INVISID LIMIT OF THE COMPRRESSIBLE NAVIER–STOKES EQUATIONS FOR ASYMPTOTICALLY ISOTHERMAL GASES

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ABSTRACT. We prove the existence of relative finite-energy vanishing viscosity solutions of the one-dimensional, isentropic Euler equations under the assumption of an asymptotically isothermal pressure law, that is, \( p(\rho)/\rho = O(1) \) in the limit \( \rho \to \infty \). This solution is obtained as the vanishing viscosity limit of classical solutions of the one-dimensional, isentropic, compressible Navier–Stokes equations. Our approach relies on the method of compensated compactness to pass to the limit rigorously in the nonlinear terms. Key to our strategy is the derivation of hyperbolic representation formulas for the entropy kernel and related quantities; among others, a special entropy pair used to obtain higher uniform integrability estimates on the approximate solutions. Intricate bounding procedures relying on these representation formulas then yield the required compactness of the entropy dissipation measures. In turn, we prove that the Young measure generated by the classical solutions of the Navier–Stokes equations reduces to a Dirac mass, from which we deduce the required convergence to a solution of the Euler equations.

1. INTRODUCTION

The dynamics of an inviscid barotropic fluid are modelled by the compressible Euler equations. In one spatial dimension, these read as

\[
\begin{align*}
\rho_t + m_x &= 0, \\
m_t + \left( \frac{m^2}{\rho} + p(\rho) \right)_x &= 0,
\end{align*}
\]

(1.1)

where \((t, x) \in \mathbb{R}^2_+ = (0, \infty) \times \mathbb{R}\) are the variables of time and space, respectively, while \(\rho \geq 0\) is the density of the fluid and \(m = \rho u\) its momentum, with \(u\) its velocity. The pressure \(p(\rho)\) is a quantity that depends solely on the density. On the other hand, when one also takes into account the effects of viscosity, the equations governing the motion of the fluid are the one-dimensional, isentropic, compressible Navier–Stokes equations,

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p(\rho))_x &= \varepsilon u_{xx},
\end{align*}
\]

(1.2)

where \(\varepsilon > 0\) is the viscosity of the fluid. The work of Hoff [9] shows that, provided we supply appropriate initial data \((\rho_0, u_0) = (\rho, u)|_{t=0}\) and the pressure law \(p(\rho)\) satisfies modest physical assumptions, there exists a unique regular classical solution of this latter system. It is then physically relevant (cf. [17]) to ask whether such solutions of the Navier–Stokes equations (1.2) converge to a solution of the Euler equations (1.1) in an appropriate topology. Moreover, the question of convergence of the vanishing viscosity limit is a crucial one for the well-posedness of the Euler equations, where admissibility criteria are conjectured to provide uniqueness of the weak solutions. This paper addresses this question of convergence and provides an affirmative answer, provided particular constitutive assumptions are satisfied by the pressure.

Much of the theory of gas dynamics revolves around so-called \(\gamma\)-law gases, that is, fluids that incorporate a pressure law of the form \(p(\rho) = \kappa \rho^\gamma\), where \(\kappa > 0\) and \(\gamma \geq 1\). When \(\gamma \in (1, 3)\), we say that the gas is polytropic. When \(\gamma = 1\), the fluid is said to be isothermal, in accordance

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Remark 1.2. Assumption (1.3) ensures the strict hyperbolicity of the system and genuine non-linearity of the characteristic fields for \( \rho > 0 \). Assumption (1.4) guarantees the existence of the entropy kernel, see (1.10) below. Indeed, an inspection of the results of [3, 4] shows that the bound \( \rho^{\alpha-j} \) for \( P^{(j)} \) may be replaced with \( \rho^{(\delta+j)\theta-j} \) for any \( \delta > 0 \). However, for clarity of exposition, we choose not to make this assumption here.

Note that the class of gases considered in the previous definition is significantly wider than the one introduced in [15]. Indeed, therein the authors considered the vanishing viscosity limit of solutions of (1.2) under the assumption of a pressure law satisfying the first two hypotheses of Definition 1.1, but with \( p(\rho) = c_\rho \rho \) for \( \rho \geq R \) instead of (1.5). This precursor was called an approximately isothermal pressure law in the doctoral thesis of the second author, [16]. This assumption for large density allowed the authors to provide an explicit representation for entropies in this regime (cf. [15, Theorem 2.6]). Such a representation is no longer possible without the explicit form of the pressure.

The convergence of the vanishing physical viscosity limit from the Navier–Stokes equations to the Euler equations is a difficult problem with a rich history. Prior to the contribution of the authors in [15], Chen and Perepelitsa proved in [5] that, for a \( \gamma \)-law gas of index \( \gamma \in (1, \infty) \) and given any initial data of relative finite-energy, a solution of (1.1) could be obtained as the inviscid limit of solutions of (1.2). Earlier than that, the existence of \( L^\infty \) entropy solutions of the isentropic Euler equations had been obtained by means of vanishing artificial viscosities and from finite-difference schemes by DiPerna [8], Chen [2], Ding, Chen, and Luo [7], Lions, Perthame, and Tadmor [13], and Lions, Perthame, and Souganidis [12] for polytropic gases, by Chen and LeFloch [3, 4] for general pressure laws, and by Huang and Wang [10] for isothermal gases. In view of the fact that the Navier–Stokes equations do not admit natural invariant regions, this \( L^\infty \) framework may not be applied directly, and so we work in the finite-energy framework, originally introduced by LeFloch and Westdickenberg in [11], the notion of which we now develop.

Recall that an entropy pair is a pair of functions \((\eta, q) : \mathbb{R}^2_+ \to \mathbb{R}^2\) such that
\[
\nabla q(\rho, m) = \nabla \eta(\rho, m) \nabla \left( \frac{m^2}{\rho} + p(\rho) \right),
\]
where \( \nabla \) is the gradient with respect to the phase-space coordinates \((\rho, m)\). An important such entropy pair is the mechanical energy and its flux, \((\eta^*, q^*)\), given by the formulas
\[
\eta^*(\rho, m) = \frac{1}{2} \frac{m^2}{\rho} + \rho e(\rho), \quad q^*(\rho, m) = \frac{1}{2} \frac{m^3}{\rho^2} + me(\rho) + pm e'(\rho),
\]
where \( e(\rho) := \int_0^\rho \frac{p(y)}{y} \, dy \) is the internal energy. In what follows, we will consider the energy of the solutions relative to nontrivial constant end-states \((\rho_\pm, u_\pm)\). Correspondingly, let \((\tilde{\rho}(x), \tilde{u}(x))\) be smooth, monotone functions such that, for some \(L_0 > 1\),
\[
(\tilde{\rho}(x), \tilde{u}(x)) = \begin{cases} 
(\rho_+, u_+), & \text{for } x \geq L_0, \\
(\rho_-, u_-), & \text{for } x \leq -L_0.
\end{cases}
\]
These reference functions are fixed at this point and remain the same throughout the paper. We now define the relative mechanical energy with respect to \((\tilde{\rho}(x), \tilde{m}(x)) = (\tilde{\rho}(x), \tilde{\rho}(x)\tilde{u}(x))\) as
\[
\overline{\eta}(\rho, m) := \eta^*(\rho, m) - \eta^*(\tilde{\rho}, \tilde{m}) - \nabla \eta^*(\tilde{\rho}, \tilde{m}) \cdot (\rho - \tilde{\rho}, m - \tilde{m}),
\]
where \(e^*(\rho, \tilde{\rho}) = \rho e(\rho) - \tilde{\rho} e(\tilde{\rho}) - (\tilde{\rho} e'(\tilde{\rho}) + e(\tilde{\rho}))(\rho - \tilde{\rho}) \geq 0\). Then we define
\[
E[\rho, u](t) := \int_\mathbb{R} \overline{\eta}(\rho, \rho u)(t, x) \, dx
\]
to be the total relative mechanical energy, relative to the end states \((\rho_\pm, u_\pm)\) and say that a pair \((\rho, m)\) with \(m = \rho u\) is said to be of relative finite-energy if \(E[\rho, u] < \infty\).

It is apparent from the definition of entropy pair, and the requirement that mixed partial derivatives commute, that any \(C^2\) entropy function satisfies the \textit{entropy equation}, i.e.,
\[
\eta_{\rho\rho} - \frac{p'(\rho)}{\rho^2} \eta_{uu} = 0. \tag{1.8}
\]
It is well known (see e.g. [3, 8]) that any regular weak entropy (i.e. one that vanishes at \(\rho = 0\)) may be generated by the integral of a test function \(\psi \in C^2(\mathbb{R})\) against a fundamental solution \(\chi(\rho, u, s)\) of the entropy equation, i.e.,
\[
\eta^\psi(\rho, \rho u) = \int_\mathbb{R} \psi(s) \chi(\rho, u, s) \, ds, \tag{1.9}
\]
where this fundamental solution, the \textit{entropy kernel}, solves
\[
\begin{aligned}
\chi_{\rho\rho} - k'(\rho)^2 \chi_{uu} &= 0, \\
\chi(0, u, s) &= 0, \\
\chi(0, u, s) &= \delta_{u=u},
\end{aligned} \tag{1.10}
\]
where
\[
k(\rho) := \int_0^\rho \frac{\sqrt{p'(y)}}{y} \, dy. \tag{1.11}
\]
The entropy kernel admits the Galilean invariance \(\chi(\rho, u, s) = \chi(\rho, u-s, 0) = \chi(\rho, 0, s-u)\), and so we write (with slight abuse of notation) \(\chi = \chi(\rho, u-s)\). The entropy kernel has corresponding flux \(\sigma(\rho, u, s)\), the \textit{entropy flux kernel}, which behaves according to
\[
\begin{aligned}
(s - u \chi)_{\rho\rho} - k'(\rho)^2 (s - u \chi)_{uu} &= \frac{p'(\rho)}{\rho} \chi_u, \\
(s - u \chi)(0, u, s) &= 0, \\
(s - u \chi)_{\rho}(0, u, s) &= 0.
\end{aligned} \tag{1.12}
\]
In what follows, we use the notation \(h(\rho, u, s) = \sigma(\rho, u, s) - u \chi(\rho, u, s)\), and the same invariance \(h = h(\rho, u-s)\) holds. One then generates the entropy flux \(q^\psi\) corresponding to the entropy \(\eta^\psi\) via
\[
q^\psi(\rho, \rho u) = \int_\mathbb{R} \psi(s) \sigma(\rho, u, s) \, ds. \tag{1.13}
\]

**Definition 1.3.** Let \((\rho_0, u_0)\) be locally integrable initial data of relative finite-energy, that is, \((\rho_0, u_0) \in L^1_{\text{loc}}(\mathbb{R}^2)\) and \(E[\rho_0, u_0] \leq E_0 < \infty\). We call \((\rho, u) \in L^1_{\text{loc}}(\mathbb{R}^2)\), with \(\rho \geq 0\) almost everywhere, a \textit{relative finite-energy entropy solution} of the Euler equations (1.1) if:
(1) There exists a positive constant $M(E_0, t)$, increasing and continuous with respect to the variable $t$, such that
\[ E[r, u](t) \leq M(E_0, t) \text{ for almost every } t \geq 0; \]

(2) The pair $(r, u)$ solves the Cauchy problem (1.1) in the sense of distributions;

(3) There exists a bounded Radon measure $\mu(t, x, s)$ on $\mathbb{R}_+^2$ such that
\[ \mu(U \times \mathbb{R}) \geq 0 \text{ for any open set } U \subset \mathbb{R}_+^2, \]
and the entropy kernel $\chi$ and entropy flux kernel $\sigma$ associated to the problem via (1.10)–(1.12) satisfy the kinetic equation
\[ \partial_t \chi(r(t, x), u(t, x), s) + \partial_r \sigma(r(t, x), u(t, x), s) = \partial_x^2 \mu(t, x, s), \tag{1.14} \]
in the sense of distributions on $\mathbb{R}_+^2 \times \mathbb{R}$.

Our main result is the following.

**Theorem 1.4.** Let $(\rho_0, u_0)$ be locally integrable initial data of relative finite-energy, that is, $(\rho_0, u_0) \in L^{1}_{\text{loc}}(\mathbb{R}_+^2)$ and $E[\rho_0, u_0] \leq E_0 < \infty$, with $\rho_0 \geq 0$ almost everywhere. Suppose additionally that the pressure $p(\rho)$ satisfies the criteria for an asymptotically isothermal gas, in the sense of Definition 1.1. Then there exists a sequence of regularised initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ such that the unique, smooth solutions $(\rho^\varepsilon, u^\varepsilon)$ of (1.2) with this initial data converge almost everywhere as $\varepsilon \to 0$, $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \to (\rho, pu)$, to a relative finite-energy entropy solution of the Euler equations (1.1) with initial data $(\rho_0, \rho_0 u_0)$, in the sense of Definition 1.3.

**Remark 1.5.** In the proof of this theorem, we will obtain convergence of $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ in measure. In conjunction with the uniform estimates of Proposition 5.2, we will therefore also obtain $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \to (\rho, pu)$ in $L^p_{\text{loc}}(\mathbb{R}_+^2) \times L^q_{\text{loc}}(\mathbb{R}_+^2)$ strongly for $p \in [1, 2)$ and $q \in [1, 3/2)$ and weakly for $p = 2, q = 3/2$.

The proof of this theorem relies on the techniques of compensated compactness, initiated in this context by DiPerna [8], and the finite-energy framework begun by LeFloch–Westdickenberg [11] and extended by the authors in [15]. Our approach is the following. The global existence of a unique entropy kernel $\chi$ solving (1.10) is guaranteed by [3, Theorem 2.1], however without estimates as $\rho \to \infty$. To obtain improved estimates in this regime, we seek a good characterisation of the kernel, and so generate the solution of this linear wave equation in two steps. First, in order to deal with the singularity in $k'(\rho)^2$ near the vacuum, we solve (1.10) in the interval $[0, \rho_*]$, for some $\rho_*$ large to be chosen later, using the results of [3, Theorem 2.1]. Then, as a second step, we produce new estimates for $\chi$ in the high density regime by solving again from $\rho_*:\n$}
\begin{align*}
\chi_{\rho \rho} - k'(\rho)^2 \chi_{uu} &= 0, & (\rho, u) \in (\rho_*, \infty) \times \mathbb{R}, \\
\chi(\rho_*, u) &= \chi(\rho_*, u), & u \in \mathbb{R}, \\
\chi_{\rho}(\rho_*, u) &= \chi_{\rho}(\rho_*, u), & u \in \mathbb{R}. 
\end{align*}
\tag{1.15}

In order to do this, we split the entropy kernel into two distinct quantities: a re-scaled version of the kernel obtained in [15], which we call $\chi^*$, and a perturbation, called $\tilde{\chi}$. To this end, we make the following definitions.

**Definition 1.6.** We define kernels $g^s_r(\rho, u - s)$ and $g^s_u(\rho, u - s)$ to be the solutions of
\[ \begin{cases} 
  g^s_{\rho \rho} - \frac{\partial}{\partial \rho} g^s_u = 0, \\
  g^s_{\rho} = 0 \\
  g^s_{\rho \rho} = 0, \\
  g^s_{\rho \rho} = 0 
\end{cases} \]
\tag{1.16}
Recalling the kernels $\chi^t$ and $\chi^s$, introduced in [15, Theorem 1.3], we deduce that for $(\rho, u) \in [\rho_*, \infty) \times \mathbb{R}$, $g^t$ and $g^s$ satisfy the explicit formulas
\[ g^t(\rho, u) := \rho_* \chi^t \left( \frac{\rho}{\rho_*}, \frac{u}{\sqrt{k_2}} \right) \quad \text{and} \quad g^s(\rho, u) := \chi^s \left( \frac{\rho}{\rho_*}, \frac{u}{\sqrt{k_2}} \right). \tag{1.17} \]
We are now in a position to define $\chi^*$, the re-scaled approximately isothermal kernel.
Definition 1.7. We define, for \((\rho, u) \in [\rho_*, \infty) \times \mathbb{R}\),
\[
\chi^*(\rho, u) := \int_{\mathbb{R}} \chi_\rho(\rho_*, s)g^\rho(\rho, u - s) \, ds + \int_{\mathbb{R}} \chi(\rho_*, s)g^\rho(\rho, u - s) \, ds.
\] (1.18)

We deduce from (1.16) and from [3, Theorem 2.1] that \(\chi^*\) is the unique solution of
\[
\begin{cases}
\chi^{**}_\rho - \frac{k_2}{\rho^2} \chi_{uu} = 0, & \text{for } (\rho, u) \in (\rho_*, \infty) \times \mathbb{R}, \\
\chi^*(\rho_*, u) = \chi(\rho_*, u), & \text{for } u \in \mathbb{R}, \\
\chi^*_\rho(\rho_*, u) = \chi_\rho(\rho_*, u), & \text{for } u \in \mathbb{R}.
\end{cases}
\] (1.19)

Definition 1.8. We define the perturbation kernel \(\tilde{\chi}\) as the difference between \(\chi\) and \(\chi^*\):
\[
\tilde{\chi}(\rho, u) := \chi(\rho, u) - \chi^*(\rho, u) \quad \text{for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R}.
\] (1.20)

Direct computation shows that \(\tilde{\chi}\) satisfies the following linear wave equation,
\[
\begin{cases}
\tilde{\chi}_{\rho\rho} - k'(\rho)^2 \tilde{\chi}_{uu} = (k'(\rho)^2 - \frac{k_2}{\rho^2})\chi^*_u, & \text{for } (\rho, u) \in (\rho_*, \infty) \times \mathbb{R}, \\
\tilde{\chi}(\rho_*, u) = 0, & \text{for } u \in \mathbb{R}, \\
\tilde{\chi}_{\rho}(\rho_*, u) = 0, & \text{for } u \in \mathbb{R}.
\end{cases}
\] (1.21)

The key idea of this paper is to represent \(\tilde{\chi}(\rho, u)\) using a representation formula through integrals (of \(\tilde{\chi}\)) along backwards characteristics (see Lemma 3.4 and (3.9) below). Such a representation formula allows us to make precise estimates on the growth of \(\tilde{\chi}(\rho, u)\) and its derivatives as \(\rho \to \infty\). Using the estimates of [15] for the principal part, \(\chi^*(\rho, u)\), we hence also obtain estimates on \(\chi(\rho, u)\). Moreover, the representation formula technique turns out to be very well adapted to giving estimates as \(\rho \to \infty\) to many of the quantities associated to entropies and entropy fluxes and will be used extensively throughout this paper. All of the uniform estimates required to establish the compactness of the entropy dissipation measures and to reduce the support of the Young measure generated by the viscous approximations \((\rho^\varepsilon, u^\varepsilon)\) are then verified. These observations are enough to prove Theorem 1.4 rigorously.

