On the singularities of solutions to singular perturbation problems

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Abstract. We consider a singularly perturbed complex first order ODE
\( \varepsilon u' = \Phi(x, u, a, \varepsilon) \), \( x, u \in \mathbb{C}, \varepsilon > 0 \) is a small complex parameter and \( a \in \mathbb{C} \) is a control parameter. It is proven
that the singularities of some solutions are regularly spaced and that they move from one to the
next as \( a \) runs about a loop of index one around a value of overstability. This gives a positive
answer to a question of J. L. Callot.

1. Introduction.
In the analytic theory of nonlinear first order singularly perturbed ODEs, equations of the form
\[ \varepsilon u' = \Phi(x, u, a, \varepsilon) \] (1)
are considered, where \( \Phi \) is analytic in some domain \( D_\Phi \) of \( \mathbb{C}^4 \), \( \varepsilon \) is a small complex parameter and \( a \) is a control parameter. The prime denotes the differentiation with respect to \( x \). In this article we will always suppose \( \varepsilon > 0 \).

Solutions of interest are families of pairs \((a(\varepsilon), u(\varepsilon))\), where \( a(\varepsilon) \) tends to some fixed \( a_0 \) as \( \varepsilon \to 0 \), and \( u(\varepsilon) \) is a solution of (1) for the values \( \varepsilon \) and \( a(\varepsilon) \) of the parameters, and \( u(\varepsilon) \) tends,
as \( \varepsilon \to 0 \), to some slow curve, i.e. a curve of equation \( u = u_0(x) \) where \( u_0 \) is analytic in some
simply connected domain \( D \) and satisfies for all \( x \in D \): \((x, u_0(x), a_0, 0) \in D_\Phi \) and
\[ \Phi(x, u_0(x), a_0, 0) = 0 , \]
for some fixed \( a_0 \). Without loss of generality, we may suppose \( a_0 = 0 \) and \( u_0(x) = 0 \) for all
\( x \in D \).

For brevity, we will call a family of solutions simply \textit{a solution} and will not always mention
the dependence on \( \varepsilon \). Sometimes a solution will refer to the pair \((a, u)\).

An important object related to a slow curve is the so-called relief function. Consider
\[ f_0(x) := \frac{\partial \Phi}{\partial u}(x, u_0(x), a_0, 0) \]
and \( F_0(x) := \int_{x_0}^{x} f_0(t) dt \) with \( x_0 \) arbitrary (in the sequel \( x_0 = 0 \)). The relief function associated with the slow curve then is

\[
R(x) := \text{Re} F_0(x) .
\]

(2)

Roughly speaking, a solution which is close to the slow curve at some point \( x_1 \in D \) will exist and remain close to it at any point \( x \) which can be reached from \( x_1 \) by a path in \( D \) where \( R \) decreases, see [4, 7, 8]. The behaviour of solutions of (1) is therefore well described by the “landscape” made up of the level curves of \( R \).

An interesting situation is when the relief function \( R \), seen as a real valued function over \( \mathbb{R}^2 \), has a critical point at some \( x_0 \). This corresponds to \( F_0(x_0) = f_0(x_0) = 0 \). We call such a point a turning point. In this article, only simple turning points will be considered, that is to say we assume \( f'(x_0) \neq 0 \). Without loss of generality we can suppose that \( x_0 = 0 \) and \( f'(x_0) \) is real and positive. Hence the landscape has a saddle point at \( x_0 = 0 \) with two mountains, where \( R > 0 \), and the mountains contain (at least) a part of the real axis. In a natural way, we call the mountains East and West, and the valleys North and South, and denote them respectively \( M_E, M_W, V_N \) and \( V_S \). These mountains and valleys are defined as connected open subsets of \( D \) and, restricting \( D \) if necessary, we may suppose that the remaining part of \( D \) is the separatrix

\[
S = \{ x \in D : R(x) = 0 \} .
\]

Consider now a solution \( u^+ \) starting from a point on the East mountain, say \( x_1 > 0 \), with initial value \( u^+(x_1) = u_0(x_1) \). In the examples we will choose, if possible, \( u^+ \) as the “exceptional” solution defined on the whole mountain, in the sense of [6], i.e. \( x_1 = +\infty \). If \( a = a(\varepsilon) \to a_0 \) as \( \varepsilon \to 0 \), this solution \( u^+ \) is defined and close to \( u_0 \) in a region bounded on the right by the level curve of \( R \) that passes through \( x_1 \) and which contains at least the part of the East mountain below \( x_1 \) and the two valleys. If \( a(\varepsilon) \) is not an overstability value, this region is bounded on the left by \( \partial M_W \cap S \). Overstability values correspond to values \( a(\varepsilon) \) such that \( u^+ \) remains close to \( u_0 \) in a full neighborhood of the turning point \( x = 0 \).

