PARAMETER ESTIMATION OF MCKEAN-VLASOV STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article, we study a class of McKean-Vlasov stochastic differential equations with unknown parameters. First, we prove the existence and uniqueness of these equations under non-Lipschitz conditions. Second, we construct maximum likelihood estimators of these parameters and then discuss their properties. Third, a numerical simulation method for the class of McKean-Vlasov stochastic differential equations is offered. Moreover, we estimate the errors between solutions of these equations and that of their numerical equations. Finally, we give an example to explain our result.

1. Introduction

McKean-Vlasov stochastic differential equations (MVSDEs in short) are a kind of special stochastic differential equations whose coefficients depend on probability distributions of their solutions. They were first initiated by Henry P. McKean [9] in 1966, and then were gradually studied by a lot of researchers. At present, there have been many results about MVSDEs, such as the well-posedness of the solutions in [5, 6], the stability of strong solutions in [7], the well-posedness of the mild solutions and their Euler approximation in infinite dimension Hilbert spaces in [10], and the particle approximations method in [3].

As the research of MVSDEs develops, the fields of their application are becoming larger and larger. This leads to new problems. Estimation of unknown parameters in MVSDEs is one of these problems. Now, there are many results about parameter estimation of stochastic differential equations. Let us mention some works. Liptser and Shiryaev [8] considered the maximum likelihood estimation of Itô diffusions under continuous observations, while Yoshida [14] estimated these diffusion processes with the maximum likelihood estimation based on discrete diffusions. In [2], Bishwal obtained the exponential bound of the large deviation rate for the maximum likelihood estimator of the drift coefficients. Other methods of parameter estimation like martingale function estimators, nonparametric methods can be found in [12, 11].

However, because of the distributions in the drift coefficients and diffusion coefficients, the previous methods and results may not well be applied to MVSDEs. In [11], Ren and...
Wu proposed the least squares estimators for a class of path-dependent MVSDEs. Wen et al. \[13\] discussed the maximum likelihood estimators on MVSDEs with the following form assuming that $\vartheta \in \mathbb{R}$ is known and $\sigma = 1$,

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} b(\vartheta, X_s, y) \mu_s(dy) dt + \int_0^t \int_{\mathbb{R}} \sigma(\vartheta, X_s, y) \mu_s(dy) dW_s, \quad X_0 = x_0 \in \mathbb{R},$$

where $\theta$ is a unknown parameter and $\mu_t$ is the probability distribution of $X_t$.

In this paper, we focus on the following MVSDE in a more general form

$$dX_t = b(\theta, X_t \wedge \cdot, \mu_t) dt + \sigma(X_t \wedge \cdot, \mu_t) dW_t, \quad X_0 = \xi,$$

(1)

where $\xi$ is a random vector. We not only construct a maximum likelihood estimator $\hat{\theta}$ for $\theta$ but also prove the consistency of $\hat{\theta}$. And then we discretize Eq.(1) and also obtain the numerical simulation of $\hat{\theta}$.

The rest of the paper is organized as follows. In Section 2, we prove the existence and uniqueness of strong solutions for Eq.(1) under non-Lipschitz conditions. The maximum likelihood estimators are constructed in Section 3. In Section 4, a numerical equation of Eq.(1) is given by interacting particles and the Euler method, and then the error between the MVSDE and its approximation is calculated, followed by giving a maximum likelihood estimator of the numerical equation. Finally, in Section 5, we apply the method to a specific equation as an example, and explain our results.

The following convention will be used throughout the paper: $C$ with or without indices will denote different positive constants whose values may change from one place to another.

2. THE EXISTENCE AND UNIQUENESS OF MVSDEs

In the section, we prove the existence and uniqueness of the solutions for Eq.(1).

Fix $T > 0$. Let $C_T^d$ be the collection of all the continuous functions from $[0, T]$ to $\mathbb{R}^d$. And then we equip it with the compact uniform convergence topology. Let $\mathcal{B}_T^d$ be the $\sigma$-field generated by the topology. For $w \in C_T^d$, set

$$\|w\|_T := \sup_{0 \leq t \leq T} |w(t)|.$$

Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel $\sigma$-field on $\mathbb{R}^d$. Let $\mathcal{P}_2(\mathbb{R}^d)$ denote the space of probability measures on $\mathcal{B}(\mathbb{R}^d)$ with finite second moments. That is, if $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, then

$$\|\mu\|_{\mathcal{M}_2}^2 := \int_{\mathbb{R}^d} (1 + |x|)^2 \mu(dx) < \infty.$$

And the distance of $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ is defined as

$$\mathbb{W}^2_2(\mu, \nu) := \inf_{\pi \in \mathcal{E}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy),$$

where $\mathcal{E}(\mu_1, \mu_2)$ denotes the set of all the probability measures whose marginal distributions are $\mu_1$ and $\mu_2$, respectively. Thus, $(\mathcal{P}_2(\mathbb{R}^d), \mathbb{W}_2)$ is a Polish space.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a complete filtered probability space and $\{W_t, t \geq 0\}$ be a $m$-dimensional standard Brownian motion. Consider the following MVSDE on $\mathbb{R}^d$:

$$\begin{cases} X_t = \xi + \int_0^t b(\theta, X_s \wedge \cdot, \mu_s) ds + \int_0^t \sigma(X_s \wedge \cdot, \mu_s) dW_s, \\
\mu_s = \text{the probability distribution of } X_s, \end{cases}$$

(2)
where $\xi$ is a $\mathcal{F}_0$-measurable random vector, $\theta \in \Theta \subset \mathbb{R}^k$ is a unknown parameter, $b : \Theta \times C^d_T \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : C^d_T \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times m}$ are Borel measurable. We assume:

