ON CROSS-SECTIONS OF PARTIAL WREATH PRODUCT OF INVERSE SEMIGROUPS

EUGENIA KOCHUBINSKA

ABSTRACT. We classify \( R \)- and \( L \)-cross-sections of partial wreath product of inverse semigroups. As a corollary, we get the description of \( R \)- and \( L \)-cross-sections of the semigroup \( \text{PAut} T \) of partial automorphisms of finite regular rooted tree \( T \) and compute also the number of different \( R \)-(\( L \))-cross-sections in this semigroup.

1. INTRODUCTION

Green’s relations are basic relations introduced on a semigroup. Therefore it is natural that the problem of description of cross-sections of Green’s relations has been arisen. During the last decade cross-sections of Green’s relations for some classical semigroups were studied by different authors. In particular, all \( H \)-cross-sections of inverse symmetric semigroup \( IS_n \) were studied in [1]. All \( R \)- and \( L \)-cross-sections were classified in paper [3].

In the present paper we describe all cross-sections of \( R \) and \( L \) Green’s relations of partial wreath product of inverse semigroups. We also count the number of different \( R \)-(\( L \))-cross-sections of this semigroup. The paper is organized as follows. In Section 2 we collect all necessary basic definitions and propositions. In Section 3 we provide a description of all \( R \) and \( L \)-cross-sections and compute the number of different \( R \)-(\( L \))-cross-sections.

2. BASIC DEFINITIONS

For a set \( X \), let \( IS(X) \) denote the set of all partial bijections on \( X \). On the set \( IS(X) \) define a composition law: \( f \circ g : \text{dom}(g \cap g^{-1}\text{dom}(f)) \) \( f \circ g = g(f(x)), \ x \in \text{dom}(g \cap g^{-1}\text{dom}(f)) \), where \( f, g \in IS(X) \). Under this operation set \( (IS(X), \circ) \) forms a semigroup. This semigroup is called the full inverse symmetric semigroup on \( X \). If \( X = \mathbb{N}_n \), where \( \mathbb{N}_n = \{1, \ldots, n\} \), then semigroup \( IS(\mathbb{N}_n) \) is called the full inverse symmetric semigroup of rank \( n \) and is denoted \( IS_n \).

It is possible to introduce for elements of \( IS_n \) an analogue of cyclic decomposition for elements of symmetric group \( S_n \). We start with introducing two classes of elements. Let \( A = \{x_1, x_2, \ldots, x_k\} \subset \mathbb{N}_n \) be an ordered subset. Denote by \( (x_1, x_2, \ldots, x_k) \) the unique element \( f \in IS_n \) such that \( f(x_i) = x_{i+1}, \ i = 1, 2, \ldots, k-1, \ f(x_k) = x_1 \) \( f(x) = x, x \notin A \). Assume that \( A \neq \emptyset \) and denote by \( [x_1, x_2, \ldots, x_k] \) the unique element \( f \in IS_n \) such that \( f(x_i) = x_{i+1}, \ i = 1, 2, \ldots, k-1, \ x_k \notin \text{dom}(x) \) \( f(x) = x, x \notin A \). The element \( (x_1, x_2, \ldots, x_k) \) is called a cycle and the element \( [x_1, x_2, \ldots, x_k] \) is called a chain. Any element

Key words and phrases. Inverse semigroup, partial transformation semigroup, partial wreath product, Green’s relations, cross-section, rooted tree, partial automorphism.
of $\mathcal{IS}_n$ decomposes uniquely into the product of disjoint cycles and chains. This decomposition is called a *chain decomposition* \([2]\).

