Parity Problem of the Scotogenic Neutrino Model

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Abstract

We show that Ma’s scotogenic model, which is arguably one of the simplest settings containing a Dark Matter candidate and generating a naturally suppressed active neutrino mass at 1-loop level, suffers from a severe hierarchy-type problem. In case the right-handed neutrinos involved have large enough masses, these may via loop effects drive the mass parameter of the inert scalar contained in the model towards negative values. This behaviour leads to a breaking of the $Z_2$ parity symmetry built into the model which is paramount to keep the setting consistent, because without it the model would lose its Dark Matter candidate and the neutrino mass would not be naturally suppressed. Trying to avoid this consistency problem leads to a new constraint on the model parameter space which has not yet been described in the literature.
1 Introduction

The smallness of neutrino masses and the identity of the Dark Matter (DM) observed in space are among the greatest puzzles of modern elementary particle physics, along with the general question of how to extend our Standard Model (SM) and how to probe this extension at colliders. Among the most attractive candidate theories beyond the SM are those models which can at the same time address several of the known open problems, which is particularly true for models generating a light neutrino mass only at loop level (i.e., radiatively) – see e.g. Ref. [1] for a general study of how such settings can be constrained from both, low and high energy phenomenology. Depending on the particle content, there are models known which generate an active neutrino mass at 1-loop [2], 2-loop [3, 4], or 3-loop [5, 6] level, but probably the simplest extension still compatible with all data is Ma’s scotogenic model [7].

The scotogenic model just adds three right-handed neutrinos and an additional scalar doublet to the SM, all of which are charged under an additional $\mathbb{Z}_2$ parity symmetry. This symmetry is absolutely crucial for the model to work since without it, neutrino masses would already be generated at tree level and none of the possible DM candidates of the model would be stable. However, if the $\mathbb{Z}_2$ is intact, the scotogenic model cannot only account for phenomenologically valid neutrino masses [8–10] and potentially for DM [8, 9, 11–13], too, but it can also lead to a variety of interesting phenomena in low-energy experiments such as lepton flavour and/or number violation [8, 9, 14–17] or in high-energy collider searches [18–21], as well as to new aspects for neutrino model building [10, 22, 23].

Just a few years ago, the first study of the renormalisation group running of the scotogenic model has appeared in Ref. [24], which has shown that running effects in particular of the 1-loop neutrino mass can indeed be large. More generally, the running may have a (“good” or “bad”) influence on the model, e.g., it could happen that radiative corrections change certain properties of the model given at tree level. This is particularly true for the scalar potential, as is well-known from settings similar to the scotogenic model. For example, it has been shown in Ref. [25] that the existence of an inert scalar can trigger electroweak symmetry breaking at 1-loop level, even if it was not present at tree level – a general fact that remains true if the setting is extended [26] or even discretised [27]. Ultimately, all these observations are based on the well-known fact that scalar mass parameters are in general known to be sensitive to the large scales of a theory [28], which is particularly true for models with two Higgs doublets [29].

In this paper, we apply exactly the same logic to the inert doublet, i.e., we investigate corrections to the mass parameter of the additional scalar of the model. This could lead

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1Due to the $\mathbb{Z}_2$ symmetry restricting its interactions, the new scalar doublet is usually called inert.
to a fundamental problem of the setting since, as we will see, the contributions from the heavy right-handed neutrinos can drive the mass parameter of the inert scalar towards negative values. If that was the case, the new scalar would develop a vacuum expectation value (VEV) and by that break the crucial $Z_2$ parity symmetry, which is why we call this observation the parity problem of the scotogenic model. This would be disastrous given that, if the parity was not a conserved global quantum number, not only would the model lose the stability of its DM candidates but it would in fact be entirely pointless because of the light neutrino mass already being generated at tree level, by the VEV of the inert doublet. This observation reveals a serious consistency problem of the scotogenic model which, as we will see, leads to a strong constraint on the allowed parameter space.

Somewhat surprisingly, to our knowledge this observation has not yet been reported, even though it has been pointed out already several years ago that such possibilities could arise in particular in the context of loop-models [30]. While recently even a specific study of the high-scale validity of the scotogenic model had been presented [31], it only treated the question of whether or not the potential is stable, i.e., has a global minimum. However, the simplest consideration of studying in how far the $Z_2$ parity of the model is threatened by radiative effects, has not been performed. We aim to close this gap and will in turn use the resulting consistency constraint to derive a phenomenologically relevant upper bound on the right-handed neutrino mass, in a way somewhat similar to how such a bound is found in settings featuring a tree level seesaw type [32].

This paper is organised as follows. After giving a brief overview of the scotogenic model in Sec. 2 we discuss in Sec. 3 the general possibilities for its possible vacuum configurations. Sec. 4 illustrates the approximate constraints arising from avoiding a violation of the $Z_2$ parity in the scotogenic model at any energy scale, which will lead us to a simple but astonishingly accurate formula that considerably limits the parameter space of the model. A more advanced numerical analysis of three different benchmark scenarios, which yields qualitatively consistent results, is presented in Sec. 5. We end our considerations with a brief discussion of how to exploit the constraint arising from the parity problem in future collider searches or in cosmology, see Sec. 6 before finally concluding in Sec. 7. Technical details such as the relevant renormalisation group equations and their approximate solutions for the decisive limiting cases can be found in Appendices A and B, respectively.

# Model Overview & Constraints

The scotogenic model [7] is certainly one of the simplest frameworks combining a naturally small neutrino mass at 1-loop level with several potential DM candidate particles. The
particle content is basically that of the SM, supplemented by (typically) three right-handed (RH) neutrinos $N_R^i$ (with $i = 1, 2, 3$) and a second scalar doublet called $\eta$ with SM quantum numbers identical to that of the Higgs. However, the crucial addendum is an additional $\mathbb{Z}_2$ (parity) symmetry, under which all SM particles are neutral whereas the new fields carry odd charges. It is this symmetry which simultaneously leads to the light neutrino mass being generated at 1-loop level only and to the stability of the potential DM candidates.

Related to the new fields and to the $\mathbb{Z}_2$ parity, several qualitatively new terms appear in the Lagrangian:

- The RH neutrinos get a direct Majorana mass term $\frac{1}{2}N_R^i M_{ij} N_R^j + h.c.$, which leads to the masses $M_{1,2,3}$ upon diagonalisation.

- A neutrino Yukawa coupling $\mathcal{L}_{\text{Yukawa}} \supset -h_{ij} \bar{N}_R^i \tilde{\eta} \ell_L^j + h.c.$ (where $\tilde{\eta} = i \sigma_2 \eta^*$) involving the new scalar and the RH neutrinos in addition to the SM lepton doublets $\ell_L^j$. It is crucial to observe that this term does not lead to a tree level neutrino mass term, as long as the $\mathbb{Z}_2$ is unbroken and thus prevents the field $\eta$ from obtaining a VEV.

- The full scalar potential involving both the SM Higgs $\phi$ as well as the “inert” Higgs $\eta$ is given by:

$$V = m_1^2 \phi^\dagger \phi + m_2^2 \eta^\dagger \eta + \frac{\lambda_1}{2} (\phi^\dagger \phi)^2 + \frac{\lambda_2}{2} (\eta^\dagger \eta)^2$$

$$+ \lambda_3 (\phi^\dagger \phi) (\eta^\dagger \eta) + \lambda_4 (\phi^\dagger \phi) (\eta^\dagger \eta) + \frac{\lambda_5}{2} ((\eta^\dagger \phi)^2 + h.c.).$$

In this expression both mass parameters $m_{1,2}^2$ must be real, as need to be the couplings $\lambda_{1,2,3,4}$. In turn, $\lambda_5$ can be chosen real (and positive) by absorbing a possible phase into $\eta$.

