Anti- (Conjugate) Linearity

Armin Uhlmann

University of Leipzig, Institute for Theoretical Physics
Germany, D-04009 Leipzig, PB 100920

key words: operators, canonical form, antilinear (skew) hermiticity, acq–lines
PACS-codes: 02.30 Tb, 03.10 Ud, 03.65 Fd, 03.65 Ud

Abstract

This is an introduction to antilinear operators. In following E. P. Wigner the terminus “antilinear” is used as it is standard in Physics. Mathematicians prefer to say “conjugate linear”.

By restricting to finite-dimensional complex-linear spaces, the exposition becomes elementary in the functional analytic sense. Nevertheless it shows the amazing differences to the linear case.

Basics of antilinearity is explained in sections 2, 3, 4, 7 and in subsection 1.2: Spectrum, canonical Hermitian form, antilinear rank one and two operators, the Hermitian adjoint, classification of antilinear normal operators, (skew) conjugations, involutions, and acq–lines, the antilinear counterparts of 1–parameter operator groups. Applications include the representation of the Lagrangian Grassmannian by conjugations, its covering by acq–lines, as well as results on equivalence relations. After remembering elementary Tomita–Takesaki theory, antilinear maps, associated to a vector of a two-partite quantum system, are defined. By allowing to write modular objects as twisted products of pairs of them, they open some new ways to express EPR and teleportation tasks. The appendix presents a look onto the rich structure of antilinear operator spaces.

Contents

1 Introduction
1.1 Time reversal operations and beyond ............................. 3
1.2 Choosing notations .............................................. 5

1SCIENCE CHINA Physics, Mechanics & Astronomy, 59, 3, (2016) 630301; doi: 10.1007/s11433-015-5777-1
2Sincere thanks to ShaoMing Fei (Associate editor) for his friendly invitation and to the editorial staff for their patient help.
2 Anti- (or conjugate) linearity

2.1 Definition ......................................................... 9
2.2 Eigenvalues ......................................................... 9
  2.2.1 dim \( L = 2 \) .............................................. 10
2.3 Rank-one operators ............................................... 11
2.4 The canonical Hermitian form ................................... 13
2.5 Pauli operators and their antilinear partners ................. 14
2.6 Matrix representation ............................................ 17

3 Antilinearity in Hilbert spaces .................................... 18

3.1 The Hermitian adjoint ............................................ 19
  3.1.1 The field of values ........................................... 22
3.2 Antilinear rank one operators ................................... 23
3.3 Linear and antilinear Pauli operators ......................... 24
3.4 Antilinear maps between Hilbert spaces ....................... 26

4 Antilinear normal operators ....................................... 29

4.1 Antiunitaries acting on \( H_2 \) .................................... 31
4.2 Decompositions of normal antilinear operators .............. 32
  4.2.1 Conjugations ................................................. 35
  4.2.2 Skew conjugations .......................................... 36
  4.2.3 The number of orthogonal (skew) conjugations ........... 37

5 A look at elementary symplectic geometry ..................... 38

5.1 Conjugations and Symplectic Geometry ....................... 39
  5.2 The canonical differential 1-form .............................. 41

6 Equivalence relations .............................................. 43

6.1 Similarity, Congruence .......................................... 44
6.2 Unitary equivalence .............................................. 45
6.3 Low dimensions ................................................ 46
6.4 UET and beyond .................................................. 48
  6.4.1 Length one .................................................. 51
  6.4.2 Examples .................................................... 53

7 Involutions .......................................................... 55

7.1 Polar decomposition ............................................. 56
7.2 Similarity of involutions ....................................... 57
7.3 Involutions and the geometric mean ............................ 58

8 Modular objects ..................................................... 60

8.1 Maximal commutative subalgebras ................................ 63
8.2 Bipartite quantum systems ..................................... 64
1 Introduction

The topic of this manuscript is the phenomenon of antilinearity. It is partly an introduction and partly a collection of examples to show its use.

An operator $\theta$ acting on a complex-linear space is called “antilinear” or “conjugate linear” if for any two vectors $\phi_1, \phi_2,$ and complex numbers $c_1, c_2,$

$$\theta (c_1 \phi_1 + c_2 \phi_2) = c_1^* \theta \phi_1 + c_2^* \theta \phi_2$$

is valid. In Mathematics the term “conjugate linearity” is preferred while in Physics the notation “antilinearity”, as advocated by E. P. Wigner, is in use. I follow the latter convention. See also the remarks on notation below.

I restrict myself to complex-linear spaces of finite dimension mostly, starting in section 2 with some basic definitions and facts. Remarkably, though quite trivial, the eigenvalues of antilinear operators form circles centered at zero in the complex plane. Therefore the trace and other symmetric functions of the eigenvalues are undefined for antilinear operators. As a "compensation" $\text{Tr} \theta_1 \theta_2$ is an Hermitian form with signature equal to the dimension of the linear space on which the operators act.

While the set of linear operators is naturally an algebra, say $\mathcal{B}(\mathcal{L})$, the antilinear operators form a $\mathcal{B}(\mathcal{L})$–bimodule $\mathcal{B}(\mathcal{L})_{\text{anti}}$. Hence their direct sum is a 2-graded algebra. An active domain of research are operator functions, for instance power series of elements of this algebra, see M. Huhtanen and A. Perämäki in [43, 44] for example. Please notice: This rich theory is not under consideration here.

Beginning with section 3 the basic spaces are equipped with scalar products making them Hilbert spaces $\mathcal{H}$. As usual, $\mathcal{B}(\mathcal{H})$ denotes the algebra
of linear operators. The linear space of antilinear operators will be called $B(\mathcal{H})_{\text{anti}}$. The scalar product allows the definition of the Hermitian adjoint $\vartheta^\dagger$ of any antilinear operator $\vartheta$. It is a linear operation.

Given a scalar product it becomes routine to define Hermitian (self-adjoint), skew Hermitian, unitary, and normal antilinear operators, including conjugations and skew conjugations.

There is a strong connection between matrices and antilinear operators. While matrix analysis is operator theory with a distinguished basis, antilinear operators allow to formulate several parts of it in a transparent and basis independent way. The said connection is mediated by a basis $\phi_1, \ldots, \phi_d$ of $\mathcal{H}$. Depending on the chosen basis, one associates to a matrix $a_{jk}$ an antilinear operator according to

$$\sum_j c_j \phi_j \to \sum_{jk} c^*_j a_{kj} \phi_k$$

From a symmetric (skew symmetric) matrix one gets an antilinear Hermitian (skew Hermitian) operator. The said operator is (skew) Hermitian in every basis of $\mathcal{H}$.

It seems tempting to translate theorems concerning symmetric or skew symmetric matrices into the language of antilinearity. For example, the question whether a matrix is unitary equivalent to its transpose can be "translated" into: Is the Hermitian adjoint $X^\dagger$ antunitarily equivalent to $X$? Some cases are reported in section 6.

In exploring properties of classes of antilinear operators, the finiteness assumption renders a lot of sophisticated functional analysis to triviality. On the other side it makes it much simpler to grasp the ideas coming with antilinearity! That is particular transparent in E. P. Wigner’s classification of antiunitary operators, [81], and in its extension to antilinear normal operators, [39], by F. Herbut and M. Vujičić discussed in section 4.

Section 5 points to the role of antilinearity in symplectic geometry. The maximal real Hilbert subspaces of $\mathcal{H}$ are the Lagrangian subspaces and, therefore, the points of the Lagrangian Grassmannian $\Lambda$. The bridge to antilinearity: Every Lagrangian subspace is the fix-point set of a conjugation and vice versa: The manifold of all conjugations is simplectomorphic to $\Lambda$. It will be shown how the Maslov index of a closed curve in $\Lambda$ can be expressed by the help of an operator-valued differential 1-form defined on the manifold of all conjugations.

In section 6, as already said, some equivalence relations are considered. There is a rich literature concerning complex symmetric matrices. Important sources are textbooks like [40] by R. A. Horn and C. R. Johnson, and newer research papers by L. Balayan, S. R. Garcia, D. E. Poore, E. Prodan, M. Putinar, J. E. Tener, and others. Only a few, hopefully typical
ones, of their results are reported. In addition I call attention to a class of
superoperators to point to some related problems.

The definition of involutions does not depend on a scalar product, see
section 7. But if there is one, their polar decompositions is of interest.

The Hermitian adjoint is an involution within the space of linear opera-
tors. Varying the scalar product varies the Hermitian adjoint. The relations
between them reflect nicely the geometric mean of S. L. Woronowicz between
different scalar products.

There are well known theories in which antilinear operators play an im-
portant role and in which the finiteness assumption is not appropriate. To
one of them, the handling of time reversal operators, a few words will be said
below. Another one is the famous Tomita–Takesaki theory, see section 8.
The finite dimensional case allows for applications to the Einstein–Podolski–
Rosen effect [24] as explained in section 9.

Some peculiar features of “quantum teleportation” are described in sec-
section 10. A first promise of quantum teleportation is already in [1]. The
very origin of all the matter is in the paper [12] by C. Bennett, G. Brassard,
C. Crepeau, R. Jozsa, A. Peres, and W. Wootters.

The purpose of the appendix is to point at antilinear operator spaces.

As already said, the paper remains within finite dimensional Hilbert and
other linear spaces. But even within this strong restriction, only a selected
part of the theory could be exposed.

Concerning (skew) conjugations and their applications, see also the re-
view [28] of S. R. Garcia, E. Prodan and M. Putinar. For much of their
work there is a natural counterpart within the language of antilinearity.

There seems to be no monograph dedicated essentially to conjugate lin-
ear, i.e. antilinear operators and maps in finite dimensional complex-linear
spaces. As already stressed, there are important parts of Linear Algebra
with a “hidden” antilinearity. The treatment of symmetric matrices, for
example, can be “translated” into that of antilinear Hermitian operators.
The transpose of a matrix can be viewed as the transform of its Hermitian
adjoint by a conjugation, and so on. These and some other examples are
explained in the main text. To reach completeness could by no means be
the aim.

1.1 Time reversal operations and beyond

The first explicit use of antilinearity in Physics is in Wigner’s 1932 paper
”¨Uber die Operation der Zeitumkehr in der Quantenmechanik” [80]. It
describes the prominent role of antilinearity in time reversal operations. In
Mathematics the idea of conjugate linearity goes probably back to E. Cartan,
[16]. The time–reversal symmetry has seen a lot of applications all over
Physics and it is well described in many textbooks. A survey can be found in \[1\]. In what follows, only a particular feature will be in the focus.

Assume $\psi(\vec{x}, t)$ satisfies a Schrödinger equation with real Hamiltonian $H$,

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi, \quad (H\psi)^* = H\psi^*.$$  

For any instant $s$ of time the time reversal operator $T_s$ is defined by

$$(T_s\psi)(\vec{x}, t - s) = \psi(\vec{x}, s - t)^*, \quad s \in \mathbb{R}.$$  

If $\psi$ is a solution of the Schrödinger equation, so it is $T_s\psi$. For physical reasons one cannot avoid antilinearity: $H$ must be bounded from below. Hence, generally, if $H$ is an Hamilton operator, $-H$ is not. Computing

$$(T_sT_r\psi)(\vec{x}, t) = \psi(\vec{x}, t - 2(s - r)), \quad T_sT_r = U(2r - 2s),$$

one gets the unitary evolution operator. It shifts any solution in time by the amount $2(s - r)$. Hence time translation operators can be written as products of two antilinear operators. This is a salient general feature: Some physically important operations can be written as products of two antilinear ones.

There is a further structure in the game. Using the equation above one shows

$$T_rT_{(r+s)/2} = T_{(r+s)/2}T_s, \quad r, s \in \mathbb{R}.$$  

(\textcolor{red}{\ast})

To stress the abstraction from time–reversal symmetry, the last equation is rewritten as

$$\vartheta_r\vartheta_{(r+s)/2} = \vartheta_{(r+s)/2}\vartheta_s, \quad r, s \in \mathbb{R}.$$  

(1)

Curves $t \to \vartheta_t$ satisfying (1) should be seen as antilinear surrogates of 1-parameter groups. The ad hoc notation “acq-line” will be used for them: “ac” stands for antilinear conjugate, and “q” for quandle. (Linear realizations of (1) may be called cq-lines.)

A set $\mathcal{M}$ of invertible antilinear operators will be called an “antilinear conjugate quandle” or an “ac-quandle” if, given $\vartheta, \vartheta' \in \mathcal{M}$, it follows $\vartheta^{-1}\vartheta'\vartheta \in \mathcal{M}$. The sets of all (skew) conjugations and of all (skew) involutions are examples of ac-quandles.

These notations follow that of rack, quandle, and kei, see S. R. Blackburn \[14\]. The fruitfulness of quandles and analogue structures has been discovered in knot theory by D. Joyce \[47\]. See also E. Nelson \[53\].

Returning now to an acq–line (1). Defining $\vartheta'_s = \vartheta_{as+b}$, $a, b$ real, $a \neq 0$, one gets a new acq–line which is a new parameterization of the original one. Notice the change of orientation if $a < 0$. However, the peculiar feature is the invariance of (1) by a change $s \to s + b$. Indeed, in contrast to groups, there is neither an identity nor an otherwise distinguished element in an acq–line.
Using an idea of M. Hellmund [38], an important class of acq–lines is gained:

**Lemma 1.1** [Hellmund] Let $\vartheta_0$ antilinear, $H$ linear and Hermitian. If $H$ commutes with $\vartheta_0$, then

$$t \rightarrow \vartheta_t := U(-t) \vartheta U(t), \quad U(t) = \exp \, itH$$

(2)

is an acq–line. Moreover, $B = \vartheta_0^2$ does not depend on $s$ and commutes with all $U(t)$.

Proof: $B = \vartheta_0^2$ is linear, commutes with $H$, and hence with $U(t)$. By $\vartheta_0^2 = U(-t) \vartheta_0^2 U(t)$ one concludes $B = \vartheta_0^2$ for all $t$. Again by $\vartheta_0 H = H \vartheta_0$ one obtains

$$U(-t) \vartheta_0 U(t) = U(2t) \vartheta_0 = \vartheta_0 U(-2t).$$

For two arguments one obtains

$$\vartheta_s \vartheta_t = U(2s) B U(-2t) = U(2s - 2t) B.$$  

(3)

Substituting either $s \rightarrow s$, $t \rightarrow (r + s)/2$ or $s \rightarrow (r + s)/2$, $t \rightarrow r$, one gets $U(s - r)B$ in both cases. Thus $t \rightarrow \vartheta_t$ fulfills [1].

A further example: The CPT operators [82], combinations of particle–antiparticle conjugation, parity and time reversal, constitute physically important ac-quandles. R. Jost [48] could prove that CPT operators are genuine symmetries of any relativistic quantum field theory satisfying Wightman’s axioms.

A CPT operator is defined up to the choice of a point $x$ in Minkowski space, which is the unique fix point of the map $x' - x \mapsto x - x'$, where $x'$ runs through all world points. Let $\Theta_x$ be the CPT-operator with this kinematical action. Then

$$\Theta_x \Theta_y = U(2y - 2x)$$

(4)

where $U$ denotes the unitary representation of the translation group as part of representations of the Poincaré group or its covering group. Notice the relation

$$\Theta_x \Theta_{(x+y)/2} = \Theta_{(x+y)/2} \Theta_y.$$  

(5)

It implies that, given two points $x$ and $y$, there is an acq–line $s \mapsto \Theta_s$ with $\Theta_0 = \Theta_x$ and $\Theta_1 = \Theta_y$.

In the same spirit the “physical” representations of the Poincaré group, respectively its covering group, can be gained by products of antilinear CT operations, provided the theory is CT-symmetric. The geometric part of a CT operations is a reflection on a space–like hyperplane in Minkowski space.

However, this is just an example within the world of Hermitian symmetric spaces [17], their automorphism groups and Shilov boundaries.

In subsection 7.3 there is a variant of [1]: The arithmetic mean is replaced be the geometric one.
1.2 Choosing notations

There are some differences in notations in Mathematics and Physics. This is somewhat unfortunate. I mostly try to follow notations common in the physical literature and in Quantum Information Theory. A mathematically trained person should not get into much troubles by a change in notations anyway.

1. The complex conjugate of a complex number $c$ is denoted by $c^*$ (and not by $\bar{c}$).

2a. If not explicitly said all linear spaces are assumed complex-linear and of finite dimension. Sometimes real linear spaces become important. These cases will be explicitly noticed by saying “real Hilbert space”, “real linear space”.

2b. Given a Hilbert space $H$, its scalar product is written $\langle \phi', \phi'' \rangle$ with “Dirac brackets” [22]. It is assumed linear in the second argument $\phi''$, and antilinear (conjugate linear) in its first one $\phi'$. This goes back to Schrödinger (1926).

2b. The symbol $(.,.)$ with arguments in $B(L)_{\text{anti}}$ or in $B(H)_{\text{anti}}$ represents the canonical form, see subsection 2.4.

2c. To avoid dangerous notations like $\langle \phi|\theta \rangle$ with $\theta$ antilinear, I do not use Dirac’s bra convention $\langle \phi|$. Schrödinger’s way of writing elements of Hilbert spaces say, $\phi$ or $\psi$, will be used mostly. Of course, Dirac’s “ket” notation, say $|\phi\rangle$ or $|12\rangle$ does not make any harm as it is not “dangerous” in the sense explained in subsection 3.1.

2d. The expressions $|\phi_1\rangle\langle\phi_2|$ and $|\phi_1\rangle\langle\phi_2|_{\text{anti}}$ will be used as entities. See subsection 3.2.

3a. $B(H)$, $B(H, H')$ stand for the set of all linear maps from $H$ into $H$ respectively $H'$. These maps are usually called operators. Similarly, $B(H)_{\text{anti}}$ and $B(H, H')_{\text{anti}}$ denote the sets of antilinear maps. They are also called antilinear operators.

3b. The Hermitian adjoint of an operator $X$, whether linear or antilinear, will be called $X^\dagger$ and not $X^*$. The latter notation is essential for antilinear operators as explained in 3.1. (Because of $\dim H < \infty$ finer functional analytic issues become irrelevant.)

3c. $X$ is called Hermitian (or self-adjoint) if $X^\dagger = X$ and skew Hermitian if $X^\dagger = -X$. The latter notation is essential for antilinear operators as explained in 3.1. (Because of $\dim H < \infty$ finer functional analytic issues become irrelevant.)

3d. A linear operator $A$ is called positive semi-definite if $\langle \psi, A\psi \rangle$ is real and non-negative for all $\psi \in H$. If no danger of confusion, such an operator is simply called “positive”.

4. The terminus technicus “superoperator”: A linear map from $B(H)$ into itself (or from $B(H)$ into $B(H')$) will be called superoperator. Similarly there are antilinear superoperators. While a linear map from $H$ into itself is usual called “operator”, the term “superoperator” shall remember that they are
linear maps between linear operators. In this picture $\mathcal{H}$ is the “floor”, $B(\mathcal{H})$ the “first etagere”, while superoperators live at the “second etagere”.

Speaking about a superoperator implies another understanding of positivity: A superoperator $\Phi$ is a positive one if $X \geq 0$ always implies $\Phi(X) \geq 0$.

2 Anti- (or conjugate) linearity

Here $L$ is a complex-linear space without a distinguished scalar product. Some elementary facts about antilinear operators are gathered.

If not said otherwise, we always assume $\dim L = d < \infty$. $\dim L$ is the dimension of $L$ as a complex-linear space.

2.1 Definition

**Definition 2.1** An operator $\vartheta$ acting on a complex linear space $L$ is called antilinear or, equivalently, conjugate linear if it obeys for complex numbers $c_j$ and vectors $\phi_j \in L$ the relation

$$\vartheta( c_1 \phi_1 + c_2 \phi_2 ) = c_1^* \vartheta \phi_1 + c_2^* \vartheta \phi_2, \quad c_j \in \mathbb{C}. \quad (6)$$

Important: An antilinear operator acts from the left to the right: Whether $X$ respectively $Y$ is linear or antilinear, $XY := X(Y\phi)$.

The product of $n$ linear and of $m$ antilinear operators in arbitrary positions is linear for even $m$ and antilinear for odd $m$.

Let $\vartheta$ be antilinear. By setting $H = 1/2$ in lemma 1.1 one gets the acq-lines

$$s \rightarrow \vartheta_s := e^{is} \vartheta, \quad s \rightarrow \vartheta_s^{-1} = e^{is} \vartheta^{-1}$$

provided $\vartheta^{-1}$ does exist in the latter case. Notice that the set of all invertible antilinear operators is an ac-quandle.

2.2 Eigenvalues

A particular case of (6) is the commutation relation $c\vartheta = \vartheta c^*$, $c$ a complex number. Hence, if $\phi$ is an eigenvector of $\vartheta$ with eigenvalue $a$, one concludes

$$\vartheta \phi = a \phi \Rightarrow \vartheta z \phi = z^* \vartheta \phi = a z^* \phi$$

which can be rewritten as

$$\vartheta \phi = a \phi \Rightarrow \vartheta(z \phi) = a z^* (z \phi). \quad (7)$$

**Proposition 2.1** The (non-zero) eigenvalues of an antilinear operator $\vartheta$ form a set of circles with $\theta$ as their common center.
The square $\vartheta^2$ of an antilinear operator is linear. If, as above, $\phi$ is an eigenvector of $\vartheta$ with eigenvalue $a$, then

$$\vartheta^2 \phi = \vartheta(a \phi) = aa^* \phi .$$

**Proposition 2.2** If $\phi$ is an eigenvector of $\vartheta$, then the corresponding eigenvalue of $\vartheta^2$ is real and not negative, see also [44]

**Corollary 2.1** Let $\vartheta$ be diagonalizable. Then $\vartheta^2$ is diagonalizable and its eigenvalues are real and not negative.

As the example below shows, antilinear operator does not necessarily have eigenvectors.

Because non-zero eigenvalues gather in circles, the unitary invariants of linear operators are mostly undefined for antilinear ones. The trace, for example, does not exist for conjugate linear operators.

2.2.1 $\text{dim } L = 2$

Let $\text{dim } L = 2$ and choose two linearly independent vectors, $\phi_1$ and $\phi_2$. Define

$$\theta_F (c_1 \phi_1 + c_2 \phi_2) = i(c_1^* \phi_2 - c_2^* \phi_1). \quad (8)$$

The $i$ is by convention. The index “F” is to honor E. Fermi for his pioneering work about spin $\frac{1}{2}$ particles. $[8]$ is also called “spin flip operator”.

From the definition (8) one gets

$$\theta_F^2 = -1 . \quad (9)$$

Clearly, the spectrum of $\theta_F$ must be empty.

Let $z \neq 0$ be a complex number and $\phi_1, \phi_2$ linear independent. An interesting set of antilinear operators is defined by

$$\theta_z (c_1 \phi_1 + c_2 \phi_2) = z c_1^* \phi_2 + z^* c_2^* \phi_1 . \quad (10)$$

For $z = i$ one recovers $\theta_F$. Short exercises yield

$$\theta_z^2 (c_1 \phi_1 + c_2 \phi_2) = (z^*)^2 c_1 \phi_1 + z^2 c_2 \phi_2 \quad (11)$$

and

$$\theta_z \theta_{z^*} = zz^* \mathbf{1}, \quad \theta_{z}^{-1} = |z|^{-2} \theta_{z^*} . \quad (12)$$

Assume that $\phi$ is an eigenvector of $\theta_z$ with eigenvalue $\lambda$. Then $\phi$ is an eigenvector of $\theta_z^2$ with eigenvalue $|\lambda|^2$. It follows $z^2 = (z^*)^2 = |\lambda|^2 > 0$, and $z$ must be real. This allows to conclude

$$\theta_z^2 = |z|^2 \mathbf{1}, \quad \theta_z = z \theta_1, \quad z \in \mathbb{C}$$

whenever $\theta_z$ possesses an eigenvector. Furthermore, if $z$ in (10) is real, there are eigenvectors, $\phi_1 \pm \phi_2$ with eigenvalues $\pm z$.  

Lemma 2.1 With a linear basis $\phi_1, \phi_2$ let $\theta \phi_1 = z \phi_2$ and $\theta \phi_2 = z^* \phi_1$.

a) If $z$ is real, then $\theta^2 = z^* 21$. $\phi_1 \pm \phi_2$ are eigenvectors with eigenvalues $\pm|z|$.

b) If $z$ is not real, then $\theta$ does not possess an eigenvector.

Remarks

1. Things are perfect if $\mathcal{L}$ is finite dimensional. If $\dim \mathcal{L} = \infty$, antilinear operators become as sophisticated as in the linear case, perhaps even more.

2. Assume there is a norm $\| \cdot \|$ defined in $\mathcal{L}$. Then the definition of an operator norm extends to the antilinear case.

$$
\| \vartheta \|_{\text{op}} := \sup \frac{\| \vartheta \phi \|}{\| \phi \|}
$$

is the norm of $\vartheta$ with respect to the norm given on $\mathcal{L}$ or simply the operator norm. In (13) $\phi$ runs through $\mathcal{L}$ with the exception of its zero element. The absolute values of the eigenvalues of $\vartheta$ are bounded from above by $\| \vartheta \|_{\text{op}}$.

3. The operator norm (13) mimics the linear case and it obeys

$$
\| R_1 R_2 \|_{\text{op}} \leq \| R_1 \|_{\text{op}} \cdot \| R_2 \|_{\text{op}}
$$

where $R_1$ and $R_2$ can be chosen linear or antilinear independently one from another.

2.3 Rank-one operators

Completely similar to the linear case, rank-one operators can be used as building blocks to represent antilinear operators.

The rank of an operator is the dimension of its range (or output space),

$$
\text{rank } X = \dim X \mathcal{L}
$$

whether $X$ is linear or antilinear.

Assume $\vartheta$ is an antilinear rank-one operator with range generated by $\phi'$. Then there is a linear function $\phi \mapsto l''(\phi) \in \mathbb{C}$ from $\mathcal{L}$ into the complex numbers such that

$$
\vartheta \phi = l''(\phi)^* \phi'.
$$

Remarks.

1. Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathcal{L}$. Then there is $\phi'' \in \mathcal{L}$ such that $l''(\phi) = \langle \phi'', \phi \rangle$ on $\mathcal{L}$. Then (15) is equivalent to

$$
\vartheta \phi = \langle \phi, \phi'' \rangle \phi'.
$$

This operator will be denoted by $\vert \phi', \phi'' \rangle_{\text{ant}}$ in subsection 3.2.

\footnote{Remind that $\langle \cdot, \cdot \rangle$ is assumed antilinear in the first argument}
Similar to the linear case one proves:

Let $\vartheta$ be antilinear and of rank $k$. Let $\phi'_1, \ldots, \phi'_k$ be vectors generating $\vartheta \mathcal{L}$. Then there is exactly one set of $k$ linear functionals $l''_1, \ldots, l''_k$ such that

$$
\vartheta \phi = \sum_{j=1}^k l''_j (\phi)^* \phi'_j, \quad \phi \in \mathcal{L}.
$$

(16)

2. If, as above, a scalar product is given. Then there are vectors $\phi''_1, \ldots, \phi''_k$ such that

$$
\vartheta \phi = \sum_{j=1}^k \langle \phi, \phi''_j \rangle \phi'_j.
$$

3. Comparing (16) with the definition of $\theta_F$, the vectors $\phi_1$ and $\phi_2$ generate $\mathcal{L}$ by definition. The two linear forms $l_j$ such that

$$
l_j (\phi_k) = \delta_{j,k}, \quad j, k = 1, \ldots, d.
$$

(17)

The $d^2$ antilinear rank-one operators

$$
\vartheta^j_k \phi := l^j (\phi)^* \phi_k, \quad \phi \in \mathcal{L}
$$

(18)

form a linear basis of the antilinear operators on $\mathcal{L}$. Hence, any antilinear operator on $\mathcal{L}$ can be written as

$$
\vartheta \phi = \sum_{j,k} a^k_j l^j (\phi)^* \phi_k, \quad \vartheta = \sum_{j,k} a^k_j \vartheta^j_k.
$$

(19)

The coefficients $a^k_j$ are gained by

$$
\text{Tr} \vartheta \vartheta^m_n = \sum_{j,k} a^k_j \text{Tr} \vartheta^j_k \vartheta^m_n = a^m_n.
$$

(20)

Indeed, the trace of $\vartheta^j_k \vartheta^m_n$ is equal to one for $j = m, k = n$ and vanishes otherwise.

Notice also that

$$
\text{Tr} A = \sum_l l^l (A \phi_l)
$$

is valid for linear operators. Indeed, [15], [19] mimic standard bi-orthonormal bases as seen from the following remark:

4. Given a scalar product on $\mathcal{L}$. As shown by 1 above, there are $\phi^j$ such that

$$
l^j (\phi) = \langle \phi^j, \phi \rangle, \quad \phi \in \mathcal{L}.
$$

The $2d$ vectors $\phi_k, \phi^j$, are bi-orthonormal:

$$
\langle \phi^j, \phi_k \rangle = \delta_{j,k}, \quad j, k = 1, \ldots, d.
$$
2.4 The canonical Hermitian form

There is a further important fact. Antilinear operators come naturally with an Hermitian form: The product of two antilinear operators is linear. Its trace

\[(\vartheta_1, \vartheta_2) := \text{Tr} \vartheta_2 \vartheta_1 \] (21)

will be called the canonical Hermitian form, or just the canonical form on the space of antilinear operators.

The canonical form \((21)\) is conjugate linear in the first and linear in the second argument. Remembering \((18)\), it follows

\[(\vartheta_j^k, \vartheta_m^n)^* = (\vartheta_m^n, \vartheta_j^k). \] (22)

Because every antilinear operator is a linear combination \((19)\) of the \(\vartheta_j^k\), the canonical form \((21)\) is Hermitian:

\[(\vartheta_1, \vartheta_2)^* = (\vartheta_2, \vartheta_1). \] (23)

**Theorem 2.1** \((21)\) is Hermitian, non-degenerate, and of signature \(\text{dim}\mathcal{H}\). If \(S\) is linear and invertible, respectively \(\vartheta\) invertible and antilinear, then

\[(\vartheta_1, \vartheta_2) = (S^{-1}\vartheta_1 S, S^{-1}\vartheta_2 S) = (\vartheta^{-1} \vartheta_2 \vartheta^{-1} \vartheta_1 \vartheta). \] (24)

**Proof:** That \((21)\) is Hermitian has already be shown. Its symmetry properties: As \(\vartheta_2 \vartheta_1\) is a linear operator,

\[\text{Tr} \vartheta_2 \vartheta_1 = \text{Tr} S \vartheta_2 S^{-1} S \vartheta_1 S^{-1} = \text{Tr} (S \vartheta_2 S^{-1})(S \vartheta_1 S^{-1})\]

is true. The case of an antilinear \(\vartheta\) is similar.

To prove non-degeneracy and the asserted signature of \((21)\), a linear basis of the space of antilinear operators is constructed by using \((17)\) and \((18)\). Let \(j, k = 1, \ldots, d\) under the condition \(j < k\). The elements of the desired basis are

\[\vartheta_j^k, \ \vartheta_j^k + \sqrt{2}\vartheta_j^k, \ \vartheta_j^k - \sqrt{2}\vartheta_j^k. \] (25)

Representing a general antilinear operator by that basis,

\[\vartheta = \sum_k a_k \vartheta_j^k + \sum_{j<k} a_{jk} \frac{\vartheta_j^k + \vartheta_j^k}{\sqrt{2}} + \sum_{j<k} b_{jk} \frac{\vartheta_j^k - \vartheta_j^k}{\sqrt{2}} \] (26)

one obtains by the help of \((20)\)

\[\langle \vartheta, \vartheta \rangle = \sum_k |a_k|^2 + \sum_{j<k} |a_{jk}|^2 - \sum_{j<k} |b_{jk}|^2. \] (27)
There are \( d(d+1)/2 \) positive and \( d(d-1)/2 \) negative terms in these representations proving

\[
\text{rank} (\langle , \rangle) = d^2, \quad \text{signature} (\langle , \rangle) = \frac{d(d+1)}{2} - \frac{d(d-1)}{2} = d. \tag{28}
\]

Now the theorem has been proved.

