Theory of non-local point transformations - Part 3: Theory of NLPT-acceleration and the physical origin of acceleration effects in curved space-times

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This paper is motivated by the introduction of a new functional setting of General Relativity (GR) based on the adoption of suitable group non-local point transformations (NLPT). Unlike the customary local point transformation usually utilized in GR, these transformations map in each other intrinsically different curved space-times. In this paper the problem is posed of determining the tensor transformation laws holding for the 4-acceleration with respect to the group of general NLPT. Basic physical implications are considered. These concern in particular the identification of NLPT-acceleration effects, namely the relationship established via general NLPT between the 4-accelerations existing in different curved-space times. As a further application the tensor character of the EM Faraday tensor with respect to the NLPT-group is established.

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1 - INTRODUCTION

Following the program outlined in Ref. [1] (Part 1) and Ref. [2] (Part 2) concerning the formulation of the theory of non-local point transformations (NLPT) for classical tensorial calculus, in this paper the problem is posed of determining the tensor transformation law for the 4-acceleration defined in different curved-space-times and to ascertain the consequent possible physical implications in general relativity (GR) which are inspired by the related transformation theory.

The issue is in some sense intrinsically built in GR and is intimately connected with the historical origins of the discipline. In particular it is closely related with the now one-century old foundations of GR dating back to Albert Einstein’s 1915 article on the field equation bringing his name [3] (see also Refs. [4–8]). The solution of the problem in the context of GR is, in fact, almost trivial when formulated in the customary functional setting initially chosen by Einstein himself [9, 10], namely when GR-reference frames (i.e., coordinate systems) are connected by means of local point transformations (LPT) only and the transformation laws of the related 4-accelerations are considered. However, the issue of how to connect the 4-accelerations defined in different curved space-time represents in itself a formidable study case and, to this date, a theoretical challenge [12, 13]. This conclusion actually seems to unveil an amazing aspect of history science and particularly of GR, namely that the mainstream literature has effectively ignored the potential of the problem. In the following we intend to show that the determination of the 4-acceleration non-local transformation laws is actually of foremost importance for its implications in GR and all relativistic theories.

In particular, we intend to show that such a problem can be conveniently formulated in the context of the theory of non-local point transformations (NLPT), and its group of general NLPT \( \{P_g\} \), presented and discussed in detail in Parts 1 and 2.

In fact, although the mathematical adequacy of the classical theory of tensor calculus on manifolds, lying as the basis of GR, remains paramount, compelling theoretical motivations suggest the actual physical inadequacy of the traditional setting of LPT adopted both in GR and relativistic theories. Nevertheless, as shown in Part 1, in certain physical problems including in particular Einstein’s Teleparallel approach to the theory of gravitation [11] (see also Refs. [14–16]) and the problem of diagonalization of metric tensors in GR [17–19] (see also Refs. [20, 21]), the introduction of the NLPT-setting is found to be mandatory. From the investigation carried out in Parts 1 and 2 it follows that the group of general NLPT \( \{P_g\} \) can map in each other two in principle arbitrary connected and time-oriented Lorentzian curved space-times \((Q^4, g)\) and \((Q^4, g')\). In Lagrangian and Eulerian forms these maps are respectively of the type

\[
\begin{align*}
P_S: r'^\mu(s) &= r^\mu(s_0) + \int_{s_0}^s ds' M^\mu_{\nu}(s', r(r')) u^\nu(s'), \\
P_S^{-1}: r'^\mu(s) &= r^\mu(s_0) + \int_{s_0}^s ds' (M^{-1}_g)_{\nu}^\mu(r, r'(s)) u^\nu(s'),
\end{align*}
\]

(1)
and

\[
\begin{align*}
P_S: & \quad r^\mu = r_o^\mu + \int_{r_o}^{r'} \, dr'' \, M^\mu_{(g)\nu}(r'', r(r')) , \\
P_S^{-1}: & \quad r'^\mu = r_o'^\mu + \int_{r_o}^{r'} \, dr'' \, \left( M^{-1}_{(g)} \right)_{\nu}^\mu (r, r'(r)).
\end{align*}
\]

(2)

Here the notations are those pointed out in Part 2. Thus, in particular \( r^\mu(s) \) and \( r'^\mu(s) \) denote suitably smooth world-lines referred to arbitrary curvilinear coordinate systems \( r \equiv \{ r^\mu(s) \} \) and \( r' \equiv \{ r'^\mu(s) \} \) of \((Q^4, g)\) and \((Q^4, g')\) respectively, while \( u^\nu(\bar{s}) = dr^\mu/ds \) and \( u'^\nu(\bar{s}) = dr'^\mu/ds \) are the corresponding 4-velocities. The corresponding 4-velocity transformations are then

\[
\begin{align*}
u(s) = M^\mu_{(g)\nu}(r', r(r'))u'^\nu(s), \\
u'^{(s)} = \left( M^{-1}_{(g)} \right)_{\nu}^\mu (r, r'(r))u'^{\nu}(s).
\end{align*}
\]

(3)

Here \( M^\mu_{(g)\nu}(r', r(r')) \left( M^{-1}_{(g)} \right)_{\nu}^\mu (r, r'(r)) \) denote the Jacobian matrix and its inverse, both to be assumed of non-gradient type (see related definitions in Part 1). In particular, these can be represented respectively as the two real Jacobian matrices

\[
\begin{align*}
M^\mu_{(g)\nu}(r', r') & = \frac{\partial g^\mu_{(r')}}{\partial r'^\nu} + A^\mu_{(g)\nu} (r', [r', u']), \\
\left( M^{-1}_{(g)} \right)_{\nu}^\mu (r, r') & = \frac{\partial f^\mu_{(r)}}{\partial r'^\nu} + B^\mu_{(g)\nu} (r, [r, u]),
\end{align*}
\]

(4) \hspace{1cm} (5)

where the functions on the rhs of the previous equations are suitably defined (see Paper 2).

However, a basic issue that arises in GR, but which includes also classical and quantum mechanics as well as the theory of classical and quantum fields, is the role of non-local effects arising due to the choice of the extended GR-frames, namely of the state vectors

\[
\begin{align*}
x(s) & \equiv \{ r^\mu(s), u^\mu(s) \}, \\
x'(s) & \equiv \{ r'^\mu(s), u'^{\mu}(s) \},
\end{align*}
\]

(6) \hspace{1cm} (7)

which are defined at the same prescribed proper time \( s \) and for all \( s \in I \subseteq \mathbb{R} \). The physical motivations are based on the Einstein principle of equivalence, namely ultimately on the equivalence between accelerating reference frames and the occurrence of gravitational fields, in connection with intrinsically different space-times. For this purpose in this paper we intend to focus our attention on the role of acceleration effects in GR, arising specifically due to NLPT transformations between different space-times. In fact, the precise mathematical formulation and physical mechanisms by which non-locality should manifest itself between accelerating frames must still be fully understood.

**Goals of the paper**

Following the recent proposal of extending the class of reference frames on which GR relies, to include the treatment of a suitably-prescribed group \( \{ P_g \} \) of non-local point transformations, in this paper the problem is posed of analyzing the mathematical properties of the corresponding Jacobian matrix and the related tensor transformation laws. As a consequence, it is shown that in such a setting, namely for arbitrary NLPT of the group \( \{ P_g \} \), the relativistic 4-acceleration is covariant, i.e., is endowed with 4-vector transformation laws. More precisely, the goals and structure of the paper are as follows.

1. **GOAL #1** - The first one, discussed in Section 2, concerns the analysis of the differential properties of the Jacobian matrix \( M^\mu_{(g)\nu} \) associated with the general NLPT-group \( \{ P_g \} \) (see Lemmas 1 and 2).

2. **GOAL #2** - In Section 3 the transformation properties of the Christoffel symbols with respect of the same group are displayed (THM.1). The mapping existing between the Christoffel symbols corresponding to different curved space-times connected by a general NLPT is displayed. As a special example the particular case of special NLPT mapping the Minkowski space-time in a generic curved space-time is considered.

3. **GOAL #3** - In Section 4, the 4-vector transformation properties of the 4-acceleration with respect to the general NLPT-group \( \{ P_g \} \) are determined (THM.2). Remarkably the 4-acceleration are shown to transform with respect to general NLPT as 4-vectors.
4. GOAL #4 - In Section 5, as a first application of THM.2, the acceleration transformation equations are determined by means of NLTP having purely diagonal-Jacobian matrices which map Schwarzschild or Reissner-Nordström space-times either onto a flat Minkowski or Schwarzschild-analog space-times.

