COMMENSURABILITY AND QUASI-ISOMETRIC CLASSIFICATION
OF HYPERBOLIC SURFACE GROUP AMALGAMS

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Abstract. Let $C_S$ denote the class of groups isomorphic to the fundamental group of two closed hyperbolic surfaces identified along an essential simple closed curve in each. We construct a bi-Lipschitz map between the universal covers of these spaces equipped with a CAT($-1$) metric, proving all groups in $C_S$ are quasi-isometric. The class $C_S$ has infinitely many abstract commensurability classes, which we characterize in terms of the ratio of the Euler characteristic of the two surfaces and the topological type of the curves identified. We characterize the groups in $C_S$ that contain a maximal element in their abstract commensurability class restricted to $C_S$.

1. Introduction

To study finitely generated infinite groups, one may use algebraic and geometric classifications. Abstract commensurability defines an algebraic equivalence relation on the class of groups, where we say two groups are abstractly commensurable if they contain isomorphic subgroups of finite index. The geometry of finitely generated groups may be studied by identifying a group with a family of metric spaces, which is well-defined up to quasi-isometric equivalence. That is, a group together with a finite generating set yields a Cayley graph, equipped with the path metric. While a change in finite generating set may produce a non-isometric metric space, any two such Cayley graphs are quasi-isometric. Since a finitely generated group is quasi-isometric to any finite index subgroup, if two finitely generated groups are abstractly commensurable, then they are quasi-isometric. Two fundamental questions in geometric group theory are to classify the abstract commensurability and quasi-isometry classes within a class of finitely generated groups and to understand for which classes of groups the characterizations coincide. In this paper, we address these questions restricted to the class, $C_S$, of groups isomorphic to the fundamental group of two closed hyperbolic surfaces identified along an essential simple closed curve in each.

Our first theorem gives the quasi-isometric classification.

Theorem 1. If $X_1 = S_{g_1} \cup \gamma S_{g_1}$ and $X_2 = S_{g_2} \cup \rho S_{g_2}$ where $S_{g_i}$ and $S_{g_i}'$ are closed hyperbolic surfaces for $i = 1, 2$ and $\gamma : S^1 \to X_1$ and $\rho : S^1 \to X_2$ are the images of essential simple closed curves under identification, then there exists a bi-Lipschitz equivalence $\Phi : \tilde{X}_1 \to \tilde{X}_2$ mapping lifts of $\gamma$ to lifts of $\rho$ bijectively.
**Corollary 2.** If $G_1, G_2 \in \mathcal{C}_S$, then $G_1$ and $G_2$ are quasi-isometric.

Corollary 2 also follows from Theorem 4.14 of Malone’s unpublished work in [7]; groups in $\mathcal{C}_S$ are examples of geometric amalgamations of free groups with the same degree refinement, a matrix associated to the graph of groups splitting. The proof in [7] follows the techniques of Behrstock and Neumann in [1]: a model space is constructed by identifying the universal cover of a surface with boundary with a fattened tree and gluing these trees along boundary curves. The bi-Lipschitz equivalence of these spaces follows from [1], and thus these groups are shown to be quasi-isometric.

In our result, we study the CAT($-1$) structure of these groups, constructing an explicit bi-Lipschitz equivalence between the universal covers of two closed hyperbolic surfaces identified along essential simple closed curves. The class $\mathcal{C}_S$ is a subclass of the class of limit groups, described in [2]; in particular, all groups in $\mathcal{C}_S$ are constructible limit groups of level 1. The quasi-isometric classification within the class of limit groups in full generality remains open, and we expect the techniques of this paper may be used towards this classification.

The class $\mathcal{C}_S$ is also a subclass of the class $\mathcal{C}$ of groups isomorphic to the fundamental group of two hyperbolic surfaces identified along a primitive closed curve in each. We prove that within $\mathcal{C}$, any group quasi-isometric to a group in $\mathcal{C}_S$ is in $\mathcal{C}_S$.

**Proposition 3.** If $G \in \mathcal{C}_S$ and $G' \in \mathcal{C} \setminus \mathcal{C}_S$, then $G$ and $G'$ are not quasi-isometric.

The abstract commensurability classes of groups in $\mathcal{C}_S$ are finer than the quasi-isometry classes; while there is a unique quasi-isometry class in $\mathcal{C}_S$, there are infinitely many abstract commensurability classes. Whyte, in [14], proves a similar result for free products of hyperbolic surface groups, which may be thought of as fundamental groups of closed hyperbolic surfaces identified along null-homotopic curves in each, and are also a subclass of limit groups.

**Theorem 4.** ([14], Theorem 1.6, 1.7) Let $\Sigma_g$ be the fundamental group of a surface of genus $g \geq 2$ and let $m, n \geq 2$. Let $\Gamma_1 \cong \Sigma_{a_1} \ast \Sigma_{a_2} \ast \ldots \ast \Sigma_{a_n}$ and $\Gamma_2 \cong \Sigma_{b_1} \ast \Sigma_{b_2} \ast \ldots \ast \Sigma_{b_m}$. Then $\Gamma_1$ and $\Gamma_2$ are quasi-isometric, and $\Gamma_1$ and $\Gamma_2$ are abstractly commensurable if and only if

$$\frac{\chi(\Gamma_1)}{n-1} = \frac{\chi(\Gamma_2)}{m-1}.$$
ratio of the Euler characteristic of the surfaces identified and the topological type of
the curves identified are obstructions to the existence of homeomorphic finite sheeted
covers, and we construct a common cover in the absence of these obstructions. Our
result is the following.

**Theorem 5.** If \( G_1, G_2 \in C \), then \( G_1 \) and \( G_2 \) are abstractly commensurable if and
only if \( G_1 \) and \( G_2 \) may be expressed as \( G_1 \cong \pi_1(S_{g_1}) \ast \langle a_1 \rangle \) \( \pi_1(S_{g_1}') \) \( \pi_1(S_{g_2}) \ast \langle a_2 \rangle \) \( \pi_1(S_{g_2}') \in C \), given by the monomorphisms \( a_i \mapsto [\gamma_i] \in \pi_1(S_{g_i}) \) and \( a_i \mapsto [\gamma_i'] \in \pi_1(S_{g_i}') \), and the following conditions hold.

(a) \( \chi(S_{g_1}) \chi(S_{g_1}') = \chi(S_{g_2}) \chi(S_{g_2}') \),
(b) \( t(\gamma_1) = t(\gamma_2) \),
(c) \( t(\gamma_1') = t(\gamma_2') \).

The result in Theorem 5 is related to the abstract commensurability classification of
the right-angled Coxeter groups considered by Crisp and Paoluzzi in [4],

\[ W_{m,n} = W(\Gamma_{m,n}) , \]

where \( \Gamma_{m,n} \) denotes the graph consisting of a circuit of length \( m + 4 \) and a circuit
of length \( n + 4 \) which are identified along a common subpath of edge-length 2. For
all \( m \) and \( n \), the group \( W_{m,n} \) is the orbifold fundamental group of a 2-dimensional
reflection orbi-complex \( O_{m,n} \) which is finitely covered by a space consisting of two
hyperbolic surfaces identified along a single non-separating essential simple closed
curve. Conversely, all amalgams of surface groups over non-separating essential simple
closed curves are finite index subgroups of \( W_{m,n} \) for some \( m \) and \( n \), dependent on
the Euler characteristic of the two surfaces. Thus, our theorem extends their result.

**Corollary 6.** ([4] Theorem 1.1) Let \( 1 \leq m \leq n \) and \( 1 \leq k \leq \ell \). Then \( W_{m,n} \) and
\( W_{k,\ell} \) are abstractly commensurable if and only if \( \frac{m}{n} = \frac{k}{\ell} \).

Dani and Thomas, in [5], generalize the commensurability classification of Crisp and
Paoluzzi and prove a quasi-isometric classification within a class of right-angled Coxeter
groups which includes \( W_{m,n} \). Our result in Theorem 1 also gives the following.

**Corollary 7.** ([5] Theorem 1.5) For all \( m, n, k, \ell \geq 1 \), \( W_{m,n} \) and \( W_{k,\ell} \) are quasi-
isometric.

A maximal element in the abstract commensurability class of \( G \in C \) restricted to \( C \) is
a group \( G_0 \in C \) that contains every other element in the abstract commensurability
class of \( G \) in \( C \) as a finite index subgroup. In this case, a \( K(G_0, 1) \) space is finitely
covered by a \( K(G_i, 1) \) space for all \( G_i \in C \) in the abstract commensurability class of \( G \).
A classic result in this setting is that of Margulis, who proved that if \( H \leq PSL(2, \mathbb{C}) \)
is a discrete subgroup of finite covolume, then there exists a maximal element in
the abstract commensurability class of \( H \) if and only if \( H \) is non-arithmetic [8]. For
recent surveys on commensurability, see [10] and [13]. Within the class $\mathcal{C}_S$, we prove the following.

**Proposition 8.** Let $G \cong \pi_1(S_g) *_{\langle \gamma \rangle} \pi_1(S_{g'}) \in \mathcal{C}_S$ be given by the monomorphisms $\gamma \mapsto [\gamma] \in \pi_1(S_g)$, and $\gamma \mapsto [\gamma'] \in \pi_1(S_{g'})$. Then there exists a maximal element in the abstract commensurability class of $G$ restricted to $\mathcal{C}_S$ if and only if $\gamma$ and $\gamma'$ are separating simple closed curves with $S_g = S_r \cup S_s$, $S_{g'} = S_{r'} \cup S_{s'}$ and $r \neq s$, $r' \neq s'$.

The outline of the paper is as follows. In Section 2, we define the spaces and groups we consider. In Section 3, we describe the geometry and quasi-isometry classes of groups in $\mathcal{C}_S$; we construct a bi-Lipschitz map between the universal covers of specified $K(G, 1)$ spaces for any $G \in \mathcal{C}_S$ and prove the quasi-isometry class containing $\mathcal{C}_S$ in $\mathcal{C}$ does not contain any group not in $\mathcal{C}_S$. In Section 4, we describe the topology of a set of $K(G, 1)$ spaces, prove the abstract commensurability classification, and characterize the groups in $\mathcal{C}_S$ that contain a maximal element in their abstract commensurability class.

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2. Preliminaries

An orientable surface of genus $g$ with $b$ boundary components will be denoted $S_{g,b}$; we use $S_g$ to denote $S_{g,0}$. A hyperbolic surface is a surface that admits a metric of constant negative curvature. A (hyperbolic) surface group refers to the fundamental group of a (hyperbolic) surface. A closed curve in a surface $S_{g,b}$ is a continuous map $S^1 \to S_{g,b}$, and we typically identify a closed curve with its image in the surface. An essential closed curve $\gamma$ is primitive if it cannot be written as $[\gamma] = [\gamma_0^n]$ for some closed curve $\gamma_0$. A closed curve is simple if it is embedded. A homotopy class of simple closed curves is a homotopy class in which there exists a simple closed curve representative. A multicurve in $S_{g,b}$ is the union of a finite collection of disjoint simple closed curves in $S_{g,b}$.

