Scaling, scattering, and blackbody radiation in classical physics

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Abstract
Here we discuss blackbody radiation within the context of classical theory. We note that nonrelativistic classical mechanics and relativistic classical electrodynamics have contrasting scaling symmetries which influence the scattering of radiation. Also, nonrelativistic mechanical systems can be accurately combined with relativistic electromagnetic radiation only provided the non-relativistic mechanical systems are the low-velocity limits of fully relativistic systems. Application of the no-interaction theorem for relativistic systems limits the scattering mechanical systems for thermal radiation to relativistic classical electrodynamic systems, which involve the Coulomb potential. Whereas the naive use of nonrelativistic scatterers or nonrelativistic classical statistical mechanics leads to the Rayleigh–Jeans spectrum, the use of fully relativistic scatterers leads to the Planck spectrum for blackbody radiation within classical physics.

Keywords: scaling symmetry, blackbody radiation, relativistic systems, no-interaction theorem, Planck spectrum, Rayleigh–Jeans spectrum

1. Introduction

The connections between classical and quantum physics are badly misunderstood today. For example, the physics literature and the modern physics textbooks claim that classical physics is incapable of accounting for the spectrum of blackbody radiation and rather leads only to the divergent Rayleigh–Jeans spectrum \(^1\). Actually, classical physics leads to the Planck spectrum for blackbody radiation provided that one uses relativistic rather than nonrelativistic classical physics and allows for classical zero-point radiation, which is an intrinsic possibility of classical electrodynamics and of thermodynamics.

\(^1\) This view is found throughout the research and textbook literature. In the textbook literature, see, for example \([1]\).
Nonrelativistic classical mechanics was developed in the 17th and 18th centuries, whereas classical electrodynamics was developed throughout the 19th century into the early 20th. The years around the turn of the 20th century saw the development of the theory of relativity. Electrodynamics was relativistic whereas nonrelativistic mechanics did not satisfy the requirements of special relativity. In the early 20th century, despite the clear assessment that all of classical electrodynamics satisfies the ideas of special relativity, physicists largely ignored the ideas of relativity in connection with the unsolved problems of physics of that era. Current textbooks of modern physics still teach the connections between classical and quantum physics as though contemporary physicists were no better informed than the physicists of the first half of the 20th century. Today textbooks of modern physics often begin with a discussion of special relativity. However, they fail to mention that the theory applies to all of relativistic classical electrodynamics. Rather they claim\(^2\), as was the viewpoint of the physicists of the early 20th century, that special relativity needs to be taken into account only in the physics of particles moving at a significant fraction \(v/c\) of the speed of light \(c\). Thus when particles have high velocity, one simply includes more terms in the ratio of \(v/c\) for the mechanical energy and momentum. However, this erroneous point of view ignores the implications of the no-interaction theorem of Currie \textit{et al}\(^2\) which restricts the allowed interactions between relativistic particles even when the particles are moving at small velocities.

In the present article, we emphasise the contrast in scaling behaviour between nonrelativistic classical physics and relativistic classical electrodynamics. We then show that this contrast in scaling behaviour leads immediately to contrasts in the scattering of electromagnetic radiation. The contrasts in scattering behaviour are tied directly to the ideas of thermal equilibrium for blackbody radiation.

2. Scaling behaviour within classical theory

Any physical theory envisions a collection of elements which form the basis for the theory. For example, nonrelativistic classical mechanics envisions a set of masses \(m_i\) at locations \(\mathbf{r}_i\) moving with velocity \(v_i\) which can interact through arbitrary potential functions \(V(\mathbf{r}_i - \mathbf{r}_j)\). In this theory, the masses, lengths, and times all scale separately and continuously from 0 to \(\infty\). Thus the theory imagines the possibility of replacing any mass \(m\) by a mass \(m' = \sigma m\), a length \(l\) by a length \(l' = \sigma l\), and a time \(t\) by a time \(t' = \sigma t\) where \(\sigma_m\), \(\sigma_l\), and \(\sigma_t\) are arbitrary positive constants. Under such a replacement, for example, a particle velocity \(v = l/t\) becomes \(v' = l'/t' = \sigma l/(\sigma t) = (\sigma/l)v\). Kinetic energy \(U = (1/2)m v^2\) is transformed as \(U' = (\sigma_m \sigma_l^2 / \sigma_t^2) U\). Thus in nonrelativistic mechanical theory, any mechanical system can be replaced by a new mechanical system which, for example, is twice as large, moves three times as fast, and has four times the energy.

