High-Velocity Estimates for the Scattering Operator and Aharonov-Bohm Effect in Three Dimensions *

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Abstract

We obtain high-velocity estimates with error bounds for the scattering operator of the Schrödinger equation in three dimensions with electromagnetic potentials in the exterior of bounded obstacles that are handlebodies. A particular case is a finite number of tori. We prove our results with time-dependent methods. We consider high-velocity estimates where the direction of the velocity of the incoming electrons is kept fixed as its absolute value goes to infinity. In the case of one torus our results give a rigorous proof that quantum mechanics predicts the interference patterns observed in the fundamental experiments of Tonomura et al. that gave a conclusive evidence of the existence of the Aharonov-Bohm effect using a toroidal magnet. We give a method for the reconstruction of the flux of the magnetic field over a cross-section of the torus modulo 2π. Equivalently, we determine modulo 2π the difference in phase for two electrons that travel to infinity, when one goes inside the hole and the other outside it. For this purpose we only need the high-velocity limit of the scattering operator for one direction of the velocity of the incoming electrons. When there are several tori -or more generally handlebodies- the information that we obtain in the fluxes, and on the difference of phases, depends on the relative position of the tori and on the direction of the velocities when we take the high-velocity limit of the incoming electrons. For some locations of the tori we can determine all the fluxes modulo 2π by taking the high-velocity limit in only one direction. We also give a method for the unique reconstruction of the electric potential and the magnetic field outside the handlebodies from the high-velocity limit of the scattering operator.

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1 Introduction

The Aharonov-Bohm effect is a fundamental quantum mechanical phenomenon wherein charged particles, like electrons, are physically influenced, in the form of a phase shift, by the existence of magnetic fields in regions that are inaccessible to the particles. This genuinely quantum mechanical phenomenon was predicted by Aharonov and Bohm [3]. See also Ehrenberg and Siday [9]. This phenomenon has been extensively studied both, from the theoretical, and the experimental points of view. For a review of the literature see [29] and [30]. There has been a large controversy, involving over three hundred papers, concerning the existence of the Aharonov-Bohm effect. For a detailed discussion of this controversy see [30]. The issue was finally settled by the fundamental experiments of Tonomura et al. [37, 38], who used toroidal magnets to enclose a magnetic flux inside them. In remarkable experiments they were able to superimpose behind the magnet an electron beam that traveled inside the hole of the magnet with another electron beam that traveled outside the magnet, and they measured the phase shift produced by the magnetic flux enclosed in the magnet, giving a conclusive evidence of the existence of the Aharonov-Bohm effect.

In this paper we give a rigorous mathematical analysis of this scattering problem with time-dependent methods. In particular, we give a rigorous mathematical proof that quantum mechanics predicts the phase shifts observed in the Tonomura et al. experiments [37, 38].

We consider bounded obstacles, $K$, whose connected components are handlebodies. In particular, they can be the union of a finite number of bodies diffeomorphic to tori or to balls. Some of them can be patched through the boundary.

We study the high-velocity limit of the scattering operator in the complement, $\Lambda$, of the obstacle, $K$, for the Schrödinger equation with magnetic field and electric potential in $\Lambda$ and with magnetic fluxes enclosed in the obstacle $K$. We obtain high-velocity estimates with error bounds for the scattering operator using the time-dependent method of [14]. We consider high-velocity limits where the direction of the velocity of the incoming electrons is kept fixed as its absolute value goes to infinity.

The leading term of our estimate gives us a reconstruction formula that allows us to reconstruct the circulation of the magnetic potential modulo $2\pi$ along lines in the direction of the velocity (the X-ray transform). From these line integrals we uniquely reconstruct the magnetic field in some region of $\Lambda$. The error term for the leading order goes to zero as a constant divided by the absolute value of the velocity.

The next term in our high-velocity estimate allows us to reconstruct the integral of the electric potential along lines in the direction of the velocity (the X-ray transform). We uniquely reconstruct the electric potential in a region of $\Lambda$ from these lines integrals. The error term for this high-velocity estimate goes to zero as a constant divided by a power of the absolute value of the velocity, that depends on the decay rate at infinity of the magnetic field and of the electric potential. If we have enough decay this power is one, as for the leading order.

The leading-order high-velocity estimate is given in Theorem 5.7 and the next term in our high-velocity estimate
is given in Theorem 5.9. The unique reconstruction of the magnetic field and the electric potential in a region of $\Lambda$ is given in Theorem 6.3. The reconstruction method is summarized in Remark 6.4.

Then, we consider the Aharonov-Bohm effect. We assume that the magnetic field in $\Lambda$ is identically zero. On the contrary, the electric potential is not assumed to be zero. In other words, we analyze the Aharonov-Bohm effect in the presence of an electric potential. We use for reconstruction only the leading-order high-velocity estimate. As for high-velocities the electric potential gives a lower-order contribution, it plays no role in the Aharonov-Bohm effect. However, to allow for a non-trivial electric potential could be of interest from the experimental point of view.

In Theorem 7.1 we reconstruct the circulation of the magnetic potential, modulo $2\pi$, over a set of closed paths in $\Lambda$ and in Remark 7.3 we reconstruct the projection of the de Rham cohomology class of the magnetic potential onto a subspace of $H^1_{\text{de R}}(\Lambda)$ in the sense that we reconstruct, modulo $2\pi$, the expansion coefficients of the projection into the subspace of the de Rham cohomology class of the magnetic potential in any basis of the subspace. The set of circulations and the projection of the de Rham cohomology class of the magnetic potential that we can reconstruct depend on the relative position of the handlebodies and on the direction of the velocity of the incoming electrons. In Theorem 7.11, Corollary 7.12 and Remark 7.13 we give our method for the reconstruction of the fluxes inside the obstacle $K$, modulo $2\pi$. Since the scattering operator is invariant under short-range gauge transformations that change the fluxes by multiples of $2\pi$, the fluxes can only be reconstructed modulo $2\pi$. Again, the fluxes that we reconstruct depend on the relative position of the handlebodies and on the direction of the velocity of the incoming electrons. In Example 7.14 we give obstacles that consist of a finite number of tori and manifolds diffeomorphic to balls, where from the high-velocity limit of the scattering operator in only one direction we reconstruct modulo $2\pi$ all the circulations in $\Lambda$ of the magnetic potential, its de Rham cohomology class modulo $2\pi$, and the flux modulo $2\pi$ of the magnetic field over the cross section of all the tori.

Finally, we discuss the fundamental experiments of Tonomura et al. [37, 38] in Section 8. We show that our results give a rigorous proof that quantum mechanics predicts the interference patterns between electron beams that go inside and outside the torus, that where observed in these remarkable experiments.

The paper is organized as follows. In Section 2 we give a precise definition of the obstacle, $K$, and we study in a detailed way the homology and the cohomology of $K$ and $\Lambda$. This allows us to construct homology and cohomology basis that have a clear physical significance. Using these results we construct in Section 3 classes of magnetic potentials characterized by the magnetic field in $\Lambda$ and by the fluxes of the magnetic field in the cross sections of the components of $K$ that have holes. We construct classes of magnetic potentials where the fluxes are fixed, and classes where the fluxes are only fixed modulo $2\pi$. We study the gauge transformations between these magnetic potentials. In Section 4 we define the Hamiltonian of our system. In Section 5 we study our direct scattering problem. We prove the existence of the wave operators and we define the scattering operator. We analyze how the wave and scattering operators change under the change of the magnetic potential when the fluxes are only fixed modulo $2\pi$. We also prove our high-velocity
estimates. In Section 6 we give our method for the reconstruction of the magnetic field and the electric potential in a region of Λ. In Section 7 we obtain our results in the Aharonov-Bohm effect and in Sections 8 we discuss the Tonomura et al. experiments [37, 38]. In Appendixes A and B we prove results in homology that we need.

For the Aharonov-Bohm effect in scattering in two dimensions see [28] and [40]. For inverse scattering by magnetic fields in all space see [41, 42, 43, 20, 21, 22]. For properties of the scattering matrix for scattering by Aharonov-Bohm potentials in all space see [33, 34] and [42, 43]. For the Aharonov-Bohm effect in inverse boundary-value problems see [10, 11, 12, 13, 24] and [25].

Finally, some words in our notations and definitions. We use notions of homology and cohomology as defined, for example, in [7], [16], [17], [8] and [41]. In particular, we consider homology and cohomology groups on open sets of \( \mathbb{R}^n \), with coefficients in \( \mathbb{Z} \) and in \( \mathbb{R} \). As these singular homology and cohomology groups are isomorphic to the \( \mathcal{C}^\infty \) homology and cohomology groups, [7] page 291, we will identify them. We also use differential forms, or just forms, in open sets of \( \mathbb{R}^3 \) with regular boundary -or in their closure- with the Euclidean metric, as defined, for example, in [8], [35], [41]. For such a set, \( O \), we denote by \( \Omega^k(O) \) the set of all \( k \)-forms in \( O \).

We use the standard identification between concepts of vector calculus and differential forms in three dimensions in the interior of \( O \), that we denote by \( \tilde{O} \). [35]. Let \( \{x^i\}_{i=1}^3 \) be the Euclidean coordinates of \( \mathbb{R}^3 \).

We identify vectors and 1-forms as
\[
(A_1, A_2, A_3) \iff \sum_{i=1}^{3} A_i dx^i.
\]
We identify vectors and 2-forms as
\[
(B_1, B_2, B_3) \iff B_3 dx^1 \wedge dx^2 - B_2 dx^1 \wedge dx^3 + B_1 dx^2 \wedge dx^3.
\]
We identify scalars and 3-forms as
\[
f \iff f dx^1 \wedge dx^2 \wedge dx^3.
\]

The exterior derivative, \( d \), in 1-forms is equivalent to the curl of the associated vector, and in 2-forms is equivalent to the divergence of the associated vector. In particular, a 1-form, \( A \), is closed if \( dA = 0 \), or equivalently, if the associated vector has curl zero, and a 2-form, \( B \), is closed if \( dB = 0 \), or equivalently, if the associated vector has divergence zero. For 0-forms the exterior derivative coincides with the gradient \( \nabla \).

We will always assume that the coefficients of our forms are at least locally integrable in any coordinate chart. Hence, they define distributions or currents [8]. We say that a form belongs to some space if its coefficients in any coordinate chart belong to that space. For example, we say that a form is continuous if it has continuous coefficients or that is \( L^p \) if its coefficients are in \( L^p \). In the case of a 2-form, \( B \), we will say that \( B \in L^p \Omega^2(O) \), or, equivalently, that the associated vector \( B \in L^p(\tilde{O}) \). For forms defined in \( O \) that are not differentiable in the classical sense the derivatives are taken in distribution sense in \( O \), if \( O \) is open, or in \( \tilde{O} \) if it is closed.
For any $x \in \mathbb{R}^3, x \neq 0$, we denote, $\hat{x} := x/|x|$. By $B^r_+(x_0), n = 2, 3$ we denote the open ball of center $x_0$ and radius $r$. By $S^2$ we denote the unit sphere in $\mathbb{R}^3$. For any set $O$ we denote by $F(x \in O)$ the operator of multiplication by the characteristic function of $O$. The symbol $\cong$ means isomorphism, the symbol $\approx$ means homotopic equivalence and the symbol $\simeq$ means homeomorphism.

We define the Fourier transform as a unitary operator on $L^2(\mathbb{R}^3)$ as follows,

$$\hat{\phi}(p) := F\phi(p) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ipx} \phi(x) \, dx.$$ 

We define functions of the operator $p := -i\nabla$ by Fourier transform,

$$f(p)\phi := F^*f(p)F\phi, D(f(p)) := \{ \phi \in L^2(\mathbb{R}^3) : f(p)\hat{\phi}(p) \in L^2(\mathbb{R}^3) \},$$

for every measurable function $f$.

## 2 The Obstacle

### 2.1 Handlebodies

Let us designate by $S^1$ the unit circle. We denote by $T := S^1 \times B^2_+(0)$ the solid torus of dimension 3. We orient $T$ assuming that the inverse of the following function is a chart that belongs to the orientation of $T$,

$$\mathcal{U} : (0, 1) \times B^2_+(0) \to T, \mathcal{U}(t, x, y) = (e^{2\pi it}, x, y), \quad (2.1)$$

The boundary sum of $T$ with itself is defined as follows. See [13], page 19. Let $D_1 \subseteq \partial T$ be a disc contained in a chart, $(U_1, \phi_1)$, belonging to the orientation of $T$ and let $D_2 \subseteq \partial T$ be a disc contained in a chart, $(U_2, \phi_2)$, belonging to the opposite orientation. We define the boundary sum $T\sharp T$ as the disjoint union of $T$ with itself, identifying $D_1$ in the first torus with $D_2$ in the second torus by means of the charts, in such a way that $T\sharp T$ is an oriented differentiable manifold, the inclusion $I_1 : T \hookrightarrow T\sharp T$ in the first torus is an homeomorphism onto its image whose restriction to $\tilde{T}$ is a diffeomorphism that preserves orientation and the inclusion $I_2 : T \hookrightarrow T\sharp T$ in the second torus is an homeomorphism onto its image whose restriction to $\tilde{T}$ is a diffeomorphism that inverts orientation. We define the boundary sum of $k$ tori by induction. Suppose that we already defined the boundary sum $\sharp(k-1)T := T\sharp T \cdots \sharp T$, $k - 1$ times of $k - 1$ tori. Let $l_j, j = 1, 2, \cdots, k - 1$ be the inclusion of $T$ on the $j$th torus. As before, Let $D_1 \subseteq \partial T$ be a disc contained in a chart $(U_1, \phi_1)$ belonging to the orientation of $T$ if $k - 1$ is odd, or belonging to the opposite orientation if $k - 1$ is even. Moreover, we assume that $l_{k-1}(U_1)$ does not intersect any of the union charts in $\sharp(k-1)T$. This is always possible choosing the union charts small enough. Let $D_2 \subseteq \partial T$ be a disc contained in a chart $(U_2, \phi_2)$ belonging to the opposite orientation of $T$. Then, the boundary sum $\sharp kT := T\sharp T \cdots \sharp T$, $k$ times is obtained from $\sharp T \cdots \sharp T$, $k - 1$ times identifying $l_{k-1}(D_1)$ with $D_2$ by means of the charts $(l_{k-1}(U_1), \phi_1 \circ l_{k-1}^{-1})$ and $(U_2, \phi_2)$ in such a way that $\sharp kT$ is an
oriented differentiable manifold, the inclusion $\natural (k - 1)T \hookrightarrow \natural kT$ in the first $k - 1$ tori is an homeomorphism onto its image whose restriction to the interior is a diffeomorphism that preserves orientation and the inclusion $l_k : T \hookrightarrow \natural kT$ in the last torus is an homeomorphism onto its image whose restriction to $T$ is a diffeomorphism that inverts orientation. The structure of $\natural kT$ as oriented differentiable manifold does not depend on the discs used to join the tori [15], page 19. We will say that any oriented differentiable manifold diffeomorphic to $\natural kT$ is a handlebody with $k$ handles, where the diffeomorphism is oriented. We will denote by $\natural 0T$ any oriented manifold that is diffeomorphic to the closed ball in $\mathbb{R}^3$ of center zero and radius one. Note that the inclusions $l_j : T \hookrightarrow \natural kT$ onto the $j$th torus are homeomorphisms onto their images whose restriction to the interior are diffeomorphisms that preserve orientation if $j$ is odd and change orientation if $j$ is even.

2.2 Homology of Handlebodies

We define the functions $\gamma_\pm : [0, 1] \rightarrow T : \gamma_\pm(t) = (e^{\pm 2\pi it}, 0, 0)$ and

$$Z_j(t) := \begin{cases} l_j \circ \gamma_+(t) & \text{if } j \text{ is odd,} \\ l_j \circ \gamma_-(t) & \text{if } j \text{ is even.} \end{cases}$$

(2.2)

For any $\xi \in S^1$ we define

$$B_\xi := \left( \{\xi\} \times B_1(0) \right) \subseteq T.$$  

(2.3)

We orient $B_\xi$ by requiring that inverse of the inclusion $B_1(0) \hookrightarrow B_\xi$ belongs to the orientation of $B_\xi$, i.e., the inverse of the inclusion is a chart.

The image of $Z_j$ in $\natural kT$ is a submanifold that we orient by means of the curve $Z_j$. We assume that $l_j(B_\xi)$ does not intersect any of the union charts, what is always possible if the union charts are small enough. We orient the submanifold $l_j(B_\xi)$ by the orientation of $B_\xi$. Let $v_1 \in T_{l_j(\xi, 0, 0)}(Z_j([0, 1])) \subseteq T_{l_j(\xi, 0, 0)}(\natural kT)$ be a tangent vector in the orientation of $Z_j([0, 1])$, and let $v_2, v_3 \in T_{l_j(\xi, 0, 0)}(l_j(B_\xi)) \subseteq T_{l_j(\xi, 0, 0)}(\natural kT)$ with $(v_2, v_3)$ in the positive orientation of $l_j(B_\xi)$. Then, $(v_1, v_2, v_3)$ is positively oriented in the tangent space $T_{l_j(\xi, 0, 0)}(\natural kT)$. This means that $Z_j([0, 1])$ and $l_j(B_\xi)$ intersect in a positive way.

Let us denote by $H_1(\natural kT; \mathbb{R})$ the first group of singular homology of $\natural kT$ with coefficients in $\mathbb{R}$. See [10], page 47. In Appendix A we give a proof, for the reader’s convenience, that $\{[Z_j]\}_{j=1}^k$ is a basis of $H_1(\natural kT; \mathbb{R})$.

2.3 Definition of the Obstacle

**Assumption 2.1.** We assume that the obstacle $K$ is a compact submanifold of $\mathbb{R}^3$ of dimension three oriented with the orientation of $\mathbb{R}^3$. Moreover, $K = \bigcup_{j=1}^L K_j$ where $K_j, 1 \leq j \leq L$ are the connected components of $K$. We assume that the $K_j$ are handlebodies.

By our assumption there exist numbers $m_j \in \mathbb{N} \cup \{0\}$ and oriented diffeomorphisms $F_j : \natural m_j T \rightarrow K_j, 1 \leq j \leq L$. 


We denote by \( \mathcal{J}_j \) the inclusion \( K_j \hookrightarrow K \). The diffeomorphisms \( F_j \) induce a diffeomorphism

\[ G : \bigvee_{j=1}^{L} \natural m_j T \to K, \]

where the symbol \( \bigvee \) means disjoint union. We denote,

\[
J := \{ j \in \{ 1, 2, \cdots, L \} : m_j > 0 \}, \quad m := \sum_{j=1}^{L} m_j, \]

\[
\{ \gamma_k \}_{k=1}^{m} := \{ \mathcal{J}_j \circ F_j \circ Z_i | j \in J, i \in \{ 1, 2, \cdots, m_j \} \}. \tag{2.4}\]

Choose a \( \xi \in S^1 \) such that \( l_i(B_\xi) \) does not intersect any chart of union in \( \natural m_j T, \forall j \in J, \forall i \in \{ 1, 2, \cdots, m_j \} \). This is always possible by choosing the charts of union in a proper way. If \( \gamma_k = \mathcal{J}_j \circ F_j \circ Z_i \) we define \( B_k := \mathcal{J}_j \circ F_j \circ l_i(B_\xi). \) \( B_k \) is a manifold that we orient by means of the orientation of \( B_\xi \). As \( F_j \) is a oriented diffeomorphism and \( Z_i \) intersects \( l_i(B_\xi) \) in a positive way, it follows that \( \gamma_k \) intersects \( B_k \) in a positive way.

We define, \( W_\xi : [0, 1] \to T : W_\xi(t) := (\xi, \cos t, \sin t) \) and

\[
\tilde{\gamma}_k := \mathcal{J}_j \circ F_j \circ l_i \circ W_\xi. \tag{2.5}\]

Take \( \varepsilon > 0 \) such that \( \{ x | \text{distance}(x, \partial K) < \varepsilon \} \) is diffeomorphic to \( \partial K \times (-\varepsilon, \varepsilon) \). This is possible by the tubular neighborhood theorem. See theorem 11.4, page 93 of [7]. We define,

\[
\hat{\gamma}_k(t) := \tilde{\gamma}_k(t) + \frac{\varepsilon}{2} N(\tilde{\gamma}_k(t)), \tag{2.6}\]

where \( N(\tilde{\gamma}_k(t)) \) is the exterior normal to \( K \) at the point \( \tilde{\gamma}_k(t) \). Note that \( \partial B_k = \hat{\gamma}_k([0, 1]) \), the orientation on \( \hat{\gamma}_k([0, 1]) \) induced by \( B_k \) is the orientation induced by the curve \( \tilde{\gamma}_k \).

2.4 The Homology of the Obstacle

**Proposition 2.2.** \( \{ [\gamma_k]_{H_1(K; \mathbb{R})} \}_{k=1}^{m} \) is a basis of \( H_1(K; \mathbb{R}) \).

*Proof:* As \( G : \bigvee_{j=1}^{L} \natural m_j T \to K \) is a diffeomorphism and since

\[
H_1 \left( \bigvee_{j=1}^{L} \natural m_j T ; \mathbb{R} \right) \cong \bigoplus_{j=1}^{L} H_1(\natural m_j T ; \mathbb{R}),
\]

by Proposition 9.5, page 47 of [16], it follows from Proposition 9.3 of Appendix A that \( \{ [\gamma_k]_{H_1(K; \mathbb{R})} \}_{k=1}^{m} \) is a basis of \( H_1(K; \mathbb{R}) \).

2.5 The Cohomology of the Obstacle

As \( K \) is an ANR (absolute neighborhood retract, page 225 and Theorem 26.17.4 of [16]) we have that

\[
\hat{H}^1(K; \mathbb{R}) \cong H^1(K; \mathbb{R}), \tag{2.7}\]
by Proposition 27.1, page 230 of [16] (see also page 347, Theorem 7.15 of [17]).

By Alexander’s duality theorem (see Theorem 27.5, page 233 of [16])

\[
\hat{H}^1(K; \mathbb{R}) \cong H_2(\mathbb{R}^3, \mathbb{R}^3 \setminus K; \mathbb{R}).
\] (2.8)

By Theorem 14.1, page 75 of [16] we have the following exact sequence.

\[
H_2(\mathbb{R}^3; \mathbb{R}) \to H_2(\mathbb{R}^3, \mathbb{R}^3 \setminus K; \mathbb{R}) \to H_1(\mathbb{R}^3 \setminus K; \mathbb{R}) \to H_1(\mathbb{R}^3; K).
\]

As \(\mathbb{R}^3\) is homotopically equivalent to a point, it follows from Theorem 11.3, page 59 and Example 9.4, page 47 of [16] that \(H_2(\mathbb{R}^3; \mathbb{R}) = 0\) and \(H_1(\mathbb{R}^3; \mathbb{R}) = 0\). Then, we have that the exact sequence

\[
0 \to H_2(\mathbb{R}^3, \mathbb{R}^3 \setminus K; \mathbb{R}) \to H_1(\mathbb{R}^3 \setminus K; \mathbb{R}) \to 0,
\]

and then,

\[
H_1(\mathbb{R}^3 \setminus K; \mathbb{R}) \cong H^1(\mathbb{R}^3 \setminus K; \mathbb{R}).
\] (2.9)

By (2.7, 2.8, 2.9),

\[
H^1(K; \mathbb{R}) \cong H_1(\mathbb{R}^3 \setminus K; \mathbb{R}).
\] (2.10)

By the theorem of universal coefficients, page 198 of [17], \(H^1(K; \mathbb{R}) \cong \text{Hom}_\mathbb{R}(H_1(K; \mathbb{R}), \mathbb{R})\). Then, it follows that,

\[
\dim H_1(K; \mathbb{R}) = \dim H_1(\mathbb{R}^3 \setminus K; \mathbb{R}) = m.
\] (2.11)

We denote,

\[
\Lambda := \mathbb{R}^3 \setminus K.
\]

We will prove in Corollary [2.4] that \(\{	ilde{\gamma}_k\}_{k=1}^m\) is a basis of \(H_1(\Lambda; \mathbb{R})\).

### 2.6 de Rham Cohomology of \(\Lambda\)

Let us define,

\[
G^{(j)}(x) := \text{curl} \frac{1}{4\pi} \int_{\gamma_j} \frac{1}{|x-y|} d\tilde{\gamma}_j := \text{curl} \frac{1}{4\pi} \int \frac{1}{|x - \gamma_j(t)|} \gamma_j(t) dt.
\] (2.12)

Then, \(\text{curl} G^{(j)}(x) = 0, x \in \mathbb{R}^3 \setminus \gamma_j\) and

\[
\int_{\gamma_k} G^{(j)} = \delta_{k,j}, j, k = 1, 2, \ldots, m.
\] (2.13)

Equation (2.12) is the law of Biot-Savart that gives the magnetic field created by a current in \(\gamma_j\) and (2.13) is Ampere’s law. For a proof see Satz 1.4, page 33, of [26].
PROPOSITION 2.3. \[ \left\{ [G^{(j)}]_{H^1_{\text{de R}}} (\Lambda) \right\}_{j=1}^m \] is a basis of \( H^1_{\text{de R}} (\Lambda) \).