Let $\mathcal{X}$ denote the class of spaces homeomorphic to two hyperbolic surfaces identified along a primitive closed curve in each. Let $\mathcal{X}_S \subset \mathcal{X}$ be the subclass in which the curves that are identified are simple. Let $\mathcal{C}$ be the class of groups isomorphic to the fundamental group of a space in $\mathcal{X}$, and let $\mathcal{C}_S \subset \mathcal{C}$ be the subclass of groups isomorphic to the fundamental group of a space in $\mathcal{X}_S$. If $G \in \mathcal{C}$ then $G \cong \pi_1(S_g) *_{\langle \gamma \rangle} \pi_1(S_h)$, the amalgamated free product of two hyperbolic surface groups over $\mathbb{Z}$. We
suppress in our notation the monomorphisms \( i_g : \langle \gamma \rangle \to \pi_1(S_g) \) and \( i_h : \langle \gamma \rangle \to \pi_1(S_h) \) given by \( i_g : \gamma \mapsto \gamma g \), \( i_h : \gamma \mapsto \gamma h \), where \( \gamma_g : S^1 \to S_g \) and \( \gamma_h : S^1 \to S_h \).

3. Quasi-isometric classification

3.1. The hyperbolic structure of groups in \( \mathcal{C}_S \). Let \( G \in \mathcal{C}_S \) so that \( G \cong \pi_1(X) \), where \( X = S_g \cup \gamma S_h \subset \mathcal{X}_S \), and \( \gamma \) is the image of \( \gamma_g : S^1 \to S_g \) identified to \( \gamma_h : S^1 \to S_h \) in \( X \). One can choose hyperbolic metrics on \( S_g \) and \( S_h \) so that the length of the geodesic representatives of \( [\gamma_g] \) and \( [\gamma_h] \) is equal. Gluing by an isometry yields a piecewise hyperbolic complex with \( \text{CAT}(-1) \) universal cover \( \tilde{X} \).

For details on gluing constructions, see ([3], Section II.11.) The space \( \tilde{X} \) consists of convex subspaces of \( \mathbb{H}^2 \) that are the preimages of hyperbolic surfaces with boundary, identified along geodesic lines that are the preimages of the curve \( \gamma \). The universal cover \( \tilde{X} \) is a geometric graph of spaces in the following sense. For more details on graphs of groups and graphs of spaces, see [11], [12].

**Definition 9.** A geometric graph of spaces is a graph of spaces, \( G \), consisting of a set of vertex spaces, \( \{V_i\}_{i \in I} \), and a set of edge spaces, \( \{E_{i,j}\}_{(i,j) \in J} \), with \( J \subset I \times I, \ i < j \), so that the vertex and edge spaces are geodesic metric spaces and there are isometric embeddings

\[
E_{i,j} \to V_i,
\]

\[
E_{i,j} \to V_j,
\]

as convex subsets. The geometric realization of \( G \) is the metric space \( X \) given by the disjoint union of the vertex and edge spaces, identified according to the relations of \( G \), and given the induced path metric. The underlying graph of the graph of spaces \( G \) is the abstract graph specifying \( G \).

To prove that all groups in \( \mathcal{C}_S \) are quasi-isometric, we exhibit a bi-Lipschitz equivalence between the universal covers of the corresponding spaces in \( \mathcal{X}_S \).

**Definition 10.** A map \( f : (X, d_X) \to (Y, d_Y) \) is \( K \)-bi-Lipschitz if there exists \( K \geq 1 \) with

\[
\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)
\]

and a \( K \)-bi-Lipschitz equivalence if, in addition, it is a homeomorphism. A map is said to be a bi-Lipschitz equivalence if it is a \( K \)-bi-Lipschitz equivalence for some \( K \). Two spaces \( X \) and \( Y \) are bi-Lipschitz equivalent if there exists a bi-Lipschitz equivalence from \( X \) to \( Y \).

We will need the following two lemmas.

**Lemma 11.** Suppose \( P \) and \( P' \) are compact convex hyperbolic polygons. Then \( P \) and \( P' \) are bi-Lipschitz equivalent.
Lemma 12. Let $T_1$ and $T_2$ be tilings of convex regions $R_1, R_2 \subset \mathbb{H}^2$ by compact convex polygons of finitely many isometry types. If $T_1$ and $T_2$ have isomorphic dual graphs, then $R_1$ and $R_2$ are bi-Lipschitz equivalent.

Proof. The tilings $T_1$ and $T_2$ of convex subsets of $\mathbb{H}^2$ are examples of locally finite geometric graphs of spaces, where the vertex spaces are the tiles, the underlying graph is the dual graph to the tiling, and $R_1$ and $R_2$ are the geometric realizations of $T_1$ and $T_2$, respectively. In this language, suppose $T_1$ and $T_2$ consist of a set of vertex and edge spaces $\{V_i\}_I$, $\{E_{i,j}\}_J$ and $\{W_i\}_I$, $\{F_{i,j}\}_J$, respectively. Since $T_1$ and $T_2$ have isomorphic dual graphs and there are finitely many isometry types of polygons, by Lemma 11, there exists a $K$-bi-Lipschitz equivalence $\phi_i : V_i \rightarrow W_i$ for all $i \in I$ such that

$$\phi_i(E_{i,j}) = \phi_j(E_{i,j}) = F_{i,j}$$

for all $(i,j) \in J$. There is an induced bijection

$$\Phi : R_1 \rightarrow R_2$$

defined by $\Phi|_{V_i} = \phi_i(V_i)$, which is continuous by the pasting lemma.

Let $x, y \in R_1$, and let $p$ be the geodesic path from $x$ to $y$. The path $p$ can be decomposed into a union of geodesic segments $\{[x_i, x_{i+1}]\}_{i=0}^{n-1}$, with $x_0 = x$ and $x_n = y$ so that the interior of each subpath $[x_i, x_{i+1}]$ is contained entirely in a vertex space $V_i$ since there are finitely many isometry types of compact polygons in the tiling. If a segment is contained in an edge space $E_{i,j}$, we may choose either $V_i$ or $V_j$ since $\phi_i(E_{i,j}) = \phi_j(E_{i,j})$. By assumption, there exists a $K$-bi-Lipschitz equivalence $\phi_i : V_i \rightarrow W_i$ for all $i$. Since $\Phi(p)$ is a path connecting $\Phi(x)$ and $\Phi(y)$,

$$d(\Phi(x), \Phi(y)) = \sum_{i=0}^{n-1} d(\phi_i(x_i), \phi_i(x_{i+1}))$$

$$\leq \sum_{i=0}^{n-1} Kd(x_i, x_{i+1})$$

$$= K \sum_{i=0}^{n-1} d(x_i, x_{i+1})$$

$$= Kd(x, y).$$

The other inequality follows similarly. Namely, suppose $q$ is a geodesic path from $\Phi(x)$ to $\Phi(y)$. The path $q$ can be decomposed into a union of geodesic segments $\{[w_i, w_{i+1}]\}_{i=0}^{m-1}$ where $w_0 = \Phi(x)$, $w_m = \Phi(y)$ and the interior of $[w_i, w_{i+1}]$ is contained entirely in a vertex space, $W_i$. Then, since $\Phi^{-1}(q)$ is a path from $x$ to $y$, and
\( \phi_i \) is a \( K \)-bi-Lipschitz equivalence for all \( i \),

\[
d(\Phi(x), \Phi(y)) = \sum_{i=0}^{m-1} d(w_i, w_{i+1}) \geq \sum_{i=0}^{m-1} \frac{1}{K} d(\phi_i^{-1}(w_i), \phi_i^{-1}(w_{i+1})) = \frac{1}{K} \sum_{i=0}^{m-1} d(\phi_i^{-1}(w_i), \phi_i^{-1}(w_{i+1})) \geq \frac{1}{K} d(x, y).
\]

Thus, \( \frac{1}{K} d(x, y) \leq d(\Phi(x), \Phi(y)) \leq K d(x, y) \), so \( \Phi \) is a \( K \)-bi-Lipschitz equivalence. \( \square \)

### 3.2. Bi-Lipschitz equivalence of spaces \( \tilde{X} \) with \( X \in \mathcal{X}_S \)

In this section, we construct a bi-Lipschitz equivalence between spaces \( \tilde{X} \) and \( \tilde{X}' \) with \( X, X' \in \mathcal{X}_S \) in two steps. In Proposition 13 we construct a bi-Lipschitz equivalence between connected hyperbolic regions in the complement of the set of lifts in the universal covers of \( X \) and \( X' \), giving an explicit map from the boundary lifts of one region onto the boundary lifts in another region. Then, in Theorem 14 we extend this map to the entire universal covers, \( \tilde{X} \) and \( \tilde{X}' \).

**Proposition 13.** Let \( S_g \) and \( S_h \) be closed hyperbolic surfaces each equipped with a hyperbolic metric. Let \( \gamma : S^1 \to S_g \) and \( \rho : S^1 \to S_h \) be essential simple closed curves. Let \( L_\Gamma \) be the set of lifts of \( \gamma \) in \( \mathbb{H}^2 \) and let \( L_P \) be the set of lifts of \( \rho \) in \( \mathbb{H}^2 \). If \( R_\gamma \) is the closure of a component of \( \mathbb{H}^2 \backslash L_\Gamma \) and \( R_\rho \) is the closure of a component of \( \mathbb{H}^2 \backslash L_P \), then there exists a bi-Lipschitz equivalence

\[
\tilde{\phi} : R_\gamma \to R_\rho,
\]

mapping the lifts bounding \( R_\gamma \) bijectively onto the lifts bounding \( R_\rho \).
Proof. Let $L_\gamma$ be the set of lifts bounding $R_\gamma$ and $L_\rho$ be the set of lifts bounding $R_\rho$. If $\tilde{\gamma}$ denotes the axis of $[\gamma]$ and $\tilde{\rho}$ denotes the axis of $[\rho]$, we may assume $\tilde{\gamma} \in L_\gamma$ and $\tilde{\rho} \in L_\rho$ since $\pi_1(S_g)$ and $\pi_1(S_h)$ act transitively on the set of lifts.

The outline of the proof is as follows. We first restrict to specific hyperbolic metrics on $S_g$ and $S_h$ and specify a generating set $S$ for $\pi_1(S_g)$ and a generating set $T$ for $\pi_1(S_h)$. We define a map between the lifts, $\phi : L_\gamma \to L_\rho$, recursively, as a union of bijective partial functions, $\phi_n$. The recursion hypothesis is if $|g|_S \leq n$, where $|g|_S$ denotes the word length of $g$ with respect to the generating set $S$, then $g\tilde{\gamma} \in L_\gamma$ is included in the domain of $\phi_n$ and if $|h|_T \leq n$, then $h\tilde{\rho} \in L_\rho$ is included in the image of $\phi_n$. Thus, the map $\phi$ is exhaustive on $L_\gamma$ and $L_\rho$.

The construction of $\phi$ induces a tiling of the regions $R_\gamma$ and $R_\rho$ by compact convex hyperbolic polygons and a homeomorphism $\bar{\phi} : R_\gamma \to R_\rho$.

Note that this tiling is determined by $\phi$ and is different from the tiling of $\mathbb{H}^2$ by fundamental domains; in general, the tiling contains more than one isometry type of tile and is not preserved by $\pi_1(S_g)$. We prove there are finitely many isometry types of tiles in $R_\gamma$ and $R_\rho$ and the dual graph to each tiling is a tree. We show $\bar{\phi}$ maps tiles in $R_\gamma$ to tiles in $R_\rho$ and induces an isomorphism between the dual trees. Therefore, we may apply Lemma \[12\] to conclude that $R_\gamma$ and $R_\rho$ are bi-Lipschitz equivalent.

