K3 SURFACES WITH AN ORDER 50 AUTOMORPHISM

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Abstract. In any characteristic \( p \) different from 2 and 5, Kondō gave an example of a K3 surface with a purely non-symplectic automorphism of order 50. The surface was explicitly given as a double plane branched along a smooth sextic curve. In this note we show that, in any characteristic \( p \neq 2, 5 \), a K3 surface with a cyclic action of order 50 is isomorphic to the example of Kondō.

Let \( X \) be a K3 surface over an algebraically closed field \( k \) of characteristic \( p \geq 0 \). An automorphism \( g \) of \( X \) is called symplectic if it preserves any regular 2-form on \( X \), and purely non-symplectic if no power of \( g \) is symplectic except the identity.

In any characteristic \( p \geq 0, p \neq 2, 5 \), Kondō [8] gave an example of a K3 surface \( X_{50} \) with a purely non-symplectic automorphism \( g_{50} \) of order 50:

\[
(0.1) \quad X_{50} = (w^2 = x^6 + xy^5 + yz^5) \subset \mathbb{P}(1, 1, 1, 3),
\]

\[
(0.2) \quad g_{50}(x, y, z, w) = (x, \zeta_{50}^4 y, \zeta_{50}^2 z, \zeta_{50}^{25} w)
\]

where \( \zeta_{50} \in k \) is a primitive 50th root of unity. In characteristic \( p = 2 \) and \( 5 \) the automorphism degenerates and the equation does not even define a K3 surface.

The result of this short note is the following.

**Theorem 0.1.** Let \( k \) be an algebraically closed field of characteristic \( p \neq 2, 5 \). Let \( X \) be a K3 surface defined over \( k \) with an automorphism \( g \) of order 50. Then

1. \( g \) is purely non-symplectic;
2. the pair \((X, \langle g \rangle)\) is isomorphic to the pair \((X_{50}, \langle g_{50} \rangle)\).

The first statement of Theorem 0.1 was proved in a previous paper [4, Lemmas 4.2 and 4.7].

Over \( k = \mathbb{C} \) the second statement of Theorem 0.1 was proved by Machida and Oguiso [9], under the assumption that \( g \) is purely non-symplectic. Our proof is characteristic free, does not use lattice theory and the holomorphic Lefschetz formula.

A similar characterization of K3 surfaces with a tame cyclic action of order 60 (resp. 66) was given in [5] (resp. [6]), where it was proven that for
such a pair \((X, \langle g \rangle)\) the K3 surface \(X\) admits a \(g\)-invariant elliptic fibration, thus can be given by a \(g\)-invariant Weierstrass equation. The case of order 50 is similar to the case of order 40 in [7], and the K3 surface admits a \(g\)-invariant double plane presentation.

**Remark 0.2.** (1) In characteristic 5 it was shown in the previous paper [4, Main Theorem and Lemma 9.6] that no K3 surface admits a cyclic action of order 25.

(2) In characteristic 2 there is a K3 surface with a cyclic action of order 50:

\[(0.3) \quad Y = (w^2 + x^3w = x^6 + xy^5 + y^5) \subset \mathbb{P}(1, 1, 1, 3),\]

\[(0.4) \quad f_{50}(x, y, z, w) = (x, \zeta_{25}^0 y, \zeta_{25} z, w + x^3)\]

where \(\zeta_{25} \in k\) is a primitive 25th root of unity. Is this the unique pair up to isomorphism in characteristic 2?

