Abstract

We study the multifield inflationary models where the cosmological perturbation is sourced by light scalar fields other than the inflaton. The corresponding perturbations are both scale invariant and special conformally invariant. We exploit the operator product expansion technique of conformal field theories to study the inflationary correlators enjoying the symmetries present during the de Sitter epoch. The operator product expansion is particularly powerful in characterizing inflationary correlation functions in two observationally interesting limits, the squeezed limit of the three-point correlator and the collapsed limit of the four-point correlator. Despite the fact that the shape of the four-point correlators is not fixed by the symmetries of de Sitter, its exact shape can be found in the collapsed limit making use of the operator product expansion. By employing the fact that conformal invariance imposes the two-point cross-correlations of the light fields to vanish unless the fields have the same conformal weights, we are able to show that the Suyama-Yamaguchi inequality relating the coefficients $f_{NL}$ of the bispectrum in the squeezed limit and $\tau_{NL}$ of the trispectrum in the collapsed limit also holds when the light fields are intrinsically non-Gaussian. In fact, we show that the inequality is valid irrespectively of the conformal symmetry, being just a consequence of fundamental physical principles, such as the short-distance expansion of operator products. The observation of a strong violation of the inequality will then have profound implications for inflationary models as it will imply either that multifield inflation cannot be responsible for generating the observed fluctuations independently of the details of the model or that some new non-trivial degrees of freedom play a role during inflation.
1 Introduction

One of the basic ideas of modern cosmology is that there was an epoch early in the history of the universe when potential, or vacuum, energy associated to a scalar field, the inflaton, dominated other forms of energy density such as matter or radiation. During such a vacuum-dominated era the scale factor grew exponentially (or nearly exponentially) in time. During this phase, dubbed inflation [1, 2], a small, smooth spatial region of size less than the Hubble radius could grow so large as to easily encompass the comoving volume of the entire presently observable universe. If the universe underwent such a period of rapid expansion, one can understand why the observed universe is so homogeneous and isotropic to high accuracy.

Inflation has also become the dominant paradigm for understanding the initial conditions for structure formation and for Cosmic Microwave Background (CMB) anisotropy. In the inflationary picture, primordial density and gravity-wave fluctuations are created from quantum fluctuations “redshifted” out of the
horizon during an early period of superluminal expansion of the universe, where they are “frozen” [3,7]. Perturbations at the surface of last scattering are observable as temperature anisotropy in the CMB. The last and most impressive confirmation of the inflationary paradigm has been recently provided by the data of the Wilkinson Microwave Anistropy Probe (WMAP) mission which has marked the beginning of the precision era of the CMB measurements in space [8]. The WMAP collaboration has produced a full-sky map of the angular variations of the CMB, with unprecedented accuracy. WMAP data con- firm the inflationary mechanism as responsible for the generation of curvature (adiabatic) superhorizon fluctuations.

Despite the simplicity of the inflationary paradigm, the mechanism by which cosmological adiabatic perturbations are generated is not yet fully established. In the standard picture, the observed density perturbations are due to fluctuations of the inflaton field itself. When inflation ends, the inflaton oscillates about the minimum of its potential and decays, thereby reheating the universe. As a result of the fluctuations each region of the universe goes through the same history but at slightly different times. The final temperature anisotropies are caused by the fact that inflation lasts different amounts of time in different regions of the universe leading to adiabatic perturbations. Under this hypothesis, the WMAP dataset already allows to extract the parameters relevant for distinguishing among single-field inflation models.

An alternative to the standard scenario is represented by the curvaton mechanism [9,11] where the final curvature perturbations are produced from an initial isocurvature perturbation associated to the quantum fluctuations of a light scalar field (other than the inflaton), the curvaton, whose energy density is negligible during inflation. The curvaton isocurvature perturbations are transformed into adiabatic ones when the curvaton decays into radiation much after the end of inflation. Alternatives to the curvaton model are those models characterized by the curvature perturbation being generated by an inhomogeneity in the decay rate [12] or the mass [13] of the particles responsible for the reheating after inflation. Other opportunities for generating the curvature perturbation occur at the end of inflation [14] and during preheating [15].

All these alternative models to generate the cosmological perturbations have in common that the comoving curvature perturbation in generated on superhorizon scale when the isocurvature perturbation, which is associated to the fluctuations of these light scalar fields different from the inflaton, is converted into curvature perturbation after (or at the end) of inflation. The very simple fact that during inflation the fluctuation associated to these light fields is of the isocurvature type, that is the energy density
stored in these fields is small compared to the vacuum energy responsible for inflation, implies that
the de Sitter isometries are not broken by the presence of these light fields. Therefore their statistical
 correlators should enjoy all the symmetries present during the de Sitter epoch and therefore be not
only scale invariant, but also conformal invariant. Building up on the results of Ref. [16] (where the
most general three-point function for gravitational waves produced during a period of exactly de Sitter
expansion was studied) and of Ref. [17], in Ref. [18] the consequences of scale invariance and special
conformal symmetry of scalar perturbations were discussed. Further extensions appeared in Ref. [19]
where conformal consistency relations for single-field inflation have been investigated and in Ref. [20]
where the existence of non-linearly realized conformal symmetries for scalar adiabatic perturbations in
cosmology has been pointed out.

In this paper we are concerned with the large class of multifield models where the non-Gaussianity
(NG) of the curvature perturbation is sourced by light fields other than the inflaton. By the \( \delta N \) formalism
[21], the comoving curvature perturbation \( \zeta \) on a uniform energy density hypersurface at time \( t_f \) is, on
sufficiently large scales, equal to the perturbation in the time integral of the local expansion from an initial
flat hypersurface \( (t = t_*) \) to the final uniform energy density hypersurface. On sufficiently large scales,
the local expansion can be approximated quite well by the expansion of the unperturbed Friedmann
universe. Hence the curvature perturbation at time \( t_f \) can be expressed in terms of the values of the
relevant scalar fields \( \sigma^I(t_*, \vec{x}) \) at \( t_* \)

\[
\zeta(t_f, \vec{x}) = N_I \sigma^I + \frac{1}{2} N_{IJ} \sigma^I \sigma^J + \cdots ,
\]

(1.1)

where \( N_I \) and \( N_{IJ} \) are the first and second derivative, respectively, of the number of e-folds

\[
N(t_f, t_*, \vec{x}) = \int_{t_*}^{t_f} dt \, H(t, \vec{x}).
\]

(1.2)

with respect to the field \( \sigma^I \). From the expansion (1.1) one can read off the \( n \)-point correlators. For
instance, the three- and four-point correlators of the comoving curvature perturbation, the so-called
bispectrum and trispectrum respectively, is given by

\[
B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3) = N_I N_J N_K B^{IJK}_{\vec{k}_1 \vec{k}_2 \vec{k}_3} + N_I N_J N_K N_L \left( P^{IK} P^{JL}_{\vec{k}_1 \vec{k}_2} + 2 \text{ permutations} \right)
\]

(1.3)
\begin{align*}
T_{\zeta}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) &= N_{IJ}N_{KL}N_{MN}T_{k_1k_2k_3k_4}^{IJKL} \\
&+ N_{IJ}N_{KL}N_{MN}(P_{k_1}^{IL}P_{k_2}^{JM}P_{k_3}^{KN} + 11 \text{ permutations}) \\
&+ N_{IJ}N_{KL}N_{MN}(P_{k_1}^{IL}P_{k_2}^{JM}P_{k_3}^{KN} + 3 \text{ permutations}),
\end{align*}

where

\begin{align*}
\langle \sigma^I_{k_1} \sigma^J_{k_2} \rangle &= (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2) P_{k_1}^{IJ}, \\
\langle \sigma^I_{k_1} \sigma^J_{k_2} \sigma^K_{k_3} \rangle &= (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_{k_1k_2k_3}^{IJK}, \\
\langle \sigma^I_{k_1} \sigma^J_{k_2} \sigma^L_{k_3} \sigma^K_{k_4} \rangle &= (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) T_{k_1k_2k_3k_4}^{IJKL},
\end{align*}

and $\vec{k}_{ij} = (\vec{k}_i + \vec{k}_j)$. We see that the three-point correlator (and similarly for the four-point one) of the comoving curvature perturbation is the sum of two pieces. One, proportional to the three-point correlator of the $\sigma^I$ fields, is model-dependent and present when the fields $\sigma^I$ are intrinsically NG. The second one is universal and is generated when the modes of the fluctuations are superhorizon and is present even if the $\sigma^I$ fields are gaussian. One should keep in mind that the relative magnitude between the two contributions is model-dependent and that the constraints imposed by the symmetries present during the de Sitter stage apply separately to both the first and the second contribution\footnote{The reason is that although the scalar fields $\sigma^I$ may have specific scaling dimension and may transform irreducibly under the conformal group, the comoving curvature perturbations $\zeta$ does not have specific scaling dimension as it is the sum of operators with different dimensions. In other words, $\zeta$ is a reducible representation of the conformal group. However, its $n$-point functions may be specified by the conformal properties of its irreducible components of the conformal group.}. Even though the intrinsically NG contributions to the $n$-point correlators are model-dependent, their forms are dictated by the conformal symmetry of the de Sitter stage (although their amplitudes remain model-dependent). This is the subject of the present paper.

After a brief summary in of the symmetries of the de Sitter geometry in section 2, we will discuss in section 3 the constraints imposed by scale invariance and conformal symmetry on the two- and three-point correlators. In particular, we will demonstrate that the two-point cross-correlations of the light fields vanish unless their conformal weights are equal. This is a in fact a standard result of field theories enjoying conformal symmetry.
We then turn out attention to the operator product expansion technique of conformal field theories to investigate which kind of informations we can gather on inflationary correlations for fields considered at coincidence points. The operator product expansion is very powerful to analyze the squeezed limit of the bispectrum and the collapsed limit of the trispectrum. These limits are particularly interesting from the observationally point of view because they are associated to the local model of NG (for a review see [22]) which leads to pronounced effects of NG on the clustering of dark matter halos and to strongly scale-dependent bias [23].

We use the techniques developed by Ferrara, Gatto and Grillo in the early 70’s to find the model-independent shape of the three- and point-correlators in the squeezed and collapsed limit, respectively. While conformal symmetry does not fix uniquely the shape of the four-point correlator, we show that its shape can be indeed computed in the collapsed limit by using the so-called conformal blocks. This allows us to prove that the contribution to the three- and four-point correlators of the curvature perturbation from the connected three- and four-point correlators of the $\sigma^I$ fields (originated from the fact that these fields can be intrinsically NG) have the same shapes of the universal and model-independent contribution generated when the modes of the fluctuations are superhorizon and present even if the $\sigma^I$ fields are gaussian. This is done in section 4. This result allows us to extend in section 5 the so-called Suyama-Yamaguchi inequality [24] which relates the coefficient of the trispectrum $\tau_{\text{NL}}$ of the curvature perturbation in the collapsed limit to the coefficient $f_{\text{NL}}$ of the squeezed limit of its bispectrum and was proved under the condition that the fluctuations of the scalar fields $\sigma^I$ at the horizon crossing are scale invariant and gaussian. A generalization of this inequality was provided in Refs. [25] to the case of NG $\sigma^I$ fields. However there the crucial assumption was made that the coefficients $f_{\text{NL}}$ and $\tau_{\text{NL}}$ were momentum-independent, which is not automatically guaranteed if the fields are NG. Based on our results stemming from scale invariance and special conformal symmetry, we can show that indeed $f_{\text{NL}}$ and $\tau_{\text{NL}}$ are momentum-independent in the squeezed and collapsed limits respectively and therefore we are able to show that the Suyama-Yamaguchi inequality is valid when the light fields $\sigma^I$ are NG. In fact, we will take a further step and, based on the operators’ short-distance expansion, we will prove that that the Suyama-Yamaguchi inequality holds on general grounds. It is consequence of fundamental physical principles (like positivity of the two-point function) and its hard violation would required some new non-trivial physics to be involved. The observation of a strong violation of the inequality will then have profound implications for inflationary models as it will imply either that multifield inflation cannot be responsible for generating the observed fluctuations independently of the details of the model or that
some new non-trivial degrees of freedom play a role during inflation.

In section 6 we study, even though briefly, the possible implications of another class of conformal theories, namely the logarithmic conformal field theories, which can be of interest from the cosmological point of view. These are theories characterized by the appearance of logarithms in correlation functions due to logarithmic short-distance singularities in the operator product expansion. As a consequence, the spectral index of the curvature perturbation power spectrum gets a new contribution due to logarithmic short-distance singularities in the OPE. This contribution is present even if the fields light involved are massless. Finally, section 7 present our conclusions.

2 Symmetries of the de Sitter geometry

Conformal invariance in three-dimensional space $\mathbb{R}^3$ is connected to the symmetry under the group $SO(1, 4)$ in the same way conformal invariance in a four-dimensional Minkowski spacetime is connected to the $SO(2, 4)$ group. As $SO(1, 4)$ is the isometry group of de Sitter spacetime, a conformal phase during which fluctuations were generated could be a de Sitter stage. In such a case, the kinematics is specified by the embedding of $\mathbb{R}^3$ as flat sections in de Sitter spacetime. The de Sitter isometry group acts as conformal group on $\mathbb{R}^3$ when the fluctuations are superhorizon. It is in this regime that the $SO(1, 4)$ isometry of the de Sitter background is realized as conformal symmetry of the flat $\mathbb{R}^3$ sections $[17, 18]$. Correlators are expected to be constrained by conformal invariance. All these reasonings apply in the case in which the cosmological perturbations are generated by light scalar fields other than the inflaton. Indeed, it is only in such a case that correlators inherit all the isometries of de Sitter.

It is also important to stress that the two-point correlator cannot capture the full conformal invariance and is only sensitive to the scale invariance symmetry. To reveal the full conformal symmetry one needs to consider higher-order correlators. This is what we will do in the following. Before though, and for the sake of self-completeness, we would like to remind the reader of some basic geometrical and algebraic properties of de Sitter spacetime and group $[26]$. The expert reader may skip the following two sections many details of which are contained already in, for instance, Ref. $[17]$.

