Strongly $t$-logarithmic $t$-generating sets:
Geometric properties of some soluble groups

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Abstract

We introduce the concept of a strongly $t$-logarithmic $t$-generating set for a $\mathbb{Z} \langle t, t^{-1} \rangle$-module, which enables us to prove that a large class of soluble groups are not almost convex. We also prove some results about dead-end depth.

1 Introduction

For an arbitrary finitely generated group $G$ with finite generating set $A$, the depth of an element $g \in G$ is defined to be the distance (in the word metric with respect to $A$) from $g$ to the nearest element farther away from the identity. More formally, if $d(1, g) = n$ then the depth of $g$ is the least integer $d$ such that $B_3(d) \nsubseteq B_1(n)$. (If there is no such integer, then we say the depth of $g$ is infinite; this can happen only if $G$ is finite.)

The depth of an element can depend on the choice of generating set; a classic example of this dependence is $\mathbb{Z} = \langle a \rangle$, which has depth identically 1 with respect to the given generating set $\{a\}$ but in which the depth of $a$ is 2 with respect to the set $\{a^2, a^3\}$. However, for hyperbolic groups it was shown by Bogopol’skii in [1] that the depth is always bounded for any given generating set. This is not true for all groups, though; Cleary and Taback showed in [4] that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z} = \langle a, t \mid t^2, [t, t^i], i \in \mathbb{N} \rangle$ has unbounded depth with respect to the given generating set. In this paper, we give conditions on a group guaranteeing the existence of generating sets with unbounded depth, which can be constructed. These conditions also guarantee that the group is not almost convex with respect to the generating set so constructed. Using this result,

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we show that a large class of soluble groups (including, for example, the soluble Baumslag-Solitar groups) have unbounded depth with respect to suitable generating sets, extending a result in [8].

Let \( n \in \mathbb{N} \). A group \( G \) is said to be \( n \)-almost convex with respect to some generating set if for some \( N \in \mathbb{N} \) any \( g_1, g_2 \in G \) at distance at most \( n \) from each other are connected by a path in \( G \) of length at most \( N \) whose points are at least as close to the identity as at least one of \( g_1 \) or \( g_2 \). If a group is 2-almost convex then it is \( n \)-almost convex for any \( n \); this was shown by Cannon in [2]. In this case we say simply that it is almost convex. Cannon, Floyd, Grayson and Thurston have shown in [3] that cocompact lattices in Sol are not almost convex with respect to any generating set. Later, Miller and Shapiro extended this result in [6] to the soluble Baumslag-Solitar groups. We will extend these results by showing that the same conditions as in the preceding paragraph guarantee that the group is not almost convex with respect to any generating set.

The question has arisen whether the property of having unbounded depth might be independent of the generating set. In joint work with Riley (see [7]), we resolved this question in the negative, producing a finitely generated group with unbounded depth with respect to one generating set but depth bounded by 2 with respect to another [4]. In this paper, we give more examples of groups satisfying the conditions of the preceding two results but which also have generating sets with respect to which their depth is bounded. These groups include the lamplighter group and the soluble Baumslag-Solitar groups.

The paper is organized as follows. In Section 2, we define some important terms and state the main results. In Sections 3 and 4, we prove that appropriate conditions on the group imply that it has deep pockets with respect to some generating set and is not almost convex with respect to any generating set, respectively. In Section 5, we prove that a different set of conditions on the group implies that it lacks deep pockets with respect to some other generating set. Finally, in Section 6, we show that a large class of soluble groups (including the nonabelian soluble Baumslag-Solitar groups) satisfy the conditions of the first two results. It will follow that those Baumslag-Solitar groups satisfy the conditions of the third result as well.

2 Definitions

We want to generalize the notion of lamplighter groups. The standard lamplighter group \( L_2 = \mathbb{Z}_2 \rtimes \mathbb{Z} \) is given, as relevant here, by the short exact sequence \( 0 \twoheadrightarrow \mathbb{Z}_2 \twoheadrightarrow L_2 \twoheadrightarrow \mathbb{Z} \twoheadrightarrow 0 \). (By \( \mathbb{Z}_2^\infty \) we mean the collection of biinfinite sequences of elements of \( \mathbb{Z}_2 \) such that all but finitely many terms are the identity.) Since \( \mathbb{Z} \) is a free group, this sequence necessarily splits, so \( L_2 \) may be seen as the semidirect product of \( \mathbb{Z}_2^\infty \) with the infinite cyclic group on one letter, say \( t \). In the case of \( L_2 \), the action of \( t \) on \( \mathbb{Z}_2^\infty \) may be taken to be by right shift.

1The published version of this paper contained an error, pointed out by Lehnert; however, the proof of Theorem 3 contained in Sections 2 and 3 is unaffected.
Thus, we may regard $Z_2^2$ as a $Z[t, t^{-1}]$-module, where the action of $t$ is again by right shift. As a module, it is generated by the generator of one copy of $Z_2$; we call that generator $a$. But this generation is in a stronger sense than the usual, for every element of $Z_2^2$ may be expressed as $\sum_{i \in I} t^i a$ for some finite set $I$. (In contrast, the usual notion of a cyclic $Z[t, t^{-1}]$-module would have every element expressible as $\sum_{i=-\infty}^{\infty} n_i t^i a$ for all $n_i \in Z$ and all but finitely many $n_i$ 0.)

The following definitions generalize the above picture. It is convenient to include 0 in all our module generating sets.

**Definition 1.** Let $K$ be an $Z[t, t^{-1}]$-module. Let $0 \in A \subset K$. Then a formal expression $\sum_{i=0}^{\infty} t^i a_i$ is a $t$-word in $A$ (or just a $t$-word if the choice of $A$ is clear) if all the $a_i \in A$ and all but finitely many $a_i$ are 0. More generally, a generalized $t$-word in $A$ is an element of the free abelian group on $\bigcup_{i=0}^{\infty} t^i A$. Any generalized $t$-word $v$ (hence, in particular, any $t$-word) represents an element of $K$, denoted $\pi(v)$.

**Remark.** Note that any $t$-word is a generalized $t$-word, but not conversely.

**Definition 2.** The length of a generalized $t$-word is its word length. The length of a $t$-word is its length as a generalized $t$-word. A (generalized) $t$-word is minimal if it is minimal under inclusion over all (generalized) $t$-words representing the same element of $K$.

**Remark.** Note that it is possible, at least a priori, for a (generalized) $t$-word to be minimal yet not of minimal length over all (generalized) $t$-words representing the same element of $K$.

**Definition 3.** Let $K$ be a $Z[t, t^{-1}]$-module. Let $0 \in A \subset K$. The $t$-span of $A$ is the set of all elements of $K$ which are represented by some $t$-word in $A$. A subset of $K$ whose $t$-span is all of $K$ is a $t$-generating set. If $A$ is a $t$-generating set for $K$, it is symmetrized if $a \in A$ iff $-a \in A$.

We will next fix some notations. In the case of the lamplighter group $Z_2 t Z$, an element $g$ is uniquely specified by two data: its effect on the position of the lamplighter and its effect on the set of illuminated lamps. These are generally denoted by $L(g) \in Z$ and $I(g) \in P_{fin}(Z)$, respectively, where $P_{fin}(Z)$ is the set of finite subsets of $Z$. We cannot define a concept equivalent to $I(g)$ in general, since many different $t$-words may represent the same element of $K$. We can, however, make the following definition.

**Definition 4.** Let $K$ be a $Z[t, t^{-1}]$-module and $A$ a symmetrized $t$-generating set for $K$, with $0 \in A$. Let $v$ be a generalized $t$-word in $A$. Then $I_{max}(v)$ is the maximal $i$ such that $t^i$ has a nonzero coefficient. Similarly, $I_{min}(v)$ is the minimal $i$ such that $t^i$ has a nonzero coefficient.

Let $k \in K$. Then $I_{max}(k)$ is the minimal $I_{max}(v)$ over all minimal-length $t$-words $v \in \pi^{-1}(k)$. Similarly, $I_{min}(k)$ is the maximal $I_{min}(v)$ over all minimal-length $t$-words $v \in \pi^{-1}(k)$.

Note that all of these may in general be $\pm \infty$.

One good property for $A$ to have would be for there to be a unique $t$-word representing any element of $K$. However, as mentioned above, this property will hold only rarely. Thus we settle for a weakened version.

**Definition 5.** Let $K$ be a $Z[t, t^{-1}]$-module and $A$ a symmetrized $t$-generating set for $K$, with $0 \in A$. Then $A$ is $t$-efficient if there is $C \in N$
with the following property. Let \( I_{\text{max}} \) and \( I_{\text{min}} \) be defined with respect to \( A \). Let \( v, w \) be minimal-length \( t \)-words in \( A \) with \( \pi(v) = \pi(w) \). Then 
\[
|I_{\text{max}}(v) - I_{\text{max}}(w)|, |I_{\text{min}}(v) - I_{\text{min}}(w)| < C.
\]

Remark. In the case of the lamplighter group, \( \{a\} \) is \( t \)-efficient with \( C = 1 \).

Note that the formal sum of two \( t \)-words is a generalized \( t \)-word, but (even in the case of the lamplighter group) not necessarily a \( t \)-word. The next property we will define for \( A \) relates the possible \( t \)-word representations of the sum of two elements of \( K \) to the elements’ individual \( t \)-word representations.

**Definition 6.** Let \( K \) be a \( \mathbb{Z}[t, t^{-1}] \)-module and \( A \) a symmetrized \( t \)-generating set for \( K \), with \( 0 \in A \). Then \( A \) is \( t \)-logarithmic if there is \( C \in \mathbb{N} \) with the following property. Let \( I_{\text{max}} \) and \( I_{\text{min}} \) be defined with respect to \( A \). Then, for all \( k_1, k_2 \in \mathbb{K} \),
\[
I_{\text{max}}(k_1 + k_2) \leq \max(I_{\text{max}}(k_1), I_{\text{max}}(k_2)) + C
\]
and
\[
I_{\text{min}}(k_1 + k_2) \geq \min(I_{\text{min}}(k_1), I_{\text{min}}(k_2)) - C.
\]

Remark. In the case of the lamplighter group, \( \{a\} \) is \( t \)-logarithmic with \( C = 0 \).

The point of the name is that, with respect to a \( t \)-logarithmic \( t \)-generating set, \( I_{\text{max}}(2^k) \leq I_{\text{max}}(k) + C \) and \( I_{\text{min}}(2^k) \geq I_{\text{min}}(k) - C \).

The properties of \( t \)-efficiency and \( t \)-logarithmicity seem insufficient for some of our purposes. We will thus define a stronger property which will give us what we want. To this end, for \( n \in \mathbb{N} \) and \( w \) a generalized \( t \)-word, let \( \|w\|_n \) denote \( \sum_{i = -\infty}^{\infty} \max(||w_i|| - n, 0) \), where \( w_i \) is that subword of \( w \) consisting of letters in \( t^iA \). (Note that \( \|\cdot\|_n \) is not a norm, since it is not multiplicative.) Then we make the following

**Definition 7.** Let \( K \) be a \( \mathbb{Z}[t, t^{-1}] \)-module and \( A \) a symmetrized \( t \)-generating set for \( K \), with \( 0 \in A \). Then we call \( A \) strongly \( t \)-logarithmic if for every \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \cup \{0\} \) there are \( B_{m,n} \) and \( C_{m,n} \in \mathbb{Z} \) with the following property. Let \( I_{\text{max}} \) and \( I_{\text{min}} \) be defined with respect to \( A \). Let \( w \) be a nonempty generalized \( t \)-word in \( A \). Let \( w' \) be a minimal \( t \)-word in \( A \) within \( n \) of minimal length among all \( t \)-words representing \( \pi(w) \). Then
\[
I_{\text{max}}(w') - I_{\text{max}}(w) < B_{m,n} \log(||w||_m + 1) + C_{m,n}
\]
and
\[
I_{\text{min}}(w) - I_{\text{min}}(w') < B_{m,n} \log(||w||_m + 1) + C_{m,n}.
\]

Remark. In the case of the lamplighter group, \( \{a\} \) is strongly \( t \)-logarithmic with \( B_{m,n} = 0, C_{m,n} = 1 \) for all \( m \) and \( n \).

Remark. For the consequences of strong \( t \)-logarithmicity that we will prove, the right-hand side of the above inequalities may be replaced by any function \( f(m, n, \|w\|_m) \) depending only on \( A \) such that, for every \( m \) and \( n \), \( f \) grows more slowly than every linear function of \( \|w\|_m \). However, this definition will suffice to include a significant class of groups.

We will now justify our assertion that this is a stronger property by showing that strong \( t \)-logarithmicity is a generalization of \( t \)-efficiency and \( t \)-logarithmicity.
Proposition 2.1. If a t-generating set is strongly t-logarithmic, then it is t-efficient.

Proof. The case that \( v \) and \( w \) are both empty is trivial. Otherwise, in the definition of strong t-logarithmicity, let \( w \) also be a minimal-length t-word. Then \( \|w\|_1 = 0 \), so \( I_{\text{max}}(w') - I_{\text{max}}(w), I_{\text{min}}(w) - I_{\text{min}}(w') < C_{1,0} \). \( \square \)

Proposition 2.2. If a t-generating set is strongly t-logarithmic, then it is t-logarithmic.

Proof. The cases that any of \( k_1, k_2 \) or \( k_1 + k_2 \) are 0 are trivial. Otherwise, let \( w_1 \in \pi^{-1}(k_1) \) and \( w_2 \in \pi^{-1}(k_2) \) be t-words such that \( I_{\text{max}}(w_1) = I_{\text{max}}(k_1) \) and \( I_{\text{min}}(w_2) = I_{\text{min}}(k_2) \). In the definition of strong t-logarithmicity, let \( w \) be the formal sum of \( w_1 \) and \( w_2 \) and let \( w' \) be a minimal-length t-word in \( \pi^{-1}(k_1 + k_2) \). Note that

\[
I_{\text{max}}(w) \leq \max(I_{\text{max}}(w_1), I_{\text{max}}(w_2)) = \max(I_{\text{max}}(k_1), I_{\text{max}}(k_2)).
\]

Then \( \|w\|_2 = 0 \), so

\[
I_{\text{max}}(k_1 + k_2) \leq I_{\text{max}}(w') < I_{\text{max}}(w) + C_{2,0} \leq \max(I_{\text{max}}(k_1), I_{\text{max}}(k_2)) + C_{2,0}.
\]

The proof for \( I_{\text{min}} \) is analogous. \( \square \)

We will prove the following theorems.

Theorem 2.3 (deep pockets for some generating set). Let \( K \) be a nontrivial abelian group and let \( T \) be an automorphism of \( K \). Let \( G = K \rtimes \langle t \rangle \), where \( t \) acts by \( T \), so that \( K \) has the structure of a \( \mathbb{Z}[t, t^{-1}] \)-module. Let \( A \) be a finite symmetrized strongly t-logarithmic t-generating set for \( K \), with \( 0 \in A \). Let \( S = \{ ta \mid a \in A \} \cup A \). Then there are elements of \( G \) with arbitrarily large depth with respect to \( S \).

