ENERGY DECAY RATES FOR SOLUTIONS OF THE WAVE EQUATION
WITH LINEAR DAMPING IN EXTERIOR DOMAIN

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Abstract. In this paper we study the behavior of the energy and the $L^2$ norm of solutions of the wave equation with localized linear damping in exterior domain. Let $u$ be a solution of the wave system with initial data $(u_0, u_1)$. We assume that the damper is positive at infinity then under the Geometric Control Condition of Bardos et al [3] (1992), we prove that:

1. The total energy $E_u(t) \leq C_0(1 + t)^{-1}I_0$ and $\|u(t)\|_{L^2}^2 \leq C_0I_0$ if $(u_0, u_1)$ belong to $H^1_0(\Omega) \times L^2(\Omega)$, where $I_0 = \|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2$.

2. The total energy $E_u(t) \leq C_2(1 + t)^{-2}I_1$ and $\|u(t)\|_{L^2}^2 \leq C_2(1 + t)^{-1}I_1$, if the initial data $(u_0, u_1)$ belong to $H^1_0(\Omega) \times L^2(\Omega)$ and verifies $\|d(\cdot)(u_1 + au_0)\|_{L^2} < +\infty$, where $I_1 = \|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2 + \|d(\cdot)(u_1 + au_0)\|_{L^2}^2$.

1. Introduction and Statement of the result

Let $O$ be a compact domain of $\mathbb{R}^d$ ($d \geq 2$) with $C^\infty$ boundary $\Gamma = \partial \Omega$ and $\Omega = \mathbb{R}^d \setminus O$. Consider the following wave equation with localized linear damping

$$
\begin{cases}
\partial^2_t u - \Delta u + a(x) \partial_t u = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\
u = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \\
u(0, x) = u_0 & \text{and } \partial_t \nu(0, x) = u_1.
\end{cases}
$$

(1.1)

Here $\Delta$ denotes the Laplace operator in the space variables. $a(x)$ is a nonnegative function in $L^\infty(\Omega)$.

Let

$$A = \begin{pmatrix} 0 & I \\ \Delta & -a \end{pmatrix},$$

and $H = H_D(\Omega) \times L^2(\Omega)$, the completion of $(C^\infty_0(\Omega))^2$ with respect to the norme

$$\|(\varphi_0, \varphi_1)\|_H^2 = \frac{1}{2} \int_{\Omega} |\nabla \varphi_0|^2 + |\varphi_1|^2 \, dx,$$

then the domain of $A$

$$D(A) = \left\{(u_0, u_1) \in H, A \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in H \right\}.$$
Let $n \in \mathbb{N}$ and $(u_0, u_1) \in D(A^n)$. Linear semigroup theory applied to (1.1), provides existence of a unique solution $u$ in the class

$$(u, \partial_t u) \in C^k \left( \mathbb{R}_+, D \left( A^{n-k} \right) \right), \text{ with } k \leq n.$$ 

Moreover, if $(u_0, u_1)$ is in $H^1_0(\Omega) \times L^2(\Omega)$, then the system (1.1), admits a unique solution $u$ in the class

$$u \in C^0 \left( \mathbb{R}_+, H^1_0(\Omega) \right) \cap C^1 \left( \mathbb{R}_+, L^2(\Omega) \right).$$

With (2.2) we associate the energy functional given by

$$E_u(t) = \frac{1}{2} \int_{\Omega} \left( |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 \right) dx.$$ 

The energy functional satisfies the following identity

$$E_u(T) + \int_0^T \int_{\Omega} a(x) |\partial_t u|^2 dx dt = E_u(0),$$

for every $T > 0$.

Zuazua [16], Nakao [13], Dehman et al [7] and Aloui et al [1] have considered the problem for the Klein-Gordon type wave equations with localized dissipations. For the Klein-Gordon equations the energy functional itself contains the $L^2$ norm and boundedness of $L^2$ norm of solution is trivial. Thus under a geometric condition we can show that the energy decays exponentially while for the system (1.1) the energy decay rate is weaker and more delicate.

In the case when $a(x) \geq \epsilon_0 > 0$ in all of $\Omega$ we know that

$$E_u(t) \leq C_0 (1 + t)^{-1} I_0 \text{ and } \|u(t)\|_{L^2}^2 \leq C_0 I_0, \text{ for all } t \geq 0,$$

for weak solution $u$ to the system (1.1) with initial data in $H^1_0(\Omega) \times L^2(\Omega)$.

Nakao in [13] obtained the same estimates in (1.3) for a damper $a$ which is positive near some part of the boundary (Lions’s condition) and near infinity.

On the other hand, Dan-Shibata [5] studied the local energy decay estimates for the compactly supported weak solutions of (1.1) with $a(x) = 1$

$$\int_{\Omega \cap B_R} \left( |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 \right) dx \leq C (1 + t)^{-d},$$

where $B_R = \{ x \in \mathbb{R}^d, |x| < R \}$.

Furthermore Ikehata and Matsuyama in [10] obtained a more precise decay estimate for the energy of solutions of the problem (1.1) with $a(x) = 1$ and for weighted initial data

$$E_u(t) \leq C_2 (1 + t)^{-2} I_1 \text{ and } \|u(t)\|_{L^2}^2 \leq C_2 (1 + t)^{-1} I_1 \text{ for all } t \geq 0.$$ (1.5)

Especially this estimate seems sharp for $d = 2$ as compared with that of [5].

Ikehata in [9] derived a fast decay rate like (1.5) for solutions of the system (1.1) with weighted initial data and assuming that $a(x) \geq \epsilon_0 > 0$ at infinity and $O = \mathbb{R}^d \setminus \Omega$ is star shaped with respect to the origin.

For another type of total energy decay property we refer the reader to [8, 11, 15, 2] and reference therein.

Before introducing our results we shall state several assumptions:

**Hyp A:** There exists $L > 0$ such that

$$a(x) > \epsilon_0 > 0 \text{ for } |x| \geq L.$$
Definition 1. \((\omega, T)\) geometrically controls \(\Omega\), i.e. every generalized geodesic travelling with speed 1 and issued at \(t = 0\), enters the set \(\omega\) in a time \(t < T\).

This condition is called Geometric Control Condition (see e.g. [3]). We shall relate the open subset \(\omega\) with the damper \(a\) by
\[
\omega = \{ x \in \Omega; a(x) > \epsilon_0 > 0 \}.
\]

We note that according to [3] and [4] the Geometric Control Condition of Bardos et al. and for a damper \(a\) positive near infinity, the estimates in (1.3) hold for all solutions of the system (1.1) with weighted initial data and to show that the estimates in (1.5) hold for all solutions of the system (1.1) with weighted initial data. Moreover we show that for every \(p \in \mathbb{N}\) there exists a initial data in \(H^1_0(\Omega) \times L^2(\Omega)\) such that the solution \(v\) of (1.1) verifies
\[
E_v(t) \leq C(1 + t)^{-p} \text{ and } \|v(t)\|_{L^2}^2 \leq C(1 + t)^{-p+1}, \text{ for all } t \geq 0
\]
and for some \(C > 0\) depending on the initial data.

Theorem 1. We assume that Hyp A holds and \((\omega, T)\) geometrically controls \(\Omega\). Then there exists \(C_0 > 0\) such that the following estimates
\[
E_u(t) \leq C_0 (1 + t)^{-1} I_0 \text{ and } \|u(t)\|_{L^2}^2 \leq C_0 I_0, \text{ for all } t \geq 0,
\]
hold for every solution \(u\) of (1.1) with initial data \(u_0, u_1\) in \(H^1_0(\Omega) \times L^2(\Omega)\), where
\[
I_0 = \|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2.
\]

As a corollary of theorem 1 we have:

Proposition 1. Let \(n \in \mathbb{N}\). We assume that Hyp A holds and \((\omega, T)\) geometrically controls \(\Omega\). Let \((u_0, u_1)\) in \(D(A^n)\), such that \(u_0 \in L^2(\Omega)\). Then the solution \(u\) of (1.1) satisfies
\[
E_{\partial^n_t} u(t) \leq C_n (1 + t)^{-n-1} I_{0,n} \text{ for all } t \geq 0,
\]
\[
\|\partial^n_t u(t)\|_{L^2}^2 \leq C_{n-1} (1 + t)^{-n} I_{0,n-1} \text{ for all } t \geq 0,
\]
and
\[
\|\Delta \partial^{n-1}_t u(t)\|_{L^2}^2 \leq C_n (1 + t)^{-n} I_{0,n} \text{ for all } t \geq 0.
\]

where \(C_p\) is a positive constant independent of the initial data and
\[
I_{1,p} = \sum_{i=0}^p \|A^i (u_0, u_1)\|_{H}^2 + \|u_0\|_{L^2}^2, \text{ for } p \in \mathbb{N}.
\]

