THE SPECTRAL DENSITY OF THE SCATTERING MATRIX FOR HIGH ENERGIES

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ABSTRACT. We determine the density of eigenvalues of the scattering matrix of the Schrödinger operator with a short range potential in the high energy asymptotic regime. We give an explicit formula for this density in terms of the X-ray transform of the potential.

1. Introduction and Main Result

1.1. Introduction. The object of study in this paper is the (on-shell) scattering matrix $S(\lambda)$ corresponding to the scattering of a $d$-dimensional quantum particle on an external short range potential $V$ at the energy $\lambda > 0$. The scattering matrix $S(\lambda)$ is a unitary operator in $L^2(S^{d-1})$ and the difference $S(\lambda) - I$ is compact. Thus, the eigenvalues of $S(\lambda)$ can be written as

$\exp(i\theta_n(\lambda)), \quad \theta_n(\lambda) \in [-\pi, \pi), \quad n \in \mathbb{N}$

and $\theta_n(\lambda) \to 0$ as $n \to \infty$. The quantities $\theta_n(\lambda)$ are known as scattering phases. The scattering phases are usually discussed in physics literature (see e.g. [6, Section 123]) under the additional assumption of the spherical symmetry of $V$; here we do not need this assumption.

Our aim is to study the asymptotic distribution of scattering phases $\{\theta_n(\lambda)\}_{n=1}^\infty$ when $\lambda \to \infty$. It turns out that after an appropriate scaling the asymptotic density of scattering phases can be described by a simple explicit formula involving the X-ray transform (see (1.5)) of the potential $V$. This formula has a semiclassical nature.

The key idea of this paper goes back to the work of M. Sh. Birman and D. R. Yafaev [1] (see also [2]), where the asymptotics of $\theta_n(\lambda)$ for a fixed $\lambda$ and $n \to \infty$ was determined for some class of potentials $V$. This asymptotic behaviour is not uniform in $\lambda$, and thus our results cannot be derived from those of [1]. However, both the results of [1] and our results are based on the following observation. The leading term of the asymptotics of $\theta_n(\lambda)$ (in both asymptotic regimes) is determined by the Born approximation of the scattering matrix. The Born approximation is essentially a pseudodifferential operator ($\Psi$DO) on the sphere $S^{d-1}$ with the symbol given by the X-ray transform of $V$. Standard $\Psi$DO results can be used to give spectral asymptotics for operators of such type. In both asymptotic regimes, the desired spectral asymptotics are given by a Weyl type formula involving the symbol of the $\Psi$DO.

1.2. The scattering matrix. Let us briefly recall the relevant definitions. Let $H_0 = -\Delta$ and $H = -\Delta + V$ be the Schrödinger operators in $L^2(\mathbb{R}^d)$, $d \geq 2$, where $V$ is the operator of
multiplication by a real-valued potential $V \in C(\mathbb{R}^d)$ which is assumed to satisfy the estimate
\begin{equation}
|V(x)| \leq \frac{C}{(1 + |x|)^\rho}, \quad \rho > 1
\end{equation}
with some constant $C > 0$. It is one of the fundamental facts of scattering theory [4, 5] that under the assumption (1.2) the wave operators
$$W_\pm = \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}$$
exist and are complete; the scattering operator $S = W_+ W_-$ is unitary in $L^2(\mathbb{R}^d)$ and commutes with $H_0$. Let $F : L^2(\mathbb{R}^d) \to L^2((0, \infty); L^2(S^{d-1}))$ be the unitary operator
$$\langle Fu \rangle(\lambda, \omega) = \frac{1}{\sqrt{2}} \lambda^{(d-2)/4} \hat{u}(\sqrt{\lambda} \omega), \quad \lambda > 0, \quad \omega \in S^{d-1},$$
where $\hat{u}$ is the usual (unitary) Fourier transform of $u$. The operator $F$ diagonalises $H_0$, i.e.
$$\langle FH_0 u \rangle(\lambda, \omega) = \lambda \langle Fu \rangle(\lambda, \omega), \quad \forall u \in C_0^\infty(\mathbb{R}^d).$$

The commutation relation $SH_0 = H_0 S$ implies that $F$ also diagonalises $S$, i.e.
$$\langle FSu \rangle(\lambda, \cdot) = S(\lambda) \langle Fu \rangle(\lambda, \cdot),$$
where $S(\lambda) : L^2(S^{d-1}) \to L^2(S^{d-1})$ is the unitary operator known as the (on-shell) scattering matrix. See e.g. the book [9] for the details.

