approach to show that PoPce and PoEdCe (which will be described in Section 4) are near-optimal with finite-sample frequentist FDR control.
1.3 Summary of contributions and paper outline

• Optimal procedures with mFDR and BFDR control. In Section 3.1, we introduce two procedures: TPoP (Truncating the Posterior Probability) and CPoP (Cumulative Posterior Probability). Both methods are based on the truncation of local fdrs, the posterior probabilities of the hypotheses being null. We show that TPoP is the optimal test with mFDR control, while CPoP is the optimal test with BFDR control. It should be noted that neither TPoP nor CPoP are entirely new procedures [MPRR04, SC07, XCML11], and their optimality proofs are primarily based on Bayesian decision theory.

• Asymptotic power of the Bayesian optimal procedures. In Section 3.2, we examine the Bayesian linear model with isotropic Gaussian covariates and obtain the analytical formula for the asymptotic TPP and FDP associated with the optimal procedures, CPoP and TPoP. The derivation of this formula is primarily heuristic, leveraging the replica method [MM09], a useful tool originating from spin-glass theory within statistical physics. The validity of the derived analytical formula is subsequently demonstrated through numerical simulations.

• Optimal procedures with frequentist FDR control. In Section 4, we establish procedures that control frequentist FDR in the Model-X setting and are conjecturally near-optimal in the Bayesian linear model, suggesting an affirmative answer to our question. Specifically, we introduce the PoPCE procedure (Posterior Probability + Conditional randomization test + eBH) along with its computationally efficient variant, PoEdCe (Posterior Expectation + distilled Conditional randomization test + eBH). We prove that both of these procedures control the frequentist FDR from finite samples under any data-generating model $P \in \mathcal{M}(P_X)$, even in instances of model misspecification. When the data originates from a Bayesian linear model with isotropic covariates, we propose the conjecture that PoPCE and PoEdCe achieve near-optimal power. We arrive at this conjecture through heuristic calculations, which we subsequently confirm via numerical simulations.

• Bayesian linear model as a benchmark. This result establishes the Bayesian linear model as a benchmark for power comparison amongst various multiple testing procedures. In other words, the efficacy of these multiple testing procedures can be evaluated in relation to the optimal power of PoEdCe within the Bayesian linear model.

Practical implications We would like to emphasize that while the PoPCE and PoEdCe procedures serve as theoretical tools for the power analysis of FDR-controlling procedures, we do not recommend their direct application in practical scenarios. Indeed, these procedures could possibly be powerless under model misspecification in real-world datasets. An interesting open question is thus whether one can enhance the power under model misspecification while maintaining finite-sample validity and Bayes optimality.

1.4 Notations and conventions

Through the paper, for an integer $n$, we denote $[n] = \{1, 2, \ldots, n\}$. We denote the samples by $\{(x_i, y_i)\}_{i \in [n]} \subseteq \mathcal{X} \times \mathcal{Y}$, the response vector by $Y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$ and the covariate matrix by $X \in \mathbb{R}^{n \times d}$. We also denote the rows of $X$ by $\{x_i\}_{i \in [n]}$ and the columns by $\{x_j\}_{j \in [d]}$. We use $\beta_{-j} \in \mathbb{R}^{d-1}$ to denote the sub-vector of $\beta \in \mathbb{R}^d$ with the $j$-th coordinate removed, use $x_{i,-j} \in \mathbb{R}^{d-1}$ to denote the sub-vector of $x_i \in \mathbb{R}^d$ with the $j$-th coordinate removed, and use $X_{-,j} \in \mathbb{R}^{n \times (d-1)}$ to denote the sub-matrix of $X$ with the $j$-th column removed.

For a measurable space $S$, we let $\mathcal{P}(S)$ denote the set of all Borel probability measures on the space. For a distribution $P \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$, we use $(X, Y) \in \mathbb{R}^d \times \mathbb{R}$ to denote the random variables that follow the distribution $P$, and use $X_{-,j} \in \mathbb{R}^{d-1}$ to denote sub-vector of $X$ with the $j$-th coordinate removed. We use $\mathcal{L}(X)$ to denote a distribution over the random vector $X$, and use $\mathcal{L}(X_j|X_{-,j} = x_{-,j})$ to denote the conditional distribution of $X_j$ given $X_{-,j} = x_{-,j}$ under the law $\mathcal{L}(X)$.

In this paper, a mathematical statement is termed as a formalism if it can be derived heuristically and confirmed numerically. In the appendix, we present several formalisms which support the intuitions behind our conjectures.
2 Other related works

Multiple testing is a central topic of statistical inference and has inspired numerous studies. However, due to space constraints, we will focus on the most relevant works.

The concept of the false discovery rate (FDR) was introduced in the frequentist context by [BH95], who also proposed a procedure (the Benjamini-Hochberg procedure, hereafter BH) to control the FDR given independent p-values. The original BH procedure is conservative by a factor of $\pi_0$, where $\pi_0$ is the proportion of null hypotheses. To address this, the same authors [BH00] recommended estimating $\pi_0$ using large p-values. Later, [BY01] demonstrated that the BH procedure also controls the FDR when the p-values satisfy the PRDS condition, which is less stringent than independence. However, the BH procedure fails to control the FDR with arbitrarily correlated p-values unless adjusted by a log factor, leading to a loss of power. As a solution, [WR22] introduced the eBH procedure (Benjamini-Hochberg with e-values), which controls the FDR even in the presence of arbitrary correlations among e-values [VW21, Sha19], a more manageable mathematical tool than p-values.

There is an alternate line of research that studies optimal procedures concerning Bayes FDR control. [ETST01, Efr05] introduced the local fdr framework, which applies the empirical Bayes approach to multiple testing problems. Here, the local fdr represents the posterior probability that the null hypothesis holds. [SC07, XCML11] demonstrated that truncating the local fdr is the optimal procedure for controlling the marginal FDR (mFDR) within a Bayes setting, and they suggested truncating the cumulative local fdr for adaptivity. [MPRR04, MPR06, Sto07] proved that truncating the local fdr in certain adaptive manners is optimal for specific Bayesian criteria. Conversely, other studies, such as [ABDJ06, MTCL20, ZMCL20], derived FDR control procedures in the super-sparse regime. These works derived the “optimal” procedures in the regime such that the optimal power can asymptotically approach one, a different context than what we consider in this paper.

Several FDR-controlling methods based on the Knockoff filter and its variants have been recently introduced. [BC15] presented the fixed design Knockoff procedure, which controls the finite-sample FDR in linear models. However, this procedure is applicable only when the sample size exceeds the dimension. [CFJL18] introduced the Model-X Knockoff procedure, demonstrating remarkable flexibility by controlling the finite-sample FDR in any probabilistic model, as long as the distribution of covariates $X$ is known. Inspired by the Knockoff procedure, [XZL21, KLM20, DLXL22, DLXL20] proposed to control FDR in an asymptotic sense using mirror statistics. They reveal that these methods attain greater power when features are highly correlated and when the parameter vector is less sparse. Furthermore, [SJ22] enhanced the power of Model-X Knockoff procedures by creating knockoffs that minimize the reconstructability of the features.

Under the assumption of a linear model and high-dimensional proportional asymptotics, a series of works [SBC17, WWM20, WBC17, WSB⁺20, WYBS20, LR19, HL19, BKRS21, WJ22] derive the precise limit of the FDP and TPP for various variable selection methods, such as LASSO, $\ell_p$-ridge regression, SLOPE, and their respective Knockoff variations. Specifically, [WBC17] computes the asymptotic power of the Knockoff procedure for the LASSO statistics, while [WSB⁺20] extends this result to the Knockoff procedure for the truncated LASSO coefficient statistics. Moreover, [HL19] investigates the trade-off curve and optimal regularization for the SLOPE procedure. The precise calculations in these studies are primarily built upon recent advancements in the high-dimensional asymptotics of $M$-estimators, as demonstrated in works such as [DMM09, Ran11, BM11, Kar13, DM16, BMN20].

The conditional randomization test (CRT) [CFJL18] is a Model-X procedure closely associated with the Knockoff filter. CRT generates a valid p-value by calculating the rank of base statistics among its resampled variants. [CFJL18, WJ22] show that CRT outperforms Model-X Knockoff in terms of power within certain statistical models, although CRT comes with a higher computational load. Variants of CRT include the conditional permutation test [BWBS20], the holdout randomization test [TVZ⁺22], and the distilled conditional randomization test (dCRT) [LKJR22]. Among these, dCRT is especially notable for its significant reduction in the computational cost of CRT, making it a key point of interest in this paper.

From a technical viewpoint, our main results and conjectures exploit the asymptotics of Bayes estimators of high dimensional models, as outlined in references such as [BKM⁺19, BM19, BDMK16, DAM15, LM19, BCPS21]. Furthermore, we borrow heuristic tools from statistical physics literature, including the replica method and the interpolation method [MM09, Tal10]. We should note that it is unclear whether local fdrs used in our procedures are efficiently computable: further exploration of these computational aspects can be
3 Statistical limits of Bayesian procedures

We begin by deriving the limiting statistical power of FDR controlling procedures within a Bayesian framework. We show that truncating the local fdr (TPoP) is the optimal procedure with mFDR control, and truncating the cumulative local fdr (CPoP) is the optimal procedure with BFDR control (Section 3.1). We then consider the Bayesian linear model with isotropic covariates, wherein we derive the asymptotic limits of FDP and TPP for both TPoP and CPoP (Section 3.2). Numerical simulations are provided for comparing TPoP and CPoP with the thresholding LASSO procedure (Section 3.3). The statistical limits of CPoP and TPoP will be used to establish the frequentist optimality of PoPCE and PoEdCE, to be introduced in Section 4.

3.1 The optimal Bayesian procedures

The local false discovery rate (local fdr) [ETST01, Efr05] is a widely-used tool in multiple hypothesis testing. Assuming a Bayesian model, the local fdr calculates the posterior probability of a hypothesis being null. Following the setup of Section 1.2, we denote $P_j(D)$ to be the $j$-th local fdr, under the Bayesian model $D \sim P_{Z|\beta_0}$, with prior $\beta_0 \sim \Pi$,

$$P_j(D) = \mathbb{P}(j \in H_0(P)|D) = \mathbb{P}(\beta_{0,j} = 0|D). \quad (13)$$

Intuitively, a larger local fdr suggests a higher likelihood of the corresponding hypothesis being null under the Bayesian model.

**The TPoP procedure for mFDR control** The TPoP procedure $T_P(D; t)$, which represents Truncating the Posterior Probability, truncates the local fdr at level $t$, i.e., rejecting the hypotheses that are unlikely to be null,

$$T_{P,j}(D; t) = 1\{P_j(D) < t\}, \quad T_P = (T_{P,1}, \ldots, T_{P,d}). \quad (14)$$

Prior studies have demonstrated that TPoP gives the optimal mTPR with mFDR control (c.f. Definition 3) in specific statistical models [SC07, XCML11]. Extending these results, we next present a general regularity assumption under which we can show the optimality of TPoP.

**Assumption 1** (Regularity). Under the Bayesian model $D \sim P_{Z|\beta_0}$ with prior $\beta_0 \sim \Pi$, the conditional densities $p(D|\beta_{0,j} = 0)$ and $p(D|\beta_{0,j} \neq 0)$ exist. Furthermore, $\mathbb{P}(P_j(D) < t|\beta_{0,j} = 0)$ and $\mathbb{P}(P_j(D) < t|\beta_{0,j} \neq 0)$ are continuous in $t$ for each $j \in [d]$.

We next show the optimality of TPoP under the regularity assumption (proof in Appendix A).

**Proposition 1** (Optimality of TPoP). Let $A = (\inf \text{mFDR}(T_P(\cdot; t), \Pi), \sup \text{mFDR}(T_P(\cdot; t), \Pi))$ and let Assumption 1 hold. Then for any $\alpha \in A$, there exists $t = t(\alpha) \in (0, 1)$ such that

$$\text{mFDR}(T_P(\cdot; t(\alpha)), \Pi) = \alpha. \quad (15)$$

Moreover, for any test statistics $T : \Omega \rightarrow \{0, 1\}^d$ with $\text{mFDR}(T, \Pi) \leq \alpha$, we have

$$\text{mTPR}(T_P(\cdot; t(\alpha)), \Pi) \geq \text{mTPR}(T, \Pi). \quad (16)$$

That is, $T_P(\cdot; t(\alpha))$ is an $(\alpha, \Pi, 0)$-optimal procedure with mFDR control (c.f. Definition 3).

We should note that Proposition 1 does not constitute an entirely new discovery; the optimality of TPoP has previously been established in specific statistical models. As an example, [SC07] proved that TPoP attains the smallest marginal false negative ratio in the case of a mixture model $X_i|\theta_i \sim \theta_i F_0 + (1 - \theta_i) F_1$, where $\{\theta_i\}_{i \in [d]}$ are independent Bernoulli random variables. Proposition 1 extends and adapts these results to general Bayesian models.
The CPoP procedure for BFDR control The CPoP procedure \( C_P \), which represents Cumulative Posterior Probability, truncates the local fdr \( \{ P_j(D) \}_{j \in [d]} \) (c.f. Eq. (13)) at some data dependant threshold that is determined by the cumulative local fdr,

\[
C_{P,j}(D; \lambda) = 1\{ P_j(D) < P_{\hat{K}(\lambda,D)}(D) \}, \quad C_P = (C_{P,1}, \ldots, C_{P,d}),
\]

where \( \hat{K}(\lambda, D) \in [d] \) is the number of rejections given by

\[
\hat{K}(\lambda, D) = \arg \max_{K \in [d]} \left( K - (1 - \lambda N/(K \vee 1)) \sum_{j=1}^K P_j(D) \right),
\]

\[
N \equiv E_{\beta_0 \sim \Pi}[\# \{ j : j \notin H_0(P) \}] = E_{\beta_0 \sim \Pi}[\# \{ j : \beta_{0,j} \neq 0 \}].
\]

Here \( N \) stands for the expected number of nonnulls, and \( \{ P_j(D) \}_{j \in [d]} \) are the order statistics of the local fdr \( \{ P_j(D) \}_{j \in [d]} \) in increasing order \( P_1(D) \leq P_2(D) \leq \cdots \leq P_d(D) \).

Proposition 2 below shows that the CPoP procedure with a properly chosen \( \lambda \) is the optimal test with BFDR control (c.f. Definition 2), under the following continuity assumption of the model distribution.

**Assumption 2** (Continuity). Under the Bayesian model \( D \sim P_{Z|\beta_0} \) with prior \( \beta_0 \sim \Pi \), the distribution of the random vector \( (P_1(D), P_2(D), \ldots, P_d(D)) \) is absolutely continuous to the Lebesgue measure on \( \mathbb{R}^d \).

**Proposition 2** (Optimality of CPoP). Let \( A = (\inf_\lambda \text{BFDR}(C_P(\cdot; \lambda), \Pi), \sup_\lambda \text{BFDR}(C_P(\cdot; \lambda), \Pi)) \) and let Assumption 2 hold. Then for any \( \alpha \in A \), there exists \( \lambda = \lambda(\alpha) \in \mathbb{R}_{\geq 0} \) such that

\[
\text{BFDR}(C_P(\cdot; \lambda(\alpha)), \Pi) = \alpha.
\]

Moreover, for any test statistics \( T : \Omega \to \{0, 1\}^d \) with BFDR(\( T, \Pi \)) \( \leq \alpha \), we have

\[
\text{mTPR}(C_P(\cdot; \lambda(\alpha)), \Pi) \geq \text{mTPR}(T, \Pi).
\]

That is, \( C_P(\cdot; \lambda(\alpha)) \) is an \((\alpha, \Pi, 0)-optimal\) procedure with BFDR control (c.f. Definition 2).

The proof of Proposition 2 is contained in Appendix B. We note that the continuity condition (Assumption 2) is technical, ensuring the existence of \( \lambda(\alpha) \) satisfying Eq. (20) for any \( \alpha \in A \). This assumption is mild and is satisfied as long as \( D \) admits a continuous probability density function, and the map \( D \mapsto (P_1(D), \ldots, P_d(D)) \) is non-degenerate almost everywhere. A concrete example satisfying this assumption is the Bayesian linear model (see Assumption 3 for details). Furthermore, we note that results similar to Proposition 2 have also been shown in specific statistical models [MPRR04, MPR06]. Proposition 2 extends and adapts these results to general Bayesian models.

### 3.2 The limiting power in Bayesian linear model

We next derive the limiting power of TPoP and CPoP within the Bayesian linear model. The precise statement of the Bayesian linear model is presented in the forthcoming assumption.

**Assumption 3** (Bayesian linear models). Assume that we observe \( n \) samples \( D = \{ (x_i, y_i) \}_{i \in [n]} \), wherein a linear relationship is formed between the response and covariates \( y_i = (x_i, \beta_0) + \varepsilon_i \). In this equation, \( \varepsilon_i i.i.d. N(0, \sigma^2) \) are Gaussian noises, and \( \beta_0 = (\beta_{0,1}, \ldots, \beta_{0,d})^T \in \mathbb{R}^d \) is the coefficient vector. In matrix form, we have \( Y = X\beta_0 + \varepsilon \) where \( Y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n \) and \( X = (x_1, \ldots, x_n)^T = (x_1, \ldots, x_d) \in \mathbb{R}^{n \times d} \). We further assume a product prior on \( \beta_0 \), wherein \( (\beta_{0,j})_{j \in [d]} \sim i.i.d. \Pi \in \mathcal{P}(\mathbb{R}) \). It is also assumed that the prior distribution gives \( \Pi = \pi_0 \delta_0 + (1 - \pi_0) \Pi_* \), where \( \delta_0 \) is the Dirac-delta distribution at 0, \( \pi_0 \in (0, 1) \) is the proportion of null variables, and \( \Pi_* \in \mathcal{P}(\mathbb{R}) \) is a general distribution without any mass at 0. Finally, we assume that the covariates follow the isotropic Gaussian distribution \( (x_i)_{i \in [n]} \sim i.i.d. \mathcal{N}(0, (1/n)I_d) \).

Despite this strong model assumption, it is worth noting that we will later develop procedures that control frequentist FDR under much weaker assumptions of the linear model. But for now, our focus is deriving the power of TPoP and CPoP, under the Bayesian linear model with this strong assumption.
The Bayes risk in the proportional limit regime The Bayesian linear model holds a special interest in the high-dimensional regime, where \( n, d \to \infty \) and \( n/d \to \delta \in (0, \infty) \). This specific regime has been widely examined in the literature [Tan02, DMM09, Ran11, BM11, DM16, BKM+19, BM19, BDMK16, CMW20a]. Prior studies have focused on deriving the asymptotic Bayes risk, defined as

\[
R(\delta) = \lim_{n,d \to \infty, n/d \to \delta} \frac{1}{d} \left\| \beta_0 - \mathbb{E}[\beta_0|D] \right\|^2_2,
\]

where \( \mathbb{E}[\beta_0|D] \) is the posterior expectation of the coefficient vector, the Bayes optimal estimator. Notably, Tanaka [Tan02] used heuristic statistical physics methods to provide a simple formula for the asymptotic Bayes risk of the high-dimensional linear model, which coincides with the Bayes risk of a scalar Bayes estimation problem. The validity of this formula was first rigorously proved by [BDMK16] through the interpolation method.

More specifically, the high dimensional Bayesian linear model is tightly connected to the following scalar Bayes estimation problem: we consider a scalar signal, \( \beta_0 \), having a prior distribution \( \Pi \), and we obtain a noisy observation, \( Y \), of the signal via an additive Gaussian channel, as expressed in:

\[
\text{Signal} : \beta_0 \sim \Pi, \quad \text{Observation} : \ Y = \beta_0 + \tau G \in \mathbb{R}, \quad \text{Noise} : \ G \sim \mathcal{N}(0, 1),
\]

where \( \tau \in \mathbb{R}_{>0} \) represents the noise level to be determined. In this scalar model, given the observation \( Y \), the Bayes optimal estimator, considering the squared loss, is the posterior expectation estimator, as presented in:

\[
\hat{\beta} = \mathcal{E}(Y; \Pi, \tau) \in \mathbb{R}, \quad \text{where} \quad \mathcal{E}(y; \Pi, \tau) = \mathbb{E}_{(\beta_0, Z) \sim \Pi \times \mathcal{N}(0, 1)}[\beta_0|\beta_0 + \tau Z = y].
\]

As a result, the Bayes risk of the scalar model yields:

\[
R(\tau, \delta) = \mathbb{E}_{(\beta_0, Z) \sim \Pi \times \mathcal{N}(0, 1)}[(\beta_0 - \mathcal{E}(\beta_0 + \tau Z; \Pi, \tau))^2].
\]

Tanaka [Tan02] shows that the limiting risk (22) of the high dimensional Bayesian linear model, under Assumption 3, coincides with the limiting risk of the scalar Bayes estimation problem

\[
R(\delta) = R(\tau_*, \delta).
\]

Here, the noise level \( \tau_* \) is given by the global minimizer of a potential function \( \phi \)

\[
\tau_* = \arg \min_{\tau \geq 0} \phi(\tau^2; \Pi, \delta, \sigma^2) = \arg \min_{\tau \geq 0} \left\{ \frac{\delta \sigma^2}{2\tau^2} - \frac{\delta}{2} \log \left( \frac{\delta \sigma^2}{\tau^2} \right) + \text{MI}(\Pi, \tau^2) \right\},
\]

where \( \text{MI} \) is the mutual information between \( \beta_0 \) and \( Y \) in the model (23),

\[
\text{MI}(\Pi, \tau^2) = \mathbb{E}_{\beta_0, Y} \left[ \log \left( \frac{p(Y|\beta_0)}{p(Y)} \right) \right] = -\frac{1}{2} \mathbb{E}_{\beta_0, Y} \log \left\{ \int e^{-\frac{(Y-\beta)^2}{2\tau^2}} \Pi(d\beta) \right\}.
\]

Taking derivative of \( \phi \) with respect to \( \tau^2 \), we deduce that \( \tau_* \) satisfies the following self-consistent equation

\[
\tau^2 = \sigma^2 + \frac{1}{\delta} R(\tau, \delta).
\]

The limiting FDP and TPP Our primary focus here is the asymptotic FDP and TPP (c.f. Eq. (2)) of the TPoP and CPoP, which depend on the joint empirical distribution of \( \{(\beta_{0,j}, P_j(D))\}_{j \in [d]} \) (recall that \( P_j(D) = \mathbb{P}(\beta_{0,j} = 0|D) \) gives the local fdr). Using the heuristic replica calculation in Appendix E, we demonstrate that the joint empirical distribution of \( \{(\beta_{0,j}, P_j(D))\}_{j \in [d]} \) is likewise linked to its counterpart in the scalar model. Specifically, we reconsider the scalar model (23), and consider the associated hypothesis testing problem: test the null hypothesis that \( \beta_0 = 0 \) given the observation \( Y \). According to the Neyman-Pearson lemma, the optimal test corresponds to the likelihood ratio test, equivalent to truncating the local fdr \( \Phi \) of the scalar model, as given by

\[
\Phi = \mathcal{P}(Y; \Pi, \tau_*), \quad \mathcal{P}(y; \Pi, \tau) = \mathbb{P}_{(\beta_0, Z) \sim \Pi \times \mathcal{N}(0, 1)}(\beta_0 = 0|\beta_0 + \tau Z = y).
\]
The replica calculations suggest that for any sufficiently smooth function $\psi : \mathbb{R} \times [0, 1] \mapsto \mathbb{R}$, there is

$$
\lim_{d \to \infty, n/d \to \delta} \frac{1}{\delta} \sum_{j=1}^{d} \psi(\beta_{0,j}, P_{j}(\mathcal{D})) = E[\psi(\hat{\beta}_0, \Phi)],
$$

(30)

where $(\beta_0, \Phi)$ follows the joint distribution specified by Eq. (23) and (29). This equation gives rise to the subsequent conjecture, stating that the asymptotic FDP and TPP of TPoP (and also CPoP) correspond to the type-I error and the power of the scalar hypothesis testing problem.