Proof: We first prove that they are linearly independent. Suppose that \( \sum \alpha_j G^{(j)} = 0 \) then, \( \sum \alpha_j G^{(j)} = d\lambda \) for some 0-form \( \lambda \). Hence,

\[
\int \gamma_k \sum \alpha_j G^{(j)} = \alpha_k = 0.
\]

By de Rham’s Theorem (Theorem 4.17, page 154 of [41]) the dual space to \( H^1_{\text{de R}} (\Lambda) \) is isomorphic to \( H^1(\Lambda; \mathbb{R}) \) and vice versa. The isomorphisms are given by

\[
\langle [\alpha]_{H^1(\Lambda; \mathbb{R})}, [A]_{H^1_{\text{de R}}(\Lambda)} \rangle := \int A.
\]

Then, by (2.11)

\[
dim H^1_{\text{de R}} (\Lambda) = \dim H^1(\Lambda; \mathbb{R}) = m,
\]

and this proves the Proposition.

COROLLARY 2.4. \( \left\{ [\hat{\gamma}_r]_{H^1(\Lambda; \mathbb{R})} \right\}_{r=1}^m \) is a basis of \( H^1(\Lambda; \mathbb{R}) \).

Proof: By (2.13) \( \left\{ [\hat{\gamma}_r]_{H^1(\Lambda; \mathbb{R})} \right\}_{r=1}^m \) is the dual basis -in the sense of de Rham’s Theorem- to the basis \( \left\{ [G^{(r)}]_{H^1_{\text{de R}}}(\Lambda) \right\}_{r=1}^m \) of \( H^1_{\text{de R}} (\Lambda) \).

PROPOSITION 2.5. Let \( A \) be a closed 1-form with continuous coefficients defined in \( \Lambda \) and such that

\[
\int \gamma_r A = 0, r = 1, 2, \ldots, m.
\]

Then, there is a continuously differentiable 0-form, \( \lambda \), such that \( A = d\lambda \). Moreover, we can take \( \lambda(x) := \int_{C(x_0, x)} A \) where \( x_0 \) is any fixed point in \( \Lambda \) and \( C(x_0, x) \) is any curve from \( x_0 \) to \( x \).

Proof: By Theorem 12, page 68, of [8] there is a regularization \( R(\epsilon) \) and an operator \( \Gamma(\epsilon) \) such that if \( \alpha \) is a continuous \( k \)-form on \( \Lambda \), \( R\alpha \) is a \( C^\infty \) \( k \)-form on \( \Lambda \) and \( \Gamma \alpha \) is a continuous \( (k - 1) \)-form on \( \Lambda \). Moreover, \( \lim_{\epsilon \to 0} R\alpha = \alpha \) uniformly on compact sets in \( \Lambda \). Furthermore,

\[
R\alpha - \alpha = b\Gamma \alpha + b\alpha,
\]

where \( b\alpha := (-1)^{\text{grade} \alpha - 1} d \). Multiplying (2.14) on the left by \( b \) and applying it to \( b\alpha \) we prove that \( RB = bR \). As \( A \) is closed, it follows from (2.14) that \( RA - A = b\Gamma A \). In particular, this implies that \( b\Gamma A \) is continuous. Let \( C \) be a closed curve. Then, by Stokes theorem,
\[
\int_C b\Gamma A = \lim_{\epsilon \to 0} \int_C Rb\Gamma A = \lim_{\epsilon \to 0} \int_C bR\Gamma A = 0,
\]
and then,
\[
\int_C RA = \int_C A,
\]
and in particular,
\[
\int_{\tilde{\gamma}_r} RA = \int_{\tilde{\gamma}_r} A = 0, \quad r = 1, 2, \cdots, m.
\]
As \( RA \) is \( C^\infty \) and closed, and since \( \{ [\tilde{\gamma}_r]_{H_1(\Lambda; \mathbb{R})} \}_{r=1}^m \) is a basis of \( H_1(\Lambda; \mathbb{R}) \) it follows from de Rham’s Theorem (Theorem 4.17, page 154, [11]) that there is an infinitely differentiable 0–form \( \alpha \) such that \( RA = b\alpha \). But then, using Stokes theorem again,
\[
\int_C RA = \int_C b\alpha = 0,
\]
and we obtain that,
\[
\int_C A = 0,
\]
for any closed curve \( C \) and we can define \( \lambda := \int_{C(x_0, x)} A \). Clearly, \( A = d\lambda \).

\[
\square
\]

Recall that \( \{ K_j \}_{j=1}^L \) is the set of connected components of \( K \). For each \( j \in \{ 1, 2, \cdots, L \} \) we choose a \( x_j \) in the interior of \( K_j \). We define the vector,

\[
D_j := -\text{grad} \frac{1}{4\pi \| x - x_j \|}, \quad x \in \mathbb{R}^3 \setminus \{ x_j \}, \tag{2.15}
\]

and according to our convention, we denote by the same symbol the associated 2–form. Note that \( \text{div} D_j(x) = dD_j = -\Delta \frac{1}{4\pi \| x - x_j \|} = 0, \quad x \neq x_j, \quad j = 1, 2, \cdots, L \) and that,

\[
|D_j(x)| \leq C(1 + |x|)^{-2}, \quad x \in \Lambda. \tag{2.16}
\]

For any \( r > 0 \) such that \( K \subset B_r^{\mathbb{R}^3}(0) \) we denote,

\[
\Lambda_r := \Lambda \cap B_r^{\mathbb{R}^3}(0), \quad \Lambda_\infty := \Lambda.
\]

**Proposition 2.6.** \( \{ [D_j]_{H^2_{\text{de R}}(\Lambda_r)} \}_{j=1}^L \) is a basis of \( H^2_{\text{de R}}(\Lambda_r) \) for \( r \leq \infty \).

**Proof:** Let us first consider the case \( r = \infty \). As in the proof of (2.11) we prove that

\[
\dim H_0(K; \mathbb{R}) = \dim H_2(\Lambda; \mathbb{R}).
\]

But by Proposition 9.6, page 48 of [16]

\[
H_0(K; \mathbb{R}) \cong \oplus_{j=1}^L \mathbb{R}.
\]
Moreover, by de Rham’s Theorem (Theorem 4.17, page 154 of [41])

\[ H^2_{\text{de R}}(\Lambda) \cong H_2(\Lambda; \mathbb{R}). \]  

(2.17)

Then,

\[ \dim H^2_{\text{de R}}(\Lambda) = \dim H_2(\Lambda; \mathbb{R}) = L. \]

(2.18)

Let us now consider \( r < \infty \). We define, \( f : \Lambda \rightarrow \Lambda_r \)

\[
  f(x) := \begin{cases}
    r_1 \frac{x}{|x|}, & \text{if } |x| \geq r_1, \\
    x, & \text{if } |x| \leq r_1,
  \end{cases}
\]

and \( H(x, t) : (\Lambda \times [0, 1]) \rightarrow \Lambda \)

\[
  H(x, t) := \begin{cases}
    x + t(r_1 \frac{x}{|x|} - x), & \text{if } |x| \geq r_1, \\
    x, & \text{if } |x| \leq r_1,
  \end{cases}
\]

where \( r_1 < r \) and \( K \subset B^3_{r_1}(0) \). Let \( l \) be the inclusion \( l : \Lambda_r \hookrightarrow \Lambda \). Then as \( l \circ f(x) = l \circ H(x, 1) = H(x, 1) \) and \( H(x, 0) = I(x) \), we have that \( l \circ f \) is homotopic to the identity. Let us denote by \( H(x, t) \) the restriction of \( H(x, t) \) to \( \Lambda_r \). Then, \( f \circ l(x) = H(l(x), 1) = H(x, 1) \), and as \( H(x, 0) = I(x) \) we also have that \( f \circ l \) is homotopic to the identity. Hence, by Theorem 11.3, page 59 [16] the inclusion \( l \) induces an isomorphism in homology. In particular, \( H_2(\Lambda_r; \mathbb{R}) \cong H_2(\Lambda; \mathbb{R}) \) and then,

\[ \dim H_2(\Lambda_r; \mathbb{R}) = \dim H_2(\Lambda; \mathbb{R}) = L. \]  

(2.19)

It follows from Stoke’s theorem and as \(-\Delta \frac{1}{4\pi |x-x_j|} = \text{div} \, D_j(x) = \delta(x - x_j) \) that,

\[ \int_{\partial K_i} D_j = \int_{\partial B^3_{\rho}(x_i)} D_j = \delta_{i,j}, \]

(2.20)

for \( \rho \) small enough and \( i, j = 1, 2, \ldots, L \). This easily implies that the set \( \left\{ [D_j]_{H^2_{\text{de R}}(\Lambda)} \right\}_{j=1}^L \) is linearly independent.

**Lemma 2.7.** Suppose that \( \{ [S_j]_{H_2(\Lambda_r; \mathbb{R})} \}_{j=1}^L \) is a basis of \( H_2(\Lambda_r; \mathbb{R}) \) for \( r \leq \infty \). Let \( D \) be a closed 2-form with continuous coefficients in \( \overline{\Lambda}_r \). Then,

\[ \int_{\partial K_j} D = 0, \forall j \in \{1, 2, \ldots, L\} \iff \int_{S_j} D = 0, \forall j \in \{1, 2, \ldots, L\}. \]

(2.21)

**Proof:** Denote \( K_{j, \varepsilon} := \{ x \in \mathbb{R}^3 : \text{dist}(x, K_j) < \varepsilon \} \) where \( \varepsilon \) is so small that the tubular neighborhood theorem applies and let \( R \) be the regularization operator. Suppose that the left side of (2.21) holds. Then, as \( D \) is closed we prove using the Stokes theorem that,
\[ \int_{\partial K_{j,\varepsilon}} RD = 0, \forall j \in \{1, 2, \cdots, L\}. \]

As \( RD \) is \( C^\infty \) and closed, since \( bR = Rb \), there are coefficients \( \lambda_j, j = 1, 2, \cdots, L \) and a 1-form \( \alpha \) such that,

\[ RD = \sum_{j=1}^{L} \lambda_j D_j + d\alpha. \]

Then, it follows from (2.20) (with \( K_{j,\varepsilon} \) instead of \( K_j \)) and Stoke’s theorem that

\[ 0 = \int_{\partial K_{j,\varepsilon}} RD = \lambda_j, \]

and we obtain that

\[ RD = d\alpha. \]

Furthermore, using the regularization operator and Stoke’s theorem we prove that,

\[ \int_{S_j} D = \int_{S_j} RD = \int_{S_j} d\alpha = 0, j \in \{1, 2, \cdots, L\}. \]

Assume now that \( \int_{S_j} D = 0, j \in \{1, 2, \cdots, L\} \). We prove as above that, \( \int_{S_j} RD = 0, j \in \{1, 2, \cdots, L\} \), and by de Rham’s Theorem (Theorem 4.17, page 154 of [11]) there is a 1-form \( \alpha \) such that

\[ RD = d\alpha. \]

Hence,

\[ \int_{\partial K_{j,\varepsilon}} D = \int_{\partial K_{j,\varepsilon}} RD = \int_{\partial K_{j,\varepsilon}} d\alpha = 0, \]

and then,

\[ \int_{\partial K_{j}} D = \lim_{\varepsilon \to 0} \int_{\partial K_{j,\varepsilon}} D = 0, j \in \{1, 2, \cdots, L\}. \]

### 3 Magnetic Field and Magnetic Potentials

In this section we introduce the class of magnetic fields that we consider and we construct a class of associated magnetic potentials with nice behaviour at infinity that will allows us to solve our scattering problems.

**DEFINITION 3.1.** We say that a form \( B \) in \( \overline{K} \) is continuous in a neighborhood of \( \partial K \) if there is a \( \varepsilon > 0 \) such that the coefficients of \( B \) are continuous in \( \overline{K} \cap K_\varepsilon \) where \( K_\varepsilon := \{ x \in \mathbb{R}^3 : \text{dist}(x, K) < \varepsilon \} \).

Below we assume that the magnetic field, \( B \), is a 2-form that is continuous in a neighborhood of \( \partial K \) and satisfies

\[ \int_{\partial K_j} B = 0, j \in \{1, 2, \cdots, L\}. \quad (3.22) \]
This condition means that the total contribution of magnetic monopoles inside each component \( K_j \) of the obstacle is 0. In a formal way we can use Stokes theorem to conclude that

\[
\int_{\partial K_j} B = 0 \iff \int_{K_j} \text{div} B = 0, \ j \in \{1, 2, \cdots, L\}.
\]

As \( \text{div} B \) is the density of magnetic charge, \( \int_{\partial K_j} B \) is the total magnetic charge inside \( K_j \), and our condition (3.22) means that the total magnetic charge inside \( K_j \) is zero, this condition in fulfilled if there is no magnetic monopole inside \( K_j, j \in \{1, 2, \cdots, L\} \).

**THEOREM 3.2.** Let \( B \) be a 2-form in \( L^p_{\text{loc}}(\Omega^2(\overline{\Lambda})) \), \( p \geq 2 \) that is continuous in a neighborhood of \( \partial K \) and satisfies (3.22). Suppose that the restriction of \( B \) to \( \Lambda \) is closed (\( dB|_{\Lambda} = 0 \)) as a distribution (or current [35]). Then, \( B \) has an extension to a closed 2-form \( \overline{B} \in L^p_{\text{loc}}(\Omega^2(\mathbb{R}^n)) \) such that, \( \overline{B}|_{\Lambda} = B \).

**Proof:** Let us denote \( M := \overline{\Lambda} \cap B, r < \infty. M \) is a compact manifold. We denote by \( B_M \) the restriction of \( B \) to \( M \). As \( dB|_{\Lambda} = 0 \), it follows from Green’s formula (Proposition 2.12, page 60, [35]) that

\[
\langle \langle B_M, \delta \eta \rangle \rangle = 0, \ \forall \eta \in C^0_{0\Omega^3}(M).
\]  
(3.23)

We denote (Definition 2.4.1, page 80 [35])

\[
C^k(M) := \{ \delta \eta|\eta \in H^1_{\Omega^{k+1}}(M) \},
\]

and (Definition 2.2.1, page 67 [35])

\[
H^1_{\Omega^{k}}(M) := \{ \eta \in H^1_{\Omega^{k}}(M)|\eta = 0 \}.
\]

Let us recall (page 27 [35]) that given \( \eta \in \Omega^3(M) \) and tangent vectors \( v_i \in T_x(M), x \in \partial M, i \in \{1, 2, 3\}, \)

\[
\tau \eta(v_1, v_2, v_3) = \eta(v_1^\parallel, v_2^\parallel, v_3^\parallel),
\]

where \( v_i^\parallel \) is the projection of \( v_i \) into \( T_x(\partial M) \). As \( \eta \) is a multi-linear function and \( \{v_1, v_2, v_3^\parallel\} \) are linearly dependent,

\[
\tau \eta = 0.
\]

By the definition in page 27 [35],

\[
\pi \eta := \eta - \tau \eta = \eta.
\]

It follows that

\[
\pi \eta = \eta, \ \eta \in H^1_{\Omega^3}(M).
\]

Let \( \eta \in H^1_{\Omega^3}(M) \), then there exists \( f \in W^{1,2}(M) \) such that

\[
\eta|_M = f|_M dx^1 \wedge dx^2 \wedge dx^3.
\]
As \( n\eta = \eta = 0 \), it follows that \( f|_{\partial M} = 0 \) in trace sense. Hence (Theorem 4.7.1, page 330, [36]), \( f \) can be approximated in the \( W^{1,2}(M) \) norm by functions in \( C_0^\infty \left( \overset{\circ}{M} \right) \), and then \( \eta \) can be approximated in the \( H^1\Omega^3(M) \) norm by forms in \( \Omega^3(M) \) with compact support. Whence, it follows from (3.23) that

\[
\langle \langle BM, \delta \eta \rangle \rangle = 0, \quad \forall \eta \in C^2(M).
\]  

(3.24)

By Corollary 2.4.9, page 87 [35]

\( BM = d\alpha + \delta\beta + de + \gamma \in \mathcal{E}^2(M) \oplus C^2(M) \oplus L^2\mathcal{H}_{\text{ext}}^2(M) \oplus \mathcal{H}_3^1(M), \)  

(3.25)

where (Definition 2.4.1, page 80 [35])

\( \mathcal{E}^k(M) := \{ d\alpha | \alpha \in H^1\Omega^{k-1}_D(M) \} \)

and (Definition 2.2.1, page 67 [35])

\( H^1\Omega^k_D(M) := \{ \eta \in H^1\Omega^k(M) | t\eta = 0 \} \).

Furthermore (page 86 [35]),

\( \mathcal{H}^k_{\text{ext}}(M) := \{ \eta \in \mathcal{H}^k(M) | \eta = de \} \),

and (Definition 2.2.1, page 67 [35])

\( \mathcal{H}^k(N) := \{ \eta \in H^1\Omega^k_0(M) | d\eta = 0, \delta\eta = 0 \} \).

are the harmonic fields, and

\( \mathcal{H}^k_N(M) := \mathcal{H}^k(M) \cap H^1\Omega^k_N(M). \)

Note that Theorem 2.2.7, page 72 [35] implies that \( \mathcal{H}^2_N(M) \) consists of \( C^\infty \) forms. Furthermore by Lemma 2.4.11 page 90 [35] we can choose \( \alpha \in W^{1,p}\Omega^1_D(M) \), and by Theorem 2.4.8, page 86 and Theorems 2.2.6 and 2.2.7, page 72 [35] \( \epsilon \in W^{1,p}\Omega^1_N(M) \). Moreover, the decomposition (3.25) is orthogonal in \( L^2(M) \), and then by (3.24) \( \delta\beta = 0 \).

Let \( R \) be the regularization operator in \( \Lambda_{\epsilon} = \overset{\circ}{M} \). Then, as in the proof of Lemma 2.7 we prove that

\[
\int_{\partial K_j,\epsilon} R B = 0.
\]

Hence,

\[
0 = \int_{\partial K_j,\epsilon} R B = \int_{\partial K_j,\epsilon} d(R\alpha + R\epsilon) + \int_{\partial K_j,\epsilon} R\gamma = \int_{\partial K_j,\epsilon} R\gamma.
\]

Then, \( \int_{\partial K_j,\epsilon} R\gamma = 0, \quad j \in \{1, 2, \cdots, L\} \) and when the parameter of the regularization tends to zero we obtain \( \int_{\partial K_j,\epsilon} \gamma = 0, \quad j \in \{1, 2, \cdots, L\} \).

As \( \gamma \) is harmonic it is closed and it follows from Stokes theorem that
$$\int_{\partial K_j} \gamma = 0, j \in \{1, 2, \ldots, L\}.$$ 

Then, by Lemma 2.7, $$\int_{S_j} \gamma = 0, j \in \{1, 2, \ldots, L\}.$$ By de Rham’s Theorem, $$\gamma = d\lambda, \lambda \in \Omega^1(\mathcal{O}).$$ Denote $$M_\varepsilon := \{x \in M : \text{dist}(x, \partial M) \geq \varepsilon\}$$. Let $$\gamma_\varepsilon$$ be the restriction of $$\gamma$$ to $$M_\varepsilon$$. Then $$\gamma_\varepsilon$$ is exact and by Lemma 3.2.1, page 119 [35], and its proof, $$\gamma_\varepsilon = d\omega_\varepsilon$$ with $$\omega_\varepsilon \in H^1\Omega^1(M_\varepsilon)$$ and

$$\|\omega_\varepsilon\|_{H^1\Omega^1(M_\varepsilon)} \leq C\|\gamma_\varepsilon\|_{L^2\Omega^2(M_\varepsilon)} \leq C\|\gamma\|_{L^2\Omega^2(M)},$$

where the constant $$C$$ can be taken independent of $$\varepsilon$$ for $$0 < \varepsilon < \varepsilon_0$$ small enough. Let us denote by $$\Lambda^k(M), \Lambda^k(M_\varepsilon),$$ respectively, the exterior $$k$$–form bundle of $$M, M_\varepsilon$$ (see Definition 1.3.8 in page 39 of [35]). For any vector bundle, $$\mathcal{F},$$ over a manifold $$N$$ we denote by $$\Gamma(\mathcal{F})$$ the space of all smooth sections of $$\mathcal{F}$$ (see Definition 1.1.9, page 17 of [35]). Note that the norm, $$C_1,$$ of the trace operator (Theorem 1.3.7, page 38 [35]) from $$H^1\Omega^1(M_\varepsilon)$$ into $$\Gamma(\Lambda^1(M_\varepsilon)|_{\partial M_\varepsilon})$$ can be taken independent of $$\varepsilon$$ for $$0 < \varepsilon < \varepsilon_0$$. By Green’s formula and as $$\delta \gamma_\varepsilon = 0$$,

\begin{equation}
\langle \langle \gamma_\varepsilon, \gamma_\varepsilon \rangle \rangle = \langle \langle d\omega_\varepsilon, \gamma_\varepsilon \rangle \rangle = \int_{\partial M_\varepsilon} t\omega_\varepsilon \wedge *n_\gamma_\varepsilon. \tag{3.26}
\end{equation}

But as $$\|t\omega_\varepsilon\|_{L^2(\Lambda^1(M_\varepsilon)|_{\partial M_\varepsilon})} \leq C_1 \|\omega_\varepsilon\|_{H^1\Omega^1(M_\varepsilon)} \leq C_1 C \|\gamma_\varepsilon\|_{L^2\Omega^2(M_\varepsilon)} \leq C_1 C \|\gamma\|_{L^2\Omega^2(M)},$$

and

$$\lim_{\varepsilon \to 0} \|n_\gamma_\varepsilon\|_{L^2(\Lambda^1(M_\varepsilon)|_{\partial M_\varepsilon})} = 0,$$

it follows from (3.26) and Schwarz inequality that

$$\|\gamma\|^2_{L^2\Omega^2(M)} = \lim_{\varepsilon \to 0} \|\gamma_\varepsilon\|^2_{L^2\Omega^2(M)} = 0.$$

Then $$\gamma = 0$$ and we have that

$$B_M = dA_M \tag{3.27}$$

where $$A_M := \alpha + \varepsilon \in W^{1,p}\Omega^1(\mathcal{O}).$$ It follows from Theorem 4.2.2, page 311 [36] that there is $$\overline{A_M} \in W^{1,p}\Omega^1(\overline{B^3_r(0)})$$ such that $$\overline{A_M}|_M = A_M.$$ We define

$$\overline{B}(x) = \begin{cases} 
    d\overline{A_M}(x), & \text{if } x \in B^3_r(0), \\
    B(x), & \text{if } x \in \mathbb{R}^3 \setminus B^3_r(0). 
\end{cases} \tag{3.28}$$

Hence, $$\overline{B}$$ is the required extension.

Recall that the functions $$\{\hat{\gamma}_j\}_{j=1}^m$$ where defined in (2.7). We introduce now a function that gives the magnetic flux across surfaces that have $$\{\hat{\gamma}_j\}_{j=1}^m$$ as their boundaries.
DEFINITION 3.3. The flux, \( \Phi \), is a function \( \Phi : \{ \hat{\gamma}_j \}_{j=1}^m \to \mathbb{R} \).

We now define a class of magnetic potentials with a given flux.

DEFINITION 3.4. Let \( B \in L^p(\Omega; \mathbb{R})^2 \), \( p > 3 \), be a closed 2- form that is continuous in a neighborhood of \( \partial K \) where \( K \) is as in Assumption 2.1. Assume, furthermore, that (3.22) holds. We denote by \( A_{\Phi}(B) \) the set of all continuous 1- forms in \( \Lambda \) that satisfy.

1. 

\[
|A(x)| \leq C \frac{1}{1 + |x|}, \quad a(r) := \max_{x \in \Lambda, |x| \geq r} \{|A(x) \cdot \hat{x}|\} \in L^1(0, \infty).
\]

(3.29)

2. 

\[
\int_{\hat{\gamma}_j} A = \Phi(\hat{\gamma}_j), \quad j \in \{1, 2, \cdots, m\}.
\]

(3.30)

3. 

\[
dA|_{\Lambda} = B|_{\Lambda}.
\]

(3.31)

The definition of the flux \( \Phi \) depends on the particular choice of the curves \( \{\hat{\gamma}_j\}_{j=1}^m \). However, the class \( A_{\Phi}(B) \) is independent of this particular choice as we prove below.

