Symplectic cohomology rings of affine varieties in the topological limit

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Abstract. We construct a multiplicative spectral sequence converging to the symplectic cohomology ring of any affine variety $X$, with first page built out of topological invariants associated to strata of any fixed normal crossings compactification $(M, D)$ of $X$. We exhibit a broad class of pairs $(M, D)$ (characterized by the absence of relative holomorphic spheres or vanishing of certain relative GW invariants) for which the spectral sequence degenerates, and a broad subclass of pairs (similarly characterized) for which the ring structure on symplectic cohomology can also be described topologically. Sample applications include: (a) a complete topological description of the symplectic cohomology ring of the complement, in any projective $M$, of the union of sufficiently many generic ample divisors whose homology classes span a rank one subspace, (b) complete additive and partial multiplicative computations of degree zero symplectic cohomology rings of many log Calabi-Yau varieties, and (c) a proof in many cases that symplectic cohomology is finitely generated as a ring. A key technical ingredient in our results is a logarithmic version of the PSS morphism, introduced in our earlier work [GP].

1. Introduction

Symplectic cohomology, a version of Hamiltonian Floer homology for exact convex symplectic manifolds (such as affine varieties and more generally Stein manifolds), has attracted widespread attention as a powerful invariant of symplectic manifolds. A landmark result of Viterbo [V2, SW, AbSc1] gives a topological description of symplectic cohomology in the case of cotangent bundles and it is known that the invariant vanishes under particular hypotheses on the underlying Stein topology (such as being “subcritical” or “flexible”, see [C, BEE2, MS2]). Outside of these cases, there are relatively few explicit computations, which tend to be difficult and rely, in one way or another, on the enumeration of pseudoholomorphic curves (see §1.3 for a comparison of other computational approaches with the methods developed here).

The present paper gives a new technique for computing symplectic cohomology rings of affine varieties via reduction to algebraic topology and algebraic geometry. To set notation, let $X$ be a smooth complex affine variety, and let us fix any smooth projective compactification $M$ of $X$ by a simple normal crossings divisor $D = D_1 \cup \cdots \cup D_k$ supporting an ample
line bundle, which is guaranteed to us by Hironaka’s theorem. One can encode much of the combinatorics and algebraic topology of the pair \((M,D)\) in a ring called the \(\log(\text{arithmic})\) cohomology of \((M,D)\), which we denote \(H^*_\log(M,D)\) (the underlying abelian group was introduced in our earlier work [GP]). See \(\S 3.2\) for a precise definition; roughly, \(H^*_\log(M,D)\) is the direct sum of the cohomology \(H^*(X)\) and classes of the form \(\alpha \nu\), where \(\nu \in (\mathbb{Z}^\geq)^k\) is a multiplicity vector, \(\alpha\) is a cohomology class in the normal torus bundle \(\tilde{S}_I\) to the (open part of the) stratum \(D_I = D_{I_1} \cap \cdots \cap D_{I_m}\) of \((M,D)\); the subset \(I \subset \{1,\ldots,k\}\) is required to consist exactly of the indices \(i\) for which \(v_i \neq 0\). The product is given by adding multiplicity vectors and restricting the cohomology classes to a common stratum where they can be multiplied.

Our first main result relates the logarithmic cohomology of the pair \(H^*_\log(M,D)\) to the symplectic cohomology of the complement \(SH^*(X)\), by way of a spectral sequence:

**Theorem 1.1.** (Theorem 4.32) There is a multiplicative spectral sequence converging to the symplectic cohomology ring

\[
\{E^{p,q}_r, d_r\} \Rightarrow SH^*(X).
\]

whose first page is isomorphic as rings to the logarithmic cohomology of \((M,D)\):

\[
H^*_\log(M,D) \cong \bigoplus_{p,q} E^{p,q}_1.
\]

For a precise relationship between the spectral sequence gradings \(p,q\) and the (bi-)grading on \(H^*_\log(M,D)\), see Theorem 4.32 below. We content ourselves here by observing that while \(SH^*(X)\) is typically \(\mathbb{Z}_2\)-graded, it can be made \(\mathbb{Z}\)-graded when \(c_1(X) = 0\), and can additionally be equipped with a grading by \(H_1(X)\) measuring homology classes of generating orbits; all of these choices can be realized on the level of log cohomology and the identification (1.2). We have opted to describe our proof in the \(\mathbb{Z}\)-graded setting for notational simplicity when defining moduli spaces and operations, but our methods are grading-independent and apply immediately to the above (as well as other) graded situations with minor modifications to definitions; see the discussion below Theorem 4.32.

There are easy examples where the spectral sequence (1.1) fails to degenerate at the \(E_1\) page (for example when \(X = \mathbb{C}\), \(SH^*(X)\) vanishes). On the other hand, one of the main themes of the present paper is that there are many cases in which (1.1) does degenerate at \(E_1\). In particular, the multiplicative structure in Theorem 1.1 gives a powerful computational tool for proving degeneration (or more generally, analyzing differentials). Note that the \(E_1\) page \(H^*_\log(M,D)\) is generated as a \(\mathbb{k}\)-algebra by classes of the form \(y \in H^*(X)\) or \(\alpha \nu\), where for any subset \(I \subset \{1,\ldots,k\}\),

\[
(v_I)_i := \begin{cases} 1 & i \in I \\ 0 & \text{otherwise.} \end{cases}
\]

i.e., \(v_I = \sum_{i \in I} e_i\) is the primitive multiplicity 1 vector supported on elements of \(I\). The multiplicatively of (1.1) then implies that the spectral sequence (1.1) degenerates if \(d_r(\alpha \nu) = d_r(y) = 0\) for all \(\alpha \nu, y \in H^*(X), r \geq 1\). Here is an easy corollary of this observation, which follows from analyzing homology classes of possible differentials on such generators:

**Corollary 1.2** (Corollary 5.20). Suppose that there is a divisor \(H\) such that for each smooth component \(D_i \subset D, O(D_i) \cong O(n_i H)\) for \(n_i \in \mathbb{Z}^\geq\). If any of the \(n_i > 1\), then the spectral sequence (1.1) degenerates at the \(E_1\) page.
In addition to being a useful computational tool, Theorem 1.1 is also very useful for proving general qualitative results, for example:

**Theorem 1.3.** *(Theorem 5.30)* Assume that $D = D_1 \cup \cdots \cup D_k$ is an ample divisor and all of the strata $D_I$ are connected. Then $SH^*(X)$ is finitely generated as a graded algebra over $k$.

A few comments on the proof of Theorem 1.1 are in order. The spectral sequence in Theorem 1.1 is induced by a version of the action filtration on (the cochain complex computing) $SH^*(X)$, specifically a nice integer-valued version of this filtration arising from the compactification $(M, D)$ (constructed in §2). The product operation on the symplectic cohomology cochain complex can be made to respect this filtration, giving us the multiplicative structure of the spectral sequence. Additively, the identification of the first page (1.2) can be thought of as a consequence of a model of Reeb (or Hamiltonian) flow near $D$ for which the orbit sets come in Morse-Bott families associated to normal bundles to strata $S_I$.

The construction of this model and the explicit description of its periodic orbits uses our earlier work [GP] and, like that work, relies extensively on the study of symplectic structures and Liouville geometry near $D$ as developed by McLean [M4,M2]. The families of orbit sets produced are manifolds with corners, but nevertheless we can apply a careful version of Morse-Bott analysis to them to produce (1.2) additively (compare [M6, §1.1] for a spectral sequence in a related situation). However, such analysis does not make it easy to see that the multiplicative structure on the first page is compatible with log cohomology via (1.2), and a new argument is needed.

The basic idea, coming from our earlier work [GP] and inspired by Pliunikhin-Salamon-Schwarz’s [PSS] classic construction, is the introduction of log PSS moduli spaces. Roughly speaking, these moduli spaces count maps from a punctured sphere to $M$ which are holomorphic near a marked point $z_0$ and solve Floer’s equation along a cylindrical end around the puncture. At the marked point $z_0$, we place tangency and jet constraints on the intersection of $z_0$ with strata of $D$, and away from $z_0$ we require the map to land in $X$, allowing us to interpolate between log cohomology classes for $(M, D)$ and Hamiltonian Floer cochains in $X$. In the presence of sphere bubbling, counting such log PSS solutions does not necessarily produce a well-defined cochain map from log cochains to Floer cohomology. However, energy considerations show that such bubbling does not arise for low energy solutions. Thus, by counting these low energy solutions and suitably quotienting the Floer complexes, we can define the map (1.2).

A considerable amount of work is then required to prove that this map is an isomorphism. However, a benefit of this approach is that one may use a standard TQFT argument to prove that the map (1.2) intertwines product structures. In fact, this argument closely follows Pliunikhin-Salamon-Schwarz’s original argument [PSS] that the PSS map intertwines product structures, adapted in a non-trivial way to our (“relative $D$”) setup.

### 1.1. Computations in the absence of pointed relative holomorphic spheres.

We next describe how, in the absence of certain relative holomorphic spheres, we can use similar methods to the proof of Theorem 1.1 to directly define an isomorphism from log cohomology of $(M, D)$ to symplectic cohomology that splits the spectral sequence (1.1). More precisely, we obtain geometric and often checkable criteria (i) under which the spectral sequence (1.1) degenerates and (ii) under which we can further topologically compute (in

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1When $D = D$ is a smooth divisor this is standard and in the literature, compare [S3, eq. (3.2)].
terms of $H^*_{\log}(M, D)$) the ring structure on $SH^*$. To state these criteria, let $J$ be an almost complex structure on $M$, and let $m$ be a non-negative integer. An $m$-pointed relative $J$-holomorphic sphere in $(M, D)$ is a non-constant $J$-holomorphic map $u : \mathbb{CP}^1 \to M$ whose image lies generically in some open stratum $(X \times D_I)$ and which intersects a deeper stratum $(D \cup \cup \cup_{j \not\in I} D_I \cap D_j)$ in exactly $m$ distinct points. We work with a class of almost complex structures $\mathfrak{J}(M, D)$ (see Definition 2.11) that tame the symplectic form $\omega$, preserve $D$ and satisfy a certain integrability condition in the normal directions to $D$; for such an almost complex structure $J \in \mathfrak{J}(M, D)$, every $J$-holomorphic sphere in $M$ is $m$-pointed for some $m$. Our criterion for degeneration is:

**Theorem 1.4.** (Theorem 5.10) Suppose that $(M, D)$ has no 0 or 1-pointed relative $J_0$-holomorphic spheres, for some $J_0 \in \mathfrak{J}(M, D)$. Then the spectral sequence (1.1) degenerates. More precisely, there is a canonical splitting of the spectral sequence, i.e., a filtered isomorphism of $k$-modules

$$PSS_{\log} : H^*_{\log}(M, D) \xrightarrow{\cong} SH^*(X).$$

such that the induced map $\oplus_{p,q} E_{p,q}^1 \cong H^*_{\log}(M, D) \to \oplus_{p,q} F^pSH^*(X)/F^{p+1}SH^*(X) \cong \oplus_{p,q} F^\infty_{p,q}$ is the isomorphism specifying collapse of the spectral sequence.

The map (1.3), which we call the Log PSS morphism, was introduced in our earlier work [GP] and is constructed in a similar way to (1.2), except we no longer restrict to low-energy PSS solutions. The main work is showing that such counts now define a cochain map in the absence of 0 and 1-pointed relative spheres (note there may be other spheres in $M$, but the point is to show they do not arise in the compactification of log PSS moduli spaces). From there, given that the “associated graded” of the map (1.3) is easily seen to be (1.2), Theorem 1.4 becomes a straightforward consequence of Theorem 1.1.

Under the hypothesis in which Theorem 1.4 applies, the multiplicative structure of the spectral sequence (1.1) implies that the Log PSS morphism (1.3) induces a ring isomorphism between the log cohomology $H^*_{\log}(M, D)$ and the associated graded symplectic cohomology ring $gr_FSH^*(X)$. Our next result gives a criterion under which this can be promoted to a ring isomorphism with symplectic cohomology $SH^*(X)$:

**Theorem 1.5.** Suppose that $(M, D)$ has no 0, 1, or 2-pointed relative $J_0$-holomorphic spheres, for some $J_0 \in \mathfrak{J}(M, D)$. Then, the additive isomorphism (1.3) given in Theorem 1.4 is a ring isomorphism.

Once more, this follows from running the same TQFT argument which showed the isomorphism (1.2) in Theorem 1.1 was a ring map, applied now to the “global” (rather than just low energy) log PSS moduli spaces that arise in the construction of (1.3). The only possible obstruction to our TQFT argument applying is the failure of the relevant interpolating moduli spaces to be compact, and we show the only possible problems are the bubbling of 0, 1 or 2-pointed relative spheres. With such spheres excluded, the proof then straightforwardly reduces to earlier methods.

\[\text{2}\]We remind the reader that for any (convergent) spectral sequence associated to a filtered dg or $A_\infty$ algebra $F^pC^*$, the algebra structure on the final $E_\infty$ page, the associated graded algebra $gr_F(H^*(C^*))$, need not be isomorphic as rings to $H^*(C^*)$ even if it is additively isomorphic: in general $H^*(C^*)$ could be a non-trivial deformation of $gr_F(H^*(C^*))$.

\[\text{3}\]with respect to the (homological shadow of) action filtration inducing the spectral sequence (1.1)
Throughout this paper, we will call \((M, D)\) a \textit{topological pair}\(^4\), respectively a \textit{multiplicatively topological pair} if, for some almost complex structure \(J_0 \in J(M, D)\), \((M, D)\) has no \(m\)-pointed relative \(J_0\)-holomorphic spheres for \(m \leq 1\), respectively \(m \leq 2\), i.e., if \((M, D)\) satisfy the hypotheses of Theorems 1.4, respectively 1.5.\(^5\)

Theorems 1.4 and 1.5 allow us to deduce many complete topological computations of symplectic cohomology groups and rings. An incomplete list of examples of topological and multiplicatively topological pairs (for which the relevant Theorems apply) are provided in Examples 5.1 and 5.2. As a first example, we note that for any smooth projective \(M\), the pair \((M, D)\) will be topological (respectively multiplicatively topological) if \(D\) consists of at least \(\dim \mathbb{C} M + 1\) (respectively \(2 \dim \mathbb{C} + 1\)) generic ample divisors which are powers of the same line bundle.

1.2. Computations in the presence of pointed spheres. We expect that, for general pairs \((M, D)\), by counting the relative spheres that arise in the compactification of log PSS moduli spaces (using a cochain level version of log Gromov-Witten theory), one could write down a corrected differential on the log cohomology cochain complex, along with a corrected product, for which counts of (compactified) log PSS moduli spaces induce both a cochain map and a ring map (up to chain homotopy); if such a cochain map were constructed, Theorem 1.1 would immediately implies that it would be a quasi-isomorphism. In the case that \(D = D\) is a smooth divisor, under suitable monotonicity assumptions Diogo and Lisi [D, DL] have given a related additive cochain level model for symplectic cohomology in terms of (relative) GW-invariants, and there is work towards a model for the product in this setting [D].

At the level of the spectral sequence (1.1), the existence of such a model would imply that the differentials \(d_r\) on the \(E_r\) pages of the spectral sequence could be calculated in terms of \textit{cohomological} log GW invariants. When all of these invariants vanish, the spectral sequence would degenerate. There would be further (analogous) vanishing criteria under which the ring structure could also be computed in terms of log cohomology.

The foundations of symplectic log Gromov-Witten theory are still under active development (see [I], [P1], [FT] for different approaches) and constructing such a cochain level model goes beyond the scope of the present article. Nevertheless, motivated by these ideas, we show that in a broad class of examples, additive and multiplicative computations of symplectic cohomology can be reduced to only counting (rather, exhibiting the vanishing of counts of) the \textit{simplest kinds of relative curves} (those which intersect each component of \(D\) at most once). We focus on two distinct situations: (a) computing the ring structure topologically in the absence of 0 or 1-pointed spheres but presence of 2-pointed spheres (meaning when we already understand \(H^*_\text{log}(M, D) \cong SH^*(X)\) additively), and (b) computing \(SH^*(X)\) additively in terms of \(H^*_\text{log}(M, D)\) in the possible presence of 0 or 1-pointed spheres.

1.2.1. Trivializing deformations of rings in the presence of 2-pointed relative spheres. For topological pairs, there is an additive (but not necessarily multiplicative) isomorphism \(SH^*(X) \cong H^*_\text{log}(M, D)\); moreover the product structure on \(SH^*(X)\) induces a deformation

\(^4\)The terminology “topological pair” was introduced in our earlier work [GP], where its usage is slightly less general than what is used here.

\(^5\)An alternative naming convention would be to say a pair \((M, D)\) is \(r\)-topological if (for some \(J_0\)) it contains no \(m\)-pointed relative \(J_0\)-holomorphic spheres for all \(m \leq r\). We expect such generalized notions to be useful in the study of higher-arity operations on symplectic cohomology, such as \(A_\infty\) or \(L_\infty\) structures.
of the product structure on $H^*_\log(M,D)$ which is in general non-trivial. For a number of such pairs $(M,D)$ (under topological hypotheses and hypotheses on the vanishing of certain two-point Gromov-Witten counts), we have an a posteriori argument establishing a ring isomorphism $SH^*(X) \cong H^*_\log(M,D)$, via showing that the deformation of the product structure on $H^*_\log(M,D)$ is trivializable; see Theorem 5.17 for such a criterion.

Example 1.1 (Symplectic cohomology rings of $\mathbb{P}^n$ minus generic hyperplanes). Let $M = \mathbb{P}^n$ and $D_k$ be a union of $k$ generic planes, and $X_k = M \setminus D_k$ the complement; note that $X_{n+2}$ is Mikhalkin’s generalized pair of pants [M8]. Our results lead to a complete computation of $SH^*(X_k)$ as a ring for all $k$, extending well-understood computations when $k \leq n + 1$:

- when $k \leq n$ then $X_k = M \setminus D_k = (\mathbb{C}^*)^{k-1} \times \mathbb{C}^{n-k+1}$ with $n - k + 1 > 0$. Since $SH^*(\mathbb{C}) = 0$, the Künneth formula [O1] implies that $SH^*(X_k) = 0$.
- When $k = n+1$, $X_{n+1} \cong (\mathbb{C}^*)^n \cong T^*(T^n)$, so Viterbo’s formula implies $SH^*(X_{n+1}) \cong H_{n-\cdot}(LT^n)$.
- For $k > n+1$, $(M,D_k)$ is a topological pair, so Theorem 1.4 gives an additive isomorphism $SH^*(X_k) \cong H^*_\log(M,D_k)$. In fact, this is a ring isomorphism by the following argument: $(M,D_k)$ is multiplicatively topological when $k \geq 2n+1$ (see Example 5.2) so Theorem 1.5 says that $H^*_\log(M,D) \to SH^*(X_k)$ is a ring isomorphism in that range. For the remaining intermediate range $n+1 < k < 2n+1$, Theorem 5.17 applies to argue that, while in principle there could have been a deformation of the product structure on $H^*_\log(M,D_k)$ contributing to $SH^*(X_k)$, the deformation was in fact trivial.

1.2.2. Degeneration arguments in the presence of spheres. We show that that for many pairs $(M,D)$, the multiplicative structure on the spectral sequence allows us to reduce (for the purposes of degeneration arguments) to counting moduli spaces of relative spheres which intersect each component of $D$ at most once. We will show that for these pairs, if the relevant cohomological count is zero, then the spectral sequence (1.1) degenerates, albeit without a canonical splitting.

More precisely, we look at “admissible” pairs $(M,D)$, which are pairs for which the differentials on primitive cohomology classes $\alpha t^{v_I}$ are tightly controlled—given an admissible pair, a vector $v_I$, and $\alpha t^{v_I} \in H^*_\log(M,D)$, there is single possible non-vanishing differential, $d_{w(v_I)}(\alpha t^{v_I})$. We show that this differential can be encoded in Gromov-Witten type invariants (called “obstruction classes” in [GP])

$$GW_{v_I} : H^*(\mathcal{S}_I) \to H^*(X)$$

Theorem 1.6. (Theorem 5.26) Let $(M,D)$ be an admissible pair and assume $k = \mathbb{Z}$. For any primitive vector $v_I$, we have an equality

$$d_{w(v_I)}(\alpha t^{v_I}) = GW_{v_I}(\alpha).$$

In particular, the above discussion shows that for admissible pairs, these invariants determine when the spectral sequence degenerates:

Corollary 1.7. (Corollary 5.28) Assume $k = \mathbb{Z}$. Given an admissible pair $(M,D)$, suppose the maps (1.4) vanish for all $I \in \{1, \ldots, k\}$. Then the spectral sequence (1.1) degenerates at the $E_1$ page.
While Corollary 1.7 is more technical to state than Corollary 1.2, it is likely significantly more general. For example, consider cases which fall outside of the purview of Corollary 1.2; however the Gromov-Witten counts vanish and the spectral sequence degenerates. Other related examples can easily be constructed and it appears likely that many more cases could be treated by slightly more elaborate Gromov-Witten theory computations. As a concrete question, we ask:

**Question 1.8.** Given a smooth hypersurface \( M \subset \mathbb{C}P^{n+1} \) and a collection of hyperplanes \( D = D_1 \cup \cdots \cup D_k \), with \( X = M \setminus D \), is it the case that either \( SH^*(X) = 0 \) or the spectral sequence (1.1) degenerates at the \( E_1 \) page?

We can also consider variants of Theorem 1.6 where we focus on the spectral sequence in a given degree. The most interesting example of this is the following theorem.

**Theorem 1.9.** (Theorem 5.31) Let \( (M, D) \) be a pair with \( M \) a Fano manifold and \( D \) an anticanonical divisor. Assume that all strata \( D_I \) are connected. Then the spectral sequence degenerates in degree zero. With respect to the standard filtration, we have an isomorphism

\[
gr_F SH^0(X) \cong \mathcal{SR}(M, D)
\]

where \( \mathcal{SR}(M, D) \) is the Stanley-Reisner ring on the dual intersection complex of \( D \), defined in (3.17) (roughly, the subalgebra consisting of \( e \in H^0(X) \) and \( \alpha t^\nu \) where \( \alpha \in H^0(\hat{S}_I) \)).

The setting of Theorem 1.9 (or more generally log Calabi-Yau pairs) is of special interest in mirror symmetry, where it is expected that under suitable circumstances, the mirror variety to \( X \) is birational to the affine variety \( \text{Spec}(SH^0(X)) \) \cite{GHK2}. It is suggested there that there should exist a flat one parameter degeneration with general fiber \( \text{Spec}(SH^0(X)) \) and central fiber \( \text{Spec}(\mathcal{SR}(M, D)) \). Theorem 1.9 validates this expectation in a large number of cases. It is worth noting that the resulting deformation is usually non-trivial and should be expressible in terms of log Gromov-Witten invariants of the pair \( (M, D) \) \cite{GS}.

As suggested in the previous paragraph, it is likely that Theorem 1.9 holds without the assumptions that \( M \) is Fano and that the strata of \( D \) are all connected and we expect that the methods developed in this paper can be extended to treat the general case. As evidence for this, we use our methods to prove this when \( \dim_{\mathbb{C}}(M) = 2 \) in Theorem 5.37, recovering a result of James Pascaleff \cite{P2}. Our method of proof bears some similarities to Pascaleff’s argument in that both use knowledge of “low energy” product operations in symplectic cohomology. In our case, these computations of the product are implied by basic considerations in TQFT, whereas Pascaleff’s argument is more geometric.

### 1.3. Comparison with other methods for computing symplectic cohomology.

Most methods of explicitly computing symplectic cohomology \( SH^*(X) \) begin by explicitly describing a (Weinstein) handle presentation of \( X \) (possibly, but not necessarily, one that comes from a Lefschetz fibration), or alternatively a Lagrangian skeleton for \( X \), any of which we might collectively call “a presentation of the core” of \( X \). From such presentations, one can attempt to extract a computation of associated “open-string” Floer-theoretic or pseudo-holomorphic curve invariants, such as the Chekanov-Eliashberg DGA, an object of symplectic field theory associated to the Legendrians in the handle presentation, or the wrapped Fukaya category, associated to non-compact Lagrangians in \( X \). Finally, one takes the Hochschild homology and/or cohomology of such a computation, which relates to \( SH^*(X) \) via surgery formulae and/or open-closed maps \cite{BEE2, BEE1, A1, G1}. While this strategy has been carried out in interesting examples (see e.g. \cite{BEE2, BEE1, EN, EL}) and...
remains a remarkably effective tool for identifying instances when $SH^*(X)$ vanishes, general computations can run into two difficult issues: computing the relevant open-string invariant (a process that in some cases can be combinatorialized), and computing its Hochschild invariants (which is known to be a difficult algebraic problem except in special circumstances). Also, realizing an affine variety as an explicit handlebody or calculating a skeleton may be challenging in practice, though there is some work to systematize the process [CM1].

The results in this paper, which instead make use of a “presentation at infinity” of $X$ (in terms of the compactification $(M, D)$), give many cases where calculations of $SH^*(X)$ can be performed purely topologically. It seems likely that any elaboration of these models to include non-trivial counts of relative spheres will frequently produce models for symplectic cohomology with far fewer generators, where the differentials may be approached using algebraic geometry. Of course, applying these techniques for a given $X$ requires finding an explicit compactification $(M, D)$, which is also known to be a non-trivial problem [BGMW]. On the other hand, affine varieties arising in mirror symmetry are typically presented as divisor complements, which provides a major impetus for developing these methods.

Remark 1.10. The interplay between the “compactification” and “core” approaches to studying $SH^*(X)$ has been explored in [N]. It would likely be profitable to pursue this interplay further, in light of the fact that frequently one of the two presentations is much simpler than the other for the purpose of computing $SH^*(X)$.

1.4. Overview of paper. In §2 we review the symplectic geometry of normal crossings compactifications, and construct a ‘normal-crossings adapted’ model of the action-filtered symplectic cohomology co-chain complex (this builds off of a model constructed in [GP]) with its ring structure, inducing the multiplicative spectral sequence (1.1). In §3, we define the log cohomology ring $H^*_\log(M, D)$, construct the low-energy Log PSS map (1.2) between log cohomology and the first page of the spectral sequence, and show that low-energy Log PSS is a ring map. In §4, we prove that this map is an isomorphism, and complete the proof of Theorem 1.1. In §5, we establish the various geometric criteria under which the spectral sequence (1.1) degenerates and the ring structure can be computed, and apply these results to give a number of computations and qualitative results: in more detail, degeneration for topological pairs and ring structure computations for multiplicatively topological pairs are discussed in §5.1, trivializing ring deformations in the presence of 2-pointed spheres are discussed in §5.2, degeneration criteria in the presence of (0 and 1-pointed) spheres are discussed in §5.3, and the special case of log Calabi-Yau pairs is discussed in §5.4.

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Conventions. Our grading conventions for symplectic cohomology follow [A1]. We work over an arbitrary ground ring $k$ (unless otherwise stated).

2. A divisorially-adapted model of filtered symplectic cohomology

In this section, we show that the symplectic cohomology complex $SC^*(X)$ of any affine variety $X$ admits a particularly nice “tailored to the divisor $D$” model in the presence of a
projective compactification \((M, D)\), which satisfies two key properties: first, it is generated by (small perturbations of) Morse-Bott families (with corners) of orbits associated to strata of \((M, D)\) and secondly, it possesses an an integral refinement of its (a priori real) action filtration, with integer weights measuring (a weighted sum of) winding of orbits around divisors in \(D\) at infinity. This filtration is compatible with products and induces the desired multiplicative spectral sequence converging to \(SH^*(X)\).

Achieving both of the above properties (and in particular the second property) entails a substantial refinement of our earlier work \([GP, \S 3]\): roughly speaking, to achieve the above integral filtration, orbits contributing to \(SC^*(X)\) which wind a large number of times around \(D\) are made to to occur arbitrarily close (depending on the amount of winding) to \(D\), in order to to compensate for action errors (between the usual symplectic action and the relevant divisorial winding number) that would otherwise magnify with winding. This integral filtration can also be thought of as a limit of the action filtrations on (symplectic cohomologies of) a family of Liouville domains which exhaust \(X\), see Remark 2.20.

### 2.1. Normal crossings symplectic geometry.

**Definition 2.1.** A log-smooth compactification of a smooth complex \(n\)-dimensional affine variety \(X\) is a pair \((M, D)\) with \(M\) a smooth, projective \(n\)-dimensional variety and \(D \subset M\) a divisor satisfying

\[(2.1) \quad X = M \setminus D;\]

\[(2.2) \quad \text{The divisor } D \text{ is normal crossings in the strict sense, e.g., } D := D_1 \cup \cdots \cup D_i \cup \cdots \cup D_k \text{ where } D_i \text{ are smooth components of } D; \text{ and}\]

\[(2.3) \quad \text{There is an ample line bundle } \mathcal{L} \text{ on } M \text{ together with a section } s \in H^0(\mathcal{L}) \text{ whose divisor of zeroes is } \sum_i \kappa_i D_i \text{ with } \kappa_i \in \mathbb{Z}_{>0}.\]

Going forward, fix a log-smooth compactification \((M, D = D_1 \cup \cdots \cup D_k)\). There is a natural induced “divisorial” stratification of \(M\), with strata indexed by subsets of \(\{1, \ldots, k\}\): for \(I \subset \{1, \ldots, k\}\), define

\[(2.4) \quad D_I := \cap_{i \in I} D_i.\]

We refer to the associated open parts of the stratification induced by \(D\) as

\[(2.5) \quad \tilde{D}_I = D_I \setminus \cup_{j \notin I} D_j,\]

and allow \(I = \emptyset\), with the convention

\[D_\emptyset := M\]

\[\tilde{D}_\emptyset = M \setminus D = X.\]

Denote by \(i : X \hookrightarrow M\) the natural inclusion map.

We equip \(M\) with a symplectic form \(\omega\) which is a Kähler form associated to some positive Hermitian metric \(|| \cdot ||\) on \(\mathcal{L}\). On \(X\), consider the potential \(h = -\log ||s||\), where \(s\) is the section given in (2.3). Restricting to \(X\), we have that \(\omega := -dd^c h\). Using \(\theta = -d^c h\), and we further equip the submanifold \(X\) with the structure of a finite type convex symplectic manifold (see e.g., \([M4, \S A]\) for a definition) which, up to deformation, is independent of the compactification or the choice of ample line bundle \(\mathcal{L}\), and equivalent to the canonical structure obtained from an embedding \(X \hookrightarrow \mathbb{C}^N\) \([S3]\).
As is typically done, we find it convenient (for understanding geometry at infinity) to further deform this finite type convex symplectic structure on $X$ to one which is “nice” or “adapted to $D$”, meaning it admits a system of suitably compatible (symplectic, punctured) tubular neighborhoods around each $D_i$ [S3, M4], see Theorems 2.3 and 2.6 below. We first review compatibility properties for such systems of symplectic tubular neighborhoods which make them “nice”, following the comprehensive notion of an $\omega$-regularization developed in [FTMZ] (see also [M6] for a slightly different exposition).

We recall the local standard model for symplectic forms near a symplectic submanifold $Z$ which is either of codimension-2 or an intersection of codimension-2 submanifolds. Let $\pi : L \to Z$ be a real-oriented rank-2 vector bundle equipped with a Hermitian structure $(\rho, \nabla)$, meaning a pair such that, with respect to the complex structure on $j := j_\rho$ uniquely induced by the associated Riemannian metric $\text{Re}(\rho)$ (recall $L$ is rank-2 oriented and $SO(2) = U(1)$), $\rho$ is a Hermitian metric and $\nabla$ is a $\rho$-compatible connection. We use the notation $L_z := \pi^{-1}(z)$ for the fibers of $\pi$, and (by a standard abuse of notation) we also use the abbreviation $\rho(v) := \rho(v, v)$ to refer to the norm-squared function. If $L$ is symplectic vector bundle with symplectic structure $\omega$, we say a Hermitian structure $(\rho, \nabla)$ is compatible with $\omega$ if $\Omega(-, j\rho(-)) = \text{Re}(\rho)$ as usual. The Hermitian structure induces a splitting of the tangent bundle of $L$: $T_L \cong T^\text{vert}_L \oplus T^\text{horiz}_L$ where $(T^\text{vert}_L)$ denotes the vertical sub-bundle of the tangent bundle, which has fibers $\ker(d\pi_p) \cong L_\pi(p)$ (and in particular has a complex structure). Recall that there is a standard angular one-form on the vertical tangent space $d\varphi \in \Gamma(T^\text{vert}^*(L \setminus Z))$ (which, for instance, admits the global definition $d(\log p) \circ j$, where $j$ is the complex structure on the vertical tangent space). Using the Hermitian structure, we lift $d\varphi$ to a connection 1-form $\theta_e \in \Omega^1(L \setminus Z)$ by the prescription that for each $p \in L \setminus Z$,

\begin{align}
(\theta_e)|_{T^\text{vert}_L p} &= (d\varphi) \\
(\theta_e)|_{T^\text{horiz}_L p} &= 0.
\end{align}

If we are further given a symplectic structure $\omega_Z$ on $Z$, we can associate a 2-form on $L$

\begin{equation}
\omega(\rho, \nabla) := \pi^* \omega_Z + \frac{1}{2} d(\rho \theta_e),
\end{equation}

which is symplectic in a neighborhood of $Z$. Similarly, given a collection of Hermitian line bundles $\{L_i = (L_i, \rho_i, \nabla_i)\}_{i \in I}$, we have connection 1-forms $\{\theta_{e,i}\}_{i \in I}$, and we can associate a 2-form on $(\oplus_{i \in I} L_i)$ which is symplectic in a neighborhood of zero:

\begin{equation}
\omega_{\{\rho_i, \nabla_i\}_{i \in I}} := \pi^* \omega_Z + \frac{1}{2} \sum_{i \in I} \pi_{T,i}^* d(\rho_i \theta_{e,i})
\end{equation}

(above, $\pi_{T,i} : \oplus_{i \in I} L_i \to L_j$ is the canonical projection map).

Next we study systems of tubular neighborhoods in the smooth category. For any smooth submanifold $Z \subset M$, let

\begin{equation}
N_M Z
\end{equation}

(or $NZ$ if $M$ is implicit) denote its normal bundle. We implicitly identify $Z$ with its zero section in $NZ$. In [FTMZ, Def. 2.8], a slight strengthening of the notion of a tubular neighborhood is introduced: a regularization of $Z$ is a tubular neighborhood $\psi : U \subset NZ \to M$ such that, under the identification $N_{NZ} Z \cong N_M Z$,

\begin{equation}
N_M Z
\end{equation}

The (normal component of) the derivative $(D\psi)|_Z : N_{NZ} Z = N_M Z \to N_M Z$ is $id_{N_M Z}$.
We recall the normal component of the derivative can be defined as the following composition, where \( \pi : NZ \to Z \) denotes the vector bundle structure:

\[
N_{NZ}Z = \pi^*(NZ)|_Z \xrightarrow{\sim} T^{vert}(NZ)|_Z \hookrightarrow T(NZ)|_Z \xrightarrow{D\psi|_Z} TM|_Z \to N_{MZ}.
\]

Now, let \( \{Z_i\}_{i \in S} \) be a collection of transversely intersecting submanifolds, and denote by \( Z_I := \cap_{i \in I} Z_i \) for \( I \subseteq S \), with \( Z_\emptyset := M \). Note that there are inclusions of normal bundles \( NZ_{Z_I} \subseteq NZ_{Z_{I'}} \) for any \( I'' \subseteq I' \subseteq I \), and in particular the sub-bundles \( NZ_{Z_I} \subseteq N_{MZ} \) ("the local model for \( Z_{I'} \) near \( Z_I \)) give a stratification of \( N_{MZ} \), also inducing a splitting

\[
N_{MZ}|_{Z_{I'}} \cong \bigoplus_{i \in I} N_{Z_{I-I'}}|_{Z_I} \cong \bigoplus_{i \in I} (N_{MZ}|_{Z_I})|_{Z_I}
\]

One can ask for a collection of regularizations for each \( Z_I, \{\psi_I : U_I \to M\}_{I \in S} \) to preserve the stratifications above them in the sense that

\[
(2.13) \quad \psi_{I'} \circ \pi_{I;I'} : NZ_{Z_I} \to Z_{I'}
\]

(c.f., [FTMZ, Def. 2.10]). In order to impose some compatibilities between the \( \psi_I \)'s, we introduce some more notation: for \( I' \subset I \), let

\[
(2.14) \quad \pi_{I;I'} : NZ_{Z_I} \to Z_{I'}
\]

denote the vector bundle projection map. There are canonical identifications

\[
(2.15) \quad \pi_{I;I'} : NZ_{Z_I} \to Z_{I'}
\]

\[
(2.16) \quad N_{MZ}|_{Z_I} = \pi_{I;I'}^*NZ_{Z_{I-I'}}|_{Z_I} = N_{MZ}|_{Z_I}N_{Z_{I-I'}}|_{Z_I}
\]

\[
(2.17) \quad N_{Z_{I-I'}}|_{Z_I} = N_{MZ}|_{Z_I'}|_{Z_I}
\]

and using this, a canonical map

\[
(2.18) \quad i : N_{MZ}|_{Z_I}N_{Z_{I-I'}}|_{Z_I} = \pi_{I;I'}^*NZ_{Z_{I-I'}}|_{Z_I} \hookrightarrow \pi_{I;I'}^*N_{MZ}|_{Z_I} = \pi_{I;I'}^*N_{MZ}|_{Z_I} \to T(N_{MZ})|_{Z_I}
\]

where the last inclusion is by the fact that \( \pi_{I;I'}^*N_{MZ}|_{Z_I} \) is the vertical tangent bundle of \( N_{MZ} \).

Now for any collection of regularizations \( \{\psi_I : U_I \to M\}_{I \in S} \) satisfying (2.14), and any \( I' \subset I \subset S \), \( \psi_I \) induces a diffeomorphism from \( NZ_{Z_I} \cap U_I \) to its image \( Z_{I'} \cap \psi_I(U_I) \), and hence \( d\psi_I \) induces an isomorphism of vector bundles

\[
(2.19) \quad T(N_{MZ})|_{Z_{I-I'}} \cap U_I \xrightarrow{\sim} TM|_{Z_{I-I'}} \cap \psi_I(U_I)
\]

and preserves the sub-tangent bundles associated to the the submanifolds on each side, hence there is a map of normal bundles (or normal derivative) given by

\[
(2.20) \quad D\psi_{I;I'} : N_{MZ}|_{Z_I}N_{Z_{I-I'}}|_{Z_I} \cap U_I = \pi_{I;I'}^*NZ_{Z_{I-I'}}|_{Z_I}N_{Z_{I-I'}}|_{Z_I} \cap U_I \xrightarrow{i} T(N_{MZ})|_{Z_{I-I'}} \cap U_I
\]

\[
\xrightarrow{d\psi_I} TM|_{Z_{I-I'}} \cap \psi_I(U_I) \to N_{MZ}|_{Z_{I-I'}} \cap \psi_I(U_I),
\]

where \( i \) is as in (2.18) the last map is the canonical projection onto the normal bundle (compare also (2.12)). This map allows us to move between the domains of the different tubular neighborhoods, and we can now ask that a collection or regularizations satisfying (2.14) further be compatible with passing to substrata, meaning that

\[
(2.21) \quad D\psi_{I;I'}(U_I) = U_I|_{Z_I \cap \psi_I(U_I)}
\]

\[
\psi_I = \psi_I \circ D\psi_{I;I'}|_{U_I}.
\]
(compare [FTMZ, Def. 2.11]).

Finally we come to the main definition of a “nice system of tubular neighborhoods for symplectic divisors”. For the definition that follows recall that, for a symplectic submanifold \( Z \subset (M, \omega) \), the normal bundle \( N_MZ \) inherits the structure of a symplectic vector bundle (and in particular is canonically oriented), with symplectic structure denoted \( \omega_{N_MZ} \).

**Definition 2.2 ([FTMZ], Def. 2.12).** Let \( \{Z_i\}_{i \in S} \) be a collection of codimension-2 symplectic manifolds of \((M, \omega)\). An \( \omega \)-regularization of \( \{Z_i\}_{i \in S} \) consists of a collection of

- tubular neighborhoods \( \{\psi_i: U_i \to M\}_{i \in S} \) for each \( \{Z_i = \cap_{\ell \in I} Z_i\}_{i \in S} \); and
- Hermitian structures \( \{(\rho_{i,\ell}, \nabla_{I,\ell})\}_{i \in I} \) on the normal bundles \( \{(N_MZ_i)|_{Z_i} = N_{Z_i-(i)}Z_i\}_{i \in I} \)

which are compatible with the canonically induced symplectic structures \( \langle \omega_{N_MZ_i}\rangle_{Z_i} \).

These structures satisfy the following conditions:

1. Each \( \psi_i \) is a regularization in the sense of (2.11);
2. Each \( \psi_i \) preserves the strata above them in the sense of (2.14);
3. The collection \( \{\psi_i: U_i \to M\}_{i \in S} \) is compatible with passing to substrata, in the sense of (2.21);
4. With respect to the decomposition \( N_MZ_i \cong \oplus_{i \in I} N_{Z_i-(i)}Z_i \), \( \psi_i \) is a regularization in the sense of (2.21);
5. The maps \( D\psi_{i,\ell} \) are product Hermitian isomorphisms (meaning a vector bundle isomorphism respecting the direct sum decomposition on both sides and intertwining Hermitian structures on each summand), with respect to the natural splittings of the source and target

\[
(\psi_i)^*\omega = \omega_{\{(\rho_{i,\ell}, \nabla_{I,\ell})\}_{i \in I}}.
\]

This deformation does not change the symplectomorphism type of the complement \( X = M \setminus D \). For notational simplicity, we replace \( \omega \) by \( \hat{\omega} \) going forward, i.e., assume that our divisors \( D \) admit (and come equipped with) an \( \omega \)-regularization \( \{\psi_i: U_i \to M\}_{i \in S} \), which we will henceforth refer to as simply a regularization. Note that the existence of a regularization implies that the divisors \( \{D_i\}_{i=1}^k \) are symplectically orthogonal. In what follows, except when necessary we will drop the parameterizations \( \psi_i \) from our notation and identify the source \( U_i \) with its image in \( M \). We have projection maps

\[
\pi_i: U_i \to D_i
\]
such that on a $|I|$-fold intersection of tubular neighborhoods

$$U_I := \cap_{i \in I} U_i = U_{i_1} \cap \cdots \cap U_{i_{|I|}},$$

iterated projection $|I|$ times

$$\pi_I := \pi_{i_1} \circ \pi_{i_2} \circ \cdots \circ \pi_{i_{|I|}} : U_I \to D_I$$

is a symplectic fibration with structure group $U(1)^{|I|}$ and with fibers symplectomorphic to a(n open subset of a) product of standard discs

$$\pi_I^{-1}(p) \hookrightarrow \prod_{i \in I} D_{\epsilon_i}. \tag{2.26}$$

We also have the radial (norm-squared distance to $U$) functions

$$\rho_i : U_i \to \mathbb{R}. \tag{2.27}$$

An elementary but important consequence of having such an $\omega$-regularization is:

**Lemma 2.4.**

1. The symplectic orthogonal to the tangent space of any fiber $\pi_I^{-1}(p)$ is contained in a level set of the radial function $\rho_i : U_i \to \mathbb{R}$.

2. In particular, if $I = \{i_1, \ldots, i_s\}$, any smooth function $f$ of the corresponding radial functions $f(\rho_{i_1}, \ldots, \rho_{i_s}) : U_I \to \mathbb{R}$, has Hamiltonian vector field $X_f$ tangent to the fibers of $\pi_I$.

3. For $f$ as in (2), if $F = \pi_I^{-1}(p)$ denotes any fiber with its standard (induced by (2.26)) symplectic form,

$$X_f|_F = X_f|_F = \sum_{i \in I} 2 \frac{\partial f}{\partial \rho_i} \partial \varphi_i \tag{2.28}$$

4. Let $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, k\}$ be any subset. In the associated neighborhood $U_I$ of $D_I$, any two functions of the radial coordinates $f(\rho_{i_1}, \ldots, \rho_{i_s})$, $g(\rho_{i_1}, \ldots, \rho_{i_s})$ have commuting Hamiltonian vector fields: $\omega(X_f, X_g) = df(X_g) = -dg(X_f) = 0$. In particular, $d\rho_i(X_f) = 0$ for any $t$ and $f$ as above.

**Proof.** By the regularization property, we can check (1) using the model symplectic form for $D_I$ (2.8), where the result is immediate. To see (2), note that $df = \omega(X_f, -)$ must be zero on vectors tangent to the intersection of level sets of $\rho_{i_1}, \ldots, \rho_{i_s}$. Hence, $X_f$ is orthogonal to the symplectic orthogonal to the tangent space to the fibers, i.e., tangent to the fibers. (3) is an immediate corollary of (2), and the specific form of $X_f|_F$ is a calculation in the fiber $F \subset \prod D_{\epsilon_i}$. (4) can be deduced from (2.28) and the fact that $d\rho_i(\partial \varphi_i) = 0$.

Finally, we recall what it means for the 1-form $\theta$ on $X = M \setminus D$ to be compatible with the regularization chosen above, in the sense of [M4].