Recall the definition of partial wreath product of semigroups. Let $S$ be a semigroup, $(P,X)$ be a semigroup of partial transformations of the set $X$. Define the set $S^{P,X}$ as a set of partial functions from $X$ to semigroup $S$:

$$S^{P,X} = \{ f : A \to S | \text{dom}(f) = A, A \subseteq X \}.$$

Given $f,g \in S^{P,X}$, the product $fg$ is defined in a following way:

$$\text{dom}(fg) = \text{dom}(f) \cap \text{dom}(g), (fg)(x) = f(x)g(x) \text{ for all } x \in \text{dom}(fg).$$

For $a \in P, f \in S^{P,X}$, define $f^a$ as:

$$(f^a)(x) = f(xa), \text{ dom}(f^a) = \{ x \in \text{dom}(a) ; xa \in \text{dom}(f) \}.$$

**Definition 1.** Partial wreath product of semigroup $S$ with semigroup $(P,X)$ of partial transformations of the set $X$ is a set

$$\{(f,a) \in S^{P,X} \times (P,X) | \text{dom}(f) = \text{dom}(a)\}$$

with composition defined by $(f,a) \cdot (g,b) = (fg^a, ab)$. We will denote partial wreath product of semigroups $S$ and $(P,X)$ by $S \wr_p P$.

It is known \([3]\) that partial wreath product of semigroups is a semigroup. Moreover, partial wreath product of inverse semigroups is an inverse semigroup. An important example of inverse semigroup is the semigroup $\text{PAut}_T$ over, partial wreath product of inverse semigroups is an inverse semigroup. An $S$-product of semigroups mean a root-preserving tree homomorphism defined on a connected subtree of $T^n$.

It is shown in \([4]\) that $\text{PAut}_{T_n^k} \simeq \mathcal{IS}_n \wr_p \mathcal{IS}_n \wr_p \cdots \wr_p \mathcal{IS}_n$.

This is an analogue of the well-known fact that $\text{Aut}_{T_n^k} \simeq S_n \wr \cdots \wr S_n$.

### 3. Description of $\mathcal{R}$- and $\mathcal{L}$-Cross Sections of Semigroup $S \wr_p \mathcal{IS}_n$

In this section we study cross-sections of partial wreath product of finite inverse semigroup $S$ with semigroup $\mathcal{IS}_n$. Denote by $0$ and $1$ correspondingly the zero and the unit of the semigroup $S$.

Recall that Green’s $\mathcal{R}$-relation on inverse semigroup $H$ is defined by $a \mathcal{R} b \iff aH^1 = bH^1$, similarly Green’s $\mathcal{L}$-relation is defined by $a \mathcal{L} b \iff H^1a = H^1b$. Note that every $\mathcal{R}$- ($\mathcal{L}$-) equivalence class contains exactly one idempotent. It is well-known (see for example \([2]\)) that Green’s relations on $\mathcal{IS}_n$ can be described as follows: $a \mathcal{R} b \iff \text{dom}(a) = \text{dom}(b)$; $a \mathcal{L} b \iff \text{ran}(a) = \text{ran}(b)$.

$\mathcal{R}$- and $\mathcal{L}$-relations on $S \wr_p \mathcal{IS}_n$ are described in the next proposition.

**Proposition 1.**

1. $(f,a) \mathcal{R} (g,b)$ if and only if $\text{dom}(a) = \text{dom}(b)$ and for any $z \in \text{dom}(a)$ $f(z) \mathcal{R} g(z)$;
2. $(f,a) \mathcal{L} (g,b)$ if and only if $\text{ran}(a) = \text{ran}(b)$ and for any $z \in \text{ran}(a)$ $g^{a^{-1}}(z) \mathcal{L} f^{b^{-1}}(z)$, where $a^{-1}$ is an inverse for $a$.

**Proof.** The proof is completely analogous to the one for $\mathcal{IS}_n \wr_p \mathcal{IS}_n$ in \([1]\). \hfill \Box
Now let \( \rho \) be an equivalence relation on a semigroup \( H \). A subsemigroup \( T \subset H \) is called \textit{cross-section with respect to} \( \rho \) provided that \( T \) contains exactly one element from every equivalence class. The cross-sections with respect to \( \mathcal{R} \) (\( \mathcal{L} \)) Green’s relations are called \( \mathcal{R} \) (\( \mathcal{L} \)) cross-sections. Note that every \( \mathcal{R} \) (\( \mathcal{L} \)) equivalence class contains exactly one idempotent. Then the number of elements in every cross-section is \(|E(H)|\), where \( E(H) \) is the subsemigroup of all idempotents of \( H \).

