Bulk reconstruction from a scalar CFT at the boundary by the smearing with the flow equation

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Abstract We explain our proposal for an alternative bulk reconstruction of AdS/CFT correspondences from a scalar field by the flow method. By smearing and then normalizing a primary field in a $d$ dimensional CFT, we construct a bulk field, through which a $d+1$ dimensional AdS space emerges.

1 Introduction

The AdS/CFT correspondence[1] is a key to understand a holographic nature of gravity and may give a hint for quantum gravity. Although a plenty of circumstance evidences exist, an essential mechanism why AdS/CFT correspondence realizes has not been fully established yet. While the correspondence may be explained by the string theory, an alternative but more universal mechanism might exist.

One of key questions is how an additional dimension of the AdS emerges from the CFT at the boundary. One approach, called the HKLL bulk

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reconstruction\[2, 3\], gives a relation between a bulk local operator in the AdS and CFT operators at the boundary with Lorentzian signature.

Recently we have proposed an alternative bulk reconstruction by the so-called flow method by the path-integral with Euclidean signature\[4, 5, 6, 7, 8, 9\], which is explained in this talk.

2 Bulk reconstruction by the flow

2.1 From conformal symmetry to bulk symmetry

Let us consider a non-singlet primary field of a conformal field theory (CFT) on a d dimensional Euclidean space, whose 2-pt function behaves as

\[ \langle \phi^a(x) \phi^b(y) \rangle = \delta^{ab} \frac{1}{|x-y|^{2\Delta}}, \]

where \( \Delta \) is a conformal dimension of \( \phi^a \) and \( a, b \ldots \) represent indices of a global symmetry such as an \( O(N) \) symmetry.

We first make one parameter extension of the field \( \phi^a \) via the generalized flow equation as

\[ (-\alpha \eta \partial^2 + \beta \partial_\eta) \phi^a(x, \eta) = \Box \phi^a(x, \eta), \quad \phi^a(x, 0) = \phi^a(x), \]

where \( \alpha, \beta \) are parameters to control this extension. We regard this process as the smearing of the field \( \phi^a \), since the flow equation becomes the heat equation at \( \alpha = 0 \). We then normalize \( \phi^a \) as

\[ \sigma^a(X) = \frac{\phi^a(x, \eta)}{\sqrt{\langle \phi(x, \eta) \cdot \phi(x, \eta) \rangle}} \Rightarrow \langle \sigma(X) \cdot \sigma(X) \rangle = 1, \]

where the average is taken for the CFT in \( d \) dimensions as in \[11] . \( X := (z, x) \) is a \( d+1 \) dimensional coordinate with \( \eta := \alpha \eta^2 / 4 \), and \( F \cdot G := \sum_a F^a G^a \).

The smearing followed by normalization may be interpreted as a kind of the renormalization group transformation. The field \( \sigma^a \) is expressed in the smeared form by the integral as

\[ \sigma^a(X) = \int d^d y h(z, x-y) \phi^a(y). \]

The smearing kernel \( h(z, x) \) is determined by conditions that the conformal transformation \( U \) to \( \phi^a \), given by

\[ U \phi^a(y) U^\dagger := \bar{\phi}^a(y) = J^\Delta(y) \phi^a(y), \]

is explained here.
Bulk reconstruction by the smearing with the flow equation generates the coordinate transformation to $\sigma^a$ in $d + 1$ dimensions as

$$U \sigma^a(X) U^\dagger := \tilde{\sigma}^a(X) = \sigma^a(\tilde{X}), \quad (6)$$

where

1. translation $\tilde{y}^\mu = y^\mu + a^\mu, \ J(y) = 1, \ \tilde{x}^\mu = x^\mu + a, \ \tilde{z} = z$,
2. rotation $\tilde{y}^\mu = \Omega^\mu_\nu y^\nu, \ J(y) = 1, \ \tilde{x}^\mu = \Omega^\mu_\nu x^\nu + a, \ \tilde{z} = z$,
3. dilatation $\tilde{y}^\mu = \lambda y^\mu, \ J(y) = \lambda, \ \tilde{X}^A = \lambda X^A$,
4. inversion $\tilde{y}^\mu = y^\mu / y^2, \ J(y) = 1 / y^2, \ \tilde{X}^A = X^A / X^2$,

which generate SO($d + 1, 1$) transformation. Transformations 1–3 imply

$$\sigma^a(X) = z^{\Delta - d} \int d^d y \Sigma \left(1 + \frac{(x - y)^2}{z^2}\right) \varphi^a(y), \quad (7)$$

where $\Sigma$ is an arbitrary function. The transformation 4 to the CFT operator in the above equation generates

$$z^{\Delta - d} \int d^d y \Sigma \left(1 + \frac{(x - y)^2}{z^2}\right) \varphi(q) = \int d^d q \Sigma \left(1 + \frac{(x - q)^2}{z^2}\right) (q^2)^{\Delta - d} \varphi(q)$$

while the bulk operator transforms as

$$\sigma^a(\tilde{X}) = \tilde{z}^{\Delta - d} \int d^d q \Sigma \left(1 + \frac{(\tilde{x} - q)^2}{\tilde{z}^2}\right) \varphi(q). \quad (9)$$