5. GOAL #5 - In Section 6, the application concerns the treatment of the Friedmann-Lemaître-Robertson-Walker (FLRW) curved space-time. This is found to be mapped onto the flat Minkowski space-time by means of NLPT having a non-diagonal Jacobian matrix.

6. GOAL #6 - In Section 7, the application is considered of the Kerr-Newman space-time, similarly mapped on Schwarzschild-analog space-times.

7. GOAL #7 - In Section 8, the application concerns the treatment of the Friedmann-Lemaître-Robertson-Walker (FLRW) curved space-time. This is found to be mapped onto the flat Minkowski space-time by means of NLTP having purely diagonal-Jacobian matrices which map Schwarzschild or Reissner-Nordström space-times either onto a flat Minkowski or Schwarzschild-analog space-times.

8. GOAL #8 - Finally, in Section 9 the main conclusions of the paper are drawn.

2 - MATHEMATICAL PRELIMINARIES: DIFFERENTIAL PROPERTIES OF \( M_{(g)\nu}^\mu \)

In this section the relevant properties of the Jacobian \( M_{(g)\nu}^\mu \) (see Eq.(4) and Eq.(5) for the corresponding inverse matrix) which is associated with a generic transformation of the group \( \{ P_g \} \) are summarized.

For the sake of reference, let us consider first the case in which the point transformations given by Eqs.(1) reduce to the customary form of local point transformations. This case occurs manifestly if the matrices \( A_{(g)\nu}^\mu \) and \( B_{(g)\nu}^\mu \) vanish identically so that the transformations reduce to the \( C^{(k)} \)-diffeomorphism (with \( k \geq 3 \))

\[
\begin{align*}
  r^\mu(s) &= g^\mu_A(r'(s)), \\
  r'^\mu(s) &= f^\mu_B(r(s)),
\end{align*}
\]  

(8) while the corresponding Jacobian \( M_{(g)\nu}^\mu \) becomes a local \( C^{(k-1)} \)-function of the form \( M_{(g)\nu}^\mu = M_{(g)\nu}^\mu(r') \). Hence, the differential of \( M_{(g)\nu}^\mu \) takes the form prescribed by the (Leibnitz) chain rule of differentiation. The following proposition holds.

**LEMMA 1 - Differential identity for LPT**

Given validity of Eqs.(5) and (6) the Jacobian \( M_{(g)\nu}^\mu = M_{(g)\nu}^\mu(r') \) is identified with the \( C^{(k-1)} \)-function

\[
M_{(g)\nu}^\mu(r') = \frac{\partial g^\mu_A(r')}{\partial r'^\nu}
\]  

(10) so that the differential of the Jacobian \( M_{(g)\nu}^\mu(r') \) reads

\[
dM_{(g)\nu}^\mu(r') = dr'^\mu \frac{\partial M_{(g)\nu}^\mu(r')}{\partial r'^\mu},
\]

(11) where on the rhs \( \frac{\partial M_{(g)\nu}^\mu(r', r'(s))}{\partial r'^\mu} \) denotes the partial derivative with respect to \( r'^\mu \).

Next, let us consider the case of an arbitrary NLPT, for which the Jacobian \( M_{(g)\nu}^\mu \) is instead of the form \( M_{(g)\nu}^\mu(r', r) \) (see Eqs.(8)), where \( r \equiv r(r') \equiv \{ r^\mu(r') \} \) and the implicit (and non-local) dependence in terms of \( r' \equiv \{ r'^\mu \} \) occurring via \( r \equiv \{ r^\mu \} \) is considered as prescribed via the NLPT. As an example, in the case of a special NLPT (see Paper 1) it follows that

\[
M_{(g)\nu}^\mu(r', r) \equiv M_{(g)\nu}^\mu(r', r' + \Delta r'(s)),
\]

(12) where \( \Delta r'^\mu(s) \equiv \Delta r'^\mu \) takes the form

\[
\Delta r'^\mu \equiv \int_{r'^\mu}^{r'^\mu} dr'^\nu A_{(g)\nu}^\mu(r', r' + \Delta r').
\]

(13)
As a consequence, invoking in particular Eq. (2), one obtains respectively:

\[
\frac{\partial r^\beta}{\partial r^\alpha} = \frac{\partial}{\partial r^\alpha} \left[ r^\beta + \Delta r^\beta(s) \right] = \delta^\beta_\alpha + A^\beta_\alpha(r', r' + \Delta r'), \quad (14)
\]

\[
\frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\beta} \Bigg|_{r'} = \frac{\partial r^\beta}{\partial r^\alpha} \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\alpha} \Bigg|_{r'} = \left( M^{-1} \right)^\beta_\alpha \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\beta} \Bigg|_{r'}, \quad (15)
\]

where we have denoted symbolically \( r(r') \equiv r' + \Delta r' \). As a consequence, the following proposition, analogous to that warranted by Lemma 1 in the case of LPTs, holds.

**LEMMA 2 - Differential identity for NLPT**

Given validity of THM.1 in Part 2 the differential of the Jacobian \( M_{(g)\nu}^\mu(r', r) = M_{(g)\nu}^\mu(r', r(r')) \) reads

\[
dM_{(g)\nu}^\mu(r', r(r')) = dr^\alpha \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\alpha}, \quad (16)
\]

where on the rhs \( \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\alpha} \) denotes the "total" partial derivative with respect to \( r^\alpha \), namely defined such that the differential \( dM_{(g)\nu}^\mu(r', r(r')) \) is written explicitly as

\[
dM_{(g)\nu}^\mu(r', r) \equiv dr^\alpha \left[ \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\alpha} \bigg|_{r(r')} + \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\alpha} \bigg|_{r'} \right], \quad (17)
\]

where the partial derivatives on the rhs of the previous equation are performed respectively at constant \( r(r') \) the first one and at constant \( r' \) the other one.

**Proof** - In fact, thanks to Eq. (15), the partial derivative of the Jacobian \( M_{(g)\nu}^\mu(r', r) \) with respect to \( r^\alpha \) becomes

\[
\frac{\partial}{\partial r^\alpha} M_{(g)\nu}^\mu(r', r) = \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\alpha} \bigg|_r + \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\beta} \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\beta} \bigg|_r, \quad (18)
\]

while its differential is just

\[
dM_{(g)\nu}^\mu(r', r) = dr^\alpha \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\alpha} \bigg|_r + dr^\beta \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\beta} \bigg|_r,
\]

\[
= dr^\alpha \left[ \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\alpha} \bigg|_r + M_{(g)\nu}^\mu(r', r) \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\beta} \bigg|_r \right]. \quad (19)
\]

Hence due to Eq. (15) it follows that

\[
dM_{(g)\nu}^\mu(r', r) = dr^\alpha \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\alpha} \bigg|_r + dr^\beta \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\beta} \bigg|_r
\]

\[
= dr^\alpha \left[ \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\alpha} \bigg|_r + M_{(g)\nu}^\mu(r', r) \left( M^{-1} \right)^\beta_\alpha \frac{\partial M_{(g)\nu}^\mu(r', r(r'))}{\partial r^\beta} \bigg|_r \right], \quad (20)
\]

which manifestly implies Eq. (17). The rhs of the same equation coincides then with the rhs of Eq. (16) so that the thesis is proved. Q.E.D.

Let us now inspect further mathematical implications which are implied in the construction of the NLPTs.
3 - NLPT-TRANSFORMATION PROPERTIES OF THE CHRISTOFFEL SYMBOLS

In this section we intend to determine the transformations properties of the Christoffel symbols with respect to the general-NLPT-group \( \{P_g\} \). For definiteness, let us denote as \( \Gamma^\nu_{\alpha\beta} \) and \( \Gamma'^\nu_{\alpha\beta} \) the (initial and transformed) Christoffel symbols when referred to the two GR-reference frames \( r^\mu \) ("initial frame") and \( r'^\mu \) ("transformed frame") respectively. The issue is the determination of the relationship between \( \Gamma^\nu_{\alpha\beta} \) and \( \Gamma'^\nu_{\alpha\beta} \) when the coordinate transformation relating the coordinates \( r^\mu \) and \( r'^\mu \) is suitably prescribed.