Converting \( (A\vartheta_2, \vartheta_1) = (\vartheta_1, A\vartheta_2)^* \) into a trace equation gives

\[
\text{Tr} \vartheta_1 A \vartheta_2 = (\text{Tr} A \vartheta_2 \vartheta_1)^\dagger. \tag{29}
\]

Remarks

1. According to the theorem, there are decompositions of the space of antilinear operators into a direct sum of two subspaces, one with dimension \( d(d+1)/2 \) and one with dimension \( d(d-1)/2 \). Choosing such a decomposition, the one with the larger dimension becomes a Hilbert space. The other one is a Hilbert space with respect to \(-\langle , \rangle\).

2. The canonical form can be regarded as “antilinear polarization” of the trace,

\[
\text{Tr} A \rightarrow \text{Tr} \vartheta_1 \vartheta_2.
\]

It is tempting to apply this idea to any symmetric function of the characteristic values. Examples are \( \text{det} \vartheta_1 \vartheta_2 \) and

\[
\text{Tr} A^2 \rightarrow \text{Tr} A_1 A_2 \rightarrow \text{Tr} \vartheta_1 \vartheta_2 \vartheta_3 \vartheta_4.
\]

There seems to be no systematic studies of these problems.

2.5 Pauli operators and their antilinear partners

The way from matrices to operators is mediated by choosing a general basis of \( \mathcal{L} \). A particular simple and nevertheless important example comes with \( \dim \mathcal{L} = 2 \) to which we now stick. We choose two linear independent vectors \( \phi_1 \) and \( \phi_2 \) as the preferred basis. With respect to the chosen basis, Pauli operators are defined by

\[
\sigma_1(c_1 \phi_1 + c_2 \phi_2) := c_1 \phi_2 + c_2 \phi_1, \tag{30}
\]

\[
\sigma_2(c_1 \phi_1 + c_2 \phi_2) := ic_1 \phi_2 - ic_2 \phi_1, \tag{31}
\]

\[
\sigma_3(c_1 \phi_1 + c_2 \phi_2) := c_1 \phi_1 - c_2 \phi_2. \tag{32}
\]

They are related by

\[
\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{j,k} \mathbf{1}, \quad \sigma_1 \sigma_2 \sigma_3 = i \mathbf{1},
\]

These Pauli operators have antilinear counterparts:

\[
\tau_0(c_1 \phi_1 + c_2 \phi_2) := c_1^* \phi_2 - c_2^* \phi_1, \tag{33}
\]

\[
\tau_1(c_1 \phi_1 + c_2 \phi_2) := -c_1^* \phi_1 + c_2^* \phi_2, \tag{34}
\]

\[
\tau_2(c_1 \phi_1 + c_2 \phi_2) := ic_1^* \phi_1 + ic_2^* \phi_2, \tag{35}
\]

\[
\tau_3(c_1 \phi_1 + c_2 \phi_2) := c_1^* \phi_2 + c_2^* \phi_1. \tag{36}
\]
It is $\theta_F = i\tau_0$ by (8). The spectrum of $\tau_0$ is empty. The spectrum of $\tau_j$, $j = 1, 2, 3,$ is a doubly covered circle of radius 1. The eigenvectors of $\tau_1$ and of $\tau_2$ are $z\phi_1$ and $z\phi_2$ with $z \in \mathbb{C}$, $z \neq 0$. The eigenvectors of $\tau_3$ are the multiples of the two vectors $\phi_1 \pm \phi_2$.

The four “Pauli-like” antilinear operators satisfy some nice commutation relations. By the help of the matrix

$$\{g_{jk}\} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

they can be written

$$\tau_j \tau_k + \tau_k \tau_j = 2g_{jk}1, \quad j, k \in \{0, 1, 2, 3\}.$$  \hspace{1cm} (38)

Remark the invariance of these and the following equations against an “abelian gauge” $\tau_m \rightarrow (\exp is)\tau_m$, $m = 0, 1, 2, 3$. The following can be checked explicitly:

$$\tau_1 \tau_2 \tau_3 = -i\tau_0, \quad \tau_1 \tau_2 = i\sigma_3,$$  \hspace{1cm} (39)

$$\tau_2 \tau_3 = i\sigma_1, \quad \tau_3 \tau_1 = i\sigma_2.$$ \hspace{1cm} (40)

and, for $j = 1, 2, 3$,

$$\tau_0 \sigma_j = \tau_j, \quad \tau_j \tau_0 = \sigma_j.$$  \hspace{1cm} (41)

From (39), (41) and $\theta_F = i\tau_0$, see (8), one also obtains

$$\tau_j \tau_k = \sigma_j \sigma_k, \quad \tau_j \theta_F = \theta_F \tau_j$$  \hspace{1cm} (42)

for all $j, k = 1, 2, 3$.

The trace of $\tau_j \tau_k$ vanishes if $j \neq k$, and one gets from (38)

$$(\tau_j, \tau_k) = 2g_{jk}, \quad j, k \in \{0, 1, 2, 3\}.$$  \hspace{1cm} (43)

Example. Consider a real vector $\{p_0, p_1, p_2, p_3\}$. Define

$$\tilde{\vec{p}} := \sum_{j=0}^{3} p_j \tau_j.$$  

Then

$$(\tilde{\vec{p}})^2 = (p_1^2 + p_2^2 + p_3^2 - p_0^2)1.$$  

Therefore, if $\{p_j\}$ is a space-like vector with respect to the Minkowski structure (37), $\tilde{\vec{p}}$ can be diagonalized with (doubly counted) eigenvalues $(\exp is)\sqrt{p_1^2 + p_2^2 + p_3^2 - p_0^2}$ with real $s$. If $\{p_j\}$ is not space-like, $\tilde{\vec{p}}$ does not have eigenvectors. $\tilde{\vec{p}}$ is nilpotent for light-like vectors $\{p_j\}$.  

---

15
Depending on the sign of $\sqrt{\cdot}$, one can define antilinear operators $S_\pm$ by

$$\tilde{p} = \sqrt{p_1^2 + p_2^2 + p_3^2 - p_0^2} S_+, \quad S_+^2 = 1 .$$ (44)

for space-like $\{p_j\}$. For time-like $\{p_j\}$ one gets

$$\tilde{p} = \sqrt{p_0^2 - p_1^2 - p_2^2 - p_3^2} S_-, \quad S_-^2 = -1 .$$ (45)

Mention the sign ambiguity in the definitions (44) and (45). $S_0$ is undefined. $S_+$ is an involution, $S_-$ a skew involution. See section 7.

**Lemma 2.2** Let $\dim \mathcal{H} = 2$ and $\vartheta$ antilinear. If and only if $\vartheta^2 = \lambda 1_2$ there are real numbers $x_0, \ldots, x_3$ and a unimodular number $\epsilon$ such that

$$\vartheta = \epsilon \sum_{j=0}^{3} x_j \tau_j .$$ (46)

Proof: It has been shown already that (46) implies $\vartheta^2 = \lambda 1_2$. This conclusion does not depend on the choice of $\epsilon$. For the other direction one assumes general complex numbers $c_j$ instead of the $x_j$ in (46). Then

$$\lambda = (|c_1|^2 + |c_2|^2 + |c_3|^2 - |c_0|^2) ,$$ (47)

$\lambda$ is real, and

$$\vartheta^2 = \lambda 1 + \sum_{j \neq k} c_j c_k^* \tau_j \tau_k .$$

This is equivalent to

$$\vartheta^2 = \lambda 1 + \sum_{j \neq k} (c_j c_k^* - c_k c_j^*) \tau_j \tau_k .$$ (48)

Assuming at first $c_0 \neq 0$, and that $\epsilon$ in (46) has been chosen such that $c_0$ is real. The right hand side of (48) can be written as a linear combination of $1$ and the Pauli operators. The coefficients $z_j$ of the Pauli operators must vanish. Consider for example $z_3$. To get this coefficient, $\tau_1 \tau_2 = i \sigma_3$ and $\tau_0 \tau_3 = -\sigma_3$ can be used, see (39) and (41):

$$z_3 = 2i(c_1 c_2^* - c_2 c_1^*) - 2(c_0 c_3^* - c_3 c_0^*)$$

There are purely imaginary numbers within the parentheses. Therefore $z_3 = 0$ implies $c_0 c_3^* = c_3 c_0$, i.e. $c_3$ is real. The same way one proves $c_1$ and $c_3$ real.

If $c_0 = 0$, one may assume $c_1 \neq 0$ and real. Then $z_3 = 0$ implies $c_1 c_2^* = c_1 c_2$. In the same manner one shows $c_3$ real.
2.6 Matrix representation

There is a bijection between matrices and linear, respectively antilinear operators, mediated by a fixed linear basis of \( L \). This can be done either by an isomorphism or else by an “anti-isomorphism” which reverses the direction of actions. Formally the two possibilities differ by a transposition with respect to the given basis.

Here the first possibility, the isomorphism, is used throughout! In doing so, operators have to act always from left to right. Hence \( R_1 R_2 \phi := R_1(R_2 \phi) \).

Let \( \phi_j, j = 1, 2, \ldots, d \) denote an arbitrary linear basis of \( L \). The are linear functionals \( \mathcal{L} \) satisfying \( \mathcal{L}(\phi_k) = \delta_{j,k} \) as in (17). Every vector \( \phi \in L \) can be uniquely decomposed according to

\[
\phi = c_1 \phi_1 + \cdots + c_d \phi_d , \quad c_j = \mathcal{L}(\phi) .
\] (49)

To enhance clarity, the coefficients in linear combinations like (49) are written left of the vectors. In doing so, the action of a linear or antilinear operator becomes

\[
A \phi = \sum c_j A \phi_j , \quad \vartheta \phi = \sum c_j^* \vartheta \phi_j .
\] (50)

Let \( \{A\}_{jk} \) and \( \{\vartheta\}_{jk} \) denote the matrices which are to be associated to the linear operator \( A \) and to the antilinear operator \( \vartheta \). They are defined by

\[
A \phi_j = \sum_k \{A\}_{kj} \phi_k , \quad \vartheta \phi_j = \sum_k \{\vartheta\}_{kj} \phi_k .
\] (51)

It follows

\[
A \phi = \sum c_j \{A\}_{kj} \phi_k , \quad \vartheta \phi = \sum c_j^* \{\vartheta\}_{kj} \phi_k
\] (52)

To see the mechanism in converting products, consider

\[
\vartheta_1 \vartheta_2 \phi = \vartheta_1 \sum c_j^* \{\vartheta_2\}_{kj} \phi_k
\]

\[
= \sum c_j \{\vartheta_2\}^*_{kj} \vartheta_1 \phi_k
\]

\[
= \sum c_j \{\vartheta_2\}^*_{kj} \{\vartheta_1\}_{lk} \phi_l
\]

\[
= \sum c_j \{\vartheta_1\}_{lk} \{\vartheta_2\}^*_{kj} \phi_l
\]

\[
= \sum c_j \{\vartheta_1 \vartheta_2\}_{lj} \phi_l .
\]
This way one checks: The product of two operators, whether linear or antilinear, can be reproduced by matrix multiplication. Indeed, let $A_i$ and $\vartheta_i$ be linear and antilinear operators respectively. It is

$$\{A_1 A_2\}_{ik} = \sum_j \{A_1\}_{ij} \{A_2\}_{jk}, \quad \{\vartheta_1 \vartheta_2\}_{ik} = \sum_j \{\vartheta_1\}_{ij} \{\vartheta_2\}_{jk}^*, \quad (53)$$

$$\{\vartheta_1 A_2\}_{ik} = \sum_j \{\vartheta_1\}_{ij} \{A_2\}_{jk}^*, \quad \{A_1 \vartheta_2\}_{ik} = \sum_j \{A_1\}_{ij} \{\vartheta_2\}_{jk}. \quad (54)$$

Let us now look at the equations (49), (50), and (52) from the matrix point of view. A matrix, say $M$, with matrix entries $m_{jk}$ can be converted as well into a linear as into an antilinear operator, provided we have distinguished a basis $\{\phi_j\}$ in advance. The need for a notational rule is obvious.

We propose to distinguish the two cases by the following notation:

$$\{M \vec{c}\}_k = \sum_j m_{kj} c_j, \quad \{M_{\text{anti}} \vec{c}\}_k = \sum_j m_{kj} c_j^*. \quad (55)$$

In these two equations $\vec{c}$ represents the column vector built from the coefficients $c_j$ in $\phi = \sum c_j \phi_j$.

Given two complex matrices, $M'$ and $M''$, with matrix elements $M'_{jk}$ and $M''_{jk}$, Accordingly to the rules above we can perform four different products. The first two below define linear, the second two antilinear operations. Their matrix entries are

$$(M'M'')_{jk} = \sum_i M'_{ji} M''_{ik}, \quad (M'_{\text{anti}} M''_{\text{anti}})_{jk} = \sum_i M'_{ji} (M''_{ik})^*, \quad (56)$$

$$(M'_\text{anti} M'')_{jk} = \sum_i M'_j (M''_{ik})^*, \quad (M' M''_{\text{anti}})_{jk} = \sum_i M'_j M''_{ik}. \quad (57)$$

Notice that the right hand sides of the equations (56), respectively of (57), define matrices acting linearly, respectively antilinearly, in the sense of (55).

For illustration, a simple example is added:

$$\begin{pmatrix} 0 & z^* \\ z & 0 \end{pmatrix}_{\text{anti}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} z^* c_2^* \\ z c_1^* \end{pmatrix}, \quad \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}_{\text{anti}} \begin{pmatrix} 0 & z^* \\ z & 0 \end{pmatrix}_{\text{anti}} = z^* z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (58)$$

3 Antilinearity in Hilbert spaces

If a finite linear space $L$ carries a distinguished positive definite scalar product $\langle ., . \rangle$ it becomes a Hilbert space, now denoted by $\mathcal{H}$.

---

\(^4\)For comparison we start with the well known linear case.
A basis of $\mathcal{H}$ is a set of vectors, say $\{\phi_1, \ldots, \phi_d\}$, $\dim \mathcal{H} = d < \infty$, satisfying $\langle \phi_j, \phi_k \rangle = \delta_{jk}$. Sometimes, to definitely distinguish from a general linear basis, such a basis is called an Hilbert basis.

The set of all linear operators mapping $\mathcal{H}$ into itself is denoted by $\mathcal{B}(\mathcal{H})$. To refer to the linear space of all antilinear operators from $\mathcal{H}$ into $\mathcal{H}$ we write $\mathcal{B}(\mathcal{H})_{\text{anti}}$. Their dimensions as linear spaces is $d^2$.

For both, $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})_{\text{anti}}$, the dominant news is the Hermitian adjoint introduced below. In the antilinear case it shows essential differences to the linear operator case.

3.1 The Hermitian adjoint

The antilinearity requires an extra definition of the Hermitian adjoint:

**Definition 3.1 (Wigner)** The Hermitian adjoint, $\vartheta^\dagger$, of $\vartheta \in \mathcal{B}(\mathcal{H})_{\text{anti}}$ is defined by

$$\langle \phi_1, \vartheta \vartheta^\dagger \varphi_2 \rangle = \langle \vartheta \varphi_2, \vartheta \varphi_1 \rangle, \quad \phi_1, \varphi_2 \in \mathcal{H}. \quad (58)$$

The following conclusions from (58) are immediate:

$$(\vartheta^\dagger)^\dagger = \vartheta, \quad (\vartheta_1 \vartheta_2)^\dagger = \vartheta_2^\dagger \vartheta_1^\dagger \quad (59)$$

and similarly, with $A$ linear, we get $(\vartheta A)^\dagger = A^\dagger \vartheta^\dagger$ and $(A \vartheta)^\dagger = \vartheta^\dagger A^\dagger$.

An important fact is seen by setting $A = c1$, namely $(c \vartheta)^\dagger = \vartheta^\dagger c^* = c \vartheta^\dagger$.

**Proposition 3.1** $\vartheta \rightarrow \vartheta^\dagger$ is a linear operation,

$$\left( \sum_j c_j \vartheta_j \right)^\dagger = \sum_j c_j \vartheta_j^\dagger. \quad (60)$$

As with linear operators we notice (respectively define)

$$\vartheta^\dagger \vartheta \geq 0, \quad |\vartheta| := (\vartheta \vartheta^\dagger)^{1/2} \geq 0. \quad (61)$$

Indeed, $\langle \varphi, \vartheta^\dagger (\vartheta \varphi) \rangle = \langle \vartheta \varphi, \vartheta \varphi \rangle \geq 0$ by (3.1). Next, a look at

$$\text{Tr} \vartheta_2 \vartheta_1 = [\text{Tr} (\vartheta_2 \vartheta_1)]^* = [\text{Tr} (\vartheta_1^\dagger \vartheta_2^\dagger)]^*.$$ 

proves the validity of the relation

$$(\vartheta_1, \vartheta_2) = (\vartheta_1^\dagger, \vartheta_2^\dagger) \quad (62)$$

Main classes of antilinear operators are defined as in the linear case. However, their properties can be quite different.

An antilinear operator $\vartheta$ is said to be Hermitian or self-adjoint if $\vartheta^\dagger = \vartheta$. $\vartheta$ is said to be skew Hermitian or skew self-adjoint if $\vartheta^\dagger = -\vartheta$. [31]; see also [64].
We denote the set of antilinear Hermitian and the set of skew Hermitian operators by $\mathcal{B}(\mathcal{H})^{+}_{\text{anti}}$ and by $\mathcal{B}(\mathcal{H})^{-}_{\text{anti}}$ respectively.

An antilinear $\vartheta$ can be written uniquely as a sum $\vartheta = \vartheta^{+} + \vartheta^{-}$ of an Hermitian and a skew Hermitian operator with

$$\vartheta \rightarrow \vartheta^{+} := \frac{\vartheta + \vartheta^\dagger}{2}, \quad \vartheta \rightarrow \vartheta^{-} := \frac{\vartheta - \vartheta^\dagger}{2}.$$  (63)

Relying on (22) and (63) one concludes

$$(\vartheta^{+}, \vartheta^{+}) \geq 0, \quad (\vartheta^{-}, \vartheta^{-}) \leq 0, \quad (\vartheta^{+}, \vartheta^{-}) = 0.$$  (64)

The essential difference to the linear case is caused by (60), saying that taking the Hermitian adjoint is a linear operation.

In particular, equipped with the canonical form, $\mathcal{B}(\mathcal{H})^{+}_{\text{anti}}$ becomes a Hilbert space. Completely analogue, $-\langle \ldots \rangle$ is a positive definite scalar product on $\mathcal{B}(\mathcal{H})^{-}_{\text{anti}}$.

**Proposition 3.2** An antilinear operator $\vartheta$ is Hermitian respectively skew Hermitian if and only if its matrix $\{\vartheta\}_{jk}$ is symmetric respectively skew symmetric with respect to any Hilbert basis. The Hilbert space $\mathcal{B}(\mathcal{H})^{+}_{\text{anti}}$ is of dimension $d(d+1)/2$, the dimension of $\mathcal{B}(\mathcal{H})^{-}_{\text{anti}}$ is equal to $d(d-1)/2$.

The first assertion follows directly from (58). Clearly a symmetric (a skew symmetric) matrix depends on exactly $d(d+1)/2$, respectively $d(d-1)/2$ complex numbers. Because $\mathcal{B}(\mathcal{H})^{+}_{\text{anti}}$ and $\mathcal{B}(\mathcal{H})^{-}_{\text{anti}}$ are Hilbert spaces with $\langle \ldots \rangle$ respectively $-\langle \ldots \rangle$, any chosen bases of them provides antilinear operators satisfying (25), (26) and (27).

**Remarks:**

1.) Do not apply an antilinear operator to a bra in the usual Dirac manner! By (58) one gets absurd results: The map $\langle \phi | \rightarrow \vartheta | \phi \rangle$ maps the dual of $\mathcal{H}$ linearly onto $\mathcal{H}$.

2.) Notice that

$$\vartheta_1, \vartheta_2 \implies (\vartheta_2, \vartheta_1^\dagger)$$  (65)

is a positive definite scalar product a la Frobenius and von Neumann.

3.) Matrix representation:

With two matrices, $M'$ and $M''$, and a given basis a look at (56) shows

$$M'(M'')^\dagger = M'^{\text{anti}}(M'')^{\text{anti}}.$$

(66)

In the second expression the $\dagger$-operation results in a change $(M''^{\text{anti}})_{jk} \rightarrow (M''^{\text{anti}})_{kj}$. The action of $M'^{\text{anti}}$ provides the complex conjugation $(M''^{\text{anti}})_{kj} \rightarrow (M''^{\text{anti}})^*_{kj}$ before the matrix $M'$ is multiplied on as in the linear case. Denoting by $M^T$ the transpose of the matrix $M$ on gets similarly

$$(M^{\text{anti}})^\dagger = (M^T)^{\text{anti}}.$$  (67)
4.) As we have seen, knowledge about Hermitian antilinear Operators can be translated into properties of symmetric Matrices and vice versa, just by choosing an appropriate basis. This procedure depends on that basis. Indeed, with the exception of the multiples of 1, a linear operator cannot be symmetric (or skew symmetric) in all Hilbert bases.

About symmetric and skew symmetric matrices see [40] or any other reasonable book on matrix algebra.

**Lemma 3.1** If the antilinear operator \( \vartheta \) is Hermitian (self-adjoint) then there exists a basis of eigenvectors, If \( \vartheta \) is skew Hermitian there does not exist any eigenvector.

**Proof:** Let \( \vartheta \) be Hermitian. Then \( B = \vartheta^2 \) is positive semi-definite. Let \( \lambda \geq 0 \) and \( \lambda^2 \) an eigenvalue of \( B \). The space \( \mathcal{H}_\lambda \) of all eigenvectors of \( B \) with eigenvalue \( \lambda^2 \) is \( \vartheta \)-invariant. (Because \( \vartheta \) and \( B \) commute.) Let \( \mathcal{H}_\lambda^+ \), respectively \( \mathcal{H}_\lambda^- \), the real subspaces of \( \mathcal{H}_\lambda \), \( \lambda > 0 \), consisting of \( \vartheta \)-eigenvectors with eigenvalues \( \lambda \) respectively \( -\lambda \). It is \( \mathcal{H}_\lambda^- = i \mathcal{H}_\lambda^+ \). (Because \( \vartheta \) is antilinear.) Hence they have the same real dimensions. Now for any \( \phi \in \mathcal{H}_\lambda \) there is a decomposition \( \phi = \phi^+ + \phi^- \) given by

\[
\phi^\pm = \frac{\phi \pm \lambda^{-1} \vartheta \phi}{2} \in \mathcal{H}_\lambda^\pm.
\]

It follows

\[
\mathcal{H}_\lambda = \mathcal{H}_\lambda^+ + \mathcal{H}_\lambda^- \quad \mathcal{H}_\lambda^- \cap \mathcal{H}_\lambda^+ = \{0\},
\]

and the real dimension of \( \mathcal{H}_\lambda^+ \) is equal to \( \dim \mathcal{H}_\lambda \). Therefore one can choose within every \( \mathcal{H}_\lambda \) a basis of eigenvectors of \( \vartheta \). Hence the first assertion is true. If \( \vartheta \) is skew Hermitian then \( \vartheta^2 \leq 0 \). Therefore, any eigenvector \( \phi \) is annihilated by \( \vartheta \), \( \vartheta \phi = 0 \).

Another way to prove the existence of a basis of eigenvectors for any Hermitian antilinear operators is in showing: Every \( \vartheta \)-irreducible subspace is 1-dimensional, see subsection 4.

**Proposition 3.3** For an antilinear operator \( \vartheta \) the following properties are equivalent:

a) \( \vartheta \) can be diagonalized.

b) There is a linear \( Z \) such that \( Z \vartheta Z^{-1} \) is Hermitian.

c) There is a positive linear operator \( A \) such that \( A \vartheta A^{-1} \) is Hermitian.

d) There is a positive linear operator \( B \) such that \( \vartheta^\dagger = B \vartheta B^{-1} \)

**Proof:** Let \( \vartheta \) be diagonalizable. If \( Z \) is invertible then \( Z \vartheta Z^{-1} \) can be made diagonal. (a) means the existence of \( d = \dim \mathcal{H} \) linear independent vectors \( \phi_j \) such that \( \vartheta \phi_j = \lambda_j \phi_j \). One can choose \( Z \) such that \( \phi_1 = Z \phi_1, \ldots, \phi_d = Z \phi_d \) is a basis of \( \mathcal{H} \). Then \( A \vartheta A^{-1} \phi_j = \lambda_j \phi_j \). Thus \( Z \vartheta Z^{-1} \) can be diagonalized by an Hilbert basis and, therefore, it is Hermitian, proving (a) \( \implies \) (b). Next,
Z can be written \( Z = UA \) with a unitary \( U \) and a positive operator \( A \). Because hermiticity of an operator is conserved by unitary transformations, (b) \( \rightarrow \) (c). Now (c) \( \rightarrow \) (a) follows from lemma 3.1. More explicitly (c) reads
\[
A\vartheta A^{-1} = (A\vartheta A^{-1})^\dagger = A^{-1}\vartheta^\dagger A .
\]
With \( B = A^2 \) this means \( \vartheta^\dagger = B\vartheta B^{-1} \) with \( B \) positive. Hence (c) \( \rightarrow \) (d).

On the other hand, if \( A \) is the positive root of \( B \), then one inversely sees that \( A\vartheta A^{-1} \) is Hermitian.

**Corollary 3.1** An antilinear operator \( \vartheta \) is diagonalizable if and only if there is a scalar product with respect to which \( \vartheta \) becomes Hermitian.

Proof: In the setting above, the scalar product reads \( \langle \phi, \vartheta \phi' \rangle_A = \langle \phi, A\vartheta \phi' \rangle \).

**3.1.1 The field of values**

The field of values is the set of all expectation values \( \langle \phi, X\phi \rangle, \langle \phi, \phi \rangle = 1, \) of an operator \( X \). According to the Toeplitz-Hausdorff theorem, it is a compact and convex set if \( X \) is a linear operator, [41]. The field of values of an antilinear operator is a disk with center at 0.

**Proposition 3.4** Let \( 2 \leq \dim \mathcal{H} < \infty \) and \( \vartheta \) antilinear. Then
\[
\{ z : z = \langle \phi, \vartheta \phi \rangle, \langle \phi, \phi \rangle = 1 \} = \{ z : |z| \leq r \} \tag{68}
\]
and \( r \) is the operator norm of \( \vartheta^+ \), which is the largest eigenvalue of \( |\vartheta^+| \).
\[
r = \sup |\langle \phi, \vartheta^+ \phi \rangle|, \quad \vartheta^+ = \frac{1}{2}(\vartheta + \vartheta^\dagger) \tag{69}
\]
and the sup runs over all unit vectors.

Proof: For the time being denote by \( F(\vartheta) \) the left of (68). \( F(\vartheta) \) is a connected and compact set of complex numbers. By proposition 2.1 it consists of circles. Hence \( F(\vartheta) \) is a set of the form \( 0 \leq r_0 \leq |z| \leq r \). By (68) the expectation values of \( \vartheta \) and \( \vartheta^\dagger \) coincide. Hence \( F(\vartheta) \) equals \( F(\vartheta^+) \). Now, \( \vartheta^+ \) being Hermitian, there is a basis of eigenvectors with non-negative eigenvalues. It follows that the largest eigenvalue of \( |\vartheta^+| \) is the radius \( r \) in (68), i.e. \( r \) is the operator norm of \( |\vartheta^+| \) or, equivalently, of \( \vartheta^+ \).

If \( \vartheta^+ \) is not invertible, then \( r_0 = 0 \) trivially. In the remaining case, there are at least two orthogonal unit eigenvectors, say \( \phi_1, \phi_2 \), with real eigenvalues \( s_1 < 0 < s_2 \). With \( 0 \geq s \geq 1 \) let \( \phi = \sqrt{s}\phi_1 + \sqrt{1-s}\phi_2 \). Then \( \langle \phi, \vartheta \phi \rangle = s_1 s + s_2 (1-s) \). The expectation value becomes negative if \( s \approx 1 \) and positive if \( s \approx 0 \). Hence the expectation value becomes zero for a certain \( s \). Hence \( r_0 = 0 \).
3.2 Antilinear rank one operators

Now we compare with subsection 2.2 and introduce some convenient description of rank-one operators a la Dirac. Any linear function \( l \) on \( \mathcal{H} \) can be expressed by \( l(\phi) = \langle \phi'' , \phi \rangle \) with unique \( \phi'' \in \mathcal{H} \). Having this in mind, the rank-one linear and antilinear operators can be described by

\[
(|\phi'\rangle\langle\phi''|) \phi := \langle \phi'' , \phi \rangle \phi', \quad (|\phi'\rangle\langle\phi''|_{\text{anti}}) \phi := \langle \phi , \phi'' \rangle \phi',
\]

which projects any vector \( \phi \) onto a multiple of \( \phi' \). (We assume \( \phi' \) and \( \phi'' \) different from the null-vector. We do not give a rank to the null-vector of \( \mathcal{H} \).

Attention: \( |\phi'\rangle\langle\phi''| \) and \( |\phi'\rangle\langle\phi''|_{\text{anti}} \) are defined by (70). I do not use \( |\phi''| \) decoupled from its other part as Dirac did, and I do not give any meaning to \( |\phi''|_{\text{anti}} \) as an standing alone expression. (Though one could do so as a conjugate linear functional).

To get the Hermitian conjugate look at

\[
\langle \phi_1, (|\phi'\rangle\langle\phi''|)_{\text{anti}}\phi_2 \rangle = \langle \phi_2, \phi'' \rangle \langle \phi_1, \phi' \rangle
\]

which is symmetric by interchanging \( \{ \phi_1, \phi' \} \) and \( \{ \phi_2, \phi'' \} \). Hence

\[
(|\phi'\rangle\langle\phi''|_{\text{anti}})^\dagger = |\phi''\rangle\langle\phi'_{\text{anti}}|, \quad |\phi'\rangle\langle\phi''|_{\text{anti}} = |\phi''\rangle\langle\phi'_{\text{anti}}|.
\]

Here and below the linear case is mentioned for comparison.

For the time being, \( A_L, A_R \) denote linear and \( \vartheta_L \) and \( \vartheta_R \) antilinear operators to write down some useful identities:

\[
A_L|\phi'\rangle\langle\phi''| = |A_L\phi'\rangle\langle\phi''|, \quad |\phi'\rangle\langle\phi''|A_R = |\phi'\rangle\langle\phi''|_{\text{anti}},
\]

\[
\vartheta_L|\phi'\rangle\langle\phi''| = |\vartheta_L\phi'\rangle\langle\phi''|_{\text{anti}}, \quad |\phi'\rangle\langle\phi''|\vartheta_R = |\phi'\rangle\langle\phi''|_{\text{anti}}\vartheta_R.
\]

The first two are well known. Concerning the others, one proceeds as follows.

\[
\vartheta_L|\phi'\rangle\langle\phi''|\phi = \vartheta_L|\phi'', \phi\rangle\phi' = \langle \phi , \phi'' \rangle \vartheta_L\phi',
\]

and by the definition (70) one obtains the third relation of (72). The proofs of the other one and of the following equations is similar.

\[
A_L|\phi'\rangle\langle\phi''|_{\text{anti}} = |A_L\phi'\rangle\langle\phi''|_{\text{anti}}, \quad |\phi'\rangle\langle\phi''|_{\text{anti}}A_R = |\phi'\rangle\langle\phi''|_{\text{anti}}A_R, \quad
\]

\[
\vartheta_L|\phi'\rangle\langle\phi''|_{\text{anti}} = |\vartheta_L\phi'\rangle\langle\phi''|_{\text{anti}}, \quad |\phi'\rangle\langle\phi''|_{\text{anti}}\vartheta_R = |\phi'\rangle\langle\phi''|_{\text{anti}}\vartheta_R.
\]

Applying \( \vartheta_L \) to the last relation and looking at the second one in (72) yields

\[
\vartheta_L|\phi'\rangle\langle\phi''|_{\text{anti}} = |\vartheta_L\phi'\rangle\langle\vartheta_R\phi''|
\]

as one of further possibilities to combine (72) and (73). The chaining

\[
|\phi_1\rangle\langle\phi_2|_{\text{anti}} \langle\phi_3, \phi_4\rangle = \langle \phi_3, \phi_2 \rangle |\phi_1\rangle\langle\phi_4|
\]
is straightforwardly. By definition (21) it follows
\[
\text{Tr} (|\phi_1\rangle\langle\phi_2|_{\text{anti}} |\phi_3\rangle\langle\phi_4|_{\text{anti}}) = \langle\phi_3, \phi_2\rangle \langle\phi_4, \phi_1\rangle .
\] (76)

**Remark:**
Let \(\phi_1, \ldots, \phi_d\) be a basis of \(\mathcal{H}\). (18) becomes
\[
\vartheta_k^j \phi := \langle\phi, \phi_j\rangle \phi_k = |\phi_j\rangle\langle\phi|_{\text{anti}} \phi_k .
\] (77)
One gets a basis of \(\mathcal{B}(\mathcal{H})_{\text{anti}}\) in terms of Dirac symbols.