**Conjecture 1** (Limiting FDP and TPP of TPoP and CPoP). Let Assumption 3 hold. The FDP and TPP of the TPoP procedure (Eq. (14)) with parameter $t$ gives

$$
\lim_{d \to \infty, n/d \to \delta} \text{FDP}(T_p(\cdot; t)) = P(\beta_0 = 0|\Phi < t),
$$

$$
\lim_{d \to \infty, n/d \to \delta} \text{TPP}(T_p(\cdot; t)) = P(\Phi < t|\beta_0 \neq 0).
$$

The FDP and TPP of the CPoP procedure (Eq. (17)) with parameter $\lambda$ gives

$$
\lim_{d \to \infty, n/d \to \delta} \text{FDP}(C_p(\cdot; \lambda)) = P(\beta_0 = 0|\Phi < \tau_*(\lambda)),
$$

$$
\lim_{d \to \infty, n/d \to \delta} \text{TPP}(C_p(\cdot; \lambda)) = P(\Phi < \tau_*(\lambda)|\beta_0 \neq 0),
$$

(31)

where $\tau_*(\lambda)$ is given by

$$
\tau_*(\lambda) = \arg \max_{t \in [0, 1]} \left( \frac{P(\Phi < t)}{P(\Phi < t)} - \frac{1 - \lambda(1 - \pi_0)}{P(\Phi < t)} \right) \cdot E[\Phi, 1\{\Phi < t\}]\right).
$$

(32)

The intuitions of the conjecture are provided in Appendix E. Notably, under similar assumptions of the Bayesian linear model, the analogous asymptotics of FDP and TPP have been rigorously derived for the thresholded LASSO procedure in [WSB+20], leveraging the approximate message passing (AMP) machinery. Applying the AMP machinery to our conjecture is not a straightforward task. The proof of this conjecture poses an intriguing open question and is a topic we plan to explore in future works.

Conjecture 1 immediately reveals that, despite being different procedures, TPoP and CPoP yield asymptotically identical FDP-TPP tradeoff curves. This is not unexpected: CPoP corresponds to truncating the local fdr at a certain data-dependent threshold; in high dimension, this threshold will concentrate and coincide with the threshold employed in TPoP.

### 3.3 Numerical simulations

We next perform numerical simulations illustrating the FDP-TPP tradeoff curves of TPoP/CPoP and the thresholded LASSO procedure, considering two distinct values of $\delta = n/d$. It is important to note that the trade-off curve for CPoP aligns with that of TPoP as both procedures reject hypotheses with a small local fdr.

The thresholded LASSO procedure rejects the hypotheses with large absolute value of the corresponding LASSO coefficient. More specifically, thresholded LASSO rejects the $j$-th hypothesis when $|\hat{\beta}_j(\lambda)| > t$ for some cutoff $t$, where $\hat{\beta}(\lambda) = \arg \min_\beta \frac{1}{2} \|Y - X\beta\|_2^2 + \lambda \|\beta\|_1$. [WSB+20] derives the asymptotic FDP and TPP of the thresholded LASSO procedure by deriving the following formula: in the limit of $n, d \to \infty$ and $n/d \to \infty$, for any sufficiently smooth function $\psi : \mathbb{R} \times [0, 1] \mapsto \mathbb{R}$, we have

$$
\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, \hat{\beta}_j(\lambda)) = E[\psi(\beta_0, Z)_{\sim \Pi \times \mathcal{N}(0, 1)}].
$$

(33)

Here, $\alpha'$ and $\tau'$ are the unique solutions of the self-consistent equation given by:

$$
\tau'^2 = \sigma^2 + \frac{1}{\delta} E(\beta_0, Z)_{\sim \Pi \times \mathcal{N}(0, 1)} (\eta(\alpha') (\beta_0 + \tau' Z) - \beta_0)^2,
$$

$$
\lambda = \left( 1 - \frac{1}{\delta} E(\beta_0, Z)_{\sim \Pi \times \mathcal{N}(0, 1)} (|\beta_0 + \tau' Z| \geq \alpha' \tau') \right) \alpha' \tau',
$$

(35)
where \( \eta_\theta(x) \equiv \text{sign}(x) \cdot (|x| - \theta)_+ \) gives the soft-thresholding operator. The characterization of the joint empirical distribution of \( \{ (\hat{\beta}_{0,j}, \hat{\beta}_{j}(\lambda)) \}_{j \in [d]} \) mirrors our Formalism 1, instrumental in deducing Conjecture 1. This characterization can then be employed to derive the limiting FDP and TPP of the thresholded LASSO procedure. Finally, [WSB+20] demonstrates that the optimal choice of \( \lambda \) in the thresholded LASSO procedure, yielding the optimal FDP-TPP trade-off curve, is provided by:

\[
\lambda_* = \arg \min_{\lambda > 0} \lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \| \hat{\beta}(\lambda) - \beta_0 \|_2^2.
\]  

(36)

Figure 1 showcases the analytical predictions of FDP and TPP for TPoP and thresholded LASSO, alongside numerically simulated curves. We simulate 10 instances of \((Y, X)\) from the Bayesian linear model, as detailed in Assumption 3, for \( \delta = 0.8, 1.25 \) at both \( n = 2000, 2500 \) and \( d = 2500, 2000 \) respectively. We choose \( \sigma = 0.25 \) and \( \Pi = 0.6 \cdot \delta_0 + 0.2 \cdot \delta_{-1} + 0.2 \cdot \delta_1 \) (so that \( \pi_0 = 0.6 \), and \( \Pi_* = 0.5 \cdot \delta_{-1} + 0.5 \cdot \delta_1 \)). For each TPoP simulated curve, we fix a linear model instance, gradually raise the cutoff \( t \) from 0 to 1, and compute the empirical (FDP, TPP) for each cutoff \( t \). For analytical curves of TPoP, we first solve the self-consistent equation outlined in Eq. (28) to obtain \( \tau_* \) for each \( t \), and then derive the asymptotic FDP(\( T_P(\cdot; t)) \) and TPP(\( T_P(\cdot; t) \)) following Conjecture 1. For each simulated curve of the thresholded LASSO, we first determine the optimal regularization parameter \( \lambda_* \), according to Eq. (36), then increase the cutoff \( t \) from 0 to 3 and calculate the empirical (FDP, TPP) at each cutoff \( t \). For thresholded LASSO analytical curves, we use Eq. (34) and (35) to calculate the limiting (FDP, TPP).

As shown in Figure 1, the simulated curves closely align with the corresponding analytical curves, which are derived using Conjecture 1. Moreover, at the same FDP level, TPoP consistently achieves a higher TPP than the optimally regularized thresholded LASSO. These findings are in line with the optimality results presented for TPoP and CPoP in Proposition 1 and 2.

4 Achieving the optimal power with frequentist FDR control

We have established that TPoP and CPoP are Bayes-optimal under the condition of correct model specification, though they may not control the frequentist FDR in cases of misspecified models. As Lemma 1 illustrates, the power of (mTPR) of the optimal test with frequentist FDR control cannot exceed the power of the optimal test with BFDR control, which aligns with the power of CPoP. This naturally leads us to question whether the
inequality of Lemma 1 is tight. More precisely, can a test with frequentist FDR control achieve the optimal power of CPoP? This section introduces two testing procedures that affirmatively answer this question.

In Section 4.1, we first revisit three recently proposed methodologies designed to control the FDR in finite samples. Building on these methodologies, in Section 4.2, we devise procedures PoPCe and PoEdCe that control the frequentist FDR in linear models within the Model-X setting, given a known $\mathcal{L}(X)$. We demonstrate that these two procedures achieve the power of CPoP asymptotically, assuming correctly specified models, in Section 4.3. Numerical simulations of the proposed procedures can be found in Section 4.4.

4.1 Building blocks

PoPCe and PoEdCe are built upon three multiple testing methodologies: conditional randomization test (CRT), distilled conditional randomization test (dCRT), and Benjamini-Hochberg with e-values (eBH). A brief review of these methodologies is provided below.

**Conditional Randomization Test (CRT) [CFJL18]** Consider the model setup as described in Section 1.1. Given the dataset $(Y, X)$ and access to the joint distribution of the covariates $\mathcal{L}(X)$, the conditional randomization test transforms a base statistic into a valid p-value. Here, the base statistic, denoted as $T_j = T(Y, X_{-j}, x_j)$, provides an estimate of the contribution of covariate $X_j$ to the outcome $Y$. This p-value will be valid under the null hypothesis $H_{0j}: Y \perp X_j | X_{-j}$. Specifically, CRT calculates a p-value $p_j$ by executing the subsequent three steps:

1. Generate $K$ conditionally independent covariates $\tilde{x}^{(k)}_{-j} \sim \mathcal{L}(X_j | X_{-j})$ for $k \in [K]$, where $\mathcal{L}(X_j | X_{-j}) \in \mathcal{P}(\mathbb{R})$ is the conditional distribution induced by $\mathcal{L}(X)$.
2. Compute the associated statistics $T_j^{(k)} = T(Y, X_{-j}, \tilde{x}^{(k)}_{-j})$ for $k \in [K]$.
3. Take $p_j = [1 + \sum_{k=1}^{K} I(T_j^{(k)} \leq T_j)]/(1 + K)$ as the proportion of $\{T_j^{(k)}\}_{k \in [K]}$ that are smaller than or equal to $T_j$.

Under the null hypothesis $H_{0j}$, since $(Y, X_{-j}, \tilde{x}^{(k)}_{-j})$ and $(Y, X_{-j}, x_j)$ are identically distributed, $p_j$ follows the uniform distribution over $\{1/(K + 1), 2/(K + 1), ..., 1\}$, confirming it as a valid p-value.

One potential concern is that the CRT p-values $\{p_j\}_{j \in [d]}$ are not necessarily independent. Consequently, the application of the Benjamini-Hochberg (BH) procedure on $\{p_j\}_{j \in [d]}$ may not guarantee control over the FDR from finite samples. Another limitation of CRT is its computational burden. The procedure necessitates computing the base statistics function, $T$, a total of $K \times d$ times. This can be computationally intensive when $T$ represents a complicated statistic, such as the LASSO estimator.

**Distilled Conditional Randomization Test (dCRT) [LKJR22]** The Distilled Conditional Randomization Test (dCRT) is similar to CRT, but it alleviates the computational burden by employing a specialized base statistic $T$. This is represented as $T(Y, X_{-j}, x_j) = \hat{T}(Y, x_j, d_y, d_x)$, where $d_y = d_y(Y, X_{-j})$ and $d_x = d_x(X_{-j})$ are distilled statistics, encoding the information of $X_{-j}$ contained in $Y$ and $x_j$ respectively. Given this structure, $T$ depends on $X_{-j}$ only through $d_y, d_x$, allowing us to reduce the repetitive computations seen in step (2) of CRT.

For example, consider a scenario where we intend to use the LASSO estimator to derive a base statistic. We can choose $d_y \equiv X_{-j} \hat{\beta}$, where $\hat{\beta} = \hat{\beta}_\lambda(Y, X_{-j})$ is the LASSO solution for fitting $Y$ on $X_{-j}$ with regularization parameter $\lambda$. Concurrently, we let $d_x \equiv \mathbb{E}[x_j | X_{-j}]$. Then we can choose $T(Y, X_{-j}, x_j) \equiv \|Y - d_y - x_j - d_x\|$ as the base statistics.

**Benjamini-Hochberg procedure with e-values (eBH) [WR22]**. eBH is designed for finite-sample FDR control. This method is a variant of the Benjamini-Hochberg (BH) procedure [BH95], using e-values as substitutes for p-values. Specifically, a random variable, $e$, is termed a valid e-value if the expectation under the null hypothesis satisfies $\mathbb{E}[e|e] \leq 1$. Given $d$ hypotheses and their corresponding $d$ e-values, denoted as $\{e_{(j)}\}_{j \in [d]}$, the eBH procedure rejects the $k$ hypotheses with the largest e-values (ordered from largest to smallest as $\{e_{(j)}\}_{j \in [d]}$). Here

$$k = \max \left\{ m \in \{0, ..., d \} : \frac{d_0}{m e_{(m)}} \leq \alpha \right\}, \quad d_0 \text{ is the number of true null hypothesis}. \quad (37)$$
Contrasting with the BH procedure, which necessitates additional structural assumptions on the p-values (e.g., PRDS) to ensure finite-sample FDR control [BY01], eBH consistently controls FDR at level $\alpha$, irrespective of correlation among the e-values [WR22].

A noteworthy point is that one can use any p-to-e calibrator [VW21, Sha19] to convert a valid p-value to a valid e-value, hence providing eBH with great flexibility for valid FDR control. However, applying a naive p-to-e calibrator might result in a power loss, and special treatments are required to make eBH as powerful as the traditional BH procedure.

### 4.2 PoPCe and PoEdCe procedures

The PoPCE (Posterior Probability + Conditional randomization test + eBH) procedure employs TPoP as the base statistics, wrapping it using CRT (in the Model-X setting with known $L(X)$) and eBH. Specifically, we first apply CRT to TPoP to generate p-values, denoted by $\{p_j\}_{j \in [d]}$. Subsequently, we construct valid e-values from these p-values using a carefully chosen p-to-e calibrator. Eventually, we implement the eBH procedure, which controls FDR from finite samples. The full algorithm is presented in Algorithm 1. Each step is explained as follows:

- **Line 2-3 (Compute the p-to-e calibration threshold):** We first compute the p-to-e calibration threshold $q = \Psi(t(\alpha - \varepsilon))$. Here, $\Psi$ represents the cumulative distribution function (CDF) of $P(\tau, Z; \Pi, \tau_*)$ when $Z \sim \mathcal{N}(0, 1)$ (c.f. Eq. (29) for the definition of $P$). This is inherently the asymptotic CDF of the local fdr of a null coordinate in the Bayesian linear model. Furthermore,

$$t(\alpha - \varepsilon) \equiv \max \left\{ s \in [0, 1] : \lim_{d \to \infty, n/d \to b} \text{FDP}(T_P(\cdot; s); \Pi) \leq \alpha - \varepsilon \right\}$$

represents the effective truncation threshold of the TPoP procedure, required for calibrating the effective FDR at level $\alpha - \varepsilon$ in the Bayesian linear model. Notice that the second equality above is due to the limiting formula of FDP for TPoP, which is explicitly given in Conjecture 1. The p-to-e calibration threshold $q$ is used to calculate e-values in Line 13.

- **Line 6-11 (Conditional randomization test):** We first compute the local fdr $u_j = P_j(Y, X) = \mathbb{P}(\beta_{0,j} = 0|Y, X)$ for each coordinate $j \in [d]$. Subsequently, for each coordinate $j \in [d]$, we generate conditionally independent covariates $\{\tilde{x}_{j}^{(k)}\}_{k \in [K]}$ given $X_{-j}$ from the conditional distribution $L(X_j|X_{-j})$. We then compute the corresponding local fdrs denoted by $\{u_j^{(k)} = P_j(Y, X_{-j}, \tilde{x}_{j}^{(k)})\}_{k \in [K]}$. Finally, we let $p_j$ be the proportion of $\{u_j^{(k)}\}_{k \in [K]}$ that are smaller than $u_j$. When the null hypothesis $H_{0j} : Y \perp X_j|X_{-j}$ holds, $u_j$ and $u_j^{(k)}$ have the same distribution since

$$(Y, X_{-j}, \tilde{x}_{j}^{(k)}) \overset{d}{=} (Y, X_{-j}, x_j).$$

Hence, $\{p_j\}_{j \in [d]}$ are valid p-values.

- **Line 13 (Construct e-values):** We construct valid e-values $\{e_j\}_{j \in [d]}$ from p-values $\{p_j\}_{j \in [d]}$ using the p-to-e calibrator $e_j = 1/p_j - q)/q$. Here, $q$ is previously computed in Line 2-3.

- **Line 16 (eBH):** We finally implement eBH on $\{p_j\}_{j \in [d]}$, which ensures frequentist FDR control at level $\alpha$. Simple algebra demonstrates that eBH is equivalently rejecting the hypotheses $\{j : p_j \leq q\}$ if

$$q \leq d/|\{j : p_j \leq q\}| < \alpha,$$

and rejecting nothing otherwise. As will be demonstrated later, when the Bayesian linear model is well-specified, the choice of $q$ results in the concentration of $|\{j : p_j \leq q\}|/d$, implying that $q \leq d/|\{j : p_j \leq q\}| \approx \alpha - \varepsilon$. Thereby, condition (39) will be satisfied with high probability, leading PoPCE to reject the hypotheses $\{j : p_j \leq q\}$. We will later show that this is further asymptotically equivalent to rejecting the hypotheses $\{j : u_j \leq t(\alpha - \varepsilon)\}$. Therefore, PoPCE asymptotically rejects the same hypotheses as TPoP, thereby possessing near-optimal power.
Hyperparameters \((K, \varepsilon)\): We remark that the choice of \((K, \varepsilon)\) will not affect the validity of PoPCE: irrespective of the chosen hyperparameters, PoPCE ensures finite-sample FDR control. However, this choice does impact the asymptotic optimality. In particular, \(K\) is the number of times to re-sample the base statistics in CRT. A larger \(K\) brings the p-values closer to the uniform distribution under the null hypothesis, albeit at a higher computational cost. Furthermore, in Line 2, we choose \(\varepsilon > 0\) to be a small number to calibrate FDP closer to \(\alpha\) and thereby attain a better power. Nevertheless, we do not want \(\varepsilon\) to be excessively small, ensuring that condition (39) happens with high probability.

Algorithm 1: The PoPCE procedure

\begin{algorithm}
\textbf{Require:} \(\{(x_i, y_i)\}_{i \in [n]} = (X, Y)\); FDR level \(\alpha \in (0, 1)\); distribution \(P_X\); null proportion \(\pi_0\); prior II and noise level \(\sigma^2\); hyperparameters \(K \in \mathbb{N}\), and \(\varepsilon > 0\).
\begin{enumerate}
\item \{Compute the p-to-e calibration threshold\}
\item Compute \(\tau^2\) which solves Eq. (27) with prior II and noise level \(\sigma^2\). Compute \(t = \max\{s \in [0, 1] : \lim_{d \to \infty, n/d \to \delta} \text{FDP}(T_p(\cdot; s); \Pi) \leq \alpha - \varepsilon\}\) (c.f. Eq. (38)).
\item Compute \(q = \Psi(t)\) where \(\Psi\) is the CDF of \(P(\tau, Z; \Pi, \tau)\) when \(Z \sim N(0, 1)\).
\item for \(j \in [d]\) do
\begin{enumerate}
\item \{Conditional randomization test\}
\item Denote \(P_j(Y, X) = \mathbb{P}(\beta_{0j} = 0 | Y, X)\). Compute \(u_j = P_j(Y, X)\).
\item for \(k \in [K]\) do
\begin{enumerate}
\item Sample \(\hat{x}^{(k)} = (\hat{x}^{(k)}_{ij}, \ldots, \hat{x}^{(k)}_{nj})^T\) where \(\hat{x}^{(k)}_{ij} \sim \mathcal{L}(X_j | X_{-j} = x_{i,-j})\) independently.
\item Compute \(u_j^{(k)} = P_j(Y, X_{-j}, \hat{x}^{(k)}_j)\).
\end{enumerate}
\end{enumerate}
\item end for
\item Compute \(p_j = (1/(K + 1))(1 + \sum_{k=1}^{K} 1\{u_j \geq u_j^{(k)}\})\).
\item \{Compute e-values\}
\item Compute \(e_j = 1\{p_j \leq q\}/q\).
\item end for
\item \{eBH\}
\end{enumerate}
16. Reject the hypotheses with the \(\hat{k}\) largest e-values, where
\[\hat{k} = \max\left\{k : \frac{\pi_{0d}}{k\varepsilon(k)} \leq \alpha\right\}.
\end{algorithm}

Computational cost: Line 6-11 of Algorithm 1 necessitate the computation of local fdrs of the high dimensional Bayesian linear model. This process can be computationally demanding if Markov Chain Monte Carlo is utilized. In our numerical implementation, we opt to calculate local fdrs using the approximate message passing (AMP) algorithm [DMM09] with subsequent post-processing. The AMP algorithm has been demonstrated to converge to the true posterior in several models [MM09, BM11, DAM15, BKM+19]. However, it is also worth noting that there are statistical models in which AMP does not reach the true posterior [BKM+19]. Despite this, the potential non-convergence of the AMP algorithm should not raise concern for the following reasons: (1) the finite-sample control of FDR is valid for any base statistics and is thus applicable even if AMP does not converge to the correct posterior; (2) in our simulation configurations, AMP does indeed converge to the correct posterior, thereby achieving optimal power in these models.

The PoEdCe procedure: In PoPCE, the local fdr (posterior probability) needs to be computed for \((K+1) \times d\) times. To alleviate the computational burden, we propose a similar procedure PoEdCe (Posterior Expectation + dCRT + eBH), where CRT is replaced by dCRT [LKJ JR22]. PoEdCe only requires a single computation of the posterior expectation for each coordinate, and thereby reducing computational costs by a factor of \(K + 1\) (assuming the computational costs for posterior expectation and posterior probability are equal).

Specifically, we illustrate the difference between PoEdCe and PoPCE in Algorithm 2: the only difference lies in the construction of the base statistics \(\{u_j\}_{j \in [d]}\) and their resampled version \(\{u_j^{(k)}\}_{k \in [K]}_{j \in [d]}\) (see
Theorem 1

(Frequentist PoPCe) In PoPCe, the base statistics is taken to be 
\[ s_j = (Y - X_j \beta_{-j}, x_j). \]
Here, \( \beta_{-j} \) represents the posterior expectation of \( \theta_0 \in \mathbb{R}^{d-1} \) given observation \( (Y, X_{-j}) \), assuming the statistical model \( Y = X_j \theta_0 + \varepsilon \in \mathbb{R}^n \), where \( \theta_{0,i} \sim i.i.d. \Pi \) and \( \varepsilon_i \sim i.i.d. \mathcal{N}(0, \sigma^2) \) (see Line 2-4). The resampled version of the base statistics possesses a similar form (see Line 6-8). Similar to PoPCe, PoEdCe also ensures control over frequentist FDR. We will later demonstrate that the asymptotic distribution of \( \{u_j, \{u_j^{(k)}\}_{k \in [K]}\}_{j \in [d]} \) in PoEdCe aligns with those in PoPCe, implying that PoEdCe and PoPCe possess approximately equal power.

Similar to PoPCe, in our numerical simulations, we employ the AMP algorithm to compute the posterior expectation of the Bayesian linear model (see Line 2 of Algorithm 2). By the same argument, the lack of a convergence guarantee for the AMP algorithm does not compromise the finite-sample control of FDR.

\begin{algorithm}
\begin{enumerate}
\item {Distilled conditional randomization test}
\item Compute \( \hat{\beta}_{-j} \), the posterior expectation of \( \theta_0 \in \mathbb{R}^{d-1} \) given observation \( (Y, X_{-j}) \), assuming the statistical model \( Y = X_j \theta_0 + \varepsilon \in \mathbb{R}^n \), where \( \theta_{0,i} \sim i.i.d. \Pi \) and \( \varepsilon_i \sim i.i.d. \mathcal{N}(0, \sigma^2) \).
\item Compute \( s_j = (Y - X_j \hat{\beta}_{-j}, x_j) \).
\item Compute \( u_j = \mathcal{P}(\tau_2^2 / \sigma^2) s_j; \Pi, \tau_\alpha \).
\item for \( k \in [K] \) do
\item Sample \( x_j^{(k)} = (x_{1j}^{(k)}, \ldots, x_{nj}^{(k)})^T \) where \( x_j^{(k)} \sim \mathcal{L}(X_j | X_{-j} = x_{-i,j}) \) independently.
\item Compute \( s_j^{(k)} = (Y - X_j \hat{\beta}_{-j}, x_j^{(k)}) \).
\item Compute \( u_j^{(k)} = \mathcal{P}(\tau_2^2 / \sigma^2) s_j^{(k)}; \Pi, \tau_\alpha \).
\item end for
\end{enumerate}
\end{algorithm}

**Algorithm 2** The PoEdCe procedure (replacing Line 5-10 of Algorithm 1 with the following)

Empirical Bayes for estimating the prior The implementation of PoPCe and PoEdCe presumes knowledge of the prior distribution \( \Pi \) and the noise level \( \sigma^2 \). We also consider a situation where both the prior distribution \( \Pi \) and the noise level \( \sigma^2 \) are unknown and, in response, propose an Empirical Bayes variant of PoEdCe, named EPoEdCe. We demonstrate that EPoEdCe also controls FDR from finite samples, and attains near-optimal power whenever the data are generated from a Bayesian linear model with unknown prior and noise levels. Detailed discussions about EPoEdCe can be found in Appendix C.

