Mean spin entanglement of two massive Dirac particles under Lorentz transformations

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(Dated: August 23, 2010)

We have studied the relativistic effects on the mean spin entanglement of two massive Dirac particles using the simultaneous eigen-spinors of the Foldy-Wouthuysen mean spin operator and the Dirac Hamiltonian. We have obtained the transformation matrix from the spinor with specific momentum to the spinor with a transformed momentum under an arbitrary Lorentz transformation. Using the transformation matrix we have shown the consistent monotonic behavior between the concurrence and the maximum value of Bell parameter in Bell inequality of transformed spin states.

PACS numbers:

I. INTRODUCTION

Quantum entanglement is a very important physical resource in quantum computing[1]. A separated pair of entangled quantum system can be used to transfer quantum information. The measurement on one system of an entangled pair determines the outcome of the measurement on the other system instantaneously. Entanglement has a non-local nature in its essence. The special theory of relativity, however, tells us that there is no agreement on simultaneity for distant events among observers in inertial reference frames moving relative to each other. There are no causal relations between two spatially separate events. In this aspect relativity is local in its nature. Hence it is an intriguing question how the entanglement of a pair changes among observers in relative motions.

The spin state of elementary spin 1/2 particles, such as electrons, are good candidates for qubits. Massive spin 1/2 particles, however, have momentum degrees of freedom other than the spin degrees of freedom. Moreover it has been shown the relativistic transformation entangles the spin and momentum degrees of freedom and the spin entropy determined by the reduced density matrix for the spin is not covariant under Lorentz transformation[2]. Then the spin entanglements would also depend on the motion of observers and it is interesting how the spin entanglements between massive spin 1/2 particles would be changed under relativistic transformations. Several authors have investigated the effect of the entanglement under relativistic transformations[3-5].

Gingrich and Adamai[6] concluded that the spin entanglement of a pair of particles is not invariant under relativistic motion, although the whole entanglement of spin and momentum degrees of freedom is Lorentz invariant. They have founded that the special pairs of spin which are initially separable can be maximally entangled in the special reference frame. They also noted that this property could lead to simplified state preparation and purification protocols. Later Czachor et. al. have pointed out the definition of reduced density matrix with traced-out momenta in Ref. [6] is not justified. The results of later works have not reached the same conclusion depending on whether only the change of states are considered [7] or what kinds of relativistic spin observables are used [8-10]. The main reason for the discrepancy is the lack of clear understanding for the proper spin operator for massive Dirac particles.

In this letter, we will clarify the problems associated with the relativistic effects on quantum spin entanglement. The discrepancy in the previous works will be cured by considering a Dirac equation and spin observables consistently. Dirac equation is a successful theory to reconcile quantum mechanics with special relativity. Therefore we will use the Dirac theory for a massive spin 1/2 particle, so-called a massive Dirac particle, to obtain a consistent covariant approach for the relativistic effect on the spin entanglement of a pair of Dirac particles. We use ‘mean spin angular momentum’ operators, defined by Foldy and Wouthuysen[11], to describe a good quantum observables we will call these operators as mean spin operators. The mean spin operators commute with Dirac Hamiltonian unlike the spin operator in the conventional representation and since the time evolution of the state of Dirac particle is generated by Dirac Hamiltonian unlike the spin operator in the conventional representation and since the time evolution of the state of Dirac particle is generated by Dirac Hamiltonian, it is described by the time-dependent Dirac equation[12],

\[ i \frac{\partial}{\partial t} \Psi(x,t) = \mathcal{H}_D \Psi(x,t) = (\beta m + \alpha \cdot P) \Psi(x,t), \]

(1)

where \( \alpha \) and \( \beta \) are 4 \times 4 Dirac matrices in standard Dirac representation, \( P \) is a momentum operator and a natural unit defined by \( c = \hbar = 1 \) is used. Here we are interested...
Therefore it is possible to find simultaneous eigenstates of the Dirac Hamiltonian \( H_D \) as follows

\[
(\beta m + \alpha \cdot \mathbf{p}) \psi^\pm(\mathbf{p}, \lambda) = \pm E \psi^\pm(\mathbf{p}, \lambda),
\]