Under the assumption (1.2) the operator $S(\lambda) - I$ is compact and consequently the eigenvalues of $S(\lambda)$ (enumerated with multiplicities taken into account) can be written as (1.1) with $\theta_n(\lambda) \to 0$ as $n \to \infty$.

1.3. The purpose of the paper. Our purpose is to describe the asymptotic density of the eigenvalues of $S(\lambda)$ as $\lambda \to \infty$. We recall the estimate (see e.g. [9, Section 8.1])
\begin{equation}
\|S(\lambda) - I\| = O(\lambda^{-1/2}), \quad \lambda \to \infty.
\end{equation}
This estimate is sharp, i.e. $O(\lambda^{-1/2})$ cannot be replaced by $o(\lambda^{-1/2})$; this can be seen by considering the case of a spherically symmetric potential and using the separation of variables. Thus, the spectrum of the scattering matrix $S(\lambda)$ for large $\lambda$ consists of a cluster of eigenvalues located on an arc of length $O(\lambda^{-1/2})$ around 1. Let us define the eigenvalue counting measure for $S(\lambda)$. The estimate (1.3) suggests the following scaling: for $\lambda \geq 1$, set $k = \sqrt{\lambda} > 0$ and define (using notation (1.1))
\begin{equation}
\mu_k([\alpha, \beta]) = \# \{ n \in \mathbb{N} : \alpha \leq k \theta_n(k^2) \leq \beta \}, \quad [\alpha, \beta] \subset \mathbb{R}\setminus\{0\}
\end{equation}
where $\#$ denotes the number of elements in the set. We will study the weak asymptotics of $\mu_k$ as $k \to \infty$, i.e. we consider the asymptotics of the integrals
$$\int_{-\infty}^{\infty} \psi(t) d\mu_k(t), \quad k \to \infty$$
for test functions $\psi \in C_0^\infty(\mathbb{R}\setminus\{0\})$. 
1.4. Main result. In order to describe the weak limit of the measures \( \mu_k \), we need to fix some notation. For any \( \omega \in \mathbb{S}^{d-1} \), let \( \Lambda_\omega \subset \mathbb{R}^d \) denote the hyperplane passing through the origin and orthogonal to \( \omega \). We equip both \( \mathbb{S}^{d-1} \) and \( \Lambda_\omega \) with the standard \((d-1)\)-dimensional Lebesgue measure (=Euclidean area). We set

\[
(1.5) \quad X(\omega, \eta) = -\frac{1}{2} \int_{-\infty}^{\infty} V(t\omega + \eta)dt, \quad \omega \in \mathbb{S}^{d-1}, \quad \eta \in \Lambda_\omega.
\]

The function \( X \) (up to a multiplicative factor) is known as the X-ray transform of \( V \) in the inverse problem literature. The following elementary estimate is a direct consequence of (1.2):

\[
(1.6) \quad |X(\omega, \eta)| \leq C(V)(1 + |\eta|)^{1-\rho}, \quad \omega \in \mathbb{S}^{d-1}, \quad \eta \in \Lambda_\omega
\]

with some constant \( C(V) \). We define a measure \( \mu \) on \( \mathbb{R}\{0\} \) by

\[
\mu([\alpha, \beta]) = (2\pi)^{1-d} \text{meas}\{(\omega, \eta) \in \mathbb{S}^{d-1} \times \Lambda_\omega : \alpha \leq X(\omega, \eta) \leq \beta\}, \quad [\alpha, \beta] \subset \mathbb{R}\{0\},
\]

where \( \text{meas} \) denotes the usual product measure. By the boundedness of \( V \), the measure \( \mu \) has a compact support. The measure \( \mu \) need not be absolutely continuous. The measure \( \mu \) may be weakly singular at zero in the following sense: \( \mu((0, \infty)) \) or \( \mu((\infty, 0)) \) may be infinite, but, by the estimate (1.6) we have

\[
(1.7) \quad \int_{-\infty}^{\infty} |t|^{\ell}d\mu(t) < \infty, \quad \forall \ell > (d-1)/(\rho - 1).
\]

Our main result is as follows:

**Theorem 1.1.** Let \( V \in C(\mathbb{R}^d) \) be a potential satisfying (1.2). Then for any test function \( \psi \in C_0^\infty(\mathbb{R}\{0\}) \),

\[
(1.8) \quad \lim_{k \to \infty} k^{1-d} \int_{-\infty}^{\infty} \psi(t)d\mu_k(t) = \int_{-\infty}^{\infty} \psi(t)d\mu(t).
\]

This can be more succinctly put as the weak convergence of the measures

\[
(1.9) \quad k^{1-d} \mu_k \to \mu, \quad k \to \infty.
\]

Much of the inspiration for both the content of this paper and the proofs may be found in [7], where similar asymptotics are determined for the spectrum of the Landau Hamiltonian (i.e. two-dimensional Schrödinger operator with a constant homogeneous magnetic field) perturbed by a potential which obeys the same condition (1.2).

