A Perspective on External Field QED

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Abstract

In light of the conference Quantum Mathematical Physics held in Regensburg in 2014, we give our perspective on the external field problem in quantum electrodynamics (QED), i.e., QED without photons in which the sole interaction stems from an external, time-dependent, four-vector potential. Among others, this model was considered by Dirac, Schwinger, Feynman, and Dyson as a model to describe the phenomenon of electron-positron pair creation in regimes in which the interaction between electrons can be neglected and a mean field description of the photon degrees of freedom is valid (e.g., static field of heavy nuclei or lasers fields). Although it may appear as second easiest model to study, it already bares a severe divergence in its equations of motion preventing any straightforward construction of the corresponding evolution operator. In informal computations of the vacuum polarization current this divergence leads to the need of the so-called charge renormalization. In an attempt to provide a bridge between physics and mathematics, this work gives a review ranging from the heuristic picture to our rigorous results in a way that is hopefully also accessible to non-experts and students. We discuss how the evolution operator can be constructed, how this construction yields well-defined and unique transition probabilities, and how it provides a family of candidates for charge current operators without the need of removing ill-defined quantities. We conclude with an outlook of what needs to be done to identify the physical charge current among this family.

1 Heuristic introduction

We begin with a basic and informal introduction inspired by Dirac’s original work [9] to provide a physical intuition for the external field QED model. Specialists among the readers are referred directly to Section 1.1. As it is well-known, the free one-particle Dirac equation, in units such that \( \hbar = 1 \) and \( c = 1 \),

\[
(i\partial - m)\psi(x) = 0, \quad \text{for } \psi \in \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4),
\]

(1)
was originally suggested to describe free motion of single electrons. Curiously enough, it allows for wave functions in the negative part \((-\infty, -m]\) of the energy spectrum \(\sigma(H^0) = (-\infty, -m] \cap [+m, \infty)\) of the corresponding Hamiltonian \(H^0 = \gamma^0(-i\gamma \cdot \nabla + m)\). As the spectrum is not bounded from below, physicists rightfully argue [17] that a Dirac electron coupled to the electromagnetic field may cascade to ever lower and lower energies by means of radiation; the reason for this unphysical instability is that the electromagnetic field is an open system, which may transport energy to spacial infinity. Other peculiarities stemming from the presence of a negative energy spectrum are the so-called Zitterbewegung first observed by Schrödinger [29] and Klein’s paradox [20]. As Dirac demonstrated [9], those peculiarities can be reconciled in a coherent description when switching from the one-particle Dirac equation (1) to a many, in the mathematical idealization even infinitely many, particle description known as the second-quantization of the Dirac equation. Perhaps the most striking consequence of this description is the phenomenon of electron-positron pair creation, which only little later was observed experimentally by Anderson [1].

In order to get rid of peculiarities due to the negative energy states, Dirac proposed to introduce a “sea” of electrons occupying all negative energy states. The Pauli exclusion principle then acts to prevent any additional electron in the positive part of the spectrum to dive into the negative one. Let us introduce the orthogonal projectors \(P^+\) and \(P^-\) onto the positive and negative energy subspaces \(\mathcal{H}^+\) and \(\mathcal{H}^-\), respectively, i.e., \(\mathcal{H}^+ = P^+\mathcal{H}\) and \(\mathcal{H}^- = P^-\mathcal{H}\). Dirac’s heuristic picture amounts to introducing an infinitely many-particle wave function of this sea of electrons, usually referred to as Dirac sea,

\[
\Omega = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \ldots, \quad (\varphi_n)_{n \in \mathbb{N}} \text{ being an orthonormal basis of } \mathcal{H}^-,
\]

where \(\wedge\) denotes the antisymmetric tensor product w.r.t. Hilbert space \(\mathcal{H}\). Given a one-particle evolution operator \(U : \mathcal{H} \otimes \), such a Dirac sea may then be evolved with an operator \(L_U\) according to

\[
L_U \Omega = U \varphi_1 \wedge U \varphi_2 \wedge U \varphi_3 \wedge \ldots.
\]

Such an ansatz may seem academic and ad-hoc. First, the Coulomb repulsion between the electrons is neglected (not to mention radiation), second, the choice of \(\Omega\) is somewhat arbitrary. These assumptions clearly would have to be justified starting from a yet to be found full version of QED. For the time being we can only trust Dirac’s intuition that the Dirac sea, when left alone, is so homogeneously distributed that effectively every electron in it feels the same net interaction from each solid angle, and in turn, moves freely so that it lies near to neglect the Coulomb repulsion; see also [3] for a more detailed discussion. Since then none of the particle effectively “sees” the others, physicists refer to such a state as the “vacuum”. A less ad-hoc candidate for \(\Omega\) would of course be the ground state of a fully interacting theory. Even though the net interaction may cancel out, electrons in the ground state will be highly entangled. The hope in using the product state (2) instead, i.e., the ground state of the free theory, to model the vacuum is that in certain regimes the particular entanglement and motion deep down in the sea might be irrelevant. The success of QED in arriving at predictions which are in astonishing agreement with experimental data substantiates this hope.
As a first step to introduce an interaction one allows for an external disturbance of the electrons in \( \Omega \) modeled by a prescribed, time-dependent, four-vector potential \( A \). This turns the one-particle Dirac equation into

\[
(i\partial - m)\psi(x) = eA(x)\psi(x).
\]

(4)

The potential \( A \) may now allow for transitions of states between the subspaces \( \mathcal{H}^+ \) and \( \mathcal{H}^- \). Heuristically speaking, a state \( \varphi_1 \in \mathcal{H} \) in the Dirac sea \( \Omega \) may be bound by the potential and over time dragged to the positive energy subspace \( \chi \in \mathcal{H}^+ \). For an (as we shall see, oversimplified) example, let us assume that up to a phase the resulting state can be represented as

\[
\Psi = \chi \wedge \varphi_2 \wedge \varphi_3 \wedge \ldots
\]

(5)

in which \( \varphi_1 \) is missing. Due to (4), states in \( \mathcal{H}^+ \) move rather differently as compared to the ones in \( \mathcal{H}^- \). Thus, an electron described by \( \chi \in \mathcal{H}^+ \) will emerge from the “vacuum” and so does the “hole” described by the missing \( \varphi_1 \in \mathcal{H}^- \) in the Dirac sea (5), which is left behind. Following Dirac, the hole itself can be interpreted as a particle, which is referred to as positron, and both names can be used as synonyms. If, as in this example, the electrons deeper down in the sea are not affected too much by this disturbance, it makes sense to switch to a more economic description. Instead of tracking all infinitely many particles, it then suffices to describe the motion of the electron \( \chi \), of the corresponding hole \( \varphi_1 \), and of the net evolution of \( \Omega \) only. Since the number of electron-hole pairs may vary over time, a formalism for variable particle numbers is needed. This is provided by the Fock space formalism of quantum field theory, i.e., the so-called “second quantization”. One introduces a so-called creation operator \( a^* \) that formally acts as

\[
a^*(\chi)\varphi_1 \wedge \varphi_2 \wedge \ldots = \chi \wedge \varphi_1 \wedge \varphi_2 \wedge \ldots,
\]

