Abstract. This paper proves the existence of potentials of the first and second kind of a Frobenius like structure in a frame which encompasses families of arrangements.

Surprisingly the proof is based on the study of finite sets of vectors in a finite-dimensional vector space $V$. Given a natural number $m$ and a finite set $(v_i)$ of vectors we give a necessary and sufficient condition to find in the set $(v_i)$ $m$ bases of $V$. If $m$ bases in $(v_i)$ can be selected, we define elementary transformations of such a selection and show that any two selections are connected by a sequence of elementary transformations.

1. Introduction and main results

A Frobenius manifold comes equipped locally with a potential. If one gives a definition which does not mention this potential explicitly, one nevertheless obtains it immediately by the following elementary fact: Let $z_i$ be the coordinates on $\mathbb{C}^n$ and $\partial_i = \frac{\partial}{\partial z_i}$ be the coordinate vector fields. Let $M$ be a convex open subset of $\mathbb{C}^n$ and $T_M$ be the holomorphic tangent bundle of $M$. Let $A : T^3_M \to O_M$ be a symmetric map such that also $\partial_i A(\partial_j, \partial_k, \partial_l)$ is symmetric in $i, j, k, l$. Then a potential $F \in O_M$ with $\partial_i \partial_j \partial_k F = A(\partial_i, \partial_j, \partial_k)$ exists. On Frobenius manifolds see [D1, D2, M].

This paper is devoted to a nontrivial generalization of this fact. The generalization turns up in the theory of families of arrangements as in [V2, ch. 3]. Theorem 1.2 below gives the main result. Definition 1.1 prepares the frame and the used notions.

Definition 1.1. (a) A Frobenius like structure of order $(n, k, m) \in \mathbb{Z}^3_{\geq 0}$ with $n \geq k$ is a tuple $(M, K, \nabla^K, C, \zeta, V, (v_1, \ldots, v_n))$ with the following properties. $M$ is an open subset of $\mathbb{C}^n$, $K \to M$ is a holomorphic vector bundle on $M$ with flat connection $\nabla^K$, $C$ is a Higgs field on $K$ with $\nabla^K(C) = 0$, $S$ is a $\nabla^K$-flat $m$-linear form $S : O(K)^m \to O_M$, which is Higgs field invariant, i.e.

$$S(C_X s_1, s_2, \ldots, s_m) = S(s_1, C_X s_2, \ldots, s_m) = \ldots = S(s_1, s_2, \ldots, C_X s_m) \quad (1.1)$$

for $s_1, s_2, \ldots, s_m \in O(K)$ and $X \in T_M$, $\zeta$ is a global section in $K$, $V$ is a $k$-dimensional $\mathbb{C}$-vector space, and $(v_1, \ldots, v_n)$ is a tuple of vectors in $V$ with $\langle v_1, \ldots, v_n \rangle = V$ such that the following holds. Denote $J := \{1, \ldots, n\}$. For any $\{i_1, \ldots, i_k\} \subset J$ with $(v_{i_1}, \ldots, v_{i_k})$ a basis of $V$, the section $C_{\partial_{i_1}} \ldots C_{\partial_{i_k}} \zeta$ is $\nabla^K$-flat.

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(b) Some notations: In the situation of (a), a subset \( I = \{i_1, \ldots, i_k\} \subset J \) is called independent if \((v_{i_1}, \ldots, v_{i_k})\) is a basis of \(V\). Then the differential operator \( \partial_I := \partial_{i_1} \cdots \partial_{i_k} \) and the endomorphism \( C_I := C_{\partial_{i_1}} \cdots C_{\partial_{i_k}} : \mathcal{O}(K) \to \mathcal{O}(K) \) are well defined (they do not depend on the chosen order of the elements \(i_1, \ldots, i_k\)). Independent subsets exist because of \( \langle v_1, \ldots, v_n \rangle = V \).

(c) In the situation of (a), a potential of the first kind is a function \( Q \in \mathcal{O}_M \) with
\[
\partial_{i_1} \cdots \partial_{i_m} Q = S(C_{I_1} \zeta, \ldots, C_{I_m} \zeta) \tag{1.2}
\]
for any \(m\) independent subsets \(I_1, \ldots, I_m \subset J\). A potential of the second kind is a function \( L \in \mathcal{O}_M \) with
\[
\partial_{i} \partial_{i_1} \cdots \partial_{i_m} L = S(C_{\partial_i} C_{I_1} \zeta, \ldots, C_{I_m} \zeta) \tag{1.3}
\]
for any \(m\) independent subsets \(I_1, \ldots, I_m \subset J\) and any \(i \in J\).

Notice that the right-hand side of (1.2) does not depend on a point in \(M\) and the right-hand side of (1.3) can be interpreted as the matrix element of the operator \(C_{\partial_i}\).

**Theorem 1.2.** Let \((M, K, \nabla^K, C, S, \zeta, V, (v_1, \ldots, v_n))\) be a Frobenius like structure of some order \((n, k, m) \in \mathbb{Z}_+^3\). Then locally (i.e. near any \(z \in M \subset \mathbb{C}^n\)) potentials of the first and second kind exist.

At the end of the paper, several remarks discuss the case of arrangements and the relation to Frobenius manifolds and F-manifolds. But the detailed study of the case of arrangements is left for the future. Remark 4.4 (iii) states some other wishes for the future.

The proof of theorem 1.2 uses a fact from linear algebra which has (to our knowledge) not been noticed before.

**Theorem 1.3.** Let \(k\) and \(m \in \mathbb{Z}_{>0}\). Let \(V\) be a \(k\)-dimensional vector space over some field \(K\). Let \((v_1, \ldots, v_{mk})\) be a tuple of vectors in \(V\). It can be split into \(m\) bases of \(V\) if and only if the following condition holds: For any vector subspace \(U \subset V\)
\[
|\{i \in \{1, \ldots, mk\} \mid v_i \in U\}| \leq m \cdot \dim U. \tag{1.4}
\]

The theorem is proved in section 2. The proof is surprisingly nontrivial. Section 3 applies an implication of it to a combinatorial situation which in turn is needed in the proof of the main theorem 1.2 in section 4. Section 4 concludes with some remarks.

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2. **Linear algebra: tuples of vectors giving several bases**

In this section theorem 1.3 and some consequences of if will be proved. Part (a) of theorem 2.2 below is a slight generalization of theorem 1.3. Part (b) is a consequence of (a). Only part (b) will be used later, and only in the case \(l = 1\).
**Notations 2.1.** Let $\mathcal{A} \neq \emptyset$ be some set. An unordered tuple of elements of $\mathcal{A}$ is an element of $\mathbb{Z}_{\geq 0}[\mathcal{A}]$. It is denoted $T = \sum_{a \in \mathcal{A}} T(a) \cdot [a]$, with $T(a) \in \mathbb{Z}_{\geq 0}$ and $|T| := \sum_{a \in \mathcal{A}} T(a) < \infty$. For simplicity, it is called system of elements of $\mathcal{A}$ instead of unordered tuple. If $|T| = k \in \mathbb{Z}_{\geq 0}$, then it is also called a $k$-system. Its support is the finite set $\text{supp } T := \{a \in \mathcal{A} \mid T(a) > 0\} \subset \mathcal{A}$. Define for any subset $\mathcal{B} \subset \mathcal{A}$ the number

$$\mu(T, \mathcal{B}) := \sum_{a \in \mathcal{B} \cap \text{supp } T} T(a) \in \mathbb{Z}_{\geq 0}. \quad (2.1)$$

The set $\mathbb{Z}_{\geq 0}[\mathcal{A}]$ is a monoid and is contained in the group $\mathbb{Z}[\mathcal{A}]$. The map

$$d_H : \mathbb{Z}[\mathcal{A}] \times \mathbb{Z}[\mathcal{A}] \to \mathbb{Z}_{\geq 0}, \quad (T_1, T_2) \mapsto \sum_{a \in \mathcal{A}} |T_1(a) - T_2(a)| \quad (2.2)$$

is a metric on $\mathbb{Z}[\mathcal{A}]$. On $\mathbb{Z}[\mathcal{A}]$ and $\mathbb{Z}_{\geq 0}[\mathcal{A}]$ one has the partial ordering $\leq$ with

$$S \leq T \iff S(a) \leq T(a) \quad \forall a \in \mathcal{A}. \quad (2.3)$$

If $S$ and $T$ are systems with $S \leq T$ then $S$ is a subsystem of $T$.