We would like to mention also the paper [10] where the high energy asymptotic distribution of the phases \( \theta_n(\lambda) \) was studied for scattering problems on manifolds of a certain special class. The results of [10] are much more detailed than ours and include the asymptotics of the pair correlation measure.

1.5. Comparison with [1]. In [1], the case of potentials with the power asymptotics at infinity of the type

\[
(1.10) \quad V(x) = v(x/|x|)|x|^{-\rho}(1 + o(1)), \quad |x| \to \infty, \quad \rho > 1,
\]

was considered. Using our notation \( \mu_k, \mu \), the result of [1] can be written as

\[
(1.11) \quad k^{1-d} \mu_k((\varepsilon, \infty)) \sim \mu((\varepsilon, \infty)), \quad k^{1-d} \mu_k((\infty, -\varepsilon)) \sim \mu((\infty, -\varepsilon)),
\]

where \( \varepsilon > 0 \).
when $k > 0$ is fixed and $\varepsilon \to +0$. Here $a \sim b$ means $\frac{a}{b} \to 1$.

Clearly, our main result (1.9) is expressed by the same formula as (1.11), but the asymptotic regimes are different. Neither of the results (1.9), (1.11) implies the other one.

1.6. **Semiclassical interpretation.** By the definition of the scattering operator $S$, for any $\psi \in L^2(\mathbb{R}^d)$ we have

$$i((S - I)\psi, \psi) = i \lim_{t \to \infty} \left((e^{-2itH}e^{itH_0}\psi, e^{-itH_0}\psi) - \|\psi\|^2\right) = i \int_0^\infty \frac{d}{dt}(e^{-2itH}e^{itH_0}\psi, e^{-itH_0}\psi)dt$$

$$= \int_0^\infty (Ve^{-2itH}e^{itH_0}\psi, e^{-itH_0}\psi)dt + \int_0^\infty (Ve^{itH_0}\psi, e^{2itH}e^{-itH_0}\psi)dt.$$  

If $\psi$ corresponds to large energies, the right hand side can be approximated by the first term in its expansion in powers of $V$. This means that we can replace $e^{itH}$ by $e^{itH_0}$ in the above expressions, and so

$$(1.12) \quad i((S - I)\psi, \psi) \approx \int_{-\infty}^\infty (Ve^{-itH_0}\psi, e^{-itH_0}\psi)dt, \quad \psi \in L^2(\mathbb{R}^d),$$

which is exactly the Born approximation in the time-dependent picture.

In order to write down the classical analogue of the right hand side of (1.12), assume that $\psi$ is concentrated near $x$ in the coordinate representation and near $p$ in the momentum representation. Then $\psi$ represents a particle with the coordinate $x$ and momentum $p$, and in the same way $e^{-itH_0}\psi$ represents a particle with the coordinate $x + 2pt$ and momentum $p$. Thus, the classical analogue of the right hand side of (1.12) is

$$\int_{-\infty}^\infty V(x + 2pt)dt = \frac{1}{2|p|} \int_{-\infty}^\infty V(x + \omega t')dt', \quad (x, p) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where $\omega = \frac{p}{|p|} \in \mathbb{S}^{d-1}$. This calculation explains the appearance of the X-ray transform in the asymptotics of $S(\lambda)$.

1.7. **Key steps of the proof.** First we recall the stationary representation for the scattering matrix. For $k > 0$ and $\rho > 1$, we define the operator $\Gamma_\rho(k) : L^2(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ by

$$(\Gamma_\rho(k)u)(\omega) = \frac{1}{\sqrt{2}} k^{(d-2)/2}(2\pi)^{-d/2} \int_{\mathbb{R}^d} u(x)\langle x \rangle^{-\rho/2} e^{-ik(x, \omega)}dx, \quad \omega \in \mathbb{S}^{d-1},$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. By the Sobolev trace theorem, $\Gamma_\rho(k)$ is bounded for $\rho > 1$. Next, let

$$T(z) = \langle x \rangle^{-\rho/2}(H - zI)^{-1}\langle x \rangle^{-\rho/2}, \quad \text{Im } z > 0;$$

according to the limiting absorption principle, the limits $T(k^2 \pm i0)$ exist in the operator norm for all $k > 0$. Denote by $J$ the bounded operator of multiplication by $\langle x \rangle^\rho V(x)$ in $L^2(\mathbb{R}^d)$. The stationary representation for the scattering matrix can be written as (see e.g. [9] Section 6.6)

$$S(k^2) = I - 2\pi i \Gamma_\rho(k)(J - JT(k^2 + i0)J)\Gamma_\rho(k)^*, \quad k > 0.$$  

