Intertwining Operators Associated with Dihedral Groups

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Abstract

The Dunkl operators associated with a dihedral group are a pair of differential-difference operators that generate a commutative algebra acting on differentiable functions in \( \mathbb{R}^2 \). The intertwining operator intertwines between this algebra and the algebra of differential operators. The main result of this paper is an integral representation of the intertwining operator on a class of functions. As an application, closed formulas for the Poisson kernels of \( h \)-harmonics and sieved Gegenbauer polynomials are deduced when one of the variables is at vertices of a regular polygon, and similar formulas are also derived for several other related families of orthogonal polynomials.

Keywords

Intertwining operator · Dunkl operators · Dihedral group · Orthogonal polynomials · Generating function

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1 Introduction

Let \( G \) be a reflection group with a reduced root system \( R \). Let \( v \mapsto \kappa_v \) be a nonnegative multiplicity function defined on \( R \) with the property that it is a constant on each conjugate class of \( G \). Then the Dunkl operators [7] are defined by

\[
D_i f(x) = \partial_i f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, \sigma_v \rangle} v_i, \quad i = 1, 2, \ldots, d, \quad (1.1)
\]

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where $x \sigma_v := x - 2 \langle x, v \rangle v / \| v \|^2$, $R_+$ is a set of positive roots, and $\langle x, v \rangle$ is the dot product in $\mathbb{R}^d$. These first-order differential-difference operators commute in the sense that $D_i D_j = D_j D_i$ for $1 \leq i, j \leq d$. A linear operator, denoted by $V_\kappa$, that satisfies the relations

$$D_i V_\kappa = V_\kappa \partial_i, \quad 1 \leq i \leq d,$$

is called an intertwining operator, which is uniquely determined over all polynomials if it also satisfies $V_\kappa 1 = 1$ and $V_\kappa \mathcal{P}_n^d \subset \mathcal{P}_n^d$, where $\mathcal{P}_n^d$ denotes the space of homogeneous polynomials of degree $n$ in $d$ variables.

The remarkable commuting property of the Dunkl operators allows a far reaching generalization of classical analysis from $L^2$ with respect to the Lebesgue measure to weighted $L^2$ space on either $\mathbb{R}^d$ or the unit sphere $S^{d-1}$, where the weight function, multiplied by $e^{-\|x\|^2/2}$ on $\mathbb{R}^d$, is defined by

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v}. \quad (1.3)$$

The intertwining operator plays an essential role in the generalization. For example, the weighted Fourier transform has $V_\kappa [e^{i \langle \cdot, x \rangle}](y)$ in places of $e^{i \langle x, y \rangle}$ and the zonal harmonics in the weighted setting is given by $V_\kappa [C_\lambda^\kappa (\langle x, \cdot \rangle)](y)$, where $C_\lambda^\kappa$ denotes the Gegenbauer polynomial and $\lambda_\kappa = \sum_{v \in R_+} \kappa_v + d - 2 \kappa_v / 2$.

In recent years, the Fourier analysis associated with reflection groups has attracted considerable attention; see, for example, [3,12] and references therein. A large portion of the classical Fourier analysis has been extended to the weighted setting. However, much of finer analysis that relies essentially on reflection symmetry requires detailed knowledge of the intertwining operator. This is the reason that finer analysis has been carried out up to now mainly in the case of $G = \mathbb{Z}_2^d$, for which the intertwining operator is given explicitly as an integral operator.

There have been several attempts at finding an integral representation of $V_\kappa f$ for other reflection groups. Partial results have been obtained for the symmetric group $S^3$ [9] and the dihedral group $I_4$ [10,17], but the results in these works are not as satisfactory since the weight function in the integral operator may not be nonnegative and, as a consequence, the positivity of the integral is not evident. In the $I_4$ case, the results are established for polynomials, not verified directly via (1.2). Without integral representation, the action of the intertwining operator on polynomials has been studied in [11,16] for Dihedral groups.

In the present paper we consider the intertwining operator associated with dihedral groups. Let $I_k$ denote the dihedral group defined as the symmetric group of the $k$-th regular polygon. We choose the positive root system $R_+ = \{v_j : 0 \leq j \leq k - 1\}$ by

$$v_j = \left( \sin \left( \frac{j\pi}{k} \right), -\cos \left( \frac{j\pi}{k} \right) \right). \quad (1.4)$$
The reflection $x \sigma_j$ of $x = (x_1, x_2)$ in $v_j$ is given by
\[
x \sigma_j = \left( \cos \left( \frac{2j \pi}{k} \right) x_1 + \sin \left( \frac{2j \pi}{k} \right) x_2, \sin \left( \frac{2j \pi}{k} \right) x_1 - \cos \left( \frac{2j \pi}{k} \right) x_2 \right).
\] (1.5)

We consider the case when the multiplicity function $\kappa$ is a constant, which we denote by $\lambda$. In this setting, the Dunkl operators are given by
\[
D_1 f(x) = \frac{\partial f}{\partial x_1} + \lambda \sum_{j=0}^{k-1} \frac{f(x) - f(x \sigma_j)}{\langle x, v_j \rangle} \sin \left( \frac{j \pi}{k} \right),
\]
\[
D_2 f(x) = \frac{\partial f}{\partial x_2} - \lambda \sum_{j=0}^{k-1} \frac{f(x) - f(x \sigma_j)}{\langle x, v_j \rangle} \cos \left( \frac{j \pi}{k} \right).
\] (1.6)

Throughout this paper, we let $T^{k-1}$ denote the simplex defined by
\[
T^{k-1} := \{ u \in \mathbb{R}^{k-1} : u_1 \geq 0, \ldots, u_{k-1} \geq 0, \ u_1 + \cdots + u_{k-1} \leq 1 \}.
\]

Our main result gives an integral representation for the intertwining operator on a class of functions.

**Theorem 1.1** Let $f$ be a differentiable function on $\mathbb{R}$ so that the integral in (1.7) is finite. For $0 \leq p \leq 2k - 1$, define
\[
F_p(x_1, x_2) := f \left( \cos \left( \frac{p \pi}{k} \right) x_1 + \sin \left( \frac{p \pi}{k} \right) x_2 \right).
\]

Then, for $k = 2, 3, 4, \ldots$ and $\lambda > 0$, the intertwining operator $V_\lambda$ for the dihedral group $I_k$ with one parameter $\lambda$ satisfies
\[
V_\lambda F_p(x_1, x_2) = a_\lambda^{(k)} \int_{T^{k-1}} f \left( \cos \left( \frac{p \pi}{k} \right) (c(u)x_1 + s(u)x_2) + \sin \left( \frac{p \pi}{k} \right) (c(u)x_2 - s(u)x_1) \right)
\]
\[
\times u_0^{k-1} \prod_{i=1}^{k-1} u_i^{k-1} du,
\] (1.7)

where $u_0 := 1 - u_1 - \cdots - u_{k-1}$,
\[
c(u) := \sum_{j=0}^{k-1} \cos \left( \frac{2j \pi}{k} \right) u_j \quad \text{and} \quad s(u) := \sum_{j=0}^{k-1} \sin \left( \frac{2j \pi}{k} \right) u_j,
\]
and $a_\lambda^{(k)}$ is chosen so that $V_\lambda 1 = 1$ or, given explicitly,
\[
a_\lambda^{(k)} = \frac{\lambda \Gamma(\lambda)^k}{\Gamma(k\lambda + 1)}.
\]
Although the result does not give a full integral representation of the intertwining operator for the dihedral group, the explicit formula of the integral in (1.7) is notable and suggestive for a possible final form. The integral over the simplex has been well studied, see the classical paper [5] and its application in orthogonal polynomials of several variables [12]; it also appears in the study of spherical functions associated with the root system of $A_d$ in the Dunkl setting [14]. In general, there is little methodology for identifying an integral transform that satisfies (1.2); the discovery of our (1.7) is motivated by an integral formula in [18], see Lemma 4.2 below, and is the result of trial and error, starting from $I_4$. Once the formula is identified, a proof can be given by a direct verification of (1.2).

As an application, we obtain a closed form formula for the Poisson kernel of the $h$-harmonics associated with the dihedral group $I_k$ when one of the variables is at vertices of a regular $k$-gon. With one parameter, the $h$-harmonics associated with $I_k$ can be written in terms of the sieved Gegenbauer polynomials studied in [1], which are orthogonal polynomials on $[-1, 1]$ with respect to the weight function

$$w_{\lambda+\frac{1}{2}}^{(k)}(t) = |U_{k-1}(t)|^{2\lambda} (1-t^2)^{\lambda-\frac{1}{2}}, \quad k = 1, 2, 3, \ldots, \quad (1.8)$$

where $U_k$ denotes the $k$th Chebyshev polynomial of the second kind. Our result leads to a closed form formula for the Poisson kernels of the sieved Gegenbauer polynomials, which can also be used to derive properties of these polynomials. Furthermore, $h$-harmonics can also be related, when $k$ is even, to another family of polynomials, which leads us to study orthogonal polynomials with respect to

$$w_{\lambda-\frac{1}{2}}^{(k), \pm 1}(t) = (1 \pm t)|U_{k-1}(t)|^{2\lambda} (1-t^2)^{\lambda-\frac{1}{2}}, \quad k = 1, 2, 3, \ldots, \quad (1.9)$$

for which we can also derive the closed formulas for their Poisson kernels.

The paper is organized as follows. In the next section, we recall essential results on the $h$-harmonics associated with the dihedral group. The main result, Theorem 1.1, is proved in Sect. 3. As applications of the main result, we derive closed form formulas for $h$-harmonics and sieved Gegenbauer polynomials in Sect. 4 and explicit bases of orthogonal polynomials in Sect. 5, and study orthogonal polynomials for the weight functions (1.9) in Sect. 6. Finally, in Sect. 7, we discuss product formulas of orthogonal polynomials and their relation with the intertwining operator.