In the proof of theorem 2.4 the following notation will be useful. There an $m \in \mathbb{Z}_{\geq 0}$ will be fixed. Then for $j \in \mathbb{Z}_{\geq 0}$ let $[j] \in \{1, \ldots, m\}$ be the unique number with $j \equiv [j]$ mod $m$.

**Definition 2.2.** Let $k, m \in \mathbb{Z}_{\geq 0}$, let $l \in \mathbb{Z}_{\geq 0}$, let $K$ be a field, let $V$ be a $k$-dimensional $K$-vector space. In the following, system means a system of elements of $V$.

(a) A system $T$ generates the subspace $\langle T \rangle := \langle v \mid v \in \text{supp } T \rangle \subset V$.

(b) A system $T$ is a basis of $V$ if $\langle T \rangle = V$ and if $|T| = k$ (so the support $\text{supp } T$ is a basis of $V$, and all $T(a) \in \{0; 1\}$).

(c) A strong decomposition of an $(mk+l)$-system $T$ is a decomposition $T = T^{(1)} + \ldots + T^{(m+1)}$ into $m$ $k$-systems $T^{(1)}, \ldots, T^{(m)}$ and one $l$-system $T^{(m+1)}$ such that $T^{(1)}, \ldots, T^{(m)}$ are bases of $V$ (and $T^{(m+1)}$ is an arbitrary $l$-system; e.g. if $l = 0$ then $T^{(m+1)} = 0$ automatically).

(d) An $(mk+l)$-system is strong if it admits a strong decomposition.

(e) An $(mk+l)$-system $T$ is qualified if it satisfies the conditions:

$$\mu(T, U) \leq l + m \cdot \dim U \quad \forall \text{ subspaces } U \subset V. \quad (2.4)$$

(f) In the case of an $(mk+l)$-system $T$ with $l \geq 1$, define the subset

$$A_1(T) := \{a \in \text{supp } T \mid \exists \text{ a strong decomposition } T = T^{(1)} + \ldots + T^{(m+1)} \text{ with } a \in \text{supp } T^{(m+1)}\}. \quad (2.5)$$

Of course, if $l \geq 1$, then $A_1(T) \neq \emptyset \iff T$ is strong.

**Lemma 2.3.** Let $k, m \in \mathbb{Z}_{\geq 0}$, let $l \in \mathbb{Z}_{\geq 0}$, let $K$ be a field, let $V$ be a $k$-dimensional $K$-vector space. In the following, system means a system of elements of $V$.

(a) A strong $(mk+l)$-system is qualified, i.e. the conditions (2.4) are necessary for being strong.

(b) Let $T$ be a qualified $(mk+l)$-system. If $U_1, U_2 \subset V$ are two subspaces with $\mu(T, U_1) = l + m \dim U_1$, then also $\mu(T, U_1 \cap U_2) = l + m \dim U_1 \cap U_2$. Therefore there is a unique minimal (with respect to inclusion) subspace
with this property. The intersection of it with supp $T$ is called $A_2(T)$. The subspace itself is $\langle A_2(T) \rangle$.

(c) Let $T$ be a qualified $(mk + l)$-system with $l \geq 1$. Then $A_1(T) \subset A_2(T)$.

Proof: (a) Let $T^{(1)} + \ldots + T^{(m+1)}$ be a strong decomposition of $T$. Each of the bases $T^{(1)}, \ldots, T^{(m)}$ contains at most $\dim U$ elements of $U$. Therefore $\mu(T, U) \leq l + m \dim U$.

(b) The assumption $\mu(T, U_1 \cap U_2) \leq (l - 1) + m \dim U_1 \cap U_2$ leads to a contradiction as follows. If it holds, then

$$
\mu(T, U_1 + U_2) = \mu(T, U_1 + U_2) - (U_1 \cup U_2) + \mu(T, U_1) + \mu(T, U_2) - \mu(T, U_1 \cap U_2)
\geq 0 + (l + m \dim U_1) + (l + m \dim U_2) - (l - 1 + m \dim U_1 \cap U_2)
= l + 1 + m \dim(U_1 + U_2),
$$

which contradicts the condition (2.4) for the qualified $(mk + l)$-system $T$. Because the subspace is minimal with the given property, it is generated by its intersection $A_2(T)$ with supp $T$.

(c) This follows from (a): If no strong decomposition exists then $A_1(T) = \emptyset$. If $T^{(1)} + \ldots + T^{(m+1)}$ is a strong decomposition of $T$ then each of the bases $T^{(1)}, \ldots, T^{(m)}$ contains at most $\dim(A_2(T))$ elements of $\langle A_2(T) \rangle$. Because of $\mu(T, A_2(T)) = l + m \dim(A_2(T))$, the $l$-system $T^{(m+1)}$ is completely filled with elements of $A_2(T)$.

\[\square\]

Theorem 2.4. Let $k, m \in \mathbb{Z}_{>0}$, let $l \in \mathbb{Z}_{\geq 0}$, let $K$ be a field, let $V$ be a $k$-dimensional $K$-vector space. In the following, system means a system of elements of $V$.

(a) An $(mk + l)$-system $T$ is strong if and only if it is qualified, i.e. the conditions (2.4) are necessary and sufficient for being strong.

(b) Let $T$ be a strong $(mk + l)$-system with $l \geq 1$. Then $A_1(T) = A_2(T)$.

Part (a) of the theorem says that the conditions (2.4) are also sufficient for the system to be strong. As the proof of this fact is surprisingly hard, it is called a theorem. Part (a) for $l = 0$ is theorem 2.3. First the case $l = 0$ is proved. The generalization to $l \geq 1$ is an easy consequence. Part (b) of theorem 2.4 is an easy consequence of (the proof of) part (a). It improves part (c) of lemma 2.3.

Proof: (a) Because of lemma 2.3 (a), it rests to show that a qualified $(mk + l)$-system is strong. Let $T$ be a qualified $(mk + l)$-system. We can suppose $0 \notin \text{supp} T$, i.e. $T(0) = 0$. If $T(0) > 0$, then $T(0) = \mu(T, \{0\}) \leq l$ by (2.4). Then the system $T - T(0) \cdot [0]$ is a qualified $(mk + (l - (T(0)))$-system. It is sufficient to prove that this system is strong. Then also $T$ is strong. Therefore suppose $0 \notin \text{supp} T$. Furthermore, we can suppose that $T(b) = 1$ for all $b \in \text{supp} T$. Because if this does not hold, one can rescale the vectors in $T = \sum_{j=1}^{mk+l} [b_j]$ with suitable scalars $\lambda_j \in K^*$ so that the new system $\tilde{T} := \sum_{j=1}^{mk+l} [\lambda_j b_j]$ satisfies $\tilde{T}(b) = 1$ for all $b \in \text{supp} \tilde{T}$. The new system $\tilde{T}$
is still qualified. It is sufficient to prove that \( \tilde{T} \) is strong. Then also the old system \( T \) is strong.

In a first step, the case \( l = 0 \) will be proved, i.e. theorem 1.3. In a second step, the cases \( l \geq 1 \) will be proved inductively on \( l \). The first step is more difficult than the second step.

**First step:** Let \( T \) be a qualified \( mk \)-system, so \( \{2.4\} \) holds with \( l = 0 \). As above, we can assume \( 0 \notin \text{supp} \) and \( \tilde{T} \). Let \( T = T(1) + \ldots + T(m) \) be a decomposition of \( T \) into \( m \) \( k \)-systems \( T(1), \ldots, T(m) \). Suppose that it is not a strong decomposition. We will construct in the following a new decomposition \( T = S(1) + \ldots + S(m) \) into \( m \) \( k \)-systems \( S(1), \ldots, S(m) \) such that

\[
\sum_{j=1}^{m} \dim \langle S(j) \rangle > \sum_{j=1}^{m} \dim \langle T(j) \rangle.
\]

(2.6)

Iterating this construction, one arrives at a strong decomposition of \( T \).

