Defect Chaos of Oscillating Hexagons in Rotating Convection

Blas Echebarria and Hermann Riecke

Department of Engineering Sciences and Applied Mathematics,
Northwestern University, 2145 Sheridan Rd, Evanston, IL, 60208, USA

(March 31, 2022)

Using coupled Ginzburg-Landau equations, the dynamics of hexagonal patterns with broken chiral symmetry are investigated, as they appear in rotating non-Boussinesq or surface-tension-driven convection. We find that close to the secondary Hopf bifurcation to oscillating hexagons the dynamics are well described by a single complex Ginzburg-Landau equation (CGLE) coupled to the phases of the hexagonal pattern. At the bandcenter these equations reduce to the usual CGLE and the system exhibits defect chaos. Away from the bandcenter a transition to a frozen vortex state is found.

The interplay between theory and experiments is at the heart of the progress made in recent years in the study of pattern-forming systems far from equilibrium. Thus, investigations of specific physical systems that allow a quantitative comparison with precise experiments are essential to check available theories and often also to suggest new theoretical venues. This has been particularly successful for ordered patterns (e.g. in Rayleigh-Bénard convection and Taylor-Couette flow [1]). For disordered patterns and spatio-temporal chaos a number of different systems have been under investigation with the goal to achieve detailed comparison between experiments and theoretical work. Spiral-defect chaos in Rayleigh-Bénard convection with low Prandtl number exhibits very rich dynamics (e.g. [2]). However, they do not arise at small amplitudes and are therefore not accessible by weakly nonlinear theory. In the presence of rotation the Kuppers-Lortz instability of convection rolls induces another type of spatio-temporal chaos (e.g. [3]). Although it occurs directly at onset a systematic theoretical treatment within weakly nonlinear theory has not been successful due to the isotropy of the system. In anisotropic electroconvection of nematic liquid crystals spatio-temporal chaos [4] can be systematically described by coupled Ginzburg-Landau equations. The derivation of these equations from the microscopic equations is, however, extremely involved and not yet complete [5].

From a theoretical point of view the most attractive and therefore most extensively studied canonical equation exhibiting spatio-temporal chaos is the complex Ginzburg-Landau equation (CGLE). It describes the onset of oscillations in a spatially extended system. Despite its simplicity it exhibits an extraordinary variety of complex dynamics, including phase chaos, defect chaos, and frozen structures [6]. In the one-dimensional case many aspects of its dynamics have been observed experimentally (e.g. localized solutions [7]). In two dimensions, however, no detailed comparison of any of the theoretical regimes of complex dynamics with experiments is available.

In this Letter we suggest that weakly nonlinear hexagon patterns in rotating convection are a good candidate to compare the theoretical results for spatio-temporal chaos in the two-dimensional CGLE with experiments. Due to the rotation the hexagons typically undergo a transition to oscillating hexagons, which are described by a complex Ginzburg-Landau equation coupled to two phase equations. We show that in a sufficiently large system the oscillating hexagons exhibit a state of defect chaos that is well described by the usual two-dimensional CGLE. Depending on the wavenumber of the underlying hexagons we find a transition to a frozen vortex state, as it is also the case in the CGLE.

We consider three coupled amplitude equations, describing the dynamics of hexagonal patterns with broken chiral symmetry, as they appear in rotating non-Boussinesq or surface-tension-driven convection. These equations can be obtained from the corresponding physical equations (e.g. Navier-Stokes) for small amplitudes with the usual scalings \( \partial_t \sim O(A_i^2) \) and \( \nabla \sim O(A_i) \), provided that the coefficient of the quadratic resonant term is small. After rescaling they can be written as:

\[
\partial_t A_1 = \mu A_1 + (i\mathbf{n}_1 \cdot \nabla)^2 A_1 + A_1^* A_2^* - A_1 |A_1|^2 - (\nu + \gamma) A_1 |A_2|^2 - (\nu - \gamma) A_1 |A_3|^2. \tag{1}
\]

The equations for the other two amplitudes are obtained by cyclic permutation of the indices. The broken chiral symmetry manifests itself by the cross-coupling coefficients not being equal. The coefficient \( \gamma \) is, therefore, a measure of the rotation.

