Optical Aharonov–Bohm effect due to toroidal moment inspired by general relativity

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Keywords: Aharonov–Bohm effect, spacetime index of refraction, gravitoelectromagnetism, toroidal moment

Abstract

We study the analogy between propagation of light rays in a stationary curved spacetime and in a toroidal (meta-)material. After introducing a novel gravitational analog of the index of refraction of a magneto-electric medium, it is argued that light rays not only feel a Lorentz-like force in a magneto-electric medium due to the non-vanishing curl of the toroidal moment, but also there exists an optical analog of Aharonov–Bohm effect for the rays traveling in a region with a curl-free toroidal moment. Experimental realization of this effect could utilize either a multiferroic material or a toroidal metamaterial.

1. Introduction

Apart from yet being the most successful theory of gravity, general relativity and the exact solutions of Einstein field equations, in recent years have turned into a very powerful tool to study linear and nonlinear phenomena in other branches of physics. An interesting context, in which general relativity has already proved amazingly fruitful in this regard, was formed around the analogy between a curved background and a material medium with respect to light propagation.

The idea could be traced back the study of electromagnetism and light propagation in curved spacetimes and the observation that the underlying spacetime could be taken as a medium assigned with an index of refraction [1]. The same interpretation was reinforced by noting that Maxwell’s equations in a curved spacetime lead to constitutive equations with magneto-electric coupling, in which electric and magnetic fields and their corresponding features are intertwined through the geometry of the underlying spacetime. In this way, one could establish a correspondence between geometric entities assigned to a curved spacetime and the electromagnetic features of a medium such as electric permittivity and magnetic permeability [1, 2].

Meanwhile studies, both theoretical and experimental, on metamaterials have flourished with the beginning of the 21st century, mainly due to the advancements in small-scale technologies, including nanotechnology. Metamaterials are artificially designed matter which have very interesting electromagnetic properties, not found in simple or composite matter in nature. Their nontrivial properties, such as negative refractive index [3], originate not from the composition of their constituents, but from the way these subwavelength macroscopic constituents are assembled.

Another key development was the introduction of transformation optics [4], which formulates how electromagnetic field lines and the corresponding light rays could be redirected by deforming the underlying space in which they are embedded, very much like rerouting a river by deforming its bed. This process could be looked upon as a coordinate transformation from the initial configuration of the electromagnetic fields in a Cartesian coordinate to their final configuration in a distorted coordinate system, justifying the nickname optical conformal transformation or mapping [5, 6].

Putting the above elements together, it was proposed that one could control the light propagation by a carefully designed metamaterial through the application of the above transformation. This simple idea is the key...
idea behind employing metamaterials to build cloaking devices in which, light is directed by a carefully assembled metamaterial to avoid a designated region of space [4, 5].

On the other hand, the geometric nature of general relativity, introducing gravity not as a force but as the geometry of spacetime, leads to a machinery, based on Riemannian geometry which allows one to calculate paths of massive and massless particles, in particular light paths, in a given spacetime. Hence, for a given spacetime, regardless of the matter distribution producing it, one is able in principle to calculate its null geodesics and the metamaterial analog of this spacetime, designed with the equivalent electromagnetic specifications, will reproduce the same optical paths.

Obviously the more exotic a spacetime may look, the more interesting is expected to be the behavior of its light trajectories and that is exactly the case with light paths in the geometry of black holes. Consequently in the metamaterial analogs of black holes, one could duplicate the same light paths, however strange they may seem. For example in the simplest spherically symmetric black hole, namely the Schwarzschild black hole, there is a photon sphere which is a stable circular photon orbit outside the black hole at a radius 1.5 times that of the black hole’s event horizon [7]. Interestingly enough, it is found, through simulations, that the metamaterial analog of this black hole exhibits the same photon sphere property [8, 9]. Later experimental demonstration of gravitational lensing effects near the photon sphere was carried out in [10]. Metamaterial designs are suggested that could form an optical black hole in the sense that it is a complete light absorber, and are hoped to find application in many areas [11]. Optical analogs of wormholes were also suggested by designing metamaterial according to the electromagnetic parameters mimicking that of a wormhole solution of Einstein field equations, connecting two distant regions in a spacetime continuum [12]. Even it is suggested that one could mimic Hawking radiation in optical analogs of a black hole [13, 14].