**Definition 2.5.** Fix $(M, D)$ be as above, as well as an $\omega$-regularization $\{\psi_I : U_I \to M\}_{I \subset S}$ for $D$. We say a convex symplectic symplectic structure $\theta$ on $X = M \setminus D$ is nice (with respect to the fixed data) if on each $\pi_I := U_I \to D_I$ described in (2.25), $\theta$ restricted to each fiber $(\pi_I)^{-1}(pt)$ agrees with

$$\sum_{i \in I} \frac{1}{2} \rho_i - \frac{\kappa_i}{2\pi})d\varphi_i. \tag{2.26},$$

with respect to the identification of the fibers (2.26), where $\kappa_i \in \mathbb{Z}^{>0}$ as in (2.3).
Theorem 2.6 ([M4, Thm. 5.20]). After possibly shrinking the neighborhoods $U_i$ (and $U_1$), there exists a deformation of the canonical convex symplectic structure $(X, \theta)$ to one $(X, \hat{\theta})$ which is nice.

Going forward, we replace $(X, \theta)$ by the corresponding nice structure, a process which leaves symplectic cohomology unchanged (recall that symplectic cohomology is unchanged under deformations such as those appearing in Theorem 2.6, compare [M4, Lemma 4.11]).

2.2. Normal crossings-adapted Liouville domains and Hamiltonians. This section describes the exhaustive family of Liouville domains as well as the Hamiltonian functions which will be used to define Floer cohomology.

Denote by $UD \subset M$ the union of the neighborhoods of $D_i$ in our regularization (thought of as living in $M$)

$$UD := \cup_i U_i.$$ (2.29)

There is some $\epsilon$ such that $\rho_i^{-1}[0, \epsilon^2] \subset U_i$ (where now we think of $U_i \subset N_M D_i$ and $\rho_i : N_M D_i \to \mathbb{R}$) for all $i$, i.e., each $U_i$ contains a tube of size $\epsilon$. For any $\rho_0 \leq \epsilon^2$ such that $\rho_0 \leq \min_i \frac{2\pi \epsilon^2}{\kappa_i}$, we define the subregion

$$U_{i, \rho_0} := \{ \frac{\rho_i}{\kappa_i}/2\pi \leq \rho_0 \} \subset U_i$$ (2.30)

and

$$UD_{\rho_0} := \cup_i U_{i, \rho_0}.$$ (2.31)

For any $\epsilon_1 \leq \sqrt{\rho_0}$ sufficiently small, we can associate a smoothing of the hypersurface with corners, $\Sigma_{\epsilon_1} := \partial(M \setminus UD_{\epsilon_1})$ which depends on a vector of sufficiently real small numbers $\vec{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$. More precisely, let $q(s) = q_{\epsilon_1, \epsilon_2}(s) : [0, \epsilon_1^2] \to \mathbb{R}$ be a non-negative function (implicitly depending on $\epsilon_1, \epsilon_2$) satisfying:

1. There is some $\epsilon_2 \in (\epsilon_1 - \epsilon_3^2/2, \epsilon_1)$ such that $q(s) = 0$ iff $s \in [\epsilon_2^2, \epsilon_1^2]$.
2. The derivative of $q(s)$ is strictly negative when $q(s) \neq 0$.
3. $q(s) = 1 - s^2$ near $s = 0$.
4. There is a unique point $s = s_0$ with $q''(s_0) = 0$ and $q(s_0) \neq 0$.

(Compare [M2, proof of Theorem 5.16] and [GP, §3.1]) Define

$$S_{\epsilon_1, \epsilon_2}(\rho_1, \cdots, \rho_k) : UD_{\epsilon_1^2} \to \mathbb{R}$$ (2.32)

$$S_{\epsilon_1, \epsilon_2}(\rho_1, \cdots, \rho_k) = \sum_i q\left(\frac{2\pi \rho_i}{\kappa_i}\right).$$ (2.33)

where we implicitly smoothly extend $q\left(\frac{2\pi \rho_i}{\kappa_i}\right)$ to be 0 outside of the region where $\rho_i$ is defined. We let $\partial \hat{X}_\tau = \Sigma_{\epsilon_1} := S_{\epsilon_1, \epsilon_2}^{-1}(\epsilon_3)$ (meaning $\hat{X}_\tau$ is the region bounded by $\Sigma_{\epsilon_1}$ for $\epsilon_3$ sufficiently small). The fourth hypothesis implies that $q''(s) > 0$ when $q$ is sufficiently small, which in turn implies that the function $S_{\epsilon_1, \epsilon_2}$ is convex near $S_{\epsilon_1, \epsilon_2}^{-1}(\epsilon_3)$; this will be useful in understanding the Reeb dynamics on the contact manifolds defined below. Let $Z$ be the Liouville vector field, that is to say the canonical vector field on $X$ determined by the equation:

$$i_Z \omega = \theta$$ (2.34)

Lemma 2.7. For $\epsilon_3$ sufficiently small the Liouville vector field $Z$ associated to $\theta$ is strictly outward pointing along $\Sigma_{\epsilon_1}$; in particular $(\hat{X}_{\tau}, \theta)$ is a Liouville domain.
Proof. This is the content of [GP, Lemma 3.7] and uses the “nice” property of \( \theta \) and the \( \omega \)-regularization of \( D \) (specifically Lemma 2.4); see also [M2, Theorem 5.16] for a similar discussion in the case of concave boundaries. \( \square \)

Remark 2.8. By choosing \( \epsilon_3 \) and \( |\epsilon_2 - \epsilon_1| \) sufficiently small, we can ensure that the rounding \( \partial X_\epsilon \) is arbitrarily \( C^0 \) close to the original cornered domain \( \tilde{\Sigma}_{\epsilon_1} \), a fact which will allow us to obtain explicit estimates on the actions of Hamiltonian orbits.

It follows also that \( \Sigma_\epsilon = \partial X_\epsilon \) admits a contact structure with contact form \( \alpha = \theta |_{\Sigma_\epsilon} \). We recall that any contact manifold equipped with a contact form \( (Y, \alpha) \) possesses a canonical Reeb vector field \( X_{\text{Reeb}} \) determined by \( \alpha(X_{\text{Reeb}}) = 1 \), \( d\alpha(X_{\text{Reeb}}, -) = 0 \). The spectrum of \( (Y, \alpha) \), denoted \( \text{Spec}(Y) \) if \( \alpha \) is implicit, is the set of real numbers which are lengths of closed Reeb orbits of \( X_{\text{Reeb}} \), and is discrete if \( \alpha \) is sufficiently generic.

Any such choice of \( X_\epsilon \) induces a Liouville coordinate defined as
\[
R_\epsilon(x) = e^t,
\]
where \( t \) is the time it takes to flow along \( Z \) from the hypersurface \( \partial X_\epsilon \) to \( x \). Flowing for some small negative time \( t_\epsilon^0 \) defines a collar neighborhood of \( \partial X_\epsilon = \Sigma_\epsilon \),
\[
C(\Sigma_\epsilon) \subset X_\epsilon.
\]
Letting \( X_\epsilon^0 \) denote the complement of this collar in the domain \( X_\epsilon \)
\[
X_\epsilon^0 := X_\epsilon \setminus C(\Sigma_\epsilon),
\]
we see that \( R_\epsilon \) may be viewed as a function \( R_\epsilon : X \setminus X_\epsilon^0 \to \mathbb{R} \). As shown in [GP] (see Lemma 3.10 and the preceding discussion), the fact that our convex symplectic structure is “nice” with respect to our fixed regularization implies that \( R_\epsilon \) is a function of \( \rho_1, \ldots, \rho_k \)
\[
R_\epsilon = R_\epsilon(\rho_1, \rho_2, \ldots, \rho_k)
\]
(meaning in each \( U_I \) it is a function of \( \rho_i \) for \( i \in I \)) and moreover that \( R_\epsilon \) extends smoothly across the divisors \( D \), hence can be viewed as a function (which by abuse of notation we use the same name for):
\[
R_\epsilon : M \setminus X_\epsilon^0 \to \mathbb{R}.
\]
We say a function \( h(r) : \mathbb{R} \to \mathbb{R} \geq 0 \) is linear adapted to \( R_\epsilon \) of slope \( \lambda \) if
\begin{enumerate}
  \item \( h \) vanishes for \( r \leq e^{-t_\epsilon^0} \) (where \( t_\epsilon^0 \) is as before).
  \item \( (h)' \geq 0 \);
  \item \( (h)'' \geq 0 \); and crucially,
  \item (linearity at \( \infty \) of slope \( \lambda \)) for some \( K_\epsilon \) much closer to 1 than \( \min_D R_\epsilon \),
\end{enumerate}
\[
(2.37) \quad h(r) = \lambda(r - 1) \quad \forall r \geq K_\epsilon
\]
Note that because of condition (1), the composition \( h(R_\epsilon) \) is linear outside the compact subset of \( X \) given by \( X_\epsilon^0 \cup (R_\epsilon)^{-1}(-\infty, K_\epsilon] \), and extends smoothly to a Hamiltonian on all of \( M \), which we also call \( h \):
\[
(2.38) \quad h := h(R_\epsilon) : M \to \mathbb{R} \geq 0.
\]
\[\text{Recall } \min_D R_\epsilon > 1, \text{ because } R_\epsilon = 1 \text{ along } \partial X_\epsilon.\]
We often fix the \( K_{\ell} \) for which (2.37) occurs, and call it the linearity level of \( h \). Often we also fix an auxiliary \( \mu_{\ell} \in (K_{\ell}, \min D R^\ell) \) in order to define open neighborhoods of \( D \) \( V_{0,\ell} = (R^\ell)^{-1}(\mu_{\ell}, \infty) \), \( V_{\ell} = (R^\ell)^{-1}(K_{\ell}, \infty) \) and a “(contact) shell” region
\[
V_{\ell} \setminus V_{0,\ell}
\]
along which we require almost complex structures to have a specified (contact-type) form. We often also perform \( C^2 \)-small (time-dependent) perturbations of the functions \( h \) to functions \( H \), taking care that on some shell of the form (2.39), the function is unperturbed. If \( \mathcal{L}X \) denotes the free loop space of \( X \), and \( H : S^1 \times X \to \mathbb{R} \) a (possibly time-dependent) Hamiltonian, recall that the \((H\text{-perturbed})\) action functional \( A_{\ell} : \mathcal{L}X \to \mathbb{R} \) is defined to be
\[
A_{H}(x_0) = -\int_{S^1} x_0^0(\theta) + \int_0^1 H(t, x_0(t)) dt
\]
Now we consider a sequence of these Liouville domains indexed by \( \ell \in \mathbb{Z}^{>0} \). Choose a sequence of integers \( w_\ell \in \mathbb{Z}^{\geq0} \) with \( w_{\ell+1} > w_\ell \) and fix for each \( \ell \), a collection of parameters \( \tilde{\epsilon}_\ell \) needed to define \( \Sigma_{\ell} \). Let
\[
Sigma_{\ell} := \partial \tilde{X}_\ell := \Sigma_{\epsilon_{\ell}}
\]
be the corresponding hypersurface, let \( \tilde{X}_\ell \) be the corresponding domain bounded by this hypersurface, and let \( \epsilon_{\ell} := \epsilon_{1,\ell} \). We assume that our choices are made so that
\[
\tilde{X}_{\ell_1} \subseteq \tilde{X}_{\ell_2} \text{ whenever } \ell_1 < \ell_2.
\]
Let \( R^\ell \) denote the (induced by \( \tilde{X}_\ell \)) Liouville coordinate as in (2.35), and \( X^0_\ell \) the complement in \( \tilde{X}_\ell \) of the time \(-t_{\ell}^0\) flow collar (with respect to \( Z \)) of \( \partial \tilde{X}_\ell \). As before, \( R^\ell \) is a function \( X \setminus X^0_\ell \to \mathbb{R} \) extending smoothly across the divisors \( D \) to a function
\[
R^\ell : M \setminus X^0_\ell \to \mathbb{R},
\]
and moreover is a function of the radial coordinates \( \rho_1, \ldots, \rho_k \). We further choose for each \( \ell \in \mathbb{Z} \) a real number \( \lambda_{\ell} \notin \text{Spec}(\partial \tilde{X}_\ell) \), satisfying
\[
w_\ell < \lambda_{\ell} < w_\ell + 1.
\]
Fix a collection of tuples \( \{(w_\ell, \lambda_{\ell}, \tilde{X}_\ell)\}_{\ell \in \mathbb{Z}^{>0}} \) satisfying the conditions of the previous paragraph. For each \( \ell \), we choose a function \( h^\ell(r) : \mathbb{R} \to \mathbb{R}^{\geq0} \) which is linear adapted to \( R^\ell \) of slope \( \lambda_{\ell} \) with linearity level \( K_{\ell} \), in the sense defined earlier. As before, the composition \( h^\ell(R^\ell) \) extends smoothly to a Hamiltonian on all of \( M \), which we also call \( h^\ell \):
\[
h^\ell := h^\ell(R^\ell) : M \to \mathbb{R}^{\geq0}.
\]
Also as before, we choose an auxiliary constant \( \mu_{\ell} \in (K_{\ell}, \min D R^\ell) \), in order to define two open sets \( V_{0,\ell} \subset V_{\ell} \subset UD \) containing \( D \) by
\[
V_{0,\ell} := (R^\ell)^{-1}(\mu_{\ell}, \infty),
\]
\[
V_{\ell} := (R^\ell)^{-1}(K_{\ell}, \infty)
\]
and the associated contact shell \( V_{\ell} \setminus V_{0,\ell} \). It is straightforward to ensure (and henceforth we assume) that
\[
\text{for } \ell \neq \ell', \text{ the contact shells' } V_{\ell} \setminus V_{0,\ell} \text{ and } V_{\ell'} \setminus V_{0,\ell'} \text{ are disjoint.}
\]
Compared to Section 3 of \([GP]\), we will shortly take \( \tilde{X}_\ell \) to be an exhaustive family of Liouville manifolds for \( X \) (that is let \( \epsilon_{\ell} \to 0 \)). Note that, in particular, the \( C^0 \)-norm of Hamiltonian \( h^\ell \) can be assumed arbitrarily small (for a given slope \( \lambda_{\ell} \)) by making \( \epsilon_{\ell} \)
sufficiently small and $K_\ell$ sufficiently close to 1. In fact, along the divisors, we have the following estimate for $h^\ell$:

**Lemma 2.9.** If $\Sigma_\ell$ is sufficiently $C^0$ close to $\tilde{\Sigma}_\ell$ (as in Remark 2.8), then on $D$ the following estimate holds:

\[
h^\ell \approx \lambda_\ell \left( \frac{1}{1 - \frac{1}{2} \epsilon_\ell^2} - 1 \right).\tag{2.48}
\]

**Proof.** Given (2.37), the estimate (2.48) is equivalent to the following estimate for $R$ at points of $D$:

\[
R \approx \frac{1}{1 - \frac{1}{2} \epsilon_\ell^2}. \tag{2.49}
\]

To see this, let $p \in D$ be any point; it is contained $D_I \subset U_I$ for some $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, k\}$. By the discussion in [GP, p. 15] the time it takes for $p$ to flow by $-Z$ to $\Sigma_\ell$ is the time it takes for $p$ to flow by $-Z_{\text{vert}}$ to $\Sigma_\ell \cap U_I$, where $Z_{\text{vert}}$, the vertical component of $Z$ with respect to the symplectic fibration $\pi_I : U_I \to D_I$ is given by:

\[
Z_{\text{vert}} = \sum_{i \in I} (\rho_i - \frac{\kappa_i}{\pi}) \partial \rho_i. \tag{2.50}
\]

This time now only (approximately) depends on a computation in the coordinates $(\rho_{i_1}, \ldots, \rho_{i_s}) \in \mathbb{R}^{|I|}$ as we note that for any $x \in \Sigma_\ell \cap U_I \approx \tilde{\Sigma}_\ell \cap U_I$, $(\rho_i(x), \ldots, \rho_{i_s}(x)) \approx (\epsilon_\ell^2 \frac{\kappa_{i_1}}{2\pi}, \ldots, \epsilon_\ell^2 \frac{\kappa_{i_s}}{2\pi})$ (and $(\rho_{i_1}(p), \ldots, \rho_{i_s}(p)) = 0$). Thus it (approximately) suffices to compute the (exponential of the) time it takes to flow in $\mathbb{R}^{|I|}$ from 0 to $(\epsilon_\ell^2 \frac{\kappa_{i_1}}{2\pi}, \ldots, \epsilon_\ell^2 \frac{\kappa_{i_s}}{2\pi})$ by $-Z_{\text{vert}}$; this can be done one component at a time as everything is split. For $j \in I$, let $\gamma(t)$ be an integral curve of $Z_{\text{vert}}^j = (\rho_j - \frac{\kappa_j}{\pi}) \partial \rho_j$ starting at 0, i.e., $\gamma(t)$ solves the ODE $\gamma'(t) = \gamma(t) - \frac{\kappa_j}{\pi}$ with initial condition $\gamma(0) = 0$, i.e., $\gamma(t) = -\frac{\kappa_j}{\pi} e^{-t} + \frac{\kappa_j}{\pi}$. We find $\gamma(t) = \epsilon_\ell^2 \frac{\kappa_j}{2\pi} \frac{t}{\epsilon_\ell}$ precisely when $e^t = \frac{1}{1 - \frac{1}{2} \epsilon_\ell^2}$.

Because the Hamiltonian flow of $h^\ell$ preserves $D$, the time-1 orbits of this flow are either completely contained in $D$ or completely contained in $X$ (in fact $X \setminus V_\ell$, in light of the form (2.37) of $h^\ell$ in $V_\ell \setminus D$). We will refer to the set of (time-1) orbits contained in $D$ as divisorial orbits and denote them by

\[
\mathcal{X}(D; h^\ell) \tag{2.51}
\]

and all other (time-1) orbits by

\[
\mathcal{X}(X; h^\ell). \tag{2.52}
\]

We first describe the orbits of the Hamiltonian flow of $h^\ell$ inside of $M \setminus V_\ell$. Let

\[
R_{0,\ell} := \max \{ R^\ell \mid h^\ell(R^\ell) = 0 \}. \tag{2.53}
\]

be the largest value of $R^\ell$ for which $h^\ell(R^\ell) = 0$. The time-1 orbits of the Hamiltonian $h^\ell$ lying in the region $M \setminus V_\ell$ come in two types of families. The first is the set of constant orbits which is the complement

\[
J_0 := M \setminus \{ R^\ell \geq R_{0,\ell} \}. \tag{2.54}
\]

This is a manifold with boundary.
The second type of Hamiltonian orbit corresponds to non-constant orbits. By Lemma 2.4, over each stratum $U_I$ the Hamiltonian flow is tangent to the fibers of the projections $\pi_I : U_I \to D_I$ and has the explicit form
\[
X_{h^\ell} = \sum_{i \in I} 2 \frac{\partial h^\ell}{\partial \rho_i} \partial \varphi_i.
\] (2.55)

The orbits of (2.55) in $U_I$ correspond to (possibly multiply-covered) circles in any fiber of $\pi_I$ where
\[
X_{h^\ell} = \sum_{i \in I} -2\pi v_i \partial \varphi_i
\] (2.56)

for an integer vector $v = (v_1, \ldots, v_k) \in \mathbb{Z}_{\geq 0}^k$ which has non-zero $i$th component precisely when $i \in I \subset \{1, \ldots, k\}$; in other words, wherever $\frac{\partial h^\ell}{\partial \rho_i} = -\pi v_i$. These sets, denoted by
\[
\mathcal{F}_v
\] (2.57)
are connected in view of item (4) in the definition of $(q$ and hence) $S_{\ell_1,\ell_2}$ and are homeomorphic to manifolds with corners. For proofs of these assertions, see Step 2 of the proof of Theorem 5.16 of [M2]. The multiplicity vector associated to an orbit $x_0$, $v(x_0)$, is the unique $v$ so that $x_0 \in \mathcal{F}_v$ (where $v = 0$ if $x_0 \in \mathcal{F}_0$). We define the weighted winding number of a multiplicity vector $v$ to be
\[
w(v) := \sum_{i \in \{1, \ldots, k\}} \kappa_i v_i = \sum_{i \in I = \text{support}(v)} \kappa_i v_i,
\] (2.58)

and the weighted winding number of an orbit $x_0$ to be
\[
w(x_0) := w(v(x_0))
\] (2.59)
(note these are integers). The weight $w(v)$ will appear geometrically in our setup as the limiting action of (meaning integral of $\theta$ over) any sufficiently small loop in $X$ winding $v_i$ times around $D_i$ (compare [GP, Lemma 2.11] or Lemma 3.14 below).

We can make all of the orbits nondegenerate by a $C^2$ small time-dependent perturbation $H^\ell : M \to \mathbb{R}$. Describing careful choices of $C^2$ small time-dependent perturbations for the divisorial orbits as well as near orbits in $X$ would take us on a small detour and so we postpone this to §4.1. For now we just state the two most important properties of these $C^2$-small perturbations:

- The perturbation is disjoint from $V_\ell \setminus V_{0,\ell}$ and inside of $M \setminus V_\ell$, it is supported in the disjoint union of small isolating neighborhoods $U_v$ of the orbit sets $\mathcal{F}_v$.
- The Hamiltonian flow of $H^\ell$ preserves each divisor $D_i$.

As before we let
\[
\mathcal{X}(D; H^\ell)
\] (2.60)
denote the time-1 orbits of $H^\ell$ contained in $D$ and let
\[
\mathcal{X}(X; H^\ell)
\] (2.61)
denote the remaining time-1 orbits of $H^\ell$, which are disjoint from $D$ (and lie in $X \setminus V_\ell$). Let $A_\ell := A_{H^\ell}$ denote the ($H^\ell$-perturbed) action functional as in (2.40) If we assume that $\Sigma_\ell$
is taken sufficiently $C^0$ close to $\hat{\Sigma}_\ell$, we also obtain good estimates on the actions of our Hamiltonian orbits:

**Lemma 2.10.** By taking $\Sigma_\ell$ sufficiently $C^0$ close to $\hat{\Sigma}_\ell$, $K_\ell$ sufficiently close to 1 and $\ell^2_0$ sufficiently small, the action of each orbit set $F_\nu$ can be made arbitrarily close to

$$A_\ell(x_0) \approx -w(x_0)(1 - \epsilon_\ell^2/2)$$

**(Proof.** The orbit set $F_\nu$ lies in the region where $R_\ell \leq K_\ell$. By assuming $K_\ell$ is sufficiently close to 1, the Hamiltonian term can be taken arbitrarily small and so we focus on the contact term. Because the orbits lie in the fibers, the action may be calculated by:

$$\sum_i \int_{x_0} \left(\frac{1}{2}\rho_i - \frac{\kappa_i}{2\pi}\right)d\varphi_i$$

$$= -\sum_i \left(\frac{1}{2}\rho_i - \frac{\kappa_i}{2\pi}\right)(-2\pi v_i)$$

By taking $\ell^2_0$ and $K_\ell$ close to zero, we also have that $\rho_i$ can be taken arbitrarily close to $(\kappa_i/2\pi)(\epsilon_\ell)$ thus completing the computation. □

**Definition 2.11.** Define $\mathcal{J}(M,D)$ to be the space of $\omega$-tamed almost complex structures $J$ which preserve $D$ and such that

1. for any $i \in \{1, \cdots, k\}$, $p \in D_i$, and tangent vectors $\eta_1, \eta_2 \in T_pM$, the Nijenhuis tensor $N_J(\eta_1, \eta_2) \in T_pD_i$.

**Definition 2.12.** For any choice of Liouville domain and shells $\bar{X}_\epsilon, V_\epsilon, V_0, \epsilon$, define $\mathcal{J}(\bar{X}_\epsilon, V) \subset \mathcal{J}(M,D)$ to be the space of $\omega$-compatible almost complex structures which are of contact type on the closure of $V_\epsilon \setminus V_0, \epsilon$, meaning on this region

$$\theta \circ J = -dR^\epsilon.$$  

For the case when $\bar{X}_\epsilon = \bar{X}_\ell, V_\epsilon = V_\ell, V_0, \epsilon = V_0, \ell$, we simplify the notation by

$$\mathcal{J}_\ell(V) := \mathcal{J}(\bar{X}_\ell, V).$$

It follows by standard arguments that these spaces are non-empty (see e.g., [I, §A]) and contractible.

### 2.3. Floer cohomology

Choose a Hamiltonian $H^\ell$ for each $\ell$, by perturbing the Hamiltonians $h^\ell$ as in the previous section. For each Hamiltonian orbit $x \in X(X; H^\ell)$, a choice of trivialization $\gamma$ of $x^*(TX)$ determines a 1-dimensional real vector space

$$\mathfrak{o}_{x},$$

the determinant line associated to a local Cauchy-Riemann operator $D_{\gamma}$. Implicitly this depends on the choice of trivialization $\gamma$, but for notational simplicity we remove this ambiguity (assuming $c_1(X) = 0$) by working in the $\mathbb{Z}$-graded context: we fix a holomorphic volume form $\Omega_{M,D}$ on $M$ which is non-vanishing on $X$, and to define (2.66) choose the unique trivialization $\gamma$ of $x^*TX$ compatible with the trivialization of $\Lambda^\ast_TTX$ induced by $\Omega_{M,D}$. Recall also that the $k$-normalization of any vector space $W$, denoted

$$|W|,$
is the free $k$ module generated by the set of orientations of $W$, modulo the relation that the sum of the orientations vanishes. We call $|o_x|$ the orientation line associated to $x$, and define

\begin{equation}
CF^*(X \subset M; H^\ell) := \bigoplus_{x \in X(X; H^\ell)} |o_x|.
\end{equation}

**Definition 2.13.** Let $J_{F, \ell}(V)$ denote the space of $S^1$ dependent complex structures, $C^\infty(S^1; J_{F, \ell}(V))$.

Choose a generic $S^1$-dependent almost complex structure $J_F \in J_{F, \ell}(V)$. For pairs of orbits $x_0, x_1 \in X(X; H^\ell)$, let $\widetilde{M}(x_0, x_1)$ denote the moduli space of Floer trajectories between $x_1$ and $x_0$, namely the space of solutions to the following PDE with asymptotics:

\begin{equation}
\begin{aligned}
&\left\{ \begin{array}{ll}
u: \mathbb{R} \times S^1 & \to X, \\
&\lim_{s \to -\infty} u(s, -) = x_0 \\
&\lim_{s \to +\infty} u(s, -) = x_1 \\
&\partial_s u + J_F(\partial_t u - X_{H^\ell}) = 0.
\end{array} \right.
\end{aligned}
\end{equation}

For generic $J_F$, $\widetilde{M}(x_0, x_1)$ is a manifold of dimension

\[\deg(x_0) - \deg(x_1)\]

where $\deg(x)$, the index of the local operator $D_x$ associated to $x$, is equal to $n - CZ(x)$, where $CZ(x)$ is the Conley-Zehnder index of $x$ [FHS]. There is an induced $\mathbb{R}$-action on the moduli space $\tilde{M}(x_0, x_1)$ given by translation in the $s$-direction, which is free for non-constant solutions. Whenever $\deg(x_0) - \deg(x_1) \geq 1$, for a generic $J_F$ the quotient space

\begin{equation}
M(x_0, x_1) := \widetilde{M}(x_0, x_1)/\mathbb{R}
\end{equation}

is a manifold of dimension $\deg(x_0) - \deg(x_1) - 1$. Whenever $\deg(x_0) - \deg(x_1) = 1$ (and $J_F$ is generically chosen), standard Gromov compactness arguments show that the moduli space (2.70) is compact of dimension 0, provided elements of $\widetilde{M}(x_0, x_1)$ have image contained in some compact subset $K \subset X$ only depending on $x_0$ and $x_1$. In turn this latter a priori $C^0$ bound (away from $D$) is a consequence of standard maximum principle arguments, which prevent Floer solutions from crossing the region where the Hamiltonian has the special form (2.37) and $J_F$ is also contact type; see e.g., [AS, Lem. 7.2]. Now orientation theory associates, to every rigid element $u \in \mathcal{M}(x_0, x_1)$ an isomorphism of orientation lines $\mu_u : o_{x_1} \xrightarrow{\sim} o_{x_0}$ and hence an induced map $\mu_u : |o_{x_1}| \to |o_{x_0}|$. Using this, one defines the $|o_{x_1}| - |o_{x_0}|$ component of the differential

\begin{equation}
(\partial_{CF})_{x_1, x_0} = \sum_{u \in \mathcal{M}(x_0, x_1)} \mu_u
\end{equation}

whenever $\deg(x_0) = \deg(x_1) + 1$. A standard analysis of the boundary of the (compactified by adding broken trajectories) 1-dimensional components of (2.70) implies that $\partial_{CF}^2 = 0$.\footnote{Once more, this requires establishing that relevant Floer trajectories are a priori bounded away from $D$, which is a consequence of the maximum principle mentioned earlier.} We define $HF^*(X \subset M; H^\ell)$ to be the cohomology of the complex $(CF^*(X \subset M; H^\ell), \partial_{CF})$.\footnote{Once more, this requires establishing that relevant Floer trajectories are a priori bounded away from $D$, which is a consequence of the maximum principle mentioned earlier.}
• the choice of $H^\ell$ satisfying (2.37) with fixed slope $\lambda_\ell$ and $K_\ell$ as well as the choice of (generic) $J_\ell$ in $\mathcal{F}_{\ell}(V)$ (with respect to a fixed contact shell).

• Moreover, it is independent of the choice of $K_\ell$ arising in the definition of $H^\ell$ and the choice of contact shell $V_\ell \setminus V_\ell, 0$ arising in the definition of $\mathcal{F}_{\ell}(V)$.

Proof. The first assertion is a consequence of standard arguments. The second is too, but we give a brief discussion: first, the maximum principle implies there is a bijection of chain complexes if we shrink the shell region along which $J_\ell$ is contact type to $V_\ell \setminus V_\ell, 0$ (the point being that Floer cylinders for any such $J$ cannot even cross $\partial V_\ell$). Next, given two different shells $S = V_\ell \setminus V_\ell, 0$ and $T = V_\ell' \setminus V_\ell', 0$, if $V_\ell = V_\ell'$, the shrinking argument implies we are done; otherwise, we can shrink $V_\ell, 0, V_\ell', 0$ so that the two shell regions are disjoint, without loss of generality say $V_\ell, 0 \subset V'_\ell \subset V_\ell, 0 \subset V_\ell$. Now, starting with a $J$ which is contact type for $S$, we can simultaneously make it contact type for $T$ without changing the Floer complex at all (as Floer cylinders don’t cross $S$). Finally, we can turn off the contact type condition on $S$, which could possibly change the Floer complex on the chain level, but does not change the cohomology, by the first assertion in the Lemma. \hfill \Box

For any $w$, let $F_w CF^\ast(X \subset M; H^\ell)$ denote the sub-$k$-module generated by those orbits $x(X; H^\ell) \leq w$ with $w(x_0) \leq w$:

\begin{equation}
F_w CF^\ast(X \subset M; H^\ell) := \bigoplus_{x \in \chi(X; H^\ell) \leq w} |\partial x|
\end{equation}

It follows from Equation (2.62) and the well known fact that $\partial_C F$ strictly increases action that

Lemma 2.15. If $\epsilon_\ell$ is sufficiently small, $\Sigma_\ell$ is sufficiently $C^0$ close to $\tilde{\Sigma}_\ell$, and if $C^2$ small perturbations are used when defining $H^\ell$, then the differential $\partial_C F$ preserves the submodule $F_w CF^\ast(X \subset M; H^\ell)$. In particular, (2.72) is filtration of chain complexes. \hfill \Box

We let $F_w HF^\ast(X \subset M; H^\ell)$ denote the filtration on $HF^\ast(X \subset M; H^\ell)$ induced by the cochain level filtration $F_w CF^\ast(X \subset M; H^\ell)$. Throughout the rest of the paper,

Choose $\epsilon_\ell$, $\Sigma_\ell$, and $H^\ell$ so that Lemma 2.15 holds for each $\ell \in \mathbb{N}$.

The $\epsilon_\ell$ guaranteed by the proof of Lemma 2.15 tend to zero as $\ell \to \infty$ and the $\tilde{X}_\ell$ form an exhaustive family of domains. We next turn to defining continuation maps:

\begin{equation}
\begin{aligned}
c_{\ell_1, \ell_2} : & HF^\ast(X \subset M; H^{\ell_1}) \to HF^\ast(X \subset M; H^{\ell_2}) \\
& \text{(note we will also use $c_{\ell_1, \ell_2}$ to refer to the chain-level maps). Typically, continuation maps between Floer complexes associated to Hamiltonians $H_a$ and $H_b$ are given by counting solutions to Floer’s equation with respect to a domain dependent Hamiltonian (and complex structure) varying between $H_a$ and $H_b$ (as well as the respective complex structures). Of course, in this non-compact setting, some care must be taken to ensure that solutions are $C^0$ bounded away from $D$, and hence that the requisite Gromov compactness results hold, and the situation is slightly requires slightly more subtle arguments than usual (in which one notes a maximum principle holds provided the surface dependent Hamiltonian is monotone), in light of the fact that even when the slope of $H_a$ is less than the slope of $H_b$, it may not be possible to ensure monotonicity of the interpolation. To constrain the situation above with the usual construction of symplectic cohomology, or rather its variant given in [GP], recall that in loc. cit. we fixed once and for all a single Liouville domain $\overline{X}_{\epsilon_0} \subset X$ (which determined a single function $R : M \times X_{\epsilon_0}^\ast \to \mathbb{R}$), and considered

We let $F_w HF^\ast(X \subset M; H^\ell)$ denote the filtration on $HF^\ast(X \subset M; H^\ell)$ induced by the cochain level filtration $F_w CF^\ast(X \subset M; H^\ell)$. Throughout the rest of the paper,
\begin{itemize}
\item functions $G^\ell$ which are $C^2$ small perturbations of the functions $g^\ell$ which are linear adapted to $R$ of slope $\lambda_\ell$, and moreover, say, equal to $\lambda_1 h^1(R)$ for a fixed $h^1(R)$ linear adapted to $R$ of slope 1; and
\item (sufficiently generic) $S^1$-dependent almost complex structures $J_\ell \in J(M, D)$ which are contact type with respect to the function $R$ on a fixed shell $V \setminus V_0$, where $V = R^{-1}(K, \infty)$ and $V_0 = R^{-1}(\mu, \infty)$ for some $\mu$.
\end{itemize}

This data defines a chain complex $CF^*(X \subset M; G^\ell)$ as above (and note that $R$ and $J$ do not depend on $\ell$). Continuation maps for any $\ell_1 \leq \ell_2$ were constructed by counting solutions to Floer’s equation with respect to a monotone homotopy of Hamiltonians $G_{s,t}$ between $G^{\ell_1}$ and $G^{\ell_2}$ (i.e. a family of functions satisfying $\partial_s G_{s,t} \leq 0$) which take on the standard form above $R = K$,

\begin{equation}
G_{s,t} = \lambda_s (R - 1),
\end{equation}

for $\lambda_s$ a monotone homotopy between $\lambda_{\ell_1}$ and $\lambda_{\ell_2}$, as well as generic families of almost complex structures satisfying the same conditions as above. In light of the standard form (2.75) and the contact-type condition of the almost complex structures chosen, and monotonicity, solutions of the continuation map equation satisfy a maximum principle (see e.g., [AS, Lem. 7.2]), implying by the usual analysis that the counts of such solutions give a chain map. Using this system of maps, we set

$$SH^*(X, \ell) := \lim_{\ell \to} HF^*(X \subset M; G^\ell).$$

The situation we will need to consider in the present paper somewhat more delicate because each $H^\ell$ is constructed using a different $X_\ell$ (and hence different radial function $R^\ell$) for a sequence of exhaustive domains $X_\ell$. Hence the standard form (2.75) for a single function $R$ (on a region where $J$ is also contact type for the same $R$) may be impossible to arrange; i.e., it may not be possible to construct strictly monotone homotopies for pairs $\ell_1 < \ell_2$. Nevertheless, we can use homotopies which are “monotone up to a sufficiently small error”, as we now describe. We will forget for a moment about our family of $H^\ell$ chosen and give slightly more general criteria under which a continuation map exists.

To do so, let $X_a$ and $X_b$ denote any pair of Liouville domains in $X$ constructed as in the previous section using parameters $\vec{c}_a$, $\vec{c}_b$; and let $R^a$ and $R^b$ be respective induced Liouville coordinates as in (2.35). For simplicity, we assume that $\partial X_a$ and $\partial X_b$ are disjoint, so that one domain strictly contains the other. We let $R^\text{out}$, $K^\text{out}$ denote the Liouville coordinate and $K^\text{out}$ constant corresponding to the bigger domain.

Pick Hamiltonians $h^a$, $h^b$ which are linear adapted to $R^a$ respectively $R^b$ of (generic) slope $\lambda^a$ respectively $\lambda^b$, with linearity levels $K^a$ and $K^b$ respectively, and denote by $H^a$ and $H^b$ $C^2$-small perturbations of these respective Hamiltonians so that all orbits are non-degenerate, and so that the perturbation is trivial on certain contact shells $V_0 \setminus V_{a,0}$ and $V_0 \setminus V_{b,0}$. The discussion so far gives, for generic time-dependent almost complex structures $J^a_t, J^b_t \in J(M, D)$ (which are contact type on the respective shell-regions) Floer complexes $CF^*(X \subset M, H^a)$ and $CF^*(X \subset M, H^b)$ respectively. Using the analysis in Lemma 2.14, it is safe to assume that

\begin{itemize}
\item the $K^\ell$ (hence $V_\ell$) and $V_0, V_\ell$ on the inner domain have been shrunk so that the contact shell for the inner domain is disjoint from the contact shell for the outer domain.
\end{itemize}
Now, let $\rho(s)$ be a non-negative, monotone non-increasing cutoff function such that
\begin{equation}
(2.76) \quad \rho(s) = \begin{cases} 
0 & s \gg 0 \\
1 & s \ll 0 
\end{cases}
\end{equation}
Set
\begin{equation}
(2.77) \quad H_{s,t} = (1 - \rho(s))H^a + \rho(s)H^b
\end{equation}
Up to a small perturbation near $D$, we claim that
\begin{align}
(2.78) \quad & \theta(X_{H_{s,t}}) = H_{s,t} + \lambda_s, \\
(2.79) \quad & dR^{\text{out}}(X_{H_{s,t}}) = 0
\end{align}
whenever $R^{\text{out}} \geq K^{\text{out}}$, where
\begin{equation}
(2.80) \quad \lambda_s := (1 - \rho(s))\lambda_a + \rho(s)\lambda_b
\end{equation}
is monotone if $\lambda_b \geq \lambda_a$. The first equation (2.78) is a basic consequence of linearity (2.37) of both Hamiltonians and the fact that for any radial coordinate $R$ as above, $\theta(X_{H(R)}) = \omega(X_H, Z) = dH(Z) = RH'(R)$. The second equation (2.79) follows from the fact that in each $U_I$ above $R^{\text{out}} = K^{\text{out}}$, $H_{s,t}$ only depends on the radial coordinates $\rho_i$ for $i \in I$, given that it is a linear function of $R^a$ and $R^b$ in this region, hence $df(X_{H_{s,t}}) = 0$ is zero for any function $f$ of $\{\rho_i\}_{i \in I}$ (see Lemma 2.4 item (4)).

Let $J_{s,t}$ be a generic compatible $\mathbb{R} \times S^1$ dependent almost complex structure which
\begin{itemize}
  \item is of contact type on the outer contact shell for all $s, t$.
  \item agrees at $s = \pm \infty$ with the choices of $J^a_t$ and $J^b_t$ defined earlier.
\end{itemize}
If $x_1$ is an orbit of $H^a$ and $x_2$ is an orbit of $H^b$, let $M_s(x_2, x_1)$ denote the moduli space of maps $u : \mathbb{R} \times S^1 \to X$ satisfying Floer’s equation for $H_{s,t}$ and $J_{s,t}$:
\begin{equation}
\partial_s u + J_{s,t}(\partial_t u - X_{H_{s,t}}) = 0
\end{equation}
which in addition satisfy requisite asymptotics:
\begin{equation}
(2.81) \quad \begin{cases}
\lim_{s \to -\infty} u(s, -) = x_2 \\
\lim_{s \to +\infty} u(s, -) = x_1
\end{cases}
\end{equation}
As usual, one defines the $|x_1| - |x_2|$ component of the continuation map (2.74) by counting rigid elements $u \in M_s(x_2, x_1)$ (for a suitably generic $J_{s,t}$).

To establish the necessary estimates, note that up to arbitrarily small error, we have that for any map $u : \mathbb{R} \times S^1 \to M$ and any closed $S \subset \mathbb{R} \times S^1$,
\begin{equation}
(2.82) \quad \int_S u^*(\partial_s H_{s,t}) ds dt \leq \sup(H^a - H^b) < \sup(H^a).
\end{equation}
We now establish the necessary Gromov compactness result for these continuation solutions:

**Lemma 2.16.** Let $H_{s,t}$ be as above, let $\deg(x_2) - \deg(x_1) \leq 1$, and suppose that either
\begin{enumerate}
  \item[(a)] (strict monotonicity at $\infty$) $\lambda_b > \lambda_a$ (meaning specifically that $\lambda_b > \lambda_a \frac{R^a - 1}{R^b - 1}$) on the region $R^{\text{out}} \geq K^{\text{out}}$ and the action of $x_1$ with respect to $A_a := A_{H^a}$ satisfies
  \begin{equation}
  (2.83) \quad -A_a(x_1) < \lambda_a,
  \end{equation}
  \item[(b)] (monotonicity of slopes with bounded error) $\lambda_b \geq \lambda_a$ and
  \begin{equation}
  (2.84) \quad -A_a(x_1) + \sup(H^a) < \lambda_a.
  \end{equation}
\end{enumerate}
Let $\overline{\mathcal{M}}_s(X; x_2, x_1)$ denote the Gromov compactification of $\mathcal{M}_s(x_2, x_1)$ in $M$. Then,

- if $\deg(x_2) - \deg(x_1) = 0$, $\overline{\mathcal{M}}_s(X; x_2, x_1) = \mathcal{M}_s(X; x_2, x_1)$ i.e. the moduli space is compact.
- if $\deg(x_2) - \deg(x_1) = 1$, $\partial \overline{\mathcal{M}}_s(x_2, x_1) = \partial_1 \overline{\mathcal{M}}_s(x_2, x_1) \cup \partial_2 \overline{\mathcal{M}}_s(x_2, x_1)$ where

$$\partial_1 \overline{\mathcal{M}}_s(x_2, x_1) = \bigcup_{y, \deg(y) - \deg(x_1) = 1} \mathcal{M}(x_2, y) \times \mathcal{M}_s(y, x_1) \tag{2.85}$$

$$\partial_2 \overline{\mathcal{M}}_s(x_2, x_1) = \bigcup_{y, \deg(y) - \deg(x_1) = 0} \mathcal{M}(x_2, y) \times \mathcal{M}_s(y, x_1) \tag{2.86}$$

**Proof.** Gromov compactness in $M$ implies that the Gromov-Floer compactification of these moduli spaces as maps to $M$ are compact. One needs to show therefore that this compactification only contains broken solutions in $X$ (not intersecting $D$), i.e., elements of these moduli spaces do not limit to broken solutions intersecting $D$ (After this, standard transversality and gluing arguments imply the desired result).

The argument, like several others in this paper, follows the pattern of [GP, Lemma 4.13]. Namely, we first make the key claim that

(i) a broken Floer solution cannot break along an orbit in $X(D, H^f)$.

If this is true, it follows that any broken Floer curve $\tilde{u}$ in $M$ in the limit of trajectories above has (all asymptotics in $X$ and hence) a well-defined total topological intersection number with $D$, equal to 0 (the original intersection number) and additive over its components, and positive over all components not completely contained in $D$ (which includes all Floer trajectories), see e.g. Lemma 4.13 of *loc. cit.*. If there were holomorphic sphere bubbles in $\tilde{u}$, the disjoint union of all of these bubbles necessarily give an $H_2$ class which has positive symplectic area hence positive intersection with some $D_i$, implying the remaining broken Floer trajectory must have negative intersection number with $D_i$, a contradiction. So,

(ii) No holomorphic sphere bubbles can form in the limit of broken Floer trajectories.

Finally, the remaining broken Floer trajectories satisfy positivity of intersection with $D$, hence (since the total intersection number is 0) do not intersect $D$ as desired.

