TRIPLE CROSSING NUMBERS OF GRAPHS

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Abstract. We introduce the triple crossing number, a variation of the crossing number of a graph, which is the minimal number of crossing points in all drawings of the graph with only triple crossings. It is defined to be zero for planar graphs, and to be infinite for non-planar graphs which do not admit a drawing with only triple crossings. In this paper, we determine the triple crossing numbers for all complete multipartite graphs which include all complete graphs.

1. Introduction

Let \( G \) be a graph. A drawing of \( G \) means a representation of the graph in the Euclidean plane or the 2-sphere, where vertices are points and edges are simple arcs joining their end-vertices. Since each edge is simple, no edge admits self crossings. Furthermore, we assume that the interiors of edges do not contain vertices, and that two edges do not intersect if they have a common vertex, and that two edges without common end-vertex intersect at most once, and if so, then they intersect transversally. These requirements are essential in this paper. A drawing is called a regular drawing (resp. semi-regular drawing) if it has only double (resp. triple) crossing points. From the requirements, we know that a graph has at least 6 vertices if it admits a semi-regular drawing with at least one triple crossing point.

The crossing number \( \text{cr}(G) \) of \( G \) is defined to be the minimal number of crossing points over all regular drawings of \( G \). In particular, \( \text{cr}(G) = 0 \) if \( G \) is planar. In this paper, we introduce a new variation of the crossing number. The triple crossing number \( \text{tcr}(G) \) is zero if \( G \) is planar, and \( \infty \) if \( G \) does not admit a semi-regular drawing. Otherwise, \( \text{tcr}(G) \) is defined to be the minimal number of triple crossing points over all semi-regular drawings of \( G \). In particular, \( \text{tcr}(G) = 0 \) if and only if \( G \) is planar.

The triple crossing number can be regarded as a specialization of the degenerate crossing number introduced by Pach and Tóth [4]. In addition, for example, the Petersen graph is known to have the crossing number two (and thus non-planar), and hence has the triple crossing number one from Figure [4]. In general, we have the inequality \( \text{cr}(G) \leq 3 \text{tcr}(G) \) for these two notions, since we obtain a regular drawing from a semi-regular drawing by perturbing each triple crossing point into three double crossing points.

In this paper, we determine the triple crossing numbers for all complete multipartite graphs. A complete multipartite graph is a graph whose vertex set can be partitioned into at least two, mutually disjoint non-empty sets, called the partite sets, so that two vertices \( u \) and \( v \) are adjacent if and only if \( u \) and \( v \) belong to different sets of the partition. If the partite sets are of sizes \( n_1, \ldots, n_t \) (\( n_i \geq 1 \)), then the graph is denoted by \( K_{n_1, \ldots, n_t} \). We always assume that \( n_i \geq n_j \) if \( i < j \). In particular, if all \( n_i = 1 \), then the graph \( K_{1, \ldots, 1} \) is the complete graph \( K_t \) with \( t \) vertices.

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Here is how the paper is organized. After we describe basic lemmas, used in the paper repeatedly, in Section 2 we show that the triple crossing number of a complete $t$-partite graph is $\infty$ if $t \geq 5$ in Section 3. In the successive sections, we work on the cases when $t \leq 4$. Here we should mention that the hardest part is the case where $t = 2$, in particular, long, but elementary, geometric arguments are needed to show that $K_{5,4}$, $K_{4,1}$, $K_{5,3}$ and $K_{n,3}$ with $n \geq 7$ do not admit a semi-regular drawing. This is treated in Sections 4, 5 and 6. After concluding the case where $t = 2$ in Section 7, the cases where $t = 4$ and $t = 3$ are established in Sections 8 and 9, respectively. Section 10 contains some remarks on our requirements for drawings and a generalization of triple crossing number.

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2. Basic lemmas

Basic terms of graph theory can be found in textbooks such as [1, 6].

**Lemma 2.1.** The complete bipartite graph $K_{3,3}$ and the complete graph $K_5$ with five vertices are non-planar. Also, a graph is non-planar if it contains $K_{3,3}$ or $K_5$ as a subgraph.

**Lemma 2.2.** Let $G$ be a plane graph with $p \geq 3$ vertices and $q$ edges. Then $G$ has a vertex of degree less than 6 and the inequality $q \leq 3p - 6$ holds. Furthermore, $q = 3p - 6$ if and only if each region of $G$ is 3-sided.

For the proofs of these lemmas, see [1].

**Lemma 2.3.** Let $G$ be a graph with $p \geq 3$ vertices and $q$ edges. If $G$ admits a semi-regular drawing, then $q \leq 3p - 6$. Thus, if $q > 3p - 6$, then $tcr(G) = \infty$.

*Proof.* Let $D$ be a semi-regular drawing of $G$, and let $k$ be the number of triple crossing points in $D$. If a new vertex is added to each triple crossing point, then we obtain a (simple) plane graph $G'$. Since $G'$ has $p + k$ vertices and $q + 3k$ edges, we have $q + 3k \leq 3(p + k) - 6$ by Lemma 2.2, from which we have the conclusion. \(\square\)

**Lemma 2.4.** Let $G$ be a connected plane graph with $p \geq 3$ vertices, $q$ edges and $r$ faces. Let $d = 3p - q - 6$.

1. If $d = 1$, then one face is 4-sided, and the others are 3-sided.
2. If $d = 2$, then either
   a. one face is 5-sided, and the others are 3-sided; or
   b. two faces are 4-sided, and the others are 3-sided.
3. If $d = 3$, then either
(a) one face is 6-sided, and the others are 3-sided; or
(b) one face is 5-sided, another face is 4-sided, and the others are 3-sided; or
(c) three faces are 4-sided, and the others are 3-sided.

We remark that $d \geq 0$, and $d = 0$ if and only if all faces are 3-sided by Lemma 2.2.

*Proof.* By Euler’s formula ([1]), $p - q + r = 2$. Let $r_i$ denote the number of $i$-sided faces of $G$. Then

\[3r_3 + 4r_4 + 5r_5 + 6r_6 + \sum_{i \geq 7} ir_i = 2q.\]  

Thus $7r - 4r_3 - 3r_4 - 2r_5 - r_6 \leq 2q$. Since $q = p - r - 2$ and $d = 2p - r - 4$, we have

\[4r - d \leq 4r_3 + 3r_4 + 2r_5 + r_6 \leq 4(r_3 + r_4 + r_5 + r_6) \leq 4r.\]

In particular, the difference between the second and third terms, which is $r_4 + 2r_5 + 3r_6$, is at most $d$. We remark that $d \equiv r \pmod{2}$.

When $d \in \{1, 2, 3\}$, $4r$ is the only multiple of four within the interval $[4r - d, 4r]$. Since $4(r_3 + r_4 + r_5 + r_6)$ is a multiple of four, we see $4(r_3 + r_4 + r_5 + r_6) = 4r$. Furthermore, if $r_5 = r_6 = 0$, then (2.1) reduces to $3r_3 + 4r_4 = 2q$. Combining this with $r_3 + r_4 = r$ gives $r_4 = 2q - 3r = d$. Thus if $r_4 + 2r_5 + 3r_6 \leq d = 1$, we have $r_5 = r_6 = 0$. Then $r_4 = 1$, and thus $r_3 = r - 1$.

(2) If $r_4 + 2r_5 + 3r_6 \leq 2$, we have $r_6 = 0$ and $r_5 \leq 1$. If $r_5 = 1$, then $r_4 = 0$, giving $r_3 = r - 1$. This is the conclusion (a). If $r_5 = 0$, then $r_4 = 2$, and thus $r_3 = r - 2$. This is the conclusion (b).

(3) If $r_4 + 2r_5 + 3r_6 \leq 3$, we have $r_6 \leq 1$. If $r_6 = 1$, then $r_4 = r_5 = 0$, and thus $r_3 = r - 1$. This is the conclusion (a).

Suppose $r_6 = 0$. Since $r_4 + 2r_5 \leq 3$, we see $r_5 \leq 1$.