where \( E = \sqrt{m^2 + \mathbf{p}^2} \) and the superscript \( \pm \) represents positive-energy and negative-energy solutions. The state \( \psi^\pm(\mathbf{p}, \lambda) \) can also be understood as the Lorentz transformed state from the state \( \psi^\pm(0, \lambda) \) in the rest frame of the particle such as

\[
\psi^\pm(\mathbf{p}, \lambda) = \frac{1}{\sqrt{m E}} S(L(\mathbf{p})) \psi^\pm(0, \lambda),
\]

where \( p^\mu = \sum_\nu L_\nu^\mu k^\nu \). \( L_\nu^\mu \) is a standard Lorentz transformation depending on \( p^\mu \) and \( S(L(\mathbf{p})) \) is a 4-spinor representation for \( L(\mathbf{p}) \). We use the following normalization

\[
(\psi^+(\mathbf{p}', \lambda'), \psi^+(\mathbf{p}, \lambda)) = \delta_{\lambda \lambda'} \delta^3(\mathbf{p}' - \mathbf{p})
\]

\[
(\psi^-(\mathbf{p}', \lambda'), \psi^-(\mathbf{p}, \lambda)) = \delta_{\lambda \lambda'} \delta^3(\mathbf{p}' - \mathbf{p}),
\]

where \( (\cdot \cdot) \) is a scalar product in Hilbert space and the scalar products between other states are zero. Note that \( p^0 \delta^3(\mathbf{p}' - \mathbf{p}) \) is a Lorentz invariant delta function.

Here the label \( \lambda \) represents other degrees of freedom than momentum degrees of freedom. It must be related to the spin of the particle, however at this stage, the meaning of this label is not clear since the conventional 4 \( \times \) 4 spin operators \( \frac{1}{2} \sigma = \frac{1}{2i} \alpha \times \alpha \) does not commute with the Hamiltonian \( H_D \). The proper spin operator must commute with the Hamiltonian which governs the dynamics of the particle and satisfies the SU(2) algebra. The operator \( \frac{1}{2} \sigma \) commutes with the Hamiltonian \( \beta m \) of the rest frame of the particle and satisfies SU(2) algebra, hence the proper spin operator must be related to \( \frac{1}{2} \sigma \) in the rest frame of the particle. Foldy and Wouthyne-FW have defined the mean spin operator by the use of the “canonical” transformation such as

\[
\frac{1}{2} \Sigma = \sum_{\nu} L_\nu^\mu \psi^\pm(0, \lambda),
\]

where \( U_{FW}(0) \) is the unitary operator as

\[
U_{FW}(\mathbf{p}) = \frac{m + \beta \mathbf{p} + E}{\sqrt{2E(E + m)}}.
\]

The mean spin operators satisfy SU(2) algebra and commute with the Dirac Hamiltonian, i.e., \( \frac{1}{2} \Sigma, H_D = 0 \). Therefore it is possible to find simultaneous eigenstates of \( H_D \) and \( \Sigma_3 \), such as

\[
H_D \psi^\pm_{FW}(\mathbf{p}, \lambda) = \pm E \psi^\pm_{FW}(\mathbf{p}, \lambda),
\]

\[
\Sigma_3 \psi^\pm_{FW}(\mathbf{p}, \lambda) = \lambda \psi^\pm_{FW}(\mathbf{p}, \lambda), \quad (\lambda = \pm 1).
\]

It is now clear that the variable \( \lambda \) in \( \psi^\pm_{FW}(\mathbf{p}, \lambda) \) represents the \( z \) component of the mean spin operator \( \Sigma \). The expectation value of the mean spin operator describes the average spin of the particle which is really measured. This mean spin operator is the same as the spin operator in the non-relativistic Pauli Hamiltonian when it is represented in the representation (so-called Foldy-Wouthysen representation) where the Dirac Hamiltonian has the diagonal form. Note that for positive-energy states the conventional Dirac spinor \( \psi^+(\mathbf{p}, \lambda) \) is the same as the Foldy-Wouthysen spinor \( \psi^+_{FW}(\mathbf{p}, \lambda) \). This implies the mean spin for a particle with positive energy in its rest frame does not change under the Lorentz transformation \( L(\mathbf{p}) \).

