Non-decay of the energy for a system of semilinear wave equations

Yoshinori Nishii*

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Abstract: We consider the global Cauchy problem for a two-component system of cubic semilinear wave equations in two space dimensions. We give a criterion for large time non-decay of the energy for small amplitude solutions in terms of the radiation fields associated with the initial data.

Key Words: Semilinear wave equation; Large time behavior; Energy non-decay.

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1 Introduction

This paper is intended to be a continuation of [5]. We are interested in large time behavior of solutions to the Cauchy problem for

\[
\begin{align*}
\Box u_1 &= - (\partial_t u_2)^2 \partial_t u_1, \\
\Box u_2 &= - (\partial_t u_1)^2 \partial_t u_2,
\end{align*}
\tag{1.1}
\]

with the initial condition

\[
u_j(0, x) = \varepsilon f_j(x), \quad \partial_t u_j(0, x) = \varepsilon g_j(x), \quad x \in \mathbb{R}^2, \quad j = 1, 2,
\tag{1.2}
\]

where \(\Box = \partial_t^2 - \triangle = \partial_0^2 - (\partial_1^2 + \partial_2^2)\) with \(\partial_0 = \partial_t = \partial/\partial t, \quad \partial_1 = \partial/\partial x_1, \quad \partial_2 = \partial/\partial x_2\). We assume that \(f = (f_1, f_2), g = (g_1, g_2)\) are compactly supported \(C^\infty\) functions on \(\mathbb{R}^2\) and \(\varepsilon\) is a small positive parameter. The following result has been obtained in [5]:

**Proposition 1.1** ([5]). The Cauchy problem (1.1)-(1.2) admits a unique global \(C^\infty\) solution if \(\varepsilon\) is suitably small. Moreover, there exists \((f^+, g^+) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)\) such that

\[
\lim_{t \to +\infty} \|u(t) - u^+(t)\|_E = 0,
\tag{1.3}
\]

*Department of Mathematics, Graduate School of Science, Osaka University. 1-1 Machikaneyama-cho, Toyonaka, Osaka 560-0043, Japan. (E-mail: y-nishii@cr.math.sci.osaka-u.ac.jp)
where \( u^+ = (u_1^+, u_2^+) \) solves the free wave equation \( \Box u^+ = 0 \) with \( (u^+, \partial_t u^+) |_{t=0} = (f^+, g^+) \) and the energy norm \( \| \cdot \|_E \) is defined by

\[
\| \phi(t) \|_E^2 := \frac{1}{2} \int_{\mathbb{R}^2} |\partial_\phi(t, x)|^2 \, dx.
\]

As emphasized in [5], this is not a trivial result because cubic nonlinearities must be regarded as long-range perturbation for two-dimensional wave equations in general. Note that the global existence part of this assertion follows from the earlier result [3] directly and that the system (1.1) possesses two conservation laws

\[
\frac{d}{dt} \left( \| u_1(t) \|_E^2 + \| u_2(t) \|_E^2 \right) = -2 \int_{\mathbb{R}^2} (\partial_1 u_1(t, x))^2(\partial_1 u_2(t, x))^2 \, dx
\]

and

\[
\frac{d}{dt} \left( \| u_1(t) \|_E^2 - \| u_2(t) \|_E^2 \right) = 0. \tag{1.4}
\]