Theorem 2.4 (not almost convex). Let \( K \) be a nontrivial abelian group and let \( T \) be an automorphism of \( K \). Let \( G = K \rtimes \langle t \rangle \), where \( t \) acts by \( T \), so that \( K \) has the structure of a \( \mathbb{Z}[t, t^{-1}] \)-module. Suppose there is a finite strongly t-logarithmic t-generating set for \( K \). Let \( S \) be any finite generating set for \( G \). Then \( G \) is not almost convex with respect to \( S \).

The next theorem requires somewhat different conditions. However, these conditions are also satisfied by the lamplighter group.

Theorem 2.5 (no deep pockets for some generating set). Let \( K \) be an abelian group and let \( G = K \rtimes \langle t \rangle \), so that \( K \) has the structure of a \( \mathbb{Z}[t, t^{-1}] \)-module. Suppose \( K \) has a finite symmetrized t-efficient t-logarithmic t-generating set \( A \), with \( 0 \in A \). Let \( N_K^{ab} \) denote the abelianization of the normal closure of \( A \) in \( \mathbb{Z}[t, t^{-1}] \). Let \( \pi \) denote the projection from \( N_K^{ab} \) to \( K \). Suppose there are \( I \) and \( J \in \mathbb{N} \) such that for every \( n \in \mathbb{N} \) there is \( k_n \in K \) such that

- \( I_{\text{min}}(k_n) \geq 1 \)
- \( I_{\text{max}}(k_n) \leq n - 1 \)

and
• for any generalized t-word $w \in \pi^{-1}(k_n)$, $|N_n(w)| \leq I(l(w) - n) + J$, where $l(w)$ is the length of $w$ and $N_n(w)$ is the set of $i$ in the range $0 < i < n$ such that $w$ does not contain any letter of $t'A$.

Let

$$S = \{ ta_1t^2 a_2 t \mid a_1, a_2 \in A \cup A^{-1} \cup \{0\} \} \cup A \cup \{t\}.$$ 

Then a bound exists on the depth of dead ends in $G$ with respect to $S$.

One set of examples of the above theorems (less trivial than the lamp-lighter groups) are the soluble Baumslag-Solitar groups $B(1, m)$, $m > 2$. These may be expressed as $\mathbb{Z}[1/m] \times \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts by multiplication by $m$. In Section 6, we will prove a general result which implies, in particular, that $A = \{-[m/2], \ldots, [m/2]\}$ is a finite strongly $t$-logarithmic $t$-generating set. Thus in particular it is $t$-efficient and $t$-logarithmic.

It remains to construct the $k_n$ in the statement of Theorem 2.3. We let $k_n = \sum_{i=1}^{n-1} m' = m(m^{n-2} - 1)/(m - 1)$. The first two conditions hold clearly (possibly after changing $n$ by a bounded amount, which makes no difference). Let $w \in \mathbb{Z}^A$ be a word representing $k_n$. Regard $w$, read with letters of greatest absolute value first, as a path through the elements of $\mathbb{Z}[1/m] \subset \mathbb{R}$. Then, for $0 < i < n$, the last multiple of $m^i$ crossed by the path must be $m^i$ or $2m^i$ modulo $m^{i+1}$. Thus the portion of the path from the last crossing of a multiple of $m^{i+1}$ (exclusive) to the last crossing of a multiple of $m^i$ (inclusive) must contain the endpoints of at least one letter, since by the end of the portion we must be reading letters of $w$ of absolute value $< m^{i+1}$. If there is no letter of $t'A$ in $w$ then the portion will contain the endpoints of at least 2 letters. It follows that $l(w) \geq n - 1 + N_n(w)$, whence $N_n(w) \leq l(w) - n + 1$, so the conditions of Theorem 2.3 are satisfied for $I = J = 1$.

Given $A$, we can define t-words without direct reference to $\mathbb{Z}[t, t^{-1}]$-modules. Let $F_A$ be the free group on $A$ and let $N_A$ be the normal closure of $A$ in $F_A \ast \langle t \rangle$. Then $N_A^{ab}$ is generated as an abelian group by $t^i a$ for $i \in \mathbb{Z}$ and $a \in A$, where the action of $t$ on $A$ is by conjugation. We say that an element of $N_A^{ab}$ is a t-word if it is of the form $\sum_{i=-\infty}^{\infty} t^i a_i$ for $a_i \in A$. Similarly, we can define generalized t-words, t-generating sets, and so forth.

## 3 Unbounded depth

This section is devoted to the proof of Theorem 2.8. Before we proceed with the proof, we make some more definitions. Let $w \in F_S$, where $F_S$ is the free group on $S$. Define $\phi(w)$ as follows. First, map $w$ to $w' \in F_A \ast \langle t \rangle$ in the obvious way, where $F_A$ denotes the free group on the set $A$. Express $w'$ as $t^i w''$, with $i \in \mathbb{Z}$ and $w'' \in N_A$, where $N_A$, as above, denotes the normal closure of $A$ in $F_A \ast \langle t \rangle$. Then $N_A^{ab}$ is a free abelian group on $\bigcup_{i=-\infty}^{\infty} t^i A$, where $t^i A$ means the image of $t^i$ under conjugation by $t'$. Define $\phi(w)$ to be the image of $w''$ in $N_A^{ab}$. Let $\sigma: F_S \to G$ and $\pi: N_A^{ab} \to K$ be the natural projections. Then the following diagram

\begin{diagram}
\end{diagram}
where the bottom arrow represents a map taking \( t^i k \) to \( k \) for \( i \in \mathbb{Z} \) and \( k \in K \). (Note that neither horizontal arrow represents a group homomorphism.)

**Proof of Theorem 2.3.** Let \( \alpha : G \to \langle t \rangle \) be the natural projection.

Let \( a \neq 0 \in A \); such an \( a \) exists since \( K \) is nontrivial. Choose \( n \in \mathbb{N} \) large and let \( g = t^n a t^{-2n} a t^n \in G \). Then \( g \in K \) and equals \( t^n a + t^{-n} a \), where we use additive notation since \( K \) is abelian.

I claim that, for \( n \) sufficiently large, \( t^n a + t^{-n} a \) is a minimal \( t \)-word. Note that neither \( t^n a \) nor \( t^{-n} a \) represents 0, since \( T \) is an automorphism.

Thus they are both minimal-length \( t \)-words. Denote the elements of \( K \) they represent by \( \overline{t^n a} \) and \( \overline{t^{-n} a} \), respectively. Suppose \( \overline{t^n a} = \overline{-t^{-n} a} \). Then

\[
-n = I_{\max}(t^{-n} a) \geq I_{\max}(t^{-n} a) = I_{\max}(t^n a) \geq I_{\max}(t^n a) - C = n - C
\]

since \( A \) is \( t \)-efficient, where \( C \) is (in this paragraph only) as in the definition of \( t \)-efficiency. This is a contradiction for \( n > C/2 \), proving the claim. Furthermore, the length of \( t^n a + t^{-n} a - 2 \) is within 1 of being minimal.

Let \( w \in F_S \) be a minimal-length element of \( \sigma (w) = g \). Then, by the above commutative diagram, \( \pi (\phi (w)) = t^n a + t^{-n} a \). Since \( A \) is strongly \( t \)-logarithmic, this implies

\[
n - I_{\max}(\phi (w)) < B_{2,1} \log (\| w \|_2 + 1) + C_{2,1}
\]

and

\[
n + I_{\min}(\phi (w)) < B_{2,1} \log (\| w \|_2 + 1) + C_{2,1},
\]

where \( B_{2,1} \) and \( C_{2,1} \) are as in the definition of strong \( t \)-logarithmicity. Let \( I(w) \) denote the greater of \( n - I_{\max}(\phi (w)) \) and \( n + I_{\min}(\phi (w)) \).

We thus get

\[
\| g \|_2 \geq \| w \|_2 + 4(n - I(w)) > e^{\frac{I(w) - C_{2,1}}{B_{2,1}}} - 4(n - I(w)) > 4n - F,
\]

where \( F \) is some number dependent only, through \( B_{2,1}, C_{2,1} \) and \( E \), on \( A \). (In the above chain of inequalities, the first step is by the construction of \( S \) and \( w \)'s being of minimal length and the second step by the preceding paragraph. The third step is an application of first-year calculus to the result of the second step, viewed as a function of \( I(w) \).)

Recall that, since \( A \) is strongly \( t \)-logarithmic, it is \( t \)-logarithmic. Let \( h \in G \), with \( \| h \|_S < (n - E)/(1 + C) \), where \( C \) is as in the definition of \( t \)-logarithmicity and \( E \) is the variable called \( C \) in the definition of \( t \)-efficiency. Let \( k \in K \) and \( i \in \mathbb{Z} \) be such that \( h = t^i k \). By the construction of \( S \), there is a generalized \( t \)-word \( t^i \in \pi^{-1}(k) \) of length \( \leq \| h \|_S \) and
with \( I_{\max}(v'') \leq \|h\|_S \) and \( I_{\min}(v'') \geq -\|h\|_S \). Since \( A \) is \( t \)-logarithmic, it follows that
\[
I_{\max}(k) \leq (1 + C)\|h\|_S < n - E
\]
and
\[
I_{\min}(k) \geq -(1 + C)\|h\|_S > E - n.
\]
Since \( A \) is \( t \)-efficient, there is a (minimal-length) \( t \)-word \( v \in \pi^{-1}(k) \) with \( I_{\max}(v) \leq I_{\max}(k) + E < n \) and \( I_{\min}(v) \geq I_{\min}(k) - E > -n \). It follows that \( hv = t'k' \), where \( k' \in K \) and there is a \( t \)-word \( v' \in \pi^{-1}(k') \) (namely \( v + t^a + t^{-a} \)) with \( I_{\max}(v') = n \) and \( I_{\min}(v') = -n \). Then it is clear that \( \|hg\|_S \leq 4n < \|g\|_S + F \). Since \( (n - E)/(1 + C) \) goes to infinity as \( n \) does, we are done by the Fuzz Lemma from [8].

4 Not almost convexity

This section is devoted to proving Theorem 2.4. This proof is modeled on that in [6] for the solvable Baumslag-Solitar groups, which was in turn modeled on that in [3] for lattices in Sol. Note that we only use strong \( t \)-logarithmcity for a restricted class of words; this is much easier to show, which simplified the work in those cases, since they could dispense with most of the work in Section 6.

We begin the proof with the following

**Lemma 4.1** (Triangle Lemma). Let \( K \) be an \( \mathbb{Z}[t, t^{-1}] \)-module. Let \( A \) be a \( t \)-efficient \( t \)-logarithmic \( t \)-generating set for \( K \). Then for every \( B \in \mathbb{R} \) there is \( D \in \mathbb{N} \) with the following property. Let \( w \) be a generalized \( t \)-word in \( A \). Then there is \( n \in \mathbb{N} \cup \{0\} \) such that \( w \) has more than \( Bn \) letters in \( \bigcup_{i = I_{\max}(\pi(w))}^{\infty} D - n t^i A \) and more than \( Bn \) letters in \( \bigcup_{i = -\infty}^{t_{\min}(\pi(w)) + D + n} t^i A \).

**Proof.** We will lose nothing if we assume \( B \in \mathbb{N} \). Also, we will prove only the clause involving \( I_{\max} \); the proof of the other, involving \( I_{\min} \), is analogous.

Let \( C \) be as in the definition of \( t \)-logarithmcity. Then we choose subwords \( w_1, w_2, \ldots \) of \( w \) inductively as follows. Let \( w_1 \) be a subword formed by choosing up to \( BC \) letters of \( \bigcup_{i = I_{\max}(\pi(w))}^{\infty} BC^2 - C \) \( BC \) \( t^i A \) for every \( j \in \mathbb{N} \). (Thus, for each \( j \) in order, if there are no more than \( BC \) letters in that range not yet chosen then take all of them; otherwise choose \( BC \) of them.) Then choose \( w_2 \) similarly as a subword of the remaining letters, and so on. Each word \( w_i \) is finite since \( w \) is finite. This process must terminate for the same reason; suppose \( w_k \) is the last nonempty subword. Then \( w = \sum_{i=1}^{k} w_i \), since each \( w_i \) can only be empty if all the letters are already taken.

Thus we have \( w = \sum_{i=1}^{k} w_i \) for some \( k \in \mathbb{N} \cup \{0\} \), where each \( w_i \) is a generalized \( t \)-word. Suppose (for a contradiction) that, for all \( n \in \mathbb{N} \cup \{0\} \), \( w \) has at most \( Bn \) letters in \( \bigcup_{i = I_{\max}(\pi(w))}^{\infty} BC^2 - n t^i A \). Then, by the construction of the \( w_i \), \( I_{\max}(w_i) < I_{\max}(\pi(w)) - BC^2 - C \). Since each \( w_i \) contains at most \( BC \) letters in \( \bigcup_{i = I_{\max}(\pi(w))}^{\infty} BC^2 - C \), it is the sum of at most \( BC \) \( t \)-words, which we denote \( w_{i,1}, \ldots, w_{i,BC} \). Since each \( w_{i,j} \) contains at most 1 letter in \( \bigcup_{i = I_{\max}(\pi(w))}^{\infty} BC^2 - C \) \( t^i A \) for
each \( j \in \mathbb{Z} \) and satisfies \( I_{\max}(w_{i,j}) \leq I_{\max}(w_{i}) < I_{\max}(\pi(w)) - BC^2 - C_i \), we have \( I_{\max}(\pi(w_{i,j})) < I_{\max}(\pi(w)) - BC^2 - C_i + C \) by \( t \)-logarithmicity. Thus

\[
I_{\max}(\pi(w_{i})) \leq \max_{j=1}^{BC} I_{\max}(\pi(w_{i,j})) + BC^2 - C
\]

\[
< I_{\max}(\pi(w)) - BC^2 - C_i + C + BC^2 - C = I_{\max}(\pi(w)) - C_i,
\]

again by \( t \)-logarithmicity.

We claim that \( I_{\max}(\pi(\sum_{i=k-j}^{k} w_{i})) < I_{\max}(\pi(w)) - C(k - j - 1) \). The proof is by induction on \( j \). If \( j = 0 \), then the claim just says \( I_{\max}(\pi(w_k)) < I_{\max}(\pi(w)) - C(k - 1) \), which is weaker than what we already know. Otherwise, we have

\[
I_{\max} \left( \pi \left( \sum_{i=k-j}^{k} w_{i} \right) \right) = I_{\max} \left( \pi \left( \sum_{i=k-j+1}^{k} w_{i} \right) + \pi(w_{k-j}) \right)
\]

\[
< I_{\max}(\pi(w)) - C(k - j) + C = I_{\max}(\pi(w)) - C(k - j - 1),
\]

where the inequality is by \( t \)-logarithmicity and induction. The claim is proven.

Letting \( j = k - 1 \) in the claim gives

\[
I_{\max}(\pi(\sum_{i=k-j}^{k} w_{i})) = I_{\max}(\pi(w)) < I_{\max}(\pi(w)),
\]

a contradiction. The lemma follows if we let \( D = BC^2 \). Note that this depends only on \( A \) (via \( C \)) and \( B \), not on \( w \).

We will want to extend the definition of \( \phi \) to the new \( F_S \), that is the free group on the now arbitrary generating set \( S \). The only step which is not obvious is the definition of the map from \( S \cup S^{-1} \) to \( F_A \ast \langle t \rangle \). We simply choose the map once and for all, requiring only that each element \( s \in S \cup S^{-1} \) be mapped to a word representing \( s \) and that inverses be mapped to inverses. Then the definition goes through without change.