We can suppose that \( T^{(1)} \) is not a basis, so \( \dim \langle T^{(1)} \rangle < k \). For any \( j \in \{1, \ldots, m\} \), choose a subset \( B^{(j)} \subset \text{supp} \langle T^{(j)} \rangle \) such that \( (v_{j} | j \in B^{(j)}) \) is a basis of \( \langle T^{(j)} \rangle \). For any \( b \in \langle T^{(j)} \rangle = \langle B^{(j)} \rangle \) denote by \( \lambda_{j}(b, c) \in \mathbb{C} \) for \( j \in \{1, \ldots, m\} \) and \( c \in B^{(j)} \) the unique coefficient with

\[
b = \sum_{c \in B^{(j)}} \lambda_{j}(b, c) \cdot c.
\]

(2.7)

Define

\[
R^{(0)} := \text{supp} \langle T^{(1)} \rangle - B^{(1)} \neq \emptyset.
\]

(2.8)

Now, define a sequence \( (R^{(j)})_{j=1, \ldots, N} \) of subsets of \( \text{supp} \langle T \rangle \) with maximal \( N \in \mathbb{Z}_{>0} \cup \{\infty\} \) in the following way: If \( R^{(0)}, \ldots, R^{(j-1)} \) for some \( j \in \mathbb{Z}_{>0} \) are defined and \( \langle R^{(j-1)} \rangle \subsetneq \langle T^{[j]} \rangle \) (see the notations \( \{2.7\} \) for \( [j] \)) then stop and set \( N := j - 1 \). But if \( \langle R^{(j-1)} \rangle \subset \langle T^{[j]} \rangle \) then define

\[
R^{(j)} := \{c \in B^{[j]} | \exists b \in R^{(j-1)} \text{ with } \lambda_{[j]}(b, c) \neq 0\}
\]

(2.9)

\[
= \text{the minimal subset } R^{(j)} \subset B^{[j]} \text{ with } \langle R^{(j)} \rangle \subset \langle R^{(j-1)} \rangle \subset \langle R^{(j)} \rangle.
\]

(2.10)

This defines a unique sequence, a priori with finite or infinite length \( N \). The length satisfies \( N \geq 1 \), so \( R^{(1)} \) exists, because \( R^{(0)} \subset \langle T^{(1)} \rangle \).

We claim that the length is finite, so \( N \in \mathbb{Z}_{>0} \), and prove this indirectly. So suppose that \( N = \infty \). Because of \( \{2.10\} \), there is a \( \theta \in \mathbb{Z}_{>0} \) such that \( \dim \langle R^{(j)} \rangle \) is constant for \( j \geq \theta \) and the spaces \( \langle R^{(j)} \rangle \subset V \) coincide for all \( j \geq \theta \). Then

\[
\mu(T, \langle R^{(\theta)} \rangle) \geq |R^{(0)}| + \sum_{j=0}^{m-1} |R^{(\theta+j)}| \geq |R^{(0)}| + m \cdot \dim \langle R^{(\theta)} \rangle \geq 1 + m \cdot \dim \langle R^{(\theta)} \rangle.
\]

(2.11)

But this is a contradiction to \( T \) being a qualified \( mk \)-system, i.e. to \( \{2.4\} \). Therefore \( N \in \mathbb{Z}_{>0} \).
It rests to construct a new decomposition \( T = S^{(1)} + \ldots + S^{(m)} \) with \((2.16)\). For this, we show now that a sequence \((b_j)_{j=0,\ldots,N}\) of elements of \(\text{supp} \, T\) with the following properties can be chosen:

\[
\begin{align*}
    b_j &\in R^{(j)} \quad \text{for } 0 \leq j \leq N, \\
    b_j &\notin R^{(j-m)} \quad \text{for } m+1 \leq j \leq N, \\
    \lambda_{[j]}(b_{j-1}, b_j) &\neq 0 \quad \text{for } 1 \leq j \leq N, \\
    b_N &\notin \langle T^{([N+1])} \rangle.
\end{align*}
\]

(2.12) \hspace{0.5cm} (2.13) \hspace{0.5cm} (2.14) \hspace{0.5cm} (2.15)

We construct the elements in the order \(b_N, b_{N-1}, \ldots, b_0\) and start with \(b_N\): By construction \(R^{(N)} \not\subset \langle T^{([N+1])} \rangle\). Choose \(b_N \in R^{(N)} - \langle T^{([N+1])} \rangle\). \((2.12)\) and \((2.15)\) hold. If \(N \geq m+1\) then \(R^{(N-m)} \subset \langle T^{([N+1])} \rangle\), so \((2.13)\) holds.

If \(b_N, \ldots, b_j\) are constructed with \((2.12) - (2.15)\) for some \(j \geq 1\), then choose \(b_{j-1} \in R^{(j-1)}\) with \((2.14)\). This is possible because of \((2.9)\). If \(b_{j-1} \in R^{(j-1-m)}\) in the case \(j-1 \geq m+1\), then by \((2.9)\) also \(b_j \in R^{(j-m)}\), a contradiction. This shows the existence of \(b_N, b_{N-1}, \ldots, b_0\) as claimed. They are all different because of \((2.12)\) and \((2.13)\).

Now define a sequence \((\tilde{R}^{(j)})_{j=1,\ldots,N}\) of subsets of \(\text{supp} \, T\) as follows.

\[
\tilde{R}^{(j)} := \left( R^{(j)} - \bigcup_{i \in \{1, \ldots, j\} : [i] = j} \{b_i\} \right) \cup \bigcup_{i \in \{0, \ldots, j-1\} : [i+1] = j} \{b_i\},
\]

(2.16)

so that

\[
\tilde{R}^{(j)} = \left( \left( R^{(j)} - \{b_j\} \cup \{b_{j-1}\} \right) - R^{(j-m)} \right) \cup \tilde{R}^{(j-m)}
\]

(2.17)

for \(j \geq m+1\).

We claim that for all \(j \in \{1, \ldots, N\}\)

\[
\langle \tilde{R}^{(j)} \rangle = \langle R^{(j)} \rangle
\]

(2.18)

holds. For \(1 \leq j \leq \min(m, N)\) this follows from \((2.14)\). If \(N \geq m+1\), then for \(m+1 \leq j \leq N\) this follows inductively: The induction hypothesis gives

\[
\langle \tilde{R}^{(j-m)} \rangle = \langle R^{(j-m)} \rangle
\]

(2.19)

Thus

\[
\langle \tilde{R}^{(j)} \rangle = \langle \left( \left( R^{(j)} - \{b_j\} \cup \{b_{j-1}\} \right) - R^{(j-m)} \right) \cup \tilde{R}^{(j-m)} \rangle
\]

\[
= \langle \left( \left( R^{(j)} - \{b_j\} \cup \{b_{j-1}\} \right) - R^{(j-m)} \right) \rangle + \langle \tilde{R}^{(j-m)} \rangle
\]

\[
= \langle \left( \left( R^{(j)} - \{b_j\} \cup \{b_{j-1}\} \right) - R^{(j-m)} \right) \rangle + \langle R^{(j-m)} \rangle
\]

\[
= \langle \left( \left( R^{(j)} - \{b_j\} \cup \{b_{j-1}\} \right) \rangle
\]

\[
= \langle R^{(j)} \rangle.
\]

The last equality uses \((2.14)\).

Because \(R^{(N)} \not\subset \langle T^{([N+1])} \rangle\), \(\langle T^{([N+1])} \rangle \not\subset V\) and \(\text{supp} \, T^{([N+1])} - B^{([N+1])} \neq \emptyset\). Choose

\[
b_N+1 \in \text{supp} \, T^{([N+1])} - B^{([N+1])}
\]

(2.20)

arbitrarily.
Now we will define $k$-systems $S^{(1)}, \ldots, S^{(m)}$ with
\[ S^{(1)} + \ldots + S^{(m)} = T \] (2.21)
and (2.6). We have to distinguish two cases.

**First case**, $N \leq m - 1$:

\[
S^{(j)} := T^{(j)} - [b_j] + [b_j-1] \quad \text{for } 2 \leq j \leq N + 1, \quad (2.22)
\]
\[
S^{(1)} := T^{(1)} - [b_1] + [b_{N+1}], \quad (2.23)
\]
\[
S^{(j)} := T^{(j)} \quad \text{for } N + 2 \leq j \leq m. \quad (2.24)
\]

These are $m$ $k$-systems. (2.21) holds obviously. The subspaces generated by the $k$-systems $S^{(j)}$ are as follows.

\[
\langle S^{(j)} \rangle := \langle T^{(j)} \rangle \quad \text{for } 2 \leq j \leq N, \quad \text{by (2.14)},
\]
\[
\langle S^{(N+1)} \rangle := \langle T^{(N+1)} \rangle + \langle b_N \rangle \supseteq \langle T^{(N+1)} \rangle \quad \text{by (2.20) and (2.15)},
\]
\[
\langle S^{(1)} \rangle := \langle T^{(1)} \rangle + \langle b_{N+1} \rangle \supset \langle T^{(1)} \rangle \quad \text{by (2.14)},
\]
\[
\langle S^{(j)} \rangle := \langle T^{(j)} \rangle \quad \text{for } N + 2 \leq j \leq m.
\]

Together these give (2.6).

