Stability of Major Constraint Programming

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Keywords: Nonlinear programming, Major constraint programming, Stability, Continuity and semicontinuity.

Abstract. In this paper, the stability of major constraint programming is studied. With the help of the representation of major constraint set structure, the continuity of the optimal value with the perturbation variables for perturbation major constraint programming is given when the objective function and constraint functions with the independent variables and perturbation variables are continuous. At the same time, the semicontinuity of the optimal solution set of perturbation major constraint programming is obtained.

Introduction

Nonlinear programming is an important part of mathematical programming, its theory and method have become an important means of modern quantitative decision-making. Since the 1940’s, a number of studies have examined about the theory and application of nonlinear programming and its various special model. Its many new research fields and discipline branches are produced[1]. While establishing actual nonlinear optimization model, the various constraint conditions of nonlinear programming model are given by the policy makers and analysts under the different needs and angle, thus it inevitably occurs that the inequality constraints of the proposed model are incompatible. So we consider a new kind of nonlinear programming, major constraint programming, relatives to the general nonlinear programming model, where the major constraints condition is compatible.

In Ref.[2], the concepts of major cone and major order in Euclid space have been introduced, and a major constraints programming problem has been presented. In Ref.[3], the major constraint programming problem is studied. Using the representation of major constraint set structure and some constraint qualification conditions for the problem, the necessary optimality condition of major constraint optimal solutions is proved. At the same time, the sufficient optimality condition of major constraint optimal solutions is also given.

As is known to all, when using mathematical programming to solve practical problem, we must first turn the nature of the problem to mathematical programming model. Since the model data often comes from measured value, its true value has certain error with the data. At the same time, the environment or human factors as a practical problem often change parameters. The studies of these data error or parameter change with respect to the authenticity of the solution of mathematical programming model are the stability analysis of the mathematical programming problem. In recent decades, a lot of people at home and abroad have studied on the stability of mathematical programming and related problems. Some good results were obtained[4–9]. In this paper, the stability of major constraint programming have been studied. With the help of the representation of major constraint set structure, the continuity of the optimal value with the perturbation variables for perturbation major constraint programming has been studied when the objective function and constraint functions with the independent variables and perturbation variables are continuous. At the same time, the semicontinuity of the optimal solution set of perturbation major constraint programming is obtained.

Definitions and Lemmas

Consider the following form of parametric major constraint programming problem
where \( t = (t_1, \cdots, t_r)^\top \in \mathbb{R}^r, \ r \geq 1, \ T = \{t || t|| \leq T_0 \} (T_0 > 0), \ f(x,t), g_i(x,t), \ i = 1, \cdots, p \) are the real functions in \( R^p \times T \). \( \text{(MCP}_0) \) is the original major constraint programming problem, \( \text{(MCP}_t) \) \( (t \neq 0, || t|| \text{ sufficiently small} \) is a major constraint programming problem after a little perturbation. \( clH \) is the closure of major cone \( H \). The major constraints condition of problem \( \text{(MCP}_t) \) \( (g_1(x,t), \cdots, g_p(x,t)) \leq 0 \) means that for any \( t \in T \) there exist at least \( \lceil \frac{p+1}{2} \rceil \) numbers \( i_1, \cdots, i_{\frac{p+1}{2}} \in \{1, \cdots, p\} \) such that
\[
 g_{i_1}(x,t) \leq 0, \cdots, g_{i_{\frac{p+1}{2}}}(x,t) \leq 0.
\]

Denote the major set of parameterized major constraint programming problem \( \text{(MCP}_t) \) for \( t \in T \)
\[
 X^t_M = \{x \in R^p \mid g(x,t) \leq 0\}, \tag{1}
\]
the optimal value function
\[
y(t) = \min_{x \in X^t_M} f(x,t),
\]
and the corresponding optimal solution set
\[
 S(t) = \arg \min_{x \in X^t_M} f(x,t).
\]
So \( X^t_M \) and \( S(t) \) define the point-set mapping from \( T \) to \( R^n \) respectively.

In order to discuss the representation of major constraint set (1), there are the following two lemmas by the Ref. [3].

Let \( v = (v_1, \cdots, v_p)^\top \in R^p \). Let \( |v|_+, |v|_- \) and \( |v|_0 \) be the number of components of \( v \) greater than zero, the less than zero and the equal to zero respectively.

**Lemma1** Let \( H \subset R^p \) be major cone. Then its closed cone is
\[
 clH = \{v \in R^p \mid |v|_+ + |v|_0 \geq \lceil \frac{p+1}{2} \rceil \}.
\]

Let \( i_1, \cdots, i_{\frac{p+1}{2}} \in \{1, \cdots, p\}, K_{i_1, \cdots, i_{\frac{p+1}{2}}} = \{v \in R^p \mid v_{i_1} \geq 0, \cdots, v_{i_{\frac{p+1}{2}}} \geq 0 \} \).