Thus the symmetry implies $\Sigma(u) = \Sigma_0 u^{\Delta - d}$, which finally gives

$$\sigma^a(X) = \int d^d y h(z, x - y) \varphi^a(y), \quad h(z, x) = \Sigma_0 \left(\frac{z}{\tilde{z}^2 + x^2}\right)^{d - \Delta}, \quad (10)$$

where $\Delta < d$ is necessary for this expression to be regarded as a smearing. After the Wick rotation, this smearing kernel is identical to the one in the HKLL[3], though $\Delta > d - 1$ is required for the HKLL.

### 2.2 Some properties

It is easy to see the kernel $h$ satisfies $(\square_{\text{AdS}} - m^2) h(X) = 0$, where $\square_{\text{AdS}} := z^2 (\partial_z^2 + \square) - (d - 1) z \partial_z$, and $m^2 R^2 := (\Delta - d) \Delta$ with some length scale $R$, which turns out to be the AdS radius. Furthermore, $h(X)$ corresponds to a solution to the flow equation (2) with $\beta / \alpha = (d - 2) / 2 - \Delta$. As a result, $\sigma^a(X)$ satisfies equations of motion for a scalar field in the Euclidean AdS
space as
\[(\mathbb{□}_{\text{AdS}} - m^2)\sigma^a(X) = 0.\] (11)

Since, for \(\Delta < d/2\),
\[
\lim_{z \to 0} \frac{z^{d-2\Delta}}{(x^2 + z^2)^{d-\Delta}} \sim \begin{cases} 
  z^{d-2\Delta} \to 0, & x \neq 0, \\
  z^{-d} \to \infty, & x = 0,
\end{cases}
\] (12)
\[
\int d^d x \frac{z^{d-2\Delta}}{(x^2 + z^2)^{d-\Delta}} = \frac{\pi^{d/2} \Gamma(d/2 - \Delta)}{\Gamma(d - \Delta)} := \frac{1}{A}.
\] (13)
we can write
\[
\lim_{z \to 0} \frac{z^{d-2\Delta}}{x^2 + z^2} = \frac{\delta^{(d)}(x)}{A}.
\] (14)

Thus the smearing function satisfies
\[
\lim_{z \to 0} h(z, x) = \Sigma_0 \frac{\delta^{(d)}(x)}{A},
\] (15)
which leads to a BDHM relation\[10\] as
\[
\lim_{z \to 0} z^{-\Delta} \sigma^a(X) = \frac{\Sigma_0}{A} \varphi^a(x).
\] (16)

Using a singlet bulk composite scalar field \(S(x) := \sum_a \sigma^a(X) \sigma^a(x)\), we can define a bulk to boundary correlation function as
\[
\mathcal{F}_O(X, y) := \langle S(X)O(y) \rangle
\] (17)
where \(O(y)\) is an arbitrary singlet scalar primary field with a conformal dimension \(\Delta_O\) at the boundary. A combination of the conformal symmetry and the bulk symmetry in the previous subsection leads to
\[
\mathcal{F}_O(X, y) = C_O \left( \frac{z}{(x - y)^2 + z^2} \right)^{\Delta_O},
\] (18)
which is exact in the sense that only unknown constant \(C_O\) depends on the detail of the boundary CFT such as coupling constants, and reproduces the standard prediction by the AdS/CFT correspondence. Furthermore, \(\mathcal{F}\) satisfies
\[
(\mathbb{□}_{\text{AdS}} - m_O^2)\mathcal{F}(X, y) = 0, \quad m_O^2 := \frac{\Delta_O(\Delta_O - d)}{R^2}.
\] (19)
3 Symmetries and bulk geometry

In this section we investigate what kind of space the $d+1$ dimensional coordinate $X$ describes.

