Criterion for the Buchstaber invariant of simplicial complexes to be equal to two.

Nickolai Erokhovets *

Abstract

In this paper we study the Buchstaber invariant of simplicial complexes, which comes from toric topology. With each simplicial complex $K$ on $m$ vertices we can associate a moment-angle complex $\mathcal{Z}_K$ with a canonical action of the compact torus $T^m$. Then $s(K)$ is the maximal dimension of a toric subgroup that acts freely on $\mathcal{Z}_K$. We develop the Buchstaber invariant theory from the viewpoint of the set of minimal non-simplices of $K$. It is easy to show that $s(K) = 1$ if and only if any two and any three minimal non-simplices intersect. For $K = \partial P^*$, where $P$ is a simple polytope, this implies that $P$ is a simplex. The case $s(P) = 2$ is much more complicated. For example, for any $k \geq 2$ there exists an $n$-polytope with $n + k$ facets such that $s(P) = 2$. Our main result is the criterion for the Buchstaber invariant of a simplicial complex $K$ to be equal to two.

1 Introduction.

For the introduction to toric topology see [BP02]. Moment-angle space is a key notion of toric topology. It was introduced by M. Davis and T. Januszkiewicz in [DJ91]. In our paper we use the following construction (see [BP02]).

Let $K = \{ \sigma \subset \{1, \ldots, m\} \}$ be a simplicial complex on $m$ vertices. For the pair of topological spaces $(X, A)$, $A \subseteq X$, define the $K$-power as

$$(X, A)^K = \{(x_1, \ldots, x_m) \in X^m : \{i : x_i \notin A\} \in K\}.$$ 

In particular cases $(D^2, S^1)$, where $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$, $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and $(D^1, S^0)$, where $D^1 = \{x \in \mathbb{R} : |x| \leq 1\}$, $S^0 = \{\pm 1\}$, we obtain a moment-angle complex $\mathcal{Z}_K$ and a real moment-angle complex $\mathbb{R}\mathcal{Z}_K \subset \mathcal{Z}_K$.

There are canonical coordinate actions of $T^m = (S^1)^m$ on $\mathcal{Z}_K$, and $(S^0)^m$ on $\mathbb{R}\mathcal{Z}_K$. We will use the isomorphisms

$$(\mathbb{R}^m / \mathbb{Z}^m) \simeq T^m : (\varphi_1, \ldots, \varphi_m) \mapsto (e^{2\pi i \varphi_1}, \ldots, e^{2\pi i \varphi_m}),$$

$$\mathbb{Z}^m_2 \simeq (S^0)^m : (a_1, \ldots, a_m) \mapsto ((-1)^{a_1}, \ldots, (-1)^{a_m}),$$

where $\mathbb{Z}_2 = \{0, 1\}$.

For the simplex $\sigma \in K$ define the coordinate subgroup

$$T^\sigma = \{(t_1, \ldots, t_m) \in T^m : \{i : t_i \neq 1\} \subset \sigma\}.$$ 

Let $x = (x_1, \ldots, x_m) \in \mathcal{Z}_K$. Define $\sigma(x) = \{i \in [m] : x_i = 0\} \in K$. Then the stabilizer $T^\sigma_x$ of the point $x$ is $T^\sigma(x)$, and $K = \{\sigma(x) : x \in \mathcal{Z}_K\}$.

Definition 1. A Buchstaber invariant $s(K)$ is the maximal dimension $s$ of the toric subgroup $H \subset T^m$, $H \cong T^s$, that acts freely on $\mathcal{Z}_K$. A real Buchstaber invariant $s_\mathbb{R}(K)$ is the maximal dimension $s$ of the subgroup $H_2 \subset \mathbb{Z}^m_2$ that acts freely on $\mathbb{R}\mathcal{Z}_K$.

For a simple polytope $P$ define $s(P)$ as the Buchstaber invariant $s(K)$ of the boundary complex $K = \partial P^*$ of the polar simplicial polytope. Similarly, $s_\mathbb{R}(P) = s_\mathbb{R}(\partial P^*)$.

If the subgroup $H \subset T^m$ acts freely on $\mathcal{Z}_K$, then the subgroup $H_2 = H \cap (S^0)^m$ acts freely on $\mathbb{R}\mathcal{Z}_K$, therefore $s(K) \leq s_\mathbb{R}(K)$ for all $K$. It can be shown that $s_\mathbb{R}(K) \leq m - \dim K - 1$.

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Problem 2 (Victor M. Buchstaber, 2002). To find an EFFECTIVE combinatorial description of \( s(K) \).

The Buchstaber invariant has been studied since 2001. The problem was originally formulated and studied for simple polytopes. In the case of simple \( n \)-polytope \( P \) with \( m \) facets we have \( 1 \leq s(P) \leq m-n \). I. Izmestiev \cite{Iz01a, Iz01b} proved the estimate \( s(P) \leq m - \gamma(P) \), where \( \gamma(P) \) is the chromatic number of \( P \), and found the lower bound in terms of the group of projectivities (see \cite{IO01}) of \( P \). The case of simplicial complexes that are skeleta of a simplex was considered by M. Masuda and Y. Fukukawa \cite{FM09}. A. Ayzenberg \cite{Ayz10} proved that \( s(\Gamma) = m - \lfloor \log_2(\gamma(\Gamma) + 1) \rfloor \) for any graph \( \Gamma \). For the theory of the Buchstaber invariant see \cite{Ayz10, Ayz11, Er08, Er09, Er11}. In this article we develop the idea that appears after reading \cite{FM09}: to consider the problem from the viewpoint of the set of minimal non-simplices of \( K \).