We prove in Section 2 that, under additional conditions on \( \Phi \), the solution \( u^+ \) has singularities near \( \partial M_W \cap S \) that are regularly spaced. More precisely, in a small neighborhood of some \( x \in \partial M_W \cap S, x \neq 0 \), these singularities can be numbered ..., \( x_{-1}, x_0, x_1, ... \) such that they satisfy for any fixed \( k \in \mathbb{Z} \):

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (F_0(x_{k+1}(\varepsilon)) - F_0(x_k(\varepsilon))) = 2\pi i.
\]

We now suppose that \( \lambda := \partial \Phi / \partial a(0,0,0,0) \) is non-zero. This is the transversality condition in [2, 5] and ensures that there exists an overstability value \( a^*(\varepsilon) \), and that \( a = a(\varepsilon) \) is another overstability value if and only if \( a - a^* \) is exponentially small with respect to \( \varepsilon \), see Section 3 for details. Once again, without loss of generality, we will consider \( a^*(\varepsilon) = 0 \). Only the analyticity of \( \Phi \) in \( \varepsilon \) has to be removed from the hypotheses.

Our main result in this article is that, as \( a \) goes around \( a^* = 0 \), the singularities of \( u^+ \) move from one to the next, see Section 5, Theorem 11. This was first observed numerically by J.L. Callot on the “Riccati-Weber” equation

\[
\varepsilon u' = x^2 - u^2 - \varepsilon a .
\]

(3)

With the use of transseries techniques, S. Aït Mokhtar proved this fact recently in [1] for equation (3). Her method, however, may be applied only to singularly perturbed equations of the Riccati type which come from classical Riccati equations \( dU/dX = F(X,U,a) \) through a macroscope \( x = \varepsilon^a X, u = \varepsilon^b U \), for suitable \( a,b \). In contrast, our method is applicable to other Riccati equations that do not fit the framework of [1], and also to higher degree nonlinear equations such as the forced van der Pol equation.
2. The lattice of singularities.

We consider some \( x_0 \in D, x_0 \neq 0 \) near enough to 0 (not depending on \( \varepsilon \)) and we denote by \( u^+ = u^+(x, a, \varepsilon) \) the solution of (1) with initial \( u^+(x_0) = \overline{u} \). Then, as \( a, \varepsilon \to 0 \), \( u^+ \) is close to \( u_0 \equiv 0 \) in the region below \( x_0 \) with respect to the relief function \( R \), but \( u^+ \) cannot be close to 0 in any neighborhood of any point on the level curve \( \{ x \in D ; R(x) = R(x_0) \} \); otherwise, choosing \( x'_0 \) above \( x_0 \) in such a neighborhood, we would get \( u^+(x_0) \to 0 \) as \( \varepsilon \to 0 \).

Except for very particular equations (e.g. linear ones) \( u^+ \) will then have singularities in any such neighborhood. We now describe how these singularities arise. To analyze what occurs near \( x_0 \) we use the change of variables (magnifying glass) \( x = x_0 + \varepsilon X \). Hence \( u \) is a solution of (1) if and only if the function \( U : X \mapsto u(x_0 + \varepsilon X) \) is a solution of

\[
\frac{dU}{dX} = \Phi(x_0 + \varepsilon X, U, a, \varepsilon) .
\]

This is a regular perturbation of the equation

\[
\frac{dU}{dX} = \Phi(x_0, U, 0, 0) .
\]

By the regularity of the dependence of solutions of ODEs on parameters, solutions of (4) and (5) are related by the following.

Proposition 1. Let \( U_0 \) be the solution of (5) with initial condition \( U_0(0) = \overline{u} \), and suppose that \( U_0 \) is analytic on some simply connected domain \( \Delta \). Suppose also that \( \Phi \) is analytic in

\[
\Omega = \{ (x_0 + \varepsilon X, u, a, \varepsilon) \in \mathbb{C}^4 ; X \in \Delta, |u - U_0(X)| < r, |a| < r, |\varepsilon| < r \}
\]

for some \( r > 0 \). Let \( K \) be a compact subset of \( \Delta \). Then there exists \( \rho > 0 \) such that for any \( (a, \varepsilon) \in D(0, \rho) \times [0, \rho] \), the solution \( U_{a,\varepsilon} \) of (4) with the same initial condition \( U_{a,\varepsilon}(0) = \overline{u} \) is defined on \( K \) and satisfies:

\[
\lim_{a,\varepsilon \to 0} \| U_{a,\varepsilon} - U_0 \|_K = 0 ,
\]

where \( \| \cdot \|_K \) is the supremum norm.