(6)

and also its corresponding adjoint \( a \), which is called annihilation operator. The state \( \Psi \) from example in (5) can then be written as \( \Psi = a^*(\chi)a(\varphi_1)\Omega \). With the help of \( a^* \), one-particle operators like the evolution operator \( U^A \) generated by (4) can be lifted to an operator \( \tilde{U} \) on \( \mathcal{F} \) in a canonical way by requiring that

\[
\tilde{U}^A a^*(f)(\tilde{U}^A)^{-1} = a^*(U^A f).
\]

(7)

This condition determines a lift up to a phase as can be seen from the left-hand side of (7). Since the operator \( a^*(f) \) is linear in its argument \( f \in \mathcal{H} \), it is commonly split into the sum

\[
a^*(f) = b^*(f) + c^*(f) \quad \text{with} \quad b^*(f) := a^*(P^+ f), \quad c^*(f) := a^*(P^- f).
\]

(8)

Hence, \( b^* \) and \( c^* \) and their adjoints are creation and annihilation operators of electrons having positive and negative energy, respectively. In order to be able to disregard the infinitely many-particle wave function \( \Omega \) in the notation, one introduces the following change in language. First, the space generated by states of the form \( b^*(f_1)b^*(f_2)\ldots b^*(f_n)\Omega \) for \( f_k \in \mathcal{H}^+ \) is identified with the electron Fock space

\[
\mathcal{F}_e = \bigoplus_{n \in \mathbb{N}_0} (\mathcal{H}^+)^\wedge n.
\]

(9)
Second, the space generated by the states of the form \( c(g_1)c(g_2)\ldots c(g_n)\Omega \) for \( g_k \in \mathcal{H}^- \) is identified with the hole Fock space
\[
\mathcal{F}_h = \bigoplus_{n \in \mathbb{N}_0} (\mathcal{H}^-)^{\wedge n}.
\] (10)

Note that this time the annihilation operator of negative energy states is employed to generate the Fock space. To make this evident in the notation, one usually replaces \( c(g) \) by a creation operator \( d^*(g) \). However, unlike creation operators, \( c(g) \) is anti-linear in its argument \( g \in \mathcal{H}^- \). Thus, in a third step one replaces \( \mathcal{H}^- \) by its complex conjugate \( \overline{\mathcal{H}^-} \), i.e., the set \( \mathcal{H}^- \) equipped with the usual \( \mathbb{C} \)-vector space structure except for the scalar multiplication \( \cdot^* : \mathbb{C} \times \overline{\mathcal{H}^-} \to \overline{\mathcal{H}^-} \) which is redefined by \( \lambda \cdot^* g = \lambda^* g \) for all \( \lambda \in \mathbb{C} \) and \( g \in \overline{\mathcal{H}^-} \). This turns \( \mathcal{F}_h \) into
\[
\overline{\mathcal{F}_h} = \bigoplus_{n \in \mathbb{N}_0} (\overline{\mathcal{H}^-})^{\wedge n},
\] (11)
and the hole creation operator \( d^*(g) = c(g) \) becomes linear in its argument \( g \in \overline{\mathcal{H}^-} \). To treat electrons and holes more symmetrically, one also introduces the anti-linear charge conjugation operator \( C : \mathcal{H} \to \mathcal{H}, C\psi = i\gamma^2 \psi^* \). This operator exchanges \( \mathcal{H}^+ \) and \( \mathcal{H}^- \), i.e., \( C\mathcal{H}^\pm = \mathcal{H}^\mp \), and thus, gives rise to a linear map \( C : \overline{\mathcal{H}^-} \to \mathcal{H}^+ \). A hole wave function \( g \in \overline{\mathcal{H}^-} \) living in the space negative states can then be represented by a wave function \( Cg \in \mathcal{H}^+ \) living in the positive energy space. Our discussion of the Dirac sea above may appear to break the charge symmetry as \( \Omega \) is represented by a sea of electrons in \( \mathcal{H}^- \). However, an equivalent description that makes the charge symmetry explicit is possible by representing the vacuum \( \Omega \) through a pair of two seas, one in \( \mathcal{H}^+ \) and one in \( \mathcal{H}^- \). Nevertheless, as the charge symmetry will not play a role in this overview we will continue using Dirac’s picture with a sea of electrons in \( \mathcal{H}^- \).

By definition (6) it can be seen that \( b, b^* \) and \( d, d^* \) fulfill the well-known anti-commutator relations:
\[
\{b(g), b(h)\} = 0 = \{b^*(g), b^*(h)\}, \quad \{b^*(g), b(h)\} = \langle g, P^+h \rangle \text{id}_{\mathcal{F}_e},
\]
\[
\{d(g), d(h)\} = 0 = \{d^*(g), d^*(h)\}, \quad \{d^*(g), d(h)\} = \langle g, P^-h \rangle \text{id}_{\mathcal{F}_h}.
\] (12)

The full Fock space for the electrons and positrons is then given by
\[
\mathcal{F} = \mathcal{F}_e \otimes \mathcal{F}_h.
\] (13)

In this space the vacuum wave function \( \Omega \) in (2) is represented by \( |0\rangle = 1 \otimes 1 \) and the pair state \( \Psi \) in (5) by \( a^*(\chi)d^*(\phi_1)|0\rangle \). Thus, in this notation one only describes the excitations of the vacuum, i.e., those electrons that deviate from it. The infinitely many other electrons in the Dirac sea one preferably would like to forget about are successfully hidden in the symbol \( |0\rangle \). Here, however, the story ends abruptly.

### 1.1 The problem and a program for a cure

For a prescribed external potential \( A \), one would be inclined to compute transition probabilities for the creation of pairs, as for example for a transition from \( \Omega \) to \( \Psi \) as in (2) and (5),
right away. Given the one-particle Dirac evolution operator \( U^A = U^A(t_1, t_0) \) generated by (4) and any orthonormal basis \((\chi n)_n \) of \( \mathcal{H}^+ \), the first order of perturbation of the probability of a possible pair creation is given by

\[
\sum_{nm} |\langle \chi_n, U^A \varphi_m \rangle|^2 = \| U^A_{+} \|_{I_2}, \tag{14}
\]

where \( I_2(\mathcal{H}) \) denotes the space of bounded operators with finite Hilbert-Schmidt norm \( \| \cdot \|_{I_2} \), and we use the notation \( U^A_{\pm} = P^\pm U^A P^\mp \). For quite general potentials \( A = (A_0, A) \), it turns out that:

**Theorem 1.1** ([26]). Term (14) \(<\infty\) for all times \( t_0, t_1 \in \mathbb{R} \iff A = 0 \).

In view of (14), the transition probability is thus only defined for external potentials \( A \) that have zero spatial components \( \mathbf{A} \). Even worse, the criterion for the well-definedness of a possible lift \( \tilde{U} \) of any unitary one-particle operator \( U \) according to (7) is given by:

**Theorem 1.2** ([30]). There is a unitary operator \( \tilde{U} : \mathcal{F} \ni \) that fulfills (7) \( \iff U_{+}, U_{+} \in I_2(\mathcal{H}) \).

Applying this result to the evolution operator \( U^A \), (14) and Theorem 1.1 imply that the criterion in Theorem 1.2 is only fulfilled for external potentials \( A \) with zero spatial components \( \mathbf{A} \). Even more peculiar, the given criterion is not gauge covariant (not to mention the Lorentz covariance). Although the free evolution operator \( U^{A=0} \) has a lift, in the case that some spatial derivatives of a scalar field \( \Gamma \) are non-zero, the gauge transformed \( U^{A=0\Gamma} \) does not. This indicates that an unphysical assumption must have been made.