(H$_1$) There exists a nonnegative constant $K_1$ such that for any $w, v \in C^d_T$, $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

(i) $|b(\theta, w, \mu) - b(\theta, v, \nu)|^2 + \|\sigma(w, \mu) - \sigma(v, \nu)\|^2 \leq K_1 \left( \kappa_1(\|w - v\|_T^2) + \kappa_2\left( \frac{W^2_2(\mu, \nu)}{\kappa_2(x)} \right) \right)$,

where $\| \cdot \|$ denotes the Hilbert-Schmidt norm of a matrix, and $\kappa_i(x), i = 1, 2$ are two positive, strictly increasing, continuous concave function and satisfy $\kappa_i(0) = 0$, $\int_0^{\infty} \frac{1}{\kappa_1(x) + \kappa_2(x)} dx = \infty$;

(ii) $|b(\theta, w, \mu)|^2 + \|\sigma(w, \mu)\|^2 \leq K_1 \left( 1 + \|w\|_T^2 + \|\mu\|_2^2 \right)$.

**Theorem 2.1.** Suppose that (H$_1$) holds and $\mathbb{E}[|\xi|^2] < \infty$. Then Eq.(2) has a unique strong solution $X$ and

$$\mathbb{E}\left( \sup_{0 \leq t \leq T} |X(t)|^2 \right) < \infty.$$  

**Proof.** First of all, set

$$\begin{align*}
X_t^{(0)} &= \xi, \\
X_t^{(n+1)} &= \xi + \int_0^t b(\theta, X_s^{(n)}, \mu_s^{(n)}) ds + \int_0^t \sigma(X_s^{(n)}, \mu_s^{(n)}) dW_s, \quad n \in \mathbb{N} \cup \{0\},
\end{align*}$$

(3)

where $\mu_s^{(n)}$ is the probability distribution of $X_s^{(n)}$. We make use of Eq.(3) to prove the well-posedness of Eq.(2).

**Step 1.** We prove that the definition of Eq.(3) is reasonable.

For $n = 0$, $\mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t^{(0)}|^2 \right) = \mathbb{E}[|\xi|^2] < \infty$. Assume that for $n \in \mathbb{N}$,

$$\mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t^{(n)}|^2 \right) < \infty.$$  

And then by the Hölder inequality, the Burkholder-Davis-Gundy inequality and (H$_1$), we get that

$$\begin{align*}
\mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t^{(n+1)}|^2 \right) &\leq 3\mathbb{E}[|\xi|^2] + 3\mathbb{E}\left( \sup_{0 \leq t \leq T} \left| \int_0^t b(\theta, X_s^{(n)}, \mu_s^{(n)}) ds \right|^2 \right) + 3\mathbb{E}\left( \sup_{0 \leq t \leq T} \left| \int_0^t \sigma(X_s^{(n)}, \mu_s^{(n)}) dW_s \right|^2 \right) \\
&\leq 3\mathbb{E}[|\xi|^2] + 3T \mathbb{E}\int_0^T \left| b(\theta, X_s^{(n)}, \mu_s^{(n)}) \right|^2 ds + 3C \mathbb{E}\int_0^T \|\sigma(X_s^{(n)}, \mu_s^{(n)})\|^2 ds \\
&\leq 3\mathbb{E}[|\xi|^2] + 3(T + C)K_1 \mathbb{E}\int_0^T \left( 1 + \|X_s^{(n)}\|_T^2 + \|\mu_s^{(n)}\|_2^2 \right) ds \\
&\leq 3\mathbb{E}[|\xi|^2] + 9(T + C)K_1 T \left( 1 + \mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t^{(n)}|^2 \right) \right),
\end{align*}$$

(4)
where the last inequality is based on the fact that \( \|\mu_s^{(n)}\|_2^2 \leq \mathbb{E}(1 + |X_s^{(n)}|)^2 \leq 2\mathbb{E}(1 + |X_s^{(n)}|^2) \). From induction on \( n \), it follows that
\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t^{(n)}|^2 \right) < \infty, \quad n \in \mathbb{N} \cup \{0\}.
\]

**Step 2.** We prove the existence of the solutions to Eq. (2).

By the same deduction to that of (4), it holds that for \( m, n \in \mathbb{N} \)
\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t^{(n+1)} - X_t^{(m+1)}|^2 \right) \leq 2T\mathbb{E} \int_0^T \|b(\theta, X_{s\wedge}, \mu_s^{(n)}) - b(\theta, X_{s\wedge}, \mu_s^{(m)})\|^2 ds + 2C\mathbb{E} \int_0^T \|\sigma(X_{s\wedge}, \mu_s^{(n)}) - \sigma(X_{s\wedge}, \mu_s^{(m)})\|^2 ds
\leq 2(T + C)K_1 \mathbb{E} \int_0^T \left( \kappa_1(\|X_s^{(n)} - X_s^{(m)}\|_T^2) + \kappa_2(\mathbb{W}_2^2(\mu_s^{(n)}, \mu_s^{(m)})) \right) ds
\leq 2(T + C)K_1 \int_0^T \left[ \kappa_1 \left( \mathbb{E} \left( \sup_{0 \leq u \leq s} |X_u^{(n)} - X_u^{(m)}|^2 \right) \right) + \kappa_2 \left( \mathbb{E} \left( \sup_{0 \leq u \leq s} |X_u^{(n)} - X_u^{(m)}|^2 \right) \right) \right] ds,
\]

where the last step is based on the Jensen inequality and the fact that
\[
\mathbb{W}_2^2(\mu_s^{(n)}, \mu_s^{(m)}) \leq \mathbb{E}|X_s^{(n)} - X_s^{(m)}|^2 \leq \mathbb{E} \left( \sup_{0 \leq u \leq s} |X_u^{(n)} - X_u^{(m)}|^2 \right).
\]