The four-dimensional de Sitter spacetime of radius $H^{-1}$ is described by the hyperboloid

$$\eta_{AB}X^AX^B = -X_0^2 + X_i^2 + X_5^2 = \frac{1}{H^2} \quad (i = 1, 2, 3),$$

(2.1)

embedded in five-dimensional Minkowski spacetime $\mathbb{M}^{1, 4}$ with coordinates $X^A$ and flat metric $\eta_{AB} =$
A particular parametrization of the de Sitter hyperboloid is provided by

$$
X^0 = \frac{1}{2H} \left( H\eta - \frac{1}{H\eta} \right) - \frac{1}{2} \frac{x^2}{\eta},
$$

$$
X^i = \frac{x^i}{H\eta},
$$

$$
X^5 = -\frac{1}{2H} \left( H\eta + \frac{1}{H\eta} \right) + \frac{1}{2} \frac{x^2}{\eta},
$$

which may easily be checked that satisfies Eq. (2.1). The de Sitter metric is the induced metric on the hyperboloid from the five-dimensional ambient Minkowski spacetime

$$
ds_5^2 = \eta_{AB} dX^A dX^B.
$$

For the particular parametrization (2.2), for example, we find

$$
ds^2 = \frac{1}{H^2\eta^2} (-d\eta^2 + d\vec{x}^2).
$$

The group $SO(1,4)$ acts linearly on $M^{1,4}$. Its generators are

$$
J_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} \quad A, B = (0,1,2,3,5)
$$

and satisfy the $SO(1,4)$ algebra

$$
[J_{AB}, J_{CD}] = \eta_{AD} J_{BC} - \eta_{AC} J_{BD} + \eta_{BC} J_{AD} - \eta_{BD} J_{AC}.
$$

We may split these generators as

$$
J_{ij}, \quad P_0 = J_{05}, \quad \Pi_i^+ = J_{i5} + J_{0i}, \quad \Pi_i^- = J_{i5} - J_{0i},
$$

which act on the de Sitter hyperboloid as

$$
J_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i},
$$

$$
P_0 = \eta \frac{\partial}{\partial \eta} + x^i \frac{\partial}{\partial x^i},
$$

$$
\Pi_i^- = -2H \eta x^i \frac{\partial}{\partial \eta} + H (x^2 \delta_{ij} - 2x_i x_j) \frac{\partial}{\partial x_j} - H \eta^2 \frac{\partial}{\partial x_i},
$$

$$
\Pi_i^+ = \frac{1}{H} \frac{\partial}{\partial x_i},
$$

8
and satisfy the commutator relations

\[ [J_{ij}, J_{kl}] = \delta_{il}J_{jk} - \delta_{ik}J_{jl} + \delta_{jk}J_{il} - \delta_{jl}J_{ik}, \]
\[ [J_{ij}, \Pi^\pm_k] = \delta_{ik}\Pi^\pm_j - \delta_{jk}\Pi^\pm_i, \]
\[ [\Pi^\pm_k, P_0] = \mp\Pi^\pm_k, \]
\[ [\Pi^-_i, \Pi^+_j] = 2J_{ij} + 2\delta_{ij}P_0. \] (2.9)

This is nothing else that the conformal algebra. Indeed, by defining

\[ L_{ij} = iJ_{ij}, \quad D = -iP_0, \quad P_i = -i\Pi^+_i, \quad K_i = i\Pi^-_i, \] (2.10)

we get

\[ P_i = -i\frac{\dot{H}}{H}\partial_i, \]
\[ D = -i\left(\eta\frac{\partial}{\partial\eta} + x^i\partial_i\right), \]
\[ K_i = -2iHx_i\left(\eta\frac{\partial}{\partial\eta} + x^i\partial_i\right) - iH(-\eta^2 + x^2)\partial_i, \]
\[ L_{ij} = i\left(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right). \] (2.11)

These are also the Killing vectors of de Sitter spacetime corresponding to symmetries under space translations \((P_i)\), dilations \((D)\), special conformal transformations \((K_i)\) and space rotations \((L_{ij})\). They satisfy the conformal algebra in its standard form

\[ [D, P_i] = iP_i, \] (2.12)
\[ [D, K_i] = -iK_i, \] (2.13)
\[ [K_i, P_j] = 2i\left(\delta_{ij}D - L_{ij}\right), \] (2.14)
\[ [L_{ij}, P_k] = i\left(\delta_{jk}P_i - \delta_{ik}P_j\right), \] (2.15)
\[ [L_{ij}, K_k] = i\left(\delta_{jk}K_i - \delta_{ik}K_j\right), \] (2.16)
\[ [L_{ij}, D] = 0, \] (2.17)
\[ [L_{ij}, L_{kl}] = i\left(\delta_{il}L_{jk} - \delta_{ik}L_{jl} + \delta_{jk}L_{il} - \delta_{jl}L_{ik}\right). \] (2.18)

The de Sitter algebra \(SO(1, 4)\) has two Casimir invariants

\[ C_1 = -\frac{1}{2}J_{AB}J^{AB}, \] (2.19)
\[ C_2 = W_AW^A, \quad W^A = \epsilon^{ABCDE}J_{BC}J_{DE}. \] (2.20)
Using Eqs. (2.7) and (2.10), we find that
\[ C_1 = D^2 + \frac{1}{2} (P_i, K_i) + \frac{1}{2} L_{ij} L^{ij}, \]  
which turns out to be, in the explicit representation Eq. (2.11),
\[ H^{-2} C_1 = -\frac{\partial^2}{\partial \eta^2} - \frac{2}{\eta} \frac{\partial}{\partial \eta} + \nabla^2. \]  
As a result, \( C_1 \) is the Laplace operator on the de Sitter hyperboloid and for a scalar field \( \phi(x) \) we have
\[ C_1 \phi(x) = \frac{m^2}{H^2} \phi(x). \]

Let us now consider the case \( H \eta \ll 1 \). The parametrization (2.2) turns out then to be
\[ X^0 = \frac{1}{2H^2 \eta} - \frac{1}{2} \frac{x^2}{\eta}, \]
\[ X^i = \frac{x^i}{H \eta}, \]
\[ X^5 = \frac{1}{2H^2 \eta} + \frac{1}{2} \frac{x^2}{\eta}, \]
and we may easily check that the hyperboloid has been degenerated to the hypercone
\[ -X_0^2 + X_5^2 + X_5^2 = 0. \]

We identify points \( X^A \equiv \lambda X^A \) (which turns the cone (2.25) into a projective space). As a result, \( \eta \) in the denominator of the \( X^A \) can be ignored due to projectivity condition. Then, on the cone, the conformal group acts linearly, whereas induces the (non-linear) conformal transformations \( x_i \to x'_i \) with
\[ x'_i = a_i + M^j_i x_j, \]
\[ x'_i = \lambda x_i, \]
\[ x'_i = \frac{x_i + b_i x^2}{1 + 2b_i x_i + b_i x^2}, \]
on Euclidean \( \mathbb{R}^3 \) with coordinates \( x^i \). These transformations correspond to translations and rotations (generated by \( P_i, L_{ij} \)), dilations (generated by \( D \)) and special conformal transformations (generated by \( K_i \)), respectively, acting now on the constant time hypersurfaces of de Sitter spacetime. It should be noted that special conformal transformations can be written in terms of inversion
\[ x_i \to x'_i = \frac{x_i}{x^2} \]
as inversion×translation×inversion.
2.1 Representations

The representations of the $SO(1, 4)$ algebra are constructed by employing the method of induced representations. Let us consider the stability subgroup at $x^i = 0$ which is the group $G$ generated by $(L_{ij}, D, K_i)$. It is easy to see from the conformal algebra, that $P_i$ and $K_i$ are actually raising and lowering operators for the dilation operator $D$. Therefore there should be states which will be annihilated by $K_i$. Every irreducible representation will then be specified by an irreducible representation of the rotational group $SO(3)$ (i.e. its spin) and a definite conformal dimension annihilated by $K_i$. Representations $\phi_s(\vec{0})$ of the stability group at $\vec{x} = \vec{0}$ with spin $s$ and dimension $\Delta$ are specified by

$$ [L_{ij}, \phi_s(\vec{0})] = \Sigma^{(s)}_{ij} \phi_s(\vec{0}), $$

$$ [D, \phi_s(\vec{0})] = -i\Delta \phi_s(\vec{0}), $$

$$ [K_i, \phi_s(\vec{0})] = 0, $$

(2.30)

where $\Sigma^{(s)}_{ij}$ is a spin-$s$ representation of $SO(3)$. Those representations $\phi_s(\vec{0})$ that satisfy the relations (2.30) are called primary fields. Once the primary fields are known, all other fields, the descendants, are constructed by taking derivatives of the primaries $\partial_i \cdots \partial_j \phi_s(\vec{0})$. For scalars in particular, the momentum $P_i$ generates translations so that for a scalar $\phi(\vec{x})$ we have

$$ [P_i, \phi(\vec{x})] = -i\partial_i \phi(\vec{x}). $$

(2.31)

Denoting collectively any generator of the stability subgroup $G$ as $J = (L_{ij}, D, K_i)$ and taking into account that $\phi(\vec{x}) = e^{i\vec{P} \cdot \vec{x}} \phi(\vec{0}) e^{-i\vec{P} \cdot \vec{x}}$, we find that

$$ [J, \phi(\vec{x})] = e^{i\vec{P} \cdot \vec{x}} [\hat{J}, \phi(\vec{0})] e^{-i\vec{P} \cdot \vec{x}} $$

(2.32)

where

$$ \hat{J} = e^{-i\vec{P} \cdot \vec{x}} J e^{i\vec{P} \cdot \vec{x}} = \sum_n \frac{(-i)^n}{n!} x^i_1 x^i_2 \cdots x^i_n [P_{i_1} [P_{i_2} \cdots [P_{i_n}, J], \ldots ] $$

(2.33)

and $\phi(\vec{0})$ is a representation of the stability subgroup. Specifying for $J = L_{ij}, D$ and and $J = K_i$ we find

$$ \hat{L}_{ij} = L_{ij} + x_i P_j - x_j P_i, $$

(2.34)

$$ \hat{D} = D + x^i P_i, $$

(2.35)

$$ \hat{K}_i = K_i + 2(x_i D - x^j L_{ij}) + 2x_i x^j P_j - x^2 P_i. $$

(2.36)
For a scalar, the right-hand side of the first equation in (2.30) vanishes, therefore we find that for a scalar \( \phi(\vec{x}) \)

\[
i[L_{ij}, \phi(\vec{x})] = (x_i \partial_j - x_j \partial_i) \phi(\vec{x}), \tag{2.37}
\]

\[
i[K_i, \phi(\vec{x})] = (2\Delta x_i + 2x_ix^j \partial_j - x^2 \partial_i) \phi(\vec{x}), \tag{2.38}
\]

\[
i[D, \phi(\vec{x})] = (x^i \partial_i + \Delta) \phi(\vec{x}), \tag{2.39}
\]

\[
i[P_i, \phi(\vec{x})] = \partial_i \phi(\vec{x}). \tag{2.40}
\]

Note that Eq. (2.21) gives, for example,

\[
[C_1, \phi(\vec{0})] = -\Delta(\Delta - 3)\phi(\vec{0}), \tag{2.41}
\]

which implies that

\[
m^2 = -\Delta(\Delta - 3)H^2. \tag{2.42}
\]

It can be shown that the scalar representations of the de Sitter group \( SO(1,4) \) actually splits into three distinct series \([27–29]\): the principal series with masses \( m^2 \geq 9H^2/4 \), the complementary series with masses in the range \( 0 < m^2 < 9H^2/4 \) and the discrete series. It is the principal representations which survive the Winger-Inonü contraction \((H \to 0)\) to the Poincaré group.

The method of the induced representations used above for the scalar can be employed to include higher-spin fields as well. For a higher-spin field described by a symmetric-traceless tensor \( \phi_{i_1...i_s} \) we get

\[
i[L_{ij}, \phi_{k_1...k_s}] = (x_i \partial_j - x_j \partial_i + i\Sigma^{(s)}_{ij}) \phi_{k_1...k_s}, \tag{2.43}
\]

\[
i[K_i, \phi_{k_1...k_s}] = (2\Delta x_i + 2x_ix^j \partial_j - x^2 \partial_i + 2ix^j\Sigma^{(s)}_{ji}) \phi_{k_1...k_s}, \tag{2.44}
\]

\[
i[D, \phi_{k_1...k_s}] = (x^i \partial_i + \Delta_s) \phi_{k_1...k_s}, \tag{2.45}
\]

\[
i[P_i, \phi_{k_1...k_s}] = \partial_i \phi_{k_1...k_s}, \tag{2.46}
\]

where the spin operator \( \Sigma^{(s)}_{ij} \) acts as

\[
\Sigma^{(s)}_{ij} \phi_{k_1...k_s} = \sum_{\{a\}} (\phi_{k_1...k_a-1ik_a+1...k_s} \delta_{jk_a} - \phi_{k_1...k_a-1jk_a+1...k_s} \delta_{ik_a}). \tag{2.47}
\]

It is then easy to verify that

\[
C_1 = \frac{m^2}{H^2} = -\Delta_s(\Delta_s - 3) - s(s + 1) \quad \text{since} \quad \frac{1}{2} \Sigma^{(s)}_{ij} \Sigma^{(s)}_{ij} = s(s + 1). \tag{2.48}
\]
3 Symmetry Constraints

Let us consider now the constraints imposed by scale and conformal invariance to the $n$-point correlators. These constraints should be imagined to be applied to the light scalar fields $\sigma^I$ generating the comoving curvature perturbations after (or at the end) of inflation when the isocurvature modes they carry become the curvature mode.

3.1 Scale Invariance

Rotations, translations and dilations form a subgroup of the full conformal group. We would like first to explore the constraints this subgroup imposes on the correlators. Obviously, rotation and translation invariance require correlators of the operators at points $\vec{x}_1$ and $\vec{x}_2$ to depend on $|\vec{x}_1 - \vec{x}_2|$. As is well known, the correlator of two operators is completely determined by their scale dimensions whereas the functional form of three-point correlator is also determined by their dimensions. Taking into account also the scaling of operators under dilations, one finds that two- and three-point functions are specified to be

$$\langle \sigma^I(\vec{x}_1) \sigma^J(\vec{x}_2) \rangle = \frac{c_{IJ}}{|\vec{x}_1 - \vec{x}_2|^{\Delta_I + \Delta_J}},$$  \hspace{1cm} (3.1)$$

$$\langle \sigma^I(\vec{x}_1) \sigma^J(\vec{x}_2) \sigma^K(\vec{x}_3) \rangle = \frac{c_{IJK}}{|\vec{x}_1 - \vec{x}_2|^{w_K} |\vec{x}_2 - \vec{x}_3|^{w_I} |\vec{x}_3 - \vec{x}_1|^{w_J}},$$  \hspace{1cm} (3.2)$$

where $c_{IJ}$ and $c_{IJK}$ are constants setting the amplitude of the correlators, $\sigma^I, \sigma^J, \sigma^K$ are operators of dimensions $\Delta_I, \Delta_J, \Delta_K$ and $(w_I + w_J + w_K) = \Delta_I + \Delta_J + \Delta_K = 3\Delta$. It is straightforward to write two- and three-point correlators in momentum space\footnote{The prime denotes correlators without the $(2\pi)^3\delta^{(3)}(\sum \vec{k})$ factors.}

$$\langle \sigma^I_{\vec{k}_1} \sigma^J_{\vec{k}_2} \rangle' = c_{IJ} k_1^{\Delta_I + \Delta_J - 3},$$  \hspace{1cm} (3.3)$$

$$\langle \sigma^I_{\vec{k}_1} \sigma^J_{\vec{k}_2} \sigma^K_{\vec{k}_3} \rangle' = c_{IJK} \prod_{i=1}^{3} \frac{2^{3-w_I} \pi^{3/2} \Gamma\left(\frac{3-w_I}{2}\right)}{\Gamma\left(\frac{w_i}{2}\right)} \times \int d^3\vec{q} \frac{|\vec{q}|^{w_K-3} |\vec{q} - \vec{k}_1|^{w_I-3} |\vec{q} + \vec{k}_2|^{w_J-3}}{((1-u)X + uY)^{3-\Delta_I/2}} \, 2F_1\left(3-\Delta_I/2, 3-\Delta_K/2, 3; Z\right) + \text{cyclic},$$  \hspace{1cm} (3.4)$$
where we have used the definitions

\[
X = \frac{k_2^2}{k_1^2}, \quad Y = \frac{k_3^2}{k_1^2}, \quad Z = 1 - \frac{u(1-u)}{(1-u)X + uY}.
\]

Eq. (3.4) for the case \( \Delta I = \Delta J = \Delta K = w \) appeared in Ref. [17]. However, we note that the hypergeometric function in (3.4) converges in \(|Z| \leq 1\) for \( \Delta I + \Delta J > 3 \) (3.6) and similarly for any pair of \( \Delta \)'s. Recalling that the scaling dimensions are related to the masses as in Eq. (2.42),

\[
\Delta_{I,J,K} = 3 \frac{2}{9} \frac{1 - \sqrt{1 - 4m_{I,J,K}^2}}{9H^2},
\]

we see that \( 0 \leq \Delta_{I,J,K} \leq 3/2 \) and Eq. (3.6) is never satisfied. What we can do is to use Euler’s transformation of the hypergeometric function

\[
2F_1(a, b, c, z) = (1 - z)^{c-a-b} 2F_1(c - a, c - b, c, z)
\]

(3.8) to express the three-point function in Eq. (3.4) as

\[
\langle \sigma_{k_1}^I \sigma_{k_2}^J \sigma_{k_3}^K \rangle' = c_{IJK} 2^{7-3\Delta} \frac{27}{16} \frac{\Gamma(3 - \frac{3\Delta}{2})\Gamma(\frac{3-\Delta_K}{2})}{\Gamma(\frac{3-\Delta_I}{2})\Gamma(\frac{3-\Delta_J}{2})} \times
\]

\[
\times k_1^{3\Delta-6} \int_0^1 du \frac{(1-u)^{\frac{3\Delta}{2}-1} u^{\frac{3\Delta_J}{2}-1}}{[(1-u)X + uY]^{3-\Delta_K/2}} 2F_1\left(\frac{3\Delta}{2}, 3 - \frac{3 - \Delta_K}{2}, \frac{3}{2}, Z\right) + \text{cyclic},
\]

(3.9)

which is now converging for \( 0 \leq \Delta_{I,J,K} \leq 3/2 \). We stress that the forms (3.3) and (3.9) of the two- and three-point functions respectively, are dictated simply by scale invariance and not by special conformal symmetry. Full conformal invariance give additional constraints.

