Stationary analysis for
coupled nonlinear Klein-Gordon equations

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Abstract. The initial and boundary value problem for a system of nonlinear Klein-Gordon equations with cubic nonlinearity is considered. In this setting, two waves possibly co-exist, and they are nonlinearly interacting with each other. Depending on the choice of parameters and also on the choice of initial data, the resonance and the competition take place. Consequently bounded stationary solutions conditionally appear. In this paper, by analyzing the stationary problem of coupled Klein-Gordon equations, existence condition for bounded stationary solutions is presented.

1. Introduction

We consider one-dimensional wave equations with cubic nonlinearity. Since the model equations correspond exactly to the $\phi^4$-theory in the quantum field theory (for a textbook, see [1]), it is associated essentially with the Higgs mechanism. For some nonlinear Klein-Gordon equations such as Sine-Gordon equations, the existence of breather solution has been known [2, 3, 4, 5], and that solution are expected to behave as a kind of building unit of our universe. Indeed, the breather solution is a kind oscillation, and behaves as time-periodic solution, so that it forms a kind of closed curve in the phase space.

In addition to the standard setting of breather solution, the periodic boundary condition is also imposed to the spatial direction $x$. Therefore, in this paper, a periodic solution for both time and space, which is likely to be called the space-time breather solution, is studied in nonlinear Klein-Gordon equation with cubic nonlinearity. For the preceding result showing the existence of space-time periodic breather solution in a single Klein-Gordon equations, see [6].

In this paper, in terms of clarifying the stationary solutions of the initial and boundary value problem for coupled nonlinear Klein-Gordon equations with cubic nonlinearity, several simplified ordinary differential equations are investigated to clarify a static property of dynamical systems. As a result, conditions for the existence of bounded solutions are obtained. It is remarkable here that the existence of bounded solution is associated with the necessary condition for the existence of co-existing breather solutions for the coupled Klein-Gordon equations.

2. Mathematical model

Let $x \in [0, L]$ be a finite domain of space. The evolution problem is studied for the positive time ($t \geq 0$). Let $\partial/\partial t$ and $\partial/\partial x$ be denoted by $\partial_t$ and $\partial_x$ respectively. We begin with coupled
nonlinear Klein-Gordon equations with the cubic nonlinearity,

\[ \begin{align*}
\partial_t^2 u + \alpha_1 \partial_x^2 u - m_1 u + k_1 u^3 + k'_1 u^2 w &= 0, \\
\partial_t^2 w + \alpha_2 \partial_x^2 w - m_2 w + k_2 w^3 + k'_2 uw^2 &= 0,
\end{align*} \]

where \( \alpha_1, \alpha_2, m_1, m_2, k_1, k_2, k'_1 \) and \( k'_2 \) are real constants. In addition to the third order self-interactions \( u^3 \) and \( w^3 \), mutual interactions \( u^2 w \) and \( uw^2 \) of the same order are introduced. The initial functions \( f(x) \) and \( g(x) \) are given as \( L^2 \)-functions, and the periodic boundary condition is imposed for \( x \)-direction. The breather solution has been shown to exist for Eq. (2), if it only consists of the single equation (i.e., \( k'_1 = \alpha_2 = m_2 = k_2 = k'_2 = 0 \) \[6\].

In terms of clarifying fundamental units of coexisting states of breather solution, let us limit ourselves to the cases when \( \alpha_1 = \alpha_2 = 0 \). The master equations are reduced to a system of ordinary differential equations

\[ \begin{align*}
\partial_t u(x, 0) &= f(x), \quad u(0, t) = u(L, t), \\
\partial_t w(x, 0) &= g(x), \quad w(0, t) = w(L, t),
\end{align*} \]

\[ \text{(QT)} \]