However, these are not enough to conclude that the solution \( u(t) \) is asymptotically free in the sense of (1.3). The novelty of the previous work [5] is to address this point, but we must say that the problem on (non-)triviality of \( u^+ \) is still obscure. What we are going to do in the present work is to investigate it in more detail. To make our concern clearer, let us focus on the special case \( f_1 = f_2, g_1 = g_2 \) first. By the uniqueness of the solution, the problem is reduced to the single equation \( \Box v = -(\partial_t v)^3 \), whence we can adapt the result of [2], [3], [6] to see that the total energy \( \| u(t) \|_E \) decays like \( O((\log t)^{-1/4+\delta}) \) as \( t \to +\infty \) for arbitrarily small \( \delta > 0 \). In other words, (1.3) holds with the trivial free solution \( u^+ \equiv 0 \) in this case. However, we remark that this is an exceptional case. Indeed, it follows from the conservation law (1.4) that at least one component \( u_1 \) or \( u_2 \) tends to a non-trivial free solution if \( \| u_1(0) \|_E \neq \| u_2(0) \|_E \). In other words, at least \( u_1^+ \) or \( u_2^+ \) given in Proposition (1.1) does not vanish for generic initial data. Moreover, it is far from obvious whether both \( u_1^+ \) and \( u_2^+ \) can behave like non-trivial free solutions as \( t \to +\infty \) in a certain case. This is why the problem on (non-)triviality of scattering state for (1.1) is of our interest.

Before going further, let us review the strategy of the proof of Proposition (1.1) briefly. The key in [5] is to focus on the function \( V = (V_1, V_2) \) defined by \( V_j(t; \sigma, \omega) = U_j(t, (t+\sigma)\omega) \), where \( U_j(t, x) = D(|x|^{1/2} u_j(t, x)), D = 2^{-1}(\partial_r - \partial_t), \partial_r = \omega_1 \partial_1 + \omega_2 \partial_2, r = |x| \) and \( \omega = (\omega_1, \omega_2) = x/|x| \). Roughly speaking, what have been seen in [5] is that the leading part of \( \partial u_j(t, x) \) as \( t \to +\infty \) can be given by \( |x|^{-1/2} \hat{\omega}(x) V_j(t; |x| - t, x/|x|) \), where \( \hat{\omega} = (\partial_0 u, \partial_1 u, \partial_2 u) \) and \( \hat{\omega}(x) = (\omega_0, \omega_1, \omega_2) = (-1, \omega) \). Moreover, the evolution of \( V = (V_1, V_2) \) can be characterized by the system

\[
\partial_t V_1 = \frac{-1}{2t} V_1 V_2^2 + K_1, \quad \partial_t V_2 = \frac{-1}{2t} V_1^2 V_2 + K_2, \tag{1.5}
\]

where \( K_1 \) and \( K_2 \) are harmless remainder terms (see Proposition (1.3) below). These allow us to reduce the problem to investigating the behavior of the solution \( V \) to (1.5) as \( t \to +\infty \). As for the asymptotic behavior of \( V \), we already know the following two propositions.
**Proposition 1.2** ([5]). The limit \( V_j^+(\sigma, \omega) = \lim_{t \to +\infty} V_j(t; \sigma, \omega) \) exists for each fixed \((\sigma, \omega) \in \mathbb{R} \times \mathbb{S}^1\) and \(j = 1, 2\), where \(\mathbb{S}^1\) denotes the unit circle in \(\mathbb{R}^2\). Moreover, we have

\[
\lim_{t \to \infty} \|\partial u_j(t, \cdot) - \hat{\omega}(\cdot) V_j^+(t, \cdot)\|_{L^2(\mathbb{R}^2)} = 0, \tag{1.6}
\]

where \( V_j^+(t, x) = |x|^{-1/2} V_j^+(|x| - t, x/|x|) \).

**Proposition 1.3** ([5]). Let \( V_j^+ \) be as above. There exists a function \( m : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R} \) such that the following holds for each \((\sigma, \omega) \in \mathbb{R} \times \mathbb{S}^1\):

- \( m(\sigma, \omega) > 0 \) implies \( V_1^+(\sigma, \omega) \neq 0 \) and \( V_2^+(\sigma, \omega) = 0 \);
- \( m(\sigma, \omega) < 0 \) implies \( V_1^+(\sigma, \omega) = 0 \) and \( V_2^+(\sigma, \omega) \neq 0 \);
- \( m(\sigma, \omega) = 0 \) implies \( V_1^+(\sigma, \omega) = V_2^+(\sigma, \omega) = 0 \).

