Stability analysis for delayed neural networks based on a generalized free-weighting matrix integral inequality

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ABSTRACT
This paper investigates the stability problem of neural networks (NNs) with time-varying delay. Firstly, a new augmented vector and suitable Lyapunov–Krasovskii Functional (LKF) considering activation function are constructed by using more information of time delay. Secondly, a generalized free-weighting matrix integral inequality (GFMII) is chosen to estimate the derivative of single integral terms more accurately. Meanwhile, Jensen integral inequality and improved convex combination are combined to estimate integral terms with activation function; as a result, a novel stability criterion with less conservatism is established. Finally, two numerical examples are employed to illustrate the effectiveness of proposed methods.

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1. Introduction
Past several decades have witnessed a wide range of applications of neural networks in diverse science and engineering fields, such as signal processing, image recognition, pattern recognition, associative memory, optimization problem and other fields (Boyd et al., 1994; Chua & Yang, 1988). As we all know, NN is composed of neurons, in which each neuron is connected directly to other neurons (Kwon et al., 2014; Yang et al., 2017). In the real NNs, the communications between neurons cannot be simultaneous because of the inherent transmission time and limited network resources (Shao et al., 2018; Yang et al., 2017). Therefore, time delay is inevitable, which often causes oscillation or deterioration of system, so the stability analysis is always a hot issue of NNs (Zing et al., 2015).

The main purpose of stability analysis is to get the maximum allowable delay region where the system can remain asymptotically stable. In the research of stability of NNs, how to construct the appropriate LKF has attracted considerable attention. In recent years, different methods of constructing LKF have been proposed for delayed NNs, for example, $s$-dependent LKF (Yang et al., 2019), double integral LKF (Hua et al., 2019), triple integral LKF (Wu et al., 2018; Yang et al., 2019), multiple integral LKF (Ding et al., 2016), delay-partitioning LKF (Chen et al., 2016), and discrete LKF (Chen et al., 2019).

Another efficient approach to reduce the conservation of stability criterion is the bounding techniques of estimating the derivative of LKF. Many integral inequalities were presented: Jensen inequality, Wirtinger inequality (Seuret & Gouaisbaut, 2013), auxiliary function inequality (Park et al., 2015), and free-matrix-based integral inequality (Zeng et al., 2019). Very recently, several new inequalities were presented, such as Wirtinger-based multiple inequality (Ding et al., 2016), Bessel-Legendre inequality (Duan et al., 2019), Nonorthogonal Polynomials inequality (Zhang et al., 2018), and generalized free-weighting matrix integral inequality (Hua et al., 2019). The $L_2 - L_\infty$ state estimation of NNs was conducted in (Qian et al., 2019) by Jensen inequalities. In (Lin et al., 2016), Jensen inequality and reciprocally convex optimization were utilized simultaneously to study the stability of NNs. By Wirtinger inequality, the stability of NNs with stochastic distributed delays was investigated in (Chen et al., 2018). The free-weighting-based inequality was utilized to establish the stability condition of recurrent NNs with interval time-varying delay in (Sun & Chen, 2013). In (Park et al., 2018), the stability and passivity for uncertain NNs with time-varying delays were further studied, in which the generalized free-weighting-matrix inequality approach was used to find more tighter bounds. In order to obtain more lower bounds, the generalized free-weighting matrix integral inequality in (Zhang et al., 2017) was proposed.
Motivated by the above discussion, the aim of this paper is to further study the stability of delayed NNs. A new augmented vector and LKF considering activation function are constructed, which can use more information of time delays and the systems. GFMII is chosen to estimate the derivative of single integral terms, Jensen integral inequality and improved convex combination are combined to estimate integral terms with activation function, which can lead to less conservative results. As a result, a novel stability criterion is established and the reduced conservatism of the result is demonstrated through two examples.