To make these notions precise, we begin by restricting to specific hyperbolic metrics on $S_g$ and $S_h$ and giving the non-standard generating sets $S$ and $T$, defined with
Figure 3. The generating set $S$ is defined so that $d_S(g, h) = 1$ since $g$ and $h$ lie in fundamental domains that share a vertex.

respect to these metrics and chosen to make the recursion hypotheses hold. The following, when described for $\gamma$ and $\pi_1(S_g)$, holds likewise for $\rho$ and $\pi_1(S_h)$.

Any two hyperbolic metrics on $S_g$ are bi-Lipschitz equivalent, so assume the hyperbolic metric on $S_g$ is given by a representation $\pi_1(S_g) \to PSL(2, \mathbb{R})$ with a fundamental domain for the action of $\pi_1(S_g)$ on the hyperbolic plane a $4g$-gon and so that the axis of $[\gamma]$, $\tilde{\gamma} : \mathbb{R} \to \mathbb{H}^2$, intersects each translate of the fundamental domain in a single geodesic segment. Parametrize $\tilde{\gamma}$ so that $\tilde{\gamma}(n)$ is the midpoint of a geodesic segment in a translate of the fundamental domain for all $n \in \mathbb{Z}$. Since $\pi_1(S_g)$ acts on $\mathbb{H}^2$ by isometries, $g\tilde{\gamma}(n)$ is the midpoint of a geodesic segment of $g\gamma$ for all $g \in \pi_1(S_g)$ and all $n \in \mathbb{Z}$. Give the set of lifts a cell structure by declaring $g\tilde{\gamma}(n)$ a 0-cell for all $g \in \pi_1(S_g)$ and $n \in \mathbb{Z}$.

Let $S$ be the generating set for $\pi_1(S_g)$ that consists of the $16g^2 - 8g$ elements of $\pi_1(S_g)$ that map $\tilde{\gamma}(0)$ onto the fundamental domains that share a vertex with the fundamental domain containing $\tilde{\gamma}(0)$. Let $d_S$ be the word metric on $\pi_1(S_g)$ with generating set $S$. Since there is a one-to-one correspondence between 0-cells in $\mathbb{H}^2$ and elements of $\pi_1(S_g)$, define $d_S(g\gamma^s, g_0\gamma^r)$ to be $d_S(g\gamma^s, g_0\gamma^r)$.

The map between the set of lifts must preserve the ordering on the boundary, so orient $\partial_\infty \mathbb{H}^2$, the visual boundary of $\mathbb{H}^2$. View $\partial_\infty \mathbb{H}^2 \cong S^1$ as the unit interval $[0, 1]$ with the identification $0 \sim 1$. A geodesic line, $\ell(t)$, defines two points on the boundary, which we denote $\ell^+\infty$ and $\ell^-\infty$. Orient the boundary clockwise so that $\tilde{\gamma}^+\infty = 0$. We may assume that for all lifts $g\tilde{\gamma} \in L_\gamma$, $\partial_\infty(g\tilde{\gamma}) \subset S^1 \setminus [0, \tilde{\gamma}^-\infty]$. Since $\gamma$ and $\rho$ are simple closed curves, the order on the boundary gives a total order on the lifts in $L_\gamma$ and $L_\rho$; we say $g\tilde{\gamma} < h\tilde{\gamma}$ if $\partial_\infty(g\tilde{\gamma}) < \partial_\infty(h\tilde{\gamma})$.

Construction:

In this section, we specify a map between the lifts, $\phi : L_\gamma \to L_\rho$, and a map between the regions $\bar{\phi} : R_\gamma \to R_\rho$, recursively, with $\phi = \bigcup_n \phi_n$ and $\bar{\phi} = \bigcup_n \bar{\phi}_n$. The maps $\phi_n$
induce a tiling of $R_\gamma$ and $R_\rho$ by compact convex hyperbolic polygons and the map $\bar{\phi}_n : R_\gamma \to R_\rho$. The recursion hypotheses are:

1. If $|g|_S \leq n$, then $g\tilde{\gamma} \in L_\gamma$ is included in the domain of $\phi_n$, and if $|h|_T \leq n$, then $h\tilde{\rho} \in L_\rho$ is included in the image of $\phi_n$.

2. Each tile has at most $M = 3 + \max\{64g^2 - 32g, 64h^2 - 32h\}$ edges, and each edge constructed in the interior of $R_\gamma$ connects vertices $g_0\tilde{\gamma}(r)$ and $g_\tilde{\gamma}(s)$ with $d_S(g_0\tilde{\gamma}(r), g_\tilde{\gamma}(s)) = 1$.

3. The dual graph to each tiling is a tree, and $\bar{\phi}$ maps tiles in $R_\gamma$ to tiles in $R_\rho$, inducing an isomorphism between the dual trees.

**Stage one:**

Begin by choosing the set of lifts of $\gamma$ and $\rho$ in the domain and image of $\phi_1$. Let $W_\gamma^1$ and $W_\rho^1$ be the set of all vertices in $R_\gamma$ and $R_\rho$ at distance 1 from $\tilde{\gamma}(0)$ and $\tilde{\rho}(0)$ in the metrics $d_S$ and $d_T$, respectively, and select and label every lift that intersects these sets as follows. No lift intersects two edges of any fundamental domain that share a vertex, so a lift intersects $W_\gamma^1$ in at most three vertices, $\{g_1\tilde{\gamma}(k-1), g_\tilde{\gamma}(k), g_\tilde{\gamma}(k+1)\}$ for some $k \in \mathbb{Z}$, as illustrated in Figure 4.

Let

$$S_\gamma^1 = \{g_1\tilde{\gamma}(0), \ldots, g_m\tilde{\gamma}(0)\} \subset W_\gamma^1$$

$$S_\rho^1 = \{h_1\tilde{\rho}(0), \ldots, h_n\tilde{\rho}(0)\} \subset W_\rho^1$$

be the subset of vertices of $W_\gamma^1$ and $W_\rho^1$ so that for $S_\gamma^1$:

(a) One vertex of each lift that intersects $W_\gamma^1$ is included.

(b) If a lift intersects $W_\gamma^1$ in three vertices, the middle vertex is chosen.
(c) The vertices are labeled so that \( g_i \tilde{\gamma} < g_j \tilde{\gamma} \) if \( i < j \).

Likewise for \( S^1_{\rho} \). Condition (a) ensures recursion hypothesis (1) is satisfied, and condition (b) ensures \( d_{S}(g_i \tilde{\gamma}(0), g_{i+1} \tilde{\gamma}(0)) = 1 \) and \( d_{T}(h_i \tilde{\rho}(0), h_{i+1} \tilde{\rho}(0)) = 1 \), and condition (c) is used to preserve the ordering on the boundary.

If \( m = n \), continue to the definition of \( \phi_1 \). Otherwise, if \( m < n \), let \( W_{\gamma}^{1r} \) be the set of all vertices in \( R_{\gamma} \) at distance 1 from \( \{ \tilde{\gamma}(1), \tilde{\gamma}(2), \ldots, \tilde{\gamma}(r) \} \), and select the \( n - m \) lifts that intersect \( W_{\gamma}^{1r} \) lowest in the ordering from the boundary. If there do not exist \( n - m \) such lifts that intersect \( W_{\gamma}^{1r} \), increase \( r \). Choose and label one vertex of each of these lifts so that if a lift intersects \( W_{\gamma}^{1r} \) in three vertices, the middle vertex is chosen, and the selected vertices are labeled \( g_{m+1} \tilde{\gamma}(0), \ldots, g_n \tilde{\gamma}(0) \) so that \( g_i \tilde{\gamma} < g_j \tilde{\gamma} \) if \( i < j \). Then \( d_{S}(g_i \tilde{\gamma}(0), g_{i+1} \tilde{\gamma}(0)) = 1 \) for \( 1 \leq i < n \).

We define \( \phi_1 : g_i \tilde{\gamma} \mapsto h_i \tilde{\rho} \); to give the map explicitly, preserving the orientation on the boundary, re-label the lifts by

\[
L^{1}_{\gamma} = \{ \ell_{e_{g}}(t), \ell_{g_1}(t), \ldots, \ell_{g_n}(t) \} \subset L_{\gamma}
\]

\[
L^{1}_{\rho} = \{ \ell_{e_{h}}(t), \ell_{h_1}(t), \ldots, \ell_{h_n}(t) \} \subset L_{\rho}
\]

where \( \ell_{g_i} = g_i \tilde{\gamma}, e_g \) and \( e_h \) denote the identity elements, and \( \ell_{g_1} \) is parametrized so that \( \ell_{g_i}(0) = g_i \tilde{\gamma}(0), d_{S}(\ell_{g_i}(p), \ell_{g_i}(q)) = |p - q| \) for \( p, q \in \mathbb{Z} \), and \( \ell^{\infty}_{g_i} < \ell^{\infty}_{g_j} \); likewise for \( L^{1}_{\rho} \). Define

\[
\phi_1 : L^{1}_{\gamma} \to L^{1}_{\rho}
\]

by

\[
\phi_1(\ell_{e_{g}}(t)) = \ell_{e_{h}}(t),
\]

\[
\phi_1(\ell_{g_i}(t)) = \ell_{h_i}(t)
\]

for all \( 1 \leq i \leq n \).

If \( m \leq n \), construct a hyperbolic polygon, \( P^1_{\gamma} \), in \( R_{\gamma} \) with vertices

\[
V^1_{\gamma} = \{ \tilde{\gamma}(-1), g_1 \tilde{\gamma}(0), \ldots, g_n \tilde{\gamma}(0), \tilde{\gamma}(r) \},
\]

edges connecting vertices adjacent in the list, and an edge connecting \( \tilde{\gamma}(r) \) and \( \tilde{\gamma}(-1) \), where \( r \) is given above if \( m < n \) and \( r = 1 \) if \( m = n \). Similarly, construct a hyperbolic polygon, \( P^1_{\rho} \), in \( R_{\rho} \) with vertices

\[
V^1_{\rho} = \{ \tilde{\rho}(-1), h_1 \tilde{\rho}(0), \ldots, h_n \tilde{\rho}(0), \tilde{\rho}(1) \},
\]

edges connecting vertices adjacent in the list, and an edge connecting \( \tilde{\rho}(1) \) and \( \tilde{\rho}(-1) \). If \( m > n \), the process is carried out in reverse, and the final vertices in the sets \( V^1_{\gamma} \) and \( V^1_{\rho} \) are \( \tilde{\gamma}(1) \) and \( \tilde{\rho}(s) \) for some \( s \). Define

\[
\tilde{\phi}_1 : P^1_{\gamma} \mapsto P^1_{\rho}
\]
Figure 5. A sketch of the lifts mapped during stage 1, and, below, the first two polygons constructed in the tilings of $R_\gamma$ and $R_\rho$.

by mapping the vertices in $V^1_\gamma$ to the vertices in $V^1_\rho$, in order, and extending to a homeomorphism on the edges and interior.