I’m grateful to Victor M. Buchstaber for the discussion of the results of this paper. During the discussion he suggested to consider the following modification of his problem.

Problem 2*. For any \( r \) to find a combinatorial criterion for the simplicial complex \( K \) to have \( s(K) = r \).

2 Combinatorial descriptions

2.1 Minimal non-simplices

The set \( \omega \subset [m] \) is called a non-simplex, if \( \omega \notin K \). Non-simplex \( \omega \) is minimal, if it’s any proper subset belongs to \( K \). Denote by \( N(K) \) the set of all minimal non-simplices. We have \( \sigma \in K \) if and only if it does not contain any \( \omega \in N(K) \), therefore \( N(K) \) determines \( K \) in a unique way.

Minimal non-simplex description of \( K \) is convenient for many reasons. For example, \( K \) is a simplex itself if and only if \( N(K) = \emptyset \). \( K \) is flag if and only if \( |\omega| = 2 \) for any \( \omega \in N(K) \). The Stanley-Reisner ring is also defined in these terms:

\[
\mathbb{Z}[K] = \mathbb{Z}[v_1, \ldots, v_m]/(v_i \ldots v_k : \{i_1, \ldots, i_k \} \in N(K)).
\]

It was proved in \cite{Er09} that if \( [m] = \omega_1 \cup \cdots \cup \omega_t \), where \( \omega_i \) are non-simplices, then

\[
s(K) \geq m - \sum_{i=1}^{l} |\omega_i| + l.
\]

In particular, \( s(K) \geq l \), if the non-simplices are pairwise disjoint.

2.2 Buchstaber invariant

Any subgroup \( H \subset T^m \), \( H \cong T^k \), can be described in two dual ways:

1) Parametrically:

\[
H = \{(e^{2\pi i (s_1^1 \psi_1 + \cdots + s_k^1 \psi_k)}, \ldots, e^{2\pi i (s_1^m \psi_1 + \cdots + s_m^m \psi_k)} : (\psi_1, \ldots, \psi_k) \in \mathbb{R}^k/\mathbb{Z}^k \},
\]

where \( S = \{S_i^j\} \in \text{Mat}_{m \times k}(\mathbb{Z}) \), and \( S \) has \( k \) units on the diagonal in the canonical form.

2) As a kernel of the mapping \( T^m \rightarrow T^{m-k} \):

\[
(e^{2\pi i \varphi_1}, \ldots, e^{2\pi i \varphi_m}) \rightarrow (e^{2\pi i (\lambda_1^1 \varphi_1 + \cdots + \lambda_m^1 \varphi_m)}, \ldots, e^{2\pi i (\lambda_{m-k}^1 \varphi_1 + \cdots + \lambda_{m-k}^m \varphi_m)}),
\]

where \( \Lambda = \{A_i^j\} \in \text{Mat}_{(m-k) \times m}(\mathbb{Z}) \), and \( \Lambda \) has \( m-k \) units on the diagonal in the canonical form.

These two descriptions, and two matrices \( S \) and \( \Lambda \) fit into the exact sequences:

\[
\begin{array}{lclll}
\{1\} & \longrightarrow & T^k & \longrightarrow & T^m & \longrightarrow & T^{m-k} & \longrightarrow & \{1\} \\
0 & \longrightarrow & \mathbb{Z}^k & \overset{S}{\longrightarrow} & \mathbb{Z}^m & \overset{\Lambda}{\longrightarrow} & \mathbb{Z}^{m-k} & \longrightarrow & 0
\end{array}
\]

The subgroup \( H \) acts freely if and only if \( H \cap T^m = \{1\} \) for all \( x \in \mathbb{Z}_K \). It is enough to consider the points \( x \) such that the simplex \( \sigma(x) \) is maximal. Since \( T^m_x = T^{\sigma(x)}_x \), we obtain that \( H \) acts freely if and only if \( H \cap T^\sigma = \{1\} \) for all maximal simplices \( \sigma \in K \). This leads to two dual combinatorial descriptions of \( s(K) \).
Let us make the following notations:
- $A^i$ – the $i$-th row of the matrix $A$;
- $A_j$ – the $j$-th column of the matrix $A$;
- $A^1$ – the matrix, consisting of the rows $\{A^i : i \in \omega\}$;
- $A_1$ – the matrix, consisting of the columns $\{A_j : j \in \omega\}$;
- $A^0$ – the matrix, obtained from $A$ by deletion of the rows $\{A^i : i \in \omega\}$;
- $A_0$ – the matrix, obtained from $A$ by deletion of the columns $\{A_j : j \in \omega\}$.

**Proposition 3 (BP02, Er09).** We have:

(A) $s(K)$ is the maximal $k$ that admits a matrix $S \in \text{Mat}_{m \times k}(\mathbb{Z})$ satisfying the condition: for any maximal simplex $\sigma \in K$, $|\sigma| = r$, the columns of the matrix $S^\sigma$ form part of a basis in $\mathbb{Z}^{m-r}$ (equivalently, the rows $\{S^i : i \notin \sigma\}$ span $\mathbb{Z}^k$);

(B) $s(K)$ is the maximal $k$ that admits a matrix $\Lambda \in \text{Mat}_{(m-k) \times m}(\mathbb{Z})$ satisfying the condition: for any maximal simplex $\sigma \in K$, $|\sigma| = r$, the columns $\{\Lambda_j : j \in \sigma\}$ form part of a basis in $\mathbb{Z}^{m-k}$ (equivalently, the rows $\{\Lambda^i : i \notin \sigma\}$ span $\mathbb{Z}^r$);

Similarly in the real case.