We also have following expression of \( m(\sigma, \omega) \):

\[
m(\sigma, \omega) = (V_1(t_0, \sigma; \sigma, \omega))^2 - (V_2(t_0, \sigma; \sigma, \omega))^2 + 2 \int_{t_0, \sigma}^{\infty} \rho(t; \sigma, \omega) d\tau, \tag{1.7}
\]

where \( t_0, \sigma = \max\{2, -2\sigma\} \),

\[
\rho(t; \sigma, \omega) = V_1(t; \sigma, \omega) K_1(t; \sigma, \omega) - V_2(t; \sigma, \omega) K_2(t; \sigma, \omega),
\]

\[
K_j(t; \sigma, \omega) = H_j(t, (t + \sigma)\omega), \quad j = 1, 2,
\]

\[
H_1(t, x) = \frac{1}{2} \left[ |x|^{1/2}(\partial_t u_2)^2(\partial_t u_1) + \frac{1}{t} U^2_2 U_1 \right] - \frac{1}{8|x|^{3/2}} (4(x_1 \partial_x - x_2 \partial_1)^2 + 1) u_1,
\]

\[
H_2(t, x) = \frac{1}{2} \left[ |x|^{1/2}(\partial_t u_1)^2(\partial_t u_2) + \frac{1}{t} U^2_1 U_2 \right] - \frac{1}{8|x|^{3/2}} (4(x_1 \partial_x - x_2 \partial_1)^2 + 1) u_2.
\]

From Proposition 1.2, we see that the vanishing of \( V_j^+(\sigma, \omega) \) implies the energy decay of \( u_j(t) \) (or, equivalently, the triviality of \( u_j^+ \)). Note also that the function \( m(\sigma, \omega) \) appearing in Proposition 1.3 gives us much information on the vanishing of \( V_j^+(\sigma, \omega) \). Accordingly, it is natural to expect that the better understanding of \( m(\sigma, \omega) \) provides us with more precise information on the energy decay. The aim of this paper is to specify the leading term of \( m(\sigma, \omega) \) for sufficiently small \( \varepsilon \). As a consequence, we will find a criterion for non-decay of the energy for small amplitude solution to (1.1)–(1.2) in the terms of the radiation fields associated with the initial data. In particular, we will see that both \( u_1(t) \) and \( u_2(t) \) can behave like non-zero free solutions as \( t \to +\infty \) with a suitable choice of \((f, g)\).
2 Main results

Before stating the main result, we introduce several notations. For $\phi, \psi \in C^\infty_0$, we define the Friedlander radiation field $F_0[\phi, \psi]$ by

$$F_0[\phi, \psi](\sigma, \omega) := -\partial_\sigma \mathcal{R}_2[\phi](\sigma, \omega) + \mathcal{R}_2[\psi](\sigma, \omega), \quad (\sigma, \omega) \in \mathbb{R} \times S^1,$$

where

$$\mathcal{R}_2[\phi](s, \omega) := \frac{1}{2\sqrt{2\pi}} \int_\sigma^\infty \mathcal{R}[\phi](s, \omega) \frac{\sqrt{s-\sigma}}{ds},$$

For simplicity, we write $F_j(\sigma, \omega) = F_0[f_j, g_j](\sigma, \omega)$, $j = 1, 2$.

Our main theorem is as follows.

**Theorem 2.1.** Let $m = m(\sigma, \omega)$ be the function given in Proposition 1.3 and $0 < \mu < 1/10$. Then we have

$$m(\sigma, \omega) = \varepsilon^2 \left( (\partial_\sigma F_1(\sigma, \omega))^2 - (\partial_\sigma F_2(\sigma, \omega))^2 \right) + O(\varepsilon^{5/2 - 2\mu})$$

uniformly in $(\sigma, \omega) \in \mathbb{R} \times S^1$ as $\varepsilon \to +0$.

This asymptotic expression of $m(\sigma, \omega)$ yields the following criterion for non-decay of $\|u_1(t)\|_E$ and $\|u_2(t)\|_E$.