In the present article, motivated by the gravitational Aharonov–Bohm effect, first we use the 1 + 3 (or threading) formulation of spacetime decomposition to exhibit an interesting analogy between electromagnetic wave propagation in a stationary curved spacetime and in a magneto-electric (meta-)material, and then employ this analogy to show that there should be an optical analog of this effect in (meta-)materials with toroidal moment. The magneto-electric coupling in (meta-)materials mimicking stationary spacetimes, are constrained to those due to the toroidal moment originating from the antisymmetric part of the linear magneto-electric tensor. A natural and direct consequence of this analogy is the existence of a Lorentz-like force acting on light, similar to what we have for charged particles moving in a magnetic field. Also employing the same analogy, we predict an optical analog of the Aharonov–Bohm effect in a toroidal (meta-)material. This effect is different from the dynamic version of the usual Aharonov–Bohm effect which is expected to affect beams of charged particles in the presence of non-radiating sources formed by a special combination of interfering toroidal and electric dipoles [15].

The outline of the paper is as follows. In the next section, in the context of gravito-electromagnetism, we introduce the idea of a spacetime as a medium assigned with an index of refraction with respect to the light trajectories. For a stationary spacetime we arrive at a novel form for the spacetime index of refraction, introduced here for the first time in the literature, which enables us to interpret it as the gravitational analog of the magneto-electric index of refraction in a material with toroidal moment. In section three this analogy is extended to Maxwell equations in a stationary spacetime and their corresponding constitutive equations which include magneto-electric terms In section four we discuss in more detail stationary spacetime metrics and their magneto-electric (Meta-)material analogs with respect to their corresponding magneto-electric tensor.

In section five after a brief introduction of the gravitational Aharonov–Bohm effect we discuss our prediction of the optical analog of the Aharonov–Bohm effect in toroidal (meta-)materials with any value for the toroidal moment and show that it reduces to the previous result in the limit of small toroidal moment. Finally we discuss and summarize our results in the last section.

Throughout we use the (+, −, −, −) signature for spacetime metric and our convention for indices is such that the Latin indices run from 1 to 3 while the Greek ones run from 0 to 3.

2. Curved spacetime as a medium

Any physical realization of 3-dimensional quantities in a curved background requires a decomposition of spacetime into spatial and temporal sections. There are two main decomposition formalisms called 1 + 3 and 3 + 1 or alternatively threading and foliation formalisms respectively [4]. To motivate the idea of a curved spacetime as a medium, we start with introducing the threading formalism which will naturally lead to two interesting analogies between a curved spacetime and a medium with respect to light propagation and electromagnetic constitutive equations in either of them.

4 For a comparison between the two formalisms refer to [16].
In the threading formulation of spacetime decomposition, the spatial and temporal distances between two infinitesimally close fundamental observers are obtained by sending and receiving light signals on their worldlines. The outcome is the following decomposed spacetime metric,

\[ ds^2 = dx^2 - dt^2 = g_{00}(dx^0 - A_{0i}dx^i)^2 - \gamma_{ij}dx^idx^j \]  \hspace{1cm} (1)

where \( A_{0i} \equiv -\frac{\phi_i}{\phi_0} \) and

\[ \gamma_{ij} = -g_{ij} + \frac{g_{00}g_{0j}}{g_{00}}, \]

(2)

is the spatial metric of a 3-space called a quotient space/manifold, on which \( dl \) gives the element of spatial distance between any two nearby events. Also the so called synchronized proper time between any two events is given by

\[ dt_{\text{syn}} = \sqrt{g_{00}}(dx^0 - A_{0i}dx^i), \]

so that any two simultaneous events have a coordinate-time difference of

\[ dx^0 = A_{0i}dx^i. \]

The spatial three-metric is invariant under the special coordinate transformation defined by \( \frac{\partial x^i}{\partial x^0} = 0 \) and \( \frac{\partial x^0}{\partial x^i} = 1 \). In addition, we may define the three-tensors, vectors, and scalars under the same coordinate transformation. Specifically, \( g_{00} \) and \( g^{00} \) are three-scalars while \( A_{0i} \equiv -\frac{\phi_i}{\phi_0} \) are the components of a covariant 3-vector called gravitomagnetic potential (see equation (5) below), and \( \gamma^{0j} = -g^{0j} \) is the inverse of the 3-metric. We can consistently raise or lower three dimensional indexes using the spatial three-metric and its inverse. Moreover, the curl of a three-vector is defined using the three-dimensional anti-symmetric tensor, \( \eta_{ijk} = \sqrt{\gamma}e_{ijk} \) (and \( \eta^{ijk} = e^{ijk} \)), with \( \gamma_1 \) the determinant of the 3-metric and \( e_{ijk} = 1 \), the Cartesian Levi-Civita symbol.