If $r_5 = 1$, then $r_4 \leq 1$. From (2.1), $3r_3 + 4r_4 = 2q - 5$. Combining this with $r_3 + r_4 = r - 1$ gives $r_4 = 2q - 3r - 2$. Since $r \equiv 1 \pmod{2}$, we have $r_4 = 1$. Hence $r_3 = r_5 = 1$ and $r_4 = r - 2$. This is the conclusion (b).

Finally suppose $r_5 = 0$. Then $r_4 = 3$, and thus $r_3 = r - 3$. This is the conclusion (c).

\[\square\]

3. Complete $t$-partite graphs ($t \geq 5$)

**Theorem 3.1.** If $t \geq 5$, then no complete $t$-partite graph $G$ admits a semi-regular drawing. Thus, $\text{tc}(G) = \infty$.

*Proof.* Assume for a contradiction that $G$ admits a semi-regular drawing $D$. Let $t \geq 7$. If a new vertex is added to each triple crossing point, then we have a plane graph $G'$. However, the original vertices have degree at least $t - 1 (\geq 6)$, and the new vertices have degree 6. This contradicts Lemma 2.2.

Let $G = K_{n_1, n_2, \ldots, n_6}$. Then $G$ has $p = \sum n_i$ vertices and $q = \sum_{i<j} n_i n_j$ edges. Then

\[q - 3p + 6 = (n_1 + n_4 - 3)(n_2 + n_3 - 3) + n_1 n_4 + n_2 n_3 + (n_5 + n_6)(n_1 + n_2 + n_3 + n_4 - 3) + n_5 n_6 - 3 \geq (2n_4 - 3)^2 + 2n_4^2 \geq 3.\]

This contradicts Lemma 2.3.
This contradicts Lemma 2.3 again. □

Let $\text{Corollary 3.2.}$

If $n \leq 4$, then $K_n$ is planar, and thus $\text{tcr}(K_n) = 0$ by definition. The rest follows from Theorem 3.1. □

4. $K_{5,4}$

Throughout this section, we will assume that $G = K_{5,4}$ admits a semi-regular drawing. We will show that this is impossible.

Let $V_1 = \{x_1, x_2, x_3, x_4, x_5\}$ and $V_2 = \{A, B, C, D\}$ be the partite sets of $G$. For convenience, we refer to vertices of $V_1$ (resp. $V_2$) as black (resp. white) vertices. We denote the edge $Ax_i$ by $a_i$ for $1 \leq i \leq 5$. These are called $A$-lines. Similarly, define $b_i, c_i, d_i$, and call them $B$-, $C$-, $D$-lines, respectively. In particular, each black vertex is of degree four, and is incident with four distinct classes of lines.

We fix a semi-regular drawing of $G$ hereafter, which is denoted by the same symbol $G$. Notice that any line in $G$ connects a white vertex and a black vertex, and there may be triple crossing points on it. From our requirements for drawings, each triple crossing point of $G$ arises from three distinct classes of lines. This fact will be referred to as property (+) throughout the paper. Property (+) is very useful and powerful. For example, if an $A$-line and a $B$-line intersect at a triple crossing point, then we can conclude that the remaining line through the triple crossing point is either a $C$- or $D$-line.

Let $k$ be the number of triple crossing points. Add a new vertex to each triple crossing point. Then we have a plane graph $G'$ with $9 + k$ vertices and $20 + 3k$ edges. Since $3(9 + k) - (20 + 3k) - 6 = 1$, the faces of the plane graph $G'$ are all 3-sided, except a single 4-sided face by Lemma 2.4. For the semi-regular drawing $G$, a face means that of $G'$, although it is an abuse of words. A 3-sided face is also called a triangle.

Take a look around vertex $A$. There are five faces of $G$, since $A$ is not a cut vertex. We may assume that all five faces around vertex $A$ are triangles without loss of generality, since the 4-sided face is incident with at most two white vertices. There are two types of triangle around vertex $A$ as shown in Figure 2. A type I triangle is incident with two triple crossing points, and a type II triangle is incident with a black vertex and a triple crossing point.

Notice that type II triangles appear in pairs. More precisely, this means that every type II triangle at $A$ shares an $A$-line fully with another type II triangle. Such a pair of type II triangle is referred to as an adjoint pair of type II triangles. See Figure 2. Hence the number of type II triangles at vertex $A$ is either 0, 2 or 4. We will eliminate these three possibilities.

Lemma 4.1. The number of type II triangles at vertex $A$ is not four.

Proof. Suppose that there are four type II triangles at $A$. Then we can assume that the local configuration at $A$ is as shown in Figure 2(1) by renaming $B, C, D,$ if necessary. (Recall that each black vertex is incident with four distinct classes of
Figure 2. Two types of triangles at A and an adjoint pair of type II triangles

Figure 3. Four type II triangles at A

lines.) Then, by property (\(*\)), the horizontal line is a B- or D-line, and the right lower line is a C- or D-line. See Figure 2(2), where the symbol B/D, for example, indicates the class of the horizontal line. It does not mean that vertex B or D locates the left side of the horizontal line. Thus there are four cases as shown in Figure 4. Here, the class of the right upper line is determined by property (\(*\)) and the fact that each black vertex is incident with four distinct classes of lines. For Figure 4(4), the right upper line may be a C-line. But then, it can be reduced to (3) by renaming B and D and symmetry.

Claim 4.2. Figure 4(1) is impossible.

Proof. Consider the B-line b_2. Let f_1 and f_2 be the faces incident with b_2 and vertex x_2. See Figure 5.

If f_1 is 4-sided, then f_2 is 3-sided, because there is only one 4-sided face. Since two B-lines cannot intersect at a triple crossing point by property (\(*\)), the horizontal B-line and b_2 meet at vertex B as shown in the second of Figure 5 in order to make f_2 3-sided. However, then f_1 cannot be 4-sided. Thus we can conclude that f_1 is 3-sided. Then vertex B is located around f_1 as shown in the first of Figure 6 by the same reason as above. Also, then f_2 cannot be 4-sided, and thus 3-sided. Hence the B-line, intersecting two A-lines a_3 and a_4, turns out to be b_1. That is, it goes to black vertex x_3 as shown in the third of Figure 6. (At this point, b_1 may contain triple crossing points on it after crossing a_4.)

Consider the face f_3, which is adjacent to f_1 along the B-line b_5. If f_3 is 3-sided, then the B-line b, which must be b_3 or b_4, intersects the A-line a_1. Then b can reach neither x_3 nor x_4, because b cannot cross b_1 or meet a_1 twice. Therefore, we found that f_3 is the only 4-sided face. Thus f_3 is 3-sided. We see that the line going through the upper triple crossing point of f_4 is either an A- or B-line by property (\(*\)). Since f_4 is also 3-sided, the B-line b' goes to x_3, or crosses the A-line a_3. In any case, f_6 cannot be 3-sided as described in Figure 7. This is a contradiction. □
Figure 4. Four cases where there are four type II triangles

Claim 4.3. Figure 4(2) is impossible.

Proof. First, since the horizontal $D$-line intersects $a_3$, $a_4$, $c_5$, it is either $d_1$ or $d_2$ from our requirements for drawings. By symmetry, we can assume that $f_1$ is 3-sided as in Figure 8. Then vertex $D$ is located around $f_1$. Then the horizontal $D$-line turns out to be $d_1$. However, neither $f_2$ nor $f_3$ is 3-sided, which contradicts that there is only one 4-sided face. □

Claim 4.4. Figure 4(3) is impossible.

Proof. In Figure 9 either $f_3$ or $f_4$ is 4-sided. (For, if $f_3$ is not 4-sided, then it is 3-sided. Then vertex $D$ is located there, which implies that $f_4$ is not 3-sided.) Thus both $f_1$ and $f_2$ are 3-sided. Then the $B$-line $b_1$ is determined. Since $f_5$ is 3-sided, the $B$-line $b$, which is $b_3$ or $b_4$, crosses the $A$-line $a_1$. Then $b$ can reach neither $x_3$ nor $x_4$, a contradiction. □

Claim 4.5. Figure 4(4) is impossible.
Figure 6. $f_1$ is 3-sided

Figure 7. $b'$ goes to $x_3$ or crosses $a_3$

Figure 8. $f_1$ is 3-sided

Proof. In Figure 10, if $f_1$ is not 3-sided, then $f_2$ is 3-sided, and then vertex $D$ is located there. But then, $f_1$ cannot be 4-sided. Hence $f_1$ is 3-sided. Similarly, so is $f_2$. Then the $D$-line $d_1$ is determined.