It is not difficult to observe that a subsemigroup \( H \) of semigroup \( \mathcal{I}S_n \) is an \( \mathcal{R} \)-cross-section if and only if for every subsemigroup \( A \subseteq N_n \) it contains exactly one element \( a \) such that \( \text{dom}(a) = A \).

Before describing \( \mathcal{R} \) and \( \mathcal{L} \)-cross-sections in semigroup \( S_I\mathcal{L} \mathcal{S}_n \), we recall first the description of \( \mathcal{R} \) and \( \mathcal{L} \)-cross-sections in semigroup \( \mathcal{I} \mathcal{S}_n \) presented in [2]. Let now \( N_n = M_1 \cup M_2 \ldots \cup M_k \) be an arbitrary decomposition of \( N_n = \{1,2,\ldots,n\} \) into disjoint union of non-empty blocks, where the order of blocks is not important. Assume that a linear order is fixed on the elements of every block: 
\[
M_i = \{m_{i1}^1, m_{i2}^1, \ldots, m_{i|M_i|}^1\}.
\]
For each pair \( i, j \), \( 1 \leq i \leq k \), \( 1 \leq j \leq |M_i| \) denote by \( a_{i,j} \) the element in \( D \)-class \( D_{n-1} \) of rank \( n-1 \) of semigroup \( \mathcal{I} \mathcal{S}_n \), containing chain \( [m_{i1}^1, m_{i2}^1, \ldots, m_{i|M_i|}^1] \), that acts as identity on the set \( N_n \setminus \{m_{i1}^1, m_{i2}^1, \ldots, m_{i|M_i|}^1\} \). Denote by \( R = R(M_1^1, M_2^1, \ldots, M_k^1) \) the semigroup \( (a_{i,j})_{1 \leq i \leq k, 1 \leq j \leq |M_i|} \cup \{e\} \).

**Theorem 1.** [2] For an arbitrary decomposition \( N_n = M_1 \cup M_2 \ldots \cup M_k \) and arbitrary linear orders on the elements of every block of this decomposition the semigroup \( R(M_1^1, M_2^1, \ldots, M_k^1) \) is an \( \mathcal{R} \)-cross-section of \( \mathcal{I} \mathcal{S}_n \). Moreover, every \( \mathcal{R} \)-cross-section is of the form \( R(M_1^1, M_2^1, \ldots, M_k^1) \) for some decomposition \( N_n = M_1 \cup M_2 \ldots \cup M_k \) and some linear orders on the elements of every block.

Since map \( a \mapsto a^{-1} \) is an anti-isomorphism of semigroup \( \mathcal{I} \mathcal{S}_n \) that sends \( \mathcal{R} \)-cross-sections to \( \mathcal{L} \)-cross-sections, then \( \mathcal{L} \)-cross-sections is described similarly.

Now we turn to description of \( \mathcal{R} \) and \( \mathcal{L} \)-cross-sections of semigroup \( S_I\mathcal{L} \mathcal{S}_n \). It follows from Proposition [4] that a subsemigroup \( H \subset S_I\mathcal{L} \mathcal{S}_n \) is an \( \mathcal{R} \)-cross-section if and only if for any \( A \subset N_n \) and any collection of idempotents \( e_1, \ldots, e_{|A|} \in E(S) \) there exists exactly one element \( (f,a) \in H \) satisfying \( \text{dom}(a) = A \) and \( f(x_i) \in e_i \) for all \( x_i \in A \). Later we will use this fact frequently.

We start the proof of the main result with a sequel of lemmas.

**Lemma 1.** Let \( R \) be an \( \mathcal{R} \)-cross-section of semigroup \( S_I\mathcal{L} \mathcal{S}_n \). Then
\[
R_1 = \{a \in \mathcal{I} \mathcal{S}_n|(f,a) \in R\}
\]
is an \( \mathcal{R} \)-cross-section of semigroup \( \mathcal{I} \mathcal{S}_n \).

**Proof.** Let \((f,a),(g,b)\) be elements of \( \mathcal{R} \)-cross-section \( R \), that is, \( a,b \in R_1 \). The product of these elements \((f,a)(g,b) = (fg^a,ab)\) is again in \( \mathcal{R} \)-cross-section \( R \), hence if \( a,b \in R_1 \), then also \( ab \in R_1 \). Then \( R_1 \) is a semigroup.