The paper is structured as follows. In Section 2, we introduce some elementary quantities linked to the pressure, and compute estimates on them that are essential throughout. Some of the proofs of these estimates are contained in Appendix A, which also contains auxiliary results on the internal energy function \(e(\rho)\). In Section 3, we obtain a representation formula for \(\tilde{\chi}\), which enables us to obtain a uniform \(L^\infty\) estimate via a Grönwall type argument. With this, in Section 4, we estimate a special entropy \(\tilde{\eta}\) and its flux \(\tilde{q}\) (see Lemma 4.1) which are necessary to obtain higher uniform integrability estimates on the viscous approximations, and we also bound entropies generated by compactly supported test functions (cf. Lemma 4.13). Similar procedures give rise to estimates on the derivatives of these entropies, which we also outline in detail in Section 4. In Section 5, we collect together the necessary uniform estimates on \((\rho^\varepsilon, u^\varepsilon)\) and the compactness of the entropy dissipation measures. In Section 6, we calculate the structure of the singularities of the entropy kernel, and prove a result akin to Lemma 2.7 of [15], which was used crucially in [15, Section 5]. Having established this, we are able to prove the reduction of support of the Young measure generated by the approximate solutions using compensated compactness techniques and hence deduce the strong convergence of the viscous solutions. This is contained in Section 7, which ends with a proof of the main result.

2. Elementary quantities

In this section, we define elementary quantities related to the pressure, and make note of some of their properties which will be essential in the proof of the main result. We fix \(\rho_* > 0\) here, to be determined later. To begin with, by analogy with the definition of \(k(\rho)\) given in (1.11), we make the following definition.
Definition 2.1. We define the quantity
\[ k_*(\rho) := \int_{\rho_*}^{\rho} \frac{\sqrt{\kappa_2}}{y} \, dy + k(\rho_*) = \sqrt{\kappa_2} \log(\rho/\rho_*) + k(\rho_*) \quad \text{for all } \rho \geq \rho_*. \quad (2.1) \]

Note that \( k'_*(\rho) \) is the speed of propagation for the entropy equation of a gas with an isothermal pressure law \( p(\rho) = \kappa_2 \rho \). Meanwhile, \( k'(\rho) \) is the speed of propagation for the actual pressure law \( p(\rho) \), that is, the asymptotically isothermal gas, as can be seen directly from (1.10).

Remark 2.2. Observe that \( k'_*(\rho) = \frac{\sqrt{\kappa_2}}{\rho} \) and \( k(\rho_*) = k_*(\rho_*) \). Thus,
\[ k(\rho) - k_*(\rho) = \int_{\rho_*}^{\rho} \frac{\sqrt{\kappa_2}(y) - \sqrt{\kappa_2}}{y} \, dy \quad \text{for all } \rho \geq \rho_. \quad (2.2) \]

Since \( \kappa_2 = \lim_{\rho \to \infty} p'(\rho) \), and \( p \in C^4((0, \infty)) \), we can rewrite the above as
\[ k(\rho) - k_*(\rho) = -\int_{\rho_*}^{\rho} \frac{1}{\rho} \left( \int_y^\infty \frac{p''(z)}{2\sqrt{p'(z)}} \, dz \right) \, dy. \quad (2.3) \]

Definition 2.3. We define the quantities \( d(\rho) \) and \( d_*(\rho) \) by
\[ d(\rho) := 2 + (\rho - \rho_*) \frac{k''(\rho)}{k'(\rho)} \quad \text{and} \quad d_*(\rho) := 2 + (\rho - \rho_*) \frac{k''_*(\rho)}{k'_*(\rho)} \quad \text{for } \rho \geq \rho_. \quad (2.4) \]

Note that both of these quantities are strictly positive on the interval \([\rho_*, \infty)\), due to assumption (1.3).

Quantities \( d(\rho) \) and \( d_*(\rho) \) will appear in the representation formulas below. The following key lemma will play an important role throughout Sections 3 and 4.

Lemma 2.4. For all \( \rho \geq R \), we have
\[ |p'(\rho) - \kappa_2| \leq 2Cp\rho^{-\alpha}, \quad |p^{(j)}(\rho)| \leq (j + 1)Cp\rho^{-\alpha-(j-1)} \quad \text{for } j = 2, 3, 4. \quad (2.5) \]
As such, choosing \( \rho_* \geq \max\{R, (4Cp/\kappa_2)^{1/\alpha}\} \),
\[ \rho^2 \leq \frac{4\kappa_2}{3} pp(\rho), \quad \frac{\kappa_2}{2} \leq p'(\rho) \leq \frac{3\kappa_2}{2}, \quad \sqrt{\frac{\kappa_2}{2}} \rho^{-1} \leq k'(\rho) \leq \sqrt{3\kappa_2} \rho^{-1} \quad \text{for } \rho \geq \rho_, \quad (2.6) \]
and
\[ |k(\rho) - k_*(\rho)| \leq \frac{3Cp}{\alpha^2 \sqrt{2\kappa_2}} \rho_*^{\alpha} \quad \text{for } \rho \geq \rho_. \quad (2.7) \]

Proof. Observe that, for \( j \geq 1 \),
\[ (p(\rho) - \kappa_2)^{(j)}(\rho) = \rho \left( \frac{p(\rho)}{\rho} - \kappa_2 \right)^{(j)} + j \left( \frac{p(\rho)}{\rho} - \kappa_2 \right)^{(j-1)}. \]
Thus, using the bounds provided by (1.5), we obtain the result. Also, from (2.3),
\[ |k(\rho) - k_*(\rho)| \leq \frac{3Cp}{\alpha \sqrt{2\kappa_2}} \int_{\rho_*}^{\rho} y^{-\alpha-1} \, dy \leq \frac{3Cp}{\alpha^2 \sqrt{2\kappa_2}} \rho_*^{-\alpha}. \]
\[
\square
\]
Corollary 2.5. Assume that \( \rho_* \geq \max\{R, (4Cp/\kappa_2)^{1/\alpha}\} \). Then there exists \( M = M(\alpha, \kappa_2, C_p) \) such that
\[ |k'(\rho) - \kappa_2| + \left| -\frac{\sqrt{\kappa_2}}{\rho^2} - k''(\rho) \right| \leq M \rho^{-\alpha-2} \quad \text{for } \rho \geq \rho_. \quad (2.8) \]
Meanwhile,
\[ \frac{2\sqrt{\kappa_2}}{\rho^3} - k^{(3)}(\rho) + p \left| -\frac{6\sqrt{\kappa_2}}{\rho^4} - k^{(4)}(\rho) \right| \leq M \rho^{-\alpha-3} \quad \text{for } \rho \geq \rho_. \quad (2.9) \]
It follows that there exists a positive constant \( M = M(\alpha, \kappa_2, C_p, \rho_*) \) such that
\[ |k''(\rho) + p k^{(3)}(\rho) + \rho^2 k^{(4)}(\rho)| \leq M \rho^{-2} \quad \text{for } \rho \geq \rho_. \quad (2.10) \]
The proof is omitted to Appendix A, which also contains a variety of other useful estimates on these quantities, along with \(d(\rho)\) and \(d_*(\rho)\).

3. Entropy and Entropy Flux Kernels

We begin this section by recalling the following theorem on the existence and regularity of the entropy and entropy flux kernels, \(\chi(\rho, u - s)\) and \(\sigma(\rho, u, s)\) due to Chen and LeFloch [3].

**Theorem 3.1** ([3, Theorems 2.1–2.3]). Assume that the pressure \(p \in C^4([0, \infty)) \cap C^4((0, \infty))\) satisfies assumptions (1.3) and (1.4). Then (1.10) and (1.12) admit global unique Hölder continuous solutions \(\chi(\rho, u, s) = \chi(\rho, u - s)\) and \(\sigma(\rho, u, s) = u\chi(\rho, u - s) + h(\rho, u - s)\). Additionally,

\[
\sup h(\rho, \cdot), \sup h(\rho, \cdot) = [-k(\rho), k(\rho)] \quad \text{for } \rho \geq 0, \tag{3.1}
\]

and, for each \(\rho > 0\), \(\chi(\rho, \cdot)\) is strictly positive on \((-k(\rho), k(\rho))\). Moreover, for any fixed \(\rho_* > 0\), the entropy kernel satisfies, for \(0 < \rho \leq \rho_*\),

\[
\|\chi(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq C(\rho_*) \rho^{1-\theta}, \quad [\chi(\rho, \cdot)]_{C^1(\mathbb{R})} \leq C(\rho_*), \tag{3.2}
\]

where \([\cdot]_{C^1(\mathbb{R})}\) is the Hölder seminorm, \(\lambda = \frac{3 - \gamma}{2(\gamma - 1)}, \tilde{\lambda} = \min(\lambda, 1)\), and \(C(\rho_*)\) is a positive constant.

We recall that the exponent \(\lambda\) is related to \(\theta\) by \(2\lambda\theta = 1 - \theta\).

In this section, we derive two representation formulas for the kernel \(\tilde{\chi}\). We obtain the first representation by taking the difference of the representation formulas for \(\chi\) and \(\chi^*\) (see the proof of Lemma 3.4), and the second by directly considering the linear wave equation for \(\tilde{\chi}\), namely (2.1). The first representation formula is required to estimate \(\|\tilde{\chi}(\rho, \cdot)\|_{L^\infty(\mathbb{R})}\) in Lemma 3.6 below. Armed with this bound, we are able to compute the derivatives with respect to \(u\) of the entropies generated by the perturbation, which is required to show that the second representation is well-defined in Lemma 3.9.

At this point, we fix \(\rho_* \geq \max\{R_\rho(4C_p/\kappa_*)^{1/\alpha}\}\), so that all of the estimates of Section 2 hold. We also fix \(\alpha \in (0, 1)\) for clarity of exposition, though we note that Theorem 1.4 holds true for any \(\alpha > 0\) and that better estimates are available for \(\alpha \geq 1\) (cf. [16, Chapter 3]). We also make note of the following lemma, which outlines some important properties of \(\chi^*\). These follow directly from the analysis of the entropy kernel considered in [15], and will be indispensable for our later estimates.

**Proposition 3.2.** There exists a positive constant \(M\) depending on \(\rho_*\) such that, with \(\tilde{\lambda} := \min(\lambda, 1)\),

\[
\|\chi^*(\rho, \cdot)\|_{L^\infty(\mathbb{R})} + [\chi^*(\rho, \cdot)]_{C^1(\mathbb{R})} \leq M \frac{\rho}{\sqrt{k(\rho)}} \quad \text{for all } \rho \geq \rho_*. \tag{3.3}
\]

**Remark 3.3.** By Theorem 3.1,

\[
\sup h(\rho, \cdot) = [-k(\rho), k(\rho)] =: \mathcal{K}. \tag{3.4}
\]

Similarly, for \(\rho \geq \rho_*\),

\[
\sup \chi^*(\rho, \cdot) = [-k_*(\rho), k_*(\rho)] =: \mathcal{K}^*. \tag{3.5}
\]

We therefore deduce from (1.20) that

\[
\sup \tilde{\chi}(\rho, \cdot) \subset [-\max\{k(\rho), k_*(\rho)\}, \max\{k(\rho), k_*(\rho)\}] =: \tilde{\mathcal{K}}. \tag{3.6}
\]

### 3.1. First representation formula and uniform estimate on the perturbation

We begin with a representation formula for the entropy kernel, representing the kernel \(\chi\) in terms of its integral along characteristic curves (note that the curves \(u \pm k(\rho) = \text{const. are characteristics}\), cf. [3, Equation (3.38)].

Lemma 3.4. Given any \((\rho_0, u_0) \in \mathcal{K}\) and any \(0 < \rho_* < \rho_0\), we have
\[
\chi(\rho_0, u_0) = \frac{1}{2(\rho_0 - \rho_*)k'(\rho_0)} \int_{\rho_*}^{\rho_0} k'(\rho) d\rho \chi(\rho, u_0 + k(\rho) - k(\rho_0)) d\rho + \frac{1}{2(\rho_0 - \rho_*)k'(\rho_0)} \int_{\rho_*}^{\rho_0} k'(\rho) d\rho \chi(\rho, u_0 - k(\rho_0) + k(\rho)) d\rho
\]
\[+ \frac{1}{2(\rho_0 - \rho_*)k'(\rho_0)} \int_{-(k(\rho_0) - k(\rho_*))}^{k(\rho_0) - k(\rho_*)} \chi(\rho_*, u_0 - y) dy. \tag{3.7} \]

Proof. Fix any \((\rho_0, u_0) \in \mathcal{K}\) with \(\rho_0 > \rho_*\). Then from applying the Fourier transform with respect to \(u\) to the equation \((1.10)\) for the entropy kernel \(\chi\) we see that, for any \(\rho \in (\rho_*, \rho_0)\),
\[
(\rho - \rho_*)k'(\rho)^2 \mathcal{F}_\chi(\rho, \xi) = - (\rho - \rho_*)\xi^{-2} \mathcal{F}_{\chi_{pp}}(\rho, \xi).
\]
Multiplying this by \(\sin((k(\rho) - k(\rho_0))\xi)\) and integrating in the interval \(\rho \in [\rho_*, \rho_0]\) we get
\[
\int_{\rho_*}^{\rho_0} (\rho - \rho_*)k'(\rho)^2 \sin((k(\rho) - k(\rho_0))\xi) \mathcal{F}_\chi(\rho, \xi) d\rho = - \xi^{-2} \int_{\rho_*}^{\rho_0} (\rho - \rho_*) \sin((k(\rho) - k(\rho_0))\xi) \mathcal{F}_{\chi_{pp}}(\rho, \xi) d\rho,
\]
\[
= \xi^{-2} \int_{\rho_*}^{\rho_0} \partial_\rho ((\rho - \rho_*) \sin((k(\rho) - k(\rho_0))\xi)) \mathcal{F}_{\chi_{pp}}(\rho, \xi) d\rho
\]
\[- \xi^{-2} (\rho - \rho_*) \sin((k(\rho) - k(\rho_0))\xi) \mathcal{F}_{\chi_{pp}}(\rho, \xi) \big|_{\rho_*}^{\rho_0},
\]
\[
= - \xi^{-2} \int_{\rho_*}^{\rho_0} \partial_\rho^2 [(\rho - \rho_*) \sin((k(\rho) - k(\rho_0))\xi)] \mathcal{F}_\chi(\rho, \xi) d\rho
\]
\[+ \xi^{-2} \left[ \partial_\rho [(\rho - \rho_*) \sin((k(\rho) - k(\rho_0))\xi)] \mathcal{F}_\chi(\rho, \xi) \right]_{\rho_*}^{\rho_0}, \]
and we find that the final line can be written as
\[
- \xi^{-1} \int_{\rho_*}^{\rho_0} \left( k' + ((\rho - \rho_*)k')' \right) \cos((k(\rho) - k(\rho_0))\xi) \mathcal{F}_\chi(\rho, \xi) d\rho
\]
\[+ \int_{\rho_*}^{\rho_0} (\rho - \rho_*)k'(\rho)^2 \sin((k(\rho) - k(\rho_0))\xi) \mathcal{F}_\chi(\rho, \xi) d\rho
\]
\[+ \xi^{-1} (\rho_0 - \rho_*)k'(\rho_0) \mathcal{F}_\chi(\rho_0, \xi) - \xi^{-2} \sin((k(\rho_*) - k(\rho_0))\xi) \mathcal{F}_\chi(\rho_*, \xi).
\]
Hence, we have
\[
(\rho_0 - \rho_*)k'(\rho_0) \mathcal{F}_\chi(\rho_0, \xi) = \xi^{-1} \sin((k(\rho_*) - k(\rho_0))\xi) \mathcal{F}_\chi(\rho_*, \xi)
\]
\[+ \int_{\rho_*}^{\rho_0} \left( k' + ((\rho - \rho_*)k')' \right) \cos((k(\rho) - k(\rho_0))\xi) \mathcal{F}_\chi(\rho, \xi) d\rho.
\]
Now, observe that for \(a > 0\) and \(g \in \mathcal{S}'(\mathbb{R}) \cap C(\mathbb{R})\), we have
\[
\mathcal{F}^{-1} (\xi^{-1} \sin(a\xi) \mathcal{F}(g(\xi))) (x) = \frac{1}{2} \int_{-a}^{a} g(x - y) dy,
\]
and
\[
\mathcal{F}^{-1} (\cos(a\xi) \mathcal{F}(g(\xi))) (x) = \frac{1}{2} (g(x + a) + g(x - a)).
\]
So, applying the inverse Fourier transform we obtain the result as claimed. \(\square\)

Remark 3.5. The same representation formula as \((3.7)\) holds for \(\chi^*\), except \(k\) should be replaced with \(k_*\), \(d\) with \(d_*\), and \(\mathcal{K}\) with \(\mathcal{K}^*\).

The most significant result of this section is the following estimate on the growth rate of \(\chi(\rho, u - s)\) as \(\rho \to \infty\).
Lemma 3.6. There exists a constant $M > 0$ such that
\[
\|\tilde{\chi}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq M \rho^{1 - \tilde{\lambda}} \quad \text{for} \ \rho \geq \rho_*,
\] (3.8)
where we recall that $\tilde{\lambda} = \min\{\lambda, 1\}$.

