On a Local Version of the Bak–Sneppen Model

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Abstract
A major difficulty in studying the Bak–Sneppen model is in effectively comparing it with well-understood models. This stems from the use of two geometries: complete graph geometry to locate the global fitness minimizer, and graph geometry to replace the species in the neighborhood of the minimizer. Over the years a number of models inspired by Bak–Sneppen were studied, usually by introducing different or new features (e.g. discretizing fitness, randomized neighbors or population size). We present a variant that only uses features present in Bak–Sneppen, and whose difference from the Bak–Sneppen is that only the graph geometry is used for the evolution. This allows to obtain the stationary distribution through random walk dynamics while preserving the geometric nature of the model. We use this to show that for constant-degree graphs, the stationary fitness distribution converges to an IID law as the number of vertices tends to infinity. We also discuss exponential ergodicity through coupling, and avalanches for the model.

Keywords Bak–Sneppen · Species · Fitness · Stationary distribution

Mathematics Subject Classification 60K35 · 60J05 · 92D15

1 Introduction

1.1 Background

The Bak–Sneppen model was introduced in [3] as a toy model for biological evolution. Let $G = (V, E)$ be a finite connected undirected graph. For $u, v \in V$, we write $u \sim v$ if $\{u, v\} \in E$ or if $u = v$. The model is a discrete time Markov chain on the state space $S$ of nonnegative one-to-one functions on the vertex set $V$. A state $\Gamma$ is to be interpreted as a

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collection of fitness values \((\Gamma(v), v \in V)\) assigned to the species at the vertices of \(G\). From state \(\Gamma\), the system samples the next state \(\Gamma'\) according to the following rules:

A-1. Let \(v = \text{argmin } \Gamma\).
A-2. \(\Gamma'(w) = \Gamma(w)\) if \(w \not\sim v\).
A-3. \((\Gamma'(w) : w \sim v)\) are IID Exp(1) independent of past.

In other words, species with lowest fitness is removed from the system, along with its graph neighbors, and those are replaced by new species with IID Exp(1) fitnesses instead. Note that the dynamics of the process guarantee that (with probability 1), a species can be identified with its fitness. The model is a Markov chain, and if \(G\) is connected, it is also exponentially ergodic. In the original model \(G\) was the \(N\)-cycle, \(V = \{0, \ldots, N-1\}\) and \(E = \{(n, n+1 \mod N) : n \in V\}\). Also, in the original model, the fitnesses assigned were \(U[0,1]\) rather than Exp(1), which, as is easy to see, has no effect on the dynamics. Furthermore, a deterministic transformation maps one model in a one-to-one fashion into the other (e.g. map fitness \(b\) in Exp(1)-model to \(1 - e^{-b}\) to obtain the \(U[0,1]\)-model). We choose to work with exponential distribution, even when referring to the original model. Numerical simulations for the original model (see for instance [3,13]) show that the stationary distribution converges weakly to a product law as \(N \to \infty\), where the marginals are Exp(1), conditioned to be above a certain threshold \(b_c\) (or, equivalently, exponentials shifted by that threshold), with \(e^{-b_c}\) estimated around \(\frac{1}{3}\). This is an open problem. The physics literature contains other interesting and concise predictions based on numerical simulations. Deep mathematically rigorous results on the Bak–Sneppen model were obtained in [15,16] through graphical representation and identification of non-trivial thresholds for the model. We also note that the stationary distribution for the original Bak–Sneppen model was computed for \(N = 4\) and \(N = 5\) in [18,19] respectively, as well as in [2].

Over the years, several additional variations of the Bak–Sneppen model were studied in the mathematics and physics literature. Here we name a few. A discrete-fitness variant on the \(N\)-cycle was introduced in [5]. In this process, a configuration is a binary word with \(N\)-bits, and at each step one chooses a random bit of minimum value and replaces it and its two neighbors by independent Bernoulli random variables with parameter \(p \in (0, 1)\). The authors obtain bounds on the average number of ones in the stationary distribution as \(N \to \infty\). In [4] it is shown that this discrete version of the Bak–Sneppen is conjugate to the classical contact process. Another variant of the Bak–Sneppen model was introduced in [12] and studied in [7,8,17]. Here the number of species follows a path of reflected random walk transient to infinity. When the population increases, new species with IID \(U[0,1]\)-fitness are added, and when the population decreases, the species with lowest fitnesses are removed from the population. For this model, the empirical fitness distribution tends to a uniform law on \([p_c, 1]\), similar to observed empirical behavior of Bak–Sneppen on the cycle as \(N \to \infty\). Note that the geometric aspect of the Bak–Sneppen model is lost in this model, since the population has no spatial structure. Other variations of the Bak–Sneppen can be found in [1,6,10,11,20] and the references within. We emphasize that all these models change or introduce new features to the Bak–Sneppen model.

The main obstacle to rigorous analysis on the Bak–Sneppen model is the incompatibility between complete graph geometry used to identify the vertex of minimal fitness and the given graph geometry, used in updating fitnesses. In this work we present a model whose only difference from the Bak–Sneppen is in using only one geometry, the graph geometry. We study the effect of this choice, highlighting differences from and similarities to the Bak–Sneppen model.
1.2 Local Bak–Sneppen

To further motivate our work, we present a class of models which contains both the Bak–Sneppen model as well as our new model, and which stresses the role of geometry in the dynamics. Recall our finite undirected graph \( G = (V, E) \). Construct a second undirected graph \( G' = (V, E') \). As before, for \( u, v \in V \), write \( u \sim v \) if \( \{u, v\} \in E' \) or if \( u = v \). We consider a new Markov chain on state space \( V \times S \), where \( S \) is the space of nonnegative one-to-one functions on the vertex set \( V \), by slightly tweaking the dynamics of the Bak–Sneppen model as follows. From state \((u, \Gamma)\) the next state \((v, \Gamma')\) is sampled according to the rules presented above, but with A-1 replaced by

\[
A' - 1. \quad v = \operatorname{argmin} \{\Gamma(w) : w \sim u\},
\]

and keeping A-2, A-3. In other words, instead of looking for the global minimum, we look for the local minimum in the \( G' \)-neighborhood of \( u \). Or: we look for minima in the \( G' \)-neighborhood, and update fitnesses in the \( G \)-neighborhood. The Bak–Sneppen model corresponds to \( G' \) being the complete graph, in which case there is no need to track the local-minimum: it coincides with the global minimum which is a function of the state \( \Gamma \).

We will study the case where there is only one geometry, namely \( G' = G \), and call it the local Bak–Sneppen model. In the sequel, we will always assume that \( G \) is connected. Our main result provides a complete description of the stationary measure for the local Bak–Sneppen model.

We now highlight some of the similarities and differences between the Bak–Sneppen and the local Bak–Sneppen model, as seen through our results. First and foremost, a general or asymptotic formula for the stationary distribution for the Bak–Sneppen model is an open problem. One of the main results in this paper (Theorem 2) is a formula for the local Bak–Sneppen model. Our formula does provide asymptotic IID structure for the fitnesses for \( d \)-regular graphs as the number of vertices tends to infinity (Proposition 4), a feature which is expected to hold for the original Bak–Sneppen model. The limiting fitness distribution in our model is exponential, conditioned not to be the minimum among \((d + 1)\)-IID \( \text{Exp}(1) \) random variables (or \( \text{Exp}(1) \) conditioned to be above a random threshold, which is the same). This differs from the expected expression for Bak–Sneppen, which is \( \text{Exp}(1) \) conditioned to be above a deterministic threshold. This difference has a simple heuristic explanation due to the difference in the geometry: in the original Bak–Sneppen, fitnesses at all vertices are considered at every step when looking for the global (complete graph) minimum, effectively eliminating all small values, while in our case, once assigned, fitness is considered at most once before being replaced, a mechanism that cannot exclude very small fitnesses.