Note that it is the combination of the Majorana mass term, of the new Yukawa coupling, and of the $\lambda_5$-term in Eq. (1) which violates lepton number. If any of those coefficients was zero, a global $U(1)$ lepton number could be consistently defined and the symmetry of the Lagrangian would be increased. Thus, by virtue of ’t Hooft naturalness [33], the RGEs for those quantities will only allow for changes proportional to the couplings themselves, so that they remain small everywhere if they are small at some energy scale, cf. Appendix [A].

The scalar potential (1) needs to yield electroweak symmetry breaking (EWSB) with-
out compromising the $\mathbb{Z}_2$ parity. This suggests the following parametrisation:

$$\phi = \begin{pmatrix} 0 \\ v + \frac{h}{\sqrt{2}} \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta^+ \\ \eta^0 \end{pmatrix},$$

where only the physical fields are written down explicitly. Splitting the complex scalar field into real and imaginary parts, $\eta^0 = \frac{1}{\sqrt{2}} (\eta_R + i \eta_I)$, gives rise to the following physical scalar masses \[34]:

$$m_h^2 = 2 \lambda_1 v^2,$$

$$m_{\pm}^2 = m_2^2 + v^2 \lambda_3,$$

$$m_R^2 = m_2^2 + v^2 (\lambda_3 + \lambda_4 + \lambda_5),$$

$$m_I^2 = m_2^2 + v^2 (\lambda_3 + \lambda_4 - \lambda_5),$$

like for a general two Higgs doublet model (THDM) \[35,36]. Note that the parameters in the scalar potential are subject to a number of theoretical constraints originating from the requirement that the scalar potential be bounded from below \[35,37,38]:

$$\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_3 > -\sqrt{\lambda_1 \lambda_2}, \quad \lambda_3 + \lambda_4 - |\lambda_5| > -\sqrt{\lambda_1 \lambda_2},$$

which also affects the valid mass spectra resulting from Eqs. (3a) to (3d). In addition, we can impose the condition that we remain in a perturbative regime, that is $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \lesssim \mathcal{O}(1)$. In our numerical analysis, we demand that the above conditions remain valid for the 1-loop corrected running couplings up to the grand unification (GUT) scale, $M_{\text{GUT}} = 10^{16}$ GeV.

The 1-loop renormalisation group equations (RGEs) can be found in Appendix A.

Given that active neutrino masses cannot be generated at tree level, one has to search for a higher order contribution, which necessarily must yield a Majorana mass term due to the lepton number violation (LNV) being present and it thus has to contain all couplings $h_{ij}, M_i$, and $\lambda_5$. The lowest order diagram doing this job is depicted in Fig. 1, which indeed does fulfill all requirements. The resulting light neutrino mass matrix in our conventions\[3]

\[\text{Note that there are some ambiguities in the definition of perturbativity } \text{[39,40]. For definiteness, we have chosen the criterion } \lambda_{1,2,3,4,5} < 4\pi \text{ in our numerical computations.}\]

\[\text{The deviation to } \text{[7] originates from the normalisations of the scalar field } \eta^0 \text{ and of the Yukawa couplings.}\]
Figure 1: One-loop diagram of neutrino mass generation through $\eta^0$-exchage.

is given by \((7)\):

\[
M_{\nu ij} = \sum_{k=1}^{3} \frac{M_k h_{ki} h_{kj}}{32\pi^2} \left\{ \frac{m_R^2}{m_R^2 - M_k^2} \ln \left( \frac{m_R^2}{M_k^2} \right) - \frac{m_I^2}{m_I^2 - M_k^2} \ln \left( \frac{m_I^2}{M_k^2} \right) \right\}, \tag{5}
\]

which is zero in case one of the LNV couplings is zero.

3 Vacuum Structure of the scotogenic model

The scalar sector of the scotogenic model is a particular realisation of a THDM with an additional $\mathbb{Z}_2$ symmetry imposed on the new fields. In a general THDM where both scalar doublets, called $\phi_1$ and $\phi_2$ for the moment to simplify the comparison, may acquire a VEV, we find the tree level vacuum of the theory by minimising Eq. (1). Replacing the fields by their VEVs $v_1$ and $v_2$, which can be chosen real if both electric charge and CP are conserved \([35,36]\), leads to two equations:

\[
v_1 (m_1^2 + \lambda_1 v_1^2 + \lambda v_2^2) = 0 \quad \text{and} \quad v_2 (m_2^2 + \lambda_2 v_2^2 + \lambda v_1^2) = 0, \tag{6}
\]

where we have abbreviated $\lambda \equiv \lambda_3 + \lambda_4 + \lambda_5$.

These equations allow four qualitatively different sets of VEVs\(^4\)

\begin{enumerate}
\item $v_1 = v_2 = 0$,
\item $v_1^2 = -\frac{m_1^2}{\lambda_1}$, \quad $v_2^2 = 0$,
\item $v_1 = v_2$,
\item each have two real solutions, which are physically equivalent due to the global symmetries of the theory. However, (4) gives rise to four solutions, which reduce to two physically \textit{inequivalent} solutions \([41]\).
\end{enumerate}
\( v_1^2 = 0, \quad v_2^2 = -\frac{m_1^2}{\lambda_1^2}, \)
\( v_1^2 = \frac{\lambda_2 m_1^2 - \lambda m_2^2}{\lambda^2 - \lambda_1 \lambda_2}, \quad v_2^2 = \frac{\lambda_1 m_2^2 - \lambda m_1^2}{\lambda^2 - \lambda_1 \lambda_2}. \)

While (1) and (2) respect the \( Z_2 \) parity symmetry, (3) and (4) lead to its spontaneous breaking. In our numerical analysis, we have investigated whether (1) or (2) are stable minima of the potential \( ^5 \) as otherwise the \( Z_2 \) parity will be broken in any case since the potential must have a minimum and only \( Z_2 \)-breaking vacua are left.

The Hessians for (1) and (2) are given by
\[
\frac{\partial^2 V}{\partial v_i \partial v_j} \bigg|_{(1)} = \begin{pmatrix} 2m_1^2 & 0 \\ 0 & 2m_2^2 \end{pmatrix}, \quad \frac{\partial^2 V}{\partial v_i \partial v_j} \bigg|_{(2)} = \begin{pmatrix} -4m_1^2 & 0 \\ 0 & 2m_2^2 - 2\frac{\lambda_1}{\lambda_1} m_1^2 \end{pmatrix}. \tag{7}
\]

Therefore we find that, if \( m_1^2 > 0 \), we have a stable, \( Z_2 \) symmetric vacuum if \( m_2^2 \geq 0 \), while in the case \( m_1^2 < 0 \), we need
\[
m_2^2 \geq \frac{\lambda_1}{\lambda_1} m_1^2. \tag{8}
\]

Note that this equation is equivalent to the condition \( m_R^2 \geq 0 \), i.e. the field \( \eta_R \) will develop a VEV if this condition is violated. Had we allowed for a relative phase between \( v_1 \) and \( v_2 \), we would find that now \( \text{Im} \phi_2 \), corresponding to \( \eta_1 \), may develop a VEV. The condition to exclude this scenario is \( m_2^2 \geq 0 \). Similarly, in order not to break electric charge, we need \( m_{\pm}^2 \geq 0 \).

If we ignore a possible instability of the \( Z_2 \)-symmetric vacua, we may expand the theory around the wrong vacuum. Expanding around the correct vacuum however, will alter the phenomenological predictions of the model, and destroy the \( Z_2 \) symmetry.