So it remains to show (i). For example suppose that there is $y \in X(D, H^f)$ together with a sequence in $\overline{\mathcal{M}}_s(X; x_2, x_1)$ limiting to a configuration $u_1 \in \mathcal{M}(y, x_1)$ and $u_2 \in \overline{\mathcal{M}}_s(x_2, y)$. Consider the piece of the curve $\tilde{S} := u_2^{-1}((R^\text{out})^{-1}[K^\text{out}, \infty))$ which lives above the slice where $R^\text{out} = K^\text{out}$ (note in this region, we have both $R^a \geq K^a$ and $R^b \geq K^b$ and both of our functions are linear with respect to their respective coordinates). Then, if $\tilde{S}$ denotes the union of $\tilde{S}$ with the domain of $u_1$, the geometric energy of $\tilde{S}$ (see [GP] or the discussion around (2.100) below for a review of this concept) satisfies the following inequality

$$E_{\text{geo}}(u_2|\tilde{S}) \leq E_{\text{geo}}(u|\tilde{S}) \tag{2.87}$$

$$\leq E_{\text{top}}(u|\tilde{S}) + \int_{\tilde{S}} u^*(\partial_s H_{s,t}) ds dt \tag{2.88}$$

$$- A_a(x_1) + \int_{\partial \tilde{S}} u^* \theta - H_{s,t} dt + \int_{\tilde{S}} u^*(\partial_s H_{s,t}) ds dt \tag{2.89}$$

$$\leq - A_a(x_1) + \{0, \sup(H^a)\} + \int_{\partial \tilde{S}} u^* \theta - H_{s,t} dt \tag{2.90}$$

$$= - A_a(x_1) + \{0, \sup(H^a)\} + \int_{\partial \tilde{S}} u^* \theta - \theta(X_{H,s,t}) dt + \int_{\tilde{S}} \lambda_{s,t} dt \tag{2.91}$$
where $E_{\text{top}}$ denotes the topological energy of a map, defined in [GP] or (2.101) below, (2.89) follows from Stokes’ theorem\(^8\), and the terminology \{0, sup$(H^a)$\} in (2.90) means one should use 0 in case (a) of the Lemma (by strict monotonicity of $H_{s,t}$ in the region $R_{\text{out}} \geq K_{\text{out}}$ in this case) and sup $H^a$ in case (b) by (2.82). Going forward we will just assume that term is sup $H^a$, as that case is strictly more difficult.

By Stokes’ theorem we have that
\[(2.92) \quad \int_{\partial S} \lambda_s dt = -\int_y \lambda_s dt + \int_S d(\lambda_s dt) \leq -\lambda_a\]
where the last inequality used $\lambda_a \leq \lambda_b$ and monotonicity of $\lambda_s$, as defined in (2.80). Therefore it follows from (2.84) that
\[(2.93) \quad -A_a(x_1) + \sup(H^a) + \int_{\partial S} u^* \theta - \theta(X_{H_{s,t}}) dt + \int_S \lambda_s dt \leq \int_{\partial S} u^* \theta - \theta(X_{H_{s,t}}) dt\]
The rest proceeds as in the proof of Lemma 4.13 of [GP] or [AS, Lem. 7.2], where it is shown that, under the hypotheses of $J$ being contact type along $R_{\text{out}} = K_{\text{out}}$, this last expression is non-positive and so $u|_{\bar{S}}$ must have 0 energy and hence be constant, a contradiction. □

Returning to our system of Hamiltonians (and Liouville domains, etc.) $H^\ell$, (2.62) shows that by taking $\epsilon^\ell$ sufficiently small (and $\Sigma^\ell$ taken sufficiently $C_0$ close to $\hat{\Sigma}^\ell$), we may also assume that
\[(2.94) \quad \mathcal{F}_v \in X(X; h^\ell) \text{ whenever } w(v) \leq w^\ell; \text{ and } \lambda_\ell(1 - \epsilon^2_\ell) > w^\ell(1 - \epsilon^2_\ell/2)^2.\]
We do this throughout the rest of the paper. In view of the estimate (2.48), Lemma 2.16 and the above inequality then implies that (2.84) holds for all orbits $x_1$, hence implies the existence of continuation maps $c_{\ell_1,\ell_2}$ between our Hamiltonians $H^\ell_1$ and $H^\ell_2$ as desired.

A standard elaboration of the above argument then shows that continuation maps compose as expected (homologically). Also, any such continuation map from a chain complex to itself is homologically the identity. In particular

**Corollary 2.17.** When $\epsilon_\ell$ is sufficiently small (and as usual $\Sigma_\ell$ is sufficiently $C^0$ close to $\hat{\Sigma}_\ell$), $HF^*(X \subset M; H^\ell)$ is independent of $\epsilon_\ell$. □

Define
\[(2.96) \quad SH^*(X) := \lim_{\ell \to \ell^*} HF^*(X \subset M; H^\ell)\]

Using continuation maps whose existence is guaranteed by Lemma 2.16, we also deduce that

**Lemma 2.18.** The natural map
\[(2.97) \quad SH^*(X) \to SH^*(\hat{X}_\ell)\]
defined using the (monotone) homotopies in (2.77) is an isomorphism. □

\(^8\)Let $\mathfrak{S} := u_2 \setminus \bar{S}$. By Stokes, $E_{\text{top}}(\mathfrak{S}) = A_a(x_2) - (\int_{\partial \mathfrak{S}} u^* \theta - H_{s,t} dt)$. As the topological energy of $u_1 \cup u_2$ is $A_a(x_2) - A_a(x_1)$, the equation holds.
Proof. At the expense of possibly increasing the slope, Lemma 2.16 says one can construct a continuation map $HF^*(X \subset M; H^1)$ to some $HF^*(X \subset M; G^{\ell_1,N})$ for each $\ell$, compatibly with maps in both systems, getting a map on direct limits. One can also go the other way, and naturality properties of continuation maps imply the composition in either direction is the identity (on the direct limits). □

Corollary 2.19. $SH^*(X)$ as defined in (2.96) coincides with the usual symplectic cohomology of $X$, as defined by taking $SH^*(\bar{X}_\ell)$ for any $\bar{X}_\ell$ in the sense of [S3]. □

In view of Equation (2.82), we have that after possibly shrinking $\epsilon_\ell$ further, the continuation maps can be made to induce maps of filtered subcomplexes

$$c_{\ell_1,\ell_2} : F_w CF^*(X \subset M; H^{\ell_1}) \to F_w CF^*(X \subset M; H^{\ell_2})$$

This enables us to equip $SH^*(X)$ with a filtration $F_w SH^*(X)$.

Remark 2.20. As remarked earlier the filtration $F_w SH^*(X)$ is a limit of the natural action filtrations on the various $SH^*(\bar{X}_\ell)$ where $\bar{X}_\ell$ is our sequence of exhausting domains. A variant of our construction would be to define $SH^*(X) := \lim_{\ell \to -}$ where $G^{\lambda,\ell}$ denote Hamiltonians which agree with $\lambda(R^\ell - 1)$ when $R^\ell \geq K^\ell$ and the inverse limit is formed using monotone continuation maps in (2.77).

Of course all of the maps in the inverse limit are isomorphisms. However the natural action filtration in the inverse limit is given by $F_w$. The equivalence between our definition and this one amounts to the fact that in the present setting one may commute the limit and the colimit.

A third natural possibility would be to take the inverse limit with respect to the system of Viterbo functoriality maps $SH^*(\bar{X}_{\ell_2}) \to SH^*(\bar{X}_{\ell_1})$ (for $\ell_2 > \ell_1$) as in [S3, Eq. (7.2)]. Doing this would require checking that our PSSlog maps are compatible with Viterbo’s construction, and involve further technical detours.

Our final task is to put a product structure on $SH^*(X)$. We again describe this first for the directed system $G^{\ell}$. Namely we may define a product operation on $SH^*(\bar{X}_\ell)$ by considering a variant of Floer’s equation defined over a pair of pants $\Sigma$ equipped with standard cylindrical ends $\epsilon_i$. For any $\ell_1$ and $\ell_2$ we choose $\ell_3$ such that

$$\lambda_{\ell_3} \geq \lambda_{\ell_1} + \lambda_{\ell_2}$$

To each cylindrical end associate a time dependent Hamiltonian $H_1$. Let $K \in \Omega^1(\Sigma, C^\infty(X))$ be a 1-form on $\Sigma$ which along the cylindrical ends, satisfies:

$$\epsilon_i^*(K) = (G^{\ell_i}) \otimes dt$$

whenever $|s|$ is large. We also require that outside of a compact set in $X$ we have that

$$K = h^1 \otimes \mathcal{K}_{pert}$$

for some subclosed $\beta$ and $\mathcal{K}_{pert}$ is supported near the divisor $D$ and along the cylindrical ends. To such a $K$, we may associate a Hamiltonian one form $X_K \in \Omega^1(\Sigma, C^\infty(TX))$ which is characterized by the property that for any tangent vector at a point $z \in \Sigma$, $\pi_z$, we have
that $X_{\mathcal{K}}(\vec{r}_z)$ is the Hamiltonian vector-field of $\mathcal{K}(\vec{r}_z)$. Over $\Sigma$, we consider a generalized form of Floer’s equation:

$$
\begin{aligned}
\left\{ u: \Sigma \rightarrow X, \\
(du - X_{\mathcal{K}})^{0,1} = 0.
\right. 
\end{aligned}
$$

To such data we can associate the geometric energy

$$
E_{\text{geo}}(u) := \frac{1}{2} \int_{\Sigma} ||du - X_{\mathcal{K}}||
$$

as well as the topological energy

$$
E_{\text{top}}(u) = \int_{\Sigma} u^* \omega - d(u^* \mathcal{K}).
$$

We say that a perturbation $\mathcal{K}$ is monotonic if its curvature (Equation 8.12 of [S4]) is nonnegative (standard examples are perturbations of the form $H(R^\ell) \otimes \beta$ with $H \geq 0$ and $\beta$ subclosed). When $\mathcal{K}$ is monotonic, we have an energy inequality

$$
E_{\text{geo}}(u) \leq E_{\text{top}}(u).
$$

As usual, we assume that when $R \geq K$, $\mathcal{K}$ is monotonic and satisfies a suitable variant of Equation (2.75) near $R_H$. By counting solutions to this equation we may define an associative and commutative product

$$
SH^*(X, \vec{\epsilon}) \otimes SH^*(X, \vec{\epsilon}) \rightarrow SH^*(X, \vec{\epsilon})
$$

To define appropriate Floer data $X_{\mathcal{K}}$ for our system $H^\ell$, we take $X_{\vec{\epsilon}} = X_{\ell,3}$ and assume that $G_{\ell,3} = H_{\ell,3}$. Along the incoming cylindrical ends we glue in the homotopies used to defined continuation maps from $H^\ell_i$ to $G^\ell_i$. It follows that the product operations so defined will respect the filtration $F_w$.

### 2.4. Action spectral sequences.

We define the low energy Floer cohomology of weight $w$, $HF^*(X \subset M; H^\ell)_w$, by the formula

$$
HF^*(X \subset M; H^\ell)_w := H^*(\frac{F_w CF^*(X \subset M; H^\ell)}{F_{w-1} CF^*(X \subset M; H^\ell)}).
$$

We consider the corresponding descending filtrations:

$$
F^p CF^*(X \subset M; H^\ell) := F_{-p} CF^*(X \subset M; H^\ell).
$$

By definition, the filtration $F^p$ on the cochain complex gives rise to a spectral sequence

$$
\{ E_{p,q}^{\ell,r}, d_r \} \Rightarrow HF^*(X \subset M; H^\ell)
$$

where the first page is by definition identified with

$$
\bigoplus_q E_{p,q}^{\ell,1} := HF^*(X \subset M; H^\ell)_{w=-p}
$$

As is customary, we set $E_{p,1} = \bigoplus_{p,q} E_{p,q}^{\ell,1}$. We have seen that the continuation map can be made to respect the filtration by $w(\mathbf{v})$ and thus induce a filtration on $SH^*(X)$. For the

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9 We do this so that our conventions for cohomological spectral sequences match those found in standard textbooks such as [M1].
purposes of constructing a spectral sequence, it is convenient to use a (co)chain-level direct limit construction. We define
\[
SC^\ast(X) := \bigoplus_{\ell} CF^\ast(X \subset M; H^\ell)[q]
\]
where \(q\) is a formal variable of degree \(-1\) such that \(q^2 = 0\). For \(a + qb \in CF^\ast(X \subset M; H^\ell)[q]\), the differential on this complex is given by the formula
\[
\partial(a + qb) = (-1)^{\deg(a)}\partial(a) + (-1)^{\deg(b)}(q\partial(b) + \zeta_{\ell,\ell+1}(b) - b)
\]
The fact that \(\partial^2 = 0\) relies on the fact that the differential on each complex \(CF^\ast(X \subset M; H^\ell)\) satisfies \(\partial^2 = 0\) and the fact that the \(\zeta_{\ell,\ell+1}\) are chain maps. It is an algebraic consequence of the definition that \(SH^\ast(X) \cong H^\ast(SC^\ast(X))\). The benefit of working with \(SC^\ast(X)\) is that the filtrations by \(w(v)\) gives it the structure of a filtered complex. We again consider the corresponding descending filtration \(F^pSC^\ast(X)\), which are bounded above and exhaustive. As a result (see e.g. [M1] Theorem 3.2), the descending filtration \(F^pSC^\ast(X)\) gives rise to a convergent cohomological spectral sequence:
\[
\{E^{p,q}_r, d_r\} \Rightarrow SH^\ast(X)
\]
Recall that we have chosen our data in such a way that the continuation maps \(\zeta_{\ell_1,\ell_2}\) respect the filtrations and thus give rise to induced continuation maps:
\[
\zeta_{\ell_1,\ell_2} : HF^\ast(X \subset M; H^{\ell_1})_w \to HF^\ast(X \subset M; H^{\ell_2})_w.
\]
By construction, the \(E_0\) page is again a cochain level direct limit of low energy Floer complexes. It follows that the \(E_1\) page may be concretely described as:
\[
\bigoplus_q E^{p,q}_1 := \lim_{\ell} HF^\ast(X \subset M; H^\ell)_{w = -p}
\]
where the maps in the direct limit are the maps (2.110). The product operation on \(SH^\ast(X)\) can similarly be lifted to a map of filtered co-chain complexes
\[
SC^\ast(X) \otimes SC^\ast(X) \to SC^\ast(X).
\]
Setting \(E_r := \bigoplus_{p,q} E^{p,q}_r\), the theory of spectral sequences shows that this induces a (bi-graded) product operation
\[
E_r \otimes E_r \to E_r
\]
which satisfies the Leibnitz rule with respect to the differential \(d_r\). Because the map (2.112) is well known to be associative up to filtered homotopies, the induced multiplications are associative for \(r = 1\) and consequently for all \(r\).

3. The low energy PSS map

The first goal of this Section is to define the low energy PSS\(_{log}\) map from log cohomology to the \(E_1\) page (2.111) of the action spectral sequence and to prove that it is a ring homomorphism. To prepare for this, in §3.1, we recall the notion of the real-oriented blow ups, which will be useful in providing an elegant model for describing the restrictions between strata involved in the definition of the product on log cohomology. We then introduce log cohomology in §3.2 and a Morse model for the product structure (Morse theory is a convenient model for carrying out cochain level constructions, but one could use other versions of cochains such as singular cochains or various flavors of geometric (co)chains as well).
In §3.3, we describe the construction of the low energy PSS map, (3.57). The main result of §3.4 is that this map is a ring homomorphism (Theorem 3.18). Finally, §3.5, shows that after perturbing our symplectic form and Hamiltonians slightly, we may further refine the PSS\textsubscript{log} map to a map (3.118) between log cohomology classes with multiplicity vector \( v \) and a Floer cohomology group generated by orbits that wind around the divisors \( D_i \) with multiplicity \( v \).

### 3.1. The real blowup

Recall the notation from the previous section \( D_I, ˚D_I \), for the stratified components of \( D \) and their open parts. Let \( S_I \) denote the \( T|I| \) torus bundle associated to \( ND_I \), 

\[
S_I = (ND_I \setminus \cup_i D_i)/(\mathbb{R}^+)^I, 
\]

and set 

\[
\tilde{S}_I := (S_I)|_{\hat{D}_I}
\]

to be the torus bundle restricted to the open stratum \( \hat{D}_I \), with \( S_0 = M \) and \( \tilde{S}_0 = X \). Note that all of these manifolds can all be oriented. For \( S_I \), this comes from the exact sequence at each tangent space 

\[
TT^I \to T_p S_I \to T_{\pi(p)} D_I.
\]

Our convention is that each circle in the torus is oriented clockwise (this convention is the opposite of the usual one) and that \( T^I \) is oriented lexicographically.

We now form the real oriented blow up of \( M \) along the divisor \( D \), 

\[
M^{\log},
\]

which canonically realizes the torus bundles (3.2) as strata of a space. There are several possible constructions for this; the most expedient for us is in terms of local coordinate charts (see [M7] for another possible construction based on the tubular neighborhood theorem).

We first consider the linear/local situation. Let \( V := \mathbb{C}^n \) with coordinates \( y_1, \ldots, y_n \) and let \( H \) be the union of first \( a \) coordinate hyperplanes \( H_1, \ldots, H_a \) for some \( n \geq a \geq 0 \). Define the real oriented blow-up of \( V \) along \( H \) to be given by \( V^{\log} := (\mathbb{R}^{\geq 0} \times S^1)^a \times \mathbb{C}^{n-a} \). There is a canonical morphism \( V^{\log} \to V \) given by sending 

\[
((r_1, \theta_1), \ldots, (r_a, \theta_a), x) \to (r_1 \theta_1, \ldots r_a \theta_a, x).
\]

To globalize this blowup construction, we need to show that local diffeomorphisms of \( V \) can be lifted uniquely to the blowup. Given a diffeomorphism \( F : V \to V \) and any \( x \in \mathbb{C}^{n-a} \), we let \( F_{a,x} \) denote the induced map 

\[
F_{a,x} : \mathbb{C}^a \xrightarrow{z \mapsto (z, x)} \mathbb{C}^n \xrightarrow{F} \mathbb{C}^n \xrightarrow{(y,w) \mapsto y} \mathbb{C}^a
\]

and we also let \( \pi_a \) denote the projection \( \pi_a : V^{\log} \to (S^1)^a \). The key computation is then the following:

**Lemma 3.1.** Given a diffeomorphism \( F : V \to V \) which preserves the hyperplanes \( H_1, \ldots H_a \), there is a unique diffeomorphism \( \tilde{F} : V^{\log} \to V^{\log} \) lifting \( F \), i.e. such that
we have a commutative diagram:

\[
\begin{array}{ccc}
V_{\log} & \xrightarrow{\tilde{F}} & V_{\log} \\
\downarrow & & \downarrow \\
V & \xrightarrow{F} & V
\end{array}
\]

Restricted to the preimage of the locus where \(y_1 = y_2 = \cdots y_a = 0\), we have the following explicit formula for \(\pi_a \circ \tilde{F}\),

\[
(3.5) \quad \pi_a \circ \tilde{F} : ((0, \theta_1), (0, \theta_2), \cdots , (0, \theta_a), x) \to [D_0 F_{a,x}(\theta_1, \cdots , \theta_a)],
\]

where \([D_0 F_{a,x}(\theta_1, \cdots , \theta_a)]\) denotes the equivalence class of the differential of \(F_{a,x}\) at \(y_1 = y_2 = \cdots y_a = 0\) applied to (any positive real multiple of) \((\theta_1, \cdots , \theta_a)\), modulo rescaling by \((\mathbb{R}^+)^a\).

**Proof.** As the oriented blow-up is an isomorphism away from the coordinate hyperplanes, the lifts \(\tilde{F}\) are unique if they exist. The existence and explicit formula for the extension follows from the case of a single smooth divisor by taking fibre products. The calculation in the case of a single smooth divisor is given in Lemma 2.8 of [AK], see also of [KM, §2.5]. \(\square\)

Returning to the global situation, we may cover \(M\) by charts \(W \subset V \to M\) on which \(D \cap W\) is the locus where at least one of \(y_1, \cdots , y_a = 0\) (\(a\) here is equal to the depth of the stratum of \(D\) at which our chart is centered). By uniqueness of lifts, we may glue together the local models \(W_{\log}\) to form a manifold with corners \(M_{\log}\) which admits a canonically defined map \(M_{\log} \to M\).

For every non-empty stratum of \(D_I\) of the normal crossings divisor \(D\), the preimage of \(D_I\) under the blow-up map defines a closed a closed stratum of \(M_{\log}\), which is canonically isomorphic (by equation (3.5)) to the blow up

\[
(3.6) \quad S^I_{\log}
\]

of \(S_I\) along the preimages of the strata \(D_j \cap D_I\) for \(j \notin I\). In particular, the fiber of the map \(M_{\log} \to M\) over any point in \(m \in \tilde{D}_I\) is a rank \(|I|\) torus, \(T^I\). Away from \(D\), there is an open inclusion \(X \to M_{\log}\) which is easily seen to induce a canonical homotopy equivalence; similarly, the open inclusions \(S_I \to S^I_{\log}\) are homotopy equivalences.

**Remark 3.2.** Algebro-geometrically, the blow-up can be constructed as the closure of the graph \(X \to \bigoplus_i S_i\) given by the defining sections \(s_i\). More intrinsically, the real oriented blow-up may be viewed as a special case of the Kato-Nakayama construction in logarithmic geometry [KN].

In what follows, we will identify \(H^*(M_{\log})\) and \(H^*(X)\) without mention and similarly for the torus bundles over lower dimensional strata. Note that if \(I \subset K\), then \(S^{I}_{\log} \subset S^K_{\log}\), inducing a restriction map on cohomology

\[
(3.7) \quad r^*_IK : H^*(\hat{S}_I) \to H^*(\hat{S}_K),
\]

which will factor into our definition of product structures on log cohomology below.

**Remark 3.3.** Up to non-canonical diffeomorphism, the real blowup \(M_{\log}\) is diffeomorphic to the complement \(M \setminus \bigcup_i U_i\) of choices of disc-bundle tubular neighborhoods for each \(D_i\).
chosen previously. In such a model, the restriction maps (3.7) can be explained by observing first that $S_I = \partial U_I \subset X$ for any $I$ and more generally for $I \subset K$, there are inclusions $S_K \subset S_I$ living over the inclusion $\partial(\text{hood}(D_K \cap D_I)) \subset D_I$ (where these inclusions are implicitly using the tubular neighborhood identifications).

3.2. Logarithmic cohomology. We now turn to recalling the log(arithmetic) cohomology ring of $(M, D)$,

\begin{equation}
H^*_{\log}(M, D),
\end{equation}

which was defined additively in [GP]. To define it, we use standard multi-index notation, i.e., we have fixed formal variables $t_1, \ldots, t_k$, and for any vector $\mathbf{v} = (v_1, \ldots, v_k) \in \mathbb{Z}_{\geq 0}^k$,

\[ t^\mathbf{v} := t_1^{v_1} \cdots t_k^{v_k}. \]

Next, denote by $v_I$ the multiplicity vector $(v_1, \ldots, v_k)$ whose components are non-zero precisely when they are in $I$, in which case they are 1:

\begin{equation}
v_I := (v_1, \ldots, v_k) \text{ where } v_i := \begin{cases} 1 & i \in I \\ 0 & \text{otherwise.} \end{cases}
\end{equation}

In the case that $I$ consists of a single element $\{i\}$, we will use the notation $v_i := v_{\{i\}}$. We refer to the vectors $v_I$ as primitive vectors. In terms of the primitive vectors $v_I$, log cohomology can be described as follows:

\begin{equation}
H^*_{\log}(M, D) := \bigoplus_{I \subset \{1, \ldots, k\}} t^v_I H^*(\hat{S}_I)[I \mid i \in I]
\end{equation}

where $S_{\emptyset} = X$, and $S_I = \emptyset$ if the intersection $\cap_{i \in I} D_i$ is empty.

$H^*_{\log}(M, D)$ is generated as a $k$-module by elements of the form $\alpha t^\mathbf{v}$, where $\mathbf{v} \in \mathbb{Z}_{\geq 0}^k$ is a multiplicity vector, $I = \{i \in \{1, \ldots, k\} \mid v_i \neq 0\}$ denotes the indices of $\mathbf{v}$ which are non-zero, and $\alpha$ is an element of the $k$-module $H^*(\hat{S}_I)$.

These groups also come equipped with natural filtrations: the logarithmic cohomology of $(M, D)$ of slope $< w$, denoted $F_w H^*_{\log}(M, D)$, is the sub $k$-module generated by those elements of the form $\alpha t^\mathbf{v}$ for some subset $I \subset \{1, \ldots, k\}$, such that

\[ w(\mathbf{v}) < w \]

(recall the definition of $w(\mathbf{v})$ in (2.58)). By definition, whenever $w_1 \leq w_2$ there is an inclusion

\[ i_{w_1, w_2} : F_{w_1} H^*_{\log}(M, D) \rightarrow F_{w_2} H^*_{\log}(M, D). \]

and hence $F_w H^*_{\log}(M, D)$ defines a canonically split ascending filtration. Finally, we also define the associated graded with respect to this filtration

\begin{equation}
H^*_w(M, D)_w := \frac{F_w H^*_{\log}(M, D)}{F_{w-1} H^*_{\log}(M, D)}
\end{equation}

and the multiplicity $\mathbf{v}$ submodule of $H^*_w(M, D)$

\begin{equation}
H^*_w(M, D)_v := H^*(\hat{S}_I)t^\mathbf{v} \text{ where } I = \{i \mid v_i \neq 0\}.
\end{equation}

We say a multiplicity vector $\mathbf{v}$ is supported on $I$ if $I = \{i \mid v_i \neq 0\}$. 
Given a choice of holomorphic volume form $\Omega_{M,D}$ on $M$ which is non-vanishing on $X$ with poles of order $a_i$ along $D_i$, i.e., a fixed isomorphism
\begin{equation}
\wedge^n_c T^* M \cong \mathcal{O}(\sum_i -a_i D_i),
\end{equation}
the vector space $H^{\ast}_{\text{log}}(M,D)$ inherits a cohomological grading given by
\begin{equation}
\deg(\alpha t^v) = \deg(\alpha) + 2 \sum_{v=1}^k (1 - a_v) v_i.
\end{equation}

For $\alpha \in H^{\ast}(\tilde{S}_I)$ and $\beta \in H^{\ast}(\tilde{S}_J)$, let $K = I \cup J$ and define
\begin{equation}
\alpha \ast \beta := r^*_I \alpha \cup r^*_J \beta \in H^{\ast}(\tilde{S}_K),
\end{equation}
where $r^*_I$ and $r^*_J$ are as in (3.7) (note that if $I = J$, this is just the usual cup product in $H^{\ast}(\tilde{S}_I)$). Using (3.15), we observe that there is a natural convolution product on $H^{\ast}_{\text{log}}(M,D)$:

**Definition 3.4.** The ring structure on $H^{\ast}_{\text{log}}(M,D)$ is by definition the unique (graded-) commutative ring structure additively extending the following product rule:
\begin{equation}
\alpha_1 t^{v_1} \cdot \alpha_2 t^{v_2} := (\alpha_1 \ast \alpha_2) t^{v_1 + v_2}.
\end{equation}
for any $\alpha_1 \in H^{\ast}(\tilde{S}_I)$, $\alpha_2 \in H^{\ast}(\tilde{S}_J)$, and $v_1, v_2$ supported on $I$, $J$ respectively, where $\ast$ is as in (3.15).

With respect to this product there is a subalgebra
\begin{equation}
\mathcal{S}\mathcal{R}^{\ast}(M,D) = \bigoplus_{I \subset \{1, \ldots, k\}} t^v H^0(\tilde{S}_I)[t_i | i \in I] \subset H^{\ast}_{\text{log}}(M,D).
\end{equation}
In cases where all of the strata $D_I$ are connected this is a graded version of the Stanley-Reisner ring on the dual intersection complex of $D$. The log cohomology $H^{\ast}_{\text{log}}(M,D)$ is generated by cohomology classes $\alpha t^{v_I}$ as a module over $\mathcal{S}\mathcal{R}^{\ast}(M,D)$; in particular, $H^{\ast}_{\text{log}}(M,D)$ is a finitely generated $\mathcal{S}\mathcal{R}^{\ast}(M,D)$-module.

**Remark 3.5.** The Kato-Nakayama space $M^{\log}$ (see Remark 3.2) arises naturally as the target of evaluation maps of punctured stable log curves $[GS]$ decorated with trivializations of the unit normal bundles to each marked point (see Lemma 3.23 for a special case). In particular, the standard “push-pull” formalism allows one to construct the ring structure on $H^{\ast}_{\text{log}}(M,D)$ using 3-pointed genus zero punctured curves in $(M,D)$ whose underlying stable map is constant. In view of this, it may be useful for algebraic geometers to view $H^{\ast}_{\text{log}}(M,D)$ as a kind of orbifold cohomology of the log pair $(M,D)$.

It will prove useful to have Morse co-chain level model for $H^{\ast}_{\text{log}}(M,D)$ with its product structure. Fix Riemannian metrics $g_I$ on the compactified strata $S^{\log}_I$, and also fix Morse functions $f_I : S^{\log}_I \to \mathbb{R}$ which point outwards along the preimages of the tubular neighborhoods $S_I \cap \pi_I^{-1}(U_J \cap D_I)$. Let $\phi^s_{f_I}$ denote the time $s$ flow of the negative gradient $-\nabla(f_I)$.

---

We thank Alessio Corti and Nicolo Sibilla for suggesting this point of view.
For any critical point $c$ of $f_I$, let
\begin{align}
W^u(f_I, c) &:= \{ x \in S_I^{log}, \lim_{s \to -\infty} \phi_{f_I}^s(x) = c \} \\
W^s(f_I, c) &:= \{ x \in S_I^{log}, \lim_{s \to +\infty} \phi_{f_I}^s(x) = c \}
\end{align}
denote the unstable and stable manifolds of $f_I$ respectively, with respect to the metric $g_I$. The regularising tubular neighborhoods $U_I$ are diffeomorphic to open subsets of $ND_I$ and hence inherit partial $(\mathbb{R}^+)^I$ actions which in turn lift to the blow-ups $U_I^{log}$. We assume that, inside of $U_I^{log}$, our metrics and functions are chosen so that the stable and unstable manifolds are invariant under these partial actions.

We recall the definition of Morse cohomology in order to fix our conventions. As all critical points and all flowlines between critical points lie in the interior $\tilde{S}_I$, we will often blur the distinction between doing Morse theory on $\tilde{S}_I$ and on $S_I^{log}$ (the notable exception being when we construct the convolution product below). We denote the set of critical points of $f_I$ on $\tilde{S}_I$ by $\mathcal{X}(\tilde{S}_I, f_I)$. To each critical point $c \in \mathcal{X}(\tilde{S}_I, f_I)$, we let the orientation line, $|\mathfrak{o}_c|$, denote the $k$-module generated by choices of orientations of the manifolds $W^u(c)$ modulo the relation that the sum of generators associated to opposite orientations vanishes. Let
\begin{equation}
CM^*(\tilde{S}_I, f_I) := \bigoplus_{c \in \mathcal{X}(\tilde{S}_I, f_I)} |\mathfrak{o}_c|
\end{equation}
to the vector space generated by $k$-orientation lines associated to each critical point. For a critical point $c \in \mathcal{X}(\tilde{S}_I, f_I)$, let
\[\deg(c) = \dim W^u(f_I, c).\]
Let $c_0$ and $c_1$ be two critical points such that $\deg(c_0) - \deg(c_1) = 1$ and let $\tilde{M}(c_0, c_1)$ denote the moduli space of flow lines of $-\nabla(f_I)$
\begin{equation}
u : \mathbb{R} \to \tilde{S}_I, \quad \dot{u}(t) = -\nabla(f_I)
\end{equation}
with asymptotics given by $c_1$ and $c_0$ respectively at $\pm \infty$:
\begin{align}
\lim_{s \to -\infty} u & = c_1 \\
\lim_{s \to +\infty} u & = c_0.
\end{align}
(this can equivalently be described as the intersection $W^u(f_I, c_0) \cap W^s(f_I, c_1)$). For a generic metric $g_I$ this moduli space is a compact 1-dimensional manifold with free $\mathbb{R}$ action, and quotienting by $\mathbb{R}$ induces a compact 0-dimensional manifold, the moduli space of unparametrized flowlines
\begin{equation}
M(c_0, c_1) := \tilde{M}(c_0, c_1)/\mathbb{R}
\end{equation}
Any rigid $u \in M(c_0, c_1)$ induces an isomorphism of orientation lines
\begin{equation}
\mu_u : |\mathfrak{o}_{c_1}| \to |\mathfrak{o}_{c_0}|,
\end{equation}
and as usual counting such solutions (using the orientation isomorphisms above) induces the Morse differential.

Let $C^*_I(M, D)$ denote the cochain complex generated by elements of the form $\alpha_v^\nu$, where $\alpha_v$ is a (co)-chain in $CM^*(\tilde{S}_I, f_I)$ for some subset $I \subset \{1, \ldots, k\}$, and $\nu = (v_1, \ldots, v_k)$ is a vector of non-negative integer multiplicities strictly supported on $I$, meaning $v_i > 0$ if and only if $i \in I$. 

\[\]
The differential on $C^*_\text{log}(M, D)$ is induced from the differential on $CM^*(\hat{S}_I, f_I)$, and as before we can give an efficient description of $C^*_\text{log}(M, D)$ as follows:

\begin{equation}
C^*_\text{log}(M, D) := \bigoplus_{I \subset \{1, \ldots, k\}} t^I CM^*(\hat{S}_I, f_I)[t_i \mid i \in I]
\end{equation}

\begin{equation}
= \bigoplus_{I \subset \{1, \ldots, k\}} \bigoplus_{v \text{ supported on } I} CM^*(\hat{S}_I, f_I).
\end{equation}

where $S_0 = X$, and $S_I = \emptyset$ if the intersection $\cap_{i \in I} D_i$ is empty, with differential induced by (3.27). With respect to the grading defined earlier, the degrees of each 1-dimensional subspace associated to the $t^I$ copy of a critical point $c \in \chi(\hat{S}_I, f_I)$ is

\begin{equation}
\deg(\lfloor a_c | t^I \rfloor) = \deg(c) + 2 \sum_{i=1}^k (1 - a_i)v_i.
\end{equation}

Observe that indeed this gives a cochain model for logarithmic cohomology, i.e.,

\begin{equation}
H^*_{\text{log}}(M, D) \cong H^*(C^*_\text{log}(M, D)).
\end{equation}

To implement a co-chain level product inducing the cohomological product from Definition 3.4, we need to recall the construction of pullbacks in Morse cohomology. If $I \subset K$, and given any $c_1 \in \chi(\hat{S}_I, f_I)$, define

\begin{equation}
W^s_K(f_I, c_1) := W^s(f_I, c_1) \cap S^\text{log}_K
\end{equation}

**Definition 3.6.** For any $c_2 \in \chi(\hat{S}_K, f_K)$ with Morse index $\deg(c_2) = \deg(c_1)$, consider the moduli space

\begin{equation}
M_{IK}(c_2, c_1) := W^s_K(f_I, c_1) \cap W^u(f_K, c_2)
\end{equation}

Note that by our assumptions on the Morse function, this intersection lies in $\hat{S}_K$ and in fact outside of tubular neighborhood of the lower dimensional strata. It therefore follows from Appendix A.2. of [AbSc2] that for generic metrics and Morse functions on $S^\text{log}_I$, and $S^\text{log}_K$, (3.31) is a compact zero dimensional manifold (Note that condition (A.2.) of loc. cit. is vacuous in our case and (A.3.) can be achieved by a generic perturbation of the function $f_I$). To orient moduli spaces of rigid solutions, note that a choice of orientation of $W^u(f_I, c_1)$, induces an orientation of the normal bundle to $W^s_K(f_I, c_1)$ inside of $S^\text{log}_K$ via the canonical isomorphism

\begin{equation}
N^\text{log}_K W^s_K(f_I, c_1) \cong N^\text{log}_I W^s(f_I, c_1)|_{W^s_K(f_I, c_1)}
\end{equation}

A choice of orientation on $W^u(f_K, c_2)$ then allows us to orient the intersection $M_{IK}(c_2, c_1)$ and we view this as giving a map on the corresponding orientation lines. By counting the induced maps on orientation lines associated to such solutions, we obtain a map

\begin{equation}
r^*_I : CM^*(\hat{S}_I, f_I) \rightarrow CM^*(\hat{S}_K, f_K)
\end{equation}

Standard arguments in Morse theory prove that this defines a cochain map. Upon identifying Morse cohomology and singular cohomology, this operation corresponds to the usual pull-back on singular cochains. We next recall the construction of the cup-product in Morse cohomology by counting Y-shaped trajectories, a special case of more general operations on Morse complexes induced by metrized ribbon trees.
For any tree $T$, let $E(T)$ denote the set of edges and $V(T)$ the set of vertices. In the Morse context, edges are allowed to come in two types; infinite (external) edges $E_{\text{ext}}(T)$ and finite (internal) edges $E_{\text{int}}(T)$. Operations in Morse theory are defined using rooted metrized ribbon (Stasheff) trees. For such graphs, there is a distinguished outgoing external edge $\vec{e}_0$ and the remaining external edges $\vec{e}_1, \ldots, \vec{e}_{|E_{\text{ext}}(T)|-1}$ (called incoming edges) acquire a linear ordering. Furthermore, each edge $e \in E(T)$ may be canonically oriented and comes with a canonical length and orientation preserving preserving map $e \to \mathbb{R}$. We let $t_e$ denote the induced coordinate on each edge.

The simplest example of such a graph is $T_{2,1}$, the unique trivalent Stasheff tree with three external edges, two of which are incoming and one which is outgoing. Each of the two incoming edges $\vec{e}_1, \vec{e}_2$ is identified with $[0, \infty)$ and the outgoing edge is identified with $(-\infty, 0]$.

Rather than equip each edge of $T_{2,1}$ with different Morse function, we will ensure transversality of gradient flow solutions for maps from $T_{2,1}$ by perturbing the gradient flow equation (for a single Morse function). We make use of the following special case of [A2, Def. 2.6]:

**Definition 3.7.** A gradient flow perturbation datum on $T_{2,1}$ (on the stratum $S_K$) is a choice, for each edge $e \in E(T_{2,1})$ of a smoothly varying family of vector fields,

$$X_e : e \to C^\infty(S_K^{\log})$$

vanishes away from a bounded subset of $e$ and which is invariant under the local action of $\Pi_j \mathbb{R}^+$ in each tubular neighborhood.

Given a gradient flow perturbation datum as above, for each edge $e \in E(T_{2,1})$ and any map $u : e \to S_K^{\log}$, one can ask for $u$ to solve the perturbed gradient flow equation (for $f_K$ with respect to $X_e$):

$$du_e(\partial_e) = -\nabla f_K + X_e.$$  

(3.34)

**Definition 3.8.** Let $f_K : S_K^{\log} \to \mathbb{R}$, be a Morse function and fix gradient flow perturbation data perturbation data $\{X_e\}_{i=0,1,2}$ as above. Suppose that $c_0, c_1, c_2$ lie in $\mathcal{X}(S_K, f_K)$ and satisfy $\deg(c_0) = \deg(c_1) + \deg(c_2)$. With respect to this data, let

$$M_Y(c_0, c_1, c_2)$$

denote the moduli space of continuous maps $u : T_{2,1} \to S_K^{\log}$ whose restriction to each edge is (smooth and) a solution to (3.34).

Note near the vertex, the perturbation data can be arbitrary. It is not difficult to show that for a generic choice of perturbation data, our moduli spaces are cut out transversally. Somewhat informally, this corresponds to the fact that infinitesimally, solutions to the perturbed gradient flow equations correspond to intersections of the unstable and stable manifolds under perturbations by the diffeomorphisms $\phi_e$ given by integrating the vector fields $X_e$. As the vector fields $X_e$ can be chosen arbitrarily, these diffeomorphisms are essentially arbitrary (for a complete proof, see [A2, §7]). Furthermore, when our data is chosen generically, the zero dimensional components of the moduli spaces above induce maps between orientation lines as before, hence an operation on Morse complexes. This map can be composed with the restriction maps (3.31) to obtain a chain model for the product map (3.15). Equivalently, the constituent product maps giving (3.15)

$$t^{v_1}CM^*(\hat{S}_I, f_I) \otimes t^{v_2}CM^*(\hat{S}_J, f_J) \to t^{v_1+v_2}CM^*(\hat{S}_K, f_K)$$
are defined by counting (in the usual signed sense) rigid elements of the fiber product of moduli spaces from (3.31) and Definition 3.8:

\[
\prod_{w,w' \in X(\hat{S}_K,f_K)} M_Y(z,w,w') \times M_{JK}(w,x) \times M_{IK}(w',y),
\]

for each triple \(x \in X(\hat{S}_I,f_I), y \in X(\hat{S}_J,f_J), z \in X(\hat{S}_K,f_K)\).

### 3.3. Moduli spaces and operations.

Here we relate log cohomological structures to the Floer theoretic constructions of §2, by constructing the low-energy log(arithmetic) PSS map:

\[
\text{PSS}^\text{low log} := \bigoplus_v \text{PSS}_v^\log : H^\log_\ast(M,D) \to \bigoplus_{p,q} E_1^{p,q},
\]

where the right hand side is the first page of the spectral sequence converging to \(SH^\ast(X)\) from §2.4.

Consider the domain

\[
S = \mathbb{CP}^1 \setminus \{0\},
\]

thought of as a punctured Riemann sphere, with a distinguished marked point \(z_0 = \infty\) and a negative cylindrical end

\[
\epsilon : (-\infty,0] \times S^1 \to S
\]

near \(z = 0\) given by

\[
\epsilon : (s,t) \mapsto e^{s+it}
\]

The map (3.38) can evidently be defined on all of \(\mathbb{R} \times S^1\), and correspondingly the coordinates \((s,t)\) extend to all of \(S \setminus z_0\). Let \(\rho(s)\) be a cutoff function as in (2.76) and let

\[
\beta = \rho(s) dt
\]

(implicitly smoothly extended across \(z_0\)).

Observe that \(\beta\) restricts to \(dt\) on the negative cylindrical end (at least for \(s \ll 0\)) and restricts to 0 in a neighborhood of \(z_0\). At \(z_0\), we also fix a distinguished tangent vector which points towards the positive real axis.

**Definition 3.9.** Let \(I_S(M,D)\) denote the space of \(S\)-dependent families of complex structures \(J \in J_\ast\) for all \(z \in S\);

- Near \(z_0\), \(J_S\) agrees with some fixed surface-independent almost complex structure \(J_0 \in J(M,D)\); and
- along the negative cylindrical end, \(J_S = J_F\) for some \(J_F \in J_F(M,D)\).

**Definition 3.10.** Let \(I_{S,\ell}(V) \subset I_S(M,D)\) be the space of almost complex structures such that along the negative cylindrical end, \(J_S = J_F\) for some \(J_F \in I_{F,\ell}(V) \subset I_F(M,D)\).

We first recall the definition of the classical PSS moduli space:

**Definition 3.11.** Fix \(\ell\) and \(J_S \in I_S(M,D)\). For every orbit \(x_0\) of \(\mathfrak{X}(X;H^\ell)\) or \(\mathfrak{X}(D,H^\ell)\) define

\[
\mathcal{M}(x_0)
\]
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(which implicitly depends on \( H^\ell, J_S \)) to be the moduli space of maps

\[ u : S \to M \]

satisfying Floer’s equation

\[ (du - X_H^\ell \otimes \beta)^{0,1} = 0 \]

(where \((0, 1)\) is taken with respect to \( J_S \)) with asymptotics

\[ \lim_{s \to -\infty} u(\varepsilon(s, t)) = x_0 \]

In \([GP]\), we introduced a relative version of these moduli spaces, where one imposes tangency conditions at the marked points:

**Definition 3.12.** Fix \( J_S \in \mathfrak{J}_{\mathbb{S}}(M, D) \). For any orbit \( x_0 \) of \( X(\mathbb{S}; H^\ell) \) and any multiplicity vector \( v = (v_1, \ldots, v_k) \in \mathbb{Z}_{\geq 0}^k \), let

\[ M(v, x_0) \]

denote the moduli space of maps

\[ u : S \to M \]

satisfying Floer’s equation (3.41) (with respect to \( H^\ell, J_S \)), with asymptotics (3.42), and the following additional tangency/intersection constraints:

\[ u(z) \notin D \text{ for } z \neq z_0; \]

\[ u(z_0) \text{ intersects } D_i \text{ with multiplicity } v_i \text{ for } i \in I. \]

As explained in \([GP, \S3.5]\), when (3.44) and (3.45) hold, the (real-oriented) projectivization of the \( v_i \) normal jets of the map \( u \) (with respect to a fixed real tangent ray in \( T_{z_0}C \)) give an enhanced evaluation map

\[ \text{Ev}_{z_0}^\gamma : M(v, x_0) \to \hat{\mathbb{S}}_I \]

(compare \([CM2, \S6]\), \([I, \S3]\)).