Scaling in classical electrodynamics is quite different\(^3\). Classical electrodynamics involves particles of charge \(e\) and various masses \(m_i\) interacting through electromagnetic fields. Because of certain universal constants found in nature, the relativistic classical electrodynamics theory envisions elements which allow only a single \(\sigma_{\text{U}^{-1}}\)-scaling, which maps length \(l\) to \(l' = \sigma_{\text{L}^{-1}} l\), time \(t\) to \(t' = \sigma_{\text{M}^{-1}} t\), and energy \(U\) to \(U' = U/\sigma_{\text{E}^{-1}}\) where \(\sigma_{\text{U}^{-1}}\) is a positive constant. The independent scalings of length \(\sigma_l\) and of time \(\sigma_t\) envisioned within nonrelativistic mechanics are restricted because of the existence of the universal constant \(c\) (with dimensions of length/time) corresponding to the speed of electromagnetic waves in

\(^2\) See, for example, the last three texts in [1].
vacuum. The independent scalings of energy \( \sigma_U (\sigma_U = \sigma_m \sigma_r^2 / \sigma_l^2) \) and of length \( \sigma_l \) envisioned within nonrelativistic mechanics are restricted because of the existence of the universal constant \( a_S / k_B^3 \) (with dimensions of \( 1/(\text{energy} \times \text{length})^3 \)) corresponding to Stefan’s constant \( a_S \) divided by Boltzmann’s constant \( k_B \) raised to the fourth power, and because of the existence of a universal smallest electric charge \( e \) (with dimensions of \( \text{energy} \times \text{length})^{1/2} \).

Thus any scaling of smallest length \( l \) and time \( t \) where \( l/t = c \) involves replacing \( l/t \) by \( l'/t' = \sigma_{\text{lu}}^{-1} l / (\sigma_{\text{lu}}^{-1} t) = l/t = c \). Also any scaling of potential energy \( U \) and separation length \( l \) where \( U = e^2 / l \) replaces \( UL \) by \( U'l' = (U / \sigma_{\text{lu}}^{-1})(\sigma_{\text{lu}}^{-1} l) = Ul = e^2 \). It turns out that relativistic classical electrodynamics allows a complete decoupling of the \( \sigma_{\text{lu}}^{-1} \)-scale-invariant quantities (such as angular momentum and velocity) from the quantities which are transformed by a \( \sigma_{\text{lu}}^{-1} \)-scale transformation (such as length, frequency, mass, and energy).

When discussing questions of scaling within relativistic classical electrodynamics, one turns to angular momentum \( J \) as the natural \( \sigma_{\text{lu}}^{-1} \)-scale-invariant parameter of choice. Of the three familiar conserved quantities in the theory (energy \( U \), linear momentum \( p \), and angular momentum \( J \)), only angular momentum is \( \sigma_{\text{lu}}^{-1} \)-scale invariant. Angular momentum is also an adiabatic invariant.\(^3\) If one imagines electromagnetic radiation confined to a spherical cavity with perfectly conducting walls, then a uniform adiabatic compression (which changes the radius of the spherical cavity) will leave the electromagnetic field angular momentum unchanged. Such an adiabatic compression is often considered within thermodynamic analyses and so is relevant for an understanding of blackbody radiation.