We may also consider particular limits of the three-point function. As we wrote in the introduction, the so-called squeezed limit \( k_1 \ll k_2 \sim k_3 \) of the three-point function is particularly interesting from the observationally point of view because it is associated to the simplest model of NG, the so-called local one in which the total initial adiabatic curvature is a local function of its gaussian counterpart \( \zeta_g \), e.g.

\[
\zeta = \zeta_g + 3f_{NL}/5(\zeta_g^2 - \langle \zeta_g^2 \rangle) + \cdots
\]

where \( f_{NL} \) the nonlinear coefficient parametrizing the amplitude of NG (for a review see [22]). The local model leads to pronounced effects of NG on the clustering of dark matter halos and to strongly scale-dependent bias [23].
In the squeezed limit the three-point correlator (3.9) for $X \sim Y \gg 1$ turns out to be

$$
\langle \sigma^I_{k_1} \sigma^J_{k_2} \sigma^K_{k_3} \rangle' = c_{IJK} \gamma_s \frac{1}{k_1^{3-\Delta_I-\Delta_J}} \frac{1}{k_2^{3-\Delta_K}} + \text{cyclic},
$$

(3.10)

where

$$
\gamma_s = (2\pi)^3 \frac{\Gamma\left(\frac{3}{2} - \frac{\Delta_K}{2}\right)}{2^{3\Delta-1} \Gamma\left(\frac{3}{2}\right)} \Gamma\left(\frac{3}{2} - \frac{\Delta_K}{2} - \frac{3\Delta}{2}\right).
$$

(3.11)

For the case of scalars of equal dimensions $\Delta_I = \Delta_J = \Delta_K = w$, Eq. (3.9) is written as

$$
\langle \sigma^I_{k_1} \sigma^J_{k_2} \sigma^K_{k_3} \rangle' = c_{IJK} 2^{7-3\Delta} \pi^\frac{5}{2} \frac{\Gamma\left(3 - \frac{3w}{2}\right) \Gamma\left(\frac{3-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)^2} \times \frac{1}{k_1^{3w-6} k_2^{3w-6} k_3^{3w-6}} \int_0^1 du \frac{(1-u)^{\frac{w-1}{2}-1} u^{\frac{w-1}{2}-1}}{[(1-u)X+uY]^\frac{3w}{2}} 2F1\left(\frac{3w}{2} - \frac{3}{2}, \frac{3-w}{2}, \frac{3}{2}, Z\right) + \text{cyclic},
$$

(3.12)

which coincides with the corresponding expression in Ref. [17] after the Euler’s transformation (3.8) in order the hypergeometric function to converge in $|Z| \leq 1$.

Applying the squeezed limit to the expression (3.12) we find that the generic NG three-point function has the form

$$
\langle \sigma^I_{k_1} \sigma^J_{k_2} \sigma^K_{k_3} \rangle' \sim \gamma_s c_{IJK} k_1^{3-2w} k_2^{3-2w} + \text{cyclic} \quad (k_1 \ll k_2 \sim k_3).
$$

(3.13)

This result is dictated by simple scale invariance and not by full conformal symmetry and fixes the shape of the three-point configuration in the squeezed limit up to a model-dependent coefficient $c_{IJK}$. We note that the result (3.13) does not coincide with the squeezed limit found in Ref. [17] for the generic three-point function in the squeezed limit. The reason is that, the authors of Ref. [17] took the squeezed limit of (3.12) before Euler transforming the hypergeometric function in (3.4) which is not defined for scalar masses within the unitarity bounds $0 \leq m_{I,J,K} < 3/2H$. Here, after Euler transforming, the integral in (3.12) is well defined for masses in the unitarity region and the expected behaviour (3.13) is recovered.

If one is interested just in the squeezed limit of the three-point correlator, this may be found quite easily in another way. Let us consider again, for simplicity, the case $\Delta_I = \Delta_J = \Delta_K = w$ so that the three-point function (3.2) is written as

$$
\langle \sigma^I_{k_1} \sigma^J_{k_2} \sigma^K_{k_3} \rangle = \frac{c_{IJK}}{k_1^{3-2w} k_2^{3-2w} k_3^{3-2w}}.
$$

(3.14)

In the limit $x_{23} \sim 0$, $x_{13} = x_{12} \gg x_{23}$ where $x_{ij} = |\vec{x}_i - \vec{x}_j|$, (3.14) may be expressed as

$$
\langle \sigma^I_{k_1} \sigma^J_{k_2} \sigma^K_{k_3} \rangle = \frac{c_{IJK}}{x_{23}^{2w} x_{13}^{2w}}.
$$

(3.15)
By using
\[ \frac{1}{|\vec{x}|^w} = \frac{\Gamma(\frac{3-w}{2})}{2^{w/2}\pi^{3/2}\Gamma(\frac{w}{2})} \int d^3k \, |\vec{k}|^{w-3}e^{-i\vec{k} \cdot \vec{x}}, \]  
(3.16)
for each factor in the denominator of (3.15) and Fourier transforming it, we get
\[ \langle \sigma^I_{\vec{k}_1}\sigma^J_{\vec{k}_2}\sigma^K_{\vec{k}_3} \rangle \propto \int d^3\vec{x}_1d^3\vec{x}_2d^3\vec{x}_3e^{i\vec{k}_1 \cdot \vec{x}_1+i\vec{k}_2 \cdot \vec{x}_2+i\vec{k}_3 \cdot \vec{x}_3} \int d^3\vec{q}_1d^3\vec{q}_2 \frac{e^{-i\vec{q}_1 \cdot (\vec{x}_2-\vec{x}_3) - i\vec{q}_2 \cdot (\vec{x}_1-\vec{x}_3)}}{|\vec{q}_1|^{3-w}|\vec{q}_2|^{3-2w}}. \]  
(3.17)
The integration of space specify the external momenta \( \vec{k}_i \) are given by
\[ \vec{k}_1 = \vec{q}_2, \quad \vec{k}_2 = \vec{q}_1, \quad \vec{k}_3 = \vec{q}_1 + \vec{q}_2. \]  
(3.18)
Finally, integration over the internal momenta \( \vec{q}_i \), specifies the three-point function in the squeezed limit to be
\[ \langle \sigma^I_{\vec{k}_1}\sigma^J_{\vec{k}_2}\sigma^K_{\vec{k}_3} \rangle' \sim \frac{1}{k_1^{3-2w}k_2^{3-w}}. \]  
(3.19)
This is what we found above in Eq. (3.13) by using the exact expression of the three-point correlator. Now, since \( |\vec{q}_1| \gg |\vec{q}_2| \) as \( \vec{q}_1, \vec{q}_2 \) are conjugate momenta of \( x_{23} \) and \( x_{13} = x_{12} \) (with \( x_{23} \to 0, \ x_{13} = x_{12} \gg x_{23} \)), we get that (3.19) is valid in the \( k_1 \ll k_2 \sim k_2 \). In other words, the local shape of the three-point function corresponds to two point close and one remote point in three-dimensional space as shown in Fig. 1. In

![Figure 1](attachment:image.png)

**Figure 1:** (a) Squeezed three-point configuration with two points \( \vec{x}_2 \) and \( \vec{x}_3 \) very close (O being the origin) and the third one \( \vec{x}_1 \) far away from the rest, i.e. \( x_{23} \simeq 0, \ x_{13} = x_{12} \gg x_{23} \). (b) Local shape in \( k \)-space with \( k_1 \ll k_2 \sim k_2 \).

the equilateral case \( k_1 = k_2 = k_3 = k \), we have \( X = Y = 1 \) and \( Z = 1 - u(1 - u) \) and thus
\[ \langle \sigma^I_{\vec{k}_1}\sigma^J_{\vec{k}_2}\sigma^K_{\vec{k}_3} \rangle' \sim c_{IJK}\gamma e k^{3\Delta-6}, \]  
(3.20)
where
\[
\gamma_c = 2^{10-3\Delta} \frac{\Gamma(3-\frac{3\Delta}{2})\Gamma(\frac{3-\Delta K}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{3-\Delta K}{2})} \times 
\int_0^1 du \left(1-u\right)^{\frac{1}{2}} \frac{\Delta \pi}{\Delta} u^{\frac{3\Delta}{2}} \frac{\Delta}{\pi} \left(3-\frac{3\Delta}{2}, \frac{\Delta K}{2}, \frac{3}{2}, 1-u\right) \right). \tag{3.21}
\]

Finally, let us comment on the massless limit
\[
w \sim \frac{3}{2} \left(1 - \sqrt{\frac{4m_I^2}{9H^2}}\right) \ll 1. \tag{3.22}
\]

The two- and three-point functions are to be obtained in the $\Delta_I \sim 0$ limit. This limit can be smoothly found by employing Eq. (3.16), which expanded around $w_I = 0$, gives
\[
\lim_{w_I \to 0} \frac{1}{|\vec{x}|^{w_I}} = \lim_{w_I \to 0} \frac{\Gamma(3-w_I)}{(2\pi)^{3/2}\Gamma(w_I/2)} \int d^3k |\vec{k}|^{w_I-3} e^{-i\vec{k} \cdot \vec{x}} \left(\gamma - 2 \ln 2 - \psi \left(\frac{3}{2}\right)\right) \left(\ln |\vec{x}| - \frac{1}{2}\gamma\right) + \frac{w_I^2}{8\pi} \ln^2 |\vec{x}| - \gamma \ln |\vec{x}| + \frac{1}{24} (6\gamma + \pi^2), \tag{3.23}
\]
where we have used
\[
\int d^3k e^{i\vec{k} \cdot \vec{x}} \frac{4\pi}{|\vec{x}|^{2n+3}} = \frac{(-1)^n \pi^{3/2}}{n!2^n \Gamma(n+\frac{3}{2})} \left(|\vec{x}|^{2n} \ln |\vec{x}| + \psi(n+1)\right). \tag{3.24}
\]
As a result, the most singular behaviour of the two- and three-point functions in the $w_I \approx m_I^2/3H^2 \ll 1$ limit is
\[
\langle \sigma^I(\vec{x}_1)\sigma^I(\vec{x}_2) \rangle \sim \ln |\vec{x}_1 - \vec{x}_2|, \tag{3.25}
\]
\[
\langle \sigma^I(\vec{x}_1)\sigma^I(\vec{x}_2)\sigma^I(\vec{x}_3) \rangle \sim \ln |\vec{x}_1 - \vec{x}_2| \ln |\vec{x}_1 - \vec{x}_3| \ln |\vec{x}_3 - \vec{x}_2|. \tag{3.26}
\]

### 3.2 Conformal Invariance

Let us now enhance the symmetry by demanding also invariance under special conformal transformations generated by $K_i$. In this case, the symmetry turns out to be the full conformal invariance generated by rotations, translations, dilatations and special conformal transformations. Correlators are more restricted now as special conformal transformations gives additional conditions. Since, special conformal transformations are reduced to inversion, it is enough to consider just transformations under space inversion.
correlated at the level of two-point correlators if their masses are different. This seems to have passed i.e. special conformal transformations, operators with different dimensions and, second, the three-point functions are completely specified by the special conformal symmetry has two consequences. First, the two-point functions are zero for conformal dimensions $\Delta_I$ and $\Delta_J$ respectively, transforms as

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2) \rangle \rightarrow \left| \frac{\partial x'_i}{\partial x_j} \right|^{\Delta_I/3} \left| \frac{\partial x'_i}{\partial x_j} \right|^{\Delta_J/3} \langle \sigma^I(\vec{x}'_1)\sigma^J(\vec{x}'_2) \rangle$$

where $|\partial x'_i/\partial x_j|$ is the Jacobian of the transformation. For the space inversion (2.29), the two-point function (3.1), the form of which was forced by scale invariance, transforms as

$$\langle \sigma^I(\vec{x}_1)\sigma^I(\vec{x}_2) \rangle \rightarrow \frac{(r_1r_2)^{\Delta_I+\Delta_J}}{r_1^{2\Delta_I}r_2^{2\Delta_J}} \langle \sigma^I(\vec{x}_1)\sigma^I(\vec{x}_2) \rangle,$$

where for $\vec{x}' = \vec{x}/|\vec{x}|^2$ we have used that

$$\left| \frac{\partial x'_i}{\partial x_j} \right| = \frac{1}{|\vec{x}|^6}, \quad |\vec{x}_1 - \vec{x}_2| = \frac{|\vec{x} - \vec{x}'_2|}{r_1^2 r_2^2},$$

and the notation $|\vec{x}_1| = r_1, |\vec{x}_2| = r_2$. Thus, space inversion leaves the two point function invariant if

$$\Delta_I = \Delta_J.$$  (3.30)

Similarly, the three-point function transforms as

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3) \rangle \rightarrow \left| \frac{\partial x'_i}{\partial x_j} \right|^{\Delta_I/3} \left| \frac{\partial x'_i}{\partial x_j} \right|^{\Delta_J/3} \left| \frac{\partial x'_i}{\partial x_j} \right|^{\Delta_K/3} \langle \sigma^I(\vec{x}'_1)\sigma^J(\vec{x}'_2)\sigma^K(\vec{x}'_3) \rangle$$

and using (3.29), we get that (3.2) is invariant if

$$w_K = \Delta_I + \Delta_J - \Delta_K, \quad w_I = \Delta_J + \Delta_K - \Delta_I, \quad w_J = \Delta_I + \Delta_K - \Delta_J.$$  (3.32)

As a result, two- and three-point function are conformal invariant if they have the form

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2) \rangle = \begin{cases} \frac{c_{IJ}}{|\vec{x}_1 - \vec{x}_2|^{\Delta_I + \Delta_J}}, & \Delta_I = \Delta_J, \\ 0, & \Delta_I \neq \Delta_J, \end{cases}$$

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3) \rangle = \frac{c_{IJ}}{|\vec{x}_1 - \vec{x}_2|^{\Delta_I + \Delta_J - \Delta_K}|\vec{x}_2 - \vec{x}_3|^{\Delta_J + \Delta_K - \Delta_I}|\vec{x}_3 - \vec{x}_1|^{\Delta_I + \Delta_K - \Delta_J}},$$

where again $\sigma^{I,J,K}$ are operators of dimensions $\Delta_{I,J,K}$. In other words, enhancing the symmetry including the special conformal symmetry has two consequences. First, the two-point functions are zero for operators with different dimensions and, second, the three-point functions are completely specified by special conformal transformations, i.e. by the full conformal symmetry.