In the sequel, we use
\[
d(x) = \begin{cases} 
|x| & d \geq 3, \\
|x| \ln(B|x|) & d = 2,
\end{cases}
\]
with \(B \inf_{x \in \Omega} |x| \geq 2\).
Theorem 2. We assume that Hyp A holds and \((\omega,T)\) geometrically controls \(\Omega\). Then there exists \(C_2 > 0\) such that the following estimates

\[
E_u (t) \leq C_2 \left( 1 + t \right)^{-2} I_1 \quad \text{and} \quad \| u(t) \|_{L^2}^2 \leq C_2 \left( 1 + t \right)^{-1} I_1 \quad \text{for all} \quad t \geq 0,
\]

hold for every solution \(u\) of \((\mathbf{I})\) with initial data \((u_0, u_1)\) in \(H^1_0 (\Omega) \times L^2 (\Omega)\) which satisfies

\[
\| d (\cdot) (u_1 + au_0) \|_{L^2} < +\infty,
\]

where

\[
I_1 = \| u_0 \|_{H^1}^2 + \| u_1 \|_{L^2}^2 + \| d (\cdot) (u_1 + au_0) \|_{L^2}^2.
\]

As a corollary we have:

Proposition 2. Let \(n \in \mathbb{N}^*\). We assume that Hyp A holds and \((\omega,T)\) geometrically controls \(\Omega\). Let \((u_0, u_1)\) in \(D (A^n)\), such that \(u_0 \in L^2 (\Omega)\) and

\[
\| d (\cdot) (u_1 + au_0) \|_{L^2} < +\infty
\]

Then the solution \(u\) of \((\mathbf{I})\) satisfies

\[
E_{\partial^n t} u (t) \leq C_n \left( 1 + t \right)^{-n-2} I_{1,n}, \quad \text{for all} \quad t \geq 0,
\]

\[
\| \partial^n_t u (t) \|_{L^2}^2 \leq C_{n-1} \left( 1 + t \right)^{-n-1} I_{1,n-1}, \quad \text{for all} \quad t \geq 0,
\]

and

\[
\| \Delta \partial^{n-1}_t u (t) \|_{L^2}^2 \leq C_{n+1} \left( 1 + t \right)^{-n-1} I_{1,n}, \quad \text{for all} \quad t \geq 0.
\]

where \(C_p\) is a positive constant independent of the initial data and

\[
I_{1,p} = \sum_{i=0}^{p} \| A^i (u_0, u_1) \|_{H^1}^2 + \| u_0 \|_{L^2}^2 + \| d (\cdot) (u_1 + au_0) \|_{L^2}^2, \quad \text{for} \quad p \in \mathbb{N}.
\]

2. Proof of Theorem 1

In order to prove theorem 1 we need some preliminary results.

Proposition 3. We assume that Hyp A holds and \((\omega,T)\) geometrically controls \(\Omega\). Let \(M = \Omega \cap B_{2L}\). Setting

\[
\omega_1 = \omega \cap B_{2L} = \{ x \in \Omega \cap B_{2L}; a (x) > \epsilon_0 > 0 \}.
\]

Then \((\omega_1, T)\) geometrically controls \(M\). Therefore there exists a positive constant \(C^1_T\), such that the following estimate

\[
E_w (t) \leq C^1_T \left( \int_{t}^{t+T} \int_{M} a | \partial_t w |^2 + | f (s, x) |^2 \, dx \, ds \right), \quad (2.1)
\]

holds for every \(t \geq 0\), for every solution \(w\) of

\[
\begin{cases}
\partial^2_t w - \Delta w + a (x) \partial_t w = f (t, x) & \mathbb{R}_+ \times M, \\
w = 0 & \mathbb{R}_+ \times \partial M,
\end{cases}
\]

\[
(w (0), \partial_t w (0)) = (w_0, w_1), \quad (2.2)
\]

with initial data in the energy space \(H^1_0 (M) \times L^2 (M)\), and for every \(f\) in \(L^2_{\text{loc}} (\mathbb{R}_+, L^2 (M))\).
Proof. We remind that 
\[ a(x) > \epsilon_0 > 0 \text{ for } |x| \geq L. \]
Let 
\[ \omega_1 = \{ x \in \Omega \cap B_{2L}; a(x) > \epsilon_0 > 0 \}. \]
To prove that \( (\omega_1, T) \) geometrically controls \( M \), we have only to show that every geodesic starting from \( B_L \) enters the set \( \omega_1 \) in a time \( t < T \). Let \( \gamma \) a geodesic starting from \( B_L \), then we have the following two cases

- \( \gamma \) stays in \( B_L \) for every \( t \in [0, T[. \) Since \( (\omega = \{ x \in \Omega; a(x) > \epsilon_0 > 0 \}, T) \) geometrically controls \( \Omega \), we infer that \( \gamma \) enters the set \( \{ x \in \Omega \cap B_L; a(x) > \epsilon_0 > 0 \} \subset \omega_1 \).
- \( \gamma \) leaves the ball \( B_L \) for some \( t \in [0, T[. \) Therefore \( \gamma \) enters the set \( \{ x \in \Omega \cap B_L; |x| \geq L \} \), since 
\[ a(x) > \epsilon_0 > 0 \text{ for } |x| \geq L \]
we deduce that \( \gamma \) enters the set \( \omega_1 \).

Therefore using \([3, \text{ proposition 3}]\), we obtain \([2.1]\). \( \square \)

Proposition 4. We assume that Hyp A holds and \((\omega, T)\) geometrically controls \( \Omega \). Let \( \delta > 0 \) and \( \chi \in C_0^\infty (\mathbb{R}^d) \). There exists \( C_{T, \delta, \chi} > 0 \), such that the following inequality
\[
\int_t^{t+T} \int_\Omega \chi^2(x) \left( |\nabla u|^2 + |\partial_t u|^2 \right) dx ds \leq C_{T, \delta, \chi} \left( \int_t^{t+T} \int_\Omega a(x) |\partial_t u|^2 dx ds \right) + \delta E_u(t), \tag{2.3}
\]
holds for every \( t \geq 0 \) and for all \( u \) solution of \((1.1)\) with initial data \((u_0, u_1)\) in \( H \).

Proof. To prove this result we argue by contradiction: If \((2.3)\) was false, there would exist a sequence of numbers \((t_n)\) and a sequence of solutions \((u_n)\) such that
\[
\int_{t_n}^{t_n+T} \int_\Omega \chi^2(x) \left( |\nabla u_n|^2 + |\partial_t u_n|^2 \right) dx ds \geq n \left( \int_{t_n}^{t_n+T} \int_\Omega a(x) |\partial_t u_n|^2 dx dt \right) + \delta E_{u_n}(t_n). \tag{2.4}
\]
Setting 
\[ \lambda_n^2 = \int_{t_n}^{t_n+T} \int_\Omega \chi^2(x) \left( |\nabla u_n|^2 + |\partial_t u_n|^2 \right) dx ds \text{ and } v_n = \frac{u_n(t_n + \cdot)}{\lambda_n}, \]
It is clear that \((2.4)\), gives
\[
\int_0^T \int_\Omega a(x) |\partial_t v_n|^2 dx dt \xrightarrow{n \to +\infty} 0 \text{ and } E_{v_n}(0) \leq \frac{1}{\delta}. \tag{2.5}
\]
Let \( Z_n \) be the solution of the following system
\[ \begin{align*}
& \partial^2_t Z_n - \Delta Z_n = 0 \quad \mathbb{R}_+ \times \Omega, \\
& Z_n = 0 \quad \mathbb{R}_+ \times \Gamma, \\
& (Z_n(0), \partial_t Z_n(0)) = \frac{1}{\lambda_n} (u_n(t_n), \partial_t u_n(t_n)).
\end{align*} \]
The hyperbolic energy inequality gives 
\[
\sup_{[0,T]} E_{v_n-Z_n}^{1/2}(s) \leq 2 \| a(x) \partial_t v_n \|_{L^1([0,T], L^2(\Omega))}, \tag{2.6}
\]
Now using \((2.5)\), we infer that 
\[
\sup_{[0,T]} E_{v_n-Z_n}(s) \xrightarrow{n \to +\infty} 0. \tag{2.6}
\]
On the other hand,
\[ \frac{1}{2} \int_0^T \int_\omega |\partial_t Z_n|^2 \, dx \, dt \leq 2T \sup_{[0,T]} \|E_{v_n} - Z_n(s)\| + \int_0^T \int_\omega |\partial_t v_n|^2 \, dx \, dt. \]

Since \( a(x) \geq \epsilon_0 > 0 \) on \( \omega \), from (2.5), we deduce that
\[ \int_0^T \int_\omega |\partial_t v_n|^2 \, dx \, dt \longrightarrow n \rightarrow +\infty 0. \]

(2.6) combined with the result above, gives
\[ \int_0^T \int_\omega |\partial_t Z_n|^2 \, dx \, dt \longrightarrow n \rightarrow +\infty 0. \quad (2.7) \]

It is clear that\[ \sup_{[0,T]} E_{Z_n}(t) = E_{Z_n}(0) = E_{v_n}(0) \leq \frac{1}{\delta}. \]
Therefore, along a subsequence, \((Z_n)\) is convergent to a function
\[ L^2 \in C([0,T];H_D(\Omega)) \text{ and } \partial_t Z \in L^2 ([0,T];L^2(\Omega)), \]
with respect to the weak topology. Since \( Z \) satisfies
\[ \left\{ \begin{array}{ll}
\partial^2_t Z - \Delta Z = 0 & \text{in } [0,T] \times \Omega, \\
Z = 0 & \text{on } [0,T] \times \Gamma, \\
(Z_0, Z_1) \in H_D(\Omega) \times L^2(\Omega) \\
\partial_t Z(t,x) = 0 & \text{on } [0,T] \times \omega.
\end{array} \right. \]

Therefore we have
\[ Z \in C([0,T];H_D(\Omega)) \text{ and } \partial_t Z \in C([0,T];L^2(\Omega)), \]
We remind that \( \{x \in \mathbb{R}^d, |x| \geq L\} \subset \omega \). By a classical result of unique continuation, we obtain that \( \partial_t Z \equiv 0 \) on \( [0,T] \times \Omega \). This means that \( Z(t,x) = Z(x) \) is independent of \( t \). Therefore, we have
\[ \Delta Z = 0 \text{ and } Z \in H_D(\Omega), \]
we conclude from this that \( Z \equiv 0 \) on \( [0,T] \times \Omega \) (cf. theorem 2.2 p 145).

