Patterns of primes in arithmetic progressions

János Pintz

1 Introduction

In their ground-breaking work Green and Tao [GT 2008] proved the existence of infinitely many \( k \)-term arithmetic progressions in the sequence of primes for every integer \( k > 0 \). I showed a conditional strengthening of it [Pin 2010] according to which if the primes have a distribution level \( \vartheta > 1/2 \) (for the definition of the distribution level see (1.1) below), then there exists a constant \( C(\vartheta) \) such that we have a positive even \( d \leq C(\vartheta) \) with the property that \( 0 < d \leq C(\vartheta) \) and for every \( k \) there exist infinitely many arithmetic progressions \( \{ p^*_i \}_{i=1}^k \) of length \( k \) with \( p^*_i \in \mathcal{P} \) (\( \mathcal{P} \) denotes the set of primes) such that \( p^*_i + d \) is a prime too, in particular, the prime following \( p^*_i \). After the proof of Zhang [Zhang 2014], proving the unconditional existence of infinitely many bounded gaps between primes (this was proved earlier in our work [GPY 2009] under the condition that primes have a distribution level \( \vartheta > 1/2 \)) I showed this without any unproved hypotheses [Pin 2015].

We say that \( \vartheta \) is a distribution level of the primes if

\[
\sum_{q \leq x^\vartheta} \max_{(a,q)=1} \left| \frac{\pi(x, q, a) - \pi(x)}{\varphi(q)} \right| \ll A \frac{x}{(\log x)^A}
\]

holds for any \( A > 0 \) where the \( \ll \) symbol of Vinogradov means that \( f(x) = O(g(x)) \) is abbreviated by \( f(x) \ll g(x) \).

In his recent work James Maynard [May 2015] gave a simpler and more efficient proof of Zhang’s theorem. In particular he gave an unconditional proof of a weaker version of Dickson’s conjecture [Dic 1904] which we abbreviate as Conjecture DHL since Hardy and Littlewood formulated a stronger quantitative version of it twenty years later [HL 1923].

*Supported by OTKA Grants NK104183, K100291 and ERC-AdG. 321104.

Conjecture DHL (Prime $k$-tuples Conjecture). Let $\mathcal{H} = \{h_1, \ldots, h_k\}$ be admissible, which means that for every prime $p$ there exists an integer $a_p$ such that for any $i a_p \not\equiv h_i (\text{mod } p)$. Then there are infinitely many integers $n$ such that all of $n + h_1, \ldots, n + h_k$ are primes.

The weaker version showed by Maynard (and simultaneously and independently by T. Tao (unpublished)) was that Conjecture DHL $(k, k_0)$ (formulated below) holds for $k \gg k_0^2 e^{4k_0}$.

Conjecture DHL($k, k_0$). If $\mathcal{H}$ is admissible of size $k$, then there are infinitely many integers $n$ such that \{$n + h_i\}_{i=1}^k$ contains at least $k_0$ primes.

A brief argument, given by Maynard [May 2015] (see Theorem 1.2 of his work) shows that if there exists a $C(k_0)$ such that DHL($k, k_0$) holds for $k \geq C(k_0)$, then a positive proportion of all admissible $m$-tuples satisfy the prime $m$-tuple conjecture for every $m$ (for the exact formulation see Theorem 1.2 of [May 2015]).

The purpose of the present work is to show a common generalization of the result of Maynard (and Tao) and that of Green–Tao.

Theorem 1. Let $m > 0$ and $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a set of $r$ distinct integers with $r$ sufficiently large depending on $m$. Let $N(\mathcal{A})$ denote the number of integer $m$-tuples $\{h_1, \ldots, h_m\} \subseteq \mathcal{A}$ such that there exist for every $\ell$ infinitely many $\ell$-term arithmetic progressions of primes $\{p_i^*\}_{i=1}^\ell$ where $p_i^* + h_j$ is also prime for each pair $i, j$. Then

\[(1.2) \quad N(\mathcal{A}) \gg_m \# \{(h_1, \ldots, h_m) \in \mathcal{A}\} \gg_m |\mathcal{A}|^m = r^m.\]

This is an unconditional generalization of the result in [Pin 2010].