We deduce that in multi field models, conformal symmetry imposes that the scalar fields are uncorrelated at the level of two-point correlators if their masses are different. This seems to have passed
unnoticed in the recent literature on the subject and usually the fact that the $\sigma^I$ fields are not correlated is taken as an assumption. We see that in fact it is a consequence of the conformal symmetry: if the conformal weights of two light fields are different, then their cross-correlation vanishes:

\[
\text{If } \Delta_I \neq \Delta_J \Rightarrow \langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2) \rangle = 0.
\] (3.35)

Although this is a classical result, quantum corrections may induce anomalous dimensions to the fields. However, as long as two fields have different dimensions at some order in perturbation theory (or even non-perturbatively), their two-point function vanishes by conformal invariance at that order. On the other hand, it may happen that two fields have the same dimension at the classical level, for example due to the same tree-level mass. However, if there is no symmetry to protect this tree-level relation, interactions will spoil it by introducing different dimensions to the fields. In this case, although classically they have a non-zero two-point correlator, the latter will vanish at the quantum level (as long as there is no conformal anomaly).

4 The operator product expansion and the NG correlators

After this excursion on the the symmetries present in a de Sitter geometry and the constraints they impose on the two-and three-point correlators of light fields and their shapes, let us proceed with the more original part of the work and consider the informations we can gather using the Operator Product Expansion (OPE) for fields considered at coincidence points when the system enjoys the symmetries of the de Sitter geometry. As we shall see, the OPE is particularly useful and powerful to characterize in their full generality the squeezed limit of the three-point correlator and the collapsed limit of the four-point correlator which are interesting limits from the observationally point of view.

The OPE has been established in perturbative quantum field theories. It is by now a standard tool in the analysis of gauge theories such as QCD and Wilson’s OPE [30] is the basis of virtually all calculations of nonperturbative effects in analytical QCD. It is believed that all quantum field theories with well-behaved ultraviolet behavior have an operator product expansion (OPE) [30–32]. This has been proven for conformally invariant quantum field theories [33,34]. In particular, OPE in two-dimensional conformal field theories has played a major role in the development of string theory and critical phenomena [35]. In addition, in search for a more solid foundation of OPE expansion, formal mathematical proofs of its existence and validity have also been given within various axiomatic settings [36] for quantum field
theory on Minkowski spacetime. There are also formulations of the operator product expansion of local operators in curved spacetimes \[37\].

Let us consider two generic operators \(\sigma^I(\vec{x})\) and \(\sigma^J(\vec{y})\) at the points \(\vec{x}\) and \(\vec{y}\) on a \(\eta = \text{const.}\) hypersurface of de Sitter spacetime. Then, we expect that the product of local operators are distances small compared to the characteristic length of the system should look like a local operator. As a result, we expect that the product of \(\sigma^I(\vec{x})\sigma^J(\vec{y})\) of the two operators \(\sigma^I(\vec{x})\) and \(\sigma^J(\vec{y})\), located at nearby points \(\vec{x}\) and \(\vec{y}\), will have a short-distance expansion of the form \[30\]

\[
\sigma^I(\vec{x})\sigma^J(\vec{y}) \sim_{\vec{x} \to \vec{y}} \sum_n C_n(\vec{x} - \vec{y})O_n(\vec{y}),
\]

where \(C_n(\vec{x} - \vec{y})\) are c-number functions (in fact distributions) and \(O_n\) local operators. Moreover, for \(H\eta \ll 1\) we expect the OPE \[4.1\] to respect the symmetries of the de Sitter spacetime realized non-linearly on the \(\eta = \text{const.}\) hypersurface. In other words, we expect \[4.1\] to enjoy conformal three-dimensional symmetry. Note that the OPE above can also be written as

\[
\sigma^I(\vec{x})\sigma^J(\vec{0}) \sim_{\vec{x} \to \vec{0}} \sum_{n,s} C_{ns}(\vec{x})x^{i_1}x^{i_2}\ldots x^{i_s}O^{(ns)}_{i_1i_2\ldots i_s}(\vec{0}),
\]

since due to translational invariance we have taken \(\vec{y} = \vec{0}\) and

\[
O_n(\vec{x}) = e^{iP_ix^i}O_n(\vec{0})e^{-iP_ix^i} = \sum_{s} \frac{1}{s!} x^{i_1}x^{i_2}\ldots x^{i_s}O^{(ns)}_{i_1i_2\ldots i_s}(\vec{0}),
\]

where

\[
O^{(ns)}_{i_1i_2\ldots i_s}(\vec{0}) = (-i)^n[P_{i_1}, [P_{i_2}, \ldots [P_{i_s}, O^{(n)}(\vec{0})], \ldots].
\]

We assume now that the local operators \(O^{(ns)}_{i_1\ldots i_s}(\vec{x})\) transforms under rotations, translations and dilations as \[38\]

\[
i[L_{ij}, O^{(ns)}_{i_1\ldots i_s}] = (x_i\partial_j - x_j\partial_i + i\Sigma^{(s)}_{ij})O^{(ns)}_{i_1\ldots i_s},
\]

\[
i[D, O^{(ns)}_{i_1\ldots i_s}] = (x^i\partial_i + \Delta_s)O^{(ns)}_{i_1\ldots i_s},
\]

\[
i[K_i, O^{(ns)}_{i_1\ldots i_s}] = (2\Delta x_i + 2x_i x^j \partial_j - x^2 \partial_i + 2ix^j\Sigma^{(s)}_{ji})O^{(ns)}_{i_1\ldots i_s},
\]

\[
i[P_i, O^{(ns)}_{i_1\ldots i_s}] = \partial_i O^{(ns)}_{i_1\ldots i_s},
\]

where, as before \(\Sigma^{(s)}_{ij}, \Delta, K_i\) are representations of the stability group at \(\vec{x} = \vec{0}\) and the index \(n\) just labels the representations of the latter. The operators \(O^{(ns)}_{i_1\ldots i_s}(\vec{x})\) are the lowest dimensional operator (of
dimension $w_n$, they commute with $K_i$ and are the primary fields in the theory. The action of $\Delta$ on a
representation $\mathcal{O}^{(ns)}_{11\ldots i_s}(\vec{x})$ is

$$[\mathcal{O}^{(ns)}_{11\ldots i_s}(\vec{x}), \Delta] = i(w_n + m - n)\mathcal{O}^{(ns)}_{11\ldots i_s}(\vec{x}).$$

(4.9)

For operators $A$ and $B$ of dimensions $w_I$ and $w_J$ respectively, scale invariance provides the condition

$$[\sigma^I(\vec{x})\sigma^J(\vec{0}), D] = (x^i\partial_i + w_I + w_J)\sigma^I(\vec{x})\sigma^J(\vec{0}).$$

(4.10)

By employing Eqs. (4.6) and (4.9), we obtain the following equation for $C_{ns}(\vec{x})$

$$x^i\partial_iC_{ns} + (w_n - n + w_I + w_J)C_{ns} = 0.$$  

(4.11)

Thus, scale invariance specifies $C_{ns} \sim x^{w_n-n-w_I-w_J}$, so that Eq. (4.2) turns out to be

$$\sigma^I(\vec{x})\sigma^J(\vec{0}) \sim \left( \frac{1}{|\vec{x}|} \right)^{w_I+w_J-w_n+n}\sum_s C_{ns}x^{i_1}x^{i_2}\ldots x^{i_s}\mathcal{O}^{(ns)}_{11i_2\ldots i_s}(\vec{0}).$$

(4.12)

The contribution of the scalar $s = 0$ sector is, for example,

$$\sigma^I(\vec{x})\sigma^J(\vec{0}) \sim \left( \frac{1}{|\vec{x}|} \right)^{w_I+w_J} \left\{ C_0 + |\vec{x}|^{-w_0} \left( C_1\mathcal{O}(\vec{0}) + C_1x^i\partial_i\mathcal{O}(\vec{0}) + \ldots \right) \right\},$$

(4.13)

where the dots stand for the contributions of higher spin descendents. By demanding simply scale
invariance, we have arrived in the short-distance expansion (4.12). We may continue and investigate
the constraints which full conformal symmetry further imposes. To do that, we have just to impose
invariance under special conformal transformations. Although it is a straightforward process, it is quit
involved and we point out here only the main steps repeating basically the corresponding steps taken by
Ferrara, Gatto and Grillo in Refs. 39, 40 for the $SO(2,4)$ case. Commuting (4.12) with the generator of
special conformal transformations, we get

$$[\sigma^I(\vec{x})\sigma^J(\vec{0}), K_i] = -i \sum_n \left( \frac{1}{|\vec{x}|} \right)^{w_I+w_J-w_n+n}\sum_s C_{ns}x^{i_1}x^{i_2}\ldots x^{i_s}[\mathcal{O}^{(ns)}_{11i_2\ldots i_s}(\vec{0}), K_i].$$

(4.14)

Then, by using Eqs. (2.47) and (4.7), we get

$$[\sigma^I(\vec{x})\sigma^J(\vec{0}), K_i] = -i \sum_{n=0} \left( \frac{1}{|\vec{x}|} \right)^{w_I+w_J-w_n+n}\sum_{s=n} C_{ns}(w_I+2s-w_J+n-w_0)x_i^{i_1}x^{i_2}\ldots x^{i_s}\mathcal{O}^{(ns)}_{11i_2\ldots i_s}(\vec{0})$$

$$= -i \sum_{n=0} \left( \frac{1}{|\vec{x}|} \right)^{w_I+w_J-w_n+n}\sum_{s=0} C_{ns}(s+1-n)(w_n+s+1)x_i^{i_1}x^{i_2}\ldots x^{i_s}\mathcal{O}^{(ns)}_{11i_2\ldots i_s}(\vec{0}).$$

(4.15)
From the above relation, we obtain the recurrence equation

$$2k(w_n + n + k)C_{n,n+k} - (w_I - w_J + w_n + n - 2k - 2)C_{n,n+k} = 0,$$

which is solved by

$$C_{n,n+k} = \frac{\Gamma \left( \frac{1}{2}(w_I - w_J + n) + k \right) \Gamma (w_n + n)}{k! \Gamma \left( \frac{1}{2}(w_I - w_J + n) \right) \Gamma (w_n + n + k)} C_{n,n}.$$  

(4.17)

By recalling that

$$\sum_{m=n}^{\infty} C_{nm} z^{m-n} = \sum_{k=0}^{\infty} C_{n,n+k} z^k = \sum_{k=0}^{\infty} \frac{\Gamma \left( \frac{1}{2}(w_I - w_J + n) + k \right) \Gamma (w_n + n + k)}{k! \Gamma \left( \frac{1}{2}(w_I - w_J + n) \right) \Gamma (w_n + n + k)} C_{n,n} z^n,$$

we get finally

$$\sigma^I(x) \sigma^J(0) \sim \sum_n C_n \frac{1}{x^{\ell_1+\ell_2} x^{i_1} x^{i_2} \cdots x^{i_n}} \Phi \left( \frac{1}{2}(\ell_1 - \ell_2); w_n + n; x^i \partial_i \right) O_{i_1 i_2 \cdots i_n}(\vec{0}).$$  

(4.19)

We have indicated here

$$\ell_1 = w_I + n, \quad \ell_2 = w_J - w_n,$$

(4.20)

and \(\Phi \left( \frac{1}{2}(w_I - w_J + w_n + n); w_n + n; x^i \partial_i \right)\) is the confluent hypergeometric function defined as

$$\Phi(a; b; x^i \partial_i) = \sum_n \frac{(a)_n}{(b)_n} (x^i \partial_i)^n.$$  

(4.21)

Moreover, \((a)_n, (b)_n\) are the Pochhammer symbols for \(a = \frac{1}{2}(w_I - w_J + w_n + n)\) and \(b = w_n + n\). By using the integral representation of the confluent hypergeometric function

$$\Phi(a; b; z) = \frac{\Gamma(b)}{\Gamma(b-a) \Gamma(a)} \int_0^1 du u^{a-1} (1 - u)^{b-a-1} e^{uz},$$  

(4.22)

and

$$e^{ux^i \partial_i} f(\vec{0}) = f(ux^i),$$  

(4.23)

we may express the OPE (4.19) as

$$\sigma^I(x) \sigma^J(0) \sim \sum_n C_n \frac{1}{x^{\ell_1+\ell_2} x^{i_1} x^{i_2} \cdots x^{i_n}} \int d^3k \Phi \left( \frac{1}{2}(\ell_1 - \ell_2); w_n + n; ik \cdot x \right) \tilde{O}_{i_1 i_2 \cdots i_n}(\vec{k}),$$  

(4.24)
where $\tilde{O}_{i_1 i_2 \cdots i_n}(\k)$ is the Fourier transform of $O_{i_1 i_2 \cdots i_n}(\vec{x})$. It should be stressed that the OPE also determines the structure of the $n$-point functions. For instance, the two point-function of two scalar operators $\sigma^I$ and $\sigma^J$ of dimensions $w_I$ and $w_J$ respectively is given by Eq. (4.19) as

$$\langle \sigma^I(\vec{x})\sigma^J(\vec{0}) \rangle \sim \sum_n C_n \frac{1}{x_1^{\ell_1 + \ell_2} x^{i_1} x^{i_2} \cdots x^{i_n}} \int \frac{d^3 k}{(2\pi)^3} \Phi \left( \frac{1}{2}(\ell_1 - \ell_2); w_n + n; i\vec{k} \cdot \vec{x} \right) \langle \tilde{O}_{i_1 i_2 \cdots i_n}(\k) \rangle$$

$$= \frac{1}{x^{w_I + w_J}} \int \frac{d^3 k}{(2\pi)^3} \Phi \left( \frac{1}{2}(w_I - w_J); 0; i\vec{k} \cdot \vec{x} \right) \langle \tilde{O}(\k) \rangle$$

since from rotational invariance of the vacuum, only $SO(3)$ singlets will contribute to the right-hand side of Eq. (4.25), that is only operators for which $n = 0$ and $w_O = 0$ operators. Then, it is clear that only for $w_I \neq w_J$, the integral in the right-hand side of Eq. (4.25) is a function of $\vec{x}$. However, by scale invariance, the integral should be $\vec{x}$-independent and, in fact, it can only be a numerical constant. This is however possible only for $w_I = w_J$ for which $\Phi \left( 0; 0; i\vec{k} \cdot \vec{x} \right) = 1$. For $w_I \neq w_J$, $\langle \tilde{O} \rangle = 0$ for a dimensionful operator since otherwise conformality will be lost. As result, with $C_O$ the coefficient of the identity operator in the operator product expansion, we get

$$\langle \sigma^I(\vec{x})\sigma^J(\vec{0}) \rangle = \begin{cases} \frac{C_O}{x^{w_I + w_J}} & w_I = w_J, \\ 0 & w_I \neq w_J. \end{cases}$$

i.e. we recover Eq. (3.33) as expected.