Recall that by Corollary 2.4 \( \beta := \{[\hat{\gamma}_j]_{H_1(\Lambda; \mathbb{R})}\}_{j=1}^m \) is a basis of \( H_1(\Lambda; \mathbb{R}) \). Let \( \beta' := \{[C_j]_{H_1(\Lambda; \mathbb{R})}\}_{j=1}^m \) be another basis of \( H_1(\Lambda; \mathbb{R}) \). We define \( \Phi_{\beta'} : \{C_j\}_{j=1}^m \to \mathbb{R} \) as follows. As \( \beta \) is a basis of \( H_1(\Lambda; \mathbb{R}) \) there are real numbers \( b_j \) and chains \( \sigma_j \) such that

\[
C_j = \sum_{i=1}^m b_j^i \hat{\gamma}_i + \partial \sigma_j.
\]

(3.32)

We define,

\[
\Phi_{\beta'}(C_j) := \sum_{i=1}^m b_j^i \Phi(\hat{\gamma}_i) + \int_{\sigma_j} B.
\]

(3.33)

We denote by \( A_{\Phi_{\beta'}}(B) \) the set of continuous 1- forms \( A \) in \( \bar{\Lambda} \) that satisfy 1 and 3 of Definition 3.3 and moreover,

\[
\int_{C_j} A = \Phi_{\beta'}(C_j), \quad j = 1, 2, \cdots, m.
\]

PROPOSITION 3.5. \( A_{\Phi_{\beta'}}(B) = A_{\Phi}(B) \).

Proof: Let \( A \in A_{\Phi}(B) \). Then, by (3.32)

\[
\int_{C_j} A = \sum_{i=1}^m b_j^i \int_{\hat{\gamma}_i} A + \int_{\sigma_j} dA = \Phi_{\beta'}(C_j), \quad j = 1, 2, \cdots, m,
\]

and it follows that \( A \in A_{\Phi_{\beta'}}(B) \).
Suppose now that $A \in A\Phi(B)$. As $\beta$ and $\beta'$ are basis, the numbers $b^i_j$, $i, j = 1, 2, \cdots, m$ determine an invertible matrix. We denote by $\tilde{b}^i_j$ the entries of the inverse matrix. Hence,

$$\tilde{\gamma}_i = \sum_{j,s=1}^{m} \tilde{b}^i_j b^j_s \gamma_s = \sum_{j=1}^{m} \tilde{b}^i_j (C_j - \partial \sigma_j),$$

and then by (3.33),

$$\int_{\tilde{\gamma}_i} A = \sum_{j=1}^{m} \tilde{b}^i_j \left( \Phi_{\beta'}(C_j) - \int_{\sigma_j} B \right) = \Phi(\gamma_i).$$

This implies that $A \in A\Phi(B)$.

By Stoke’s theorem the circulation $\int_{\tilde{\gamma}_j} A$ of a potential $A \in A\Phi(B)$ represents the flux of the magnetic field $B$ in any surface whose boundary is $\tilde{\gamma}_j$, $j = 1, 2, \cdots, m$. As the magnetic field is $a$ priori known outside the obstacle, it is natural to specify the magnetic potentials fixing fluxes of the magnetic field in surfaces inside the obstacle. This is accomplished fixing the circulations $\int_{\tilde{\gamma}_j} A$ instead of the circulations $\int_{\gamma_j} A$, as we prove below. Recall that $\tilde{\gamma}_j$ is defined in (2.5). With $\varepsilon$ as in (2.6) we define,

$$S_j := \left\{ \tilde{\gamma}_j(t) + s \frac{s}{2} N(\tilde{\gamma}_j(t)) \mid t \in [0, 1], s \in [0, 1] \right\}.$$

We give $S_j$ the structure of an oriented surface with boundary $\tilde{\gamma}_j - \tilde{\gamma}_j$. By Stoke’s theorem and regularizing we prove that,

$$\int_{\tilde{\gamma}_j} A = \int_{\gamma_j} A - \int_{S_j} B.$$

We define the fluxes $\hat{\Phi} : \{\tilde{\gamma}_j\}_{j=1}^{m} \rightarrow \mathbb{R}$ accordingly,

$$\hat{\Phi}(\tilde{\gamma}_j) = \Phi(\tilde{\gamma}_j) - \int_{S_j} B.$$

We denote by $\hat{A}_\Phi(B)$ the set of continuous 1-forms, $A$, in $\mathbb{X}$ that satisfy 1 and 3 of Definition 3.4 and moreover,

$$\int_{\tilde{\gamma}_j} A = \hat{\Phi}(\tilde{\gamma}_j), j = 1, 2, \cdots, m.$$

**PROPOSITION 3.6.** $A_{\Phi}(B) = A_{\Phi}(B)$.

*Proof:* Let $A \in A_{\Phi}(B)$. By Stoke’s theorem and regularizing,

$$\int_{\tilde{\gamma}_j} A = \int_{\gamma_j} A - \int_{S_j} B = \hat{\Phi}_j.$$

Then, $A \in A_{\Phi}(B)$. We prove in the same way that $A \in A_{\Phi}(B) \Rightarrow A \in A_{\Phi}(B)$.

Note that for 1-forms $A = \sum_{i=1}^{3} A_i dx^i$, $\delta A = - \sum_{i=1}^{3} \frac{\partial}{\partial x^i} A_i = - \text{div} A$. We use the definition of divergence of a vector field, $A$, as it is usual in vector calculus. The definition given in [35] differs from our’s in a $-\text{sign}$. 

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THEOREM 3.7. (Coulomb Potential) Let \( B \in L^p \Omega^2(\mathbb{R}^3), p > 3 \), be a closed 2–form that is continuous in a neighborhood of \( \partial K \), where \( K \) is as in Assumption 2.1. Assume, furthermore, that (3.22) holds and that for some \( r \) with \( K \subset B_r(0) \),
\[
|B(x)| \leq C(1 + |x|)^{-\eta}, |x| \geq r, \mu > 2.
\] (3.34)
Then, for any flux, \( \Phi \), there is a potential \( A_C \in A_\Phi(B) \) such that \( A_C = A_{(C,1)} + A_{(C,2)} \) where \( A_{(C,1)} \) is continuous on \( \overline{K} \), \( A_{(C,2)} \) is \( C^\infty \) on \( \overline{K} \), and \( \delta A_{(C,j)} = -\text{div} A_{(C,j)} = 0, j = 1, 2. \) Furthermore,
\[
|A_{(C,1)}(x)| \leq C(1 + |x|)^{-\min(2-\varepsilon, \mu-1)}, \forall \varepsilon > 0,
\] (3.35)
\[
|A_{(C,2)}(x)| \leq C(1 + |x|)^{-2}.
\] (3.36)

Proof: Let \( \overline{B} \) be the extension to \( \mathbb{R}^3 \) of \( B \) given by Theorem 3.2 by Proposition 2.6 of [22] and its proof we can take as \( A_{(C,1)} \) the Coulomb gauge of \( B \).

\[
A_{(C,1)} := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \times \overline{B}(y) \, dy,
\] (3.37)
where we use the notation of vector calculus. We define \( A_{(C,2)} \) as follows,
\[
A_{(C,2)} := \sum_{j=1}^m \left( \Phi(\gamma_j) - \int_{\gamma_j} A_{(C,1)} \right) G^{(j)},
\] (3.38)
where \( G^{(j)}, j = 1, 2, \cdots, m \) are defined in (2.12) and we used (2.13). Clearly, \( G^{(j)} \in C^\infty(\overline{\Lambda}) \) and \( |G^{(j)}(x)| \leq C(1+|x|)^{-2} \).}

Note that in \( \mathbb{R}^3 \) \( A_C \) is the Coulomb potential that corresponds to the magnetic field
\[
\overline{B} + \sum_{j=1}^m \left( \Phi(\gamma_j) - \int_{\gamma_j} A_{(C,1)} \right) \delta(x - \gamma_j) d\gamma_j,
\]
with
\[
\langle \delta(x - \gamma_j) d\gamma_j, \phi \rangle := \int_{\gamma_j} \phi d\gamma_j.
\]

The div-curl problem in exterior domains in the case of \( C^1 \) vector fields with Hölder continuous first derivatives was considered in [39].

LEMMA 3.8. (Gauge Transformations) Suppose that \( A, \tilde{A} \in A_\Phi(B) \). Then, there is a \( C^1 \) 0– form \( \lambda \) in \( \overline{\Lambda} \) such that, \( \tilde{A} - A = d\lambda \). Moreover, we can take \( \lambda(x) := \int_{C(x_0,x)} (\tilde{A} - A) \) where \( x_0 \) is any fixed point in \( \Lambda \) and \( C(x_0,x) \) is any curve from \( x_0 \) to \( x \). Furthermore, \( \lambda_\infty(x) := \lim_{r \to \infty} \lambda(rx) \) exists and it is continuous in \( \mathbb{R}^3 \setminus \{0\} \) and homogeneous of
order zero, i.e. \( \lambda_\infty(rx) = \lambda_\infty(x), r > 0, x \in \mathbb{R}^3 \setminus \{0\} \). Moreover,
\[
|\lambda_\infty(x) - \lambda(x)| \leq \int_{|x|}^\infty b(|x|), \text{for some } b(r) \in L^1(0, \infty),
\]
and \( |\lambda_\infty(x + y) - \lambda_\infty(x)| \leq C|y|, \forall x : |x| = 1, \text{and } \forall y : |y| < 1/2. \) 

**Proof:** The existence of \( \lambda \) follows from Proposition 2.5. The existence of \( \lambda_\infty \) and the first equation in (3.39) follow from condition 1 in Definition 3.4. The homogeneity follows from the definition. Denote \( G := \tilde{A} - A \). Take \( m > 1 \) such that \( K \subset B_m^{\mathbb{R}^3}(0) \). Suppose that \( |x| = 1 \) and that \( |y| < 1/2 \).

Denote, \( x' := mx, y' := m \frac{x + y}{|x + y|} - x' \). Then, \( \lambda_\infty(x) = \lambda_\infty(x'), \lambda_\infty(x + y) = \lambda_\infty(x' + y') \). Hence,
\[
\lambda_\infty(x + y) - \lambda_\infty(x) = \lambda_\infty(x' + y') - \lambda_\infty(x') = \int_{x'}^{x' + y'} G + \int_{x'}^\infty G - \int_{x'}^\infty G = \lim_{r \to \infty} \int_{rx}^{rm(x + y)/|x + y|} G,
\]
where we used Stoke’s theorem and \( dG = 0 \). Then,
\[
|\lambda_\infty(x' + y') - \lambda_\infty(x')| \leq \lim_{r \to \infty} \int_{rx}^{rm(x + y)/|x + y|} |G| \leq C|y|.
\]
This proves (3.39).

We now consider potentials that satisfy the flux condition modulo \( 2\pi \).

**DEFINITION 3.9.** Let \( B \) be as in Definition 3.4. We denote by \( A_{\Phi, 2\pi}(B) \) the set of all continuous 1-forms in \( \overline{X} \) that satisfy 1 and 3 of Definition 3.4 and moreover,
\[
\int_{\gamma_j} A = \Phi(\gamma_j) + 2\pi n_j(A), n_j(A) \in \mathbb{Z}, j \in \{1, 2, \ldots, m\}.
\]

Given \( A \in A_{\Phi, 2\pi}(B) \) we define,
\[
A_\Phi := A - \sum_{j=1}^m 2\pi n_j(A) G^{(j)}. \tag{3.40}
\]

By (2.13) \( A_\Phi \in A_\Phi(B) \).

Suppose that \( A, \tilde{A} \in A_{\Phi, 2\pi}(B) \). Then, \( A_\Phi, \tilde{A}_\Phi \in A_\Phi(B) \), and by Lemma 3.8
\[
\tilde{A}_\Phi - A_\Phi = d\lambda \tag{3.41}
\]
and it follows that,
\[
\tilde{A} - A = d\lambda + A_Z, \tag{3.42}
\]
where
\[
A_Z := \sum_{j=1}^m 2\pi (n_j(\tilde{A}) - n_j(A)) G^{(j)}. \tag{3.43}
\]

Let \( C \) be any closed curve in \( \Lambda \). Then, by Proposition 10.1 in Appendix B,
\[
C := \sum_{j=1}^m n_j(C) \hat{\gamma}_j + \partial \sigma, n_j(C) \in \mathbb{Z}.
\]
Hence,

\[ \int_C (\hat{A} - A) = 2\pi N, \text{ for some } N \in \mathbb{Z}. \]  \hspace{1cm} (3.44)

Whence, we can define the non-integrable factors [41],

\[ U_{\hat{A}, A}(x) := e^{i \int_{C(x_0, x)} (\hat{A} - A)} = e^{i (\lambda(x) + \int_{C(x_0, x)} A_z)}, \]  \hspace{1cm} (3.45)

where \( x_0 \) is any fixed point in \( \Lambda \) and \( C(x_0, x) \) is any curve in \( \Lambda \) from \( x_0 \) to \( x \). Clearly, \( U_{\hat{A}, A} \in C^1(\overline{\Lambda}) \). Moreover, if \( \hat{A}, A \in \mathcal{A}_\Phi(B) \) we have that \( A_\infty = 0 \), and then,

\[ U_{\hat{A}, A}(x) = e^{i \lambda(x)}, \hat{A}, A \in \mathcal{A}_\Phi(B). \]  \hspace{1cm} (3.46)

**Lemma 3.10.** Suppose that \( \hat{A}, A \in \mathcal{A}_{\Phi, 2\pi}(B) \). Then, for \( x \neq 0 \),

\[ \lim_{r \to \infty} U_{\hat{A}, A}(rx) = e^{i (\lambda_\infty(x) + C_{\hat{A}, A})}, \]  \hspace{1cm} (3.47)

with \( \lambda_\infty(x) := \lim_{r \to \infty} \lambda(rx) \) given by Lemma 3.5, with \( \lambda \) as in \( (3.41) \), and where \( C_{\hat{A}, A} \) is a real number that is independent of \( x \). Furthermore,

\[ \left| U_{\hat{A}, A}(x) - e^{i (\lambda_\infty(x) + C_{\hat{A}, A})} \right| \leq \int_0^\infty c(|x|), \text{ for some } c(r) \in L^1(0, \infty). \]  \hspace{1cm} (3.48)

Moreover, if \( \hat{A}, A \in \mathcal{A}_\Phi(B) \) we have that \( C_{\hat{A}, A} = 0 \).

**Proof:** Let \( r_0 \) be such that \( K \subset B_{r_0}^R(0) \). Take in \( (3.45) \) any curve from \( x_0 \) to \( r_0 \hat{x} \) and then the straight line from \( r_0 \hat{x} \) to \( r \hat{x} \) with \( r_0 \leq r < \infty \). By \( (2.12, 3.29, 3.39, 3.42) \)

\[ \lim_{r \to \infty} U_{\hat{A}, A}(rx) = e^{i \lambda_\infty(x)} \lim_{r \to \infty} e^{i \int_{C(x_0, x)} A_z} \]

and

\[ \left| U_{\hat{A}, A}(x) - e^{i \lambda_\infty(x)} \lim_{r \to \infty} e^{i \int_{C(x_0, x)} A_z} \right| \leq \int_0^\infty c(|x|), \text{ for some } c(r) \in L^1(0, \infty). \]

For any \( y \neq 0, y \neq \pm x \) let \( C(r \hat{x}, r \hat{y}) \) be the straight line from \( r \hat{x} \) to \( r \hat{y} \). Then,

\[ \lim_{r \to \infty} e^{i \int_{C(x_0, x)} A_z} = \lim_{r \to \infty} e^{-i \int_{C(x_0, x)} A_z} = 1, \]

and it follows that,

\[ \lim_{r \to \infty} e^{i \int_{C(x_0, x)} A_z} = \lim_{r \to \infty} e^{i \int_{C(x_0, x)} A_z} = e^{i C_{\hat{A}, A}} \]

for some \( C_{\hat{A}, A} \in \mathbb{R} \) that is independent of \( x \). If \( \hat{A}, A \in \mathcal{A}_\Phi(B) \), \( n_j(\hat{A}) = n_j(A) = 0, j = 1, 2, \ldots, m \) and hence, \( A_\infty = 0 \), what implies that \( C_{\hat{A}, A} = 0 \).
4 The Hamiltonian

Let us denote \( p := -i \nabla \). The Schrödinger equation for an electron in \( \Lambda \) with electric potential \( V \) and magnetic field \( B \) is given by

\[
i \hbar \frac{\partial}{\partial t} \phi = \frac{1}{2m} (p - \frac{q}{c} A)^2 + q V,
\]

where \( \hbar \) is Planck’s constant, \( p := \hbar p \) is the momentum operator, \( c \) is the speed of light, \( M \) and \( q \) are, respectively, the mass and the charge of the electron and \( A \) a magnetic potential with curl\( A = B \). To simplify the notation we multiply both sides of (4.1) by \( \frac{1}{\hbar} \) and we write Schrödinger’s equation as follows

\[
i \frac{\partial}{\partial t} \phi = \frac{1}{2m} (p - A)^2 \phi + V \phi,
\]

with \( m := M/\hbar, A = \frac{2}{m} A \) and \( V := \frac{q}{\hbar} V \). Note that since we write Schrödinger’s equation in this form our Hamiltonians below are the physical Hamiltonians divided by \( \hbar \). We fix the flux modulo \( 2\pi \) by taking \( A \in A_{\phi, 2\pi} \), where \( B := \frac{q}{2\hbar c} B \). Note that this corresponds to fixing the circulations of \( A \) modulo \( \frac{h c}{q} 2\pi \), or equivalently, to fixing the fluxes of the magnetic field \( B \) modulo \( \frac{2\pi}{c} 2\pi \).

For any open set, \( O \), we denote by \( H_s(O), s = 1, 2, \cdots \) the Sobolev spaces [1] and by \( H_{s, 0}(O) \) the closure of \( C_0^\infty (O) \) in the norm of \( H_s(O) \). We define the quadratic form,

\[
h_0(\phi, \psi) := \frac{1}{2m} (p\phi, p\psi), \ D(h_0) := H_{1, 0}(\Lambda).
\]

The associated positive operator in \( L^2(\Lambda) \) [23, 31] is \( \frac{1}{2m} \Delta_D \) where \( \Delta_D \) is the Laplacian with Dirichlet boundary condition on \( \partial \Lambda \). We define \( H(0, 0) := \frac{1}{2m} \Delta_D \). By elliptic regularity [2], \( D(H(0, 0)) = H_2(\Lambda) \cap H_{1, 0}(\Lambda) \).

For any \( A \in A_{\phi, 2\pi} (B) \) we define,

\[
h_A(\phi, \psi) := \frac{1}{2m} ((p - A)\phi, (p - A)\psi) = h_0(\phi, \psi) + \frac{1}{2m} ((-p\phi, A\psi) - (A\phi, p\psi)) + \frac{1}{2m} (A\phi, A\psi), \ D(h_A) = H_{1, 0}(\Lambda).
\]

As the quadratic form \(-\frac{1}{2m} ((p\phi, A\psi) + (A\phi, p\psi)) + \frac{1}{2m} (A\phi, A\psi) \) is \( h_0 \)- bounded with relative bound zero, \( h_A \) is closed and positive. We denote by \( H(A, 0) \) the associated positive self-adjoint operator [23, 31]. \( H(A, 0) \) is the Hamiltonian with magnetic potential \( A \). Note that as the operator \( \frac{1}{2m} (-2A_C \cdot p + A_C^2) \) is \( H(0, 0) \) compact we have that \( H(0, 0) = \frac{1}{m} A_C \cdot p + \frac{1}{2m} A_C^2 \) is self-adjoint on the domain of \( H(0, 0) \) and then,

\[
H(A_C, 0) = H(0, 0) = \frac{1}{m} A_C \cdot p + \frac{1}{2m} A_C^2, \ D(H(A_C, 0)) = H_2(\Lambda) \cap H_{1, 0}(\Lambda).
\]

The electric potential \( V \) is a measurable real-valued function defined on \( \Lambda \). We assume that \( |V| \) is \( h_0 \)- bounded with relative bound zero. Under this condition [23, 31] the quadratic form,

\[
h_{A,V}(\phi, \psi) := h_A(\phi, \psi) + (V\phi, \psi), \ D(h_{A,V}) = H_{1, 0}(\Lambda),
\]

is self-adjoint on the domain of \( H(0, 0) \) and then, \( H_{A,V}(0, 0) = H_{A,V}(0, 0) \).
is closed and bounded from below. The associated operator, \( H(A, V) \), is self-adjoint and bounded from below. \( H(A, V) \) is the Hamiltonian with magnetic potential \( A \) and electric potential \( V \). If furthermore, \( V = -\Delta_D \) compact, the operator 
\[
H(0, 0) - \frac{1}{m} A_C \cdot p + \frac{1}{2m} A_C^2 + V, \quad D(H(A_C, V)) = \mathcal{H}_2(\Lambda) \cap \mathcal{H}_{1,0}(\Lambda).
\]  
(4.7)

We will denote by \( U_{\tilde{A}, A} \) the operator of multiplication by \( U_{\tilde{A}, A}(x) \). See (3.45). Note that \( U_{\tilde{A}, A} \) is unitary in \( L^2(\Lambda) \) and that \( U_{\tilde{A}, A}^* \) is the operator of multiplication by \( U_{A, \tilde{A}}(x) \).

**Theorem 4.1.** Suppose that \( \tilde{A}, A \in \mathcal{A}_{\Phi, 2\pi}(B) \). Then \( H(\tilde{A}, V) \) and \( H(A, V) \) are unitarily equivalent,
\[
H(\tilde{A}, V) = U_{\tilde{A}, A} H(A, V) U_{\tilde{A}, A}^*, \quad D(H(\tilde{A}, V)) = U_{\tilde{A}, A} D(H(A, V)).
\]  
(4.8)

**Proof:** As \( U_{\tilde{A}, A} \) and \( U_{\tilde{A}, A}^* \) are bijections on \( \mathcal{H}_{1,0}(\Lambda) \) we have that
\[
h_{\tilde{A}, V}(\phi, \psi) = h_{A, V}\left(U_{\tilde{A}, A}^* \phi, U_{\tilde{A}, A}^* \psi\right), \quad \phi, \psi \in \mathcal{H}_{1,0}(\Lambda).
\]
Suppose that \( \phi \in D(H(\tilde{A}, V)) \). Then, for every \( \chi \in \mathcal{H}_{1,0}(\Lambda) \),
\[
(U_{\tilde{A}, A}^* H(\tilde{A}, V) \phi, \chi) = h_{A, V}(U_{\tilde{A}, A}^* \phi, \chi).
\]
This implies that \( U_{\tilde{A}, A}^* \phi \in D(H(A, V)) \) and that
\[
H(A, V) U_{\tilde{A}, A}^* \phi = U_{\tilde{A}, A}^* H(\tilde{A}, V) \phi.
\]
what proves the theorem.

**5 Scattering**

In the following assumptions we summarize the conditions on the magnetic field and the electric potential that we use. We denote by \( \Delta \) the self-adjoint realization of the Laplacian in \( L^2(\mathbb{R}^3) \) with domain \( \mathcal{H}_2(\mathbb{R}^3) \). Below we assume that \( V \) is \( \Delta \)-bounded with relative bound zero. By this we mean that the extension of \( V \) to \( \mathbb{R}^3 \) by zero is \( \Delta \)-bounded with relative bound zero. Using an extension operator from \( \mathcal{H}_2(\Lambda) \) to \( H_2(\mathbb{R}^3) \) we prove that this is equivalent to require that \( V \) is bounded from \( \mathcal{H}_2(\Lambda) \) into \( L^2(\mathbb{R}^3) \) with relative bound zero. We denote by \( \| \cdot \| \) the operator norm in \( L^2(\mathbb{R}^3) \).