As we intend to show here, the solution of such a problem is closely related to the requirement, already included in the prescription of the group of NLPT \( \{P_g\} \), that the (initial and transformed) metric tensors \( g^{ij}(r) \) and \( g'^{ij}(r') \) defined respectively for the two Riemannian manifolds \( \{Q^4, g\} \) and \( \{Q'^4, g'\} \) are extremal, namely they satisfy identically the extremal conditions

\[
\nabla_k g^{ij}(r) = 0, \\
\nabla'_k g'^{ij}(r') = 0. 
\]

Here \( \nabla \) and \( \nabla' \) denote as usual the covariant derivatives, defined as

\[
\nabla_k g^{ij}(r) = \frac{\partial g^{ij}(r)}{\partial r^k} + \Gamma^j_{kl}(r) g^{ij}(r), \\
\nabla'_k g'^{ij}(r') = \frac{\partial g'^{ij}(r')}{\partial r'^k} + \Gamma'^j_{kl}(r') g'^{ij}(r'),
\]

where \( \Gamma^j_{kl} \) and \( \Gamma'^j_{kl} \) denote the initial and transformed Christoffel symbols defined on \( \{Q^4, g\} \) and \( \{Q'^4, g'\} \) respectively.

We notice that Eqs. (21) and (22) are manifestly equivalent to the ODE’s

\[
\frac{D}{Ds} g^{ij}(r(s)) = 0, \\
\frac{D'}{Ds} g'^{ij}(r'(s)) = 0,
\]

once \( \frac{D}{Ds} \) and \( \frac{D'}{Ds} \) are identified with the covariant derivatives of \( g^{ij}(r(s)) \) and \( g'^{ij}(r'(s)) \). These are defined respectively in \( Q^4 \) and \( Q'^4 \) in terms of the differential operators acting on the covariants components \( g^{ij}(r(s)) \) and \( g'^{ij}(r'(s)) \) as

\[
\frac{D}{Ds} g^{ij}(r(s)) = u^k \nabla_k g^{ij}(r), \\
\frac{D'}{Ds} g'^{ij}(r'(s)) = u'^k \nabla'_k g'^{ij}(r'),
\]

with \( u^k \) and \( u'^k = (M^{-1})^k_j u' \) denoting the 4—velocities in the corresponding tangent spaces. Then the following proposition holds.

**THM.1 - NLPT-transformation laws for the Christoffel symbols**

Within the group \( \{P_g\} \) the following two propositions hold:

L1) The extremal conditions (21) and (22) holding for the metric tensors \( g^{ij}(r) \) and \( g'^{ij}(r') \) are equivalent to require that the initial and transformed Christoffel symbols \( \Gamma^k_{\gamma\nu} \) and \( \Gamma'^k_{\gamma\nu} \) defined respectively on \( \{Q^4, g\} \) and \( \{Q'^4, g'\} \) satisfy the constraint differential equation

\[
dr^s \frac{\partial}{\partial r^s} M^\mu_{(g)\nu} (r', r) + \Gamma^\nu_{\alpha\beta} M^\beta_{(g)\nu} (r', r) M^\alpha_{(g)s} (r', r) dr^s = M^\mu_{(g)\gamma} (r', r) \Gamma'^\gamma_{\nu s} dr^s. \tag{29}
\]

L2) The previous equation in turn is equivalent to the equation

\[
\Gamma'^k_{\gamma\nu} = (M^{-1})^k_\mu (r', r) \frac{\partial M^\mu_{(g)\nu} (r', r)}{\partial r^s} + (M^{-1})^k_\mu (r', r) \Gamma^\nu_{\alpha\beta} M^\alpha_{(g)\nu} (r', r) M^\beta_{(g)s} (r', r), \tag{30}
\]

which determines the transformation laws for the transformed Christoffel symbol \( \Gamma'^k_{\gamma\nu} \).

**Proof -** We first prove proposition L2). Notice for this purpose that Eq. (29), due to the arbitrariness of the differential displacement \( dr^s \), implies also that

\[
\frac{\partial}{\partial r^s} M^\mu_{(g)\nu} (r', r) + \Gamma^\nu_{\alpha\beta} M^\alpha_{(g)\nu} (r', r) M^\beta_{(g)s} (r', r) = M^\mu_{(g)\gamma} (r', r) \Gamma'^\gamma_{\nu s}, \tag{31}
\]
which, after multiplying it term by term by \((M^{-1})^k_{\mu}(r,r')\), exchanging the indexes \(s \leftrightarrow \gamma\) and recalling that 
\[M^\mu_{(g)^s}(r,r') (M^{-1})^k_{\mu}(r,r') = \delta^k,\] 
reduces to Eq. (36).

Next we address the proof of proposition L1, i.e., that Eq. (21) is equivalent to Eq. (30). To start with, let us consider the definition of the covariant derivative recalled above [see Eq. (23)]. Then, Eq. (21) delivers necessarily:

\[\nabla_k g^{ij}(r) = \frac{\partial g^{ij}(r)}{\partial r^k} + \Gamma^i_{kl} g^{lj}(r) + \Gamma^j_{kl} g^{ik}(r) = 0.\]  

Invoking now the identity

\[\frac{\partial}{\partial r^k} g^{\alpha\beta}(r') = \left(M^{-1}\right)^s_{k} (r,r') \frac{\partial}{\partial r^s} g^{\alpha\beta}(r'),\]

thanks to the chain rule, it follows that the first term on the rhs of Eq. (32) becomes

\[\frac{\partial}{\partial r^k} \left[ M^i_{(g)^{\alpha}} (r', r) M^j_{(g)^{\beta}} (r', r) g^{\alpha\beta}(r') \right] = g^{\alpha\beta}(r') \frac{\partial}{\partial r^k} \left[ M^i_{(g)^{\alpha}} (r, r) M^j_{(g)^{\beta}} (r, r) \right] + M^i_{(g)^{\alpha}} (r', r) M^j_{(g)^{\beta}} (r', r) \left(M^{-1}\right)^s_{k} (r,r') \frac{\partial}{\partial r^s} g^{\alpha\beta}(r').\]  

Therefore, noting that thanks to Eqs. (24) it must be

\[\frac{\partial}{\partial r^k} g^{\alpha\beta}(r') = \nabla_s g^{\alpha\beta}(r') - \Gamma^s_{\sigma\beta} g^{\rho\beta}(r') + \Gamma^s_{\sigma\alpha} g^{\rho\alpha}(r'),\]

it follows that Eq. (32) requires necessarily the validity of the following constraint equation, obtained also upon exchanging summations indexes, namely

\[g^{\alpha\beta}(r') M^i_{(g)^{\alpha}} (r', r) \frac{\partial}{\partial r^k} M^j_{(g)^{\beta}} (r', r) + g^{\alpha\beta}(r') M^j_{(g)^{\beta}} (r', r) \frac{\partial}{\partial r^k} M^i_{(g)^{\alpha}} (r', r) - M^i_{(g)^{q}} (r', r) M^j_{(g)^{\beta}} (r', r) \left(M^{-1}\right)^s_{k} (r,r') \Gamma^q_{\alpha s} g^{\alpha\beta}(r') + M^i_{(g)^{q}} (r', r) M^j_{(g)^{p}} (r', r) \left(M^{-1}\right)^s_{k} (r,r') \Gamma^p_{\beta s} g^{\alpha\beta}(r') + \Gamma^i_{kl} M^i_{(g)^{\alpha}} (r', r) M^j_{(g)^{\beta}} (r', r) g^{\alpha\beta}(r') + \Gamma^j_{kl} M^i_{(g)^{\alpha}} (r', r) M^j_{(g)^{\beta}} (r', r) g^{\alpha\beta}(r') = 0.\]  

Considering now \(g^{\alpha\beta}(r')\) as independent of the Jacobian matrix of the transformation, then thanks to the symmetry of the indexes \(\alpha\) and \(\beta\) the previous equation delivers

\[M^j_{(g)^{\beta}} (r', r) \left[ \frac{\partial}{\partial r^k} M^i_{(g)^{\alpha}} (r', r) - M^i_{(g)^{q}} (r', r) \left(M^{-1}\right)^s_{k} (r,r') \Gamma^q_{\alpha s} \right] + \Gamma^i_{kl} M^i_{(g)^{\alpha}} (r', r) M^j_{(g)^{\beta}} (r', r) = 0.\]  

