Adventures in Harmonic Analysis

Stephen William Semmes
Rice University
Houston, Texas

Contents

1 Power Series 4
2 The exponential function 7
3 Laurent series 9
4 Polynomials 11
5 Harmonic functions 13
6 Fourier series 16
7 Fourier transforms 17
8 Uniform continuity 18
9 Inner product spaces 20
10 Fourier series, continued 23
11 Almost periodic functions 25
12 Step functions 27
13 Norms 29
14 Some computations 32
15 Fourier inversion
16 Several complex variables
17 Multiple Fourier series
18 Invariant means
19 Banach spaces
20 $\ell^p(Z^n)$
21 Measures on $T^n$
22 Measures on $R$
23 Convolutions on $T^n$
24 Convolutions on $R$
25 Smooth functions
26 Gaussians
27 Plancherel’s theorem
28 Bounded functions
29 Subharmonic functions
A Metric spaces
B Compact sets
C Continuous functions
D Lipschitz functions
E Ultrametric spaces
As usual, the integers are denoted \( \mathbb{Z} \), and the real and complex numbers are denoted \( \mathbb{R} \), \( \mathbb{C} \), respectively. If \( x \) is a real number, then its absolute value is denoted \( |x| \) and defined to be equal to \( x \) when \( x \geq 0 \) and to \(-x \) when \( x \leq 0 \). Thus \( |x| \geq 0 \) for every \( x \in \mathbb{R} \) and \( |x| = 0 \) if and only if \( x = 0 \). One can check that
\[
|x + y| \leq |x| + |y|
\]
and
\[
|x y| = |x| |y|
\]
for all \( x, y \in \mathbb{R} \).

A complex number \( z \) can be expressed in a unique way as \( x + yi \), where \( x, y \) are real numbers and \( i^2 = -1 \). We may refer to \( x \) and \( y \) as the real and imaginary parts of \( z \), denoted \( \text{Re} z \), \( \text{Im} z \), respectively. The complex conjugate of \( z \) is denoted \( \overline{z} \) and defined by
\[
\overline{z} = x - yi.
\]
Thus
\[
\text{Re} z = \frac{z + \overline{z}}{2}
\]
and
\[
\text{Im} z = \frac{z - \overline{z}}{2i}.
\]
Also, for all \( z, w \in \mathbb{C} \),
\[
z + w = \overline{z} + \overline{w}
\]
and
\[
z w = \overline{z} \overline{w}.
\]
If \( z = x + yi \) is a complex number, \( x, y \in \mathbb{R} \), then \( z \overline{z} = x^2 + y^2 \). The modulus of \( z \) is denoted \( |z| \) and is defined by
\[
|z| = \sqrt{x^2 + y^2}.
\]
In particular, \( |\text{Re} z|, |\text{Im} z| \leq |z| \). Because \( |zw|^2 = z \overline{z} w \overline{w} \), we get that
\[
|zw| = |z| |w|
\]
for every $z, w \in C$. Also,

$$(0.10) \quad |z + w| \leq |z| + |w|$$

for every $z, w \in C$, because

$$(0.11) \quad |z + w|^2 = (z + w)(\overline{z} + \overline{w})$$

$$= |z|^2 + 2 \text{Re} z \overline{w} + |w|^2$$

$$\leq |z|^2 + 2 |z| |w| + |w|^2$$

$$= (|z| + |w|)^2.$$ 

## 1 Power Series

By a *power series* in one complex variable we mean a series

$$(1.1) \quad \sum_{n=0}^{\infty} a_n z^n$$

where the coefficients $a_n$ are complex numbers, $z$ is a complex variable, and $z^n$ is interpreted as being equal to 1 for every $z \in C$ when $n = 0$. Such a series converges trivially when $z = 0$, and may or may not converge elsewhere.

For example, consider the *geometric series*

$$(1.2) \quad \sum_{n=0}^{\infty} z^n.$$ 

For every $z \in C$ and nonnegative integer $r$ we have that

$$(1.3) \quad (1 - z) \sum_{n=0}^{r} z^n = 1 - z^{r+1},$$

and hence

$$(1.4) \quad \sum_{n=0}^{r} z^n = \frac{1 - z^{r+1}}{1 - z}$$

when $z \neq 1$. If $|z| < 1$, then the geometric series converges and

$$(1.5) \quad \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z},$$
while if \(|z| \geq 1\) then \(|z|^n \geq 1\) for all \(n\) and the series diverges because the terms do not tend to 0 as \(n \to \infty\).

If (1.1) converges for some \(z_0 \in \mathbb{C}\), then
\[
\lim_{n \to \infty} a_n z_0^n = 0.
\]
In particular, \(\{a_n z_0^n\}_{n=0}^\infty\) is a bounded sequence of complex numbers, which is to say that there is an \(A \geq 0\) such that
\[
|a_n| |z_0|^n \leq A
\]
for every positive integer \(n\). This implies that (1.1) converges absolutely when \(|z| < |z_0|\), by comparison with a geometric series. Moreover, the partial sums converge uniformly on the set of \(z \in \mathbb{C}\) with \(|z| \leq r\) for every \(r \geq 0\) such that \(r < |z_0|\), as a consequence of the Weierstrass M-test. The radius of convergence of (1.1) is the unique \(R, 0 \leq R \leq +\infty\), such that (1.1) converges absolutely when \(|z| < R\) and does not converge when \(|z| > R\).

If (1.1) has radius of convergence \(R\), then the power series defines a continuous complex-valued function on the open disk
\[
\{z \in \mathbb{C} : |z| < R\},
\]
which is the whole complex plane when \(R = +\infty\). If \(R < +\infty\) and
\[
\sum_{n=0}^\infty |a_n| R^n
\]
converges, then (1.1) converges absolutely for every \(z \in \mathbb{C}\) such that \(|z| \leq R\), the partial sums of the series converges uniformly on the closed disk
\[
\{z \in \mathbb{C} : |z| \leq R\},
\]
and (1.1) defines a continuous function on this disk.

One can multiply a pair of power series \(\sum_{j=0}^\infty a_j z^j, \sum_{l=0}^\infty b_l z^l\) formally to get a new power series \(\sum_{n=0}^\infty c_n z^n\), where
\[
c_n = \sum_{j=0}^n a_j b_{n-j}.
\]
This is the Cauchy product, and one can just as well say that the Cauchy product of
\[
\sum_{j=0}^\infty a_j \quad \text{and} \quad \sum_{l=0}^\infty b_l
\]
is the series

\[ \sum_{n=0}^{\infty} c_n. \]  

(1.13)

It is easy to check that

\[ \sum_{n=0}^{r} |c_n| \leq \left( \sum_{j=0}^{r} |a_j| \right) \left( \sum_{l=0}^{r} |b_l| \right) \]  

(1.14)

for all nonnegative integers \(r\). Hence the absolute convergence of (1.12) implies the absolute convergence of (1.13). One can show that the product of the sums in (1.12) is equal to (1.13) in this case.

Let \(U\) be an open set in the complex plane \(\mathbb{C}\), and let \(f(z)\) be a complex-valued function on \(U\) which is differentiable at some \(z = x + yi \in U, x, y \in \mathbb{R}\). This means that there are complex numbers

\[ \frac{\partial f}{\partial x}(z), \quad \frac{\partial f}{\partial y}(z) \]  

(1.15)

such that

\[ f(\zeta) = f(z) + \frac{\partial f}{\partial x}(z) (\xi - x) + \frac{\partial f}{\partial y}(z) (\eta - y) + e_z(\zeta) |\zeta - z|, \]  

(1.16)

where \(\zeta = \xi + \eta i \in U, \xi, \eta \in \mathbb{R}\), and

\[ \lim_{\zeta \to z} e_z(\zeta) = 0. \]  

(1.17)

If we put

\[ \frac{\partial f}{\partial z}(z) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(z) - \frac{\partial f}{\partial y}(z) i \right), \quad \frac{\partial f}{\partial \bar{z}}(z) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(z) + \frac{\partial f}{\partial y}(z) i \right), \]  

(1.18)

then we can rewrite (1.16) as

\[ f(\zeta) = f(z) + \frac{\partial f}{\partial z}(z) (\zeta - z) + \frac{\partial f}{\partial \bar{z}}(z) (\bar{\zeta} - \bar{z}) + e_z(\zeta) |\zeta - z|, \]  

(1.19)

where \(e_z(\zeta)\) satisfies (1.17) as before. We say that \(f(z)\) is holomorphic on \(U\) if it is differentiable everywhere in \(U\) and

\[ \frac{\partial f}{\partial \bar{z}}(z) = 0. \]  

(1.20)
for every $z \in U$. In this event we put

$$f'(z) = \frac{\partial f}{\partial z}(z),$$

(1.21)

the complex derivative of $f$ at $z$.

Constant functions are trivially holomorphic with derivative equal to 0. The identity function $f(z) = z$ is holomorphic on the whole complex plane, with derivative equal to 1. The product rule implies that $z^n$ is a holomorphic function on $\mathbb{C}$ for each positive integer $n$, with derivative equal to $nz^{n-1}$. If the power series (1.1) has radius of convergence $R$, then the series

$$\sum_{n=0}^{\infty} na_n z^n$$

(1.22)

also has radius of convergence equal to $R$, and the function defined for $|z| < R$ by (1.1) is holomorphic and its derivative is given by the power series (1.22). This follows from standard results about differentiating power series term by term.

2 The exponential function

Consider the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!},$$

(2.1)

where $n!$ or “$n$ factorial” is the product of the positive integers from 1 to $n$, which is interpreted as being equal to 1 when $n = 0$. This series converges absolutely for every complex number $z$, and the sum is denoted $\exp z$.

By the remarks in the previous section, the exponential function $\exp z$ is a continuous complex-valued function on $\mathbb{C}$ which is holomorphic and whose derivative is equal to itself. The exponential function is characterized by these properties and the normalization $\exp 0 = 1$.

One can show that

$$\exp(z + w) = (\exp z) (\exp w)$$

(2.2)

for all complex numbers $z, w$. More precisely, the binomial theorem gives

$$\exp(z + w) = \sum_{n=0}^{\infty} \frac{(z + w)^n}{n!} = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{z^l}{l!} \frac{w^{n-l}}{(n-l)!},$$

(2.3)
which is the same as the Cauchy product of the series defining \( \exp z \), \( \exp w \). Because the series converge absolutely there is no problem in identifying the Cauchy product with the product of the exponentials.

Because \( \exp 0 = 1 \), we get that

\[
(\exp z)(\exp -z) = 1. \tag{2.4}
\]

In particular, \( \exp z \neq 0 \) for every \( z \in C \).

If \( x \in R \), then \( \exp x \in R \), since the coefficients in the series expansion for the exponential function are real numbers. Observe that \( \exp x \geq 1 \) when \( x \geq 0 \), since all the terms in the series expansion for the exponential function are nonnegative real numbers. Consequently,

\[
\exp x > 0 \tag{2.5}
\]

for every \( x \in R \).

We also have that \( \exp z = \exp \bar{z} \) for every \( z \in C \), because the coefficients of the series expansion are real. Hence

\[
|\exp z|^2 = (\exp z)(\exp \bar{z}) = \exp(z + \bar{z}) = \exp(2 \text{Re } z) \tag{2.6}
\]

and therefore

\[
|\exp z| = \exp \text{Re } z. \tag{2.7}
\]

In particular, if \( t \in R \), then

\[
|\exp(it)| = 1. \tag{2.8}
\]

One can show that

\[
\exp(it) = \cos t + i \sin t, \tag{2.9}
\]

by comparing series expansions. Alternatively, one can view this in terms of differential equations, using

\[
\frac{d}{dt} \exp(it) = i \exp(it). \tag{2.10}
\]

Geometrically, \( \exp(it) \) wraps around the unit circle at unit speed counterclockwise. Thus \( t \in R \) is the same as the length of the oriented arc from 1 to \( \exp(it) \) on the unit circle, and \( \cos t, \sin t \) are the projections of this point on the unit circle onto the real and imaginary axes, as usual.
Observe in particular that

\begin{equation}
\exp(2 \pi i) = 1, \tag{2.11}
\end{equation}

because the length of the unit circle is equal to \(2 \pi\). It follows that

\begin{equation}
\exp(z + 2 \pi i) = \exp z \tag{2.12}
\end{equation}

for every \(x \in \mathbb{C}\).

## 3 Laurent series

A *Laurent series* has the form

\begin{equation}
\sum_{n=-\infty}^{\infty} a_n z^n \tag{3.1}
\end{equation}

for some coefficients \(a_n \in \mathbb{C}\), where \(n\) runs through all integers, positive and negative. This can be considered as a combination of the two series

\begin{equation}
\sum_{n=0}^{\infty} a_n z^n, \quad \sum_{n=1}^{\infty} a_{-n} z^{-n}, \tag{3.2}
\end{equation}

which are power series in \(z\) and \(1/z\), respectively.

The convergence of the Laurent series (3.1) can be defined in terms of the convergence of the two series (3.2) individually. The radii of convergence of the two component power series lead to a maximal open annulus

\begin{equation}
\{z \in \mathbb{C} : r < |z| < R\} \tag{3.3}
\end{equation}

on which the Laurent series converges absolutely and uniformly on compact subsets to a holomorphic function, where

\begin{equation}
0 \leq r \leq R \leq +\infty. \tag{3.4}
\end{equation}

The series

\begin{equation}
\sum_{n=-\infty}^{\infty} n a_n z^n \tag{3.5}
\end{equation}

converges absolutely and uniformly on compact subsets of the same open annulus, and is equal to the complex derivative of the function defined by (3.1).
If \( r = 0 \) and \( R > 0 \), then the annulus is actually a punctured disk when \( R < \infty \) and a punctured plane when \( R = +\infty \). If \( a_n = 0 \) for every \( n < 0 \), then the Laurent series is a power series, and one can include 0 in the domain of convergence to get a disk or plane. If \( a_n = 0 \) for every \( n > 0 \), then one can include \( z = \infty \) in the domain of convergence of the series, with the convention that \( 1/z = 0 \). If \( a_n \neq 0 \) for at least one and only finitely many \( n < 0 \), then one can interpret the series as taking the value \( \infty \) at \( z = 0 \). Similarly, if \( a_n \neq 0 \) for at least one and only finitely many \( n > 0 \), then one can interpret the series as taking the value \( \infty \) at \( z = \infty \).

Suppose that \( \sum_{j=-\infty}^{\infty} a_j \) is absolutely convergent, i.e., that \( \sum_{j=-\infty}^{\infty} |a_j| \) converges, which is equivalent to the condition that the partial sums \( \sum_{j=-N}^{N} |a_j|, N \geq 0 \), be uniformly bounded. This implies that the series (3.1) is absolutely convergent for every \( z \in \mathbb{C} \) with \( |z| = 1 \), and that the partial sums converge uniformly to a continuous complex-valued function on the unit circle.

Suppose also that \( \sum_{l=-\infty}^{\infty} b_l \) is an absolutely convergent doubly-infinite series of complex numbers. The Cauchy product of \( \sum a_j, \sum b_l \) is given by

\[
\sum_{n=-\infty}^{\infty} c_n, \quad c_n = \sum_{j=-\infty}^{\infty} a_j b_{n-j}. 
\]

Using the absolute convergence of \( \sum a_j, \sum b_l \) one can check that the series defining \( c_n \) converges absolutely for every \( n \), that the series \( \sum c_n \) converges absolutely, and that

\[
\sum_{n=-\infty}^{\infty} |c_n| \leq \left( \sum_{j=-\infty}^{\infty} |a_j| \right) \left( \sum_{l=-\infty}^{\infty} |b_l| \right). 
\]

Moreover, under these conditions we have that

\[
\sum_{n=-\infty}^{\infty} c_n = \left( \sum_{j=-\infty}^{\infty} a_j \right) \left( \sum_{l=-\infty}^{\infty} b_l \right). 
\]

One can show this by approximating the infinite series by finite sums.

For the same reasons, the product of \( \sum a_j z^j, \sum b_l z^l \) is equal to \( \sum c_n z^n \) for every \( z \in \mathbb{C} \) with \( |z| = 1 \). Analogous statements hold for products of holomorphic functions defined on an annulur region (3.3) by absolutely convergent Laurent series.
4 Polynomials

A holomorphic polynomial on $\mathbb{C}$ can be expressed as

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where the coefficients $a_0, \ldots, a_n$ are complex numbers. Such a polynomial defines a holomorphic function on the complex plane.

Let us think of the complex variable $z$ as being $x + yi$, where $x, y$ are independent real variables, which amounts to identifying the complex plane with $\mathbb{R}^2$. In general polynomials on $\mathbb{R}^2$ are given by finite linear combinations of the real monomials $x^j y^l$, $j, l \geq 0$. Let us continue to use complex coefficients here, which ensures that holomorphic polynomials in $z$ are polynomials in the two real variables $x, y$.

Equivalently, a general polynomial of $x, y$ is a finite linear combination of monomials of the form $z^j \bar{z}^l$ with complex coefficients. The holomorphic polynomials are then just those for which $\bar{z}$ is not necessary.

The Laplace operator $\Delta$ on the plane is defined as usual by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

which is the same as

$$\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}.$$

A twice continuously-differentiable complex-valued function $f$ on an open set $U$ in the plane is said to be harmonic if

$$\Delta f = 0$$

on $U$. Holomorphic functions are continuously differentiable of all orders, by general results, and automatically harmonic.

Conjugate holomorphic functions on open sets in the plane are defined in the same way as holomorphic functions but with the differential equation

$$\frac{\partial}{\partial \bar{z}} f(z) = 0.$$

Equivalently, $f(z)$ is conjugate holomorphic if $\overline{f(z)}$ is holomorphic, which is also equivalent to $f(\overline{z})$ being holomorphic on the complex conjugate of the domain of $f$. 

11
Let us note too that $f(z)$ is holomorphic if and only if $\overline{f(\overline{z})}$ is holomorphic on the complex conjugate of the domain of $f$. At any rate, both holomorphic and conjugate holomorphic functions are automatically harmonic.

A harmonic polynomial on $\mathbb{C}$ is a polynomial which defines a harmonic function on $\mathbb{C}$. Because holomorphic and conjugate holomorphic functions are harmonic, every polynomial which is a linear combination of $z^j$’s and $\overline{z}^l$’s is harmonic. Conversely every harmonic polynomial is of this form.

A general polynomial on $\mathbb{C}$ is a linear combination of monomials of the form $z^j \overline{z}^l$, which is the same as a linear combination of monomials $z^j |z|^{2r}$, $\overline{z}^l |z|^{2r}$, $j, l, r \geq 0$. It follows that every polynomial on $\mathbb{C}$ agrees with a harmonic polynomial on the unit circle.

Let us write $T$ for the unit circle, i.e., the set of $z \in \mathbb{C}$ with $|z| = 1$. If $f(z)$ is a continuous complex-valued function on $T$, then $f(\exp(it))$ is a continuous function on the real line which is periodic with period $2\pi$. Every $2\pi$-periodic continuous function on the real line corresponds to a continuous function on the unit circle in this way.

For a continuous complex-valued function $f(z)$ on $T$, the integral

$$\int_T f(z) \, |dz|$$

of $f$ over the unit circle with respect to arc length is the same as

$$\int_0^{2\pi} f(\exp(it)) \, dt$$

as an ordinary Riemann integral on the real line. We shall typically normalize these integrals by dividing by $2\pi$.

If $f(z) = z^n$ for some $n \geq 1$, then $f(\exp(it)) = \exp(in \cdot t)$ can be expressed as $\left(1/\pi n\right)(d/dt) \exp(in \cdot t)$, and therefore

$$\int_T z^n \, |dz| = 0.$$  

Similarly,

$$\int_T \overline{z}^n \, |dz| = 0$$

for every $n \geq 1$. For $n = 0$ we note that the normalized integral of the constant function 1 is equal to 1.

The standard integral Hermitian inner product on the vector space $\mathcal{C}(T)$ of continuous complex-valued functions on the unit circle is defined by

$$\langle f_1, f_2 \rangle_T = \frac{1}{2\pi} \int_T f_1(z) \overline{f_2(z)} \, |dz|,$$
Suppose that
\begin{equation}
(4.11) \quad h(z) = a_n z^n + \cdots + a_1 z + a_0 + a_{-1} \overline{z} + \cdots + a_{-n} \overline{z}^n
\end{equation}
is a harmonic polynomial on \(\mathbb{C}\). The coefficients of \(h\) can be expressed as
\begin{equation}
(4.12) \quad a_j = \frac{1}{2\pi} \int_T h(z) \overline{z}^j |dz| = \frac{1}{2\pi} \int_T h(z) z^{-j} |dz|,
\end{equation}
where the two integrals are the same because \(\overline{z} = 1/z\) when \(|z| = 1\). In particular, \(h(z)\) is uniquely determined by its restriction to the unit circle.

## 5 Harmonic functions

Let \(\sum_{n=-\infty}^{\infty} a_n\) be a doubly-infinite series of complex numbers, and consider
\begin{equation}
(5.1) \quad \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \overline{z}^n.
\end{equation}

One can view this as a sum of two power series, one in \(z\) and the other in \(\overline{z}\).

Suppose that these power series have radii of convergence \(\geq 1\). This means that \(|a_n| r^n|\) is uniformly bounded over all integers \(n\) for every \(r > 0\) such that \(r < 1\). Equivalently,
\begin{equation}
(5.2) \quad \sum_{n=-\infty}^{\infty} |a_n| t^n
\end{equation}
converges for every \(t > 0\) such that \(t < 1\).

It follows that (5.1) converges absolutely at every point in the open unit disk
\begin{equation}
(5.3) \quad \{ z \in \mathbb{C} : |z| < 1 \}.
\end{equation}

Moreover, the partial sums of the series converge uniformly on compact subsets of the open unit disk, and defines a continuous complex-valued function \(h(z)\) there.

By standard results about power series, \(h(z)\) is infinitely differentiable on the open unit disk. One can differentiate the series (5.1) term by term, and it follows that \(h(z)\) is a harmonic function on the open unit disk.
For each $r > 0$ such that $r < 1$ and every integer $\ell$,

$$a_\ell r^{\ell} = \frac{1}{2\pi} \int_T h(r\,z)\,z^{-\ell}\,|dz| = \frac{1}{2\pi} \int_T h(r\,z)\,z^\ell\,|dz|.$$  

(5.4)

This follows by expanding $h(r\,z)$ into a series and interchanging the order of integration and summation, which is allowed because of uniform convergence. The sum of integrals reduces to a single term as in the previous section.

Suppose in addition that $h(z)$ has a continuous extension to the closed unit disk

$$\{z \in \mathbb{C} : |z| \leq 1\}.$$  

(5.5)

A sufficient condition for this to occur is that the series

$$\sum_{n=-\infty}^{\infty} |a_n|$$  

(5.6)

converges, in which event (5.1) converges absolutely at every point in the closed unit disk and the partial sums converge uniformly on the closed unit disk. Note that the series (5.1) is the same as (3.1) when $|z| = 1$. At any rate, if $h(z)$ has a continuous extension to the closed unit disk, also denoted $h(z)$, then

$$a_\ell = \frac{1}{2\pi} \int_T h(z)\,z^{-\ell}\,|dz| = \frac{1}{2\pi} \int_T h(z)\,z^\ell\,|dz|$$  

(5.7)

for every integer $\ell$. This follows by sending $r \to 1$ in the earlier formula, although if the series of coefficients converge absolutely then one can just apply the same argument directly with $r = 1$.

Conversely, let $f(z)$ be a continuous complex-valued function on the unit circle $T$. For every integer $\ell$, put

$$a_\ell = \frac{1}{2\pi} \int_T f(w)\,w^{-\ell}\,|dw| = \frac{1}{2\pi} \int_T f(w)\,w^\ell\,|dw|.$$  

(5.8)

Thus

$$|a_\ell| \leq \frac{1}{2\pi} \int_T |f(w)|\,|dw|$$  

(5.9)

for each $\ell$, and hence (5.1) converges absolutely at every point in the open unit disk with these choices of coefficients. Let $h(z)$ be the value of (5.1) at $z$ when $|z| < 1$. We would like to show that $h(z)$ extends continuously to the closed unit disk with $h(z) = f(z)$ when $|z| = 1$. 