The sequence \((Z_n)\) is bounded in \( H^{1,loc}_\text{loc}([0,T] \times \Omega) \), so eventually after extracting a subsequence we can associate to the sequence \((Z_n)\) a microlocal defect measure \( \mu \). This measure satisfies these properties: The support of \( \mu \) is contained in the characteristic set of the wave operator and it propagates along the geodesics of \( \Omega \).

Now using (2.7), we deduce that
\[ \mu = 0, \text{ on } (0,T) \times \omega. \]

Since \((\omega,T)\) geometrically controls \( \Omega \) and \( \mu \) propagates along geodesic flow, therefore we obtain
\[ \mu = 0, \text{ on } (0,T) \times \Omega. \]

Let \( R > 0 \) such that the support of \( \chi \) is contained in \( B_R \) and \( \varphi \in C_0^\infty(0,T) \) such that
\[ \int_0^T \varphi(s) \, ds \geq \epsilon > 0. \]
Using the finite speed propagation property, we obtain
\[
\int_0^T \varphi (s) \int_{\Omega \cap B_{R+T}} |\nabla Z_n (s)|^2 + |\partial_t Z_n (s)|^2 \, dx \, ds \geq \epsilon \int_{\Omega \cap B_R} |\nabla Z_n (t)|^2 + |\partial_t Z_n (t)|^2 \, dx,
\]
for all \( t \in [0, T] \). Now passing to the limit and using the fact that \( \mu = 0 \), on \((0, T) \times \Omega\), we deduce that
\[
\int_{\Omega \cap B_R} |\nabla Z_n (t)|^2 + |\partial_t Z_n (t)|^2 \, dx \longrightarrow 0, \text{ for all } t \in [0, T].
\]
Using the result above and (2.6) we deduce that
\[
\int_{\Omega \cap B_R} |\nabla v_n (t)|^2 + |\partial_t v_n (t)|^2 \, dx \longrightarrow 0, \text{ for all } t \in [0, T].
\]
So we conclude that
\[
\int_{\Omega} \chi^2 (x) \left( |\nabla v_n (t)|^2 + |\partial_t v_n (t)|^2 \right) \, dx \longrightarrow 0, \text{ for all } t \in [0, T].
\]
The fact that the energy of \( v_n \) is decreasing, gives
\[
\int_{\Omega} \chi^2 (x) \left( |\nabla v_n (t)|^2 + |\partial_t v_n (t)|^2 \right) \, dx \leq C, \text{ for all } t \in [0, T].
\]
By the dominated convergence theorem we infer that
\[
1 = \int_0^T \int_{\Omega} \chi^2 (x) \left( |\nabla v_n (t)|^2 + |\partial_t v_n (t)|^2 \right) \, dx \, dt \longrightarrow 0.
\]
\( \square \)

As a corollary we have,

**Corollary 1.** We assume that Hyp A holds and \((\omega, T)\) geometrically controls \( \Omega \). Let \( \delta, R > 0 \). There exists \( C_{T, \delta, R} > 0 \), such that the following inequality
\[
\int_t^{t+T} \int_{\Omega \cap B_R} \left( |\nabla u|^2 + |\partial_t u|^2 \right) \, dx \, ds \leq C_{T, \delta, R} \left( \int_t^{t+T} \int_{\Omega} a (x) |\partial_t u|^2 \, dx \, ds \right) + \delta E_u (t), \quad (2.8)
\]
holds for every \( t \geq 0 \) and for all \( u \) solution of (1.1) with initial data \((u_0, u_1)\) in \( H \).

In order to prove theorem 1 we need the following result

**Lemma 1.** Let \( \psi \in C^\infty_0 (\mathbb{R}^d) \) such that \( 0 \leq \psi \leq 1 \) and
\[
\psi (x) = \begin{cases} 
1 & \text{for } |x| \leq L \\
0 & \text{for } |x| \geq 2L 
\end{cases}
\]
Setting \( w = \psi u \) and \( v = (1 - \psi) u \) where \( u \) is a solution of (1.1) with initial data in \( H^1_0 (\Omega) \times L^2 (\Omega) \). Let
\[
X (t) = \int_{\Omega} v (t) \partial_t v (t) \, dx + \frac{1}{2} \int_{\Omega} a (x) |v (t)|^2 \, dx + kE_u (t),
\]
where \( k \) is a positive constant. We have
\[
X(t+T) - X(t) + \int_t^{t+T} E_u(s) \, ds + \left( k - \frac{2}{\epsilon_0} \right) \int_t^{t+T} \int_\Omega a |\partial_t u|^2 \, dx \, ds
\]
\[
\leq 2 \int_t^{t+T} E_u(s) \, ds + \int_t^{t+T} \int_\Omega |\nabla \psi|^2 |u|^2 \, dx \, ds.
\] (2.9)

**Proof.** Noting that for each \((u_0, u_1)\) in \( H^1_0(\Omega) \times L^2(\Omega) \) the solution \( u \) of (1.1) are given as a limit of smooth solution \( u_n \) with initial data \((u_{n,0},u_{n,1})\) smooth such that \((u_{n,0},u_{n,1}) \xrightarrow{n \to +\infty} (u_0,u_1)\) in \( H^1_0(\Omega) \times L^2(\Omega) \). Note that
\[
\|u_n(t,\cdot) - u(t,\cdot)\|_{H^1} + \|\partial_t u_n(t,\cdot) - \partial_t u(t,\cdot)\|_{L^2} \xrightarrow{n \to +\infty} 0,
\]
uniformly on each closed interval [0, T] for any \( T > 0 \). Therefore we may assume that \( u \) is smooth.

We have \( v = (1 - \psi) u \). Then \( v \) is a solution of
\[
\begin{cases}
\partial_t^2 v - \Delta v + a(x) \partial_t v = f(t,x) & \text{in } \mathbb{R}^+ \times \Omega, \\
v = 0 & \text{in } \mathbb{R}^+ \times \Gamma, \\
(v(0), \partial_t v(0)) = (1 - \psi)(u_0,u_1),
\end{cases}
\] (2.10)

with
\[
f(t,x) = 2 \nabla \psi \nabla u + u \Delta \psi.
\]

Using the fact that \( v \) is a solution of (2.10) and that
\[
\frac{d}{dt} E_u(t) = - \int_\Omega a(x) |\partial_t u(t)|^2 \, dx,
\]
we deduce that
\[
\frac{d}{dt} X(t) = \int_\Omega |\partial_t v(t)|^2 - |\nabla v(t)|^2 \, dx - k \int_\Omega a(x) |\partial_t u(t)|^2 \, dx + \int_\Omega f(t,x) v \, dx.
\]

Since
\[
\int_\Omega f(t,x) v \, dx = \int_\Omega (2 \nabla \psi \nabla u + u \Delta \psi) (1 - \psi) u \, dx
\]
\[
= \int_\Omega \nabla \psi \nabla u^2 + u^2 \Delta \psi \, dx - \frac{1}{2} \int_\Omega \nabla \psi^2 \nabla u^2 + u^2 \psi \Delta \psi \, dx
\]
\[
= \int_\Omega |\nabla \psi|^2 |u|^2 \, dx.
\]

Thus we obtain
\[
\frac{d}{dt} X(t) = \int_\Omega |\partial_t v(t)|^2 - |\nabla v(t)|^2 \, dx - k \int_\Omega a(x) |\partial_t u(t)|^2 \, dx + \int_\Omega |\nabla \psi|^2 |u|^2 \, dx
\]
\[
= 2 \int_\Omega |\partial_t v(t)|^2 \, dx - 2 E_v(t) - k \int_\Omega a(x) |\partial_t u(t)|^2 \, dx + \int_\Omega |\nabla \psi|^2 |u|^2 \, dx
\]
\[
\leq 2 \int_\Omega |\partial_t v(t)|^2 \, dx - E_u(t) + 2 E_w(t) - k \int_\Omega a(x) |\partial_t u(t)|^2 \, dx + \int_\Omega |\nabla \psi|^2 |u|^2 \, dx.
\]

Using the fact that the support of \((1 - \psi)\) is contained in the set \( \{ x \in \Omega, a(x) > \epsilon_0 \} \), we infer that
\[
\int_\Omega |\partial_t v(t)|^2 \, dx = \int_\Omega |(1 - \psi) \partial_t u|^2 \, dx \leq \frac{1}{\epsilon_0} \int_\Omega a(x) |\partial_t u(t)|^2 \, dx.
\]
This gives
\[ \frac{d}{dt} X(t) + E_u(t) + \left( k - \frac{2}{\varepsilon_0} \right) \int_\Omega a(x) |\partial_t u(t)|^2 \, dx \leq 2E_w(t) + \int_\Omega |\nabla \psi|^2 |u|^2 \, dx. \] (2.11)
Integrating the estimate above between \( t \) and \( t + T \) we get (2.9).