If for element \((\emptyset,e) \in R \) we have \( \text{dom}(e) = N_n \), then \( \text{dom}(e^2) = \text{dom}(e) \), and since an \( \mathcal{R} \)-cross-section contains only one element, domain of which is equal \( N_n \), then \( e^2 = e \), and hence \( e = id_{N_n} \). Then for every element \((f,a) \in R \) the product \((f,a)(\emptyset,e) = (\emptyset,e)(f,a) = (\emptyset,a) \in R \).

As \( R \) is an \( \mathcal{R} \)-cross-section, then for every subset \( A \subset N_n \) there exists exactly one element \( a \in R_1 \) such that \( \text{dom}(a) = A \). So, \( R_1 \) is an \( \mathcal{R} \)-cross-section of semigroup \( \mathcal{I} \mathcal{S}_n \). \( \Box \)
Lemma 2. Let $R$ be an $\mathcal{R}$-cross-section of semigroup $S \wr_p \mathcal{IS}_n$. Then
$$R_2 = \{f(1) | (f, a) \in R, a = id_{N_a}, f(x) = 0, \text{ for all } x \neq 1\}$$
is an $\mathcal{R}$-cross-section of semigroup $S$.

Proof. Let $(f, a), (g, b) \in R$ be such that $a = b = id_{N_a}$, $f(x) = g(x) = 0$, for all $x \neq 1$, that is, $(f(1), g(1)) \in R_2$. Product $(f, a)(g, b) = (fg^a, ab)$ satisfies condition $ab = id_{N_{ab}}$, $fg^a(x) = 0$ for all $x \neq 1$. Since $fg^a(1) \in R_2$ and $fg^a(1) = f(1)g(1)$, then for $(f, a), (g, b) \in R_2$ their product also belongs to $R_2$. Hence $R_2$ is a semigroup. As for every $f(1) \in R_2$ the corresponding element $(f, a)$ is an element of an $\mathcal{R}$-cross-section of semigroup $S \wr_p \mathcal{IS}_n$, then for every idempotent $e \in E(S)$ exists exactly one element $f(1) \in R_2$ such that $e \mathcal{R} f(1)$. Thus $R_2$ is indeed an $\mathcal{R}$-cross-section of semigroup $S$. \hfill \Box

Lemma 3. Let $S$ be an inverse semigroup, $\psi : S \rightarrow S$ be an automorphism, $R$ be an $\mathcal{R}$-cross-section of $S$. Then $\psi(R)$ is also an $\mathcal{R}$-cross-section of $S$.

Proof. For any idempotent $e \in E(S)$ of semigroup $S$ there exists unique element $a \in \psi(R)$ such that $a^{-1}a = e$. Let $a = \psi(b)$ for $b \in R$. Since $\psi$ is an automorphism, then $\psi(bb^{-1}) = e$ if and only if $a^{-1} = e$. An element $\psi^{-1}(e)$ is also an idempotent of semigroup $S$. Then $bb^{-1} = \psi^{-1}(e)$ if and only if $\psi(bb^{-1}) = e$. The uniqueness of element $b$ such that $bb^{-1} = \psi^{-1}(e)$ follows from the uniqueness of element $a$. Thus $\psi(R)$ is an $\mathcal{R}$-cross-section of $S$. \hfill \Box

Lemma 4. Let $R$ be an $\mathcal{R}$-cross-section of semigroup $S \wr_p \mathcal{IS}(M)$. If $R_1 = R(M)$, then $R \simeq R_2 \wr_p R_1$.

Proof. The following holds for a partial wreath product $P = P_2 \wr_p P_1$ of $\mathcal{R}$-cross-sections.

(1) If $(f, a) \in R$ is such that $f(i) \mathcal{R} 1$ for $i \in \text{dom}(a)$, then $f(i) = 1$.

We will show now that the general case when $f(i) \neq 1$ reduces to this one.

We may assume $M = \{1, 2, \ldots, m\}$ with the usual order. Let $(f_i, a_i), i = 1, \ldots, m$ be such elements of an $R$ that $a_i(1) = i, f_i(1) \mathcal{R} 1$. Put $\varphi_i = f_i(1)$.