A study of the steady hexagons with a wavenumber \( \mathbf{k} \) slightly different from the critical wavenumber \( \mathbf{k}_c = \mathbf{k}_e^* + \mathbf{q}_a \), \( |\mathbf{q}_a| = q \) and their bandside instabilities has been undertaken in [8]. We focus here on the secondary Hopf bifurcation that appears for larger values of the control parameter \( \mu = \mu_c \equiv (2 + \nu)/(\nu - 1)^2 + q^2 \). It gives rise to what we call oscillating hexagons, in which the three amplitudes of the hexagonal pattern oscillate with frequency \( \omega = 2\sqrt{3}\gamma/(\nu - 1)^2 \) and a phase shift of \( 2\pi/3 \) among them. As \( \mu \) is increased further, eventually a point \( \mu = \mu_{\text{het}} \) is reached at which the branch of oscillating hexagons ends on the branch corresponding to an unstable mixed-mode solution in a global bifurcation involving a heteroclinic connection [9]. Above this point the only stable solution is the roll solution. When
\(|\gamma| > \nu - 1\) the rolls are never stable and the limit cycle persists for arbitrarily large values of \(\mu\). In the absence of the quadratic term in Eq. (1) this condition corresponds to the K"{u}ppers-Lortz instability of rolls \([20]\). Further above the Hopf bifurcation the periodic orbit is expected to become anharmonic, and may somewhat resemble the state encountered in the K"{u}ppers-Lortz regime of rotating Boussinesq Rayleigh-B"{e}nard convection \([21,22]\).

To study the stability of the oscillating hexagons close to the Hopf bifurcation point, the amplitudes \(A_n\) are expanded as:

\[
A_n = (R + |e^{2\pi ni/3}\sqrt{\epsilon}He^{i\omega t} + \text{c.c.}| + \mathcal{O}(\epsilon)) e^{i\mathbf{\alpha} \cdot \mathbf{x} + \sqrt{\gamma} \phi_n}.
\]

(2)

The amplitude of the steady hexagons at the bifurcation point, \(R = 1/(\nu - 1)\), is independent of \(\mu\) and \(q\). Since we are considering a secondary bifurcation the amplitude \(\mathcal{H}\) will couple to the phases of the hexagonal pattern \([23]\). From the phases \(\phi_n\) of each of the modes it is possible to construct a phase vector \(\mathbf{\phi} = (\phi_x, \phi_y) = (-\phi_2 - \phi_3, (\phi_2 - \phi_3)/\sqrt{3})\), whose components are related to translations in the \(x\)- and \(y\)-directions \([24–26]\). Eliminating the fast variables, we arrive at order \(\epsilon^{3/2}\) an equation for the amplitude of the oscillation \(\mathcal{H}\), coupled to the phases of the underlying hexagonal pattern,

\[
\partial_t \mathcal{H} = \epsilon \delta_1 \mathcal{H} + \xi \nabla^2 \mathcal{H} - \delta_2 \mathcal{H} \nabla \cdot \mathbf{\phi} - \rho \mathcal{H}|\mathcal{H}|^2,
\]

\[
\partial_t \mathbf{\phi} = D_{\perp} \nabla^2 \mathbf{\phi} + D_\parallel \nabla (\nabla \cdot \mathbf{\phi}) + D_{\times 1} (\hat{e}_z \times \nabla^2 \mathbf{\phi})
\]

\[
+ D_{\times 2} (\hat{e}_z \times \nabla) (\nabla \cdot \mathbf{\phi}) + \alpha \nabla |\mathcal{H}|^2 + \beta_1 (\hat{e}_z \times \nabla) |\mathcal{H}|^2 - i \beta_2 (\mathcal{H} \nabla \mathcal{H}^* - \mathcal{H}^* \nabla \mathcal{H}) + i \eta |\mathcal{H}|^4 - \mathcal{H}^* (\hat{e}_z \times \nabla) \mathcal{H},
\]