What singles out the spatial components of \( A \)? Mathematically, they appear in the Hamiltonian, \( H^A = \gamma^0(\gamma \cdot \Delta + m) + A_0 - \gamma^0 \gamma \cdot \mathbf{A} \), preceded by the spinor matrix \( \gamma^0 \gamma \) whereas \( A_0 \) is only a multiple of the identity. Heuristically, if \( \mathbf{A} \) is non-zero then the \( \gamma^0 \gamma \) matrix transforms the negative energy states \( \varphi_n \) in spinor space to develop components in \( \mathcal{H}^+ \). There is no mechanism that would limit this development, not even smallness of \( |A| \), so there is no reason why the infinite sum (14) should be finite – and in general this is also not the case as Theorem 1.1 shows. In other words, for \( A \neq 0 \), instantly infinitely many electron-positron pairs are created from the vacuum state \( \Omega \). Therefore, the picture is not nearly as peaceful as suggested by example state (5). However, if \( A \) is switched off at some later time one can expect that almost all of these pairs disappear again, and only a few excitations of the vacuum as in (5) will remain (hence, the name virtual pairs that is used by physicists). Assuming that at initial and final times \( A = 0 \), it can indeed be shown that the scattering matrix \( S^A \) fulfills the conditions of Theorem 1.2. The physical reason why the spatial components are singled out is due to the use of equal-time hyperplanes and will be discussed more geometrically in Section 2; see Theorem 2.8 below.

In conclusion, the problem lies in the fact that even the “vacuum” \( \Omega \) consists of infinitely many particles. In the formalism of the free theory this fact is usually hidden by the use of normal ordering. Without it the ground state energy of \( \Omega \) would be the infinite sum of all negative energies, or the charge current operator expectation value \( \langle \Omega, \bar{\pi} \gamma^a \pi \rangle \) of the vacuum would simply be the infinite sum of all one-particle currents \( \bar{\varphi}_n \gamma^a \varphi_n \) – both quantities
that diverge. The rational behind the ad-hoc introduction of normal ordering of, e.g., the charge current operator is again the assumption that in the vacuum state these currents are effectively not observable since the net interaction between the particles vanishes.

The incompatibility of Theorem 1.2 with the gauge freedom however shows that, although the choice of $\Omega$ may be distinguished for $A = 0$ by the ground state property, it is somehow arbitrary when $A \neq 0$, and so is the choice in the splitting of $\mathcal{H}$ into $\mathcal{H}^+$ and $\mathcal{H}^-$, which is usually referred to as polarization. As a program for a cure of these divergences, one may therefore attempt to carefully adapt the choice of the polarization depending on the evolution of $A$ instead of keeping it fixed. Several attempts have been made to give a definition of a more physical polarization, one of them being the Furry picture. It defines the polarization according to the positive and negative parts of the spectrum of $H^A$ given a fixed $A$. Unfortunately, none of the proposed choices are Lorentz invariant as it is shown in [10] since the vacuum state w.r.t. one of such choices in one frame of reference may appear as a many-particle state in another. This is due to the fact that the energy spectrum is obviously not invariant under Lorentz boosts.

Although a fully developed QED may be able to distinguish a class of states that can be regarded as physical vacuum states, simply by verifying the assumption above that the net interaction between the particles vanishes, the external field QED model has no mathematical structure to do so. Nevertheless, whenever a distinction between electrons and positrons by means of a polarization is not necessary, e.g., in the case of vacuum polarization in which the exact number of pairs is irrelevant, it should still be possible to track the time evolution $\tilde{U}^A \Omega$ and study the generated dynamics – not only asymptotically in scattering theory but also at intermediate times. The choice in admissible polarizations can then be seen to be analogous to the choice of a convenient coordinate system to represent the Dirac seas. Since the employed Fock space $\mathcal{F}$ depends directly on the polarization of $\mathcal{H}$ into $\mathcal{H}^+$ and $\mathcal{H}^-$, see (9)-(10) and (13), the standard formalism has to be adapted to allow the Fock space to also vary according to $A$, and the evolution operator $\tilde{U}^A$ must be implemented mapping one Fock space into another. While the idea of varying Fock space may be unfamiliar from the non-relativistic setting, it is natural when considering a relativistic formalism. A Lorentz boost, for example, tilts an equal-time hyperplane to a Cauchy surface $\Sigma$ which requires a change from the standard Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ to one that is attached to $\Sigma$, and likewise, for the corresponding Fock spaces. Hence, a Lorentz transform will naturally be described by a map from one Fock space into another [5]. In the special case of equal-time hyperplanes, parts of this program have been carried out in [21, 22] and [4]. In the former two works the time evolution operator is nevertheless implemented on standard Fock space $\mathcal{F}$ by conjugation of the evolution operator with a convenient (non-unique) unitary “renormalization” transformation. In the latter work it is implemented between time-varying Fock spaces, so-called infinite wedge spaces, and furthermore, the degrees of freedom in the construction have been identified. These latter results have been extended recently to allow for general Cauchy surfaces in [5, 6] and are presented in Section 2. All these results ensure the existence of an evolution operator by a quite abstract argument. Therefore, we review a construction of it in Section 3 based on [4]. It utilizes a notation that is very close to Dirac’s original view of a sea of electrons as in (2). Though it is canonically equivalent to the Fock space formalism, it provided us a more intuitive view of the problem and helped
in identifying the degrees of freedom involved in the construction. In Section 4 we conclude with a discussion of the unidentified phase of the evolution operator and its meaning for the charge current in. Beside the publications cited so far, there are several recent contributions which also take up on Dirac’s original idea. As a more fundamental approach we want to mention the one of the so-called “Theory of Causal Fermion Systems” [11, 12, 13], which is based on a reformulation of quantum electrodynamics from first principles. The phenomenon of adiabatic pair creation was treated rigorously in [24]. Furthermore, there is a series of works treating the Dirac sea in the Hartree-Fock approximation. The most general is [16] in which the effect of vacuum polarization was treated self-consistently for static external sources.

2 Varying Fock spaces

In order to better understand why the spatial components of $A$ had been singled out in the discussion above, it is helpful to consider the Dirac evolution not only on equal-time hyperplanes but on more general Cauchy surfaces.

**Definition 2.1.** A Cauchy surface $\Sigma$ in $\mathbb{R}^4$ is a smooth, 3-dimensional submanifold of $\mathbb{R}^4$ that fulfills the following two conditions:

(a) Every inextensible, two-sided, time- or light-like, continuous path in $\mathbb{R}^4$ intersects $\Sigma$ in a unique point.

(b) For every $x \in \Sigma$, the tangent space $T_x \Sigma$ of $\Sigma$ at $x$ is space-like.

To each Cauchy surface $\Sigma$ we associated a Hilbert space $H_\Sigma$.