3. Thermal equilibrium for electromagnetic radiation

In this article, we will consider only the classical physics of blackbody radiation. Blackbody radiation involves the radiation in a cavity which has been brought to thermal equilibrium. The radiation in a cavity with perfectly conducting walls will never achieve thermal equilibrium on its own, since the scattering of electromagnetic radiation from a perfectly reflecting wall may change the direction but not the frequency of the radiation in the inertial frame at rest with respect to the walls of the cavity. Rather, we must introduce radiation scatterers into the mirror-walled cavity in order bring about thermal equilibrium. Crucially, it is the character of the radiation scatterers which determines the spectrum of radiation equilibrium. (1) If nonrelativistic mechanical scatterers are used \(^4\), or if nonrelativistic statistical mechanics is applied to the scatterers or to the electromagnetic wave modes themselves, then one arrives at the Rayleigh–Jeans spectrum \([1]\). This result is noted in the textbooks of modern physics and throughout the physics literature. (2) If relativistic electromagnetic scatterers are used and one allows the natural possibility of classical electromagnetic zero-point radiation, then one arrives at the Planck spectrum \([5]\). This result is noted at a very few places in the physics literature and in none of the textbooks. (3) If quantum mechanical scatterers are used, then one arrives at the Planck spectrum. Indeed, the early rules of quantum physics were developed as ad hoc postulates introduced precisely in order to obtain the Planck spectrum \([6]\).

In the present article, we will carry out simple scattering calculations which illustrate the contrasting aspects when nonrelativistic mechanical systems are used as scatterers compared to when relativistic electromagnetic scatterers are employed. In order to keep the analysis as simple and transparent as possible, we consider the steady-state situation for a circularly

\(^3\) Indeed, all action variables (appearing in the action-angle variables of mechanics) have the dimensions of angular momentum and are both \( \sigma_{\text{lu}}^{-1} \)-scale invariants and also adiabatic invariants.

\(^4\) Van Vleck intended to publish this result in the mid 1920s but was diverted by the appearance of Schrödinger’s quantum mechanics. (Private communication to the author from Van Vleck.)
polarised plane wave falling on a circular particle orbit for various particle masses $m$. Furthermore, in the interests of simplicity, we will focus on the scaling aspects of the scattering.

4. Radiation scattering in classical physics

4.1. Scattering of a plane wave by a nonrelativistic mechanical system

We start with the traditional formulation of radiation scattering in classical physics. The traditional treatment of the interaction of radiation and matter in classical electrodynamics involves a nonrelativistic mechanical system interacting with electromagnetic fields. Here we consider a nonrelativistic particle of mass $m$ and charge $e$ moving in a circular orbit of radius $r$ and frequency $\omega$ in a nonrelativistic central potential $V(r)$. For simplicity, we take the potential as a power-law potential $V(r) = \alpha r^{n+1}/(n + 1)$, giving an attractive radial force of magnitude $F(r) = -\alpha r^n$. Accordingly, Newton’s second law for the particle in the circular orbit takes the form

$$m \omega^2 r = \alpha r^n. \quad (1)$$

The mass $m$ and the constant $\alpha$ are associated with the fundamental mechanical system itself while the frequency $\omega$ and the radius $r$ depend upon the energy or angular momentum contained within the system. We will choose the angular momentum $J$ as the parameter for use in our analysis, since angular momentum is a $\sigma_{\text{LTU}}$-scale invariant and is also an adiabatic invariant. In the nonrelativistic formulation, we have the angular momentum $J$ for a circular orbit given by

$$J = mr^2 \omega. \quad (2)$$

Solving equations (1) and (2) for $r$ and $\omega$ in terms of $m$, $\alpha$, and $J$, we find

$$r = \left( \frac{J^2}{m \alpha} \right)^{1/(n+3)} \quad (3)$$

and

$$\omega = \left( \frac{\alpha^2 J^{n-1}}{m^{n+1}} \right)^{1/(n+3)}. \quad (4)$$

The velocity $v$ and energy $U$ of the orbiting charge are

$$v = r \omega = \frac{\alpha^{1/(n+3)} J^{(n+1)/(n+3)}}{m^{(n+2)/(n+3)}} \quad \text{and}$$

$$U = \frac{1}{2} m v^2 + \frac{\alpha r^{n+1}}{n + 1} = \frac{n + 3}{2n + 2} \frac{\alpha^{2/(n+3)} J^{(2n+2)/(n+3)}}{m^{(n+1)/(n+3)}}. \quad (5)$$

A circularly polarised electromagnetic plane wave of minimum amplitude $E_0$ falls on the mechanical system. If the circular orbit of the mechanical system is in the $xy$-plane and the circular orbit is centred on the origin, then we may take the plane wave as travelling up the $z$-axis [7].