The polygons $P^1_\gamma$ and $P^1_\rho$ have an equal number of sides, $M$, by construction. We claim

$$3 \leq M \leq 2 + \max\{16g^2 - 8g, 16h^2 - 8h\}.$$  

By the choice of representation, each translate of the fundamental domain that intersects $R_\gamma$ intersects $R_\gamma$ in at least two vertices, so $m,n \geq 1$, proving the lower bound. On the other hand, $m,n \leq \max\{16g^2 - 8g, 16h^2 - 8h\}$ by the choice of generating sets, proving the upper bound. In addition, each edge constructed in the interior of $R_\gamma$ satisfies $d_S(g_i\tilde{\gamma}(0),g_{i+1}\tilde{\gamma}(0)) = 1$ for $1 \leq i \leq n - 1$. Likewise for $P^1_\rho$. The dual tree after stage one is a single point, thus, all recursion hypotheses are satisfied.

Stage $n$:

Suppose $\phi_{n-1}$, $n-1$ stages of the tiling, and $\tilde{\phi}_{n-1}$ have been constructed, satisfying the hypothesis of the recursion. For each of the $K$ edges constructed in the interior of $R_\gamma$ during stage $n-1$, extend the maps $\phi_{n-1}$ and $\tilde{\phi}_{n-1}$ and the tiling as follows.

Suppose $\{g_0\tilde{\gamma}(0),g\tilde{\gamma}(0)\}$ is the $i^{th}$ edge constructed during stage $n-1$ in the interior of $R_\gamma$, $g_0\tilde{\gamma} < g\tilde{\gamma}$, and $\phi_{n-1}$ induces the map $\tilde{\phi}_{n-1} : \{g_0\tilde{\gamma}(0),g\tilde{\gamma}(0)\} \mapsto \{h_0\tilde{\rho}(0),h\tilde{\rho}(0)\}$. 
Consider

\[(1) \quad \{g_0 \tilde{\gamma}(-1), g_0 \tilde{\gamma}(0), g_0 \tilde{\gamma}(1)\}, \]
\[(2) \quad \{h_0 \tilde{\rho}(-1), h_0 \tilde{\rho}(0), h_0 \tilde{\rho}(1)\}.\]

In a manner similar to that of stage 1, first choose the lifts of \(\gamma\) and \(\rho\) in the domain and image of \(\phi_{n_i}\). Let \(W^{n_i}_\gamma\) and \(W^{n_i}_\rho\) be the set of all vertices in \(R_\gamma\) and \(R_\rho\) at distance 1 from (1) and (2) that do not lie on a lift included in the domain or image of \(\phi_{n-1}\), respectively. Select and label each lift that intersects one of these sets as follows. Let

\[S_{n_i}^\gamma = \{g_{i,1} \tilde{\gamma}(0), \ldots, g_{i,r} \tilde{\gamma}(0)\} \subset W^{n_i}_\gamma\]
\[S_{n_i}^\rho = \{h_{i,1} \tilde{\rho}(0), \ldots, h_{i,s} \tilde{\rho}(0)\} \subset W^{n_i}_\rho\]

be the subset of vertices of \(W^{n_i}_\gamma\) and \(W^{n_i}_\rho\) chosen so that for \(S_{n_i}^\gamma\):

(a) One vertex of each lift that intersects \(W^{n_i}_\gamma\) is included.

(b) If a lift intersects \(W^{n_i}_\gamma\) in three vertices, the middle vertex is chosen.

(c) The vertices are labeled so that \(g_i \tilde{\gamma} < g_j \tilde{\gamma}\) if \(i < j\).

Likewise for the vertices in \(S_{n_i}^\rho\). Condition (b) ensures \(d_S(g_{i,k} \tilde{\gamma}(0), g_{i,k+1} \tilde{\gamma}(0)) = 1\) and \(d_T(h_{i,k} \tilde{\rho}(0), h_{i,k+1} \tilde{\rho}(0)) = 1\).

If \(r = s\), continue to the definition of \(\phi_{n_i}\). Otherwise, if \(r < s\), let \(W^{n_i,k}_\gamma\) be the set of all vertices in \(R_\gamma\) at distance 1 from \(\{g_{2} \tilde{\gamma}(2), g_{3} \tilde{\gamma}(3), \ldots, g_{k} \tilde{\gamma}(k)\}\) that do not lie on a lift included in the domain of \(\phi_{n-1}\), and select the \(s - r\) lifts that intersect \(W^{n_i,k}_\gamma\) lowest in the ordering from the boundary. If there do not exist \(s - r\) such lifts, increase \(k\). Choose and label one vertex of each of these lifts so that if a lift intersects \(W^{n_i,k}_\gamma\)
in three vertices, the middle vertex is chosen, and the vertices selected are labeled $g_{i,r+1} \tilde{\gamma}(0), \ldots, g_{i,s} \tilde{\gamma}(0)$ so that $g_{i,j} \tilde{\gamma} < g_{i,k} \tilde{\gamma}$ if $j < k$. Then $d_S(g_{i,j} \tilde{\gamma}(0), g_{i,j+1} \tilde{\gamma}(0)) = 1$ for $1 \leq j < s$.

As before, we map $g_{i,j} \tilde{\gamma}$ to $h_{i,j} \tilde{\rho}$; to define the map $\phi_{n,i}$, label these lifts

\[ L_{\gamma}^{n_i} = \{\ell_{g_{i,1}}(t), \ldots, \ell_{g_{i,s}}(t)\} \subset L_\gamma, \]
\[ L_{\rho}^{n_i} = \{\ell_{h_{i,1}}(t), \ldots, \ell_{h_{i,s}}(t)\} \subset L_\rho \]

where $\ell_{g_{i,k}} = g_{i,k} \tilde{\gamma}$ and is parametrized so that $\ell_{g_{i,k}}(0) = g_{i,k} \tilde{\gamma}(0)$, $d_S(\ell_{g_{i,k}}(p), \ell_{g_{i,k}}(q)) = |p - q|$ for $p, q \in \mathbb{Z}$ and $\ell_{g_{i,k}}^{+\infty} < \ell_{g_{i,k}}^{-\infty}$; likewise for $L_{\rho}^{n_i}$. Define

\[ \phi_{n,i} : L_{\gamma}^{n_i} \to L_{\rho}^{n_i} \]

by

\[ \phi_{n,i}(\ell_{g_{i,j}}(t)) = \ell_{h_{i,j}}(t) \]

for all $1 \leq j \leq s$ and let

\[ \phi_n = \phi_{n-1} \cup \bigcup_{i=1}^{K} \phi_{n,i}. \]

If $r \leq s$, construct a hyperbolic polygon, $P_{\gamma}^{n_i}$ in $R_\gamma$ with vertices

\[ V_{\gamma}^{n_i} = \{g_0 \tilde{\gamma}(0), g_0 \tilde{\gamma}(-1), g_{i,1} \tilde{\gamma}(0), \ldots, g_{i,s} \tilde{\gamma}(0), g_{i,k} \tilde{\gamma}(k), g_{i}(1), g_{i}(0)\}, \]
edges connecting adjacent vertices in the list, and an edge connecting \(g_0\gamma(0)\) and \(g\gamma(0)\), where \(k\) is given above if \(r < s\) and \(k = 2\) if \(r = s\). Construct a hyperbolic polygon, \(P^{n_i}_{\rho}\) in \(R_\rho\) with vertices
\[
V_{\rho}^{n_i} = \{h_0\tilde{\rho}(0), h_0\tilde{\rho}(-1), h_{i, 1}\tilde{\rho}(0), \ldots, h_{i, s}\tilde{\rho}(0), h\tilde{\rho}(2), h\tilde{\rho}(1), h\tilde{\rho}(0)\},
\]
diagram edges connecting vertices adjacent in the list, and an edge connecting \(h_0\tilde{\rho}(0)\) and \(h\tilde{\rho}(0)\). If \(r > s\), the process is carried out in reverse. Define
\[
\tilde{\phi}_{n_i} : P^{n_i}_{\gamma} \mapsto P^{n_i}_{\rho}
\]
by mapping vertices in \(V^{n_i}_{\gamma}\) to vertices in \(V^{n_i}_{\rho}\), in the order given in the lists, and extending to the edges and interior by a homeomorphism. Let
\[
\tilde{\phi}_n = \tilde{\phi}_{n-1} \cup \bigcup_{i=1}^{K} \tilde{\phi}_{n_i}.
\]
To prove the first recursion hypothesis holds after stage \(n\), consider \(\Gamma\), the Cayley graph for \(\pi_1(S_g)\) with generating set \(S\), which maps into the hyperbolic plane with the 0-cells, \(\{g\gamma(0)\}\), previously described and edges that are geodesic segments. A more standard Cayley graph, \(\Gamma'\), for \(\pi_1(S_g)\) maps into \(H^2\) dual to the tiling by \(4g\)-gons; there is a \(4g\)-gon in \(\Gamma'\) about each vertex in the tiling. The Cayley graph \(\Gamma\) is obtained from the Cayley graph \(\Gamma'\) by replacing each \(4g\)-gon with \(K_{4g}\), the complete graph on \(4g\) vertices. Let \(\Gamma_R\) be the subgraph of \(\Gamma\) induced by vertices contained in \(R_\gamma\). That is, \(\Gamma_R\) consists of all vertices that lie on the lifts in \(L_\gamma\) and all the edges in \(\Gamma\) between these vertices. The subgraph \(\Gamma_R\) is connected since any fundamental domain that intersects \(R_\gamma\) intersects \(R_\gamma\) in at least two vertices. Now, let \(g\gamma \in R_\gamma\) with \(|g|_S = n\). Since the subgraph \(\Gamma_R\) is connected, there exists \(g' \in \Gamma_R\) with \(d_S(g\gamma(0), g\gamma(0)) = 1\) and \(|g'|_S \leq n - 1\). By the recursion hypothesis, \(g\gamma\) is in the domain of \(\phi_{n-1}\), so \(g\gamma(0)\) lies on the edge of a polygon in the interior of \(R_\gamma\). Hence \(g\gamma\) is in the domain of \(\phi_n\) by the choice of elements in the set (1) and \(W^{n_i}_{\gamma}\).

The bound on the number of sides of the polygons \(P^{n_i}_{\gamma}\) and \(P^{n_i}_{\rho}\) holds by the choice of elements in \(W^{n_i, 1}_{\gamma}\), \(W^{n_i, 1}_{\rho}\) and the size of the generating sets, which imply
\[
(3) \quad r \leq 64g^2 - 32g \quad \text{and} \quad s \leq 64h^2 - 32h.
\]
Each edge of \(P^{n_i}_{\gamma}\) in the interior of \(R_\gamma\) satisfies
\[
d_S(g_{i,j}\gamma(0), g_{i, j+1}\gamma(0)) = 1, \ d_S(h_{i,j}\tilde{\rho}(0), h_{i, j+1}\tilde{\rho}(0)) = 1
\]
for \(1 \leq j \leq n - 1\) by construction.