### 4 Constraints due to radiative symmetry breaking

For illustrative purposes, we now show that a lower bound on the mass parameter \( m_2 \) can be obtained from demanding that the \( Z_2 \) symmetry be unbroken up to the GUT scale in a simplified model: we only consider one generation of fermions and assume \( m_1^2(\mu) > 0 \) for all \( \mu \), i.e. we study under which conditions only \( m_2^2(\mu) \) becomes negative. Assuming sufficiently small quartic scalar couplings, we ignore their contributions to the RGEs altogether and consider two simple but illustrative limiting cases, where all quantities under

\(^5\)Of course this does not exclude the case that the \( Z_2 \)-symmetric vacua are not the global minima of the potential. We do not consider this additional subtlety since there is some freedom in choosing such local minima if their decay time is larger than the age of the Universe, see e.g. \([41]\).
consideration are assumed to be real. In addition, we shall assume that the Majorana mass $M$ does not run at all, which is justified due to the suppression factor of $(16\pi^2)^{-1}$ in the RGE, cf. Eq. (A-5) and Ref. [24].

Case 1: Large neutrino Yukawa coupling ($h(\mu) \gg g_i(\mu)$)

In this limiting case, neglecting any gauge and also all non-neutrino Yukawa couplings, we can approximate the coupled RGEs (A-4b) and (A-6b) for $h$ and $m_2^2$, respectively, as follows:

$$Dh \simeq \frac{5}{2} h^3, \quad Dm_2^2 \simeq 2h^2m_2^2 - 4h^2M^2,$$

where $D \equiv 16\pi^2\mu \frac{d}{d\mu}$ and we have suppressed explicit scale-dependences for brevity. These differential equations can be solved exactly (see Appendix B):

$$h(t) = \frac{h(0)}{\sqrt{1 - \frac{5}{16\pi^2} h^2(0)t}}, \quad m_2^2(t) = 2M^2 + \frac{m_2^2(0) - 2M^2}{\left(1 - \frac{5}{16\pi^2} h^2(0)t\right)^{2/5}},$$

with $t \equiv \ln(\mu/\mu_0)$ where $\mu_0$ is a reference scale.

From this, we get a condition for the mass squared to become negative, i.e. for symmetry breaking to occur at some point $t = t_*$,

$$m_2^2(t_*) = 0 \iff \left(\frac{5}{16\pi^2} h^2(0)\right) t_* = 1 - \left(1 - \frac{m_2^2(0)}{2M^2}\right)^{5/2},$$

which yields in the linear approximation (where $m_2^2(0) \ll 2M^2$):

$$t_* \simeq \frac{4\pi^2 m_2^2(0)}{M^2 h^2(0)}.$$

Case 2: Small neutrino Yukawa coupling ($h(\mu) \ll g_i(\mu)$)

This case involves the gauge coupling RGEs (A-1), which in this limit look like:

$$Dh \simeq -\frac{3}{4} \left(g_1^2 + 3g_2^2\right) h, \quad Dm_2^2 \simeq -\frac{3}{2} \left(g_1^2 + 3g_2^2\right) m_2^2 - 4h^2M^2.$$

The solutions are simple to find (see Appendix B for more details):

$$h(t) = h(0)\sqrt{f(t)}, \quad m_2^2(t) = \left[m_2^2(0) - \frac{M^2 h^2(0)}{4\pi^2} t\right] f(t),$$

7
where we have defined the auxiliary function

\[ f(t) \equiv \exp \left( -\frac{3}{2} \int_0^t \frac{g_1^2(t') + 3g_2^2(t')}{16\pi^2} \, dt' \right). \]  

(15)

For \( m_2^2(t_*) \uparrow 0 \), we identify

\[ t_* = \frac{4\pi^2 m_2^2(0)}{M^2 h^2(0)}. \]  

(16)

Note that the conditions for both cases agree if \( m_2^2(0) \ll 2M^2 \), which is just the case we are interested in. If we now impose that the crossing to negative values of \( m_2^2 \) shall not occur until \( M_{\text{GUT}} \), where new physics may kick in to cancel this dangerous behaviour, we find for \( \mu_0 = M_Z \):

\[ m_2(0) \geq \sqrt{\frac{\ln (10^{16} \, \text{GeV}/M_Z)}{4\pi^2}} h(0) M \approx 0.905 M h(0). \]  

(17)

This example illustrates that, in the case of only one generation, we get a lower bound on the low scale input value of the inert mass parameter. Since the quartic couplings are themselves constrained and we hence cannot tune them freely to obtain small physical scalar masses in Eqs. (3b) to (3d), we may suspect that the physical masses themselves become constrained. To exemplify this will be the purpose of the following section.

5 Numerical Analysis

We now turn to the numerical analysis of the full system of RGEs for all three generations. Since many of the parameters of the scotogenic model are essentially unconstrained, we choose for some of them sensible input values. While it has to be emphasised that the results shown in this section depend on this choice, the crucial point is the qualitative statement that there is a non-trivial bound from avoiding the \( Z_2 \) breaking by radiative corrections. The main message is that, in an explicit study, one needs to take this bound into account for any given choice of parameters – as otherwise one would be in danger to consider a point in the parameter space which suffers from an internal inconsistency. The goal of this section is to illustrate that keeping the \( Z_2 \) intact at all energy scales invalidates points which would otherwise be allowed by all other constraints.
5.1 General Strategy

The model parameters “beyond the SM” are

\[ \{h_{ij}, M_k, m_2, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}, \]  

some of which are constrained by neutrino oscillation experiments and possibly by cosmological observations. In spite of that, we chose to treat these numbers as free parameters just to keep the computation time under control. Since a particular leptonic mixing does not really touch our argumentation, we can neglect this point for the purpose of illustrating the parity problem of the scotogenic model.

We fix \( \lambda_5 = 10^{-9} \) at the input scale (for which we use the electroweak scale, i.e., the \( Z \)-pole at \( \mu = M_Z \)), whose value is kept small by the corresponding RGE (A-7e). The conditions (4) also require that we use large enough values for \( \lambda_2 \), such that it is positive for all energy scales. We find that, at the electroweak scale, \( \lambda_2 = 0.1 \) gives good results while the computation time can be limited.

For the neutrino Yukawa couplings \( h_{ij} \), we input a bimaximally mixed setup [42] at the GUT scale, which is known to potentially yield phenomenologically valid leptonic mixing at low energies [24]. To make sensible choices for the Majorana masses, we must first understand the relation between the two scalar mass parameters, \( m_2^1 \) and \( m_2^2 \).

The connection between the different mass parameters in the model is intricate: at the input scale, we must choose \( m_2^2 \) large enough in order for it not to be driven to negative values by the corrections originating from the Majorana masses, cf. Eq. (A-6b), as we increase the renormalisation scale. At the same time, a large \( m_2^2 \) will generate large positive corrections to \( m_2^1 \) (provided that \( 2\lambda_3 + \lambda_4 > 0 \), see Eq. (A-6a)), which could even spoil EWSB. This is where the actual tension resides: the running mass parameter \( m_2^2 \) must lie within the range of the electroweak scale and at the same time it must not be too small to avoid \( \mathbb{Z}_2 \) symmetry breaking.

However, there is one more player in the game: the RH neutrino Majorana masses, which drive this tension by their appearance in the RGE for \( m_2^2 \), Eq. (A-6b). If chosen too heavy (we will see that 10 TeV is sufficiently large), demanding an unbroken \( \mathbb{Z}_2 \) symmetry at all scales requires \( m_2^2(M_Z) \) to be larger than allowed by the obligation to achieve EWSB.