2 Dihedral Symmetry

The dihedral group $I_k$ is the symmetric group of a regular polygon of $k$ sides. We consider the case of even $k$ and odd $k$ separately.
2.1 Dihedral Group \( I_{2k} \)

For the dihedral group \( I_{2k} \), we choose the positive root system as

\[
v_j = \left( \sin \left( \frac{j\pi}{2k} \right), -\cos \left( \frac{j\pi}{2k} \right) \right), \quad j = 0, 1, \ldots, 2k - 1.
\]

For \( x = r \cos \theta, \sin \theta \), we have \( \langle x, v_j \rangle = r \sin \left( \frac{j\pi}{2k} - \theta \right) \). The reflection \( \sigma_j \) is given by

\[
x \sigma_j = x - 2 \langle x, v_j \rangle v_j = (\cos(\frac{j\pi}{k} - \theta), \sin(\frac{j\pi}{k} - \theta)).
\]

The group has two conjugacy classes, described by \( v_{2j} \) and \( v_{2j+1} \), respectively. The Dunkl operators associated with the dihedral group with parameters \( \lambda \geq 0 \) and \( \mu \geq 0 \) are given by

\[
D_1 f(x) = \frac{\partial f}{\partial x_1} + \lambda \sum_{j=0}^{k-1} \frac{f(x) - f(x\sigma_2j)}{\langle x, v_{2j} \rangle} \sin \left( \frac{j\pi}{k} \right)
\]

\[
+ \mu \sum_{j=0}^{k-1} \frac{f(x) - f(x\sigma_{2j+1})}{\langle x, v_{2j+1} \rangle} \sin \left( \frac{(2j+1)\pi}{2k} \right),
\]

\[
D_2 f(x) = \frac{\partial f}{\partial x_2} - \lambda \sum_{j=0}^{k-1} \frac{f(x) - f(x\sigma_2j)}{\langle x, v_{2j} \rangle} \cos \left( \frac{j\pi}{k} \right)
\]

\[
- \mu \sum_{j=0}^{k-1} \frac{f(x) - f(x\sigma_{2j+1})}{\langle x, v_{2j+1} \rangle} \cos \left( \frac{(2j+1)\pi}{2k} \right).
\]

In this case, the weight function \( h_\kappa \) in (1.3) is given by

\[
h^{(2k)}_{\lambda, \mu}(x_1, x_2) = r^{k(\lambda+\mu)} |\sin(k\theta)|^{\lambda} |\cos(k\theta)|^{\mu}
\]

in polar coordinates \((x_1, x_2) = (r \cos \theta, r \sin \theta)\). In particular, for \( I_2 = \mathbb{Z}_2^2 \), the weight function is \( h^{(2)}_{\lambda, \mu}(x_1, x_2) = |x_1|^{\mu} |x_2|^{\lambda} \) and, with \( k = 1 \),

\[
D_1 f(x) = \frac{\partial f}{\partial x_1} + \mu \frac{f(x) - f(-x_1, x_2)}{x_1},
\]

\[
D_2 f(x) = \frac{\partial f}{\partial x_2} + \lambda \frac{f(x) - f(x_1, -x_2)}{x_2}.
\]

The intertwining operator \( V_{\lambda, \mu} \) for \( I_2 \) is given by an integral transform [12, p. 232]

\[
V_{\lambda, \mu} f(x_1, x_2) = c_\lambda c_\mu \int_{-1}^{1} \int_{-1}^{1} f(sx_1, tx_2)(1+s)(1-t^2)^{\mu-1}(1+t)(1-t^2)^{\lambda-1} ds dt.
\]

(2.1)
No other satisfactory integral representation of the intertwining operator is known for other dihedral groups.

Let \( P_n^2 \) be the space of homogeneous polynomials of two variables of degree \( n \). A polynomial \( Y \in P_n^2 \) is called an \( h \)-harmonic if \( \Delta_h Y = 0 \) where \( \Delta_h = D_1^2 + D_2^2 \). Let \( \mathcal{H}_n(h_{\lambda,\mu}^{(2k)}) \) be the space of \( h \)-harmonics of degree \( n \) associated with the dihedral group \( I_k \). It is known that \( h \)-harmonics of different degrees are orthogonal; that is,

\[
\int_{S^1} Y_n(\xi) Y_m(\xi) \left[ h_{\lambda,\mu}^{(2k)}(\xi) \right]^2 \, d\sigma(\xi) = 0, \quad Y_n \in \mathcal{H}_n(h_{\lambda,\mu}^{(2k)}), \quad Y_m \in \mathcal{H}_m(h_{\lambda,\mu}^{(2k)}).
\]

(2.2)

As in the case of ordinary spherical harmonics, we know that \( \dim \mathcal{H}_0(h_{\lambda,\mu}^{(2k)}) = 1 \) and \( \dim \mathcal{H}_n(h_{\lambda,\mu}^{(2k)}) = 2 \) for \( n \geq 1 \). An orthogonal basis of \( \mathcal{H}_n(h_{\lambda,\mu}^{(2k)}) \) can be given explicitly in terms of the Jacobi polynomials \( P_n^{(\alpha,\beta)} \). Let us define first the generalized Gegenbauer polynomials \( C_n^{(\lambda,\mu)} \) by

\[
C_{2n}^{(\lambda,\mu)}(t) = \frac{(\lambda + \mu)_n}{(\mu + \frac{1}{2})_n} P_n^{(\lambda - \frac{1}{2},\mu + \frac{1}{2})}(2t^2 - 1),
\]

\[
C_{2n+1}^{(\lambda,\mu)}(t) = \frac{(\lambda + \mu)_{n+1}}{(\mu + \frac{1}{2})_{n+1}} t P_n^{(\lambda - \frac{1}{2},\mu + \frac{1}{2})}(2t^2 - 1),
\]

(2.3)

where \((a)_n := a(a + 1) \cdots (a + n - 1)\) stands for the Pochhammer symbol, which are orthogonal with respect to the weight function

\[
w(t) = |t|^{2\mu} (1 - t^2)^{\lambda - \frac{1}{2}}, \quad t \in [-1, 1].
\]

In the polar coordinates \( x_1 = r \cos \theta \) and \( x_2 = r \sin \theta \), an \( h \)-harmonic \( Y \) can be written as \( Y(x_1, x_2) = r^n \tilde{Y}(\theta) \), where \( \tilde{Y}(\theta) = Y(\cos \theta, \sin \theta) \). We state an explicit orthogonal basis for \( \mathcal{H}_n(h_{\lambda,\mu}^{(2k)}) \) in \( \tilde{Y}_{n,1} \) and \( \tilde{Y}_{n,2} \) [8].

**Proposition 2.1** For \( n = mk + j \) with \( 0 \leq j \leq k - 1 \), define

\[
\tilde{Y}_{mk+j,1}(\theta) = \frac{m + 2\lambda + \delta_m}{2\lambda + 2\mu} \cos j\theta \ C_m^{(\lambda,\mu)}(\cos k\theta) - \sin j\theta \sin k\theta \ C_{m-1}^{(\lambda+1,\mu)}(\cos k\theta),
\]

\[
\tilde{Y}_{mk+j,2}(\theta) = \frac{m + 2\lambda + \delta_m}{2\lambda + 2\mu} \sin j\theta \ C_m^{(\lambda,\mu)}(\cos k\theta) - \cos j\theta \sin k\theta \ C_{m-1}^{(\lambda+1,\mu)}(\cos k\theta),
\]

(2.4)

where \( \delta_m = 2\mu \) if \( m \) is even and \( \delta_m = 0 \) if \( m \) is odd. Then \( \{Y_{n,1}, Y_{n,2}\} \) is an orthogonal basis of \( \mathcal{H}_n(h_{\lambda,\mu}^{(2k)}) \).

Let \( H_{n,i} = H_{n,i}^{(\lambda,\mu)} \) denote the norm square of \( Y_{n,i} \), defined by

\[
H_{n,i} := c_{\lambda,\mu} \int_0^{2\pi} \left| \tilde{Y}_{n,i}(\theta) \right|^2 \left[ h_{\lambda,\mu}^{(2k)}(\cos \theta, \sin \theta) \right]^2 \, d\theta.
\]

(2.5)
where \( c_{\lambda,\mu} \) is chosen so that
\[
c_{\lambda,\mu} \int_0^{2\pi} \left| h^{(2k)}_{\lambda,\mu} \left( \cos \theta, \sin \theta \right) \right|^2 d\theta = 1.
\] Its value is independent of \( k \) and, as can be easily verified,
\[
c_{\lambda,\mu} = \frac{\Gamma(\lambda + \mu + 1)}{2\Gamma(\lambda + 1)\Gamma(\mu + 1)}.
\] (2.6)

Let \( \xi(\theta) := (\cos \theta, \sin \theta) \). Denote by \( P_n(h^{(2k)}_{\lambda,\mu}; \cdot, \cdot) \) the reproducing kernel of \( \mathcal{H}_n(h^{(2k)}_{\lambda,\mu}) \), which is uniquely determined by the property
\[
c_{\lambda,\mu} \int_0^{2\pi} P_n \left( h^{(2k)}_{\lambda,\mu}; \xi(\theta), \xi(\phi) \right) \tilde{Y}(\theta) \left[ h^{(2k)}_{\lambda,\mu}(\xi(\theta)) \right]^2 d\theta = \tilde{Y}(\phi), \quad \forall Y \in \mathcal{H}_n(h^{(2k)}_{\lambda,\mu})
\]
and, as a function of either one of its variables, \( P_n(h^{(2k)}_{\lambda,\mu}; \cdot, \cdot) \in \mathcal{H}_n(h^{(2k)}_{\lambda,\mu}) \). In terms of an orthogonal basis, the kernel can be written as
\[
P_n \left( h^{(2k)}_{\lambda,\mu}; x, y \right) = \frac{Y_{n,1}(x)Y_{n,1}(y)}{H_{n,1}} + \frac{Y_{n,2}(x)Y_{n,2}(y)}{H_{n,2}},
\]
where \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \). The kernel satisfies a closed form in terms of the intertwining operator, denoted by \( V_{\lambda,\mu} \), and the Gegenbauer polynomial [15],
\[
P_n \left( h^{(2k)}_{\lambda,\mu}; x, y \right) = \frac{n + k(\lambda + \mu)}{k(\lambda + \mu)} V_{\lambda,\mu} \left[ C_n^{(\lambda+\mu)k}(\cdot, y) \right](x)
\] (2.7)
for \( \|x\| = \|y\| = 1 \). The Poisson kernel of the \( h \)-harmonics, denoted by \( P(h^{(2k)}_{\lambda,\mu}; \cdot, \cdot) \), is defined by the property that
\[
c_{\lambda,\mu} \int_0^{2\pi} P \left( h^{(2k)}_{\lambda,\mu}; \xi(\theta), \xi(\phi) \right) \tilde{Y}(\theta) d\theta = \tilde{Y}(\phi), \quad \forall Y \in \mathcal{H}_n(h^{(2k)}_{\lambda,\mu}), \quad \forall n \in \mathbb{N}_0.
\]
It satisfies a closed formula for \( \|x\| = \|y\| = 1 \) and \( 0 \leq r < 1 \),
\[
P \left( h^{(2k)}_{\lambda,\mu}, x, ry \right) = \sum_{n=0}^{\infty} P_n \left( h^{(2k)}_{\lambda,\mu}; x, y \right) r^n = V_{\lambda,\mu} \left[ \frac{1 - r^2}{(1 - 2r\langle \cdot, y \rangle + r^2(\lambda+\mu)k+1)} \right](x).
\] (2.8)

2.2 The Dihedral Group \( I_{2k+1} \)

For the dihedral group \( I_{2k+1} \), we choose the positive root system as
\[
v_j = (\sin(\frac{j\pi}{2k+1}), -\cos(\frac{j\pi}{2k+1})), \quad j = 0, 1, \ldots, 2k.
\]
There is only one conjugacy class and one parameter $\lambda \geq 0$. The Dunkl operators in this case can be derived from those of $I_{2k}$ by setting $\mu = 0$ and replacing $k$ by $2k + 1$. All other results discussed in the previous subsection also hold under the same conversion.

