CHARACTERIZATION OF THE GENERALIZED CHEBYSHEV POLYNOMIALS OF FIRST KIND

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ABSTRACT. Orthogonal polynomials have very useful properties in the solution of mathematical problems, so recent years have seen a great deal in the field of approximation theory using orthogonal polynomials. In this paper, we characterize the generalized Chebyshev orthogonal polynomials of the first kind $T_{n}^{M,N}(x)$, then we provide a closed form of the constructed polynomials in term of the Bernstein polynomials $B_{n}^{k}(x)$. We conclude the paper with some results on the integration of the product of the weighted generalized Chebyshev polynomials with the Bernstein polynomials.

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1. INTRODUCTION AND BACKGROUND

Approximation is essential to many numerical techniques, since it is possible to approximate arbitrary continuous function by a polynomial, at the same time polynomials can be represented in many different bases such as monomial and Bernstein basis.

1.1. Univariate Chebyshev-I polynomials. Chebyshev polynomial of degree $n \geq 0$ is defined as $T_{n}(x) = \cos(n \arccos x), \ x \in [-1,1]$. The univariate classical orthogonal polynomials are traditional defined on $[-1,1]$. However, it is more convenient to use $[0,1]$.

The Chebyshev polynomials of the first kind $T_{n}(x)$ are a special case of Jacobi polynomials $P_{n}^{(\alpha,\beta)}(x)$ and the interrelation is given by

\begin{equation}
T_{n}(x) = \left(n - \frac{1}{2}\right)^{-1} P_{n}^{(-\frac{1}{2},-\frac{1}{2})}(x).
\end{equation}

Since authors are not uniform on notations, and for the convenience we recall different representations of univariate Chebyshev-I polynomials. To do this we need to introduce the double factorial notation.

1.1.1. Double Factorials. The double factorial of an integer $m$ is a generalization of the usual factorial $m!$, the notation $m!!$ appears not to be widely known and defined as

\begin{equation}
m!! = (m)(m-2)(m-4)\ldots(4)(2) \quad \text{if } m \text{ is even}
\end{equation}

\begin{equation}
(2m-1)!! = (2m-1)(2m-3)(2m-5)\ldots(3)(1) \quad \text{if } m \text{ is odd},
\end{equation}

where $0!! = (-1)!! = 1$. From (1.2), we have the following definition.
Definition 1.1. For an integer \( n \), the double factorial is defined as

\[
\begin{align*}
n!! &= \begin{cases} 
2^\frac{n}{2} \frac{(\frac{n}{2})!}{2^{\frac{n}{2}}(\frac{n-1}{2})!} & \text{if } n \text{ is even} \\
\frac{n!}{2^{\frac{n}{2}}(\frac{n-1}{2})!} & \text{if } n \text{ is odd}
\end{cases}.
\end{align*}
\]

From the definition, we can derive the factorial of an integer minus half as

\[
(\begin{array}{c}
r - \frac{1}{2} \\
n - k
\end{array})! = \frac{r! (2r - 1)!! \sqrt{\pi}}{(2r)!!}.
\]

Using the double factorial notation, the univariate Chebyshev-I polynomials of degree \( n \) in \( x \) can be written as

\[
T_n(x) := \frac{(2n)!!}{(2n - 1)!!} \sum_{k=0}^{n} \binom{n}{n-k} (\binom{n-\frac{1}{2}}{k}) (\binom{n-\frac{1}{2}}{n-k}) (x - \frac{1}{2})^k (x + \frac{1}{2})^{n-k},
\]

which it can be transformed in terms of Bernstein basis on \( x \in [0,1] \) as

\[
T_n(2x - 1) := \frac{2^{2n}(n!)^2}{(2n)!} \sum_{k=0}^{n} (-1)^{n+1} \binom{n}{n-k} (\binom{n-\frac{1}{2}}{n-k}) B_n(x),
\]

where \( B_n(x) \) are the Bernstein polynomials of degree \( n \).

Definition 1.2. The \( n+1 \) Bernstein polynomials \( B_n(x) \) of degree \( n \), \( x \in [0,1] \), \( k = 0, 1, \ldots, n \), are defined by:

\[
B_n(x) = \begin{cases} 
\binom{n}{k} x^k (1-x)^{n-k} & k = 0, 1, \ldots, n \\
0 & \text{else}
\end{cases},
\]

where the binomial coefficients \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \), \( k = 0, 1, \ldots, n \).