Because equation (5) is autonomous, we can describe its solutions. For convenience, we introduce the following notation:

\[
\alpha = f_0(x_0) \quad \quad H_\alpha = \{ X \in \mathbb{C} ; \text{Re}(\alpha X) < 0 \} .
\]

Proposition 2. If \( |\overline{u}| \) is small enough, then the solution \( U_0 \) of (5) with initial condition \( U_0(0) = \overline{u} \) is defined on \( H_\alpha \). Moreover \( U_0 \) is periodic, of period \( 2\pi i/\alpha \), and satisfies

\[
U_0(X) \to 0 \quad \text{as Re}(\alpha X) \to -\infty .
\]

Proof. Since \( \Phi(x, 0, 0, 0) = 0 \) for all \( x \in D \), equation (5) is of the form

\[
\frac{dU}{dX} = U\Psi(U) ,
\]

with \( \Psi \) analytic, \( \Psi(0) = \alpha \), and \( U_0 \) is the solution with initial condition \( U_0(0) = \overline{u} \).
Put $\delta = \pi/8$ and $c = \sin \delta$, and consider $r > 0$ small enough such that $|\Psi(U) - \alpha| \leq c|\alpha|$ for all $|U| \leq r$; hence

$$|\arg(\Psi(U) / \alpha)| \leq \delta \quad \text{and} \quad (1 - c)|\alpha| \leq |\Psi(U)| \leq (1 + c)|\alpha| \quad \text{if} \quad |U| \leq r.$$  \hfill (8)

By Gronwall’s inequality and the well known majorization principle in ODEs, it follows that the solution $U_0$ is defined on the disk $D(0, \pi/|\alpha|)$ and satisfies $|U_0(X)| \leq r$ on this disk when

$$|\Psi(U)| \leq r e^{(1+c)\pi/2}.$$  

Consider now the sector

$$S = \{X \in \mathbb{C} : |\arg(-\alpha(X - X_0))| < \pi/2 - 2\delta\}$$

with $X_0 := \pi/\alpha$, see Figure 1. In other words, $S$ has vertex $X_0$, opening $\pi - 4\delta = \pi/2$ and is bisected by $-\infty / \alpha, X_0]$. By the first inequality of (8), we have $|\arg(-\Psi(U)(X - X_0))| < \pi/2 - \delta$ for all $X \in S$ and $|U| < r$.

Writing $X = X_0 + se^{i\theta}$ with $s \in \mathbb{R}^+$ and $|\theta - \arg(-1/\alpha)| \leq \pi/4$, we get

$$\frac{d(U \overline{U})}{ds} = 2 \text{Re} \left( \frac{dU}{ds} \overline{U} \right) = 2U \overline{U} \text{Re} \left( \Psi(U)e^{i\theta} \right) \leq 2U \overline{U}(1 - c)|\alpha| \cos(\arg[\Psi(U)(X - X_0)]) ,$$

hence

$$\frac{d|U|}{ds} \leq -|U|(1 - c)|\alpha|c .$$

The same majorization principle now shows that the solution $U_0$ is defined on the whole sector $S$ and satisfies

$$|U_0(X)| \leq |U_0(X_0)| \exp(-c(1 - c)|\alpha| |X - X_0|) .$$

This shows that, for $\overline{U}$ small enough, $U_0$ is defined at least on the half strip

$$B = \{X \in \mathbb{C} : \text{Re}(\alpha X) < 0, |\text{Im}(\alpha X)| \leq \pi\}$$

and satisfies (6) there, since this half strip is contained in $S$. That $U_0$ is defined and satisfies (6) on the whole $H_\alpha$ will therefore follow from the periodicity of $U_0$.

This periodicity is proved by considering the reciprocal function of $U_0$. Roughly speaking, equation (7) is rewritten

$$dX = \frac{dU}{U \Psi(U) ,}$$
which yields a reciprocal of \( U_0 \) as a primitive of \((U \Psi(U))^{-1}\). To be more precise, we have to deal with coverings of punctured disks.

Recall that \( \Psi \) is analytic and does not vanish on the disk \( D(0, r) \). Denote by \( \widetilde{D}^* \) the universal covering of the punctured disk \( D^* := D(0, r) \setminus \{0\} \) with base point \( \pi \), i.e. the set of equivalence classes by homotopy of paths in \( D^* \) starting from \( \pi \) and with a fixed end point. As usual, each class is identified with its end point. Denote by \( \tilde{D}^* \) the covering without a base point, i.e. the set of classes of paths with fixed extremities.

Given \( X \) in \( S \), denote by \( \gamma_X \) the “standard” path on \([0, X]\) given by
\[
\gamma_X : [0, 1] \to [0, X], \ t \mapsto tX,
\]
and consider the function \( \tilde{U} : S \to \widetilde{D}^*_\pi \), \( X \mapsto \text{cl}(U_0 \circ \gamma_X) \) which, to \( X \in S \), associates the class of the image under \( U_0 \) of the standard path from \( 0 \) to \( X \).

Notice that, because the sector \( S \) is simply connected, for all \( X \) in \( S \), any path in \( S \) from \( 0 \) to \( X \) is homotopic to \( \gamma_X \). Therefore in the definition of \( \tilde{U} \) the standard path may be replaced by any path in \( S \) from \( 0 \) to \( X \).