**Proposition 4 (BP02, Er09).** We have

(A2) $s_\mathbb{R}(K)$ is the maximal $k$ that admits a matrix $S \in \text{Mat}_{m \times k}(\mathbb{Z}_2)$ satisfying the condition: for any maximal simplex $\sigma \in K$, the columns of the matrix $S^\sigma$ are linearly independent (equivalently, the rows $\{S^i : i \notin \sigma\}$ span $\mathbb{Z}_2^k$);

(B2) $s_\mathbb{R}(K)$ is the maximal $k$ that admits a matrix $\Lambda \in \text{Mat}_{(m-k) \times m}(\mathbb{Z}_2)$ satisfying the condition: for any maximal simplex $\sigma \in K$, $|\sigma| = r$, the columns $\{\Lambda_j : j \in \sigma\}$ are linearly independent (equivalently, the rows $\{\Lambda^i : i \notin \sigma\}$ span $\mathbb{Z}_2^r$);

We call the matrix $A$ a 0/1-matrix, if its entries are zeroes and units.

**Lemma 5 (Er09).** For a 0/1-matrix $A$ of sizes $2 \times 2$ or $3 \times 3$ equality $\det A = 1 \mod 2$ implies that $\det A = \pm 1$.

**Proof.** For $2 \times 2$-matrices one of the rows should contain 0, therefore we come to the case $1 \times 1$.

For a $3 \times 3$-matrix $A$ if one of the rows has two zeroes, we come back to the case $2 \times 2$. If all the rows are different and have one zero and two units, then their sum is equal to zero modulo two. Hence up to a transposition of rows and columns we come to the case $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\det A = -1$. \hfill $\square$

Let us mention that for $k \times k$-matrices, $k \geq 4$, lemma is not valid. For even $k$ a counterexample is given by the matrix

$$A_k = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}, \quad \det A_k = (-1)^{k-1}(k - 1),$$

and for odd – by the matrix $\begin{pmatrix} \frac{1}{0} & 0 & \ldots & 0 \\ 0 & \frac{1}{0} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{0} \end{pmatrix}$.

**Lemma 6.** (Er02) For $r = 1, 2, 3$ we have: $s(K) \geq r$ if and only if $s_\mathbb{R}(K) \geq r$.

**Proof.** If the vectors $a_1, \ldots, a_r \in \mathbb{Z}_2^t$ are linearly independent, then there is an $r \times r$-minor equal to 1. It follows from lemma 5 that it is equal to $\pm 1$ over $\mathbb{Z}$, therefore these vectors form part of a basis in $\mathbb{Z}^t$. Hence the 0/1-matrix $S$ for $s_\mathbb{R}(K)$ satisfies condition (A) for $s(K)$, therefore $s_\mathbb{R}(K) \geq r$ implies $s(K) \geq r$. The opposite implication follows from the fact that $s(K) \leq s_\mathbb{R}(K)$. \hfill $\square$

\footnote{This condition was used by HyunWoong Cho and JinHong Kim}

\footnote{This condition in a special case was used in PM09.
2.3 Description of the Buchstaber invariant in terms of minimal non-simplices

**Proposition 7.** Condition (A) is equivalent to following condition (A\*): for any prime \( p \) and any nonzero vector \( a \in \mathbb{Z}_p^k \) there exists \( \omega(a) \in N(K) \) such that \( \langle a, S^i \rangle \neq 0 \) mod \( p \) for all \( i \in \omega(a) \).

**Proof.** Let condition (A) hold but (A\*) fail. Then there exists prime \( p \) and \( a \in \mathbb{Z}_p^k \setminus \{0\} \) such that for any \( \omega \in N(K) \) there is \( i_\omega \) with \( \langle a, S^{i_\omega} \rangle = 0 \) mod \( p \). Set \( \sigma = \{ m \} \setminus \{ i_\omega : \omega \in N(K) \} \). Then \( \sigma \neq \emptyset \), since \( \{ S^i, i \in [m] \} \) span \( \mathbb{Z}_p^k \). Moreover, \( \sigma \in K \). Otherwise there is \( \omega \in N(K) \) such that \( \omega \subset \sigma \), therefore \( i_\omega \in \sigma \), which is a contradiction. Then all the rows \( \{ S^i : i \notin \sigma \} \) lie in the proper subgroup \( \{ x : (a, x) = 0 \} \setminus \{ \} \subset \mathbb{Z}_p^k \). This contradicts to the fact that they span \( \mathbb{Z}_p^k \).