### 2.3 The Dihedral Group $I_k$ with One Parameter

We are interested in the case of one parameter, that is, $\mu = \lambda$ when $k$ is even. With one parameter, our setting for $I_{2k}$ and $I_{2k+1}$ can be unified. For the dihedral group $I_k$ with $k = 2, 3, 4, \ldots$, the Dunkl operators with one parameter $\lambda$ are those given in (1.6) with the root system given by (1.4) and the reflections given by (1.5). The corresponding weight function is given by

$$h^{(k)}_{\lambda}(x_1, x_2) := r^\lambda |\sin(k \theta)|^{\lambda}, \quad \lambda \geq 0,$$

for $k = 2, 3, 4, \ldots$. Accordingly, we denote the intertwining operator by $V_\lambda$. In this setting, the Poisson kernel (2.8) becomes

$$P \left( h^{(k)}_{\lambda} ; x, ry \right) = \sum_{n=0}^{\infty} P_n \left( h^{(k)}_{\lambda} ; x, y \right) r^n = V_\lambda \left[ \frac{1 - r^2}{(1 - 2r \langle \cdot, y \rangle + r^2)k^{\lambda+1}} \right] (x)$$

for $\|x\| = \|y\| = 1$. For $I_{2k}$, the parameter $2k \lambda$ agrees with $k(\lambda + \mu)$ in (2.8) when $\mu = \lambda$, whereas for $I_{2k+1}$ the parameter $\mu = 0$ in (2.8).

### 3 The Intertwining Operator

We first give a proof of Theorem 1.1, which we reformulate below, using the polar coordinates $x_1 = r \cos \theta, x_2 = r \sin \theta$.

**Theorem 3.1** Let $f$ be a differentiable function on $\mathbb{R}$. For $0 \leq p \leq 2k - 1$, define

$$F_p(x_1, x_2) := f \left( \cos \left( \frac{p \pi}{k} \right) x_1 + \sin \left( \frac{p \pi}{k} \right) x_2 \right).$$

Then, for $k = 2, 3, 4, \ldots$ and $\lambda > 0$, the intertwining operator $V_\lambda$ for the dihedral group $I_k$ with one parameter $\lambda$ satisfies

$$V_\lambda F_p(x_1, x_2) = a^{(k)}_\lambda \int_{T^{k-1}} f \left( r \sum_{j=0}^{k-1} \cos \left( \theta - \frac{p \pi}{k} - \frac{2j \pi}{k} \right) u_j \right) u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} du,$$

where $u_0$ and $u_i$ are the coordinates on $T^{k-1}$.
where \( u_0 = 1 - u_1 - \ldots - u_{k-1} \) and \( a^{(k)}_\lambda \) is chosen so that \( V_\lambda 1 = 1 \) or, given explicitly,

\[
a^{(k)}_\lambda = \frac{\lambda \Gamma(\lambda) k}{\Gamma(k\lambda + 1)}.
\]

**Proof** We only need to consider \( 0 \leq p \leq k - 1 \), since we can replace \( f(t) \) by \( f(-t) \) if \( k \leq p \leq 2k - 1 \) by \( \sin(\pi + \theta) = -\sin \theta \) and \( \cos(\pi + \theta) = -\cos \theta \). Let \( p \) be fixed, \( 0 \leq p \leq k - 1 \). For simplicity, we shall write

\[
\Psi(\theta, u) := \sum_{j=0}^{k-1} \cos \left( \theta - \frac{p\pi}{k} - \frac{2j\pi}{k} \right) u_j.
\]

Let \( \hat{V}_\lambda \) denote the right-hand side of (3.1). Our goal is to verify that

\[
D_1 \hat{V}_\lambda F_p = \hat{V}_\lambda \partial_1 F_p \quad \text{and} \quad D_2 \hat{V}_\lambda F_p = \hat{V}_\lambda \partial_2 F_p,
\]

(3.2)

where \( D_1 \) and \( D_2 \) are the Dunkl operators in (1.6). With a slight abuse of notation, we write \( V_\lambda \) instead of \( \hat{V}_\lambda \) for the right-hand side of (3.1) below. First we consider the difference part for each reflection.

Fix \( \ell, 0 \leq \ell \leq k - 1 \). In polar coordinates, (1.5) becomes

\[
x \sigma_j = r \left( \cos(\frac{2j\pi}{k} - \theta), \sin(\frac{2j\pi}{k} - \theta) \right), \quad 0 \leq j \leq k-1,
\]

so that the reflection in \( v_\ell \) is simply a shift in \( \theta \) variable. In particular,

\[
V_\lambda F_p(x \sigma_\ell) = a^{(k)}_\lambda \int_{T^{k-1}} f \left( r \sum_{j=0}^{k-1} \cos \left( \frac{2\ell\pi}{k} - \theta - \frac{p\pi}{k} - \frac{2j\pi}{k} \right) u_j \right) u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} \, du
\]

\[
= a^{(k)}_\lambda \int_{T^{k-1}} f \left( r \sum_{j=0}^{k-1} \cos \left( \theta - \frac{2(\ell - j)\pi}{k} + \frac{p\pi}{k} \right) u_j \right) u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} \, du.
\]

We need to consider two cases.

**Case 1.** \( \ell \geq p \). Here we write the sum inside \( f \) as

\[
\sum_{j=0}^{k-1} \cos \left( \theta - \frac{2(\ell - j)\pi}{k} + \frac{p\pi}{k} \right) u_j = \sum_{j=0}^{k-1} \cos \left( \theta - \frac{p\pi}{k} - \frac{2(\ell - p - j)\pi}{k} \right) u_j
\]

\[
= \sum_{j=0}^{\ell-p} \cos \left( \theta - \frac{p\pi}{k} - \frac{2j\pi}{k} \right) u_{\ell-p-j} + \sum_{j=\ell-p+1}^{k-1} \cos \left( \theta - \frac{p\pi}{k} - \frac{2j\pi}{k} \right) u_{k+\ell-p-j}.
\]
Making a change of variables \( u_j = u_{\ell - p - j} \) for \( 0 \leq j \leq \ell - p \) and \( u_j = u_{k + \ell - p - j} \) for \( \ell - p + 1 \leq j \leq k - 1 \), we see that

\[
V_{\lambda} F_p(x) = a_{\lambda}^{(k)} \int_{T^{k-1}} f(r \Psi(\theta, u)) u_{\ell - p} \prod_{i=0}^{k-1} u_i^{\lambda - 1} du.
\]

Consequently, we conclude that

\[
V_{\lambda} F_p(x) - V_{\lambda} F_p(x_{\sigma \ell}) = a_{\lambda}^{(k)} \int_{T^{k-1}} f(r \Psi(\theta, u))(u_0 - u_{\ell - p}) \prod_{i=0}^{k-1} u_i^{\lambda - 1} du.
\]

Evidently, \( V_{\lambda} F_p(x) - V_{\lambda} F_p(x_{\sigma p}) = 0 \). For \( \ell > p \), we use the relation

\[
\frac{d}{du_{\ell - p}} \left[ u_0 u_{\ell - p} \prod_{i=0}^{k-1} u_i^{\lambda - 1} \right] = \lambda(u_0 - u_{\ell - p}) \prod_{i=0}^{k-1} u_i^{\lambda - 1},
\]

which follows from \( u_0 = 1 - u_1 - \cdots - u_{k-1} \), so that integrating by parts gives

\[
\lambda \left[ V_{\lambda} F_p(x) - V_{\lambda} F_p(x_{\sigma \ell}) \right] = -a_{\lambda}^{(k)} \int_{T^{k-1}} f'(r \Psi(\theta, u))
\times r \left( \cos \left( \theta - \frac{(2\ell - p)\pi}{k} \right) - \cos \left( \theta - \frac{p\pi}{k} \right) \right) u_0 u_{\ell - p} \prod_{i=0}^{k-1} u_i^{\lambda - 1} du.
\]

By (1.4), we have \( \langle x, v_\ell \rangle = r \sin \left( \frac{\ell \pi}{k} - \theta \right) \) in polar coordinates. Hence, using

\[
\cos \left( \theta - \frac{(2\ell - p)\pi}{k} \right) - \cos \left( \theta - \frac{p\pi}{k} \right) = 2 \sin \left( \theta - \frac{\ell \pi}{k} \right) \sin \left( \frac{(\ell - p)p\pi}{k} \right),
\]

we conclude that

\[
\frac{\lambda}{\langle x, v_\ell \rangle} V_{\lambda} F_p(x) - V_{\lambda} F_p(x_{\sigma \ell})
= 2a_{\lambda}^{(k)} \int_{T^{k-1}} f'(r \Psi(\theta, u)) \sin \left( \frac{(\ell - p)p\pi}{k} \right) u_0 u_{\ell - p} \prod_{i=0}^{k-1} u_i^{\lambda - 1} du.
\]
Case 2. \( \ell < p \). Here we write the sum inside \( f \) as

\[
\sum_{j=0}^{k-1} \cos \left( \theta - \frac{2(\ell - j)\pi}{k} + \frac{p\pi}{k} \right) u_j = \sum_{j=0}^{k-1} \cos \left( \theta - \frac{p\pi}{k} + \frac{2(\ell - p - j + k)\pi}{k} \right) u_j
\]

\[
= \sum_{j=0}^{k-p+\ell} \cos \left( \theta - \frac{p\pi}{k} - \frac{2j\pi}{k} \right) u_{k-p+\ell-j} + \sum_{j=k-p+\ell+1}^{k-1} \cos \left( \theta - \frac{p\pi}{k} - \frac{2j\pi}{k} \right) u_{2k-p+\ell-j}.
\]

Making a change of variables \( u_j = u_{k-p+\ell-j} \) for \( 0 \leq j \leq k - p + \ell \) and \( u_j = 2k - p + \ell - j \) for \( k - p + \ell + 1 \leq j \leq k - 1 \), we see that

\[
V_\lambda F_p(x\sigma_\ell) = a_\lambda^{(k)} \int_{T^{k-1}} f(r\Psi(\theta, u)) u_{k-p+\ell} \prod_{i=0}^{k-1} u_i^{\lambda-1} du.
\]