Also, Chebyshev polynomials of the first kind \( T_n(x) \) may be represented in terms of Gauss hypergeometric series as follows \[7\]:

\[
T_n(x) = _2F_1\left(-\frac{n}{2}, \frac{1-x}{2}, 1 - x, \frac{1}{2}\right).
\]

The Chebyshev-I polynomials \( T_n(x) \) of degree \( n \) are orthogonal polynomials, except for a constant factor, with respect to the weight function

\[
W(x) = \frac{1}{\sqrt{1-x^2}}.
\]

Moreover, the Chebyshev-I polynomials satisfy the orthogonality relation \[7\]

\[
\int_0^1 (x - x^2)^{-\frac{1}{2}} T_n(x) T_m(x) dx = \begin{cases} 
0 & \text{if } m \neq n \\
\frac{\pi}{2} & \text{if } m = n = 0 \\
\frac{n}{\pi} & \text{if } m = n = 1, 2, \ldots
\end{cases}.
\]

2. Main Results

In this section we characterize the generalized Chebyshev-I polynomials, then we provide a closed form for generalized Chebyshev-I polynomials \( \mathcal{T}_{(M,N)}(x) \) as a linear combination of the Bernstein polynomials \( B^r_n(x) \). We conclude this section with the closed form of the integration of the weighted generalized Chebyshev-I with respect to the Bernstein polynomials.
2.1. The generalized Chebyshev-I polynomials Characterization. Using the fact \((1.1)\) and similar construction of the results in \([4, 6]\). For \(M, N \geq 0\), we define the generalized Chebyshev-I polynomials \(\mathcal{T}_{n}^{(M,N)}(x)\) as

\[
\mathcal{T}_{n}^{(M,N)}(x) = \frac{(2n)!}{2^{2n}(n!)^3} T_n(x) + MQ_n(x) + NR_n(x) + MNS_n(x), \quad n = 0, 1, 2, \ldots
\]

where \(n = 1, 2, 3, \ldots\)

\[
Q_n(x) = \frac{(2n)!}{2^{2n-1}(n!)^3} \left[ n^2 T_n(x) - \frac{1}{2}(x-1)D T_n(x) \right], \quad n = 1, 2, 3, \ldots
\]

\[
R_n(x) = \frac{(2n)!}{2^{2n-1}(n!)^3} \left[ n^2 T_n(x) - \frac{1}{2}(x+1)D T_n(x) \right], \quad n = 1, 2, 3, \ldots
\]

and

\[
S_n(x) = \frac{(2n)!}{2^{2n-2}(n!)^3(n-1)!} [n^2 T_n(x) - xDT_n(x)], \quad n = 1, 2, 3, \ldots
\]

By using \((x^2 - 1)D^2 T_n(x) = n^2 T_n(x) - xDT_n(x)\), we find that

\[
S_n(x) = \frac{4(2n-1)!!}{n!(n-1)!(2n)!!} (x^2 - 1)D^2 T_n(x), \quad n = 1, 2, 3, \ldots
\]

It is clear that \(Q_0(x) = R_0(x) = S_0(x) = 0\). Note that the generalized Chebyshev-I polynomials satisfy the symmetry relation \([5]\),

\[
\mathcal{T}_{n}^{(M,N)}(x) = (-1)^n \mathcal{T}_{n}^{(N,M)}(-x), \quad n = 0, 1, 2, \ldots
\]

which implies that for \(n = 0, 1, 2, \ldots\)

\[
Q_n(x) = (-1)^n R_n(-x) \quad \text{and} \quad S_n(x) = (-1)^n S_n(-x).
\]

From \([2.2]\) and \([2.3]\) if follows that for \(n = 1, 2, 3, \ldots\)

\[
Q_n(1) = \frac{2(2n-1)!!}{(n-1)!(2n)!!} T_n(1) \quad \text{and} \quad R_n(-1) = \frac{2(2n-1)!!}{(n-1)!(2n)!!} T_n(-1).
\]

Note that \([2.2]\) and \([2.3]\) and \([2.4]\) imply that for \(n = 1, 2, 3, \ldots\), we have

\[
Q_n(x) = \sum_{k=0}^{n} \frac{(2k)!}{2^{2k}(k!)^2} \frac{I_k}{2} T_k(x) \quad \text{with} \quad q_n = \frac{4}{(2n-3)(n-1)!},
\]

\[
R_n(x) = \sum_{k=0}^{n} \frac{(2k)!}{2^{2k}(k!)^2} \frac{I_k}{2} T_k(x) \quad \text{with} \quad r_n = \frac{4}{(2n-3)(n-1)!},
\]

and

\[
S_n(x) = \sum_{k=0}^{n} \frac{(2k)!}{2^{2k}(k!)^2} \frac{I_k}{2} T_k(x) \quad \text{with} \quad s_n = \frac{4}{(n-1)!(n-2)!}.
\]

