FUNCTIONAL EQUATIONS FOR DOUBLE SERIES OF EULER TYPE WITH COEFFICIENTS

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Abstract. We prove two types of functional equations for double series of Euler type with complex coefficients. The first one is a generalization of the functional equation for the Euler double zeta-function, proved in a former work of the second-named author. The second one is more specific, which is proved when the coefficients are Fourier coefficients of cusp forms and the modular relation is essentially used in the course of the proof. As a consequence of functional equation we are able to determine trivial zero divisors.

1. Introduction

Inspired by two source, namely, the theory of multiple zeta values on the one hand, and the theory of modular symbols and periods of cusp forms on the other, Manin in [7], [8] extended the theory of periods of modular forms replacing integration along geodesics in the complex upper half plane by iterated integrations to set up the foundation of the theory of ”iterated noncommutative modular symbols”. In particular Manin[7, 8] considered the following iterated Mellin transform

\[ I^0_{\infty}(\omega_{s_j}, ..., \omega_{s_1}) := \int_{\infty}^{0} \omega_{s_j}(\tau_{\ell}) \int_{\infty}^{\tau_{\ell}} \omega_{s_{\ell-1}}(\tau_{\ell-1}).. \int_{\infty}^{\tau_2} \omega_{s_1}(\tau_1) \]

of a finite sequence of cusp forms \( f_1, ..., f_{\ell} \) of weight \( k_j \in \mathbb{N} \) with respect to a congruence subgroup \( \Gamma \) of \( SL_2(\mathbb{Z}) \) and \( \omega_{s_j}(\tau) := f_j(\tau)\tau^{s_j-1}d\tau, s_j \in \mathbb{C}, j = \ldots \)

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When $\ell = 1$ then

$$I^0_{i\infty}(\omega_s) = \int_{i\infty}^{0} f(\tau) \tau^{s-1} d\tau$$

is the classical Mellin transform of a cusp form $f \in S_k(\Gamma)$ satisfying the following functional equation:

$$I^0_{i\infty}(\omega_s) = -\epsilon_f e^{\pi i s N \frac{k}{2} - s} I^0_{i\infty}(\omega_{k-s})$$

if $f$ is an eigenform with eigenvalue $\epsilon_f = \pm 1$ with respect to the involution $\omega_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ (see [16]).

However it seems that it is not anymore true to expect a simple functional equation if $\ell \geq 2$. Manin [8] said "since a neat functional equation can be written not for these individual integrals but for their generating series...", so the functional equation of the "total Mellin transform" associated to the finite family $\{f_j | j = 1, \ldots, \ell, \ldots\}$ of cusp forms was derived.

Now consider the case when $\ell = 2$: for $f_j(\tau) = \sum_{n \geq 1} a_j(n) e^{2\pi in\tau}, j = 1, 2$, we have

$$I^0_{i\infty}(\omega_{s_2}, \omega_{s_1}) = \int_{i\infty}^{0} f_2(\tau_2) \tau_2^{s_2-1} \int_{i\infty}^{\tau_2} f_1(\tau_1) \tau_1^{s_1-1} d\tau_1 d\tau_2$$

$$= \int_{i\infty}^{0} f_2(\tau_2) \tau_2^{s_2-1} d\tau_2 \int_{i\infty}^{0} f_1(\tau_1 + \tau_2)(\tau_1 + \tau_2)^{s_1-1} d\tau_1.$$

If $s_1 \geq 2$ is a positive integer, then $I^0_{i\infty}(\omega_{s_2}, \omega_{s_1})$ is a finite linear combination of the following multiple Dirichlet series:

$$\sum_{n,m \geq 1} \frac{a_1(n)a_2(m)}{(n + m)^{s_2}m^r}, \quad 0 \leq r \leq s_1,$$

where $s_2 \in \mathbb{C}$ whose real part is sufficiently large.

In this paper we study more general type of multiple Dirichlet series, motivated by the above iterated Mellin transform of Manin. Take $\mathfrak{A} = \{a(n)\}_{n \geq 1}$ be a sequence of complex numbers, and define

$$L_2(s_1, s_2; \mathfrak{A}) = \sum_{m,n \geq 1} \frac{a(n)}{m^{s_1}(m + n)^{s_2}}, \quad (1.1)$$
where \( s_j = \sigma_j + it_j \) \((j = 1, 2)\) be two complex variables. The purpose of the present paper is to prove two types of functional equations for this double series. This is more general situation since we take \( a_1(n) \) be an arbitrary complex number and allow \( a_2(n) = 1 \) for any \( n \geq 1 \).

Before stating our main results, we recall functional equations for classical zeta-functions, and for the double zeta-function without coefficients. It is well-known that the Riemann zeta-function \( \zeta(s) \) \((s = \sigma + it \in \mathbb{C})\) has the beautiful symmetric functional equation

\[
\pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-(1-s)/2} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s).
\]

(1.2)

For Hurwitz zeta-functions \( \zeta(s, \alpha) = \sum_{n \geq 0} (n + \alpha)^{-s} \) \((\alpha > 0)\), however, this symmetricity is no longer valid in general. In fact, the functional equation for \( \zeta(s, \alpha) \) is of the form

\[
\zeta(s, \alpha) = \frac{\Gamma(1-s)}{i(2\pi)^{1-s}} \left\{ e^{\pi is/2} \phi(1-s, \alpha) - e^{-\pi is/2} \phi(1-s, -\alpha) \right\},\]

(1.3)

where \( \phi(s, \alpha) = \sum_{n \geq 1} e^{2\pi in\alpha} n^{-s} \) is the Lerch zeta-function (Titchmarsh [18, (2.17.3)]). See also a recent work of Lagarias and Li [6]. Those functional equations are very fundamental in the study of \( \zeta(s) \) and \( \zeta(s, \alpha) \).

The theory of multiple zeta-functions has been studied very actively in recent decades. Searching for some kind of functional equations is a quite natural problem in order to develop further analytic studies of multiple zeta-functions. In the simplest double zeta case, the following functional equation is already known: consider the case \( \mathcal{A}_1 := \{ a(n) = 1 \text{ for all } n \} \) in the definition (1.1), that is

\[
\zeta_2(s_1, s_2) = \sum_{m, n \geq 1} \frac{1}{m^{s_1} (m+n)^{s_2}}.
\]

(1.4)

This is called the Euler double zeta-function, and satisfies the following functional equation:

\[
\zeta_2(s_1, s_2) = \frac{\Gamma(1-s_1) \Gamma(s_1 + s_2 - 1)}{\Gamma(s_2)} \zeta(s_1 + s_2 - 1) + \Gamma(1-s_1) \left\{ F_+(1-s_2, 1-s_1; \mathcal{A}_1) + F_-(1-s_2, 1-s_1; \mathcal{A}_1) \right\},
\]

(1.5)
where
\[(1.6)\quad F_{\pm}(s_1, s_2; \mathfrak{A}_1) = \sum_{k \geq 1} \sigma_{s_1+s_2-1}(k) \Psi(s_2, s_1 + s_2; \pm 2\pi ik),\]
with \(\sigma_{c}(k) = \sum_{0 < d | k} d^{c}\) and
\[
\Psi(a, b; x) = \frac{1}{\Gamma(a)} \int_{0}^{e^{i\phi}} e^{-xy} y^{a-1} (y + 1)^{b-a-1} dy
\]
(1.7) (the confluent hypergeometric function), where \(\Re a > 0, -\pi < \phi < \pi\), and \(|\phi + \arg x| < \pi/2\).

Formula (1.5) may be regarded as a double analogue of (1.3). In fact, since the asymptotic expansion
\[
\Psi(a, b; x) = \sum_{j=0}^{M-1} \frac{(-1)^j (a - b + 1)_j (a)_j}{j!} x^{-a-j} + \rho_M(a, b; x)
\]
(1.8) (where \((a)_j = \Gamma(a + j)/\Gamma(a)\) and \(\rho_M(a, b; x)\) is the remainder term; see [2 formula 6.13.1(1)]) is known, taking the first term of the right-hand side of (1.8), we can “approximate” \(F_{\pm}(s_1, s_2; \mathfrak{A}_1)\) by the Dirichlet series \(\sum_{k \geq 1} \sigma_{s_1+s_2-1}(k) (\pm 2\pi ik)^{-s_2}\). Therefore \(F_{\pm}(s_1, s_2; \mathfrak{A}_1)\) is a kind of “generalized Dirichlet series”.

Moreover, (1.5) gives a symmetric form of functional equation, similar to (1.2), on some hyperplanes (see Remark 6.1 at the end of Section 6).

Formula (1.5) was essentially included in [12], and first explicitly stated in [14] (in a generalized form with certain shifting parameters). In [4] a generalization of formula (1.5) to the case where the denominator includes certain complex parameters. Moreover in [5] a formula analogous to (1.5) was shown for double \(L\)-functions whose numerator includes Dirichlet characters.

The purpose of the present paper is to discuss such kind of functional equations in a more general setting. As a consequence we are able to determine trivial zero divisors of double series. We will state the main results in the next section.