**Definition 3.13.** For \( c \in \mathcal{X}(\hat{\mathbb{S}}_I, f_I) \) a critical point of \( f_I \) on \( \hat{\mathbb{S}}_I \), let \( M(v, c, x_0) \) denote the moduli space

\[ M(v, c, x_0) := M(v, x_0) \times_{\text{Ev}_{z_0}^\gamma} W^s(f_I, c). \]

A straightforward index computation (see e.g., \([GP, \text{Lemma 4.11}]\)) shows that the virtual dimension of \( M(v, c, x_0) \) is given by

\[ \text{vdim}(M(v, c, x_0)) = \deg(x_0) - \deg(|o_c|t^\gamma), \]

where \( \deg(|o_c|t^\gamma) \) is as in (3.28). It follows from the same arguments explained in \([GP, \text{Lemma 4.15}]\) that for a generic choice of almost complex structure\(^\text{11}\), the moduli space (3.47) is a manifold of dimension equal to its virtual dimension (3.48). In general, the Gromov compactification of this moduli space necessarily contains holomorphic sphere bubbles, which could obstruct counts of the (compactified) moduli space from (being well-defined or) giving a chain map (see Lemma 4.21 of \([GP]\)). The following Lemma gives us an estimate for the energy of the PSS moduli space, which will be useful for further analysis:

\(^{11}\text{In } [GP], \text{ we do not make any assumptions on the Nijenhuis tensor of our almost complex structure, however the arguments explained there apply without modification.}\)
Lemma 3.14. Fix a multiplicity vector $v$ as well as an orbit $x_0$. If $\epsilon_\ell$ and $\|H^\ell - h^\ell\|_{C^2}$ are each sufficiently small and if $\Sigma_\ell$ is sufficiently $C^0$-close to $\tilde{\Sigma}_\ell$, then the (topological) energy of any $u \in M(v, c, x_0)$ is approximately (i.e., becomes arbitrarily close to)

$$E_{\text{top}}(u) \approx w(v) - w(x_0)(1 - \epsilon_\ell^2/2). \tag{3.49}$$

Proof. One obtains (3.49) by taking the limit as $\delta \to 0$ of the topological energy of $u$ restricted to the complement $S_\delta := S\setminus B_\delta(z_0)$. Now $u$ restricted to $S_\delta$ maps to $X$ where $\omega$ is exact, so Stokes' theorem applies and this energy can be calculated as the integral of $\theta$ pulled back along $u|_{\partial B_\delta(z_0)}$ minus the $H^\ell$ action of $x_0$, which we have seen in Equation (2.62) is $\approx w(x_0)(1 - \epsilon_\ell^2/2)$ under the above smallness hypothesis. The term $w(v)$ arises as limit of the former term, i.e., the limiting integral of $\theta$ over a small loop winding around each $D_i$ with multiplicity $v_i$, compare [GP, Lemma 2.11].

Using this, the next Lemma shows that the “low energy” PSS moduli spaces (3.47) with $w(x_0) = w(v)$ admit a nice compactification (i.e., without sphere bubbles):

Lemma 3.15. Suppose that

$$w(v) < \lambda_\ell$$

and that $\epsilon_\ell, \|H^\ell - h^\ell\|_{C^2}$ are both sufficiently small and $\Sigma_\ell$ is sufficiently $C^0$-close to $\tilde{\Sigma}_\ell$. Consider $\{o_c\}_c^v \in C_{\log}^*(M, D)$ and take $x_0 \in \mathcal{X}(X; H^\ell)$ to be an orbit such that $w(x_0) = w(v)$:

- If $\deg(x_0) - \deg([o_c]^v) = 0$, then for generic $J_S \in \mathcal{J}_{S, \ell}(V)$, the moduli space $M(v, c, x_0)$ is compact.

- If $\deg(x_0) - \deg([o_c]^v) = 1$, then for generic $J_S \in \mathcal{J}_{S, \ell}(V)$, $M(v, c, x_0)$ admits a compactification (in the sense of Gromov-Floer convergence) $\overline{M}(v, c, x_0)$, such that

$$\partial \overline{M}(v, c, x_0) = \partial M \sqcup \partial F$$

where

$$\partial F := \bigcup_{x', \deg(x_0) - \deg(x') = 1} M(x_0, x') \times M(v, c, x') \tag{3.50}$$

and

$$\partial M := \bigcup_{c', \deg(c') - \deg(c) = 1} M(v, c', x_0) \times M(c', c) \tag{3.51}$$

Proof. We consider the closure $\overline{M}(v, x_0) \subset \overline{M}(x_0)$. The argument of Lemma 4.13 rules out Floer cylinder breaking along $D$, so the only “bad” possibilities which prevent the compactification from being as stated are sphere bubbling (and cylinders breaking in $X$ but touching $D$, but this is excluded by positivity of intersection). Now under our assumptions, Lemma 3.14 implies that the (topological) energy of a PSS solution is approximately $w(v)\epsilon_\ell^2/2$. If $\epsilon_\ell$ is sufficiently small, then this quantity can be made much smaller than the minimal energy of a non-constant $J$-holomorphic sphere in $M$. Hence, assuming all of the parameters in the statement of the Lemma are small enough to make the estimate (3.49) have sufficiently small error, it follows that the energy of a PSS solution will be smaller than the minimal energy of a non-constant $J$-holomorphic sphere. This implies that such sphere bubbling is impossible. The same argument shows that breaking can only occur along orbits with $w(x') = w(v)$.

For any $v$ with $w(v) \leq w_\ell$, choose generic $J_S \in \mathcal{J}_{S, \ell}(V)$ and define the $t^v$ component of $\text{PSS}^{v, \ell}_{\text{log}}$

$$\text{PSS}^{v, \ell}_{\text{log}} : H^*(\tilde{S}_t)t^v \to HF^*(X \subset M; H^\ell)w(v) \tag{3.52}$$
by the following prescription: as usual a rigid element $u \in M(v, c, x_0)$ induces an isomorphism of orientation lines $\mu_u : |o_c| \to |o_{x_0}|$; then following a standard pattern, we define $\text{PSS}^v_{log}$ by using such isomorphisms to give signed counts of rigid elements (which is a well-defined finite count by Lemma 3.15): for $z \in |o_c|,$

\[
\text{PSS}^v_{log}(zt^v) := \sum_{w(v), \text{vdim}(M(v, c, x_0)) = 0} \sum_{u \in M(v, c, x_0)} \mu_u(z) \quad (3.53)
\]

Under the smallness conditions of $\epsilon_{\ell},$ this is a well-defined finite count and gives a chain map

\[
\text{PSS}^v_{log} : C^*_v(M, D) \to CF^*(X \subset M; H^\ell)_w(v)
\]

Taking a direct sum of these maps over all multiplicity vectors $v$ with $w(v) = w,$ we obtain a map on cohomology

\[
\text{PSS}^w_{log} : H^w_{log}(M, D) \to HF^*(X \subset M; H^\ell)_w
\]

for any $w \leq w_\ell.$ A straightforward variation of the above analysis (implemented in Lemma 4.18 of [GP]) verifies that (3.54), hence (3.55), is compatible with the continuation maps (2.110):

\[\text{Lemma 3.16.}\quad \text{For } \ell_2 \geq \ell_1, \text{ we have a commutative triangle:}
\]

\[
\begin{array}{ccc}
H^*(\hat{S}_1)t^v & \xrightarrow{\text{PSS}^v_{log}} & HF^*(X \subset M; H^{\ell_1})_w(v) \\
& \text{PSS}^v_{log} & \xrightarrow{\text{PSS}^v_{log}} & HF^*(X \subset M; H^{\ell_2})_w(v)
\end{array}
\]

In view of this, we drop the superscript $\ell$ from the notations $\text{PSS}^v_{log}$ and $\text{PSS}^w_{log}.$ Summarizing the above considerations, we have the following:

\[\text{Lemma 3.17.}\quad \text{There are canonical maps:}
\]

\[
(3.56) \quad \left( \bigoplus_{v, w(v) \leq w} \text{PSS}^v_{log} \right) : F^w \to \bigoplus_{p \geq -w, q} E^p_{1,1}
\]

\[
(3.57) \quad \text{PSS}^\text{low}_{log} := \left( \bigoplus_v \text{PSS}^v_{log} \right) : H^*_v \to \bigoplus_{p, q} E^p_{1,1}
\]

\[\text{Proof.}\quad \text{The first statement is immediate from the above discussion and the second follows from that statement in view of the explicit description of the } E_1 \text{ page from (2.111) and Lemma 3.16.}\]

We refer to the map defined in (3.57) as the \textit{low energy} PSS map.

\[\text{3.4. Low energy PSS is a ring map.}\quad \text{The aim of this subsection is to prove that}
\]

\[\text{Theorem 3.18.}\quad \text{The low energy PSS map (3.57) is a ring homomorphism.}\]

A proof of Theorem 3.18 appears at the bottom of this subsection, using intermediate results we now describe.

In order to simplify our proof, we begin by observing that the ring structure on $H^*_v(M, D)$ from Definition 3.4 is determined uniquely by the value of the product on a much smaller
collection of (pairs of) elements. Given a stratum \( \hat{S}_J \) with connected components \( \hat{S}_{J,m} \), \( m \in \{1, \ldots, \pi_0(\hat{S}_J)\} \), we denote the fundamental cycle (i.e., the unit) in \( H^0(\hat{S}_{J,m}) \) by \([\hat{S}_{J,m}]\). For what follows below, recall the definition (3.9) of the primitive vector \( v_I \) supported on a given \( I \subset \{1, \ldots, k\} \).

**Lemma 3.19.** The product on \( H^*_{\log}(M,D) \) is the unique (graded-) commutative ring structure such that

(i) For any \( I, J, \alpha_1 \in H^*(\hat{S}_I) \) and \( \alpha_2 \in H^*(\hat{S}_J) \),

\[
\alpha_1 t^{v_I} \cdot \alpha_2 t^{v_J} = (\alpha_1 \ast \alpha_2) t^{v_I+v_J}.
\]

(ii) For any \( I, J \subseteq I, \alpha_1 \in H^*(\hat{S}_I) \) and multiplicity vector \( v_1 \) supported on \( I \),

\[
\alpha_1 t^{v_1} \cdot [\hat{S}_{J,m}] t^{v_J} = (\alpha_1 \ast [\hat{S}_{J,m}]) t^{v_1+v_J}.
\]

**Proof.** It is a straightforward exercise to see that these relations imply the multiplication rule given in Definition 3.4. \( \square \)

Thus Theorem 3.18 will follow from verifying the following special cases:

(i) For any \( I, J, \alpha_1 \in H^*(\hat{S}_I) \) and \( \alpha_2 \in H^*(\hat{S}_J) \),

\[
PSS_{\log}^\text{low}(\alpha_1 t^{v_I}) \cdot PSS_{\log}^\text{low}(\alpha_2 t^{v_J}) = PSS_{\log}^\text{low}((\alpha_1 \ast \alpha_2) t^{v_I+v_J}).
\]

(ii) For any \( I, J \subseteq I, \alpha_1 \in H^*(\hat{S}_I) \) and multiplicity vector \( v_1 \) supported on \( I \),

\[
PSS_{\log}^\text{low}(\alpha_1 t^{v_1}) \cdot PSS_{\log}^\text{low}([\hat{S}_{J,m}] t^{v_J}) = PSS_{\log}^\text{low}((\alpha_1 \ast [\hat{S}_{J,m}]) t^{v_1+v_J}).
\]

The proof of Theorem 3.18 in these cases follows the standard pattern in TQFT of using interpolating families of moduli spaces to give a cobordism (relative “chain homotopy terms”) between the operations either side of the equality. In our case, the argument bears a resemblance to Piunikhin-Salamon-Schwarz’s original argument [PSS] that the PSS map intertwines product structures, adapted in a non-trivial way to our (“relative D”) setup.

To begin, for some small \( b > 0 \) and any \( q \in [0,b] \), let \( S_q = \mathbb{C}P^1 \setminus \{0\} \), with a negative cylindrical end (3.38) as before, along with two additional marked points at \( z_1 = -1/q \) and \( z_2 = \infty \). The family of \( \{S_q\}_{q \in [0,b]} \) limits to a stable domain \( S_0 \) as \( q \to 0 \), and including this domain gives a family of stable curves over \( q \in [0,b] \). It will be convenient to work in coordinates centered about \( \infty \), which we denote by

\[
y = z^{-1}.
\]

As usual, we will equip these domains with Floer data. To simplify the discussion a bit, note that our product operation from \( \Sect 2.3 \) is defined by first continuing both inputs to a single Liouville domain and then defining the product on a fixed Liouville domain. As \( PSS_{\log} \) is compatible with these continuation maps, we can and will assume that our Liouville domain \( \tilde{X}_t \) used to define the Hamiltonians remains fixed. The Floer data on domains \( S_q, q \neq 0 \) is determined by a sub-closed one form \( \beta \) satisfying

\[
\beta = 0 \text{ in a neighborhood of the points } y(z_1) = -q \text{ and } y(z_2) = 0.
\]

In the next definition, we fix vectors \( v_1, v_2 \) as well as an orbit \( x_0 \) of \( H^\ell \). We also fix a family of domain-dependent almost complex structures \( J_{S_q} \in \mathcal{J}_S(\tilde{X}_t, V) \), which varies smoothly in \( q \) and so that \( J_{S_q} \) is domain independent in a neighborhood of \( y(z_1) = -q \) as

\[
\begin{align}
(3.62) & \quad \text{The form } \beta \text{ restricts to } 2dt \text{ on the cylindrical end (3.38).} \\
(3.63) & \quad \beta = 0 \text{ in a neighborhood of the points } y(z_1) = -q \text{ and } y(z_2) = 0.
\end{align}
\]
well. We assume that as $q \to 0$, $J_{S_q}$ converges to a complex structure $J_S \in \mathcal{J}_S(\bar{X}_f, V)$ which we used to define low-energy PSS.

**Definition 3.20.** Define $\mathcal{M}(v_1, v_2; x_0)$ to be the moduli space of pairs

$$\{(q, u)|q \in (0, b], \ u : S_q \to M\}$$

whose associated map $u$ satisfying Floer’s equation

$$(du - X_{H_f} \otimes \beta)^{0,1} = 0$$

(where $0, 1$ component is taken with respect to $J_{S_q}$), with asymptotics given by $x_0$

$$(\text{3.65}) \quad \lim_{s \to -\infty} u(\epsilon(s, t)) = x_0$$

and tangency/intersection conditions along $D$ at $(z_1, z_2)$ as specified by $(v_1, v_2)$:

$$(\text{3.66}) \quad u(x) \notin D \text{ for } x \neq z_i;$$

$$(\text{3.67}) \quad u(z_1) \text{ intersects } D \text{ with multiplicity } v_1.$$

$$(\text{3.68}) \quad u(z_2) \text{ intersects } D \text{ with multiplicity } v_2.$$

We define $\mathcal{M}_s(v_1, v_2; x_0)$ to consist of the subspace of maps $(q, u)$ as above with $q = s$, i.e., the space of maps $u : S_s \to M$ satisfying the conditions above.

The virtual dimension of these moduli spaces $\mathcal{M}_s(v_1, v_2; x_0)$ is

$$(\text{3.69}) \quad \text{vdim}(\mathcal{M}_s(v_1, v_2; x_0)) = \deg(x_0) - 2 \sum_{i=1}^{k} (1 - a_i)v_{1,i} - 2 \sum_{i=1}^{k} (1 - a_i)v_{2,i}.$$

For generic choices of $J_{S_q}$, these are moduli spaces of the expected dimension by arguments similar to those in Lemma 4.14 of [GP].

As a degenerate case of this (when $q = 0$) we study maps from $S_0$. For what follows, write $S_0 = S_{\text{sphere}} \cup_{z' = z_0} S_{\text{plane}}$, where $S_{\text{sphere}}$ is the sphere bubble containing $z_1$, $z_2$ and another (nodal) point $z'$ and $S_{\text{plane}} = \mathbb{CP}^1 \setminus \{0\}$ at the point $z_0$ (which is connected to $z'$).

**Definition 3.21.** Define $\mathcal{M}_0(v_1, v_2; x_0)$ to be the moduli space of maps $u : S_0 \to M$ satisfying the following conditions:

$$(\text{3.70}) \quad u|_{S_{\text{sphere}}} \text{ is a constant map;}$$

$$(\text{3.71}) \quad u|_{S_{\text{plane}}} \text{ is an element of the moduli space } \mathcal{M}(v_1 + v_2; x_0).$$

Using this ($q = 0$) moduli space, we can state the key result about the Gromov compactification of $\mathcal{M}(v_1, v_2; x_0)$:

**Lemma 3.22.** If $w(x_0) = w(v_1) + w(v_2)$ and the constants $\epsilon_{\ell}$, etc. are chosen to be small as in Lemma 3.14, then the Gromov compactification of $\mathcal{M}(v_1, v_2; x_0)$

$$(\text{3.72}) \quad \overline{\mathcal{M}(v_1, v_2; x_0)}$$

has codimension-1 boundary covered by the images of the natural inclusions:

$$(\text{3.73}) \quad \overline{\mathcal{M}_b(v_1, v_2; x_0)} \to \partial \overline{\mathcal{M}(v_1, v_2; x_0)} \ (q \to b)$$

$$(\text{3.74}) \quad \overline{\mathcal{M}_0(v_1, v_2; x_0)} \to \partial \overline{\mathcal{M}(v_1, v_2; x_0)} \ (q \to 0)$$

$$(\text{3.75}) \quad \overline{\mathcal{M}(v_1, v_2; x_1)} \times \overline{\mathcal{M}(x_1, x_2)} \to \partial \overline{\mathcal{M}(v_1, v_2; x_0)}.$$
Proof. The proof involves identical analysis as in Lemma 3.15 (except that there are new boundaries to consider when \( q \to 0, b \) and no Morse flowlines). In particular, the condition that \( w(x_0) = w(v_1) + w(v_2) \) implies, under the smallness hypotheses of Lemma 3.14 (and by a straightforward adaptation of the energy approximation in Lemma 3.14 to \( M(v_1, v_2; x_0) \)) that the topological energy of any \( u \in M(v_1, v_2; x_0) \) is smaller than the minimal energy of a non-constant \( J \)-holomorphic sphere in \( \bar{M} \). Hence, there can be no sphere bubbles that arise, including when \( q \to 0 \) and the domain \( S_q \) degenerates into the stable domain \( S_0 \), meaning any limiting map from \( S_0 \) must be a constant map on the sphere component \( S_{\text{sphere}} \), hence an element of \( M_0(v_1, v_2; x_0) \).

We further equip the marked points \( z_1 \) and \( z_2 \) with asymptotic markers, in the positive real direction at \( z_1 \) and in the negative real direction at \( z_2 \). As before, doing so gives rise to evaluation maps:

\[
\begin{align*}
(3.76) & \quad \text{Ev}_{z_1}^v : M(v_1, v_2; x_0) \to \bar{S}_I \\
(3.77) & \quad \text{Ev}_{z_2}^v : M(v_1, v_2; x_0) \to \bar{S}_J,
\end{align*}
\]

where \( I \) denotes the support of \( v_1 \) and \( J \) denotes the support of \( v_2 \). There are natural embeddings \( \bar{S}_I, \bar{S}_J \to M_{\text{log}} \) which allow us to view both of the above evaluation maps as landing in \( M_{\text{log}} \). Given a limiting map from the stable domain \( S_0 \) (which as we have seen above, intersects \( D \) at \( z_0 \) with multiplicity \( v_1 + v_2 \)), we set

\[
(3.78) \quad \text{Ev}_{z_i}^v(u) = \text{Ev}_{z_0}^{v_1+v_2}(u)
\]

viewed as a point in \( M_{\text{log}} \). Doing this allows us to extend the evaluation map over points in \( \bar{M}(v_1, v_2; x_0) \) where \( q = 0 \):

\[
\begin{align*}
(3.79) & \quad \text{Ev}_{z_1}^v : \bar{M}(v_1, v_2; x_0) \to M_{\text{log}} \\
(3.80) & \quad \text{Ev}_{z_2}^v : \bar{M}(v_1, v_2; x_0) \to M_{\text{log}}
\end{align*}
\]

Lemma 3.23. Assume that \( v_2 = v_J \) is primitive.

(i) If \( v_1 = v_I \) is primitive, then the evaluation maps (3.79), (3.80) are continuous.

(ii) If \( J \subseteq I \), then the evaluation (3.79) is continuous.

Proof. For what follows, let \( K = I \cup J \) denote the support of \( v_1 + v_2 \). Consider a convergent sequence of Floer curves \( u_q : S_q \to M \), with \( q \to 0 \) with limit \( u_0 \). We first discuss assertion (i), where \( v_1 = v_I \). To simplify our notation, we may assume that, after reordering the divisors, \( K = \{1, \ldots, |K|\} \). Choose the data of a chart \( W \subset C^n, W \xrightarrow{\phi} M \) (as usual we suppress \( \phi \)) centered about \( u_0(z_0) \), such that the divisors are sent to standard linear subspaces: \( D_k \cap W = \{y_k = 0\} \cap W \) for any \( k \in K \). We can assume for specificity that \( W := (B^2(0))^{\{K\}} \times W \subset C^{\{K\}} \times C^{n-|K|} \) and that \( W_{\text{log}} := (B^2(0)_{\text{log}})^{\{K\}} \times W \).

There exists an open set \( U_S \subset S \) containing \( y = -q \) and \( y = 0 \) with \( u_q(U_S) \subset W \) for all \( q \) sufficiently small. We will show that \( \text{Ev}_{z_0}(u_0) \) is the limit as \( q \to 0 \) of the evaluations \( \text{Ev}_{z_0} \) viewed as lying inside of \( W_{\text{log}} \). It suffices to prove this convergence in each of the \( (B^2(0)_{\text{log}})^{\{K\}} \) factors as the other factors are not modified by the blow-up and hence there is nothing to check. To this end, let \( \pi_k \) denote the coordinate projections \( \pi_k : W \to C \) and set \( u_k,q = \pi_k \circ u_q : U_S \to C \). We first consider the case when \( k \in I \cap J \). [IP, Lemma 3.4]
allows us to determine the leading order terms of the Taylor expansions of \( \bar{u}_{k,q} \):

\[
\bar{u}_{k,0} = b_0 y^2 + O(|y|^3)
\]
\[
\bar{u}_{k,q} = a(q)y + O(|y|^2), \quad q \neq 0
\]

where \( b_0, a(q) \neq 0 \). Because there is no bubbling, \( C^\infty \) convergence together with the fact that \( \bar{u}_{k,q}(-q) = 0 \) implies that

\[
-a(q)q + b(q)q^2 + O(q^3) = 0
\]

with \( b(q) \to b_0 \). In particular, we have used that the third order derivatives are uniformly bounded in \( q \) to conclude that the remainder term is \( O(q^3) \). It therefore follows that

\[
\lim_{q \to 0} a(q)/q = b_0.
\]

Performing a Taylor expansion about \( z_1 \) instead and writing \( \bar{u}_{k,q}(y) = \tilde{a}(q)(y + q) + O(|y + q|^2) \), the same reasoning shows that \( \lim_{q \to 0} \tilde{a}(q)/q = -b_0 \). In the case that \( k \in J \) but not in \( I \) we have that (3.82) holds, this time when \( q = 0 \) as well. \( C^\infty \) convergence of the maps \( u_0 \) imply that that \( a(q) \to a_0 \neq 0 \) and \( \lim_{q \to 0} \bar{u}_{k,q}(z_1)/q = a_0 \).

Assertion (ii) of the Lemma follows similarly. Suppose that \( k \notin J \), then the result follows immediately by \( C^\infty \) convergence, so we consider the other case. Then, employing the notation from the previous paragraph and taking \( q \to 0 \), we have

\[
\bar{u}_{k,q} = a(q)(y + q)\nu_1 + b(q)(y + q)^{\nu_1+1} + O(|y + q|^{\nu_1+2})
\]

As before we have that \( a(q)/q \to -b_0 \).

\[ \square \]

We now incorporate Morse flow-lines into the picture. To formulate our next definition, fix two infinite incoming edges \( \vec{e}_1, \vec{e}_2 \cong [0, \infty) \). When dealing with moduli spaces where \( I = J = K \), we fix perturbation data \( X_{\vec{e}_i} : \vec{e}_i \to C^\infty(S_{K}^{\log}) \) on each edge in order to assure that regularity holds.

**Definition 3.24.** Fix two critical points \( c_1 \in \mathcal{X}(\hat{S}_I, f_I), \ c_2 \in \mathcal{X}(\hat{S}_J, f_J) \). Let

\[ \mathcal{M}(v_1, v_2; c_1, c_2; x_0) \]

denote the moduli of triples \( (q, u), \phi_1, \phi_2 \), where \( (q, u) \in \mathcal{M}(v_1, v_2; x_0) \) is an element of the moduli space from Definition 3.20, \( \phi_1 : \vec{e}_1 \to S_{I}^{\log} \) and \( \phi_2 : \vec{e}_2 \to S_{J}^{\log} \) solve the gradient flow equation for \( f_I \) and \( f_J \) respectively (perturbed by \( X_{\vec{e}_i} \) in the case \( I = J = K \)), satisfying the following additional asymptotics and incidence conditions:

\[
\lim_{t_{\vec{e}_i} \to \infty} \phi_i(t_{\vec{e}_i}) = c_i;
\]
\[
\phi_i(0) \in \hat{S}_K;
\]
\[
\text{Ev}^X_{\vec{e}_i}(u) = \phi_i(0).
\]

Let us assume that \( f_J : \hat{S}_{J,m} \to \mathbb{R} \) is chosen to have a unique critical point of index 0, which we denote by \( c_0 \). In view of the defining relations given in (3.60), (3.61), it suffices to consider only those moduli spaces where \( v_2 = v_J \) and furthermore that either

A.1 \( v_1 = v_I \).
A.2 \( c_2 = c_0 \) and \( J \subseteq I \).

We will do this for the remainder of this section without further mention. As before, we choose our data so that these spaces are cut out transversally. Next consider the moduli
Lemma 3.23 that given a \( u \) space \( \mathcal{M}_b(M, \nu_1, \nu_2; c_1, c_2; x_0) \) which is the restriction of the above moduli space to domains \( S_0 \). By counting rigid elements \( u \in \mathcal{M}_b(M, \nu_1, \nu_2; c_1, c_2; x_0) \), we may define a map
\[
H^*(\hat{S}_I) t^{x_1} \otimes H^*(\hat{S}_J) t^{x_2} \to SH^*(X)
\]

**Lemma 3.25.** The operation defined in (3.88) agrees with \( \text{PSS}^\text{low}_{\log}(\alpha_1 t^{x_1}) \cdot \text{PSS}^\text{low}_{\log}(\alpha_2 t^{x_2}) \).

**Proof.** This is a very standard gluing argument in TQFT and so we will only sketch the main idea. Let \( \Sigma \) denote the pair of pants, with three standard cylindrical ends attached. We consider the broken domain \( S_n \) of the form
\[
S \cup_\epsilon \Sigma \cup_\epsilon S
\]
where the negative cylindrical ends of \( S \) are glued to the positive cylindrical ends of \( \Sigma \) and gradient flow lines are attached at the two marked points. Maps from \( S_n \to \hat{M} \) (glued to appropriate gradient flow lines) are given by the fiber product of moduli spaces given by:
\[
\prod_{x_1, x_2} \mathcal{M}(\nu_1, x_1, c_1) \times \mathcal{M}(\Sigma, x_0, x_1, x_2) \times \mathcal{M}(\nu_2, c_2, x_2)
\]

We construct a homotopy between this moduli space and \( \mathcal{M}_b(M, \nu_1, \nu_2; x_0) \) in two steps. First, we perform a finite connect sum along the cylindrical ends, producing a domain with two distinguished marked points and one negative cylindrical end. The precise complex modulus of this domain and the Floer datum over it are determined by the gluing parameter. Then, we can further homotopy the complex structure and Floer datum to the domain \( S_{b,2} \) above. We thus reach the desired conclusion. \( \square \)

Let \( T_{2,0} \) denote the graph with two infinite edges \( e_1 \) and \( e_2 \) as above are glued along their vertices. Let \( \mathcal{M}_V(c_1, c_2) \) denote the moduli space of continuous maps \( \phi : T_{2,0} \to M^\text{log} \) where along each edge \( \phi_i = \phi_{e_i} \) is a (possibly perturbed) gradient trajectory in the appropriate stratum and asymptotic to \( c_i \) such that \( \phi_1(0) = \phi_2(0) \in \hat{S}_K \).

**Definition 3.26.** Define the moduli space \( \partial V \mathcal{M}(\nu_1 + \nu_2, c_1, c_2, x_0) \) to be the space of pairs \( (\phi, u), \phi \in \mathcal{M}_V(c_1, c_2), u \in \mathcal{M}(\nu_1 + \nu_2, x_0) \) with \( E \nu_1 u + \nu_2 = \phi_i(0) \).

**Lemma 3.27.** Suppose that \( \nu_2 = \nu_1 \) holds and furthermore either A.1 or A.2 hold. For generic choices, when the moduli space from Definition (3.24) is 1-dimensional, it may be completed over \( q \to 0 \) to a 1-dimensional manifold with boundary whose fiber over \( q = 0 \) is given by \( \partial V \mathcal{M}(\nu_1 + \nu_2, c_1, c_2, x_0) \).

**Proof.** Suppose first that we are in case A.1 above with \( \nu_1 = \nu_I \). It follows from Lemma 3.23 that given a \( u_0 \) which is the limit of curves \( u_q \in \mathcal{M}(\nu_1, \nu_2; c_1, c_2; x_0) \),
\[
E \nu_0 \in W^s_K(f, c_1) \cap W^s_K(f, c_2)
\]
As in Definition (3.24), in the case where \( I = J = K \), this should be interpreted as stable manifolds of the perturbed gradient flow. To complete the proof of the lemma (in the case A.1), requires that the topology near a curve satisfying (3.90) may be identified with that near an endpoint of a closed interval. This is an elementary gluing result (involving gluing in a constant sphere bubble) which follows the arguments of Chapter 10 of [MS1] quite closely and which we regard as standard.

It remains to consider the remaining cases when \( c_2 = c_u \), in which case the stable manifold \( W^s(f, c_2) \) has closure all of \( S_{J,m}^\text{log} \) and \( S_{J,m}^\text{log} \setminus W^s(f, c_2) \) has codimension 1. In this case, by choosing our data generically, we may assume that an element of the moduli
space $M(v_1 + v_2, W_K^*(f_1, c_1))$ is disjoint from $S_{J,m}^{log} \setminus W^*(f_J, c_2)$. The rest proceeds as in the preceding paragraph. \hfill \Box

Lemma 3.28. Suppose that $v_2 = v_J$ holds and furthermore either A.1 or A.2 hold. The operation defined in (3.88) is equal to $\text{PSS}^{log}(\alpha_1 v_1 - \alpha_2 v_2)$

Proof. This follows another homotopy argument. We can consider configurations depending on a parameter $p \in [0, \infty]$. Let us first describe the case where both $I, J \neq K$. For any such $p$, let $T^{(p)}_{2,0}$ denote the metrized tree with two incoming external edges $\bar{e}_1, \bar{e}_2$, two internal edges $\bar{e}_1, \bar{e}_2 \cong [0, p]$ of length $p$ and one outgoing edge $\bar{e}_0$ of length $p$. We glue these edges by setting

- $\bar{e}_1(0) = \bar{e}_1(0), \bar{e}_2(0) = \bar{e}_2(0)$ and
- $\bar{e}_1(p) = \bar{e}_2(p) = \bar{e}_0(0)$.

In the limit as $p \to 0$, this becomes $T_{2,0}$ above. For all $p$, we equip this tree with perturbation data $e \to C^\infty(S_K^{log})$ on each of the finite edges $\bar{e}_1, \bar{e}_2, \bar{e}_0$ which is supported in a neighborhood of the vertex $\bar{e}_0(0)$. A $T^{(p)}_{2,0}$ shaped flow line is a continuous map $\phi : T^{(p)}_{2,0} \to M^{log}$ where $\phi$ restricted to each of the finite edges $\bar{e}_1, \bar{e}_2, \bar{e}_0$ are perturbed gradient trajectories of $f_K$ and $\phi_1 = \phi_{|\bar{e}_1}$ and $\phi_2 = \phi_{|\bar{e}_2}$ are gradient trajectories of $f_I$ and $f_J$ respectively. Note that this implies in particular that we have that $\phi(e_i(0)) \in S_K^{log}$.

Let $M_p(v_1 + v_2, c_1, c_2, x_0)$ denote the moduli space of pairs $(\phi, u)$ where $\phi$ is a $T^{(p)}_{2,0}$ shaped flow line and $u$ is a solution in $M(v_1 + v_2, x_0)$ such that $\text{Ev}_{x_0} v_1 + v_2 = \phi(\bar{e}_0(p))$. At $p \to \infty$ solutions limit to elements in (3.89). When $p = 0$, this moduli space reduces to the moduli space of Definition 3.26. This gives rise to a cobordism (up to boundary components which give rise to chain homotopy terms) between these two moduli spaces. Combining Lemmas 3.27 and 3.25 gives rise to a cobordism (again up to chain homotopy) between the moduli space at $p = 0$ and the moduli spaces which appear in (3.89).

The other cases where either $I$ or $J$ coincide with $K$ are an easy modification of the above argument. To handle it, we modify the graph $T^{(p)}_{2,0}$ above by keeping the edges $\bar{e}_1, \bar{e}_2,$ $\bar{e}_0$ the same as above, but only introducing an edge $\bar{e}_1$ (respectively $\bar{e}_2$) when $I$ (respectively $J$) is not equal to $K$. The rest proceeds as before. \hfill \Box

Proof of Theorem 3.18. Combine Lemmas 3.25 and 3.28 to verify the low energy PSS map is compatible with ring structure for inputs of the form specified in (i) and (ii) of Lemma 3.19; the aforementioned Lemma then implies compatibility for all inputs. \hfill \Box

3.5. Separating actions. Now we vary our symplectic form slightly on $M$ in order to ensure that the following nice properties hold:

1. the (families of) Hamiltonian orbits (below a fixed slope) we consider below will all have distinct actions, furthermore so that
2. the resulting weight of a cohomology class in $F_{w_i}H^*_\text{log}(M, D)$ uniquely determines the multiplicity vector $v$, and show
3. the low energy log PSS map, defined with respect to this perturbed symplectic form, preserves these newly refined filtrations and crucially agrees with the previously defined log PSS map.

More precisely, fixing an $\ell$ in our sequence, we will perturb the divisorial weights $\kappa_i$ (as they appear in $[\omega]$ and $w(v)$) to rational numbers $\kappa_{i,\ell} \in \mathbb{Q}$ very close to $\kappa_i$, which in turn leads
to a perturbed weight of a vector \( \mathbf{v} \)

\[
w_p(\mathbf{v}) := \sum \kappa_{i,\ell} v_i \in \mathbb{Q}
\]

(this depends on \( \ell \), though we have suppressed that from the notation). In terms of this perturbed weight, we require our perturbation \( \{\kappa_{i,\ell}\} \) to separate actions below level \( w_\ell \) in the sense that:

\[
(3.92) \quad \text{Given any } \mathbf{v}_1 \neq \mathbf{v}_2 \text{ with } w(\mathbf{v}_1) \leq w_\ell, w_p(\mathbf{v}_1) \neq w_p(\mathbf{v}_2).
\]

We will also want to assume that our perturbations \( \kappa_{i,\ell} \) are sufficiently close to \( \kappa_i \), so that

\[
(3.93) \quad \text{if } w(\mathbf{v}) \in [w_\ell + 1, \infty), \text{ then } w_p(\mathbf{v}) \in [w_\ell + 1 - \delta_p, \infty) \text{ for some very small } \delta_p.
\]

\[
(3.94) \quad \text{if } w(\mathbf{v}) \in [0, w_\ell], \text{ then } |w(\mathbf{v}) - w_p(\mathbf{v})| < \delta_p.
\]

The first condition roughly says that we don’t necessarily require the perturbed weight \( w_p(\mathbf{v}) \) to be that close to \( w(\mathbf{v}) \) above the critical action window \([0, w_\ell]\); in that range the perturbation simply needs to stay out of the critical action window.

To define our perturbed symplectic form, choose two very small constants \( \epsilon_p^\ell, \bar{\epsilon}_p^\ell \) with \( \epsilon_p^\ell < \bar{\epsilon}_p^\ell \ll \epsilon_\ell \) (recall \( \epsilon_\ell \) was fixed earlier). Fix a monotone increasing function \( g_p : [0, \epsilon_\ell] \to [-1, 0] \) with

\[
g_p(x) = \begin{cases} 
-1 & x \in [0, (\epsilon_p^\ell)^2] \\
0 & x \in [(\bar{\epsilon}_p^\ell)^2, \epsilon_\ell].
\end{cases}
\]

Because the Hamiltonian flow of \( X_{H_\ell} \) preserves the divisors, it follows that on any Hamiltonian orbit (either divisorial or in \( X \)), either \( \rho_i = 0 \) or \( \rho_i \) is bounded away from zero, by say some \( \tau_i > 0 \). We can therefore assume that by choosing our constants \( \epsilon_p^\ell < \bar{\epsilon}_p^\ell \) suitably (i.e., so that \( \bar{\epsilon}_p^\ell < \min_i \tau_i \)),

\[
(3.95) \quad \text{the functions } \frac{dg_p(\rho_i)}{d\rho_i} \text{ vanish in open neighborhoods of}
\]

the periodic orbits of \( X_{H_\ell} \).

We choose \( \kappa_{i,p}^\ell \in \mathbb{Q} \) (so that \( \kappa_{i,\ell} = \kappa_i + \kappa_{i,p}^\ell \) are also rational) and set

\[
(3.96) \quad \theta_{\ell} = \theta + \sum_i \frac{\kappa_{i,p}^\ell}{2\pi} g_p(\rho_i) \theta_{e,i}
\]

Here we use the coordinates \( \rho_i \) and forms \( \theta_{e,i} \) which are part of the regularization data chosen in §2.1. Finally, we assume that \( \kappa_{i,p}^\ell \) are sufficiently small so that

\[
(3.97) \quad \omega_{\ell} := d\theta_{\ell}
\]

remains symplectic. The above condition on the support of \( \frac{dg_p(\rho_i)}{d\rho_i} \) implies that:

the perturbation \( \alpha = \omega_{\ell} - \omega \) vanishes in an open neighborhood

of all the time-1 orbits of the Hamiltonian vector field \( X_{H_\ell} \).

This will be used in Lemma 3.30 below. Moreover, it is easy to see that for \( \kappa_{i,p}^\ell \) sufficiently small, all time-1 Hamiltonian orbits of \( H_\ell \) with respect to the two symplectic forms \( \omega \) and \( \omega_{\ell} \) coincide and the almost-complex structure \( J_S \) used to define (3.55) is tamed by both \( \omega \) and \( \omega_{\ell} \) (recall that being tamed is an open condition in the symplectic form).
Let $X^p_{H^\ell}$ denote the Hamiltonian vector field of $H^\ell$ with respect to $\omega^\ell$. Fix $J_S$ used to define the map (3.55), and consider solutions to the equation

\[(du - X^p_{H^\ell} \otimes \beta)^{0,1} = 0.\]

Define the moduli space of PSS solutions with respect to $J$ as in Definition 3.13. In view of the fact that our complex structure remains of contact type only non-zero when $q$, and quotiented) Floer group of $\omega$ using the obvious modification of (3.53). Implicit in this formula is the fact that the (filtered and quotiented) Floer group of $H^\ell$ with respect to $\omega^\ell$ agrees with $HF^*(X \subset M; H^\ell)_{\omega(\nu)}$ canonically. This is because the symplectic forms $\omega$ and $\omega^\ell$ both agree on $X_\ell$ and by the maximum principle, Floer trajectories do not escape this region.

While the Floer groups coincide automatically it is not a priori clear that the maps $PSS^V_{log}$ and $PSS^V_{log}$ coincide. To prove this, we let $\kappa(s) \geq 0$ be a nondecreasing function such that

- $\kappa(s) = 0$, for $s \leq -1/2$
- $\kappa(s) = 1$, for $s \geq 1/2$

The symplectic forms $\omega^\ell$ are connected by a one parameter family of symplectic forms

\[\omega_s = \omega + \kappa(s) \alpha.\]

The members of this family of symplectic forms all agree in a neighborhood of the orbits, which as before enables us to identify Floer cohomologies $HF^*(X \subset M; H^\ell)_{\omega(\nu)}$. For each $s$, let $X_H^s(s)$ denote the Hamiltonian vector field taken with respect to the symplectic form $\omega_s$.

For what follows, recall that $\rho(r)$ denotes a cutoff function as in (2.76).

**Definition 3.29.** Fix a complex structure $J_S \in J_{S,\ell}(V)$, an orbit $x_0 \in U_\nu$ of $X_H^\ell$, and a critical point of $c \in X(\tilde{S}_I, f_I)$. For a parameter $q \in \mathbb{R}$, define the moduli space $\mathcal{M}_q(\nu, c, x_0)$ to be the set of solutions $u : S \to M$ to the following differential equation

\[(du - \rho(s-q) X_{H^\ell}(s) \otimes dt)^{0,1} = 0\]

with asymptotics (3.42), incidence/tangency conditions (3.45), and enhanced evaluation constraint $Ev_{\nu_0}^V(u) \in W^s(f_I, c)$.

By choosing $J_S$ generically, one can ensure that the parameterized moduli space of pairs $(q, u)$, $u \in \mathcal{M}_q(\nu, c, x_0)$ is cut out transversely and also that the same holds for $\mathcal{M}_q(\nu, c, x_0)$ when $q \in \mathbb{R}$ is chosen generically. For $q \ll 0$, this equation coincides with the standard PSS solution moduli space for the symplectic form $\omega$ (note that the cutoff function $\rho(s-q)$ is only non-zero when $s \ll q \ll 0$, in which region $X_{H^\ell}(s)$ is simply the usual $X_{H^\ell}$ computed with respect to $\omega$).

We define the geometric energy of such a solution $u \in \mathcal{M}_q(\nu, c, x_0)$ to be the usual

\[E_q(u) = \int_S \|du - \rho(s-q) X_{H^\ell}(s) \otimes dt\|\]
where the norm is taken with respect to the symplectic form $\omega_s$ and the family of complex structures $J_S = \{J_z\}_{z \in S}$. By compactness of $M$, the fact that our homotopy $\omega_s$ is compactly supported, and the fact that our space of complex structures $J_z$ depends only on $t$ along the cylindrical end, there exists a smallest constant $|\alpha|$ such that for all $z \in S$ we have

$$\alpha(v,w) \leq |\alpha||v|_z|w|_z$$

for any two tangent vectors $v, w$ at any point $m \in M$. In this equation the norms $|v|_z, |w|_z$ are again the norms associated with $\omega_s$ and $J_z$. Of course, the constant $|\alpha|$ becomes smaller as the $\kappa_{i,p}$ all tend to zero. We may also make use of the following energy which is defined by analogy with the topological energy

$$E_b(u) := \int_S \omega_s(\partial_s u, \partial_t u) - \int S (H \wedge \beta_q)$$

where in this equation we have $\beta_q = \rho(s-q)dt$. As before, we have that

$$E_g(u) \leq E_b(u)$$

A slight complication is that this integral $E_b(u)$ is no longer topological, i.e. it depends on the curve $u$ and not just the multiplicity vector $v$ and the output $x_0$. Nevertheless, we still have:

**Lemma 3.30.** For any $u \in M_q(v,c,x_0)$, we have the following bounds

$$E_g(u) \leq w(v) + A_\ell(x_0) + C|\alpha|E_g(u)$$

$$E_b(u) \leq w(v) + A_\ell(x_0) + C|\alpha|E_b(u)$$

for a constant $C$ which is independent of the curve $u$ or $q$. Moreover the constant $C$ can be taken independent of the $\kappa_{i,p}$.