**Assumption 5.1.** We assume that the magnetic field, \( B \), is a real-valued, bounded 2-form in \( \overline{\mathbb{R}^3} \), that is continuous in a neighborhood of \( \partial K \), where \( K \) satisfies Assumption 2.4 and furthermore,

1. \( B \) is closed : \( dB|_\Lambda \equiv \text{div}B = 0 \).
2. There are no magnetic monopoles in $K$:

$$
\int_{\partial K_j} B = 0, \ j \in \{1, 2, \ldots, L\}. \quad (5.1)
$$

3. 

$$
|B(x)| \leq C(1 + |x|)^{-\mu}, \ \text{for some } \mu > 2. \quad (5.2)
$$

4. $d * B|_\Lambda \equiv \text{curl } B$ is bounded and, 

$$
|\text{curl } B| \leq C(1 + |x|)^{-\mu}. \quad (5.3)
$$

5. The electric potential, $V$, is a real-valued function, it is $\Delta-$bounded, and 

\[ 
\| F(|x| < r)V(-\Delta + I)^{-1} \| \leq C(1 + |x|)^{-\alpha}, \ \text{for some } \alpha > 1. 
\] 

Note that (5.4) implies that $V$ is $h_0-$bounded with relative bound zero. Furthermore, condition (5.4) is equivalent to the following assumption [32]

\[ 
\| V(-\Delta + I)^{-1}F(|x| \geq r) \| \leq C(1 + |x|)^{-\alpha}, \ \text{for some } \alpha > 1. 
\] 

Condition (5.4) has a clear intuitive meaning, it is a condition on the decay of $V$ at infinity. However, in the proofs below we use the equivalent statement (5.5).

Let us define,

$$
H_0 := -\frac{1}{2m}\Delta, \ \mathcal{D}(H_0) = \mathcal{H}_2(\mathbb{R}^3).
$$

Let $J$ be the identification operator from $L^2(\mathbb{R}^3)$ onto $L^2(\Lambda)$ given by multiplication by the characteristic function of $\Lambda$. The wave operators are defined as follows,

$$
W_ \pm(A, V) := \lim_{t \to \pm \infty} e^{itH(A, V)} J e^{-itH_0}, \quad (5.6)
$$

provided that the strong limits exist. We first prove that they exist in the Coulomb gauge.

**PROPOSITION 5.2.** Suppose that $B$ and $V$ satisfy Assumption [5.1]. Then, the wave operators $W_ \pm(A_C, V)$ exist and are isometric.

**Proof:** Let $\chi \in C^\infty(\mathbb{R}^3)$ satisfy $\chi(x) = 0$ in a neighborhood of $K$ and $\chi(x) = 1$ for $|x| \geq r_0$ with $r_0$ large enough. Then, since $(1 - \chi(x))(H_0 + I)^{-1}$ is compact,

$$
W_ \pm(A_C, V) = \lim_{t \to \pm \infty} e^{itH(A_C, V)} \chi(x) e^{-itH_0}.
$$

By Duhamel’s formula, for $\phi \in \mathcal{D}(H_0), $
\[ W_{\pm}(A_C, V)\phi = \chi(x)\phi(x) + \int_0^{\pm\infty} i e^{itH(A_C, V)} [H(A_C, V)\chi(x) - \chi(x)H_0] \phi(x) \, dt. \] (5.7)

By Theorem 3.7 the proof that the integral in the right-hand side of (5.7) is absolutely convergent is standard. For example, it follows from Lemma 2.2 of [14] taking \( \phi = e^{imv \cdot x}\varphi \), with \( v \in \mathbb{R}^3, |v| \geq 4\eta > 0 \), and \( \hat{\varphi} \in C_0^\infty(B_{m\eta}^c(0)) \), what is a dense set in \( L^2(\mathbb{R}^3) \).

\[ \square \]

**Lemma 5.3. (Gauge Transformations)** Suppose that Assumption 5.1 is true. Then, for every \( A \in A_{\Phi,2\pi}(B) \) the wave operators \( W_{\pm}(A, V) \) exist and are isometric. Moreover, if \( \tilde{A} \in A_{\Phi,2\pi}(B) \), then,

\[ W_{\pm}(\tilde{A}, V) = e^{-iC_{\tilde{A},A}} U_{\tilde{A},A} W_{\pm}(A, V) e^{-i\lambda_\infty(\pm p)}. \] (5.8)

**Proof:** Since we already know that \( W_{\pm}(A_C, V) \) exist and are isometric it is enough to prove the gauge transformation formula (5.8). We argue as in the proof of Lemma 2.3 of [40]. By (4.8)

\[ W_{\pm}(\tilde{A}, V) = U_{\tilde{A},A} s- \lim_{t \to \pm \infty} e^{itH(A,V)} U_{A, \tilde{A}} J e^{-itH_0} = U_{\tilde{A},A} s- \lim_{t \to \pm \infty} e^{itH(A,V)} J e^{-i(\lambda_\infty(x) + C_{\tilde{A},A})} e^{-itH_0}, \]

where we used that by Lemma 3.10 and Rellich selection theorem \( U_{A, \tilde{A}} - e^{-i(\lambda_\infty(x) + C_{\tilde{A},A})} \) is a compact operator from \( D(H_0) \) into \( L^2(\mathbb{R}^3) \). We finish the proof of the lemma as in the proof of equation (2.29) of [40], using the second equation in (3.39).

The scattering operator is defined as

\[ S(A, V) := W_+^*(A, V) W_-(A, V). \]

By (5.8)

\[ S(\tilde{A}, V) = e^{i\lambda_\infty(\pm p)} S(A, V) e^{-i\lambda_\infty(\pm p)}, \tilde{A}, A \in A_{\Phi,2\pi}(B). \] (5.9)

**Definition 5.4.** We say that \( A \in A_{\Phi,2\pi}(B) \) is short-range if

\[ |A(x)| \leq C(1 + |x|)^{-1-\varepsilon}, \text{ for some } \varepsilon > 0. \] (5.10)

We denote the set of all short-range potentials in \( A_{\Phi,2\pi}(B) \) by \( A_{\Phi,2\pi,SR}(B) \).

Note that if \( \tilde{A}, A \in A_{\Phi,2\pi}(B) \) and \( \tilde{A} - A \) satisfies (5.10), \( \lambda_\infty \) is constant, and then,

\[ S(\tilde{A}, V) = S(A, V), \tilde{A}, A \in A_{\Phi,2\pi}(B) \text{ and } \tilde{A} - A \text{ satisfies (5.10)}. \] (5.11)
This implies that,
\[ S(A_{\Phi}, V) = S(A, V), \text{ for any } A \in A_{\Phi, 2\pi}(B), \]
where \( A_{\Phi} \) is defined in (5.10). Remark that (5.11) holds if \( \tilde{A}, A \in A_{\Phi, 2\pi, \text{SR}}(B) \).

We quote below the following result of [40] that we will often use.

**Lemma 5.5.** For any \( f \in C_0^\infty(B_{m\eta}(0)), 0 \leq \rho < 1 \), and for any \( j = 1, 2, \cdots \) there is a constant \( C_j \) such that
\[
\left\| F \left( |x - vt| > \frac{|vt|}{4} \right) e^{-itH_0} f \left( \frac{p - mv}{v^\rho} \right) F (|x| \leq |vt|/8) \right\| \leq C_j (1 + |vt|)^{-j},
\]
for \( v := |v| > (8\eta)^{1/(1-\rho)} \).

**Proof:** Corollary 2.2 of [40].

### 5.1 High-Velocity Estimates I. The Magnetic Potential

We denote,
\[
\Lambda_{\hat{v}} := \{ x \in \Lambda : x + \tau \hat{v} \in \Lambda, \forall \tau \in \mathbb{R} \}, \text{ for } \hat{v} \neq 0.
\]

\[
L_{A, \hat{v}}(t) := \int_0^t \hat{v} \cdot A(x + \tau \hat{v}) d\tau, -\infty \leq t \leq \infty.
\]

Remark that under translation in configuration or momentum space generated, respectively, by \( p \) and \( x \) we obtain

\[
e^{ip \cdot vt} f(x) e^{-ip \cdot vt} = f(x + vt),
\]
\[
e^{-imv \cdot x} f(p) e^{imv \cdot x} = f(p + mv),
\]
and, in particular,
\[
e^{-imv \cdot x} e^{-itH_0} e^{imv \cdot x} = e^{-imv^2 t/2} e^{-ip \cdot vt} e^{-itH_0}.
\]

The purpose of the obstacle \( K \) is to shield the incoming electrons from the magnetic field inside the obstacle. In order to separate the scattering effect of the magnetic potential from that of the magnetic field inside the obstacle \( K \), we consider asymptotic configurations that have negligible interaction with \( K \) for all times in the high-velocity limit. For any non-zero \( v \in \mathbb{R}^3 \) we take asymptotic configurations \( \phi \) with compact support in \( \Lambda_{\hat{v}} \). The free evolution boosted by \( \hat{v} \) is given by (5.18) and -to a good approximation- in the limit when \( v \to \infty \) with \( \hat{v} \) fixed this can be replaced (modulo an unimportant phase factor) by the classical translation \( e^{-ip \cdot vt} \). Then, in the high-velocity limit it is a good approximation to assume that the free evolution of our asymptotic configuration is given by \( e^{-ip \cdot vt} \phi_0 = \phi_0(x - vt) \), and as \( \phi_0 \) has support in \( \Lambda_{\hat{v}} \), it has negligible interaction with \( K \) for all times. Note that instead of boosting the observables we can boost the asymptotic configurations and consider the high-velocity asymptotic configurations
\[
\phi_{\hat{v}} := e^{imv \cdot x} \phi_0.
\]
LEMMA 5.6. Suppose that $B, V$ satisfy Assumption [5.7]. Let $\Lambda_0$ be a compact subset of $\Lambda_\varphi$, with $\nu \in \mathbb{R} \setminus \{0\}$. Then, for all $\Phi$ and all $A \in \mathcal{A}_{0,2}\nu(B)$ there is a constant $C$ such that,

$$
\left\| \left( e^{-i\nu \cdot x} W_\pm (A, V) e^{i\nu \cdot x} - e^{-iL_A \varphi (\pm \infty)} \right) \Phi \right\|_{L^2(\mathbb{R}^3)} \leq C \frac{1}{\nu} \| \Phi \|_{\mathcal{H}_2(\mathbb{R}^3)},
$$

(5.19)

and if moreover, $\text{div} A \in L^2_{\text{loc}}(\mathbb{R})$,

$$
\left\| \left( e^{-i\nu \cdot x} W_\pm (A, V) e^{i\nu \cdot x} - e^{iL_A \varphi (\pm \infty)} \right) \Phi \right\|_{L^2(\mathbb{R}^3)} \leq C \frac{1}{\nu^2} \| \Phi \|_{\mathcal{H}_2(\mathbb{R}^3)},
$$

(5.20)

for all $\Phi \in \mathcal{H}_2(\mathbb{R}^3)$ with support $\Phi \subset \Lambda_0$.

Proof: We follow the proof of Lemma 2.4 of [40]. We first give the proof in the case of the Coulomb potential $A_C$. We give the proof for $W_+ (A_C, V)$. The proof for $W_- (A_C, V)$ follows in the same way.

Let $g \in C^\infty_0(\mathbb{R}^3)$ satisfy $g(p) = 1, |p| \leq 1, g(p) = 0, |p| \geq 2$. Denote

$$
\tilde{\phi} := g(p/\nu^p) \phi, \quad \frac{1}{2} \leq \rho < 1.
$$

(5.21)

Then,

$$
\left\| \tilde{\phi} - \phi \right\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{\nu^{2p}} \| \phi \|_{\mathcal{H}_2(\mathbb{R}^3)}.
$$

(5.22)

Hence, it is enough to prove (5.19) for $\tilde{\phi}$.

By our assumption there is a function $\chi \in C^\infty(\mathbb{R}^3)$ such that $\chi \equiv 0$ in a neighborhood of $K$ and $\chi(x) = 1, x \in \{ x : x = y + \tau \hat{v}, y \in \text{support } \phi, \tau \in \mathbb{R} \} \cup \{ x : |x| \geq M \}$ for some $M$ large enough. We use the following notation,

$$
H_1 := \frac{1}{\nu} e^{-i\nu \cdot x} H_0 e^{i\nu \cdot x}, \quad H_2 := \frac{1}{\nu} e^{-i\nu \cdot x} H(A_C, V) e^{i\nu \cdot x}.
$$

(5.23)

Note that,

$$
\left( e^{-i\nu \cdot x} W_+ (A_C, V) e^{i\nu \cdot x} - \chi(x) e^{-iL_{A_C} \varphi (\infty)} \right) \tilde{\phi} = \lim_{t \to \infty} \left[ e^{itH_2} \chi(x) e^{-itH_1} - \chi(x) e^{-itL_{A_C} \varphi (t)} \right] \tilde{\phi}.
$$

(5.24)

Denote,

$$
P(t, \tau) := e^{itH_2} \left[ H_2 e^{-iL_{A_C} \varphi(t-\tau)} \chi(x) - e^{-iL_{A_C} \varphi(t-\tau)} \chi(x) (H_1 - \hat{v} \cdot A_C (x + (t-\tau)\hat{v})) \right] e^{-itH_1} \tilde{\phi}.
$$

(5.25)

Then, by Duhamel’s formula,

$$
\left[ e^{itH_2} \chi(x) e^{-itH_1} - \chi(x) e^{-itL_{A_C} \varphi (t)} \right] \tilde{\phi} = \int_0^t d\tau \ P(t, \tau).
$$

(5.26)

We designate,

$$
b(x, t) := A_C (x + t\hat{v}) + \int_0^t (\hat{v} \times B)(x + \tau\hat{v}) d\tau.
$$

(5.27)

For $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ with $f_t(x) := f(x, t) \in L^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ we define,

$$
\Xi_f(x, t) := \frac{1}{2m} \chi(x) \left[ -p \cdot f(x, t) - f(x, t) \cdot p + (f(x, t))^2 \right].
$$

(5.28)
We have that \([40]\),

\[ P(t, \tau) = T_1 + T_2 + T_3, \tag{5.29} \]

with

\[ T_1 := \frac{1}{v} e^{i\tau H_2} e^{-iL_{AC} \psi(x, t-\tau)} (\Xi_b(x, t-\tau) + \chi V(x)) e^{-i\tau H_1} \tilde{\phi}, \tag{5.30} \]

\[ T_2 := \frac{1}{2\nu v} e^{i\tau H_2} e^{-iL_{AC} \psi(x, t-\tau)} \left\{ -\left( \Delta \chi \right) + 2(p\chi) : p - 2b(x, t-\tau) \cdot (p\chi) \right\} e^{-i\tau H_1} \tilde{\phi}, \tag{5.31} \]

\[ T_3 := e^{i\tau H_2} e^{-iL_{AC} \psi(x, t-\tau)} ([p\chi] \cdot \tilde{v}) e^{-i\tau H_1} \tilde{\phi}. \tag{5.32} \]

Note that \([4], \text{equation (2.18)}\)

\[ \left\| \int_0^{t-\tau} d\nu (\tilde{v} \times B)(x + \nu \tilde{v}) F(|x - \tau \tilde{v}| \leq |\tau|/4) \right\|_{L^\infty(\mathbb{R}^3)} \leq C \frac{1}{(1 + |\tau|)^\mu - 1}, \tag{5.33} \]

\[ \left\| \int_0^{t-\tau} d\nu (\nabla \cdot (\tilde{v} \times B))(x + \nu \tilde{v}) F(|x - \tau \tilde{v}| \leq |\tau|/4) \right\|_{L^\infty(\mathbb{R}^3)} \leq \int_0^{t-\tau} d\nu (\tilde{v} \cdot \text{curl} B)(x + \nu \tilde{v}) \]

\[ F(|x - \tau \tilde{v}| \leq |\tau|/4) \right\|_{L^\infty(\mathbb{R}^3)} \leq C \frac{1}{(1 + |\tau|)^\mu - 1}. \tag{5.34} \]

Using Theorem \([3.7], \text{Lemma 5.5, 5.2, 5.3, 5.5, 5.33, 5.34}\) we prove as in the proof of Lemma 2.4 of \([40]\) that,

\[ \|T_1(\tau)\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{v} \frac{1}{(1 + |\tau|)^{(\min(2-\mu, \mu - 1, \alpha))}} \|\phi\|_{\mathcal{H}_2(\mathbb{R}^3)}, \tag{5.35} \]

\[ \|T_2(\tau)\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{v} \frac{1}{(1 + |\tau|)^j} \|\phi\|_{\mathcal{H}_2(\mathbb{R}^3)}, j = 1, 2, \ldots, \tag{5.36} \]

\[ \int_{-\infty}^{\infty} d\tau \|T_3(\tau)\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{v} \|\phi\|_{\mathcal{H}_2(\mathbb{R}^3)}. \tag{5.37} \]

For the reader’s convenience we estimate one of the terms. Denote by

\[ \eta(x, t) := \int_0^t (\tilde{v} \times B)(x + \tau \tilde{v}) d\tau. \tag{5.38} \]

Then, by Lemma \([5.5]\) and \([5.33]\),

\[ \left\| \frac{1}{\nu v} e^{-iL_{AC} \psi(x, t-\tau)} \eta(x, t - \tau) e^{-i\tau H_1} \cdot p\tilde{\phi} \right\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{v} \left[ \|\eta(x, t - \tau)\|_{L^\infty(\mathbb{R}^3)} \|F(|x - \tau \tilde{v}| \geq |\tau|/4) e^{-i\tau H_0 \tau/v} \right] \]

\[ g(\frac{p - p\chi}{v}) F(|x| \leq |\tau|/8) \|\phi\|_{\mathcal{H}_2(\mathbb{R}^3)} + \|\eta(x, t - \tau) F(|x - \tau \tilde{v}| \leq |\tau|/4) \|_{L^\infty(\mathbb{R}^3)} \|\phi\|_{\mathcal{H}_2(\mathbb{R}^3)} + \|F(|x| \geq |\tau|/8) \|p \cdot \tilde{v}\|_{L^2(\mathbb{R}^3)} \right] \]

\[ \leq \frac{C}{(1 + |\tau|)^\mu - 1} \|\phi\|_{\mathcal{H}_2(\mathbb{R}^3)}. \]

By \([5.26, 5.29, 5.33, 5.36, 5.37]\)

\[ \left\| e^{i\tau H_2} \chi(x) e^{-i\tau H_1} - \chi(x) e^{-iL_{AC} \psi(t)} \right\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{v} \|\phi\|_{\mathcal{H}_2(\mathbb{R}^3)}. \tag{5.39} \]
By (5.24) this proves (5.19) for \( A \). Given \( A \in A_{\Phi,2\pi}(B) \) we define \( A_\Phi \) as in (3.40). As \( A_\Phi \in A_\Phi(B) \), we prove that (5.19) holds for \( A_\Phi \) as in the proof of Lemma 2.4 of [40] using the formulae for change of gauge (5.8). Then, we prove that it is true for \( A \) using the gauge transformation formulae between \( A \) and \( A_\Phi \), note that in this case \( \lambda \equiv \lambda_\infty \equiv 0 \), observing that

\[
e^{-iC_{A,A_\Phi}} = e^{-i \int_{C(z_0,\epsilon)} A + \int_0^{\infty} \tau \cdot A_\Phi(x + \tau \nu) d\tau} = (U_{A,A_\Phi})^* e^{-i \int_0^{\infty} \tau \cdot A_\Phi(x + \tau \nu) d\tau},
\]

and using (3.22) with \( \lambda \equiv 0 \).

We now prove (5.20). Note that ([3], equation 2.12)

\[
(p - A(x))e^{-iL A,\phi(t)} = e^{-iL A,\phi(t)}
\]

Then, since \( \text{div} A \in L^2_{\text{loc}}(\Lambda) \) if follows from Sobolev’s imbedding theorem [1] that, \( \|e^{iL A,\phi(\pm \infty)}\|_{H^2(\mathbb{R}^3)} \leq C\|\phi\|_{H^2(\mathbb{R}^3)} \).

For simplicity we denote below \( W_{\pm}(A,V) \) by \( W_{\pm} \) and we define,

\[
W_{\pm,\nu} := e^{-im\nu \cdot x} W_{\pm} e^{im\nu \cdot x}.
\]

As the wave operators are isometric, \( W_{\pm,\nu}^* W_{\pm,\nu} = I \) and then,

\[
\left\| \left( W_{\pm,\nu}^* - e^{iL A,\phi(\pm \infty)} \right) \phi \right\|_{L^2(\mathbb{R}^3)} = \left\| W_{\pm,\nu} \phi - W_{\pm,\nu}^* W_{\pm,\nu} e^{iL A,\phi(\pm \infty)} \phi \right\|_{L^2(\mathbb{R}^3)} \\
\leq \left\| \left( W_{\pm,\nu} - e^{-iL A,\phi(\pm \infty)} \right) e^{iL A,\phi(\pm \infty)} \phi \right\|_{L^2(\mathbb{R}^3)} \leq C_\nu \|\phi\|_{H^2(\mathbb{R}^3)}.
\]

We now state the main result of this subsection.

**THEOREM 5.7. (Reconstruction Formula I)** Suppose that \( B,V \) satisfy Assumption 5.7. Let \( \Lambda_0 \) be a compact subset of \( \Lambda_\nu \), with \( \nu \in \mathbb{R} \setminus \{0\} \). Then, for all \( \Phi \) and all \( A \in A_{\Phi,2\pi}(B) \) there is a constant \( C \) such that,

\[
\left\| \left( e^{-im\nu \cdot x} S(A,V) e^{im\nu \cdot x} - e^{i \int_{-\infty}^{\infty} \tau \cdot A_\Phi(x + \tau \nu) d\tau} \right) \phi \right\|_{L^2(\mathbb{R}^3)} \leq C_{\nu} \|\phi\|_{H^2(\mathbb{R}^3)},
\]

\[
\left\| \left( e^{-im\nu \cdot x} S(A,V)^* e^{im\nu \cdot x} - e^{-i \int_{-\infty}^{\infty} \tau \cdot A_\Phi(x + \tau \nu) d\tau} \right) \phi \right\|_{L^2(\mathbb{R}^3)} \leq C_{\nu} \|\phi\|_{H^2(\mathbb{R}^3)},
\]

for all \( \phi \in H^2(\mathbb{R}^3) \) with support \( \phi \subset \Lambda_0 \).
Proof: We use the same notation as in the end of the proof of Lemma 5.16

First we prove (5.44) and (5.45) for $A_C$,

$$\left\| \left( e^{-im\cdot x} S(A_C, V) e^{im\cdot x} - e^{i \int_{-\infty}^{\infty} \hat{\psi}_C(x+\tau \hat{\psi}) d\tau} \right) \phi \right\|_{L^2(\mathbb{R}^3)} = \| W^*_+, \hat{\psi}_C \phi - W^*_-, \hat{\psi}_C e^{i(L_{A_C}, \psi(\infty) - L_{A_C}, \psi(-\infty))} \phi \|_{L^2(\mathbb{R}^3)} \leq \| \phi \|_{H^2(\mathbb{R}^3)}.$$  

The proof for $S(A_C, V)^*$ follows in the same way.

Now we prove (5.44) for $A \in \mathcal{A}_{\Phi, 2\pi}(B)$, the proof of (5.45) follows in the same way.

By (5.42), $S(A, V) = S(A_{\Phi}, V)$. From (5.40) it follows that

$$e^{i \int_{-\infty}^{\infty} \hat{\psi}_C A(x+\tau \hat{\psi}) d\tau} = e^{-iC_A A_{\Phi}} e^{iC_A A_{\Phi}} = 1,$$

and thus,

$$e^{i \int_{-\infty}^{\infty} \hat{\psi}_C A(x+\tau \hat{\psi}) d\tau} = e^{i \int_{-\infty}^{\infty} \hat{\psi}_C A(x+\tau \hat{\psi}) d\tau}. \quad (5.46)$$

Then it is enough to prove (5.44) for $A = A_C + \nabla \lambda$.

By (5.9), (5.17) and as $\lambda$ in homogenous of order zero,

$$\left\| \left( e^{-im\cdot x} S(A, V) e^{im\cdot x} - e^{i \int_{-\infty}^{\infty} \hat{\psi}_C A(x+\tau \hat{\psi}) d\tau} \right) \phi \right\|_{L^2(\mathbb{R}^3)} =$$

$$\left\| \left( e^{i\lambda_\Phi} e^{-im\cdot x} S(A_C, V) e^{im\cdot x} e^{-i\lambda_\Phi(-\frac{\partial}{\partial x} - \hat{\psi}) - e^{i \int_{-\infty}^{\infty} \hat{\psi}_C A(x+\tau \hat{\psi}) d\tau} \phi \right) \|_{L^2(\mathbb{R}^3)} \leq$$

$$\| \left( e^{i\lambda_\Phi} e^{-im\cdot x} S(A_C, V) e^{im\cdot x} e^{-i\lambda_\Phi(-\frac{\partial}{\partial x} - \hat{\psi}) - e^{i \int_{-\infty}^{\infty} \hat{\psi}_C A(x+\tau \hat{\psi}) d\tau} \phi \right) \|_{L^2(\mathbb{R}^3)} +$$

$$\left\| \left( e^{i\lambda_\Phi} e^{-im\cdot x} S(A_C, V) e^{im\cdot x} e^{i \int_{-\infty}^{\infty} \hat{\psi}_C A(x+\tau \hat{\psi}) d\tau} e^{-i\lambda_\Phi(-\frac{\partial}{\partial x} - \hat{\psi})} \phi \right) \|_{L^2(\mathbb{R}^3)} \leq C \frac{1}{e|\ell|} \| \phi \|_{H^2(\mathbb{R}^3)}.$$

The last inequality follows from (5.32), (5.42) and (5.44) for $A_C$.