2. Statement of results

Our first main result in the present paper is a further generalization of formula (1.5). We assume that \(\mathfrak{A} = \{a(n)\}_{n \geq 1}\) be a sequence of complex numbers
satisfying

(i) \( a(n) \ll n^{(\kappa - 1)/2 + \varepsilon} \) with a certain constant \( \kappa \geq 1 \), where \( \varepsilon \) is an arbitrarily small positive number,

(ii) the Dirichlet series \( L(s, \mathfrak{A}) = \sum_{n \geq 1} a(n)n^{-s} \) (which is absolutely convergent for \( \Re(s) > (\kappa + 1)/2 \) by (i)) can be continued to the whole complex plane as a meromorphic function which has only finitely many poles.

Let \( \mathcal{H} \) be the complex upper half plane and let

\[
(2.1) \quad f(\tau) = \sum_{n \geq 1} a(n)q^n,
\]

where \( q = e^{2\pi i\tau}, \tau \in \mathcal{H} \). It is obvious by (i) that \( f(\tau) \) is convergent for \( \tau \in \mathcal{H} \) and holomorphic in \( \tau \). Moreover, using (i) we find that the right-hand side of (2.1) is

\[
\ll \sum_{m,n \geq 1} n^{(\kappa - 1)/2 + \varepsilon} m^{-\sigma_1} (m+n)^{-\sigma_2} \leq \sum_{m,n \geq 1} m^{-\sigma_1} (m+n)^{(\kappa - 1)/2 + \varepsilon - \sigma_2},
\]

so, using [10] Theorem 3, we see that (1.1) is convergent absolutely in the region

\[
\sigma_2 > \frac{\kappa + 1}{2}, \quad \sigma_1 + \sigma_2 > \frac{\kappa + 3}{2}.
\]

Under assumption (ii), using [15] we can show that (1.1) has meromorphic continuation to the whole complex space \( \mathbb{C}^2 \).

Let

\[
A_{\varepsilon}(l) = \sum_{0 < n \ll l} n^\varepsilon a(n),
\]

and

\[
F_{\pm}(s_1, s_2; \mathfrak{A}) = \sum_{l \geq 1} A_{s_1 + s_2 - 1}(l) \Psi(s_2, s_1 + s_2; \pm 2\pi il).
\]
Also put
\[ L_1(s_1, s_2; \mathcal{A}) = \frac{\Gamma(1 - s)\Gamma(s_1 + s - 1)}{\Gamma(s)} L(s_1 + s_2 - 1; \mathcal{A}). \quad (2.5) \]

**Theorem 2.1.** *(The first form of the functional equation)* Under the above assumptions (i) and (ii), The functions \( F_{\pm}(s_1, s_2; \mathcal{A}) \) can be continued meromorphically to the whole space \( \mathbb{C}^2 \), and for any \( s_1, s_2 \in \mathbb{C} \), except for singularity points, it holds that
\[
L_2(s_1, s_2; \mathcal{A}) = L_1(s_1, s_2; \mathcal{A}) + \Gamma(1 - s_1) \{ F_+(1 - s_2, 1 - s_1; \mathcal{A}) + F_-(1 - s_2, 1 - s_1; \mathcal{A}) \}. \quad (2.6)
\]

**Remark 2.2.** We can determine the location of singular locus of \( L_2(s_1, s_2; \mathcal{A}) \) from the right-hand side of (2.6). In fact, the explicit form of \( L_1(s_1, s_2; \mathcal{A}) \) is given by (2.5), while the explicit information on the singular locus of \( F_{\pm} \) can be obtained from (6.8) and (6.10) in Section 6.

The proof of Theorem 2.1, which will be described in Sections 3 to 6, is analogous to that in [12], the basic idea of which goes back to Motohashi [17] and Katsurada and Matsumoto [3].

Since the assumptions (i) and (ii) for \( \mathcal{A} \) is very general, we may discuss various specific examples. For instance, by replacing \( \mathcal{A} \) by \( \mathcal{A}_1 \) in (2.1) we recover the functional equation (1.5) of double zeta function. Further let us consider the very special situation that
\[ \mathcal{A}_0(n) = \{ a(n) = 1 \text{ for only one fixed } n, \text{ and } a(n) = 0 \text{ for all other } n \}. \]

In this case (2.1) is reduced to
\[
L_2(s_1, s_2; \mathcal{A}_0(n)) = \sum_{m \geq 1} \frac{1}{m^{s_1}(m + n)^{s_2}}, \quad (2.7)
\]
a single series in two variables. This is a special case of the series
\[
\xi(s_1, s_2; (\alpha, \beta)) := \sum_{m \geq 0} \frac{1}{(m + \alpha)^{s_1}(m + \beta)^{s_2}} \quad (\beta \geq \alpha > 0),
\]
which was used in [11]. From the above theorem we immediately obtain the following “two-variables analogue” of (1.3).
Corollary 2.3. For any $s_1, s_2 \in \mathbb{C}$, except for singularity points, it holds that
\[
L_2(s_1, s_2; \mathcal{A}_0(n)) = \frac{\Gamma(1 - s_1) \Gamma(s_1 + s_2 - 1)}{\Gamma(s_2)} \cdot \frac{1}{n^{s_1 + s_2 - 1}} + \Gamma(1 - s_1) \{ F_+(1 - s_2, 1 - s_1; \mathcal{A}_0(n)) + F_-(1 - s_2, 1 - s_1; \mathcal{A}_0(n)) \},
\]
where
\[
F_\pm(s_1, s_2; \mathcal{A}_0(n)) = n^{s_1 + s_2 - 1} \sum_{k \geq 1} \Psi(s_2, s_1 + s_2; \pm 2\pi i kn).
\]

Remark 2.4. It is to be noted that
\[
L_2(s_1, s_2; \mathcal{A}) = \sum_{n \geq 1} a(n)L_2(s_1, s_2; \mathcal{A}_0(n)).
\]
Therefore, multiplying the both sides of (2.8) by $a(n)$ and adding with respect to $n$, we obtain (2.6). From this observation we may say that Corollary 2.3 is a “refinement” or ”decomposition” of Theorem 2.1.

Another important example, closely related with our original motivation on periods, is the case that $\mathcal{A}$ is the set of Fourier coefficients of a certain cusp form. Now assume that (2.1) is a holomorphic cusp form of weight $\kappa$ with respect to the Hecke congruence subgroup $\Gamma_0(N)$. In this case the assumptions (i) and (ii) are surely satisfied; (i) is Deligne’s estimate and $\kappa$ is the weight. In this case we write $L_2(s_1, s_2; \mathcal{A})$, $L_1(s_1, s_2; \mathcal{A})$, $L(s, \mathcal{A})$ and $F_\pm(s_1, s_2; \mathcal{A})$ by $L_2(s_1, s_2; f)$, $L_1(s_1, s_2; f)$, $L(s, f)$ and $F_\pm(s_1, s_2; f)$, respectively. Then (2.6) can be written as
\[
L_2(s_1, s_2; f) = L_1(s_1, s_2; f) + \Gamma(1 - s_1) \{ F_+(1 - s_2, 1 - s_1; f) + F_-(1 - s_2, 1 - s_1; f) \}.
\]
Since this formula is proved under the above very general setting, no property of cusp form is used in the proof. When $f$ is a cusp form, it is natural to expect some different type of results, for which the modularity is essentially used. Our second main result gives such a functional equation. Let
\[
\tilde{f}(\tau) = (f |_\kappa \omega_N)(\tau) = \left(\sqrt{N} \tau\right)^{-\kappa} f \left( -\frac{1}{N\tau} \right).
\]
This $\tilde{f}$ is again a cusp form of weight $\kappa$ with respect to $\Gamma_0(N)$, and especially $\tilde{f} = f$ when $N = 1$. We write the Fourier expansion of $\tilde{f}$ at $\infty$ as $\tilde{f}(\tau) = \sum_{n \geq 1} \tilde{a}(n) q^n$. Define

$$H_{2, N}^{\pm}(s_1, s_2; \tilde{f}) = \sum_{m, n \geq 1} m^{-s_1-s_2} \tilde{a}(n) \Psi(s_1 + s_2, s_2; \pm 2\pi i n / N m)$$

(2.11)

Theorem 2.5. (The second form of the functional equation) When $f(\tau)$ is a cusp form of weight $\kappa$ with respect to $\Gamma_0(N)$, the functions $H_{2, N}^{\pm}(s_1, s_2; \tilde{f})$ can be continued meromorphically to the whole space $\mathbb{C}^2$, and we have

$$L_2(s_1, s_2; f) = L_1(s_1, s_2; f)$$

$$+ \frac{(2\pi)^{s_1+s_2-1} \Gamma(1-s_1)}{\Gamma(s_2)} \Gamma(\kappa-s_1-s_2+1)$$

$$\times N^{-\kappa/2} \left\{ e^{\pi i (1-s_1-s_2)/2} H_{2, N}^{+}(-s_1, \kappa-s_2+1; \tilde{f})$$

$$+ e^{\pi i (s_1+s_2-1)/2} H_{2, N}^{-}(-s_1, \kappa-s_2+1; \tilde{f}) \right\}.$$

The proof of this theorem will be given in Sections 7 and 8.

Recall that an important application of the classical functional equation (1.2) is that from which we can find the ”trivial zeros” at negative even integer points of $\zeta(s)$. The above Theorem 2.5 has the same type of application. In fact, as we will see in Section 9, we can show the following

Corollary 2.6. For any non-negative integer $l$, the hyperplane $\Re s_2 = -l$ is a zero-divisor of $L_2(s_1, s_2; f)$.