If \( w_1, w_2 \in F_S \), then \( \phi(w_1), \phi(w_2) \) and \( \phi(w_1w_2) \) are all generalized \( t \)-words. If \( \alpha: G \to \mathbb{Z} \) is the projection, then

\[
I_{\max}(\phi(w_1w_2)) \leq \max(I_{\max}(\phi(w_1)) + \alpha(w_2), I_{\max}(\phi(w_2)))
\]

and

\[
I_{\min}(\phi(w_1w_2)) \geq \min(I_{\min}(\phi(w_1)) + \alpha(w_2), I_{\min}(\phi(w_2))).
\]

Similarly, \( \phi(w_1^{-1}) \) is a generalized \( t \)-word. We have

\[
I_{\max}(\phi(w_1^{-1})) = I_{\max}(\phi(w_1)) - \alpha(w_1)
\]

and

\[
I_{\min}(\phi(w_1^{-1})) = I_{\min}(\phi(w_1)) - \alpha(w_1).
\]

Let \( g \in G \). Let \( i \in \mathbb{Z} \) and \( k \in K \) be such that \( g = t^k \). Define \( I_{\max}(g) = I_{\max}(k) \) and \( I_{\min}(g) = I_{\min}(k) \).
Corollary 4.2. Let $K$ be a nontrivial abelian group and let $R$ be an automorphism of $K$. Let $G = K \times (t)$, where $t$ acts by $R$, so that $K$ has the structure of a $\mathbb{Z} [t, t^{-1}]$-module. Let $\alpha: G \to \mathbb{Z}$ be the projection. Suppose there is a finite $t$-efficient $t$-logarithmic $t$-generating set for $K$. Let $S$ be any finite generating set for $G$ and let $\sigma: F_S \to G$ be the projection. Let $z = \max \{ \alpha(s) \mid s \in S \cup S^{-1} \}$. For $w \in F_S$, let $l(w)$ denote the length of $w$ as a word. Then there is $F \in \mathbb{N}$ such that, for every $w \in \ker(\alpha \circ \sigma)$, either $I_{\max}(\sigma(w)) \leq l(w)z/4 + F$ or $I_{\min}(\sigma(w)) \geq -l(w)z/4 - F$.

**Proof.** Let $n = l(w)$. Let $w = s_1s_2 \ldots s_n$, $s_1, \ldots, s_n \in S \cup S^{-1}$. Consider the sequence of $n + 1$ integers

$$(0, \alpha(\sigma(s_1)), \alpha(\sigma(s_2s_n)), \ldots, \alpha(\sigma(s_2 \ldots s_n)), 0 = \alpha(\sigma(w))),$$

where $\alpha$ again denotes the projection from $G$ to $\mathbb{Z}$. Either at most $n/2$ of them are positive or at most $n/2$ are negative. Assume without loss of generality that at most $n/2$ are positive. But consecutive members of the sequence differ by at most $z$. It follows that, for each $i \in \mathbb{N} \cup \{0\}$, at most $n/2 - 2i$ are greater than $iz$.

Decompose $\phi(w)$ into the generalized $t$-words $v^+$ and $v^-$, where $v^+$ consists of all letters of $\phi(w)$ coming from letters $s_i$ of $w$ where

$$\alpha(\sigma(s_{i+1} \ldots s_n)) > 0$$

and $v^-$ consists of all other letters of $\phi(w)$. Let $N$ denote the maximal length (as a generalized $t$-word) of the $\phi(s')$ for all $s' \in S \cup S^{-1}$. Let

$$I = \max \{ I_{\max}(\phi(s')) \mid s' \in S \cup S^{-1} \}.$$ 

Then the length of $v^-$ is at most $Nn$ and $I_{\max}(v^-) \leq I$. Let $C$ be as in the definition of $t$-logarithmic. Then $I_{\max}(\pi(v^-)) \leq I + C(\log_2(Nn) + 1)$.

But, by the first paragraph and the definitions of $v^+$, $N$ and $I$, for each $i \in \mathbb{Z}$, at most $N(n/2 - 2i)$ letters of $v^+$ are in $\bigcup_{j=\lfloor nz/4 + j + 1 \rfloor}^{\infty} t^jA$. Let $k = [n/4 - i + 1]$ in the preceding sentence. Then, for each $k \in \mathbb{Z}$, at most $2Nk$ letters of $v^+$ are in $\bigcup_{j=\lfloor nz/4 + j + 1 - k \rfloor}^{\infty} t^jA$.

Apply Lemma 4.3 with $B = 2N/z$. Note that $B$ is independent of $w$ and $n$. Then there are $D \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ also independent of $w$ and $n$ such that more than $2Nm/z$ letters of $v^+$ are in $\bigcup_{j=\lfloor nz/4 + D - m \rfloor}^{\infty} t^jA$.

Thus there is $k = [m/z]$ such that more than $2Nk$ letters of $v^+$ are in

$$\bigcup_{j=I_{\max}(\pi(v^+)) - D - z - k}^{\infty} t^jA.$$ 

By the preceding paragraph, this is only possible if

$$I_{\max}(\pi(v^+)) - D - z - k < nz/4 + z + I + 1 - k,$$

that is if $I_{\max}(\pi(v^+)) < nz/4 + I + D + 2z + 1$.

But, by $t$-logarithmic,

$$I_{\max}(\sigma(w)) = I_{\max}(\pi(\phi(w)))$$

$$= I_{\max}(\pi(v^+) + \pi(v^-)) \leq \max(I_{\max}(\pi(v^+)), I_{\max}(\pi(v^-))) + C$$

$$\leq \max \left( \frac{nz}{4} + I + D + 2z + 1, I + C(\log_2(Nn) + 1) \right) + C < \frac{nz}{4} + F,$$
where $C$ is again as in the definition of $t$-logarithmicity and $F$ is a constant depending only, via $z$, $I$, $D$, $C$ and $N$, on $K$, $R$ and $S$ (not on $n$ or $w$).

We are now ready for the

Proof of Theorem 2.4 Let $a$ again denote the projection from $G$ to $Z$. Let $a \neq 0 \in A$ and let $s \in S \cup S^{-1}$ be chosen such that $\alpha(s) \geq \alpha(s')$ for all $s' \in S \cup S^{-1}$. Let $t = \alpha(s)$. For ease of notation, let $C$ be the greater of the $C$s from the definitions of $t$-efficiency and $t$-logarithmicity.

For $s \in N \cup \{0\}$ and $i \in Z$, let $g_n(i) = s^{n+i}a as^{-n} = s^{-n}a as^{n}$. Then, for $|i| \leq n$, $\|g_n(i)\| \leq 4n - |i| + 2\|s\|_S$.

For $s \in N \cup \{0\}$, define $h_n^+ = g_n(J)$ and $h_n^- = g_n(-J)$, where $J \in N \cup \{0\}$ is a constant to be chosen later. Let $n \geq J$. We have

$$\|h_n^+\| \leq 4n - J + 2\|s\|_S$$

and

$$\|h_n^-\| \leq 4n - J + 2\|s\|_S.$$  

Since $h_n^+(h_n^-)^{-1} = s^{2J}$, we have

$$\|h_n^+(h_n^-)^{-1}\| \leq 2J.$$  

Note that this depends only on our choice of $J$. Let $N$ be the length of $\phi(s)$ as a generalized $t$-word. Let $v_n = \phi(s^{n+J}a as^{-n}) = \phi(s^{-n}a as^{n})$. Then $t^na + t^{-n}a$ is a $t$-word representing the same element as $s_n$, and it is of minimal length so long as $n > C$. We can ensure this by picking $J > C$. Thus $I_{max}(v_n) - C = nz - C \leq I_{max}(h_n^+)$.

Every edge of the (left) Cayley graph of $G$ connects some $g_1$ and $g_2 \in G$ with $|\alpha(g_1) - \alpha(g_2)| \leq z$. Thus any path in the (left) Cayley graph of $G$ connecting $h_n^-$ and $h_n^+$ must contain some $g \in G$ with $|\alpha(g)| < z$. Suppose $\|g\|_S \leq 4n - J + 2\|s\|_S$. Then Corollary 4.2 says that either $I_{max}(g) \leq nz - Jz/4 - \|s\|_S z/2 + F$ or $I_{min}(g) \geq Jz/4 + \|s\|_S z/2 - nz - F$, where $F$ is as in the corollary. Without loss of generality, we assume the former.

Since $A$ is $t$-logarithmic,

$$I_{max}(h_n^+) \leq \max(I_{max}(h_n^+ g^{-1}) + \alpha(g), I_{max}(g)) + C.$$  

We want to choose $J$ so that

$$I_{max}(h_n^+) > I_{max}(g) + C.$$  

Since $I_{max}(g) + C \leq nz - Jz/4 - \|s\|_S z/2 + F + C$, it will suffice to take

$$nz - Jz/4 - \|s\|_S z/2 + F + C < nz - C,$$

that is $J > (4z)(F - \|s\|_S z/2 + 2C)$. Note that this is independent of $n$. Then we will have $I_{max}(g) + C < nz - C \leq I_{max}(h_n^+)$, as desired. It will follow that

$$nz \leq I_{max}(h_n^+) + C \leq I_{max}(h_n^+ g^{-1}) + \alpha(g) + 2C.$$  

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that is \( I_{\text{max}}(h_n^+g^{-1}) \geq nz - \alpha(g) - 2C > nz - z - 2C \).

Let \( w \in F_S \) be of minimal length in \( \sigma^{-1}(h_n^+g^{-1}) \). For \( v \) a generalized \( t \)-word in \( A \) and \( i \in \mathbb{Z} \), let \( v_i \) be the number of letters of \( v \) in \( t^iA \). Let \( M = \sum_{i=-\infty}^{\infty} \max_{s' \in S \cup S^{-1}} \| \phi(s') \|_{t^iA} \); this sum is finite since \( S \) is finite and, for each \( s' \in S \cup S^{-1} \), \( \phi(s') \) has finitely many letters. Let \( B_{M,0} \) and \( C_{M,0} \) be as in the definition of strong \( t \)-logarithmicity. As in the proof of Corollary 4.2, let \( N \) denote the maximal length (that is number of nonzero terms) of the \( \phi(s') \) for all \( s' \in (\cup S^{-1}) \). Let \( I \) denote the maximal \( I_{\text{max}}(\phi(s')) \) for all such \( s' \). Then

\[
l(w) \geq \frac{\| \phi(w) \|_M}{N} + \frac{2I_{\text{max}}(\phi(w)) - I - \left| \alpha(h_n^+g^{-1}) \right|}{z}.
\]

By strong \( t \)-logarithmicity and the result of two paragraphs ago,

\[
\| \phi(w) \|_M > e^\frac{I_{\text{max}}(h_n^+g^{-1}) - I_{\text{max}}(\phi(w)) - C_{M,0}}{B_{M,0}} - 1.
\]

Also, \( |\alpha(h_n^+g^{-1})| \leq |\alpha(h_n^+) + \alpha(g)| < J + z \). Putting this all together, we have

\[
\|h_n^+g^{-1}\|_S > e^\frac{nz - I_{\text{max}}(\phi(w)) - z - 2C - C_{M,0}}{B_{M,0}} + 2(I_{\text{max}}(\phi(w)) - nz) \cdot \frac{I + J}{z} - 2.
\]

This expression is bounded below as a function of \( nz - I_{\text{max}}(\phi(w)) \), say by \( F \in \mathbb{Z} \). Note that \( F \) depends only on \( A, S \) and the one-time choices we made in defining the map \( \phi \) (provided \( J \) is chosen appropriately). In particular, it is independent of \( n \). So \( \|h_n^+g^{-1}\|_S \geq 2n + F \). Since \( 2n + F \) goes to infinity as \( n \) does, we are done.

\[\square\]

### 5 Bounded depth

This section is devoted to the proof of Theorem 2.3. To prove this theorem, we will use a general lemma about groups obtained as the semidirect product of an abelian group with \( Z \). We begin with the following

**Definition 8.** Let \( G \) be an indicable group and let \( \phi: G \to Z \). Let \( K = \ker \phi \). Fix a splitting \( \alpha \) for \( \phi \), so every element of \( G \) can be expressed uniquely as a product \( \alpha(n)k \), \( k \in K \), \( n \in Z \). Let \( A \) be a generating set for \( G \). Then we call \( A \) symmetrized about \( Z \) if \( \alpha(n)k \in A \) if \( \alpha(n)k^{-1} \in A \) for all \( k \in K \) and \( n \in Z \).

**Notation.** If \( A \) is symmetrized about \( Z \), then we have an involution on \( A \), which we denote with an overbar; thus \( \alpha(n)k = \alpha(n)k^{-1} \).

We extend the map \( \tau \) to \( A^{-1} \) so that it will be an involution on \( A \cup A^{-1} \). We then extend it to \( F_A \) so that

\[\bar{a_1a_2 \ldots a_m} = a_1a_2 \ldots a_m \]

for \( m \in \mathbb{N} \) and \( a_1, \ldots, a_m \in A \cup A^{-1} \).
Lemma 5.1. Let $G$ be an indicable group and let $\phi : G \to \mathbb{Z}$ with $K = \ker \phi$ abelian. Let $S$ be a generating set for $G$ symmetrized about $\mathbb{Z}$ with respect to the splitting $\alpha$ and let $\pi : F_S \to G$. Let $w \in \pi^{-1}(K)$. Then $\pi(w) = \pi(w^{-1})$.

Proof. Let $w = \alpha(n_1)k_1 \ldots \alpha(n_m)k_m$ with $n_1, \ldots, n_m \in \mathbb{Z}$, $k_1, \ldots, k_m \in K$ and $\alpha(n_1)k_1, \ldots, \alpha(n_m)k_m \in S \cup S^{-1}$. Then $\pi(w) = \sum_{j=1}^m k_j \sum_{i=1}^{n_i} n_i$, where we use additive notation because $K$ is abelian. (We continue to denote the action of $\mathbb{Z}$ by exponentiation in order to avoid confusion with the natural action of $\mathbb{Z}$ on $K$ as an abelian group.)

On the other hand,

$$\pi(w^{-1}) = \pi(k_m \alpha(-n_m) \ldots k_1 \alpha(-n_1)) = \sum_{j=1}^m k_j^{-1} \sum_{i=1}^{n_i} n_i = \sum_{j=1}^m k_j \sum_{i=1}^{n_i} n_i,$$

where the last inequality is because $\sum_{i=1}^m n_i = 0$ since $w \in \pi^{-1}(K)$. Since this is the same as $\pi(w)$, we are done. \(\square\)

Lemma 5.2. Let $G$ be an indicable group and let $\phi : G \to \mathbb{Z}$ with $K = \ker \phi$ abelian. Let $S$ be a generating set for $G$ symmetrized about $\mathbb{Z}$ and let $\pi : F_S \to G$. Then there exist $B$ and $C \in \mathbb{N}$ with the following property. Let $g \in G$ and $l \in \mathbb{N} \cup \{0\}$ with $l \leq \phi(g)$. Then there exist $w$, $w_1$ and $w_2 \in F_S$ with the following properties:

- $w = w_1w_2$ as words (that is without any cancellation),
- $\pi(w) = g$,
- $l(w) \leq \|g\|_S + C$,
- $|\phi(\pi(w_2)) - l|, |\phi(\pi(w_1)) + l - \phi(g)| \leq B$ and
- if $v$ is a prefix of $w_2$ or a suffix of $w_1$ then $\phi(\pi(v)) \leq 2B$.