Consider now \( X \) defined by
\[
\tilde{X} : \widetilde{D}^* \to \mathbb{C}, \ \tilde{\gamma} \mapsto \int_{\tilde{\gamma}} \frac{ds}{s \Psi(s)},
\]
where \( \gamma \) is any element of the class \( \tilde{\gamma} \).

Because \((\tilde{X} \circ \tilde{U})' = 1\) and \( \tilde{X} \circ \tilde{U}(0) = 0 \), by connectivity \( \tilde{X} \circ \tilde{U} = \text{id}_S \), the identity in \( S \). This shows that \( \tilde{U} \) is one-to-one in \( S \).

Moreover, since \( 0 \) is the only singularity of \( s \mapsto (s \Psi(s))^{-1} \) in \( D(0, r) \), with residue \( 1/\alpha \), we get that for any \( \tilde{\gamma} \in \widetilde{D}^*_\pi \) and any loop \( \tilde{l} \in \widetilde{D}^* \):
\[
\tilde{X}(\tilde{\gamma}\tilde{l}) - \tilde{X}(\tilde{\gamma}) = \int_{\tilde{l}} \frac{ds}{s \Psi(s)} = \frac{2\pi i}{\alpha} \text{ind}(\tilde{l}),
\]
where \( \tilde{\gamma}\tilde{l} \) is the class of the concatenation of two paths, \( \gamma \in \tilde{\gamma} \) and \( l \in \tilde{l} \). For brevity we denote
\[
T := 2\pi i/\alpha.
\]

Now fix \( X, Y \in S \) such that \( U_0(X) = U_0(Y) \) and consider \( \gamma : [0, 1] \to D^* \), \( t \mapsto U_0(tX) \), i.e. \( \gamma = U_0 \circ \gamma_X \) with the above notation. Similarly, denote by \( c \) the image under \( U_0 \) of the standard path in \( S \) from \( X \) to \( Y \) : \( c : [0, 1] \to D^* \), \( t \mapsto U_0((1-t)X + tY) \). Since \( U_0(X) = U_0(Y) \), \( c \) is a loop. Denoting by \( \tilde{\gamma} \) and \( \tilde{c} \) as the corresponding classes in \( \widetilde{D}^*_\pi \), \( \widetilde{D}^*_r \), respectively, we have
\[
Y - X = \tilde{X}(\tilde{\gamma}\tilde{c}) - \tilde{X}(\tilde{\gamma}) = T \text{ ind}(\tilde{c}) \text{ by } (9).
\]

This shows that \( U_0 \) cannot have a smaller period than \( T = 2\pi i/\alpha \). Its remains to show that \( U_0 \) is actually periodic of period \( T \), i.e. that for any \( X \in S \) with \( X + T \in S \), \( U_0(X + T) = U_0(X) \). This amounts to proving that, if \( U_0(X) \) and \( U_0(Y) \) are sufficiently small and \( U_0(X) \neq U_0(Y) \), then \( X - Y \notin T \mathbb{Z} \). In other words, this amounts to proving that for \( r \) small enough, if \( u_1, u_2 \in D^* \) are such that \( p(u_1) \neq p(u_2) \) then
\[
\int_{u_1}^{u_2} \frac{ds}{s \Psi(s)} \neq 0,
\]
where \( p \) is the projection from \( \widetilde{D}^*_r \) to its base \( D^*_r = D(0, r) \setminus \{0\} \).

We first write \( \frac{1}{s \Psi(s)} = \frac{1}{\alpha s} + \phi(s) \) with \( \phi \) analytic near 0. Let \( C > 0 \) be a strict majorant of \( |\phi| \) in \( D(0, r) \). Without loss of generality we may suppose that \( r \leq \frac{1}{2C|\alpha|} \).
Inequality (10) in therefore proved if
\[
\left| \int_{u_1}^{u_2} \frac{ds}{s} \right| \geq C|\alpha| |u_1 - u_2|, \tag{11}
\]
where we define \(|u_1 - u_2| := |p(u_1) - p(u_2)|\).

We consider separately the cases \(|\arg u_1 - \arg u_2| > \pi/2\) and \(|\arg u_1 - \arg u_2| \leq \pi/2\).

In the first case we get
\[
\left| \int_{u_1}^{u_2} \frac{ds}{s} \right| \geq \frac{\pi}{2} \geq \frac{\pi}{4r}|u_1 - u_2| \text{ which gives (11).}
\]

In the second case, with \(\theta := \frac{1}{2}(\arg u_1 + \arg u_2)\) and any \(s \in [u_1, u_2]\), \(\Re \left( \frac{e^{i\theta}}{s} \right) \geq \frac{1}{\sqrt{2|s|}}\), and hence
\[
\left| \int_{u_1}^{u_2} \frac{ds}{s} \right| \geq \frac{|u_1 - u_2|}{\sqrt{2r}} \text{ which also gives (11). This completes the proof of Proposition 2. □}
\]

As a consequence of Propositions 1 and 2, and the local inversion theorem, we get the following.

