Towards a 3D reduction of the N-body Bethe-Salpeter equation.

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1 Introduction.

The Bethe-Salpeter equation is the usual tool for computing bound states of relativistic particles. The principal difficulty of this equation comes from the presence of N-1 (for N particles) unphysical degrees of freedom: the relative time-energy degree of freedom. In the two-body problem, the relative energy is usually eliminated by replacing the free two-body propagator by an expression combining a delta fixing the relative energy and a 3D propagator. The exact equivalence (in what concerns the physically measurable quantities of the pure two-fermion problem) with the original Bethe-Salpeter equation can be obtained by recuperating the difference with the original free propagator in a series of correction terms to the 3D potential. It is not possible to generalize this constraining propagator-based reduction method to three or more particles, because of the unconnectedness of the two-body terms of the Bethe-Salpeter kernel, which are in fact the more important terms and often the only ones to be considered.

A less often used 3D reduction method is based on the replacement of the Bethe-Salpeter kernel by an ”instantaneous” (i.e. independent of the relative energy) approximation (kernel-based reduction). In this case, the resulting 3D potential is not manifestly symmetric (i.e. hermitian when the total energy on which it depends is treated as a parameter). In the two-fermion problem, we obtained a symmetric 3D potential by performing a supplementary series expansion at the 3D level and combining it with the first 3D reducing expansion. We found that the starting instantaneous approximation of the Bethe-Salpeter kernel disappears from the final 3D potential. In fact, this potential can be obtained directly by a new integrating propagator-based reduction method, in which the relative energy is integrated on, instead of being fixed by a $\delta$-function (or constraint).

This integrating propagator-based reduction can easily be generalized to a system of N particles, consisting in any mixing of bosons and fermions.

2 Inhomogeneous and homogeneous Bethe-Salpeter equations for 2 fermions:

$$G = G^0 + G^0 KG, \quad \Phi = G^0 K \Phi$$

$\Phi$: Bethe-Salpeter amplitude
$K$: Bethe-Salpeter kernel

( must give $G$ via the inhomogeneous equation)

$G \equiv G^0 + G^0 TG^0$: Full propagator (Feynman graphs)

$G^0$: Free propagator:

$$G^0 = G^0_1 G^0_2, \quad G^0_i = \frac{1}{\gamma_i \cdot p_i - m_i + i\epsilon} = \frac{1}{p_{i0} - h_i + i\epsilon \beta_i}$$

$$h_i = \vec{\alpha}_i \cdot \vec{p}_i + \beta_i m_i \quad (i = 1, 2)$$

The self-energy parts of the propagator were transferred to the kernel. We shall neglect them here for simplicity. $K$ is then the sum of the irreducible two-fermion Feynman graphs.
Notations for the following:

\[ P = p_1 + p_2 , \quad p = \frac{1}{2}(p_1 - p_2) \]

\[ E = E_1 + E_2 , \quad E_i = \sqrt{h_i^2} = (p_i^0 + m_i^2)^\frac{1}{2} \]

\[ \Lambda^+ = \Lambda_1^+ \Lambda_2^+, \quad \Lambda_i^+ = \frac{E_i + h_i}{2E_i} , \quad \beta = \beta_1 \beta_2 \]

3 3D reduction by expansion around a positive-energy instantaneous approximation of \( K \).

Write \( K = K^0 + K^R \) with \( K^0 = \Lambda_+ \beta K^0 \Lambda_+ \) (positive-energy) and \( K^0(p_0', p_0) \) independent of \( p_0', p_0 \) (instantaneous). The Bethe-Salpeter equation becomes

\[ \Phi = G^0 K^0 \Phi + G^0 K^R \Phi \rightarrow \]

\[ \Phi = (1 - G^0 K^R)^{-1} G^0 K^0 \Phi \rightarrow \Phi = (G^0 + G^{KR}) K^0 \Phi \]

with

\[ G^{KR} = G^0 K^R (1 - G^0 K^R)^{-1} G^0. \]

Integrate with respect to \( p_0' \) and apply \( \Lambda^+ \rightarrow 3D \) equation:

\[ \psi = (g^0 + g^{KR}) V^0 \psi \]

with

\[ \Lambda^+ \int dp_0 G^0(p_0) = -2i\pi \Lambda^+ g^0 \beta , \quad g^0 = \frac{1}{P_0 - E + i\epsilon} \]

\[ \psi = \Lambda^+ \int dp_0 \Phi(p_0) , \quad V^0 = -2i\pi \beta K^0 \]

\[ g^{KR} = -\frac{1}{2i\pi} \Lambda^+ \int dp_0' dp_0 G^{KR}(p_0', p_0) \beta \Lambda^+. \]

4 Render the potential symmetric.

The 3D potential \( (g^0)^{-1}(g^0 + g^{KR}) V^0 \) is not symmetric. In Phillips and Wallace’s method [3], one computes \( K^0 \) in order to make \( g^{KR} \) vanish. Here, we shall write

\[ g^{KR} = g^0 T^{KR} g^0 \]

\[ \rightarrow \psi = (1 + g^0 T^{KR}) g^0 V^0 \psi \rightarrow (1 + g^0 T^{KR})^{-1} \psi = g^0 V^0 \psi \]

\[ \rightarrow \psi = \left[ g^0 V^0 + 1 - (1 + g^0 T^{KR})^{-1} \right] \psi \rightarrow \psi = g^0 V \psi \]

with

\[ V = V^0 + T^{KR}(1 + g^0 T^{KR})^{-1}. \]