If $f_3$ is 3-sided, then the $D$-line $d$, which is $d_3$ or $d_4$, crosses the $A$-line $a_1$. Then $d$ cannot reach any black vertex as before. Hence $f_3$ is 4-sided. As in the proof of Claim 4.2, examining $f_4$, $f_5$, $f_6$ leads to a contradiction. □
This completes the proof of Lemma 4.1. □

**Lemma 4.6.** The number of type II triangles at vertex A is not two.

**Proof.** Suppose that there are two type II triangles at A. Then we can assume that the local configuration at A is as shown in Figure 11(1), up to renaming. By property (*), the left upper line is a B- or D-line. Similarly, the right upper line is a C- or D-line. See Figure 11(2). Then there are three cases, up to symmetry and relabeling of vertices, as shown in Figure 12, where the class of the horizontal line is determined by property (*).

**Claim 4.7.** Figure 12(1) is impossible.

**Proof.** First, assume that $f_1$ is 4-sided in Figure 13. Then the others are all 3-sided. Thus $f_2$ and $f_3$, and then $f_4$, $f_5$ are determined as in Figure 13. (If an A-line goes through the left triple crossing point of $f_2$, then the face sharing a D-line with $f_2$ cannot be 3-sided. Similarly for $f_3$.)

Consider the B-line $b$. It goes to $x_2$ or crosses the A-line $a_2$. Suppose that the former happens. Then $f_6, \ldots, f_9$ are determined as in Figure 14. Moreover, the D-line $d_5$ is also determined.

Then the C-line $c$ cannot go to $x_5$, since it crosses $d_5$. Hence it crosses the A-line $a_5$. This forces the D-line $d$ to cross the same $a_5$. Then it cannot reach any black
Figure 11. Two type II triangles at $A$

Figure 12. Three cases where there are two type II triangles

Figure 13. Assume that $f_1$ is 4-sided

vertex, a contradiction. Therefore, $b$ crosses the $A$-line $a_2$. By the same reason, $c$ crosses $a_5$.

Repeating the same argument, we obtain the configuration as shown in Figure 15. If the $B$-line $b'$ crosses the $A$-line $a_4$, then both $b'$ and $b''$ go to $x_3$, a contradiction. Thus $b'$ goes to $x_4$. Similarly, the $C$-line $c_3$ is determined. See the second of Figure 15.

Then $f_1$ can be incident with neither vertex $B$ nor $C$. For example, if $f_1$ is incident with $B$, then the left face of $f_1$ cannot be 3-sided. Hence $f_1$ is incident with two more triple crossing points. Then the upper horizontal line of $f_1$ is an $A$- or $D$-line by property ($\ast$). From our requirements, it cannot be an $A$-line. Thus we have the third of Figure 15 but then vertex $D$ cannot be located.

Next, assume that $f_1$ is 3-sided. We see that a $D$-line goes through the upper triple crossing point of $f_1$ from our requirements and property ($\ast$).
By symmetry, we can assume that $f_2$ is 3-sided. See Figure 16. If $f_3$ is also 3-sided, then the $B$-line $b'''$ and the $C$-line $c'$ meet twice, a contradiction. Hence $f_3$ turns out to be 4-sided. Also, no $A$-line is adjacent to $f_3$, because each of the four $A$-lines $a_2, \ldots, a_5$ meets $b'''$ or $c'$. Thus vertex $B$ is located as in Figure 16. Again, examining $f_4, f_5, f_6$ leads to a contradiction as in the proof of Claim 4.2. \hfill $\Box$

**Claim 4.8.** Figure 12(2) is impossible.

**Proof.** In Figure 17 suppose that $f_1$ is 4-sided. Then $f_2$ is 3-sided, and so vertex $D$ appears there. Then $f_3$ is not 3-sided, a contradiction. Hence $f_1$ is 3-sided.
If $f_2$ is 3-sided, then $f_3$ is 4-sided as above. Otherwise, $f_2$ is 4-sided. In any case, $f_4$ is 3-sided, and vertex $C$ appears. Also, $f_5$ and $f_6$ are 3-sided. But this is impossible as in the proof of Claim 4.2 again. □

Claim 4.9. Figure 12(3) is impossible.

Proof. In Figure 18, at least two of $f_1$, $f_2$, $f_3$ are 3-sided. If $f_1$ and $f_2$ are 3-sided, then vertex $D$ cannot be located correctly. Similarly for the case where $f_1$ and $f_3$ are 3-sided. Hence $f_2$ and $f_3$ are 3-sided. Then the $D$-line $d$ goes to $x_2$ or $x_3$, and another $D$-line $d'$ goes to $x_4$ or $x_5$. But this is impossible. □

This completes the proof of Lemma 4.10.

Lemma 4.10. The number of type II triangles at vertex $A$ is not zero.

Proof. Assume that there are no type II triangles at $A$. Up to symmetry and relabeling of vertices, the local configuration at $A$ can be assumed as in the first of Figure 19.

By symmetry, we can assume that the right hand side does not contain a 4-sided face. More precisely, $f_1, \ldots, f_4$ are all 3-sided. Thus vertex $B$ is located. Then examining $f_2, f_3, f_4$, as in the proof of Claim 4.2 leads to a contradiction. (In this case, $f_3$ can be incident with $x_5$. Then $f_4$ cannot be 3-sided likewise.) □

Theorem 4.11. $K_{5,4}$ does not admit a semi-regular drawing.

Proof. This follows from Lemmas 4.1, 4.6 and 4.10. □
Throughout this section, we will assume that $G = K_{4,4}$ admits a semi-regular drawing. We will show that this is impossible.

Let $V_1 = \{x_1, x_2, x_3, x_4\}$ and $V_2 = \{A, B, C, D\}$ be the partite sets of $G$. As in Section 4 we refer to vertices of $V_1$ (resp. $V_2$) as black (resp. white) vertices, and use the same notions as $A$-lines, type I or II triangles, so on. Also, property $(\ast)$ holds from our requirements.

We fix a semi-regular drawing of $G$, which is denoted by $G$ again. Let $k$ be the number of triple crossing points. Add a new vertex to each triple crossing point. Then we have a plane graph $G'$ with $8 + k$ vertices and $16 + 3k$ edges. Since $3(8 + k) - (16 + 3k) - 6 = 2$, either 

1. one face of $G'$ is 5-sided, and the others are 3-sided; or 
2. two faces of $G'$ are 4-sided, and the others are 3-sided 

by Lemma 2.4. As in Section 4 a face of $G$ means that of $G'$.

5.1. Case (1). We treat the case where one face of $G$ is 5-sided, and the others are 3-sided. At most two white vertices appear in the 5-sided face. Hence we can assume that four faces at vertex $A$ are all 3-sided. Thus the number of type II triangles at vertex $A$ is either 0, 2 or 4. We will eliminate these three possibilities.

**Lemma 5.1.** The number of type II triangles at vertex $A$ is not four.

**Proof.** Assume that there are four type II triangles at $A$. We may assume that the local configuration at $A$ is as shown in Figure 20(1), up to renaming. By property $(\ast)$, the right upper line is a $C$- or $D$-line, and the left upper line is a $B$- or $D$-line (see Figure 20(2)). Up to symmetry and relabeling, there are two possibilities as shown in Figure 21.

**Claim 5.2.** Figure 21(1) is impossible.
Figure 20. Four type II triangles at A

Figure 21. Two cases where there are two type II triangles

Proof. By symmetry, we can assume that both $f_1$ and $f_2$ are 3-sided. Since $f_1$ is 3-sided, vertex $D$ is located. Then $f_2$ cannot be 3-sided, a contradiction. □

Claim 5.3. Figure 21(2) is impossible.