Proof. In view of Lemma 3.4 and Remark 3.5, by subtracting $\chi^*(\rho_0, u_0)$ from $\chi(\rho_0, u_0)$ and recalling that $\chi(\rho_*, \cdot) = \chi^*(\rho_*, \cdot)$, we arrive at the first representation formula for the perturbation. Given any $(\rho_0, u_0) \in \tilde{K}$,
\[
2k'(\rho_0)(\rho_0 - \rho_*)\tilde{\chi}(\rho_0, u_0) = \int_{\rho_*}^{\rho_0} k'(\rho) d\rho \left[ \tilde{\chi}(\rho, u_0 + k(\rho_0) - k(\rho)) + \tilde{\chi}(\rho, u_0 - k(\rho_0) + k(\rho)) \right] d\rho
\]
\[+ \int_{\rho_*}^{\rho_0} k'(\rho) d\rho \left[ \chi^*(\rho, u_0 + k(\rho_0) - k(\rho)) - \chi^*(\rho, u_0 + k(\rho_0) - k(\rho)) \right] d\rho
\]
\[+ \int_{\rho_*}^{\rho_0} k'(\rho) d\rho \left[ \chi^*(\rho, u_0 - k(\rho_0) + k(\rho)) - \chi^*(\rho, u_0 - k(\rho_0) + k(\rho)) \right] d\rho
\]
\[+ \int_{\rho_*}^{\rho_0} k'(\rho) d\rho \left[ k'(\rho_0) \right] \chi^*(\rho, u_0 - k(\rho_0) + k(\rho)) d\rho
\]
\[+ \int_{\rho_*}^{\rho_0} k'(\rho) d\rho \left[ k'(\rho_0) \right] \chi^*(\rho, u_0 - k(\rho_0) + k(\rho)) d\rho
\]
\[= I_1 + \cdots + I_6.
\]

We then bound $|I_1|$ by
\[
|I_1| \leq \int_{\rho_*}^{\rho_0} 2d\rho \|k'(\rho)\tilde{\chi}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} d\rho.
\]

$I_2$ is bounded by
\[
|I_2| \leq \int_{\rho_*}^{\rho_0} d\rho \|k'(\rho)\|_{C^*(\mathbb{R})} \|(k(\rho_0) - k(\rho)) - (k_*(\rho_0) - k_*(\rho))\| \tilde{\lambda} d\rho
\]
\[\leq M \int_{\rho_*}^{\rho_0} \rho^{-\alpha} \|k'(\rho)\|_{C^*(\mathbb{R})} d\rho,
\]
where we have applied Lemma A.4 and the bound (A.3), and $M = M(\alpha, \kappa_2, C_p)$. Notice that $I_4$ can be bounded in exactly the same way. Next, $I_3$ is bounded by
\[
|I_3| \leq \int_{\rho_*}^{\rho_0} d\rho \left| d(\rho) - d_*(\rho) \frac{k'(\rho_0)}{k'(\rho_0)} \frac{k_*(\rho)}{k'(\rho)} \right| \|k'(\rho)\chi^*(\rho, \cdot)\|_{L^\infty(\mathbb{R})} d\rho,
\]
and we bound the first term in the integrand by
\[
|d(\rho) - d_*(\rho)| \leq |d(\rho) - d_*(\rho)| \frac{k'(\rho_0)}{k'(\rho_0)} \frac{k_*(\rho)}{k'(\rho)} \leq M \rho^{-\alpha}
\]
using Lemma A.2 for the right-hand term and Lemma A.3 for the left-hand term, respectively, along with (A.3). Thus we see that
\[
|I_3| \leq M \int_{\rho_*}^{\rho_0} \rho^{-\alpha} \|k'(\rho)\chi^*(\rho, \cdot)\|_{L^\infty(\mathbb{R})} d\rho,
\]
where $M = M(\alpha, \kappa_2, C_p)$. Once again, $I_5$ may be bounded in exactly the same way. We split $I_6$ into two terms as
\[
I_6 = \int_{(k(\rho_0) - k(\rho_0))}^{k(\rho_0) - k(\rho_0)} \chi^*(\rho_*, u_0 - y) dy - \int_{(k_*(\rho_0) - k_*(\rho_0))}^{k_*(\rho_0) - k_*(\rho_0)} \chi^*(\rho_*, u_0 - y) dy
\]
\[+ \left( 1 - \frac{k'(\rho_0)}{k'_*(\rho_0)} \right) \int_{(k_*(\rho_0) - k_*(\rho_0))}^{k_*(\rho_0) - k_*(\rho_0)} \chi^*(\rho_*, u_0 - y) dy.
\] (3.10)
We concentrate on the first line of (3.10). Observe that the two intervals, \([-k(\rho_0) - k(\rho_*)], k(\rho_0) - k(\rho_*)]\) and \([-k_*(\rho_0) - k_*(\rho_*)], k_*(\rho_0) - k_*(\rho_*)]\), are always nested within one another; one interval is always entirely contained in the other. Hence, since the same quantity is being integrated, we may bound this line by

\[
2 \left| k(\rho_0) - k(\rho_*) \right| \cdot \left| \| \chi^*(\rho_* \cdot) \|_{L^\infty(\mathbb{R})} \right| \leq M \rho_*^{-\alpha} \min \left( \frac{\rho_0}{\rho_*}, 1 \right) \| \chi^*(\rho_* \cdot) \|_{L^\infty(\mathbb{R})},
\]

by Lemma A.4. On the other hand, the final term of (3.10) is bounded by

\[
\left| 1 - \frac{k'(\rho_0)}{k'_*(\rho_0)} \right| \cdot 2 \left| k_*(\rho_0) - k_*(\rho_*) \right| \cdot \left| \| \chi^*(\rho_* \cdot) \|_{L^\infty(\mathbb{R})} \right| \leq M \rho_0^{-\alpha} \log \left( \frac{\rho_0}{\rho_*} \right) \| \chi^*(\rho_* \cdot) \|_{L^\infty(\mathbb{R})},
\]

by Lemma A.5. We emphasise that it has been sufficient to assume \(\alpha > 0\).

In total, by combining these estimates for \(I_1, \ldots, I_6\) and then dividing through (3.9) by the factor \((\rho_0 - \rho_*)\), we obtain

\[
\| k'(\rho_0) \chi(\rho_0 \cdot) \|_{L^\infty(\mathbb{R})} \leq \frac{1}{(\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} d\rho \| k'(\rho) \chi(\rho \cdot) \|_{L^\infty(\mathbb{R})} d\rho
\]

\[
+ \frac{M}{(\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} \rho^{-\alpha} \| k'(\rho) \chi^*(\rho \cdot) \|_{C^1(\mathbb{R})} d\rho
\]

\[
+ \frac{M}{(\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} \rho^{-\alpha} \| k'(\rho) \chi^*(\rho \cdot) \|_{L^\infty(\mathbb{R})} d\rho
\]

\[
+ \frac{M}{1 + (\rho_0 - \rho_*)} \rho_0^\alpha \log \left( 1 + \frac{\rho_0}{\rho_*} \right) \| \chi^*(\rho_* \cdot) \|_{L^\infty(\mathbb{R})}.
\]

We now apply Grönwall’s lemma to (3.11) and divide by \(k'(\rho_0)\), which yields

\[
\| \chi(\rho_0 \cdot) \|_{L^\infty(\mathbb{R})} \leq \frac{M}{(\rho_0 - \rho_*) k'(\rho_0)} \int_{\rho_*}^{\rho_0} \rho^{-\alpha} \| k'(\rho) \chi^*(\rho \cdot) \|_{C^1(\mathbb{R})} d\rho
\]

\[
+ \frac{M}{(\rho_0 - \rho_*) k'(\rho_0)} \int_{\rho_*}^{\rho_0} \rho^{-\alpha} \| k'(\rho) \chi^*(\rho \cdot) \|_{L^\infty(\mathbb{R})} d\rho
\]

\[
+ \frac{M}{k'(\rho_0)} \frac{1}{1 + (\rho_0 - \rho_*)} \rho_0^\alpha \| \chi^*(\rho_* \cdot) \|_{L^\infty(\mathbb{R})} \exp \left( \frac{1}{(\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} d\rho \right).
\]

It is clear that, in view of the bound on \(d(\rho)\) provided by Lemma A.2, the exponential on the right-hand side is bounded above by \(e^3\). Now recall from Proposition 3.2 that

\[
\| \chi^*(\rho \cdot) \|_{L^\infty(\mathbb{R})} + [\chi^*(\rho \cdot)]_{C^1(\mathbb{R})} \leq M \rho \quad \text{for all } \rho \geq \rho_*.
\]

Then, using also Lemma A.5, we have

\[
\| k'(\rho) \chi^*(\rho \cdot) \|_{L^\infty(\mathbb{R})} + [k'(\rho) \chi^*(\rho \cdot)]_{C^1(\mathbb{R})} \leq M \quad \text{for all } \rho \geq \rho_*.
\]

We are now in a position to bound each of the terms in the brackets of (3.12). We begin with the first term,

\[
\frac{M}{(\rho_0 - \rho_*) k'(\rho_0)} \int_{\rho_*}^{\rho_0} \rho^{-\alpha} \| k'(\rho) \chi^*(\rho \cdot) \|_{C^1(\mathbb{R})} d\rho \leq \frac{M}{(\rho_0 - \rho_*) k'(\rho_0)} \int_{\rho_*}^{\rho_0} \rho^{-\alpha} d\rho \leq M \rho_0^{-\alpha},
\]

since \(\alpha \in (0, 1)\) implies \(\alpha \lambda \in (0, 1)\).

Similarly, for the second term of (3.12), noting again that \(\alpha \in (0, 1)\),

\[
\frac{M}{(\rho_0 - \rho_*) k'(\rho_0)} \int_{\rho_*}^{\rho_0} \rho^{-\alpha} \| k'(\rho) \chi^*(\rho \cdot) \|_{L^\infty(\mathbb{R})} d\rho \leq \frac{M}{(\rho_0 - \rho_*) k'(\rho_0)} \int_{\rho_*}^{\rho_0} \rho^{-\alpha} d\rho \leq M \rho_0^{-\alpha}.
\]
For the third and fourth terms of (3.12), observe that, by Lemma 2.4,
\[ \frac{1}{k'(\rho_0)[1 + (\rho_0 - \rho_*)]} \leq M. \] (3.15)
Thus, collecting the results in (3.13)–(3.15), we have that, for some positive \( M = M(\alpha, \kappa_2, C_\rho, \rho_*), \)
\[ \|\tilde{\chi}(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq M \left( \rho_0^{1-\alpha} + \rho_0^{1-\alpha} + 1 \right), \]
which (by making \( M \) larger if necessary) straightforwardly yields the result.

**Remark 3.7.** We note that, if \( \alpha \lambda > 1 \), the proof of Lemma 3.6 shows that we have the stronger estimate
\[ \|\tilde{\chi}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq M \quad \text{for } \rho \geq \rho_* \]

### 3.2. Generating entropies and second representation formula

In what follows, we generate entropies by convolving with the entropy kernel. Specifically, given a suitably integrable test function \( \psi \), we may generate an entropy according to the formula
\[ \eta^\psi(\rho, u) := \int_\mathbb{R} \chi(\rho, u - s)\psi(s) \, ds. \] (3.16)
Thus, for \( \rho \geq \rho_* \),
\[ \eta^\psi(\rho, u) = \int_\mathbb{R} \chi(\rho, u - s)\psi(s) \, ds + \int_\mathbb{R} \tilde{\chi}(\rho, u - s)\psi(s) \, ds =: \eta^\psi(\rho, u). \] (3.17)

**Remark 3.8.** Since \( \chi \) is given as a function of \((\rho, u)\), we will also generate entropies as functions of \((\rho, u)\), even though these are technically functions of \((\rho, m)\). We adopt this convention throughout this subsection and Section 4 only, in order to simplify computations.

In particular, when we choose the special test function \( \hat{\psi}(s) = \frac{1}{2}|s| \), for \( \rho \geq \rho_* \),
\[ \hat{\eta}(\rho, u) := \eta^\psi(\rho, u) = \eta^\psi(\rho, u) + \tilde{\eta}^\psi(\rho, u) =: \hat{\eta}(\rho, u). \] (3.18)

**Lemma 3.9** (Second representation formula for the perturbation). Let \( \psi \in C^2(\mathbb{R}) \) be such that \( \tilde{\eta}^\psi, \eta^\psi, \nu^\psi \) are well-defined continuous functions. Then, for any \((\rho_0, u_0) \in K, \)
\begin{align*}
2(\rho_0 - \rho_*)k'(\rho_0)\tilde{\eta}^\psi(\rho_0, u_0) \\
&= \int_{\rho_*}^{\rho_0} d(\rho)k'(\rho) \left( \tilde{\eta}^\psi(\rho, u_0 + k(\rho)) - \tilde{\eta}^\psi(\rho, u_0 + k(\rho)) \right) \, d\rho \\
&+ \int_{\rho_*}^{\rho_0} (\rho - \rho_*)k'(\rho) \left( \frac{K_2}{\rho^2} \right) \eta^\psi_u(\rho, u_0 + k(\rho)) \, d\rho \\
&+ \int_{\rho_*}^{\rho_0} (\rho - \rho_*)k'(\rho) \left( \frac{K_2}{\rho^2} \right) \nu^\psi_u(\rho, u_0 - k(\rho)) \, d\rho.
\end{align*} (3.19)

The proof of Lemma 3.9 is similar to that of Lemma 3.4, and makes use of the fact that \( \tilde{\eta}^\psi \) satisfies an analogous wave equation to (1.21).

### 3.3. Generating entropy fluxes

Recall from Theorem 3.1 that we decompose the entropy flux kernel \( \sigma(\rho, u, s) \) for our problem as
\[ \sigma(\rho, u, s) = u \chi(\rho, u, s) + h(\rho, u, s). \] (3.20)
Then the entropy flux corresponding to an entropy \( \eta^\psi \) is given by the representation
\[ q^\psi(\rho, u) = \int_\mathbb{R} \sigma(\rho, u, s)\psi(s) \, ds = u \int_\mathbb{R} \chi(\rho, u - s)\psi(s) \, ds + \int_\mathbb{R} h(\rho, u - s)\psi(s) \, ds, \]
\[ = u\eta^\psi(\rho, u) + \int_\mathbb{R} h(\rho, u - s)\psi(s) \, ds. \]
In turn, defining

\[ H^\psi(\rho, u) := \int h(\rho, u - s) \psi(s) \, ds, \tag{3.21} \]

we see that \( q^\psi(\rho, u) = u w^\psi(\rho, u) + H^\psi(\rho, u) \). As such, any entropy flux can be generated from the kernels \( \chi \) and \( h \). Analogously to the decomposition of the entropy kernel, we decompose the kernel \( h \) as an isothermal kernel, and a perturbation. To this end, we make the following definition.

**Definition 3.10.** Define \( h^\ast \) by

\[ h^\ast(\rho, u) := \int h_\rho(\rho_s, s) g^x(\rho, u - s) \, ds + \int h(\rho_s, s) g^y(\rho, u - s) \, ds, \tag{3.22} \]

where \( g^x \) and \( g^y \) were introduced in Definition 1.6 and define also

\[ \bar{h}(\rho, u) := h(\rho, u) - h^\ast(\rho, u) \quad \text{for } \rho \geq \rho_s. \tag{3.23} \]

Note that (3.22) and Theorem 3.1 imply that \( h^\ast \) is the unique solution of

\[
\begin{cases}
  h^\ast_{\rho\rho} - \frac{k_2}{\rho^2} h^\ast_{uu} = 0 & \text{for } (\rho, u) \in (\rho_s, \infty) \times \mathbb{R}, \\
  h^\ast(\rho_s, u) = h(\rho_s, u), \\
  h^\ast_{\rho}(\rho_s, u) = h_\rho(\rho_s, u).
\end{cases}
\tag{3.24}
\]

Thus \( h^\ast(\rho, \cdot) \) is Hölder continuous, and

\[ \text{supp } h^\ast(\rho, \cdot) \subset [-k_\ast(\rho), k_\ast(\rho)] \quad \text{for } (\rho, u) \in [\rho_s, \infty) \times \mathbb{R}. \tag{3.25} \]

Moreover,

\[ \sigma^\ast(\rho, u, s) = u \chi^\ast(\rho, u - s) + h^\ast(\rho, u - s), \]

where \( \sigma^\ast \) is in fact the entropy flux kernel considered in [15] (up to a re-scaling in terms of \( \rho_s \)). This observation makes use of the fact, see [15], that the isothermal flux term \( h^\ast \) is itself an entropy.

**Lemma 3.11.** There exists a positive constant \( M \) depending also on \( \rho_s \) such that

\[ \| h^\ast(\rho, \cdot) \|_{L^\infty(\mathbb{R})} \leq M \frac{\rho}{\sqrt{k(\rho)}} \quad \text{for } \rho \geq \rho_s. \tag{3.26} \]

**Proof.** Using the kernels \( g^x \) and \( g^y \) (see Definition 1.6), we see that

\[ h^\ast(\rho, u) = \int h_\rho(\rho_s, s) g^x(\rho, u - s) \, ds + \int h(\rho_s, s) g^y(\rho, u - s) \, ds. \tag{3.27} \]

The result is now easily deduced using the explicit forms of \( g^x \) and \( g^y \), as was done for \( \chi^\ast \) in Proposition 3.2. \qed

A simple calculation verifies the following lemma.