**Definition 2.2.** Let $H_\Sigma = L^2(\Sigma, \mathbb{C}^4)$ denote the vector space of all 4-spinor valued measurable functions $\phi : \Sigma \to \mathbb{C}^4$ (modulo changes on null sets) having a finite norm $\|\phi\| = \sqrt{\langle \phi, \phi \rangle} < \infty$ w.r.t. the scalar product

$$\langle \phi, \psi \rangle = \int_\Sigma \phi(x)i_\gamma(d^4x)\psi(x). \quad (15)$$

Here, $i_\gamma(d^4x)$ denotes the contraction of the volume form $d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ with the spinor-matrix valued vector $\gamma^\mu$, $\mu = 0, 1, 2, 3$. The corresponding dense subset of smooth and compactly supported functions will be denoted by $C_\Sigma$.

The well-posedness of the initial value problem related to (4) for initial data on Cauchy surfaces has been studied in the literature; e.g., see [18, 31] for general hyperbolic systems and more specifically for wave equations on Lorentzian manifolds [8], [2], [25], [14], and [7]. For the purpose of our study we furthermore introduced generalized Fourier transforms for the Dirac equation in [5] and extended the standard Sobolev and Paley-Wiener methods in $\mathbb{R}^n$ to the geometry given by the Cauchy surfaces and the mass shell of the Dirac equation. These methods were required for the analysis of solutions. They play along nicely with Lorentz and gauge transforms and allow for the introduction of an interaction picture. As a byproduct, these methods also ensure existence, uniqueness, and causal structure of strong
solutions. Since we avoid technicalities in this paper, we assume \( A \) is a smooth and compactly supported (although sufficient strong decay would be sufficient), and the following theorem will suffice to discuss the one-particle Dirac evolution.

**Theorem 2.3** (Theorem 2.23 in [5]). Let \( \Sigma, \Sigma' \) be two Cauchy surfaces and \( \psi_{\Sigma} \in C_{\Sigma} \) the initial data. There is a unique strong solution \( \psi \in C^\infty(\mathbb{R}^4, \mathbb{C}^4) \) to (4) being supported in the forward and backward light cone of \( \text{supp} \psi_{\Sigma} \) such that \( \psi|_{\Sigma} = \psi_{\Sigma} \) holds. Furthermore, there is an isometric isomorphism \( U^{A}_{\Sigma,\Sigma'} : C_{\Sigma} \to C_{\Sigma'} \) fulfilling \( \psi|_{\Sigma'} = U^{A}_{\Sigma,\Sigma'} \psi_{\Sigma} \). Its unique extension to a unitary map \( U^{A}_{\Sigma,\Sigma'} : H_{\Sigma} \to H_{\Sigma'} \) is denoted by the same symbol.

Similarly to the standard Fock space (13) we define the Fock space for a Cauchy surface on the basis of a polarization.

**Definition 2.4.** Let \( \text{Pol}(H_{\Sigma}) \) denote the set of all closed, linear subspaces \( V \subset H_{\Sigma} \) such that \( V \) and \( V^\perp \) are both infinite dimensional. Any \( V \in \text{Pol}(H_{\Sigma}) \) is called a polarization of \( H_{\Sigma} \). For \( V \in \text{Pol}(H_{\Sigma}) \), let \( P^V_\Sigma : H_{\Sigma} \to V \) denote the orthogonal projection of \( H_{\Sigma} \) onto \( V \).

The Fock space attached to Cauchy surface \( \Sigma \) and corresponding to polarization \( V \in \text{Pol}(H_{\Sigma}) \) is defined by

\[
F(V, \Sigma) := \bigoplus_{c \in \mathbb{Z}} F_c(V, H_{\Sigma}), \quad F_c(V, \Sigma) := \bigoplus_{n,m \in \mathbb{N}_0} (V^\perp)^n \otimes V^\wedge m. \tag{16}
\]

Note that the standard Fock space is included in this definition by choosing \( \Sigma = \{0\} \times \mathbb{R}^3 \) and \( V = H^- \).

Given two Cauchy surfaces \( \Sigma \) and \( \Sigma' \), polarizations \( V \in \text{Pol}(H_{\Sigma}) \) and \( V' \in \text{Pol}(H_{\Sigma'}) \), and the one-particle evolution operator \( U^{A}_{\Sigma,\Sigma'} : H_{\Sigma} \to H_{\Sigma'} \), we need a condition analogous to (7) that allows us to find an evolution operator \( \tilde{U}^{A}_{V',\Sigma',\Sigma} : F(V, \Sigma) \to F(V', \Sigma') \). For the discussion, let \( a^*_\Sigma \) and \( a_\Sigma \) denote the corresponding creation and annihilation operators on any \( F(W, \Sigma) \) for \( W \in \text{Pol}(H_{\Sigma}) \); note that the defining expression of \( a^*_\Sigma \) in (6) does not depend on the choice of a polarization \( W \). In this notation, the lift requirement reads

\[
\tilde{U}^{A}_{V',\Sigma';\Sigma} a^*_\Sigma(f) \left( \tilde{U}^{A}_{V',\Sigma';\Sigma} \right)^{-1} = a^*_\Sigma(U^{A}_{\Sigma,\Sigma'} f), \quad \forall f \in H_{\Sigma}. \tag{17}
\]

The condition under which such a lift of the one-particle evolution operator \( U^{A}_{\Sigma,\Sigma'} \) exists can be inferred from a slightly rewritten version of the Shale-Stinespring Theorem 1.2:

**Corollary 2.5.** Let \( \Sigma, \Sigma' \) be Cauchy surfaces, \( V \in \text{Pol}(H_{\Sigma}) \), and \( V' \in \text{Pol}(H_{\Sigma'}) \). Then the following statements are equivalent:

(a) There is a unitary operator \( \tilde{U}^{A}_{V',\Sigma';\Sigma} : F(V, \Sigma) \to F(V', \Sigma') \) which fulfills (17).

(b) The off-diagonals \( P^{-1}_{\Sigma'} U^{A}_{\Sigma,\Sigma'} P^V_{\Sigma} \) and \( P^V_{\Sigma'} U^{A}_{\Sigma,\Sigma'} P^{-1}_{\Sigma} \) are Hilbert-Schmidt operators.

Note again that if such a lift exists, its phase is not fixed by (17) and the corollary above does not provide any information about it. Therefore, we will discuss a direct construction of the lifted operator \( \tilde{U}^{A}_{V',\Sigma';\Sigma} \) in Section 3, which makes the involved degrees of freedom apparent.
Coming back to the question which polarizations \( V \in \text{Pol}(\mathcal{H}_\Sigma) \) and \( V' \in \text{Pol}(\mathcal{H}_{\Sigma'}) \) guarantee the existence of a lifted evolution operator \( \tilde{U}_{\Sigma \Sigma'}^A : \mathcal{F}(V, \Sigma) \to \mathcal{F}(V', \Sigma') \), one readily finds a trivial choice. Let us pick a Cauchy surface \( \Sigma_{\text{in}} \) in the remote past fulfilling:

\[
\Sigma_{\text{in}} \text{ is a Cauchy surface such that } \text{supp} \, A \cap \Sigma_{\text{in}} = \emptyset. \tag{18}
\]

