I summarize recent progress in lattice gauge theory, with particular emphasis on results from numerical simulations. A major success has been the determination of the light hadron spectrum in the quenched approximation with sufficient accuracy to expose statistically significant disagreements with the experimental spectrum. The differences are, however, fairly small, $\sim 5 - 10\%$. The data are also accurate enough to show evidence for artifacts of quenching predicted by chiral perturbation theory. I give an update on results for light quark masses, the kaon $B$-parameter, and the decay constants and $B$-parameters of heavy-light mesons. Most of these are known in the quenched approximation to $\sim 10\%$ accuracy or better, and preliminary estimates of quenching errors are of comparable size. One exception is the light quark masses, for which the quenching errors appear to be larger. I discuss the computational requirements for simulations of QCD with all approximations controlled, and argue that they will likely begin once computers sustain about 10 Teraflops. This is 30-40 times faster than present state-of-the-art machines. This estimate assumes that improvements in the discretization of lattice fermions are sufficient to allow continuum extrapolations to be made with a minimum lattice spacing of $\approx 0.1$ fm. I review results obtained with improved discretizations and conclude that they satisfy this requirement in most cases. Examples of successful improvement include the calculation of the glueball spectrum and excited heavy-quark potentials in pure Yang-Mills theory. Finally, I discuss recent developments which may allow simulations of QCD with full chiral symmetry even at finite lattice spacing.

1 Introduction

In this talk I aim to give an overview of recent progress in lattice gauge theory, to report on the status of calculations of various quantities of phenomenological interest, and to give my expectations for future progress. I mainly consider results from numerical simulations of QCD, although towards the end I discuss some “analytical” advances. Given constraints of time and space, I have been highly selective, and I apologize in advance to those whose work I have not included. An excellent source for more details is the proceedings of the annual lattice conference, the most recent of which was held in July 1998. Previous reviews in the ICHEP and Lepton-Photon series (by Lüscher last year, and by Flynn in 1996), have different emphases than mine and are also highly recommended.

The most important feature of the lattice is that it provides a non-perturbative regularization of field theory. This is necessary in order to investigate the properties of strongly coupled theories. The prime example is QCD at low energies or temperatures, and this is the major focus of numerical lattice simulations. There is, however, another non-perturbative sector of the standard model, namely the electroweak sector close to the finite temperature symmetry-restoration transition. The properties of this transition are being actively investigated using lattice methods. These simulations are simpler than those for QCD, since fermions can, to a good approximation, be integrated out analytically. The status of calculations is described in the recent review of Laine and Rummukainen, and I will not discuss them here.

The ultimate goals of simulations of QCD include the following:

- To test that QCD is indeed the correct theory of the strong interactions;
- To make predictions for exotic mesons and baryons;
- To determine the host of hadronic matrix elements which are needed to extract the underlying quark parameters (masses and weak mixing angles) from experimental results.

These are the topics I discuss in this talk. There are other important goals which I will not discuss:

- The attempt to gain at least a qualitative understanding of the physics of confinement and chiral symmetry breaking (a recent review is that by Negele);
- To understand the behavior of QCD at finite temperature and density (see reviews by Kanaya and Laermann).

Another interesting topic I will cover only in passing is the status of attempts to discretize chiral gauge theories. For reviews see Refs.

The outline of this review is as follows. Following a lightning description of lattice calculations, which serves also to introduce some common terminology, I explain why simulations of lattice QCD are so computationally intensive, and indicate how close we are to controlling all sources of error. In sec. I describe the status of quenched calculations of the spectrum. Section gives an update on the successes of the “improvement program” at reducing discretization errors, successes that are illustrated using the examples of the spectrum of glueballs...
and potentials in the $SU(3)$ Yang-Mills theory. Section 1 gives a brief update on the status of spectrum calculations including dynamical quarks. I then give a summary of the results for those hadronic matrix elements which are calculated most reliably. Section 2 gives an overview of the recent progress in formulating lattice QCD with exact chiral symmetry.