2 Preparation. First part of the proof of Theorem 2

The arguments in the last three paragraphs of Section 4 of [May 2015] can be applied here practically without any change and so, similarly to Theorems 1.1 and 1.2 of [May 2015], our Theorem 1 will also follow in essentially the same way from (the weaker)

Theorem 2. Let $m$ be a positive integer, $\mathcal{H} = \{h_1, \ldots, h_k\}$ be an admissible set of $k$ distinct non-negative integers $h_i \leq H$, $k = \lceil C m^2 e^{4m} \rceil$ with a
sufficiently large absolute constant $C$. Then there exists an $m$-element subset
\begin{equation}
\{h'_1, h'_2, \ldots, h'_m\} \subseteq \mathcal{H}
\end{equation}
such that for every positive integer $\ell$ we have infinitely many $\ell$-element non-trivial arithmetic progressions of primes $p^*_i$ such that $p^*_i + h'_j \in \mathcal{P}$ for $1 \leq i \leq \ell$, $1 \leq j \leq m$, further $p^*_i + h'_j$ is always the $j$-th prime following $p^*_i$.

**Remark.**

(i) For $\ell = m = 1$ this is Zhang’s theorem,

(ii) for $\ell = 1$, $m$ arbitrary this is the Maynard–Tao theorem,

(iii) for $m = 0$, $\ell$ arbitrary this is the Green–Tao theorem,

(iv) for $m = 1$, $\ell$ arbitrary this was proved under the condition that primes have a distribution level $\theta > 1/2$ in [Pin 2010], unconditionally (using Zhang’s method) in [Pin 2015].

In order to show our Theorem 2 we will follow the scheme of [May 2015]. We therefore emphasize just a few notations here, but we will use everywhere Maynard’s notation throughout our work. Similarly to his work, $k$ will be a fixed integer, $\mathcal{H} = \{h_1, \ldots, h_k\} \subseteq [0, H]$ a fixed admissible set. Any constants implied by the $\ll$ and $0$ notations may depend on $k$ and $H$. $N$ will denote a large integer and asymptotics will be understood as $N \to \infty$. Most variables will be natural numbers, $p$ (with or without subscripts) will denote always primes, $[a, b]$ the least common multiple of $[a, b]$ (however, sometimes the closed interval $[a, b]$). We will weight the integers with a non-negative weight $w_n$ which will be zero unless $n$ lies in a fixed residue class $\nu_0$ (mod $W$) where $W = \prod_{p \leq D_0} p$. $D_0$ tends in [May 2015] slowly to infinity with $N$. His choice is actually $D_0 = \log \log \log N$. However, it is sufficient to choose
\begin{equation}
D_0 = C^*(k),
\end{equation}
with a sufficiently large constant $C^*(k)$, depending on $k$.

The proof runs similarly in this case as well just we lose the asymptotics then, but the dependence on $D_0$ is explicitly given in [May 2015]. The weights $w_n$ are defined in (2.4) of [May 2015] as
\begin{equation}
w_n = \left( \sum_{d_i | n+h_i \forall i} \lambda_{d_1, \ldots, d_k} \right)^2.
\end{equation}
The choice of \( \lambda_{d_1,...,d_k} \) will be through the choice of other parameters \( y_{r_1,...,r_k} \) by the aid of the identity

\[
\lambda_{d_1,...,d_k} = \left( \prod_{i=1}^{k} \mu(d_i) d_i \right) \sum_{r_1,...,r_k \atop d_i|r_i \forall i} \frac{\sum_{i=1}^{k} \varphi(r_i)}{\prod_{i=1}^{k} \varphi(r_i)} y_{r_1,...,r_k}
\]

whenever \( \prod_{i=1}^{k} d_i W = 1 \) and \( \lambda_{d_1,...,d_r} = 0 \) otherwise. Here \( y_{r_1,...,r_k} \) will be defined by the aid of a piecewise differentiable function \( F \), the distribution \( \theta > 0 \) of the primes, with \( R = N^{\theta/2-\varepsilon} \) as

\[
y_{r_1,...,r_k} = F \left( \frac{\log r_1}{\log R}, \frac{\log r_k}{\log R} \right)
\]

where \( F \) will be real valued, supported on

\[
R_k = \left\{ \left( x_1, \ldots, x_k \right) \in [0,1]^k : \sum_{i=1}^{k} x_i \leq 1 \right\}.
\]

All this is in complete agreement with the notation of Proposition 1 and (6.3) of [May 2015].