For illustrative purposes, we present three qualitatively different scans with the fol-
lowing input masses, where all off-diagonal elements vanish:

\begin{align*}
\text{a)} & \quad \{M_1, M_2, M_3\} (M_{\text{GUT}}) = \{900, 1500, 2000\} \text{ GeV}, \\
\text{b)} & \quad \{M_1, M_2, M_3\} (M_{\text{GUT}}) = \{2000, 5000, 10000\} \text{ GeV}, \\
\text{c)} & \quad \{M_1, M_2, M_3\} (M_{\text{GUT}}) = \{20, 200, 2000\} \text{ GeV}.
\end{align*}

Each setup captures a qualitatively different behaviour:

\begin{itemize}
  \item[a)] There is essentially no mass hierarchy – all masses lie around the TeV scale. This will already be sufficient to constrain the mass parameter \(m_2^2\) from below.
  
  \item[b)] This setting represents a mild hierarchy between the RH neutrinos, which creates a too strong tension between the two requirements of enabling EWSB and keeping the \(\mathbb{Z}_2\) intact.
  
  \item[c)] Here the hierarchy between the RH neutrino masses is strong and we include a relatively light RH neutrino. In such a scenario no global constraints on \(m_2^2\) arise.
\end{itemize}

For each input scenario a) – c) we have generated a random set of \(10^5\) input values within

\[0 \text{ TeV} \leq m_2(M_Z) \leq 3 \text{ TeV}, \quad -1 \leq \lambda_3(M_Z), \lambda_4(M_Z) \leq 1.\]  

We then calculate the running. Eqs. (A-1) to (A-5) can be solved at once, since they do not depend on the choice of scalar input parameters. The SM input values have been chosen according to [43]. We then solve the RGEs for the running couplings of the scalar sector \((\lambda_1, 2, 3, 4, 5)\), Eqs. (A-7a) to (A-7c), and check the consistency criteria for all energy scales between \(M_Z\) and \(M_{\text{GUT}}\). If these are not violated for any scale below the GUT scale, we solve the scalar mass parameter RGEs (A-6a) and (A-6b). If we do not find EWSB below 1 TeV, i.e. if \(m_1^2(\mu) \geq 0\), we reject the input values. Otherwise we distinguish two subcases depending on the sign of \(m_1^2(\mu)\), see the discussion below Eq. (7) which is considered for the running couplings now. In case we find that all vacua that respect the \(\mathbb{Z}_2\) parity are unstable, we reject the input values, too. By adhering to this approach, we can avoid rejecting input values because of a broken \(\mathbb{Z}_2\) symmetry although some other criterion fails as well, i.e., we determine exactly those points which would not be rejected if one did not know about the bound arising from radiative symmetry breaking.

\footnote{The \(M_i\) are fixed at the GUT scale, which however has no influence due to their weak running [24].}

\footnote{In this way, we can ensure that we do not have to consider further particle thresholds and all SM fermions can be safely neglected. This even holds for the top quark given that its mass is so close to the lower input scale.}
There is one subtlety involved: since we are working in the MS scheme, the decoupling of heavy degrees of freedom is not manifest. If we want to vary the renormalisation scale down to small scales, we must integrate out the heavy fields in case we wish to lower the renormalisation scale below their mass thresholds \cite{[44]}. We have implemented the decoupling of the heavy Majorana fields by hand. At any threshold $\mu_\ast = M_i(\mu_\ast)$, we have to diagonalise the RH neutrino mass matrix, remove the row and column corresponding to the largest eigenvalue, and resume the running to lower scales in the corresponding effective field theory \cite{[24]}. Similarly, in the neutrino Yukawa matrix $h$, we must remove the row corresponding to couplings of the heaviest Majorana field to lighter fields and match the remaining entries at the threshold. This matching must be carried out in the same basis where the Majorana mass matrix is diagonal \cite{[24],[45]}, in order to clearly identify which entries to neglect at lower energies.

Since the leading effective operator, the so-called Weinberg operator \cite{[46]}, does not give any contributions to the scalar or lepton self-energies due to hypercharge conservation and (at this order) vanishing lepton mass, the situation is simpler for the remaining couplings: here it suffices to remove a given RH neutrino contribution from an RGE at the threshold scale $\mu_\ast = M_i(\mu_\ast)$, which can be technically done by a simple $\theta$-function. Since higher-order effective operators are at least suppressed by two inverse powers of the heavy Majorana masses, this is in fact a very good approximation.

Note that, since we are interested in the physical masses of the scalar particles, we do not integrate them out – which would be a much more involved task. Besides, the effect on physics below this mass scale can be assumed to be small if we do not lower $\mu$ by many orders of magnitude.

Finally, one might object that having two input scales ($M_Z$ for the SM couplings and $M_{\text{GUT}}$ for neutrino Yukawa couplings and Majorana masses) must introduce errors upon integrating out fields. This is in principle true, since we start at the high scale and upon matching at the first threshold, we get a new prediction for the SM couplings, which may very well deviate from the previously chosen input values. However, in practice this effect is negligible, because the dominant contribution to all quantities is the top quark Yukawa coupling and its RGE (A-3a) is only influenced very indirectly by the matching, by the (very small) lepton Yukawa coupling. For consistency, we have compared our results to those obtained by only considering the top quark (so that it decouples from the matching) and find no deviation.

As mentioned in our discussion of the vacuum structure of a THDM, the inert doublet acquiring a VEV may not only lead to breaking of $Z_2$, but it could also cause electric charge and/or CP violation. Note however that, since $\lambda_5$ is small for all renormalisation scales, we will have (to a high accuracy) degenerate CP-even and -odd neutral scalar masses:
\( m_R \approx m_1 \), such that CP violation is inseparable from the breaking of \( \mathbb{Z}_2 \). To also exclude such scenarios, we have made use of the equivalence of a squared mass becoming negative to the corresponding symmetry being broken (see Sec. 3). This has been investigated by replacing the Lagrangian parameters in the mass relations in Eqs. (3b) to (3d) by the running couplings and the tree level VEV by a running VEV (cf., e.g., Ref. 47):

\[
v^2 = -\frac{m_1^2}{\lambda_1} \rightarrow v^2(\mu) = -\frac{m_1^2(\mu)}{\lambda_1(\mu)},
\]

where \( m_1(\mu) \) and \( \lambda_1(\mu) \) are the running 1-loop corrected mass parameter and quartic scalar coupling, respectively. This can be done because, in the broken phase, we only need counterterms that are invariant under the spontaneously broken symmetry group [48]. This means that we can obtain the counterterms in the broken phase from those in the symmetric phase. We can use this fact to construct running quantities in one phase from running quantities in the other [49].

Before we turn to the results of our numerical analysis, we would like to address one last theoretical aspect. In defining the “running VEV” in Eq. (21), we have argued that this is equivalent to a broken-phase calculation. In such calculations, one usually chooses a renormalised VEV such that the tadpole graphs vanish at a given order in perturbation theory, meaning that the shifted field \( h \) in Eq. (2) acquires no VEV itself [50]. Since we have adhered to the MS scheme, this is not true and the tadpoles would be non-zero (and divergent), had we calculated them in the broken phase. As shown in [51], the correct tadpole contributions can be derived from an effective Coleman-Weinberg (CW) potential [52], meaning that the three approaches are equivalent. This is not the full picture, however, since there exist cases, such as massless scalar QED, where there simply are no tadpoles at 1-loop level. In such cases, the CW potential can point towards minima of the potential, which are not present at tree level. Since in the present work, we are only interested in the stability of the existing vacua, we shall not pursue an effective potential approach – which could however reveal the existence of further minima of the potential not present at tree level.