Proof. By symmetry, we can assume that $f_1$, $f_2$ are 3-sided again. Then the $B$-line $b_3$ meets the $C$-line $c_3$, a contradiction. □

This complete the proof of Lemma 5.1.

Lemma 5.4. The number of type II triangles at vertex A is not two.

Proof. Assume that there are two type II triangles at A. As before, we may assume that the local configuration at A is as shown in Figure 22(1). By property (*), the right upper line is a C- or D-line, and the left upper line is a B- or D-line (see Figure 22). Up to symmetry and relabeling, there are two possibilities as shown in Figure 22.

Claim 5.5. Figure 22(1) is impossible.

Proof. Assume that $f_2$ is not 3-sided. Then $f_1$ is 3-sided, and so vertex $B$ is located. Thus the $B$-line $b_2$ is determined. See the first of Figure 23. Another $B$-line $b$ crosses the $A$-line $a_2$, but then it cannot reach any black vertex. Hence $f_2$ is 3-sided, so vertex $D$ is located. Then $f_3$ cannot be 3-sided. Thus $f_1$ is 3-sided, so vertex $B$ is located, and the $B$-line $b_2$ is determined again as in the second of Figure 23. Examining $b$ leads to a contradiction as above. □
Claim 5.6. Figure 23(2) is impossible.

Proof. By symmetry, we may assume that both \( f_1 \) and \( f_2 \) are 3-sided. Then vertex \( B \) is located. See the first of Figure 25.

If \( f_3 \) is 3-sided, then the \( B \)-line \( b \) and the \( C \)-line \( c \) meet twice, a contradiction. Hence \( f_3 \) is not 3-sided, and \( b \) goes to \( x_2 \) as in Figure 25. Then a line \( b' \) turns out to be a \( B \)-line. But this \( B \)-line cannot reach any black vertex, otherwise it crosses \( a_2 \) twice. \( \Box \)

This completes the proof of Lemma 5.4. \( \Box \)
Lemma 5.7. The number of type II triangles at vertex $A$ is not zero.

Proof. Assume that there is no type II triangle at $A$. Up to symmetry and relabeling, there are two possibilities as shown in Figure 26.

In any case, we can assume that $f_1$ is 3-sided by symmetry. Then vertex $B$ is located there. The $B$-line $b$ goes to $x_1$ or $x_2$, and the $B$-line $b'$ goes to $x_3$ or $x_4$. This is impossible. □

Proposition 5.8. Case (1) is impossible.

Proof. This immediately follows from Lemmas 5.1, 5.4 and 5.7. □

5.2. Case (2). We treat the case where two faces of $G$ are 4-sided, and the others are 3-sided. There are two subcases.

(2-1) All white vertices are incident with a 4-sided face.

(2-2) There is a white vertex which is not incident with a 4-sided face.

5.2.1. Subcase (2-1). Around each 4-sided face, just two white vertices appear. Hence there is just one 4-sided face around each white vertex. Also, it implies that if a face is incident with two adjacent triple crossing points or an adjacent pair of a triple crossing point and a black vertex, then it must be 3-sided.

In this subcase, then the number of type II triangle at vertex $A$ is 0, 1, 2 or 3.

Lemma 5.9. The number of type II triangles at vertex $A$ is not three.
**Proof.** Assume that there are three type II triangle at $A$. Up to symmetry and renaming, the situation is as shown in Figure 27(1), where $f$ is 4-sided.

As remarked above, $f_1$ is 3-sided. If $f$ is incident with vertex $D$, then $f_1$ is incident with vertex $D$. This is impossible. Thus $f$ is incident with vertex $C$. See Figure 27(2). Since $f_2$ is also 3-sided, the left upper line of $f_2$ is a $B$-line. We have the configuration as in Figure 28(1). Then $f_3$ is 3-sided, but this forces $f_4$ to be neither 3-sided nor 4-sided, a contradiction. See Figure 28(2). □

**Lemma 5.10.** The number of type II triangles at $A$ is not two.

**Proof.** Assume that there are two type II triangles at $A$. Up to symmetry and renaming, there are two possibilities as shown in Figure 29, where $f$ is 4-sided.

Notice that $f_1$ and $f_2$ are 3-sided.

For Figure 29(1), either vertex $C$ or $D$ appears around $f$ by property (*). Since $f_1$ is 3-sided, the former is impossible. The latter is also impossible, because $f_2$ is 3-sided. For Figure 29(2), vertex $C$ appears around $f$ by property (*). Then $f_1$ gives a contradiction, again. □

**Lemma 5.11.** The number of type II triangles at $A$ is not one.

**Proof.** Suppose that there is one type II triangle at $A$. Up to symmetry and renaming, the configuration is as shown in Figure 30, where $f$ is 4-sided.
Notice that $f_1$ is 3-sided. If $f$ is incident with vertex $C$, then $f_1$ is incident with $C$, an impossible. Hence $f$ is incident with vertex $D$. Thus we have the configuration as in Figure 30. The fact that $f_2$ is 3-sided forces $f_3$ to be 4-sided. Then $f_3$ is incident with vertices $C$ and $D$, which contradicts the fact that $D$ is incident with only one 4-sided face.

□

Lemma 5.12. The number of type II triangles at $A$ is not zero.

Proof. Assume that there is no type II triangle at $A$. Up to symmetry and renaming, there are two possibilities as shown in Figure 31 where $f$ is 4-sided.

For Figure 31(1), $f_1$ is 3-sided, so vertex $B$ is located there. Then $b$ goes to $x_1$ or $x_2$, and $b'$ goes to $x_3$ or $x_4$. This is impossible.
Figure 32. For (a) and (b)

For Figure 31(2), vertex $C$ appears around $f$ by property (∗). On the other hand, $f_1$ is 3-sided. This is impossible. □

5.2.2. Subcase (2-2). In this subcase, we may assume that vertex $A$ is not incident with a 4-sided face without loss of generality. Thus the number of type II triangles at $A$ is 0, 2 or 4.

Lemma 5.13. The number of type II triangles at $A$ is not four.

Proof. Suppose that there are four type II triangles at $A$. Then there are two possibilities as in Figure 21.

Claim 5.14. Figure 21(1) is impossible.

Proof. Among the four faces $f_1, \ldots, f_4$, at least two are 3-sided. Furthermore, if $f_1$ (resp. $f_3$) is 3-sided, then $f_2$ (resp. $f_4$) is 4-sided, and vice versa. Up to symmetry, there are three possibilities:

(a) $f_1$ and $f_3$ are 3-sided.
(b) $f_1$ and $f_4$ are 3-sided.
(c) $f_2$ and $f_3$ are 3-sided.

(a) In this case, $f_2$ and $f_4$ are 4-sided. Thus the others are all 3-sided. See Figure 32(1). By examining $f_5$, the left upper line of $f_4$ is not an $A$-line. Since $f_5$ and $f_6$ are 3-sided, two $D$-lines $d$ and $d'$ go to $x_2$, or cross, a contradiction.
(b) In this case, $f_2$ and $f_3$ are 4-sided. By examining $d$ and $d'$ shown in Figure 32(2), the same argument as (a) leads to a contradiction.
(c) In this case, $f_1$ and $f_4$ are 4-sided. As above, we see that neither $f_1$ nor $f_4$ is incident with an $A$-line. See the first of Figure 33. Thus we have two $B$-lines $b$ and $b'$ as shown there. Since $f_5$ and $f_6$ are 3-sided, vertex $B$ is located as in the second of Figure 33. But then $f_7$ cannot be 3-sided, a contradiction. □

Claim 5.15. Figure 21(2) is impossible.

Proof. If both $f_1$ and $f_2$ are 3-sided, then we have a contradiction as in the proof of Claim 5.3. Hence either of $f_1$ or $f_2$ is 4-sided. Similarly, either $f_3$ or $f_4$ is 4-sided. Then there are two possibilities, up to symmetry.
(d) $f_1$ and $f_3$ are 4-sided.
(e) $f_1$ and $f_4$ are 4-sided.