3.1 Constraints to a generic correlation function

Let us consider a generic correlation function, which contains $m$ bulk fields and $s$ boundary fields, given by

$$\left\langle m \prod_{i=1}^{m} G_{i}^{A_{1}^{i} A_{2}^{i} \cdots A_{n_{i}}^{i}}(X_{i}) \prod_{j=1}^{s} O_{j}^{\mu_{1}^{j} \mu_{2}^{j} \cdots \mu_{l_{j}}^{j}}(y_{j}) \right\rangle, \quad (20)$$

where $A_{1}^{i} A_{2}^{i} \cdots A_{n_{i}}^{i}$ is a tensor index of a bulk operator $G_{i}$, while $\mu_{1}^{j} \mu_{2}^{j} \cdots \mu_{l_{j}}^{j}$ is that for a boundary operator $O_{j}$. The conformal symmetry at the boundary with the associated coordinate transformation in the bulk gives an exact quantum relation as

$$\left\langle m \prod_{i=1}^{m} G_{i}^{A_{1}^{i} A_{2}^{i} \cdots A_{n_{i}}^{i}}(\tilde{X}_{i}) \prod_{j=1}^{s} O_{j}^{\mu_{1}^{j} \mu_{2}^{j} \cdots \mu_{l_{j}}^{j}}(\tilde{y}_{j}) \right\rangle = m \prod_{i=1}^{m} \frac{\partial X_{i}^{B_{1}^{i}}}{\partial X_{i}^{A_{1}^{i}}} \frac{\partial X_{i}^{B_{2}^{i}}}{\partial X_{i}^{A_{2}^{i}}} \cdots \frac{\partial X_{i}^{B_{n_{i}}^{i}}}{\partial X_{i}^{A_{n_{i}}^{i}}}$$

$$\times \prod_{j=1}^{s} J(y_{j})^{-\Delta_{j}} \frac{\partial y_{j}^{\nu_{1}^{j}}}{\partial \tilde{y}_{j}^{\mu_{1}^{j}}} \frac{\partial y_{j}^{\nu_{2}^{j}}}{\partial \tilde{y}_{j}^{\mu_{2}^{j}}} \cdots \frac{\partial y_{j}^{\nu_{l_{j}}^{j}}}{\partial \tilde{y}_{j}^{\mu_{l_{j}}^{j}}} \left\langle m \prod_{i=1}^{m} G_{i}^{B_{1}^{i} B_{2}^{i} \cdots B_{n_{i}}^{i}}(X_{i}) \prod_{j=1}^{s} O_{j}^{\nu_{1}^{j} \nu_{2}^{j} \cdots \nu_{l_{j}}^{j}}(y_{j}) \right\rangle. \quad (21)$$

3.2 Metric field

We define the bulk metric field as

$$g_{AB}(X) := \ell^{2} \sum_{a=1}^{N} \partial_{A} \sigma^{a}(X) \partial_{B} \sigma^{a}(X), \quad (22)$$

which is the simplest among singlet and symmetric 2nd rank tensors in the bulk, where $\ell$ is some length scale. It is worth noting that this metric field is finite without ultra-violate (UV) divergence thanks to the smearing.

The metric becomes classical in the large $N$ limit due to the large $N$ factorization as
\[
\langle g_{AB}(X_1)g_{CD}(X_2) \rangle = \langle g_{AB}(X_1) \rangle \langle g_{CD}(X_2) \rangle + O(1/N),
\]
which make a vacuum expectation value (VEV) of the quantum Einstein tensor \( G_{AB} \) classical:
\[
\langle G_{AB}(g_{CD}) \rangle = G_{AB}(\langle g_{CD} \rangle) + O(1/N).
\]
A classical geometry appears after quantum averages.

The VEV of the metric can be interpreted as the (Bures) information metric\[^4\], which defines a distance between (mixed state) density matrices \( \rho \) and \( \rho + d\rho \) as
\[
d^2(\rho, \rho + d\rho) := \frac{1}{2} \text{tr}[d\rho G],
\]
where \( G \) satisfies \( \rho G + G\rho = d\rho \). In our case, \( \rho \) is given by a mixed state as
\[
\rho(X) := \sum_{a=1}^{N} |\sigma^a(X)\rangle \langle \sigma^a(X)|,
\]
which represents \( N \) entangled pairs, where the inner product between states is defined by \( \langle \sigma^a(X)|\sigma^b(Y) \rangle = \delta^{ab}\langle \sigma(X)\cdot \sigma(Y)\rangle/N \). Using this definition, we obtain
\[
\ell^2 d^2(\rho, \rho + d\rho) = \langle g_{AB}(X) \rangle dX^A dX^B.
\]
Thus the VEV of the metric field \( g_{AB}(X) \) defines a distance in the bulk space through \( d^2(\rho, \rho + d\rho) \) in unit of \( \ell^2 \).