Finally, the dual tree associated to \(\tilde{\phi}_{n_i}\) has one additional edge, connecting the tile constructed in stage \(n - 1\) containing \(g_0\gamma(0)\) and \(g\gamma(0)\) to the tile constructed associated to \(\phi_{n_i}\). Since \(\tilde{\phi}\) still maps the edge \(\{g_0\gamma(0), g\gamma(0)\}\) to the edge \(\{h_0\tilde{\rho}(0), h\tilde{\rho}(0)\}\), \(\tilde{\phi}\) extends the isomorphism between the dual trees.
Finitely many isometry types of tiles:

By this construction, \(R_\gamma\) and \(R_\rho\) are tiled by compact convex hyperbolic polygons and \(\phi\) induces an isomorphism between the dual trees of the tilings. The final claim needed to conclude the proof is that there are only finitely many isometry types of tiles constructed.

By the recursion hypothesis, each polygon in \(R_\gamma\) has at most \(M\) edges, where \(M = 3 + \max\{64g^2 - 32g, 64h^2 - 32\}\). To bound the edge lengths, observe that all but possibly one edge of the polygon connects vertices \(v\) and \(w\) with \(d_S(v, w) = 1\). There are at most \(16g^2 - 8g\) edge lengths of these edges; let \(L\) be the length of the longest. The last edge connects \(g\tilde{\gamma}(k)\) and \(g\tilde{\gamma}(1)\) and \(k \leq 64h^2 - 32h\). So the maximum length of this edge is \(k \times d\), where \(d\) is the translation length of \([\gamma]\). Let \(N = \max\{L, n \times d\}\).

Let \(P\) be a polygon constructed in the tiling and fix a vertex \(g\tilde{\gamma}(0)\) in the polygon. All other vertices lie in \(B_{g\tilde{\gamma}(0)}(r)\), a ball of radius \(r = M \times N\) in the hyperbolic plane with center \(g\tilde{\gamma}(0)\). The element \(g^{-1} \in \pi_1(S_g)\) is an isometry mapping \(g\tilde{\gamma}(0)\) to \(\tilde{\gamma}(0)\), and \(B_{g\tilde{\gamma}(0)}(r)\) to \(B_{\tilde{\gamma}(0)}(r)\), a ball of radius \(r\) about \(\tilde{\gamma}(0)\). Since the group action preserves the \(0\)-cells, \(g^{-1}\) maps \(P\) to a polygon in \(B_{\tilde{\gamma}(0)}(r)\) with vertices \(g_i\tilde{\gamma}(0)\) and \(d_S(\tilde{\gamma}(0), g_i\tilde{\gamma}(0)) \leq r\). There are only finitely many polygons in this ball with vertices \(g_i\tilde{\gamma}(0)\), so there are only finitely many isometry types of polygons constructed in the tiling. A similar argument holds for the tiling of \(R_\rho\), concluding the proof of Proposition 13.

To prove the entire universal covers of \(X_1, X_2 \in \mathcal{X}_2\) are bi-Lipschitz equivalent, we map the regions incident to the axis of \([\gamma]\) to the regions incident to the axis of \([\rho]\) using the bi-Lipschitz equivalence from Proposition 13. We use the action of the groups by deck transformations on the universal covers to extend the map recursively along regions incident to the lifts that bound a region already in the domain.

**Theorem 14.** If \(X_1 = S_{g_1} \cup \gamma S_{g_1}^\prime\) and \(X_2 = S_{g_2} \cup \rho S_{g_2}^\prime\) where \(S_{g_1}\) and \(S_{g_1}^\prime\) are closed hyperbolic surfaces for \(i = 1, 2\) and \(\gamma : S^1 \to X_1\) and \(\rho : S^1 \to X_2\) are the images of essential simple closed curves under identification, then there exists a bi-Lipschitz equivalence \(\Phi : \tilde{X}_1 \to \tilde{X}_2\) mapping lifts of \(\gamma\) to lifts of \(\rho\) bijectively.

**Proof.** Let \(\tilde{\gamma}\) and \(\tilde{\rho}\) be the axes for \([\gamma]\) and \([\rho]\), respectively. Then \(\tilde{\gamma}\) is contained in a hyperbolic plane stabilized by \(\pi_1(S_{g_1})\) and a hyperbolic plane stabilized by \(\pi_1(S_{g_1}^\prime)\), and \(\tilde{\rho}\) is contained in a hyperbolic plane stabilized by \(\pi_1(S_{g_2})\) and a hyperbolic plane stabilized by \(\pi_1(S_{g_2}^\prime)\). Let \(R_1\) and \(R_2\) be the convex regions bounded by lifts incident to \(\tilde{\gamma}\) in the plane stabilized by \(\pi_1(S_{g_1})\) and let \(R_3\) and \(R_4\) be the convex regions bounded by lifts incident to \(\tilde{\gamma}\) in the plane stabilized by \(\pi_1(S_{g_1}^\prime)\). Similarly, let \(S_1\) and \(S_2\) be the convex regions bounded by lifts incident to \(\tilde{\rho}\) in the plane stabilized by \(\pi_1(S_{g_2})\) and let \(S_3\) and \(S_4\) be the convex regions bounded by lifts incident to \(\tilde{\rho}\) in the plane stabilized by \(\pi_1(S_{g_2}^\prime)\).
By Proposition 13 there exist $K_i$-bi-Lipschitz equivalences

$$\bar{\phi}_i : R_i \to S_i$$

that are bijective on the lifts bounding $R_i$ and $S_i$ and each map $\tilde{\gamma}$ to $\tilde{\rho}$. Let $K = \max\{K_i\}$ so that $\phi_i$ is a $K$-bi-Lipschitz equivalence for $1 \leq i \leq 4$. Define

$$\Phi|_{R_i} = \bar{\phi}_i.$$

To extend the map coherently along the regions incident to a lift already included in the domain, color the convex regions bounded by lifts in $\tilde{X}_1$ and $\tilde{X}_2$ with colors $\{C_1, C_2, C_3, C_4\}$ as follows. Let $C(R)$ denote the color of region $R$. Two-color the plane stabilized by $\pi_1(S_{g_1})$ with $C_1$ and $C_2$ so that $C(R_i) = C_i$ and so that adjacent regions have distinct colors. Do the same for the planes stabilized by $\pi_1(S_{g_1}'), \pi_1(S_{g_2})$, and $\pi_1(S_{g_2}')$ so that $C(R_i) = C(S_i) = C_i$. Extend the coloring to a coloring of $\tilde{X}_1$ and $\tilde{X}_2$ recursively as follows. Suppose a hyperbolic plane, $H$, stabilized by a conjugate of $\pi_1(S_{g_1})$ or $\pi_1(S_{g_1}')$ has not been colored and that $H$ intersects, in a lift $g\tilde{\gamma}$, a hyperbolic plane that has been colored. Then $g^{-1}H$ contains $\tilde{\gamma}$, and has been colored. For every region $R \subset H$, let $C(R) = C(g^{-1}R)$. Color $\tilde{X}_2$ in analogously.

Observe that, regardless of the topological type of $\gamma_i$ and $\gamma_i'$, $\pi_1(X_i)$ is transitive on the regions restricted to one color in $\tilde{X}_1$ and $\tilde{X}_2$.

Define the map $\Phi$ recursively. Suppose $R$ is a region that intersects, in a lift $g\tilde{\gamma}$, the boundary of a region already mapped by $\Phi$ to $\tilde{X}_2$. Then $\Phi : g\tilde{\gamma} \mapsto g'\tilde{\rho}$ for some $g' \in \pi_1(X_2)$. Let $S$ be the region in $\tilde{X}_2$ intersecting $g'\tilde{\rho}$ with $C(R) = C(S) = C_i$ for some $i$, $1 \leq i \leq 4$. Then, map $R$ to $S$ by the map

$$\Phi|_R = g' \circ \phi_i \circ g^{-1},$$

a $K$-bi-Lipschitz equivalence since $g$ and $g'$ are isometries. \hfill $\Box$

3.3. Amalgams over immersed curves. All groups in $C_S$ are quasi-isometric, and the quasi-isometric classification of groups of the form $\pi_1(S_g) * \langle a \rangle \pi_1(S_h)$ in full generality depends on the image of $a$ in the groups $\pi_1(S_g)$ and $\pi_1(S_h)$.

**Proposition 15.** If $G \in C_S$ and $G' \in C \setminus C_S$, then $G$ and $G'$ are not quasi-isometric.

**Proof.** All groups in $C$ are $\delta$-hyperbolic, so any quasi-isometry between groups in this class induces a homeomorphism between their visual boundaries. The boundary of a lift of the curves identified is the intersection of two circles in the boundary of the group, $\partial G$. The number of components in the complement of the boundary of a lift depends on whether the curves identified have simple representatives on the surface. That is, if both curves identified are simple, there are four components in the complement of the boundary of a lift in $\partial G$, while if at least one curve has no simple representative, there are fewer than four components in the complement of the boundary of a lift in $\partial G$. A homeomorphism preserves the number of components in the complement of the boundary of a lift, so the fundamental group of a
space obtained by identifying closed hyperbolic surfaces along two essential simple closed curves is not quasi-isometric to the fundamental group of a space obtained by identifying along at least one curve with no simple representative.

Corollary 16. If \( G \in \mathcal{C}_S \) and \( G' \in \mathcal{C}\setminus\mathcal{C}_S \), then \( G \) and \( G' \) are not abstractly commensurable.

The quasi-isometric and abstract commensurability classification of groups in \( \mathcal{C} \) remains open.

4. Abstract Commensurability Classes

In this section, we give the abstract commensurability classification for groups in \( \mathcal{C}_S \) and characterize the groups in \( \mathcal{C}_S \) that contain a maximal element in their abstract commensurability class restricted to \( \mathcal{C}_S \). We use \( \text{lcm}(a, b) \) to denote the least common multiple of \( a \) and \( b \) and \( \gcd(a, b) \) to denote the greatest common divisor of \( a \) and \( b \).

4.1. The topology of spaces in \( \mathcal{X}_S \). The subgroup structure of an amalgamated product is described in the following theorem of Scott and Wall.

Theorem 17. ([11], Theorem 3.7) If \( G \cong A \ast_C B \) and if \( H \leq G \), then \( H \) is the fundamental group of a graph of groups, where the vertex groups are subgroups of conjugates of \( A \) or \( B \) and the edge groups are subgroups of conjugates of \( C \).

Topologically, Theorem 17 implies that any finite sheeted cover of the space \( X = S_g \cup \gamma S_h \), where \( \gamma \) is the image of \( \gamma_g : S^1 \to S_g \) and \( \gamma_h : S^1 \to S_h \) under identification, consists of a set of surfaces which cover \( S_g \) and a set of surfaces which cover \( S_h \). These surfaces are identified along multicurves that are the preimages of \( \gamma_g \) and \( \gamma_h \). These covers are examples of simple, thick, 2-dimensional hyperbolic \( P \)-manifolds (see [6], Definition 2.3.) So, the following topological rigidity theorem of Lafont allows us to address the abstract commensurability classification for members in \( \mathcal{C}_S \) from a topological point of view. Corollary 19 also follows from the proof of Proposition 3.1 in [4].

Theorem 18. ([6], Theorem 1.2) Let \( X_1 \) and \( X_2 \) be a pair of simple, thick, 2-dimensional hyperbolic \( P \)-manifolds, and assume that \( \phi : \pi_1(X_1) \to \pi_1(X_2) \) is an isomorphism. Then there exists a homeomorphism \( \Phi : X_1 \to X_2 \) that induces \( \phi \) on the level of fundamental groups.

