On form-factors in Sin(h)-Gordon theory.

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Abstract

We present here an explicit classical solution of the type of perturbiner in Sin(h)-Gordon model. This solution is a generating function for form-factors in the tree approximation.

Perturbiner is a solution of classical field equations which is a generating function for tree form-factors – that is objects of the type of $\langle 1,\ldots,N|\phi(x)|0\rangle$ – in the theory. A convenient definition of this solution – formally independent of considering Feynman diagrams – was given in [1]. Verbally, it sounds as follows. Take a solution of the free (linear) field equation in the theory under consideration in the form of superposition of a set of plane waves with every plane wave multiplied by a nilpotent coefficient. The corresponding perturbiner is a complex solution of the full (nonlinear) field equation which is polynomial in the (nilpotent) plane waves, first order term of the polynomial being the solution of the free field equation. The plane waves are nothing but the asymptotic wave functions of the one-particle states included in the form-factors which the perturbiner is the generating function for. The nilpotency assumes that the perturbiner is the generating function for form-factors without identical one-particle states. Obviously, there is no loss of generality in the assumption of nilpotency. Of course, it is very well known that tree amplitudes (form-factors) can in principle be obtained from classical field equations, see, for instance, [2], [3], where the appropriate solution was defined by the Feynman-type boundary conditions. Our definition is different and has an advantage that an infinite-dimensional space of space-time functions is substituted by the finite-dimensional space of polynomials.
in a finite number of nilpotent variables. In particular, the existence and uniqueness are almost automatic in our case.

Of course, in generic theory one cannot go further than the uniqueness and existence theorem and the common perturbation procedure. Nevertheless, the perturbiner happens to be very easy to construct in cases when the field equations are integrable, that is, essentially, when there is a zero-curvature representation. Thus we have constructed perturbiner for the self-duality equations in Yang-Mills theory [4], in gravity [5] and in Yang-Mills theory interacting with gravity [6]. The zero-curvature representation in the above cases is provided by the twistor construction [6], [7], [8]. The way of reducing from the generic perturbiner to the self-dual one is obvious: one includes into the plane wave solutions of the linearized field equations only self-dual plane waves, which is equivalent to describing only amplitudes with on-shell particles of a given helicity (say, of the positive one). The particles of the opposite helicity can then be added one-by-one perturbatively (see [4] and [10]).

In this letter we describe construction of the perturbiner in the \( \text{Sin}(h) \)-Gordon model (apparently, since perturbiner is a complex solution, there is no difference between \( \text{Sin} \)-Gordon and \( \text{Sinh} \)-Gordon cases). Thus we directly obtain the generating function for the tree form-factors in the theory. Although, the form-factors in the model are known exactly [16], not only in the tree approximation, we still think that our construction makes some sense. Firstly, we could not find in the literature a closed explicit expressions for the tree limit of the form-factors, so Eq. (25) below fills this gap. Say, in [17] the exact expressions for the form-factors include some polynomials for which only a recursion relation without explicit general solution is given. And it seems not to be completely trivial to extract tree factors from general formulae in [17]. Notice that what is called the classical limit in [18] is different from the tree approximation. Secondly, the popular approach to the exact form-

\footnote{In the case of Yang-Mills theory, the idea to use the self-duality equations to describe the so-called like-helicity amplitudes was first formulated in [13] and, independently, in [14]. In [13] it was basically shown that the self-duality equations reproduce the recursion relations for tree like-helicity gluonic form-factors (also called “currents”), obtained originally in ref. [12] from the Feynman diagrams; the corresponding solution of self-duality equations was then obtained in terms of the known solutions refs. [12] of the recursion relations for the “currents”. In ref. [14] an example of self-dual perturbiner was obtained in the SU(2) case by a ’tHooft ansatz upon further restriction on the on-shell particles included. A bit later a similar solution was studied in ref. [5], where consideration was based on solving recursion relations analogous to refs. [12].}
factors is based on a set of axioms [16]. Upon obtaining an exact solution for this set of axioms it becomes a separate task to identify the corresponding operator or even the corresponding field theory while our approach starts from the field equations and is absolutely straightforward. It is yet only a tree approximation, but notice, that the tree contribution always serves as a skeleton for the complete quantum expressions and sometimes factorizes out. For example, in [19] the exact (with all quantum corrections) formulae for the Sin(h)-Gordon form-factors are given in such a form with the factorized (not given explicitly) tree contribution so that our results complement the results of [19]. In [20] the Yang-Mills amplitudes in the so-called multi-Regge asymptotics are also given in such a factorized form with explicit tree factor.