**Second case**, $N \geq m$: Define

\[
S^{(j)} := T^{(j)} - \sum_{i \in \{1, \ldots, N+1\}} [b_i] + \sum_{i \in \{1, \ldots, N\}} [b_i] \quad \text{for } 2 \leq j \leq m, \quad (2.25)
\]
\[
S^{(1)} := T^{(1)} - \sum_{i \in \{1, \ldots, N+1\}} [b_i] + \sum_{i \in \{1, \ldots, N\}} [b_i] + [b_{N+1}]. \quad (2.26)
\]

These are $m$ $k$-systems. (2.21) holds because of (2.25) and (2.26). For $j \in \{1, \ldots, m\}$ let $\alpha(j)$ be the unique integer in $\{N-m+1, N-m+2, \ldots, N\}$ with $[\alpha(j)] = j$. Especially $\alpha([N+1]) = N - m + 1$. Then for $j \in \{1, \ldots, m\} - \{1, [N+1]\}$
\[
S^{(j)} = T^{(j)} - \sum_{b \in R^{(\alpha(j))}} [b] + \sum_{b \in R^{(\alpha(j))}} [b] \quad (2.27)
\]

and
\[
\langle S^{(j)} \rangle = \langle T^{(j)} \rangle \quad \text{by (2.18)}. \quad (2.28)
\]

If $1 \neq [N+1]$ then
\[
S^{(1)} = T^{(1)} - \sum_{b \in R^{(\alpha(1))}} [b] + \left( \sum_{b \in R^{(\alpha(1))}} [b] - [b_0] \right) + [b_{N+1}], \quad (2.29)
\]
\[
S^{([N+1])} = T^{([N+1])} - \sum_{b \in R^{(N-m+1)}} [b] + \sum_{b \in R^{(N-m+1)}} [b] - [b_{N+1}] + [b_N] \quad (2.30)
\]

and
\[
\langle S^{(1)} \rangle = \langle T^{(1)} \rangle + \langle b_{N+1} \rangle \supset \langle T^{(1)} \rangle \quad \text{by (2.18)}, \quad (2.31)
\]
\[
\langle S^{([N+1])} \rangle = \langle T^{([N+1])} \rangle + \langle b_N \rangle \supset \langle T^{([N+1])} \rangle \quad (2.32)
\]

by (2.18), (2.20) and (2.15).
If \(1 = [N + 1]\) then
\[
S^{(1)} = T^{(1)} - \sum_{b \in R^{(N-m+1)}} [b] + (\sum_{b \in R^{(N-m+1)}} [b] - [b_0]) + [b_N] \tag{2.33}
\]
(in the case \(1 = [N + 1]\), \(b_{N+1}\) is not used) and
\[
\langle S^{(1)} \rangle = \langle T^{(1)} \rangle + \langle b_N \rangle \not= \langle T^{(1)} \rangle \quad \text{by (2.18) and (2.15).} \tag{2.34}
\]
In both cases, \(1 \not= [N + 1]\) and \(1 = [N + 1]\), (2.6) holds. This finishes the first step and proves part (a) of theorem 2.4 for \(l = 0\).

**Second step:** The cases \(l \geq 1\) are proved by induction in \(l\). Fix some \(l \geq 1\). Suppose that any qualified \((mk + l)\)-system with \(k \in \mathbb{Z}_{>0}\) arbitrary and \(l \in \{0, 1, \ldots, l - 1\}\) is strong. Let \(T\) be a qualified \((mk + l)\)-system. As in the beginning of the proof we can suppose \(0 \not\in \text{supp} T\) and \(T(b) = 1\) for all \(b \in \text{supp} T\).

Define \(g := \dim \langle A_2(T) \rangle\). Choose an arbitrary element \(a \in A_2(T)\). The subsystem

\[
R := \sum_{b \in A_2(T) - \{a\}} [b] \tag{2.35}
\]

of \(T\) is an \((mg + l - 1)\)-system by definition of \(A_2(T)\). Furthermore, it is a qualified \((mg + l - 1)\)-system with respect to the vector space \(\langle A_2(T) \rangle\), again by definition of \(A_2(T)\): For any \(U \not\subseteq \langle A_2(T) \rangle\), \(\mu(T, U) \leq l - 1 + m \dim U\) by the minimality of \(\langle A_2(T) \rangle\), thus also \(\mu(R, U) \leq l - 1 + m \dim U\). For \(U = \langle A_2(T) \rangle\), this holds because \(R\) is an \((mg + l - 1)\)-system.

By induction hypothesis, \(R\) is a strong \((mg + l - 1)\)-system with respect to the vector space \(\langle A_2(T) \rangle\), so it has a strong decomposition \(R = R^{(1)} + \ldots + R^{(m+1)}\). Therefore \(R + [a] = \sum_{b \in A_2(T)} [b]\) is a strong \((mg + l)\)-system with the strong decomposition \(R + [a] = R^{(1)} + \ldots + R^{(m)} + (R^{(m+1)} + [a])\).

Consider the system

\[
S := \sum_{b \in \text{supp} T - A_2(T)} [b + \langle A_2(T) \rangle] \tag{2.36}
\]

of vectors in the quotient space \(V/\langle A_2(T) \rangle\). It is a qualified \(m(k - g)\)-system with respect to the vector space \(V/\langle A_2(T) \rangle\) because \(T\) is a qualified \((mk + l)\)-system and \(R + [a]\) is a qualified \((mg + l)\)-system. By the first step in this proof, \(S\) has a strong decomposition \(S = S^{(1)} + \ldots + S^{(m)}\).

The vectors in \(\text{supp} S \subset V/\langle A_2(T) \rangle\) lift uniquely to vectors in \(\text{supp} T\), because \(T(b) = 1\) for all \(b \in \text{supp} T\). Let \(\tilde{S} = \tilde{S}^{(1)} + \ldots + \tilde{S}^{(m)}\) be the corresponding lift to \(V\) of \(S\) and its decomposition. Then

\[
T = (R^{(1)} + \tilde{S}^{(1)}) + \ldots + (R^{(m)} + \tilde{S}^{(m)}) + (R^{(m+1)} + [a]) \tag{2.37}
\]

is a strong decomposition of \(T\). This finishes the second step and the proof of part (a).

(b) Because of lemma 2.3 (c), it rests to show \(A_2(T) \subseteq A_1(T)\). But this follows from the second step above and especially the strong decomposition (2.37) of \(T\). There \(a \in A_2(T)\) was arbitrary. \(\square\)
Remarks 2.5. Part (a) of theorem 2.4 has some similarity with the marriage theorem of Hall:

Let $A$ and $B$ be nonempty finite sets with $|A| \leq |B|$, and let $f : A \rightarrow \mathcal{P}(B) (:= \text{the set of subsets of } B)$ be a map. Then a map $g : A \rightarrow B$ with $g(a) \in f(a)$ exists if and only

$$| \bigcup_{c \in C} f(c) | \geq |C| \quad \text{for all } C \subset A \text{ with } C \neq \emptyset. \quad (2.38)$$

In theorem 2.4 and in the marriage theorem of Hall, the conditions (2.4) respectively (2.38) are obviously necessary, but that they are sufficient requires a nontrivial proof.

3. An equivalence between index systems

This section prepares the proof of the main result, theorem 4.2. It builds on section 2.

Start with three positive integers $k$ and $n$ and $m$ with $k < n$ and $m \geq 2$, with a field $K$, a $K$-vector space $V$ of dimension $k$ and a map

$$v : J \rightarrow V, \quad i \mapsto v(i) =: v_i, \quad \text{where } J := \{1, \ldots, n\}, \quad (3.1)$$

with the property

$$\langle v_1, \ldots, v_n \rangle = V. \quad (3.2)$$

Definition 3.1. (a) A system $T = \sum_{i \in J} T(i) \cdot [i]$ of elements of $J$ induces the system

$$v^{sys}(T) := \sum_{i \in J} T(i) \cdot [v_i] \quad (3.3)$$

of elements of $V$. Of course $|T| = |v(T)|$.

