A joint limit theorem for zeta-functions of newforms

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Abstract. In the paper a joint limit theorem for zeta-functions of newforms on the complex plane is proved.

Keywords: limit theorem, zeta-function, newform.

Let $SL(2, \mathbb{Z})$ be the full modular group, and for $q \in \mathbb{Z}$,

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S(2\mathbb{Z}) : c \equiv 0 \pmod{q} \right\}$$

be its Hecke subgroup.

Suppose that $F(z)$ is a holomorphic function on the upper half plane $\text{Im} z > 0$, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$ satisfies the functional equation

$$F\left(\frac{az + b}{cz + d}\right) = (cz + d)^k F(z), \quad k \in 2\mathbb{N},$$

and is holomorphic and vanishing at cusps. Then $F(z)$ is called a cusp form of weight $k$ and level $q$, and has the following Fourier series expansion at infinity

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz}.$$

Denote the space of all cusp forms of weight $k$ and level $q$ by $S_k(\Gamma_0(q))$. For every $d|q$, the element of the space $S_k(\Gamma_0(d))$ can also be considered as an element of the space $S_k(\Gamma_0(q))$. The form $F \in S_k(\Gamma_0(q))$ is called a newform if it is not a cup form of level less than $q$, and if it is an eigenfunction of all Hecke operators. Then we have that $c(1) \neq 0$, therefore, we may assume that $F$ is a normalized newform, i.e.,

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz}, \quad c(1) = 1.$$

Let $s = \sigma + it$ be a complex variable. To a newform $F$, we attach the $L$-function $L(s, F)$ defined, for $\sigma > \frac{k+1}{2}$, by

$$L(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$
Moreover, $L(s, F)$ has, for $\sigma > \frac{k_j + 1}{2}$, the Euler product over primes

$$L(s, F) = \prod_{p \nmid q} \left( 1 - \frac{c(p)}{p^s} \right)^{-1} \prod_{p \nmid q} \left( 1 - \frac{c(p)}{p^s} + \frac{1}{p^{2s+1-k}} \right)^{-1},$$

is analytically continuable to an entire function and satisfies the functional equation

$$q^{s/2}(2\pi)^{-s} \Gamma(s)L(s, F) = \varepsilon(-1)^{k/2} q^{(k-s)/2}(2\pi)^{k-s-k} \Gamma(k-s)L(k-s, F),$$

where $\varepsilon = \pm 1$.

A. Laurinčikas, K. Matsumoto and J. Steuding [1] obtained a limit theorem for the function $L(s, F)$ and applied it for the investigation of the universality of $L(s, F)$. Let $D = \{ s \in \mathbb{C} : \frac{k_j}{2} < \sigma < \frac{k_j + 1}{2} \}$, and $H(D)$ denote the space of analytic functions on $D$ equipped with the topology of uniform convergence on compacta. Let $B(S)$ stand for the class of Borel sets of the space $S$. Define

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{ s \in \mathbb{C} : |s| = 1 \}$ for all primes $p$. With the product topology and pointwise multiplication, the torus $\Omega$ is a compact topological Abelian group, therefore, on $(\Omega, B(\Omega))$ the probability Haar measure $m_H$ can be defined. This gives the probability space $(\Omega, B(\Omega))$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space $\gamma_p$, and define on the probability space $(\Omega, B(\Omega), m_H)$ the $H(D)$-valued random element $L(s, \omega, F)$ by

$$L(s, \omega, F) = \prod_{p \nmid q} \left( 1 - \frac{\omega(p)c(p)}{p^s} \right)^{-1} \prod_{p \nmid q} \left( 1 - \frac{\omega(p)c(p)}{p^s} + \frac{\omega^2(p)}{p^{2s+1-k}} \right)^{-1}.$$

Let $P_L$ be the distribution of $L(s, \omega, F)$, i.e.,

$$P_L(A) = m_H(\omega \in \Omega : L(s, \omega, F) \in A), \quad A \in B(H(D)).$$

Denote by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the following statement holds.

**Theorem 1.** (See [1].) The probability measure

$$\frac{1}{T} \text{meas}\{ \tau \in [0, T] : L(s + i\tau, F) \in A \}, \quad A \in B(H(D)),$$

converges weakly to $P_L$ as $T \to \infty$.