Proof. Let $B = \max_{s \in S} |\phi(s)| + 1$. Let $w' \in F_S$ with $l(w') = \|g\|_S$ and $\pi(w') = g$. Let $T$ be the set of all $w'' \in F_S$ such that there exist $w_1$ and $w_2 \in F_S$ with the following properties:

- $w'' = w_1w_2w_r$ as words,
- $|\phi(\pi(w_r)) - l| \leq B$,
- $|\phi(\pi(w''w_r)) - l| \leq B$ and
- there is no nonempty proper suffix $w_a$ of $w''$ with $|\phi(\pi(w_a)) - l| \leq B$.

Then $w'$ decomposes uniquely as $w_aw_1'' \ldots w_k''w_b$, where

- $k \in \mathbb{N} \cup \{0\}$,
- $T = \{w_i'' \mid 1 \leq i \leq k\}$,
- $w_b$ is the minimal suffix of $w'$ with $|\phi(\pi(w_b)) - l| \leq B$ and
- $w_a$ is the minimal prefix with $|\phi(\pi(w_a^{-1}w')) - l| \leq B$. 

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Choose $j_0, j_1, \ldots, j_k \in \pi^{-1}(\alpha(\{-B, \ldots, B\}))$ such that

$$\phi(\pi(j_k^{-1} w_b)) = l$$

and $\pi(j_{k-1}^{-1} w''_b j_1) \in K$. Let $w'_a = w_{a,j_0}$, $w''_i = j_{i-1}^{-1} w''_i j_i$ and $w''_b = j_k^{-1} w_b$. Then

$$\pi(w'_a w''_i \ldots w''_b w'_b) = \pi(w_{a,j_0} j_0^{-1} w''_j j_1^{-1} \ldots j_k^{-1} w_k j_k^{-1} w_b) = \pi(w_a w''_1 \ldots w''_k w_b) = g,$$

and the $w''_i$ commute with each other, since $K$ is abelian.

For $m, n \in \{-B, \ldots, B\}$, let $w''_m \in T$ be an $(m,n)$-word if $\pi(j_i) = \alpha(m)$ and $\pi(j_{i-1}) = \alpha(n)$. Let $W_{m,n}$ be the (possibly multi-) set of all $(m,n)$-words. Let the finite sequence $(w''_{m,n,i})$ be an enumeration of $W_{m,n}$. Let

$$w_{m,n} = w'_a \prod_{0 < m,n \leq B} w_{m,n} \prod_{|m|, |n| \leq B, m \leq n} w_{m,n} \prod_{-B \leq m,n < 0} w_{m,n} w''_b.$$

Then $\pi(w) = g$.

Let $J$ be at least the maximal length of any of the $j_i$; $J$ clearly can be taken independent of $g$. The length of $w_{m,n}$ is at most $2J + \text{the sum of the lengths of all the } (m,n)\text{-words in } W_{m,n}$. Then

$$l(w) \leq l(w') + 2 \left[(2B + 1)^2 + 1\right] J = ||g||_S + 2 \left[(2B + 1)^2 + 1\right] J.$$

Then we are done if we set $C = 2 \left[(2B + 1)^2 + 1\right] J$ and let $w_1$ and $w_2$ be as marked in (1). Note that $C$ is also independent of $g$.

**Corollary 5.3.** Let $G$ be a group such that there is $\phi: G \to \mathbb{Z}$ with ker $\phi$ abelian. Let $S$ be a generating set for $G$ symmetrized about $\mathbb{Z}$ and let $\pi: F_S \to G$. Then there exist $B$ and $C \in \mathbb{N}$ with the following property. Let $g \in G$ with $\phi(g) \geq 0$. Let $l_1, l_2 \in \mathbb{Z}$ with $0 \leq l_1 \leq l_2 \leq \phi(g)$. Then there exists $w, w_1, w_2$ and $w_3 \in F_S$ with the following properties:

- $w = w_1 w_2 w_3$ as words,
- $\pi(w) = g$,
- $l(w) \leq ||g||_S + C$,
- $|\phi(\pi(w_3)) - l_1|, |\phi(\pi(w_2)) + l_1 - l_2|$ and $|\phi(\pi(w_1)) + l_2 - \phi(g)| \leq B$ and
- if $v$ is a prefix of $w_2$ or $w_3$ or a suffix of $w_1$ or $w_2$ then $\phi(\pi(v)) \leq 2B$.
Proof. Let $w'$, $w'_1$ and $w'_2$ be the words $w$, $w_1$ and $w_2$ from Lemma 5.2 with $l = l_1$. Let $w_r$ be the minimal suffix of $w'$ such that $\phi(\pi(w_r)) > l_1 + B$, where $a$ is the letter of $w'$ next to the left of $w_r$. Note that $w_r$ contains $w'_2$ by construction, so we can let $w_r = w'_r w'_2$. Let $w_1, w_{11}$ and $w_{12}$ be the words $w$, $w_1$ and $w_2$ from Lemma 5.2 with $g = \pi(w'w_r^{-1})$ and $l = l_2 - \phi(\pi(w_r))$. We are done if we let $B$ be as in Lemma 5.2 $C$ be twice the $C$ from Lemma 5.2 $w = w_1w_r$, $w_1 = w_{11}$, $w_2 = w_{12}w_r$ and $w_3 = w'_2$.

For $g \in G$, as in Section 4 we define

$$I_{\max}(g) = I_{\max}(\alpha(\phi(g))^{-1}g)$$

and

$$I_{\min}(g) = I_{\min}(\alpha(\phi(g))^{-1}g).$$

This makes sense since $\alpha(\phi(g))^{-1}g \in K$. As there, if $g$, $h \in G$ and $C$ is as in the definition of $t$-logarithmicity,

$$I_{\max}(gh) \leq \max(I_{\max}(g) + \phi(h), I_{\max}(h)) + C,$$

$$I_{\min}(gh) \geq \min(I_{\min}(g) + \phi(h), I_{\min}(h)) - C,$$

$$I_{\max}(g^{-1}) = I_{\max}(g) - \phi(g)$$

and

$$I_{\min}(g^{-1}) = I_{\min}(g) - \phi(g).$$

Furthermore, $I_{\max}(g^h) = I_{\max}(g) + \phi(h)$ and $I_{\min}(g^h) = I_{\min}(g) - \phi(h)$.

**Lemma 5.4.** Let $G$ be an indicable group and $\phi: G \to \mathbb{Z}$ with $K = \ker \phi$ abelian. Let $\alpha: \mathbb{Z} \to G$ be the splitting map. Let $t = \alpha(1)$, so that $t$ acts on $K$ by conjugation and makes it into a $\mathbb{Z} \cdot [t, t^{-1}]$-module. Let $A$ be a finite symmetrized $t$-efficient $t$-logarithmic $t$-generating set for $K$, with $0 \in A$. Let

$$S = \{tat^2a2t \mid a_1, a_2 \in A\} \cup A \cup \{t\}.$$

For every $B \in \mathbb{N}$ there is $L \in \mathbb{N}$ such that the following holds. Let $n \in \mathbb{N}$,

- $g$, $h$, $h_1$, $h_2$ and $h_3 \in G$ such that
  - $\phi(hg^{-1}) = 4n$,
  - $I_{\min}(hg^{-1}) \geq 0$,
  - $I_{\max}(hg^{-1}) \leq 4n$,
  - $h = h_1h_2h_3$,
  - $|\phi(g^{-1}h_3)| \leq B$,
  - $|\phi(h_1)| \leq B$,
  - $I_{\min}(h_1), I_{\min}(h_2) \geq -B$,
  - $I_{\max}(h_2) \leq 4n + B$ and
  - $I_{\max}(h_3) \leq \phi(g) + B$.

Then $\|g\|_S \leq \|h_1\|_S + \|h_3\|_S + 2n + L$.

Proof. Let $C$ be the constant with respect to which $A$ is $t$-logarithmic. Let $w_1, w_2 \in F_S$ be as given by Lemma 5.2 with $l = \phi(g)$. Let $g_1 = \pi(w_1)$ and $g_2 = \pi(w_2)$. Then (possibly after increasing $B$)
\begin{itemize}
\item $g = g_1g_2$.
\item $|\phi(g_1)| \leq B$.
\item $I_{\min}(g_1) \geq -B$ and
\item $I_{\max}(g_2) \leq \phi(g) + B$.
\end{itemize}

I claim that $\|g_2\|_S - \|h_3\|_S \leq F$. It follows from $t$-logarithmicity and the conditions on the $h_i$ and $g_i$ that

$$I_{\min}(h_1h_2g_i^{-1}) \geq \min(\min(I_{\min}(h_1) + \phi(h_2), I_{\min}(h_2)) - C - \phi(g_i), I_{\min}(g_i) - \phi(g_1)) - C$$

$$\geq \min(\min(4n - 3B, -B) - C - B, -2B - C)$$

$$\geq \min(-4B - C, -2B) - C = -D,$$

where $D \in \mathbb{N}$ depends only on $A$ and $B$. But

$$\|\phi(h_3g_2^{-1})\| \leq |\phi(g^{-1}h_3)| + |\phi(g_1)| \leq 2B.$$

Also,

$$h_3g_2^{-1} = [(h_1h_2g_1^{-1})^{-1}hg_i^{-1}]^g_1$$

and $\phi(h_1h_2g_1^{-1}) = \phi(h) - \phi(h_3) - \phi(g_1) \leq 4n + 2B$. Thus, we have

$$I_{\min}(h_2g_1^{-1}) = I_{\min}((h_1h_2g_1^{-1})^{-1}hg_i^{-1} + \phi(g_1))$$

$$\geq \min(I_{\min}(hg_i^{-1}), I_{\min}(h_1h_2g_1^{-1}) - \phi(h_1h_2g_1^{-1}) + \phi(h_1g_i^{-1})) - C + \phi(g_1)$$

$$\geq \min(0, -D - 2B) - C - B \geq -E,$$

where $E \in \mathbb{N}$ also depends only on $A$ and $B$. But, by $t$-logarithmicity,

$$I_{\max}(h_3g_2^{-1}) \leq \max(I_{\max}(h_3) - \phi(g_2), I_{\max}(g_2) - \phi(g_1)) + C$$

$$\leq \max(2B, 2B) + C \leq 2B + C \leq F,$$

where $F \in \mathbb{N}$ also depends only on $A$ and $B$. Thus there is $H \in \mathbb{N}$ (depending, via $C$, $D$, $E$, $F$ and the $t$-efficiency constant, only on $A$ and $B$) such that $\|h_3g_2^{-1}\|_S \leq F$. The claim follows.

For $j \in \{1, 2\}$ and $i \in \mathbb{Z}$, let the $a_{ji} \in A$ be such that

$$\alpha(\phi(g_1))^{-1}g_1 = \sum_{i=-B}^{\infty} t^i a_{1(i+\phi(g_2))},$$

$$\alpha(\phi(g_2))^{-1}g_2 = \sum_{i=-\infty}^{\phi(g)+B} t^i a_{2i},$$

and these are minimal-length $t$-words representing

$$\alpha(\phi(g_1))^{-1}g_1$$

and

$$\alpha(\phi(g_2))^{-1}g_2.$$
(For ease of notation, let all $a_j$, not referenced in the above sums be 0.) Then
\[ \sum_{i=1}^{4n-1} t^i a_1(i + \phi(g)) \]
and
\[ \sum_{i=1}^{4n-1} t^i a_{2i} \]
are minimal-length $t$-words, so so are
\[ \sum_{i=1}^{4n-1} t^i a_{1(i + \phi(g))} \]
and
\[ \sum_{i=1}^{4n-1} t^i a_{2(i + \phi(g))}. \]

Let $J$ be the constant referred to as $C$ in the definition of $t$-efficiency. For each $n \in \mathbb{N}$, since $A$ is $t$-efficient and $t$-logarithmic, there are $w_r(n)$ and $w_l(n) \in F_\pi$ such that $l(w_r(n)), l(w_l(n)) \leq n + C + J + 3, \phi(w_r(n)) = 4n$, $\phi(w_l(n)) = -4n$ and
\[ \pi(w_l(n)) = \sum_{i=1}^{4n-1} t^i(a_1(i + \phi(g)) + a_2(i + \phi(g))). \]

Let $v_1 \in \pi^{-1}(i^{-\phi(h_1)})$. For $i \in \mathbb{Z}, i \geq -B$, let $h_{1i} \in A$ be such that $h_1 = \sum_{i=-B}^{\infty} t^i h_{1i}$ and this is a minimal-length $t$-word representing $h_1$; this is possible by the assumption (in the hypothesis of the lemma) that $I_{min}(h_1) \geq -B$. Let
\[ v_2 \in \pi^{-1}\left( \sum_{i=0}^{\infty} t^i(a_1(i + \phi(g) + 4n) + a_2(i + \phi(g) + 4n)) - \sum_{i=-\phi(h_1) - B}^{\infty} t^i h_{1(i + \phi(h_1))} \right). \]

By $t$-logarithmicity, $I_{min}(\pi(v_2)) \geq \min(0, -\phi(h_1) - B) - C \geq -2B - C$. Although the indices of both sums individually go to infinity, note that
\[ \pi(t^{\phi(g) + 4n} v_2) = \sum_{i=\phi(g) + 4n}^{\infty} t^i(a_{1i} + a_{2i}) - \sum_{i=-B}^{\phi(g) + 4n - \phi(h_1)} t^i a_{1i} + a_{2i} \]
\[ = \alpha(\phi(g))^{-1} g - \sum_{i=-\infty}^{\phi(g) + 4n - \phi(h_1)} t^i a_{1i} + a_{2i} \]
\[ - t^{\phi(g) + 4n - \phi(h_1)} \alpha(\phi(h_1))^{-1} h_1 + \sum_{i=\infty}^{1-B} t^{\phi(g) + 4n - \phi(h_1)} h_{1i} \]
\[ = \alpha(\phi(g))^{-1} g - t^{\phi(h_2 h_3)} \alpha(\phi(h_1))^{-1} h_1 + P \]
\[ = t^{\phi(g)} \alpha(\phi(h_1))^{-1} h_2 + \alpha(\phi(h_2 h_3))^{-1} h_2 h_3 + P, \]

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where \( P \in K \). Note that, by \( t \)-logarithmicity (and recalling that \( |\phi(h_1)| \leq B \)), \( I_{\max}(P) \leq \phi(g) + 4n + 1 + 2C \). It follows (again by \( t \)-logarithmicity) that

\[
I_{\max}(\pi(v_2)) \\
\leq \max(I_{\max}(hg^{-1} + \phi(g)), I_{\max}(h_2h_3), \phi(g) + 4n + 1 + 2C) + 2C - \phi(g) - 4n \\
\leq \max(\phi(g) + 4n + 1 + 2C, \phi(g) + 4n + 2B + C) - \phi(g) - 4n + 2C \\
\leq \max(4C + 1, 2B + 3C);
\]

in particular, it has a bound depending only on \( A \) and \( B \) (not \( n \)). Finally, let

\[
v_3 \in \pi^{-1} \left( \alpha(\phi(g_1)) \left[ \sum_{i=-B}^{\phi(g_1)} t^i a_1(i+\phi(g_2)) - \sum_{i=\phi(g_1)+1}^{\phi(g_2)} t^i a_2(i+\phi(g_2)) \right] \right).
\]

Recall that \( |\phi(g_1)| \leq B \) also.