### 4.1 The three-point function from the OPE and its squeezed limit

Similar considerations also may apply to three-point, or generally to $n$-point correlators. By employing the integral representation (4.22) of the confluent hypergeometric function in the OPE (4.19), we get

$$\sigma^I(\vec{x})\sigma^J(\vec{0}) \frac{\vec{x} \cdot \vec{0}}{3} \sim \sum_n C_n \frac{1}{x_1^{\ell_1 + \ell_2} x^{i_1} x^{i_2} \cdots x^{i_n}} \int_0^1 du u^{a-1}(1 - u)^{b-a-1} O_{i_1 \cdots i_n}(u\vec{x}) \tag{4.27}$$

This form of the OPE is quite appropriate to calculate the three-point correlator $\langle \sigma^I(\vec{x})\sigma^J(\vec{0})\sigma^K(\vec{y}) \rangle$ of three scalar operators $A, B$ and $C$, with dimensions $w_I, w_J$ and $w_C$ respectively. We find

$$\langle \sigma^I(\vec{x})\sigma^J(\vec{0})\sigma^K(\vec{y}) \rangle \frac{\vec{x} \cdot \vec{0}}{3} \sim \sum_n C_n \frac{1}{x_1^{\ell_1 + \ell_2} x^{i_1} x^{i_2} \cdots x^{i_n}} \int_0^1 du u^{a-1}(1 - u)^{b-a-1} \langle O_{i_1 \cdots i_n}(u\vec{x}) \sigma^K(\vec{y}) \rangle \tag{4.28}$$

We now use the orthogonality of the two-point correlator. This means, in particular, that only scalar operators $O$ ($n = 0$) will contribute to the right-hand side of Eq. (4.28)

$$\langle \sigma^I(\vec{x})\sigma^J(\vec{0})\sigma^K(\vec{y}) \rangle \frac{\vec{x} \cdot \vec{0}}{3} \sim C_O \frac{1}{x^{w_I + w_J - w_O}} \int_0^1 du u^{a-1}(1 - u)^{w_O - a-1} \langle O(u\vec{x})\sigma^K(\vec{y}) \rangle \tag{4.29}$$
The two-point function \( \langle O(u\vec{x})\sigma^K(\vec{y}) \rangle \) is given by
\[
\langle O(u\vec{x})\sigma^K(\vec{y}) \rangle \sim \begin{cases} 
\frac{1}{|\vec{y} - u\vec{x}|^{2w_C}} & w_C = w_C, \\
\frac{1}{0} & w_C \neq w_C.
\end{cases}
\] (4.30)
and Eq. (4.29) turns out to be
\[
\langle \sigma^I(\vec{x})\sigma^J(\vec{0})\sigma^K(\vec{y}) \rangle \sim \frac{1}{x_{w_I + w_J - w_C}} \int_0^1 du \frac{u^{a_1 - 1}(1 - u)^{w_C - a_1}}{(y^2 - 2u \vec{x} \cdot \vec{y})^{w_C}}.
\] (4.31)
Using Feynman parameters
\[
\int_0^1 du \frac{u^{a_1 - 1}(1 - u)^{a_2 - 1}}{(uD_1 + (1 - u)D_2)^{a_1 + a_2}} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} \frac{1}{D_1^{a_1}D_2^{a_2}},
\] (4.32)
we have
\[
\int_0^1 du \frac{u^{a_1 - 1}(1 - u)^{w_C - a_1}}{(y^2 - 2u \vec{x} \cdot \vec{y})^{w_C}} = \int_0^1 du \frac{u^{a_1 - 1}(1 - u)^{w_C - a_1}}{(u(y^2 - 2u \vec{x} \cdot \vec{y}) + (1 - u)y^2)^{w_C}}
\] = \[
\frac{\Gamma(w_C - a)}{\Gamma(w_C^C)} \frac{1}{\Gamma(w_C)} \frac{1}{(y^2 - 2u \vec{x} \cdot \vec{y})^{w_C - a}},
\] (4.33)
and thus, the three-point function as specified by the OPE (4.31) is found to be
\[
\langle \sigma^I(\vec{x})\sigma^J(\vec{0})\sigma^K(\vec{y}) \rangle \sim \frac{1}{|\vec{x}|^{w_I + w_J - w_C}} \frac{1}{|\vec{y} - \vec{x}|^{w_I + w_J + w_C}} \frac{1}{|\vec{y}|^{w_J + w_C - w_I}}.
\] (4.34)
The OPE’s are particularly appropriate for calculating the squeezed limit of correlators. Let us consider for example a single scalar whose two-point correlator is
\[
\langle \sigma(\vec{x})\sigma(\vec{y}) \rangle \sim \frac{1}{|\vec{x} - \vec{y}|^{2w}}.
\] (4.35)
Let us consider now the OPE
\[
\sigma(\vec{x})\sigma(\vec{0}) \sim \sum_n C_n \frac{1}{x_{\ell_1 + \ell_2}x_{i_1}x_{i_2} \cdots x_{i_n}} \int d^3k \Phi\left(\frac{1}{2}(\ell_1 - \ell_2); w_n + n; ik \cdot \vec{x}\right) \tilde{O}_{i_1 \cdots i_n}(\vec{k})
\] (4.36)
and multiply the above expression by \( \sigma(\vec{y}) \) (with \( |\vec{y}| \gg |\vec{x}| \approx 0 \)). Fourier transforming it we get
\[
\langle \sigma_{k_1}\sigma_{k_2}\sigma_{k_3} \rangle \sim \sum_n C_n \int d^3x e^{ik_1 \cdot \vec{x}} \frac{1}{x_{\ell_1 + \ell_2}x_{i_1}x_{i_2} \cdots x_{i_n}}
\times \int d^3k \Phi\left(\frac{1}{2}(\ell_1 - \ell_2); w_n + n; ik \cdot \vec{x}\right) \langle \tilde{O}_{i_1 \cdots i_n}(\vec{k})\sigma_{E_1} \rangle,
\] (4.37)
since \( \vec{k}_1 \) and \( \vec{k}_2 \) are the dual vectors to \( \vec{y} \) and \( \vec{x} \), respectively. Using again the orthogonality of the two-point function, we get that only the scalar \( O = \sigma \) will contribute to the right-hand side of (4.37), i.e.\( n = 0, w_n = w, \ell_2 = 0, \ell_1 = w \). Eq. (4.37) turns out to be
24
dependent amplitude. For instance, this happens in the cubic model with interaction $L \supset (m/3)\sigma^3$. They originate from the fact that the perturbation mode, after horizon crossing, has a nontrivial evolution due to the nonlinearities. Note that the OPE expansion is not sensitive to contact terms so factors of the form $\ln(-k_i \eta)$ cannot be detected in the squeezed limit, however, this time dependence is likely to disappear at the level of correlators of the light fields when a consistent resummation of the IR effects is performed [42].

\[
\langle \sigma_{k_1} \sigma_{k_2} \sigma_{k_3} \rangle' \sim (2\pi)^3 \frac{\Gamma(\frac{3-w}{2}) \Gamma(w)}{\Gamma(\frac{3}{2} - w) k_{k_1}^w} P_{k_1} P_{k_2} \left\{ 1 + \alpha_0 \left( \frac{k_1^2}{k_2^2} + (w-5) \frac{(k_1 \cdot k_2)^2}{k_2^4} \right) \right\} \] (k_1 \ll k_2 \sim k_3).
\] (4.42)

Footnote: Tree-level computations of the three-point correlator (as well as of the four-point correlator) may lead to the presence of logarithmic factors $\ln(-k_i \eta)$, where $k_i = (k_1 + k_2 + k_3)$. For instance, this happens in the cubic model with interaction $L \supset (m/3)\sigma^3$. They originate from the fact that the perturbation mode, after horizon crossing, has a nontrivial evolution due to the nonlinearities. Note that the OPE expansion is not sensitive to contact terms so factors of the form $\ln(-k_i \eta)$ cannot be detected in the squeezed limit, however, this time dependence is likely to disappear at the level of correlators of the light fields when a consistent resummation of the IR effects is performed [42].
where the cyclic terms have been taken into account, reproducing the result \(3.13\). Note that terms linear in \(|\vec{k}_1|\) from \(A_2\) do not contribute as they cancel when cyclicity is considered. For an almost scale invariant spectrum \(w \approx 0\) the above expression reduces to

\[
\langle \sigma_{\vec{k}_1} \sigma_{\vec{k}_2} \rangle' \sim P_{\vec{k}_1} P_{\vec{k}_2} \left( 1 + \frac{3}{4} \left( \frac{k_1^2}{k_2^2} - 5 \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_2^4} \right) \right) \quad (k_1 \ll k_2 \sim k_3).
\]

(4.44)

4.2 The four-point function from the OPE and its collapsed limit

We have seen above that the OPE encodes the conformal structure of the conformal field theory. In particular, for primary operators the OPE forms a kind of algebra in the sense that the product of two primaries at short distance may be expressed as a series of local operators. For example, the product of two operators \(\sigma^I\) and \(\sigma^J\) of conformal dimensions \(w_I\) and \(w_J\) respectively can be expanded in terms of local operators, collectively denoted by \(O_{(m)}\) of dimension \(w_O\) as

\[
\sigma^I(\vec{x}) \sigma^J(\vec{0}) \approx \sum_{O} f_{IJ} O \left\{ C^{(m)}(\vec{x}) O_{(m)}(\vec{0}) + \ldots \right\},
\]

(4.45)

where only primaries are needed to be included in the right-hand side of (4.45). The coefficient \(C^{(m)}(\vec{x})\) are generally given by

\[
C^{(m)}(\vec{x}) = \frac{x^{i_1} \ldots x^{i_m}}{|\vec{x}|^\ell}, \quad \ell = w^I + w^J - w_O + m.
\]

(4.46)

and the dots in (4.45) represent less singular contributions. The structure of the OPE (4.45) is similar to a Lie algebra. There, an arbitrary product of generators (the enveloping algebra), can be reduced to a product of two generators by employing the commutation relation. The same happens here. A generic \(n\)-point correlator \(\langle \sigma^I(\vec{x}_1) \sigma^J(\vec{x}_2) \cdots \sigma^K(\vec{x}_n) \rangle\) can be reduced to a three-point function, which is specified by conformal invariance, by employing continuously the operator product expansion (4.45) \([43,44]\). However, such a procedure requires the knowledge of the OPE (4.45) and in particular of the coefficients \(f_{IJ}\) of (4.45). Of course this is as difficult as the original problem.

This program can be explicitly seen in the case of the four-point function. Contrary to the three-point correlators, four-point correlators are not fully specified by conformal invariance. In particular, conformal invariant four-point functions for arbitrary operators \(O_I\) of dimension \(w_I\) takes the form

\[
\langle O^I(\vec{x}_1) O^J(\vec{x}_2) O^K(\vec{x}_3) O^L(\vec{x}_4) \rangle = \left( \frac{x_{12}}{x_{14}} \right)^{w_I - w_J} \left( \frac{x_{14}}{x_{13}} \right)^{w_K - w_L} g(u, v) \frac{1}{x_{12}^{w_I + w_J} x_{34}^{w_K + w_L}},
\]

(4.47)
where

\[
\begin{align*}
  u &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \\
  v &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}
\end{align*}
\]

(4.48)

are the so-called anharmonic ratios. Therefore, the four-point functions are determined up to an unknown function \( g(u, v) \). Nevertheless, this function has to satisfy certain conditions, following basically from the associativity of Eq. (4.45). In fact, we can employ an OPE expansion along the (12)(34) (or (14)(23)) channel. It is easy to see that we get in this case the consistency condition

\[
g(u, v) = \sum_{f} f_{120} f_{340} G_{w_0, l}(u, v),
\]

(4.49)

where the sum is over primaries (belonging both to the (12) and (34) channels) and \( G_{w, l}(u, v) \) are the so-called conformal blocks. We now restrict ourselves to the case of scalar operators \( \sigma^I \) of equal dimensions \( w_I = w \). In such a case, the four-point function turns out to be

\[
\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4) \rangle = g_{JJKL}^{I}(u, v) x_{12}^{w_1} x_{34}^{w_2}.
\]

(4.50)

The conformal blocks \( G_{w, l}(u, v) \) have a closed form in \( d = 2 \) and 4 dimensions \[45-48\]. However, in \( d = 3 \) their structure is determined by the following equations \[48\]

\[
\begin{align*}
  G_{w, l} &= \frac{(2w - 1)l}{2l - 1} G_{w, l - 2} + \frac{1}{2} \left\{ (w - 1) \mathcal{F}_0 + \frac{\mathcal{F}_2}{l - 1} \right\} G_{w + 1, l - 1} \\
  &\quad - \frac{4w^2 (w - 1)}{4w^2 - 1} b G_{w + 2, l - 2}, \\
  \frac{(2w - 1)l}{2l - 1} G_{w, l} &= \left\{ \frac{1}{2} (w + l - 3) \mathcal{F}_0 + \mathcal{F}_1 \right\} G_{w + 1, l - 1} \\
  &\quad - \frac{4w^2 (l - 1)}{4w^2 - 1} b G_{\Delta + 2, l - 2} - \frac{2(w + l - 1)(l - 1)}{2l - 1} G_{w, l - 2}.
\end{align*}
\]

(4.51)

We have defined

\[
\begin{align*}
  \mathcal{F}_0 &= \frac{1}{z} + \frac{1}{\bar{z}} - 1, \\
  \mathcal{F}_1 &= (1 - z) \partial_z + (1 - \bar{z}) \partial_{\bar{z}}, \\
  \mathcal{F}_2 &= \frac{z - \bar{z}}{z \bar{z}} \left\{ z^2 (1 - z) \partial_z - \bar{z}^2 (1 - \bar{z}) \partial_{\bar{z}} + z^2 (1 - \bar{z}) \partial_z - \bar{z}^2 \partial_{\bar{z}} \right\}, \\
  b &= \frac{(w - l + 1)^2}{16(w - l + 1)(w - l + 2)}
\end{align*}
\]

(4.53)

and \( z, \bar{z} \) are given in terms of \( u, v \) as

\[
\begin{align*}
  z &= \frac{1}{2} \left( 1 - v + u + \sqrt{(1 - v + u)^2 - 4u} \right), \\
  \bar{z} &= \frac{1}{2} \left( 1 - v + u - \sqrt{(1 - v + u)^2 - 4u} \right).
\end{align*}
\]

(4.55)
or, equivalently,
\[ u = z z\overline{z}, \quad v = (1 - z)(1 - \overline{z}). \] (4.56)

In principle, Eqs. (4.51) and (4.52) is a system of equations which can be solved recursively. Although this is complicated system, it simplifies considerably for \( z = \overline{z} \). In this case, \( F_2 = 0 \) and one may solve either (4.51) or (4.52) recursively, which gives

\[ G_{w,0}(z) = \left( \frac{z^2}{1 - z} \right)^{w/2} \frac{3F_2\left( \frac{w}{2}, \frac{w - 1}{2}; \frac{1}{2}, \frac{z^2}{4(z - 1)} \right)}{3F_2(2, 1 - z, 1)} \] (4.57)

\[ G_{w,1}(z) = \frac{2 - z}{2z} \left( \frac{z^2}{1 - z} \right)^{w+1/2} \frac{3F_2\left( \frac{w + 1}{2}, \frac{w + 2}{2}; \frac{1}{2}, \frac{w + 2 - 1}{2}, \frac{z^2}{4(z - 1)} \right)}{3F_2(2, 1 - z, 1)} \] (4.58)

It is noticeable that there is a further simplification in the \( z = \overline{z} \rightarrow 0 \) limit. In this case, we get to leading order in \( z \)

\[ G_{w,l} = c_{w,l} z^w + \cdots. \] (4.59)

