Quantum entanglement, unitary braid representation and Temperley-Lieb algebra

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Abstract – Important developments in fault-tolerant quantum computation using the braiding of anyons have placed the theory of braid groups at the very foundation of topological quantum computing. Furthermore, the realization by Kauffman and Lomonaco that a specific braiding operator from the solution of the Yang-Baxter equation, namely the Bell matrix, is universal implies that in principle all quantum gates can be constructed from braiding operators together with single qubit gates. In this paper we present a new class of braiding operators from the Temperley-Lieb algebra that generalizes the Bell matrix to multi-qubit systems, thus unifying the Hadamard and Bell matrices within the same framework. Unlike previous braiding operators, these new operators generate directly, from separable basis states, important entangled states such as the generalized Greenberger-Horne-Zeilinger states, cluster-like states, and other states with varying degrees of entanglement.

Introduction. – Recent developments in fault-tolerant quantum computation using the braiding of anyons [1], have stimulated interest in applying the theory of braid groups to the fields of quantum information and quantum computation. In this respect, an interesting result is the realization that a specific braiding operator is a universal gate for quantum computing in the presence of local unitary transformations [2]. This operator involves a unitary matrix $R$ that generates the four maximally entangled Bell states from the standard basis of separable states. This has led to further investigation on the possibility of generating other entangled states by appropriate braiding operators [3–5]. In [4], unitary braiding operators were used to realize entanglement swapping and generate the Greenberger-Horne-Zeilinger (GHZ) state [6], as well as the linear cluster states [7]. Further generalizations of the braiding operators to bipartite quantum systems with states of arbitrary dimension, i.e., qudits, were obtained by the approach of Yang-Baxterization [8,9].

The GHZ state was not directly generated by the braiding operator in [4]. The resulting state was transformed, by use of a local unitary transformation, to the GHZ state. We argue here that this state does not, in fact, possess the same entanglement properties as the GHZ state. In this note we show how the Bell states, the generalized GHZ states and some cluster-like states may be generated directly from a braiding operator. We adopt a different approach, based on the Temperley-Lieb algebra (TLA) [10], to obtain a class of unitary representations of the braid group, and with it the required braiding operator. We first obtain an explicit representation of the TLA, and then find the braid group representation via the Jones representation [11].

Braid group and quantum entanglement. – The $m$-stranded braid group $B_m$ is generated by a set of elements $\{b_1, b_2, \ldots, b_{m-1}\}$ with defining relations:

\[
\begin{align*}
    b_i b_j &= b_j b_i, \quad |i - j| > 1; \\
    b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}, \quad 1 \leq i < m.
\end{align*}
\]

(1)

Quantum computing requires that quantum gates be represented by unitary operators. Thus, for applications of
the braid group in quantum computation, one requires its unitary representations. For an m-qubit system the usual
$2^m \times 2^m$ unitary representation of $B_m$ employed in the literature is

$$b_i = I \otimes \ldots \otimes I \otimes R \otimes I \otimes \ldots \otimes I \quad (i = 1 \ldots m - 1), \quad (2)$$

where $I$ is the 2 × 2 unit matrix and $R$ is a 4 × 4 unitary matrix that acts on both the $i$-th and $(i + 1)$-th qubits;
that is, occupying the $(i, i + 1)$ position. The first of the two braid group relations in (1) is automatically satisfied
by the form (2). To fulfill the second relation, $R$ must satisfy

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R). \quad (3)$$

This relation is sometimes called the (algebraic) Yang-Baxter equation. One of the simplest solutions of (3) that
produces entanglement of states is the matrix

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \quad (4)$$

When acting on the standard basis \{000, 011, 101, 110\}, $R$ generates the four maximally entangled Bell states
\((00 \pm 11) / \sqrt{2}\) and \((01 \pm 10) / \sqrt{2}\). Here we adopt the convention \(0 = (1, 0)^t\) and \(1 = (0, 1)^t\),
where $t$ denotes the transpose. Following [5] and [8], we shall call $R$ the Bell matrix\(^1\). In the presence of local unitary transformations, $R$ is a universal gate [2].