5 Since we are dealing with foundational questions of electrodynamics, we will use Gaussian units, which are natural units for the theory.
\[ E(z, t) = \hat{E}_0 \cos(kz - \omega t) + \hat{J}_0 \sin(kz - \omega t), \]
\[ B(z, t) = \hat{J}_0 \cos(kz - \omega t) - \hat{E}_0 \sin(kz - \omega t). \]  
(6)

For minimum amplitude \( E_0 \) of the incident wave, the motion of the orbiting charge must be in phase with the plane wave, with the direction of the electric field in the direction of the particle velocity. Assuming a steady-state situation, the power delivered to the orbiting charged particle is \( eE_0 v = eE_0 r\omega \), and this must balance the power radiated by the orbiting charge \( P = (2/3)(e^2/c^3)\omega^5r^2 \). Therefore we have
\[
eE_0 r\omega = \frac{2e^2}{3c^3}\omega^4r^2, \quad \text{or} \quad E_0 = \frac{2e}{3c^3}\omega^3r, \]
(7)

where \( r \) and \( \omega \) are given in equations (3) and (4), so that
\[
E_0 = \frac{2e}{3c^3} \alpha^{-5/(n+3)} j^{(3n-1)/(n+3)}.
\]

The magnetic field \( B \) of the plane wave places a \( z \)-component of force on the orbiting charge; however, we will imagine the orbiting charge as confined to the \( xy \)-plane by a frictionless surface and so will ignore this force.

### 4.2. Scattering of energy

The plane wave in equation (6) can be regarded as scattered by the charge moving in a circular orbit. Energy and angular momentum are removed from the incident wave and scattered into new directions. The total scattering cross-section can be computed as the power \( P \) radiated by the orbiting charge divided by the power crossing per unit area \( S = cE_0^2/(4\pi) \) in the plane wave,
\[
cross-section = \frac{P}{S} = \left( \frac{2e^2}{3c^3}\omega^4r^2 \right) \left( \frac{4\pi}{cE_0} \right) = \left( \frac{2e^2}{3c^3}\omega^4r^2 \right) \left( \frac{4\pi}{c} \left( \frac{3c^3}{2e\omega^3r} \right)^2 \right) = 6\pi e^2 \omega^2 = 6\pi e^2 \left( \frac{m^{n+1}}{\alpha^2 J^{n-1}} \right)^{5/2},
\]
(8)

where we have used equation (7).

In the low-velocity (nonrelativistic) limit, all the scattered radiation goes into the fundamental mode which is at the same frequency \( \omega \) as both the particle orbital frequency and the circularly polarised plane wave frequency.

The two most notable special cases are the simple harmonic oscillator potential and the Coulomb potential. The simple harmonic potential \( V(r) = \alpha r^2/2 \) involves \( n \) and \( \alpha \) corresponds to the spring constant. In this case, the scattering cross-section in equation (8) becomes cross-section = \( 6\pi e^2 (m/\alpha) = 6\pi e^2 / \omega_0^2 \) which depends upon the natural frequency of the oscillator \( \omega_0 = (\alpha/m)^{1/2} \) but is completely independent of the value of the particle’s angular momentum \( J \). The particle velocity \( v \) is \( v = \alpha^{1/2} J^{1/2}/m^{1/4} \) and the particle energy is \( U = (\alpha/m)^{1/2}J = \omega_0 J \). The electric field \( E_0 \) is related to the parameters of the orbiting charge as \( E_0 = (2/3)(e/c^3)(\alpha/m)^{5/4}(J/m)^{1/2} = (2/3)(e/c^3)(\omega_0)^{5/2}(J/m)^{1/2} \). We see that, for fixed angular momentum \( J \), different oscillators of the same natural frequency \( \omega_0 = (\alpha/m)^{1/2} \) will have the same energy \( U \), but will involve completely different values of velocity \( v \) and electric field \( E_0 \) depending upon the choice of the mass \( m \).