We now give a formal description of the model. First, assume that \( G = (V, E) \) is a finite, undirected and connected graph. Let \( \hat{\Omega} = V \times [0, \infty)^{|V|} \), equipped with the Borel \( \sigma \)-algebra induced by the product of the discrete topology on \( V \) and the Euclidean metric on each of the \(|V|\) copies of \([0, \infty)\). We write \( \Omega \ni \omega = (v_\omega, \Gamma_\omega) \), where \( v_\omega \in V \) and \( \Gamma_\omega : V \to [0, \infty) \), and let \( \bar{\Omega} \subset \hat{\Omega} \) denote the set consisting of all elements \( \Omega \ni \omega = (v_\omega, \Gamma_\omega) \) with \( \Gamma_\omega \) one-to-one, and equip it with the Borel \( \sigma \)-algebra, induced by the subspace topology. The set \( \Omega \) will serve as the state space for our process, and the restriction on \( \Gamma_\omega \) is to be understood as distinct fitnesses at distinct vertices, a requirement needed for the dynamics to be well-defined. Also, for \( v \in V \), we let \( A_v = \{u \in V : u \sim v\} \cup \{v\} \), that is \( A_v \) consists of all neighbors of \( v \) in \( G \) and \( v \) itself.
The process is denoted by $\Xi = (\Xi_t : t \in \mathbb{Z}_+)$, with $\Xi_t = (X_t, \Gamma_t) \in \Omega$. Conditioned on $(\Xi_s : s \leq t)$, $\Xi_{t+1} = (X_{t+1}, \Gamma_{t+1})$ is defined as follows:

$$
\begin{align*}
X_{t+1} &= \text{argmin}_{u \in AX_t} \Gamma_t(u), \\
(\Gamma_{t+1}(u) : u \in AX_{t+1}) &\text{ are IID Exp}(1), \text{ independent of } \Xi_0, \ldots, \Xi_t, \text{ and} \\
\Gamma_{t+1}(v) &= \Gamma_t(v), \hspace{1em} v \notin AX_{t+1}.
\end{align*}
$$

Clearly, $\Xi$ is a discrete time Markov process. Observe that it follows directly from the definition that the vertex process $X = (X_t : t \in \mathbb{Z}_+)$ is a random walk on $G$, with transition function $p(v, v') = \frac{1}{|A|}$. In particular, $X$ is ergodic with stationary measure $\mu$, given by

$$
\mu(v) = \deg(v) + 1 = \frac{|A_v| + \sum_{u \in V} |A_u|}{S_G}, \tag{1}
$$

where $S_G = \sum_{u \in V} |A_u|$. Unless clearly specified, whenever a random walk is mentioned in this paper, the reader should have in mind a process with the above transition probabilities.

## 2 Results

### 2.1 Ergodicity

If $\gamma$ is a distribution on $\Omega$, write $P_\gamma$ for the distribution of $\Xi$ when $\Xi_0$ is $\gamma$-distributed. If $\eta \in \Omega$, we abuse notation and write $P_\eta$ for $P_{\delta_\eta}$, where $\delta_\eta$ is the Dirac-delta measure at $\eta$.

Let $Q_1, Q_2$ be two probability measures on $\Omega$. We define the total variation distance between $Q_1$ and $Q_2$ as

$$
\|Q_1 - Q_2\|_{TV} = \sup_{A} |Q_1(A) - Q_2(A)|.
$$

We write $P_\gamma(\Xi_t \in \cdot)$ for the distribution of $\Xi_t$ under $P_\gamma$. We write

$$
\bar{d}_t = \sup_{\eta, \eta' \in \Omega} \|P_\eta(\Xi_t \in \cdot) - P_{\eta'}(\Xi_t \in \cdot)\|_{TV}.
$$

All the results in this section hold for any finite, undirected and connected graph $G$. We have the following exponential ergodicity statement.

**Theorem 1** There exist constants $c > 0$ and $\beta \in (0, 1)$, depending only on $G$, such that for any $t \in \mathbb{Z}_+$,

$$
\bar{d}_t \leq c \beta^t. \tag{2}
$$

Before we prove Theorem 1, we state a standard corollary whose proof we leave to the “Appendix”.

**Corollary 1** $\Xi = (\Xi_t : t \in \mathbb{Z}_+)$ has a unique stationary measure $\pi$.

We note that much of our discussion in the following sections will focus on a detailed description of the stationary distribution $\pi$, and, in particular, obtaining an explicit formula for it that allows to study the model along a sequence of graphs.

Let

$$
d_t = \sup_{\eta} \|P_\eta(\Xi_t \in \cdot) - \pi\|_{TV}.
$$
Then a standard convexity argument and the triangle inequality imply
\[ d_t \leq \tilde{d}_t \leq 2d_t. \]

Here is another useful yet standard corollary, obtained from the exponential ergodicity and whose proof is left to the “Appendix”.

**Corollary 2** Let \( f : \Omega \to \mathbb{R} \) be bounded and measurable. Then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} f(\Xi_t) = \int f \, d\pi.
\]
in \( P_\eta \)-probability for every \( \eta \).

In order to prove Theorem 1 we remember several notions that will be used in the sequel. Recall that if \( Y = (Y_t : t \in \mathbb{Z}_+) \) and \( Y' = (Y'_t : t \in \mathbb{Z}_+) \) are both Markov processes defined on a common probability space having the same transition kernel, then the pair \((Y, Y')\) is called a coupling for \( Y \). Given a coupling, we write
\[
\tau_{coup}(Y, Y') = \inf\{t \in \mathbb{Z}_+ : Y_t = Y'_t\}
\]
for the coupling time. A coupling is called successful if \( \tau_{coup}(Y, Y') < \infty \) a.s. All couplings we will construct and consider in the sequel will be successful, and, in addition, will be also coalescing, that is
\[ Y_t = Y'_t, \quad \text{for all} \quad t \geq \tau_{coup}(Y, Y'). \quad (3) \]

Finally, suppose that \( X \) is a random walk on a finite connected graph \( G \). We define the cover time for \( X \) as
\[
\tau_{cover}(X) = \inf\{t \in \mathbb{Z}_+ : \{X_0, \ldots, X_t\} = V\}.
\]

**Remark 1** (Aldou’s Inequality) The basic coupling inequality (see Chap. 5 of [14] for example) reads as follows: let \((Y_t, Y'_t)\) be a coupling satisfying (3) for which \( Y_0 \) is \( \eta \)-distributed and \( Y'_0 \) is \( \eta' \)-distributed. Then
\[
|| P_\eta(Y_t \in \cdot) - P_{\eta'}(Y'_t \in \cdot)||_{TV} \leq P_\eta,_{\eta'}(Y_t \neq Y'_t).
\]

We are now ready to prove exponential ergodicity.

**Proof of Theorem 1** The idea is to construct a coupling and employ Aldous’ inequality. The coupling has two stages. In the first stage we are running the two copies independently until the coupling between the respective random walks on \( G \) is successful (the fact that the graph is finite and connected and that the probability to stay in each vertex is positive guarantee that the two random walks will eventually meet). This completes the first stage. In the second stage, the two random walks move together by assigning the same fitnesses in the local neighborhoods for both copies. This guarantees a successful coupling for the local Bak–Sneppen model on or before the cover time of the random walk, starting from the vertex where the first stage ended.

Fix two states \( \eta \) and \( \eta' \). We will construct a coupling \((\Xi, \Xi')\), where each of the processes is a local Bak–Sneppen model on \( G \), and \( \Xi_0 = \eta \) and \( \Xi'_0 = \eta' \). To this end, for each \( t \geq 0 \) and \( v \in V \), let \( K(t, \nu)(\cdot) \) be a random vector of \( \text{Exp}(1) \) random variables, indexed by \( u \in A_v \). We will assume that all random variables \( K(\cdot, \cdot)(\cdot) \) are independent. Given \( \Xi_t = (X_t, \Gamma_t) \) and \( \Xi'_t = (X'_t, \Gamma'_t) \), we continue as follows. Let

\[ \Xi \text{ Springer} \]
\[ X_{t+1} = \arg\min_{u \in A_{X_t}} \Gamma_t(u) \]

\[ \Gamma_{t+1}(u) = \begin{cases} K(t + 1, X_{t+1})(u) & u \in A_{X_{t+1}} \\ \Gamma_t(u) & \text{otherwise.} \end{cases} \]

\[ X'_{t+1} = \arg\min_{u \in A_{X'_t}} \Gamma'_t(u) \]

\[ \Gamma'_{t+1}(u) = \begin{cases} K(t + 1, X'_{t+1})(u) & u \in A_{X'_{t+1}} \\ \Gamma'_t(u) & \text{otherwise.} \end{cases} \]

Then the resulting process \((\Xi, \Xi')\) is a coupling. Let \(P_{\eta, \eta'}\) denote the distribution of \((\Xi, \Xi')\) as constructed above. Observe that \((X, X')\) is a coupling of the irreducible and aperiodic random walk on \(G\), and by construction up to time \(\tau_{\text{coup}}(X, X')\), \(X\) and \(X'\) are independent. This implies that \(\tau_{\text{coup}}(X, X') < \infty\) a.s. and that \(\tau_{\text{coup}}(X, X')\) has a geometric tail. From the definition of \(\Xi, \Xi'\) it also follows that \(\Gamma_{\tau_{\text{coup}}(X, X')(u)}(u) = \Gamma'_{\tau_{\text{coup}}(X, X')(u)}(u)\) for all \(u \in A_{X_{\tau_{\text{coup}}(X, X')}}\), and, consequently, that \(\Gamma_{\tau_{\text{coup}}(X, X')+t}(u) = \Gamma'_{\tau_{\text{coup}}(X, X')+t}(u)\) for all \(u \in \cup_{s \leq t} A_{X_{\tau_{\text{coup}}(X, X')+s}}\).