The representation (2) can also be used to generate maximally entangled $n$-qubit states which are equivalent,
up to local unitary transformation, to the GHZ states [4]. To see this, let us take the $n = 3$ qubit case, and consider
the action of $b_1b_2$ on the separable state \(000\):

$$|\psi\rangle = b_1b_2|000\rangle = \frac{1}{2} \left( |000\rangle + |011\rangle + |101\rangle + |110\rangle \right). \quad (5)$$

$|\psi\rangle$ is related to the GHZ state \(|GHZ\rangle = |000\rangle + |111\rangle / \sqrt{2}\) by a local unitary transformation as

$$|\psi\rangle = H \otimes H \otimes H|GHZ\rangle, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (6)$$

where $H$ is the Hadamard matrix (or gate).

That the state $|\psi\rangle$ is said to be equivalent to the GHZ state is based on the fact that local unitary transformations
do not alter the degree of entanglement\(^2\). Nevertheless, it is evident that they have very different entanglement properties. For instance, after making a measurement on any one of the three qubits, the other two

qubits of the GHZ state become separable, whereas those of $|\psi\rangle$ are still in one of the maximally entangled Bell states! It would be more desirable if one could generate the GHZ states directly from the braiding operators without recourse to any local unitary transformation.

A common feature of the Bell states and the GHZ states is that they have the form of the superposition of a separable
product state $|a_1a_2 \ldots a_n\rangle \equiv |a_1\rangle |a_2\rangle \cdots |a_n\rangle$ with its conjugate state $|\overline{a_1a_2 \ldots a_n}\rangle$, which has all $a_k$’s changed from 0 to 1, and 1 to 0, i.e., $a_k = 1$ if $a_k = 0$, 1, respectively. Thus, the state |00\rangle is conjugate to |11\rangle, |01\rangle is conjugate to |10\rangle, etc. As pointed out after eq. (4), the Bell matrix essentially superimposes each two-qubit basis state on its conjugate, as does the Hadamard matrix in the one-qubit case.

We wish to generalize the Hadamard and Bell matrices to higher dimensions (i.e., to $n$-qubits), so that they generate
generalized GHZ states from separable states directly. We want these matrices to be representatives of certain
braiding operators of the braid group. Hence the main task is to find an appropriate unitary representation of the
braid group, and to determine the correct combination of the braid generators that gives the required matrix. We
find that a very simple way to achieve this task is by means of the Jones representation of the braid group, which
we describe below.

Unitary Jones representation of $B_3$. — In his construction of certain polynomial invariants, the Jones polynomials,
for knots and links, Jones [11] provided a new representation of the braid group based on what is essentially the TLA. The TLA, more specifically denoted by $TL_m(d)$, is defined, for an integer $m$ and a complex number $d$, to be the algebra generated by the unit element $I$ and the elements $h_1, h_2, \ldots, h_{m - 1}$ satisfying the relations

$$h_i h_{i+1} h_i = h_i, \quad 1 \leq i < m, \quad (7)$$

$$h_i^2 = dh_i. \quad (8)$$

Given a TLA, the Jones representation of the braid group is defined by (see e.g., [12])

$$b_i = Ah_i + A^{-1}I, \quad b_i^{-1} = A^{-1}h_i + AI, \quad (8)$$

where $A$ is a complex number given by $d = -A^2 - A^{-2}$. It is easily checked that the $b_i$’s so defined do satisfy the
braid group relation (1).

In general the Jones representation is not unitary. However, it is obvious from (8) that if $A = e^{i\theta}$ ($\theta \in [0, 2\pi]$)
and all the $h_i$’s are Hermitian ($h_i = \overline{h_i}$), then indeed the Jones representation is unitary\(^3\).

\(^1\)Not to be confused with the Bell matrix of combinatorial mathematics (after E.T. Bell).

\(^2\)A more precise statement is that for bipartite states entanglement is preserved under LOCC (local operations and classical communication).

\(^3\)This representation is not faithful in that more than one group element can be represented by the same matrix. It is easily checked using the TLA and the binomial theorem that if $m$ is the least integer such that $A^m = 1$, then $b_i^m = (\frac{1 - e^{i\theta}}{1 - e^{2i\theta}})h_i + I$. Hence $b_i^m = I$ for
Based on this fact, in what follows we shall provide a class of unitary representation of the 3-stranded braid group $B_3$, and show that a subclass of it gives nonlocal unitary entanglements that generate conjugate-state entanglements from separable basis states.

