Multi-scale structural complexity of natural patterns

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Complexity of patterns is a key information for human brain to differ objects of about the same size and shape. Like other innate human senses, the complexity perception cannot be easily quantified. We propose a transparent and universal machine method for estimating structural (effective) complexity of two- and three-dimensional patterns that can be straightforwardly generalized onto other classes of objects. It is based on multistep renormalization of the pattern of interest and computing the overlap between neighboring renormalized layers. This way, we can define a single number characterizing the structural complexity of an object. We apply this definition to quantify complexity of various magnetic patterns and demonstrate that not only does it reflect the intuitive feeling of what is “complex” and what is “simple”, but also can be used to accurately detect different phase transitions. When employed for that, the proposed scheme is much simpler and numerically cheaper than the standard methods based on computing correlation functions or using machine learning techniques.

I. INTRODUCTION

Complexity is one of the most fundamental properties of the world around us and a key subject for many natural and social sciences; in some of them like biology the origin of complexity is one of the central issues [1,7]. Despite numerous attempts to give a formal definition of complexity [4,8–13], our understanding of these matters is still far from being complete. The famous motto “I know it when I see it” is definitely applicable to complexity but to formalize this feeling is a very nontrivial problem. One of the first and the most famous definitions, the Kolmogorov complexity which is given in terms of the minimal instruction length required to describe the object [14], characterizes rather randomness and irregularity of the object than its structural non-triviality. Importantly, there is no general way to calculate the Kolmogorov complexity [4]. A different approach was taken by P. Bak and coauthors [3,15–17] who introduced a concept of self-organized criticality (SOC) as a universal root of structural complexity. Despite a definite relevance of this concept to a number of natural and social phenomena such as [15,20] and to the emergence of biological complexity [4], it does not give however a satisfactory full solution of the problem. Our intuitive perception of complexity is based on a tiny balance between how many different elements and connections the system has and how recognizable it is. The latter is usually related to having a reasonable number of distinguishable features at several well-separated characteristic scales. In other words, complexity assumes hierarchy. If we consider, for instance, “complex” structures in metallurgy, like pearlite colonies in steel [21], we deal with essentially different pictures at the atomistic scale within different phases (ferrite and cementite), at the scale of interphase boundaries, and at the scale of mesoscopic structure which is directly related to mechanical properties. Coexistence of essentially different structural levels and competing constraints at these levels is also of crucial importance in biology [7] and social sciences [22]. This poses a natural question of how to account for this property quantitatively.

While there are many definitions of structural (or effective) complexity [4,23,24], most of them have a common weakness: in each particular case, one must decide subjectively what is essential structural features, and what is mere a noise which must be ignored. In principle, there is nothing wrong with complexity being context-dependent and a bit subjective. Still, it is tempting to find a way to define complexity as a more “observer-independent” quantity that can be used in different contexts with only slight modifications. With this in mind, a natural list of requirements for a proper notion of structural complexity can be formulated:

- It must aggregate information about different scales present in the problem.
- It must be well-defined analytically, so that for the selected class of objects (patterns, texts, melodies etc.) the protocol of computing it can be executed with a little need to make subjective choices and decisions.
- Within the same class of objects, it must be robust and stable upon reasonably mild deformations of an object.
- It should be small both for trivially ordered and fully disordered structures.
Among other things, a promising view on these matters was formulated in [25, 26], where, for a broad variety of structures, a clearly defined and computable measure of self-dissimilarity complexity was given (see also [24] for a similar in spirit approach). It was suggested that a structure is the more complex the more it differs from itself when considered at different spatial and temporal scales.

The idea of relating complexity of a pattern or a structure to a certain functional over all scales has also been discussed more pragmatically in concrete physical contexts. For example, in the theory of polymers, it was suggested to study conformational properties of proteins by analyzing how certain observables scale upon Renormalization Group (RG) transformations and keeping track of the whole RG flow profile, not only the deep infrared behavior [28]. Another research area where the concept of complexity has attracted considerable attention is the anti de Sitter/Conformal Field Theory correspondence (also known as holography). There it was conjectured that computational complexity of a quantum state should be related to the volume of dual bulk space which, in holographic terms, means integration over all the involved energy scales [29, 30], and possible conceptual connections to SOC were discussed [31].