2 Why are lattice QCD calculations hard?

Numerical simulations of gauge theories have been under way since Creutz’s seminal work almost twenty years ago (see, e.g., Ref. 1). The first simulations including fermions (in the quenched approximation) followed soon thereafter. Yet we are still not at the point of having reliable results for QCD. In this section I will describe why the simulations are so difficult, and explain what it will take to get reliable results.

I begin with an quick review of lattice methodology. Wilson taught us how to discretize QCD while maintaining gauge invariance: fermion fields are placed on the sites of a lattice, while the gauge fields live on the intervening links. The simplest lattice, used in most numerical simulations, is a hypercubic array, with equal spacings in all directions. The symmetry between sites makes the simulations, is a hypercubic array, with equal spacings in all directions. The symmetry between sites makes the system well suited for numerical implementation on parallel computers. It is advantageous for some applications to make one or more lattice spacings different, and I will mention an example of this below. For numerical simulations one must also truncate to finite volume. What results is a finite dimensional quantum mechanical system that can be simulated using Feynman’s sum over paths. This is only practical if the time direction is imaginary, so that one is in effect studying 4-dimensional statistical mechanics.

The fermionic path integral is over anticommuting variables, and is usually done by hand resulting in the partition function

$$Z_{QCD} = \int [dU] \exp(-S_g) \prod_q \det(\Phi_{\text{lat}} + m_q^{\text{dyn}}).$$

(1)

In other words, the path integral is over all gauge fields $U$ (one $SU(3)$ matrix per link), weighted by the gauge action $S_g$ and the effective action resulting from fermion loops. $S_g$ is a discretization of the continuum gauge action, and is commonly taken to be that proposed by Wilson. This has discretization errors of $O(a^2)$. $\Phi_{\text{lat}}$ is a discretization of the Dirac operator. The most common choices are Wilson fermions (or improvements thereof which have smaller discretization errors), and staggered fermions. Unimproved Wilson fermions have discretization errors of $O(a)$, whereas staggered fermions have errors of $O(a^2)$.

Physical quantities are obtained from correlation functions. The simplest examples are two-point functions such as that of the pseudoscalar density,

$$C_P(x) = - \langle \bar{u} \gamma_5 d(x) \bar{D} \gamma_5 u(0) \rangle.$$  

(2)

From this one can extract the mass of the $\pi^-$ (and, less easily, of its excited states). Integrating out the fermions analytically gives the following expression

$$C_P(x) = Z_{QCD}^{-1} \int [dU] \exp(-S_g) \prod_q \det(\Phi_{\text{lat}} + m_q^{\text{dyn}})$$

$$\times \text{Tr} \left[ \gamma_5 \left( \frac{1}{\Phi_{\text{lat}} + m_d^{\text{val}}} \right)_{x,0} \gamma_5 \left( \frac{1}{\Phi_{\text{lat}} + m_u^{\text{val}}} \right)_{0,x} \right].$$

(3)

which is suitable for numerical evaluation. Thus, to calculate a correlation function, one has to calculate quark “propagators” (i.e. inverses of the discretized Dirac operator plus mass term) in the background gluon configuration, connect them together as shown by the trace, and average the result over gauge configurations weighted by the total effective action. The average is done using Monte-Carlo methods.

A very important distinction in the following is between “dynamical” and “valence” quark masses. The former are those which appear in the determinant, while that latter occur in the propagators. I have labeled the quark masses in Eq. 3 according to this terminology. In QCD, these two masses are the same, flavor by flavor, since they both arise from the same original action. But once one has integrated out the fermions, one has the option of picking the masses to be different. This is called the “partially quenched” approximation, and is made in most large scale calculations at present. Typically, the dynamical quark masses are heavier than the valence quark masses. The fully quenched limit is that in which $m^{\text{dyn}} \to \infty$, in which case the fermion determinant is a constant and drops out of the action. In summary, the partially quenched approximation means taking $m^{\text{val}} \neq m^{\text{dyn}}$, and the limit $m^{\text{dyn}} \to \infty$ with finite $m^{\text{val}}$ is the quenched approximation.