We can now follow the procedure in Case 1 to conclude that

\[
\lambda \frac{V_\lambda F_p(x) - V_\lambda F_p(x\sigma_\ell)}{\langle x, v_\ell \rangle} = 2a_\lambda^{(k)} \int_{T^{k-1}} f'(r\Psi(\theta, u)) \sin \left( \frac{(\ell - p)\pi}{k} \right) u_0 u_{k-p+\ell} \prod_{i=0}^{k-1} u_i^{\lambda-1} du.
\]

We are now ready to verify (3.2). We consider the intertwining identity for \( D_1 \) first. Putting the two cases together, we obtain

\[
\lambda \sum_{\ell=0}^{k-1} \frac{V_\lambda F_p(x) - V_\lambda F_p(x\sigma_\ell)}{\langle x, v_\ell \rangle} \sin \left( \frac{\ell\pi}{k} \right) = a_\lambda^{(k)} \int_{T^{k-1}} f'(r\Psi(\theta, u)) S(u) u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} du,
\]

where

\[
S(u) = 2 \sum_{\ell=0}^{p-1} \sin \left( \frac{\ell\pi}{k} \right) \sin \left( \frac{(\ell - p)\pi}{k} \right) u_{k-p+\ell} + 2 \sum_{\ell=p+1}^{k-1} \sin \left( \frac{\ell\pi}{k} \right) \sin \left( \frac{(\ell - p)\pi}{k} \right) u_{\ell-p}.
\]
For $0 \leq \ell \leq p - 1$, we use the identity
\[
2 \sin \left( \frac{\ell \pi}{k} \right) \sin \left( \frac{(\ell - p)\pi}{k} \right) = 2 \sin \left( \frac{(k - p + \ell)\pi}{k} \right) \sin \left( \frac{(k - p + \ell)\pi}{k} \right) \\
= \left( 1 - \cos \left( \frac{2(k - p + \ell)\pi}{k} \right) \right) \cos \left( \frac{p\pi}{k} \right) + \sin \left( \frac{2(k - p + \ell)\pi}{k} \right) \sin \left( \frac{p\pi}{k} \right),
\]
where the index $k - p + \ell$ matches that of $u_{k-p+j}$ in the first sum of $S(u)$, whereas for $p + 1 \leq \ell \leq k - 1$, we use the identity
\[
2 \sin \left( \frac{\ell \pi}{k} \right) \sin \left( \frac{(\ell - p)\pi}{k} \right) = 2 \sin \left( \frac{(\ell - p)\pi}{k} \right) \sin \left( \frac{(\ell - p)\pi}{k} \right) \\
= \left( 1 - \cos \left( \frac{2(\ell - p)\pi}{k} \right) \right) \cos \left( \frac{p\pi}{k} \right) + \sin \left( \frac{2(\ell - p)\pi}{k} \right) \sin \left( \frac{p\pi}{k} \right),
\]
where the index $\ell - p$ agrees with that of $u_{k-p}$ in the second sum of $S(u)$. Putting together, we conclude that
\[
S(u) = \cos \left( \frac{p\pi}{k} \right) \sum_{\ell=0}^{k-1} \left( 1 - \cos \left( \frac{2\ell\pi}{k} \right) \right) u_\ell + \sin \left( \frac{p\pi}{k} \right) \sum_{\ell=0}^{k-1} \sin \left( \frac{2\ell\pi}{k} \right) u_\ell \\
= \cos \left( \frac{p\pi}{k} \right) - \sum_{\ell=0}^{k-1} \cos \left( \frac{p\pi}{k} + \frac{2\ell\pi}{k} \right) u_\ell,
\]
since $\sum_{\ell=0}^{k-1} u_\ell = 1$. Furthermore, taking the derivative in (3.1), we obtain
\[
\frac{\partial}{\partial x_1} V_\lambda F_p(x_1, x_2) = a_2^{(k)} \int_{T^{k-1}} f'(r \Psi(\theta, u)) \sum_{j=0}^{k-1} \cos \left( \frac{p\pi}{k} + \frac{2j\pi}{k} \right) u_j u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} du.
\]
Hence, by the definition of $D_1$ in (1.6), we conclude that
\[
D_1 V_\lambda F_p(x) = \cos \left( \frac{p\pi}{k} \right) \int_{T^{k-1}} f'(r \Psi(\theta, u)) u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} du \\
= \int_{T^{k-1}} \frac{\partial}{\partial x_1} F_p(r \Psi(\theta, u)) u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} du = V_\lambda \partial_1 F_p(x),
\]
since $\partial_1 F_p(x_1, x_2) = \cos \left( \frac{p\pi}{k} \right) f'(\cos \left( \frac{p\pi}{k} \right) x_1 + \sin \left( \frac{p\pi}{k} \right) x_2).$ This verfies the first identity in (3.2).
The second identity in (3.2) is verified similarly. From the two cases that we consider for individual difference operator, we obtain

$$\lambda \sum_{\ell=0}^{k-1} \frac{V_\lambda F_p(x) - V_\lambda F_p(x \sigma_\ell)}{(x, v_\ell)} \cos \left( \frac{\ell \pi}{k} \right) = a^{(k)}(x) \int_{x^{(k-1)}} f' (r \Psi(\theta, u)) C(u) u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} du,$$

where

$$C(u) = 2 \sum_{\ell=0}^{p-1} \cos \left( \frac{\ell \pi}{k} \right) \sin \left( \frac{(\ell - p) \pi}{k} \right) u_{k-p+\ell} + 2 \sum_{\ell=p+1}^{k-1} \cos \left( \frac{\ell \pi}{k} \right) \sin \left( \frac{(\ell - p) \pi}{k} \right) u_{\ell-p}.$$

For $0 \leq \ell \leq p - 1$, we use the identity

$$2 \cos \left( \frac{\ell \pi}{k} \right) \sin \left( \frac{(\ell - p) \pi}{k} \right) = \cos \left( \frac{p \pi}{k} \right) \sin \left( \frac{2(k - p + \ell) \pi}{k} \right) - \sin \left( \frac{p \pi}{k} \right) \left( 1 - \cos \left( \frac{2(k - p + \ell) \pi}{k} \right) \right),$$

whereas for $p + 1 \leq \ell \leq k - 1$, we use the identity

$$2 \cos \left( \frac{\ell \pi}{k} \right) \sin \left( \frac{(\ell - p) \pi}{k} \right) = \cos \left( \frac{p \pi}{k} \right) \sin \left( \frac{2(\ell - p) \pi}{k} \right) - \sin \left( \frac{p \pi}{k} \right) \left( 1 - \cos \left( \frac{2(\ell - p) \pi}{k} \right) \right),$$

together they lead to

$$C(u) = - \sin \left( \frac{p \pi}{k} \right) + \sum_{\ell=0}^{k-1} \sin \left( \frac{p \pi}{k} + \frac{2\ell \pi}{k} \right).$$

Furthermore, taking the derivative in (3.1), we obtain

$$\frac{\partial}{\partial x_2} V_\lambda F_p(x_1, x_2) = a^{(k)}(x) \int_{x^{(k-1)}} f' (r \Psi(\theta, u)) \sum_{j=0}^{k-1} \sin \left( \frac{p \pi}{k} + \frac{2j \pi}{k} \right) u_j u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} du.$$
Hence, by the definition of $D_2$ in (1.6), we conclude that

$$D_2 V_\lambda P_p(x) = \cos \left( \frac{p\pi}{k} \right) \int_{t_{k-1}} f'(r\Psi(\theta, u))u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} \, du$$

$$= \int_{t_{k-1}} \frac{d}{dx_2} P_p(r\Psi(\theta, u))u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1} \, du = V_\lambda \partial_2 P_p(x),$$

since $\partial_2 P_p(x_1, x_2) = \sin \left( \frac{p\pi}{k} \right) f'(\cos \left( \frac{p\pi}{k} \right)x_1 + \sin \left( \frac{p\pi}{k} \right)x_2).$ This verifies the second identity in (3.2). The proof is completed. \(\square\)

Let us mention some consequences of our main theorem. Let $f$ be a function of one variable. We define $f(|\cdot|)$ by $f(|\cdot|)(x_1, x_2) = f(x_i)$ for $i = 1, 2$.

**Corollary 3.2** Let $f$ be a differentiable function on $\mathbb{R}$. Then, for $k = 2, 3, 4, \ldots$,

$$Vf(|\cdot|)(x_1, x_2) = a^{(k)}_\lambda \int_{t_{k-1}} f(c(u)x_1 + s(u)x_2)u_0^{\lambda} \prod_{i=1}^{k-1} u_i^{\lambda-1} \, du$$

$$= a^{(k)}_\lambda \int_{t_{k-1}} f \left( r \sum_{j=0}^{k-1} \cos \left( \theta - \frac{2j\pi}{k} \right) u_j \right) u_0^{\lambda} \prod_{i=1}^{k-1} u_i^{\lambda-1} \, du. \quad (3.3)$$

Furthermore, if $k$ is even, then

$$Vf(|\cdot|)(x_1, x_2) = a^{(k)}_\lambda \int_{t_{k-1}} f(c(u)x_1 - s(u)x_2)u_0^{\lambda} \prod_{i=1}^{k-1} u_i^{\lambda-1} \, du$$

$$= a^{(k)}_\lambda \int_{t_{k-1}} f \left( r \sum_{j=0}^{k-1} \sin \left( \theta - \frac{2j\pi}{k} \right) u_j \right) u_0^{\lambda} \prod_{i=1}^{k-1} u_i^{\lambda-1} \, du. \quad (3.4)$$

**Proof** The first identity is (1.7) with $p = 0$. The second identity is (1.7) with $p = m/2$, which is an integer only if $m$ is even. \(\square\)

The result and the proof may seem to suggest that we only need to prove the theorem for $V_\lambda f(|\cdot|)$. However, a moment of reflection shows that this is not the case since $V_\lambda [f(a|\cdot| + b|\cdot|)](x) \neq V_\lambda [f(|\cdot|)](ax_1 + bx_2)$ in general.

The statement of the theorem and its proof may suggest a possible formula for $V_\lambda f(|\cdot|, |\cdot|)$. However, the obvious choice does not work out, and a full formula is still elusive at the time of this writing.