(b) A strong decomposition of an $(mk + l)$-system $T$ of elements of $J$ for $l \in \{0; 1\}$ is a decomposition $T = T^{(1)} + \ldots + T^{(m+1)}$ into $m + 1$ systems such that the induced decomposition $v^{sys}(T) = v^{sys}(T^{(1)}) + \ldots + v^{sys}(T^{(m+1)})$ of $v^{sys}(T)$ is strong (definition 2.2 (c)).

(c) An $(mk + l)$-system $T$ of elements of $J$ for $l \in \{0; 1\}$ is strong if it admits a strong decomposition. Of course, this holds if and only if the system $v^{sys}(T)$ is strong (definition 2.2 (d)).

(d) A good decomposition of an $N$-system $T$ of elements of $J$ with $N \geq mk + 1$ is a decomposition $T = T_1 + T_2$ into two systems such that $T_2$ is a strong $(mk + 1)$-system of elements of $J$.

(e) Two good decompositions $T_1 + T_2 = T$ and $S_1 + S_2 = T$ of an $N$-system $T$ of elements of $J$ with $N \geq mk + 1$ are locally related, notation: $(S_1, S_2) \sim_{loc} (T_1, T_2)$, if there are strong decompositions $S_2^{(1)} + \ldots + S_2^{(m+1)} = S_2$ of $S_2$ and $T_2^{(1)} + \ldots + T_2^{(m+1)} = T_2$ of $T_2$ with $S_2^{(j)} = T_2^{(j)}$ for $1 \leq j \leq m$. Of course, $\sim_{loc}$ is a reflexive and symmetric relation.

(f) Two good decompositions $T_1 + T_2 = T$ and $S_1 + S_2 = T$ of an $N$-system $T$ of elements of $J$ with $N \geq mk + 1$ are equivalent, notation: $(S_1, S_2) \sim (T_1, T_2)$, if there is a sequence $\sigma_1, \sigma_2, \ldots, \sigma_r$ for some $r \in \mathbb{Z}_{\geq 1}$
of good decompositions of $T$ such that $\sigma_1 = (S_1, S_2)$, $\sigma_r = (T_1, T_2)$ and
$\sigma_j \sim_{\text{loc}} \sigma_{j+1}$ for $j = 1, \ldots, r - 1$. Of course, $\sim$ is an equivalence relation.

(g) The distance $d_H(S, T)$ between two systems $S$ and $T$ of elements of $J$ is the number

$$d_H(S, T) := \sum_{i \in J} |S(i) - T(i)| \in \mathbb{Z}_{\geq 0}. \quad (3.4)$$

This defines a metric on the set of systems of elements of $J$.

The main result of this section is the following theorem 3.2.

**Theorem 3.2.** Let $T$ be an $N$-system of elements of $J$ for some $N \geq mk + 1$
which has good decompositions. Then all its good decompositions are equiva-

The theorem will be proved after the proof of lemma 3.3.

**Lemma 3.3.** Let $S$ and $T$ be two strong $(mk + 1)$-systems of elements of $J$.
At least one of the following two alternatives holds.

(a) $T$ has a strong decomposition $T = T^{(1)} + \ldots + T^{(m+1)}$ with $T^{(m+1)} = [i]
$ for some $i \in \text{supp } T$ with $T(i) > S(i)$.

(b) $T$ and $S$ have strong decompositions $T = T^{(1)} + \ldots + T^{(m+1)}$ and
$S = S^{(1)} + \ldots + S^{(m+1)}$ with $T^{(m+1)} = S^{(m+1)}$.

The lemma builds on section 2 especially on part (b) of theorem 2.4.

**Proof of lemma 3.3:** Define $A_1(T) := \{i \in \text{supp } T \mid v_i \in A_1(v^{\text{sys}}(T))\}$
and analogously $A_1(S)$. Then

$$A_1(T) = \{i \in \text{supp } T \mid \exists \text{ a strong decomposition } T^{(1)} + \ldots + T^{(m+1)}
\text{ with } T^{(m+1)} = [i]\}, \quad (3.5)$$

$$\sum_{i \in A_1(T)} T(i) = 1 + m \dim\langle A_1(v^{\text{sys}}(T))\rangle \quad (3.6)$$

$$= 1 + m \dim\langle v_i \mid i \in A_1(T)\rangle,$$ and analogously for $A_1(S)$. Here (3.6) follows from theorem 2.4 (b).

Suppose that (a) does not hold. Then for any $i \in A_1(T)$ $S(i) \geq T(i) > 0$.

Especially

$$\sum_{i \in A_1(T)} S(i) \geq 1 + m \dim\langle v_i \mid i \in A_1(T)\rangle. \quad (3.7)$$

and the argument in the proof of lemma 2.3 (c) show $A_1(S) \subset A_1(T)$
(and $S(i) = T(i)$ for $i \in A_1(T)$). Thus (β) holds.

**Proof of theorem 3.2:** Let $(S_1, S_2)$ and $(T_1, T_2)$ be two different good
decompositions of an $N$-system $T$ of elements of $J$ (with $N \geq mk + 1$). Then
$S_2$ and $T_2$ are strong $(mk + 1)$-systems of elements of $J$. At least one of the
two alternatives (α) and (β) in lemma 3.3 holds for $S_2$ and $T_2$.

**First case, (α) holds:** Let $T_2 = T_2^{(1)} + \ldots + T_2^{(m+1)}$ be a strong decomposition
with $T_2^{(m+1)} = [i]$ for some $i \in \text{supp } T_2$ with $T_2(i) > S_2(i)$. Then a $j \in \text{supp } T$
with \( T_1(j) > S_1(j) \) and \( T_2(j) < S_2(j) \) exists. The decomposition

\[
T = R_1 + R_2 \quad \text{with} \quad R_1 = T_1 - [j] + [i], \quad R_2 = T_2 + [j] - [i]
\]  

(3.8)
is a good decomposition of \( T \) because \( T_2^{(1)} + \ldots + T_2^{(m)} + [j] \) is a strong decomposition of \( R_2 \). The good decompositions \( (R_1, R_2) \) and \( (T_2, T_2) \) are locally related, \( (R_1, R_2) \sim_{\text{loc}} (T_1, T_2) \), and thus equivalent,

\[
(R_1, R_2) \sim (T_1, T_2).
\]

Furthermore,

\[
d_H(R_2, S_2) = d_H(T_2, S_2) - 2.
\]

(3.10)

Second case, \( (\beta) \) holds: Let \( T_2 = T_2^{(1)} + \ldots + T_2^{(m+1)} \) and \( S_2 = S_2^{(1)} + \ldots + S_2^{(m+1)} \) be strong decompositions of \( T_2 \) and \( S_2 \) with \( T_2^{(m+1)} = S_2^{(m+1)} = [a] \) for some \( a \in \text{supp} T \). Two elements \( b, c \in \text{supp} T \) with \( T_1(b) > S_1(b), T_2(b) < S_2(b) \), and \( T_1(c) < S_1(c), T_2(c) > S_2(c) \) exist. Consider the decompositions of \( T \) and \( S \),

\[
T = R_1 + R_2 \quad \text{with} \quad R_1 = T_1 - [b] + [a], R_2 = T_2 + [b] - [a], \quad (3.11)
\]

\[
S = Q_1 + Q_2 \quad \text{with} \quad Q_1 = S_1 - [c] + [a], Q_2 = S_2 + [c] - [a]. \quad (3.12)
\]

They are good decompositions because \( R_2 \) has the strong decomposition \( R_2 = T_1^{(1)} + \ldots + T_2^{(m)} + [b] \) and \( Q_2 \) has the strong decomposition \( Q_2 = S_1^{(1)} + \ldots + S_2^{(m)} + [c] \). The local relations

\[
(R_1, R_2) \sim_{\text{loc}} (T_1, T_2) \quad \text{and} \quad (Q_1, Q_2) \sim_{\text{loc}} (S_1, S_2)
\]

and the equivalences

\[
(R_1, R_2) \sim (T_1, T_2) \quad \text{and} \quad (Q_1, Q_2) \sim (S_1, S_2)
\]

(3.13)

hold. Furthermore

\[
d_H(R_2, Q_2) = d_H(T_2, S_2) - 2.
\]

(3.14)

The properties \((3.10), (3.11), (3.13) \) and \((3.14) \) show that in both cases the equivalence classes of \( (S_1, S_2) \) and \( (T_1, T_2) \) contain good decompositions whose second members are closer to one another with respect to the metric \( d_H \) than \( T_2 \) and \( S_2 \). This shows that \( (S_1, S_2) \) and \( (T_1, T_2) \) are in one equivalence class.