### 4.3 Frequentist validity and statistical optimality

It is guaranteed that PoPCe and PoEdCe will control the frequentist FDR from finite samples, as stated in Theorem 1 below. The frequentist FDR control is ensured by the eBH procedure in Line 16 of Algorithm 1, and the validity of the \( p \)-values of CRT from finite samples, as stated in Theorem 11 (see Appendix D for the detailed proof).

**Theorem 1** (Frequentist FDR control of PoPCe and PoEdCe). For any joint distribution \( P \in \mathcal{M}(\mathcal{P}_X) \) (c.f. Eq. (6)), suppose that \( \{(x_i, y_i)\}_{i \in [n]} \) are i.i.d. from \( P \). Let \( T^*_\alpha \) be either PoPCe or PoEdCe. Then we have the frequentist FDR control (c.f. Eq. (3))

\[ \text{FDR}(T^*_\alpha, P) \leq \alpha. \]

We subsequently introduce a conjecture proposing that PoPCe and PoEdCe are asymptotically optimal procedures with frequentist FDR control (c.f. Definition 1). We will present a heuristic argument supporting this conjecture in Appendix F, and we will validate the conjecture numerically in Section 4.4.

**Conjecture 2** (Optimality of PoPCe and PoEdCe). In the asymptotic regime where \( n, d \to \infty, n/d \to \delta, K = K_n \to \infty, \) and \( \varepsilon = \varepsilon_n \to 0 \), and under the conditions of the Bayesian linear model as per Assumption 3, both PoPCe and PoEdCe have the same asymptotic power as CPoP (for \( T^*_\alpha \) to be either PoPCe or PoEdCe)

\[ \lim_{n \to \infty} \frac{1}{d} \mathbb{E}_{T^*}(\text{mTPR}(T^*_\alpha, \Pi)) = \lim_{n \to \infty} \frac{1}{d} \mathbb{E}_{T^*}(\text{mTPR}(C_P(\cdot: \lambda(\alpha)), \Pi)). \]  

(40)

Subsequently, as per Proposition 2, both procedures are asymptotically BFDR optimal (c.f. Definition 2):

\[ \lim_{n \to \infty} \frac{1}{d} \mathbb{E}_{T^*}(\text{mTPR}(T^*_\alpha, \Pi)) \geq \lim_{n \to \infty} \frac{1}{d} \max_T \left\{ \mathbb{E}_{T^*}(\text{mTPR}(T, \Pi) : \text{BFDR}(T, \Pi) \leq \alpha) \right\}. \]
Consequently, according to Lemma 1, PoPCE and PoEdCe are both asymptotically \((\alpha, M(P_X), \Pi, o_n(1))\)-optimal procedures with frequentist FDR control (c.f. Definition 1).

Generally speaking, the conjecture builds on the intuition that the PoPCE procedure (as well as PoEdCe) will, in an asymptotic sense, reject the same set of hypotheses as CPoP, when the model is well-specified. This intuition stems from the derivation of the asymptotic empirical distribution of local fdrs \(\{P_j(D)\}\) and \(p\)-values \(\{p_j\}\) in PoPCE under the Bayesian linear model (c.f. Formalism 1, 2 and 3). More specifically, in the limit as \(n, p \to \infty\), we can “marginally” treat

\[
(\beta_{0,j}, P_j(D)) \sim (\Pi, \mathcal{P}(\Pi + \tau_* Z)),
\]

\[
(\beta_{0,j}, p_j) \sim (\Pi, \Psi(\mathcal{P}(\Pi + \tau_* Z))),
\]

and we use a dot above the \(\sim\) symbol to indicate that the approximation holds only in a restricted sense. Above, \(Z \sim \mathcal{N}(0, 1)\), \(\tau_*\) is the constant determined through (27), \(\mathcal{P}(\cdot) = \mathcal{P}(\cdot; \Pi, \tau_*)\) is given by (29), and \(\Psi(t) = \mathbb{P}_{Z \sim \mathcal{N}(0, 1)}(\mathcal{P}(\tau_* Z) \leq t)\) is the CDF of \(\mathcal{P}(\tau_* Z)\), a strictly increasing function.

Given that CPoP and PoPCE (as well as PoEdCe), respectively, threshold \(\{P_j(D)\}\) and \(\{p_j\}\) at some levels that are calibrated to control the asymptotic BFDR level \(\alpha\), the monotonicity of \(\Psi\) dictates that both procedures asymptotically reject lower values of \(\{P_j(D)\}\) at the same level, hence rejecting the same set of hypotheses (c.f. Appendix F for details).

### 4.4 Numerical simulations

We perform numerical simulations, comparing the predicted and simulated FDP and TPP of PoPCE, PoEdCe, and EPoEdCe. We first focus on well-specified Bayesian linear models and demonstrate that the simulated curves align with analytical predictions. Following this, we examine misspecified models to verify that these procedures maintain control over the frequentist FDR from finite samples.

#### 4.4.1 FDP and TPP in well-specified models

Figure 2 displays the realized FDP and TPP of PoPCE, PoEdCe, and EPoEdCe against the nominal level \(\alpha\), using data generated from a well-specified Bayesian linear model. We simulate 10 instances of \((Y, X)\) with \(n = 500, d = 400, \delta = 1.25, \sigma = 0.25\), and prior distribution \(\Pi = 0.6 \cdot \delta_0 + 0.26 \delta_1 + 0.26 \delta_{-1}\). For each simulated instance, we apply each PoPCE, PoEdCe and EPoEdCe across various FDR control levels \(\alpha\) from 0 to 0.6. For each \(\alpha\), we select \(\varepsilon = 0.1\alpha\), expecting that the FDP will concentrate at level \(\alpha - \varepsilon = 0.9\alpha\). We set the number of repetitions \(K = 1000\) for the CRT sub-routine. For EPoEdCe, we choose \(M = 50\), the number of blocks in the empirical Bayes procedure. The realized FDP and TPP against the level \(\alpha\) for all three procedures are then plotted. The analytical prediction curve of TPP and the line \(y = 0.9\alpha\) serving as the analytical prediction curve of FDP are also included. As demonstrated in Figure 2, the TPP tightly concentrates around the optimal TPR level, while the FDP closely adheres to the predicted FDR level of 0.9\(\alpha\), for all three procedures.
Figure 3: Realized FDP and TPP of PoEdCε with two $\delta = 0.8, 1.25$ ($n = 500, d = 400$ and $n = 400, d = 500$ respectively). The Bayesian linear model is generated with model parameters $\sigma = 0.25$ and $\Pi = 0.4 \cdot \delta_0 + 0.5 \cdot \delta_1 + 0.1 \cdot \delta_{-1}$. Hyperparameters are chosen to be $\epsilon = 0.1\alpha$ and $K = 1000$. The inputs of PoEdCε are three sets of model parameters: well-specified model with correct $(\Pi, \sigma)$, misspecified model with noise level $\sigma = 0.5$, and misspecified model with prior $\Pi = 0.6 \cdot \delta_0 + 0.2\delta_1 + 0.2\delta_{-1}$. Upper panel: Blue, red, and yellow dots are (FDP, $\alpha$) for different model parameters. The black line is the FDR level $y = 0.9\alpha$. Lower panel: Blue, red, and yellow dots are (TPP, $\alpha$) for different model parameters. The black curve is the analytical prediction of TPP.

4.4.2 FDP and TPP of PoEdCε with well-specified and misspecified models

Figure 3 displays the realized FDP and TPP of PoEdCε against the nominal level $\alpha$ under well-specified and misspecified models. We simulate 10 instances of $(Y, X)$ for $\delta = 0.8, 1.25$ (corresponding to $n = 400, d = 500$ and $n = 500, d = 400$ respectively), setting $\sigma = 0.25$ and $\Pi = 0.4 \cdot \delta_0 + 0.5 \cdot \delta_1 + 0.1 \cdot \delta_{-1}$. For each simulated instance, we employ the PoEdCε procedure under three different model assumptions (considering three sets of model parameters $(\Pi, \sigma)$ as inputs of PoEdCε): (1) a well-specified model, where PoEdCε utilizes the true model parameters $(\Pi, \sigma)$ that generate the data; (2) a misspecified model with an incorrect noise level $\sigma = 0.5$; (3) a misspecified model with an inaccurate prior $\Pi = 0.6 \cdot \delta_0 + 0.2\delta_1 + 0.2\delta_{-1}$. In all scenarios, we select $\epsilon = 0.1\alpha$ and set the repetition number to $K = 1000$.

The upper panel demonstrates that FDP is controlled under level $0.9\alpha$, even in instances of model misspecification. It also illustrates that the FDP concentrates around level $0.9\alpha$ when the model is well-specified. The lower panel shows that the TPP concentrates on the analytical prediction when the model is well-specified, thereby aligning with our conjecture. However, in model instances with a misspecified prior or an erroneous noise level, the power of PoEdCε falls below the optimal TPP-FDP tradeoff curve.
5 Conclusion and discussion

In this paper, we proposed multiple testing procedures with frequentist FDR control, which are also near-optimal under Bayesian linear models. We begin by calculating an upper bound of power, for any procedure with finite-sample FDR control. This statistical limit is demonstrated as asymptotically achievable by two testing procedures, PoPce and PoEdCe. These procedures control FDR from finite samples under the Model-X framework and are conjectured to be near-optimal when the Bayesian linear model is well-specified. We provide the intuition behind these conjectures and employ numerical simulations to corroborate the validity and optimality of the proposed procedures.

Our work establishes the Bayesian linear model as a reference point for power comparisons among various multiple testing procedures (for example, knockoffs [CFJL18], mirror statistics [XZL21], dBH [FL20], etc). In other words, the effectiveness of a multiple testing procedure can be assessed relative to the power of PoEdCe within the Bayesian linear model. On the other hand, we would like to emphasize that while the PoPce and PoEdCe procedures serve as theoretical tools for the power analysis of FDR-controlling procedures, we do not recommend their direct application in practical scenarios. Indeed, these procedures could possibly be powerless under model misspecification in practice.

This paper presents several important questions for further exploration. Firstly, our optimality conjecture for PoPce and PoEdCe is based on heuristic calculations, and a significant challenge would be to formally prove this conjecture. An essential step towards this goal is the derivation of the asymptotics for the joint empirical distribution of the parameters and local fdrs. An approach to consider might be the application of advanced Gaussian interpolation techniques, such as those utilized in [BKM+19].

Furthermore, an intriguing question is the design of optimal procedures that extend beyond the model assumptions of isotropic Gaussian covariates and the Bayesian linear model. For example, a natural extension is Bayesian generalized linear models with anisotropic Gaussian covariates. Under this assumption, designing procedures with finite-sample FDR control is a straightforward problem; the challenging question, however, is how to achieve near-optimality under well-specified models.

Finally, we notice that when the model has certain misspecification, PoPce and PoEdCe are often too conservative and do not reject any hypothesis. This outcome arises because we select the p-to-e calibrator to be a truncation function with an estimated truncation threshold, and eBH might reject nothing if the estimated threshold is inaccurate. A compelling open question, therefore, is whether one can enhance or optimize the power under model misspecification while maintaining finite-sample validity and Bayes optimality.

Acknowledgement

This project is supported by NSF grant DMS-2210827 and CCF-2315725. We thank Will Fithian and his group for helpful discussions.

References

[ABDJ06] Felix Abramovich, Yoav Benjamini, David L Donoho, and Iain M Johnstone. Adapting to unknown sparsity by controlling the false discovery rate. *The Annals of Statistics*, 34(2):584–653, 2006.

[BC15] Rina Foygel Barber and Emmanuel J Candès. Controlling the false discovery rate via knockoffs. *The Annals of Statistics*, 43(5):2055–2085, 2015.

[BCPS21] Jean Barbier, Wei-Kuo Chen, Dmitry Panchenko, and Manuel Sáenz. Performance of bayesian linear regression in a model with mismatch. *arXiv preprint arXiv:2107.06936*, 2021.

[BDMK16] Jean Barbier, Mohamad Dia, Nicolas Macris, and Florent Krzakala. The mutual information in random linear estimation. In *2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 625–632. IEEE, 2016.
[BH95] Yoav Benjamini and Yosef Hochberg. Controlling the false discovery rate: a practical and powerful approach to multiple testing. *Journal of the Royal statistical society: series B (Methodological)*, 57(1):289–300, 1995.

[BH00] Yoav Benjamini and Yosef Hochberg. On the adaptive control of the false discovery rate in multiple testing with independent statistics. *Journal of educational and Behavioral Statistics*, 25(1):60–83, 2000.

[BKM+19] Jean Barbier, Florent Krzakala, Nicolas Macris, Léo Miolane, and Lenka Zdeborová. Optimal errors and phase transitions in high-dimensional generalized linear models. *Proceedings of the National Academy of Sciences*, 116(12):5451–5460, 2019.

[BKRS21] Zhiqi Bu, Jason Klusowski, Cynthia Rush, and Weijie J Su. Characterizing the slope trade-off: A variational perspective and the donoho-tanner limit. *arXiv preprint arXiv:2105.13302*, 2021.

[BM11] Mohsen Bayati and Andrea Montanari. The dynamics of message passing on dense graphs, with applications to compressed sensing. *IEEE Transactions on Information Theory*, 57(2):764–785, 2011.

[BM19] Jean Barbier and Nicolas Macris. The adaptive interpolation method: a simple scheme to prove replica formulas in bayesian inference. *Probability theory and related fields*, 174(3):1133–1185, 2019.

[BMN20] Raphael Berthier, Andrea Montanari, and Phan-Minh Nguyen. State evolution for approximate message passing with non-separable functions. *Information and Inference: A Journal of the IMA*, 9(1):33–79, 2020.

[BWBS20] Thomas B Berrett, Yi Wang, Rina Foygel Barber, and Richard J Samworth. The conditional permutation test for independence while controlling for confounders. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 82(1):175–197, 2020.

[BY01] Yoav Benjamini and Daniel Yekutieli. The control of the false discovery rate in multiple testing under dependency. *Annals of statistics*, pages 1165–1188, 2001.

[CFJL18] Emmanuel Candes, Yingying Fan, Lucas Janson, and Jinchi Lv. Panning for gold: ‘model-x’ knockoffs for high dimensional controlled variable selection. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80(3):551–577, 2018.

[CMW20a] Michael Celentano, Andrea Montanari, and Yuting Wei. The lasso with general gaussian designs with applications to hypothesis testing. *arXiv preprint arXiv:2007.13716*, 2020.

[CMW20b] Michael Celentano, Andrea Montanari, and Yuchen Wu. The estimation error of general first order methods. In *Conference on Learning Theory*, pages 1078–1141. PMLR, 2020.

[DAM15] Yash Deshpande, Emmanuel Abbe, and Andrea Montanari. Asymptotic mutual information for the two-groups stochastic block model. *arXiv preprint arXiv:1507.08685*, 2015.

[DLXL20] Chenguang Dai, Buyu Lin, Xin Xing, and Jun S Liu. A scale-free approach for false discovery rate control in generalized linear models. *arXiv preprint arXiv:2007.01237*, 2020.

[DLXL22] Chenguang Dai, Buyu Lin, Xin Xing, and Jun S Liu. False discovery rate control via data splitting. *Journal of the American Statistical Association*, (just-accepted):1–38, 2022.

[DM16] David Donoho and Andrea Montanari. High dimensional robust m-estimation: Asymptotic variance via approximate message passing. *Probability Theory and Related Fields*, 166(3):935–969, 2016.

[DMM09] David L Donoho, Arian Maleki, and Andrea Montanari. Message-passing algorithms for compressed sensing. *Proceedings of the National Academy of Sciences*, 106(45):18914–18919, 2009.
[Efr05] Bradley Efron. Local false discovery rates, 2005.

[ETST01] Bradley Efron, Robert Tibshirani, John D Storey, and Virginia Tusher. Empirical bayes analysis of a microarray experiment. *Journal of the American statistical association*, 96(456):1151–1160, 2001.

[FL20] William Fithian and Lihua Lei. Conditional calibration for false discovery rate control under dependence. *arXiv preprint arXiv:2007.10438*, 2020.

[GW02] Christopher Genovese and Larry Wasserman. Operating characteristics and extensions of the false discovery rate procedure. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 64(3):499–517, 2002.

[HL19] Hong Hu and Yue M Lu. Asymptotics and optimal designs of slope for sparse linear regression. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pages 375–379. IEEE, 2019.

[Kar13] Noureddine El Karoui. Asymptotic behavior of unregularized and ridge-regularized high-dimensional robust regression estimators: rigorous results. *arXiv preprint arXiv:1311.2445*, 2013.

[KLM20] Zheng Tracy Ke, Jun S Liu, and Yucong Ma. Power of fdr control methods: The impact of ranking algorithm, tampered design, and symmetric statistic. *arXiv preprint arXiv:2010.08132*, 2020.

[KLJR22] Molei Liu, Eugene Katsevich, Lucas Janson, and Aaditya Ramdas. Fast and powerful conditional randomization testing via distillation. *Biometrika*, 109(2):277–293, 2022.

[LKJR22] Marc Lelarge and Léa Miolane. Fundamental limits of symmetric low-rank matrix estimation. *Probability Theory and Related Fields*, 173(3):859–929, 2019.

[LR19] Jingbo Liu and Philippe Rigollet. Power analysis of knockoff filters for correlated designs. *Advances in Neural Information Processing Systems*, 32, 2019.

[MM09] Marc Mezard and Andrea Montanari. *Information, physics, and computation*. Oxford University Press, 2009.

[MPR06] Peter Müller, Giovanni Parmigiani, and Kenneth Rice. Fdr and bayesian multiple comparisons rules. 2006.

[MPRR04] Peter Müller, Giovanni Parmigiani, Christian Robert, and Judith Rousseau. Optimal sample size for multiple testing: the case of gene expression microarrays. *Journal of the American Statistical Association*, 99(468):990–1001, 2004.

[MTCL20] Rong Ma, T Tony Cai, and Hongzhe Li. Global and simultaneous hypothesis testing for high-dimensional logistic regression models. *Journal of the American Statistical Association*, pages 1–15, 2020.

[Ran11] Sundeep Rangan. Generalized approximate message passing for estimation with random linear mixing. In *2011 IEEE International Symposium on Information Theory Proceedings*, pages 2168–2172. IEEE, 2011.

[Rob50] Herbert Robbins. A generalization of the method of maximum likelihood-estimating a mixing distribution. In *Annals of Mathematical Statistics*, volume 21, pages 314–315. INST MATHEMATICAL STATISTICS IMS BUSINESS OFFICE-SUITE 7, 3401 INVESTMENT . . . , 1950.
Weijie Su, Malgorzata Bogdan, and Emmanuel Candes. False discoveries occur early on the lasso path. *The Annals of statistics*, pages 2133–2150, 2017.

Wenguang Sun and T Tony Cai. Oracle and adaptive compound decision rules for false discovery rate control. *Journal of the American Statistical Association*, 102(479):901–912, 2007.

Glenn Shafer. The language of betting as a strategy for statistical and scientific communication. *arXiv preprint arXiv:1903.06991*, 2019.

Asher Spector and Lucas Janson. Powerful knockoffs via minimizing reconstructability. *The Annals of Statistics*, 50(1):252–276, 2022.

John D Storey. The optimal discovery procedure: a new approach to simultaneous significance testing. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 69(3):347–368, 2007.

Michel Talagrand. *Mean field models for spin glasses: Volume I: Basic examples*, volume 54. Springer Science & Business Media, 2010.

Toshiyuki Tanaka. A statistical-mechanics approach to large-system analysis of cdma multiuser detectors. *IEEE Transactions on Information theory*, 48(11):2888–2910, 2002.

Wesley Tansey, Victor Veitch, Haoran Zhang, Raul Rabadan, and David M. Blei. The holdout randomization test for feature selection in black box models. *Journal of Computational and Graphical Statistics*, 31(1):151–162, 2022.

Vladimir Vovk and Ruodu Wang. E-values: Calibration, combination and applications. *The Annals of Statistics*, 49(3):1736–1754, 2021.

Asaf Weinstein, Rina Barber, and Emmanuel Candes. A power and prediction analysis for knockoffs with lasso statistics. *arXiv preprint arXiv:1712.06465*, 2017.

Wenshuo Wang and Lucas Janson. A high-dimensional power analysis of the conditional randomization test and knockoffs. *Biometrika*, 109(3):631–645, 2022.

Ruodu Wang and Aaditya Ramdas. False discovery rate control with e-values. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 84:822 – 852, 2022.

Asaf Weinstein, Weijie J Su, Malgorzata Bogdan, Rina F Barber, and Emmanuel J Candès. A power analysis for knockoffs with the lasso coefficient-difference statistic. *arXiv preprint arXiv:2007.15346*, 2020.

Shuaiwen Wang, Haolei Weng, and Arian Maleki. Which bridge estimator is the best for variable selection? *The Annals of Statistics*, 48(5):2791 – 2823, 2020.

Hua Wang, Yachong Yang, Zhiqi Bu, and Weijie Su. The complete lasso tradeoff diagram. *Advances in Neural Information Processing Systems*, 33, 2020.

Jichun Xie, T Tony Cai, John Maris, and Hongzhe Li. Optimal false discovery rate control for dependent data. *Statistics and its Interface*, 4(4):417, 2011.

Xin Xing, Zhigen Zhao, and Jun S Liu. Controlling false discovery rate using gaussian mirrors. *Journal of the American Statistical Association*, pages 1–20, 2021.

Lenka Zdeborová and Florent Krzakala. Statistical physics of inference: Thresholds and algorithms. *Advances in Physics*, 65(5):453–552, 2016.

Linjun Zhang, Rong Ma, T Tony Cai, and Hongzhe Li. Estimation, confidence intervals, and large-scale hypotheses testing for high-dimensional mixed linear regression. *arXiv preprint arXiv:2011.05598*, 2020.

Xinyi Zhong, Chang Su, and Zhou Fan. Empirical bayes pca in high dimensions. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 84:853 – 878, 2022.
A Proof of Proposition 1

Throughout the proof of this proposition, all the probability and expectation are with respect to the randomness in $\beta_0$ and $D$. We first prove the existence of $t = t(\alpha)$ such that Eq. (15) holds.

Recall the definition of $mFDR$ as in Eq. (5). For any $t \in (0, 1)$, note that we can rewrite $mFDR(T_p(\cdot; t), \Pi)$ as

$$mFDR(T_p(\cdot; t), \Pi) = \frac{\sum_{j \in [d]} P(P_j(D) < t, \beta_{0,j} = 0) + P(P_j(D) < t, \beta_{0,j} \neq 0)}{\sum_{j \in [d]} P(P_j(D) < t, \beta_{0,j} = 0) + P(P_j(D) < t, \beta_{0,j} \neq 0)}.$$  

By Assumption 1, $P(P_j(D) < t|\beta_{0,j} = 0)$ and $P(P_j(D) < t|\beta_{0,j} \neq 0)$ are continuous in $t$, so that we have $P(P_j(D) < t, \beta_{0,j} = 0)$ and $P(P_j(D) < t, \beta_{0,j} \neq 0)$ are also continuous in $t$. Thus $mFDR(T_p(\cdot; t), \Pi)$ is also
continuous in $t$. Now for any $\alpha \in A = (\inf_t \text{mFDR}(T_p(\cdot; t), \Pi), \sup_t \text{mFDR}(T_p(\cdot; t), \Pi))$, by intermediate value theorem, there exists $t(\alpha)$ such that $\text{mFDR}(T_p(\cdot; t(\alpha)), \Pi) = \alpha$, i.e., Eq. (15) holds.

