Multiplicity of solutions for a nonlocal elliptic PDE involving singularity

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Abstract
In this paper we prove the existence of multiple solutions for a nonlinear nonlocal elliptic PDE involving a singularity which is given as

\((-\Delta_p)^s u = \frac{\lambda}{u^\gamma} + u^q \text{ in } \Omega,\)
\[ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \]
\[ u > 0 \text{ in } \Omega, \]

where \(\Omega\) is an open bounded domain in \(\mathbb{R}^N\) with smooth boundary, \(N > ps, s \in (0, 1), \lambda > 0, 0 < \gamma < 1, 1 < p < \infty, p - 1 < q \leq \frac{Np}{N - ps}.\) We employ variational techniques to show the existence of multiple positive weak solutions of the above problem.

Keywords: Elliptic PDE, PS condition, Mountain Pass theorem, Gâteaux derivative.

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1. Introduction

The study of elliptic PDEs involving fractional \(p\)-Laplacian operator plays an important role in many field of sciences, like in the field of finance, optimization, electromagnetism, astronomy, water waves, fluid dynamics, probability theory, phase transitions etc. The application to Lévy processes in probability theory can be seen in [7, 11] and that in finance can be seen in [45]. For further details on applications, one can refer to [46].

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and the references therein. In the recent past, a vast investigation has been done for the following local problem.

\[-\Delta_p u = \frac{\lambda a(x)}{u^q} + Mu^q \text{ in } \Omega,\]
\[u = 0 \text{ in } \partial\Omega,\]
\[u > 0 \text{ in } \Omega,\]  

(1.1)

where \(1 < p < N, M \geq 0, a : \Omega \to \mathbb{R}\) is a nonnegative bounded function. For \(M = 0\), the existence of weak solutions and regularity of solutions for singular problem as in (1.1) has been widely studied in [10, 18, 20, 33, 39] and the references therein. Recently, for \(M \neq 0\) the problem (1.1) has been studied to show the existence of multiple solutions in [1, 2, 5, 17, 18, 19, 25, 26, 27, 29, 30] and the references therein. In most of these references, the multiplicity results were proved by the variational methods, Nehari manifold method, perron method, concentration compactness and the moving hyperplane method.

For \(p = 2\), \(a(x) \equiv 1\) and \(M = 1\), Haitao [29] used the variational method to show that for \(0 < \lambda < \Lambda < \infty\), the problem (1.1) has two solutions. For \(a(x) \equiv 1\) and \(M = 1\) Giacomoni & Sreenadh [27] had studied the problem (1.1) for \(1 < p < \infty\), to show the existence of at least two solutions by using shooting method. Later Giacomoni et al. [26] had proved the multiplicity result using the variational method. In [5] the authors showed the multiplicity of solutions using the moving hyperplane method. In [19] the authors applied the concentration compactness method to establish the multiplicity results. The Nehari manifold method is used in [48] to show the existence of multiple solutions.

Recently, the study of nonlocal elliptic PDEs involving singularity with Dirichlet boundary condition has drawn interest by many researchers. The investigation for the existence of weak solutions for a nonlocal elliptic pdes with concave-convex type nonlinearity, i.e. \(u^p + \lambda u^q\), for \(p, q > 0\) has been extensively studied in [6, 8, 9, 12, 16, 47] and the references therein. The existence results for the Brezis-Nirenberg type problem has been studied in [8, 35, 41]. The eigenvalue problem for fractional \(p\)-Laplacian and the properties of first and second eigen values can be found in [13, 22, 34].

The following nonlocal problem has been studied by many authors,

\[(-\Delta_p)^s u = \frac{\lambda a(x)}{u^q} + M f(x, u) \text{ in } \Omega,\]
\[u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,\]
\[u > 0 \text{ in } \Omega,\]

(1.2)

where \(N > ps, M \geq 0, a : \Omega \to \mathbb{R}\) is a nonnegative bounded function. For \(p = 2\), \(M = 0, \lambda = 1\) and \(a(x) \equiv 1\) the problem (1.2) was studied by [21]. In [21] the author
had shown that for $0 < \gamma < 1$, the problem (1.2) has a unique solution $u \in C^{2,\alpha}(\Omega)$ for $0 < \alpha < 1$. For $1 < p < \infty$, $M = 0$ and $\lambda = 1$ the problem (1.2) was studied by Canino et al. [15]. For $0 < \gamma < 1$ and $\lambda = 1$, Ghanmi & Saoudi [24] established the existence of two solutions by Nehari manifold method for fractional Laplacian operator. For $p = 2$, $f(x, u) = u^{2s-1}$, Mukherjee & Sreenadh [37] studied the problem (1.2) by variational method. Ghanmi & Saoudi [23] guaranteed the existence of multiple weak solutions to the problem (1.2), for $0 < \gamma < 1$ and $1 < p - 1 < q \leq p_s^*$. The authors in [23] have used the Nehari manifold method.

In the present article we will prove the existence of multiple solutions for the following nonlocal problem

\[ (-\Delta_p)^s u = \frac{\lambda}{u^\gamma} + u^q \quad \text{in } \Omega, \]

\[ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \]

\[ u > 0 \quad \text{in } \Omega, \]

where $\Omega$ is a open bounded domain of $\mathbb{R}^N$ with smooth boundary, $N > ps$, $s \in (0, 1)$, $\lambda > 0$, $0 < \gamma < 1$, $1 < p < \infty$, $p - 1 < q \leq p_s^* = \frac{Np}{N - ps}$ and $(-\Delta_p)^s$ is the fractional $p$-Laplacian operator which is defined as

\[ (-\Delta_p)^s u(x) = C_{n,s,p,v} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy, \forall p \in [1, \infty) \]

with $C_{n,s,p,v}$ being the normalizing constant.

Similar problems to that in (1.3) has been studied by a few authors like Mukherjee & Sreenadh [38], Saoudi [40]. In [38], the authors established the existence of multiple solutions by using the Nehari manifold method. In [40], for $p = 2$ the multiplicity result for the problem (1.3) is proved with the help of the variational method, where the author proved the existence result by converting the nonlocal problem to a local problem.

In this article we show the existence of multiple solutions to the nonlocal problem (1.3) by combining some variational techniques developed in [3]. We first show the existence of a weak solution using sub-super solution method. To show the existence of a second solution, we use a modified version of the Mountain Pass lemma due to Ambrosetti & Rabinowitz [3], which can be found in Ghoussoub & Preiss [28].

The article is organised in the following sequence. In Section 2 we give the mathematical formulation with the appropriate functional analytic setup. Section 3 is devoted to establish the existence of a weak solution. In Section 4 we prove the multiplicity of solutions using the Ekeland’s variational principle.

The main results proved in this manuscript are the followings
Theorem 1.1. There exists $\Lambda \in (0, \infty)$ such that,

(i) $\forall \lambda \in (0, \Lambda)$, the problem (1.3) has a minimal solution.

(ii) For $\lambda = \Lambda$ the problem (1.3) has at least one solution.

(iii) $\forall \lambda \in (\Lambda, \infty)$ the problem (1.3) has no solution.

Theorem 1.2. For every $\lambda \in (0, \Lambda)$, the problem (1.3) has multiple solutions.

2. Mathematical formulation and Space setup

This section is entirely devoted to a brief discussion about a few definitions, notations and function spaces which will be used henceforth in this manuscript. We begin by defining the following function space. Let $\Omega \subset \mathbb{R}^N$ and $Q = \mathbb{R}^{2N} \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega))$, then the space $(X, \| \cdot \|_X)$ is defined by

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable}, u|_{\Omega} \in L^p(\Omega) \text{ and } \frac{|u(x) - u(y)|}{|x-y|^{N+ps}} \in L^p(Q) \right\}$$

equipped with the Gagliardo norm

$$\|u\|_X = \|u\|_p + \left( \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$ 

Here $\|u\|_p$ refers to the $L^p$-norm of $u$. We further define the space $X_0 = \{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}$ equipped with the norm

$$\|u\| = \left( \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$ 

The best Sobolev constant is defined as

$$S = \inf_{u \in X_0 \setminus \{0\}} \frac{\int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy}{(\int_\Omega |p_s^*|dx)^{\frac{1}{p}}} \tag{2.1}$$

We now define a weak solution to the problem defined in (1.3).

Definition 2.1. A function $u \in X_0$ is a weak solution to the problem (1.3), if

(i) $u > 0$, $u^{-\gamma} \phi \in L^1(\Omega)$ and

(ii) $\int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}} dxdy - \int_\Omega \frac{\lambda}{u^\gamma} \phi - \int_\Omega u^q \phi = 0$

for each $\phi \in X_0$. 
Following are the definitions of a sub and a super solution to the problem (1.3).

**Definition 2.2.** A function $u \in X_0$ is a weak subsolution to the problem (1.3), if

$(i)$ $u > 0$, $u^{-\gamma}\phi \in L^1(\Omega)$ and

$(ii)$ $\int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+ps}} dxdy - \int_{\Omega} \frac{\lambda}{u^\gamma} \phi - \int_{\Omega} u^q \phi \leq 0$

for every nonnegative $\phi \in X_0$.

**Definition 2.3.** A function $\bar{u} \in X_0$ is a weak supersolution to the problem (1.3), if

$(i)$ $\bar{u} > 0$, $\bar{u}^{-\gamma}\phi \in L^1(\Omega)$ and

$(ii)$ $\int_Q \frac{|\bar{u}(x) - \bar{u}(y)|^{p-2}(\bar{u}(x) - \bar{u}(y))(\phi(x) - \phi(y))}{|x - y|^{N+ps}} dxdy - \int_{\Omega} \frac{\lambda}{\bar{u}^\gamma} \phi - \int_{\Omega} \bar{u}^q \phi \geq 0$

for all nonnegative $\phi \in X_0$.

We now list out the embedding results in the form of a Lemma pertaining to the function space $X_0$ [42, 43].

**Lemma 2.4.** The following embedding results holds for the space $X_0$.

1. If $\Omega$ has a Lipschitz boundary, then the embedding $X_0 \hookrightarrow L^q(\Omega)$ for $q \in [1, p_\star^s)$, where $p_\star^s = \frac{Np}{N-ps}$.