Inspired by these attempts, we give a quantitative definition of structural complexity of patterns in terms of RG flow. A pattern can be regarded as a function \( f(x) \) defined on a certain domain \( D \). For example, a gray-scale picture is a real-valued function on a two-dimensional rectangle. For such an object, RG transformation can be defined in a natural way. E.g., if \( D \) is a discret set of pixels or lattice sites, a coarse-grained pattern can be obtained by means of Kadanoff decimation. If it is a continuous domain, RG transformation can be implemented as convolution of \( f(x) \) with some scale-dependent filter. It is natural to say that scale \( \Lambda \) contributes some features to the pattern if there is a difference between the coarse-grained patterns \( f_\Lambda(x) \) and \( f_{\Lambda+d\Lambda}(x) \). The latter can be measured as deviation

\[
\Delta_\Lambda = |\langle f_\Lambda(x)|f_{\Lambda+d\Lambda}(x)\rangle| - \frac{1}{2} \left( \langle f_\Lambda(x)|f_\Lambda(x)\rangle + \langle f_{\Lambda+d\Lambda}(x)|f_{\Lambda+d\Lambda}(x)\rangle \right) = \frac{1}{2} |\langle f_{\Lambda+d\Lambda}(x) - f_\Lambda(x)|f_{\Lambda+d\Lambda}(x) - f_\Lambda(x)\rangle|,
\]

where \( \langle f(x)|g(x)\rangle = \int_D df(x)g(x) \) is a non-normalized overlap of two patterns\(^1\). Summing up this over all scales, we obtain a number that we call *multi-scale structural complexity* \( C \):

\[
C = \sum_\Lambda \frac{1}{d\Lambda} \Delta_\Lambda = \int |\frac{\partial f}{\partial d\Lambda}|d\Lambda, \quad \text{as } d\Lambda \to 0
\]

where we introduced additional \( 1/d\Lambda \) factor to make the continuous limit well defined.

While this approach is quite generic and allows to estimate complexity of almost any structure for which the coarse-graining procedure can be defined, here we focus on three concrete examples to demonstrate how the concept of structural complexity can be utilized to address physical problems. First, we study the phase transitions in the 2d and 3d classical Ising model, and demonstrate that complexity of the critical point is indeed higher than that of the fully ordered ferromagnetic phase or fully random paramagnetic one. Moreover, we show that one can compute \( T_c \) with high accuracy simply by looking at the temperature dependence of complexity. For each value of \( T \) it is enough to compute \( C(T) \) just for a single snapshot of the system, without any need to compute correlation functions and average over multiple Monte Carlo samples.

From that we proceed to a more complicated classical Heisenberg model with Dzyaloshinskii-Moriya interactions which hosts a variety of phases that cannot be characterized with a local order parameter but appear to be non-trivial patterns, such as spin spirals, bimerons, and skyrmion crystals. Again, not only the suggested multi-scale structural complexity maximizes on the most visually non-trivial spin spirals (magnetic labyrinths) and minimizes on the ordered ferromagnetic configurations, but transition lines between the phases can be easily determined by computing complexity of mere single realizations of the spin configuration at each point of the phase diagram.

Finally, we consider time evolution of complexity of a dye drop dissolving in water. This is an archetypical example of a process where entropy of a system grows steadily, but the apparent structural complexity evolves in a non-monotonous way. Computing \( C \) of snapshots of this process made at different moments of time, we show that the multi-scale complexity we defined indeed attains its maximum not on the most random configurations and demonstrates very appealing robustness of the pattern of temporal evolution for different runs of the experiment.

**II. METHOD**

To demonstrate how complexity of a pattern can be computed, we shall consider a photo of \( L \times L \) pixels as an example [32]. Fig.\( \text{I} \). Position of each pixel is given by its row and column indices \( i,j \), and its state is characterized in general case by some vector \( s_{ij} \). The meaning of the vector depends on the context. For example, in the case of a color picture, it is a three-dimensional vector

\[^1\text{Independently normalizing each scale to the same number can be a cause of undesired artifacts. For example, it might happen that the overall “intensity” of a picture is decreased in the coarse-graining procedure, and in that case we want to regard large-scale contributions to the overall complexity value as small. In that case, normalization of each scale will make the less intense patterns more important than they should be.}\]
The space of parameters might require extensive Monte Carlo simulations. The situation becomes much trickier if the order parameter is unknown, or if the transition is of unconventional nature (e.g. topological phase transitions). Recently, an automatic way to detecting phase boundaries based on machine learning methods has been suggested [33, 34]. Since a neural network is dealing directly with patterns, the success of this approach poses a natural question whether states of a system belonging to different phases can be distinguished by calculating their structural complexity.

To check this, we first consider the classical Ising model with nearest-neighbor ferromagnetic exchange interaction on square (2d) and cubic (3d) lattices:

$$H = J \sum_{n,n'} S_n^z S_{n'}^z, \quad J < 0,$$

and consider the paramagnetic/ferromagnetic phase transitions both in two and three dimensions. Then we study how complexity changes across the transition point.