Let

\[ \tilde{\sigma} = \inf\{t \in \mathbb{Z}_+ : \cup_{s \leq t} A_{X_{\tau_{\text{coup}}(X, X')+s}} = V\}. \]

It is clear that \(\tilde{\sigma}\) is stochastically dominated by the cover time of the random walk on \(G\) starting from \(X_{\tau_{\text{coup}}(X, X')}\). As a result of the finiteness of the graph, \(\tilde{\sigma} < \infty\) a.s. and it has a geometric tail. From the construction, \(\Xi_t = \Xi'_t\) for all \(t \geq \tau_{\text{coup}}(X, X') + \tilde{\sigma}\). By Aldous’ inequality we have

\[ \|P_{\eta}(\Xi_t \in \cdot) - P_{\eta'}(\Xi'_t \in \cdot)\|_{TV} \leq P_{\eta, \eta'}(\tau_{\text{coup}}(X, X') + \tilde{\sigma} > t) \leq c\beta^t, \quad t \geq 0, \]

for some constant \(c > 0\) and \(\beta \in (0, 1)\), whose dependence on \(\eta\) and \(\eta'\) is only through \((X_0, X'_0)\). Since the graph is finite, we may choose \(c\) and \(\beta\) so that the inequality holds for all \(\eta, \eta'\).

**Remark 2** The choice of an independent coupling is a generic recipe that works for any graph. It is easy to see from the proof that the independent coupling in the first stage can be replaced by any coupling satisfying all of the following three conditions:

1. Successful coupling for the random walks \(X, X'\) for any choice of \((X_0, X'_0) = (u, v), u, v \in V\).
2. For every \(t \in \mathbb{Z}_+\), \(\Gamma_{\tau_{\text{coup}}(X, X')+t}(u) = \Gamma'_{\tau_{\text{coup}}(X, X')+t}(u)\) for all \(u \in \cup_{s \leq t} A_{X_{\tau_{\text{coup}}(X, X')+s}}\).
3. For every \(t \in \mathbb{Z}_+\), the conditional distribution of \((X_{\tau_{\text{coup}}(X, X')+t+1}, X'_{\tau_{\text{coup}}(X, X')+t+1})\) on \(\{(X_s, X'_s) : s \leq \tau_{\text{coup}}(X, X') + t\}\)

is a function of \((X_{\tau_{\text{coup}}(X, X')+t}, X'_{\tau_{\text{coup}}(X, X')+t})\).

We now derive two-sided bounds on \(\bar{d}_t\) for a general graph that are somewhat more explicit than the upper bound of Theorem 1. The idea is to get an upper bound expressible as the tail of a sum of two independent random variables. The method works for any coupling satisfying the conditions of Remark 2. The last condition guarantees that conditioned on \(X_{\tau_{\text{coup}}(X, X')}\) (or \(X'_{\tau_{\text{coup}}(X, X')}, \text{which is the same}\), \(\tau_{\text{coup}}(X, X')\) and \(\tilde{\sigma}\) are independent. Nevertheless, if the graph is not vertex transitive, there is no guarantee that \(X_{\tau_{\text{coup}}(X, X')}\) and \(\tilde{\sigma}\) are independent. We wish to eliminate this in order to obtain an easier-to-handle upper bound, as the tail of
the sum of two independent random variables. The idea is not to modify the coupling, but instead to replace $\bar{\sigma}$ with a random variable which stochastically dominates $\bar{\sigma}$ conditioned on $X_{\text{coup}}(X, X')$, whatever the latter may be. To accomplish that, consider a random walk $X$ on $G$ starting from $v$, i.e., $X_0 = v$. Let

$$\tilde{\sigma}_v = \inf\{t \geq 0 : \cup_{s \leq t} AX_s = V\},$$

and denote by $F_v$ the distribution function of $\tilde{\sigma}_v$ and $F^\max = \min_{v \in V} F_v$. Let $\sigma^\max$ be a random variable with distribution function $F^\max$, independent of $\tau_{\text{coup}}(X, X')$. Then $\sigma^\max$ stochastically dominates $\tilde{\sigma}_v$ for every choice of $v$, and we obtain the following upper bound:

$$\tilde{d}_t \leq \sup_{u,v} P(\tau_{\text{coup}}(X, X') + \sigma^\max > t | (X_0, X'_0) = (u, v)).$$  \hspace{1cm} (4)

We turn to a lower bound. Let $x \in V$ and suppose now that $\eta = \Xi_0 = (x, \Gamma_0), \eta' = \Xi'_0 = (x, \Gamma'_0) \in \Omega$, with $\Gamma_0(v) = \Gamma'_0(v)$ for all $v \in A_x$ and $\Gamma_0(v) \neq \Gamma'_0(v)$ for all remaining vertices $v \in V - A_x$. We will make a more specific choice of $\eta, \eta'$. Let $\epsilon > 0$ and $\delta \in (0, 1)$. We will choose our initial state to satisfy $\Gamma_0(v) > 1$ for all $v \in A_x$, and $\Gamma'_0(v) < \delta$ for all $v \in V - A_x$. Let $C_{\delta,t}$ denote the event that all fitnesses sampled in the coupling up to time $t$ are $\geq \delta$. By choosing $\delta$ sufficiently small, we can guarantee that $P(C_{\delta,t}) > 1 - \epsilon$. With these initial states, our coupling gives $\tau_{\text{coup}}(X, X') = 0$, hence $\tau_{\text{coup}}(\Xi, \Xi') = \bar{\sigma}$. Therefore it follows that on the event $C_{\delta,t} \cap \{\tau_{\text{coup}}(\Xi, \Xi') > t\}$ all fitness in $\Gamma_t$ are $\geq \delta$, and there exists at least one $v \in V$ such that $\Gamma'_t(v) < \delta$. Now let $B_{\delta}$ be the event that all fitnesses are $\geq \delta$. Clearly,

$$\tilde{d}_t \geq (P_{\eta}(\Xi_{t} \in B_{\delta}) - P_{\eta}(\Xi'_{t} \in B_{\delta})$$

$$= E_{\eta,\eta'}[1_{B_{\delta}}(\Xi_{t}) - 1_{B_{\delta}}(\Xi'_{t}), \tau_{\text{coup}}(\Xi, \Xi') > t]$$

$$\geq E_{\eta,\eta'}[1_{B_{\delta}}(\Xi_{t}) - 1_{B_{\delta}}(\Xi'_{t}), C_{\delta,t} \cap \tau_{\text{coup}}(\Xi, \Xi') > t] - P(C^c_{\delta,t})$$

$$= P_{\eta,\eta'}(C_{\delta,t} \cap \tau_{\text{coup}}(\Xi, \Xi') > t) - P(C^c_{\delta,t})$$

$$\geq P(\tau_{\text{coup}}(\Xi, \Xi') > t) - 2P(C^c_{\delta,t})$$

$$\geq P_{\eta,\eta'}(\bar{\sigma} > t) - 2\epsilon = (*)$$

Let $F^\min = \max_{v \in V} F_v$ and $\sigma^\min$ a random variable with distribution function $F^\min$. Then, since the distribution of $\bar{\sigma}$ depends only on $x$, $\bar{\sigma}$ dominates $\sigma^\min$ for every choice of $x$, and therefore

$$(*) \geq P(\sigma^\min > t) - 2\epsilon.$$ 

Since $\epsilon$ is arbitrary, we conclude that

$$\tilde{d}_t \geq P(\sigma^\min > t).$$  \hspace{1cm} (5)

Clearly, the upper and the lower bounds in Expressions (4) and (5) are not tight. When $G$ is vertex transitive, then $F^\min = F^\max$, and if, in addition, we know that the exponential tail of $\tau_{\text{coup}}(X, X')$ is lighter than that of $\sigma^\min$, then it is easy to see that the bounds decay at the same exponential tail (a quick way to see this is through the moment generating function of $\sigma^\min$ and of $\tau_{\text{coup}}(X, X') + \sigma^\max$ which blow up at the same value).

2.1.1 Coupling for the N-Cycle

We will improve the lower bound for the case where $G$ is the $N$-cycle. We will assume $N \in 4\mathbb{N}$. Let $\eta = (0, \Gamma_0)$ and $\eta' = (N/2, \Gamma'_0)$ with $\Gamma_0(v) > \delta^{-1}$ for all $v$, and $\Gamma'_0(v) < \delta$
for all \( \delta \), for some \( \delta \in (0, 1) \). For \( v \in V \), let \( \tilde{v} \) denote its “reflection” about the “equator” \( \{N/4, 3N/4\} \) of \( v = (N/2 - v) \mod N \). We will further assume that the minima of \( \Gamma_0(\tilde{v}) \) and \( \Gamma_0(v) \) over \( v \in A_0 \) are attained at the same vertex. Unlike the generic coupling for \( X \) and \( X' \) we have used for a general graph, here we will consider a reflection coupling, which satisfies the conditions on Remark 2. Specifically, assuming that \( (\Xi_s, \Xi'_s) \) is defined for all \( s \leq t \), we define \( (\Xi_{s+1}, \Xi'_{s+1}) \) inductively. Our induction hypothesis is that if \( X'_s \neq X_t \), then \( X'_t = \tilde{X}_t \) and \( \Gamma'_t(\tilde{u}) = \Gamma_t(u) \) for \( u \in A_{X_t} \). Let \( X_{t+1} \) be the vertex minimizing \( \Gamma_t(u) \) among the three vertices in \( A_{X_t} \). Sample three IID Exp(1) random variables \( U_{-1}, U_0 \) and \( U_1 \), independent of the past, and let \( \Gamma_{t+1}(X_{t+1} + e \mod N) = U_e, \ e \in \{-1, 0, 1\} \).