Clearly,

\[
\phi(\pi(w_1(n)))^{-1} h_1^{\pi(v_3)} \pi(v_2w_r(n)v_3) \\
= \phi(\pi(w_1(n))) - \phi(h_1) + \phi(h_1) + \phi(\pi(v_2)) + \phi(\pi(w_r(n))) + \phi(v_3) \\
= 4n - \phi(h_1) - 4n + \phi(g_1) = \phi(g_1).
\]
Also,

\[
\alpha(\phi(g_1))^{-1} \pi(w_1(n)) \alpha(\phi(h_1))^{-1} h_1^{\nu(v_1)} \pi(v_2 w_r(n) v_3) = \\
= \sum_{i = 1}^{B} t^i a_1(i + \phi(g_2)) - \sum_{i = \phi(g_1) + 1}^{B} t^i a_2(i + \phi(g_2)) \\
+ \sum_{i = 1}^{4n - 1} t^i \phi(g_1) (a_1(i + \phi(g)) + a_2(i + \phi(g))) \\
+ \sum_{i = 0}^{\infty} t^i \phi(g_1 + 4n) (a_1(i + \phi(g) + 4n) + a_2(i + \phi(g) + 4n)) \\
- \sum_{i = -\phi(h_1) - B}^{\infty} t^i \phi(g_1) (a_1(i + \phi(g_2)) - \sum_{i = \phi(g_1) + 1}^{B} t^i a_2(i + \phi(g_2))) \\
+ \sum_{i = \phi(g_1) + 4n}^{\infty} t^i (a_1(i + \phi(g_2)) + a_2(i + \phi(g_2))) \\
- \sum_{i = \phi(g_1) + 4n - \phi(h_1) - 4n}^{\infty} t^i h_1(i + \phi(h_1) - \phi(g_1) - 4n) \\
+ \sum_{i = \phi(g_1) + 4n - \phi(h_1) - 4n}^{\infty} t^i h_1(i + \phi(h_1) - \phi(g_1) - 4n) \\
= \sum_{i = -B}^{\infty} t^i a_1(i + \phi(g_2)) = \alpha(\phi(g_1))^{-1} g_1.
\]

Thus

\[
\pi(w_1(n)) \alpha(\phi(h_1))^{-1} h_1^{\nu(v_1)} \pi(v_2 w_r(n) v_3) = g_1.
\]

For all \(i \in \{1, 2, 3\},\)

\[
|\phi(\pi(v_i))| \leq B,
\]

\[
I_{\max}(\pi(v_i)) \leq B + 2C
\]

and

\[
I_{\min}(\pi(v_i)) \geq -2B - C.
\]

Thus there is \(I \in \mathbb{N}\) (again depending only on \(A\) and \(B\)) such that all the \(v_i\) are of length (as words in \(F_S\)) \(\leq I\). Possibly increasing \(I\), we may arrange that \(\|\alpha(\phi(h_1))\|_S \leq I\) as well; since \(|\phi(h_1)| \leq B\) by hypothesis, \(I\) still depends only on \(A\) and \(B\). It follows that

\[
\|g_1\|_S \leq \|h_1\|_S + 5I + 8 + 2n + 2C + 2J.
\]
Putting this all together, we get
\[
\|g\|_S \leq \|g_1\|_S + \|g_2\|_S \\
\leq \|h_1\|_S + 5I + 8 + 2n + 2C + 2J + \|h_3\|_S + F \leq \|h_1\|_S + \|h_3\|_S + 2n + L,
\]
where we take \( L = 5I + 8 + 2C + 2J + F \).

 Lemma 5.5. Let \( G \) be an indicable group and \( \phi: G \to \mathbb{Z} \) with \( K = \ker \phi \) abelian. Let \( \alpha \) be the splitting map. Let \( t \) represent the image of \( 1 \in \mathbb{Z} \) under \( \alpha \), so that \( t \) acts on \( K \) by conjugation and makes it into a \( \mathbb{Z} [t, t^{-1}] \)-module. Let \( A \) be a finite \( t \)-efficient \( t \)-logarithmic \( t \)-generating set for \( K \). Let \( S \) be a finite generating set for \( G \), symmetrized about \( \mathbb{Z} \). Let \( \pi: F_S \to G \). Then there is a \( C \subset \mathbb{N} \) with the following property. Let \( g \in G \) and let \( w \in F_S \) be of minimal length in \( \pi^{-1}(g) \). Let \( v = \beta(w) \in N_A^k \), where \( \beta \) is the same as \( \tilde{\phi} \) in Section 4. Then for every \( i \in \mathbb{Z} \), the length (with respect to \( A^i \)) of the component of \( v \) in \( \mathbb{Z} A^i \) is \( \leq C \).

Remark. It follows that, under the above conditions, there is a bound on \( |I_{\text{max}}(v) - I_{\text{min}}(g)| \) and \( |I_{\text{max}}(v) - I_{\text{min}}(g)| \).

Proof. Since \( S \) is finite, it will suffice to bound \( n \in \mathbb{N} \) such that there exist \( s \in S \cup S^{-1} \) and \( w_0, w_1, \ldots, w_n \in F_S \) with \( w = w_0sw_1s \ldots sw_{n-1}sw_n \) and \( \phi(w_1) = \cdots = \phi(w_{n-1}) = \phi(s) \).

Suppose \( n, s, w_0, \ldots, w_n \) are as above. We may assume without loss of generality that \( n \) is even; this will simplify the notation. We will bound \( n \). By Lemma 5.1
\[
w_0sw^{-1}w_1^{-1}w_2sw^{-1}w_3^{-1} \ldots sw_{n-1}^{-1}w_n \in \pi^{-1}(g)
\]
also, and it is of minimal length since \( w \) is. But \( s \pi^{-1} \in K \) and, for any \( i \) with \( 1 \leq i \leq n/2 - 1 \), \( \pi(w_{2i-1}^{-1}w_{2i}) \in K \), since \( \phi(\pi(w_{2i-1})) = \phi(\pi(w_{2i})) \).

Since \( K \) is abelian,
\[
w_0w_1^{-1}w_2w_3^{-1}w_4 \ldots w_{n-3}^{-1}w_{n-2} \cdot (s \pi^{-1}) \cdot w_{n-1}^{-1}w_n \in \pi^{-1}(g)
\]
again, and again it is of minimal length since its length is the same as that of \( w \). But it follows from the \( t \)-logarithmicity of \( A \) that, for sufficiently large \( n \), \( (s \pi^{-1})^{n/2} \) is not of minimal length. The lemma is thus proven.

We next prove the following proposition, from which Theorem 2.5 follows trivially.

Proposition 5.6. Let \( G \) be an indicable group and \( \phi: G \to \mathbb{Z} \) with \( K = \ker \phi \) abelian. Let \( \alpha: \mathbb{Z} \to G \) be the splitting map. Let \( t = \alpha(1) \), so that \( t \) acts on \( K \) by conjugation and makes it into a \( \mathbb{Z} [t, t^{-1}] \)-module. Let \( A \) be a finite symmetrized \( t \)-efficient \( t \)-logarithmic \( t \)-generating set for \( K \), with \( 0 \in A \). Let
\[
S = \{ ta_1t^2a_2t \mid a_1, a_2 \in A \} \cup A \cup \{ t \}.
\]
For all \( i \in \mathbb{N} \), let \( k_i \) be as in the statement of Theorem 2.5. Then there is \( n \in \mathbb{N} \) with the following property. Let \( g \in G \) with \( \phi(g) \geq 0 \). Let \( C \) be the greatest of
- \( B \) from Lemma 0.7.6

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• $B$ from Corollary 5.2
• the $t$-logarithmicity constant,
• the $t$-efficiency constant and
• the bound from the remark following the statement of Lemma 5.6

Let $h_{g}(n) \in G$, $v$, $v_1$ and $v_2$ $t$-words and $a_i, b_i \in A$ for $i \in \mathbb{Z}$ have the following properties:

- $\phi(h_{g}(n)) = \phi(g) + 4n,$
- $v$ represents $\alpha(\phi(h_{g}(n)g^{-1}))^{-1}h_{g}(n)g^{-1},$
- $I_{\min}(v) \geq 0,$
- $I_{\min}(v) \leq 4n,$
- $v_1 + v_2$ represents

$$\alpha(\phi(h_{g}(n)))^{-1}h_{g}(n) - t^{\phi(g) + C - 1}k_{4n - 2C + 2},$$

- $I_{\max}(v_1) < \phi(g) + C$ and
- $I_{\min}(v_2) > \phi(g) + 4n - C.$

Then $\|h_{g}(n)\|_{S} > \|g\|_{S}.$

**Proof.** Let $w$, $w_1$ and $w_2 \in F_S$ be the words given by Lemma 5.2 with $l = \phi(g)$. For $n \in \mathbb{N}$, let $(w(n), w_1(n), w_2(n))$ and $w_3(n) \in F_S$ be the words given by Corollary 5.3 with $g = h_{g}(n)$, $l_1 = \phi(g)$ and $l_2 = \phi(g) + 4n = \phi(h_{g}(n)).$ We remind the reader of three salient facts about these words, namely

- no prefix or suffix $w'$ of any $w_i$ or $w_i(n)$, except possibly a suffix of $w_2$ or $w_3(n)$ or a prefix of $w_1(n)$, has $\phi(\pi(w')) < -2C$,
- $|\phi(\pi(w_2)) - \phi(g)|, |\phi(\pi(w_3(n))) - \phi(g)| \leq C$ and
- $|\phi(\pi(w_2(n))) - 4n| \leq 2C.$

For $i \in \{0, 1, 2\}$, let $m_i(n)$ be the number of letters of $w_2(n)$ with $i$ of $a_1, a_2$ nontrivial, where $a_1$ and $a_2$ are as in the definition of $S$. Let $\beta$ be as in Lemma 5.4. Let $z(n) \in \mathbb{N}$ be the number of $i \in \{\phi(g), \ldots, \phi(g) + 4n - 1\}$ such that the coefficient of $\beta^{i-\phi(w_2(n))}$ in $\beta(w_2(n))$ is 0. Then

$$4n - z(n) \leq m_1(n) + 2m_2(n) = 2\|\pi(w_2(n))\|_{S} - 2m_0(n) - m_1(n) + 2C,$$

where for the latter inequality we remind the reader that $w_2(n)$ is within $C$ of minimal length. It follows that $m_0(n) + m_1(n) \leq 2m_0(n) + m_1(n) \leq 2\|\pi(w_2(n))\|_{S} - 4n + z(n) + 2C.$

The word $w_2(n)$ corresponds to a path between points of $Z$ connecting $\phi(\pi(w_3(n)))$ with $\phi(\pi(w_2(n)w_3(n))).$ For $i \in \{0, 1, 2\}$, we let $s_i(n)$ be the set of edges of this path corresponding to letters $b \in S \cup S^{-1}$ with $i$ of $a_1, a_2$ nontrivial. Thus $s_i(n)$ has cardinality $m_i(n).$ By a stretch we mean a maximal (under inclusion) set $T$ of adjacent elements of

$$\{l + 1, \ldots, l + 4n - 1\}$$

such that
• $T$ lies entirely between $l + C$ and $l + 4n - C$,
• for every $m \in T$, every edge of the path incident to or passing over $m$ is in $s_2(n)$ and
• for every $m \in T$, the coefficient of $t^{m-\phi(p_{w_1}(n))}$ in $\beta(w_2(n))$ is non-trivial.

The path must traverse each stretch an odd number of times, since it begins at $\phi(\pi(w_3(n)))$ and ends at $\phi(\pi(w_2(n)w_3(n)))$. But, if a stretch of $k$ integers is traversed at least three times, this will take at least $k/\phi$ letters of $S$. If a stretch is traversed only once, then by the second condition in the definition of a stretch the edges of this traverse are only incident to integers of one parity, say even. Thus, by the third condition in the definition of a stretch, the path must also, if the stretch has any even integers in it, either enter the stretch from one end, reach the integer one beyond the even integer in it furthest from that end, and return, or else enter the stretch from both ends and reach the same odd integer.

This will take at least $2(k/4 - 2)$ letters, which, added to the $k/4 - 2$ letters consumed by the one traverse, again makes $3(k/4 - 2)$ letters of $S$.

Let $N$ be the total number of integers in all stretches and $N_s$ the number of stretches. Then

$$N \geq 4n - z(n) - 5(m_0(n) + m_1(n)) - 2C$$

and

$$N_s \leq m_0(n) + m_1(n) + z(n) + 1.$$

Then we have (again using that $w_2(n)$ is within $C$ of having minimal length)

$$\|\pi(w_2(n))\|_S \geq \frac{3N}{4} - 6N_s - C$$

$$\geq 3n - \frac{7z(n)}{4} - \frac{39}{4}(m_0(n) + m_1(n)) - \frac{5C}{2} - 6$$

$$\geq 42n - \frac{39}{2}\|\pi(w_2(n))\|_S \cdot \frac{23z(n)}{2} - 22C - 6.$$

We thus get $E \in \mathbb{N}$ (depending, via $C$, only on $A$) such that

$$\|\pi(w_2(n))\|_S \geq \frac{84n}{41} - \frac{23z(n)}{41} - E.$$

I claim there is $F \in \mathbb{N}$ independent of $n$ such that

$$t^{\phi(g) + C - 1}k_{4n - 2C + 2} - t^{\phi(\pi(w_3(n)))}a(\phi(\pi(w_2(n))))^{-1}\pi(w_2(n))$$

is represented by a generalized $t$-word of length at most $F$. To see this, note that, by assumption, $v_1$ and $v_2$ are $t$-words such that $-v_1 - v_2$ represents

$$t^{\phi(g) + C - 1}k_{4n - 2C + 2} - t^{\phi(\pi(w_3(n)))}a(\phi(\pi(w_2(n))))^{-1}\pi(w_2(n))$$

$$- a(\phi(\pi(w_3(n))))^{-1}\pi(w_3(n))$$

$$- t^{\phi(\pi(w_2(n)w_3(n)))}a(\phi(\pi(w_1(n))))^{-1}\pi(w_1(n)).$$
But, by assumption and Lemma 5.5, there is a $t$-word $v_3$ with $I_{\text{max}}(v_3) \leq \phi(g) + 2C$ representing $\alpha(\phi(\pi(w_3(n))))^{-1}\pi(w_3(n))$. Also, there is a $t$-word $v_4$ with $I_{\text{min}}(v_4) \geq \phi(g) + 4n - 2C$ representing $\alpha(\phi(\pi(w_4(n))))^{-1}\pi(w_4(n))$.