Therefore, for \(M, N \geq 0\) the generalized Chebyshev-I polynomials \(\mathcal{T}_{n}^{(M,N)}(x)\) are orthogonal on the interval \([-1, 1]\) with respect to the weight function

\[
\frac{1}{\pi} (1-x)^{-\frac{1}{2}} (1+x)^{-\frac{1}{2}} + M\delta(x+1) + N\delta(x-1),
\]
and can be written as

\[ J_n^{(M,N)}(x) = \frac{(2n-1)!!}{(2n)!!} T_n(x) + \sum_{k=0}^{n} \frac{(2k)! \lambda_k}{2^{2k}(k!)^2} T_k(x) \]

where

\[ \lambda_k = Mq_k + Nr_k + MNs_k. \]

2.2. Generalized Chebyshev-I Polynomials using Bernstein basis. The Bernstein polynomials have been studied thoroughly and there exist many great enduring works on theses polynomials. There are known for their analytic and geometric properties \[1, 3\], where the basis are known to be optimally stable.

They are all non-negative, \( B_i^n(x) \geq 0, x \in [0,1] \), form a partition of unity (normalization) \( \sum_{k=0}^{n} B_i^n(x) = 1 \), satisfy symmetry relation \( B_i^n(x) = B_{n-i}^n(1-x) \), have a single unique maximum of \( (\frac{n}{n-i})^i n^{-n} (n-i)^{n-i} \) at \( x = \frac{1}{n} \), \( i = 0, \ldots, n \), and their roots are \( x = 0, 1 \) with multiplicities. The Bernstein polynomials of degree \( n \) can be defined by combining two Bernstein polynomials of degree \( n-1 \). That is, the \( k \)th \( n \)th-degree Bernstein polynomial defined by the following recurrence relation

\[ B_k^n(x) = (1-x)B_k^{n-1}(x) + xB_k^{n-1}(1-x), \quad k = 0, \ldots, n; n \geq 1 \]

where \( B_0^0(x) = 0 \) and \( B_k^0(x) = 0 \) for \( k < 0 \) or \( k > n \).

Moreover, the product of two Bernstein polynomials is also a Bernstein polynomial and given by

\[ B_i^n(x)B_j^m(x) = \frac{\binom{n}{i} \binom{m}{j}}{\binom{n+m}{i+j}} B_{i+j}^{n+m}(x). \]

In addition, it is possible to write Bernstein polynomial of degree \( r \) where \( r \leq n \) in terms of Bernstein polynomials of degree \( n \) using the following degree elevation \[2\]:

\[ B_k^r(x) = \sum_{i=k}^{n-r+k} \frac{\binom{r}{k-i} \binom{n-r}{i}}{\binom{n}{i}} B_i^n(x), \quad k = 0, 1, \ldots, r. \]

Now, to write a generalized Chebyshev-I polynomial \( J_r^{(M,N)}(x) \) of degree \( r \) as a linear combination of the Bernstein polynomial basis \( B_i^n(x) \), \( i = 0, 1, \ldots, r \) of degree \( r \) in explicit closed form, we begin with substituting (2.7) into (2.10) to get

\[ J_r^{(M,N)}(x) = \frac{(2r-1)!!}{(2r)!!} \sum_{i=0}^{r} \frac{\binom{r-i}{i} \binom{r}{i}}{\binom{r}{i}} B_{r-i}^i(x) + \sum_{k=0}^{r} \frac{(2k)! \lambda_k}{2^{2k}(k!)^2} \sum_{j=0}^{k} \frac{\binom{k}{j} \binom{k-j}{j}}{\binom{k}{j}} B_{k-j}^j(x) \]

\[ = \frac{(2r-1)!!}{(2r)!!} \sum_{i=0}^{r} (-1)^{r-i} \eta_{i,r} B_i^r(x) + \sum_{k=0}^{r} \frac{(2k)! \lambda_k}{2^{2k}(k!)^2} \sum_{j=0}^{k} (-1)^{k-j} \eta_{j,k} B_j^k(x). \]

where

\[ \eta_{i,r} = \frac{\binom{r-i}{i} \binom{r}{i}}{\binom{r}{i}}, \quad i = 0, 1, \ldots, r. \]