Consider now a map $\Theta : S \wr_p \mathcal{IS}(M) \rightarrow S \wr_p \mathcal{IS}(M)$, which acts as: $(f, a) \mapsto (g, a)$, where for $x \in \text{dom}(a)$ we define $g(x) = \varphi_x f(x) \varphi_x^{-1}$. It is easy to check that this map is an isomorphism. From Lemma 5 it follows that isomorphic image $\Theta(R)$ of an $\mathcal{R}$-cross-section $R$ is an $\mathcal{R}$-cross-section too. Moreover, the next paragraph shows that for $\Theta(R)$ the property (1) is true.

Let $\psi_j = (f_{\psi_j}, b_{\psi_j})$ be such an element of $\mathcal{R}$-cross-section that $b_{\psi_j}(j) = m, f_{\psi_j}(j) \mathcal{R} 1$. Let $(f, a) \in R$ be an element such that $f(x) \mathcal{R} 1$ for some $x \in \text{dom}(a)$. Consider now the product of elements $(f, a), (f, a)$ and $(\psi_x, b_x)$. We obtain $\Theta(f, a)(f, a)(\psi_x, b_x) = (f_x f_x^{a_x} \psi_x^{a_x}, a_x b_x)$. Domain of component $a_x b_x$ is the set $\{1\}$ and $a_x b_x(1) = m$. Also $f_x f_x^{a_x} \psi_x^{a_x} \mathcal{R} 1$, and $(f_x f_x^{a_x} \psi_x^{a_x})(1) = f_x(1)f_x(1) = f_x(1)\psi_x(1) = f_x(1)f(x) \psi_x(x)$. It is obvious that $(f_x, a_x)(f_x, a)(\psi_x, b_x) = (f_m, a_m)$, hence $(\varphi_x f_x^{a_x} \psi_x^{a_x})(1) = \varphi_m$. Then we have $(\varphi_x f_x^{a_x} \varphi_x^{-1} \varphi_x^{a_x})(1) = \varphi_m$, but $\varphi_x f_x \varphi_x^{-1} = \varphi_m$. Thus $g(x) = \varphi_x f(x) \varphi_x^{-1} = 1$.

As $R \simeq \Theta(R)$, we may assume that for $R$ itself the property (1) holds. In this case we will show $R = R_2 \wr_p R_1$.

Let $\varphi = (f_\varphi, a_\varphi)$ be some element of $\mathcal{R}$-cross-section $R$. Then $a_\varphi \in R_1$. We want to show that $f_\varphi(i) \in R_2$ for arbitrary $i \in \text{dom}(a_\varphi)$.
For that we put \( j = a(i) \) and define three groups of elements of semigroup \( S_{lp}I\!\!S(M) \): element \( \psi_i = (f_i, a_{\psi_i}) \), where \( a_{\psi_i} = [1, i, i + 1, \ldots, m - 1, m] \). \( f_{\psi_i}(1) = 1 \), \( f_{\psi_i} = 0 \), \( j \geq i \); element \( \sigma = (f_{\sigma}, a_{\sigma}) \), where \( a_{\sigma} = id_M \), and \( f_{\sigma}(1) \not\in f_{\psi_i}(i) \), and \( f_{\sigma}(i) = 0 \) when \( x \neq 1 \); element \( \tau_j = (f_j, a_{\tau_j}) \), where \( a_{\tau_j} = [j, m] \), \( f_{\tau_j}(x) = 1 \) for \( x \in dom(a) \). All of them are in \( R \), because they are the only possible elements for corresponding domains and idempotents.

Consider product of elements \( \psi_i, \varphi \), and \( \tau_j \). Then we obtain \( \psi_i \cdot \varphi \cdot \tau_j = (f_{\psi_i}, f_{\varphi}, f_{\tau_j}, a_{\psi_i}a_{\varphi}a_{\tau_j}) \). Domain of component \( a_{\psi_i}a_{\varphi}a_{\tau_j} \) is the set \( \{1\} \). Then \( dom(f_{\psi_i}, f_{\tau_j}) = dom(a_{\psi_i}a_{\varphi}a_{\tau_j}) = \{1\} \) and \( f_{\psi_i}(f_{\varphi}\cdot f_{\tau_j}, a_{\psi_i}a_{\varphi}a_{\tau_j}) = \{1\} \).