2. Problem formulation

Consider the following NNs:

\[ \dot{y}(t) = -Ay(t) + W_0g(y(t)) + W_1g(y(t-h(t))) + J \tag{1} \]

where \( y(t) = [y_1(t), \ldots, y_n(t)]^T \in \mathbb{R}^n \) is the state vector, \( n \) is the number of neurons; \( g(y(t)) = [g_1(y_1(t)), \ldots, g_n(y_n(t))]^T \in \mathbb{R}^n \) denotes the activation function; \( J = [J_1, \ldots, J_n] \) is the constant input vector; \( A = \text{diag}[a_1, \ldots, a_n] > 0 \) is the known system matrix; \( W_0, W_1 \in \mathbb{R}^n \) are the connection weight matrices representing the weight coefficients among the neurons. Time-delay signal \( h(t) \) is a time-varying differentiable function and satisfies

\[ 0 \leq h(t) \leq h, \dot{h}(t) \leq \mu < 1 \tag{2} \]

where \( h \) and \( \mu \) are known constants.

**Assumption 1:** The neuron activation functions \( g_i(\cdot), (i = 1, 2, \ldots, n) \) are continuous, bounded and satisfy

\[ k_i^- \leq \frac{g_i(u) - g_i(v)}{u - v} \leq k_i^+, \forall u \neq v \tag{3} \]

where \( k_i^- \) and \( k_i^+ \) are known real constants.

Based on Assumption 1, there exists an equilibrium point \( y^* = [y_1^*, \ldots, y_n^*]^T \) for the neural networks (1) and the uniqueness has been proved by Brouwer’s fixed point Theorem. By using the transformation \( x(t) = y(t) - y^* \), one can shift the equilibrium point \( y^* \) of (1) to the origin, and consequently system (1) can be converted into the following form:

\[ \dot{x}(t) = -Ax(t) + W_0f(x(t)) + W_1f(x(t-h(t))) \tag{4} \]

where \( x(t) = [x_1(t), \ldots, x_n(t)]^T \) is the state vector of the transformed neural networks system; \( f(x(t)) = [f_1(x_1(t)), \ldots, f_n(x_n(t))]^T \) and \( f_i(x_i(t)) = g_i(x_i(t) + y_i^*) - g_i(y_i^*) \) with \( f_i(0) = 0 (i = 1, 2, \ldots, n) \). Activation functions \( f_i(\cdot) \) satisfy the following:

\[ k_i^- \leq \frac{f_i(u) - f_i(v)}{u - v} \leq k_i^+, \forall u \neq v \tag{5} \]

If \( v = 0 \) in (5), then we can obtain

\[ k_i^- \leq \frac{f_i(u)}{u} \leq k_i^+, \forall u \neq 0 \tag{6} \]

Therefore, based on the inequalities (5) and (6), the following inequality holds

\[ [f_i(u) - f_i(v) - k_i^-(u - v)](v - f_i(u) + f_i(v)) \geq 0 \tag{7} \]

\[ [f_i(u) - k_i^- u][k_i^+ v - f_i(v)] \geq 0 \tag{8} \]

This paper aims to derive new stability criteria with less conservatism. To prove the main results, the following main lemmas are necessary.

**Lemma 1:** (Li et al., 2019). For a real scalar \( \alpha \in (0, 1) \), symmetric matrices \( R_i = R_i^T > 0 (i = 1, 2) \), and any matrix \( S \), the following matrix inequality holds

\[ \begin{bmatrix} 1 - R_1 & 0 \\ \alpha & 0 \\ 0 & 1 - \alpha \end{bmatrix} \preceq \begin{bmatrix} R_1 + (1 - \alpha)T_1 & S \\ ST & R_2 + \alpha T_2 \end{bmatrix} \]

where \( T_1 = R_1 - SR_2^{-1}S^T, T_2 = R_2 - S^TR_1^{-1}S \).