Now let condition (A\*) hold but (A) fail. Then for some \( \sigma \subset K \) the rows \( \{ S^i : i \notin \sigma \} \) do not span \( \mathbb{Z}_p^k \). Therefore the matrix \( S^\sigma \) in the canonical form has nonnegative number \( c \neq 1 \) on the diagonal. This means that there exists a primitive vector \( b \) in \( \mathbb{Z}_p^k \) such that \( \langle b, S^\sigma \rangle \) is either 0 (if \( c = 0 \)), or is divided by \( c \) (if \( c > 0 \)) for all \( i \notin \sigma \). Set \( p = 2 \), if \( c = 0 \); or any prime divisor of \( c \), if \( c > 0 \). Set \( a = b \) mod \( p \). Then \( a \neq 0 \), and \( \langle a, S^i \rangle = 0 \) mod \( p \) for all \( i \notin \sigma \). On the other hand, \( \omega(a) \cap ([m] \setminus \sigma) \neq \emptyset \), therefore for any \( i \in \omega(a) \setminus \sigma \) we have \( \langle a, S^i \rangle \neq 0 \) mod \( p \), which is a contradiction. \( \square \)

**Proposition 8.** Condition (A2) is equivalent to the following condition (A2\*): for any nonzero vector \( a \in \mathbb{Z}_2^k \) there exists \( \omega(a) \in N(K) \) such that \( \langle a, S^i \rangle = 1 \) in \( \mathbb{Z}_2 \) for all \( i \in \omega(a) \).

The proof is similar to the previous one. We give it here for the completeness.

**Proof.** Let condition (A2) hold but (A2\*) fail. Then there exists \( a \in \mathbb{Z}_2^k \setminus \{0\} \) such that for any \( \omega \in N(K) \) there is \( i_\omega \) with \( \langle a, S^{i_\omega} \rangle = 0 \). Set \( \sigma = [m] \setminus \{ i_\omega : \omega \in N(K) \} \). Then \( \sigma \neq \emptyset \), since \( \{ S^i, i \in [m] \} \) span \( \mathbb{Z}_2^k \). Moreover, \( \sigma \in K \). Otherwise there is \( \omega \in N(K) \) such that \( \omega \subset \sigma \), therefore \( i_\omega \in \sigma \), which is a contradiction. Then all the rows \( \{ S^i : i \notin \sigma \} \) lie in the hyperplane \( \langle a, x \rangle = 0 \). This contradicts to the fact that they span \( \mathbb{Z}_2^k \).

Now let condition (A2\*) hold and (A2) fail. Then for some \( \sigma \) the rows \( \{ S^i : i \notin \sigma \} \) do not span \( \mathbb{Z}_2^k \). Then they lie in some hyperplane \( \langle a, x \rangle = 0 \). On the other hand, \( \omega(a) \cap ([m] \setminus \sigma) \neq \emptyset \), therefore for any \( i \in \omega(a) \setminus \sigma \) we have \( \langle a, S^i \rangle = 1 \), which is a contradiction. \( \square \)

Let us call the linear dependence \( a_1 + \cdots + a_t = 0 \) in the vector space \( \mathbb{Z}_2^k \) minimal if any proper subset of vectors in \( \{ a_1, \ldots, a_t \} \) is linearly independent.

**Proposition 9.** We have \( s_\mathbb{R}(K) \geq k \) if and only if there exists a mapping \( \xi : \mathbb{Z}_2^k \setminus \{0\} \to N(K) \) such that \( \xi(a_1) \cap \cdots \cap \xi(a_{2r+1}) = \emptyset \) for any minimal linear dependence \( a_1 + \cdots + a_{2r+1} = 0 \).

**Proof.** Let \( s_\mathbb{R}(K) \geq k \). Set \( \xi(a) = \omega(a) \). Let \( i \in \xi(a_1) \cap \cdots \cap \xi(a_{2r+1}) \). Then \( \langle a_j, S^{i} \rangle = 1 \) for all \( j = 1, \ldots, 2r+1 \). Hence, \( a_1 + \cdots + a_{2r+1}, S^i \rangle = 1 \), therefore \( a_1 + \cdots + a_{2r+1} = 0 \).

Now let us prove the "if" part. For any \( i \in [m] \) set \( M_i = \{ a : \xi(a) \ni i \} \). Consider the system of equations \( \{ \langle a, x \rangle = 1 : a \in M_i \} \). Let \( a_1, \ldots, a_t \) be a maximal linearly independent subset in \( M_i \). Since any minimal dependence in \( M_i \) contains even number of vectors, we obtain \( a = a_1 + \cdots + a_{2r+1} \) for any \( a \in M_i \). Therefore all the equations are expressed in terms of basic equations, hence the system has solutions. Let \( S^i \) be some of them. Consider the matrix \( S \) consisting of rows \( S^i \). From construction we have \( \langle a, S^i \rangle = 1 \) for any \( i \in \xi(a) \), therefore \( s_\mathbb{R}(K) \geq k \). \( \square \)

3 Criteria for \( s(K) \geq 1, 2, 3 \)

Lemma 8 implies that it is enough to consider \( s_\mathbb{R}(K) \). The following proposition easily follows from Proposition 7 or Proposition 8.

**Proposition 10 (Condition (S1)).** We have \( s_\mathbb{R}(K) \geq 1 \) if and only if \( N(K) \neq \emptyset \), i.e. \( K \neq \Delta^a \).