Set
\[
g(t) := \lim_{n, m \to \infty} \mathbb{E} \left( \sup_{0 \leq u \leq t} |X_u^{(n)} - X_u^{(m)}|^2 \right),
\]

and then (5) admits us to have that
\[
g(T) \leq 2(T + C)K_1 \int_0^T \left( \kappa_1(g(s)) + \kappa_2(g(s)) \right) ds.
\]

Thus, by [6, Lemma 3.6], one can get \( g(T) = 0 \). That is, \( \{X^{(n)}\} \) is a Cauchy sequence in the space \( L^2(\Omega, \mathcal{F}, \mathbb{P}, C_T^2) \). From this, we know that there exists a \( X \in L^2(\Omega, \mathcal{F}, \mathbb{P}, C_T^2) \) such that
\[
\lim_{n \to \infty} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^{(n)} - X_t|^2 \right) = 0.
\]

Note that
\[
\sup_{0 \leq t \leq T} \mathbb{W}_2^2(\mu_t, \mu_t) \leq \sup_{0 \leq t \leq T} \mathbb{E}|X_t^{(n)} - X_t|^2 \leq \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^{(n)} - X_t|^2 \right).
\]

So, we conclude that
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} \mathbb{W}_2^2(\mu_t^{(n)}, \mu_t) = 0.
\]
And then (6)-(7) imply that for $\forall t \in [0, T]$,
\[
\int_0^t b(\theta, X_s^{(n)}, \mu_s^{(n)})ds \to \int_0^t b(\theta, X_s, \mu_s)ds, \quad a.s.,
\]
\[
\int_0^t \sigma(X_s^{(n)}, \mu_s^{(n)})dW_s \to \int_0^t \sigma(X_s, \mu_s)dW_s \quad \text{in} \quad L^2(\Omega, \mathcal{F}_t, \mathbb{P}).
\]
Therefore, taking the limit on two hand sides of Eq.(3) as $n \to \infty$, we have that
\[
X_t = \xi + \int_0^t b(\theta, X_s, \mu_s)ds + \int_0^t \sigma(X_s, \mu_s)dW_s,
\]
that is, $X$ is a solution of Eq.(2).

**Step 3.** We prove the uniqueness of the solutions to Eq.(2).

Suppose that $X$ and $\hat{X}$ are two solutions to Eq.(2). And then by the similar calculation to that of (5), it holds that
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t - \hat{X}_t|^2\right) \leq 2T \mathbb{E}\left(\int_0^T |b(\theta, X_s, \mu_s) - b(\theta, \hat{X}_s, \hat{\mu}_s)|^2 ds\right)
\]
\[
+ 2C \mathbb{E}\left(\int_0^T \|\sigma(X_s, \mu_s) - \sigma(\hat{X}_s, \hat{\mu}_s)\|^2 ds\right)
\]
\[
\leq 2(T + C)K_1 \mathbb{E}\int_0^T \left(\kappa_1(\|X_s - \hat{X}_s\|_T^2) + \kappa_2(\mathbb{W}_2(\mu_s, \hat{\mu}_s))\right) ds
\]
\[
\leq 2(T + C)K_1 \kappa_1 \left(\mathbb{E}\left(\sup_{0 \leq u \leq s} |X_u - \hat{X}_u|^2\right)\right)
\]
\[
+ \kappa_2 \left(\mathbb{E}\left(\sup_{0 \leq u \leq s} |X_u - \hat{X}_u|^2\right)\right)\right) ds,
\]
which together with [6, Lemma 3.6] yields that
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t - \hat{X}_t|^2\right) = 0.
\]
That is, $X_t = \hat{X}_t$ for all $t \in [0, T]$ and almost all $\omega$. The proof is complete. \qed

3. The maximum likelihood estimation of MVSDEs

In the section, we assume (H$_1$) and $d = m = k = 1$. And then Eq.(2) has a unique solution $X^\theta$. We construct a maximum likelihood estimator of $\theta$ and prove its properties.

Let $C_T := C_T^1$.

Assume:

(H$_2$) For any $w \in C_T, \mu \in \mathcal{P}_2(\mathbb{R})$, $\sigma(w, \mu) \neq 0$ and
\[
\left|\frac{b(\theta, w, \mu)}{\sigma(w, \mu)}\right| \leq K_2,
\]
where $K_2 \geq 0$ is a constant.
Let $\theta_0$ be the true value of $\theta$. Let $\mathbb{P}_T^\theta, \mathbb{P}_0^T$ be the distributions of $(X_t^\theta)_{t \in [0,T]}$ and $(X_t^{\theta_0})_{t \in [0,T]}$, respectively. Thus, under $(\text{H}_2)$, it follows from [8, Theorem 7.19, P. 294] that $\mathbb{P}_T^\theta \ll \mathbb{P}_0^T$. Define a maximum likelihood function of $\theta$ as

$$L_T(\theta) := \frac{d\mathbb{P}_T^\theta}{d\mathbb{P}_0^T}$$

$$= \exp \left\{ \int_0^T \frac{1}{\sigma^2(X_t^{\theta_0}, \mu_t^{\theta_0})} \left( b(\theta, X_t^{\theta_0}, \mu_t^{\theta_0}) - b(\theta_0, X_t^{\theta_0}, \mu_t^{\theta_0}) \right) dX_t^{\theta_0} \right. $$