Corollary 19. Let \( G, G' \in \mathcal{C}_S \) with \( G \cong \pi_1(X) \), \( G' \cong \pi_1(X') \) with \( X, X' \in \mathcal{X}_S \). Then \( G \) and \( G' \) are abstractly commensurable if and only if \( X \) and \( X' \) have homeomorphic finite-sheeted covering spaces.

To study finite-sheeted covering spaces, we use the following well-known fact.
Lemma 20. If \( X \) is a CW-complex and \( X' \) is a degree \( n \) cover of \( X \), then \( \chi(X') = n\chi(X) \), where \( \chi \) denotes Euler characteristic.

A converse to Lemma 20 holds for hyperbolic surfaces with one boundary component, which we use to construct covers of spaces in \( \mathcal{X} \).

Lemma 21. For \( g_i \geq 1 \), if \( \chi(S_{g_i,1}) = n\chi(S_{g_1,1}) \), then \( S_{g_2,1} \) \( n \)-fold covers \( S_{g_1,1} \).

Proof. Let
\[
\pi_1(S_{g_i,1}) = \langle a_1, b_1, \ldots, a_{g_i}, b_{g_i} \rangle \cong F_{2g_i}
\]
be presentations for the fundamental groups of \( S_{g_i,1} \). Then, the homotopy class of the boundary element \( \gamma_i : S^1 \to S_{g_i,1} \) corresponds to the element \([a_1, b_1] \ldots [a_{g_i}, b_{g_i}] \in \pi_1(S_{g_i,1})\).

We exhibit \( \pi_1(S_{g_2,1}) \) as an index \( n \) subgroup of \( \pi_1(S_{g_1,1}) \) so that in the corresponding cover, \( \gamma_2 \) has preimage a single curve that \( n \)-fold covers \( \gamma_1 \).

Realize \( \pi_1(S_{g_1,1}) \) as the fundamental group of a wedge of \( 2g_1 \) oriented circles labeled by the generating set. Construct an \( n \)-fold cover of this space as a graph, \( \Gamma \), on \( n \).
vertices labeled \( \{0, \ldots, n - 1\} \). For every generator besides \( a_1 \), construct an oriented \( n \)-cycle on the \( n \) vertices with each edge labeled by the generator. Since \( \chi(S_{g_1,1}) \) and \( \chi(S_{g_2,1}) \) are both odd, \( n \) must be odd as well. Let \( \{i, i + 1\} \) and \( \{i + 1, 1\} \) be directed edges labeled by \( a_1 \) for \( i < n \) and \( i \) odd. Construct a directed loop labeled \( a_1 \) at vertex \( \{0\} \). By construction, \( \Gamma \) covers the wedge of circles given above.

To see that \( \gamma_1 \) has a preimage with one component, choose a vertex \( v \) in the graph \( \Gamma \) and consider the edge path \( p \) with edges labeled \( ([a_1, b_1] \ldots [a_n, b_n])^k \), a preimage of a representative of \( \gamma_1 \). Then \( n \) is the smallest non-zero \( k \) for which \( p \) terminates at \( v \). To see this, note that it suffices to consider the path \( p' = [a_1, b_1]^k \) since every other segment \([a_j, b_j]\) returns to its initial vertex. Starting at vertex \( \{0\} \), observe that the path \([a_1, b_1]^k\) terminates at the vertex labeled

\[
\begin{cases}
2k - 1 & \text{if } 0 < k < \left\lfloor \frac{n}{2} \right\rfloor \mod n \\
2n - 2k & \text{if } \left\lfloor \frac{n}{2} \right\rfloor \leq k < n \mod n \\
0 & \text{if } k = 0 \mod n,
\end{cases}
\]

proving the claim.

If \( \gamma : S^1 \to S_g \) is an essential simple closed curve, then up to a homeomorphism of \( S_g \), \( \gamma \) is either a non-separating curve, or a separating curve so that \( S_g = S_{r,1} \cup \gamma S_{s,1} \). We characterize the topological type of \( \gamma \) with the following notation.

**Definition 22.** Let \( \gamma : S^1 \to S_g \) be an essential simple closed curve. Define the topological type of \( \gamma \), \( t(\gamma) \), by \( t(\gamma) = 1 \) if \( \gamma \) is non-separating and \( t(\gamma) = \frac{\chi(S_{r,1})}{\chi(S_{s,1})} \) if \( \gamma \) is separating so that \( S_g = S_{r,1} \cup \gamma S_{s,1} \) and \( \chi(S_{r,1}) \leq \chi(S_{s,1}) \).

**Definition 23.** If \( S_g \) and \( S_h \) are closed hyperbolic surfaces, \( \gamma \) is a multicurve on \( S_g \) and \( \rho \) is a multicurve on \( S_h \), we say \((S_g, \gamma) \) covers \((S_h, \rho) \) if there exists a covering map \( p : S_g \to S_h \) so that \( \gamma \) is the full preimage of \( \rho \) in \( S_g \), \( \gamma = p^{-1}(\rho) \).

**Proposition 24.** Let \( S_{g_1} \) and \( S_{g_2} \) be closed hyperbolic surfaces and \( \gamma_1 : S^1 \to S_{g_1} \) and \( \gamma_2 : S^1 \to S_{g_2} \) essential simple closed curves. There exists \((S_g, \gamma) \) which covers both \((S_{g_1}, \gamma_1) \) and \((S_{g_2}, \gamma_2) \) if and only if \( t(\gamma_1) = t(\gamma_2) \).

**Proof.** Suppose there exists \((S_{g_0}, \gamma_0)\) which covers both \((S_{g_1}, \gamma_1)\) and \((S_{g_2}, \gamma_2)\). Then there exists \((S_g, \gamma)\) a regular cover of \((S_{g_1}, \gamma_1)\) and a cover of \((S_{g_2}, \gamma_2)\).

First suppose that \( \gamma_1 \) is a non-separating simple closed curve; we claim that \( t(\gamma_2) = 1 \).

Either \( \gamma_2 \) is non-separating, or \( \gamma_2 \) is separating so that \( S_{g_2} = S_{r,1} \cup \gamma_2 S_{s,1} \). Since the cover of \( S_{g_1} \) is regular, \( S_g \setminus \gamma \) consists of a set of homeomorphic surfaces with boundary. Color \( S_{r,1} \) red and \( S_{s,1} \) blue, which lifts to a coloring of the components of \( S_g \setminus \gamma \). Let \( C_R \) denote the red components of \( S_g \setminus \gamma \) and let \( C_S \) denote the blue components of \( S_g \setminus \gamma \); then, \( C_R \) covers \( S_{r,1} \) and \( C_S \) covers \( S_{s,1} \). The colored boundary components of the surfaces in \( S_g \setminus \gamma \) come in red-blue pairs that project to \( \gamma_2 \) under the covering map. Since all components of \( S_g \setminus \gamma \) are homeomorphic, there are an equal number of red and blue components. Thus, \( \chi(C_R) = \chi(C_S) \). So,
\[ \chi(C_S) = \chi(C_R) = n\chi(S_{r,1}) = n\chi(S_{s,1}), \]

and therefore \( r = s \), and \( t(\gamma_2) = 1 \). Likewise, if \( \gamma_2 \) is non-separating, then \( t(\gamma_1) = 1 \).

Otherwise, both \( \gamma_1 \) and \( \gamma_2 \) are separating simple closed curves with \( S_{g_1} = S_{r_{1,1}} \cup \gamma_1 \) \( S_{s_{1,1}} \) and \( S_{g_2} = S_{r_{2,1}} \cup \gamma_2 \) \( S_{s_{2,1}} \) and \( r_i \leq s_i \). Since the cover of \( S_{g_1} \) is regular, \( S_g \setminus \gamma \) consists of two homeomorphism classes of surfaces with boundary: \( S_{g_r,b_r} \) which cover \( S_{r_{1,1}} \) and \( S_{g_s,b_s} \) which cover \( S_{s_{1,1}} \). Since there is a homeomorphism of \( S_g \) taking any lift of \( \gamma_2 \) to any other lift of \( \gamma_2 \), we may assume the \( S_{g_r,b_r} \) cover \( S_{r_{2,1}} \) and the \( S_{g_s,b_s} \) cover \( S_{s_{2,1}} \), as well.

Suppose that \( S_{g_r,b_r} \) forms a \( k_i \) degree cover of \( S_{r_{1,1}} \) and that \( S_{g_s,b_s} \) forms a \( \ell_i \) degree cover of \( S_{s_{1,1}} \). Further, suppose that \( S_g \) contains \( K \) subsurfaces homeomorphic to \( S_{g_r,b_r} \) and \( L \) subsurfaces homeomorphic to \( S_{g_s,b_s} \). If \( S_g \) is an \( n_i \) degree cover of \( S_{g_i} \), then

\[ n_i = L\ell_i = Kk_i. \]

By Lemma 20

\[ \chi(S_{g_r,b_r}) = k_1\chi(S_{r_{1,1}}) = k_2\chi(S_{r_{2,1}}), \]

\[ \chi(S_{g_s,b_s}) = \ell_1\chi(S_{s_{1,1}}) = \ell_2\chi(S_{s_{2,1}}). \]

Combining (4) and (5), we have

\[ \frac{\chi(S_{r_{1,1}})}{\chi(S_{s_{1,1}})} = \frac{\chi(S_{r_{2,1}})}{\chi(S_{s_{2,1}})}, \]

so \( t(\gamma_1) = t(\gamma_2) \).

For the converse, let \( L = \text{lcm}(|\chi(S_{g_1})|, |\chi(S_{g_2})|) \) and let \( k_i = \frac{L}{\chi(S_{g_i})} \). We construct \((S_g, \gamma)\) so that \( \chi(S_g) = 2L \) and \( \gamma \) has two components, each of which covers \( \gamma_i \) by degree \( k_i \).

First suppose that \( t(\gamma_1) = t(\gamma_2) = 1 \). Let \( S_g \) be the closed surface with Euler characteristic \( 2L \). Let \( \gamma \) be a multicurve on \( S_g \) so that \( S_g \setminus \gamma \) consists of two surfaces with Euler characteristic \( L \) and two boundary components. We prove \((S_g, \gamma)\) covers both \((S_{g_1}, \gamma_1)\); the proof that \((S_g, \gamma)\) covers \((S_{g_2}, \gamma_2)\) is similar. If \( \gamma_1 \) is non-separating, the pair \((S_g, \gamma)\) covers \((S_{g_1}, \gamma_1)\) via an intermediate cover

\[ (S_g, \gamma) \xrightarrow{2} (\widetilde{S}_{g_1}, \tilde{\gamma}_i) \xrightarrow{k_i} (S_{g_1}, \gamma_1). \]

Let \( \widetilde{S}_{g_1} \) be the closed surface with Euler characteristic \( L \). Arranging the handles symmetrically about a central handle, choose \( \tilde{\gamma}_i : S^1 \to \widetilde{S}_{g_1} \) a non-separating simple closed curve so that upon rotation by \( \frac{2\pi}{k_i} \), \( \tilde{\gamma}_i \) projects to \( \gamma_1 \) and \((\widetilde{S}_{g_1}, \tilde{\gamma}_i)\) is a cyclic cover of \((S_{g_1}, \gamma_1)\) of degree \( k_i \). Then \((S_g, \gamma)\) covers \((\widetilde{S}_{g_1}, \tilde{\gamma}_i)\): arrange the handles symmetrically about the multicurve \( \gamma \) and rotate by \( \pi \).
That is, since $\chi$ The first intermediate cover, ($S$, $\gamma$) via an intermediate cover

$$(S_g, \gamma) \xrightarrow{\chi(S_g)} (\tilde{S}_{g_1}, \gamma_1) \xrightarrow{k_1} (S_{g_1}, \gamma_1),$$

where $k_1 = \frac{L}{\chi(S_{g_1})}$. Since $g_2$ is even, $L$ is congruent to 2 modulo 4, so $\frac{L}{2}$ is odd.