Note that this corollary cannot be deduced from Theorem 2.1. These zero-divisors may be regarded as ”trivial zeros” of $L_2(s_1, s_2; f)$.

Remark 2.7. It is to be noted that, when $f(\tau)$ is a cusp form, $L_1(s_1, s_2; f)$ also satisfies a functional equation. Let

$$L_1^*(s_1, s_2; f) = (2\pi)^{-s_1-s_2} \Gamma(s_1) \Gamma(s_2) L_1(s_1, s_2; f).$$

Then from the functional equation for $L(s, f)$ (see, e.g., [16] Theorem 4.3.6) we can deduce

$$L_1^*(s_1, \kappa-2s_1-s_2+2; \tilde{f}) = (-1)^{\kappa/2} N^{s_1+s_2-\kappa/2-1} L_1^*(s_1, s_2; f).$$
Remark 2.8. A very different type of functional equation for certain iterated integrals related with certain multiple Hecke $L$-series has been proved by [1].

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3. An integral expression

Now we assume (i) and (ii) in the introduction and start the proof of Theorem [2]. In this section we prove the following integral expression of $L_2(s_1, s_2; \mathfrak{A})$.

Proposition 3.1. In the region

$$\sigma_1 > 0, \quad \sigma_2 > \frac{\kappa + 1}{2}, \quad \sigma_1 + \sigma_2 > \frac{\kappa + 3}{2},$$

(3.1)

the double integral

$$\Lambda(s_1, s_2; \mathfrak{A}) = \int_0^\infty f(iy) \int_0^\infty \frac{1}{e^{2\pi i(x+y)} - 1} x^{s_1-1} y^{s_2-1} dx dy$$

(3.2)

converges, and we have

$$L_2(s_1, s_2; \mathfrak{A}) = \frac{(2\pi)^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)} \Lambda(s_1, s_2; \mathfrak{A}).$$

(3.3)

Proof. Put

$$g_0(\tau) = \sum_{m \geq 1} q^m = \sum_{m \geq 1} e^{2\pi i \tau m} = \frac{e^{2\pi i \tau}}{1 - e^{2\pi i \tau}} = \frac{1}{e^{-2\pi i \tau} - 1}.\quad (3.4)$$

Let $\delta > 0$, and at first assume $y \geq \delta$. Then

$$\int_0^\infty g_0(i(x+y)) x^{s_1-1} dx = \int_0^\infty \sum_{m \geq 1} e^{2\pi i \tau (x+y)m} x^{s_1-1} dx$$

(3.5)

$$= \sum_{m \geq 1} e^{-2\pi ym} \int_0^\infty e^{-2\pi x m} x^{s_1-1} dx$$

$$= (2\pi)^{-s_1} \Gamma(s_1) \sum_{m \geq 1} \frac{e^{-2\pi y m}}{m^{s_1}}$$
if the gamma integral converges, that is, if $\sigma_1 > 0$ holds. The change of integration and summation in the above can be justified by absolute convergence (because $y > 0$). Therefore

$$
\int_\delta^\infty f(iy) \int_0^\infty g_0(i(x+y)) x^{s_1-1} dx \; y^{s_2-1} dy
$$

(3.6)

$$
= (2\pi)^{-s_1} \Gamma(s_1) \int_\delta^\infty \sum_{n \geq 1} a(n) e^{-2\pi y n} \sum_{m \geq 1} \frac{1}{m^{s_1}} e^{-2\pi y m} y^{s_2-1} dy
$$

$$
= (2\pi)^{-s_1} \Gamma(s_1) \sum_{m,n \geq 1} a(n) \frac{1}{m^{s_1}} \int_\delta^\infty e^{-2\pi y (m+n)} y^{s_2-1} dy,
$$

if we can again change the integration and summation. The series on the right-hand side is

$$
\leq \sum_{m,n \geq 1} \frac{|a(n)|}{m^{\sigma_1}} \int_\delta^\infty e^{-2\pi y (m+n)} y^{\sigma_2-1} dy
$$

$$
\leq \sum_{m,n \geq 1} \frac{|a(n)|}{m^{\sigma_1}} \int_0^\infty e^{-2\pi y (m+n)} y^{\sigma_2-1} dy
$$

$$
= (2\pi)^{-\sigma_2} \Gamma(\sigma_2) \sum_{m,n \geq 1} \frac{|a(n)|}{m^{\sigma_1} (m+n)^{\sigma_2}}
$$

if $\sigma_2 > 0$ holds. The resulting infinite series is convergent if (2.2) holds. Therefore, if (2.2) (which includes the condition $\sigma_2 > 0$) holds, then the change of integration and summation in the course of (3.6) is justified, and moreover, on the right-hand side of (3.6), we can take the limit $\delta \to 0$ termwisely. Then the right-hand side tends to

$$
(2\pi)^{-s_1-s_2} \Gamma(s_1) \Gamma(s_2) \sum_{m,n \geq 1} \frac{a(n)}{m^{s_1} (m+n)^{s_2}},
$$

(3.7)

while the left-hand side of (3.6) tends to $\Lambda(s_1, s_2; \mathfrak{A})$. This completes the proof.

4. Separating a single series factor

The integrand of the inner integral of the right-hand side of (3.3) is singular at $x+y = 0$. The next step is to “separate” the contribution of this singularity.
Let
\[ h(z) = \frac{1}{e^{2\pi z} - 1} - \frac{1}{2\pi z}. \] (4.1)

Using this function, we rewrite (3.3) as follows:
\[
\Lambda(s_1, s_2; \mathfrak{A}) = \int_0^\infty f(iy) \int_0^\infty h(x+y)x^{s_1-1}y^{s_2-1}dxdy \\
+ \int_0^\infty f(iy) \int_0^\infty \frac{x^{s_1-1}y^{s_2-1}}{2\pi(x+y)}dxdy \\
= I_1 + I_2,
\] say. To verify this decomposition, we have to check the absolute convergence of \( I_1 \) and \( I_2 \). Consider \( I_2 \) under the condition
\[ 0 < \sigma_1 < 1. \] (4.3)

It is known that
\[
\int_0^\infty \frac{x^{s_1-1}}{x+y}dx = y^{s_1-1}\Gamma(s_1)\Gamma(1-s_1)
\] (4.4)
holds for \( 0 < \sigma_1 < 1 \) and \( y > 0 \). Therefore, under (4.3), we have
\[
I_2 = \lim_{\delta \to 0} \int_\delta^\infty f(iy)yg^{s_2-1}y^{s_1-1}\frac{\Gamma(s_1)}{2\pi(1-s_1)}dy \\
= \frac{1}{2\pi}\Gamma(s_1)\Gamma(1-s_1)\int_0^\infty f(iy)y^{s_1+s_2-2}dy,
\] if the last integral is convergent. But the last integral is
\[
= \int_0^\infty \sum_{n \geq 1} a(n)e^{-2\pi yn}y^{s_1+s_2-2}dy \\
= \sum_{n \geq 1} a(n) \int_0^\infty e^{-2\pi yn}y^{s_1+s_2-2}dy \\
= (2\pi)^{-s_1-s_2+1}\Gamma(s_1+s_2-1) \sum_{n \geq 1} \frac{a(n)}{n^{s_1+s_2-1}}
\] if \( \sigma_1 + \sigma_2 > 1 \), and the last sum is absolutely convergent if \( \sigma_1 + \sigma_2 > (\kappa + 3)/2 \) and is equal to \( L(s_1 + s_2 - 1, \mathfrak{A}) \). This verifies the change of integration and
summation in the course of (4.6), and the convergence of the last integral of (4.5). Therefore we obtain

\[ I_2 = (2\pi)^{-s_1-s_2}\Gamma(s_1)\Gamma(1-s_1)\Gamma(s_1+s_2-1)L(s_1+s_2-1, \mathfrak{A}) \] (4.7)

in the region

\[ 0 < \sigma_1 < 1, \quad \sigma_1 + \sigma_2 > \frac{\kappa + 3}{2}. \] (4.8)

As for \( I_1 \), we first note that \( h(z) \) is holomorphic at \( z = 0 \), so it is \( O(1) \) when \( |z| \) is small. If the real part of \( z \) is large, then clearly \( h(z) = O(|z|^{-1}) \). Therefore

\[ I_1 \ll \int_0^\infty |f(iy)| \left\{ \int_0^1 x^{\sigma_1-1} dx + \int_1^\infty \frac{x^{\sigma_1-1}}{x+y} \right\} y^{\sigma_2-1} dy, \] (4.9)

and the quantity in the curly bracket is \( O(1) \), uniformly in \( y \), if (4.3) holds.

Under the condition (i) at the beginning of Section 3, it is known that

\[ f(iy) \ll \begin{cases} y^{-(\kappa+1)/2-\varepsilon} & \text{as } y \to 0, \\ e^{-2\pi y} & \text{as } y \to \infty \end{cases} \] (4.10)

(see [16, Lemma 4.3.3]). Using these estimates we find that the right-hand side of (4.9) is convergent absolutely if (4.3) and \( \sigma_2 > (\kappa + 1)/2 \) holds.

Therefore now we verify the decomposition (4.2) under the condition (4.8).

