EQUIVARIANT COMPACTIFICATIONS OF A NILPOTENT GROUP BY 
$G/P$

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Abstract. Let $G$ be a simple complex algebraic group, $P$ a parabolic subgroup of $G$ and $N$ the unipotent radical of $P$. The so-called equivariant compactification of $N$ by $G/P$ is given by an action of $N$ on $G/P$ with a dense open orbit isomorphic to $N$. In this article, we investigate how many such equivariant compactifications there exist. Our result says that there is a unique equivariant compactification of $N$ by $G/P$, up to isomorphism, except $P^n$.

1. Introduction

Throughout, we will work over complex numbers. Let $H$ be an algebraic group. Then a projective variety $M$ is called an equivariant compactification of $H$ if there is an algebraic $H$-action $H \times M \rightarrow M$ with a Zariski open orbit $O$, equivariantly biregular to $H$. In this situation, the action of $H$ is called an Equivariant Compactification structure (for short, EC-structure) on $M$. An isomorphism between two EC-structures $A_i : H \times M \rightarrow M$ for $i = 1, 2$ consists of an automorphism $F : H \rightarrow H$ and a biregular morphism $\Psi : M \rightarrow M$ such that the diagram

$$
\begin{array}{ccc}
H \times M & \xrightarrow{A_1} & M \\
\downarrow{(F,\Psi)} & & \downarrow{\Psi} \\
H \times M & \xrightarrow{A_2} & M
\end{array}
$$

commutes.

In [10], Hassett and Tschinkel studied EC-structures on $\mathbb{P}^n$ of the vector group $\mathbb{G}_a^n$. They showed that there is a correspondence between EC structures of the vector group and Artin local algebras of length $n$. Then, by classifying such algebras, they showed that there are finitely (resp. infinitely) many isomorphism classes of EC-structures on $\mathbb{P}^n$ for $n \geq 2$ (resp. $n \geq 6$).

Arzhantsev considered EC-structures of a vector group on a homogeneous space $G/P$ for a semisimple algebraic group $G$ and a parabolic subgroup $P$ of $G$ ([11]). His result, restricted to simple groups $G$, says that $G/P$ is an equivariant compactification of a vector group exactly when the unipotent radical $P_u$ of $P$ is commutative, or when $P_u$ is not commutative and there is a pair $(\tilde{G}, \tilde{P})$ such that $G/P = \tilde{G}/\tilde{P}$ and the unipotent radical $\tilde{P}_u$ of $\tilde{P}$ is commutative; see Subsection 2.3 for $\tilde{G}$ and $\tilde{P}$. Arzhantsev also raised the question of how many such EC-structures there are on $G/P$, which was mentioned again by Arzhantsev and Sharoyko ([2]).

Very recently, Fu and Hwang proved a more general result on EC-structures of a vector group and posed the question whether the analogue of their result holds for other linear algebraic groups ([7]). Roughly, their result says that if $M$ is a Fano manifold, different from $\mathbb{P}^n$, of Picard number 1 having smooth VMRT, then all EC-structures on $M$ are isomorphic. See Section 4 for the definition of VMRT. There are various manifolds that

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satisfy the necessary condition on VMRT. They include all irreducible Hermitian symmetric homogeneous spaces and some non-homogeneous spaces (see Proposition of 6.14 of [6] for examples of non-homogeneous spaces). In particular, their result answers the question of Arzhantsev and Sharonyko ([1], [2]) on the classification of EC-structures of a vector group on $G/P$ for $P$ maximal.

Lastly, Devyatov also studied EC-structures on $G/P$ of a vector group, using relevant Lie algebra representations. To classify EC-srtuctures, he first set up a correspondence between EC-structures on $G/P$ and multiplications with a certain property on Lie algebra representations, and then solved the initial EC-problem by classifying these multiplications.

In this article, we generalize the results on vector groups above to some non-commutative nilpotent groups $N$. To make a precise statement, let us fix some notations. Let $g = \bigoplus_{k \in \mathbb{Z}} g_k$ be a simple graded Lie algebra. Then there is a triple $(G, P, N)$ of groups associated with $g = \bigoplus_{k \in \mathbb{Z}} g_k$, where $G$ is an (adjoint) algebraic group with Lie algebra $g$, $P$ is a parabolic subgroup of $G$ with Lie algebra $p = \bigoplus_{k \geq 0} g_k$, and $N$ is a unipotent radical of $P$, so that the Lie algebra of $N$ equals $n = \bigoplus_{k > 0} g_k$. Then our main theorem can be stated as follows.

**Theorem 1.1.** Let $g = \bigoplus_{k \in \mathbb{Z}} g_k$ be a simple graded Lie algebra, and let $G, P$ and $N$ be given as above. Suppose that the homogeneous space $G/P$ is different from $\mathbb{P}^n$. Then there exists a unique EC-structure of $N$ on $G/P$, up to isomorphism.

We remark that existence of an EC-structure on $G/P$ follows directly from the Bruhat decomposition of $G/P$([14]). To prove uniqueness, we mostly follow a proving scheme Fu and Hwang made in [7]. A main difference lies in which tool to use in order to obtain an extension of a locally defined map to the entire space, which is the most essential in proving the uniqueness; Fu and Hwang used the Cartan-Fubini extension theorem which is applicable only to Fano manifolds of Picard number 1, while we use the Yamaguchi’s result on the prolongation of simple graded Lie algebras. For reader’s convenience, let us give a bit more detailed sketch of proof of the uniqueness.

First of all, we divide all simple graded Lie algebras $g = \bigoplus_{k \in \mathbb{Z}} g_k$ except for two cases $(A_l, \{\alpha_1\})$ and $(C_l, \{\alpha_1\})$) into three types $I, II$ and $III$ (Subsection $5.1$).

1. The simple graded Lie algebras of type $I$ are all but the simple graded Lie algebras of types $II$ and $III$ right below.
2. The simple graded Lie algebras of type $II$ are ones of depth one, or contact gradation, but not isomorphic with $(A_l, \{\alpha_1, \alpha_k\})$.
3. The simple graded Lie algebras of type $III$ are ones isomorphic with $(A_l, \{\alpha_1, \alpha_k\})$ or $(C_l, \{\alpha_1, \alpha_k\})$.

We note that for a simple graded Lie algebra $g = \bigoplus_{k \in \mathbb{Z}} g_k$, there are the natural differential system $D \subset T(G/P)$ and the so called VMRT $\mathcal{C}$ on $G/P$, and, in particular, for the type $III$ there is the natural decomposition $D = D^{(1)} \oplus D^{(2)}$ on $G/P$. See Subsections $3.1$, $4.2$ and $5.6$ for the differential system $D$, VMRT $\mathcal{C}$, and the decomposition $D = D^{(1)} \oplus D^{(2)}$, respectively. Let $A$ (resp. $A_\phi$) be the sheaf of Lie algebras of infinitesimal automorphisms preserving $D$ (resp. $\mathcal{C}$) on $G/P$, and for simple graded Lie algebra of type $III$, let $A_\oplus$ be the sheaf of Lie algebras of infinitesimal automorphisms on $G/P$ preserving the decomposition $D = D^{(1)} \oplus D^{(2)}$.

In order to prove the uniqueness, we suppose that there are two EC-structures of $N$ on $G/P$

$$A_i : N \times G/P \to G/P \text{ for } i = 1, 2.$$  

We only have to construct an isomorphism between two EC-structures $A_i$. As the first step toward a construction of the isomorphism, we construct a local biholomorphic map on $G/P$.
as follows. For each $i = 1, 2$, $O_i$ be the Zariski open orbit for the EC-structure $A_i$, and, after fixing a point $x_i \in O_i$, define a biholomorphic map $a_i : N \to O_i$ by $a_i(h) = A_i(h, x_i)$. Then using two pull-backed differential systems $a_i^*(D(O_i))$ on $N$, we construct a Lie algebra isomorphism $f : n \to n$ which induces a group isomorphism $F : N \to N$. Then, through the biholomorphic maps $a_i$, the group isomorphism $F$, in turn, gives rise to a biholomorphic map $\Psi : O_1 \to O_2$ that preserves $D$, $G$, or the decomposition $D = D^{(1)} \oplus D^{(2)}$ for types $I, II$ or $III$, respectively. This is a content of Proposition 5.3.

As the second step, we need to extend the local biholomorphic $\Psi : O_1 \to O_2$. To do this, we first characterize $g$ as a subalgebra of the Lie algebra of holomorphic vector fields on any connected open subsets $U \subset G/P$ as follows.

**Proposition 1.2.** (Propositions 5.4, 5.7, 5.8) Let $g = \bigoplus_{k \in \mathbb{Z}} g_k$ be a simple graded Lie algebra, $G/P$ its associated homogeneous space. Then for any connected open subset $U \subset G/P$, the Lie algebra of sections $\mathcal{A}(U)$ (resp. $\mathcal{A}_G(U)$, $\mathcal{A}_\mathbb{C}(U)$) is isomorphic to $g$ if $g = \bigoplus_{k \in \mathbb{Z}} g_k$ is type I (resp. type II, type III).

We note that since $\Psi : O_1 \to O_2$ preserves $D$, $G$, or the decomposition $D = D^{(1)} \oplus D^{(2)}$ for types $I, II$ or $III$, respectively, the differential $d\Psi$ sends $\mathcal{A}(O_1)$, $\mathcal{A}_G(O_1)$ or $\mathcal{A}_\mathbb{C}(O_1)$ isomorphically onto $\mathcal{A}(O_2)$, $\mathcal{A}_G(O_2)$ or $\mathcal{A}_\mathbb{C}(O_2)$ for types $I, II$ or $III$, respectively, i.e., $d\Psi$ send $g$ to $g$ for all types $I, II$ and $III$. Furthermore, it turns out that $d\Psi$ sends a parabolic subalgebra $p$ onto another parabolic subalgebra $p'$. Therefore we have an isomorphism $d\Psi : (g, p) \to (g, p')$, which induces an automorphism $\tilde{\Psi} : G/P \to G/P'$, extending $\Psi : O_1 \to O_2$. This is a content of Proposition 5.9.

As a final step, we show that the biholomorphic map $\tilde{\Psi}$ is equivariant with respect to the actions $A_i$, which implies that $\tilde{\Psi}$ is an isomorphism between two EC-structures $A_i$ on $G/P$.

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## 2. Simple graded Lie algebras and its associated homogeneous spaces

### 2.1. Gradations of simple Lie algebras

In this subsection, we collect basics on complex simple Lie algebras. We refer to [11] for more detailed accounts.

A **gradation** of a Lie algebra $g$ is a direct sum $g = \bigoplus_{k \in \mathbb{Z}} g_k$ of subspaces of $g$ such that $[g_r, g_s] \subset g_{r+s}$. A graded Lie algebra is a Lie algebra equipped with a gradation $g = \bigoplus_{k \in \mathbb{Z}} g_k$. Throughout the paper, we will restrict ourselves to (complex) simple graded Lie algebra $g = \bigoplus_{k \in \mathbb{Z}} g_k$ satisfying an additional condition

$$g_k = [g_{k+1}, g_{-1}]$$

for all $k < -1$.

Such simple graded Lie algebras can be characterized as follows.