1. If \( X'_t \neq X_t \), then by induction hypothesis, \( X'_t = \tilde{X}_t \), and the minima of \( \Gamma'_t(u) \) and \( \Gamma_t(u) \) over \( u \in A_{X_t} \) are attained at the same vertex. We then let \( X'_{t+1} \) be its reflection about the equator, that is \( X'_{t+1} = \tilde{X}_{t+1} \). As for \( \Gamma'_{t+1} \):
   
   (a) If \( X'_{t+1} \neq X_{t+1} \), let \( \Gamma'_{t+1}(\tilde{v}) = \Gamma_{t+1}(v) \) for \( v \in A_{X_{t+1}} \); otherwise
   
   (b) Let \( \Gamma'_{t+1}(v) = \Gamma_{t+1}(v) \) for all \( v \in A_{X_{t+1}} \).

2. If \( X'_t = X_t \), let \( X'_{t+1} = X_{t+1} \) and let \( \Gamma'_{t+1}(u) = \Gamma_{t+1}(u) \) for all \( u \in A_{X_{t+1}} \).

With this coupling, we have that \( \tau_{\text{coup}}(X, X') \) is the time \( X \) exits the upper semi-circle \( \{3N/4 + 1, \ldots, N - 1, 0, 1, \ldots, N/4 - 1\} \), which is equal in distribution to the exit time of the RW \( Z \) in Lemma 1 below from the interval \( \{1, \ldots, N/2 - 1\} \) when starting from \( N/4 \).

Let \( B_X \) denote the vertices \( u \) such that \( \Gamma_{\text{coup}}(X, X')(u) = \Gamma_0(u) \). Then by construction, \( \Gamma_{\text{coup}}(X, X')(u) \neq \Gamma_{\text{coup}}(X, X')(u) \) for all \( u \in B_X \), a.s. and, \( B_X \) a.s. contains all elements in the lower semicircle, \( \{N/2 + 1, \ldots, 3N/4 - 1\} \), except for the neighbor of \( X_{\text{coup}}(X, X') \) in the lower semicircle (this is either \( N/2 + 1 \) or \( 3N/4 - 1 \)). Define \( B_{X'} \) analogously. Then again, \( \Gamma_{\text{coup}}(X, X')(u) \neq \Gamma_{\text{coup}}(X, X')(u) \) for all \( u \in B_{X'} \), a.s. and, \( B_{X'} \) a.s. contains all elements of the upper semicircle \( \{3N/4 + 1, \ldots, 3N/4 - 1\} \) except the neighbor of \( X_{\text{coup}}(X, X') \) in the upper semicircle. Write \( u_P \) for \( X_{\text{coup}}(X, X') \) and \( \tilde{u}_P \) for the second element in the equator, \( \tilde{u}_P = N - u_P \). Thus \( B = B_X \cup B_{X'} \) contains \( V - (A_{u_P} \cup \{\tilde{u}_P\}) \), and clearly \( \Xi \) and \( \Xi' \) will be coupled only after all fitnesses in \( B \) are updated. In other words, letting

\[
\tilde{\sigma}_B = \inf\{t \geq 0 : B \subset \bigcup_{s \leq t} A_{X_{\text{coup}}(X, X') + 1}\} = \inf\{t \geq 0 : V - \{\tilde{u}_P\} \subset \bigcup_{s \leq t} A_{X_{\text{coup}}(X, X') + 1}\},
\]

then we showed that

\[
\tau_{\text{coup}}(\Xi, \Xi') \geq \tau_{\text{coup}}(X, X') + \tilde{\sigma}_B.
\]

Suppose that \( t < \tau_{\text{coup}}(X, X') + \tilde{\sigma}_B \). Then there exists an element \( v \) in the upper semicircle such that \( \tilde{v} \in B_X \) or \( v \in B_{X'} \), with the property that, respectively, \( \Gamma_t(\tilde{v}) = \Gamma_0(\tilde{v}) \) or \( \Gamma'_t(v) = \Gamma'_0(v) \). Let \( f_\pm \) be the indicator of the event that at least one of the fitnesses in the lower semicircle is larger than \( \frac{1}{\delta} \), and \( f_- \) be the indicator that at least one of the fitnesses in the upper semicircle is less than \( \delta \), and let \( f = f_+ - f_- \). Let \( f_{\delta, t} \) be the event that up to time \( t \) all fitnesses sampled are in \( [\delta, \frac{1}{\delta}] \). Observe that on the event \( \{t < \tau_{\text{coup}}(X, X') + \tilde{\sigma}_B\} \cap C_{\delta,t}, f(\Xi_t) - f(\Xi'_t) \geq 1 \). Indeed, since all sampled fitnesses are in \( [\delta, \frac{1}{\delta}] \), and all initial fitnesses for \( \Xi \) are \( > \frac{1}{\delta} \), \( f_- (\Xi_t) = 0 \), and similarly \( f_+ (\Xi'_t) = 0 \). Therefore, on this event, \( f(\Xi_t) - f(\Xi'_t) = f_+(\Xi_t) + f_-(\Xi'_t) \geq 1 \), because for some \( v \) in the lower semicircle the fitness \( \Gamma_t(v) = \Gamma_0(v) > \frac{1}{\delta} \), hence \( f_+(\Xi_t) = 1 \), or for some vertex \( v \) in the upper semicircle \( \Gamma'_t(v) = \Gamma'_0(v) < \delta \), hence \( f_-(\Xi'_t) = 1 \). Also, on the event
\[ \{ t \geq \tau_{coup} + \bar{\sigma}_B \} \cap C_{\delta,t}, \ f(\Xi_t) = f(\Xi'_t) = 0. \text{ Therefore,} \]
\[ 2\bar{d}_t \geq E_{\eta,\eta^t}[f(\Xi_t) - f(\Xi'_t)] \]
\[ \geq E_{\eta,\eta^t}[f(\Xi_t) - f(\Xi'_t), C_{\delta,t}] - 2P(C_{\delta,t}^c) \]
\[ = E_{\eta,\eta^t}[f(\Xi_t) - f(\Xi'_t), C_{\delta,t} \cap \{ t < \tau_{coup}(X,X') + \bar{\sigma}_B \}] - 2P(C_{\delta,t}^c) \]
\[ \geq P(\tau_{coup}(X,X') + \bar{\sigma}_B > t) - 3P(C_{\delta,t}^c). \]

The distribution of \( \tau_{coup}(X,X') + \bar{\sigma}_B \) is independent of the choice of \( \delta \), as it only depends on the path of the random walk \( X \). Furthermore, since \( G \) is vertex transitive, \( \tau_{coup}(X,X') \) and \( \bar{\sigma}_B \) are independent. Since \( P(C_{\delta,t}) \to 1 \) as \( \delta \to 0 \), we conclude with
\[ \bar{d}_t \geq \frac{1}{2} P(\tau_{coup}(X,X') + \bar{\sigma}_B > t). \]

As for the upper bounds, we observe that since the graph is vertex-transitive, for any coupling satisfying the conditions of Remark 2, \( \bar{\sigma} \) is independent of the history up to and including the coupling time. In addition, we have the following two lemmas:

**Lemma 1** Let \( Z \) be a random walk on \( \mathbb{Z} \) with \( P(Z_{t+1} - Z_t = \pm 1|Z_0, \ldots, Z_t) = \frac{1}{2} \), and \( P(Z_{t+1} = Z_t|Z_0, \ldots, Z_t) = \frac{1}{2} \). Fix \( M = 2, 3, \ldots \) and let \( I_M = \{1, \ldots, M\} \). Then the exit time of the random walk from \( I_M \) starting from \( x \in I_M \) is stochastically dominated by the exit time of the random walk starting from \( \lfloor M/2 \rfloor \) (equivalently \( \lceil M/2 \rceil \)).

The proof is in the “Appendix”.

**Lemma 2** Let \( N \in 4\mathbb{N} \). Fix two distinct points \( u, v \in \{0, \ldots, N - 1\} \). Then there exists a coupling \( (X, X') \) of the random walk on the \( N \)-cycle satisfying the conditions in Remark 2, such that \( X_0 = u, X' = v \), and \( \tau_{coup}(X,X') \) is stochastically dominated by the exit time of the random walk in Lemma 1 from the interval \( \{1, \ldots, N/2\} \) starting from \( N/4 \).

The proof is in the “Appendix”.

Note: in our proof we actually show that if \( u \) and \( v \) have the same parity, then the domination is by the exit time from the smaller interval \( \{1, \ldots, N/2 - 1\} \).