**Proposition 11 (Condition (S2)).** We have \( s_\mathbb{R}(K) \geq 2 \) if and only if \( N(K) \) contains one of the subsets of the form:

1. \( \{ r_1, r_2, r_3 \} : \ r_1 \cap r_2 \cap r_3 = \emptyset ; \)
2. \( \{ r_1, r_2 \} : \ r_1 \cap r_2 = \emptyset . \)
Proof. Proposition 2 implies that condition \( s_{\mathbb{R}}(K) \geq 2 \) is equivalent to the existence of the mapping \( \xi : \mathbb{Z}^2_2 \setminus \{0\} \to N(K) \) with \( \xi(1,0) \cap \xi(0,1) \cap \xi(1,1) = \emptyset \). There are two possibilities:

1. The mapping is injective. Set \( \tau_1 = \xi(1,0), \tau_1 = \xi(0,1), \tau_3 = \xi(1,1) \).

2. Exactly two vectors have the same images. Take them as a basis. Set \( \tau_1 = \xi(1,0) = \xi(0,1), \) and \( \tau_2 = \xi(1,1) \).

This proves the "only if" part and gives the mappings for the "if" part. \( \square \)

**Proposition 12** (Condition (S3)). We have \( s_{\mathbb{R}}(K) \geq 3 \) if and only if \( N(K) \) contains one of the subsets of the form:

1. \( \{ \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7 \} : \tau_1 \cap \tau_2 \cap \tau_4 = \emptyset; \quad \tau_1 \cap \tau_3 \cap \tau_5 = \emptyset; \quad \tau_1 \cap \tau_6 \cap \tau_7 = \emptyset; \quad \tau_2 \cap \tau_3 \cap \tau_6 = \emptyset; \quad \tau_2 \cap \tau_4 \cap \tau_7 = \emptyset; \quad \tau_3 \cap \tau_4 \cap \tau_6 = \emptyset; \quad \tau_4 \cap \tau_5 \cap \tau_6 = \emptyset; \quad \tau_4 \cap \tau_5 \cap \tau_7 = \emptyset. \)

2. \( \{ \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6 \} : \tau_1 \cap \tau_3 = \emptyset; \quad \tau_1 \cap \tau_2 \cap \tau_4 = \emptyset; \quad \tau_1 \cap \tau_2 \cap \tau_5 = \emptyset; \quad \tau_1 \cap \tau_3 \cap \tau_6 = \emptyset; \quad \tau_2 \cap \tau_3 \cap \tau_6 = \emptyset; \quad \tau_3 \cap \tau_4 \cap \tau_5 = \emptyset; \quad \tau_3 \cap \tau_4 \cap \tau_6 = \emptyset; \quad \tau_4 \cap \tau_5 \cap \tau_6 = \emptyset; \quad \tau_4 \cap \tau_5 \cap \tau_7 = \emptyset. \)

3. \( \{ \tau_1, \tau_2, \tau_3, \tau_4, \tau_5 \} : \tau_1 \cap \tau_2 = \emptyset; \quad \tau_1 \cap \tau_5 = \emptyset; \quad \tau_1 \cap \tau_3 \cap \tau_4 = \emptyset; \quad \tau_2 \cap \tau_3 \cap \tau_5 = \emptyset; \quad \tau_2 \cap \tau_4 \cap \tau_5 = \emptyset. \)

4. \( \{ \tau_1, \tau_2, \tau_3, \tau_4 \} : \tau_1 \cap (\tau_2 \cup \tau_3 \cup \tau_4) = \emptyset; \quad \tau_2 \cap \tau_3 \cap \tau_4 = \emptyset. \)

5. \( \{ \tau_1, \tau_2, \tau_3 \} : \tau_1 \cap \tau_2 = \tau_1 \cap \tau_3 = \tau_2 \cap \tau_3 = \emptyset. \)

Proof. Proposition 2 implies that condition \( s_{\mathbb{R}}(K) \geq 3 \) is equivalent to the existence of the mapping \( \xi : \mathbb{Z}^2_2 \setminus \{0\} \to N(K) \) such that \( \xi(\mathbf{a}) \cap \xi(\mathbf{b}) \cap \xi(\mathbf{c}) = \emptyset \) for any triple of pairwise distinct vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) with \( \mathbf{a} + \mathbf{b} + \mathbf{c} = 0 \). There are exactly 7 such triples and they correspond to two-dimensional subspaces:

\[
(1,0,0) + (0,1,0) + (1,1,0) = (1,0,0) + (0,0,1) + (1,0,1) = (1,0,0) + (0,1,1) + (1,1,1) = (0,1,0) + (0,0,1) + (1,0,1) = (1,1,0) + (1,1,1) = (0,0,1) + (1,1,0) + (1,1,1) = (1,1,0) + (0,1,1) + (0,1,1) = 0.
\]

Set \( \mathbf{a}_1 = (1,0,0), \mathbf{a}_2 = (0,1,0), \mathbf{a}_3 = (0,0,1), \mathbf{a}_4 = (1,1,0), \mathbf{a}_5 = (1,0,1), \mathbf{a}_6 = (0,1,1), \mathbf{a}_7 = (1,1,1). \) Then

\[
\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_4 = 0; \quad \mathbf{a}_1 + \mathbf{a}_3 + \mathbf{a}_5 = 0; \quad \mathbf{a}_1 + \mathbf{a}_6 + \mathbf{a}_7 = 0; \quad \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_6 = 0; \quad \mathbf{a}_2 + \mathbf{a}_5 + \mathbf{a}_7 = 0; \quad \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_7 = 0; \quad \mathbf{a}_4 + \mathbf{a}_5 + \mathbf{a}_6 = 0.
\]

Now the proof is obtained by enumeration of all the possible cases. In each case we choose a basis \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) in \( \mathbb{Z}^2_2 \) and denote the sets in the image by \( \omega_\sigma \), where \( \sigma = \{ i : \xi(\mathbf{e}_i) = \omega_\sigma \} \). We use the fact that no four vectors can have the same image. Redundant equalities are enclosed in brackets.