For this ordinary differential equations, the initial values are given by real numbers; we take \( v(x, 0) = f(x) = u_0 \), \( w(x, 0) = g(x) = w_0 \), and \( \partial_t u(x, 0) = \partial_t w(x, 0) = 0 \). In the following, we make systematics by choosing the values of \( u_0 \) and \( w_0 \). Here \( u_{tt} \) means the second order derivative of \( u \) with respect to a variable \( t \), and the same notation is true for \( w_{tt} \). Note that Eq. (QT') is essentially the same as the master equation of stationary problem

\[ \begin{align*}
\alpha_1 u_{xx} - m_1 u + k_1 u^3 + k'_1 u^2 w &= 0, \\
\alpha_2 w_{xx} - m_2 w + k_2 w^3 + k'_2 uw^2 &= 0
\end{align*} \]

with nonzero \( \alpha_1 \) and \( \alpha_2 \). In this paper the initial value problem (QT') is investigated. For a further simplification, the coefficients are fixed to be symmetric: \( m_1 = m_2 = 1, \quad k_1 = k_2 = 1, \quad k'_1 = k'_2 = 1 \). Consequently, the wave propagation in the interacting two media is symmetric for the exchange between \( u \) and \( w \). This symmetric case will provide a good starting point of analyzing complex interaction of coupled nonlinear Klein-Gordon equations including asymmetrically interacting cases.

3.1. Four special solutions as essential pieces of the dynamical system

3.1. Cases with \( u_0 = 0 \) or \( w_0 = 0 \)

Much attention is paid to clarify the sensitivity to the initial condition. For the time evolution of the solution of Eq. (QT'), we see the different time evolution depending on the choice of initial value \( (u_0, w_0) \). Let us begin with the cases with \( w_0 = 0 \). The right-hand side of the second equation of Eq. (QT') is represented by

\[ w - w^3 - uw^2 = w(1 - w^2 - uw), \]

\[ (1) \]
so that $w = 0$ is the fixed point of the second equation. That is, $w(t) = 0$, $t > 0$ is a stationary solution, and it is realized if we take $w_0 = 0$. On the other hand, by substituting $w(t) = 0$, the right-hand side of the first equation of Eq. (QT') can be represented by

$$u - u^3.$$  \hfill (2)

In this situation Eq. (QT') is reduced to

$$u_{tt} = u - u^3, \quad w = 0. \quad (QT'_1)$$

The solutions of (QT'$_1$) with the initial condition $u(0) = u_0$ and $u_t(0) = 0$ are necessarily located on the horizontal axis $(u, 0)$ in the $u-w$ plane, and all we have to do is to solve the single equation $u_{tt} = u - u^3$. For the behavior of the solution to this single equation, see [6].

Equation (QT') is symmetric for the exchange between $u$ and $w$. In case with $u_0 = 0$, Eq. (QT') is also reduced to the single equation.

$$u_{tt} = u - u^3, \quad w = 0. \quad (QT'_1)$$

In the same manner, the solutions of (QT'$_2$) with the initial condition $w(0) = w_0$ and $w_t(0) = 0$ are necessarily located on the vertical axis $(0, w)$ in the $u-w$ plane. The solutions of (QT'$_1$) and (QT'$_2$) play a role of basic units in the phase space analysis.

3.2. Cases with $u_0 = w_0$ or $u_0 = -w_0$

Let us move on to the cases with identical initial value satisfying $u_0 = w_0$. Due to the symmetry of (QT'), the functions $u(t)$, $w(t)$ behave identically for any time by taking such an identical initial value. That is, the solution is necessarily represented by $(u, u)$, and the master equation of Eq. (QT') is reduced to an essentially-single equation.

$$u_{tt} = u - 2u^3, \quad w = u. \quad (QT'_3)$$

The solutions of (QT'$_3$) with the initial condition $u(0) = u_0$ and $u_t(0) = 0$ are necessarily located on the line $u = w$ in the $u-w$ plane. Similarity of the master equation to (QT'$_1$) and (QT'$_2$) are noticed in this case.