The Coulomb potential \( V(r) = e^2/r \) involves \( n = -2 \) and \( \alpha = e^2 \). In this case, we have cross-section = \( 6\pi e^2 J^2/(me^2)^3 \) = \( 6\pi [e^2/(me^2)]^2 (J/e^2)^6 \), where \([e^2/(me^2)]\) is the classical radius of the electron, and \([e^2/(Jc)]\) is a \( \sigma_{BU}^{1+} \)-scale-invariant constant involving the
angular momentum \( J \). The particle velocity \( v \) is \( v = e^2/J \) and the particle energy is \( U = -(1/2)mc^2[e^2/(Jc)^2] \). The amplitude of the circularly polarised plane wave is \( E_\theta = (2e/3)(mc^2/e^2)^{1/2}[e^2/(Jc)^2] \). For the Coulomb potential with fixed particle angular momentum, all quantities scale with the mass \( m \). Thus the quantities \( v, U/m, \) cross-section \( \times m^2 \) and \( E_\theta/m^2 \) depend upon only \( \sigma_{BU}^{-1} \)-scale invariant quantities.

5. Nonrelativistic physics as a limit of relativistic physics

In the analysis just presented, we have used nonrelativistic mechanics for the motion of the charged particle although we have used relativistic classical electromagnetic theory for the circularly polarised plane wave. Now we want to check that the textbook claim that these scattering calculations are justified in the sense that they represent low-velocity motion where \( v/c \ll 1 \) within a relativistic system. Thus we want to see that indeed our scattering calculations represent low-velocity limits for a fully relativistic system. Clearly the fully relativistic system must involve relativistic expressions for the particle momentum \( p = m\gamma v \) (where \( \gamma = (1 - v^2/c^2)^{-1/2} \)) and angular momentum \( J = \mathbf{r} \times \mathbf{p} \) (or in magnitude for a circular orbit \( J = mr\gamma vr \)). Then for a relativistic circular orbit, the equations (1) and (2) become

\[
m\gamma^2r = \omega r^n
\]

and

\[
J = m\gamma r^2\omega,
\]

where

\[
\gamma = [1 - (r\omega/c)^2]^{-1/2}.
\]

However, despite the introduction of these relativistic expression for momentum and angular momentum, our analysis is not relativistic. As emphasised in the no-interaction theorem by Currie et al [2], any relativistic mechanical system which goes beyond point interactions between particles must involve a field theory. The circular orbit for our charged particle involves a potential energy \( V(r) \) of interaction between the particle of mass \( m \) and a hypothetical particle of very large mass \( M \rightarrow \infty \) at the coordinate origin. In order to become a relativistic theory, this potential energy function \( V(r) \) must be extended to a full relativistic field theory. In the case of the Coulomb potential \( V(r) = \epsilon^2/r \), the extension is thoroughly familiar as relativistic classical electrodynamics. However, the relativistic field-theory extensions of the other potentials, and in particular the harmonic oscillator potential, are lacking. It is not sufficient to take the low-velocity limits of relativistic expressions for particle energy and momentum; we must also deal with interactions between particles which allow extension to a relativistic field theory.

6. Radiation scattering in relativistic classical electrodynamics

6.1. Special aspects of scattering from a charge in a Coulomb potential

Having emphasised that the only familiar relativistic analysis involves the use of relativistic classical electrodynamics, we now return to the scattering calculation from a fully relativistic point of view. This means using fully relativistic mechanical expressions and also limiting ourselves to the Coulomb potential which is part of a fully relativistic field theory. We find that there are now strong constraints on the behaviour of the system. In this electromagnetic case, the relativistic equation (9) becomes
The energy of the particle is

\[ U = mc^2 - \frac{e^2}{r} = mc^2 \left( 1 - \left( \frac{e^2}{Jc} \right)^2 \right)^{1/2} \]

and the velocity \( v = r\omega \) of the particle in its circular orbit is

\[ v = r\omega = \frac{e^2}{J}. \]