### 3.3 Linear and antilinear Pauli operators

In subsection 2.5 the operators \(\sigma_j, \tau_k\), have been defined relative to two linear independent but otherwise arbitrary vectors. From now on, \(\phi_1, \phi_2\) denotes a Hilbert basis of \(\mathcal{H}_2\). Then \(\tau_1, \tau_2, \tau_3\) are Hermitian. They form a basis of \(\mathcal{B}(\mathcal{H}_2)^{\text{anti}}\). \(\tau_0\) is skew Hermitian. Every element of \(\mathcal{B}(\mathcal{H}_2)^{-\text{anti}}\) is a multiple of \(\tau_0\).

An antilinear Hermitian operators, the square of which is \(1\) is called a conjugation. It is a skew conjugation if its square is \(-1\). See the next section for more.

The following identities are useful.
\[
\tau_0 = |\phi_2\rangle\langle\phi_1|_{\text{anti}} - |\phi_1\rangle\langle\phi_2|_{\text{anti}}, \quad \tau_1 = |\phi_1\rangle\langle\phi_1|_{\text{anti}} - |\phi_2\rangle\langle\phi_2|_{\text{anti}},
\] (78)
\[
\tau_2 = i|\phi_1\rangle\langle\phi_1|_{\text{anti}} + i|\phi_2\rangle\langle\phi_2|_{\text{anti}}, \quad \tau_3 = |\phi_1\rangle\langle\phi_2|_{\text{anti}} + |\phi_2\rangle\langle\phi_1|_{\text{anti}},
\] (79)
\[
|\phi_1\rangle\langle\phi_1|_{\text{anti}} = \frac{1}{2} (\tau_1 - i\tau_2), \quad |\phi_1\rangle\langle\phi_2|_{\text{anti}} = \frac{1}{2} (\tau_3 - \tau_0),
\] (80)
\[
|\phi_2\rangle\langle\phi_1|_{\text{anti}} = \frac{1}{2} (\tau_3 + \tau_0), \quad |\phi_2\rangle\langle\phi_2|_{\text{anti}} = -\frac{1}{2} (\tau_1 + i\tau_2).
\] (81)

Miscellaneous facts about the \(\tau_j\)-operators will be gathered: Remember of \(\theta_F = i\tau_0\), see (8), and consider the equation
\[
\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{\text{anti}} \begin{pmatrix} a_{00} & a_{10} \\ a_{01} & a_{11} \end{pmatrix}_{\text{anti}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{\text{anti}} = -\begin{pmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{pmatrix}
\] (82)
which can be verified by direct calculation. The operator between the two spin flips is the matrix representation of the Hermitian adjoint of a general linear operator \(A\). The matrix equation above can be converted into a basis independent operator equation:
\[
\tau_0 A^\dagger \tau_0^{-1} = (\text{Tr} \ A) \mathbf{1} - A = (\det A) A^{-1}.
\] (83)
The last equality sign supposes \(A\) invertible. Another form of (83) reads
\[
A \tau_0 A^\dagger = (\det A) \tau_0 .
\] (84)
It is obtained by multiplying (83) by $A$ from the left and by $\tau_0$ from the right. Note that these equations remain valid if $\tau_0$ is replaced by $\theta_F$.

An application of the above is a description of the special unitary group $SU(2)$. Multiplying (83) from the left by $\tau_0^{-1}$ yields

$$\tau_0 A = A \tau_0 \iff AA^\dagger = (\det A) 1 .$$

(85)

Hence, $U \in SU(2)$ if and only if $U$ commutes with $\tau_0$ and has determinant one.

The determinants of the Pauli operators (Pauli matrices) are $-1$. To get $SU(2)$ operators the multiplication by $i$ is sufficient:

$$\{1, i\sigma_1, i\sigma_2, i\sigma_3\} \in SU(2) .$$

(86)

These operators commute with $\tau_0$, and they generate the real linear space of operators commuting with $\tau_0$ (respectively $\theta_F$). Hence one gets:

**Lemma 3.2** Let $A \in \mathcal{B}(\mathcal{H}_2)$. The following three conditions are equivalent:

(i) $A$ has a representation

$$A = x_0 1_2 + i \sum_1^3 x_j \sigma_j , \quad x_0, x_1, x_2, x_3 \in \mathbb{R} .$$

(87)

(ii) It is $A = (\det A) U$ with $U \in SU(2)$ and $\det A$ real.

(iii) $A$ commutes with $\tau_0$.

The set of all operators (87) is a real $\dagger$-involutive subalgebra of $\mathcal{B}(\mathcal{H}_2)$ isomorphic to the field of quaternions.

An antilinear operator, $\vartheta$, can be decomposed similar to (87):

$$\vartheta = c_0 \tau_0 + c_1 \tau_1 + c_2 \tau_2 + c_3 \tau_3 .$$

(88)

The first term is $\vartheta^-$, the sum of the last three gives $\vartheta^+$. Note the typical Minkowskian structure

$$\frac{1}{2}(\vartheta, \vartheta) = (|c_1|^2 + |c_2|^2 + |c_3|^2 - |c_0|^2) ,$$

(89)

**Lemma 3.2** can be “antilinearly” rewritten. At first one multiply from the left with a unimodular number $\epsilon$ to represent all unitaries. Then one replaces $\sigma_j$ by $\tau_j \tau_0$ and multiplies from the left by $\tau_0$. One gets

**Lemma 3.3** The following two conditions are equivalent:

(i) The antiunitary operator $\vartheta$ allows for a representation

$$\vartheta = \epsilon (y_0 \tau_0 + i \sum_1^3 y_j) , \quad y_0, \ldots y_3 \in \mathbb{R} .$$

(90)
(ii) There is an antiunitary operator $\Theta$ and a non-negative real number $\lambda$ such that

$$\vartheta = \lambda \Theta.$$  \hspace{1cm} (91)

By letting $\epsilon$ constant, the set of all operators $\Theta$ becomes a $\dagger$-invariant real linear space.

The lemma is an impressive example that the two-dimensional case is often special. This is seen from

**Lemma 3.4** If $\dim \mathcal{H} \geq 3$ then a linear operator commutes with $\mathcal{B}(\mathcal{H}_{\text{anti}})$ if and only if it is a real multiple of $1$.

Proof: After choosing a basis $\{\phi_j\}$. Let the antilinear operators $\theta_{jk}$, $j < k$ acting on the subspace generated by $\phi_j$ and $\phi_k$ like $\tau_0$, and annihilating all other elements of the basis. These operators form a general basis of $\mathcal{B}(\mathcal{H}_{\text{anti}})$.

It is to prove that $A\theta_{jk} = \theta_{jk}A$ for all $j < k$ if and only if the linear operator $A$ is a real multiple of the unit operator $1$.

The following proof assumes $\dim \mathcal{H} = 3$ for transparency. Let $A$ be a linear operator with matrix elements $a_{jk}$. The non-zero matrix elements of $\theta_{12}$ are $+1$ and $-1$ in the positions 12 and 21 respectively. The equation $A\theta_{12} = \theta_{12}A$, written in the $\{\phi_i\}$ matrix representation, reads

$$\begin{pmatrix}
-a_{12} & a_{11} & 0 \\
-a_{22} & a_{21} & 0 \\
-a_{32} & a_{21} & 0
\end{pmatrix} = \begin{pmatrix}
a_{21}^* & a_{22}^* & a_{23}^* \\
-a_{11}^* & -a_{12}^* & -a_{13}^* \\
0 & 0 & 0
\end{pmatrix}.$$  \hspace{1cm} (92)

Firstly this implies $a_{j3} = a_{3j} = 0$ for $j = 1, 2$. Using $\theta_{23}$ or $\theta_{13}$ instead of $\theta_{12}$ gives $a_{j1} = a_{1j} = 0$ for $j = 2, 3$ or $a_{j2} = a_{2j} = 0$ for $j = 1, 3$. Therefore, $A$ must be diagonal in the chosen basis.

Secondly one gets $a_{11}^* = a_{22}$ from the pre-proposed commutativity between $A$ and $\theta_{12}$. The same procedure with $\theta_{23}$ and $\theta_{13}$ implies $a_{22}^* = a_{33}$ and $a_{11}^* = a_{33}$. All together one arrives at $a_{11} = a_{22} = a_{33} \in \mathbb{R}$. Thus, $A = a 1$, $a$ real. The extension to higher dimensions is obvious. That $A = a 1$, $a$ real, is in the commutant of $\mathcal{B}(\mathcal{H}_{\text{anti}})$ is trivial.

Concerning the antilinear Hermitian operators one can prove the following

**Lemma 3.5** A linear operator commuting with all antilinear Hermitian operators is a real multiple of $1$ for any $\dim \mathcal{H} \geq 1$.

### 3.4 Antilinear maps between Hilbert spaces

As in the linear case the Hermitian adjoint can be defined not only within $\mathcal{B}(\mathcal{H}_{\text{anti}})$ but also for antilinear maps between Hilbert spaces. The following is a mini-introduction to this topic. Something more will be said in the sections 9 and 10.
Let $\mathcal{H}^A$, $\mathcal{H}^B$ denote two finite dimensional Hilbert spaces and $\vartheta$ an antilinear map from $\mathcal{H}^A$ into $\mathcal{H}^B$. Its Hermitian adjoint, $\vartheta^\dagger$, is an antilinear map from $\mathcal{H}^B$ into $\mathcal{H}^A$ and it is defined by

$$\langle \phi^B, \vartheta \phi^A \rangle = \langle \phi^A, \vartheta^\dagger \phi^B \rangle, \quad \phi^A \in \mathcal{H}^A, \, \phi^B \in \mathcal{H}^B.$$  \hspace{0.5cm} (92)

The Hermitian adjoint from $\mathcal{H}^B$ into $\mathcal{H}^A$ is defined similarly. With respect to these definitions, the rules (60), and (61) remain valid also in this setting.

In particular $\vartheta^{\dagger \dagger} = \vartheta$, In this spirit, also the second part of (59) has its counterpart:

$$(\vartheta_{12}^\dagger \vartheta_{12}^\dagger)^\dagger = \vartheta_{12}^\dagger \vartheta_{21}^\dagger, \quad \vartheta_{12} \in \mathcal{B}(\mathcal{H}^B, \mathcal{H}^A)_{\text{anti}}, \quad \vartheta_{21} \in \mathcal{B}(\mathcal{H}^A, \mathcal{H}^B)_{\text{anti}}$$

The linear space of all antilinear maps from $\mathcal{H}^A$ into $\mathcal{H}^B$ will be denoted by $\mathcal{B}(\mathcal{H}^A, \mathcal{H}^B)_{\text{anti}}$. In the same manner $\mathcal{B}(\mathcal{H}^B, \mathcal{H}^A)_{\text{anti}}$ is the linear space of all antilinear maps from $\mathcal{H}^B$ into $\mathcal{H}^A$. Their dimensions are $(\dim \mathcal{H}^A)(\dim \mathcal{H}^B)$. The Hermitian adjoint (92) induces a linear isomorphism

$$\mathcal{B}(\mathcal{H}^A, \mathcal{H}^B)_{\text{anti}} \overset{\vartheta^\dagger}{\leftrightarrow} \mathcal{B}(\mathcal{H}^B, \mathcal{H}^A)_{\text{anti}}.$$  \hspace{0.5cm} (93)

The space of antilinear maps from one Hilbert space into another one is itself Hilbertian in a natural way. Before coming to that, it is helpful to introduce the antilinear rank one operators acting between a pair of Hilbert spaces. Following (70), definition and Hermitian adjoint are given by

$$|\phi_1^A\rangle\langle \phi_1^B|_{\text{anti}} \phi^B := \langle \phi^B, \phi_1^B \rangle \phi_1^A, \hspace{0.5cm} (94)$$

$$|\phi_1^B\rangle\langle \phi_1^A|_{\text{anti}} \phi^A := \langle \phi^A, \phi_1^A \rangle \phi_1^B, \hspace{0.5cm} (95)$$

$$\langle \phi_1^A \rangle_{\text{anti}} \langle \phi_1^B \rangle_{\text{anti}} = \langle \phi_1^B \rangle_{\text{anti}} \langle \phi_1^A \rangle_{\text{anti}} \phi^A = \langle \phi_1^B \rangle_{\text{anti}} \langle \phi_1^A \rangle_{\text{anti}} \phi^A.$$  \hspace{0.5cm} (96)

The following relations can be checked:

$$|\phi_1^A\rangle_{\text{anti}} |\phi_2^B\rangle_{\text{anti}}|\phi^A\rangle_{\text{anti}} = \langle \phi_2^B, \phi_1^B \rangle |\phi_2^A\rangle_{\text{anti}}.$$  \hspace{0.5cm} (97)

$$|\phi_1^B\rangle_{\text{anti}} |\phi_2^A\rangle_{\text{anti}}|\phi^B\rangle_{\text{anti}} = \langle \phi_2^A, \phi_1^A \rangle |\phi_2^B\rangle_{\text{anti}}.$$  \hspace{0.5cm} (98)

An intermediate step in proving (97) is in

$$|\phi_1^A\rangle_{|\phi_1^B\rangle_{\text{anti}}|\phi_2^A\rangle_{\text{anti}}|\phi^A\rangle_{\text{anti}} = \langle \phi_2^A, \phi_2^A \rangle |\phi_2^B\rangle_{\text{anti}} \phi_1^A.$$  

Let $\vartheta$ and $\tilde{\vartheta}$ be chosen from $\mathcal{B}(\mathcal{H}^A, \mathcal{H}^B)_{\text{anti}}$ arbitrarily. Assume

$$\vartheta = \sum a_{jk} |\phi_k^B\rangle_{\text{anti}}|\phi_j^A\rangle_{\text{anti}}, \quad \tilde{\vartheta} = \sum b_{mn} |\phi_m^A\rangle_{\text{anti}}|\phi_n^B\rangle_{\text{anti}}.$$  \hspace{0.5cm} (99)

Using (97) one derives the identities

$$\tilde{\vartheta} \vartheta^\dagger = \sum a_{jk}^* b_{mn} |\phi_j^A\rangle_{\text{anti}} |\phi_m^B\rangle_{\text{anti}}, \hspace{0.5cm} (100)$$

$$\vartheta^\dagger \tilde{\vartheta} = \sum a_{jk}^* b_{mn} |\phi_k^B\rangle_{\text{anti}} |\phi_n^A\rangle_{\text{anti}}.$$  \hspace{0.5cm} (101)
Taking traces one gets on $\mathcal{B}(\mathcal{H}^A, \mathcal{H}^B)_{\text{anti}}$

$$\text{Tr } \tilde{\theta}^\dagger \tilde{\vartheta} = \text{Tr } \tilde{\vartheta}^\dagger \tilde{\theta} = \sum a_{jkmn}^{*} \langle \phi_j^A, \phi_m^A \rangle \langle \phi_k^B, \phi_n^B \rangle . \quad (102)$$

These relations define the **natural scalar product**

$$\langle \tilde{\vartheta}, \vartheta \rangle_{ab} := \text{Tr } \tilde{\vartheta}^\dagger \tilde{\vartheta} = \text{Tr } \tilde{\vartheta} \vartheta^\dagger \quad (103)$$

so that $\mathcal{B}(\mathcal{H}^A, \mathcal{H}^B)_{\text{anti}}$ becomes an Hilbert space.

Exchanging the roles of $A$ and $B$ by setting $\vartheta' = \vartheta^\dagger$, $\tilde{\vartheta}' = \tilde{\vartheta}^\dagger$ one gets

the **natural scalar product**

$$\langle \tilde{\vartheta}', \vartheta' \rangle_{ba} := \text{Tr } \tilde{\vartheta}'^\dagger \vartheta' = \text{Tr } (\tilde{\vartheta}'^\dagger \vartheta'), \vartheta', \tilde{\vartheta}' \in \mathcal{B}(\mathcal{H}^B, \mathcal{H}^A)_{\text{anti}} \quad (104)$$

on $\mathcal{B}(\mathcal{H}^B, \mathcal{H}^A)_{\text{anti}}$.

Thus (93) becomes an isometry.

The next aim is in construction further remarkable isometries.

The direct product $\mathcal{H}^A \otimes \mathcal{H}^B$ of two finite dimensional Hilbert spaces is the linear span of all symbols $\phi^A \otimes \phi^B$ with $\phi^A \in \mathcal{H}^A$, $\phi^B \in \mathcal{H}^B$ modulo the defining relations

$$c(\phi^A \otimes \phi^B) = (c \phi^A) \otimes \phi^B = \phi^A \otimes (c \phi^B),$$

$$\phi_1^A \otimes \phi^B + \phi_2^A \otimes \phi^B = (\phi_1^A + \phi_2^A) \otimes \phi^B,$$

$$\phi^A \otimes \phi_1^B + \phi^A \otimes \phi_2^B = \phi^A \otimes (\phi_1^B + \phi_2^B).$$

An antilinear operator $|\phi^B\rangle \langle \phi^A|_{\text{anti}}$ acts according to the rule (95). It is indexed by a pair of vectors, $\phi^A$ and $\phi^B$ from which it depends bilinearly. Therefore the defining relations of $\mathcal{H}^A \otimes \mathcal{H}^B$ are fulfilled. Thus the map

$$\phi^A \otimes \phi^B \rightarrow |\phi^B\rangle \langle \phi^A|_{\text{anti}}$$

induces a linear isomorphism

$$\psi = \sum c_{jk} \phi_j^A \otimes \phi_k^B \rightarrow \theta = \sum c_{jk} |\phi_k^B\rangle \langle \phi_j^A|_{\text{anti}} \quad (105)$$

from $\mathcal{H}^A \otimes \mathcal{H}^B$ onto $\mathcal{B}(\mathcal{H}^A, \mathcal{H}^B)_{\text{anti}}$. The map is onto because both linear spaces are of the same dimension.

(105) is an isometry. To see it one starts with $\langle \psi, \psi \rangle = \sum |c_{jk}|^2$ and computes $\langle \theta, \theta \rangle_{ab}$ which is given by (103). To do so, it is sufficient to establish

$$\langle \phi^A \otimes \phi^B, \phi^A \otimes \phi^B \rangle = \text{Tr } |\phi^A\rangle \langle \phi^B|_{\text{anti}} |\phi^A\rangle \langle \phi^B|_{\text{anti}} .$$

In the same way one proves that

$$\sum c_{kj} \phi_k^B \otimes \phi_j^A \rightarrow \sum c_{kj} |\phi_k^B\rangle \langle \phi_j^A|_{\text{anti}} \quad (106)$$

is a linear isometry from $\mathcal{H}^B \otimes \mathcal{H}^A$ onto $\mathcal{B}(\mathcal{H}^B, \mathcal{H}^A)_{\text{anti}}$. 

28
Proposition 3.5 The Hilbert spaces

\[ \mathcal{H}^A \otimes \mathcal{H}^B, \quad \mathcal{B}(\mathcal{H}^A, \mathcal{H}^B)_{\text{anti}}, \quad \mathcal{B}(\mathcal{H}^B, \mathcal{H}^A)_{\text{anti}}, \quad \mathcal{H}^B \otimes \mathcal{H}^A \] (107)

are mutually canonically isometrical equivalent. The isometries are described by (93), (105), (106), and by

\[ \phi^A \otimes \phi^B \leftrightarrow \phi^B \otimes \phi^A . \] (108)

Remark: In order to handle some quantum theoretical problems, a more systematic treatment starts with section 8. Also further notations will be introduced: The map (105), for instance, will be written

\[ \psi \equiv \sum c_{jk} \phi_j^A \otimes \phi_k^B \rightarrow s_{\psi}^{ba} := \sum c_{jk} |\phi_k^B \rangle \langle \phi_j^A|_{\text{anti}} . \] (109)

4 Antilinear normal operators

An antilinear operator is called normal if it commutes with its Hermitian adjoint:

\[ \vartheta^\dagger \vartheta = \vartheta \vartheta^\dagger . \] (110)

Starting with the decomposition \( \vartheta = \vartheta^+ + \vartheta^- \), the operators is normal if

\[ (\vartheta^+ + \vartheta^-) (\vartheta^+ - \vartheta^-) = (\vartheta^+ - \vartheta^-) (\vartheta^+ + \vartheta^-) . \]

Multiplying out the products, we see:

\[ \vartheta \text{ is normal if and only if} \]

\[ \vartheta^+ \vartheta^- = \vartheta^- \vartheta^+ . \] (111)

Important subclasses of the normal antilinear operators are the already defined Hermitian and the skew Hermitian ones. A further essential class constitute the unitary antilinear operators, also called antunitaries.

As in the linear case an antiunitary is characterized by

\[ \Theta^\dagger = \Theta^{-1} . \]

Antiunitaries are isometric, i.e.

\[ \langle \Theta \phi, \Theta \phi' \rangle = \langle \phi', \phi \rangle . \] (112)

The set \( \mathcal{U}_{\text{anti}}(\mathcal{H}) \) of all antiunitarities is not a group. Yet it contains with \( \Theta \) also \( \Theta^{-1} \). Thus its adjoint representation

\[ \Theta \rightarrow \Theta^{-1} X \Theta \] (113)

\footnote{A notation due to E. P. Wigner}
is well defined for linear and for antilinear $X$. Assume
\[ \Theta_1^\dagger \vartheta \Theta_1 = \Theta_2^\dagger \vartheta \Theta_2 \quad \forall \vartheta \in \mathcal{B}(\mathcal{H}_{\text{anti}}) \]
and applying (60) it follows
\[ \Theta_2 = \pm \Theta_1 . \]

Notice also that $\mathcal{U}(\mathcal{H}) \cup \mathcal{U}_{\text{anti}}(\mathcal{H})$ is a group. $\mathcal{U}(\mathcal{H})$ is a normal subgroup of it.

A conjugation is an antilinear operator which is both, unitary and Hermitian, i.e. $\theta$ is a conjugation if and only if it is antilinear and satisfies
\[ \theta^\dagger = \theta = \theta^{-1} . \]
(114)

It follows $\theta^2 = 1$. On the other hand, $\theta^2 = 1$ together with either $\theta^\dagger = \theta$ or, alternatively, with $\theta^\dagger = \theta^{-1}$ implies (114).

Conjugations form an important class of operators as will be seen later on. In many papers they appear “masked” by notations like $|\psi^*\rangle$ where the complex conjugation refers to a distinguished basis.

In the same spirit, a skew conjugation is a skew Hermitian antiunitary operator:
\[ \theta^\dagger = -\theta = \theta^{-1} , \quad \theta^2 = -1 . \]
(115)

**Proposition 4.1 (Polar decomposition)** Let $\vartheta$ be an antilinear operator and $\mathcal{H}$ finite dimensional. There are antiunitaries $\theta_L$, $\theta_R$ such that
\[ \vartheta = \theta_L \sqrt{\vartheta^\dagger \vartheta} = \sqrt{\vartheta \vartheta^\dagger} \theta_R \equiv \vartheta \theta_R . \]
(116)

A proof is by transforming the assertion into the corresponding linear case: let us choose an antiunitary $\theta_0$. Then there is a unitary $U_0$ such that
\[ \theta_0 \vartheta = U_0 \sqrt{\vartheta^\dagger \vartheta} , \quad \theta_L := \theta_0^{-1} U_0 . \]

The other case is similar.

**Remark:** A partial isometry is an antilinear operator $\theta$ for which $\theta^\dagger \theta$ and $\theta \theta^\dagger$ are projection operators. The choice of $\theta_L$ in the polar decomposition (116) is unique up to its action onto the kernel of $\vartheta^\dagger \vartheta$. If $\vartheta^{-1}$ does not exist, a unique polar decomposition requires $\theta_L$ (respectively $\theta_R$) to be partially isometric. In particular, the supports of $\theta^\dagger L \theta_L$ and $\vartheta^\dagger \vartheta$ must coincide.

That sharpening of the polar decomposition is obligatory if dim $\mathcal{H} = \infty$.

For such a more general treatment one may consult [30].

30
Proposition 4.2 If \( \vartheta \) is normal, there is an antiunitary \( \theta \) such that
\[
\vartheta = \theta \rvert \vartheta \rvert = \rvert \vartheta \rvert \theta, \quad \vartheta \rvert \vartheta \rvert = \rvert \vartheta \rvert \theta, \quad \vartheta \dagger \rvert \vartheta \rvert = \rvert \vartheta \rvert \theta^\dagger.
\]  
(117)
Indeed, from (116), one deduces
\[
\vartheta = \theta_L \rvert \vartheta \rvert = \rvert \vartheta \rvert \theta_R, \quad \rvert \vartheta \rvert^2 = \vartheta \theta^\dagger = \theta_L \rvert \vartheta \rvert^2 \theta_L^{-1},
\]
saying that \( \theta_L \) commutes with \( \rvert \vartheta \rvert^2 \) and, therefore, with \( \rvert \vartheta \rvert \). Hence \( \theta = \theta_L \) does the job.

Corollary 4.1 If \( \vartheta \) is Hermitian, then the restriction of \( \theta \) to the support of \( \rvert \vartheta \rvert \) is a conjugation. If \( \vartheta \) is skew Hermitian, then the restriction of \( \theta \) to the support of \( \rvert \vartheta \rvert \) is a skew conjugation.

More information can be drawn from the WHV-theorem 4.1 below.

Let \( \Theta \) be an antiunitary and \( X \) a linear operator. Then
\[
\text{Tr } \Theta^\dagger X \Theta = \text{Tr } X^\dagger.
\]  
(118)
This is because
\[
\langle \phi, \Theta^\dagger X \Theta \phi \rangle = \langle \Theta \phi, X \Theta \phi \rangle^* = \langle \Theta \phi, X^\dagger \Theta \phi \rangle,
\]
as with \( \{ \phi_j \} \) also \( \{ \Theta \phi_j \} \) is a basis, this proves (118).

Corollary 4.2 Let \( \vartheta \) be antilinear and invertible. Then
\[
\text{Tr } \vartheta X \vartheta^{-1} = \text{Tr } X^\dagger.
\]  
(119)
By (116), there is a positive invertible operator \( A \) such that \( \vartheta = \Theta^\dagger A \) and \( \vartheta^{-1} = A^{-1} \Theta \). Inserting into (118) proves (119).

Let \( \vartheta = \Theta \rvert \vartheta \rvert \) a polar decomposition (116). Replacing \( X \) by \( \rvert \vartheta \rvert X^* \rvert \vartheta \rvert \) in (118) yields
\[
\text{Tr } \vartheta X^* \vartheta^* = \text{Tr } X \vartheta^* \vartheta.
\]  
(120)

4.1 Antiunitaries acting on \( \mathcal{H}_2 \)

An applications of (83) is as following:

Proposition 4.3 Let \( A \in \mathcal{B}(\mathcal{H}_2) \) and \( z \neq 0 \) a complex number. 
Then \( \theta_F A \theta_F^{-1} = z A \) if and only if \( A = a U \) with \( U \in SU(2) \) and \( z = a^*/a \). In particular, \( A \) commutes with \( \theta_F \) if and only if \( a \in \mathbb{R} \).

With \( A \) also \( A^\dagger \) commutes with \( \tau_F \). Hence (83) provides \( (\det A) A^{-1} = z^* A^\dagger \). Hence there is a complex number \( a \) such that \( A = a U \) with \( U \) unitary, \( \det U = 1 \). Then \( a = z^* a^* \).

Remark: There is no antilinear operator different from 0 commuting with \( SU(n) \) if \( \dim \mathcal{H} > 2 \).
Lemma 4.1 Let \( \theta \) be an antiunitary operator on \( \mathcal{H}_2 \). Either \( \theta \) is a conjugation or \( \theta \) does not possess an eigenvector.

At first, the assertion is invariant with respect to transformations \( \theta' = V \theta V^\dagger \) where \( V \in \text{SU}(2) \). These transformations commute with \( \theta_F \) and we get \( \theta' = U \theta_F \). With a properly chosen \( V \) we get \( U' \) diagonal with respect to a given basis. Then \( \theta' = \epsilon_0 \theta_\epsilon \) with \( |\epsilon_0| = |\epsilon| = 1 \) and \( \theta_\epsilon \) is defined as in example 1. Now the lemma 2.1 in example 1 proves the assertion.

Proposition 4.4 For any antiunitary operator \( \theta \) there is a basis \( \phi_1, \phi_2 \) such that

\[
\theta(c_1 \phi_1 + c_2 \phi_2) = \epsilon c_1^* \phi_2 + \epsilon^* c_2^* \phi_1.
\] (121)

\( \theta \) is a conjugation if and only if \( \epsilon \) is real.

For the last assertion see lemma 2.1. Using Euler’s formula, \( \epsilon = \exp(i \alpha) = \cos \alpha + i \sin \alpha \), one obtains

Corollary 4.3 There is a representation

\[
\theta = \cos \alpha \cdot \theta' + \sin \alpha \cdot \theta_F, \quad \theta'(c_1 \phi_1 + c_2 \phi_2) = c_1^* \phi_2 + c_2^* \phi_1
\] (122)

or, in matrix language,

\[
\begin{pmatrix} 0 & \epsilon^* \\ \epsilon & 0 \end{pmatrix}_{\text{anti}} = \cos \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\text{anti}} + \sin \alpha \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{\text{anti}} \] (123)

4.2 Decompositions of normal antilinear operators

The structure of antilinear unitary operators has been clarified by E. P. Wigner, 81, who gave credit to a method due to E. Cartan, 16. The extension of Wigner’s classification to antilinear normal operators is due to F. Herbut and M. Vujčić, 39.

As all these authors we restrict ourselves to the finite dimensional case. A complete classification for infinite dimensional Hilbert spaces is not known to me.

We derive the decomposition of an antilinear normal operator into elementary parts, a “surrogate” of the spectral decomposition of linear normal operators.

It will be seen that the “1-qubit case”, \( \dim \mathcal{H} = 2 \), provides a key ingredient in handling the problem.

At first, however, subspace decompositions of \( \mathcal{H} \) are considered which are related to a general antilinear operator.

A subspace \( \mathcal{H}_0 \) of \( \mathcal{H} \) will be called \( \varphi \)-normal if it is \( \varphi \) and \( \varphi^\dagger \) invariant, i. e. \( \varphi \mathcal{H}_0 \subset \mathcal{H}_0 \) and \( \varphi^\dagger \mathcal{H}_0 \subset \mathcal{H}_0 \) is valid. Let \( (\mathcal{H}_0)^\perp \) be the orthogonal complement of \( \mathcal{H}_0 \). It contains, by definition, all vectors \( \varphi \) which are orthogonal
to the vectors of $\mathcal{H}_0$. As in the linear case it follows $\vartheta^\dagger(\mathcal{H}_0)^\perp \subset \mathcal{H}_0$ from $\vartheta\mathcal{H}_0 \subset \mathcal{H}_0$. Therefore, the complement of a $\vartheta$-normal subspace is $\vartheta$-normal.

The next step is again a standard one: If a $\vartheta$-normal subspace $\mathcal{H}_0$ contains a proper $\vartheta$-normal subspace $\mathcal{H}_1$ then $\mathcal{H}_2 = (\mathcal{H}_1)^\perp \cap \mathcal{H}_0$ is a further $\vartheta$-normal subspace contained in $\mathcal{H}_0$ and orthogonal to $\mathcal{H}_1$. A minimal $\vartheta$-normal subspace is a $\vartheta$-normal subspace which does not contain any proper $\vartheta$-normal subspace. Because $\mathcal{H}$ is finite dimensional, there are minimal $\vartheta$-normal subspaces.

**Proposition 4.5** Let $\vartheta$ be antilinear and $\mathcal{H}$ of finite dimension. Then there is an orthogonal decomposition of $\mathcal{H}$ into minimal $\vartheta$-normal subspaces.