Namely, multiplying term by term by \((M^{-1})^s_{j} (r,r')\)

\[\delta^s_{\beta} \left[ \frac{\partial}{\partial r^k} M^i_{(g)^{\alpha}} (r', r) - M^i_{(g)^{q}} (r', r) \left(M^{-1}\right)^s_{k} (r,r') \Gamma^q_{\alpha s} \right] + \Gamma^i_{kl} M^i_{(g)^{\alpha}} (r', r) M^j_{(g)^{\beta}} (r', r) = 0,\]

which yields

\[\delta^s_{\beta} \left[ \frac{\partial}{\partial r^k} M^i_{(g)^{\alpha}} (r', r) - M^i_{(g)^{q}} (r', r) \left(M^{-1}\right)^s_{k} (r,r') \Gamma^q_{\alpha s} + \Gamma^i_{kl} M^i_{(g)^{\alpha}} (r', r) \right] = 0.\]
Therefore, one has that
\[ \frac{\partial}{\partial r \partial r'} M^{(g)\alpha}_{\mu\nu}(r', r) - M^{(g)}_{(q)\mu\nu}(r', r) \left( M^{-1}_{(g)} \right)_{k}^{r}(r, r') \Gamma^{r}_{\alpha \gamma} + \Gamma_{k \mu}^{i} M^{(g)\alpha}_{i}(r', r) = 0, \]  
implying also
\[ \left( M^{-1}_{(g)} \right)_{k}^{r}(r, r') \frac{\partial}{\partial r \partial r'} M^{(g)\alpha}_{(q)\mu\nu}(r', r) - M^{(g)\alpha}_{(q)\mu\nu}(r', r) \left( M^{-1}_{(g)} \right)_{k}^{r}(r, r') \Gamma^{r}_{\alpha \gamma} + \Gamma_{k \mu}^{i} M^{(g)\alpha}_{i}(r', r) = 0. \]  
Hence it follows
\[ M^{(g)\alpha}_{\mu\nu}(r', r) \left( M^{-1}_{(g)} \right)_{k}^{r}(r, r') \frac{\partial}{\partial r \partial r'} M^{(g)\alpha}_{(q)\mu\nu}(r', r) - M^{(g)\alpha}_{(q)\mu\nu}(r', r) M^{(g)\alpha}_{\mu\nu}(r', r) \left( M^{-1}_{(g)} \right)_{k}^{r}(r, r') \Gamma^{r}_{\alpha \gamma} + \Gamma_{k \mu}^{i} M^{(g)\alpha}_{i}(r', r) = 0, \]  
so that
\[ \delta^{r}_{r} \frac{\partial}{\partial r \partial r'} M^{(g)\alpha}_{i}(r', r) - M^{(g)\alpha}_{(q)\mu\nu}(r', r) \delta^{r}_{r} \Gamma^{r}_{\alpha \gamma} + \Gamma_{k \mu}^{i} M^{(g)\alpha}_{i}(r', r) = 0. \]  
Finally, replacing the index \( x \) with \( k \) one gets
\[ \frac{\partial}{\partial r \partial r} M^{(g)\alpha}_{\mu\nu}(r', r) - M^{(g)\alpha}_{(q)\mu\nu}(r', r) \Gamma^{r}_{\alpha \gamma} + \Gamma_{k \mu}^{i} M^{(g)\alpha}_{i}(r', r) \Gamma^{r}_{\nu \gamma} \equiv 0. \]  
Straightforward algebra shows that this equation coincides with Eq. (31) and hence Eq. (29) too. Finally, one can show that in a similar way Eq. (30) implies Eq. (29) too. In view of the equivalence between Eqs. (30), (29) and (30) the thesis is reached. Q.E.D.

The implication of THM.1 is therefore that the requirements that both the initial and transformed metric tensors are extremal, i.e., in the sense that the corresponding covariant derivatives vanish identically in both cases (see Eqs. (26) and (27)), is necessarily equivalent to impose between the initial and transformed Christoffel symbols - i.e., \( \Gamma^{\nu}_{\mu \gamma} \) and \( \Gamma'^{\nu}_{\mu \gamma} \) which are defined respectively on \( \{ Q^4, g \} \) and \( \{ Q'^4, g' \} \) - the transformation law (30). The conclusion, as shown in the next section, is important to establish the tensor transformation laws which hold for the 4-acceleration for arbitrary NLPTs belonging to the group \( \{ P_4 \} \).

As a final comment, we remark that if the space-time \( \{ Q'^4, g' \} \) is identified with the flat Lorentzian Minkowski space-time \( \{ M^4, \eta \} \) then the following proposition holds.

**COROLLARY TO THM.1 - Case of Minkowski space-time**

*Within the group \( \{ P_4 \} \) if the space-time \( \{ Q'^4, g' \} \) coincides with the flat Lorentzian Minkowski space-time \( \{ M^4, \eta \} \) expressed in orthogonal Cartesian coordinates, then Eq. (31) reduces to
\[ 0 = \frac{\partial M^{(g)\alpha}_{\mu\nu}(r', r)}{\partial r \partial r} + \Gamma^{\mu}_{\alpha \beta} M^{(g)\alpha}_{\nu \gamma}(r', r) M^{(g)\gamma}_{\nu \beta}(r', r), \]  
which provides a representation for the Christoffel symbol \( \Gamma^{\mu}_{\alpha \beta} \).

*Proof* - Assume in fact that the space-time \( \{ Q'^4, g' \} \) coincides with the flat Minkowski space-time \( \{ M, \eta \} \). Then by construction in Eq. (30) the transformed Christoffel symbols \( \Gamma'^{\mu}_{\alpha \beta} \) in such a space-time necessarily vanish identically. Then the same equation reduces to Eq. (35). Q.E.D.

**4 - NLPT-TRANSFORMATION PROPERTIES OF THE 4-ACCELERATION**

THM.1 and in particular Eq. (30) can be used to determine also the relationships holding between the 4-accelerations defined in the two Riemannian manifolds \( \{ Q^4, g \} \) and \( \{ Q'^4, g' \} \) respectively. In fact, let us identify the 4-accelerations with the covariant derivatives of \( u^{\nu} \) and \( u'^{\nu} \) defined in \( Q^4 \) and \( Q'^4 \) as
\[ a^{\mu} \equiv \frac{D}{Ds} u^{\mu}, \]
\[ a'^{\mu} \equiv \frac{D'}{Ds} u'^{\mu}, \]
and where $\frac{D}{Ds}$ and $\frac{D'}{Ds}$ are identified with the ordinary differential operators (24) and (28). This means that in the two space-times they must be identified respectively as

$$\frac{D}{Ds} u^\nu = \frac{d}{ds} u^\nu + u^\alpha u^\beta \Gamma^\nu_{\alpha\beta}, \quad (48)$$

$$\frac{D'}{Ds} u^\nu = \frac{d}{ds} u^\nu + u'^\alpha u'^\beta \Gamma'^\nu_{\alpha\beta}, \quad (49)$$

where $\Gamma^\nu_{\alpha\beta}$ and $\Gamma'^\nu_{\alpha\beta}$ denote the corresponding standard connections defined in the same space-times. Let us consider for definiteness Eq. (48), the other one being uniquely dependent from it (as will be obvious from the subsequent considerations). Invoking the tensor transformation laws for the 4-velocity (3) then Eq. (48) implies that

$$\frac{d}{ds} u^\mu = M'^{\mu}_{\nu}(r', r) \frac{D'}{Ds} u^\nu, \quad (50)$$

Then it is immediate to prove that, thanks to the validity of THM.1, the 4-accelerations $\frac{D}{Ds} u^\nu$ and $\frac{D'}{Ds} u'^\nu$ are linearly related. In particular the following result holds.

**THM.2 - NLPT-transformation law for the 4-acceleration**

If $M'^{\mu}_{\nu}$ is the Jacobian of an arbitrary NLPT, defined according to Eq. (4), and $\frac{D}{Ds} u^\nu$ and $\frac{D'}{Ds} u'^\nu$ are the 4-accelerations defined according to Eqs. (48) and (49), then with respect to an arbitrary NLPT of the group $\{P_g\}$ it follows that they are related by means of the tensor transformation laws

$$\frac{D}{Ds} u^\mu = M^{\mu}_{(g)\nu}(r', r) \frac{D'}{Ds} u'^\nu, \quad (51)$$

$$\frac{D'}{Ds} u'^\mu = (M^{-1}_{(g)})^\mu_{\nu}(r, r') \frac{D}{Ds} u^\nu. \quad (52)$$

The result is analogous to that holding for arbitrary LPTs belonging to the group $\{P_g\}$.