**Notation**

For a variety \(X\) with an automorphism \(g\), we use the following notation:

- \(\text{NS}(X)\) : the Néron-Severi group of \(X\);
- \(X^g = \text{Fix}(g)\) : the fixed locus of \(g\) in \(X\);
- \(e(g) := e(\text{Fix}(g)), \) the Euler characteristic of \(\text{Fix}(g)\) for \(g\) tame;
- \(\text{Tr}(g^*|H^*(X)) := \sum_{j=0}^{2\dim X} (-1)^j \text{Tr}(g^*|H^j_{et}(X, \mathbb{Q}_l))\);
- \([g^*] = [\lambda_1, \ldots, \lambda_{b_2}]\) : the list of eigenvalues of \(g^*|H^2_{et}(X, \mathbb{Q}_l)\) where \(b_2\) is the second Betti number of \(X\);
- \(\zeta_a\) : a primitive \(a\)-th root of unity in \(\mathbb{Q}_l\);
- \([\zeta_a : \phi(a)] \subset [g^*]\) : all primitive \(a\)-th roots of unity appear in \([g^*]\) where \(\phi(a)\) indicates the number of them;
- \([\lambda, r] \subset [g^*]\) : the eigenvalue \(\lambda\) repeats \(r\) times in \([g^*]\);
- \([\zeta_a : \phi(a)], r \subset [g^*]\) : the list \(\zeta_a : \phi(a)\) repeats \(r\) times in \([g^*]\).

1. Preliminaries

We first recall the following basic result used in the paper [4].

**Proposition 1.1.** (See [4] Proposition 2.1.) Let \(g\) be an automorphism of a projective variety \(X\) over an algebraically closed field \(k\) of characteristic \(p > 0\). Let \(l\) be a prime \(\neq p\). Then the following hold true.
(1) (See [3, 3.7.3].) The characteristic polynomial of $g^*|H^j_{\text{et}}(X, \mathbb{Q}_l)$ has integer coefficients for each $j$. The characteristic polynomial does not depend on the choice of cohomology, $l$-adic or crystalline. In particular, if a primitive $m$-th root of unity appears with multiplicity $r$ as an eigenvalue of $g^*|H^j_{\text{et}}(X, \mathbb{Q}_l)$, then so does each of its conjugates.

(2) If $g$ is of finite order, then $g$ has an invariant ample divisor, and $g^*|H^2_{\text{et}}(X, \mathbb{Q}_l)$ has 1 as an eigenvalue.

(3) If $X$ is a K3 surface, $g$ is tame and $g^*|H^0(X, \Omega^2_X)$ has $\zeta_n \in k$ as an eigenvalue, then $g^*|H^2_{\text{et}}(X, \mathbb{Q}_l)$ has $\zeta_n \in \overline{\mathbb{Q}}_l$ as an eigenvalue.

The following is well known, see e.g., [1, Theorem 3.2].

**Proposition 1.2.** (Lefschetz fixed point formula) Let $X$ be a smooth projective variety over an algebraically closed field $k$ of characteristic $p > 0$ and let $g$ be a tame automorphism of $X$. Then $X^g = \text{Fix}(g)$ is smooth and

$$e(g) := e(X^g) = \text{Tr}(g^*|H^*(X)).$$

**Lemma 1.3.** (See [5, Lemma 1.6].) Let $X$ be a K3 surface in characteristic $p \neq 2$, admitting an automorphism $h$ of order 2 with $\dim H^2_{\text{et}}(X, \mathbb{Q}_l)^h = 2$. Then $h$ is non-symplectic and has an $h$-invariant elliptic fibration $\psi : X \to \mathbb{P}^1$, $X/(h) \cong F_e$

a rational ruled surface, and $X^h$ is either a curve of genus 9 which is a 4-section of $\psi$ or the union of a section and a curve of genus 10 which is a 3-section. In the first case $e = 0, 1$ or 2, and in the second $e = 4$. Each singular fibre of $\psi$ is of type $I_1$ (nodal), $I_2$, $II$ (cuspidal) or $III$, and is intersected by $X^h$ at the node and two smooth points if of type $I_1$, at the two singular points if of type $I_2$, at the cusp with multiplicity 3 and a smooth point if of type $II$, at the singular point tangentially to both components if of type $III$. If $X^h$ contains a section, then each singular fibre is of type $I_1$ or $II$.