2. The embedding $X_0 \hookrightarrow L^{p_\star^s}(\Omega)$ is continuous.

The main goal achieved in this article is the existence of two distinct, positive weak solutions to the problem (1.3) in $X_0$. To establish that we will engage ourselves to find the existence of two distinct critical points to the following energy functional, $I_{\lambda} : X_0 \rightarrow \mathbb{R}$ defined as

$$I_{\lambda}(u) = \frac{1}{p} \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy - \frac{\lambda}{1 - \gamma} \int_{\Omega} (u^+)^{1-\gamma} dx - \frac{1}{q+1} \int_{\Omega} (u^+)^{q+1} dx.$$ 

Here, $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. It is easy to observe that $I_{\lambda}$ is not $C^1$ due the presence of the singular term in it. Therefore the usual approach by Mountain Pass lemma [3] fails. So, we will proceed with a cut off functional argument as in Ghoussoub & Preiss [28]. Let us define

$$\Lambda = \inf\{\lambda > 0 : \text{ The problem (1.3) has no weak solution}\}$$
3. Existence of weak solutions

We begin the section by considering the problem

\[(−Δ_p)^s w = λw^{−γ} \text{ in } Ω,\]
\[w > 0 \text{ in } Ω,\]
\[w = 0 \text{ in } R^N \setminus Ω.\]  
(3.1)

We now state an existence result due to [15].

**Lemma 3.1.** Assume \(0 < γ < 1\) and \(λ > 0\). Then the problem (3.1) has a unique solution, \(u_λ \in W_0^{1,p}(Ω)\), such that for every \(K \subset⊂ Ω\), \(\text{ess.}\inf_K u_λ > 0\).

With this Lemma in consideration, we now prove our first major theorem.

**Lemma 3.2.** Assume \(0 < γ < 1 < q \leq p^*_s − 1\). Then \(0 < Λ < ∞\).

**Proof.** Define,

\[\bar{T}(x,t) = \begin{cases} 
  f(t), & \text{if } t > u_λ \\
  f(u_λ), & \text{if } t \leq u_λ
\end{cases}\]

where, \(f(u) = \frac{λ}{u^γ} + u^q\) and \(u_λ\) is the solution to (3.1). Let \(\bar{F}(x,s) = \int_0^s \bar{T}(x,t)dt, λ > 0\). Define a function \(\bar{T}_λ : X_0 \to R\) as follows.

\[\bar{T}_λ(u) = \frac{1}{p} \int_Q \frac{|u(x) − u(y)|^p}{|x − y|^{N+ps}} \, dx dy − \int_Ω \bar{F}(x,u) \, dx.\]  
(3.2)

The functional is \(C^1\) (refer to Lemma 5.7 in the Appendix) and weakly lower semicontinuous. From the Hölder’s inequality and Lemma (2.4), we obtain

\[\bar{T}_λ(u) = \frac{1}{p} \|u\|^p − \int_Ω \bar{F}(x,u) \, dx \geq \frac{1}{p} \|u\|^p − \lambda c_1 \|u|^{1−γ} \geq c_2 \|u|^{q+1}
\]

where, \(c_1, c_2\) are constants. We choose \(r > 0\) small enough and \(λ > 0\) sufficiently small so that the term \(\frac{1}{2} \|u\|^2 − \lambda c_1 \|u|^{1−γ} > c_2 \|u|^{q+1} > 0\). Thus we have a pair of \((λ, r)\) such that

\[\min_{u \in ∂B_r} \{\bar{T}_λ(u)\} > 0.\]

Now, for \(φ > 0 \in X_0\), we have

\[\bar{T}_λ(φ) = \frac{|t|^p}{p} \|φ\|^p − \frac{|t|^{1−γ}}{1 − γ} \int_Ω |φ|^{1−γ} \, dx − \frac{|t|^{q+1}}{q+1} \int_Ω |φ|^{q+1} \, dx.\]  
(3.3)
The above equation (3.3) holds since $1 - \gamma < 1 < q + 1$. Thus $\overline{T}_{\lambda}(t\phi) \to -\infty$ as $t \to \infty$. Therefore we have $\inf_{\|u\|_{X_0} \leq r} \overline{T}_{\lambda}(u) = c < 0$. By the definition of infimum, we consider a minimizing sequence $\{u_n\}$ for $c$. By the reflexivity of $X_0$ there exists a subsequence, still denoted by $\{u_n\}$, which weakly converges to, say, $u$. Such that

\begin{align*}
  u_n &\to u \text{ weakly in } L^{p^*_s}(\Omega) \\
  u_n &\to u \text{ strongly in } L^r(\Omega) \text{ for } 1 \leq r < p^*_s \\
  u_n &\to u \text{ pointwise a.e. in } \Omega.
\end{align*}

(3.4)

Therefore, by the Brezis-Lieb lemma [14], we get

\begin{align*}
  \|u_n\|^p &= \|u\|^p + \|u_n - u\|^p + o(1) \\
  \|u_n\|^{q+1}_{q+1} &= \|u\|^{q+1}_{q+1} + \|u_n - u\|^{q+1} + o(1).
\end{align*}

(3.5)

On using the Hölder’s inequality and passing the limit $n \to \infty$, we obtain

\begin{align*}
  \int_\Omega u_n^{1-\gamma} dx &\leq \int_\Omega u^{1-\gamma} dx + \int_\Omega |u_n - u|^{1-\gamma} dx \\
  &\leq \int_\Omega u^{1-\gamma} dx + c_1 \|u_n - u\|_{p}^{1-\gamma} \\
  &= \int_\Omega u^{1-\gamma} dx + o(1).
\end{align*}

(3.6)

Therefore, on similar lines, we have

\begin{align*}
  \int_\Omega u^{1-\gamma} dx &\leq \int_\Omega u_n^{1-\gamma} dx + \int_\Omega |u_n - u|^{1-\gamma} dx \\
  &\leq \int_\Omega u_n^{1-\gamma} dx + c_1 \|u_n - u\|_{p}^{1-\gamma} \\
  &= \int_\Omega u^{1-\gamma} dx + o(1).
\end{align*}

(3.7)

Combining (3.6) and (3.7), we obtain the following

\begin{align*}
  \int_\Omega u_n^{1-\gamma} dx &= \int_\Omega u^{1-\gamma} dx + o(1).
\end{align*}

(3.8)

Thus, clubbing equations (3.5), (3.6) and (3.7), we deduce that

\begin{align*}
  \overline{T}_{\lambda}(u_n) = \overline{T}_{\lambda}(u) + \frac{1}{p} \|u_n - u\|^p - \frac{1}{q+1} \|u_n - u\|_{q+1}^{q+1} + o(1).
\end{align*}

(3.9)
We also observe from (3.5) that for \( n \) sufficiently large \( u \), \( u_n - u \in B_r \) and \( \frac{1}{p} \| u_n - u \|^p - \frac{1}{q + 1} \| u_n - u \|_{q+1}^{q+1} \geq o(1) \). Since \( r > 0 \) was chosen to be sufficiently small, we have

\[
\frac{1}{p} \| u_n - u \|^p - \frac{1}{q + 1} \| u_n - u \|_{q+1}^{q+1} > 0 \text{ on } \partial B_r.
\]

As a consequence, we can conclude that

\[
\frac{1}{p} \| u_n - u \|^p - \frac{1}{q + 1} \| u_n - u \|_{q+1}^{q+1} \geq o(1).
\]

(3.10)

Therefore on passing the limit \( n \to \infty \) to (3.9), we obtain \( \mathcal{I}_\lambda(u_n) \geq \mathcal{I}_\lambda(u) + o(1) \). Since \( \inf_{\|u\|_{X_0} \leq r} \mathcal{I}_\lambda(u) = c \) we have \( u \neq 0 \) which is a minimizer of \( \mathcal{I}_\lambda \) over \( X_0 \). Thus we have

\[
(-\Delta_p)\phi_1 = f(x, u) \text{ in } \Omega,
\]

\[
u > 0 \text{ in } \Omega,
\]

\[
u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\]

(3.12)

By the comparison principle (refer to lemma 5.1 in the Appendix) of fractional \( p \)-Laplacian we conclude that \( \phi_1 \leq u \) in \( \Omega \). Thus \( \Lambda > 0 \) since the choice \( \lambda > 0 \) has been made.

We now claim that \( \Lambda < \infty \). We let \( \lambda_1 \) to denote the principal eigenvalue of \( (-\Delta_p)^s \) in \( \Omega \) and let \( \phi_1 > 0 \) be the associated eigenfunction. In other words, we have

\[
(-\Delta_p)^s \phi_1 = \lambda_1 \phi_1 \text{ in } \Omega,
\]

\[
\phi_1 > 0 \text{ in } \Omega,
\]

\[
\phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\]

(3.13)

We choose, \( \phi_1 \) as a test function in the weak formulation of (1.3), to get

\[
\lambda_1 \int_{\Omega} u \phi_1 dx = \int_{\Omega} (-\Delta_p)^s \phi_1 u dx
\]

\[
= \int_{\Omega} \left( \frac{\lambda}{u^\gamma} + u^q \right) \phi_1 dx.
\]

(3.14)

Let \( \bar{\Lambda} \) be any constant such that \( \bar{\Lambda} t^{-\gamma} + t^q > 2\lambda_1 t \forall t > 0 \). This leads to a contradiction to the equation (3.14). Hence we conclude that \( \Lambda < \infty \).

Lemma 3.3. Let \( 0 < \gamma < 1 \). Suppose that \( \underline{u} \) is a weak subsolution while \( \bar{u} \) is a weak supersolution to the problem (1.3) such that \( \underline{u} \leq \bar{u} \), then for every \( \lambda \in (0, \Lambda) \) there exists a weak solution \( u_\lambda \) such that \( \underline{u} \leq u_\lambda \leq \bar{u} \) a.e. in \( \Omega \). This \( u_\lambda \) is a local minimizer of \( \mathcal{I}_\lambda \) defined over \( X_0 \).
Proof. We begin by showing that $u \leq \bar{u}$. For this let us consider the problem (1.3). Let $\mu \in (0, \Lambda)$. By the definition of $\Lambda$, there exists $\lambda_0 \in (\mu, \Lambda)$ such that (1.3) with $\lambda = \lambda_0$ has a solution by the Lemma 3.2, say $u_{\lambda_0}$. Then $\bar{u} = u_{\lambda_0}$ happens to be a supersolution of the problem (1.3). Consider the function $\phi_1$ an eigenfunction of $(-\Delta_p)^s$ corresponding to the smallest eigenvalue $\lambda_1$. Thus $\phi_1 \in L^\infty(\Omega)$ \cite{33} and
\[
(-\Delta_p)^s \phi_1 = \lambda_1 |\phi_1|^{p-2} \phi_1, \\
\phi_1 > 0 \text{ in } \Omega, \\
\phi_1 = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\] (3.15)
Choose, $t > 0$ such that $t \phi_1 \leq \bar{u}$ and $t^{p+q-1} \phi_1^p + t^q \phi_1^q \leq \frac{1}{\lambda_1}$. On defining $u = t \phi_1$ we have
\[
(-\Delta_p)^s u = \lambda_1 t^{p-1} \phi_1^{p-1} \\
\leq \lambda t^{-q} \phi_1^{-q} + t^q \phi_1^q \\
= \lambda u^{-q} + u^q,
\] (3.16)
i.e., $u$ is a subsolution of the problem (1.3). This implies that $u \leq \bar{u}$.