**Lemma 2:** (Hua et al., 2019; Yang et al., 2017). For symmetric positive definite matrix \( R \in \mathbb{R}^{n \times n} \), any matrices \( L, M \) and vector \( \omega : [ab] \rightarrow \mathbb{R}^n \) such that the integrations concerned are well defined, then the following inequality holds

\[ \int_a^b \omega^T(s)R\omega(s)ds \geq -\text{Sym}[\chi_0^TLX_1 + \chi_0^TMX_2] - (b - a)\chi_0^TLR^{-1}L^T\chi_0 - \frac{(b - a)}{3}\chi_0^TM^{-1}L^TM^{-1}\chi_0 \]

where \( \chi_0 \) is any vector, \( \chi_1 = \int_a^b \omega(s)ds \) and \( \chi_2 = -\int_a^b \omega(s)ds + \int_a^b \int_a^s \omega(u)du \) ds.

**Lemma 3:** (Hua et al., 2019). For a quadratic function \( f(x) = a_2x^2 + a_1x + a_0 (a_0, a_1, a_2) \in \mathbb{N} \), if the following inequalities hold

\( (i) f(0) < 0, (ii) f(h) < 0, (iii) -h^2a_2 + f(0) < 0 \)

then \( f(x) < 0, \forall x \in [0, h] \).
3. Main results

In this section, a new stability criterion for NNs (4), based on an augmented LKF, is proposed. For simplicity, some notations for matrices and vectors are introduced as follows:

Firstly, let:

\[ \eta_1(t) = \begin{bmatrix} x^T(t) & x^T(t - h) \int_{t-h}^t x^T(s) ds \end{bmatrix}^T \]

\[ \eta_2(t) = \begin{bmatrix} x^T(t) & \dot{x}(t) \end{bmatrix}^T \]

\[ \varepsilon_1(t,s) = \begin{bmatrix} x^T(t) & x^T(s) & f^T(x(s)) \int_s^t x^T(u) du \end{bmatrix} \]

\[ \varepsilon_2(t,s) = \begin{bmatrix} x^T(t) & x^T(s) & f^T(x(s)) \end{bmatrix}^T \]

\[ \Phi_1 = \text{Sym} \left[ \begin{bmatrix} e_1 & e_3 & h(t)e_5 + (h - h(t))e_6 \end{bmatrix} \times \begin{bmatrix} e_5 & e_4 & e_1 - e_3 \end{bmatrix}^T \right] \]

\[ \Phi_2 = \begin{bmatrix} e_1 & e_1 & e_5 & e_7 & e_0 \end{bmatrix} R_1 \begin{bmatrix} e_1 & e_5 & e_7 & e_0 \end{bmatrix}^T \]

\[ - \begin{bmatrix} e_1 & e_3 & e_4 & e_9 \end{bmatrix} h(t)e_5 + (h - h(t))e_6 \]

\[ + \text{Sym} \left[ \begin{bmatrix} h(t)e_5 & (h - h(t))e_6 \end{bmatrix} \right] \]

\[ \Phi_3 = \text{Sym} \left[ \begin{bmatrix} e_1 & e_3 & h(t)e_5 + (h - h(t))e_6 \end{bmatrix} \times \begin{bmatrix} e_5 & e_4 & e_9 \end{bmatrix} \right] \]

\[ \quad \times \begin{bmatrix} e_1 & e_5 & h^2(t)e_{12} + (h - h(t))^2e_{13} + h(t) \end{bmatrix} \]

\[ (h - h(t))e_5 \right] R_1 \begin{bmatrix} e_5 & 0 & 0 & 0 & e_1 \end{bmatrix}^T \]

\[ \Phi_4 = \begin{bmatrix} e_1 & e_3 \end{bmatrix} Z \begin{bmatrix} e_1 & e_7 \end{bmatrix} + e_1 P_1 e_1^T - e_2 P_1 e_2^T \]

\[ + e_2 P_2 e_2^T - e_3 P_2 e_3^T, \quad \Phi_4 \]

\[ \Phi_5 = \begin{bmatrix} N_1 Z_a^{-1} N_1^T + \frac{1}{3} N_2 Z_a^{-1} N_2^T \end{bmatrix} \]

\[ + (h - h(t)) \begin{bmatrix} M_1 Z_b^{-1} M_1^T + \frac{1}{3} M_2 Z_b^{-1} M_2^T \end{bmatrix} \]