(d) Then $f_2$ and $f_4$ are 3-sided as in the first of Figure 34. By property (*), the right upper line $\ell$ of $f_3$ is an $A$- or $B$-line. If $\ell$ is an $A$-line, then it is either $a_2$ or $a_4$. But if $\ell$ is $a_4$, then $\ell$ meets $c_1$ twice, impossible. If $\ell$ is $a_2$, then $\ell$ meets $b_1$ twice, impossible. Thus $\ell$ is a $B$-line. By the same reason, the right lower line of $f_1$ is not an $A$-line, and so a $C$-line.

Thus vertices $B$ and $C$ are located as in the second of Figure 34. After locating $f_5$ and $f_6$ as in the third of Figure 34, consider the $B$-line $b$ and the $C$-line $c$. If $b$ crosses the $A$-line $a_2$, then so does $c_1$ through the same triple crossing point on $a_2$. Then $f_7$ cannot be 3-sided. Hence $b$, and then $c$, go to $x_2$. Then $f_7$ and $f_8$ cannot be 3-sided simultaneously.
(e) Then $f_2$ and $f_3$ are 3-sided. See Figure 35. By the same argument as (d), the right lower line of $f_1$ turns out to be a $C$-line. Similarly, the upper line of $f_4$ is a $C$-line. Since both $f_1$ and $f_4$ are 4-sided, vertex $C$ is located as in the second of Figure 35. Then $f_5$ is not 3-sided, a contradiction. □

This completes the proof of Lemma 5.13. □

Lemma 5.16. The number of type II triangles at $A$ is not two.

Proof. Suppose that there are two type II triangles at $A$. Then the local configuration at $A$ is Figure 23(1) or (2).

Claim 5.17. Figure 23(1) is impossible.

Proof. We claim that $f_1$ is 4-sided. Assume that $f_1$ is 3-sided. Then vertex $B$ is located, and the $B$-line $b_2$ is determined as in Figure 36.

Suppose further that $f_2$ is 4-sided. If vertex $D$ is incident with $f_2$, then $f_2$ cannot be 4-sided. Hence $f_2$ is incident with two more triple crossing points. Then the fourth line of $f_2$ is an $A$-, $B$- or $C$-line by property (*). However, the existence of
Figure 37. The case where $f_1$ is 3-sided (continued)

$b_2$ implies that it is neither an $A$- nor $B$-line. Thus the right lower line of $f_2$ is a $C$-line. Then the second of Figure 38 is the only possible configuration for $f_2$. But this is impossible, because two lines meet at most once.

Thus we see that $f_2$ is 3-sided, so vertex $D$ is located as in Figure 37. Then $f_3$ is 4-sided, and the $D$-lines $d_3$ and $d_4$, and thus $d_2$, are determined. But, $f_4$ can be neither 3-sided nor 4-sided, because the existence of $b_2$ disturbs an $A$-line and a $B$-line as above. We have thus shown that $f_1$ is 4-sided.

Next, we claim that $f_2$ is 4-sided. Assume not. Then vertex $D$ is located, and thus $d_4$ is determined. Also, $f_3$ is 4-sided. If $f_3$ is incident with $x_2$, then the $D$-line $d_2$ is determined, and thus $d_3$ cannot be drawn. Hence $f_3$ is incident with another triple crossing point as in the first of Figure 38 where an $A$- or $C$-line goes through. If it is a $C$-line, then $f_4$ cannot be 3-sided. Hence it is an $A$-line, in particular, $a_2$. See the second of Figure 38. Then $d_2$ cannot be drawn. Thus we have specified two $4$-sided faces $f_1$ and $f_2$.

Now, $f_3$ is 3-sided. By property ($*$), the right lower line of $f_2$ is an $A$- or $C$-line. See the first of Figure 39. If it is an $A$-line, then it is $a_2$. But this is impossible, because the right upper line of $f_2$ already meets $a_2$. Thus the right lower line of $f_2$ is a $C$-line. Furthermore, if $f_2$ is incident with $x_4$, then the situation is drawn as in the second of Figure 39. Then $b_4$ is determined, and thus vertex $B$ is located. However, $b_1$ cannot be drawn.

Thus $f_2$ is incident with a triple crossing point at its right. See the first of Figure 40. By property ($*$), an $A$- or $B$-line goes through the triple crossing point. But it cannot be an $A$-line by examining the (3-sided) face right above $f_2$. After locating vertex $B$, the $B$-line $b_2$ is determined. Then $b_1$ cannot be drawn.

\[\square\]

Claim 5.18. Figure 38 (2) is impossible.
Figure 39. $f_1$ and $f_2$ are 4-sided

Figure 40. $f_1$ and $f_2$ are 4-sided (continued)

Figure 41. The case where $f_1$ is 3-sided

Proof. We claim that $f_1$ and $f_4$ are 4-sided. Assume that $f_1$ is 3-sided. Then vertex $B$ is located, and then the $B$-line $b_2$ is determined. See the first of Figure 39.

Suppose further that $f_2$ is 4-sided. Then $f_2$ is incident with either vertex $C$, or vertex $D$, or two more triple crossing points. If $f_2$ is incident with vertex $C$, then $c_1$ is determined, and then $c_1$ cannot be drawn. If $f_2$ is incident with vertex $D$, then an $A$- or $B$-line appears at the triple crossing point where a $C$-line meets a $D$-line. But this is impossible by the existence of $b_2$ as in the proof of Claim 5.17. If $f_2$ is
incident with two more triple crossing points, then a similar argument to the proof of Claim 5.17 gives a contradiction again (see Figure 41). Thus $f_2$ is 3-sided.

Then the situation is as in the proof of Claim 5.6, leading to a contradiction. Thus $f_1$ is 4-sided. By the same argument, $f_3$ and $f_4$ are 4-sided. Then $f_2$ and $f_3$ are 3-sided. But this is impossible, because there are a $B$-line and $C$-line meeting twice. □

This completes the proof of Lemma 5.16. □

Lemma 5.19. The number of type II triangles at $A$ is not zero.

Proof. Suppose that there is no type II triangle at $A$. Then the local configuration at $A$ is Figure 26(1) or (2).

In Figure 26(1), if $f_1$ or $f_2$ is 3-sided, then we have a contradiction as in the proof of Lemma 5.7. Thus both are 4-sided. By the same reason, $f_3$ and $f_4$ are 4-sided, a contradiction.

In Figure 26(2), if $f_1$ or $f_2$ is 3-sided, then we have a contradiction as above. Hence $f_1$ and $f_2$ are 4-sided. Thus $f_3$ and $f_4$ are 3-sided. Then the $C$-line $c$ meets the $D$-line $d$ twice, a contradiction. □

Proposition 5.20. Case (2) is impossible.

Proof. This follows from Lemmas 5.9, 5.10, 5.11, 5.12, 5.13, 5.16 and 5.19. □

Theorem 5.21. $K_{4,4}$ does not admit a semi-regular drawing.

Proof. This follows from Propositions 5.9 and 5.20. □

6. $K_{n,3}$

Let $G = K_{n,3}$ with $n \geq 5$. In this section, we show that if $n \neq 6$ then $G$ does not admit a semi-regular drawing. Hereafter, we assume that $n \geq 5$ and $n \neq 6$.

6.1. Exceptional faces. Let $V_1$ and $V_2 = \{A, B, C\}$ be the partite sets of $G$. As before, we refer to a vertex of $V_1$ (resp. $V_2$) as a black (resp. white) vertex. Any black vertex is incident with an $A$-line, $B$-line and $C$-line.

Suppose that $G$ admits a semi-regular drawing. Fix such a drawing, denoted by $G$ again. Property (*) holds. That is, at each triple crossing point, an $A$-line, a $B$-line and a $C$-line meet. Let $k$ be the number of triple crossing points. Add a new vertex to each triple crossing point. Then we have a plane graph $G'$ with $n + 3 + k$ vertices and $3n + 3k$ edges. Since $3(n + 3 + k) - (3n + 3k) - 6 = 3$, the faces of $G'$ are 3-sided, except at most three faces, by Lemma 2.4. We refer to a non-triangular face as an exceptional face. More precisely, Lemma 2.4 claims that either

(1) $G'$ has only one exceptional face, which is 6-sided; or
(2) $G'$ has just two exceptional faces, which are 5-sided and 4-sided, respectively; or
(3) $G'$ has just three exceptional faces, which are 4-sided.