**Proof** - First it is obvious that Eqs. (51) and (52) mutually imply each other so that it is sufficient to prove that one of the two actually holds. Consider then the proof of Eq. (51). First, let us invoke the transformation law for the 4-velocity (3) and invoke Eq. (48) to give

$$\frac{D}{Ds} u^\mu = \frac{D}{Ds} \left[ M^{\mu}_{(g)\nu}(r', r) u'^\nu \right] = \frac{d}{ds} \left[ M^{\mu}_{(g)\nu}(r', r) u'^\nu \right] + u^h u^k M^{\alpha}_{(g)\nu}(r', r) M^{\beta}_{(g)\gamma}(r', r) \Gamma^\nu_{\alpha\beta}, \quad (53)$$

where the chain rule delivers

$$\frac{d}{ds} \left[ M^{\mu}_{(g)\nu}(r', r) u'^\nu \right] = M^{\mu}_{(g)\nu}(r', r) \frac{d}{ds} u'^\nu + u'^\nu \frac{d}{ds} \left[ M^{\mu}_{(g)\nu}(r', r) \right].$$

Its is immediate to show that Lemma and Eq. (49) of THM.1 then imply identically the identity

$$dM^{\mu}_{(g)\nu}(r', r) + \Gamma'^{\alpha}_{\nu\beta} M^{\alpha}_{(g)\nu}(r', r) M^{\beta}_{(g)\gamma}(r', r) dr^\gamma = M^{\mu}_{(g)\nu}(r', r) \Gamma'^{\gamma}_{\nu\beta} dr^\gamma. \quad (54)$$

Hence the thesis is proved. Incidentally, thanks to Lemma 1, it is obvious that the same conclusion holds in the case of arbitrary LPTs belonging to group $\{P\}$. Q.E.D.

**5 - APPLICATION #1: NLPT-ACCELERATION EFFECTS IN SCHWARZSCHILD, REISSNER-NORDSTROM AND SCHWARZSCHILD-ANALOG SPACE-TIMES**

The first application to be considered concerns the construction of a NLPT mapping two connected and time-oriented space-times $(Q^1, g)$ and $(Q^2, g')$ both having diagonal form with respect to suitable sets of coordinates. More precisely we shall require that:
Indeed, from Eqs. (56) one finds Schwartzchild-analog space-times. In pseudo-spherical coordinates *** (A) Schwarzschild and (B) Reissner-Nordström space-times to be mapped onto either the (C) Minkowski or (D) M \text{transformations.}

In validity of Eqs. (55) the tensor transformation equation for the metric tensor (see Eq. (24) in Part 2) take the general form:

\[
\begin{align*}
S_\alpha(r) &= \begin{pmatrix} M_{(g)}^{-1} \end{pmatrix}_{\mu}^{\alpha}(r, r') \left( M_{(g)}^{-1} \right)_{\mu}^{\lambda}(r, r') S_\lambda(r'), \\
S_\alpha'(r') &= M_{(g)}^\alpha_{\mu}(r', r) M_{(g)\mu}^\alpha(r', r) S_\alpha(r),
\end{align*}
\]

where manifestly \( M_{(g)}^\alpha_{\mu}(r', r) \equiv M_{\mu}^\alpha(r', r) \) and \( \left( M_{(g)}^{-1} \right)_{\mu}^{\alpha}(r, r') \equiv M_{\mu}^{-1} \) \( (r, r') \) as corresponds to the case of a special NLTP. For such a type of space-times in the following we intend to display a number of explicit particular solutions of Eqs. (56) for the Jacobian \( M_{\mu}^\alpha \) and its inverse \( (M^{-1})_{\mu}^\alpha \) and to construct also the corresponding NLTP-phase-space transformations.

In the case in which \((Q^4, g)\) and \((Q^4, g')\) have the same signatures a particular solution of Eqs. (56) in the accessible subsets of \((Q^4, g)\) and \((Q^4, g')\) is provided, as shown in Part 2, by a diagonal Jacobian matrix, i.e., of the form

\[
M_{\mu}^\alpha(r', r) = M_{\mu}^\alpha(r', r) \delta^\alpha_{\mu} = \left[ \delta^\alpha_{\mu} + A_{\mu}^\alpha(r', r) \right] \delta^\alpha_{\mu},
\]

which corresponds to a diagonal NLPT of the form

\[
\begin{align*}
\left\{ \begin{array}{l}
\text{dr}^i = F_{(g)}(r')dr'^{(i)} \\
i = 0, 1, 2, 3,
\end{array} \right.
\end{align*}
\]

Indeed, from Eqs. (56) one finds

\[
M_{(g)\mu}^\alpha(r', r) = \frac{1}{\left( M_{(g)}^{-1} \right)_{\mu}^{\alpha}} = \frac{S_\alpha'(r)}{S_\alpha(r)},
\]

where \( \frac{S_\alpha'(r)}{S_\alpha(r)} > 0 \) in the accessible subsets. In terms of Eqs. (51) and (52) one obtains the acceleration transformation laws

\[
\begin{align*}
\frac{D'}{D_s} u^\mu &= M_{(g)\mu}^\alpha(r', r) \frac{D'}{D_s} u'^{\alpha} \\
\frac{D'}{D_s} u'^{\mu} &= \left( M_{(g)}^{-1} \right)^{\frac{\mu}{\alpha}} \frac{D'}{D_s} u^\alpha,
\end{align*}
\]

Let us now consider a possible physical realizations for the space-times \((Q^4, g)\) and \((Q^4, g')\) and the corresponding metric tensors \(g_{\mu\nu}(r)\) and \(g'_{\mu\nu}(r')\) respectively. Here we consider the examples pointed out in Part 2, namely concerning (A) Schwarzschild and (B) Reissner-Nordström space-times to be mapped onto either the (C) Minkowski or (D) Schwartzchild-analog space-times. In pseudo-spherical coordinates *** \((r, r^2, r^3)\) (see Part 1) the following generic representation is assumed to hold for all of them of the form \(g_{\mu\nu}(r) \equiv diag((S_0(r), -S_1(r), -S_2(r), -S_3(r)))\),

\[
\begin{align*}
S_0(r) &= a(r), \\
S_1(r) &= b(r), \\
S_2(r) &= g(r), \\
S_3(r) &= g(r).
\end{align*}
\]
In particular, the Schwarzschild, Reissner-Nordström and Schwartzchild-analog cases are obtained letting:

\[
\begin{cases}
a(r) = f(r) \\
b(r) = \frac{1}{f(r)} \\
g(r) = 1
\end{cases}
\]  

(62)

where respectively

\[
\begin{cases}
f(r) = \left( 1 - \frac{r_s}{r} \right) \quad \text{case A} \\
f(r) = \left( 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} \right) \quad \text{case B} \\
f(r) = \prod_{i=1,n} \left( 1 - \frac{r_i}{r} \right) \quad \text{case C}
\end{cases}
\]

(63)

Here, \( r_s, r_Q \) and \( r_i \) (for \( i = 1, n \)) are suitably-prescribed characteristic scale lengths, in particular

\[r_s = \frac{2GM}{c^2},\]

(64)

\[r_Q = \sqrt{\frac{Q^2G}{4\pi\varepsilon_0 c^4}},\]

(65)

are respectively the Schwarzschild and Reissner-Nordström radii, with \( Q \) being the electric charge and \( 1/4\pi\varepsilon_0 \) the Coulomb coupling constant. In all cases A, B and C we shall require that the function \( f(r) \) defined according to Eq.(63) is strictly positive, i.e., \( r > r_s \) and \( r > r_n \), with \( r_n \) denoting the largest root of the equation \( f(r) = \left( 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} \right) = 0 \) or \( \prod_{i=1,n} \left( 1 - \frac{r_i}{r} \right) = 0 \). In all cases, the transformed space-time \((Q'^4, g')\) when expressed in the same pseudo-spherical coordinates is identified either with the Minkowski space-time or with the Schwartzchild-analog space-time, so that respectively either for all \( \mu = 0, 3 \):

\[S'_\mu(r') = 1,\]

(66)

or where (61), (62) and case C of Eq. (63) applies. In the subsets where \( f(r) > 0 \) and for case C \( f'(r') > 0 \) the transformation matrix \( M^\mu_\nu(r', r) \) becomes:

\[
\begin{cases}
M^0_0(r', r) = \sqrt{\frac{a'(r')}{a(r)}} \\
M^1_1(r', r) = \sqrt{\frac{b'(r')}{b(r)}} \\
M^2_2(r', r) = M^3_3(r', r) = 1
\end{cases}
\]

(67)

where in the first terms on the rhs of the previous equations the positive values of the square roots have been taken. Let us briefly analyze the physical implications of Eqs.(67).