For consistency, we have investigated how our results would change if we included the CW correction to the scalar potential and we have found that it generates only small corrections to the running masses. For a possible treatment of this issue see for example references [53, 54], which treat radiative symmetry breaking in the minimal supersymmetric Standard Model (MSSM) but otherwise resemble our situation qualitatively (apart from

\[Similarly, the \( \beta \)-function of the mass parameter (which is set to zero in the massless case) is proportional to the mass parameter itself so that, if it is zero at one particular energy scale, it will remain zero at any scale.\]
from, of course, the tendency of the running being just opposite to our case, since the heavy super-partners of the quarks and charged leptons are bosons rather than fermions).

5.2 Results

Let us start with the results of the scan for case a), with three Majorana fermions of very similar masses. Fig. 2 shows the allowed and rejected input values at the electroweak scale. Here, all data points violating vacuum stability or perturbativity have already been excluded from the plot. The colour code is as follows:

- **black dots**: These are the points which fulfill all the constraints. All other points are excluded for various reasons.

- **yellow dots**: These points are excluded due to failing to produce EWSB below 1 TeV.

- **red/green dots**: These points are excluded only because they would lead to radiative breaking of the $Z_2$ symmetry, where $m_R^2 < 0$ is signified by red dots and $m_\pm^2 < 0$ by green ones respectively. Thus, if not taking into account this constraint, we would erroneously classify these data points as being valid.

For a large range of input values for $\lambda_3$ and $\lambda_4$, the criterion that we must encounter EWSB somewhere below 1 TeV translates into an upper bound on $m_2(M_Z)$, which is a function of $\lambda_3$ and $\lambda_4$. The reason is that, as we have already discussed at the beginning of this section, too large values of $m_2^2$ may cause $m_1^2$ to become positive below 1 TeV. Only if $2\lambda_3 + \lambda_4 < 0$, EWSB can be maintained below 1 TeV while allowing for a large $m_2$. This manifests itself in the narrow columns of valid (black) data points ranging up to very high input values of $m_2$, which are visible in both panels of Fig. 2.

On the other hand, one can see that including the criterion of an unbroken $Z_2$ symmetry is essentially equivalent to a lower bound on $m_2(M_Z)$ – in our example about 550 GeV, indicated by the gray area in the plots. As can be seen particularly well in the left Fig. 2, we have indeed found red and green points below this bound, i.e., if we did not impose the consistency requirement of the $Z_2$ symmetry to be unbroken at all scales, we would erroneously consider these points to be phenomenologically valid. Even above this bound it is visible that, for certain choices of $\lambda_{3,4}$, there are some red and green dots whose failure we can only detect by taking into account the parity.

For the physical scalar masses in the MS scheme, we obtain values in the ranges

$$554.2 \text{ GeV} \leq m_\pm(\mu = m_\pm) \leq 2780.6 \text{ GeV},$$

$$558.5 \text{ GeV} \leq m_{R/I}(\mu = m_{R/I}) \leq 2781.6 \text{ GeV},$$

(22)
clearly confirming our expectation of a lower bound on the physical masses.

Generalising our naïve estimate from Eq. (17) to the case of three generations, we find good agreement with the exact result

$$0.905 \sqrt{\max_{i} |M_{i}^{2} \left[h(M_{Z})h(M_{Z})\right]|_{ii}} \approx 625 \text{ GeV},$$

(23)

to be compared to the estimated value of 550 GeV. It looks nearly surprising that the simple estimates outlined in Sec. 4 leads to such a good agreement. However, given that there is quite a range possible for both Yukawa couplings and RH neutrino masses, it is in fact to be expected that, rather generically, one of the products of the form (neutrino Yukawa coupling) × (RH neutrino mass) will in most cases dominate over the others, thereby effectively mimicking the situation of only one RH neutrino being present.

Glancing at Eq. (A-6b), one might be led to the assumption that, by raising the Majorana masses, the transition to negative values of $m_{2}^{2}$ can be pushed beyond $M_{\text{GUT}}$, since below their mass thresholds the RH neutrino fields are integrated out. However such an attempt must fail since it is always overcompensated by the quadratic term in the RGE for $m_{2}^{2}$, Eq. (A-6b). Only if we chose all $M_{i} \geq M_{\text{GUT}}$, this could be achieved, but at latest at that point some other new physics would probably appear which may completely change the situation. Scenario b) with a mild hierarchy between the RH neutrino masses, whose outcome is shown in Fig. 3 illustrates this point very well: for this case we have used input values of $m_{2}(M_{Z})$ up to 8000 GeV. One can see that out of the huge number of points only very few are valid (i.e., black), which are situated around $(\lambda_{3}, \lambda_{4}, m_{2}) \approx (0.15, -0.26, 2400 \text{ GeV})$ – note that $2\lambda_{3} + \lambda_{4} < 0$. This clearly
Figure 3: Scenario b) Parameter scan for Majorana masses up to 10 TeV. The resulting valid parameter space is very limited, see the few scattered black points. The strong tension for scenario b) would be missed if the parity problem was not taken into account.

illustrates that the tension between having a broken electroweak phase below 1 TeV and an unbroken $Z_2$ symmetry, as discussed above, is too large for the validity of the model to be maintained for the bulk of the parameter space. This reveals how remarkably difficult it can be in reality to find valid parameter points for such a simple model. In addition, for case b), we obtain no physical scalar mass below 2 TeV, which is very undesireable from a phenomenological point of view. This is in agreement with the naïve estimate, which predicts $m_2(M_Z) \gtrsim 2100$ GeV.

In the input scenario c), we find in Fig. 4 that again a similar picture as in case a) arises, but this time with a lower bound of $m_2(M_Z) \gtrsim 200$ GeV. While this is true for a large part of the $\lambda_3 - \lambda_4$ plane, there exist a few parameter constellations that allow for $m_2(M_Z) < 200$ GeV. Again this can be understood from the RGEs (A-6a) and (A-6b), which show that a small $m_2$ will keep $m_1^2$ at negative values which vary only very little, and we have to fulfill the broken phase condition (8) in order not to spontaneously break the $Z_2$. As it turns out, $\lambda_1$ is quickly running towards smaller, positive values as $\mu$ grows (i.e. the negative contribution from the top quark Yukawa coupling dominates), such that Eq. (8) is never violated. This time, the physical scalar masses lie in the ranges

$$41.2 \text{ GeV} \leq m_\pm(\mu = m_\pm) \leq 2762.7 \text{ GeV},$$

$$37.6 \text{ GeV} \leq m_{R/I}(\mu = m_{R/I}) \leq 2763.2 \text{ GeV},$$

and are less constrained than before. In this case, the naïve estimate from Sec. 4 Eq. (17), yields

$$0.905 \sqrt{\max_i |M_i^2 (h(M_Z)h^\dagger(M_Z))_{ii}|} \approx 225 \text{ GeV},$$

$$15$$
which does not capture the full picture, in which this bound may be circumvented.

6 Implications for collider physics and cosmology

Before we conclude, let us briefly mention a few potential applications of our considerations both in near future collider detections and in cosmology.

Suppose that, within the next few years, a further (seemingly fundamental) scalar doublet is detected at LHC with a mass of, say, 500 GeV, and without measurable couplings to quarks. Such a scalar could be produced by its couplings to ordinary gauge bosons [18], even if inert.