where \(\hat{e}_z\) is the unit vector in the direction perpendicular to the plane, \(\epsilon = \mu - \mu_c\), \(\omega = 2\sqrt{3\gamma R^2}\) and

\[
v = 3R(1 + 2R),
\]

\[
\delta_1 = \frac{2R}{v} - \frac{2\omega}{v}, \quad \delta_2 = q \delta_1,
\]

\[
\xi = \frac{1}{2} - \frac{3q^2 R^2}{9R^2 + \omega^2} - \frac{q^2 R^2}{9R^2 + \omega^2} + \frac{iq^2 R^4}{9R^2 + \omega^2} - \frac{q^2 R^4}{9R^2 + \omega^2},
\]

\[
\rho = \frac{8(3R + 1)}{v} - \frac{4i\omega(1 + 4R)}{Rv} - \frac{32i}{3\omega},
\]

\[
D_{\perp} = \frac{1}{4}, \quad D_\parallel = \frac{1}{2}, \quad D_{\times 1} = \frac{q^2}{\omega}, \quad D_{\times 2} = 0,
\]

\[
\alpha = - \frac{2q^2 R^2}{9R^2 + \omega^2} - \frac{q^2 (1 + 6R)}{Rv},
\]

\[
\beta_1 = \frac{6q^2}{R(9R^2 + \omega^2)}, \quad \beta_2 = \beta_1, \quad \eta = \frac{18q}{9R^2 + \omega^2}.
\]

It is worth pointing out that the phase-amplitude equations (3,4) can be deduced by means of symmetry arguments alone and are, therefore, generic to this order in \(\epsilon\). In fact, they could be derived directly from the fluid equations without the use of the Ginzburg-Landau equations (1). Thus, keeping higher order terms in (1) would change the values of the coefficients in (3,4), but not their form.

Central to the results presented in this Letter is the observation that for hexagons with rotation as described by (1) the phase-amplitude equations (3,4) decouple at the bandcenter, i.e. for \(q = 0\). In this case they reduce to the usual CGLE for the amplitude of the oscillation. Scaling the time, space and amplitude, Eq. (3) can be written in the more usual form (5).

\[
\partial_t \mathcal{H} = H + (1 + ib_1)\nabla^2 \mathcal{H} - (b_3 - i \text{sign}(\omega))|\mathcal{H}|^2
\]

(5)

where \(b_1 = \xi_i/\xi_r = 0\) and \(b_3\) is given by:

\[
b_3 = \frac{\rho_r}{|\rho_i|} = \frac{2|\omega|R(3R + 1)}{\omega^2(1 + 4R) + 8R^2(1 + 2R)}.
\]

(6)

As \(\omega\) is increased by increasing the rotation rate \(\gamma\), \(b_3\) reaches a maximum \(b_3^{\max}\) at \(\gamma^{\max} = \sqrt{2(\nu - 1)^2(\nu + 1)/3(\nu + 3)}\). For any value of \(\nu\) the coefficient \(b_3\) will move across the range \(0 < b_3 < b_3^{\max}\) as the rotation rate is varied. Note that the dependence of \(b_3^{\max}\) on \(\nu\) is only weak and is limited to the range \(0.354 < b_3^{\max} < 0.375\), as indicated by the dotted lines in Fig. 1. Thus, independent of \(\nu\), which depends on the physical properties of the system, the oscillating hexagons are always in an intermittent regime in which stable plane waves and defect chaos coexist (see Fig. 1).

![FIG. 1. Phase diagram for the CGLE (after [6]). Defect chaos is found to the left of line T. Above the Benjamin-Fair (BF) line, all plane waves are longwave unstable, resulting in a state of phase chaos, which is bistable with defect chaos or a frozen vortex state up to the line L. The lines S1 and S2 represent the convective and absolute stability limits of plane waves emitted by spirals. The vertical dotted lines represent the limits for the range of \(b_3^{\max}\) (cf. (4)). As \(q\) is increased \(b_1\) decreases.](image-url)
which is chirally symmetric. Going to higher order we would expect terms like \( \mathbf{e}_2 \cdot (\nabla H \times \nabla H^*) H \) to appear in Eq. (6), breaking the symmetry between spirals with opposite topological charge \( \mp 1 \).