Our proof will also make use of the main pillars of Maynard’s proof, his Propositions 1–3, which we quote now with the above notations as Proposition 1’.

**Proposition 1’.** With the above notation let

\[
S_1 := \sum_{N \leq n < 2N \atop n \equiv \nu_0 \pmod{W}} w_n, \quad S_2 := \sum_{N \leq n < 2N \atop n \equiv \nu_0 \pmod{W}} \left( w_n \sum_{i=1}^{k} \chi_P(n+h_i) \right),
\]

where \( \chi_P(n) \) denotes the characteristic function of the primes. Then we have as \( N \to \infty \)

\[
S_1 = \frac{\left( 1 + O\left( \frac{1}{D_0} \right) \right) \varphi(W)^k N (\log R)^k}{W^{k+1}} I_k(F),
\]

\[
S_2 = \frac{\left( 1 + O\left( \frac{1}{D_0} \right) \right) \varphi(W)^k N (\log R)^{k+1}}{W^{k+1}} \sum_{j=1}^{k} J_k^{(j)}(F),
\]
provided $I_k(F) \neq 0$ and $J_k^{(j)}(F) \neq 0$ for each $j$, where

$$I_k(F) = \int_0^1 \ldots \int_0^1 F(t_1, \ldots, t_k)^2 dt_1 \ldots dt_k,$$

(2.10)

$$J_k^{(j)}(F) = \int_0^1 \ldots \int_0^1 \left( \int_0^1 F(t_1, \ldots, t_k) dt_j \right)^2 dt_1 \ldots dt_{j-1} dt_{j+1} \ldots dt_k.$$  

(2.11)

**Proposition 2’**. Let $S_k$ denote the set of piecewise differentiable functions with the earlier given properties, including $I_k(F) \neq 0$ and $J_k^{(j)}(F) \neq 0$ for $1 \leq j \leq k$. Let

$$M_k = \sup \frac{\sum_{j=1}^k J_k^{(j)}(F)}{I_k(F)}, \quad r_k = \left\lceil \frac{\theta M_k}{2} \right\rceil$$

(2.12)

and let $\mathcal{H}$ be a fixed admissible sequence $\mathcal{H} = \{h_1, \ldots, h_k\}$ of size $k$. Then there are infinitely many integers $n$ such that at least $r_k$ of the $n + h_i$ $(1 \leq i \leq k)$ are simultaneously primes.

**Proposition 3’**. $M_{105} > 4$ and $M_k > \log k - 2 \log \log k - 2$ for $k > k_0$.

**Remark**. In the proof Maynard will use for every $k$ an explicitly given function $F = F_k$ satisfying the above inequality. Therefore the additional dependence on $F$ will be actually a dependence on $k$.

The main idea (beyond the original proof of Maynard–Tao) is that in the weighted sum $S_1$ in (2.7) all those weights $w_n$ for numbers $n \in [N, 2N]$ are in total negligible for which any of the $n + h_i$ terms $(1 \leq i \leq k)$ has a small prime factor $p$ (i.e. with a sufficiently small $c_1(k)$ depending on $k$, $p \mid n + h_i$, $p < n^{c_1(k)}$).

To make it more precise let $c_1(k)$ be a sufficiently small fixed constant (to be determined later and fixed for the rest of the work). Let $P^-(n)$ be the smallest prime factor of $n$. Then we have

**Lemma 1**. We have

$$S_1^{-} = \sum_{n \equiv r_0 \pmod{W}} \sum_{n \leq N < 2N} w_n \ll_{k,H} c_1(k) \frac{\log N}{\log R} S_1.$$  