### 4 Poisson Kernel of $h$-Harmonics and Orthogonal Polynomials

The closed formula of $V_\lambda$ in the previous section has implications on $h$-harmonics and their associated orthogonal polynomials. We start with a short subsection on the connection of $h$-harmonics and orthogonal polynomials.
4.1 $h$-Harmonics and Orthogonal Polynomials

Let $w$ be a nonnegative weight function on $[-1, 1]$. Let $p_n(w)$ denote the orthogonal polynomial of degree $n$ and

$$c_\lambda \int_{-1}^{1} p_n(w; x) p_m(w; x) w(x) dx = h_n(w) \delta_{n,m},$$

where $c_\lambda$ is the normalization constant of $w_\lambda$. The Poisson kernel associated with $w$ is defined by

$$\phi_r(w; x, y) = \sum_{n=0}^{\infty} \frac{p_n(w; x) p_n(w; y)}{h_n(w)} r^n, \quad 0 \leq r < 1.$$

For the weight function $w_\lambda(x) = (1-x^2)^{\lambda - \frac{1}{2}}, \lambda > -\frac{1}{2}$, the corresponding orthogonal polynomials are the Gegenbauer polynomials $C_n^\lambda$, which are known to satisfy two generating functions,

$$\frac{1}{(1-2rx + r^2)^\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) r^n, \quad 0 \leq r < 1, \quad (4.1)$$

and, for $\lambda > 0$,

$$\frac{1-r^2}{(1-2rx + r^2)^{\lambda+1}} = \sum_{n=0}^{\infty} \frac{n+\lambda}{\lambda} C_n^\lambda(x) r^n, \quad 0 \leq r < 1. \quad (4.2)$$

The right-hand side of the second one, (4.2), is in fact the Poisson kernel $\phi_r(w_\lambda; x, 1)$.

For the $h$-spherical harmonics associated with $h_n^{(k)}(x)$ defined in (2.9), an orthogonal basis $\{Y_n, \ldots, Y_n^{(k)}\}$ of $H_n(h_n^{(k)})$ can be given in terms of the orthogonal polynomials on $[-1, 1]$ with respect to the weight functions $w_{\lambda-\frac{1}{2}}^{(k)}$ and $w_{\lambda+\frac{1}{2}}^{(k)}$, respectively, where

$$w_{\lambda\pm\frac{1}{2}}^{(k)}(t) = |U_{k-1}(t)|^{2h} (1-t^2)^{\lambda \mp \frac{1}{2}}, \quad -1 < t < 1. \quad (4.3)$$

The normalization constant $b_{\lambda\pm\frac{1}{2}} = 1/ \int_{-1}^{1} w_{\lambda\pm\frac{1}{2}}^{(k)}(t) dt$ is easily seen to be

$$b_{\lambda-\frac{1}{2}} = 2c_\lambda = \frac{\Gamma(\lambda + \frac{1}{2})^2}{\Gamma(2\lambda + 1)} \quad \text{and} \quad b_{\lambda+\frac{1}{2}} = 2b_{\lambda-\frac{1}{2}} = \frac{2\Gamma(\lambda + \frac{1}{2})^2}{\Gamma(2\lambda + 1)}.$$
Proposition 4.1  For $k = 2, 3, 4, \ldots$ and $\lambda \geq 0$,

\[
Y_{n,1}^{\lambda,k}(x_1, x_2) = r^n p_n \left( w_{\lambda - \frac{1}{2}}^{(k)}; x_1 \right), \quad n = 0, 1, 2, \ldots, \tag{4.4}
\]

\[
Y_{n,2}^{\lambda,k}(x_1, x_2) = r^n x_2 p_{n-1} \left( w_{\lambda + \frac{1}{2}}^{(k)}; x_1 \right), \quad n = 1, 2, \ldots.
\]

Furthermore, the norm $H_{n,i}^{\lambda,k}$ of $Y_{n,i}^{\lambda,k}$ satisfies

\[
H_{n,1}^{\lambda,k} = h_n \left( w_{\lambda - \frac{1}{2}}^{(k)} \right) \quad \text{and} \quad H_{n,2}^{\lambda,k} = \frac{1}{2} h_n \left( w_{\lambda + \frac{1}{2}}^{(k)} \right). \tag{4.5}
\]

Proof  The relation (4.4) is known in a much more general setting, see [12, Section 4.2]. We compute the norm of $Y_{n,2}^{\lambda,k}$,

\[
H_{n,2}^{\lambda,k} = c_\lambda \int_{-\pi}^{\pi} \left| Y_{n,2}^{\lambda,k}(\cos \theta, \sin \theta) \right|^2 | \sin(k\theta)|^{2\lambda} d\theta
\]

\[
= 2c_\lambda \int_{0}^{\pi} \left| p_{n-1} \left( w_{\lambda + \frac{1}{2}}^{(k)}; \cos \theta \right) \right|^2 (\sin \theta)^2 | \sin(k\theta)|^{2\lambda} d\theta
\]

\[
= b_{\lambda - \frac{1}{2}} \int_{-1}^{1} \left| p_{n-1} \left( w_{\lambda + \frac{1}{2}}^{(k)}; t \right) \right|^2 w_{\lambda + \frac{1}{2}}^{(k)}(t) d\theta = \frac{1}{2} h_n \left( w_{\lambda + \frac{1}{2}} \right),
\]

where we have used $b_{\lambda + \frac{1}{2}} = 2b_{\lambda - \frac{1}{2}}$ in the last step. The case of $H_{n,1}^{\lambda,k}$ can be verified similarly and is easier.

The polynomials $p_n(\cdot) := p_n \left( w_{\lambda + \frac{1}{2}}^{(k)}; \cdot \right)$ are called sieved Gegenbauer polynomials since their three-term relation possesses a structure that can be viewed as if a sieve is operated on the recurrence relations of the Gegenbauer polynomials. These polynomials are studied in [1], where they are defined by their recurrence relations.

An explicit formula of $p_n(\cdot)$ is given in Proposition 2.1. Indeed, it is easy to see that $\hat{Y}_{mk+j}$ in (2.4) is a polynomial of degree $n$ in $t = \cos \theta$ and $\hat{Y}_{mk+j}$ in (2.4) is equal to $\sin \theta$ multiple of a polynomial of degree $n - 1$ in $t = \cos \theta$. It turns out that a simpler basis can be given in this case; see Proposition 5.1 in the following section.

4.2 Poisson Kernels for $h$-Harmonics and Sieved Gegenbauer Polynomials

The formula of the intertwining operator in Theorem 1.1 can be used to derive a closed form formula for the Poisson kernels (2.8) when one argument is at the vertexes of a regular polygon. We need two lemmas.
Lemma 4.2 Let $k = 2, 3, \ldots$ and $\lambda = (\lambda_0, \ldots, \lambda_{k-1})$ with $\lambda_i > 0$, $0 \leq i \leq k - 1$. For $(x_0, x_1, \ldots, x_{k-1}) \in \mathbb{R}^k$ and $r \geq 0$ such that $r|x_i| < 1$, $0 \leq i \leq k - 1$,

$$k-1 \prod_{i=0}^{k-1} \frac{1}{(1 - 2rx_i + r^2)^{\lambda_i}} = \frac{\Gamma(|\lambda|)}{\prod_{i=0}^{k-1} \Gamma(\lambda_i)} \int_{T^{k-1}} \frac{1}{1 - 2r \sum_{i=0}^{k-1} x_i u_i + r^2 |\lambda|} \prod_{i=0}^{k-1} u_i^{\lambda_i - 1} du.$$ 

This lemma is established recently in [18] but it follows immediately from a classical work by Dixon [5].\(^\dagger\) The second lemma is elementary; a proof is outlined for completeness.

Lemma 4.3 For $k = 2, 3, 4, \ldots$,

$$1 - 2r^k \cos(k\theta) + r^{2k} = \prod_{j=0}^{k-1} \left(1 - 2r \cos \left(\theta - \frac{2j\pi}{k}\right) + r^2\right).$$

Proof Using $z^k - 1 = \prod_{i=0}^{k-1} (z - e^{\frac{2\pi i j}{k}})$, it is easy to see that

$$1 - r^k e^{i\theta} = \prod_{j=0}^{k-1} \left(1 - r e^{i\theta - \frac{2\pi ij}{k}}\right).$$

The stated formula then follows from $(1 - r e^{i\theta})(1 - r e^{-i\theta}) = 1 - 2r \cos \theta + r^2$. □

Our first result gives a closed formula for the Poisson kernel (2.10) associated with the dihedral group $I_k$ with $k = 2, 3, 4, \ldots$.

Theorem 4.4 For $k = 2, 3, \ldots$ and $p = 0, 1, \ldots, k - 1$, let $y_{p,k} = (\cos \frac{p\pi}{k}, \sin \frac{p\pi}{k})$. Then, for $\|x\| = 1$ and $0 \leq r < 1$,

$$P(h; x, r y_{p,k}) = \sum_{n=0}^{\infty} \left(\frac{Y_{n,1}(x) Y_{n,1}(y_{p,k})}{H_{n,1}} + \frac{Y_{n,2}(x) Y_{n,2}(y_{p,k})}{H_{n,2}}\right) r^n = \frac{1 - r^2}{(1 - 2r(\cos \frac{p\pi}{k}) x_1 + \sin (\frac{p\pi}{k}) x_2) + r^2(1 - 2(-1)^p r^k T_k(x_1) + r^{2k})},$$

(4.6)

where $T_k$ denotes the $k$th Chebyshev polynomial of the first kind.

\(^\dagger\) I’m grateful to David Chow for bringing Dixon’s paper to my attention after my paper was posted on the arXiv. Dixon’s paper was published in 1905.
**Proof** Applying (1.7) to the function $f(t) = (1 - 2rt + r^2)^{-(k\lambda + 1)}$, the Poisson kernel in (2.10) becomes

$$P(h_{\lambda, \mu}; x, ry_p, k) = a^{(k)}_\lambda \int_{T^{k-1}} \frac{1 - r^2}{(1 - 2r \sum_{j=0}^{k-1} \cos \left( \theta - \frac{2j\pi}{k} - \frac{\rho\pi}{k} \right) u_j + r^2)^{k\lambda + 1}} \times u_0^\lambda \prod_{i=1}^{k-1} u_i^{\lambda - 1} du.$$  

Applying the identity in Lemma 4.2, then the identity in Lemma 4.3 with $\theta$ replaced by $\theta - \frac{\rho\pi}{k}$, we see that the above integral is equal to

$$\frac{1 - r^2}{(1 - 2r \cos (\theta - \frac{\rho\pi}{k}) + r^2)^{k\lambda + 1}} \prod_{j=0}^{k-1} (1 - 2r \cos \left( \theta - \frac{2j\pi}{k} - \frac{\rho\pi}{k} \right) + r^2)^{\lambda} = \frac{1 - r^2}{(1 - 2r \cos (\theta - \frac{\rho\pi}{k}) + r^2)(1 - 2r^k \cos (k\theta - p\pi) + r^{2k})^\lambda},$$

which leads to the stated result, since $\cos (k\theta - p\pi) = (-1)^p \cos (k\theta)$, after setting $x_1 = \cos \theta$ and $x_2 = \sin \theta$.  