The asymptotic density of eigenvalues of $S(k^2)$ for large $k$ is determined by the *Born approximation* of the scattering matrix, defined as

$$(1.14) \quad S_B(k^2) = I - 2\pi i \Gamma_\rho(k)J \Gamma_\rho(k)^*, \quad k > 0.$$
The key observation due to M. Birman and D. Yafaev [1] is that the operator $\Gamma_\rho(k)J\Gamma_\rho(k)^*$ in $L^2(S^{d-1})$ with the integral kernel

$$
2^{-1}k^{d-2}(2\pi)^{-d}\int_{\mathbb{R}^d} e^{-ik(\omega-\omega',x)}V(x)dx, \quad \omega, \omega' \in S^{d-1}
$$

can be represented as a $\Psi$DO on the sphere with the symbol (up to inessential constants) $X(\omega, \eta)$. We combine this observation with the standard semiclassical pseudodifferential techniques to obtain the spectral asymptotics of $S_B(k^2)$. In this way we prove the asymptotic formula (see Lemma 3.2)

$$
(1.15) \lim_{k\to \infty} k^{1-d} \text{Tr}(\text{Im} k S_B(k^2))^{\ell} = \int_{-\infty}^{\infty} t^{\ell} d\mu(t)
$$

for any natural number $\ell > (d-1)/(\rho-1)$; note that the r.h.s. of (1.15) is finite by (1.7).

Using (1.15) and the estimates for the Schatten norm of $S(k^2) - S_B(k^2)$ we prove that (1.8) holds true for test functions $\psi(t)$ which coincide with $t^{\ell}$ for all sufficiently small $t$. Theorem 1.1 follows by an application of the Weierstrass approximation theorem.

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2. Preliminary statements

2.1. The limiting absorption principle and its consequences. First we need some notation. We denote by $S_p$, $p \geq 1$, the usual Schatten class and by $\| \cdot \|_p$ the associated norm. Let $X_\rho$ be the normed linear space of all potentials $V \in C(\mathbb{R}^d)$ satisfying (1.2) with the norm

$$
\|V\|_{X_\rho} = \sup_{x \in \mathbb{R}^d} |V(x)|\langle x \rangle^\rho.
$$

We recall the following estimates from [9]:

**Proposition 2.1.** Let $V \in X_\rho$ with some $\rho > 1$. Then for any $\ell \geq 1$ satisfying $\ell > \frac{d-1}{\rho-1}$, one has

$$
(2.1) \quad \sup_{k \geq 1} k^{\frac{1-\rho}{\ell}} \|k \text{Im} S_B(k^2)\|_\ell \leq C(\ell, \rho, d)\|V\|_{X_\rho},
$$

$$
(2.2) \quad \sup_{k \geq 1} k^{\frac{1-\rho}{\ell}} \|k^2 \text{Im} (S_B(k^2) - S(k^2))\|_\ell \leq C(\ell, \rho, d, V),
$$

$$
(2.3) \quad \sup_{k \geq 1} k^{\frac{1-\rho}{\ell}} \|\text{Im} S(k^2)\|_\ell \leq C(\ell, \rho, d, V).
$$

The estimate (2.1) is a direct consequence of [9, Proposition 8.1.3]. The estimate (2.2) is proven in [9, Proposition 8.1.4]. The estimate (2.3) is a direct consequence of (2.1) and (2.2).