It is important to realize that (partially) quenched theories are unphysical—for example, they do not have an hermitian Hamiltonian. These approximations are made simply out of necessity because of the difficulty in simulating QCD with light dynamical masses. To illustrate this difficulty I will use an approximate scaling law for the number of floating-point operations (Flops) that are required to generate a statistically independent gauge configuration using 2 flavors of dynamical staggered fermions and present algorithms.

$$\frac{\# \text{ Flops}}{\text{Indep. Config.}} \approx \frac{6000 \times N_s^4 N_t}{(am^{\text{dyn}})^{3/2}}.$$  

(4)
I use the scaling of present algorithms for my estimates. The critical slowing down as part is due to shortcomings of present algorithms (e.g. of (unquenched) QCD. Simulations at this quark mass to the lightest quark masses used in present simulations and the expansion parameter is the middle row, because we can use chiral perturbation probably do not need to use masses smaller than those of QCD. Finally, the middle row corresponds to a quark mass which should be light enough that one can extrapolate from it to the physical values. In other words, we mass which should be light enough that one can extrapolate, etc. Thus I expect that reliable results for QCD itself will begin to emerge once we have machines sustaining 10 Teraflops. I want to stress that this is only 30-40 times faster than the present state-of-the-art machines. These are the CP-PACS machine at the University of Tsukuba, Japan, which peaks at 0.6 Tflops and sustains 0.3 Tflops; and the two Columbia-Riken/BNL machines, which together have a peak speed of 1 Tflop, and present efficiencies of about 25%. CP-PACS has been in production mode for nearly 2 years, while the Columbia machine (0.4 Tflop peak) has been in production mode for 6 months, and the Riken/BNL machine (0.6 Tflop peak) is just becoming fully operational.

Table 1: Time for 100 QCD configurations on a Teraflop machine.

| $m^{dyn}$ | $m_\pi/m_\rho$ | $L_{2.5\text{fm}}$ | $a=0.2\text{fm}$ | $a=0.1\text{fm}$ | $a=0.05\text{fm}$ |
|-----------|----------------|------------------|-----------------|----------------|----------------|
| $m_\pi/2$ | 0.56           | 2.5              | 7.5f mins       | 11f hrs        | 42f days       |
| $m_\pi/8$ | 0.3            | 3.5              | 15f hrs         | 58f days       | 1.4f yrs       |
| $m_\pi/25$| 0.18           | 6.7              | 0.4f yrs        | 3.7f yrs       | 3.30f yrs      |

Here $N_s$ and $N_t$ are the number of points in the spatial and Euclidean time directions. Assuming a typical value for their ratio, $N_t/N_s = 2$, this converts to

$$\# \text{Tflop-hours per 100 Configs} \approx 11f \left( \frac{L}{2.5\text{fm}} \right)^4 \left( \frac{0.1\text{fm}}{a} \right)^6.5 \left( \frac{m_\pi/2}{m^{dyn}} \right)^{2.5} \, ,$$

where $L = N_s a$ is the spatial size of the lattice in physical units, and $m_\pi$ is the physical strange quark mass. I choose 100 independent configurations in the denominator since this is roughly the minimal number needed to obtain small statistical errors in masses. I have introduced a factor $f$ to account for differences in simulation times between different discretizations of fermions. Numerical examples of this formula are give in Table 1. Note that the lattice size is chosen to satisfy both $L \geq 2.5\text{fm}$ and $m_\pi L \geq 4.5$ to avoid significant finite volume effects.

The top row in the Table corresponds approximately to the lightest quark masses used in present simulations of (unquenched) QCD. Simulations at this quark mass are being done with the lattice spacings down to 0.1 fm. The bottom row corresponds to the physical average light-quark mass. This is not quite QCD, since a dynamical strange quark is not included, but it gives an indication of the CPU time required for a direct simulation of QCD. Finally, the middle row corresponds to a quark mass which should be light enough that one can extrapolate from it to the physical values. In other words, we probably do not need to use masses smaller than those of the middle row, because we can use chiral perturbation theory to guide us in doing extrapolations. The chiral expansion parameter is $m_\pi^2/m_\rho^2$, and this is 10% for the middle row (but 30% for the top row).

A striking feature of these numbers is how it becomes rapidly more difficult to simulate as the lattice spacing and quark mass decrease. This follows directly from Eq. 5. Part of this increase is unavoidable (e.g. the number of lattice points increases as $a$ decreases) but part is due to shortcomings of present algorithms (e.g. the critical slowing down as $m_\pi \to 0$). Although I expect algorithms with improved scaling laws to be developed, I use the scaling of present algorithms for my estimates.