The particle velocity is completely independent of the particle mass \( m \). Indeed, only the Coulomb potential has the orbiting particle velocity \( v \) independent of the mass \( m \). This independence from \( m \) is consistent with the \( \sigma_{btv}^{-1} \)-scale invariance of the velocity \( v \) in relativistic classical electrodynamics. Since the constants \( e^2 \) and \( c \), and also the parameter \( J \), are all \( \sigma_{btv}^{-1} \)-scale invariant, the constant \( e^2/(Jc) \) is \( \sigma_{btv}^{-1} \)-scale invariant. Thus the only parameter giving a scale to the particle orbit is the mass \( m \) which gives a length scale corresponding to the classical radius of the electron \( e^2/(mc^2) \). The mass \( m \) also gives the scale of time in terms of the classical radius of the electron divided by the speed of light \( e^2/(mc^2) \), and the frequency scale depends upon the inverse of this time. The electric field of the incident plane wave also has its scale determined by the particle mass \( m \). The power radiated in the relativistic treatment of the circular orbit [8] is

\[ P = \frac{2}{3} (e^2/c^3) \gamma^4 \omega^4 r^2 \]

so that the relativistic version of equation (7) is

\[ E_0 = \frac{2e}{3c^3} \gamma^3 r = \frac{2e}{3} \left( \frac{mc^2}{e^2} \right)^2 \left( \frac{e^2}{Jc} \right)^3 \left[ 1 - \left( \frac{e^2}{Jc} \right)^2 \right]^{-3}. \]

Again this electric field magnitude is consistent with \( \sigma_{btv}^{-1} \)-scaling since for a point charge the electric field behaves as \( E = r e/r^2 \), and \( E_0 \) in equation (15) involves the inverse of the classical radius of the electron squared times \( \sigma_{btv}^{-1} \)-scale-invariant quantities.

In the analysis involving a nonrelativistic low-velocity limit for the orbiting charged particle, all the scattered radiation went into the radiation at the same fundamental frequency \( \omega \) as the orbital motion and the incident plane wave. Such scattering cannot lead to thermal equilibrium since there is no exchange of energy involving different frequencies. However, in the fully relativistic calculation which is not restricted to the limit of low velocities, the orbiting charged particle indeed shifts the radiation of the incident wave into new frequencies; the scattering of the plane wave moves part of the radiation to multiples of the fundamental frequency and so acts like the sort of scatterer envisioned in discussions of radiation thermal equilibrium. For a charged particle moving in a circular orbit of radius \( r \) at frequency \( \omega \), the power radiated per unit solid angle into the \( j \)th harmonic is given by

\[ \frac{dP_j}{d\Omega} = \frac{e^2 \omega^4 r^2}{2\pi c^3} k^2 \left( \frac{dJ_j(k\beta \sin \theta)}{d(k\beta \sin \theta)} \right)^2 + \left( \frac{\cot \theta}{\beta} J_j(k\beta \sin \theta) \right)^2, \]

See Jackson in [8, p 702].
where here $\beta = rw/c$, and $J_k$ is the Bessel function of order $k$. The ratio of the power radiated into the various harmonics depends on only the velocity $\beta = v/c = c^2/(Jc)$ and is completely independent of the mass $m$ of the charge in the circular orbit. This is the sort of behaviour which makes equilibrium thermal radiation independent of the details of the relativistic electromagnetic scatterer.

6.2. Stability of the orbiting charge

The total electromagnetic fields $E(r, t) = -\nabla \Phi - c^{-1}\partial A/\partial t$ and $B(r, t) = \nabla \times A$ in space involve a homogeneous (source-free) solution of Maxwell’s equations (here given by the circularly polarised plane wave) plus the fields arising from the orbiting charge using the retarded Green function for the scalar wave equation. In terms of the vector potential $A$, these expressions take the general form

$$A(r, t) = A^i(r, t) + \int \int d^3r' dt' \frac{\delta(t - t' - |r - r'|/c) J(r', t')}{|r - r'| c}, \quad (17)$$

where here $A^i(r, t)$ corresponds to the incident circularly polarised plane wave and $J(r', t')$ corresponds to the current source arising from the orbiting charge. In our example involving the circularly polarised plane wave, what prevents the collapse of the orbiting charge into the centre of the Coulomb potential is the driving force of the homogeneous in-fields. The charged particle loses energy and angular momentum due to radiation, but picks up energy and angular momentum from the homogeneous (source-free) radiation fields, which in our example are those of the circularly polarised plane wave.