I. There are no triples of vectors with the same image. Then there are at most three pairs of vectors with the same images.

1. There are no pairs. Then \( \text{Im} \xi = \{ \omega_i : i = 1, \ldots, 7 \} \), and

\[
\omega_1 \cap \omega_2 \cap \omega_4 = \emptyset; \quad \omega_1 \cap \omega_3 \cap \omega_5 = \emptyset; \quad \omega_1 \cap \omega_6 \cap \omega_7 = \emptyset; \quad \omega_2 \cap \omega_3 \cap \omega_5 = \emptyset; \quad \omega_2 \cap \omega_4 \cap \omega_7 = \emptyset; \quad \omega_3 \cap \omega_4 \cap \omega_6 = \emptyset; \quad \omega_4 \cap \omega_5 \cap \omega_6 = \emptyset.
\]

Set \( \tau_i = \omega_i \).

2. There is exactly one pair. Choose \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) to be the vectors of this pair, and \( \mathbf{e}_3 \notin \text{Ls}\{\mathbf{e}_1, \mathbf{e}_2\} \). Then \( \text{Im} \xi = \{ \omega_{12}, \omega_3, \omega_5, \omega_6, \omega_7 \} \), and

\[
\omega_1 \cap \omega_4 = \emptyset; \quad \omega_1 \cap \omega_3 \cap \omega_5 = \emptyset; \quad \omega_1 \cap \omega_6 \cap \omega_7 = \emptyset; \quad \omega_2 \cap \omega_3 \cap \omega_6 = \emptyset; \quad \omega_2 \cap \omega_5 \cap \omega_7 = \emptyset; \quad \omega_3 \cap \omega_4 \cap \omega_6 = \emptyset; \quad \omega_4 \cap \omega_5 \cap \omega_7 = \emptyset; \quad \omega_5 \cap \omega_6 \cap \omega_7 = \emptyset.
\]

Set \( \tau_1 = \omega_{12}, \) and \( \tau_i = \omega_{i+1} \) for \( i \geq 2 \).
3. There are exactly two pairs.

(a) One pair contains the sum of the vectors of the other. Choose \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) to be the vectors of the second pair, and \( \mathbf{e}_3 \) to be the vector paired to \( \mathbf{e}_1 + \mathbf{e}_2 \). Then \( \text{Im} \xi = \{ \omega_{12}, \omega_{34}, \omega_5, \omega_6, \omega_7 \} \), and

\[
\begin{align*}
\omega_{12} \cap \omega_{34} &= \emptyset; \\
(\omega_{12} \cap \omega_{34} \cap \omega_5 &= \emptyset); \\
\omega_{12} \cap \omega_6 \cap \omega_7 &= \emptyset; \\
(\omega_{12} \cap \omega_{34} \cap \omega_6 &= \emptyset);
\end{align*}
\]

Set \( \tau_1 = \omega_{34}, \tau_2 = \omega_5, \tau_3 = \omega_6, \tau_4 = \omega_7, \tau_5 = \omega_{12} \).

(b) No pair contains the sum of the vectors of the other. Choose \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) to be the vectors of the first pair, and \( \mathbf{e}_3 \) to be any vector of the second. The only possible case is:

\[
\xi(0, 0, 1) = \xi(1, 1, 1).
\]

Then \( \text{Im} \xi = \{ \omega_{12}, \omega_{34}, \omega_4, \omega_5, \omega_6 \} \), and

\[
\begin{align*}
\omega_{12} \cap \omega_4 &= \emptyset; \\
\omega_{12} \cap \omega_{34} \cap \omega_5 &= \emptyset; \\
\omega_{12} \cap \omega_6 \cap \omega_{37} &= \emptyset; \\
(\omega_{12} \cap \omega_{34} \cap \omega_6 &= \emptyset);
\end{align*}
\]

Set \( \tau_1 = \omega_4, \tau_2 = \omega_{12}, \tau_3 = \omega_5, \tau_4 = \omega_6, \tau_5 = \omega_{37} \).

4. There are exactly three pairs.

(a) The seventh vector is not equal to the sum of the vectors of any pair. Choose \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) to be the vectors of any pair, and \( \mathbf{e}_3 \) to be the vector paired to \( \mathbf{e}_1 + \mathbf{e}_2 \). We have \( \xi(0, 0, 1) = \xi(1, 1, 0) \). Then the vector \( (1, 1, 1) = (0, 0, 1) + (1, 1, 0) \) belongs to the third pair. The remaining vector of it's pair up to the transposition of \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) is \( (1, 0, 1) \): \( \xi(0, 0, 1) = \xi(1, 1, 1) \). Then \( \text{Im} \xi = \{ \omega_{12}, \omega_{34}, \omega_{35}, \omega_6 \}, \) and

\[
\begin{align*}
\omega_{12} \cap \omega_4 &= \emptyset; \\
(\omega_{12} \cap \omega_{34} \cap \omega_{35} &= \emptyset); \\
\omega_{12} \cap \omega_6 \cap \omega_{37} &= \emptyset; \\
(\omega_{12} \cap \omega_{34} \cap \omega_6 &= \emptyset);
\end{align*}
\]

Set \( \tau_1 = \omega_{12}, \tau_2 = \omega_{34}, \tau_3 = \omega_{35}, \tau_4 = \omega_6 \).