2.1. Proof of Theorem 1. In the sequel \( C, C_T \) and \( C_{T,\delta} \) denote a generic positive constants and any changes from one derivation to the next will not be explicitly outlined.

Let \( \psi \in C_0^\infty(\mathbb{R}^d) \) such that \( 0 \leq \psi \leq 1 \) and
\[ \psi(x) = \begin{cases} 1 & \text{for } |x| \leq L \\ 0 & \text{for } |x| \geq 2L \end{cases} \]
Setting \( w = \psi u \) and \( v = (1 - \psi) u \) where \( u \) is a solution of (1.1) with initial data in \( H_0^1(\Omega) \times L^2(\Omega) \). Let
\[ X(t) = \int_\Omega v(t) \partial_t v(t) \, dx + \frac{1}{2} \int_\Omega a(x) |v(t)|^2 \, dx + kE_u(t). \]
According to lemma 1
\[ X(t + T) - X(t) + \int_t^{t+T} E_u(s) \, ds + \left( k - \frac{2}{\varepsilon_0} \right) \int_t^{t+T} \int_\Omega a |\partial_t u|^2 \, dxds \]
\[ \leq 2 \int_t^{t+T} E_w(s) \, ds + \int_t^{t+T} \int_\Omega |\nabla \psi|^2 |u|^2 \, dxds. \]
We have \( w = \psi u \). Then \( w \) is a solution of
\[ \begin{cases} \partial_t^2 w - \Delta w + a(x) \partial_t w = f(t,x) & \mathbb{R}_+ \times M, \\ w = 0 & \mathbb{R}_+ \times \partial M, \\ (w(0), \partial_t w(0)) = (\psi u_0, \psi u_1), \end{cases} \]
with \( M = \Omega \cap B_{2L} \) and
\[ f(t,x) = -2 \nabla \psi \nabla u - u \Delta \psi \in L^2_{\text{loc}}(\mathbb{R}_+, L^2(M)). \]
Since \( (\psi u_0, \psi u_1) \in H_0^1(M) \times L^2(M) \), using (2.11) we infer that the following inequality
\[ E_w(t) \leq C_T \left( \int_s^{s+T} \int_\Omega a |\partial_t w|^2 + |f(t,x)|^2 \, dx \, d\tau \right), \] (2.12)
holds for every \( t \geq 0 \). According to (6, proposition 2)
\[ E_w(s) \leq 2e^{s-t} \left( E_w(t) + \int_t^s \int_\Omega |f(t,x)|^2 \, dx \, d\tau \right) \]
\[ \leq 2e^T \left( E_w(t) + \int_t^{t+T} \int_\Omega |f(t,x)|^2 \, dx \, d\tau \right), \text{ for } t \leq s \leq t + T. \]
The estimate above combined with (2.12) gives
\[ \int_t^{t+T} E_w(s) \, ds \leq \bar{C}_T \left( \int_t^{t+T} \int_\Omega a |\partial_t w|^2 + |f(t,x)|^2 \, dx \, d\tau \right), \]
since
\[ \int_t^{t+T} \int_\Omega a |\partial_t w|^2 \, dx \, d\tau \leq \int_t^{t+T} \int_\Omega a |\partial_t u|^2 \, dx \, d\tau, \]
we obtain

$$\int_t^{t+T} E_u (s) \, ds \leq \tilde{C}_T \left( \int_t^{t+T} \int_\Omega a |\partial_t u|^2 + |f (\tau, x)|^2 \, d\tau d\tau \right),$$

On the other hand, using Poincare’s inequality we obtain

$$\int_t^{t+T} \int_\Omega |f (\tau, x)|^2 \, d\tau d\tau \leq C_L \int_t^{t+T} \int_{\Omega \cap B_{2L}} |\nabla u|^2 \, dx ds,$$

for some $C_L > 0$. The estimate (2.8), gives

$$\int_t^{t+T} E_u (s) \, ds \leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a |\partial_t u|^2 \, d\tau d\tau \right) + C_L \tilde{C}_T \delta E_u (t),$$

since

$$E_u (t) \leq \int_t^{t+T} \int_\Omega a |\partial_t u|^2 \, d\tau d\tau + \frac{1}{T} \int_t^{t+T} E_u (s) \, ds,$$

we deduce that

$$\int_t^{t+T} E_u (s) \, ds \leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a |\partial_t u|^2 \, d\tau d\tau \right) + C_T \delta \int_t^{t+T} E_u (s) \, ds.$$  (2.13)

Using (2.8) and (2.13), we infer that

$$\int_t^{t+T} \int_\Omega |\nabla \psi|^2 |u|^2 \, dx ds \leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a |\partial_t u|^2 \, d\tau d\tau \right) + C_T \delta \int_t^{t+T} E_u (s) \, ds.$$  

Now (2.9) and the estimates above gives

$$X (t + T) - X (t) + (1 - 3C_T \delta) \int_t^{t+T} E_u (s) \, ds + \left( k - \frac{2}{\epsilon_0} - 3C_T \delta \right) \int_t^{t+T} \int_\Omega a |\partial_t u|^2 \, dx ds \leq 0.$$  

We have

$$X (t) \leq \frac{3}{2} \|a\|_{\infty} \int_\Omega |v (t)|^2 \, dx + \left( k + \frac{2}{\epsilon_0} \right) E_u (t) \quad \text{and}$$

$$X (t) \geq \frac{\epsilon_0}{4} \int_\Omega |v (t)|^2 \, dx + \left( k - \frac{8}{\epsilon_0} \right) E_u (t).$$  (2.14)

We choose $\delta$ and $k$ such that

$$1 - 3C_T \delta = \frac{1}{2},$$

$$k - \frac{2}{\epsilon_0} - 3C_T \delta \geq \epsilon > 0 \quad \text{and} \quad k - \frac{8}{\epsilon_0} \geq \epsilon,$$

therefore we obtain

$$X (t + T) - X (t) + \frac{1}{2} \int_t^{t+T} E_u (s) \, ds + \epsilon \int_t^{t+T} \int_\Omega a |\partial_t u|^2 \, dx ds \leq 0,$$  (2.15)

this gives

$$X (nT) + \frac{1}{2} \int_0^{nT} E_u (s) \, ds \leq X (0), \quad \text{for all } n \in \mathbb{N}.$$  

So there exists a positive constant $C$ such that

$$\sup_{\mathbb{R}^+} \int_0^{t+\infty} E_u (s) \, ds \leq CX (0) \leq CI_0,$$  (2.16)
with

\[ I_0 = \|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2. \]

Since \(1 - \psi \equiv 1\) for \(|x| \geq 2L\) and

\[ X(t) \geq \frac{\epsilon_0}{4} \int_{\Omega} |v(t)|^2 \, dx \geq \frac{\epsilon_0}{4} \int_{\{|x| \geq 2L\}} |u(t)|^2 \, dx, \]

therefore using (2.16) we obtain

\[ \sup_{\mathbb{R}^+} \int_{\{|x| \geq 2L\}} |u(t)|^2 \, dx \leq \frac{4C}{\epsilon_0} I_0. \]

Poincaré’s inequality and the fact that the energy of \(u\) is decreasing gives

\[ \int_{\Omega \cap B_{2L}} |u(t)|^2 \, dx \leq C_L \int_{\Omega} |\nabla u(t)|^2 \, dx \leq C_L E_u(0). \]

Combining the last two estimates, we get

\[ \sup_{\mathbb{R}^+} \int_{\Omega} |u(t)|^2 \, dx \leq \frac{4C}{\epsilon_0} I_0 + C_L E_u(0) \leq C I_0. \]

The energy decay estimate follows from (2.16) and the fact that

\[ (1 + t) E_u(t) \leq E_u(0) + \int_0^{+\infty} E_u(s) \, ds \leq C I_0, \text{ for all } t \geq 0. \]  
(2.17)

This finishes the proof of theorem 1, now we give the proof of proposition 1.