**Lemma 3.12.** The perturbation \( \bar{h}(\rho, u) \) solves, for \( (\rho, u) \in (\rho_s, \infty) \times \mathbb{R} \),

\[
\begin{cases}
  \bar{h}_{\rho\rho} - k'(\rho)^2 \bar{h}_{uu} = (k'(\rho)^2 - \frac{k_2}{\rho^2}) h^\ast_{uu}(\rho, u) + \frac{\rho''(\rho)}{\rho} \chi_u(\rho, u), \\
  \bar{h}(\rho_s, u) = 0, \\
  \bar{h}_{\rho}(\rho_s, u) = 0.
\end{cases}
\tag{3.28}
\]

For \( \psi \in C^2_\nu(\mathbb{R}) \) and \( \tilde{\psi}(s) = \frac{\sqrt{\rho}}{\rho} s |s| \), we define, for \( (\rho, u) \in [\rho_s, \infty) \times \mathbb{R} \),

\[ H^{\ast, \tilde{\psi}}(\rho, u) := \int h^\ast(\rho, u - s) \tilde{\psi}(s) \, ds, \quad \bar{H}^\ast(\rho, u) := \int \bar{h}(\rho, u - s) \tilde{\psi}(s) \, ds. \tag{3.29} \]

Now the entropy flux corresponding to \( \eta^\psi \) is given by

\[ q^\psi(\rho, u) = q^{\psi, \ast}(\rho, u) + \tilde{q}^\psi(\rho, u) = u \eta^\psi, \ast(\rho, u) + u \tilde{\eta}^\psi(\rho, u) + H^{\psi, \ast}(\rho, u) + \bar{H}^\ast(\rho, u) \]
and the flux corresponding to \( \bar{\eta} \) is given by
\[
\bar{q}(\rho, u) = q^*(\rho, u) + \bar{q}(\rho, u) = u\bar{\eta}(\rho, u) + u\bar{\eta}(\rho, u) + \bar{H}^*(\rho, u) + \bar{H}(\rho, u)
\]
with obvious notation for \( \bar{H}^* \) and \( \bar{H} \).

4. Estimating entropy pairs

In this section, we use the representation formulas obtained in Section 3 and the estimate of Lemma 3.6 to estimate the entropy pairs generated by the special function \( \hat{\psi}(s) = \frac{1}{2}s|s| \) and by compactly supported test functions.

4.1. Special entropy pair. We first consider the special entropy, \( \hat{\eta}(\rho, u) \), generated by \( \hat{\psi}(s) = \frac{1}{2}s|s| \). This was defined by the formula (3.18) for \( \rho \geq \rho_* \),
\[
\hat{\eta}(\rho, u) = \frac{1}{2} \int_{\mathbb{R}} \chi^*(\rho, u - s)|s||s| ds + \frac{1}{2} \int_{\mathbb{R}} \chi(\rho, u - s)|s||s| ds,
\]
(4.1)
The goal of this subsection is to prove the following lemma, which is the analogue of [15, Lemma 3.5].

Lemma 4.1. Let \( \hat{\psi}(s) = \frac{1}{2}s|s| \), and \( (\hat{\eta}, \hat{q}) \) its associated entropy pair via (3.18) and (3.30). There exists a positive constant \( M \) such that
\[
|\hat{\eta}(\rho, m)| \leq M\eta^*(\rho, m), \quad |\hat{\eta}_m(\rho, m)| \leq M(|u| + \sqrt{\log(\rho/\rho_*)}), \quad |\hat{\eta}_{mm}(\rho, m)| \leq M, \quad (4.2)
\]
where \( \eta^*(\rho, m) \) is the physical entropy defined in (1.6) and
\[
\bar{q}(\rho, m) \geq M^{-1}|u|^3 - M(\rho|u|^2 + \rho + \rho(\log(\rho/\rho_*))^4)
\]
(4.3)
for all \( \rho \geq \rho_* \). For the mixed derivatives \( \hat{\eta}_{mu}(\rho, pu) = \partial_u \hat{\eta}_m(\rho, pu) \) and \( \hat{\eta}_{mp}(\rho, pu) = \partial_p \hat{\eta}_m(\rho, pu) \), we have
\[
|\hat{\eta}_{mu}(\rho, pu)| \leq M \frac{1}{1 + \sqrt{\log(\rho/\rho_*)}}, \quad |\hat{\eta}_{mp}(\rho, pu)| \leq M \rho^{-1}.
\]
(4.4)
Moreover, on the complement region \( \rho \leq \rho_* \), we have
\[
|\hat{\eta}(\rho, m)| \leq M\eta^*(\rho, m), \quad \hat{\eta}(\rho, m) \geq M^{-1}(\rho|u|^3 + \rho^{\gamma+\theta}) - M(\rho|u|^2 + \rho^{\gamma}),
\]
\[
|\hat{\eta}_m(\rho, m)| \leq M(|u| + \rho^\gamma), \quad |\hat{\eta}_{mm}(\rho, m)| \leq M, \quad (4.5)
\]
\[
|\hat{\eta}_{mu}(\rho, pu)| \leq M, \quad |\hat{\eta}_{mp}(\rho, pu)| \leq M \rho^{-1},
\]
where in the final line we consider \( \bar{\eta}_m(\rho, pu) \) as a function of \( \rho \) and \( u \), as in (4.4). Finally,
\[
\rho|\hat{\eta}_m(\rho, 0) - \bar{\eta}_m(\rho, 0)|^2 \leq Me^*(\rho, \rho) \quad \text{for} \ \rho, \bar{\rho} \geq 0.
\]
(4.6)
By [15, Lemma 3.5], the estimates of Lemma 4.1 hold for the principal parts \( \hat{\eta}^* \) and \( \hat{q}^* \). In addition, the estimates (4.5) in the lower region are by now standard and may be seen, for example, in [15, Lemma 3.5]. Thus we will focus in the sequel on the error terms, \( \bar{\eta} \) and \( \bar{q} \). As these estimates are lengthy, the following two subsections will be devoted to their proofs. As a guide to the reader, we point out here that estimates (4.2) may be found in Lemmas 4.3 and 4.4 while (4.4) may be found in Lemma 4.6. Finally, (4.3) is proved in Corollary 4.11.

4.1.1. Higher differentiability for the special entropy. Surprisingly, it is the case that the special entropy \( \hat{\eta} \) of (4.1) is three times continuously differentiable in its second variable, as demonstrated in the next lemma. This property will be used repeatedly in the proofs of (4.2)-(4.4).
Lemma 4.2. The entropies \( \tilde{\eta}^* \) and \( \tilde{\eta} \) are three times continuously differentiable in their second variable, i.e., \( \tilde{\eta}^*(\rho, \cdot), \tilde{\eta}(\rho, \cdot) \in C^3(\mathbb{R}) \) for each \( \rho \in [\rho_*, \infty) \). In fact,
\[
\tilde{\eta}_{uuu}(\rho, u) = 2\tilde{\chi}^*(\rho, u), \quad \tilde{\eta}_{uu}(\rho, u) = 2\tilde{\chi}(\rho, u)
\] for \( (\rho, u) \in [\rho_*, \infty) \times \mathbb{R} \). (4.7)
Correspondingly, there exists a positive constant \( M \) such that
\[
\|\tilde{\eta}_{uuu}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{M\rho}{\sqrt{k(\rho)}} \quad \|\tilde{\eta}_{uu}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq M\rho^{1-\alpha}\lambda \quad \text{for } \rho \geq \rho_*. \quad (4.8)
\]
Proof. Recall that, by definition,
\[
\tilde{\eta}(\rho, u) = \frac{1}{2} \int_{-\infty}^u \tilde{\chi}(\rho, s)(u-s)^2 \, ds - \frac{1}{2} \int_u^\infty \tilde{\chi}(\rho, s)(u-s)^2 \, ds.
\]
The rest of the proof now follows from direct calculation and Lemma 3.6.

4.1.2. Bounds on the special entropy and its \( u \) derivatives. Below is the first estimate on the special entropy, which shows that it is controlled by the mechanical energy, \( \eta^* \).

Lemma 4.3. There exists a positive constant \( M \) such that
\[
|\tilde{\eta}(\rho, u)| \leq M \eta^*(\rho, u).
\]
(4.10)
Proof. From Remark 3.3, supp \( \tilde{\chi}(\rho, \cdot) \subset [-\max\{k(\rho), k_*(\rho)\}, \max\{k(\rho), k_*(\rho)\}] \). Using the bound \( \max\{k(\rho), k_*(\rho)\} \leq Mk(\rho) \) due to Lemma A.6, we see from (4.9) and the Cauchy–Schwarz inequality that
\[
|\tilde{\eta}(\rho, u)| \leq Mk(\rho)u^2\|\tilde{\chi}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} + Mk(\rho)^3\|\tilde{\chi}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq \rho^{1-\alpha}\lambda (k(\rho)u^2 + k(\rho)^3)
\]
(4.11)
using Lemma 3.6. The result now follows easily as \( \eta^*(\rho, u) = \frac{1}{2}\rho u^2 + \rho \epsilon(\rho) \) and as \( \rho \to \infty, \frac{\epsilon(\rho)}{\log(\rho)} \to \text{const.} > 0 \).

Arguing similarly for the derivatives of \( \tilde{\eta} \), we also obtain the following lemma.

Lemma 4.4. There exists a positive constant \( M \) such that
\[
|\tilde{\eta}_u(\rho, u)| \leq \frac{M}{1 + \log(\rho/\rho_*)} (|u| + 1) \quad \text{for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R},
\]
\[
|\tilde{\eta}_{uu}(\rho, u)| + |\rho \tilde{\eta}_{u0}(\rho, u)| \leq \frac{M}{1 + \log(\rho/\rho_*)} \quad \text{for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R}.
\]
(4.12)
The last estimate that we need for \( \tilde{\eta} \) is that for \( \tilde{\eta}_{u0} \). To prove this, we first show an improved estimate on \( \tilde{\eta}_m \). This is the subject of the next result, Lemma 4.5. Convoluting (1.21) with the special test function \( \psi = \frac{1}{2}e|s| \), we find that \( \tilde{\eta} \) satisfies
\[
\begin{cases}
\tilde{\eta}_{pp} - k'(\rho)^2\tilde{\eta}_{uu} = (k'(\rho)^2 - \frac{K_2}{\rho^2})\tilde{\eta}_{uu}^* & \text{for } (\rho, u) \in (\rho_*, \infty) \times \mathbb{R},
\tilde{\eta}(\rho_*, u) = 0,
\tilde{\eta}_{u}(\rho_*, u) = 0.
\end{cases}
\]
(4.13)
Proceeding exactly as was done in the proof of Lemma 3.4, we obtain
\[
2(\rho_0 - \rho_*)k'(\rho_0)\tilde{\eta}(\rho_0, u_0)
= \int_{\rho_0}^{\rho_u} d(\rho)k'(\rho)(\tilde{\eta}(\rho, u_0 + k(\rho_0) - k(\rho)) + \tilde{\eta}(\rho, u_0 - k(\rho_0) + k(\rho))) \, d\rho
+ \int_{\rho_0}^{\rho_u} (\rho - \rho_*)(k'(\rho)^2 - \frac{K_2}{\rho^2})\tilde{\eta}_{u}(\rho, u_0 + k(\rho_0) - k(\rho)) \, d\rho
- \int_{\rho_0}^{\rho_u} (\rho - \rho_*)(k'(\rho)^2 - \frac{K_2}{\rho^2})\tilde{\eta}_{u}(\rho, u_0 - k(\rho_0) + k(\rho)) \, d\rho.
\]
(4.14)
Thus, noting that $\tilde{\eta}_m^* = \rho\tilde{\eta}_m^*$ and taking a derivative with respect to $u_0$,
\[
2(\rho_0 - \rho_*)\rho_0 k'(\rho_0)\tilde{\eta}_m(\rho_0, u_0) \\
= \int_{\rho_*}^{\rho_0} d(\rho)\rho k'(\rho)\left(\tilde{\eta}_m(\rho, u_0 + k(\rho_0) - k(\rho)) + \tilde{\eta}_m(\rho, u_0 - k(\rho_0) + k(\rho))\right) d\rho \\
+ \int_{\rho_*}^{\rho_0} (\rho - \rho_*)\rho k'(\rho)^2 \frac{\kappa_2}{\rho^2} \tilde{\eta}_{m\mu}(\rho, u_0 + k(\rho_0) - k(\rho)) d\rho \\
- \int_{\rho_*}^{\rho_0} (\rho - \rho_*)\rho k'(\rho)^2 \frac{\kappa_2}{\rho^2} \tilde{\eta}_{m\mu}(\rho, u_0 - k(\rho_0) + k(\rho)) d\rho.
\]  
(4.15)

**Lemma 4.5.** There exists a positive constant $M$ such that
\[
\|\tilde{\eta}_m(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq M \rho^{-\alpha} \quad \text{for } \rho \geq \rho_*.
\]  
(4.16)

**Proof.** Observe that, after dividing by $2(\rho_0 - \rho_*)$, the first term on the right-hand side of (4.15) is bounded by
\[
\frac{1}{(\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} d(\rho)\rho k'(\rho)\tilde{\eta}_m(\rho, \cdot)\|_{L^\infty(\mathbb{R})} d\rho.
\]
On the other hand, the sum of the second and third terms on the right-hand side of (4.15) is bounded by
\[
\frac{M}{(\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} |k'(\rho)^2 - \frac{\kappa_2}{\rho^2}|(\rho - \rho_*)d\rho,
\]
where we made use of the bound on $\tilde{\eta}_{m\mu}^*$ provided by Proposition 3.2 (cf. [15, Lemma 3.5]). Thus, appealing to Corollary 2.5 to control the $k'(\rho)^2$ term, the sum of the second and third terms on the right-hand side of (4.15) is bounded by
\[
\frac{M}{\rho_0 - \rho_*} \int_{\rho_*}^{\rho_0} \rho^{-\alpha} d\rho \leq M \rho_0^{-\alpha},
\]
provided $\alpha \in (0, 1)$. In summary, we get
\[
\|\rho_0 k'(\rho_0)\tilde{\eta}_m(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq M \rho_0^{-\alpha} + \frac{1}{(\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} d(\rho)\rho k'(\rho)\tilde{\eta}_m(\rho, \cdot)\|_{L^\infty(\mathbb{R})} d\rho,
\]
from which an application of Grönwall’s lemma yields
\[
\|\rho_0 k'(\rho_0)\tilde{\eta}_m(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq M \rho_0^{-\alpha} \exp \left(\frac{1}{\rho_0 - \rho_*} \int_{\rho_*}^{\rho_0} d(\rho) \right).
\]
Using the bound on $d(\rho)$ provided by Lemma A.2 concludes the proof. \hfill \Box

Armed with the previous result, we are in a position to prove the required estimate on the mixed derivative $\tilde{\eta}_{m\rho}$, contained in the next lemma.

**Lemma 4.6.** There exists a positive constant $M$ such that
\[
\|\tilde{\eta}_{m\rho}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq Mk(\rho)^{\frac{1}{1-\alpha\lambda}} \quad \text{for } \rho \geq \rho_*.
\]  
(4.17)

**Proof.** We begin by differentiating (4.15) with respect to $\rho_0$. We get
\[
2(\rho_0 - \rho_*)\rho_0 k'(\rho_0)\tilde{\eta}_{m\rho}(\rho_0, u_0) + \partial_{\rho_0} \left(2(\rho_0 - \rho_*)\rho_0 k'(\rho_0)\right)\tilde{\eta}_m(\rho_0, u_0) \\
= k'(\rho_0) \int_{\rho_*}^{\rho_0} d(\rho)\rho k'(\rho)\left(\tilde{\eta}_{m\rho}(\rho, u_0 + k(\rho_0) - k(\rho)) - \tilde{\eta}_{m\rho}(\rho, u_0 - k(\rho_0) + k(\rho))\right) d\rho \\
+ 2d(\rho_0)\rho_0 k'(\rho_0)\tilde{\eta}_m(\rho_0, u_0) \\
+ k'(\rho_0) \int_{\rho_*}^{\rho_0} (\rho - \rho_*)\rho k'(\rho)^2 \frac{\kappa_2}{\rho^2} \tilde{\eta}_{m\mu}(\rho, u_0 + k(\rho_0) - k(\rho)) d\rho \\
+ k'(\rho_0) \int_{\rho_*}^{\rho_0} (\rho - \rho_*)\rho k'(\rho)^2 \frac{\kappa_2}{\rho^2} \tilde{\eta}_{m\mu}(\rho, u_0 - k(\rho_0) + k(\rho)) d\rho.
\]  
(4.18)
Using the bounds of Lemmas 4.4 and 4.5, we may apply a Grönewall argument, similarly to the proof of Lemma 4.5 in order to conclude the desired result. Note that the second term on the left-hand side of (4.18) is controlled by expanding the derivative term and using the bounds provided by Lemmas 2.4 and 4.5, and Corollary 2.5. Further details may be found in [16, Chapter 3]. □

We conclude this subsection with a stronger estimate on the entropy error, $\tilde{\eta}$.