Now we prove the second part of the proposition. First we define $p(D|\beta_{0,j} = 0)$ to be the conditional density of $D$ given $\beta_{0,j} = 0$ and $p(D|\beta_{0,j} \neq 0)$ to be the conditional density of $D$ given $\beta_{0,j} \neq 0$ as in Assumption 1. By the Bayes formula, we have

$$P_j(D) = \frac{p(D|\beta_{0,j} = 0) \mathbb{P}(\beta_{0,j} = 0)}{p(D|\beta_{0,j} = 0) \mathbb{P}(\beta_{0,j} = 0) + p(D|\beta_{0,j} \neq 0) \mathbb{P}(\beta_{0,j} \neq 0)},$$

so that $P_j(D) < t$ is equivalent to

$$p(D|\beta_{0,j} = 0) \mathbb{P}(\beta_{0,j} = 0) \times (1-t) < p(D|\beta_{0,j} \neq 0) \mathbb{P}(\beta_{0,j} \neq 0) \times t.$$  \hfill (41)

For any test $T: \Omega \to \{0,1\}^d$, we have

$$t \cdot \mathbb{E}[TP(T)] - (1-t) \cdot \mathbb{E}[FD(T)] = \sum_{j=1}^{d} \left( t \cdot \mathbb{P}(T_j = 1, \beta_{0,j} \neq 0) - (1-t) \cdot \mathbb{P}(T_j = 1, \beta_{0,j} = 0) \right)$$

$$= \sum_{j=1}^{d} \int T_j(D) \left( t \cdot p(D|\beta_{0,j} \neq 0) \mathbb{P}(\beta_{0,j} \neq 0) - (1-t) \cdot p(D|\beta_{0,j} = 0) \mathbb{P}(\beta_{0,j} = 0) \right) dD. \hfill (42)$$

Therefore, by the definition of $TP(\cdot; t)$ as in Eq. (14) and by Eq. (41), we have that $TP(\cdot; t)$ maximizes $t \cdot \mathbb{E}[TP(T)] - (1-t) \cdot \mathbb{E}[FD(T)]$, i.e., for any $T: \Omega \to \{0,1\}^d$, we have

$$t \cdot \mathbb{E}[TP(T)] - (1-t) \cdot \mathbb{E}[FD(T)] \leq t \cdot \mathbb{E}[TP(T_p(\cdot; t))] - (1-t) \cdot \mathbb{E}[FD(T_p(\cdot; t))]. \hfill (43)$$

We next show that $t = t(\alpha) > \alpha$. Define $T_\alpha = \mathbb{E}[TP(T_p(\cdot; t(\alpha)))]$ and $F_\alpha = \mathbb{E}[FD(T_p(\cdot; t(\alpha)))].$ Then by the definition of $TP$ as in Eq. (14) and by Eq. (42), we have

$$t \cdot T_\alpha - (1-t) \cdot F_\alpha \geq 0. \hfill (44)$$

We first show that $t \cdot T_\alpha - (1-t) \cdot F_\alpha > 0$. We use proof by contradiction. Assume the contrary holds which gives $t \cdot T_\alpha - (1-t) \cdot F_\alpha = 0$. Then the integrand in Eq. (42) is 0 almost everywhere for all $j \in [d]$. Thus, by the fact that $P_j(D) < t$ is equivalent to Eq. (41), we have $TP(\cdot; t) = 0$ almost everywhere in $D$. Then $T_\alpha = F_\alpha = 0$, and so that $\text{mFDR}$ of $TP(\cdot; t(\alpha))$ is equal to 0, which contradicts the definition of $t(\alpha)$. As a consequence, we have

$$0 < (1-\alpha)(t \cdot T_\alpha - (1-t) \cdot F_\alpha) = (t-\alpha)T_\alpha + (1-t)(\alpha T_\alpha + F_\alpha) - F_\alpha = (t-\alpha)T_\alpha,$$

where the last equality uses the fact that $\text{mFDR}(T_p(\cdot; t(\alpha)), \Pi) = \alpha$ so that $\alpha(T_\alpha + F_\alpha) - F_\alpha = 0$. Since $T_\alpha \geq 0$, we have from the equation above that $t > \alpha$.

Now for any test $T_0: \Omega \to \{0,1\}^d$ with $\text{mFDR}(T_0, \Pi) \leq \alpha$, we define $T = \mathbb{E}[TP(T_0)], F = \mathbb{E}[FD(T_0)].$ Since $\text{mFDR}(T_0, \Pi) \leq \alpha$, we have

$$\alpha(T + F) - F \geq 0. \hfill (45)$$

As a consequence, for $t = t(\alpha)$, we have

$$(t-\alpha)T \leq (t-\alpha)T + (1-t)(\alpha(T + F) - F) \hfill (45)$$

$$= (1-\alpha)(t \cdot T - (1-t) \cdot F) \hfill (43)$$

$$\leq (1-\alpha)(t \cdot T_\alpha - (1-t) \cdot F_\alpha) \hfill (43)$$

$$= (t-\alpha)T_\alpha + (1-t)(\alpha T_\alpha + F_\alpha) - F_\alpha \hfill (43)$$

$$= (t-\alpha)T_\alpha, \hfill \text{since } \text{mFDR}(T_p(\cdot; t(\alpha)), \Pi) = \alpha.$$ 

Since we have shown that $t - \alpha > 0$, it follows from the equation above that $T \leq T_\alpha$. This further implies

$$\text{mTPR}(T_0, \Pi) = T/\mathbb{E}_{\beta_0}[\# \{j : j \notin \text{null} \}] \leq T_\alpha/\mathbb{E}_{\beta_0}[\# \{j : j \notin \text{null} \}] = \text{mTPR}(T_p(\cdot; t(\alpha)), \Pi),$$

which proves Eq. (16).
B Proof of Proposition 2

To prove the proposition, we start with the following lemmas.

Lemma 2. For any fixed \( \lambda > 0 \), we define
\[
U(T, \lambda) \equiv mTPR(T, \Pi) - \lambda \cdot BFDR(T, \Pi).
\]
Then \( T = C_P(\cdot; \lambda) \) maximizes \( U(T, \lambda) \) over \( T : \Omega \rightarrow \{0, 1\}^d \), where the definition of \( C_P \) is given by Eq. (17).

Proof of Lemma 2. Recall the definition of \( C_P \) in Eq. (17), \( N \) in (19), \( mTPR \) in (4), and \( mTPR \) in (5). Note that for \( T : \Omega \rightarrow \{0, 1\}^d \), we have
\[
U(T, \lambda) = E_{D, \beta_0} \left[ \frac{TP(T)}{N} - \lambda \frac{FD(T)}{N} \right]
= E_{D, \beta_0} \left[ \sum_{K=0}^{d} \left( \frac{TP(T)}{N} - \lambda \frac{FD(T)}{K \lor 1} \right) \right]
= E_{D, \beta_0} \left[ \sum_{K=0}^{d} \sum_{j=1}^{d} \left( \frac{TP(T)}{N} - \lambda \frac{FD(T)}{K \lor 1} \right) \right]
= E_D \left[ \sum_{K=0}^{d} \sum_{j=1}^{d} \left( \frac{TP(T)}{N} - \lambda \frac{FD(T)}{K \lor 1} \right) \right]
= E_D \left[ \sum_{K=0}^{d} \sum_{j=1}^{d} \left( \frac{TP(T)}{N} - \lambda \frac{FD(T)}{K \lor 1} \right) \right]
= E_D \left[ \sum_{K=0}^{d} \sum_{j=1}^{d} \left( \frac{TP(T)}{N} - \lambda \frac{FD(T)}{K \lor 1} \right) \right].
\]

Now, define \( \bar{U} : \{0, 1\}^d \times \Omega \rightarrow \mathbb{R} \) as (for \( \bar{T} \in \{0, 1\}^d \) and \( D \in \Omega \))
\[
\bar{U}(\bar{T}, \bar{D}) = \sum_{K=0}^{d} \sum_{j=1}^{d} \left( \frac{TP(T)}{N} - \lambda \frac{FD(T)}{K \lor 1} \right)
= E_D \left[ \sum_{K=0}^{d} \sum_{j=1}^{d} \left( \frac{TP(T)}{N} - \lambda \frac{FD(T)}{K \lor 1} \right) \right]
\]
where \( \bar{R}(\bar{T}) \) is the number of 1 in \( \bar{T} \). In order to maximize \( U(T, \lambda) \) over \( T : \Omega \rightarrow \{0, 1\}^d \), we just need to maximize \( \bar{U}(\bar{T}, \bar{D}) \) over \( \bar{T} \in \{0, 1\}^d \) for any fixed \( D \), and set \( \bar{T}(\bar{D}) = \bar{T} \).

To do this, note that for any fixed \( K \), we have
\[
\max_{\bar{T}, \bar{D}(\bar{T}) = K} \sum_{j=1}^{d} \left( \frac{TP(T)}{N} - \lambda \frac{FD(T)}{K \lor 1} \right) = K/N - (1/N + \lambda)(K \lor 1),
\]
where \( \{P(j)(\bar{D})\}_{j=1}^{d} \) are the order statistics of the local fdr \( \{P_j(\bar{D})\}_{j=1}^{d} \) with \( P(1)(\bar{D}) \leq P(2)(\bar{D}) \leq \cdots \leq P(d)(\bar{D}) \). To maximize \( \bar{U}(\bar{T}, \bar{D}) \), we should select the number of rejection \( K \) such that the right hand side of the equation above is maximized, which give rise to the \( \hat{K} \) as in Eq. (18). This implies that \( C_P \) attains the maximum of \( U \).

\[\square\]

Lemma 3. For functions \( A, B : [-1, +1]^d \rightarrow \mathbb{R} \) and \( \lambda > 0 \), let \( x_\lambda \in \arg\max_{x \in [-1, +1]^d} \{A(x) - \lambda B(x)\} \).

Then if \( \lambda_1 < \lambda_2 \), we have \( B(x_{\lambda_1}) \geq B(x_{\lambda_2}) \).

Proof of Lemma 3. For any \( \lambda_1 < \lambda_2 \), by definition of \( x_{\lambda_1}, x_{\lambda_2} \)
\[
\lambda_2 (B(x_{\lambda_1}) - B(x_{\lambda_2})) \geq A(x_{\lambda_1}) - A(x_{\lambda_2}) \geq \lambda_1 (B(x_{\lambda_1}) - B(x_{\lambda_2})),
\]
Thus \( B(x_{\lambda_1}) \geq B(x_{\lambda_2}) \) since \( \lambda_2 > \lambda_1 \).

\[\square\]

Lemma 4. Let \( Q \) be a probability measure on a measurable space \( S \). Suppose that \( f(x, \lambda) \) is a bounded function on \( S \times \mathbb{R} \) such that (1) \( f(x, \lambda) \) is a monotonic, right-continuous step function in \( \lambda \) for any fixed \( x \in S \); (2) \( Q\{\{x|f(x, \cdot) \text{ is discontinuous at } \lambda\} = 0 \) for any \( \lambda \in \mathbb{R} \). Then \( E_{x \sim Q} f(x, \lambda) \) is continuous in \( \lambda \).
Proof of Lemma 4. Without loss of generality, we assume that $f(x, \lambda)$ is non-decreasing in $\lambda$ and $f(x, 0) = 0$ for all $x \in \mathcal{S}$. Denote the set of discontinuous points of $f(x, \cdot)$ by $D_x$ and let $V_x : \mathcal{S} \mapsto \mathbb{R}$ be $V_x(a) \equiv \lim_{\lambda \to a^+} f(x, \lambda) - \lim_{\lambda \to a^-} f(x, \lambda)$. Since $f(x, \cdot)$ is monotonic, the left and right limits exist and hence $V_x$ is well-defined. Note that $V_x(a)$ is non-negative and $V_x(a) > 0$ iff $f(x, \cdot)$ is discontinuous at $a$. For any $x \in \mathcal{S}$, we also define a measure $\Lambda_x((\infty, \lambda_0]) \equiv \sum_{\lambda \leq \lambda_0} V_x(\lambda)$. We further define a finite measure \( \tilde{Q} \) on $\mathcal{S} \times \mathbb{R}$ by $\tilde{Q}(A, B) \equiv \int_{x \in A} \int_{\lambda \in B} d\Lambda_x dQ$.

Now we prove the lemma. By definition, it suffices to show $\lim_{\lambda \to \lambda_0} \mathbb{E}_{x \sim Q} f(x, \lambda) = \mathbb{E}_{x \sim Q} f(x, \lambda_0)$ for all $\lambda_0 \in \mathbb{R}$. For the right limit, we have

$$
\lim_{\lambda \to \lambda_0} \mathbb{E}_{x \sim Q} f(x, \lambda) = \lim_{\varepsilon \to 0^+} \int_{x \in \mathcal{S}} [f(x, \lambda + \varepsilon) - f(x, \lambda)] dQ
$$

where the last convergence equality comes from the dominated convergence theorem. For the left limit,

$$
\lim_{\lambda \to \lambda_0} \mathbb{E}_{x \sim Q} f(x, \lambda) = \lim_{\varepsilon \to 0^+} \int_{x \in \mathcal{S}} [f(x, \lambda - \varepsilon) - f(x, \lambda)] dQ
$$

where the third equality uses the dominated convergence theorem and the last equality follows from the definition of $\tilde{Q}$ and $\Lambda_x$ and Fubini’s theorem. Since we assume $f(x, \lambda)$ is bounded and the set of $x$ at which $f(x, \cdot)$ is discontinuous at $\lambda$ has zero measure, $0 \leq \int_{x \sim Q} V_x(\lambda) dQ \leq 2 \sup_{x, \lambda} |f(x, \lambda)| Q(|x| f(x, \cdot) \text{ is discontinuous at } \lambda) = 0$ for all $\lambda$. Thus (48) equals zero and it completes the proof. \(\square\)

Proof of Proposition 2. We first prove that for any $\alpha \in A$, there exists $\lambda = \lambda(\alpha)$ such that $\text{BFDR}(C_P(\cdot; \lambda(\alpha)), \Pi) = \alpha$. It suffices to show that $\text{BFDR}(C_P(D; \lambda), \Pi)$ is continuous in $\lambda$. For any fixed $D$, we have $C_P(D; \lambda) \in \arg \sup_{T} \mathbb{U}(T, D)$ as shown in the proof of Lemma 2 and note that $\max_{T} \mathbb{U}(T, D) = \max_{T} \mathbb{E}_{\beta_{\lambda}}[\frac{\mathbb{U}(T, D)}{\Pi_\lambda V_{\Pi}}]$. Then it follows from Lemma 3 that $\mathbb{E}_{\beta_{\lambda}}[\mathbb{D}\text{FDP}(C_P(D; \lambda))]$ is non-increasing in $\lambda$. Moreover, in the proof of Lemma 2 we have shown that $\mathbb{E}_{\beta_{\lambda}}[\mathbb{D}\text{FDP}(C_P(D; \lambda))] = \sum_{j=1}^{K_1(\lambda, \mathcal{D})} P_j / (\hat{K}(\lambda, \mathcal{D}) + 1)$. Therefore $\hat{K}(\lambda, \mathcal{D})$ is non-increasing in $\lambda$ and $\mathbb{E}_{\beta_{\lambda}}[\mathbb{D}\text{FDP}(C_P(D; \lambda))]$ has at most $d$ discontinuous points as a non-increasing function in $\lambda$. For any $\lambda > 0$, define

$$M(\lambda) \equiv \{ D | \lambda \text{ is a discontinuous point of } \mathbb{E}_{\beta_{\lambda}}[\mathbb{D}\text{FDP}(C_P(D; \lambda))] \}.$$

Note that for fixed $D$, $\lambda$ being a discontinuous point of $\mathbb{E}_{\beta_{\lambda}}[\mathbb{D}\text{FDP}(C_P(D; \lambda))]$ implies that $\hat{K}(\lambda, \mathcal{D})$ is discontinuous at $\lambda$. Thus $\mathcal{D} \in M(\lambda)$ implies that $f_{K_1(\lambda, \mathcal{D})} \equiv K/N - (1/N + \lambda/(K \vee 1)) \sum_{j=1}^{K_1(\lambda, \mathcal{D})} P_j$ are equal for some different $K$’s. Therefore $\{ (P_1(D), ..., P_d(D)) : D \in M(\lambda) \} \subseteq \bigcup_{1 \leq k_1 < k_2 \leq d} \{ (P_1(D), ..., P_d(D)) \}$. Thus $\mathcal{D}$ is a union of solutions of linear equation, is a measure zero set with respect to the Lesbegue measure. Let $\hat{P}(\cdot)$ be the induced probability measure of $(P_1(D), ..., P_d(D))$ when $D \sim P$. Then we have $P(M(\lambda)) = \hat{P}(\{ (P_1(D), ..., P_d(D)) : D \in M(\lambda) \}) = 0$ following directly from Assumption 2. Applying Lemma 4 to $\mathbb{E}_{\beta_{\lambda}}[\mathbb{D}\text{FDP}(C_P(D; \lambda))]$ then gives the desired result that $\text{BFDR}(C_P(D; \lambda))$ is continuous in $\lambda$. This proves the existence of $\lambda(\alpha)$ satisfying Eq. (20).

Now we prove the second part of the proposition: given $T$ with $\text{BFDR}(T, \Pi) \leq \alpha$, we show Eq. (21). Note
that

\[ mTPR(T, \Pi) = U(T, \lambda) + \lambda \cdot BFDR(T, \Pi) \]
\[ \leq U(T, \lambda) + \lambda \alpha \]
\[ \leq U(C_P(\cdot; \lambda(\alpha)), \lambda) + \lambda \cdot BFDR(C_P(\cdot; \lambda(\alpha)), \Pi) \]

by Lemma 2
\[ = mTPR(C_P(\cdot; \lambda(\alpha)), \Pi). \]

This proves Eq. (21) and hence concludes the proof. □

C The empirical Bayes variant: the EPoEdCe procedure

We have shown that PoPCe and PoEdCe control frequentist FDR from finite-samples, and attain near-optimal power when the data are generated from a Bayesian linear model with a known prior \( \Pi \) and a known noise level \( \sigma^2 \). In this section, we consider the setting when we do not know the prior and the noise level, and propose Empirical Bayes PoEdCe (EPoEdCe). EPoEdCe also controls FDR from finite-samples, and attains near-optimal power whenever the data are generated from a Bayesian linear model with unknown prior and noise level. The full algorithm is presented in Algorithm 3.

At a high level, EPoEdCe first estimates the prior and the noise level using nonparametric methods, and then applies the PoEdCe procedure. To ensure that the computed p-values are valid under the null hypothesis, we need to use a covariate-splitting method to estimate the prior and the noise level. In the following, we give a line-by-line description of EPoEdCe (Algorithm 3):

- Line 2-4 (Split the covariates and estimate the prior and the noise level): We first split the indices of covariates into \( M \) equal-sized blocks \( C_m \subseteq \{d\}, m \in [M] \). For each \( m \in [M] \), we use the dataset \( \{(x_{ij})_{j \in C_m, \ y_j}\}_{i \in [n]} \) to estimate the prior and the noise level. This can be done using standard nonparametric procedures, e.g., nonparametric maximum likelihood estimate (NPML) in [Rob50, KW56]. The details of our numerical implementation of this part are presented in Appendix C.1.

- Line 6-7 (Compute the p-to-e calibration threshold using the estimated prior and noise level): We take \( q \) for each block \( m \in [M] \), using the same approach as PoPCe (Line 2-3 of Algorithm 1). In Line 6, we take

\[ BFDR(s; \Pi, \tau) = \mathbb{P}(\beta_0, G) \sim \Pi \times N(0,1) (\beta_0 = 0) | \mathbb{P}(\beta_0 + \tau G; \Pi, \tau) < s) \]

(49)

where \( \mathbb{P}(y; \Pi, \tau) = \mathbb{P}(\beta_0, z) \sim \Pi \times N(0,1) (\beta_0 = 0 | \beta_0 + \tau Z = y) \) (as defined in Eq. 29).

- Line 10-22 (Apply PoEdCe using the estimated prior and noise level): For each coordinate \( j \), compute the e-value \( e_j \) using the same steps as in PoEdCe, with \( (\Pi, \sigma^2) \) replaced by \( (\tilde{\Pi}_m, \tilde{\sigma}_m^2) \), and threshold \( q \) replaced by \( q_m \). Here \( m \) is the block that the coordinate \( j \) belongs to.

- Hyperparameter \( M \): We remark that the choice of \( M \) will not affect the validity of EPoEdCe: for any choice of \( M \), EPoEdCe has frequentist FDR control. The choice of \( M \) will also have a small effect on the asymptotic power as long as the estimated prior and noise level are consistent. In numerical simulations, we choose \( M = 50 \).

The covariate-splitting method for estimating the prior ensures that for each null coordinate \( j, \tilde{\beta}_{-j} \) only depends on \( (Y, X_{-j}) \). Therefore, \( (Y = X_{-j} \tilde{\beta}_{-j}) \) and \( x_j \) are independent. This further ensures that \( p_j \) is a valid p-value, and hence ensures the validity of EPoEdCe. This gives the following theorem with proof in Section D.

**Theorem 2** (Frequentist FDR control of EPoEdCe). For any joint distribution \( P \in \mathcal{M}(P_X) \) (c.f. Eq. 6), suppose that \( \{(x_i, y_i)\}_{i \in [n]} \) are i.i.d. from \( P \), then the EPoEdCe procedure \( T_* \) (Algorithm 3) controls the frequentist FDR,

\[ FDR(T_*, P) \leq \alpha. \]
Algorithm 3 The EPoEdCe procedure

Require: \((\{x_i, y_i\})_{i \in [n]} = (Y, X); \) FDR level \(\alpha \in (0, 1);\) distribution \(P_X;\) null proportion \(\pi_0;\) hyperparameters \(K, M \in \mathbb{N}, \) and \(\varepsilon > 0.\)

1: {Split the covariates and estimate the prior and noise level}
2: Partition \([d]\) into equal-sized blocks \(\{C_m\}_{m \in [M]}.$$ Let \(i : [d] \to [M]\) with \(i(j) = m\) iff \(j \in C_m.\)
3: for \(m \in [M]\) do
4: {Estimate the prior \(\hat{\Pi}_{-m}\) and noise level \(\hat{\sigma}_m^2\) using the dataset \(\{(x_{ij}, y_i)\}_{i \in [n]}.$$}
5: {Compute the p-to-e calibration threshold using the estimated prior and noise level}
6: Compute \(\hat{\tau}_{-m}^2\) which solves the self-consistent equation (28) with prior \(\Pi = \hat{\Pi}_{-m},\) noise level \(\sigma^2 = \hat{\sigma}_{-m}^2,\) and \(\delta\) re-scaled to be \(\delta m/(m - 1).\) Apply \(P \sim N(\hat{\tau}_{-m}, \hat{\tau}_{-m}^2) \leq \alpha - \varepsilon,\) where \(\text{FDR}\) is as defined in Eq (49).
7: Compute \(q_m = \Psi(t_m)\) where \(\Psi\) is the CDF of \(P(\hat{\tau}_{-m}; Z, \hat{\Pi}_{-m}, \hat{\tau}_{-m})\) where \(Z \sim N(0, 1).\)
8: end for
9: {Apply PoEdCe using the estimated prior and noise level}
10: for \(j \in [d]\) do
11: Compute \(\hat{\beta}_{-j},\) the posterior expectation of \(\theta_0 \in \mathbb{R}^{d-1}\) given observation \((Y, X_{-j});\) assuming the statistical model \(Y = X_{-j}\theta_0 + \varepsilon \in \mathbb{R}^n,\) where \(\theta_0, \varepsilon \sim i.i.d. \hat{\Pi}_{-i(j)}\) and \(\varepsilon_i \sim i.i.d. N(0, \sigma^2_{u(i)}).\)
12: Compute \(s_j = (Y - X_{-j}\hat{\beta}_{-j}, x_j).\)
13: Compute \(u_j = P((\hat{\tau}_{-i(j)}/\hat{\sigma}_{-i(j)})s_j; \hat{\Pi}_{-i(j)}, \hat{\tau}_{-i(j)}).\)
14: for \(k \in [K]\) do
15: Sample \(\hat{x}_{i(j)}^{(k)} = (\hat{x}_{i(j)}^{(k)}, \ldots, \hat{x}_{n(j)}^{(k)})^T\) where \(\hat{x}_{i(j)}^{(k)} \sim \mathcal{L}(X_{i, -j} = x_{i, -j})\) independently.
16: Compute \(s_j^{(k)} = (Y - X_{-j}\hat{\beta}_{-j}, \hat{x}_{j}^{(k)}).\)
17: Compute \(u_j^{(k)} = P((\hat{\tau}_{-i(j)}/\hat{\sigma}_{-i(j)})s_j^{(k)}; \hat{\Pi}_{-i(j)}, \hat{\tau}_{-i(j)}).\)
18: end for
19: Compute \(p_j = (1/(K + 1))(1 + \sum_{k=1}^K 1\{u_j \geq u_j^{(k)}\}).\)
20: Compute \(e_j = 1\{p_j \leq q_{U(j)}\}/q_{U(j)}.\)
21: end for
22: Reject the hypotheses with the \(\hat{k}\) largest \(e\)-values, where
\[
\hat{k} = \max \left\{ k : \frac{\pi_0 d}{\hat{k} e(\hat{k})} \leq \alpha \right\}.
\]
If \((\hat{\Pi}_{-m}, \hat{\sigma}^2_{-m})\) is a consistent estimator of \((\Pi, \sigma^2)\) (e.g., we believe that the NPMLE estimator is consistent \([\text{ZF22}]\)), \text{EPoEdCe} will have asymptotically the same power as \text{PoEdCe} and hence is also near-optimal. Indeed, we have the following conjecture for the asymptotic optimality of \text{EPoEdCe}, which follows from Conjecture 2 and the continuity of \((t_m, \hat{\beta}_j, \hat{\tau}_{-ij}(\mathcal{P}))\) (and hence \((s_j, u_j, p_j, e_j)\)) as functions of \((\Pi, \sigma^2)\).