5.2 High-Velocity Estimates II. The Electric Potential

Recall that $\tilde{\phi}$ is defined in (5.21) and that $H_1$ is given by (5.28).

LEMMA 5.8. Let $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a bounded function with compact support contained in $\mathbb{R}^3 \setminus \Lambda_\psi$, and let $\phi$ be a function in $H^2_0(\mathbb{R}^3)$ with compact support contained in $\Lambda_\psi$. Then, for any $l \in \mathbb{N}$ there exists constant $C_l$ such that following inequalities hold:

i) $\| h e^{-i\tau H_1} \tilde{\phi} \|_{L^2(\mathbb{R}^3)} \leq C_l \frac{1}{(1+|\tau|)^l} e^\frac{1}{\epsilon e^\tau} \| \phi \|_{H^2(\mathbb{R}^3)} \forall \epsilon > 0.$

ii) $\| h pe^{-iH_1} \tilde{\phi} \|_{L^2(\mathbb{R}^3)} \leq C_l \frac{1}{(1+|\tau|)^l} e^\frac{1}{\epsilon e^\tau} \| \phi \|_{H^2(\mathbb{R}^3)} \forall \epsilon > 0.$
Proof: We prove i), ii) follows in a similar way.

Clearly,
\[ \| \tilde{\phi} - \phi \|_{L^2(\mathbb{R}^3)} \leq \frac{1}{\nu \rho} \| \phi \|_{\mathcal{H}_0(\mathbb{R}^3)}, \quad \text{where } \rho \geq 1/2. \] (5.47)

It follows from (5.18) and the properties of the support of \( h \) and \( \phi \) that
\[ \| h e^{-i\tau H_1} \phi \|_{L^2(\mathbb{R}^3)} = \| h e^{-i\tau p \cdot \hat{v}}(e^{-i\tau \frac{p^2}{2mv}} - I - (-i\tau \frac{p^2}{2mv}) - \frac{1}{2}(-i\tau \frac{p^2}{2mv})^2)\phi \|. \]
Observing that
\[ \| (e^{-i\tau \frac{p^2}{2mv}} - I - (-i\tau \frac{p^2}{2mv}) - \frac{1}{2}(-i\tau \frac{p^2}{2mv})^2) \| \leq C |\tau|^3 \frac{p^6}{(2mv)^3}, \]
we obtain,
\[ \| h e^{-i\tau H_1} \phi \|_{L^2(\mathbb{R}^3)} \leq C \frac{(1 + |\tau|^3 \frac{p^6}{(2mv)^3})}{|\tau|^3} \| \phi \|_{\mathcal{H}_0(\mathbb{R}^3)}. \] (5.48)

We prove as in (5.36) that there exist a constant \( C_l \) such that,
\[ \| h e^{-i\tau H_1} \tilde{\phi} \|_{L^2(\mathbb{R}^3)} \leq C_l \frac{1}{(1 + |\tau|^3)} \| \phi \|_{L^2(\mathbb{R}^3)}. \] (5.49)

Finally we obtain i) from (5.47) and interpolating (5.48, 5.49).
\[ \square \]

We denote,
\[ a(\hat{v}, x) := \int_{-\infty}^{\infty} A(x + \tau \hat{v}) \cdot \hat{v} d\tau, \] (5.50)
and for \( \phi_0 \in \mathcal{H}_0(\mathbb{R}^3) \) with compact support in \( \Lambda_{\hat{v}} \),
\[ \phi_{\hat{v}} := e^{im\hat{v} \cdot x} \phi_0. \]

Recall that \( \Lambda_{\hat{v}} \) is defined in (5.14), that \( \eta \) is defined in (5.38) and that \( A_{\Phi, 2\pi} (B) \) is defined in Definition 5.4.

**THEOREM 5.9. (Reconstruction Formula II)** Suppose that \( B, V \) satisfy Assumption [5.7]. Let \( \Lambda_0 \) be a compact subset of \( \Lambda_{\hat{v}} \), with \( v \in \mathbb{R} \setminus \{0\} \). Then, for all \( \Phi \) and all \( A \in A_{\Phi, 2\pi, SR}(B) \)
\[ v \left( [S(A,V) - e^{ia(\hat{v}, x)}] \phi_{\hat{v}}, \psi_{\hat{v}} \right) = \left( -ie^{ia(\hat{v}, x)} \int_{-\infty}^{\infty} V(x + \tau \hat{v}) d\tau \phi_0, \psi_0 \right) \]
\[ + \left( -ie^{ia(\hat{v}, x)} \int_{-\infty}^{0} \Xi_{\eta}(x + \tau \hat{v}, -\infty) d\tau \phi_0, \psi_0 \right) + \left( i \int_{0}^{\infty} \Xi(x + \tau \hat{v}, \infty) d\tau e^{ia(x, \hat{v})} \phi_0, \psi_0 \right) + R(v, \phi_0, \psi_0), \] (5.51)
where,

\[
|R(v, \phi_0, \psi_0)| \leq C\|\phi\|_{\mathcal{H}_0(\mathbb{R}^3)} \|\psi\|_{\mathcal{H}_0(\mathbb{R}^3)} \left\{ \begin{array}{ll}
\frac{1}{v \min(\mu-2, \alpha-1)}, & \text{if } \min(\mu-3, \alpha-2) < 0, \\
\frac{|\ln v|}{v}, & \text{if } \min(\mu-3, \alpha-2) = 0, \\
\frac{1}{v}, & \text{if } \min(\mu-3, \alpha-2) > 0,
\end{array} \right.
\]

for some constant C and all \(\phi_0, \psi_0 \in \mathcal{H}_0(\mathbb{R}^3)\) with compact support in \(\Lambda_0\).

Proof: We first prove the theorem in the Coulomb gauge \(A_C\). Note that,

\[
v \left( [S(A, V) - e^{ia}] \phi_V, \psi_V \right) = v \left( e^{-iL_{AC}\cdot\psi(-\infty)} \phi_0, R_+ \psi_0 \right) + v \left( R_- \phi_0, e^{-iL_{AC}\cdot\psi(\infty)} \psi_0 \right) + v \left( R_- \phi_0, R_+ \psi_0 \right),
\]

where,

\[
R_{\pm} := e^{-imv\cdot x} W_{\pm}(A_C, V) e^{imv\cdot x} - e^{-iL_{AC}\cdot\psi(\pm\infty)}.
\]

By Lemma 5.6

\[
v |(R_- \phi_0, R_+ \psi_0)| \leq C \frac{1}{v} \|\phi\|_{\mathcal{H}_0(\mathbb{R}^3)} \|\psi\|_{\mathcal{H}_0(\mathbb{R}^3)}. \tag{5.54}
\]

We prove below that,

\[
v \left( e^{-iL_{AC}\cdot\psi(-\infty)} \phi_0, R_+ \psi_0 \right) = -i \int_0^\infty (\Xi_0(x + \tau \hat{v}, \infty) + \chi V(x + \tau \hat{v})) d\tau e^{ia} \phi_0, \psi_0 \right) + R_+(v, \phi_0, \psi_0), \tag{5.55}
\]

\[
v \left( R_- \phi_0, e^{-iL_{AC}\cdot\psi(\infty)} \psi_0 \right) = -ie^{ia} \int_{-\infty}^0 (\Xi_0(x + \tau \hat{v}, -\infty) + \chi V(x + \tau \hat{v})) d\tau \phi_0, \psi_0 \right) + R_-(v, \phi_0, \psi_0), \tag{5.56}
\]

where \(R_{\pm}\) satisfy (5.52). Note that (5.56) follows from (5.55) by time inversion and charge conjugation in the magnetic potential, i.e., by taking complex conjugates and changing \(A_C\) to \(-A_C\). It can also be proved as in the proof of (5.55) that we give below in seven steps.

We use the notation of the proof of Lemma 5.6 For simplicity we denote by \(O(r)\) a term that satisfies

\[
|O(r)| \leq C\|\phi\|_{\mathcal{H}_0(\mathbb{R}^3)} \|\psi\|_{\mathcal{H}_0(\mathbb{R}^3)} r.
\]

Step 1

\[
v \left( e^{-iL_{AC}\cdot\psi(-\infty)} \phi_0, R_+ \psi_0 \right) = \left( e^{-iL_{AC}\cdot\psi(-\infty)} \phi_0, \right.
\]

\[
\lim_{t \to -\infty} \int_0^t d\tau e^{iH_2 \tau} e^{-iL_{AC}\cdot\psi(t-\tau)} \left[ \Xi_0(x, t-\tau) + \chi V(x) e^{-i\tau H_1} \tilde{\psi} \right] + O(1/v). \tag{5.57}
\]

Equation (5.57) follows from (5.24), (5.26), (5.29) and the following formula that is easily obtained from Lemma 5.8

\[
\|T_2 + T_3\|_{L^2(\mathbb{R}^3)} \leq C_{\epsilon} \frac{\|\phi\|_{\mathcal{H}_0(\mathbb{R}^3)}}{v^{1-\epsilon}(1 + |\tau|)^l}, \quad \forall \epsilon > 0, l = 1, 2, \ldots, \tag{5.58}
\]

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that improves \[ (5.36, 5.37). \]

**Step 2**

\[ \lim_{t \to \infty} \int_0^t d\tau e^{i\tau H_2} e^{-iL_{AC} \cdot \varphi(t-\tau)} [\Xi_0(x, t - \tau) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi} = \] 
\[ \lim_{t \to \infty} \int_0^t d\tau e^{i\tau H_2} e^{-iL_{AC} \cdot \varphi(t-\tau)} [\Xi_0(x, t - \tau) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi}. \]  
(5.59)

This follows from Lebesgue’s dominated convergence theorem and as
\[ \lim_{t \to \infty} \left\| \Xi_0(x, t - \tau) - \Xi_0(x, t - \tau) e^{-itH_1} \tilde{\psi} \right\|_{L^2(\mathbb{R}^3)} = 0, \]
and, moreover,
\[ \left\| \Xi_0(x, t - \tau) - \Xi_0(x, t - \tau) e^{-itH_1} \tilde{\psi} \right\|_{L^2(\mathbb{R}^3)} \leq h(\tau), \text{ for some } h(\tau) \in L^1(0, \infty). \]

This estimate is proven as in the proof of Lemma 5.6 using Lemma 5.5.

**Step 3**

\[ v \left( e^{-iL_{AC} \cdot \varphi(-\infty)} \phi_0, R_x \psi_0 \right) = \int_0^\infty d\tau \left( e^{-iL_{AC} \cdot \varphi(-\infty)} \phi_0, \right. \]
\[ e^{i\tau H_2} e^{-iL_{AC} \cdot \varphi(\infty)} [\Xi_0(x, \infty) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi} + O(1/v) + 0(1/(1 + |t|)^{\min(\mu - 2, 2, \alpha - 1)}). \]  
(5.60)

This follows from Steps 1 and 2, and from the following argument. As in the proof of Lemma 5.6 we prove that
\[ \left\| \Xi_0(x, \infty) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi} \right\|_{L^2(\mathbb{R}^3)} \leq C \left( (1/1 + |\tau|)^{\min(\mu - 1, \alpha)} \right) \| \psi \|_{H_2(\mathbb{R}^3)}. \]  
(5.61)

Then by Fatou’s lemma
\[ \left\| \Xi_0(x, \infty) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi} \right\|_{L^2(\mathbb{R}^3)} \leq C \left( (1/1 + |\tau|)^{\min(\mu - 1, \alpha)} \right) \| \psi \|_{H_2(\mathbb{R}^3)}. \]  
(5.62)

Hence, by Lebesgue’s dominated convergence theorem,
\[ \lim_{t \to \infty} \int_0^t d\tau e^{i\tau H_2} e^{-iL_{AC} \cdot \varphi(t-\tau)} [\Xi_0(x, t - \tau) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi} \]
\[ = \int_0^\infty d\tau e^{i\tau H_2} e^{-iL_{AC} \cdot \varphi(\infty)} [\Xi_0(x, \infty) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi}, \]
where the limit is on the strong topology of $L^2(\mathbb{R}^3)$. We complete the proof of (5.60) using (5.62).

We now estimate the integrand in (5.60).

**Step 4**

\[ \left( e^{-iL_{AC} \cdot \varphi(-\infty)} \phi_0, e^{i\tau H_2} e^{-iL_{AC} \cdot \varphi(\infty)} i[\Xi_0(x, \infty) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi} \right) = \]
\[ \left( e^{iL_{AC} \cdot \varphi(\tau) - iL_{AC} \cdot \varphi(-\infty)} \phi_0, e^{i\tau H_1} e^{-iL_{AC} \cdot \varphi(\infty)} i[\Xi_0(x, \infty) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi} \right) + \frac{1}{v} O(\frac{1}{(1 + |\tau|)^{\min(\mu - 2, 2, \alpha - 1)}}). \]  
(5.63)

Denote by $\chi_\Lambda$ the characteristic function of $\Lambda$. Then,
\[ \left( e^{-iL_{AC} \cdot \varphi(-\infty)} \phi_0, e^{i\tau H_2} e^{-iL_{AC} \cdot \varphi(\infty)} i[\Xi_0(x, \infty) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi} \right) = \]
\[ \left( e^{i\tau H_1} \chi_\Lambda e^{-i\tau H_2} e^{-iL_{AC} \cdot \varphi(-\infty)} \phi_0, e^{i\tau H_1} e^{-iL_{AC} \cdot \varphi(\infty)} i[\Xi_0(x, \infty) + \chi V(x)] e^{-i\tau H_1} \tilde{\psi} \right). \]
Hence, (5.63) will be proved if we can replace $e^{i\tau H_1} \chi \Lambda e^{-i\tau H_2}$ by $\chi e^{iL_{AC} \cdot \varphi(\tau)}$ adding the error term. But, this follows from (5.62) and the estimate,

$$\left\| \left( e^{i\tau H_1} \chi \Lambda e^{-i\tau H_2} - \chi e^{iL_{AC} \cdot \varphi(\tau)} \right) e^{-iL_{AC} \cdot \varphi(\tau)} \phi_0 \right\| \leq C \frac{1 + |\tau|}{\nu} \| \phi_0 \|_{H_2(\mathbb{R}^3)}, \tag{5.64}$$

that we prove below.

We designate,

$$\varphi_\tau := e^{i(L_{AC} \cdot \varphi(\tau) - L_{AC} \cdot \varphi(\tau))} \phi_0.$$  

We have that,

$$\left( e^{i\tau H_1} \chi \Lambda e^{-i\tau H_2} - \chi e^{iL_{AC} \cdot \varphi(\tau)} \right) e^{-iL_{AC} \cdot \varphi(\tau)} \phi_0 = e^{i\tau H_1} \chi \Lambda e^{-i\tau H_2} \left( e^{-iL_{AC} \cdot \varphi(\tau)} - e^{-i\tau H_1} \right) \varphi_\tau + \left( e^{i\tau H_1} \chi e^{-i\tau H_1} - \chi \right) \varphi_\tau. \tag{5.65}$$

By (5.41)

$$\| \varphi_\tau \|_{H_2(\mathbb{R}^3)} \leq C \| \phi_0 \|_{H_2(\mathbb{R}^3)}. \tag{5.66}$$

Hence, using

$$\left| e^{-i\tau(p + mv)/2mv} - e^{-i\tau(p + \nu v^2/2mv)} \right| \leq C \frac{|\tau|^2}{2mv},$$

we prove that,

$$\left\| \left( e^{-i\tau H_1} - e^{-i\tau(p + \nu v^2/2mv)} \right) \varphi_\tau \right\|_{L^2(\mathbb{R}^3)} \leq C \frac{|\tau|}{\nu} \| \phi_0 \|_{H_2(\mathbb{R}^3)}, \tag{5.67}$$

and since $\chi - 1 \equiv 0$ on the support of $e^{-i\tau(p + \nu v^2/2mv)} \varphi_\tau$,

$$\left\| \left( e^{i\tau H_1} \chi e^{-i\tau H_1} - \chi \right) \varphi_\tau \right\|_{L^2(\mathbb{R}^3)} = \left\| e^{i\tau H_1} (\chi - 1) \left( e^{-i\tau H_1} - e^{-i\tau(p + \nu v^2/2mv)} \right) \varphi_\tau \right\|_{L^2(\mathbb{R}^3)} \tag{5.68} \leq C \frac{|\tau|}{\nu} \| \phi_0 \|_{H_2(\mathbb{R}^3)}.$$  

Then, 5.64 follows from 5.39, 5.65, 5.66, 5.68.

Step 5.

We now replace $e^{\pm i \tau H_1}$ by $e^{\pm i(\tau p + \nu v^2/2mv)}$. We will prove that,

$$\left( e^{i(L_{AC} \cdot \varphi(\tau) - L_{AC} \cdot \varphi(\tau))} \phi_0, e^{i\tau H_1} e^{-iL_{AC} \cdot \varphi(\tau)} \right) e^{i\tau(\tau p + \nu v^2/2mv)} \psi \right) = \left( e^{i(L_{AC} \cdot \varphi(\tau) - L_{AC} \cdot \varphi(\tau))} \phi_0, e^{i\tau p \cdot \varphi(\tau)} e^{-iL_{AC} \cdot \varphi(\tau)} \psi \right) + O \left( \frac{1}{(1 + |\tau|)^{\min(p - 2, n - 1)}} \right) \tag{5.69}$$

Recall that $\varphi_\tau$ is defined below (5.64). By (5.62) and (5.67),

$$\left( e^{-i\tau H_1} \varphi_\tau, e^{-iL_{AC} \cdot \varphi(\tau)} \psi \right) = \left( \varphi_\tau, e^{-i\tau p \cdot \varphi(\tau) + \nu v^2/2mv} \psi \right) + O \left( \frac{1}{(1 + |\tau|)^{\min(p - 2, n - 1)}} \right). \tag{5.70}$$

The first equality in (5.69) follows from (5.67) and as,

$$\left\| \psi \right\|_{L^2(\mathbb{R}^3)} \leq C \frac{1}{(1 + |\tau|)^{\min(p - 2, n - 1)}} \| \phi_0 \|_{H_2(\mathbb{R}^3)}, \tag{5.71}$$
because $\phi_0$ has compact support, $e^{-i\tau P \hat{\psi}}$ is just a translation and the decay properties of $V(x)$ and $\Xi_\eta(x,\infty)$ (in the direction $\hat{\psi}$). The second equality is immediate.

By \([5.60, 5.69, 5.69]\)

$$v \left( e^{-i L_{\Lambda_C} \cdot v (-\infty) \phi_0, R_+ \psi_0} \right) = \int_0^t d\tau \left( \phi_0, e^{-i a(\hat{\psi}, x)} i [\Xi_\eta(x + \tau \hat{\psi}, \infty) + \chi V(x + \tau \hat{\psi})] \psi_0 \right) + O \left( \frac{1}{1+|\tau|} \right),$$

if $\min(-2, \alpha - 1) = 1$, otherwise.

(5.72)

Step 6

We now prove that,

$$\int_0^t d\tau \left( \phi_0, e^{-i a(\hat{\psi}, x)} i [\Xi_\eta(x + \tau \hat{\psi}, \infty) + \chi V(x + \tau \hat{\psi})] \psi_0 \right)$$

(5.73)

As $\phi_0$ has compact support,

$$\left\| \Xi_\eta(x + \tau \hat{\psi}, \infty) + \chi V(x + \tau \hat{\psi}) e^{ia(\hat{\psi}, x) \phi_0} \right\|_{L^2(\mathbb{R}^3)} \leq C \frac{1}{(1+|\tau|)^{\min(-1, \alpha)}} \|\phi_0\|_{H^2(\mathbb{R}^3)}, \tau > 0.$$  

(5.74)

Equations \([5.22, 5.74]\) prove \([5.73]\).

By \([5.72, 5.73]\)

$$v \left( e^{-i L_{\Lambda_C} \cdot v (-\infty) \phi_0, R_+ \psi_0} \right) = \int_0^\infty d\tau \left( \phi_0, e^{-i a(\hat{\psi}, x)} i [\Xi_\eta(x + \tau \hat{\psi}, \infty) + \chi V(x + \tau \hat{\psi})] \psi_0 \right) +$$

if $\min(-2, \alpha - 1) = 1$, otherwise.

(5.75)

Finally, taking $t = v$ we obtain \([5.55]\) in the Coulomb gauge, and then, \([5.51]\) is proven for $A_C$.

Suppose that $A \in A_{\Phi, 2\pi}SR(B)$. By \([5.11]\) $S(A, V) = S(A_C, V)$. As $\lambda_\infty$ is constant, $e^{i \int_{-\infty}^\infty A(x + \tau \hat{\psi}) \cdot \psi d\tau} = e^{i \int_{-\infty}^\infty A_C(x + \tau \hat{\psi}) \cdot \psi d\tau}$, and it follows that \([5.51]\) holds for $A \in A_{\Phi, 2\pi}(B)$.

6 Reconstruction of the Magnetic Field and the Electric Potential Outside the Obstacle

In this section we obtain a method for the unique reconstruction of the magnetic field and the electric potential outside the obstacle, $K$, from the high-velocity limit of the scattering operator. The method is given in the proof of Theorem \([5.3]\) and is summarized in Remark \([5.3]\)

**DEFINITION 6.1.** We denote by $\Lambda_{\text{rec}}$ the set of points $x \in \Lambda$ such that for some two-dimensional plane $P_x$ we have that $x + P_x \subset \Lambda$. 

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Note that if $K$ is convex $\Lambda_{rec} = \Lambda$.

**LEMMA 6.2.** For every $A \in \mathcal{A}_{\Phi,tw}(B)$ and every unit vector, $\hat{v}$, in $\mathbb{R}^3$, we have that

$$
\nabla \int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v}) \, d\tau = \int_{-\infty}^{\infty} \hat{v} \times B(x + \tau \hat{v}) \, d\tau,
$$

in distribution sense in $\Lambda_\Phi$.

**Proof:**

The following identity holds in distribution sense in $\Lambda_\Phi$ (this is just the triple vector product formula),

$$
\hat{v} \times (\nabla \times A) = \nabla(\hat{v} \cdot A) - (\hat{v} \cdot \nabla)A.
$$

(6.2)

Then, for every $\phi \in C^\infty_0(\Lambda_\Phi)$

$$
\int_{-\infty}^{\infty} \hat{v} \times B(x + \tau \hat{v}) \, d\tau \, [\phi] = \int_{\mathbb{R}^3} dx \int_{-\infty}^{\infty} \hat{v} \times B(x + \tau \hat{v}) \, d\tau \, \phi(x) = \int_{\mathbb{R}^3} dx \lim_{r \to \infty} \int_{-r}^{r} d\tau \hat{v} \times B(x) \phi(x - \tau \hat{v}) = \int_{\mathbb{R}^3} dx \lim_{r \to \infty} \int_{-r}^{r} (\nabla \times (\hat{v} \cdot A(x)(\nabla \phi)(x - \tau \hat{v}) + A(x) (\hat{v} \cdot \nabla \phi)(x - \tau \hat{v})) = \\
\left( \nabla \int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v}) \, d\tau \right) [\phi] + \lim_{r \to \infty} \int_{\mathbb{R}^3} A(x) (\phi(x - r \hat{v}) - \phi(x + r \hat{v})) = \left( \nabla \int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v}) \, d\tau \right)[\phi],
$$

where in the last equality we used the decay of $A$ and the fact that $\phi$ has compact support.

**THEOREM 6.3.** (Reconstruction of the Magnetic Field and the Electric Potential) Suppose that $B, V$ satisfy Assumption $[\mathcal{A}]$. Then, for any flux, $\Phi$, and all $A \in \mathcal{A}_{\Phi,tw}(B)$, the high-velocity limits of $S(A, V)$ in (5.44) known for all $\Lambda_0$, all unit vectors $\hat{v}$ and all $\phi_0 \in \mathcal{H}_2(\mathbb{R}^3)$ with support $\phi_0 \subset \Lambda_0$, uniquely determine $B(x)$ for almost every $x \in \Lambda_{rec}$. Furthermore, for any flux, $\Phi$, and all $A \in \mathcal{A}_{\Phi,tw,SR}(B)$, the high-velocity limits of $S(A, V)$ in (5.51) known for all $\Lambda_0$, all unit vectors $\hat{v}$ and all $\phi_0, \psi_0 \in \mathcal{H}_0(\mathbb{R}^3)$ with support $\phi_0$, support $\psi_0 \subset \Lambda_0$, uniquely determine $V(x)$ for almost every $x \in \Lambda_{rec}$.