**Lemma 2.2.** Let $V \in X_\rho$ with $\rho > 1$. Then for any integer $\ell \geq 1$ satisfying $\ell > \frac{d-1}{\rho-1}$, one has

$$
|\text{Tr}(k \text{Im} S(k^2))^{\ell} - \text{Tr}(k \text{Im} S_B(k^2))^{\ell}| = O(k^{d-2}), \quad k \to \infty.
$$
Proof. From
\[ A^\ell - B^\ell = \sum_{j=0}^{\ell-1} A^j (A - B) B^{\ell-1-j} \]
one easily obtains
\[ |\text{Tr}(A^\ell - B^\ell)| \leq \ell \|A - B\|_\ell \max\{\|A\|^{\ell-1}_\ell, \|B\|^{\ell-1}_\ell\}, \quad A, B \in S_\ell. \]
Thus, it suffices to prove the relation
\[ \|k \text{Im} (S(k^2) - S_B(k^2))\|_\ell \max\{\|k \text{Im} S(k^2)\|^{\ell-1}_\ell, \|k \text{Im} S_B(k^2)\|^{\ell-1}_\ell\} = O(k^{d-2}), \quad k \to \infty. \]
The latter relation follows by combining (2.1)–(2.3). \hfill \square

2.2. Semiclassical $\Psi DO$ on the sphere. A semiclassical $\Psi DO$ in $L^2(\mathbb{S}^{d-1})$ can be defined in a variety of ways; below we describe a slightly non-standard approach to this definition, which will simplify our exposition in Section 3.

For $\omega, \omega' \in \mathbb{S}^{d-1}$ such that $\omega + \omega' \neq 0$, we set
\[ \kappa = \kappa(\omega, \omega') = \frac{\omega + \omega'}{|\omega + \omega'|} \in \mathbb{S}^{d-1}. \]
Clearly, $\kappa(\omega, \omega')$ is a smooth function of $(\omega, \omega') \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ away from the anti-diagonal
\[ \text{AD} = \{ (\omega, \omega') \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} | \omega + \omega' = 0 \}. \]
In order to overcome the (inessential) difficulties related to the singularity of $\kappa$ at the anti-diagonal, we will assume that our amplitudes vanish in an open neighbourhood of AD. We will say that a function $b = b(\omega, \omega', \eta), \omega, \omega' \in \mathbb{S}^{d-1}, \eta \in \Lambda_\kappa(\omega, \omega')$, is an admissible amplitude, if:
\begin{itemize}
  \item $b$ is a $C^\infty$-smooth function of its arguments;
  \item $b(\omega, \omega', \eta) = 0$ if $|\eta|$ is sufficiently large;
  \item $b(\omega, \omega', \eta) = 0$ if $(\omega, \omega')$ are in an open neighbourhood of AD.
\end{itemize}

For an admissible amplitude $b$ and a semiclassical parameter $k > 0$, we define the operator $\text{Op}_k[b]$ in $L^2(\mathbb{S}^{d-1})$ via its integral kernel
\[ \text{Op}_k[b](\omega, \omega') = \left(\frac{k}{2\pi}\right)^{d-1} \int_{\Lambda_\kappa(\omega, \omega')} e^{-ik(\omega - \omega', \eta)} b(\omega, \omega', \eta) d\eta, \]
where $\omega, \omega' \in \mathbb{S}^{d-1}$. It is easy to see that for $\omega \neq \omega'$ one has
\[ \text{Op}_k[b](\omega, \omega') = O(k^{-\infty}), \quad k \to \infty. \]
This shows that the values of the amplitude $b(\omega, \omega', \eta)$ away from an open neighbourhood of the diagonal $\omega = \omega'$ do not affect the asymptotic properties of the operator $\text{Op}_k[b]$ as $k \to \infty$. \hfill \square

Proposition 2.3. For any admissible amplitude $b$ and any $k > 0$, the operator $\text{Op}_k[b]$ is trace class, and for any $\ell \in \mathbb{N}$ one has
\[ \lim_{k \to \infty} \left(\frac{k}{2\pi}\right)^{-d+1} \text{Tr}(\text{Op}_k[b])^\ell = \int_{\mathbb{S}^{d-1}} \int_{\Lambda_\omega} b(\omega, \omega, \eta)^\ell d\eta d\omega. \]
Sketch of proof. The proof follows standard methods of ΨDO theory; see e.g. [3, Theorem 9.6] for a similar statement in the context of operators in \( \mathbb{R}^n \). Here we only outline the main steps.