For the product \( \sigma \cdot \psi_m = (f_{\sigma}, f_{\psi_m}, a_{\sigma}a_{\psi_m}) \) we have that domain of \( a_{\sigma}a_{\psi_m} \) is the set \( \{1\} \), then \( dom(f_{\sigma}f_{\psi_m}) = dom(a_{\sigma}a_{\psi_m}) = \{1\} \) and \( (f_{\sigma}f_{\psi_m})(f_{\psi_m}, a_{\psi_m}) = \{1\} \).

Thus we obtain that \( dom(a_{\sigma}a_{\psi_m}) \) and \( f_{\psi_m}(f_{\psi_m}(1) \cdot f_{\sigma}(f_{\psi_m}(1)) = \{1\} \).

As element \( f_{\psi}(1) \) lays in \( R \)-cross-section \( R_2 \), then \( f_{\psi}(1) \cdot f_{\sigma}(f_{\psi}(1)) = \{1\} \).

The number of elements of \( R \)-cross-section of inverse semigroup is equal to the number of idempotents of this semigroup. The element \( (f, a) \) of the semigroup \( S_{lp}I\!\!S(M) \) is idempotent iff all \( a \) and \( f(i) \) are idempotents. Then number of idempotents of this wreath product equals \((|E(S)| + 1)^m \). The number of elements of partial wreath product \( R_{2} \cdot R_1 \) equals \( \sum_{i=1}^{m} |R_2|^i \cdot (m) = (|E(S)| + 1)^m \). Therefore \( R = R_{2} \cdot R_1 \).

\[ \square \]

**Theorem 2.** Let \( R(M_1, M_2, \ldots, M_k) \) be \( R \)-cross-section of semigroup \( I\!\!S_n, R_1, \ldots, R_k \) be \( R \)-cross-sections of semigroup \( S \). Then

\( R = (R_1 \cdot R_2 \cdot \ldots \cdot R_k) \times (R_{2} \cdot R_3 \cdot \ldots \cdot R_k) \times \ldots \times (R_{k-1} \cdot R_k) \)

is an \( R \)-cross-section of semigroup \( S \). Moreover, every \( R \)-cross-section is isomorphic to \( (R_1 \cdot R_2 \cdot \ldots \cdot R_k) \times (R_{2} \cdot R_3 \cdot \ldots \cdot R_k) \times \ldots \times (R_{k-1} \cdot R_k) \).

**Proof.** Let \( R_1, \ldots, R_k \) be \( R \)-cross-sections of semigroup \( S \). It is obvious that \( (R_1 \cdot R_2 \cdot \ldots \cdot R_k) \) is a semigroup.

Let \( h = (f_h, a_h) \) be an element of \( S_{lp}I\!\!S_n \). Now show that there exists only one element \( g = (f_g, a_g) \in R \) such that \( h \cdot R \cdot g \). Define \( g \) in a following way. Put \( a_g = b_i \), where \( b_i = R_1(M_1) \), \( dom(b_i) = dom(a_h) \cap M_i \). For every \( x_i \in M_i \cap M_i \) put \( f_g(x_i) = y_i \), where \( y_i \in R_1, y_i \cdot R \cdot h(x_i) \). It follows from definition of \( g \) that \( g \cdot R \cdot h \) is a group.

Now prove that every \( R \)-cross-section is obtained in this way. Let \( R \) be an \( R \)-cross-section of semigroup \( S_{lp}I\!\!S_n \). According to Lemma 1, the set \( R_1 = \{ a | f(a) \in R \} \) is an \( R \)-cross-section of semigroup \( I\!\!S_n \), hence \( R_1 = R(M_1, M_2, \ldots, M_n) \) for some decomposition \( M_1 \sqcup M_2 \sqcup \ldots \sqcup M_n \) of \( N_n \).

Let \( (g_i, e_i) \in R \) be such that \( dom(e_i) = M_i \), \( g_i(x) \cdot R \cdot 1 \) for all \( x \in M_i \). Then analogously to Lemma 1, \( e_i = id_{M_i}, g_i(x) = 1 \). This element is the element of \( R \)-cross-section \( R \).