14
For $z, w \in \mathbb{C}$ with $|z| < 1$ and $|w| = 1$, put

\begin{equation}
P(z, w) = \frac{1}{2\pi} \sum_{n=0}^{\infty} z^n \overline{w}^n + \frac{1}{2\pi} \sum_{n=1}^{\infty} z^n w^n. \tag{5.10}
\end{equation}

This is the Poisson kernel associated to the unit disk. By construction,

\begin{equation}
h(z) = \int_{\mathbb{T}} f(w) P(z, w) |dw| \tag{5.11}
\end{equation}

when $|z| < 1$. Observe that

\begin{equation}
P(z, w) = \frac{1}{2\pi} + \frac{1}{\pi} \text{Re} \sum_{n=1}^{\infty} z^n \overline{w}^n, \tag{5.12}
\end{equation}

since $z^n \overline{w}^n$ and $\overline{w}^n w^n$ are complex conjugates of each other. In particular, $P(z, w)$ is actually real for every $z, w$.

Therefore

\begin{equation}
P(z, w) = \frac{1}{2\pi} + \frac{1}{\pi} \text{Re} \frac{z \overline{w}}{1 - z \overline{w}}, \tag{5.13}
\end{equation}

by summing the geometric series. Equivalently,

\begin{equation}
P(z, w) = \frac{1}{2\pi} \text{Re} \frac{1 + z \overline{w}}{1 - z \overline{w}}. \tag{5.14}
\end{equation}

Using

\begin{equation}
\frac{1}{1 - z \overline{w}} = \frac{1 - \overline{w} w}{|1 - z \overline{w}|^2}, \tag{5.15}
\end{equation}

we get that

\begin{equation}
P(z, w) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - w|^2}. \tag{5.16}
\end{equation}

This also employs the fact that $z \overline{w} - \overline{w} w$ is imaginary and the restriction to $|w| = 1$. Consequently,

\begin{equation}
P(z, w) > 0 \tag{5.17}
\end{equation}

for every $z, w$.

It follows from the definition of the Poisson kernel as a series that

\begin{equation}
\int_{\mathbb{T}} P(z, w) |dw| = 1 \tag{5.18}
\end{equation}

for every $z \in \mathbb{C}$, $|z| < 1$. Thus $h(z)$ is an average of the values of $f$ on the unit circle, weighted according to the Poisson kernel. For a fixed $\zeta \in \mathbb{T}$,
$P(z, w)$ is concentrated near $\zeta$ when $z$ is close to $\zeta$. For instance, if $\delta > 0$, $|z - \zeta| < \delta$, and $|w - \zeta| \geq 2\delta$, then $|z - w| > \delta$ and hence

$$P(z, w) \leq \frac{1}{2\pi \delta^2} (1 - |z|^2),$$

which tends to 0 as $|z| \to 1$. One can use this and the continuity of $f$ to show that

$$\lim_{z \to \zeta} \frac{|z|}{h(z)} < 1,$$

which implies that the extension of $h(z)$ to the closed unit disk defined by setting $h(\zeta) = f(\zeta)$ when $|\zeta| = 1$ is continuous, as desired.

### 6 Fourier series

Let $f(z)$ be a continuous complex-valued function on the unit circle $T$. For each integer $\ell$, the $\ell$th Fourier coefficient of $f$ is given by

$$a_\ell = \frac{1}{2\pi i} \int_T f(w) \bar{w}^\ell |dw|,$$

and the associated Fourier series is

$$\sum_{\ell=-\infty}^{\infty} a_\ell z^\ell, \quad z \in T.$$

More precisely, the Fourier series is defined as a formal series, and convergence of the series is one of the principal questions. There are various extensions of Fourier series and versions of convergence to consider too.

If $\sum_{n=-\infty}^{\infty} c_n$ is any series of complex numbers, then we say that this series is *Abel summable* if $\sum_{n=-\infty}^{\infty} |c_n| r^{|n|}$ converges for every $r > 0$ such that $r < 1$ and

$$\lim_{r \to 1^-} \sum_{n=-\infty}^{\infty} c_n r^{|n|}$$

exists, in which event the limit is said to be the Abel sum of $\sum_{n=-\infty}^{\infty} c_n$.

It is easy to check that absolutely convergent series are Abel summable, with the Abel sum equal to the ordinary sum. One can be a bit more careful and get a similar conclusion for series which may converge conditionally.
For a continuous function $f$ on the unit circle, the Fourier coefficients satisfy the bound
\begin{equation}
|a_\ell| \leq \frac{1}{2\pi} \int_T |f(w)| \, |dw|
\end{equation}
for all $\ell$, and hence $\sum |a_\ell| r^{\ell}$ converges absolutely for every $r > 0$ with $r < 1$. The remarks at the end of the previous section show that for every $z \in T$, the Fourier series for $f$ at $z$ has Abel sum equal to $f(z)$.

\section{Fourier transforms}

Let us write $C(\mathbb{R})$ for the vector space of continuous complex-valued functions on the real line.

We say that $f \in C(\mathbb{R})$ is \textit{integrable} if
\begin{equation}
\int_\mathbb{R} |f(x)| \, dx < \infty,
\end{equation}
which is to say that the integrals $\int_a^b |f(x)| \, dx$ of $|f|$ over bounded intervals $[a, b]$ are uniformly bounded over all $a, b \in \mathbb{R}$, $a < b$. The integral in (7.1) can be defined as the supremum of these integrals over bounded intervals.

The integrable functions in $C(\mathbb{R})$ form a linear subspace denoted $\mathcal{I}C(\mathbb{R})$. If $f \in \mathcal{I}C(\mathbb{R})$, then the integral $\int_\mathbb{R} f(x) \, dx$ can be defined as an improper integral, analogous to the way that the partial sums of an absolutely summable series converge.

Let us write $C_{00}(\mathbb{R})$ for the linear subspace of $C(\mathbb{R})$ consisting of functions with \textit{bounded support}, which is to say continuous functions $f(x)$ on the real line for which there are $a, b \in \mathbb{R}$ such that $f(x) = 0$ when $x \leq a$ and when $x \geq b$. Continuous functions on $\mathbb{R}$ with bounded support are automatically integrable, with the integrals reducing to ones on bounded intervals.

A continuous function $f(x)$ on the real line is said to be \textit{bounded} if there is an $A \geq 0$ such that $|f(x)| \leq A$ for every $x \in \mathbb{R}$. The bounded functions in $C(\mathbb{R})$ form a linear subspace which is denoted $\mathcal{B}C(\mathbb{R})$ and which contains the continuous functions with bounded support.

Note that the product of a bounded continuous function on $\mathbb{R}$ and an integrable continuous function on $\mathbb{R}$ is integrable. The product of an arbitrary continuous function on $\mathbb{R}$ and a continuous function on $\mathbb{R}$ that has bounded support also has bounded support.
Let \( f(x) \) be an integrable continuous complex-valued function on the real line. The Fourier transform is the function \( \hat{f}(\xi) \) on the real line given by

\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x) \exp(-\xi x i) \, dx,
\]

where the integral makes sense because \( \exp(-\xi x i) \) is a bounded continuous function on the real line as a function of \( x \) for every \( \xi \in \mathbb{R} \), and hence its product with \( f(x) \) is integrable. Observe that the Fourier transform of an integrable function is bounded, and more precisely

\[
|\hat{f}(\xi)| \leq \int_{\mathbb{R}} |f(x)| \, dx
\]

for every \( \xi \in \mathbb{R} \).

For each \( y \in \mathbb{R} \), put \( f_y(x) = f(x-y) \). Thus \( f_y \) is an integrable continuous function on the real line too, and it is easy to see that its Fourier transform is given by

\[
\hat{f}_y(\xi) = \hat{f}(\xi) \exp(-\xi y i).
\]

## 8 Uniform continuity

A complex-valued function \( f(x) \) on the real line is said to be uniformly continuous if for each \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) for every \( x, y \in \mathbb{R} \) with \( |x-y| < \delta \). There is a general theorem which states that continuous functions are uniformly continuous on compact sets, and hence a continuous function on the real line with bounded support is uniformly continuous.

Uniformly continuous functions are automatically continuous, and we let \( \mathcal{UC}(\mathbb{R}) \) be the linear subspace of \( \mathcal{C}(\mathbb{R}) \) consisting of uniformly continuous functions. Let \( \mathcal{BUC}(\mathbb{R}) \) be the space of bounded uniformly continuous complex-valued functions on \( \mathbb{R} \), a linear subspace of \( \mathcal{BC}(\mathbb{R}) \).

The product of two continuous functions on the real line is also a continuous function, and thus \( \mathcal{C}(\mathbb{R}) \) is actually a commutative algebra with respect to pointwise multiplication. Clearly \( \mathcal{BC}(\mathbb{R}) \) is a subalgebra of \( \mathcal{C}(\mathbb{R}) \), since the product of two bounded functions is a bounded function. One can check that the product of two bounded uniformly continuous functions is uniformly continuous, which implies that \( \mathcal{BUC}(\mathbb{R}) \) is a subalgebra of \( \mathcal{BC}(\mathbb{R}) \).
In general the product of two uniformly continuous functions is not uniformly continuous. The sum of two uniformly continuous functions is uniformly continuous and the product of a uniformly continuous function with a constant is uniformly continuous, which is why \( \mathcal{UC}(\mathbb{R}) \) is a vector space.

The \textit{supremum norm} of a bounded continuous function \( f \) on the real line is defined by
\[
\|f\|_\infty = \sup\{ |f(x)| : x \in \mathbb{R} \}.
\]

Observe that
\[
\|f_1 + f_2\|_\infty \leq \|f_1\|_\infty + \|f_2\|_\infty
\]
and
\[
\|f_1 f_2\|_\infty \leq \|f_1\|_\infty \|f_2\|_\infty
\]
for every \( f_1, f_2 \in \mathcal{BC}(\mathbb{R}) \). The \textit{supremum metric} on \( \mathcal{BC}(\mathbb{R}) \) is defined by
\[
d_\infty(f_1, f_2) = \|f_1 - f_2\|_\infty
\]
for \( f_1, f_2 \in \mathcal{BC}(\mathbb{R}) \).

It is well known that \( \mathcal{BC}(\mathbb{R}) \) is \textit{complete} as a metric space with respect to the supremum metric, in the sense that every Cauchy sequence in \( \mathcal{BC}(\mathbb{R}) \) converges. Also, \( \mathcal{BUC}(\mathbb{R}) \) is a closed subspace of \( \mathcal{BC}(\mathbb{R}) \) with respect to the supremum metric, which implies that \( \mathcal{BUC}(\mathbb{R}) \) is complete with respect to the supremum metric too.

Let \( f \) be a bounded continuous function on the real line, and for every \( y \in \mathbb{R} \) put \( f_y(x) = f(x - y) \), as in the previous section. It is easy to see that \( f \) is uniformly continuous if and only if \( y \mapsto f_y \) is continuous as a mapping from \( \mathbb{R} \) into \( \mathcal{BC}(\mathbb{R}) \) with respect to the supremum metric. There is a similar statement for functions which may not be bounded using the mapping \( y \mapsto f_y - f \).

A complex-valued function \( f(x) \) on the real line is said to be \textit{Lipschitz} if there is a \( C \geq 0 \) such that
\[
|f(x) - f(y)| \leq C |x - y|
\]
for every \( x, y \in \mathbb{R} \). Lipschitz functions are clearly uniformly continuous, and the space of Lipschitz functions on \( \mathbb{R} \) is a linear subspace of \( \mathcal{UC}(\mathbb{R}) \). A continuously differentiable function on the real line is Lipschitz if and only if its derivative is bounded.
For every $x \in \mathbb{R}$, $\exp(-\xi x i)$ is a Lipschitz function of $\xi$ on the real line, and more precisely

\begin{equation}
|\exp(-\xi_1 x i) - \exp(-\xi_2 x i)| \leq |x| |\xi_1 - \xi_2|
\end{equation}

for every $\xi_1, \xi_2 \in \mathbb{R}$. This follows from the fact that the derivative of $\exp(-\xi x i)$ in $\xi$ is equal to $-i x \exp(-\xi x i)$, which has modulus equal to $|x|$ everywhere.

Let $f(x)$ be an integrable continuous function on the real line. For every $a, b \in \mathbb{R}$ with $a \leq b$ one can check that

\begin{equation}
\int_a^b f(x) \exp(-i \xi x) \, dx
\end{equation}

is Lipschitz as a function of $\xi$, using the Lipschitz estimates for the complex exponentials mentioned in the previous paragraph.

For each $\rho > 0$ there are $a, b \in \mathbb{R}$ such that $a \leq b$ and

\begin{equation}
\int_{\mathbb{R}} |f(x)| \, dx < \int_a^b |f(x)| \, dx + \rho.
\end{equation}

This implies that

\begin{equation}
\left| \hat{f}(\xi) - \int_a^b f(x) \exp(-\xi x i) \, dx \right| < \rho
\end{equation}

for every $\xi \in \mathbb{R}$. It is easy to see from here that $\hat{f}(\xi)$ is uniformly continuous.

## 9 Inner product spaces

Let $V$ be a vector space with complex numbers as scalars. An inner product on $V$, or Hermitian inner product, is a complex-valued function $\langle v, w \rangle$ defined for $v, w \in V$ which satisfies the following properties: (i) $\langle v, w \rangle$ is linear in $v$ for each fixed $w \in V$, which is to say that

\begin{equation}
\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle
\end{equation}

for every $v_1, v_2 \in V$, and

\begin{equation}
\langle \alpha v, w \rangle = \alpha \langle v, w \rangle
\end{equation}

for every $\alpha$. \vfill
for every $\alpha \in \mathbb{C}$ and $v \in V$; (ii) $\langle v, w \rangle$ is Hermitian symmetric in the sense that
\begin{equation}
\langle w, v \rangle = \overline{\langle v, w \rangle}
\end{equation}
for every $v, w \in V$; and (iii) $\langle v, w \rangle$ is positive-definite, which means that
\begin{equation}
\langle v, v \rangle \geq 0
\end{equation}
for every $v \in V$, and $\langle v, v \rangle = 0$ if and only if $v = 0$.

Observe that the Hermitian symmetry of the inner product implies that $\langle v, v \rangle \in \mathbb{R}$ for every $v \in V$.

For example, if $n$ is a positive integer, consider $V = \mathbb{C}^n$, the space of $n$-tuples of complex numbers, using coordinatewise addition and scalar multiplication. The standard inner product on $\mathbb{C}^n$ is defined by
\begin{equation}
\langle v, w \rangle = \sum_{j=1}^{n} v_j \overline{w_j},
\end{equation}
$v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n)$.

Let $V$ be a complex vector space equipped with an inner product $\langle v, w \rangle$, and for every $v \in V$ put
\begin{equation}
\|v\| = \langle v, v \rangle^{1/2}.
\end{equation}
If $V = \mathbb{C}^n$ with the standard inner product, as in the previous paragraph, then
\begin{equation}
\|v\| = \left( \sum_{j=1}^{n} |v_j|^2 \right)^{1/2},
\end{equation}
v = (v_1, \ldots, v_n), is the standard norm.

In general, the Cauchy–Schwarz inequality states that
\begin{equation}
|\langle v, w \rangle| \leq \|v\| \|w\|
\end{equation}
for every $v, w \in V$. One can show this using the fact that
\begin{equation}
0 \leq \langle \alpha v + w, \alpha v + w \rangle = |\alpha|^2 \|v\|^2 + 2 \Re \alpha \langle v, w \rangle + \|w\|^2
\end{equation}
for every $\alpha \in \mathbb{C}$.

By the definition of $\|v\|$, 
\begin{equation}
\|\alpha v\| = |\alpha| \|v\|
\end{equation}
for every $\alpha \in \mathbb{C}$ and $v \in V$. Moreover,
\begin{equation}
\|v + w\| \leq \|v\| + \|w\|
\end{equation}
for every $v, w \in V$, because
\begin{equation}
\|v + w\|^2 = \langle v + w, v + w \rangle = \|v\|^2 + 2 \text{Re}\langle v, w \rangle + \|w\|^2 \\
\leq (\|v\| + \|w\|)^2
\end{equation}
by the Cauchy–Schwarz inequality.

A pair of vectors $v, w \in V$ are said to be orthogonal if
\begin{equation}
\langle v, w \rangle = 0,
\end{equation}
in which event we write $v \perp w$. Observe that $v \perp w$ is equivalent to
\begin{equation}
\|v + w\|^2 = \|v\|^2 + \|w\|^2.
\end{equation}

A collection of vectors $u_1, \ldots, u_n \in V$ is said to be orthonormal if $\|u_j\| = 1$ for each $j$ and $u_j \perp u_l$ for every $j, l$ with $j \neq l$. If $u_1, \ldots, u_n$ are orthonormal and $\alpha_1, \ldots, \alpha_n$ are complex numbers, then
\begin{equation}
\left| \sum_{j=1}^n \alpha_j u_j \right|^2 = \sum_{j=1}^n |\alpha_j|^2.
\end{equation}

Let $u_1, \ldots, u_n$ be a collection of orthonormal vectors in $V$, and let $W$ be the linear subspace of $V$ consisting of linear combinations of the $u_j$’s. By definition this means that $W$ consists of the vectors of the form
\begin{equation}
w = \sum_{j=1}^n \alpha_j u_j
\end{equation}
for some complex numbers $\alpha_1, \ldots, \alpha_n$. Because of orthonormality,
\begin{equation}
\alpha_j = \langle w, u_j \rangle
\end{equation}
for each $j$ in this case.

For every $v \in V$ put
\begin{equation}
P(v) = \sum_{j=1}^n \langle v, u_j \rangle u_j.
\end{equation}
Thus $P(v) \in W$ and $P(w) = w$ for every $w \in W$ by the remarks of the preceding paragraph.

By construction, $v - P(v) \perp u_j$ for each $j$, and hence $v - P(v) \perp w$ for every $w \in W$. In particular, $v - P(v) \perp P(v)$, and therefore

$$
(9.19) \quad \|v\|^2 = \|v - P(v)\|^2 + \|P(v)\|^2 = \|v - P(v)\|^2 + \sum_{j=1}^{n} |\langle v, u_j \rangle|^2.
$$

Consequently,

$$
(9.20) \quad \sum_{j=1}^{n} |\langle v, u_j \rangle|^2 \leq \|v\|^2.
$$

If $v \in V$ and $w \in W$, then $v - w$ can be expressed as the sum of $v - P(v)$ and $P(v) - w$, and these two vectors are orthogonal to each other since $P(v) - w \in W$. Hence

$$
(9.21) \quad \|v - w\|^2 = \|v - P(v)\|^2 + \|P(v) - w\|^2,
$$

which implies that

$$
(9.22) \quad \|v - P(v)\| \leq \|v - w\|
$$

with equality exactly when $w = P(v)$.

### 10 Fourier series, continued

As in Section 4,

$$
(10.1) \quad \langle f_1, f_2 \rangle_T = \frac{1}{2\pi} \int_T f_1(z) \overline{f_2(z)} |dz|
$$

defines an inner product on the vector space $C(T)$ of continuous complex-valued functions on $T$. The functions $z^n$ on $T$, where $n$ runs through the integers, are orthonormal with respect to this inner product.

Let $f$ be a continuous complex-valued function on the unit circle, and let $a_\ell$ be the $\ell$th Fourier coefficient of $f$. Equivalently,

$$
(10.2) \quad a_\ell = \langle f, z^\ell \rangle_T.
$$

For $N \geq 0$ put

$$
(10.3) \quad f_N(z) = \sum_{\ell=1-N}^{N} a_\ell \overline{z^\ell},
$$

23
as a continuous function on the unit circle. Because of orthonormality,

\[ \frac{1}{2\pi} \int_T |f_N(z)|^2 \, dz = \sum_{\ell=-N}^N |a_\ell|^2. \]  

(10.4)

Moreover,

\[ \frac{1}{2\pi} \int_T |f(z)|^2 \, dz = \sum_{\ell=-N}^N |a_\ell|^2 + \frac{1}{2\pi} \int_T |f(z) - f_N(z)|^2 \, dz. \]  

(10.5)

Thus

\[ \sum_{\ell=-N}^N |a_\ell|^2 \leq \frac{1}{2\pi} \int_T |f(z)|^2 \, dz \]  

for each \( N \). Hence

\[ \sum_{\ell=-\infty}^\infty |a_\ell|^2 \leq \frac{1}{2\pi} \int_T |f(z)|^2 \, dz, \]  

(10.6)

where the sum of the \( |a_\ell|^2 \)'s converges in particular.

Consider

\[ \frac{1}{2\pi} \int_T |f(z) - p(z)|^2 \, dz. \]  

(10.8)

This is the same as the minimum of

\[ \frac{1}{2\pi} \int_T |f(z) - p(z)|^2 \, dz \]  

(10.9)

over all functions \( p(z) \) on \( T \) which are linear combinations of \( z^j, -N \leq j \leq N \), with equality exactly when \( p(z) = f_N(z) \).

Every continuous function on the unit circle is uniformly continuous, by compactness. The arguments described at the end of Section 5 imply that the Abel sums of the Fourier series of \( f \) converge to \( f \) uniformly on \( T \). Using this one can check that \( f \) can be uniformly approximated by finite linear combinations of the \( z^j \)'s, which is a version of the Lebesgue – Weierstrass – Stone approximation theorem.

More explicitly, for each \( \epsilon > 0 \) there is an \( N_\epsilon \geq 0 \) and a function \( p_\epsilon(z) \) on \( T \) such that \( p_\epsilon(z) \) is a linear combination of \( z^j, -N_\epsilon \leq j \leq N_\epsilon \), and

\[ |f(z) - p_\epsilon(z)| < \epsilon \]  

(10.10)
for every \( z \in T \). Hence

\[
\frac{1}{2\pi} \int_T |f(z) - p_\epsilon(z)|^2 |dz| < \epsilon^2.
\]

(10.11)

Because of the minimization property,

\[
\frac{1}{2\pi} \int_T |f(z) - f_N(z)|^2 |dz| < \epsilon^2
\]

(10.12)

when \( N \geq N_\epsilon \). Therefore

\[
\lim_{N \to \infty} \frac{1}{2\pi} \int_T |f(z) - f_N(z)|^2 |dz| = 0.
\]

(10.13)

Furthermore,

\[
\frac{1}{2\pi} \int_T |f(z)|^2 |dz| = \sum_{\ell=-\infty}^{\infty} |a_\ell|^2.
\]

(10.14)

11 Almost periodic functions

Let \( f \) be a bounded continuous complex-valued function on the real line. We say that \( f \) is almost periodic if for each \( \epsilon > 0 \) there are \( y_1, \ldots, y_n \in \mathbb{R} \) such that for every \( y \in \mathbb{R} \),

\[
|f(x - y) - f(x - y_j)| < \epsilon
\]

(11.1)

for some \( j \) and all \( x \in \mathbb{R} \).

Equivalently, \( f \) is almost periodic if for each \( \epsilon > 0 \) there are \( h_1, \ldots, h_n \) in \( \mathcal{BC}(\mathbb{R}) \) such that for every \( y \in \mathbb{R} \),

\[
|f(x - y) - h_j(x)| < \epsilon
\]

(11.2)

for some \( j \) and all \( x \in \mathbb{R} \). More precisely, one can check that this condition for \( \epsilon/2 \) implies the previous one for \( \epsilon \).

In more abstract terms, \( f \) is almost periodic if the collection of translates \( f_y(x) = f(x - y), y \in \mathbb{R} \), is a precompact set in \( \mathcal{BC}(\mathbb{R}) \) with respect to the supremum metric. Because \( \mathcal{BC}(\mathbb{R}) \) is complete, a set of bounded continuous functions is precompact if and only if it is totally bounded, which means that for each \( \epsilon > 0 \) the set is contained in the union of finitely many balls of radius \( \epsilon \). One can also ask that the balls have their centers contained in the
set, because a ball with radius $\epsilon/2$ which intersects the set is contained in a ball with radius $\epsilon$ centered on the set.

If $f$ is a continuous periodic function on the real line, then $f$ is uniformly continuous, since any continuous function is uniformly continuous on compact sets. Thus $y \mapsto f_y$ is a continuous periodic mapping from the real line into $\mathcal{BC}(\mathbb{R})$, whose image is therefore compact, which implies that $f$ is an almost periodic function.

The space of almost periodic functions on $\mathbb{R}$ is denoted $\mathcal{AP}(\mathbb{R})$. Observe that every translate of an almost periodic function is almost periodic, since the set of translations would be the same.

One can check that the sum or product of two almost periodic functions on the real line is almost periodic. In particular, the sum or product of two periodic functions is almost periodic, without restrictions on the periods.

Thus $\mathcal{AP}(\mathbb{R})$ is a subalgebra of $\mathcal{BC}(\mathbb{R})$. One can check that $\mathcal{AP}(\mathbb{R})$ is closed with respect to the supremum metric, which basically means that the limit of a sequence of almost periodic functions on the real line which converges uniformly is almost periodic.