Fix a Cartan subalgebra $h \subset g$ and a simple root system $\Delta := \{\alpha_1, ..., \alpha_l\}$ of the root system $\Phi$ with respect to $h$, and denote by $\Phi^+$ the set of positive roots. Let $\Delta_1$ be a nonempty subset of $\Delta$. For each $k \geq 0$, let $\Phi^+_k$ be the set of positive roots $\alpha \in \Phi^+$ that can be written as $\alpha = \sum_{i=1}^{l} n_i(\alpha)\alpha_i$ for $n_i(\alpha)$ with $\sum_{\alpha_i \in \Delta_1} n_i(\alpha) = k$. Then we put

$$g_0 = h \oplus \bigoplus_{\alpha \in \Phi^+_0} (g_\alpha \oplus g_{-\alpha}),$$

$$g_k = \bigoplus_{\alpha \in \Phi^+_k} g_\alpha, \; g_{-k} = \bigoplus_{\alpha \in \Phi^+_{-k}} g_{-\alpha} \quad (k > 0).$$
Then it is easy to check that the direct sum \( \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \) forms a gradation of \( \mathfrak{g} \) and satisfies the condition \( \mathfrak{g}_k = [\mathfrak{g}_{k+1}, \mathfrak{g}_{-1}] \) for all \( k < -1 \). When \( \mathfrak{g} \) is of Lie type \( X_l \), we often write \((X_l, \Delta_1)\) for the simple graded Lie algebra \( \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \) obtained in this way. The converse is true by Theorem 3.12 of [16]. Namely, if \( \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \) is a gradation of a simple Lie algebra \( \mathfrak{g} \) satisfying \( \mathfrak{g}_k = [\mathfrak{g}_{k+1}, \mathfrak{g}_{-1}] \) for each \( k < -1 \) and \( \mathfrak{g} \) is of Lie type \( X_l \), then it is conjugate to \((X_l, \Delta_1)\) for a subset \( \Delta_1 \) of the simple root system \( \Delta \) of \( X_l \).

A gradation \( \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \) is symmetric in the sense that \( \mathfrak{g}_k \neq 0 \) if and only if \( \mathfrak{g}_{-k} \neq 0 \). More precisely, the dimension of \( \mathfrak{g}_k \) is equal to the dimension of \( \mathfrak{g}_{-k} \). This follows from the fact that the restriction of Killing form \( B \) to \( \mathfrak{g}_k \times \mathfrak{g}_{-k} \) is nondegenerate (Lemma 3.1 of [16]). Therefore for a simple graded Lie algebra \( \mathfrak{g} \), there is a unique integer \( \mu > 0 \) such that \( \mathfrak{g}_k \neq 0 \) for any \( -\mu \leq k \leq \mu \), and \( \mathfrak{g} = \bigoplus_{k=-\mu}^\mu \mathfrak{g}_k \). Such an integer \( \mu \) is called the \textit{depth} of \( \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \).

Let \( \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \) be a simple graded Lie algebra, and let \( \mathfrak{g}^i = \bigoplus_{k \geq i} \mathfrak{g}_k \). Then subalgebra \( \mathfrak{g}^0 \) is a parabolic subalgebra \( \mathfrak{p} \) associated with \( \Delta_1 \), and the subalgebra \( \mathfrak{g}^1 \) is the nilradical \( \mathfrak{n} \) of \( \mathfrak{p} \). Let \( G \) be the adjoint algebraic group of \( \mathfrak{g} \), and let \( P \) and \( N \) be subgroups of \( G \) whose Lie algebras are \( \mathfrak{p} \) and \( \mathfrak{n} \), respectively. Then \( P \) is a parabolic subgroup of \( G \) and the nilpotent subgroup \( N \) is a unipotent radical of \( P \). In this situation, we say that \( G/P \) is \textit{associated with} the graded simple Lie algebra \( \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \).

2.2. Examples of simple graded Lie algebras. We give an explicit description of gradations of some simple graded Lie algebras which will be used in later sections.

2.2.1. \( A_l \) type \((l \geq 1)\). For \( \mathfrak{g} = \mathfrak{sl}(l+1) \), we choose Cartan subalgebra \( \mathfrak{h} \) consisting of all diagonal matrices \( D(a_1, ..., a_{l+1}) \) in \( \mathfrak{sl}(l+1) \), where \( D(a_1, ..., a_{l+1}) \) is the diagonal matrix with the entries \( a_1, ..., a_{l+1} \) on the diagonal. For \( i = 1, ..., l+1 \), let \( \lambda_i \) be linear forms defined by \( \lambda_i : D(a_1, ..., a_{l+1}) \mapsto a_i \). Let \( E_{i,j} \) be the \((l+1) \times (l+1)\) matrix whose \((i,j)\)-th entry is 1 and other entries are all 0. Then we have

\[
[H, E_{i,j}] = (\lambda_i - \lambda_j)(H)(E_{i,j}) \quad \text{for } H \in \mathfrak{h}.
\]

Thus, we have the root system \( \Phi = \{\lambda_i - \lambda_j \mid 1 \leq i, j \leq l+1, i \neq j\} \). Take a simple root system \( \Delta = \{\alpha_1, ..., \alpha_l\} \), where \( \alpha_i = \lambda_i - \lambda_{i+1} \). For \((A_1, \{\alpha_k\})\), the gradation of \((A_1, \{\alpha_k\})\) is given by \( \mathfrak{sl}(l+1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \);

\[
\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \mid C \in M(j,k) \right\}, \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \mid C \in M(k,j) \right\},
\]

\[
\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in M(k,k), B \in M(j,j) \text{ and } \text{tr}A + \text{tr}B = 0 \right\},
\]

where \( j = l + 1 - k \) and \( M(p, q) \) denotes the set of \( p \times q \) matrices. This gradation can be described schematically by the diagram:

\[
\begin{array}{c|c|c}
& k & j \\
\hline
k & 0 & 1 \\
\hline
j & -1 & 0 \\
\end{array}
\]

(j = l + 1 - k)

The diagram of \((A_1, \{\alpha_k\})\) is obtained by superposing two diagrams corresponding to \((A_1, \{\alpha_k\})\) and \((A_1, \{\alpha_k\})\);
In general, the diagram of \((\mathcal{A}_1, \{\alpha_{k_1}, ..., \alpha_{k_m}\})\) is obtained by superposing \(m\) diagrams of \((\mathcal{A}_1, \{\alpha_{k_1}\}), ..., (\mathcal{A}_1, \{\alpha_{k_m}\})\).

2.2.2. \(C_l\) type \((l \geq 2)\). Let \((V, \langle , \rangle)\) be a symplectic vector space of dimension \(2l\). Let \(\mathfrak{g} = \mathfrak{sp}(V)\). Choose a symplectic basis \(\{e_1, ..., e_l, f_1, ..., f_l\}\) of \(V\) such that \(\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0\) and \(\langle f_i, e_{l+1-j} \rangle = \delta_{i,j}\) for \(i, j = 1, ..., l\). Then using this basis, elements of \(\mathfrak{g}\) can be written as \(2l \times 2l\)-matrices of the form

\[
X = X(A, B, C) = \begin{pmatrix} A & B \\ C & -A' \end{pmatrix}.
\]

Here \(A, B, C\) are \(l \times l\) matrices, and \(B, C\) are matrices such that \(B = B'\) and \(C = C'\), where \((\cdot)'\) means taking “transpose” with respect to the anti-diagonal line.

In this case, the root systems is \(\Phi = \{\lambda_i - \lambda_j \ (i \neq j), \pm(\lambda_i + \lambda_j) \ (1 \leq i \leq j \leq l)\}\). Note that the roots \(\lambda_i - \lambda_j, \lambda_i + \lambda_j\) and \(-\lambda_i - \lambda_j\) have the root vectors \(X(e_{ij}, 0, 0), X(0, f_{l+1-i-j}, 0)\) and \(X(0, 0, f_{l+1-i-j})\), respectively, where \(f_{l+1-i-j} := E_{i,j} + E'_{i,j}\). By putting \(\alpha_i = \lambda_i - \lambda_j\) for \(1 \leq i \leq l - 1\) and \(\alpha_l = 2\alpha_1\), we take a simple root system \(\Delta = \{\alpha_i \ | \ i = 1, ..., l\}\).

Then we see that the gradation of \((\mathcal{A}_1, \{\alpha_k\})\) is given by the following diagrams;

\[
\begin{array}{ccc}
  \text{k} & 0 & 1 & 2 \\
-1 & 0 & 1 & \text{1 \leq k < l} \\
-2 & -1 & 0 & \text{k = l}
\end{array}
\]

As in Lie type \(A_l\), the diagram of \((\mathcal{A}_1, \{\alpha_{k_1}, ..., \alpha_{k_m}\})\) is obtained by superposing \(m\) diagrams of \((\mathcal{A}_1, \{\alpha_{k_1}\}), ..., (\mathcal{A}_1, \{\alpha_{k_m}\})\).

2.3. Automorphism group of \(G/P\). Let \(G/P\) be the homogeneous space associated with \(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k\). Then it is well-known that \(G\) (of adjoint type) coincides with the identity component \(\text{Aut}^0(G/P)\) of the group of automorphisms of \(G/P\) except when \(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k\) is isomorphic with \((\mathcal{A}_1, \{\alpha_1\})\) \((l \geq 2)\), \((B_l, \{\alpha_l\})\) \((l \geq 3)\) or \((G_2, \{\alpha_1\})\); see [13]. In these exceptions, the Lie algebra of the algebraic group \(\text{Aut}^0(G/P)\) is of type \(A_{2l-1}, D_{l+1}\) or \(B_3\), respectively. Let \(\tilde{G} = \text{Aut}^0(G/P)\). Then there is a parabolic subgroup \(\tilde{P} \subset \tilde{G}\) such that \(G/P = \tilde{G}/\tilde{P}\). Indeed, the simple graded Lie algebra with which \(\tilde{G}/\tilde{P}\) is associated is \((A_{2l-1}, \{\alpha_1\}), (D_{l+1}, \{\alpha_{l+1}\})\) or \((B_3, \{\alpha_1\})\), respectively.

3. Differential systems

In this section, we give an overview on differential systems on complex manifolds \(M\).

3.1. Generalities on differential systems. Materials in this subsection are taken from [16]. For more detailed accounts or proofs, also see [15]. A differential system on \(M\) is a subbundle \(D\) of the tangent bundle \(TM\) on a complex manifold \(M\). We will simply write \((M, D)\) for a differential system \(D\) on \(M\). A differential system \((M, D)\) is completely integrable if \([D, D] \subset D\), where \(D\) is the sheaf of sections of \(D\). In this paper, we only deal with non-integrable differential system \((M, D)\) which we simply refer to as a differential
system. A differential system \((M, D)\) gives rise to the \(k\)-th weak derived system \(D^k\) which is a subsheaf of sections of \(TM\) defined inductively as follows. Define
\[
D^{-1} = D, \quad D^k = D^{k+1} + [D, D^{k+1}] \quad \text{for} \quad k < -1.
\]
In general, \(D^k\) is not locally free. But if \(D^k\) is locally free for each \(k \leq -1\), then the differential system \(D\) is called regular, and we denote by \(D^k\) the bundle corresponding to \(D^k\). We often call \(D^k\) the \(k\)-th weak derived system of \(D\) for \(k \leq -1\), too.