(b) The seventh vector is the sum of two vectors of exactly one pair. Choose \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) to be the vectors of this pair, and \( \mathbf{e}_3 \) to be any of the vectors of the remaining two pairs. The vector paired to \( \mathbf{e}_3 \) can not be \( (1, 1, 1) \), therefore up to a transposition of \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) we obtain:

\[
\xi(0, 0, 1) = \xi(1, 0, 1), \xi(0, 1, 1) = \xi(1, 1, 1).
\]

Then \( \text{Im} \xi = \{ \omega_{12}, \omega_{35}, \omega_4, \omega_5, \omega_{37} \}, \) and

\[
\begin{align*}
\omega_{12} \cap \omega_4 &= \emptyset; \\
(\omega_{12} \cap \omega_{35} &= \emptyset); \\
\omega_{12} \cap \omega_{37} &= \emptyset; \\
(\omega_{12} \cap \omega_{35} \cap \omega_6 &= \emptyset);
\end{align*}
\]

Set \( \tau_1 = \omega_{12}, \tau_2 = \omega_{35}, \tau_3 = \omega_4, \tau_4 = \omega_5 \).

(c) The seventh vector is the sum of vectors of at least two pairs. Choose \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) to be the vectors of the first pair, and \( \mathbf{e}_3 \) to be any of the vectors of the second. For the second pair we obtain: \( \xi(0, 0, 1) = \xi(1, 1, 1), \) and for the third pair: \( \xi(1, 0, 1) = \xi(0, 1, 1) \). Then \( \text{Im} \xi = \{ \omega_{12}, \omega_{37}, \omega_4, \omega_5, \omega_{35} \}, \) and

\[
\begin{align*}
\omega_{12} \cap \omega_4 &= \emptyset; \\
(\omega_{12} \cap \omega_{37} \cap \omega_5 &= \emptyset); \\
\omega_{12} \cap \omega_{35} \cap \omega_{37} &= \emptyset; \\
(\omega_{12} \cap \omega_{35} \cap \omega_{37} &= \emptyset);
\end{align*}
\]

Set \( \tau_1 = \omega_4, \tau_2 = \omega_{12}, \tau_3 = \omega_{37}, \tau_4 = \omega_{35} \).

II. There is a triple of vectors with the same image. These vectors are linearly independent and we can choose them as \( \mathbf{e}_1, \mathbf{e}_2, \) and \( \mathbf{e}_3 \). Consider the rest four vectors.

1. There are no triples of vectors with the same image.

(a) There are no pairs of vectors with the same image. Then \( \text{Im} \xi = \{ \omega_{123}, \omega_4, \omega_5, \omega_6, \omega_7 \} \), and

\[
\begin{align*}
\omega_{123} \cap \omega_4 &= \emptyset; \\
\omega_{123} \cap \omega_5 &= \emptyset; \\
\omega_{123} \cap \omega_6 \cap \omega_7 &= \emptyset; \\
\omega_{123} \cap \omega_6 &= \emptyset.
\end{align*}
\]

Set \( \tau_1 = \omega_{123}, \tau_2 = \omega_4, \tau_3 = \omega_5, \tau_4 = \omega_6 \).
(b) There is exactly one pair of vectors with the same image.

i. One of the vectors of the pair is \( e_1 + e_2 + e_3 \). Up to a transposition of \( e_1, e_2, \) and \( e_3 \) we obtain: \( \xi(1, 1, 0) = \xi(1, 0, 1) \). Then \( \text{Im } \xi = \{\v_1, \v_4, \v_5, \v_6\} \), and

\[
\begin{align*}
\v_1 \cap \v_4 &= \emptyset; \\
\v_2 \cap \v_4 &= \emptyset; \\
\v_2 \cap \v_5 &= \emptyset; \\
\v_3 \cap \v_4 &= \emptyset; \\
\v_4 \cap \v_5 &= \emptyset; \\
\v_5 \cap \v_6 &= \emptyset; \\
\v_1 \cap \v_5 &= \emptyset; \\
\v_1 \cap \v_6 &= \emptyset; \\
\v_2 \cap \v_6 &= \emptyset.
\end{align*}
\]

Set \( \tau_1 = \v_1, \tau_2 = \v_4, \tau_3 = \v_5, \tau_4 = \v_6 \).

ii. Both vectors of the pair are sums of two basis vectors. Up to a transposition of \( e_1, e_2, \) and \( e_3 \) we obtain: \( \xi(1, 1, 0) = \xi(1, 0, 1) \). Then \( \text{Im } \xi = \{\v_1, \v_4, \v_5, \v_6\} \), and

\[
\begin{align*}
\v_1 \cap \v_4 &= \emptyset; \\
\v_2 \cap \v_4 &= \emptyset; \\
\v_2 \cap \v_5 &= \emptyset; \\
\v_3 \cap \v_4 &= \emptyset; \\
\v_4 \cap \v_5 &= \emptyset; \\
\v_5 \cap \v_6 &= \emptyset; \\
\v_1 \cap \v_5 &= \emptyset; \\
\v_1 \cap \v_6 &= \emptyset; \\
\v_2 \cap \v_6 &= \emptyset.
\end{align*}
\]

Set \( \tau_1 = \v_1, \tau_2 = \v_4, \tau_3 = \v_5, \tau_4 = \v_6 \).