A continuous function $f(x)$ on $\mathbb{R}$ is uniformly continuous if and only if the family of translations $f_y(x)$, $y \in \mathbb{R}$, is equicontinuous at $x = 0$, which means that for each $\epsilon > 0$ there is a $\delta > 0$ such that

\begin{equation}
|f_y(x) - f_y(0)| < \epsilon
\end{equation}

for every $x, y \in \mathbb{R}$ with $|x| < \delta$. Using this characterization one can check that almost periodic functions are uniformly continuous.

Let $\{\xi_j\}_{j=1}^\infty$ be an arbitrary sequence of real numbers, let $\sum_{j=1}^\infty a_j$ be a sequence of complex numbers which is absolutely convergent, and consider

\begin{equation}
\sum_{j=1}^\infty a_j \exp(\xi_j x i).
\end{equation}

The partial sums of this series of functions converges uniformly, by the Weierstrass $M$-test. It follows that the sum is almost periodic.

Let $\mathcal{C}_0(\mathbb{R})$ be the space of continuous complex-valued functions $f(x)$ on the real line which “vanish at infinity” in the sense that for each $\epsilon > 0$ there is an $R \geq 0$ such that $|f(x)| < \epsilon$ for every $x \in \mathbb{R}$ with $|x| \geq R$. Continuous functions on the real line with bounded support have this property trivially.

One can check that the sum and product of two continuous functions on the real line which vanish at infinity also vanishes at infinity. Continuous
function on the real line which vanish at infinity are bounded and uniformly continuous, since continuous functions are bounded and uniformly continuous on compact sets.

One can also check that \( C_0(\mathbb{R}) \) is closed with respect to the supremum metric. In other words, if a sequence of continuous functions on the real line vanish at infinity and converge uniformly, then the limit vanishes at infinity too. Continuous functions on the real line with bounded support are dense among continuous functions which vanish at infinity with respect to the supremum metric, which is to say that \( C_0(\mathbb{R}) \) is the closure of \( C_{00}(\mathbb{R}) \) in \( BC(\mathbb{R}) \) with respect to the supremum metric.

If \( f \) is a continuous function on the real line which vanishes at infinity and is almost periodic, then \( f(x) = 0 \) for every \( x \in \mathbb{R} \). For if \( \epsilon > 0 \) and \( y_1, \ldots, y_n \in \mathbb{R} \) have the property that for every \( y \in \mathbb{R} \) there is a \( j \) such that \( \| f_y - f_{y_j} \|_\infty \leq \epsilon \), then one can apply this with \( |y| \) very large to get that for every \( r \geq 0 \) there is a \( j \) such that \( |f(x - y_j)| < 2 \epsilon \) when \( |x| \leq 2 r \), and hence that \( |f(x)| < 2 \epsilon \) when \( |x| \leq r \) and \( r \geq \max(|y_1|, \ldots, |y_n|) \).

## 12 Step functions

By an interval in the real line we mean a set \( I \subseteq \mathbb{R} \) such that for every \( x, z \in I \) and \( y \in \mathbb{R} \) such that \( x < y < z \), we have that \( y \in I \).

This includes open intervals

\[(12.1) \quad (a, b) = \{ x \in \mathbb{R} : a < x < b \},\]

closed intervals

\[(12.2) \quad [a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \},\]

and the half-open, half-closed intervals

\[(12.3) \quad (a, b] = \{ x \in \mathbb{R} : a < x \leq b \}\]

and

\[(12.4) \quad [a, b) = \{ x \in \mathbb{R} : a \leq x < b \},\]

\( a, b \in \mathbb{R} \), \( a \leq b \). More precisely, these are bounded intervals, and there are unbounded intervals which are open and closed half-lines as well as the real line.

In general, if \( X \) is a set and \( E \subseteq X \), then the indicator function associated to \( E \) on \( X \) is the function \( 1_E(x) \) which is equal to \( 1 \) when \( x \in E \) and to 0
when \( x \in X \setminus E \). A *step function* on the real line is a function \( f(x) \) which can be expressed as
\[
(12.5) \quad f(x) = \sum_{j=1}^{n} \alpha_j 1_{I_j}(x)
\]
for some complex numbers \( \alpha_1, \ldots, \alpha_n \) and intervals \( I_1, \ldots, I_n \).

Thus step functions are automatically bounded. If a function \( f(x) \) can be expressed as a sum as in (12.5) where the intervals \( I_j \) are bounded, then we say that \( f(x) \) is an *integrable step function*.

If \( f(x) \) is a step function on the real line and \( a, b \in \mathbb{R}, a \leq b \), then the integral
\[
(12.6) \quad \int_a^b f(x) \, dx
\]
can be defined in the usual way, since
\[
(12.7) \quad \int_a^b 1_f(x) \, dx = |I \cap [a, b]|
\]
for any interval \( I \), where \(|A|\) denotes the length of an interval \( A \). If \( f(x) \) is an integrable step function, then
\[
(12.8) \quad \int_{\mathbb{R}} f(x) \, dx
\]
makes sense, using
\[
(12.9) \quad \int_{\mathbb{R}} 1_f(x) \, dx = |I|
\]
when \( I \) is a bounded interval.

More generally, one can consider piecewise-continuous functions on \( \mathbb{R} \), in order to have a class of functions which includes step functions and continuous functions.

If \( a, b \in \mathbb{R} \) and \( a < b \), then
\[
(12.10) \quad \int_a^b \exp(-\xi x i) \, dx = \frac{i}{\xi} (\exp(-\xi b i) - \exp(-\xi a i))
\]
when \( \xi \neq 0 \) and to \( b - a \) when \( \xi = 0 \). This is a continuous function on the real line which vanishes at infinity, and in particular it follows that integrals of families of continuous periodic functions may not be almost periodic.
If \( f \) is an integrable step function on the real line, then the Fourier transform of \( f \) is defined as usual by
\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x) \exp(-\xi x i) \, dx.
\]
(12.11)
This can be expressed as a linear combination of integrals as in the previous paragraph, and hence \( \hat{f} \in C_0(\mathbb{R}) \).

Let \( f \) be a continuous integrable function on the real line, and let \( \epsilon > 0 \) be given. Let \( a, b \) be real numbers such that \( a < b \) and
\[
\int_{\mathbb{R}} |f(x)| \, dx < \int_{a}^{b} |f(x)| \, dx + \frac{\epsilon}{2}.
\]
(12.12)
Because \( f \) is uniformly continuous on \([a, b]\), there is an integrable step function \( f_1(x) \) on \( \mathbb{R} \) such that
\[
|f_1(x) - f(x)| < \frac{\epsilon}{2(b - a)}
\]
(12.13)
when \( x \in I \) and \( f_1(x) = 0 \) when \( x \notin I \), and therefore
\[
\int_{\mathbb{R}} |f(x) - f_1(x)| \, dx < \epsilon.
\]
(12.14)
This implies that
\[
|\hat{f}(\xi) - \hat{f_1}(\xi)| < \epsilon
\]
(12.15) for every \( \xi \in \mathbb{R} \). Since \( \hat{f_1} \in C_0(\mathbb{R}) \) and \( C_0(\mathbb{R}) \) is closed with respect to the supremum metric, it follows that \( \hat{f} \in C_0(\mathbb{R}) \), which is a version of the Riemann–Lebesgue lemma.

### 13 Norms

Let \( V \) be a complex vector space and let \( N(v) \) be a nonnegative real-valued function on \( V \). We say that \( N(v) \) is a norm on \( V \) if \( N(v) = 0 \) if and only if \( v = 0 \), \( N(v) \) is homogeneous of degree 1 in the sense that
\[
N(\alpha v) = |\alpha| \, N(v)
\]
(13.1)
for every \( \alpha \in \mathbb{C} \) and \( v \in V \), and
\[
N(v + w) \leq N(v) + N(w)
\]
(13.2)
for every $v, w \in V$.

A set $E \subseteq V$ is said to be \textit{convex} if for every $v, w \in E$ and $t \in \mathbb{R}$ with $0 \leq t \leq 1$,
\begin{equation}
(13.3) \quad t v + (1 - t) w \in E.
\end{equation}
If $N(v)$ is a norm on $V$, then
\begin{equation}
(13.4) \quad \{ v \in V : N(v) \leq 1 \}
\end{equation}
is convex. Conversely, the triangle inequality (13.2) is implied by the combination of the other conditions on $N$ described in the previous paragraph and the convexity of the closed unit ball (13.4).

For example, let $n$ be a positive integer, and suppose that $V = \mathbb{C}^n$. Put
\begin{equation}
(13.5) \quad \|v\|_p = \left( \sum_{j=1}^{n} |v_j|^p \right)^{1/p}
\end{equation}
for $p > 0$ and $v \in \mathbb{C}^n$. When $p = \infty$ put
\begin{equation}
(13.6) \quad \|v\|_{\infty} = \max(|v_1|, \ldots, |v_n|).
\end{equation}

Observe that
\begin{equation}
(13.7) \quad \|v\|_{\infty} \leq \|v\|_p
\end{equation}
for every $p > 0$ and $v \in \mathbb{C}^n$. Since
\begin{equation}
(13.8) \quad \|v\|_q = \sum_{j=1}^{n} |v_j|^q \leq \|v\|_{\infty}^{q-p} \sum_{j=1}^{\infty} |v_j|^p = \|v\|_{\infty}^{q-p} \|v\|_p
\end{equation}
when $0 < p \leq q < \infty$, we get that
\begin{equation}
(13.9) \quad \|v\|_q \leq \|v\|_p
\end{equation}
for every $v \in \mathbb{C}^n$ when $0 < p \leq q$.

It is easy to see directly from the definitions that $\|v\|_p$ defines a norm on $\mathbb{C}^n$ when $p = 1, \infty$. When $1 < p < \infty$ one can use the convexity of the function $r^p$ on the nonnegative real numbers to show that the closed unit ball associated to $\|v\|_p$ is convex, and hence that $\|v\|_p$ is a norm.

If $0 < p < 1$ and $n \geq 2$, then the closed unit ball associated to $\|v\|_p$ is not convex and $\|v\|_p$ is not a norm. Using the inequality $(a + b)^p \leq a^p + b^p$ for nonnegative real numbers $a, b$ when $0 < p \leq 1$, one can check that
\begin{equation}
(13.10) \quad \|v + w\|_p \leq \|v\|_p + \|w\|_p
\end{equation}
for every \( v, w \in \mathbb{C}^n \), which is a natural alternative to convexity. As another alternative, the closed unit ball associated to \( \|v\|_p \) is “pseudoconvex” in the sense of several complex variables, and \( \|v\|_p \) is a “plurisubharmonic” function instead of a convex function.

If \( V \) is a complex vector space equipped with an inner product, then the norm associated to the inner product as in Section 9 is a norm in the general sense considered here. In particular, \( \|v\|_2 \) is the norm associated to the standard inner product on \( \mathbb{C}^n \).

Now let \( V = C(T) \), the space of continuous complex-valued functions on the unit circle. For \( 1 \leq p < \infty \), put

\[
\|f\|_p = \|f\|_{p,T} = \left( \frac{1}{2\pi} \int_T |f(z)|^p |dz| \right)^{1/p}.
\]

(13.11)

We can extend this to \( p = \infty \) using the supremum norm,

\[
\|f\|_\infty = \|f\|_{\infty,T} = \sup\{|f(z)| : z \in T\}.
\]

(13.12)

Observe that

\[
\|f\|_p \leq \|f\|_\infty
\]

(13.13)

for every continuous function \( f \) on the unit circle. More generally,

\[
\|f\|_p \leq \|f\|_q
\]

(13.14)

when \( 1 \leq p \leq q < \infty \), because of the convexity of the function \( r^{q/p} \) on the nonnegative real numbers.

It is easy to see directly that \( \|f\|_p \) defines a norm on \( C(T) \) when \( p = 1, \infty \). When \( p = 2, \|f\|_2 \) is the norm associated to the integral inner product considered in Section 4. One can show that \( \|f\|_p \) defines a norm on \( C(T) \) for every \( p, 1 < p < \infty \), by showing that the associated closed unit ball is a convex set in \( C(T) \) using the convexity of the function \( r^p \) on the nonnegative real numbers.

Similarly, if \( f \) is a continuous complex-valued function on the real line with bounded support, put

\[
\|f\|_p = \|f\|_{p,R} = \left( \int_R |f(x)|^p dx \right)^{1/p}
\]

(13.15)

when \( 1 \leq p < \infty \), and let \( \|f\|_\infty = \|f\|_{\infty,R} \) be the supremum norm of \( f \) as in Section 8. One can check that \( \|f\|_p \) defines a norm on \( C_{00}(\mathbb{R}) \) for every \( p, 1 \leq p \leq \infty \), and that

\[
\langle f_1, f_2 \rangle = \langle f_1, f_2 \rangle_R = \int_R f_1(x) f_2(x) dx
\]

(13.16)
defines an inner product on $C_{00}(\mathbb{R})$ for which the associated norm is equal to $\|f\|_2$.

As in Section 8, the supremum norm can be defined in the same way on the space $\mathcal{BC}(\mathbb{R})$ of bounded continuous complex-valued functions on the real line. The norm $\|f\|_1$ can be defined in the same way on the space $\mathcal{IC}(\mathbb{R})$ of integrable continuous complex-valued functions on $\mathbb{R}$.

For $1 \leq p < \infty$, let $\mathcal{IC}^p(\mathbb{R})$ be the space of continuous complex-valued functions $f(x)$ on the real line such that $|f(x)|^p$ is integrable on $\mathbb{R}$. One can check that $\mathcal{IC}^p(\mathbb{R})$ is a linear subspace of $C(\mathbb{R})$, and it clearly contains the continuous functions with bounded support. The norm $\|f\|_p$ can be extended to $\mathcal{IC}^p(\mathbb{R})$ using the same formula as for functions with bounded support. The product of two square-integrable continuous functions on the real line is an integrable continuous function on the real line, and the integral inner product on $C_{00}(\mathbb{R})$ described above extends to an inner product on $\mathcal{IC}^2(\mathbb{R})$ by the same formula, whose associated norm is equal to $\|f\|_2$.

As another variant of the same notions, let $a, b$ be real numbers with $a < b$, and let $C([a, b])$ be the vector space of continuous complex-valued functions on the interval $[a, b]$. For every $f \in C([a, b])$, put

\begin{equation}
\|f\|_p = \|f\|_{p,[a,b]} = \left( \int_a^b |f(x)|^p \, dx \right)^{1/p}
\end{equation}

when $1 \leq p < \infty$ and

\begin{equation}
\|f\|_\infty = \|f\|_{\infty,[a,b]} = \sup\{|f(x)| : a \leq x \leq b\}.
\end{equation}

These are norms on $C([a, b])$ for the same reasons as in the previous situations. Furthermore,

\begin{equation}
\langle f_1, f_2 \rangle = \langle f_1, f_2 \rangle_{[a,b]} = \int_a^b f_1(x) \overline{f_2(x)} \, dx
\end{equation}

defines an inner product on $C([a, b])$ for which the associated norm is equal to $\|f\|_2$.

14 Some computations

For each $\eta > 0$, put

\begin{equation}
A_\eta(x) = \exp(-\eta |x|).
\end{equation}
Thus $A_\eta$ is a continuous integrable function on the real line, with

(14.2) \[ \int_{\mathbb{R}} A_\eta(x) \, dx = 2 \int_{0}^{\infty} \exp(-\eta x) \, dx = \frac{2}{\eta}. \]

Observe that

(14.3) \[ \int_{0}^{\infty} \exp(-\eta x - \xi x i) \, dx = \frac{1}{\eta + \xi i}. \]

Similarly,

(14.4) \[ \int_{-\infty}^{0} \exp(\eta x - \xi x i) \, dx = \int_{0}^{\infty} \exp(-\eta x + \xi x i) \, dx = \frac{1}{\eta - \xi i}. \]

Hence

(14.5) \[ \hat{A}_\eta(\xi) = \frac{1}{\eta - \xi i} + \frac{1}{\eta + \xi i} = \frac{2\eta}{\xi^2 + \eta^2}. \]

Using this formula one can check directly that $\hat{A}_\eta(\xi)$ is an integrable continuous function on the real line, and that $\int_{\mathbb{R}} \hat{A}_\eta(\xi) \, d\xi$ does not depend on $\eta$. We would like to determine its precise value.

For every $\zeta \in \mathbb{C}$ with $\text{Re}\, \zeta > 0$ there is a unique $\log \zeta \in \mathbb{C}$ such that

(14.6) \[ -\pi < \text{Im}\, \log \zeta < \pi \]

and

(14.7) \[ \exp \log \zeta = \zeta. \]

Moreover, $\log \zeta$ is a holomorphic function on the right half-plane and

(14.8) \[ \left( \frac{\partial}{\partial \zeta} \log \zeta \right) = \frac{1}{\zeta}. \]

If $a, b \in \mathbb{R}$, $a \leq b$, and $\eta > 0$, then

(14.9) \[ \int_{a}^{b} \frac{1}{\eta + \xi i} \, d\xi = -i \int_{a}^{b} \frac{\partial}{\partial \xi} \log(\eta + \xi i) \, d\xi = -i(\log(\eta + b i) - \log(\eta + a i)). \]

Since $\hat{A}_\eta(\xi) = 2\text{Re} \, 1/(\eta + \xi i)$,

(14.10) \[ \int_{a}^{b} \hat{A}_\eta(\xi) \, d\xi = 2\text{Im}(\log(\eta + b i) - \log(\eta + a i)). \]
Therefore
\[
\int_{\mathbb{R}} \widehat{A}_\eta(\xi) \, d\xi = 2\pi,
\]
because \( \text{Im} \log(\eta + \xi i) \) tends to \(-\pi/2\) as \( \xi \to -\infty \) and to \( \pi/2\) as \( \xi \to \infty \).

This approach to the computation of the integral is quite pleasant in the way that the integrand is explicitly a derivative. Alternatively, the computation of the integral is a standard exercise in basic calculus using trigonometric substitution.

One can also view the integral geometrically, by projecting the line \( \eta + \xi i, \xi \in \mathbb{R} \), in the complex plane onto the right half of the unit circle, and checking that the integrand corresponds to twice the element of arclength on the circle. The integral can be treated as well by applying Cauchy’s theorem in complex analysis to convert the integral on a line to a simpler integral on a circle.

### 15 Fourier inversion

Let \( f(x) \) be an integrable continuous complex-valued function on the real line. We would like to make sense of
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) \exp(\xi x i) \, d\xi
\]
and show that it is equal to \( f(x) \).

As an analogue of Abel sums, consider the integrals
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) A_\eta(\xi) \exp(\xi x i) \, d\xi
\]
for \( \eta > 0 \). These integrals make sense, because \( A_\eta(\xi) \) is integrable for each \( \eta > 0 \) and because \( \hat{f} \) is a bounded continuous function. If \( \hat{f}(\xi) \) happens to be integrable, then one can check that (15.1) is equal to the limit of (15.2) as \( \eta \to 0 \).

Using the definition of the Fourier transform we can rewrite (15.2) as a double integral
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) A_\eta(\xi) \exp(\xi (x - y) i) \, dy \, d\xi.
\]
Integrating in \( \xi \) first and using the computations from the previous section, we get that this is equal to
\[
\int_{\mathbb{R}} f(y) P_\eta(x - y) \, dy.
\]
where
\begin{equation}
  P_t(y) = \frac{1}{\pi} \frac{t}{y^2 + t^2}
\end{equation}
is the Poisson kernel associated to the real line.

We would like to show that the Poisson integral (15.4) of \( f \) converges to \( f(x) \) as \( \eta \to 0 \). If we can do this, then it follows that (15.2) converges to \( f(x) \) as \( \eta \to 0 \).

A key point is that
\begin{equation}
  \int_{\mathbb{R}} P_t(y) \, dy = 1
\end{equation}
for each \( t > 0 \), by the computations in the previous section. We would therefore like to show that
\begin{equation}
  \lim_{\eta \to 0} \int_{\mathbb{R}} (f(y) - f(x)) P_{\eta}(x - y) \, dy = 0.
\end{equation}
The basic idea is that \( P_{\eta}(y - x) \) is concentrated near \( x \) and \( f(y) - f(x) \) is small when \( y \) is close to \( x \) by continuity.

For each \( \delta > 0 \),
\begin{equation}
  \lim_{t \to 0} \sup_{|y| \geq \delta} \{ P_t(y) : |y| \geq \delta \} = 0.
\end{equation}
One can also check that
\begin{equation}
  \lim_{t \to 0} \int_{|y| \geq \delta} P_t(y) \, dy = 0.
\end{equation}

Let \( \epsilon > 0 \) be given, and suppose that \( \delta > 0 \) has the property that
\begin{equation}
  |f(y) - f(x)| < \epsilon
\end{equation}
when \( |y - x| < \delta \). It follows that
\begin{equation}
  \int_{|y-x|<\delta} |f(y) - f(x)| P_{\eta}(y-x) \, dy < \epsilon \int_{\mathbb{R}} P_{\eta}(y-x) \, dy = \epsilon.
\end{equation}
Using the concentration properties mentioned in the previous paragraph, one can check that
\begin{equation}
  \lim_{\eta \to 0} \int_{|y-x|\geq\delta} (|f(y)| + |f(x)|) P_{\eta}(y-x) \, dy = 0.
\end{equation}
Thus (15.7) holds. Similar arguments show that the Poisson integral (15.4) of \( f \) converges to \( f(x) \) uniformly on bounded subsets of the real line, and uniformly on the whole real line if \( f \) is bounded and uniformly continuous.
16 Several complex variables

Fix an integer \( n \geq 1 \). By a multi-index we mean an \( n \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_n) \) of nonnegative integers.

For \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), put

\[
    z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.
\]

(16.1)

We interpret \( z^\alpha_j \) as being equal to 1 when \( \alpha_j = 1 \), and hence \( z^\alpha = 1 \) for every \( z \in \mathbb{C}^n \) when \( \alpha = 0 \).

A holomorphic polynomial on \( \mathbb{C}^n \) is given by a sum

\[
    \sum_{\alpha \in A} a_\alpha z^\alpha,
\]

(16.2)

where \( A \) is a set of finitely many multi-indices and the \( a_\alpha \)'s are complex numbers.

If \( U \) is an open set in \( \mathbb{C}^n \) and \( f(z) \) is a continuously-differentiable complex-valued function on \( U \), then the partial derivatives \( (\partial/\partial z_j)f \), \( (\partial/\partial \overline{z}_j)f \) are defined by applying the partial derivatives in one complex variable described in Section 1 to \( f \) as a function of \( z_j \), \( 1 \leq j \leq n \). We say that \( f(z) \) is holomorphic on \( U \) if \( (\partial/\partial \overline{z}_j)f(z) = 0 \) for every \( z \in U \) and \( 1 \leq j \leq n \).

Note that \( (\partial/\partial z_j)z_l^p = (p-1)z_l^{p-1} \) when \( j = l \) and \( p \) is a positive integer, and is 0 when \( j \neq l \). For every \( j, l, \) and \( p \), \( (\partial/\partial \overline{z}_j)z_l^p = 0 \), and thus holomorphic polynomials are holomorphic functions on \( \mathbb{C}^n \).

An infinite sum \( \sum_\alpha a_\alpha \) of nonnegative real numbers using all multi-indices \( \alpha \) is said to converge if the sums \( \sum_{\alpha \in E} a_\alpha \) over arbitrary sets \( E \) of finitely many multi-indices \( \alpha \) are bounded. In this event \( \sum_\alpha a_\alpha \) is defined to be the supremum over all such sums \( \sum_{\alpha \in E} a_\alpha \).

Similarly, an infinite sum \( \sum_\alpha a_\alpha \) of complex numbers is said to converge absolutely if \( \sum_\alpha |a_\alpha| \) converges as a sum of nonnegative real numbers. One can then make sense of the sum \( \sum_\alpha a_\alpha \) by reducing it to a linear combination of convergent sums of nonnegative real numbers.

Alternatively, if \( \sum_\alpha a_\alpha \) converges absolutely, then for every \( \epsilon > 0 \) there is a set \( E_\epsilon \) of finitely many multi-indices \( \alpha \) such that

\[
    \sum_\alpha |a_\alpha| < \sum_{\alpha \in E_\epsilon} |a_\alpha| + \epsilon.
\]

(16.3)

If \( A, B \) are sets of finitely many multi-indices \( \alpha \) such that \( E_\epsilon \subseteq A, B \), then

\[
    \left| \sum_{\alpha \in A} a_\alpha - \sum_{\alpha \in B} a_\alpha \right| < \epsilon.
\]

(16.4)
The sum $\sum_{\alpha} a_{\alpha}$ can be characterized by the property that

$$
|\sum_{\alpha \in A} a_{\alpha} - \sum_{\alpha} a_{\alpha}| \leq \epsilon \quad (16.5)
$$

for every set $A$ of finitely many multi-indices such that $E_{\epsilon} \subseteq A$.