We now show the existence of a $u_\lambda$. For this, we define
\[
\tilde{f}(x,t) = \begin{cases} f_\lambda(x,u), & \text{if } t \geq \bar{u} \\ f_\lambda(x,t), & \text{if } u \leq t \leq \bar{u} \\ f_\lambda(x,u), & \text{if } t \leq u. \end{cases}
\]
We further define $\tilde{I}(u) = \frac{1}{p} \|u\|^p - \int_\Omega \tilde{F}(x,u)dx$, where $\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)ds$. Let $u_\lambda$ be a global minimizer of the functional $\tilde{I}$ due to the definition of $\tilde{f}$. We first observe that the $C^1$ functional $\tilde{I}$ is sequentially weakly lower semicontinuous and coercive. This can be seen from the dominated convergence theorem and the Sobolev embedding. Due to the monotonicity of $\tilde{f}$ we have,
\[
(-\Delta_p)^s (\bar{u} - u_\lambda) \geq f(x,u) - \tilde{f}(x,u)
\] (3.17)
along with $\bar{u} - u_\lambda \geq 0$, in $\mathbb{R}^N \setminus \Omega$. We now refer to a result proved in the Lemma 5.2 in the Appendix that $\bar{u} - u_\lambda \geq 0$ and is a weak supersolution to the problem (1.3).

On using the Lemma 2.7 in \cite{32}, we have $\frac{u-u_\lambda}{s^\alpha} \geq c_2 > 0$ in $\bar{\Omega}$. Similarly we prove $\frac{\bar{u}}{s^\alpha} \geq c_2 > 0$ in $\bar{\Omega}$. Then $u_\lambda$ is a weak solution to the problem (1.3).

Now, we prove that $u_\lambda$ is a local minimizer of $\tilde{I}_\lambda$. Due to Theorem 4.4 in \cite{31}, we have $u_\lambda \in C^3_0(\Omega)$. Thus for any $u \in B_\lambda^d(u_\lambda)$, we obtain $\bar{u} - u_\lambda = \bar{u} - u + u - u_\lambda \geq c_2 - \frac{c_2}{2}$ in $\tilde{\Omega}$. Hence, by the maximum principle we get $\bar{u} - u_\lambda > 0$ in $\Omega$. using similar argument, it follows that $u - u_\lambda > 0$ in $\Omega$. Therefore, $\tilde{I}$ and $\tilde{I}_\lambda$ becomes identical over $B_\lambda^d(u_\lambda) \cap X_0$. Further we have $u_\lambda$ is local minimizer of $\tilde{I}_\lambda$ in $C^0_\text{loc}(\Omega) \cap X_0$. Hence, Theorem 1.1 in \cite{32}, implies that $u_\lambda$ is a local minimizer of $\tilde{I}_\lambda$. 

\hfill \Box
We now prove the following theorem.

**Theorem 3.4.** The problem in (1.3) has at least one solution if \( \lambda = \Lambda \).

**Proof.** Consider an increasing sequence \( \{\lambda_n\} \), which converges to \( \Lambda \), as \( n \to \infty \). Let \( u_n = u_{\lambda_n} \) be a weak solution to the problem (1.3) for \( \lambda = \lambda_n \). Thus

\[
\int_Q \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+ps}} dx dy - \lambda_n \int_\Omega u_n^{-\gamma} \phi dx - \int_\Omega u_n^p \phi dx = 0, \quad \forall \phi \in X_0.
\] (3.18)

Hence putting \( \phi = u_n \), we have

\[
\int_Q \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}} dx dy - \lambda_n \int_\Omega u_n^{1-\gamma} dx - \int_\Omega u_n^{-1+1} dx = 0. \tag{3.19}
\]

From the Lemma 3.3, the energy functional

\[
I(u_n) = \frac{1}{p} \int_Q \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}} dx dy - \frac{\lambda_n}{1 - \gamma} \int_\Omega u_n^{1-\gamma} dx - \frac{1}{q + 1} \int_\Omega u_n^{q+1} dx
\leq B
\] (3.20)

for every \( 0 < \gamma < 1 \). Using (3.19) in (3.20) we get

\[
\frac{1}{p} \left( \lambda_n \int_\Omega u_n^{1-\gamma} dx + \int_\Omega u_n^{-1+1} dx \right) - \frac{\lambda_n}{1 - \gamma} \int_\Omega u_n^{1-\gamma} dx - \frac{1}{q + 1} \int_\Omega u_n^{q+1} dx \leq B. \tag{3.21}
\]

From (3.21) we get

\[
\left( \frac{1}{p} - \frac{1}{q + 1} \right) \int_\Omega u_n^{q+1} dx \leq B + \lambda_n \left( \frac{1}{1 - \gamma} - \frac{1}{p} \right) \int_\Omega u_n^{1-\gamma} dx. \tag{3.22}
\]

Using (3.22) in (3.19) we obtain

\[
\|u_n\|_{X_0}^{p-1+\gamma} \leq A_1 + \frac{A_2}{\|u_n\|_{X_0}^{1-\gamma}}. \tag{3.23}
\]

From the inequality in (3.23), it is easy to see that \( \sup_{n \in \mathbb{N}} \|u_n\|_{X_0} < \infty \). Thus by the reflexivity of \( X_0 \), we have a subsequence, which will still be denoted by \( \{u_n\} \), such that \( u_n \rightharpoonup u \) weakly in \( X_0 \), as \( n \to \infty \). This establishes that \( u \) is a weak solution corresponding to \( \Lambda \).

We now prove a corollary based on the Theorem 3.4.
Corollary 3.5. Let $1 < q \leq p^*_s - 1$, $0 < \gamma < 1$ and $0 < \lambda \leq \Lambda$. Then there exists a smallest solution in $X_0$ to the problem (1.3).

Proof. From Lemma 3.3, we guarantee the existence of a weak solution $u_\lambda$ to the problem (1.3) for $\lambda \in (0, \Lambda)$. We now define a sequence $\{v_n\}$ by the following iterative sequence of problems. Define $v_1 = u$, a subsolution of (1.3). The remaining terms of the sequence can be defined by the following iterative scheme.

\[
(-\Delta_p)^s v_n - \frac{\lambda}{v_n^q} = v_{n-1}^q \quad \text{in } \Omega,
\]
\[
v_n > 0 \quad \text{in } \Omega,
\]
\[
v_n = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\]

for each $n \in \mathbb{N}$. By the choice of $v_1$ we have $v_1 \leq u$, where $u$ is a weak solution to (1.3), whose existence is again attributed to the Lemma 3.3. By the weak comparison principle (refer Lemma 5.1 in the Appendix), it is clear that $u_1 \leq u_2 \leq \ldots \leq u$. Owing to the Theorem 6.4 in [37], we have $u$ is in $L^\infty(\Omega)$, which further implies that the sequence $\{u_n\}$ is bounded in $X_0$. Thus we have a subsequence such that $u_n \rightharpoonup \hat{u}$. To conclude that $\hat{u}$ is the minimal solution, we let $\hat{v}$ to be a solution to (1.3). We have $u_n \leq \hat{v}$ which on passing the limit $n \to \infty$ we get $\hat{u} \leq \hat{v}$.

4. Multiplicity of weak solutions

This section is devoted to show the existence of a critical point $v_\lambda$ of the functional $\bar{I}_\lambda$ since, the functional $I_\lambda$ fails to be $C^1$. The critical point $v_\lambda$ of $\bar{I}_\lambda$ is also a point where the Gâteaux derivative of the functional $I_\lambda$ vanishes. Therefore, $v_\lambda$ will solve the problem (1.3). We will prove $v_\lambda \neq u_\lambda$, where, $u_\lambda$ is the solution to the problem (1.3) as proved in the Lemma 3.3. We have the following theorem proved in Ghoussoub-Preiss [28].

Theorem 4.1 (Ghoussoub-Preiss). Let $\varphi : X \to \mathbb{R}$ be a continuous and Gâteaux differentiable function on a Banach space $X$ such that $\varphi : X \to X^*$ is continuous from the norm topology on $X$ to the weak* topology of $X^*$. Take two point $u_\lambda$ and $v_\lambda$ in $X$ and consider the number

\[
c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} \varphi(g(t))
\]

where $\Gamma = \{g \in C([0, 1], X) : g(0) = u_\lambda \& g(1) = v_\lambda\}$. Suppose $F$ is a closed subset of $X$ such that $F \cup \{x \in X : \varphi(x) \geq c\}$ separates $u_\lambda$ and $v_\lambda$, then, there exists a sequence $\{x_n\}$ in $X$ verifying the following:

(i) $\lim_{n \to \infty} \text{dist} (x_n, F) = 0$
(ii) \( \lim_{n \to \infty} \varphi(x_n) = c \)

(iii) \( \lim_{n \to \infty} \| \varphi'(x_n) \| = 0 \)

**Definition 4.2.** Let \( F \subset \Omega \), be closed and \( c \in \mathbb{R} \). Then a sequence \( \{v_n\} \subset X_0 \) is said to be a Palais Smale sequence [in short \((PS)_{F,c}\)] for the functional \( \bar{I}_\lambda \) around \( F \) at the level \( c \), if

\[
\lim_{n \to \infty} \text{dist}(x_n, F) = 0, \quad \lim_{n \to \infty} \varphi(x_n) = c \quad \& \quad \lim_{n \to \infty} \| \varphi'(x_n) \| = 0
\]

Every \((PS)_{F,c}\) sequence for \( \bar{I}_\lambda \) have the following compactness property.

**Lemma 4.3.** Let \( F \subset \Omega \) be closed and \( c \in \mathbb{R} \). Let \( \{v_n\} \subset X_0 \) be a \((PS)_{F,c}\) sequence for the functional \( \bar{I}_\lambda \), then \( \{v_n\} \) is bounded in \( X_0 \) and there exists a subsequence \( \{v_{n_k}\}\) such that \( v_{n_k} \rightharpoonup v_\lambda \) in \( X_0 \), where \( v_\lambda \) is a weak solution of the problem (1.3).