**Proposition 3.1.** (Proposition 1.1 of [15]) Let \((M, D)\) be a regular differential system. Then the weak derived system of \(D\) satisfies

1. There exists a unique integer \(\mu > 0\) such that
\[
TM = D^{-\mu} \supset D^{-\mu+1} \supset \cdots \supset D^{-1} = D, \quad \text{and} \quad TM \neq D^{-\mu+1}.
\]
2. \([D^k, D^l] \subset D^{k+l}\) for all \(k, l < 0\).

To a regular differential system \((M, D)\), we associate a nilpotent graded Lie algebra \(\mathfrak{m}(x)\) for each \(x \in M\) defined as follows. Put \(\mathfrak{g}_{-1}(x) = D^{-1}(x)\) and \(\mathfrak{g}_k(x) = D^k(x)/D^{k+1}(x)\) for \(k < -1\). Then define
\[
\mathfrak{m}(x) = \bigoplus_{k=-1}^{\mu} \mathfrak{g}_k(x).
\]
The Lie bracket defined on local vector fields around \(x\) induces a bracket product on the graded vector space \(\mathfrak{m}(x)\) which gives a graded Lie algebra structure on \(\mathfrak{m}(x)\) and satisfies \(\mathfrak{g}_k(x) = [\mathfrak{g}_{k+1}(x), \mathfrak{g}_{-1}(x)]\) for \(k < -1\). Note that \(\mathfrak{m}(x)\) depends on the chosen \(D\), and \(\mathfrak{m}(x)\) is nilpotent by Proposition 3.1. The nilpotent graded Lie algebra \(\mathfrak{m}(x)\) is called the symbol algebra of \(D\) at \(x\). Let \(\mathfrak{m}\) be a fundamental graded Lie algebra of \(\mu\)-th kind, i.e., \(\mathfrak{m} = \bigoplus_{k=-1}^{\mu} \mathfrak{g}_k\) be a nilpotent graded Lie algebra such that \(\mathfrak{g}_k = [\mathfrak{g}_{k+1}, \mathfrak{g}_{-1}]\) for \(k < -1\). Then the differential system \(D\) on \(M\) is called of type \(\mathfrak{m}\) if the symbol algebra \(\mathfrak{m}(x)\) is isomorphic with \(\mathfrak{m}\) for each \(x \in M\).

3.2. **Natural regular differential system on homogeneous spaces.** Let \(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k\) be a simple graded Lie algebra, and \(G/P\) its associated homogeneous space. Then the tangent bundle \(T(G/P)\) is identified with the associated bundle \(G \times_P (\mathfrak{g}/\mathfrak{g}^0)\). Under this identification, \(D := G \times_P (\mathfrak{g}^{-1}/\mathfrak{g}^0)\) forms a regular differential system on \(G/P\), and the weak derived system \(D^k\) for \(k \leq -1\) induced by \(D\) is identified with \(G \times_P (\mathfrak{g}^k/\mathfrak{g}^0)\). Another way of constructing \(D\) is as follows. First, identify \(\mathfrak{g}\) with the Lie algebra of left invariant vector fields on \(G\). Then the subspace \(\mathfrak{g}^{-1} \subset \mathfrak{g}\) defines a left invariant subbundle of the tangent bundle \(TG\). Note that this subbundle is preserved under the right action of \(P\) on \(G\). Therefore \(\mathfrak{g}^{-1}\) induces a \(G\)-invariant differential system \(D\) on \(G/P\). The following is a standard fact about a differential system on \(G/P\) associated with \(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k\).

**Proposition 3.2.** ([13]) Let \(\mathfrak{g} = \bigoplus_{k=-\mu}^{\mu} \mathfrak{g}_k\) be a simple graded algebra such that \(\mathfrak{m} = \bigoplus_{k=-1}^{\mu} \mathfrak{g}_k\) is fundamental. Then the associated homogeneous space \(G/P\) carries a natural regular differential system \(D \subset T(G/P)\) of type \(\mathfrak{m}\).

3.3. **Local picture of the natural differential system \(D\) on \(G/P\).** Let \(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k\) be a simple graded Lie algebra. We describe a local picture of the differential system \(D\) on the associated homogeneous space \(G/P\). We fix \(S = G/P\) in case that there may be a different presentation \(G/P\) for a different parabolic subgroup \(P\) involved in some explanation. First of all, let us construct ‘principal’ dense open subsets of \(S\) indexed by parabolic subgroups of \(G\). Note that there is a principal dense open subset of \(S\) associated with the given \(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k\) (or its parabolic \(P\)). To be precise, let \(N^-\) a subgroup of \(G\) with Lie algebra
A generic member of a minimal rational component is standard (11). A rational curve where $\tau = \pi P \circ \tau P : G/P \to S$ is an open subset of $S$. A rational component (Section 1 members are minimal rational curves. Note that there are only finitely many minimal curves. A canonical degree, i.e., the degree of $\xi = \pi P \circ \tau P$ for all nonnegative integers $n$. The relation $\tau = \pi P \circ \tau P$ induces an isomorphism $\tau P : G/P \to S$ defined by $\tau P(gP') = (h^{-1}gh)P$ for $gP' \in G/P'$, and a new gradation $\xi = \bigoplus_{k \in \mathbb{Z}} g_k'$, where $g_k' = hgh^{-1}$. Then $M(m')$ constructed as above for the new gradation $\xi = \bigoplus_{k \in \mathbb{Z}} g_k'$ is an open subset of $G/P'$. By abuse of notation, let $M(m') \subset S$ denote the image of $\tau P : G/P' \to S$.

Now, given $\xi = \bigoplus_{k \in \mathbb{Z}} g_k$, we construct a model differential system $D_m$ on the manifold $N^-$ defined as follows (p. 420 of [16]). Identifying $m$ with the Lie algebra of left invariant vector fields on $N^-$, $g_{-1}$ defines a left invariant subbundle $D_m$ of $T(N^-)$. Then $D_m$ is a regular differential system on $N^-$ of type $m$. Furthermore, we see that the differential system $D_m$ on $N^-$ is identified with the differential system $D_{|M(m)}$ on $M(m) \subset S$ via the isomorphism $\tau P : N^- \to M(m)$, where $D$ is the natural differential system on $G/P$ of Proposition 3.2 associated with $\xi = \bigoplus_{k \in \mathbb{Z}} g_k$.

More generally, let $P'$ and $\xi = \bigoplus_{k \in \mathbb{Z}} g_k'$ be given as above. Let $N' := (N')^-$ be the subgroup of $G$ with Lie algebra $m' = \bigoplus_{k < 0} g_k'$. As above, for $\xi = \bigoplus_{k \in \mathbb{Z}} g_k'$ (or $P'$), we can construct a model differential system on $D_{m'}$ on $N'$, which is identified with the differential system $D_{|M(m')}$ on $M(m') \subset G/P'$, where $D'$ is the natural differential system on $G/P'$ associated with $\xi = \bigoplus_{k \in \mathbb{Z}} g_k'$. But since the differential system $D'$ on $G/P'$ is identified with the differential system $D$ on $N^-$ via $\tau P : G/P' \to S$, the differential system $D_m$ on $N^-$ is identified with the differential system $D_{|M(m')}$. This construction shows how the differential systems on principal open subsets of $S$ look like.

4. Varieties of minimal rational tangents

This section is devoted to giving an overview of a variety of minimal rational tangent of $M$. We refer to [9] for details on VMRT. Throughout this section, we will assume that $M$ is a Fano manifold of dimension $n$ and of Picard number 1.

4.1. Minimal rational curves. A parametrized rational curve, or simply a rational curve on $M$ is a morphism $f : \mathbb{P}^1 \to M$ that is birational over its image. It is well-known that for each $x$ on a Fano manifold $M$ of Picard number 1, there is a rational curve $f : \mathbb{P}^1 \to M$ through $x$ ([12] and [11]). Recall that a vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles. A rational curve $f : \mathbb{P}^1 \to M$ is free if the pull-back of the tangent bundle $TM$ of $M$ splits as

$$f^*TM \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$$

for all nonnegative integers $a_i$. A free rational curve $f : \mathbb{P}^1 \to M$ is minimal if its anticanonical degree, i.e., the degree of $f^*(K_X^{-1})$ is minimal. A minimal rational curve on $M$ is a rational curve that can be obtained as a deformation of minimal free rational curves. A minimal rational component is a component of the Chow space of $M$ whose members are minimal rational curves. Note that there are only finitely many minimal rational components (Section 1.3 of [9]). A free rational curve $f : \mathbb{P}^1 \to M$ is standard if

$$f^*TM \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^{n-1-p},$$

where $p + 2$ is the anti-canonical degree of $f$. By the so-called bend-and-break of Mori, a generic member of a minimal rational component is standard ([11]).
4.2. Variety of minimal rational tangents. Choose a minimal rational component \( \mathcal{X} \), and for a general element \( x \in M \), let \( \mathcal{X}_x \) be the normalization of the Chow space of members of \( \mathcal{X} \) through \( x \). Then it was proved in [12] that \( \mathcal{X}_x \) is the union of finitely many smooth algebraic varieties of dimension \( p \), where \( p \) is in \([1,1]\). Note that a generic member of each component of \( \mathcal{X}_x \) is a standard rational curve that is smooth at \( x \). Thus, by associating to a member of \( \mathcal{X}_x \) smooth at \( x \) its tangent, we define a rational map called the tangent map at \( x \)

\[ \Theta_x : \mathcal{X}_x \to \mathbb{P}(T_x M). \]

Let \( \mathcal{C}_x \) be the strict image of \( \Theta_x \). Then \( \mathcal{C}_x \) is called the variety of minimal rational tangents, VMRT for short, at \( x \). We will denote by \( \mathcal{C} \) the variety obtained by taking the union of \( \mathcal{C}_x \).

The following proposition makes it possible to compute the VMRT at a general point for various manifolds.

**Proposition 4.1.** (Proposition of [9]) Suppose that \( M \) can be embedded in a projective space \( \mathbb{P}^N \) so that for each \( x \in M \), \( M \) contains a line of \( \mathbb{P}^N \) through \( x \). Then the tangent map at a general point \( y \in M \) is an embedding, and \( \mathcal{C}_y \) is smooth.

4.3. Examples. We give examples of VMRT for some homogeneous spaces.

4.3.1. Projective space. For \( \mathbb{P}^n \), the Chow space \( \mathcal{X}_x \) of all lines through \( x \) is isomorphic to \( \mathbb{P}^{n-1} \) for every \( x \in \mathbb{P}^n \). From Proposition 4.1, we have the isomorphism

\[ \Theta_x : \mathcal{X}_x = \mathbb{P}^{n-1} \to \mathbb{P}(T_x \mathbb{P}^n). \]

So we have \( \mathcal{C}_x = \mathbb{P}(T_x \mathbb{P}^n) \).