In this region, combining with (3.3) and (4.7), we obtain

\[ L_2(s_1, s_2; \mathfrak{A}) = J_2(s_1, s_2; \mathfrak{A}) + L_1(s_1, s_2; \mathfrak{A}) \] (4.11)

where

\[ J_2(s_1, s_2; \mathfrak{A}) = \frac{(2\pi)^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)} \int_0^\infty f(iy) \int_0^\infty h(x+y)x^{s_1-1}y^{s_2-1} dx dy. \] (4.12)

5. Contour integration

In this section we show an infinite series expression of \( J_2(s_1, s_2; \mathfrak{A}) \), whose terms can be written in terms of confluent hypergeometric functions.

Let \( \mathcal{C} \) be the contour which starts at \( +\infty \), goes along the real axis to a small positive number, rounds the origin counterclockwise, and then goes back to
+∞ again along the real axis. At first we assume (4.3). Then, since $\sigma_1 > 0$, we can replace the inner integral of (4.12) by the integral along $C$ to obtain

$$J_2(s_1, s_2; A) = \frac{(2\pi)^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)} I_3,$$

(5.1)

where

$$I_3 = \int_0^\infty f(iy)y^{s_2-1} \int_C h(x+y)x^{s_1-1}dx dy.$$  

(5.2)

The inner integral of (5.2) is absolutely convergent for any $s_1$ with $\sigma_1 < 1$, and is $O(1)$ uniformly in $y$. Therefore, using (4.10), we find that the double integral on the right-hand side of (5.2) is absolutely convergent when

$$\sigma_1 < 1, \quad \sigma_2 > \frac{\kappa+1}{2}.$$  

(5.3)

Our assumption (ii) implies that $L(s, A)$ is meromorphic in $\mathbb{C}$, and $L_2(s_1, s_2; A)$ is meromorphic in the whole space $\mathbb{C}^2$, as was mentioned in the introduction. Therefore we can now conclude that formula (4.11) is valid in the region (5.3).

Next we replace the contour $C$ by

$$C_R = \{ x = -y + 2\pi(R + 1/2)e^{i\phi} \mid 0 \leq \phi < 2\pi \} \quad (R \in \mathbb{N}),$$

and let $R \to \infty$. Since $h(x+y) = O(1)$ on $C_R$ (9 formula (5.2)), we see that

$$\int_C h(x+y)x^{s_1-1}dx \to 0$$

(as $R \to \infty$) if $\sigma_1 < 0$, which we now assume. That is, we are now in the subregion

$$\sigma_1 < 0, \quad \sigma_2 > \frac{\kappa+1}{2}$$

(5.4)

of (5.3). Then by the residue calculus we have

$$\int_C h(x+y)x^{s_1-1}dx = -2\pi i \sum_{m \in \mathbb{Z}, m \neq 0} \text{Res}_{x=-y+im} (h(x+y)x^{s_1-1}),$$

(5.5)

and the value of the residue at $x = -y + im \ (m \neq 0)$ is given by

$$\lim_{\delta \to 0} \delta \left( \frac{1}{e^{2\pi i(m+\delta)} - 1} - \frac{1}{2\pi (im + \delta)} \right) (-y + im + \delta)^{s_1-1}$$

$$= \lim_{\delta \to 0} \frac{1}{2\pi} \frac{2\pi \delta}{e^{2\pi \delta} - 1} (-y + im + \delta)^{s_1-1} = \frac{1}{2\pi} (-y + im)^{s_1-1}.$$
Substituting these results into (5.2) we obtain

\[ I_3 = -i \int_0^{\infty} f(iy) \sum_{m \in \mathbb{Z}, m \neq 0} (-y + im)^{s_1 - 1} y^{s_2 - 1} dy. \] (5.7)

When \( m > 0 \), we see that

\[ (-y + im)^{s_1 - 1} = (e^{\pi i} y + e^{\pi i/2} m)^{s_1 - 1} = (e^{\pi i/2} m(z + 1))^{s_1 - 1}, \] (5.8)

where \( z \) is defined by \( y = me^{-\pi i/2} z \). Similarly, when \( m < 0 \),

\[ (-y + im)^{s_1 - 1} = (e^{\pi i} y + e^{3\pi i/2} |m|^{s_1 - 1} = (e^{\pi i/2} |m|(z + 1))^{s_1 - 1}, \] (5.9)

where \( z \) is defined by \( y = |m| e^{\pi i/2} z \). Therefore, if the change of integration and summation is possible, from (5.7) we obtain

\[ I_3 = -i(I_{31} + I_{32}), \] (5.10)

where

\[ I_{31} = \sum_{m \geq 1} (e^{\pi i/2} m)^{s_1 - 1} \int_0^{i\infty} f(mz) (me^{-\pi i/2} z)^{s_2 - 1} (z + 1)^{s_1 - 1} me^{-\pi i/2} dz, \]

\[ = \sum_{m \geq 1} e^{\pi i(s_1 - s_2 - 1)/2} m^{s_1 + s_2 - 1} \int_0^{i\infty} f(mz) z^{s_2 - 1} (z + 1)^{s_1 - 1} dz \]

and (rewriting \( |m| \) as \( m \))

\[ I_{32} = \sum_{m \geq 1} (e^{3\pi i/2} m)^{s_1 - 1} \int_0^{-i\infty} f(-mz) (me^{\pi i/2} z)^{s_2 - 1} (z + 1)^{s_1 - 1} me^{\pi i/2} dz \]

\[ = \sum_{m \geq 1} e^{\pi i(3s_1 + s_2 - 3)/2} m^{s_1 + s_2 - 1} \int_0^{-i\infty} f(-mz) z^{s_2 - 1} (z + 1)^{s_1 - 1} dz. \] (5.12)

Substitute the definition of \( f(mz) \) into (5.11) and change the integration and summation again to obtain

\[ I_{31} = \sum_{m,n \geq 1} e^{\pi i(s_1 - s_2 - 1)/2} m^{s_1 + s_2 - 1} a(n) \int_0^{i\infty} e^{2\pi imnz} z^{s_2 - 1} (z + 1)^{s_1 - 1} dz. \]

Putting \( mn = l \), this is equal to

\[ e^{\pi i(s_1 - s_2 - 1)/2} \sum_{l \geq 1} A_{s_1 + s_2 - 1}(l) \int_0^{i\infty} e^{2\pi ilz} z^{s_2 - 1} (z + 1)^{s_1 - 1} dz, \] (5.13)
where

\[ A_0^0(l) = \sum_{mn=l} m^\sigma a(n). \]  

(5.14)

Then we see that the integral on the right-hand side of [5.13] is \( \Gamma(s_2)\Psi(s_2, s_1 + s_2; -2\pi il) \) (because \( \sigma_2 > 0 \) is satisfied by (5.4), and \( \phi = \pi/2 \) so \( \phi + \arg(-2\pi il) = 0 \)), hence

\[ I_{31} = e^{\pi i(s_1-s_2-1)/2} \Gamma(s_2) \sum_{l \geq 1} A^0_{s_1+s_2-1}(l) \Psi(s_2, s_1 + s_2; -2\pi il). \]  

(5.15)

Similarly we obtain

\[ I_{32} = e^{\pi i(3s_1+s_2-3)/2} \Gamma(s_2) \sum_{l \geq 1} A^0_{s_1+s_2-1}(l) \Psi(s_2, s_1 + s_2; 2\pi il). \]  

(5.16)

To verify the above changing process (twice) of integration and summation, we check the absolute convergence of the resulting expression. Putting \( lz = i\xi \), we see that the integral on the right-hand side of (5.13) is

\[ = \int_0^\infty e^{-2\pi \xi} \left( \frac{i\xi}{l} \right)^{s_2-1} \left( 1 + \frac{i\xi}{l} \right)^{s_1-1} \frac{i}{l^2} \xi d\xi \]  

\[ \ll \int_0^\infty e^{-2\pi \xi} \left( \frac{\xi}{l} \right)^{\sigma_2-1} \left| \left( 1 + \frac{i\xi}{l} \right)^{s_1-1} \right| \frac{d\xi}{l}, \]  

(5.17)

where the implied constant depends on \( s_2 \). Further,

\[ \left| \left( 1 + \frac{i\xi}{l} \right)^{s_1-1} \right| = \left| 1 + \frac{i\xi}{l} \right|^{\sigma_1-1} e^{-\tau_1 \arg(1+i\xi/l)}. \]  

(5.18)

The first factor on the right-hand side is \( \leq 1 \), because \( |1 + i\xi/l| \geq 1 \) and \( \sigma_1 < 0 \) by (5.4), while the second factor is \( O_{\tau_1}(1) \) because \( \arg(1+i\xi/l) \leq \pi/2 \). Hence the right-hand side of (5.17) is

\[ \ll \int_0^\infty e^{-2\pi \xi} \left( \frac{\xi}{l} \right)^{\sigma_2-1} \frac{d\xi}{l} \ll l^{-\sigma_2}. \]

Therefore (5.13) is

\[ \ll \sum_{l \geq 1} |A^0_{s_1+s_2-1}(l)|l^{-\sigma_2} \ll \sum_{m,n \geq 1} m^{\sigma_1+\sigma_2-1}|a(n)|(mn)^{-\sigma_2} \]

\[ \ll \sum_{m \geq 1} m^{\sigma_1-1} \sum_{n \geq 1} n^{(\kappa-1)/2+\varepsilon-\sigma_2}, \]
which is convergent in the region (5.4). Therefore the whole step of the above procedure is verified.