**Corollary 2.1.** Suppose that there exist $(\sigma^*, \omega^*), (\sigma_*, \omega_*) \in \mathbb{R} \times S^1$ satisfying

$$|\partial_\sigma F_1(\sigma^*, \omega^*)| > |\partial_\sigma F_2(\sigma^*, \omega^*)|$$

and

$$|\partial_\sigma F_1(\sigma_*, \omega_*)| < |\partial_\sigma F_2(\sigma_*, \omega_*)|,$$

respectively. Then we have $\lim_{t \to +\infty} \|u_1(t)\|_E > 0$ and $\lim_{t \to +\infty} \|u_2(t)\|_E > 0$ for suitably small $\varepsilon$.

**Remark 2.1.** From Corollary 2.1, we can construct the solution $u = (u_1, u_2)$ to (1.1)–(1.2) whose energy does not decay. Consequently, Corollary 2.1 shows that both $u_1(t)$ and $u_2(t)$ can behave like non-trivial free solutions as $t \to +\infty$ in a certain case.

**Remark 2.2.** In [4], analogous results have been obtained for the nonlinear Schrödinger system of the form

$$\begin{align*}
(i\partial_t + \frac{1}{2}\partial_x^2)u_1 &= -i|u_2|^2u_1, \\
(i\partial_t + \frac{1}{2}\partial_x^2)u_2 &= -i|u_1|^2u_2,
\end{align*}$$

$t > 0, \ x \in \mathbb{R}$.

Many parts of our proof below are similar to those of [4], but we need several modifications to fit for the wave equation case.

The rest part of this paper is organized as follows. In Section 3 we introduce preliminary estimates for the small amplitude solution $u$ to (1.1)–(1.2). We also recall basic estimates for solutions to the free wave equation and the radiation fields associated with its initial data. We prove Theorem 2.1 and Corollary 2.1 in Section 4.
3 Preliminaries

In this section, we collect various estimates which will be used in the next section. First of all, we introduce several notations. For $z \in \mathbb{R}$, we write $\langle z \rangle = \sqrt{1 + |z|^2}$. We define

$$S := t \partial_t + x_1 \partial_{x_1} + x_2 \partial_{x_2}, \quad L_1 := t \partial_{x_1} + x_1 \partial_t, \quad L_2 := t \partial_{x_2} + x_2 \partial_t, \quad \Omega := x_1 \partial_{x_2} - x_2 \partial_{x_1},$$

and we set $\Gamma = (\Gamma_j)_{0 \leq j \leq 6} = (S, L_1, L_2, \Omega, \partial_{x_1}, \partial_{x_2})$. For a multi-index $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_6) \in \mathbb{Z}_+^7$, we write $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_6$ and $\Gamma^\alpha = \Gamma_0^{\alpha_0} \Gamma_1^{\alpha_1} \cdots \Gamma_6^{\alpha_6}$, where $\mathbb{Z}_+ = \{n \in \mathbb{Z}; n \geq 0\}$. We also define $| \cdot |_s$ and $\| \cdot \|_s$ by

$$|\phi(t, x)|_s = \sum_{|\alpha| \leq s} |\Gamma^\alpha \phi(t, x)|, \quad \|\phi(t, \cdot )\|_s = \sum_{|\alpha| \leq s} \|\Gamma^\alpha \phi(t, \cdot )\|_{L^2(\mathbb{R}^2)}.$$

From the argument of Section 3 in [3], we already know that the following estimates are satisfied by the global small amplitude solution $u$ to (1.1)–(1.2).

**Lemma 3.1.** Let $k \geq 4$, $0 < \mu < 1/10$ and $0 < (8k + 7)\nu < \mu$. If $\varepsilon > 0$ is suitably small, then the solution $u$ to (1.1)–(1.2) satisfies

$$|u(t, x)|_{k+1} \leq C\varepsilon (t + |x|)^{-1/2 + \mu}, \quad (3.1)$$

$$|\partial u(t, x)| \leq C\varepsilon (t + |x|)^{-1/2} (t - |x|)^{1/2 - \mu}, \quad (3.2)$$

$$|\partial u(t, x)|_k \leq C\varepsilon (t + |x|)^{-1/2 + \nu} (t - |x|)^{1/2 - \nu}, \quad (3.3)$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^2$ and

$$\|\partial u(t)\|_k \leq C\varepsilon (1 + t)^{\mu - \nu}. \quad (3.4)$$

for $t \geq 0$, where $C$ is a positive constant independent of $\varepsilon$.