6.3. Circular particle orbit in a spherical cavity

In the discussion above, we considered the scattering of a circularly polarised plane wave by a charged particle in a circular orbit in a Coulomb potential. The use of a circularly polarised plane wave was made for considerations of simplicity and familiarity. However, a still more relevant calculation would involve a charged particle in steady-state motion in a circular orbit in a Coulomb potential inside a spherical cavity with perfectly conducting walls. Any accelerating electric charge would radiate so as to introduce electromagnetic fields into the cavity. Thus steady-state motion requires the presence of precisely those electromagnetic fields which meet the boundary conditions at both the conducting walls and at the position of the orbiting charged particle. The electromagnetic fields can be obtained by modifying the calculations of Burko for a charged particle in a circular orbit in free space [9]. For a charge of mass $m$ in steady-state circular orbit in a Coulomb potential at angular momentum $J$, the radius $R$ of the cavity corresponds to certain discrete values related to the wavelength $\lambda = 2\pi/\omega$. There is radiation at all the harmonics of the fundamental frequency, and the ratios of the radiation energies at the harmonics are independent of the fundamental frequency $\omega$. If the charged particle is replaced by a charged particle of the same charge and different mass in the ratio $m'/m$, the spherical cavity could be compressed in the ratio $R'/R = m/m'$ so as to again bring the orbital motion and radiation back into steady-state balance. The energy in the radiation modes has the adiabatic invariant $U/\omega$ which is $\sigma_{4\pi}\omega$-scale invariant.

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7 See Jackson in [8, p 245], equation (6.45) for the basic form.
7. Classical zero-point radiation and blackbody radiation

7.1. Classical zero-point radiation

In the scattering analysis used above, we have always worked with coherent radiation connected to the steady-state motion of a charged particle. However, thermal radiation involves not coherent but rather random radiation over a spectrum of frequencies. As pointed out in an earlier analysis [10] of the thermodynamics of the harmonic oscillator (or of radiation normal modes), the principles of thermodynamics allow the possibility of zero-point energy. Zero-point energy is random energy which exists even at the absolute zero of temperature and which takes the form

\[ U_0(\omega) = \text{const} \times \omega \]  

for each normal mode of frequency \( \omega \). We note that this zero-point energy satisfies \( \sigma_{\mu U^{-1}} \)-scaling since energy \( U \) and frequency \( \omega \) transform in the same fashion under the scaling. Since energy divided by frequency \( U/\omega \) is an adiabatic invariant for any oscillator mode, the spectrum of zero-point radiation in a spherical cavity is invariant under adiabatic compression provided that the constant \( \text{const} \) is the same for every radiation mode. Thus under adiabatic compression, a mode of frequency \( \omega \) becomes a mode of frequency \( \omega' \); however, the original energy \( U \) of the mode becomes energy \( U' \) of the new mode so that the spectrum of the radiation in the compressed cavity is still that of zero-point radiation, since it still takes the form \( U_0(\omega') = \text{const} \times \omega' \).

The motion of a relativistic charged particle in a Coulomb potential at the centre of the cavity is described by the action variables \( J_1, J_2, J_3 \) which have the units of angular momentum and are therefore \( \sigma_{\mu U^{-1}} \)-scale invariant. The energy of the relativistic particle is given by [11]

\[ U = m c^2 - \frac{e^2}{r} = mc^2 \left( 1 + \frac{e^2}{[(J_3 - J_2)c + (J_2c^2 - e^3/2)^{1/2}]^2} \right)^{-1} \]  

which again satisfies \( \sigma_{\mu U^{-1}} \)-scaling behaviour with the particle mass \( m \). The particle radiates away energy and angular momentum (just as our charged particle in a steady-state circular orbit radiated away energy and angular momentum into the radiation field), and picks up energy and angular momentum out of the random radiation field (just as our orbiting particle picked up energy and angular momentum from the circularly polarised plane wave). The constant \( \text{const} \) appearing in the spectrum for the random classical zero-point radiation is an adiabatic invariant and a \( \sigma_{\mu U^{-1}} \)-scale invariant; it takes the same value for each mode of the radiation field. We expect that (just as in our example involving coherent radiation above) the balance between energy pick up and loss for any charged particle will lead to an average particle behaviour which is related to that of the random radiation. In this classical view, it is the random classical zero-point radiation which prevents atomic collapse in a Coulomb potential.