The gravitational 3-force acting on a test particle moving on a spacetime geodesic, in a stationary background is given by \[ f = \frac{e_0}{\sqrt{g_{00}}}[-\nabla \ln (\sqrt{g_{00}}) + \sqrt{g_{00}}\vec{v} \times (\nabla \times \vec{A}_g)] \]

\[ = \frac{e_0}{\sqrt{g_{00}}} (E_g + \vec{v} \times \sqrt{g_{00}}B_g) \]

in which \( \vec{v} \) is the three-velocity of the test particle, \( e_0 = -e \frac{\partial \phi}{\partial x^0} \) is its conserved energy with action \( S \), and the gravitoelectric and gravitomagnetic 3-fields are defined as follows:

\[ B_g = \nabla \times \vec{A}_g \quad E_g = -\nabla \ln (\sqrt{g_{00}}). \]

(5)

In the case of null rays (photons), \( \vec{v} = c\hat{k} \) where \( \hat{k} \) is the unit vector along the direction of propagation. Compared with electromagnetism, the first term in the right hand side of (3) could be interpreted as the gravitoelectric force due to the gravitoelectric potential \( \phi_g = \ln (\sqrt{g_{00}}) \) while the second term is interpreted as the gravitomagnetic force due to the gravitomagnetic field \( B_g \) with \( \vec{A}_g \) as the gravitomagnetic vector potential. In the same spirit \( e_0 \) could be interpreted as the gravitoelectric charge of the test particle \[ f \].

Now starting with Fermat’s principle \( \delta \int \sqrt{g_{00}}dx^0 = 0 \), and using the threading formulation of spacetime decomposition \[ 1 \], one arrives at the following relation

\[ \delta \int \left( \frac{dl}{\sqrt{g_{00}}} + A_{0i}dx^i \right) = 0 \]

(6)

in which \( dl \) is the spatial line element along the light path. Restricting the above result to the case of static spacetimes by setting \( A_{0i} = 0 \), one could obviously assign an index of refraction, \( n_0 = \frac{1}{\sqrt{g_{00}}} \) to the underlying static spacetime in analogy with a material medium \[ 1 \]. Interestingly enough, in what follows, we show that an index of refraction could also be assigned to stationary spacetimes. To this end we rewrite (6) in the following way

\[ \delta \int \left( \frac{1}{\sqrt{g_{00}}} + A_{0i}\frac{dx^i}{dl} \right) dl \equiv \delta \int n dl = 0 \]

(7)

from which the index of refraction could be read as

\[ n = \frac{1}{\sqrt{g_{00}}} + A_{0i}\frac{dx^i}{dl} \]

(8)

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5 All the differential operations in these relations are defined in the 3-space with metric \( \gamma_{ij} \). Specifically the curl and divergence of a 3-vector \( \textbf{V} \) are given by \( (\nabla \times \textbf{V})^i = \frac{1}{\sqrt{\gamma}}\epsilon^{ijk}V_j \) and \( (\nabla \cdot \textbf{V}) = \frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}V^i) \) respectively.
Using the facts that \( k^2 = \frac{k^i k^j}{g_{ij}} - \gamma_0 k^j = 0 \) and that the wave vector \( k^j \) is along the spatial displacement vector, \( dx^j \), leads to \( k^i = \frac{k_0}{\sqrt{g_{00}}} \) and \( |k| = \frac{k_0}{\sqrt{g_{00}}} \), whereby the above relation could be written in the following illuminating simple form
\[
n = n_0 + A_{g} \hat{k} \tag{9}
\]

On the other hand it has been shown, interestingly enough, that there exists a Lorentz-like force acting on light rays in materials with toroidal moment corresponding to the optical analog of the Lorentz force acting on a charged particle in a magnetic field. This was realized in the optical magnetoelectric effect in multiferroic materials [18]. The optical magnetoelectric effect in a material with toroidal moment \( T(r) \), in the geometrical optics limit, is characterized by the following space-dependent index of refraction
\[
n(r) = n_0(r) + \alpha T(r) \hat{k} \tag{10}
\]

Comparing the above results shows that (9) is an interesting gravitational analog of the above magnetoelectric index of refraction (for \( \alpha = 1 \)). In this analogy the above introduced gravitomagnetic potential \( A_g \) [17], plays the role of the toroidal moment and in the same way its presence accounts for the breaking of both time-reversal and parity symmetries. To the best of our knowledge, both the relation (9), and the above mentioned analogy between the two refractive indices (i.e. (9) and (10)), appear here for the first time in the literature. Now if we turn the analogy around, the quasi-Lorentz force (3) acting on test particles or light rays in stationary spacetimes implies that there should exist a Lorentz-like force acting on light rays in a toroidal (meta-)material with the curl of the toroidal moment acting as a magnetic-type field.