In the case of the scotogenic model, Eqs. (3c) and (3d) would then immediately imply that the scalar mass parameter $m_2$ should also be roughly of the order of 500 GeV, in case the couplings $\lambda_{3,4,5}$ are so small that the mass parameter basically determines the physical inert scalar mass. Also in case $\lambda_{3,4}$ were positive ($\lambda_5$ is expected to be small anyway), an upper bound on $m_2$ would be implied. This relatively generic conclusion would immediately allow us to use Eq. (17) to estimate the value of $hM$ in case we want $m_2^2$ not be driven negative at any energy scale. Thus, for Yukawa couplings of $h \sim 1$ ($h \sim 0.1$, $h \sim 0.01$), we would expect one of the RH neutrinos in the scotogenic model to at most have a mass of roughly around 550 GeV (5.5 TeV, 55 TeV), i.e., we would obtain an educated guess on the rough mass region of one of the RH neutrinos of the model.

Obviously this conclusion requires knowledge on the Yukawa couplings, but they are
quite generally constrained by both neutrino physics and other low energy phenomenology. Having more information on the scalar potential would in addition help us to refine this rough upper bound so that, with a bit of luck and a sufficient amount of information from complementary sectors, we may be able to derive solid hints on where to look for signs of the scotogenic model at a collider.

Furthermore, even if the mass bounds of for the RH neutrinos might potentially be too large to be observed in ground-based experiments, they might have cosmological impact and e.g. influence leptogenesis within the scotogenic model [55–57]. This is known to impose a non-trivial constraint for tree level seesaw-type settings [32], and qualitatively the same holds for the scotogenic model (taking into account the constraints discussed in Secs. [4] and [5]).

Considering the DM side of the scotogenic model, the parity problem quite generally implies that neither the scalar mass nor the RH neutrino mass should be too large, cf. the discussion in Sec. [5.1]. This could in particular hint towards DM consisting of comparatively light sterile neutrinos, which is known to work very well (see e.g. Refs. [58–71] for suitable production mechanisms). While such settings are discussed from a phenomenological point of view in the context of the scotogenic model [8, 13], it should however be noted that it is non-trivial to motivate such scenarios within the model from a theoretical point of view, i.e., it is not easy to find a convincing explanation for why the three RH neutrinos should be light in this particular framework [72].

### 7 Conclusions

We have illustrated that the scotogenic neutrino mass model suffers from a parity problem, i.e., it is in danger that its intrinsic built-in $\mathbb{Z}_2$ parity symmetry, which is absolutely crucial for the model to yield viable active neutrino masses and possibly a Dark Matter candidate, is broken by quantum effects driven by the heavy right-handed neutrinos involved. This issue imposes a visible constraint on the parameter space available, in particular because the most generic solution, i.e., simply pushing the corresponding mass parameter in the Lagrangian to large enough values to avoid the breaking, does not work due to electroweak symmetry breaking being threatened. Thus, the scotogenic model suffers from tension from two different sides which considerably shrinks the allowed parameter ranges.

After introducing the scotogenic model and its general vacuum structure, we have shown both analytically and by an exact numerical evolution that non-trivial constraints arise from trying to avoid the parity problem. In particular we have shown that there exist regions in parameter space which are consistent with all other constraints and which would hence be erroneously regarded as valid if the parity problem was not taken into
consideration. However, the fact that quantum corrections are unavoidable makes it necessary to ensure for any parameter point that the $\mathbb{Z}_2$ symmetry is not broken so that the consistency of the model is not spoiled. We have derived a resulting approximate relation between the inert scalar and right-handed neutrino masses, which can even have implications for future collider searches or for cosmology. Our considerations are based on the 1-loop renormalisation group equations of the scotogenic model which we have re-derived and, in passing, updated compared to previous versions found in the literature.

Summing up, we have revealed a somewhat subtle but non-trivial constraint on the scotogenic model which however is able to strongly reduce the allowed parameter space. This makes it necessary for future considerations to check whether the parity problem exists for a certain choice of parameters, or not, in order to avoid the trap of studying physically irrelevant regions of the model.

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A Renormalisation group equations of the scotogenic model

The 1-loop RGEs for the scotogenic model have first been computed in Ref. [24]. We have re-derived those equations needed for the purpose of this paper, and we have in passing taken the opportunity to update part of the earlier results.

For convenience, we define the differential operator $\mathcal{D} \equiv 16\pi^2\mu \frac{d}{d\mu}$. The 1-loop RGEs for the gauge couplings are those of a generic THDM [73]:

$$\mathcal{D}g_i = b_i g_i^3 \text{ (no sum!)}, \quad (A-1)$$

with $b = (7, -3, -7)$. These equations can be solved exactly by

$$g_i(t) = \frac{g_i(0)}{\sqrt{1 - \frac{b_i}{8\pi^2}g_i(0)^2t}}, \quad (A-2)$$

where $t \equiv \ln(\mu/\mu_0)$, where $\mu_0$ is a reference scale, such that $\frac{d}{d\mu} = \frac{d}{dt}$. 

18
The quark sector of the scotogenic model is the same as that of the SM, such that the corresponding RGEs do not change:

\[ \mathcal{D} Y_u = Y_u \left\{ \frac{3}{2} Y_u^\dagger Y_u - \frac{3}{2} Y_d^\dagger Y_d + T - \frac{17}{12} g_1^2 - \frac{9}{4} g_2^2 - 8 g_3^2 \right\}, \quad (A-3a) \]

\[ \mathcal{D} Y_d = Y_d \left\{ \frac{3}{2} Y_d^\dagger Y_d - \frac{3}{2} Y_u^\dagger Y_u + T - \frac{5}{12} g_1^2 - \frac{9}{4} g_2^2 - 8 g_3^2 \right\}. \quad (A-3b) \]

The RGEs for the Yukawa couplings of the lepton sector are given by:

\[ \mathcal{D} Y_e = Y_e \left\{ \frac{3}{2} Y_e^\dagger Y_e + \frac{3}{2} Y_e^\dagger Y_e + T - \frac{15}{4} g_1^2 - \frac{9}{4} g_2^2 \right\}, \quad (A-4a) \]

\[ \mathcal{D} h = h \left\{ \frac{3}{2} h^\dagger h + \frac{3}{2} Y_e^\dagger Y_e + T_\nu - \frac{3}{4} g_1^2 - \frac{9}{4} g_2^2 \right\}, \quad (A-4b) \]

where \( T_\nu \equiv \text{Tr}(h^\dagger h) \) and \( T \equiv \text{Tr} \left( Y_e^\dagger Y_e + 3 Y_u^\dagger Y_u + 3 Y_d^\dagger Y_d \right) \). For the Majorana mass matrix, one finds \[24, 45\]:

\[ \mathcal{D} M = \left\{ (h h^\dagger) M + M (h h^\dagger)^* \right\}. \quad (A-5) \]

The scalar mass parameters obey the following RGEs:

\[ \mathcal{D} m_1^2 = 6 \lambda_1 m_1^2 + 2 (2 \lambda_3 + \lambda_4) m_2^2 + m_1^2 \left[ 2 T - \frac{3}{2} (g_1^2 + 3 g_2^3) \right], \quad (A-6a) \]

\[ \mathcal{D} m_2^2 = 6 \lambda_2 m_2^2 + 2 (2 \lambda_3 + \lambda_4) m_1^2 + m_2^2 \left[ 2 T_\nu - \frac{3}{2} (g_1^2 + 3 g_2^3) \right] - 4 \sum_{i=1}^3 M_i^2 (h h^\dagger)_{ii}, \quad (A-6b) \]

where the last term in Eq. (A-6b) is characteristic for a scalar field coupled to Majorana fermions (see, e.g., Ref. \[30, 74\]). This term is in fact the crucial point of our study. Conveniently, since it is nothing but a trace, it is invariant under the transformation that diagonalises \( M \) \[45\], such that we do not have to perform this diagonalisation explicitly. Furthermore, the decoupling of the heavy Majorana fields must be carried out by hand, since we are working in a mass-independent renormalisation scheme. To this end, we match the neutrino Yukawa couplings and Majorana mass matrices as described in Sec. 5. Apart from these two quantities, the only appearance of the Majorana masses

\[ ^9 \text{Note the implicit changes in } g_{1,2}, \text{ though, by virtue of Eq. (A-1).} \]
is Eq. (A-6b), where it suffices to remove the corresponding contribution by a suitable $\theta$-function (see the discussion in Sec. 5.1 for more details).