- The first is that Eqs.(67) generate a diagonal special NLPT in which non-local effects are carried only by the time and radial components of the 4-displacement, i.e., of 4-velocity and correspondingly of the 4-acceleration.

- The corresponding NLPT which map respectively either the Schwarzschild (A) or the Reissner-Nordström (B) space-times onto the Minkowski (C) or Schwartzchild-analog (D) space-times are provided in all cases by Eqs.(58). In particular, the acceleration transformation (60) implies that a point particle endowed with a 4-acceleration \( \frac{D}{Dt}u^\mu \) with respect to the space-time C or D, in the space time A or B mapped via Eqs.(58) is necessarily endowed with an acceleration \( \frac{D}{Dt}u^\mu \) given by the same equations (i.e., Eqs.(60)).

6 - APPLICATION #2: NLPT-ACCELERATION EFFECTS IN THE FLRW SPACE-TIME

Let us now consider the case of the Frieman-Lemaitre-Robertson-Walker (FLRW) space-time, which incidentally is again of the type (61). Indeed this is obtained identifying respectively

\[
\begin{cases}
a(r) = 1 \\
b(r) = b^2_0(t) \\
g(r) = b^2_0(t) \left( \frac{7}{t} \right)^2
\end{cases}
\]

(68)
where \( b_0(t) > 0 \) is a smooth function of the coordinate time, \( r \) is the radial-like coordinate and \( \tau \) is prescribed respectively as

\[
\tau = \begin{cases} 
R_C \sinh (r/R_C) & \text{for negative curvature } R_C \\
r & \text{for vanishing curvature } R_C \\
R_C \sin (r/R_C) & \text{for positive curvature } R_C 
\end{cases}
\]

with \( R_C \) being the curvature associated with the isotropic Ricci tensor. Let us pose also in this case the problem of the construction of a mapping onto the Minkowski (C) or the Schwartzchild-analog (D) space times. In difference with Application \#1 however, here we consider the case of a non-diagonal special NLPT of the form:

\[
\begin{align*}
\{ d\tau^0 & = M_0^0 dr^0 + M_1^0 dr^1 \\
\quad & \quad
\} \\
\quad & \quad
\{ dr^1 & = M_0^1 dr^0 + M_1^1 dr^1 \\
\quad & \quad
\} \\
\quad & \quad
\{ dr^i & = M_0^i dr^0 + M_1^i dr^1, \\
\quad & \quad
\} \\
\quad & \quad
\{ (i = 2, 3) \}
\end{align*}
\]

with the corresponding inverse transformations given by

\[
\begin{align*}
\{ dr^0 & = \frac{dr^0}{M_0^0} - \frac{M_0^0 M_1^0 dr^1 - M_1^0 dr^0}{M_0^1 M_1^0 dr^0 + M_0^0 M_1^0 dr^1} \\
\quad & \quad \\
\quad & \quad \\
\quad & \quad
\{ dr^1 & = \frac{dr^1}{M_1^0} - \frac{M_0^0 M_1^0 dr^1 - M_1^0 dr^0}{M_0^1 M_1^0 dr^0 + M_0^0 M_1^0 dr^1} \\
\quad & \quad
\} \\
\quad & \quad
\{ dr^i & = \frac{dr^i}{M_0^i} - \frac{M_0^0 M_1^0 dr^1 - M_1^0 dr^0}{M_0^1 M_1^0 dr^0 + M_0^0 M_1^0 dr^1} \\
\quad & \quad
\} \\
\quad & \quad
\{ (i = 2, 3) \}
\end{align*}
\]

Then, in validity of the constraint

\[ S_0^0 (r') M_0^0 M_0^0 = S_1^1 (r') M_1^1 M_1^1, \]

the tensor transformation for the metric tensor yields for case D the roots

\[
\begin{align*}
M_0^0 & = \sqrt{f'(r') + b_0^2(t) (M_0^1)^2} \\
M_1^1 & = \left( \frac{1}{b_0(t)} \right) \sqrt{f'(r') + b_0^2(t) (M_0^1)^2} \\
M_0^0 & = \frac{b_0(t) M_0^0}{b_0(t) M_0^0} \\
M_1^1 & = M_0^0 \left( \frac{D'}{D_s} M_0^0 + D' \right)
\end{align*}
\]

with \( M_0^1 \) to be still suitably prescribed. The corresponding solutions for case C (Minkowski space-time) are obtained letting \( f'(r') = 1 \) in the previous equations. Let us briefly analyze the physical implications of Eqs. (73):

1) First, we stress that the matrix element \( M_0^1 \) is still undetermined both in magnitude and sign.

2) In this case the time-components of both the 4-velocity and the 4-acceleration in the FLRW space-time are generated by two "components" acting in the Minkowski (or Schwartzchild-analog) space-time, respectively the time and radial components of the 4-velocity and 4-acceleration. In particular one obtains that

\[
\frac{D}{D_s} u^0 = \sqrt{f'(r') + b_0^2(t) (M_0^1)^2} \frac{D'}{D_s} u^0 + \frac{b_0(t)}{f'(r')} M_0^0 \frac{D'}{D_s} u^1.
\]

3) We remark in particular that the second term on the rhs depends in turn linearly in terms of the arbitrary quantity \( M_0^1 \) as well as the time-dependent factor \( b_0(t) \). Depending also on the behavior of the function \( b_0(t) \) such a contribution may therefore give rise in principle both to an increase or a decrease in time of the same components.

4) Finally, the radial component of the 4-acceleration in the FLRW space-time is generated again by two components in the mapped space-times C or D, since

\[
\frac{D}{D_s} u^1 = M_0^1 \frac{D'}{D_s} u^0 + \frac{1}{b_0(t)} \sqrt{f'(r') + b_0^2(t) (M_0^1)^2} \frac{D'}{D_s} u^1.
\]

Notice that the radial component of the acceleration in the Minkowski or the Schwartzchild-analog space-time (see, i.e., the 2nd term on the rhs of the previous equation), gives rise to a corresponding time-dependent acceleration in the FLRW space-time. The time component contribution on the rhs is instead proportional to the matrix element \( M_0^1 \). As a consequence also this contribution can in principle significantly affect the radial acceleration.
7 - APPLICATION #3: NLPT-ACCELERATION EFFECTS IN KERR-NEWMAN AND KERR SPACE-TIMES

As a further example, let us consider the case of Kerr-Newman and Kerr space-times, identified here for definiteness with the primed space-time \((Q'^4, g'(r'))\). In both cases, when cast in spherical coordinates \((r', \theta', \varphi')\) the corresponding metric tensor is of the generic non-diagonal form

\[
g'_{\mu\nu}(r') = \begin{pmatrix}
S_0(r') & S_0'(r') & S_{03}(r') \\
0 & S_1(r') & S_{23}(r') \\
0 & 0 & S_3(r')
\end{pmatrix}.
\] (76)

For definiteness, let us first introduce the standard notations

\[
\begin{align*}
\alpha &= \frac{J}{\mathcal{M}}, \\
\rho^2 &= r^2 + \alpha^2 \cos^2 \theta, \\
\Delta &= r^2 - \mathcal{M} r + \alpha^2 + \frac{\mathcal{Q}^2}{\rho^2},
\end{align*}
\] (77)

where \(\alpha\) identifies a constant scale length and \(\mathcal{M}\) and \(\mathcal{Q}\) the Schwarzschild and Reissner-Nordström radii (see Eqs. (64) and (65)). Then, the Kerr-Newman metric is defined:

\[
\begin{align*}
S_0'(r') &= \frac{\Delta + \alpha^2 \sin^2 \theta}{\rho^2}, \\
S_1'(r') &= \frac{\Delta}{\rho^2}, \\
S_2'(r') &= \frac{\Delta \rho^2}{\rho^2}, \\
S_3'(r') &= \alpha (r'^2 + \alpha^2 \sin^2 \theta').
\end{align*}
\] (78)

Instead the Kerr metric is prescribed requiring:

\[
\begin{align*}
S_0'(r') &= 1 - \frac{\mathcal{M}^2}{r^2}, \\
S_1'(r') &= \frac{\Delta}{\rho^2}, \\
S_2'(r') &= \frac{\Delta \rho^2}{\rho^2}, \\
S_3'(r') &= \left( r'^2 + \alpha^2 + \frac{\mathcal{M}^2}{\rho^2} - \frac{\mathcal{Q}^2}{\rho^2} \sin^2 \theta' \right) \sin^2 \theta'.
\end{align*}
\] (79)

