A NEW OBSTRUCTION TO MINIMAL ISOMETRIC IMMERSIONS INTO A REAL SPACE FORM

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Abstract

In the theory of minimal submanifolds, the following problem is fundamental: when does a given Riemannian manifold admit (or does not admit) a minimal isometric immersion into an Euclidean space of arbitrary dimension? S.S. Chern, in his monograph Minimal submanifolds in a Riemannian manifold, remarked that the result of Takahashi (the Ricci tensor of a minimal submanifold into a Euclidean space is negative semidefinite) was the only known Riemannian obstruction to minimal isometric immersions in Euclidean spaces. A second solution to this problem was obtained by B.Y. Chen as an immediate application of his fundamental inequality [1]: the scalar curvature and the sectional curvature of a minimal submanifold into a Euclidean space satisfies the inequality $\tau \leq k$. In this paper we prove that the sectional curvature of a minimal submanifold into a Euclidean space also satisfies the inequality $k \leq -\tau$.

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1. OPTIMIZATIONS ON RIEMANNIAN MANIFOLDS

Let \((N, \tilde{g})\) be a Riemannian manifold, \((M, g)\) a Riemannian submanifold, and \(f \in \mathcal{F}(N)\). To these ingredients we attach the optimum problem

\[
\text{(1) } \min_{x \in M} f(x).
\]

Let’s remember the result obtained in [6].

**Theorem 1.1.** If \(x_0 \in M\) is the solution of the problem (1), then

i) \((\text{grad } f)(x_0) \in T_{x_0}^\perp M,\)

ii) the bilinear form

\[
\alpha : T_{x_0}M \times T_{x_0}M \rightarrow \mathbb{R},
\]

\[
\alpha(X,Y) = \text{Hess}_x f(X,Y) + \tilde{g}(h(X,Y), (\text{grad } f)(x_0))
\]

is positive semidefinite, where \(h\) is the second fundamental form of the submanifold \(M\) in \(N\).

2. THE CHEN’S INEQUALITY

Let \((M, g)\) be a Riemannian manifold of dimension \(n\), and \(x\) a point in \(M\). We consider the orthonormal frame \(\{e_1, e_2, \ldots, e_n\}\) in \(T_x M\).

The scalar curvature at \(x\) is defined by

\[
\tau = \sum_{1 \leq i < j \leq n} R(e_i, e_j, e_i, e_j).
\]

We denote

\[
\delta_M = \tau - \min(k),
\]

where \(k\) is the sectional curvature at the point \(x\). The invariant \(\delta_M\) is called the Chen’s invariant of Riemannian manifold \((M, g)\).
The Chen’s invariant can be estimated, if \((M, g)\) is a Riemannian submanifold in a real space form \(\tilde{M}(c)\), varying with \(c\) and the mean curvature of \(M\) in \(\tilde{M}(c)\).

**Theorem 2.1.** Consider \((\tilde{M}(c), \tilde{g})\) a real space form of dimension \(m\), \(M \subset \tilde{M}(c)\) a Riemannian submanifold of dimension \(n \geq 3\). The Chen’s invariant of \(M\) satisfies

\[
\delta_M \leq \frac{n - 2}{2} \left( \frac{n^2}{n - 1} \|H\|^2 + (n + 1)c \right),
\]

where \(H\) is the mean curvature vector of submanifold \(M\) in \(\tilde{M}(c)\). The equality is attained at the point \(x \in M\) if and only if there is an orthonormal frame \(\{e_1, ..., e_n\}\) in \(T_xM\) and an orthonormal frame \(\{e_{n+1}, ..., e_m\}\) in \(T_x^\perp M\) in which the Weingarten operators have the following form

\[
A_{n+1} = \begin{pmatrix}
    h_{11}^{n+1} & 0 & 0 & 0 \\
    0 & h_{22}^{n+1} & 0 & 0 \\
    0 & 0 & h_{33}^{n+1} & 0 \\
    0 & 0 & 0 & h_{nn}^{n+1}
\end{pmatrix},
\]

with \(h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = ... = h_{nn}^{n+1}\) and

\[
A_r = \begin{pmatrix}
    h_{11}^r & h_{12}^r & 0 & 0 \\
    h_{12}^r & -h_{11}^r & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}, \quad r \in \overline{n+2, m}.
\]