(2.13)
Since $R = N^\frac{\theta}{2} - \varepsilon$, $S_1^-/S_1$ will be arbitrarily small if $c_1(k)$ is chosen sufficiently small. The proof of Lemma 1 will be postponed to Section 3. This means that during the whole proof we can neglect those numbers $n$ for which $P^{-\left(\prod_{i=1}^{k}(n+h_i)\right)} < n^{c_1(k)}$ and it is sufficient to deal with numbers $n$ with $n + h_i$ being almost primes for each $i = 1, 2, \ldots, k$ (by which we mean that $n + h_i$ has only prime factors at least $n^{c_1(k)}$). A trivial consequence of this fact is that for such numbers $n$ $\prod_{i=1}^{k}(n+h_i)$ has a bounded number of prime factors. Consequently we have for these numbers $n$ by (5.9) and (6.3)

\begin{equation}
(2.14)
  w_n \ll_{c_1(k), k} \lambda_{\text{max}}^2 \ll_{c_1(k), k} y_{\text{max}}^2 (\log R)^{2k} \ll_{c_1(k), k, F} (\log R)^{2k} \ll (\log R)^{2k}
\end{equation}

with the convention that the constants implied by the $\ll$ and $O$ constants can depend on $k$ and both $c_1(k)$ and $F = F_k$ will only depend on $k$.

The essence of Maynard’s proof is that (see (4.1)–(4.4) of [May 2015])

\begin{equation}
(2.15)
  S_2 > \left(\left(\frac{\theta}{2} - \varepsilon\right)(M_k - \varepsilon) + O\left(\frac{1}{D_0}\right)\right) S_1
\end{equation}

which directly implies the existence of infinitely many values $n$ such that there are at least

\begin{equation}
(2.16)
  r_k = \left\lfloor \frac{\theta M_k}{2} \right\rfloor
\end{equation}

primes among $n + h_i$ $(1 \leq i \leq k)$.

Let us denote, in analogy with (2.7)

\begin{equation}
(2.17)
  S_1^+ := \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod{W} \\ P^{-\left(\prod_{i=1}^{k}(n+h_i)\right)} \geq n^{c_1(k)}}} w_n, \quad S_2^+ := \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod{W} \\ P^{-\left(\prod_{i=1}^{k}(n+h_i)\right)} \geq n^{c_1(k)}}} w_n \left(\sum_{i=1}^{k} \chi_P(n+h_i)\right).
\end{equation}

Then Lemma 1 i.e. (2.13) implies together with (2.15) that (if $c_1(k)$ and $\varepsilon$ are chosen sufficiently small, $D_0$ sufficiently large, then)

\begin{equation}
(2.18)
  S_2^+ > \left(\left(\frac{\theta}{2} - \varepsilon\right)(M_k - \varepsilon) + O(c_1(k)) + O\left(\frac{1}{D_0}\right) + o(1)\right) S_1,
\end{equation}
which implies the existence of a large number of $n$ values in $[N, 2N)$, $n \equiv \nu_0 \pmod{W}$, with at least $r_k$ primes among them and additionally almost primes with $P^-(n + h_i) > n^{\nu_1(k)}$ in all other components $i \in [1, k]$.

Together with (2.14) this implies

\begin{equation}
S^*_1 := \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod{W}}} 1 \gg \frac{S_1}{(\log R)^2k} = \left(1 + O \left(\frac{1}{D_0}\right)\right) \frac{\varphi(W)^k N I_k(F)}{W^{k+1}(\log R)^k}.
\end{equation}

(2.19)

Since $D_0 = C^*(k)$ we have $\varphi(W)^k/W^{k+1} \geq C'(k)$. Thus a positive proportion (depending on $k$) of the integers $n \in [N, 2N)$ with $n \equiv \nu_0 \pmod{W}$ and $P^-(\prod_{i=1}^{k} (n + h_i)) > n^{c_1(k)}$ contain at least $r_k$ primes among $n + h_i$ ($1 \leq i \leq k$). This follows from (2.19) and

\begin{equation}
\sum_{\substack{N \leq n < 2N, n \equiv \nu_0 \pmod{W}}} 1 \ll \frac{N}{\log^k N}
\end{equation}

(2.20)

where the implied constant in the $\ll$ symbol depends only on $k$, $H$ and $c_1(k)$, therefore only on $k$, finally. (2.20) is a consequence of Selberg’s sieve (see, for example, Theorem 5.1 of [HR 1974] or Theorem 2 in §2.2.2 of [Gre 2001]).