**Proof.** We use the Definition 4.2, which says, there exists \( K > 0 \) such that the following holds

\[
\frac{1}{p} \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^p}{|x - y|^{N+ps}} dx \, dy - \int_{v_n > u_\lambda} \left[ \left( \frac{\lambda}{1-\gamma} v_n^{1-\gamma} + \frac{v_n^{q+1}}{q+1} \right) - \left( \frac{\lambda}{1-\gamma} u_\lambda^{1-\gamma} + \frac{u_\lambda^{q+1}}{q+1} \right) \right] dx - \int_{v_n \leq u_\lambda} v_n (\lambda u_\lambda^{-\gamma} + u_\lambda^q) dx \leq K
\]

this implies,

\[
\frac{1}{p} \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^p}{|x - y|^{N+ps}} dx \, dy - \int_{v_n > u_\lambda} \left( \frac{\lambda}{1-\gamma} v_n^{1-\gamma} + \frac{v_n^{q+1}}{q+1} \right) dx \leq K \tag{4.1}
\]

Again, by using the Definition 4.2 we get

\[
\frac{1}{p} \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^p}{|x - y|^{N+ps}} dx \, dy = \int_{v_n > u_\lambda} (\lambda v_n^{1-\gamma} + v_n^{q+1}) dx + \int_{v_n \leq u_\lambda} v_n (\lambda u_\lambda^{-\gamma} + u_\lambda^q) dx + o_n(1) \|v_n\| \tag{4.2}
\]

Therefore, from (4.1) and (4.2), we have

\[
\|v_n\|^p + O_n(\|v_n\|) \geq \int_{v_n > u_\lambda} v_n^{q+1} dx \geq \frac{q+1}{p} \|v_n\|^p - K \tag{4.3}
\]
By using (4.3) we can conclude that the sequence \( \{v_n\} \) is bounded in \( X_0 \). Since the space \( X_0 \) is reflexive, there exists \( v_\lambda \in X_0 \) such that \( v_n \rightharpoonup v_\lambda \) in \( X_0 \) upto a subsequence. Thus

\[
\int_Q \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}} dxdy \rightarrow \int_Q \frac{|v_\lambda(x) - v_\lambda(y)|^{p-2}(v_\lambda(x) - v_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}} dxdy, \quad \forall \phi \in X_0.
\]

Passing the limit as \( n \to \infty \) and applying the embedding result in Lemma 2.1 we have, for \( \phi \in X_0 \)

\[
\int_Q \frac{|v_\lambda(x) - v_\lambda(y)|^{p-2}(v_\lambda(x) - v_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}} dxdy - \int_\Omega (\frac{\lambda}{v_\lambda} - v_\lambda^q)\phi = 0 \tag{4.4}
\]

Therefore, using the strong maximum principle and (4.4) we conclude that \( v_\lambda \) is a weak solution of the problem (1.3). This completes the proof. \( \square \)

We observe from Lemma 3.3 and the fact that \( \bar{I}_\lambda(tu) \to -\infty \) as \( t \to \infty \) for all \( u \in X_0, u > 0 \), we can conclude that \( \bar{I}_\lambda \) has a Mountain pass geometry near \( u_\lambda \). Therefore, we may fix \( e \in X_0, e > 0 \) such that \( \bar{I}_\lambda(e) < \bar{I}_\lambda(u_\lambda) \). Let \( R = \|e - u_\lambda\| \) and \( r_0 > 0 \) be small enough such that \( u_\lambda \) is a minimizer of \( \bar{I}_\lambda \) on \( B(u_\lambda, r_0) \). Consider the following complete metric space consisting of paths which is defined as

\[
\Gamma = \left\{ \eta \in C \left( \left[ 0, \frac{1}{2} \right], X_0 \right) : \eta(0) = u_\lambda, \eta \left( \frac{1}{2} \right) = e \right\}
\]

and the min-max value for mountain pass level

\[
\delta_0 = \inf_{\eta \in \Gamma} \max_{0 \leq t \leq \frac{1}{2}} I_\lambda(\eta(t))
\]

Let us distinguish between the following two cases.

**Case 1:** (Zero altitude case). There exists \( R_0 > 0 \), such that

\[
\inf \left\{ \bar{I}_\lambda(\bar{u}) : \bar{u} \in X_0, \|\bar{u} - u_\lambda\| = r \right\} \leq \bar{I}_\lambda(u_\lambda), \text{ for all } r < R_0. \tag{4.5}
\]

**Case 2:** There exists \( r_1 < r_0 \) such that

\[
\inf \left\{ I_\lambda(\bar{u}) : \bar{u} \in X_0 \text{ and } \|\bar{u} - u_\lambda\| = r_1 \right\} > I_\lambda(u_\lambda). \tag{4.6}
\]

**Remark 4.4.** Observe that, (4.5) implies \( \delta_0 = \bar{I}_\lambda(u_\lambda) \), whereas (4.6) implies \( \delta_0 > \bar{I}_\lambda(u_\lambda) \).
For the “Zero altitude case” let us consider $F = \partial B(u_\lambda, r)$ with $r \leq R_0$. We can then construct a $(PS)_{F,\delta_0}$ sequence and get a second weak solution. We have the following result.

**Lemma 4.5.** Suppose Case 1 holds, then for $1 < p < \infty$, $p-1 < q \leq q^*_s - 1$, $0 < \gamma < 1$ and $\lambda \in (0, \Lambda)$, there exists a weak solution $v_\lambda$ of the problem (1.3) such that $v_\lambda \neq u_\lambda$.

**Proof.** From Theorem 4.1 we can guarantee the existence of a $(PS)_{F,\delta_0}$ sequence $\{v_n\}$ for every $r \leq R_0$. From Lemma 4.3, we can conclude that the sequence $\{v_n\}$ is bounded in $X_0$ and it converges, up to a subsequence, to a weak solution $v_\lambda$ of the problem (1.3). To show $v_\lambda \neq u_\lambda$, it is enough to show the strong convergence of $\{v_n\}$ to $v_\lambda$, i.e. $v_n \rightharpoonup v_\lambda$ strongly in $X_0$ as $n \to \infty$.

Further, by the Sobolev embedding theorem, we get

$$\int_{v_n \geq \Omega} |v_n^{1-\gamma} - v_\lambda^{1-\gamma}| dx = o_n(1) \quad \text{as} \quad n \to \infty$$

(4.8)

Since $v_\lambda$ is a weak solution to the problem (1.3), we get

$$\|v_\lambda\|^p - \|v_\lambda\|_{L^{q+1}(\Omega)}^{q+1} - \lambda \int_\Omega v_\lambda^{1-\gamma} dx = 0$$

(4.9)

Therefore, by passing the limit $n \to \infty$ we obtain

$$\int_{\Omega} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(v_n - v_\lambda)(x) - (v_n - v_\lambda)(y))}{|x - y|^{N+ps}} dxdy$$

$$= \lambda \int_{v_n \geq \Omega} v_n^{1-\gamma}(v_n - v_\lambda) dx + \int_\Omega v_n^{q}(v_n - v_\lambda) dx + o_n(1)$$

(4.10)

Hence, by using (4.7), (4.10) and (4.9) the following holds as $n \to \infty$

$$\|v_n - v_\lambda\|^p = \int_\Omega |v_n - v_\lambda|^{q+1} dx + o_n(1).$$

(4.11)

We now consider the following two cases

(a). $\bar{I}_\lambda(v_\lambda) \neq \bar{I}_\lambda(u_\lambda)$

(b). $\bar{I}_\lambda(v_\lambda) = \bar{I}_\lambda(u_\lambda)$
In case (a) holds, then we are through. Otherwise, from (4.7) we get,
\[ \bar{I}_\lambda(v_n - v_\lambda) = \bar{I}_\lambda(v_n) - \bar{I}_\lambda(v_\lambda) + o_n(1), \quad n \to \infty. \quad (4.12) \]

Consequently, from (4.9) we have
\[ \frac{1}{p} \|v_n - v_\lambda\|^p \leq \frac{1}{q + 1} \|v_n - v_\lambda\|^{q + 1}_{L^{q+1}(\Omega)} + o_n(1), \quad n \to \infty. \quad (4.13) \]

Therefore, from (4.11) and (4.13), we get \( \|v_n - v_\lambda\| \to 0 \) as \( n \to \infty \). Hence \( \|u_\lambda - v_\lambda\| = r \) and \( v_\lambda \neq u_\lambda \). This completes the proof.

Before we state the multiplicity result for Case 2, let us accumulate the necessary tools for this. Let \( U(x) = (1 + |x|^{p'})^{-\frac{N-sp}{p}} \) and \( U_\epsilon(x) = \epsilon^{-\frac{N-sp}{p}} U\left(\frac{|x|}{\epsilon}\right) \), where \( \epsilon > 0 \), \( x \in \mathbb{R}^N \) and \( p' = \frac{p}{p-1} \). Therefore,
\[ U_\epsilon(x) = \frac{\epsilon^{(N-sp)/(p')}}{(\epsilon^{p'} + |x|^{p'})^{\frac{N-sp}{p}}}. \quad (4.14) \]

For a fixed \( r > 0 \) such that,
\[ B_{4r} \subset \Omega \quad (4.15) \]

let us consider, \( \phi \in C_0^\infty(\mathbb{R}^N) \) as,
\[
\begin{cases}
0 \leq \phi \leq 1, & \text{in } \mathbb{R}^N \\
\phi \equiv 0, & \text{in } \mathbb{R}^N \setminus B_{2r} \\
\phi \equiv 1 & \text{in } B_r.
\end{cases}
\]

Henceforth, \( r \) will denote any such number satisfying (4.15). Consider the following nonnegative family of truncated functions
\[ \eta_\epsilon(x) = U_\epsilon(x) \phi(x). \quad (4.16) \]

We now prove the following proposition.

**Proposition 4.6.** Let \( \rho > 0 \). Then for every \( \epsilon > 0 \) and for any \( x \in \mathbb{R}^N \setminus B_\rho \), the following holds
\[
(a) \quad |\eta_\epsilon(x)| \leq C\epsilon^{\frac{(N-ps)}{p}}(\frac{p'}{p})
\]
\[
(b) \quad |\nabla \eta_\epsilon(x)| \leq C\epsilon^{\frac{(N-ps)}{p}}(\frac{p'}{p})
\]
Proof.  (a) We have that, \( U_\epsilon(x) = \epsilon \frac{N - \alpha_\rho}{p} U\left(\frac{|x|}{\epsilon}\right) \). Thus, for \( x \in B_\rho^c \) we have

\[
|\eta_\epsilon(x)| \leq U_\epsilon(x) \\
\leq \epsilon \frac{(N - \alpha_\rho)}{p} \left( 1 + \frac{\rho |x|^{p'}}{\epsilon} \right)^{-\frac{N - \alpha_\rho}{p}} \\
\leq C \epsilon \frac{(N - \alpha_\rho)}{p} \left( \rho^{p'} - 1 \right) \\
= C \epsilon \frac{(N - \alpha_\rho)}{p} \left( \frac{\rho^{p'}}{p} \right) \tag{4.17}
\]

This proves (a).