This shows that the generalized Chebyshev-I polynomial \( J_r^{(M,N)}(x) \) of degree \( r \) can be written in the Bernstein basis form.
Now, by expanding the right-hand side and using [14] with some simplifications, we have
\[
\eta_{i,r} = \frac{(r - \frac{1}{2})!(r - \frac{1}{2})!(i - \frac{1}{2})!(r - i - \frac{1}{2})!}{(2r - 1)!!(2r - 1)!!(2i - 1)!!(2r - 2i - 1)!!}
\]
\[
= \frac{2^r r!(i - 1)!!(2r - 2i - 1)!!}{2^r r!(2i - 1)!!(2r - i - 1)!!}
\]
Using the fact \( (2n)! = (2n - 1)!!2^n n! \) we get
\[
\eta_{i,r} = \frac{\binom{2r}{i}}{2^r \binom{n}{i}}.
\]
It is clear that \( \eta_{0,r} = \frac{1}{2r} \binom{2r}{r} \). With simple combinatorial identities simplifications we have
\[
\eta_{i-1,r} = \frac{(i - \frac{1}{2})}{(r - i + \frac{1}{2})} \eta_{i,r}.
\]
Thus we have the following theorem.

**Theorem 2.1.** For \( M, N \geq 0 \), the generalized Chebyshev-I polynomials \( \mathcal{F}^{(M,N)}_n(x) \) of degree \( n \) have the following Bernstein representation:
\[
\mathcal{F}^{(M,N)}_n(x) = \frac{(2n - 1)!!}{(2n)!!} \sum_{i=0}^{n} (-1)^{n-i} \eta_{i,n} B^n_i(x) + \sum_{k=0}^{n} \frac{(2k)! \lambda_k}{2^{2k} (k!)^2} \sum_{j=0}^{k} (-1)^{k-j} \eta_{j,k} B^j_k(x)
\]
where \( \lambda_k = M q_k + N r_k + M N s_k \) and \( \eta_{i,n} = \frac{\binom{2n}{i}}{2^n \binom{n}{i}} \), \( i = 0, 1, \ldots, n \) where \( \eta_{0,n} = \frac{1}{2\pi} \binom{2n}{n} \). Moreover, the coefficients \( \eta_{i,n} \) satisfy the recurrence relation
\[
\eta_{i,n} = \frac{(2n - 2i + 1)}{(2i - 1)} \eta_{i-1,n}, \quad i = 1, \ldots, n.
\]
It is worth mentioning that Bernstein polynomials can be differentiated and integrated easily as
\[
\frac{d}{dx} B^n_k(x) = n[B^n_{k-1}(x) - B^n_{k-1}(x), \quad n \geq 1, \quad \text{and} \quad \int_0^1 B^n_k(x) dx = \frac{1}{n+1}, \quad k = 0, 1, \ldots, n.
\]
Rababah [8] provided some results concerning integrals of univariate Chebyshev-I and Bernstein polynomials. In the following we consider integration of the product of the weighted generalized Chebyshev-I with Bernstein polynomials we get
\[
I = \int_0^1 x^{-\frac{d}{2}} (1 - x)^{-\frac{d}{2}} B^n_k(x) \mathcal{F}^{(M,N)}_i(x) dx.
\]
By using (2.13), the integral can be simplified to
\[
I = \int_0^1 (1 - x)^{-\frac{d}{2}} x^{-\frac{d}{2}} \frac{(n)}{r} (1 - x)^{n-r} \frac{(2i)!}{2^{2i}(i!)^2} \sum_{k=0}^{i} (-1)^{i-k} \binom{i-k}{k} B^i_k(x)
\]
\[
+ \sum_{d=0}^{i} \lambda_d \int_0^1 (1 - x)^{n-r-\frac{d}{2}} x^{-\frac{d}{2}} \frac{(n)}{r} \frac{(2d)!}{2^{2d}(d!)^2} \sum_{j=0}^{d} (-1)^{d-j} \binom{d-j}{j} 2^d B^d_j(x) dx,
\]
where \( \lambda_d \) defined in (2.11). By reordering the terms we get

\[
I = \binom{n}{r} \frac{(2i)!}{2^{2i}(i)!^2} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \binom{i-k}{i-k} \int_0^1 x^{r+k-\frac{3}{2}} (1-x)^{n+i-r-k-\frac{3}{2}} dx \\
+ \sum_{d=0}^{i} \binom{n}{r} \frac{(2d)!}{2^{2d}(d)!^2} \sum_{j=0}^{d} (-1)^{d-j} \binom{d}{j} \binom{d-j}{d-j} \int_0^1 x^{r+j-\frac{3}{2}} (1-x)^{n+d-r-j-\frac{3}{2}} dx.
\]

The integrals in the last equation are the Beta functions \( B(x_1, y_1) \) with \( x_1 = r+k+\frac{1}{2}, y_1 = n+i-r-k+\frac{1}{2}, x_2 = r+j+\frac{1}{2}, \) and \( y_2 = n+d-r-j+\frac{1}{2} \).