A power series in the $n$ complex variables $z_1, \ldots, z_n$ is a sum of the form $\sum_{\alpha} a_{\alpha} z^{\alpha}$, where the $a_{\alpha}$'s are complex numbers and the sum extends over all multi-indices $\alpha$. If $\sum_{\alpha} |a_{\alpha}| t^{\alpha}$ converges for some $t = (t_1, \ldots, t_n)$, where each $t_j$ is a nonnegative real number, then $\sum_{\alpha} a_{\alpha} z^{\alpha}$ converges absolutely for every $z \in \mathbb{C}^n$ such that $|z_j| \leq t_j$ for each $j$, and the sum defines a continuous function on this set for reasons of uniform convergence.

Suppose that $r = (r_1, \ldots, r_n)$ is an $n$-tuple of positive real numbers, and that $\sum_{\alpha} a_{\alpha} z^{\alpha}$ converges absolutely for every $z \in \mathbb{C}^n$ in the open polydisk where $|z_j| < r_j$ for each $j$. The sum then defines a smooth function on the polydisk with the derivatives given by differentiating the power series term-by-term, again by considerations of uniform convergence of these power series on compact subsets of the polydisk. In particular the power series defines a holomorphic function on the open polydisk.

If $\sum_{\alpha} a_{\alpha}, \sum_{\beta} b_{\beta}$ are two sums of complex numbers extending over all multi-indices $\alpha, \beta$, then the Cauchy product of these two sums is the sum $\sum_{\gamma} c_{\gamma}$, where

$$
c_{\gamma} = \sum_{\alpha + \beta = \gamma} a_{\alpha} b_{\beta}. \quad (16.6)
$$

More precisely, the sum extends over all multi-indices $\alpha, \beta$ such that

$$
\alpha_j + \beta_j = \gamma_j \quad (16.7)
$$

for $j = 1, \ldots, n$, and there are only finitely many such $\alpha, \beta$ since the coordinates of these multi-indices are nonnegative integers.

If all but finitely many of the $a_{\alpha}$'s and $b_{\beta}$'s are equal to 0, then the same holds for the $c_{\gamma}$'s, and

$$
\sum_{\gamma} c_{\gamma} = \left( \sum_{\alpha} a_{\alpha} \right) \left( \sum_{\beta} b_{\beta} \right). \quad (16.8)
$$

If the $a_{\alpha}$'s and $b_{\beta}$'s are nonnegative real numbers and $\sum_{\alpha} a_{\alpha}, \sum_{\beta} b_{\beta}$ converge, then the $c_{\gamma}$'s are nonnegative real numbers and one can show that $\sum_{\gamma} c_{\gamma}$ converges and is equal to the product of the sums of the $a_{\alpha}$'s and $b_{\beta}$'s. If
\[ \sum_{\alpha} a_{\alpha}, \sum_{\beta} b_{\beta} \] are absolutely convergent sums of complex numbers, then one can show that \[ \sum_{\gamma} c_{\gamma} \] converges absolutely and is equal to the product of the sums of the \( a_{\alpha} \)'s and \( b_{\beta} \)'s.

The Cauchy product of two power series \( \sum_{\alpha} a_{\alpha} z^{\alpha}, \sum_{\beta} b_{\beta} z^{\beta} \) is equal to \( \sum_{\gamma} c_{\gamma} z^{\gamma} \), where the \( c_{\gamma} \)'s are obtained from the Cauchy product of the \( a_{\alpha} \)'s and \( b_{\beta} \)'s as in the previous paragraphs. If \( \sum_{\alpha} a_{\alpha} z^{\alpha}, \sum_{\beta} b_{\beta} z^{\beta} \) converge absolutely then it follows that \( \sum_{\gamma} c_{\gamma} z^{\gamma} \) converges absolutely and is equal to the product of \( \sum_{\alpha} a_{\alpha} z^{\alpha}, \sum_{\beta} b_{\beta} z^{\beta} \).

As a basic example, consider the power series \( \sum_{\alpha} a_{\alpha} z^{\alpha} \), where all of the coefficients are equal to 1. Formally this is the same as the product of the geometric series \( \sum_{j_1=0}^{\infty} z_1^{j_1}, \ldots, \sum_{j_n=0}^{\infty} z_n^{j_n} \) in the variables \( z_1, \ldots, z_n \).

In particular, for each \( l \geq 0 \), the sum of \( z^{\alpha} \) over all multi-indices \( \alpha \) with \( 0 \leq \alpha_j \leq l \) for each \( j \) is equal to the product

\[
(\sum_{j_1=0}^{l} z_1^{j_1}) \cdots (\sum_{j_n=0}^{l} z_n^{j_n})
\]

of the corresponding partial sums of the \( n \) geometric series in the variables \( z_1, \ldots, z_n \). Using this one can check that \( \sum_{\alpha} z^{\alpha} \) converges absolutely when \( |z_j| < 1 \) for each \( j \), as a consequence of the absolute convergence of the classical geometric series \( \sum_{j=0}^{\infty} a^j \) when \( |a| < 1 \). Moreover,

\[
\sum_{\alpha} z^{\alpha} = \prod_{j=1}^{n} \frac{1}{1 - z_j}
\]

for every \( z \in \mathbb{C}^n \) such that \( |z_j| < 1 \) for each \( j \).

### 17 Multiple Fourier series

Fix a positive integer \( n \), and in this section let us use arbitrary \( n \)-tuples of integers as multi-indices, i.e., elements of \( \mathbb{Z}^n \). For \( z \in \mathbb{C}^n \) and \( \alpha \in \mathbb{Z}^n \), put

\[
\tilde{z}^{\alpha} = \tilde{z}_1^{\alpha_1} \cdots \tilde{z}_n^{\alpha_n},
\]

where \( \tilde{z}_j^{\alpha_j} \) is equal to \( z_j^{\alpha_j} \) when \( \alpha_j > 0 \), to \( \tilde{z}_j^{-\alpha_j} \) when \( \alpha_j < 0 \), and to 1 when \( \alpha_j = 0 \). Let \( T^n \) be the \( n \)-dimensional torus, which is the set of \( z \in \mathbb{C}^n \) such that \( |z_j| = 1 \) for \( j = 1, \ldots, n \). When \( z \in T^n \), \( \tilde{z}^{\alpha} \) is equal to \( z^{\alpha} \), the product of \( z_j^{\alpha_j}, 1 \leq j \leq n \), for every \( \alpha \in \mathbb{Z}^n \).
A general polynomial on \( \mathbb{C}^n \) can be expressed as a sum of finitely many terms, where each term is a complex multiple of a product of nonnegative powers of the the real and imaginary parts of the coordinates of \( z = (z_1, \ldots, z_n) \). Equivalently, a general polynomial can be expressed as a sum of finitely many complex multiples of products of nonnegative powers of the \( z_j \)'s and \( \overline{z}_j \)'s. A twice-continuously differentiable complex-valued function on an open set in \( \mathbb{C}^n \) is said to be polyharmonic if it is harmonic in each \( z_j \) separately, \( 1 \leq j \leq n \). The polyharmonic polynomials are the polynomials which can be expressed as the sum of finitely many complex multiples of \( \overline{z}^\alpha \)'s, \( \alpha \in \mathbb{Z}^n \), and every general polynomial on \( \mathbb{C}^n \) agrees with a polyharmonic polynomial on \( T^n \), as one can see by removing factors of \( |z_j|^2 \) whenever possible.

Let \( \mathcal{C}(T^n) \) be the vector space of continuous complex-valued functions on \( T^n \). If \( f_1, f_2 \in \mathcal{C}(T^n) \), then put

\[
\langle f_1, f_2 \rangle_{T^n} = \frac{1}{(2 \pi)^n} \int_{T^n} f_1(z) \overline{f_2(z)} |dz|,
\]

where the integral over \( T^n \) is equivalent to an iterated integral over \( T \) in each of the \( n \) variables. This is the standard integral inner product for continuous functions on the \( n \)-dimensional torus. One can check that the restrictions of the monomials \( \overline{z}^\alpha \), \( \alpha \in \mathbb{Z}^n \), to \( T^n \) are orthonormal with respect to this inner product.

For every \( f \in \mathcal{C}(T^n) \) and \( \alpha \in \mathbb{Z}^n \), \( a_\alpha = \langle f, z^\alpha \rangle_{T^n} \) is the Fourier coefficient of \( f \) associated to \( \alpha \), and the corresponding Fourier series is given by

\[
\sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha.
\]

A priori the Fourier series is a formal expression whose convergence properties are to be investigated. Observe that

\[
|a_\alpha| \leq \frac{1}{(2 \pi)^n} \int_{T^n} |f(z)| |dz|
\]

for every \( \alpha \in \mathbb{Z}^n \). Because of orthonormality of the \( z^\alpha \)'s, \( \alpha \in \mathbb{Z}^n \),

\[
\sum_{\alpha \in A} |a_\alpha|^2 \leq \frac{1}{(2 \pi)^n} \int_{T^n} |f(z)|^2 |dz|
\]

for every set \( A \) with finitely many multi-indices \( \alpha \).
By a polyharmonic power series we mean a series of the form \( \sum_{\alpha \in \mathbb{Z}^n} a_\alpha \tilde{z}^\alpha \) with complex coefficients \( a_\alpha \). One can think of this as a combination of \( 2^n \) power series in the \( z_j \)'s and their complex conjugates, according to the signs of the coordinates of \( \alpha \). If the \( a_\alpha \)'s are bounded, for instance, then the series converges absolutely for every \( z \in \mathbb{C}^n \) such that \(|z_j| < 1\) for \( 1 \leq j \leq n \). In this event the series defines a smooth polyharmonic function on the open unit polydisk, for reasons of uniform convergence on compact sub-polydisks.

In particular we can apply this to the Fourier coefficients \( a_\alpha \) of a continuous function \( f \) on \( T^n \).

For \( z \) in the open unit polydisk and \( w \in T^n \), the Poisson kernel \( P_n(z, w) \) can be defined as \( 1/(2\pi)^n \) times the sum over \( \alpha \in \mathbb{Z}^n \) of \( \tilde{z}^\alpha \) times \( \tilde{w}^{-\alpha} = w^{-\alpha} \). This is the same as the product of the 1-dimensional Poisson kernels \( P(z_j, w_j) \), \( 1 \leq j \leq n \). The main point is that the polyharmonic power series \( \sum_{\alpha \in \mathbb{Z}^n} a_\alpha \tilde{z}^\alpha \) with coefficients equal to the Fourier coefficients of \( f \) is equal to the integral of \( f(w) \) times \( P_n(z, w) \) over \( w \in T^n \) for each \( z \) in the open unit polydisk. One can use this to show that as \( z \) approaches a point \( \zeta \in T^n \), the value of the polyharmonic power series at \( z \) approaches \( f(\zeta) \), which is another version of Abel summability.

As in Section 10 one can show that \( \sum_{\alpha \in \mathbb{Z}^n} |a_\alpha|^2 \) is equal to \( 1/(2\pi)^n \) times the integral of \( |f|^2 \) over \( T^n \), where the sum is defined to be the supremum of all subsums over finitely many \( \alpha \). For any set \( A \) of finitely many \( \alpha \in \mathbb{Z}^n \), let \( f_A \) be the function on \( T^n \) which is the sum of \( a_\alpha \tilde{z}^\alpha \) over \( \alpha \in A \). One also has that \( 1/(2\pi)^n \) times the integral of \( |f - f_A|^2 \) over \( T^n \) is equal to the sum of \( |a_\alpha|^2 \) over \( \alpha \in \mathbb{Z}^n \setminus A \), which is as small as one would like for suitably-large sets \( A \).

Fix a positive integer \( n \), and let \( \phi \) be a continuous complex-valued function on the \( n \)-dimensional torus \( T^n \). For every \( a \in \mathbb{R}^n \) and \( z \in T^n \), put

\[
(17.6) \quad f_{a,z}(t) = \phi(z_1 \exp(a_1 t i), \ldots, z_n \exp(a_n t i)).
\]

By construction, \( f_{a,z} \) is a bounded uniformly continuous function on the real line. If there is a \( b \in \mathbb{R} \) such that every coordinate of \( a \) is an integer multiple of \( b \), then \( f_{a,z} \) is a periodic function on \( \mathbb{R} \).

In general,

\[
(17.7) \quad f_{a,z}(t - v) = f_{a,z'}(t),
\]

where \( z'_j = z_j \exp(-a_j v i) \) for every \( v \in \mathbb{R} \). Using this one can check that \( f_{a,z} \) is almost periodic for every \( a \in \mathbb{R}^n \) and \( z \in T^n \). For instance, a sum or product of \( n \) continuous periodic functions on the real line can be expressed in this way.
18  Invariant means

Let \( f(x) \) be a bounded continuous complex-valued function on the real line which is almost periodic. For each \( \epsilon > 0 \) there is an \( L > 0 \) such that

\[
\left| \frac{1}{|I|} \int_I f(x-y) \, dx - \frac{1}{|I|} \int_I f(x) \, dx \right| < \epsilon
\]

for every bounded interval \( I \) with \( |I| \geq L \) and every \( y \in \mathbb{R} \), since every translate of \( f \) can be approximated uniformly by one of finitely many translates.

Equivalently, if \( I, I' \subseteq \mathbb{R} \) are bounded intervals with \( |I| = |I'| \geq L \), then

\[
\left| \frac{1}{|I|} \int_I f(x) \, dx - \frac{1}{|I'|} \int_{I'} f(x) \, dx \right| < \epsilon.
\]

If \( I, I' \subseteq \mathbb{R} \) are bounded intervals such that \( |I'| = r |I| \) for some positive integer \( r \), then \( I' \) is the union of \( r \) subintervals \( I'_1, \ldots, I'_r \) with length equal to \( |I| \)

\[
\frac{1}{|I'|} \int_{I'} f(x) \, dx = \frac{1}{r} \sum_{p=1}^{r} \frac{1}{|I'_p|} \int_{I'_p} f(x) \, dx,
\]

and we can apply the preceding estimate to \( I'_p, 1 \leq p \leq r \), to get that (18.2) holds when \( |I| \geq L \).

If \( I, I' \subseteq \mathbb{R} \) are any bounded intervals such that \( |I|, |I'| \geq L \), then

\[
\left| \frac{1}{|I|} \int_I f(x) \, dx - \frac{1}{|I'|} \int_{I'} f(x) \, dx \right| < 3 \epsilon,
\]

since the averages of \( f \) over \( I, I' \) are close to the averages of \( f \) over expanded intervals whose lengths are integer multiples of the lengths of \( I, I' \), and the averages of \( f \) on these expanded intervals are approximately the same if the expanded intervals are sufficiently large and approximately the same.

To summarize, the averages of \( f \) over sufficiently large intervals are all approximately the same. It follows that there is an average \( \mu(f) \) in the limit, which is characterized by the property that for every \( \eta > 0 \) there is an \( L_\eta > 0 \) such that

\[
\left| \mu(f) - \frac{1}{|I|} \int_I f(x) \, dx \right| < \eta
\]

whenever \( I \subseteq \mathbb{R} \) is a bounded interval which satisfies \( |I| \geq L_\eta \).

If \( f \) is a continuous periodic function on the real line with period \( p \), then the averages of \( f \) over arbitrary intervals of length \( p \) are the same. These
averages are also the same as the averages of \( f \) over intervals whose lengths are positive integer multiples of \( p \), and hence their common value is equal to \( \mu(f) \).

If \( f \) is a continuous real-valued almost periodic function on \( \mathbb{R} \), then

\[
(18.6) \quad \mu(f) \in \mathbb{R}.
\]

If in addition \( f(x) \geq 0 \) for every \( x \in \mathbb{R} \), then

\[
(18.7) \quad \mu(f) \geq 0.
\]

It is easy to see that \( \mu \) defines a linear mapping from the space \( \mathcal{AP}(\mathbb{R}) \) of bounded continuous almost periodic functions on the real line into the complex numbers. Furthermore,

\[
(18.8) \quad |\mu(f)| \leq \|f\|_{\infty}
\]

for every \( f \in \mathcal{AP}(\mathbb{R}) \), where \( \|f\|_{\infty} \) denotes the supremum norm of \( f \) on \( \mathbb{R} \).

If \( f \in \mathcal{AP}(\mathbb{R}) \) and \( h \) is any continuous complex-valued function on \( \mathbb{C} \), then \( h(f(x)) \) is almost periodic too. One can check this directly from the definitions, using the fact that \( h \) is uniformly continuous on bounded subsets of \( \mathbb{C} \). In particular, \( |f(x)| \) is almost periodic.

If \( \mu(|f|) = 0 \), then \( f(x) = 0 \) for every \( x \in \mathbb{R} \). For if \( f(x_0) \neq 0 \) for some \( x_0 \in \mathbb{R} \), then there are \( \eta, t > 0 \) such that

\[
(18.9) \quad |f(x)| \geq 2\eta
\]

when \( |x - x_0| \leq t \). Because of almost periodicity, every point in \( \mathbb{R} \) is at a bounded distance from some \( w \in \mathbb{R} \) such that

\[
(18.10) \quad |f(z)| \geq \eta
\]

when \( |z - w| \leq t \). This leads to a positive lower bound for the averages of \( |f| \) on sufficiently large intervals, and hence \( \mu(|f|) > 0 \). It follows that \( \mu(|f|^p)^{1/p} \) defines a norm on \( \mathcal{AP}(\mathbb{R}) \) when \( 1 \leq p < \infty \).

Let us focus now on the case where \( p = 2 \). Using the invariant mean \( \mu \) we get an inner product

\[
(18.11) \quad \langle f_1, f_2 \rangle_{\mathcal{AP}(\mathbb{R})} = \mu(f_1 \overline{f_2})
\]

on \( \mathcal{AP}(\mathbb{R}) \) for which the associated norm is \( \mu(|f|^2)^{1/2} \).
For every $\xi \in \mathbb{R}$, put
\begin{equation}
  e_\xi(x) = \exp(\xi x i).
\end{equation}

One can check that $\mu(e_\xi) = 0$ when $\xi \neq 0$. Therefore the functions $e_\xi$, $\xi \in \mathbb{R}$, are orthonormal with respect to the inner product $\langle f_1, f_2 \rangle_{AP}(\mathbb{R})$.

If $f \in \mathcal{A}(\mathbb{R})$ and $\xi_1, \ldots, \xi_n \in \mathbb{R}$, then
\begin{equation}
  \sum_{j=1}^n |\langle f, e_{\xi_j} \rangle_{AP(\mathbb{R})}|^2 \leq \mu(|f|^2),
\end{equation}

because of the orthonormality of the $e_{\xi_j}$'s. For each $\epsilon > 0$, the set of $\xi \in \mathbb{R}$ such that
\begin{equation}
  |\langle f, e_\xi \rangle_{AP(\mathbb{R})}| \geq \epsilon
\end{equation}

has $\leq \mu(|f|^2)/\epsilon^2$ elements, and in particular the set of $\xi \in \mathbb{R}$ such that
\begin{equation}
  \langle f, e_\xi \rangle_{AP(\mathbb{R})} \neq 0
\end{equation}

has only finitely or countably many elements.

\section{19 \ Banach spaces}

Let $V$ be a complex vector space equipped with a norm $\|v\|$. We say that $V$ is a \textit{Banach space} if $V$ is complete as a metric space with the associated metric $\|v - w\|$. A complete inner product space is a \textit{Hilbert space}.

For each positive integer $n$, $\mathcal{C}^n$ equipped with the norm $\|v\|_p$, $1 \leq p \leq \infty$, as in Section 13 is a Banach space, and a Hilbert space when $p = 2$. The space $\mathcal{C}(\mathbb{T}^n)$ of continuous complex-valued functions on the $n$-dimensional torus $\mathbb{T}^n$, and the space $\mathcal{B}\mathcal{C}(\mathbb{R})$ of bounded continuous functions on the real line, are Banach spaces with respect to the supremum norm.

A closed subspace of a Banach or Hilbert space is a Banach or Hilbert space, as appropriate. In particular, the space $\mathcal{B}\mathcal{U}\mathcal{C}(\mathbb{R})$ of bounded uniformly continuous functions and the space $\mathcal{A}\mathcal{P}(\mathbb{R})$ of almost periodic functions are closed linear subspaces of the space $\mathcal{B}\mathcal{C}(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$, and hence Banach spaces with respect to the supremum norm.

There is a nice characterization of completeness of $V$ in terms of infinite series. As usual an infinite series $\sum_{j=1}^\infty v_j$ with terms $v_j \in V$ for each $j$ is said to converge if the sequence of partial sums $\sum_{j=1}^n v_j$ converges in $V$. 

43
An infinite series \(\sum_{j=1}^{\infty} v_j\) with terms in \(V\) is said to converge absolutely if \(\sum_{j=1}^{\infty} \|v_j\|\) converges as an infinite series of nonnegative real numbers. Equivalently, this holds when the sums \(\sum_{j=1}^{n} \|v_j\|\) are bounded.

If \(\sum_{j=1}^{\infty} v_j\) converges absolutely, then for each \(\epsilon > 0\) there is an \(L \geq 0\) such that
\[
\sum_{j=l}^{n} \|v_j\| < \epsilon \quad (19.1)
\]
when \(n \geq l \geq L\). Hence
\[
\sum_{j=l}^{n} v_j < \epsilon \quad (19.2)
\]
when \(n \geq l \geq L\).

Thus absolute convergence of \(\sum_{j=1}^{\infty} v_j\) implies that the sequence of partial sums \(\sum_{j=1}^{n} v_j\) forms a Cauchy sequence in \(V\). If \(V\) is complete, then \(\sum_{j=1}^{\infty} v_j\) converges in \(V\).

Conversely, suppose that every absolutely convergent series in \(V\) converges. Let \(\{v_j\}_{j=1}^{\infty}\) be a Cauchy sequence in \(V\), which we would like to show converges.

Because \(\{v_j\}_{j=1}^{\infty}\) is a Cauchy sequence, there is a subsequence \(\{v_{ji}\}_{i=1}^{\infty}\) of \(\{v_j\}_{j=1}^{\infty}\) such that
\[
\|v_{ji+1} - v_{ji}\| \leq 2^{-l} \quad (19.3)
\]
for each \(l\). Consequently,
\[
\sum_{i=1}^{\infty} (v_{ji+1} - v_{ji}) \quad (19.4)
\]
converges absolutely.

If every absolutely convergent series in \(V\) converges, then \(\{v_{ji}\}_{i=1}^{\infty}\) converges. Since
\[
\sum_{i=1}^{r} (v_{ji+1} - v_{ji}) = v_{j_{r+1}} - v_{j_1} \quad (19.5)
\]
for every positive integer \(r\), this means that \(\{v_{ji}\}_{i=1}^{\infty}\) converges in \(V\).

A Cauchy sequence with a convergent subsequence also converges, and thus \(\{v_j\}_{j=1}^{\infty}\) converges in \(V\), as desired.

\[\ell^p (\mathbb{Z}^n)\]

Fix a positive integer \(n\). In the present section it is again convenient to let multi-indices be \(n\)-tuples of arbitrary integers, i.e., elements of \(\mathbb{Z}^n\).
Let $C(Z^n)$ be the space of families $a = \{a_\alpha\}_{\alpha \in Z^n}$ of complex numbers indexed by $Z^n$, which amount to complex-valued functions on $Z^n$. The support of $a \in C(Z^n)$ is the set of $\alpha \in Z^n$ such that $a_\alpha \neq 0$, and we let $C_{00}(Z^n)$ be the space of $a \in C(Z^n)$ whose support has only finitely many elements.

For every $a = \{a_\alpha\}_{\alpha \in Z^n} \in C_{00}(Z^n)$, put
\begin{equation}
\|a\|_p = \|a\|_{p,Z^n} = \left( \sum_{\alpha \in Z^n} |a_\alpha|^p \right)^{1/p}
\end{equation}
when $0 < p < \infty$ and
\begin{equation}
\|a\|_\infty = \|a\|_{\infty,Z^n} = \sup\{|a_\alpha| : \alpha \in Z^n\}
\end{equation}
when $p = \infty$. As in Section 13, $\|a\|_p$ defines a norm on the complex vector space $C_{00}(Z^n)$ when $1 \leq p \leq \infty$, and when $0 < p \leq 1$ one has
\begin{equation}
\|a + b\|_p \leq \|a\|_p + \|b\|_p,
\end{equation}
a, b \in C_{00}(Z^n), as an alternative version of the triangle inequality. We also have the inner product
\begin{equation}
\langle a, b \rangle = \langle a, b \rangle_{Z^n} = \sum_{\alpha \in Z^n} a_\alpha \overline{b_\alpha}
\end{equation}
for $a, b \in C_{00}(Z^n)$, which satisfies $\langle a, a \rangle = \|a\|_2^2$.