**Proof:** We proceed as in the proof of Theorem 1.1 of [14] (see also the proof of Theorem 1.4 [40]) with the modifications that are necessary to take the obstacle into account and to reconstruct the magnetic field.

Let us fix a $x_0 \in \Lambda_{rec}$. For each $j = 1, 2, 3$ we take vectors unit vectors $\hat{u}_j, \hat{v}_j$ and $\varepsilon > 0$ such that the following conditions are satisfied.

1. $\hat{u}_j \cdot \hat{v}_j = 0, i,j \in \{1,2,3\}$.

2. The unit vectors $\hat{n}_j := \hat{u}_j \times \hat{v}_j, j = 1, 2, 3,$ are linearly independent.
For any \( z = (z_1, z_2) \in \mathbb{R}^2 \) we define,
\[
\phi_j(z) := e^{-i(z_1 \hat{u}_j + z_2 \hat{v}_j)} p \phi_0, \quad \psi_j(z) := e^{-i(z_1 \hat{u}_j + z_2 \hat{v}_j)} p \psi_0, \quad j = 1, 2, 3, \quad \phi_0, \psi_0 \in C_c^\infty \left( B^{3_\infty}_r (x_0) \right).
\]
(6.3)

From the limit (5.44) we uniquely reconstruct
\[
e^i \int_{-\infty}^{\infty} \nabla \cdot A(x + \tau \hat{v}) \, d\tau
\]
for all \( x \in \Lambda \hat{v} \) and then, we reconstruct \( \int_{-\infty}^{\infty} \nabla \cdot A(x + \tau \hat{v}) \, d\tau + 2\pi n(x, \hat{v}) \) with \( n(x, \hat{v}) \) an integer that is locally constant.

By Lemma 6.2 we also reconstruct uniquely
\[
\int_{-\infty}^{\infty} \hat{v} \times B(x + \tau \hat{v}) \, d\tau
\]
(6.4)
for a.e. \( x \in \Lambda \hat{v} \).

Take now \( \hat{v} \in p(\hat{u}_j, \hat{v}_j) \). Hence, we uniquely reconstruct
\[
\int_{-\infty}^{\infty} \hat{u}_j \cdot B(x + \tau \hat{v}) \, d\tau = -\hat{u}_j \cdot \left( \hat{v} \times \int_{-\infty}^{\infty} \hat{v} \times B(x + \tau \hat{v}) \, d\tau \right),
\]
(6.5)
for a.e. \( x \in \Lambda \hat{v} \). We used the triple vector product formula, \( a \times (b \times c) = (a \cdot c)b - (a \cdot b)c \). We now define \( F_j : \mathbb{R}^2 \to \mathbb{C} \),
\[
F_j(z) := (\hat{u}_j \cdot B(x) \phi_j(z), \psi_j(z)).
\]

\( F_j \) is continuous and
\[
|F_j(z)| \leq C(1 + |z|)^{-\mu}, \quad j = 1, 2, 3.
\]
Moreover, we uniquely reconstruct from (5.44) the Radon transforms,
\[
\hat{F}_j(\hat{w}; z) := \int_{-\infty}^{\infty} F_j(z + \tau \hat{w}) \, d\tau = \left( \int_{-\infty}^{\infty} \hat{u}_j \cdot B(x + \tau (\hat{w}_1 \hat{u}_j + \hat{w}_2 \hat{v}_j)) \, d\tau \phi_j(z), \psi_j(z) \right),
\]
where \( z \in \mathbb{R}^2 \) and \( \hat{w} := (\hat{w}_1, \hat{w}_2) \in \mathbb{R}^2 \) has modulus one.

Inverting this Radon transform (see Theorem 2.17 of [18], [19], [27]) we uniquely reconstruct \( F_j(z) \) and in particular \( F_j(0) = (\hat{u}_j \cdot B \phi_0, \psi_0) \) and hence, we uniquely reconstruct \( \hat{u}_j \cdot B(x) \), \( j = 1, 2, 3 \) for a.e. \( x \in B^{3_\infty}_r (x_0) \) and as the \( \hat{u}_j \) are linearly independent we uniquely reconstruct \( B(x) \) for a.e. \( x \in B^{3_\infty}_r (x_0) \). Since \( x_0 \in \Lambda_{\text{rec}} \) is arbitrary we uniquely reconstruct \( B(x) \) for a.e. \( x \in \Lambda_{\text{rec}} \).

We now uniquely reconstruct \( V \). Take any \( x_0 \in \Lambda_{\text{rec}} \). Let \( \hat{u}, \hat{w} \) be orthonormal vectors such that \( B^{3_\infty}_r (x_0) + p(\hat{u}, \hat{w}) \subset \Lambda \hat{v} \). We define,
\[
\phi(z) := e^{-i(z_1 \hat{u} + z_2 \hat{w})} p \phi_0, \quad \psi(z) := e^{-i(z_1 \hat{u} + z_2 \hat{w})} p \psi_0, \quad \phi_0, \psi_0 \in C_c^\infty \left( B^{3_\infty}_r (x_0) \right),
\]

where \( p(\hat{u}_j, \hat{v}_j) \) is the two-dimensional plane generated by \( \hat{u}_j, \hat{v}_j \).
and the function $F : \mathbb{R}^2 \to \mathbb{C},$

$$F(z) := (V(x)\phi(z), \psi(z)).$$

$F$ is continuous and

$$|F(z)| \leq C(1 + |z|)^{-\alpha}.$$

Moreover, since $B$ is already known in $\Lambda_{\text{rec}},$ we uniquely reconstruct from (5.51) the Radon transforms,

$$\tilde{F}(\hat{y}; z) := \int_{-\infty}^{\infty} F(z + \tau \hat{y})d\tau = \left( \int_{-\infty}^{\infty} V(x + \tau(\hat{y}_1 \hat{u} + \hat{y}_2 \hat{w}))d\tau \phi(z), \psi(z) \right),$$

where $z \in \mathbb{R}^2$ and $\hat{y} := (\hat{y}_1, \hat{y}_2) \in \mathbb{R}^2$ has modulus one.

As above inverting these Radon transforms we uniquely reconstruct $F(z),$ and in particular $F(0) = (V\phi_0, \psi_0)$ what uniquely determines $V(x)$ for a.e. $x \in B_{\mathbb{R}^3}(x_0).$ Since $x_0 \in \Lambda_{\text{rec}}$ is arbitrary, $V(x)$ is uniquely reconstructed for a.e. $x \in \Lambda_{\text{rec}}.$

**REMARK 6.4.** Let us summarize the reconstruction method given by Theorem 6.3. From the high-velocity limit (5.44) we uniquely reconstruct,

$$e^{i \int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v})d\tau},$$

and from this we uniquely reconstruct,

$$\int_{-\infty}^{\infty} \hat{v} \times B(x + \tau \hat{v})d\tau, x \in \Lambda_{\hat{v}},$$

what gives us the Radon transform

$$\tilde{F}_j(\hat{w}; z) := \int_{-\infty}^{\infty} F_j(z + \tau \hat{w})d\tau = \left( \int_{-\infty}^{\infty} \hat{n}_j \cdot B(x + \tau(\hat{w}_1 \hat{u}_j + \hat{w}_2 \hat{v}_j))d\tau \phi_j(z), \psi_j(z) \right),$$

where $z \in \mathbb{R}^2$ and $\hat{w} := (\hat{w}_1, \hat{w}_2) \in \mathbb{R}^2$ has modulus one.

Inverting this Radon transform we uniquely reconstruct $F_j(z)$ and in particular $F_j(0) = (\hat{n}_j \cdot B\phi_0, \psi_0)$ and hence, we uniquely reconstruct $\hat{n}_j \cdot B(x), j = 1, 2, 3$ for a.e. $x \in B_{\mathbb{R}^3}(x_0)$ and as the $\hat{n}_j$ are linearly independent we uniquely reconstruct $B(x)$ for a.e. $x \in B_{\mathbb{R}^3}(x_0).$ Since $x_0 \in \Lambda_{\text{rec}}$ is arbitrary we uniquely reconstruct $B(x)$ for a.e. $x \in \Lambda_{\text{rec}}.$ Note that to reconstruct $B$ almost everywhere in a neighborhood of a point $x_0$ we only need the high-velocity limit of the scattering operator a neighborhood of three two-dimensional planes. For the inversion of the Radon transform see Theorem 2.17 of [18] and [19], [27].

Remember that given any $A \in A_{\Phi,2\pi}(B)$ we can always find an $A \in A_\Phi(B)$ with the same scattering operator. We can take, for example, $A_4.$ See equation (5.12). Then, there is no loss of generality taking $A \in A_\Phi(B).$ Note that
\[ \tilde{A} \] is not a gauge invariant quantity. If \( \tilde{A}, A \in \mathcal{A}_\Phi(B) \) and \( \tilde{A} = A + d\lambda \), then,

\[ \int_{-\infty}^{\infty} \dot{v} \cdot \tilde{A}(x + \tau \dot{v}) d\tau = \int_{-\infty}^{\infty} \dot{v} \cdot A(x + \tau \dot{v}) d\tau + \lambda_\infty(\dot{v}) - \lambda_\infty(-\dot{v}). \]

We can, however, reconstruct (6.7) from the gauge invariant quantity,

\[ \mathcal{R}(x, y) := e^{i \int_{-\infty}^{\infty} \dot{v} \cdot [A(x + \tau \dot{v}) - A(y + \tau \dot{v})] d\tau}, \]

\[ x, y \in \Lambda_{\dot{v}}. \]

We have that,

\[ \frac{1}{i} \frac{\partial}{\partial x} \mathcal{R}(x, y) = \nabla_x \int_{-\infty}^{\infty} \dot{v} \cdot A(x + \tau \dot{v}) d\tau = \int_{-\infty}^{\infty} \dot{v} \times B(x + \tau \dot{v}) d\tau, \]

\[ x \in \Lambda_{\dot{v}}. \]

We now uniquely reconstruct \( V \). Since \( B \) is already known in \( \Lambda_{\text{rec}} \), for any \( \dot{v} \in p(\dot{u}, \dot{w}) \) we uniquely reconstruct from (5.51) the Radon transforms,

\[ \hat{F}(\tilde{y}; z) := \int_{-\infty}^{\infty} F(z + \tau \tilde{y}) d\tau = \left( \int_{-\infty}^{\infty} V(x + \tau (\dot{y}_1 \dot{u} + \dot{y}_2 \dot{w})) d\tau \phi(z), \psi(z) \right), \]

where \( z \in \mathbb{R}^2 \) and \( \tilde{y} := (\dot{y}_1, \dot{y}_2) \in \mathbb{R}^2 \) has modulus one.

As above inverting these Radon transforms we uniquely reconstruct \( F(z) \), and in particular \( F(0) = (V \phi_0, \psi_0) \) what uniquely determines \( V(x) \) for a.e. \( x \in B_{\mathbb{R}^3}^+(x_0) \). Since \( x_0 \in \Lambda_{\text{rec}} \) is arbitrary, \( V(x) \) is uniquely reconstructed for a.e. \( x \in \Lambda_{\text{rec}} \).

7 The Aharonov-Bohm Effect

In this section we assume that \( B \equiv 0 \), i.e., that there is no magnetic field in \( \Lambda \). On the contrary, the electric potential, \( V \), is not assumed to be zero. In other words, we will analyze the Aharonov-Bohm effect in the presence of an electric potential. As we will show, for high-velocities the electric potential gives a lower-order contribution that plays no role in the Aharonov-Bohm effect. However, it could be of interest to allow for a non-trivial electric potential from the experimental point of view.

For any \( x \in \mathbb{R}^3 \) and any unit vector \( \dot{v} \in S^2 \) we denote

\[ L(x, \dot{v}) := x + \mathbb{R} \dot{v}, \]

and we give to \( L(x, \dot{v}) \) the orientation of \( \dot{v} \). Suppose that \( x, y \in \mathbb{R}^3, \dot{v}, \dot{w} \in S^2 \) satisfy \( \dot{v} \cdot \dot{w} \geq 0 \) and that

\[ L(x, \dot{v}) \cup L(y, \dot{w}) \subset \Lambda. \]

Take \( \rho > 0 \) so large that

\[ \text{convex } \left( (x + (-\infty, -\rho] \dot{v}) \cup (y + (-\infty, -\rho] \dot{w}) \right) \cup \text{convex } ((x + [\rho, \infty) \dot{v}) \cup (y + [\rho, \infty) \dot{w})) \subset B_{\mathbb{R}^3}^+(0), \]
where $K \subset B_{r}^{R^{3}}(0)$, $B_{r}^{R^{3}}(0)^{c}$ is the complement of $B_{r}^{R^{3}}(0)$ and the symbol convex(·) denotes the convex hull of the indicated set.

We denote by $\gamma(x, y, \hat{v}, \hat{w})$ the continuous, simple, oriented and closed curve with sides, $x + [-\rho, \rho] \hat{v}$, oriented in the direction of $\hat{v}$, $y + [-\rho, \rho] \hat{w}$, oriented in the direction of $-\hat{w}$ and the oriented straight lines that join the points $x + \rho \hat{v}$ with $y + \rho \hat{w}$ and $y - \rho \hat{w}$ and $x - \rho \hat{v}$.

Suppose that $A$ is short-range (see Definition 5.4). For example, we can take $A = A_{C}$. We denote $x_{\perp, \hat{v}} := x - (x, \hat{v}) \hat{v}$. It follows from Stoke’s theorem that if $|x_{\perp, \hat{v}}| \geq r$,

$$
\int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v}) d\tau = \lim_{s \to \infty} \int_{-\infty}^{\infty} \hat{v} \cdot A(sx_{\perp} + \tau \hat{v}) d\tau = 0. \quad (7.1)
$$

By Stoke’s theorem and arguing as in the proof of (7.1) we prove that for short-range $A$

$$
\int_{\gamma(x, y, \hat{v}, \hat{w})} A = \int_{L(x, \hat{v})} A - \int_{L(y, \hat{w})} A. \quad (7.2)
$$

Take any $z \in R^{3}$ such that, $|(x + z)_{\perp, \hat{v}}| \geq r$, $|(y + z)_{\perp, \hat{w}}| \geq r$. By Stoke’s theorem and (7.1),

$$
\int_{L(x + z, \hat{v})} A = \int_{L(y + z, \hat{w})} A = 0.
$$

Then, adding zero we write (7.2) as

$$
\int_{\gamma(x, y, \hat{v}, \hat{w})} A = \left( \int_{L(x, \hat{v})} A - \int_{L(x + z, \hat{v})} A \right) - \left( \int_{L(y, \hat{w})} A - \int_{L(y + z, \hat{w})} A \right). \quad (7.3)
$$

The point is that the left- and right-hand sides of (7.3) are gauge invariant, and in consequence (7.3) holds for any $A \in A_{\Phi, 2\pi}(0)$.

It follows that from the high-velocity limit (5.44) we can reconstruct $\int_{\gamma(x, y, \hat{v}, \hat{w})} A$, modulo $2\pi$. We have proven the following theorem.

**THEOREM 7.1.** Suppose that $B \equiv 0$ and that $V$ satisfies Assumption 7.1. Then, for any flux, $\Phi$, and all $A \in A_{\Phi, 2\pi}(0)$, the high-velocity limits of $S(A, V)$ in (5.44) known for $\hat{v}$ and $\hat{w}$ determines the fluxes

$$
\int_{\gamma(x, y, \hat{v}, \hat{w})} A \quad (7.4)
$$

modulo $2\pi$, for all curves $\gamma(x, y, \hat{v}, \hat{w})$.

**REMARK 7.2.** Theorem 7.1 implies that from the high-velocity limit (5.44) for $\hat{v}$ and $\hat{w}$ we can reconstruct the fluxes

$$
\int_{\alpha} A
$$

for any closed curve $\alpha$ such that there is a surface (or chain) $S$ in $\Lambda$ with $\partial S = \alpha - \gamma(x, y, \hat{v}, \hat{w})$, because by Stoke’s theorem,
\[ \int_a A = \int_{\gamma(x,y,\hat{v},\hat{w})} A + \int_S B = \int_{\gamma(x,y,\hat{v},\hat{w})} A. \]

Remember also that given any \( A \in A_{\Phi,2\pi}(B) \) we can always find an \( A \in A_{\Phi}(B) \) with the same scattering operator. We can take, for example, \( A_{\Phi} \). See equation (5.12). Then, there is no loss of generality taking \( A \in A_{\Phi}(0) \). Furthermore, notice that we can at most reconstruct the fluxes modulo \( 2\pi \) because by (5.12) \( S(A_{\Phi},V) = S(A,V) \) and the fluxes of \( A_{\Phi} \) and \( A \) differ by integer multiples of \( 2\pi \). For general \( A \in A_{\Phi,2\pi}(0) \) we recuperate the fluxes from equation (7.3). However if \( A \) is short-range we can use the simpler formula (7.2).

**REMARK 7.3.** As \( \gamma(x,y,\hat{v},\hat{w}) \) is a cycle, the homology class \( [\gamma(x,y,\hat{v},\hat{w})]_{H_1(\Lambda;\mathbb{R})} \) is well defined.

We denote,

\[ H_{1,rec}(\Lambda;\mathbb{R}) := \left\{ [\gamma(x,y,\hat{v},\hat{w})]_{H_1(\Lambda;\mathbb{R})} : L(x,\hat{v}) \cup L(x,\hat{w}) \subset \Lambda \right\}. \tag{7.5} \]

\( H_{1,rec}(\Lambda;\mathbb{R}) \) is a vector subspace of \( H_1(\Lambda;\mathbb{R}) \). Let us denote by \( H^1_{de\; R,\; rec}(\Lambda) \) the vector subspace of \( H^1_{de\; R}(\Lambda) \) that is the dual to \( H_{1,rec}(\Lambda;\mathbb{R}) \), given by de Rham’s Theorem. Then, for all \( \Phi \) and all \( A \in A_{\Phi,2\pi}(0) \), from the high-velocity limit (5.44) known for all \( \hat{v},\hat{w} \) we reconstruct the projection of \( A \) into \( H^1_{de\; R,\; rec}(\Lambda) \) modulo \( 2\pi \), as we now show.

Let

\[ \left\{ [\sigma_j]_{H_1(\Lambda;\mathbb{R})} \right\}_{j=1}^m, \]

be a basis of \( H_{1,rec}(\Lambda;\mathbb{R}) \), and let

\[ \left\{ [A_j]_{H^1_{de\; R,\; rec}(\Lambda)} \right\}_{j=1}^m, \]

be the dual basis, i.e.,

\[ \int_{\sigma_j} A_k = \delta_{j,k}, j,k = 1,2,\cdots,m. \]

Let us denote by \( P_{rec} \) the projector onto \( H^1_{de\; R,\; rec}(\Lambda) \). Hence, for any \( A \in A_{\Phi,2\pi}(B) \)

\[ P_{rec}[A]_{H^1_{de\; R}(\Lambda)} = \sum_{j=1}^m \lambda_j [A_j]_{H^1_{de\; R,\; rec}(\Lambda)}, \]

and, furthermore, as

\[ \lambda_j = \int_{\sigma_j} A, \]

we reconstruct \( \lambda_j, j = 1,2,\cdots,m \) (modulo \( 2\pi \)) from the high-velocity limit (5.44) known for all \( \hat{v},\hat{w} \).

□

We now give a precise definition of when a line \( L(x,\hat{v}) \) goes through a hole of \( K \). Take \( r > 0 \) such that \( K \subset B_{2r}^3(0) \).

Suppose that \( L(x,\hat{v}) \subset \Lambda \), and \( L(x,\hat{v}) \cap B_{r}^3(0) \neq \emptyset \). we denote by \( c(x,\hat{v}) \) the curve consisting of the segment
DEFINITION 7.4. A line $L(x, \hat{v}) \subset \Lambda$ goes through a hole of $K$ if $L(x, \hat{v}) \cap B_R^3(0) \neq \emptyset$ and $[c(x, \hat{v})]_{H_1(\Lambda; \mathbb{R})} \neq 0$. Otherwise we say that $L(x, \hat{v})$ does not go through a hole of $K$.

Note that this characterization of lines that go or do not go through a hole of $K$ is independent of the $r$ that was used in the definition. This follows from the homotopic invariance of homology. See Theorem 11.2, page 59 of [16].

In an intuitive sense $[c(x, \hat{v})]_{H_1(\Lambda; \mathbb{R})} = 0$ means that $c(x, \hat{v})$ is the boundary of a surface (actually of a chain) that is contained in $\Lambda$ and then it can not go through a hole of $K$. Obviously, as $K \subset B^3_R(0)$, if $L(x, \hat{v}) \cap B^3_R(0) = \emptyset$ the line $L(x, \hat{v})$ can not go through a hole of $K$.

DEFINITION 7.5. Two lines $L(x, \hat{v}), L(y, \hat{w}) \subset \Lambda$ that go through a hole of $K$ go through the same hole if $[c(x, \hat{v})]_{H_1(\Lambda; \mathbb{R})} = \pm [c(y, \hat{w})]_{H_1(\Lambda; \mathbb{R})}$. Furthermore, we say that the lines go through the hole in the same direction if $[c(x, \hat{v})]_{H_1(\Lambda; \mathbb{R})} = [c(y, \hat{w})]_{H_1(\Lambda; \mathbb{R})}$.

LEMMA 7.6. Let $A, A_0 \in A_0(0)$ with $A_0$ short-range and let $\lambda$ be such that $A_0 = A + d\lambda$. Assume that $L(x, \hat{v})$ and $L(y, \hat{w})$ go through the same hole of $K$. Then,

$$
\int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v}) \, d\tau + \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) = \pm \left( \int_{-\infty}^{\infty} \hat{w} \cdot A(y + \tau \hat{w}) \, d\tau + \lambda_\infty(\hat{w}) - \lambda_\infty(-\hat{w}) \right),
$$

(7.6)

if $[c(x, \hat{v})]_{H_1(\Lambda; \mathbb{R})} = \pm [c(y, \hat{w})]_{H_1(\Lambda; \mathbb{R})}$.

Moreover,

$$
\int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v}) \, d\tau + \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) = \int_{-\infty}^{\infty} \hat{v} \cdot A_0(x + \tau \hat{v}) \, d\tau = \int_{c(x, \hat{v})} A_0 = \int_{c(x, \hat{v})} A.
$$

(7.7)

Proof: By (7.1) and Stoke's theorem,

$$
\int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v}) \, d\tau + \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) = \int_{-\infty}^{\infty} \hat{v} \cdot A_0(x + \tau \hat{v}) \, d\tau = \int_{c(x, \hat{v})} A_0 = \pm \int_{c(y, \hat{w})} A_0 = \pm \left( \int_{-\infty}^{\infty} \hat{w} \cdot A(y + \tau \hat{w}) \, d\tau + \lambda_\infty(\hat{w}) - \lambda_\infty(-\hat{w}) \right).
$$

LEMMA 7.7. Let $A, A_0 \in A_0(0)$ with $A_0$ short-range and let $\lambda$ be such that $A_0 = A + d\lambda$. Assume that $L(x, \hat{v})$ does not go through a hole of $K$. Then,

$$
\int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v}) \, d\tau + \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) = 0.
$$

(7.8)

Proof: If $L(x, \hat{v}) \cap B^3_R(0) = \emptyset$ it follows from (7.1) and Stoke's theorem that (7.8) holds. Otherwise, $[c(x, \hat{v})]_{H_1(\Lambda; \mathbb{R})} = 0$, and then, by Stoke's theorem,

$$
\int_{c(x, \hat{v})} A = 0.
$$
Take \( z \in \partial B_2^3(0) \cap c(x, \hat{v}) \) such that \( L(z, \hat{v}) \) is tangent to \( \partial B^3_2(0) \). By the argument above,

\[
\int_{-\infty}^{\infty} \hat{v} \cdot A(z + \tau \hat{v}) \, d\tau + \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) = 0.
\]

Finally, using once more Stoke's theorem we obtain that,

\[
0 = \int_{c(x, \hat{v})} A = \int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v}) \, d\tau - \int_{-\infty}^{\infty} \hat{v} \cdot A(z + \tau \hat{v}) \, d\tau,
\]

and then, (7.8) is proven.

**REMARK 7.8.** If \((x, \hat{v}) \in \Lambda \times S^2\), there are neighborhoods \( B_x \subset \mathbb{R}^3, B_{\hat{v}} \subset S^2 \) such that \((x, \hat{v}) \in B_x \times B_{\hat{v}}\) and if \((y, \hat{w}) \in B_x \times B_{\hat{w}}\) then, the following is true: if \(L(x, \hat{v})\) does not go true a hole of \(K\), then, also \(L(y, \hat{w})\) does not go through a hole of \(K\). If \(L(x, \hat{v})\) goes through a hole of \(K\), then, \(L(y, \hat{w})\) goes through the same hole and in the same direction. This follows from the homotopic invariance of homology, Theorem 11.2, page 59 of [16].