Notice: $\mathcal{H}$ may not contain any proper $\vartheta$-normal subspace.

The intersection of two $\vartheta$-normal subspaces is a $\vartheta$-normal subspace. If one of the two subspaces is minimal, then their intersection is either the minimal $\vartheta$-normal subspace itself or it consists of the zero vector only, see also [43]. Hence:

**Proposition 4.6** Any $\vartheta$-normal subspace can be decomposed into mutually orthogonal minimal $\vartheta$-normal subspaces.

Now let $\vartheta$ be an antilinear normal operator.

From $\vartheta\phi = 0$ it follows $\vartheta^\dagger\phi = 0$ and vice versa: The null space ker$[\vartheta]$ of $\vartheta$ is $\vartheta$-normal. Hence the orthogonal complement ker$[\vartheta]^{\perp}$ of the null space is $\vartheta$-normal. It follows, because $\mathcal{H}_0^{\perp}$ is finite dimensional, that $\vartheta$ is invertible on ker$[\vartheta]^{\perp}$. By that fact we can ignore the kernel of $\vartheta$ in what follows and start, without loss of generality, with the assumption of an invertible and normal $\vartheta$.

This assumption implies the uniqueness of the polar decomposition: There is a unique antunitary operator $\theta$ such that $\vartheta = \theta|\vartheta|$ and $\theta^2 = |\vartheta|$. Therefore, $\vartheta$, $\vartheta^\dagger$, $|\vartheta|$, and the unitary operator $\theta^2$ is a set of mutually commuting operators.

Hence there is a complete set of common eigenvectors of the linear operators $|\vartheta|$, $\vartheta^2$, and $\theta^2$. Let $\phi$ a unit eigenvector for these operators. Now we can assume

$$
|\vartheta|\phi = s\phi, \quad \vartheta^2\phi = \epsilon^2\phi, \quad \theta^2\phi = \epsilon^2s^2\phi \quad (124)
$$

with $s > 0$ and a unimodular $\epsilon$, determined up to a sign by [124]. Both numbers can be described by $z = \epsilon s$, again up to a sign. This ambiguity is respected in the following notation:

$$
\mathcal{H}_{\pm z} = \{ \phi \in \mathcal{H} \mid |\vartheta|\phi = s\phi, \theta^2\phi = \epsilon^2\phi, z = \epsilon s \} \quad (125)
$$

As the notation indicates, these subspaces can be characterized also by

$$
\mathcal{H}_{\pm z} = \{ \phi \in \mathcal{H} \mid \vartheta^2\phi = z^2\phi \} \quad (126)
$$
Indeed, (126) is $\vartheta^2$-invariant and the linear operator $\vartheta^2$ is normal. Hence (126) is necessarily $|\vartheta^2|$-invariant and $\pm \epsilon$ fulfills $z^2 = \epsilon^2 |z|^2$. This proves:

**Proposition 4.7** The spaces defined by (126) are $\vartheta$-normal for normal $\vartheta$.

There is a unique orthogonal decomposition

$$\mathcal{H} = \bigoplus_z \mathcal{H}_{\pm z}, \quad z^2 \in \text{spec}(\vartheta^2),$$

(127)

Every minimal $\vartheta$-invariant subspace belongs to just one of these subspaces.

The last assertion is evident: A minimal $\vartheta$-invariant subspace is either contained in a given $\vartheta$-invariant subspace or their intersection contains the 0-vector only.

The antilinear operator $\vartheta$ is reduced to $|z|\theta$ on $\mathcal{H}_{\pm z}$, i.e. to a positive multiple of an antiunitary operator. By this observation the classification of normal antilinear operators becomes the classification of antiunitary operators supported by subspaces of type $\mathcal{H}_{\pm z}$.

A subspace, invariant under the action of an antiunitary $\theta$, is $\theta^\dagger$-invariant: Being finite dimensional, the subspace is mapped onto itself by $\theta$. So it does $\theta^\dagger = \theta^{-1}$. In particular, any minimal $\vartheta$-invariant subspace is $\vartheta$-normal.

Combining with corollary 4.6 results in one of the possible forms of Wigner’s classification of antiunitaries, [81], and its extension to normal antiunitary operators by Herbut and Vujičić, [39].

**Theorem 4.1 (Wigner; Herbut, Vujičić)** Let $\vartheta$ be normal and $\dim \mathcal{H} < \infty$. Every $\vartheta$-invariant subspace is $\vartheta^\dagger$-invariant. $\mathcal{H}$ can be decomposed into an orthogonal sum of minimal $\vartheta$-invariant subspaces. A minimal invariant subspace is either 1-dimensional, generated by an eigenvector of $\vartheta$ and contained in a subspace $\mathcal{H}_{\pm z}$ with $z = z^*$, or it is 2-dimensional, contained in a subspace $\mathcal{H}_{\pm z}$ with $z \neq z^*$ allowing for a basis $\phi', \phi''$ such that $\vartheta \phi' = z^* \phi''$, $\vartheta \phi'' = z \phi'$. It remains to show that a minimal subspace, say $\mathcal{H}_{\text{min}}$, is either 1- or 2-dimensional. It is obvious that a 1-dimensional subspace is minimal and generated by an eigenvector of $\vartheta$. Now let $\dim \mathcal{H}_{\text{min}} > 1$ and $\phi'$ one of its unit vectors. Define $\phi''$ by $\vartheta \phi' = z^* \phi''$. Now $\vartheta^2 \phi' = z \phi''$ and, by assumption, $\vartheta^2 \phi' = z^2 \phi'$. Hence $\vartheta \phi'' = z \phi'$. The subspace $\mathcal{H}_{\text{min}}$ should not contain an eigenvector of $\vartheta$. Because $|z|\theta = \vartheta$ on $\mathcal{H}_{\text{min}}$ we can rely on proposition 4.4 to exclude $z = z^*$.

**Corollary 4.4 (Wigner; Herbut, Vujičić)** There is a unique orthogonal decomposition $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$ into $\vartheta$-normal subspaces with the following properties: If restricted to $\mathcal{H}'$, $\vartheta$ is Hermitian and can be diagonalized. $\mathcal{H}''$ is even dimensional and there is no eigenvector of $\vartheta$ in $\mathcal{H}''$. In degenerate cases one of the two subspaces is absent and the other one is the whole of $\mathcal{H}$.
Corollary 4.5 (Wigner; Herbut, Vujičić) If \( \vartheta \) is normal, it allows for a block matrix representation with blocks of dimensions not exceeding two. The \( 1 \times 1 \) blocks contain eigenvalues of \( \vartheta \), the \( 2 \times 2 \) block are filled with zeros in the diagonal and with pairs \( z, z^* \), \( z \neq z^* \), as off-diagonal entries.

4.2.1 Conjugations

The decomposition of conjugations a la Wigner is a simple particular case. A conjugation \( \theta \) allows for a basis \( \phi_1, \ldots, \phi_d \) such that \( \theta \phi_j = t_j \phi_j \) with positive real \( t_j \) because \( \theta \) is Hermitian. As \( \theta^2 = 1 \), all \( t_j = 1 \). It will be shown in section 5 that the real linear hull of the basis \( \{ \phi_j \} \) uniquely characterizes \( \theta \). \[ \text{(14)} \]

The next topic are relations between two and more conjugations. Let \( \theta \) and \( \theta' \) be conjugations. Then \( U = \theta' \theta \) is unitary and its \( d \) eigenvalues are unimodular numbers. The trace of \( U \) is the sum of these eigenvalues and its absolute sum is bounded by \( \dim \mathcal{H} \).

\[ | \text{Tr} \theta' \theta | \leq \dim \mathcal{H} . \quad \text{(128)} \]

If this bound is reached, the unitary \( \theta' \theta \) is a multiple of \( 1 \). Thus

\[ | \text{Tr} \theta' \theta | = \dim \mathcal{H} \iff \theta = \epsilon \theta' \quad \text{(129)} \]

with \( |\epsilon| = 1 \).

If \( \theta \) and \( \theta' \) commute, \( \mathcal{H}_\theta \) is \( \theta' \)-invariant. With \( \phi \in \mathcal{H}_\theta \) one gets \( \theta' (\phi \pm \theta' \phi) = \theta' \phi \pm \phi \). Hence \( \mathcal{H}_\theta \) splits into the real Hilbert subspace of the vectors \( \phi \) satisfying \( \theta \phi = \theta' \phi = \phi \) and into the subspace of all vectors fulfilling \( \theta \phi = -\theta' \phi = \phi \), showing

\[ \theta \theta' = \theta' \theta \Rightarrow \text{Tr} \theta' \theta = \dim(\mathcal{H}_\theta \cap \mathcal{H}_{\theta'}) - \dim(\mathcal{H}_\theta \cap \mathcal{H}_{-\theta'}) \quad \text{(130)} \]

One observes that the trace of the product of two commuting conjugations is an integer.

One can associate to a given basis \( \phi_1, \phi_2, \ldots \) a set of \( 2^d \) mutually commuting conjugations: For any subset \( E \) of basis vectors one defines \( \theta_E \) by

\[ \theta_E \phi_j = \phi_j \text{ if } \phi_j \in E, \text{ and by } \theta_E \phi_k = -\phi_k \text{ if } \phi_j \notin E. \]

Going through all the subsets one gets mutually commuting conjugations.

Some further easy relations: With two conjugations \( \theta_1 \) and \( \theta_2 \), \( U = \theta_1 \theta_2 \) is unitary, \[ \text{(133)} \]. If \( \phi_j \), \( j = 1, \ldots, d \), is an eigenvector basis of \( U \), let \( \theta_3 \) the conjugation satisfying \( \theta_3 \phi_j = \phi_j \) for all \( j \). Then \( \theta_3 U^\dagger \theta_3 \) = \( U \) and \( \theta_3 \theta_2 \theta_1 \theta_3 = \theta_1 \theta_2 \). Thus

\[ \text{Lemma 4.2} \quad \text{Given two conjugations, } \theta_1 \text{ and } \theta_2, \text{ There is a third one, } \theta_3, \text{ such that} \]

\[ \theta_3 \theta_2 \theta_1 = \theta_1 \theta_2 \theta_3 \quad \text{(131)} \]

and \( \theta := \theta_1 \theta_2 \theta_3 \) is a conjugation.
A further elementary fact is stated by

**Lemma 4.3** Every unitary operator is the product of two conjugations. Every antiunitary operator is the product of three conjugations.

Indeed, let $\phi_1, \ldots$ denote an eigenbasis of the unitary $U$, $U \phi_j = \epsilon_j \phi_j$. There are two conjugations satisfying $\theta_1 \phi_j = \epsilon_j \phi_j$ and $\theta_2 \phi_j = \phi_j$ for all $j$. Now the linear operator $\theta_1 \theta_2$ transforms $\phi_j$ into $\epsilon \phi_j$. Therefore it must be the linear operator $U$. The other assertion is now trivial as any antiunitary is a product of a unitary operator and a conjugation.

**Corollary 4.6** Given $2n$ conjugations $\theta_1, \ldots, \theta_{2n}$. There exists a conjugation $\theta_{2n+1}$ such that

$$\theta = \theta_1 \cdots \theta_{2n} \theta_{2n+1} \quad (132)$$

is a conjugation.

To see it write $U = \theta_1 \cdots \theta_{2n}$. Being a unitary it is a product of two conjugation: $U = \theta \theta_{2n+1}$. Eliminating $U$ yields (132).

### 4.2.2 Skew conjugations

Let $\theta$ be a skew conjugation. Then the dimension of the Hilbert space must be even, $d = 2n$. By Wigner’s theorem $\mathcal{H}$ can be decomposed as an orthogonal direct sum

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n, \quad \dim \mathcal{H}_j = 2, \quad (133)$$

of irreducible $\theta$-invariant subspaces,

$$\theta \mathcal{H}_j = \mathcal{H}_j, \quad j = 1, 2, \ldots, n. \quad (134)$$

From $\theta^2 = -1$ follows: $\theta_j$ acts on $\mathcal{H}_j$ as a multiple $(\exp is) \tau_0$ of $\tau_0$. Choosing in every $\mathcal{H}_j$ a unit vector $\phi_{2j}$, then $\psi_{2j+1} := \theta \psi_{2j}$ is a unit vector orthogonal to $\psi_j$. Hence the pair $\psi_{2j}, \psi_{2j+1}$ is a basis of $\mathcal{H}_j$ such that

$$\theta \psi_{2j} = \psi_{2j+1}, \quad \theta \psi_{2j+1} = -\psi_{2j}, \quad (135)$$

saying that $\theta$ is an orthogonal direct sum of $\tau_0$-operators with respect to a suitably chosen basis:

**Proposition 4.8** Let $\dim \mathcal{H} = 2n$ and let $\theta$ be a skew conjugation. Then there is a basis $\{\phi_j\}$ of $\mathcal{H}$ such that

$$\theta \phi_{2k} = -\phi_{2k-1}, \quad \theta \phi_{2k-1} = \phi_{2k} \quad (136)$$

for all $k = 1, \ldots n$.

From the proposition follows
Corollary 4.7 Let $\theta$ be a skew conjugation. There are orthogonal decompositions $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_b$ such that for all $\phi_a \in \mathcal{H}_a$ and $\phi_b := \theta \phi_a$ one gets $\theta \phi_b = -\phi_a$.

Indeed, with a basis fulfilling (136) let $\mathcal{H}_a$ be the complex linear hull of the $n$ vectors $\phi_{2k}$ and $\mathcal{H}_b$ the complex linear hull of the $n$ vectors $\phi_{2k-1}$. Clearly, $\mathcal{H}$ is an orthogonal sum of these two Hilbert subspaces. Furthermore, for all $\phi_a \in \mathcal{H}_a$

$$\phi_a = \sum_{k=1}^{n} c_{2k} \phi_{2k}, \quad \phi_b := \theta \phi_a = \sum_{k=1}^{n} c'_{2k} \phi_{2k-1}$$

with $\phi_b \in \mathcal{H}_b$. Now $\theta^2 = -1$ shows $\theta \phi_b = -\phi_a$.

Analogue to lemma 4.3 one can show

Lemma 4.4 Every antiunitary operator can be represented by a product of two conjugations and a skew conjugation. The position of the skew conjugation can be fixed in advance.

4.2.3 The number of orthogonal (skew) conjugations

For the purpose of just this subsection let $N_+(d)$ be the maximal possible number $n$ of conjugations, $\theta_1, \ldots, \theta_n$, such that $(\theta_j, \theta_k) = \delta_{jk}d$. Similarly $N_-(d)$ is the maximal possible number of skew conjugations, $\theta_{-1}, \ldots, \theta_{-m}$, with $(\theta_{-j}, \theta_{-k}) = -\delta_{jk}d$. These numbers are bounded from above by (25):

$$N_+(d) \leq \frac{d(d+1)}{2}, \quad N_-(d) \leq \frac{d(d-1)}{2}.$$  \hfill (137)

There are bounds from below too. Let $\theta_1$ and $\theta_2$ denote antilinear operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of dimensions $d_1$ and $d_2$ respectively. The operator $\theta = \theta_1 \otimes \theta_2$ is an antilinear operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $\theta_1$ and $\theta_2$ are conjugations, or if both are skew conjugations, then $\theta$ is a conjugation. If one of them is a conjugation and the other a skew conjugation, $\theta$ is a skew conjugation.

It is now possible to conclude

$$N_+(d_1d_2) \geq N_+(d_1)N_+(d_2) + N_-(d_1)N_-(d_2), \quad N_-(d_1d_2) \geq N_+(d_1)N_-(d_2) + N_-(d_1)N_+(d_2). \hfill (138)$$

Assuming now that equality holds in (137), it follows from (138), (139)

$$N_+(d_1d_2) \geq \frac{(d_1 + d_2)(d_1 + d_2 + 1)}{2}, \quad N_-(d_1d_2) \geq \frac{(d_1 + d_2)(d_1 + d_2 - 1)}{2}$$

as a straightforward calculation establishes. Again by (137), the last two inequalities must be equalities.
Proposition 4.9 The set of Hilbert spaces for which equality holds in \((137)\) is closed under performing direct products.

If the dimension \(d\) of an Hilbert space \(\mathcal{H}\) is a power of 2, \(d = 2^n, n > 1\), then there are \(d(d + 1)/2\) conjugations, \(\theta_1, \theta_2, \ldots\), and \(d(d - 1)/2\) skew conjugations, \(\theta'_1, \theta'_2, \ldots\), such that

\[
(\theta_j, \theta_k) = d \delta_{jk}, \quad (\theta'_j, \theta'_k) = -d \delta_{jk}, \quad (\theta_j, \theta'_k) = 0 .
\] (140)

The first part of the proposition has already been proved. For its second part it suffices that the assertion is true for \(d = 2\). Indeed, the conjugations \(\tau_j, j = 0, 2, 3, 4\), show it.

Choosing for every \(\theta'_j, j = 1, \ldots, N_+(d)\) an invariant basis, \(\phi_{j1}, \ldots, \phi_{jd}\), the orthogonality relations (140) can be rewritten by the use of (137). It follows

Corollary 4.8 There are \(N_+(d)\) bases \(\phi_{j1}, \ldots, \phi_{jd}\) such that

\[
\sum_{n,m} \langle \phi_{nj}, \phi_{mk} \rangle^2 = d \delta_{jk}
\] (141)

is true for all \(j, k\) in \(\{1, 2, \ldots, N_+(d)\}\).

5 A look at elementary symplectic geometry

Let \(\theta\) be a conjugation and define

\[
\mathcal{H}_\theta := \{ \phi \in \mathcal{H} : \theta \phi = \phi \} .
\] (142)

By (112) we see that \(\phi_1, \phi_2 \in \mathcal{H}_\theta\) implies \(\langle \phi_1, \phi_2 \rangle = \langle \phi_2, \phi_1 \rangle\). Hence \(\mathcal{H}_\theta\) is a real Hilbert subspace. The real dimension of \(\mathcal{H}_\theta\) is \(d = \dim \mathcal{H}\). Therefore it is a maximal real Hilbert subspace of \(\mathcal{H}\) consisting of all vectors of the form \(\phi = (\psi + \theta \psi), \psi \in \mathcal{H}\). Any maximal real Hilbert subspace is of the form \(\mathcal{H}_\theta\) with a unique conjugation \(\theta\).

Proposition 5.1 There is a one-to-one correspondence

\[
\theta \leftrightarrow \mathcal{H}_\theta,
\] (143)

between the set of conjugations and the set of maximal real Hilbert subspaces.

This simple observation opens the door from conjugations to elementary symplectic geometry.

Note also

\[
\mathcal{H} = \mathcal{H}_\theta + i \mathcal{H}_\theta, \quad i \mathcal{H}_\theta = \mathcal{H}_{-\theta},
\]

38
5.1 Conjugations and Symplectic Geometry

Symplectic Geometry is an eminent topic in its own. See [63], [70], [35], or any monograph on symplectic geometry. Only some elementary comments, concerning the correspondence (143) in proposition 5.1, can be given here.

\[ \mathcal{H}, \text{if considered as a real linear space, is naturally equipped with the symplectic form} \]
\[ \Omega(\phi, \phi') := \frac{\langle \phi, \phi' \rangle - \langle \phi', \phi \rangle}{2i}. \]  

(144)

A real linear subspace of \( \mathcal{H} \) on which this form vanishes, is called isotropic. If it is a maximal isotropic one, its real dimension is \( d \), and it is called a Lagrangian one or a Lagrangian plane. By this definition the Lagrangian subspaces are just the real Hilbert subspaces \( \mathcal{H}_\theta \) with \( \theta \) a conjugation.

The set of all Lagrange subspaces is a compact smooth manifold called the Lagrangian Grassmannian of \( \mathcal{H} \). The Lagrangian Grassmannian is usually denoted by \( \Lambda_d \), \( \Lambda[H] \), or Lag\[H\]. By (143) Proposition 5.2 \( \Lambda[H] \) is symplectomorphic to the manifold of all conjugations on \( \mathcal{H} \).

Fixing a Lagrangian subspace \( \mathcal{H}_\theta \), its isometries form an orthogonal group \( O(\mathcal{H}_\theta) \simeq O(d) \). Because the unitary group \( U(\mathcal{H}) \simeq U(d) \) acts transitively on the Lagrangian subspaces, as all bases are unitary equivalent, one gets the well known isomorphism \( \Lambda_d \simeq U(d)/O(d) \). The dimension of the manifold \( \Lambda_d \) is \( (d^2 + d)/2 \). A useful relation is

\[ U \mathcal{H}_\theta = \mathcal{H}_{\theta'} \iff \theta' = U \theta U^{-1}, \quad U \in U(\mathcal{H}). \]  

(145)

Next is to show: The acq–lines going through \( \theta_0 \) cover a neighborhood of \( \theta_0 \).  

Lemma 5.1 Given a conjugation \( \theta_0 \) and a tangent \( \eta_0 \) at \( \theta_0 \). There is an Hermitian \( H \) commuting with \( \theta_0 \) such that the tangent of

\[ t \mapsto \theta_s = U_H(t)\theta_0 U_H(-t), \quad U_H(t) = \exp(itH), \]  

(146)

at \( \theta_0 \) is equal to \( \eta_0 \).

Proof: There is an Hermitian \( A \) such that

\[ t \mapsto \theta'_s = U_A(t)\theta_0 U_A(-t), \quad U_A(t) = \exp(itA) \]

has tangent \( \eta_0 \) at \( \theta_0 \). This implies \( \eta_0 = i(A\theta_0 + \theta_0A) \). Set \( 2H := A + \theta_0A\theta_0 \) for the path (146). Then \( \theta_0H = H\theta_0 \) and \( \eta_0 = i(H\theta_0 + \theta_0H) = 2iH\theta_0 \).
Proposition 5.3 Given a conjugation $\theta$ and an Hermitian operator $H$ such that $H \mathcal{H}_\theta \subset \mathcal{H}_\theta$ or, equivalently, $H \theta = \theta H$. Let

$$t \mapsto U_H(t) = \exp i t H, \quad \theta_t = e^{itH} \theta e^{-itH}.$$  \hspace{0.5cm} (147)

Then $H$ commutes with $\theta_t$ for all $t \in \mathbb{R}$ and

$$\frac{d\theta_t}{dt} = i(\theta_t H + H \theta_t) = 2iH \theta_t.$$  \hspace{0.5cm} (148)

Moreover, $t \to \theta_t$ is an acq–line, i.e.

$$\theta_r \theta_{(r+s)/2} = \theta_{(r+s)/2} \theta_r, \quad \theta_r \theta_t = U_H(2r - 2t)$$  \hspace{0.5cm} (149)

for all $r, s, t \in \mathbb{R}$.

The core of the proof is lemma 1.1, saying that (149) follows if $\theta = \theta_0$ commutes with $H$. The first equality sign in (148) is true for all Hermitian $H$, while the second one is due to the lemma above. Finally, $\theta_t H = H \theta_t$ by definition (147).

With $t \to \theta_t$ also $t \to \theta_{bt}$ is an acq–line. Hence, if this path returns to $\theta_0$ for some $t' \neq 0$, one can assume $t' = \pi$ for the parameter of return.

Corollary 5.1 Let $t \to \theta_t$ be as in the proposition. The equality $\theta_0 = \theta_\pi$ takes place if and only if all eigenvalues of $H$ are integers.

Let $\theta_0 \phi = \phi$ and $H \phi = a \phi$. If $\theta_\pi = \theta_0$ then $(\exp i \pi H) \phi = \pm \phi$. Hence $a$ must be an integer. Hence all eigenvalues of $H$ must be integers.

Let $H$ and $t \to \theta_t$ be as in proposition 5.3 and $\theta_\pi = \theta_0$. Then there are mutually orthogonal rank one projection operators $P_j$ and $n_j \in \mathbb{Z}$ such that

$$H = \sum n_j P_j, \quad P_k \theta_s = \theta_s P_k$$  \hspace{0.5cm} (150)

for $k = 1, \ldots, d$. As one knows that $[146]$ is a generator for $H_1(\Lambda, \mathbb{Z})$ if $H$ is a projection operator of rank one, the map

$$\{ H : \text{spec} H \in \mathbb{Z} \} \to [H] \in H_1(\Lambda, \mathbb{Z})$$  \hspace{0.5cm} (151)

is onto. If $H$ is mapped by (151) onto $[H]$, then

$$[H] = \sum n_j [P_j].$$  \hspace{0.5cm} (152)

One can find a closed curve generating $H_1[\Lambda, \mathbb{Z}]$ as follows, [63]. Let $\phi_1, \ldots, \phi_d$ be a basis of $\mathcal{H}$. Define the conjugation $\theta$ by $\theta \phi_k = \phi_k$ for all $k$, the unitaries $U(t)$ by $U(t) \phi_1 = (\exp i t) \phi_1$, and by $U(t) \phi_j = \phi_j$ if $j > 1$. Then $H$ is the projection operator $P = [\phi_1] \langle \phi_1 |$ onto $\phi_1$, i.e.

$$t \to \theta_t := U(t) \theta U(-t), \quad U(t) = \exp i t P$$  \hspace{0.5cm} (153)
is a generator for $H_1[\Lambda, \mathbb{Z}]$.

An inequality. Let $\lambda_1, \ldots$ denote the eigenvalues of $H$. The general acq–line in $\mathcal{B}(\mathcal{H})_{\text{anti}}^+$ is contained in the sphere of radius $\sqrt{d}$ around the null vector. Any piece of it has a well defined length

$$\int_{t'}^{t''} \text{Tr} \sqrt{\dot{\theta}_s^2} ds = 2(t'' - t') \text{Tr} \sqrt{H^2} = 2(t'' - t') \sqrt{\sum a_j^2}. \quad (154)$$

Indeed, $\dot{\theta}_s^2 = (H\theta_s + \theta_s H)^2$ which is equal to $4H^2$ by virtue of (147). Taking into account (150) one obtains (154).

Assuming now $\theta_0 = \theta_\pi$, the numbers $a_j$ become integers $n_j$. Then

$$\int_0^\pi \sqrt{\dot{\theta}_s^2} ds = 2\pi \sqrt{\sum n_j^2}. \quad (155)$$

The shortest closed acq–lines are of length $2\pi$. They are generators for $H_1[\Lambda, \mathbb{Z}]$.

5.2 The canonical differential 1-form

Seeing $\Lambda_d$ is a submanifold of the Hermitian part $\mathcal{B}(\mathcal{H})_{\text{anti}}^+$ of $\mathcal{B}(\mathcal{H})_{\text{anti}}$, the differential $d\theta$ is well defined on the Lagrangian Grassmannian: It is the restriction onto $\Lambda$ of $d\vartheta$, $\vartheta \in \mathcal{B}(\mathcal{H})_{\text{anti}}^+$. The operator valued differential 1-form

$$\nu := \theta d\theta \quad (156)$$

is skew symmetric, $\nu + \nu^\dagger = 0$, as following from $\theta^2 = 1$ and $\theta^\dagger = \theta$.

**Proposition 5.4** The differential 1-form

$$\bar{\nu} := \frac{1}{2\pi i} \text{Tr} \nu = \frac{1}{2\pi i} \text{Tr} \theta d\theta, \quad \bar{\nu}^\dagger = \nu, \quad (157)$$

is closed, but not exact. It is a unitary invariant. It changes its sign by antiunitary transformations.

Proof: Fix $U$. The unitary invariance follows from $U\theta d\theta U^\dagger = (U\theta U^\dagger)d(U\theta U^\dagger)$ by taking the trace. If $U$ is replaced by an antiunitary, the sign change is seen from (118). To show $\text{d} \bar{\nu} = 0$, the representation $U \rightarrow \vartheta := U\theta_0 U^\dagger$ with $U$ varying in $U(d)$ and an arbitrarily chosen conjugation $\theta_0$ is inserted,

$$\theta d\theta = U\theta_0 U^\dagger d(U\theta_0 U^\dagger) = U(\theta_0 U^\dagger)(dU\theta_0)U^\dagger + UdU^\dagger.$$

Taking the trace one finds

$$\text{Tr} \theta d\theta = 2\text{Tr} UdU^\dagger. \quad (158)$$

The equality of both summands follows by setting $X = U^\dagger dU$ and $\Theta = \theta_0$ in (118). The right hand side of (158) can simplified further: In the vicinity
of the identity map there is a unique logarithm $iH = \ln U$ with $H = H^\dagger$. Then
\[
\text{Tr} UdU^\dagger = -i\text{d} \text{Tr} H, \quad U = \exp iH .
\] (159)
Hence, by (158), the differential 1-form $\tilde{\nu}$ is closed.

**Proposition 5.5** The 1–form
\[
\tilde{\nu} := \frac{1}{2\pi i} \text{Tr} \nu
\] (160)
generates the first integer–valued cohomology group $H^1(\Lambda, \mathbb{Z})$.

Let $H$ and the curve $\gamma : t \to \theta_t$ be as in proposition 5.3. Assuming $\theta_0 = \theta_\pi$, $\gamma$ becomes a closed curve. By (148)
\[
\int_\gamma \nu = \int_0^\pi \theta_\pi \dot{\theta}_t ds = 2\pi i H .
\] (161)
The eigenvalues of $H$ are integers, $n_j$, for closed acq–lines. Taking the trace yields
\[
\int_\gamma \tilde{\nu} = \frac{1}{2\pi i} \text{Tr} \int_\gamma \nu = \sum n_j .
\] (162)
One knows that $\gamma$ generates the first homology group if $H$ is a rank one projection operator [63]. Hence $\tilde{\nu}$ is a generator of the first integer–valued cohomology group. According to V. I. Arnold its integral over a closed curve provides its *Maslov Index* [35].

**Corollary 5.2** $\tilde{\nu}$ is a generator of $H^1(\Lambda, \mathbb{Z})$. For a closed curve $\gamma$ in $\Lambda_d$
\[
\text{Maslov}[\gamma] = \int_\gamma \tilde{\nu} \in \mathbb{Z}
\] (163)
is the *Maslov index* of $\gamma$.

The right hand side of (158) can be rewritten in terms of unitary operators. By (158)
\[
\text{Tr} \nu = 2\text{Tr} UdU^{-1} = -2i\text{d} \text{Tr} H .
\]
In small enough open sets one has $\det U = \det \exp iH = \exp \text{Tr} iH$. Hence
\[
\text{d} \det U = \text{d} \exp \text{Tr} iH = i(\text{d} \text{Tr} H)(\det U) .
\]
One gets one of the known expressions for the Maslov index:
\[
\text{Maslov}[\gamma] = i \pi \int_\gamma \frac{\text{d} \det U}{\det U}, \quad \theta = U \theta_0 U^\dagger .
\] (164)
Remark:
ν is not closed for \( d \geq 2 \). Therefore the closed operator valued differential 2-form
\[
\omega := d\nu = d\theta \wedge d\theta
\]  
may be of interest. The case of \( \Lambda_2 \) will be examined: According to lemma 2.2, see also (46), a conjugation can be written as a real linear combination of the antilinear Hermitian Pauli matrices \( \tau_j, j = 1, 2, 3 \), multiplied by a phase \( \epsilon \) in the form
\[
\theta = \epsilon \sum_{j=1}^{3} x_j \tau_j, \quad \sum_{j=1}^{3} x_j^2 = 1
\]  
Varying \( \theta \) within these constraints defines a space topological equivalent to the product \( S^1 \times S^2 \) of a circle and a 2-sphere. Locally this remains true for \( \Lambda_2 \). However, by (166), every \( \theta \) is represented by two couples, \( \epsilon, \vec{x} \) and \(-\epsilon, -\vec{x}\). One gets the well known fact: \( \Lambda_2 \) is topological the product of a 1- and a 2-sphere on which the points \( \epsilon, \vec{x} \) and \(-\epsilon, -\vec{x}\) are identified. Writing \( \theta = \epsilon \theta_\vec{x} \), one obtains
\[
\nu = \epsilon d\epsilon^\ast \mathbf{1}_2 + \theta_\vec{x} d\theta_\vec{x}
\]  
Using now (167), the differential form (165) becomes \((d\theta_\vec{x}) \wedge (d\theta_\vec{x})\). Therefore,
\[
\omega = \sum_{j \neq k} dx_j \wedge dx_k \tau_j \tau_k = 2 \sum_{j < k} \tau_j \tau_k dx_j \wedge dx_k
\]  
By (39) one obtains
\[
\omega = 2i(\sigma_3 dx_1 \wedge dx_2 + \sigma_1 dx_2 \wedge dx_3 + \sigma_2 dx_3 \wedge dx_1)
\]  
6 Equivalence relations
There is a lot of literature on equivalence relations between matrices. A good part can be found in Horn and Johnson’s “Matrix Analysis”, [40]. Concerning more recent results, much of the following is based on papers by L. Balayan, S. R. Garcia, D. E. Poore, M. Putinar, J. E. Tener, and W. R. Wogen, in particular on [8], [31], [74], [32].

