Wilson Coefficients in the Operator Product Expansion of Scalar Currents at Finite Temperature

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In this paper, we have investigated operator product expansion for thermal correlation function of the two scalar currents. Due to breakdown of Lorentz invariance at finite temperature, more operators of the same dimension appear in the operator product expansion than those at zero temperature. We calculated Wilson coefficients in the short distance expansion and obtain operator product expansion for thermal correlation function in terms of quark condensate $\langle \bar{\psi} \psi \rangle$, gluon condensate $\langle G_{\mu\nu}^a G^{a\mu\nu} \rangle$, quark energy density $\langle u \Theta^f u \rangle$ and gluon energy density $\langle u \Theta^g u \rangle$.

I. INTRODUCTION

The Shifman, Vainshtein and Zakharov (SVZ) sum rules, proposed about three decades ago [1], is one of the powerful method for investigating the properties of hadrons. This method has been extensively used as an efficient tool to study properties of resonances, decay form factors and so on [2]. In this approach, hadrons are represented by their interpolating quark currents taken at large virtualities and the correlation function of these currents is investigated in the framework of operator product expansion (OPE).

The SVZ sum rules method is extended to the finite temperature in the paper [3]. In extending these sum rules to finite temperature, two sources of complications arise. One is the interaction of the current with the particles of the medium. The other complication is the breakdown of Lorentz invariance by the choice of the reference frame [4, 5]. Taking into account both complications, OPE of vector currents at finite temperature first investigated in [6] and application of these results to the temperature dependence of the $\rho$-meson parameters are done in [7, 8]. Also, modifications of meson parameters due to nuclear medium are widely discussed in the literature [9, 10].

In this paper we studied OPE for thermal correlation function, necessary for the phenomenological investigation of scalar meson parameters. We calculated Wilson coefficients in the OPE of scalar currents at finite temperature.

II. THERMAL CORRELATION FUNCTION IN SHORT DISTANCE EXPANSION

The starting point for SVZ sum rules is the OPE [1] and it gives a general form of the considered quantities in terms of operators. We begin by considering the thermal correlation function,

$$ T(q) = i \int d^4 x e^{i q \cdot x} \langle T(J(x)J(0)) \rangle, $$

where $J(x) = :\bar{\psi}(x)\psi(x):$ is the scalar current and $T$ indicates the time ordered product. The thermal average of an operator is defined as follows [11]

$$ \langle A \rangle = \frac{\text{tr} e^{-\beta H} A}{\text{tr} e^{-\beta H}}, $$

where $H$ is the QCD Hamiltonian and $\beta = 1/T$ stands for the inverse of the temperature. Traces are carried out over any complete set of states. Using Wick’s theorem and making some transformations, we get

$$ T(J(x)J(0)) = \text{tr} S(x, 0) S(0, x) - \frac{1}{2\pi^2 x^4} [m_\pi^2 : \bar{\psi} \psi : + 2(\bar{\phi})_{ab} x^\mu : \bar{\psi}_a i D_\mu \psi_b : ] , $$

where $a$ and $b$ are spinor indices, and $D_\mu$ is covariant derivative. The fundamental assumption of Wilson expansion is that the product of operators at different points can be expanded as the sum of local operators with momentum dependent coefficients in the form:

$$ T(q) = \sum C_n(q^2) \langle O_n \rangle, $$
where \( C_n(q^2) \) are called Wilson coefficients and \( O_n \) are a set of local operators. In this expansion, the operators are ordered according to their dimension \( d \). The lowest dimension operator with \( d = 0 \) is the unit operator associated with the perturbative contribution. In the vacuum sum rules low dimension operators composed of quark and gluon fields are quark condensate \( \langle \overline{\psi}\psi \rangle \) and gluon condensate \( \langle G_{\mu\nu}^a G^{a\mu\nu} \rangle \). At finite temperature Lorentz invariance is broken by the choice of a preferred frame of reference and new operators appear in the Wilson expansion. To restore Lorentz invariance in thermal field theory, four-vector velocity of the medium \( u^\mu \) is introduced. Using four-vector velocity and quark/gluon fields, we can construct a new set of low dimension operators \( \langle u\Theta^f u \rangle \) and \( \langle u\Theta^g u \rangle \) with dimension \( d = 4 \). So, we can write thermal correlation function in terms of operators up to dimension four:

\[
T(q) = C_1 I + C_2 \langle \overline{\psi} \psi \rangle + C_3 \langle G_{\mu\nu}^a G^{a\mu\nu} \rangle + C_4 \langle u\Theta^f u \rangle + C_5 \langle u\Theta^g u \rangle.
\]

In our calculations, we used the following expansion of quark fields \( \psi(x) \)

\[
\psi(x) = \psi(0) + x^\mu D_\mu \psi(0) + ..., \tag{6}
\]

which takes place in Fock-Schwinger gauge \( x^\mu A_\alpha^\mu(x) = 0 \). In order to calculate Wilson coefficients in \([4]\), we use massless quark field propagator in the background gauge field \([6]\)

\[
S(x, 0) = \frac{1}{4\pi^2} \left[ -\frac{2}{x^4} \frac{ig}{4x^2} \gamma^\mu \gamma^\nu G_{\mu\nu} + \frac{B}{48} (a \cdot x + b \cdot y) \right], \tag{7}
\]

where \( G_{\mu\nu} = \frac{1}{2} \lambda^a G_{\mu\nu}^a \) is the gluon strength tensor, coefficients \( a \) and \( b \) have the following forms

\[
a = 1 - \frac{4}{x^2} (u \cdot x)^2 - 2 \ln(-4x^2 \mu^2), \tag{8}
\]

\[
b = 8 (u \cdot x) \ln(-4x^2 \mu^2). \tag{9}
\]

In our calculations we get expressions of kind \( T_{\alpha\beta\lambda\sigma} = \langle tr G_{\alpha\beta} G_{\lambda\sigma} \rangle \), where trace is carry out over color matrices. \( T_{\alpha\beta\lambda\sigma} \) is four rank tensor and must be expressed in terms of the metric tensor \( g_{\alpha\beta} \) and four-vector velocities \( u_\alpha \). The Lorentz covariance at finite temperature allows us to write general structure of this tensor in the following form:

\[
T_{\alpha\beta\lambda\sigma} = B_1 g_{\alpha\beta} g_{\lambda\sigma} + B_2 g_{\alpha\lambda} g_{\beta\sigma} + B_3 g_{\alpha\sigma} g_{\beta\lambda} + B_4 g_{\alpha\beta} u_\lambda u_\sigma + B_5 g_{\alpha\lambda} u_\beta u_\sigma + B_6 g_{\alpha\sigma} u_\beta u_\lambda + B_7 g_{\beta\lambda} u_\alpha u_\sigma + B_8 g_{\beta\sigma} u_\alpha u_\lambda + B_9 g_{\lambda\sigma} u_\alpha u_\beta + B_{10} u_\alpha u_\beta u_\lambda u_\sigma \tag{10}
\]

where \( B_i \) are unknown scalar coefficients. Note that \( T_{\alpha\beta\lambda\sigma} \) is antisymmetric under the interchange of indices \( \alpha \) and \( \beta \), as well as indices \( \lambda \) and \( \sigma \): \( T_{\alpha\beta\lambda\sigma} = -T_{\beta\alpha\lambda\sigma} \) and \( T_{\alpha\beta\lambda\sigma} = -T_{\alpha\beta\lambda\sigma} \). Using these properties we obtain that \( B_1 = B_4 = B_9 = B_{10} = 0 \), \( B_6 = B_7 = -B_5 = -B_8 \) and \( B_2 = -B_3 \). Therefore \( T_{\alpha\beta\lambda\sigma} = \langle tr G_{\alpha\beta} G_{\lambda\sigma} \rangle > 0 \) is expressed in terms of two scalar functions of \( B_2 = A \) and \( B_8 = B \). Contracting indices on both sides and using gluonic part of energy momentum tensor \( \Theta_{\mu\nu} \), these coefficients can be expressed with gluon condensate and gluon energy densities as follows

\[
A = \frac{1}{24} \langle G_{\mu\nu}^a G^{a\mu\nu} \rangle + \frac{1}{6} \langle u\Theta_{\mu\nu}^g u^\nu \rangle, \tag{11}
\]

\[
B = \frac{1}{3} \langle u\Theta_{\mu\nu}^g u^\nu \rangle. \tag{12}
\]