When transporting the standard polarization along with the Dirac evolution we get

\[
V = U_{\Sigma \Sigma_{\text{in}}}^A P_{\Sigma_{\text{in}}}^- \mathcal{H}_{\Sigma_{\text{in}}} \in \text{Pol}(\mathcal{H}_\Sigma), \quad V' = U_{\Sigma' \Sigma_{\text{in}}}^A P_{\Sigma_{\text{in}}}^- \mathcal{H}_{\Sigma_{\text{in}}} \in \text{Pol}(\mathcal{H}_{\Sigma'}), \tag{19}
\]

which automatically fulfills condition (b) of Theorem 2.5 as then the off-diagonals \( (U_{\Sigma \Sigma_{\text{in}}}^A)^{\pm \pm} \) become zero. This choice is usually called the interpolation picture. Its drawback is that the polarizations \( V \) and \( V' \) depend on the whole history of \( A \) between \( \Sigma_{\text{in}} \) and \( \Sigma \) and \( \Sigma' \). Moreover, such \( V \) and \( V' \) are rather implicit. Luckily, there are other choices. Statement (b) in Theorem 2.5 allows to differ from the projectors \( P_V^\Sigma \) and \( P_W^{\Sigma'} \) by a Hilbert-Schmidt operator. Hence, all admissible polarizations can be collected and characterized by means of the following classes:

**Definition 2.6.** For a Cauchy surface \( \Sigma \) we define the class

\[
C_\Sigma(A) := \{ W \in \text{Pol}(\mathcal{H}_\Sigma) \mid W \approx U_{\Sigma \Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}} \}
\]

where for \( V, W \in \text{Pol}(\mathcal{H}_\Sigma) \), \( V \approx W \) means that the difference of the corresponding orthogonal projectors \( P_V^\Sigma - P_W^\Sigma \) is a Hilbert-Schmidt operator.

As simple implication of Corollary 2.5 one gets:

**Corollary 2.7.** Let \( \Sigma, \Sigma' \) be Cauchy surfaces and polarizations \( V \in C_\Sigma(A) \) and \( W \in C_{\Sigma'}(A) \). Then up to a phase there is a unitary operator \( \tilde{U}_{\Sigma \Sigma'}^A : \mathcal{F}(V, \mathcal{H}_\Sigma) \to \mathcal{F}(W, \mathcal{H}_{\Sigma'}) \) obeying (17).

We emphasize again that any other possible polarization than the choice in (19) is comprised in the respective class \( C_\Sigma(A) \) as Corollary 2.5 only allows for the freedom encoded in the equivalence relation \( \approx \). Although the polarization (19) depends on the history of the evolution it turns out that the classes \( C_\Sigma(A) \) are independent thereof. The sole dependence of the classes \( C_\Sigma(A) \) is on the tangential components of \( A \), which can be stated as follows.

**Theorem 2.8** (Theorem 1.5 in [6]). Let \( \Sigma \) be a Cauchy surface and let \( A \) and \( \tilde{A} \) be two smooth and compactly supported external fields. Then

\[
C_\Sigma(A) = C_\Sigma(\tilde{A}) \quad \iff \quad A|_{T\Sigma} = \tilde{A}|_{T\Sigma}, \tag{21}
\]

where \( A|_{T\Sigma} = \tilde{A}|_{T\Sigma} \) means that for all \( x \) in \( \Sigma \) and all vectors \( y \) in the tangent space \( T_x \Sigma \) of \( \Sigma \) at \( x \), the relation \( A_\mu(x)y^\mu = \tilde{A}_\mu(x)y^\mu \) holds.

This theorem is a generalization of Ruijsenaar’s result [27] and helps to understand why on equal-time hyperplanes the spatial components of \( A \) appeared to play such a special role. The spatial components \( A \) are the tangential ones w.r.t. such Cauchy surfaces. Furthermore, the classes \( C_\Sigma(A) \) transform nicely under Lorentz and gauge transformations:
Theorem 2.9 (Theorem 1.6 in [6]).

(i) Consider a Lorentz transformation given by $L_{\Sigma}^{(S,A)} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Lambda\Sigma}$ for a spinor transformation matrix $S \in \mathbb{C}^{4 \times 4}$ and an associated proper orthochronous Lorentz transformation matrix $\Lambda \in \text{SO}^+(1,3)$, see for example [5, Section 2.3]. Then:

$$V \in \mathcal{C}_\Sigma(A) \iff L_{\Sigma}^{(S,A)} V \in \mathcal{C}_{\Lambda\Sigma}(\Lambda A (\Lambda^{-1})) .$$

(ii) Consider a gauge transformation $A' = A + \partial \Gamma$ for some $\Gamma \in \mathcal{C}_c^\infty(\mathbb{R}^4, \mathbb{R})$ given by the multiplication operator $e^{-i\Gamma} : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$, $\psi \mapsto \psi' = e^{-i\Gamma} \psi$. Then:

$$V \in \mathcal{C}_\Sigma(A) \iff e^{-i\Gamma} V \in \mathcal{C}_\Sigma(A + \partial \Gamma).$$

As an analogy from geometry one could think of the particular polarization as a particular choice of coordinates to represent the Dirac sea. Corollary 2.5 and Theorem 2.9 explain why gauge transformations that introduce spatial components in the external fields do not comply with the condition to the Shale-Stinespring Theorem 1.2 in which the “coordinates” $\mathcal{H}^+$ and $\mathcal{H}^-$ were fixed.

The key idea in the proofs of Theorem 2.8 and 2.9 is to guess a simple enough operator $P^A_{\Sigma} : \mathcal{H}_\Sigma \otimes$ depending only on the restriction $A|_\Sigma$ so that

$$U_{\Sigma\Sigma_{in}} A^4_{\Sigma_{in} \Sigma} - P^A_{\Sigma} \in I_2(\mathcal{H}_\Sigma), \quad \text{and} \quad (P^A_{\Sigma})^2 - P^A_{\Sigma} \in I_2(\mathcal{H}_\Sigma).$$

The claims about the properties of the polarization classes $C_\Sigma(A)$ can then be inferred directly from the properties of $P^A_{\Sigma}$. This is due to the fact that (24) is compatible with the Hilbert-Schmidt operator freedom encoded in the $\approx$ equivalence relation. The intuition behind the guess of $P^A_{\Sigma}$ used in the proofs presented in [6] comes from the gauge transform. Imagine the special situation in which an external potential $A$ could be gauged to zero, i.e., $A = \partial \Gamma$ for a given scalar field $\Gamma$. In this case $e^{-i\Gamma} P^A_{\Sigma} e^{i\Gamma}$ is a good candidate for $P^A_{\Sigma}$. Now in the case of general external potentials $A$ that cannot be attained by a gauge transformation of the zero potential, the idea is to implement gauge transforms locally at each space-time point. For example, if $p_-(x,y)$ denotes the informal integral kernel of the operator $P^-_{\Sigma}$, one could try to define $P^A_{\Sigma}$ as the operator corresponding to the informal kernel $p^A_-(x,y) = e^{-i\lambda_\Lambda(x,y)} p_-(x,y)$ for the choice $\lambda_\Lambda(x) = A(x)_{\mu}(y - x)^\mu$. The effect of $\lambda_\Lambda(x,y)$ on the projector can be interpreted as a local gauge transform of $p_-(x,y)$ from the zero potential to the potential $A_{\mu}(x)$ at space-time point $x$. A careful analysis of $P^A_{\Sigma}$, which was conducted in Section 2 of [6], shows that $P^A_{\Sigma}$ fulfills (24).