3. Maxwell equations in a curved background and the Magnetoelectric constitutive relations

Apart from the novel analogy introduced in the last section, there is another well known clue hinting towards the same analogy between a curved spacetime and a so called biaxial isotropic medium in a flat background spacetime. This is achieved by looking at the equations of electrodynamics in a curved background employing the 1+3 spacetime decomposition through the introduction of the following electromagnetic fields [1],
\[
E_i = F_{0i}, \quad D^i = -\sqrt{g_{00}} F^{0i}, \quad B^i = -\frac{1}{2\sqrt{g}} \epsilon^{ijk} F_{jk}, \quad H_i = -\frac{\sqrt{g}}{2} \epsilon^{ijk} F_{jk} \tag{11}
\]

which coincide, in the absence of curvature, with their definitions in flat spacetime. Using these definitions, Maxwell’s equations in the absence of charges and currents in a stationary spacetime are
\[
\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} (\sqrt{g_0} \vec{B}), \quad \nabla \cdot \vec{D} = 0, \quad \nabla \times \vec{H} = \frac{1}{c} \frac{\partial}{\partial t} (\sqrt{g_0} \vec{D}) \tag{12}
\]

Now, we rewrite the above decomposed, non-covariant Maxwell equations in the following form which is formally equivalent to the Maxwell equations in a biaxial isotropic medium in a flat background [2],
\[
\frac{1}{\sqrt{g}} \frac{\partial}{\partial t} \left[ \sqrt{g_0} \right] = 0, \quad \frac{1}{\sqrt{g}} \frac{\partial}{\partial t} \left[ \sqrt{g} B^i \right] = 0, \quad \frac{\epsilon^{ijk} \partial_j E_k}{\sqrt{g}} = -\frac{1}{c} \frac{\partial}{\partial t} \left( \sqrt{g_0} B^i \right), \quad \frac{\epsilon^{ijk} \partial_j H_k}{\sqrt{g}} = \frac{1}{c} \frac{\partial}{\partial t} \left( \sqrt{g_0} D^i \right)
\]

where, \( \gamma_0 \) is the determinant of the spatial metric \( \gamma_{ij} \) in flat space and in the same orthogonal curvilinear coordinate system used to define \( \gamma \) [7]. Defining the new 3-vectors \( b^i = \sqrt{g} \frac{\partial}{\partial t} \) and \( d^i = \sqrt{g} \frac{\partial}{\partial t} \), the Maxwell equations are now given by
\[
\nabla_0 \cdot \vec{b} = 0, \quad \nabla_0 \times \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{b}, \quad \nabla_0 \cdot \vec{d} = 0, \quad \nabla_0 \times \vec{H} = \frac{1}{c} \frac{\partial}{\partial t} \vec{d} \tag{13}
\]

in which the \textit{curl} and \textit{divergence} (with \( \nabla_0 \)) are defined with respect to the orthogonal curvilinear coordinates in flat space. (For a detailed discussion on the relation between the background metric of a medium and its analog spacetime metric refer to [19]). Using the above definitions of the 3-vectors \( b \) and \( d \) and the fact that the fields introduced in (11) are not independent, one is led to the following constitutive equations [1],

\[ \text{It should be noted that the spatial contravariant components of the 4-vector } \gamma_0^k \text{ are taken as a 3-vector in the } \gamma \text{-space with the corresponding covariant 3-vector given by } k_i = \gamma_0 \gamma^k \text{.} \]

\[ \text{We notice that the above definitions of } D^i \text{ and } B^i \text{, differ from those introduced in [1] by a } \sqrt{g} \text{ factor.} \]
\[ d^i = \frac{\gamma^0 \sqrt{\gamma}}{\sqrt{g_{00}}} E^i + \frac{\epsilon^{i\beta} A^\beta_k}{\sqrt{g}} H_j, \quad b^i = \frac{\gamma^0 \sqrt{\gamma}}{\sqrt{g_{00}}} H_j - \frac{\epsilon^{i\beta} A^\beta_k}{\sqrt{g}} E^i \]  

(14)

Comparison of the above equations with the electromagnetic constitutive equations, which describe the behavior of matter under the influence of electromagnetic fields, once again shows that the spacetime metric is formally playing the role of a medium. This analogy is in accordance with the assignment of an index of refraction to a stationary spacetime with respect to light propagation, studied in the previous section. In other words solving Maxwell equations (12) with respect to the above constitutive relations in the geometric optics limit should lead to the null trajectories in the underlying spacetime. The above analogy establishes an optogeometric correspondence between a spacetime and a material medium through which one is able to replicate the same light paths in a material or metamaterial medium with equivalent properties (e.g. electric and magnetic susceptibilities) read from the above spacetime constitutive relations.