**Remark 1.4.** If $e \neq 0$, the $h$-invariant elliptic fibration $\psi$ is the pull-back of the unique ruling of $F_e$. If $e = 0$, either ruling of $F_0$ lifts to an $h$-invariant elliptic fibration.

The following easy lemmas also will be used frequently.

**Lemma 1.5.** (See [1, Lemma 2.10].) Let $S$ be a set and $\text{Aut}(S)$ be the group of bijections of $S$. For any $g \in \text{Aut}(S)$ and positive integers $a$ and $b$,

1. $\text{Fix}(g) \subset \text{Fix}(g^a)$;
2. $\text{Fix}(g^a) \cap \text{Fix}(g^b) = \text{Fix}(g^d)$ where $d = \gcd(a, b)$;
3. $\text{Fix}(g) = \text{Fix}(g^a)$ if $\text{ord}(g)$ is finite and prime to $a$. 

Lemma 1.6. (See [1] Lemma 2.11.) Let $R(n)$ be the sum of all primitive $n$-th root of unity in $\mathbb{Q}$ or in $\mathbb{Q}_l$, where $\gcd(l, n) = 1$. Then

$$R(n) = \begin{cases} 0 & \text{if } n \text{ has a square factor,} \\ (-1)^t & \text{if } n \text{ is a product of } t \text{ distinct primes.} \end{cases}$$

For an automorphism $g$ of finite order of a K3 surface $X$, tame or wild, we write

$$\text{ord}(g) = m.n$$

if $g$ is of order $mn$ and the natural homomorphism $\langle g \rangle \to \text{GL}(H^0(X, \Omega^2_X))$ has kernel of order $m$ and image of order $n$.

2. Proof: the Tame Case

Throughout this section, we assume that the characteristic $p > 0$, $p \neq 2, 5$. Let $g$ be an automorphism of order 50 of a K3 surface. We first determine the list of eigenvalues of $g^*$ acting on the second cohomology of $X$.

Lemma 2.1. $[g^*] \neq [1, 1, \zeta_{50} : 20]$.

Proof. Suppose that $[g^*] = [1, 1, \zeta_{50} : 20]$. Then

$$[g^{25*}] = [1, 1, -1.20], \quad e(g^{25}) = -16.$$ 

Thus $\text{Fix}(g^{25})$ is either a curve $C_9$ of genus 9 or the union of a smooth rational curve and a curve $C_{10}$ of genus 10. Using Lemma 1.6 we compute

$$e(g) = 4$$

and

$$[g^{5*}] = [1, 1, (\zeta_{10} : 4)5], \quad e(g^5) = 9.$$ 

Note that

$$\text{Fix}(g^d) \subset \text{Fix}(g^{25})$$

for any $d$ dividing 25. If $\text{Fix}(g^{25})$ is a curve $C_9$ of genus 9, then the action of $g$ (resp. $g^5$) on $C_9$ has 4 (resp. 9) fixed points, hence the degree 25 cover $C_9 \to C_9/\langle g \rangle$ has 4 points of ramification index 25 and 5 points of ramification index 5, which contradicts the Hurwitz formula. By Lemma 1.3 the quotient surface $X/\langle g^{25} \rangle$ is isomorphic to the rational ruled surface $F_4$, $X$ has a $g^{25}$-invariant elliptic fibration

$$\psi : X \to \mathbb{P}^1$$

and $\text{Fix}(g^{25})$ is the union of a section $R$ and a curve $C_{10}$ of genus 10 which is a 3-section of $\psi$. We also know that a fibre of $\psi$ is of type $I_0$ (smooth), $I_1$ or $II$. The automorphism $\bar{g}$ of $F_4$ induced by $g$ preserves the unique ruling, so $g$ preserves the elliptic fibration. Since $\bar{g}^{25}$ acts trivially on $F_4$, $g^{25}$ acts trivially on the base $\mathbb{P}^1$ and the orbit of a singular fibre under the action of $g|\mathbb{P}^1$ has length 1 or 5. Thus $g^5|\mathbb{P}^1$ fixes all singular fibres and $g^5$ fixes the singular points of all singular fibres. Since $\psi$ has at least 12 singular fibres, $g^5$ fixes at least 12 points, contradicting $e(g^5) = 9$. □