First note that for \( \ell = 1 \) the result of Proposition 2.3 is trivial, since by a direct evaluation of trace we have the identity

\[
\left( \frac{k}{2\pi} \right)^{-d+1} \text{Tr}(\text{Op}_k[b]) = \int_{S^{d-1}} \int_{\Lambda_\omega} b(\omega, \eta) d\eta d\omega.
\]

Next, using the local coordinates on the sphere and the composition formula for symbols of ΨDOs (see e.g. [3, Proposition 7.7]), we obtain the following statement. For any \( N > 0 \) there exists \( M > 0 \) such that \( (\text{Op}_k[b])^\ell \) can be represented as

\[
(\text{Op}_k[b])^\ell = \sum_{m=0}^{M} k^{-m} \text{Op}_k[b_m] + R_k,
\]

where \( b_m \) are admissible symbols, \( b_0 \) is such that

\[
b_0(\omega, \omega, \eta) = b(\omega, \eta)^\ell, \quad \forall \omega \in S^{d-1}, \quad \forall \eta \in \Lambda_\omega,
\]

and \( R_k \) is an integral operator with a smooth kernel which satisfies

\[
\sup_{\omega, \omega'} |R_k(\omega, \omega')| = O(k^{-N}), \quad k \to \infty.
\]

Now taking \( N > d - 1 \) and evaluating the trace in (2.8), we get

\[
\lim_{k \to \infty} \left( \frac{k}{2\pi} \right)^{-d+1} \text{Tr}(\text{Op}_k[b])^\ell = \lim_{k \to \infty} \left( \frac{k}{2\pi} \right)^{-d+1} \text{Tr}(\text{Op}_k[b_0]) = \int_{S^{d-1}} \int_{\Lambda_\omega} b_0(\omega, \omega, \eta) d\eta d\omega.
\]

In view of (2.9), this proves the required identity. \( \square \)

3. Proof of Theorem 1.1

3.1. The Born approximation with \( V \in C^\infty_0(\mathbb{R}^d) \).

Lemma 3.1. Let \( V \in C^\infty_0(\mathbb{R}^d) \). Then for any \( \ell \in \mathbb{N} \), one has

\[
\lim_{k \to \infty} k^{1-d} \text{Tr}(k \text{Im} S_B(k^2))^\ell = \int_{-\infty}^{\infty} t^\ell d\mu(t).
\]

Proof. 1. For ease of notation we write \( Q(k) = k \text{Im} S_B(k^2) \). By (1.13) and (1.14), \( Q(k) \) is the integral operator in \( L^2(S^{d-1}) \) with the integral kernel

\[
Q(k)(\omega, \omega') = -\frac{1}{2} \left( \frac{k}{2\pi} \right)^{d-1} \int_{\mathbb{R}^d} e^{-ik(\omega - \omega', x)} V(x) dx, \quad \omega, \omega' \in S^{d-1}.
\]

For fixed \( \omega, \omega' \), \( \omega + \omega' \neq 0 \), let \( \kappa = \kappa(\omega, \omega') \) be as in (2.5). Write any \( x \in \mathbb{R}^d \) as \( x = \kappa t + \eta \) with \( t \in \mathbb{R}, \eta \in \Lambda_\kappa \). Note that by the orthogonality relation \( (\omega - \omega') \perp \kappa \), one has

\[
\langle \omega - \omega', x \rangle = \langle \omega - \omega', \eta \rangle.
\]
Thus, the integral kernel of $Q(k)$ can be rewritten as
\begin{equation}
Q(k)(\omega', \omega) = -\frac{1}{2} \left( \frac{k}{2\pi} \right)^{d-1} \int_{\Lambda_{\omega, \omega'}} \int_{-\infty}^{\infty} e^{-ik(\omega - \omega', \eta)} V(\kappa(\omega, \omega') t + \eta) dt d\eta
= \left( \frac{k}{2\pi} \right)^{d-1} \int_{\Lambda_{\omega, \omega'}} e^{-ik(\omega - \omega', \eta)} X(\kappa(\omega, \omega'), \eta) d\eta,
\end{equation}
where $X$ is given by (1.5). From here we directly obtain the required identity (3.1) in the case $\ell = 1$ by integrating the kernel of $Q(k)$ over the diagonal. Now it remains to prove (3.1) for $\ell \geq 2$.