(b) For any \( x \in B_\rho^c \) we have,

\[
|\nabla \eta_\epsilon(x)| \leq C \epsilon \frac{(N - \alpha_\rho)}{p} \left[ \left( 1 + \frac{|x|^{p'}}{\epsilon} \right)^{-\frac{N - \alpha_\rho}{p}} + \frac{1}{\rho} \frac{|x|^{p'}}{\epsilon} \left( 1 + \frac{|x|^{p'}}{\epsilon} \right)^{-\frac{N - \alpha_\rho}{p}} \right] \\
\leq C \epsilon \frac{(N - \alpha_\rho)}{p} \left[ \left( 1 + \frac{|x|^{p'}}{\epsilon} \right)^{-\frac{N - \alpha_\rho}{p}} + \frac{1}{\rho} \frac{|x|^{p'}}{\epsilon} \left( 1 + \frac{|x|^{p'}}{\epsilon} \right)^{-\frac{N - \alpha_\rho}{p}} \right] \\
\leq C \epsilon \frac{(N - \alpha_\rho)}{p} \left( \frac{1 + \frac{1}{\rho}}{\epsilon} \right) \left( 1 + \frac{|x|^{p'}}{\epsilon} \right)^{-\frac{N - \alpha_\rho}{p}} \\
\leq C \epsilon \frac{(N - \alpha_\rho)}{p} \left( \frac{\epsilon}{\rho} \right)^{\frac{(N - \alpha_\rho)p'}{p}} \\
\leq C \epsilon \frac{(N - \alpha_\rho)}{p} \left( \frac{\rho^{p'}}{p} \right) \tag{4.18}
\]

Hence the proof. \( \square \)

**Proposition 4.7.** Let \( r > 0 \) be as chosen in \((4.15)\). Then we have the following

(a) For every \( \epsilon > 0 \) and any \( x \in \mathbb{R}^N \), \( y \in \mathbb{R}^N \setminus B_r \) with \( |x - y| \leq \frac{r}{2} \)

\[
|\eta_\epsilon(x) - \eta_\epsilon(y)| \leq C \epsilon \frac{(N - \alpha_\rho)}{p} \left( \frac{\rho^{p'}}{p} \right) |x - y|.
\]

(b) For every \( \epsilon > 0 \) and any \( x, y \in \mathbb{R}^N \setminus B_r \),

\[
|\eta_\epsilon(x) - \eta_\epsilon(y)| \leq C \epsilon \frac{(N - \alpha_\rho)}{p} \left( \frac{\rho^{p'}}{p} \right) \min\{1, |x - y|\}.
\]

**Proof.** (a) For \( x \in \mathbb{R}^N \), \( y \in \mathbb{R}^N \setminus B_r \) with \( |x - y| \leq \frac{r}{2} \), let \( z \) be any point on the line segment joining \( x \) and \( y \), i.e. \( z = tx + (1 - t)y \) for some \( t \in [0, 1] \). Observe that

\[
|z| = |tx + (1 - t)y| \geq |y| - |t(x - y)| \geq r - t \frac{r}{2} \geq \frac{r}{2}. \tag{4.19}
\]
Therefore, with the help of (4.18), (4.19), we have $|\nabla \eta(x)| \leq C\epsilon \frac{(N-ps)(p')}{p}$ for $\rho = \frac{r}{2}$. Hence,

$$|\eta(x) - \eta(y)| \leq C\epsilon \frac{(N-ps)(p')}{p}|x - y|$$  \hspace{1cm} (4.20)

This proves the (a).

(b) We may assume $|x - y| \geq \frac{r}{2}$, for otherwise the proof follows from part (a). Therefore, from (4.17), we have

$$|\eta(x) - \eta(y)| \leq |\eta(x)| + |\eta(y)| \leq C\epsilon \frac{(N-ps)(p')}{p}.$$  \hspace{1cm} (4.21)

This completes the proof of (b). \hfill \square

**Proposition 4.8.** For every sufficiently small $\epsilon > 0$ we have,

$$\int_{Q} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \leq 2^p S^\frac{N}{p} + o(\epsilon^{(N-ps)(p')})$$

where, $S$ is the best Sobolev constant.

**Proof.** We will use the previous propositions to establish this estimate. Let $r > 0$ be chosen as in (4.15). Then, on using (4.16), we have

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+ps}} \, dx \, dy = \int_{B_r \times B_r} \frac{|U_{\epsilon}(x) - U_{\epsilon}(y)|^p}{|x - y|^{N+ps}} \, dx \, dy$$

$$+ 2 \int_{A} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+ps}} \, dx \, dy$$

$$+ 2 \int_{B} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+ps}} \, dx \, dy$$

$$+ \int_{B_r^c \times B_r^c} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \hspace{1cm} (4.22)$$

where, $A = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \in B_r, y \in B_r^c$ and $|x - y| > \frac{r}{2}\}$ and $B = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \in B_r, y \in B_r^c$ and $|x - y| \leq \frac{r}{2}\}$. We will try to estimate the last three terms of (4.22). From (4.21), we have

$$\int_{B_r^c \times B_r^c} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \leq C\epsilon \frac{(N-ps)(p')}{p} \int_{B_{2r} \times \mathbb{R}^N} \min\{1, |x - y|^p\} \, dx \, dy$$

$$= o(\epsilon^{(N-ps)(p')}), \text{ as } \epsilon \to 0.$$  \hspace{1cm} (4.23)
On the other hand, from (4.20) we have,

\[
\int_{\mathbb{R}} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^p}{|x - y|^{N + ps}} \, dxdy \leq C\varepsilon^{(N - ps)(\frac{p}{p'})} \int_{\{x \in B_r, \ y \in B_r^c \mid |x - y| \leq \frac{x}{r}\}} \frac{|x - y|^p}{|x - y|^{N + ps}} \, dxdy \leq \frac{1}{C} \|U_\varepsilon - \eta_\varepsilon\|_{L^{p'}(\mathbb{R}^n)} \leq C \varepsilon^{(N - ps)(\frac{p}{p'})} \int_{|x| \leq r} \int_{|z| \leq \frac{x}{p}} \frac{1}{|z|^{N + ps - p}} \, dz \quad (4.24)
\]

Now, the only estimate remains to be proved is the integral over \( A \) in (4.22), which is the following

\[
\int_{A} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^p}{|x - y|^{N + ps}} \, dxdy \quad (4.25)
\]

Since, \( \eta_\varepsilon(x) = U_\varepsilon(x) \) in \( B_r \), we have

\[
(|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^p) \leq |U_\varepsilon(x) - \eta_\varepsilon(y)|^p = |U_\varepsilon(x) - U_\varepsilon(y) + U_\varepsilon(y) - \eta_\varepsilon(y)|^p \leq (|U_\varepsilon(x) - U_\varepsilon(y)| + |U_\varepsilon(y) - \eta_\varepsilon(y)|)^p \leq 2^{p-1}(|U_\varepsilon(x) - U_\varepsilon(y)|^p + |U_\varepsilon(y) - \eta_\varepsilon(y)|^p) \quad (4.26)
\]

On using (4.26) in (4.25), the integral becomes

\[
\int_{A} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^p}{|x - y|^{N + ps}} \, dxdy \leq 2^{p-1} \int_{A} \frac{|U_\varepsilon(x) - U_\varepsilon(y)|^p}{|x - y|^{N + ps}} \, dxdy + 2^{p-1} \int_{A} \frac{|U_\varepsilon(y) - \eta_\varepsilon(y)|^p}{|x - y|^{N + ps}} \, dxdy \quad (4.27)
\]

We will estimate the last term of (4.27). From (4.17) for \( \rho = r \), when \( \varepsilon \to 0 \) we have

\[
\int_{A} \frac{|U_\varepsilon(y) - \eta_\varepsilon(y)|^p}{|x - y|^{N + ps}} \, dxdy \leq \int_{A} \frac{|U_\varepsilon(y)| + |\eta_\varepsilon(y)|}{|x - y|^{N + ps}} \, dxdy \leq C \int_{A} \frac{|U_\varepsilon(y)|}{|x - y|^{N + ps}} \, dxdy \leq C \varepsilon^{(N - ps)(\frac{p}{p'})} \int_{\{x \in B_r, \ y \in B_r^c \mid |x - y| > \frac{x}{p}\}} \frac{|x - y|^p}{|x - y|^{N + ps}} \, dxdy \quad (4.28)
\]

\[
\leq C \varepsilon^{(N - ps)(\frac{p}{p'})} \int_{|x| \leq r} \int_{|z| > \frac{x}{p}} \frac{1}{|z|^{N + ps - p}} \, dz = o(\varepsilon^{(N - ps)(\frac{p}{p'})}).
\]
Therefore, by using (4.23), (4.24), (4.27) and (4.28), we have

\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta_\epsilon(x) - \eta_\epsilon(y)|^p}{|x - y|^{N+ps}} \, dx \, dy = \int_{B_r \times B_r} \frac{|U_\epsilon(x) - U_\epsilon(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \\
+ 2^{p-1} \int_{A} \frac{|U_\epsilon(x) - U_\epsilon(y)|^p}{|x - y|^{N+ps}} \, dx \, dy + o(\epsilon^{(N-ps)}(|x'|^p)) \\
\leq 2^p \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|U_\epsilon(x) - U_\epsilon(y)|^p}{|x - y|^{N+ps}} \, dx \, dy + o(\epsilon^{(N-ps)}(|x'|^p))
\]

For every \( \epsilon > 0 \), the functions \( U_\epsilon(x) \) are the minimizer of the problem

\[
(-\Delta_p)^s u = |u|^{p^*_s - 2}u, \quad \text{in } \Omega \\
u = 0, \quad \text{on } \partial \Omega
\]

and hence satisfies the following equality

\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|U_\epsilon(x) - U_\epsilon(y)|^p}{|x - y|^{N+ps}} \, dx \, dy = \int_{\mathbb{R}^N} |U_\epsilon(x)|^{p^*_s} \, dx = S^\frac{N}{p^*_s}
\]

Hence, we get

\[
\int_{Q} \frac{|\eta_\epsilon(x) - \eta_\epsilon(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \leq 2^p S^\frac{N}{p^*_s} + o(\epsilon^{(N-ps)}(|x'|^p)) \quad (4.29)
\]

This completes the proof. \( \square \)

**Proposition 4.9.** For a sufficiently small \( \epsilon > 0 \) we have,

(a) \( \int_{\Omega} |\eta_\epsilon|^\beta \, dx \leq C \epsilon^{(\frac{N-ps}{p})(\frac{p'}{p})^\beta} \)