**Proof.** These estimates are implicit in [Z] (Proof of Theorem 1.6) and can be treated in a very similar fashion to the proof of Proposition 1.5 of loc. cit. (see also [BR]). For the reader’s convenience and to illustrate how the crucial assumption ($*$) on the perturbation is used, we sketch how to modify the proof of Proposition 1.5 of loc.cit. First, note that

$$E_b(u) = \int_S u^*(\omega) + \int S \kappa(s)u^*\alpha - \int S H_{s,t}dsdt.$$

We have also seen that (as described in e.g., the proof of Lemma 3.14)

$$\int S u^*(\omega) - \int S H_{s,t}dsdt = w(v) + A_\ell(x_0).$$

Thus, in view of (3.105), the essential point is to estimate $|\int \kappa(s)\alpha(\partial_s u, \partial_t u)dsdt|$ for any PSS solution $u$ in terms of the energy $E_g$ and $|\alpha|$. To begin, we have the upper bound

$$|\int \kappa(s)\alpha(\partial_s u, \partial_t u)dsdt| \leq \int S |\alpha(\partial_s u, \partial_t u)|dsdt.$$ 

Floer’s equation and the triangle inequality allow us to further bound the right hand side of (3.110) by

$$\int S |\alpha(\partial_s u, \partial_t u)|dsdt \leq \int S |\alpha(\partial_s u, J\partial_s u)|dsdt + \int S |\alpha(\partial_s u, \rho(s-q)X_{H^t})(s)|dsdt.$$ 

Observe that the first term of the right hand side of (3.111) is bounded by $|\alpha|E_g(u)$ (where $|\alpha|$ is the constant in (3.103)); hence the problem may be reduced to bounding the second
term of this equation. Let \( s_0^* \) denote the minimum \( s \) for which \( \rho(s - q) = 0 \) and let \( s_1^* \) denote the maximum \( s \) for which \( \rho(s - q) = 1 \). Then

\[
(3.112) \quad \int_S |\alpha(\partial_s u, \rho(s - q)X_{H^0}(s))| ds dt = \int_{-\infty}^{s_0^*} \int_0^1 |\alpha(\partial_s u, \rho(s - q)X_{H^0}(s))| ds dt,
\]

seeing as the integrand vanishes for \( s \geq s_0^* \). By assumption (\(*\)), there is an isolating set \( N_E \) about the union of all period orbits where \( \alpha \) vanishes. Let

\[
J := \{ s \in (-\infty, s_1^*], u(s, -) \notin \mathcal{L}N_E \}.
\]

Note that the integrand (3.112) is further 0 when \( s \notin J \cup [s_1^*, s_0^*] \); hence

\[
\int_{-\infty}^{s_0^*} \int_0^1 |\alpha(\partial_s u, \rho(s - q)X_{H^0}(s))| ds dt = \int_{\mathcal{L}[s_1^*, s_0^*]} \int_0^1 |\alpha(\partial_s u, \rho(s - q)X_{H^0}(s))| ds dt.
\]

Now

\[
|\alpha(\partial_s u, \rho(s - q)X_{H^0}(s))| \leq C' |\alpha| |\partial_s u|
\]

where \( C' \) is the sup of the norm of the Hamiltonian vectorfields \( X_{H^0}(s) \). Thus,

\[
\int_{\mathcal{L}[s_1^*, s_0^*]} \int_0^1 |\alpha(\partial_s u, \rho(s - q)X_{H^0}(s))| ds dt \leq C' |\alpha| \int_{\mathcal{L}[s_1^*, s_0^*]} \int_0^1 |\partial_s u| ds dt
\]

The crucial point is that, as shown by Zhang in [Z, Proof of Prop. 1.5], the condition (\(*\)) implies that the Lebesgue measure of \( J \) is bounded by \( E_g(u)/N \) for some constant \( N \) which is independent of the PSS solution. Then by Cauchy Schwarz

\[
(\int_{\mathcal{L}[s_1^*, s_0^*]} \int_0^1 |\partial_s u| ds dt)^2 \leq (s_0^* - s_1^* + E_g(u)/N) E_g(u).
\]

Putting all of this together, the first energy estimate (3.106) follows from the fact that either \( E_g(u) \) is smaller than \( w(v) + A_{\ell}(x_0) \) (in which case the estimate trivially holds) or there will be some fixed constant for which \( E_g \leq C'' E_g^2 \). The second inequality (3.107) may be deduced similarly. Note also that since \( s_0^* - s_1^* \) is independent of \( q \), so are the final estimates. \( \square \)

This provides the necessary energy bounds which are needed for Gromov compactness arguments to go through. Expanding on this, we have:

**Lemma 3.31.** Suppose that \( |\alpha| \) is taken sufficiently small \( \epsilon_{\ell}, ||H^\ell - h^\ell||_{C^2} \) are both sufficiently small and \( \Sigma_d \) is sufficiently \( C^0 \)-close to \( \Sigma_{\epsilon}\) . Counting rigid index 0 solutions to (3.101) for a generic \( J_S \in J_{S, \ell}(V) \) and a generic \( q \in \mathbb{R} \) defines a chain map, inducing the following cohomological map:

\[
(3.113) \quad \text{PSS}_{\log}^*: H^*(\mathcal{S}_\ell)t^V \rightarrow HF^*(X \subset M; H^\ell)w(v)
\]

**Proof.** This again follows the pattern of Lemma 4.13 of [GP] (or Lemma 3.15 of the present text) and to avoid being repetitive, we only mention the two key points. The first is that one can use the second equation of Lemma 3.30 to obtain a suitable replacement for Equation (4.34) of [GP]. This allows one to handle breaking along orbits in the divisor just as in loc. cit.

The second point concerns sphere bubbling: observe that this Lemma also implies that when \( |\alpha| \) is sufficiently small, \( E_b(u) \) is still approximately \( w(v)\epsilon_{\ell}^2/2 \) and thus no sphere bubbling can occur at the point \( z_0 \) as the minimal energy of any such non-constant sphere
bubble is larger than $E_0(u)$ (compare proof of Lemma 3.15). It follows that given a sequence of curves $u_n$ limiting to a curve $u_\infty$, the limiting configuration intersects $D$ with multiplicity $v$ at the point $z_0$ (here we identify the domain $S$ as one component of such a conjectural limiting configuration). To rule out sphere bubbling elsewhere, note that the $\omega_s$-energy of the union of any putative sphere bubbles must be positive and hence the collection of these sphere bubbles must intersect at least one of the divisors $D_i$ positively. By positivity of intersection with $D$, any Floer cylinder or PSS solution must intersect $D_i$ with non-negative multiplicity. This however contradicts the fact that the sum of all intersections with $D_i$ away from $z_0$ must be zero. It follows that no sphere bubbling can occur. With these points in place, the rest of the proof follows as in the proof of Lemma 4.13 of loc. cit. \hfill \Box

**Lemma 3.32.** Let $x_0, x_1$ be orbits in $U_\nu$ such that $\deg(x_0) = \deg(x_1)$. Let $u$ be a solution to
\begin{equation}
(du - X_{H^\ell}(s) \otimes dt)^{0,1} = 0
\end{equation}
For $|\alpha|$ sufficiently small, generic $J_t$, and $||H^\ell - h^\ell||$ sufficiently $C^2$ small, $u$ is $s$-independent.

**Proof.** Again assuming that we can show that solutions do not develop limits at orbits along $D$, the usual compactness argument shows that all such solutions lie in $U_\nu$ where the symplectic form is constant (and the result follows by noticing that, by applying $\mathbb{R}$-translations, $s$-dependent solutions come in non-rigid families). In order to show solutions do not break along orbits in $D$, similarly to the construction of the map (3.113), we adapt the proof of Lemma 2.16. Note that this Lemma gets off the ground via the estimate in Equations (2.87)-(2.91). Using the notation of that Lemma, one can in the present situation use \cite{Z, Proposition 1.5} to obtain the following modified version of these equations:
\begin{equation}
E_q(\tilde{S}) \leq \frac{-A_{\ell}(x_1) + C|\alpha|A_{\ell}(x_0)}{1 - C|\alpha|} + \int_{\partial S} u^* \theta - \theta (X_{H^\ell}) dt + \int_{\partial S} \lambda^\ell dt
\end{equation}
When $|\alpha|$ is sufficiently small, this is enough to carry out the rest of the argument of that Lemma to rule out breaking along $D$. With this established, when $||H^\ell - h^\ell||$ sufficiently $C^2$ small, solutions to (3.114) are therefore just Floer trajectories and the only index 0 solutions are constant in $s$. \hfill \Box

**Lemma 3.33.** The maps $\text{PSS}^\nu_{\log}$ and $\text{PSS}^{\nu,p}_{\log}$ (cohomologically) coincide.

**Proof.** As observed above when $q \ll 0$, $\text{PSS}^\nu_{\log} = \text{PSS}^\nu_{\log}$. As $q \to +\infty$, the fact that Gromov compactness applies to our setting shows that solutions tend to broken configurations consisting of PSS solutions elements followed by cylinders solving (3.114); appealing to Lemma 3.32, this broken moduli space can be expressed as:
\begin{equation}
\tilde{M}(v, c, X_{H^\ell}, x_0) \times \tilde{M}(x_0, x_0).
\end{equation}

The right hand side of the above product (which is the space of parametrized Floer trajectories prior to any quotient by $\mathbb{R}$) consists of a single trivial constant solution; in particular, the associated count gives the identity map, and the operation associated to counting elements of (3.116) is simply $\text{PSS}^{\nu,p}_{\log}$. It follows by standard methods and what we have seen previously that counts associated to the (rigid elements of the compactification of the) parametrized moduli space $\{(q, u) | u \in \mathcal{M}_q(v, c, x_0)\}$ give, by looking at the boundary of the 1-dimensional components of the same moduli space, a chain homotopy between $\text{PSS}^\nu_{\log}$ and $\text{PSS}^{\nu,p}_{\log}$.

\hfill \Box
For the remainder of §3 and all of §4, all Hamiltonian vector fields, Floer cohomology groups, PSS maps, etc. will be defined using the symplectic form $\omega_\ell$.

We now complete the process of separating out the Hamiltonian actions of our orbits by perturbing our Hamiltonian to a new Hamiltonian $H_\ell^\ell$ so that all time-1 Hamiltonian orbits with respect to $\omega_\ell$ are now supported in the region where $g_p = -1$. Fix $\epsilon_\ell^p < \epsilon_\ell^p$ and consider a new Liouville boundary $\Sigma_\ell^p = \Sigma_{\epsilon_\ell^p}$ which is even closer to the divisor by taking $\epsilon_\ell^p = \epsilon_\ell^p$ and rounding the boundary as before. Next, fix functions $h_p(R_\ell^p)$ of the Liouville coordinate $R_\ell^p$ (with respect to $\theta\ell$) which are of slope $\lambda_\ell$ and satisfy the same conditions as $h_\ell(R_\ell)$ from Section 2. Finally, we perturb these Hamiltonians in small regions about the Hamiltonian orbits which, in a slight abuse of notation we also denote by $U_\ell$, to obtain Hamiltonians $H_\ell^\ell$.

For any $w_p \leq w_\ell + \delta_p$, we may form Floer cochain groups $F_{w_p} CF^*(X \subset M; H_\ell^\ell)$ and cohomology groups $F_{w_p} HF^*(X \subset M; H_\ell^\ell)$ by complete analogy with the unperturbed case. Continuation maps give rise to isomorphisms:

$$F_{w} HF^*(X \subset M; H_\ell^\ell) \cong F_{w_\ell + \delta_p} HF^*(X \subset M; H_\ell^\ell)$$

These isomorphisms are compatible with the respective PSS maps. Shrinking $\epsilon_\ell^p$ as needed, the complexes $F_{w_p} CF^*(X \subset M; H_\ell^\ell)$ admit a finer filtration than $F_{w} CF^*(X \subset M; H_\ell^\ell)$ because the function $w_p$ uniquely determines the winding vector $v$. Namely, for any vector $v$ with $w(v) \leq w$, we may set

$$CF^*(X \subset M; H_\ell^\ell)_v = \frac{F_{w_p(v)} CF^*(X \subset M; H_\ell^\ell)}{F_{w_p(v) - d_v} CF^*(X \subset M; H_\ell^\ell)}$$

for a sufficiently small constant $d_v > 0$ so that this quotient is generated by orbits in the single isolating set $U_v$. We therefore have a “local” PSS map

$$\text{PSS}_v^\ell : H^\ell(\tilde{S}_1) t^v \to HF^*(X \subset M; H_\ell^\ell)_v$$

where the right hand side denotes the cohomology of the cochain complex defined in (3.117). The following lemma follows immediately from the above considerations and allows us to completely localize the task of proving that the map (3.55) is an isomorphism:

**Lemma 3.34.** The map (3.55) is an isomorphism iff for every $v$ with $w(v) \leq w$, (3.118) is an isomorphism. 

4. Low energy PSS is an isomorphism

In this section, we complete the proof of Theorem 1.1. So far, we have constructed the relevant action-filtered spectral sequence for $SH^*(X)$, a low energy log PSS ring map from $H^*_\text{log}(M, D)$ to the first page, and separated actions so that this map splits into localized maps (3.118) each containing one multiplicity vector $v$. Here, we construct a one-sided inverse to the maps (3.118) and use this to deduce that the low energy PSS map (3.57) is an isomorphism.

In some more detail: the first step is to perform a Morse-Bott analysis of the associated graded Hamiltonian Floer complex, in order to argue that the source and target of the localized (3.118) are abstractly isomorphic finitely generated $k$-modules; as explained in the proof of Corollary 4.30 it therefore suffices to construct a one-sided inverse to each localized
map (3.118). The relevant Morse-Bott analysis is begun in §4.1 but only completed later in the section, in Lemma 4.17. Next, we show in §4.1.4.2 using methods of local Floer homology, monotonicity for pseudo-holomorphic curves, and nice choices of “split” $J$ with respect to the chosen regularization of $\mathcal{D}$, that (if necessary shrinking our various choices of small constants further) each associated graded (low-energy) Floer moduli space and localized low-energy log PSS moduli space from the previous section can be geometrically confined in $M$ to a small neighborhood of some stratum $D_I$ near the relevant Hamiltonian output orbit. In particular, we can identify these moduli spaces spaces (and operations) with corresponding ($\rho$) strata where $\rho \in \mathcal{R}$ and we again refer to Step 2 of the proof of Theorem 5.16 of §3.3.

Using the fact that we are now in projective bundles, in §4.6 we can finally exhibit a one-sided inverse, the “local SSP map;” the relevant operation is built out of maps from a punctured sphere into the various projective bundles $PD_I$ with incidence conditions now along $\infty$-divisors (i.e., divisors where fiber coordinates are equal $\infty$).

Combining everything we have done so far, the proof of Theorem 1.1 appears at the end of §4.7 as Theorem 4.32.

4.1. Local Floer cohomology. To begin, we describe careful perturbations of our Hamiltonian $h^\ell_p$ for a given $\ell \in \mathbb{N}$. For nonzero values of $v$, the orbit sets $\mathcal{F}_v$ occur at points $U_I$ where $\rho_i = \rho_{i,v}$ for some $\rho_{i,v} \in \mathbb{R}$ and $i \in I$. The manifold also has boundary and corner strata where $\rho_i = \rho_{i,v}^\ell$ for some $\rho_{i,v}^\ell \in \mathbb{R}$ and $i \notin I$. For later use, we now note the following proposition

**Lemma 4.1.** For $v \neq 0$, the manifolds with corners $\mathcal{F}_v$ are homeomorphic to the manifolds $S_{I,\log}^\ell$ defined in §3.1.

**Proof.** This follows from the above description of $\mathcal{F}_v$ and the discussion in Remark 3.3. $\square$

All of the $\mathcal{F}_v$ are Morse-Bott in their interiors $\mathcal{F}_v \setminus \partial \mathcal{F}_v$ (for $v = 0$, this is obvious and we again refer to Step 2 of the proof of Theorem 5.16 of [M2] for this result in the case $v \neq 0$) and thus Morse-Bott type perturbations are required to make the orbits non-degenerate. Compared to genuine Morse-Bott situations, where the orbit sets form genuine closed manifolds, we must pay a bit of extra attention near the boundaries of our orbit sets.

We next choose the isolating neighborhoods $U_v$ for these critical sets for each critical set $\mathcal{F}_v$, that is to say neighborhoods of $\mathcal{F}_v$, which contain only the orbit set $\mathcal{F}_v$ and no others. For $\mathcal{F}_0$, choose our isolating neighborhood $U_0$ to be the complement of neighborhood where

$$U_0 := M \setminus \{ R_p^\ell \geq R_{0,\ell} + c_0 \}$$

for a sufficiently small constant $c_0$. For the other orbits, let $D_I^{c_0}$ denote the open manifold $D_I^{c_0} := D_I \setminus \cup_{i \notin I} U_i \rho_{i,v}^{\ell} - c_0$ and let $S_{I,\log}^{c_0}$ denote the induced $T^I$ bundle over $D_I^{c_0}$ where $\rho_i = \rho_{i,v}^{\ell}$. After possibly shrinking $c_0$, the isolating sets are then chosen to be the neighborhoods $U_v \subset U_I$ such that $\pi_I(x) \in D_I^{c_0}$ and $\rho_{i,v} - c_0 < \rho_i < \rho_{i,v} + c_0$, $i \in I$. We choose $c_0$ sufficiently small so that these neighborhoods do not pairwise intersect, e.g. $U_v \cap U_{v'} = \emptyset$ for $v \neq v'$. We let $U'_v$ to be slightly smaller subsets such that $U'_v \subset U_v$ which are of the same form (to construct them just take a constant $c'_0$ which is slightly smaller than $c_0$). Choose a Morse function $h_I : S_I^{c_0} \to \mathbb{R}$ such that near the corners the function $h_I$
are functions of the \( \rho_i \) and point outwards along the boundary. Finally, we choose cutoff functions \( \rho_v \), such that

- \( \rho_v(x) = 0, x \in M \setminus U_v \)
- \( \rho_v(x) = 1, x \in U'_v \)

Next we recall the *spinning* construction in Morse-Bott theory (see e.g. [KvK, Proof of Prop. B.4]). For non-constant orbits, observe that on all of the orbit sets \( \mathcal{F}_v \), the Reeb flow generates an \( S^1 \)-action on \( \mathcal{F}_v \) which extends canonically to \( \mathcal{S}_I^0 \) and \( U_v \). The inverse circle action on \( U_v \) is a Hamiltonian flow with associated Hamiltonian function \( K: U_v \to \mathbb{R} \), where

\[
(4.1) \quad K = \sum_i \pi v_i,0 \rho_i
\]

We denote the associated time-\( t \) flow of \( K \) by \( \Delta_t(x) \). If \( x(t) \) is a one periodic orbit of the Hamiltonian vector field, then \( \dot{x}(t) = \Delta_t \circ x(t) \) is a one periodic orbit corresponding to the Hamiltonian

\[
(4.2) \quad \hat{h}_p^\ell = h_p^\ell + K(x)
\]

and similarly for Floer trajectories. Here we have used the fact \( h_p^\ell(t, \Delta_t^{-1}(x)) = h_p^\ell \) by the local invariance of the function \( h_p^\ell \). We thus obtain a new Hamiltonian system in which the Hamiltonian \( \hat{h}_p^\ell \) is constant on \( \mathcal{F}_v \) and hence has constant orbits. Define \( h_I \) to be the time dependent function \( h_I := \hat{h}(t, \Delta_I(x)) \). For \( v \neq 0 \), let \( h_v \) denote the pull-back of \( h_I \) to \( U_v \) under the projection map. We also set \( h_0 \) to be a function which near \( R_p^\ell = R_{0,\ell} \) is a function of \( R_p^\ell \) with positive derivative.

We also recall how to choose perturbing data for the divisorial orbits (see [GP, §4.1] for more details on this perturbation), even though the details of this will be less important for our applications. The divisorial orbits come in strata which are also manifolds with corners. The orbits which lie in \( D_I \) correspond to the points where \( \rho_i = 0 \) for all \( i \in I \). The Hamiltonian vector field restricted to the fibers of \( U_I \) at these points is of the form \( \sum_{i \in I} -\lambda_i \partial_{\rho_i} \) with \( \lambda_i > 0 \) which infinitesimally generates a non-trivial rotation of the fibers fixing the points where \( \rho_i = 0 \).

It follows that these orbits are transversely nondegenerate over the open parts. Similar to what we have seen above, choose an outward pointing Morse function \( \hat{h}_D \) on the disjoint union of these submanifolds and set \( h_D = \hat{h}(t, \Delta_I(x)) \) as before. Choose cutoff functions \( \rho_D \) supported in a small neighborhood of the divisors (and in particular inside \( V_{0,\ell} \)). For sufficiently small constants \( \delta_v \) and \( \delta_D \) define

\[
(4.3) \quad H_p^\ell = \sum_v \delta_v \rho_v + h_p^\ell + \delta_D \rho_D h_D
\]

It follows from the analysis in [GP, §4.1] that for suitable choices of \( \rho_D \) and \( h_D \) that the Hamiltonian flow of this perturbed function preserves the divisor \( D \). For the remainder of this section, we focus on the orbits which lie in \( M \setminus V_I \) in each open set \( U_v \) there are obvious time-1 orbits corresponding to critical points \( x_i \) of the function \( \hat{h}_I \) on \( \mathcal{F}_v \).

**Lemma 4.2.** For \( \delta_v \) sufficiently small, all time-1 orbits of \( H_p^\ell \) are those created as critical points from the manifolds \( \mathcal{F}_v \) as critical points of \( h_v \).

**Proof.** For sufficiently small \( \delta_v \), it follows by compactness that such orbits must be contained in some \( U'_v \) where \( \rho_v = 1 \). There are also no fixed points near \( \rho_{x,i} \) because
the derivative of \( \hat{h}_t \) points outwards along the strata. Thus, the lemma reduces to the well-known case in Morse-Bott theory as covered in e.g. [KvK].

We will also consider a very closely related version of Floer cohomology known as \emph{local Floer cohomology}, which is defined whenever \( \delta_\nu \) above are taken sufficiently small. The abelian group underlying the complex is the same as \( CF^*(X \subset M; H_p^\ell) \). However, the differential \( \partial_{\text{loc}} \) counts local solutions to Floer’s equation which lie in \( W_\ell := \cup_\nu U_\nu \). Let \( J_\ell \in \mathcal{J}(M, D) \), be a (generic) time-dependent complex structure, which we will always assume is \( \omega_\ell \)-compatible in an open neighborhood of all Hamiltonian orbits \( x_0 \in X(X; H_p^\ell) \).

For any two orbits \( x_0, x_1 \), let \( \mathcal{M}(W_\ell; x_0, x_1) \) denote the moduli space of solutions to Floer’s equation (2.69) for the Hamiltonian \( H_\ell \), modulo \( \mathbb{R} \)-translation, which additionally satisfies the following “locality” and asymptotic conditions:

\[
\left\{ \begin{array}{c}
u : \mathbb{R} \times S^1 \to \cup_\nu U_\nu, \\
\lim_{s \to -\infty} u(s, -) = x_0 \\
\lim_{s \to +\infty} u(s, -) = x_1 \end{array} \right. \]

The following basic result ensures that such local Floer curves are confined i.e., stay away from the boundary of \( \cup_\nu U_\nu \):

\textbf{Lemma 4.3.} ([M5, Lemma 2.3]) \textit{For sufficiently small} \( \delta_\nu \), \textit{those Floer curves contained in} \( \cup_\nu U_\nu \) \textit{are in fact contained in} \( \cup_\nu U_\nu' \).

From here, Gromov compactness applies and the usual Floer-theoretic arguments allow us to, for small \( \delta_\nu \) as above, define the \( |\sigma_{x_1}|-|\sigma_{x_0}| \) component of the differential as

\[
(\partial_{\text{loc}})_{x_1, x_0} = \sum_{u \in \mathcal{M}(W_\ell; x_0, x_1)} \mu_u \]

where \( \mu_u : \sigma_{x_1} \sim \sigma_{x_1} \) is the induced isomorphism of orientation lines for a rigid \( u \) (which occurs whenever \( \text{deg}(x_0) - \text{deg}(x_1) = 1 \)). Also, standard methods show \( \partial_{\text{loc}}^2 = 0 \). We denote the resulting cohomology theory by \( HF^*(W_\ell \subset M, H_p^\ell) \). We similary form the group \( HF^*(U_\nu \subset M, H_p^\ell) \) which is the local Floer subcomplex consisting of orbits just in \( U_\nu \). There is an obvious decomposition

\[
HF^*(W_\ell \subset M, H_p^\ell) \cong \bigoplus_{\mathcal{F}_\nu \in \mathcal{F}(X, h_p^\ell)} HF^*(U_\nu \subset M, H_p^\ell). 
\]

\textbf{Lemma 4.4.} \textit{Fix a generic} \( J_\ell \in \mathcal{J}(X_p^\ell, V) \) \textit{and let} \( x_0, x_1 \in U_\nu \). For \( \epsilon_\ell, \delta_\nu \) sufficiently small and \( \Sigma^p_\epsilon \) sufficiently \( C^0 \)-close to \( \Sigma^p_\nu \), any two Floer trajectories in \( X \) connecting \( x_0, x_1 \) lie in \( U_\nu \). As a consequence, we have canonical isomorphisms

\[
HF^*(X \subset M; H_p^\ell)_\nu \cong HF^*(U_\nu \subset M, H_p^\ell)_\nu \\
\bigoplus_{w(\nu) \leq u_\ell} HF^*(X \subset M; H_p^\ell)_{w(\nu)} \cong HF^*(W_\ell \subset M, H_p^\ell)
\]

\textbf{Proof.} Because we have chosen our \( \kappa_{i, \ell} \) so that our orbits have distinct Hamiltonian actions, this follows from [M5, Lemma 2.8].

\footnote{We do this to rely on standard gluing results in the literature on (local) Floer cohomology. It is likely not necessary.}
4.2. Controlling the low energy PSS solutions.

(§) From this point until the end of §4.6, we work with a fixed non-zero vector \( \mathbf{v} = \mathbf{v}_0 \).

Now that we have localized the source and target of our PSS\(_{\text{log}}\) map, we want to choose our data so that the PSS solutions themselves are constrained to lie in a small neighborhood of the stratum corresponding to \( \mathbf{v} \). To do so, we will enlarge the class of complex structures allowed, and specifically show that our local PSS\(_{\text{log}}\) map is defined with respect to arbitrary complex structures preserving the divisors. This will enable us to use "split" complex structures (c.f. Definition 4.9) which have simple local models (in our chosen tubular neighborhoods) near the various strata, and allow for a direct appeal to monotonicity arguments to deduce the desired locality phenomena.

Let \( H^S_2(M) \) denote the image \( \text{im}(\pi_2(M)) \subset H_2(M) \). Since by assumption our symplectic form \( \omega_\ell \) is rational, it follows there is a minimum quantity \( \omega_{\text{min}} \) such for any class \( A \in H^S_2(M) \), \( |\omega_\ell([A])| > \omega_{\text{min}} \) whenever \( \omega_\ell([A]) \neq 0 \); by convention we set \( \omega_{\text{min}} = \infty \) if \( \omega_\ell([A]) = 0 \) for all \( A \in H^S_2(M) \). In particular, we also know that \( \omega_\ell(A) < -\omega_{\text{min}} \) whenever \( \omega_\ell(A) < 0 \).

Choose \( J_S \in \mathcal{J}_S(M, \mathbf{D}) \) which restricts on the cylindrical end to some \( J_0 \) that we use to define local Floer cohomology. For any \( x_0 \in \mathcal{U}_\mathbf{v} \) and \( c \in X(\hat{S}_1, f_1) \), let \( \mathcal{M}(\mathbf{v}, c, H_\ell, x_0) \) denote the moduli space of PSS solutions with respect to the Hamiltonian \( H_\ell \). The next Lemma gives the basic compactness results for low energy PSS moduli spaces, for generic such \( J_S \):

**Lemma 4.5.** Fix any critical point \( c \in X(\hat{S}_1, f_1) \) and an orbit \( x_0 \in \mathcal{U}_\mathbf{v} \) of \( H_\ell \) such that \( \text{vdim}(\mathcal{M}(\mathbf{v}, c, H_\ell, x_0)) = 1 \). Then for generic \( J_S \in \mathcal{J}_S(M, \mathbf{D}) \), \( \ell_\ell, ||H_\ell - h_\ell||_{C^2} \) sufficiently small, and \( \Sigma_\ell \) sufficiently \( C^0 \) close to \( \hat{\Sigma}_\ell \), the Gromov-Floer compactification of \( \mathcal{M}(\mathbf{v}, c, H_\ell, x_0) \) is a compact 1-manifold with boundary \( \partial \mathcal{M}(\mathbf{v}, c, H_\ell, x_0) = \partial_F \bigcup \partial_M \) where

\begin{align*}
\partial_F := & \bigcup_{x, |x_0| - |x'| = 1} \mathcal{M}(\mathbf{v}, c, H_\ell, x') \times \mathcal{M}(x_0, x') \\
\partial_M := & \bigcup_{c', \deg(c') - \deg(c) = 1} \mathcal{M}(c', c) \times \mathcal{M}(\mathbf{v}, c', H_\ell, x_0)
\end{align*}

**Proof.** The energy of a solution \( u \in \mathcal{M}(\mathbf{v}, c, H_\ell, x_0) \) is arbitrarily close to \( \frac{1}{2} w_p(\mathbf{v})(\ell_\ell^p)^2 \).

Recall from Lemma 2.9 that by taking \( \Sigma_\ell \) sufficiently \( C^0 \) close to \( \hat{\Sigma}_\ell \), we can assume that \( H_\ell \approx \lambda_\ell \left( \frac{1}{1 - \frac{1}{2} (\ell_\ell^p)^2} - 1 \right) \) along the divisors. Assume that \( \ell_\ell^p \) is chosen sufficiently small so that

\begin{equation}
\lambda_\ell \left( \frac{1}{1 - \frac{1}{2} (\ell_\ell^p)^2} - 1 \right) < \omega_{\text{min}}.
\end{equation}

The Lemma as usual follows from Gromov compactness provided we exclude other possible "bad" limits from occurring in the compactification. In light of our total energy being very small, sphere bubbling cannot occur, so it suffices to rule out cylinder breaking along orbits in \( \mathbf{D} \). So, suppose \( y \) is in \( X(\mathbf{D}; H_\ell) \), and that the curve limits to a broken curve \( (u_1, u_2) \) with \( u_1 \) a PSS solution and \( u_2 \in \overline{\mathcal{M}}(x_0, y) \). The class of \( u_1 \) in relative homology must be the connect sum of

- a (multiply covered) fiber disc \( F \) (the canonical capping disc of \( y \) in \( \mathbf{D} \) which is a product of discs and constant discs in the fibers of the regularization), oriented so that the boundary is \( y \) with the opposite orientation; with
• some absolute homology class $A \in H_2(M)$.

We have that
\begin{equation}
E_{\text{top}}(u_1) = \int_{u_1} \omega(\gamma) + \lambda\ell\left(\frac{1}{1 - \frac{1}{2}(\epsilon^p_{\ell})^2} - 1\right)
\end{equation}

Observe that $\int_{\ell} \omega_{\ell} \geq 0$ is small, at most $\frac{1}{2}(w_{\ell} + \delta_p)(\epsilon^p_{\ell})^2$. In particular, it is not possible for $\omega_{\ell}(A) < 0$ because otherwise it would be less than $-\omega_{\text{min}}$, implying the topological energy of $u_1$ would be negative. It follows that the topological energy of $u_1$ is at least $\lambda\ell(\frac{1}{1 - \frac{1}{2}(\epsilon^p_{\ell})^2} - 1)$, which is bigger than the total energy $\frac{1}{2}w_p(v)(\epsilon^p_{\ell})^2$. Thus the topological energy of $u_2$ must be negative, a contradiction. \hfill \Box

The process of geometrically confining (or localizing) PSS moduli spaces has two steps: first, we take a limit where we turn off most of the Hamiltonian perturbations supported anywhere except for our particular $U_{v_0}$. The resulting Hamiltonian is degenerate, but has the nice property that for “split” almost complex structures (Definition 4.9), its Floer curves are genuinely holomorphic curves outside $U_{v_0}$ (after possibly projecting to whichever divisorial stratum it is near). In particular an iterative application of the the usual monotonicity Lemma for pseudo-holomorphic curves (over strata) gives the desired geometric control over “degenerate PSS” solutions. By a version of Gromov compactness (Lemma 4.6) this implies confinement for “small” perturbations. This argument and the appeal to monotonicity appears in Lemma 4.10.

To begin this process, for any $v' \neq v_0$ (where $v_0$ is our fixed vector ($\hat{\phi}$)), we fix a sequence of $\delta_{v,n} \to 0$. We also fix a sequence of $\delta_{D,n} \to 0$ and we let $H^\epsilon_{\ell}$ denote the associated Hamiltonians constructed by Equation (4.3). Let $H^\epsilon_{\infty}$ denote the limiting Hamiltonian. By construction, we have that $H^\epsilon_{\infty} - h^\epsilon_p$ is supported in $U_{v_0}$. For each $n$, we fix $J_{S,n} \in J_\ell(M,D)$ which converge to some $J_S \in J_\ell(M,D)$. Let $M(v_0, J_{S,n}, H^\epsilon_{\ell}, x_0)$ denote the moduli space of PSS solutions for this sequence of Hamiltonians. We will make use of the general fact about solutions to Floer’s equation using a possibly degenerate Hamiltonian (such as the limiting Hamiltonian $H^\epsilon_{\infty}$):

**Lemma 4.6** (compare [O2] Proposition 18.4.10 or [S1] proof of Prop. 4.2). Let $H : S^1 \times X \to \mathbb{R}$ be any Hamiltonian and $\Sigma$ be a domain decorated with suitable cylindrical ends and perturbation data $K$. Suppose that $u : \Sigma \to X$ be a finite energy solution to Floer’s equation (2.99). Restrict $u$ to a cylindrical end $\epsilon_+ : [0, \infty) \times S^1$ or $\epsilon_- : (-\infty, 0] \times S^1$. Then for any $s_m \to \pm\infty$ there is a subsequence $s_m$, so that $u(s_m, -) \to \gamma(t)$ in $C^\infty(S^1, X)$, where $\gamma$ is some periodic orbit of the Hamiltonian vector field.

When the Hamiltonian is degenerate there may exist sequences $s_m$ and $s'_m$ giving rise to different limits $\gamma$ and $\gamma'$. Returning to our sequence of moduli spaces, the relevant version of Gromov compactness we need is:

**Lemma 4.7.** Fix $x_0 \in U_{v_0}$. For $\epsilon^p_\ell$, $\delta_{v_0}$ sufficiently small and $\Sigma^p_\ell$ sufficiently $C^0$ close to $\Sigma^p_\ell$ and given any sequence of solutions $u_\ell \in M(v_0, J_{S,n}, H^\epsilon_{\ell}, x_0)$, there is a subsequence which converges in the Gromov topology to some
\begin{equation}
(4.12) \quad u_\infty \in M(v_0, J_S, H^\epsilon_{\infty}, y_0) \times M(y_1, y_0) \times \cdots \times M(x_0, y_k)
\end{equation}
for $y_i \in U_{v_0}$.

\[13\] It is still true that $\gamma$ and $\gamma'$ must have the same Hamiltonian action.
Proof. Observe that by our assumptions no bubbling can occur by arguments already given (namely, that the energies of the $u_n$ are smaller than the minimal energy of $J$-holomorphic spheres). It follows that

$$|\nabla u_n|_{L^\infty} < \infty$$

remains bounded and hence that we have that the solution converges in $C^\infty_{\text{loc}}$ to some solution

$$\hat{u}_\infty : S \to M$$

By Lemma 4.6 it follows that there are sequences $s_m \to -\infty$ and an orbit $y'$ such that

$$\hat{u}_\infty(s_m, t) \to y' \in C^\infty(S^1, M)$$

Suppose that $y'$ is a divisorial orbit in $D$. Then there would be a pair $(n, s_0)$ for which $u_n(s_0, t)$ is arbitrarily close to $y' \in C^\infty$. The argument of Lemma 4.5 shows that this is not possible.

Suppose there is some orbit $y'$ in $X$ not in $U_{V_0}$ together with a sequence of $s_m \to -\infty$ such that $\hat{u}_\infty(s_m, t) \to y' \in C^\infty(S^1, M)$. Then there would again be a pair $(n, s_0)$ for which $u_n(s_0, t)$ is arbitrarily close to $y' \in C^\infty$. Let $\hat{u}_n = u_n^{-1}(-\infty, s_0)$. Then

$$E(\hat{u}_n) \approx -A_t(y') - (1 - \frac{1}{2} \epsilon_t^p)w(v_0)$$

which by our assumptions would imply that either

- $E(\hat{u}_n) < 0$; or
- $E(\hat{u}_n) \gg \frac{1}{2} w(v_0) \epsilon_t^p/2$.

Thus we conclude that $\hat{u}_\infty$ limits uniquely to some orbit $y_0$ in $U_{V_0}$. Likewise, the same arguments show that after rescaling by suitable $s_m \to -\infty$ we obtain Floer trajectories with asymptotes only in $U_{V_0}$. \qed

The version of monotonicity for pseudoholomorphic curves we will appeal to is:

Lemma 4.8 (Monotonicity lemma, compare [BEH+] Lemma 5.2 or [S7] Proposition 4.3.1). Let $(W, J)$ be a compact almost complex manifold and suppose that $J$ is tamed by some $\omega$. Then there exists a positive constant $C_0$ having the following property: For any compact $J$-holomorphic curve $f : (S, j) \to (W, J)$, point in the domain $s_0 \in S \setminus \partial S$, and $r > 0$ smaller than the injectivity radius of $W$, if the boundary $f(\partial S)$ is contained in the complement of the ball $B_r(s_0)$ about $s_0$ of radius $r$, then the area\footnote{Both the ball $B_r(s_0)$ and definition of area are with respect to the the metric induced by $\omega$ and $J$.} of the portion of $f$ mapping to $B_r(s_0)$ is bounded below by:

$$A(f^{-1}(B_r(s_0))) \geq C_0 r^2.$$

Definition 4.9. We say that $J_0 \in \mathfrak{J}(M, D)$ is split if over each $\pi_I : U_I \to D_I$

- the map $\pi_I$ is $J_0$-holomorphic;
- for every point $p$, the complex structure respects the decomposition $H_p \oplus F_p$;
- on each $F_p$, the complex structure is split with respect to the product decomposition on $D_p^{\vert I}$.

For any $U_I$, we may consider the the horizontal piece of the symplectic form

$$\omega_H = \omega_I - \omega_{\text{vert}}$$
This is symplectic on each horizontal subspace $H$. Observe that for a split $J_0$, the energy of a curve which lies in $U_I$ can be split into horizontal and vertical pieces.

$$E(u) = E_{\text{hor}} + E_{\text{vert}}$$

Let $g_{\omega_H}$ and $g_{\omega_D}$ denote the corresponding metrics on $D_I$ and the horizontal subspace of the tangent space to $U_I$. Since the metric $g_{\omega_H}$ can be made to extend to a compactification $\bar{U}_I$ of $U_I$ (by shrinking $U_I$ if necessary), it follows that there exists a constant $G_I > 0$ such that

$$g_{\omega_D} < G_I \cdot g_{\omega_H}$$

as positive bilinear forms on each $H_p$.

We fix once and for all an almost-complex structure $J_0$ which is split outside of a small neighborhood of $U_{\mathcal{V}_0}$ and compatible inside of $U_{\mathcal{V}_0}$. All of our almost-complex structures in this section will be arbitrarily close to this fixed complex structure. Let $C_I$ be the monotonicity constant associated to the complex structures over $D_I$.

**Lemma 4.10.** Fix $\mathcal{V}_0$ as above, let $I$ denote the support of $\mathcal{V}_0$, and fix some $\epsilon' > 0$ smaller than the size $\epsilon$ of all of the tubular neighborhoods $U_i$. For $\epsilon'_0$, $\delta_D$, $\delta_\omega$, $\delta_{\mathcal{V}_0}$ sufficiently small (depending on $J_0$), $S'_\epsilon$ sufficiently $C^0$ close to $\Sigma_{\epsilon'}$ and $J_S$ sufficiently close to the split $J_0$ above, any PSS$^{\epsilon_0}$ solution $u \in M(\mathcal{V}_0, x_0)$ must lie in $U_{I,(\epsilon')}^2$.

**Proof.** By the compactness statement of Lemma 4.7, it suffices to prove that any limiting broken curve $u_\infty$ appearing in the statement of Lemma 4.7 with $J_S = J_0$ may not escape $U_{I,(\epsilon')}^2$, for some sufficiently small $\epsilon'_0$ (or equivalently large constant $C_\epsilon = \epsilon'/\epsilon''_0$) determined in the proof. For sufficiently small $\delta_{\mathcal{V}_0}$ we have seen that any Floer trajectory remains in $U_{\mathcal{V}_0}$ by Lemma 4.4. Hence it only remains to confine the PSS component $\hat{u}_\infty$ of the broken curve, which as we have seen has energy approximately $\frac{1}{2}w(\mathcal{V}_0)(\epsilon'_0)^2$, which becomes arbitrarily small as we decrease $\epsilon'_0$.

For what follows, it suffices to replace the curve $\hat{u}_\infty$ with its intersection with $U_I$. Our argument will proceed by inductively decreasing $\epsilon'_0$ (and implicitly the other constants so that the confinement of Floer trajectories continues to hold) in order to confine this PSS curve away from various strata.

Note that $U_I \cap U_K = \emptyset$ for any subset $K$ with $D_I \cap D_K = \emptyset$. Now assume by contradiction that this curve contains a point $p$ in $U_I \setminus U_{I,(\epsilon')}^2$. We let $K$ range over all subsets such that $D_K \cap D_I \neq \emptyset$ and $I \not\subset K$ and consider the ordering where $K_1 < K_2$ if $\#K_1 < \#K_2$ or if $\#K_1 = \#K_2$ and the (lexicographically) first distinct digits $k_1$ and $k_2$ between $K_1$ and $K_2$ satisfy $k_1 < k_2$. Define $\hat{U}_K := U_{K,(\epsilon'')}^2 \setminus \bigcup_i g_K U_{I,(\epsilon'')}^2$ (as usual $U_0 = U_{\emptyset,(\epsilon'')}^2 = M$ for any $\epsilon''$). These sets cover $U_I \setminus U_{I,(\epsilon')}^2$ and, throughout the rest of the proof we will, in an abuse of notation, replace $U_{K,(\epsilon'')}^2$ with its intersection with $U_I \setminus U_{I,(\epsilon')}^2$.

Our proof will inductively in $K$ (using the above ordering) show that, by taking $\epsilon'_0$ sufficiently small, the curve cannot intersect $\hat{U}_K$. First let us consider the base case, which is to show that for $\epsilon'_0$ small, the curve cannot have a point $p$ in $U_0 \setminus U_{I,(\epsilon'')}^2$ (implicitly intersected with $U_I \setminus U_{I,(\epsilon')}^2$). This follows from monotonicity (Lemma 4.8) since for $\epsilon'_0 \ll \frac{1}{\rho_{\epsilon'}} \epsilon'_0$ the curve is holomorphic in the larger region $M \setminus \bigcup_i U_i(\epsilon'')^2$, and its intersection with this region has boundary on the loci where one or more of the $\rho_i = \frac{\kappa}{\epsilon''_0}(\epsilon'_0)^2$. This loci is metrically bounded away from the smaller region $U_0 \setminus \bigcup_{i=1}^k U_i(\epsilon'')^2$, and hence...
for some \( r > 0 \) smaller than the injectivity radius, any point in the smaller region has \( r \)-ball contained in the larger region, a fact that continues to hold if \( \epsilon^p_\ell \) is further shrunk. Appealing to monotonicity (Lemma 4.8), if there is a point \( p \in U \setminus \bigcup_{i=1}^k U_i(\epsilon'_i/2n-1)^2 \) in the image of the curve, the energy of the curve must be bounded below by \( C_{\eta} \epsilon^p \) for all \( \epsilon^p \) sufficiently small, a quantity that becomes greater than the energy of a PSS solution when \( \epsilon^p \) is small, a contradiction.

Now we turn to the inductive step in the proof. Suppose that there is a point \( p \) where the curve intersects \( \hat{U}_K \). Observe that \( \hat{U}_K \subset U_{K_i(\epsilon'_i/2n-|K_i|-1)^2} \setminus \cup_{i \notin K}(U_i(\epsilon'_i/2n-|K_i|)^2) \). Consider \( u_K = \hat{u}_{\infty}^{-1}(U_{K_i(\epsilon'_i/2n-|K_i|-1)^2} \setminus \cup_{i \notin K}(U_i(\epsilon'_i/2n-|K_i|)^2)) \), the intersection of \( \hat{u}_{\infty} \) with this larger set. By the inductive hypothesis, this curve can have no boundary along the loci where \( \rho_i = \frac{\kappa_i}{\pi} (\epsilon'_i/2n-|K_i|-1)^2 \) for any \( i \in K \) (as such points are contained in some \( \hat{U}_K \), for \( K' < K \). Using the fact that the cylindrical end of our curves lie in \( U_\nu \) and that \( K \) does not contain \( I \), we see that the projection of this curve \( \pi_K(u_K) \) is then a holomorphic curve in \( D \setminus \cup_{i \notin K}(U_i(\epsilon'_i/2n-|K_i|)^2) \) with boundary along the boundary of this region.

The point \( \pi_K(p) \) lies in \( D \setminus \cup_{i \notin K}(U_i(\epsilon'_i/2n-|K_i|-1)^2) \), a region which is bounded away from the boundary of the larger region \( D \setminus \cup_{i \notin K}(U_i(\epsilon'_i/2n-|K_i|)^2) \). It follows that we can find a ball \( B_K(\pi_K(p)) \) about \( \pi_K(p) \) of some non-zero radius that is independent of the particular point \( p \) in the region (depending on the metric distance from \( D \setminus \cup_{i \notin K}(U_i(\epsilon'_i/2n-|K_i|-1)^2) \) to the boundary of the larger region) which is disjoint from the boundary. In particular, monotonicity (Lemma 4.8) implies that, since \( u \) has image containing this point \( p \), the energy of \( \pi_K(u) \) must be bounded below by \( C_K r_K^2 \) for \( r_K \) the lesser of the size of the ball \( B_K \) and the injectivity radius of \( D_K \).