Thus,
\[ \bar{d}_t \leq \sup_{v,u} P(\tau_{coup}(X,X') + \bar{\sigma} > t|(X_0, X'_0) = (v, u), \]

and by independence of \( \bar{\sigma} \) from the history of \( (X, X') \) up to and including \( \tau_{coup}(X, X') \), and Lemma 2, we can replace \( \tau_{coup}(X, X') \) with a random variable independent of \( \bar{\sigma} \) and distributed as the exit time in the lemma. Thus, we have obtained the following two-sided bounds:

**Proposition 1** Consider the local Bak–Sneppen model on the \( N \)-cycle, with \( N \in 4\mathbb{N} \). Let \( X \) be a random walk on the \( N \)-cycle, starting from 0, and let \( \tau_1, \tau_2, \tau_3, \tau_4 \) be independent random variables with the following distributions:

- \( \tau_1 \) is equal in distribution to the exit time of \( Z \) in Lemma 1 from interval \( \{1, \ldots, N/2 - 1\} \) starting from \( N/4 \).
- \( \tau_2 \overset{\text{dist}}{=} \inf\{t \geq 0 : \cup_{s \leq t} A_{X_s} \supseteq \{0, \ldots, N - 1\} - \{N/2\} \} \).
- \( \tau_3 \) is equal in distribution to the exit time of \( Z \) in Lemma 1 from the interval \( \{1, \ldots, N/2\} \) starting from \( N/4 \).
Then $\tau_4 \overset{dist}{=} \inf\{t \geq 0 : \cup_{s \leq t} A_{X_s} = \{0, \ldots, N - 1\}\}$. 

\[
\frac{1}{2} P(\tau_1 + \tau_2 > t) \leq \tilde{d}_t \leq P(\tau_3 + \tau_4 > t).
\]

### 2.2 The Stationary Distribution

We start with some additional notation. If $Y_1, \ldots, Y_n$ are IID Exp(1), write $\text{Exp}^+(n)$ for the distribution of $Y_1$ conditioned that $Y_1$ is not the minimum of $(Y_1, \ldots, Y_n)$. A straightforward calculation shows that if $Y \sim \text{Exp}^+(n)$, then $Y$ has density

\[
\rho_n(t) = \frac{n}{n - 1} e^{-t} (1 - e^{-(n-1)t}) , \quad t > 0.
\]

Also, the minimum of $(Y_1, \ldots, Y_n)$ has distribution $\text{Exp}(n)$. Our main result is a description of the stationary distribution for the local Bak–Sneppen model on any finite, undirected and connected graph $G = (V, E)$.

**Theorem 2** Let $X_0$ be a $\mu$-distributed (as in Eq. 1) random variable, and for each $u \in V$, let $Z^u = (Z^u_t : t \in \mathbb{Z}_+)$ be a random walk on $V$, starting at $u$, independent of $X_0$. Next for each $u, v \in V$, let

\[
\tau_{u,v} = \inf\{t \in \mathbb{Z}_+ : Z^u_t \in A_v\},
\]

and partition $V$ into the sets

\[
V_i = \{v \in V : \tau_{X_0,v} = i\}.
\]

Conditioned on $X_0$ and $Z^{X_0}$, define a random vector $\Gamma_0$ indexed by elements of $V$ according to the following rules:

1. Given the $V_i$’s, the random vectors $((\Gamma_0(v) : v \in V_i) : V_i \neq \emptyset)$ are independent.
2. For each nonempty $V_i$, the random variables $(\Gamma_0(v) : v \in V_i)$ are IID with
   
   (a) $\text{Exp}(1)$-distribution if $i = 0$; and
   
   (b) $\text{Exp}^+(|A_{Z_{X_0}}|)$ otherwise.

Then $\Xi_0 = (X_0, \Gamma_0)$ is distributed according to the stationary distribution for $\Xi$.

**Proof** In order to prove the theorem, it is convenient to take time backwards. More precisely, we let $X_{-j} = Z^X_j$. Let $\hat{Z}_0 = X_1$ and let $\hat{Z}_t = Z^X_{t-1}$ for all $t \geq 1$. By reversibility, $\hat{Z}$ is a random walk on $G$ starting from initial distribution $\mu$ at time 0. Let $\tilde{\tau}_v = \inf\{t \in \mathbb{Z}_+ : \hat{Z}_t \in A_v\}$, and for $i \in \mathbb{Z}_+$, let $\check{V}_i = \{v \in V : \tilde{\tau}_v = i\}$. We observe that $\check{V}_0 = A_{X_1}$, and that by construction, $\check{V}_i = V_{i-1} - \check{V}_0$. Furthermore, the distribution of $(X_1, \hat{Z})$ and of $(X_0, Z^{X_0})$ is the same. Let $(f_\nu : v \in V)$ be bounded real-valued continuous functions on $\mathbb{R}$. We need to show that

\[
E \left[ \prod_{v \in V} f_\nu(\Gamma_1(v)) \right] = E \left[ \prod_{v \in V} f_\nu(\Gamma_0(v)) \right].
\]

By construction, this is the same as proving

\[
E \left[ \prod_{v \in V} f_\nu(\Gamma_1(v)) \right] = E \left[ \prod_{w \in V_0} E[f_w(\text{Exp}(1))] \times \prod_{i \geq 1} \prod_{w \in V_i} E[f_w(\text{Exp}^+(|A_{Z_{W_i}}|))] Z^{X_0} \right].
\]
We do this by decomposing the left-hand side according to the values taken by \( f_w(\Gamma_0(w)) \) on the sets \( \{\bar{V}_i, i \geq 0\} \). We have

\[
E \left[ \prod_{v \in V} f_v(\Gamma_1(v)) \right] = E \left[ \prod_{i \in \bar{V}_i} \prod_{v \in V} f_v(\Gamma_1(v)) \right]. \tag{6}
\]

Since, according to the process rules, \( f_w(\Gamma_1(w)) \) coincides with \( f_w(\Gamma_0(w)) \) on \( V_{i-1} - A_{X_1}, i \geq 1 \), we obtain

\[
E \left[ \prod_{i \in \bar{V}_i} \prod_{v \in V} f_v(\Gamma_1(v)) \right] = \sum_{u, v} E \left[ \prod_{w \in A_v} f_w(\Gamma_1(w)) \times \prod_{i \geq 1} \prod_{w \in V_{i-1} - A_v} f_w(\Gamma_0(w)); X_1 = v, X_0 = u \right].
\]

Now, on the event \( \{X_1 = v\} \), \( \Gamma_1(w) \) is \( \text{Exp}(1) \) for all \( w \in A_v \). Hence, the equation above is equal to

\[
\sum_{u, v} \left( \prod_{w \in A_v} E[f_w(\text{Exp}(1))] \times E \left[ \prod_{i \geq 1} \prod_{w \in V_{i-1} - A_v} f_w(\Gamma_0(w)); X_1 = v, X_0 = u|Z_{X_0} \right] \right)
\]

\[
= \sum_{v \in V} \left( \prod_{w \in A_v} E[f_w(\text{Exp}(1))] \times \sum_{u \in A_v} E \left[ \prod_{i \geq 1} \prod_{w \in V_{i-1} - A_v} f_w(\Gamma_0(w)); X_1 = v, X_0 = u|Z_{X_0} \right] \right).
\]

We now handle the conditional expectation above. Note that

\[
E \left[ \prod_{i \geq 1} \prod_{w \in V_{i-1} - A_v} f_w(\Gamma_0(w)); X_1 = v, X_0 = u|Z_{X_0} \right] = E \left[ \prod_{i \geq 1} \prod_{w \in A_{\gamma_{i-1}^w} - A_v} f_w(\Gamma_0(w)); \Gamma_0(v) = \min_{w \in A_u} \Gamma_0(w)|X_0 = u, Z_{X_0} \right] P(X_0 = u).
\]

Observe that, given \( X_0 \) and \( Z \), the fitnesses are independent random vectors. This implies that the above expression equals

\[
\mu(u) E \left[ \prod_{w \in A_u - A_v} f_w(U_w); U_v = \min_{w \in A_u} U_w \right] \times E \left[ \prod_{i \geq 2} \prod_{w \in A_{\gamma_{i-1}^w} - A_v} f_w(\text{Exp}^+(A_{\gamma_{i-1}^w})) \right].
\]
where \((U_w : w \in V)\) are IID \(\text{Exp}(1)\). But,

\[
E \left[ \prod_{w \in A_u - A_v} f_w(U_w); U_v = \min_{w \in A_u} U_w \right] = E \left[ \prod_{w \in A_u - A_v} f_w(U_w); U_v = \min_{w \in A_u} U_w \right] \frac{1}{|A_u|} \prod_{w \in A_u - A_v} E[f_w(\text{Exp}(|A_u|))].
\]