7.2. Classical blackbody radiation

Random classical zero-point radiation is the unique spectrum of random classical radiation which is \( \sigma_{\mu U^{-1}} \)-scale invariant, Lorentz invariant, invariant under adiabatic compression, and isotropic in every inertial frame [12]. Within classical electromagnetic theory, zero-point radiation is crucial for understanding blackbody radiation.
Based upon experimental measurements of Casimir forces [13], the constant const appearing in the spectrum of classical zero-point radiation takes the value

\[
\text{const} = \left( \frac{\pi^2 k_B}{120 c^3 a_S} \right)^{1/3} = 0.527 \times 10^{-34} \text{ J} \cdot \text{s},
\]

where \(a_S\) is Stefan’s constant of 1789 and \(k_B\) is Boltzmann’s constant. This constant takes the numerical value that is more familiarly denoted as \(\hbar/2\) where \(2\pi\hbar = \hbar\) is Planck’s constant. This constant value sets the scale for the energy \(U_0(\omega)\) at temperature \(T = 0\). It will also set the scale for the particle energy in equation (19).

The classical understanding of thermal radiation is as follows. The divergent spectrum of classical zero-point radiation is always present at any temperature \(T\). Thermal radiation at temperature \(T > 0\) represents a finite density of radiation energy above the zero-point radiation. In a volume \(V\), the total thermal energy equation of the blackbody spectrum \(U = a_S T^4 V\) (referring to the energy above the zero-point radiation) is invariant under \(\sigma_{\mu U}^{-1}\)-scaling. Thus the quantity \(k_B T\) (like the quantity \(U\)) has the dimensions of energy while the volume \(V\) has the dimensions of length cubed, and the universal constant \(a_S/k_B^4\) is invariant under \(\sigma_{\mu U}^{-1}\)-scaling. Therefore the equation is \(\sigma_{\mu U}^{-1}\)-scale invariant.

The presence of zero-point energy is contrary to the assumptions of nonrelativistic classical statistical mechanics. But then too, the ideas of special relativity are completely contrary to the ideas of nonrelativistic classical statistical mechanics. It is nonrelativistic statistical mechanics which suggests energy equipartition and the Rayleigh–Jeans spectrum.

Classical zero-point radiation shares the \(\sigma_{\mu U}^{-1}\)-scaling symmetry of relativistic classical electrodynamics. Classical zero-point radiation also provides the basis for understanding the Planck spectrum in classical physics from a number of points of view. If we consider the thermodynamics of a harmonic oscillator and ask for the smoothest interpolation between zero-point energy \((1/2)\hbar \omega\) at low temperature and \(k_B T\) at high temperature, then we derive the Planck blackbody spectrum within classical physics [10]. If we compare paramagnetic behaviour with diamagnetic behaviour in classical zero-point radiation while using relativistic limits consistently, then we derive the Planck spectrum within classical physics [14]. If we consider time-dilating conformal transformations of thermal radiation in a Minkowski coordinate frame and in a Rindler frame, then the Planck spectrum is derived within classical physics based upon the structure of relativistic spacetime [5].

8. Discussion

In our introductory courses on special relativity, we often discuss collisions between relativistic particles. When we do this, we use only the mechanical aspects of special relativity, only the expressions for mechanical energy \(U = m c^2\) and momentum \(p = m \gamma v\). It is rarely mentioned that if we attempt to introduce an interaction between the particles which is not a point interaction, then relativity forces us to go to a full field theory. This is the content of the no-interaction theorem of Currie et al [2]. Mixtures of relativistic and nonrelativistic physics follow neither the rules of relativistic nor of nonrelativistic physics and are full of pitfalls for the unsuspecting physicist [15].

Electromagnetic radiation cannot bring itself to thermal equilibrium. Therefore the analysis of blackbody radiation forces us to introduce interactions between radiation and matter in order to describe thermal equilibrium. However, if we hope to explain nature, we must use theories which describe nature accurately. Use of nonrelativistic mechanics for the
scatterers of radiation leads to the same energy equipartition as appears in nonrelativistic classical statistical mechanics. Accurate treatment of thermal equilibrium within classical physics requires the use of a fully relativistic analysis. Relativistic analysis leads to the Planck spectrum for blackbody radiation within classical physics.

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