For one-parameter, the following formula for the Poisson kernel is given in [8, Theorem 2.1, (2.3)],

$$P(h_\lambda; x, y) = \frac{1 - |zw|^2}{|1 - zw|^2 |1 - zk w^k|^2} 2F_1 \left( \lambda, \lambda; \frac{4(z^k \lambda^k)(\lambda w^k)}{2\lambda + 1}; \frac{1 - zk w^k}{z^k} \right),$$

where $z = x_1 + iy_2$ and $w = y_1 + iy_2$. Specialize to $w = r e^{i\frac{\rho\pi}{k}}$ so that $w^k = r^k(-1)^p$, and this agrees with (4.6). The Poisson kernel for $I_k$ can be written in terms of the Poisson kernel associated with $I_2$ and the latter has an integral expression [12, Theorem 7.6.11]. This is used to derive a complicated integral formula for the kernel in [2]. It is worth mentioning that there has also been an attempt on explicit expression for the kernel $V[e^{i(\cdot, y)}]$$\chi$ in the dihedral group setting [4], but the result is in series rather than in integral. Our partial closed form of the intertwining operator gives satisfactory formulas when $y = y_{p,k}$ in both cases.

The definition of the Poisson kernel $P(h_\lambda; x, y)$ is independent of the choice of bases of $\mathcal{H}_{n}(h_\lambda)$. For the basis given in Proposition 4.1, $Y_{n,1}^{\lambda, k}$ is a function of $x_1$ only and $Y_{n,2}^{\lambda, k}$ contains a single $x_2$. As a consequence, we can separate the series for $Y_{n,1}^{\lambda, k}$ and $Y_{n,2}^{\lambda, k}$ by considering either the addition or the difference of $P(h_\lambda; (x_1, x_2), y_{p,m})$ and $P(h_\lambda; (x_1, -x_2), y_{p,m})$. Given the relation (4.4), this leads to the Poisson kernels for $w_{\lambda \pm \frac{1}{2}}^{(k)}$.  

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Theorem 4.5  For $k = 1, 2, 3 \ldots, 0 \leq p \leq k - 1$, $\|x\| = 1$ and $0 \leq r < 1$,

\[
\phi_r \left( w_{\lambda - \frac{1}{2}}^{(k)}(t), \cos \left( \frac{p\pi}{k} \right) \right) = \sum_{n=0}^{\infty} p_n \left( w_{\lambda - \frac{1}{2}}^{(k)}(t), \cos \left( \frac{p\pi}{k} \right) \right) h_n \left( w_{\lambda - \frac{1}{2}}^{(k)}(t) \right) r^n
\]

\[
= \frac{(1 - r^2)(1 - 2r \cos(p\pi/k)t + r^2)}{\left((1 - 2r \cos(p\pi/k)t + r^2)^2 - 4r^2 \sin^2(p\pi/k)(1 - t^2)\right)(1 - 2(-1)^prkT_k(t) + r^{2k})^\lambda},
\]

and

\[
\phi_r \left( w_{\lambda + \frac{1}{2}}^{(k)}(t), \cos \left( \frac{p\pi}{k} \right) \right) = \sum_{n=0}^{\infty} p_n \left( w_{\lambda + \frac{1}{2}}^{(k)}(t), \cos \left( \frac{p\pi}{k} \right) \right) h_n \left( w_{\lambda + \frac{1}{2}}^{(k)}(t) \right) r^n
\]

\[
= \frac{1 - r^2}{\left((1 - 2r \cos(p\pi/k)t + r^2)^2 - (2r \sin(p\pi/k))^2(1 - t^2)\right)(1 - 2(-1)^prkT_k(t) + r^{2k})^\lambda}.
\]

Proof  The first identity \((4.7)\) follows precisely from \((4.6)\) since

\[
\phi_r \left( w_{\lambda - \frac{1}{2}}^{(k)}(t), \cos \left( \frac{p\pi}{k} \right) \right) = \frac{1}{2} \left[ P \left( h_{\lambda,\mu}; (x_1, x_2), ry_{p,m} \right) + P \left( h_{\lambda,\mu}; (x_1, -x_2), ry_{p,m} \right) \right]
\]

when we use \((4.4)\) and \((4.5)\). The second identity \((4.8)\) is a bit more complicated, since

\[
2x_2 \sin(p\pi/k)\phi_r \left( w_{\lambda - \frac{1}{2}}^{(k)}(t), \cos \left( \frac{p\pi}{k} \right) \right) = \frac{1}{2} \left[ P \left( h_{\lambda,\mu}; (x_1, x_2), ry_{p,m} \right) - P \left( h_{\lambda,\mu}; (x_1, -x_2), ry_{p,m} \right) \right]
\]

by \((4.4)\) and \((4.5)\), where the factor 2 comes from \((4.5)\). Working out the right-hand side by \((4.6)\) and canceling $2x_2 \sin(p\pi/k)$ that appears in the numerator, we obtain \((4.8)\).

The statement of the above theorem includes the case $k = 1$, for which \((4.7)\) is the classical identity \((4.2)\) for the Gegenbauer polynomials. For $k > 1$, these results are new except in the case of $p = 0$ for $w_{\lambda - \frac{1}{2}}$. We state the case $p = 0$ as a corollary.

Corollary 4.6  For $k = 1, 2, 3 \ldots, t \in [-1, 1]$ and $0 \leq r < 1$,

\[
\sum_{n=0}^{\infty} \frac{p_n \left( w_{\lambda - \frac{1}{2}}^{(k)}(t) \right)}{h_n \left( w_{\lambda - \frac{1}{2}}^{(k)}(t) \right)} r^n = \frac{1 - r^2}{(1 - 2rt + r^2)(1 - 2rkT_k(t) + r^{2k})^\lambda}.
\]
and

\[
\sum_{n=0}^{\infty} \frac{p_n(w^{(k)}_{\lambda+\frac{1}{2}}; t) p_n(w^{(k)}_{\lambda+\frac{1}{2}}; 1)}{h_n(w^{(k)}_{\lambda+\frac{1}{2}})} r^n = \frac{1 - r^2}{(1 - 2rt + r^2)^2(1 - 2r^k T_k(t) + r^{2k})}. \tag{4.10}
\]

The identity (4.9) appeared in [1] for the basis defined recursively therein; it was treated as a generating function of \(p_n(w^{(k)}_{\lambda-\frac{1}{2}}; t)\) but was not identified as the Poisson kernel. The identity (4.10) is new; a different generating function is given for \(p_n(w^{(k)}_{\lambda+\frac{1}{2}}; t)\) in [1], see (5.6) below.

5 Sieved Gegenbauer Polynomials

Using the connection between \(h\)-harmonics and orthogonal polynomials in Proposition 4.1, an explicit basis of orthogonal polynomials with respect to \(w^\lambda_{\lambda-\frac{1}{2}}\) can be derived from Proposition 2.1. A simpler form of the basis can be derived from the relation (4.9).

**Proposition 5.1** For \(k = 2, 3, \ldots, n = mk + j\) with \(j = 0, 1, \ldots, k - 1\), a basis of orthogonal polynomial with respect to \(w^{(k)}_{\lambda-\frac{1}{2}}\) is given by

\[
p_{km+j}(w^{(k)}_{\lambda-\frac{1}{2}}; \cos \theta) = \cos(j\theta) C^{\lambda+1}_m(\cos(k\theta)) - \cos((k - j)\theta) C^{\lambda+1}_{m-1}(\cos(k\theta)).
\]  

(5.1)

Moreover, the \(L^2\) norm of these polynomials are given by

\[
h_{km}(w^{(k)}_{\lambda-\frac{1}{2}}) = \frac{m + 2\lambda}{2(m + \lambda)} p_{km}(w^{(k)}_{\lambda-\frac{1}{2}}; 1) = \frac{m + 2\lambda}{m + \lambda} \frac{(2\lambda + 1)m}{2m!} \tag{5.2}
\]

and, for \(1 \leq j \leq k - 1\),

\[
h_{km+j}(w^{(k)}_{\lambda-\frac{1}{2}}) = \frac{1}{2} p_{km+j}(w^{(k)}_{\lambda-\frac{1}{2}}; 1) = \frac{(2\lambda + 1)m}{2m!}. \tag{5.3}
\]

**Proof** Using (4.1), we can verify directly, or with the help of a computer algebra system, that
\[
2 \sum_{j=1}^{k-1} \sum_{m=0}^{\infty} p_{km+j}(u_{\lambda - \frac{1}{2}}^{(k)}; \cos \theta)r^{km+j}
\]
\[
= 2 \sum_{m=0}^{\infty} \frac{C_{m+1}^{\lambda+1}(\cos(k\theta))r^{km}}{(1 + r^k \cos(k\theta) + r^{2k})^{\lambda+1}} \left(\sum_{j=0}^{k-1} \cos(j\theta)r^j - \sum_{j=1}^{k-1} \cos((k - j)\theta)r^{k+j}\right)
\]
\[
= \frac{1}{(1 + r^k \cos(k\theta) + r^{2k})^{\lambda+1}} \left(\frac{1 - r^2}{1 - 2r \cos \theta + r^2} - (1 - r^{2k})\right).
\]

For \(k = 0\), we use the identity
\[
C_{m+1}^{\lambda+1}(t) - tC_{m}^{\lambda+1}(t) = \frac{m + 2\lambda}{2\lambda} C_{m}(t),
\]
which can be easily verified by using the \(2F1\) expansion of these polynomials, so that
\[
\sum_{m=0}^{\infty} \frac{2(m + \lambda)}{m + 2\lambda} p_{km}(u_{\lambda - \frac{1}{2}}^{(k)}; \cos \theta)r^{km} = \sum_{m=0}^{\infty} \frac{m + \lambda}{\lambda} C_{m}^{\lambda}(\cos k\theta)r^{km}
\]
\[
= \frac{1 - r^{2k}}{(1 - 2r \cos k\theta + r^{2k})^{\lambda+1}}
\]
by (4.2). Together, we conclude that
\[
\sum_{m=0}^{\infty} \frac{2(m + \lambda)}{m + 2\lambda} p_{km}(w_{\lambda - \frac{1}{2}}^{(k)}; \cos \theta)r^{km} + 2 \sum_{j=1}^{k-1} \sum_{m=0}^{\infty} p_{km+j}(w_{\lambda - \frac{1}{2}}^{(k)}; \cos \theta)r^{km+j}
\]
\[
= \frac{1 - r^{2}}{(1 - 2r \cos \theta + r^{2})(1 - 2r^k \cos(k\theta) + r^{2k})^{\lambda}}.
\]

Consequently, by (4.9), we see that
\[
\frac{p_{km}(w_{\lambda - \frac{1}{2}}^{(k)}; 1)}{h_{km}(w_{\lambda - \frac{1}{2}}^{(k)})} = \frac{2(m + \lambda)}{m + 2\lambda} \quad \text{and} \quad \frac{p_{km+j}(w_{\lambda - \frac{1}{2}}^{(k)}; 1)}{h_{km+j}(w_{\lambda - \frac{1}{2}}^{(k)})} = 2, \quad j \geq 1,
\]
from which the norm \(h_n(w_{\lambda - \frac{1}{2}}^{(k)})\) can be deduced from \(C_n^2(1) = \frac{(2\lambda)_n}{n!}\). This completes the proof.