\( \square \)

4. Potentials of the First and Second Kind

The main part of this section is devoted to the proof of theorem 1.2. At the end some remarks on the relation to families of arrangements, Frobenius manifolds, F-manifolds and possible extensions of the work here are made.

**Remark 4.1.** This is a reminder of the notion of a Higgs field and the meaning of the condition \( \nabla^K(C) = 0 \) in definition 1.1. There a Higgs field is an \( \mathcal{O}_M \)-linear map

\[
C : \mathcal{O}(K) \rightarrow \Omega^1_K \otimes \mathcal{O}(K)
\]

(4.1)
such that all the endomorphisms \( C_X, X \in \mathcal{T}_M \), commute. \( \nabla^K(C) \in \Omega^2_M \otimes \mathcal{O}(K) \) is the 2-form on \( M \) with values in \( K \) such that for \( X, Y \in \mathcal{T}_M \)

\[
\nabla^K(C)(X, Y) = \nabla^K_X(C_Y) - \nabla^K_Y(C_X) - C_{[X,Y]}.
\]

(4.2)

Now \( \nabla^K(C) = 0 \) says

\[
\nabla^K_\partial_i(C_\partial_j) = \nabla^K_\partial_j(C_\partial_i).
\]

(4.3)

**Proof of theorem 1.2**: Let \((M, K, \nabla^K, C, S, \zeta, V, (v_1, \ldots, v_n))\) be a Frobenius like structure of some order \((n, k, m) \in \mathbb{Z}_0^3\).

We need some notations. If \( T \in \mathbb{Z}_0^J \) is a system of elements of \( J \), then

\[
(z - x)^T := \prod_{i \in J} (z_i - x_i)^{T(i)} \quad \text{for any } x \in \mathbb{C}^n,
\]

\[
T! := \prod_{i \in J} T(i)!,
\]

\[
\partial_T := \prod_{i \in J} \partial^{T(i)}_{z_i},
\]

\[
C_T := \prod_{i \in J} C^{T(i)}_{\partial z_i}.
\]

Thus, if \( S \) and \( T \) are systems of elements of \( J \), then

\[
\partial_T (z - x)^S = \begin{cases} 
0 & \text{if } T \preceq S, \\
\frac{S_1!}{(S-T)!} \cdot (z - x)^{S-T} & \text{if } T \preceq S,
\end{cases}
\]

(4.4)

for any \( x \in \mathbb{C}^n \).

The existence of a (not just local, but even global) potential \( Q \) of the first kind is trivial. The function

\[
Q := \sum_{T \text{ with } (*)} \frac{1}{T!} \cdot S(C_T \zeta, \zeta, \ldots, \zeta) \cdot z^T \quad (m \text{ times } \zeta),
\]

(4.5)

\((*) : T \in \mathbb{Z}_0^J \) is a strong \( mk \)-system (definition 3.1(c)).

works. It is a homogeneous polynomial of degree \( mk \) and contains only monomials which are relevant for \((1.2)\). In fact, one can add to this \( Q \) an arbitrary linear combination of the monomials \( z^T \) for the \( mk \)-systems \( T \) which are not strong, so which are not relevant for \((1.2)\).

The existence of a potential \( L \) of the second kind is not trivial. Let some \( x \in M \) be given. We make the power series ansatz

\[
L := \sum_{T \in \mathbb{Z}_0^J} a_T \cdot (z - x)^T,
\]

(4.6)

where the coefficients \( a_T \) have to be determined. If \( T \) satisfies \(|T| \leq mk \) or if it satisfies \(|T| \geq mk + 1 \), but does not admit a good decomposition (definition 3.1(d)), then the conditions (1.3) are empty for \( a_T(z - x)^T \) because of (4.4), so then \( a_T \) can be chosen arbitrarily, e.g. \( a_T := 0 \) works.
Now consider $T$ with $|T| \geq mk+1$ which admits good decompositions. Then each good decomposition $T = T_1 + T_2$ gives via (1.3) a candidate

$$a_T(T_1, T_2) := \frac{1}{T!} \cdot (\partial_T S(C_{T_2}, \zeta, \ldots, \zeta))(x), \quad (4.7)$$

for the coefficient $a_T$ of $(z - x)^T$ in $L$. We have to show that the candidates $a_T(T_1, T_2)$ for all good decompositions $(T_1, T_2)$ of $T$ coincide.

Suppose that two good decompositions $(T_1, T_2)$ and $(S_1, S_2)$ are locally related, $(T_1, T_2) \sim_{loc} (S_1, S_2)$ (definition 3.1 (e)), but not equal. Then there are strong decompositions $T_2 = T_2^{(1)} + \ldots + T_2^{(m)} + [a]$ and $S_2 = T_2^{(1)} + \ldots + T_2^{(m)} + [b]$ with $a \neq b$, and thus also $T_1 - [b] = S_1 - [a] \in \mathbb{Z}_{\geq 0}[J]$ holds. Because any $T_2^{(j)}$, $j \in \{1, \ldots, m\}$, is independent, $C_{T_2^{(j)}}$ is $\nabla^K$-flat. This and (4.3) give

$$\partial_{x_i} S(C_{T_2}, \zeta, \ldots, \zeta) = \partial_{x_i} S(C_{\partial_{x_a} C_{T_2}}, C_{T_2^{(1)}} \zeta, \ldots, C_{T_2^{(m)}} \zeta) = (\nabla^K \partial_{x_a} C_{T_2})(C_{T_2^{(1)}} \zeta, \ldots, C_{T_2^{(m)}} \zeta)$$

This implies

$$a_T(T_1, T_2) = a_T(S_1, S_2), \quad (4.9)$$

so the locally related good decompositions $(T_1, T_2)$ and $(S_1, S_2)$ give the same candidate for $a_T$. Thus all equivalent (definition 3.1 (f)) good decompositions give the same candidate for $a_T$. By theorem 3.2 all good decompositions of $T$ are equivalent. Therefore they all give the same candidate for $a_T$. Thus a potential $L$ of the second kind exists as a formal power series as in (4.3).

It is in fact a convergent power series because of the following. There are finitely many strong $mk$-systems $T_2$. Each determines the coefficients $a_T$ for all $T \geq 2k$. We put $a_T := 0$ for $T$ which do not admit good decompositions. The part of $L$ in (4.6) which is determined by some strong $mk$-system $T_2$ is a convergent power series. Thus $L$ is the union of finitely many overlapping convergent power series. It is easy to see that it is itself convergent. This finishes the proof of theorem 1.2.

**Remark 4.2.** In [V2, ch. 3] families or arrangements are considered which give rise to Frobenius like structures $(M, K, \nabla^K, C, S, \zeta, V, (v_1, \ldots, v_n))$ of order $(n, k, 2)$, see the special case of generic arrangements in [V1, V3].

Start with two positive integers $k$ and $n$ with $k < n$ and with a matrix $B := (b^j_i)_{i=1, \ldots, n; j=1, \ldots, k} \in M(n \times k, \mathbb{C})$ with rank $B = k$. Define $J := \{1, \ldots, n\}$. Here the vector space $V$ and the vectors $v_1, \ldots, v_n$ are

$$V = M(1 \times k, \mathbb{C}), \quad (4.10)$$

$$v_i = (b^j_i)_{j=1, \ldots, k} \in V \quad \text{for } i = 1, \ldots, n. \quad (4.11)$$

We assume that $B$ is such that all vectors $v_i$ are nonzero.
Consider \( \mathbb{C}^n \times \mathbb{C}^k \) with the coordinates \((z, t) = (z_1, ..., z_n, t_1, ..., t_k)\) and with the projection \( \pi : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n \). Define the functions

\[
g_i := \sum_{j=1}^k b_j^i \cdot t_j, \quad f_i := g_i + z_i \quad \text{for } i \in J
\]

on \( \mathbb{C}^n \times \mathbb{C}^k \).