**Corollary 4.7.** There exists a positive constant $M$ such that

$$|\tilde{\eta}(\rho, u)| \leq M |u| \quad \text{for } (\rho, u) \in \rho_s \times \mathbb{R}.$$  \hfill (4.19)

**Proof.** Firstly, recall from the definition of $\tilde{\eta}$ that

$$\tilde{\eta}(\rho, 0) = \frac{1}{2} \int_\mathbb{R} \tilde{\chi}(\rho, 0 - s)s|s|ds = 0 \quad \text{for } \rho \geq \rho_s,$$  \hfill (4.20)

since the integrand is odd, as $\tilde{\chi}(\rho, \cdot)$ is even (cf. [16, Lemma 2.7]). We also have

$$\tilde{\eta}_m(\rho_s, u) = \rho_s^{-1} \frac{\partial}{\partial u} \tilde{\eta}(\rho_s, u) = 0 \quad \text{for all } u \in \mathbb{R}.$$  \hfill (4.21)

Additionally, since $k(\rho) \leq M(1 + \log(\rho/\rho_s))$ by Corollary A.1, there exists $M > 0$ such that

$$\rho^{-\alpha \lambda/2}k(\rho) \leq M \quad \text{for } \rho \geq \rho_s.$$  

The fundamental theorem of calculus and (4.21) show that

$$\tilde{\eta}_m(\rho, u) - \tilde{\eta}_m(\rho_s, u) = \int_{\rho_s}^{\rho} \frac{\partial}{\partial \rho} \tilde{\eta}_m(y, u) dy = \int_{\rho_s}^{\rho} \tilde{\eta}_m(y, u) dy,$$

where the final integrand is the mixed derivative $\partial_\rho \tilde{\eta}_m(\rho, pu)$. Therefore, integrating in $\rho$ yields

$$|\rho^{-1} \tilde{\eta}_u(\rho, u)| = |\tilde{\eta}_m(\rho, u)| \leq \int_{\rho_s}^{\rho} |\tilde{\eta}_m(y, u)| dy \leq M \int_{\rho_s}^{\rho} y^{-1-\alpha \lambda}k(y) dy,$$

$$\leq M \int_{\rho_s}^{\rho} y^{-1-\alpha \lambda/2} dy,$$

using Lemma 4.6, so that $\tilde{\eta}_u(\rho, u) \leq M \rho$ for all $\rho \geq \rho_s$. Integrating the above in $u$, using the fact that $\tilde{\eta}(\rho, 0) = 0$ from (4.20), yields the result. □

4.1.3. Lower bound for the special entropy flux. The final estimate of Lemma 4.1 that we need to prove is (4.3). This is the purpose of this subsection. We recall from (3.30) that

$$\tilde{q}^*(\rho, u) = u \tilde{\eta}^*(\rho, u) + \tilde{H}^*(\rho, u)$$

and begin this subsection by collecting some estimates on $\tilde{H}^*$.

**Lemma 4.8.** The function $\tilde{H}^*$ is three times continuously differentiable in its second variable, i.e., $\tilde{H}^*(\rho, \cdot) \in C^3(\mathbb{R})$. In fact,

$$\tilde{H}^*_{uuu}(\rho, u) = 2h^*(\rho, u) \quad \text{for } (\rho, u) \in \rho_s \times \mathbb{R}.$$  \hfill (4.22)

In addition, there exists $M > 0$ depending on $\rho_s$ such that

$$\|\tilde{H}^*_{uu}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} + k(\rho)\|\tilde{H}^*_u(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq M \rho \sqrt{k(\rho)} \quad \text{for } \rho \geq \rho_s.$$  \hfill (4.23)

The proof of Lemma 4.8 is similar to that of Lemma 4.2.

In order to bound $\tilde{q}(\rho, u)$, we require an estimate on $\tilde{H}(\rho, u)$. This is the main content of the following lemma.
Lemma 4.9. There exists $M > 0$ such that, for all $(\rho, u) \in [\rho_*, \infty) \times \mathbb{R}$,

$$|\hat{H}(\rho, u)| \leq M(\rho|u| + \rho + \rho\sqrt{\log(\rho/\rho_*)}).$$

(4.24)

In addition, for all $\rho \geq \rho_*$, we have the bounds

$$\|\hat{H}_u(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq Mk(\rho)^{1-\alpha},$$

(4.25)

$$\|k'(\rho)\hat{H}_{uu}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq M\rho^{-\alpha}. $$

Proof. We begin with the estimates on the derivatives of $\hat{H}$ and subsequently apply these to deduce the main bound for $\hat{H}$.

By convolving (3.28) with the special function $\hat{\psi}(s) = \frac{1}{2}s|s|$, we obtain

$$\begin{align*}
\hat{H}_{pp} - k'(\rho)^2\hat{H}_{uu} &= (k'(\rho)^2 - \frac{\kappa_2}{\rho^2})\hat{H}_{uu}^*(\rho, u) + \frac{\rho''(\rho)}{\rho}\hat{\eta}(\rho, u), \\
\hat{H}(\rho_*, u) &= 0, \\
\hat{H}_u(\rho_*, u) &= 0.
\end{align*}$$

(4.26)

Therefore by arguments similar to those in Lemma 3.4 (see also Lemma 3.9), we find the following representation formula for $\hat{H}$:

$$2(\rho_0 - \rho_*)k'(\rho_0)\hat{H}(\rho_0, u_0)$$

$$= \int_{\rho_*}^{\rho_0} d(\rho)k'(\rho)(\hat{H}(\rho, u_0 + k(\rho_0) - k(\rho)) + \hat{H}(\rho, u_0 - k(\rho_0) + k(\rho))) d\rho$$

$$+ \int_{\rho_*}^{\rho_0} \rho - \rho_*)(k'(\rho)^2 - \frac{\kappa_2}{\rho^2})\hat{H}_u^*(\rho, u_0 + k(\rho_0) - k(\rho)) d\rho$$

$$- \int_{\rho_*}^{\rho_0} (\rho - \rho_*)((k'(\rho)^2 - \frac{\kappa_2}{\rho^2})\hat{H}_u^*(\rho, u_0 - k(\rho_0) + k(\rho)) d\rho$$

$$+ \int_{\rho_*}^{\rho_0} \frac{\rho''(\rho)}{\rho}(\rho - \rho_*)(\hat{\eta}(\rho, u_0 + k(\rho_0) - k(\rho)) - \hat{\eta}(\rho, u_0 - k(\rho_0) + k(\rho))) d\rho,$$

(4.27)

which has a similar structure to (4.14). The proof of estimates (4.25) now follows the same lines as the proof of Lemma 4.5, and so we omit the details.

To conclude the final estimate, (4.24), we convolve (3.28) with the function $\hat{\psi} = \frac{1}{2}s|s|$, to obtain

$$\hat{H}_{pp}(\rho, u) = k'(\rho)^2\hat{H}_{uu}(\rho, u) + (k'(\rho)^2 - \frac{\kappa_2}{\rho^2})\hat{H}_{uu}^*(\rho, u) + \frac{\rho''(\rho)}{\rho}\hat{\eta}(\rho, u),$$

from which we may directly estimate, using (2.8) and (4.23), and the bound on $\hat{\eta}$ provided by Lemmas 3.2 (cf. [15, Lemma 3.5]) and 4.4,

$$|\hat{H}_{pp}(\rho, u)| \leq M\left(\rho^{-\alpha - 1} + \rho^{-\alpha - 1}(|u| + \sqrt{k(\rho)})\right).$$

We now integrate in $\rho$, making use of the fact that $\hat{H}_\rho(\rho_*, \cdot) = 0$ and the monotonicity of $k$, and obtain

$$|\hat{H}_\rho(\rho, u)| \leq M\left(\rho_\star^{-\alpha} + \rho_\star^{-\alpha}(|u| + \sqrt{\log(\rho/\rho_*)})\right).$$

Integrating once more in $\rho$, and making use of the fact that $\hat{H}(\rho_*, \cdot) = 0$, gives the result. \qed

Using this result, we are able to bound $\hat{q}$ in absolute value.

Corollary 4.10. There exists a positive constant $M$ such that

$$|\hat{q}(\rho, u)| \leq M(\rho|u|^2 + \rho + \rho\sqrt{\log(\rho/\rho_*)})$$

for $(\rho, u) \in [\rho_*, \infty) \times \mathbb{R}$.  

(4.28)
Proof. Given the decomposition for $\tilde{q}$ given in (3.30),
\[ |\tilde{q}(\rho, u)| \leq |u\tilde{\eta}(\rho, u)| + |\tilde{H}(\rho, u)|, \]
\[ \leq M|\rho||u|^2 + \rho + \rho\sqrt{\log(\rho/\rho_*)}, \]
where we used the results of Corollary 4.7, Lemma 4.9, and the Cauchy–Schwarz inequality.

We now state the lower bound on the entropy flux, (4.3), from Lemma 4.11.

**Corollary 4.11.** There exists a positive constant $M$ such that
\[ \tilde{q}(\rho, u) \geq M^{-1}|\rho||u|^3 - M|\rho||u|^2 + \rho + \rho\log(\rho/\rho_*)^4 \]
for $(\rho, u) \in [\rho_*, \infty) \times \mathbb{R}$.

**Proof.** Recall that, as we saw in (3.30), we may write $\tilde{q}(\rho, u) = \tilde{q}^\ast(\rho, u) + \tilde{q}(\rho, u)$, where $\tilde{q}^\ast$ satisfies the lower bound (4.3). The result follows from Corollary 4.10.

**Remark 4.12.** The results of Subsections 4.1.2 and 4.1.3 prove Lemma 4.1 in full except for (4.6), which is contained in Appendix A.

### 4.2. Entropies generated by compactly supported test functions.

We now consider entropies generated by functions $\psi \in C^\infty_c(\mathbb{R})$. Recall from (3.16) that, for $\rho \geq \rho_*$,
\[ \eta^\psi(\rho, u) = \int_\mathbb{R} \chi^\ast(\rho, u - s)\psi(s)\,ds + \int_\mathbb{R} \tilde{\chi}(\rho, u - s)\psi(s)\,ds, \]
\[ = \eta^\ast \psi(\rho, u) + \tilde{\eta}^\psi(\rho, u). \]

The goal of this subsection is to prove the following lemma, which is the analogue of [15, Lemma 3.8].

**Lemma 4.13.** Let $\psi \in C^\infty_c(\mathbb{R})$ and $(\eta^\psi, q^\psi)$ the associated entropy pair via (1.9) and (1.13). There exists a positive constant $M_\psi$ such that
\[ |\eta^\psi(\rho, m)| \leq M_\psi \frac{\rho}{\sqrt{\log(\rho/\rho_*) + 1}}, \]
\[ |q^\psi(\rho, m)| \leq M_\psi \rho \quad \text{for } (\rho, m) \in \mathbb{R}^2_+. \]

Also, with $\eta^\psi_{m*}(\rho, \rho u) = \partial_u \eta^\psi_m(\rho, \rho u)$ the usual mixed derivative,
\[ |\eta^\psi_{m*}(\rho, m)| + |\eta^\psi_m(\rho, m)| + |\rho \eta^\psi_{mm}(\rho, m)| \leq M_\psi \frac{1}{\sqrt{\log(\rho/\rho_*) + 1}} \quad \text{for } (\rho, m) \in \mathbb{R}^2_+. \]

Finally, with $\eta^\psi_{mp}(\rho, \rho u) = \partial_{\rho u} \eta^\psi_m(\rho, \rho u)$ the usual mixed derivative,
\[ |\eta^\psi_{mp}(\rho, m)| \leq M_\psi \frac{\sqrt{\rho'}}{\rho} \quad \text{for } (\rho, m) \in \mathbb{R}^2_+. \]

We recall that [15, Lemma 3.8] proved these inequalities for the principal parts, $\tilde{\eta}^\psi$ and $\tilde{q}^\psi$. To begin with, in view of Lemma 3.6, we can bound the error as
\[ |\tilde{\eta}^\psi(\rho, u)| \leq M \rho^{1-\alpha\lambda} \int_\mathbb{R} |\psi(s)|\,ds, \]
\[ \leq M \rho^{1-\alpha\lambda}. \]

Notice also that, if supp $\psi \subset [a, b]$, then, in view of (3.6),
\[ \tilde{\eta}^\psi(\rho, u) = \int_\mathbb{R} \tilde{\chi}(\rho, s)\psi(u - s)\,ds = \int_{\max\{k(\rho), k_1(\rho)\}}^{\max\{k(\rho), k_2(\rho)\}} \tilde{\chi}(\rho, s)\psi(u - s)\,ds, \]
and the right-hand side vanishes for $u \notin [a - \max\{k(\rho), k_2(\rho)\}, b + \max\{k(\rho), k_1(\rho)\}]$. Additionally, differentiating in $u$, we get
\[ \tilde{\eta}^\psi_{u}(\rho, u) = \int_\mathbb{R} \tilde{\chi}(\rho, u - s)\psi'(s)\,ds. \]
Taking further derivatives and arguing as in (4.33), it easily follows from Lemma 3.6 that
\[ |\tilde{\eta}_m^\psi(\rho, u)| + |\tilde{\eta}_m^\psi(\rho, u)| + |\rho\tilde{\eta}_m^{\psi, m}(\rho, u)| \leq M \rho^{-\alpha \lambda}, \]
where \( \tilde{\eta}_m^{\psi, u}(\rho, u) = \partial_u \tilde{\eta}_m^\psi(\rho, u) \) is the usual mixed derivative. Hence, we have proved the following lemma.

**Lemma 4.14.** Let \( \psi \in C_c^2(\mathbb{R}) \) be such that \( \text{supp } \psi \subset [a, b] \). Then,
\[ \text{supp } \tilde{\eta}_m^\psi(\rho, \cdot) \subset [a - \max\{k(\rho), k_*(\rho)\}, b + \max\{k(\rho), k_*(\rho)\}] \]  
Also, there exists a positive constant \( M_\psi \) such that
\[ \| \tilde{\eta}_m^\psi(\rho, \cdot) \|_{L^\infty(\mathbb{R})} + \| \rho \tilde{\eta}_m^{\psi, m}(\rho, \cdot) \|_{L^\infty(\mathbb{R})} \leq M_\psi \rho^{1-\alpha \lambda} \]
for \( \rho \geq \rho_* \), (4.35)
and
\[ \| \tilde{\eta}_m^{\psi, u}(\rho, \cdot) \|_{L^\infty(\mathbb{R})} + \| \rho \tilde{\eta}_m^{\psi, m}(\rho, \cdot) \|_{L^\infty(\mathbb{R})} \leq M_\psi \rho^{-\alpha \lambda} \]
for \( \rho \geq \rho_* \). (4.36)

**Corollary 4.15.** For \( \psi \in C_c^2(\mathbb{R}) \), there exists a positive constant \( M_\psi \) such that
\[ |w\tilde{\eta}_m^\psi(\rho, u)| \leq M_\psi k(\rho) \rho^{1-\alpha \lambda} \]
for \( (\rho, u) \in [\rho_*, \infty) \times \mathbb{R} \). (4.37)

**Proof.** The result now follows easily from Lemma 4.14, using the compact support of \( \tilde{\eta}_m^\psi \) and the estimate (4.35), along with the bound on \( \max\{k(\rho), k_*(\rho)\} \) provided by Lemma A.6.

On the other hand, [15, Lemma 3.8] showed that
\[ |\eta^{*, \psi}(\rho, u)| \leq \frac{M_\psi \rho}{\sqrt{\log(\rho/\rho_* + 1)}} \]
for \( \rho \geq \rho_* \).

Thus, adding the contributions from \( \eta^{*, \psi} \) and \( \tilde{\eta}_m^\psi \),
\[ \| \tilde{\eta}_m^\psi(\rho, \cdot) \|_{L^\infty(\mathbb{R})} \leq \frac{M_\psi \rho}{\sqrt{k(\rho)}} \]
for \( \rho \geq \rho_* \). (4.38)

We now consider the mixed derivative of the full entropy, \( \eta_m^{\psi, m} \), for which the following result holds.

**Lemma 4.16.** Let \( \psi \in C_c^2(\mathbb{R}) \) and \( \eta^\psi \) be generated by (1.13). Then, there exists a positive constant \( M_\psi \) such that
\[ \| \eta_m^{\psi, m}(\rho, \cdot) \|_{L^\infty(\mathbb{R})} \leq M_\psi \sqrt{\rho'(\rho)} \rho \]
for \( \rho \geq \rho_* \). (4.39)

**Proof.** We first use the representation formula (3.7) to write down
\[ \eta_m^{\psi}(\rho_0, u_0) = \frac{1}{2\rho_0(\rho_0 - \rho_*) k'(\rho_0)} \int_{\rho_*}^{\rho_0} k'(\rho) d(\rho) \eta_m^{\psi}(\rho, u_0 + k(\rho_0) - k(\rho)) d\rho \]
\[ + \frac{1}{2\rho_0(\rho_0 - \rho_*) k'(\rho_0)} \int_{\rho_*}^{\rho_0} k'(\rho) d(\rho) \eta_m^{\psi}(\rho, u_0 - k(\rho_0) + k(\rho)) d\rho \]
\[ - \frac{1}{2\rho_0(\rho_0 - \rho_*) k'(\rho_0)} \int_{k(\rho_0) - k(\rho_0)}^{k(\rho_0) - k(\rho_0)} \eta_m^{\psi}(\rho_0, u_0 - y) dy, \]
\[ =: I + II + III. \]

where \( \eta_m^{\psi}(\rho, u) = \int_{\mathbb{R}} \chi(\rho, u - s) \psi'(s) ds \). Hence, the mixed derivative \( \eta_m^{\psi, m} \) is given by
\[ \eta_m^{\psi, m}(\rho_0, u_0) = \partial_{\rho_0} I + \partial_{\rho_0} II + \partial_{\rho_0} III. \]

We begin by controlling the term \( \partial_{\rho_0} I \).
\[ \partial_{\rho_0} I = \partial_{\rho_0} \left( \frac{1}{2\rho_0(\rho_0 - \rho_*) k'(\rho_0)} \right) \int_{\rho_*}^{\rho_0} k'(\rho) d(\rho) \eta_m^{\psi}(\rho, u_0 + k(\rho_0) - k(\rho)) d\rho \]
\[ + \frac{1}{2\rho_0(\rho_0 - \rho_*)} d(\rho_0) \eta_m^{\psi}(\rho_0, u_0) + \frac{1}{2\rho_0(\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} k'(\rho) d(\rho) \eta_m^{\psi}(\rho, u_0 + k(\rho_0) - k(\rho)) d\rho, \]
\[ =: I_1 + I_2 + I_3. \]
We expand the first term in big brackets,
\[
\partial_{\rho_0} \left( \frac{1}{2\rho_0 (\rho_0 - \rho_*) k'(\rho_0)} \right) = -\frac{1}{2\rho_0 (\rho_0 - \rho_*) k'(ho_0)} \cdot \frac{1}{\rho_0 (\rho_0 - \rho_*)^2 k''(\rho_0)} - \frac{k''(\rho_0)}{2\rho_0 (\rho_0 - \rho_*) k'(\rho_0)^2}.
\]