The general situation appears when the observer to see a particle with momentum \( p^\mu \) moves relativistically such that the momentum of the particle becomes \( \sum_\nu L_\nu^\mu \eta^\mu \), where \( \lambda \) is an arbitrary orthochronous Lorentz transformation. The spinor state \( \psi^\pm_{FW}(\mathbf{p}, \lambda) \) described by the observer in old reference frame will be transformed as \( S(\Lambda) \psi^\pm_{FW}(\mathbf{p}, \lambda) \) to the observer in the new reference frame, where \( S(\Lambda) \) is the spinor representation for \( \Lambda \).

It is not trivial to represent \( S(\Lambda) \psi^\pm_{FW}(\mathbf{p}, \lambda) \) with eigenstates \( \psi^\pm_{FW}(\Lambda \mathbf{p}, \lambda) \) in the new reference frame since the two successive Lorentz transformations \( S(\Lambda) \) and \( S(L(\mathbf{p})) \) does not become one Lorentz transformation in general. Here \( \Lambda \mathbf{p} \) is a spatial component for the transformed momenta. A FW-representation will be used to represent the transformed state explicitly. In the FW-representation, the transformed Dirac Hamiltonian and the mean spin operator take simple forms

\[
\mathcal{H}_D = U_{FW}(\mathbf{p}) H_D U_{FW}(\mathbf{p})^\dagger = \beta(m^2 + \mathbf{p}^2),
\]

\[
\frac{1}{2} \Sigma'^\dagger = U_{FW}(\mathbf{p}) \frac{1}{2} \Sigma U_{FW}(\mathbf{p}) = \frac{1}{2} \sigma.
\]

Note that both operators are block diagonal. This fact makes that the eigenstates in FW-representation can be written as the product form

\[
\psi^\pm(\mathbf{p}, \lambda) = (\mathbf{p}) \otimes |\lambda\rangle^\pm.
\]

Now the eigenstate in the original representation can be obtained as \( \psi^\pm_{FW}(\mathbf{p}, \lambda) = U_{FW}(\mathbf{p}) \psi^\pm(\mathbf{p}, \lambda) \). It is convenient to use the new notations for the spin state \( |\lambda\rangle^\pm \) in FW-representation such as \( |+\rangle^+ = (1 0 0 0)^T \equiv |0\rangle, \quad |+\rangle^- = |1\rangle, \quad |+\rangle^+ = |2\rangle, \text{ and } |+\rangle^- = |3\rangle \).

We define a transformation matrix \( T^{(\Lambda \mathbf{p})} \) which represents the transformation of a FW-spinor state under an arbitrary Lorentz transformation \( \Lambda \):

\[
S(\Lambda) \psi_{FW}(\mathbf{p}, \mu) = \sum_\nu T^{(\Lambda \mathbf{p})}_{\mu \nu} \psi_{FW}(\Lambda \mathbf{p}, \nu),
\]

where \( \mu, \nu = 0, 1, 2, 3 \). This relation is described in the FW-representation as follows

\[
S(\Lambda) U_{FW}(\mathbf{p}) \otimes |\mu\rangle = \sum_\nu T^{(\Lambda \mathbf{p})}_{\mu \nu} U_{FW}(\mathbf{p}) |\Lambda \mathbf{p}\rangle \otimes |\nu\rangle.
\]

Therefore the transformation matrix is obtained as

\[
T^{(\Lambda \mathbf{p})} = \langle \Lambda \mathbf{p}| U_{FW}(\mathbf{p}) S(\Lambda) U_{FW}(\mathbf{p})^\dagger |\mathbf{p}\rangle.
\]
To find an explicit form of $T^{(x, p)}$, we consider simple Lorentz boost $\Lambda$ in $x_3$ direction with rapidity $\xi$. It is clear that a rotation of the space does not have any effect on the entanglement. Therefore we consider only a Lorentz boost without loss of generality. Writing the space part of momentum vector in spherical coordinate $p = \{E, p \sin \theta \cos \phi, p \sin \theta \sin \phi, p \cos \theta\}$, where $\theta$ is a polar angle from the positive $x_3$-axis and $\phi$ is the azimuthal angle in the $x_1x_2$ plane from the $x_1$-axis. We get

$$T^{(\Lambda, p)} = \sqrt{E'} \frac{E}{E'} \begin{pmatrix} A & B e^{-i\phi} & C & D e^{-i\phi} \\ -B e^{i\phi} & A & D e^{i\phi} & C \\ 0 & 0 & -\tilde{A} & \tilde{B} e^{i\phi} \\ 0 & 0 & \tilde{A} & -\tilde{B} e^{-i\phi} \end{pmatrix},$$