**DEFINITION 7.9.** For any \( \hat{v} \in S^2 \) we denote by \( \Lambda_{\hat{v}, \text{out}} \) the set of points \( x \in \Lambda_{\hat{v}} \) such that \( L(x, \hat{v}) \) does not go through a hole of \( K \). We call this set the region without holes of \( \Lambda_{\hat{v}} \). The holes of \( \Lambda_{\hat{v}} \) is the set \( \Lambda_{\hat{v}, \text{in}} := \Lambda_{\hat{v}} \setminus \Lambda_{\hat{v}, \text{out}} \).

We define the following equivalence relation on \( \Lambda_{\hat{v}, \text{in}} \). We say that \( xR_\hat{v}y \) if and only if \( L(x, \hat{v}) \) and \( L(y, \hat{v}) \) go through the same hole and in the same direction. By \([x]\) we designate the classes of equivalence under \( R_\hat{v} \).

We denote by \( \{\Lambda_{\hat{v}, h}\}_{h \in \mathcal{I}} \) the partition of \( \Lambda_{\hat{v}, \text{in}} \) given by this equivalence relation. It is defined as follows.

\[
\mathcal{I} := \{[x]\}_{x \in \Lambda_{\hat{v}, \text{in}}}.
\]

Given \( h \in \mathcal{I} \) there is \( x \in \Lambda_{\hat{v}, \text{in}} \) such that \( h = [x] \). We denote,

\[
\Lambda_{\hat{v}, h} := \{y \in \Lambda_{\hat{v}, \text{in}} : yR_\hat{v}x\}.
\]

Then,

\[
\Lambda_{\hat{v}, \text{in}} = \bigcup_{h \in \mathcal{I}} \Lambda_{\hat{v}, h}, \quad \Lambda_{\hat{v}, h_1} \cap \Lambda_{\hat{v}, h_2} = \emptyset, \quad h_1 \neq h_2.
\]

We call \( \Lambda_{\hat{v}, h} \) the hole \( h \) of \( K \) in the direction of \( \hat{v} \). Note that

\[
\{\Lambda_{\hat{v}, h}\}_{h \in \mathcal{I}} \cup \{\Lambda_{\hat{v}, \text{out}}\} \quad (7.9)
\]

is an open disjoint cover of \( \Lambda_{\hat{v}} \).

**DEFINITION 7.10.** For any \( \Phi, A \in \mathcal{A}_{\Phi, 2\pi}(B), \hat{v} \in S^2, \) and \( h \in \mathcal{I} \) we define,

\[
F_h := \int_{c(x, \hat{v})} A,
\]

where \( x \) is any point in \( \Lambda_{\hat{v}, h} \). Note that \( F_h \) is independent the \( x \in \Lambda_{\hat{v}, h} \) that we choose. \( F_h \) is the flux of the magnetic field over any surface (or chain) in \( \mathbb{R}^3 \) whose boundary is \( c(x, \hat{v}) \). We call \( F_h \) the magnetic flux on the hole \( h \) of \( K \).
Let us take \( \phi_0 \in \mathcal{H}_2(\mathbb{R}^3) \) with compact support in \( \Lambda_\varphi \). Then, since \((7.10)\) is a disjoint open cover of \( \Lambda_\varphi \),

\[
\phi_0 = \sum_{h \in I} \varphi_h + \varphi_{\text{out}}, \tag{7.10}
\]

with \( \varphi_h, \varphi_{\text{out}} \in \mathcal{H}_2(\mathbb{R}^3) \), \( \varphi_h \) has compact support in \( \Lambda_{\varphi,h}, h \in \mathcal{H} \), and \( \varphi_{\text{out}} \) has compact support in \( \Lambda_{\varphi,\text{out}} \). The sum is finite because \( \phi \) has compact support. We denote,

\[
\phi_\varphi := e^{imv \cdot x} \phi_0, \; \varphi_{h,\varphi} := e^{imv \cdot x} \varphi_h, \; \varphi_{\text{out},\varphi} := e^{imv \cdot x} \varphi_{\text{out}}.
\]

**THEOREM 7.11.** Suppose that \( B \equiv 0 \) and that \( V \) satisfies Assumption \( [5.7] \). Then, for any \( \Phi \) and any \( A \in \mathcal{A}_\Phi(0) \),

\[
S(A,V) \phi_\varphi = e^{-i(\lambda_\infty(\hat{v})-\lambda_\infty(-\hat{v}))} \left( \sum_{h \in I} e^{iF_h} \varphi_{h,\varphi} + \varphi_{\text{out},\varphi} \right) + O \left( \frac{1}{v} \right). \tag{7.11}
\]

**Proof:** The theorem follows from Theorem \( 5.7 \) and Lemmas \( 7.6 \) \( 7.7 \).

**COROLLARY 7.12.** Under the conditions of Theorem \( 7.11 \)

\[
(S(A,V) \phi_\varphi, \varphi_{h,\varphi}) = e^{-i(\lambda_\infty(\hat{v})-\lambda_\infty(-\hat{v}))} e^{iF_h} + O \left( \frac{1}{v} \right), \; h \in I, \tag{7.12}
\]

\[
(S(A,V) \phi_\varphi, \varphi_{\text{out},\varphi}) = e^{-i(\lambda_\infty(\hat{v})-\lambda_\infty(-\hat{v}))} + O \left( \frac{1}{v} \right). \tag{7.13}
\]

Moreover, the high-velocity limit of \( S(A,V) \) in the direction \( \hat{v} \) determines \( \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) \) and the fluxes \( F_h, h \in I \), modulo \( 2\pi \).

**Proof:** The corollary follows immediately from Theorem \( 7.11 \).

**REMARK 7.13.** Equations \( 7.12 \) \( 7.13 \) are reconstruction formulae that allows us to reconstruct \( \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) \) and the fluxes \( F_h, h \in I \), modulo \( 2\pi \), from the high-velocity limit of the scattering operator in the direction \( \hat{v} \). Recall that \( \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) \) is independent of the particular short-range potential that we use to define \( \lambda \). Remember also that given any \( A \in \mathcal{A}_{\Phi,2\pi}(B) \) we can always find an \( A \in \mathcal{A}_\Phi(B) \) with the same scattering operator. We can take, for example, \( A_\Phi \). See equation \( 5.12 \). Then, there is no loss of generality taking \( A \in \mathcal{A}_\Phi(0) \).

Note that it is quite remarkable that we can determine \( \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) \) since it is not a gauge invariant quantity. According to the standard interpretation of quantum mechanics only gauge invariant quantities are physically relevant.

Note that if \( A \) is short-range \( \lambda_\infty \) is constant. In this case \( \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) \equiv 0 \) and it drops out from all our formulae. We see that one possibility is to consider that only short-range potentials are physically admissible. This is consistent with the usual interpretation of quantum mechanics in three dimensions. However, we can also go beyond the standard interpretation of quantum mechanics and consider the class of long-range potentials \( \mathcal{A}_\Phi(B) \) as physically admissible. This raises the interesting question of what is the physical significance of the \( \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}) \).
EXAMPLE 7.14. Here we consider a simple example where we give an explicit description of the holes. Furthermore, the fluxes of the holes are the fluxes of the magnetic field over cross sections of the tori. We reconstruct all the fluxes modulo $2\pi$ and also we determine the cohomology class of the magnetic potential modulo $2\pi$, from the high-velocity limit of the scattering operator in only one direction.

Given a vector $z \in \mathbb{R}^3$ and $a > b > 0$ we denote by $T(z, a, b)$ the following set

$$T(z, a, b) := \left\{ z + a(\cos \theta, \sin \theta, 0) + b(x(\cos \theta, \sin \theta, 0) + y(0, 0, 1)) : \theta \in [0, 2\pi], (x, y) \in B^2_1(0) \right\}.$$

The map $F_{z,a,b} : T \to T(z, a, b)$ given by

$$F_{z,a,b}((\cos \theta, \sin \theta), (x, y)) \mapsto z + a(\cos \theta, \sin \theta, 0) + b(x(\cos \theta, \sin \theta, 0) + y(0, 0, 1))$$

is a diffeomorphism.

The Obstacle

We now define the obstacle $K$. We assume that $v = (0, 0, 1)$.

As before the connected components of $K$ are $K_j, j = 1, 2, \cdots, L$. Let us denote $J = \{1, 2, \cdots, m\}$ and $I = \{m + 1, \cdots, L\}$. If $m = L$, then, $I = \emptyset$. We assume that $K$ satisfy the following assumptions.

1. There are vectors $z_j \in \mathbb{R}^3$ and numbers $a_j > b_j, j = 1, 2, \cdots, m$ such that,

$$K_j = T(z_j, a_j, b_j), \forall j \in J, K_j \cong B^2_1(0), j \in I.$$

2. 

$$(\text{convex}(K_j) + \mathbb{R}v) \cap \text{convex}(K_l) + \mathbb{R}v) = \emptyset, j, l \in J, (\text{convex}(K_j) + \mathbb{R}v) \cap (K_l + \mathbb{R}v) = \emptyset, j \in J, l \in I.$$

We denote as before by $\text{convex}(\cdot)$ the convex hull of the indicated set.

The Curves $\gamma_j, \tilde{\gamma}_j, \hat{\gamma}_j$

Let $\theta_j$ be such that $z_j = r_j(\cos(\theta_j), \sin(\theta_j), 0) + (0, 0, (z_j)_3)$.

The curves $\gamma_j, j \in J$ are given by

$$\gamma_j(t) := z_j + a_j(\cos t, \sin t, 0),$$

and the curves $\tilde{\gamma}_j, j \in J$, are

$$\tilde{\gamma}_j := z_j + a_j(\cos \theta_j, \sin \theta_j, 0) + b_j(\cos t(\cos \theta_j, \sin \theta_j, 0) + \sin t(0, 0, 1)).$$

Furthermore, the curves $\hat{\gamma}_j, j \in J$, are

$$\hat{\gamma}_j := z_j + a_j(\cos \theta_j, \sin \theta_j, 0) + (b_j + \delta/2)(\cos t(\cos \theta_j, \sin \theta_j, 0) + \sin t(0, 0, 1)).$$
where \( \delta > 0 \) so small that, \( \delta < a_j - b_j \), and

\[
(\text{convex}(K_j, \delta) + \mathbb{R}v) \cap (\text{convex}(K_l, \delta) + \mathbb{R}v) = \emptyset, j, l \in J,
\]

\[
((\text{convex } K_j, \delta) + \mathbb{R}v) \cap (K_l, \delta + \mathbb{R}v) = \emptyset, j \in J, l \in I.
\]

The subindex \( \delta \) denotes the set of points that are at distance up to \( \delta \) of the indicated set.

**The Flux \( \Phi \)**

We define the following sets

\[
h_j := \{z_j + t(\cos \theta, \sin \theta) : \theta \in [0, 2\pi], t \in [0, a_j - b_j]\} + \mathbb{R}\hat{v}, j \in J.
\]

We have that,

\[
[c(x, \hat{v})]_{H_1(\Lambda; \mathbb{R})} = [c(y, \hat{v})]_{H_1(\Lambda; \mathbb{R})}, \forall x, y \in h_j, j \in J.
\]  
(7.14)

Since \( c(x, \hat{v}) \) and \( c(y, \hat{v}) \) are homotopic in \( \Lambda \), this follows from the homotopic invariance of homology. See Theorem 11.2, page 59 of [16]. Then, we can associate a flux \( \Phi_j \) to each \( h_j, j \in J \) as follows,

\[
\Phi_j = \int_{c(x, \hat{v})} A, \text{ for some } x \in h_j, j \in J.
\]

We have that,

\[
[c(y, \hat{v})]_{H_1(\Lambda; \mathbb{R})} = 0, \forall y \in (\Lambda \setminus (\cup_{j \in J} h_j)) \cap \left( B^3_r(0) + \mathbb{R}\hat{v} \right).
\]  
(7.15)

Let us prove this. As the segment of straight line in \( c(y, \hat{v}) \) does not belong to any of the sets convex \((K_j) + \mathbb{R}\hat{v}, j \in J\), we have that, for any \( j \in J \) there is a surface (or a chain) \( \sigma_j \) contained in the complement of \( K_j \) such that \( \partial \sigma_j = c(y, \hat{v}) \).

Let \( \left\{ [G^{(j)}]_{H_1(\Lambda; \mathbb{R})} \right\}_{j=1}^m \) be the basis of \( H_1(\Lambda; \mathbb{R}) \) constructed in Proposition 2.3. Then, as \( dG^{(j)} = 0 \) it follows from Stokes' theorem that

\[
\int_{c(y, \hat{v})} G^{(j)} = 0, \forall j \in J.
\]

Hence, (7.15) follows from de Rham's Theorem, Theorem 4.17, page 154 of [41].

Let us prove now that,

\[
\Phi_j = \Phi(\hat{\gamma}_j), j \in J.
\]  
(7.16)

For any \( j \in J \) we define,

\[
x_j := z_j + a_j(\cos \theta_j, \sin \theta_j, 0) - (b_j + \delta/2)(\cos \theta_j, \sin \theta_j, 0),
\]

\[
y_j := z_j + a_j(\cos \theta_j, \sin \theta_j, 0) + (b_j + \delta/2)(\cos \theta_j, \sin \theta_j, 0).
\]

We choose the curves \( c(x_j, \hat{v}), c(y_j, \hat{v}) \) in such a way that the arc in \( c(y_j, \hat{v}) \) is contained in the arc in \( c(x_j, \hat{v}) \). Let \( c_j \) be the curve obtained by taking the segments of straight line in \( c(x_j, \hat{v}) \) and in \( c(y_j, \hat{v}) \) and the two arcs that are
obtained by cutting from the arc in \( c(x_j, \hat{v}) \) the arc in \( c(y_j, \hat{v}) \). We orient \( c_j \) in such a way that the segment of straight line in \( c(x_j, \hat{v}) \) has the orientation of \( \hat{v} \). Then, in homology,

\[
[c_j]_{H_1(\Lambda; \mathbb{R})} = [c(x_j, \hat{v})]_{H_1(\Lambda; \mathbb{R})} - [c(y_j, \hat{v})]_{H_1(\Lambda; \mathbb{R})}.
\]  

(7.17)

This follows from de Rham’s Theorem -Theorem 4.17, page 154 of [41]- since for any closed 1-form \( D \),

\[
\int_{c_j} D = \int_{c(x_j, \hat{v})} D - \int_{c(y_j, \hat{v})} D.
\]

The curves \( \hat{\gamma}_j \) and \( c_j \) are homotopically equivalent in \( \Lambda \). Hence, by the homotopical invariance of homology, Theorem 11.2, page 59 of [16],

\[
[c_j]_{H_1(\Lambda; \mathbb{R})} = [\hat{\gamma}_j]_{H_1(\Lambda; \mathbb{R})}.
\]  

(7.18)

Then, by (7.15) (7.17),

\[
[\hat{\gamma}_j]_{H_1(\Lambda; \mathbb{R})} = [c(x_j, \hat{v})]_{H_1(\Lambda; \mathbb{R})},
\]

(7.19)

and hence,

\[
\int_{c(x_j, \hat{v})} A = \int_{\hat{\gamma}_j} A, \ j \in J,
\]

what proves (7.16).

**The Holes of \( K \)**

Recall that \( \Lambda_{\hat{v}, \text{out}} \) and \( \Lambda_{\hat{v}, \text{in}} \) were defined in Definition 7.9 that the holes of \( K \) are the sets \( \Lambda_{\hat{v}, h}, h \in \mathcal{I} \), that \( F_h \) is the flux over the hole \( \Lambda_{\hat{v}, h}, h \in \mathcal{I} \), that \( \Lambda_{\hat{v}, \text{in}} = \bigcup_{h \in \mathcal{I}} \Lambda_{\hat{v}, h} \).

Then, we have that,

1. The index set \( \mathcal{I} \) can be taken as \( \mathcal{I} = \{h_j\}_{j \in J} \sim J \). Moreover, denoting \( \Lambda_{\hat{v}, j} = \Lambda_{\hat{v}, h_j} \), we have that \( \Lambda_{\hat{v}, j} = h_j \) and \( \Lambda_{\hat{v}, \text{in}} = \bigcup_{j \in J} h_j \).

2. We designate, \( F_j := F_{h_j} \). Then,

\[
F_j = \Phi(\hat{\gamma}_j), \ j \in J.
\]

3. \( \Lambda_{\hat{v}, \text{out}} = \mathbb{R}^3 \setminus \left( \Lambda_{\hat{v}, \text{in}} \cup_{j=1}^J (K_j + \hat{v} \mathbb{R}) \right) \).

Let us prove this. By (7.15) \( [c(y, \hat{v})]_{H_1(\Lambda; \mathbb{R})} = 0, \forall y \in (\Lambda_{\hat{v}} \setminus \cup_{j \in J} h_j) \) such that \( B_{r}^\mathbb{R}(0) \cap L(y, \hat{v}) \neq \emptyset \). Then,

\[
(\Lambda_{\hat{v}} \setminus \cup_{j \in J} h_j) \cap \left( B_{r}^\mathbb{R}(0) + \mathbb{R}\hat{v} \right) \subset \Lambda_{\hat{v}, \text{out}}.
\]
But by our definition the complement in $\Lambda_{\psi}$ of $B^{\mathbb{R}_{3}}_{r}(0) + \mathbb{R}\hat{v}$ is contained in $\Lambda_{\psi,\text{out}}$. It follows that,

$$(\Lambda_{\psi} \setminus \bigcup_{j \in J} h_{j}) \subset \Lambda_{\psi,\text{out}}.$$  

Moreover, if $x \in h_{j}$ for some $j \in J$, since $[\gamma_{j}]_{H_{1}(\Lambda;\mathbb{R})} \neq 0$, it follows from (7.14) and (7.19) that $[c(x, \hat{v})]_{H_{1}(\Lambda;\mathbb{R})} = [c(x, \hat{v})]_{H_{1}(\Lambda;\mathbb{R})} \neq 0$, and then, $x \notin \Lambda_{\psi,\text{out}}$. Then, we have proven that,

$$(\Lambda_{\psi} \setminus \bigcup_{j \in J} h_{j}) = \Lambda_{\psi,\text{out}},$$  

(7.20)

and hence,

$$\Lambda_{\psi,\text{in}} = \bigcup_{j \in J} h_{j}.$$  

Item 3 is now obvious. By (7.14) if $x, y \in h_{j}$, then, $[c(x, \hat{v})]_{H_{1}(\Lambda;\mathbb{R})} = [c(y, \hat{v})]_{H_{1}(\Lambda;\mathbb{R})}$. Hence, $xR_{\psi}y$ what implies that $h_{j}$ is contained in some hole of $K$. But by (7.14) and (7.19) if $x \in h_{j}, y \in h_{l}, j \neq l$, then, $[c(x, \hat{v})]_{H_{1}(\Lambda;\mathbb{R})} \neq [c(y, \hat{v})]_{H_{1}(\Lambda;\mathbb{R})}$ because as the $[\gamma_{j}]_{H_{1}(\Lambda;\mathbb{R})}, j \in J$ are a basis of $H_{1}(\Lambda;\mathbb{R})$ they are different. In consequence, $x$ and $y$ belong to different holes of $K$. Then, since (7.20) holds, we have proven item 1. Item 2 follows from (7.16).

By Corollary 7.12 and Remark 7.13 this proves that from the high-velocity limit of $S(A, V)$ in the direction of $\hat{v}$ we reconstruct all the fluxes $\Phi(\gamma_{j}), j \in J$, modulo $2\pi$.

Let us now prove that from the high-velocity limit of $S(A, V)$ in the direction of $\hat{v}$ we also reconstruct the cohomology class $[A]_{H_{1}^{\text{de R}}(\Lambda)}$ modulo $2\pi$, in the sense that we reconstruct modulo $2\pi$ the coefficients of $[A]_{H_{1}^{\text{de R}}(\Lambda)}$ in any basis of $H_{1}^{\text{de R}}(\Lambda)$.

Let $\left\{[M_{j}]_{H_{1}^{\text{de R}}(\Lambda)}\right\}_{j=1}^{m}$ be any basis of $H_{1}^{\text{de R}}(\Lambda)$ and let $\left\{[\Gamma_{j}]_{H_{1}(\Lambda;\mathbb{R})}\right\}_{j=1}^{m}$ be the dual basis of $H_{1}(\Lambda;\mathbb{R})$ given by de Rham’s Theorem,

$$\int_{\Gamma_{j}} M_{l} = \delta_{j,l}, j,l \in J.$$  

Let $\left\{\alpha_{j}\right\}_{j \in J}$ be the expansion coefficients of $A$,

$$[A]_{H_{1}^{\text{de R}}(\Lambda)} = \sum_{j \in J} \alpha_{j} [M_{j}]_{H_{1}^{\text{de R}}(\Lambda)}.$$  

$$\alpha_{j} = \int_{\Gamma_{j}} A.$$  

By Proposition 10.1, $\left\{[\gamma_{j}]_{H_{1}(\Lambda;\mathbb{Z})}\right\}_{j=1}^{m}$ is a basis of $H_{1}(\Lambda;\mathbb{Z})$. Then,

$$[\Gamma_{j}]_{H_{1}(\Lambda;\mathbb{Z})} = \sum_{l \in J} n(j,l) [\gamma_{l}]_{H_{1}(\Lambda;\mathbb{Z})},$$

where the coefficients $n(j,l)$ are integers. Finally,
\[ \alpha_j = \int_{\Gamma_j} A = \sum_{l \in L} n(j, l) \int_{\hat{\gamma}_l} A = \sum_{l \in L} n(j, l) \Phi(\hat{\gamma}_l), j \in J, \]

and since we have already determined the \( \Phi(\hat{\gamma}_l) \) modulo \( 2\pi \), the coefficients \( \alpha_j, j \in J \) are determined modulo \( 2\pi \).

### 8 The Tonomura et al. Experiments

The fundamental experiments of Tonomura et al. \[37, 38\], gave a conclusive evidence of the existence of the Aharonov-Bohm effect. For a detailed account see \[30\].

Tonomura et al. \[37, 38\] did their experiments in the case of toroidal magnets. This corresponds to our Example 7.14 with only one torus, i.e., \( L = 1, J = \{1\} \). In very careful and precise experiments they managed to superimpose behind the toroidal magnet two electron beams. One of them traveled inside the hole of the toroidal magnet and the other -the reference beam- outside it. They measured the interference fringes between the two beams produced by the magnetic flux inside the torus.

We show now that our results give a rigorous mathematical proof that quantum mechanics predicts the interference fringes observed by Tonomura et al. \[37, 38\] in their remarkable experiment.

An equivalently description of these experiments is to consider that both electron beams traveled inside the hole of the torus, one of them with a nonzero magnetic flux inside the torus, and the other -the reference beam- with the magnetic flux inside the torus set to zero. Since long-range magnetic potentials add a global constant phase that does not affect the interference pattern, we take, for simplicity, a short-range magnetic potential. According to Theorem 7.11 for the particle that goes inside the hole with the magnetic flux present, up to an error of order \( 1/v \), we have that,

\[ S(A, V)\phi_0 = e^{i\frac{2}{\hbar} \Phi} \phi_0, \quad (8.1) \]

where we have taken physical units, with \( \Phi \) the flux of the physical magnetic field \( B \) and \( \phi_0 = e^{i\frac{2}{\hbar} \hat{V} \cdot x} \phi_0 \). See Section 4. For the particle that goes outside the hole of the magnet, or equivalently inside the hole with the magnetic field set to zero,

\[ S(A, V)\phi_0 = \phi_0, \quad (8.2) \]

If we superimpose both asymptotic states we obtain the wave function,

\[ \left(1 + e^{i\frac{2}{\hbar} \Phi}\right) \phi_0, \quad (8.3) \]
up to an error of order $1/v$. This shows the interference patterns that were observed experimentally by Tonomura et al. \[37, 38\]. For example, if $\frac{\Phi}{\pi}$ is an odd multiple of $\pi$ there is destructive interference and there is a dark zone behind the hole of the magnet, as observed experimentally.