To account for the other side of this correspondence, consider the matter interacting with the time-independent electromagnetic field to be in thermodynamic equilibrium. A systematic way to derive the general response of such a material to the electromagnetic field is through the following expansion of the free energy [20] in terms of the electromagnetic fields as well as spontaneous polarization and magnetization, \( P_i^{(s)} \) and \( M_j^{(s)} \), as the secondary sources;

\[ F(\vec{E}, \vec{H}) = F_0 - P_i^{(s)} E_i - M_j^{(s)} H_j \]
\[ - \frac{\epsilon^{ij} E_i E_j}{8\pi} - \frac{\mu^{ij} H_i H_j}{8\pi} - \alpha^{ij} E_i H_j - \text{Higher order terms} \]

in which \( \epsilon^{ij} \) and \( \mu^{ij} \) are the electric permittivity and the magnetic permeability respectively, and \( \alpha^{ij} \) is the magneto-electric tensor of matter inducing either polarization by a magnetic field or magnetization by an electric field. Note that the higher order terms only come into play when the fields acting on matter are strong enough to be comparable to the Coulomb potential in the atomic scale. Here we only consider the linear terms which are enough to mimic a medium’s response to an electromagnetic field with that of a spacetimes’ response to the same field.

The thermodynamic stability condition imposes the following constraint on the magneto-electric tensor [20]

\[ \alpha^{ij} \leq \frac{\sqrt{\epsilon^{ij} \mu^{ij}}}{4\pi}. \]

(15)

We have the following general expansion for the electric displacement field and the magnetic intensity

\[ D^i = -4\pi \frac{\partial F}{\partial E_i} = \epsilon^{ij} E_j + 4\pi \alpha^{ij} H_j + 4\pi P_i^{(s)} \]

(16)

\[ B^i = -4\pi \frac{\partial F}{\partial H_i} = \mu^{ij} H_j - 4\pi \alpha^{ij} E_j + 4\pi M_i^{(s)} \]

(17)

Although the above equations are obtained for time-independent EM fields, they are also valid for low frequency time-dependent EM fields [21].

4. Stationary spacetimes and their magneto-electric (meta-)material analogs

In the absence of spontaneous polarization and magnetization, comparing (16) and (17) with (14), we conclude that the Maxwell equations in a material medium and in a curved spacetime are equivalent if the following correspondences are held;

\[ \mu^{ij} = \epsilon^{ij} \sim \frac{\gamma^0 \sqrt{\gamma}}{\sqrt{g_{00}}} \]
\[ = - \frac{g^{ij} \sqrt{\gamma}}{\sqrt{g_{00}}} \]

(18)

\[ 4\pi \alpha^{ij} = -4\pi \alpha^{ij} \sim \frac{\epsilon^{i\beta} A^\beta_k}{\sqrt{g}} \]
\[ = - \frac{\epsilon^{i\beta} A^\beta_k}{\sqrt{g_{00}}} \]

(19)

We note, on passing, that the nonvanishing spontaneous polarization and magnetization could be taken as the electromagnetic analogs of the gravitoelectric and gravitomagnetic fields originating from the presence of the charges and currents in exact stationary solutions of Einstein-Maxwell equations such as the Kerr-Newmann spacetime. Here our study is restricted to the analog responses of a vacuum stationary spacetime and a medium to a test electromagnetic field.

Also by the non-covariant nature of the spacetime decomposition, leading to equations (14) and (19), it is expected that the gravitational analog of the magneto-electric tensor depends, not only on the employed
coordinates and different coordinate patches, but also on the definitions of the 3-dimensional electromagnetic fields [22]. For a covariant approach to transformation optics in linear dielectric materials refer to [23].

The general form of the magneto-electric tensor of a (meta-)material medium is given by [24],

$$\alpha^{ij} = \mathbf{S}^{ij} + \epsilon^{ijk} \mathbf{A}_k + \gamma \delta^{ij},$$

(20)
in which $\mathbf{S}^{ij}$ is the symmetric traceless part of the magneto-electric tensor, $\gamma$ is a 3-vector dual of its antisymmetric part, and $\gamma$ is a pseudo-scalar representing the trace of the magneto-electric tensor. Therefore, comparing equations (19) and (20), one can conclude that the magneto-electric response of a time independent spacetime to an electromagnetic wave is the analog of the magneto-electric response of a material medium due to the 3-vector $\gamma$. In other words $\gamma$ of a material medium is the analog of the gravitomagnetic potential $A^t$ of a stationary spacetime. A non-vanishing $A^t$ in stationary spacetimes is a manifestation of the breaking of the time reversal symmetry. This is the gravitational analog of breaking of the same symmetry in material media with a non-vanishing $\gamma$ such as in the case of multiferroic metamaterials in which $\gamma$ is the so called toroidal moment.