(b) \( \int_{\Omega} |\eta_\epsilon|^{q+1} \, dx \geq C \epsilon^{N-(\frac{N-ps}{p})(q+1)} \)

**Proof.** (a) From (4.17), we have

\[
\int_{\Omega} |\eta_\epsilon(x)|^\beta \, dx \leq C \epsilon^{(\frac{N-ps}{p})(\frac{p'}{p})^\beta} \quad (4.30)
\]
(b) We have
\[
\int_{\Omega} |\eta_\epsilon(x)|^{q+1} dx = C \int_{|x|<r} U_\epsilon^{q+1}(x) dx \\
= C \int_{|x|<r} \epsilon^{(N-\frac{sp}{p})(\frac{q}{p})(q+1)} dx \\
= C \epsilon^{(N-\frac{sp}{p})(\frac{q}{p})(q+1)} \int_{|x|<r} \left( \epsilon^{p'} + |x|^{p'} \right)^{-\frac{sp}{p}(q+1)} \]
\[
= C \epsilon^{N-(\frac{sp}{p})(\frac{q}{p})(q+1)(p-1)} \int_{r}^{\bar{r}} \frac{y^{N-1}}{\left( 1 + y^{p'} \right)^{\frac{sp}{p}(q+1)}} dy \\
\geq C \epsilon^{N-(\frac{sp}{p})(\frac{q}{p})(q+1)(p-1)} \int_{1}^{\bar{r}} y^{N-1-(N-ps)(q+1)} dy \\
= \frac{C \epsilon^{N-(\frac{sp}{p})(q+1)}}{L} \left[ 1 - \left( \frac{1}{\bar{r}} \right)^{L} \right] \\
\geq C' \epsilon^{N-(\frac{sp}{p})(q+1)}, \text{ for some } C' > 0. \quad (4.31)
\]

where, \( L = -(N - (N - ps)(q + 1)) > 0 \). Hence the proof is complete. \( \square \)

We now prove the following Lemma, when Case 2 holds.

**Lemma 4.10.** Suppose Case 2 holds, then for \( 1 < p < \infty, \ p-1 < q \leq p^*_s - 1, \ 0 < \gamma < 1 \) and \( \lambda \in (0, \Lambda) \), there exists a weak solution \( v_\lambda \) of the problem (1.3) such that \( v_\lambda \neq u_\lambda \).

**Proof.** From \( \ref{88} \), we have the weak solution \( u_\lambda \) of the problem (1.3) is bounded. Therefore, there exists positive real numbers \( m \) and \( M \) such that \( m \leq u_\lambda(x) \leq M, \ \forall x \in \Omega \). We have, by Mosconi et al. \( \ref{35} \), that the Palais Smale(PS) condition is satisfied, if

\[
\delta_0 < I_\lambda(u_\lambda) + \frac{s}{N} S^{\frac{N}{ps}}.
\]

**Claim.** \( \sup_{0 \leq t \leq \frac{1}{2}} I_\lambda(u_\lambda + t\eta_v) < I_\lambda(u_\lambda) + \frac{s}{N} S^{\frac{N}{ps}} \).

By the Definitions of \( I_\lambda \) and \( \tilde{I}_\lambda \), we have \( \tilde{I}_\lambda(u_\lambda + t\eta_v) = I_\lambda(u_\lambda + t\eta_v) \) and \( \tilde{I}_\lambda(u_\lambda) = I_\lambda(u_\lambda) \).

Using the estimates given in the page 946, of Azorero and Alonso \( \ref{4} \), one can conclude that

\[
\tilde{I}_\lambda(u_\lambda + t\eta_v) \leq \tilde{I}_\lambda(u_\lambda) + pt \left[ \int_{Q} |u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\eta_v(x) - \eta_v(y)) |x - y|^{N+ps} dxdy \right] + o(\epsilon^\alpha)
\]
for every $\alpha > \frac{N - ps}{p}$. Therefore, we have

$$I_\lambda(u_\lambda + t\eta_\epsilon) - \bar{I}_\lambda(u_\lambda)$$

$$= I_\lambda(u_\lambda + t\eta_\epsilon) - I_\lambda(u_\lambda) - pt \left[ \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\eta_\epsilon(x) - \eta_\epsilon(y))}{|x - y|^{N+ps}} dx dy \right]$$

$$- \lambda \int_\Omega u_\lambda^\gamma \eta_\epsilon(x) dx - \int_\Omega u_\lambda^q \eta_\epsilon(x) dx$$

$$\leq \frac{tp^p}{p} \int_Q \frac{|\eta_\epsilon(x) - \eta_\epsilon(y)|^p}{|x - y|^{N+ps}} dx dy$$

$$+ \lambda \left[ t \int_\Omega u_\lambda^\gamma \eta_\epsilon(x) dx + \frac{1}{1 - \gamma} \int_\Omega |u_\lambda|^{1-\gamma} dx - \frac{1}{1 - \gamma} \int_\Omega |u_\lambda + t\eta_\epsilon|^{1-\gamma} dx \right]$$

$$- \left[ t \int_\Omega u_\lambda^q \eta_\epsilon(x) dx + \frac{1}{q+1} \int_\Omega |u_\lambda|^{q+1} dx - \frac{1}{q+1} \int_\Omega |u_\lambda + t\eta_\epsilon|^{q+1} dx \right] + o(\epsilon^\frac{N-ps}{p}).$$

(4.32)

The following two inequalities holds true [35]. For every $a, b \geq 0$ with $a \geq m$, we have

$$\lambda \left( a^{-\gamma}b + \frac{a^{1-\gamma}}{1 - \gamma} - \frac{(a + b)^{1-\gamma}}{1 - \gamma} \right) \leq C_1 b^\beta, \text{ for some constant } C_1 > 0. \quad (4.33)$$

and

$$\left( \frac{(a + b)^{q+1}}{q+1} - \frac{a^{q+1}}{q+1} - a^q b \right) \geq \frac{t\beta^{q+1}}{q+1}, \text{ for some constants } a, b \geq 0. \quad (4.34)$$

By using the above two inequalities (4.33) and (4.34), the inequality (4.32) becomes

$$I_\lambda(u_\lambda + t\eta_\epsilon) - \bar{I}_\lambda(u_\lambda) \leq \frac{tp^p}{p} \int_Q \frac{|\eta_\epsilon(x) - \eta_\epsilon(y)|^p}{|x - y|^{N+ps}} dx dy$$

$$- \frac{tp^{q+1}}{q+1} \int_\Omega |\eta_\epsilon|^{q+1} dx + C_1 t^\beta \int_\Omega |\eta_\epsilon|^\beta dx$$

(4.35)

Hence, from (4.29), (4.30) and (4.31), we get

$$I_\lambda(u_\lambda + t\eta_\epsilon) - I_\lambda(u_\lambda) \leq \left( \frac{2t^p}{p} \left( S^{\frac{N}{ps}} + o(\epsilon^{(N-4ps)(\frac{q+1}{p})}) \right) \right)$$

$$- C_2 \epsilon^{\frac{N-4ps}{p}(q+1)} + C_3 \epsilon^{\frac{N-4ps}{p}(\frac{q+1}{p})^\beta}$$

(4.36)

Therefore, we can conclude that

$$\sup_{0 \leq t \leq \frac{1}{2}} \{ I_\lambda(u_\lambda + t\eta_\epsilon) - I_\lambda(u_\lambda) \} \leq \frac{s}{N} S^{\frac{N}{ps}}$$

$$\Rightarrow \sup_{0 \leq t \leq \frac{1}{2}} I_\lambda(u_\lambda + t\eta_\epsilon) < I_\lambda(u_\lambda) + \frac{s}{N} S^{\frac{N}{ps}}$$

Hence, by the result in [35], $\{v_n\}$ is a (PS) sequence. Thus the sequence $\{v_n\}$ has a strongly convergent subsequence, from which we conclude that $\delta_0 = I_\lambda(v_\lambda) > I_\lambda(u_\lambda)$. Therefore $v_\lambda \neq u_\lambda$. This completes the proof.
5. Appendix

We begin this section with the following two comparison results.

**Lemma 5.1 (Weak Comparison Principle).** Let \( u, v \in X_0 \). Suppose, \((-\Delta_p)^s u - \frac{\lambda}{v^q} \geq (-\Delta_p)^s v - \frac{\lambda}{u^q}\) weakly with \( v = u = 0 \) in \( \mathbb{R}^N \setminus \Omega \). Then \( v \geq u \) in \( \mathbb{R}^N \).

**Proof.** Since, \((-\Delta_p)^s v - \frac{\lambda}{v^q} \geq (-\Delta_p)^s u - \frac{\lambda}{u^q}\) weakly with \( u = v = 0 \) in \( \mathbb{R}^N \setminus \Omega \), we have
\[
\langle (-\Delta_p)^s v, \phi \rangle - \int_{\Omega} \frac{\lambda \phi}{v^q} dx \geq \langle (-\Delta_p)^s u, \phi \rangle - \int_{\Omega} \frac{\lambda \phi}{u^q} dx, \forall \phi \geq 0 \in X_0. \tag{5.1}
\]
In particular choose \( \phi = (u - v)^+ \). To this choice, the inequality in (5.1) looks as follows.
\[
\langle (-\Delta_p)^s v - (-\Delta_p)^s u, (u - v)^+ \rangle - \int_{\Omega} \lambda (u - v)^+ \left( \frac{1}{v^q} - \frac{1}{u^q} \right) dx \geq 0. \tag{5.2}
\]
Let \( \psi = u - v \). The identity
\[
|b|^{p-2}b - |a|^{p-2}a = (p-1)(b-a) \int_0^1 |a + t(b-a)|^{p-2}dt \tag{5.3}
\]
with \( a = v(x) - v(y), \ b = u(x) - u(y) \) gives
\[
|u(x) - u(y)|^{p-2}(u(x) - u(y)) - |v(x) - v(y)|^{p-2}(v(x) - v(y)) = (p-1)\{(u(y) - v(y)) - (u(x) - v(x))\}Q(x, y) \tag{5.4}
\]
where
\[
Q(x, y) = \int_0^1 |(u(x) - u(y)) + t((v(x) - v(y)) - (u(x) - u(y)))|^{p-2}dt. \tag{5.5}
\]
We choose the test function \( \phi = (u - v)^+ \). We express,
\[
\psi = u - v = (u - v)^+ - (u - v)^-
\]
to further obtain
\[
[\psi(y) - \psi(x)][\phi(x) - \phi(y)] = -(\psi^+(x) - \psi^+(y))^2. \tag{5.6}
\]
The equation in (5.6) implies
\[
0 \geq \langle (-\Delta_p)^s v - (-\Delta_p)^s u, (v - u)^+ \rangle
= -(p-1) \frac{Q(x, y)}{|x-y|^{N+sp}} (\psi^+(x) - \psi^+(y))^2
\geq 0. \tag{5.7}
\]
This leads to the conclusion that the Lebesgue measure of \( \Omega^+ \), i.e., \( |\Omega^+| = 0 \). In other words \( v \geq u \) a.e. in \( \Omega \). \qed
Lemma 5.2. Consider the problem

\[ (-\Delta_p)^s u = f \text{ in } \Omega \]
\[ u = g \text{ in } \mathbb{R}^N \setminus \Omega, \]  

where \( f \in L^\infty(\Omega) \) and \( g \in W^{s,p}(\mathbb{R}^N) \). If \( u \in W^{s,p}(\mathbb{R}^N) \) is a weak super-solution of (5.8) with \( f = 0 \) and \( g \geq 0 \), then \( u \geq 0 \) a.e. and admits a lower semicontinuous representation in \( \Omega \).