where

$$A = \sqrt{\frac{m + E}{m + E'}} \left[ \cosh \left( \frac{\xi}{2} \right) + \frac{p \cos(\theta)}{m + E} \sinh \left( \frac{\xi}{2} \right) \right],$$

$$B = \frac{p \sin(\theta)}{\sqrt{(m + E)(m + E')}} \sinh \left( \frac{\xi}{2} \right),$$

$$C = \sinh \left( \frac{\xi}{2} \right) \left[ \cosh^2 \left( \frac{\xi}{2} \right) \left\{ (m + E)^2 \right. \right.$$  $$- p^2 \cos(2\phi) \right. + mp \cos(\phi) \sinh(\xi) \biggr] \left.$$  $$- E' \sqrt{(E + m)(E' + m)} \left[ E \sinh \left( \frac{\xi}{2} \right) \right.$$  $$+ p \cos(\phi) \cosh \left( \frac{\xi}{2} \right) \biggr],$$

$$\tilde{A} = \frac{E}{E'} A, \quad \tilde{B} = \frac{E}{E'} B,$$

$$E' = \frac{p \sinh(\xi) \cos(\theta) + E \cosh(\xi)}{1}. \quad (19)$$

Since the Dirac Hamiltonian in the FW-representation is in diagonal form, positive energy subspace is spanned by the upper two components of the state and negative energy subspace is spanned by the lower two components of the state. The sign of energy is invariant under Lorentz transformation so that we do not have to consider negative energy state for a particle with positive mass $m$. Here we will study the entanglement of particles with positive energies and all considerations can be given in the two-dimensional positive energy subspace. Therefore the concurrence is a good measure of the spin entanglement for two massive positive energy Dirac particles. In the positive energy subspace the transformation matrix becomes $2 \times 2$ matrix

$$T^{(\Lambda, p)}_2 = \sqrt{E'} \frac{E}{E'} \begin{pmatrix} A & B e^{-i\phi} \\ -B e^{i\phi} & A \end{pmatrix}. \quad (20)$$

A general positive energy state in consideration can be written as

$$\Psi = \int d^3 p \sum_\lambda a_{p, \lambda} \psi^{(+)}_{FW}(p, \lambda). \quad (21)$$

Where the normalization condition is satisfied as $\langle \Psi, \Psi \rangle = \int d^3 p \sum_\lambda |a_{p, \lambda}|^2$. The density matrix $\rho$ is defined by $\Psi \Psi^\dagger$. Then the $\sigma^\prime \cdot \sigma$ component of the reduced spin density matrix $\rho^r$ is obtained by

$$\rho^r_{\sigma, \sigma'} = \int d^3 p (\psi^{(+)}_{FW}(p, \sigma), \rho(\psi^{(+)}_{FW}(p, \sigma'))) \quad (22)$$

Note that this reduced density matrix is $2 \times 2$ matrix as expected for the positive energy state. The spin entanglement of this reduced state is obtained from the concurrence of Wootters formula $C = \max \{ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0 \}$ \cite{[13],[14]}, where $\lambda_i$’s are the square roots of the eigenvalues of the matrix $\rho \tilde{\rho}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^r (\sigma_y \otimes \sigma_y)$.

It is interesting to study the effect of Lorentz boost on the state whose reduced spin state is a Bell state initially,

$$\Psi_{AB} = \frac{1}{2} (|\psi_{FW}(p_1, -p_1; 1, -1) + \psi_{FW}(p_1, -p_1; -1, 1) \rangle \\ + \psi_{FW}(p_2, -p_2; 1, -1) + \psi_{FW}(p_2, -p_2; -1, 1) \rangle \quad (23)$$

where $\Psi_{p A}, -p B: \lambda_A, \lambda_B) = \psi^{(+)}_{FW}(p_A, \lambda_A) \otimes \psi^{(+)}_{FW}(p_B, \lambda_B)$ and $p_1$, $p_2$ are in the $x_1x_2$ plane with the same magnitude $p$ and spin variables represents the eigenvalues of mean spin operators. The angle between $p_1$ and $p_2$ is $\phi$. By writing the state as a density matrix and tracing over the momentum variable, we get a maximally entangled spin state ($C = 1$). For an observer moving to the positive $x_3$ axis with rapidity $\xi$, the state changes according to the transformation matrix \cite{[20]} and, therefore, the concurrence would change also.