Let us now pose the problem of mapping either the Kerr-Newman or the Kerr space-times onto the Minkowski space-time \((M^4, \eta)\). For convenience let us represent also the latter space-time in spherical coordinates. This gives for the Minkowski metric the customary diagonal representation

\[
\eta_{\mu\nu}(r) \equiv \text{diag} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (S_0(r) = 1, -S_1(r) = 1, -S_2(r) = -r^2, -S_3(r) = -r^2 \sin^2 \theta).
\] (80)

A possible non-unique realization of the NLPT between the two space-times \((Q'^4, g')\) and \((M^4, \eta)\) indicated above, in some sense analogous to the one developed here for the FLRW space-time (see Section 6), is proposed here. This is provided by the non-diagonal special NLPT of the form

\[
\begin{align*}
dr^0 &= M_0^0 dr^0 + M_0^1 dr^1 + M_0^2 dr^2 + M_0^3 dr^3, \\
dr^1 &= M_1^0 dr^0 + M_1^1 dr^1 + M_1^2 dr^2 + M_1^3 dr^3, \\
dr^2 &= M_2^0 dr^0 + M_2^1 dr^1 + M_2^2 dr^2 + M_2^3 dr^3, \\
dr^3 &= M_3^0 dr^0 + M_3^1 dr^1 + M_3^2 dr^2 + M_3^3 dr^3,
\end{align*}
\] (81)

subject to the validity of the constraints

\[
\begin{align*}
S_0(r) M_0^0 M_1^1 - S_1(r) M_1^0 M_0^1 &= 0, \\
S_0(r) M_2^0 M_3^3 - S_3(r) M_3^0 M_2^3 &= 0.
\end{align*}
\] (82)

One can readily show that Eqs. (81) indeed realize a NLPT which mutually maps in each other the two space-times \((Q'^4, g')\) and \((M^4, \eta)\). For this purpose, in validity of Eqs. (81), let us require that the Jacobian matrix \(M^\nu_\mu\) satisfies
the further tensor equations

\[
\begin{align*}
(M_0^0)^2 - (M_1^0)^2 &= S_0'(r') \\
(M_0^0)^2 - (M_1^1)^2 - r^2 \sin^2 \theta (M_3^3)^2 &= -S_1'(r') \\
r^2 (M_2^2)^2 &= S_2'(r') \\
(M_1^3)^2 - r^2 \sin^2 \theta (M_2^3)^2 &= -S_3'(r')
\end{align*}
\]  
(83)

From Eqs. (83) and (82) elementary algebra gives the general solution

\[
\begin{align*}
M_0^0 &= \frac{M_1^1}{M_0^0} M_0^1 \\
M_3^3 &= \frac{r^2 \sin^2 \theta M_0^0 M_0^3}{M_1^1 M_0^0} M_0^3 \\
M_0^0 &= \sqrt{S_0'(r') + (M_1^1)^2} \\
M_1^1 &= \sqrt{S_1'(r') - r^2 (M_3^3)^2} \sqrt{S_0'(r')} + (M_0^3)^2} \\
M_3^3 &= \sqrt{S_3'(r') + (M_1^3)^2}
\end{align*}
\]  
(84)

where the matrix elements \(M_0^1\) and \(M_1^3\) still remain in principle arbitrary. Notice that the ratios \(\frac{S_1'(r') M_1^1}{S_0'(r') M_0^0}\) and \(\frac{S_0'(r') M_0^0}{S_1'(r') M_1^3}\) read then

\[
\begin{align*}
\frac{M_1^1}{M_0^0} &= \sqrt{\frac{S_1'(r') - r^2 (M_3^3)^2}{S_0'(r')}}, \\
r^2 \sin^2 \theta \frac{M_0^0}{M_1^1} &= r^2 \sin^2 \theta \sqrt{\frac{S_1'(r') - r^2 (M_3^3)^2}{S_0'(r')}}.
\end{align*}
\]  
(85, 86)

Hence it follows that

\[
\begin{align*}
M_0^0 &= \sqrt{\frac{S_1'(r') - r^2 (M_3^3)^2}{S_0'(r')}} M_0^1, \\
M_3^3 &= r^2 \sin^2 \theta \sqrt{\frac{S_1'(r') - r^2 (M_3^3)^2}{S_0'(r')}} \frac{M_1^1}{M_0^0} M_1^3.
\end{align*}
\]  
(87)

Notice, however, that the existence of the solution (81) demands manifestly

\[
S_1'(r') - r^2 (M_3^3)^2 > 0,
\]  
(88)

to be interpreted as solubility condition. A number of remarks can be made:

1) Notice that when letting in particular \(M_1^0 = M_1^3\), Eqs. (81) reduce to the non-diagonal special NLPT considered in Part 2.

2) Eqs. (81) imply that the time-component of the 4-acceleration in the Minkowski space-time is generated by time-, radial and tangential components in the Kerr-Newman and Kerr space-times respectively, namely \(\frac{D'}{D s} w^0\), \(\frac{D'}{D s} w^1\) and \(\frac{D'}{D s} w^3\).

3) Similarly, the tangential component \(\frac{D'}{D s} w^3\) depends also on the radial component \(\frac{D'}{D s} w^1\) arising in the Kerr-Newman or Kerr space-times, besides \(\frac{D'}{D s} w^3\).

4) Let us now consider the inverse transformations following from Eqs. (81). In analogy with Application #2 (see Section 5) also in this case both the time and radial components arising in the Kerr-Newman or Kerr space-times, namely respectively \(\frac{D'}{D s} w^0\) and \(\frac{D'}{D s} w^1\) generally depend on the analogous components of the 4-acceleration arising in the Minkowski space-time, namely \(\frac{D}{D s} w^0\) and \(\frac{D}{D s} w^1\), as well as the tangential component \(\frac{D}{D s} w^3\). Thus, for example one obtains that

\[
\frac{D'}{D s} w^0 = \left[ M_3^3 M_1^1 - (M_0^0 M_1^1 - M_0^3 M_1^3) \frac{D}{D s} w^1 - M_1^1 M_0^0 \frac{D}{D s} w^3 \right] \frac{M_0^1}{M_1^1 - M_0^1 M_0^0} M_1^3 M_1^3 M_1^0.
\]  
(89)
and similarly the radial component reads

\[
\frac{D'_{\nu}}{D_{s}} u'_{\alpha} = \frac{1}{M'_{1}} \left[ \frac{D}{D_{s}} u_{\alpha} - M'_{0} \frac{D'}{D_{s}} u'_{\alpha} \right].
\]  

(90)

5) In the previous equations the matrix elements $M'_{0}$ and $M'_{3}$ remain still in principle arbitrary, with the second one required to fulfill the inequality [53] indicated above. A further solubility condition is provided, however, by the equation for $M'_{3}$ in Eqs. [54]. In fact, this is only defined provided also

\[
M'_{0} \neq 0.
\]  

(91)

6) Since due to Eqs. [52] the matrix elements $M'_{1}$ and $M'_{3}$ become linear functions of $M'_{0}$ and $M'_{3}$ respectively, it follows that both the time and radial accelerations [59] and [60] strongly depend on the choices of the same parameters. Notice that, in particular, the coupling of $\frac{D'_{\nu}}{D_{s}} u'_{\alpha}$ with non-radial components occurs always due to the solubility condition [51].