The coefficients \( c_{w,l} \) satisfy the recursion relation

\[ c_{w,l} = \frac{3(w + l - 1)}{(2w - 1)l} c_{w,l-2}, \] (4.60)

which is solved by

\[ c_{w,l} = \left( \frac{3}{2w - 1} \right)^{l/2} \frac{\Gamma\left( \frac{w + l + 1}{2} \right)}{\Gamma\left( \frac{w + 1}{2} \right)} \left( 1 + c_0((-)^l - 1) \right), \]

\[ c_0 = \frac{1}{12} \left\{ 6 - \frac{(3\pi)^{1/2} (2w - 1)^{1/2} \Gamma\left( \frac{1 + w}{2} \right)}{\Gamma\left( 1 + \frac{w}{2} \right)} \right\}. \] (4.61)

Since for \( z = \overline{z} \) we have \( u = z^2 \) and \( v = (1 - z)^2 \), we get that to leading order \( g^{IJKL}(u,v) \) is given by

\[ g^{IJKL}(u,v) = g_0^{IJKL} \left( \frac{x_{12}x_{34}}{x_{13}x_{24}} \right)^w + \cdots (u \simeq 0, v \simeq 1). \] (4.62)

Therefore we find that the four-point function in (4.50) has the following form in the \( u \simeq 0 \) and \( v \simeq 1 \) limit (\( g_0^{IJKL} \) being model-dependent coefficients)

\[ \langle \sigma^I(x_1)\sigma^J(x_2)\sigma^K(x_3)\sigma^L(x_4) \rangle \sim \frac{g_0^{IJKL}}{x_{12}x_{13}x_{24}x_{34}w}. \] (4.63)

By Fourier transforming (4.63) we get the four-point function in momentum space

\[ \langle \sigma^{I_{k_1}}(x_{k_1})\sigma^{J_{k_2}}(x_{k_2})\sigma^{K_{k_3}}(x_{k_3})\sigma^{L_{k_4}}(x_{k_4}) \rangle = \int \left( \prod_{i=1}^{4} d^4x_i \right) e^{i \sum_{i=1}^{4} k_i \cdot x_i} \langle \sigma^I(x_1)\sigma^J(x_2)\sigma^K(x_3)\sigma^L(x_4) \rangle. \] (4.64)
Using Eqs. (3.16) and (4.63) we get

\[
\langle \sigma_{k}^{I} \sigma_{k_{2}}^{J} \sigma_{k_{3}}^{K} \sigma_{k_{4}}^{L} \rangle \sim \left( \frac{\Gamma(\frac{3-w}{2})}{2^{w-3/2} \pi^{2}} \right)^{4} \int \left( \prod_{i=1}^{4} d^{3} x_{i} \right) \left( \prod_{i=1}^{4} d^{3} q_{i} \right) \times e^{i \sum \vec{k}_{i} \cdot \vec{x}_{i}} e^{-i \vec{q}_{1} \cdot \vec{x}_{12}} e^{-i \vec{q}_{2} \cdot \vec{x}_{23}} e^{-i \vec{q}_{3} \cdot \vec{x}_{34}} e^{-i \vec{q}_{4} \cdot \vec{x}_{43}} \frac{|\vec{q}_{1}|^{3-w}|\vec{q}_{2}|^{3-w}|\vec{q}_{3}|^{3-w}|\vec{q}_{4}|^{3-w}}. \tag{4.65}
\]

It is clear that the internal momenta \( \vec{q}_{i} \) are the eigenvalues of the operators

\[
\vec{q}_{1} = -i \vec{\partial}_{12}, \quad \vec{q}_{2} = -i \vec{\partial}_{31}, \quad \vec{q}_{3} = -i \vec{\partial}_{24}, \quad \vec{q}_{4} = -i \vec{\partial}_{43}. \tag{4.66}
\]

Performing the \( \vec{x}_{i} \) integrations in (4.65), we get \( \delta \)-functions which specify the external momenta \( \vec{k}_{i} \) as

\[
\vec{k}_{1} = \vec{q}_{1} - \vec{q}_{3}, \quad \vec{k}_{2} = \vec{q}_{2} - \vec{q}_{1}, \quad \vec{k}_{3} = \vec{q}_{3} - \vec{q}_{4}, \quad \vec{k}_{4} = \vec{q}_{4} - \vec{q}_{2}. \tag{4.67}
\]

Now, what does the limit \( u \approx 0 \) and \( v \approx 1 \) correspond to in terms of space distances? Using Eqs. (4.66) and (4.67) we may find the limit \( u \approx 0 \) and \( v \approx 1 \) in momentum space; then from (4.48) we get that these limits correspond to

\[
x_{12} x_{34} \ll x_{13} x_{24} \quad \text{and} \quad x_{14} x_{23} \sim x_{13} x_{24}. \tag{4.68}
\]

We can satisfy the relations (4.68) by taking

\[
x_{12} \ll \min(x_{13}, x_{24}) \quad \text{and} \quad x_{34} \ll \min(x_{13}, x_{34}). \tag{4.69}
\]

For the internal momenta \( \vec{q} \) this implies that

\[
|\vec{q}_{1}| \gg \max(|\vec{q}_{2}|, |\vec{q}_{3}|) \quad \text{and} \quad |\vec{q}_{4}| \gg \max(|\vec{q}_{2}|, |\vec{q}_{3}|), \tag{4.70}
\]

which, for the external momenta, implies

\[
|\vec{k}_{12}| = |\vec{q}_{2} - \vec{q}_{3}| \ll \min(|\vec{k}_{i}|) \quad (i = 1, \cdots, 4). \tag{4.71}
\]

This particular configuration is indicated in Fig. 2(a) in real space and in Fig. 3(a) in momentum space, respectively. It is known in the literature as the collapsed configuration. Therefore we conclude that the generic NG four-point correlator in the collapsed configuration is of the form (as \( x_{13} \approx x_{24} \))

\[
\langle \sigma^{I}(\vec{x}_{1}) \sigma^{J}(\vec{x}_{2}) \sigma^{K}(\vec{x}_{3}) \sigma^{L}(\vec{x}_{4}) \rangle = \frac{g_{0}}{x_{12}^{w} x_{13}^{2w} x_{34}^{w}} \tag{4.72}
\]

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Figure 2: (a) Collapsed configuration projected on a plane in space where $x_{12} \approx x_{34} \approx 0$ with $x_{13} \gg x_{12}, x_{34}$. (b) Double squeezed configuration where $x_{34} \approx x_{13} \gg x_{24} \gg x_{12} \approx 0$.

Figure 3: (a) Collapsed and (b) double squeezed shapes in momentum space.

By Fourier transforming (using again (3.16)) we find that the generic NG four-point correlator in momentum space in the collapsed limit is (up to a model-dependent coefficient)

$$\langle \sigma_{k_1}^I \sigma_{k_2}^I \sigma_{k_3}^K \sigma_{k_4}^L \rangle' \sim \frac{1}{|k_{12}|^{2-2w}|k_2|^{2-w}|k_4|^{2-w}} + \text{permutations} \ (|k_{12}| \to 0),$$ (4.73)

or

$$\langle \sigma_{k_1}^I \sigma_{k_2}^I \sigma_{k_3}^K \sigma_{k_4}^L \rangle' \sim |k_2|^{-w}|k_4|^{-w} P_{k_{12}} P_{k_2} P_{k_4} + \text{permutations} \ (|k_{12}| \to 0).$$ (4.74)

Before closing this section, we notice that there is another possibility to realize the condition (4.68), namely we can consider the configuration

$$x_{34} \approx x_{13} \gg x_{24} \gg x_{12} \approx 0.$$ (4.75)

The internal momenta $\vec{q}_i$ then obey

$$|\vec{q}_2| \approx |\vec{q}_4| \ll |\vec{q}_3| \ll |\vec{q}_1|$$ (4.76)
and correspondingly the external momenta $\vec{k}_i$ satisfy the relation

$$|\vec{k}_1| \approx |\vec{k}_2| \gg |\vec{k}_3| \gg |\vec{k}_4|. \quad (4.77)$$

This double squeezed configuration is drawn in Fig. 2(b) in real space and in Fig. 3(b) in momentum space. Since $x_{13} \approx x_{34}$, we get for the four-point function

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4)\rangle \sim \frac{g_0}{x_{12}^{w_1}x_{24}^{w_2}x_{34}^{2w_3}}. \quad (4.78)$$

By Fourier transforming and employing Eq. (3.16) we find

$$\langle \sigma^I_{\vec{k}_1}\sigma^J_{\vec{k}_2}\sigma^K_{\vec{k}_3}\sigma^L_{\vec{k}_4}\rangle' \sim \frac{1}{|\vec{k}_1|^{3-w}|\vec{k}_2|^{3-2w}|\vec{k}_3|^{3-w}} = |\vec{k}_1|^{-w} |\vec{k}_4|^{-w} P_{\vec{k}_1} P_{\vec{k}_3} P_{\vec{k}_4} \quad (|\vec{k}_1| \approx |\vec{k}_2| \gg |\vec{k}_4| \gg |\vec{k}_3|). \quad (4.79)$$

## 5 On the Suyama-Yamaguchi inequality

The collapsed limit of the four-point correlator is particularly important because, together with the squeezed limit of the three-point correlator, it may lead to the so-called Suyama-Yamaguchi (SY) inequality [24]. Consider a class of multi-field models which satisfy the following conditions: a) scalar fields are responsible for generating curvature perturbations and b) the fluctuations in scalar fields at the horizon crossing are scale invariant and Gaussian. The second condition amounts to assuming that the connected three- and four-point correlations of the $\sigma^I$ fields vanish and that the NG is generated at superhorizon scales. If so, the three- and four-point correlators of the comoving curvature perturbation (1.3) and (1.4) respectively reduce to

$$B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3) = N_I N_J K N_L \left( P^{IK}_{\vec{k}_1} P^{JL}_{\vec{k}_2} + 2 \text{ permutations} \right) \quad (5.1)$$

and

$$T_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = N_{IJ} N_{KL} N_M N_N \left( P^{IL}_{\vec{k}_1} P^{JM}_{\vec{k}_2} P^{KN}_{\vec{k}_3} + 11 \text{ permutations} \right) + N_{IJK} N_L N_M N_N \left( P^{IL}_{\vec{k}_1} P^{JM}_{\vec{k}_2} P^{KN}_{\vec{k}_3} + 3 \text{ permutations} \right), \quad (5.2)$$
Notice in particular that in the collapsed limit the last term of the four-point correlator (5.2) is subleading. By defining the nonlinear parameters $f_{NL}$ and $\tau_{NL}$ as

\[
 f_{NL} = \frac{5}{12} \frac{\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle}{P_{k_1}^2 P_{k_2}^2} \quad (k_1 \ll k_2 \sim k_3),
\]

\[
 \tau_{NL} = \frac{1}{4} \frac{\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle}{P_{k_1}^2 P_{k_2}^2 P_{k_3}^2 P_{k_4}^2} \quad (\vec{k}_{12} \simeq 0),
\]

and making use of the Cauchy-Schwarz inequality one can prove the SY inequality (at the tree-level) $\tau_{NL} \geq (6 f_{NL}/5)^2$, where the equality holds in the case of a single scalar field [24].

A crucial question is if the SY inequality will still hold if the NG correlators of the fields $\sigma^I$ do not vanish. Indeed, the conformal symmetry imposes that the squeezed limit of the three-point correlator as well as the collapsed limit of the four-point correlator have the same shapes of those present in Eqs. (5.1) and Eqs. (5.2), respectively. Therefore, one might expect a contamination of the inequality if the light scalar fields are NG at horizon crossing. Since being NG at horizon crossing requires simply that the light fields have self-interactions, a contamination of the SY inequality is rather plausible. However, the SY inequality is still valid even in the case of intrinsically NG fields. A step towards this proof was taken in Refs. [25] were the generic case of NG fields was considered. Nevertheless, it was assumed there that the coefficients $f_{NL}$ and $\tau_{NL}$ in Eq. (5.3) were momentum-independent, see e.g. the discussion between Eqs. (2.13) and (2.14) of Assassi et al. [25].

To explicitly demonstrate the SY inequality for NG fields, let us consider the OPE expansions for the two fields $\sigma^I$ and $\sigma^J$ in the (12) channel at the coincident point

\[
 \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2) = \left( \frac{C_{IJ}^0(w)}{x_{12}^{2w}} + \frac{C_{IJ}^M(w)}{x_{12}^w} \sigma^M(\vec{x}_2) + \cdots \right) = \sum_{n,s} \frac{C_{IJ ns}^M(w)}{x_{12}^{2w+2w+n+s}} x_{12}^{i_1} x_{12}^{i_2} \cdots x_{12}^{i_s} O_M^{(ns)}(\vec{x}_2) \quad (x_{12} \simeq 0)
\]

(5.4)

and similarly in the (34) channel at the coincident point

\[
 \sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4) = \left( \frac{C_{KL}^0(w)}{x_{34}^{2w}} + \frac{C_{KL}^M(w)}{x_{34}^w} \sigma^M(\vec{x}_4) + \cdots \right) = \sum_{n,s} \frac{C_{KL ns}^M(w)}{x_{34}^{2w+2w+n+s}} x_{34}^{i_1} x_{34}^{i_2} \cdots x_{34}^{i_s} O_M^{(ns)}(\vec{x}_4) \quad (x_{34} \simeq 0).
\]

(5.5)
The four-point function in the collapsed limit

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4) \rangle \quad (x_{12} \simeq 0 \text{ and } x_{34} \simeq 0)$$  \hfill (5.6)$$
can be expressed, using the OPE’s (4.45) and (5.5), as

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4) \rangle = \left( \sum_{n,s} C^{IJ}_{ns M} x_{12}^{i_1}\cdots x_{34}^{i_4} \sigma^M(\vec{x}_2) \right) \times \left( \sum_{n',s'} C^{KL}_{ns N} x_{34}^{i_1'}\cdots x_{34}^{i_4'} \sigma^N(\vec{x}_4) \right).$$  \hfill (5.7)$$

Due to the orthogonality of the two point function

$$\langle \mathcal{O}^{M(ns)}_{i_1 i_2 \cdots i_n}(\vec{x}_2) \mathcal{O}^{N(n's')}_{j_1 j_2 \cdots j_s}(\vec{x}_4) \rangle = \frac{C^{MN}_{ns i_1 i_2 \cdots i_n j_1 j_2 \cdots j_s}}{x_{24}^{2w}} \delta_{ss'} \delta_{nn'},$$  \hfill (5.8)$$
where \( t_{i_1 \cdots i_n j_1 \cdots j_s} \) is a positive tensor build up from Kronecker deltas, the four-point function in the collapsed limit may be expressed as

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4) \rangle = \frac{C^{IJ}_{00} A_{CL}}{x_{12}^{2w} x_{34}^{2w}} + \frac{C^{IJ}_{AB} A_{KL}}{x_{12}^{2w} x_{34}^{2w}} \langle \sigma^A(\vec{x}_2)\sigma^B(\vec{x}_4) \rangle + \cdots.$$  \hfill (5.9)$$