In the same manner, let us consider the case with $u_0 = -w_0$. The solution is represented by $(u, -u)$ in the $u-w$ plane, and the master equation of Eq. (QT') is also reduced to an essentially-single equation.

$$u_{tt} = u, \quad w = -u. \quad (QT'_4)$$

The solutions of (QT'$_4$) with the initial condition $u(0) = u_0$ and $u_t(0) = 0$ are necessarily located on the line $u = -w$ in the $u-w$ plane. It should be noted here that, unlike (QT'$_1$), (QT'$_2$) and (QT'$_3$) cases, $u$ diverges in infinite time in this case (cf. grow-up of solution).

3.3. Cross section of dynamical system

All the four model cases (QT'$_1$) to (QT'$_4$) are shown by four lines in Fig. 1. Solutions of (QT') with general initial data evolve between the four lines in Fig. 1. Note that in order to complete the dynamical system representation, we have to add two additional variables $\partial_t u$ and $\partial_t w$ in Fig. 1 so that two-dimensional cross section of the full four-dimensional dynamical system is shown in Fig. 1. As shown in Fig. 1, the dynamical system, which consists of the phase space and the solution orbits, becomes symmetric with respect to the line $u = -w$. This fact arises from the symmetric setting of coefficients in this paper. The purpose of this paper is to draw detail structures inside Fig. 1.
Figure 1. Dynamical system of \((QT')\) are shown in \(u - w\) plane. Four model cases are shown by arrows (shown as arrows), where red thick arrows mean oscillatory motions, and blue dashed arrows show the divergent motions. The black points are the fixed points. Solution orbits of \((QT'_1)\), \((QT'_2)\), and \((QT'_3)\) show the oscillatory motion around the fixed points. Solution orbits of \((QT'_4)\) diverge monotonically to \((+\infty, -\infty)\) or \((-\infty, +\infty)\).

4. Result
4.1. Divergent and bounded solution
The time evolution of the solution to Eq. \((QT')\) has a wide variety of cases other than the four fundamental cases. Based on the four model cases, solution orbits starting from general initial values are studied. Much attention is paid to find the initial condition to hold bounded solutions. In the present paper, the initial values of first derivative is always given by \(u_t(0) = w_t(0) = 0\).

The solution \(u(t)\) is called divergent if there exists a certain positive real number \(t_M < \infty\) such that
\[
|u(t_M)| > M
\]
for an arbitrary real number \(M > 0\). On the other hand, the solution \(u(t)\) is called bounded if there exists a certain positive real number \(M < \infty\) such that
\[
\max_{t \in [0, \infty)} |u(t)| \leq M.
\]

Depending on the initial data, divergent/bounded solutions appear. In this paper, these solutions are examined with a sufficiently long time interval \(t \in [0, 1024]\).

For example, as shown in the left panel of Fig. 2 the solution diverges when the initial value is chosen as \((u_0, w_0) = (0.125, 0.750)\). Similarly, as shown in the right panel of Fig. 2 the solution diverges when the initial value is chosen as \((u_0, w_0) = (0.025, 0.750)\). Here it is clear that, depending on the initial settings, some solutions possibly diverge. By comparing the left and right panels of Fig. 2 small difference results in a different kind of divergence. The coupled system inherently holds the sensitivity to the initial data; one goes to the upper left side, and the other goes to the lower right side.

Depending on the choice of initial data, some solutions of Eq. \((QT')\) remain in a finite domain. Those solutions are called the bounded solutions in this paper. For example, as shown in the left panel of Fig. 3 the solution is bounded when the initial value is chosen as \((u_0, w_0) = (0.800, 0.900)\). Similarly, as shown in the right panel of Fig. 3 the solution is also bounded when the initial value is chosen as \((u_0, w_0) = (0.100, 0.900)\). Although the examination
4.2. Condition for bounded solution
4.2.1. Stationary solution We have seen four trajectories with and without divergence; for the initial values \((u_0, w_0, u_t(0), w_t(0)) = (0.025, 0.750, 0.000, 0.000), (0.100, 0.900, 0.000, 0.000), (0.125, 0.750, 0.000, 0.000), (0.800, 0.900, 0.000, 0.000)\), the second and fourth cases result in bounded solutions, and the first and third cases result in different divergence. This fact implies

has been done for a finite duration time, no divergent modes appear in these cases, and modes like time-periodic oscillation appear instead. It is a supportive evidence for the existence of bounded solutions.