Next, we recall some estimates relevant to the free wave equation

$$\begin{cases}
\Box \phi = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^2, \\
\phi(0) = \phi_0, \quad \partial_t \phi(0) = \phi_1, & x \in \mathbb{R}^2.
\end{cases} \quad (3.5)$$

**Lemma 3.2.** For $\phi_0, \phi_1 \in C^\infty_0(\mathbb{R}^2)$ and $\alpha \in \mathbb{Z}_+^2$, there is a positive constant $C = C_\alpha(\phi_0, \phi_1)$ such that the smooth solution $\phi$ to (3.5) satisfies

$$|\partial^\alpha \phi(t, x)| \leq C(t + |x|)^{-1/2} |t - |x||^{-|\alpha|/2}, \quad (t, x) \in [0, \infty) \times \mathbb{R}^2. \quad (3.6)$$

**Lemma 3.3.** For $\phi_0, \phi_1 \in C^\infty_0(\mathbb{R}^2)$, there is a positive constant $C = C(\phi_0, \phi_1)$ such that the smooth solution $\phi$ to (3.5) satisfies

$$||x|^{1/2} \partial_x \phi(t, x) - \omega_a (\partial_x F_0[\phi_0, \phi_1])(|x| - t, \omega)\rangle \leq C(t + |x|)^{-1/2} (t - |x|)^{-1/2} \quad (3.7)$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^2 \setminus \{0\}$, $a = 0, 1, 2$, with the convention $\omega_0 = -1$, $\omega_1 = x_1/|x|$, $\omega_2 = x_2/|x|$.

**Lemma 3.4.** For $\phi_0, \phi_1 \in C^\infty_0(\mathbb{R}^2)$, there is a positive constant $C = C(\phi_0, \phi_1)$ such that

$$|\partial_x F_0[\phi_0, \phi_1](\sigma, \omega)| \leq C(\sigma)^{-3/2}, \quad (\sigma, \omega) \in \mathbb{R} \times S^1. \quad (3.8)$$

For the proof of Lemmas 3.2, 3.3 and 3.4 see Section 3 in [1].
4 Proof of the main results

In this section, we prove the Theorem 2.1 and Corollary 2.1. In what follows, we denote several positive constants by the same letter \( C \), which may be different from one line to another.

4.1 Proof of Theorem 2.1

First we note that \( t_0, \sigma \) in (1.7) can be replaced by \( t_1, \sigma := \max\{\varepsilon - 1, -2\sigma\} \) since we have

\[
\begin{align*}
(V_1(t_1, \sigma))^2 - V_2(t_1, \sigma)^2 & - (V_1(t_0, \sigma))^2 - V_2(t_0, \sigma)^2 \\
& = 2 \int_{t_0}^{t_1} (V_1(\tau)\partial_\tau V_1(\tau) - V_2(\tau)\partial_\tau V_2(\tau)) \, d\tau \\
& = 2 \int_{t_0}^{t_1} \rho(\tau; \sigma, \omega) \, d\tau.
\end{align*}
\]

(4.1)

We also recall the following estimate for \( \rho \) obtained in Section 4 in [5]:

\[
\left| \int_{t_1}^{\infty} \rho(\tau; \sigma, \omega) \, d\tau \right| \leq C\varepsilon^2 \langle \sigma \rangle^{-3/2} \int_{\varepsilon - 1}^{\infty} \tau^{2\mu - 3/2} \, d\tau \leq C\varepsilon^{5/2 - 2\mu}.
\]

(4.2)