Proof. We first show that \( u \geq 0 \in W^{s,p}(\mathbb{R}^N) \). We already have \( u = g \geq 0 \) in \( \mathbb{R}^N \setminus \Omega \).

Thus by an elementary inequality \((a - b)(a_+ - b_+) \leq -(a_+ - b_+)^2\) we have

\[
0 \leq \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p-2}(u(x) - u(y))(u_+(x) - u_+(y)) \frac{dxdy}{|x-y|^{N+ps}} \leq - \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p-2}(u_+(x) - u_+(y))^2 \frac{dxdy}{|x-y|^{N+ps}}.
\]  

(5.9)

From (5.9) we have \( u \geq 0 \) a.e. in \( \Omega \).

We now define the following

\[ u^*(x_0) = \liminf_{x \to x_0} u(x). \]  

(5.10)

Clearly \( u^* \geq 0 \) a.e. in \( \Omega \) since \( u \geq 0 \) a.e. in \( \Omega \). Let \( x_0 \in \Omega \) be a Lebesgue point. By the definition of a Lebesgue point we have

\[
u(x_0) = \lim_{r \to 0^+} \frac{1}{B_r(x_0)} \int_{B_r(x_0)} u(x) dx = \lim_{r \to 0^+} \inf_{B_r(x_0)} u(x) = u^*(x_0).
\]  

(5.11)

We now prove the reverse inequality, i.e. \( u \leq u^* \) a.e. in \( \Omega \). Consider the function \(-u\), which serves as a weak subsolution to (5.8), with \( k = u(x_0) \) to obtain

\[
\text{ess sup}_{B_{r/2}(x_0)} (-u) \leq -u(x_0) + \text{Tail}((u(x_0) - u)_+; x_0, r/2) + C \left( \int_{B_r(x_0)} (u(x_0) - u(x))^p_+ dx \right)^{1/p}.
\]  

(5.12)

Passing the limit \( r \to 0^+ \), we have

\[
\lim_{r \to 0^+} \left( \int_{B_r(x_0)} (u(x_0) - u(x))^p_+ dx \right)^{1/p} = 0.
\]  

(5.13)
since, \( x_0 \in \Omega \) is a Lebesgue point. Also by the Hölder’s inequality we have
\[
\text{Tail}((u(x_0) - u)_+; x_0, r/2) \leq r^{2s} \left( \int_{\mathbb{R}^{2N}\setminus B_r(x_0)} \frac{(u(x_0) - u(x))^p}{|x_0 - x|^{N+ps}} \, dx \, dy \right)^{1/p}
\times \left( \int_{\mathbb{R}^N \setminus B_r(x_0)} \frac{1}{|x_0 - x|^{N+ps}} \, dx \, dy \right)^{1/q}
\leq C r^s \left( \int_{\mathbb{R}^N} \frac{|u(x_0) - u(x)|^p}{|x - x_0|^{N+ps}} \, dx \right)^{1/p} \to 0 \text{ as } r \to 0^+. \quad (5.14)
\]
Thus we have
\[
\lim_{r \to 0^+} \text{ess sup}_{B_{r/2}(x_0)} (-u) \leq -u(x_0). \quad (5.15)
\]
This implies \( u^*(x_0) \geq u(x_0) \) for every Lebesgue point in \( \Omega \) and hence for almost all \( x \in \Omega \). From (5.11) and (5.15), we obtain the lower semicontinuous representation of \( u \).

**Corollary 5.3** (Strong Maximum Principle). Let \( u \in X_0 \). Suppose \( u \geq 0 \in \mathbb{R}^N \setminus \Omega \), and for all \( \phi \in X_0 \) with \( \phi \geq 0 \), we have
\[
\int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} (\phi(x) - \phi(y)) \, dx \, dy \geq 0.
\]
Then \( u \) has a lower semicontinuous representation in \( \Omega \), such that either \( u \equiv 0 \) or \( u > 0 \).

**Proof.** See Lemma 2.3 of [36]. \qed

We now prove the following two Lemmas with the help of Lemma 5.4, to establish the Gâteaux differentiability of the functional \( I_\lambda : X_0 \to \mathbb{R} \), for \( 0 < \gamma < 1 \).

**Lemma 5.4.** For every \( 0 < \gamma < 1 \), there exists \( C_\gamma > 0 \), depending on \( \gamma \), such that the following inequality holds true
\[
\int_0^1 |a + tb|^{-\gamma} \, dt \leq C_\gamma \left( \max_{t \in [0,1]} |a + tb| \right)^{-\gamma} \quad (5.16)
\]

**Proof.** The proof of this can be found in Lemma A.1. of [44]. \qed

**Lemma 5.5.** Let \( 0 < \gamma < 1 \), \( 1 < p < \infty \), \( p - 1 < q \leq p_*^\gamma - 1 \) and \( \phi_1 \) be the first eigenvector of the fractional \( p \)-Laplacian operator. Suppose \( u, v \in X_0 \) with \( u \geq \epsilon \phi_1 \), for some \( \epsilon > 0 \). Then we have
\[
\lim_{t \to 0} \frac{I_\lambda(u + tv) - I_\lambda(u)}{t} = \int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} (v(x) - v(y)) \, dx \, dy
- \lambda \int_\Omega u^{-\gamma} v \, dx - \int_\Omega u^q v \, dx \quad (5.17)
\]
Proof. In order to estimate (5.17), it is enough to prove the convergence of the singular term \( \int_\Omega u^{-\gamma} v dx \). Let \( v \in X_0 \) and \( t > 0 \) be sufficiently small. Then we have

\[
0 \leq \frac{I_\lambda(u + tv) - I_\lambda(u)}{t} = \frac{1}{p} \left( \frac{\|u + tv\|^p - \|u\|^p}{t} \right) - \lambda \left( \frac{F(u + tv) - F(u)}{t} \right)
\]

(5.18)

where,

\[
F(u) = \frac{1}{1 - \gamma} \int_\Omega (u^+)^{1-\gamma} dx, \quad \text{for all } x \in X_0.
\]

We see that as \( t \to 0^+ \), we get

\[
(a) \quad \frac{\|u + tv\|^p - \|u\|^p}{t} \to \int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} (v(x) - v(y)) dx dy
\]

\[
(b) \quad \frac{1}{q + 1} \int_\Omega \left( \frac{|u + tv|^{q+1} - |u|^{q+1}}{t} \right) dx \to \int_\Omega |u|^q v dx
\]

We now define for \( z \in \mathbb{R} \setminus \{0\} \),

\[
V(x) = \frac{1}{1 - \gamma} \frac{d}{dz} (z^+)^{1-\gamma} = \begin{cases} 
    z^{-\gamma}, & \text{if } z > 0 \\
    0, & \text{if } z < 0
\end{cases}
\]

(5.19)

Therefore, for every \( x \in \Omega \)

\[
\frac{F(u + tv) - F(u)}{t} = \int_\Omega \left( \int_0^1 V(u + stv) ds \right) v dx
\]

(5.20)

Hence we get for all \( x \in \Omega, \ u(x) > 0, \) and

\[
\lim_{t \to 0^+} \int_0^1 V(u(x) + stv(x)) ds = V(u(x)) = u(x)^{-\gamma}
\]

Also we have

\[
\left| \int_0^1 V(u(x) + stv(x)) ds \right| \leq \int_0^1 |u(x) + stv(x)| ds
\]

Now using the estimate in the previous Lemma 5.16 we get

\[
\left| \int_0^1 V(u(x) + stv(x)) ds \right| \leq C_\gamma \left( \max_{s \in [0,1]} |u(x) + stv(x)| \right)^{-\gamma}
\]

\[
\leq C_\gamma u(x)^{-\gamma}
\]

\[
\leq C_\gamma (\epsilon \phi_1(x))^{-\gamma}
\]

\[
= C_{\epsilon, \gamma} \phi_1(x)^{-\gamma}
\]
where, the constant \( C_{\epsilon, \gamma} > 0 \) is independent of \( x \in \Omega \). Therefore, by the Hardy’s inequality and for all \( v \in X_0 \), we have \( v\phi_1^{-\gamma} \in L^1(\Omega) \). Hence the Lemma follows by applying Lesbegue dominated convergence theorem in (5.20) and taking the limit as \( t \to 0^+ \). In fact we have the following Corollary to the Lemma 5.5 \( \Box \)

**Corollary 5.6.** Let \( 0 < \gamma < 1 \), \( 1 < p < \infty \), \( p - 1 < q \leq p_s^* - 1 \). If \( u \in X_0 \) is such that \( u \geq \epsilon \phi_1 \), for some \( \epsilon > 0 \). Then the functional \( I_\lambda : X_0 \to \mathbb{R} \) is Gâteaux differentiable at \( u \). The Gâteaux derivative \( I_\lambda(u) \) at \( u \) is given by

\[
\langle I_\lambda(u), v \rangle = \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}(v(x) - v(y))\,dxdy
\]

(5.21)

\[-\lambda \int_{\Omega} u^{-\gamma}vdx - \int_{\Omega} u^qvdx, \text{ for all } v \in X_0.\]

**Lemma 5.7.** Let \( 0 < \gamma < 1 \), \( 1 < p < \infty \), \( p - 1 < q \leq p_s^* - 1 \). Let \( w \in X_0 \) is such that \( w \geq \epsilon \phi_1 \), for some \( \epsilon > 0 \). For each \( x \in \Omega \), we consider

\[
f_\lambda(x, s) = \begin{cases} \lambda w(x)^{-\gamma} + w(x)^q, & \text{if } s < w(x) \\ \lambda s^{-\gamma} + s^q, & \text{if } s \geq w(x) \end{cases}
\]

with \( F_\lambda(x, s) = \int_0^s f_\lambda(x, t)dt \). For each \( u \in X_0 \) we define

\[
\bar{I}_\lambda(u) = \frac{1}{p} \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\,dxdy - \int_{\Omega} F_\lambda(x, u)dx.
\]

Then the energy functional \( \bar{I}_\lambda \) belongs to \( C^1(X_0, \mathbb{R}) \).