As before, a face of $G$ means that of $G'$. Let $N$ be the number (counted with multiplicities) of white vertices which are incident with exceptional faces. Then $0 \leq N \leq 6$, because two white vertices are not adjacent in $G'$. Since a white vertex is not a cut-vertex of $G'$, a white vertex cannot appear around one exceptional face twice.

Lemma 6.1. Two exceptional faces are not incident with the same pair of white vertices.
Figure 42. Two exceptional faces with the same pair of white vertices

Figure 43. A type I-B triangle and a type I-C triangle

Proof. Suppose that two exceptional faces $f$ and $f'$ are incident with white vertices $A$ and $B$, say. Then both $f$ and $f'$ are 4-sided, or one is 4-sided and the other 5-sided. See Figure 42. Recall that any black vertex is incident with a $C$-line. Thus, in any case, we cannot place $C$-lines. □

Recall that there are two types of triangles at a white vertex as shown in Figure 2. Let $X = B$ or $C$. At vertex $A$, if a type I triangle is bounded by two $A$-lines and an $X$-line, then it is said to be of type I-$X$. See Figure 43. Furthermore, a type I-$X$ triangle is said to be good if the face sharing the $X$-line with the type I-$X$ triangle is 3-sided. Otherwise, it is bad. In particular, a bad type I triangle is adjacent to an exceptional face, which is referred to as its associated exceptional face.

Lemma 6.2. Let $\{X, Y\} = \{B, C\}$. If there is a good type I-$X$ triangle at vertex $A$, then there is neither an exceptional face incident with both $A$ and $Y$, nor another good type I-$X$ triangle at $A$. In particular, the number of good type I triangles is at most two.

Proof. Let $f_1$ be a good type I-$C$ triangle at $A$. Then the face $f_2$ sharing the $C$-line with $f_1$ is 3-sided, so vertex $B$ is located there. Suppose that an exceptional face $f$ is incident with vertex $A$ and $B$. Then we have a similar situation to the proof of Lemma 6.1. Hence we cannot place $C$-lines. The existence of another good type I-$C$ triangle is excluded by a similar argument. See Figure 44. Here, we cannot place the $C$-lines going to the left upper black vertex and the left lower black vertex simultaneously. □

Lemma 6.3. Suppose that there is a bad type I triangle $f$ at vertex $A$. Let $g$ be the associated exceptional face of $f$. If $g$ is $k$-sided ($4 \leq k \leq 6$), then $g$ is incident with at most $k - 4$ white vertices.
Proof. We may assume that \( f \) is a bad type I-B triangle without loss of generality. As shown in the first of Figure 43, the boundary of \( g \) contains a sequence of

- a \( C \)-line,
- a triple crossing point,
- a \( B \)-line,
- a triple crossing point,
- a \( C \)-line.

The existence of this sequence forces \( g \) to admit at most \( k - 4 \) white vertices.

\[ \square \]

Lemma 6.4. Two bad type I triangles at vertex \( A \) cannot have the same associated exceptional face.

Proof. Let \( f_1 \) and \( f_2 \) be bad type I triangles at \( A \) whose associated exceptional faces coincide. Let \( g \) be the common associated exceptional face. When \( g \) is 4- or 5-sided, the situation is as shown in Figure 45 where labels \( B \) and \( C \) may be exchanged. Then we cannot place \( C \)-lines as before. If \( g \) is 6-sided, then the situation is as shown in Figure 46. Similarly, we cannot draw \( C \)-lines.

\[ \square \]

Lemma 6.5. Suppose that \( G \) satisfies one of the following conditions.

1. \( G \) has a 5-sided exceptional face incident with exactly one white vertex and a 4-sided exceptional face incident with at least one white vertex.
2. \( G \) has three 4-sided exceptional faces, only one of which is incident with no white vertex.

Then there is at most one bad type I triangle at vertex \( A \).

Proof. Let \( f_i \) be a bad type I triangle at \( A \), and let \( g_i \) be the associated exceptional face for \( i = 1, 2 \). By Lemma 6.3 we have \( g_1 = g_2 \). But this contradicts Lemma 6.4.

\[ \square \]
Figure 46. Two bad type I triangles with the same associated 6-sided exceptional face

Figure 47. Type II triangles at A

Lemma 6.6. There is at most three bad type I triangles at vertex A.

Proof. Suppose that there are four bad type I triangles at vertex A. Since G has at most three exceptional faces and each bad type I triangle is adjacent to an exceptional face, there exist two bad type I triangles whose associated exceptional faces coincide, contradicting Lemma 6.4. □

Recall that an adjoint pair of type II triangles at vertex A is a pair of type II triangles sharing an A-line fully. See Figure 47(1).

Lemma 6.7. Suppose that G satisfies one of the following conditions.

(1) G has a single 6-sided exceptional face, which is incident with all white vertices.
(2) G has a 5-sided exceptional face incident with two white vertices and a 4-sided exceptional face incident with a white vertex.
(3) G has three 4-sided exceptional faces, each of which is incident with a white vertex.

Then there is no adjoint pair of type II triangles at vertex A. Hence, if there is a type II triangle at A, then it shares an A-line fully with an exceptional face (Figure 47(2)).

Proof. Suppose that there is an adjoint pair of type II triangle at A. Then f, indicated in Figure 47(1), is not 3-sided. However, f cannot be an exceptional face from the assumption, a contradiction. □

Lemma 6.8. There is at most one adjoint pairs of type II triangle at vertex A.
Figure 48. Two adjoint pairs of type II triangles sharing the same exceptional face.

Figure 49. An adjoint pair of type II triangles adjacent to a 4-sided face.

Proof. As in the proof of Lemma 6.7, each adjoint pair of type II triangles yields an exceptional face. If two such pairs share the same exceptional face, then the exceptional face must be a 6-sided face without a white vertex. The situation is as shown in Figure 48. Then we cannot draw $C$-lines to the left two black vertices from vertex $C$.

Hence the number of adjoint pairs of type II triangles is no greater than the number of exceptional faces. When $G$ has a 6-sided exceptional face, we have the conclusion.

Assume that $G$ has at least two, then two or three, exceptional faces. Suppose that there are two adjoint pairs of type II triangle. Then these pairs correspond to distinct exceptional faces as above. In any case, there exists an adjoint pair of type II triangles which yields a 4-sided exceptional face $f$. Then the situation is as shown in Figure 49. This implies that both $g$ and $h$ are exceptional. Another adjoint pair of type II triangles yields one more exceptional face. Thus $G$ would have 4 exceptional faces, a contradiction. □

Lemma 6.9. If $n$ is odd, then each white vertex is incident with an exceptional face, hence $N \geq 3$.

Proof. Assume that only triangles appear at a white vertex, $A$, say. Each triangle at $A$ is incident with either a $B$-line or a $C$-line. Moreover, such triangles appear alternatively around $A$. Hence $n$ must be even. □

Lemma 6.10. Suppose that vertex $A$ is incident with only one exceptional face $f$. If $f$ is 4-sided and incident with two white vertices, then $n$ is even.
Proof. We may assume that \( f \) is incident with \( A \) and \( B \). Then each triangle adjacent to \( f \) at \( A \) is incident with a \( C \)-line. By the same reason as the proof of Lemma 6.9, \( n \) is even. \( \square \)

6.2. Reduction.

Lemma 6.11. \( N \neq 6 \).

Proof. Let \( N = 6 \). This happens only when \( G \) has three 4-sided exceptional faces, each of which is incident with two white vertices. In particular, a face incident with two adjacent triple crossing points is 3-sided. By Lemma 6.1, there is one exceptional face for each pair of \( A, B, C \). We examine the local configuration at vertex \( A \).