Let $v_5$ and $v_6$ be minimal-length $t$-words representing the same element as $v_3 - v_1$ and $v_4 - v_2$ respectively. Then

$$I_{\text{max}}(v_5) \leq \phi(g) + 4C$$

and

$$I_{\text{min}}(v_6) \geq \phi(g) + 4n - 4C.$$  

Thus, for sufficiently large $n$, $v_5 + v_6$ is a minimal-length (by $t$-efficiency) $t$-word representing $\alpha(\phi(\pi(w_2(n))))^{-1}\pi(w_2(n))$.

But, by the first two conditions of Theorem 2.5 and Lemma 5.5, this means $I_{\text{min}}(v_5 + v_6) \geq \phi(g) - 4C$ and $I_{\text{max}}(v_5 + v_6) \leq \phi(g) + 4n + 4C$. Since $C$ is independent of $n$, the claim follows.

By the last condition of Theorem 2.5, it follows that there are $I, J \in \mathbb{N}$ depending only on $A$ such that $23z(n)/41 \leq I(\|\pi(w_2(n))\|_S - 2n) + J$. Then $\|\pi(w_2(n))\|_S \geq 84n/41 + 2In - I(\|\pi(w_2(n))\|_S - E - J$, so

$$\|\pi(w_2(n))\|_S \geq \frac{84n + 2In - E - J}{1 + I} \geq Kn - L,$$

where $K > 2$ and $L$ depend only on $A$. Putting this together with Lemma 5.4 yields that

$$\|g\|_S \geq \|\pi(w_1(n))\|_S + \|\pi(w_3(n))\|_S + 2n + H$$

$$\leq \|h_g(n)\|_S - \|\pi(w_2(n))\|_S + 2n + C + H$$

$$\leq \|h_g(n)\|_S - (K - 2)n + C + H + L,$$

where $H$ is the $L$ from the statement of Lemma 5.4 and thus also depends only on $A$. Rearranging yields $\|h_g(n)\|_S \geq \|g\|_S + (K - 2)n - C - H - L > \|g\|_S$ for sufficiently large $n$, where the definition of “sufficiently large” depends only on $A$.

### 6 Hyperbolic actions on abelian groups

Suppose $K$ to be any nontrivial finite-rank torsion-free abelian group, and let $t$ act by a hyperbolic automorphism $T$ whose matrix has integer coefficients. We will construct a finite strongly $t$-logarithmic $t$-generating set for $K$. Theorem 2.4 will then give us a generating set for $K \rtimes \langle t \rangle$ with respect to which it has unbounded depth. Also, Theorem 2.3 will tell us $K \rtimes \langle t \rangle$ is not almost convex for any generating set.

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6.1 Construction of the $t$-generating set

To construct our $t$-generating set, we let $A$ be a basis for a maximal-rank lattice in $K$. Then the $t$-generating set will be a certain finite set of words in $A$. We will choose this set of words so that it will be a $t$-generating set.

**Lemma 6.1.** Let $K$ be a real finite-dimensional vector space and let $T$ be an automorphism of $K$. Suppose the (complex) eigenvalues of $K$ all have absolute value $> 1$. Let $B$ be a basis of $K$ and let $\overline{B}$ be the (closed) convex hull of $B \cup B^{-1}$. Let $\|\cdot\|$ be a norm on $K$. Then there are $C_1 > 0$ and $C_2 > 1$ such that, for all $n \in \mathbb{N}$, $T^nB$ contains the ball of radius $C_1 C_2^n$ with respect to $\|\cdot\|$.

**Proof.** Since all norms on a finite-dimensional vector space are equivalent, we may restrict attention to a norm such that $\|T^{-1}\| < 1$; such a norm exists since all (complex) eigenvalues of $T$ have absolute value $> 1$, so all eigenvalues of $T^{-1}$ have absolute value $< 1$.

Let $C_1 > 0$ be such that $\overline{B}$ contains all $k \in K$ with $\|k\| \leq C_1$; this is possible since $B$ is a basis and $K$ is finite-dimensional. Let $n \in \mathbb{N}$ and $k \in K$ be such that $\|k\| < C_1/\|T^{-1}\|^n$. Then I claim that $k \in T^nB = T^n\overline{B}$.

This will be so iff $T^{-n}k \in \overline{B}$. But if $n \in \mathbb{N}$ we have

$$\|T^{-n}k\| \leq \|T^{-1}\|^n \|k\| < C_1.$$  

We are done by our choice of $C_1$ if we let $C_2 = 1/\|T^{-1}\|$; this is $> 1$ by our choice of norm $\|\cdot\|$.

**Proposition 6.2.** Let $K$ be a finite-rank torsion-free abelian group and let $T$ be an endomorphism of $K$. Then $T \otimes \mathbb{R}$ is an endomorphism of $K \otimes \mathbb{R}$. Suppose it is an automorphism. Suppose none of the (complex) eigenvalues of $T$ have absolute value $1$. Suppose further there is a finite set $A \subseteq K$ such that

- $A$ is an $\mathbb{R}$-basis for $K \otimes \mathbb{R}$,
- $(TA) \subseteq \langle A \rangle$ and
- $(B) = K$, where $B = \bigcup_{t=0}^{\infty} T^t A$.

Then (using additive notation in $K$) for every $k \in K$ there are $C_1$ and $C_2 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\|nk\|_B \leq C_1 \log n + C_2$.

**Proof.** Since none of the eigenvalues of $T$ have absolute value $1$, $K \otimes \mathbb{R}$ decomposes as the direct sum of an expanding subspace $K_e$ and a contracting subspace $K_c$. Let $A_e \subseteq K \otimes \mathbb{R}$ denote the projection of $A$ to $K_e$, and similarly let $A_c \subseteq K \otimes \mathbb{R}$ denote the projection of $A$ to $K_c$. For every $k \in K$, let $k_e$ and $k_c \in K \otimes \mathbb{R}$ denote the projections of $k$. Over $\mathbb{R}$, $A_e$ spans $K_e$ and $A_c$ spans $K_c$. Let $d \in \mathbb{N}$ be the maximal dimension of any generalized eigenspace. (If $K$ is trivial, let $d = 1$.) Then trivially $B_e = \bigcup_{i=0}^{d-1} T^i A_e$ spans $K_e$, and similarly $B_c = \bigcup_{i=0}^{d-1} T^i A_c$ spans $K_c$.

As in Lemma 6.1 we use the overbar to denote the symmetrized closed convex hull of a set. For $m \in \mathbb{Z}$, let

$$E_m = \{ k' \in K \otimes \mathbb{R} \mid k'_e \in T^m B_e, k'_c \in T^{-m} B_c \}.$$
Let $\| \cdot \|$ denote the $\mathbb{R}$-norm with respect to $A$. It follows from Lemma 6.1 that there are $C_1 > 0$ and $C_2 > 1$ such that, for every $k \in K$ and $m$, $n \in \mathbb{N}$, if $n\|k\| < C_1C_2^n$ then $nk \in E_m$. It follows that there are $D_1 > 0$ and $D_2 > 1$ such that, for every $k \in K$ and $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $nk \in E_m$ and $D_1D_2^n < n\|k\|$. We can choose $m$ for all $k$ and $n$ so that $k \in \langle T^{-m}A \rangle$; assume this done.

Let $D_e$ (respectively $D_c$) be the Hausdorff distance with respect to $B_e$ (resp. $B_c$) between $TB_e$ (resp. $T^{-1}B_e$) and $B_e$ (resp. $B_c$). Let $D$ be the greater of $D_e$ and $D_c$. Then any element of $T^mB_e$ is within distance $D$ of $T^{-m}B_c$ with respect to $T^{-m}B_c$. Similarly, any element of $T^{-m}B_e$ is within distance $D$ of $T^{1-m}B_c$ with respect to $T^{1-m}B_c$. It follows that any element $k \in E_m$ is within distance $D$ with respect to $\bigcup_{i=0}^{d-1} T^{i+1-m}A$ of some $k'_e \in K$ such that $k'_e \in T^{m-1}B_c$. Similarly, $k$ is within distance $D$ with respect to $\bigcup_{i=0}^{d-1} T^{i+1-m}A$ of some $k'_c \in K$ such that $k'_c \in T^{1-m}B_c$.

But all the eigenvalues of the restriction of $T$ to $K_e$ have absolute value $< 1$, and all the eigenvalues of its restriction to $K_c$ have absolute value $> 1$. Thus there is $F$ depending only on $K$, $T$ and $A$ such that $k'_e$ is within $F$ of $k_e$, with respect to $A$, and $k'_c$ is within $F$ of $k_e$, with respect to $A_c$. Let $k' = k'_e + k'_c - k$. Then $k' \in K$ is within $2D$ of $k$ with respect to $\bigcup_{i=0}^{d-1} T^{i+1-m}A$. Also, $k'_e$ is within $F$ of $T^{m-1}B_e$ with respect to $A_e$, and $k'_c$ is within $F$ of $T^{1-m}B_c$ with respect to $A_c$. We thus have $H$ depending only on $K$, $T$ and $A$ such that $k'$ is within $H$ of $E_m$ with respect to $\| \cdot \|$. (Recall that $\| \cdot \|$ is the $\mathbb{R}$-norm with respect to $A$.)

Repeating this process $m$ times, we find that any $k \in E_m$ is within $2mD$ with respect to $\bigcup_{i=0}^{m+d-2} T^mA$ of some $k'' \in K$ in within $mH$ of $E_0$ with respect to $\| \cdot \|$. Let $I$ denote the radius of $E_0$ with respect to $\| \cdot \|$. Then $\|k''\| \leq I + mH$. Thus, by another application of Lemma 6.1 there exist $D_3, D_4 \in \mathbb{N}$ depending only on $K$, $T$ and $A$ such that $k'' \in E_{D_3 \log m + D_4}$. For $m$ large, $D_3 \log m + D_4 < m$. Thus, we can repeat the procedure in this and the preceding two paragraphs to find $D_5, D_6 \in \mathbb{N}$ depending only on $K$, $T$ and $A$ such that, for all $m \in \mathbb{N}$, any $k \in E_m$ is, with respect to $\bigcup_{i=0}^{m+d-2} T^mA \subseteq B$, within $D_5 m$ of some $k''' \in K$ with $\|k'''\| < D_6$.

Putting the preceding paragraph together with the second paragraph tells us that there exist $D_1, D_2, D_5, D_6 \in \mathbb{N}$ depending only on $K$, $T$ and $A$ with the following property. Let $k \in K$ and $n \in \mathbb{N}$. Then there is $m \in \mathbb{N}$ such that $D_1D_2^n < n\|k\|$. Also, $nk$ is, with respect to $\bigcup_{i=0}^{m+d-2} T^mA \subseteq B$, within $D_5 m$ of some $k''_n \in K$ with $\|k''_n\| < D_6$. Finally, $k''_n \in \langle nk, T^{1-m}A \rangle = \langle T^{-m}A \rangle$, since $\langle T \rangle \subseteq \langle A \rangle$.

I claim there is $L \in \mathbb{N}$ with the following property. Let $a \in \langle A \rangle$ with $\|a\| \leq D_6$. Then there is $a' \in \langle T \rangle$ within $H$ of $a$ with respect to $A$ and with $\|a'\| \leq D_6$. To prove this claim, let $a \in \langle A \rangle$ with $\|a\| \leq D_6$. If $K_e$ is trivial, then $a_e = a$, so $\|a\| \leq D_6$, and, for sufficiently large $i$, we have $\|T^i a\| < \|a\| \leq D_6$. Since $T^i a \in \langle T \rangle$ for $i \geq 1$, we can assume $K_e$ is not trivial. There is $J$ depending only on $K$, $T$ and $A$ such that, for every $a$, there is $a' \in \langle T \rangle$ within $J$ of $a$ with respect to $A$. Then $\|a'\| \leq D_6 + J$. If $K_c$ is trivial, then in particular $\|a''\| = 0 \leq D_6$, which is what we want, so suppose $K_e$ is also not trivial. The projection of $\langle T \rangle$ is dense in $K_e$ since $T \otimes \mathbb{R}$ is an automorphism. Thus there is $a'' \in \langle T \rangle$ depending only on $K$, $T$ and $A$ with $0 < \|a''\| < 2D_6$. Then there is $i \in \mathbb{Z}$ with
\[ |i| \leq (J + D_0)/\|a''_i\| + 1 \text{ such that } \|a''_i + ia''_{i+1}\| \leq D_0. \] The claim is proven.

Applying the claim \( m \) times to the situation of the preceding paragraph, we see that there exist \( D_1, D_2, D_3, D_4, D_5 \in \mathbb{N} \) depending only on \( K, T, A \) with the following property. Let \( k \in K \) and \( n \in \mathbb{N} \). Then there is \( m \in \mathbb{N} \) such that \( D_1D_2^m < n\|k\| \). Also, \( nk \) is, with respect to \( \bigcup_{i=1}^{m+1} T^iA \subseteq B \), within distance \( D_3m \) of some \( k'_n \in K \) with \( \|k'_n\| < D_0 \).

Finally, \( k'_n \) is, with respect to \( \bigcup_{i=1}^{m+1} T^iA \subseteq B \), within distance \( D_3m \) of some \( k''_n \in \langle A \rangle \) such that \( \|k''_n\| \leq D_0 \).

But, since \( A \) is finite, there is \( M \in \mathbb{N} \) such that, for all \( i \in \mathbb{N} \) and \( a \in A \), \( \|T^{-i}(a)\| < M \). It follows that \( \|k'_n - k''_n\| \leq D_7Mm \). Thus there are \( D_8, D_9 \in \mathbb{N} \) depending only on \( K, T, A \) such that

\[ \|k''_n\| \leq \|k'_n\| + \|k'_n - k''_n\| < D_8m + D_9. \]

Thus there is \( D_{10} \in \mathbb{N} \) such that \( \|k''_n\| \leq \|k'_n\| + \|k'_n - k''_n\| < D_8m + D_{10} \).

Recall that \( \|\cdot\| \) denotes the norm with respect to \( A \subseteq B \). Thus, by the triangle inequality, there is \( D_{11} \in \mathbb{N} \) depending only on \( K, T, A \) such that \( nk \) is within \( D_{11}m + D_{10} \) of 0 with respect to \( B \), so we are done.

The following is an easy consequence.

**Corollary 6.3.** Let \( K \) be a finite-rank torsion-free abelian group and let \( T \) be an endomorphism of \( K \). Then \( T \otimes \mathbb{R} \) is an endomorphism of \( K \otimes \mathbb{R} \). Suppose it is an automorphism. Suppose none of the (complex) eigenvalues have absolute value 1. Suppose finally there is a finite set \( A \subseteq K \) such that

- \( A \) is an \( \mathbb{R} \)-basis for \( K \otimes \mathbb{R} \).
- \( \langle TA \rangle \subseteq \langle A \rangle \) and
- \( \langle B \rangle = K \), where \( B = \bigcup_{i=-\infty}^{\infty} T^iA \).

Let \( \pi : \mathbb{Z}^B \to K \) be the projection. Then there is \( n \in \mathbb{N} \) such that, for any \( k \in K \) and \( w \in \mathbb{Z}^B \) of minimal length in \( \pi^{-1}(k) \), the coefficient of every letter of \( B \) in \( w \) has absolute value less than \( n \). In particular, \( \{ \sum_{a \in A} ia \mid |ia| < n, a \in A \} \) is a \( t \)-generating set for \( K \), if \( t \) acts by \( T \).