We obtain on \( \mathbb{C}^n \times \mathbb{C}^k \) the arrangement \( \mathcal{C} = \{H_i\}_{i \in J} \), where \( H_i \) is the zero set of \( f_i \). Let \( U(\mathcal{C}) := \mathbb{C}^n \times \mathbb{C}^k - \bigcup_{i \in J} H_i \) be the complement. For every \( x \in \mathbb{C}^n \), the arrangement \( \mathcal{C} \) restricts to an arrangement \( \mathcal{C}(x) \) on \( \pi^{-1}(x) \cong \mathbb{C}^k \). For almost all \( x \in \mathbb{C}^k \) the arrangement \( \mathcal{C}(x) \) is essential (definition in [V2]) with normal crossings. The subset \( \Delta \subset \mathbb{C}^n \) is this does not hold, is a hypersurface and is called the discriminant, see [V2] 3.2. Define \( M := \mathbb{C}^n - \Delta \).

A set \( I = \{i_1, ..., i_k\} \subset J \) is independent, i.e. \((v_{i_1}, ..., v_{i_k})\) is a basis of \( V \), if and only if (for some or equivalently for any \( x \in \mathbb{C}^n \)) the hyperplanes \( H_{i_1}(x), ..., H_{i_k}(x) \) are transversal.

Let \( a = (a_1, ..., a_n) \in (\mathbb{C}^*)^n \) be a system of weights such that for any \( x \in M \) the weighted arrangement \( (\mathcal{C}(x), a) \) is unbalanced: See [V2] for the definition of unbalanced, e.g. \( a \in \mathbb{R}_{>0}^n \) is unbalanced, also a generic system of weights is unbalanced. The master function of the weighted arrangement \( (\mathcal{C}, a) \) is

\[
\Phi_a(z, t) := \sum_{i \in J} a_i \log f_i.
\]

Several deep facts are related to this master function. We use some of them in the following. See [V2] for references.

For \( z \in M \) all critical points of \( \Phi_a \) are isolated, and the sum \( \mu \) of their Milnor numbers is independent of the unbalanced weight \( a \) and the parameter \( z \in M \). The bundle

\[
K := \bigcup_{z \in M} K_z \quad \text{with } K_z := \mathcal{O}(U(\mathcal{C}) \cap \pi^{-1}(z))/\left(\frac{\partial \Phi_a}{\partial t_j} \mid j = 1, ..., k\right)
\]

over \( M \) is a vector bundle of \( \mu \)-dimensional algebras.

It comes equipped with the section \( \zeta \) of unit elements \( \zeta(z) \in K_z \), a Higgs field \( C \), a combinatorial connection \( \nabla^K \) and a pairing \( S \). The Higgs field \( C : \mathcal{O}(K) \to \Omega^1_M \otimes \mathcal{O}(K) \) is defined with the help of the period map

\[
\Psi : TM \to K, \quad \partial_{z_i} \mapsto \left[\frac{\partial \Phi_a}{\partial z_i}\right] = \left[\frac{a_i}{f_i}\right] =: p_i
\]

by

\[
C_{\partial_{z_i}}(h) := p_i \cdot h \quad \text{for } h \in K_z.
\]

Because of

\[
0 = \left[\frac{\partial \Phi_a}{\partial t_j}\right] = \sum_{i=1}^n b_j^i p_i,
\]

the Higgs field vanishes on the vector fields \( X_j := \sum_{i=1}^n b_j^i \partial_{t_i}, j \in \{1, ..., k\}, \)

\[
C_{X_j} = 0 \quad \text{for } j \in \{1, ..., k\}.
\]
In fact the whole geometry of the family of arrangements is invariant with respect to the flows of these vector fields.

The sections $\det(b_{ij})_{i \in I, j = 1, \ldots, k} \cdot C_I \zeta$ for all independent sets $I = \{i_1, \ldots, i_k\} \subset J$ generate the bundle $K$, and they satisfy only relations with constant coefficients in $\mathbb{Z}$. The combinatorial connection $\nabla^K$ is the unique flat connection such that the sections $C_I \zeta$ for $I \subset J$ independent are $\nabla^K$-flat. The sections $\det(b_{ij})_{i \in I, j = 1, \ldots, k} \cdot C_I \zeta$ for $I \subset J$ independent generate a $\nabla^K$-flat $\mathbb{Z}$-lattice structure on $K$.

The pairing $S$ comes from the Grothendieck residue with respect to the volume form $dt_1 \wedge \ldots \wedge dt_k$.

It is a symmetric, nondegenerate, $\nabla^K$-flat, multiplication invariant and Higgs field invariant.

The existence of potentials of the first and second kind for families of arrangements was conjectured in [V1]. If all the $k \times k$ minors of the matrix $B = (b_{ij})$ are nonzero, the potentials were constructed in [V1], cf. [V3]. The potentials are given by explicit formulas in terms of the linear functions defining the hyperplanes in $\mathbb{C}^n$ composing the discriminant.

Remarks 4.3. (i) The situation in remark 4.2 is in several aspects richer than a Frobenius like structure of type $(n, k, m)$. The bundle $K$ is a bundle of algebras. The sections $C_I \zeta$ for independent sets $I \subset J$ generate the bundle. The sections $\det(b_{ij})_{i \in I, j = 1, \ldots, k} \cdot C_I \zeta$ generate a flat $\mathbb{Z}$-lattice structure in $K$. The Higgs field vanishes on the vector fields $X_1, \ldots, X_k$. The $m$-linear form $S$ is a pairing $(m = 2)$ and is nondegenerate. We will not discuss the $\mathbb{Z}$-lattice structure, but we will discuss some logical relations between the other enrichments and some implications of them.

(ii) Let $(M, K, \nabla^K, C, S, \zeta, V, (v_1, \ldots, v_n))$ be a Frobenius like structure of order $(n, k, m)$. Suppose that it satisfies the generation condition

\begin{equation}
\text{(GC) } \quad \text{The sections } C_I \zeta \text{ for independent sets } I \subset J \quad (4.20)
\end{equation}

generate the bundle $K$.

Let $\mu$ be the rank of $K$. Then for any $x \in M$, the endomorphisms $C_X, X \in T_x M$, generate a $\mu$-dimensional commutative subalgebra $A_x \subset \text{End}(K_x)$. And any endomorphism which commutes with them is contained in this subalgebra. This gives a rank $\mu$ bundle $A$ of commutative algebras. And the map

\begin{equation}
A \to K, \quad B \mapsto B \zeta, \quad (4.21)
\end{equation}

is an isomorphism of vector bundles and induces a commutative and associative multiplication on $K_x$ for any $x \in M$, with unit field $\zeta(x)$. Therefore the special section $\zeta$ and the generation condition (GC), which exist and hold in remark 4.2, give the multiplication on the bundle $K$ there.

(iii) In the situation in (ii) with the condition (GC), the $m$-linear form is multiplication invariant because it is Higgs field invariant. The condition (GC)
implies also that it is symmetric:

\[ S(C_{I_1}\zeta, C_{I_2}\zeta, \ldots, C_{I_n}\zeta) = S(C_{I_{\sigma(1)}}\zeta, C_{I_{\sigma(2)}}\zeta, \ldots, C_{I_{\sigma(m)}}\zeta) \]

for any independent sets \( I_1, \ldots, I_m \) and any permutation \( \sigma \in S_m \).

(iv) For a Frobenius like structure \((M, K, \nabla^K, C, S, \zeta, V, (v_1, \ldots, v_n))\) of order \((n, k, m)\) define the following \( k \)-dimensional space of linear combinations of the coordinate vector fields \( \partial_1, \ldots, \partial_n \),

\[ F^{inv} := \{ \sum_{i=1}^n \lambda(v_i)\partial_i \mid \lambda \in V^* \}. \quad (4.22) \]

It embeds into the tangent space \( T_xM \) for any \( x \in M \). In remark 4.2 the space of these vector fields is the space \( \sum_{i=1}^k C \cdot X_i \).

The weak injectivity condition is the condition for any \( x \in M \):

\[ (wIC) \quad \{ X \in T_xM \mid C_X = 0 \} = F^{inv} \subset T_xM. \quad (4.23) \]

We expect that the potentials \( Q \) and (locally) \( L \) in theorem 4.2 can be chosen to be invariant with respect to the flows of the vector fields \( X \in F^{inv} \) if the conditions (wIC) and (GC) hold. This is relevant for part (v) below.