4.3.2. Grassmannian. Let \( \mathbb{G}(m, N) \) be the Grassmannian of \( m \)-dimensional subspaces \( W \) in a vector space \( V \) of dimension \( N \). Let \( x \) be a point of \( \mathbb{G}(m, N) \) corresponding to a subspace \( W \). Under the Plücker embedding, \( \mathbb{G}(m, N) \) satisfies the hypothesis of Proposition 4.1 and so \( \Theta_x \) is an embedding. To find \( \mathcal{X}_x \), we note that a line \( L \) on \( \mathbb{G}(m, N) \) is completely determined by an \( (m - 1) \)-dimensional subspace \( W' \subset W \) and an \( (m + 1) \)-dimensional subspace \( W'' \subset V \) such that \( W \subset W'' \), and points on the line \( L \) correspond to subspaces of \( V \) containing \( W' \) contained in \( W'' \). But \( (m - 1) \)-dimensional subspaces of \( W \) and \( (m + 1) \)-dimensional subspaces of \( V \) containing \( W \) are parametrized by \( \mathbb{P}(W^*) \) and \( \mathbb{P}(V/W) \), respectively. Thus, \( \mathcal{X}_x \) is isomorphic to \( \mathbb{P}^{m-1} \times \mathbb{P}^{N-m-1} \). Then we can identify the VMRT \( \mathcal{C}_x \) at \( x \) through the tangent map

\[ \Theta_x : \mathcal{X}_x = \mathbb{P}^{m-1} \times \mathbb{P}^{N-m-1} \to \mathbb{P}(T_x M) = \mathbb{P}(W^* \otimes V/W). \]

Note that \( \mathbb{G}(m, N) \) is the homogeneous space associated with the depth 1 simple graded Lie algebra \( \{ A_N, \{ \alpha_m \} \} \).

For other examples, let us take simple graded Lie algebras \( (B_l, \{ \alpha_l \}) \) \( (l \geq 3) \) and \( (D_l, \{ \alpha_l \}) \) \( (l \geq 4) \). Then they have the contact gradation, and the associated homogeneous spaces \( G/P \) have \( \mathbb{P}^1 \times Q^{2l-5} \) and \( \mathbb{P}^1 \times Q^{2l-6} \) as VMRTs, respectively, where \( Q \) denotes the quadric. Details on this can be found in [8].

5. Prolongation of \( m \)

In this section, we study the prolongation introduced originally by Tanaka which provide us with a main tool to prove our result. References are [16] and [15].
5.1. **Definition of the prolongation.** Let \( m = \bigoplus_{k<0} g_k \) be a fundamental graded Lie algebra of \( \mu \)-th kind. We construct a graded Lie algebra depending on \( m \)

\[
g(m) = \bigoplus_{k \in \mathbb{Z}} g_k(m)
\]
as follows. For \( k < 0 \), put \( g_k(m) = g_k \). Let \( g_0(m) \) be the Lie algebra of gradation preserving derivations on the graded Lie algebra \( m = \bigoplus_{k<0} g_k \). For \( u \in g_0(m) \) and \( X \in m \), define

\[
[u, X] = -[X, u] = u(X).
\]
We easily see that this bracket satisfies the Jacobi identity on the direct sum \( \bigoplus_{k<0} g_k(m) \), and so \( \bigoplus_{k<0} g_k(m) \) is a graded Lie algebra. For \( k > 0 \), we proceed inductively. Suppose \( g_p(m) \) are defined for all \( p \leq k \). Then let \( g_{k+1}(m) \) be the vector space of degree \((k+1)\) linear maps \( u : m \to \bigoplus_{j \leq k} g_j(m) \) satisfying

\[
u([X, Y]) = [u(X), Y] + [X, u(Y)] \quad \text{for } X, Y \in m.
\]

So we have a direct sum \( g(m) = \bigoplus_{k \in \mathbb{Z}} g_k(m) \) of vector spaces. Now we define a Lie bracket on \( g(m) = \bigoplus_{k \in \mathbb{Z}} g_k(m) \). For \( X \in m \) and \( u \in g_r(m) \) with \( r \geq 0 \), define \( [u, X] = u(X) \). For \( u_1 \in g_r(m) \) and \( u_2 \in g_s(m) \), define a bracket \([u_1, u_2]\) to be a degree \((r+s)\) linear map \([u_1, u_2] : m \to \bigoplus_{j \leq r+s-1} g_j(m) \) given by

\[
[u_1, u_2](X) = [u_1(X), u_2] + [u_1, u_2(X)].
\]

We can easily check that this bracket satisfies the Jacobi identity, and so \( g(m) \) becomes a graded Lie algebra. The graded Lie algebra \( g(m) \) is called the prolongation of \( m \). There is a more general notion of prolongation: Let \( g_0 \) be a subalgebra of the Lie algebra \( g_0(m) \). For \( k \geq 1 \), define a subspace of \( g_k(m) \) inductively as

\[
g_k = \{ u \in g_k(m) | [u, g_{-1}] \subset g_{k-1} \}.
\]

Then \( g(m, g_0) := m \oplus \bigoplus_{k \geq 0} g_k \) is a graded subalgebra of \( g(m) \). \( g(m, g_0) \) is called the prolongation of \((m, g_0)\).

The prolongation \( g(m) \) satisfies the following property (15):

\[
(5.2) \quad \text{For } k \geq 0, \text{ if } X \in g_k(m) \text{ and } [X, m] = 0, \text{ then } X = 0.
\]

The following result of Yamaguchi is essential in proving our result.

**Proposition 5.1.** (Theorem 5.2 of [16]) Let \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) be a simple graded Lie algebra satisfying \( g_k = [g_{k+1}, g_{-1}] \) for \( k < -1 \). Then \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) is the prolongation of \( m = \bigoplus_{k<0} g_k \) except for the following three cases.

1. \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) is of depth one, i.e., \( g = g_{-1} \oplus g_0 \oplus g_1 \).
2. \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) has a contact gradation, i.e., of depth two with \( g_{-2} \) having dimension one.
3. \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) is isomorphic with \((A_l, \{\alpha_1, \alpha_k\}) (1 < k < l) \) or \((C_l, \{\alpha_1, \alpha_l\})\).

Furthermore, \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) is the prolongation of \((m, g_0)\) except when \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) is isomorphic with \((A_l, \{\alpha_1\})\) or \((C_l, \{\alpha_1\})\).

**Definition 1.** We divide simple graded Lie algebras \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) not isomorphic with \((A_l, \{\alpha_1\})\) or \((C_l, \{\alpha_1\})\) into three types I, II and III as follows.

1. \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) is of type I if it is the prolongation of \( m = \bigoplus_{k<0} g_k \).
2. \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) is of type II if it is of depth one or contact gradation, and not isomorphic with \((A_l, \{\alpha_1, \alpha_l\})\).
3. \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) is of type III if it is isomorphic with \((A_l, \{\alpha_1, \alpha_k\}) (2 \leq k \leq l) \) or \((C_l, \{\alpha_1, \alpha_l\})\).

The associated homogeneous space \( G/P \) is said to be of types I, II or III if \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) is of types I, II or III, respectively.
5.2. Infinitesimal automorphisms and the prolongation. Let \( \mathfrak{g} = \bigoplus_{k=-\mu}^{\mu} \mathfrak{g}_k \) be a simple graded Lie algebra. We will explain how local vector fields on \( G/P \) preserving the differential system \( D \) are related to the prolongation. Recall that the differential system \( (G/P, D) \) is locally isomorphic with the differential system \( (N', D_m') \). Thus we may work with \( (N', D_m') \) to give a local picture of \( D \) on \( G/P \). We only give a sketch of a relationship between these two objects. We refer to Section 2 of [16] for a complete description for the relation. We begin with the ‘special chart’ \( (N', D_m) \).

Let \( \omega : N' \to \mathfrak{m} \) be the Maurer-Cartan form on \( N' \). Given a vector field \( X \) on \( N' \), we have a \( \mathfrak{m} \)-valued function \( f_X : N' \to \mathfrak{m} \) defined by \( f_X(h) = \xi(X_h) \) for \( h \in N' \). By the gradation \( \mathfrak{m} = \bigoplus_{k=-\mu}^{\mu} \mathfrak{g}_k \), we can write

\[
f_X = \sum_{k=-\mu}^{-1} f^k_X,
\]

where \( f^k_X \in \mathfrak{g}_k \). Furthermore, if a vector field \( X \) on \( N' \) is an infinitesimal automorphism of \( (N', D_m) \), then we can associate to the vector field \( X \) a sequence of functions

\[
\{ f^k_X : N' \to \mathfrak{g}_k(\mathfrak{m}) \}_{k \geq 0},
\]

where \( f^k_X \) are inductively defined; we omit the explicit definition of \( f^k_X \); see Page 427 of [16]. Therefore, given an infinitesimal automorphism \( X \) of \( (N', D_m) \), we obtain \( \prod_{k=-\mu}^{\mu} \mathfrak{g}_k(\mathfrak{m}) \)-valued function \( f_X \) defined by

\[
f_X := \prod_{k=-\mu}^{\mu} f^k_X.
\]

We call \( f_X = \prod_{k=-\mu}^{\mu} f^k_X \) the presentation of \( X \) with respect to \( \mathfrak{m} \). The function \( f_X \) encodes higher derivative of \( X \) in the sense that the sequence \( \{ f^k_X \}_{k \geq -\mu} \) satisfies, for each \( k \),

\[
df^k_X = \sum_{r=-\mu}^{-1} [f^{k-r}_X, \xi^r].
\]

Conversely, given \( a = \sum_{k=-\mu}^{l} a^k \) of \( \mathfrak{g}(\mathfrak{m}) \) with \( a^k \in \mathfrak{g}_k(\mathfrak{m}) \) and \( h_0 \in N' \), there is a unique infinitesimal automorphism \( X \) preserving \( D_m \) such that \( f^k_X(h_0) = a^k \) for \( k \leq l \), and \( f^k_X \equiv 0 \) for \( k > l \). Indeed, \( f_X \) (and so \( X \)) exists as a solution to the following differential equation for \( \mathfrak{g}_k(\mathfrak{m}) \)-valued functions \( u^k = f^k_X (\mu \leq k \leq l) \)

\[
du^k = \sum_{k<s \leq l} [u^s, \xi^{k-s}] \quad \text{for} \quad k = -\mu, \ldots, l,
\]

subject to the initial condition \( u^k(h_0) = a^k \in \mathfrak{g}_k(\mathfrak{m}) \).

We remark that the solution \( X \) to the differential equation \( (5.4) \) is unique: For the infinitesimal automorphism \( X \) constructed above, and any \( g_0 \in N' \) with \( g_0 \neq h_0 \), let \( b = X(g_0) \in \mathfrak{g}(\mathfrak{m}) \). Then an infinitesimal automorphism \( Y \) on \( N' \) constructed as above for \( b \in \mathfrak{g}(\mathfrak{m}) \) and \( g_0 \in N' \) coincides with the infinitesimal automorphism \( X \) on \( N' \).

Recall that \( \mathcal{A} \) is the sheaf of Lie algebras of infinitesimal automorphisms on \( N' \) preserving \( D_m \).