As \( M^R_1 = M_1 \) and \( N_n = M_1 \sqcup M_2 \sqcup \ldots \sqcup M_k \) we have monomorphism from \( R \) to \( \prod_{i=1}^{k}(S_{lp}I\!\!S(M_i)) \) defined by \( (f, a) \mapsto \left( f \mid _{M_1}, a \mid _{M_1}, \ldots, f \mid _{M_k}, a \mid _{M_k} \right) \). Similarly to Lemma 1, multiplying by \( (g_i, e_i) \), we get that each component \( R(g_i, e_i) \) of image
of $R$ is an $\mathcal{R}$-cross-section of $SI\mathcal{S}(M_i)$. From Lemma 4 it follows that every component is isomorphic to $R_i\ell_pR(\overrightarrow{M_i})$ for some $\mathcal{R}$-cross-section $R_i$ of semigroup $S$. Now the statement of theorem is obvious. □

A map $a \mapsto a^{-1}$ is an anti-isomorphism of semigroup $SI\mathcal{S}_n$, that sends $\mathcal{R}$-classes to $\mathcal{L}$-classes. It is also clear that it maps $\mathcal{R}$-cross-sections to $\mathcal{L}$-cross-sections and vice-versa. Hence dualizing Theorem 2 one gets description of $\mathcal{L}$-cross-sections.

**Corollary 1.** Let $R(\overrightarrow{M_1}, \overrightarrow{M_2}, \ldots, \overrightarrow{M_k})$, $R_1, \ldots, R_k$ be $\mathcal{R}$-cross-sections of semigroup $\mathcal{I}\mathcal{S}_n$. Then $R = (R_1\ell_pR(\overrightarrow{M_1})) \times (R_2\ell_pR(\overrightarrow{M_2})) \times \ldots \times (R_k\ell_pR(\overrightarrow{M_k}))$ is an $\mathcal{R}$-cross-section of semigroup $SI\mathcal{S}_n\ell_pSI\mathcal{S}_n$. Moreover, every $\mathcal{R}$-cross-section is isomorphic to $(R_1\ell_pR(\overrightarrow{M_1})) \times (R_2\ell_pR(\overrightarrow{M_2})) \times \ldots \times R_k\ell_p(R(\overrightarrow{M_k}))$.

Starting from the last corollary and iterating Theorem 2 one gets full description of $\mathcal{R}$-cross-sections of the semigroup $PAut T_n^k$ of partial automorphisms of a rooted tree.

**Corollary 2.** Semigroup $SI\mathcal{S}_n\ell_pSI\mathcal{S}_n$ contains

$$\sum_{k=1}^{n} \frac{(n-1)!}{k} \left( \sum_{i=1}^{k} \frac{1}{i} \frac{(n-1)!}{i!} \right)^k$$

different $\mathcal{R}$-$L$-cross-sections.

**References**

[1] Cowan D. F., Reilly N. R. *Partial cross-sections of symmetric inverse semigroups*. Int. J. Algebra Comput. 5 (1995), no. 3, pp. 259–287.

[2] Ganyushkin O., Mazorchuk V. *The full finite inverse symmetric semigroup $\mathcal{I}\mathcal{S}_n*.* Preprint 2001:37, Chalmers University of Technology and Göteborg University, Göteborg, 2001.

[3] Ganyushkin O., Mazorchuk V. *L- and R-Cross-Sections in $\mathcal{I}\mathcal{S}_n*.* Communications in Algebra, vol. 31 (2003), no. 9, pp. 4507–4523.

[4] Kochubinska Ye. *Combinatorics of partial wreath power of finite inverse symmetric semigroup $\mathcal{I}\mathcal{S}_d*.* Algebra and Discrete Mathematics, to appear.

[5] Meldrum J.P.D. *Wreath products of groups and semigroups*. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 74. Harlow, Essex: Longman Group Ltd., 1995.

Taras Shevchenko National University of Kyiv, Faculty of Mechanics and Mathematics, Volodymyrska str. 64, 01601, Kyiv, Ukraine.