### 6.2 Proof of strong \( t \)-logarithmicity

We will next show the \( t \)-generating set just constructed is strongly \( t \)-logarithmic. We will use that, roughly speaking, each element of \( K \) is represented by finitely many minimal-length \( t \)-words. More precisely, we have

**Proposition 6.4.** Let \( K \) be a finite-rank torsion-free abelian group and let \( L \) be a full-rank lattice in \( K \). Let \( T \) be a hyperbolic automorphism of \( K \) which acts on \( L \) by an endomorphism and gives \( K \) the structure of a \( \mathbb{Z}[t, t^{-1}] \)-module. Let \( A \) be a finite \( t \)-generating set for \( K \). Then for every \( k \in K - \{0\} \) and \( l \in \mathbb{N} \) there are only finitely many minimal \( t \)-words \( w \) in \( A \) of length at most \( l \) representing \( k \).

The proof is deferred to Subsection 6.3.
Proposition 6.5. Let $K$ be a finite-rank torsion-free abelian group and let $T$ be a hyperbolic automorphism of $K$, so that $K$ has the structure of a $\mathbb{Z} [t, t^{-1}]$-module. Choose a norm on $K \otimes \mathbb{R}$ and let $d_c$ (respectively $d_e$) denote distance with respect to this norm from the contracting (resp. expanding) subspace. Let $A$ be a finite $t$-generating set for $K$. For $w$ a $t$-word in $A$, let $\pi(w)$ be the element of $K$ it represents. Let $L \supseteq A$ be a full-rank lattice in $K$ and $\pi$ the generator of $L$. Then $\sum_{i=1}^{\infty} \max \{ d_c(T^i a) \mid a \in A \}$ is finite. Also, for every $D \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$, there is $E_1 \in \mathbb{Z}$ with the following property. Let $w \in \pi^{-1}(L)$ be a $t$-word. Suppose $d_c(\pi(w)) \leq D$ and $w$ is minimal and within $n$ of minimal length in $\pi^{-1}(\pi(w))$ subject to the condition $I_{\min}(w) \geq 0$. Then $I_{\max}(w) < E_1$.

Similarly, $\sum_{i=1}^{\infty} \max \{ d_e(T^{-i} a) \mid a \in A \}$ are finite. Also, for every $D \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$, there is $E_2 \in \mathbb{Z}$ with the following property. Let $w \in \pi^{-1}(L)$ be a $t$-word. Suppose $d_e(\pi(w)) \leq D$ and $w$ is minimal and within $n$ of minimal length in $\pi^{-1}(\pi(w))$ subject to the condition $I_{\min}(w) \leq 0$. Then $I_{\min}(w) > E_2$.

Proof. If $w = 0$, then any $E_1$ and $E_2$ at all will work. We thus may assume $w \neq 0$. It follows that $w \notin \ker \pi$, since the only minimal element of $\ker \pi$ is 0.

We prove the first paragraph of the proposition; the proof of the second is exactly analogous.

Since $T$ is hyperbolic, there are $C_1 > 0$ and $C_2 < 1$ such that, for every $k \in K$, $d_c(T^i k) < C_1 C_2^i d_e(k)$. In particular, this is so for every $k \in A$. Thus

$$D' = \sum_{i=1}^{\infty} \max \{ d_e(T^i a) \mid a \in A \}$$

is finite, proving the first claim.

For the second claim, let $w$ be a $t$-word as in the condition of the proposition. Since $I_{\min}(w) \geq 0$, it follows from the first claim that $d_e(\pi(w)) < D'$. Then $d_e(\pi(w)) < D'$ and $d_e(\pi(w)) \leq D$. Thus $\pi(w)$ is constrained to lie within a bounded region. Since $\pi(W) \subseteq L$ and $L$ is discrete, there are finitely many possible values of $\pi(w)$. None of these is 0, since we are assuming $w \notin \ker \pi$. By Lemma 6.4 each of them is represented by finitely many minimal $t$-words $w$ of any given length, in particular of any length within $n$ of that length which is least subject to the condition that $I_{\min}(w) \geq 0$. Thus there are finitely many possibilities for $w$. We are done if we let $E_1$ be the maximum of their $I_{\max}$'s.

Proposition 6.6. Let $K$ be a finite-rank torsion-free abelian group and let $T$ be a hyperbolic automorphism of $K$, so that $K$ has the structure of a $\mathbb{Z} [t, t^{-1}]$-module. Choose a norm on $K \otimes \mathbb{R}$ and let $d_c$ (respectively $d_e$) denote distance with respect to this norm from the contracting (resp. expanding) subspace. Let $A$ be a $t$-generating set for $K$. Let $L \supseteq A$ be a full-rank lattice in $K$ on which $T$ acts by an endomorphism. For $w$ a $t$-word, let $\pi(w)$ denote the element of $K$ it represents. Then $d_1 = \sum_{i=1}^{\infty} \max \{ d_c(T^{-i} a) \mid a \in A \}$ and $d_2 = \sum_{i=1}^{\infty} \max \{ d_e(T^i a) \mid a \in A \}$ are finite. Let $N_1$ and $N_2$ be the radius-$d_1$ and $-d_2$ closed neighborhoods, respectively, of the contracting (resp. expanding) subspace of $K \otimes \mathbb{R}$. Let
$n \in \mathbb{N} \cup \{0\}$. Then there are $E_1$ and $E_2 \in \mathbb{Z}$ such that, for any $k \in K \cap N_1$ (resp. $L \cap N_2$), any minimal $t$-word $w$ representing $k$ and within $n$ of minimal length in $\pi^{-1}(\pi(w))$ has $I_{max}(w) < E_1$ (resp. $I_{min}(w) > E_2$). In particular, $I_{max}(k) < E_1$ (resp. $I_{min}(k) > E_2$), for $E_1$ and $E_2$ independent of $n$.

Proof. The sums $d_1$ and $d_2$ are finite by Proposition 6.5.

Let $E_1$ and $E_2$ be given by Proposition 6.5 with $D = 2 \max(d_1, d_2)$ and $n$ as in this proposition. Let $w$ be a minimal $t$-word within $n$ of minimal length in $\pi^{-1}(\pi(w))$ subject to the condition that $I_{min}(w) \geq 0$ (resp. $I_{max}(w) \leq 0$). Thus, by Proposition 6.5, $d_c(\pi(w)) > D \geq 2d_1$ (resp. $d_c(\pi(w)) > D \geq 2d_2$). But clearly $d_c(\pi(w) - \pi(w_i)) = d_c(\pi(w_i)) \leq d_1$ (resp. $d_c(\pi(w) - \pi(w)) \leq d_2$) since $w_i$ is a $t$-word. By the triangle inequality, it follows that $d_c(\pi(w)) > d_1$ (resp. $d_c(\pi(w)) > d_2$). Thus $k \notin N_1$ (resp. $N_2$), as claimed.

The last sentence follows trivially.

Corollary 6.7. Let $K$ be a finite-rank torsion-free abelian group and let $T$ be a hyperbolic automorphism of $K$, so that $K$ has the structure of a $\mathbb{Z}[t, t^{-1}]$-module. Choose a norm on $K \otimes \mathbb{R}$ and let $d_c$ (respectively $d_1$) denote distance with respect to this norm from the contracting (resp. expanding) subspace. Let $A$ be a $t$-generating set of $K$. Let $L \supseteq A$ be the full-rank lattice in $K$ on which $T$ acts by an endomorphism. For $w$ a $t$-word, let $\pi(w)$ denote the element of $K$ it represents. Then $d_1 = \sum_{i=1}^{\infty} \max\{d_c(T^{-a}) \mid a \in A\}$ and $d_2 = \sum_{i=1}^{\infty} \max\{d_c(T^a) \mid a \in A\}$ are finite. Let $N_1$ and $N_2$ be the radius-$d_1$ and $-d_2$ closed neighborhoods, respectively, of the contracting (resp. expanding) subspace of $K \otimes \mathbb{R}$. Let $n \in \mathbb{N} \cup \{0\}$. Then there are $E_1$ and $E_2 \in \mathbb{Z}$ such that, for any $i \in \mathbb{Z}$ and any $k \in K \cap T^i N_1$ (resp. $T^i L \cap T^i N_2$), any minimal $t$-word $w$ representing $k$ and within $n$ of minimal length in $\pi^{-1}(\pi(w))$ has $I_{max}(w) < E_1 + i$ (resp. $I_{min}(w) > E_2 + i$). In particular, $I_{max}(k) < E_1 + i$ (resp. $I_{min}(k) > E_2 + i$), for $E_1$ and $E_2$ independent of $n$ (and $i$).

Proof. Just apply $T^i$ to the statement of Proposition 6.6.

Proposition 6.8. Let $K$ be a nontrivial finite-rank torsion-free abelian group and $T$ a hyperbolic automorphism of $K$. Let $L$ be a full-rank lattice in $K$ on which $T$ acts by an endomorphism. Suppose $\bigcup_{i=-\infty}^{\infty} T^i L = K$. Let $G = K \times \langle t \rangle$, where $t$ acts by $T$, so that $K$ has the structure of a
Then $K$ has a finite strongly $t$-logarithmic $t$-generating set.

Proof. Let $A$ be a basis of $L$. Then the hypotheses of Corollary 6.9 are clearly satisfied, so let $A'$ be the finite $t$-generating set given by that result. We will show $A'$ is strongly $t$-logarithmic.

Choose a norm on $K \otimes \mathbb{R}$ and let $d_\circ$ (respectively $d_e$) denote distance with respect to this norm from the contracting (resp. expanding) subspace. Let $N_1$, $N_2$, $E_1$ and $E_2$ be defined as in Corollary 6.7 with $A = A'$ and with respect to the above norm. Let $B = \bigcup_{i=-\infty}^\infty T^i A'$ and let $\pi : \mathbb{Z}^d \to K$ be the projection. Let $m \in \mathbb{N}$, $k \in K$ and $w \in \pi^{-1}(k) - \{0\}$. (Note that $w$ is a generalized $t$-word, in the terminology of Section 2, but not necessarily a $t$-word.) Let $\|w\|_m$ be as in the definition of strong $t$-logarithmicity. Then

$$d_e(T^{-i_{\max}(w)} k) \leq m \sum_{i=0}^\infty \max \{ d_e(T^{-i} a) \mid a \in A' \} \leq (m + \|w\|_m) \sum_{i=0}^{\infty} \max \{ d_e(T^{-i} a) \mid a \in A' \},$$

where the first inequality is by the definition of $\|w\|_m$. Thus

$$k \in (m + \|w\|_m) T^{i_{\max}(w)} N_1,$$

It follows that there are $C_1 > 0$ and $C_2 \in \mathbb{R}$ (depending only on $K$ and $T$) such that

$$k \in T^{i_{\max}(w)} N_1 + C_1 \log(m + \|w\|_m) + C_2 N_1.$$

Let $w'$ be a minimal $t$-word representing $k$ and within $n$ of minimal length among all $t$-words representing $k$. (The set of all such $t$-words is $\pi^{-1}(\pi(w))$ in the notation of Corollary 6.7.) Let $E_1$ and $E_2$ be given by Corollary 6.7 with $n$ as in this proof. Then, by Corollary 6.7

$$I_{\max}(w') < I_{\max}(w) + C_1 \log(m + \|w\|_m) + C_2 + E_1.$$

In exactly the same way,

$$k \in T^{i_{\min}(w)} - C_1 \log(m + \|w\|_m) + C_2 N_2.$$

Also, since every letter of $w$ is in $T^{i_{\min}(w)} L$, $k \in T^{i_{\min}(w)} L$. By Corollary 6.7 again,

$$I_{\min}(w') > I_{\min}(w) - C_1 \log(m + \|w\|_m) + C_2 + E_2.$$

Corollary 6.9. Let $K$ be a nontrivial finite-rank torsion-free abelian group and $T$ a hyperbolic automorphism of $K$. Let $L$ be a full-rank lattice in $K$ on which $T$ acts by an endomorphism. Suppose $\bigcup_{i=-\infty}^\infty T^i L = K$. Let $G = K \rtimes \langle t \rangle$, where $t$ acts by $T$. Then $G$ has a generating set with respect to which it has deep pockets. Furthermore, it is not almost convex with respect to any generating set.

Proof. Combine Theorems 2.3 and Proposition 5.8.
6.3 Proof of Proposition 6.3

In this subsection, we complete the proof of Proposition 6.3 by proving Proposition 6.3 whose statement we repeat here for the convenience of the reader.

**Proposition 6.3.** Let $K$ be a finite-rank torsion-free abelian group and let $L$ be a full-rank lattice in $K$. Let $T$ be a hyperbolic automorphism of $K$ which acts on $L$ by an endomorphism and gives $K$ the structure of a $\mathbb{Z}[t, t^{-1}]$-module. Let $A$ be a finite $t$-generating set for $K$. Then for every $k \in K - \{0\}$ and $i \in \mathbb{N}$ there are only finitely many minimal $t$-words $w$ in $A$ of length at most $l$ representing $k$.

The plan of the proof is that $K$ (or at least a finite-index submodule of $K$) can be split as a direct sum of two pieces: a finitely generated piece, which we call $K_d$, and a piece $K_c$ with \( \bigcap_{n=-\infty}^{\infty} T^n k_c = \{0\} \). This splitting is given by Lemma 6.10. Then we will deal with $K/K_c$ in Lemma 6.11 and with $K/K_d$ in Corollary 6.14.

**Lemma 6.10 (splitting).** Let $K$ be a finite-rank torsion-free abelian group and let $T$ be a hyperbolic automorphism of $K$. Let $L$ be a full-rank lattice in $K$ on which $T$ acts as an endomorphism. Suppose $\bigcap_{n=-\infty}^{\infty} T^n L = K$. Let $K_d = \bigcap_{n=-\infty}^{\infty} T^n L$. Then $K$ has a finite-index $T$-submodule $K'$ such that $K_d$ is a complemented $T$-submodule of $K'$. Furthermore, $K_d$ has a complement $K_c$ in $K'$ such that $K/K_d$ and $K/K_c$ are both torsion-free.

**Proof.** It is clear that the actions by $T$ and $T^{-1}$ preserve $K_d$. It remains to show that it is complemented in some finite-index submodule of $K$.

First, I claim that $K/K_d$ is torsion-free. To this end, let $L_d$ denote the set of all $l \in L$ such that there is $n \in \mathbb{N}$ with $nl \in K_d$. Then $K_d \subseteq L_d$ and, for all $i \in \mathbb{N} \cup \{0\}$, $T^i L_d \subseteq L_d$. Let $i \in \mathbb{N} \cup \{0\}$, $l \in L$ and $T^i l \in L_d$. Then there is some $n \in \mathbb{N}$ with $nT^i l = T^i (nl) \in K_d$. It follows that $nl \in K_d$, so $l \in L_d$. Thus also $T^i l \in L_d$. We have thus shown that $T^i L \cap L_d \subseteq T^i L_d$. Since clearly $T^i L_d \subseteq T^i L$, we have $T^i L \cap L_d = T^i L_d$. Thus

$$K_d = K_d \cap L_d = \bigcap_{i=0}^{\infty} (T^i L \cap L_d) = \bigcap_{i=0}^{\infty} T^i L_d.$$ 

But $K_d$ must be a full-rank subgroup of $L_d$, by the definition of $L_d$. Since $L$ is finitely generated, so is $L_d$, so $L_d/K_d$ is finite. By the above equation, this implies there is $i \in \mathbb{N}$ such that $T^i L_d = K_d$. Now let $k \in K - K_d$ and $n \in \mathbb{N}$ such that $nk \in K_d$. Let $j \in \mathbb{Z}$ be such that $k \in T^j L$. Then $T^{i-j} k \in T^i L - K_d$, but $nT^{i-j} k = T^{i-j} (nk) \in K_d$. Thus $T^{i-j} k \in T^i L_d - K_d$, a contradiction. Our claim is proven.