From (1.7), (4.1) and (4.2), we get

\[
\left| m(\sigma, \omega) - ((V_1(t_1, \sigma; \sigma, \omega))^2 - (V_2(t_1, \sigma; \sigma, \omega))^2) \right| \leq C\varepsilon^{5/2 - 2\mu}
\]

for \( (\sigma, \omega) \in \mathbb{R} \times S^1 \). Thus, to prove Theorem 2.1, it suffices to show

\[
V_j(t_1, \sigma; \sigma, \omega) = \varepsilon \partial_\sigma F_j(\sigma, \omega) + O(\varepsilon^{2-\mu})
\]

(4.3)

as \( \varepsilon \to +0 \) uniformly in \( (\sigma, \omega) \in \mathbb{R} \times S^1 \) for \( j = 1, 2 \). The rest part of this subsection is devoted to the proof of (4.3). We divide the argument into the following two cases.

• **Case 1:** \( \sigma \leq -1/(2\varepsilon) \). If we assume \( |x| \leq t/2 \) and \( t \geq \varepsilon^{-1} \), we have \( \varepsilon^{-1} \leq t \leq (t + |x|) \leq C(t - |x|) \). It follows from (3.1) and (3.2) that

\[
\begin{align*}
|U(t, x)| \leq C|x|^{-1/2} |u(t, x)| + C|x|^{1/2} |\partial u(t, x)| \\
& \leq C\varepsilon|x|^{-1/2}(t + |x|)^{-1/2+\mu} + C\varepsilon|x|^{1/2}(t + |x|)^{-1/2}(t - |x|)^{\mu-1} \\
& \leq C\varepsilon^{2-\mu}
\end{align*}
\]

for \( |x| \leq t/2 \) and \( t \geq \varepsilon^{-1} \). Then we obtain

\[
|V(t, \sigma, \omega)| = |U(t, (t + \sigma)\omega)| \leq C\varepsilon^{2-\mu}
\]

(4.4)
for $t + \sigma \leq t/2$ and $t \geq \varepsilon^{-1}$. In the case $\varepsilon^{-1} \leq -2\sigma$, we have $t_{1,\sigma} + \sigma = t_{1,\sigma}/2$ and $t_{1,\sigma} \geq \varepsilon^{-1}$. Therefore, from (3.8), (4.1) and $|\sigma| \geq 1/(2\varepsilon)$, we get

$$
|V_j(t_{1,\sigma}; \sigma, \omega) - \varepsilon \partial_\sigma \mathcal{F}_j(\sigma, \omega)| \leq |V_j(t_{1,\sigma}; \sigma, \omega)| + \varepsilon |\partial_\sigma \mathcal{F}_j(\sigma, \omega)|

\leq C\varepsilon^{2-\mu} + C\varepsilon (\sigma)^{-3/2}

\leq C\varepsilon^{2-\mu}.

\textbf{Case 2:} $\sigma > -1/(2\varepsilon)$. For $j = 1, 2$, let $u_j^0 = u_j^0(t, x)$ be the solution to the free wave equation $\Box u_j^0 = 0$ with the initial data $u_j^0(0) = f_j$, $\partial_t u_j^0(0) = g_j$ and we put $u_j^1(t, x) := u_j(t, x) - \varepsilon u_j^0(t, x)$, so that $u_j^1$ solves

$$
\Box u_j^1(t, x) = F_j(\partial u), \quad (t, x) \in (0, \infty) \times \mathbb{R}^2,

u_j^1(0, x) = \partial u_j(0, x) = 0, \quad x \in \mathbb{R}^2.

$$

We also define $U^l(t, x) = D(|x|^{1/2}u^l(t, x))$ and $V^l(t, \sigma, \omega) := U^l(t, (t+\sigma)\omega)$, for $l = 0, 1$, respectively. It follows from (3.6) and (3.7) that

$$
|U_j^0(t, x) - \partial_\sigma \mathcal{F}_j(|x| - t, x/|x|)|

\leq \frac{1}{2} \sum_{\sigma = 0}^2 \left| |x|^{1/2} \partial_\sigma u_j^0(t, x) - \omega_\sigma \partial_\sigma \mathcal{F}_j(|x| - t, x) \right| + \frac{1}{4|x|^{1/2}} |u_j^0(t, x)|