\[ \Phi_5(t) = e^T(h(t)L_1 + (h - h(t))L_2)e - \frac{1}{h} \begin{bmatrix} e_{10} \end{bmatrix}^T \Omega_1 \begin{bmatrix} e_{10} \end{bmatrix} \]

\[ \Phi_5(t) = -\frac{1}{h} \begin{bmatrix} e_{10} \end{bmatrix}^T \Omega_2 \begin{bmatrix} e_{10} \end{bmatrix} \]

\[ \Theta_1 = \text{Sym} \left[ \begin{bmatrix} e_7 & e_8 & (e_1 - e_2)K_m \end{bmatrix} \right] \]

\[ \times U_1 \left[ \begin{bmatrix} e_7 & e_8 & (e_1 - e_2)K_m \end{bmatrix} \right]^T \]

\[ + \left[ e_8 - e_9 \right] (e_2 - e_3)K_m \]

\[ \times U_2 \left[ \begin{bmatrix} e_8 & e_9 & (e_2 - e_3)K_m \end{bmatrix} \right]^T \]

\[ + \left[ e_7 - e_9 \right] (e_1 - e_3)K_m \]

\[ \times U_3 \left[ \begin{bmatrix} e_7 & e_9 & (e_1 - e_3)K_m \end{bmatrix} \right]^T \]

\[ \Theta_2 = \text{Sym} \left[ \begin{bmatrix} e_7 & e_1 & K_m H_1 \end{bmatrix} \right] \]

\[ + \left[ e_8 - e_9 \right] (e_2 - e_3)K_m \]

\[ + \left[ e_9 \right] (e_3)K_m \]

\[ \text{The following LKF is constructed for system (4):} \]

\[ V(t) = \sum_{i=1}^{5} V_i(t) \]

\[ \text{where} \]

\[ V_i(t) = \eta_i^T(t)P \eta_i(t) \]

\[ V_2(t) = \int_{t-h}^{t} \varepsilon_1^T(t,s) R_1 \varepsilon_1(t,s) ds \]

\[ + \int_{t-h}^{t} \varepsilon_2^T(t,s) R_2 \varepsilon_2(t,s) ds \]

\[ V_3(t) = 2 \sum_{i=1}^{n} \left[ d_{1i} \int_{0}^{W_{2i}(x)} (f_i(s) - k_i^{-1}s) ds \right] \]

\[ + d_{2i} \int_{0}^{W_{2i}(x)} (k_i^{-1}s - f_i(s)) ds \]
\[
+ \int_{-h}^{t-h} f^T(x(u))L_2 f(x(u)) duds
\]

and \( P = P^T > 0, R_1 = R_1^T > 0, R_2 = R_2^T > 0, Z = Z^T > 0, \)
\( L_1 = L_1^T > 0, L_2 = L_2^T > 0 \)

Remark 1: An appropriate LKF is crucial to obtain less conservative stability criteria of NNs. In this paper, \( V_2(t) \) containing non-integral terms, activation function, and single integral terms is introduced. Meanwhile, the \( \int_{-h}^{t-h} \int_{-h}^{t-h} x(u) duds \) derivative of \( V_2(t) \) is divided to obtain more cross terms such as \( \int_{-h}^{t-h} x(u) duds \int_{-h}^{t-h} x(u) duds \)
\( \int_{-h}^{t-h} \int_{-h}^{t-h} x(u) duds \). The constructed LKF takes more delay information into account, which helps to derive less conservative stability results.

Taking the derivatives of \( V(t) \) along the trajectory of system (4) yields:

\[
\dot{V}_1(t) = 2\eta_1^T(t)P_2\dot{e}_1(t) = \xi^T(t)\Phi_1\xi(t) \tag{9}
\]

where

\[
\xi(t) = \begin{bmatrix}
\xi_1^T(t) \\
\xi_2^T(t) \\
\frac{1}{h(t)} \int_{-h(t)}^{t-h(t)} \int_{-h(t)}^{t-h(t)} x^T(u) duds \\
\frac{1}{h(t)} \int_{t-h(t)}^{t-h(t)} x^T(s) ds \end{bmatrix}
\]