Putting all together we obtain

\[
E \prod_{i \geq 1} \prod_{w \in V_{i-1} - A_v} f_w(\Gamma_0(w)); X_1 = v, X_0 = u|Z^{X_0} = 1
\]

\[
= \frac{\mu(u)}{|A_u|} \prod_{w \in A_u - A_v} E[f_w(\text{Exp}^+(|A_u|))] \times \prod_{i \geq 2} \prod_{w \in A_{V_{i-1}}} E[f_w(\text{Exp}^+(A_{Z_{i-1}}))].
\]

Now we make use of the time reversed random walk \(\tilde{Z}\). Since the distribution of \(X_0\) given \(X_1 = v\) is equal to

\[
P(X_0 = u|X_1 = v) = P(X_1 = v|X_0 = u)P(X_0 = u)/P(X_1 = v) = \frac{\mu(u)}{|A_u|} \frac{1}{\mu(v)},
\]

summing over \(u \in A_v\), we obtain

\[
\sum_{u \in A_v} E \left[ \prod_{i \geq 1} \prod_{w \in V_{i-1} - A_v} f_w(\Gamma_0(w)); X_1 = v, X_0 = u|Z^{X_0} = 1 \right]
\]

\[
= \mu(v) \prod_{w \in V_1} E[f_w(\text{Exp}^+(|A_{\tilde{Z}_1}|))]|\tilde{Z}_0 = v] \times \prod_{i \geq 2} \prod_{w \in A_{\tilde{V}_i}} E[f_w(\text{Exp}^+(|A_{\tilde{Z}_i}|))]|\tilde{Z}_0 = v]
\]

\[
= \mu(v) \prod_{i \geq 1} \prod_{w \in V_i} E[f_w(\text{Exp}^+(|A_{\tilde{Z}_i}|))]|\tilde{Z}_0 = v].
\]

Plugging this into the righthand side of Eq. (6), we obtain

\[
E \left[ \prod_{v \in V} f_v(\Gamma_1(v)) \right] = E \left[ \prod_{w \in V_0} E[f_w(\text{Exp}(1))] \times \prod_{i \geq 1} \prod_{w \in V_i} E[f_w(\text{Exp}^+(|A_{\tilde{Z}_i}|))]|\tilde{Z}^{X_1} \right].
\]

Since the distribution of \(\tilde{Z}\) coincides with the distribution of \(Z\), the result follows. \(\square\)

Consider a graph \(G\) of constant degree \(d \geq 2\). Then \(\mu\) is uniform. Also, since \(|A_v| = d + 1\) for all \(v \in V\), the conditional distribution of \((\Gamma_0(v) : v \notin A_{X_0})\) on \(X_0\) and \(Z\), is IID \(\text{Exp}^+(d + 1)\). In other words,

**Corollary 3** Suppose that \(G\) is of constant degree \(d \geq 2\). That is \(|A_v| = d + 1\) for all \(v \in V\). Let \(X_0\) be uniformly distributed on \(V\), \((U_v : v \in V)\) IID \(\text{Exp}(1)\) and \((Z_v : v \in V)\) IID \(\text{Exp}^+(d + 1)\), both independent of \(X_0\) and each other. Set

\[
\Gamma_0(v) = \begin{cases} U_v & v \in A_{X_0} \\ Z_v & v \notin A_{X_0}. \end{cases}
\]

Then the distribution of \(Z_0 = (X_0, \Gamma_0)\) is stationary for the local Bak–Sneppen model on \(G\).
Corollary 4 If $G$ is of constant degree $d$, then under the stationary distribution

1. The fitness distribution at every site is the convex combination \( (1 - \frac{d + 1}{|V|}) \exp^+(d + 1) + \frac{d + 1}{|V|} \exp(1) \).

2. Suppose that $u, v \in V$ are such that $v \notin A_u$ (equivalently $u \notin A_v$). Then the joint distribution of $(\Gamma_0(u), \Gamma_0(v))$ is given by

\[
\frac{d + 1}{|V|} \exp(1) \otimes \exp^+(d + 1) + \frac{d + 1}{|V|} \exp^+(d + 1) \otimes \exp(1) + \left(1 - \frac{2(d + 1)}{|V|}\right) \exp^+(d + 1) \otimes \exp^+(d + 1)
\]

Finally, we obtain a formula for the fitness density in a general graph. To simplify notation, define

\[
\sigma_v = \inf\{t \in \mathbb{Z}_+: X_t \in A_v\}.
\]

We also write $P_u$ for the distribution of the random walk $X$, starting from $u \in V$. If $X_0 = u$, then the distribution of $(X_{\sigma_v}, \sigma_v)$ coincides with that of $(Z_{\tau_{u,v}}^v, \tau_{u,v})$ defined above. Also for $v \in V$, let

\[
\partial A_v = \{z \in A_v : A_z \cap A_v^c \neq \emptyset\}.
\]

We have the following formula for the stationary density at vertex $v$:

Proposition 2 Let $G = (V, E)$ be a finite, undirected and connected graph. The stationary distribution for $\Gamma_0(v), v \in V$, is equal to

\[
\sum_{u \in A_v} \frac{|A_u|}{S_G} \exp(1) + \sum_{u \notin A_v} \frac{|A_u|}{S_G} \sum_{z \in \partial A_v} P_u(X_{\sigma_v} = z) \exp^+(|A_z|).
\]

Proof Suppose that $\Xi_0$ has the stationary distribution as given in Theorem 2. We clearly have

\[
E[f(\Gamma_0(v))] = E[E[f(\Gamma_0(v))|X_0, Z]].
\]

Observe:

\[
E[f(\Gamma_0(v))]; X_0 \in A_v|X_0, Z] = E[f(\exp(1))|L_{A_v}(X_0)].
\]

Since for $u \in A_v$, $\mu(u) = \frac{|A_u|}{S_G}$, this explains the first summand in the expression. It remains to evaluate

\[
E[f(\Gamma_0(v)); X_0 \notin A_v|X_0, Z] = E[f(\Gamma_0(v)|X_0, Z)]\mathbb{1}_{A_v^c}(X_0).
\]

(7)

Suppose then that $u \notin A_v$. Then $v$ belongs to some $V_i, i > 1$, characterized by $i = \inf\{t \geq 0 : Z_t^u \in A_v\}$, and the (conditional) distribution of $\Gamma_0(v)$ is $\exp^+(|A_{Z_t^u}|)$. That is

\[
E[f(\Gamma_0(v))|X_0 = u, Z^u] = E[f(\exp^+(|A_{Z_{\tau_{u,v}}^v}|))|X_0 = u, Z^u].
\]

The distribution of $Z_{\tau_{u,v}}^v$ coincides with $P_u(X_{\sigma_v} \in \cdot)$. Since $u \in A_v^c$, and $X_{\sigma_v} \in A_v$, it follows that $X_{\sigma_v} \in \{z \in A_v : A_z \cap A_v^c \neq \emptyset\} = \partial A_v$. Taking expectation with respect to $Z^u$ gives

\[
E[E[f(\exp^+(|A_{Z_{\tau_{u,v}}^v}|))|X_0 = u, Z^u)] = \sum_{z \in \partial A_v} P_u(X_{\tau_{u,v}} = z) E[f(\exp^+(|A_z|))].
\]
Using this in Eq. (7) gives
\[
E[f(\Gamma_0(v)); X_0 \not\in A_v|X_0, Z)] = \sum_{u \notin A_v} \mu(u) E[f(\Gamma_0(v)); |X_0 = u, Z^u)]
\]
\[
= \sum_{u \notin A_v} \frac{|A_u|}{S_G} \sum_{z \in \partial A_v} P_u(X_{\tau_v} = z) E[f(\text{Exp}^+|A_z))].
\]

\[\square\]

2.3 The \(\alpha\)-Avalanches

We now study another aspect of the model, namely its avalanches. Here, the definition of an avalanche is relaxed in the following way: given thresholds \(b > 0\) and \(\alpha \in [0, 1)\), we define an \(\alpha\)-avalanche from threshold \(b\) as a portion of path of the process since the proportion of the vertices whose fitness is at least \(b\) exceed \(\alpha\) until, but not including, the next time this happens. The reason for introducing the \(\alpha\)-avalanches is because the local Bak–Sneppen is “slow” in replacing low fitnesses as it follows a random walk on the graph, and so in typical conditions, the expected duration of avalanches \((\alpha = 1)\) from any threshold \(b > 0\) will tend to infinity as the graph becomes larger. Note also that the distribution of the duration of an \(\alpha\)-avalanche depends on the initial configuration. Yet, the ergodicity of the local Bak–Sneppen model allows to replace the expectation by the limit of time averages of the durations along the sequence of consecutive \(\alpha\)-avalanches, a limit which exists a.s. and is equal to a deterministic constant.