We investigate the defect chaos regime by numerical simulations of Eq. (4) using a pseudospectral method with a 4th-order Runge-Kutta/integrating factor time-stepping scheme and periodic boundary conditions. To allow for regular hexagonal patterns we take a rectangular box of aspect ratio \( L_x/L_y = \sqrt{3}/2 \). Fig. 3 shows a picture of the hexagonal pattern in this regime. It consists of patches of slightly roll-like hexagons, whose preferred direction oscillates on a fast time scale. The patches change shape and size on a slow time scale. The regions where almost perfect hexagons can be observed (e.g., on the bottom left part of the figure) correspond to the zeros (defects) of the oscillation amplitude. In order to extract this complex amplitude \( \mathcal{H} \) from the amplitudes \( A_n \) we use that close to onset \( \mathcal{H} e^{i\omega t} + \mathcal{H}^* e^{-i\omega t} \approx |A_n| - (\sum_{j=1}^3 |A_j|)/3 \). The amplitude \( \mathcal{H} \) is obtained multiplying the former expression by \( e^{i\omega t} \) and taking the average over each period. A snapshot of the magnitude \( |\mathcal{H}| \) as obtained from Fig. 2 is given in Fig. 3a, while in Fig. 3b we show the corresponding lines \( \text{Re}(\mathcal{H}) = 0 \) and \( \text{Im}(\mathcal{H}) = 0 \), whose intersection points correspond to the defects of \( \mathcal{H} \).

Away from the bandcenter, \( q \neq 0, b_1 \) becomes non-zero and is given by:

\[
  b_1 = \frac{2(9R^2 + 2\omega^2)q^2}{(6q^2R - 9R^2 - \omega^2)\omega} \quad (7)
\]

The coefficient \( b_3 \) remains unchanged. Depending on the value of \( q \) the system can therefore cross the line \( T \) (see Fig. 4), beyond which defect chaos is no longer an attractor within the CGLE (5). But, for \( q \neq 0 \), the oscillation amplitude \( \mathcal{H} \) is coupled to the phase \( \phi \). In order to study the influence of this coupling in the defect chaotic regime we measure the density of defects for Eqs. (1) and Eqs. (3,4) as a function of \( q \) (accordingly \( b_1 \)) and compare it with the results for the CGLE (5).

We begin with a perfect steady hexagonal pattern with wavenumber \( q \) \( (A_n = \text{Re}^{n\pi\times}, |q_n| = q, \) in Eq. (1), \( \mathcal{H} = 0, \phi = 0 \) in Eqs. (3,4), \( H = 0 \) in Eq. (2)) and add noise of zero mean. If the system is large enough the resulting oscillating state is not homogeneous but ends up in a persistent chaotic state. We also checked numerically that a perfect oscillating state in Eq. (4), \( A_n = (R + \mathcal{H} e^{i\omega t} + c.c.) e^{n\pi\times} \), is linearly stable, although sufficiently strong perturbations can destabilize it, giving rise to defect chaos. In the simulations we use \( 64 \times 64 \) modes, in a box of length \( L_x = 200 \) for Eqs. (1) and (3,4), and \( L = 50 \) for the CGLE, which roughly corresponds to the former after rescaling. As the scaling depends on \( q \) we take the same box size \( L_x, L_y \) in all the simulations of Eqs. (1) and (3,4) and then scale the density of defects appropriately to compare with the results from the CGLE. After a transient time, the number of defects is measured for a sufficiently long time so the system is statistically stationary. The results corresponding to the CGLE are the average of three independent runs.