On the other hand,

\[
\frac{1}{2} w(\nu)(\epsilon^p_\ell)^2 \approx E(u) \geq E(u_K) \geq E_{\text{horiz}}(u_K) \geq \frac{E(\pi_K(u))}{G_K}
\]

where the last inequality uses (4.17). In particular, by taking \( \epsilon^p_\ell \) sufficiently small, \( E(\pi_K(u)) \) can be made smaller than \( C_K r_K^2 \), implying such a \( p \) cannot exist. \( \square \)

### 4.3. Projective bundle compactifications.

Fix an \( \epsilon'_i \) as in the statement of Lemma 4.10. Given a smooth component \( D \subset D \), consider the standard projective bundle \( PD = PD(ND \oplus \mathcal{O}_D) \) over \( D \), and let \( \pi_P : PD \to D \) denote the projection to \( D \). There are two natural holomorphic sections \( D_0 \) and \( D_\infty \) and we may algebraically identify

\[
PD \setminus D_\infty = ND = \text{Tot}(\mathcal{O}(D))
\]
\[
PD \setminus D_0 \cong \text{Tot}(\mathcal{O}(-D))
\]

Observe also that

\[
PD \setminus (D_\infty \cup D_0) = ND \setminus D = \text{Tot}(\mathcal{O}(D)) \setminus D \cong SD \times \mathbb{R}
\]

where the last isomorphism makes sense in the smooth category only. Turning to symplectic forms and letting \( p \) denote the norm in the fiber with respect to a Hermitian metric on \( ND \),
we equip $PD$ with the standard symplectic form
\begin{equation}
\omega_{PD} = \frac{\kappa(e_i')^2}{2\pi} d\left(\frac{p^2}{1 + p^2} \theta\right) + \pi_P(\omega_D)
\end{equation}
After setting
\begin{equation}
\rho_{\text{loc}} = \frac{\kappa(e_i')^2}{\pi} \frac{p^2}{1 + p^2}
\end{equation}
we see that the complement of the divisor $D_\infty$ can be identified with a standard (open) symplectic disc bundle $U_{D_\theta}$ of radius $\sqrt{\pi} e_i'$. Symmetrically, we can identify a neighborhood of $D_\infty$ inside of $PD$ with a disc bundle and the projective bundle as arising from the gluing of these two disc bundles. We may embed $U_{(\epsilon_i')^2} \subset U_{D_\theta} \subset PD$. Denote its image by $U_{\ell}^{\text{loc}}$.

Over higher dimensional strata we let $PD_I$ be the $(\mathbb{C}P^1)^{|I|}$ bundle which is the fiber product over $D_I$ of the $PD_i$ for $i \in I$. For $i \notin I$, denote by $D_{i,I}$ the divisors $D_i \cap D_I$, so $D_I = D_I \setminus \cup_{i \notin I} D_{i,I}$. Let $\bar{N}D_I \to \bar{D}_I$ denote the complement (in $PD_I$) of the sections $D_{i,0}$, $D_{i,\infty}$ for $i \in I$, restricted to the open stratum of the base $\bar{D}_I$. This is a $(\mathbb{C}^*)^I$ bundle over $\bar{D}_I$. As $PD_I$ is a fiber product, for any subset $J \subset I$ (changed $I \in J$ to $J \subset I$) we also have maps $\pi_J : PD_I \to PD_J^{[J]}$, where $PD_J^{[J]}$ is the $(\mathbb{C}P^1)^{|J|}$ bundle over $D_I$ given by taking fiber products of $PD_i$ for $i \in I \setminus J$, and $\pi_J$ projects away from the fibers corresponding to elements of $J$.

We equip $PD_I$ with the fiber product symplectic form where on each factor we use Hermitian connections $\theta_{e,i}$. More precisely we set $\bar{\kappa}_{i,\ell} = \kappa_i(\epsilon_i')^2$ and let
\begin{equation}
\omega_{\text{loc}} := \sum_{i \in I} \frac{\bar{\kappa}_{i,\ell}}{2\pi} d\left(\frac{p_i^2}{1 + p_i^2} \theta_{e,i}\right) + \pi_I(\omega_{D_I})
\end{equation}
As before, for any $i \in I$ we set
\begin{equation}
\rho_{i,\text{loc}} = \frac{\bar{\kappa}_{i,\ell}}{\pi} \frac{p_i^2}{1 + p_i^2}.
\end{equation}
For $i \notin I$, we set $\rho_{i,\text{loc}} := \rho_i \circ \pi_I$ where $\rho_i$ are the functions defined earlier in §2. The cohomology class of the symplectic form (4.25) is Poincaré dual to the divisors
\begin{equation}
[\omega_{\text{loc}}] = \sum_{i \in I} \kappa_{i,\ell}D_{i,0} - (\kappa_{i,\ell} - \bar{\kappa}_{i,\ell})D_{i,\infty} + \sum_{i \notin I} \kappa_{i,\ell}D_{i,\ell}.
\end{equation}
We let
\begin{equation}
X_I^{\text{loc}}
\end{equation}
be the symplectic manifold which is the complement of the divisors $D_{i,0}$, $D_{i,\infty}$, $D_{i,\ell}$. Of course this is diffeomorphic to $\bar{N}D_I$ as smooth manifolds. We denote by $U_{\ell}^{\text{loc}}$ the regions of $PD_I$ where $\rho_{i,\text{loc}} \leq \bar{\kappa}_{i,\ell} \frac{1}{2\pi}$, and set
\begin{equation}
U_{\ell}^{\text{loc}} := \cap_{i \in I} U_{i,\ell}^{\text{loc}}.
\end{equation}
We may assume that $\theta_{e,i}$ agree with those fixed in the regularization from §2 and that $\rho_{i,\text{loc}}$ agrees with $\rho_i$. Doing so gives rise to a symplectic identification:
\begin{equation}
U_{\ell}^{\text{loc}} \cong U_{i,\ell}(\epsilon_i')^2
\end{equation}
The choice of $\epsilon'_\ell$ above was somewhat arbitrary, and it will be useful for some arguments to introduce another symplectic form enabling us to vary $\epsilon'_\ell$. To this end, for some very small constant $\eta$, introduce a new symplectic form

$$
\omega'_\text{loc} = \sum_{i \in I} d(b(\rho_{i,\text{loc}})\theta_{\epsilon,i}) + \pi_i^*(\omega_{D_1})
$$

such that $b(\rho_{i,\text{loc}}) = \frac{1}{2}\rho_{i,\text{loc}}$ for $\rho_{i,\text{loc}} \leq \frac{3\eta_i}{4\pi}$ and whose cohomology class is Poincaré dual to the divisors

$$
[\omega'_\text{loc}] = \sum_{i \in I} \kappa_{i,\ell}D_{i,0} - (\kappa_{i,\ell} - (\frac{3}{4} + \eta)\kappa_{i,\ell})D_{i,\infty} + \sum_{i \notin I} \kappa_{i,\ell}D_{i,1}.
$$

The utility of this additional symplectic form comes from the following simple observation. Let $f : (\mathbb{C}P^1)[t] \to PD_1$ denote the inclusion of any fiber. Let $H_2^S(PD_1) \subset H_2(PD_1)$ denote the image of the Hurewicz map. We may decompose any homology class $A \in H_2^S(PD_1)$ as

$$
A = A_0 + A_1
$$

where $A_0 \in \text{im}(f)$ and $A_1 \cdot D_{i,\infty} = 0$ for all $i \in I$.

**Lemma 4.11.** Fix any $C_0 > 0$ and a $C_1 \gg C_0$. There exists $\eta, \epsilon'_\ell$ sufficiently small, so that for any class $A \in H_2^S(PD_1)$ such that $\omega_{\text{loc}}(A_1) \neq 0$ and $|\omega_{\text{loc}}(A)| < C_0\epsilon'^2_\ell$ we have that

$$
|\omega'_\text{loc}(A)| > C_1\epsilon'^2_\ell.
$$

**Proof.** Let $\omega_{\text{min}}$ be the quantity defined in §4.2 coming from rationality of the $\kappa_{i,\ell}$. By choosing $\epsilon'_\ell$ sufficiently small, we can assume that

$$
\omega_{\text{min}} > (5C_1 + C_0)(\epsilon'_\ell)^2
$$

Now suppose a class existed satisfying the hypotheses of the Lemma. We have $\omega_{\text{loc}}(A_1) = \sum_i \kappa_{i,\ell}d_i$ for some $d_i \in \mathbb{Z}$. We assume that $\omega_{\text{loc}}(A_1) \neq 0$ and hence we must have that $|\omega_{\text{loc}}(A_1)| = |\omega'_\text{loc}(A_1)| \geq \omega_{\text{min}}$. Now Equation (4.35) together with the assumption that $|\omega_{\text{loc}}(A)| < C_0\epsilon'^2_\ell$ forces:

$$
|\omega_{\text{loc}}(A_0)| > 5C_1(\epsilon'_\ell)^2.
$$

We also have that

$$
\omega_{\text{loc}}(A_0) - \omega'_\text{loc}(A_0) = (\frac{1}{4} - \eta)\omega_{\text{loc}}(A_0)
$$

$$
|\omega_{\text{loc}}(A) - \omega'_\text{loc}(A)| = |\omega_{\text{loc}}(A_0) - \omega'_\text{loc}(A_0)| > 5(\frac{1}{4} - \eta)C_1(\epsilon'_\ell)^2
$$

For $C_1 \gg C_0$, this implies the result. \hfill \Box

### 4.4. Local Floer cohomology in the compactification.

Consider the function $h'_\ell(\rho_{1,\text{loc}}, \cdots, \rho_{n,\text{loc}})$ where $h'_\ell$ are the perturbed versions of the functions $h^\ell$ defined in §3.5. This is a well-defined function on all of $PD_1$ and following previous conventions we label its orbit sets by $\mathcal{O}^\text{loc}$ and fix isolating sets $U^\text{loc}_\mathcal{V}$. As in the earlier construction of the Hamiltonians $H^\ell$, we may perturb this function to a function $H^\ell_{\text{loc}} : PD_1 \to \mathbb{R}$:

$$
H^\ell_{\text{loc}} = \sum_{\mathcal{V}} \delta_{\mathcal{V}} \rho_{\mathcal{V}} h_{\mathcal{V}} + h'_\ell
$$

where each of the $h_{\mathcal{V}}$ are supported in $U^\text{loc}_{\mathcal{V}}$. However, we no longer require these functions to be Morse except in $U^\text{loc}_{0\mathcal{V}}$. The key properties of our perturbation are:
Within $U_{\text{loc}}^\ell I$, the Hamiltonian $H^\ell_{\text{loc}}$ should coincide with the Hamiltonian $H_{p}^\ell$ under the identification (4.30);

• In the neighborhood of $D_{i,\infty}$ where $\rho_{i,\text{loc}} \leq \frac{3\kappa_{i,\ell}}{4\pi}$, $H^\ell_{\text{loc}} - h^\ell_{\text{loc}}$ and, depends only on $\rho_{i,\text{loc}}, \cdots, \rho_{j,\text{loc}}, \theta_{i,\text{loc}}, \cdots, \theta_{j,\text{loc}}$ for some collection $J \in I J := \{i_1, \cdots, i_J\}$, with $i \notin J$;

• There are no orbits in the region where $\frac{\kappa_{i,\ell}}{2\pi} \leq \rho_{i,\text{loc}} \leq \frac{3\kappa_{i,\ell}}{4\pi}$.

Notice in particular that these conditions imply that in the neighborhood of $D_{i,\infty}$ where $\rho_{i,\text{loc}} \leq \frac{3\kappa_{i,\ell}}{4\pi}$ for all $i \in I$ and $\rho_j \geq \frac{\kappa_j}{2\pi} (\epsilon_{p})^2$, we have that $H^\ell_{\text{loc}} = 0$. It is very easy to see that such a perturbation is possible. In $U^\ell_{I,\text{loc}}$, we perturb according to (4.3) and whenever $\rho_{i,\text{loc}} \leq \frac{3\kappa_{i,\ell}}{4\pi}$, we can assume all perturbations are independent of the variables $\rho_{i,\text{loc}}, \theta_{i,\text{loc}}$ as well as any variables in the base of the projective bundle. It is not difficult to arrange for the third condition to be achieved either.

All of the Hamiltonian orbits $x_0$ have canonical capping discs $F(x_0)$: In the case of constant orbits, we take these orbits to be constant capping discs and in the case of orbits which wind non-trivially around the divisor, we take these orbits to be multiply covered fiber discs passing through the divisors $D_{i,0}$ oriented so that the boundary of $F(x_0)$ is $x_0$.

**Remark 4.12.** While these Hamiltonians have some degenerate orbits, we will only be concerned with local versions of Floer cohomology between degenerate orbits and will be able to use Lemma 4.6, as in the proof of Lemma 4.7, to rule out undesirable breakings.

We note for later use that several of our other constructions from §2 and §3.5 have obvious analogues in the projective bundle, for example we let the hypersuface $\Sigma_{\ell}^{p}, \Sigma_{\ell}^{p}$ denotes the local analogues of the hypersufaces $\Sigma_{\ell}, \Sigma_{\ell}^{p}$ in $X$. In the projective bundle $PD_I$, we work relative to the normal crossings divisor

$$D = \bigcup_{i \in I} (D_{i,0} \cup D_{i,\infty}) \cup \bigcup_{i \notin I} D_{i,I}. \tag{4.40}$$

We will need to adapt the Floer theoretic structures introduced in the previous sections to this local setting.

**Definition 4.13.** Let $J_c(PD_I, D)$ denote the space of complex structures which are

• split outside of $U^\ell_{I,\text{loc}}$, and

• split in a neighborhood $V^\text{loc,}^\ell$ of the divisors which is disjoint from all Hamiltonian orbits.

**Definition 4.14.** Let $J_{c,\text{loc}}^\ell(PD_I, D) \subset C^\infty(S^1, J_c(PD_I, D))$ denote the space of $S^1$-dependent almost complex structures in $J_c(PD_I, D)$ that are time independent outside of $U^\ell_{I,\text{loc}}$.

Observe that our class of complex structures are also tamed by $\omega^\ell_{\text{loc}}$ and furthermore for our Hamiltonians $H^\ell_{\text{loc}}$, we also have that

$$\omega^\ell_{\text{loc}}(-, X_{H^\ell_{\text{loc}}}) = dH^\ell_{\text{loc}}(-) \tag{4.41}$$
We will assume our complex structures are $\omega_{\text{loc}}$ compatible in $U^{\text{loc}}_{\mathcal{V}_0}$. We can consider Floer’s equation for these Hamiltonians:

\begin{equation}
\begin{cases}
  u: \mathbb{R} \times S^1 \to PD_I, \\
  \partial_s u + j^{\text{loc}}_F(\partial_t u - X_{H^I_{\text{loc}}}) = 0.
\end{cases}
\end{equation}

subject to the usual asymptotic constraints. For any two orbits $x_0, x_1 \in U^{\text{loc}}_{\mathcal{V}_0}$, let $\tilde{M}(X^I_{\text{loc}}; x_0, x_1)$ denote the moduli space of these solutions which satisfy

\begin{equation}
u \cdot D = 0\end{equation}

and let $M(X^I_{\text{loc}}; x_0, x_1)$ denote this moduli space modulo $\mathbb{R}$-translations.

**Lemma 4.15.** Fix $\delta_{\mathcal{V}_0}$ sufficiently small. For any two orbits $x_0 \in U^{\text{loc}}_{\mathcal{V}_0}$ and $x_1 \in U^{\text{loc}}_{\mathcal{V}_0}$, we have that any $u \in M(X^I_{\text{loc}}; x_0, x_1)$ lies in $U^{\text{loc}}_{\mathcal{V}_0}$.

**Proof.** The symplectic form is exact on $X^I_{\text{loc}}$ and so action considerations imply that the energy of such a solution must be very small. The argument then follows from the same Gromov compactness argument as Lemma 4.3. \qed

For $\delta_{\mathcal{V}_0}$ sufficiently small, we therefore may define the Floer cochain groups

\begin{equation}
CF^*(U^{\text{loc}}_{\mathcal{V}_0} \subset PD_I, H^I_{\text{loc}}) := \bigoplus_{x \in U^{\text{loc}}_{\mathcal{V}_0}} |\sigma_x| \nabla_{|\sigma_x|} \\sigma_x
\end{equation}

The differential is defined by counting solutions in (a compactification of) $M(X^I_{\text{loc}}; x_0, x_1)$ as we have seen previously. Denote the resulting cohomology groups by $HF^*(U^{\text{loc}}_{\mathcal{V}_0} \subset PD_I, H^I_{\text{loc}})$. In fact, for $\delta_{\mathcal{V}_0}$ sufficiently small, we have a bijection between $M(X^I_{\text{loc}}; x_0, x_1)$ and $M(x_0, x_1)$ where in the latter case the orbits and Floer trajectories are thought of as lying in $X$. Hence we have a canonical isomorphism:

\begin{equation}
HF^*(U^{\text{loc}}_{\mathcal{V}_0} \subset PD_I, H^I_{\text{loc}}) \cong HF^*(U^{\text{loc}}_{\mathcal{V}_0} \subset M, H^I_{\text{loc}})
\end{equation}

We now turn to analyzing the cohomology groups $HF^*(U^{\text{loc}}_{\mathcal{V}_0} \subset PD_I, H^I_{\text{loc}})$. For this, we need to recall one more construction. Consider the Hamiltonian on $PD_I$

\begin{equation}
\tilde{H}^I_{\rho}(\rho_{1,\text{loc}}, \cdots, \rho_{k,\text{loc}}) = h^I_{\rho}(x) + K(x)
\end{equation}

where $K(x)$ is the Hamiltonian inducing the circle action considered in Equation (4.2). For $0 < \delta' \leq 1$ and $\delta_{\mathcal{V}_0}$ sufficiently small (depending on $\delta'$) consider the family of functions

\begin{equation}
\hat{H}^I_{\delta'} := \delta' \tilde{h}^I_{\rho}(\rho_{1,\text{loc}}, \cdots, \rho_{k,\text{loc}}) + \delta_{\mathcal{V}_0} \rho_{\mathcal{V}_0} \hat{H}_I.
\end{equation}

In $X^I_{\text{loc}}$, this defines a Morse function whose critical points lie entirely in $U^{\text{loc}}_{\mathcal{V}_0}$. Set $\hat{H}^I_{\text{loc}} = \hat{H}^I_{\delta'}$. As discussed in Section 4.1, the spinning construction induces a bijection of Hamiltonian orbits and Floer trajectories. To prove that it induces an isomorphism of local Floer cohomologies, we also need that this bijection preserves orientation theories. The differences in determinant lines is measured by a local system defined on the interior of $\mathcal{F}^{\text{loc}}_{\mathcal{V}_0}$, where our Hamiltonians orbits are Morse-Bott. In the present situation, this local system is trivial (see Lemma 8.7 of [KvK] or Section 8 of [DL], for a careful discussion of closely related situations) giving rise to an isomorphism

\begin{equation}
HF^*(U^{\text{loc}}_{\mathcal{V}_0} \subset PD_I, H^I_{\text{loc}}) \cong HF^*(U^{\text{loc}}_{\mathcal{V}_0} \subset PD_I, \hat{H}^I_{\text{loc}})
\end{equation}
The family of Hamiltonians $\hat{H}_{\delta'}$ form an isolated deformation i.e. in the neighborhood $U_{\psi_0}^{\text{loc}}$, $\mathcal{F}_{\delta_0}^{\text{loc}}$, are the only family of orbits for all $\delta'$. Local Floer cohomology is invariant under such isolated deformations [M5, Lemma 2.5] or [G2, (LF1) of Section 3.2] and so as a consequence we have that

$$HF^*(U_{\psi_0}^{\text{loc}} \subset PD_I, \hat{H}_{\delta'}^{\text{loc}}) \cong HF^*(U_{\psi_0}^{\text{loc}} \subset PD_I, \hat{H}_{\delta_0}^{\text{loc}}).$$

for any $\delta'$. For $\delta'$ sufficiently small $\hat{H}_{\delta'}$ is a $C^2$ small Morse function and we claim that:

**Lemma 4.16.** We have an isomorphism

$$HF^*(U_{\psi_0}^{\text{loc}} \subset PD_I, \hat{H}_{\delta'}^{\text{loc}}) \cong H^*(U_{\psi_0}^{\text{loc}})$$

**Proof.** For any $q \geq 1,$ we set $\hat{H}_q = \frac{1}{q} \hat{H}_{\delta'}$. We define $HF^*(X_I^{\text{loc}}, \hat{H}_q)$ to be the Floer cohomology of $\hat{H}_q$ in $X_I^{\text{loc}}$ (defining this may require further shrinking $\delta_{\psi_0}$). In order for this to be well-defined, we need the usual compactness results to hold, which in our case reduces to showing that:

**Subclaim:** Trajectories cannot break along Hamiltonian orbits in $D$.

**Proof of subclaim:** For $D_{i,\infty},$ this follows by monotonicity using the fact that the energies of solutions are small and a simpler version of the argument of Lemma 4.10 applied to the space $PD_I$. For $D_{i,0}$ or $D_{i,1},$ this follows by action considerations as in Lemma 4.5. More precisely, note that by shrinking $\delta_{\psi_0}$ we can suppose the energy of the Floer trajectories is arbitrarily small(independently of $q$). Suppose that a trajectory from $x_1$ to $x_0$ breaks along some $y$ in $D_{i,0}$ or $D_{i,1}$. Using a fiber capping for $x_1,$ $-F(x_1)$, we can cap this trajectory to obtain a capping of $y$ which differs from the fiber capping $-F(y)$ by some element $A$ with $A \cdot D_{i,\infty}$ i.e. in the decomposition (4.33), $A = A_1$. So, we have that either $\omega_{\psi_0}(A) = 0$ or $|\omega_{\psi_0}(A)| > \omega_{\text{min}}$. The rest proceeds as in the proof of Lemma 4.5. **End of proof of subclaim.**

It follows by definition that when $q = 1$,

$$HF^*(U_{\psi_0}^{\text{loc}} \subset PD_I, \hat{H}_{\delta'}^{\text{loc}}) \cong HF^*(X_I^{\text{loc}}, \hat{H}_{\delta'}^{\text{loc}}).$$

We aim to show that these coincide with the cohomologies $H^*(U_{\psi_0}^{\text{loc}}) \cong H^*(X_I^{\text{loc}})$. To do this, we adapt a well-known argument of Hofer and Salamon [HS, Lemma 7.1] to show that for some $q \gg 0$ sufficiently large, all of the Floer trajectories are $t$-independent and coincide with Morse trajectories. We explain briefly why their proof carries over to our non-compact setting. The first step in their proof is a compactness argument. Given a sequence of curves $u_q$ with $q \to \infty$ which are not $t$-independent, Hofer and Salamon employ a compactness argument to show that one may produce a sequence of curves $v_q$ converging to a possibly broken $t$-independent solution $v_{\infty}$. This argument applies here for the same reasons that Floer cohomology in $X_I^{\text{loc}}$ is well-defined i.e. by excluding undesired breakings along $D$ as in the proof of the above.

In the next step of their proof, they observe that by comparing Fredholm theories, rigid gradient flow lines are rigid as Floer trajectories and hence the $v_q$ for large $q$ actually agree with $v_{\infty}$. From the construction it follows that the $u_q$ are $t$-independent as well. In our situation, all $t$-independent trajectories actually lie in $U_{\psi_0}^{\text{loc}},$ where our complex structure is taken $\omega$-compatible and $t$-independent solutions of Floer’s equation are (negative) gradient flow solutions. In this open subset, the proof that the Fredholm theories coincide carries over without change. We conclude that there is an isomorphism:

$$HF^*(X_I^{\text{loc}}, \hat{H}_q) \cong H^*(X_I^{\text{loc}}).$$
Finally, the usual continuation argument shows that the above cohomologies are independent of $q$ and in particular that there is an isomorphism
\[(4.53)\quad HF^*(X_{I_{\ell}}^{\text{loc}}, \hat{H}_{q}) \cong HF^*(X_{I_{\ell}}^{\text{loc}}, \hat{H}_{q})\]
\[\square\]

As a byproduct of our local analysis, we now complete the Morse-Bott (additive) identification of the first page with log cohomology:

**Corollary 4.17.** For $\delta_{v_0}$ sufficiently small, there is an identification of cohomologies
\[(4.54)\quad HF^*(X \subset M, H_p^\ell)_{v_0} \cong HF^*(U_{v_0} \subset M, H_p^\ell) \cong H^*_\log(M, D)_{v_0}\]

**Proof.** As an immediate consequence of Lemma 4.16 and (4.45) we have that
\[HF^*(X \subset M, H_p^\ell)_{v_0} \cong HF^*(U_{v_0} \subset M, H_p^\ell) \cong H^*_\log(U_{v_0}).\]
But the latter deformation retracts onto $\mathcal{F}_{v_0}^\ell \cong \mathcal{S}_I^\log$, (by the analogue of Lemma 4.1, where $I$ is the support of $v_0$) whose cohomology by definition agrees with $H^*_\log(M, D)_{v_0}$.  
\[\square\]

**Remark 4.18.** A systematic description of the types of (smooth but possibly with corners) families of orbits which induce Morse-Bott spectral sequences, generalizing the situation considered in Corollary 4.17, has been given by McLean [M3].

4.5. Local PSS in the compactification. For this section and the next, we assume for convenience that the ratio $C_e = \epsilon' \ell > w_p(v_0)$ (this can be achieved by shrinking $\epsilon_p$ as needed).

**Definition 4.19.** Let $J^\text{loc}_S(PD_I, D) \subset C^\infty(S, J_{c}(PD_I, D))$ denote the space of almost complex structures which

- are time independent and split (in the sense of Definition 4.9) in a neighborhood of $z = \infty$, as well as in $(X_{I_{\ell}}^{\text{loc}} \setminus U_{I_{\ell}}^{\text{loc}}) \cup V_{\text{loc}, \ell}$; and
- agree with some $J_{F_{\ell}}^\text{loc}$ along the cylindrical end.

For what follows recall the domain $S = \mathbb{C}P^1 \setminus \{0\}$ with distinguished marked point $z_0 = \infty$, equipped with a cylindrical end and a distinguished tangent vector to $z_0$ described in §3.3. Recall also that $v_0 \in \mathbb{Z}_{\geq 0}^k$ is a fixed non-zero multiplicity vector and $I$ is its support.

**Definition 4.20.** Fix $J^\text{loc}_S \in \mathcal{J}_S(PD_I, D)$. For every orbit $x_0$ of in $U_{v_0}^{\text{loc}}$ define a moduli space
\[(4.55)\quad \mathcal{M}_{PSS}(v_0, x_0)\]
to be the space of maps
\[u : S \to PD_I\]
satisfying Floer’s equation
\[(4.56)\quad (du - X_{H_p^\ell} \otimes \beta)^{0,1} = 0\]
with incidence and tangency conditions
\[(4.57)\quad u(z) \notin D \text{ for } z \neq z_0;\]
\[(4.58)\quad u(z_0) \text{ intersects } D_{i,0} \text{ with multiplicity } (v_0)_i \text{ for } i \in I;\]
\[(4.59)\quad u(z_0) \text{ does not intersect } D_{i,\infty} \text{ or } D_{j,I} \text{ for any } i \in I \text{ or } j \notin I,\]
and asymptotics
\[(4.60)\quad \lim_{s \to -\infty} u(s,t) = x_0.\]

As before the incidence and tangency conditions give us an enhanced evaluation map
\[(4.46)\quad \Ev_{s_0}^\nu : \cM_{PSS}^\loc(v_0,x_0) \to \tilde{S}_I.\]

**Definition 4.21.** Let \( c \in \mathcal{X}(\tilde{S}_I,f_I) \) be a critical point of \( f_I \). The moduli space \( \cM_{PSS}^\loc(v_0,c,x_0) \) is defined to be the fiber product
\[(4.61)\quad \cM_{PSS}^\loc(v_0,x_0) \times_{\Ev_{s_0}^\nu} W^s(f_I,c).\]

As we have seen previously, for generic \( J_S^\loc \) these moduli spaces are manifolds of dimension (3.48) and have a natural Gromov compactification. In the present case, the structure of the compactification is very simple:

**Lemma 4.22.** For generic \( J_S^\loc \in J_S^\loc(PD_I,D) \) and \( \epsilon_I^\nu \), all parameters \( \delta_\nu \) are sufficiently small, \( \Sigma_\nu^\epsilon \) is \( C^0 \)-close to \( \tilde{\Sigma}_\nu^\epsilon \).

- If \( \deg(x_0) - \deg(|\omega_c|\nu) = 0 \), then \( \cM_{PSS}^\loc(v_0,c,x_0) \) is compact.
- If \( \deg(x_0) - \deg(|\omega_c|\nu) = 1 \), then \( \cM_{PSS}^\loc(v_0,c,x_0) \) admits a compactification \( \overline{\cM}_{PSS}^\loc(v_0,c,x_0) \) such that \( \partial \overline{\cM}_{PSS}^\loc(v_0,c,x_0) := \partial M \bigcup F \), where

\[(4.62)\quad \partial F := \bigcup_{x' \in U_0^\loc,\deg(x_0) - \deg(x') = 1} \cM_{PSS}^\loc(v_0,c,x') \times M(x_0,x').\]

\[(4.63)\quad \partial M := \bigcup_{c' \in U_0^\loc,\deg(c') - \deg(c) = 1} M(c',c) \times \cM_{PSS}^\loc(v_0,c',x_0).\]

**Proof.** The proof is an adaptation of Lemma 4.5. The energy of such a solution is arbitrarily close to \( \frac{1}{2} w_p(v_0)(\epsilon_I^\nu)^2 \). Assume that \( \epsilon_I^\nu \) is chosen sufficiently small so that \( \lambda_\nu \left( 1 - \frac{1}{1 - \frac{1}{2}(\epsilon_I^\nu)^2} \right) \ll \omega_{\min} \). Then we proceed by contradiction, supposing that the above moduli space is not compact. Because the minimum area of any \( J \)-holomorphic sphere is bigger than \( \frac{1}{2} w_p(v_0)(\epsilon_I^\nu)^2 \), no bubbling can occur in the moduli space and so we may extract a sequence of curves converging in \( C^\infty(S,PD_I) \) to a curve \( u_1 \) with \( u_1 : S \to PD_I \) together with \( s_n \to -\infty \) such that \( u_1(s_n,-) \to y \) for some \( y \in \mathcal{X}(D_1,H^1_{loc}). \)

The class of \( u_1 \) must be the connect sum of a fiber disc \( -F \) and some other class \( A \in H_2(PD_I) \). We claim, in the notation of the decomposition (4.33), that
\[(4.64)\quad \sum_{i \in I} \omega^\loc(A_i) \geq 0.\]

\[(4.65)\quad A \cdot D_{i,\infty} \geq 0 \quad \text{for all} \quad i.\]

The first equation follows because otherwise, by Lemma 4.11, we would have that \( |\omega^\loc(A)| \gg \lambda_\nu \left( 1 - \frac{1}{1 - \frac{1}{2}(\epsilon_I^\nu)^2} \right) \) contradicting the fact the geometric energy of \( u_1 \) with respect to \( \omega^\loc \) is also nonnegative.

To establish the second equation, note that if \( y \) is disjoint from \( D_{i,\infty} \), this follows simply by positivity of intersection so assume that \( y \in D_{i,\infty} \). For \( K \cap I = \emptyset \), we let \( U_K^\loc \) denote the region where \( \rho_i,\loc \leq \frac{2\pi(\epsilon_I^\nu)^2}{\kappa_i} \) for all \( i \in K \). So assume that \( y \subset D_{i,\infty} \cap U_K \) where \( K \cap I = \emptyset \), we have that near \( y \), the Hamiltonian depends on \( \rho_i, \theta_i \) for \( i \in J = \{i_1, \ldots, i_J\} \).
and some further $\rho_i$ where $i \in K$. By construction we have that $i \notin J$. Let $PD^1_I$ denote the natural $(\mathbb{C}P^1)^{|I|-|J|}$ bundle over $D_K$. Projecting away from these variables, $\pi_{(JK)} : PD_I \to PD^1_I$, we locally obtain a holomorphic curve $\pi_{(JK)}(u_1)$ whose (end) compactification intersects $D_{i,\infty}$ positively. Let $\tau_i$ be a standard Thom form for $D_i$ inside of $PD^1_I$. We have that

\begin{equation}
(4.66) \quad \int_{(\pi_{(JK)}(u_1))} \tau_i = \int_{u_1} \pi^{*}_{(J)}(\tau_i) > 0.
\end{equation}

As $\pi^{*}_{(JK)}(\tau_i)$ is a Thom form for $D_i$ inside of $PD_I$ which integrates trivially along the capping disc $-F$, the claim follows.

We therefore have that $\omega_{loc}(A) \geq 0$ as in the proof of Lemma 4.5. When $H^\ell_{loc}(y) \approx \lambda_{\ell}(\frac{1}{2}(\epsilon_\ell^p)^2 - 1)$, the result now follows as before. Otherwise $y$ is disjoint from all $D_i$ or $D_{i,0}$, $H^\ell_{loc}(y)$ is small in norm, and the second equation of (4.64) shows that all $v_i(y) \leq v_{0,i}$. From our assumption that $C_e > w_p(v_0)$, we obtain that fiber spheres have area at least $w_p(v_0)(\epsilon_\ell^p)^2$ and again that the geometric energy of $u_1$ is larger than $\frac{1}{2} w_p(v_0)(\epsilon_\ell^p)^2$.

Choose a generic $J_S \in \mathfrak{F}S(V)$ as above. It follows that for any $\alpha_c \in \mathfrak{o}_c \subset CM^*(\hat{S}_I)$, we can define $PSS_{loc,v_0}(\alpha_c) \in CF^*(U^loc_{v_0} \subset PD_I, H^\ell_{loc})$ by

\begin{equation}
(4.67) \quad PSS_{loc,v_0}(\alpha_c) = \sum_{x_0, vdim(M^loc_{PSS}(v_0,c,x_0))=0} \sum_{u \in M^loc_{PSS}(v_0,c,x_0)} \mu_u(\alpha_c)
\end{equation}

where $\mu_u : \mathfrak{o}_c \to \mathfrak{o}_{x_0}$ is the induced isomorphism induced on orientation lines we have seen previously. Lemma 4.22 implies that (linearly extending the above definition to all of $CM^*(\hat{S}_I)$) $PSS_{loc,v_0}$ defines a cochain map, giving rise to a well-defined cohomological map (with the same name, by standard abuse of notation):

\begin{equation}
(4.68) \quad PSS_{loc,v_0} : H^*(\hat{S}_I)^t v_0 \to HF^*(U^loc_{v_0} \subset PD_I, H^\ell_{loc}).
\end{equation}

The following confinement Lemma is proven exactly as in Lemma 4.10.

**Lemma 4.23.** Fix $x_0 \in U^loc_{v_0}$. For $\Sigma^loc_\ell$ sufficiently $C^0$ close to $\hat{\Sigma}^loc_\ell$, $\epsilon_\ell^p$ and all $\delta_e$ sufficiently small, and $J_S$ sufficiently close to a split $J_0$, any $PSS^v_{log}$ solution $u \in M^loc_{PSS}(v_0,c,x_0)$ must lie in $U^loc_{J_S}$. 

\[\square\]

**Corollary 4.24.** Under the canonical identification (4.45), there is an equality of operations

\begin{equation}
(4.69) \quad PSS_{loc,v_0} = PSS^v_{log}.
\end{equation}

\[\square\]

**4.6. Local SSP.** We now construct a moduli space (and operations) which enable us to invert the local PSS maps (on one side). Choose $\delta_e$ small enough and $J_{F,loc}$ sufficiently close to split so that Lemma 4.23 holds. For this section, we will further assume that the slope $\lambda_\ell$ has been chosen sufficiently large so that

\begin{equation}
(4.70) \quad \lambda_\ell > (2C_e)^2 w_p(v_0)
\end{equation}
(where \(C_e = \frac{\epsilon'}{\epsilon'}\) is as before), in order to rule out certain breakings in the moduli spaces defined below. There is an orbit set of constant orbits \(\mathcal{F}_0\) of \(H_{loc}^I\) which is a submanifold with corners in \(PD_I\) (and which contains a closed submanifold with corners of \(D_{I,\infty}\)). Let \(\mathcal{F}_0\) denote its interior and consider Floer trajectories \(\tilde{u}\) in \(X_I^{loc}\) asymptotic as \(s \to -\infty\) to an orbit \(x_0 \in \mathcal{F}_0\). As Floer curves are \(J\)-holomorphic curves in a neighborhood \(x_0\) (for some time independent \(J\), recall Definition 4.14), Gromov’s removal of singularities theorem implies that by setting

\[
\tag{4.71}
u(0) := x_0
\]

any such trajectory \(\tilde{u}\) extends to a smooth map \(u\) from the “thimble domain” \(S' := \mathbb{R} \times S^1 \cup \{0\} \cong \mathbb{C}P^1 \setminus \{\infty\}\) \(^{15}\) which is \(J\)-holomorphic in some neighborhood of the compactification point \(0 \in \mathbb{C}P^1 \setminus \{\infty\}\).

**Definition 4.25.** For \(x_1 \in U_{v_0}\), we let \(\tilde{M}^{loc}_{SSP}(x_1)\) denote the moduli space of maps \(u : S' \to PD_I\) with \(u(0) \in \mathcal{F}_0\) such that \(u\) solves Floer’s equation (4.42) on the complement of \(z = 0\) (and hence extends holomorphically to zero) with asymptotic condition

\[
\tag{4.72}\lim_{s \to +\infty} u(s,-) = x_1.
\]

We define \(M^{loc}_{SSP}(x_1)\) to be the quotient of this moduli space by \(\mathbb{R}\) translations of the domain cylinder.

We again place incidence conditions on this moduli space.

**Definition 4.26.** We let \(M^{loc}_{SSP}(v_0,x_1)\) denote the moduli space of maps \(u \in M^{loc}_{SSP}(x_1)\) such that \(u(0) \in \mathcal{F}_0 \cap D_{I,\infty}\) and such that

\[
\tag{4.73}u(z) \notin D \text{ for } z \neq 0;
\]

\[
\tag{4.74}u(0) \text{ intersects } D_{i,\infty} \text{ with multiplicity } (v_0)_i \text{ for } i \in I;
\]

\[
\tag{4.75}u(0) \text{ does not intersect } D_{i,0} \text{ or } D_{j,I} \text{ for any } i \in I \text{ or } j \notin I.
\]

Maps \(u : S' \to PD_I\) satisfying the conditions of Definition 4.25 fit into a natural Fredholm theory and are cut out regularly for generic choices of \(J_I\). \(^{16}\) As in the case of \(PSS_{log}\) moduli spaces, the moduli spaces \(M^{loc}_{SSP}(v_0,x_1)\) are obtained by placing incidence conditions with \(D\) and transversality for these moduli spaces can be handled by similar methods as those explained in [GP, §4.4]. It follows that for generic choices of almost complex structure, \(M^{loc}_{SSP}(v_0,x_1)\) is a manifold of dimension

\[
\tag{4.76}vdim(M^{loc}_{SSP}(v_0,x_1)) = 2n - 1 + 2 \sum_{i} v_i (1 - a_i) - \deg(x_1).
\]

Let \(i_{v_0} : \mathbb{R}^+ \to (\mathbb{R}^+)^I\) denote the embedding \(a \to a^{v_0} := (a^{(v_0)_i})_{i \in I}\) for any \(a \in \mathbb{R}^+\). The group \(\mathbb{R}^+\) acts by scaling (by different factors) in each fiber coordinate \(N D_I\) via the embedding \(i_{v_0}\), and we denote by \(N D_I / \mathbb{R}^+\) the resulting smooth quotient. Set \(G := (\mathbb{R}^+)^I / \mathbb{R}^+\). The natural projection map \(N D_I / \mathbb{R}^+ \to \hat{S}_I\) is a homotopy equivalence

\(^{15}\)Here as in §3.3 we are viewing \(\mathbb{R} \times S^1\) as embedded in \(\mathbb{C}P^1\) via (3.38).

\(^{16}\)By restricting to the Floer solutions in the complement of \(z = 0\), one can also place these maps into the more general framework of Morse-Bott Fredholm theory as explained in §6.1 of [DL].
which we use to identify the cohomology of the two spaces. More precisely, there is a $G$-equivariant isomorphism
\begin{equation}
ND_I/\mathbb{R}^+ \cong G \times \hat{S}_I.
\end{equation}
and in particular there are projection maps $\pi_G : ND_I/\mathbb{R}^+ \to G$ and $\pi_{S_I} : ND_I/\mathbb{R}^+ \to \hat{S}_I$.

Let $f_G : G \to \mathbb{R}$ be an outward pointing (at infinity) Morse function with a single critical point at the identity element of $G$. On $ND_I/\mathbb{R}^+$, we may consider the “split” (with respect to (4.77)) Morse function $\hat{f}_I$ which given by
\begin{equation}
\hat{f}_I := \pi_G^*(f_G) + \pi_{S_I}^*(f_I),
\end{equation}
(where $f_I$ is the Morse function we have previously fixed on $\hat{S}_I$ in §3.2) and similarly consider the product metric $\hat{g}_I = \pi_G^*(g_G) + \pi_{S_I}^*g_I$ where $g_I$ is the metric on $\hat{S}_I$ fixed previously and $g_G$ is any metric making $(f_G, g_G)$ a Morse-Smale pair. We let $\hat{f}_I^{pert}$ be another (currently unspecified) Morse function that is a small compactly supported perturbation (which is compactly supported in fiber directions and supported outside of the neighborhoods $U_j \cap D_I$ for $j \notin I$) of $\hat{f}_I$.

Denote the marked point at $z = 0$ on $S^\vee$ by $z_1$. We equip this with an asymptotic marker that points in the positive real direction. As with the PSS$_{loc}$ map, we have an enhanced evaluation map defined at this marked point:
\begin{equation}
\text{Ev}_{z_1}^L : M_{SSP}^0(v_0, x_1) \to ND_{I,\infty}/\mathbb{R}^+.
\end{equation}
which is a lift of the enhanced evaluation map defined previously in (3.46) given by only quotienting the tuple of all $v_i$ normal jets of the map by the diagonal $\mathbb{R}^+$ action (instead of real-oriented projectivizing). We can compose this with the natural diffeomorphism
\begin{equation}
\tau : ND_{I,\infty}/\mathbb{R}^+ \xrightarrow{\pi_{z_1}} ND_I/\mathbb{R}^+
\end{equation}
which in each fiber copy of $\mathbb{C}^*$ is given by the map $z \mapsto z^{-1}$ to obtain:
\begin{equation}
\hat{\text{Ev}}_{z_1} := \tau \circ \text{Ev}_{z_1}^L : M_{SSP}^0(v_0, x_1) \to ND_I/\mathbb{R}^+.
\end{equation}
Let $b$ be any critical point of the function $\hat{f}_I^{pert}$. By construction we know that $b \in ND_I \setminus \bigcup_{j \notin I} \pi_{z_1}^{-1}(U_j \cap D_I)$. Using $b$ and the map (4.81), we form the moduli space:
\begin{equation}
M_{SSP}^{loc}(v_0, x_1, b) := M_{SSP}^{loc}(v_0, x_1) \times_{\hat{\text{Ev}}_{z_1}} W^u(\hat{f}_I^{pert}, b),
\end{equation}
where the unstable manifold $W^u$ is taken with respect to the metric $\hat{g}_I$ defined above. For generic choices of unstable manifold and complex structure (making the constituents of the fiber product and the fiber product itself transversely cut out), this is a manifold of dimension
\begin{align}
\text{vdim}(M_{SSP}^{loc}(v_0, x_1, b)) &= \text{vdim}(M_{SSP}^{loc}(v_0, x_1)) - (2n - 1 - \deg(b)) \\
&= \deg(b) + 2 \sum_{i} v_i(1 - a_i) - \deg(x_1).
\end{align}

We observe that any point in $W^u(\hat{f}_I^{pert}, b)$ projects to a point inside of $D_{I,\infty} \setminus \bigcup_{j \notin I} U_j \cap D_{I,\infty}$ and in particular lies far away from $\partial \mathcal{F}_0 := \mathcal{F}_0 \setminus \hat{\mathcal{F}}_0$. We fix a small neighborhood $U_\infty \subset \hat{\mathcal{F}}_0$ which contains $D_{I,\infty} \setminus \bigcup_{j \notin I} U_j \cap D_{I,\infty}$. 