**Definition 2.** For a fixed \( h_0 \in N' \), define \( \psi_{h_0} : \mathfrak{g}(\mathfrak{m}) \to \mathcal{A}(N') \) by \( \psi_{h_0}(a) = X \), where \( X \) is an infinitesimal automorphism constructed as above. When \( \psi_{h_0}(a) = X \), we say that \( X \) is determined by \( a \) at \( h_0 \).
Convention 5.2. So far we have constructed an infinitesimal automorphism $X$ on $N^−$ determined by $a \in \mathfrak{g}(m)$ at $h_0 \in N^−$, and the map $\psi_{h_0} : \mathfrak{g}(m) \to \mathcal{A}(N^−)$. When identifying $(N^−, D_m)$ and $(M(m), D|_{M(m)})$, we use the same notations for corresponding objects: an infinitesimal automorphism $X$ on $M(m)$ determined by $a \in \mathfrak{g}(m)$ and $x_0 \in M(m)$, and the map $\psi_{x_0} : \mathfrak{g}(m) \to \mathcal{A}(M(m))$, where $x_0 = \iota_P(h_0)$ for $\iota_P : N^− \cong M(m)$.

Note that we can do the same construction of the above objects $X$ and $\psi_{x_0}$ on other principal open subsets $M(m')$ corresponding to $P'$ by working with $(N'^−, D_{m'})$.

5.3. Extension of infinitesimal automorphisms.

Lemma 5.3. Let $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$ be a simple graded Lie algebra, and let $S = G/P$ be its associated homogeneous space. Let $(M(m)$ and $M(m')$ be the principal dense open subset of $S$ corresponding to parabolic subgroups $P$ and $P'$, respectively. If $X$ is an infinitesimal automorphism preserving $D$ on $M(m)$ determined by $a = \sum a^k \in \mathfrak{g}$ at $x_0 \in M(m)$, then $X$ extends to $M(m')$.

Proof. Choose $h \in G$ such that $P' = hPh^{-1}$, so that $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k'$, $m'$ and $N'^− = (N')^−$ are given as before. Let us work with two ‘charts’ $(N^−, D_m)$ and $(N'^−, D_{m'})$. Let $f_X^m$ and $f_X^{m'}$ be the presentations of $X$ with respect to $m$ and $m'$, respectively. Fix $x_1 \in M(m) \cap M(m')$. Let $g_1$ and $g_1'$ be the points of $N^−$ and $N'^−$, respectively, corresponding to $x_1$ (so we have $g_1' = hgh^{-1}$). Put $b := f_X^m(g_1)$. Then we see that from the construction of $X$, $f_X^m$ is $\mathfrak{g}$-valued function on $N^−$, and hence $b$ belongs to $\mathfrak{g}$. Thus, the vector field $X$ is determined by $b \in \mathfrak{g}$ at $g_1 \in N^−$ for the chart $(N^−, D_m)$, too. Note that, by the ‘coordinate change’, the element $b$ of $\mathfrak{g}$ on the chart $N^−$ changes into the element $b' := hbh^{-1}$ of $\mathfrak{g}$ on the chart $N'^−$. Therefore, letting $X'$ be an infinitesimal automorphism on $N'^−$ determined by $b' \in \mathfrak{g}$ at $g_1' \in N'^−$, by the uniqueness of solutions to the differential equation (5.4), we have

$$X'_{M(m) \cap M(m')} = X_{M(m) \cap M(M')}$$

Therefore the vector field $X$ extends to a vector field, denoted $X$, on $M(m) \cup M(m')$. □

5.4. Prolongation of infinitesimal automorphisms for $G/P$ of type I.

Proposition 5.4. (Corollary 5.4 of [16]) Let $S = G/P$ be a homogeneous space of type I, and let $U$ be any connected open subset of $S$. Then $\mathcal{A}(U) = \mathfrak{g}$.

Proof. First, suppose that $U \subset G/P$ is an open subset of $M(m)$ (with $o \in U$). Let $X \in \mathcal{A}(U)$. Then $f_X$ is a $\mathfrak{g}(m)$-valued function on $U$. Indeed, by the hypothesis that $\mathfrak{g}(m) = \mathfrak{g}$ is finite dimensional, there is some $l$ such that if $k \geq l$, then $\mathfrak{g}_k(m) = 0$. Thus if $k \geq l$, then $f_X^k(x) = 0$ for all $x \in U$. Thus $f_X$ is $\mathfrak{g}(m)$-valued function. Now fix $x$, say $x = o$. Let $a = X(o)$. It is obvious that $X = \psi_o(a)$. Since $X$ is arbitrary, $\psi_o$ is onto and hence an isomorphism. Therefore we have shown that for $U \subset M(m)$

$$\mathcal{A}(U) = \mathcal{A}(M(m)) = \mathfrak{g}(m) = \mathfrak{g}.$$ 

Similarly, for any $U \subset M(m')$ (corresponding to $P'$), we have

$$\mathcal{A}(U) = \mathcal{A}(M(m')) = \mathfrak{g}.$$ 

Now if we are given a vector field $X \in \mathcal{A}(M(m))$, then using Lemma 5.3, we can extend it to the entire space $G/P$. Thus the Lie algebra of global sections $\mathcal{A}(G/P)$ is isomorphic to $\mathfrak{g}$. Let $V$ be any connected open subset of $G/P$. Choose $U \subset M(m')$ so that $U \subset V$. Then we have $\mathcal{A}(V) = \mathcal{A}(G/P) = \mathcal{A}(U) = \mathfrak{g}$. This completes the proof. □
5.5. Prolongation of infinitesimal automorphism for \(G/P\) for type II. Note that tangent space \(T_o(G/P)\) is canonically identified with \(g_{-\mu} \oplus \cdots \oplus g_{-1}\). Via this identification, the VMRT \(\mathcal{C}_o\) at \(o\) is inside \(\mathbb{P}(g^{-1})\)

**Lemma 5.5.** Given a simple graded Lie algebra \(g = \bigoplus_{k \in \mathbb{Z}} g_k\) of type II, let \(G'_0 \subset \text{Aut}(g_{-1})\) be a group of automorphisms preserving \(\mathcal{C}_o \subset \mathbb{P}(g_{-1})\), and \(g'_0\) be the Lie algebra of \(G'_0\). Then the Lie algebra \(g_0\) coincides with the Lie algebra \(g'_0\).

**Proof.** For simple graded Lie algebras with depth one, we refer to Section 3 of \([5]\). For the contact cases, see Section 2 of \([5]\).

**Proposition 5.6.** Let \(G/P\) be a homogeneous space of type II, and \(U\) be any connected open subset of \(G/P\). Then \(\mathcal{A}_{\mathcal{E}}(U) = g\).

**Proof.** Suppose that \(U \subset G/P\) is an open subset of \(M(m)\) (with \(o \in U\)). Then by the definitions of \(g_0\) and \(\mathcal{A}_{\mathcal{E}}(U)\), \(\psi_x : g(m) \rightarrow \mathcal{A}(U)\) descends to \(\psi'_x : g(m, g'_0) \rightarrow \mathcal{A}_{\mathcal{E}}(U)\) for any \(x \in U\). But since \(g_0 = g'_0\), we have that \(g(m, g'_0) = g(m, g_0) = g\) is finite dimensional. Then, as in the proof of Proposition 5.4 by fixing \(x\), say \(x = o\), \(\psi'_x\) is an isomorphism. Thus we have

\[
\mathcal{A}_{\mathcal{E}}(U) = \mathcal{A}_{\mathcal{E}}(M(m)) = g(m, g'_0) = g(m, g_0) = g.
\]

The proof for a general connected open subset \(U \subset G/P\) is similar to the proof of Proposition 5.4.

5.6. Prolongation of infinitesimal automorphisms for \(G/P\) type III. Let \(g = \bigoplus_{k \in \mathbb{Z}} g_k\) be a simple graded Lie algebra of type III, that is, \(g = \bigoplus_{k \in \mathbb{Z}} g_k\) is isomorphic with \((A_l, \{\alpha_1, \alpha_k\})\) (2 \leq k \leq l) or \((C_l, \{\alpha_1, \alpha_l\})\). Let \(P_1\) and \(P_2\) be the parabolic subgroups corresponding to the first root \(\alpha_1\) and the second root \(\alpha_k\), respectively. Then there are two natural projections from the associated homogeneous space

\[
\pi_i : G/P \rightarrow G/P_i,
\]

For \(i = 1, 2\), let \(g^{(i)}_1\) be the kernel of the differential \(d\pi_i : T_o(G/P) \rightarrow T_o(G/P_i)\). Then under the identification, \(T_o(G/P) = g_{-\mu} \oplus \cdots \oplus g_{-1}\). \(g_{-1}\) decomposes into \(g_{-1} = g_{-1}^{(1)} \oplus g_{-1}^{(2)}\). To be concrete, \(g_{-1}^{(1)}\) and \(g_{-1}^{(2)}\) have the decompositions of root spaces;

\[
g_{-1}^{(1)} = \bigoplus_{\alpha} g_{\alpha}, \quad g_{-1}^{(2)} = \bigoplus_{\beta} g_{\beta},
\]

where for \((A_l, \{\alpha_1, \alpha_k\})\), \(\alpha = \lambda_i - \lambda_j\) runs over \(k+1 \leq i \leq l+1\) and \(2 \leq j \leq k\), and \(\beta = \lambda_i - \lambda_1\) over \(2 \leq i \leq k\) and for \((C_l, \{\alpha_1, \alpha_l\})\), \(\alpha = -\lambda_i - \lambda_j\) over \(2 \leq i \leq j \leq l\) and \(\beta = \lambda_i - \lambda_1\) over \(2 \leq i \leq l\).

More generally, by the construction of \(D \subset TM\), \(D\) decomposes into \(D = D^{(1)} \oplus D^{(2)}\) such that \((D^{(i)})_o = g^{(i)}_1\) for \(i = 1, 2\).

**Lemma 5.7.** For a simple graded Lie algebra \(g = \bigoplus_{k \in \mathbb{Z}} g_k\) of type III, let \(g'_0 \subset g_0(m)\) be the Lie algebra of elements preserving the decomposition \(g_{-1} = g_{-1}^{(1)} \oplus g_{-1}^{(2)}\). Then we have \(g_0 = g'_0\).

**Proof.** We refer to Section 5 of \([5]\) for a proof for \(g = \bigoplus_{k \in \mathbb{Z}} g_k\) isomorphic to \((A_l, \{\alpha_1, \alpha_k\})\) (2 \leq k \leq l-1) or \((C_l, \{\alpha_1, \alpha_l\})\).