Thirdly, the present construction of the perturbiner in Sin(h)-Gordon theory is analogous to the construction of the self-dual perturbiner in Yang-Mills theory and in gravity, in particular all these constructions are based on the same universal solution for a sort of Riemann-Hilbert problem (see the last paragraph in Section 2). This, hopefully, will open a new field of application for the exact 2d results. Finally, solutions of the type of perturbiner form another class of physically motivated solutions, in addition to the solitonic solutions and to the finite-gap solutions. Probably, they deserve study on their own right.

The rest of this paper is organized as follows. In Section 1 we formulate what type of solutions for the Sin(h)-Gordon equation we are looking for. In Section 2 we lift this formulation to the zero-curvature problem associated with the Sin(h)-Gordon equation. In Section 3 we present solution of the problem.

We would like to point out that our construction generalizes straightforwardly to the 2d Toda theories.

\section{Definition}

Let us consider the field equation

\begin{equation}
\partial \bar{\partial} \phi + \frac{m^2}{\beta} \sinh(\beta \phi) = 0
\end{equation}

where $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$, and $z$, $\bar{z}$ are two coordinates. In what follows we put $m^2 = 1$ since $m^2$-dependence can easily be restored. To define the perturbiner
one picks up a solution
\[ \phi^{(1)} = \sum_{j=1}^{N} a_j e^{ik_jz + i\frac{1}{2}k_j\bar{z}} = \sum_j \mathcal{E}_j \] (2)
of the free field equation
\[ \partial \bar{\partial} \phi^{(1)} + \phi^{(1)} = 0 \] (3)
The coefficients \( a_j \) are assumed to be commuting and nilpotent formal variables, \( a_j a_l = a_l a_j, \ a_j^2 = 0 \). Index \( j = 1, \ldots, N \) numbers the one-particle states in the given set.

The perturbiner \( \phi^{ptb} \) is a (complex) solution of Eq.(1) which is polynomial in the variables \( \mathcal{E}_j \) entering Eq.(2), first order term of the polynomial being just \( \phi^{(1)} \) Eq.(2). One can see that when the set of momenta \( k_j \) is generic this solution exists and is unique.

\( \phi^{ptb}(z, \bar{z}, \{k\}, \{a\}) \) thus defined is a generating function for the tree form-factors in the theory, that is its expansion in powers of \( a \)'s reads
\[ \phi^{ptb}(z, \bar{z}, \{k\}, \{a\}) = \sum_{J_d} <k_{j_1}, \ldots, k_{j_d}|\phi(z, \bar{z})|0 >_{tree} a_{j_1} \ldots a_{j_d} \] (4)
where the sum runs over all subsets \( J_d \) of the set \( 1, \ldots, N \). The nilpotency of \( a \)'s means that the form-factors with identical one-particle states will not appear in the expansion Eq.(4).

2 Zero-curvature representation

Zero-curvature representation for the Sin(h)-Gordon theory (see the book \[21\] and references therein) can be taken in the form
\[ A_z = -\frac{\beta}{4} \sigma_1 \partial \phi + \frac{\lambda}{2} \sigma_3 \cosh \frac{\beta \phi}{2} + \frac{\lambda}{2} i \sigma_2 \sinh \frac{\beta \phi}{2} \]
\[ A_{\bar{z}} = \frac{\beta}{4} \sigma_1 \bar{\partial} \phi - \frac{1}{2\lambda} \sigma_3 \cosh \frac{\beta \phi}{2} + \frac{1}{2\lambda} i \sigma_2 \sinh \frac{\beta \phi}{2} \] (5)
where \( \lambda \) is a non-homogeneous coordinate on an auxiliary \( CP^1 \) space, the so-called spectral parameter, and \( \sigma_i \) are Pauli matrixes. The Sin(h)-Gordon

\[ ^2\text{Since we work in the tree approximation we do not care about separation of the positive- and negative-energy one-particle states. We just consider all particles in the out-states while, more precisely, those with negative energy should be considered as in-states.} \]
equation (1) is equivalent to
\[ \partial A_{\bar{z}} - \bar{\partial} A_z + [A_z, A_{\bar{z}}] = 0 \] (6)

The connection form Eq.(5) is meromorphic on the auxiliary \( CP^1 \) space with simple poles at \( \lambda = 0 \) and \( \lambda = \infty \). Correspondingly, the zero-curvature condition Eq.(6) consists in fact of a number of equations - at different powers of \( \lambda \) - most of which are automatically resolved when the connection form is taken in the form Eq.(5), independently of the field \( \phi(z, \bar{z}) \). The only nontrivial equation arises at \( \lambda^0 \) and is equivalent to Eq.(1).