$$- \frac{1}{2} \int_0^T \frac{1}{\sigma^2(X_t^{\theta_0}, \mu_t^{\theta_0})} \left( b^2(\theta, X_t^{\theta_0}, \mu_t^{\theta_0}) - b^2(\theta_0, X_t^{\theta_0}, \mu_t^{\theta_0}) \right) dt \right\}$$

$$= \exp \left\{ \int_0^T \frac{1}{\sigma(X_t^{\theta_0}, \mu_t^{\theta_0})} \left( b(\theta, X_t^{\theta_0}, \mu_t^{\theta_0}) - b(\theta_0, X_t^{\theta_0}, \mu_t^{\theta_0}) \right) dW_t \right. $$

$$- \frac{1}{2} \int_0^T \frac{1}{\sigma^2(X_t^{\theta_0}, \mu_t^{\theta_0})} \left( b(\theta, X_t^{\theta_0}, \mu_t^{\theta_0}) - b(\theta_0, X_t^{\theta_0}, \mu_t^{\theta_0}) \right)^2 dt \right\},$$

where $\mu_t^\theta, \mu_t^{\theta_0}$ are the distributions of $X_t^\theta, X_t^{\theta_0}$, respectively. So, the maximum likelihood estimator of $\theta$ is given by

$$\theta_T := \arg \max_{\theta \in \Theta} L_T(\theta).$$

Next, we study some properties of the maximum likelihood estimator $\theta_T$. To do this, we assume more:

$(\text{H}_3)$ For any $w \in C_T, \mu \in \mathcal{P}_2(\mathbb{R})$, $b(\theta, w, \mu)$ is differentiable in $\theta$, $b'(\theta, w, \mu)$ denotes the first order partial derivative of $b(\theta, w, \mu)$ in $\theta$, $b'(\theta, w, \mu)$ is continuous in $\theta$ and

$$\frac{b'(\theta, w, \mu)}{\sigma(w, \mu)} \neq 0, \quad \left| \frac{b'(\theta, w, \mu)}{\sigma(w, \mu)} \right| \leq K_3,$$

where $K_3 > 0$ is a constant.

Now, set

$$m_T := \mathbb{E} \int_0^T \left( \frac{b'(\theta_0, X_t^{\theta_0}, \mu_t^{\theta_0})}{\sigma(X_t^{\theta_0}, \mu_t^{\theta_0})} \right)^2 dt,$$

and then by $(\text{H}_3)$ we know that $m_T < \infty$ for $T < \infty$ and $\lim_{T \to \infty} m_T^{-1} = 0$.

**Theorem 3.1.** Suppose that $(\text{H}_1)$-$(\text{H}_3)$ hold. Then for $\theta = \theta_0 + l m_T^{-\frac{1}{2}}, l \in \mathbb{R}$,

$$\log \frac{d\mathbb{P}_T^{\theta_0}}{d\mathbb{P}_0^T} = l \Phi_T(\theta_0) - \frac{1}{2} l^2 \Sigma_T(\theta_0) + \Psi_T(\theta_0, l),$$

where

$$\Phi_T(\theta_0) := m_T^{-\frac{1}{2}} \int_0^T \frac{1}{\sigma(X_t^{\theta_0}, \mu_t^{\theta_0})} b'(\theta_0, X_t^{\theta_0}, \mu_t^{\theta_0}) dW_t,$$

$$\Sigma_T(\theta_0) := m_T^{-1} \int_0^T \frac{1}{\sigma^2(X_t^{\theta_0}, \mu_t^{\theta_0})} \left( b'(\theta_0, X_t^{\theta_0}, \mu_t^{\theta_0}) \right)^2 dt,$$

$$\Psi_T(\theta_0, l) := l m_T^{-\frac{1}{2}} \int_0^T \frac{1}{\sigma(X_t^{\theta_0}, \mu_t^{\theta_0})} \left( b'(\theta_0 + l m_T^{-\frac{1}{2}}, X_t^{\theta_0}, \mu_t^{\theta_0}) - b'(\theta_0, X_t^{\theta_0}, \mu_t^{\theta_0}) \right) dW_t.$$
and \( \eta, \zeta \in [0, 1] \). Moreover, \( \mathbb{E} \Phi_T(\theta_0) = 0, \mathbb{E} |\Phi_T(\theta_0)|^2 = 1 \) and \( \mathbb{E} \Sigma_T(\theta_0) = 1 \).

**Proof.** By (H₃) and the mean value theorem, it holds that

\[
\log \frac{d\mathbb{P}_T^\theta}{d\mathbb{P}_{\theta_0}} = \int_0^T \frac{1}{\sigma(X_{t,T}, \mu_t)} \left( b(\theta, X_{t,T}, \mu_t) - b(\theta_0, X_{t,T}, \mu_t) \right) dW_t
\]

and

\[
\log \frac{d\mathbb{P}_T^\theta}{d\mathbb{P}_{\theta_0}} = \int_0^T \frac{1}{\sigma(X_{t,T}, \mu_t)} \left( b(\theta + l \eta m_T^{-\frac{1}{2}}, X_{t,T}, \mu_t) - b(\theta, X_{t,T}, \mu_t) \right)^2 dt
\]

Proof. By (H₃) and the mean value theorem, it holds that

\[
\log \frac{d\mathbb{P}_T^\theta}{d\mathbb{P}_{\theta_0}} = \int_0^T \frac{1}{\sigma(X_{t,T}, \mu_t)} \left( b(\theta + l \eta m_T^{-\frac{1}{2}}, X_{t,T}, \mu_t) - b(\theta, X_{t,T}, \mu_t) \right)^2 dt
\]

Next, we know that \( \mathbb{E} \Phi_T(\theta_0) = 0 \) and

\[
\mathbb{E} |\Phi_T(\theta_0)|^2 = \frac{1}{m_T^2} \mathbb{E} \left( \int_0^T \frac{1}{\sigma(X_{t,T}, \mu_t)} b(\theta, X_{t,T}, \mu_t) dW_t \right)^2 = \frac{1}{m_T} \mathbb{E} \int_0^T \frac{|b(\theta, X_{t,T}, \mu_t)|^2}{\sigma(X_{t,T}, \mu_t)} dt = 1.
\]

For \( \Sigma_T(\theta_0) \), it holds that \( \mathbb{E} \Sigma_T(\theta_0) = 1 \). The proof is complete.