For $0 < p < \infty$, $\ell^p(Z^n)$ is defined to be the space of $a \in C(Z^n)$ such that
\begin{equation}
\sum_{\alpha \in Z^n} |c_\alpha|^p < +\infty.
\end{equation}
More precisely, this means that the sums
\begin{equation}
\sum_{\alpha \in A} |a_\alpha|^p
\end{equation}
over finite sets $A \subseteq Z^n$ are bounded, and the sum over all $\alpha \in Z^n$ is defined to be the supremum of these finite sums.

Let $\ell^\infty(Z^n)$ be the space of $a \in C(Z^n)$ such that the $a_\alpha$'s are bounded. We say that $a \in C(Z^n)$ vanishes at infinity if for each $\epsilon > 0$ there is a finite set $A \subseteq Z^n$ such that
\begin{equation}
|a_\alpha| < \epsilon
\end{equation}
for $\alpha \not\in A$. The space of $a \in C(Z^n)$ which vanish at infinity is denoted by $C_0(Z^n)$. A bounded sequence $\{a_\alpha\}_{\alpha \in Z^n}$ converges to $a_\alpha$, written $a_\alpha \Omega \rightarrow a$, if and only if
\begin{equation}
\sum_{\alpha \in Z^n} |a_\alpha - a_\alpha'|^p < +\infty
\end{equation}
for all $a_\alpha' \in C(Z^n)$. Hence, $\|a\|_{C_0(Z^n)} = \lim_{N \rightarrow \infty} \|a|_{\infty,N}\| = \lim_{N \rightarrow \infty} \|a\|_{1,N}$.
when $\alpha \not\in A$. The space of $a \in \mathcal{C}(\mathbb{Z}^n)$ which vanish at infinity is denoted $\mathcal{C}_0(\mathbb{Z}^n)$, and is contained in $\ell^\infty(\mathbb{Z}^n)$.

Clearly $\mathcal{C}_{00}(\mathbb{Z}^n)$ is contained in $\ell^p(\mathbb{Z}^n)$ for every $p$, $0 < p \leq \infty$, and in $\mathcal{C}_0(\mathbb{Z}^n)$.

Suppose that $a \in \ell^p(\mathbb{Z}^n)$ for some $p$, $0 < p \leq \infty$. For each $\epsilon > 0$, there is a set $A_\epsilon$ of finitely many $\alpha \in \mathbb{Z}^n$ such that

$$\sum_{\alpha \in \mathbb{Z}^n} |a_\alpha|^p < \sum_{\alpha \in A_\epsilon} |a_\alpha|^p + \epsilon^p. \quad (20.8)$$

This implies that $|a_\alpha| < \epsilon$ when $\alpha \not\in A_\epsilon$, and hence that $a \in \mathcal{C}_0(\mathbb{Z}^n)$.

Note that (20.1), (20.2) carry over to all $a \in \ell^p(\mathbb{Z}^n)$, $0 < p \leq \infty$. One can check that $\ell^p(\mathbb{Z}^n)$, $0 < p \leq \infty$, and $\mathcal{C}_0(\mathbb{Z}^n)$ are linear subspaces of $\mathcal{C}(\mathbb{Z}^n)$, and that $\|a\|_p$ defines a norm on $\ell^p(\mathbb{Z}^n)$ when $1 \leq p \leq \infty$ and (20.3) holds for $a, b \in \ell^p(\mathbb{Z}^n)$ when $0 < p < 1$, by reducing to the earlier inequalities for finite sums. If $a \in \ell^p(\mathbb{Z}^n)$, $0 < p < \infty$, then

$$\|a\|_\infty \leq \|a\|_p. \quad (20.9)$$

As in Section 13,

$$(\sum_{\alpha \in A} |a_\alpha|^q)^{1/q} \leq \left( \sum_{\alpha \in A} |a_\alpha|^p \right)^{1/p} \quad (20.10)$$

when $a \in \mathcal{C}(\mathbb{Z}^n)$, $0 < p < q < \infty$, and $A \subseteq \mathbb{Z}^n$ has only finitely many elements, and it follows that $\ell^p(\mathbb{Z}^n) \subseteq \ell^q(\mathbb{Z}^n)$ and

$$\|a\|_q \leq \|a\|_p \quad (20.11)$$

when $0 < p < q < \infty$ and $a \in \ell^p(\mathbb{Z}^n)$.

Let us think of $\ell^p(\mathbb{Z}^n)$ as a metric space with the metric

$$\|a - b\|_p \quad (20.12)$$

when $1 \leq p \leq \infty$ and

$$\|a - b\|_p^p \quad (20.13)$$

when $0 < p \leq 1$. If a sequence in $\ell^p(\mathbb{Z}^n)$ is a Cauchy sequence, then the corresponding sequence of values at any $\alpha \in \mathbb{Z}^n$ is a Cauchy sequence in $\mathbb{C}$ and hence converges. One can show that the limit is an element of $\ell^p(\mathbb{Z}^n)$ and that the sequence converges to the limit in $\ell^p(\mathbb{Z}^n)$. In other words, $\ell^p(\mathbb{Z}^n)$ is complete for each $p$. 
For $0 < p < \infty$ one can also check that $C_{00}(\mathbb{Z}^n)$ is dense in $\ell^p(\mathbb{Z}^n)$. Similarly, $C_{00}(\mathbb{Z}^n)$ is dense in $C_0(\mathbb{Z}^n)$ with respect to the supremum metric. Furthermore, $C_0(\mathbb{Z}^n)$ is equal to the closure of $C_{00}(\mathbb{Z}^n)$ in $\ell^\infty(\mathbb{Z}^n)$. In particular, $C_0(\mathbb{Z}^n)$ is complete with respect to the supremum metric. There is a more general sense in which $C_{00}(\mathbb{Z}^n)$ is dense in $\ell^\infty(\mathbb{Z}^n)$, i.e., for every $a \in \ell^\infty(\mathbb{Z}^n)$ there is a sequence in $C_{00}(\mathbb{Z}^n)$ which is bounded in $\ell^\infty(\mathbb{Z}^n)$ and which converges to $a$ pointwise on $\mathbb{Z}^n$.

If $a \in \ell^1(\mathbb{Z}^n)$, then one can make sense of the sum

$$\sum \alpha \in \mathbb{Z}^n a_\alpha. \tag{20.14}$$

When the $a_\alpha$'s are nonnegative real numbers, (20.14) can be defined as the supremum over all finite subsums. Every $a \in \ell^1(\mathbb{Z}^n)$ can be expressed as a linear combination of nonnegative real-valued elements of $\ell^1(\mathbb{Z}^n)$, which permits one to get (20.14) as a linear combination of sums of nonnegative real numbers. The resulting value of (20.14) can be characterized uniquely in terms of approximation by finite sums. One can also think of (20.14) as a complex-valued function on $C_{00}(\mathbb{Z}^n)$ which is uniformly continuous with respect to the $\ell^1$-metric, and which therefore has a unique extension to a uniformly continuous function on all of $\ell^1(\mathbb{Z}^n)$.

If $a, b \in \ell^2(\mathbb{Z}^n)$, then the product $a \overline{b}$ is an element of $\ell^1(\mathbb{Z}^n)$ and

$$\sum \alpha \in \mathbb{Z}^n |a_\alpha| |b_\alpha| \leq \|a\|_2 \|b\|_2. \tag{20.15}$$

This follows by applying the Cauchy–Schwarz inequality to sums over finite subsets of $\mathbb{Z}^n$. Hence the inner product (20.14) carries over to all $a, b \in \ell^2(\mathbb{Z}^n)$. The norm $\|a\|_2$ is the norm associated to this inner product.

More generally, if $0 < p, q, r \leq \infty$, $a \in \ell^p(\mathbb{Z}^n)$, $b \in \ell^q(\mathbb{Z}^n)$, and

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \tag{20.16}$$

then the product $a b \in \ell^r(\mathbb{Z}^n)$, and

$$\|a b\|_r \leq \|a\|_p \|b\|_q. \tag{20.17}$$

This is a version of Hölder’s inequality, which is trivial when one of the exponents is infinite, in which event its reciprocal is interpreted as being equal to 0. To prove it we therefore suppose that $p, q < \infty$, and one can also
make the reductions $r = 1$ and $\|a\|_p = \|b\|_q = 1$. If $x$, $y$ are nonnegative real numbers, then one can check that
\[
xy \leq \frac{x^p}{p} + \frac{y^q}{q}.
\]
This implies that
\[
\sum_{\alpha \in \mathbb{Z}^n} |a_\alpha| |b_\alpha| \leq \sum_{\alpha \in \mathbb{Z}^n} \frac{|a_\alpha|^p}{p} + \sum_{\alpha \in \mathbb{Z}^n} \frac{|b_\alpha|^q}{q} = 1,
\]
as desired.

## 21 Measures on $\mathbb{T}^n$

Fix a positive integer $n$. By a measure on $\mathbb{T}^n$ we mean a linear function $\mu$ on the vector space $C(\mathbb{T}^n)$ of continuous complex-valued functions on $\mathbb{T}^n$ into the complex numbers which is bounded in the sense that there is an $A \geq 0$ such that
\[
|\mu(\phi)| \leq A \|\phi\|_{\infty}
\]
for every $\phi \in C(\mathbb{T}^n)$, where
\[
\|\phi\|_{\infty} = \|\phi\|_{\infty, \mathbb{T}^n} = \sup\{|\phi(z)| : z \in \mathbb{T}^n\}
\]
is the supremum norm for continuous functions on $\mathbb{T}^n$. In this event we put
\[
\|\mu\|_* = \|\mu\|_{*, \mathbb{T}^n} = \sup\{|\mu(\phi)| : \phi \in C(\mathbb{T}^n), \|\phi\|_{\infty} \leq 1\}.
\]
Equivalently, (21.1) holds with $A = \|\mu\|_*$, and this is the smallest choice of $A$ with this property.

It is easy to see that the space of measures on $\mathbb{T}^n$ is a vector space with respect to addition and scalar multiplication of linear functionals on $C(\mathbb{T}^n)$. That is to say, if $\mu_1$, $\mu_2$ are measures on $\mathbb{T}^n$, then we get another measure $\mu_1 + \mu_2$ defined by
\[
(\mu_1 + \mu_2)(\phi) = \mu_1(\phi) + \mu_2(\phi)
\]
for every $\phi \in C(\mathbb{T}^n)$. Similarly, if $\mu$ is a measure on $\mathbb{T}^n$ and $\alpha \in \mathbb{C}$, then we get another measure $c \mu$ defined by
\[
(c \mu)(\phi) = c \mu(\phi)
\]
for every \( \phi \in \mathcal{C} \left( \mathbb{T}^n \right) \). The vector space of measures on \( \mathbb{T}^n \) is denoted \( \mathcal{M} \left( \mathbb{T}^n \right) \).

One can also check that \( \| \mu \|_* \) defines a norm on \( \mathcal{M} \left( \mathbb{T}^n \right) \).

If \( f \) is a continuous complex-valued function on \( \mathbb{T}^n \), then

\[
\mu(\phi) = \frac{1}{(2 \pi)^n} \int_{\mathbb{T}^n} \phi(w) f(w) \, |dw| \tag{21.6}
\]

defines a measure on \( \mathbb{T}^n \), and we have that

\[
\| \mu \|_* = \frac{1}{(2 \pi)^n} \int_{\mathbb{T}^n} |f(w)| \, |dw|. \tag{21.7}
\]

One can also allow discontinuous functions \( f \) here, as long as one can make sense of the integrals. If \( z \in \mathbb{T}^n \), then

\[
\delta_z(\phi) = \phi(z) \tag{21.8}
\]
defines a measure on \( \mathbb{T}^n \), the Dirac mass at \( z \), and

\[
\| \delta_z \|_* = 1. \tag{21.9}
\]

Suppose that \( \{ \mu_j \}_{j=1}^\infty \) is a sequence of measures on \( \mathbb{T}^n \) which is a Cauchy sequence, which means that for each \( \epsilon > 0 \) there is an \( L \geq 1 \) such that

\[
\| \mu_j - \mu_l \|_* < \epsilon \tag{21.10}
\]

for every \( j, l \geq L \). It follows that \( \{ \mu_j(\phi) \}_{j=1}^\infty \) is a Cauchy sequence in \( \mathbb{C} \) for every \( \phi \in \mathcal{C} \left( \mathbb{T}^n \right) \), and hence that \( \{ \mu_j \}_{j=1}^\infty \) converges as a sequence of complex numbers. One can check that the limit defines a measure on \( \mathbb{T}^n \) and that \( \{ \mu_j \}_{j=1}^\infty \) converges to this measure in \( \mathcal{M} \left( \mathbb{T}^n \right) \). Therefore \( \mathcal{M} \left( \mathbb{T}^n \right) \) is complete and hence a Banach space.

Let \( \mu \) be a measure on \( \mathbb{Z}^n \). For \( \alpha \in \mathbb{Z}^n \), the corresponding Fourier coefficient of \( \mu \) is given by

\[
a_\alpha = \mu(\psi_\alpha), \tag{21.11}
\]

where \( \psi_\alpha(w) = w^{-\alpha} \). Observe that

\[
|a_\alpha| \leq \| \mu \|_* \tag{21.12}
\]

for every \( \alpha \in \mathbb{Z}^n \).

Because the \( a_\alpha \)'s are bounded, the polyharmonic power series

\[
\sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha \tag{21.13}
\]
converges absolutely for every \( z \in \mathbb{C}^n \) such that \( |z_j| < 1, 1 \leq j \leq n \). Here 
\[
\tilde{z}^\alpha = \tilde{z}_1^\alpha_1 \cdots \tilde{z}_n^\alpha_n,
\]
with \( \tilde{z}_j^\alpha_j = \tilde{z}_j^\alpha_j \) when \( \alpha_j \geq 0 \) and \( = \tilde{z}_j^{-\alpha_j} \) when \( \alpha_j \leq 0 \).

If \( \{ \phi_j \}_{j=1}^\infty \) is a sequence of continuous complex-valued functions on \( T^n \) which converges uniformly to the continuous function \( \phi \), then

\[
\lim_{j \to \infty} \mu(\phi_j) = \mu(\phi).
\]

This follows from the boundedness of \( \mu \), since

\[
|\mu(\phi_j) - \mu(\phi)| = |\mu(\phi_j - \phi)| \leq \|\mu\|_* \|\phi_j - \phi\|_\infty
\]

for every \( j \geq 1 \).

Let \( P_n(z, w) \) be the \( n \)-dimensional Poisson kernel, as in Section 17. For each \( z \) in the open unit polydisk, put

\[
P_{n,z}(w) = (2 \pi)^n P_n(z, w),
\]

considered as a continuous function on \( T^n \). One can check that

\[
\sum_{\alpha \in \mathbb{Z}^n} a_\alpha \tilde{z}^\alpha = \mu(P_{n,z})
\]

for all \( z \) in the open unit polydisk. Basically this is the same as in the case of Fourier coefficients of continuous functions. Technical matters of applying \( \mu \) to a nice sum and getting the same answer as applying \( \mu \) to the individual terms and then summing can be handled as in the previous paragraph.

Fix \( r_1, \ldots, r_n \in (0, 1) \), and put

\[
r \circ \zeta = (r_1 \zeta_1, \ldots, r_n \zeta_n)
\]

for \( \zeta \in \mathbb{T}^n \). For every continuous complex-valued function \( f \) on \( \mathbb{T}^n \), consider the expression

\[
\frac{1}{(2 \pi)^n} \int_{\mathbb{T}^n} \mu(P_{n,r \circ \zeta}) f(\zeta) \, |d\zeta|,
\]

which makes sense because \( \mu(P_{n,r \circ \zeta}) \) is continuous as a function of \( \zeta \) on \( \mathbb{T}^n \). The same quantity can be obtained by applying \( \mu \) to

\[
\int_{\mathbb{T}^n} P_n(r \circ \zeta, w) f(\zeta) \, |d\zeta|
\]

as a function of \( w \) on \( \mathbb{T}^n \). We can rewrite this last integral as

\[
\int_{\mathbb{T}^n} P_n(r \circ w, \zeta) f(\zeta) \, |d\zeta|,
\]
since
\[(21.22)\quad P_n(r \circ \zeta, w) = P_n(r \circ w, \zeta)\]
for every \(w, \zeta \in T^n\). These Poisson integrals of \(f\) converge uniformly to \(f\) on \(T^n\) as \(r_j \to 1, 1 \leq j \leq n\), and hence \(21.19\) converges to \(\mu(f)\) as \(r \to (1, \ldots, 1)\), which gives a version of \textit{Abel summability} for the Fourier series \(\sum_{\alpha \in \mathbb{Z}^n} a_{\alpha} z^\alpha\) of \(\mu\) in this situation.

22 Measures on \(\mathbb{R}\)

A \textit{measure} on the real line is a linear mapping \(\mu\) from the vector space \(C_{00}(\mathbb{R})\) of continuous complex-valued functions on \(\mathbb{R}\) with bounded support into the complex numbers which is bounded in the sense that there is an \(A \geq 0\) such that
\[(22.1)\quad |\mu(\phi)| \leq A \|\phi\|_{\infty}\]
for every \(\phi \in C_{00}(\mathbb{R})\), where \(\|\phi\|_{\infty}\) is the supremum norm of \(\phi\) on \(\mathbb{R}\). If we put
\[(22.2)\quad \|\mu\|_* = \|\mu\|_{*,\mathbb{R}} = \sup\{|\mu(\phi)| : \phi \in C_{00}(\mathbb{R}), \|\phi\|_{\infty} \leq 1\},\]
then the previous inequality holds with \(A = \|\mu\|_*\), and this is the smallest value of \(A\) which works.

The space of measures on \(\mathbb{R}\) is denoted \(\mathcal{M}(\mathbb{R})\) and is a vector space with respect to addition and scalar multiplication of linear functionals. It is easy to see that \(\|\mu\|_*\) defines a norm on \(\mathcal{M}(\mathbb{R})\). One can also show that \(\mathcal{M}(\mathbb{R})\) is complete and hence a Banach space.

If \(f\) is a continuous integrable complex-valued function on \(\mathbb{R}\), or an integrable step function, then
\[(22.3)\quad \mu(\phi) = \int_{\mathbb{R}} f(x) \phi(x) \, dx\]
defines a measure on \(\mathbb{R}\) with norm given by
\[(22.4)\quad \|\mu\|_* = \int_{\mathbb{R}} |f(x)| \, dx.\]
For every \(y \in \mathbb{R}\), the Dirac mass
\[(22.5)\quad \delta_y(\phi) = \phi(y)\]
defines a measure on \(\mathbb{R}\) with norm equal to 1.
More generally, one might consider linear functionals $\mu$ on $\mathcal{C}_{00}(\mathbb{R}^n)$ which are bounded when restricted to functions that are supported in a bounded interval in $\mathbb{R}$. In other words, one would ask that for each bounded interval $I \subseteq \mathbb{R}$ there is an $A_I \geq 0$ such that

$$|\mu(\phi)| \leq A_I \|\phi\|_{\infty} \tag{22.6}$$

whenever $\phi$ is a continuous complex-valued function on $\mathbb{R}$ which satisfies $\phi(x) = 0$ for every $x \in \mathbb{R}\setminus I$. For example,

$$\mu(\phi) = \int_{\mathbb{R}} \phi(x) \, dx \tag{22.7}$$

has this property with $A_I = |I|$.

Let us say that a measure $\mu$ on $\mathbb{R}$ has bounded support if there is a closed and bounded interval $I \subseteq \mathbb{R}$ such that $\mu(\phi) = 0$ whenever $\phi \in \mathcal{C}_{00}(\mathbb{R})$ satisfies $\phi(x) = 0$ for every $x \in I$. In this event $\mu(\phi_1) = \mu(\phi_2)$ whenever $\phi_1, \phi_2 \in \mathcal{C}_{00}(\mathbb{R})$ are equal on $I$, and $\mu(\phi)$ can be defined for any continuous complex-valued function $\phi$ on $I$ by extending $\phi$ to a continuous function with bounded support on $\mathbb{R}$ and applying $\mu$ to the extension. One can choose the extension of $\phi$ in such a way that the supremum norm of the extension is equal to the supremum norm of $\phi$ on $I$. In particular, $|\mu(\phi)|$ is less than or equal to $\|\mu\|_* \times \text{the supremum norm of } \phi$ on $I$.

It turns out that the measures on $\mathbb{R}$ with bounded support are dense in $\mathcal{M}(\mathbb{R})$.

To see this, let $\mu \in \mathcal{M}(\mathbb{R})$ and $\epsilon > 0$ be given. Suppose that $\psi \in \mathcal{C}_{00}(\mathbb{R})$, $\|\psi\|_* \leq 1$, and

$$\|\mu\|_* < |\mu(\psi)| + \epsilon. \tag{22.8}$$

Let $\rho$ be a real-valued continuous function on the real line with bounded support such that

$$0 \leq \rho(x) \leq 1 \tag{22.9}$$

for every $x \in \mathbb{R}$ and $\rho(x) = 1$ when $\psi(x) \neq 0$. One can show that

$$\mu'(\phi) = \mu(\rho \phi) \tag{22.10}$$

defines a measure on $\mathbb{R}$ with bounded support and $\|\mu - \mu'\|_* < \epsilon$.

Let us say that a measure $\mu$ on $\mathbb{R}$ has a regular extension if there is an extension of $\mu$ as a linear functional to the vector space of bounded continuous functions on the real line, also denoted $\mu$, such that

$$\lim_{j \to \infty} \mu(\phi_j) = \mu(\phi) \tag{22.11}$$
whenever \( \{ \phi_j \}_{j=1}^\infty \) is a uniformly bounded sequence of continuous complex-valued functions on the real line which converges uniformly on bounded intervals to the continuous function \( \phi \). Such an extension would be unique, because for every bounded continuous complex-valued function \( \phi \) on \( \mathbb{R} \) there is a uniformly bounded sequence \( \{ \phi_j \}_{j=1}^\infty \) of continuous functions on \( \mathbb{R} \) such that each \( \phi_j \) has bounded support and the sequence converges uniformly to \( \phi \) on every bounded interval in \( \mathbb{R} \). By choosing \( \phi_j \)'s with \( \| \phi_j \|_\ast \leq \| \phi \|_\ast \) for each \( j \) we also get that
\[
|\mu(\phi)| \leq \| \mu \|_\ast \| \phi \|_\ast \quad (22.12)
\]
for every bounded continuous function \( \phi \) on the real line.

If a measure \( \mu \) on \( \mathbb{R} \) has bounded support, then \( \mu \) has a regular extension, as a consequence of the earlier remarks. Suppose that \( \{ \mu_j \}_{j=1}^\infty \) is a sequence of measures on \( \mathbb{R} \) with regular extensions which converges in the norm on \( \mathcal{M}(\mathbb{R}) \) to a measure \( \mu \). In this case one can check that \( \mu \) has a regular extension, obtained by taking the limit of \( \{ \mu_j(\phi) \}_{j=1}^\infty \) for every bounded continuous complex-valued function \( \phi \) on \( \mathbb{R} \). It follows that every measure on \( \mathbb{R} \) has a regular extension.

Let \( \mu \) be a measure on \( \mathbb{R} \), which has a regular extension to bounded continuous functions on \( \mathbb{R} \), as in the previous paragraph, also denoted \( \mu \). For every \( \xi \in \mathbb{R} \),
\[
e_\xi(x) = \exp(\xi x i)
\]
is a bounded continuous complex-valued function on \( \mathbb{R} \). The Fourier transform of \( \mu \) is the function \( \hat{\mu} \) on \( \mathbb{R} \) given by
\[
\hat{\mu}(\xi) = \mu(e^{-\xi}),
\]
which satisfies
\[
|\hat{\mu}(\xi)| \leq \| \mu \|_\ast \quad (22.15)
\]
for every \( \xi \in \mathbb{R} \). If \( \mu \) has bounded support, then one can check that \( \hat{\mu} \) is a Lipschitz function on the real line. Because the measures with bounded support are dense in \( \mathcal{M}(\mathbb{R}) \), it follows that \( \hat{\mu} \) is uniformly continuous for every measure \( \mu \) on \( \mathbb{R} \).