8 - APPLICATION #4: NLPT-TRANSFORMATION LAWS OF THE EM FARADAY TENSOR

Let us now consider as an application of NLPT-theory the dynamics of a charged point particle of rest-mass $m_{\nu}$ and electric charge $q$ immersed in an external EM field. As it is well known in the curved space-times $(Q^{4}, g)$ and $(Q'^{4}, g')$ the relativistic equation of motion takes respectively the forms

\[
m_{\nu} \frac{D u_{\mu}}{D_{s}} = qF_{\nu}^{(ext)\mu}(r)u_{\nu},
\]  

(92)

\[
m_{\nu} \frac{D' u'_{\mu}}{D_{s}} = qF_{\nu}^{(ext)\mu}(r')u'_{\nu}
\]  

(93)

with $u_{\mu}$, $u'_{\mu}$ and respectively $F_{\nu}^{(ext)\mu}(r)$, $F_{\nu}^{(ext)\mu}(r')$ denoting the in the same space-times the 4–velocities and the Faraday tensors generated by an externally-produced EM field. Assuming that a general NLPT maps in each other $(Q^{4}, g)$ and $(Q'^{4}, g')$ the NLPT-transformation law for the 4–acceleration, and in particular Eqs. [51] and [52] in THM.2, since

\[
\frac{D}{D_{s}} u_{\mu} = M'_{(g)\nu}(r, r') \frac{D'}{D_{s}} u'_{\nu}
\]

it must be identically that

\[
qF_{\alpha}^{(ext)\mu}(r)u_{\nu} = M'_{(g)\nu}(r, r')qF_{\nu}^{(ext)\beta}(r') (M^{-1})_{(g)\alpha}^{\nu}(r, r')u'_{\alpha}.
\]  

(94)

Therefore, due to the arbitrariness of the 4–vector $u_{\nu}$, the quantity $F_{\alpha}^{(ext)\mu}(r)$ necessarily satisfies the 4–tensor NLPT (direct) transformation law

\[
F_{\alpha}^{(ext)\mu}(r) = M'_{(g)\nu}(r, r')F_{\nu}^{(ext)\beta}(r') (M^{-1})_{(g)\alpha}^{\nu}(r, r')
\]  

(95)

Hence by construction it follows

\[
(M^{-1})_{(g)\nu}^{k} F_{\alpha}^{(ext)\mu} M'_{(g)\alpha}^{\nu} = (M^{-1})_{(g)\nu}^{k} M'_{(g)\beta} F_{\nu}^{(ext)\beta} (M^{-1})_{(g)\alpha}^{\nu} M'_{(g)\beta},
\]

which yields the corresponding inverse transformation law too

\[
F_{\alpha}^{(ext)k}(r') = (M^{-1})_{(g)\mu}^{k}(r, r') F_{\alpha}^{(ext)\mu}(r) M'_{(g)\alpha}(r, r').
\]  

(96)

Eqs. [53] and [54] provide the transformation equations connecting for the Faraday tensors $F_{\alpha}^{(ext)\mu}(r)$ and $F_{\alpha}^{(ext)\mu}(r')$ which are defined respectively in the two space-times $(Q^{4}, g)$ and $(Q'^{4}, g')$. In particular we stress that on the rhs of the first equation $r \equiv \{ r^{\mu} \}$ must be regarded as a non-local function of $r' \equiv \{ r'^{\mu} \}$ whose form is determined by the same NLPT. This means that $F_{\nu}^{(ext)\mu}(r)$ (and conversely $F_{\alpha}^{(ext)\beta}(r')$ when represented via the inverse transformation [94]) must be regarded in turn as a non-local function of $r'$ too.
There remains an important question to answer, i.e., whether the transformed Faraday tensor $F_{\alpha}^{\beta(\text{ext})}$ can be identified or not with an exact solution of the Maxwell equations defined in the space-time $(Q^4, g)$, being $F_{\nu}^{(\text{ext})\mu}(r)$ an exact solution of the same equations in $(Q^4, g')$. The answer to this question requires to prove that $F_{\nu}^{(\text{ext})\mu}(r)$ and $F_{\nu}^{(\text{ext})\mu}(r')$ are respectively solutions of the Maxwell equations in the two space-times, i.e., that these equations are endowed with a tensor transformation law with respect to the group of general NLPT $\{P_g\}$. The proof of this statement will be reported elsewhere.

9 - PHYSICAL IMPLICATIONS AND CONCLUSIONS

In this paper implications of NLPT-theory presented in Parts 1 and 2 have been investigated with particular reference to the establishment of the 4–acceleration tensor transformations. In our view the analysis here performed can help reaching a novel interpretation and a deeper physical interpretation of the standard formulation of GR (SF-GR).

The NLPT-theory here developed is based on the extension of the customary functional setting which lays at the basis of the same SF-GR. This involves the construction of a suitable invertible coordinate transformation, to be identified with a suitable class of non-local point transformation (NLPT) which map in each other two in principle arbitrary curved space-times $(Q^4, g)$ and $(Q^4, g')$ possibly parametrized in terms of different curvilinear coordinate systems. In other words, the two transformed space-times may exhibit, as a consequence, intrinsically-different non-vanishing Riemann curvature tensors. As pointed out in Part 1 the new transformations have the distinctive property that the corresponding Jacobians are endowed with a characteristic non-gradient form. In particular these transformation, identified with the general NLPT-group $\{P_g\}$, involve also to the adoption of a new kind of phase-space reference frames, denoted as extended GR-frames involving the prescription of both 4–position and corresponding 4–velocities (see related discussion in Part 1).

In this paper in particular the fundamental issue has been addressed of the establishment of the tensor transformation laws relating the 4–acceleration arising in different space-times as well as of the investigation of the main related physical implications.

In order to reach the goal indicated above in this paper a systematic analysis of the mathematical implications of NLPT-theory has been performed. This has been realized, first, by inspecting the differential properties of the Jacobian of general NLTP, established by Lemmas 1 and 2 (see Section 2). Second, the transformation properties of the Christoffel symbols with respect to the NLPT-group $\{P_g\}$ have been determined (see THM.1 in Section 3). Notably, this includes also as a particular possible realization in the case of NLPT mapping a curved space-time onto the Minkowski space-time (Corollary to THM.1). Based on these results, the tensor transformation laws of the 4–accelerations within the general NLPT-group $\{P_g\}$ have been established (see THM.2, Section 4), thus enabling one to determine the relationships holding between the 4–accelerations which are defined in the two different curved space-times $\{Q^4, g\}$ and $\{Q^4, g'\}$ mutually mapped in each other by an arbitrary NLPT.

Physical insight about the meaning and implications of the tensor transformation laws determined here emerges from the applications considered in Sections 5, 6, 7 which have concerned the analysis of NLPT-acceleration effects arising:

- Between the Schwarzschild, Reissner-Nordström and the Minkowski or Schwarzschild-analog space-times.
- Between the Friedman-Lemaitre-Robertson-Walker and the Minkowski or Schwarzschild-analog space-times.
- Between the Kerr-Newman or Kerr space-times and the Minkowski space-time.

as well as, notably, Section 8 dealing with:

- the determination of the NLPT-tensor transformation laws of the EM Faraday tensor, a result which is directly implied by the 4–acceleration transformation law here established.

These results are undoubtedly in qualitative consistency with the Einstein famous equivalence principle (EEP, \cite{4}; see also related discussion in Part 1) and, more precisely with Einstein’s key related conjecture which actually lays at the basis of GR, namely that “local effects of motion in a curved space (produced by gravitation)” should be considered as “indistinguishable from those of an accelerated observer in flat space” \cite{3, 10}.

In order to further elucidate the issue let us in fact consider a NLPT transforming in each other a curved space-time $(Q^4, g)$ and the flat Minkowski space-time $(M^4, \eta)$ . In addition, for definiteness let us require that in both cases the
same coordinate systems are adopted (so that the NLPT identifies a special NLPT). Then denoting $\mathbf{a}_\mu$ and $\mathbf{a}'_\mu$ the accelerations in the two space-times - in view of the discussions of the applications #1-#3 discussed above - it is always possible to construct the NLPT in such a way that for example the equation

$$0 \equiv a_1^1 = M_{10}^1 a_0^0 + M_{11}^1 a_1^1$$

(97)

holds identically, i.e., the spatial component of the acceleration in the curved space-time ($a_1^1$) vanishes identically, while the corresponding component in the Minkowski space-time is non-vanishing. In other words, the acceleration effect arising in the flat space-time apparently disappears in the curved space-time (i.e., is effectively hidden in the gravitational-curvature effects of the space-time $(Q^1, g)$).

To conclude, it should also be mentioned that also the transformation law of the EM Faraday tensor and the NLPT-covariance law of Maxwell equations here discovered are actually consistent with the same Einstein’s viewpoint. In fact, in accordance with Einstein, exclusively the effects on particle motion which are due to gravitation should be considered indistinguishable from those of an accelerated observer.

As discussed at length also in Parts 1 and 2 of the present investigation, both acceleration effects are found to be realized in the framework of NLPT-theory. In turn, this requires the adoption of phase-space reference frames, denoted as extended GR-frames, and a suitably-defined set of phase-space maps, which involve in particular the introduction of appropriate non-local coordinate transformations identified with the group of general NLPT $\{P_g\}$.

These conclusions strongly support the crucial importance of non-locality effects in physics. In particular, in our view, the NLPT-theory here presented appears susceptible of a plethora of potential applications from classical relativistic mechanics and electrodynamics \[24–29\], general relativity quantum theory of extended particle dynamics \[30, 32–35\], relativistic kinetic theory \[28\], to cosmology as well as relativistic quantum mechanics \[31\] and quantum gravity.

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