To construct $S_{g_1}$, let $\tilde{S}_{r,1}$ be the surface with one boundary component and Euler characteristic $\frac{L}{2}$ and let $\tilde{S}_{s,1}$ be the surface with one boundary component and Euler characteristic $\frac{L}{2}$. By Lemma 21 $\tilde{S}_{r,1}$ covers $S_{r,1}$ by degree $k_1$ and $\tilde{S}_{s,1}$ covers $S_{s,1}$ by degree $k_1$. Let $\tilde{S}_{g_1} = \tilde{S}_{r,1} \cup \tilde{S}_{s,1}$ be the surface obtained by identifying $\tilde{S}_{r,1}$ and $\tilde{S}_{s,1}$ along their boundary curves. Then $\gamma_1$ is a separating curve that projects to $\gamma_1$ and $(\tilde{S}_{g_1}, \gamma_1)$ covers $(S_{g_1}, \gamma_1)$ by degree $k_1$. To see that $(S_g, \gamma)$ covers $(\tilde{S}_{g_1}, \gamma_1)$, cut $\tilde{S}_{g_1}$ along a non-separating curve in $\tilde{S}_{r,1}$ and a non-separating curve in $\tilde{S}_{s,1}$ and double the resulting surface with four boundary components. Re-gluing the boundary components in pairs yields $(S_g, \gamma)$, as illustrated in Figure 9.

If $g_2$ is odd, $(S_g, \gamma)$ covers $(S_{g_1}, \gamma_1)$ by degree $\frac{\chi(S_g)}{\chi(S_{g_1})} = \frac{2L}{\chi(S_{g_1})} = 2^n d$, where $n \geq 1$ and $d$ is an odd integer. The covering is given via intermediate covers:

$$(S_g, \gamma) \xrightarrow{2} \ldots \xrightarrow{2} (\tilde{S}_{g_1}, \gamma_1) \xrightarrow{2} (\tilde{S}_{g_1}, \gamma_1) \xrightarrow{d} (S_{g_1}, \gamma_1).$$

The first intermediate cover, $(\tilde{S}_{g_1}, \gamma_1) \xrightarrow{d} (S_{g_1}, \gamma_1)$, is constructed similarly to above. That is, since $\chi(S_{g_1})$ is congruent to 2 modulo 4, $\frac{d \chi(S_{g_1})}{2}$ is odd. If $d > 1$, let $\tilde{S}_{r,1}$ be the surface with one boundary component and Euler characteristic $\frac{d \chi(S_{g_1})}{2}$ and let $\tilde{S}_{s,1}$ be the surface with one boundary component and Euler characteristic $\frac{d \chi(S_{g_1})}{2}$. By Lemma 21 $\tilde{S}_{r,1}$ covers $S_{r,1}$ by degree $d$ and $\tilde{S}_{s,1}$ covers $S_{s,1}$ by degree $d$. Let $\tilde{S}_{g_1} = \tilde{S}_{r,1} \cup \gamma_2$ $\tilde{S}_{s,1}$ be the surface obtained by identifying $\tilde{S}_{r,1}$ and $\tilde{S}_{s,1}$ along their boundary curves. Then $(\tilde{S}_{g_1}, \gamma_1)$ covers $(S_{g_1}, \gamma_1)$ by degree $d$. 

**Figure 9.** An example of a multicurve that projects to a non-separating curve and a separating curve under different covering maps.
The second intermediate cover, \((\tilde{S}_g, \tilde{1}) \xrightarrow{2} (\tilde{S}_g, \tilde{1})\), is the degree two cover of \((\tilde{S}_g, \tilde{1})\) obtained by cutting \(\tilde{S}_g\) along a non-separating curve in \(\tilde{S}_{g,1}\) and a non-separating curve in \(S_{g,1}\), doubling the resulting surface with four boundary components, and re-gluing along the boundary curves as illustrated in Figure 9. Then, \(\tilde{1}\) is a multicurve that bounds two subsurfaces in \(\tilde{S}_g\), each with two boundary components and the same Euler characteristic.

If \(n = 1\), then \((S_g, 1) = (\tilde{S}_g, \tilde{1})\). Otherwise, \((S_g, \gamma)\) covers \((\tilde{S}_g, \tilde{1})\) by degree \(2^{n - 1}\): cut \(\tilde{S}_g\) along a non-separating curve that intersects each of the components of \(\tilde{1}\) once. Double the resulting surface with two boundary components and re-glue the boundary components in pairs to form a double cover of \((\tilde{S}_g, \tilde{1})\). Repeat this process \(n - 1\) times to obtain the cover \((S_g, \gamma) \xrightarrow{2} \ldots \xrightarrow{2} (\tilde{S}_g, \tilde{1})\).

Suppose now that \(t(\gamma_1) = t(\gamma_2) \neq 1\) so that \(S_{g_1} = S_{r,1} \cup_{\gamma_1} S_{s,1}, S_{g_2} = S_{r,2} \cup_{\gamma_2} S_{s,2}\), and \(\frac{\chi(S_{s,1})}{\chi(S_{s,1})} = \frac{\chi(S_{s,2})}{\chi(S_{s,2})}\). Construct \((S_g, \gamma)\) via an intermediate cover

\[ (S_g, \gamma) \xrightarrow{2} (S_{g,0}, \gamma_0) \xrightarrow{k_i} (S_{g, \gamma_i}). \]

To construct \((S_{g,0}, \gamma_0)\), let

\[ L_R = -\ellcm(\chi(S_{r,1}), \chi(S_{s,1})); \]

then \(L_R\) and \(L_S\) are both odd integers. Let \(S_{R,1}\) be the surface with one boundary component and Euler characteristic \(L_R\) and let \(S_{S,1}\) be the surface with one boundary component and Euler characteristic \(L_S\). By Lemma 1.2, \(S_{R,1}\) covers \(S_{r,1}\) with degree \(L_R \chi(S_{r,1})\) and \(S_{S,1}\) covers \(S_{s,1}\) with degree \(L_S \chi(S_{s,1})\). Since \(\frac{\chi(S_{r,1})}{\chi(S_{s,1})} = \frac{\chi(S_{s,1})}{\chi(S_{s,1})}\), \(L_R = L_S \chi(S_{s,1})\). Let \(S_{g,0} = S_{R,1} \cup_{\gamma_0} S_{S,1}\), be the surface obtained by identifying \(S_{R,1}\) and \(S_{S,1}\) along their boundary curves. Then, \((S_{g,0}, \gamma_0)\) covers \((S_g, \gamma_i)\) by degree \(k_i\).

Let \((S_g, \gamma)\) be the 2-fold cover of \((S_{g,0}, \gamma_0)\) obtained by cutting \(S_{g,0}\) along a non-separating curve in \(S_{R,1}\) and a non-separating curve in \(S_{S,1}\) and doubling the resulting surface with four boundary components. Re-gluing the boundary components in pairs yields \((S_g, \gamma)\), as illustrated in Figure 9. Then \((S_g, \gamma)\) covers \((S_g, \gamma_i)\) and \(\gamma\) is a multicurve with two components, each of which covers \(\gamma_i\) by degree \(k_i\).

4.2. Abstract commensurability classification. In this section we give the complete abstract commensurability classification for groups in \(C_S\).

**Theorem 25.** If \(G_1, G_2 \in C_S\), then \(G_1\) and \(G_2\) are abstractly commensurable if and only if \(G_1\) and \(G_2\) may be expressed as \(G_1 \cong \pi_1(S_{g_1}) \ast \langle a_1 \rangle \pi_1(S_{g_1}'), G_2 \cong \pi_1(S_{g_2}) \ast \langle a_2 \rangle \pi_1(S_{g_2}') \in C_S\), given by the monomorphisms \(a_i \mapsto [\gamma_i] \in \pi_1(S_{g_i})\) and \(a_i \mapsto [\gamma_i] \in \pi_1(S_{g_i}')\), and the following conditions hold.
Proof. Suppose that the groups $G_1$ and $G_2$ are abstractly commensurable and that $H \cong H_1 \cong H_2$, where $H_1 \leq G_1$ is a subgroup finite index. Let $X_i$ be the $K(G_i,1)$ space given by identifying the surfaces $S_{g_i}$ and $S_{g'_i}$ along the curves $\gamma_i$ and $\gamma'_i$. Suppose $\tilde{X}$ is the common finite sheeted cover of $X_1$ and $X_2$ with fundamental group $H$ given by Corollary 19. Since $H_1 \leq G_1$ is a subgroup of finite index, the intersection of its conjugates in $G_1$, $H'_1 = \bigcap_{g \in G_1} gH_1g^{-1}$, is a finite index normal subgroup of $G_1$. If $\phi : H_1 \to H_2$ is an isomorphism, then $\phi(H'_1)$ is a finite index subgroup of $G_2$. So, we may suppose that $\tilde{X}$ is a regular cover of $X_1$.

The cover $\tilde{X}$ consists of a set of surfaces and multicurves covering $(S_{g_i}, \gamma_i)$ and $(S_{g'_i}, \gamma'_i)$ for $i = 1, 2$. Since $\tilde{X}$ is a regular cover of $X_1$, there are two homeomorphism classes of surfaces and multicurves in $\tilde{X}$, $(S_g, \gamma)$ and $(S_{g'}, \gamma')$, where components of $\gamma$ are identified to components of $\gamma'$ and $(S_g, \gamma)$ covers $(S_{g_i}, \gamma_i)$ and $(S_{g'}, \gamma')$ covers $(S_{g'_i}, \gamma'_i)$. Up to relabeling, we may also assume $(S_g, \gamma)$ covers $(S_{g_2}, \gamma_2)$ and $(S_{g'}, \gamma')$ covers $(S_{g'_2}, \gamma'_2)$ since there is a homeomorphism of $X$ taking any lift of $\gamma_2$ and $\gamma'_2$ to any other lift of $\gamma_2$ and $\gamma'_2$. Claims (b) and (c) follow from Proposition 24.