Therefore, using Lemma 2.4, \[ |\partial_{\rho_0} \left( \frac{1}{2\rho_0 (\rho_0 - \rho_*) k'(\rho_0)} \right) | \leq \frac{M}{\rho_0 (\rho_0 - \rho_*)^2}. \] Thus, using [15, Lemma 3.8] and Lemma 4.14,
\[
|I_1| \leq \frac{M}{\rho_0 (\rho_0 - \rho_*)^2} \int_{\rho_*}^{\rho_0} dp = \frac{M}{\rho_0 - \rho_*}.
\]

For \( I_2 \), we have \[ \frac{1}{2\rho_0 (\rho_0 - \rho_*)} d\rho_0 \eta^\psi_{\rho_0} (\rho_0, u_0) \leq \frac{M}{\rho_0 - \rho_*}. \] The last term in the expansion of \( \partial_{\rho_0} I \) is controlled as
\[
|I_3| \leq \frac{M}{\rho_0 (\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} dp \leq \frac{M}{\rho_0 - \rho_*}.
\]

Thus, for \( \rho_0 \geq \rho_* + 1 \),
\[
|\partial_{\rho_0} I| \leq \frac{M}{\rho_0 - \rho_*} \leq \frac{M}{1 + (\rho_0 - \rho_*)}.
\]

On the other hand, it is straightforward to check that \( \eta^\psi_{\rho_0} \) also remains bounded for \( \rho_0 \in (\rho_*, \rho_* + 1) \). Hence, we conclude that
\[
|\partial_{\rho_0} I| \leq \frac{M}{1 + (\rho_0 - \rho_*)},
\]
and the same bound holds for \( |\partial_{\rho_0} III| \). We now bound the final term,
\[
\partial_{\rho_0} III = -\partial_{\rho_0} \left( \frac{1}{2\rho_0 (\rho_0 - \rho_*) k'(\rho_0)} \right) \int_{-(k(\rho_0) - k(\rho_*))}^{k(\rho_0) - k(\rho_*)} \eta^\psi_{\rho_0} (\rho_0, u_0 - y) dy
\]
\[
- \frac{1}{2\rho_0 (\rho_0 - \rho_*)} (\eta^\psi_{\rho_0} (\rho_0, u_0 - k(\rho_0) + k(\rho_*)) + \eta^\psi_{\rho_0} (\rho_0, u_0 + k(\rho_0) - k(\rho_*))).
\]

Hence, using (4.38), we get
\[
|\partial_{\rho_0} III| \leq \frac{M\rho_*}{\rho_0 (\rho_0 - \rho_*)^2} 2(k(\rho_0) - k(\rho_*)) + \frac{M\rho_*}{\rho_0 (\rho_0 - \rho_*)} \leq \frac{M}{\rho_0 - \rho_*}.
\]

Thus, checking again that no singularity arises at \( \rho_0 = \rho_* \),
\[
|\partial_{\rho_0} III| \leq \frac{M}{1 + (\rho_0 - \rho_*)}.
\]

Meanwhile, using Lemma 2.4, we have \( \sqrt{p'(\rho)} \rho^{-1} \geq \sqrt{\frac{\pi}{2}} \rho^{-1} \). The result follows at once. \( \square \)

In fact, the same procedure can be followed so as to obtain the same estimates for \( \eta^{*,\psi}_{\rho_0} \), from which we see that \( |\tilde{\eta}^\psi_{\rho_0} (\rho_0, u_0)| \leq M\rho^{-1}_0 \). In view of this, we have the following corollary.

**Corollary 4.17.** There exists a positive constant \( M_{\psi} \) such that
\[
\|\eta^{*,\psi}_{\rho_0} (\rho, \cdot)\|_{L^\infty(\mathbb{R})} + \|\tilde{\eta}^\psi_{\rho_0} (\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq M_{\psi} \frac{\sqrt{\rho'(\rho)}}{\rho} \quad \text{for } \rho \geq \rho_*.
\]

We have yet to inspect the entropy flux. To this end, we recall the equation for \( \hat{H}^\psi \), obtained by convolving (3.28) with \( \psi \in C^2_b(\mathbb{R}) \),
\[
\begin{align*}
\hat{H}^\psi_{\rho_0} - k'(\rho_0)^2 \hat{H}^\psi_{uu} &= f^\psi (\rho, u), \quad \text{for } (\rho, u) \in (\rho_*, \infty) \times \mathbb{R}, \\
\hat{H}^\psi (\rho_*, u) &= 0, \\
\hat{H}^\psi_{\rho_0} (\rho_*, u) &= 0,
\end{align*}
\]
where
\[
f^\psi (\rho, u) := (k'(\rho_0)^2 - \frac{1}{\rho^2}) H^s u^\psi (\rho, u) + \frac{p''(\rho)}{\rho} \eta^\psi_{\rho_0} (\rho, u) \quad \text{for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R}.
\]

This equation allows us to make a similar representation formula and Grönwall argument to those above to obtain the following lemma, the proof of which we omit.
Lemma 4.18. There exists a positive constant $M_0$ such that
\[
\|\tilde{H}^0(\rho, u)\|_{L^\infty(\mathbb{R})} \leq M_0 k(\rho)^{1-\alpha} \text{ for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R}.
\] (4.43)

By recalling $\tilde{q}^0 = w\tilde{q}^0 + \tilde{H}^0$ and combining Corollary 4.15 and Lemma 4.18 to bound the entropy flux, the proof of Lemma 4.13 is now complete.

5. Uniform estimates and compactness of the entropy dissipation measures

Finally, we prove the uniform estimates that are required for the compactness of the entropy dissipation measures. To begin with, we collect the assumptions required on the initial data for the desired energy estimates to hold, and for the main theorem of Hoff in [9] to be valid.

Definition 5.1. We say that a family of functions $\{(\rho_0^\varepsilon, u_0^\varepsilon)\}_{0 < \varepsilon \leq \varepsilon_0}$ for some $\varepsilon_0 > 0$ is an admissible sequence of initial data if the following assumptions hold:

- The total relative mechanical energies are uniformly bounded, i.e.,
  \[
  \sup_{\varepsilon \in (0, \varepsilon_0]} E[\rho_0^\varepsilon, u_0^\varepsilon] \leq E_0 < \infty;
  \]
- For the initial densities, there holds the weighted derivative uniform bound
  \[
  \sup_{\varepsilon \in (0, \varepsilon_0]} \varepsilon^2 \int_{\mathbb{R}} \frac{|\rho_0^\varepsilon(x)|^2}{\rho_0^\varepsilon(x)^3} \, dx \leq E_1 < \infty;
  \]
- The relative total initial momenta are uniformly bounded, i.e.,
  \[
  \sup_{\varepsilon \in (0, \varepsilon_0]} \int_{\mathbb{R}} \rho_0^\varepsilon(x)|u_0^\varepsilon(x)| - \bar{u}(x) | \, dx \leq M_0 < \infty;
  \]
- The initial densities are bounded away from the vacuum, i.e., there exists $c_0^\varepsilon > 0$ such that $\rho_0^\varepsilon(x) \geq c_0^\varepsilon > 0$ for all $x \in \mathbb{R}$.

Following similar proofs to those of [15, Lemmas 3.2–3.4], making use of the results of Lemma 4.1, we obtain the following result.

Proposition 5.2 (Uniform estimates). Assume that $\{(\rho_0^\varepsilon, u_0^\varepsilon)\}_{0 < \varepsilon \leq \varepsilon_0}$ is an admissible sequence of initial data, in the sense of Definition 5.1, and let $\{(\rho^\varepsilon, u^\varepsilon)\}_{0 < \varepsilon \leq \varepsilon_0}$ be the sequence of viscous solutions of (1.2) that it generates. Then, for any $T > 0$, there exist constants $M_1, M_2, \ldots > 0$, independent of $\varepsilon$, such that, for all $\varepsilon \in (0, \varepsilon_0]$,

\[
\begin{align*}
(i) & \quad \sup_{t \in [0, T]} E[\rho^\varepsilon, u^\varepsilon](t) + \int_0^T \int_{\mathbb{R}} \varepsilon |u^\varepsilon|^2 \, dx \, dt \leq M_1. \\
(ii) & \quad \varepsilon^2 \int_{\mathbb{R}} \frac{|\rho^\varepsilon(T, x)|^2}{\rho^\varepsilon(T, x)^3} \, dx + \varepsilon \int_0^T \int_{\mathbb{R}} \frac{p'(\rho^\varepsilon)}{(\rho^\varepsilon)^2} |\dot{\rho'}|^2 \, dx \, dt \leq M_2.
\end{align*}
\] (5.1) \hspace{1cm} (5.2)

Moreover, for any compact $K \subset \mathbb{R}$, there exist constants $M_3, M_4 > 0$, depending on $T$ and $K$, but independent of $\varepsilon$, such that, for all $\varepsilon \in (0, \varepsilon_0]$,

\[
\begin{align*}
(iii) & \quad \int_0^T \int_K \rho^\varepsilon p(\rho^\varepsilon) \, dx \, dt \leq M_3. \\
(iv) & \quad \int_0^T \int_K \rho^\varepsilon |u^\varepsilon|^3 \, dx \, dt \leq M_4.
\end{align*}
\] (5.3) \hspace{1cm} (5.4)

The above key lemma, Lemma 4.1, is essential for the proof of the final higher integrability estimate of this Proposition (see [15, Lemma 3.4]). Analogously, following the strategy of proof of [15, Proposition 3.9] while appealing to Lemma 4.13, we obtain the following compactness result.
Proposition 5.3 (Compactness of the entropy dissipation measures). Let $\psi \in C^2_c(R)$ and let $(\eta^0, q^0)$ be the associated entropy pair via (1.9) and (1.13). Additionally, let $\{(\rho_0^\varepsilon, u_0^\varepsilon)\}_{\varepsilon > 0}$ be a sequence of admissible initial data, in the sense of Definition 5.1, and let $\{(\rho^\varepsilon, u^\varepsilon)\}_{\varepsilon > 0}$ be the associated viscous solutions of (1.2). Then, we have that the entropy dissipation measures
\[
\eta^0 (\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)_x + q^0 (\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)_x
\]
are confined to a compact subset of $W^{-1,p}_{loc}(R^d_+)$ for any $p \in (1, 2)$.

6. Singularities of the entropy kernel

The main result of this section is Lemma 6.4, which is the analogue of [15, Lemma 2.7], and is concerned with the fractional derivatives of the kernels $\chi$ and $\sigma$ (cf. [15, Section 2.2]). This result is indispensable for the proof of the reduction of the Young measure, which is contained in the next section. We recall that for a distribution $g(s)$, the fractional derivative of order $\lambda$ of $g$ is defined by the following formula, where $\Gamma$ is the gamma function,
\[
\partial^\lambda g(s) = \Gamma(-\lambda) g(s) * [s]^{-\lambda-1}.
\]
We require sharp bounds on the fractional derivatives of $\chi$ and $\sigma$. We begin with the expansions of these kernels from [3, Theorems 2.1–2.3].

Theorem 6.1. The entropy kernel admits globally the expansion
\[
\chi(\rho, u) = a_2(\rho) G_\chi(\rho, u) + b_2(\rho) G_\chi(\rho, u) + g_1(\rho, u) \quad \text{for } (\rho, u) \in R^d_+,
\]
where $G_\chi(\rho, u) = [k(\rho^2 - u^2)]^\lambda$, and $g_1(\rho, \cdot)$ and its fractional derivative $\partial^\lambda_+ g_1(\rho, \cdot)$ are Hölder continuous. There exists a constant $M > 0$ such that
\[
|a_2(\rho)| + |b_2(\rho)| \leq M \sqrt{k(\rho)}^{-\lambda} \quad \text{for } \rho \geq \rho_*,
\]
and
\[
\|\partial^\lambda_+ g_1(\rho, \cdot)\|_{L^\infty} \leq M \sqrt{k(\rho)^2} \quad \text{for } \rho \geq \rho_*.\]

Proof. Recall from [3, Proposition 2.1] that $a_2(\rho) = a_2(\rho) k(\rho)^{-2\lambda - 1}$, where
\[
a_2(\rho) = M k(\rho)^{\lambda + 1} k'(\rho)^{-1/2},
\]
while $b_2(\rho) = a_2(\rho) k(\rho)^{-2\lambda - 3}$, with
\[
a_2(\rho) = -\frac{1}{4(\lambda + 1)} k(\rho)^{\lambda + 2} k'(\rho)^{-1/2} \int_0^\rho k(\tau)^{-(\lambda + 1)} k'(\tau)^{-1/2} \rho^\prime(\tau) d\tau.
\]
A calculation (cf. [3, Lemma 3.1]) then shows that there exists a constant $M = M(\gamma, r)$ such that
\[
\rho^{-1}|a_2(\rho)| + |\alpha_2''(\rho)| + |\alpha_2''\prime(\rho)| + \rho |\alpha_2^{(3)}(\rho)| \leq M \quad \text{for } \rho \in [0, r),
\]
where $r > 0$ is as in Definition 1.1. From this, it follows by direct calculation that
\[
\rho^{-1}|a_2(\rho)| + |\alpha_2''(\rho)| + |\alpha_2''\prime(\rho)| \leq M \quad \text{for } \rho \in [0, r).
\]
Moreover, calculating the derivatives explicitly and using the bounds on the derivatives of $k$ provided by (2.10), we obtain
\[
|\alpha_2''(\rho)| + \rho |\alpha_2''(\rho)| + \rho^2 |\alpha_2''\prime(\rho)| + \rho^3 |\alpha_2^{(3)}(\rho)| \leq M \sqrt{k(\rho)^{\lambda + 1}} \quad \text{for } \rho \geq \rho_*,
\]
for some $M > 0$ and also
\[
|\alpha_2''(\rho)| + \rho^2 |\alpha_2''(\rho)| + \rho^2 |\alpha_2''(\rho)| \leq M \sqrt{k(\rho)^{\lambda + 3}} \quad \text{for } \rho \geq \rho_*,
\]
and therefore the bounds on $a_2$ and $a_3$ in (6.2) follow easily.

For the remainder term, one may check (cf. [3, Proof of Theorems 2.1 and 2.2]) that
\[
\begin{cases}
g_{1,pp} - k'(\rho)^2 g_{1,uu} = A(\rho) k(\rho)^{-1} f_{\lambda+1} \left( \frac{u}{k(\rho)} \right), \\
g_1(0, \cdot) = 0, \\
g_{1,\rho}(0, \cdot) = 0,
\end{cases}
\]
where \( f_\lambda(y) = [1 - y^2]^\lambda \) and
\[
A(\rho) = -\left( \alpha''(\rho) + \frac{2\lambda + 3}{2(\lambda + 1)} \alpha''(\rho) \right) \quad \text{for } \rho \geq 0, \tag{6.11}
\]
which is locally bounded. Thus there exists \( M > 0 \) such that \(|A(\rho)| \leq M \) for \( \rho \in [0, \rho_*) \). Arguing as in the proof of Lemma 3.9, we get the representation
\[
k'(\rho_0)g_1(\rho_0, u_0) = \frac{1}{2\rho_0} \int_{\rho_0}^{\rho_0} d(\rho)k'(\rho)(g_1(\rho, u + k(\rho_0) - k(\rho)) + g_1(\rho_0, u_0) - k(\rho_0) + k(\rho)) d\rho
\]
\[
+ \frac{1}{2\rho_0} \int_{\rho_0}^{\rho_0} \rho A(\rho)k(\rho)^{-1}\left( \int_{\rho_0 + k(\rho_0) - k(\rho)}^{\rho_0 + k(\rho_0) + k(\rho)} f_{\lambda+1}\left( \frac{s}{k(\rho)} \right) ds \right) d\rho.
\]
As \( \rho A(\rho)k(\rho)^{-1} \) is bounded on \( [0, \rho_*) \), the second integral is well-defined. So, taking derivatives in \( u_0 \), we get
\[
k'(\rho_0)\partial_{\rho_0}g_1(\rho_0, u_0)
\]
\[
= \frac{1}{2\rho_0} \int_{\rho_0}^{\rho_0} d(\rho)k'(\rho)(\partial_{\rho_0}g_1(\rho, u_0 + k(\rho_0) - k(\rho)) + \partial_{u_0}g_1(\rho, u_0 - k(\rho_0) + k(\rho))) d\rho
\]
\[
+ \frac{1}{2\rho_0} \int_{\rho_0}^{\rho_0} \rho A(\rho)k(\rho)^{-1}\left( \int_{\rho_0 + k(\rho_0) - k(\rho)}^{\rho_0 + k(\rho_0) + k(\rho)} \partial_{\rho_0}f_{\lambda+1}\left( \frac{s}{k(\rho)} \right) ds \right) d\rho. \tag{6.12}
\]
By the fundamental theorem of calculus, we see that the fractional derivative \( \partial_{\rho_0}^\lambda f_{\lambda+1} \) satisfies
\[
\partial_{\rho_0}^\lambda f_{\lambda+1}(s) - \partial_{\rho_0}^\lambda f_{\lambda+1}(-1) = \int_{-1}^{s} \partial_{\rho_0}^{\lambda+1}f_{\lambda+1}(y) dy. \tag{6.13}
\]
Note that supp \( f_\lambda = [-1, 1] \) so that the second term on the left-hand side vanishes. Also, from [15, Section 2.2],
\[
\partial_{\rho_0}^{\lambda+1} f_{\lambda+1}(s) = A_{1}^{\lambda+1}(H(s + 1) + H(s - 1)) + A_{2}^{\lambda+1}(\text{Ci}(s + 1) - \text{Ci}(s - 1)) + \tilde{\rho}^{\lambda+1}(s),
\]
where \( A_{1}^{\lambda+1}, A_{2}^{\lambda+1} \in C \) and \( \tilde{\rho}^{\lambda+1} \) is a compactly supported Hölder continuous function. Hence, substituting this equality into (6.13), we get
\[
\partial_{\rho_0}^\lambda f_{\lambda+1}(s) = A_{1}^{\lambda+1}(s + 1 + s - 1) + A_{2}^{\lambda+1}\left( \int_{-1}^{s} \text{Ci}(y + 1) dy - \int_{-1}^{s} \text{Ci}(y - 1) dy \right)
\]
\[
+ \int_{-1}^{s} \tilde{\rho}^{\lambda+1}(y) dy. \tag{6.14}
\]
Given the terms in (6.14) and the support of \( f_\lambda \), we see that \( \partial_{\rho_0}^\lambda f_{\lambda+1} \) is a continuous function with compact support, and it is therefore uniformly bounded. In view of this, by applying the fractional derivative \( \partial_{\rho_0}^\lambda \) to the representation formula (6.12), we get
\[
\|k'(\rho_0)\partial_{\rho_0}^{\lambda+1} g_1(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\rho_0} \int_{\rho_0}^{\rho_0} d(\rho)\|k'(\rho)\partial_{\rho_0}^{\lambda+1} g_1(\rho, \cdot)\|_{L^\infty(\mathbb{R})} d\rho
\]
\[
+ \frac{M}{\rho_0} \int_{\rho_0}^{\rho_0} \rho A(\rho)k(\rho)^{-1}\|f_{\lambda+1}\|_{L^\infty(\mathbb{R})} d\rho, \tag{6.15}
\]
where we have taken into account the homogeneity of the fractional derivative in the final term. We now bound this final term. Suppose that \( \rho_0 \in (0, r) \), then
\[
\int_{\rho_0}^{\rho_0} \rho A(\rho)k(\rho)^{-1}\|f_{\lambda+1}\|_{L^\infty(\mathbb{R})} d\rho \leq M \int_{\rho_0}^{\rho_0} \rho \frac{1}{\rho_0} \frac{\rho_0^2 - \frac{\rho^2}{2}}{\rho_0^2} d\rho \leq M \rho_0^\frac{2}{3} - \frac{2}{3},
\]
and since \( \frac{3}{2} - \frac{2}{3} > \frac{1}{2} \), the right-hand side is dominated by \( M \rho_0^\frac{2}{3} \). If \( \rho_0 \in [r, \rho_*) \), only a bounded contribution is added, since the integrand is continuous on this interval. Finally, if \( \rho_0 \geq \rho_* \),
\[
\int_{\rho_0}^{\rho_0} |\rho A(\rho)k(\rho)^{-1}| d\rho \leq M \int_{\rho_0}^{\rho_*} \rho^\frac{1}{\rho_0} \frac{\rho_0^2 - \frac{\rho^2}{2}}{\rho_0^2} d\rho + M \int_{\rho_*}^{\rho_0} d\rho + M \int_{\rho_*}^{\rho_0} k(\rho)^2 \rho^{-1/2} d\rho,
\]
\[
\leq M\left(1 + \sqrt{\rho_0}k(\rho_0)^2\right).
\]
Thus, applying Grönwall’s lemma to (6.15), we obtain
\[
\|k'(\rho_0)\partial_{\rho_0}^{\lambda+1} g_1(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq M \rho_0^\frac{1}{2} - \frac{\rho_0^2}{2}, \quad \text{for } \rho_0 \in [0, r), \tag{6.16}
\]
while
\[ \|k'(\rho_0)\partial_u^{\lambda+1} g_1(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{Mk(\rho_0)^2}{\sqrt{\rho_0}} \] for \( \rho_0 \geq \rho_* \), from which the result is easily deduced using (2.8).