Lemma 2.2. (1) $[g^*] = [1, -1, \zeta_{50} : 20]$

where the first eigenvalue corresponds to a $g^*$-invariant ample class;
In the same way as above, we see that 

\[ C_{\subspace} \text{ of } g \]

\[ \text{Since } e \]

\[ \text{then by Hodge index theorem} \]

\[ \text{smooth rational curve. Since } e \]

\[ \text{hence } d < \]

\[ \text{points, } p \]

\[ \text{Since } C \]

\[ \text{Let } \]

\[ \text{the result follows from Lemma 2.1.} \]

Proof. (1) By [4, Lemmas 4.2 and 4.7], \( g \) cannot be of order 2.25 or 5.10, hence is purely non-symplectic. By Proposition 1.1 the action of \( g^* \) on \( H^2_{et}(X, \mathbb{Q}) \) has \( \zeta_{50} \in \mathbb{Q}^* \) as an eigenvalue. Thus \( [g^*] = [1, \pm 1, \zeta_{50} : 20] \) and the result follows from Lemma 2.1.

(2) follows from (1), since \( e(g^{25}) = -18 \) and the invariant subspace of \( g^*|H^2_{et}(X, \mathbb{Q}) \) has dimension 1.

(3)-(5) We compute

\[ [g^{5*}] = [1, -1, (\zeta_{10} : 4), 5], \quad e(g^5) = 7, \]

\[ [g^{10*}] = [1, 1, (\zeta_5 : 4), 5], \quad e(g^{10}) = -1. \]

Since \( e(g^{10}) < 0 \), \( \text{Fix}(g^{10}) \) contains a curve of genus > 1. Since the invariant subspace of \( g^{5*}|H^2_{et}(X, \mathbb{Q}) \) has dimension 2, \( \text{Fix}(g^{10}) \) contains at most one smooth rational curve. Since \( e(g^5) = 7 \), \( \text{Fix}(g^5) \) consists of 7 points of \( C_{10} = \text{Fix}(g^{25}) \). Suppose \( \text{Fix}(g^{10}) \) contains a rational curve \( R \). Then

\[ \text{Fix}(g^{10}) = R \cup D_{d+3} \cup \{2d + 1 \text{ points}\}, \quad d \geq 0. \]

Since \( C_{10} \cap D_{d+3} \subset \text{Fix}(g^{25}) \cap \text{Fix}(g^{10}) = \text{Fix}(g^5) \), we have

\[ C_{10}D_{d+3} \leq 7, \]

then by Hodge index theorem

\[ 18(2d + 4) = C_{10}^2D_{d+3}^2 \leq (C_{10}D_{d+3})^2 \leq \tau^2, \]

hence \( d < 0 \), absurd. Thus \( \text{Fix}(g^{10}) \) cannot contain a rational curve and

\[ \text{Fix}(g^{10}) = D_{d+2} \cup \{2d + 1 \text{ points}\}, \quad d \geq 0. \]

In the same way as above, we see that \( C_{10}D_{d+2} \leq 7. \) Then by the Hodge index theorem

\[ 18(2d + 2) = C_{10}^2D_{d+2}^2 \leq (C_{10}D_{d+2})^2 \leq \tau^2, \]

hence \( d = 0 \) and \( 6 \leq C_{10}D_2 \). Let

\[ q \in \text{Fix}(g^{10}) \]

be the isolated point. Then \( g(q) = q \) and \( g^5 \) fixes 6 points on \( D_2 \). Then \( C_{10}D_2 \leq 6 \) as \( C_{10} \cap D_2 \subset \text{Fix}(g^{25}) \cap \text{Fix}(g^{10}) = \text{Fix}(g^5) \). Thus \( C_{10}D_2 = 6. \) Let \( p_1, \ldots, p_6 \) be the 6 intersection points of \( C_{10} \) and \( D_2 \). Then \( g^5 \) fixes the 7 points, \( p_1, \ldots, p_6 \) and \( q \). This proves (3) and (4). Since \( e(g) = 2 \), the action of \( g \) on \( \{p_1, \ldots, p_6\} \) fixes one and rotates five, proving (5) \( \square \)
Proof of the second statement of Theorem 0.1