**Corollary 2.1.** If the Riemannian manifold \((M, g)\), of dimension \(n \geq 3\), admit a minimal isometric immersion into a real space form \(\tilde{M}(c)\), then

\[
k \geq \tau - \frac{(n - 2)(n + 1)c}{2}.
\]

By using the constrained extremum method we give another Riemannian obstruction to minimal isometric immersions in real space forms

\[
k \leq -\tau + \frac{(n^2 - n + 2)c}{2}.
\]
3. A NEW OBSTRUCTION TO MINIMAL ISOMETRIC IMMERSIONS INTO A REAL SPACE FORM

Let \((M, g)\) be a Riemannian manifold of dimension \(n\). We define the following invariants

\[
\delta^a_M = \begin{cases} 
\tau - a \inf k, & \text{for } 0 \leq a < 1, \\
\tau - a \sup k, & \text{for } -1 < a < 0 
\end{cases}
\]

\[
\delta'_M = \tau + \sup k,
\]

where \(\tau\) is the scalar curvature, and \(k\) is the sectional curvature.

With these ingredients we obtain

**Theorem 3.1.** For any real number \(a \in (-1, 1)\), the invariant \(\delta^a_M\) of a Riemannian submanifold \((M, g)\) of dimension \(n \geq 3\), into a real space form \(\tilde{M}(c)\), of dimension \(m\), verifies the inequality

\[
\delta^a_M \leq \frac{(n^2 - n - 2a)c}{2} + \frac{n(a + 1) - 3a - 1}{n(a + 1) - 2a} \frac{n^2 \|H\|^2}{2},
\]

where \(H\) is the mean curvature vector of submanifold \(M\) in \(\tilde{M}(c)\). The equality is attained at the point \(x \in M\) if and only if there is an orthonormal frame \(\{e_1, \ldots, e_n\}\) in \(T_x M\) and an orthonormal frame \(\{e_{n+1}, \ldots, e_m\}\) in \(T_x^\perp M\) in which the Weingarten operators have the following form

\[
A_r = \begin{pmatrix}
  h_{11}^r & 0 & 0 & 0 \\
  0 & h_{22}^r & 0 & 0 \\
  0 & 0 & h_{33}^r & 0 \\
  \vdots & \vdots & \vdots & \ddots \\
  0 & 0 & 0 & h_{nn}^r
\end{pmatrix},
\]

with \((a + 1)h_{11}^r = (a + 1)h_{22}^r = h_{33}^r = \ldots = h_{nn}^r, \forall r \in \overline{n+1,m}\).

**Proof.** Consider \(x \in M\), \(\{e_1, e_2, \ldots, e_n\}\) an orthonormal frame in \(T_x M\), \(\{e_{n+1}, e_{n+2}, \ldots, e_m\}\) an orthonormal frame in \(T_x^\perp M\) and \(a \in (-1, 1)\).
From Gauss equation it follows
\[
\tau - ak(e_1 \wedge e_2) = \frac{(n^2 - n - 2a)c}{2} + \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} (h^r_{ii}h^r_{jj} - (h^r_{ij})^2) - \\
-a \sum_{r=n+1}^{m} (h^r_{11}h^r_{22} - (h^r_{12})^2).
\]

Using the fact that \(a \in (-1, 1)\), we obtain
(1) \(\tau - ak(e_1 \wedge e_2) \leq \frac{(n^2 - n - 2a)c}{2} + \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} h^r_{ii}h^r_{jj} - a \sum_{r=n+1}^{m} h^r_{11}h^r_{22}\).

For \(r \in n+1, m\), let us consider the quadratic form

\[
f_r : \mathbb{R}^n \to \mathbb{R},
\]

\[
f_r(h^r_{11}, h^r_{22}, ..., h^r_{nn}) = \sum_{1 \leq i < j \leq n} (h^r_{ii}h^r_{jj}) - ah^r_{11}h^r_{22}
\]

and the constrained extremum problem

\[
\max f_r,
\]

subject to \(P : h^r_{11} + h^r_{22} + ... + h^r_{nn} = k^r\),

where \(k^r\) is a real constant.