As in Section 15 consider
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mu}(\xi) A_\eta(\xi) \exp(\xi x i) \, d\xi \quad (22.16)
\]
as a version of Abel sums for the inverse Fourier transform of \( \hat{\mu} \), where \( A_\eta(x i) = \exp(-\eta |x|), \eta > 0 \). If \( P_t(y) = (1/\pi) \frac{t}{(y^2 + t^2)} \) is the Poisson kernel
associated to the real line, then this integral is equal to \( \mu(p_{\eta,x}), p_{\eta,x}(y) = P_\eta(x - y) \). One can show that this is an integrable continuous function on the real line which satisfies

\[
\int_\mathbb{R} \mu(p_{\eta,x}) \phi(x) \, dx = \mu(P_\eta(\phi))
\]

for every bounded continuous function \( \phi \) on \( \mathbb{R} \), where \( P_t(\phi) \) is the Poisson integral of \( \phi \),

\[
P_t(\phi)(w) = \int_\mathbb{R} P_t(w - x) \phi(w) \, dw.
\]

This amounts to interchanging the order of integration. Consequently,

\[
\lim_{\eta \to 0} \int_\mathbb{R} \mu(p_{\eta,x}) \phi(x) \, dx = \mu(\phi),
\]

because \( P_t(\phi) \) is a uniformly bounded family of continuous functions on the real line when \( \phi \) is a bounded continuous function on \( \mathbb{R} \) which converges to \( \phi \) uniformly on bounded intervals as \( t \to 0 \).

\section{Convolutions on \( T^n \)}

Fix a positive integer \( n \), and let \( f(z), g(z) \) be continuous complex-valued functions on the \( n \)-dimensional torus. The \emph{convolution} of \( f, g \) is the function on \( T^n \) defined by

\[
(f * g)(z) = \frac{1}{(2\pi)^n} \int_{T^n} f(w) g(z \circ w) \, |dw|.
\]

Here

\[
(23.2) \quad u \circ v = (u_1 v_1, \ldots, u_n v_n)
\]

and

\[
(23.3) \quad w = (w_1, \ldots, w_n)
\]

for \( u, v, w \in T^n \).

Observe that

\[
(23.4) \quad f * g = g * f,
\]

as one can check using a change of variables. Because the torus is compact, \( f, g \) are uniformly continuous, and this implies that the convolution \( f * g \) is a continuous function on \( T^n \).
Let $a_\alpha, b_\alpha, c_\alpha$ be the Fourier coefficients of $f$, $g$, and $f * g$. A key feature of the convolution is the identity

$$c_\alpha = a_\alpha b_\alpha$$

for every $\alpha \in \mathbb{Z}^n$. Basically, $c_\alpha$ can be expressed as a double integral which reduces to a product of integrals after a change of variables.

For every $z \in \mathbb{T}^n$,

$$|f * g(z)| \leq \frac{1}{(2 \pi)^n} \int_{\mathbb{T}^n} |f(w)||g(z \circ w)| \, |dw|.$$  

One can check that

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1,$$

by converting a double integral into a product of integrals again, and that

$$\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty, \|f\|_\infty \|g\|_1.$$

Let $\mu$ be a measure and $f$ be a continuous complex-valued function on $\mathbb{T}^n$. For every $z \in \mathbb{T}^n$, put

$$f_z(w) = f(z \circ w).$$

The convolution of $\mu$ and $f$ is the function on $\mathbb{T}^n$ defined by

$$\mu * f(z) = \mu(f_z).$$

Because $f$ is uniformly continuous on $\mathbb{T}^n$, $\mu * f$ is a continuous function. We also have the estimate

$$\|\mu * f\|_\infty \leq \|\mu\|_1 \|f\|_\infty.$$  

More generally, suppose that $\mu, \nu$ are measures on $\mathbb{T}^n$. We would like to begin by defining a measure $\mu \times \nu$ on $\mathbb{T}^{2n} \cong \mathbb{T}^n \times \mathbb{T}^n$. Let $f(z, w)$ be a continuous function on $\mathbb{T}^n \times \mathbb{T}^n$, and let us say how to evaluate $(\mu \times \nu)(f)$.

For every $z \in \mathbb{T}^n$, we can apply $\nu$ to $f(z, w)$ as a function of $w$, and get a function of $z$ which is continuous because of uniform continuity. If we apply $\mu$ to the resulting function of $z$, then we get the first definition of $(\mu \times \nu)(f)$.

Alternatively, for every $w \in \mathbb{T}^n$ we can apply $\mu$ to $f(z, w)$ as a function of $z$, and then apply $\nu$ to the resulting function of $w$. We would like to show that these two definitions of $(\mu \times \nu)(f)$ are the same.
If \( f(z, w) = f_1(z) f_2(w) \) for continuous functions \( f_1, f_2 \) on \( T^n \), then both definitions yield
\[
(\mu \times \nu)(f) = \mu(f_1) \nu(f_2).
\]
By linearity, both definitions agree for sums of products of functions of \( z \), \( w \), separately. Every continuous function on \( T^n \times T^n \) can be uniformly approximated by such sums, and one can use this to show that the two definitions of \( (\mu \times \nu)(f) \) agree for every continuous function \( f(z, w) \).

Both definitions imply that
\[
|((\mu \times \nu)(f))| \leq \|\mu\|_{*, T^n} \|\nu\|_{*, T^n} \|f\|_{\infty, T^{2n}},
\]
and hence that \( \mu \times \nu \) is a measure on \( T^{2n} \) with norm less than or equal to the product of the norms of \( \mu \) and \( \nu \) on \( T^n \). In fact we have that
\[
\|\mu \times \nu\|_{*, T^{2n}} = \|\mu\|_{*, T^n} \|\nu\|_{*, T^n},
\]
because of (23.12).

The convolution of two measures \( \mu, \nu \) on \( T^n \) is the measure \( \mu \ast \nu \) defined by applying \( \mu \times \nu \) to \( f(z \circ w) \) for every continuous function \( f \) on \( T^n \). It follows from the previous discussion that
\[
|((\mu \ast \nu)(f))| \leq \|\mu\|_{*} \|\nu\|_{*} \|f\|_{\infty}
\]
for every \( f \in \mathcal{C}(T^n) \), which is to say that \( \mu \ast \nu \) is bounded and that
\[
\|\mu \ast \nu\|_{*} \leq \|\mu\|_{*} \|\nu\|_{*}.
\]

If \( \mu \) or \( \nu \) are given by \( (2\pi)^{-n} \) times integration with a continuous density, then the convolution \( \mu \ast \nu \) is given by \( (2\pi)^{-n} \) times integration with a continuous density, where the density corresponds to one of the previous definitions of the convolution, as appropriate.

For every \( \alpha \in \mathbb{Z}^n \), one can check that the \( \alpha \)th Fourier coefficient of \( \mu \times \nu \) is equal to the product of the \( \alpha \)th Fourier coefficients of \( \mu, \nu \).

The convolution of the Dirac mass at \((1, 1, \ldots, 1)\) with any function or measure is equal to that function or measure.

## 24 Convolutions on \( \mathbb{R} \)

If \( f(x), g(x) \) are continuous complex-valued functions on the real line, then, under suitable additional conditions, the convolution of \( f, g \) is defined by
\[
f \ast g(x) = \int_{\mathbb{R}} f(y) g(x - y) \, dx.
\]
For instance, this makes sense if $f, g$ have bounded support, in which event $f * g$ has bounded support. Uniform continuity of $f, g$ imply that $f * g$ is a continuous function on the real line.

If at least one of $f, g$ has bounded support and the other is an arbitrary continuous function on $\mathbb{R}$, then the convolution $f * g$ is defined as a function on $\mathbb{R}$. One can also show that $f * g$ is continuous in this case.

If $\mu$ is a measure and $h$ is a bounded continuous function on $\mathbb{R}$, then the convolution $\mu * h$ is defined as a function on $\mathbb{R}$ by the formula

\[(\mu * h)(x) = \mu(h_x), \quad h_x(u) = h(x - u).\]

(24.2) This is the same as the previous formula when $\mu$ is given by integration with a continuous integrable density. One can check that $\mu * h$ is continuous using the continuity properties of the regular extension of $\mu$ to bounded continuous functions on $\mathbb{R}$. If $h$ is bounded and uniformly continuous, then $\mu * h$ is uniformly continuous. At any rate, for a bounded continuous function $h$ on $\mathbb{R}$,

\[\|\mu * h\|_\infty \leq \|\mu\|_* \|h\|_\infty.\]

(24.3)

Now suppose that $\mu, \nu$ are measures on $\mathbb{R}$. If $f(x, y)$ is a continuous complex-valued function on $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}^2$, then there are two ways to try to make sense of $(\mu \times \nu)(f)$, by applying $\mu$ or $\nu$ to $f$ as a function of $x$ or $y$ and then to the other variable. Because $f$ has bounded support in the plane, the functions on $\mathbb{R}$ obtained by applying $\mu$ or $\nu$ to $f(x, y)$ in $x$ or $y$ each have bounded support. They are also continuous, for the usual reasons of uniform continuity. Hence one can apply $\mu$ or $\nu$ to the resulting function of one variable.

After applying $\mu$ or $\nu$ to $f(x, y)$ as a function of $x$ or $y$, the resulting function has supremum norm less than or equal to $\|\mu\|_*$ or $\|\nu\|_*$ times $\|f\|_\infty$, as appropriate. For both definitions of $(\mu \times \nu)(f)$ one gets

\[|(\mu \times \nu)(f)| \leq \|\mu\|_* \|\nu\|_* \|f\|_\infty.\]

(24.4)

If $f(x, y)$ is of the form $f_1(x) f_2(y)$, where $f_1, f_2$ are continuous functions on the real line with compact support, then both definitions of $(\mu \times \nu)(f)$ are equal to $\mu(f_1) \nu(f_2)$. The two definitions of $\mu \times \nu$ agree on functions $f$ which are sums of products of this type, and hence on all continuous functions on the plane with bounded support, by approximation arguments. Furthermore,

\[\sup\{|(\mu \times \nu)(f)| : f \in C^00(\mathbb{R}^2), \|f\|_\infty, \mathbb{R}^2\} = \|\mu\|_* \|\nu\|_*.\]

(24.5)
where $C^0_0(R^2)$ is the space of continuous complex-valued functions on the plane with bounded support, and $\|f\|_{\infty, R^2}$ is the supremum norm for such functions.

Just as for measures on $\mathbb{R}$, one can consider the notion of a regular extension for $\mu \times \nu$ to bounded continuous functions on $\mathbb{R}^2$. One can also start with the regular extensions of $\mu$, $\nu$ to bounded continuous functions on the real line and use them to deal with $(\mu \times \nu)(f)$ when $f$ is a bounded continuous function on $\mathbb{R}^2$. If $\mu$, $\nu$ have bounded support in $\mathbb{R}$, then the product $\mu \times \nu$ has bounded support in $\mathbb{R}^2$, and $(\mu \times \nu)(f)$ makes sense for arbitrary continuous functions $f$ on the plane. In general there are approximations of $\mu$, $\nu$ by measures with bounded support, which lead to approximations of $\mu \times \nu$ by measures with bounded support. At any rate, $\mu \times \nu$ has a version of a regular extension to bounded continuous functions on the plane.

The convolution $\mu * \nu$ of $\mu$, $\nu$ is the measure on $\mathbb{R}$ defined by saying that $(\mu * \nu)(f)$ is equal to $\mu \times \nu$ applied to $f(x+y)$ as a continuous function on the plane. In particular, this is a bounded linear functional, and more precisely $\|\mu * \nu\|_* \leq \|\mu\|_* \|\nu\|_*$. The Fourier transform of $\mu * \nu$ is equal to the product of the Fourier transforms of $\mu$ and $\nu$.

### 25 Smooth functions

Let $C^\infty(R)$ be the space of complex-valued functions on the real line which are continuously differentiable of all orders, and let $C^\infty_0(R)$ be the subspace of $C^\infty(R)$ consisting of functions with bounded support. For instance, the function defined by $\exp(-1/x)$ when $x > 0$ and equal to 0 when $x \leq 0$ is in $C^\infty(R)$, and one can use this to get nontrivial functions in $C^\infty_0(R)$. The Schwartz class $S(R)$ consists of the functions $f \in C^\infty(R)$ such that $f$ and all of its derivatives are rapidly decreasing in the sense that for each pair of nonnegative integers $j$, $l$ there is a $C(j, l) \geq 0$ such that

\begin{equation}
|f^{(j)}(x)| \leq \frac{C(j, l)}{(1 + |x|)^l}
\end{equation}

for every $x \in R$, where $f^{(j)}$ denotes the $j$th order derivative of $f$, with $f^{(0)} = f$. In particular,

\begin{equation}
C^\infty_0(R) \subseteq S(R).
\end{equation}

If $f \in S(R)$, then $f$ is an integrable continuous function, and its Fourier transform $\hat{f}$ is a bounded continuous function. One can show that the Fourier
transform of \( f \) is continuously differentiable of all orders, and more precisely that the \( l \)th derivative of \( \hat{f} \) is equal to \((-i)^l \) times the Fourier transform of \( x^l \ f(x) \) for every positive integer \( l \). Similarly, \( \xi^l \ \hat{f}(\xi) \) is equal to \( i^l \) times the Fourier transform of the \( l \)th derivative of \( f \) for each positive integer \( l \) and hence is bounded. Continuing in this way one can show that

\[
\hat{f} \in \mathcal{S}(\mathbb{R}).
\]

Similarly, the inverse Fourier transform maps the Schwartz class into itself, and it follows that the Fourier transform is a one-to-one mapping of \( \mathcal{S}(\mathbb{R}) \) onto itself.

For each positive integer \( n \), let \( \mathcal{C}^\infty(T^n) \) be the space of complex-valued functions on the \( n \)-dimensional torus which are continuously-differentiable of all orders. One can show that the Fourier coefficients \( a_\alpha \) of such a function decay rapidly in the sense that for every positive integer \( l \) there is a \( C(l) \geq 0 \) such that

\[
|a_\alpha| \leq \frac{C(l)}{(1 + |\alpha_1| + \cdots + |\alpha_n|)^l}
\]

for every \( \alpha \in \mathbb{Z}^n \), by expressing the Fourier coefficients of derivatives of \( f \) in terms of the \( a_\alpha \)'s times products of powers of the \( \alpha_j \)'s. Conversely, if \( a_\alpha \in \mathbb{C} \), \( \alpha \in \mathbb{Z}^n \), is a family of complex numbers which are rapidly decreasing in this sense, then \( \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha \) is continuously differentiable of all orders on \( T^n \), and the derivatives are given by differentiating the sum term-by-term.

Suppose that \( \Phi \in \mathcal{S}(\mathbb{R}) \) and \( \Phi(0) = 1 \). Put \( \phi = \hat{\Phi} \) and

\[
\phi_\eta(w) = \frac{1}{2 \pi \eta} \phi\left(\frac{w}{\eta}\right),
\]

which is \( 1/(2 \pi) \) times the Fourier transform of \( \Phi(\eta \cdot) \) at \( w \). For each \( \eta > 0 \),

\[
\int_{\mathbb{R}} \phi_\eta(w) \, dw = 1,
\]

because \( \Phi(0) = 1 \). For every integrable continuous function \( f \) on the real line, consider

\[
\frac{1}{2 \pi} \int_{\mathbb{R}} \hat{f}(\xi) \, \Phi(\eta \xi) \exp(\xi \cdot x \, i) \, d\xi
\]

as an extension of the Abel sums (15.2) for the inverse Fourier transform (15.1) applied to \( \hat{f} \). As usual, the limit of (25.7) as \( \eta \to 0 \) is equal to (15.1) when \( \hat{f} \) is integrable. In general, we can rewrite (25.7) as a double integral.
using the definition of \( \hat{f} \). By interchanging the order of integration, we get that (25.7) is equal to

\[
(25.8) \quad \int_{\mathbb{R}} f(y) \phi_{\eta}(y-x) \, dy.
\]

As in the previous situation, one can show that the limit of (25.8) as \( \eta \to 0 \) is equal to \( f(x) \).

## 26 Gaussians

For each positive real number \( a \), the corresponding Gaussian function is defined by

\[
G_a(x) = \exp(-a x^2).
\]

This is an integrable continuous function on the real line which is an element of the Schwartz class \( S(\mathbb{R}) \).

It is a well-known fact from integral calculus that

\[
(26.2) \quad \int_{\mathbb{R}} \exp(-\pi x^2) \, dx = 1.
\]

The trick for showing this is to observe that the integral is a positive real number whose square can be expressed as the double integral

\[
(26.3) \quad \int_{\mathbb{R}^2} \exp(-\pi (x^2 + y^2)) \, dx \, dy.
\]

Using polar coordinates this double integral reduces to

\[
(26.4) \quad \int_0^{2\pi} \int_0^\infty \exp(-\pi r^2) r \, dr \, d\theta = \int_0^\infty 2\pi r \exp(-\pi r^2) \, dr = 1,
\]

since the derivative of \( \exp(-\pi r^2) \) is equal to \(-2\pi r \exp(-\pi r^2)\).

It follows that

\[
(26.5) \quad \int_{\mathbb{R}} \exp(-a x^2) \, dx = \sqrt{\frac{\pi}{a}}.
\]

Specifically, one can use the change of variables \( x = \sqrt{\frac{\pi}{a}} w \) to get this from the case where \( a = \pi \).

For every \( b \in \mathbb{R} \),

\[
(26.6) \quad \int_{\mathbb{R}} \exp(-a (x + (2a)^{-1} b)^2) \, dx = \sqrt{\frac{\pi}{a}},
\]
by translation-invariance of the integral. Hence

\[ \int_\mathbb{R} \exp(-a x^2 - b x) \, dx = \sqrt{\frac{\pi}{a}} \exp((4a)^{-1} b^2). \]

This suggests that

\[ \int_\mathbb{R} \exp(-a x^2 - \pi \xi x i) \, dx = \sqrt{\frac{\pi}{a}} \exp(- (4a)^{-1} \xi^2), \]

which is to say that

\[ \widehat{G}_a(\xi) = \sqrt{\frac{\pi}{a}} G_{(4a)^{-1}}(\xi), \]

\[ \xi \in \mathbb{R}. \]

One can show this rigorously using complex analysis, through Cauchy’s theorem or analytic continuation in \( b \).

Alternatively one can use differential equations, observing a linear relation between the derivative of \( G_a(x) \) and \( x G_a(x) \) and a similar relationship for the Fourier transform.

### 27 Plancherel’s theorem

If \( \phi_1, \phi_2 \) are continuous integrable functions on the real line with integrable Fourier transforms, then

\[ \int_\mathbb{R} \hat{\phi}_1(\xi) \overline{\hat{\phi}_2(\xi)} \, d\xi = \int_\mathbb{R} \int_\mathbb{R} \hat{\phi}_1(\xi) \overline{\hat{\phi}_2(x)} \exp(\xi x i) \, dx \, d\xi. \]

Interchanging the order of integration and using the formula for the inverse Fourier transform, we get that

\[ \int_\mathbb{R} \hat{\phi}_1(\xi) \overline{\hat{\phi}_2(\xi)} \, d\xi = 2\pi \int_\mathbb{R} \phi_1(x) \overline{\phi_2(x)} \, dx. \]

In particular,

\[ \int_\mathbb{R} |\hat{\phi}(\xi)|^2 \, d\xi = 2\pi \int_\mathbb{R} |\phi(x)|^2 \, dx
\]
when \( \phi \) is a continuous integrable function on the real line with integrable Fourier transform.

More generally, a continuous integrable function on the real line whose square is integrable has square-integrable Fourier transform, and the identities \((27.2)\) and \((27.3)\) carry over to these functions. Observe that a continuous
integrable function with integrable Fourier transform is bounded and hence square integrable.

In order to show this extension, one can regularize the integrals in the preceding computations as in the Abel summability techniques employed several times now. One can also approximate an integrable and square-integrable function on the real line simultaneously in the norms \( \| f \|_1 \) and \( \| f \|_2 \) by integrable continuous functions with integrable Fourier transforms, e.g., by smooth functions with bounded support.

### 28 Bounded functions

If \( f, \phi \) are continuous integrable functions on the real line, then

\[
(28.1) \quad \int_\mathbb{R} \hat{f}(\xi) \phi(\xi) \, d\xi = \int_\mathbb{R} f(x) \hat{\phi}(x) \, dx,
\]

basically because they are both equal to the double integral

\[
(28.2) \quad \int_\mathbb{R} \int_\mathbb{R} f(x) \phi(\xi) \exp(-\xi x i).
\]

Let \( \mathcal{E}(\mathbb{R}) \) be the vector space of continuous integrable functions \( \phi \) on \( \mathbb{R} \) such that the Fourier transform \( \hat{\phi} \) of \( \phi \) is integrable too, which implies that \( \phi, \hat{\phi} \in \mathcal{C}_0(\mathbb{R}) \). If \( f \) is a bounded continuous function on the real line, then

\[
(28.3) \quad L_f(\phi) = \int_\mathbb{R} f(x) \hat{\phi}(x) \, dx
\]
defines a linear mapping from \( \mathcal{E}(\mathbb{R}) \) into the complex numbers which is the same as

\[
(28.4) \quad L_f(\phi) = \int_\mathbb{R} \hat{f}(\xi) \phi(\xi) \, d\xi
\]
when \( f \) is integrable.

We can think of \( L_f \) as a kind of *generalized Fourier transform* which makes sense for bounded continuous functions on the real line. For example, if

\[
(28.5) \quad f(x) = \exp(ax i)
\]
for some \( a \in \mathbb{R} \), then

\[
(28.6) \quad L_f(\phi) = 2\pi \phi(a)
\]
for every $\phi \in \mathcal{E}(\mathbb{R})$. As a version of Abel sums for the inverse Fourier transform of a function $f \in \mathcal{BC}(\mathbb{R})$, we can apply $L_f$ to

$$
\frac{1}{2\pi} A_\eta(\xi) \exp(\xi v i)
$$

as a function of $\xi$ for every $v \in \mathbb{R}$ and $\eta > 0$. As usual this is equal to the Poisson integral $P_\eta(f)(v)$ of $f$, and we recover $f(v)$ as $\eta \to 0$. One can also use other functions instead of $A_\eta(\xi)$ as in Section 25.

Let $\sigma(x)$ be the function on the real line which is equal to $-1$ when $x < 0$, to 0 when $x = 0$, and to $+1$ when $x > 0$. Although this bounded function is not quite continuous,

$$
L_\sigma(\phi) = \int_{\mathbb{R}} \sigma(x) \hat{\phi}(x) \, dx
$$

still makes sense as a generalized Fourier transform of $\sigma$, and it will be convenient for us to restrict our attention now to $\phi \in \mathcal{S}(\mathbb{R})$. For each $\eta > 0$, put

$$
L_{\sigma,\eta}(\phi) = \int_{\mathbb{R}} \sigma(x) A_\eta(x) \hat{\phi}(x) \, dx,
$$

where $A_\eta(x) = \exp(-\eta |x|)$. Clearly

$$
\lim_{\eta \to 0} L_{\sigma,\eta}(\phi) = L_\sigma(\phi)
$$

for every $\phi \in \mathcal{S}(\mathbb{R})$.

Because

$$
B_\eta(x) = \sigma(x) A_\eta(x)
$$

is integrable, with a jump discontinuity at $x = 0$, its Fourier transform can be defined in the usual way, and

$$
L_{\sigma,\eta}(\phi) = \int_{\mathbb{R}} \hat{B}_\eta(\xi) \phi(\xi) \, d\xi.
$$

Observe that

$$
\hat{B}_\eta(\xi) = -\int_{-\infty}^{0} \exp((\eta - \xi i) x) \, dx + \int_{0}^{\infty} \exp(-(\eta + \xi i) x) \, dx
$$

$$
= \frac{-1}{\eta - \xi i} + \frac{1}{\eta + \xi i} = \frac{2\xi i}{\eta^2 + \xi^2},
$$

which is an odd function,

$$
\hat{B}_\eta(-\xi) = -\hat{B}_\eta(\xi).
$$
We can rewrite $L_{\sigma, \eta}(\phi)$ as

$$L_{\sigma, \eta}(\phi) = \int_{|\xi| \leq 1} \tilde{B}_\eta(\xi) (\phi(\xi) - \phi(0)) \, d\xi + \int_{|\xi| > 1} \tilde{B}_\eta(\xi) \phi(\xi) \, d\xi.$$  

(28.15)

Because of the smoothness and integrability of $\phi$,

$$\lim_{\eta \to 0} L_{\sigma, \eta}(\phi) = 2i \int_{|\xi| \leq 1} \frac{\phi(\xi) - \phi(0)}{\xi} \, d\xi + 2i \int_{|\xi| > 1} \frac{\phi(\xi)}{\xi} \, d\xi.$$  

(28.16)

Equivalently,

$$\lim_{\eta \to 0} L_{\sigma, \eta}(\phi) = \lim_{\epsilon \to 0} 2i \int_{|\xi| > \epsilon} \frac{\phi(\xi)}{\xi} \, d\xi.$$  

(28.17)

29 Subharmonic functions

Let $u(z)$ be a twice-continuously differentiable real-valued function defined on an open set in the complex plane. We say that $u$ is subharmonic if

$$\Delta u(z) \geq 0$$  

(29.1)

for every $z$ in the domain of $u$, where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$  

(29.2)

is the usual Laplace operator, and $x, y$ are the real and imaginary parts of $z$. We say that $u$ is strictly subharmonic if

$$\Delta u(z) > 0$$  

(29.3)

for every $z$ in the domain of $u$.