Lemma 2.2 plays a key role in the proof. We modify the proof of [9, Section 4]. The quotient

$$X/\langle g^{25} \rangle$$

is a smooth rational surface with Picard number 1, hence is isomorphic to $P^2$. The branch curve

$$B := \bar{C}_{10} \subset P^2$$

is a smooth sextic and the image

$$L := \bar{D}_2 \subset P^2$$

of $D_2$ is a line. Let

$$\bar{p}_i, \bar{q} \in P^2$$

be the images of $p_i$ and $q$. Choose coordinates $x, y, z$ of $P^2$ such that

$$L = (z = 0), \quad \bar{q} = (0, 0, 1).$$

Our automorphism $g$ induces an automorphism $\bar{g}$ of $P^2$. The fixed locus of $\bar{g}^5$ is the image of

$$\text{Fix}(g^5) \cup \text{Fix}(g^{20}) = \text{Fix}(g^{20}) = \text{Fix}(g^{10}),$$

thus

$$\text{Fix}(\bar{g}^5) = L \cup \{ \bar{q} \}.$$ 

The fixed locus of $\bar{g}$ is the image of

$$\text{Fix}(g) \cup \text{Fix}(g^{24}) = \text{Fix}(g^{24}) = \text{Fix}(g^{2}) = \{ p_6, q, q_1, q_2 \}$$

where $q_1$ and $q_2$ are two points of $D_2$ which are interchanged by $g$, thus

$$\text{Fix}(\bar{g}) = \{ \bar{p}_6, \bar{q}, \bar{q}_1 \}.$$ 

We further may assume that

$$\bar{p}_6 = (0, 1, 0), \quad \bar{q}_1 = \bar{q}_2 = (1, 0, 0).$$

From these, we infer that

$$\bar{g}(x, y, z) = (x, \zeta^{20}_{25}y, \zeta^{j}_{25}z)$$

for some $j$. Since $\bar{g}$ has order 25, $5 \nmid j$. The monomials $x^6$ and $xy^5$ are $\bar{g}$-invariant. We know that the branch $B = \bar{C}_{10}$ is a smooth sextic. Thus there must exist a $\bar{g}$-invariant monomial of the form $y^a z^{6-a}$. Then $20a + j(6-a) \equiv 0 \mod 25$. Since $5 \nmid j$, $a = 1$ and $j = 5i + 1$ for some $i$. Then

$$(\zeta^{j}_{25})^{20} = (\zeta^{5i+1}_{25})^{20} = \zeta^{20}_{25},$$

we may assume that $j = 1$. The branch $B$ is defined by $x^6 + \alpha xy^5 + \beta yz^5 = 0$ for some non-zero $\alpha$ and $\beta$. Replacing $y$ and $z$ by a scalar multiple, we may assume that $\alpha = \beta = 1$. Now the surface $X$ and the automorphism $g$ are defined by

$$X : w^2 = x^6 + xy^5 + yz^5$$

$$g(x, y, z, w) = (x, \zeta^{20}_{25}y, \zeta^{25}_{25}z, -w).$$
3. Proof: the Complex Case

We may assume that $X$ is projective, since a non-projective complex K3 surface cannot admit a non-symplectic automorphism of finite order ([11], [10]) and its automorphisms of finite order are symplectic, hence of order $\leq 8$. Now the same proof goes, once $H^2_{et}(X, \mathbb{Q}_l)$ is replaced by $H^2(X, \mathbb{Z})$ and Proposition 1.2 by the usual topological Lefschetz fixed point formula.

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