**Conjecture 3** (Optimality of \text{EPoEdCe}). Consider the asymptotic regime \(n, d \to \infty, n/d \to \delta, K = K_n \to \infty, \) and \(\varepsilon = \varepsilon_n \to 0\) slow enough. If \((\hat{\Pi}_{-m}, \hat{\sigma}^2_{-m})\) is consistent, namely \(\hat{\Pi}_{-m}\) converges weakly to \(\Pi\) in probability and \(\hat{\sigma}^2_{-m}\) converges to \(\sigma^2\) in probability for all \(m \in [M]\), then under the conditions of the Bayesian linear model as per Assumption 3, \text{EPoEdCe} has the same asymptotic power as \text{CPoP} (for \(T\) to be \text{EPoEdCe})

\[
\lim_{n \to \infty} \frac{1}{d} \mathbb{E} \text{TPR}(T, \Pi) = \lim_{n \to \infty} \frac{1}{d} \mathbb{E} \text{TPR}(C_{P}(:, \lambda(\alpha)): \Pi).
\]

Subsequently, as per Proposition 2, the \text{EPoEdCe} procedure is asymptotically BFDR optimal (c.f. Definition 2):

\[
\lim_{n \to \infty} \frac{1}{d} \mathbb{E} \text{TPR}(T, \Pi) \geq \lim_{n \to \infty} \frac{1}{d} \max_{T} \left\{ \mathbb{E} \text{TPR}(T, \Pi): \text{BFDR}(T, \Pi) \leq \alpha \right\}.
\]

Consequently, according to Lemma 1, \text{EPoEdCe} is asymptotically \((\alpha, \mathcal{M}(P_X), \Pi, o_n(1))-\text{optimal procedure with frequentist FDR control (c.f. Definition 1).}

**C.1 Estimation of the prior \(\Pi\) in \text{EPoEdCe}\)**

The log-likelihood function of observing \((X, Y)\) given prior \(\Pi \in \mathcal{P}(\mathbb{R})\) and noise level \(\sigma^2 \in \mathbb{R}\) is given by

\[
\log p_{\Pi, \sigma^2}(Y, X) = \log \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{-\frac{||Y - X\beta_0||^2}{2\sigma^2}\right\} \Pi(d\beta_0).
\]

In principle, we can jointly estimate \((\Pi, \sigma^2)\) using nonparametric maximum likelihood estimate (NPMLE; \([\text{RF50}, \text{KW56}]\)). However, in our numerical implementation of \text{EPoEdCe}, due to the heavy computational burden of NPMLE, we consider a simpler parametric setting instead. In particular, we assume that the noise level \(\sigma^2\) is known and the true prior \(\Pi\) is a three point distribution supported on \([-1, 0, 1]\), i.e., \(\Pi = \pi_0\delta_0 + \pi_1\delta_1 + (1 - \pi_0 - \pi_1)\delta_{-1}\). We further consider the setting when the null proportion \(\pi_0\) is given, and we only estimate a single parameter \(\pi_1\).

In order to estimate \(\pi_1\), we further use a heuristic method as following. First, we choose some \(\lambda > 0\) and compute the ridge regression estimator

\[
\hat{\beta}_{\text{ridge}} = (X^\top X + 2\lambda I_d)^{-1} X^\top Y.
\]

Under Assumption 3 (the bayesian linear model), in the limit of \(n, d \to \infty, n/d \to \delta\), the empirical distribution of entries of \(\hat{\beta}_{\text{ridge}}\) satisfies that for any sufficiently smooth function \(\psi\), we have

\[
\lim_{n,d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\hat{\beta}_{\text{ridge},j}) = \lim_{n,d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(u_j),
\]

where \(u_j = \beta_{0,j} + \tau G_j\) with \((\beta_{0,j}, G_j)_{j \in [d]} \sim \text{i.i.d.} \ N(0, 1), \) and \(\tau = \tau(\delta, \sigma^2, \lambda), v = v(\delta, \sigma^2, \lambda)\) are the unique solution to the equations

\[
\delta(v - 1 - 2\lambda)v = v - 1,
\]

\[
\delta(v^2 - \sigma^2) = \frac{\tau^2}{v^2} + \delta^2(v - 1 - 2\lambda)^2(1 - \pi_0).
\]

Notice that the likelihood function of observing \(u = \{u_j\}_{j \in [d]}\) given prior \(\Pi\) is given by

\[
\log q_{\Pi}(u; \tau^2) = \sum_{i=1}^{d} \log \int \frac{1}{\sqrt{2\pi\tau^2}} \exp \left\{-\frac{(u_j - \theta)^2}{2\tau^2}\right\} \Pi(d\theta).
\]
This suggests that we can take an estimator of form \( \hat{\pi}_1 = \hat{\pi}_1 \delta_1 + \pi_0 \delta_0 + (1 - \hat{\pi}_1 - \pi_0) \delta_1 \) where \( \hat{\pi}_1 \) is given by

\[
\hat{\pi}_1 = \arg \max_{\pi_1 \in [0, 1 - \pi_0]} \log q_{\Pi}(v \times \hat{\beta}_{\text{ridge}}; \tau^2).
\]

(52)

Note that this is essentially a one-dimensional optimization problem, and computing \( q_{\Pi} \) only involves one-dimensional integrations. Hence this is numerically non-expensive to be solved.

In summary, our algorithm to estimate \( \hat{\Pi} = \hat{\pi}_1 \delta_1 + \pi_0 \delta_0 + (1 - \hat{\pi}_1 - \pi_0) \delta_1 \) is given as following: (1) Compute \( \hat{\beta}_{\text{ridge}} \) via Eq. (50) for a fixed \( \lambda > 0 \); (2) Solve \((\tau, v)\) which is the unique solution to Eq. (51) (assuming \( \sigma^2 \) and \( \pi_0 \) is known); (3) Compute \( \hat{\pi}_1 \) being the solution of Eq. (52).

### D Proofs of Theorem 1, 2

**Proof of Theorem 1.** Here we give the proof for PoEdCe. The proof for PoPCe is almost the same.

By Theorem 5.1 in [WR22], it suffices to show that \((\epsilon_j)_{j \in [d]}\) defined in PoEdCe are valid e-values. Recall that we denote \((y_i)_{i \in [n]}\) by \( Y \in \mathbb{R}^n \), \((x_i)_{i \in [n]}\) by \( X \in \mathbb{R}^{n \times d} \), the \( j \)-th column of \( X \) by \( x_j \). We also let \( X_{-j} \) be the matrix obtained by removing \( j \)-th column of \( X \). For any \( j \in [d] \), under the null hypothesis \( H_{j0} : Y \perp \perp X_j | X_{-j} \), we have

\[
x_j | (X_{-j}, Y) \overset{d}{=} x_j | X_{-j} = x_j | X_{-j} \overset{d}{=} x_j | (X_{-j}, Y)
\]

holds for each \( k \in [K] \). Moreover, \( x_j \) and \( x_j^{(k)} \) are independent conditional on \((X_{-j}, Y)\) by construction.

Since the posterior expectation \( \hat{\beta}_{-j} \) is a function of \((X_{-j}, Y)\), it follows from the conditional independence of \( x_j \) and \((X_j^{(k)})_{k \in [K]}\) that \( s_j \) and \((s_j^{(k)})_{k \in [K]}\) are i.i.d. conditional on \((X_{-j}, Y)\), and therefore \( u_j \) and \((u_j^{(k)})_{k \in [K]}\) are also i.i.d. conditional on \((X_{-j}, Y)\). Combining this with symmetry and taking expectation over \((X_{-j}, Y)\), we obtain \( P(p_j \leq c/(K + 1)) \leq c/(K + 1) \) for \( c = 1, 2, ..., K + 1 \), for any \( P \) such that \( H_{j0} \) holds. Hence \( p_j \) is a valid p-value for each \( j \in [d] \). Finally, letting \( e_j = 1\{p_j \leq t\}/t \) converts a valid p-value into a valid e-value since \( \int_0^t \frac{1}{x} \cdot dx = 1 \).

**Proof of Theorem 2.** Similar to the proof of Theorem 1, it suffices to show that \((e_j)_{j \in [d]}\) defined in Algorithm 3 are valid e-values. Since \( \hat{\tau}_{-i(j)}, \hat{\sigma}_{-i(j)} \) and \( \hat{\Pi}_{-i(j)} \) are constructed using \( Y, X_{-j} \), \( \hat{\beta}_{-j} \) is independent of \( x_j \) under the \( j \)-th null hypothesis \( H_{j0} : Y \perp \perp X_j | X_{-j} \). Then following the same argument as in the proof of Theorem 1, we have that for any \( j \in [d] \), under the null, \( s_j \) and \((s_j^{(k)})_{k \in [K]}\) are i.i.d. conditional on \((X_{-j}, Y)\).

Using again the fact that \( \hat{\tau}_{-i(j)}, \hat{\sigma}_{-i(j)} \) and \( \hat{\Pi}_{-i(j)} \) are constructed using \( Y, X_{-j} \), \( u_j \) (or \((u_j^{(k)})_{k \in [K]}\)) can be viewed as a function of \( \hat{\tau}_{-i(j)}, \hat{\sigma}_{-i(j)} \), \( \hat{\Pi}_{-i(j)} \) and \( s_j \) (or \((s_j^{(k)})_{k \in [K]}\)), it follow that \( u_j \) and \((u_j^{(k)})_{k \in [K]}\) are i.i.d. conditional on \((X_{-j}, Y)\) under the null hypothesis. Combining this with symmetry, we obtain \( P(p_j \leq c/(K + 1)|Y, X_{-j}) \leq c/(K + 1) \) for \( c = 1, 2, ..., K + 1 \), for any \( P \) such that \( H_{j0} \) holds. Finally,

\[
\mathbb{E}_{H_{j0}}[e_j] = \mathbb{E}_{H_{j0}} \left[ \frac{1\{p_j \leq t_{-i(j)}\}}{t_{-i(j)}} \right] = \mathbb{E}_{H_{j0}} \left[ \frac{1\{p_j \leq t_{-i(j)}\}}{t_{-i(j)}} Y, X_{-j} \right] \leq 1,
\]

where the last step follows from the fact that \( p_j \) is a valid p-value conditional on \((Y, X_{-j})\), and \( t_{-i(j)} \) is a function of \((Y, X_{-j})\). Therefore, \( e_j \) is a valid e-value and we conclude the proof.

### E Intuitions of Conjecture 1

Conjecture 1 is based on the following heuristic formalism which we derive using the replica method in statistical physics. This formalism gives the asymptotic joint empirical distributions of \{\( \hat{\beta}_j, P_j(D) \)\}_{j \in [d]} and \{\( \hat{\beta}_j, \beta_j(D) \)\}_{j \in [d]}. These results are not rigorous proofs, and we leave the proofs for future work. We will provide numerical verifications of the formalism in Appendix G.
Formalism 1. Let $\mathcal{D} = (\mathbf{X}, \mathbf{Y})$ be generated from the Bayesian linear model (Assumption 3). Let $\hat{\beta}_j$ be the posterior expectation of $\beta_{0,j}$ and let $P_j$ be the posterior probability of $\beta_{0,j} = 0$, i.e.,

$$\hat{\beta}_j(\mathcal{D}) = \mathbb{E}[\hat{\beta}_{0,j} | \mathcal{D}], \quad P_j(\mathcal{D}) = \mathbb{P}(\beta_{0,j} = 0 | \mathcal{D}).$$

Then for any sufficiently smooth function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we have

$$\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, P_j(\mathcal{D})) = \mathbb{E}(\beta_{0}, Z) \sim \Pi \times \mathcal{N}(0,1) \mathbb{E}(\beta_{0} + \tau_* Z)],$$

$$\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, \hat{\beta}_j(\mathcal{D})) = \mathbb{E}(\beta_{0}, Z) \sim \Pi \times \mathcal{N}(0,1) \mathbb{E}(\beta_{0} + \tau_* Z]],$$

where $\tau_*$ is the unique minimizer to the potential $\phi$ in Eq. (27), and $\mathcal{E}$ and $\mathcal{P}$ are given by Eq. (24) and (29) respectively.

We first use this formalism to give the intuitions of Conjecture 1.

**Part (1). Limiting FDP and TPP of TPoP.** Given this formalism, we first derive the limiting formula for FDP and TPP of TPoP as given in Eq. (31). Recall the definition of FD, TD, R and $S$ as in Eq. (1), rewritten here for convenience:

$$\text{FD}(T_P(t; \cdot)) = \sum_{j=1}^{d} I(\beta_{0,j} = 0, P_j(\mathcal{D}) < t), \quad \text{R}(T_P(t; \cdot)) = \sum_{j=1}^{d} I(P_j(\mathcal{D}) < t),$$

$$\text{TD}(T_P(t; \cdot)) = \sum_{j=1}^{d} I(\beta_{0,j} \neq 0, P_j(\mathcal{D}) < t), \quad S = \sum_{j=1}^{d} I(\beta_{0,j} \neq 0).$$

Note that these quantities are functions of the joint empirical distribution $\{(\beta_{0,j}, P_j(\mathcal{D}))\}_{j \in [d]}$. Then applying Eq. (53) in Formalism 1 gives

$$\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \text{FD}(T_P(t; \cdot)) = \mathbb{P}(\beta_{0} = 0, \Phi < t), \quad \lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \text{R}(T_P(t; \cdot)) = \mathbb{P}(\Phi < t),$$

$$\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \text{TD}(T_P(t; \cdot)) = \mathbb{P}(\beta_{0} \neq 0, \Phi < t), \quad \lim_{d \to \infty, n/d \to \delta} \frac{1}{d} S = \mathbb{P}(\beta_{0} \neq 0),$$

where $\Phi = \mathbb{P}(\beta_{0} + \tau_* Z)$ is as defined in Eq. (29), and the probabilities to the right hand side of the equations are taken with respect to $(\beta_{0}, Z) \sim \Pi \times \mathcal{N}(0,1)$. Finally, by the definitions of FDP and TPP as in Eq. (2), we have

$$\lim_{d \to \infty} \text{FDP}(T; \mathcal{D}, P) = \lim_{d \to \infty} \frac{\text{FD}(T; \mathcal{D}, P)}{\text{R}(T; \mathcal{D}) \lor 1} = \frac{\mathbb{P}(\beta_{0} = 0, \Phi < t)}{\mathbb{P}(\Phi < t)} = \mathbb{P}(\beta_{0} = 0 | \Phi < t),$$

$$\lim_{d \to \infty} \text{TPP}(T; \mathcal{D}, P) = \lim_{d \to \infty} \frac{\text{TD}(T; \mathcal{D}, P)}{|S(P)| \lor 1} = \frac{\mathbb{P}(\beta_{0} \neq 0, \Phi < t)}{\mathbb{P}(\beta_{0} \neq 0)} = \mathbb{P}(\Phi < t | \beta_{0} \neq 0).$$

This justifies Eq. (31).

**Part (2). Limiting FDP and TPP of CPoP.** We next provide the intuitions for the limiting formula for FDP and TPP of the CPoP procedure as in Eq. (32). To show this, note that CPoP (Eq. (17)) can be viewed as TPoP (Eq. (14)) with data dependant rejection threshold $t(\mathcal{D}) = P_{\hat{K}(\lambda, \mathcal{D})}(\mathcal{D})$. Since we already have the limiting FDP and TPP for TPoP, we just need to show that the data-dependent rejection threshold $P_{\hat{K}(\lambda, \mathcal{D})}(\mathcal{D})$ converges to $t_*(\lambda)$, and then apply the limiting formula for TPoP. That is, it suffices to show that

$$\lim_{d \to \infty} P_{\hat{K}(\lambda, \mathcal{D})}(\mathcal{D}) = t_*(\lambda),$$

where $t_*(\lambda)$ is given by Eq. (33).

Note that $\hat{K}(\lambda, \mathcal{D})$ is given by the solution of an optimization problem as in Eq. (18), and $t_*(\lambda)$ is given by the solution of another optimization problem as in Eq. (33). To show the asymptotic correspondence
of $P_{\hat{K}(\lambda, D)}(D)$ and $t_*(\lambda)$, we just need to build connections between the objective functions of these two optimization problems.

In order to build the connection between these two optimization problems, we define

$$h_d(t) := \mathbb{P}(\Phi < t) - (1 - \lambda(N/d)/(\mathbb{P}(\Phi < t))) \frac{1}{d} \sum_{j=1}^{\lfloor d\mathbb{P}(\Phi < t) \rfloor} P(j)(D),$$

$$h(t) := \mathbb{P}(\Phi < t) - (1 - \lambda(1 - \pi_0)/(\mathbb{P}(\Phi < t))) \mathbb{E}[\Phi \cdot 1\{\Phi < t\}].$$

Note that $h_d(t)$ depends on the empirical distribution of $\{P_j(D)\}_{j \in [d]}$, so Eq. (53) in Formalism 1 shows that

$$\lim_{d \to \infty, n/d \to \delta} h_d(t) = h(t).$$

Define $K(t) \equiv d \cdot F_\Phi(t)$ where $F_\Phi$ is cumulative distribution function of $\Phi$ as defined in Eq. (29), we have

$$h_d(t) = K(t) - (1 - \lambda N/(K(t) \vee 1)) \sum_{j=1}^{K(t)} P(j)(D),$$

which has the same form as the objective function as in Eq. (18). Furthermore, by Formalism 1 again, for any $t \in [0, 1]$, the $t$ quantile of $\{P_j(D)\}_{j \in [d]}$ should converge to $F_\Phi^{-1}(t)$, i.e., we have $\lim_{d \to \infty, n/d \to \delta} P_{d,t}(D) = \lim_{d \to \infty, n/d \to \delta} F_\Phi^{-1}(t)$. Combining the arguments above, we have

$$\lim_{d \to \infty, n/d \to \delta} P_{\hat{K}(\lambda, D)}(D) = \lim_{d \to \infty, n/d \to \delta} F_\Phi^{-1}(K(\lambda, D))/d = \lim_{d \to \infty, n/d \to \delta} \arg \max_{t \in [0, 1]} h_d(t) = \arg \max_{t \in [0, 1]} h(t) = t_*(\lambda).$$

This gives Eq. (56), which gives the desired result Eq. (32).

\[ \square \]

### E.1 Intuitions of Formalism 1

We next provide the intuitions for Formalism 1. Throughout this section, we let $\lim_{d \to \infty}$ or $\lim_{n \to \infty}$ both mean that $n, d \to \infty$ and $n/d \to \delta$.

We use the free energy trick and the replica method to calculate the asymptotic empirical distribution of $\{[\beta_{0,j}, \hat{\beta}_j(Y, X)]\}_{j \in [d]}$ and $\{[\beta_{0,j}], P_j(Y, X)\}_{j \in [d]}$. Taking a test function $g : \mathbb{R} \to \mathbb{R}$ eventually we will take $g$ to be $g(x) = x$ and $g(x) = 1[\{x = 0\}]$, a test function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and a scalar parameter $\lambda$, we define a perturbed Hamiltonian $H_{\lambda, N}$ which is a function of $\beta = \{\beta^1, \ldots, \beta^N\} \in \mathbb{R}^{N \times d}$

$$H_{\lambda, N}([\beta]) := -\frac{1}{2\sigma^2} \sum_{b=1}^{N} \| Y - X\beta^b \|^2_2 + \lambda \sum_{i=1}^{d} \frac{1}{N} \sum_{b=1}^{N} g(b^b, \beta_{0,i}).$$

We further define $Z_n$ to be the perturbed partition function

$$Z_n(\lambda, N) := \int_{\mathbb{R}^{N \times d}} \exp[H_{\lambda}(\beta)] \Pi(d\beta),$$

where $\Pi(d\beta)$ stands for $\prod_{b=1}^{N} \Pi(d\beta^b)$ with some abuse of notations. We then define the free energy density

$$\phi(\lambda, N) := \lim_{d \to \infty} \frac{1}{d} \mathbb{E}_{\lambda, N}[\log Z_n(\lambda, N)].$$

Here the expectation is with respect to the covariate matrix $X$, and the noise vector $\epsilon$ (recall that $Y = X\beta_0 + \epsilon$ as in Assumption 3). Taking the derivative of the free energy density and using a heuristic change of limit with derivative, we have

$$\partial_{\lambda} \phi(\lambda, N) = \lim_{d \to \infty} \frac{1}{d} \mathbb{E}_{X, \epsilon} \left[ \left( \sum_{i=1}^{d} \psi\left( \frac{1}{N} \sum_{b=1}^{N} g(b^b, \beta_{0,i}) \right) \right)_{H_{\lambda, N}} \right].$$
where \( \langle \cdot \rangle_{H_{\lambda,N}} \) stands for the expectation with respect to \( \beta \sim Z_n(\lambda, N)^{-1} \exp \{ H_{\lambda,N}(\beta) \} \). We then take the \( N \to \infty \) limit and set \( \lambda = 0 \). Using again a heuristic change of limits, and by the law of large numbers, we have

\[
\lim_{N \to \infty} \partial_\lambda \phi(\lambda, N)|_{\lambda=0} = \lim_{d \to \infty} \mathbb{E}_{X,\epsilon} \left[ \frac{1}{d} \sum_{i=1}^{d} \psi(\langle g(\beta_i) \rangle_\mu, \beta_{0,i}) \right],
\]

where we used \( \lim_{N \to \infty} \langle N^{-1} \sum_{b=1}^{N} g(\beta_i^b) \rangle_{H_{\lambda=0,N}} = \langle g(\beta_i) \rangle_\mu \), and \( \langle \cdot \rangle_\mu \) stands for the expectation with respect to \( \mu \in \mathcal{P}(\mathbb{R}^d) \), where

\[
\mu(d\beta) \propto \exp \left\{ -\frac{1}{2\sigma^2} \| Y - X^\top \beta \|_2^2 \right\} \Pi(d\beta).
\]

We would expect that the right hand side of Eq. (61) concentrates well around its expectation, so that as \( d \to \infty \), we can remove the expectation operator, i.e.

\[
\lim_{N \to \infty} \partial_\lambda \phi(\lambda, N)|_{\lambda=0} = \lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(\langle g(\beta_i) \rangle_\mu, \beta_{0,i}).
\]

Note that the right hand side of Eq. (62) above is what we are interested in. Our goal is thus to calculate the left hand side of Eq. (62), and we claim that the following equation holds.

**Claim 1.** Under the same setup as Formalism 1, we have

\[
\lim_{N \to \infty} \partial_\lambda \phi(\lambda, N)|_{\lambda=0} = \mathbb{E}(\beta_0,G)_{-\Pi \times \mathcal{N}(0,1)} \left[ \psi(\mathbb{E}[g(\beta_0)]\beta_0 + \tau_\ast G, \beta_0) \right].
\]

Combining Eq. (62) and (63), taking \( g(x) = x \), we get

\[
\lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(\langle \beta_i \rangle_\mu, \beta_{0,i}) = \mathbb{E}(\beta_0,G)_{-\Pi \times \mathcal{N}(0,1)} \left[ \psi(\mathbb{E}(\beta_0 + \tau_\ast G), \beta_0) \right],
\]

and taking \( g(x) = 1\{x = 0\} \), we get

\[
\lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \psi(\langle 1\{\beta_i = 0\} \rangle_\mu, \beta_{0,i}) = \mathbb{E}(\beta_0,G)_{-\Pi \times \mathcal{N}(0,1)} \left[ \psi(\mathcal{P}(\beta_0 + \tau_\ast G), \beta_0) \right].
\]

These are the desired equations (53) and (54) in Formalism 1.