**Corollary 3.** For all \(N \in \mathbb{N}\) there exists \(\rho > 0\) such that for each \(k \in \{-N, \ldots, N\}\) there is a function \(x_k : D(0, \rho) \times [0, \rho] \to D\) which satisfies \(u^+(x_k(a, \varepsilon), a, \varepsilon) = \pi\) and \(x_{k+1}(a, \varepsilon) - x_k(a, \varepsilon) = \frac{2\pi i \varepsilon}{\alpha}(1 + o(1))\) as \(a, \varepsilon \to 0\).

This means that — locally — the loci where \(u^+\) assumes a given value \(\pi\) are regularly spread over a lattice. We now extend this result to singularities, under an additional assumption.

**Theorem 4.** If \(U_0\) has an isolated singularity at \(X = S_0\) and no other singularity on the ray \(\{X = S_0 + s\pi; s \in \mathbb{R}^-\}\) then, for any fixed \(n \in \mathbb{Z}\), \(u_{n,\varepsilon}\) has a singularity at \(x = s_{n,a,\varepsilon}\) which satisfies
\[
s_{n,a,\varepsilon} = x_0 + \varepsilon S_0 + (n + o(1)) \frac{2\pi i \varepsilon}{\alpha}, \quad a, \varepsilon \to 0. \tag{12}
\]

**Proof.** We first prove by contradiction that there exists a singularity \(S_{a,\varepsilon}\) of \(U_{a,\varepsilon}\) which tends to \(S_0\) as \(a, \varepsilon \to 0\). Suppose that \(U_{a,\varepsilon}\) is analytic in a fixed neighborhood \(V\) of \(S_0\). Removing a small disk \(W\) centered in \(S_0\) and writing \(V \setminus W\) as a union of two simply connected domains \(\Delta_1 \cup \Delta_2\), we get by Proposition 1 that \(U_{a,\varepsilon}\) tends to \(U_0\) on each \(\Delta_i\). Therefore the family \((U_{a,\varepsilon})\) is uniformly bounded on \(V \setminus W\), and hence on the whole \(V\) by the maximum principle. By the well known theorem on normal families, a sequence \((a_n, \varepsilon_n)_{n \in \mathbb{N}}\) exists such that \((U_{a_n, \varepsilon_n})\) converges to an analytic function on the whole \(V\), which must be \(U_0\), which contradicts the assumption. This means that \(S_0\) is a removable singularity of \(U_0\).

This gives a singularity \(x_{0,a,\varepsilon} = x_0 + \varepsilon X_{\varepsilon}\) for \(u_{a,\varepsilon}\). Since \(U_0\) is \(2\pi i/\alpha\) periodic, it has a singularity at \(S_n = S_0 + 2n\pi i/\alpha\) for all \(n \in \mathbb{Z}\), and the same proof gives a singularity \(s_{n,a,\varepsilon}\) for \(u_{a,\varepsilon}\) that satisfies (12).

3. Some basic tools.

We now recall two results of [2, 3]. Then we give a description of the derivative of a solution \(u^+\) with respect to \(a\). Recall the decomposition of \(D\) into mountains, valleys and the separatrix, see Figure 2 below:

\[
D = M_E \cup M_W \cup V_N \cup V_S \cup S.
\]

**Proposition 5.** There exists \(r > 0\) such that for all \(x_1 \in M_E, x_2 \in M_W\) and all \(u_1, u_2 \in D(0, r)\) there exists \(\rho > 0\) that satisfies the following: For all \(\varepsilon \in [0, \rho]\) there is a unique \(a = a(\varepsilon) \in D(0, r)\) and a unique solution \(u\) of (1) with boundary conditions
\[
u(x_1, a, \varepsilon) = u_1, \quad u(x_2, a, \varepsilon) = u_2.
\]
We refer the reader to [2] for a proof. Restricting the domain $D$ if necessary, we may suppose that there is $a^* = a^*(\varepsilon)$ and $u^* = u^*(x, a^*(\varepsilon), \varepsilon)$ which tend both to 0 as $\varepsilon \to 0$, uniformly for $x \in D$. Without loss of generality we can suppose $a^*(\varepsilon) = 0$ and $u^*(x, a^*(\varepsilon), \varepsilon) = 0$ for all $x, \varepsilon$. Only the assumption that $\Phi$ is analytic in $\varepsilon$ in a neighborhood of $0 \in \mathbb{C}$ has to be changed to: $\Phi$ is analytic in $\varepsilon$ in a small sector containing $0, r$ for $r$ small enough.

We now fix $x_1 \in M_F$ and we denote by $u^+ = u^+(x, a, \varepsilon)$ the solution with initial condition $u^+(x_1, a, \varepsilon) = 0$. Restricting $D$ once again, if necessary, we also suppose that $F_0(x) < F_0(x_1) - \delta$ for some $\delta > 0$ and for all $x \in M_W$.