Finally, given Cauchy surface $\Sigma$, there is also an explicit representative of the polarization class $C_\Sigma(A)$ which can be given in terms of the bounded operator $Q^A_{\Sigma} : \mathcal{H}_\Sigma \otimes$ defined by

$$Q^A_{\Sigma} := P^A_{\Sigma} (P^4_{\Sigma} - P^-_{\Sigma}) P^A_{\Sigma} - P^-_{\Sigma} (P^4_{\Sigma} - P^-_{\Sigma}) P^A_{\Sigma}.$$

With it, the polarization class can be identified as follows:

Theorem 2.10 (Theorem 1.7 in [6]). Given Cauchy surface $\Sigma$, $C_\Sigma(A) = [e^{Q_{\Sigma}(A)} \mathcal{H}_{\Sigma}^-]_\Sigma$. 

10
The implications of these results on the physical picture can be seen as follows. The Dirac sea on Cauchy surface $\Sigma$ can be described in any Fock space $\mathcal{F}(V, H_\Sigma)$ for any choice of polarization $V \in C_{\Sigma}(A)$. The polarization class $C_{\Sigma}(A)$ is uniquely determined by the tangential components of the external potential $A$ on $\Sigma$. When regarding the Dirac evolution from one Cauchy surface $\Sigma$ to $\Sigma'$, another choice of “coordinates” $V' \in C_{\Sigma'}(A)$ has to be made. Then one yields an evolution operator $\tilde{U}^A_{\Sigma'\Sigma}: \mathcal{F}(V, H_\Sigma) \to \mathcal{F}(V', H_{\Sigma'})$ which is unique up to an arbitrary phase. Transition probabilities $|\langle \Psi, \tilde{U}^A_{\Sigma'\Sigma} \Phi \rangle|^2$ for $\Psi \in \mathcal{F}(V', H_{\Sigma'})$ and $\Phi \in \mathcal{F}(V, H_{\Sigma})$ are well-defined and unique without the need of a renormalization method. Finally, for a family of Cauchy surfaces $(\Sigma_t)_{t \in \mathbb{R}}$ that interpolates smoothly between $\Sigma$ and $\Sigma'$ one can also infer an infinitesimal version of how the external potential $A$ changes the polarization in terms of the flow parameter $t$; see Theorem 2.6 in [6].

We remark that the kernel of the orthogonal projector corresponding to a polarization in $C_{\Sigma}(A)$, which can be interpreted as a distribution, is frequently called two-point function. Two kernels belonging to two polarizations in the same class $C_{\Sigma}(A)$ may differ by a square-integrable kernel. This stands in contrast to the so-called Hadamard property (see, e.g., [19]) which allows changes with $C^\infty$ kernels as freedom in two-point functions.

3 An explicit construction of the evolution operator

The argument in Section 2 that ensures the existence of dynamics on varying Fock spaces is quite abstract. In this section we present a more direct approach that is also closer to Dirac’s original picture in describing infinite particle wave functions like in (2). As discussed, the infinitely many particles are also present in the usual Fock space formalism but commonly hidden by use of normal ordering. But since the very obstacle in a straightforward construction of the evolution operator is due to their presence, it seems to make sense to work with a formalism that makes them apparent. One such formalism, introduced in Section 2 of [4], employs so-called infinite wedge spaces and will be used in the following.

To leave our discussion general, let $\mathcal{H}$ be a one-particle Hilbert space (e.g., $\mathcal{H} = H_\Sigma$ as in Section 2) and let $V \in \text{Pol}(\mathcal{H})$ be a polarization thereof. The Dirac sea corresponding to that choice of polarization can be represented, using any orthonormal basis $(\varphi_n)_{n \in \mathbb{N}}$ that spans $V$, by the infinite wedge product

$$\wedge \Phi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \ldots,$$

i.e., the anti-symmetric product of all wave functions $\varphi_n$, $n \in \mathbb{N}$. Slightly more general, it suffices if $(\varphi_n)_{n \in \mathbb{N}}$ is only asymptotically orthonormal in the sense that the infinite matrix $(\langle \varphi_n, \varphi_m \rangle)_{n, m \in \mathbb{N}}$ has a (Fredholm) determinant, i.e., that it differs from the identity only by a matrix that has a trace. The reason for this property will become clear when introducing the scalar product of two infinite wedge products.

In order to keep the formalism short, we encode the basis $(\varphi_n)_{n \in \mathbb{N}}$ by a bounded linear operator

$$\Phi: \ell \to \mathcal{H}, \quad \Phi e_n = \varphi_n$$

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on a Hilbert space $\ell$. The role of $\ell$ is only that of an index space, and one example we have in mind is $\ell = \ell^2(\mathbb{N})$, i.e., the space of square summable sequences where the vectors $e_n, n \in \mathbb{N}$, denote the canonical basis. In this language, the asymptotic orthonormality requirement from above can be rewritten as $\Phi^*\Phi \in I_\ell + I_1(\ell)$, where $I_1(\ell)$ is the space of bounded linear maps $\ell \to \ell$ which have a trace, the so-called trace class. We will also write $\Lambda\Phi = \varphi_1 \land \varphi_2 \land \ldots$ which denotes the infinite wedge product (26) and refer to all such $\Phi$ as Dirac seas.

Given another Dirac sea $\Psi$ with $\psi_n = \Psi e_n, n \in \mathbb{N}$, the pairing that will later become a scalar product

$$\langle \Lambda\Psi, \Lambda\Phi \rangle = \langle \psi_1 \land \psi_2 \land \ldots, \varphi_1 \land \varphi_2 \land \ldots \rangle = \det(\langle \psi_n, \varphi_m \rangle)_{nm} = \det \Psi^*\Phi \quad (28)$$

is well-defined if $\Psi^*\Phi$ has a determinant, which is the case if $\Psi^*\Phi \in I_\ell + I_1(\ell)$. Thus, it makes sense to build a Fock space, referred to as “infinite wedge space $\mathcal{F}_{\Lambda\Phi}$”, based on a basis encoded by $\Phi$. It is defined by the completion w.r.t. the pairing (28) of the space of formal linear combinations of all such $\Psi$; see Section 2.1 in [4] for a rigorous construction. This space consists of the sea wave function $\Lambda\Phi$, its excitations $\Lambda\Psi$ that form a generating set, and superpositions thereof. An example excitation analogous to (5) representing an electron-positron pair with electron wave function $\chi \in V^\perp$ and positron wave function $\varphi_1 \in V$ is given by

$$\Lambda\Psi = \chi \land \varphi_2 \land \varphi_3 \land \varphi_4 \land \ldots \quad (29)$$

Note, however, that mathematically $\Phi$ is not distinguished as “the one vacuum” state as it turns out that $\mathcal{F}_{\Lambda\Psi} = \mathcal{F}_{\Lambda\Phi}$ if and only if $\Psi^*\Phi$ has a determinant, i.e., if the scalar product $\langle \Lambda\Psi, \Lambda\Phi \rangle$ in (28) is well-defined. This is due to the fact $\Phi \sim \Phi := \Psi^*\Phi \in I_\ell + I_1(\ell)$ is an equivalence relation on the set of all Dirac seas; see Corollary 2.9 in [4].