Finally, we obtain for the quartic scalar couplings, in accordance with Ref. [75]:

\[
D\lambda_1 = 12\lambda_1^2 + 4\lambda_3^2 + 4\lambda_3\lambda_4 + 2\lambda_4^2 + 2\lambda_5^2 + \frac{3}{4} \left( g_1^4 + 2g_2^2g_2^2 + 3g_3^4 \right) \\
- 3\lambda_1 \left( g_1^2 + 3g_2^2 \right) + 4\lambda_1 T - 4T_4,
\]

(A-7a)

\[
D\lambda_2 = 12\lambda_2^2 + 4\lambda_3^2 + 4\lambda_3\lambda_4 + 2\lambda_4^2 + 2\lambda_5^2 + \frac{3}{4} \left( g_1^4 + 2g_2^2g_2^2 + 3g_3^4 \right) \\
- 3\lambda_2 \left( g_1^2 + 3g_2^2 \right) + 4\lambda_1 T_{\nu} - 4T_{4\nu},
\]

(A-7b)

\[
D\lambda_3 = 2(\lambda_1 + \lambda_2)(3\lambda_3 + \lambda_4) + 4\lambda_2^2 + 2\lambda_4^2 + 2\lambda_5^2 + \frac{3}{4} \left( g_1^4 - 2g_2^2g_2^2 + 3g_3^4 \right) \\
- 3\lambda_3 \left( g_1^2 + 3g_2^2 \right) + 2\lambda_3 \left( T + T_{\nu} \right) - 4T_{\nu e},
\]

(A-7c)

\[
D\lambda_4 = 2(\lambda_1 + \lambda_2)\lambda_4 + 8\lambda_3\lambda_4 + 4\lambda_4^2 + 8\lambda_5^2 + 3g_2^2 \\
- 3\lambda_4 \left( g_1^2 + 3g_2^2 \right) + 2\lambda_3 \left( T + T_{\nu} \right) + 4T_{\nu e},
\]

(A-7d)

\[
D\lambda_5 = \lambda_5[2(\lambda_1 + \lambda_2) + 8\lambda_3 + 12\lambda_4 - 3(\lambda_1^2 + 3g_2^2) + 2( T + T_{\nu})],
\]

(A-7e)

where we have used the abbreviations $T_4 \equiv \text{Tr} \left( Y_e^\dagger Y_e Y_e^\dagger Y_e + 3Y_u^\dagger Y_u Y_u^\dagger Y_u + 3Y_d^\dagger Y_d Y_d^\dagger Y_d \right)$, $T_{4\nu} \equiv \text{Tr} \left( h^\dagger h h^\dagger h \right)$ and $T_{\nu e} \equiv \text{Tr} \left( h^\dagger Y_e Y_e^\dagger \right)$.

As to be expected, glancing at Eqs. (A-4b), (A-5), and (A-7e) we can see that the changes in all LNV couplings are proportional to the couplings themselves. This in fact provides a motivation for all the corresponding quantities to be “small”.

**B Solutions to the RG equations in the two limiting cases**

In the case of a large Yukawa coupling, where the approximate RGEs are given by Eq. (9), we immediately find $h(t) = \frac{h(0)}{\sqrt{1 - \frac{5}{16\pi^2} h^2(0) t}}$. The equation for $y(t) \equiv m_2^2(t)$ can be tackled by first finding a general solution to the homogeneous problem,

\[
y_0'(t) = \frac{h^2(0)}{8\pi^2} \frac{1}{\frac{5}{2}h^2(0) t} y_0(t) \quad \Rightarrow \quad y_0(t) = \frac{C_1}{\left(1 - \frac{5}{16\pi^2} h^2(0) t\right)^{2/5}},
\]

(B-1)
where $C_1$ is an integration constant. For the inhomogeneous problem, a particular solution is given by $y_i(t) = 2M^2$. Putting both solutions together and observing that $y(0) = C_1 + 2M^2$, we obtain

$$m_2^2(t)_{\text{large } h} = 2M^2 + \frac{m_2^2(0) - 2M^2}{(1 - \frac{5}{16\pi^2} h^2(0)t)^{2/5}}. \quad (B-2)$$

In the second case, where the Yukawa coupling is small, the solution for $h$ by given by $h(t) = h(0)\sqrt{f(t)}$, cf. Eqs. (14) and (15), which can be readily verified by plugging it into Eq. (13). The homogeneous part of the RGE for $y$ is solved by $y(t) = C_2 f(t)$. We find a particular solution of the inhomogeneous equation by varying the constant $C_2 \to C(t)$, which results into an equation for $C$:

$$16\pi^2 C'(t) = -4M^2 h^2(0) \quad \Rightarrow \quad C(t) = C_3 - \frac{M^2 h^2(0)}{4\pi^2} t. \quad (B-3)$$

Again putting both pieces together, we arrive at Eq. (14):

$$m_2^2(t)_{\text{small } h} = \left[m_2^2(0) - \frac{M^2 h^2(0)}{4\pi^2} t\right] \exp\left(-\frac{3}{2} \int_0^t \frac{g_1^2(t') + 3g_2^2(t')}{16\pi^2} \, dt'\right). \quad (B-4)$$

**References**

[1] S. F. King, A. Merle, and L. Panizzi, JHEP 1411, 124 (2014), [1406.4137](http://arxiv.org/abs/1406.4137).

[2] A. Zee, Phys. Lett. B93, 389 (1980).

[3] A. Zee, Nucl. Phys. B264, 99 (1986).

[4] K. S. Babu, Phys. Lett. B203, 132 (1988).

[5] M. Gustafsson, J. M. No, and M. A. Rivera, Phys. Rev. Lett. 110(21), 211802 (2013), [1212.4806](http://arxiv.org/abs/1212.4806).

[6] M. Gustafsson, J. M. No, and M. A. Rivera, Phys. Rev. D90(1), 013012 (2014), [1402.0515](http://arxiv.org/abs/1402.0515).

[7] E. Ma, Phys. Rev. D73, 077301 (2006), [hep-ph/0601225](http://arxiv.org/abs/hep-ph/0601225).

[8] D. Aristizabal Sierra, J. Kubo, D. Restrepo, D. Suematsu, and O. Zapata, Phys. Rev. D79, 013011 (2009), [0808.3340](http://arxiv.org/abs/0808.3340).
[9] D. Suematsu, T. Toma, and T. Yoshida, Phys. Rev. D79, 093004 (2009), 0903.0287

[10] Y. H. Ahn and H. Okada, Phys. Rev. D85, 073010 (2012), 1201.4436

[11] L. Lopez Honorez, E. Nezri, J. F. Oliver, and M. H. G. Tytgat, JCAP 0702, 028 (2007), hep-ph/0612275.

[12] E. M. Dolle and S. Su, Phys. Rev. D80, 055012 (2009), 0906.1609

[13] G. B. Gelmini, E. Osoba, and S. Palomares-Ruiz, Phys. Rev. D81, 063529 (2010), 0912.2478.