### E.2 Intuitions of Claim 1

We are thus left to give the intuitions for Claim 1. To calculate the left hand side of Eq. (63), we need to first calculate \( \phi(\lambda, N) = \lim_{d \to \infty} \mathbb{E}[\log Z_n(\lambda, N)] \). We calculate \( \phi(\lambda, N) \) using the replica trick \( \mathbb{E}[\log Z] = \lim_{k \to 0} \log \mathbb{E}[Z^k]/k \) [MM09], and using a heuristic exchange of limits \( d \to \infty \) and \( k \to 0 \). The calculation of \( \lim_{N \to \infty} \partial_\lambda \phi(\lambda, N)|_{\lambda=0} \) is thus divided into three steps as below.

**S1.** The \( d \to \infty \) limit. For fixed integer \( k \), \( N \), and scalar \( \lambda \in \mathbb{R} \), we calculate

\[
S(k, \lambda, N) \equiv \lim_{d \to \infty} \frac{1}{d} \log \mathbb{E}_{X,\epsilon}[Z_n(\lambda, N)^k].
\]

**S2.** The \( k \to 0 \) limit. For fixed integer \( N \) and scalar \( \lambda \in \mathbb{R} \), we calculate

\[
\phi(\lambda, N) = \lim_{k \to 0} \left[ \frac{1}{k} S(k, \lambda, N) \right].
\]

**S3.** The \( \lambda \) differentiation. We calculate the derivative with respect to \( \lambda \), and take \( N \to \infty \)

\[
\psi_\ast \equiv \lim_{N \to \infty} \partial_\lambda \phi(\lambda, N)|_{\lambda=0}.
\]
Step S1. The $d \to \infty$ limit. Throughout the rest of this section, the indices $a, b, i$ under the summation or product operators run over $a \in [k], b \in [N]$, and $i \in [d]$. For example, we write $\sum_{a,b} = \sum_{a \in [k], b \in [N]}$ in short. We start with calculating $E_{X, \epsilon}[Z_n(\Lambda, N)^k]$. Recall that Assumption 3 gives $\beta_{0,b} \sim_{i.i.d.} \Pi, \, x_{ij} \sim_{i.i.d.} \mathcal{N}(0, 1/n), \, \varepsilon_i \sim_{i.i.d.} \mathcal{N}(0, \sigma^2)$, for $i \in [n]$ and $j \in [d]$. Using the fact that $(\int_{\mathbb{R}^N} f(\beta) \Pi(\beta))^k = \int_{\mathbb{R}^{KN}} \prod_a f((\beta(a^b))_b) \prod_a \Pi(d\beta(a^b))$, we obtain

$$
E_{X, \epsilon}[Z_n^k] = E_{X, \epsilon} \left[ \int_{\mathbb{R}^{kN}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{a,b} \|Y - X\beta(a^b)\|^2/2 + \lambda \sum_i \psi \left( \left( \frac{1}{N} \sum_b \beta_i(b^b), \beta_0, i \right) \right) \right\} \prod_{a,b} \Pi(d\beta(a^b)) \right]
$$

$$
= \int_{\mathbb{R}^{kN}} E_{X, \epsilon} \left[ \exp \left\{ -\frac{1}{2\sigma^2} \sum_{a,b} \|Y - X\beta(a^b)\|^2/2 \right\} \right] \times \exp \left\{ \lambda \sum_i \psi \left( \frac{1}{N} \sum_b \beta_i(b^b), \beta_0, i \right) \right\} \prod_{a,b} \Pi(d\beta(a^b)),
$$

(67)

where we denote $\beta_\square = \{ \beta(a^b)_{a,b}, \beta_0 \}$. We simplify $E(\beta_\square)$ as follows:

$$
E(\beta_\square) = E_{X, \epsilon} \left[ \exp \left\{ -\frac{1}{2\sigma^2} \sum_{a,b} \|Y - X\beta(a^b)\|^2/2 \right\} \right]
$$

$$
= E_X \left[ \frac{1}{(2\pi)^{N/2}} \exp \left\{ -\frac{1}{2\sigma^2} kN \|\varepsilon\|^2/2 - \frac{1}{2\sigma^2} \left( \sum_{a,b} X(\beta_0 - \beta(a^b)) \right) \right\} \right] \times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{a,b} \|X(\beta_0 - \beta(a^b))\|^2/2 \right\}
$$

$$
= \left( kN + 1 \right)^{-\frac{N}{2}} \cdot E_X \left[ \exp \left\{ \frac{1}{2\sigma^2} \left( \sum_{a,b} X(\beta_0 - \beta(a^b)) \right)^2 + \frac{1}{2\sigma^2} \left( \sum_{a,b} X(\beta_0 - \beta(a^b)) \right)^2 \right\} \right]
$$

$$
= c_n E_{x \sim \mathcal{N}(0, \frac{1}{2} I_d)} \left[ \exp \left( \frac{1}{2\sigma^2} \left( \sum_{a,b} X(\beta_0 - \beta(a^b)) \right)^2 + \frac{1}{2\sigma^2} \left( \sum_{a,b} X(\beta_0 - \beta(a^b)) \right)^2 \right) \right]^n,
$$

where we denote $\bar{\sigma}^2 = \sigma^2/(1 + Nk)$ and $c_n = (kN + 1)^{-\frac{N}{2}}$.

To further simplify the expression above, we define the overlaps of $\beta_\square$ given by $(1, kN) \in \mathbb{R}^{kN}$ is the all one vector

$$
\bar{Q}(\beta_\square) = \left( \beta(a^b), \beta(a^b)' \right)_{(a,b) \in [k] \times [N]} \in \mathbb{R}^{kN \times kN},
$$

$$
\bar{\mu}(\beta_\square) = \left( \beta(a^b), \beta_0 \right)_{(a,b) \in [k] \times [N]} \in \mathbb{R}^{kN},
$$

$$
p(\beta_\square) = \|\beta_0\|^2/d \rightarrow p = \|\beta_\square\|^2 \in \mathbb{R},
$$

$$
\Sigma(Q(\beta_\square), \bar{\mu}(\beta_\square)) = (Q(\beta_\square) - \bar{\mu}(\beta_\square)1_{kN} - 1_{kN} \bar{\mu}(\beta_\square)^T + p(\beta_0)1_{kN}1_{kN}^T)/\delta \in \mathbb{R}^{kN \times kN},
$$

Now for a Gaussian random vector $x \sim \mathcal{N}(0, (1/n)I_d)$, we define $G(a|b) = x^T(\beta_0 - \beta(a^b))$. Then these $\{G(a^b)_{a,b}\}$ are multi-variate Gaussian variables with mean 0 and covariance $E[G(a|b)G(a'|b')] = (\beta_0 - \beta(a^b)^T(\beta_0 - \beta(a'|b'))/\Sigma(\bar{Q}(\beta_\square), \bar{\mu}(\beta_\square))_{(a,b), (a',b')}.

Then we can further simplify $E(\beta_\square)$ as follows

$$
E(\beta_\square) = c_n \left[ \frac{1}{(2\pi)^{kN/2} \det(\Sigma)^{kN/2}} \exp \left\{ \frac{1}{2\sigma^2} \left( \sum_{a,b} G(a|b) \right)^2 - \frac{1}{2\sigma^2} \sum_{a,b} G(a|b)^2 - G^T \Sigma^{-1} G/2 \right\} \right]^n
$$

$$
= c_n \left[ \det(\Sigma^{-1} + I_{kN}/\sigma^2 - \bar{\sigma}^2 1_{kN}1_{kN}^T) \det(\Sigma)/\sigma^4 \right]^{-\frac{N}{2}}.
$$

Note that $E(\beta_\square)$ depend on $\beta_\square$ only through $\Sigma(Q(\beta_\square), \bar{\mu}(\beta_\square))$. Thus defining

$$
\tilde{E}(Q, \mu) = c_n \left[ \det(\Sigma(Q, \mu)^{-1} + I_{kN}/\sigma^2 - \bar{\sigma}^2 1_{kN}1_{kN}^T) \det(\Sigma(Q, \mu))/\sigma^4 \right]^{-\frac{N}{2}},
$$

(68)
we have \(E(\beta_\Xi) = \bar{E}(\bar{Q}(\beta_0), \mu(\beta_0))\).

Now plugging the expression of \(E(\beta_\Xi)\) into Eq. (67) and using the delta identity formula 1 = \(\int \delta(\bar{Q} - Q)\delta(\bar{\mu} - \mu)dQd\mu\), we have

\[
\mathbb{E}_{X_\epsilon}[Z_n^k] = \int_{\mathbb{R}^d \times k \times N} \bar{E}(\bar{Q}(\beta_0), \bar{\mu}(\beta_0)) \times \exp \left\{ \lambda \sum_{i,a} \psi \left( \frac{1}{N} \sum_b g(\beta_i^{(a|b)}), \beta_{0,i} \right) \right\} \Pi(d\beta^{(a|b)})
\]

\[
= \int dQd\mu \bar{E}(Q, \mu) \times \text{Ent}(Q, \mu),
\]

where

\[
\text{Ent}(Q, \mu) = \int \delta(\bar{Q} - Q)\delta(\bar{\mu} - \mu) \times \exp \left\{ \lambda \sum_{i,a} \psi \left( \frac{1}{N} \sum_b g(\beta_i^{(a|b)}), \beta_{0,i} \right) \right\} \Pi(d\beta^{(a|b)}).
\]

Using the Laplace method \(\lim_{n \to \infty} \frac{1}{d} \log \int \exp \{nf_n(Q, \mu)\}dQd\mu = \sup_{Q,\mu} \lim_{n \to \infty} f_n(Q, \mu)\), we have

\[
S(k, \lambda, N) \equiv \lim_{d \to \infty} \frac{1}{d} \log \mathbb{E}_{X_\epsilon}[Z_n(\lambda, N)]^k = \sup_{Q,\mu} \left[ \lim_{d \to \infty} \frac{1}{d} \log \bar{E}(Q, \mu) + \frac{1}{d} \log \text{Ent}(Q, \mu) \right].
\]

It is straightforward to calculate \(\lim_{d \to \infty} d^{-1} \log \bar{E}(Q, \mu)\). Denoting

\[
e(Q, \mu) := -\frac{\delta}{2} \log(1 + kN) - \frac{\delta}{2} \log \det(I_{kN} + (Q - \mu \mathbf{1}^T - 1\mu^T + p11^T)(I_{kN}/\sigma^2 - \bar{\sigma}^211^T/\sigma^4)).
\]

Then it is straightforward to see that

\[
e(Q, \mu) := \lim_{d \to \infty} \frac{1}{d} \log \bar{E}(Q, \mu).
\]

To calculate \(\lim_{d \to \infty} d^{-1} \log \text{Ent}(Q, \mu)\), using the delta identity formula

\[
\delta(\bar{Q} - Q)\delta(\bar{\mu} - \mu) = \int \exp \{i \cdot d \cdot \langle r, (Q - Q) + (\xi, \mu - \mu) \rangle \}d(r/(2\pi))d(\xi/(2\pi)),
\]

and saddlepoint approximation (here ext stands for the extremum)

\[
\lim_{n \to \infty} \frac{1}{n} \log \int \mathbb{R} \exp \{nf_n(ir, i\xi)\}d\lambda = \text{ext}_{r \in \mathbb{C}^N \times k \times N, \xi \in \mathbb{R}^{kN}} \lim_{n \to \infty} f_n(r, \xi),
\]

we have

\[
\text{Ent}(Q, \mu) \approx \text{ext}_{r, \xi} \int_{\mathbb{R}^d \times k \times N} \exp \left\{ - \sum_{a,b,a',b'} \left( r_{(a|b), (a'|b')} - (\beta_i^{(a|b)}, \beta_i^{(a'|b')}) \right)^2 - \sum_{a,b} \xi_{(a|b)}(d \cdot \mu_{(a|b)} - (\beta_i^{(a|b)}, \beta_{0,i})) + \lambda \sum_{i,a} \psi \left( \frac{1}{N} \sum_b g(\beta_i^{(a|b)}), \beta_{0,i} \right) \right\} \Pi(d\beta^{(a|b)}).
\]

\[
\approx \text{ext}_{r, \xi} \left[ \prod_{i=1}^d \left( \int_{\mathbb{R}^k \times N} \exp \left\{ - \sum_{a,b,a',b'} \left( r_{(a|b), (a'|b')} - (\beta_i^{(a|b)}, \beta_i^{(a'|b')}) \right)^2 - \sum_{a,b} \xi_{(a|b)}(\mu_{(a|b)} - \beta_i^{(a|b)}\beta_{0,i}) + \lambda \sum_{a} \psi \left( \frac{1}{N} \sum_b g(\beta_i^{(a|b)}), \beta_{0,i} \right) \right\} \Pi(d\beta^{(a|b)}) \right] \right].
\]

Then by the law of large numbers uniform in \(r, \xi\), and recall that \((\beta_{0,i}) \sim i.i.d.\), we have

\[
\lim_{d \to \infty} \frac{1}{d} \log \text{Ent}(Q, \mu) = \text{ext}_{r, \xi} \text{ent}(Q, \mu, r, \xi),
\]
where
\[
\text{ent}(Q, \mu, r, \xi) = \mathbb{E}_{\beta_0 \sim \mathcal{N}} \left[ \log \int_{\mathbb{R}^{k \times N}} \exp \left\{ - \sum_{a,b,a',b'} r_{(a|b),(a'|b')} (Q_{(a|b),(a'|b')}) - \beta_{(a|b)} \beta_{(a'|b')} \right\} / 2 \right.
\]
\[
- \sum_{a,b} \xi_{(a|b)} (\mu_{(a|b)} - \beta_{(a|b)} \beta_0) + \lambda \sum_{a} \psi \left( \frac{1}{N} \sum_{b} g(\beta_{(a|b)}), \beta_0 \right) \left\{ \prod_{a,b} \mathbb{E} \left[ \delta_{\beta_{(a|b)}} \right] \right\}. \tag{74}
\]

Therefore, combining Eq. (70), (72), (73), we conclude that \( S(k, \lambda, N) \) as defined in Eq. (64) gives
\[
S(k, \lambda, N) = \text{ext}_{Q, \mu} \left[ \lim_{d \to \infty} \frac{1}{d} \log \mathcal{E}(Q, \mu) + \lim_{d \to \infty} \frac{1}{d} \log \text{Ent}(Q, \mu) \right]
\]
\[
= \text{ext}_{Q, \mu, r, \xi} [e(Q, \mu) + \text{ent}(Q, \mu, r, \xi)], \tag{75}
\]
where \( e \) and \( \text{ent} \) are as defined in Eq. (71) and (74) respectively.

**Step S2. The \( k \to 0 \) limit.** We next calculate the \( k \to 0 \) limit in Eq. (65). The difficulty lies in the fact that the dimension of \((Q, \mu, r, \xi)\) depends on \( k \). Following the replica trick in statistical physics, we use the replica symmetric ansatz to simplify the expression of \( S \), and then calculate the \( k \to 0 \) limit. Using the replica symmetric ansatzs, we assume that the variables \((Q, \mu, r, \xi)\) achieving the extremum of Eq. (75) are replica symmetric in the following sense: there exists variables \((q_0, q_1, q_2, r_0, r_1, r_2, \mu, \xi)\) such that \( Q \) and \( r \) have the block form (where each block is of size \( k \times k \))

\[
Q = \begin{pmatrix}
  q_1 & q_0 \\
  \vdots & \ddots \\
  q_0 & q_1 \\
q_2 & q_1 & \cdots & q_2 \\
  \vdots & \vdots & \ddots & \vdots \\
q_2 & q_1 & \cdots & q_0 \\
\end{pmatrix}, \quad r = \begin{pmatrix}
  r_1 & r_0 \\
  \vdots & \ddots \\
  r_0 & r_1 \\
  r_2 & r_1 & \cdots & r_2 \\
  \vdots & \vdots & \ddots & \vdots \\
r_2 & r_1 & \cdots & r_0 \\
\end{pmatrix},
\]

and \( \mu \) and \( \xi \) have the following form
\[
\mu = [\mu, \ldots, \mu]^T, \quad \xi = [\xi, \ldots, \xi]^T.
\]

We further reparametrize these variables and introduce \((q, w_1, w_2, \rho_1, \rho_2, \nu, \zeta)\) satisfying
\[
q_2 = q, \quad q_0 = q_0 + \sigma^2 w_2, \quad q_1 = q_0 + \sigma^2 w_1,
\]
\[
r_2 = \frac{1}{\sigma^2} \rho_1, \quad r_0 = r_2 + \frac{1}{\sigma^2} \rho_2, \quad r_1 - r_0 = -\nu / \sigma^2, \quad \xi = \zeta / \sigma^2.
\](77)

Using this parametrization, the \( e \) function defined as in Eq. (71), and take \( k \to 0 \) limit, we have
\[
\lim_{k \to 0} \frac{1}{k} e(Q, \mu) = e(q, w_1, w_2, \mu) \equiv -\frac{\delta N}{2} \left[ \log (1 + \frac{w_1}{\delta}) + \frac{w_2}{\delta + w_1} + \frac{1}{2\sigma^2 (\delta + w_1)} (p - 2\mu + q + \delta \sigma^2) \right].
\]

Moreover, using this parameterization, the ent function as defined in Eq. (73) gives
\[
\text{ent}(Q, \mu, r, \xi) = -\frac{1}{2} \left( N^2 k^2 q \rho / \sigma^4 + N k^2 (\rho_1 w_2 / \sigma^2 + \rho_2 w_2 / \sigma^2 + \rho_2 q / \sigma^4) \right.
\]
\[
+ N k (-\nu w_1 - \nu w_2 - \nu q / \sigma^2 + w_1 \rho_1 / \sigma^2 + w_1 \rho_2 / \sigma^2) \right) - N k \rho_1 \zeta / \sigma^2
\]
\[
+ \mathbb{E}_{\beta_0} \left[ \log \int_{\mathbb{R}^{k \times N}} \exp \left\{ \frac{\rho_1}{2\sigma^4} \left( \sum_{a,b} \beta_{(a|b)}^2 \right)^2 + \frac{\rho_2}{2\sigma^4} \sum_{a} \left( \sum_{b} \beta_{(a|b)} \beta_0 \right)^2 - \frac{\nu}{\sigma^2} \sum_{a,b} \beta_{(a|b)} \right\} \left\{ \prod_{a,b} \mathbb{E} \left[ \delta_{\beta_{(a|b)}} \right] \right\} \right]
\]
\[
+ \frac{\zeta}{\sigma^2} \sum_{a,b} \beta_{(a|b)} \beta_0 + \lambda \sum_{a} \psi \left( \frac{1}{N} \sum_{b} g(\beta_{(a|b)}), \beta_0 \right) \left\{ \prod_{a,b} \mathbb{E} \left[ \delta_{\beta_{(a|b)}} \right] \right\}. \tag{78}
\]
To further simplify the equation above, we use the fact that for \( G \sim N(0, 1) \), \( \mathbb{E}[\exp(\lambda G)] = \exp(\lambda^2/2) \). So that we introduce Gaussian random variables \( G_0, \ldots, G_N \sim i.i.d. \mathcal{N}(0, 1) \), then the \( \mathbb{E}_{\beta_0} \) part of the equation above becomes

\[
\mathbb{E}_{\beta_0} \left[ \log \mathbb{E}_{G_0, \ldots, G_N} \left[ \int_{\mathbb{R}^{k \times N}} \exp \left\{ \frac{\sqrt{\nu_1}}{\sigma^2} \sum_{a,b} \beta_0(\beta(a,b)G_0 + \sqrt{\rho_2} \sum_{b} (\sum_{a} \beta(a,b)G_b) \right\} ight] \right] 
- \frac{\nu}{2\sigma^2} \sum_{ab}(\beta(a,b))^2 + \frac{\zeta}{\sigma^2} \sum_{ab} (\beta(a,b)) \beta_0 + \lambda \sum_a \psi \left( \frac{1}{N} \sum_b g(\beta(a,b), \beta_0) \right) \prod_b (d\beta(a,b)) \right] 
= \mathbb{E}_{\beta_0} \left[ \log \mathbb{E}_{G_0, \ldots, G_N} \left[ \int_{\mathbb{R}^{k \times N}} \exp \left\{ \frac{1}{\sigma^2} \sum_b \beta_0(\sqrt{\nu_1}G_0 + \sqrt{\rho_2}G_b) \right\} \prod_b (d\beta(b)) \right] \right].
\]

To take the \( k \to 0 \) limit, using the replica formula \( \mathbb{E}[\log Z]=\lim_{k \to 0}(1/k) \log \mathbb{E}[Z^k] \) in a reverse way, we obtain

\[
\lim_{k \to 0} \frac{1}{k} \text{ent} = \overline{\text{ent}}(q, w_1, w_2, \mu, \rho_1, \rho_2, \nu, \zeta) 
= N \left\{ -\frac{\nu}{2} \left( -\nu w_1 - \nu w_2 - \nu/\sigma^2 q + w_1 \rho_1 / \sigma^2 + w_1 \rho_2 / \sigma^2 \right) - \mu \zeta / \sigma^2 \right\} 
+ \mathbb{E}_{\beta_0, G_0, \ldots, G_N} \left[ \log \mathbb{E}_{G_0, \ldots, G_N} \left[ \int_{\mathbb{R}^{k \times N}} \exp \left\{ \frac{1}{\sigma^2} \sum_b \beta^b(\sqrt{\nu_1}G_0 + \sqrt{\rho_2}G_b) \right\} \prod_b (d\beta(b)) \right] \right].
\]

Then, combining Eq. (65), (75), (78), and (79), we have

\[
\phi(\lambda, N) = \text{ext}_{q,w_1,w_2,\mu,\rho_1,\rho_2,\nu,\zeta} \left[ \overline{\text{ent}}(q, w_1, w_2, \mu, \rho_1, \rho_2, \nu, \zeta) \right] 
= \text{ext}_{q,w_1,w_2,\mu,\rho_1,\rho_2,\nu,\zeta} \left\{ -N \left\{ -\frac{\nu}{2} \left( -\nu w_1 - \nu w_2 - \nu/\sigma^2 q + w_1 \rho_1 / \sigma^2 + w_1 \rho_2 / \sigma^2 \right) - \mu \zeta / \sigma^2 \right\} 
+ \mathbb{E}_{\beta_0, G_0, \ldots, G_N} \left[ \log \mathbb{E}_{G_0, \ldots, G_N} \left[ \int_{\mathbb{R}^{k \times N}} \exp \left\{ \frac{1}{\sigma^2} \sum_b \beta^b(\sqrt{\nu_1}G_0 + \sqrt{\rho_2}G_b + \zeta \beta_0) \right\} \prod_b (d\beta(b)) \right] \right] \right\}. 
\]

Taking derivatives with respect to \( (q, \mu) \) in the equation above and setting them to be zero, we obtain that the extremum will take place at \( \delta / (\delta + w_1) = \nu = \zeta \). Plugging in this equality, we get a simplified equation for \( \phi(\lambda, N) \)

\[
\phi(\lambda, N) = \text{ext}_{\rho_1,\rho_2,\nu} \left\{ -N \left\{ \frac{\nu}{2}(p + \delta^2) - \frac{\delta \sigma^2}{2} \log \nu + \frac{\delta}{2}(1 - 1)(\rho_1 + \rho_2 - \nu \sigma^2) \right\} \right\} 
+ \mathbb{E}_{\beta_0, G_0, \ldots, G_N} \left[ \log \mathbb{E}_{G_0, \ldots, G_N} \left[ \int_{\mathbb{R}^{k \times N}} \exp \left\{ \frac{1}{\sigma^2} \left( \sum_b \beta^b(\sqrt{\nu_1}G_0 + \sqrt{\rho_2}G_b + \nu \beta_0) \right) - \frac{1}{2} \nu \sum_b (\beta^b)^2 \right\} \right] \right]\prod_b (d\beta(b)) \right] \right\}. 
\]