Using these expressions for coefficients and making some transformations, we obtain the gluonic part of correlation function

\[
\langle tr S(x, 0) S(0, x) \rangle = -\frac{1}{16\pi^4} \left[ \frac{16}{3x^6} - \frac{aB}{3x^2} + \frac{g^2}{6x^2} \langle u\Theta^g u \rangle + \frac{g^2}{8x^2} \langle G_{\mu\nu}^a G^{a\mu\nu} \rangle + \frac{(u \cdot x) B}{3x^4} + \frac{2g^2}{3x^4} (u \cdot x)^2 \langle u\Theta^g u \rangle \right]. \tag{13}
\]
Let us consider second term in eq. (3) for zero chemical potential case. Expanding the quark bilinear matrix in terms of complete set of Dirac matrices and taking into account parity conservation, the most general finite temperature decomposition of two operators in spinor space becomes

\[
\langle \bar{\psi}_a \psi_b \rangle = D_1 \delta_{ab} + D_2 (\not{\partial})_{ab},
\]

where \( D_1 \) and \( D_2 \) are scalar coefficients. By contracting indices on both sides and using fermionic part of energy momentum tensor \( \Theta_{\mu\nu} \), we get

\[
E_1 = \frac{m}{16} \langle \bar{\psi} \psi \rangle - \frac{1}{12} \langle u^\mu \Theta_{\mu\nu} u^\nu \rangle,
\]

\[
E_2 = \frac{1}{3} \langle u^\mu \Theta_{\mu\nu} u^\nu \rangle,
\]

Substituting these equalities in eq. (3), the contributions of operators up to dimension four to correlation function in the coordinate space can be written as

\[
\langle T(J(x) J(0)) \rangle = - \frac{1}{2\pi^2} \int \frac{3}{2} mx^2 \langle \bar{\psi} \psi \rangle - \frac{2}{3} x^2 \langle u \Theta^I u \rangle + \frac{8}{3} \langle u \cdot x \rangle^2 \langle u \Theta^I u \rangle \bigg[ - \frac{1}{16\pi} \int \frac{16}{r^5} \frac{aB}{3x^2} - \frac{(u \cdot x)B}{3x^4} + \frac{g^2}{8x^2} G_{\mu\nu}^a G^{a\mu\nu} \bigg]
\]

\[
- \frac{g^2}{6x^2} \langle u \Theta^I u \rangle + \frac{2g^2}{3x^4} \langle u \cdot x \rangle^2 \langle u \Theta^I u \rangle \bigg].
\]

Applying the Fourier transformation to eq. (18), correlation function may be written in the momentum space. As can be seen in Fourier transformation, we meet different kind of integrals such as

\[
I = \int d^4x \frac{\ln(-x^2)}{x^2} e^{iq \cdot x}.
\]

Up to now, we evaluated expressions in the Minkowski space. After that, in order to calculate integrals we proceed to the Euclidean space. Using the well known identity

\[
\ln x^2 = - \lim_{\lambda \to 0} \frac{d}{d\lambda} \frac{1}{(x^2)^{\lambda}},
\]

the integral in eq. (19) can be transformed to the following form

\[
I = -i \lim_{\lambda \to 0} \frac{d}{d\lambda} \int d^4x \frac{e^{-iQ \cdot x}}{(x^2)^{1+\lambda}},
\]

where \( Q \) is Euclidean momentum and \( Q^2 = -q^2 \). Using the standard Fourier transformation and expansion formula \( \Gamma(1+\varepsilon) = 1 - \gamma \varepsilon + O(\varepsilon^2) \), we get