### 3.3 VEV of the metric field

Let us calculate the VEV of the metric field. The constraint (21) applied to \( g_{AB} \) leads to
\[
\langle 0|g_{AB}(X)|0 \rangle = R^2 \frac{\delta_{AB}}{\ell^2},
\]
which describes the AdS space in the Poincare coordinate, where an unknown constant \( R \) is the radius of the AdS space. The explicit form of the 2-pt function of CFT in (1) gives \( R^2 = \ell^2 \Delta(d-\Delta)/(d+1) \), which is positive since \( \Delta < d/2 \). The boundary CFT generates the bulk AdS, and the bulk symmetry in the previous section is equal to the AdS isometry on \( \langle 0|g_{AB}(X)|0 \rangle \). Thus the AdS/CFT correspondence is realized by the flow construction.
3.4 Scalar excited state contribution

We finally consider how excited states modify the AdS structure described by the VEV of the metric field. As the simplest example, we calculate the VEV of the metric in the presence of the source $J$ coupled to the singlet CFT scalar field at the origin in the radial quantization as

$$\bar{g}_{AB}(X) = \langle 0| g_{AB}(X)e^{iO(0)}|0 \rangle = \langle 0| g_{AB}(X)|0 \rangle + J\langle 0| g_{AB}(X)|S \rangle + O(J^2),$$

where we need to calculate

$$\langle 0| g_{AB}(X)|S \rangle = \lim_{y^2 \to 0} G_{AB}(X, y), \quad G_{AB}(X, y) := \langle 0| g_{AB}(X)O(y)|0 \rangle.$$  

(29)

The constraint (21) to $G_{AB}$ reads

$$J(y)^{\Delta O} G_{AB}(X, y) = \frac{\partial \bar{X}^C}{\partial X^A} \frac{\partial \bar{X}^D}{\partial X^B} G_{CD}(\bar{X}, \bar{y}),$$

(31)

which leads to

$$G_{AB}(X, y) = T^{\Delta O}(X, y) \left[ a_1 \frac{\delta_{AB}}{z^2} + a_2 \frac{T_A(X, y)T_B(Y, y)}{T^2(X, Y)} \right],$$

(32)

where $a_1$ and $a_2$ are unknown constants, and

$$T(X, y) := \frac{z}{(x - y)^2 + z^2}, \quad T_A(X, y) := \partial_A T(X, y).$$

(33)

We thus obtain

$$\bar{g}_{AB}(X) = \frac{\delta_{AB}}{z^2} \left[ R^2 + a_1 J \left( \frac{z}{x^2 + z^2} \right)^{\Delta O} \right] + a_2 JT_A T_B \left( \frac{z}{x^2 + z^2} \right)^{\Delta O - 2}$$

(34)

at the 1st order in $J$, where

$$T_z := \frac{x^2 - z^2}{(x^2 + z^2)^2}, \quad T_\mu := -\frac{2x_\mu z}{(x^2 + z^2)^2}.$$  

(35)

This metric describes the asymptotically AdS space since

$$\bar{g}_{AB}(X) = R^2 \frac{\delta_{AB}}{z^2} \left[ 1 + O(z^{\Delta O}) \right],$$

(36)

as $z \to 0$. 


4 Summary and discussion

Smearing and normalizing the non-singlet primary field $\varphi^a$ with $\Delta < d/2$ at the boundary give the bulk field $\sigma^a$. Through this process, the conformal symmetry turns into the bulk coordinate transformation, which leads to the expected behavior for the bulk to boundary correlation function. As the VEV of the metric field describes the AdS space, the AdS/CFT correspondence is naturally realized by our method.

As one of future problems, let us consider a bulk to bulk correlation function for the scalar, which is evaluated as

$$
\langle S(X_1)S(X_2) \rangle = 1 + \frac{2}{N} G^2(X_1, X_2) + \langle S(X_1)S(X_2) \rangle_c,
$$

where

$$
G(X_1, X_2) = \frac{\Delta}{2} \frac{d - \Delta}{2} F_1 \left( \frac{d + 1}{2}, \frac{1 - R_{12}^2}{4} \right),
$$

$$
R_{12} := \left( \frac{x_1 - x_2}{z_1 z_2} \right)^2 + \frac{z_1^2 + z_2^2}{R_{12}^2},
$$

which is non-singular at $X_1 = X_2$. Since the connected part given by the last term is absent for the free CFT, the corresponding bulk theory is non-local (stringy). Thus the interaction in the CFT is required to recover the locality of the bulk theory.

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