For $A = e^{i\theta}, \ d = -2\cos 2\theta$ is real. A simple unitary representation of $B_3$ is given by the Jones representation with TLA elements $h_i = dE_i, \ (i = 1, 2)$, where

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} a^2 & e^{-i\phi}ab \\ e^{i\phi}ab & b^2 \end{pmatrix}, \quad a^2 + b^2 = 1.$$  \hfill (9)

Here $\phi$ is a phase angle. The $E_i$’s satisfy

$$E_1^2 = E_1, \quad E_1E_2E_1 = a^2E_1, \quad E_2E_1E_2 = a^2E_2.$$  \hfill (10)

With $a^2 = d^{-2}$, $h_i$’s as constructed from $E_i$’s satisfy the TLA. Now as $d$ and $a$ are real, in order that $h_i$’s be Hermitian, we must have $b^2 = 1 - 1/d^2 > 0$. This implies $d^2 > 1$, and hence $\theta ( \text{mod } 2\pi)$ is restricted to be in the range $|\theta| \leq \pi/6$ or $|\theta - \pi| \leq \pi/6$. We shall assume $\theta$ to be in these domains below. The special case of this representation with $\phi = 0$ was employed previously in exploring the relation between quantum computing and the Jones polynomials [12] (see also [13]).

A very simple way to generalize the above representation of TLA to higher dimensions is as follows. Let

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}.$$  \hfill (11)

Define

$$E_1^{(n,k)} \equiv \otimes_{j=1}^{k-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes e_1 \otimes_{j=k+1}^n I,$$  \hfill (12)

$$E_2^{(n,k)} \equiv \otimes_{j=1}^{k-1} \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \otimes e_2 \otimes_{j=k+1}^n I + ab \otimes_{j=1}^{k-1} s_j \otimes e_3 \otimes_{j=k+1}^n s_j, \quad (13)$$

where $\otimes_{j=1}^m s_j = s_1 \otimes s_2 \otimes \cdots \otimes s_m$. Here $s_j$ is any Hermitian operator satisfying $s_j^2 = 1$. For example, $s_j$ can be $I$, any one of the Pauli matrices $\sigma_m (m = 1, 2, 3)$, or the Hadamard matrix $H$. The integer $n$ is the number of $2 \times 2$ matrices in the tensor products, and $k$ indicates the position of $e_1, e_2$ and $e_3$. The $E_i^{(n,k)}$’s are $2^n \times 2^n$ matrices, and they reduce to (9) in the case $n = k = 1$. One can easily check that $E_i^{(n,k)}$’s satisfy (10). Hence, the operators $h_i^{(n,k)} = dE_i^{(n,k)}$ form a $2^n \times 2^n$ matrix realization of $TL_3(d)$ (see footnote 4). A unitary braid group representation is then obtained from the $h_i$’s by the Jones representation.

Our new unitary braid representation generalizes the $2 \times 2$ matrices of (9) to $2^n \times 2^n$ matrices of (13) within the TLA $TL_3(d)$. Other routes of generalization are possible. For instance, in [14] the $2 \times 2$ representation of $TL_3(d)$ were generalized to higher dimensional matrices for $TL_m(d)$ with $m > 3$, where the dimension of representation varies with the number of strands $m$ according to the Fibonacci numbers, or with the number of independent bit-strings of certain path model proposed in [15].

**Generalized GMZ states.** – From now on we will be mainly concerned with the unitary braiding transformation representing the action of the braid $b_1b_2$. This braiding operator can be evaluated to be

$$b_i^{(n,k)}b_j^{(n,k)} = \begin{pmatrix} \langle 1 \rangle & 0 \\ 0 & \langle 0 \rangle \end{pmatrix}$$

$$+ \otimes_{j=1}^n s_j \otimes e^{i\phi \langle 1 \rangle} \otimes_{j=k+1}^n I \bigotimes_{j=k+1}^n I \bigotimes_{j=k+1}^n I.$$  \hfill (14)