2.2. Proof of proposition 1. Let \(n \in \mathbb{N}^*\) and \(u\) solution of (1.1) with initial data \((u_0, u_1)\) in \(D(A^n)\) such that \(u_0 \in L^2(\Omega)\). We set \(u_n = \partial^n_t u\). First we prove

\[ \int_0^{+\infty} (1 + s)^n E_{u_n}(s) \, ds \leq C_n I_{0,n}, \text{ for all } n \in \mathbb{N} \]  
(2.18)

where

\[ I_{0,n} = \sum_{i=0}^{n} \left\| A^i \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right) \right\|_{H^1}^2 + \|u_0\|_{L^2}^2. \]

Let \(u\) be a solution of (1.1) with initial data \((u_0, u_1)\) in \(D(A^0)\) such that \(u_0 \in L^2(\Omega)\). From (2.16) we infer that

\[ \int_0^{+\infty} E_u(s) \, ds \leq C_0 I_0. \]

We assume that the following estimate

\[ \int_0^{+\infty} (1 + s)^p E_{u_p}(s) \, ds \leq C_p I_{0,p}, \]  
(2.19)

holds, for all solution \(u\) of (1.1) with initial data \((u_0, u_1)\) in \(D(A^p)\) such that \(u_0 \in L^2(\Omega)\). Let \(u\) be a solution of (1.1) with initial data \((u_0, u_1)\) in \(D(A^{p+1})\) such that \(u_0 \in L^2(\Omega)\).
We have \( u_{p+1} = \partial_t^p (\partial_t u) \). Since \( (\partial_t u (0), \partial_t^2 u (0)) = A \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right) \in D (A^p) \) and \( \partial_t u (0) = u_1 \in L^2 (\Omega) \). According to (2.19), we have

\[
\int_0^{+\infty} (1 + s)^p E_{u_{p+1}} (s) \, ds \leq C_p \left( \sum_{i=0}^p \left\| A^{i+1} \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right) \right\|_H^2 + \| u_1 \|_L^2 \right)
\leq C_p I_{0,p+1}.
\]  

(2.20)

Let \( \psi \in C_0^\infty (\mathbb{R}^d) \) such that \( 0 \leq \psi \leq 1 \) and

\[
\psi (x) = \begin{cases} 
1 & \text{for } |x| \leq L \\
0 & \text{for } |x| \geq 2L 
\end{cases}
\]

Setting \( w = \psi u_{p+1} \) and \( v = (1 - \psi) u_{p+1} \). Let

\[
X (t) = \int_\Omega v (t) \partial_t v (t) \, dx + \frac{1}{2} \int_\Omega a (x) |v (t)|^2 \, dx + k E_{u_{p+1}} (t),
\]

where \( k \) is a positive constant. \( u_{p+1} \) satisfies

\[
\begin{cases}
\partial_t^2 u_{p+1} - \Delta u_{p+1} + a (x) \partial_t u_{p+1} = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
u_{p+1} = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\
(u_{p+1} (0, x), \partial_t u_{p+1} (0, x)) \in H^1_0 (\Omega) \times L^2 (\Omega),
\end{cases}
\]

Then we know from (2.15) that

\[
X (t + T) - X (t) + \frac{1}{2} \int_t^{t+T} E_{u_{p+1}} (s) \, ds + \epsilon \int_t^{t+T} \int_\Omega a \partial_t u_{p+1}^2 \, dx \, ds \leq 0. 
\]  

(2.21)

Multiplying the estimate above by \((1 + t + T)^{p+1}\), we obtain

\[
(1 + t + T)^{p+1} X (t + T) - (1 + t)^{p+1} X (t) + \frac{1}{2} \int_t^{t+T} (1 + s)^{p+1} E_{u_{p+1}} (s) \, ds \leq C_T (1 + t)^p X (t).
\]

Therefore using (2.19), (2.20) and the fact that

\[
X (t) \leq \frac{3}{2} \| a \|_\infty E_{u_p} (t) + \left( k + \frac{2}{\epsilon_0} \right) E_{u_{p+1}} (t),
\]
we deduce that for any $q \in \mathbb{N}^*$

$$
\frac{1}{2} \int_0^{qT} (1 + s)^{p+1} E_{u_{p+1}} (s) \, ds \leq CT\sum_{i=0}^{q-1} (1 + iT)^p X (iT)
$$

$$
\leq CT\sum_{i=0}^{q-1} (1 + iT)^p \left( E_{u_{p+1}} (iT) + E_{u_p} (iT) \right)
$$

$$
\leq CT\sum_{i=0}^{q-1} \int_{iT}^{(i+1)T} (1 + s)^p \left( E_{u_{p+1}} (s) + E_{u_p} (s) \right) \, ds
$$

$$
+ C_T \left( E_{u_{p+1}} (0) + E_{u_p} (0) \right)
$$

$$
\leq CT \int_0^{+\infty} (1 + s)^p \left( E_{u_{p+1}} (s) + E_{u_p} (s) \right) \, ds
$$

$$
+ C_T \left( E_{u_{p+1}} (0) + E_{u_p} (0) \right)
$$

$$
\leq CT_p I_{0,p+1}.
$$

We deduce that

$$
\int_0^{+\infty} (1 + s)^{p+1} E_{u_{p+1}} (s) \, ds \leq C_{p+1} I_{0,p+1}
$$

We remind that we have proved that

$$
\int_0^{+\infty} (1 + s)^n E_{u_n} (s) \, ds \leq C_n I_{0,n}.
$$

Now the energy decay estimate follows from the fact that

$$(1 + t)^{n+1} E_{u_n} (t) \leq E_{u_n} (0) + (n + 1) \int_0^t (1 + s)^n E_{u_n} (s) \, ds$$

$$\leq C_n I_{0,n},$$

for all $t \geq 0$. Now using the estimate above, we infer that,

$$(1 + t)^n \| \partial^n_t u (t) \|^2_{L^2} \leq 2 (1 + t)^n E_{u_{n-1}} (t)$$

$$\leq C_1 I_{0,n-1}$$

for all $t \geq 0$.

We have $\partial^{n-1}_t u$ is a solution of the following system

$$
\begin{cases}
\partial^{n+1}_t u - \Delta \partial^{n-1}_t u + a (x) \partial^n_t u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
\partial^{n-1}_t u = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\
(\partial^{n-1}_t u (0, x), \partial^n_t u (0, x)) \in D (A)
\end{cases}
$$

(2.22)

therefore

$$(\partial^{n-1}_t u, \partial^n_t u) \in C^1 (\mathbb{R}_+, H).$$

Using Eq (2.22), we infer that

$$(1 + t)^n \| \Delta \partial^{n-1}_t u (t) \|^2_{L^2} \leq C (1 + t)^n (E_{u_n} (t) + E_{u_{n-1}} (t))$$

$$\leq C I_{0,n}.$$
3. Proof of Theorem 2

This section is devoted to the proof of theorem 2 and proposition 2, we begin by giving some preliminary results.

Proposition 5. We assume that Hyp A holds and \((\omega, T)\) geometrically controls \(\Omega\). Let \(\delta > 0\) and \(\chi \in C_0^\infty (\mathbb{R}^d)\). There exists \(C_{T, \delta, \chi} > 0\), such that the following inequality

\[
\int_t^{t+T} \int_\Omega \chi^2 (x) (1 + s) |\nabla u|^2 dxds \leq C_{T, \delta, \chi} \left( \int_t^{t+T} \int_\Omega a (x) (1 + s) |\partial u|^2 dxds \right) + \delta \left( (1 + t) E_u(t) + \int_t^{t+T} \int_\Omega |u|^2 dxds \right).
\]

holds for every \(t \geq 0\) and for all \(u\) solution of (1.1) with initial data \((u_0, u_1)\) in \(H_0^1 \times L^2\).

Proof. To prove this result we argue by contradiction: If (3.1) was false, there would exist a sequence of numbers \((t_n)\) and a sequence of solutions \((u_n)\) such that

\[
\int_{t_n}^{t_n+T} \int_\Omega \chi^2 (x) (1 + s) |\nabla u_n|^2 dxds \geq n \left( \int_{t_n}^{t_n+T} \int_\Omega a (x) (1 + s) |\partial u_n|^2 dxds \right)
+ \delta \left( (1 + t_n) E_{u_n}(t_n) + \int_{t_n}^{t_n+T} \int_\Omega |u_n|^2 dxds \right).
\]

We may assume that \(t_n \to +\infty\) (if the sequence \(t_n\) is bounded we can argue as in the proof of proposition 4). Setting

\[
\lambda_n^2 = \int_{t_n}^{t_n+T} \int_\Omega \chi^2 (x) (1 + s) |\nabla u_n|^2 dxds \quad \text{and} \quad v_n = \frac{(1 + t_n + \cdot)^{1/2}}{\lambda_n} u_n(t_n + \cdot).
\]

We have

\[
\frac{1}{\lambda_n^2} (1 + t_n) E_{u_n}(t_n) \leq \frac{1}{\delta} \quad \text{and} \quad \frac{1}{\lambda_n^2} \int_0^T \int_\Omega |u_n(t_n + s)|^2 dxds \leq \frac{1}{\delta}, \tag{3.2}
\]

moreover

\[
\frac{1}{\lambda_n^2} \int_{t_n}^{t_n+T} \int_\Omega a (x) (1 + s) |\partial u_n|^2 dxds \to 0 \quad \text{as} \quad n \to +\infty, \tag{3.3}
\]