Our aim is a joint limit theorem for newforms. For $j = 1, \ldots, r$, let $F_j$ be a new form of weight $k_j$ and level $q_j$, and $L(s, F_j)$ be the corresponding $L$-function given, for $\sigma > \frac{k_j + 1}{2}$, by

$$L(s, F_j) = \prod_{m=1}^{\infty} \frac{c_j(m)}{m^s} \prod_{p | q_j} \left( 1 - \frac{c_j(p)}{p^s} \right)^{-1} \prod_{p | q_j} \left( 1 - \frac{c_j(p)}{p^s} + \frac{1}{p^{2s+1-k_j}} \right)^{-1},$$

where $c_j(p) = p^{\frac{k_j}{2}}$ for $p \nmid q_j$ and $c_j(p) = \frac{p^{k_j} - 1}{p-1}$ for $p | q_j$. See [2].
where
\[ F_j(z) = \sum_{m=1}^{\infty} c_j(m) e^{2\pi i m z}, \quad c_j(1) = 1. \]

On the probability space \((\Omega, \mathcal{B}(\Omega), m_H)\), define \(C^r\)-valued random element \(L(\sigma, \omega, F)\) by the formula
\[
L(\sigma, \omega, F) = (L(\sigma_1, \omega, F_1), \ldots, L(\sigma_r, \omega, F_r)),
\]
where
\[
L(\sigma_j, \omega, F_j) = \prod_{p|q_j} \left(1 - \frac{\omega(p)c_j(p)}{p^{\sigma_j}}\right)^{-1} \prod_{p \not| q_j} \left(1 - \frac{\omega(p)c_j(p)}{p^{\sigma_j}} + \frac{\omega^2(p)}{p^{2\sigma_j+k_j}}\right)^{-1},
\]
and \(\sigma = (\sigma_1, \ldots, \sigma_r), F = (F_1, \ldots, F_r)\). Denote by \(P_L\) the distribution of \(L(\sigma, \omega, F)\).

Then we have the following theorem.

**Theorem 2.** Suppose that \(\sigma_j > \frac{k_j}{2}, j = 1, \ldots, r\). Then the probability measure
\[
P_T(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\{t \in [0, T]: (L(\sigma_1 + it, F_1), \ldots, L(\sigma_r + it, F_r)) \in A\},
\]
\(A \in \mathcal{B}(C^r)\),

converges weakly to \(P_L\) as \(T \to \infty\).

A generalization of Theorem 2 to the space of analytic functions is also possible.

We will give only a sketch of the proof of Theorem 2. Let \(P\) denote the set of all prime numbers.

**Lemma 1.** The probability measure
\[
Q_T(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\{t \in [0, T]: (p^{-it}: p \in P) \in A\}, \quad A \in \mathcal{B}(\Omega),
\]

converges weakly to the Haar measure \(m_H\) on \((\Omega, \mathcal{B}(\Omega))\) as \(T \to \infty\).

Proof of the lemma is given in [1].

Now let \(\sigma_1 > \frac{1}{2}\) be fixed, and, for \(m, n \in \mathbb{N}\),

\[
v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}.
\]

For \(j = 1, \ldots, r\) define
\[
L_n(s, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m)v_n(m)}{m^s},
\]
and, for \(\hat{\omega} \in \Omega\),
\[
L_n(s, \hat{\omega}, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m)\hat{\omega}(m)v_n(m)}{m^s}.
\]

Then the series for \(L_n(s, F_j)\) and \(L_n(s, \omega, F_j)\) converge absolutely for \(\sigma > \frac{k_j}{2}\).
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Lemma 2. Suppose that \( \sigma_j > \frac{k_j}{2}, \ j = 1, \ldots, r. \) Then the probability measures

\[
P_{T,n}(A) \overset{\text{def}}{=} \frac{1}{T} \meas \{ t \in [0, T] : (L_n(\sigma_1 + it, F_1), \ldots, L_n(\sigma_r + it, F_r)) \in A \},
\]

\( A \in \mathcal{B}(\mathbb{C}^r), \)

and

\[
\tilde{P}_{T,n}(A) \overset{\text{def}}{=} \frac{1}{T} \meas \{ t \in [0, T] : (\tilde{L}_n(\sigma_1 + it, \tilde{\omega}, F_1), \ldots, \tilde{L}_n(\sigma_r + it, \tilde{\omega}, F_r)) \in A \},
\]

\( A \in \mathcal{B}(\mathbb{C}^r), \)

both converge weakly to the same probability measure on \((\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))\) as \( T \to \infty. \)

Proof. The lemma easily follows Lemma 1, continuity of mappings \( \tilde{h}_n : \Omega \to \mathbb{C}^r \) and \( h_n : \Omega \to \mathbb{C}^r \) given by the formulae

\[
h_n(\omega) = (L_n(\sigma_1, \omega, F_1), \ldots, L_n(\sigma_r, \omega, F_r))
\]

and

\[
\tilde{h}_n(\omega) = (L_n(\sigma_1, \tilde{\omega}, F_1), \ldots, L_n(\sigma_r, \tilde{\omega}, F_r)),
\]

respectively, and of Theorem 5.1 from [2]. The limit measure in both the cases is of the form \( m_Hh_n^{-1}. \) This follows from the invariance of the Haar measure \( m_H. \)