For the following construction it is important that a generic zero-curvature connection obeying the “reduction condition”
\[ A(-\lambda) = \sigma_1 A(\lambda) \sigma_1 \] (7)
is equivalent to the connection Eq.(5) modulo gauge transformations, choice of coordinates \( z, \bar{z} \) and redefinition of the field \( \phi \) [22]. The gauge transformations are transformations
\[ A \rightarrow h^{-1} Ah + h^{-1} dh \] (8)
where \( h \) is an \( SL(2, \mathbb{C}) \) matrix independent of \( \lambda \) and commuting with the reduction Eq.(7), \( d \) is external derivative. Notice that the field \( \phi \) is a gauge invariant object.

The zero-curvature condition Eq.(6) is (locally) solved as
\[ A = g^{-1} dg \] (9)
where, unlike \( h \) in Eq.(5), \( g \) is a nontrivial function of \( \lambda \) subject to the condition that the connection form \( A(\lambda) \) has simple poles at \( \lambda = 0 \) and \( \lambda = \infty \) and also that \( A(\lambda) \) obeys the reduction condition Eq.(7) which for \( g \) gives
\[ g(-\lambda) = \sigma_1 g(\lambda) \sigma_1 \] (10)
The gauge transformation Eq.(8) acts on \( g(\lambda) \) as multiplication by an independent of \( \lambda \) matrix \( h \) from the right.

Since \( \phi^{ptb} \) is polynomial in \( \{E\}_j \), the corresponding connection form \( A^{ptb} \) is also polynomial. It is convenient to split off explicitly the zeroth order part of it,
\[ A^{ptb}(\lambda, \{E\}) = A^{(0)}(\lambda) + A'(\lambda, \{E\}) \], (11)
to define a derivative
\[ \nabla^{(0)} = d + A^{(0)}, \] (12)
to introduce \( g^{(0)}(\lambda) \) such that
\[ A^{(0)} = g^{(0)-1}dg^{(0)}, \] (13)
and to define \( g'(\lambda) \) according to
\[ A_{\rho b} = g_{\rho b}^{(0)-1}dg_{\rho b}, \]
\[ g_{\rho b}(\lambda) = g^{(0)}(\lambda)g'(\lambda) \] (14)

In terms of the introduced notations the local solution of the zero-curvature condition rewrites as
\[ A' = g'^{-1}\nabla^{(0)}g' \] (15)
where the non-derivative term in \( \nabla^{(0)} \) acts on \( g' \) as commutator.

Like \( \phi_{\rho b} \), \( g' \) can be sought for as polynomial in \( \mathcal{E} \)'s. We stress that, like in \( \phi_{\rho b} \), the space-time dependence of \( g' \) comes only via polynomials in the nilpotent variables \( \mathcal{E} \); that is why it was convenient to split out \( g^{(0)} \) (which has different type of the space-time dependence). First order terms \( g'^{(1)} \) of the polynomial \( g' \) are fixed by the ones in \( A' \) which, in turn, are defined by the ones in \( \phi_{\rho b} \), Eq. (2),
\[ g'^{(1)}(\lambda) = \sum_{j=1}^{N} \frac{\beta}{4} \mathcal{E}_j^{\sigma_{+}} \frac{\lambda + q_j - 2ik_j}{\lambda + ik_j - q_j} + \frac{\beta}{4} \mathcal{E}_j^{\sigma_{+}} \frac{\lambda - q_j - 2ik_j}{\lambda - ik_j} \] (16)
where \( \sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2) \).

The parameters \( q_j \) in Eq. (16) are present due to the gauge freedom, Eq. (8). To get the gauge condition used in Eq. (5) one should put \( q_j = -ik_j \), however there might be other convenient choice, as we are anyway interested in the gauge invariant object, field \( \phi \). We remind that that index \( j = 1, \ldots, N \) numbers the one-particle states in the given set.

These first order terms, \( g'^{(1)}(\lambda) \) Eq. (16), possess simple poles at \( \lambda = \pm ik_j \).