Its action on the separable $n$-qubit states $|a_1a_2\ldots a_{k-1}a_{k+1}\ldots a_n\rangle$ and $|a_1a_2\ldots a_{k-1}a_{k+1}\ldots a_n\rangle$ is given by

$$b_1^{(n,k)}b_2^{(n,k)}|a_1a_2\ldots a_{k-1}0a_{k+1}\ldots a_n\rangle =$$

$$= (|d^2a_1a_2\ldots a_{k-1}0a_{k+1}\ldots a_n\rangle =$$

$$+ (e^{i\phi \langle 1 \rangle})|a_1a_2\ldots a_{k-1}a_{k+1}a_{k+1}\ldots a_n\rangle,$$  \hfill (15)

and

$$b_1^{(n,k)}b_2^{(n,k)}|a_1a_2\ldots a_{k-1}1a_{k+1}\ldots a_n\rangle =$$

$$= (|d^2a_1a_2\ldots a_{k-1}1a_{k+1}\ldots a_n\rangle =$$

$$+ (e^{i\phi \langle 1 \rangle})|a_1a_2\ldots a_{k-1}a_{k+1}\ldots a_n\rangle,$$  \hfill (16)

where $|a_j\rangle \equiv s_j|a_j\rangle \ (j = 1, \ldots, k-1, k+1, \ldots, n)$, $|1\rangle$ and $|0\rangle$ are the $k$-th qubit $|a_k\rangle$ is either the form (15) or (16), depending on whether the $k$-th qubit $|a_k\rangle$ is $|0\rangle$ or $|1\rangle$. The states in (15) and (16) are normalized, as $\langle d^2a_1^2a_2^2\ldots a_{k-1}^20^2a_{k+1}^2\ldots a_n^2|d^2a_1^2a_2^2\ldots a_{k-1}^21^2a_{k+1}^2\ldots a_n^2\rangle = 1$, which can be easily checked. Depending on the choice of the set of $s_j$’s, the resulting state (15) or (16) will have varying degrees of entanglement. In particular, if all $s_j = I$, then the resulting state is separable, and $b_1^{(n,k)}b_2^{(n,k)}$ is simply a local unitary transformation.

4See [9] for an $n^2 \times n^2$ matrix realization of the TLA. The braiding operator (called the Yang-Baxter matrix in these works) was obtained there through a Yang-Baxterization process. This latter process was also employed in [8], but not related to TLA, to generalize the Bell matrix to $(2n)^2 \times (2n)^2$ braid matrices.
We now consider a subclass of the representation obtained by setting $\phi = 0$ in (13) (i.e., $e_3 = \sigma_1$), $s_j = 1$ for $j < k$, and $s_j = \sigma_1$ for $j > k$. In this case, $|a_j\rangle = |a_j\rangle$ for $j < k$ and $|a_j\rangle = \sigma_1|a_j\rangle = |a_j\rangle$ for $j > k$. Hence, under the action of $B(n,k) \equiv b_1^{(n,k)}b_2^{(n,k)}$ (with the above-mentioned choice of the $s_j$'s in $b_1^{(n,k)}$ understood), the separable $n$-qubit state $|a_1a_2\ldots a_k\ldots a_n\rangle$ is superimposed on the state $|a_1a_2\ldots a_{k-1}a_1a_{k+1}\ldots a_n\rangle$ in either the form (15) or (16) (with the appropriate change in the $a_j$), depending on whether the $k$-th qubit $|a_k\rangle$ is $|0\rangle$ or $|1\rangle$. The resulting states are separable in the first $(k-1)$ qubits, but entangled in the other $(n-k+1)$ qubits. In particular, for $k = 1$, the operator $B(n,1)$ entangles the state $|a_1a_2\ldots a_k\ldots a_n\rangle$ with its conjugate state $|\bar{a}_1\bar{a}_2\ldots \bar{a}_k\ldots \bar{a}_n\rangle$, thus giving the generalized GHZ states. We see that these states can indeed be obtained from separable basis states by the braiding operator.