To prove claim (a), suppose that $\tilde{X}$ is a $k_1$ degree cover of $X_1$ and a $k_2$ degree cover of $X_2$, $S_g$ is an $n_i$ degree cover of $S_{g_i}$, and $S_{g'}$ is an $n'_i$ degree cover of $S_{g'_i}$. Suppose $\tilde{X}$ consists of $m$ copies of $S_g$ and $m'$ copies of $S_{g'}$. Then,

$$k_i = n_im = n'_im'$$
Figure 11. A common cover of $S_5 \cup S_3$ and $S_8 \cup S_4$ in the case that the identified curves are non-separating.

so that

\begin{equation}
\frac{k_1}{k_2} = \frac{n_1}{n_2} = \frac{n_1'}{n_2'}.
\end{equation}

Also, by Lemma 20,

\begin{equation}
\chi(S_g) = n_1\chi(S_{g_1}) = n_2\chi(S_{g_2}),
\end{equation}

\begin{equation}
\chi(S_{g'}) = n'_1\chi(S_{g'_1}) = n'_2\chi(S_{g'_2}).
\end{equation}

Combining (6) and (7) we have

\begin{equation}
\frac{\chi(S_{g_1})}{\chi(S_{g_2})} = \frac{\chi(S_{g'_1})}{\chi(S_{g'_2})}.
\end{equation}

For the converse, we construct a common cover of $X_1$ and $X_2$ when the conditions of the theorem hold. Let $(S_g, \gamma)$ be the cover of $(S_{g_1}, \gamma_1)$ and $(S_{g_2}, \gamma_2)$ and let $(S_{g'}, \gamma')$ be the cover of $(S_{g'_1}, \gamma'_1)$ and $(S_{g'_2}, \gamma'_2)$ given by Proposition 24. Let $L = -\ell cm(|\chi(S_{g_1})|, |\chi(S_{g_2})|)$ and let $L' = -\ell cm(|\chi(S_{g'_1})|, |\chi(S_{g'_2})|)$. By construction, $(S_g, \gamma)$ covers $(S_{g_1}, \gamma_1)$ by degree $\frac{2L}{\chi(S_{g_1})}$ and each component of $\gamma$ covers $\gamma_1$ by degree $\frac{L}{\chi(S_{g_1})}$. Likewise, $(S_{g'}, \gamma')$ covers $(S_{g'_1}, \gamma'_1)$ by degree $\frac{2L'}{\chi(S_{g'_1})}$ and each component of $\gamma'$ covers $\gamma'_1$ by degree $\frac{L'}{\chi(S_{g'_1})}$. Let $X$ be the space obtained by identifying $S_g$ and $S_{g'}$ along the two components of $\gamma$ and the two components of $\gamma'$. By condition (a),

\begin{equation}
\frac{L}{\chi(S_{g_1})} = \frac{L'}{\chi(S_{g'_1})} = n_i.
\end{equation}

Thus, $X$ forms a $2n_1$-fold cover of $X_1$ and a $2n_2$-fold cover of $X_2$. Therefore, $G_1$ and $G_2$ are abstractly commensurable.

4.3. Maximal element in an abstract commensurability class.

Proposition 26. Let $G \cong \pi_1(S_g) *_{[\gamma]} \pi_1(S_{g'}) \in C_S$ be given by the monomorphisms

$\gamma \mapsto [\gamma_g] \in \pi_1(S_g)$, and $\gamma \mapsto [\gamma_{g'}] \in \pi_1(S_{g'})$. Then there exists a maximal element in the abstract commensurability class of $G$ restricted to $C_S$ if and only if $\gamma_g$ and $\gamma_{g'}$
There exist relatively prime $p$ and $a$ space $X$ denote the abstract commensurability class of $C$.

Let $H$ be the surface obtained by identifying $−$ boundary component and Euler characteristic $S$ are separating simple closed curves with $r \neq s$, $r' \neq s'$.

Proof. We first exhibit a maximal element when the conditions of the proposition hold. Suppose $G \cong \pi_1(S_g) \ast_\gamma \pi_1(S_{g'}) \in C_S$ where $\gamma_g$ and $\gamma_{g'}$ are separating simple closed curves with $S_g = S_{r,1} \cup \gamma_g S_{s,1}$ and $S_{g'} = S_{r',1} \cup \gamma_{g'} S_{s',1}$ where $r \neq s$ and $r' \neq s'$. Let $X_g$ denote the space obtained by identifying $S_g$ and $S_{g'}$ along $\gamma_g$ and $\gamma_{g'}$. Let $C$ denote the abstract commensurability class of $G$ restricted to $C_S$. We construct a space $X \in X_S$ that $X_g$ covers and prove that if $H \in C$ with $H \cong \pi_1(X_h)$ and $X_h \in X_S$, then $X_h$ covers $X$ as well. Then, $\pi_1(X)$ is a maximal element in $C$.

There exist relatively prime $p$ and $q$ and relatively prime $p'$ and $q'$ so that

$$\frac{\chi(S_{r,1})}{\chi(S_{s,1})} = \frac{p}{q} \quad \text{and} \quad \frac{\chi(S_{r',1})}{\chi(S_{s',1})} = \frac{p'}{q'}.$$

So, $\chi(S_{r,1}) = −dp$, $\chi(S_{s,1}) = −dq$, $\chi(S_{r',1}) = −d'p'$, and $\chi(S_{s',1}) = −d'q'$, for some $d,d' \in \mathbb{N}$. Let $k = gcd(d,d')$. Let $S_{u,1}$ be the surface with one boundary component and Euler characteristic $−\frac{d}{k}p$, let $S_{v,1}$ be the surface with one boundary component and Euler characteristic $−\frac{d}{k}q$, let $S_{u',1}$ be the surface with one boundary component and Euler characteristic $−\frac{d'}{k}p'$, and let $S_{v',1}$ be the surface with one boundary component and Euler characteristic $−\frac{d'}{k}q'$. Let $S = S_{u,1} \cup \gamma S_{v,1}$ be the surface obtained by identifying $S_{u,1}$ and $S_{v,1}$ along their boundary curves and let $S' = S_{u',1} \cup \gamma' S_{v',1}$ be the surface obtained by identifying $S_{u',1}$ and $S_{v',1}$ along their boundary curves. Then, $(S_g, \gamma_g)$ covers $(S, \gamma)$ by degree $k$ and $(S_{g'}, \gamma_{g'})$ covers $(S', \gamma')$ by degree $k$; so, $X_g$ covers $X$ by degree $k$.

To see that $\pi_1(X)$ is a maximal element $C$, let $H \in C$. By Theorem 25, $H$ may be expressed as $H \cong \pi_1(S_h) \ast_\rho \pi_1(S_{h'}) \in C$ where $\rho \mapsto [\gamma_h] \in \pi_1(S_h)$ and $\rho \mapsto [\gamma_{h'}] \in \pi_1(S_{h'})$.
\[ \pi_1(S_h), \gamma_h \text{ and } \gamma_{h'} \text{ are separating simple closed curves so that } S_h = S_{m,1} \cup_{\gamma_h} S_{n,1} \text{ and } S_{h'} = S_{m',1} \cup_{\gamma_{h'}} S_{n',1}, \text{ and} \]

\[ \frac{\chi(S_{m,1})}{\chi(S_{n,1})} = \frac{p}{q} \text{ and } \frac{\chi(S_{m',1})}{\chi(S_{n',1})} = \frac{p'}{q'}. \]

Then \( \chi(S_{m,1}) = -fp, \chi(S_{n,1}) = -fq, \chi(S_{m',1}) = -f'p', \text{ and } \chi(S_{n',1}) = -f'q' \) for some \( f, f' \in \mathbb{N}. \) By Theorem 25 (a), \( \frac{\chi(S_h)}{\chi(S_{h'})} = \frac{\chi(S_h)}{\chi(S_{h'})} \), hence \( -\frac{d(p+q)}{d'(p'+q')} = -\frac{f(p+q)}{f'(p'+q')} \), so \( \frac{d}{d'} = \frac{f}{f'}. \) Let \( \ell = \gcd(f, f'). \) By a fact of basic number theory, \( \frac{d}{\ell} = \frac{f}{\ell} \) and \( \frac{d'}{\ell} = \frac{f'}{\ell}. \)

Therefore, \((S_h, \gamma_h)\) covers \((S, \gamma)\) by degree \( \ell \) and \((S_{h'}, \gamma_{h'})\) covers \((S', \gamma')\) by degree \( \ell; \) thus, \( X_h \text{ covers } X \) by degree \( \ell \) as desired.

If \( G \in C_S \) does not satisfy the conditions of the proposition, then there are two groups, \( H_1 \) and \( H_2, \) in the abstract commensurability class of \( G \) in \( C_S, \) where \( H_1 \cong \pi_1(S_{h_1}) \ast_{\langle \gamma \rangle} \pi_1(S_{h_1}'), H_2 \cong \pi_1(S_{h_2}) \ast_{\langle \rho \rangle} \pi_1(S_{h_2}) \) and, up to relabeling, \( \gamma \mapsto [\gamma_{h_1}] \in \pi_1(S_{h_1}), \rho \mapsto [\gamma_{h_2}] \in \pi_1(S_{h_2}), \) where \( \gamma_{h_1} \) is an essential non-separating simple closed curve and \( \gamma_{h_2} \) is a separating simple closed curve. Thus, \((S_{h_1}, \gamma_{h_1})\) and \((S_{h_2}, \gamma_{h_2})\) cannot cover the same pair \((S, \gamma)\), so there is no maximal element in the abstract commensurability class of \( G \) in \( C_S. \)

\[ \square \]

REFERENCES

[1] J. Behrstock and W. Neumann, Quasi-isometric classification of graph manifold groups. Duke Math. Jour. 141 (2008), pgs. 217–240.
[2] M. Bestvina and M. Feighn, Notes on Sela’s work: limit groups and Makanin-Razborov diagrams. Geometric and Cohomological Methods in Group Theory, LMS Lecture Note Series 358 (2009), pgs. 1-29.
[3] M. Bridson and A. Haefliger, Metric spaces of non-positive curvature. Springer-Verlag, Berlin, 1999.
[4] J. Crisp and L. Paoluzzi, Commensurability classification of a family of right-angled Coxeter groups. Proc. Amer. Math. Soc. 136 (2008), pgs. 2343–2349.
[5] P. Dani and A. Thomas. Quasi-isometry classification of certain right-angled Coxeter groups. (2014) arXiv:1402.6224v2.
[6] J.F. Lafont, Diagram rigidity for geometric amalgamations of free groups. J. Pure Appl. Algebra 209 (2007), pgs. 771–780.
[7] W. Malone, Topics in geometric group theory. Ph.D. thesis, University of Utah, (2010), pgs. 1–79.
[8] G. Margulis, Discrete groups of motions of manifolds of nonpositive curvature. Proceedings of the Int. Cong. of Math., Vancouver, B.C. 2 (1975), pgs. 21–34.
[9] C. McMullen, Kleinian Groups. [http://www.math.harvard.edu/~ctm/programs/index.html]
[10] L. Paoluzzi, The notion of commensurability in group theory and geometry. RIMS Kkyroku 1836 (2013), pgs. 124–137.
[11] P. Scott and C.T. Wall, Topological methods in group theory. Lon. Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge 36 (1979), pgs. 137–203.
[12] J. Serre, Trees, Springer, New York, 1980.
[13] G. Walsh, Orbifolds and commensurability. Interactions Between Hyperbolic Geometry, Quantum Topology and Number Theory, Contemporary Mathematics 541 (2011), pgs. 221–231.
[14] K. Whyte, Amenability, Bilipschitz equivalence, and the Von Neumann conjecture. Duke Math. Jour. 99 (1999), pgs. 93–112.