For the sake of concreteness let us suppose that $u$ is a continuous real valued function on the closed unit disk \{ $z \in \mathbb{C} : |z| \leq 1$ \} which is twice-continuously differentiable on the open unit disk \{ $z \in \mathbb{C} : |z| < 1$ \}. If $u$ is strictly subharmonic on the open unit disk, then the second-derivative test from calculus implies that $u$ does not have any local maxima in the open unit disk. However, the maximum of $u$ on the closed unit disk is attained, because the closed unit disk is compact and $u$ is continuous. Hence the maximum is attained on the unit circle.
Therefore
\[(29.4) \quad \max\{u(z) : z \in \mathbb{C}, |z| \leq 1\} = \max\{u(z) : z \in \mathbb{C}, |z| = 1\}.\]

This also works when \(u\) is subharmonic on the open unit disk. For in this event \(u_\epsilon(z)u(z) + \epsilon |z|^2\) is strictly subharmonic for every \(\epsilon > 0\). By applying the maximum principle to \(u_\epsilon\) and sending \(\epsilon \to 0\) we recover the maximum principle for \(u\).

If \(u\) is harmonic on the open unit disk, then \(u\) and \(-u\) are subharmonic. In particular, if \(u(z) = 0\) for every \(z\) in the unit circle, then \(u\) vanishes everywhere. Equivalently, two continuous real-valued functions on the closed unit disk which are harmonic on the open unit disk and are equal at every point in the unit circle are equal everywhere. The same statement also holds for complex-valued functions, by considering the real and imaginary parts.

\section{Metric spaces}

A \textit{metric space} is a nonempty set \(M\) together with a distance function \(d(x, y)\) defined for \(x, y \in M\) such that \(d(x, y)\) is a nonnegative real number for every \(x, y \in M\),
\[(A.1) \quad d(x, y) = 0\]
if and only if \(x = y\),
\[(A.2) \quad d(y, x) = d(x, y)\]
for every \(x, y \in M\), and
\[(A.3) \quad d(x, z) \leq d(x, y) + d(y, z)\]
for every \(x, y, z \in M\). The last property is known as the triangle inequality.

For example, the real line \(\mathbb{R}\) is a metric space with the standard metric \(|x - y|\), and the complex numbers \(\mathbb{C}\) form a metric space with the standard metric \(|z - w|\). If \(V\) is a vector space equipped with a norm \(N(v)\), then \(N(v - w)\) defines a metric on \(V\). On \(\mathbb{C}^n\), \(\|v - w\|_p\) is a metric when \(0 < p \leq 1\). Here \(\|v\|_p\) is as in Section 13 and is a norm on \(\mathbb{C}^n\) when \(p \geq 1\). If \((M, d(x, y))\) is any metric space and \(E \subseteq M, E \neq \emptyset\), then \(E\) is a metric space using the restriction of \(d(x, y)\) to \(E\) as the metric.

Let \((M, d(x, y))\) be a metric space. For every \(x \in M\) and \(r > 0\), the open ball with center \(x\) and radius \(r\) is given by
\[(A.4) \quad B(x, r) = \{y \in M : d(x, y) < r\},\]
and the closed ball with center $x$ and radius $r$ is given by

\[(A.5) \quad \overline{B}(x, r) = \{ y \in M : d(x, y) \leq r \}.\]

A set $U \subseteq M$ is said to be open if for every $x \in U$ there is an $r > 0$ such that $B(x, r) \subseteq U$. One can check that open balls are open sets.

If $A \subseteq M$ and $p \in M$, then $p$ is a limit point of $A$ if for every $r > 0$ there is a $q \in A$ such that $q \neq p$ and $d(p, q) < r$. Similarly, $p$ is an accumulation point of $A$ if for every $r > 0$ there is a $q \in A$ such that $d(p, q) < r$. Every accumulation point of $A$ is an element of $A$, or a limit point of $A$, or both.

We say that $E \subseteq M$ is closed if every $p \in M$ which is a limit point of $E$ is also an element of $E$, which is the same as saying that every accumulation point of $E$ is an element of $E$. One can check that closed balls are closed sets.

The complement of a set $A \subseteq M$ is the set $M \setminus A$ of $x \in M$ such that $x$ is not an element of $A$. One can show that $A \subseteq M$ is an open set if and only if $M \setminus A$ is closed.

The union of any collection of open subsets of $M$ is an open set. The intersection of any collection of closed subsets of $M$ is a closed set. These statements are easy to check, just from the definitions, and they also correspond to each other by taking complements as in the previous paragraph.

The intersection of finitely many open subsets of $M$ is an open set. It follows from the fact about complements that the union of finitely many closed subsets of $M$ is closed.

The closure of $E \subseteq M$ is denoted $\overline{E}$ and defined to be the set of accumulation points of $E$ in $M$, which is the same as the union of $E$ and the set of limit points of $E$ in $M$. Observe that $E$ is closed if and only if $\overline{E} = E$. One can check that $\overline{E}$ is automatically closed, which is to say that an accumulation point of the set of accumulation points of $E$ is an accumulation point of $E$.

We say that $E \subseteq M$ is bounded if there are $p \in M$ and $r > 0$ such that

\[(A.6) \quad E \subseteq B(p, r),\]

in which event the diameter of $E$ is given by

\[(A.7) \quad \text{diam } E = \sup \{ d(x, y) : x, y \in E \},\]

with $\text{diam } E = 0$ when $E = \emptyset$. The closure of a bounded set is bounded and has the same diameter.
Let \( \{x_j\}_{j=1}^{\infty} \) be a sequence of points in \( M \). We say that \( \{x_j\}_{j=1}^{\infty} \) converges to \( x \in M \) if for each \( \epsilon > 0 \) there is an \( L \geq 1 \) such that
\[
d(x_j, x) < \epsilon
\]
for every \( j \geq L \). We say that \( \{x_j\}_{j=1}^{\infty} \) is a Cauchy sequence if for each \( \epsilon > 0 \) there is an \( L \geq 1 \) such that
\[
d(x_j, x_l) < \epsilon
\]
for every \( j, l \geq L \). It is easy to see that convergent sequences are Cauchy sequences. If \( \{x_j\}_{j=1}^{\infty} \) converges to \( x \) in \( M \), then \( x \) is said to be the limit of the sequence, also expressed by
\[
\lim_{j \to \infty} x_j = x,
\]
and one can check that the limit \( x \) is unique.

A point \( p \in M \) is an accumulation point of \( E \subseteq M \) if and only if there is a sequence of elements of \( E \) which converges to \( p \). Hence \( E \subseteq M \) is closed if and only if every sequence of elements of \( E \) which converges in \( M \) has its limit in \( E \).

If \( \{z_j\}_{j=1}^{\infty}, \{w_j\}_{j=1}^{\infty} \) are sequences of real or complex numbers which converge to the real or complex numbers \( z, w \), respectively, then the sequences
\[
\{z_j + w_j\}_{j=1}^{\infty}, \ {z_j w_j\}_{j=1}^{\infty}
\]
of sums and products converge to \( z + w, zw \), respectively.

More generally, if \( V \) is a complex vector space equipped with a norm and thus a metric, and if \( \{v_j\}_{j=1}^{\infty}, \ {w_j\}_{j=1}^{\infty} \) are sequences of vectors in \( V \) converging to \( v, w \in V \), then
\[
\lim_{j \to \infty} v_j + w_j = v + w.
\]
If \( \{\alpha_j\}_{j=1}^{\infty} \) is a sequence of complex numbers converging to \( \alpha \in \mathbb{C} \) and \( \{v_j\}_{j=1}^{\infty} \) is a sequence of vectors in \( V \) converging to \( v \in V \), then
\[
\lim_{j \to \infty} \alpha_j v_j = \alpha v.
\]

A metric space is said to be complete if every Cauchy sequence in the space converges. The real and complex numbers are complete with respect to their standard metrics.

If \((M, d(x, y)) \) is a metric space, then \( E \subseteq M \) is said to be dense in \( M \) if \( \overline{E} = M \). Equivalently, \( E \) is dense in \( M \) if for every \( x \in M \) there is a sequence \( \{x_j\}_{j=1}^{\infty} \) of elements of \( E \) which converges to \( x \). For example, the rational numbers are dense in \( \mathbb{R} \).
Let \( \{x_j\}_{j=1}^{\infty} \) be a sequence with terms in any set. A subsequence of \( \{x_j\}_{j=1}^{\infty} \) is a sequence of the form \( \{x_{j_i}\}_{i=1}^{\infty} \), where \( \{j_i\}_{i=1}^{\infty} \) is a strictly increasing sequence of positive integers. In particular, \( \{x_j\}_{j=1}^{\infty} \) is a subsequence of itself.

In a metric space \((M, d(x, y))\), every subsequence of a convergent sequence converges to the same point in \( M \). If a Cauchy sequence in \( M \) has a convergent subsequence, then the Cauchy sequence converges to the same point in \( M \).

A set \( E \subseteq M \) is said to be sequentially compact if every sequence of elements of \( E \) has a subsequence which converges to an element of \( E \). In particular, if a sequence of elements of \( E \) converges in \( M \), then the limit has to be in \( E \). Hence a sequentially compact set is closed.

As a partial converse, if \( E \subseteq M \) is closed, \( E_1 \subseteq M \) is sequentially compact, and \( E \subseteq E_1 \), then \( E \) is sequentially compact. For sequential compactness of \( E_1 \) implies that every sequence in \( E \) has a convergent subsequence, whose limit is in \( E \) since \( E \) is closed.

Sequentially compact sets are bounded. For if \( E \subseteq M \) is not bounded and \( p \in M \), then for each positive integer \( j \) there is an \( x_j \in E \) such that \( d(x_j, p) \geq j \), and it is easy to see that \( \{x_j\}_{j=1}^{\infty} \) has not convergent subsequence.

A set \( E \subseteq M \) is said to be totally bounded if for each \( \epsilon > 0 \) there are finitely many points \( p_1, \ldots, p_n \in E \) such that \( E \) is contained in the union of the balls \( B(p_1, \epsilon), \ldots, B(p_n, \epsilon) \). If \( E \) is not totally bounded, then there is a sequence \( \{x_j\}_{j=1}^{\infty} \) of elements of \( E \) such that \( d(x_j, x_i) \geq \epsilon \) when \( j \neq l \). It follows that sequentially compact sets are totally bounded.

If a sequence has its terms in the union of two sets, then there is a subsequence whose terms are all in one of the sets. It follows that the union of two sequentially compact sets is sequentially compact.

A set \( E \subseteq M \) has the limit point property if every infinite set \( A \subseteq E \) has a limit point contained in \( E \). If \( A \subseteq E \) is infinite, then there is a sequence \( \{a_j\}_{j=1}^{\infty} \) with \( a_j \in A \) for all \( j \) and \( a_j \neq a_l \) when \( j \neq l \), and the limit of any convergent subsequence of this sequence is a limit point of \( A \). Hence sequential compactness implies the limit point property. Conversely, if \( \{x_j\}_{j=1}^{\infty} \) is a sequence of elements of \( E \) and \( A \) is the set of all the \( x_j \)'s, then either \( A \) has only finitely many elements and the sequence has a constant subsequence, or \( A \) is infinite and one can check that any limit point of \( A \) is the limit of a subsequence of \( \{x_j\}_{j=1}^{\infty} \). Therefore the limit point property implies sequential compactness.

68
In these notes we shall use the term “compact” to refer to a set which is sequentially compact or has the limit point property. Equivalently, \( E \subseteq M \) is compact if every covering of \( E \) by open subsets of \( M \) can be reduced to a covering by finitely many of the open sets, but we shall not use this here.

Suppose that \( E \subseteq M \) has the property that every sequence of elements of \( E \) has a subsequence which is a Cauchy sequence. The same argument as above shows that \( E \) is totally bounded. Conversely, one can show that a totally bounded set has this property.

As a consequence, if \((M,d(x,y))\) is a complete metric space, and \( E \subseteq M \) is closed and totally bounded, then \( E \) is sequentially compact. The converse works in any metric space by the earlier remarks.

\section{Continuous functions}

Let \((M,d(x,y))\) be a metric space, and let \( f \) be a complex-valued function on \( M \). We say that \( f \) is \textit{continuous} at a point \( x \in M \) if for every \( \epsilon > 0 \) and \( x \in M \) there is a \( \delta > 0 \) such that
\[
|f(x) - f(y)| < \epsilon
\]
for each \( y \in M \) such that \( d(x,y) < \delta \). Equivalently, \( f \) is continuous at \( x \) if for every sequence \( \{x_j\}_{j=1}^{\infty} \) of elements of \( M \) which converges to \( x \) we have that
\[
\lim_{j \to \infty} f(x_j) = f(x).
\]

The space of complex-valued functions on \( M \) which are continuous at every point in \( M \) is denoted \( \mathcal{C}(M) \). Constant functions are obviously continuous, and one can show that sums and products of continuous functions are continuous, which implies that \( \mathcal{C}(M) \) is a commutative algebra over \( \mathbb{C} \).

A continuous complex-valued function \( f \) on \( M \) is said to be \textit{bounded} if there is an \( A \geq 0 \) such that
\[
|f(x)| \leq A
\]
for every \( x \in M \). The space of bounded continuous complex-valued functions on \( M \) is denoted \( \mathcal{BC}(M) \) and is a subalgebra of \( \mathcal{C}(M) \). The supremum norm of \( f \in \mathcal{BC}(M) \) is given by
\[
\|f\|_\infty = \|f\|_{\infty,M} = \sup\{|f(x)| : x \in M\},
\]
which one can check is a norm on $BC(M)$ as a complex vector space and also satisfies
\[ \|f_1 f_2\|_\infty \leq \|f_1\|_\infty \|f_2\|_\infty \]
for every $f_1, f_2 \in BC(M)$.

If $f$ is a continuous complex-valued function on $M$, $E \subseteq M$ is sequentially compact, and $\{x_j\}_{j=1}^{\infty}$ is any sequence of elements of $E$, then there is a subsequence $\{x_{j_l}\}_{l=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ which converges to a point $x \in E$, and $\{f(x_j)\}_{j=1}^{\infty}$ converges in $\mathbb{C}$ to $f(x)$. One can use this to show that $f$ is bounded on $E$ and $|f|$ attains its maximum on $E$ when $E \neq \emptyset$.

A complex-valued function $f$ on $M$ is said to be uniformly continuous if for each $\epsilon > 0$ there is a $\delta > 0$ such that (C.1) holds for every $x, y \in M$ with $d(x, y) < \delta$. Uniformly continuous functions are automatically continuous. The space of uniformly continuous complex-valued functions on $M$ is denoted $UC(M)$ and is a linear subspace of $C(M)$.

Suppose that $f$ is a continuous function on $M$ which is not uniformly continuous. Then there is an $\epsilon > 0$ and sequences $\{x_j\}_{j=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty}$ of elements of $M$ such that
\[ \lim_{j \to \infty} d(x_j, y_j) = 0 \]
and
\[ |f(x_j) - f(y_j)| \geq \epsilon. \]

If $M$ is sequentially compact, then every continuous function on $M$ is uniformly continuous. Otherwise there would be a subsequence $\{x_{j_l}\}_{l=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ as in the previous paragraph which converges to a point $x \in M$, the corresponding subsequence $\{y_{j_l}\}_{l=1}^{\infty}$ would also converge to $x$, and continuity of $f$ at $x$ would imply that $\{f(x_{j_l})\}_{l=1}^{\infty}$, $\{f(y_{j_l})\}_{l=1}^{\infty}$ converge to $f(x)$ as sequences of complex numbers, a contradiction.

Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of complex-valued continuous functions on $M$ which converges uniformly to a complex-valued function $f$ on $M$ in the sense that for each $\epsilon > 0$ there is an $L \geq 1$ such that
\[ |f_j(x) - f(x)| < \epsilon \]
for every $x \in M$ and $j \geq L$. In this event one can show that the limiting function $f$ is also a continuous function on $M$. Similarly, if the $f_j$’s are uniformly continuous, then $f$ is uniformly continuous.
If the \( f_j \)'s are bounded, then \( f \) is bounded too. For bounded continuous functions, uniform convergence is equivalent to convergence in \( BC(M) \) with respect to the supremum metric.

One can check that \( BC(M) \) is complete with respect to the supremum metric. For if \( \{f_j\}_{j=1}^\infty \) is a Cauchy sequence in \( BC(M) \) with respect to the supremum metric, then \( \{f_j(x)\}_{j=1}^\infty \) is a Cauchy sequence of complex numbers for every \( x \in M \). The completeness of the complex numbers implies that \( \{f_j(x)\}_{j=1}^\infty \) converges in \( \mathbb{C} \) for every \( x \in M \), and \( f(x) \) denotes the limit, then one can check that \( \{f_j\}_{j=1}^\infty \) converges uniformly to \( f \) on \( M \).

Let \( BUC(M) \) be the space of bounded uniformly continuous complex-valued functions on \( M \), i.e.,

\[
BUC(M) = BC(M) \cap UC(M).
\]

One can check that \( BUC(M) \) is a closed subalgebra of \( BC(M) \).

If \( f \) is a continuous complex-valued function on \( M \), then the zero set of \( f \), consisting of \( x \in M \) such that \( f(x) = 0 \), is a closed set. If \( f_1 \), \( f_2 \) are continuous complex-valued functions on \( M \), then the set of \( x \in M \) such that \( f_1(x) = f_2(x) \) is closed, since this is the same as the zero set of \( f_1 - f_2 \). In particular, if two continuous functions on \( M \) are equal at every element of a dense set, then they are equal at every point in \( M \).

If \( E \subseteq M \) is dense and \( f \) is a uniformly continuous complex-valued function on \( E \), then there is an extension of \( f \) to a uniformly continuous complex-valued function on \( M \). The main point is that for each \( x \in M \) there is a sequence \( \{x_j\}_{j=1}^\infty \) of elements of \( E \) which converges to \( x \) in \( M \), and which is a Cauchy sequence as a sequence in \( E \). Uniform continuity of \( f \) on \( E \) implies that \( \{f(x_j)\}_{j=1}^\infty \) is a Cauchy sequence and hence converges as a sequence of complex numbers. If \( \{y_j\}_{j=1}^\infty \) is another sequence of elements of \( E \) which converges to \( x \), then uniform continuity of \( f \) on \( E \) also implies that

\[
\lim_{j \to \infty} f(x_j) = \lim_{j \to \infty} f(y_j).
\]

For every \( x \in M \), the value of the extension of \( f \) at \( x \) is defined to be any such limit, and it is easy to check that uniform continuity carries over to the extension.
D Lipschitz functions

Let \((M,d(x,y))\) be a metric space. A complex-valued function \(f\) on \(M\) is said to be \textit{Lipschitz} if there is a \(C \geq 0\) such that

\[
|f(x) - f(y)| \leq Cd(x,y)
\]

for every \(x, y \in M\). If \(f\) is real-valued, then this condition is equivalent to

\[
f(x) \leq f(y) + Cd(x,y)
\]

for every \(x, y \in M\). In particular, this holds for \(f_p(x) = d(x,p)\) for every \(p \in M\) with \(C = 1\), by the triangle inequality.

More generally, if \(a\) is a positive real number, then a complex-valued function \(f\) on \(M\) is said to be \textit{Lipschitz of order} \(a\) if there is a \(C \geq 0\) such that

\[
|f(x) - f(y)| \leq Cd(x,y)^a.
\]

A Lipschitz function of any order \(a > 0\) is uniformly continuous.

On any metric space a constant function is Lipschitz of order \(a\) for each \(a > 0\). On the real line or the \(n\)-dimensional torus, for instance, one can show that a Lipschitz function of order \(a > 1\) is constant.

The sum of two Lipschitz functions of order \(a\) is a Lipschitz function of order \(a\), and the product of a Lipschitz function of order \(a\) with a constant is a Lipschitz function of order \(a\). The product of two bounded Lipschitz functions of order \(a\) is a Lipschitz function of order \(a\). If \(f_1, f_2\) are two real-valued Lipschitz functions of order \(a\) on \(M\), both with constant \(C\), then max\((f_1, f_2)\) and min\((f_1, f_2)\) are Lipschitz functions on \(M\) of order \(a\) and with constant \(C\).

When \(0 < a \leq 1\),

\[
(r + t)^a \leq r^a + t^a
\]

for every \(r, t \geq 0\), because

\[
r + t \leq \max(r, t)^{1-a} (r^a + t^a) \\
\leq (r^a + t^a)^{(1-a)/a} (r^a + t^a) \\
= (r^a + t^a)^{1/a}.
\]

This implies that \(d(x,y)^a\) satisfies the triangle inequality and is therefore a metric on \(M\), and a Lipschitz function on \(M\) of order \(a\) with respect to \(d(x,y)\) is the same as a Lipschitz function on \(M\) of order 1 with respect to \(d(x,y)^a\).
Let us restrict our attention now to Lipschitz functions of order 1. Suppose that \( f \) is a bounded continuous real-valued function on \( M \). For each positive integer \( j \), put
\[
(D.6) \quad f_j(x) = \inf \{ f(y) + j \, d(x, y) : y \in M \}.
\]
If \( c \in \mathbb{R} \) and \( f(w) \geq c \) for every \( w \in M \), then \( f(y) + j \, d(x, y) \geq c \) for every \( x, y \in M \) and \( j \geq 1 \), and hence
\[
(D.7) \quad f_j(x) \geq c
\]
for every \( x \in M \) and \( j \geq 1 \). Since we can take \( y = x \) in the infimum in the definition of \( f_j(x) \), we get that
\[
(D.8) \quad f_j(x) \leq f(x)
\]
for every \( x \in M \) and \( j \geq 1 \).

For each \( j \), \( f_j(x) \) is a Lipschitz function on \( M \) with constant \( j \). One can check this using the fact that \( f(y) + j \, d(x, y) \) is Lipschitz with constant \( j \) as a function of \( x \) for every \( y \in M \) and positive integer \( j \).

Because \( f \) is bounded, only \( y \in M \) with \( d(x, y) = O(1/j) \) are important in the definition of \( f_j(x) \), and one can use this and the continuity of \( f \) to show that
\[
(D.9) \quad \lim_{j \to \infty} f_j(x) = f(x)
\]
for every \( x \in M \). If \( f \) is uniformly continuous, then one can show that the convergence is uniform. By applying this to the real and imaginary parts of bounded uniformly continuous complex-valued functions on \( M \), it follows that the bounded Lipschitz functions are dense in \( \mathcal{BUC}(M) \).

### E Ultrametric spaces

An ultrametric space is a metric space \((M, d(x, y))\) in which the distance function \( d(x, y) \) satisfies the stronger version of the triangle inequality,
\[
(E.1) \quad d(x, z) \leq \max(d(x, y), d(y, z))
\]
for every \( x, y, z \in M \). In this case, \( d(x, y)^a \) is an ultrametric on \( M \) for every \( a > 0 \), and these ultrametrics determine the same topology on \( M \).
Let \((M, d(x, y))\) be an ultrametric space, and consider the open ball \(B(x, r)\) for some \(x \in M\) and \(r > 0\). Because of the ultrametric version of the triangle inequality, if \(d(x, y) < r\) and \(d(x, z) \geq r\), then \(d(y, z) \geq r\), which implies that \(B(x, r)\) is a closed set. Similarly, for every \(y \in M\) which satisfies \(d(x, y) \leq r\), \(\overline{B}(y, r)\) is contained in \(\overline{B}(x, r)\), and therefore \(\overline{B}(x, r)\) is an open set in \(M\).