As can be seen in Fig. 4 the simulations of (1), (3,4) and (5) agree very well up to the line \( S_2 \), where the spiral defects go from being absolutely to convectively unstable. Since for \( q \neq 0 \) the amplitude \( \mathcal{H} \) is no longer decoupled from the phases, the agreement indicates that the phase does not have a strong influence on the dynamics in this regime. Beyond line \( S_2 \), after a chaotic transient, the system settles down in a frozen vortex state in which the density of defects remains unchanged for long periods of time. Since this final state is history dependent a quantitative agreement is harder to achieve. In particular it would require a more extensive sampling of initial conditions. However, the agreement remains good. Beyond line \( T \) still a small number of defects remain, probably because our simulations are not long enough given the weak interaction of defects. If and to what extent the coupling with the phase affects the lines \( S_2 \) and \( T \) is not
clear from these simulations, but the qualitative picture remains the same as in the CGLE. Note also that, for these parameters, additional long-wave instabilities occur beyond line $T$.\cite{28}

We gratefully acknowledge interesting discussions with F. Sain and M. Silber. The numerical simulations were performed with a modification of a code by G.D. Granzow. This work was supported by the Engineering Research Program of the Office of Basic Energy Sciences at the Department of Energy (DE-FG02-92ER14303) and NASA Grant NAG3-2113.

[1] G. Ahlers, Physica D 51, 421 (1992).
[2] S. Morris, E. Bodenschatz, D. Cannell, and G. Ahlers, Physica D 97, 164 (1996).
[3] Y. Hu, W. Pesch, G. Ahlers, and R. Ecke, Phys. Rev. E 58, 5821 (1998).
[4] M. Dennin, G. Ahlers, and D. Cannell, Science 272, 388 (1996).
[5] M. Treiber and L. Kramer, Phys. Rev. E 58, 1973 (1998).
[6] H. Chaté and P. Manneville, Physica A 224, 348 (1996).
[7] P. Manneville and H. Chaté, Physica D 96, 30 (1996).
[8] G. Grinstein, C. Jayaprakash, and R. Pandit, Physica D 90, 96 (1996).
[9] R. Montagne, E. Hernández-García, and M. S. Miguel, Physica D 96, 47 (1996).
[10] R. Montagne, E. Hernández-García, A. Amengual, and M. S. Miguel, Phys. Rev. E 56, 151 (1997).
[11] A. Torcini, Phys. Rev. Lett. 77, 1047 (1996).
[12] A. Torcini, H. Frauenkron, and P. Grassberger, Phys. Rev. E 55, 5073 (1997).
[13] S. Popp, O. Stiller, I. Aranson, and L. Kramer, Physica D 84, 398 (1995).
[14] G. Huber, P. Alstrom, and T. Bohr, Phys. Rev. Lett. 69, 2380 (1992).
[15] J. Burguete, H. Chaté, F. Daviaud, and N. Mukolobwiez, Phys. Rev. Lett. 82, 3252 (1999).
[16] B. Echebarria and H. Riecke, preprint, [matt-sol/9906008].
[17] J. Swift, in Contemporary Mathematics Vol. 28 (American Mathematical Society, Providence, 1984), p. 435.
[18] A. Soward, Physica D 14, 227 (1985).
[19] J. Millán-Rodríguez et al., Phys. Rev. A 46, 4729 (1992).
[20] G. Küppers and D. Lortz, J. Fluid Mech. 35, 609 (1969).
[21] F. Busse and R. Cleaver, Recent Developments in Theoretical and Experimental Fluid Mechanics: Compressible and Incompressible Flows (Springer-Verlag, New York, 1979), Vol. 94, p. 376.
[22] F. Busse and K. Heikes, Science 208, 173 (1980).
[23] P. Coullet and G. Iooss, Phys. Rev. Lett. 64, 866 (1990).
[24] J. Lanzera, S. Metens, and D. Walgraef, Europhys. Lett. 24, 707 (1993).
[25] R. Hoyle, Appl. Math. Lett. 9, 81 (1995).
[26] B. Echebarria and C. Pérez-García, Europhys. Lett. 43, 35 (1998).
[27] K. Nam, E. Ott, M. Gabbay, and N. Guzdar, Physica D 118, 69 (1998).
[28] B. Echebarria and H. Riecke, unpublished.