To prove the other property of the maximum likelihood estimator \( \theta_T \), we assume:

(H₄) For any \( w \in C_T, \mu \in \mathcal{P}_2(\mathbb{R}) \), \( b(\theta, w, \mu) \) is one-to-one and continuous in \( \theta \).

**Theorem 3.2.** *(The consistency)*

Under the assumptions (H₁)-(H₂) (H₃), it holds that

\[
\theta_T \xrightarrow{a.s.} \theta_0, \quad T \to \infty.
\]

**Proof.** Set

\[
l_T(\theta) := \log L_T(\theta) = \log \frac{d\mathbb{P}_T^\theta}{d\mathbb{P}_{\theta_0}}.
\]

And then it holds that for \( \delta > 0 \),

\[
l_T(\theta_0 + \delta) - l_T(\theta_0) = \log \frac{d\mathbb{P}_{\theta_0+\delta}}{d\mathbb{P}_{\theta_0}}
\]

\[
= \int_0^T \frac{1}{\sigma(X_{t,T}, \mu_t)} \left( b(\theta_0 + \delta, X_{t,T}, \mu_t) - b(\theta_0, X_{t,T}, \mu_t) \right) dW_t
\]

\[
- \frac{1}{2} \int_0^T \frac{1}{\sigma^2(X_{t,T}, \mu_t)} \left( b(\theta_0 + \delta, X_{t,T}, \mu_t) - b(\theta_0, X_{t,T}, \mu_t) \right)^2 dt
\]
where
\[ \Gamma_t^{\theta_0} := \frac{1}{\sigma(X_t^{\theta_0}, \mu_t^{\theta_0})} \left( b(\theta_0 + \delta, X_t^{\theta_0}, \mu_t^{\theta_0}) - b(\theta_0, X_t^{\theta_0}, \mu_t^{\theta_0}) \right). \]

Note that
\[ \int_0^T \Gamma_t^{\theta_0} dW_t = \int_0^T |\Gamma_t^{\theta_0}|^2 dt, \]
where \([-\cdot\] stands for the quadratic variation of \(-\cdot\). Thus, by the time change, we know that
\[ \tilde{W}_t := \int_0^{A_t} \Gamma_s^{\theta_0} dW_s \]
is a \((\mathcal{F}_{A_t})_{t \geq 0}\)-adapted Brownian motion, where \(A_t\) is the inverse function of \(\int_0^t |\Gamma_s^{\theta_0}|^2 ds\). So,
\[ \frac{l_T(\theta_0 + \delta) - l_T(\theta_0)}{\int_0^T (\Gamma_t^{\theta_0})^2 dt} = \frac{1}{2} - \frac{1}{2} \frac{\tilde{W}_{A_t^{-1}}}{A_t^{-1}} \xrightarrow{a.s.} \frac{1}{2} \quad T \to \infty, \]  
(8)

where the last step is based on the strong law of large numbers for Brownian motions. By the same deduction to that of (8), one can get that
\[ \frac{l_T(\theta_0 - \delta) - l_T(\theta_0)}{\int_0^T (\Gamma_t^{\theta_0})^2 dt} \xrightarrow{a.s.} \frac{1}{2}, \quad T \to \infty, \]  
(9)

Combining (8) with (9), we obtain that
\[ \frac{l_T(\theta_0 \pm \delta) - l_T(\theta_0)}{\int_0^T (\Gamma_t^{\theta_0})^2 dt} \xrightarrow{a.s.} \frac{1}{2}, \quad T \to \infty. \]  
(10)

Next, we observe (10). It follows from (10) that for \(\delta\) and \(\theta_0\), there exists some \(t_0 > 0\) such that
\[ l_T(\theta_0 \pm \delta) < l_T(\theta_0), \quad T \geq t_0, \quad a.s. \]  
(11)

Besides, by \((H_3)\), we know that \(l_T(\theta)\) is continuous on \([\theta_0, \theta_0 + \delta]\). So, there exists a \(\theta^* \in [\theta_0, \theta_0 + \delta]\) such that \(l_T(\theta^*)\) is the maximum value of \(l_T(\theta)\) on \([\theta_0 - \delta, \theta_0 + \delta]\). That is, \(\theta_T = \theta^*\) for \(\Theta = [\theta_0 - \delta, \theta_0 + \delta]\). Based on (11), it holds that \(\theta_T \neq \theta_0 \pm \delta\) for \(T \geq t_0\). Thus, \(\theta_T \to \theta_0\) as \(T \to \infty\). The proof is over. \(\square\)

4. The Numerical Simulation of MVSDEs

In the section, we introduce the numerical simulation of Eq. (2) under \((H_1)\) and estimate the error between the solution of Eq. (2) and that of the numerical equation under Lipschitz conditions.