**Lemma2** Let \( H \subset R^p \) be major cone. Then its closed cone is
\[
 clH = \bigcup_{i_1, \cdots, i_{\frac{p+1}{2}} \in \{1, \cdots, p\}} K_{i_1, \cdots, i_{\frac{p+1}{2}}}.
\]

Finally, we give the representation of major constraint set (1) structure. Denote
\[
 X^t_{i_1, \cdots, i_{\frac{p+1}{2}}} = \{x \in R^p \mid -g(x,t) \in K_{i_1, \cdots, i_{\frac{p+1}{2}}} \}
\]
\[
 = \{x \in R^p \mid g_{i_1}(x,t) \leq 0, \cdots, g_{i_{\frac{p+1}{2}}}(x,t) \leq 0 \}.
\]

**Lemma3** Let \( X^t_M \) be the parameterized major constraint set of \( \text{(MCP}_t) \) for \( t \in T \). Then
$$X_i' = \bigcup_{t \in [0, T)} X_i^{t-\mu_i^1}.$$ 

Let us recall now some definitions of continuity for point set mapping.

Let $$T = \{ t || t || \leq T_0 \} (T_0 > 0)$$ be a nonempty set in $$R^n$$. Let $$\psi : T \rightarrow 2^{R^n}, t \mapsto \psi(t)$$ be a point set mapping. Denote 

$$\text{graf}\psi = \{(t, y) \in T \times Y : y \in \psi(t), t \in T\}.$$ 

**Definition 1** Let $$t_0 \in \text{int } T$$.

1. $$\psi$$ is said to be upper semicontinuous at $$t_0$$, if for any neighborhood $$V \subset Y$$ of $$\psi(t_0)$$ there exists a neighborhood $$U \subset T$$ of $$t_0$$ such that $$\psi(t) \subset V$$ for each $$t \in U$$.
2. $$\psi$$ is said to be lower semicontinuous at $$t_0$$ if for any $$y \in \psi(t_0)$$ and any neighborhood $$V$$ of $$y$$ there exists a neighborhood $$U \subset T$$ of $$t_0$$ such that $$\psi(t) \cap V \neq \emptyset$$ for each $$t \in U$$.
3. $$\psi$$ is said to be continuous at $$t_0$$ if it is both upper semicontinuous and lower semicontinuous at that point.

**Definition 2** Let $$t_0 \in \text{int } T$$. $$\psi$$ is said to be closed at $$t_0$$ if whenever a sequence $$\{ (t_k, y_k) \}$$ in the graph $$\text{graf}\psi$$ of $$\psi$$ has a limit $$(t_0, y_0) \in \text{int } T \times Y$$ then $$y_0 \in \psi(t_0)$$.

For any $$y \in Y$$, if $$F(y)$$ is a compact set in $$X$$, then $$F$$ is upper semicontinuous in $$Y$$ if and only if it is a closed set-valued mapping in $$Y$$.

The result about the sets operation of point set mapping is given as follows:

**Lemma 4** Let the point set mappings $$A_j : T \rightarrow 2^X, t \mapsto A_j(t) (j = 1, \cdots, m)$$ be continuous in $$T$$.

If $$A_j(t) (j = 1, \cdots, m)$$ are compact set in $$X$$, then the set-valued mapping $$A : T \rightarrow 2^X, t \mapsto A(t) = \bigcup_{j=1}^m A_j(t)$$ is continuous in $$T$$.

**Proof** Only prove that $$A$$ is a closed set mapping in $$T$$. Let sequences $$\{t_k\} \subset T$$, $$\{x_k\} \subset X$$ to satisfy $$x_k \in A(t_k), t_k \rightarrow \overline{t}, x_k \rightarrow \overline{x}$$. By $$x_k \in A(t_k)$$, it is easy to prove that there exists an $$M \in \{1, \cdots, m\}$$ and a subsequence $$\{t_{k_p}\}$$ of sequences $$\{t_k\}$$ such that $$x_{k_p} \in A_M(t_{k_p})$$. From $$t_k \rightarrow \overline{t}$$, we have $$t_{k_p} \rightarrow \overline{t}$$. Since point set mapping $$A_M : T \rightarrow 2^X, t \mapsto A_M(t)$$ is upper semicontinuous in $$T$$ and $$A_M(t)$$ is a compact set in $$X$$, it follows that $$\overline{x} \in A_M(\overline{t})$$. Then $$\overline{x} \in \bigcup_{j=1}^m A_j(\overline{t}) = A(\overline{t})$$, that is $$A$$ is a closed set-valued mapping in $$T$$.