Taking limit $p \to \infty$ and $\xi \to \infty$, the concurrence yields the simple form

$$C = | \cos(\phi) |. \quad (24)$$

In this limit the final reduced spin state has all possible concurrence depending on the angle $\phi$. When $p_1$ and $p_2$ are perpendicular, the spin entanglement vanishes entirely.

For completeness we consider the Bell inequality \cite{[13],[14]} as a parallel approach. The Bell inequality violation is given only for states with non-zero concurrence, since the entanglement is needed to violate the Bell inequality. Moreover for two qubit systems the amount of violation of Bell inequality can be an entanglement measure to some extent \cite{[17]}. Therefore we expect the consistent violation of Bell inequality is obtained with the concurrence. The previous results \cite{[8],[9],[10]}, however, shows the maximal violation of Bell inequality is obtained in case the proper direction of spin measurement is chosen. We will show the consistent results are obtained independent on the direction of spin measurements different from the previous results. The main reason for this is that we consider both the transformation of the state and a spin operator correctly.

The generalized Bell inequality is determined by the following Bell parameter,

$$B = |C(a_1, b_1) + C(a_1, b_2) + C(a_2, b_1) - C(a_2, b_2)| \quad (25)$$
where the spin correlation function is given by $C(\mathbf{a}, \mathbf{b}) = \text{Tr}[\left\{ \frac{1}{2} \Sigma_A \cdot \mathbf{a} \otimes \left( \frac{1}{2} \Sigma_B \cdot \mathbf{b} \right) \rho^r \right\}$, where $\Sigma$ is the mean spin operator and $\mathbf{a}(\mathbf{b})$ is the direction for spin measurement.

For a given state the maximum value of Bell parameter, $B_{\text{max}}$, is obtained by adjusting the directions $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{a}_2$, and $\mathbf{b}_2$ for spin measurement. Fig. 1 shows $B_{\text{max}}$ versus the concurrence in Eq. (24) for the reduced spin state in the limit of $p, \xi \to \infty$. This shows one-to-one correspondence between two quantities, specifically, a monotonic behavior. The maximum value of Bell parameter becomes less than 2 even though the entanglement remains ($C \neq 0$). This is because the more entanglement is required to give the same Bell parameter for mixed states [17]. The reduced spin state for $\phi = \pi/2$ in the limit of $p, \xi \to \infty$ is the mixed state, $\frac{1}{2} |\Psi_-\rangle \langle \Psi_-| + \frac{1}{2} |\Phi_+\rangle \langle \Phi_+|$, where $|\Psi_-\rangle$ and $|\Phi_+\rangle$ are Bell states. The concurrence and the maximum value of Bell parameter for this state are both zero.

FIG. 1: The maximum value of Bell parameter $B_{\text{max}}$ versus the concurrence for the reduced spin states in the limit of $p, \xi \to \infty$.

We have studied the relativistic effects on the spin entanglement of two massive Dirac particles using the simultaneous eigenstate of the Dirac Hamiltonian and the mean spin operator. In this formalism the meaning of spin operators as quantum observables is clear. The mean spin operators commute with the Dirac Hamiltonian and become the conventional spin operators of the Pauli Hamiltonian in non-relativistic limit [11]. This means the mean spin operator is the good quantum variable involved in spin measurement. We have found the transformation matrix which represents the transformation of a spinor state with given momentum to spinor states with the transformed momentum under an arbitrary Lorentz transformation. We have shown the consistent monotonic relation between the concurrence and Bell inequality of the final reduced spin states transformed from the special initial states. We expect this consistent behavior is achieved for other states considering our mean spin correlations. We have clarified the meaning of trace over momentum for a 4-spinor state, so our approach makes it possible to study the relation between the entanglement of momentum and spin of two Dirac particles under the arbitrary Lorentz transformation in clear manner.

This work was supported by National Research Foundation of Korea Grant funded by the Korean Government(KRF-2008-331-C00073) and by National Research Foundation of Korea Grant funded by the Korean Government(2010-0004855). We gratefully acknowledge KIAS members for helpful discussions.

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