By Lemmas 6.2 and 6.3, there is no type I triangle. By Lemma 6.7, each type II triangle is adjacent to an exceptional face.

If two exceptional faces at \( A \) share an \( A \)-line, then there are at most two type II triangles. This implies that \( n \leq 4 \), a contradiction. (We remark that this situation can happen when \( n = 4 \).) Otherwise, there are at most four type II triangles. In fact, both sides of a type II triangle cannot be exceptional faces, because there is only one exceptional (4-sided) face for each pair of \( A, B, C \). (See Figure 47(2).) Hence there are exactly four type II triangles and two exceptional faces around \( A \), giving \( n = 6 \), a contradiction. \( \square \)

Lemma 6.12. \( N \neq 5 \).

Proof. Assume \( N = 5 \). This happens only when \( G \) has three 4-sided exceptional faces \( f_1, f_2, f_3 \), two of which are incident with two white vertices, the other to one white vertex. By Lemma 6.1, we may assume that \( f_1 \) is incident with \( A \) and \( B \), \( f_2 \) is incident with \( B \) and \( C \), and \( f_3 \) is incident with \( B \) or \( C \).

We examine the local configuration at \( A \). By Lemma 6.3 there is no bad type I triangle. Assume that \( f_3 \) is incident with \( B \). By Lemma 6.2, a good type I-C triangle is impossible, and at most one good type I-B triangle is possible. By Lemma 6.7, there are at most two type II triangles, which are adjacent to \( f_1 \). Hence we have \( n \leq 4 \), a contradiction.

The case where \( f_3 \) is incident with \( C \) is similar. \( \square \)

Lemma 6.13. \( N \neq 4 \).

Proof. Assume \( N = 4 \). Then \( G \) has at least two exceptional faces.

First, suppose that \( G \) has a 5-sided face \( f \) and a 4-sided face \( f' \). Then both \( f \) and \( f' \) are incident with two white vertices. We may assume that \( f \) are incident with \( A \), \( B \), and \( f' \) to \( B \), \( C \). This case is handled by the same argument as in the second paragraph of the proof of Lemma 6.12.

Next, suppose that \( G \) has three 4-sided faces \( f_1, f_2, f_3 \). By Lemma 6.1, there are five possibilities for three exceptional faces as shown in Table 1 up to renaming. By Lemmas 6.9 and 6.10 \( n \geq 8 \) in any case.
From (b) to (e), there is neither bad type I triangle (by Lemma 6.3) nor adjoint pair of type II triangle (by Lemma 6.7). Moreover, there are at most two good type I triangles (by Lemma 6.2) and at most two type II triangles (by Lemma 6.7). This gives \( n \leq 5 \), a contradiction.

Consider (a). By Lemma 6.2, there is no good type I-C triangle at \( A \), and at most one good type I-B triangle is possible. By Lemma 6.5, there is at most one bad type I triangle. In total, there are at most two type I triangles. By Lemma 6.8, there is at most one adjoint pair of type II triangles, and further at most two type II triangles, which are adjacent to \( f_1 \), can be possible. Hence we have \( n \leq 7 \), a contradiction. □

**Lemma 6.14.** \( N \neq 3 \).

**Proof.** Assume \( N = 3 \). We divide the proof into three cases, according to the set of exceptional faces of \( G' \).

**Case 1.** \( G \) has a single 6-sided exceptional face.

Let \( f \) be the exceptional face. Then each white vertex is incident with \( f \). By Lemmas 6.2 and 6.3, there is no type I triangle. By Lemma 6.7, there are at most two type II triangles adjacent to \( f \). Hence \( n \leq 3 \), a contradiction.

**Case 2.** \( G \) has a 5-sided exceptional face and a 4-sided exceptional face.

Let \( f_1 \) and \( f_2 \) be the 5-sided, 4-sided exceptional faces, respectively. According to white vertices incident with them, there are four possibilities as in Table 2 up to renaming.

By Lemmas 6.9 and 6.10, we have \( n \geq 8 \), except case (b).

(a) There is neither bad type I triangle (by Lemma 6.3) nor type II triangle (by Lemma 6.7). By Lemma 6.2, there are at most two good type I triangles. So, \( n \leq 2 \), a contradiction.

(b) By Lemma 6.7, there are at most two type II triangles, which are incident with \( f_1 \). There is neither bad type I triangle nor good type I-C triangle. Thus \( n \leq 4 \), a contradiction.

(c) There can be a good type I-X triangle at \( A \) for \( X \in \{B, C\} \). Hence there are at most two good type I triangles by Lemma 6.2. By Lemma 6.5, there is at most one bad type I triangle. There is at most one adjoint pair of type II triangles by Lemma 6.8.

**Claim 6.15.** A bad type I triangle does not coexist with an adjoint pair of type II triangles.

**Proof.** Suppose that there is an adjoint pair of type II triangles. Then it is adjacent to \( f_1 \) (see Figure 47(1)). If there is a bad type I triangle, then its associated exceptional face is also \( f_1 \) by Lemma 6.3. Hence \( f_1 \) is not incident with a white vertex, a contradiction. □

In any case, we have \( n \leq 4 \), a contradiction.
(d) There is no good type I-C triangle. By Lemma 6.5, there is at most one bad type I triangle. Claim 6.15 holds again. There are at most two type II triangles, which are incident with \( f_2 \), by Lemma 6.8. In any case, we have \( n \leq 6 \), a contradiction.

**Case 3.** \( G \) has three 4-sided exceptional faces.

Let \( f_1, f_2, f_3 \) be the exceptional faces. There are five possibilities as in Table 3. Again, we have \( n \geq 8 \), except case (e), by Lemmas 6.9 and 6.10.

The number of good type I triangles is at most two, except (b), by Lemma 6.2. For (b), there is no good type I-C triangle by Lemma 6.2. For (c), (d) and (e), there is neither bad type I triangle (by Lemma 6.3) nor adjoint pair of type II triangles (by Lemma 6.7). Thus (c) and (d) are settled as in Table 3. For (a) and (b), there is at most one bad type I triangle (by Lemma 6.5) and at most one adjoint pair of type II triangles (by Lemma 6.8). Thus these cases are also settled as in Table 3.

The remaining case is (e). If there is no type II triangle, then we have \( n \leq 3 \), a contradiction. Otherwise, let \( g \) be a type II triangle. Then \( g \) is adjacent to the exceptional face \( f_1 \). We may assume that \( g \) is incident with a \( B \)-line. Let \( h \) be the face sharing this \( B \)-line with \( g \). Then \( h \) is 3-sided, so vertex \( C \) is located there. This implies that \( f_1 \) is incident with \( A \) and \( C \), a contradiction. □

**Lemma 6.16.** \( n \geq 3 \).

**Proof.** Assume \( n \leq 2 \). We can assume that vertex \( A \) is not incident with an exceptional face. By Lemma 6.7, \( n \) is even, so \( n \geq 8 \). We estimate the number of triangles at \( A \) as before. By Lemmas 6.2 and 6.6 there are at most two good type I triangles and at most three bad type I triangles. Since \( A \) is not incident with an exceptional face, type II triangles appear as adjoint pairs. By Lemma 6.8 there is at most one adjoint pair of type II triangles. Then we have \( n \leq 7 \), a contradiction. □

**Theorem 6.17.** Let \( n \geq 5 \) and \( n \neq 6 \). Then \( K_{n,3} \) cannot admit a semi-regular drawing.

**Proof.** By Lemma 6.16, \( n \geq 3 \). However, this is impossible by Lemmas 6.11, 6.12, 6.13, and 6.14. □

### 7. Complete bipartite graphs

We have already shown that \( K_{4,4}, K_{5,4}, K_{5,3} \) and \( K_{n,3} \) with \( n \geq 7 \) do not admit a semi-regular drawing in Sections 4, 5 and 6.