\[
\xi_1(t) = \begin{bmatrix}
x^T(t) \\
x^T(t-h(t)) \\
x^T(t-h) \\
x^T(t-h) \\
\frac{1}{h(t)} \int_{t-h(t)}^{t-h(t)} x^T(s) ds \\
\frac{1}{h(t)} \int_{t-h(t)}^{t-h(t)} x^T(s) ds \\
\end{bmatrix}
\]

\[
\xi_2(t) = \begin{bmatrix}
f^T(x(t)) \\
f^T(x(t-h(t))) \\
f^T(x(t-h)) \\
\int_{t-h(t)}^{t-h(t)} f^T(x(s)) ds \\
\int_{t-h(t)}^{t-h(t)} f^T(x(s)) ds \\
\end{bmatrix}
\]

\[
\dot{V}_2(t) = \xi_1^T(t)R_1\xi_1(t) - \xi_1^T(t-h)R_1\xi_1(t-h) \\
+ \xi_2^T(t)R_2\xi_2(t) \\
- \xi_2^T(t-h)R_2\xi_2(t-h(t)) \\
+ 2 \int_{t-h}^{t} \xi_1^T(t,s)R_1 \frac{\partial \xi_1(t,s)}{\partial t} ds \\
+ 2 \int_{t-h}^{t} \xi_2^T(t,s)R_2 \frac{\partial \xi_2(t,s)}{\partial t} ds \\
= \xi^T(t) (\Phi_2 + \Phi_2[h(t)]) \xi(t) \tag{10}
\]

where

\[
\xi_1^T(t) = \text{col} \{ x(t), x(t-h), x(t-h), f(x(t-h)) \}, \\
\xi_2^T(t) = \int_{t-h}^{t} x(s) ds + \int_{h(t)}^{t-h} x(s) ds \\
\xi_3^T(t) = \text{col} \{ x(t), x(t-h), f(x(t-h)) \}, \\
\xi_4^T(t) = \int_{t-h}^{t} x(s) ds + \int_{h(t)}^{t-h} x(s) ds
\]

For symmetric matrices \( P_1 \) and \( P_2 \), we have:

\[
0 = x^T(t)P_1 x(t) - x^T(t-h(t))P_1 x(t-h(t)) \\
- 2 \int_{t-h(t)}^{t} x^T(s)P_1 x(s) ds \tag{13}
\]

\[
0 = x^T(t-h(t))P_2 x(t-h(t)) - x^T(t-h)P_2 x(t-h) \\
- 2 \int_{t-h}^{t-h(t)} x^T(s)P_2 x(s) ds \tag{14}
\]

Dividing the integral interval \(-h \int_{t-h}^{t} \dot{x}(s) Z\dot{x}(s) ds \) in \( \dot{V}_4 \), we can obtain:

\[
\dot{V}_4(t) = \xi_4^T(t)\Phi_4\xi(t) - \int_{t-h}^{t} \eta_2^T(s)Z\eta_2(s) ds \\
- \int_{t-h(t)}^{t-h} \eta_2^T(s)Z\eta_2(s) ds \\
- 2 \int_{t-h(t)}^{t} x^T(t)P_1\dot{x}(t) ds \\
- 2 \int_{t-h}^{t-h(t)} x^T(t)P_2\dot{x}(t) ds \\
= \xi^T(t) (\Phi_4 + \Phi_4[h(t)]) \xi(t)
\]
where
\[
Z_a = Z + \begin{bmatrix} 0 & P_1 \\ P_1 & 0 \end{bmatrix} Z_b = Z + \begin{bmatrix} 0 & P_2 \\ P_2 & 0 \end{bmatrix}
\]