To pass from the functions \( L_n(s, F_j) \) to \( L(s, F_j) \), the following approximation is used. Let, for \( \tilde{\omega}_1 = (z_{11}, \ldots, z_{1r}) \) and \( \tilde{\omega}_2 = (z_{21}, \ldots, z_{2r}), \)

\[
g(\tilde{\omega}_1, \tilde{\omega}_2) = \left( \sum_{k=1}^{r} |z_{1k} - z_{2k}|^2 \right)^{1/2},
\]

\[
L_n(\sigma + it, \tilde{\omega}, E) = (L_n(\sigma_1 + it, F_1), \ldots, L_n(\sigma_r + it, F_r)),
\]

\[
L_n(\sigma + it, \omega, E) = (L_n(\sigma_1 + it, \omega, F_1), \ldots, L_n(\sigma_r + it, \omega, F_r)),
\]

\[
\tilde{L}(\sigma + it, F) = (L(\sigma_1 + it, F_1), \ldots, L(\sigma_r + it, F_r)).
\]

Lemma 3. Suppose that \( \sigma_j > \frac{k_j}{2}, \ j = 1, \ldots, r. \) Then

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T g(L_n(\sigma + it, F), \tilde{L}(\sigma + it, F)) \, dt = 0
\]

and, for almost all \( \omega \in \Omega, \)

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T g(L_n(\sigma + it, \omega, F), \tilde{L}(\sigma + it, \omega, F)) \, dt = 0.
\]

Proof of lemma follows from the corresponding one-dimensional statements, and from the definition of the metric \( g. \)

Define one more probability measure

\[
\tilde{P}_T(A) \overset{\text{def}}{=} \frac{1}{T} \meas \{ t \in [0, T] : \tilde{L}(\sigma + it, \omega, F) \in A \}, \ A \in \mathcal{B}(\mathbb{C}^r).
\]
Lemma 4. Suppose that $\sigma_j > \frac{N_j}{T}$, $j = 1, \ldots, r$. Then the measures $P_T$ and $\tilde{P}_T$ both converge weakly to the same probability measure $P$ on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ as $T \to \infty$.

Proof. Let $\theta$ be a random variable defined in a certain probability space $(\hat{\Omega}, \mathcal{F}, \mu)$ and uniformly distributed on $[0, 1]$. Define

$$X_{T,n}(\sigma) = L_n(\sigma + i\theta T, F).$$

Then, by Lemma 4,

$$X_{T,n} \xrightarrow{D} X_n,$$  \hspace{1cm} (1)

where $X_n$ is the random element with the distribution $P_n$, and $P_n$ is the limit measure in Lemma 4. After this, it is proved that the family of probability measures $\{P_n; n \in \mathbb{N}\}$ is tight. Hence, by the Prokhorov theorem, it is relatively compact. Thus, there exists a sequence $\{P_{n_k}\} \subset \{P_n\}$ such that $P_{n_k}$ converges weakly to a certain probability measure $P$. In other words,

$$X_{n_k} \xrightarrow{D} P.$$  \hspace{1cm} (2)

Define

$$X_T(\sigma) = L(\sigma + i\theta T, F).$$

Then, in view of Lemma 5, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \mu(\phi(X_T(\sigma), X_{T,n}(\sigma)) \geq \varepsilon) \leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\varepsilon T} \int_0^T \phi(L(\sigma + it, F), L_n(\sigma + it, F)) \, dt = 0.$$

This, (1), (2) and Theorem 4.2 of [2] show that

$$X_T(\sigma) \xrightarrow{D} P.$$  \hspace{1cm} (3)

Thus, $P_T$ converges weakly to $P$ as $T \to \infty$.

Repeating the above arguments for the random elements

$$\tilde{X}_{T,n}(\sigma) = L_n(\sigma + iT, \omega, F)$$

and

$$\tilde{X}_T(\sigma) = L(\sigma + i\theta T, \omega, F),$$

we obtain that $\tilde{P}_T$ also converges weakly to $P$ as $T \to \infty$.

Proof of Theorem 2. In view of Lemma 4, it suffices to prove that $P$ coincides with $P_\perp$. For this, elements of the ergodic theory is applied.
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References

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REZIUMĖ

Jungtinė ribinė teorema naujųjų formų dzeta funkcijoms

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Straipsnyje įrodyta jungtinė ribinė teorema kompleksinėje plokštumoje naujųjų formų dzeta funkcijoms.

Raktiniai žodžiai: dzeta funkcija, naujoji forma, ribinė teorema.