Fix a finite connected graph \(G = (V, E)\) and consider the local Bak–Sneppen model on \(G\). For a threshold \(b > 0\) and time \(t \in \mathbb{Z}_+\), let
\[
\Psi_t(b) = \frac{\sum_{v \in V} \mathbb{1}_{[b, \infty)}(\Gamma_t(v))}{|V|}
\]
denote the proportion of vertices with fitness \(\geq b\) at time \(t\). Let \(T_0 = 0\), and continue inductively by letting
\[
T_{n+1} = \inf \{t > T_n : \Psi_t(b) \geq \alpha\}.
\]
The sequence \(\{T_n\}_{n \in \mathbb{N}}\) consists of all times when the proportion of vertices with fitness greater or equal than \(b\) is at least \(\alpha\), and so, represent the sequence of times where \(\alpha\)-avalanches begin, with the possible exception of an \(\alpha\)-avalanche starting at time \(T_0 = 0\). Observe that if \(t > 0\), then \(t = T_n\) for some \(n\) if and only if \(\Psi_t(b) \geq \alpha\). Let \(N_t\) count the number of \(\alpha\)-avalanches from threshold \(b\) completed by time \(t\). Then for any state \(\eta\), it follows from Corollary 2 that
\[
\lim_{t \to \infty} \frac{N_t}{t} = \pi(A_{\alpha,b}),
\]
in \(P_\eta\)-probability, where \(A_{\alpha,b}\) is the set of all state \(\gamma\) such that \(\Psi_0(b) \geq \alpha\) when \(X_0 = \gamma\). That is, all states exhibiting at least a proportion \(\alpha\) of the vertices with fitness \(\geq b\). Letting \(t = T_n\), we have the following immediate corollary from Theorem 1:

Corollary 5 Let \(G = (V, E)\) be any finite connected graph and set \(D(\alpha, b) = \frac{1}{\pi(A_{\alpha,b})}\). Then for any state \(\eta\),
\[
\lim_{m \to \infty} \frac{T_m}{m} = D(\alpha, b),
\]
in \(P_\eta\)-probability.
We now study the limiting behavior of the $\alpha$-avalanches. In what follows we assume that $d \geq 2$ and that $(G_n = (V_n, E_n) : n \in \mathbb{N})$ is a sequence of finite connected $d$-regular graphs, satisfying $V_1 \subseteq V_2 \subseteq \cdots$ and $\lim_{n \to \infty} |V_n| = \infty$. Write $\pi_n$ for the stationary distribution for the local Bak–Sneppen model on $G_n$ and $D_n = \frac{1}{\pi_n(A_{\alpha,b})}$.

Now let

$$p_{d,b} \equiv P(\text{Exp}^+ (d+1) \geq b) = \int_b^\infty \rho_{d+1}(t)dt = \frac{d+1}{d} \left( e^{-b} - \frac{1}{d+1} e^{-(d+1)b} \right).$$

We have the following:

**Proposition 3**  Fix $\alpha \in (0, 1)$. Let $b_c \in (0, \infty)$ be the unique solution to

$$\alpha = p_{d,b}.$$  

Then

1. If $b < b_c$, then for some $\rho_1 > 0$, $|D_n(\alpha, b) - 1| \leq e^{-\rho_1 n}.$
2. if $b = b_c$, then for some $c > 0$, $|D_n(\alpha, b) - 2| \leq \frac{c}{\sqrt{n}}$.
3. If $b > b_c$, then for some $\rho_2 > 0$, $D_n(\alpha, b) \geq e^{\rho_2 n}$.

**Proof**  Letting $m = |\{V_n|\}$, the event $A_{\alpha,b}$ coincides with the event that at least $m$ of the $|V_n|$ sites have fitness above $b$. Conditioning on $X_0$ we obtain

$$\pi_n(A_{\alpha,b}) = \sum_{k=1}^{|V_n|} \pi_n(A_{\alpha,b}|X_0 = v_k)\pi_n(X_0 = v_k),$$

(8)

where $v_1, v_2, \ldots, v_{|V_n|}$ is some fixed ordering of the elements of $V_n$.

Given $X_0$, the event $A_{\alpha,b}$ contains the event that at least $m$ among the $|V_n|-(d+1)$ vertices, not neighbors of $X_0$, have fitness above $b$. Furthermore, given $X_0$, it follows from Corollary 3 that, under the stationary distribution $\pi_n$, the fitness of vertices outside the neighborhood of $X_0$ are IID $\text{Exp}^+(d+1)$ random variables. Now given $v \in V_n$, and $u \in V_n - A_v$, we have

$$p_{d,b} = \pi_n(\Gamma_0(u) \geq b|X_0 = v).$$

Hence, if $I_1, I_2, \ldots$ are IID Ber($p_{d,b}$), and $S_n = I_1 + \cdots + I_n$, then from Eq. (8) we obtain

$$\pi_n(A_{\alpha,b}) \geq \sum_{k=1}^{|V_n|} P(S_{|V_n|-(d+1)} \geq m)\pi_n(X_0 = v_k).$$

Since $G_n$ is $d$-regular, $\pi_n(X_0 = v_k)$ is uniform on the vertices of $G_n$. Therefore

$$\pi_n(A_{\alpha,b}) \geq P(S_{|V_n|-(d+1)} \geq m).$$

On the other hand, again conditioning on $X_0$, the event $A_{\alpha,b}$ is contained in the event that at least $m - (d+1)$ vertices in $V_n - A_{X_0}$ have fitness above $b$, and so we have

$$\pi_n(A_{\alpha,b}) \leq P(S_{|V_n|-(d+1)} \geq m - (d + 1)).$$

It follows from the Law of Large Numbers and from the Central Limit Theorem that

$$\lim_{n \to \infty} \pi_n(A_{\alpha,b}) = \begin{cases} 1 & \alpha < p_{d,b}, \\ \frac{1}{2} & \alpha = p_{d,b}, \\ 0 & \alpha > p_{d,b}. \end{cases}$$
Furthermore, from large deviations we know that the convergence in the first and last cases occurs at an exponential rate, while for the case $\alpha = p_{d,b}$, Berry–Essen theorem (see [9] for example) guarantees that

$$ \left| \pi_n(A_{\alpha,b}) - \frac{1}{2} \right| = O(n^{-1/2}). $$

Finally, observe that $b \to p_{d,b}$ is a continuous decreasing function, with $p_{d,0} = 1$ and $\lim_{b \to \infty} p_{d,b} = 0$. Therefore for any given $\alpha \in (0, 1)$, there exists a unique $b_c > 0$, such that $p_{d,b_c} = \alpha$, $\alpha < p_{d,b}$ if $b < b_c$ and $\alpha > p_{d,b}$ if $b > b_c$. This completes the proof. 

The next result establishes the limiting fitness distribution in the local Bak–Sneppen model for $d$-regular graphs.

**Proposition 4** The limit $\lim_{n \to \infty} \pi_n$ exists and is equal to the distribution of IID Exp$^+$($d+1$)-distributed random variables labeled by $V_\infty = \bigcup_n V_n$, and where the convergence is in the weak topology on probability measures on the product space $[0, \infty)^{V_\infty}$.

**Proof** let $v_1, v_2, \ldots, v_m \in V_\infty = \bigcup_n V_n$ be fixed, and let $(f_i : i \leq m)$ be bounded real-valued continuous functions on $[0, \infty)$. We run the local Bak–Sneppen model on $G_n$ from its stationary distribution $\pi_n$. Then $X_0$ is uniformly distributed on $V_n$. This gives (for some constant $K > 0$),

$$ \left| E \left[ \prod_{i=1}^m f_i(\Gamma_0(v_i)) \right] - E \left[ \prod_{i=1}^m f_i(\Gamma_0(v_i)); X_0 \notin \{v_1, \ldots, v_m\} \right] \right| 
\leq KP(X_0 \in \{v_1, \ldots, v_m\}) = \frac{m}{|V_n|} \to 0, $$

as $n$ goes to infinity. However,

$$ E \left[ \prod_{i=1}^m f_i(\Gamma_0(v_i)); X_0 \notin \{v_1, \ldots, v_m\} \right] 
= E \left[ \prod_{i=1}^m f_i(\Gamma_0(v_i)) | X_0 \notin \{v_1, \ldots, v_m\} \right] \left( 1 - \frac{m}{|V_n|} \right) 
\geq \prod_{i=1}^m E[f_i(\text{Exp}^+(d+1))](1 + o(1)). $$

This proves convergence of finite dimensional distributions. Finally, tightness is obtained by observing that if $V_\infty = \{1, 2, \ldots\}$, and since Exp$^+(d+1)$ stochastically dominates Exp$^+(d+1)$, then

$$ P(\cap_{i=1}^\infty [\Gamma_0(i) \in [0, N + i]]) 
\geq \prod_{i=1}^\infty (1 - P(\text{Exp}^+(d+1) > N + i)) 
\geq \prod_{i=1}^\infty \left( 1 - \frac{d+1}{d} e^{-(N+i)} \right) $$

for all $N \in \mathbb{N}$. Although the lefthand side depends on $\pi_n$, the righthand side does not. By dominated convergence ($N = 0$), the righthand side converges to 1 as $N \to \infty$. This completes the proof. 