Using Lemma 2 to deal with \( \int_{t-h(t)}^{t} \eta_2^T(s)Z_a\eta_2(s)ds \) and \( \int_{t-h(t)}^{t} \eta_2^T(s)Z_b\eta_2(s)ds \), we can obtain:
\[
- \int_{t-h(t)}^{t} \eta_2^T(s)Z_a\eta_2(s)ds - \int_{t-h}^{t-h(t)} \eta_2^T(s)Z_b\eta_2(s)ds \\
\leq \zeta^T(t) \left\{ h(t) \left( N_1Z_a^{-1}N_a^T + \frac{1}{3}N_2Z_a^{-1}N_2^T \right) \\
+ (h - h(t)) \left( M_1Z_b^{-1}M_1^T + \frac{1}{3}M_2Z_b^{-1}M_2^T \right) \\
+ \text{Sym}[N_1T_1 + N_2T_2] \\
+ \text{Sym}[M_1F_1 + M_2F_2] \right\} \zeta(t) \\
= \zeta^T(t) (\Phi_4 + \Phi_{4[h(t)]}) \zeta(t) 
\]

(16)

**Remark 2:** When dealing with the derivative of \( V_4(t) \), in order to cooperate with GFMII, two zero equalities are introduced to relax the matrix conditions. On the other hand, GFMII can make more state as augmented vectors and get some new cross terms, which can efficiently reduce the conservativeness of stability criteria.

\[
\dot{V}_5(t) = f^T(x(t))(h(t)L_1 + (h - h(t))L_2)f(x(t)) \\
- \int_{t-h(t)}^{t} f^T(x(s)L_a f(x(s))ds \\
- \int_{t-h}^{t-h(t)} f^T(x(s)L_2 f(x(s))ds 
\]

(17)

By applying Jensen inequality and Lemma 1 to \( \int_{t-h(t)}^{t} f^T(x(s)L_a f(x(s))ds \) and \( \int_{t-h}^{t-h(t)} f^T(x(s)L_2 f(x(s))ds \), we can deduce:
\[
- \int_{t-h(t)}^{t} f^T(x(s)L_a f(x(s))ds \\
- \int_{t-h}^{t-h(t)} f^T(x(s)L_2 f(x(s))ds \\
\leq - \frac{1}{h} \left( \left( \int_{t-h(t)}^{t} f^T(x(s))ds L_a \int_{t-h(t)}^{t} f(x(s))ds \right) \\
+ \left( \int_{t-h}^{t-h(t)} f^T(x(s))ds L_2 \int_{t-h}^{t-h(t)} f(x(s))ds \right) \right) \\
\leq - \frac{1}{h} \left[ \begin{array}{c} \int_{t-h(t)}^{t} f(x(s))ds \\ \int_{t-h}^{t-h(t)} f^T(x(s))ds \end{array} \right]^T 
\]

So, the upper bound of \( V_5(t) \) can be obtained as follows:
\[
\dot{V}_5(t) \leq \zeta^T(t) (\Phi_5 + \Phi_{5[h(t)]}) \zeta(t) 
\]

(19)

**Remark 3:** In order to further reduce the conservativeness of the stability condition, when disposing of the derivative of \( V_5(t) \), Jensen inequality is employed to estimate the single integral terms such as \( \int_{t-h(t)}^{t} f^T(x(s))L_a f(x(s))ds \) and \( \int_{t-h}^{t-h(t)} f^T(x(s))L_2 f(x(s))ds \). Furthermore, the improved convex combination is used to estimate integral terms in \( V_5(t) \). Noting that the upper bound of time delay is a quadratic function rather than a linear one, we apply Lemma 3 to check whether the derivative of the LKF is a negative definite or not.

By taking into account the assumption of the activation functions (7) and (8), the following inequalities hold:
\[
l_i(s_1, s_2) := 2[f(x(s_1)) - f(x(s_2)) - K_m(x(s_1) - x(s_2))]^T \\
\times U_i[K_p(x(s_1) - x(s_2)) - f(x(s_1)) + f(x(s_2))] \geq 0 
\]

(20)

\[
k_i(s) := 2[f(x(s)) - K_m x(s)]^T H_i[K_p x(s) - f(x(s))] \geq 0 
\]

(21)

where \( U_i = \text{diag}\{u_1, \ldots, u_n\}, H_i = \text{diag}\{h_1, \ldots, h_n\}, i = 1, 2, 3 \)