First of all, for \(N \in \mathbb{N}\) consider these following MVSDEs
\[ \begin{cases} dX_t^{i,N} = b(\theta, X_t^{i,N}, \mu_t^{i,N}) dt + \sigma(X_t^{i,N}, \mu_t^{i,N}) dW_t^i, \\ X_0^{i,N} = \xi_i, \quad i = 1, 2, \ldots, N, \end{cases} \]  
(12)

where \(\mu_t^{i,N} := \frac{1}{N} \sum_{j=1}^N \delta_{X_t^{j,N}}, \delta_{X_t^{j,N}}\) is the Dirac measure at \(X_t^{i,N}\), and \(W_t^i, i = 1, 2, \ldots, N\) are \(N\) mutually independent one dimensional Brownian motions. By Theorem 2.1 under
we know that Eq. (12) has a unique solution \( X_{t}^{i, N} \). And then we construct the following numerical simulation equation: for \( M \in \mathbb{N} \)

\[
\begin{align*}
Y_{0}^{i} &= \xi, \\
Y_{t}^{i} &= Y_{t_{k}}^{i} + b(\theta, Y_{t_{k}}^{i}, \mu_{t_{k}}^{M}) (t - t_{k}) + \sigma(Y_{t_{k}}^{i}, \mu_{t_{k}}^{M}) (W_{t}^{i} - W_{t_{k}}^{i}), & t \in [t_{k}, t_{k+1}],
\end{align*}
\]  

(13)

where \( t_{k} := k \frac{T}{M} \), \( \mu_{t_{k}}^{M} := \frac{1}{N} \sum_{i=1}^{N} \delta_{y_{t_{k}}^{i}} \) for \( k = 0, ..., M - 1 \). In order to estimate the error between the solution of Eq. (13) and the solution of Eq. (2), we also introduce the following MVSDE:

\[
X_{t}^{i} = \xi + \int_{0}^{t} b(\theta, X_{s}^{i}, \mu_{s}^{i}) ds + \int_{0}^{t} \sigma(X_{s}^{i}, \mu_{s}^{i}) dW_{s}^{i},
\]  

(14)

where \( \mu_{s}^{i} \) is the distribution of \( X_{s}^{i} \). Note that the solution of Eq. (14) has the same distribution to that of the solution for Eq. (2). Therefore, we compute the distance between \( X_{t}^{i} \) and \( Y_{t}^{i} \) to estimate the error between \( X_{t} \) and \( Y_{t}^{i} \). To do this, we need stronger assumptions than (H1). Assume:

(H1) There exists a nonnegative constant \( K_{1}^{i} \) such that for any \( w, v \in C_{T}^{d}, \mu, \nu \in \mathcal{P}_{2}(\mathbb{R}^{d}) \)

(i)

\[
|b(\theta, w, \mu) - b(\theta, v, \nu)|^{2} + \|\sigma(w, \mu) - \sigma(v, \nu)\|^{2} \leq K_{1}^{i} \left( \|w - v\|^{2} + \mathbb{W}_{2}^{2}(\mu, \nu) \right),
\]

(ii)

\[
|b(\theta, w, \mu)|^{2} + \|\sigma(w, \mu)\|^{2} \leq K_{1}^{i} \left( 1 + \|w\|^{2} + \|\mu\|_{2}^{2} \right).
\]

Theorem 4.1. Suppose that (H1) holds and \( \mathbb{E}|\xi|^{p} < \infty \) for \( p > 4 \). Then it follows that

\[
\sup_{1 \leq i \leq N} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{t}^{i} - Y_{t}^{i}|^{2} \right] \leq C \Gamma_{N} + C \frac{T}{M} \left( \frac{T}{M} + C \right),
\]

(15)

where the constant \( C > 0 \) is independent of \( N, M \) and

\[\Gamma_{N} := \begin{cases} N^{-1/2}, & d < 4, \\ N^{-1/2} \log N, & d = 4, \\ N^{-1/d}, & d > 4. \end{cases}\]

Proof. Note that

\[
\sup_{1 \leq i \leq N} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{t}^{i} - Y_{t}^{i}|^{2} \right] \leq 2 \sup_{1 \leq i \leq N} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{t}^{i} - X_{t}^{i, N}|^{2} \right] + 2 \sup_{1 \leq i \leq N} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{t}^{i, N} - Y_{t}^{i}|^{2} \right] =: I_{1} + I_{2},
\]

(16)

For \( I_{1} \), it follows from the same deduction as that of (4) that for \( \forall i = 1, \ldots, N \),

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_{t}^{i} - X_{t}^{i, N}|^{2} \right) \leq 2 \mathbb{E} \sup_{0 \leq t \leq T} t \int_{0}^{t} \left| \left( b(\theta, X_{s}^{i}, \mu_{s}^{i}) - b(\theta, X_{s}^{i, N}, \mu_{s}^{N}) \right)^{2} \right| ds + 2C \mathbb{E} \int_{0}^{T} \left( \left\| \sigma(X_{s}^{i}, \mu_{s}^{i}) - \sigma(X_{s}^{i, N}, \mu_{s}^{N}) \right\|^{2} \right) ds
\]

\[
\leq 2(T + C) K_{1} \mathbb{E} \int_{0}^{T} \left( \left\| X_{s}^{i} - X_{s}^{i, N} \right\|^{2} + \mathbb{W}_{2}^{2}(\mu_{s}^{i}, \mu_{s}^{N}) \right) ds
\]
Gronwall’s inequality admits us to obtain that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^i - X_t^{i,N}|^2 \right) \leq 2(T + C)K_1 \int_0^T \mathbb{E} \left( \sup_{0 \leq r \leq s} |X_r^i - X_r^{i,N}|^2 \right) ds + 2(T + C)K_1 \int_0^T \mathbb{E} W_2^2(\mu_s^i, \mu_s^N) ds.
\]
where the last inequality is based on [4, Theorem 5.8, P. 362], and furthermore
\[
I_1 \leq CT_N,
\]
where \( I_1 \) is defined in (17).