A similar proof shows that we also have the equivalent results for the entropy flux kernel.

**Theorem 6.2.** Recall from [3, Theorem 2.3] that \( h(\rho, u - s) = \sigma(\rho, u, s) - u\chi(\rho, u - s) \) admits globally the expansion
\[ h(\rho, u) = -u(b_2(\rho)G_\lambda(\rho, u) + b_4(\rho)G_{\lambda+1}(\rho, u)) + g_2(\rho, u) \quad \text{for } (\rho, u) \in \mathbb{R}_+^2, \] (6.17)
where \( G_\lambda(\rho, u) = [k(\rho)^2 - u^2]_+^{\lambda-1} \) and \( g_2(\rho, \cdot) \) and its fractional derivative \( \partial_u^{\lambda+1} g_2(\rho, \cdot) \) are Hölder continuous. There exists a constant \( M > 0 \) such that
\[ |b_2(\rho)| + |b_4(\rho)| \leq M\sqrt{\rho k(\rho)^{-\lambda-1}} \quad \text{for } \rho \geq \rho_*, \] (6.18)
and
\[ \|\partial_u^{\lambda+1} g_2(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq M\sqrt{\rho k(\rho)^2} \quad \text{for } \rho \geq \rho_. \] (6.19)

We recall the following lemma from [3, Proposition 2.4] for future use.

**Lemma 6.3.** The coefficients in the expansions (6.1) and (6.17) satisfy
\[ D(\rho) := a_2(\rho)b_2(\rho) - k(\rho)^2(a_2(\rho)b_4(\rho) - a_4(\rho)b_4(\rho)) > 0. \] (6.20)

We now arrive at the main result of this section, which will be used in key steps of the Young measure reduction in Section 7.

**Lemma 6.4.** For all values of \((\rho, u, s) \in \mathbb{R}_+^2 \times \mathbb{R}\), the fractional derivatives \( \partial_u^{\lambda+1} \chi \) and \( \partial_u^{\lambda+1} h \) admit the expansions:
\[ \partial_u^{\lambda+1} \chi(\rho, u - s) = \sum_{\pm} \left( A_{1,\pm}(\rho)\delta(s - u \pm k(\rho)) + A_{2,\pm}(\rho)H(s - u \pm k(\rho)) \right. \]
\[ \left. + A_{3,\pm}(\rho)\text{PV}(s - u \pm k(\rho)) + A_{4,\pm}(\rho)\text{Ci}(s - u \pm k(\rho)) \right) + r_\chi(\rho, u - s), \] (6.21)
and
\[ \partial_u^{\lambda+1} h(\rho, u - s) = \sum_{\pm} (s - u) \left( B_{1,\pm}(\rho)\delta(s - u \pm k(\rho)) + B_{2,\pm}(\rho)H(s - u \pm k(\rho)) \right. \]
\[ \left. + B_{3,\pm}(\rho)\text{PV}(s - u \pm k(\rho)) + B_{4,\pm}(\rho)\text{Ci}(s - u \pm k(\rho)) \right) \]
\[ + \sum_{\pm} \left( B_{5,\pm}(\rho)H(s - u \pm k(\rho)) + B_{6,\pm}(\rho)\text{Ci}(s - u \pm k(\rho)) \right) + r_\sigma(\rho, u - s), \] (6.22)
where \( \delta, \text{PV}, H \) and \( \text{Ci} \) denote the Dirac mass, Principal Value, Heaviside function and Cosine integral, respectively, and the remainder terms \( r_\chi \) and \( r_\sigma \) are Hölder continuous functions. Additionally, in the large density regime, there exists a positive constant \( M \) such that
\[ \sum_{j=1,\pm}^4 |A_{j,\pm}(\rho)| + \sum_{j=1,\pm}^6 |B_{j,\pm}(\rho)| \leq M\sqrt{\rho}(1 + \log(\rho/\rho_*)) \quad \text{for } \rho \geq \rho_*. \] (6.23)

and
\[ \|r_\chi(\rho, \cdot)\|_{L^\infty(\mathbb{R})} + \|r_\sigma(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq M\sqrt{\rho}(1 + (\log(\rho/\rho_*))^2) \quad \text{for } \rho \geq \rho_. \] (6.24)
Proof. By taking fractional derivatives of (6.1), we have, as in [15, Section 2.2],
\[
\partial_s^{\lambda+1}\chi(\rho, u - s) = a_4(\rho)\partial_s^{\lambda+1}G_\lambda(\rho, u - s) + a_5(\rho)\partial_s^{\lambda+1}G_{\lambda+1}(\rho, u - s) + \partial_s^{\lambda+1}g_1(\rho, u - s),
\]
where we know from [11, Proposition 3.4] that
\[
\partial_s^{\lambda+1}G_\lambda(\rho, u - s) = k(\rho)^\lambda A_1^s(H(s - u + k(\rho)) + H(s - u - k(\rho)))
\]
\[
+ k(\rho)^\lambda A_2^s(\mathrm{Ci}(s - u + k(\rho)) - \mathrm{Ci}(s - u - k(\rho))) + k(\rho)^\lambda \tilde{r}_1 \left(\frac{s - u}{k(\rho)}\right),
\]
and
\[
\partial_s^{\lambda+1}G_{\lambda+1}(\rho, u - s) = k(\rho)^\lambda [A_1^s(\delta(s - u + k(\rho)) + \delta(s - u - k(\rho)))
\]
\[
+ A_2^s(\PV(s - u + k(\rho)) - \PV(s - u - k(\rho)))]
\]
\[
+ k(\rho)^{\lambda-1} [A_3^s(H(s - u + k(\rho)) - H(s - u - k(\rho)))
\]
\[
+ A_4^s(\mathrm{Ci}(s - u + k(\rho)) - \mathrm{Ci}(s - u - k(\rho)))]
\]
\[
+ k(\rho)^{\lambda-1} ( - A_4^s(\log k(\rho))^2 + \tilde{q}(\frac{s - u}{k(\rho)}) ) ,
\]
where \( A_i^s \in C \) for \( i \in \{1, \ldots, 4\} \) are \( \lambda \)-dependent constants, and \( \tilde{r} \) and \( \tilde{q} \) are uniformly bounded Hölder continuous functions. We therefore deduce the form for \( \partial_s^{\lambda+1}\chi \) given in (6.21), with
\[
A_{1,\pm}(\rho) = a_4(\rho) k(\rho)^\lambda A_1^s,
A_{2,\pm}(\rho) = \pm a_5(\rho) k(\rho)^{\lambda-1} A_3^s + a_6(\rho) k(\rho)^{\lambda+1} A_{4,\pm}^s,
A_{3,\pm}(\rho) = \pm a_7(\rho) k(\rho)^{\lambda-1} A_2^s,
A_{4,\pm}(\rho) = \pm a_8(\rho) k(\rho)^{\lambda-1} A_4^s \pm a_9(\rho) k(\rho)^{\lambda+1} A_{2,\pm}^s,
\]
and
\[
r_\chi(\rho, u - s) = a_4(\rho) k(\rho)^{\lambda-1} \tilde{q}(\frac{s - u}{k(\rho)}) + a_5(\rho) k(\rho)^{\lambda+1} \tilde{r}_1(\frac{s - u}{k(\rho)})
\]
\[
- k(\rho)^{\lambda-1} A_4^s(\log k(\rho))^2 + \partial_s^{\lambda+1}g_1(\rho, u - s).
\]
The conclusion of the lemma now follows easily from Lemma 6.1, using the fact that \( k(\rho) \leq M(1 + \log(\rho/\rho_*)) \) for \( \rho \geq \rho_* \), by Corollary A.1. An identical procedure using the estimates of Lemma 6.2 yields the result for the entropy flux kernel. \( \square \)

7. THE YOUNG MEASURE AND PROOF OF THE MAIN RESULT

We begin this section by constructing a probability measure which characterises the limiting behaviour of the viscous solutions of the Navier–Stokes equations (1.2): the Young measure. We follow the approach in [11, Section 2.3], which is also common to [5, Section 5] and [15, Section 4].

Define the upper half-plane \( \mathbb{H} := \{(\rho, u) \in \mathbb{R}^2 : \rho > 0\} \), and the ring of continuous functions
\[
\mathcal{C}(\mathbb{H}) := \left\{ \phi \in C(\bar{\mathbb{H}}) \mid \phi(0, \cdot) \text{ is a constant function and such that the mapping } (\rho, u) \mapsto \lim_{s \to \infty} \phi(s\rho, su) \text{ lies in } C(S^1 \cap \bar{\mathbb{H}}) \right\},
\]
where \( S^1 \) is the unit circle. This particular subset of functions is chosen to mitigate difficulties at the vacuum and in the limit of infinitely large densities.

Since \( \mathcal{C}(\mathbb{H}) \) is a complete subring of the continuous functions on \( \mathbb{H} \) containing the constant functions, there exists a compactification \( \overline{\mathbb{H}} \) of \( \mathbb{H} \) such that \( C(\mathbb{H}) \cong C(\overline{\mathbb{H}}) \), where \( \cong \) means isometrically isomorphic to. The topology on \( \overline{\mathbb{H}} \) is the weak-star topology induced by \( C(\overline{\mathbb{H}}) \), i.e., a sequence \( (v_n)_{n \in \mathbb{N}} \subset \overline{\mathbb{H}} \) converges to \( v \in \overline{\mathbb{H}} \) if and only if \( |\phi(v_n) - \phi(v)| \to 0 \) for all functionals \( \phi \in C(\overline{\mathbb{H}}) \). This topology is both separable and metrizable (see [14] for details). Additionally, by defining \( V \) to be the weak-star closure of the set \( \{\rho = 0\} \), one can show that \( \mathbb{H} = \mathbb{H} \cup V \). We remark that the topology on \( \overline{\mathbb{H}} \) has the useful property that it does not distinguish points in the vacuum set.

By applying the fundamental theorem of Young measures for maps into compact metric spaces ([1, Theorem 2.4]) we find that, for some subsequence of the viscous solutions \( (\rho^\varepsilon, u^\varepsilon) \), there exists a
measure \( \nu_{t,x} \in \text{Prob}(\mathcal{H}) \) for almost every \((t, x) \in \mathbb{R}^2_+ \) with the property that, given any \( \phi \in C(\mathcal{H}) \),

\[
\phi(\rho^\varepsilon, u^\varepsilon) \to \int_{\mathcal{H}} \phi(\rho, u) \, d\nu_{t,x}(\rho, u) \quad \text{in } L^\infty(\mathbb{R}^2_+) \text{ as } \varepsilon \to 0.
\]

For ease of notation, we relabel this subsequence as \((\rho^\varepsilon, u^\varepsilon)\). One can in fact show that the set of test functions \( \phi \) for which such convergence holds is larger than simply \( C(\mathcal{H}) \). By following analogous arguments to those in the proof of [15, Proposition 4.1], making use of Proposition 5.2, we arrive at the following result.

**Lemma 7.1.** Let \( \nu_{t,x} \) be a Young measure associated to a sequence of solutions to \((1.2)\) generated by admissible initial data, in the sense of Definition 5.1. Then \( \nu_{t,x} \) has the following properties:

1. For almost every \((t, x)\), the measure \( \nu_{t,x} \) is a probability measure on \( \mathcal{H} \), i.e., \( \int_{\mathcal{H}} d\nu_{t,x} = 1 \);
2. The mapping \((t, x) \mapsto \int_{\mathcal{H}} \phi(\rho) \, d\nu_{t,x}(\rho) \) belongs to \( L^1_{\text{loc}}(\mathbb{R}_+^2) \);
3. The space of admissible test functions may be extended in the following way: Let \( \phi \in C(\hat{\mathcal{H}}) \) be such that \( \phi|_{\partial \mathcal{H}} = 0 \). Suppose that there exists \( a > 0 \) such that \( \supp \phi \subset \{ u + k(\rho) \geq -a, u - k(\rho) \leq a \} \). Provided that \( \phi \) also satisfies the growth condition

\[
\lim_{\rho \to \infty} \rho^{-2} |\phi(\rho, u)| = 0, \quad \text{uniformly for } u \in \mathbb{R},
\]

then \( \phi \) is integrable with respect to \( \nu_{t,x} \) for almost every \((t, x)\) and we have the convergence

\[
\phi(\rho^\varepsilon, u^\varepsilon) \to \int_{\mathcal{H}} \phi(\rho, u) \, d\nu_{t,x}(\rho, u) \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^2_+).
\]

Likewise, by arguments analogous to those in the proof of [15, Proposition 4.2], which entail an application of the div-curl lemma for sequences with divergence and curl compact in \( W^{-1,1} \) due to Conti–Dolzmann–Müller. [6], we use Proposition 5.3 in conjunction with Lemma 4.13 to obtain the following commutation relation, in the style of Tartar–Murat (cf. [18]).

**Proposition 7.2.** Let \( \{ (\rho^\varepsilon, u^\varepsilon) \}_{0 < \varepsilon \leq \varepsilon_0} \) be an admissible sequence of initial data in the sense of Definition 5.1 and let \( \{ (\rho^\varepsilon, u^\varepsilon) \}_{0 < \varepsilon \leq \varepsilon_0} \) be the associated viscous solutions of \((1.2)\). Correspondingly, let \( \nu_{t,x} \) be the Young measure generated by the family \( \{ (\rho^\varepsilon, u^\varepsilon) \}_{0 < \varepsilon \leq \varepsilon_0} \). Then, we have the constraint equation

\[
\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1) = \chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1),
\]

for all \( s_1, s_2 \in \mathbb{R} \), where we use the notation \( \mathcal{F} = \int f \, d\nu_{t,x} \) and \( \chi(s_j) = \chi(\cdot, \cdot, -s_j), \sigma(s_j) = \sigma(\cdot, \cdot, s_j) \).

The main theorem of this section is then the following reduction result.

**Theorem 7.3.** Let \( \nu \in \text{Prob}(\mathcal{H}) \) be such that \( \int_{\mathcal{H}} \rho^2 \, d\nu < \infty \) and, for all \( s_1, s_2 \in \mathbb{R} \),

\[
\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1) = \chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1).
\]

Then either \( \supp \nu \subset V \), or the support of \( \nu \) is a singleton in \( \mathcal{H} \).

The proof of this theorem follows from now standard arguments (cf. [5, 8] and see especially [15, Theorem 5.1]) once we have shown the next lemma (Lemma 7.4).