From the condition of regularity of \( A' \) Eq. (15) anywhere except \( \lambda = 0, \lambda = \infty \) one can show that the all the polynomial \( g'(\lambda) \) can have only simple poles at the same points, \( \lambda = \pm ik_j \), the higher order terms of the polynomial being defined uniquely modulo gauge transformations.

\[ ^3 \text{Of course, the zeroth order term in the polynomial } g' \text{ is unit matrix, } g'^{(0)} = 1. \]
To put the problem of finding $g'(\lambda)$ in a more universal form we introduce an index $\hat{j}$ consisting of two indices,

$$\hat{j} = (j, s); j = 1, \ldots, N; s = \pm,$$

introduce notations $\hat{E}_j$,

$$\hat{E}_{j,+} = \frac{\beta}{4} \frac{2i k_j}{ik_j - q_j} E_j \sigma_+,$$

$$\hat{E}_{j,-} = \frac{\beta}{4} \frac{2i k_j}{ik_j - q_j} E_j \sigma_-,$$

and notations $\lambda_j, q_j$, where

$$\lambda_{j,\pm} = \mp i k_j, q_{j,\pm} = \mp q_j$$

With these notations Eq. (18) becomes

$$g'(\hat{j}) = \sum_j \hat{E}_j \frac{\lambda - q_j}{\lambda - \lambda_j}$$

Introduce also $g'_j$,

$$g'_j(\lambda) = 1 + \hat{E}_j \frac{\lambda - q_j}{\lambda - \lambda_j}$$

Then the condition of regularity of the connection form $A' Eq. (19)$ in vicinity of point $\lambda = \lambda_j$ requires that

$$g'_j(\lambda) \frac{1}{g'(\lambda)}$$

is regular at $\lambda = \lambda_j$

Thus, we are looking for $g'(\lambda)$, polynomial in $\hat{E}_j Eq. (18)$, first order term of the polynomial being as in Eq. (20). $g'(\lambda)$ has simple poles at $\lambda = \lambda_j Eq. (19)$ and obeys conditions Eq. (21) at the poles. $g'(\lambda)$ must also obey the “reduction condition” Eq. (10). This problem for $g'(\lambda)$ has a unique solution modulo the gauge transformations.

### 3 Solution

The problem formulated in the last paragraph of section 2 is analogous to the ones solved in the case of Yang-Mills theory in [4] (that solution was later
used in the case of gravity in [5] and in the case of Yang-Mills+gravity in [6]. The new things are the reduction condition Eq.(10) and the condition
\[ \tilde{E}^\pm = 0 \] (a trivial consequence of the nilpotency condition, \( a^2_j = 0 \)), which are not crucial. We just give the solution,
\[ g'(\lambda) = 1 + \sum_{d=1}^{N} \sum_{j_1,\ldots,j_d} \frac{\lambda - q_{j_1} - q_{j_2} - \ldots - q_{j_d}}{\lambda - \lambda_{j_1} - \lambda_{j_2} - \ldots - \lambda_{j_d}} \tilde{E}_{j_1} \ldots \tilde{E}_{j_d}, \] (23)
all notations are introduced in Eqs.(17)-(19).

The inverse for this matrix reads
\[ g'^{-1}(\lambda) = 1 - \sum_{d=1}^{N} \sum_{j_1,\ldots,j_d} \frac{\lambda - q_{j_1} - q_{j_2} - \ldots - q_{j_d}}{\lambda - \lambda_{j_1} - \lambda_{j_2} - \ldots - \lambda_{j_d}} \tilde{E}_{j_1} \ldots \tilde{E}_{j_d}, \] (24)
With the Eqs.(23),(24) one can find the connection form \( A' \) Eq.(15), from which one obtains, finally, \( \phi^{ptb} \),
\[ \phi^{ptb} = \sum_{d \text{ odd}} 2^d \frac{\beta}{d-1} \sum_{j_1,\ldots,j_d} \frac{k_{j_1} \ldots k_{j_d}}{(k_{j_1} + k_{j_2}) \ldots (k_{j_d} + k_{j_1})} \tilde{E}_{j_1} \ldots \tilde{E}_{j_d} \] (25)
We remind that \( \tilde{E}_j = a_j e_{ik_j z + \frac{i}{2} k_j} \) (due to the nilpotency, \( \tilde{E}_j^2 = 0 \), all \( j, l = 1, \ldots, d \) must be different). \( \phi^{ptb} \) is a generating function for the tree form-factors in the sense of Eq.(14).

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\[ \text{We point out a minor subtlety concerning the solution Eq.(23). At general values of the free parameters } q_j, \text{ } g'(\lambda) \text{ does not belong to } SL(2,C), \text{ only to } GL(2,C) \text{ instead. However, one can show that } \det g'(\lambda) \text{ is, actually, independent of } \lambda, \text{ thus } g'(\lambda) \text{ is gauge equivalent to an } SL(2,C) \text{ matrix. Therefore this subtlety doesn’t actually matter since the field } \phi - \text{ which is the final aim - is a gauge invariant object. On can also show that } \det g'(\lambda) = 1 \text{ when all } q \text{'s are equal each other.} \]
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