For \( I_2 \), by the similar deduction to that of (4), it holds that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{i,N} - Y_t^i|^2 \right] \leq 2T \int_0^T \mathbb{E} \left| b(\theta, X_t^{i,N}, \mu_s^N) - b(\theta, Y_{t_k}^{i,N(s)}, \mu_{t_k}^N) \right|^2 ds + 2C \int_0^T \mathbb{E} \left| \sigma(X_t^{i,N}, \mu_s^N) - \sigma(Y_{t_k}^{i,N(s)}, \mu_{t_k}^N) \right|^2 ds.
\]
\[
\leq (2T + 2C)K_1 \int_0^T \mathbb{E} \left( \|X_t^{i,N} - Y_{t_k}^{i,N(s)}\|_T^2 + W_2^2(\mu_s^N, \mu_{t_k}^N) \right) ds.
\]
\[
\leq (2T + 2C)K_1 \int_0^T \mathbb{E} \left( \|X_t^{i,N} - Y_{t_k}^{i,N(s)}\|_T^2 + 2\|Y_{t_k}^{i,N(s)} - Y_t^{i,N(s)}\|_T^2 + 2W_2^2(\mu_s^N, \mu_{t_k}^N) \right) ds.
\]
\[
\leq 8(T + C)K_1 T \sup_{k \leq t_k \leq t_{k+1}} \mathbb{E} \left( \sup_{0 \leq r \leq s} |X_r^{i,N} - Y_r^i|^2 \right) + 8(T + C)K_1 T \sup_{k \leq t_k \leq t_{k+1}} \mathbb{E} \left( \sup_{r \leq t_k} |Y_r^i - Y_{t_k}^i|^2 \right),
\]
where \( \eta(s) = t_k, s \in [t_k, t_{k+1}] \) and the following fact is used:
\[
\mathbb{E} W_2^2(\mu_s^N, \mu_s^M) \leq \mathbb{E} \left( \frac{1}{N} \sum_{j=1}^N |X_s^{i,N} - Y_s^j|^2 \right) = \mathbb{E} |X_s^{i,N} - Y_s^j|^2.
\]

The Gronwall inequality admits us to obtain that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{i,N} - Y_t^i|^2 \right] \leq 8(T + C)K_1 T \sup_{k \leq t_k \leq t_{k+1}} \mathbb{E} \left( \sup_{t_k \leq r \leq t_{k+1}} |Y_r^i - Y_{t_k}^i|^2 \right) e^{8(T + C)K_1 T}. \tag{18}
\]

In the following, we estimate \( \mathbb{E} \left( \sup_{t_k \leq r \leq t_{k+1}} |Y_r^i - Y_{t_k}^i|^2 \right) \). By (13), it holds that
\[
\mathbb{E} \left( \sup_{t_k \leq r \leq t_{k+1}} |Y_r^i - Y_{t_k}^i|^2 \right) \leq 2\mathbb{E} \left( \sup_{t_k \leq r \leq t_{k+1}} \left| \int_{t_k}^r b(\theta, Y_{tk}^i, \mu_{tk}^M) \, du \right|^2 \right) + 2\mathbb{E} \left( \sup_{t_k \leq r \leq t_{k+1}} \left| \int_{t_k}^r \sigma(Y_{tk}^i, \mu_{tk}^M) \, dW_{ru} \right|^2 \right)
\]
where in the last second inequality we use the fact that

\[ E \| \mu_{t_k}^M \|_\lambda^2 = \frac{1}{N} \sum_{j=1}^{N} E \int_{\mathbb{R}^d} (1 + |x|)^2 \delta_{Y^i_j} (dx) = \frac{1}{N} \sum_{j=1}^{N} E(1 + |Y^i_{t_k}|)^2 = E(1 + |Y^i_{t_k}|)^2. \]

Besides, from the similar deduction to that of \([4]\), it follows that

\[
\begin{align*}
E \left( \sup_{0 \leq t \leq T} |Y^i_t|^2 \right) & \leq 3E|\xi|^2 + 3TE \int_{0}^{T} |b(\theta, Y^i_{\eta(s)}) \cdot \mu_{\eta(s)}^M)|^2 ds + 3E \int_{0}^{T} \| \sigma(Y^i_{\eta(s)}) \cdot \mu_{\eta(s)}^M)\|^2 ds \\
& \leq 3E|\xi|^2 + 3(T + 1)E \int_{0}^{T} K_1 (1 + \|Y^i_{\eta(s)}\|^2 + \|\mu_{\eta(s)}^M\|^2) ds \\
& \leq 3E|\xi|^2 + 3(T + 1) \int_{0}^{T} K_1 (1 + E \left( \sup_{0 \leq u \leq s} |Y^i_u|^2 \right) + 2E(1 + |Y^i_{\eta(s)}|^2)) ds \\
& \leq 3E|\xi|^2 + 9(T + 1)TK_1 + 9(T + 1)K_1 \int_{0}^{T} E \left( \sup_{0 \leq u \leq s} |Y^i_u|^2 \right) ds.
\end{align*}
\]

The Gronwall inequality admits us to obtain that

\[
E \left( \sup_{0 \leq t \leq T} |Y^i_t|^2 \right) \leq C.
\]