One purpose is to “translate” (and extend) some of their theorems into the language of antilinearity, in which they become basis-independent and, hopefully, of a more transparent structure.

Let \( \theta \) be a conjugation. The operator \( \theta X^\dagger \theta \) is called the \( \theta \)-transpose, or simply the transpose \( X^\top \) of \( X \) if there is no danger of confusion. Then one writes
\[
X^\top = \theta X^\dagger \theta, \quad X^\dagger = \theta X^\top \theta
\]
The transpose of an operator is defined relative to a basis \( \{ \phi_j \} \). There is a conjugation fulfilling \( \theta \phi_j = \phi_j \) for all its elements. The transpose \( X^\top \) of a linear operator \( X \) can be written

\[
X \phi_j = \sum_i x_{ji} \phi_i, \quad X^\top \phi_j = \sum_i x_{ij} \phi_i,
\]

with respect to the given basis. It follows

\[
\theta X^\top \theta \phi_j = \theta X^\top \phi_j = X^\dagger \phi_j,
\]

and \( \theta X^\top \theta = X^\dagger \) as in (170).

One may look at \( X \to \theta X^\dagger \theta \) as at a superoperator. Its fixpoints defines an important class of linear operators, see [29] and [30].

6.1 Similarity, Congruence

A bit of terminology: Let \( X, Y \) two operators, either both linear or both antilinear. \( Y \) is called cosimilar to \( X \) if there is an invertible antilinear operator \( \vartheta \) such that \( Y = \vartheta X \vartheta^{-1} \). The definition mimics the similarity relation \( Y = AXA^{-1} \) between \( X \) and \( Y \) with \( A \) linear and invertible. Proposition 3.3 is an instructive example for similarity between antilinear operators.

\( X \to \vartheta X \vartheta^{-1} \) operates antilinearly within \( B(H) \) and within \( B(H)_{\text{anti}} \), while \( X \to AXA^{-1} \) operates linearly. There are more differences: Cosimilarity is not an equivalence relation. Instead, if \( X \) is cosimilar to \( Y \) and \( Y \) cosimilar to \( Z \) then \( X \) is similar to \( Z \). Hence similarity relations could be “factorized” by a pair of cosimilar ones.

One calls \( Y \) congruent to \( X \) if there exists a congruence relation \( Y = AXA^\dagger \) with invertible linear operator \( A \). In the same manner as above, \( Y \) is cocongruent to \( X \) if \( Y = \vartheta X \vartheta^\dagger \) takes place with an invertible antilinear \( \vartheta \).

Cocongruence is not an equivalence relation: If \( X \) is cocongruent to \( Y \) and \( Y \) cocongruent to \( Z \) then \( X \) is congruent to \( Z \).

Sometimes it is necessary to weaken these concepts to open the door to another domain of research: Let \( \vartheta \) be an antilinear operator. Following Woronowicz, [83], the linear map

\[
X \mapsto T(X) := \vartheta X^\dagger \vartheta^\dagger, \quad X \in B(H),
\]

is called an elementary copositive map or an elementary copositive superoperator. A map or a superoperator of \( B(H) \) into itself is called completely copositive if it can be written as a sum of elementary copositive maps. These definitions mimic the elementary positive maps \( X \mapsto A^\dagger X A \) and the completely positive maps, which are sums of elementary positive maps, see, a. e., [55], [10].
Remarks
1. The operator $\lambda \mathbf{1}$ is cosimilar to itself if and only if $\lambda$ is real.
2. Any $X \in \mathcal{B}(\mathcal{H})$ is similar to its transpose, $X^\dagger = AXA^{-1}$. See part 3.2.3 in [40]. By (170) this translates into
   \begin{equation}
   X^\dagger = \vartheta X \vartheta^{-1}, \quad \vartheta \text{ antilinear}
   \end{equation}
   with $\vartheta = \theta A$ and $\theta$ a conjugation as in (170). Thus
   \begin{equation}
   \text{Every linear operator is cosimilar to its transpose.}
   \end{equation}
3. A remarkable result of R. A. Horn and C. R. Johnson, [40] theorem 4.4.9, reads: Every matrix is similar to a symmetric one. As in the preceding example one gets:
   \begin{equation}
   \text{Every linear operator is cosimilar to an Hermitian one.}
   \end{equation}
   Hence, given $X$, there is an invertible $A$ such that
   \begin{equation}
   \theta AXA^{-1} \theta = [AXA^{-1}]^\dagger = (A_\dagger)^{-1}X^\dagger A^\dagger.
   \end{equation}
   By defining $\vartheta := A_\dagger \theta A$ one obtains a variant of Horn and Johnson’s result:
   \begin{equation}
   \text{To every } X \in \mathcal{B}(\mathcal{H}) \text{ there is } \vartheta \in \mathcal{B}(\mathcal{H})_{\text{anti}} \text{ such that}
   \end{equation}
   \begin{equation}
   X^\dagger = \vartheta X \vartheta^{-1}, \quad \vartheta = \vartheta^\dagger.
   \end{equation}
4. Item (b) of proposition 3.3 states that every diagonalisable antilinear operator is similar to an Hermitian one. An antilinear Hermitian operator $\vartheta'$ allows for a basis $\{\phi_j\}$ of eigenvectors with non-negative real eigenvalues. The conjugation $\theta$, satisfying $\theta \phi_j = \phi_j$ for all $j$, commutes with $\vartheta'$. If, therefore, $A \vartheta A^{-1} = \vartheta'$, one gets, using the quoted results of Horn and Johnson, $\vartheta' = (\theta A)\vartheta (\theta A)^{-1}$ and $(\vartheta')^\dagger = \vartheta'$.

Lemma 6.1 An antilinear operator is similar to an Hermitian one if and only if it is cosimilar to an Hermitian one.

6.2 Unitary equivalence
A matrix is said to be “UET” if it is unitarily equivalent to its transpose. The questions which matrices are UET or “UECSM”, an acronym for unitarily equivalent to a complex symmetric matrix, goes back to P. R. Halmos, see [37]. The abbreviations “UET” and “UECSM” are used in the mathematical literature.

Unitary equivalence of an operator $X$ to its transpose means: There is a basis with respect to which the matrix representation of $X$ is UET. By (170) this is equivalent to the existence of an antiunitary such that
   \begin{equation}
   X^\dagger = \Theta X \Theta^{-1}, \quad \Theta \text{ antiunitary.}
   \end{equation}
If there is a unitary $U$ such that $Y = UXU^\dagger$ satisfies $Y = Y^\top$ in a matrix representation, then, again by (170), there is a conjugation $\theta'$ such that $Y = \theta'Y^\dagger\theta'$. This proves the first part of a lemma due to Garcia and Tener [31].

**Lemma 6.2** The following three items are equivalent:

a) $X$ is UECSM.

b) $X^\dagger$ is antiunitarily equivalent to an Hermitian operator.

c) There is a conjugation such that $X^\dagger = \theta X\theta$.

Step c) → a) is simple: By (170) $\theta X\theta$ is the $\tau$-transpose $X^\top$ of $X$. The latter is by c) equal to $X$. To be more explicit one writes $\vartheta := \theta X^\dagger = X\theta$. Conjugations are Hermitian. Hence $\vartheta = \vartheta^\dagger$. Therefore $\langle \vartheta', X\theta\vartheta'' \rangle = \langle \vartheta', X\theta\vartheta'' \rangle$ for any pair of vectors. If $\vartheta', \vartheta'' \in \mathcal{H}_\theta$, the $\vartheta$’s can be skipped, and the matrix representation of $X$ is symmetric with respect to any basis chosen from $\mathcal{H}_\theta$.

UET is less strong than UECSM. But sometimes they are equally strong. An astonishing case has been settled by R. S. Garcia, and J. E. Tener in [31]:

**Theorem 6.1 (Garcia, Tener)** If $\dim \mathcal{H} < 8$ then a linear operator is antiunitarily equivalent to an Hermitian one if and only if it is antiunitarily equivalent to its Hermitian adjoint. The assertion fails for some $X$ if $\dim \mathcal{H} = 8$.

The quoted authors could prove a decomposition theorem 9. From it the theorem comes as a corollary.

To answer the question whether a given operator is UET or UECSM is another issue. There are quite different approaches to obtain criteria.

### 6.3 Low dimensions

One of the first results concerning the 2-dimensional case is due to S. L. Woronowicz, see appendix of [83] or [71] and the monograph [72]. He called an operator $X$ almost normal if the rank of $X^\dagger X - XX^\dagger$ is not larger than two and he proved

**Proposition 6.1 (Woronowicz)** If $\dim \mathcal{H} < \infty$ and if there are vectors $\phi_1, \phi_2$ such that 

$$X^\dagger X - XX^\dagger = |\phi_2 \rangle \langle \phi_2| - |\phi_1 \rangle \langle \phi_1|$$

(175)
then there exists a conjugation $\theta$ fulfilling

$$\theta X^\dagger \theta = X, \quad \text{and} \quad \theta \phi_1 = \phi_2.$$  \hfill (176)

The proof is by constructing the algebra with defining relation (175) and showing by induction along the degree of monomials in $X$ and $X^\dagger$ the existence of $\theta$.

A completely other way is in characterizing unitary orbits by values of unitary invariants. A particular problem asks whether $X$ and its transpose $X^\top$ belong to the same unitary orbit. S. R. Garcia, D. E. Poore and J. E. Tener offer in [32] a solution for the dimensions 3 and 4 by trace criteria.

Let $U = U(\mathcal{H})$ and $U_{\text{anti}}$ denote the unitary group and the set of antunitary operators respectively. The set $\Lambda[\mathcal{H}]$ is symplectomorphic to the manifold of all One purpose is to “translate” (and extend) some few of their $U \cup U_{\text{anti}}$ is a group generated by the set $U_{\text{anti}}$. Here the interest is in the orbits of the adjoint representation of these groups.

Let $X \in B(\mathcal{H})$ and denote by $\{X\}_u$ the set of all $UXU^{-1}$, $U \in U$. To get an “orbit” $\{X\}_\text{au}$ one enlarges $\{X\}_u$ by the set of all operators $\Theta X^\dagger \Theta^{-1}$, $\Theta \in U_{\text{anti}}$. Indeed,

$$X \mapsto \Theta_2(\Theta_1 X^\dagger \Theta_1)^\dagger = \Theta_2 \Theta_1 X(\Theta_2 \Theta_1)^{-1}$$

is a unitary transformation. From (170) it follows $X^\top \in \{X\}_\text{au}$. By unitary invariance: The $\theta$-transpose $Y^\top$ is contained in $\{X\}_\text{au}$ for all $Y \in \{X\}_\text{au}$ and all conjugations $\theta$. However, $X^\top \in \{X\}_u$ if and only if $X^\dagger \in \{X\}_\text{au}$.

If $\dim \mathcal{H} = 2$, an orbit $UXU^{-1}$, $U$ unitary, is completely characterized by the numbers

$$\text{Tr} X, \quad \text{Tr} X^2, \quad \text{Tr} X^\dagger X.$$  \hfill (177)

They do not change by substituting $X \rightarrow X^\top$, i.e. by $X \rightarrow \theta X^\dagger \theta$ unitary orbits transform into itself. Hence: For all $X \in B(\mathcal{H}_2)$ one has $X^\top \in \{X\}_\text{au}$ and $\{X\}_u = \{X\}_\text{au}$.

Also the case $\dim \mathcal{H} = 3$ is manageable: The unitary orbits for $\dim \mathcal{H} = 3$ are characterized by seven numbers, by the traces of $X$, $X^2$, $X^3$, $X^\dagger X$, $XX^\dagger X$, $X^\dagger XX^\dagger X^2$, $XX^\dagger XX^\dagger X^2$. It turns out that only the last trace is not invariant against $X \rightarrow X^\top$.

**Proposition 6.2 (Garcia, Poore, Tener)** Assuming $\dim \mathcal{H} = 3$. Then $X^\top \in \{X\}_u$ if and only if

$$\text{Tr} X^\dagger XX^\dagger X^2 = \text{Tr} X^\dagger X^2 X^\dagger XX^\dagger,$$  \hfill (178)
Unitary equivalence of two operators can be expressed by traces in any finite dimension. However, their number increases rapidly with increasing dimension $d$ of $\mathcal{H}$. In the case $\dim \mathcal{H} = 4$ Djokić, \cite{23}, could list 20 trace relations in $X$ and $X^\dagger$ which present a complete description of the unitary orbits.

From them Garcia, Poore, and Tener \cite{32} could extract seven trace relations guarantying $X^\top \in \{X\}_u$ and, hence, $X^\dagger \in \{X\}_{au}$.

Already in 1970 W. R. Gordon \cite{34} could show that $X^\top$ can be expressed for any $X$ in the form $UXV$ by two unitary operators $U$ and $V$. Gordon’s result translates into:

\begin{equation}
\text{Given } X \text{ there are two antiunitaries, } \Theta_1 \text{ and } \Theta_2, \text{ such that }
X^\dagger = \Theta_1 X \Theta_2.
\end{equation}

### 6.4 UET and beyond

Remind that UET, unitary equivalence of a matrix to its transpose, can be expressed as antiunitary equivalence of the matrix to its Hermitian adjoint. The latter relation is basis independent.

New criteria for being UET or UECSM have been developed by S. R. Garcia, J. E. Tener in \cite{31}, by J. E. Tener in \cite{74}, and by L. Balayan, S. R. Garcia in \cite{8}. Following essential ideas of the latter paper, the slightly weaker UET assumption will be considered. The problem will be embedded into a more general one: An antilinear variant of a theorem of the present author \cite{75}, extended and refined by P. M. Alberti, \cite{2}. In connection with applications to QIT, a more recent paper is by A. Chefles, R. Jozsa, A. Winter, \cite{21}.

Here the starting point is a general completely copositive map

\begin{equation}
X \rightarrow T(X) = \sum_{k=1}^{r} \vartheta_k X^\dagger \vartheta_k^\dagger, \quad X \in \mathcal{B}(\mathcal{H}),
\end{equation}

where $\vartheta_j \in \mathcal{B}(\mathcal{H})_{anti}$, i.e., a sum of elementary copositive maps \cite{171}. If there is no representation of $T$ by less than $r$ terms, the number $r$ is called the length of $T$.

In the following

\begin{equation}
\phi_1, \phi_2, \ldots, \phi_d \in \mathcal{H}, \quad d = \dim \mathcal{H}
\end{equation}

denote a set of linear independent unit vectors.

\begin{equation}
\phi'_1, \phi'_2, \ldots, \phi'_d \in \mathcal{H}
\end{equation}

is assumed to be a set of $d$ unit vectors.
Theorem 6.2 The following items are equivalent:

a.) There is a map (180) fulfilling

\[ T(\langle \phi_j | \phi_j \rangle) = |\phi'_j \rangle \langle \phi'_j |, \quad j = 1, \ldots, d. \]  

(183)

b.) There is a positive semi-definite matrix

\[ \{ \beta_{jk} \} \geq 0, \quad \beta_{jj} = 1. \]  

(184)

such that

\[ T(\langle \phi_j | \phi_k \rangle) = \beta_{jk} |\phi'_k \rangle \langle \phi'_j | \]  

(185)

for all \( j, k \in \{1, \ldots, d\} \).

c.) There is a positive semi-definite matrix (184) and a positive semi-definite operator \( K \) such that

\[ \langle \phi_k, K \phi_j \rangle = \beta_{jk} \langle \phi'_j, \phi'_k \rangle. \]  

(186)

for all \( j, k \in \{1, 2, \ldots, d\} \).

Proof: The step b. \( \Rightarrow \) a.) is trivial. (All vectors are unit vectors by assumption, and \( \beta_{jj} = 1 \) necessarily.) Consider now a.) The map (180) is constrained by (183). Hence

\[ T(\langle \phi_j | \phi_j \rangle) = \sum_i \vartheta_i |\vartheta_i \rangle \langle \vartheta_i | = |\vartheta'_j \rangle \langle \vartheta'_j |. \]  

If a sum of positive operators is of rank one, every non-zero term must be proportional to it, i.e. to \( |\vartheta'_j \rangle \langle \vartheta'_j | \). Using (74) one obtains

\[ |\vartheta \phi_j \rangle \langle \vartheta \phi_j | \sim |\vartheta'_j \rangle \langle \vartheta'_j|. \]  

and \( \vartheta_i \phi_j \sim \vartheta'_j \). Hence there are numbers \( \alpha_{ij} \) such that

\[ \vartheta_i \phi_j = \alpha_{ij} \vartheta'_j. \]  

(187)

Inserting in (180) yields

\[ T(|\phi_j \rangle \langle \phi_k |) = \sum_i |\vartheta_i \phi_k \rangle \langle \vartheta_i \phi_j | = \beta_{jk} |\phi'_k \rangle \langle \phi'_j |, \]  

\[ \beta_{jk} = \sum_i \alpha_{ij}^* \alpha_{ik}, \quad \beta_{jj} = 1, \]  

(188)

so that from b.) it follows a.) Coming now to the step b. \( \Rightarrow \) c.) one observes that the right of (185) is the trace of \( T(|\phi_j \rangle \langle \phi_k |) \). Generally one gets from (180), and by the help of the (120) the identities

\[ \text{Tr} \ T(X) = \text{Tr} \ KX, \]  

(189)
\[ K := \sum \vartheta_i^\dagger \vartheta_i. \]  

(190)

This way one finds

\[ \text{Tr} T(|\phi_j\rangle\langle\phi_k|) = \text{Tr} K|\phi_j\rangle\langle\phi_k| = \langle\phi_k, K \phi_j\rangle. \]

so that (c) follows from (b) Starting now from (c) one can use an arbitrary decomposition (188) to define \( r \) antilinear operators \( \vartheta_j \) by (187). Then \( T \), constructed as in (180) with these antilinear operators, satisfies (183) and (185).

The map \( T \) described by the theorem acts as

\[ T : \sum x_{jk}|\phi_j\rangle\langle\phi_k| \mapsto \sum x_{jk}\beta_{jk}|\phi'_k\rangle\langle\phi'_j|. \]

(191)

Of use is the reconstruction of the \( \vartheta_i \) from (187). By

\[ \langle\phi_j, \phi_k\rangle = \delta_{jk} \]

(192)

the vectors \( \tilde{\phi}_j \) are uniquely determined. Together with the vectors \( \phi_k \) they define a bi-orthogonal basis. However, the \( \tilde{\phi}_j \) are not necessarily normalized. Next, for all \( \phi \in \mathcal{H} \),

\[ \vartheta_i\phi = \sum_j \alpha_{ij}\langle\phi, \tilde{\phi}_j\rangle \phi'_j. \]

(193)

Indeed, the left of (193) defines an antilinear operator, and for \( \phi = \phi_k \) one remains with (187). By (70) and (71) one rewrites (193) as

\[ \vartheta_i = \sum_j \alpha_{ij}|\phi'_j\rangle\langle\tilde{\phi}_j|_{\text{anti}}. \]

(194)

As seen from the proof of the theorem:

**Corollary 6.1** The length of a representation (180) satisfying (183) is never less than the rank of \( \{\beta_{jk}\} \). If (188) is an orthogonal decomposition, equality is reached.

**Corollary 6.2** If and only if \( K = 1 \) the map \( T \) of (180) is trace preserving, i.e. a cochannel.

A further observation: From (184) one deduces \( |\beta_{jk}| \leq 1 \). Therefore

\[ |\langle\phi_j, K\phi_k\rangle| \geq |\langle\phi'_j, \phi'_k\rangle|. \]

(195)

Equality holds if and only if all \( \beta_{jk} \) are unimodular. If in addition \( K = 1 \) is required, one gets what is called “weak angle condition” in [8].

50
6.4.1 Length one

It has already be shown that a completely copositive map $T$ is of length one if $T(X) = \varphi X^\dagger \varphi^\dagger$. If $T$ is further trace–preserving, hence a cochannel, it of the form

$$T(X) = \Theta X^\dagger \Theta^\dagger, \quad \Theta^\dagger \Theta = 1,$$

(187) becomes $\Theta \phi_j = \epsilon_j \phi'_j$ for all $j$. Hence

$$\langle \phi_k, \phi_j \rangle = \epsilon_j^\ast \epsilon_k, \quad \langle \phi'_j, \phi'_k \rangle = \epsilon_j \epsilon_k.$$

(197)

Looking at (195) one wonders whether $|\beta_{jk}| = 1$ for all $j, k$ results in $\beta_{jk} = \epsilon_j^\ast \epsilon_k$. At first one proves:

**Lemma 6.3** Assume all matrix elements of the matrix (184) are unimodular. Then every $3 \times 3$ main minor is of rank one.

Because the determinants of the $2 \times 2$ main minors vanish, it suffices to consider

$$\det [\text{main 3x3 minor}] = \epsilon + \epsilon^\ast - 2, \quad \epsilon = \epsilon_{ij} \epsilon_{jk} \epsilon_{ki}.$$  

(198)

These determinants are non-negative if and only if $\epsilon = 1$.

**Proposition 6.3** Assume (184), then the following conditions are mutually equivalent.

a.) The matrix $\{\beta_{jk}\}$ is of rank one.

b.) There are unimodular numbers $\epsilon_j$ such that $\beta_{jk} = \epsilon_j^\ast \epsilon_k$.

c.) It is

$$\beta_{ij} \beta_{jk} \beta_{ki} = 1, \quad \forall i, j, k \in \{1, \ldots, d\}$$

(199)

**Proof:** (a) $\iff$ (b) is trivial. The same is with (b) $\Rightarrow$ (c). It remains to prove (a), (b), from (c). At first, because of $|\beta_{jk}| \leq 1$ it follows $|\beta_{jk}| = 1$ from (c). Hence the preceding lemma implies that every $3 \times 3$ main minor of $\{\beta_{jk}\}$ is of rank one. Being valid for dim $\mathcal{H} = 3$ the proof proceeds by induction. Assume (a), (b), suffices to prove (c) if dim $\mathcal{H} = d$. Asking for dimension $d + 1$, the hypothesis allows to start with

$$\epsilon_{ij} = \epsilon_i^\ast \epsilon_j^*, \quad i, j, k \in \{1, \ldots, d\}$$

and $\epsilon_{1,d+1} = \epsilon_1 \epsilon_{d+1}^*$. Hence

$$\epsilon_{k,d+1} \epsilon_{d+1,1} \epsilon_{1k} = 1, \quad \epsilon_{d+1,1} = \epsilon_{d+1}^\ast \epsilon_1^*, \quad \epsilon_{1k} = \epsilon_1 \epsilon_k^*$$

Inserting this for $1 < k < n + 1$ into (199), one gets $\epsilon_{k(d+1)} = \epsilon_k \epsilon_{d+1}^*$. Hence $T$ in theorem 6.2 is of length larger than one if and only if at least one entry of $\{\beta_{jk}\}$ is not unimodular.

It is interesting to compare proposition 6.3 with the topological geometric phase of M. Berry and B. Simon. (For a survey consult the monographs [20], [10], and [18] See also the comment below.)
Lemma 6.4 Let $K = 1$. If and only if $T$ is of length one it is
\[
\langle \phi_i, \phi_j \rangle \langle \phi_j, \phi_k \rangle = \langle \phi_i', \phi_j' \rangle \langle \phi_j', \phi_k' \rangle
\]
for all $i, j, k \in \{1, \ldots, d\}$, vectors (181) and (182) in theorem 6.2.

For the proof one applies proposition 6.3 to (186) respecting $K = 1$.

Corollary 6.3 Let one of the conditions of lemma 6.3 be valid and $K = 1$. Then every operator $X = \sum x_j |\phi_j\rangle \langle \phi_j|$ is UET, i. e. $\Theta X^\dagger \Theta^\dagger = X$ with $\Theta$ antiunitary.

Indeed, one can find a cochannel $T$ of length one mapping $|\phi_j\rangle \langle \phi_j|$ onto $|\phi_j'\rangle \langle \phi_j'|$ for $j = 1, \ldots, d$. If $T$ is of the assumed form, there is a $\vartheta$ such that $\vartheta X^\dagger \vartheta^\dagger = X$. By trace preserving one gets $\vartheta^\dagger \vartheta = 1$.

The lemma is a variant of Balayan and Garcia’s “Strong Angle Test”, [8].

Some relations become more transparent by introducing the projection operators
\[
P_j = |\phi_j\rangle \langle \phi_j|, \quad Q_j = |\phi_j'\rangle \langle \phi_j'|,
\]
(200), as an example, becomes
\[
\text{Tr} \, P_i P_j P_k = \text{Tr} \, Q_k Q_j Q_i.
\]
One further extend (199) to:
\[
\text{Tr} \, P_{i_1} \cdots P_{i_n} = \text{Tr} \, Q_{i_n} Q_{i_{n-1}} \cdots Q_{i_1}, \quad (202)
\]
\[\{i_1, \ldots, i_n\} \in \{1, \ldots, d\}, \text{ for all } n.\]

To see the trick consider $\text{Tr} \, P_1 P_2 P_3 P_4$. One may convert this expression into
\[
\text{Tr} \, P_1 P_2 P_3 P_4 P_1 = (\text{Tr} \, P_1 P_2 P_3) (\text{Tr} \, P_3 P_4 P_1).
\]
This way one can rewrite any of the numbers in (202) as products of form $\text{Tr} \, P_i P_j P_k$ respectively $\text{Tr} \, Q_i Q_j Q_k$.

Comment. Given a pair of normalized vectors $\phi$ and $\phi'$, the number $\langle \phi, \phi' \rangle$ is often called the transition amplitude for the change $\phi \rightarrow \phi'$. Its absolute square is the transition probability. It is the probability of the transition $|\phi\rangle \langle \phi| \rightarrow |\phi'\rangle \langle \phi'|$ by a von Neumann - Lüders measurement which asks whether the system is either in state $|\phi'\rangle \langle \phi'|$ or in a state orthogonal to it. The symmetry of the transition probability prevents to distinguish Past and Future by the von Neumann-Lüders rule.

The phase factor of the transition amplitude becomes physically important in cyclic processes. For instance, consider an adiabatic process $P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_4 \rightarrow P_1$. Its phase $\gamma$ is defined modulo $2\pi$ by
\[
z = |z| \exp i \gamma, \quad z = \text{Tr} \, P_1 P_2 P_3 P_4
\]
(203)
if \( z \neq 0 \). Otherwise the phase is undefined. By an antiunitary transformation \( P_j \to \Theta P_j \Theta^\dagger \) the phase changes its sign and indicates the change of the processes orientation.

The phase \( \gamma \) in (203) (and in similar constructs) is an instant of the geometric or Berry phase: The pure states are the points of a complex projective space of dimension \( d - 1 \). The latter carries the Study-Fubini metric. Let be \( w \) a closed oriented curve which is the oriented boundary of a 2-dimensional submanifold \( F \). The geometric phase \( \gamma(w) \) can be computed by integrating the Kähler form of the Study-Fubini metric over \( F \).

6.4.2 Examples

1. Solutions of \( X = \Theta X^\dagger \Theta^\dagger \) for some 2-dimensional cases are summarized.  
1a. The case \( \Theta = \tau_0 \) can be read of from lemma 3.2 saying: Exactly all real multiples of unitary operators with determinant one are solutions.  
1b. Let \( \Theta = \tau_1 \). Then \( \sigma_2 \) and \( \sigma_3 \) remain unchanged while \( T(\sigma_1) = -\sigma_1 \), see subsection 2.3. The set of solutions is the real linear space generated by 1, \( \sigma_2 \), and \( \sigma_3 \).  
1c. The border between the two cases above consists of operators \( \vartheta \) with \( \vartheta^2 = 0 \). With a basis \( \phi_1, \phi_2 \) the square of the antilinear operator \( \vartheta = |\phi_1\rangle\langle \phi_2| \) vanishes. By (75)  
\[ \vartheta \vartheta^\dagger = |\phi_1\rangle\langle \phi_1|, \quad \vartheta^\dagger \vartheta = |\phi_2\rangle\langle \phi_2|, \]
so that their sum is the identity map, and  
\[ T(X) = \vartheta X^\dagger \vartheta^\dagger + \vartheta^\dagger X^\dagger \vartheta \]
is a cochannel of length two. \( T \) acts according to  
\[ \sum c_{jk} |\phi_j\rangle \langle \phi_k| \mapsto c_{11} |\phi_2\rangle \langle \phi_2| + c_{22} |\phi_1\rangle \langle \phi_1|. \]  
(204)

Therefore any fixed point is a multiples of \( |\phi_2\rangle \langle \phi_2| + |\phi_1\rangle \langle \phi_1| \).

2. It follows from theorem 6.2 that there exists a representation \( T(X) = \Theta X^\dagger \Theta^\dagger \), \( \Theta \) antiunitary, if and only if  
\[ \Theta \phi_j = \epsilon_j \phi_j', \quad |\epsilon_j| = 1, \quad j = 1, \ldots, d. \]  
(205)

This is equivalent to  
\[ \langle \phi_k, \phi_j \rangle = \epsilon_j^* \epsilon_k \langle \phi_j', \phi_k' \rangle. \]
(206)
Lemma 6.5 Let $\langle 205 \rangle$ be valid. If the unit vectors $\phi_j$ and $\phi'_j$, $j = 1, \ldots, d$, are bi-orthogonal,

$$\langle \phi_j, \phi'_k \rangle = z_j \delta_{jk}, \quad (207)$$

then there is a conjugation $\theta$ such that

$$T(X) = \theta X^\dagger \theta^\dagger. \quad (208)$$

Proof: For a bi-orthogonal system of vectors $\langle 207 \rangle$ one has for all $\phi$

$$\phi = \sum_j \frac{\langle \phi_j, \phi \rangle}{z_j} \phi'_j = \sum_j \frac{\langle \phi'_j, \phi \rangle}{z_j} \phi_j. \quad (209)$$

Now $\langle 205 \rangle$ is transformed by the second relation into

$$\Theta \phi_k = \epsilon_k \phi'_k = \epsilon_k \sum_j \frac{\langle \phi'_j, \phi'_k \rangle}{z_j^*} \phi_j.$$

Using antilinearity and again $\langle 205 \rangle$ one obtains

$$\Theta^2 \phi_k = \sum_j \epsilon_k^* \epsilon_j \frac{\langle \phi'_k, \phi'_j \rangle}{z_j} \phi'_j.$$

Positioning the unimodular numbers within the scalar product and using again (twice) $\langle 205 \rangle$ it follows

$$\Theta^2 \phi_k = \sum_j \frac{\langle \Theta \phi_k, \Theta \phi_j \rangle}{z_j} \phi'_j.$$

Knowing already that $\Theta$ is antiunitary, this means

$$\Theta^2 \phi_k = \sum_j \frac{\langle \phi_j, \phi_k \rangle}{z_j} \phi'_j = \phi_k,$$

the latter equality follows from the first relation in $\langle 209 \rangle$. Because $\Theta^2$ is linear, and the $\phi_k$ span $\mathcal{H}$, one gets $\Theta^2 = 1$.

The essence of the proof is due to Balayan and Garcia [8]. See also lemma 6.2.

3. Also the following lemma is due to Balayan and Garcia, [8].

Lemma 6.6 Let $X$ be a linear operator with non-degenerate spectrum. If $X$ is antiunitarily equivalent to $X^\dagger$ then there is a conjugation $\theta$ such that $X = \theta X^\dagger \theta$. 

54
Proof: (206) is valid. To apply the preceding lemma, the eigenvectors of \(X\) and \(X^\dagger\) should constitute a bi-orthogonal vector system. Indeed, this is the case: Let \(\phi_j, \phi'_k\) denote the eigenvectors and \(\lambda_j, \lambda'_k\) the eigenvalues of \(X\) and of \(X^\dagger\) respectively. Then

\[
\lambda_j \langle \phi'_k, \phi_j \rangle = \langle \phi'_k, X \phi_j \rangle = (\lambda'_k)^* \langle \phi'_k, \phi_j \rangle
\]

and \(\lambda_j \neq \lambda'_k\) for \(j \neq k\) shows the asserted bi-orthogonality.