Define

$$F_0^0(s_1, s_2; \mathfrak{A}) = \sum_{l \geq 1} A_{s_1+s_2-1}(l) \Psi(s_2, s_1 + s_2; \pm 2\pi il). \quad (5.19)$$

Using notation (5.19), from (5.10), (5.15) and (5.16) we obtain

$$I_3 = -i \Gamma(s_2) \left\{ e^{\pi i(s_1-s_2)/2} F_0^0(s_1, s_2; \mathfrak{A}) + e^{\pi i(3s_1+s_2-3)/2} F_0^0(s_1, s_2; \mathfrak{A}) \right\}$$

$$= i \Gamma(s_2) \left\{ e^{\pi i(s_1-s_2+1)/2} F_0^0(s_1, s_2; \mathfrak{A}) + e^{\pi i(3s_1+s_2-1)/2} F_0^0(s_1, s_2; \mathfrak{A}) \right\}$$

in the region (5.4).

Using the identity

$$\frac{1}{\Gamma(s_1)(e^{2\pi is_1} - 1)} = \frac{\Gamma(1-s_1)}{2\pi i e^{\pi is_1}} \quad (5.21)$$

we find that (5.1) is rewritten as

$$J_2(s_1, s_2; \mathfrak{A}) = \frac{(2\pi)^{s_1+s_2-1} \Gamma(1-s_1)}{i e^{\pi is_1} \Gamma(s_2)} I_3. \quad (5.22)$$

Substituting (5.20) into the above, we obtain

$$J_2(s_1, s_2; \mathfrak{A}) = \frac{(2\pi)^{s_1+s_2-1} \Gamma(1-s_1)}{e^{\pi is_1}}$$

$$\times \left\{ e^{\pi i(s_1-s_2+1)/2} F_0^0(s_1, s_2; \mathfrak{A}) + e^{\pi i(3s_1+s_2-1)/2} F_0^0(s_1, s_2; \mathfrak{A}) \right\}$$

$$= (2\pi)^{s_1+s_2-1} \Gamma(1-s_1)$$

$$\times \left\{ e^{\pi i(1-s_1-s_2)/2} F_0^0(s_1, s_2; \mathfrak{A}) + e^{\pi i(s_1+s_2-1)/2} F_0^0(s_1, s_2; \mathfrak{A}) \right\}$$

in the region (5.4).

6. Completion of the proof of Theorem 2.1

The transformation formula

$$\Psi(a, c; x) = x^{1-c} \Psi(a - c + 1, 2 - c; x) \quad (6.1)$$
of the confluent hypergeometric function is well-known ([2, formula 6.5(6)]).

Using (6.1), we see that

\[ F^0_\pm(s_1, s_2; \mathfrak{A}) = (2\pi e^{\pm \pi i/2})^{1-s_1-s_2} \sum_{l \geq 1} A^0_{s_1+s_2-1}(l) l^{1-s_1-s_2} \Psi(1-s_1, 2-s_1-s_2; \pm 2\pi il). \]  

(6.2)

Since

\[ A^0_c(l)^{-c} = \sum_{mn=l} m^c a(n) l^{-c} = \sum_{mn=l} \left( \frac{l}{m} \right)^{-c} a(n) = \sum_{0 < n | l} n^{-c} a(n) \]

which is equal to \( A_{-c}(l) \) (recall (2.3)), we obtain

\[ F^0_\pm(s_1, s_2; \mathfrak{A}) = (2\pi e^{\pm \pi i/2})^{1-s_1-s_2} F_\pm(1-s_2, 1-s_1; \mathfrak{A}). \]  

(6.3)

Substituting this into (5.23), we obtain

\[ J_2(s_1, s_2; \mathfrak{A}) = \Gamma(1-s_1) \{ F_-(1-s_2, 1-s_1; \mathfrak{A}) + F_+(1-s_2, 1-s_1; \mathfrak{A}) \}; \]

in the region (5.4). Combining with (4.11), we now obtain (2.6) in the region (5.4).

To complete the proof of the above theorem, it is sufficient to show that \( F_\pm(s_1, s_2; \mathfrak{A}) \) can be continued to the whole space \( \mathbb{C}^2 \). For this purpose we use (1.8). This implies

\[ \Psi(s_2, s_1 + s_2; \pm 2\pi il) \]

\[ = \sum_{j=0}^{M-1} \frac{(-1)^j (1-s_1)_j (s_2)_j}{j!} (\pm 2\pi il)^{-s_2-j} + \rho_M(s_2, s_1 + s_2; \pm 2\pi il), \]

so

\[ F_\pm(s_1, s_2; \mathfrak{A}) = \sum_{j=0}^{M-1} \frac{(-1)^j (1-s_1)_j (s_2)_j}{j!} \sum_{l \geq 1} A_{s_1+s_2-1}(l)(\pm 2\pi il)^{-s_2} \]

\[ + \sum_{l \geq 1} A_{s_1+s_2-1}(l) \rho_M(s_2, s_1 + s_2; \pm 2\pi il). \]  

(6.6)
We see that

\[ \sum_{l \geq 1} A_{s_1+s_2-1}(l)(\pm 2\pi i)^{-s_2-j} \]

\[ = (\pm 2\pi i)^{-s_2-j} \sum_{l \geq 1} \sum_{n|l} n^{s_1+s_2-1} a(n)l^{-s_2-j} \]

\[ = (\pm 2\pi i)^{-s_2-j} \sum_{m,n \geq 1} n^{s_1+s_2-1} a(n)(mn)^{-s_2-j} \]

\[ = (\pm 2\pi i)^{-s_2-j} \sum_{m \geq 1} m^{-s_2-j} \sum_{n \geq 1} a(n)n^{s_1-1-j}, \]

whose last two sums are convergent when \( \sigma_2 > 1 - j \) and \( \sigma_1 < j - (\kappa - 1)/2 \), and so the above is equal to \( (\pm 2\pi i)^{-s_2-j}\zeta(s_2 + j)L(1 - s_1 + j, \mathfrak{A}) \). Therefore

\[ F_\pm(s_1, s_2; \mathfrak{A}) \]

\[ = \sum_{j=0}^{M-1} \frac{(-1)^j (1 - s_1)_j (s_2)_j (\pm 2\pi i)^{-s_2-j}\zeta(s_2 + j)L(1 - s_1 + j, \mathfrak{A})}{j!} \]

\[ + \sum_{l \geq 1} A_{s_1+s_2-1}(l)\rho_M(s_2, s_1 + s_2; \pm 2\pi il). \]

The explicit form of \( \rho_M \) is

\[ \rho_M(a, c, x) = \frac{(-1)^{M}(a - c + 1)_M}{\Gamma(a)} \int_0^{e^{i\theta} \infty} e^{-xy} y^{a+M-1} \]

\[ \times \int_0^1 \frac{(1 - \tau)^{M-1}}{(M-1)!} (1 + \tau y)^{c-a-M-1} d\tau dy \]
(see [12, (3.3)]). In the present situation, \( \phi = \mp \pi / 2 \). Therefore, if the change of integration and summation is possible, we have

\[
\sum_{l \geq 1} A_{s_1 + s_2 - 1}(l) \rho_M (s_2, s_1 + s_2; \pm 2\pi i l) \tag{6.10}
\]

\[
= \frac{(-1)^{M}(1 - s_1)}{\Gamma(s_2)} \int_{0}^{\mp i \infty} \sum_{l \geq 1} A_{s_1 + s_2 - 1}(l) e^{\mp 2\pi i ly} s_2^{M - 1} \\
\times \int_{0}^{1} \frac{(1 - \tau)^{M - 1}}{(M - 1)!} (1 + \tau y)^{s_1 - M - 1} d\tau dy \\
= \frac{(-1)^{M}(1 - s_1)}{\Gamma(s_2)} \int_{0}^{\infty} \sum_{l \geq 1} A_{s_1 + s_2 - 1}(l) e^{-\eta} \left( \frac{\mp i\eta}{2\pi l} \right)^{s_2 + M - 1} \\
\times \int_{0}^{1} \frac{(1 - \tau)^{M - 1}}{(M - 1)!} \left( 1 \mp \frac{i\tau}{2\pi} \right)^{s_1 - M - 1} d\tau \frac{\mp i}{2\pi l} d\eta \\
= \frac{(-1)^{M}(1 - s_1)}{\Gamma(s_2)} \int_{0}^{\infty} \sum_{l \geq 1} A_{s_1 + s_2 - 1}(l) l^{-s_2 - M} e^{-\eta} \left( \frac{\mp i\eta}{2\pi} \right)^{s_2 + M - 1} \\
\times \int_{0}^{1} \frac{(1 - \tau)^{M - 1}}{(M - 1)!} \left( 1 \mp \frac{i\tau}{2\pi} \right)^{s_1 - M - 1} d\tau \frac{\mp i}{2\pi} d\eta.
\]

Similarly to (6.7), the sum with respect to \( l \) is \( \zeta(s_2 + M)L(1 - s_1 + M, \mathfrak{A}) \), if

\[
\sigma_1 < M + 1 - \frac{\kappa + 1}{2}, \quad \sigma_2 > 1 - M. \tag{6.11}
\]

Also, similarly to the argument around (5.18), we see that

\[
\left( 1 \mp \frac{i\tau}{2\pi l} \right)^{s_1 - M - 1} = O(1)
\]

if \( \sigma_1 < M + 1 \). Therefore the integral on the right-hand side of (6.10) is

\[
\ll_{s_1, s_2, M} \int_{0}^{\infty} e^{-\eta} \left| \left( \frac{\mp i\eta}{2\pi} \right)^{s_2 + M - 1} \right| d\eta
\]

which is convergent if \( \sigma_2 > -M \). Hence the above change is verified, and we can now conclude that the second sum on the right-hand side of (6.8) is convergent in the region (6.11). This implies that (6.8) gives the meromorphic continuation of \( F_{\pm}(s_1, s_2; \mathfrak{A}) \) to the region (6.11). Since \( M \) is arbitrary, \( F_{\pm}(s_1, s_2; \mathfrak{A}) \) can be continued meromorphically to the whole space \( \mathbb{C}^2 \). This completes the proof of Theorem 2.1.
Remark 6.1. In [4] it was pointed out that Theorem 2.1 in the special case $\mathfrak{A} = \mathfrak{A}_1$ gives a symmetric form of the functional equation on some hyperplanes ([4, Theorem 2.2]). It is desirable to deduce such a symmetric form of the functional equation for general $\mathfrak{A}$. However there is a difficulty, caused by the fact that $F^0_{\pm}$ on the left-hand side of (6.3) is different from $F_{\pm}$ on the right-hand side.