Denoting by \( \langle x_{12} | t | x_{34} \rangle = x_{12}^{i_1}x_{12}^{i_2} \cdots x_{12}^{i_s} t_{i_1 \cdots i_s j_1 \cdots j_s} x_{34}^{j_1}x_{34}^{j_2} \cdots x_{34}^{j_s} \) and \( C^A = N_I N_J C^{IJ} \), we can write

$$N_I N_J N_K N_L \langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4) \rangle = \frac{C^A C_0 A}{x_{12}^{2w} x_{34}^{2w}} + \frac{C^A C_{AB} A}{x_{12}^{2w} x_{34}^{2w}} \langle \sigma^A(\vec{x}_2)\sigma^B(\vec{x}_4) \rangle + \cdots$$

$$= \sum_{n,s} \frac{C^A (ns) A \langle x_{12} | t | x_{34} \rangle}{x_{24}^{2w}} \geq 0,$$  \hfill (5.10)$$
from which we deduce that

$$N_I N_J N_K N_L \langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4) \rangle \geq \frac{C^A C_{AB} A}{x_{12}^{2w} x_{34}^{2w}} \langle \sigma^A(\vec{x}_2)\sigma^B(\vec{x}_4) \rangle \quad (x_{12} \simeq 0 \text{ and } x_{34} \simeq 0).$$  \hfill (5.11)$$

On the other side, the two-point correlator is (up to an irrelevant overall normalization factor)

$$\langle \sigma^I(\vec{x}_2)\sigma^J(\vec{x}_4) \rangle = \frac{1}{x_{24}^{2w}} \delta^{IJ},$$  \hfill (5.12)$$
so that the expression (5.11) may be explicitly written as

$$N_I N_J N_K N_L \langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4) \rangle \geq \frac{C^A C_{AB} A}{x_{12}^{2w} x_{34}^{2w} x_{24}^{2w}}.$$

(5.13)
By Fourier transforming (5.12) we get

\[ \langle \sigma^I_{k_1} \sigma^J_{k_2} \rangle' = \frac{B(2w)}{k_3^{3-2w}} \delta^{IJ} \]

(5.14)

where

\[ B(w) = 2^{3-w} \pi^{\frac{3}{2}} \frac{\Gamma \left( \frac{3-w}{2} \right)}{\Gamma \left( \frac{w}{2} \right)}. \]

(5.15)

Therefore we find that the power spectrum of the comoving curvature perturbation is

\[ \langle \zeta_{k_1} \zeta_{k_2} \rangle' = P_{k_1}^c = N_I N_J P_{k_1}^{IJ} = \frac{N_I N^I B(2w)}{k_3^{3-2w}}. \]

(5.16)

By Fourier transforming both sides of (5.13) we get

\[ N_I N_J N_K N_L T_{k_1 k_2 k_3 k_4}^{IJKL} = N_I N_J N_K N_L \langle \sigma^I_{k_1} \sigma^J_{k_2} \sigma^k_{k_3} \sigma^l_{k_4} \rangle' \]

\[ \geq \frac{C_A C^A B^2(w) B(2w)}{|k_1|^{3-w} |k_3|^{3-w} |k_1|^{3-2w}} = \frac{C_A C^A}{(N_I N^I)^3} P_{k_1}^c P_{k_3}^c \gamma(k_1) \gamma^l(k_3), \]

(5.17)

where

\[ \gamma(k) = \frac{B(w)}{B(2w)} k^{-w}. \]

(5.18)

Similarly, the three-point correlator in the squeezed limit can be evaluated by employing the OPE (4.45) as

\[ \langle \sigma^I(x_1) \sigma^J(x_2) \sigma^K(x_3) \rangle = \left( \frac{C_{0}^{IJ}}{x_{12}^{2w}} + \frac{C_{A}^{IJ}}{x_{12}^{2w}} \sigma^A(x_2) + \cdots \right) \sigma^K(x_3). \]

(5.19)

Using again the orthogonality of the two-point functions, only the correlator (5.12) will contribute to the sum above, so that

\[ \langle \sigma^I(x_1) \sigma^J(x_2) \sigma^K(x_3) \rangle = \frac{C_{A}^{IJ}}{x_{12}^{2w}} \sigma^A(x_2) \sigma^K(x_3) = \frac{C_{A}^{IJ}}{x_{12}^{2w} x_{23}^{2w}} (x_{12} \simeq 0). \]

(5.20)

Again Fourier transforming (5.20) we get

\[ B_{k_1 k_2 k_3}^{IJ} = \frac{C_{A}^{IJ}}{x_{12}^{2w} x_{23}^{2w}} = \frac{C_{A}^{IJ}}{N_A N^A} \frac{P_{k_1}^c P_{k_3}^c \gamma(k_3)}{|k_1|^{3-2w} |k_3|^{3-2w}}. \]

(5.21)
Using Eq. (1.4) we get in the collapsed limit for the four-point function and the squeezed limit for the three-point function

\[
4\tau_{NL} = \frac{T_{\zeta} (p_1^*, p_2^*, p_3^*)}{p_1^* p_2^* p_3^*} = \frac{N_I N_J N_K N_L}{p_1^* p_2^* p_3^*} T_{IJJKL}^{IJJKL} \frac{1}{p_1^* p_2^* p_3^*} + \frac{N_I N_K N_L N_M}{p_1^* p_2^* p_3^*} \left( P_{k_1}^{IJ} B_{k_1 k_2 k_3 k_4}^{IJLM} + 11 \text{ permutations} \right) + \frac{N_I N_K N_L N_M N_N}{p_1^* p_2^* p_3^*} \left( P_{k_1}^{IJ} P_{k_2}^{JM} P_{k_3}^{KN} + 11 \text{ permutations} \right) + \frac{N_I J K N_L N_M N_N}{p_1^* p_2^* p_3^*} \left( P_{k_1}^{IJ} P_{k_2}^{JM} P_{k_3}^{KN} + 3 \text{ permutations} \right) \tag{5.22}
\]

In the squeezed limit there are four leading terms in the second and four relevant terms in the third line of (5.22) while the last line in (5.22) is subleading. Using now Eqs. (5.17) and (5.22), we may deduce the inequality

\[
4\tau_{NL} \geq \gamma(k_1) \gamma(k_3) \frac{C_A C^A}{(N_A N^A)^3} + 4\gamma(k_1) \frac{C^I N_I N^L}{(N_I N^I)^3} + 4 \frac{N^I N_I N^J K N_K}{(N_I N^I)^3} \tag{5.23}
\]

It can easily be checked that the right-hand side of the above inequality can be written as \((\gamma(k_1) \approx \gamma(k_3))\)

\[
\gamma(k_1) \gamma(k_3) \frac{C_A C^A}{(N_A N^A)^3} + 4\gamma(k_1) \frac{C^I N_I N^L}{(N_I N^I)^3} + 4 \frac{N^I N_I N^J K N_K}{(N_I N^I)^3} = 4 \frac{D_A D^A}{(N_I N^I)^3} \tag{5.24}
\]

where

\[
D_A = \frac{1}{2} \gamma(k_3) C_I + N_I J N^J. \tag{5.25}
\]

Thus we have that

\[
\tau_{NL} \geq \frac{D_A D^A}{(N_I N^I)^3}. \tag{5.26}
\]

Similarly, using Eq. (1.3) we find that \(f_{NL}\) is given in the squeezed limit \(k_1 \ll k_2 \sim k_3\)

\[
\frac{12}{5} f_{NL} = \frac{B_{\zeta} (p_1^*, p_2^*, p_3^*)}{p_1^* p_2^* p_3^*} = \frac{N_I N_J N_K}{p_1^* p_2^* p_3^*} B_{k_1 k_2 k_3}^{IJJK} + \frac{N_I N_I N^I N^J}{p_1^* p_2^* p_3^*} \left( P_{k_1}^{IJ} P_{k_2}^{IJ} + 2 \text{ permutations} \right) \tag{5.27}
\]

from which we find

\[
\frac{6}{5} f_{NL} = \frac{1}{2} \gamma(k_3) \frac{C^I N_I}{(N_I N^I)^2} + \frac{N^I N_I N^J}{(N_I N^I)^2} = \frac{D_I N^I}{(N_I N^I)^2}. \tag{5.28}
\]
Then, applying the Cauchy-Schwarz inequality

\[(D_I N^I)^2 \leq D_I D^I N_J N^J\]

(5.29)

to Eqs. (5.26) and (5.28) we finally get

\[\tau_{NL} \geq \left( \frac{6}{3} f_{NL} \right)^2\] (also for NG fields).

(5.30)

This completes the proof that the SY inequality in all multifield models where the NG comes from light scalar fields other than the inflaton even when such light scalar fields are NG at horizon crossing. Loop corrections from the universal superhorizon NG part of the comoving curvature perturbation were shown not to change the inequality \[50\]. Indeed, in an exact conformal invariant theory, this result would be robust against loop corrections as the arguments on NG correlators of the light fields we worked out through the OPE’s are valid at any order of perturbation theory.

5.1 A further generalization of the Suyama-Yamaguchi inequality

Before closing this section we would like to discuss two issues: how much the SY inequality depends on our assumption that the system enjoys the conformal symmetry and what modifications are introduced due to quantum effects.

Concerning the first point, the conformal symmetry is not really crucial. In fact, assuming that Wilsonian OPE holds, we expect the short distance behaviour of a set of fields \(\sigma^I(\vec{x})\) to be

\[\sigma^I(\vec{x})\sigma^J(\vec{y}) \sim \sum_n C_n(\vec{x} - \vec{y})O_n(\vec{y}).\]

(5.31)

On pure dimensional grounds and naive dimensional counting, the coefficients \(C_n\) should behave like

\[C_n \sim \left( \frac{1}{|\vec{x} - \vec{y}|} \right)^{w_I + w_J - w_n},\]

(5.32)

where \(w_I, w_J, w_n\) are the dimensions of the \(\sigma^I, \sigma^J\) and \(O_n\) operators, respectively. What is relevant is to specify the most singular term in this expansion. Clearly, the highest the dimension of the operators \(O_n\), the less singular the coefficient \(C_n\) \[30,32\]. Thus, we will have for example

\[\sigma^I(\vec{x}_1)\sigma^I(\vec{x}_2) = \left( \frac{C^{IJ}_0}{x_1^{w_I + w_J}} + \frac{C^{IJ}_M}{x_1^{w_I + w_J - w_M}}\sigma^M(\vec{x}_2) + \cdots \right),\]

(5.33)
where the dots stand for less singular terms. Doing the same for \( \sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4) \), we find that the four-point function, in the \( x_{12}, x_{34} \to 0 \) limit, is given again as

\[
N_1 N_J N_K N_L T_{k_1 k_2 k_3 k_4}^{IJKL} = N_1 N_J N_K N_L \langle \sigma_{k_1}^I \sigma_{k_2}^J \sigma_{k_3}^K \sigma_{k_4}^L \rangle' \geq C_A C_B C^{AB} B(w_1) B(w_2) B(2w) \frac{1}{|k_1|^{3-w_1} |k_2|^{3-w_2} |k_3|^{3-w_3} |k_4|^{3-2w}}
\]

(5.34)

where \( w_1 = w_I + w_J - w_A \), \( w_2 = w_K + w_L - w_B \), \( 2w = w_A + w_B \) and (up to a normalization constant)

\[
P^\zeta_{k_1} = N_1 N_J C_{IJ} B(w_I + w_J) \frac{k_1^{3-w_I + w_J}}{k_1^{3-2w}}.
\]

(5.35)

The coefficients \( C_{IJ} \) are defined through

\[
\langle \sigma_{k_1}^I \sigma_{k_2}^J \rangle' = C_{IJ} B(2w) \frac{k_1^{3-2w}}{k_1^{3-2w}}
\]

(5.36)

and

\[
\gamma(k_1) = B(w_1) \frac{B(2w_1)}{B(2w_1)} k_1^{-w_1}, \quad \gamma(k_3) = B(w_2) \frac{B(2w_2)}{B(2w_2)} k_3^{-w_2}.
\]

(5.37)

Similarly, we find for the bispectrum the dominant contribution

\[
B_{k_1 k_2 k_3}^{IJK} = \frac{C_{IJ} C_{JK} B(w_1) B(2w)}{|k_3|^{3-w_1} |k_1|^{3-2w}} = \frac{C_{IJ} C_{JK} B(w_1) B(2w)}{(N_A N_B C^{AB})^2} P^\zeta_{k_1} P^\zeta_{k_2} P^\zeta_{k_3} \gamma(k_3).
\]

(5.38)

Then, following the same steps from Eq. (5.22) to Eq. (5.28), the SY condition (5.30) follows when the inequality

\[
(D_1 N_J C_{IJ})^2 \leq D_1 D_J C_{IJ} N_K N_L C_{KL}
\]

(5.39)

is implemented. The above discussion avoids any reference to conformal invariance. It is based simply on the short-distance expansion of the product of two-operators in Eq. (5.33). Although, we could not use the orthogonality of the two-point function for operators of different dimensions as we did in the conformal case, based on the fact that we are interested in the collapsed limit, we have kept only the dominant most singular term. All the other terms are subleading and therefore do not contribute in the collapsed limit.

This discussion is valid for any fields with arbitrary dimension. Clearly, the inflaton field can be one of these fields and still the SY inequality is valid for this case as well. Note that we could not come to this conclusion previously in the conformal case, as a time-evolving inflaton background breaks the
special conformal scale symmetry. However, here we can deduce that SY inequality holds also when the inflation field plays a role in determining the cosmological perturbations. The SY inequality is more a consequence of fundamental physical principles rather than of pure mathematical arrangements. For example, the inequality \([5.39]\) is true only for a positive definite matrix \(C^{IJ}\). A negative definite \(C^{IJ}\) would lead to violation of the inequality, but this would require, for example, ghost-like scalars among the light \(\sigma^I\) fields. The observation of a strong violation of the inequality will then have profound implications for inflationary models as it will imply either that multifield inflation cannot be responsible for generating the observed fluctuations independently of the details of the model or that some new non-trivial degrees of freedom play a role during inflation. We will give an example of such a case in the next section.

The second point concerns quantum effects. Clearly, although at the tree-level the behaviour of the coefficients \(C_n\)'s is determined by the relation \([5.32]\), the renormalization effects will modify it. In fact, the scaling properties of the \(C_n\)'s will be given by the Callan-Symanzik equation with a particular operator mixing. For example, for asymptotically free theories, deviations from canonical scaling is characterized by multiplicative logarithmic functions. As a result, the quantum effects change the functional form of the coefficient \(C_n\)'s in the short distance expansion of the operators \([5.33]\). This change might show up in the thee- and four-point function in the collapsed limit as momentum-dependent \(f_{NL}\) and \(\tau_{NL}\). Therefore, renormalization will induce in general a different functional momentum-dependence of \(f_{NL}\) and \(\tau_{NL}\), which might lead to violation of the SY inequality for certain range of momenta. Of course, this is model-dependent problem and should be analyzed case by case.

### 6 Logarithmic Conformal Field Theories

There is another class of conformal theories, namely the logarithmic CFT’s \([51]\), which can be of interest from the cosmological point of view for two reasons. First the perturbations in exact de Sitter are not scale invariant even in the limit of zero mass. Secondly, it provides one example in which the SY is reversed.

These are theories characterized by the appearance of logarithms in correlation functions due to logarithmic short-distance singularities in the OPE. These singularities are connected to special operators having conformal dimensions degenerate with those of the usual primary operators. It is this degeneracy that is at the origin of the appearance of the logarithms \([51]\).