In geometrical optics, light behaves as particles and so in this limit equation (3) should also be applicable to light and therefore in the corresponding metamaterial analog, we expect the light ray to be affected by an electric-like force as well as a magnetic-like force. The rational behind this idea is that the wave equations in a curved spacetime and in the corresponding metamaterial analog (in the same flat space coordinates) are equal and in the geometric approximation the same behavior for light paths is expected in the analog counterparts. Therefore, one can duplicate the null geodesics of a stationary spacetime, as the light paths in the corresponding metamaterial and vice versa [25]. So through the above analogy, one expects the existence of a Lorentz-like force on light rays in a metamaterial designed with respect to (18) and (19) in which the rotation of $\gamma$ of the (meta-)material, plays the role of the gravitomagnetic field. Obviously, now the gravitomagnetic potential $A^t$ is the gravitational analog of the vector $\gamma$. Extending the above analogy to other gravitational effects and inspired by the gravitational Aharonov–Bohm effect, in the next section, we discuss the optical Aharonov–Bohm effect which could be realized in metamaterials designed with the specific optical characteristics determined from (18) and (19).

5. Optical Aharonov–Bohm effect in toroidal (meta-)material

Before discussing the optical Aharonov–Bohm effect we digress to briefly discuss the gravitational Aharonov–Bohm effect. Different versions of gravitational analogue of the Aharonov–Bohm effect are discussed in the literature all of which corresponding to a test particle moving in a region of space in which either the gravitomagnetic field ($B^t$) or the gravitoelectric field ($E^t$) are absent, but the particle is affected by the fluxes of the same fields in the regions of spacetime from which it is excluded [26]. These two versions could be called gravitomagnetoic and gravitoelectric Aharonov–Bohm effects respectively [27]. Obviously the gravitomagnetic Aharonov–Bohm effect is more analogous, specially in its form, to its electromagnetic counterpart which involves magnetic field and its potential. In the usual Aharonov–Bohm effect, for a time-independent 4-vector potential $A_{\mu} = (\Phi, A^t)$, a particle is influenced locally by an electric field $\mathbf{E} = -\nabla \Phi(\mathbf{r})$ and globally by the curl-free vector potential $A^t$ through its integral over a non-trivial closed curve, i.e

$$\oint A_\mu dx^\mu = -\oint A_t dx^t = 0,$$

(21)

where the line integral is taken over a closed path in the region where $\mathbf{B} = \nabla \times \mathbf{A} = 0$.

Employing the spacetime decomposition introduced in the last section, the existence of the gravitational analog of (21) is given by

$$\oint_C A_{\mu} dx^\mu = 0.$$

(22)

As an example of the gravitational analog of the Aharonov–Bohm effect, it is shown in [28] that the interference of light passing each side of a rotating dust cylinder, represented by the Van Stockum solution [29] of Einstein field equations, is affected by the gravitomagnetic potential of the cylinder. The effect is nothing but a phase change proportional to the above factor calculated along the light path in a region where there are no local effects due to rotation.\footnote{A more faithful gravitational analog of the Aharonov–Bohm effect for a truly confined gravitomagnetic potential could be illustrated for the toroidal metric of a toroidal shell with rolling motions around its small cross section [30].}

Since the gravitational Aharonov–Bohm effect only depends on the condition (22), its optical counterpart is also expected to depend only on the toroidal moment and be independent of the structure of either $\varepsilon^{ij}$ or $\mu^{ij}$. Therefore we expect the optical Aharonov–Bohm effect in materials which do not have a curved spacetime counterpart satisfying the correspondence (18), but do have a curl-free toroidal moment satisfying the analog
condition of (22), namely
\[ \oint_C \tau_i \, dx^i = 0. \]  
(23)

Taking this fact into account and also noting that design of metamaterials with simple \( \epsilon^\parallel \) or \( \mu^\parallel \) is more feasible, in what follows we show the existence of optical Aharonov–Bohm effect in a (meta-)material with isotropic \( \epsilon^\parallel = \epsilon \, \delta^\parallel \) and \( \mu^\parallel \) along with a toroidal moment satisfying (23).