For the case \((A_l, \{\alpha_1, \alpha_l\})\), we give an elementary proof. Let \(\text{Der}_o(m)\) be the algebra of gradation preserving derivations on \(m\), and let \(g'_0 \subset \text{Der}_o(m)\) be the algebra of derivations preserving the decomposition \(g_{-1} = g_{-1}^{(1)} \oplus g_{-1}^{(2)}\). We will show that \(g'_0 = g_0\). Since it is obvious that \(g_0 \subset g'_0\), it is enough to show that \(\dim g'_0 = \dim g_0\). For this, first note that
since \( g_{-2} \) is generated by \( g_{-1}, \varphi \in \text{Der}_0(m) \) is completely determined by its values on \( g_{-1} \). Therefore the assignment \( \varphi \mapsto \varphi|_{g_{-1}} \) defines an embedding of vector spaces

\[
i : g'_{0} \hookrightarrow \text{End}(g^{(1)}_{-1}) \oplus \text{End}(g^{(2)}_{-1}).
\]

Now we will identify the equations defining the subspace \( g_{0}' \) in \( \text{End}(g^{(1)}_{-1}) \oplus \text{End}(g^{(2)}_{-1}) \). Note that these defining equations come from the derivation conditions among generators of \( g_{-1} \). So we fix a basis of \( g_{-1} \) as follows. Recall that \( E_{i,j} \) is the \((l + 1) \times (l + 1)\) matrix whose \((i, j)\)-th entry is 1 and other entries are all 0. Write, for convenience,

\[
e_i = E_{l+1,i+1}, \quad f_i = E_{i+1,1} \quad \text{for} \quad i = l, \ldots, l - 1.
\]

Then \( \{e_1, e_2, \ldots, e_{l-1}\} \) (resp. \( \{f_1, f_2, \ldots, f_{l-1}\} \)) forms a standard basis of \( g_{-1}^{(1)} \) (resp. \( g_{-1}^{(2)} \)). Let \( \{f_1, \ldots, f_{l-1}, e_1, \ldots, e_{l-1}\} \) be an ordered basis of \( g_{-1} \). Recall that there are the relations among generators:

\[
\begin{align*}
\text{(5.5)} \quad [e_i, e_j] &= 0 = [f_i, f_j] \quad \text{for all} \quad 1 \leq i, j \leq l - 1, \\
\text{(5.6)} \quad [f_i, e_j] &= 0 \quad \text{for all} \quad 1 \leq i \neq j \leq l - 1, \\
\text{(5.7)} \quad [f_1, e_1] = [f_2, e_2] = \cdots = [f_{l-1}, e_{l-1}] &= h,
\end{align*}
\]

where \( h := E_{l+1,1} \) is the generator of the one dimensional subspace \( g_{-2} \).

Let \( \phi \in \text{End}(g_{-1}^{(1)}) \oplus \text{End}(g_{-1}^{(2)}) \). In order for \( \phi \) to become a derivation \( \varphi \) on \( m \), i.e., \( \nu(\varphi) = \varphi|_{g_{-1}} = \phi \), \( \phi \) must satisfy the derivation conditions for the relations (5.6) and (5.7) only. Note that the relations in (5.5) do not impose any restriction on \( \phi \) since \( \phi \) is an endomorphism preserving the decomposition \( g_{-1} = g_{-1}^{(1)} \oplus g_{-1}^{(2)} \).

To be explicit, we write \( \phi(e_i) = \sum_{k=1}^{l-1} a_{k,i} e_k \) and \( \phi(f_i) = \sum_{k=1}^{l-1} b_{k,i} f_k \). Then we may consider \( a_{i,j} \) and \( b_{i,j} \) as coordinate functions of \( \text{End}(g_{-1}^{(1)}) \oplus \text{End}(g_{-1}^{(2)}) \). From (5.6), we have

\[
0 = \phi([f_i, e_j]) = [\phi(f_i), e_j] + [f_i, \phi(e_j)] = (a_{i,j} + b_{j,i}) h.
\]

The last equality follows from the equations (5.6) and (5.7). Therefore we get \((l-1)(l-2)\) equations

\[a_{i,j} + b_{j,i} = 0 \quad (1 \leq i \neq j \leq l - 1).
\]

From (5.7), we fix \((l-2)\) relations

\[
\begin{align*}
\text{(5.8)} \quad [f_1, e_i] &= [f_i, e_i] \quad \text{for} \quad i = 2, 3, \ldots, l - 1.
\end{align*}
\]

Applying \( \phi \) on both sides of (5.8), we get

\[
\begin{align*}
\text{(5.9)} \quad (a_{1,1} + b_{1,1}) h &= (a_{i,i} + b_{i,i}) h,
\end{align*}
\]

which gives \((l-2)\) equations

\[a_{i,i} + b_{i,i} = a_{1,1} + b_{1,1} \quad (i = 2, 3, \ldots, l - 1).
\]

Therefore \( g_{0}' \) is defined by \( l(l-2) \) linear equations. We can easily see that all these equations are independent and hence the codimension of \( g_{0}' \) is \( l(l-2) \). Since the dimension of \( \text{End}(g_{-1}^{(1)}) \oplus \text{End}(g_{-1}^{(2)}) \) is \( 2(l-1)^2 \), the dimension of \( g_{0}' \) is \( (l-1)^2 + 1 \), which is exactly the dimension of \( g_0 \). Therefore we conclude that \( g_{0}' = g_0 \). This completes the proof.

\textbf{Proposition 5.8.} Let \( G/P \) be a homogeneous space of type III, and let \( U \) be a connected open subset of \( G/P \). Then \( \mathcal{A}_{\mathfrak{g}}(U) = \mathfrak{g} \).
Proof. The proof is similar to the proof of Proposition 5.6. Suppose $U \subset G/P$ is an open subset of $M(m)$ (with $o \in U$). Then by the definitions of $g'_0$ and $A(U)$, for any $x \in U$, $\psi_x : g(m) \to A(U)$ descends to $\overline{\psi}_x : g(m, g'_0) \to A(U)$. Since $g(m, g'_0) = g(m, g_0) = g$ is finite dimensional, as before, by fixing $x$, say, $x = o$, we obtain an isomorphism $\psi_o : g(m) \cong A(U)$. Therefore we have

$$A_{\oplus}(U) = A_{\oplus}(M(m)) = g(m, g'_0) = g(m, g_0) = g.$$ 

The proof for a general connected open subset $U \subset G/P$ is similar to that of Proposition 5.1. \qed

5.7. Extension of a local biholomorphic map.

Proposition 5.9. Let $\Psi : U_1 \to U_2$ be a biholomorphic map between two connected open subsets of $G/P$. Suppose that $\Psi$ preserves $D$, $\mathcal{C}$ or the decomposition $D = D^{(1)} \oplus D^{(2)}$ for $G/P$ of types I, II or III, respectively. Then $\Psi$ extends to an automorphism $\tilde{\Psi} : G/P \to G/P$.

Proof. We note that since $\Psi : U_1 \to U_2$ preserves $D$, $\mathcal{C}$, or the decomposition $D = D^{(1)} \oplus D^{(2)}$ for types I, II or III, respectively, the differential $d\Psi$ sends $A(U_1)$, $A_{\mathcal{C}}(U_1)$ or $A_{\oplus}(U_1)$ isomorphically onto $A(U_2)$, $A_{\mathcal{C}}(U_2)$ or $A_{\oplus}(U_2)$ for types I, II or III, respectively. Since $A(U_1)$, $A_{\mathcal{C}}(U_1)$ or $A_{\oplus}(U_1)$ are all identified with $g$ for types I, II or III, respectively, by Propositions 5.4, 5.6 and 5.8 $d\Psi$ sends $g$ isomorphically onto $g$ for all types I, II, III. Fix $x \in U_1$. Put $x' = \Psi(x)$, and let $p$ and $p'$ be subalgebras of vector fields vanishing at $x$ and $x'$, respectively. Then $p$ and $p'$ are parabolic subalgebras of $g$, and $d\Psi : g \to g$ sends $p$ onto $p'$ isomorphically. Now let $P$ and $P'$ be the parabolic subgroups of $G$ with Lie algebras $p$ and $p'$, respectively. Then the automorphism $d\Psi$ of Lie algebras, in turn, gives rise to an automorphism $G \to G$ sending $P$ to $P'$ and hence an automorphism $\tilde{\Psi} : G/P \to G/P'$, which naturally extends $\Psi : U_1 \to U_2$. \qed

6. Main result

In this section, we will complete the proof of the main theorem. We begin with the following lemma which will be used throughout this section.

Lemma 6.1. Let $g = \bigoplus_{k \in \mathbb{Z}} g_k$ be a simple graded Lie algebra and $G/P$ its associated homogeneous space. Then the natural action of $G$ on $G/P$ preserves $D$ for all simple graded Lie algebras $g = \bigoplus_{k \in \mathbb{Z}} g_k$ including $(A_l, \{\alpha_1,\})$ and $(C_l, \{\alpha_1,\})$. Furthermore, it preserves $\mathcal{C}$ or the decomposition $D = D^{(1)} \oplus D^{(2)}$ on $G/P$ for types II or III, respectively.

Proof. Let $h \in G$, and let $x$ and $\tilde{x}$ be points of $G/P$ with $\tilde{x} = h \cdot x$, where $h \cdot x$ denotes the standard action.

Case of $\mathcal{C}$ for type II:

If $f : \mathbb{P}^1 \to G/P$ is a rational curve through $x$ with the image $C$, then the curve $\tilde{C} = h \cdot C \subset G/P$ is a rational curve through $\tilde{x} = h \cdot x$ parametrized by $\tilde{f} : \mathbb{P}^1 \to G/P$ defined by $\tilde{f}(u) = h \cdot f(u)$. Moreover, the differential $dh : T_x(G/P) \to T_{\tilde{x}}(G/P)$ sends the tangent of the rational curve $f$ at $x$ to the tangent of the rational curve $\tilde{f}$ at $\tilde{x}$, i.e., $dh$ send isomorphically $\mathcal{C}_x$ to $\mathcal{C}_{\tilde{x}}$. Therefore the natural action of $G$ on $G/P$ preserves $\mathcal{C}$ for type II.

Cases of $D$ and the decomposition for type III:

WLOG, we assume that $x = o$ is the base point of $G/P$. Recall that we have the canonical identifications

$$T_o(G/P) = g_{-\mu} \oplus \cdots \oplus g_{-1},$$
$$T_o(G/P) = h(g_{-\mu} \oplus \cdots \oplus g_{-1})h^{-1} = hgh^{-1} \oplus \cdots \oplus hg_{-1}h^{-1} \text{ for } \tilde{o} = h \cdot o,$$
and note that, under these identifications, the differential $dh : T_o(G/P) \to T_o(G/P)$ is defined by $dh(X) = hXh^{-1}$. Therefore $dh$ sends $g_{i-1} = D_o$ onto $hg_{i-1}h^{-1} = D_o$ isomorphically, and hence the natural action of $G$ on $G/P$ preserves $D$. Furthermore, for type $III$, we have $D_o^{(i)} = \mathfrak{g}^{(i)}_1$ and $D_o^{(i)} = h(\mathfrak{g}^{(i)}_1)h^{-1}$ for $i = 1, 2$, and, by the definition of $dh$, $dh$ sends isomorphically $D_o^{(i)}$ onto $D_o^{(i)}$. Thus the natural action of $G$ on $G/P$ preserves the decomposition $D = D^{(1)} \oplus D^{(2)}$ for type $III$. □

6.1. Differential system on $N$ coming from an action of $N$ on $G/P$. Let $A : N \times G/P \to G/P$ be an EC-structure on $G/P$. Let $O = O_A$ be the Zariski open orbit of the action $A$. Fix $x \in O$, and define a biregular map $a : N \to O$ by $a(h) = A(h, x)$. Let $D$ be the natural differential system on $G/P$ and let

$$D^{-\mu} \supset D^{-\mu+1} \supset \cdots \supset D^{-1} = D$$

be the weak derived system induced by $D$. Let $E := a^*(D|_O)$ be a differential system on $N$ and let

$$(6.10) \quad E^{-\mu} \supset E^{-\mu+1} \supset \cdots \supset E^{-1} = E$$

be the weak derived system induced by the differential system $E$ on $O$. Note that the subbundles $E^k$ of $TN$ are equal to $a^*(D^k|_O)$ since the morphism $da$ is an isomorphism of Lie algebras of holomorphic vector fields.