[14] J. Kubo, E. Ma, and D. Suematsu, Phys. Lett. B642, 18 (2006), hep-ph/0604114

[15] A. Adulpravitchai, M. Lindner, and A. Merle, Phys. Rev. D80, 055031 (2009), 0907.2147

[16] T. Toma and A. Vicente, JHEP 1401, 160 (2014), 1312.2840

[17] A. Vicente and C. E. Yaguna (2014), 1412.2545

[18] Q.-H. Cao, E. Ma, and G. Rajasekaran, Phys. Rev. D76, 095011 (2007), 0708.2939

[19] S. Bar-Shalom, G. Eilam, T. Han, and A. Soni, Phys. Rev. D77, 115019 (2008), 0803.2835

[20] E. Dolle, X. Miao, S. Su, and B. Thomas, Phys. Rev. D81, 035003 (2010), 0909.3094

[21] A. Arhrib, R. Benbrik, and N. Gaur, Phys. Rev. D85, 095021 (2012), 1201.2644

[22] E. Ma, Phys. Lett. B671, 366 (2009), 0808.1729.

[23] A. Adulpravitchai, M. Lindner, A. Merle, and R. N. Mohapatra, Phys. Lett. B680, 476 (2009), 0908.0470

[24] R. Bouchand and A. Merle, JHEP 1207, 084 (2012), 1205.0008.

[25] T. Hambye and M. H. G. Tytgat, Phys. Lett. B659, 651 (2008), 0707.0633

[26] M. Kadastik, K. Kannike, A. Racioppi, and M. Raidal, Phys. Rev. Lett. 104, 201301 (2010), 0912.2729.

[27] R. Lewis and R. M. Woloshyn, Phys. Rev. D82, 034513 (2010), 1005.5420

[28] M. Malinsky, Eur. Phys. J. C73, 2415 (2013), 1212.4660

22
[29] A. Biswas and A. Lahiri (2014), 1412.6187.

[30] F. Vissani, Phys. Rev. D57, 7027 (1998), hep-ph/9709409.

[31] N. Chakrabarty, D. K. Ghosh, B. Mukhopadhyaya, and I. Saha (2015), 1501.03700.

[32] J. D. Clarke, R. Foot, and R. R. Volkas (2015), 1502.01352.

[33] G. ’t Hooft, NATO Sci. Ser. B 59, 135 (1980).

[34] E. M. Dolle and S. Su, Phys. Rev. D80, 055012 (2009), 0906.1609.

[35] G. C. Branco, P. M. Ferreira, L. Lavoura, M. N. Rebelo, M. Sher, et al., Phys. Rept. 516, 1 (2012), 1106.0034.

[36] M. E. Peskin and D. Schroeder, An Introduction to Quantum Field Theory, Advanced book classics (Addison-Wesley Publishing Company, 1995), ISBN 9780201503975.

[37] M. Maniatis, A. von Manteuffel, O. Nachtmann, and F. Nagel, Eur. Phys. J. C48, 805 (2006), 0605184.

[38] K. G. Klimenko, Theor. Math. Phys. 62, 58 (1985).

[39] M. Nebot, J. F. Oliver, D. Palao, and A. Santamaria, Phys. Rev. D77, 093013 (2008), 0711.0483.

[40] J. Herrero-Garcia, M. Nebot, N. Rius, and A. Santamaria, Nucl. Phys. B885, 542 (2014), 1402.4491.

[41] A. Barroso, P. M. Ferreira, I. P. Ivanov, and R. Santos, JHEP 1306, 045 (2013), 1303.5098.

[42] V. D. Barger, S. Pakvasa, T. J. Weiler, and K. Whisnant, Phys. Lett. B437, 107 (1998), hep-ph/9806387.

[43] K. A. Olive et al. (Particle Data Group), Chin. Phys. C38, 090001 (2014).

[44] T. Appelquist and J. Carazzone, Phys. Rev. D11, 2856 (1975).

[45] S. Antusch, J. Kersten, M. Lindner, and M. Ratz, Phys. Lett. B538, 87 (2002), 0203233.

[46] S. Weinberg, Phys. Rev. Lett. 43, 1566 (1979).
[47] F. Jegerlehner, M. Y. Kalmykov, and B. A. Kniehl (2014), \texttt{1412.4215}

[48] J. C. Collins, \textit{Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion} (Cambridge University Press, 1984), ISBN 9780521311779.

[49] C. Kim, J. Korean Phys. Soc. \textbf{59}, 2993 (2011), \texttt{1106.2389}

[50] T. Appelquist, J. Carazzone, J. T. Goldman, and H. R. Quinn, Phys. Rev. \textbf{D8}, 1747 (1973).

[51] S. Weinberg, Phys. Rev. \textbf{D7}, 2887 (1973).

[52] S. R. Coleman and E. J. Weinberg, Phys. Rev. \textbf{D7}, 1888 (1973).

[53] A. Dedes, A. B. Lahanas, and K. Tamvakis, Phys. Rev. \textbf{D53}, 3793 (1996), \texttt{9504239}

[54] G. Gamberini, G. Ridolfi, and F. Zwirner, Nucl. Phys. \textbf{B331}, 331 (1990).

[55] D. Suematsu, Eur. Phys. J. \textbf{C56}, 379 (2008), \texttt{0706.2401}

[56] N. Haba and O. Seto, Prog. Theor. Phys. \textbf{125}, 1155 (2011), \texttt{1102.2889}

[57] D. Suematsu, Eur. Phys. J. \textbf{C72}, 1951 (2012), \texttt{1103.0857}

[58] S. Dodelson and L. M. Widrow, Phys. Rev. Lett. \textbf{72}, 17 (1994), \texttt{hep-ph/9303287}

[59] X.-D. Shi and G. M. Fuller, Phys. Rev. Lett. \textbf{82}, 2832 (1999), \texttt{astro-ph/9810076}

[60] F. Bezrukov, H. Hettmansperger, and M. Lindner, Phys. Rev. \textbf{D81}, 085032 (2010), \texttt{0912.4415}

[61] M. Nemevsek, G. Senjanovic, and Y. Zhang, JCAP \textbf{1207}, 006 (2012), \texttt{1205.0844}

[62] A. Kusenko, Phys. Rev. Lett. \textbf{97}, 241301 (2006), \texttt{hep-ph/0609081}

[63] K. Petraki and A. Kusenko, Phys. Rev. \textbf{D77}, 065014 (2008), \texttt{0711.4646}

[64] A. Merle, V. Niro, and D. Schmidt, JCAP \textbf{1403}, 028 (2014), \texttt{1306.3996}

[65] A. Adulpravitchai and M. A. Schmidt, JHEP \textbf{1501}, 006 (2015), \texttt{1409.4330}

[66] A. Merle and M. Totzauer (2015), \texttt{1502.01011}

[67] M. Frigerio and C. E. Yaguna (2014), \texttt{1409.0659}
[68] L. Lello and D. Boyanovsky (2014), [1411.2690]

[69] A. Abada, G. Arcadi, and M. Lucente (2014), [1406.6556]

[70] D. Boyanovsky, Phys. Rev. D78, 103505 (2008), [0807.0646]

[71] B. Shuve and I. Yavin, Phys. Rev. D89(11), 113004 (2014), [1403.2727]

[72] A. Merle, Int. J. Mod. Phys. D22, 1330020 (2013), [1302.2625]

[73] B. Grzadkowski, M. Lindner, and S. Theisen, Phys. Lett. B198, 64 (1987).

[74] J. Kersten, Renormalization Group Evolution of Neutrino Masses, Diploma thesis, Technische Universität München, Germany (2001).

[75] C. T. Hill, C. N. Leung, and S. Rao, Nucl. Phys. B262, 517 (1985).