We now give a few examples of the braiding operator $B(n,1)$ for $k = 1$ and $n = 1, 2, 3$. From now on we choose $\theta = \pi/8$. This gives $d = -\sqrt{2}$, and $a, b = \pm 1/\sqrt{2}$. Without loss of generality, we take $a = b = 1/\sqrt{2}$. The four matrix elements in (14) are $da^2 = dab = -1/\sqrt{2}$ and $db^2 + A^{-2} = A^2dab = -i/\sqrt{2}$. Explicitly, $B(n,1)$ has the form

$$B(n,1) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \otimes_{j=2}^{n} I + \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \otimes_{j=2}^{n} \sigma_1. \quad (17)$$

For $n = 1$, $B(1,1) = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$ is, up to global phases, equivalent to the Hadamard gate. For $n = 2$:

$$B(2,1) = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & 0 & i \end{pmatrix}. \quad (18)$$

This is equivalent to the Bell matrix up to global phases, and it gives all four Bell states from the separable standard basis. For example, when acting on the states $|00\rangle$ and $|10\rangle$, it gives $-(|00\rangle + |11\rangle)/\sqrt{2}$ and $-i(|10\rangle - |01\rangle)/\sqrt{2}$, respectively.

Note, however, the difference between the appearance of this matrix in our approach, and the Bell matrix $R$ in (4). There the Bell matrix $R$ is the solution of the algebraic Yang-Baxter equation (3), and is the basic building block of the braid generators $b_1$ in (2). In our approach the matrix (18) is obtained from the product of the matrices representing the braid generators $b_1$ and $b_2$, i.e., it represents the braid $b_1b_2$. In a sense, we have factorized $R$.

It was mentioned in the introduction that the main impetus to using braid group representations in quantum computing is that the Bell matrix is a universal gate [2]. Since $B(2,1)$ is equivalent to $R$ in generating the Bell states, it should also be a universal gate. To prove that, it suffices to show, following [2], that the universal CNOT gate can be generated from $B(2,1)$ and local unitary transformations. This is indeed the case, as we have

$$CNOT = ((\alpha \otimes \beta)B(2,1)(\gamma \otimes \delta), \quad (19)$$

where

$$\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad \beta = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ 1 & -1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (19)$$

Generalized GHZ states for larger $n$ can be obtained accordingly.

Cluster-like states. – As mentioned in the introduction, the GHZ state is rather fragile in its entanglement, as it becomes separable after one of its qubits is measured. Multi-qubit systems which possess more robust entanglement can in fact be generated using $B(n,k)$. As an example, we consider the result of applying a braiding operator $B(n,k)$ on a generalized GHZ state $|\phi\rangle$ generated from $|00\ldots 00\rangle$ with the braiding operator $B^{-1}(n,1) = B^1(n,1)$. We have $|\Phi\rangle = B^{-1}(n,1)|00\ldots 00\rangle = (|00\ldots 00\rangle + i|11\ldots 11\rangle)/\sqrt{2}$. Upon applying $B(n,k)$ to $|\phi\rangle$, we get

$$B(n,k)|\Phi\rangle = \frac{1}{2}\left(|00\ldots 00\rangle_{k-1}|00\ldots 00\rangle_{n-k+1} + |00\ldots 00\rangle_{k-1}|11\ldots 11\rangle_{n-k+1} + |11\ldots 11\rangle_{k-1}|00\ldots 00\rangle_{n-k+1} + |11\ldots 11\rangle_{k-1}|11\ldots 11\rangle_{n-k+1}\right). \quad (20)$$

Here $|00\ldots 00\rangle_{k-1} \equiv |0\rangle_1|0\rangle_2\ldots |0\rangle_{k-1}$, $|00\ldots 00\rangle_{n-k+1} \equiv |0\rangle_1|0\rangle_2\ldots |0\rangle_{n-k+1} = \{0\}_n$, etc. This state is an entangled state for $n \geq 2$ and $k > 1$. Unlike the GHZ states, when it loses one of its qubits, the remaining state is still partially entangled when $n > 2$. For $n = 4$ and $k = 3$, the state (20) is just the 4-qubit linear cluster state given in [7].

By acting with $B(n,k)B^{-1}(n,1)$ on any one of the $2^n$ separable basis state $|a_1a_2\ldots a_n\rangle$, one can in fact generate $2^n$ orthogonal cluster-like states similar to those of (20).

Summary. – In summary, we have obtained a new class of unitary representation of the three-stranded braid group by the Jones representation. The construction is based on a new matrix realization of the Temperley-Lieb algebra. A subclass of the representation provides a braiding operator that can superimpose states on their conjugate states, thus giving the generalized GHZ states. This braiding operator becomes the Hadamard matrix and the Bell matrix in the one-qubit and two-qubit case, respectively. Certain cluster-like states with robust entanglement can also be generated from separable basis states with two such braiding operators.
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