To be more concrete, let us consider fields \(\Phi\) and \(\Psi\) on de Sitter background with action...
\[ S_2 = \int d^4x \sqrt{-g} \left( -\partial_\mu \Phi \partial^\mu \Psi - m^2 \Phi \Psi - \frac{\mu^2}{2} \Psi^2 \right). \] (6.1)

The equations of motions are simply

\[ \Box \Psi - m^2 \Psi = 0, \] (6.2)
\[ \Box \Phi - m^2 \Phi - \mu^2 \Psi = 0. \] (6.3)

As usual, the conformal dimensions can be calculated by the asymptotic form of the space-independent solution \( \Phi(\eta) \) and \( \Psi(\eta) \) of Eqs. (6.2) and (6.3), which are easily found to be

\[ \Psi \sim \eta^w, \quad \Phi \sim \eta^w \log \eta, \quad w = \frac{3}{2} \left( 1 - \sqrt{1 - \frac{4m^2}{9H^2}} \right). \] (6.4)

Clearly, the scaling of \( \Psi \) and \( \Phi \) is not conventional as they transform under rescalings as

\[ \Psi \rightarrow \lambda^w \Psi, \quad \Phi \rightarrow \lambda^w (\Phi + \ln \lambda \Psi). \] (6.5)

The fields \( \Phi \) and \( \Psi \) are what is called in two-dimensional conformal field theories a logarithmic pair \cite{51}. In the AdS literature is known as dipole pair \cite{52} and it is believed to describe singleton fields of the AdS group. This has also been confirmed in the AdS/CFT context \cite{53,54}. It should be noted that Eqs. (6.2) and (6.3) reveal some problems. In fact, it is obvious that \( \Phi \) satisfied the higher-order equation

\[ (\Box - m^2)^2 \Phi = 0 \] (6.6)

by acting with the Klein-Gordon operator on (6.3). This can also be seen, at the level of the action, by integrating out the \( \Psi \) field in (6.1). Besides this apparent problem, logarithmic field theories seem to describe rather successfully, among others, percolation \cite{55}, the quantum Hall effect \cite{56} as well as planar magnetohydrodynamics \cite{57}. As in the case of AdS, we expect that the ghost mode in this higher-derivative theory to be eliminated by appropriate gauge symmetry \cite{52,58}. Details will be given elsewhere.

The transformation (6.5) shows that, in general, two operators \( \sigma_1 \) and \( \sigma_2 \) of conformal dimension \( w \) which under dilations transform as

\[ i[D, \sigma_a] = \left( x^i \partial_i \delta_a^b + \Delta_a^b \right) \sigma_a, \quad a = 1, 2. \] (6.7)
Up to now we have considered the case where the matrix $\Delta^b_a$ is diagonal and in particular $\Delta^b_a = w \delta^b_a$. However, we may consider a more general case where $\Delta = (\Delta^b_a)$ is brought to its Jordan canonical form

$$
\Delta = \begin{pmatrix}
w & 0 \\
1 & w
\end{pmatrix}.
$$

(6.8)

This means that $\sigma_a$ transforms under dilations $\vec{x} \rightarrow \lambda \vec{x}$ as

$$
\sigma_a(\vec{x}) \rightarrow \sigma'_a(\vec{x}') = \left( \exp[\Delta \ln \lambda] \right)^b_a \sigma_b(\lambda \vec{x})
$$

(6.9)

and reproduce exactly the transformation (6.5) for $\Psi = \sigma_1, \Phi = \sigma_2$. To find the correlators $G_{ab}(\vec{x}, \vec{y}) = \langle \sigma_I(\vec{x}) \sigma_J(\vec{y}) \rangle$ we may use the Ward identities for scale and special conformal transformations. Denoting by $G$ the matrix $G_{ab}$, scale invariance requires that $G$ satisfies

$$
\Delta G + G \Delta^T + r \frac{\partial}{\partial r} G = 0,
$$

(6.10)

where $r = |\vec{x} - \vec{y}|$. Eq. (6.10) is explicitly written as

$$
\begin{align*}
(2w + x_{12} \frac{\partial}{\partial x_{12}}) G_{11} & = 0, \\
(2w + x_{12} \frac{\partial}{\partial x_{12}}) G_{12} + G_{11} & = 0, \\
(2w + x_{12} \frac{\partial}{\partial x_{12}}) G_{22} + 2G_{12} & = 0.
\end{align*}
$$

(6.11)

In addition, special conformal transformation gives the constraint

$$
\Delta G = G \Delta^T,
$$

(6.12)

which leads to

$$
G_{11} = 0, \quad G_{12} = G_{21}.
$$

(6.13)

We may then proceed to solve Eqs. (6.11), the solution of which is provided by

$$
G_{12} = \frac{c}{|\vec{x} - \vec{y}|^{2w}}, \quad G_{11} = 0,
$$

$$
G_{22} = a G_{12} + \frac{\partial}{\partial w} G_{12} = \frac{c}{|\vec{x} - \vec{y}|^{2w}} (-2 \ln |\vec{x} - \vec{y}| + a).
$$

(6.14)

Therefore the two-point functions of the logarithmic pair $\sigma_1, \sigma_2$ turn out to be

$$
\begin{align*}
\langle \sigma_1(\vec{x}) \sigma_2(\vec{y}) \rangle & = \langle \sigma_2(\vec{x}) \sigma_1(\vec{y}) \rangle = \frac{c}{|\vec{x} - \vec{y}|^{2w}}, \\
\langle \sigma_2(\vec{x}) \sigma_2(\vec{y}) \rangle & = \frac{c}{|\vec{x} - \vec{y}|^{2w}} \left( -2 \ln |\vec{x} - \vec{y}| + a \right), \\
\langle \sigma_1(\vec{x}) \sigma_1(\vec{y}) \rangle & = 0.
\end{align*}
$$

(6.15-6.17)
Let us now consider the three-point functions. Here we want to calculate the correlator
\[ G_{abc}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \langle \sigma_a(\vec{x}_1)\sigma_b(\vec{x}_2)\sigma_c(\vec{x}_3) \rangle. \] (6.18)

Again we will use Ward identities for dilations and special conformal transformations. From dilation we get
\[ \Delta_i^a G_{ibc} + \Delta_i^b G_{aic} + \Delta_i^c G_{abi} + \left( \vec{x}_1 \cdot \vec{\nabla}_1 + \vec{x}_2 \cdot \vec{\nabla}_2 + \vec{x}_3 \cdot \vec{\nabla}_3 \right) G_{abc} = 0, \] (6.19)

whereas from special conformal transformations we have \( \vec{b} \) being the parameter vector of the special conformal transformation, see Eq. (2.28)
\[ \delta\vec{b} \cdot \left\{ 2\vec{x}_1 \Delta_i^a G_{ibc} + 2\vec{x}_2 \Delta_i^b G_{aic} + 2\vec{x}_3 \Delta_i^c G_{abi} + 
\left[ (\vec{x}_1 + \vec{x}_2)x_{12} \frac{\partial}{\partial x_{12}} + (\vec{x}_1 + \vec{x}_3)x_{13} \frac{\partial}{\partial x_{13}} + (\vec{x}_2 + \vec{x}_3)x_{23} \frac{\partial}{\partial x_{23}} \right] \right\} G_{abc} = 0. \] (6.20)

Combining the two equations above we get
\[ \Delta_i^a G_{ibc} = \Delta_i^b G_{aic} = \Delta_i^c G_{abi}, \] (6.21)

which leads us to
\[ wG_{122} + x_{ij} \frac{\partial}{\partial x_{ij}} G_{122} = 0, \quad \forall \ i < j \] (6.22)
\[ wG_{222} + G_{122} + x_{ij} \frac{\partial}{\partial x_{ij}} G_{122} = 0, \quad \forall \ i < j \] (6.23)
\[ G_{111} = G_{112} = 0. \] (6.24)

The solution to Eqs. (6.22) and (6.23) is given by
\[ G_{122} = cx_{12}^{-w} x_{23}^{-w} x_{13}^{-w}, \] (6.25)
\[ G_{222} = cx_{12}^{-w} x_{23}^{-w} x_{13}^{-w} \left( 2 \ln(x_{12}x_{23}x_{13}) + a \right). \] (6.26)

As a result, the three-point functions in the theory are given by
\[ \langle \sigma_1(\vec{x}_1)\sigma_2(\vec{x}_2)\sigma_3(\vec{x}_3) \rangle = \frac{c}{x_{12}^{-w} x_{23}^{-w} x_{13}^{-w}}, \] (6.27)
\[ \langle \sigma_2(\vec{x}_1)\sigma_2(\vec{x}_2)\sigma_3(\vec{x}_3) \rangle = \frac{c}{x_{12}^{-w} x_{23}^{-w} x_{13}^{-w}} \left\{ -\ln(x_{12}x_{23}x_{13}) + a \right\}, \] (6.28)
\[ \langle \sigma_1(\vec{x}_1)\sigma_1(\vec{x}_2)\sigma_3(\vec{x}_3) \rangle = 0, \] (6.29)
\[ \langle \sigma_1(\vec{x}_1)\sigma_1(\vec{x}_2)\sigma_1(\vec{x}_3) \rangle = 0. \] (6.30)
The correlators in momentum space are easily evaluated. For example we find

\[
\langle \sigma_1(\vec{k}_1)\sigma_2(\vec{k}_2) \rangle' = \frac{C_0(w)}{k_1^{3-2w}},
\]

\[
\langle \sigma_2(\vec{k}_1)\sigma_2(\vec{k}_2) \rangle' = a\langle \sigma_1(\vec{k}_1)\sigma_2(\vec{k}_2) \rangle' + \frac{\partial}{\partial w} \langle \sigma_1(\vec{k}_1)\sigma_2(\vec{k}_2) \rangle' = \frac{C_0(w)}{k_1^{3-2w}} \left( -2 \ln k_1 + a + C_{0,w} \right),
\]

\[
\langle \sigma_1(\vec{k}_1)\sigma_1(\vec{k}_2) \rangle = 0,
\]

where \( C_{0,w} \) denotes derivative of \( C_0 \) with respect to \( w \). Similar expression holds for the three-point functions. For example we have that

\[
\langle \sigma_2(\vec{k}_1)\sigma_2(\vec{k}_2)\sigma_2(\vec{k}_3) \rangle' = a\langle \sigma_1(\vec{k}_1)\sigma_2(\vec{k}_2)\sigma_2(\vec{k}_3) \rangle' + \frac{\partial}{\partial w} \langle \sigma_1(\vec{k}_1)\sigma_2(\vec{k}_2)\sigma_2(\vec{k}_3) \rangle' = 0.
\]

In particular, in the squeezed \( k_1 \ll k_2, k_3 \) we get

\[
\langle \sigma_1(\vec{k}_1)\sigma_2(\vec{k}_2)\sigma_2(\vec{k}_3) \rangle' \sim \frac{C_1(w)}{k_1^{3-w}k_2^{3-2w}},
\]

\[
\langle \sigma_2(\vec{k}_1)\sigma_2(\vec{k}_2)\sigma_2(\vec{k}_3) \rangle' \sim \frac{C_1(w)}{k_1^{3-w}k_2^{3-2w}} \left( \ln(k_1k_2^2) + a + C_{1,w} \right),
\]

\[
\langle \sigma_1(\vec{k}_1)\sigma_1(\vec{k}_2)\sigma_2(\vec{k}_3) \rangle' = \langle \sigma_1(\vec{k}_1)\sigma_2(\vec{k}_2)\sigma_2(\vec{k}_3) \rangle' = 0.
\]

The corresponding correlators of the comoving curvature perturbations can be easily calculated. For example, by using the expression (1.1) we get for the spectrum

\[
\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2} \rangle' \sim \frac{A}{k_1^{3-2w}}(1 + 2\gamma\ln k_1) \sim \frac{A}{k_1^{3-2w-2\gamma}},
\]

where

\[
A = 2N_1N_2 + aN_2^2C_0 + C_{0,w}, \quad \gamma = \frac{C_0}{N_2^2}A,
\]

and in the last step in (6.36) we have assumed that \( \gamma \ll 1 \). We see that the spectral index of the curvature perturbation power spectrum, \( n_\zeta - 1 = d\ln k^3P^\zeta/d\ln k \), gets a new contribution equal to \( 2\gamma \) from the due to logarithmic short-distance singularities in the OPE, even if the fields involved are massless. Further considerations will be presented elsewhere.

It should be noted that logarithmic field theories have OPE which contains short distance logarithmic singularities. For example the OPE in Eq. (4.13) is modified as [51,59]

\[
\sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2) \sim \left( \frac{1}{|x_{12}|} \right)^{(w_j+w_j)} \left\{ C_{0,I} \sigma_0^I + D_0^{IJ} \ln|x_{12}| + \frac{C_{IJ,K}}{x_{12}^K} \sigma^K(\vec{x}_2) + \cdots \right\}.
\]

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Repeating the analysis of the previous section, we can calculate the four-point function at the collapsed limit by using the above OPE in the (12) and (34) channels. The only difference is that the matrix $C^{IJ}$ in Eq. (5.36) is not positive definite. In fact, a simple inspection of (6.31) reveals that, in the simplest case of two fields ($I, J = 1, 2$), $C^{IJ}$ has a positive and a negative eigenvalues. Therefore, the Cauchy-Schwarz inequality (5.39) gets inverted and leads to

$$\tau_{NL} \leq \left( \frac{6}{5} f_{NL} \right)^2.$$  

Thus, logarithmic conformal field theories provide an example, consistent with the de Sitter symmetries, which leads to violation of the SY inequality. Such theories violate unitarity but there is no obvious reason for a CFT to be unitary [29], and logarithmic CFTs is an example. It remains to be seen if logarithmic conformal field theories play a real role in cosmology or not.

7 Some considerations and conclusions

In this paper we have studied the implications of the symmetries present during a de Sitter phase for the statistical correlations of the light fields present during a multifield inflationary dynamics. In particular, we have assumed that the NG is generated by light fields other than the inflaton field. The cosmological perturbations are both scale invariant and conformally invariant.

We have first shown that, as a consequence of the conformal symmetries, the two-point cross-correlation of the light fields vanish if their conformal weights are different. Therefore, no assumption is needed on such a cross-correlation, it is simply dictated by the conformal symmetry.

Secondly, we have pointed out that the OPE technique is very suitable to analyze two interesting limits: the squeezed limit of the three-point correlator and the collapsed limit of the four-point correlator. Despite the fact that the conformal symmetry does not fix the shape of the four-point correlators of the light NG fields, we have been able to compute it in the collapsed limit. Both the resulting shapes of the squeezed limit of the bispectrum and the collapsed limit of the trispectrum of the NG light fields turn out to be of the same form of the shapes of the corresponding bispectrum and trispectrum universally generated on superhorizon scales of the comoving curvature perturbation. Thanks to this result, we have succeeded in showing that the SY inequality relating the two NG observables $f_{NL}$ and $\tau_{NL}$ is valid independently of the NG nature of the light scalar fields at horizon crossing. In fact, we have been able to show that the SY inequality is valid irrespectively of the conformal symmetry, being just a consequence of the short-distance expansion of the two-operator product expansion.
In most of this paper the working assumption was that the cosmological perturbations enjoy both the scale invariant and the conformal symmetry of pure de Sitter. The inflaton background spontaneously breaks this symmetry, so that the variation of a correlation function of the curvature perturbation under the de Sitter isometry group should always be connected with the soft emission of one or many soft inflaton perturbations [19, 60–63]. It would be interesting to understand how our results will change under the assumption of a slight breaking of the de Sitter isometries. Under the assumption that the NG is generated by scalar fields other than the inflaton, we expect that our results in the squeezed and collapsed limits of the bispectrum and trispectrum respectively are still valid up to small corrections of the order of the slow-roll parameters.

Finally, while it is clear that a detection of a non-conformal correlation function, for example an equilateral three-point function, would imply that the source of perturbations is not decoupled from the inflaton [18], it would be interesting to understand if it possible to find other cosmological observables which can robustly test the conformality of the primordial cosmological perturbations. This might be a non-trivial task as post-inflationary nonlinear evolution of the correlators contaminate such a primordial input.

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