In what follows, we consider the fractional derivative operators \( P_j := \partial_j^{\lambda_j+1} \), for \( j = 2, 3 \), where it is understood that the distributions \( P_j\chi(s_j) \) act on test functions \( \psi \in \mathcal{D}(\mathbb{R}) \) via

\[
(P_j\chi(s_j), \psi) = -\int_{\mathbb{R}} \chi(s_j)\psi'(s_j) \, ds_j \quad \text{for } j = 2, 3.
\]

Additionally, we let \( \phi_2, \phi_3 \in \mathcal{D}(\mathbb{R}) \) be standard mollifiers, with \( \int_{\mathbb{R}} \phi_j(s_j) \, ds_j = 1 \) and \( \phi_j \geq 0 \) for \( j = 2, 3 \), chosen such that

\[
Y(\phi_2, \phi_3) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\phi_2(s_2)\phi_3(s_3) - \phi_2(s_3)\phi_3(s_2)) \, ds_3 \, ds_2 > 0.
\]
As is standard, for \( \delta > 0 \), we define \( \phi_j^\delta(s_j) := \delta^{-1}\phi_j(s_j/\delta) \), and, for \( j = 2,3 \),

\[
\bar{P}_j \chi_j^\delta := P_j \chi_j \ast \phi_j^\delta(s_1) = \int R \partial_{s_j} \chi(s_j) \delta^{-2} \phi_j^\delta \frac{(s_j - s_1)}{\delta} \, ds_j.
\]

**Lemma 7.4.** For any test function \( \psi \in \mathcal{D}(\mathbb{R}) \), writing also \( \chi_1 = \chi(s_1) \),

\[
(i) \quad \lim_{\delta \to 0} \int \frac{\chi(s_1)}{\delta} \left( P_2 \chi_2^\delta \delta \frac{P_3 \sigma_3^\delta}{\delta} - P_3 \chi_3^\delta \delta \frac{P_2 \sigma_2^\delta}{\delta} \right) \psi(s_1) \, ds_1
\]

\[
= \int \mathcal{H} Y(\phi_2, \phi_3) Z(\rho) \sum_{\pm} (K^\pm)^2 \chi(u \pm k(\rho)) \psi(u \pm k(\rho)) \, d\nu(\rho, u),
\]

where \( Z(\rho) = (\lambda + 1)M^2 \chi k(\rho)^{2\lambda} D(\rho) > 0 \) for \( \rho > 0 \), and \( D(\rho) \) is as in Lemma 6.3.

\[
(ii) \quad \lim_{\delta \to 0} \int \frac{\chi(s_1)}{\delta} \left( P_2 \chi_2^\delta \delta \frac{P_3 \sigma_3^\delta}{\delta} - P_1 \chi_1^\delta \sigma_1 \right) \psi(s_1) \, ds_1 = \lim_{\delta \to 0} \int \mathcal{H} \left( P_2 \chi_2^\delta \delta \frac{P_3 \sigma_3^\delta}{\delta} - \chi_1 \sigma_1 \right) \psi(s_1) \, ds_1.
\]

The positive constants \( M_\chi \) and \( K^\pm \) of the previous lemma were introduced in [3, Section 2].

Given the results of Lemma 6.4 and the estimates of Proposition 5.2, the proof of this lemma proceeds in the same manner as [15, Lemmas 5.2–5.3]. For the convenience of the reader, we give a sketch of the proof of the first part of the lemma below.

**Sketch of Lemma 7.4 (i).** Let \( \psi(s_1) \in \mathcal{D}(\mathbb{R}) \). We recall that (see [3, Lemma 4.3] and [15, Proposition 5.5]) on sets on which \( \rho \) is bounded,

\[
P_2 \chi_2^\delta \delta \frac{P_3 \sigma_3^\delta}{\delta} - P_3 \chi_3^\delta \delta \frac{P_2 \sigma_2^\delta}{\delta} \to Y(\phi_2, \phi_3) Z(\rho) \sum_{\pm} (K^\pm)^2 \delta_{s_1 = u \pm k(\rho)}
\]

as \( \delta \to 0 \) weakly-star in measures in \( s_1 \) and uniformly in \( (\rho, u) \) on sets where \( \rho \) is bounded, where \( Y \) was defined in (7.5). Thus,

\[
\lim_{\delta \to 0} \int \frac{\chi(s_1)}{\delta} \left( P_2 \chi_2^\delta \delta \frac{P_3 \sigma_3^\delta}{\delta} - P_3 \chi_3^\delta \delta \frac{P_2 \sigma_2^\delta}{\delta} \right) \psi(s_1) \, ds_1
\]

\[
= Y(\phi_2, \phi_3) Z(\rho) \sum_{\pm} (K^\pm)^2 \chi(u \pm k(\rho)) \psi(u \pm k(\rho))
\]

pointwise for all \( (\rho, u) \in \mathcal{H} \). To interchange the pointwise convergence with the Young measure \( \nu \), we observe that the function \( \rho^2 \in L^1(\mathcal{H}, \nu) \). We therefore prove that

\[
\left| \int \frac{\chi(s_1)}{\delta} \left( P_2 \chi_2^\delta \delta \frac{P_3 \sigma_3^\delta}{\delta} - P_3 \chi_3^\delta \delta \frac{P_2 \sigma_2^\delta}{\delta} \right) \psi(s_1) \, ds_1 \mathbb{1}_{\rho > \rho_0} \right| \leq C(\rho^2 + 1),
\]

for some \( C > 0 \) independent of \( \rho, u \) and \( \delta \) and hence apply Lebesgue’s Dominated Convergence Theorem (and the local uniform convergence of (7.6)) to conclude the proof of the lemma.

In order to prove claim (7.7), we exploit the decomposition \( \sigma = u \chi + h \) by writing

\[
P_2 \chi_2^\delta P_3 \sigma_3^\delta - P_3 \chi_3^\delta P_2 \sigma_2^\delta = P_2 \chi_2^\delta P_3 (\sigma_3^\delta - u \chi_3^\delta) - P_3 \chi_3^\delta P_2 (\sigma_2^\delta - u \chi_2^\delta) = P_2 \chi_2^\delta P_3 h_3^\delta - P_3 \chi_3^\delta P_2 h_2^\delta.
\]

Using Lemma 6.4, we see that this product consists of a sum of terms of the form

\[
A_i,\pm(\rho) B_j,\pm(\rho) T_2(s_2 - u \pm k(\rho)) T_3(s_3 - u \pm k(\rho)),
\]

and

\[
A_i,\pm(\rho) B_j,\pm(\rho) (s_2 - s_3) T_2(s_2 - u \pm k(\rho)) T_3(s_3 - u \pm k(\rho)),
\]

where \( T_2, T_3 \in \{ \delta, PV, H, Ci \} \), or terms with the same structure but where \( T_2 \in \{ \delta, H, PV, Ci, r_\chi \} \) and \( T_3 \in \{ H, Ci, r_\sigma \} \) and likewise with \( s_2 \) and \( s_3 \) reversed.
A simple estimate for distributions of these types (see [11, Lemmas 3.8 and 3.9 for a proof]) yields, for any pair $T_2,T_3 \in \{\delta, PV, H, Ci\}$,

$$\left| \int_{-\infty}^{\infty} \overline{\chi(s_1)} \psi(s_1) \left( (s_2 - s_3)T_2(s_2 - u \pm k(\rho))T_3(s_3 - u \pm k(\rho)) \right) * \phi^\delta_2 * \phi^\delta_3(s_1) \, ds_1 \right| \leq C \| \overline{\chi \psi} \|_{C^{0,\alpha}(\mathbb{R})},$$

(7.8)

where we have noted that $s \mapsto \overline{\chi(s)}$ is Hölder continuous. Likewise,

$$\left| \int_{-\infty}^{\infty} \overline{\chi(s_1)} \psi(s_1) \left( (s_2 - s_3)T_2(s_2 - u \pm k(\rho))T_3(s_3 - u \pm k(\rho)) \right) * \phi^\delta_2 * \phi^\delta_3(s_1) \, ds_1 \right| \leq C \| \overline{\chi \psi} \|_{C^{0,\alpha}(\mathbb{R})} \left( 1 + \| r_{\chi} \|_{C^{0,\alpha}_1(\mathbb{R}^n)} + \| r_{\sigma} \|_{C^{0,\alpha}_1(\mathbb{R}^n)} \right),$$

for $T_2 \in \{\delta, PV, Ci, r_{\chi}\}$ and $T_3 \in \{H, Ci, r_{\sigma}\}$. In this case, $R > 0$ is such that $\text{supp} \, \psi \subset B_{R-2}(0)$ and the terms involving $r_{\chi}$ and $r_{\sigma}$ are contributed when one of $T_2,T_3 \in \{r_{\chi}, r_{\sigma}\}$. Hence, Lemma 6.4 gives

$$\left| \int_{-\infty}^{\infty} \overline{\chi(s_1)} \left( P_2 \chi_2 P_3 \sigma_3 - P_3 \chi_3 P_2 \sigma_2 \right) \psi(s_1) \, ds_1 \right| \leq C \max\{ |A_{j,\pm} B_{k,\pm}|, |A_{j,\pm}|, |r_{\chi}|_{C^{0,\alpha}_1(\mathbb{R}^n)}, |B_{j,\pm}|, |r_{\sigma}|_{C^{0,\alpha}_1(\mathbb{R}^n)} \} \leq C(\rho^2 + 1).$$

Thus we have concluded the desired convergence. \qed

Finally, we conclude this section with a proof of the main theorem.

**Proof of Theorem 1.4.** Let $\overline{\rho}_0(x):= \max\{\rho_0(x), \sqrt{\varepsilon}\}$. Now mollify $\overline{\rho}_0$ and $u_0$ suitably, such that we obtain an admissible sequence of initial data $(\overline{\rho}_0, \overline{u}_0)$, in the sense of Definition 5.1. The sequence of smooth solutions $(\overline{\rho}^\varepsilon, \overline{u}^\varepsilon)$ of (1.2) corresponding to this initial data then generates a Young measure $\nu_{t,x}$, constrained by the Tartar–Murat commutation relation, according to Proposition 7.2. An application of Theorem 7.3 then yields that $\nu_{t,x}$ is either a point mass or is supported in the vacuum set $V$. In the phase-space coordinates $(\rho, m)$, where $m = \rho u$, this measure is a Dirac mass. As such, we write $\nu_{t,x} = \delta_{(\rho(t,x), m(t,x))}$, where $\rho$ and $m$ are measurable functions. In view of this, we deduce that the convergence of the subsequence $(\rho^\varepsilon, \overline{\rho}^\varepsilon)$ occurs in measure and hence (up to subsequence) almost everywhere in $\mathbb{R}^2_{+}$. This proves that $(\rho, m)$ is a weak solution of (1.1). It remains to check that it is an entropy solution, in the sense of Definition 1.3.

Since the weak entropies are taken to be $C^2$ functions, the almost everywhere convergence guarantees that $\overline{\eta}^\varepsilon(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \to \overline{\eta}^\varepsilon(\rho, m)$ a.e. $(t,x) \in \mathbb{R}^2_{+}$. In turn, Fatou’s lemma yields

$$\int_{\mathbb{R}} \overline{\eta}^\varepsilon(\rho, m) \, dx \leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}} \overline{\eta}^\varepsilon(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \, dx$$

for almost every $t \geq 0$.

In view of Proposition 5.2, the right-hand side is bounded independently of $\varepsilon$. We thereby deduce that there exists $M = M(E_0, t)$, monotonically increasing such that

$$\int_{\mathbb{R}} \overline{\eta}^\varepsilon(\rho, m) \, dx \leq M(E_0, t)$$

for almost every $t \geq 0$. The pair $(\rho, m)$ is therefore of relative finite-energy. The last part of the Definition 1.3 can be verified by following the argument presented in the final portion of [15, Section 6]. \qed

**Appendix A. Estimates on Elementary Quantities**

We state and prove here various auxiliary estimates that we use throughout the paper to compare and estimate quantities relating to the pressure $p(\rho)$ and the function $k(\rho)$. We begin with the proof of Corollary 2.5.
Proof of Corollary 2.5. Since \( k'(\rho) = \sqrt{p'(\rho)}/\rho \), we have
\[
\left| k'(\rho) \right|^2 - \frac{\kappa_2}{\rho^2} = \left| \frac{p'(\rho) - \kappa_2}{\rho^2} \right| \leq \frac{1}{\rho^2} \int_{\rho}^{\infty} \left| p''(y) \right| dy = \frac{3C_p}{\alpha} \rho^{-\alpha - 2},
\]
while
\[
\left| k''(\rho) + \frac{\sqrt{\kappa_2}}{\rho^2} \right| = \left| \frac{p''(\rho)}{2\rho \sqrt{p'(\rho)}} + \frac{\sqrt{\kappa_2} - \frac{p''(\rho)}{2p'(\rho)} - \frac{\kappa_2}{\rho^2}}{\rho^2} \right| \leq \frac{3C_p \rho^{-\alpha - 2}}{\sqrt{2 \kappa_2}} + \frac{1}{\rho^2} \int_{\rho}^{\infty} \left| p''(y) \right| dy, \]
\[
\leq \frac{3C_p}{\sqrt{2 \kappa_2}} (1 + \alpha^{-1}) \rho^{-\alpha - 2}.
\]
For the third derivative,
\[
k^{(3)}(\rho) - \frac{2 \sqrt{\kappa_2}}{\rho^3} = \frac{p'''(\rho)}{2p'\sqrt{p'(\rho)}} - \frac{\sqrt{\kappa_2} - \frac{p''(\rho)}{2p'(\rho)}}{\rho^2} \frac{p''(\rho) - \kappa_2}{\rho^2} + \frac{2}{\rho^3} (\sqrt{p'(\rho) - \kappa_2}),
\]
from which we obtain
\[
\left| k^{(3)}(\rho) - \frac{2 \sqrt{\kappa_2}}{\rho^3} \right| \leq M \rho^{-\alpha - 3} + \frac{1}{\rho^3} \int_{\rho}^{\infty} \left| p''(y) \right| dy.
\]
In a similar vein, we have the following, from which the result is easily deduced,
\[
\left| k^{(4)}(\rho) + \frac{6 \sqrt{\kappa_2}}{\rho^4} \right| \leq M \rho^{-\alpha - 4} + \frac{3}{\rho^4} \int_{\rho}^{\infty} \left| p''(y) \right| dy.
\]
\(\square\)

We now state several other similar estimates, the proofs of which can be found in [16, Chapter 3].

**Corollary A.1.** Assume that \( \rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha} \} \). Then, there exists a positive \( M = M(\kappa_2) \) such that
\[
M^{-1} (1 + \log(\rho/\rho_*)) \leq k(\rho) \leq M (1 + \log(\rho/\rho_*)) \quad \text{for } \rho \geq \rho_*.
\]

**Lemma A.2.** Assume that \( \rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha} \} \). Then,
\[
0 < d_*(\rho) - 1 = \frac{\rho_*}{\rho} \quad \text{and} \quad |d(\rho) - d_*(\rho)| \leq M \rho^{-\alpha} \quad \text{for all } \rho \geq \rho_*.
\]
for some positive constant \( M = M(C_p, \kappa_2) \) independent of \( \rho_* \). Additionally, it follows that
\[
|d(\rho) - 1| \leq 2 \quad \text{for all } \rho \geq \rho_*.
\]

**Lemma A.3.** We have the equality
\[
\frac{k'(\rho_0)}{k_*'(\rho_0)} \frac{k_*'\rho_0)}{k'(\rho)} = \sqrt{\frac{p'(\rho_0)}{p'(\rho)}} \quad \text{for all } R \leq \rho \leq \rho_0.
\]
As such, provided \( \rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha} \} \), there is a positive \( M = M(\alpha, C_p, \kappa_2) \) such that
\[
\left| 1 - \frac{k'(\rho_0)}{k_*'(\rho_0)} \frac{k_*'\rho_0)}{k'(\rho)} \right| \leq M \rho^{-\alpha} \quad \text{for all } \rho_* \leq \rho \leq \rho_0.
\]

In turn,
\[
0 < \frac{k'(\rho_0)}{k_*'(\rho_0)} \frac{k_*'\rho_0)}{k'(\rho)} \leq 2 \quad \text{for all } \rho_* \leq \rho \leq \rho_0.
\]

**Lemma A.4.** Assume that \( \rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha} \} \). Then, there exists a positive constant \( M = M(\alpha, C_p, \kappa_2) \) such that
\[
|k(\rho_0) - k(\rho) - (k_*\rho_0) - k_*(\rho)| \leq M \rho^{-\alpha} \min\left(\frac{\rho_0 - \rho}{\rho_0}, 1\right) \quad \text{for } \rho_* \leq \rho \leq \rho_0.
\]

**Lemma A.5.** Assume that \( \rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha} \} \). Then, there exists a positive constant \( M = M(\alpha, C_p, \kappa_2) \) such that
\[
\left| 1 - \frac{k'(\rho_0)}{k_*'(\rho_0)} \frac{k_*'\rho_0)}{k'(\rho)} \right| \leq M \rho_0^{-\alpha} \quad \text{for } \rho_0 \geq \rho_*.
\]
Next is a result concerning the size of the supports of the re-scaled isothermal kernel and the perturbation, which follows immediately from (2.1) and (A.1).

**Lemma A.6.** There exists a positive constant $M = M(\alpha, \kappa_2, C_p, \rho_*)$ such that
\[
\max\{k(\rho), k_*(\rho)\} \leq M k(\rho) \quad \text{for all } \rho \geq \rho_*.
\] (A.9)

Finally, we state two technical lemmas regarding the relative internal energy.

**Lemma A.7.** There exists a positive constant $M = M(\gamma, \kappa_1, \rho_*, k(\rho_*), \bar{p})$ such that
\[
\rho + \rho \log(\rho/\rho_*) \leq M (1 + \epsilon^*(\rho, \bar{p})) \quad \text{for } \rho \geq \rho_*.
\] (A.10)

**Lemma A.8.** There exists a positive constant $M = M(\gamma, \kappa_1, \rho_*, k(\rho_*), \bar{p})$, such that
\[
0 \leq \rho + p(\rho) \leq M (1 + \epsilon^*(\rho, \bar{p})) \quad \text{for } \rho \geq 0.
\] (A.11)

The proofs of Lemmas A.7 and A.8 have a common strategy, and one verifies (4.6) using similar techniques. Complete proofs may be found in [16, Chapter 3].

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