Hence, the following theorem provides a closed form of the integration of the weighted generalized Chebyshev-I with respect to the Bernstein polynomials.

**Theorem 2.2.** Let \( B_i^n(x) \) be the Bernstein polynomial of degree \( n \) and \( \mathcal{J}_i^{(M,N)}(x) \) be the generalized Chebyshev-I polynomial of degree \( i \), then for \( i, r = 0, 1, \ldots, n \) we have

\[
\int_0^1 (x - x^2)^{-\frac{1}{2}} B_i^n(x) \mathcal{J}_i^{(M,N)}(x) dx = \binom{n}{r} \frac{(2i)!}{2^{2i}(i)!^2} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \binom{i-k}{i-k} B(r+k+\frac{1}{2}, n+i-r-k+\frac{1}{2}) \\
+ \sum_{d=0}^{i} \binom{n}{r} \frac{(2d)!}{2^{2d}(d)!^2} \sum_{j=0}^{d} (-1)^{d-j} \binom{d}{j} \binom{d-j}{d-j} B(r+j+\frac{1}{2}, n+d-r-j+\frac{1}{2})
\]

where \( \lambda_k = Mq_k + Nr_k + MNs_k \) and \( B(x,y) \) is the Beta function.

3. Applications

The analytic and geometric properties of the Bernstein polynomials made them important for the development of Bézier curves and surfaces in Computer Aided Geometric Design. The Bernstein polynomials are actually the standard basis for the Bézier representations of curves and surfaces in CAGD. However, the Bernstein polynomials are not orthogonal and could not be used effectively in the least-squares approximation, thus the calculations performed in obtaining the least-square approximation polynomial of degree \( n \) do not reduce the calculations to obtain the least-squares approximation polynomial of degree \( n+1 \).

Since then many approximation methods have been introduced and analyzed. The method of least squares approximation accompanied by orthogonal polynomials is one of these methods.

3.1. Least-square approximation. A normal and useful practice in many applications is to construct a curve that is considered to be the best fit for a given function, in some sense. Here we describe continuous least-square approximation of a function \( f(x) \).

**Definition 3.1.** For a function \( f(x) \), continuous on \([0,1]\) the least square approximation requires finding a polynomial (Least-Squares Polynomial)

\[
p_n(x) = a_0 \varphi_0(x) + a_1 \varphi_1(x) + \cdots + a_n \varphi_n(x)
\]
that minimize the error

\[ E(a_0, a_1, \ldots, a_n) = \int_0^1 [f(x) - p_n(x)]^2 \, dx. \]

For minimization, the partial derivatives must satisfy

\[ \frac{\partial E}{\partial a_i} = 0, \, i = 0, \ldots, n. \]

These conditions give rise to a system of \((n + 1)\) normal equations in \((n + 1)\) unknowns: \(a_0, a_1, \ldots, a_n\). Solution of these equations will yield the unknowns of the least-squares polynomial \(p_n(x)\). It is important to choose a suitable basis, choosing \(\varphi_i(x) = x^i\), the coefficients of the normal equations give the Hilbert matrix

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\frac{1}{2} & 1 & \cdots & 1 \\
\frac{1}{3} & \frac{1}{2} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n} & \frac{1}{n-1} & \cdots & 1
\end{pmatrix}
\]

that has round-off error difficulties and notoriously ill-condition for even modest values of \(n\). However, using a special type of polynomials, called orthogonal polynomials, can make numerous numerical methods computationally efficient.

Choosing \(\{\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)\}\) to be orthogonal greatly simplifies the least-squares approximation problem. The matrix of the normal equations will be diagonal, which simplifies calculations and gives a compact form for \(a_i, i = 0, 1, \ldots, n\). That is not all, once \(p_n(x)\) is known, it is only necessary to compute \(a_{n+1}\) to get \(p_{n+1}(x)\), which turns out to be computationally efficient. See [9] for more details on the least squares approximations.

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