If Lemma 1 will be proved (see Section 3) then Theorem 2 will follow from Theorem 5 of [Pin 2010] which we quote here as

**Main Lemma.** Let $k$ be an arbitrary positive integer and $\mathcal{H} = \{h_1, \ldots, h_k\}$ be an admissible $k$-tuple. If the set $\mathcal{N}(\mathcal{H})$ satisfies with constants $c_1(k)$, $c_2(k)$

\begin{equation}
\mathcal{N}(\mathcal{H}) \subseteq \left\{ n; P^-(\prod_{i=1}^{k} (n + h_i)) \geq n^{c_1(k)} \right\}
\end{equation}

(2.21)

and

\begin{equation}
\#\{n \leq X, n \in \mathcal{N}(\mathcal{H})\} \geq \frac{c_2(k)X}{\log^k X}
\end{equation}

(2.22)

for $X > X_0$, then $\mathcal{N}(\mathcal{H})$ contains $\ell$-term arithmetic progressions for every $\ell$.
In order to see that the extra condition that the given prime pattern occurs also for consecutive primes we have to work in the following way. For any given $\mathcal{H} = \{h_1, \ldots, h_k\}$ with $k = \lceil Cm^2 \log m \rceil$ we choose an $m$-element subset $\mathcal{H}' = \{h'_1, \ldots, h'_m\} \subseteq \mathcal{H}$ with minimal diameter $h'_m - h'_1$ such that with some constants $c'_1(k), c'_2(k) > 0$ the relations (2.21)–(2.22), more exactly (2.23)

$$
\#\left\{n \leq X; \mathcal{P} - \prod_{i=1}^{k} (n + h_i) \geq n^{c'_1(k)}, n + h'_i \in \mathcal{P} \ (1 \leq i \leq m) \right\} \geq c'_2(k)X \log^k X
$$

should hold for $X > X_0$.

By the condition that $\mathcal{H}'$ has minimal diameter we can delete from our set $\mathcal{N}(\mathcal{H})$ those $n$’s for which there exists any $h_i \in \mathcal{H} \setminus \mathcal{H}'$, $h'_1 < h_i < h'_m$ such that beyond (2.23) also $n + h_i \in \mathcal{P}$ would hold.

On the other hand we can also neglect those $n \in \mathcal{N}(\mathcal{H})$ for which with a given $h \in [1, H]$, $h \notin \mathcal{H}_k$ we would have additionally $n + h \in \mathcal{P}$ since the total number of such $h \in [1, H]$ is by (2.20) at most

$$
O_k \left( \frac{NH}{\log^{k+1} N} \right) = o \left( \frac{N}{\log^k N} \right)
$$

since our original $H$ in Theorem 2 was fixed.

We note that the above way of specifying the $m$-element sets $\mathcal{H}'_m$ for which we have arbitrarily long (finite) arithmetic progressions of $n$’s such that $n + h'_i$ ($1 \leq i \leq m$) would be a given bounded pattern of consecutive primes does not change the validity of the argument of Maynard (see Theorem 1.2 of [May 2015]) which shows that the above is true for a positive proportion of all $m$-element sets (the proportion depends on $m$).

3 Proof of Lemma 1. End of the proof of Theorem 2

The proof of Lemma 1 will be a trivial consequence of the following

**Lemma 2.** The following relation holds for any prime $D_0 < p < N^{c_1}$ and all $i \in [1, \ldots, k]$:

$$
S_{1,p}^* := \sum_{n \leq n < 2N \atop p \equiv 0 \pmod{W}} w_n \ll_{F,H,k} \frac{\log p}{p \log R} \sum_{n \leq n < 2N \atop p \equiv 0 \pmod{W}} w_n = \frac{\log p}{p \log R} S_1.
$$
Proof. It is clear that it is enough to show this for $i = 1$, for example. During the proof we will use the analogue of Lemma 6 of [GGPY 2010] for the special case $k = 1$, $\delta = p \in \mathcal{P}$ and for squarefree $n$ with

$$f(n) = n \quad f_1(n) = \mu * f(n) = \prod_{p | n} (p - 1) = \varphi(n)$$

which is as follows:

$$T_p := \sum_{d,e} \frac{\lambda_d \lambda_e}{[d,e,p]/p} = \sum_{r \in \mathbb{R}^+} \frac{\mu^2(r)}{\varphi(r)} (y_r - y_{rp})^2.$$  

This form appears as the last displayed equation on page 85 of Selberg [Sel 1991] or equation (1.9) on page 287 of Greaves [Gre 2001]. We note the general starting condition that similarly to [May 2015] the numbers $W, [d_1,e_1], \ldots, [d_k,e_k]$ will be always coprime to each other.