7. Modularity comes into play

Now we proceed to the proof of Theorem 2.5. Therefore, hereafter, we assume that $\kappa$ is an even positive integer and $f(\tau)$ is a cusp form of weight $\kappa$ with respect to $\Gamma_0(N)$.

In Section 5, we only used the estimates (4.10) to show that the double integral (5.2) is convergent in the region (5.3). However, since now we assume that $f(\tau)$ is a cusp form, we see that $f(iy)$ is also of exponential decay when $y \to 0$. Therefore $I_3$ is convergent for any $s_2 \in \mathbb{C}$, and so we can remove the second condition $\sigma_2 > (\kappa + 1)/2$ from (5.3), and also from (5.4). This implies that, if $f$ is a cusp form, the whole argument in Section 5 is valid in the region

\[(7.1) \quad \sigma_1 < 0\]

instead of (5.4). (This is important, because the former region (5.4) has no intersection with (7.19) below.)

We begin with (5.11), in the region (7.1). Using (2.10), we can rewrite (5.11) as

\[
I_{31} = \sum_{m \geq 1} e^{\pi i(s_1-s_2-1)/2} m^{s_1+s_2-1} \int_0^{i\infty} (-\sqrt{Nmz})^{-\kappa} f\left(-\frac{1}{Nmz}\right) z^{s_2-1}(z+1)^{s_1-1}dz 
\]

(7.2)

\[
= N^{-\kappa/2} \sum_{m \geq 1} e^{\pi i(s_1-s_2-1)/2} m^{s_1+s_2-1-\kappa} \int_0^{i\infty} \tilde{f}\left(-\frac{1}{Nmz}\right) z^{-\kappa+s_2-1}(z+1)^{s_1-1}dz,
\]
because \((-mz)^{-\kappa} = (mz)^{-\kappa}\) since \(\kappa\) is even. Putting \(z = 1/w\), we see that the above integral is
\[
\int_{-i\infty}^{0} \tilde{f} \left( -\frac{w}{Nm} \right) w^{\kappa-s_2+1} \left( \frac{1}{w} + 1 \right)^{s_1-1} \left( -\frac{dw}{w^2} \right)
\]
\[
= \int_{0}^{-i\infty} \sum_{n \geq 1} \tilde{a}(n) e^{2\pi i (-w/Nm)} w^{\kappa-s_1-s_2} (w + 1)^{s_1-1} dw,
\]
so, if we can change the integration and summation, we have
\[
I_{31} = N^{-\kappa/2} e^{\pi i s_1 (s_1-s_2-1)/2} \sum_{m,n \geq 1} m^{s_1+s_2-1-\kappa} \tilde{a}(n) \times \int_{0}^{-i\infty} e^{-2\pi i (n/Nm)w} w^{\kappa-s_1-s_2} (w + 1)^{s_1-1} dw. \tag{7.3}
\]
If
\[
\sigma_1 + \sigma_2 < \kappa + 1 \tag{7.4}
\]
holds, then the last integral is expressed by the confluent hypergeometric function (with \(\phi = -\pi/2\) here); that is,
\[
I_{31} = N^{-\kappa/2} e^{\pi i (s_1-s_2-1)/2} \Gamma(\kappa - s_1 - s_2 + 1) \sum_{m,n \geq 1} m^{s_1+s_2-1-\kappa} \tilde{a}(n) \times \Psi(\kappa - s_1 - s_2 + 1, \kappa - s_2 + 1; 2\pi i/Nm)
\]
\[
= N^{-\kappa/2} e^{\pi i (s_1-s_2-1)/2} \Gamma(\kappa - s_1 - s_2 + 1) H_{2,N}^{+}(-s_1, \kappa - s_2 + 1; \tilde{f}). \tag{7.5}
\]

We have to check the convergence of (7.3) in order to verify the above interchange of integration and summation. Put \(2\pi i (n/Nm)w = y\). Then the integral on the right-hand side of (7.3) is
\[
= \int_{0}^{\infty} e^{-y} \left( \frac{Nm y}{2\pi i n} \right)^{\kappa-s_1-s_2} \left( \frac{Nm y}{2\pi i n} + 1 \right)^{s_1-1} \frac{Nm}{2\pi i n} dy
\]
\[
= \left( \frac{Nm}{2\pi i n} \right)^{\kappa-s_1-s_2+1} \int_{0}^{\infty} e^{-y} y^{\kappa-s_1-s_2} \left( \frac{Nm y}{2\pi i n} + 1 \right)^{s_1-1} dy.
\]
Under (7.1) we can show
\[
\left( \frac{Nm}{2\pi i n} + 1 \right)^{s_1-1} = O(1)
\]
similarly to the argument around (5.18). But this is not sufficient to prove the convergence of the series with respect to \( m \). We should be more careful here:

Using

\[
\left( \frac{Nmy}{2\pi in} + 1 \right)^{s_1-1} = \left( \frac{Nmy + 2\pi in}{2\pi in} \right)^{s_1-1},
\]

we obtain

\[
I_{31} = N^{-\kappa/2} e^{\pi i (s_1-s_2-1)/2} \sum_{m,n \geq 1} m^{s_1+s_2-1-\kappa} \tilde{a}(n)
\left( Nm \right)^{\nu_1-\nu_2+1} (2\pi in)^{s_2} \int_0^\infty e^{-y} y^{\kappa-s_1-s_2} (Nmy + 2\pi in)^{s_1-1} dy.
\]

Denote the last integral by \( J_{31} \). Then

\[
I_{31} \ll \sum_{m,n \geq 1} |\tilde{a}(n)| (n^{-\kappa+\sigma_2} |J_{31}|) (7.7)
\]

(\( \ll_N \) means that the implied constant depends on \( N \), and

\[
J_{31} = \int_0^{n/Nm} + \int_{n/Nm}^\infty 
\ll \int_0^{n/Nm} e^{-y} y^{\kappa-\sigma_1-\sigma_2} n^{\sigma_1-1} dy + \int_{n/Nm}^\infty e^{-y} y^{\kappa-\sigma_1-\sigma_2} (Nmy)^{\sigma_1-1} dy
\ll n^{\sigma_1-1} \int_0^{n/Nm} e^{-y} y^{\kappa-\sigma_1-\sigma_2} dy + (Nm)^{\sigma_1-1} \int_{n/Nm}^\infty e^{-y} y^{\kappa-\sigma_2-1} dy
= n^{\sigma_1-1} J_{311} + (Nm)^{\sigma_1-1} J_{312},
\]

say. Note that \( J_{311} \) is convergent in the region (7.4), and \( J_{312} \) is always convergent. As for \( J_{312} \), we just use the following simple estimate:

\[
J_{312} \leq \int_0^\infty e^{-y} y^{\kappa-\sigma_2-1} dy,
\]

where the integral on the right-hand side is convergent (hence \( O(1) \)) if

\[
\sigma_2 < \kappa
\]