**Proposition 6.** Let $x_2 \in M_W$ and suppose $u_2 \neq 0$ satisfies $|u_2| < r$ for the $r$ of Proposition 5. Consider $a : ]0, \rho[ \to D(0, r)$ given by Proposition 5 such that $u^+(x_2, a(\varepsilon), \varepsilon) = u_2$. Then we have:

$$
\lim_{\varepsilon \to 0} (-\varepsilon \ln |a(\varepsilon)|) = R(x_2) .
$$

Again see [2] for a proof. Let us define a function $F$ as follows. Given $x \in D, a \in D(0, r)$ and $\varepsilon \in ]0, \rho[$ such that there exists a path from 0 to $x$ on which $u^+$ is defined and remains in $D(0, r)$, we define:

$$
F(x, a, \varepsilon) = \int_0^x \Phi_u(s, u^+(s, a, \varepsilon), a, \varepsilon)ds .
$$

We have for all $x \in D$ and all $a : ]0, \rho[ \to D(0, r)$ with $\lim_{\varepsilon \to 0} a(\varepsilon) = 0$, such that $F(x, a(\varepsilon), \varepsilon)$ is defined:

$$
\lim_{\varepsilon \to 0} F(x, a(\varepsilon), \varepsilon) = F_0(x) .
$$

We now consider $v := u^+$, the derivative of $u^+$ with respect to $a$.

**Proposition 7.** Let $x \in M_W$ and $a : ]0, \rho[ \to D(0, r)$ such that $F(x, a(\varepsilon), \varepsilon)$ is defined for all $\varepsilon \in ]0, \rho[$. Then we have the following estimate:

$$
v(x, a(\varepsilon), \varepsilon) = \sqrt{-\frac{\pi}{\varepsilon f_0'(0)}} \lambda e^{F(x, a(\varepsilon), \varepsilon)/\varepsilon} (1 + o(1)) , \quad \varepsilon \to 0 .
$$

**Proof.** For the sake of simplicity, we sometimes omit the dependence in $\varepsilon$. Differentiation of equation (1) with respect to $a$ and using $u^+(x_1, a) = 0$ shows that $v$ is the solution of the equation:

$$
\varepsilon v' = \Phi_u(x, u^+(x, a), a, \varepsilon) v + \Phi_u(x, u^+(x, a), a, \varepsilon) , \quad v(x_1, a) = 0 .
$$

By variation of constants, and using (14), we get:

$$
v(x, a) = \frac{1}{\varepsilon} e^{F(x, a)/\varepsilon} \int_{x_1}^x e^{-F(s, a)/\varepsilon} \Phi_u(s, u^+(s, a), a, \varepsilon)ds .
$$
Then the saddle point method gives
\[ v(x, a) = e^{F(x, a)/\varepsilon} \sqrt{\frac{\pi}{\varepsilon F_{xx}(0, a)}} \Phi_a(0, u^+(0, a), \varepsilon)(1 + o(1)). \]

We conclude with \( \Phi_a(0, u^+(0, a), \varepsilon) = \lambda(1 + o(1)) \) and \( F_{xx}(0, a) = f_x(0, a) = f'_0(0)(1 + o(1)) \) as \( \varepsilon \to 0 \).

\[ \square \]

4. Relationship between \( a \) and \( u \).
In this section, we fix some \( x \in M_W \).

**Lemma 8.** Let \( a = a(\varepsilon) \) be such that, for all \( \varepsilon \in [0, \rho[ \) and all \( b \in [0, a(\varepsilon)] \), \( F(x, b, \varepsilon) \) given by (14) is defined and satisfies
\[ F(x, b, \varepsilon) = F(x, 0, \varepsilon) + o(\varepsilon), \quad \varepsilon \to 0. \]
Then we have
\[ u^+(x, a(\varepsilon), \varepsilon) = a(\varepsilon)v(x, 0, \varepsilon)(1 + o(1)). \]

**Proof.** We use \( u^+(x, a, \varepsilon) = \int_0^a v(x, b, \varepsilon)db \). By hypothesis we have \( e^{F(x, b, \varepsilon)/\varepsilon} = e^{F(x, 0, \varepsilon)/\varepsilon}(1 + o(1)) \) for all \( b \in [0, a(\varepsilon)] \), hence (15) yields \( v(x, b, \varepsilon) = v(x, 0, \varepsilon)(1 + o(1)) \). We conclude with \( u^+(x, 0, \varepsilon) = u^*(x, a(\varepsilon), \varepsilon) = 0 \).

As a consequence of Lemma 8, as the parameter \( a \) moves around 0, then the value at \( x \) of the corresponding solution, \( u^+(x, a) \), moves around \( u^+(x, 0) = 0 \) in the same direction, under the condition that (16) is satisfied. This is expressed in the next Theorem, where condition (16) is removed.

**Theorem 9.** Let \( \gamma_\varepsilon : [0, 1] \to D(0, r) \) be a loop of index 1 around 0 such that for all \( t \in [0, 1] \), \( u^+(x, \gamma_\varepsilon(t)) \) exists and remains in \( D(0, r) \). Then the loop \( t \mapsto u^+(x, \gamma_\varepsilon(t)) \) has index 1 around 0.