2. Let $\chi_0 \in C_0^\infty(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ be such that $\chi_0(\omega, \omega') = 1$ in an open neighbourhood of the diagonal $\omega = \omega'$ and $\chi_0(\omega, \omega') = 0$ in an open neighbourhood of the anti-diagonal $\omega + \omega' = 0$. Denote $\chi_1 = 1 - \chi_0$, and let
\[ Q(k) = Q_0(k) + Q_1(k), \]
where $Q_j(k)$ is the operator with the integral kernel $\chi_j(\omega, \omega') Q(k)(\omega, \omega')$. By the fast decay of the Fourier transform of $V$ and by the fact that $|\omega - \omega'|$ is separated away from zero on the support of $\chi_1$, we see that
\[ \sup_{\omega, \omega'} |Q_1(k)(\omega, \omega')| = O(k^{-\infty}), \quad k \to \infty. \]
Thus, it suffices to prove that
\[ \lim_{k \to \infty} k^{1-d} \text{Tr}(Q_0(k))^\ell = \int_{-\infty}^{\infty} t^\ell d\mu(t). \]
From (3.2) it follows that $Q_0(k)$ can be represented as a semiclassical $\Psi$DO on the sphere of the type (2.6):
\[ Q_0(k) = \text{Op}_k[b], \quad \text{where} \quad b(\omega, \omega', \eta) = \chi_0(\omega, \omega') X(\kappa(\omega, \omega'), \eta), \]
b is an admissible amplitude in the sense discussed in Section 2.2, and $X$ is given by (1.5). Applying Proposition 2.3 we get
\begin{equation}
\lim_{k \to \infty} k^{1-d} \text{Tr}(Q_0(k))^\ell = (2\pi)^{1-d} \int_{\mathbb{S}^{d-1}} \int_{\Lambda_\omega} X(\omega, \eta)^\ell d\eta d\omega = \int_{-\infty}^{\infty} t^\ell d\mu(t),
\end{equation}
as required. \hfill \Box

3.2. The Born approximation with general $V$.

Lemma 3.2. Let $V \in X_\rho$ with $\rho > 1$. Then for any integer $\ell \geq 1$ satisfying $\ell > \frac{d-1}{\rho-1}$ one has
\[ \lim_{k \to \infty} k^{1-d} \text{Tr}(\text{Im} k S_B(k^2))^\ell = \int_{-\infty}^{\infty} t^\ell d\mu(t). \]
Proof. Let $X_\rho^0$ be the closure of $C_0^\infty(\mathbb{R}^d)$ in $X_\rho$. For any $\ell > \frac{d-1}{\rho-1}$, denote
\[ g_\ell^+(V) = \int_{-\infty}^{\infty} t^\ell d\mu(t), \]
\[ g_\ell^+(V) = \limsup_{k \to \infty} k^{1-d} \text{Tr}(k \text{Im} S_B(k^2))^\ell, \]
\[ g_\ell^-(V) = \liminf_{k \to \infty} k^{1-d} \text{Tr}(k \text{Im} S_B(k^2))^\ell. \]
By Lemma 3.1 for all $V \in C_0^\infty(\mathbb{R}^d)$ we have
\begin{equation}
(3.4) \quad g_\ell(V) = g_\ell^+(V) = g_\ell^-(V).
\end{equation}
Recall that $S_\rho(k^2)$ depends linearly on $V$. Using this fact and the estimates (2.1) and (2.4), it is easy to check that $g_\ell^\pm(V)$ are continuous functionals on $X_\rho$. Next, using the last equality in (3.3) and the estimate (1.6), it is easy to see that $g_\ell(V)$ is a continuous functional on $X_\rho$. Thus, by an approximation argument, (3.4) holds for any $V \in X_\rho$. Finally, for any $V \in X_\rho$ and a given $\ell > \frac{d-1}{\rho-1}$, choose $\rho_1$ such that $1 < \rho_1 < \rho$ with $\ell > \frac{d-1}{\rho_1-1}$. Then $X_\rho \subset X_{\rho_1}$ and the previous argument proves (3.4) for all $V \in X_{\rho_1}$ which suffices.

3.3. From the Born approximation to the full scattering matrix.

Lemma 3.3. Let $V \in X_\rho$ with $\rho > 1$. Then for any integer $\ell \geq 1$ satisfying $\ell + 2 > \frac{d-1}{\rho-1}$,
\begin{equation}
(3.5) \quad \lim_{k \to \infty} k^{1-d} \int_{-\infty}^{\infty} t^\ell d\mu_k(t) = \int_{-\infty}^{\infty} t^\ell d\mu(t).
\end{equation}
Proof. By Lemmas 2.2 and 3.2 it suffices to prove
\begin{equation}
(3.6) \quad \lim_{k \to \infty} k^{1-d+\ell} \left| \sum_{n=1}^{\infty} \left[ (\theta_n(k^n))^\ell - (\sin \theta_n(k^n))^\ell \right] \right| = 0.
\end{equation}
By (1.3) we have $0 < |\theta_n(k^n)| < \pi/4$ for all sufficiently large $k$ and all $n$. From the elementary estimates $|\theta_n| \leq 2|\sin \theta_n|$ and $|\theta_n - \sin \theta_n| \leq C|\sin \theta_n|^3$ which hold for $|\theta_n| < \pi/4$, it follows that for all sufficiently large $k$
\[
k^{1-d+\ell} \sum_{n=1}^{\infty} |\theta_n|^\ell - (\sin \theta_n)^\ell | \leq k^{1-d+\ell} \sum_{n=1}^{\infty} \left( |\theta_n - \sin \theta_n| \sum_{j=0}^{\ell-1} |\theta_n|^j |\sin \theta_n|^{\ell-1-j} \right)
\leq k^{1-d+\ell} C(\ell) \sum_{n=1}^{\infty} |\sin \theta_n|^{\ell+2} = k^{1-d-2} C(\ell) \| \text{Im} kS(k^2) \|_{\ell+2}^2.
\]
Now (3.6) follows by applying the estimate (2.3) for $\| \text{Im} kS(k^2) \|_{\ell+2}$ to the result just obtained.