Therefore, the following inequalities hold:
\[
l_1(t, t - h(t)) + l_2(t - h(t), t - h) + l_3(t, t - h) \geq 0 
\]

(22)

\[
k_1(t) + k_2(t - h(t)) + k_3(t - h) \geq 0 
\]

(23)

Combining (10)–(21), we can obtain:
\[
\dot{V}(t) = \sum_{i=1}^{5} \dot{V}_i(t) = \zeta^T(t) \Pi \zeta(t) \\
= \zeta^T(t) (\Psi_{[h(t)]} + \Phi_{4[h(t)]} + \Phi_{5[h(t)]}) \zeta(t) \\
= \zeta^T(t) \tilde{\Psi}_{[h(t)]} \zeta(t) 
\]

(24)
where
\[
\Pi = \Phi_1 + \Phi_2 + \Phi_{2[h(t)]} + \Phi_3 + \Phi_4 + \Phi_{4[h(t)]} + \Phi_5 + \Phi_{5[h(t)]} + \Theta_1 + \Theta_2
\]
\[
\Psi_{[h(t)]} = \Phi_1 + \Phi_2 + \Phi_{2[h(t)]} + \Phi_3 + \Phi_4 + \Phi_{4[h(t)]} + \Phi_5 + \Theta_1 + \Theta_2
\]
From (19), it can be found that there exist some \(h^2(t)\)-dependent terms in \(\tilde{\Psi}_{[h(t)]}\). Therefore, by Lemma 3, \(\tilde{\Psi}_{[h(t)]}\) can be written in the following form:
\[
\tilde{\Psi}_{[h(t)=h]} < 0, \quad \tilde{\Psi}_{[h(t)=0]} < 0, \quad \tilde{\Psi}_{[h(t)=0]} - h^2g < 0 \quad (25)
\]
where
\[
g = \text{Sym} \left[ \begin{bmatrix} e_0 & e_0 & e_0 & e_0 & e_{12} + e_{13} - e_5 \end{bmatrix} \right] \times R_1 \begin{bmatrix} e_3 & e_0 & e_0 & e_1 \end{bmatrix}^T
\]
By Lyapunov Stability Theorem, when \(\dot{V} = \zeta^T \Pi \zeta < 0\), system (4) is asymptotically stable.

Based on the above proof, the main result for the asymptotic stability of the system (4) is given as follows:

**Theorem 1**: For the given scalars \(h, \mu\), system (4) is asymptotically stable if there exist definite symmetric matrices \(P \in R^{3n \times 3n}\), matrices \(R_i \in R^{3n \times 3n}\), matrices \(Z \in R^{2n \times 2n}\), matrices \(L_1 \in R^{n \times n}\), matrices \(L_2 \in R^{n \times n}\), positive diagonal matrices \(\Delta_i = \text{diag} \{\delta_{1i}, \ldots, \delta_{ni}\}\), \(\Lambda_i = \text{diag} \{\lambda_{1i}, \ldots, \lambda_{ni}\}\), \(\Upsilon_i = \text{diag} \{u_{1i}, \ldots, u_{ni}\}\) \((i = 1, 2, 3)\), matrices \(H_i = \text{diag} \{h_{1i}, \ldots, h_{ni}\}\) \((i = 1, 2, 3)\), symmetric matrices \(P_i \in R^{n \times n}\) \((i = 1, 2)\) and any matrices \(S \in R^{n \times n}\), matrices \(N_i, M_i \in R^{3n \times 2n}\) \((i = 1, 2)\) satisfying:
\[
\begin{bmatrix}
\Psi_{[h(t)=h]} & \Gamma_1 & L_1^T S \cr
* & \Upsilon_1 & 0 \cr
* & * & -L_2
\end{bmatrix} < 0, \quad (26)
\]
\[
\begin{bmatrix}
\Psi_{[h(t)=0]} & \Gamma_2 & L_2^T S \cr
* & \Upsilon_2 & 0 \cr
* & * & -L_2
\end{bmatrix} < 0, \quad (27)
\]
\[
\begin{bmatrix}
-h^2g + \Psi_{[h(t)=0]} & \Gamma_2 & L_2^T S \cr
* & \Upsilon_2 & 0 \cr
* & * & -L_2
\end{bmatrix} < 0, \quad (28)
\]
\[
L_1 > L_2, L_3 > 0, Z_0 > 0, Z_b > 0 \quad (29)
\]