\leq C(t + |x|)^{-1} (t - |x|)^{-1/2} + C|x|^{-1/2} (t + |x|)^{-1/2} (t - |x|)^{-1/2}

\leq C\varepsilon

$$

for $|x| \geq 1/(2\varepsilon)$. Hence we get

$$
|V_j^0(t; \sigma, \omega) - \partial_\sigma \mathcal{F}_j(\sigma, \omega)| \leq C\varepsilon \quad (4.5)

$$

for $t + \sigma \geq 1/(2\varepsilon)$. We next consider the estimate for $V^1$. Note that we have $(\Gamma^0 \phi, \partial_\tau \Gamma^0 \phi)|_{t = 0} \in (C_0^\infty(\mathbb{R}^2))^2$ and $\|\Gamma^0 \phi(0)\|_{L^\infty(\mathbb{R}^2)}$, $\|\partial_\tau \Gamma^0 \phi(0)\|_{L^\infty(\mathbb{R}^2)} = O(\varepsilon^3)$ for $\alpha \in \mathbb{N}_0^3$ if $\phi$ satisfies $\Box \phi = N(\partial \phi)$ with a cubic nonlinear term $N(\partial \phi)$ and $(\phi, \partial_\tau \phi)|_{t = 0} \in (C_0^\infty(\mathbb{R}^2))^2$. By using (3.3), (3.4) and the standard energy method for $\Gamma^0 u^1$ with $|\alpha| \leq 2$, we obtain

$$
\|\partial u^1(t)\|_2 \leq C\varepsilon^3 + C \int_0^t \|\partial u(\tau, \cdot)\|_1^2 \|\partial u(\tau)\|_2 d\tau

\leq C\varepsilon^3 + C\varepsilon^3 \int_0^{\varepsilon^{-1}} (1 + \tau)^{-1+\mu+\nu} d\tau

\leq C\varepsilon^3 + C\varepsilon^3 (1 + \varepsilon^{-1})^{\mu+\nu}

\leq C\varepsilon^{3-\mu-\nu}

$$

for $0 \leq t \leq \varepsilon^{-1}$. Then, by the Klainerman-Sobolev inequality, we get

$$
\langle t + |x| \rangle^{1/2} |\partial u^1(t, x)| \leq C\varepsilon^{3-\mu-\nu} \quad (4.6)

$$

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for $0 \leq t \leq \varepsilon^{-1}, x \in \mathbb{R}^2$. It follows from (3.1) and (3.6) that
\[
|x|^{-1/2} |u^1(t, x)| \leq |x|^{-1/2} \left( |u(t, x)| + \varepsilon |u^0(t, x)| \right) \\
\leq |x|^{-1/2} \left( C\varepsilon(t + |x|^{\mu - 1/2} + C\varepsilon(t + |x|)^{-1/2}(t - |x|)^{-1/2} \right) \\
\leq C\varepsilon^{2-\mu}
\] (4.7)

for $|x| \geq 1/(2\varepsilon)$. From (4.6) and (4.7), we get
\[
|U^1(t, x)| \leq C|x|^{1/2} |\partial u^1(t, x)| + C|x|^{-1/2} |u^1(t, x)| \\
\leq C\varepsilon^{3-\mu} + C\varepsilon^{2-\mu} \\
\leq C\varepsilon^{2-\mu}
\]

for $|x| \geq 1/(2\varepsilon), 0 \leq t \leq \varepsilon^{-1}$. Therefore, we obtain
\[
|V^1(t; \sigma, \omega)| \leq C\varepsilon^{2-\mu}
\] (4.8)

for $t + \sigma \geq 1/(2\varepsilon), 0 \leq t \leq \varepsilon^{-1}$. When $\varepsilon^{-1} > -2\sigma$, we have $t_{1,\sigma} = \varepsilon^{-1}$ and $t_{1,\sigma} + \sigma > t_{1,\sigma}/2 = 1/(2\varepsilon)$. Thus, by (4.5) and (4.8), we get
\[
|V_j^1(t_{1,\sigma}; \sigma, \omega) - \varepsilon \partial_{\sigma} F_j(\sigma, \omega)| \leq |V_j^1(t_{1,\sigma}; \sigma, \omega)| + \varepsilon |V_j^0(t_{1,\sigma}; \sigma, \omega) - \partial_{\sigma} F_j(\sigma, \omega)| \\
\leq C\varepsilon^{2-\mu}.
\]