In case \(X\) is UET, there is an antiunitary \(\Theta\) such that

\[
\Theta X = X^\dagger \Theta, \quad \Theta^\dagger \Theta = 1.
\] (210)

Applying (210) to an eigenvector \(\phi_j\) of \(X\) yields \(\lambda^* \Theta \phi_j = X^\dagger \Theta \phi_j\). Thus \(\Theta \phi_j\) is an eigenvector of \(X^\dagger\) with eigenvalue \(\lambda_j^*\). Because the spectrum is non-degenerate by assumption, \(\Theta \phi_j\) must be proportional to \(\phi'_j\). As \(\Theta\) is isometric, the factor must be unimodular.

On the other hand, (206) is sufficient for the existence of an antiunitary \(\Theta\) satisfying (205) and (210).

### 7 Involutions

An antilinear operator \(S\) is an involution if \(S^2 = 1\). It is a skew involution if \(S^2 = -1\).

These definitions do depend on the linear structure of \(\mathcal{H}\) only and not on its scalar product. On the other hand, the Hilbert structure will be used to polar decompose involutions.

Clearly, the classes of involutions and of skew involutions are larger than those of (skew) conjugations. While the unitary transformations act transitively on the set of conjugations, any two involutions are similar (see below).

#### 7.1 Polar decomposition

Involutions and skew involutions come with characteristic polar decompositions. At first, with \(S\) also \(S^\dagger\) is a (skew) involution. Polar decomposing \(S\) defines an antiunitary operator \(\theta\) such that

\[
S = |S| \theta = \pm \theta^|S|^{-1}, \quad |S| := (SS^\dagger)^{1/2}.
\] (211)

The sign reflects \(S = \pm S^{-1}\). Here and below \(\pm\) means that \(S\) is an involution if \(+\) is valid and a skew involution if \(-\) takes place. From \((S^\dagger S)(SS^\dagger) = 1\) it follows \((S^\dagger S)^{1/2} = |S|^{-1}\) and, by going to the Hermitian adjoint in (211),

\[
|S^\dagger| = (S^\dagger S)^{1/2} = |S|^{-1}.
\] (212)
Proposition 7.1 Assume $S$ is an involution or a skew involution. Then $\theta$ in (211) is a conjugation or a skew conjugations respectively such that
\[ S = |S|\theta = \theta|S|^{-1}, \quad \theta^2 = S^2 = \pm 1, \quad (213) \]
As a consequence one gets
\[ S^\dagger = |S|^{-1}\theta = \theta|S|. \quad (214) \]
For the proof let $\phi_1, \phi_2, \ldots$ be an eigenvector basis of $|S|$ and $\lambda_1, \lambda_2, \ldots$ the corresponding eigenvalues. (211) can be rewritten as
\[ \langle \phi_j, S\phi_k \rangle = \lambda_j \langle \phi_j, \theta \phi_k \rangle = \pm \lambda_k^{-1} \langle \phi_j, \theta^\dagger \phi_k \rangle = \pm \lambda_k^{-1} \langle \phi_k, \theta \phi_j \rangle, \]
and we conclude
\[ \lambda_j \lambda_k \langle \phi_j, \theta \phi_k \rangle = \pm \langle \phi_k, \theta \phi_j \rangle \quad (215) \]
for all $j, k = 1, \ldots, \dim \mathcal{H}$. As all $\lambda_j$ are different from zero, there is the alternative: Either $\lambda_j \lambda_k = 1$ and $\langle \phi_j, \theta \phi_k \rangle$ is multiplied by $\pm 1$ if $j$ and $k$ are exchanged, or both expectation values in (215) vanish. Hence $\langle \phi_j, \theta \phi_k \rangle$ is for all $j, k$ symmetric respectively skew symmetric. Hence $\theta$ is Hermitian respectively skew Hermitian. A (skew) Hermitian antiunitary operator is a (skew) conjugation. Now the assertion has been proved.

There is something more to be said. The spaces
\[ \mathcal{H}[\lambda] := \{ \phi \in \mathcal{H} : |S|\phi = \lambda \phi \} \quad (216) \]
are mutually orthogonal and, by the proof above,
\[ \theta \mathcal{H}[\lambda] = \mathcal{H}[\lambda^{-1}] \quad (217) \]

Theorem 7.1 Let $S = |S|\theta = \theta|S|^{-1}$ be the polar decomposition of either an involution or of a skew involution. If $\lambda$ is an eigenvalue of $|S|$ then so is $\lambda^{-1}$. There is an orthogonal decomposition of $\mathcal{H}$ into subspaces $\mathcal{H}[\lambda]$ the vectors of which are eigenvectors of $|S|$ with eigenvalue $\lambda$. $\theta$ maps $\mathcal{H}[\lambda]$ onto $\mathcal{H}[\lambda^{-1}]$.

If $S$ is either a conjugation or a skew conjugation then $|S| = 1$ and $\mathcal{H}[1] = \mathcal{H}$. If at least one eigenvalue of $|S|$, say $\lambda$, is different from 1, then $\lambda \neq \lambda^{-1}$ and $\lambda + \lambda^{-1} > 2$, i.e. $S$ cannot be a conjugation or a skew conjugation.

In general, the eigenvectors of $|S|$ are grouped in one subspace with eigenvalue 1 and in pairs of subspaces with eigenvalues $\lambda \neq 1$ and $\lambda^{-1}$, This proves

Lemma 7.1 Let $S$ be either an involution or a skew involution. Then
\[ \text{Tr} |S| = \text{Tr} |S|^{-1} \geq \dim \mathcal{H} \quad (218) \]
and equality holds if and only if $S$ is a conjugation or a skew conjugation.

56
7.2 Similarity of involutions

Let $S$ be an involution. Then, like (142),

$$\mathcal{H}_S := \{ \phi \in \mathcal{H} : S \phi = \phi \} \quad (219)$$

is a real linear subspace of $\mathcal{H}$ of maximal dimension. It is generally not a Hilbertian one:

**Lemma 7.2** Let $S$ be an involution. The following items are equivalent.
1. $S$ is a conjugation.
2. $S$ is normal.
3. $\mathcal{H}_S$ is a real Hilbert subspace.
4. $S^\dagger \mathcal{H}_S \subseteq \mathcal{H}_S$.

From (211) one infers $|S| = 1$ if $S$ is normal. Then provides $\theta = \theta^\dagger$ and $S$ must a be conjugation. The inverse statement can easily be seen: An involution is normal if and only if it is a conjugation. Next, with any pair $\phi_1, \phi_2$ of vectors in $\mathcal{H}_S$,

$$\langle \phi_1, \phi_2 \rangle = \langle \phi_1, S \phi_2 \rangle = \langle \phi_2, S^\dagger \phi_1 \rangle.$$

If $S^\dagger \mathcal{H}_S \subseteq \mathcal{H}_S$ then $S^\dagger \mathcal{H}_S = \mathcal{H}_S$ and $S^\dagger \phi_1 = \phi_1$. Consequently, the restriction to $\mathcal{H}_S$ of the scalar product becomes symmetric and, hence, Hilbertian. It also follows $S - S^\dagger = 0$ on $\mathcal{H}_j$. This proves $S = S^\dagger$ on the whole of $\mathcal{H}$.

Of course there are scalar products making $\mathcal{H}_S$ Hilbertian and $S$ a conjugation: Let $\tilde{\phi}_1, \ldots, \tilde{\phi}_d$ be a linear basis of $\mathcal{H}_S$. There is a scalar product associated to it, say $\langle \ldots \rangle^\sim$, defined by

$$\langle \sum a_k \tilde{\phi}_k, \sum b_l \tilde{\phi}_l \rangle^\sim := \sum a_k^* b_l. \quad (220)$$

Equipped with this scalar product $\mathcal{H}_S$ becomes a real Hilbert space. Hence there is an invertible positive operator $A \in \mathcal{B}(\mathcal{H})$ such that

$$\langle \phi', \phi'' \rangle^\sim = \langle \phi', A \phi'' \rangle. \quad (221)$$

Computing the Hermitian adjoint $X^\sim$ of an operator $X$ with respect to $\langle \ldots \rangle^\sim$,

$$\langle \phi', X \phi'' \rangle^\sim = \langle \phi', A X \phi'' \rangle = \langle X^\dagger A \phi', \phi'' \rangle = \langle X \phi', \phi'' \rangle^\sim,$$

yields $X^\dagger A = AX^\sim$. Thus

$$X \Rightarrow X^\sim = A^{-1} X^\dagger A \quad (222)$$

is the Hermitian adjoint coming with the scalar product $\langle \ldots \rangle^\sim$, i.e.

$$\langle \phi', \phi'' \rangle_A := \langle \phi', \phi'' \rangle^\sim = \langle \phi', A \phi'' \rangle. \quad (223)$$

57
In the same manner one proves \((222)\) for antilinear operators \(\vartheta\),

\[ \vartheta \Rightarrow \vartheta^* = A^{-1}\vartheta^\dagger A. \]  

(224)

Looking again at the definition \((220)\), there is an invertible linear operator \(Z\) such that \(Z\phi_j = \tilde{\phi}_j, j = 1, \ldots, d\). The definition of \(Z\) and \((220)\) implies

\[ Z^\dagger Z = A. \]  

(225)

Reminding now \(S\tilde{\phi}_j = \tilde{\phi}_j\) and, defining the conjugation \(\theta\) by \(\theta\phi_j = \phi_j\), one gets \(SZ\phi_j = \tilde{\phi}_j = Z\phi_j\). Hence \(Z^{-1}SZ = \theta\), and \(S\) is similar to \(\theta\). As \(S\) stands for any involution, and because similarity is an equivalence relation, one gets

**Proposition 7.2** Any two involutions are similar. The set of all involutions forms a \(GL(\mathcal{H})\)–orbit. Any two involutions are cosimilar.

To prove cosimilarity, one multiplies both sides of \(S_1 = Z^{-1}S_2Z\) by \(S_1\) and defines \(\vartheta = ZS_1\). It follows \(S_1 = \vartheta^{-1}S_2\vartheta\).

### 7.3 Involutions and the geometric mean

The geometric mean was introduced by W. Pusz and S. L. Woronowicz \[61\]. See also \[18\] and \[7\] for its specification to positive operators.

Many of its properties and various applications are known. In accordance with the aim of the present paper only its connection with Hermitian adjoints, i.e. with certain involutions, is presented.

Given two semi-definite positive Hermitian forms, say \(\langle \cdot, \cdot \rangle_1\) and \(\langle \cdot, \cdot \rangle_2\), on an arbitrary complex-linear spaces \(\mathcal{L}\). The theorem of Pusz and Woronowicz states that within the set of semi-definite Hermitian forms \(\langle \cdot, \cdot \rangle_x\) satisfying

\[ \langle \phi_a, \phi_a \rangle_1 \langle \phi_b, \phi_b \rangle_2 \geq \left| \langle \phi_a, \phi_b \rangle_x \right|^2 \]  

(226)

for all \(\phi_a, \phi_b \in \mathcal{L}\), there is a unique largest one, say \(\langle \cdot, \cdot \rangle_{12}\), such that

\[ \langle \phi, \phi \rangle_{12} \geq \langle \phi, \phi \rangle_x, \quad \forall \phi \in \mathcal{H}. \]  

(227)

In the following we impose two strong restrictions:

1. \(\mathcal{L}\) is a finite dimensional Hilbert space \(\mathcal{H}\) with scalar product \(\langle \cdot, \cdot \rangle\),
2. Only positive definite Hermitian forms are considered.

Using \(\langle \cdot, \cdot \rangle\) as reference, scalar products can be labelled by invertible positive operators, \(A \rightarrow \langle \cdot, \cdot \rangle_A\), according to

\[ \langle \phi_1, \phi_2 \rangle_A := \langle \phi_1, A\phi_2 \rangle, \quad A > 0. \]  

(228)

All possible scalar products in \(\mathcal{H}\) can be gained this way.
Let $\langle \cdot, \cdot \rangle_C$ be the geometric mean of $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_B$. With T. Ando one then writes $C = A\#B$ for the geometric mean of $A$ and $B$.

$$C = A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}, \quad (229)$$

where the right hand side is the unique solution of

$$B = CA^{-1}C, \quad C > 0. \quad (230)$$

For more see [3, 4, 5, 49].

Now let $S_A$ be the Hermitian adjoint belonging to $\langle \phi_1, \phi_2 \rangle_A$. It is an involution acting on the linear space $B(H)$, and it is explicitly given by

$$S_A(X) = A^{-1}X^\dagger A. \quad (231)$$

Indeed, one checks

$$\langle \phi_1, X\phi_2 \rangle_A = \langle X^\dagger A\phi_1, \phi_2 \rangle = \langle (A^{-1}X^\dagger A)\phi_1, A\phi_2 \rangle.$$ 

In conclusion one has the implication

$$A \iff \langle \cdot, \cdot \rangle_A \iff S_A. \quad (232)$$

For two scalar products, indexed by $A$ and $C$ accordingly, one gets

$$S_AS_C(X) = A^{-1}(C^{-1}X^\dagger C)^\dagger A = A^{-1}XC^{-1}A. \quad (233)$$

Now given $A$ and $B$, the next aim is to solve the equation

$$S_AS_C = S_CS_B, \quad (234)$$

for $C$. By the help of (233) one gets

$$C^{-1}AC^{-1}B = X C^{-1}AC^{-1}B. \quad (235)$$

Being valid for all $X$, one obtains

$$C^{-1}AC^{-1}B = \lambda^21, \quad \lambda > 0; \quad (235)$$

Demanding $C > 0$ the solution for $C$ is unique and results in $\lambda C = A\#B$ by (230). $\lambda$ drops out in (234), and it follows

$$S_AS_{A\#B} = S_{A\#B}S_B, \quad (236)$$

Remark: Up to a positive numerical factor, $C$ is uniquely defined by the pair $S_A, S_B$ of involutions. By using the reference Hilbert scalar product, and by restricting the positive operators to those with determinant 1, the constant $\lambda$ can be fixed to $\lambda = 1$. This is consistent because of

$$\det A\#B = \sqrt{\det A \det B}. \quad (237)$$
It can be shown that through any two positive operators $A$ and $B$ there is an \textit{acq-line} $[r \rightarrow S_r]$, satisfying

$$S_tS_{(t+r)/2} = S_{(t+r)/2}S_r, \quad r, t \in \mathbb{R},$$

(238)

and going through $S_A$ for $s = 0$ and through $S_B$ for $s = 1$.

The prove starts by defining $S_q := C_q^{-1}X_q^\dagger C_q$ and

$$C_q = A^{1/2}Y_q^s A^{1/2}, \quad Y := A^{-1/2}BA^{-1/2}.$$  \hfill (239)

One gets

$$S_tS_s = C_t^{-1}(C_s^{-1}X_s^\dagger C_s)^\dagger C_t = C_t^{-1}C_sXC_t^{-1}C_t,$$

which should be equal to $S_sS_r$. Because

$$C_s^{-1}C_t = A^{-1/2}Y^{t-s}A^{1/2},$$

(240)

it suffices to choose $t = (r + s)/2$ to satisfy

$$C_rC_{(t+r)/2} = C_{(t+r)/2}C_t.$$

$A$ is determined by $S_A$ up to a positive number, $C_t$ becomes unique by assuming $\det A = \det B = 1$. Remember also $C_{(t+r)/2} = C_r\#C_t$.

8  Modular objects

In this subsection the most elementary part of the Tomita–Takesaki theory is presented; a theory which has been appreciated as the “second revolution” in the treatment of von Neumann algebras by H. J. Borchers. (The first one is the factor classification by von Neumann and Murray.) It is said that Tomita needed about 10 years to prove the polar decomposition of the modular involution for a general von Neumann algebra.

In contrast, just this task is quite simple for von Neumann algebras acting on a finite dimensional Hilbert space, thereby getting contact to the concept of entanglement. There are some standard notations [36] in the Tomita–Takesaki Theory, which are also used here.

Let $\mathcal{H}$ be a Hilbert space and $\dim \mathcal{H} = d < \infty$. An \textit{operator system} $\mathcal{A}$ on $\mathcal{H}$ is a complex linear subspace of $\mathcal{B}(\mathcal{H})$ which contains the identity operator 1 of $\mathcal{H}$, and which contains with any operator $A$ its Hermitian adjoint $A^\dagger$, [57]. A \textit{von Neumann subalgebra} of $\mathcal{B}(\mathcal{H})$ is an operator system which contains with two operators also their product, i. e. if $A, B \in \mathcal{A}$ then $AB \in \mathcal{A}$.

In the following $\mathcal{A}$ is von Neumann subalgebra.

\footnote{\textit{a line of antilinear conjugate quandles}}
It is $\dim A \leq d^2$. Presently, the main interest is focused on the case $\dim A = d$. Then, as a further input, a vector $\psi$ is chosen from $\mathcal{H}$ such that

$$A\psi = \mathcal{H}, \quad \dim A = \dim \mathcal{H} = d.$$  \hfill (241)

The left hand side is the set of all vectors which can be written as $A\psi$ with $A \in \mathcal{A}$. This set has dimension $d$ by assumption. It follows, as we are within finite dimensional linear spaces, that there is a bijection between the vectors of $\mathcal{H}$ and the operators of $\mathcal{A}$,

$$\varphi \longleftrightarrow A : \quad \varphi = A\psi, \quad A \in \mathcal{A}.$$  \hfill (242)

**Notational remark:** Generally, a vector $\psi$ is called *cyclic* with respect to $\mathcal{A}$ if $\mathcal{A}\psi = \mathcal{H}$. $\psi$ is called *separating* if $A\psi = 0$ and $A \in \mathcal{A}$ implies $A = 0$. Hence (242) is true if $\psi$ is cyclic and separating. In QIT one says instead: $\psi$ is completely entangled. This roots in the fact that there is a unique positive linear form $\omega$, $\omega(A) = \text{Tr} DA$, with $D \in \mathcal{A}$ positive and invertible, such that

$$\langle \psi, A\psi \rangle = \text{Tr} DA := \omega(A)$$

for all $A \in \mathcal{A}$. (242) is an elementary example of a *Gelfand isomorphism* within a *GNS construction*. The letters GNS are the initials of Gelfand, Ne’imark, and Segal.

The correspondence between vectors and operators, seen in (242), is used to define an involution $S_\psi$ by

$$S_\psi A\psi = A^\dagger \psi, \quad A \in \mathcal{A}.$$  \hfill (243)

Because by (242) $S_\psi$ is a well defined antilinear operator. It clearly obeys $S_\psi^2 = 1$. Thus $S_\psi$ is called the *modular involution based on $\psi$*.

Now the polar decomposition of an involution comes into play, and in this connection some Tomita–Takesaki terminology will be introduced: The polar decomposition of $S_\psi$ provides a conjugation $J_\psi$, the *modular conjugation*, and a positive operator $\Delta_\psi$, called the *modular operator*. Their definitions start with

$$\Delta_\psi := S_\psi^\dagger S_\psi, \quad \Delta^{-1}_\psi = S_\psi S_\psi^\dagger.$$  \hfill (244)

The second equation appears from rewriting (212). The polar decomposition (213) reads in the present setting

$$S_\psi = J_\psi \Delta_{\psi}^{1/2} = \Delta^{-1/2}_\psi J_\psi, \quad S_\psi^\dagger = \Delta^{1/2}_\psi J_\psi.$$  \hfill (245)

One observes $J \Delta^{1/2} J = \Delta^{-1/2}$ so that

$$\Delta_t^t J_\psi = J_\psi \Delta_t^{-t}, \quad t \in \mathbb{R}.$$  \hfill (246)
Remark: It is in use to write $F_\psi$ instead of $S_\psi^\dagger$, see (249) below.

The set of operators $B$, commuting with all $A \in A$, is denoted by $A'$. $A'$ is the commutant of $A$. The commutant is a von Neumann subalgebra of $B(H)$.

**Proposition 8.1** Let $A$ be a von Neumann subalgebra of $B(H)$ and assume (241) and (242) are valid. Then

$$A \in A \Rightarrow S_\psi AS_\psi \in A', \quad (247)$$

and this is a one-to-one map from $A$ onto $A'$.

For the proof let us write $S$ for $S_\psi$ to simplify the notation. Because of (242) the assertion (247) can be written

$$SA_1SA_2A\psi = A_2SA_1A\psi, \quad \text{for all } A_1, A_2, A \in A \in A.$$

The left side of the relation becomes $SA_1A_1^\dagger A_2^\dagger \psi = A_2AA_1^\dagger \psi$ by repeated application of (243). The right is handled similar: $A_2SA_1A_1^\dagger \psi = A_2AA_1^\dagger \psi$, and (244) becomes evident. It implies that $A'$ is at least $d$-dimensional. However, if $\dim A' > d$, there is an operator $B \in A'$ such that $B\psi = 0$ implying $BA\psi = 0$ for all $A \in A$. By (241), (242), all vectors from $H$ are null vectors of $B$, hence $B = 0$, and the assertion is proved.

The proof shows that $A$ and $A'$ are isomorphic algebras: $A \to S_\psi AS_\psi$ is an isomorphism. Both algebras can be handled completely similar with respect to the chosen vector $\psi$. In particular, there is an involution $F_\psi$ such that

$$F_\psi B\psi = B^\dagger \psi, \quad B \in A'. \quad (248)$$

The same arguments as above show that $B \to F_\psi BF_\psi$ is an isomorphism from $A'$ onto $A$. The next task is the proof of

$$F_\psi^\dagger = S_\psi, \quad S_\psi^\dagger = F_\psi. \quad (249)$$

Let $\psi_1, \psi_2 \in H$. There are $A \in A$ and $B \in A'$ such that $\psi_1 = A\psi_0$ and $\psi_2 = B\psi_0$. It follows $AB = BA$ and

$$\langle \psi_2, \psi_1 \rangle = \langle B\psi_0, A\psi_0 \rangle = \langle A^\dagger \psi_0, B^\dagger \psi_0 \rangle = \langle SA\psi_0, FB\psi_0 \rangle,$$

and by (58) the last term is $\langle B\psi_0, F^\dagger SA\psi_0 \rangle$ or $\langle \psi_2, F^\dagger S\psi_1 \rangle$. Being true for all $\psi_1, \psi_2 \in H$, we must have $F^\dagger S = 1$ and, because $S^2 = 1$, the first equation in (247) is true. The second follows by taking the Hermitian adjoint.

Knowing (247), from $SAS \in A'$ one gets $S_\psi SASS_\psi^\dagger = A$. Now (244) provides $\Delta A\Delta^{-1} \in A$ for all $A \in A$. A similar conclusion can be drawn for $A'$. All together, by functional calculus, one gets, as $\Delta$ is strictly positive,
**Proposition 8.2** Let $A \in \mathcal{A}$ and $B \in \mathcal{A}'$. Then
\[ \Delta_{\psi}^t A \Delta_{\psi}^{-t} \in \mathcal{A}, \quad \Delta_{\psi}^t B \Delta_{\psi}^{-t} \in \mathcal{A}' \] (250)
for all real numbers $t$. The 1-parameter unitary group $t \to \Delta_{\psi}^t$, $s \in \mathbb{R}$, is the so-called modular automorphism group.

Combining (250) with (247) one gets
\[ J \psi A J \psi = A', \quad J \psi A' J \psi = A, \] (251)
saying that $JAJ \in \mathcal{A}'$ if and only if $A \in \mathcal{A}$ and vice versa. (251) is more comfortable than (247) as $J$ is a conjugation, $S$ “only” an involution.

### 8.1 Maximal commutative subalgebras

Simple though illustrating is an application of the formalism to a maximal commutative von Neumann subalgebra $\mathcal{C}$ of $\mathcal{B}(\mathcal{H})$. There is a basis $\phi_1, \ldots, \phi_d$ of $\mathcal{H}$, that simultaneously diagonalize all operators of $\mathcal{C}$. The operator $S_{\psi}$ can be based on a vector of the form
\[ \psi = \sum \epsilon_j \phi_j, \quad |\epsilon_j| = 1. \] (252)

Let $a_1, \ldots, a_d$ be the eigenvalues of $A \in \mathcal{C}$. Then (243) becomes
\[ S_{\psi} A \psi = \sum a_j^* \epsilon_j \phi_j. \] (253)

In particular $S_{\psi} \phi_k = \epsilon_k \phi_k$ for all $k$. Applying rule (58) results in

**Proposition 8.3** For a maximal commuting von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$
\[ S_{\psi} = S_{\psi}^* = J_{\psi}, \quad \Delta_{\psi} = 1 \] (254)
is valid and (252) implies (253). The modular conjugations are parameterized by the points of a $d$-Torus with points $\{\epsilon_1, \ldots, \epsilon_d\}$. Given $J_{\psi'}$ and $J_{\psi''}$, there are solutions $S_{\psi}$ of
\[ S_{\psi'} S_{\psi} = S_{\psi} S_{\psi''}. \] (255)

Representing all unimodular number in the form $\exp(it)$, (255) is equivalent to
\[ s_j = \frac{s_j' + s_j''}{2} \mod \pi, \quad j = 1, \ldots, d. \] (256)

The last assertion comes from rewriting (255) as a set of the $d$ conditions $\epsilon_j' = (\epsilon_j)^2 \epsilon_j''$. 

63
8.2 Bipartite quantum systems

To see the meaning of the modular objects in bipartite quantum systems,

\[ \mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B, \quad \dim \mathcal{H}^A = \dim \mathcal{H}^B = d, \]  

(257)

is the starting assumption. The decomposition provides two von Neumann subalgebras,

\[ \mathcal{A} := \mathcal{B}(\mathcal{H}^A) \otimes \mathcal{1}^B, \quad \mathcal{A}' = \mathcal{1}^A \otimes \mathcal{B}(\mathcal{H}^B) \]  

(258)

of \( \mathcal{B}(\mathcal{H}^{AB}) \). \( \mathcal{A}' \) is the commutant of \( \mathcal{A} \). \( \mathcal{A} \) is the commutant of \( \mathcal{A}' \). A vector \( \psi \), allowing for a correspondence in the sense of (242), must be separating, i.e. \( A \in \mathcal{A} \) and \( A\psi = 0 \) only if \( A = 0 \). Because of (257) it follows \( \dim A\psi = d^2 \) and \( \psi \) is cyclic. In Quantum Information Theory one would call \( \psi \) completely entangled, i.e. the Schmidt number of \( \psi \) is \( d = \dim \mathcal{H}^A \).

Let \( \psi \in \mathcal{H}^{AB} \) denote a completely entangled vector. Then there are bases \( \phi^A_1, \phi^A_2, \ldots \) of \( \mathcal{H}^A \) and \( \phi^B_1, \phi^B_2, \ldots \) of \( \mathcal{H}^B \) such that

\[ \psi = \sum_{j=1}^d \sqrt{p_j} \phi^A_j \otimes \phi^B_j, \quad p_j > 0, \]  

(259)

is a Schmidt decomposition of \( \psi \). The reduced operators attached to \( \psi \) are the partial traces of \( |\psi\rangle\langle\psi| \) onto the two subsystems. They are

\[ \rho^A = \sum_j p_j |\phi^A_j\rangle\langle\phi^A_j|, \quad \rho^B = \sum_j p_j |\phi^B_j\rangle\langle\phi^B_j|, \]  

(260)

see [55], [10]. In case \( \psi \) is a unit vector then the partial traces are of trace one and allow for an interpretation as density operators.

To determine \( S_\psi \), the modular involution, it suffices to know its action on a basis of \( \mathcal{H}^A \otimes \mathcal{1}^B \). A good choice reads \( |\phi^A_j\rangle\langle\phi^A_k| \otimes \mathcal{1}^B \). Then the problem reduces to

\[ S_\psi (|\phi^A_j\rangle\langle\phi^A_k| \otimes \mathcal{1}^B) \psi = |\phi^A_k\rangle\langle\phi^A_j| \otimes \mathcal{1}^B \psi. \]

Performing the calculation results in

\[ S_\psi \sqrt{p_j} \phi^A_j \otimes \phi^B_k = \sqrt{p_k} \phi^A_k \otimes \phi^B_j. \]  

(261)

\( S_\psi \) is the unique antilinear operator satisfying (261). Abbreviation \( |jk\rangle := \phi^A_j \otimes \phi^B_k \), one gets

\[ S_\psi |jk\rangle = \sqrt{\frac{p_j}{p_k}} |kj\rangle, \quad S_\psi^\dagger |jk\rangle = \sqrt{\frac{p_k}{p_j}} |kj\rangle. \]  

(262)

It is now evident how to polar decompose \( S_\psi \) as in (244), (245). Having in mind that \( \Delta_\psi = S_\psi^\dagger S_\psi \) is linear and \( J_\psi \) antilinear, it follows

\[ \Delta_\psi |jk\rangle = \frac{p_j}{p_k} |jk\rangle, \quad J_\psi |jk\rangle = |kj\rangle, \]  

(263)

64
From these relations one gets evidence of
\[ \Delta \psi = \rho^A \otimes (\rho^B)^{-1}. \] (264)
Again from (262) one can construct a basis of \( \mathcal{H}_\psi \).
\[ \sqrt{p_k} | j \rangle k + \sqrt{p_j} | k \rangle j, \quad i \sqrt{p_k} | j \rangle k - i \sqrt{p_j} | k \rangle j, \quad \forall j \leq k. \] (265)
There are links to the geometric mean, see 7.3.

**Proposition 8.4** Let \( S_k = J \Delta_k \) be defined according to (245). There are modular objects fulfilling
\[ S_1 S = S S_2. \] (266)
The modular operator \( \Delta \) of \( S \) satisfies
\[ \Delta \Delta -^{1/2} \Delta = \Delta \Delta_1 \Delta^{-1/2} \Delta, \] (267)
The geometric mean \( \Delta \) of \( \Delta_1, \Delta_2 \), is uniquely defined by (266).

Proof: Multiplying (266) from the left by \( J \), one obtains \( \Delta \Delta_2 = \Delta J \Delta_1 J \Delta = \Delta \Delta_1 \Delta^{-1} \Delta \), where \( J \Delta_1 J = \Delta_1^{-1} \Delta \) has been used. To compare with (236) one rewrites the first expression in (267) as \( (\Delta_1 ^{-1/2} \Delta \Delta_1 ^{-1/2})^2 = \Delta_1 ^{-1/2} \Delta_2 \Delta_1 ^{-1/2} \) and solves for \( \Delta \):
\[ \Delta = \Delta_1 ^{1/2} (\Delta_1 ^{-1/2} \Delta_2 \Delta_1 ^{-1/2})^{1/2} \Delta_1 ^{1/2} \] (268)

A useful observation says that \( \psi \) is maximally entangled if and only if \( \Delta \psi \) is proportional to \( \mathbf{1}^{AB} \). Then \( J \psi = S \psi \) and all Schmidt numbers are mutually equal. All maximal entangled vectors can be gained by applying unitary operators of the product form \( U^A \otimes U^B \) to a chosen maximally entangled \( \psi \). In fact it suffices to apply a unitary of the form \( U^A \otimes \mathbf{1}^B \).

The set of all \( S \psi \) satisfying \( S \psi = J \psi \) is a submanifold of the symplectic space of all conjugations contained in \( B\mathcal{H}^{AB} \_{anti} \).

Question: Is this submanifold symplectic?
Question: Is there for any pair \( \psi', \psi'' \) of maximally entangled vectors at least one further maximally entangled vector \( \psi \) such that
\[ J \psi' J \psi = J \psi J \psi'' \] (269)
can be fulfilled ?

Up to my knowledge the answers to these question are unknown.

By (213) or, equivalently, (245) one shows
\[ S_\psi (\Delta^t \mathcal{H}_\psi) = \Delta^{-t-1/2} \mathcal{H}_\psi, \quad S_\psi^\dagger (\Delta^t \mathcal{H}_\psi) = \Delta^{t+1/2} \mathcal{H}_\psi. \] (270)
Notice that \( \mathcal{H}_\psi = \mathcal{H}_S \) if \( S = S_\psi \).