$\Box$
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Appendix

**Proof of Corollary 1** First we show that \( \Sigma \) has a unique stationary measure. Fix \( \eta \in \Omega \). Then for every event \( A \), \(( P_\eta(\Sigma_t \in A) : t \in \mathbb{Z}_+ )\) is a Cauchy sequence since

\[
|P_\eta(\Sigma_t \in A) - P_\eta(\Sigma_{t+s} \in A)| = E_\eta \left[ |P_\eta(\Sigma_t \in A) - P_\Sigma_s(\Sigma_t \in \Omega)| \right] \leq \bar{d}(t) \leq c \beta^t. \tag{9}
\]

(9)

As a result, for each \( A \),

\[
\pi_\eta(\Sigma_t \in \Omega) = \lim_{t \to \infty} P_\eta(\Sigma_t \in A) \exists \pi_\eta(\Sigma_t \in \Omega) = 1, \text{ and that } \pi_\eta \text{ is monotone with respect to inclusion. Assume } A_1, A_2, \ldots \text{ is a sequence of disjoint events. Fix } \epsilon > 0, \text{ and let } t \text{ be large enough so that } c \beta^t < \epsilon. \text{ Then}
\]

\[
|\pi_\eta(\bigcup_{j=1}^\infty A_j) - P_\eta(\Sigma_t \in \bigcup_{j=1}^\infty A_j)| \leq c \beta^t < \epsilon.
\]

(9)

There exists \( N \) such that \( P_\eta(\Sigma_t \in \bigcup_{j=n}^\infty A_j) < \epsilon \) whenever \( n \geq N \), and it follows from Eq. (9) that \( P_\eta(\Sigma_{t+s} \in \bigcup_{j=n}^\infty A_j) < \epsilon + c \beta^t < 2 \epsilon \) for all \( n \geq N \) and \( s \geq 0 \). This gives

\[
\pi_\eta(\bigcup_{j=1}^\infty A_j) \leq P_\eta(\Sigma_t \in \bigcup_{j=1}^\infty A_j) + \epsilon = \sum_{j=1}^n P_\eta(\Sigma_t \in A_j) + P_\eta(\Sigma_t \in \bigcup_{j=n}^\infty A_j) + \epsilon \\
\leq \sum_{j=1}^n P_\eta(\Sigma_t \in A_j) + 3 \epsilon \\
\rightarrow t \to \infty \sum_{j=1}^\infty \pi_\eta(A_j) + 3 \epsilon \\
\leq \sum_{j=1}^\infty \pi_\eta(A_j) + 3 \epsilon.
\]

On the other hand, it immediately follows from Fatou’s lemma that

\[
\pi_\eta(\bigcup_{j=1}^\infty A_j) \geq \sum_{j=1}^\infty \pi_\eta(A_j),
\]

implying that \( \pi_\eta \) is indeed a probability measure.

Next we prove that \( \pi_\eta \) is a stationary measure for \( \Sigma \). From its definition and bounded convergence, \( P_\pi_\eta(\Sigma_t \in A) = \lim_{t \to \infty} E_\eta P_\Sigma_t(\Sigma_t \in A) = \lim_{t \to \infty} P_\eta(\Sigma_{t+s} \in A) = \pi_\eta(A) \).

The final step is to show uniqueness. Observe that, for \( \eta, \eta' \in \Omega \),

\[
|\pi_\eta(A) - \pi_\eta'(A)| \leq |P_\eta(\Sigma_t \in A) - \pi_\eta(A)| + |P_\eta(\Sigma_t \in A) - \pi_\eta'(\Sigma_t \in A)| \\
+ |P_\eta'(\Sigma_t \in A) - \pi_\eta'(A)|.
\]

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Hence $|\pi_\eta(A) - \pi_{\eta'}(A)| \to 0$ when $t$ goes to infinity and $\pi_\eta = \pi_{\eta'}$. Now if $\pi$ is a stationary measure for $\Xi$, then for $s, t \geq 0$, we have
\[
\pi(A) = P_\pi(A) = P_\pi[\pi_{\Xi_t}(\Xi_t \in A)] \to_{t \to \infty} P_\pi[\pi_{\Xi}(A)],
\]
and the result follows because $\pi_{\Xi_s}$ is independent of $\Xi_s$. □

**Proof of Corollary 2** Without loss of generality, assume $\int f d\pi = 0$. Let $S_n = \sum_{t=0}^{n-1} f(\Xi_t)$. We need to prove $S_n/n \to 0$ in $P_\eta$-probability for every $\eta$. This will follow if we show $E_\eta[|S_n^2/n^2|] \to 0$ as $n \to \infty$. Now
\[
E_\eta[S_n^2] = \sum_{t=0}^{n-1} E_\eta[f^2(\Xi_t)] + 2 \sum_{t=0}^{n-1} \sum_{s=1}^{n-1-t} E_\eta[f(\Xi_t)f(\Xi_{t+s})].
\]
Observe that $E_\eta[f(\Xi_t)f(\Xi_{t+s})] = E_\eta[f(\Xi_t)]E_\eta[f(\Xi_s)]$. By Theorem 1 and Corollary 1
\[
|E_\eta[f(\Xi_t)f(\Xi_s)]| \leq cE_\eta[|f(\Xi_t)|]|f\|_\infty \beta^s = c\|f\|_\infty \beta^s.
\]
Therefore,
\[
\sum_{t=0}^{n-1} \sum_{s=1}^{n-1-t} E_\eta[f(\Xi_t)f(\Xi_{t+s})] \leq \sum_{t=0}^{n-1} \sum_{s=1}^{n-1-t} c\|f\|_\infty \beta^s \leq \frac{cn}{1 - \beta}\|f\|_\infty.
\]
Hence $E[|S_n^2/n^2|] = O(n^{-1})$, and the result follows. □

**Proof of Lemma 1** Without loss of generality, assume $x \in [M/2]$. We construct a coupling for the random walks $(Z, Z')$ so that $Z_0 = [M/2]$ and $Z'_0 = x$, and which is such that $Z'$ will exit $I_M$ before $Z$. Suppose we have $Z$ defined. We define $Z'$ to be a reflection of $Z$ (namely when $Z$ goes to the right, $Z'$ goes to the left, when $Z$ goes to the left, $Z'$ goes to the right, and when $Z$ stays put, so is $Z'$), until either:

1. $Z'$ exits the interval $I_M$ from the left by hitting 0, at which time $Z$ will be in $[M/2] + x \leq M$; or
2. (a) If $x, [M/2]$ have the same parity: $Z$ and $Z'$ meet at $(x + [M/2])/2$. From this time the two copies coalesce.
   (b) If $x, [M/2]$ have different parity: $Z'$ is at $(x + [M/2])/2 - 0.5$ and $Z$ is at $(x + [M/2])/2 + 0.5$. At this time we define $Z'_{t+1}$ as follows: If $Z$ goes to the left, $Z'$ stays put (coupling), if $Z$ stays put $Z'$ goes to the right (coupling), and if $Z$ goes to the right, $Z'$ goes to the left (making distance even). In the former two cases the copies coalesce, and in the latter case, we restart the reflection coupling with the copies having the same parity.

With this coupling scheme, we guarantee that $Z'$ will exit the interval $I_M$ before or at the same time $Z$ will exit it, and the proof is completed. □

**Proof of Lemma 2** Without loss of generality we can assume $u = 0, v \in \{1, \ldots, N/2 - 1\}$. We continue as follows. Let $X$ be a copy of the random walk starting from $X_0 = 0$, let $X'$ be another copy starting from $X'_0 = v$, and let $X''$ be another copy starting from $X''_0 = N/2$. We will assume that both $X'$ and $X''$ are reflections of $X$. Clearly $\tau_{\text{coup}}(X, X'')$ is equal to the time $X''$ hits $N/4$ or $3N/4$, which is the same as the exit time of a random walk starting at $N/4$ from $\{1, N/2 - 1\}$. " Springer
Assume that $v$ is even. In this case, $\tau_{\text{coup}}(X, X')$ coincides with the first time $X'$ hits $v/2$ or $v + (N - v)/2 = N/2 + v/2$. This time has the same distribution as the exit time of the random walk from $\{1, \ldots, N/2 - 1\}$, starting from $v/2$.

By Lemma 2, $\tau_{\text{coup}}(X, X'')$ stochastically dominates $\tau_{\text{coup}}(X, X')$. This proves the lemma when $u$ and $v$ have the same parity.

If $v$ is odd, then in order to guarantee meeting with the reflection coupling, we must modify the rule for $X'$ when the two copies are neighbors. We do this as follows:

- When $X$ moves to where $X'$ is, then $X'$ will stay put;
- When $X$ stays put, $X'$ will move towards where $X$ is;
- Otherwise both will move away from where the other was.

Therefore the coupling time will be on or before $X'$ hits $\lfloor v/2 \rfloor$ or $N/2 + \lceil v/2 \rceil$, that is the exit time of the random walk starting from $\lceil v/2 \rceil$ from the interval $\{1, \ldots, N/2\}$, and, in light of Lemma 2, the result is proven. \qed

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