Thus $K_d$ and $K/K_d$ are both finite-rank torsion-free abelian groups. In fact, $K_d$ is finitely generated since $L$ is, so the action of $T$ on $K_d$ has all its (complex) eigenvalues algebraic units. Suppose some eigenvalue of $T$ on $K/K_d$ were a unit. Then let $K_u$ be the subspace of $K/K_d \otimes \mathbb{Q}$ generated by the generalized eigenspaces of that eigenvalue and all its conjugates over $\mathbb{Q}$. But then $K_u \cap L/K_d$ is a nontrivial finitely generated subgroup of $K/K_d$ on which $T$ acts as an automorphism. This is a contradiction, so none of the eigenvalues of the action of $T$ on $K/K_d$ are units.
It follows by rational canonical form that $K_d \otimes \mathbb{Q}$ is complemented as a $T$-submodule of $K \otimes \mathbb{Q}$. Let $K_c$ be the intersection of the complement with $K$. Clearly, $K_c$ is a $T$-module, and all elements of $K$ which have a multiple in $K_c$ are themselves in $K_c$. Thus $K/K_c$ is torsion-free.

Let $K' = (K_c, K_d)$; we will show that $[K : K'] < \infty$. Note that $K'$ is of full rank in $K$, since $K \otimes \mathbb{Q} = K_d \otimes \mathbb{Q} \oplus K_c \otimes \mathbb{Q}$. Thus $[L : K' \cap L] < \infty$, since $L$ is finitely generated. Since $K'$ is invariant under $T$ and $T^{-1}$ (since $K_d$ and $K_c$ are), it follows that, for all $i \in \mathbb{Z}$, $[T^iL : K' \cap T^iL] = [L : K' \cap L]$. But then

$$[K : K'] = [K : K \cap K'] = \bigcup_{i=-\infty}^{\infty} T^iL : K' \cap \bigcup_{i=-\infty}^{\infty} T^iL = [L : K' \cap L] < \infty,$$

so we are done.

\section*{Lemma 6.11 (case of $K/K_c$)} Let $K$ be a finitely generated torsion-free abelian group and let $T$ be a hyperbolic automorphism of $K$, so that $K$ has the structure of a $\mathbb{Z}[t, t^{-1}]$-module. Let $A$ be a finite $t$-generating set for $K$ and let $B = \bigcup_{i=-\infty}^{\infty} t^iA$. Let $k \in K - \{0\}$. Then there is a finite subset $C$ of $B$ such that any $t$-word representing $k$ must contain a letter in $C$.

\begin{proof}
Since none of the eigenvalues of $T$ have absolute value 1, $K \otimes \mathbb{R}$ decomposes as the direct sum of an expanding subspace $K_\alpha$ and a contracting subspace $K_\gamma$. The restrictions of $T$ to $K_\alpha \cap K$ and of $T^{-1}$ to $K_\gamma \cap K$ must have all their eigenvalues with absolute value $< 1$. Since $T$ is an automorphism, it follows that $K_\alpha \cap K = K_\gamma \cap K = \{0\}$. For every $k \in K$, let $k = k_\alpha + k_\gamma$, where $k_\alpha \in K_\alpha$ and $k_\gamma \in K_\gamma$. Let $k \in K$, and suppose at least one of $k_\alpha$ and $k_\gamma$ is $\in K$. Then the other must be too, so $k_\alpha = k_\gamma = 0$, so $k = 0$.

Choose a norm $\|\|$ on $K \otimes \mathbb{R}$ and let $k \in K - \{0\}$. Let $H_k$ be the Hausdorff distance with respect to this norm between $k_\alpha$ and $K$. Since $K$ is finitely generated, hence discrete, and $k_\alpha \notin K$ by the preceding paragraph, $H_k > 0$.

For $w'$ a $t$-word, let $\pi(w')$ be the element of $K$ it represents. Let $w$ be a $t$-word of length $l$ in $\pi^{-1}(k)$, where $k \in K - \{0\}$. For any $m \in \mathbb{Z}$, let $w^{m} \in \bigcup_{i=-m+1}^{m-1} t^iA$ and $w^{<m} \in \bigcup_{i=-m-1}^{-1} t^iA$. Suppose $w = w^{>m} + w^{<m}$. Let $k^{>m} = \pi(w^{>m})$ and $k^{<m} = \pi(w^{<m})$. Then $k^{<m} = k_\alpha + k^{<m} - k_\gamma \in K$.

There are $D \in \mathbb{R}$ and $E > 1$ depending only on $K$, $T$ and $A$ such that, for all $a \in A$, $\|t^i(a)e\| < DE^i$ for all $i \leq 0$ and $\|t^i(a)e\| < DE^{-i}$ for all $i \geq 0$. It follows that there are $F \in \mathbb{R} \cap E > 1$ such that $\|k^{<m}\|$ and $\|k^{<m}\| < FE^{-m}$ for all $m \in \mathbb{N} \cup \{0\}$. Putting this together with the last two paragraphs, we get that $2FE^{-m} > H_k$. We are thus done if we let $M = -\lfloor \log_E[H_k/(2F)] \rfloor$ and $C = \bigcup_{m=-M}^{M} t^iA$.

\end{proof}

\section*{Lemma 6.12} Let $K$ be a finite-rank torsion-free abelian group and let $L$ be a full-rank lattice in $K$. Let $T$ be an automorphism of $K$ such that

\begin{itemize}
\item $T$ acts by an endomorphism on $L$,
\end{itemize}
• $\bigcup_{i=-\infty}^{\infty} T^i L = K$ and
• $\bigcap_{i=-\infty}^{\infty} T^i L = \{0\}$.

Let $A$ be a finite subset of $K$ not including 0 and let $B = \bigcup_{i=-\infty}^{\infty} T^i A$.

Suppose $B$ generates $K$ as a group and let $\pi: \mathbb{Z}^B \to K$ be the projection. Let $\|\cdot\|$ denote the $L_1$ norm on $\mathbb{Z}^B$. For $w \in \mathbb{Z}^B$, let $L_{\min}(w)$ denote the largest $n \in \mathbb{Z}$ such that every letter of $w$ is in $T^n L$. Then for every $l \in \mathbb{N}$ there is $n$ such that any nonempty $w \in \pi^{-1}(T^{L_{\min}(w)+n} L)$ with $\|w\| < l$ has a nonempty subword in $\ker \pi$.

Proof. The proof is by induction on $l$. The statement is obvious for $l = 1$ or $l = 2$; let $n = 1$.

For $l > 2$, suppose $w \in \mathbb{Z}^B$ is nonempty with $\|w\| < l$. If $w \in \ker \pi$, then $w$ is itself a nonempty subword of $w$ in $\ker \pi$, so assume $w \notin \ker \pi$.

For the same reason, assume none of the letters of $w$ are 0. Let $L_{\max}(w)$ denote the greatest $i$ such that some letter of $w$ is in $T^i L$. Let $b$ be a letter of $w$ in $T^{L_{\max}(w)} L$. Let $w' = w - b$, that is the word obtained by deleting $b$ from $w$.

Suppose $w' \notin \ker \pi$. Then $w'$ is nonempty, so $L_{\min}(w') = L_{\min}(w)$.

Also, since $w'$ is a subword of $w$, it has no nonempty subword in $\ker \pi$.

Thus, since $\|w'\| < l - 1$, it must, by induction, be $\notin \pi^{-1}(T^{L_{\min}(w)+n} L)$, where $n'$ is the $n$ given by applying the induction assumption.

Thus either $w \notin \pi^{-1}(T^{L_{\min}(w)+n'})$ or $b \notin \pi^{-1}(T^{L_{\min}(w)+n'})$. In the latter case, $L_{\max}(w) < L_{\min}(w) + n'$.

For $i \in \mathbb{Z}$, let

$$N_{i,i} = \{ w \in \mathbb{Z}^B \mid L_{\min}(w) = i, L_{\max}(w) < i + n', \|w\| < l \}.$$ 

Since $A$ is finite, so is $N_{i,i}$. Let $n''$ be the least integer such that $w \notin \pi^{-1}(T^{L_{\min}(w)+n''} L)$ for all $w \in N_{i,i}$. Such an integer exists since $N_{i,i}$ is finite and disjoint from $\ker \pi$. (It is clear that $n''$ does not depend on $i$.) Then the claim follows for $l$, letting $n$ be the greater of $n'$ and $n''$.

We are done by induction. \qed

Corollary 6.13. Let $K$ be a finite-rank torsion-free abelian group and let $L$ be a full-rank lattice in $K$. Let $T$ be an automorphism of $K$ such that

• $T$ acts by an endomorphism on $L$,
• $\bigcup_{i=-\infty}^{\infty} T^i L = K$ and
• $\bigcap_{i=-\infty}^{\infty} T^i L = \{0\}$.

Let $A$ be a finite subset of $K$ and let $B = \bigcup_{i=-\infty}^{\infty} T_i A$. Suppose $B$ generates $K$ as a group and let $\pi: \mathbb{Z}^B \to K$ be the projection. Let $\|\cdot\|$ denote the $L_1$ norm on $\mathbb{Z}^B$. Then for every $l \in \mathbb{N}$ and $i \in \mathbb{Z}$ there exist $n_1$ and $n_2 \in \mathbb{Z}$ such that each word $w \in \pi^{-1}(T^i L - T^{i+1} L)$ with $\|w\| < l$ contains a letter in $T^{n_1} A - T^{n_2} A$.

Proof. We may assume $0 \notin A$.

Any word $w$ satisfying the conditions contains a subword

$$w' \in \pi^{-1}(T^i L - T^{i+1} L)$$

such that
• \( \|w'\| < l \),
• no nonempty subword of \( w' \) is \( \in \ker \pi \) and
• no letter of \( w' \) is \( \in T^{i+1}L \).

By Lemma 6.12 \( w' \) contains all its letters \( \in T^{i-n+1}L \). Since \( w' \) is clearly nonempty, we are done.

**Corollary 6.14** (case of \( K/K_d \)). Let \( K \) be a finite-rank torsion-free abelian group and let \( L \) be a full-rank lattice in \( K \). Let \( T \) be an automorphism of \( K \) such that

- \( T \) acts by an endomorphism on \( L \),
- \( \bigcup_{i=-\infty}^{\infty} T^i L = K \) and
- \( \bigcap_{i=-\infty}^{\infty} T^i L = \{0\} \),

so that \( K \) has the structure of a \( \mathbb{Z}[t, t^{-1}] \)-module. Let \( A \) be a finite \( t \)-generating set for \( K \) and let \( B = \bigcup_{i=-\infty}^{\infty} t^i A \). Let \( k \in K - \{0\} \). Then for every \( l \in \mathbb{N} \) there is a finite subset \( C \) of \( B \) such that any \( t \)-word representing \( k \) of length \( < l \) must contain a letter \( \in C \).

**Proof.** This is just a special case of Corollary 6.13.

We are now ready for the Proof of Proposition 6.2. Let \( K_d \) be a complemented \( \mathbb{Z}[t, t^{-1}] \)-submodule \( K' \supseteq K_d \) such that \( K_d \) is a complemented \( \mathbb{Z}[t, t^{-1}] \)-submodule of \( K \). Let \( K_c \) be a complement for \( K_d \) in \( K' \). Then the actions of \( t \) on \( K/K_d \) and \( K/K_c \) are hyperbolic. Let \( \phi_d \colon K \to K/K_d \) and \( \phi_c \colon K \to K/K_c \) be the projections. Let \( A_d \) and \( A_c \) denote \( \phi_d(A) \) and \( \phi_c(A) \) respectively. Clearly, they are \( t \)-generating sets for \( K/K_d \) and \( K/K_c \) respectively. By a slight abuse of notation, if \( w \) is a \( t \)-word in \( A \), \( \phi_d(w) \) and \( \phi(c)(w) \) will denote the corresponding \( t \)-words in \( A_d \) and \( A_c \).

Let \( B_c = \bigcup_{i=-\infty}^{\infty} t^i A_c \) and \( B_d = \bigcup_{i=-\infty}^{\infty} t^i A_d \). For \( w \) a \( t \)-word in \( A \), \( A_c \) or \( A_d \), let \( \pi(w) \), \( \pi_c(w) \) or \( \pi_d(w) \) be the element of \( K/K_c \) or \( K/K_d \) it represents.

We prove the result by induction on \( l \). If \( l = 1 \) then there are only finitely many \( t \)-words of length 1 representing any \( k \in K - \{0\} \), since \( T \) is hyperbolic and \( A \) is finite. If \( l > 1 \), let \( k \in K - \{0\} \). Since \( K_d \cap K_c = \{0\} \), at least one of \( \phi_d(k) \) and \( \phi_c(k) \) is nonzero. If \( w \in \pi^{-1}(k) \) is a \( t \)-word of length \( l \) in \( A \) then \( \phi_d(w) \in \pi_d^{-1}(\phi_d(k)) \) and \( \phi_c(w) \in \pi_c^{-1}(\phi_c(k)) \) are \( t \)-words of length \( l \) in \( A_d \) and \( A_c \), respectively. But \( K/K_c \) is a finite extension of \( K' / K_d \supseteq K_d \), so it is finitely generated. Thus, by Lemma 6.11 if \( \phi_c(k) \neq 0 \) then there is a finite \( C \subseteq \phi_c(B) \) depending only on \( K \), \( T \), \( A \) and \( k \) (not on \( w \)) such that \( \phi_c(w) \) contains a letter from \( C \).

If \( \phi_c(k) = 0 \) then \( \phi_d(k) \neq 0 \). Since \( K/K_d \) satisfies the hypotheses of Corollary 6.13, there is again some finite \( C \subseteq \phi_d(B) \) depending only on \( K \), \( T \), \( A \), \( k \) and \( l \) (again, not on \( w \)) such that \( \phi_d(w) \) contains a letter from \( C \). Putting this together with the preceding paragraph, there is some finite \( C \subseteq B \) depending only on \( K \), \( T \), \( A \), \( k \) and \( l \) which contains a letter from each \( t \)-word \( w \in \pi^{-1}(k) \) of length at most \( l \).
If \( w \) contains a letter \( \in \pi^{-1}(k) \) then the remainder of \( w \) is a nonempty subword representing 0, so \( w \) is not minimal. We may thus assume that no letter of \( w \) represents \( k \). In particular, by the preceding paragraph, \( w \) contains a letter \( c \in C \) not representing \( k \). Then \( w - c \) is a minimal \( t \)-word of length at most \( l - 1 \) representing an element of the finite set \( \{ k - \pi(c) \mid c \in C \} - \{0\} \) with no nonempty subword representing 0. We are done by induction.

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