**Stability of Major Constrains Programming**

In order to discuss the stability of the major constrains programming, we will use some of the following hypotheses:

$$H_1$$: $$f(x, t), g_i(x, t), i = 1, \cdots, p$$ are continuous functions in $$R^n \times T$$;

$$H_2$$: Let $$i_1, \cdots, i_2, j_{\frac{\mu_i^1}{T_i}} \in \{1, \cdots, p\}$$. If $$X_{i_1, \cdots, i_2, j_{\frac{\mu_i^1}{T_i}}}^0 \neq \emptyset$$, then for any $$x^0 \in X_{i_1, \cdots, i_2, j_{\frac{\mu_i^1}{T_i}}}^0$$ and for any real number $$\delta > 0$$ there exists $$x^\delta \in X_{i_1, \cdots, i_2, j_{\frac{\mu_i^1}{T_i}}}^0$$ such that

$$\| x^\delta - x^0 \| < \delta,$$

$$\{ x \mid g_i(x, 0) < 0, \cdots, g_i(x, 0) < 0 \}$$

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that is \( cl(s - \text{int} X_{t}^{0}) = X_{t}^{0} \).

H3: There is an \( \alpha_{0} < T_{0} \) such that \( S(t) \neq \emptyset \) and are uniformly bounded for all \( t (||t|| \leq \alpha_{0}) \).

**Theorem 1** If the hypotheses H1,H2 are hold, then for any given \( x^{0} \in X_{t}^{0} \) and \( \delta > 0 \), there exists an \( \alpha > 0 \) such that \( X_{t}^{0} \neq \emptyset \) for each \( t (||t|| \leq \alpha) \) and there exists \( x^{\delta} \in X_{t}^{0} \) such that \( ||x^{\delta} - x^{0}|| < \delta \).

**Proof** For any given \( x^{0} \in X_{t}^{0} \) there exist \( i_{1}, \ldots, i_{p} \in \{1, \ldots, p\} \) such that \( x^{0} \in X_{t}^{0} \). By hypothesis H2, for any \( \delta > 0 \), there exist a \( x^{\delta} \in X_{t}^{0} \) such that \( ||x^{\delta} - x^{0}|| < \delta, g_{i_{1}}(x^{\delta},0) < 0, \ldots, g_{i_{p}}(x^{\delta},0) < 0 \).

Take \( \epsilon > 0 \) to satisfy \( g_{i_{j}}(x^{\delta},0) \leq -\epsilon < 0, \ldots, g_{i_{p}}(x^{\delta},0) \leq -\epsilon < 0 \). Since \( g_{i}(x,t), \ldots, g_{i_{p}}(x,t) \) are continuous at \( t = 0 \), there exists \( \alpha > 0 \) for each \( t (||t|| \leq \alpha) \) such that \( g_{i_{1}}(x^{\delta},t) < g_{i_{1}}(x,0) + \epsilon \leq 0, \ldots, g_{i_{p}}(x^{\delta},t) < g_{i_{p}}(x^{\delta},0) + \epsilon \leq 0 \), that is \( x^{\delta} \in X_{t}^{0} \subset X_{t}^{0} \).

**Theorem 2** If the hypotheses H1,H2 are hold and let \( S(t) \neq \emptyset (||t|| \leq \alpha) \). Then \( S(t) \) is closed at \( t = 0 \).

**Proof** For any sequence \( \{t_{k}\} \), \( t_{k} \rightarrow 0(k \rightarrow \infty) \), if \( x(t_{k}) \in S(t_{k}) \) and \( \lim x(t_{k}) = \bar{x} \), then we only prove \( \bar{x} \in S(0) \).

Take any sequence \( \{\delta_{k}\} \), \( \delta_{k} > 0 \), \( \lim_{k \rightarrow \infty} \delta_{k} = 0 \) and any \( x(0) \in S(0) \subset X_{t}^{0} \). By Theorem 3, for each \( \delta_{k} \) \( (k = 1,2, \ldots) \) there exists an \( \alpha_{k} > 0 \) and \( x^{\delta_{k}} \in X_{t}^{0} \) for each \( t (||t|| \leq \alpha_{k}) \), such that \( ||x^{\delta_{k}} - x(0)|| < \delta_{k} \), so \( \lim_{k \rightarrow \infty} x^{\delta_{k}} = x(0) \). From \( \lim_{k \rightarrow \infty} t_{k} = 0 \), there exists a subsequence \( \{t_{j_{k}}\} \) of \( \{t_{k}\} \) to satisfy \( ||t_{j_{k}}|| \leq \alpha_{j}(j = 1,2, \ldots) \). Obviously, there is still \( \lim x(t_{j_{k}}) = \bar{x} \). Since \( x(t_{j_{k}}) \in S(t_{j_{k}}) \) and \( x^{\delta_{j_{k}}} \in X_{t}^{0} \),

\[
f(x^{\delta_{j_{k}}}, t_{j_{k}}) \geq f(x(t_{j_{k}}), t_{j_{k}}).
\]

By the continuity of \( f(x,t), f(x(0),0) \geq f(\bar{x},0) \), it means that \( \bar{x} \in S(0) \) as \( x(0) \in S(0) \) and \( \bar{x} \in X_{t}^{0} \).