**Theorem 7.1.** Let \( G = K_{n_1,n_2} \). If \( n_2 \leq 2 \), then \( \text{tc}(G) = 0 \). If \( n_2 \geq 3 \), then \( \text{tc}(G) = \infty \) except \( K_{3,3}, K_{4,3}, K_{6,3}, K_{6,4} \). Moreover, \( \text{tc}(K_{3,3}) = \text{tc}(K_{4,3}) = 1 \), \( \text{tc}(K_{6,3}) = 2 \) and \( \text{tc}(K_{6,4}) = 4 \).

**Proof.** If \( n_2 \leq 2 \), then \( G \) is planar, and thus \( \text{tc}(G) = 0 \). The graph \( G \) has \( p = n_1 + n_2 \) vertices and \( q = n_1 n_2 \) edges. Then \( q - 3p + 6 = (n_1 - 3)(n_2 - 3) - 3 \). Hence
if \( n_2 \geq 5 \), or \( n_2 = 4 \) and \( n_1 \geq 7 \), then \( q - 3p + 6 > 0 \), and thus \( \text{tcr}(G) = \infty \) by Lemma 2.3.

Figure 50 (after removing three edges \( v_2v_3, v_3v_4, v_4v_2 \)) shows that \( \text{tcr}(K_{3,3}) = 1 \), since \( K_{3,3} \) and \( K_{4,3} \) are not planar. Note that \( \text{tcr}(K_{6,3}) \geq 2 \), since \( \text{cr}(K_{6,3}) = 12 \) (2) and \( \text{tcr}(K_{6,3}) \geq 2 \). Thus we have that \( \text{tcr}(K_{6,3}) = 2 \) from Figure 50 (after removing three edges \( v_2v_3, v_3v_4, v_4v_2 \)). Similarly, we have that \( \text{tcr}(K_{6,4}) = 4 \) from the fact \( \text{cr}(K_{6,4}) = 12 \) (2) and Figure 50.

Theorems 4.11, 5.21 and 6.17 show that \( G = K_{n_1, n_2} \) has no semi-regular drawing for \((n_1, n_2) = (4, 4), (5, 3), (n, 3)\) with \( n \geq 7 \).

\[ \text{tcr}(K_{n_1, n_2}) = \begin{cases} 0 & \text{if } n_1 = 1, 2, \\ 1 & \text{if } n_1 = 3, 4, \\ 2 & \text{if } n_1 = 6. \end{cases} \]

**Proof.** The graph \( G \) has \( p = \sum n_i \) vertices and \( q = \sum_{i<j} n_i n_j \) edges. If \( n_2 \geq 2 \), then

\[
q - 3p + 6 = (n_1 - 1)(n_3 - 1) + (n_1 + n_2)(n_3 + n_4 - 2) + (n_3 - 3)(n_4 - 3) - 4 \\
\geq 1 + 4(n_3 + n_4 - 2) + (n_3 - 3)(n_4 - 3) - 4 \\
= (n_3 + 1)(n_4 + 1) - 3 \geq 1.
\]

Hence we have \( \text{tcr}(G) = \infty \) by Lemma 2.3.

Consider the case where \( n_2 = n_3 = n_4 = 1 \). If \( n_1 = 1 \) or 2, then \( G \) is planar, and thus \( \text{tcr}(G) = 0 \). Suppose \( n_1 \geq 3 \). Then \( G \) is non-planar by Lemma 2.1 because \( G \) contains \( K_{3,3} \) as a subgraph. Let \( V \) be the partite set of \( G \) with \( n_1 \) elements, and let \( v_2, v_3, v_4 \) be the other vertices of \( G \). Notice that if \( G \) admits a semi-regular drawing, then no edge of the triangle \( v_2v_3v_4 \) contains a triple crossing point. By removing three edges of the triangle from \( G \), we obtain a semi-regular drawing of a complete bipartite graph \( K_{n_1, 3} \). However, this is impossible by Theorem 4.1 unless...
Figure 51. $K_{3,1,1,1}$ and $K_{4,1,1,1}$

Figure 52. $K_{6,1,1,1}$

$n_1 = 3, 4$ or $6$. Since $K_{3,1,1,1}$ and $K_{4,1,1,1}$ admit a semi-regular drawing with one triple crossing as shown in Figure 51, they have triple crossing number one. Finally, $K_{6,1,1,1}$ admits a semi-regular drawing with two triple crossings as shown in Figure 52. Since $\text{tcr}(K_{6,3}) = 2$ by Theorem 7.1, $\text{tcr}(K_{6,1,1,1}) = 2$. □

9. Complete tripartite graphs

Theorem 9.1. Let $G = K_{n_1,n_2,n_3}$. Then we have the value of the triple crossing number of $G$ as in Table 3.

Proof. The graph $G$ has $p = \sum_i n_i$ vertices and $q = \sum_{i<j} n_i n_j$ edges. Then

$$q - 3p + 6 = (n_1 + n_3 - 3)(n_2 + n_3 - 3) - n_2^2 + 3n_3 - 3 \geq (2n_3 - 3)^2 - n_2^2 + 3n_3 - 3 = 3(n_3 - 1)(n_3 - 2).$$

If $n_3 \geq 3$, then $q - 3p + 6 > 0$. Thus we obtain $\text{tcr}(G) = \infty$ by Lemma 2.3.

Let $n_3 = 2$. Then $q - 3p + 6 = (n_1 - 1)(n_2 - 1) - 1$. If $n_2 \geq 3$, then $q - 3p + 6 > 0$. If $n_2 = 2$, then $q - 3p + 6 > 0$, except when $n_1 = 2$. For these cases, $\text{tcr}(G) = \infty$ by Lemma 2.3 again. Since $K_{2,2,2}$ is planar, $\text{tcr}(K_{2,2,2}) = 0$.

Let $n_3 = 1$. We have $q - 3p + 6 = (n_1 - 2)(n_2 - 2) - 1$. If $n_2 \geq 4$, or if $n_2 = 3$ and $n_1 \geq 4$, then $q - 3p + 6 > 0$. For these cases, $\text{tcr}(G) = \infty$. Since $K_{4,3,1}$ is not
The remaining cases are when $n_2 = 1$, 2. If $n_2 = 1$ or $n_1 = n_2 = 2$, $G$ is planar. Therefore consider the case where $n_2 = 2$ and $n_1 \geq 3$. Let $V_1$ and $V_2$ be the partite sets of $G$ with $n_1$ and $n_2$ elements, respectively, and let $v_3$ be the remaining vertex. Assume that $G$ admits a semi-regular drawing. Notice that no edge connecting $v_3$ and a vertex of $V_2$ contains a triple crossing point, since $|V_2| = 2$. Thus we obtain a semi-regular drawing of a complete bipartite graph $K_{n_1, 3}$ by removing two edges between $v_3$ and $V_2$. Then we have that $n_1 = 3, 4$ or 6 from Theorem 7.1. In either case, $G$ is not planar, since $G$ contains $K_{3,3}$ as a subgraph. Thus we have that $tcr(K_{3,2,1}) = tcr(K_{4,2,1}) = 1$ by Figure 51 (after removing the edge $v_2v_2$). Since $tcr(K_{6,3}) = 2$ by Theorem 7.1 we obtain that $tcr(K_{6,2,1}) = 2$ by Figure 52 (after removing the edge $v_2v_2$).

10. Comments

In this paper, we require that two edges intersect at most once, and two edges with a common end-vertex do not intersect. This is one natural standpoint in the study of the crossing number ([3, 5]), but this might be so strong that most complete multipartite graphs do not admit semi-regular drawings. If we relax it, then $K_{4,4}$, for example, admits a semi-regular drawing.
In general, for \( n \geq 4 \), we can define the \( n \)-fold crossing number for a graph \( G \) to be the minimal number of \( n \)-fold crossing points over all drawings with only \( n \)-fold crossings. Clearly, Theorem 3.1 holds for the \( n \)-fold crossing number. Furthermore, if \( G \) is a non-planar complete \( t \)-partite graph with \( t \geq 3 \), then we can show that \( G \) does not admit a drawing with only \( n \)-fold crossings by similar arguments to those of Sections 2, 8 and 9. It might be possible to determine the values of this invariant for complete bipartite graphs.

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