holds. Consider \( J_{311} \). When \( Nm \geq n \), we see that

\[
J_{311} \leq \int_0^{n/Nm} y^{\kappa-\sigma_1-\sigma_2} dy \ll \left( \frac{n}{Nm} \right)^{\kappa-\sigma_1-\sigma_2+1} (7.10)
\]
(under \((7.4)\)), while when \(Nm < n\) we have
\[
J_{311} \leq \int_0^\infty e^{-y} y^{\kappa-\sigma_1-\sigma_2} dy \ll 1 \quad (7.11)
\]
(under \((7.4)\)). Therefore, under the conditions \((7.4)\) and \((7.9)\), we have
\[
J_{31} \ll \begin{cases} 
 n^{\sigma_1-1} \left( \frac{n}{Nm} \right)^{\kappa-\sigma_1+\sigma_2+1} + (Nm)^{\sigma_1-1} & \text{if } Nm \geq n, \\
 n^{\sigma_1-1} + (Nm)^{\sigma_1-1} & \text{if } Nm < n.
\end{cases} \quad (7.12)
\]
Substituting this into \((7.7)\), we obtain
\[
I_{31} \ll \sum_{Nm \geq n} |\tilde{a}(n)| n^{-\kappa+\sigma_2} ((Nm)^{\sigma_1+\sigma_2-\kappa-1} n^{\kappa-\sigma_2} + (Nm)^{\sigma_1-1}) + \sum_{Nm < n} |\tilde{a}(n)| n^{-\kappa+\sigma_2} \left( n^{\sigma_1-1} + (Nm)^{\sigma_1-1} \right)
\]
\[
= I_{31}' + I_{31}'' \quad (7.13)
\]
say. Using Deligne’s estimate, we find that
\[
I_{31}' \ll \sum_{m \geq 1} (Nm)^{\sigma_1+\sigma_2-\kappa-1} \sum_{n \leq Nm} n^{(\kappa-1)/2+\varepsilon} + \sum_{m \geq 1} (Nm)^{\sigma_1-1} \sum_{n \leq Nm} n^{(\kappa-1)/2+\varepsilon-\kappa+\sigma_2}. \quad (7.14)
\]
The first double sum is
\[
\ll \sum_{m \geq 1} (Nm)^{\sigma_1+\sigma_2-\kappa-1+(\kappa+1)/2+\varepsilon} \ll_{\varepsilon} \sum_{m \geq 1} m^{\sigma_1+\sigma_2-(\kappa+1)/2},
\]
which is convergent if
\[
\sigma_1 + \sigma_2 < \frac{\kappa-1}{2}. \quad (7.15)
\]
The inner sum of the second double sum of \((7.14)\) is \(O((Nm)^{\sigma_2-(\kappa-1)/2+\varepsilon})\) if \(\sigma_2 \geq (\kappa-1)/2\), and \(O(1)\) otherwise. Therefore the second double sum is convergent if \(\sigma_2 \geq (\kappa-1)/2\) and \((7.15)\) holds, or if \(\sigma_2 < (\kappa-1)/2\) and \(\sigma_1 < 0\). Consequently we find that the right-hand side of \((7.14)\) is convergent in the region
\[
\sigma_1 < 0, \quad \sigma_1 + \sigma_2 < \frac{\kappa-1}{2}. \quad (7.16)
\]
Similarly we find that

\[ I_{31}^{**} \ll \sum_{n \geq 1} n^{(\kappa - 1)/2 + \varepsilon - \kappa + \sigma_2 + \sigma_1 - 1} \sum_{Nm<n} \frac{1}{2} + \sum_{n \geq 1} n^{(\kappa - 1)/2 + \varepsilon - \kappa + \sigma_2} \sum_{Nm<n} (Nm)^{\sigma_1 - 1} \]

(7.17)

whose first double sum is

\[ \ll \sum_{n \geq 1} n^{\sigma_1 + \sigma_2 - (\kappa + 1)/2 + \varepsilon} \]

which is convergent in the region (7.15). The second double sum converges in the same region if \( \sigma_1 \geq 0 \), while if \( \sigma_1 < 0 \) it is convergent when \( \sigma_2 < (\kappa - 1)/2 \). Therefore the right-hand side of (7.17) is convergent in the region

\[ \sigma_2 < \frac{\kappa - 1}{2}, \quad \sigma_1 + \sigma_2 < \frac{\kappa - 1}{2}. \]  

(7.18)

Therefore by (7.4), (7.9), (7.16) and (7.18), we now arrive at the conclusion that the right-hand side of (7.3) is absolutely convergent in the region

\[ \sigma_1 < 0, \quad \sigma_2 < \frac{\kappa - 1}{2}. \]  

(7.19)

Next we consider \( I_{32} \). Using the modularity again, similarly to (7.2), we have

\[ I_{32} = \sum_{m \geq 1} e^{\pi i (3s_1 + s_2 - 3)/2} m^{s_1 + s_2 - 1} \int_0^{-i\infty} (\sqrt{Nm}z)^{-\kappa} \hat{f} \left( \frac{1}{Nmz} \right) z^{s_2 - 1} (z + 1)^{s_1 - 1} dz \]

(7.20)

\[ = N^{-\kappa/2} \sum_{m \geq 1} e^{\pi i (3s_1 + s_2 - 3)/2} m^{s_1 + s_2 - 1 - \kappa} \int_0^{-i\infty} \hat{f} \left( \frac{1}{Nmz} \right) z^{-\kappa + s_2 - 1} (z + 1)^{s_1 - 1} dz. \]
Putting $z = 1/w$, similarly to (7.3), we obtain

$$I_{32} = N^{-\kappa/2}e^{\pi i(3s_1+s_2-3)/2} \sum_{m,n \geq 1} m^{s_1+s_2-1-\kappa} a(n) \tag{7.21}$$

$$\times \int_0^{\infty} e^{2\pi i(n/Nm)w} w^{\kappa-s_1-s_2} (w+1)^{s_1-1} dw$$

$$= N^{-\kappa/2}e^{\pi i(3s_1+s_2-3)/2} \Gamma(\kappa-s_1-s_2+1) \sum_{m,n \geq 1} m^{s_1+s_2-1-\kappa} a(n)$$

$$\times \Psi(\kappa-s_1-s_2+1, \kappa-s_2+1; -2\pi i/nm)$$

$$= N^{-\kappa/2}e^{\pi i(3s_1+s_2-3)/2} \Gamma(\kappa-s_1-s_2+1) H_{2,N}^-(s_1, \kappa-s_2+1; \tilde{f}).$$

The convergence can be discussed exactly the same way as in the case of $I_{31}$. (This time we start with $-2\pi i(n/m)w = y$.) Hence (7.21) is also valid in the region (7.19).

Substituting (7.5) and (7.21) into (5.10), and combining with (5.22), we have

$$J_2(s_1,s_2; \tilde{f}) = -\frac{(2\pi)^{s_1+s_2-1} \Gamma(1-s_1)}{\Gamma(s_2)} \Gamma(\kappa-s_1-s_2+1) \tag{7.22}$$

$$\times N^{-\kappa/2} \left\{ e^{\pi i(1-s_1-s_2)/2} H_{2,N}^+(s_1, \kappa-s_2+1; \tilde{f}) + e^{\pi i(3s_1+s_2-3)/2} H_{2,N}^-(s_1, \kappa-s_2+1; \tilde{f}) \right\}$$

$$= \frac{(2\pi)^{s_1+s_2-1} \Gamma(1-s_1)}{\Gamma(s_2)} \Gamma(\kappa-s_1-s_2+1)$$

$$\times N^{-\kappa/2} \left\{ e^{\pi i(1-s_1-s_2)/2} H_{2,N}^+(s_1, \kappa-s_2+1; \tilde{f}) + e^{\pi i(s_1+s_2-1)/2} H_{2,N}^-(s_1, \kappa-s_2+1; \tilde{f}) \right\}.$$

This with (4.11) gives (2.12) in the region (7.19). Therefore, to complete the proof of Theorem 2.5 the remaining task is to show the meromorphic continuation of $H_{2,N}^\pm$.

8. The meromorphic continuation of $H_{2,N}^\pm(s_1,s_2; \tilde{f})$

In this section we prove that $H_{2,N}^\pm(s_1,s_2; \tilde{f})$ can be continued meromorphically to the whole space $\mathbb{C}^2$. We first consider the case $H_{2,N}^+$. Applying (1.7)
(with $\phi = -\pi/2$) to the right-hand side of (2.11) and putting $y = -i\eta$, we obtain

$$H_{2,N}^+(s_1, s_2; \tilde{f}) = \frac{-i}{\Gamma(s_1 + s_2)} \sum_{m,n \geq 1} m^{-s_1-s_2} \tilde{a}(n) \int_0^\infty e^{-2\pi(n/N)m\eta} (-i\eta)^{s_1+s_2-1}(-i\eta + 1)^{-s_1-1} d\eta \tag{8.1}$$

The argument in the preceding section shows that the double series form of $H_{2,N}^+(-s_1, \kappa - s_2 + 1; \tilde{f})$ (that is, (7.3)) is absolutely convergent in the region (7.19). This implies that the right-hand side of (8.1) is absolutely convergent in the region

$$\sigma_1 > 0, \quad \sigma_2 > \frac{\kappa + 3}{2}. \tag{8.2}$$

Therefore, if we assume (8.2), we may change the order of summation and integration on the right-hand side of (8.1) to obtain

$$H_{2,N}^+(s_1, s_2; \tilde{f}) = \frac{-i}{\Gamma(s_1 + s_2)} \int_0^\infty \sum_{m \geq 1} m^{-s_1-s_2} \tilde{f} \left( \frac{i\eta}{N m} \right) (-i\eta)^{s_1+s_2-1}(-i\eta + 1)^{-s_1-1} d\eta \tag{8.3}$$

where

$$\tilde{F}(\tau, s) = \sum_{m \geq 1} \tilde{f} \left( \frac{\tau}{N m} \right) m^{-s}. \tag{8.4}$$

**Lemma 8.1.** Let $u$ be a complex variable. We have

$$\int_0^\infty \tilde{F}(i\eta, s) \eta^{u-1} d\eta = \Gamma(u) \left( \frac{N}{2\pi} \right)^u \zeta(s-u)L(u, \tilde{f}) \tag{8.5}$$

in the region

$$\Re(s) - 1 > \Re(u) > \frac{\kappa + 1}{2}. \tag{8.6}$$
Proof. We have

\[
\int_0^\infty \tilde{\mathcal{F}}(i\eta, s)\eta^{u-1}d\eta = \sum_{m,n\geq 1} m^{-s}a(n) \int_0^\infty e^{-2\pi(n/Nm)\eta}\eta^{u-1}d\eta
\]

\[
= \sum_{m,n\geq 1} m^{-s}a(n)\Gamma(u) \left(\frac{N}{2\pi n}\right)^u
\]

\[
= \Gamma(u) \left(\frac{N}{2\pi}\right)^u \sum_{m\geq 1} m^{-s+u} \sum_{n\geq 1} a(n)n^{-u}
\]

\[
= \Gamma(u) \left(\frac{N}{2\pi}\right)^u \zeta(s-u)L(u, \tilde{f}),
\]

where in the above calculations, changes of summation and integration can be verified because of absolute convergence under condition (8.6).