In the 2d case, we perform classical Monte Carlo simulations for (5) on square lattice of $1024 \times 1024$ size scanning over temperatures $0 < T/J < 4.5$ with step $\Delta T = 0.045J$. For a lattice of this size, one can do eight renormalization steps within the proposed scheme. In 3d, we conduct the same analysis for the Ising model defined on cubic lattice of $256 \times 256 \times 256$ spins with the smallest possible $2 \times 2 \times 2$ renormalization block, and scanning over $2 < T/J < 6.5$, $\Delta T = 0.045J$.

Structural complexity as a function of temperature is presented in Figs. 2 and 3. First thing interesting to note is that structural complexity of the Ising lattice configurations is very robust. Both in 2d and 3d, for each value of $T$, we conduct the same analysis for the Ising model defined on cubic lattice of $256 \times 256 \times 256$ spins with the smallest possible $2 \times 2 \times 2$ renormalization block, and scanning over $2 < T/J < 6.5$, $\Delta T = 0.045J$. For a lattice of this size, one can do eight renormalization steps within the proposed scheme. In 3d, we conduct the same analysis for the Ising model defined on cubic lattice of $256 \times 256 \times 256$ spins with the smallest possible $2 \times 2 \times 2$ renormalization block, and scanning over $2 < T/J < 6.5$, $\Delta T = 0.045J$.

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\( T \) we generated five different Monte Carlo samples, and their complexity turned out to be the same with very high accuracy (about \( \sim 0.01\% \)), thus we do not even show the error bars on the plot.

One can see that by taking derivative with respect to \( T \) and associating the phase transition with the extremum of \( dC/dT \), the critical temperature can be estimated with very high accuracy. For the square lattice, our approach gives the value of \( T/J = 2.26 \), which is in excellent agreement with known analytical results \[35\] \( T_c/J = 2/\ln(1 + \sqrt{2}) \approx 2.269 \). For the cubic one, we obtain \( T_c \approx 4.5 \), which is very close to the results of the high-temperature series expansion \( T_c \approx 4.5103 \) \[36\] and Monte Carlo simulations \( T_c \approx 4.5 \) \[37\]. Note that sometimes MC simulations lead to metastable configurations of magnetic domains inserted into the ferromagnetic phase, and the structural complexity keeps track of that as well, Fig. 3.

A peculiar detail of the \( C(T) \) dependence is that it saturates and reaches a constant value in the paramagnetic phase. This seemingly contradicts our intention to define structural complexity because the magnetization patterns at \( T > T_c \) look visually more random and less structured than the critical point. However, two aspects should be kept in mind. First, if we neglect the contribution of the most microscopic scale that can be barely resolved visually (i.e. the \( |0_{0,1} - 1/2(O_{0,0} + O_{1,1})| \) term in (4)), the resulting complexity of paramagnet would be smaller than that of the critical point, and will be decreasing with temperature. This fact speaks in favor of the suggested definition as it is natural to expect structural complexity to depend on the resolution of a perceiver (be it a human being, a detector, or a neural network). Secondly, as we will discuss in more detail in the next section, apart from the single numerical value \( C \), another important property of a structure is how complexity is distributed between different scales. In the case of paramagnet, it comes mainly from the finest scale \( k = 0 \), while for the more non-trivial structures it resides on a number of scales.

IV. COMPLEXITY OF SPIN TEXTURES

Our next goal is to see if the notion of structural complexity can be employed to detect phase transitions of a more sophisticated nature. An illustrative example of a system where complex patterns emerge naturally is magnets with Dzyaloshinskii-Moriya (DM) interactions described by the Hamiltonian \[38 \, 39\]:

\[
H = -J \sum_{nn'} S_n S_{n'} - D \sum_{nn'} [S_n \times S_{n'}] - \sum_n BS_n, \tag{6}
\]

where \( J \) and \( D \) are the isotropic exchange and DM interactions respectively, and the sums run over links of two-dimensional square lattice. Vector \( D \) is orthogonal to the lattice links.

Depending on the relative strength of interactions and the magnetic field, the magnet exhibits clearly distinguishable textures such as spin spirals, skyrmion crystals, and bimerons. Contra to the case of ferromagnetic/paramagnetic phase transition, transition between two types of textures cannot be related to symmetry breaking and described in terms of local order parameter. At the same time, it is clearly a physical effect that should be amenable to quantification. In our analysis, we consider a square lattice of 1024\( \times \)1024 size with \( J = 1, |D| = 1 \), and perform Monte Carlo simulations at fixed temperature \( T = 0.02 \) varying the external magnetic field \( B \) in the range \( 0 < B < 1 \) with step \( \Delta B = 0.01 \).
For each value of B, we assume that the state of a lattice site (“pixel” of the corresponding pattern) is characterized only by z-component of spin. Structural complexity is then computed in the same manner as before for the Ising patterns.