(v) Let \((M, K, \nabla^K, C, S, \zeta, V, (v_1, \ldots, v_n))\) be a Frobenius like structure of type \((n, k, 2)\) with nondegenerate pairing \( S \) which satisfies the conditions (GC) and (wIC). Consider an affine linear submanifold \( N \subset M \) of dimension \( n - k \) which is transversal to the orbits of \( F^{inv} \). One can identify it with the manifold of these orbits. If potentials \( Q \) and \( L \) can be chosen to be constant on these orbits, they live on this manifold. An unfolding result in [HM] can be applied to the restriction of \((M, K, \nabla^K, C)\) to \( N \). This will be discussed in the remarks 4.4.

Remarks 4.4. (i) Let \((M, K, \nabla^K, C, S)\) be as follows. \( K \to M \) is a holomorphic vector bundle, \( \nabla^K \) is a holomorphic flat connection on \( K, C \) is a Higgs field on \( K \) with \( \nabla^K(C) = 0 \), and \( S \) is a holomorphic symmetric nondegenerate \( \nabla^K \)-flat and Higgs field invariant pairing on \( K \). Let \( p : \mathbb{P}^1 \times M \to M \) be the projection. The holomorphic vector bundle \( H := p^*K \) on \( \mathbb{P}^1 \times M \) is a family of trivial vector bundles \( H|_{\mathbb{P}^1 \times \{z\}}, z \in M, \) on \( \mathbb{P}^1 \). Extend \( \nabla^K, C \) and \( S \) canonically to \( H \). Define

\[ \nabla := \nabla^K + \frac{1}{\kappa}C, \quad (4.24) \]

\[ \nabla : \mathcal{O}(H) \to \mathcal{O}_{\mathbb{P}^1 \times M}(\{0\} \times M) \cdot \Omega^1_M \otimes \mathcal{O}(H), \]

where \( \kappa \) is the coordinate on \( \mathbb{C} \subset \mathbb{P}^1 \). Then \( \nabla \) restricts for any \( \kappa \in \mathbb{P}^1 - \{0\} \) to a flat connection on \( H|_{\{\kappa\} \times M} \) and has a pole of order 1 along \( \{0\} \times M \). Define a pairing \( P : H_{(\kappa, z)} \times H_{(-\kappa, z)} \to \mathbb{C} \) for any \( (\kappa, z) \in \mathbb{P}^1 \times M \) by

\[ P(a(\kappa, z), b(-\kappa, z)) := S(a(z), b(z)), \quad (4.25) \]

where \( a(\kappa, z) \in H_{\kappa,z} \) and \( b(-\kappa, z) \in H_{-\kappa,z} \) are the canonical lifts of elements \( a(z), b(z) \in K_{z} \). Then \( P \) is a holomorphic symmetric nondegenerate \( \nabla \)-flat pairing.

In the notation of [HM], the tuple \((H \to \mathbb{P}^1 \times M, \nabla, P)\) is a \((trTLP(0))\)-structure. One can recover \((M, K, \nabla^K, C, S)\) from it. So one has an equivalence.
of data \((M, K, \nabla^K, C, S)\) and \((H \to \mathbb{P}^1 \times M, \nabla, P)\). For slightly richer structures, such equivalences are formulated in [S ch. VII], [H 5.2], [HM theorem 4.2].

(ii) The main unfolding result theorem 2.5 in [HM] applies also to \((tr TLP(0))\)-structures, see [HM remark 3.3 (vii)]. In the situation in remark 4.3 (v), it applies, because its hypotheses are satisfied: The generation condition \((GC)\) above is a special case of the generation condition in [HM theorem 2.5]. For any \(x \in N\), the map \(T_xN \to K_x, Y \mapsto C_Y\zeta\), \((4.26)\)

is injective, so it satisfies the injectivity condition in [HM theorem 2.5]. The unfolding result reads in our situation as follows.

For any \(x \in N\), the germ of the tuple \(((N, x), K|_{(N, x)}, \nabla^K, C, S)\) has a unique (up to isomorphism) unfolding to a tuple \(((\tilde{N}, x), \tilde{K}, \nabla^{\tilde{K}}, \tilde{C}, \tilde{S})\) with the properties: \((\tilde{N}, x) \supset (N, x)\) is the germ at \(x\) of a manifold of dimension \(\mu := \text{rank} K\), \(\tilde{K} \to (\tilde{N}, x)\) is a vector bundle of rank \(\mu\), \(\nabla^{\tilde{K}}\) is a flat connection on it, \(\tilde{C}\) is a Higgs field on it with \(\nabla^{\tilde{K}}(\tilde{C}) = 0\), and \(\tilde{S}\) is a symmetric nondegenerate \(\nabla^{\tilde{K}}\)-flat and Higgs field invariant pairing on \(\tilde{K}\), and finally, the map

\[ T_x\tilde{N} \to \tilde{K}_x, \quad Y \mapsto \tilde{C}_Y\zeta, \] \((4.27)\)

is an isomorphism. On \((N, x) \subset (\tilde{N}, x)\) the tuple restricts to the tuple \(((N, x), K|_{(N, x)}, \nabla^K, C, S)\).

The Higgs field endomorphisms \(\tilde{C}_X, X \in T_{\tilde{N}}\), induce a bundle \(\tilde{A}\) of \(\mu\)-dimensional commutative subalgebras \(A_z \subset \text{End}(K_z)\) for \(z \in \tilde{N}\), and the map

\[ T_z\tilde{N} \to A_z, \quad Y \mapsto C_Y \] \((4.28)\)

is an isomorphism. It induces a multiplication on \(T_{\tilde{N}}\) which turns out to give \(T_{\tilde{N}}\) the structure of an \(F\)-manifold ([H lemma 4.1 and lemma 4.3]).

(iii) We continue with the situation in (ii). Choose an extension of the section \(\zeta\) to a section \(\tilde{\zeta}\) in the bundle \(\tilde{K} \to (\tilde{N}, x)\). The isomorphism \((4.25)\) extends to an isomorphism

\[ T_{\tilde{N}} \to \mathcal{O}(\tilde{K}), \quad Y \mapsto \tilde{C}_Y\tilde{\zeta}. \] \((4.29)\)

The section \(\tilde{\zeta}\), the flat connection \(\nabla^{\tilde{K}}\), the Higgs field \(\tilde{C}\) and the pairing \(\tilde{S}\) can be shifted to \(T_{\tilde{N}}\) with this isomorphism. The induced Higgs field gives the multiplication above: That does not depend on the choice of \(\tilde{\zeta}\). But the other induced data depend on it.

One wishes an extension \(\tilde{\zeta}\) such that the induced connection on \(T_{\tilde{N}}\) is torsion free. One wishes a natural way to extend the notion of a Frobenius like structure to the bundle \(\tilde{K} \to \tilde{N}\). And one wishes an extension \(\tilde{\zeta}\) which shifts this extension in the best possible way to \(T_{\tilde{N}}\).

(iv) The following special case gives rise to Frobenius manifolds without Euler fields. Consider a Frobenius like structure \((M, K, \nabla^K, C, S, \zeta, V, (v_1, ..., v_n))\)
of order \((n,1,2)\) with nondegenerate pairing \(S\), \(\nabla^K\)-flat section \(\zeta\), the conditions (GC) and (wIC) and all \(v_j \neq 0\) in the 1-dimensional space \(V\). Then the sections \(\phi_\partial \zeta\) generate the bundle \(K\) and are \(\nabla^K\)-flat, the map \(T_x M \to K_x\), \(Y \mapsto C_Y \zeta\), is surjective with 1-dimensional kernel, \(\mu = n - 1\), the map \([4.24]\) is an isomorphism, the tuple \(((N, x), K|_{(N, x)}, \nabla^K, C, S)\) is its own universal unfolding, and \(N = \tilde{N}\). Here \(N\) becomes a Frobenius manifold (without Euler field). The induced connection on \(\mathcal{T}_N\) is the one of the affine linear structure on \(N\) and is torsion free. It is also the Levi-Civita connection of the metric on \(\mathcal{T}_M\) which is induced by \(S\). The restriction of the potential \(L\) to \(N\) is the potential of the Frobenius manifold.

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