Therefore, \(\tilde{\mathcal{F}}(i\eta, s)\) is the inverse Mellin transform of the right-hand side of (8.5), and hence

\[
\tilde{\mathcal{F}}(i\eta, s_1 + s_2) = \frac{1}{2\pi i} \int_{(c)} \eta^{-u}\Gamma(u) \left(\frac{N}{2\pi}\right)^u \zeta(s_1 + s_2 - u)L(u, \tilde{f})du
\]

where \(c = \Re(u)\) satisfies

\[
\sigma_1 + \sigma_2 - 1 > c > \frac{\kappa + 1}{2}
\]

and the path of integration is the vertical line from \(c - i\infty\) to \(c + i\infty\).

From (8.3) and (8.7) we obtain

\[
H_{2,N}^+(s_1, s_2; \tilde{f}) = \frac{-1}{2\pi \Gamma(s_1 + s_2)} \int_0^\infty \int_{(c)} \eta^{-u}\Gamma(u) \left(\frac{N}{2\pi}\right)^u \zeta(s_1 + s_2 - u)L(u, \tilde{f})du
\]

\[
\times (-i\eta)^{s_1+s_2-1}(-i\eta + 1)^{-s_1-1}d\eta
\]

\[
= \frac{-1}{2\pi \Gamma(s_1 + s_2)} \int_{(c)} \Gamma(u) \left(\frac{N}{2\pi}\right)^u \zeta(s_1 + s_2 - u)L(u, \tilde{f})J(u)du,
\]

where

\[
J(u) = \int_0^\infty \eta^{-u}(-i\eta)^{s_1+s_2-1}(-i\eta + 1)^{-s_1-1}d\eta,
\]
if the change of the order of integration is possible. The integral (8.10) is absolutely convergent in the region
\[ \sigma_2 < c + 1, \quad \sigma_1 + \sigma_2 > c, \tag{8.11} \]
and hence the above change of integration is valid by Fubini’s theorem in this region. From (8.8) and (8.11) we see that (8.9) is valid in the region
\[ \sigma_2 < c + 1, \quad \sigma_1 + \sigma_2 - 1 > c > \frac{\kappa + 1}{2}. \tag{8.12} \]
Since the intersection of (8.2) and (8.12) is non-empty, now we find that \( H_{2,N}^+(s_1, s_2; \tilde{f}) \) is continued to the region (8.12) by the expression (8.9).

Putting \( y = -i\eta \) on the right-hand side of (8.10), and rotating the path of integration to the positive real axis (this is possible under condition (8.11)), we obtain
\[ J(u) = e^{\pi i (1-u)/2} \int_0^\infty y^{s_1+s_2-1-u} (1+y)^{-s_1-1} dy. \]
Therefore, applying the beta integral formula we obtain
\[ J(u) = e^{\pi i (1-u)/2} \frac{\Gamma(u-s_2+1)\Gamma(s_1+s_2-u)}{\Gamma(s_1+1)}. \tag{8.13} \]
Substituting this into (8.9), we now arrive at the expression
\[ H_{2,N}^+(s_1, s_2; \tilde{f}) = \frac{-1}{2\pi\Gamma(s_1+s_2)\Gamma(s_1+1)} \int_{(c)} \Gamma(u)\Gamma(u-s_2+1)\Gamma(s_1+s_2-u) \]
\[ \times e^{\pi i (1-u)/2} \left( \frac{N}{2\pi} \right)^u \zeta(s_1+s_2-u)L(u, \tilde{f}) du \tag{8.14} \]
in the region (8.11).

We prove that the right-hand side of (8.14) can be continued meromorphically to the whole space \( \mathbb{C}^2 \) by suitable modifications of the path of integration. Let \((s_1^0, s_2^0)\) be any point in the space \( \mathbb{C}^2 \). We choose a point \((s_1^*, s_2^*)\) in the region (8.11), which satisfies \( \Im s_1^* = \Im s_1^0, \Im s_2^* = \Im s_2^0 \). Then (8.14) holds for \((s_1, s_2) = (s_1^*, s_2^*)\). The poles of the integrand on the right-hand side of (8.14) are

(A) \( u = 0, -1, -2, -3, \ldots \),
(B) \( u = s_2^* - 1, s_2^* - 2, s_2^* - 3, \ldots \),
(C) \( u = s_1^* + s_2^*, s_1^* + s_2^* + 1, s_1^* + s_2^* + 2, \ldots \).
(D) \( u = s_1^* + s_2^* - 1 \).
The poles (A) and (B) are on the left of the vertical line \( \Re u = c \), while the poles (C) and (D) are on the right of \( \Re u = c \).

First consider the case when \( \Im (s_1^* + s_2^*) \neq \Im s_2^* \) and \( \Im (s_1^* + s_2^*) \neq 0 \). Let \( L_1 \) be the line segment joining \( s_2^* - 1 \) and \( s_0^2 - 1 \), and \( L_2 \) the line segment joining \( s_1^* + s_2^* - 1 \) and \( s_0^1 + s_0^2 - 1 \). We deform the original path \( \Re u = c \) to make a new path \( D \), such that \( L_1 \) is on the left of \( D \) while \( L_2 \) is on the right of \( D \) (see Fig.1).

---

Fig.1
Then we move the variables \((s_1, s_2)\) from \((s_1^*, s_2^*)\) to \((s_1^0, s_2^0)\), keeping the values of their imaginary parts. Then the location of poles moves, but during this procedure they do not encounter the new path \(D\). Therefore in this case we can continue \(H^+_{2,N}(s_1, s_2; \tilde{f})\) to the point \((s_1^0, s_2^0)\) holomorphically.

Next consider the situation when \(\Im(s_1^* + s_2^*) = \Im s_2^*\) or \(\Im(s_1^* + s_2^*) = 0\). We discuss the former case, because the latter case can be treated similarly. When \(\Im(s_1^* + s_2^*) = \Im s_2^*\), we deform \(\Re u = c\) to make \(D'\), which only requires that \(L_1\) is on the left of \(D'\). Then, when \(s_1^* + s_2^* - 1\) is moved to \(s_1^0 + s_2^0 - 1\), several poles encounter \(D'\). Therefore at the point \((s_1^0, s_2^0)\), we have to add the residue terms coming from the above poles to the right-hand side of (8.14). This new expression of \(H^+_{2,N}(s_1, s_2; \tilde{f})\) gives the meromorphic continuation to the point \((s_1^0, s_2^0)\). We thereby obtain the proof of meromorphic continuation of \(H^+_{2,N}(s_1, s_2; \tilde{f})\) to the whole space \(\mathbb{C}^2\).

The function \(H^-_{2,N}(s_1, s_2; \tilde{f})\) can be treated quite similarly. We can show the integral expression of \(H^-_{2,N}(s_1, s_2; \tilde{f})\), almost the same as (8.14), only the factor \(e^{\pi i (1-u)/2}\) is replaced by \(e^{\pi i (u-1)/2}\), which can be continued meromorphically as above. The proof of Theorem 2.5 is now complete.

**Remark 8.2.** We can discuss the location of singularities of \(H^\pm_{2,N}(s_1, s_2; \tilde{f})\) more closely, by using the more sophisticated path (like the path \(C'\) defined in [13], described in Fig.2 of [13]) instead of \(D'\).

9. **Proof of Corollary 2.6**

We conclude this paper with the proof of Corollary 2.6.

The right-hand side of the formula (2.12) given in Theorem 2.5 consists of two terms, the term \(L_1(s_1, s_2; f)\) and the other.

First consider the term \(L_1(s_1, s_2; f)\). The denominator \(\Gamma(s_2)\) on the right-hand side of the definition (2.5) of \(L_1(s_1, s_2; f)\) has poles at \(s_2 = -l\), and the other factors do not cancel those poles. Therefore \(s_2 = -l\) are zero-divisors of \(L_1(s_1, s_2; f)\).

Consider the other term on the right-hand side of (2.12). Again the denominator is \(\Gamma(s_2)\), so what we have to show is that the other factors do not cancel the poles \(s_2 = -l\) of \(\Gamma(s_2)\). This is obvious except for the terms...
As for $H_{2,N}^\pm$, in Section 8 we noticed that these are absolutely convergent, hence especially finite, in the region $(7.19)$. Since the region $(7.19)$ includes $s_2 = -l$ when $\sigma_1 < 0$, we now arrive at the conclusion that $s_2 = -l$ are zero-divisors of the second term on the right-hand side of (2.12).

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