\(v_n\) is a solution of the following system

\[
\begin{dcases}
\partial_t^2 v_n - \Delta v_n + a \partial_t v_n = f_n(t, x) & \text{in} \ \mathbb{R}_+ \times \Omega, \\
v_n(t, x) = 0 & \text{on} \ \mathbb{R}_+ \times \Gamma, \\
(v_n(0, 0), v_n(0, 1)) \in H_0^1(\Omega) \times L^2(\Omega),
\end{dcases}
\]

with

\[
f_n(t, x) = \frac{1}{2\lambda_n} (1 + t_n + t)^{-1/2} \left( a(x) - \frac{1}{2} (1 + t_n + t)^{-1} \right) u_n(t_n + t) + \frac{1}{\lambda_n} (1 + t_n + t)^{-1/2} \partial_t u_n(t_n + t).
\]

It is clear that (3.2), gives

\[
\int_0^T \int_\Omega \frac{1}{2\lambda_n} (1 + t_n + t)^{-1/2} \left( a(x) - \frac{1}{2} (1 + t_n + t)^{-1} \right) u_n(t_n + t)^2 dxdt
\leq C (1 + t_n)^{-1} \frac{1}{\lambda_n^2} \int_{t_n}^{t_n+T} \int_\Omega |u_n(s)|^2 dxds
\leq C \frac{(1 + t_n)^{-1}}{\delta}
\]

\]
and
\[ \int_0^T \int_\Omega \left| \frac{1}{\lambda_n} (1 + t_n + t)^{-\frac{1}{2}} \partial_t u_n (t_n + t) \right|^2 \, dx \, dt \leq \frac{(1 + t_n)^{-1}}{\lambda_n^2} \int_0^T \int_\Omega |\partial_t u_n (t_n + t)|^2 \, dx \, dt \]
\[ \leq \frac{(1 + t_n)^{-1}}{\lambda_n^2} \cdot 2 T E_{u_n} (t_n) \]
\[ \leq \frac{2 T (1 + t_n)^{-2}}{\delta} . \]

We conclude that
\[ \int_0^T \int_\Omega |f_n (s, x)|^2 \, dx \, ds \leq C \frac{(1 + t_n)^{-1}}{\delta} \rightarrow 0. \quad (3.4) \]

(3.2) gives
\[ \int_0^T \int_\Omega a (x) |\partial_t v_n|^2 \, dx \, dt \leq \|a\|_\infty (1 + t_n)^{-1} \frac{1}{\lambda_n^2} \int_{t_n}^{t_n+T} \int_\Omega |u_n (s)|^2 \, dx \, ds \]
\[ + \frac{2}{\lambda_n^2} \int_{t_n}^{t_n+T} \int_\Omega a (x) (1 + s) |\partial_t u_n|^2 \, dx \, ds \]
\[ \leq \frac{\|a\|_\infty (1 + t_n)^{-1}}{\delta} + \frac{2}{\lambda_n^2} \int_{t_n}^{t_n+T} \int_\Omega a (x) (1 + s) |\partial_t u_n|^2 \, dx \, ds. \]

Now using (3.3), we deduce that
\[ \int_0^T \int_\Omega a (x) |\partial_t v_n|^2 \, dx \, dt \rightarrow 0. \quad (3.5) \]

We multiply the equation satisfied by \( u_n \) by \( (1 + t) \partial_t u_n \) and integrating between \( t_n \) and \( t_n + t \), we obtain
\[ (1 + t_n + t) E_{u_n} (t_n + t) - (1 + t_n) E_{u_n} (t_n) = \int_{t_n}^{t_n+T} E_{u_n} (s) \, ds - \int_{t_n}^{t_n+T} \int_\Omega a (x) (1 + s) |\partial_t u_n|^2 \, dx \, ds \]
thus by using (3.2), we infer that
\[ \frac{1}{\lambda_n^2} (1 + t_n + t) E_{u_n} (t_n + t) \leq \frac{1}{\lambda_n^2} (1 + t_n) E_{u_n} (t_n) + \frac{1}{\lambda_n^2} \int_{t_n}^{t_n+T} E_{u_n} (s) \, ds \]
\[ \leq \frac{1}{\delta} + \frac{T}{\delta} \quad \text{for all } t \in [0, T] . \]

On the other hand, we have
\[ \int_0^T \int_\Omega E_{v_n} (t) \, dt \leq \frac{1}{\lambda_n^2} \int_0^T \left[ (1 + t_n + t) E_{u_n} (t_n + t) + (1 + t_n + t)^{-1} \int_\Omega |u_n (t_n + t)|^2 \, dx \right] \, dt \]
\[ \leq \frac{C_T}{\delta} . \]

Let \( Z_n \) be the solution of the following system
\[
\begin{align*}
\partial_t^2 Z_n - \Delta Z_n &= 0 & \quad & \mathbb{R}_+ \times \Omega, \\
Z_n &= 0 & \quad & \mathbb{R}_+ \times \Gamma, \\
(Z_n (0), \partial_t Z_n (0)) &= (v_n (0), \partial_t v_n (0)) .
\end{align*}
\]
The hyperbolic energy inequality gives
\[ \sup_{[0,T]} E_{\psi u_n} (s) \leq C_T \| a (x) \partial_t v_n + f_n (t, x) \|_{L^2([0,T], L^2(\Omega))}^2 \]

Now using (3.5) and (3.4), we deduce that
\[ \sup_{[0,T]} E_{\psi u_n} (s) \to 0. \]

Using the estimate above, we obtain
\[ T E_{\psi u_n} (0) = T E_{\psi u} (0) = \int_0^T E_{\psi u} (t) \, dt \leq 2T \sup_{[0,T]} E_{\psi u_n} (s) + 2 \int_0^T E_{\psi u} (t) \, dt \leq C_{T, \delta} \]

this gives
\[ \sup_{[0,T]} E_{\psi u} (s) = E_{\psi u} (0) \leq C_{T, \delta}. \]

Using the result above and (6 proposition 2), we infer that
\[ E_{\psi u} (t) \leq 2e^T \left( E_{\psi u} (0) + \int_0^T \int_{\Omega} | f_n (\tau, x) |^2 \, dx \, d\tau \right) \text{ for } 0 \leq t \leq T \leq C_{T, \delta}, \text{ for all } 0 \leq t \leq T. \]

To complete the proof we have only to argue as in the proof of the proposition 4. \hfill \Box

As a corollary we have

**Corollary 2.** We assume that Hyp A holds and \((\omega, T)\) geometrically controls \(\Omega\). Let \(\delta, R > 0\). There exists \(C_{T, \delta, R} > 0\), such that the following inequality
\[ \int_t^{t+T} \int_{\Omega \cap BR} (1 + s) \, | \nabla u |^2 \, dx \, ds \leq C_{T, \delta, R} \left( \int_t^{t+T} \int_{\Omega} a (x) (1 + s) \, | \partial_t u |^2 \, dx \, ds \right) + \delta \left( (1 + t) \, E_u (t) + \int_t^{t+T} \int_{\Omega} | u |^2 \, dx \, ds \right). \tag{3.6} \]

holds for every \(t \geq 0\) and for all \(u\) solution of (1.1) with initial data \((u_0, u_1)\) in \(H\).

In the sequel we need the following result

**Lemma 2.** Let \(\psi \in C_0^\infty (\mathbb{R}^d)\) such that \(0 \leq \psi \leq 1\) and
\[ \psi (x) = \begin{cases} 1 & \text{for } |x| \leq L \\ 0 & \text{for } |x| \geq 2L \end{cases} \]

Setting \(w = \psi u\) and \(v = (1 - \psi) u\) where \(u\) is a solution of (1.1) with initial data in \(H_0^1 (\Omega) \times L^2 (\Omega)\). Let
\[ X (t) = \int_{\Omega} v (t) \, \partial_t v (t) \, dx + \frac{1}{2} \int_{\Omega} a (x) \, | v (t) |^2 \, dx + k E_u (t). \]

where \(k\) is a positive constant. We have
\[ (1 + t + T) \, X (t + T) - (1 + t) \, X (t) + \int_t^{t+T} (1 + s) \, E_u (s) \, ds + \left( k - \frac{2}{c_0} \right) \int_t^{t+T} \int_{\Omega} a (1 + s) \, | \partial_t u |^2 \, dx \, ds \leq 2 \int_t^{t+T} (1 + s) \, E_w (s) \, ds + \int_t^{t+T} \int_{\Omega} \, | \nabla \psi |^2 \, | u |^2 \, dx \, ds + \int_t^{t+T} X (s) \, ds. \tag{3.7} \]
**Proof.** We may assume that \( u \) is smooth. According to (2.11)

\[
\frac{d}{dt}X(t) + E_u(t) + \left(k - \frac{2}{\epsilon_0}\right) \int_{\Omega} a(x)|\partial_t u(t)|^2 \, dx \leq 2E_w(t) + \int_{\Omega} |\nabla \psi|^2 |u|^2 \, dx.
\]

We multiply the estimate above by \(1 + t\), we obtain

\[
\frac{d}{dt}(1 + t)X(t) + (1 + t)E_u(t) + \left(k - \frac{2}{\epsilon_0}\right) \int_{\Omega} a(x)(1 + t)|\partial_t u(t)|^2 \, dx \\
\leq 2(1 + t)E_w(t) + (1 + t) \int_{\Omega} |\nabla \psi|^2 |u|^2 \, dx + X(t).
\]

We integrate the inequality above between \( t \) and \( t + T \), we obtain \((3.7)\). \( \square \)

### 3.1. Proof of Theorem 2

In the sequel \( C, C_T \) and \( C_{T,\delta} \) denote a generic positive constants and any changes from one derivation to the next will not be explicitly outlined.