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Lemma 4.27. Fix a split complex structure $J_0 \in J_c(PD_1, \mathbf{D})$. Suppose as usual that $\epsilon_\ell^p$ and all parameters $\delta_\ell$ are sufficiently small and $\Sigma_\ell^p$ is sufficiently $C^0$-close to $\Sigma_{\ell, \epsilon}$. Then there exists a generic $J_t \in J_t^{loc}(PD_1, \mathbf{D})$ which is very close to $J_0$ such that:

- If $\mathrm{vdim}(\mathcal{M}_{SSP}^{loc}(v_0, x_1, b))$ the moduli spaces $\mathcal{M}_{SSP}^{loc}(v_0, b, x_1)$ are compact.
- If $\mathrm{vdim}(\mathcal{M}_{SSP}^{loc}(v_0, x_1, b)) = 1$, then $\mathcal{M}_{SSP}^{loc}(v_0, b, x_1)$ to a (compact 1-dimensional) manifold-with-boundary such that $\partial \mathcal{M}_{SSP}^{loc}(v_0, b, x_1) := \partial_F \sqcup \partial_M$ where

\begin{equation}
\partial_F := \bigsqcup_{x' \in U_0^{loc}, \deg(x') - \deg(x_1) = 1} \mathcal{M}(x', x_1) \times \mathcal{M}_{SSP}^{loc}(v_0, b, x')
\end{equation}

\begin{equation}
\partial_M := \bigsqcup_{b', \deg(b) - \deg(b') = 1} \mathcal{M}_{SSP}^{loc}(v_0, b', x') \times \mathcal{M}(b, b').
\end{equation}

Proof. The energy of a solution $u \in \mathcal{M}_{SSP}^{loc}(v_0, x_1, b)$ is arbitrarily close to $\frac{1}{\ell} w_p(v_0)((2\epsilon_\ell^p)^2 - (\epsilon_\ell^p)^2)$. The class of $u$ must be the connect sum of the fiber capping disc $F$ (described in §4.4) and some other class $A \in H_2(PD_1)$. By the argument of Lemma 4.22 it follows that $A$ must satisfy (4.64) and (4.65). We will consider sequences $u_m \in \mathcal{M}_{SSP}^{loc}(v_0, b', x_0)$ $(m \in \mathbb{N})$ which, as in standard proofs of Gromov compactness in the $\mathbb{R}$-invariant setting, we rescale so that $s = 0$ is the first $s$ value which exits $U_\infty$. No sphere bubbles (or more precisely Floer breaking where Hamiltonian is zero) can occur in $D_{I, \infty}$ because such a sphere bubble would have too high energy in view of (4.64). Because the evaluation $Ev_{I, \infty}^{L}$ is bounded away from the zero divisors of $ND_I$ (i.e., the leading order jets going into the definition of $Ev_{z_1}$ are constrained to lie along $W^u(f_{I, \text{pert}}^{L}, b)$ which is compact and away from zero or infinity), it follows that the intersection multiplicity at $z_1$ with $D_{I, \infty}$ of any limit is still $(v_0)_i$.

We choose a sequence of parameters $\delta_{v, n} \to 0$ for $n \in \mathbb{N}$ and let $H_n^I$ denote the corresponding perturbed functions as in Equation (4.39) (which converge to $h_{loc}^I$ as $n \to \infty$). Also, choose a sequence of $J_{t, n}$ converging to $J_0$ with $J_{t, n}$ generic (meaning achieves transversality) for all elements of $\mathcal{M}_{SSP}^{loc}(v_0, x_0)$ and for all Floer trajectories (both defined with respect to the functions $H_n^I$ appearing in the statement of this Lemma. As before, by appealing to usual Gromov compactness arguments, the key claim that verifies the Lemma is the exclusion of various “undesired” breakings from any limiting stable curve, namely:

Subclaim: For all $n$ sufficiently large there are no trajectories breaking along orbits of $H_n^I$ in $D_{I, \infty}, D_{I, j}, D_{I, 0}$. More precisely, fixing such $n$ sufficiently large, there are no sequences $u_{m, n} \in \mathcal{M}_{SSP}^{loc}(v_0, x_0)$ rescaled as above which converge (modulo sphere bubbling at finitely many possible points) in $C_{loc}^{\infty}$ to solutions $u_{\infty, n}$ which asymptotically limit to any orbit $y$ in $D_{I, \infty} D_{I, 0}$ or $D_{I, j}$, in the generalized sense that there exists a sequence of $s_k \to \infty$ with $u_{\infty, n}(s_k, -) \to y$.\textsuperscript{17}

Proof of subclaim: Our argument will work to exclude breakings along each “bad” orbit set above at one time for all $n$ sufficiently large: since there are only finitely many such orbit sets, the desired result will follow. To begin, suppose such a sequence $u_{m, n}$ (as in the statement of the subclaim) existed for every $n$, for an orbit $y$ in any of the divisors. Then by diagonalizing the sequence $\{u_{m, n}\}$, we can find a collection of numbers $m_n$ and $s_n$ so that $u_{m, n}(s_n, -)$ lies in a tubular neighborhood of of size $\epsilon'/n$ for $\epsilon' > 0$ (with respect to the Riemannian metric determined by $J_0$ and $\omega_{loc}$) of the relevant divisors. After passing

\textsuperscript{17}Recall that these orbits $y$ may be degenerate; compare the discussion just below Lemma 4.6.
to a subsequence, these converge in $C^\infty_{loc}$ (a priori modulo sphere bubbling at finitely many points) to an SSP solution $u_0$ for the Hamiltonian $h^f_{loc}$ and with split complex structure $J_0$. Let us now first show that such a $u_0$ does not develop a limit along any of the relevant divisors:

First, let us show such a $u_0$ stays away (i.e., does not develop a limit along) $D_{1,j}$ for $j \notin I$. We note by definition that $Ev^z_{\mathcal{U}}(u_0)$ lies in the unstable locus $W^u(f^{pert}_I, b)$, hence projects to a point in $D_I \setminus \cup_j U_j \cap D_I$. Meanwhile the function $h^f_{loc}$ is independent of $\rho_{j,loc}$ whenever $\rho_{j,loc} \geq c^p_{\ell}$, so it follows that (since the projection map $PD_I \to D_I$ is $J_0$-holomorphic since $J_0$ is split), the projection of $u_0$ is $J$-holomorphic in $D_I \setminus \cup_j U_j, e^p_{\ell} \cap D_I$. If the projection of $u_0$ is nonconstant, and did not stay away from $D_{1,j}$, we would obtain a holomorphic curve about $ev^z_{\mathcal{U}}(u_0)$ with boundary along $\partial U_j, e^p_{\ell} \cap D_I$. Because the $\omega_{D_I}$ energy of the projection is bounded by some $g_I \cdot \frac{1}{2} w_p(v_0)((2c_{\ell}^2)(e^p_{\ell})^2)$, monotonicity (Lemma 4.8) implies that such a curve would have to energy bounded below by a constant which can be taken larger than the energy of $u$ and therefore $u_0$ (by shrinking $\epsilon^p_{\ell}$ as needed), so this is impossible. Thus, indeed $u_0$ stays away from $D_{1,j}$.

The above argument further implies that the projection of $u_0$ is (after applying removal of singularities) a closed holomorphic curve which, since it has energy less than the minimal energy of a sphere in $D_I$ (by taking $\epsilon^p_{\ell}$ small), must be constant. Hence $u_0$ in fact lies in a fiber of $PD_I$.

Next, we see that $u_0$ cannot develop a limit along any $D_{i,\infty}$ (for $i \in I$) for homology class reasons, as $u_0$ has intersection number $(v_0)_i$ with $D_{i,\infty}$.

Now we argue that $u_0$ cannot develop a limit along $D_{i,0}$ for some $i \in I$; suppose it did along some $y$ in $D_{i,0}$. Then the intersection multiplicities of the class $A(u_0)$ with $D_{i,\infty}$ imply that $\omega_{\mathcal{U}}(A(u_0)) = w_p(v_0)(e^p_{\ell})^2$. But along $D_{i,0}$, we know from equations (4.70) and (2.48) (which holds equally well in the projective bundle setting) that

\begin{equation}
H^f_{loc} > \frac{1}{2} \lambda_\ell (e^p_{\ell})^2 > 2w_p(v_0)(e^p_{\ell})^2.
\end{equation}

As the energy of the fiber disc $F$ itself is nonpositive, this implies that the energy of the solution $u_0$ would be negative and hence cannot exist.

Given that $u_0$ cannot develop to any of the $D_{i,0}$, $D_{i,\infty}$ or $D_{1,j}$, it follows that, because the function $h^f_{loc}$ is Morse-Bott in the fibers, $u_0$ must converge (in fact exponentially quickly to) to a unique orbit $z \in \mathcal{F}_V^{loc}$. This implies two key facts. The first is that because this $z$ lies in $\mathcal{F}_V^{loc}$, we have by direct calculation that

\begin{equation}
E_{top}(u_0) = \lim_n E_{top}(u_{m,n,n}).
\end{equation}

The second is that by convergence, there exists $s^*$ for which $u_0$ does not escape $U_{v_0}$ for $s \geq s^*$. Choose $n$ sufficiently large so that $u_{m,n}$ is close to $u_0$ for $s < s^*$ and so that $u_{m,n}$ lies in $U_{v_0}$. Because our SSP solution $u_0$ does not intersect $D_{i,0}$ and $U_{v_0}$ lies a bounded distance away from $D_{i,0}$, it follows that after possibly enlarging $n$ we can ensure that $s_n > s^*$. In particular, $u_{m,n}$ exits $U_{v_0}$ at some parameter $s'_n$ for $s'_n > s^*$. Then by considering the rescaling $u_{m,n}(s + s'_n)$, we obtain a Floer trajectory that passes between an orbit $z'$ in $\mathcal{F}_V^{loc}$.

\[\text{In particular, no sphere bubbling can occur in the limiting process.}\]
and through a point on $\partial U_{v_0}$. However, because $E_{\text{top}}(u_0) = \lim_n E_{\text{top}}(u_{m_n,n})$, this rescaled curve zero energy, hence must be constant in $s$, a contradiction. \textit{End proof of subclaim.}

With this established, we see that for $n$ sufficiently large, SSP solutions with respect to $H^\ell_n$ cannot develop any other intersections with the divisors $D_{i,0}, D_{i,\infty}$ or $D_{j,I}$ by positivity of intersection. This also implies that they cannot develop limits along other orbit sets in $X^\text{loc}_I \setminus U_{v_0}$ by the following argument: suppose that a broken SSP trajectory limits to some $y$ in $X^\text{loc}_I \setminus U_{v_0}$. Then the energy of such a solution is up to small error $w_p(v_0) - A_I(y)$. As our actions are separated (see §3.5), this energy can be taken larger than the initial energy of our SSP solutions, hence such a broken trajectory does not exist.

This concludes the proof of the Lemma. □

Choosing a $J_t$ as in the Lemma, for an orbit $x_1 \in U_{v_0}$ and element $\alpha_c \in |\mathfrak{o}_{x_1}|$, we define

$$\text{SSP}_{\text{loc},v_0}(\alpha_c) := \sum_{b,v \in \mathcal{M}_{SSP}^\text{loc}(v_0,x_1,b)} \sum_{u \in \mathcal{X}_{SSP}^\text{loc}(v_0,x_1,b)} \mu_u(\alpha_c) t^v_0,$$

where as usual $\mu_u : \mathfrak{o}_{x_1} \to \mathfrak{o}_b$ is the isomorphism of orientation lines induced by a rigid element of the moduli space $u$. The Lemma implies that this prescription gives rise to a (finite, well-defined) count and moreover a cochain map:

$$\text{CF}^*(U^\text{loc}_{v_0} \subset PD_I, H^\ell_{\text{loc}}) \to \text{CM}^*(\mathcal{N}D_I/\mathbb{R}, \hat{I}^\text{pert}_I) t^v_0.$$ 

This induces a well-defined cohomological map, which we call the \textit{local SSP map} with multiplicity $v_0$ (we may sometimes refer to the cochain map as local SSP too):

$$\text{SSP}_{\text{loc},v_0} : HF^*(U^\text{loc}_{v_0} \subset PD_I, H^\ell_{\text{loc}}) \to H^*(\hat{S}_I) t^v_0.$$ 

We will need one final moduli space which is constructed again from the domain $S$, from the definition of local PSS in §4.5, equipped with the Floer data from that section. We assume that the complex structure $J_S$ that we fixed then is taken surface-independent in a neighborhood of $D_{i,\infty}$, and agrees on its cylindrical end with a $J_t$ satisfying the conditions of Lemmas 4.22 and 4.27. As above, any PSS solution $\hat{u} : S \to PD_I$ which is asymptotic to an orbit $x_0$ for some $x_0 \in \mathcal{F}_0 \cap D_{i,\infty}$ can be compactified to a map $u : S^2 \to PD_I$ which is $J$-holomorphic near $0$. In particular, we can form the following moduli space:

**Definition 4.28.** We let $\mathcal{M}_{0,2}(v_0)$ denote the moduli space of maps $u : S^2 \to PD_I$ such that

$$\hat{u} : \mathbb{C}P^1 \setminus \{0\} \to PD_I$$

solves

$$(d\hat{u} - X_{H^\ell_{\text{loc}}} \otimes \beta)^{0,1} = 0$$

with $u(z) \notin D$ for $z \neq 0, \infty$ and with the following incidence conditions:

$$u(0) \text{ intersects } D_{i,\infty} \text{ with multiplicity } (v_0)_i \text{ for } i \in I;$$

$$u(0) \text{ does not intersect } D_{i,0} \text{ or } D_{j,1} \text{ for any } i \in I \text{ or } j \notin I.$$ 

$$u(\infty) \text{ intersects } D_{i,0} \text{ with multiplicity } (v_0)_i \text{ for } i \in I;$$

$$u(\infty) \text{ does not intersect } D_{i,\infty} \text{ or } D_{j,1} \text{ for any } i \in I \text{ or } j \notin I.$$ 

The techniques used to regularize PSS_{log} moduli spaces again allow us to show that for generic choices of $J_S$ this is a manifold of dimension $2n$. 

Lemma 4.29. For any class \( \alpha^{v_0} \in H^*_\text{log}(M, D) \), we have that

\[
\text{SSP}_{\text{loc}, v_0} \circ \text{PSS}_{\text{loc}, v_0} (\alpha t^{v_0}) = \alpha t^{v_0}
\]

\[
(4.99)
\]

Proof. We will consider the moduli spaces

\[
M_{0,2}(v_0, c, b) := W^s(f_I, c) \times_{E_{v_0}} M_{0,2}(v_0) \times_{\hat{E}_v} W^u(J^\text{pert}_I, b)
\]

(4.100)

For generic choices of \( J_S \) and perturbed Morse function \( J^\text{pert}_I \), this is a manifold of the correct dimension. We will be interested in cases when \( M_{0,2}(v_0, c, b) \) has dimension 1. For \( J_S \) sufficiently close to a split \( J_0 \) and all perturbing parameters sufficiently small, the Gromov compactification of \( M_{0,2}(v_0, c, b) \) has two strata \( \partial_B M_{0,2} \) and \( \partial_S M_{0,2} \) (as well as other intermediate strata where Morse trajectories of \( J^\text{pert}_I \) or \( f_I \) break off, giving chain homotopy terms whose descriptions are standard, hence omitted) corresponding respectively to breaking along Hamiltonian orbits and sphere bubbling with respect to the time independent complex structure for which \( J_S = J_0 \) near \( z = \infty \). We have that

\[
\partial_B M_{0,2} := \bigcup_{x_0, v \dim(M_{\text{PSS}}(v_0, c, x_0) = 0)} M^\text{loc}_{\text{PSS}}(v_0, c, x_0) \times M^\text{loc}_{\text{SSP}}(v_0, x_0, b).
\]

To analyze the other boundary \( \partial_S M_{0,2} \), first note that all \( J_0 \)-holomorphic sphere bubbles must lie in multiple covers of fiber classes. This follows because, as in the proof of Lemmas 4.27 and 4.22, any other configuration of sphere bubbles would either have too large area or would have to contain a component which intersects one of the divisors \( D_{i,0} \) with negative multiplicity, which we have also seen is impossible. It follows for energy reasons that the PSS component of \( (\text{the } \overline{M}_{0,2}(v_0) \text{ component of the fiber product) } \partial_S M_{0,2} \) must be constant.

We can therefore identify the moduli space \( \partial_S M_{0,2} \) with the moduli space of \( J_0 \)-holomorphic spheres with two marked points satisfying (4.95)-(4.98) modulo \( \mathbb{R} \)-translation in the domain with \( E_{v_0}(u) \in W^s(f_I, c) \) and \( \hat{E}_v(u) \in W^u(J^\text{pert}_I, b) \). The evaluation of \( \partial_B M_{0,2} \) gives rise to the left-hand side of (4.99).

We also claim that the evaluation of \( \partial_S M_{0,2} \) gives rise to the right-hand side of (4.99). To see this, note that we may deform our complex structure \( J_0 \) on \( PD_I \), so that the projection to \( D_I \) is everywhere \( J_0 \) holomorphic. This will give rise to a chain homotopic map, which will induce the same map on the level of cohomology.

For such \( J_0 \), the curves satisfying (4.95)-(4.98) genuinely lie in the fiber because any curve which projects to a non-constant \( J \)-holomorphic curve in \( D_I \) will have energy higher than \( 2w_p(v_0)(\epsilon'_j)^2 \). Let \( M^I_{0,2} \) denote the moduli space of (smooth) \( J_0 \)-holomorphic spheres with two marked points satisfying (4.95)-(4.98) modulo \( \mathbb{R} \)-translation which lie in the fiber of the projection. Regularity for such spheres follows from Proposition 6.3.B of [B1] (see also the final paragraph of the proof of Lemma 5.23 of [I]). As before, for any map \( u \in M^I_{0,2} \), let \( E_{v_{20}}(u) \) and \( E_{v_{21}}(u) \) denote the lifted higher evaluation to \( ND_I/\mathbb{R}^+ \) and \( \tilde{ND}_{I,\infty}/\mathbb{R}^+ \) respectively, given by taking the tuple of all leading order jets modulo (only) the diagonal \( \mathbb{R}^+ \) action. The higher evaluation maps \( E_{v_{2i}} \) define diffeomorphisms

\[
E_{v_{20}} : M^I_{0,2} \cong \tilde{ND}_I/\mathbb{R}^+\n\]

(4.101)

\[
E_{v_{21}} : M^I_{0,2} \cong \tilde{ND}_{I,\infty}/\mathbb{R}^+.
\]

(4.102)
This is because we can assume that in any fiber, up to reparameterization, our maps take the form $z \to (a_i z^{w_i})$ where $a_i$ ranges over $(\mathbb{C}^*)^I/\mathbb{R}^+$. The composition $E_{v_{20}}^{\ell} \circ (E_{v_{20}}^{\ell})^{-1}$ is the map $\tau$ above in (4.80).

Next, recall from of [AbSc2, Appendix A.2], that the signed count of points in (compare with the moduli space in (3.31))

$$W^s(\hat{f}_I, (id,c)) \cap W^u(\hat{f}_I^{\text{pert}}, b)$$

where $(id, c)$ denotes the critical point of $\hat{f}_I$ induced by the critical point $id \in G$ of $f_G$ and $c \in \hat{S}_I$ of $f_I$, thought of as a point of $\hat{D}_I/\mathbb{R}^+$ via (4.77) induces a map

$$CM^s(\hat{D}_I/\mathbb{R}^+, \hat{f}_I) \to CM^s(\hat{D}_I/\mathbb{R}^+, \hat{f}_I^{\text{pert}}).$$

By Remark A.2. of loc. cit., this map induces the identity on cohomology once both sides are identified with singular cohomology. Next note that having $E_{v_{20}}(u) \in W^s(f_I, c)$, is equivalent to requiring that $E_{v_{20}}(u) \in W^s(\hat{f}_I, (id, c))$, as we have made choices so that $W^s(\hat{f}_I, (id, c)) \cong G \times W^s(f_I, c)$. This gives rise to an identification between points in Equation (4.103) and $\partial S_{M_0, 2}$. It is straightforward to check that this correspondence preserves orientations and the proof of the Lemma is concluded.

**Corollary 4.30.** For any $v$, the map (3.118) induces an isomorphism:

$$\text{PSS}^v_{x} : H^s(G) \cong HF^s(X \subset M; H^0_p).$$

**Proof.** From Corollary 4.17 and Lemma 4.29 we deduce that (3.55) induces an isomorphism. Applying Lemma 4.29 again shows that $\text{PSS}^v_{x}$ is an isomorphism and the result follows from Corollary 4.24. (strictly speaking our description of the one-sided inverse $\text{PSS}^v_{x}$ used $v \neq 0$ in order to work in a projective bundle; however if $v = 0$, $\text{PSS}^0_{x}$ is the usual PSS map $H^s(X) \to HF^s(X \subset M; H^0_p)_0$, which is well known to also be an isomorphism of $k$-modules (compare [R, §15.2]).

We now complete the proof of the main result of this section:

**Theorem 4.31.** The map (3.55) is an isomorphism:

$$\text{PSS}^v_{x} : H^s(\hat{S}_I) \cong HF^s(X \subset M; H^0_p).$$

**Proof.** Using Corollary 4.30 it follows from Lemma 3.34 that (3.55) induces an isomorphism.

**4.7. Proof of Theorem 1.1.** Collecting all of the results proven so far, we prove our first main theorem:

**Theorem 4.32 (Theorem 1.1).** There is a multiplicative spectral sequence converging to the symplectic cohomology ring

$$\{E^p_{\ast}, d_{\ast}\} \Rightarrow SH^s(X).$$

whose first page is isomorphic as rings to the logarithmic cohomology of $(M, D)$:

$$H^s_{\log}(M, D) \cong \bigoplus_{p, q} E^{p, q}_{1}.$$
Proof. The spectral sequence was constructed in (2.109) of §2.4. In Equation (3.57), we constructed a map

\[ \text{PSS}_{\log}^\text{low} : H_{\log}^*(M, D) \to \bigoplus_{p,q} E_1^{p,q}. \]

The fact that this map respects ring structures was proven in Theorem 3.18. It therefore remains to show that these maps (3.57) are isomorphisms. By taking limits, showing that this is an isomorphism reduces to showing that (3.56) is an isomorphism, which in turn immediately reduces to proving that (3.55) is an isomorphism. As this is the main result of Theorem 4.31, the proof of Theorem 1.1 is complete. \( \square \)

Comparing \( \mathbb{Z} \)-gradings: Under the isomorphism (4.105), a class \( \alpha \gamma^v \) lies in bidegree

\[ (p, q) = (-w(v), \deg(\alpha \gamma^v) + w(v)) \]

Comparing \( H_1(X) \)-gradings: It is useful to observe that symplectic cohomology (and Hamiltonian Floer cohomology) admits an optional second grading by \( H_1(X) \), which assigns to any orbit \( x \) (rather its orientation line) the associated homology class \( [x] \in H_1(X) \). The differential preserves this grading and the multiplication is additive; in particular, symplectic cohomology additively splits as a direct sum over \( H_1(X) \) classes. An analogous \( H_1 \) grading can be associated to \( H_{\log}^*(M, D) \). On a generator of the form \( \alpha \gamma^v \), it is described as follows: Let \( [y_1], \ldots, [y_k] \) be the \( H_1(X) \) classes of small loops around each \( D_i \) (for instance, \( y_i = \) the boundary of a disc fiber in \( U_i \)). Then to a class of multiplicity \( v \), associate the \( H_1(X) \) class \( \sum v_i [y_i] \). (this is also the \( H_1(X) \) class of a small loop that winds \( v_i \) times around each \( D_i \)). As this \( H_1(X) \) grading depends (additively) only on the vector \( v \), we immediately see that differential on log cochains preserves this grading and the product of two elements of grading \( [x_1] \) and \( [x_2] \) has grading \( [x_1] + [x_2] \). Finally, it is straightforward to see that the low energy log PSS map is compatible with the two \( H_1(X) \) gradings, and in particular the spectral sequence and identification of its first page from Theorem 1.1 split as a direct sum over \( H_1(X) \) classes (in a manner multiplicatively compatible with adding \( H_1(X) \) classes).

Remark 4.33 (The \( \mathbb{Z}/2\mathbb{Z} \)-graded case). To simplify the exposition of the numerous moduli spaces and operations appearing here, we have described dimensions of moduli spaces, their associated operations, and the relevant cohomology groups in the \( \mathbb{Z} \)-graded setting (which applies when \( c_1(X) = 0 \) using a choice \( \Omega_{M,D} \) of a holomorphic volume form on \( M \) which is non-vanishing on \( X \)).

However, our proof remains valid in \( \mathbb{Z}/2\mathbb{Z} \)-graded\(^{19} \) settings as well, with the following adaptations to the definitions of complexes, moduli spaces, and operations. First, one defines the Floer cohomology for any Hamiltonian as in §2.3 by associating to an orbit \( x \) the determinant line associated to some trivialization \( \gamma \) of \( x^*TM \). There is an ambiguity in such a choice, but any two choices of \( \gamma \) induce canonically isomorphic vector spaces \( \sigma_x \) (compare [A3, Prop. 1.4.10]). The degrees associated to different choices of \( \gamma \) only coincide mod 2, so the complex inherits a well-defined \( \mathbb{Z}/2\mathbb{Z} \) grading. The space of Floer trajectories between \( x \) and \( x' \) now contains components of varying dimension (coinciding mod 2 with the difference of gradings) depending on the underlying homotopy class of cylinders from \( x \) to \( x' \), and we only count the 0-dimensional components when defining operations. Finally, note that any homotopy class of cylinder induces, for any choice of trivialization \( \gamma \)

\(^{19}\) or \( \mathbb{Z}/2k\mathbb{Z} \)-graded, or fractionally-graded when \( c_1(X) \) is torsion, etc. though we leave these details to the reader.
for $x^*TM$, a canonical induced trivialization $\gamma'$ for $(x')^*TM$; the pair $(\gamma, \gamma')$ can be used to compute the dimension of this component of the moduli space, and gluing theory associates, for rigid Floer trajectories in this homotopy class, isomorphisms between the determinant lines of $\gamma$ and $\gamma'$, which is the necessary input to getting signed counts.

Next we $\mathbb{Z}/2\mathbb{Z}$ grade log cohomology by defining $\deg(\alpha t^y) = \deg(\alpha) \text{ mod } 2$ (compare this to the mod 2 reduction of (3.14)). Again, the low energy log PSS moduli spaces $\mathcal{M}(\nu, x_0)$ contain components of varying dimension depending on the underlying homology class $[u]$ in $H_2(M, M \setminus D)$ of the map; these are dealt with as in the case of Floer trajectories by noting that any such homotopy class defines a trivialization $\gamma$ of $x_0^*TM$ and thus a choice of orientation line to map to (in the rigid case). The remainder of the moduli spaces and operations (local PSS, local Floer homology, and local SSP) work in a similar fashion.

In particular, the proof of Theorem 1.1 continues to hold, seeing as nothing about the proof of isomorphism used gradings: once the maps are defined as above, the same appeal to energy and action considerations to confine curves and/or argue e.g., that PSS is a ring map or SSP and PSS compose in the desired fashion go through (with suitable definitional changes as above to all of the intermediate chain homotopies).

5. Computations of symplectic cohomology

5.1. Topological and multiplicatively topological pairs. We recall definitions and give examples of topological pairs (a slight generalization of the notion used in [GP, §3]) and multiplicatively topological pairs, introduced in the introduction. In the terminology of §1.1, a pair $(M, D)$ is topological (respectively multiplicatively topological) if there is some $J_0 \in \mathcal{J}(M, D)$ such that $(M, D)$ has no 0 or 1-pointed (respectively no 0, 1, or 2-pointed) relative $J_0$-holomorphic spheres. To spell this out:

**Definition 5.1.** We say that a pair $(M, D = D_1 \cup \cdots \cup D_k)$ is topological if there exists a $J_0 \in \mathcal{J}(M, D)$ such that for any subset $I \subset \{1, \ldots, k\}$, there are no non-constant $J_0$-holomorphic curves $u : \mathbb{C}P^1 \to D_I$ which intersect $\cup_{j \notin I}(D_j \cap D_I)$ in 1 or fewer distinct points.

Note that as per our convention, we include the case $I = \emptyset$ above, with $D_\emptyset = M$.

**Example 5.1.** To illustrate that this is a reasonably broad class of pairs, we list some examples:

(1) If $\pi_2(M) = 0$ or more generally $\omega(\pi_2(M)) = 0$, any pair $(M, D)$ will be topological, as there are no $J$-holomorphic spheres in $M$ at all for any $J$.

(2) If each smooth component $D_i$ of $D$ corresponds to some power of the same line bundle and the number of components $k$ of $D$ satisfies $k \geq \dim_{\mathbb{C}} M + 1$, then $(M, D)$ is topological. To see this, note that if $[u] \in H_2(M)$ is the class of a $J$-holomorphic sphere in $M = D_\emptyset$ which isn’t contained in any $D_j$, then since $\omega([u]) > 0$, $[u] \cdot D_j > 0$ for every $j$. Thus, $u$ must intersect $D$, and moreover cannot intersect $D$ at only one point, because then that point would be contained in $\cap_{i=1}^{n+1} D_i = \emptyset$. A similar argument applies to any $[u] \in H_2(D_I)$ the class of a $J$-holomorphic sphere which is not contained in any $D_j \cap D_I$.

As a specific case, note that $(\mathbb{P}^n, D = \{ \geq n + 1 \text{ generic planes} \})$ is a topological pair.
(3) In the $\mathbb{Z}/2\mathbb{Z}$ graded setting (as in Remark 4.33), another general class of topological pairs $(M, D)$ with $D = D$ a smooth divisor can be constructed as follows: Let $M$ be a hypersurface of degree at least $2n + 1$ in $\mathbb{CP}^n$, $n \geq 2$ and let $D$ be any smooth hyperplane section. The topological pair condition holds since for any $A \in H_2(M, \mathbb{Z})$ with $\omega(A) > 0$, the virtual dimension of the moduli space of $J$-holomorphic spheres in homology class $A$ is negative (and the same applies to curves that lie entirely in $D$). As any nonconstant curve factors through a somewhere injective curve, the moduli spaces are generically empty.

As (2) shows, $M$ (and each stratum $D_I$) could contain many $J$-holomorphic spheres even if $(M, D)$ is a topological pair.

To give a slightly more elaborate example, let $M_0 = \mathbb{P}^2$ and let $D_0$ be a (possibly non-generically) hyperplane arrangement such that for every component $D_{0,i}$ of $D_0$, there are at least two distinct points $D_{0,i} \cap D_{0,j}$ and $D_{0,i} \cap D_{0,j'}$ for $j, j' \neq i$.

**Lemma 5.2.** Let $M \to M_0$ denote the blowup of $M_0$ at each of the points where $\geq 3$ components of $D_0$ meet. Let $D$ denote the union of the proper transform of the divisors of $D_0$ and the exceptional divisors. Then the pair $(M, D)$ just constructed is a topological pair.

**Proof.** Every sphere which lies in $D$ must intersect at least two of the other components in distinct points, so consider spheres $u$ which meet $D$ transversely. Either it meets more than one exceptional divisor, in which case we are done, or it meets at most one exceptional divisor. If it meets no exceptional divisors, then it must intersect all components of $D$ with the same intersection multiplicity. If it meets one exceptional divisor, there is at least one other component of $D$ disjoint from that exceptional divisor and it must meet this divisor too.

**Remark 5.3.** Similar examples can likely be constructed for higher dimensional hyperplane arrangements, but this requires tracing through the resolutions of such arrangements, which are necessarily more elaborate than in the 2-dimensional case.

There is a strengthening of the topological condition which is useful for comparing product structures.

**Definition 5.4.** We say that a pair $(M, D)$ is multiplicatively topological if there exists a $J_0 \in \mathfrak{I}(M, D)$ such that for any $I \subset \{1, \ldots, k\}$, there are no non-constant holomorphic curves $u : \mathbb{CP}^1 \to D_I$ which intersect $\bigcup_{j \notin I}(D_j \cap D_I)$ in 2 or fewer distinct points.

**Example 5.2.** Some examples of multiplicatively topological pairs include:

1. If $\pi_2(M) = 0$ or $\omega(\pi_2(M)) = 0$, any pair $(M, D)$ will be multiplicatively topological because there are no spheres in $M$ at all.

2. Whenever each smooth component $D_i$ of $D$ corresponds to powers of the same line bundle and the number of components of $D$, $k$, satisfies $k \geq 2 \dim_{\mathbb{C}} M + 1$, then $(M, D)$ is multiplicatively topological, by the same analysis as (2) of Example 5.1. For example $(\mathbb{P}^n, D = \{\geq 2n + 1 \text{ generic planes}\})$ is multiplicatively topological.

3. Let $M_0 = \mathbb{P}^2$ and let $D_0$ be a hyperplane arrangement such that every component $D_{0,i}$ meets other components of $D_0$ at least three distinct points. Let $(M, D)$ be the resolution of this hyperplane arrangement constructed in Lemma 5.2. Then $(M, D)$ is multiplicatively topological, by an analogous argument to Lemma 5.2.

4. In the $\mathbb{Z}/2\mathbb{Z}$-graded setting, we can let $M$ be a hypersurface of degree at least $2n + 1$ in $\mathbb{CP}^n$, $n \geq 2$ and let $D$ be any smooth hyperplane section as above. This
is once more multiplicatively topological because (as seen in Example 5.1) there are no spheres in \( M \) at all, for generic \( J \).

We now turn to the proofs of Theorem 1.4 (Theorem 5.10) and Theorem 1.5 (Theorem 5.12), which involves introducing some basic definitions arising in log Gromov-Witten theory. To fix notation, for \( T \) a tree with \( |E_{\text{ext}}(T)| = 2 \), we let \( \mathcal{M}_{0,2}(T, M) \) to be the moduli space of \( J_0 \)-holomorphic sphere bubbles modelled on \( T \). Denote the two external edges by \( \vec{e} \) and \( \vec{\nu} \).

**Definition 5.5.** We say that a \( J_0 \)-holomorphic sphere \( u : S^2 \to D_I \) has depth \( I \) if \( u(S^2) \subset D_I \) and \( u(S^2) \not\subset D_j \) for \( j \not\in I \).

For any \( u \in \mathcal{M}_{0,2}(T, M) \), and any vertex \( \nu \in V(T) \), let \( I_\nu \subset \{1, \ldots, k\} \) denote the depth of \( u_\nu \); over all \( \nu \in V(T) \), one has a corresponding depth function associated to \( u \)

\[
I(-) : V(T) \to \mathcal{P}(\{1, \ldots, k\}),
\]

(as usual \( \mathcal{P}(\cdot) \) denotes powerset).

Assume that for each \( \nu \in V(T) \), \( u_\nu \) is enhanced with the additional data of meromorphic sections \( \psi_i \in \Gamma_m(S^2, u_\nu^*(ND_i))/\mathbb{C}^* \) for every \( i \in I_\nu \). For any \( z \in S^2 \) and \( i \in I_\nu \), we set \( \text{ord}_{\nu, i}(z) \) to be the order of any zero or pole of \( \psi_i \) at \( z \). This function is non-vanishing at at most finitely many points \( z \in S^2 \). We may extend the definition of \( \text{ord}_{\nu, j}(z) \) to any \( j \not\in I_\nu \) by recording the intersection multiplicity \( m_i(z) \) of \( u_\nu \) with \( D_i \) at \( z \) (which is again non-vanishing at at most finitely many points); note that positivity of intersection implies that this number is strictly positive if \( u_\nu \) intersects \( D_i \). Putting these constructions together gives rise to a function

\[
\text{ord}_\nu : S^2 \to \mathbb{Z}^k
\]

\[
z \to \{\text{ord}_{\nu, i}(z)\}
\]

(which implicitly depends on the choice of extra data \( \{\psi_i \in \Gamma_m(S^2, u_\nu^*(ND_i))/\mathbb{C}^* \}_{i \in I_\nu} \).

**Definition 5.6.** A pre-logarithmic enhancement of a stable curve \( u \in \mathcal{M}_{0,2}(T, M) \) consists of, for each \( \nu \in V(T) \), a collection of meromorphic sections \( \{\psi_i \in \Gamma_m(S^2, u_\nu^*(ND_i))/\mathbb{C}^* \}_{i \in I_\nu} \) such that the associated functions \( \{\text{ord}_\nu\}_{\nu \in V(T)} \) satisfy:

(i) \( \text{ord}_\nu \) is non-vanishing only at the marked points corresponding to edges \( e \in E(T) \).

(ii) For any internal edge \( e \in E(T) \) bounding two vertices \( \nu^+ \) and \( \nu^- \), we have

\[
\text{ord}_{\nu^+}(z_e^+) = -\text{ord}_{\nu^-}(z_e^-),
\]

where \( z_e^+ \) and \( z_e^- \) are the marked points corresponding to \( e \) on \( u_{\nu^+} \) and \( u_{\nu^-} \).

**Remark 5.7.** We note that unlike the more sophisticated notion of log map from Definition 3.8 of [FT], the data and conditions constituting a pre-logarithmic enhancement are simply a fiber product of the data and conditions defined individually for (a pre-logarithmic enhancement relative) each smooth component of \( D \).

If \( z_\infty \) is the marked point on \( S^2 \) corresponding to the edge \( \vec{e} \), we have an evaluation

\[
ev_{z_\infty} : \mathcal{M}_{0,2}(T, M) \to M
\]

Let \( \mathcal{M}(T, x_0) \) be the moduli space of

\[
\mathcal{M}(T, x_0) := \mathcal{M}_{0,2}(T, M) \times_{ev_{z_\infty}} \mathcal{M}(x_0)
\]

(recall the definition of \( \mathcal{M}(x_0) \) in Definition 3.11).
The next Lemma shows that for topological pairs, the log PSS moduli spaces (not just in low energy) have a suitable compactification provided that \( \ell \) and hence \( \lambda_\ell \) is taken sufficiently large and provided we choose generic \( J_S \in \mathcal{J}_S(V) \) such that the complex structure \( J_0 \) at \( z_0 \) (recall Definition 3.9) agrees with the one from Definition 5.1.

**Lemma 5.8.** Let \((M, D)\) be a topological pair, \( \nu \) a multiplicity vector such that

\[
w(\nu) < \lambda_\ell,
\]

and let \( c \) be a critical point of the Morse function \( f_I : \tilde{S}_I \to \mathbb{R} \) where \( I \) is the support of \( \nu \). Assume that \( || H^\ell - h^\ell || \) is chosen sufficiently \( C^2 \) small and let \( x_0 \) be a Hamiltonian orbit in \( \mathcal{X}(X; H^\ell) \).

- If \( \deg(x_0) - \deg(|\nu|) = 0 \), then for generic \( J_S \in \mathcal{J}_{S, \ell}(V) \) with \( J_{z_0} = J_0 \), the moduli space \( \mathcal{M}(\nu, c, x_0) \) is compact.
- If \( \deg(x_0) - \deg(|\nu|) = 1 \), then for generic \( J_S \in \mathcal{J}_{S, \ell}(V) \) with \( J_{z_0} = J_0 \), \( \mathcal{M}(\nu, c, x_0) \) admits a compactification (in the sense of Gromov-Floer convergence) \( \overline{\mathcal{M}}(\nu, c, x_0) \), such that \( \partial \overline{\mathcal{M}}(\nu, c, x_0) = \partial \mathcal{M}(\nu, c, x_0) = \partial F \) where

\[
\begin{align*}
\partial_F &:= \bigsqcup_{x', \deg(x_0) - \deg(x') = 1} \mathcal{M}(x_0, x') \times \mathcal{M}(\nu, c, x') \\
\partial_M &:= \bigsqcup_{c', \deg(c') - \deg(c) = 1} \mathcal{M}(\nu, c', x_0) \times \mathcal{M}(c', c)
\end{align*}
\]

**Proof.** We consider the closure \( \overline{\mathcal{M}}(\nu, x_0) \subset \mathcal{M}(x_0) \). The argument of [GP, Lemma 4.13] rules out cylinders breaking along orbits in \( D \), so we are only concerned with preventing sphere bubbling (after which, positivity of intersection implies as usual that broken cylinders stay away from \( D \)). For simplicity, we first discuss sphere bubbling at the distinguished point \( z_0 \in S \), temporarily ignoring the possibility of sphere bubbles forming at other marked points along \( S \) or along Floer cylinders. Consider a subsequence \( u_n \) converging to some limit

\[
uf \in \prod_{\nu_1, \cdots, \nu_r} \mathcal{M}(x_0, \nu_1) \times \cdots \times \mathcal{M}(x_2, \nu_1) \times \mathcal{M}(T, \nu_1).
\]

As our PSS solution is \( J_0 \)-holomorphic near \( z_0 \) and in view of the condition on the Nijenhuis tensor for complex structures in \( \mathcal{J}(M, D) \), we may apply the rescaling analysis of [FT, Lemma 4.9] to conclude that the corresponding \( u_\infty, T \in \mathcal{M}_{0,2}(T, M) \) admits a pre-logarithmic enhancement such that if \( \nu_f \) is the vertex bounding \( \nuf \),

\[
\ord_\nu_f(z_\infty) = -m(z_0)
\]

where \( -m(z_0) \) is equal to the intersection multiplicity of the PSS solution at \( z_0 \).\(^{20}\) By definition \( u_{\nu_f} \) lies in the stratum \( D_{\nu_f} \), and because \( \ord_\nu_f(z_\infty) \in (\mathbb{Z} \leq 0)^k z_\infty \) cannot be an intersection point of \( u_{\nu_f} \) with \( \cup_{j \notin I_{\nu_f}} (D_j \cap D_{\nu_f}) \) by positivity of intersection. The topological pair condition therefore implies \( u_{\nu_f} \) has at least 2 other marked points where it intersects \( \cup_{j \notin I_{\nu_f}} (D_j \cap D_{\nu_f}) \), i.e., there are at least 3 marked points including \( z_\infty \). The topological pair condition similarly implies that any other component \( u_\nu \) must have at least 2 marked

\(^{20}\)The argument given in [FT, Lemma 4.9] begins by noting that it suffices to verify (5.1) for the orders relative each smooth component \( D_0 \) of \( D \) individually, immediately reducing to the case of a single smooth divisor. Once in the smooth case, the result appears in various places; see e.g., [IP, Proposition 7.3] for a separate treatment.
points, but now the stable curve has too many external marked points to arise as a bubble tree at \( z_0 \in S \).

Now we rule out the possibility of sphere bubbling occurring at some other point in \( S \) or along a Floer cylinder. We have seen that no sphere bubbling can occur at \( z_0 \) and thus \( u_\infty \) intersects \( D \) with multiplicity \( v \) at this point. If \( u_\nu, \infty \) denote the collection of non-trivial sphere bubbles, by positivity of symplectic area, there must exist a divisor \( D_i \) for which \( \sum_\nu u_\nu, \infty \cdot D_i > 0 \). By positivity of intersection with \( D \), any Floer cylinder or PSS solution must intersect \( D_i \) with non-negative multiplicity. This however contradicts the fact that the sum of all intersections with \( D_i \) away from \( z_0 \) must be zero. □

It follows from Lemma 5.8 that (after as usual choosing generic \( J_S \)) defining:

\[
(PSS^{\ell, v}_{\log}(z^v)) := \sum_{\nu \in \nu_{v, \infty}} \sum_{u_{v, \infty}} \mu_u(z)
\]

for any \( z \in \partial_c \) and extending by \( k \)-linearity induces a cochain map

\[
PSS^\ell_{\log} : F_w C^*_\log(M, D) \to CF^*(X \subset M; H^\ell)
\]

and thus a well-defined cohomological map

\[
PSS^\ell_{\log} : F_w H^*_\log(M, D) \to HF^*(X \subset M; H^\ell).
\]

The argument from [GP, Lemma 4.13] can also be similarly adapted to show that for \( \ell_2 \geq \ell_1 \), the continuation map commutes with the PSS maps:

**Lemma 5.9.** We have a commutative triangle

\[
\begin{array}{ccc}
F_{w_1} H^*_{\log}(M, D) & \xrightarrow{PSS^1_{\log}} & HF^*(X \subset M; H^{\ell_1}) \\
\downarrow & & \downarrow \\
F_{w_2} H^*_{\log}(M, D) & \xrightarrow{PSS^2_{\log}} & HF^*(X \subset M; H^{\ell_2})
\end{array}
\]

Passing to the limit, we therefore obtain a map

\[
PSS_{\log} : H^*_\log(M, D) \to SH^*(X)
\]

For topological pairs \( (M, D) \), the spectral sequence (2.109) degenerates. Moreover, the map (5.7) gives a canonical splitting of this spectral sequence:

**Theorem 5.10.** Suppose that \( (M, D) \) is a topological pair. The map (5.7) is an isomorphism.

**Proof.** Recall that we have inclusion maps

\[
i_{w_\ell, w_{\ell+1}} : F_w C^*_\log(M, D) \to F_{w_{\ell+1}} C^*_\log(M, D)
\]

Set

\[
\tilde{C}_\log(M, D) = (\bigoplus_{\ell} F_w C^*_\log(M, D)[q], \partial)
\]

(5.8)

using the chain homotopy constructed in the proof of [GP, Lemma 4.18]. This cochain level lifting is filtered and chasing through the definitions, it is easy to see that the induced map on spectral sequences is given by (3.57). It follows that the map (5.7) is an isomorphism. □
Corollary 5.11. For each of the Examples in Example 5.1 and Lemma 5.2, we have additive isomorphisms $H^*_{\log}(M, D) \cong SH^*(X)$.

We next show how this result may be strengthened for multiplicative pairs to give a description of the ring structure on $SH^*(X)$:

Theorem 5.12. For multiplicatively topological pairs, the map (5.7) is an isomorphism of rings.

Proof. We now have a global $PSS_{\log}$ map and we run the same analysis as in the proof of Theorem 3.18 but we consider the moduli spaces $M(v_1, v_2; x_0)$ for all $x_0$ (not just those where $w(x_0) = w(v_1) + w(v_2)$). We assume that our complex structures near the marked points coincide with the $J_0$ which appears in the definition of multiplicatively topological pairs. The multiplicatively topologically hypothesis then ensures, by an analysis identical to the proof of Theorem 5.8,

that even though we are looking at all $x_0$, the Gromov compactification of this moduli space continues to behave as given in Lemma 3.22; in particular, only constant spheres arise in the Gromov compactification $\overline{M}(v_1, v_2; x_0)$. As in the proof of Theorem 3.18 it suffices to verify compatibility with ring structures for inputs of the form specified in cases (i) and (ii) of Lemma 3.19.