Next, let us consider another one-particle Hilbert space $\mathcal{H}'$ and a one-particle unitary operator $U : \mathcal{H} \to \mathcal{H}'$ such as the one-particle Dirac evolution operator $U_{\Sigma^\perp\Sigma}^A$. To infer from this a corresponding evolution of the Dirac seas, we define a canonical operation from the left as follows

$$\mathcal{L}_U : \mathcal{F}_{\Lambda\Phi} \to \mathcal{F}_{\Lambda U\Phi}, \quad \mathcal{L}_U \Lambda\Psi := \Lambda U\Psi = (U\psi_1) \land (U\psi_2) \land \ldots \quad (30)$$

Here, $\Psi$ is taken from the generating set of Dirac seas fulfilling $\Psi^*\Phi \in 1 + I_1(\ell)$; see Section 2.2 in [4]. That the range of $\mathcal{L}_U$ is $\mathcal{F}_{\Lambda U\Phi}$ is due to the fact that $\Psi^*\Phi$ has a determinant if and only if $(U\Psi)^* (U\Phi)$ does. Such a map $\mathcal{L}_U$ represents an evolution operator from one infinite wedge space into another that in the sense of (6) also complies with the previously discussed lift condition (7).

Nevertheless, the construction of the evolution operator for the Dirac seas does not end here because the target space $\mathcal{F}_{\Lambda U\Phi}$ in (30) is completely implicit, and hence, $\mathcal{L}_U$ alone is not very helpful. On the contrary, relying on the observations made in Section 2, physics should allow us to decide beforehand between which infinite wedge spaces the evolution operator should be implemented. Consider the example situation of

an evolution operator $U = U_{\Sigma^\perp\Sigma}^A$ from Theorem 2.3,

$$\mathcal{H} = \mathcal{H}_\Sigma, \quad V \in \text{Pol}(\mathcal{H}_\Sigma), \quad \Phi : \ell \to \mathcal{H}_\Sigma \text{ such that } \text{range } \Phi = V; \quad (31)$$

$$\mathcal{H}' = \mathcal{H}_{\Sigma'}, \quad V' \in \text{Pol}(\mathcal{H}_{\Sigma'}), \quad \Phi' : \ell' \to \mathcal{H}_{\Sigma'} \text{ such that } \text{range } \Phi' = V'.$$
In this situation one would wish for an evolution operator of the form \( \tilde{U} : \mathcal{F}_{A\Phi} \to \mathcal{F}_{A\Phi'} \) instead of \( \hat{U} : \mathcal{F}_{A\Phi} \to \mathcal{F}_{A\Phi'} \). If we are not in the lucky case \( \mathcal{F}_{\Phi'} = \mathcal{F}_{A\Phi'} \), there are two ways in which the equality may fail. First, Corollary 2.5 suggests that polarization \( V \) and \( V' \) must be elements of the appropriate polarization classes, more precisely, \( V \in C_\Sigma(A) \) and \( V' \in C_{\Sigma'}(A) \). However, there is a more subtle obstacle as for \( \mathcal{F}_{\Phi'} \neq \mathcal{F}_{A\Phi'} \) to hold we need to ensure that \( \langle \Phi', U\Phi \rangle \) is well-defined, which even for \( \ell = \ell' \) and admissible \( V \) and \( V' \) does not need to be the case. Thus, in general \( U\Phi \) and \( \Phi' \) belong to entirely different infinite wedge spaces as the choice of orthonormal bases encoded in \( \Phi \) and \( \Phi' \) was somehow arbitrary. However, let \( \Psi : \ell \to \mathcal{H} \) be another Dirac sea with range \( \Psi = V' \), then there is a unitary \( R : \ell' \to \ell \) such that \( \Phi' = \Psi \circ R \). The action of \( R \) gives rise to a unitary operation from the right \( \mathcal{R}_R \) characterized by

\[
\mathcal{R}_R : \mathcal{F}_{A\Phi} \to \mathcal{F}_{A\Phi'}, \quad \mathcal{R}_R \Lambda \tilde{\Psi} = \Lambda (\Psi \circ R) \tag{32}
\]

for all \( \tilde{\Psi} : \ell \to \mathcal{H} \) in the generating system of \( \mathcal{F}_{A\Phi} \), which connects the infinite wedge spaces \( \mathcal{F}_{A\Phi} \) and \( \mathcal{F}_{A\Phi'} \). The spaces \( \mathcal{F}_{A\Phi} \) and \( \mathcal{F}_{A\Phi'} \) coincide if and only if \( \ell = \ell' \) and \( R \) has a determinant. Slightly more generally, it suffices if \( R \) is only asymptotically unitary in the sense that \( R^*R \) has a non-zero determinant. Then the operation from the right \( \det(R^*R)^{-1/2}\mathcal{R}_R \) is unitary. Whether there is a unitary \( R : \ell' \to \ell \) in the situation of example (31) above such that \( \mathcal{F}_{A\Phi} = \mathcal{F}_{A\Phi'} \) is answered by the next theorem. It can be seen as yet another version of the Shale and Stinespring’s Theorem:

**Theorem 3.1** (Theorem 2.26 of [4]). Let \( \mathcal{H} \), \( \ell \), \( \mathcal{H}' \), \( \ell' \) be Hilbert spaces, \( V \in \text{Pol}(\mathcal{H}) \) and \( V' \in \text{Pol}(\mathcal{H}') \) polarizations, \( \Phi : \ell \to \mathcal{H} \) and \( \Phi' : \ell' \to \mathcal{H}' \) Dirac seas such that range \( \Phi = V \) and range \( \Phi' = V' \). Then the following statements are equivalent:

(a) The off-diagonals \( P_{V'} UP_V \) and \( P_{V'}' UP_{V'} \) are Hilbert-Schmidt operators.

(b) There is a unitary \( R : \ell' \to \ell \) such that \( \mathcal{F}_{A\Phi} = \mathcal{F}_{A\Phi'} \).

Coming back to the example (31) from above, in the case \( V \in C_\Sigma(A) \) and \( V' \in C_{\Sigma'}(A) \), i.e., that the chosen polarization belong to the admissible classes of polarizations, condition (a) of Theorem 3.1 is fulfilled, which implies the existence of a unitary map \( R : V' \to V \) such that the evolution operator

\[
\tilde{U}_{V,\Sigma,V',\Sigma'}^A : \mathcal{F}_{A\Phi} \to \mathcal{F}_{A\Phi'}, \quad \tilde{U}_{V,\Sigma,V',\Sigma'}^A = \mathcal{R}_R \circ \mathcal{L}_{U_{\Sigma}} \tag{33}
\]

is well-defined and unitary. An immediate question is of course how many such maps exist, and it turns out that any other operation from the right \( \mathcal{R}_{R'} \) for which \( \mathcal{R}_{R'} \circ \mathcal{L}_U : \mathcal{F}_{A\Phi} \to \mathcal{F}_{A\Phi'} \) is well-defined and unitary fulfills \( \tilde{U}_{V,\Sigma,V',\Sigma'}^A = e^{i\theta} \mathcal{R}_{R'} \circ \mathcal{L}_U \) for some \( \theta \in \mathbb{R} \); see [4, Corollary 2.28]. Now \( \Phi \) and \( \Phi' \) are Dirac seas in which all states in \( V \) and \( V' \) are occupied, respectively. A canonical choice for their representation is to choose \( \ell = V \), \( \ell' = V' \), and to define the inclusion maps \( \Phi : V \to \mathcal{H}_\Sigma \), \( \Phi v = v \) for all \( v \in V \), and \( \Phi' : V' \to \mathcal{H}_{\Sigma'} \), \( \Phi' v' = v' \) for all \( v' \in V' \). In this case there is a canonical isomorphism between the spaces \( \mathcal{F}_{A\Phi} \) and \( \mathcal{F}_{V,\Sigma} \) as well as between \( \mathcal{F}_{A\Phi'} \) and \( \mathcal{F}_{V',\Sigma'} \). Hence, we are again in the situation of Corollary 2.5. We can identify the evolution of the Dirac seas only up to a phase \( \theta \in \mathbb{R} \). However, now we have a more direct construction at hand which identifies the involved degrees of freedom:
(a) The choice of particular polarizations \( V \in C_\Sigma(A) \) and \( V' \in C_\Sigma(A) \).