Let \( \psi \in C_0^\infty(\mathbb{R}^d) \) such that \( 0 \leq \psi \leq 1 \) and

\[
\psi(x) = \begin{cases} 
1 & \text{for } |x| \leq L \\
0 & \text{for } |x| \geq 2L
\end{cases}
\]

Setting \( w = \psi u \) and \( v = (1 - \psi) u \) where \( u \) is a solution of (1.1) with initial data in \( H_0^1(\Omega) \times L^2(\Omega) \) such that

\[
\|d(\cdot)(u_1 + au_0)\|_{L^2} < +\infty.
\]

Let

\[
X(t) = \int_{\Omega} v(t) \partial_t v(t) \, dx + \frac{1}{2} \int_{\Omega} a(x)|v(t)|^2 \, dx + kE_u(t),
\]

where \( k \) is a positive constant. According to (3.7) we have

\[
(1 + t + T)X(t + T) - (1 + t)X(t) + \int_t^{t+T}(1 + s)E_u(s) \, ds \\
+ \left(k - \frac{2}{\epsilon_0}\right) \int_t^{t+T} \int_{\Omega} a(1 + s)|\partial_t u|^2 \, dxds \\
\leq 2 \int_t^{t+T}(1 + s)E_w(s) \, ds + \int_t^{t+T} \int_{\Omega}(1 + s)|\nabla \psi|^2 |u|^2 \, dxds + \int_t^{t+T} X(s) \, ds.
\]

Let \( w_1 = (1 + t)^{\frac{1}{2}}w \). Then \( w_1 \) is a solution of

\[
\begin{cases}
\partial_t^2 w_1 - \Delta w_1 + a(x)\partial_t w_1 = \frac{1}{2}(1 + t)^{-\frac{3}{2}} \left(a(x) - \frac{1}{2}(1 + t)^{-1}\right)w + f(t, x) & \mathbb{R}_+ \times M, \\
\partial_t w_1 = 0 & \mathbb{R}_+ \times \partial M, \\
w_1(0) = (\psi u_0, \psi u_1),
\end{cases}
\]

with \( M = \Omega \cap B_{2L} \) and

\[
f(t, x) = (1 + t)^{-\frac{3}{2}} \partial_t w - (1 + t)^{\frac{1}{2}} \left(2\nabla \psi \nabla u + u \Delta \psi\right).
\]

We have

\[
\partial_t w_1 = \frac{1}{2}(1 + t)^{-\frac{1}{2}}w + (1 + t)^{\frac{1}{2}} \partial_t w
\]

therefore, using Poincare’s inequality we conclude that there exists a positive constant \( c \) such that

\[
(1 + t)E_w(t) \leq cE_{w_1}(t), \text{ for all } t \geq 0.
\]

(3.8)
Moreover we have
\[
\int_t^{t+T} \int_\Omega a \left| \partial_t w \right|^2 \, dx \, ds \leq (1 + \|a\|_\infty) \int_t^{t+T} \int_\Omega a (1 + s) \left| \partial_t w \right|^2 + |w|^2 \, dx \, ds.
\]

\((\psi u_0, \psi u_1) \in H_0^1 (M) \times L^2 (M)\) and
\[
\frac{1}{2} (1 + t)^{-\frac{1}{2}} \left( a (x) - \frac{1}{2} (1 + t)^{-1} \right) w + f (t, x) \in L^2_{loc} (\mathbb{R}_+, L^2 (M)).
\]

Therefore using (2.1), we infer that
\[
E_w (t) \leq C_T \left( \int_t^{t+T} \int_\Omega a (1 + s) \left| \partial_t w \right|^2 + |f (s, x)|^2 \, dx \, ds \right)
\]
\[
\leq C_T \left( \int_t^{t+T} \int_\Omega a (1 + s) \left| \partial_t w \right|^2 + |f (s, x)|^2 + |w|^2 \, dx \, ds \right),
\]
holds for every \( t \geq 0 \). We have (cf, [6, proposition 2])
\[
E_w (s) \leq 4 e^{s-t} \left( E_w (t) + \int_t^s \int_\Omega |f (s, x)|^2 + |w|^2 \, dx \, ds \right)
\]
\[
\leq 4 e^{(t+T-s)} \left( E_w (t) + \int_t^{t+T} \int_\Omega |f (s, x)|^2 + |w|^2 \, dx \, ds \right), \text{ for } t \leq s \leq t + T.
\]

Using the estimate above, (3.8) and (3.9), we get
\[
\int_t^{t+T} (1 + s) E_w (s) \, ds \leq \tilde{C}_T \left( \int_t^{t+T} \int_\Omega a (1 + s) \left| \partial_t w \right|^2 + |f (s, x)|^2 + |w|^2 \, dx \, ds \right).
\]

Which yields to
\[
\int_t^{t+T} (1 + s) E_w (s) \, ds \leq \tilde{C}_T \left( \int_t^{t+T} \int_\Omega a (1 + s) \left| \partial_t u \right|^2 + |f (s, x)|^2 + |w|^2 \, dx \, ds \right).
\]

Now we estimate the second and the third term of the RHS of the estimate above. For second term it is clear that
\[
\int_t^{t+T} \int_\Omega |f (s, x)|^2 \, dx \, ds \leq C_L \int_t^{t+T} \int_{\Omega \cap B_{2L}} (1 + s) |\nabla u|^2 + (1 + s)^{-1} |\partial_t u|^2 \, dx \, ds,
\]
for some \( C_L > 0 \). Using (3.6), we infer that
\[
\int_t^{t+T} \int_{\Omega \cap B_{2L}} (1 + s) |\nabla u|^2 \, dx \, ds \leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a (1 + s) |\partial_t u|^2 \, dx \, ds \right)
\]
\[
+ C_T \delta \left( \int_t^{t+T} \int_\Omega |w|^2 \, dx \, ds + (1 + t) E_u (t) \right).
\]

For the second term of the RHS of (3.11), we use (2.8)
\[
\int_t^{t+T} \int_{\Omega \cap B_{2L}} (1 + s)^{-1} |\partial_t u|^2 \, dx \\
\leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a (x) |\partial_t u|^2 \, dx \, ds \right) + \delta E_u (t)
\]
\[
\leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a (1 + s) |\partial_t u|^2 \, dx \, ds \right) + C_T \delta \left( \int_t^{t+T} \int_\Omega |w|^2 \, dx \, ds + (1 + t) E_u (t) \right),
\]
since
\[
(1 + t) E_u (t) \leq \int_t^{t+T} \int_\Omega a (1 + s) |\partial_t u|^2 \, dx \, ds + \frac{1}{T} \int_t^{t+T} (1 + s) E_u (s) \, ds,
\]
then
\[
\int_t^{t+T} \int_\Omega |f(s, x)|^2 \, dx \, ds \leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a(1+s) |\partial_t u|^2 \, dx \, ds \right) + C_T \delta \left( \int_t^{t+T} \int_\Omega |u|^2 \, dx \, ds + \int_t^{t+T} (1+s) E_u(s) \, ds \right).
\] (3.12)

To estimate the third term of the RHS of (3.10) we use (2.8) and we obtain
\[
\int_t^{t+T} \int_\Omega |u|^2 \, dx \, ds \\
\leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a(x) |\partial_t u|^2 \, dx \, ds \right) + \delta E_u(t) \]
\[
\leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a(1+s) |\partial_t u|^2 \, dx \, ds \right) + C_T \delta \left( \int_t^{t+T} \int_\Omega |u|^2 \, dx \, ds + \int_t^{t+T} (1+s) E_u(s) \, ds \right).
\] (3.13)

Combining (3.12) and (3.13), we get
\[
\int_t^{t+T} (1+s) E_u(s) \, ds \\
\leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a(1+s) |\partial_t u|^2 \, dx \, ds \right) + C_T \delta \left( \int_t^{t+T} \int_\Omega |u|^2 \, dx \, ds + \int_t^{t+T} (1+s) E_u(s) \, ds \right).
\]

On the other hand, using (3.6) we infer that
\[
\int_t^{t+T} \int_\Omega (1+s) |\nabla \psi|^2 |u|^2 \, dx \, ds \\
\leq C_{T, \delta} \left( \int_t^{t+T} \int_\Omega a(1+s) |\partial_t u|^2 \, dx \, ds \right) + C_T \delta \left( \int_t^{t+T} \int_\Omega |u|^2 \, dx \, ds + \int_t^{t+T} (1+s) E_u(s) \, ds \right).
\]