Lemma 6.2. Let $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$ be a simple graded Lie algebra not isomorphic with $(C_l, \{\alpha_l\})$ $(l \geq 2)$, $(B_l, \{\alpha_l\})$ $(l \geq 3)$ or $(G_2, \{\alpha_1\})$, and let $G, P$ and $N$ be its associated groups. Let $E$ be a differential system on $N$ defined above. Then $E$ is invariant under the natural action of $N$ on $N$.

Proof. Fix $h_1, h_2 \in N$ and choose $h \in N$ with $h_2 = hh_1$. Define $l^h : N \to N$ by $l^h(g) = hg$ for $g \in N$. To prove the lemma, we need to show that the differential $dl^h : T_{h_1}N \to T_{h_2}N$ sends $E_{h_1}$ to $E_{h_2}$. Let $y_i = A(h_i, x) \in O$ for $i = 1, 2$.

Note that the homomorphism of groups $N \to \text{Aut}(G/P)$ corresponding to the action $A : N \times G/P \to G/P$ factors through a homomorphism $N \to \text{Aut}^0(G/P) = G$ since $N$ is connected. Let $\xi \in G$ be the image of $h$ under this map $N \to \text{Aut}^0(G/P) = G$. Then the given action of $h$ on $O$, which corresponds to the natural action of $h$ on $N$, is the same as the natural action of $\xi$ on $O \subset G/P$, i.e., $A(h, z) = \xi \cdot z$ for all $z \in O$. In particular, putting $z = y_1 = A(h_1, x)$, we get $y_2 = \xi \cdot y_1$. Therefore, we have the commutative diagram of isomorphisms of differentials

$$
\begin{array}{c}
T_{h_1}N \xrightarrow{da_{h_1}} T_{y_1}O \\
\downarrow dl^h & \downarrow d\xi \\
T_{h_2}N \xrightarrow{da_{h_2}} T_{y_2}O
\end{array}
$$

Since the natural action of $G$ on $G/P$ preserves $D$, the differential $d\xi$ sends $D_{y_1}$ onto $D_{y_2}$. Since we have $dl^h = (da_{h_2})^{-1} \circ d\xi \circ da_{h_1}$, and two differentials $da_{h_i}$ send $E_{h_i}$ to $D_{y_i}$ for $i = 1, 2$, $dl^h$ sends $E_{h_1}$ onto $E_{h_2}$. This proves the lemma. □

6.2. Construction of a local biholomorphic map. Let $\mathfrak{m}(o) = \bigoplus_{k=-\mu}^{-1} \mathfrak{m}_k$ be the symbol algebra at the identity $o$ of $N$ associated with the differential system $E$. Let $\omega : T\mathbb{N} \to \mathfrak{n}$ be the Maurer-Cartan form on $N$, and $\hat{\mathfrak{m}}(o) = \bigoplus_{k=-\mu}^{-1} \hat{\mathfrak{m}}_k$ the graded Lie algebra associated with the filtration on $\mathfrak{n}$

$$\omega(E^{-\mu}) \supset \cdots \supset \omega(E^{-1}).$$
Then since the derived system \((6.10)\) is \(N\)-invariant by Lemma \(6.2\), the symbol algebra \(m(o)\) is nothing but the graded Lie algebra \(m(o)\). On the other hand, \(m(o)\) is isomorphic to \(n\) as Lie algebras. Note that there is no canonical isomorphism between \(m(o)\) and \(n\). Indeed, a choice of a subspace \(n_k \subset \omega(E^k)\) with \(\omega(E^k) = n_k \oplus \omega(E^{k+1})\) for each \(k = -1, \ldots, -\mu\) determines an isomorphism of (graded) Lie algebras

\[
\n = \bigoplus_{k=-\mu}^{-1} n_k \cong m(o) = \bigoplus_{k=-\mu}^{-1} m_k
\]

and vice versa. In the sequel, by fixing one of such isomorphisms, we will identify three Lie algebras

\((6.11)\)

\[n = m(o) = m(o).\]

For \(i = 1, 2\), let \(A_i : N \times G/P \to G/P\) be an EC-structure on \(G/P\). For these actions \(A_i\), we will use the same notation as for the action \(A\) in Subsection \(6.1\) only with the subscript \(i\). For instance, for each \(i = 1, 2\), \(O_i\) denotes the Zariski open subset of the action \(A_i\), \(a_i : N \to O_i\) a biregular map defined by \(a_i(h) = A_i(h, x_i)\), \(m_i(o)\) the symbol algebra at the identity \(o \in N\) associated with \(E_i = a^*_i(D|_{O_i})\) and etc. To avoid any confusion arising from the notations, we make the notations as simple as possible.

**Notation.** Write \(A_1(h, x) = h \ast x\), \(A_2(h, x) = h \ast x\) and \(A_0(h, x) = h \cdot x\) for \(h \in N\) and \(x \in G/P\), where \(A_0\) is the natural action of \(N\) on \(G/P\). For \(h \in N\) (or \(G\)), define

\[\phi^{i,h} : G/P \to G/P\]

by \(\phi^{i,h}(x) = A_i(h, x)\) for \(i = 0, 1, 2\). For \(h \in N\), recall that \(l^h : N \to N\) is the map defined by the left multiplication by \(h\).

**Proposition 6.3.** Let \(g = \bigoplus_{k \in \mathbb{Z}} g_k\) be a simple graded Lie algebra not isomorphic with \((C_l, \{\alpha_l\})\) \((l \geq 2)\), \((B_l, \{\alpha_l\})\) \((l \geq 3)\) or \((G_2, \{\alpha_1\})\), and \(G/P\) its associated homogeneous space. For \(i = 1, 2\), let \(A_i : N \times G/P \to G/P\) be an EC-structure on \(G/P\) of \(N\). Then there is a group automorphism \(F : N \to N\) such that

1. \(dF_o \circ dl^h = dl^{F(h)} \circ dF_o\) for all \(h \in N\),
2. the biholomorphic map \(\Psi : O_1 \to O_2\) defined by \(\Psi = a_2 \circ F \circ a_1^{-1}\) satisfies
   a. \(\Psi(x_1) = x_2\),
   b. \(\Psi(h \ast x_1) = F(h) \ast x_2\) for all \(h \in N\),
   c. for each \(u \in O_1\), the differential \(d\Psi_u\) sends \(D_u\) to \(D\Psi_u\). Furthermore, it sends \(\mathcal{C}_u\) to \(\mathcal{C}_{\Psi(u)}\) for \(G/P\) of type \(II\), and respects the decomposition \(D = D^{(1)} \oplus D^{(2)}\) for \(G/P\) of type \(III\).

**Proof.** The proof is an adaption of Proposition 2.4 of \[7\]. For \(i = 1, 2\), fix an identification \(n = m_i(o)\) of Lie algebras as in \((6.11)\). Take \(g \in G\) such that \(x_2 = g \cdot x_1\). Then the differential \(d\phi^{0,g} = d\phi^{0,g} : m(x_1) \to m(x_2)\) is an isomorphism of graded Lie algebras that sends \(D_{x_1}\) to \(D_{x_2}\) and \(\mathcal{C}_{x_1}\) to \(\mathcal{C}_{x_2}\). In particular, it respects the decomposition for \(G/P\) of type \(III\). Define a Lie algebra isomorphism \(f : n \to n\) by \(f = da^{-1}_2 \circ d\phi^{0,g} \circ da_1\), where \(da_i\) denotes \(da_i = d(a_i)_o : n = m_i(o) \to m_i(x_i)\) for \(i = 1, 2\). Then since \(N\) is a simply connected Lie group, there is an automorphism \(F\) such that \(dF_o = f\). Define \(\Psi : O_1 \to O_2\) by \(\Psi = a_2 \circ F \circ a_1^{-1}\). Now we will show that \(\Psi\) satisfies the properties \((1)\) and \((2)\).

\((1)\) is immediate from the automorphism property \(F \circ l^h(g) = l^{F(h)} \circ F(g)\) for all \(g, h \in N\). \((a)\) of \((2)\) is trivial to check. \((b)\) of \((2)\) follows from the equalities

\[
\Psi(h \ast x_1) = a_2 \circ F \circ a_1^{-1} (h \ast x_1) = a_2 \circ F(h \circ)
\]

\[= a_2(F(h \circ a_2(o)) = F(h) \ast x_2.
\]
Here $o$ is the identity of $N$. The second and fourth equalities follow from the equivariance of $a_1$ and $a_2$, respectively, and the others from the definitions.

For (c), fix $u \in O_1$ and let $h \in N$ such that $u = h \ast x_1$, i.e., $h = a_1^{-1}(u)$. Then we have

$$d\Psi_u(D_u) = da_2 \circ dF_h \circ da_1^{-1}(D_{h \ast x_1}) = da_2 \circ dF_h \circ da_1^{-1}(d\phi^{1,h}(D_{x_1}))$$

$$= da_2 \circ dF_h \circ d\alpha_1^{-1}(D_{x_1}) = da_2 \circ dF_h \circ d\alpha_1^{-1}(D_{x_1}) = da_2 \circ dF_h \circ d\alpha_1^{-1}(D_{x_1}) = da_2 \circ dF_h \circ d\alpha_1^{-1}(D_{x_1}) = da_2 \circ dF_h \circ d\alpha_1^{-1}(D_{x_1}).$$

Here the equality (2) follows from the equality $a_1 \circ l^h = \phi^{1,h} \circ a_1$ and the fact that $D$ is invariant under the action $A_1$. The equalities (2) and (3) follow from (1) and the definition of $f$, respectively. The equality (3) follows since $D$ is invariant under the natural action of $G$ on $G/P$. Note that for $y \in O_2, a_2 \circ l^F(h) \circ a_2^{-1}(y) = F(h) \ast y$. Therefore since $D$ is invariant under the action $(h, y) \rightarrow F(h) \ast y$, the equality (2) follows. The equality (3) is immediate from (b) of (2). This proves the first statement of (c) of (2).