(b) The choice of particular bases encoded in \( \Phi \) and \( \Phi' \).

The restriction of the polarizations to polarization classes in (a) has been discussed in Section 2. Moreover, choice (b) can be given a quite intuitive picture coming from Dirac’s original idea that the motion deep down in the sea should be irrelevant when studying the excitations on its “surface”. Clearly, when a sea wave function \( \Lambda \Psi \in F_{\Lambda \Phi} \), which could represent an excitation w.r.t. \( \Lambda \Phi \), is evolved from \( \Sigma \) to \( \Lambda \Psi' \) on \( \Sigma' \), clearly also the particles deep down in the sea will “move”. Since there are infinitely many it will be impossible to directly compare \( \Psi' \) with \( \Psi \) in general. Writing \( U = U_{\Sigma \Sigma}^A \) in matrix notation

\[
U = \begin{pmatrix}
U_{++} & U_{+-} \\
U_{-+} & U_{--}
\end{pmatrix} = \begin{pmatrix}
P_{\Sigma'}^\dagger U P_\Sigma^\dagger & P_{\Sigma'}^\dagger U P_\Sigma^V \\
P_{\Sigma'}^V U P_\Sigma^\dagger & P_{\Sigma'}^V U P_\Sigma^V
\end{pmatrix},
\]

(34)

the motion deep down in the sea is governed by \( U_{--} \). Now, if according to Dirac’s original idea the motion deep down in the sea can be considered irrelevant for the behavior of the excitations on its surface one should still be able to compare \( \Lambda \Psi' \) to \( \Lambda \Psi \) when reversing the motion deep down in the sea with \( (U_{--})^{-1} \). If \( U \) is for example sufficiently close to the identity this can be done explicitly since then \( U_{--} \) has an inverse \( R = (U_{--})^{-1} \). As we shall see now, the inversion of the motion deep down in the sea can be implemented by means of an operation from the right \( R_R \). For \( R \) to induce an operation from the right it has to be asymptotically orthonormal, i.e., \( R^* R \) must have a determinant. Recall that condition (a) in Theorem 3.1 states that the off-diagonals \( U_{+-} \) and \( U_{-+} \) are Hilbert-Schmidt operators. Thanks to \( U^* U = id_H \) the identity

\[
U^* U_{--} = id_V - (U^*)_{+-} U_{+-}
\]

(35)

holds, and since the product of two Hilbert-Schmidt operators has a trace, one finds \( U^* U_{--} \in id_V + I_1(V) \). Thus, \( U^* U_{--} \) and then also \( R^* R \) have determinants. Note that in general \( \det(R^* R) \neq 1 \), which implies that \( R_R \) may fail to be unitary up to the factor \( \det |R| \). By definition one finds

\[
R_R \circ L_U F_{\Lambda \Phi} = F_{\Lambda' \Phi R} = F_{\Lambda' \Phi'}
\]

(36)

because \( \Phi'^* U \Phi R = P^V (U_{++} + U_{--}) R = id_V \), and therefore, has a determinant. In consequence, we yield the unitary Dirac evolution

\[
\tilde{U}_{\Sigma; \Sigma'}^A : F_{\Lambda \Phi} \to F_{\Lambda \Phi'}, \quad \tilde{U}_{\Sigma; \Sigma'}^A = \det |(U_{\Sigma \Sigma'}^A)_{--}| R_{([U_{\Sigma \Sigma'}^A]_{--})^{-1}} \circ L_{U_{\Sigma \Sigma'}^A},
\]

(37)

which implements both the forward evolution of the whole Dirac sea and the backward evolution of the states deep down in the sea.

4 The charge current and the phase of the evolution operator

Although the construction of the second-quantized evolution operator according to the above program is successful, it fails to identify the phase. This short-coming has no effect on the
uniqueness of transition probabilities but it turns out that the charge current depends directly on this phase. One way to see that is from Bogolyubov’s formula of the current

\[ J^\mu(x) = i \tilde{U}^A_{\text{Vin}, \Sigma_{\text{in}}; \text{Vout}, \Sigma_{\text{out}}} \frac{\delta}{\delta A^\mu(x)} \tilde{U}^A_{\text{Vout}, \Sigma_{\text{out}}, \text{Vin}, \Sigma_{\text{in}}}, \]

where \( \Sigma_{\text{out}} \) is a Cauchy surface in the remote future of the support of \( A \) such that \( \Sigma_{\text{out}} \cap \text{supp } A = \emptyset \). Changing the evolution operator by an \( A \)-dependent phase generates another summand on the right hand side of (38) by the chain rule. Until some phase is distinguished, (38) has no particular physical meaning as charge current. Nevertheless, all possible currents can be derived from (38) given an evolution operator and a particular phase. Therefore, the situation is better than in standard QED. There, the charge current is a quantity whose formal perturbation series leads to several divergent integrals which have to be taken out by hand until only a logarithmic divergent is left, which in turn is remedied by means of charge renormalization. On the contrary, here, the currents are well-defined and in a sense the correct one only needs to be identified by determining the phase of the evolution operator. As already envisioned in [28] and discussed by [22, 15], this may be done by imposing extra conditions on the evolution operator. One of them is clearly the following property. For any choice of a future oriented foliation of space-time into a family of Cauchy surfaces \( (\Sigma_t)_{t \in \mathbb{R}} \) and polarization \( V_t \in \mathcal{C}_{\Sigma_t}(A) \), \( t \in \mathbb{R} \), the assigned phase of the evolution operator \( \tilde{U}^A(t_1, t_0) = \tilde{U}^A_{\Sigma_{t_1}; V_{t_1}; \Sigma_{t_0}; V_{t_0}} \) constructed in Section 3 should be required to fulfill \( \tilde{U}(t_1, t_0) = \tilde{U}(t_1, t)\tilde{U}(t, t_0) \). Other constraints come from the fact that \( J^\mu(x) \) must be Lorentz and gauge covariant, and its vacuum expectation value for \( A = 0 \) should be zero. The hope is that the collection of all such physical constraints restrict the possible currents (38) to a class which can be parametrized by a real number only, the electric charge of the electron. In the case of equal-time hyperplanes one possible choice of the phase was given by Mickelsson via a parallel transport argument [23]. On top of the nice geometric construction and despite the fact that there are still degrees of freedom left, Mickelsson’s current agrees with conventional perturbation theory up to second order. The aim of this program is to settle the question which conditions are required to identify the charge current upon changes of the value of the electric charge.

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