The first three partial derivatives of the function \(f_r\) are
(2) \(\frac{\partial f_r}{\partial h^r_{11}} = \sum_{2 \leq j \leq n} h^r_{jj} - ah^r_{22}\),
(3) \(\frac{\partial f_r}{\partial h^r_{22}} = \sum_{j \in \{1, m\} \setminus \{2\}} h^r_{jj} - ah^r_{11}\),
(4) \(\frac{\partial f_r}{\partial h^r_{jj}} = \sum_{j \in \{1, m\} \setminus \{3\}} h^r_{jj}\).

As for a solution \((h^r_{11}, h^r_{22}, ..., h^r_{nn})\) of the problem in question, the vector \((\text{grad} \ (f_1))\) is normal at \(P\), from (2) and (3) we obtain \(\sum_{j=1}^{n} h^r_{jj} - h^r_{11} - \\
-ah^r_{22} = \sum_{j=1}^{n} h^r_{jj} - h^r_{22} - ah^r_{11}\), therefore
(5) \(h^r_{11} = h^r_{22} = b^r\).
From (2) and (4), it follows \[ \sum_{j=1}^{n} h_{jj}^r - h_{11}^r - ah_{22}^r = \sum_{j=1}^{n} h_{jj}^r - h_{33}^r. \] By using (5) we obtain \( h_{33}^r = b^r(a + 1). \) Similarly one gets
\[ h_{jj}^r = b^r(a + 1), \quad \forall j \in \{3, n\}. \]

As \( h_{11}^r + h_{22}^r + \ldots + h_{nn}^r = k^r, \) from (5) and (6) we obtain
\[ b^r = \frac{k^r}{n(a + 1) - 2a}. \]

We fix an arbitrary point \( p \in P. \)
The 2-form \( \alpha : T_pP \times T_pP \to R \) has the expression
\[ \alpha(X, Y) = \Hess f_r(X, Y) + \langle h'(X, Y), (\gr f_r)(p) \rangle, \]
where \( h' \) is the second fundamental form of \( P \) in \( R^n \) and \( \langle , \rangle \) is the standard inner-product on \( R^n. \)

In the standard frame of \( R^n, \) the Hessian of \( f_r \) has the matrix
\[
\Hess f_r = \begin{pmatrix}
0 & 1 - a & 1 & 1 \\
1 - a & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
. & . & . & . \\
1 & 1 & 1 & 0
\end{pmatrix}
\]

As \( P \) is totally geodesic in \( R^n, \) considering a vector \( X \) tangent to \( P \) at the arbitrary point \( p, \) that is, verifying the relation \( \sum_{i=1}^{n} X^i = 0, \) we have
\[ \alpha(X, X) = 2 \sum_{1 \leq i < j \leq n} X^iX^j - 2aX^1X^2 = (\sum_{i=1}^{n} X^i)^2 - (\sum_{i=1}^{n} (X^i)^2) - 2aX^1X^2 =
\]
\[ = -\sum_{i=1}^{n} (X^i)^2 - a(X^1 + X^2)^2 + a(X^1)^2 + a(X^2)^2 =
\]
\[ = -\sum_{i=3}^{n} (X^i)^2 - a(1-a)(X^1)^2 - (1-a)(X^2)^2 \leq 0. \] Therefore the point \( (h_{11}^r, h_{22}^r, \ldots, h_{nn}^r), \) which satisfies (5), (6), (7) is a maximum point.

From (5) and (6) it follows
\[ f_r \leq (b^r)^2 + 2b^r(n-2)b^r(a+1) + C_{n-2}^2(b^r)^2(a+1)^2 - a(b^r)^2 =
\]
\[ = \frac{(b^r)^2}{2} [n^2(a+1)^2 - n(a+1)(5a+1) + 6a^2 + 2a] = \]

6
\[ f_r \leq \frac{(k')^2}{2(n(a+1)-2a)}[n(a+1)-3a-1][n(a+1)-2a]. \]

By using (7) and (8), we obtain
\[ (9) \quad f_r \leq \frac{(k')^2}{2n(a+1)-2a}[n(a+1)-3a-1] = \frac{n^2(H')^2}{2} \frac{n(a+1)-3a-1}{n(a+1)-2a}. \]

The relations (1) and (9) imply
\[ (10) \quad \tau - ak(e_1 \wedge e_2) \leq \frac{(n^2-n-2a)c}{2} + \frac{n(a+1)-3a-1}{n(a+1)-2a} \frac{n^2\|H\|^2}{2}. \]

In (10) we have equality if and only if the same thing occurs in the inequality (1) and, in addition, (5) and (6) occurs. Therefore
\[ (11) \quad h_{ij} = 0, \forall r \in n+1, m, \forall i, j \in 1, n, \text{ with } i \neq j \text{ and} \]
\[ (12) \quad (a+1)h_{11} = (a+1)h_{22} = h_{33} = \ldots = h_{nn}, \forall r \in n+1, m. \]

The relations (10), (11) and (12) imply the conclusion of the theorem.