To prove the second statement of (c) of (2), we note that in proving the first statement of (c) of (2) which involves $D$, we used only the fact that $D$ is invariant under any action of $N$ on $G/P$. Since $G$ (resp. the decomposition $D = D^{(1)} \oplus D^{(2)}$) is invariant under any action of $N$ for $G/P$ of type $II$ (resp. type $III$) by Lemma 6.2, the proof for the differential system $D$ works verbatim for VMRT (resp. the decomposition $D = D^{(1)} \oplus D^{(2)}$) for $G/P$ of type $II$ (resp. type $III$). This completes the proof of the proposition. \[\square\]

6.3. For cases where $\text{Aut}^0(G/P) \neq G$. When we proved Lemma 6.2 and Proposition 6.3, essential in proving our main theorem, we needed the condition $\text{Aut}^0(G/P) = G$. Now we explain what happens to the simple graded Lie algebras $g = \bigoplus_{k \in \mathbb{Z}} g_k$ with $\text{Aut}^0(G/P) \neq G$. Recall from Subsection 2.3 that the simple graded Lie algebra $g = \bigoplus_{k \in \mathbb{Z}} g_k$ for which $\text{Aut}^0(G/P) \neq G$ are

$$(C_l, \{\alpha_1\}) \ (l \geq 2), \ (B_l, \{\alpha_1\}) \ (l \geq 3), \ or \ (G_2, \{\alpha_1\}),$$

and there are corresponding the simple graded Lie algebras $\tilde{g}$ such that $\tilde{G} = \text{Aut}^0(G/P)$ and $\tilde{G}/P$ is isomorphic to $G/P$:

$$(A_{2l-1}, \{\alpha_1\}), \ (D_{l+1}, \{\alpha_{l+1}\}) \ and \ (B_3, \{\alpha_1\}).$$

We remark that even though two simple graded Lie algebras $g$ and $\tilde{g}$ give the isomorphic homogeneous spaces, they produce non-isomorphic differential systems $D$ and $\tilde{D}$ on $G/P$ and $\tilde{G}/P$, respectively. For instance, for $g = (C_l, \{\alpha_1\})$ (this case is not considered in our main theorem!), $D_0$ is not isomorphic with $\tilde{D}_0$ as the diagrams below shows.

$$\begin{array}{c|c|c|c}
1 & 2l - 1 & 1 \\
0 & 1 & 2 \\
-1 & 0 & 1 \\
-2 & -1 & 0 \\
\end{array} \quad \text{for } C_l; \quad \begin{array}{c|c|c|c}
1 & 2l - 1 & 1 \\
0 & 1 & \\
-1 & 0 & \\
\end{array} \quad \text{for } A_{2l-1}.$$

This difference in the differential systems makes things worse for the exceptions; $\tilde{G} = \text{Aut}^0(G/P)$ does not preserve $D$, and hence, for a (connected) subgroup $H \subset G$, an action of $H$ on $G/P$ does not necessarily preserve $D$. In the above example, it is not difficult to take an element $g \in \tilde{G}_0 \subset \tilde{P} \subset \tilde{G} = PSL_{2l}$ such that $g$ does not preserve $D_0 = g_{-1}$ (but $\tilde{D}_0 = \tilde{g}_{-1}$), where the parabolic subgroup $\tilde{P} \subset \tilde{G}$ is the group of automorphisms fixing
the base point \( o \in G/P \), and \( \tilde{G}_0 \) is the group of automorphisms preserving the gradation \( \tilde{g} = \tilde{g}_{-1} \oplus \tilde{g}_0 \oplus \tilde{g}_1 \). (Due to this ‘pathology’ arising on \( G/P \) for the exception cases, we will treat them separately when we prove the main theorem below.) However we observe that, despite this pathology with the exception case, there is a ‘nice’ embedding of algebraic groups

\[ J : (G, P) \hookrightarrow (\tilde{G}, \tilde{P}) \]  

(or, equivalently, an embedding \( J : (g, p) \hookrightarrow (\tilde{g}, \tilde{p}) \) of Lie algebras) such that

1. \( J(P) = \tilde{P} \cap J(G) \),
2. there is an isomorphism of groups \( \rho : J(N) \xrightarrow{\sim} \tilde{N} \) (or an isomorphism of Lie algebras \( \rho : J(n) \xrightarrow{\sim} \tilde{n} \)),
3. the induced isomorphism \( J : G/P \rightarrow \tilde{G}/\tilde{P} \) sends \( D \) into \( \tilde{D} \) (not onto).

We would like to emphasize that in general the nilpotent group \( \tilde{J} \) is isomorphic with, but isomorphic to \( \tilde{J} \) below. We also note that \( J : g \rightarrow \tilde{g} \) does not preserve the gradation.

For the above example, \( J \) is the embedding defined by \( J(X) = X \) for \( X \in g \), and \( \rho \circ J \) can be described informally as follows: The light shaded \((-1)\) part is completely determined by the dark shaded \((-1)\) part, and hence the dark shaded \((-1)\) and \((-2)\) parts, which corresponds to \( m = n^- \subset \mathfrak{sp}(2l) \), is “isomorphic” via \( \rho \circ J \) to the dark shaded \((-1)\) part that corresponds to \( \tilde{m} = (\tilde{n})^- \subset \mathfrak{sl}(2l) \).

Let us take another example; \((B_l, \{\alpha_1\})\). Note that, in the notations of \([10]\), \( \mathfrak{so}(2l + 1) \) consists of \((2l + 1) \times (2l + 1)\) matrices of the form

\[
X = \begin{pmatrix}
A & a & B \\
\xi & 0 & -a' \\
C & -\xi' & -A'
\end{pmatrix},
\]

Here \( A, B, C \) are \( l \times l \) matrices, and \( B \) and \( C \) satisfies \( B = -B' \), \( C = -C' \). \( a \) and \( \xi \) are column \( l \)-vector \( a = (a_1, \ldots, a_l)^t \) and row \( l \)-vector \( \xi = (\xi_1, \ldots, \xi_l) \) respectively such that \( a' = (a_1, \ldots, a_l) \) and \( \xi' = (\xi_1, \xi_{l-1}, \ldots, \xi_1)^t \). Note that \( \mathfrak{so}(2l) \) consists of \( 2l \times 2l \)-matrices of the form \( X^0 \), where \( X^0 \) is a matrix obtained from \( X \in \mathfrak{so}(2l + 1) \) by removing the center column and row. Given \( X \in \mathfrak{so}(2l + 1) \) above, let \( \tilde{X} \) be a \((2l + 2) \times (2l + 2)\)-matrix defined by

\[
\tilde{X} = \begin{pmatrix}
A & a & a & B \\
\xi & 0 & 0 & -a' \\
\xi & 0 & 0 & -a' \\
C & -\xi' & -\xi' & -A'
\end{pmatrix}.
\]

Then we see that \( \tilde{X} \) belongs to \( \mathfrak{so}(2l + 2) \). Associating \( \tilde{X} \) to \( X \) gives the desired embedding \( J : g = \mathfrak{so}(2l + 1) \hookrightarrow \tilde{g} = \mathfrak{so}(2l + 2) \). In this case, \( J(n) = \tilde{n} \), i.e., \( \rho : J(m) \rightarrow \tilde{m} \) is the identity map.

We will use the above isomorphisms \( J : G/P \xrightarrow{\sim} \tilde{G}/\tilde{P} \) and \( \rho \circ J : N \rightarrow \tilde{N} \) to treat EC-structures on \( G/P \) for exception cases. This idea was suggested by the referee.

6.4. Completion of the proof of the main theorem.

Theorem 6.4 (Main Theorem). Let \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) be a simple graded Lie algebra not isomorphic with \((A_l, \{\alpha_1\})\), \((C_l, \{\alpha_1\})\), and \( G/P \) its associated homogeneous space. Then there exists, up to isomorphism, a unique EC-structure of \( N \) on \( G/P \).

Proof. Since there is an EC-structure of \( N \) on \( G/P \) coming from the Bruhat decomposition of \( G/P \) \([14] \), it remains to prove uniqueness. We consider two cases separately.

Case 1: \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) not isomorphic with \((B_l, \{\alpha_1\})\) or \((G_2, \{\alpha_1\})\):
Let \( A_i : N \times G/P \rightarrow G/P \) be two EC-structures on \( G/P \) for \( i = 1, 2 \). Then by Proposition
there is a biholomorphism \( \Psi : O_1 \to O_2 \) such that the differential \( d\Psi \) preserves \( D, \mathcal{C} \) or the decomposition \( D = D^{(1)} \oplus D^{(2)} \) for \( G/P \) of types I, II or III, respectively. Then by Proposition 5.9, \( \Psi \) extends to an automorphism \( \bar{\Psi} : G/P \to G/P \). Now we have to check that the automorphism \( \bar{\Psi} \) of \( G/P \) is an isomorphism between two actions \( A_i : N \times G/P \to G/P \), extending the isomorphism \( \Psi \) between \( A_1 : N \times O_1 \to O_1 \). Since we have

\[
\bar{\Psi} \circ A_1|_{N \times O_1} = A_2 \circ (F \times \bar{\Psi})|_{N \times O_1},
\]

and \( N \times O_1 \) is Zariski open in \( N \times G/P \), the equality \( \bar{\Psi} \circ A_1 = A_2 \circ (F \times \bar{\Psi}) \) holds on \( N \times G/P \), which means that two EC-structures \( A_i \) on \( G/P \) are isomorphic.

**Case 2:** \( g = \bigoplus_{k \in \mathbb{Z}} g_k \) isomorphic with \((B_i, \{\alpha_i\})\) or \((G_2, \{\alpha_1\})\):
Recall that Lemma 6.2 and Proposition 6.3 do not work for this case, but from Subsection 6.3 we have the isomorphisms \( J : G/P \xrightarrow{\sim} \tilde{G}/\tilde{P} \) and \( \rho \circ J : N \to \tilde{N} \). These two isomorphisms induce a bijective correspondence between actions of \( N \) on \( G/P \) and actions of \( \tilde{N} \) on \( \tilde{G}/\tilde{P} \). Therefore, uniqueness of EC-structures of \( N \) on \( G/P \) follows from that of \( \tilde{N} \) on \( \tilde{G}/\tilde{P} \). \( \square \)

We remark that our result implies that if \( N_1 \) and \( N_2 \) are subgroup of \( G \) isomorphic to \( N \), and act on \( G/P \) with an open orbit, then \( N_1 \) and \( N_2 \) are conjugate. To see this, for \( i = 1, 2 \), let \( A_i : N \times G/P \to G/P \) be an EC-structure on \( G/P \) given via a homomorphism \( \varphi_1 : N \to N_1 \subset G \subset Aut^0(G/P) \). Then the automorphism \( \bar{\Psi} : G/P \to G/P \) is given by an action of an element \( h \in G \). In the notations of the proof of Proposition 5.9, this is because \( \bar{\Psi} \) is induced by a group isomorphism \( (G, P) \to (G, P') \) and this isomorphism, in turn, is given as the conjugation by the element \( h \) such that \( P' = hPh^{-1} \). Thus, letting \( F' : N_1 \to N_2 \) be a map given via the bihomorphic map \( \Psi : O_1 \to O_2 \), \( F' \) is an isomorphism of group given as the conjugation by the element \( h \).

Note that for the construction \( \Psi : O_1 \to O_2 \), it is enough to hypothesize in Proposition 6.3 that the actions \( A_i \) are given via homomorphisms \( \varphi_i : N \to G \subset Aut^0(G/P) \). This is a slightly weaker condition than \( Aut^0(G/P) = G \). Since in this case \( N_1 \) are already included in \( G \), this result holds for all simple graded Lie algebras \( g = \bigoplus_{k \in \mathbb{Z}} g_k \)不归纳。
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