The resulting dependence of complexity on magnetic field is presented in Fig. 4. Again, for each value of B the complexity appears to be very robust, fluctuating within 0.01% error range for independent Monte Carlo runs, Fig. 5. As before for the paramagnetic/ferromagnetic phase transitions, the extrema of complexity derivatives dC/dB reflect very well both the melting of spin spirals (magnetic labyrinths) into skyrmion crystals, with the transition point being exactly the bimeron phase, as well as the transition between skyrmion crystals and ferromagnets.

An intriguing feature of C(B) is that the visually most complex magnetic configurations of labyrinth type that emerge at weak magnetic fields have the largest C value, which is yet another argument in favor of the interscale approach to defining effective complexity. Transitions between spin textures in DM magnets are a clear example of truly non-trivial physical application of structural complexity. Formation of a skyrmion crystal from decaying spin spiral cannot be detected with conventional observables, such as as magnetization and skyrmion number. Of course, it can be identified by a trained neural network or by computing the Binder cumulant:

\[ U = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}, \]  

but it would require either learning a network on a large set of configuration, or computing correlation functions by averaging over a number of Monte Carlo samples at each point of the phase diagram. Instead, thanks to the robustness of C upon choosing different patterns at the same point of parametric space, one can resort to computing complexity of a single Monte Carlo sample for each value of B and find the transition point with much lesser effort.

V. INTERSCALE DISTRIBUTIONS OF COMPLEXITY

As we briefly mentioned before, the absolute value of complexity C is not the only interesting quantity. More can be learned from how different scales contribute to structural complexity of a pattern. Thus, it is instructive to look at the scale distribution of partial complexities \( C_k \) for the four studied types of 2d patterns - spin spirals, skyrmion crystals, Ising spins at criticality and paramagnets. Those are plotted in Fig. 6. One can see that the most visually non-trivial configurations (spirals and crystals) are characterized by a couple of scales, complexity of the critical point is distributed pretty homogeneously among all the involved scales, which is what one would expect for a scale-invariant system, and complexity of a random paramagnetic pattern is strongly dominated by its deep microscopics.

VI. COMPLEXITY OF TIME-DEPENDENT SYSTEMS

Finally, we would like to analyze how structural complexity evolves in time if entropy of the system is steadily increasing. The common wisdom is that computational complexity keeps increasing alongside the entropy, getting higher for more random states of the system. However, for structural complexity we should expect non-monotonous dependence on entropy.

To study this, we move aside from the magnetic pat-
FIG. 6. Partial contributions of different scales to the overall structural complexity for four types of magnetic patterns.

FIG. 7. The evolution of the complexity during the process of dissolving a food dye drop of 0.3 ml in water at 31°C.

terns case and take a look at the process of dissolving a dye drop in water. We put a 0.3 ml drop of green dye in water at 31°C and keep track of time evolution of the color spot. At every moment of time, state of the system is recorded as 2048×2048 photo which is used to compute complexity of the apparent pattern. We have conducted the experiment six times and found that complexity as a function of time obeys quite a robust curve Fig. 7 with a quick increment stage followed by slow oscillatory fall-off at larger times.

VII. CONCLUSIONS

In this paper, we have introduced a quantitative definition of effective complexity based on interscale dissimilarities of a system of interest. The system is assumed to be the more complex, the more distinctive features of different characteristic scales it has.

We exemplified this approach by computing complexity of certain 2d and 3d spatial structures, but it can be straightforwardly generalized onto any case that allows to define a coarse-graining protocol. Being a new easily computable measure, it might help to reveal some novel features of complex systems and processes. Proceeding further along the line of studying classical magnetic structures, the concept of multi-scale complexity can be employed to detect novel structural transitions on the fly using raw data of STM experiments [41]. In biology, one can think of studying how complexity of genomic sequences evolves along different branches of the phlogenetic tree, and see whether major evolutionary transitions can be quantified in this way [6]. In quantum science, it can be employed to define effective complexity of many-body wavefunctions that would complement the existent notions of computational circuit complexity [42] and provide a basis for new types of metric on Hilbert spaces.

By no means, does our study give an exhaustive answer to the problem of quantifying effective complexity. After all, it is quite unlikely, that a unique universal definition should exist. Further studies are required to demonstrate how really useful the suggested measure is, but it is already clear that a number of research lines can be initiated on the basis of this approach.

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