(c) There are exactly two pairs of vectors with the same image. Up to a transposition of \( e_1, e_2, \) and \( e_3 \) we obtain: \( \xi(1, 1, 0) = \xi(1, 0, 1), \xi(0, 1, 1) = \xi(1, 1, 1) \). Then \( \text{Im } \xi = \{\v_1, \v_4, \v_5, \v_6\} \), and

\[
\begin{align*}
\v_1 \cap \v_4 &= \emptyset; \\
\v_2 \cap \v_4 &= \emptyset; \\
\v_2 \cap \v_5 &= \emptyset; \\
\v_3 \cap \v_4 &= \emptyset; \\
\v_4 \cap \v_5 &= \emptyset; \\
\v_5 \cap \v_6 &= \emptyset; \\
\v_1 \cap \v_5 &= \emptyset; \\
\v_1 \cap \v_6 &= \emptyset; \\
\v_2 \cap \v_6 &= \emptyset.
\end{align*}
\]

Set \( \tau_1 = \v_1, \tau_2 = \v_4, \tau_3 = \v_5, \tau_4 = \v_6 \).

2. There is a triple of vectors with the same image. Their sum is nonzero, therefore up to a transposition of \( e_1, e_2, \) and \( e_3 \) we obtain: \( \xi(1, 1, 0) = \xi(1, 0, 1), \xi(1, 1, 1) \). Then \( \text{Im } \xi = \{\v_1, \v_4, \v_5, \v_6\} \), and

\[
\begin{align*}
\v_1 \cap \v_4 &= \emptyset; \\
\v_2 \cap \v_4 &= \emptyset; \\
\v_2 \cap \v_5 &= \emptyset; \\
\v_3 \cap \v_4 &= \emptyset; \\
\v_4 \cap \v_5 &= \emptyset; \\
\v_5 \cap \v_6 &= \emptyset; \\
\v_1 \cap \v_5 &= \emptyset; \\
\v_1 \cap \v_6 &= \emptyset; \\
\v_2 \cap \v_6 &= \emptyset.
\end{align*}
\]

Set \( \tau_1 = \v_1, \tau_2 = \v_4, \tau_3 = \v_5, \tau_4 = \v_6 \).

This enumeration proves the "only if" part. For the "if" part the cases II, I2, I3(a), I4(b), and II2 give the mappings for cases 1-5 respectively.

Now our main result follows from lemma 6 and propositions 10, 11, and 12.

**Theorem 13.** We have

1. \( s(K) \geq 1 \) if and only if condition \((S1)\) holds;
2. \( s(K) \geq 2 \) if and only if condition \((S2)\) holds;
3. \( s(K) \geq 3 \) if and only if condition \((S3)\) holds.

**Corollary 14.** We have

1. \( s(K) = 0 \) if and only if \( N(K) \neq \emptyset \) (equivalently, \( K = \Delta^n \));
2. \( s(K) = 1 \) if and only if any two and any three subsets in \( N(K) \) intersect;
3. \( s(K) = 2 \) if and only if some two or three subsets in \( N(K) \) do not intersect and \( N(K) \) does not contain any of 5 subsets from proposition 72.
4 Problems

Theorem 13 naturally leads to the following problems.

Problem 15. To classify all simplicial complexes $K$ such that $s(K) = 2$.

Minimal non-simplices are closely related to other combinatorial characteristics of simplicial complexes such as bigraded Betti numbers

$$\beta^{-1,2j}(K) = \text{rank } \text{Tor}_{Z[v_1, \ldots, v_m]}^{-1,2j}(Z[K], Z) = \text{rank } H^{-1,2j}[\Lambda[u_1, \ldots, u_m] \otimes Z[K], d],$$

where $\text{bideg } u_i = (-1, 2)$, $\text{bideg } v_i = (0, 2)$, $du_i = v_i$, $dv_i = 0$. For example, $\sum_j \beta^{-1,2j} = |N(K)|$.

Problem 16. To find a criterion for $s(K) = 2$ in terms of bigraded Betti numbers.

Unlike simplicial complexes for a simple $n$-polytope $P$ with $m$ facets $s(P) = 1$ if and only if $P = \Delta^n$ (equivalently, $m - n = 1$). The case $s(P) = 2$ is much more complicated. It was shown in [Er09] that

$$s(C^n(m^*)) = 2 \quad \text{for} \quad 2 \leq m - n \leq 2 + \frac{n - 13}{48},$$

where $C^n(m)$ is a cyclic polytope. In particular, for each $k \geq 2$ there exists a polytope with $m - n = k$ and $s(P) = 2$. Moreover, the estimate $s(P) \geq m - \gamma(P) + s(\Delta_{n-1}^{m-1})$ (see [Er09]) implies that if $s(P) = 2$, then one of the following holds:

1) $P = I \times I$;
2) Any two facets of $P$ intersect, and $m < \frac{7}{2}(n + 1) + 2$;
3) $\gamma(P) = m - 1$, and $m < \frac{7}{2}(n + 1) + 1$.

Problem 17. To classify all simple polytopes with $s(P) = 2$.

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