**Proof.** To establish the result we emphasize only on the singular term. Let

\[
g(x, s) = \begin{cases} w(x)^{-\gamma}, & \text{if } s < w(x) \\ s^{-\gamma}, & \text{if } s \geq w(x) \end{cases}
\]

where, \( w \in X_0 \) such that \( w \geq \epsilon \phi_1 \). Let us define \( G(x, s) = \int_0^s g(x, t)dt \) and \( J(u) = \int_{\Omega} G(x, u)dx \). Proceeding with the arguments as in Lemma 5.5 we get \( J(u) \) has a Gâteaux derivative \( J'(u) \) for all \( u \in X_0 \) and it is given by

\[
\langle J'(u), v \rangle = \int_{\Omega} (\max\{u(x), w(x)\})^{-\gamma} v(x)dx.
\]

Now, let \( u_n \in X_0 \) be such that \( u_n \to u \). Then we have, for all \( v \in X_0 \)

\[
|\langle J'(u_n) - J'(u), v \rangle| = \left| \int_{\Omega} \left[(\max\{u_n(x), w(x)\})^{-\gamma} - (\max\{u(x), w(x)\})^{-\gamma}\right]v(x)dx \right|
\]

\[\leq 2 \int_{\Omega} w^{-\gamma}|v|dx
\]

\[\leq 2\epsilon^{-\gamma} \int_{\Omega} \phi_1^{-\gamma}|v|dx.
\]
Now as in Lemma 5.5 using the Hardy’s inequality we conclude that $\phi_1^{-\gamma}v \in L^1(\Omega)$. Hence by Lesbegue dominated convergence theorem we conclude that the Gateaux derivative of $J$ is continuous which guaranties that $J \in C^1(X_0, \mathbb{R})$.

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References

[1] Adimurthi A. and Giacomoni J., Multiplicity of positive solutions for a singular and critical elliptic problem in $\mathbb{R}^2$, *Communications in Contemporary Mathematics*, 8(5), 621-656, 2006.

[2] Ambrosetti A., Brezis H. and Cerami G., Combined effects of concave and convex nonlinearities in some elliptic problems, *Journal of Functional Analysis*, 122(2), 519-543, 1994.

[3] Ambrosetti A. and Rabinowitz P.H., Dual variational methods in critical point theory and applications, *Journal of Functional Analysis*, 14(4), 349-381, 1973.

[4] Azorero J. G. and Alonso I. P., Some results about the existence of a second positive solution in a quasilinear critical problem, *Indiana University Mathematics Journal*, 43(3), 941-957, 1994.

[5] Bal K. and Garain P., Multiplicity results for a quasilinear equation with singular nonlinearity, *arXiv preprint arXiv:1709.05400*, 2017.

[6] Barrios B., Colorado E., Servadei R. and Soria F., A critical fractional equation with concave-convex power nonlinearities, *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, 32(4), 875-900, 2015.

[7] Bertoin J. Lévy Processes, *Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press*, 1998.

[8] Bisci G. M. and Servadei R., A Brezis-Nirenberg splitting approach for nonlocal fractional equations, *Nonlinear Analysis: Theory, Methods & Applications*, 119, 341-353, 2015.
[9] Bisci G. M. and Servadei R., Lower semicontinuity of functionals of fractional type and applications to nonlocal equations with critical sobolev exponent, *Advances in Differential Equations*, 20(7/8), 635-660, 2015.

[10] Boccardo L. and Orsina L., Semilinear elliptic equations with singular nonlinearities, *Calculus of Variations and Partial Differential Equations*, 37(3/4), 363-380, 2010.

[11] Bojdecki T. and Gorostiza L. G., Fractional brownian motion via fractional Laplacian, *Statistics & Probability Letters*, 44(1), 107-108, 1999.

[12] Brändle C., Colorado E., de Pablo A. and Sánchez U., A concave-convex elliptic problem involving the fractional Laplacian, *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 143(1), 39-71, 2013.

[13] Brasco L. and Parini E., The second eigenvalue of the fractional $p$-Laplacian, *Advances in Calculus of Variations*, 9(4), 323-355, 2016.

[14] Brezis H. and Lieb E., A relation between pointwise convergence of functions and convergence of functionals, *Proceedings of the American Mathematical Society*, 88(3), 486-490, 1983.

[15] Canino A., Montoro L., Sciunzi B. and Squassina M., Nonlocal problems with singular nonlinearity, *Bulletin des Sciences Mathématiques*, 141(3), 223-250, 2017.

[16] Choudhuri D. and Soni A., Existence of multiple solutions to a partial differential equation involving the fractional $p$-Laplacian, *Journal of Analysis*, 23, 33-46, 2015.

[17] Coclite M. M. and Palmieri G., On a singular nonlinear Dirichlet problem, *Communications in Partial Differential Equations*, 14(10), 1315-1327, 1989.

[18] Crandall M. G., Rabinowitz P. H. and Tartar L., On a Dirichlet problem with a singular nonlinearity, *Communications in Partial Differential Equations*, 2(2), 193-222, 1977.

[19] Dhanya R., Giacomoni J., Prashanth S. and Saoudi K., Global bifurcation and local multiplicity results for elliptic equations with singular nonlinearity of super exponential growth in $\mathbb{R}^2$, *Advances in Differential Equations*, 17(3/4), 369-400, 2012.

[20] Diaz J. I., Morel J. M. and Oswald L., An elliptic equation with singular nonlinearity, *Communications in Partial Differential Equations*, 12(12), 1333-1344, 1987.
[21] Fang Y. Existence, uniqueness of positive solution to a fractional Laplacians with singular nonlinearity, *arXiv preprint arXiv:1403.3149*, 2014.

[22] Franzina G. and Palatucci G., Fractional $p$-eigenvalues, *Rivista di Matematica della Universitá di Parma*, 5(2), 373-386, 2014.

[23] Ghanmi A. and Saoudi K., A multiplicity results for a singular problem involving the fractional $p$-Laplacian operator, *Complex variables and elliptic equations*, 61(9), 1199-1216, 2016.

[24] Ghanmi A. and Saoudi K., The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator, *Fractional Differential Calculus*, 6(2), 201-217, 2016.

[25] Giacomoni J. and Saoudi K., Multiplicity of positive solutions for a singular and critical problem, *Nonlinear Analysis: Theory, Methods & Applications*, 71(9), 4060-4077, 2009.

[26] Giacomoni J., Schindler I. and Takáč P., Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation, *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 6(1), 117-158, 2007.

[27] Giacomoni J. and Sreenadh K., Multiplicity results for a singular and quasilinear equation, *Discrete and Continuous Dynamical Systems*, 2007(special), 429-435, 2007.

[28] Ghoussoub N. and Preiss D., A general mountain pass principle for locating and classifying critical points, *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, 6(5), 321-330, 1989.

[29] Haitao Y., Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem, *Journal of Differential Equations*, 189(2), 487-512, 2003.

[30] Hirano N., Saccon C. and Shioji N., Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities, *Advances in Differential Equations*, 9(1-2), 197-220, 2004.

[31] Iannizzotto A., Mosconi S. and Squassina M., Global Hölder regularity for the fractional $p$-Laplacian, *Revista Matemática Iberoamericana*, 32(4), 1353-1392, 2016.

[32] Iannizzotto A., Mosconi S. and Squassina M., $H^s$ versus $C^0$-weighted minimizers, *Nonlinear Differential Equations and Applications NoDEA*, 22(3), 477-497, 2015.
[33] Lazer A. C. and McKenna P. J., On a singular nonlinear elliptic boundary-value problem, *Proceedings of the American Mathematical Society*, 111(3), 721-730, 1991.

[34] Lindgren E. and Lindqvist P., Fractional eigenvalues, *Calculus of Variations and Partial Differential Equations*, 49(1-2), 795-826, 2014.

[35] Mosconi S., Perera K., Squassina M. and Yang Y., The Brezis-Nirenberg problem for the fractional $p$-Laplacian, *Calculus of Variations and Partial Differential Equations*, 55(4), 105, 2016.

[36] Mosconi S. and Squassina M., Nonlocal problems at nearly critical growth, *Nonlinear Analysis: Theory, Methods & Applications*, 136, 84-101, 2016.

[37] Mukherjee T. and Sreenadh K., Fractional elliptic equations with critical growth and singular nonlinearities, *Electronic Journal of Differential Equations*, 2016(54), 1-23, 2016.

[38] Mukherjee T. and Sreenadh K., On Dirichlet problem for fractional $p$-Laplacian with singular non-linearity, *Advances in Nonlinear Analysis*, 2016.

[39] Rosen G. Minimum value for $c$ in the sobolev inequality $|\varphi^3| \leq c|\nabla \varphi|^3$, *SIAM Journal on Applied Mathematics*, 21(1), 30-32, 1971.

[40] Saoudi K., A critical fractional elliptic equation with singular nonlinearities, *Fractional Calculus and Applied Analysis*, 20(6), 1507-1530, 2017.

[41] Servadei R. and Valdinoci E., A Brezis-Nirenberg result for non-local critical equations in low dimension, *Communications on Pure and Applied Analysis*, 12(6), 2445-2464, 2013.

[42] Servadei R. and Valdinoci E., Mountain pass solutions for non-local elliptic operators, *Journal of Mathematical Analysis and Applications*, 389(2), 887-898, 2012.

[43] Servadei R. and Valdinoci E., Variational methods for non-local operators of elliptic type, *Discrete and Continuous Dynamical Systems*, 33(5), 2105-2137, 2013.

[44] Takáč P., On the fredholm alternative for the $p$-Laplacian at the first eigenvalue, *Indiana University mathematics journal*, 51(1), 187-237, 2002.

[45] Tankov P. and Cont R., Financial modelling with jump processes, *Chapman and Hall, CRC Financial Mathematics Series*, 2003.

[46] Valdinoci E., From the long jump random walk to the fractional Laplacian. *arXiv preprint arXiv:0901.3261* 2009.
[47] Wei Y. and Su X., Multiplicity of solutions for non-local elliptic equations driven by the fractional Laplacian. *Calculus of Variations and Partial Differential Equations*, 52(1-2), 95-124, 2015.

[48] Yijing S., Shaoping W. and Yiming L., Combined effects of singular and superlinear nonlinearities in some singular boundary value problems. *Journal of Differential Equations*, 176(2), 511-531, 2001.