Remark. i) Making \( a \) to converge at 1 in previous inequality, we obtain Chen’s Inequality. The conditions for which we have equality are obtained in [1] and [6].

ii) For \( a = 0 \) we obtain the well-known inequality
\[ \tau \leq \frac{n(n-1)}{2}(\|H\|^2 + c). \]

The equality is attained at the point \( x \in M \) if and only if \( x \) is a totally umbilical point.

iii) Making \( a \) to converge at \(-1\) in previous inequality, we obtain
\[ \delta'_M \leq \frac{(n^2-n+2)c}{2} + \frac{n^2\|H\|^2}{2}. \]

**Theorem 3.2.** The invariant \( \delta'_M \) of a Riemannian submanifold \((M, g)\), of dimension \( n \geq 3 \), into a real space form \( \tilde{M}(c) \), of dimension \( m \), verifies the inequality
\[ \delta'_M \leq \frac{(n^2-n+2)c}{2} + \frac{n^2\|H\|^2}{2}, \]

where \( H \) is the mean curvature vector of submanifold \( M \) in \( \tilde{M}(c) \). The equality is attained at the point \( x \in M \) if and only if there is an orthonormal
frame \( \{ e_1, \ldots, e_n \} \) in \( T_x M \) and an orthonormal frame \( \{ e_{n+1}, \ldots, e_m \} \) in \( T^\perp_x M \) in which the Weingarten operators have the following form

\[
A_r = \begin{pmatrix}
  h_{11}^r & 0 & 0 & 0 \\
  0 & h_{22}^r & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 
\end{pmatrix},
\]

with \( h_{11}^r = h_{22}^r, \) \( \forall r \in n + 1, m. \)

**Proof.** We consider the point \( x \in M \), the orthonormal frames \( \{ e_1, \ldots, e_n \} \) in \( T_x M \) and \( \{ e_{n+1}, \ldots, e_m \} \) in \( T^\perp_x M \), \( \{ e_1, e_2 \} \) being an orthonormal frame in the 2−plane which maximize the sectional curvature at the point \( x \) in \( T_x M \).

The invariant \( \delta'_M \) verifies

\[
(1) \quad \delta'_M = \frac{(n^2-n+2)c}{2} + \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2) + \sum_{r=n+1}^{m} (h_{11}^r h_{22}^r - (h_{12}^r)^2) \leq \frac{(n^2-n+2)c}{2} + \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} (h_{ii}^r h_{jj}^r) + h_{11}^r h_{22}^r.
\]

For \( r \in n + 1, m \), let us consider the quadratic form

\[
f_r : R^n \to R,
\]

\[
f_r(h_{11}^r, h_{22}^r, \ldots, h_{nn}^r) = \sum_{1 \leq i < j \leq n} (h_{ii}^r h_{jj}^r) + h_{11}^r h_{22}^r
\]

and the constrained extremum problem

\[
\max f_r ,
\]

subject to \( P : h_{11}^r + h_{22}^r + \ldots + h_{nn}^r = k^r \),

where \( k^r \) is a real constant.

The first three partial derivatives of the function \( f_r \) are

\[
(2) \quad \frac{\partial f_r}{\partial h_{11}^r} = \sum_{2 \leq j \leq n} h_{jj}^r + h_{22}^r,
\]

\[
(3) \quad \frac{\partial f_r}{\partial h_{22}^r} = \sum_{j \in [1, m \setminus \{2\}]} h_{jj}^r + h_{11}^r,
\]

\[
(4) \quad \frac{\partial f_r}{\partial h_{33}^r} = \sum_{j \in [1, m \setminus \{3\}]} h_{jj}^r.
\]
As for a solution \((h^r_{11}, h^r_{22}, ..., h^r_{nn})\) of the problem in question, the vector 
\((\text{grad}) (f_1)\) is normal at \(P\), from (2) and (3) we obtain
\[
\sum_{j=1}^{n} h^r_{jj} - h^r_{11} + h^r_{22} = \sum_{j=1}^{n} h^r_{jj} - h^r_{33},
\]
therefore
\[(5) \quad h^r_{11} = h^r_{22} = b^r.\]