For real \( t \) the linear spaces \( \Delta^t \mathcal{H}_\psi \) are real too. A bases for them can be generated by applying \( \Delta^t \) to the basis (265) of \( \mathcal{H}_J \), so that, applying definition (219) to the involutions \( S_\psi \) and \( F_\psi = S_\psi^\dagger \),
\[ \mathcal{H}_{S_\psi} = \Delta^{-1/4}_\psi \mathcal{H}_J, \quad \mathcal{H}_{F_\psi} = \Delta^{1/4}_\psi \mathcal{H}_J. \] (271)
9 Antilinearity and the Einstein-Podolski-Rosen Effect

This section aims at a particular aspect of the Einstein-Podolski-Rosen or “EPR” effect [23]: The appearance of antilinearity.

While the paper of Einstein et al. generated some wild philosophical discussion, Schrödinger [66], [67], introduced the concept of entanglement. For a general overview one may consult [58], [55], or any QIT textbook.

Far from being complete in any sense, the present paper introduces to some antilinear facets of the EPR effect. The idea, to look at these particular antilinearities, is already in [18], more elaborated, in [76], [77], [19], and, more recently, in [13]. Related in spirit is [9].

To begin with, it needs some mathematical and notational preliminaries. As in (257) the basic structure is a bipartite quantum system

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B,$$

however without requiring equal dimensionality of the factors. In the language of QIT: The subsystem given by $\mathcal{H}_A$ is in the hands of Alice, who is responsible for the actions on the $A$–system. In the same spirit Bob is the owner of the $B$–system. The two subsystems may or may be not spatially separated.

Fixing $\psi \in \mathcal{H}_{AB}$ there are Schmidt decompositions, (as in (259) but normalized,)

$$\psi = \sum_{j=1}^{k} \sqrt{p_j} \phi_j^A \otimes \phi_j^B, \quad p_j > 0, \quad \sum p_j = 1 .$$

$k$ is the Schmidt rank of $\psi$.

$$\rho^A := \text{Tr}_B |\psi\rangle\langle \psi| = \sum_{j=1}^{k} p_j |\phi_j^A\rangle\langle \phi_j^A|$$

is the partial trace over the $B$–system. $\rho^A$ encodes “what can be seen” of $\psi$ from Alice’s system. By Exchanging $A$ with $B$ one gets the partial trace over the $A$–system.

A further convention is as follows: $X^{ba}$ indicates a map from $\mathcal{H}_A$ into $\mathcal{H}_B$. Similarly, the superscript “$ab$” in $X^{ab}$ points to a map from $\mathcal{H}_B$ into $\mathcal{H}_A$. This contrasts the notion, say $X^{AB}$, for an operator acting on $\mathcal{H}_{AB}$. See also subsection 3.4.

The EPR phenomenon says: An action done by (say) Alice generally influences Bob’s system. This is the reason for saying “the state of the composed system entangles the two subsystems”.

66
As previously we assume the bipartite quantum system in a state described by the unit vector \( \psi \in \mathcal{H}^{AB} \). Alice asks whether her state is \( |\phi^A\rangle \langle \phi^A| \) or not. (The general case of a mixed state will be described in subsection 9.3.)

In the composed system Alice’s question can be seen as a Lüders’ measurement \[50\] using the projection operator \( P = |\phi^A\rangle \langle \phi^A| \otimes 1^B \) and its orthogonal complement \( P^\perp = 1^{AB} - P \). If Alice gets the answer “yes”, then the vector \( \psi' = P\psi \) and the state \( w^{-1}|\psi'\rangle \langle \psi'| \) is prepared. Here \( w = \langle \psi, P\psi \rangle \) is Alice’s success probability for getting the answer YES. Now \( P\psi = (|\phi^A\rangle \langle \phi^A| \otimes 1^B)\psi \),

and in the composed system the vector \( \psi \) is transformed into a product vector of the form \( \phi^A \otimes \phi^B \). Clearly, \( \phi^B \) depends on \( \phi^A \) and \( \psi \) only. Fixing \( \psi \) but varying \( \phi^A \) defines a map \( s_{\psi}^{ba} \) from \( \mathcal{H}^A \) into \( \mathcal{H}^B \),

\[
(|\phi^A\rangle \langle \phi^A| \otimes 1^B)\psi = \phi^A \otimes s_{\psi}^{ba}\phi^A, \quad \phi^A \in \mathcal{H}^A .
\]  

(272)

Thus, if Alice is successful in preparing \( \phi^A \), then \( \phi^B_{\text{out}} := s_{\psi}^{ba}\phi^A \) is prepared in Bob’s system.

Of course one can exchange the roles of Alice and Bob: There is a map \( s_{\psi}^{ab} \) showing the state in Alice’s system caused by a successfully preparing \( \phi^B \) by Bob.

\[
(1^A \otimes |\phi^B\rangle \langle \phi^B|)\psi = s_{\psi}^{ab}\phi^B \otimes \phi^B, \quad \phi^B \in \mathcal{H}^B .
\]  

(273)

**Proposition 9.1** \( s_{\psi}^{ba} \) and \( s_{\psi}^{ab} \) are antilinear maps from \( \mathcal{H}^A \) into \( \mathcal{H}^B \) and from \( \mathcal{H}^B \) into \( \mathcal{H}^A \) respectively. They depend linearly on \( \psi \). If \( \psi \) is represented by

\[
\psi = \sum c_{jk} \phi^A_j \otimes \phi^B_k
\]  

(274)

then

\[
s_{\psi}^{ba}\phi^A = \sum c_{jk} \langle \phi^A_j, \phi^A_k \rangle \phi^B_k, \quad s_{\psi}^{ab}\phi^B = \sum c_{jk} \langle \phi^A_j, \phi^B_k \rangle \phi^A_j .
\]  

(275)

Proof: It is immediate from (272) that \( \psi \rightarrow s_{\psi}^{ba} \) is linear in \( \psi \) and maps \( \mathcal{H}^A \) antilinearly into \( \mathcal{H}^B \), i.e. it is contained in \( \mathcal{H}^{AB} \). The map realizes the mapping \[108\] advertised in subsection 3.4 (Though without demanding equality of the subsystem’s dimensions.) By linearity in \( \psi \) it suffices to show the equivalence of (272) and (275) if \( \psi = \tilde{\phi}^A \otimes \tilde{\phi}^B \) is a product vector. (275) becomes

\[
s_{\psi}^{ba}\phi^A = \langle \phi^A, \tilde{\phi}^A \rangle \phi^B ,
\]  

67
which is consistent with (272). Exchanging the roles of Alice and Bob in
the preceding discussion gives the other part of the assertion. Also compare
with (105).

As already indicated in subsection 3.4, the maps (275) are partial isome-
tries: Rewriting (109) results in
\[
\langle \psi, \phi \rangle = \text{Tr}_{s_{\psi}^{ab}} (s_{\psi}^{ab})^{\dagger} = \text{Tr}_{s_{\psi}^{ba}} (s_{\psi}^{ba})^{\dagger}.
\] (276)
The ordering within the traces is due to the antilinearity of the maps in-
volved.

An important relation reads
\[
\langle \phi^{A} \otimes \phi^{B}, \psi \rangle = \langle \phi^{A}, s_{\psi}^{ab} \phi^{B} \rangle = \langle \phi^{B}, s_{\psi}^{ba} \phi^{A} \rangle.
\] (277)
To prove the first equality, one calculates by (272) the transition amplitudes
\[
\langle \phi^{A}, \phi^{A} \rangle \langle \phi^{A}, s_{\psi}^{ab} \phi^{B} \rangle = \langle \phi^{A} \otimes \phi^{B}, (|\phi^{A}\rangle \langle \phi^{A}| \otimes 1^{B}) \psi \rangle = \langle \phi^{A}, \phi^{A} \rangle \langle \phi^{A} \otimes \phi^{B}, \psi \rangle
\]
saying the first equality in (277) is true. The other one is seen by exchanging
the roles of Alice and Bob.

A consequence of the proposition is
\[
(s_{\psi}^{ba})^{\dagger} = s_{\psi}^{ab}, \quad (s_{\psi}^{ab})^{\dagger} = s_{\psi}^{ba}.
\] (278)
Another one is the reconstruction of \(\psi\) from \(s_{\psi}^{ba}^{\dagger}\):

**Proposition 9.2** Let
\[
\sum_{j} |\phi^{A}_{j}\rangle \langle \phi^{A}_{j}| = 1^{A}, \quad \sum_{k} |\phi^{B}_{k}\rangle \langle \phi^{B}_{k}| = 1^{B}
\] (279)
be decompositions of the unity of \(\mathcal{H}^{A}\) and \(\mathcal{H}^{A}\) respectively. Then
\[
\psi = \sum_{j} \phi^{A}_{j} \otimes s_{\psi}^{ba} \phi^{A}_{j} = \sum_{k} s_{\psi}^{ab} \phi^{B}_{k} \otimes \phi^{B}_{k}.
\] (280)
To see it, one starts with
\[
\psi = \sum \langle \phi^{A}_{j} \otimes \phi^{B}_{k}, \psi \rangle \phi^{A}_{j} \otimes \phi^{B}_{k}
\]
and applies (277) to get, for instance,
\[
\psi = \sum \langle \phi^{A}_{j}, s_{\psi}^{ab} \phi^{B}_{k} \rangle \phi^{A}_{j} \otimes \phi^{B}_{k}.
\]
Summing up over \(j\) yields \(\sum s_{\psi}^{ab} \phi^{B}_{k} \otimes \phi^{B}_{k}\). The other case is similar.

By the help of (105) one may rewrite (272) and (273) by
\[
s_{\psi}^{ba} = \sum c_{jk} |\phi^{B}_{j}\rangle \langle \phi^{A}_{j}|_{\text{anti}} \quad \text{if} \quad \psi = \sum c_{jk} \phi^{A}_{j} \otimes \phi^{B}_{k}.
\] (281)
If $\psi$ is in the Schmidt form

$$\psi = \sum \sqrt{p_j} \phi_j^A \otimes \phi_j^B,$$  \quad (282)

a simplification occurs and one gets

$$s_{\psi}^{ba} \phi_j^A = \sqrt{p_j} \phi_j^B, \quad s_{\psi}^{ab} \phi_j^B = \sqrt{p_j} \phi_j^A.  \quad (283)$$

From it one easily sees

$$s_{\psi}^{ab} s_{\psi}^{ba} \phi_j^A = p_j \phi_j^A, \quad s_{\psi}^{ba} s_{\psi}^{ab} \phi_j^B = p_j \phi_j^B,$$

in agreement with (257).

Assuming $\phi^A$ is a unit vector, the probability that Alice is successfully preparing $\phi^A$ is equal to $\langle \phi^A, \rho^A \phi^A \rangle$. On Bob’s system the transition $\rho^B \rightarrow \phi^B_{\text{out}} = s_{\psi}^{ba} \phi^A$ occurs with the same probability. Indeed,

$$\langle \phi^A, \rho^A \phi^A \rangle = \langle \phi^A, (s_{\psi}^{ba})^\dagger s_{\psi}^{ba} \phi^A \rangle = \langle \phi^B_{\text{out}}, \phi^B_{\text{out}} \rangle.$$

(285)

Exchanging the roles of Alice and Bob and setting $\phi^A_{\text{out}} = s_{\psi}^{ab} \phi^B$ one gets

$$\langle \phi^B, \rho^B \phi^B \rangle = \langle \phi^A_{\text{out}}, \phi^A_{\text{out}} \rangle.$$

Finally, the change by applying a general local transformation $\psi \rightarrow \varphi := (X^A \otimes X^B) \psi$ is described by

$$s_{\varphi}^{ab} = X^A s_{\psi}^{ab} (X^B)^\dagger, \quad s_{\varphi}^{ba} = X^B s_{\psi}^{ba} (X^A)^\dagger.$$  \quad (287)

9.1 Polar decomposition

Given $\psi$, let us denote the supporting subspaces of $\rho^A$ and $\rho^B$ in $H^A$ and in $H^B$ by $H^A_\psi$ and $H^B_\psi$ respectively. The dimensions of the supporting subspaces coincides and they are equal to the Schmidt rank of $\psi$,

$$\text{Schmidt rank}[\psi] = \text{rank}[\rho^A] = \text{rank}[\rho^B].$$  \quad (288)

It follows from (284) that $s_{\psi}^{ba}$ maps $H^A_\psi$ onto $H^B_\psi$. The vectors $\phi' \in H^A$ which are orthogonal to $H^A_\psi$ are annihilated: $s_{\psi}^{ba} \phi' = 0$. Similar statements are true for $s_{\psi}^{ab}$. 

69
Theorem 9.1 (Polar decomposition of the \(s\)-maps) There are antilinear partial isometries \(j_{\psi}^{ba}\) and \(j_{\psi}^{ab}\) fulfilling
\[
j_{\psi}^{ba} = (j_{\psi}^{ab})^\dagger, \quad j_{\psi}^{ab} = (j_{\psi}^{ba})^\dagger
\] (289)
and
\[
(j_{\psi}^{ba})^\dagger j_{\psi}^{ba} = P_{\psi}^A, \quad (j_{\psi}^{ab})^\dagger j_{\psi}^{ab} = P_{\psi}^B
\] (290)
where \(P_{\psi}^A\), respectively \(P_{\psi}^B\), is the projection operator onto \(\mathcal{H}_{\psi}^A\) respectively onto \(\mathcal{H}_{\psi}^B\), such that
\[
s_{\psi}^{ba} = j_{\psi}^{ba} \sqrt{\rho_{\psi}^A} = \sqrt{\rho_{\psi}^B j_{\psi}^{ba}}.
\] (291)

Proof: Let \(m\) be the Schmidt rank of the unit vector \(\psi\), and assume \(\psi\) in a Schmidt form (282) with \(m\) terms. The antilinear operators defined by
\[
j_{\psi}^{ba} \phi_j^A := \sum_j \langle \phi_j^A, \phi_j^A \rangle \phi_j^A, \quad j_{\psi}^{ab} \phi_j^B := \sum_j \langle \phi_j^B, \phi_j^B \rangle \phi_j^B
\] (292)
satisfy (289). Hence
\[
(j_{\psi}^{ba})^\dagger j_{\psi}^{ba} = j_{\psi}^{ab} j_{\psi}^{ab} = P_{\psi}^A, \quad (j_{\psi}^{ab})^\dagger j_{\psi}^{ab} = j_{\psi}^{ba} j_{\psi}^{ba} = P_{\psi}^B.
\]
By (292) it becomes obvious for \(1 \leq j \leq m\) that
\[
j_{\psi}^{ba} \phi_j^A = \phi_j^B, \quad j_{\psi}^{ab} \phi_j^B = \phi_j^A,
\] (293)
while \(j_{\psi}^{ab}\) annihilates the orthogonal complement of the supporting space of \(\rho_{\psi}^B\). \(j_{\psi}^{ba}\) behaves similarly. Now (283) is used to show
\[
s_{\psi}^{ba} \phi_j^A = j_{\psi}^{ba} \sqrt{\rho_{\psi}^A \phi_j^A} = j_{\psi}^{ba} \sqrt{\rho_{\psi}^B j_{\psi}^{ba} \phi_j^A}
\]
for bases which serve to Schmidt compose \(\psi\). Both sides of both equations uniquely define antilinear operators, and
\[
s_{\psi}^{ba} = j_{\psi}^{ba} \sqrt{\rho_{\psi}^A} = \sqrt{\rho_{\psi}^B j_{\psi}^{ba}}
\] has been established.

Taking the Hermitian adjoint one obtains
\[
s_{\psi}^{ab} = j_{\psi}^{ab} \sqrt{\rho_{\psi}^B} = j_{\psi}^{ab} \sqrt{\rho_{\psi}^A j_{\psi}^{ab}}.
\] (294)
Taking in account the support property (290), one obtains
\[
j_{\psi}^{ab} \sqrt{\rho_{\psi}^B j_{\psi}^{ba}} = j_{\psi}^{ab} \sqrt{\rho_{\psi}^A \rho_{\psi}^A j_{\psi}^{ab}} = \sqrt{\rho_{\psi}^B j_{\psi}^{ba}}
\] (295)

Let \(f(x)\) be real and continuous on \(0 \leq x\). It follows
\[
j_{\psi}^{ab} f(\rho_{\psi}^B) j_{\psi}^{ba} = \rho_{\psi}^A, \quad j_{\psi}^{ab} f(\rho_{\psi}^A) j_{\psi}^{ba} = \rho_{\psi}^B.
\] (296)
by functional calculus. In doing so, the 0-th powers of \(\rho_{\psi}^A\) and of \(\rho_{\psi}^B\) should be set to \(P_{\psi}^A\) and to \(P_{\psi}^B\) respectively.
9.2 Representing modular objects

In this subsection $H^A$ and $H^B$ are supposed to be of equal dimensions, and $\psi \in H^{AB}$ to be completely entangled, i.e. of maximal Schmidt rank,

$$\dim H^A = \dim H^B = d, \quad (\rho^A)^{-1}, (\rho^B)^{-1} \text{ do exist.}$$  \hfill (297)

Then $P^A \psi = 1^A$ and $P^B \psi = 1^B$, and $j^{ba}_\psi, j^{ab}_\psi$ become

$$j^{ba}_\psi^\dagger = j^{ab}_\psi = (j^{ba}_\psi)^{-1}, \quad (j^{ab}_\psi)^\dagger = j^{ba}_\psi = (j^{ab}_\psi)^{-1}.$$  \hfill (298)

As it turns out, the “modular objects” considered in subsection 9.1 of section 9 can be represented by the maps $s^{ba}_\psi, s^{ab}_\psi, j^{ba}_\psi, j^{ab}_\psi$. This will be shown in the next but next part. In the next one twisted direct products of antilinear maps will be introduced.

9.2.1 Twisted direct products

The direct product $\vartheta' \otimes \vartheta''$ of two antilinear operators or maps is well defined. It is antilinear and it is acting on product vectors as

$$(\vartheta' \otimes \vartheta'') \varphi' \otimes \varphi'' = (\vartheta' \varphi') \otimes (\vartheta'' \varphi'').$$

The main difference to the direct product of two linear operators is in the rule

$$c(\vartheta' \otimes \vartheta'') = (c\vartheta') \otimes \vartheta'' = (\vartheta' c^*) \otimes \vartheta'' = \vartheta' \otimes (c\vartheta'') = \vartheta' \otimes (\vartheta'' c^*) = (\vartheta' \otimes \vartheta'') c^*$$

Notice: There is no mathematical consistent direct product of a linear and an antilinear operator within the category of complex linear spaces.

The twisted direct product will be denoted by a “twisted cross” $\tilde{\otimes}$. Let $\vartheta^{ab}$ be an antilinear map from $H^A$ into $H^B$ and $\vartheta^{ba}$ an antilinear map from $H^B$ into $H^A$. Then $\vartheta^{ab} \tilde{\otimes} \vartheta^{ba}$ is an antilinear map defined by

$$(\vartheta^{ab} \tilde{\otimes} \vartheta^{ba})(\varphi^A \otimes \varphi^B) = (\vartheta^{ab} \varphi^B) \otimes (\vartheta^{ba} \varphi^A).$$  \hfill (299)

One of the remarkable features of the twisted direct product is the rule

$$(\vartheta^{ab} \tilde{\otimes} \vartheta^{ba})^2 = (\vartheta^{ab} \vartheta^{ba}) \otimes (\vartheta^{ba} \vartheta^{ab}),$$  \hfill (300)

saying that the square of a twisted cross product (or the product of any two of them) is an “ordinary” cross product.

9.2.2 Modular objects

For the following we use the Schmidt form \hfill (259), \hfill (282), of $\psi$

$$\psi = \sum \sqrt{p_j} \phi^A_j \otimes \phi^B_j.$$
Thus (283) and (293) are valid:

\[
\begin{align*}
    s_{ab}^{\psi} \phi_j^A &= \sqrt{p_j} \phi_j^B, \\
    s_{ba}^{\psi} \phi_j^A &= s_{ab}^{\psi} \phi_j^B = \sqrt{p_j} \phi_j^B, \\
    j_{ab}^{\psi} \phi_j^A &= j_{ba}^{\psi} \phi_j^B = \phi_j^A,
\end{align*}
\]

As a first consequence

\[
\bigl( j_{ab}^{\psi} \otimes j_{ba}^{\psi} \bigr) \phi_j^A \otimes \phi_k^B = j_{ab}^{\psi} \phi_k^A \otimes j_{ba}^{\psi} \phi_j^B = p_k \phi_k^A \otimes \phi_j^B. \tag{301}
\]

On the right one finds the action of the modular operator \( J_{\psi} \) onto \( \phi_j^A \otimes \phi_k^B \). But if the actions of two antilinear operators agree on a basis, they are equal one to another:

\[
J_{\psi} = j_{ab}^{\psi} \otimes j_{ba}^{\psi}. \tag{302}
\]

A similar reasoning, following (300), results in

\[
( j_{ab}^{\psi} \otimes j_{ba}^{\psi} ) \phi_j^A \otimes \phi_k^B = (s_{ab}^{\psi} \phi_k^B) \otimes s_{ba}^{\psi} \phi_j^A = \sqrt{p_j p_k} \phi_k^A \otimes \phi_j^B. \tag{303}
\]

and, using (302), in

\[
s_{ab}^{\psi} \otimes s_{ba}^{\psi} = (\rho_A \otimes \rho^B)^{1/2} J_{\psi}. \tag{304}
\]

This equation is symmetric with respect to the two subsystems. One observes

\[
( s_{ab}^{\psi} \otimes s_{ba}^{\psi} ) J_{\psi} = (1^A \otimes \rho^B) ( (\rho_A)^{1/2} \otimes (\rho^B)^{-1/2} ).
\]

Comparing with (264), the last cross product at the right of the equation is identified with the square root of the modular operator. Hence,

\[
s_{ab}^{\psi} \otimes s_{ba}^{\psi} = (1^A \otimes \rho^B) \Delta^{1/2}_{\psi} J_{\psi} = (1^A \otimes \rho^B) S_{\psi}. \tag{305}
\]

The last equality is gained from (245).

### 9.3 From vectors to states

**Theorem 9.2** To any \( \rho \in B(\mathcal{H}^{AB}) \) there are linear maps \( \Phi_{\rho}^{ba} \) and \( \Phi_{\rho}^{ab} \) from \( B(\mathcal{H}^A) \) into \( B(\mathcal{H}^B) \) and from \( B(\mathcal{H}^B) \) into \( B(\mathcal{H}^A) \) such that

\[
\text{Tr} \left( X^A \otimes X^B \right) \rho = \text{Tr} X^B \Phi_{\rho}^{ba} (X^A) = \text{Tr} X^A \Phi_{\rho}^{ab} (X^B) \tag{306}
\]

is valid for all \( X^A \in \mathcal{H}^A \) and for all \( X^B \in \mathcal{H}^B \).

Proof: By fixing \( X^A \) the left of (306) becomes a linear form on \( B(\mathcal{H}^B) \). Hence it can be expressed uniquely as \( \text{Tr} X^B Y, \ Y \in B(\mathcal{H}^B) \). \( Y \) depends linearly on \( X^A \) and maps \( B(\mathcal{H}^A) \) into \( B(\mathcal{H}^B) \). By denoting \( \Phi_{\rho}^{ba} (X^A) = Y \), the asserted properties are satisfied. Exchanging the roles of \( X^A \) and of \( X^B \) gives the other equation.

The map \( \rho \mapsto \Phi_{\rho}^{ba} \) is linear. Because our spaces are all finite dimensional, the map is onto. Therefore (306) induces isomorphisms

\[
B(\mathcal{H}^A, \mathcal{H}^B) \leftrightarrow \mathcal{H}^{AB} \leftrightarrow B(\mathcal{H}^A, \mathcal{H}^B). \tag{307}
\]
Theorem 9.3 Let $\rho$ in (306) be positive semi-definite. Then $\Phi_{\rho}^{ba}$ and $\Phi_{\rho}^{ab}$ in (306) are completely copositive linear mappings. If
\[
\rho = \sum |\psi_j\rangle\langle\psi_j|, \quad s_j^{ba} := s_{\psi_j}^{ba}
\]
with vectors $\psi_j \in \mathcal{H}^{AB}$, then
\[
\Phi_{\rho}^{ba}(X^A) = \sum s_j^{ba} (X^A)^\dagger s_j^{ab}, \quad (309)
\]
\[
\Phi_{\rho}^{ab}(X^B) = \sum s_j^{ab} (X^B)^\dagger s_j^{ba}, \quad (310)
\]
where $X^A \in \mathcal{B}(\mathcal{H}^A)$, $X^B \in \mathcal{B}(\mathcal{H}^B)$.

Proof: By (306) the maps are linearly dependent on $\rho$. Therefore it suffices to prove (309) and (310) in case $\rho$ is of the form $|\psi\rangle\langle\psi|$, $\psi$ a unit vector. With this specification the left of (306) becomes
\[
\langle \psi, (X \otimes Y) \psi \rangle = \sum \langle s_j^{ab} \phi_j^B \otimes \phi_j^B, (X^A \otimes X^B) s_{\psi}^{ba} \phi_j^A \otimes \phi_j^B \rangle.
\]
and the better structured equation
\[
\langle \psi, (X \otimes Y) \psi \rangle = \sum \langle (X^B)\dagger \phi_j^B, s_{\psi}^{ba} (X^A)\dagger s_{\psi}^{ab} \phi_j^B \rangle \langle \phi_j^A, X^B \phi_j^B \rangle.
\]
Next (92) will be applied to the antilinear operator $X^A s_{\psi}^{ab}$, i.e.
\[
\langle \psi, (X \otimes Y) \psi \rangle = \sum \langle \phi_j^B, s_{\psi}^{ba} (X^A)\dagger s_{\psi}^{ab} \phi_j^B \rangle \langle \phi_j^A, X^B \phi_j^B \rangle.
\]
Summing up over $k$ yields
\[
\langle \psi, (X \otimes Y) \psi \rangle = \sum \langle (X^B)\dagger \phi_j^B, s_{\psi}^{ba} (X^A)\dagger s_{\psi}^{ab} \phi_j^B \rangle
\]
and this nothing than
\[
\langle \psi, (X \otimes Y) \psi \rangle = \text{Tr} Y s_{\psi}^{ba} (X^A)\dagger s_{\psi}^{ab}.
\]
The other part of (306) is verified by a similar exercise.

Remarks: (a) The Hermitian conjugate ensures linearity.
(b) One may compare the theorem with Jamiołkowski’s isomorphism [46], [10], to notice the difference enforced by complete copositivity. The latter comes with an “hidden antilinearity”.

73
10 Antilinearity in quantum teleportation

The quantum teleportation protocol was discovered by Bennett et al [12]. The protocol has been extended in various directions and applied to build more complex quantum information tasks. An overview is in Nielsen and Chuan [55] and most other QIT textbooks. There are many papers concerning quantum teleportation. See [15], [42], [51], [79], [6], and [62] for example.

Quantum teleportation consists of some preliminaries and a description how to do certain operations.

Generally, the basic structure is a tripartite quantum system

$$\mathcal{H}^{ABC} = \mathcal{H}^{A} \otimes \mathcal{H}^{B} \otimes \mathcal{H}^{C}. \quad (311)$$

One starts with two vectors

$$\psi \in \mathcal{H}^{A} \otimes \mathcal{H}^{B}, \quad \varphi \in \mathcal{H}^{B} \otimes \mathcal{H}^{C}, \quad (312)$$

and given input vectors

$$\phi^{in} \in \mathcal{H}^{A}, \quad \varphi^{in} := \phi^{in} \otimes \varphi \in \mathcal{H}^{ABC}. \quad (313)$$

Notice $d^{A} = \text{dim} \mathcal{H}^{A}, d^{BC} = \text{dim} \mathcal{H}^{BC}$, and so on.

In what follows the main emphasis is to what will be called “teleportation map”: The vectors (312) induce the maps $s_{\psi}^{ba}$ and $s_{\varphi}^{cb}$ from $\mathcal{H}^{A}$ into $\mathcal{H}^{B}$ and from $\mathcal{H}^{B}$ into $\mathcal{H}^{C}$ respectively. Therefore there is a mapping

$$t^{ca} \equiv t^{ca}_{\varphi, \psi} := s_{\varphi}^{cb} s_{\psi}^{ba}, \quad (314)$$

which will be called the teleportation map associated to the vectors (312).

This notation will be justified below.

$t^{ca}$ transports any $\phi^{in} \in \mathcal{H}^{A}$ to an output vector $\phi^{out} \in \mathcal{H}^{C}$,

$$\phi^{in} \rightarrow \phi^{out} := t^{ca}_{\varphi, \psi}. \quad (315)$$

As a product of two antilinear maps, the teleportation map is linear. The teleportation map (314) depends linearly on $\varphi$ and antilinearly on $\psi$.

For the Hermitian adjoint one gets

$$(t^{ca}_{\varphi, \psi})^\dagger = t^{ac}_{\psi, \varphi} = s_{\psi}^{ab} s_{\varphi}^{bc}. \quad (316)$$

$\varphi$ is given in advance. The symmetry between (314) and (316) will be broken in the teleportation protocol by a measurement by which $\psi$ in (312) and (314) will be either prepared or not. This way the irreversibility of quantum teleportation comes into the game: The teleportation map only applies if $\psi$ has been prepared. See the next subsection.

The teleportation map has been introduced by the present author, see in [77] and [19]. A more recent work on these questions is in Bertlmann et al [13].
10.1 Quantum teleportation

Let the A-system be in a not necessarily known pure state, represented by $\varphi^\text{in}$ in $H_A$. In the BC-system a pure state, is given in advance by the vector $\varphi = \varphi^\text{BC}$. This vector must be known. These assumption are allowed by the mutual independence of the A- and the BC-system.

The next step is to assume a basis $\{\psi_1^{AB}, \ldots, \psi_n^{AB}\}$ defining a von Neumann measurement preparing one of the basis states. (Indeed, it would suffice to instal a L"uders measurement with the rank one projection onto $\psi$ and the projection operator onto the orthogonal complement of $\psi$.)

Let $\psi = \psi_1^{AB}$ be an element of the basis $\{\psi_j^{AB}\}$,

$$\psi \in \{\psi_1^{AB}, \psi_2^{AB}, \ldots, \psi_n^{AB}\}, \quad n = d_A d_B,$$

and assume that the measurement reports that just this state is prepared. Then, similar to \ref{eq:272},

$$\langle |\psi\rangle \langle \psi| \otimes 1^C \rangle \varphi^\text{in} = \psi \otimes \varphi^\text{out}, \quad \varphi^\text{in} = \varphi^\text{in} \otimes \varphi.$$

**Theorem 10.1 (composition law)** There is a unique linear map $H_A \to H_C$ defined by \ref{eq:318} and expressed as in \ref{eq:314} by

$$\varphi^\text{in} \to \varphi^\text{out} = s_{cb}^{\psi} s_{ba}^{\varphi} \varphi^\text{in} \equiv t_{ac}^{\varphi,\psi} \varphi^\text{in},$$

The theorem and the teleportation map is in [76].

Proof: From \ref{eq:318} one infers: The vector $\varphi^\text{out} \in H_C$ is uniquely determined by the vectors $\varphi^\text{in}$, $\varphi$, and $\psi$. The vector and $\varphi^\text{out}$ depends linearly on $\varphi^\text{in}$ and on $\varphi$. Its dependence on $\psi$ is antilinear.

Choosing in $H_B$ a basis $\{\psi_1^B, \psi_2^B, \ldots\}$ one may write

$$\varphi = \sum_j \psi_j^B \otimes s_{cb}^{\varphi_j} \psi_j^B$$

to resolve the left side of \ref{eq:318} into

$$\langle |\psi\rangle \langle \psi| \otimes 1^C \rangle \varphi^\text{in} \otimes \sum_j \psi_j^B \otimes s_{cb}^{\varphi_j} \psi_j^B = \sum_j \langle |\psi\rangle \langle \psi| \otimes \psi_j^B \rangle \langle \varphi^\text{in} \otimes \psi_j^B \rangle \otimes s_{cb}^{\varphi_j} \psi_j^B.$$
On the right one has $\psi \otimes \phi^{\text{out}}$. Hence
\[
\phi^{\text{out}} = s^b_{\varphi} \sum_j \langle \phi^{\text{in}} \otimes \psi^B_j, \psi \rangle \psi^B_j
\]
(322)

By using (277) one gets
\[
\langle \phi^{\text{in}} \otimes \psi^B_j, \psi \rangle = \langle \psi^B_j, s^b_{\varphi} \phi^{\text{in}} \rangle \psi^B_j.
\]

Now, substituting it into (322), one arrives at
\[
\phi^{\text{out}} = s^b_{\varphi} \sum_j \langle \psi^B_j, s^b_{\varphi} \phi^{\text{in}} \rangle \psi^B_j = s^b_{\varphi} s^b_{\varphi} \phi^{\text{in}}
\]
and the assertion has been proved.