On any nonempty set the discrete metric, which assigns distance 1 to every pair of distinct points, is an ultrametric. As a more complicated class of examples, let \(A\) be a nonempty set, and let \(\Sigma(A)\) be the set of sequences \(\{x_j\}_{j=1}^\infty\) with \(x_j \in A\) for each \(j\). For \(0 < \rho \leq 1\), put \(d_\rho(x, y)\) equal to 0 when \(x = y\), and put
\[
d_\rho(x, y) = \rho^l
\]
when \(x_j = y_j\) for \(j \leq l\) and \(x_{l+1} \neq y_{l+1}\), \(x, y \in \Sigma(A)\), \(l \geq 0\). One can check that \(d_\rho(x, y)\) defines an ultrametric on \(\Sigma(A)\) which is equal to the discrete metric when \(\rho = 1\), and which satisfy
\[
d_\rho^a(x, y) = d_\rho(x, y)^a
\]
for each \(a > 0\). One can also show that \(\Sigma(A)\) is compact when \(0 < \rho < 1\) if and only if \(A\) has only finitely many elements.

Let \(B\) be the space of all binary sequences, which is the same as \(\Sigma(A)\) with \(A = \{0, 1\}\). There is a mapping from \(B\) onto the unit interval \([0, 1]\) in the real line defined by
\[
x = \{x_j\}_{j=1}^\infty \mapsto \sum_{j=1}^\infty x_j 2^{-j}.
\]
This mapping is Lipschitz with respect to the metric \(d_\rho(x, y)\) described in the previous paragraph with \(\rho = 1/2\). There is a one-to-one mapping from \(B\) onto the classical Cantor middle-thirds set in the real line defined by
\[
x = \{x_j\}_{j=1}^\infty \mapsto \sum_{j=1}^\infty 2 x_j 3^{-j}.
\]
This mapping is Lipschitz and moreover bi-Lipschitz in the sense that distances in the domain and the corresponding distances in the image are each bounded by a constant multiple of the other when we use the metric described in the previous paragraph with \(\rho = 1/3\).
References

[1] M. Adams and V. Guillemin, Measure Theory and Probability, Birkhäuser, 1996.

[2] L. Ambrosio and F. Serra Cassano, editors, Lecture Notes on Analysis in Metric Spaces, Scuola Normale Superiore, Pisa, 2000.

[3] L. Ambrosio and P. Tilli, Topics on Analysis in Metric Spaces, Oxford University Press, 2004.

[4] T. Apostol, Mathematical Analysis, 2nd edition, Addison-Wesley, 1974.

[5] R. Archibald, Mathematicians and Music, American Mathematical Monthly 31 (1924), 1–25.

[6] W. Arveson, A Short Course on Spectral Theory, Springer-Verlag, 2002.

[7] J. M. Ash, editor, Studies in Harmonic Analysis, Mathematical Association of America, 1976.

[8] J. M. Ash, Uniqueness of representation by trigonometric series, American Mathematical Monthly 96 (1989), 873–885.

[9] R. Askey, Orthogonal Polynomials and Special Functions, Society for Industrial and Applied Mathematics, 1975.

[10] R. Askey and D. Haimo, Similarities between Fourier and power series, American Mathematical Monthly 103 (1996), 297–304.

[11] P. Auscher, T. Coulhon, and A. Grigoryan, editors, Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces, American Mathematical Society, 2003.

[12] S. Axler, Linear Algebra Done Right, 2nd edition, Springer-Verlag, 1997.

[13] S. Axler, P. Bourdon, and W. Ramey, Harmonic Function Theory, 2nd edition, Springer-Verlag, 2001.

[14] S. Axler, J. McCarthy, and D. Sarason, editors, Holomorphic Spaces, Cambridge University Press, 1998.
[15] R. Bartle, editor, *Studies in Functional Analysis*, Mathematical Association of America, 1980.

[16] R. Bartle, *The Elements of Integration and Lebesgue Measure*, Wiley, 1995.

[17] R. Bartle, *A Modern Theory of Integration*, American Mathematical Society, 2001.

[18] R. Bartle and D. Sherbert, *Introduction to Real Analysis*, 2nd edition, Wiley, 1992.

[19] R. Beals, *Topics in Operator Theory*, University of Chicago Press, 1971.

[20] R. Beals, *Advanced Mathematical Analysis*, Springer-Verlag, 1973.

[21] R. Beals, *Analysis: An Introduction*, Cambridge University Press, 2004.

[22] R. Beals and P. Greiner, *Calculus on Heisenberg Manifolds*, Princeton University Press, 1988.

[23] R. Beals, P. Deift, and C. Tomei, *Direct and Inverse Scattering on the line*, American Mathematical Society, 1988.

[24] S. Bell, *The Cauchy Transform, Potential Theory, and Conformal Mapping*, CRC Press, 1992.

[25] J. Benedetto, *Harmonic Analysis on TotallyDisconnected Sets*, Lecture Notes in Mathematics 202, Springer-Verlag, 1971.

[26] J. Benedetto, *Harmonic Analysis and Applications*, CRC Press, 1997.

[27] S. Berberian, *Measure and Integration*, Chelsea, 1970.

[28] S. Berberian, *Lectures in Functional Analysis and Operator Theory*, Springer-Verlag, 1974.

[29] S. Berberian, *A First Course in Real Analysis*, Springer-Verlag, 1994.

[30] S. Berberian, *Introduction to Hilbert Space*, American Mathematical Society, 1999.

[31] S. Berberian, *Fundamentals of Real Analysis*, Springer-Verlag, 1999.
[32] A. Besicovitch, *Almost Periodic Functions*, Dover, 1955.

[33] R. Boas, *A Primer of Real Functions*, 4th edition, revised and with a preface by H. Boas, Mathematical Association of America, 1996.

[34] S. Bochner, *Lectures on Fourier Integrals*, Princeton University Press, 1959.

[35] S. Bochner and K. Chandrasekharan, *Fourier Transforms*, Princeton University Press, 1949.

[36] S. Bochner and W. Martin, *Several Complex Variables*, Princeton University Press, 1948.

[37] S. Bochner, *Harmonic Analysis and the Theory of Probability*, University of California Press, 1955.

[38] S. Bochner, *Fourier series came first*, American Mathematical Monthly 86 (1979), 197–199.

[39] H. Bohr, *Almost Periodic Functions*, Chelsea, 1947.

[40] H. Bohr, *On almost periodic functions and the theory of groups*, American Mathematical Monthly 56 (1949), 595–609.

[41] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, Pitman, 1982.

[42] R. Buck, editor, *Studies in Modern Analysis*, Mathematical Association of America, 1962.

[43] R. Buck, *Advanced Calculus*, 3rd edition, with the collaboration of E. Buck, McGraw-Hill, 1978.

[44] A. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Mathematica 88 (1952), 85–139.

[45] A. Calderón and A. Zygmund, *On higher gradients of harmonic functions*, Studia Mathematica 24 (1964), 211–226.

[46] L. Capogna, C. Kenig, and L. Lanzani, *Harmonic Measure*, American Mathematical Society, 2005.

77
[47] H. Cartan, *Elementary Theory of Analytic Functions of One or Several Variables*, Dover, 1973.

[48] J. Cassels, *Local Fields*, Cambridge University Press, 1986.

[49] R. Coifman and Y. Meyer, *Au-Delà des Opérateurs Pseudo-Differentiels*, Astérisque 57, 1978.

[50] R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Mathematics 242, Springer-Verlag, 1971.

[51] R. Coifman and G. Weiss, *Transference Methods in Analysis*, American Mathematical Society, 1976.

[52] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bulletin of the American Mathematical Society 83 (1977), 569–645.

[53] R. Cooke, *The Cantor–Lebesgue theorem*, American Mathematical Monthly 86 (1979), 558–565.

[54] R. Cooke, *Almost periodic functions*, American Mathematical Monthly 88 (1981), 515–526.

[55] W. Coppel, *J. B. Fourier — on the occasion of his two hundreth birthday*, American Mathematical Monthly 76 (1969), 468–483.

[56] C. Corduneanu, *Almost Periodic Functions*, 2nd edition, American Mathematical Society, 1989.

[57] D. Curtiss, *Analytic Functions of a Complex Variable*, Mathematical Association of America, 1926.

[58] J. D’Angelo, *Inequalities from Complex Analysis*, Mathematical Association of America, 2002.

[59] P. Davis, *The Schwarz Function and its Applications*, Mathematical Association of America, 1974.

[60] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*, American Mathematical Society, 1999.
[61] R. Douglas, *Banach Algebra Techniques in Operator Theory*, 2nd edition, Springer-Verlag, 1998.

[62] R. Douglas and V. Paulsen, *Hilbert Modules over Function Algebras*, Wiley, 1989.

[63] P. Doyle and J. Snell, *Random Walks and Electrical Networks*, Mathematical Association of America, 1984.

[64] J. Duoandikoetxea, *Fourier Analysis*, translated and revised by D. Cruz-Uribe, SFO, American Mathematical Society, 2001.

[65] P. Duren, *Theory of $H^p$ Spaces*, Academic Press, 1970.

[66] P. Duren, *Univalent Functions*, Springer-Verlag, 1983.

[67] P. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press, 2004.

[68] P. Duren and A. Schuster, *Bergman Spaces*, American Mathematical Society, 2004.

[69] R. Edwards, *Fourier Series: A Modern Introduction*, volumes 1 and 2, 2nd editions, Springer-Verlag, 1979 and 1982.

[70] R. Edwards and G. Gaudry, *Littlewood–Paley and Multiplier Theory*, Springer-Verlag, 1977.

[71] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley, 1970.

[72] C. Fefferman and E. Stein, *$H^p$ spaces of several variables*, Acta Mathematica 129 (1972), 137–193.

[73] S. Fisher, *Complex Variables*, Dover, 1999.

[74] S. Fisher, *Function Theory on Planar Domains*, Wiley, 1983.

[75] G. Folland, *Lectures on Partial Differential Equations*, Springer-Verlag, 1983.

[76] G. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press, 1989.
[77] G. Folland, *Fourier Analysis and its Applications*, Wadsworth & Brooks / Cole, 1992.

[78] G. Folland, *Introduction to Partial Differential Equations*, 2nd edition, Princeton University Press, 1995.

[79] G. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.

[80] G. Folland, *Real Analysis*, 2nd edition, Wiley, 1999.

[81] G. Folland and E. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton University Press, 1982.

[82] T. Gamelin, *Uniform Algebras*, Prentice-Hall, 1969.

[83] T. Gamelin, *Uniform Algebras and Jensen Measures*, Cambridge University Press, 1978.

[84] T. Gamelin, *Complex Analysis*, Springer-Verlag, 2001.

[85] T. Gamelin and R. Greene, *Introduction to Topology*, 2nd edition, Dover, 1999.

[86] J. García-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, 1985.

[87] J. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Mathematics 297, Springer-Verlag, 1972.

[88] J. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[89] J. Garnett and D. Marshall, *Harmonic Measure*, Cambridge University Press, 2005.

[90] C. Goffman and D. Waterman, *Some aspects of Fourier series*, American Mathematical Monthly 77 (1970), 119–133.

[91] R. Goldberg, *Methods of Real Analysis*, 2nd edition, Wiley, 1976.

[92] E. González-Velasco, *Connections in mathematical analysis: The case of Fourier series*, American Mathematical Monthly 99 (1992), 427–441.
[93] F. Gouvea, *p-Adic Numbers: An Introduction*, 2nd edition, Springer-Verlag, 1997.

[94] R. Greene and S. Krantz, *Function Theory of One Complex Variable*, 2nd edition, American Mathematical Society, 2002.

[95] M. de Guzmán, *Differentiation of Integrals in \( \mathbb{R}^n \)*, with appendices by A. Córdoba, R. Fefferman, and R. Moriyón, Lecture Notes in Mathematics 481, Springer-Verlag, 1975.

[96] M. de Guzmán, *Real Variable Methods in Fourier Analysis*, North-Holland, 1981.

[97] J. Hale, editor, *Studies in Ordinary Differential Equations*, Mathematical Association of America, 1977.

[98] P. Halmos, *Measure Theory*, van Nostrand, 1950.

[99] P. Halmos, *Lectures on Ergodic Theory*, Chelsea, 1960.

[100] P. Halmos, *Finite-Dimensional Vector Spaces*, Springer-Verlag, 1974.

[101] P. Halmos, *Lectures on Boolean Algebras*, Springer-Verlag, 1974.

[102] P. Halmos, *Naive Set Theory*, Springer-Verlag, 1974.

[103] P. Halmos, *A Hilbert Space Problem Book*, 2nd edition, Springer-Verlag, 1982.

[104] P. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, American Mathematical Society, 1998.

[105] G. H. Hardy, *A Course of Pure Mathematics*, Cambridge University Press, 1958.

[106] G. H. Hardy, *Divergent Series*, Oxford University Press, 1949.

[107] G. H. Hardy and W. Rogosinski, *Fourier Series*, 2nd edition, Cambridge University Press, 1950.

[108] G. H. Hardy, J. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, 1952.
[109] V. P. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis*, Springer-Verlag, 1994.

[110] W. Hayman, *Subharmonic Functions*, volume 2, Academic Press, 1989.

[111] W. Hayman and P. Kennedy, *Subharmonic Functions*, volume 1, Academic Press, 1976.

[112] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Springer-Verlag, 2000.

[113] J. Heinonen, *Calculus on Carnot groups*, in *Fall School in Analysis (Jyväskylä, 1994)*, 1–31, Reports of the Department of Mathematics and Statistics 68, University of Jyväskylä, 1995.

[114] J. Heinonen, *Lectures on Analysis on Metric Spaces* Springer-Verlag, 2001.

[115] J. Heinonen, *Geometric embeddings of metric spaces*, Reports of the Department of Mathematics and Statistics 90, University of Jyväskylä, 2003.

[116] H. Helson, *Harmonic Analysis*, Wadsworth, 1991.

[117] E. Hernández and G. Weiss, *A First Course on Wavelets*, with a forward by Y. Meyer, CRC Press, 1996.

[118] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, volumes I, II, Springer-Verlag, 1970, 1979.

[119] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, 1975.

[120] I. Hirschman, Jr., editor, *Studies in Real and Complex Analysis*, Mathematical Association of America, 1965.

[121] K. Hoffman, *Banach Spaces of Bounded Analytic Functions*, Dover, 1988.

[122] L. Hörmander, *Notions of Convexity*, Birkhäuser, 1994.

[123] J. Horvath, *An introduction to distributions*, American Mathematical Monthly 77 (1970), 227–240.
[124] D. Jackson, *The convergence of Fourier series*, American Mathematical Monthly 41 (1934), 67–84.

[125] D. Jackson, *Fourier Series and Orthogonal Polynomials*, Mathematical Association of America, 1941.

[126] F. John, *Partial Differential Equations*, 4th edition, Springer-Verlag, 1991.

[127] F. Jones, *Lebesgue Integration on Euclidean Space*, Jones and Bartlett, 1993.

[128] J.-L. Journé, *Calderón–Zygmund Operators, Pseudodifferential Operators, and the Cauchy Integral of Calderón*, Lecture Notes in Mathematics 994, Springer-Verlag, 1983.

[129] J.-P. Kahane, *Séries de Fourier Absolument Convergentes*, Springer-Verlag, 1970.

[130] Y. Katznelson, *An Introduction to Harmonic Analysis*, 3rd edition, Cambridge University Press, 2004.

[131] C. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, American Mathematical Society, 1994.

[132] J. Kigami, *Analysis on Fractals*, Cambridge University Press, 2001.

[133] P. Koosis, *The Logarithmic Integral*, volumes 1 and 2, Cambridge University Press, 1988 and 1992.

[134] P. Koosis, *Introduction to H_p Spaces*, 2nd edition, with two appendices by V. P. Havin, Cambridge University Press, 1998.

[135] T. Körner, *A Companion to Analysis*, American Mathematical Society, 2004.

[136] T. Körner, *Fourier Analysis*, 2nd edition, Cambridge University Press, 1989.

[137] S. Krantz, *What is several complex variables?*, American Mathematical Monthly 94 (1987), 236–256.
[138] S. Krantz, *Geometric Analysis and Function Spaces*, American Mathematical Society, 1993.

[139] S. Krantz, *A Panorama of Harmonic Analysis*, Mathematical Association of America, 1999.

[140] S. Krantz, *Function Theory of Several Complex Variables*, American Mathematical Society, 2001.

[141] S. Krantz, *Partial Differential Equations and Complex Analysis*, lecture notes prepared by E. Gavosto and M. Peloso, CRC Press, 1992.

[142] S. Krantz, *Complex Analysis: The Geometric Viewpoint*, 2nd edition, Mathematical Association of America, 2004.

[143] S. Krantz, *A Handbook of Real Variables*, Birkhäuser, 2004.

[144] S. Krantz, *Real Analysis and Foundations*, 2nd edition, Chapman & Hall / CRC Press, 2005.

[145] S. Krantz and H. Parks, *The Geometry of Domains in Space*, Birkhäuser, 1999.

[146] S. Krantz and H. Parks, *A Primer of Real Analytic Functions*, 2nd edition, Birkhäuser, 2002.

[147] S. Krantz and H. Parks, *The Implicit Function Theorem: History, Theory, and Applications*, Birkhäuser, 2002.

[148] R. Langer, *Fourier’s Series: The Genesis and Evolution of a Theory*, American Mathematical Monthly 54 (1947), # 7, part 2.

[149] P. Lax, *Linear Algebra*, Wiley, 1997.

[150] P. Lax, *Functional Analysis*, Wiley, 2002.

[151] D. Li and H. Queffélec, *Introduction à l’Étude des Espaces de Banach: Analyse et Probabilités*, Société Mathématique de France, 2004.

[152] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Lecture Notes in Mathematics 338, Springer-Verlag, 1973.
[153] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces*, Springer-Verlag, 1977.

[154] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II: Function Spaces*, 1979.

[155] W. Littman, editor, *Studies in Partial Differential Equations*, Mathematical Association of America, 1982.

[156] E. Lorch, *Spectral Theory*, Oxford University Press, 1962.

[157] D. Luecking and L. Rubel, *Complex Analysis*, Springer-Verlag, 1984.

[158] R. McLeod, *The Generalized Riemann Integral*, Mathematical Association of America, 1980.

[159] M. Marcus and G. Pisier, *Random Fourier Series with Applications to Harmonic Analysis*, Princeton University Press, 1981.

[160] F. Morgan, *Real Analysis and Applications*, American Mathematical Society, 2005.

[161] A. Nagel and E. Stein, *Lectures on Pseudodifferential Operators: Regularity Theorems and Applications to Nonelliptic Problems*, Princeton University Press, 1979.

[162] U. Neri, *Singular Integrals*, Lecture Notes in Mathematics 200, Springer-Verlag, 1971.

[163] N. Nikolski, *Treatise on the Shift Operator*, with an appendix by S. Khrushchev and V. Peller, Springer-Verlag, 1986.

[164] N. Nikolski, *Operators, Functions, and Systems: An Easy Reading*, volumes 1 and 2, American Mathematical Society, 2002.

[165] R. Paley and N. Weiner, *Fourier Transforms in the Complex Domain*, American Mathematical Society, 1934.

[166] J. Partington, *An Introduction to Hankel Operators*, Cambridge University Press, 1988.

[167] A. Pelczynski, *Banach Spaces of Analytic Functions and Absolutely Summing Operators*, American Mathematical Society, 1977.
[168] V. Peller, *Hankel Operators and their Applications*, Springer-Verlag, 2003.

[169] M. Picardello and W. Woess, editors, *Random Walks and Discrete Potential Theory*, Cambridge University Press, 1999.

[170] G. Pisier, *Factorization of Linear Operators and Geometry of Banach Spaces*, American Mathematical Society, 1986.

[171] G. Polya and G. Szegö, *Problems and Theorems in Analysis*, volumes I and II, Springer-Verlag, 1998.

[172] S. Power, *Hankel Operators on Hilbert Space*, Pitman, 1982.

[173] J. Rauch, *Partial Differential Equations*, Graduate Texts in Mathematics, 1991.

[174] M. Reed, *Fundamental Ideas of Analysis*, Wiley, 1998.

[175] C. Rickart, *General Theory of Banach Algebras*, van Nostrand, 1960.

[176] C. Rickart, *Natural Function Algebras*, Springer-Verlag, 1979.

[177] F. Riesz and B. Szökefalvi-Nagy, *Functional Analysis*, Dover, 1990.

[178] M. Rosenblatt, *Studies in Probability Theory*, Mathematical Association of America, 1978.

[179] M. Rosenlicht, *Introduction to Analysis*, Dover, 1986.

[180] H. Royden, *Real Analysis*, 3rd edition, Macmillan, 1988.

[181] W. Rudin, *Function Theory in Polydisks*, Benjamin, 1969.

[182] W. Rudin, *Lectures on the Edge-of-the-Wedge Theorem*, American Mathematical Society, 1971.

[183] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976.

[184] W. Rudin, *Function Theory in the Unit Ball of C^n*, Springer-Verlag, 1980.
[185] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1987.

[186] W. Rudin, *Fourier Analysis on Groups*, Wiley, 1990.

[187] W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill, 1991.

[188] W. Rudin, *Sums of squares of polynomials*, American Mathematical Monthly 107 (2000), 813–821.

[189] C. Sadosky, *Interpolation of Operators and Singular Integrals: An Introduction to Harmonic Analysis*, Dekker, 1979.

[190] D. Sarason, *Function Theory on the Unit Circle*, Department of Mathematics, Virginia Polytechnic and State University, 1978.

[191] C. Sogge, *Fourier Integrals in Classical Analysis*, Cambridge University Press, 1993.

[192] E. Stein, *On the theory of harmonic functions of several variables II: Behavior near the boundary*, Acta Mathematica 106 (1961), 137–174.

[193] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[194] E. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Princeton University Press, 1970.

[195] E. Stein, *Boundary Behavior of Holomorphic Functions of Several Variables*, Princeton University Press, 1972.

[196] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, with the assistance of T. Murphy, Princeton University Press, 1993.

[197] E. Stein and R. Shakarchi, *Fourier Analysis*, Princeton University Press, 2003.

[198] E. Stein and R. Shakarchi, *Complex Analysis*, Princeton University Press, 2003.

[199] E. Stein and R. Shakarchi, *Real Analysis*, Princeton University Press, 2005.
[200] E. Stein and R. Shakarchi, *Functional Analysis*, forthcoming.

[201] E. Stein and G. Weiss, *On the theory of harmonic functions of several variables*, Acta Mathematica **103** (1960), 25–62.

[202] E. Stein and G. Weiss, *Generalization of the Cauchy–Riemann equations and representations of the rotation group*, American Journal of Mathematics **90** (1968), 163–196.

[203] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[204] R. Strichartz, *The Way of Analysis*, Jones and Bartlett, 1995.

[205] R. Strichartz, *A Guide to Distribution Theory and Fourier Transforms*, World Scientific, 2003.

[206] K. Stromberg, *Introduction to Classical Real Analysis*, Wadsworth, 1981.

[207] D. Stroock, *Probability Theory: An Analytic View*, Cambridge University Press, 1993.

[208] D. Stroock, *A Concise Introduction to the Theory of Integration*, 3rd edition, Birkhäuser, 1999.

[209] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, 1939.

[210] B. Szökefalvi-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, 1970.

[211] M. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, 1975.

[212] E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 3rd edition, Chelsea, 1986.

[213] C. Tomei, *Fluxos de Matrizes*, 15° Colóquio Basileiro de Matemática, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1985.

[214] A. Torchinsky, *Real Variables*, Addison-Wesley, 1988.
[215] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Dover, 2004.

[216] N. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, 1992.

[217] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society, 1998.

[218] G. Weiss, *Analisys Armonico en Varias Variables: Teoria de los Espacios H^p*, Cursos y Seminarios de Matemática 9, Universidad de Buenos Aires, 1960.

[219] G. Weiss, *Complex methods in harmonic analysis*, American Mathematical Monthly 77 (1970), 465–474.

[220] G. Weiss, *Espacios Generados por Bloques*, Publicaciones de la Sección de Matemáticas, Universidad de Extremadura 8, 1985.

[221] N. Weaver, *Lipschitz Algebras*, World Scientific, 1999.

[222] R. Wheeden and A. Zygmund, *Measure and Integral: An Introduction to Real Analysis*, Dekker, 1977.

[223] N. Wiener, *The Fourier Integral and Certain of its Applications*, with a forward by J.-P. Kahane, Cambridge University Press, 1988.

[224] W. Woess, *Random Walks on Infinite Graphs and Groups*, Cambridge University Press, 2000.

[225] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge University Press, 1991.

[226] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, Cambridge University Press, 1997.

[227] A. Zemanian, *Distribution Theory and Transform Analysis*, Dover, 1987.

[228] K. Zhu, *Operator Theory in Function Spaces*, Dekker, 1990.

[229] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, 2005.
[230] A. Zygmund, *Trigonometric Series*, volumes I and II, 3rd edition, with a foreword by R. Fefferman, Cambridge University Press, 2002.