In other words, as in the case of the gravitational Aharonov–Bohm effect, one expects a modified interference pattern for a coherent light ray which splits into two and then reunited to form a closed path where \( \pi \) is curl-free. To this end we begin with the constitutive relations (16) and (17) in the absence of the symmetric part of the magneto-electric coupling (i.e for \( \alpha^\parallel = e^{\parallel} \tau_0 \)), which could be written as follows
\[ B = \mu H + \pi \times E, \quad D = \epsilon E - \pi \times H \]  
(24)

For the sake of simplicity, the medium is taken to be isotropic, but \( \epsilon, \mu \) and \( \tau \) could be functions of spatial coordinates. Taking \( \tilde{E} \) and \( \tilde{H} \) as
\[ \tilde{E}(\vec{x}, t) = \tilde{\epsilon}(\vec{x}) e^{i[kS(\vec{x}) - \omega t]}, \quad \tilde{H}(\vec{x}, t) = \tilde{\mu}(\vec{x}) e^{i[kS(\vec{x}) - \omega t]} \]  
(25)
in which \( S(\vec{x}) \) is the spatial part of the eikonal. Now taking the variations of both amplitudes \( \tilde{\epsilon}(\vec{x}) \) and \( \tilde{\mu}(\vec{x}) \) to be much smaller than the variation of the phase, namely \( kS(\vec{x}) \), and employing the geometrical optics approximation \( k \to \infty \), Maxwell’s equations translate into [31]
\[ \nabla \times \tilde{B} = 0 \Rightarrow \mu \nabla S \tilde{H} - \pi (\nabla S \times \tilde{\epsilon}) = 0 \]  
(26)
\[ \nabla \times \tilde{E} = -\frac{1}{c} \frac{\partial}{\partial t} \tilde{B} \Rightarrow (\nabla S \times \tilde{\epsilon}) = \mu \tilde{H} + \pi \times \tilde{\epsilon} \]  
(27)
\[ \nabla \cdot \tilde{D} = 0 \Rightarrow \epsilon \nabla S \tilde{\epsilon} + \pi (\nabla S \times \tilde{H}) = 0 \]  
(28)
\[ \nabla \times \tilde{H} = \frac{1}{c} \frac{\partial}{\partial t} \tilde{D} \Rightarrow (\nabla S \times \tilde{H}) = -\epsilon \tilde{\epsilon} + \pi \times \tilde{H} \]  
(29)
equations (27) and (29) which imply that \( \tilde{H} \) and \( \tilde{\epsilon} \) are perpendicular could be rewritten in the following forms
\[ (\nabla S - \pi) \times \tilde{\epsilon} = \mu \tilde{H} \]  
(30)
\[ (\nabla S - \pi) \times \tilde{H} = -\epsilon \tilde{\epsilon}, \]  
(31)
implying that \( (\nabla S - \pi) \) and \( \tilde{\epsilon} \) are mutually orthogonal. Combination of the last two equations yields
\[ (\nabla S - \pi) \times [(\nabla S - \pi) \times \tilde{\epsilon}] = -\epsilon \mu \tilde{\epsilon}, \]  
which upon using the orthogonality of \( (\nabla S - \pi) \) and \( \tilde{\epsilon} \), we end up with the following eikonal equation
\[ (\nabla S - \pi)^2 = \epsilon(\vec{x}) \mu(\vec{x}). \]  
(32)
The above equation resembles the Hamilton-Jacobi equation for charged particles in an electromagnetic field, in which the energy flux \((\tilde{E} \times \tilde{H})\) is in the direction of \((\nabla S - \pi)\) which plays the role of physical momentum.

In this correspondence \( \pi \) and \( \epsilon(\vec{x}) \mu(\vec{x}) \) are playing the roles of the vector potential and scalar potential respectively or more precisely those of \( \vec{A}^\parallel \) and \( \epsilon(\vec{x}) \). On this basis, one expects a Lorentz-like force to be acting on a light ray due to the non-vanishing curl of \( \pi \). Now assume that a light ray travels in a region of an optical medium with \( \nabla \times \pi = 0 \). If \( S_0 \) is the solution of equation (32) in the absence of \( \pi \), the general solution (i.e in the presence of \( \pi \)) in the same region is given by
\[ S(\vec{x}) = S_0 + \int_{\vec{C}} \pi \times \vec{r} \, d\vec{r}. \]  
(33)
Integral \( \int_{\vec{C}} \pi \times d\vec{r} \) over a closed path is not necessarily zero unless \( \nabla \times \pi \) vanishes over all the area enclosed by the path. As a consequence, interference effects due to this extra term are expected when a coherent light beam splits in two and forms a closed path enclosing the area in which \( \nabla \times \pi = 0 \). In other words, while the light trajectory is not influenced by the Lorentz-like force since \( \nabla \times \pi = 0 \) along the path, we still have the interference effects due to the non-zero \( \pi \) through its line integral \( \int_{\vec{C}} \pi \times d\vec{r} \). This is the optical analog of the Aharonov–Bohm effect.