**Step S3. The \( \lambda \) Differentiation.** We finally calculate the \( \lambda \) differentiation as in Eq. (66). Using Danskin’s theorem, we get

\[
\partial_\lambda \phi(\lambda, N) \big|_{\lambda=0} = \mathbb{E}_{\beta_0, \beta_1, \ldots, \beta_N, G_0, \ldots, G_N} \psi \left( \frac{1}{N} \sum_b g(\beta^b), \beta_0 \right),
\]

(81)
where
\[ (\beta_0, G_0, \ldots, G_N) \sim \Pi \times \mathcal{N}(0, 1)^{\otimes (N+1)}, \]
and
\[ (\hat{\beta}_1, \ldots, \hat{\beta}_N) |_{\beta_0, G_0, \ldots, G_N} \sim \hat{\mu}(d\hat{\beta}) \propto \exp \left\{ -\frac{\nu}{2\sigma^2} \sum_b \left[ \beta^b - (\beta_0 + \frac{\sqrt{p_1}}{\nu} G_0 + \frac{\sqrt{p_2}}{\nu} G_b) \right]^2 \right\} \Pi(d\hat{\beta}), \]
with \((p_1, p_2, \nu)\) satisfying the following self-consistent equation, which is obtained by setting stationary of the objective in Eq. (80) with respect to \((\rho_1, \rho_2, \nu)\)
\[ N \left[ \frac{1}{2} (p + \delta \sigma^2) - \frac{\delta \sigma^2}{2\nu} - \frac{(p_1 + p_2)\delta}{2\nu^2} + \frac{1}{2} \frac{\delta \sigma^2}{\nu^2} \right] + \mathbb{E} \left[ \sum_b \left( \frac{1}{2} (\hat{\beta}^b)^2 - \beta_0^b \beta^b \right) \right] = 0, \tag{84} \]
\[ \frac{N}{2} \delta \left( \frac{1}{\nu} - 1 \right) = \frac{1}{2\nu^2} \mathbb{E} \left[ \sum_b \hat{\beta}^b G_b \right], \tag{85} \]
\[ \frac{N}{2} \delta \left( \frac{1}{\nu} - 1 \right) = \frac{1}{2\nu^2} \mathbb{E} \left[ \sum_b \hat{\beta}^b G_0 \right]. \tag{86} \]

Here the expectations in these equations are taken with respect to \((\beta_0, \hat{\beta}_1, \ldots, \hat{\beta}_N, G_0, \ldots, G_N)\). Note that we have \(p = \mathbb{E}_{\beta \sim \Pi} [\beta^2]\). Thus Eq. (84) can be simplified as
\[ \delta \left( \frac{p_1 + p_2}{\nu^2} - \sigma^2 \right) + \delta \left( \frac{1}{\nu} - 1 \right) \sigma^2 = \mathbb{E}_{\beta_0, G_0, \ldots, G_N} \left[ \frac{1}{N} \sum_b ((\beta^b - \beta_0)^2) \mu \right]. \tag{87} \]

To simplify Eq. (85) and (86), using Gaussian integration by parts, we have
\[ \mathbb{E} \left[ \sum_b \hat{\beta}^b G_b \right] = \frac{\sqrt{p_2}}{\alpha^2} \mathbb{E}_{\beta_0, G_0, \ldots, G_N} \left[ \sum_b ((\beta^b)^2) \mu - (\beta^b)^2 \mu \right], \]
\[ \mathbb{E} \left[ \sum_b \hat{\beta}^b G_0 \right] = \frac{\sqrt{p_1}}{\alpha^2} \mathbb{E}_{\beta_0, G_0, \ldots, G_N} \left[ \sum_b ((\beta^b)^2) \mu - (\beta^b)^2 \mu \right]. \]

Plugging this into Eq. (85) and (86) gives
\[ \delta \left( \frac{1}{\nu} - 1 \right) \sigma^2 = \mathbb{E}_{\beta_0, G_0, \ldots, G_N} \left[ \frac{1}{N} \sum_b ((\beta^b)^2) \mu - (\beta^b)^2 \mu \right]. \tag{88} \]

Define \(\tau^2 = (p_1 + p_2)/\nu^2\). Combining Eq. (87) and (88) gives
\[ \delta (\tau^2 - \sigma^2) = \mathbb{E}_{(\beta_0, G) \sim \Pi \times \mathcal{N}(0, 1)} [(\mathbb{E}[\beta_0 | \beta_0 + \tau G] - \beta_0)^2]. \]

This gives the same equation as Eq. (28), and we denote its fixed point to be \(\tau_*^2\). Furthermore, when \(\lambda = 0\), we can see that \((\rho_2, \nu)\) = 0 leads to a solution of the fixed point equation. Then by Eq. (88) we further get \(\nu_* / \sigma^2 = \tau_*^2\). Therefore, the distribution of \((\hat{\beta}_1, \ldots, \hat{\beta}_N) |_{\beta_0, G_0, \ldots, G_N}\) as in Eq. (83) becomes conditionally independent
\[ (\hat{\beta}_1, \ldots, \hat{\beta}_N) |_{\beta_0, G_0} \sim \hat{\mu}(d\hat{\beta}) \propto \prod_b \exp \left\{ -\frac{1}{2\tau_*^2} \left[ \beta^b - (\beta_0 + \tau G_0) \right]^2 \right\} \Pi(d\hat{\beta}). \]

That is, we have \((\hat{\beta}_1, \ldots, \hat{\beta}_N) |_{\beta_0, G_0} \sim i.i.d. \mathcal{L}(\beta, Z) \sim \Pi \times \mathcal{N}(0, 1)(\beta + \tau Z = \beta_0 + \tau_* G_0). \]

We finally calculate the \(N\) limit of Eq. (81). By the conditional independence and identical distribution of \((\hat{\beta}_1, \ldots, \hat{\beta}_N) |_{\beta_0, G_0}\), using law of large numbers, taking \(N \to \infty\) in Eq. (81) gives
\[ \psi_* = \lim_{N \to \infty} \frac{\partial}{\partial \lambda} \phi(\lambda, N) |_{\lambda = 0} = \mathbb{E}_{(\beta_0, G_0) \sim \Pi \times \mathcal{N}(0, 1)} \left[ \psi(\mathbb{E}[g(\beta_0) | \beta_0 + \tau_* G_0], \beta_0) \right]. \tag{89} \]

This gives Eq. (63) of Claim 1.
F Intuitions of Conjecture 2

In this section, we provide the intuitions of Conjecture 2. We first present Formalism 2 and 3 below, whose intuitions are contained in Section F.1 and F.2 respectively. We remark that the intuitions of these formalisms are not rigorous proofs, and we leave rigorous proofs as future work. We will provide numerical verifications of these formalisms in Appendix G.

Formalism 2. Let \((X, Y)\) be generated from the Bayesian linear model (Assumption 3). For \(j \in [d]\), let \(P_j\) be the posterior probability of \(\beta_{0,j} = 0\) given \((X, Y)\), i.e.,
\[
P_j(Y, X) = P(\beta_{0,j} = 0 | Y, X).
\]

Let \(\hat{\beta}_j\) be the CRT p-value corresponding to \(P_j(Y, X)\) in the \(K \to \infty\) limit (c.f. Line 11 of Algorithm 1), i.e.,
\[
\hat{\beta}_j(Y, X) = P_{\hat{x}_j \sim N(0,(1/n)I_n)}(P_j(Y, X) \geq P_j(Y, X_{-j}, \hat{x}_j)).
\]

Then for any sufficiently smooth function \(\psi: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}\), we have
\[
\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, \hat{\beta}_j(Y, X)) = E_{(\beta_0, Z) \sim \Pi \times N(0,1)}[\psi(\beta_0, \Psi[P(\beta_0 + \tau_* Z)])],
\]
where \(\mathcal{P}(\cdot) = \mathcal{P}(\cdot | \Pi, \tau_*)\) is as defined in Eq. (29), \(\tau_*\) is the unique minimizer to the potential \(\phi\) in Eq. (27), and \(\Psi\) is the CDF of \(\mathcal{P}(\tau_* Z)\) when \(Z \sim N(0,1)\).

Formalism 3. Let \((X, Y)\) be generated from the Bayesian linear model (Assumption 3). For \(k \in [d]\), define \(\langle \cdot \rangle_{-k}\) to be the ensemble average over the leave-one-out distribution
\[
\mu_{-k}(d\beta_{-k}) \propto \exp\left\{ -\|Y - X_{-k}\beta_{-k}\|^2/(2\sigma^2) \right\} \prod_{j \neq k} \Pi(d\beta_j).
\]

We further define base statistics
\[
U_k(Y, X) = \mathcal{P}(\tau^2_*/\sigma^2)\langle Y - X_{-k}\beta_{-k}\rangle_{-k}, x_k\rangle.
\]

Then we let \(\tilde{\beta}_j\) be the distilled CRT p-value corresponding to \(U_j\) in the \(K \to \infty\) limit in Algorithm 2, i.e.,
\[
\tilde{\beta}_j(Y, X) = P_{\hat{x}_j \sim N(0,(1/n)I_n)}(U_j(Y, X) \geq U_j(Y, X_{-j}, \hat{x}_j)).
\]

Then for any sufficiently smooth function \(\psi: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}\), we have
\[
\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, \tilde{\beta}_j(Y, X)) = E_{(\beta_0, Z) \sim \Pi \times N(0,1)}[\psi(\beta_0, \Psi[P(\beta_0 + \tau_* Z)])],
\]
where \(\mathcal{P}(\cdot) = \mathcal{P}(\cdot | \Pi, \tau_*)\) is as defined in Eq. (29), \(\tau_*\) is the unique minimizer to the potential \(\phi\) in Eq. (27), and \(\Psi\) is the CDF of \(\mathcal{P}(\tau_* Z)\) when \(Z \sim N(0,1)\).

Now we use Formalism 1, 2 and 3 to show Conjecture 2. Here we focus on showing the asymptotic optimality of the PoPCE procedure. The intuition for the PoEDCE procedure is the same.

Recall that we have shown that CPoP gives the largest \(\text{mTPR}\) given BFDR controlled at level \(\alpha\) as in Proposition 2. We have also derived the limiting FDP and TPP curve of TPoP and CPoP as in Conjecture 1, and showed that TPoP and CPoP have the same asymptotic TPP and FDP with proper choice of parameters. Therefore, in order to show that PoPCE asymptotically achieves the optimal \(\text{mTPR}\) given BFDR controlled at level \(\alpha\), it suffices to show that PoPCE has the same asymptotic \(\text{mTPR}\) as the level-\(\alpha\) TPoP procedure (so that it has the same asymptotic \(\text{mTPR}\) as the level-\(\alpha\) CPoP procedure and thus it approximately gives the largest \(\text{mTPR}\) given BFDR controlled at level \(\alpha\)).

To show the asymptotic equivalence of PoPCE and TPoP, note that when \(K_n \to \infty\), PoPCE (Algorithm 1) is equivalent to the following procedure: if \(\Psi(t_{\text{PoPCE}}(\alpha - \varepsilon)) n_{0d}/\{\{j : \Psi^{-1}(\hat{p}_j) \leq t_{\text{PoPCE}}(\alpha - \varepsilon)\}\} < \alpha\), we
reject the hypotheses \( \{ j : \Psi^{-1}(\hat{\beta}_j) \leq t_{\text{PoPCe}}(\alpha - \varepsilon) \} \), and otherwise we reject nothing. Here the truncation threshold \( t_{\text{PoPCe}} \) gives
\[
t_{\text{PoPCe}}(\alpha) = \max \left\{ s \in [0, 1] : \lim_{d \to \infty, n/d \to \delta} \text{FDP}(T_P(s; \cdot); \Pi) \leq \alpha \right\}. \tag{96}
\]
On the other hand, the TPoP procedure (Eq. (14)) rejects \( \{ j : P_j(D) \leq t_{\text{TPoP}}(\alpha) \} \), where
\[
t_{\text{TPoP}}(\alpha) \text{ is such that } \text{mFDR}(T_P(t_{\text{TPoP}}(\alpha); \cdot), \Pi) = \alpha. \tag{97}
\]
By the concentration property of mFDR and by Eq. (96) and (97), we have that \( t_{\text{TPoP}}(\alpha) \) and \( t_{\text{PoPCe}}(\alpha) \) are asymptotically the same. Furthermore, by Formalism 2, we have that
\[
\lim_{d \to \infty} \Psi(t_{\text{PoPCe}}(\alpha - \varepsilon)) \pi_{0d}/|\{ j : \Psi^{-1}(\hat{\beta}_j) \leq t_{\text{PoPCe}}(\alpha - \varepsilon) \}| = \alpha - \varepsilon.
\]
Since \( \varepsilon = \varepsilon_n \) goes to 0 slow enough, the inequality \( \Psi(t_{\text{PoPCe}}(\alpha - \varepsilon)) \pi_{0d}/|\{ j : \Psi^{-1}(\hat{\beta}_j) \leq t_{\text{PoPCe}}(\alpha - \varepsilon) \}| < \alpha \) can be satisfied with high probability. Therefore, with high probability, PoPCe rejects the hypotheses \( \{ j : \Psi^{-1}(\hat{\beta}_j) \leq t_{\text{PoPCe}}(\alpha - \varepsilon) \} \). Finally, by Formalism 1 and 2, for any test function \( \psi \), we have
\[
\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, P_j(D)) = \lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, \Psi^{-1}(\hat{\beta}_j)).
\]
The equality above implies that, in terms of limiting TPP and FDP, rejecting \( P_j(D) \) below any threshold \( t \) is equivalent to rejecting \( \Psi^{-1}(\hat{\beta}_j) \) below the same threshold \( t \). Since the rejection threshold \( t_{\text{TPoP}}(\alpha) \) of \( P_j(D) \) in TPoP are asymptotically the same as the rejection threshold \( t_{\text{PoPCe}}(\alpha - \varepsilon) \) of \( \Psi^{-1}(\hat{\beta}_j) \) in PoPCe (module a small \( \varepsilon = \varepsilon_n \) that goes to zero sufficiently slow), it follows that PoPCe and TPoP have asymptotically the same mTPR. Finally, notice that Conjecture 1 implies that CPoP and TPoP have asymptotically the same mTPR, we have that PoPCe and CPoP also have asymptotically the same mTPR. This gives Eq. (40) of Conjecture 2. The later statements of Conjecture 2 follow immediately from Proposition 2 and Lemma 1.

We next provide the intuitions of Formalism 2 and 3.

F.1 Intuitions of Formalism 2: Distribution of the CRT p-values

Recall the definition of the CRT p-value \( \hat{\beta}_j \) as in Eq. (91), and recall that \( P_j(Y, X) = P(\beta_{0,j} = 0|Y, X) \) is the local fdr of the \( j \)-th hypothesis. Based on heuristic derivations and numerical simulations, we claim that the CRT p-values \( \hat{\beta}_j(Y, X) \) are very close:
\[
\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \left( \hat{\beta}_j(Y, X) - \Psi(P_j(Y, X)) \right)^2 = 0. \tag{98}
\]
Let us admit this claim for now. Moreover, Formalism 1 gives that, for any sufficiently smooth function \( \psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \),
\[
\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, P_j(Y, X)) = \mathbb{E}_{(\beta_{0,Z}) \sim \Pi \times \mathcal{N}(0,1)}[\psi(\beta_{0}, P(\beta_{0} + \tau, Z))]. \tag{99}
\]
Therefore, for any sufficiently smooth function \( \tilde{\psi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), taking \( \psi(x, y) = \tilde{\psi}(x, \Psi(y)) \) in Eq. (99) and combining it with the claimed Eq. (98), we get
\[
\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \tilde{\psi}(\beta_{0,j}, \hat{\beta}_j(Y, X)) = \lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \tilde{\psi}(\beta_{0,j}, \Psi(P_j(Y, X))) = \mathbb{E}_{(\beta_{0,Z}) \sim \Pi \times \mathcal{N}(0,1)}[\tilde{\psi}(\beta_{0}, \Psi|P(\beta_{0} + \tau, Z))].
\]
This is the conclusion of Formalism 2.

We are thus left to provide intuitions for the claim as in Eq. (98). Note that \( \hat{p}_j \) is given by

\[
\hat{p}_j(Y, X) = \mathbb{P}(X_j \sim \mathcal{N}(0, (1/n)I_n))(P_j(Y, X) \geq P_j(Y, X, -j, \bar{x}_j)).
\]

Therefore, to show that \( \hat{p}_j(Y, X) \approx \Psi(P_j(Y, X)) \), we just need to show that, the distribution of \( P_j(Y, X, -j, \bar{x}_j) \), conditional on \( (Y, X) \), is approximately the same as \( P(\tau, Z) \) when \( Z \sim \mathcal{N}(0, 1) \).

By definition, we can rewrite \( P_j(Y, X, -j, \bar{x}_j) \) as following:

\[
P_j(Y, X, -j, \bar{x}_j) = \mathbb{P}(\beta_{0,j} = 0|Y, X, -j, \bar{x}_j) = \{1{\beta_j = 0}\}_\mu,
\]

where

\[
\mu(d\beta_j, d\beta_{-j}) \propto \exp\left\{-\|Y - X_{-j}\beta_{-j} - \bar{x}_j\beta_j\|^2 / (2\sigma^2)\right\} \Pi(d\beta_j) \prod_{i \neq j} \Pi(d\beta_i)
\]

\[
= \exp\left\{-\|X_{-j}(\beta_{0,-j} - \beta_{-j}) + \varepsilon + x_j\beta_{0,j} - \bar{x}_j\beta_j\|^2 / (2\sigma^2)\right\} \Pi(d\beta_j) \prod_{i \neq j} \Pi(d\beta_i).
\]

Applying Formalism 4 below, we have

\[
\lim_{d \to \infty, n/d \to \delta} \mathbb{E}_{X, x, \varepsilon, \beta_0} (\{1{\beta_j = 0}\}_\mu - \{1{\beta_j = 0}\}_\nu)^2 = 0,
\]

where \( \nu \) is a distribution over \( \beta_j \), defined as

\[
\nu(d\beta_j) \propto \exp\left\{\beta_j \bar{x}_j^T \left[ \frac{X_{-j}(\beta_{0,-j} - \langle \beta_{-j}\rangle_{\mu_-}) + \varepsilon}{\sigma^2} \right] - \frac{\|\bar{x}_j\|^2 - \zeta_n^2 / \sigma^2}{2\sigma^2} \beta_j^2 \right\} \Pi(d\beta_j)
\]

\[
\propto \mathbb{P}_{(\beta_j, Z) \sim \Pi \times \mathcal{N}(0, 1)} (\beta_j|\beta_j + \tau_j Z = \tau_j G_j).
\]

In the above equation, we have

\[
\bar{\varepsilon} \equiv \varepsilon + x_j\beta_{0,j},
\]

\[
\mu_-(d\beta_{-j}) \propto \exp\left\{-\|X_{-j}\beta_{0,-j} + \bar{\varepsilon} - X_{-j}\beta_{-j}\|^2 / (2\sigma^2)\right\} \prod_{i \neq j} \Pi(d\beta_{-j,i}),
\]

\( \langle \cdot \rangle_{\mu_-} \) is the ensemble average with respect to \( \mu_- \), and

\[
\zeta_n^2 = \|X_{-j}(\beta_{0,-j} - \langle \beta_{-j}\rangle_{\mu_-})\|^2 / n,
\]

\[
\tau_j \equiv 1 / \sqrt{\langle \|\bar{x}_j\|^2 - \zeta_n^2 / \sigma^2 \rangle / \sigma^2},
\]

\[
G_j \equiv \frac{\bar{x}_j^T \left[ X_{-j}(\beta_{0,-j} - \langle \beta_{-j}\rangle_{\mu_-}) + \bar{\varepsilon} \right]}{\sigma \sqrt{\|\bar{x}_j\|^2 - \zeta_n^2 / \sigma^2}}.
\]

By Eq. (4) in [BDMK16] and by Eq. (26), we have

\[
\lim_{n, d \to \infty, n/d \to \delta} \mathbb{E} \mathcal{C}_n^2 = \frac{\sigma^2 \mathbb{E}_{(\beta_0, G) \sim \Pi \times \mathcal{N}(0, 1)} [(\beta_0 - \mathcal{E}(\beta_0 + \tau G; \Pi, \tau))^2]}{\sigma^2 \delta + \mathbb{E}_{(\beta_0, G) \sim \Pi \times \mathcal{N}(0, 1)} [(\beta_0 - \mathcal{E}(\beta_0 + \tau G; \Pi, \tau))^2]} = \frac{\sigma^2 (\tau^2 - \sigma^2)}{\tau^2} \frac{\tau^2}{\tau^2}.
\]

Furthermore, by Eq. (26) in [BDMK16], we obtain

\[
\frac{\|X_{-j}(\beta_{0,-j} - \langle \beta_{-j}\rangle_{\mu_-}) + \bar{\varepsilon}\|^2}{n} = \frac{\|\bar{\varepsilon}\|^2}{n} + \zeta_n^2 + \frac{2(\bar{\varepsilon}, X_{-j}(\beta_{0,-j} - \langle \beta_{-j}\rangle_{\mu_-}))}{n} \sigma^2 \rightarrow \lim_{n, d \to \infty, n/d \to \delta} \mathbb{E} \mathcal{C}_n^2 = \frac{\sigma^4}{\tau^2}. \]
Moreover, we have \( \|\tilde{X}_j\|^2_2 \xrightarrow{p} 1 \). Combining the convergence results above implies that
\[
\lim_{j \to \infty} \tau_j = 1/\sqrt{(1 - \sigma^2(\tau^2_j - \tau^2_j)/\tau^2_j)} = \tau_*,
\]
\[
G_j \Rightarrow \mathcal{N}(0, \|X_{-j}(\beta_{0,j} - \langle \beta_{-j} \rangle_{\mu_-}) + \tilde{\epsilon}\|^2_2/(n\sigma^2\sqrt{\|X_j\|^2_2 - \zeta^2_\alpha/\sigma^2})) \Rightarrow \mathcal{N}(0, 1).
\]
As a consequence, by Eq. (101) and (103) and by the definition of \( \mathcal{P}(\cdot) = \mathcal{P}(\cdot; \Pi, \tau_*) \) as in Eq. (29), for any set \( S \subseteq [0, 1] \), we have
\[
\mathbb{P}_{\mathcal{X}_j}(\{1\{\beta_j = 0\}\}_\mu \in S|X, \epsilon, \beta_0) \approx \mathbb{P}(\{1\{\beta_j = 0\}\}_\nu \in S|X, \epsilon, \beta_0) \approx \mathbb{P}_{G \sim \mathcal{N}(0, 1)}(\mathcal{P}(\tau_* G) \in S).
\]
Taking \( S = [0, P_j(X, Y)] \), we obtain for any fixed \( j \in [d] \) that
\[
\hat{P}_j(X, Y) = \mathbb{P}_{\mathcal{X}_j}(\{1\{\beta_j = 0\}\}_\mu \leq P_j(X, Y)|X, \epsilon, \beta_0)
\]
\[
\approx \mathbb{P}(\{1\{\beta_j = 0\}\}_\nu \leq P_j(X, Y)|X, \epsilon, \beta_0) = \mathbb{P}_{G \sim \mathcal{N}(0, 1)}(\mathcal{P}(\tau_* G) \leq P_j(X, Y)) = \Psi(P_j(X, Y)).
\]
Averaging this approximation over \( j \in [d] \) gives Eq. (98).

We finally present Formalism 4 and provide its intuitions.