**Remarks:**

1.) Comparing (318) with (272) one gets the identity
\[
t^c_{\psi,\varphi} \phi^{\text{in}} = s^{ac,c}_{\chi} \chi, \quad \langle \chi \rangle := \varphi^{\text{in}} = \phi^{\text{in}} \otimes \varphi.
\]
(323)

2.) Equation (320) describes the possible result of a measurement on the rather particular tripartite state (318), $\varphi^{\text{in}} = \phi^{\text{in}} \otimes \varphi$. One may ask what happens in case of a general state vector. To see it, one can choose any product decomposition
\[
|\varphi^{ABC} \rangle := \sum_i |\phi^A_i \rangle \otimes |\varphi^{BC}_i \rangle
\]
(324)
to get
\[
(|\psi \rangle \langle \psi| \otimes 1^C |\phi^{ABC} \rangle = \sum_i |\psi \rangle \otimes t^C_i \phi^A_i
\]
or, by the composition law, and abbreviating the s–map belonging to $|\varphi^{BC}_i \rangle$ by $s^C_{\lambda}$,
\[
(|\psi \rangle \langle \psi| \otimes 1^C |\phi^{ABC} \rangle = |\psi \rangle \otimes \sum_i s^C_{\lambda} \cdot s^B_{\lambda} |\phi^A_i \rangle
\]
(325)

10.1.1 The trace norm

If the Hilbert spaces are infinite dimensional the maps of type $s^x_{\psi,\varphi}$ are Hilbert Schmidt ones. Therefore, the teleportation maps must be of trace class. Thus, also in finite dimensions, it seems quite natural to use the trace norm to estimate them. It is, [19],
\[
\| t^a_{\psi,\varphi} \|_1 := \text{tr} \left( (t^a_{\psi,\varphi})^\dagger t^a_{\psi,\varphi} \right)^{1/2}
\]
(326)

Let us call $\rho_\psi$ the reduction of $|\psi \rangle \langle \psi|$ to $H^B$, and $\rho_\varphi$ the reduced density operator of $|\varphi \rangle \langle \varphi|$ to the same Hilbert space. The aim is to prove that (320)
depends on these data only, i.e., by knowing these two density operators on $\mathcal{H}^B$ the trace norm in question can be computed. The result is the (square root) fidelity which is the square root of the generalized transition probability:

$$\| t^{ac}_{\psi,\varphi} \|_1 = \text{Tr} \left[ \sqrt{\rho_{\psi}} \rho_{\varphi} \sqrt{\rho_{\psi}} \right]^{1/2} \equiv F(\rho_{\psi}, \rho_{\varphi})$$  \hspace{1cm} (327)

For the proof the case of equal dimensions of the three Hilbert spaces and of maximal Schmidt rank of $\psi$ and $\varphi$ is assumed. One gets by (320) and by the help of (294)

$$ (t^{ac}_{\psi,\varphi})^\dagger t^{ac}_{\psi,\varphi} = s_{ab} s_{bc} s_{cb} s_{ba} = s_{ab} \rho_{\varphi} s_{ba} $$

Again, using (294) appropriately, one obtains

$$ (t^{ac}_{\psi,\varphi})^\dagger t^{ac}_{\psi,\varphi} = j^{ab}_\psi (\sqrt{\rho_{\psi}} \rho_{\varphi} \sqrt{\rho_{\psi}}) j^{ba}_\psi $$

The assumptions imply that $j^{ab}_\psi$ and $j^{ba}_\psi$ are isometries, and one of them is the inverse of the other. Therefore

$$ j^{ba}_\psi (t^{ac}_{\psi,\varphi}) j^{ab}_\psi = \sqrt{\rho_{\psi}} \rho_{\varphi} \sqrt{\rho_{\psi}}. $$

This equation proves an even stronger result than (326):

**Lemma 10.1** The singular values of $t^{ac}_{\psi,\varphi}$ and those of $t^{ca}_{\psi,\varphi}$ coincide with the eigenvalues of $\left( \sqrt{\rho_{\psi}} \rho_{\varphi} \sqrt{\rho_{\psi}} \right)^{1/2}$.

Indeed, the 1- or trace norm of $t^{ac}_{\psi,\varphi}$ is the sum of their singular values so that the lemma establishes (327).

Another estimate, seen from the proof above, reads

$$ \| \phi^{\text{out}} \| \leq \| (\sqrt{\rho_{\psi}} \rho_{\varphi} \sqrt{\rho_{\psi}})^{1/2} \|_1 \cdot \| \phi^{\text{in}} \| $$  \hspace{1cm} (328)

A comparison of different entanglement measures in quantum teleportation is in [65]. The authors allow $\varphi^{BC}$ to be mixed.

### 10.2 Distributed measurements

Looking at the quantum teleportation composition law (320) one may ask whether there is a similar structure in multi-partite systems. An obvious ansatz is the following: Assume $\mathcal{H}$ is the direct product of $n+1$ Hilbert spaces $\mathcal{H}^j$. Then any $\varphi_{k+1,k} \in \mathcal{H}^k \otimes \mathcal{H}^{k+1}$ corresponds uniquely to an antilinear map $s_{k+1,k}$ from $\mathcal{H}^k$ into $\mathcal{H}^{k+1}$. Hence there is a map $t^{n+1,1}$ defined by $t^{n+1,1} = s_{n+1,n} \ldots s_{2,1}$. This is a composition of $n$ antilinear maps.

$t^{n+1,1}$ is linear and “teleportation-like” if $n$ is even. It is antilinear and “EPR-like” for $n$ odd. Together with suitable measurements one gets something like a distributed teleportation or a distributed EPR scheme.

The case $n = 4$ has been treated in [77] and [19] and will be outlined below. In [19] and, more recently in [13], the case $n = 3$, in which $\mathcal{H}$ consists of four parts, has been considered.
10.2.1 The case of five subsystems

The quantum system in question is

\[ \mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \mathcal{H}^3 \otimes \mathcal{H}^4 \otimes \mathcal{H}^5. \]  

(329)

The input is an unknown vector \( \phi^1 \in \mathcal{H}^1 \), the ancillary vectors are selected from the 23- and the 45-system,

\[ \varphi^{2,3} \in \mathcal{H}^2 \otimes \mathcal{H}^3, \quad \varphi^{4,5} \in \mathcal{H}^4 \otimes \mathcal{H}^5, \]  

(330)

and the vector of the total system we are starting with is

\[ \varphi^{1,2,3,4,5} = \phi^1 \otimes \varphi^{2,3} \otimes \varphi^{4,5}. \]  

(331)

The channel is triggered by measurements in the 1, 2- and in the 3, 4-system. Suppose these measurements prepare successfully the vector states

\[ \psi^{1,2} \in \mathcal{H}^1 \otimes \mathcal{H}^2, \quad \psi^{3,4} \in \mathcal{H}^3 \otimes \mathcal{H}^4. \]  

(332)

Then we get the relation

\[ (|\psi^{1,2}\rangle\langle\psi^{1,2}| \otimes |\psi^{3,4}\rangle\langle\psi^{3,4}| \otimes 1^5)\varphi^{1,2,3,4,5} = \psi^{1,2} \otimes \psi^{3,4} \otimes \phi^5 \]  

(333)

and the vector \( \phi^1 \) is mapped onto \( \phi^5 \), \( \phi^5 = t^{5,1}\phi^1 \). Introducing the maps \( s^{k,k+1} \) corresponding to the vectors

\[ \psi^{1,2}, \quad \varphi^{2,3}, \quad \varphi^{3,4}, \quad \varphi^{4,5}, \]

the factorization rule becomes

\[ \phi^5 = t^{5,1}\phi^1, \quad t^{5,1} = s^{5,4}s^{4,3}s^{3,2}s^{2,1}. \]  

(334)

10.2.2 The EPR–like case of four subsystems

Remaining within the previous setting, ignoring however the Hilbert space \( \mathcal{H}^1 \). The following is an extended EPR protocol. Instead of giving an (unknown) input vector out of \( \mathcal{H}^2 \), there is a measurement on the system carried in \( \mathcal{H}^2 \). The state vector carrying the entanglement reads

\[ \varphi^{2,3,4,5} = \varphi^{2,3} \otimes \varphi^{4,5} \in \mathcal{H}^2 \otimes \mathcal{H}^3 \otimes \mathcal{H}^4 \otimes \mathcal{H}^5. \]

A test (a measurement) is performed to check whether \( \psi^{3,4} \) is prepared or not.

Let the answer be YES. Then the subsystems 2, 3 and 4, 5 become disentangled both. The state of the 3, 4 system changes to \( \psi^{3,4} \). The previously unentangled systems 2, 5 will become entangled. Indeed, the newly prepared state is

\[ \chi^{2,3,4,5} := (1^2 \otimes |\psi^{3,4}\rangle\langle\psi^{3,4}| \otimes 1^5)\varphi^{2,3,4,5}. \]  

(335)
If there is a decomposition
\[ \psi_{3,4} = \sum \lambda_j \phi_j^3 \otimes \phi_j^4 \]
of \( \psi_{3,4} \), one obtains
\[ \chi_{2,3,4,5} = \sum \lambda_j \lambda_k [(|1^2 \otimes \phi_j^3 \rangle \langle \phi_j^3 |) \otimes [(|\phi_j^4 \rangle \langle \phi_j^4 | \otimes 1^5)] \phi_{4,5}]. \]
Let \( s^{2,3} \) and \( s^{4,5} \) denote the antilinear mappings defined by \( \varphi^{2,3} \) and \( \varphi^{3,4} \) respectively. They allow to rewrite \( \chi_{2,3,4,5} \) as
\[ \chi_{2,3,4,5} = \sum \lambda_j \lambda_k (s^{2,3} \phi_k^3 \otimes \phi_j^3) \otimes (\phi_j^4 \otimes s^{5,4} \phi_k^4) \]
which is equal to
\[ \chi_{2,3,4,5} = \sum \lambda_k (s^{2,3} \phi_k^3) \otimes \psi_{3,4} \otimes (s^{5,4} \phi_k^4), \tag{336} \]
The Hilbert space \( \mathcal{H}^3 \otimes \mathcal{H}^4 \) is decoupled from \( \mathcal{H}^2 \) and \( \mathcal{H}^5 \). The vector state of the latter can be characterized by a map from \( \mathcal{H}^3 \otimes \mathcal{H}^4 \) into \( \mathcal{H}^2 \otimes \mathcal{H}^5 \).
\[ \varphi^{23} := (s^{2,3} \otimes s^{5,4}) \psi_{3,4}, \tag{337} \]
indicating how the entanglement within the 2,5-system is produced by entanglement swapping, and how the three vectors involved come together to achieve it.

11 Appendix: Antilinear operator spaces

The subspaces of \( \mathcal{B}(\mathcal{H}\text{anti}) \) and their relations to completely copositive maps are topics calling for attention and research. Though there are many similarities to spaces of linear operators, (see [57] for an introduction to operator spaces), there are remarkable differences also: There does not exist a substitute for the identity \( \mathbb{1} \) in linear spaces of antilinear operators. There is, however, the canonical Hermitian form, see [2.4] which does not depend on the scalar product of \( \mathcal{H} \). Further one may hope for interesting factorizations of operator spaces as products of antilinear ones. Here only a first impression can be gained. (To my knowledge there is no systematic exploration presently.)

Following [57] a complex-linear subspace of \( \mathcal{B}(\mathcal{H}\text{anti}) \) will be called an antilinear operator space or an AO-space for short. If the operator space contains with any \( \vartheta \) also \( \vartheta^\dagger \), it is an antilinear operator system or an AO-system.

Let \( \mathcal{M} \) be an AO-space. Its canonical form, denoted by \( (\ldots)_M \), is the restriction onto \( \mathcal{M} \) of the canonical form [21]. There are decompositions
\[ \mathcal{M} = \mathcal{M}^+ + \mathcal{M}^- + \mathcal{M}^0, \tag{a1} \]
such that the restriction of the canonical form onto $\mathcal{M}^+$ is positive definite, onto $\mathcal{M}^-$ is negative definite, and is vanishing on $\mathcal{M}^0$. Their dimensions are the inertia, see [10], of the Hermitian form $(\ldots)_M$. Hence

$$\dim \mathcal{M} = \dim \mathcal{M}^+ + \dim \mathcal{M}^- + \dim \mathcal{M}^0. \quad (a2)$$

A peculiarity is that the inertia of $(\ldots)_M$ do not depend on the scalar product of $\mathcal{H}$. Indeed the canonical form [21] posses just that property.

Fixing an arbitrary Hilbert scalar product on $\mathcal{M}$ there are $\dim \mathcal{M}$ mutual orthogonal elements $\vartheta_k \in \mathcal{M}$ fulfilling

a) The first $\dim \mathcal{M}^+$ elements constitute a basis of $\mathcal{M}^+$,
b) The next $\dim \mathcal{M}^-$ elements generate $\mathcal{M}^-$,
c) The remaining $\dim \mathcal{M}^0$ elements span $\mathcal{M}^0$.

More can be said about the decomposition if $\mathcal{M}_T$ is an antilinear operator system. In this case, $\mathcal{M}_T$ contains with $\vartheta$ necessarily $\vartheta^\dagger$ and, hence, $\vartheta \pm \vartheta^\dagger$. This simple observation results in the AO–systems

$$\mathcal{M}_T = \mathcal{M}_T^+ \oplus \mathcal{M}_T^- \quad (a3)$$

and in the nice property

$$\mathcal{M}_T^+ = \mathcal{M}_T \cap \mathcal{B}(\mathcal{H})^+, \quad \mathcal{M}_T^- = \mathcal{M}_T \cap \mathcal{B}(\mathcal{H})^- \quad (a4)$$

If an AO-space carries a scalar product $\langle \ldots \rangle_T$, it will be called antilinear operator Hilbert space or $\text{AOH}$-space for short.

**Lemma A.1.**

Let $\mathcal{M}$ be an AOH-space and $\vartheta_1, \ldots, \vartheta_m$, $m = \dim \mathcal{M}$, be a basis of $\mathcal{M}$ with respect to the scalar product $\langle \ldots \rangle_T$. Then the map

$$X \rightarrow T(X) = \sum_{j=1}^m \vartheta_j X^\dagger \vartheta_j^\dagger \quad (a5)$$

does not depend on the choice of the basis.

**Proof:** Let $\vartheta'_1, \ldots$ be another basis. There is a unitary matrix $u_{jk}$ such that $\vartheta_j = \sum_k u_{jk} \vartheta'_k$. Due to this relation the sum in (a1) is replaced by

$$\sum_j \sum_{kl} u_{jk} \vartheta_k X^\dagger u_{jl} \vartheta_l^\dagger$$

because the Hermitian adjoint acts linearly on antilinear operators. By placing $u_{jl}$ on the left changes it to $u^*_{jl}$. Now $\sum_j u_{jk} u_{jl}^\dagger = d_{kl}$ proves the assertion.

$T$ in eq. (a5) is a completely copositive map with length $\dim \mathcal{M}$. On the other hand, given such a map $T$ as in eq. (a5), there is a uniquely associated
AOH-space $\mathcal{M}_T$. It is the AO-space $\mathcal{M}$ generated by the antilinear operators $\vartheta_j$, $j = 1, \ldots, m$ in a representation eq. (a5). Requiring the $\vartheta_j$ to become an orthonormal basis fixes a scalar product which makes $\mathcal{M}$ an AOH-space $\mathcal{M}_T$. Hence

**Proposition A.2.**

There is a bijection between AOH-spaces and completely copositive maps.

\[
T \iff \{\mathcal{M}_T, \langle .. \rangle_T\}.
\]  

(a6)

If $\vartheta_j$, $j = 1, \ldots \dim \mathcal{M}_T$ is a Hilbert basis of $\mathcal{M}_T$, then $T$ as in eq. (a5) is uniquely associated to the given AOH-space and vice versa.

The correspondence mimics similar constructs for completely positive maps.

**Examples**

(1) Let $\mathcal{M}_T = \mathcal{B}(\mathcal{H})^+_\text{anti}$. The relevant scalar product is the restriction to $\mathcal{B}(\mathcal{H})^+_\text{anti}$ of the canonical form [21], i. e.

\[
\langle \vartheta'', \vartheta' \rangle_T := \text{Tr} \vartheta' \vartheta'' \equiv (\vartheta'', \vartheta') .
\]

$T^+ := T_M$ is defined by eq. (a5). According to lemma A.1., $T^+$ can be computed with any basis of $\mathcal{B}(\mathcal{H})^+_\text{anti}$. One obtains

\[
T^+(X) = \frac{(\text{Tr} X)1 + X}{2},
\]

(a7)

$T^+$ is simultaneously completely copositive and completely positive. Notice also

\[
\text{Tr } T^+(X) = \frac{d + 1}{2} \text{Tr } X, \quad T^+(1) = \frac{d + 1}{2} 1.
\]

(a8)

(2) Let $\mathcal{M} = \mathcal{B}(\mathcal{H})^-\text{anti}$ and

\[
(\vartheta'', \vartheta')_T = -\text{Tr} \vartheta' \vartheta''.
\]

The length of any basis is $d(d - 1)/2$. Let $T_M = T^-$ then

\[
T^-(X) = \frac{(\text{Tr} X)1 - X}{2}.
\]

(a9)

This completely copositive map is not even 2-positive. One knows by the work of M.-D. Choi that one can construct examples of $k$-positive maps by convexly combining $T^-$ and $T^+$ for all relevant $k$.

(3) Let $\phi_1, \ldots, \phi_d$ be a basis of $\mathcal{H}$. Let $\mathcal{M}$ be spanned by the operators

\[
|\phi_{2m}\rangle \langle \phi_{2n+1}|_{\text{anti}}, \quad 2m \leq d, \quad 2n + 1 \leq d .
\]

(a10)

Then, as see from [75], $\mathcal{M}$ is of type $\mathcal{M}^0$. If $d$ is even, then $\dim \mathcal{M} = d^2/4$. It is $\dim \mathcal{M} = (d^2 - 1)/4$ for $d$ odd.
\( \mathcal{M} \), as defined above, is an algebra. The same is with \( \mathcal{M}^{\dagger} \). Clearly \( \mathcal{M} \cap \mathcal{M}^{\dagger} \) consists of the null operator only. The product spaces \( \mathcal{M} \mathcal{M}^{\dagger} \) and \( \mathcal{M}^{\dagger} \mathcal{M} \) are ”almost” operator systems: They do not contain the identity operator.

Acknowledgement: I like to thank Bernd Crell and Meik Hellmund for helpful remarks and support, Stephan R. Garcia and Mihai Putinar for calling my attention to the “Finish School”, and, last but not least, ShaoMing Fei for inviting me to publish in SCIENCE CHINA Physics, Mechanics & Astronomy.

References

[1] Y. Aharonov, D. Rohrlich: Quantum Paradoxes: Quantum Theory for the Perplexed. WILEY-VCH Verlag, Weinheim 2005.

[2] P. M. Alberti: On the Simultaneous Transformation of Density Operators by means of completely positive, unity preserving linear maps. Publ. RIMS, Kyoto Univ. 21 (1985) 617–624.

[3] T. Ando: Topics in Operator Inequalities. Hokkaido University, Sapporo, 1978.

[4] T. Ando: Concavity of certain maps on positive definite matrices and applications to Hadamard products. Lin. Alg. Appl. 26 (1979) 203–241.

[5] T. Ando: On some operator inequalities. Math. Ann. 279 (1987) 157–159.

[6] S. Albeverio, Shao-M. Fei: Teleportation of general finite dimensional quantum systems. Phys. Lett. A 276 (2000) 8–11. quant-ph/0012035

[7] R. Bhatia: Positive Definite Matrices. Princeton Universsity Press, Princeton and Oxford, 2007.

[8] L. Balayan, S. R. Garcia: Unitary equivalence to a complex symmetric Matrix: Geometric criteria. Operators and Matrices 4 (2010) 53–76. arXiv:0907.2728v2 [Math.FA]

[9] V. P. Belavkin, M. Ohya: Entanglement and compound states in quantum information theory. Proc. R. Soc. Lond. A 458 (2002) 209–231 quant-ph/0004069

[10] I. Bengtsson and K. Życzkowski, Geometry of Quantum States. Cambridge University Press, Cambridge 2006

82
[11] Ch. H. Bennett, S. J. Wiesner, Communication via One- and Two-Particle Operators on Einstein–Podolski–Rosen States, *Phys. Rev. Lett.*, 69 (1992) 2881 – 2884.

[12] C. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, W. Wootters: Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, *Phys. Rev. Lett.*, 70 (1993) 1895–1898.

[13] R. A. Bertlmann, H. Narnhofer, W. Thirring: Time-ordering Dependence of Measurements in Teleportation. *Eur. Phys. J D* (2013) 62–67 arXiv:1210.5646v1 [quant-ph]

[14] S. R. Blackburn: Enumerating finite racks, quandles and kei. arXiv:1203.6504v1 [mathGT]

[15] G. Brassard: Teleportation as Quantum Computation. *Physica D* 120 (1998) 43–47.

[16] E. Cartan: *Lecons sur la Geometrie Projective Complexe*. Gauthier-Villars, Paris, 1931

[17] E. Cartan: Sur le domaines bornés homogénes de l’espace de n variables complexes. *Abh. Sem. Hans. Univ.* 11 (1936) 116–162

[18] B. Crell, A. Uhlmann: *Einführung in Grundlagen und Protokolle der Quanteninformatik*. Univ. of Leipzig, NTZ-preprint 33 (1998) 1–69. Unpublished

[19] B. Crell, A. Uhlmann: Geometry of state spaces. *Lect. Notes Phys.* 768 (2009) 1–60.

[20] D. Chruścienki, A. Jamiołkowski: *Phases in Classical and Quantum Mechanics*. Birkhäuser, Boston 2004.

[21] A. Chefles, R. Jozsa, A. Winter: On the existence of physical transformations between sets of quantum states. aeXiv.quant-phys./0307227

[22] P. A. M. Dirac: *The Principles of Quantum Mechanics*. Clarendon Press, Oxford 1930.

[23] D. Ž. Djoković: Poincaré series of some pure and mixed trace algebras of two generic matrices. *J. Algebra* 309 (2007) 654–671

[24] A. Einstein, B. Podolsky, N. Rosen: Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* 47 (1935) 777–780.

[25] S.R. Garcia and M. Putinar: Complex symmetric operators and applications. *Trans. Amer. Math. Soc.* 358 (2006) 67
[26] S. R. Garcia and M. Putinar, Complex symmetric operators and applications II, *Trans. Amer. Math. Soc.* 359 (2007) 3913–3931.

[27] S. R. Garcia, E. Prodan and M. Putinar, Norm estimates of complex symmetric operators applied to quantum systems. arXiv:math-ph/0501022v2

[28] S. R. Garcia, E. Prodan and M. Putinar, Mathematical and Physical Aspects of Complex Symmetric Operators, *J. Phys. A. Math. Theor.* 47 (2014) 363001 arXiv:1404.1304v1 [math.FA] 2014

[29] S. R. Garcia and W. R. Wogen: Some new classes of complex symmetric operators. arXiv:0907.3761v1 [math.FA] 2009

[30] S. R. Garcia and W. R. Wogen: Complex symmetric partial isometries. *J. Funct. Analysis* 267 (2009) 251–260 arXiv:0907.4486v1 [math.FA] 2009

[31] S. R. Garcia, J. E. Tener: Unitary Equivalence of a Matrix to its Transpose. *Lin. Alg. Appl.* 437 (2012) 271–284 arXiv:0908.2107v4 [math.FA] 2011

[32] S. R. Garcia, D. E. Poore and J. E. Tener: Unitary Equivalence to a complex symmetric Matrix: Low dimensions. *J. Math. Anal. Appl.* 341 (2008) 640–648. arXiv:0908.2201 [math.FA] 2011

[33] V. I. Godic, I. E. Lucenko: On the representation of a unitary operator in the form of a product of two involutions. Uspehi Mat. Nauk, 20 (1965) 64–65

[34] W. R. Gordon: Unitary relation between a matrix and its transpose. *Canad. Math. Bull.* 13 (1970) 279–280.

[35] M. de Gosson: *Symplectic Geometry and Quantum Mechanics.* Birkhäuser Verlag, Basel, Boston, Berlin, 2006.

[36] R. Haag: *Local Quantum Physics.* Springer Verlag, Berlin, Heidelberg, New York, 1993.

[37] P. R. Halmos: *A Linear Algebra Problem Book.* The Dolciani Mathematical Expositions, 16, Math. Ass. of America, Washington, DC, 1995.

[38] M. Hellmund, private communication. (Januar 2015)

[39] F. Herbut and M. Vujčić: Basic Algebra of Antiunitary Operators and some Applications. *J. Math. Phys.* 8 (1966) 1345–1354.
[40] R. A. Horn and C. R. Johnson: *Matrix Analysis*. Cambridge University Press: Cambridge, UK, 1990.

[41] R. A. Horn and C. R. Johnson: *Topics in Matrix Analysis*. Cambridge University Press 1991

[42] M. Horodecki, P. Horodecki and R. Horodecki: General teleportation channel, singlet fraction and quasi-distillation. quant-ph/9807091

[43] M. Huhtanen and A. Perämäki: Function theory of antilinear operators. arXiv:1212.0360v1 [math.FA]

[44] M. Huhtanen and A. Perämäki: Orthogonal Polynomials of the $R$–linear generalized minimal residual method. *J. Approx. Theory*, 126 (2013) 220–239

[45] M. Huhtanen: How real is your matrix? *Lin. Alg. Appl.*, 424 (2006) 304–319

[46] A. Jamiołkowski: Linear transformations which preserve trace and positive semi-definiteness of operators. *Rep. Math. Phys.*, 3 (1972) 275

[47] D. Joyce: A classifying invariant of knots, the knot quandle. *J. Pure Appl. Alg.*, 23 (1982) 37–65

[48] R. Jost: *The general theory of quantized fields*. American Math. Soc. 1965.

[49] J. D. Lawson, Y. Lim: The geometric mean, matrices, metrics, and more. *Am. Math. Monthly* 108 (2001) 797–812.

[50] G. Lüders, Über die Zustandsänderung durch den Meßprozeß. *Ann.d.Physik*, 8 (1951) 322–328.

[51] T. Mor and P. Horodecki: Teleportation via generalized measurements, and conclusive teleportation. quant-ph/9906039

[52] H. Narnhofer: The Role of Transposition and CPT Operation for Entanglement. Preprint UWTPh-2001-48.

[53] S. Nelson: The combinatorial revolution in knot theory. Amer. Math. Soc. 58 (2011) 1553–1561

[54] J. von Neumann: *Mathematische Grundlagen der Quantenmechanik*. Springer Verlag, Berlin, 1932.

[55] M. A. Nielsen and I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press 2000
[56] M. Ohya: Note on Quantum Probability. *Lettere al Nuovo Cim.* **38** (1983) 402–406.

[57] V. Paulsen: *Completely Bounded Maps and Operator Algebras.* Cambridge Studies in Advanced Mathematics, Vol. 78, Cambridge University Press, Cambridge, 2002.

[58] A. Peres: *Quantum Theory: Concepts and Methods.* Kluwer Academic Publ., Dortrecht 1993

[59] C. Piercy: A complete set of unitary invariants for $3 \times 3$ complex matrices. *Trans.Amer.Math.Soc.* **104** (1962) 425–429.

[60] E. Prodan, S. R. Garcia, M. Putinar: Norm estimates of complex symmetric operators applied to quantum systems. *J. Phys. A. Math. Theor.* **32** (1999) 4877–4881

[61] W. Pusz S. L. Woronowicz: Functional calculus of sesquilinear forms and the purification map. *Rep. Math. Phys.* **8** (1975) 159–170

[62] J. Rehacek, Z. Hradil, J. Fiurasek and C. Bruckner: Designing optimal CP maps for quantum teleportation. *Phys. Rev. A* **64** (2001) 060301 quant-ph/0105119

[63] G. Rudolph, M. Schmidt: *Differential Geaometry and Mathematical Physics.* Springer Science + Business Media Dordrecht, 2013.

[64] S. Ruotsalainen: *Antilinear Selfadjoint Operators.* arXiv:1203.4670v2 math.SP 2012

[65] Sk. Sazim, S. Adhikari, S. Banerjee and T. Pramanik: Quantification of Entanglement of Teleportation in arbitrary Dimensions. arXiv:1208.4200 v1 [quant-phys]

[66] E. Schrödinger: Die gegenwärtige Situation in der Quantenmechanik, *Naturwissenschaften*, **35** (1935) 807–812,823–828,844–849.

[67] E. Schrödinger: Discussion of probability relations between separated systems, *Proc. Cambr. Phil. Soc.*, **31** (1935) 555–563.

[68] K. S. Sibirskii: A minimal polynomial basis of unitary invariants of a square matrix of order three. *Math. Sametki*, **3** (1968) 291–295.

[69] R. G. Sachs: *The Physics of Time Reversal.* The University of Chicago Press, 1987.

[70] A. C. da Silva: Symplectic Geometry. In (F. J. E. Dillen and L. C. A. Verstraelen eds.) *Handbook of Differential Geometry.*
[71] E. Störmer: On anti-automorphisms of von Neumann algebras. *Pac. J. Math.* 21 (1967) 349–370.

[72] E. Störmer: *Positive Linear Maps of Operator Algebras.* Springer-Verlag Berlin Heidelberg 2013.

[73] N. Takahashi: Quandle Varieties, Generalized Symmetric Spaces and ϕ-spaces. arXiv: 1306.2396v1

[74] J. E. Tener: Unitary Equivalence of a complex symmetric Matrix: An algorithm. *Lin. Alg. Appl.* 437 (2012) 271 – 284. arXiv:0908.2201v1

[75] A. Uhlmann: Eine Bemerkung über vollständig positive Abbildungen von Dichteoperatoren. *Wiss. Z. Karl-Marx-Univ. Leipziger Math.-Nat. R.* 34 (1985) 580-582.

[76] A. Uhlmann: Quantum channels of the Einstein-Podolski-Rosen kind. In: (A. Borowiec, W. Cegla, B. Jancewicz, W. Karwowski eds.), *Proceedings of the XII Max Born Symposium FINE DE SIECLE*, Wroclaw 1998. Lecture notes in physics; Vol. 539, 93–105. Springer, Berlin 2000, quant-ph/9901027

[77] A. Uhlmann: Operators and Maps Affiliated to EPR Channels. In: (H.-D. Doebner, S. T. Ali, M. Keyl, R. F. Werner eds.), *Trends in Quantum Mechanics.* World Scientific, Singapore 2000, 138–145.

[78] A. Uhlmann: Antilinearity in bipartite quantum systems and imperfect teleportation. In: (W. Freudenberg, ed.) Quantum Probability and Infinite-dimensional Analysis. Vol. 15. World Scientific, Singapore, 2003, 255-268. arXiv: quant-ph/0407244

[79] R. F. Werner, All Teleportation and Dense Coding Schemes. In: (D. Bouwmeester, A. Ekert, A. Zeilinger, eds.) *The Physics of Quantum Information*, Springer Verlag, Berlin, Heidelberg, New York, 2000.

[80] E. P. Wigner: Über die Operation der Zeitumkehr in der Quantenmechanik. *Nachr. Ges. Wiss. Göttingen, Math.-Physikal. Klasse* 1932, 31, 546–559.

[81] E. P. Wigner: Normal form of anitunitary operators. *J. Math. Phys.* 1960, 1, 409–413.

[82] G. C. Wick, A. S. Wightman, E. P. Wigner: *Phys. Rev.* 88 (1952) 101.

[83] S. L. Woronowicz: Positive maps of low dimensional matrix algebras. *Rep. Math. Phys.* 10 (1976) 165–183.

[84] S. L. Woronowicz: Nonextendible maps. *Comm. Math. Phys.* 51 (1976) 243–282.
[85] B. Zbinden, J. Brendel, N. Gisin and W. Tittel: Experimental test of non-local quantum correlations in relativistic configurations. *Phys. Rev. Lett.*, **84** (2000) 4737–4740. quant-ph/0007009