The polynomial \(p_n(w_{\lambda - \frac{1}{2}}^{(k)}; t)\) in (5.1) differs from the orthogonal polynomial \(c_n^2(\chi; k)\) given in [1] by a multiplicative constant \(\alpha_n\) as can be seen from (4.9) and [1, (4.10)]. In particular, comparing [1, (4.13)] and our expression for \(p_{mk+j}\) leads to the following proposition, which can also be verified directly.
Proposition 5.2  For $1 \leq j \leq m$ and $m = 2, 3, \ldots$,

$$
\sum_{\ell=0}^{m} T_{\ell k+j} (\cos \theta) C_{m-\ell}^{\lambda} (\cos k \theta) = \cos (j \theta) C_{m}^{\lambda+1} (\cos (k \theta)) - \cos ((k-j) \theta) C_{m-1}^{\lambda+1} (\cos (k \theta)).
$$

(5.5)

We next consider orthogonal polynomials with respect to $w_{\lambda+\frac{1}{2}}$. These polynomials are shown in [1] to satisfy a generating function ([1, (4.3)]),

$$
\sum_{n=0}^{\infty} p_n (w_{\lambda+\frac{1}{2}}; t) r^n = \frac{1}{(1 - 2 \cos r + r^2)(1 - 2 T_k (\cos \theta) r^k + r^{2 k})}. \quad (5.6)
$$

It turns out that they can be given by analogues of those in Proposition 5.1, replacing some of the cosines by sines. We shall define $U_{-1} (t) = 0$.

Proposition 5.3  For $k = 2, 3, \ldots, n = mk + j$ with $j = 0, 1, \ldots, k - 1$, a basis of orthogonal polynomial with respect to $w_{\lambda+\frac{1}{2}}$ is given by, for $0 \leq j \leq k - 1$,

$$
p_{km+j} (w_{\lambda+\frac{1}{2}}; \cos \theta) = U_j (\cos \theta) C_{m}^{\lambda+1} (\cos (k \theta)) + U_{k-j-2} (\cos \theta) C_{m-1}^{\lambda+1} (\cos (k \theta)).
$$

(5.7)

Moreover, the $L^2$ norm of these polynomials are given by, for $0 \leq j \leq k - 2$,

$$
h_{km+j} (w_{\lambda+\frac{1}{2}}) = \frac{(2 \lambda + 1)_m}{2m!},
$$

(5.8)

and, for $j = k - 1$,

$$
h_{km+k-1} (w_{\lambda+\frac{1}{2}}) = \frac{(2 \lambda + 1)_m}{2m!} \frac{2 \lambda + m + 1}{\lambda + m + 1}.
$$

(5.9)

Proof  Just as in the previous proof, we can easily verify that

$$
\sum_{j=0}^{k-1} \left( U_{j-1} (\cos \theta) r^{j-1} + U_{k-j-1} (\cos \theta) r^{k+j-1} \right) = \frac{1 - 2 r^k \cos (k \theta) + r^{2 k}}{1 - 2 r \cos \theta + r^2}.
$$

Together with (4.1) applied to $C_{n}^{\lambda+1} (\cos (k \theta))$, we can then verify that the polynomials $p_{km+j} (w_{\lambda+\frac{1}{2}})$ defined in (5.7) satisfy the generating function (5.6). The $L^2$ norms of these polynomials are given in [1, (3.4)].

In particular, comparing [1, (4.4)] and our expression for $p_{mk+j}$ leads to the following proposition, which can also be verified directly.
Proposition 5.4 For $1 \leq j \leq m$ and $m = 2, 3, \ldots$,
\[
\sum_{\ell=0}^{m} U_{\ell k+j}(\cos \theta) C_{m-\ell}^{\lambda}(\cos k\theta) = U_j(\cos \theta) C_{m+1}^{\lambda+1}(\cos(k\theta)) - U_{k-j-2}(\cos \theta) C_{m-1}^{\lambda+1}(\cos(k\theta)).
\] (5.10)

6 Two Related Families of Orthogonal Polynomials

In the case when $k$ is an even integer, we can relate orthogonal polynomials with respect to $w_{\lambda+\frac{1}{2}}$ to another set of orthogonal polynomials associated with the weight function
\[
w^{(k),\frac{1}{2}}(t) := (1-t)w^{(k)}(\frac{1}{2} t) = (1-t)(1-t^2)^{\frac{1}{2}} |U_{k-1}(t)|^{2\lambda}.
\] (6.1)

The latter can also be related to orthogonal polynomials associate with the weight function
\[
w^{(k),-\frac{1}{2}}(t) := (1+t)w^{(k)}(\frac{1}{2} t) = (1+t)(1-t^2)^{\frac{1}{2}} |U_{k-1}(t)|^{2\lambda}.
\] (6.2)

Proposition 6.1 For $k = 1, 2, 3, \ldots, n = 0, 1, 2, \ldots$,
\[
p_{2n}(w_{\lambda-\frac{1}{2}}^{(2k)}; \cos \theta) = p_n(w_{\lambda-\frac{1}{2}}^{(k)}; \cos(2\theta)),
\]
\[
p_{2n+1}(w_{\lambda-\frac{1}{2}}^{(2k)}; \cos \theta) = \cos \theta p_n(w_{\lambda-\frac{1}{2}}^{(k),1}; \cos(2\theta)).
\] (6.3)

Furthermore, the norms of these polynomials in their respective $L^2$ space satisfy
\[
h_{2n}(w_{\lambda-\frac{1}{2}}^{(2k)}) = h_n(w_{\lambda-\frac{1}{2}}^{(k)}) \quad \text{and} \quad h_{2n+1}(w_{\lambda-\frac{1}{2}}^{(2k)}) = \frac{1}{2} h_n(w_{\lambda-\frac{1}{2}}^{(k),1}).
\] (6.4)

Proof Since the weight function $w_{\lambda-\frac{1}{2}}^{(2k)}(t)$ is even, $p_{2n}(w_{\lambda-\frac{1}{2}}^{(2k)})$ is even in $t$, so that $p_{2n}(w_{\lambda-\frac{1}{2}}^{(2k)}; t) = q_{n,1}(2t^2 - 1)$, where $q_{n,1}$ is a polynomial of degree $n$, and $p_{2n+1}(w_{\lambda-\frac{1}{2}}^{(2k)}; t)$ is odd, so that $p_{2n+1}(w_{\lambda-\frac{1}{2}}^{(2k)}; t) = tq_{n,2}(2t^2 - 1)$, where $q_{n,2}$ is also a polynomial of degree $n$. Using $\cos^2 \theta = (1 + \cos 2\theta)/2$, we obtain
\[
b_{\lambda-\frac{1}{2}} \int_0^{\pi} p_{2n+1}(w_{\lambda-\frac{1}{2}}^{(2k)}; \cos \theta) p_{2m+1}(w_{\lambda-\frac{1}{2}}^{(2k)}; \cos \theta) |\sin(2k\theta)|^{2\lambda} d\theta
\]
\[
= b_{\lambda-\frac{1}{2}} \int_0^{\pi} (\cos \theta)^2 q_{n,2}(\cos 2\theta) q_{m,2}(\cos 2\theta) |\sin(2k\theta)|^{2\lambda} d\theta
\]
\[ \lambda - \frac{1}{2} \int_0^{2\pi} \frac{1 + \cos \theta}{2} q_{n,2}(\cos \theta) q_{m,2}(\cos \theta) |\sin(k\theta)|^{2\lambda} \frac{d\theta}{2} \]
\[ = \lambda - \frac{1}{2} \int_0^{\pi} q_{n,2}(\cos \theta) q_{m,2}(\cos \theta)(1 + \cos \theta) |\sin(k\theta)|^{2\lambda} \frac{d\theta}{2}, \]
where we have changed \( \theta \rightarrow 2\pi - \theta \) in the integral over \([\pi, 2\pi]\). This establishes the orthogonality of \( q_{n,2} \) with respect to the weight function \( w^{(k),1}_{\lambda - \frac{1}{2}} \) as well as the relation on \( L^2 \) norm, since the normalization constant for \( w^{(k),1}_{\lambda - \frac{1}{2}} \) is the same as the one for \( w^{(k)}_{\lambda - \frac{1}{2}} \). The verification for \( q_{n,1} \) is similar and easier.

The relation (6.3) allows us to derive an orthogonal basis for \( w^{(k),1}_{\lambda - \frac{1}{2}} \), from which an orthogonal basis for \( w^{(k),-1}_{\lambda - \frac{1}{2}} \) also follows.

**Proposition 6.2** For \( k = 1, 2, 3, \ldots, m = 0, 1, 2, \ldots \) and \( j = 0, 1, \ldots, k - 1 \),

\[
p_{km+j} \left( w^{(k),1}_{\lambda - \frac{1}{2}}; \cos \theta \right) = \cos((j + \frac{1}{2})\theta) C_{m}^{\lambda+1}(\cos(k\theta)) - \cos((k - j + \frac{1}{2})\theta) C_{m-1}^{\lambda+1}(\cos(k\theta)) \quad (6.5)
\]

and

\[
p_{km+j} \left( w^{(k),-1}_{\lambda - \frac{1}{2}}; \cos \theta \right) = (-1)^{km+j} \times \left[ \frac{\sin((j + \frac{1}{2})\theta)}{\sin(\frac{\theta}{2})} C_{m}^{\lambda+1}(\cos(k\theta)) + \frac{\sin((k - j + \frac{1}{2})\theta)}{\sin(\frac{\theta}{2})} C_{m-1}^{\lambda+1}(\cos(k\theta)) \right]. \quad (6.6)
\]

Furthermore, \( h_n(w^{(k),1}_\lambda) = h_n(w^{(k),-1}_\lambda) \) for all \( n = 0, 1, 2, \ldots \).

**Proof** By (5.1) with \( k \) replaced by \( 2k \), we have

\[
p_{2km+2j+1} \left( w^{(2k),1}_{\lambda - \frac{1}{2}}; \cos \theta \right) = \cos((2j + 1)\theta) C_{m}^{\lambda+1}(\cos(2k\theta)) - \cos((2k - 2j - 1)\theta) C_{m-1}^{\lambda+1}(\cos(2k\theta)),
\]
from which (6.6) follows from the second identity in (6.3) by setting \( \theta \rightarrow \theta/2 \).