Now (3.7) and the two estimates above gives
\[
(1+t+T) X(t+T) - (1+t) X(t) + (1-3C_T \delta) \int_t^{t+T} (1+s) E_u(s) \, ds \\
+ \left( k - \frac{\alpha}{\epsilon_0} - 3C_{T, \delta} \right) \int_t^{t+T} \int_\Omega a(1+s) |\partial_t u|^2 \, dx \, ds \\
\leq \int_t^{t+T} X(s) \, ds + C_T \delta \int_t^{t+T} \int_\Omega |u|^2 \, dx \, ds.
\]

We have
\[
X(t) \leq \frac{3}{2} \|a\|_\infty \int_\Omega |v(t)|^2 \, dx + \left( k + \frac{2}{\epsilon_0} \right) E_u(t) \quad \text{and} \quad (3.14)
\]
\[
X(t) \geq \frac{\epsilon_0}{4} \int_\Omega |v(t)|^2 \, dx + \left( k - \frac{8}{\epsilon_0} \right) E_u(t). \quad (3.15)
\]

We choose \( \delta \) and \( k \) such that
\[
1 - 3C_T \delta = \frac{1}{2},
\]
\[
k - \frac{2}{\epsilon_0} - 3C_{T, \delta} \geq \epsilon > 0 \quad \text{and} \quad k - \frac{8}{\epsilon_0} \geq \epsilon.
\]

Thus we get
\[
(1+t+T) X(t+T) - (1+t) X(t) + \frac{1}{2} \int_t^{t+T} (1+s) E_u(s) \, ds \\
+ \epsilon \int_t^{t+T} \int_\Omega a(1+s) |\partial_t u|^2 \, dx \, ds \\
\leq \int_t^{t+T} X(s) \, ds + \frac{1}{6} \int_t^{t+T} \int_\Omega |u|^2 \, dx \, ds.
\] (3.16)
As in the proof of theorem 1, from the estimate above, we deduce that
\[
\sup_{\mathbb{R}^+} (1 + s) X(t) + \frac{1}{2} \int_0^{+\infty} (1 + s) E_u(s) \, ds \\
\leq C \left( X(0) + \int_0^{+\infty} X(s) \, ds + \int_0^{+\infty} \int_\Omega |u|^2 \, dx \, ds \right).
\] (3.17)

Using the fact that \(1 - \psi \equiv 1\) for \(|x| \geq 2L\), we obtain
\[
\sup_{\mathbb{R}^+} (1 + t) \int_{\{|x| \geq 2L\}} |u(t)|^2 \, dx \leq C \left( X(0) + \int_0^{+\infty} X(s) \, ds + \int_0^{+\infty} \int_\Omega |u|^2 \, dx \, ds \right).
\]

On the other hand, using (2.17), we obtain
\[
(1 + t) \int_{\Omega \cap B_{2L}} |u(t)|^2 \, dx \leq C_L (1 + t) \int_\Omega |\nabla u(t)|^2 \, dx \\
\leq C_L (1 + t) E_u(t) \\
\leq CI_0.
\]

Combining the estimates above we deduce that
\[
\sup_{\mathbb{R}^+} (1 + t) \int_\Omega |u(t)|^2 \, dx \leq C \left( X(0) + \int_0^{+\infty} X(s) \, ds + \int_0^{+\infty} \int_\Omega |u|^2 \, dx \, ds + I_0 \right).
\] (3.18)

Now we need the following result due to Ikehata [9, lemma 2.5]

**Lemma 3.** Let \(u\) be a solution of (1.4) with initial data \((u_0, u_1)\) in \(H_0^1(\Omega) \times L^2(\Omega)\) which satisfies
\[
\|d(\cdot) (u_1 + au_0)\|_{L^2} < +\infty
\]
where
\[
d(x) = \begin{cases} 
|x| & d \geq 3 \\
|x| \ln (B|x|) & d = 2
\end{cases}
\]
with \(B \inf_{x \in \Omega} |x| \geq 2\). Then there exists \(C > 0\), such that
\[
\|u(t)|^2|_{L^2} + \int_0^t \int_\Omega a |u(s, x)|^2 \, dx \, ds \leq C \left( \|u_0|^2_{L^2} + \|d(\cdot) (u_1 + au_0)|^2_{L^2} \right),
\] (3.19)
for all \(t \geq 0\).

We have
\[
\int_0^t \int_\Omega |u|^2 \, dx \, ds \leq \int_0^t \int_{\Omega \cap B_L} |u|^2 \, dx \, ds + \int_0^t \int_{\{|x| \geq L\}} |u|^2 \, dx \, ds \\
\leq C_L \int_0^t \int_{\Omega \cap B_L} |\nabla u|^2 \, dx \, ds + \frac{1}{c_0} \int_0^t \int_\Omega a |u(s, x)|^2 \, dx \, ds \\
\leq C_L \int_0^t E_u(s) \, ds + \frac{1}{c_0} \int_0^t \int_\Omega a |u(s, x)|^2 \, dx \, ds.
\]

Now using (3.19) and (2.16), we get
\[
\int_0^{+\infty} \int_\Omega |u|^2 \, dx \, ds \leq CI_1,
\]
with
\[
I_1 = \|u_0|^2_{H^1} + \|u_1|^2_{L^2} + \|d(\cdot) (u_1 + au_0)|^2_{L^2}.
\]
On the other hand, using (3.14) and the fact that
\[ \int_0^{+\infty} E_u(s) \, ds \leq CI_0 \]
we obtain
\[ \int_0^{+\infty} X(s) \, ds \leq C \left( \int_0^{+\infty} \int_{\Omega} |u|^2 \, dx \, ds + \int_0^{+\infty} E_u(s) \, ds \right) \]
\[ \leq CI_1. \]

Finally it is clear that,
\[ X(0) \leq CI_1. \]
Therefore from (3.18)
\[ \sup_{\mathbb{R}_+} (1 + t) \int_{\Omega} |u(t)|^2 \, dx \leq C \left( X(0) + \int_0^{+\infty} X(s) \, ds + \int_0^{+\infty} \int_{\Omega} |u|^2 \, dx \, ds + I_0 \right) \]
\[ \leq CI_1, \]
and (3.17), gives
\[ \int_0^{+\infty} (1 + s) E_u(s) \, ds \leq CI_1. \]

The energy decay estimate follows from the fact that
\[ (1 + t)^2 E_u(t) \leq E_u(0) + 2 \int_0^{+\infty} (1 + s) E_u(s) \, ds \]
\[ \leq CI_1, \text{ for all } t \geq 0. \]

This finishes the proof of theorem 2 and now we give the proof of proposition 2.

3.2. Proof of proposition 2. Let \( n \in \mathbb{N}^* \) and \( u \) the solution of (1.1) with initial data in \( D(A^n) \) such that \( u_0 \in L^2(\Omega) \) and
\[ \| d(\cdot)(u_1 + au_0) \|_{L^2} < +\infty. \]

Let \( v = \partial_t u \). Using these estimates
\[ \int_0^{+\infty} (1 + s) E_u(s) \, ds \leq CI_1 \text{ and } \int_0^{+\infty} (1 + s) E_v(s) \, ds \leq CI_{0,1}, \]
and proceeding as in the proof of proposition 1, we show that
\[ \int_0^{+\infty} (1 + s)^2 E_v(s) \, ds \leq C_1 I_{1,1}. \]

We set \( u_n = \partial_t^n u \), for \( n \geq 1 \). Using induction argument and arguing as in the proof of proposition 1, we prove that
\[ \int_0^{+\infty} (1 + s)^n E_{u_n}(s) \, ds \leq C_n I_{1,n}, \]
with
\[ I_{1,n} = \sum_{i=0}^n \| A^i (u_0, u_1) \|^2_H + \| u_0 \|^2_{L^2} + \| d(\cdot)(u_1 + au_0) \|^2_{L^2}. \]
The energy decay estimate follows from the fact that

\[(1 + t)^{n+2} E_{u_n}(t) \leq E_{u_n}(0) + (n + 2) \int_0^t (1 + s)^{n+1} E_{u_n}(s) \, ds \leq C_n I_{1,n}, \text{ for all } t \geq 0.\]

Now using the estimate above, we infer that,

\[(1 + t)^{n+1} \|\partial_n^{-1} u(t)\|_{L^2}^2 \leq 2 (1 + t)^{n+1} E_{u_{n-1}}(t) \leq C_{n-1} I_{1,n-1} \text{ for all } t \geq 0,\]

We have \(\partial_n^{-1} u\) is a solution of the following system

\[
\begin{cases}
\partial_t^{n+1} u - \Delta \partial_t^{n-1} u + a(x) \partial_t^n u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
\partial_t^{n-1} u = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\
(\partial_t^{n-1} u(0,x), \partial_t^n u(0,x)) \in D(A) &
\end{cases}
\]

therefore

\[\left(\partial_t^{n-1} u, \partial_t^n u\right) \in C^1(\mathbb{R}_+, H).\]

Using Eq (3.20), we infer that

\[(1 + t)^{n+1} \|\Delta \partial_t^{n-1} u(t)\|_{L^2}^2 \leq C (1 + t)^{n+1} (E_{u_n}(t) + E_{u_{n-1}}(t)) \leq CI_{1,n}.
\]

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