The fact that only constant sphere bubbles arise now suffices to prove as before that we have continuous evaluations to the real blow-ups as in Lemma 3.23. From here, we may construct the moduli spaces $M(v_1, v_2; c_1, c_2; x_0)$ of Definition 3.24 as well as their Gromov compactifications. The remaining arguments in §3.4 needed to show that these compactified moduli spaces define a cobordism (as usual up to intermediate boundary components which define chain homotopies) between $PSS_{\log}(\alpha_1 t^{v_1}) \cdot PSS_{\log}(\alpha_2 t^{v_2})$ and $PSS_{\log}(\alpha_1 t^{v_1} \cdot \alpha_2 t^{v_2})$ carry through without change.

Corollary 5.13. For each of the Examples in Example 5.2, we have ring isomorphisms $H^*_{\log}(M, D) \cong SH^*(X)$.

Remark 5.14. More generally, one can define, for each $r \geq 0$ the notion of a pair $(M, D)$ being “$r$-topological”: there should exist a $J_0 \in \mathcal{J}(M, D)$ such that $(M, D)$ has no $m$-pointed relative spheres for all $m \leq r$. The topological condition corresponds to $r = 1$ and multiplicatively topological to $r = 2$. It seems reasonable to expect that for such pairs, genus zero topological field theoretic operations of “arity less than or equal $r$” on symplectic cohomology can be described in terms of the operations on log cohomology (or rather cochains) of $(M, D)$.

5.2. Proving deformations are trivial using GW-invariants. For topological pairs, it follows from Lemma 3.18 and Theorem 5.10 that the product on $SH^*(X)$ is a deformation of the product on $H^*_{\log}(M, D)$ which respects the filtration. Although this deformation is often non-trivial, it can frequently be trivial, even for pairs that are not multiplicatively topological. To this end, we formulate a criterion for topological pairs which implies that the $PSS_{\log}$ map becomes a map of rings. Our criterion will be valid for pairs $(M, D)$ satisfying the following additional condition:

Condition A: All divisors $D_i, i \in \{1, \cdots, k\}$ for $k > \dim(X)$, are in the same linear system and all strata $D_I$ are connected.

---

The only difference is that, given that the points $z_1$ and $z_2$ collide, any possible stable sphere bubble will have at most 3 (rather than 2) external marked points. The same analysis thus requires the extra “multiplicatively” topological hypothesis to rule such bubbles out.
**Definition 5.15.** For any pair satisfying Condition $A$, choose an almost complex structure $J_0 \in \mathcal{J} (M, D)$. For any partition $\{1, \cdots, k\} = I \cup K$ with $I \cap K = \emptyset$ let $M_{0,3} (M, D, v_I, v_K)$ denote the space of $D_1$ and $D_J$-regular maps $u : (S^2, z_0, z_1, z_2) \to (M, D)$ such that

\[
\begin{align*}
    u^{-1}(D_i) &= z_0 \text{ for } i \in I, \\
    u^{-1}(D_j) &= z_1 \text{ for } j \in K.
\end{align*}
\]

As the classes represented by curves are primitive, all such curves $u$ are automatically somewhere injective and thus we may achieve transversality for such maps. Consider

\[
ev_{z_2}^{-1}(X) = M_{0,3} (M, D, v_I, v_K)^o.
\]

Then the evaluation map

\[
ev_{z_2} : M_{0,3} (M, D, v_I, v_K)^o \to X
\]

is a proper map. Fix a pair of Liouville domains $\tilde{X}_{\gamma}, \tilde{X}_{\iota}$ such that $\tilde{X}_{\gamma} \subset \tilde{X}_{\iota}$. After choosing $J_0 \in \mathcal{J} (\tilde{X}_{\gamma}, V)$ generically, we may define a relative pseudocycle

\[
GW(v_I, v_K) \in H^*(\tilde{X}_{\iota})
\]

by intersecting with $\tilde{X}_{\iota}$.

**Lemma 5.16.** Let $(M, D)$ be a topological pair which additionally satisfies Condition $A$. Let $\{1, \cdots, k\} = I \cup K$ be a partition as above. Then there is a (cohomological) equality:

\[
PSS_{\log} (I \partial^{v_I}) \cdot PSS_{\log} (I \partial^{v_K}) = PSS (GW(v_I, v_K))
\]

**Proof.** As this again follows the same pattern as Theorem 5.12, we will only point out the new points that arise here (the reader can also see Lemma 6.10 of [GP] for an almost identical argument). For simplicity, we assume that $J_z \in \mathcal{J} (\tilde{X}_{\gamma}, V)$ for all $z \in S$ when running this argument. The first point to note is that, as we are not putting any normal bundle constraints on the jets, we need not equip our marked points with projectivized tangent vectors or consider enhanced evaluations into the oriented blowup (hence the argument of Lemma 3.23 is not needed). The second is that there is a new stratum in the Gromov compactification compared to Lemma 3.22 which consists of configurations in the fiber product

\[
\tilde{M}_{0,3} (M, D, v_I, v_K)^o \times_{ev_x} M(\tilde{0}, x_0).
\]

The operation associated to counting rigid configurations of (5.12) for varying $x_0$ is by definition the composition

\[
PSS (GW(v_I, v_K))
\]

The proof now follows from the same cobordism arguments as before, where this extra stratum gives rise to the right-hand term. \qed

We can now state our criterion for triviality of the deformation:

**Theorem 5.17.** Let $(M, D)$ be a topological pair satisfying Condition $A$ and suppose that

- all invariants $GW(v_I, v_K)$ vanish,
- the restriction maps $H^*(X) \to H^*(\tilde{S}_I)$ are surjective for all $I$.

Then (5.7) is an isomorphism of rings.
Proof. We first observe that in view of the previous Lemma and the first bulleted assumption,

$$\text{PSS}_{\log} : \mathcal{SR}(M, D) \to SH^*(X)$$

is a ring map, where $\mathcal{SR}(M, D) \subset H^*_{\log}(M, D)$ is the subalgebra defined in (3.17). Indeed, consider the product of two elements

$$\text{PSS}_{\log}(\text{Id}_{t^v_1}) \cdot \text{PSS}_{\log}(\text{Id}_{t^v_2}).$$

There are two cases:

(i) If the stratum on which $v_1 + v_2$ is supported is nonempty, then the argument of Theorem 5.12 (using Condition A to exclude the relevant possible spheres) shows that this product agrees with $\text{PSS}_{\log}(\text{Id}_{t^v_1} \cdot \text{Id}_{t^v_2})$. In more detail: let $A \subset \{1, \ldots, k\}$ denote the support of $v_1$ and $B \subset \{1, \ldots, k\}$ the support of $v_2$. Let us suppose that, in the process of the proof of Theorem 5.12 for the product of two such elements, a limiting stable curve containing non-constant $J_0$-holomorphic sphere bubbles arose. Denote by $u$ the union of all such bubbles. Note first that Condition A (specifically that all divisors are in the same linear system) implies that $u$ has (positive symplectic area hence) positive homological intersection number with $[D_i]$ for each $i \in \{1, \ldots, k\}$. On the other hand, (by conservation of homological intersection number) the total stable curve, like the original relevant PSS moduli space, has positive homological intersection with $D_j$ for $j \in A \cup B$ and zero intersection with $D_i$ for $j \notin A \cup B$. Since $D_A \cap D_B \neq \emptyset$, it must be the case that $\#|A \cup B| \leq \dim(X)$, so Condition A again (specifically $k > \dim(X)$) implies that such a $D_i$ with $i \notin A \cup B$ exists. By additivity of homological intersection numbers over the components of the broken curve, this implies that the PSS moduli space component of this stable curve has negative intersection with $D_i$, a contradiction to positivity of intersection (seeing as the PSS moduli spaces are not contained in $D_i$).

(ii) If the stratum on which $v_1 + v_2$ is supported is empty, we aim to show that this product is zero. After factoring common supports out of the right hand term, we may assume that the the supports of $v_1$ and $v_2$ give a partition of $\{1, \ldots, k\}$. By further factoring, we can assume that $v_1 = v_1$ and $v_2 = v_2$. The claim now follows from the hypothesis that all of the invariants $GW(v_1, v_k)$ vanish and Lemma 5.16.

In view of the second hypothesis, we may next write any class in $H^*_{\log}(M, D)$ as a product of two elements $\alpha t^v_1 = \text{Id}_{t^v_1} \cdot \alpha'$ for $\alpha' \in H^*(X)$. Now again by Theorem 5.12 we have that

$$\text{PSS}_{\log}(\alpha t^v_1) = \text{PSS}_{\log}(\text{Id}_{t^v_1}) \cdot \text{PSS}(\alpha')$$

For a general pair of elements $\alpha_1 t^v_1$, $\alpha_2 t^v_2$ factoring them each as $\text{PSS}_{\log}(\text{Id}_{t^v_1}) \cdot \text{PSS}(\alpha'_1)$, we have

$$\text{PSS}_{\log}(\alpha_1 t^v_1) \cdot \text{PSS}_{\log}(\alpha_2 t^v_2) = \text{PSS}_{\log}(\text{Id}_{t^v_1}) \cdot \text{PSS}(\alpha'_1) \cdot \text{PSS}_{\log}(\text{Id}_{t^v_2}) \cdot \text{PSS}(\alpha'_2)$$

$$= \text{PSS}_{\log}(\text{Id}_{t^v_1} \cdot \text{Id}_{t^v_2}) \cdot \text{PSS}(\alpha'_1 \cup \alpha'_2)$$

$$= \text{PSS}_{\log}(\alpha_1 t^v_1 \cdot \alpha_2 t^v_2)$$

□
In the situation of Theorem 5.17, we obtain a presentation for the symplectic cohomology given by
\begin{equation}
SH^*(X) \cong \frac{S(R(M, D) \otimes H^*(X))}{(t^I \ker(r^I_{0,1}) = 0)}
\end{equation}

We now give a prototypical case where our criterion applies, the case where $M = \mathbb{C}P^n$ and $D$ is the union of $k \geq n + 2$ generic hyperplanes in $\mathbb{C}P^n$. For $k \geq 2n + 1$, these examples are multiplicatively topological and so $H^*_0(M, D) \cong SH^*(X)$ as rings by Theorem 5.12.

We now turn to the computation of cohomology of the algebra by the fact that $\Lambda^*(E_A)$ defines a cohomology class $\hat{\beta}_i$ in $H^1(M(A))$. Let $E_A$ be the vector space generated by these forms and $\Lambda^*(E_A)$ the exterior algebra. Equip this with a derivation $\delta$ which is determined by the fact that $\delta(\hat{\beta}_i) = 1$. Consider the ideal $OS(A)$ generated by
\begin{itemize}
  \item $\hat{\beta}_I = \hat{\beta}_{i_1} \wedge \hat{\beta}_{i_2} \cdots \wedge \hat{\beta}_{i_I}$ for any stratum $H_I = \emptyset$.
  \item $\delta(\hat{\beta}_I)$ for any $H_I$ such that $\text{codim}(H_I) < |I|$.
\end{itemize}

Then we have an isomorphism
\begin{equation}
H^*(M(A)) \cong \Lambda^*(E_A)/OS(A)
\end{equation}

In the present situation, we take $A$ to be the affine cone over our arrangement of divisors. We have an identification $M(A) \cong \mathbb{C}^* \times X$ and we may identify the cohomology of $X$ with the algebra
\begin{equation}
H^*(X) \cong \ker(\delta) \subset H^*(M(A))
\end{equation}

Of course, we may also view our hyperplane complement $X$ as the complement of an affine hyperplane arrangement $\bar{A} \subset \mathbb{C}^n$ by removing one of the divisors $H_j$. After choosing this $H_j$, letting $\beta_i = \hat{\beta}_i - \hat{\beta}_j$ gives an identification of between our description of the cohomology of $H^*(X)$ and the Orlik Solomon algebra associated to $\bar{A}$. The description using affine cones is, naturally, more symmetric. We now turn to the computation of $SH^*(X)$.

**Lemma 5.18.** $H^*_0(\mathbb{C}P^n, D) \cong SH^*(X)$ as rings.

**Proof.** It suffices to check that the criterion of Theorem 5.17 are satisfied. The vanishing of the obstructions $GW(\mathbf{v}_I, \mathbf{v}_J)$ follow easily from the fact that $H^*(\mathbb{C}P^n) \to H^*(X)$ vanishes except in degree 0. So it suffices to check the surjectivity of the restriction maps $r^I_{0,1}$. The higher dimensional strata $S^I_{log}$ are trivial torus bundles over the real oriented blowups of lower dimensional hyperplane complements.

Choose any $j \notin I$ and view $X$ as an affine hyperplane complement by removing $D_j$. Then the classes $\beta_i$ for $i \in I$ are represented by forms $\frac{1}{z_i} dz_i/z_i$ along the restriction maps $\mathbb{C}^n \setminus \cup_{i \in I} H_i$. Thus these classes $\beta_i$, $i \in I$ surject onto the cohomology of the fibers of the torus bundles. From the Orlik-Solomon presentation, we can see that the remaining classes $\beta_k$, $k \notin I$, generate the pull-back of classes on $\hat{D}_j$. The surjectivity of the map $H^*(X) \to H^*(\hat{S}_I)$ now follows from this. \qed
We can also describe the kernel \( \ker(r^*_0, I) \). As above, choose any \( j \) not in \( I \). Then
\[
\ker(r^*_0, I) = \text{span}(\beta^j)
\]
such that \( J \cap I = \emptyset \).

**Remark 5.19.** It would be interesting to know whether this argument could be adapted to the non-generic hyperplane arrangements of Lemma 5.2.

It is interesting to see how this calculation fits into the context of Kontsevich’s homological mirror symmetry conjecture. More precisely, we let \( \mathcal{P} \subset \mathbb{P}^{n+1} \) denote the hyperplane cut out by the equations \( \sum_i z_i = 0 \) and set \( X = \mathcal{P} \setminus \cup_i \{ z_i = 0 \} \). The affine variety \( X \) is called the generalized pair of pants \([M8]\) and is known to play an important role in the study of homological mirror symmetry for hypersurfaces in projective space \([S6, S5]\). The mirror to \( X \) is the Landau-Ginzburg model
\[
(\mathbb{A}^{n+2}, W = z_1 z_2 \cdots z_{n+2}).
\]
Let us try to identify \( SH^\bullet(X) \) (with \( \mathbb{C} \) coefficients) with the Hochschild cohomology of the matrix factorization category \( MF(\mathbb{A}^{n+2}, W = z_1 z_2 \cdots z_{n+2}) \). The first thing to observe is that the Jacobian ring is given by \( \mathcal{J}(W) \cong \mathbb{C}[\{ z_1, \cdots , z_{n+2} \}/(z_1 z_2 \cdots z_{n+2})] \) and that the map sending \( \alpha_i = \text{Id} t^{\nu_i} \rightarrow z_i \) induces an isomorphism:
\[
\mathcal{S} R^\bullet(M, D) \cong \mathcal{J}(W)
\]
Let \( \beta_i \) denote a standard set of generators for \( H^1(X) \) determined by viewing \( X \) as an affine hyperplane complement by removing the divisor \( z_{n+2} = 0 \). On the mirror side, we have corresponding cohomology classes
\[
z_i \partial z_i - z_{n+2} \partial z_{n+2} \in H^\bullet(T^\text{poly}_{\mathbb{A}^{n+2}}, [W, -])
\]
where \( i \in \{ 1, \cdots , n+1 \} \). These classes generate \( H^1(T^\text{poly}_{\mathbb{A}^{n+2}}[W, -]) \) as a module over \( \mathcal{J}(W) \).

With these observations in place, one can easily check for low \( n \) using a computer algebra package that the map
\[
H^\bullet_{\text{log}}(\mathbb{C}P^n, D) \rightarrow H^\bullet(T^\text{poly}_{\mathbb{A}^{n+2}}, [W, -])
\]
defined by sending \( \alpha_i \rightarrow z_i \) and \( \beta_i \rightarrow z_i \partial z_i - z_{n+2} \partial z_{n+2} \) is an isomorphism of rings.

### 5.3. Proving degeneration at \( E_1 \) page using GW-invariants.

Note that the ring structure on \( H^\bullet_{\text{log}}(M, D) \) is generated by primitive classes of the form \( \alpha t^{\nu_i} \). Because our spectral sequence \((2.109)\) is multiplicative, it follows that it degenerates at the \( E_1 \) page iff all differentials \( d_r(\alpha t^{\nu_i}) = 0 \) for all primitive classes and all \( r \geq 1 \). The purpose of this section is to record some consequences of this observation.

**Corollary 5.20.** Suppose that there is a divisor \( H \) such that for each smooth component \( D_i \subset D \), \( \mathcal{O}(D_i) \cong \mathcal{O}(n_i H) \) for \( n_i \in \mathbb{Z}^{>0} \). If any of the \( n_i > 1 \), then the spectral sequence \((1.1)\) degenerates at the \( E_1 \) page.

**Proof.** We equip this pair with the Kahler form in class \( \{ \sum_i D_i \} \). For any \( A \in H_2(M, \omega) \) (meaning a class with positive symplectic area), the hypotheses imply that \( \sum_i A \cdot D_i > k \). Let \( y, y' \) be a pair of Floer cochains whose weighted winding numbers (see \((2.59)\)) satisfy
\[
w(y') < w(y) \leq k.
\]
Then energy considerations imply that any Floer trajectory contributing to a differential with inputs \( y \) and output \( y' \) could be capped off using the canonical cappings of orbits to
produce a sphere of homology class with positive symplectic area \( A \in H_2(M) \). On the other hand (5.20) implies that such an \( A \) has \( \sum_i A \cdot D_i \leq k \); so it follows that no such Floer trajectories exist. In particular, for \( w(y) \leq k \), \( d_{CF}(y) \) only has terms of weight \( w(y) \), hence if \( y \) is a cocycle for the associated graded (of the action filtration) differential with non-vanishing class \([y]\) in \( E_1 \), then \( d_r([y]) = 0 \) for all \( r \geq 1 \). Under the (weight-preserving) identification of \( E_1 \) with \( H^*_{log}(M, D) \), we observe that the weight of any primitive vector \( v_I \) is at most \( k \), which from the definition of the pages \( (E^p,q, d_r) \) (see e.g., [M1, p. 35]) of a spectral sequence associated to a filtered complex, implies that \( d_r(\alpha t^v_I) = 0 \) for all primitive classes and all \( r \geq 1 \). From multiplicativity (see the discussion above the Corollary), it now follows that the spectral sequence (1.1) degenerates at \( E_1 \) as desired. \( \square \)

Simple examples of this include the cases where \( M = \mathbb{C}P^n \) and \( D \) is a smooth Calabi-Yau hypersurface. Let us consider the case where \( n = 2 \) and choose \( k \) to be a field of characteristic zero. We begin by calculating the cohomology \( H^*(M, D) \) above. We have that \( H^*(X, k) \) is
\[
H^*(X) \cong k \oplus k^2[2]
\]
where \([a]\) means shift the grading by \( a \). We have
\[
H^0(SD) \cong H^3(SD) \cong k
\]
\[
H^1_{log}(M, D) \cong H^3_{log}(M, D) \cong k[t]
\]
It is also straightforward to calculate using the Gysin exact sequence that
\[
H^1(SD) \cong H^2(SD) \cong k^2
\]
Let \( e_1 \) and \( e_2 \) denote standard generators of \( H^1 \) of the torus and call the generators of \( H^2(X), \ x, y. \) Lastly, observe that the restriction map \( H^2(X) \to H^2(SD) \) is an isomorphism. We therefore have that as \( k[t] \) modules
\[
H^1_{log}(M, D) \cong k[t] \cdot e_1 \oplus k[t] \cdot e_2
\]
\[
H^2_{log}(M, D) \cong k[t] \cdot x \oplus k[t] \cdot y
\]

**Lemma 5.21.** When \( M = \mathbb{C}P^2 \) and \( D \) is a smooth elliptic curve, \( SH^*(X) \) is a free module over \( k[t] \) of the same rank as \( H^*_{log}(M, D) \).

**Proof.** Because the spectral sequence degenerates, we have that there is a filtration on symplectic cohomology such that \( gr_F(SH^*(X)) = H^*_{log}(M, D) \). A simple result on associated graded modules (see e.g., Chapter 3, Section II of [B2]) says that generators of \( H^*_{log}(M, D) \) lift to generators of \( SH^*(X) \), giving rise to a surjection from a free module \( s : N \to SH^*(X) \). Applying the same argument to the kernel \( K \) of this surjection shows that \( s \) is an isomorphism (here we are using the fact that each \( H^*_{log}(M, D) \) page consists of free modules over \( k[t] \)). \( \square \)

It is again interesting to compare the result with what happens in mirror symmetry in the case where \( k = \mathbb{C} \). In this setting, mirror symmetry predicts (in somewhat simplified terms) that there is a mirror partner to \( X, Y \), which admits a proper algebraic map (see Lemma 5.38 for another result consistent with this)
\[
f : Y \to \mathbb{A}^1_k
\]
In the present situation, the mirror geometry is well-known and is studied in [AKO]. The mirror is a noncompact Calabi-Yau $Y$ which admits an elliptic fibration $f : Y \to \mathbb{A}^1$ with exactly three Lefschetz critical points. By properness, we have that

$$H^0(O_Y) \cong H^1(\Lambda^2 T_Y) \cong k[f]$$

An elementary sheaf theory computation shows that we have

$$H^1(O_Y) \cong H^0(T_Y) \cong H^1(T_Y) \cong H^0(\Lambda^2 T_Y)$$

are both free modules of rank two over $k[f]$.

We next consider a slightly more general case where the differentials in the spectral sequence can be calculated in terms of Gromov-Witten theory.

**Definition 5.22.** Given a class $A \in H_2(M, \mathbb{Z})$, we will denote by $A \cdot D$ the vector of multiplicities $[A \cdot D]_i \in \mathbb{Z}^k$.

**Definition 5.23.** Fix a pair $(M, D)$ and let $H_2(M)$ denote the set of integral classes $A$ for which $\omega(A) > 0$. We say that a primitive vector $\nu I$ is admissible if for all $A \in H_2(M)$,

$$\sum_i \kappa_i(A \cdot D_i) \geq w(\nu I).$$

Given a primitive cohomology class $\alpha t^{\nu I}$, corresponding to an admissible vector, there is exactly one possible non-vanishing differential on $\alpha t^{\nu I}$ in the spectral sequence, namely

$$d_w(\nu I)(\alpha t^{\nu I}) \in H^r(X).$$

**Definition 5.24.** Consider $\mathbb{C}P^1$ with two fixed marked points $z_0, z_1$ at 0, $\infty$ respectively, and let $\nu I$ be a primitive vector. For any $A \in H_2(M, \mathbb{Z})$ satisfying

$$(5.28) \quad A \cdot D = \nu I$$

set $\tilde{M}_{0,2}(M, D, \nu I, A)$ to be the moduli space of maps $u : (\mathbb{C}P^1, z_0, z_1) \to (M, D)$ with

$$(5.29) \quad u_*([\mathbb{C}P^1]) = A$$

$$(5.30) \quad u^{-1}(D_I) = z_0$$

**Definition 5.25.** From the moduli spaces in Definition 5.24 form:

$$(5.31) \quad \tilde{M}_{0,2}(M, D, \nu I) := \bigsqcup_{A, A \cdot D = \nu I} \tilde{M}_{0,2}(M, D, \nu I, A)$$

$$(5.32) \quad M_{0,2}^{S^1}(M, D, \nu I) := \tilde{M}_{0,2}(M, D, \nu I)/\mathbb{R}$$

where the quotient in the latter equation is by $\mathbb{R}$-translations.

These moduli spaces are canonically oriented. We have an evaluation map at $z_1 = \infty$

$$ev_\infty : M_{0,2}^{S^1}(M, D, \nu I) \to M.$$

Let $M_{0,2}^{S^1}(M, D, \nu I)^o$ denote the preimage $ev_\infty^{-1}(X)$. By the arguments of [GP, Lemma 3.26], for an admissible vector $\nu I$, the moduli spaces $M_{0,2}^{S^1}(M, D, \nu I)^o$ map properly to $X$. Moreover, after equipping the marked point at $z_0$ with the additional data of real positive ray in $T_{z_0} \mathbb{C}P^1$, we have as in (3.46) an enhanced evaluation map

$$E_{V_I}^Y : M_{0,2}^{S^1}(M, D, \nu I)^o \to \hat{S}_I.$$
It follows that, for any class $\alpha$ in $H^*(\mathcal{S}_I)$, we may define a Borel-Moore homology class in $H^{BM}_*(X)$ via

$$GW_{\nu_I}(\alpha) = [\text{ev}_\infty^*(\text{Ev}_0^{\nu_I,*}(\alpha))] \in H^{BM}_*(X).$$

(compare [GP, eq. (3.35)]).

**Theorem 5.26.** Assume $k = \mathbb{Z}$. For any admissible vector $\nu_I$, with respect to the identification of the first page of (1.1) with log cohomology by Theorem 1.1, there is an equality

$$d_{w(\nu_I)}(\alpha^{\nu_I}) = GW_{\nu_I}(\alpha) \in H^*(X) \subset H^*_\log(M, D).$$

**Proof.** Given an admissible vector $\nu_I$, and a cohomology class $\alpha^{\nu_I}$, choose $\alpha^{\nu_I} \in C^*_\log(M, D)$ be a cocycle which represents this class. We choose generators of the orientation lines associated to each critical point and denote them by $c$ in a slight abuse of notation. For every $c$, let $\overbar{W}_s(c)$ denote the partial compactification of $W_s(c)$ to a manifold with boundary given by adding in simply broken flow lines. Recall from [S2, §4.1] that, given a cocycle of the form $\alpha = \sum c \alpha_c$, $\alpha_c \in |\sigma_c|$ we may construct a corresponding pseudocycle $Z(\alpha)$ by gluing together ($|\sigma_c|$ copies of) the stable manifolds $W_s(c)$ (oriented by whichever generator $\alpha_c$ is a positive multiple of) along cancelling boundary components. Given a primitive admissible class and a generic complex structure, restricting the fiber product

$$\text{M}_{0,2}(M, \nu_I) \times_{\text{Ev}_0} Z(\alpha)$$

to $\tilde{X}_\ell \subset X$ defines a pseudocycle (rel boundary) $GW_{\nu_I}(\alpha)$ whose corresponding relative homology class is $GW_{\nu_I}(\alpha)$. Consider also the variant of the PSS map defined using pseudocycles:

$$\text{PSS}_0 : H_{2n-*}(\tilde{X}_\ell, \partial \tilde{X}_\ell) \to F_0 HF^*(X \subset M; H^\ell),$$

which is defined by representing elements of $H_{2n-*}(\tilde{X}_\ell, \partial \tilde{X}_\ell)$ by relative pseudocycles $P$ such that $\partial P \subset \partial \tilde{X}_\ell$. On homology this map agrees with the classical PSS map defined using Morse co-chains.

In [GP, Lemma 4.21], we proved that

$$\partial_{CF}(\text{PSS}_0(\alpha^{\nu_I})) = \text{PSS}_0(GW_{\nu_I}(\alpha)).$$

The result follows from the definition of the pages $(E_{p,q}^r, d_r)$ together with the fact that on the $E_1$ page of the spectral sequence, $[\text{PSS}_0(\alpha^{\nu_I})] = \text{PSS}_0^{\text{low}}(\alpha^{\nu_I})$ and $[\text{PSS}(GW_{\nu_I}(\alpha))] = \text{PSS}(GW_{\nu_I}(\alpha))$. □

**Remark 5.27.** The assumption that $k = \mathbb{Z}$ in Theorem 5.26 was only for simplicity and can likely be removed.

**Corollary 5.28.** Assume $k = \mathbb{Z}$. Given an admissible pair $(M, D)$, suppose the maps (1.4) vanish for all $I \in \{1, \cdots k\}$. Then the spectral sequence (1.1) degenerates at the $E_1$ page.
Remark 5.29. As mentioned in the introduction, Section 6.1.2 of [D] considered the case where $M \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a quadric in $\mathbb{P}^3$ and $D$ is a smooth hyperplane. Here, the moduli spaces $\mathcal{M}_{0,2}(M, D, \mathbf{v}_I)$ are nonempty; however the invariants $GW_{\mathbf{v}_i}(\alpha) = 0$. In this case, the relevant class lives in $H^2(X)$ and it is easy to compute the moduli spaces directly as is done in loc. cit. A more abstract way to see that it vanishes is to observe $GW_{\mathbf{v}_i}(\alpha)$ is invariant under the action which switches the two factors of $\mathbb{P}^1$ and hence vanishes when restricted to the complement. Similar reasoning involving group actions applies e.g. in the case when $D = D_1 \cup D_2$ is the union of two hyperplanes. It would be interesting to see if such reasoning applies in other cases.

Our partial description of the differentials in the spectral sequence may be used to prove the following strong finiteness result:

Theorem 5.30. Assume that all $\kappa_i = 1$ and all of the strata $D_I$ are connected. Then $SH^*(X)$ is finitely generated as a graded algebra over $k$.

Proof. Because all of the $\kappa_i = 1$, all of the vectors $\mathbf{v}_i$ are admissible for $i \in \{1, \cdots, k\}$. Choose over each $\hat{S}_i$ a Morse function with a unique critical point $c$ of degree zero and again choose a generator for the corresponding orientation line which we denote by $c$ as well. Because the cycle $\mathcal{M}_{0,2}(M, D, \mathbf{v}_I) \times_{E_V} \hat{Z}(c)$ factors through a lower dimensional manifold, it follows from [GP, Lemma 4.21] (see the proof of the previous Theorem for discussion) that

$$\partial_{CF} PSS_{\log}(c \mathbf{v}_i) = 0$$

i.e., we obtain a class $PSS_{\log}(\alpha_i t^{\mathbf{v}_i})$ in symplectic cohomology, where $\alpha_i \in H^0(\hat{S}_i)$ is the fundamental class of the stratum $\hat{S}_i$. Let $A$ denote the subalgebra of $SH^*(X)$ generated by $PSS_{\log}(\alpha_i t^{\mathbf{v}_i})$. Connectedness of all of the strata implies that the associated graded of $A$ is a quotient algebra of $\mathcal{S}(M, D)$ and that the first page of the spectral sequence is a finitely generated module over $\mathcal{S}(M, D)$. As the spectral sequence is multiplicative this means that the cohomology groups of all subsequent pages are also finitely generated modules over $\mathcal{S}(M, D)$. Because our filtrations on $A$ and $SH^*(X)$ are increasing and in nonnegative degree (and in particular induce discrete topologies), Chapter 3, Section II of [B2] again implies that $SH^*(X)$ is finitely generated over $A$ as well. \hfill \Box

5.4. Log Calabi-Yau pairs. One may also consider how the spectral sequence behaves in specific cohomological degrees. A natural situation to do this in is where $D$ is an anticanonical divisor, i.e. $M$ admits a meromorphic volume form with all $a_i = 1$. In this case, we say that $(M, D)$ is a log Calabi-Yau pair. \footnote{This notion of log Calabi-Yau pair is stricter than the usual usage in birational geometry [GHK1].} Here we wish to use our methods to derive information about $SH^0(X)$. For such pairs, we have

$$H^i_{\log}(M, D) = 0, \quad * < 0$$

and to understand the degree zero piece of the $E_\infty$ page, it suffices to analyze the differential in degree zero.

Let $(M, D)$ be a log Calabi-Yau pair such that all strata $D_I$ are connected. Then the ring $H^0_{\log}(M, D)$ is generated by classes $\alpha_i t^{\mathbf{v}_i}$ for $i \in \{1, \cdots, k\}$ where $\alpha_i \in H^0(\hat{S}_i)$ is the fundamental class. Again, to prove that $d_r$ vanishes in degree zero, it suffices to prove that $d_r(\alpha_i t^{\mathbf{v}_i}) = 0$ for all $r \geq 1$. 


Theorem 5.31. Let \((M,D)\) be a pair with \(M\) a Fano manifold and \(D\) an anticanonical divisor. Assume that all strata \(D_1\) are connected. Then the spectral sequence degenerates in degree zero. With respect to the standard filtration, we have an isomorphism
\[
gr_F SH^0(X) \cong \mathcal{SR}(M,D),
\]
where \(\mathcal{SR}(M,D)\) is the Stanley-Reisner ring on the dual intersection complex of \(D\) (see (3.17)).

Proof. Under the assumption that \(M\) is Fano, all of the vectors \(v_i\) are admissible for \(i \in \{1, \cdots, k\}\). Choose over each \(\tilde{S}_i\) a Morse function with a unique critical point \(c\) of degree zero and again choose a generator for the corresponding orientation line which we denote by \(c\) as well. As we saw previously,
\[
\partial CF_{PSS}^*(c t^{v_i}) = 0
\]
i.e. this defines a class \(PSS_{log}^*(\alpha t^{v_i})\) in symplectic cohomology. As their image in the \(E_1\) page generates \(H^0_{log}(M,D)\) as a ring, the result follows.

We next show how to adapt our methods to recover a result of Pascaleff [P2] that for any log Calabi-Yau surface, there is an isomorphism \(gr_F SH^0(X) \cong H^0_{log}(M,D)\). To prepare for this, we need to introduce a little bit more terminology. If \((T,A)\) is a tree with \(|E_{ext}(T)| = 2\) equipped with a labelling function \(A : V(T) \rightarrow H^2(M,\mathbb{Z})\),
\[
\mathcal{M}_{0,2}(T,A,M) := \bigcup_A \mathcal{M}_{0,2}(T,A,M).
\]
Let \(z_1 = \infty\) and \(z_0 = 0\) be the marked points corresponding to the external vertices \(\tilde{e}\) and \(\tilde{e}'\) respectively. We let \(\nu_f\) and \(\nu_i\) denote the vertices which are attached to each of these points.

Definition 5.32. For any \(v',v \in (\mathbb{Z}^\geq 0)^k\), we let
\[
\mathcal{M}_{0,2}(T,A,v',v) \subset \mathcal{M}_{0,2}(T,A,M)
\]
denote the subset of maps admitting a pre-logarithmic enhancement with
\[
\text{ord}_{\nu_i}(z_0) = v' \quad \text{ord}_{\nu_f}(z_\infty) = -v
\]
Notice that
\[
\mathcal{M}_{0,2}(T,A,v',v) \text{ is empty unless } A \cdot D = v' - v.
\]

Definition 5.33. Let \((M,D)\) be a log Calabi-Yau pair and let \(v_I\) denote a primitive vector. We say that the vector \(v_I\) is degree zero admissible if the moduli spaces \(\mathcal{M}_{0,2}(T,A,v_I,v) = \emptyset\) unless
- \(v = 0\); and
- there is exactly one \(\nu \in V(T)\) for which \(A(\nu) \neq 0\).
In the case $T$ has a single vertex, we write $M_{0,2}(T, A, v_I, 0) := M_{0,2}(A, v_I)$ and let $M_{0,2}(A, v_I)^o := ev_\infty^I(X)$. The moduli space $M_{0,2}(A, v_I)$ has expected dimension $2n - 2$ and, because the vector $v_I$ is primitive, $M_{0,2}(A, v_I)^o$ is a manifold of this expected dimension for generic $J$.\(^{23}\)

When $v_I$ is degree zero admissible, $|V(T)| \leq 2$ because configurations either consist of a single component with two marked points (in which case we are in the situation of the previous paragraph) or a single constant component in $D$ with three marked points glued to a non-constant component with one marked point. We can describe the configurations with two components very explicitly: the constant sphere bubble comes equipped with an $|I| - \text{tuple} \$ of meromorphic functions (well-defined up to a rescaling action) with two marked points corresponding to unique zeros and poles of order $v_I$. The zero corresponds to the point $z_0$ and the nonconstant component is glued in along the pole. The marked point $z_\infty$ corresponds to the third marked point on the constant sphere, at which the meromorphic functions has neither a zero or pole. In the case $|V(T)| = 2$, we therefore see that $ev_\infty \in D$.

Corollary 5.35 below shows that in dimension 2, all $v_I$ are degree zero admissible. It is an immediate consequence of the following stronger lemma:

**Lemma 5.34.** Let $(M, D)$ be a log Calabi-Yau pair with $\dim_\mathbb{C} M = 2$. Let $v', v$ be any pair and let $u \in M_{0,2}(T, A, v', v)$. Then any component $u_\nu$ of $u$ which is contained in $D$ must be constant.

**Proof.** Suppose that the statement of the Lemma is false and let $u_\nu$ be a non-constant component in $D$ whose corresponding node is furthest away from $\vec{e}$ (in terms of number of nodes in between that node and $\vec{e}$.) When $\dim_\mathbb{C} M = 2$, $D$ is either a cycle of $\mathbb{P}^1$s or an elliptic curve, and in particular we know that every non-constant component $u_\nu$ in $D$ has at least two marked points $z$ and $z'$ at which $\ord_{\nu,i}(z) > 0$ (respectively $\ord_{\nu,i}(z') > 0$) for some divisor $D_i$ (respectively $D'_i$). Without loss of generality, assume that $z$ corresponds to an edge $e$ which lies in a different connected component of $T \setminus \nu$ from $\vec{e}$. In $T \setminus \nu$, the connected component containing $e$ defines a tree $T_e$ beginning at $e$. Then because $\vec{e} \notin E(T_e)$, we must have that $\sum_{\nu \in T_e} A(\nu) \cdot D_i \leq -\ord_{\nu,i}(z) < 0$. But this is impossible if none of the components $u_\nu$, $\nu \in T_e$ lies in $D$, contradicting the fact that $u_\nu$ was a component the furthest away from $\vec{e}$.

**Corollary 5.35.** Let $(M, D)$ be a log Calabi-Yau pair with $\dim_\mathbb{C} M = 2$. Then every primitive vector $v_I$ is degree zero admissible.

**Proof.** Suppose some $M_{0,2}(T, A, v_I, v) \neq \emptyset$. By (5.41), observe that $A \cdot D \leq v_I$. In view of Lemma 5.34, every nonconstant curve $u$ in this moduli space intersects $D$ transversely with positive intersection multiplicites. Let us consider the component $u_\nu$ containing the marked point $z_0$. If it is non-constant, then $u_\nu$ must pass through $D_I$ and by primitivity the intersection must be at least $v_I$. It follows that it equals $v_I$ and that there can be no other intersections with the divisor or any other non-constant components. If $u_\nu$ is constant, then it is connected by a chain of constant curves to a non-constant curve $u_{\nu'}$ for which we again have $u_{\nu'} \cdot D = v_I$. In particular, $v$ must equal 0, and there is exactly one non-constant component as desired.

We now turn to describing how to modify the proof of Lemma 5.8, using the fact that sphere bubbling is relatively controlled in this setting:

\(^{23}\)This follows because each curve $u \in M_{0,2}(A, v_I)^o$ is necessarily somewhere injective.
Let \((M, D)\) be a log Calabi-Yau pair and let \(v_I\) be a degree zero admissible primitive vector. Choose a Morse function on \(\mathcal{S}_I\) with a unique degree zero critical point on each connected component. If \(c\) is one of these critical points, then for generic choices of complex structure, the count of elements in (5.5) defines a class in \(SH^*(X)\).

**Proof.** As in the proof of Lemma 5.8, for any \(x_0\) such that \(\text{vdim}(\mathcal{M}(v_I, x_0)) \leq 1\), consider the closure \(\overline{\mathcal{M}}(v_I, x_0) \subset \overline{\mathcal{M}}(x_0)\). As before, we temporarily ignore the possibility of sphere bubbles forming at other marked points other than \(z_0\) along \(S\) or along Floer cylinders. Consider a subsequence \(u_n\) converging to some limit

\[
\lim_{n \to \infty} u_n \in \prod_{r=1}^k \mathcal{M}(x_0, x_r) \times \cdots \times \mathcal{M}(x_2, x_1) \times \mathcal{M}(T, x_1).
\]

We conclude using the rescaling argument of [FT, Lemma 4.9] that the corresponding \(u_{\infty, T} \in \overline{\mathcal{M}}_{0,2}(T, M)\) admits a pre-logarithmic enhancement such that if \(\nu_I\) is the vertex bounding \(\mathcal{C}\),

\[
\text{ord}_{\nu_I}(z_\infty) = -v
\]

where \(v\) is equal to the intersection multiplicity of the PSS solution at \(z_0\). Because \(v_I\) is degree zero admissible it follows that \(v = 0\) and that the PSS solution lies in \(X\). Thus \(\text{ev}_{\infty}(u_{\infty, T}) \in \overline{\mathcal{M}}_{0,2}(A, v_I)^o\).

Meanwhile, whether such bubbling occurs or not at \(z_0\), we see that by conservation of intersection with \(D\) that there can be no further sphere bubbles at points other than \(z_0\) (again c.f. the proof of Lemma 5.8). To conclude, it suffices to observe that for generic complex structures, \(\dim(\mathcal{M}(0, x_0)) \leq 1\) while \(\dim(\mathcal{M}_{0,2}(A, v_I)^o) = 2n - 2\) and so such bubbling configurations do not exist generically. 

As in the discussion following (5.5), if \(\alpha \in H^0(\mathcal{S}_I)\) is the Morse cohomology class corresponding to \(c\), we again denote the class constructed in Lemma 5.36 by \(\text{PSS}_{\log}(\alpha v_I)\).

**Theorem 5.37 ([P2]).** Assume that \((M, D)\) is a log Calabi-Yau surface. Then the spectral sequence (2.109) degenerates in degree zero. With respect to the standard filtration, we have an isomorphism

\[
gr_F SH^0(X) \cong H^0_{\log} (M, D)
\]

**Proof.** This follows essentially as in Theorem 5.26. We have that \(H^0_{\log} (M, D)\) is generated by the classes \(\alpha v_I^i\) for \(i \in \{1, \cdots, k\}\). By Corollary 5.35 and Lemma 5.36, we have classes \(\text{PSS}_{\log}(\alpha v_I^i)\) for every \(I\), whose image generates the first page of the spectral sequence (2.109) multiplicatively. It follows that the spectral sequence degenerates.

We conclude the paper with an extended remark concerning Theorems 5.31 and 5.37. As we have seen in Theorem 5.10, in the topological case, the \(\text{PSS}_{\log}\) map defines a canonical splitting of the spectral sequence (2.109). It is very likely that in the setting of the above two Theorems, one may use the “full” PSS moduli spaces to define a similar splitting of the spectral sequence in degree zero. For a simple example to illustrate this idea, suppose that \(D = D\) is a smooth anticanonical divisor.

**Lemma 5.38.** There are canonically defined elements \(s_v\) together with an isomorphism

\[
\text{PSS}_{\log} : \bigoplus_{v \in \mathbb{N}^0} k \cdot s_v \cong SH^0(X, k)
\]
Moreover as a ring we have that

\[(5.45) \quad k[s_1] \cong SH^0(X, k).\]

**Proof.** For any \(v \geq 1\), choose a Morse function on \(SD\) with a unique critical point \(c\). Then again as before the elements \(PSS_{\log}(ct^v)\) define a class in symplectic cohomology \(SH^*(X)\). The difference between this situation and Lemma 5.8 is that in the present situation, sphere bubbling can arise near the point \(z_0\). However, in this case, after passing to the somewhere injective images of curves, this sphere bubbling occurs in codimension 2 (this works with either the standard Deligne-Mumford compactification or logarithmic/SFT enhancements). As a result, we obtain a map

\[(5.46) \quad PSS_{\log} : H^0_{\log}(M, D) \to SH^0(X).\]

The arguments of §3.5 and §4 apply without change when restricted to the degree zero pieces to show that this an isomorphism. Letting \(\alpha_v\) denote a copy of the fundamental class on either \(X\) or \(SD\) for each \(v\) and setting \(\alpha_v^sv_v\) therefore proves the first part.

The fact there is an isomorphism of rings \(k[s_1] \cong SH^0(X)\) follows from (a very special case of) Theorem 5.31 because \(gr_F(SH^0(X))\) is isomorphic to the ring for which \(s_v \cdot s_v = s_{v+v}\). As this is a polynomial ring in \(s_1\), \(k[s_1]\), it follows that \(SH^0(X)\) is as well because a polynomial ring has no commutative deformations. \(\square\)

**Remark 5.39.** Unlike the case of multiplicatively topological pairs, the map \(PSS_{\log}\) above is not compatible with the topological product on log cohomology defined in Definition 3.4, even though there is an isomorphism abstractly (5.45). For example, it is not difficult to see by a modification of Theorem 5.12 that, in the case where \(M = \mathbb{P}^2\) and \(E\) is a smooth elliptic curve, \(PSS_{\log}(s_1)^3 = PSS_{\log}(s_3) + 6\) (the number 6 arises here as the degree of the dual elliptic curve from classical algebraic geometry).

In the general case, the isomorphisms (5.44) and (5.45) contain rich enumerative geometry, worthy of further exploration. More precisely, the elements \(s_v\) correspond to canonically defined degree \(v\) polynomials whose coefficients are defined in terms of certain (relative) Gromov-Witten invariants. We leave this and generalizations of Lemma 5.38 to the normal crossings setting to future work.

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