Noting that \( \nabla S = \vec{n} \) is the local refractive index [31], the above analogy could be more clearly illustrated for \( |\pi| \ll 1 \) which is the case for natural materials. To this end we use the following identity
\[ \nabla S^2 = (\nabla S - \pi)^2 + 2\pi \cdot \nabla S - \pi^2, \]  
(34)
to arrive at the following form for the refractive index,

$$|\hat{n}| = |\hat{\nabla} S - \hat{\tau}| \left( 1 + \frac{2\hat{\nabla} \cdot \hat{\nabla} S - \tau^2}{|\hat{\nabla} S - \tau|^2} \right)^{1/2} = |\hat{\nabla} S - \hat{\tau}| + \hat{\tau} (\hat{\nabla} S - \hat{\tau}) + O(\tau^2).$$

(35)

Now from (32) with \(n_0^2 = \varepsilon(\vec{x})\mu(\vec{x})\), the above relation reads

$$|\hat{n}| \simeq |n_0| + \hat{\tau} \hat{k}$$

(36)

where we have taken \(\hat{k} = (\hat{\nabla} S - \hat{\tau})\) because the vector \((\hat{\nabla} S - \hat{\tau})\) is proportional to the Poynting vector. This clearly has the same form as (9) with \(\hat{\tau}\) playing the role of a vector potential, as expected from the established analogy. Indeed for light rays with parallel and anti-parallel propagation directions to the toroidal moment, the above relation leads to

$$|\hat{n}|_+ - |\hat{n}|_- \propto \hat{\tau} \hat{k}$$

(37)

a result in agreement with the non-reciprocal refraction which was shown to happen in a toroidal domain wall in multiferroic materials [18].

6. Summary and discussion

It would be useful to summarize our motivation and the follow up prescription, which led us to the prediction of an optical Aharonov–Bohm effect in toroidal (meta-)materials. We started with considering the gravitational analog of the usual (electromagnetic) Aharonov–Bohm effect in the context of threading formulation of spacetime decomposition. We then employed the analogy established between a curved background and a (meta-)material medium, or more specifically between a stationary spacetime’s gravitomagnetic potential and the toroidal moment of its analog medium, to predict the existence of an optical analog of the Aharonov–Bohm effect in toroidal (meta-)materials.

Indeed formally similar constitutive equations could be obtained for electromagnetic fields in a moving [21] or rotating medium [32], in which the linear velocity of that medium or its angular velocity plays the role of the magnetic potential in magnetoelectric constitutive relations, and hence one could expect different analogs of the Aharonov–Bohm effect. Considering the classical wave equation for light propagation in a moving medium, it was shown that there exists a classical optical analog of Aharonov–Bohm effect in which velocity of the medium and its vortex play the roles of magnetic potential and magnetic field in the usual Aharonov–Bohm effect respectively [33]. It is worth recalling that the more faithful gravitational analog of the Aharonov–Bohm effect arises in stationary spacetimes whose metric mixes space and time through which natural deﬁnitions of a gravitomagnetic potential and the corresponding gravitomagnetic field emerge. On the other hand if we are going to design a metamaterial medium to exhibit an optical analog effect through the so called transformation medium [4, 34], the employed transformation should mix space and time coordinates. Indeed it was shown [6] that the spacetime transformation: \(c dt = c dt' + ad\phi'\), \(dr = \frac{dr'}{n}\), \(d\phi = d\phi'\), \(dz = \frac{dz'}{n}\), with constants \(n\) and \(a\), corresponds to a moving fluid forming a vortex with velocity \(\mathbf{u} = (0, u, 0)\) which, without bending light, produces a phase change proportional to \(\int \mathbf{u} \cdot d\mathbf{r}\) on light enclosing the vortex, leading to an optical Aharonov–Bohm effect [35]. Now if we wish to design a metamaterial with a toroidal moment to exhibit an optical Aharonov–Bohm effect, we should employ, as transformation media, the same generic coordinate transformation with the velocity of the vortex replaced by the toroidal moment of the metamaterial. Obviously one could read off the corresponding optical characteristics of the metamaterial from equations (18) and (19) using the metric tensor constructed from the above transformed spacetime differential elements.

Finally, a simple experimental setup for detecting this effect will probably include a coherent light beam splitting in two, each passing through the same medium with the same depth, permeability, and permittivity but with opposite rotation-free toroidal moments, before interfering again.

As a final note we should point out that our derivation of the optical AB effect is not only inspired by GR and its mathematical machinery, but also due to the established opto-geometric duality, is not limited to the small values for the toroidal moment. Also our approach, compared to previous studies, establishes a more general and unifying picture of the different versions of the Aharonov–Bohm effect, including electromagnetic, gravitational and optical analogs of the effect.
Acknowledgments

The authors would like to thank the University of Tehran for supporting this project under the grants provided by the research council. M N-Z also thanks the high energy, cosmology, and astroparticle group at the Abdus Salam ICTP for kind hospitality during his visit when part of this study was carried out.

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