**Theorem 3** If the hypotheses H1, H2, H3 are hold, then

1. the multifunction \( S(t) \) is upper semicontinuous at \( t = 0 \);
2. the optimal value function \( y(t) \) is continuous at \( t = 0 \).

**Proof** (1) Let \( \epsilon \) be an arbitrary positive number and consider the set

\[
G_{\epsilon} = [S(0)]_{\epsilon} = \bigcup_{x(0) \in S(0)} \{x \mid ||x - x(0)|| < \epsilon\}
\]

It is easy to know that \( G_{\epsilon} \) is an open set and \( G_{\epsilon} \supset S(0) \). We now show that there exists an \( \bar{\alpha} > 0 \) such that \( S(t) \subseteq G_{\epsilon} \) when \( ||t|| \leq \bar{\alpha} \). By contradiction, if not, take \( \tilde{\alpha} > 0(\tilde{\alpha} \leq \alpha_{0}) \) to satisfy hypothesis H3, then there exists a sequence \( \{t_{j}\} \), \( t_{j} \rightarrow 0(k \rightarrow \infty) \) such that \( x(t_{j}) \in S(t_{j}) \) and \( x(t_{j}) \notin G_{\epsilon} \). By hypothesis that \( S(t) \) are uniformly bounded for all \( t (||t|| \leq \alpha_{0}) \), then there exists a constant \( c > 0 \) (independent of \( t \)) such that \( ||x|| \leq c \) for all \( x \in S(t)(||t|| \leq \bar{\alpha}) \). Denote
It is easy to know that \( G_\varepsilon \subset S, x(t^*) \in S \). So \[ S - G_\varepsilon = \{ x \mid x \in S, x \notin G_\varepsilon \}. \]
Since \( S - G_\varepsilon \) is a bounded closed set, without loss of generality, it follows that there exists a \( \bar{x} \in S - G_\varepsilon \) such that \( \lim_{k \to \infty} x(t^k) = \bar{x} \). But from Theorem 2, we have \( \bar{x} \in S(0) \subset G_\varepsilon \). It contradict with \( \bar{x} \in S - G_\varepsilon \) so \( S(t) \) is upper semicontinuous at \( t = 0 \).

(2) By the uniform continuity of \( f(x,t) \) and the uniform boundedness of \( S(t) \) on a bounded closed set, it follows that for any given \( \varepsilon > 0 \) there exist \( \delta > 0, \alpha \) such that

\[ |y(t) - y(0)| = |f(x(t),t) - f(x(0),0)| < \varepsilon \]
for each \( t \) such that \( \|t\| \leq \alpha_2, \|x(t) - x(0)\| < \delta, x(t) \in S(t), x(0) \in S(0) \).

Let \( \varepsilon_1 = \min\{\varepsilon, \delta\} \). By the upper semicontinuity of \( S(t) \) at \( t = 0 \) there exist \( \alpha_2 > 0 \) and \( x(0) \in S(0) \) such that \( \|x(t) - x(0)\| < \varepsilon_1 < \varepsilon \) for each \( t \) such that \( \|t\| \leq \alpha_2 \). Let \( \alpha = \min\{\alpha_1, \alpha_2\} \) it follows that \( \|t\| \leq \alpha_1, \|x(t) - x(0)\| < \delta \) for \( \|t\| \leq \alpha \), so

\[ |y(t) - y(0)| = |f(x(t),t) - f(x(0),0)| < \varepsilon. \]

Therefore \( y(t) \) is continuous at \( t = 0 \).

Summary

Stability and sensitivity analysis for optimization problems is a very active field [10-11]. Among important contributions that were not quoted yet, let us mention the work based on epigraphical analysis [12-14] and applications of polyhedricity theory [15] and its generalizations [16].

We have carefully reviewed an approach to stability and sensitivity analysis that uses lower and upper estimates of the optimal value function. For nonlinear programming problems, this gives strong results, and the theory seems to be more or less complete now. These results can extended to major constraint programming problems, and to semidefinite major constraint programming. At the same time, it is possible to derive Lipschitz stability of optimal solutions of semidefinite major constraint programs under the second order growth condition and a nondegeneracy assumption [17-18].

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