Tonomura et al. \[37, 38\] also considered the case when the reference beam is slightly tilted. In this case the reference beam is given by

$$\phi_{\psi + \varphi_0} = e^{i\frac{\Phi}{\pi}v_0 \cdot \varphi} \phi_{\bar{\psi}},$$

and $[\Sigma_2]$ is replaced by,

$$S(A, V)\phi_{\psi + \varphi_0} = \phi_{\psi + \varphi_0} = e^{i\frac{\Phi}{\pi}v_0 \cdot \varphi} \phi_{\bar{\psi}}.$$

In this case we obtain the wave function

$$e^{i\frac{\Phi}{\pi}v_0 \cdot \varphi} \left(1 + e^{-i\frac{\Phi}{\pi}v_0 \cdot \varphi} e^{i\frac{\Phi}{\pi}}\right) \phi_{\bar{\psi}},$$

up to an error of order $1/v$. We see that the factor,

$$\left(1 + e^{-i\frac{\Phi}{\pi}v_0 \cdot \varphi} e^{i\frac{\Phi}{\pi}}\right)$$

produces the parallel fringes that were observed experimentally by Tonomura et al. \[37, 38\].

9 Appendix A

In this appendix we prove, for the reader’s convenience, that $H_1(\mathbb{Z}, \mathbb{R}; \mathcal{R}) = 0$, $s \geq 2$, that $H_1(\mathbb{Z}, \mathbb{R}; \mathcal{R}) \cong \bigoplus_{i=1}^k \mathcal{R}$, and that $\{[Z_j]_{H_1(\mathbb{Z}, \mathbb{R}; \mathcal{R})}\}_{j=1}^k$ is a basis of $H_1(\mathbb{Z}, \mathbb{R}; \mathcal{R})$. $\mathcal{R}$ is $\mathbb{Z}$ or $\mathbb{R}$.

Recall that we defined, $\gamma_{\pm} : [0, 1] \rightarrow T : \gamma_{\pm}(t) = \left(e^{\pm 2\pi it}, 0, 0\right)$.

**Proposition 9.1.** $H_s(T; \mathbb{R}) = 0$, $s > 2$ and $H_1(T; \mathbb{Z}) \cong \mathbb{Z}$ and $\{[\gamma_{\pm}]_{H_1(T; \mathbb{Z})}\}$ are basis of $H_1(T; \mathbb{Z})$.

**Proof:** We define $\tilde{\gamma}_{\pm} : [0, 1] \rightarrow S^1 : \tilde{\gamma}_{\pm}(t) := e^{\pm 2\pi it}$ and let $I_{S^1} : S^1 \rightarrow T$ be the inclusion given by $I_{S^1}(s) := (s, 0, 0)$. Clearly, $I_{S^1} \circ \tilde{\gamma}_{\pm} = \gamma_{\pm}$. It is easy to see that $S^1$ is homotopically equivalent to $T$ and that the inclusion $I_{S^1} : S^1 \rightarrow T$ is a homotopic equivalence. It follows that $I_{S^1}$ induces an isomorphism in homology given by $H_s(I_{S^1})$ (see theorem 11.3, page 59 \[16\]). Then, $H_s(T; \mathbb{R}) \cong H_s(S^1; \mathbb{R})$ and hence, we have that $H_s(T; \mathbb{R}) = 0$, $s > 2$ by Corollary 15.5, page 84 of \[16\]. For $s = 1$ and $\mathcal{R} = \mathbb{Z}$, the isomorphism is given in the following way (see page 49 \[14\]). Let $\sigma_{i} : [0, 1] \rightarrow T$ be continuous functions and let $n_i \in \mathbb{Z}$. Let us assume that $\sum n_i \sigma_i$ is a cycle (its boundary is zero). Then, $H_1(I_{S^1})[\sum n_i \sigma_i]_{H_1(S^1; \mathbb{Z})} := [\sum n_i I_{S^1} \circ \sigma_i]_{H_1(T; \mathbb{Z})}$.

As $I_{S^1} \circ \tilde{\gamma}_{\pm} = \gamma_{\pm}$, it follows that $H_1(I_{S^1})[\tilde{\gamma}_{\pm}]_{H_1(S^1; \mathbb{Z})} = [\gamma_{\pm}]_{H_1(T; \mathbb{Z})}$. Then, to prove the Proposition it is enough to prove that $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$ and that $\{[\gamma_{\pm}]_{H_1(S^1; \mathbb{Z})}\}$ are basis of $H_1(S^1; \mathbb{Z})$.

By Theorem 12.1, page 63 of \[16\], there is a homomorphism $\Xi : \Pi_1(S^1; 1) \rightarrow H_1(S^1; \mathbb{Z})$ that sends a homotopy class to its homology class. In our case $\Xi$ is an isomorphism since $\Pi_1(S^1; 1)$ is abelian. Actually, $\Pi_1(S^1; 1) \cong \mathbb{Z}$. See
Theorem 4.4, page 17 of [16]. Then, \( Z \cong \Pi_1(S^1; 1) \cong H_1(S^1; Z) \). To prove that \( \{[\gamma_{\pm}]_{H_1(S^1; Z)} \} \) are basis of \( H_1(S^1; Z) \), it is enough to prove that \( \{[\gamma_{\pm}]_{H_1(S^1; 1)} \} \) are basis of \( \Pi_1(S^1; 1) \). The isomorphism \( \Lambda : \Pi_1(S^1; Z) \to Z \) is given (see Theorem 4.4, page 17 of [16]) as follows. Given a path \( \sigma \) with \([\sigma]_{H_1(S^1; 1)} \in \Pi_1(S^1; 1)\) let \( \sigma' : [0, 1] \to \mathbb{R} \) satisfy \( \sigma'(0) = 0 \) and \( e^{2\pi i \omega'(t)} = \sigma(t) \). Then, \( \Lambda[\sigma]_{H_1(S^1; 1)} = \sigma'(1) \). In our case, if we take \( \gamma_{\pm}^1(t) := \pm t, \gamma_{\pm}^1(0) = 0 \) and \( \gamma_{\pm}^1(1) = \pm 1 \). It follows that \( \Lambda[\gamma_{\pm}]_{H_1(S^1; 1)} = \pm 1 \). As \( \pm 1 \) are basis of \( Z \) it follows that \( \{[\gamma_{\pm}]_{H_1(S^1; 1)} \} \) are basis of \( \Pi_1(S^1; 1) \) and this concludes the proof that \( [\gamma_{\pm}]_{H_1(T; Z)} \) are basis of \( H_1(T; Z) \).

**PROPOSITION 9.2.** For \( s \geq 2 \), \( H_s(\#kT; R) = 0 \). Furthermore, \( H_1(\#kT; Z) \cong \oplus_{i=1}^k Z \), and \( \{[Z_j]_{H_i(\#kT; Z)}\}_{j=1}^k \) is a basis of \( H_1(\#kT; Z) \).

**Proof:** We prove the Proposition by induction in \( k \). For \( k = 1 \), \( Z_1 = \gamma_+ \) and the result follows from Proposition 9.1. Let us assume that \( H_s(\#(k-1)T; R) \cong \oplus_{i=1}^{k-1} R \) and that \( \{[Z_j]_{H_i(\#(k-1)T; Z)}\}_{j=1}^{k-1} \) is a basis of \( H_1(\#(k-1)T; Z) \). Let \( X_1 \) and \( X_2 \) be open subsets of \( \#kT \) such that

\[
\bigcup_{j \leq k-1} l_j(T) \subset X_1, \quad l_k(T) \subset X_2, \quad X_1 \simeq \bigcup_{j \leq k-1} l_j(T) \simeq \#(k-1)T, \quad X_2 \simeq l_k(T) \simeq T,
\]

and \( X_1 \cap X_2 \) is contractible, i.e. \( X_1 \cap X_2 \simeq \) to a single point. The symbol \( \simeq \) means homotopic equivalence and \( \cong \) means homeomorphism.

By Example 17.1, page 98 of [16] (\( \#kT, X_1, X_2 \)) is an exact triad and we can apply the sequence of Mayer-Vietoris (17.7 page 99 and 17.9 page 100 of [16]).

\[
H_s(X_1 \cap X_2; R) \to H_s(X_1; R) \oplus H_s(X_2; R) \to H_s(\#kT; Z) \to H_{s-1}^Z(X_1 \cap X_2; R).
\]

As \( X_1 \cap X_2 \) is homotopically equivalent to a point -that we denote by \( \{*\}- \) we have that \( H_s(X_1 \cap X_2; R) \cong H_s(\{*\}; R) = 0, H_{s-1}^Z(X_1 \cap X_2; R) \cong H_{s-1}^Z(\{*\}; R) = 0 \) (see Theorem 11.3, page 59, Example 9.4, page 47 and Example 9.7, page 48 of [16]). Hence, we obtain the isomorphism,

\[
H_s(X_1; R) \oplus H_s(X_2; R) \to H_s(\#kT; R). \tag{9.1}
\]

This isomorphism is given by (see 17.4, page 99 of [16])

\[
([c_1]_{H_s(X_1; R)}, [c_2]_{H_s(X_2; R)}) \mapsto -[c_1]_{H_s(\#kT; R)} + [c_2]_{H_s(\#kT; R)}. \tag{9.2}
\]

As \( \bigcup_{j \leq k-1} l_j(T) \simeq X_1, l_k(T) \simeq X_2 \), the inclusions \( \bigcup_{j \leq k-1} l_j(T) \hookrightarrow X_1, l_k(T) \hookrightarrow X_2 \) induce isomorphisms in homology (see Theorem 11.3, page 59 of [16]). We have, then, the following isomorphisms.

\[
H_s(\#(k-1)T; R) \cong H_s(\bigcup_{j \leq k-1} l_j(T); R) \cong H_s(X_1; R), \tag{9.3}
\]

\[
H_s(T; R) \cong H_s(l_k(T); R) \cong H_s(X_2; R). \tag{9.4}
\]
By our induction hypothesis and Corollary 3.1, \( H_s(\mathbb{z} k T; \mathbb{R}) = 0, s \geq 2 \). Moreover, by the induction hypothesis and (9.3), it also follows that \( \{ [Z_j]_{H_i(X_1; \mathbb{Z})} \}_{j=1}^{k-1} \) is a basis of \( H_1(X_1; \mathbb{Z}) \).

By Proposition 9.1 and (9.4), \( H_1(X_1; \mathbb{Z}) \cong \mathbb{Z} \); furthermore, by the definition of \( Z_k \) (see 2.2) and as the homeomorphism \( l_k : T \to l_k(T) \) induces an isomorphism in homology it follows from Proposition 9.1 that \( [Z_k]_{H_1(l_k(T); \mathbb{Z})} \) is a basis of \( H_1(l_k(T); \mathbb{Z}) \) and then, by (9.4) it is also a basis of \( H_1(X_2; \mathbb{Z}) \). Finally, it follows from (9.4) that \( H_1(\mathbb{z} k T; \mathbb{Z}) \cong \bigoplus_{i=1}^{k} \mathbb{Z} \) and \( \{ [Z_j]_{H_i(\mathbb{z} k T; \mathbb{Z})} \}_{j=1}^{k} \) is a basis of \( H_1(\mathbb{z} k T; \mathbb{R}) \).

**PROPOSITION 9.3.** \( H_1(\mathbb{z} k T; \mathbb{R}) \cong \bigoplus_{i=1}^{k} \mathbb{R} \) and \( \{ [Z_j]_{H_i(\mathbb{z} k T; \mathbb{R})} \}_{j=1}^{k} \) is a basis of \( H_1(\mathbb{z} k T; \mathbb{R}) \).

**Proof:** The homology group \( H_1(\mathbb{z} k T; \mathbb{R}) \) is a module over the ring \( \mathbb{R} \), i.e. it is a vector space (page 47 of [10]). If \( G \) is an abelian group we can also define the homology groups as in page 153 of [17]. In this case \( H_1(\mathbb{z} k T; G) \) is a group. As \( \mathbb{R} \) is a group and a ring we can define the homology groups as modules and as groups. To differentiate them we will denote by \( H_1(\mathbb{z} k T; \mathbb{R}) \) the homology module considering \( \mathbb{R} \) as a ring, and by \( \tilde{H}_1(\mathbb{z} k T; \mathbb{R}) \) considering \( \mathbb{R} \) as a group. Actually, \( H_1(\mathbb{z} k T; \mathbb{R}) \) and \( \tilde{H}_1(\mathbb{z} k T; \mathbb{R}) \) are equal as sets and as groups. By the theorem of universal coefficients -Corollary 3 A.4, page 264 of [17]— there is the exact sequence,

\[
0 \to \tilde{H}_s(\mathbb{z} k T; \mathbb{Z}) \otimes \mathbb{R} \to \tilde{H}_s(\mathbb{z} k T; \mathbb{R}) \to \text{Tor}(\tilde{H}_{s-1}(\mathbb{z} k T; \mathbb{Z}), \mathbb{R}) \to 0.
\]

As \( \mathbb{R} \) is torsion free, \( \text{Tor}(\tilde{H}_{s-1}(\mathbb{z} k T; \mathbb{Z}), \mathbb{R}) = 0 \). See Proposition 3A.5, page 265 of [17]. In consequence,

\[
\tilde{H}_s(\mathbb{z} k T; \mathbb{Z}) \otimes \mathbb{R} \cong \tilde{H}_s(\mathbb{z} k T; \mathbb{R}).
\]  

(9.5)

The isomorphism, \( I \), is given as follows. Let \( \sigma \) be a singular simplex and take \( r \in \mathbb{R} \). Then,

\[
I([\sigma] \otimes r) = [r\sigma].
\]

See equation (iv) and Lemma 3.A1, pages 261, 262 of [17]. By Proposition 9.2 \( H_1(\mathbb{z} k T; \mathbb{Z}) \cong \bigoplus_{i=1}^{k} \mathbb{Z} \) and \( \{ [Z_j]_{H_i(\mathbb{z} k T; \mathbb{Z})} \}_{j=1}^{k} \) is a basis of \( H_1(\mathbb{z} k T; \mathbb{Z}) \). Then,

\[
\bigoplus_{i=1}^{k} \mathbb{R} \cong \tilde{H}_1(\mathbb{z} k T; \mathbb{Z}) \otimes \mathbb{R}.
\]

The isomorphism is given by,

\[
\bigoplus_{i=1}^{k} \mathbb{R} \to \bigoplus_{j=1}^{k} (\mathbb{Z} \otimes \mathbb{R}) \to (\bigoplus_{j=1}^{k} \mathbb{Z}) \otimes \mathbb{R} \to \tilde{H}_1(\mathbb{z} k T; \mathbb{Z}) \otimes \mathbb{R}
\]

\[
(r_1, \ldots, r_k) \to (1 \otimes r_1 \cdots 1 \otimes r_k) \to (1, 0, \ldots, 0) \otimes r_1 + \cdots + (0, 0, \ldots, 1) \otimes r_k \to \sum_{j=1}^{k} [Z_j]_{\tilde{H}_1(\mathbb{z} k T; \mathbb{Z})} \otimes r_j.
\]

It follows that the morphism

\[
I' : \bigoplus_{j=1}^{k} \mathbb{R} \to \tilde{H}_1(\mathbb{z} k T; \mathbb{R}) : I'(\langle r_1, \ldots, r_k \rangle) := \sum_{j=1}^{k} [r_j Z_j]_{\tilde{H}_1(\mathbb{z} k T; \mathbb{R})}
\]

is an isomorphism of groups.
We now prove that this implies that \( \{[Z_j]_{H_1(\mathbb{R}; \mathbb{R})}\}_{j=1}^{k} \) is a basis of \( H_1(\mathbb{R}; \mathbb{R}) \) as a vector space. As \( \tilde{H}_1(\mathbb{R}; \mathbb{R}) \) and \( H_1(\mathbb{R}; \mathbb{R}) \) are equal as sets and as groups the morphism

\[
I' : \oplus_{j=1}^{k} \to H_1(\mathbb{R}; \mathbb{R}) : I'((r_1, \cdots, r_k)) := \sum_{j=1}^{k} [r_j Z_j]_{H_1(\mathbb{R}; \mathbb{R})},
\]

is an isomorphism of groups. By the structure of vector space of \( H_1(\mathbb{R}; \mathbb{R}) \) we have that, \( \sum_{j=1}^{k} [r_j Z_j]_{H_1(\mathbb{R}; \mathbb{R})} = \sum_{j=1}^{k} r_j [Z_j]_{H_1(\mathbb{R}; \mathbb{R})} \). As \( I' \) is an isomorphism of groups we have that \( \forall \sigma \in H_1(\mathbb{R}; \mathbb{R}) \) there are real numbers \( \{r_j\}_{j=1}^{k} \) such that \( \sigma = \sum_{j=1}^{k} r_j [Z_j]_{H_1(\mathbb{R}; \mathbb{R})} \). This means that \( \{[Z_j]_{H_1(\mathbb{R}; \mathbb{R})}\}_{j=1}^{k} \) generates \( H_1(\mathbb{R}; \mathbb{R}) \). Moreover, if \( 0 = \sum_{j=1}^{k} r_j [Z_j]_{H_1(\mathbb{R}; \mathbb{R})} = \sum_{j=1}^{k} [r_j Z_j]_{H_1(\mathbb{R}; \mathbb{R})} \) we have that, \( (r_1, r_2, \cdots, r_k) = 0 \) and we conclude that \( \{[Z_j]_{H_1(\mathbb{R}; \mathbb{R})}\}_{j=1}^{k} \) is a linearly independent set and since it also generates \( H_1(\mathbb{R}; \mathbb{R}) \) it is a basis.

# Appendix B

In this appendix we prove, for completeness, the following proposition.

**PROPOSITION 10.1.** \( \{(\gamma_j)_{H_1(\Lambda; \mathbb{Z})}\}_{j=1}^{m} \) is a basis of \( H_1(\Lambda; \mathbb{Z}) \).

**Proof:** For simplicity we will omit \( \mathbb{Z} \) in the homology groups in this proof.

**Step 1.**

As in the proof of [2.40] we prove that \( H_2(\mathbb{R}^3, \mathbb{R}^3 \setminus K) \cong H_1(\mathbb{R}^3 \setminus K) \). Moreover the isomorphism is given by (page 75 of [16])

\[
[\sigma]_{H_1(\mathbb{R}^3, \mathbb{R}^3 \setminus K)} \to [\partial \sigma]_{H_1(\mathbb{R}^3 \setminus K)}. \tag{10.1}
\]

**Step 2.**

Define \( K_\varepsilon := \{x \in \mathbb{R}^3 : \text{dist}(x, K) \leq \varepsilon\} \). Since \( \overline{\mathbb{R}^3 \setminus K_\varepsilon} \subset (\mathbb{R}^3 \setminus K)^\circ \) it follows from the excision theorem (page 82 of [16]) that the inclusion \( (K_\varepsilon, K_\varepsilon \setminus K) \hookrightarrow (\mathbb{R}^3, \mathbb{R}^3 \setminus K) \) induces an isomorphism in homology.

**Step 3.**

Let \( K_{\varepsilon,j}, j = 1, 2, \cdots, L \) be the connected components of \( K_\varepsilon \) for \( \varepsilon \) small enough. Then, \( K_{\varepsilon,j} = \{x \in \mathbb{R}^3 : \text{dist}(x, K_j) \leq \varepsilon\} \). By Proposition 13.9, page 72 of [16]

\[
H_2(K_\varepsilon, K_\varepsilon \setminus K) \cong \oplus_{j=1}^{L} H_2(K_{\varepsilon,j}, K_{\varepsilon,j} \setminus K_j).
\]

**Step 4.**

We have the following homotopic equivalence \( K_{\varepsilon,j} \setminus K_j \simeq \partial K_{\varepsilon,j} \), that induces the isomorphism in homology

\[
H_k(K_{\varepsilon,j} \setminus K_j) \cong H_k(\partial K_{\varepsilon,j}).
\]
Let us consider the exact homology sequences of the pairs \((K_{ε,j}, K_{ε,j} \setminus K_j)\) and \((K_{ε,j}, ∂K_{ε,j})\). The first starts at \(H_k(K_{ε,j} \setminus K_j)\) and ends at \(H_{k-1}(K_{ε,j})\) and the second starts at \(H_k(∂K_{ε,j})\) and ends at \(H_{k-1}(K_{ε,j})\). By the five lemma (page 77 of [16]) the inclusion \((K_{ε,j}, ∂K_{ε,j}) \hookrightarrow (K_{ε,j}, K_{ε,j} \setminus K_j)\) induces the isomorphism in homology,

\[
H_k(K_{ε,j}, ∂K_{ε,j}) \cong H_k(K_{ε,j}, K_{ε,j} \setminus K_j).
\]

Step 5.

By the exact homology sequence for the pair \((K_{ε,j}, ∂K_{ε,j})\) we obtain the sequence

\[
\rightarrow H_2(K_{ε,j}) \rightarrow H_2(K_{ε,j}, ∂K_{ε,j}) \overset{Δ_2}{\rightarrow} H_1(∂K_{ε,j}) \overset{I}{\rightarrow} H_1(K_{ε,j}) \rightarrow 0,
\]

where \(Δ_2\) is taking boundary and \(I\) is the inclusion. By Proposition [17], \(H_2(K_{ε,j}) = 0\). Hence we obtain the exact sequence

\[
0 \rightarrow H_2(K_{ε,j}, ∂K_{ε,j}) \overset{Δ_2}{\rightarrow} H_1(∂K_{ε,j}) \overset{I}{\rightarrow} H_1(K_{ε,j}) \rightarrow 0. \tag{10.2}
\]

Let \(Γ_j \subset \{1, 2, \cdots m\}\) be such that \([γ_i]_{H_1(K_{ε,j})}\}_{i ∈ Γ_j}\) is a basis of \(H_1(K_{ε,j})\) (see Subsection 2.4).

Let \(\{α_i\}_{i ∈ Γ_j}, \{β_i\}_{i ∈ Γ_j}\) be the curves defined in Example 2A.2, page 168 of [17]. Note that we can choose \(α_i = ˆγ_i\) (see [18], just take \(\hat{\cdot}\) instead of \(ε\) in \(K_{ε}\)). Moreover as \(γ_i \simeq β_i\) we have that (see Theorem 11.2, page 59 of [16]) \([β_i]_{H_1(K_{ε,j})} = [γ_i]_{H_1(K_{ε,j})}\). Then, by Example 2A.2, page 168 of [17],

\[
\{[γ_i]_{H_1(K_{ε,j})}, [β_i]_{H_1(K_{ε,j})}\}_{i ∈ Γ_j}
\]

is a basis of \(H_1(∂K_{ε,j})\).

It is clear that \(I([γ_i]_{H_1(K_{ε,j})}) = 0, i ∈ Γ_j\). Moreover, \(I([β_i]_{H_1(K_{ε,j})}) = [β_i]_{H_1(K_{ε,j})} = [γ_i]_{H_1(K_{ε,j})}\). Hence, \(\text{Kern } I = \langle \{[γ_i]_{H_1(K_{ε,j})}\}_{i ∈ Γ_j} \rangle\), the free \(\mathbb{Z}\)-module or the free group generated by \(\{[γ_i]_{H_1(K_{ε,j})}\}_{i ∈ Γ_j}\). We obtain then that

\[
H_2(K_{ε,j}, ∂K_{ε,j}) \overset{Δ_2}{\rightarrow} \text{Kern } I = \langle \{[γ_i]_{H_1(K_{ε,j})}\}_{i ∈ Γ_j} \rangle.
\]

It follows that to construct a basis of \(H_2(K_{ε,j}, ∂K_{ε,j})\) it is enough to compute the inverse image under \(Δ_2\) of the \(\{[γ_i]_{H_1(K_{ε,j})}\}_{i ∈ Γ_j}\). Let us take then, \([σ_i]_{H_1(K_{ε,j}, ∂K_{ε,j})}\) such that, \(∂σ_i = ˆγ_i\). Hence, \(\{[σ_i]_{H_1(K_{ε,j}, ∂K_{ε,j})}\}_{i ∈ Γ_j}\) is a basis of \(H_1(K_{ε,j}, ∂K_{ε,j})\).

Finally, by steps 4 and 5 \(\{[σ_i]_{H_2(K_{ε,j}, K_{ε,j} \setminus K_j)}\}_{i ∈ Γ_j}\) is a basis of \(H_2(K_{ε,j}, K_{ε,j} \setminus K_j)\). By step 3 \(\{[σ_i]_{H_2(K_{ε,j}, K_{ε,j} \setminus K_j)}\}_{i = 1}^m\) is a basis of \(H_2(K_{ε,j}, K_{ε,j} \setminus K_j)\). By step 2 \(\{[σ_i]_{H_2(\mathbb{R}^3, \mathbb{R}^3 \setminus K)}\}_{i = 1}^m\) is a basis on \(H_2(\mathbb{R}^3, \mathbb{R}^3 \setminus K)\). By step 1 \(\{[γ_i]_{H_1(\mathbb{R}^3 \setminus K)}\}_{i = 1}^m\) is a basis of \(H_1(\mathbb{R}^3 \setminus K)\).

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