Figure 2. Trajectory of the divergent solutions \((u, w)\), \((0 \leq t \leq 1024)\). Left and right panels correspond to the cases with the initial value \((u_0, w_0) = (0.125, 0.750)\) and that with the initial value \((u_0, w_0) = (0.025, 0.750)\), respectively.

Figure 3. Trajectory of bounded solutions \((u, w)\) \((0 \leq t \leq 1024)\). Left and the right panels correspond to the cases with the initial value \((u_0, w_0) = (0.800, 0.900)\) and that with the initial value \((u_0, w_0) = (0.100, 0.900)\), respectively.
Figure 4. The red lines show the fixed points arising from Eq. (3) when $w$ is assumed to be a constant for solving Eq. (3). The blue lines show the fixed points arising from Eq. (4) when $u$ is assumed to be a constant for solving Eq. (4). The two crossing points mean the stationary solution of the coupled equation.

4.2.2. Potential defined on the $u − w$ plane  Based on the stationary solutions, we define the potential structure on $u − w$ plane. The potential function $P(u, w)$ is defined by

$$P(u, w) = \int_u^{u^*} f(u',w)du' + \int_w^{w^*} g(u,w')dw',$$  \hspace{1cm} (5)$$  

where the ending points $u^*$ and $w^*$ are given by Eqs. (3) and (4), respectively. A method to define a potential, which is equivalent to the way of defining the energy, is similar to but not
Figure 5. The potential function $P(u, w)$ is plotted on the $u - w$ plane. The contour plot shows the height of the potential. The potential shown in the left panel gives a magnified view of a certain part of the potential shown in the right panel.

Figure 6. The potential $P(u, w)$ has two local minimums in the vicinity of $(0.7, 0.7)$ and $(-0.7, -0.7)$. If the initial values $(u_0, w_0)$ near $(0.7, 0.7)$ and $(-0.7, -0.7)$ satisfies $0 < P(u_0, w_0) < 0.1$, the solution $(u(t), w(t))$ is bounded for a long time $0 < t < 1024$.

...exactly the same as the method to define the boundary point of absorbing set [6]. Actually we take the ending points as

$$
\begin{align*}
&\begin{cases}
  u^* = -(1/2)w + \sqrt{(w/2)^2 + 1}, \\
  w^* = -(1/2)u + \sqrt{(u/2)^2 + 1}
\end{cases} \\
&\text{for the cases with } w > -u,
&\begin{cases}
  u^* = -(1/2)w - \sqrt{(w/2)^2 + 1}, \\
  w^* = -(1/2)u - \sqrt{(u/2)^2 + 1}
\end{cases}
\end{align*}
$$
Figure 7. The upper and lower limits of bounded solutions, where the terminology upper and lower is used for the vertical axis. The blue squares show the lower limit of \((u_0, w_0)\), and the black circles show the upper limit of \((u_0, w_0)\). The contour plot shows the magnitude of the potential \(P(u, w)\), \((0 < P < 0.1)\), where white region indicates that the magnitude is greater than 0.1. Relation between initial value \((u_0, w_0)\), \((w_0 > u_0 > 0)\) and potential \(P\) are shown. (see also the left panel of Fig. 5)

for the cases with \(w < -u\). As a result, \(P(u, w)\) becomes

\[
P(u, w) = \frac{1}{2} \left[(u^*)^2 + (w^*)^2 - (u^2 + w^2)\right] - \frac{1}{4} \left[(u^*)^4 + (w^*)^4 - (u^4 + w^4)\right] - \frac{1}{3} \left[u \{(w^*)^3 - w^3\} + w \{(u^*)^3 - u^3\}\right].
\]

As shown in the left panel of Fig. 5 for \(u, w > 0\), the potential \(P(u, w)\) has a minimum around \((u, w) = (0.7, 0.7)\). Solution orbits around or close to a minimum point are expected to be the bounded solutions. As shown in the right panel of Fig. 5, if we enlarge the phase space, the potential \(P(u, w)\) has two local minimums in the vicinity of \((0.7, 0.7)\) and \((-0.7, -0.7)\) in a symmetric fashion. In other words, the two bottoms exist on the \(u - w\) plane. Solution orbits around or close to a bottom are expected to be the bounded solutions.