Combing \([18]-[20]\), we have that

\[
E \left[ \sup_{0 \leq t \leq T} |X^i_t - Y^i_t|^2 \right] \leq C \frac{T}{M} \left( \frac{T}{M} + C \right),
\]

and furthermore

\[
I_2 \leq C \frac{T}{M} \left( \frac{T}{M} + C \right).
\]

Finally, from \([16]-[17]-[21]\), it follows that

\[
\sup_{1 \leq i \leq N} E \left[ \sup_{0 \leq t \leq T} |X^i_t - Y^i_t|^2 \right] \leq CT_N + C \frac{T}{M} \left( \frac{T}{M} + C \right).
\]

The proof is complete.
Next, we construct a maximum likelihood estimator of the parameter \( \theta \). Let \( d = m = k = 1 \). Assume that \( (H_1) \sim (H_2) \) hold. And then define the maximum likelihood function

\[
L_T^M(\theta) := \exp \left\{ \int_0^T \frac{1}{\sigma(Y_{\eta(t)\wedge \cdot}, \mu_{\eta(t)}^M)} \left( b(\theta, Y_{\eta(t)\wedge \cdot}, \mu_{\eta(t)}^M) - b(\theta_0, Y_{\eta(t)\wedge \cdot}, \mu_{\eta(t)}^M) \right) \, dW_t^i \\
- \frac{1}{2} \int_0^T \frac{1}{\sigma^2(Y_{\eta(t)\wedge \cdot}, \mu_{\eta(t)}^M)} \left( b(\theta, Y_{\eta(t)\wedge \cdot}, \mu_{\eta(t)}^M) - b(\theta_0, Y_{\eta(t)\wedge \cdot}, \mu_{\eta(t)}^M) \right)^2 \, dt \right\},
\]

Thus, the maximum likelihood estimator of the parameter \( \theta \) is given by

\[
\hat{\theta}_T^M := \arg \max_{\theta \in \Theta} L_T^M(\theta).
\] (22)

5. An example

In the section, we present an example to explain our results.

Consider the following MVSDE on \( \mathbb{R} \):

\[
dX_t = (\theta X_t + \beta \mathbb{E}[X_t]) \, dt + \sigma \, dW_t, \quad X_0 = x_0 \in \mathbb{R},
\] (23)

where \( \theta \in \Theta \) is a unknown parameter and \( \beta, \sigma \) are nonzero constants. Using the numerical simulation method in Section 4, we have the following numerical equation for Eq.(23)

\[
\begin{cases}
Y_0^i = x_0, \\
Y_t^i = Y_{t_k}^i + \left( \theta Y_{t_k}^i + \beta \frac{1}{N} \sum_{j=1}^N Y_{t_k}^j \right) (t - t_k) + \sigma(W_t^i - W_{t_k}^i), \quad t \in [t_k, t_{k+1}].
\end{cases}
\] (24)

In terms of the number \( N \) of particles and the step size \( M \), we draw Figure 1 and Figure 2. That is, we take \( N = 160, M = 16 \) in Figure 1, and \( N = 2560, M = 256 \) in Figure 2. Comparing Figure 1 with Figure 2, one can find that the numerical solution has higher frequency and smaller amplitude when the number of particles and the step size are larger.

![Figure 1. Comparison of approximate solution and true solution, taking \( \beta = \sigma = 1 \), \( N = 160 \), \( M = 16 \).](image)

According to (15) in Section 4, we calculate the errors between the solutions of Eq.(24) and the solution of Eq.(23) and list them in Table 1. From Table 1, one can find that the error decreases when the number of particles and the step size increase.
Figure 2. Comparison of approximate solution and true solution, taking \( \beta = \sigma = 1, N = 2560, M = 256 \).

Table 1. The errors between the numerical solution and the original solution when \( N, M \) take different values.

| N   | 16  | 32  | 64  | 128 | 256 |
|-----|-----|-----|-----|-----|-----|
| 160 | 0.0753 | 0.0389 | 0.0182 | 0.0093 | 0.0040 |
| 320 | 0.0686 | 0.0354 | 0.0176 | 0.0086 | 0.0048 |
| 640 | 0.0656 | 0.0337 | 0.0167 | 0.0077 | 0.0037 |
| 1280| 0.0670 | 0.0331 | 0.0158 | 0.0077 | 0.0034 |
| 2560| 0.0672 | 0.0323 | 0.0157 | 0.0073 | 0.0032 |

Finally, by the formula (22) in Section 4, we get the maximum likelihood estimator \( \theta_M^T \) as follows:

\[
\theta_M^T = \frac{\sum_{k=0}^{M-1} Y_{t_k}^i (Y_{t_{k+1}}^i - Y_{t_k}^i) - \sum_{k=0}^{M-1} \beta Y_{t_k}^i \frac{1}{N} \sum_{j=1}^{N} Y_{t_k}^j T}{\sum_{k=0}^{M-1} (Y_{t_k}^i)^2 \frac{T}{M}}.
\]  

(25)

In terms of \( T \), the values of \( \theta_M^T \) present in Table 2, which indicates that the value of \( \theta_T \) is closer to the true value \( \theta_0 = -0.5 \) when the time \( T \) becomes larger.

Table 2. The maximum likelihood estimator \( \theta_M^T \) with \( \beta = \sigma = 1, N = 2560, M = 256 \).

| \( T \) | 1     | 2     | 5     | 8     | 10    |
|-------|-------|-------|-------|-------|-------|
| \( \theta_M^T \) | -1.0510 | -0.7420 | -0.5107 | -0.5009 | -0.4999 |

Acknowledgements:
The authors are very grateful to Professor Xicheng Zhang for valuable discussions. The second author also thanks Professor Renming Song for providing her an excellent environment to work in the University of Illinois at Urbana-Champaign.
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