where
\[
\Gamma_1 = \begin{bmatrix} hN_1 & hN_2 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} hM_1 & hM_2 \end{bmatrix},
\]
\[
\Upsilon_1 = \text{diag} \{-hZ_a \quad -3hZ_a \},
\]
\[
\Upsilon_2 = \text{diag} \{-hZ_b \quad -3hZ_b \}
\]

### 4. Numerical simulations

In this section, two typical numerical examples are given to show the less conservatism and the effectiveness of the proposed method in this paper.

**Example 1**: Consider the systems (1) with the following parameters:
\[
A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},
\]
\[
W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}
\]

\(K_p = \text{diag} \{0.4, 0.8\}, \quad K_m = \text{diag} \{0, 0\} J = \begin{bmatrix} 0.1 & 0.6 \end{bmatrix}^T\)

With different \(\mu\), the obtained maximum delay bounds by our method and the results of (Hua et al., 2019; Yang et al., 2019; Feng Shao, & Shao, 2019) are introduced in Table 1. It can be found that Theorem 1 enhances the feasible region of stability criteria, which shows that the proposed criterion is indeed less conservative.

By setting \(x(0) = [0.2, 0.5]^T, f(x(t)) = [0.4 \tan(x_1(t)) \quad 0.8 \tan(x_2(t))], h(t) = 0.8 \sin(t) + 6.1834\), the state trajectories of the delayed NNs with \(h = 6.9834, \mu = 0.8\) are shown in Figure 1. The equilibrium point of the NNs with given parameters can be obtained as \(y^* = [0.4380 \quad 0.3000]^T\). The result shows that the NNs are asymptotically stable at their equilibrium point.

**Example 2**: Consider the systems (1) with the following parameters:
\[
A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix},
\]
\[
W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix}
\]

\(K_p = \text{diag} \{0.3, 0.8\}, \quad K_m = \text{diag} \{0, 0\} J = \begin{bmatrix} 0.4 & 0.2 \end{bmatrix}^T\)

Table 2 shows the comparison on maximum delay bounds when \(\mu = 0.4, 0.45, 0.5\) and 0.55, it can be seen that the maximum delay bounds obtained by our results are much larger than those of (Feng et al., 2019; Hua et al., 2019; Lee et al., 2018; Wu et al., 2018; Yang et al., 2017; Yang et al., 2019; Zhang et al., 2017).

| \(\mu\) | 0.8 | 0.9 |
|---|---|---|
| Yang et al. (2019) | 4.8599 | 3.1342 |
| Yang et al. (2017) | 5.4428 | 3.6482 |
| Hua et al. (2019) | 5.5098 | 3.7098 |
| Feng et al. (2019) | 6.7186 | 3.7098 |
| Theorem 1 | 6.9834 | 3.7412 |
The state trajectories of the delayed NNs with $h = 32.6955, \mu = 0.4$ are shown in Figure 2, by setting $x(0) = [0.8, 0.5]^T$, $h(t) = 0.4 \sin(t) + 32.2955$, $f(x(t)) = [0.3 \tanh(x_1(t)) \quad 0.4 \tanh(x_2(t))]$. The equilibrium point of the NNs with given parameters can be obtained as $y^* = [0.6760 \quad 0.9077]^T$. The result shows that the NNs are asymptotically stable at their equilibrium point.

5. Conclusion

The stability of NNs with time-varying delay was considered in this paper. By constructing a suitable augmented LKF with more delay information, using GFMII, improved convex combination and delay decomposing approach, the less conservative result was obtained. The effectiveness of the obtained criteria was illustrated by numerical examples.

Disclosure statement

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