Proof of Theorem 1.1. By the estimate (1.3), the supports of $\mu_k$ are bounded uniformly in $k \geq 1$. On the other hand, by the boundedness of $V$, the support of $\mu$ is also bounded. Thus, we may choose $T > 0$ such that
\[
supp \mu \subset [-T, T] \quad \text{and} \quad supp \mu_k \subset [-T, T] \quad \text{for all} \quad k \geq 1.
\]
Next, fix $\psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, and let $\ell_0$ be an even natural number satisfying $\ell_0 > \frac{d-1}{\rho-1}$. Since $\psi(t)$ vanishes near $t = 0$ by assumption, the function $\psi(t)/t^{\ell_0}$ is smooth. By the Weierstrass approximation theorem, for any $\varepsilon > 0$ there exists a polynomial $\psi_0(t)$ such that
\[
|\psi(t)t^{-\ell_0} - \psi_0(t)| \leq \varepsilon, \quad \forall t \in [-T, T].
\]
Denoting \( \psi_\pm(t) = (\psi_0(t) \pm \varepsilon)t^\ell_0 \), we obtain

\[
\psi_-(t) \leq \psi(t) \leq \psi_+(t), \quad \forall t \in [-T, T],
\]

(3.7)

\[
\psi_+(t) - \psi_-(t) = 2\varepsilon t^\ell_0.
\]

(3.8)

By (3.7), we get

\[
\int_{-\infty}^\infty \psi_-(t) d\mu_k(t) \leq \int_{-\infty}^\infty \psi(t) d\mu_k(t) \leq \int_{-\infty}^\infty \psi_+(t) d\mu_k(t).
\]

(3.9)

By construction, \( \psi_\pm(t) \) are polynomials which involve powers \( t^m \) with \( m \geq \ell_0 \). Thus, we can apply Lemma 3.3 to (3.9), which yields

\[
\lim_{k \to \infty} \sup \ k^{1-d} \int_{-\infty}^\infty \psi(t) d\mu_k(t) \leq \int_{-\infty}^\infty \psi(t) d\mu(t),
\]

\[
\lim_{k \to \infty} \inf \ k^{1-d} \int_{-\infty}^\infty \psi(t) d\mu_k(t) \geq \int_{-\infty}^\infty \psi(t) d\mu(t).
\]

On the other hand, by (3.7), (3.8),

\[
\int_{-\infty}^\infty \psi_-(t) d\mu(t) \leq \int_{-\infty}^\infty \psi(t) d\mu(t) \leq \int_{-\infty}^\infty \psi_+(t) d\mu(t)
\]

and

\[
\int_{-\infty}^\infty \psi_+(t) d\mu(t) - \int_{-\infty}^\infty \psi_-(t) d\mu(t) = 2\varepsilon \int_{-\infty}^\infty t^\ell_0 d\mu(t).
\]

By (1.6), the integral in the right hand side of the last estimate is finite; denote this integral by \( C \). Combining the above estimates, we obtain

\[
\lim_{k \to \infty} \sup \ k^{1-d} \int_{-\infty}^\infty \psi(t) d\mu_k(t) \leq \int_{-\infty}^\infty \psi(t) d\mu(t) + 2\varepsilon C,
\]

\[
\lim_{k \to \infty} \inf \ k^{1-d} \int_{-\infty}^\infty \psi(t) d\mu_k(t) \geq \int_{-\infty}^\infty \psi(t) d\mu(t) - 2\varepsilon C.
\]

Since \( \varepsilon > 0 \) can be taken arbitrary small, we obtain the required statement. \( \square \)

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