**Proof.** First of all, notice that, since \( x \) is on the West mountain, the condition that \( u^+(x, \gamma_\varepsilon(t)) \) exists and remains in \( D(0, r) \) implies that \( \gamma_\varepsilon \) has to be exponentially small, of order at most \( O(e^{-\rho(x)}) \).

Choose \( \delta = \delta(\varepsilon) \) exponentially small. We first modify the loop \( \gamma_\varepsilon \) in \( \tilde{\gamma} = s^+cs^- \) where \( s^+ \) goes from \( \gamma_\varepsilon(0) \) to \( \delta\gamma_\varepsilon(0) \), \( \delta \) is exponentially small, \( s^- \) goes from \( \delta\gamma_\varepsilon(0) \) to \( \gamma_\varepsilon(0) \), and \( c \) is the circle of radius \( |\delta\gamma_\varepsilon(0)| \). Because we can perform this modification with \( a \) avoiding 0, by the uniqueness in Proposition 5, the index of the loop \( t \mapsto u^+(x, \tilde{\gamma}(t)) \) is the same as that of \( t \mapsto u^+(x, \gamma_\varepsilon(t)) \). On \( c \), \( \tilde{\gamma}(t)e^{F_0(x)} \) is still exponentially small (of order \( e^{-\delta} \)), therefore if the path from 0 to \( x \) in (14) is suitably chosen \( u^+(s, \gamma_\varepsilon(t), \varepsilon) \) is exponentially close to \( u^+(s, 0, \varepsilon) \). As a consequence, (16) is satisfied with \( b = \gamma_\varepsilon(t) \). Because the contributions of \( s^+ \) and \( s^- \) cancel, by Lemma 8 we get that the index of \( t \mapsto u^+(x, \tilde{\gamma}(t)) \) is an integer close to the index of \( \tilde{\gamma} \), and therefore equal to 1.

\[ \square \]

5. Relationship between \( a \) and \( x \).
Our aim is to explain how the singularities of \( u^+ \) move as \( a \) moves. However, we will restrict our attention to values of \( x \) where \( u^+ \) takes an assigned value \( \overline{u} \) which is small enough rather than to a singularity. The passage from this value \( \overline{u} \) to a singularity was described in Section 2.

Besides, we find it more convenient to move \( x \) and to see how \( a \) moves. Before fixing the value of \( u \), we look at \( a \) as a function of \( x, u \) and \( \varepsilon \) together. For technical reasons we have to
define this function on a somewhat restricted subdomain $M^\delta_W$ of $M_W$. We hope to be able to extend this domain in a future work.

Let $\delta > 0$ be small enough, and consider

$$
M^\delta_W = \{ x \in M_W : R(x) > \delta \} .
$$

As in Proposition 5, there exist $r, \rho > 0$ such that for any $x \in M^\delta_W, u \in D(0, r)$ and $\varepsilon \in [0, \rho[$, there is a unique value of $a$ such that $|a| < r$ and $u^+(x, a, \varepsilon) = u$.

This defines a function

$$
A : M^\delta_W \times D(0, r) \times [0, \rho[ \times [0, \rho[ \rightarrow \mathbb{R}, (x, u, a, \varepsilon) \mapsto A(x, u, a, \varepsilon)
$$

such that

$$
u^+(x, A(x, u, a, \varepsilon), \varepsilon) = u .
$$

(17)

Because of the analytic dependence of $u^+$ on $a$, by local inversion we get that $A$ is also analytic.

Now we fix $a \in D(0, r)$ and we consider:

$$
\bar{a} : M^\delta_W \times [0, \rho[ \times [0, \rho[ \rightarrow \mathbb{R}, (x, \varepsilon) \mapsto A(x, \varepsilon)
$$

**Lemma 10.** The function $\bar{a}$ is analytic on $M^\delta_W \times [0, \rho[$. Moreover $\bar{a}$ is of the form

$$
\bar{a}(x, \varepsilon) = \exp(-G(x, \varepsilon) / \varepsilon)
$$

where $G : M^\delta_W \times [0, \rho[ \rightarrow \mathbb{C}$ is analytic and satisfies $\lim_{\varepsilon \to 0} G(x, \varepsilon) = F_0(x)$ uniformly on $M^\delta_W$.