Combining the two cases above, we arrive at the desired expression (1.3). This completes the proof of Theorem 2.1. \hfill \Box

4.2 Proof of Corollary 2.1

Let $E = \{(\sigma, \omega) \in \mathbb{R} \times \mathbb{S}^1; |\partial_{\sigma} F_1(\sigma, \omega)| > |\partial_{\sigma} F_2(\sigma, \omega)|\}$. By (2.2), $E$ is a non-empty open set. We can take a bounded open set $\mathcal{M}$ in $\mathbb{R}$ and an open set $\mathcal{N}$ in $\mathbb{S}^1$ such that $\sigma^* \in \mathcal{M}$, $\omega^* \in \mathcal{N}$ and $\mathcal{M} \times \mathcal{N} \subset E$, where $\mathcal{M} \times \mathcal{N}$ denotes the closure of $\mathcal{M} \times \mathcal{N}$ in $\mathbb{R} \times \mathbb{S}^1$. Now we put $F = \overline{\mathcal{M} \times \mathcal{N}}$ and
\[
C_1 = \min_{(\sigma, \omega) \in F} \left( (\partial_{\sigma} F_1(\sigma, \omega))^2 - (\partial_{\sigma} F_2(\sigma, \omega))^2 \right).
\]

Then we see that $F$ is compact, and thus $C_1 > 0$. By Theorem 2.1 we have
\[
m(\sigma, \omega) \geq C_1 \varepsilon^2 - C\varepsilon^{5/2 - 2\mu} > 0
\]

for $(\sigma, \omega) \in F$, if $\varepsilon > 0$ is small enough. This and Proposition 1.3 imply $V^+_1(\sigma, \omega) \neq 0$ for $(\sigma, \omega) \in F$, whence $\|V^+_1\|_{L^2(F)} > 0$. By virtue of (1.6), we can take $T_1 > 0$ such that
\[
\|\partial u_1(t, \cdot) - \hat{\omega}(\cdot) V^+_1(\cdot, \cdot)\|_{L^2(\mathbb{R})} < \frac{1}{\sqrt{2}} \|V^+_1\|_{L^2(F)}
\]
for $t > T_1$. Therefore we have
\[
\|u_1(t)\|_E \geq \left( \frac{1}{2} \int_{\mathbb{R}^2} |\dot{\omega}(x)V_1^{+,\#}(t,x)|^2 \, dx \right)^{1/2} - \left( \frac{1}{2} \int_{\mathbb{R}^2} |\partial u_1(t,x) - \dot{\omega}(x)V_1^{+,\#}(t,x)|^2 \, dx \right)^{1/2},
\]
\[
= \|V_1^{+,\#}(t,\cdot)\|_{L^2(\mathbb{R}^2)} - \frac{1}{\sqrt{2}} \|\partial u_1(t,\cdot) - \dot{\omega}(\cdot)V_1^{+,\#}(t,\cdot)\|_{L^2(\mathbb{R}^2)}
\]
\[
\geq \|V_1^+\|_{L^2(F)} - \frac{1}{2} \|V_1^+\|_{L^2(F)}
\]
\[
= \frac{1}{2} \|V_1^+\|_{L^2(F)}
\]
for $t > T_1$. Consequently,
\[
\lim_{t \to +\infty} \|u_1(t)\|_E \geq \frac{1}{2} \|V_1^+\|_{L^2(F)} > 0,
\]
as desired. Interchanging the role of $u_1$ and $u_2$, we also have $\lim_{t \to +\infty} \|u_2(t)\|_E > 0$. \qed

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