\[
I = -i\pi^2 \left( \frac{8\gamma}{Q^2} + \frac{4}{Q^2} \ln \frac{Q^2}{4} \right),
\]

where \( \gamma \) is Euler constant. After calculating the other integrals thermal correlation function in the momentum space in short distance expansion can be written as

\[
T(Q) = \frac{1}{8\pi^2} Q^2 \left( \gamma - \ln \frac{4\pi}{Q^2} \right) + \frac{3}{Q^2} m \langle \bar{\psi} \psi \rangle + \frac{4}{3} \left( \frac{4(u \cdot Q)^2}{Q^4} + \frac{1}{Q^2} \right) \langle u \Theta^I u \rangle
\]

\[
+ \frac{g^2}{32\pi^2 Q^2} \langle G_{\mu\nu}^a G^{a\mu\nu} \rangle - \frac{g^2}{12\pi^2} \left[ (5 + 8\gamma) \frac{1}{Q^2} - (1 + 8\gamma) \left( \frac{4(u \cdot Q)^2}{Q^4} + \frac{2}{Q^2} \right) - 4 \ln \left( \frac{Q^2}{16\mu^2} \right) \left( \frac{4(u \cdot Q)^2}{Q^4} + \frac{1}{Q^2} \right) + 16 \frac{(u \cdot Q)^2}{Q^4} \right] \langle u \Theta^I u \rangle.
\]
which reproduces well-known zero temperature result in the $T \to 0$ limit \cite{13}. At the end, due to breakdown of Lorentz invariance at finite temperature, two new operators appear in the Wilson expansion, in addition to the two old (Lorentz invariant) ones, already existing in the vacuum sum rules. Therefore OPE for thermal correlator is expressed with four temperature dependent quantities, quark condensate $\langle \bar{\psi}\psi \rangle$, gluon condensate $\langle G^a_{\mu\nu}G^{a\mu\nu} \rangle$, quark energy density $\langle u\Theta^f u \rangle$ and gluon energy density $\langle u\Theta^g u \rangle$. The thermal average of quark condensate $\langle \bar{\psi}\psi \rangle$ is known from chiral perturbation theory \cite{14,15}

$$\langle \bar{\psi}\psi \rangle = \langle 0|\bar{\psi}\psi|0 \rangle \left[ 1 - \frac{n_f^2 - 1}{n_f} \frac{T^2}{12F^2} + O(T^4) \right] \quad (24)$$

where $n_f$ is number of quark flavors and $F = 0.088 GeV$.

The relationship between the trace of energy momentum tensor and the gluon condensate has been studied at finite temperature in paper \cite{16}

$$\frac{g^2}{4\pi^2} \left( \langle G^a_{\mu\nu}G^{a\mu\nu} \rangle - \langle 0|G^a_{\mu\nu}G^{a\mu\nu}|0 \rangle \right) = -\frac{8}{9} \left( \Theta^a_{\mu\nu} + \sum_f m_f \left( \langle \bar{\psi}\psi \rangle - \langle 0|\bar{\psi}\psi|0 \rangle \right) \right) \quad (25)$$

For two massless quarks in the low temperature chiral perturbation expansion the trace of the total energy momentum tensor $\Theta = n_f \Theta^f + \Theta^g$ has following form \cite{17}

$$\langle \Theta^a_{\mu\nu} \rangle = \frac{\pi^2}{270} \frac{T^8}{F^4} \ln \frac{\Lambda_p}{T} + O \left( T^{10} \right) \quad (26)$$

where the pion decay constant has the value of $F_\pi = 0.093 GeV$ and the logarithmic scale factor is $\Lambda_p = 0.275 GeV$. The obtained results allow us to investigate the temperature dependence of the $\sigma$ and $a_0$ mesons parameters. The investigations of hadronic parameters at finite temperature are very important for the interpretation of heavy ion collision results and for understanding the phenomenological and theoretical aspects of thermal QCD.

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