**Proof.** We omit sometimes the dependence on $\varepsilon$. Differentiation of equation (17) with respect to $x$, rewritten $u^+(x, A(x, u, a, \varepsilon)) = u$, yields:

$$
u^+(x, \bar{a}(x), \varepsilon) + u^+_0(x, \bar{a}(x)) \bar{\varphi}'(x) = 0 .
$$

Using equation (1) for $u^+_0$ and the notation $\nu$ already used for $u^+_0$, we get:

$$
\bar{\varphi}'(x) = -\frac{\Phi(x, \bar{a}(x), \varepsilon)}{\varepsilon \nu(x, \bar{a}(x))} .
$$

With (15), this implies:

$$
\bar{\varphi}'(x) = -\frac{\Phi(x, \bar{a}(x), \varepsilon)}{\lambda^2 \nu(x, \bar{a}(x))} \exp \left(-\frac{1}{\varepsilon} F(x, \bar{a}(x)) \right) (1 + o(1)) .
$$

Now fix $x \in M_W$ and choose $x^* \in M_W$ such that there is a path from $x^*$ to $x$ where the relief function $R$ decreases. Equation (13), both for $\bar{a}(x)$ and $\bar{a}(x^*)$, shows that $\bar{a}(x^*)$ can be neglected before $\bar{a}(x)$. Therefore, using

$$
\bar{a}(x) = \bar{a}(x^*) + \int_x^{x^*} \bar{a}'(\xi) d\xi ,
$$

the Laplace method gives

$$
\bar{a}(x) = -\frac{\Phi(x, \bar{a}(x), \varepsilon)}{\lambda^2 \nu(x, \bar{a}(x))} \exp \left(-\frac{1}{\varepsilon} F(x, \bar{a}(x)) \right) (1 + o(1)) .
$$
Moreover the definition of $F$, given by equation (14), yields

$$\frac{d}{dx} F(x, \pi(x)) = \Phi_u(x, \pi(x), \epsilon) + \pi(x) \int_0^x \frac{d}{du} \Phi_u(t, u^+(t, \pi(t), \epsilon), \pi(t), \epsilon) dt .$$

Since $\pi(x)$ is exponentially small in $M_W$, by Cauchy estimates, $\pi'(x)$ is also exponentially small. Therefore $\frac{d}{dx} F(x, \pi(x)) = \Phi_u(x, \pi, 0, 0)(1 + o(1))$ and we get the formula:

$$\pi(x) = -\frac{\Phi(x, \pi, 0, 0)}{\lambda \Phi_u(x, \pi, 0, 0)} \sqrt{\frac{f_0(0)}{\epsilon \pi}} \exp \left( -\frac{1}{\epsilon} F(x, \pi(x)) \right) (1 + o(1)) .$$

It follows that the function $G(x, \epsilon) = -\epsilon \log a(x, \epsilon)$ is analytic on $M_W^0 \times ]0, \rho[$ and satisfies $\lim_{\epsilon \to 0} G(x, \epsilon) = \lim_{\epsilon \to 0} F(x, \pi(x, \epsilon, \epsilon)) = F_0(x)$ uniformly on $M_W^0$.

This allows us to prove our main result. We insist that the loop $\gamma_\epsilon$ in the statement below lies in an exponentially small neighborhood of the origin.

**Theorem 11.** Let $x \in M_W^0$ and, for any $\epsilon \in ]0, \rho[$, let $y = y(\epsilon)$ be given by Corollary 3 such that

$$y(\epsilon) - x = \frac{2\pi i \epsilon}{f_0(x)} (1 + o(1)) \quad \text{and} \quad \pi(y) = \pi(x)$$

(i.e. $x = x_0, y = x_1$ of this corollary). Then for all $\epsilon \in ]0, \rho[$, the loop

$$\gamma_\epsilon : [0, 1] \to D(0, \rho), \ t \mapsto \pi((1-t)x + ty(\epsilon))$$

has index $-1$ around $a = 0$.

**Proof.** Firstly, because Corollary 3 yields $u^+(y, \pi(x), \epsilon) = \pi$, by the uniqueness in Proposition 5, we get that $\pi(x) = \pi(y)$, i.e. $\gamma_\epsilon$ is indeed a loop.

Since $F_0$ is analytic in a neighborhood of $x$, $F_0((1-t)x + ty(\epsilon)) = F_0(x) + 2\pi i \epsilon t (1 + o(1))$ as $\epsilon \to 0$. In the same manner, since $G(\cdot, \epsilon)$ is close to $F_0$, we get $G((1-t)x + ty(\epsilon), \epsilon) = G(x, \epsilon) + 2\pi i \epsilon t (1 + o(1))$, hence by Lemma 10:

$$\pi((1-t)x + ty(\epsilon)) = \pi(x)e^{-2\pi i t} .$$

The theorem now follows straightforwardly.

Concluding remark: Numerical experiments show that the condition $x \in M_W$ is not necessary. As $x$ tends to the separatrix $S$, $\pi(x)$ escapes from the exponentially small neighborhood of the origin and takes values of order $\epsilon$. One then observes that a loop of index $-1$ for $a$ still produces a shift of a singularity.

In fact it is possible to prove that the function $\pi$ can be analytically continued until taking values of order $\epsilon$. Using arguments analogous to those in the proof of Theorem 9, it is then possible to prove this numerical observation. However this analytic continuation is beyond the scope of the present article and will be done elsewhere, together with a detailed study of examples.

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