As a general trend, for the directions \((u, w) \to (+\infty, +\infty)\) and \((u, w) \to (-\infty, -\infty)\), the potential \(P(u, w)\) holds monotonic increase. On the other hand, for \((u, w) \to (+\infty, -\infty)\) or \((u, w) \to (-\infty, +\infty)\), the potential \(P(u, w)\) holds monotonic decrease. The orbits of divergent solutions are regarded as located close to \((u, w) \to (+\infty, -\infty)\) or \((u, w) \to (-\infty, +\infty)\) on \(u - w\) plane. Therefore the potential structure shown in Fig. 7 tells us that solution orbits starting sufficiently far from \((u, w) \to (+\infty, -\infty)\) or \((u, w) \to (-\infty, +\infty)\) are expected to be bounded solutions. This issue is numerically examined in the following.

4.3. Numerical results
The validity of the proposed potential leading to the existence condition of bounded solutions is confirmed by numerical calculations. Note that the numerical calculations are carried out partially based on the high-precision numerical code [7]. In order to understand the time evolution near \((0.7, 0.7)\) in Fig. 5 we focus on the non-trivial solutions with initial values
$w_0 > u_0$. Since Eq. (QT') is symmetric with respect to the exchange between $u$ and $w$, the dynamics is symmetric with respect to the line $w = u$. Therefore it is sufficient to consider the area satisfying $w_0 > u_0 > 0$. Indeed, at a sufficiently large time $T > 0$, we examine whether $(u(T), w(T))$ is still close to $(0.7, 0.7)$ or not. Consequently, for certain $u_0$ and $w_0$, the upper and lower limits of bounded solution is determined by the bisection method.

Setting $T = 1024$, as for the upper and lower limits of bounded solutions, 40 points are determined to be the boundary points. The results are shown in Fig. 7. The area bounded by blue square points, black circle points and black line ($w = u$) shows the area for the initial values giving rise to the bounded solutions. The potential $P(u, w)$ is also shown by the contour plot. The comparison between the calculated limit and the potential shows that, in the present parameter setting, the initial values with

$$0 < P(u_0, w_0) < 0.1$$

(7)

give rise to the bounded solution, where white-colored parts indicate a potential values greater than 0.1. Consequently, it is clearly seen that the potential $P(u, w)$ well reproduces the boundary condition of the bounded solutions. We notice here that the boundary of bounded solutions around $(u, w) = (0, 1, 0)$ is slightly different between the potential boundary and the calculated boundary (Fig. 7). Since the potential is a simplified concept only taking into account the dynamics of $(u, w)$ with a fixed $u$ or $v$, such a small discrepancy appears to be reasonable.

5. Conclusion

A coupled ordinary differential equation (QT') is studied, which corresponds to the stationary problem of the initial and boundary value problems for coupled nonlinear Klein-Gordon equations (QT). By visualizing the time evolution of numerical solutions (Figs. 2 and Fig. 3), it has been confirmed that there are divergent and bounded solutions with the sensitive dependence on the initial data. By introducing the potential function $P(u, w)$ on the $u - w$ plane, the existence conditions for bounded solutions is found out to be sufficiently reproduced by the potential function (Fig. 7). The potential function is utilizable in the future studies for predicting the possible initial data for the bounded solutions. More importantly, it has been confirmed numerically and also by the potential structure that there are two local minimum points for $P(u, w)$ and different two bounded solutions exist (Fig. 6). These bounded solutions correspond to the co-existence of two waves $u(t)$ and $w(t)$, which always hold the same sign.

As can be seen from the preceding result [6], the breather solution is the solution in which two different bounded solutions coexist spatially. Therefore, based on this finding, it is expected to clarify the co-existence of breather solutions in coupled equations.

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