To establish (6.6), we observe that \( p_n \left( w^{(k),1}_{\lambda - \frac{1}{2}}; -t \right) \) is an orthogonal polynomial with respect to the weight function \( w^{(k),-1}_{\lambda - \frac{1}{2}} \), as can be seen by changing variable \( t \rightarrow -t \) in the orthogonality relation. Hence, changing variable \( \theta \rightarrow \pi - \theta \) in (6.5) gives (6.6).
We note that the right-hand sides of (6.5) and (6.6) can also be written in terms of the Jacobi polynomials by using

\[
\frac{\cos((j + \frac{1}{2})\theta)}{\cos(\frac{\theta}{2})} = \frac{2^j j!^2}{(2j)!} P_j^{(-\frac{1}{2}, \frac{1}{2})}(\cos \theta), \\
\frac{\sin((j + \frac{1}{2})\theta)}{\sin(\frac{\theta}{2})} = \frac{2^j j!^2}{(2j)!} P_j^{(\frac{1}{2}, -\frac{1}{2})}(\cos \theta).
\]

We now derive a closed formula for the Poisson kernel \( \phi_r(w^{(k),1}_{\lambda - \frac{1}{2}}) \).

**Theorem 6.3** For \( k = 2, 3, 4 \ldots, 0 \leq p \leq k - 1 \) and \( 0 \leq r < 1 \),

\[
\phi_r(w^{(k),1}_{\lambda - \frac{1}{2}}; t, \cos \left(\frac{2p\pi}{k}\right)) = \sum_{n=0}^{\infty} \frac{p_n(w^{(k),1}_{\lambda - \frac{1}{2}}; t)}{h_n(w^{(k),1}_{\lambda - \frac{1}{2}})} P_n^{(\frac{2p\pi}{k})}(\cos \theta).
\]

**Proof** From the definition of Poisson kernels, we obtain from (6.3) and (6.4) that

\[
\phi_r(w^{(2k)}_{\lambda - \frac{1}{2}}; \cos \theta, \cos \theta) = \phi_{r,2}(w^{(k)}_{\lambda - \frac{1}{2}}; \cos(2\theta), \cos(2\phi)) \\
+ 2r \cos \theta \cos \phi \phi_{r,2}(w^{(k),1}_{\lambda - \frac{1}{2}}; \cos(2\theta), \cos(2\phi)).
\]

The identity (4.7) allows us to derive a closed form for

\[
\phi_r(w^{(2k)}_{\lambda - \frac{1}{2}}; \cos \theta, \cos(\frac{p\pi}{k})) - \phi_{r,2}(w^{(k)}_{\lambda - \frac{1}{2}}; \cos(2\theta), \cos(\frac{2p\pi}{k}))
\]

as a ratio, which has the denominator as a product

\[
((1 + r^4)(1 - 4r^2 \cos(\frac{p\pi}{k}) \cos(2\theta) + r^4) + 2r^4(\cos(\frac{2p\pi}{k}) + \cos(4\theta))) \\
\times (1 - 2(-1)^p r^{2k} \cos(2k\theta) + r^{4k})^\lambda
\]

and the numerator as

\[
2r \cos \theta \cos(\frac{p\pi}{k})(1 - r^2)((1 + r^2)^2 - 2r^2(\cos(\frac{p\pi}{k}) + \cos(2\theta))).
\]

Removing \( 2r \cos \theta \cos(\frac{p\pi}{k}) \) in the numerator and changing \( r^2 \) to \( r \) and \( 2\theta \) to \( \theta \), we obtain a closed expression for \( \phi_r(w^{(2k),1}_{\lambda - \frac{1}{2}}; \cos \theta, \cos(\frac{p\pi}{k})) \), which is the stated result once we rewrite the final formula in \( t = \cos \theta \). \( \square \)
Using the fact that $p_n(w^{(k)}_{\lambda-\frac{1}{2}}; -t)$ is an orthogonal polynomial with respect to $w^{(k)}_{\lambda-\frac{1}{2}}$, we could derive a similar formula for $\phi_r(w^{(k)}_{\lambda-\frac{1}{2}}; t, \cos \left(\pi - \frac{2\pi n}{k}\right))$. We shall do so only for the case when $p = 0$, which is stated in the corollary below.

**Corollary 6.4** For $k = 2, 3, 4 \ldots$ and $0 \leq r < 1$,

$$
\sum_{n=0}^{\infty} \frac{p_n(w^{(k)}_{\lambda-\frac{1}{2}}; t) p_n(w^{(k)}_{\lambda-\frac{1}{2}}; 1)}{h_n(w^{(k)}_{\lambda-\frac{1}{2}})} r^n = \frac{1 - r}{(1 - 2rt + r^2)(1 - 2r^k T_k(t) + r^{2k})^{\lambda}},
$$

and, furthermore,

$$
\sum_{n=0}^{\infty} \frac{p_n(w^{(k)}_{\lambda-\frac{1}{2}}; -t) p_n(w^{(k)}_{\lambda-\frac{1}{2}}; -1)}{h_n(w^{(k)}_{\lambda-\frac{1}{2}})} r^n = \frac{1 - r}{(1 + 2rt + r^2)(1 - 2r^k T_k(-t) + r^{2k})^{\lambda}}.
$$

**Proof** The identity (6.7) is the case $p = 0$ of the identity in Theorem 6.3. The identity (6.8) follows from the fact that $p_n(w^{(k)}_{\lambda-\frac{1}{2}}; -t)$ is an orthogonal polynomial with respect to $w^{(k)}_{\lambda-\frac{1}{2}}$ and $h_n(w^{(k)}_{\lambda-\frac{1}{2}}) = h_n(w^{(k)}_{\lambda-\frac{1}{2}})$.

In the case $k = 1$, these formulas give simple generating functions for the Jacobi polynomials $P_n^{(\lambda+\frac{1}{2}, \lambda-\frac{1}{2})}(t)$ and $P_n^{(\lambda-\frac{1}{2}, \lambda+\frac{1}{2})}(t)$, respectively, which we state below:

$$
\sum_{n=0}^{\infty} \frac{(2\lambda + 1)_n}{(\lambda + 1)_n} P_n^{(\lambda-\frac{1}{2}, \lambda+\frac{1}{2})}(t) r^n = \frac{1 - r}{(1 - 2rt + r^2)^{\lambda+1}},
$$

$$
\sum_{n=0}^{\infty} \frac{(2\lambda + 1)_n}{(\lambda + 1)_n} P_n^{(\lambda+\frac{1}{2}, \lambda-\frac{1}{2})}(t) r^n = \frac{1 + r}{(1 - 2rt + r^2)^{\lambda+1}}.
$$

In the case of the second identity, we have used $P_n^{(\alpha, \beta)}(-1) = (-1)^n P_n^{(\beta, \alpha)}(1)$ and replaced $r$ by $-r$. These formulas can be derived from the $_2F_1$ formulas of the Poisson kernels of the Jacobi polynomials. That they can be simplified to such an elegant form when $\beta = \alpha + 1$ appears to be new as far as we are aware.

### 7 Product Formula and Intertwining Operator

The generating functions that we derived in the previous sections lead to several integral representations of the corresponding orthogonal polynomials. We state one such result as an example.
Theorem 7.1 Let $\lambda \geq 0$. For $k = 1, 2, 3, \ldots$, and $n = 0, 1, 2, \ldots$,

$$
\frac{p_n(w^{(k)}_{\lambda - \frac{1}{2}} \cos \theta) p_n(w^{(k)}_{\lambda - \frac{1}{2}} ; 1)}{h_n(w^{(k)}_{\lambda - \frac{1}{2}})} = \frac{n + k\lambda - a_{(k)}}{k\lambda} \times \int_{T^{k-1}} C^{k\lambda}_n \left( \sum_{j=0}^{k-1} \cos \left( \theta - \frac{2j\pi}{k} \right) u_j \right) u_0^k \prod_{i=1}^{k-1} u_i^{\lambda - 1} du. \tag{7.1}
$$

In particular, for $m = 0, 1, 2, \ldots$,

$$
C^\lambda_m (\cos (k\theta)) = a^\lambda_m \int_{T^{k-1}} C^{k\lambda}_m \left( \sum_{j=0}^{k-1} \cos \left( \theta - \frac{2j\pi}{k} \right) u_j \right) u_0^k \prod_{i=1}^{k-1} u_i^{\lambda - 1} du. \tag{7.2}
$$

The integral representation (7.1) follows from (4.9), Lemmas 4.2 and 4.3, and (4.2). Setting $n = km$ in (7.1) gives (7.1) by Proposition 5.1 and (5.4).

When $k = 2$, the identity (7.2) is a special case of the identity [12, Theorem 1.5.6]

$$
C_m^{(\lambda, \mu)}(t) = c_\mu \int_{-1}^{1} C_m^{\lambda+\mu} (tu)(1+u)(1-u^2)^{\mu-1} du \tag{7.3}
$$

for the generalized Gegenbauer polynomials defined in (2.3), since it can be easily verified that $C_m^{(\lambda, \mu)} (\cos \theta) = C_m^{\lambda}(\cos 2\theta)$ and $\cos \theta u_0 + \cos (\theta - \pi) u_1 = \cos \theta (1-2u_1)$ with $u_0 = 1 - u_1$. Furthermore, the generalized Gegenbauer polynomials satisfy a product formula [15, (2.10)]

$$
C_m^{(\lambda, \mu)} (\cos \theta) C_m^{(\lambda, \mu)} (\cos \phi) = \frac{m + \lambda + \mu}{\lambda + \mu} c_\lambda c_\mu \times \int_{-1}^{1} \int_{-1}^{1} C_m^{\lambda+\mu} (t \cos \theta \cos \phi + s \sin \theta \sin \phi)(1+t)(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1} dt ds, \tag{7.4}
$$

which reduces to, when $m = 2n$, the product formula for the Jacobi polynomials proved in [6, p.192, (2.5)], see also [13, p.133]. Evidently (7.4) becomes (7.3) when $\phi = 0$. The identity (7.4) can be deduced from the identity (2.7) for the $h$-harmonics and the closed formula of the intertwining operator for the group $I_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ as shown in [15].

For the dihedral group, the identity (2.7) gives a product formula for the $h$-harmonics, from which we can deduce product formulas for orthogonal polynomials $p_n(w_{\lambda - \frac{1}{2}} ; \cdot)$. The identities (7.1) and (7.2) are consequences of our partial closed form of the intertwining operator in Theorem 1.1. A full closed form of the intertwining operator would lead to a product formula for $p_n(w_{\lambda - \frac{1}{2}} ; \cdot)$ that will be a generalization of (7.4). In view of (7.3) and (7.4), however, a full closed form could be much more involved and is still not known at this point.
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