This potential \( V \) is now symmetric.
5 Expand $T^{KR}$ and recombine the series.

\[ T^{KR} = \langle K^R (1 - G^0 K^R)^{-1} \rangle \]

with

\[ < A > = \frac{1}{-2\pi} \Lambda^{+}(g^0)^{-1} \int dp'_0 dp_0 G^0(p'_0) A(p'_0, p_0) G^0(p_0) \beta \Lambda^{+}(g^0)^{-1}. \]

This leads to

\[ V = \langle K^0 \rangle + \langle K^R (1 - G^0 K^R)^{-1} \rangle (1 + g^0 < K^R (1 - G^0 K^R)^{-1} >)^{-1} \]

\[ = \langle K^0 + K^R (1 - G^0 K^R)^{-1} (1 + g^0 < K^R (1 - G^0 K^R)^{-1} >) \rangle \]

\[ = \langle K^0 + K^R (1 - G^0 K^R + g^0 < K^R >)^{-1} \rangle = \langle K^0 + K^R (1 - G^0 K^R)^{-1} \rangle \]

with the definitions

\[ G^R = G^0 - G^I, \quad G^I = > g^0 < . \]

Less formally:

\[ G^0(p'_0, p_0) = G^0(p_0) \delta(p'_0 - p_0), \quad G^I(p'_0, p_0) = G^0(p'_0) \beta \frac{\Lambda^{+}}{-2\pi g^0} G^0(p_0). \]

but

\[ K^0 G^R = G^R K^0 = 0 \rightarrow K^R (1 - G^0 K^R)^{-1} = -K^0 + K (1 - G^R K)^{-1} \rightarrow \]

\[ V = < K (1 - G^R K)^{-1} > = < K > + < KG^R K > + \cdots \]

\[ = < K > + \{ < KG^0 K > - < K > g^0 < K > \} + \cdots \]

In the relative-energy integrals, $-G^I$ cancels the leading term coming from $G^0$.

Good surprise: $V$ does not depend on the initial choice of $K^0$ anymore.

6 We made in fact an integrating propagator-based reduction.

Our final 3D equation could also be obtained directly from the Bethe-Salpeter equation by performing an expansion around an approximation $G^I$ of the propagator $G^0$ (→ integrating propagator-based reduction instead of the constraining propagator-based reduction using $\delta$–functions).

7 We could start with the equal-times retarded propagator.

Following [4] ([5] in the three-body case), we could also start by taking the retarded part of the full propagator at equal times. In momentum space, it is

\[ g = g^0 + g^0 < T > g^0. \]

The corresponding 3D potential is

\[ V = < T > (1 + g^0 < T >)^{-1}. \]

Writing then the expansion $T = K(1 - G^0 K)^{-1}$ and recombining the series gives $V = < K (1 - G^R K)^{-1} >$ again. Note that $T$ and $< T >$ are both proportional to the physical scattering amplitude when the initial and final fermions are on their positive-energy mass shell.
Generalization to systems of N particles.

Our 3D reduction method (as established in section 5 or section 6’s way) can be easily generalized to systems consisting in any number of fermions and/or bosons.

Here we shall consider only the case of N fermions. The writing of the Bethe-Salpeter equation and of the final 3D equation remains the same:

\[ \Phi = G^0 K \Phi \rightarrow \psi = g^0 V \psi \]

\[ V = < K (1 - G^R) K^{-1} >, \quad G^R = G^0 - > g^0 <, \quad g^0 = \frac{1}{P_0 - E + i \epsilon} \]

with a trivial generalization of some notations:

\[ A^+ = \Lambda^+_1 \cdots \Lambda^+_N, \quad \beta = \beta_1 \cdots \beta_N, \]

\[ P_0 = p_{01} + \cdots + p_{0N} \quad E = E_1 + \cdots + E_N \]

\[ < A > = \frac{1}{(-2i\pi)^{N-1}} \Lambda^+(g^0)^{-1} \int dp_0 dp_0' G^0(p_0') A(p_0', p_0) G^0(p_0) \beta \Lambda^+(g^0)^{-1} \]

\[ dp_0 = dp_{01} \cdots dp_{0N} \delta (p_{01} + \cdots + p_{0N} - P_0). \]

Expressions of \( K \) for \( N \geq 3 \):

\( N = 3 \):

\[ K = K_{12}(G_{03})^{-1} + K_{23}(G_{01})^{-1} + K_{31}(G_{02})^{-1} + K_{123}. \]

\( N = 4 \):

\[ K = K_{12,34} + K_{13,24} + K_{14,23} + K_{123} \]

\[ + K_{123}(G_4^0)^{-1} + K_{124}(G_2^0)^{-1} + K_{134}(G_2^0)^{-1} + K_{234}(G_1^0)^{-1} \]

\[ + K_{1234}, \]

with

\[ K_{12,34} = K_{12}(G_3^0 G_4^0)^{-1} + K_{34}(G_1^0 G_2^0)^{-1} - K_{12} K_{34}. \]

The counter-term \( K_{12} K_{34} \) cancels the double-countings which would come from the fact that two graphs containing respectively \( K_{12} K_{34} \) and \( K_{34} K_{12} \) in the expansion of \( G \) must be taken only once \([6, 7, 8, 2]\).

\( N \geq 5 \): Very complicated. For \( N \geq 5 \) (perhaps even for \( N = 4 \)) we suggest to bypass the Bethe-Salpeter equation by writing \( V = < T > (1 + g^0 < T >)^{-1} \) without expanding \( T \) in terms of \( K \), and sorting the contributing graphs by increasing number of vertexes.

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