From (2) and (4), it follows
\[
\sum_{j=1}^{n} h^r_{jj} - h^r_{11} + h^r_{22} = \sum_{j=1}^{n} h^r_{jj} - h^r_{33}.\]
By using (5) we obtain
\[(6) \quad h^r_{33} = 0. \quad \text{Similarly one gets}\]
\[(7) \quad h^r_{jj} = 0, \quad \forall \quad j \in \{3, n\}.\]

As \(h^r_{11} + h^r_{22} + ... + h^r_{nn} = k^r\), from (5) and (6) we obtain
\[(8) \quad b^r = \frac{k^r}{2}.\]

We fix an arbitrary point \(p \in P\).
The 2-form \(\alpha : T_pP \times T_pP \to \mathbb{R}\) has the expression
\[\alpha(X, Y) = \text{Hess}_{f_r}(X, Y) + \langle h'(X, Y), (\text{grad}_{f_r})(p) \rangle,\]
where \(h'\) is the second fundamental form of \(P\) in \(\mathbb{R}^n\) and \(\langle , \rangle\) is the standard inner-product on \(\mathbb{R}^n\).

In the standard frame of \(\mathbb{R}^n\), the Hessian of \(f_r\) has the matrix
\[
\text{Hess}_{f_r} = \begin{pmatrix}
0 & 2 & 1 & . & 1 \\
2 & 0 & 1 & . & 1 \\
1 & 1 & 0 & . & 1 \\
. & . & . & . & . \\
1 & 1 & 1 & . & 0
\end{pmatrix}.
\]

As \(P\) is totally geodesic in \(\mathbb{R}^n\), considering a vector \(X\) tangent to \(P\) at the arbitrary point \(p\), that is, verifying the relation \(\sum_{i=1}^{n} X^i = 0\), we have
\[
\alpha(X, X) = 2 \sum_{1 \leq i < j \leq n} X^i X^j + 2X^1 X^2 = (\sum_{i=1}^{n} X^i)^2 - \sum_{i=1}^{n} (X^i)^2 + 2X^1 X^2 = -\sum_{i=3}^{n} (X^i)^2 - (X^1 - X^2)^2.
\]
The 2-form $\alpha$ is semipositive definite. Therefore the point $(h_{r11}, h_{r22}, ..., h_{rnn})$ which satisfies (5), (6) and (7) is a global maximum point. Using this fact and (5), (6) and (7) we obtain

$$\delta'_M \leq \frac{(n^2 - n + 2)c}{2} + \frac{n^2 \|H\|^2}{2}$$

The relation

$$\delta'_M = \frac{(n^2 - n + 2)c}{2} + \frac{n^2 \|H\|^2}{2}$$

occurs if and only if we have

9. $h_{ij}^r = 0, \forall \ r \in \overline{n + 1, m}, \forall \ i, j \in \overline{1, n}$ with $i \neq j$,

10. $h_{11}^r = h_{22}^r, \forall \ r \in \overline{n + 1, m}$,

11. $h_{33}^r = ... = h_{nn}^r = 0, \forall \ r \in \overline{n + 1, m}$.

Therefore, there is an orthonormal frame $\{e_1, ..., e_n\}$ in $T_xM$ and an orthonormal frame $\{e_{n+1}, ..., e_m\}$ in $T^\perp_xM$ in which the Weingarten operators have the following form

$$A_r = \begin{pmatrix} h_{11}^r & 0 & 0 & 0 \\ 0 & h_{22}^r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with $h_{11}^r = h_{22}^r, \forall \ r \in \overline{n + 1, m}$.

**COROLLARY 3.1.** If the Riemannian manifold $(M, g)$, of dimension $n \geq 3$, admit a minimal isometric immersion into a real space form $\tilde{M}(c)$, then

$$\tau - \frac{(n - 2)(n + 1)c}{2} \leq k \leq -\tau + \frac{(n^2 - n + 2)c}{2}.$$  

**COROLLARY 3.2.** If the Riemannian manifold $(M, g)$, of dimension $n \geq 3$, admit a minimal isometric immersion into a Euclidean space, then

$$\tau \leq k \leq -\tau.$$
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