Writing $n + h_1 = pm$ we see that we have for any $\varepsilon > 0$ and denoting $\sum^*$ for the conditions $n \in [N, 2N)$, $n \equiv \nu_0 \pmod{W}$; $d_i, e_i \mod n + h_i$ ($2 \leq i \leq k$)

$$S_{1,p} = \sum_1 + \sum_2 + O(R^{2+\varepsilon})$$

where

$$\sum_1 = \sum^* \lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k} \prod_{i=1}^{k} \frac{\lambda_{d_i, e_i}}{[d_i, e_i]}$$

$$\sum_2 = \sum^* \lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k} \prod_{i=1}^{k} \frac{\lambda_{d_i, e_i}}{[d_i, e_i]} \prod_{i=1}^{k} \frac{\lambda_{d_i, e_i}}{[d_i, e_i]}$$

Distinguishing further in $\sum_2$ according to $p^2 | d_1 e_1$ or not we obtain from (3.5) and (3.6) for any $\varepsilon > 0$

$$\sum_1 = \frac{N}{pW} \sum^* \lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k} \prod_{i=1}^{k} \frac{\lambda_{d_i, e_i}}{[d_i, e_i]} + O(R^{2+\varepsilon})$$
and

\[
\sum_2 = \frac{N}{pW} \left\{ \left( \sum^{\ast} \frac{\lambda_{pd_1', \ldots, d_k} \lambda_{e_1, \ldots, e_k}}{d_1' e_1 \prod_{i=2}^{k} [d_i, e_i]} \right) + \sum^{\ast} \frac{\lambda_{d_1, \ldots, d_k} \lambda_{pe_1', \ldots, e_k}}{e_1 = pe_1' \prod_{i=2}^{k} [d_i, e_i]} \right\} + \sum^{\ast} \frac{\lambda_{pd_1', \ldots, d_k} \lambda_{e_1', \ldots, e_k}}{d_1' e_1' \prod_{i=2}^{k} [d_i, e_i]} \right\} + O(R^{2+\varepsilon}).
\]

Consequently we have

\[
S_{1,p} = \frac{N}{pW} \sum^{\ast} \frac{\lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k}}{[d_1, e_1] \prod_{i=2}^{k} [d_i, e_i]} + O(R^{2+\varepsilon}).
\]

Let us denote the sum in (3.9) analogously to (3.3) by \( T_{p,1} \). Then, similarly to (3.3) we obtain using additionally the argument of Section 5 of May 2015

\[
T_{p,1} = \sum_{u_1, \ldots, u_k} \prod_{i=1}^{k} \mu^2(u_i) \prod_{i=1}^{k} \phi(u_i) (y_{u_1, \ldots, u_k} - y_{u_{1p}, u_2, \ldots, u_k})^2.
\]

However by the choice (6.3) of May 2015 we have

\[
(y_{u_1, \ldots, u_k} - y_{u_{1p}, u_2, \ldots, u_k})^2 \leq F \left( \log \frac{u_1}{\log R}, \ldots, \log \frac{u_k}{\log R} \right)^2 - F \left( \log \frac{u_1 + \log p}{\log R}, \ldots, \log \frac{u_k}{\log R} \right)^2 \leq \frac{\log p}{\log R},
\]

since \( F \) depends only on \( k \), and hence the constant implied by the \( \ll \) symbol may depend on \( k \). Hence we have by Proposition (4.1) of May 2015

\[
T_{p,1} \ll \frac{N}{W} \frac{\log p}{p \log R} \sum_{(u_1, W) = 1}^{k} \prod_{i=1}^{k} (\mu^2(u_i)) \prod_{i=1}^{k} \phi(u_i) \ll \frac{\log p}{p \log R} \cdot S_1
\]

which proves Lemma 2 and thereby Lemma 1 and Theorem 2. \( \square \)
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