**Formalism 4 (Marginal distribution of \( \mu \) in Formalism 2).** Assume that \( \text{supp}(\Pi) \subseteq [-B, B] \) for some fixed \( 0 < B < \infty \). Let \( Z \in \mathbb{R}^{n \times (d-1)} \) with \( Z_{ij} \sim \text{iid} \mathcal{N}(0, 1/n) \), \( z \in \mathbb{R}^n \) with \( z_i \sim \text{iid} \mathcal{N}(0, \sigma^2) \), \( \xi \sim \mathcal{N}(0, \sigma^2) \), and \( \xi \in \mathbb{R} \). Let \( y = Z\xi + z \xi + \epsilon \). Define measures \( \mu \in \mathcal{P}(\mathbb{R}^d), \mu_- \in \mathcal{P}(\mathbb{R}^{d-1}) \), \( \nu \in \mathcal{P}(\mathbb{R}) \) by (with ensemble average \( \langle \cdot \rangle_\mu \), \( \langle \cdot \rangle_{\mu_-} \), and \( \langle \cdot \rangle_\nu \))
\[
\mu(\mathrm{d} \beta, \mathrm{d} \beta_-) \propto \exp \left\{ -\|y - Z\beta - z\beta\|^2_2 / (2\sigma^2) \right\} \Pi(\mathrm{d} \beta) \prod_{i=1}^{d-1} \Pi(\mathrm{d} \beta_{-i}),
\]
\[
\mu_-(\mathrm{d} \beta_-) \propto \exp \left\{ -\|Z\xi - \epsilon - Z\beta_-\|^2_2 / (2\sigma^2) \right\} \Pi(\mathrm{d} \beta_-),
\]
\[
\nu(\mathrm{d} \beta) \propto \exp \left\{ \frac{(\beta - \xi)z^T \left[ Z \left( \xi - \langle \beta_- \rangle_{\mu_-} \right) + \epsilon \right]}{\sigma^2} - \frac{(\|z\|^2 - \zeta^2_\alpha/\sigma^2)}{2\sigma^2} (\beta - \xi)^2 \right\} \Pi(\mathrm{d} \beta),
\]
where
\[
\zeta^2_\alpha = \|Z \left( \xi - \langle \beta_- \rangle_{\mu_-} \right)\|^2_2 / n.
\]
Then, fix any bounded function \( f : \mathbb{R} \to \mathbb{R} \) and \( \xi \in \mathbb{R} \), we have
\[
\lim_{d \to \infty, n/d \to 0} \mathbb{E}_{Z, z, \epsilon, \xi_-} \left( \langle f(\beta) \rangle_\mu - \langle f(\beta) \rangle_\nu \right)^2 = 0.
\]

**Intuitions of Formalism 4.** Without loss of generality, we assume \( \sigma^2 = 1 \). Define
\[
M(\beta, \beta_-) \equiv \exp \left\{ - (\beta - \xi)z^T \left( \xi - \langle \beta_- \rangle_{\mu_-} \right) - \frac{\zeta^2_\alpha}{2} (\beta - \xi)^2 \right\},
\]
\[
h(\beta, \beta_-) \equiv (\beta - \xi)z^T (Z\xi + \epsilon - Z\beta_-) - \frac{1}{2} \|z\|^2_2 (\beta - \xi)^2,
\]
\[
u(\beta) \equiv (\beta - \xi)z^T \left[ Z \left( \xi - \langle \beta_- \rangle_{\mu_-} \right) + \epsilon \right] - \frac{(\|z\|^2 - \zeta^2_\alpha/2)}{2} (\beta - \xi)^2.
\]
Then we have \( \nu(\mathrm{d} \beta) = \exp\{u(\beta)\} \Pi(\mathrm{d} \beta) \), and
\[
\exp\{h(\beta, \beta_-)\} = M(\beta, \beta_-) \exp\{u(\beta)\}.
\]
For any \( f : \mathbb{R} \to \mathbb{R} \) (as a function of \( \beta \)), we have
\[
\langle f(\beta) \rangle_\mu = \frac{\int f(\beta) \exp\{h(\beta, \beta_-)\}_\mu \Pi(\mathrm{d} \beta)}{\int \exp\{h(\beta, \beta_-)\}_\mu \Pi(\mathrm{d} \beta)} = \frac{\langle f(\beta) \rangle_{M(\beta, \beta_-)} \mu_\nu}{\langle M(\beta, \beta_-) \rangle_{\mu_\nu}}.
\]
This gives
\[ \langle f(\beta) \rangle_\mu - \langle f(\beta) \rangle_\nu = \frac{\langle f(\beta) \rangle_\mu \nu - \langle f(\beta) \rangle_\nu \nu}{\langle M \rangle_\mu \nu} \langle (1 - M) \rangle_\mu \nu + \langle f(\beta) \rangle_\nu \langle (M - 1) \rangle_\mu \nu, \]
so that
\[ \left| \langle f(\beta) \rangle_\mu - \langle f(\beta) \rangle_\nu \right| \leq 2 \|f\| \left| \langle (M - 1) \rangle_\mu \nu \right|. \tag{108} \]

Next, to upper bound \( E(\langle f(\beta) \rangle_\mu - \langle f(\beta) \rangle_\nu)^2 \), we define the event
\[ E_\epsilon = \left\{ \sup_{\beta \in [-B, B]} \langle M - 1 \rangle_\mu \epsilon \leq \epsilon \right\} \tag{109} \]
and let \( F = \sup_{\beta \in [-B, B]} |f(\beta)| < \infty \). Then
\[ E_\epsilon = \left\{ \sup_{\beta \in [-B, B]} \langle M - 1 \rangle_\mu \epsilon \leq \epsilon \right\} \tag{109} \]
where the first inequality uses Eq. (108). As a consequence, Formalism 4 holds as long as \( \lim_{d \to \infty, n/d \to 0} \mathbb{P}_{\epsilon, \epsilon_0, \epsilon_0, \epsilon_0} E_\epsilon^c = 0 \).

We show \( \lim_{d \to \infty, n/d \to 0} \mathbb{P}_{\epsilon, \epsilon_0, \epsilon_0, \epsilon_0} E_\epsilon^c = 0 \) using an interpolation method. Denote
\[ V(x) = \exp \left\{ - (\beta - \xi)x - \frac{\xi^2}{2}(\beta - \xi)^2 \right\} - 1. \]
Let \( G_1, G_2 \sim_{i.i.d.} \mathcal{N}(0, 1) \), and \( S_j(t) = \sqrt{t} z^T (Z(\beta^{(j)}_1 - \langle \beta_\cdot \rangle_\mu)) + \sqrt{1 - \xi_n G_j} \). Define
\[ \ell(t) = E_{\epsilon, G_1, G_2} \left[ \langle V(S_1(t))V(S_2(t)) \rangle_{\mu_\epsilon} \right], \]
where \( O(\beta^{(1)}_\cdot, \beta^{(2)}_\cdot) \rangle_{\mu_\epsilon} \) stands for the expectation of \( O(\beta^{(1)}_\cdot, \beta^{(2)}_\cdot) \) with respect to \( (\beta^{(1)}_\cdot, \beta^{(2)}_\cdot) \sim \mu_\cdot \times \mu_\cdot \). Then we have \( \ell(0) = 0 \), and
\[ \ell(1) = E_{\epsilon} \left[ \left( \exp \left\{ - (\beta - \xi) z^T Z(\beta_\epsilon - \langle \beta_\cdot \rangle_\mu) - \frac{\xi_n}{2}(\beta - \xi)^2 \right\} - 1 \right)^2 \right] - 1 = \langle M - 1 \rangle_\mu^2. \tag{111} \]
Furthermore, calculating the derivative of \( \ell \) and using the Stein’s formula, we have
\[ \ell'(t) = \left\{ \frac{1}{2} \sum_{i,j=1}^2 \left( \frac{1}{n} \langle Z(\beta^{(1)}_\cdot - \langle \beta_\cdot \rangle_\mu), Z(\beta^{(2)}_\cdot - \langle \beta_\cdot \rangle_\mu) \rangle - \xi_n 1 \{i = j\} \right) \times \mathbb{E}_{\epsilon, G_1, G_2} \left[ \frac{\partial^2}{\partial y^2} V(S_1(t))V(S_2(t)) \right] \right\}_{\mu_\epsilon} \]
\[ \leq 2 \left( \left( \frac{1}{n} \langle Z(\beta_\cdot - \langle \beta_\cdot \rangle_\mu), Z(\beta^{(2)}_\cdot - \langle \beta_\cdot \rangle_\mu) \rangle \right)^2 \right)_{\mu_\epsilon} + \left( \frac{1}{n^2} \langle Z(\beta^{(1)}_\cdot - \langle \beta_\cdot \rangle_\mu), Z(\beta^{(2)}_\cdot - \langle \beta_\cdot \rangle_\mu) \rangle \rangle^2 \right)_{\mu_\epsilon}^{1/2} \]
\[ \times \left( \left\{ \mathbb{E}[\frac{\partial^2}{\partial y} V(S_1(t))^4] \right\}_{\mu_\epsilon} + \mathbb{E}[\right\{ \frac{\partial^2}{\partial y} V(S_1(t))^2] \}^2 \mathbb{E}[\right\{ V(S_1(t))^2] \}^2 \right)^{1/2}. \tag{112} \]
To upper bound \( \ell'(t) \), note that when \( \Pi \) is supported in \([-B, B] \) and \( \beta, \xi \in [-B, B] \), we have (for some universal constant \( K \))
\[ \langle \mathbb{E}[\frac{\partial^2}{\partial y} V(S_1(t))^4] \rangle_{\mu_\epsilon} = (\beta - \xi)^4 \exp \left\{ - 2 \xi_n (\beta - \xi)^2 \right\} (\mathbb{E} \exp \left\{ - 4(\sqrt{t} z^T (Z(\beta^{(2)}_\cdot - \langle \beta_\cdot \rangle_\mu)) + \sqrt{1 - \xi_n G_0}) (\beta - \xi) \right\})_{\mu_\epsilon} \]
\[ \leq (\beta - \xi)^4 \exp \left\{ 6 \xi_n (\beta - \xi)^2 \right\} (\exp \left\{ 8 \| Z(\beta_\cdot - \langle \beta_\cdot \rangle_\mu) \|_2^2/n \} (\beta - \xi)^2 \right\})_{\mu_\epsilon} \]
\[ \leq KB^4 \exp \{ KB^4 \| Z \|_{op} \}. \]
Similarly, we have

$$\langle E[\partial_2^2 V(S_1(t))^2] \rangle_{\mu} \leq KB^4 \exp\{KB^4\|Z\|_2^4\}, \quad \langle E[V(S_1(t))^2] \rangle_{\mu} \leq K\left(\exp\{KB^4\|Z\|_2^4\} + 1\right).$$

Denote \( \Gamma(Z) = K\{B^2 + 1\} \exp\{KB^4\|Z\|_2^4\} + 1 \). Combining the above bounds with Eq. (110) and (111), we have

$$\mathbb{E}_{\xi, \epsilon}[(M-1)^2_{\mu}] \leq \mathbb{E}_{\xi, \epsilon}[\ell(1)] \leq \int_0^1 \mathbb{E}[\ell'(t)]dt \leq \Gamma(Z) \cdot (E_1 + E_2)^{1/2},$$

where

$$E_1 = \mathbb{E}_{\xi, \epsilon}\left(\frac{1}{n} \|Z(\beta_0 - (\beta_0)_{\mu})\|_2^2 - \mu_n^2 \right)^2_{\mu} = \mathbb{E}_{\xi, \epsilon}\left(\frac{1}{n} \|Z(\xi - (\beta_0)_{\mu})\|_2^2 - \mu_n^2 \right)^2_{\mu},$$

$$E_2 = \mathbb{E}_{\xi, \epsilon}\left(\frac{1}{n^2} \{Z(\beta_0 - (\beta_0)_{\mu}), Z(\beta_{\ast} - (\beta_0)_{\mu})\} \right)^2_{\mu}.\right.$$  

We believe that \( \frac{1}{n} \|Z(\xi - (\beta_0)_{\mu})\|_2^2 \) will concentrate around its expectation \( \mu_n^2 \), and \( \frac{1}{n^2} \langle Z(\beta_0 - (\beta_0)_{\mu}), Z(\beta_{\ast} - (\beta_0)_{\mu}) \rangle \) will concentrate around its expectation 0, so that \( E_1, E_2 \) converge to zero uniformly over \( \beta \in [-B, B] \) as \( n \to \infty \). However, due to technical difficulties, we make this as a conjecture and leave it open for future work. Then by Eq. (113) and by that \( E_1, E_2 \to 0 \), we have \( \mathbb{E}_{\xi, \epsilon}[(M-1)^2_{\mu}] \to 0 \), so that by Eq. (109), we have

$$\lim_{d \to \infty, n/d \to \delta} \mathbb{P}_{\xi, Z, \xi, \epsilon}(E_\gamma) = 0.$$

Combining with Eq. (110) gives the conclusion of the formalism. \( \square \)

### F.2 Intuitions of Formalism 3: Distribution of the distilled statistics

Since \( \tilde{X}_k \sim \mathcal{N}(0, (1/n)I_n) \) is independent of \( (X, Y) \) and note that \( \langle \cdot \rangle_{-k} \) does not depend on \( \tilde{X}_k \), it follows that

$$\langle Y - X - k(\beta_{-k}) - k, \tilde{X}_k \rangle \overset{d}{\to} \mathcal{N}(0, \|Y - X - k(\beta_{-k}) - k\|_2^2/n) \overset{d}{\to} \mathcal{N}(0, \sigma^4/\tau_*^2),$$

where the last convergence is by Eq. (107). Therefore, \( U_j \) as defined in Eq. (93) satisfies \( U_j(Y, X, k, \tilde{X}_j) \overset{d}{\to} \mathcal{P}(\tau_* Z) \), where \( Z \sim \mathcal{N}(0, 1) \). Since \( \Psi \) is the CDF of \( \mathcal{P}(\tau_* Z) \) when \( Z \sim \mathcal{N}(0, 1) \), we have that for fixed \( j \in [d] \),

$$\tilde{Y}_j(Y, X) = \mathbb{P}_{\tilde{X}, \sim \mathcal{N}(0, (1/n)I_n)}(U_j(Y, X) \geq U_j(Y, X, j, \tilde{X}_j)) \overset{d}{\approx} \Psi(U_j(Y, X)).$$

Due to this approximation, we also expect that

$$\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^d \psi(\beta_{0,j}, \tilde{Y}_j(Y, X)) = \lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^d \psi(\beta_{0,j}, \Psi(U_j(Y, X)) \right).$$

Therefore, to show Formalism 3, we just need to derive the asymptotic empirical distribution of \( \{\beta_{0,j}, \Psi(U_j(Y, X))\} \), which is given by the following formalism.

**Formalism 5.** Let \( (Y, X) \) be generated from the Bayesian linear model (Assumption 3). Define \( \langle \cdot \rangle_{-k} \) to be the ensemble average over

$$\mu_{-k}(d\beta_{-k}) \propto \exp\left\{-\|Y - X - k(\beta_{-k})\|_2^2/(2\sigma^2)\right\} \prod_{j \neq k} \Pi(\mu).$$

Define

$$S_k(Y, X) = \langle Y - X - k(\beta_{-k}) - k, X_k \rangle.$$

Then in the \( n, d \to \infty \) and \( n/d \to \delta \) asymptotics, we have for sufficiently smooth \( \psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) that

$$\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^d \psi(\beta_{0,j}, S_j(Y, X)) = \mathbb{E}_{(\beta_0, Z) \sim \Pi \times \mathcal{N}(0, 1)}[\psi(\beta_0, (\sigma^2/\tau_*^2)\beta_0 + (\sigma^2/\tau_*)Z)].$$
Using Formalism 5 and Eq. (114), we immediately obtain
\[
\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, \hat{p}_j(Y, X)) = \lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, \Psi(U_j(Y, X)))
\]
\[
= \lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_{0,j}, \Psi[\mathcal{P}(\tau^2/\sigma^2)S_j(Y, X)]) = \mathbb{E}(\beta_0, \Psi[\mathcal{P}(\beta_0 + \beta_0, Z)])].
\]

This gives Formalism 3.

**Intuitions of Formalism 5.** Here we fix a coordinate \(k \in [d]\), and provide the intuition that \((\beta_{0,k}, S_k(Y, X))\) has asymptotically the same distribution as \((\beta_0, (\sigma^2/\tau^2)\beta_0 + (\sigma^2/\tau^2)Z)\) where \((\beta_0, Z) \sim \Pi \times \mathcal{N}(0, 1)\).

We define \(\tilde{Y} = Y - x_k \cdot \beta_{0,k}\). That is, \(\tilde{Y}\) is a leave-one-out model
\[
\tilde{Y} = X_{-k} \beta_{0,-k} + x_k \cdot 0 + \varepsilon.
\]

We further define
\[
\tilde{\mu}_k(d \tilde{\beta}_k) \propto \exp \left\{- \|\tilde{Y} - X_{-k} \tilde{\beta}_k\|_2^2/(2\sigma^2)\right\} \prod_{j \neq k} \Pi(d \tilde{\beta}_j)
\]
with a shorthand notation \(\langle \cdot \rangle_{-k}\) denoting the ensemble average over \(\tilde{\mu}_k\). We next define an intermediate quantity
\[
\tilde{S}_k(\tilde{Y}, X) = \langle \tilde{Y} - X_{-k} \langle \tilde{\beta}_k \rangle_{-k}, x_k \rangle.
\]

By the independence between \(\tilde{Y} - X_{-k} \langle \tilde{\beta}_k \rangle_{-k}\) and \(x_k\), and by Eq. (107), we have
\[
\tilde{S}_k(\tilde{Y}, X) \overset{d}{=} \mathcal{N}(0, \|\tilde{Y} - X_{-k} \langle \tilde{\beta}_k \rangle_{-k}\|_2^2/n) \overset{d}{=} \mathcal{N}(0, \sigma^4/\tau^2).
\]

Furthermore, we define
\[
\Delta_k(Y, X) \equiv S_k(Y, X) - \tilde{S}_k(\tilde{Y}, X) - (\sigma^2/\tau^2)\beta_{0,k}
\]
\[
= \langle Y - X_{-k} \langle \beta_k \rangle_{-k} - \langle \tilde{Y} - X_{-k} \langle \tilde{\beta}_k \rangle_{-k}, x_k \rangle - (\sigma^2/\tau^2)\beta_{0,k}
\]
(117)

Let \(\beta_{0,k}\) to be fixed and taking expectation over remaining quantities, we obtain
\[
\lim_{n \to \infty} \mathbb{E}\Delta_k(Y, X) = \lim_{n \to \infty} \mathbb{E}\langle -X_{-k} \langle \beta_k \rangle_{-k}, x_k \rangle + \beta_{0,k}(1 - \sigma^2/\tau^2)
\]
\[
= \lim_{n \to \infty} \frac{1}{n} \text{tr}(\nabla x_k (X_{-k} \beta_{0,-k} - \beta_{0,-k})) + \beta_{0,k}(1 - \sigma^2/\tau^2)
\]
\[
= \lim_{n \to \infty} -\frac{\beta_{0,k}}{n\sigma^2} \mathbb{E}\|X_{-k} \beta_{0,-k} - \beta_{0,-k}\|_2^2 + \beta_{0,k}(1 - \sigma^2/\tau^2) = 0.
\]

Here, the first equality uses the fact that \(\mathbb{E}[\|x_k\|^2] = 1\) and \(\mathbb{E}[\langle X_k \langle \beta_k \rangle_{-k}, x_k \rangle] = 0\). The second equality uses Stein’s lemma. The third equality follows from derivative calculations and the Nishimori’s identity. The last equality is by Eq. (106).

Furthermore, we believe that \(\Delta_k(Y, X)\) will concentrate around its mean, i.e.,
\[
\Delta_k(Y, X) \overset{P}{\to} \mathbb{E}[\Delta_k(Y, X)] \to 0.
\]

(119)

This concentration phenomenon is not a simple consequence of any standard concentration inequality. We leave the rigorous proof to future work. As a consequence, by the definition of \(\Delta_k\) as in Eq. (117), and by Eq. (116) and (119), this shows that \(S_k(Y, X) \overset{d}{=} \mathcal{N}(\sigma^2/\tau^2)\beta_{0,k}, \sigma^4/\tau^2)\), and hence \((\beta_{0,k}, S_k(Y, X))\) has asymptotically the same distribution as \((\beta_0, (\sigma^2/\tau^2)\beta_0 + (\sigma^2/\tau^2)Z)\) where \((\beta_0, Z) \sim \Pi \times \mathcal{N}(0, 1)\). Hence we expect that Eq. (115) holds.
G Verification of formalisms through numerical simulations

In this section, we numerically verify Formalism 1, 2 and 3. Note that these formalisms have a common form

\[
\lim_{d \to \infty, n/d \to \delta} \frac{1}{d} \sum_{j=1}^{d} \psi(\beta_\theta, f_j(D)) = \mathbb{E}_{(\beta_\theta, Z) \sim \Pi \times \mathcal{N}(0,1)}[\psi(\beta_\theta, f(\beta_\theta, Z))],
\]  

(120)

where \{f_j(D)\}_{j \in [d]} is the object of interest, and \(f(\beta_\theta, Z)\) is the limiting version of \{f_j(D)\}_{j \in [d]}. To numerically verify equations like (120) hold, we compute the 1-Wasserstein distance between the empirical distribution of \{f_j(D)\}_{j \in [d]} and the empirical distribution of \{f(\beta_\theta, Z)\}_{j \in [d]}, where (\beta_j, Z_j) \sim \text{i.i.d.} \Pi \times \mathcal{N}(0,1). Eq. (120) implies that the 1-Wasserstein distance should converge to 0 as \(n, d \to \infty, n/d \to \delta\). To show this holds, in the following figures, we plot the 1-Wasserstein distance versus the sample size \(n\) (in the regime \(n/d \to \delta\)).

![Posterior probability](image)

![Posterior expectation](image)

Figure 4: Log-log plot of the Wasserstein distance versus the sample size \(n\) (for \(n = 250, 500, 100, 2000\)) for Formalism 1. Left panel: the distance between the local fdrs and the samples from the predicted limiting distribution (Eq. (53) in Formalism 1). Right panel: the distance between the posterior expectations and samples from the predicted limiting distribution (Eq. (54) in Formalism 1). The mean curve is averaged over 10 independent instances, and the error bars report the standard deviation across instances.

Figure 4 is log-log plots of Wasserstein distances against sample sizes \(n\) for verifying Formalism 1. For each choice of \(\delta = 0.8, 1.25, 2\) and each \(n = 250, 500, 1000, 2000\), we generate 10 instances of \((Y, X)\) from the Bayesain linear model with model parameters \(\sigma = 0.25\) and \(\Pi = 0.6 \cdot \delta_0 + 0.2 \cdot \delta_1 + 0.2 \cdot \delta_{-1}\). We also sample \((\beta_j, Z_j) \sim \text{i.i.d.} \Pi \times \mathcal{N}(0,1)\) for \(j \in [d]\). In the left panel, we plot the Wasserstein distance between the posterior means \{\mathbb{E}[\beta_\theta | D]\}_{j \in [d]} and \{\mathbb{E}(\beta_j + \tau, Z_j)\}_{j \in [d]}\). In the right panel, we plot the Wasserstein distance between the local fdrs \{\mathbb{P}(\beta_\theta_j = 0 | D)\}_{j \in [d]} and \{\mathbb{P}(\beta_j + \tau, Z_j)\}_{j \in [d]}\). Figure 4 shows that the Wasserstein distances decay to zero as \(n \to \infty\), which coincides with the predictions (53) and (54) in Formalism 1.

Figure 5 is log-log plots of Wasserstein distances against sample sizes \(n\) for verifying Formalism 2 and 3. Similar to the experiments in Figure 4, we generate 10 instances of \((Y, X)\) with the same aspect ratios \(\delta\), noise level \(\sigma\), and prior \(\Pi\). In the left panel, we plot the Wasserstein distance between the CRT p-values in PopC{\{\beta_j(D)\}_{j \in [d]} and \{\mathbb{P}(\beta_j + \tau, Z_j)\}_{j \in [d]}\}, with sample size \(n = 25, 50, 100, 250\). In the right panel, we plot the Wasserstein distance between the dCRT p-values in PopEd{\{\beta_j(D)\}_{j \in [d]} and \{\mathbb{P}(\beta_j + \tau, Z_j)\}_{j \in [d]}\}, with sample size \(n = 250, 500, 1000, 2000\). We choose smaller \(n\) in the left panel since it is computationally heavier to calculate CRT p-values. Figure 4 shows that the Wasserstein distances decay to zero as \(n \to \infty\), which coincides with the prediction (92) in Formalism 2 and prediction (95) in Formalism 3.
Figure 5: Log-log plot of the Wasserstein distance versus the sample size $n$ for Formalism 2 and 3. Left panel: the distance between the PoPCe p-values and the samples from the predicted limiting distribution (Eq. (92) in Formalism 2). The sample size grid is chosen to be $n = 25, 50, 100, 250$. Right panel: the distance between the PoEdCe p-values and the samples from the predicted limiting distribution (Eq. (95) in Formalism 3). The sample size grid is chosen to be $n = 250, 500, 100, 2000$. The mean curve is averaged over 10 independent instances, and the error bars report the standard deviation across instances.