The dependence on $a$ makes clear the importance of the “improvement program”, which aims to reduce discretization errors and thus allow simulations at larger lattice spacings. In quenched simulations with unimproved Wilson fermions, it turns out (as discussed below) that one needs to go down to $a \approx 0.05\text{fm}$ in order to extrapolate to the continuum limit. It would be a great advance to be able to work with a minimum lattice spacing of $a \approx 0.1\text{fm}$ instead. I will show later that this is likely to be possible using improved Wilson fermions. With staggered fermions, on the other hand, it appears that $a \approx 0.05\text{fm}$ will be necessary.

Let us assume, then, that we work with improved Wilson fermions, and can get away with a minimum $a$ of 0.1 fm, and a minimum quark mass of $m_\pi/8$. Then we need only consider the top left $2 \times 2$ block of the table. What does this imply for the absolute time required? To answer this I must discuss the factor $f$. This represents the extra cost of doing simulations with improved Wilson fermions compared to staggered fermions. Based on the experience of the CP-PACS collaboration this factor is at least 10, implying that the simulation with $m^{dyn} = m_\pi/8$ and $a = 0.1\text{fm}$ would take almost 2 years on a sustained Teraflop machine. To be realistic one should also include a sizeable additional factor to account for the extra cost of including the dynamical strange quark, the need to do multiple simulations in order to extrapolate, etc. Thus I expect that reliable results for QCD itself will begin to emerge once we have machines sustaining 10 Teraflops. I want to stress that this is only 30-40 times faster than the present state-of-the-art machines. These are the CP-PACS machine at the University of Tsukuba, Japan, which peaks at 0.6 Tflops and sustains 0.3 Tflops; and the two Columbia-Riken/BNL machines, which together have a peak speed of 1 Tflop, and present efficiencies of about 25%. CP-PACS has been in production mode for nearly 2 years, while the Columbia machine (0.4 Tflop peak) has been in production mode for 6 months, and the Riken/BNL machine (0.6 Tflop peak) is just becoming fully operational.

Figure 1 gives a slightly different view of the status of present simulations. It shows a two-dimensional section through parameter space (and thus hides the dependence on the lattice spacing and the strange quark mass). Theories along the diagonal are physical, while those off the diagonal are partially quenched and thus unphysical. The fully quenched theory is on the right-hand vertical axis at $m^{dyn} = \infty$. The upper shaded region roughly indicates where present simulations are done. Note that it is relatively inexpensive, for a given dynamical quark mass, to map out a range of valence quark masses. Thus progress will occur along the various arrows pointing to-

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*This estimate is somewhat more pessimistic than that I gave in the talk. Simulations with improved actions are somewhat slower than I had assumed.*
wards the physical point in the LH corner. I wish to emphasize that, because of the ease of collecting partially quenched data, it would be very useful if we could use the information from these unphysical theories to aid the extrapolation to the physical theory. This can perhaps be done using chiral perturbation theory, and work in this direction has begun. There is, however, a limitation on how small one can take $m_{\text{val}}$, because certain artifacts of quenching diverge as $m_{\text{val}} \to 0$. Examples of this in the quenched theory will be shown below.

To conclude this section, I list the important issues that affect the progress towards simulations of full QCD.

- How large are quenching errors? To what extent can we use quenched QCD as a guide to QCD?
- How small a lattice spacing is needed to extrapolate to the continuum? By how much can we increase $a_{\text{min}}$ with an improved action?
- How small a quark mass is needed to reliably extrapolate to QCD?
- Can we use (partially quenched) chiral perturbation theory to aid extrapolation to the physical parameters?

3 The spectrum of quenched QCD

The CP-PACS collaboration presented this year final results from their study of the spectrum of QCD in the quenched approximation. Their aim is to control all errors, except that due to quenching, and thus to obtain definitive results for the quenched spectrum. They use:

- the simplest discretizations of the gauge and fermion actions, both due to Wilson. Thus the dominant errors are of $O(a)$, coming from the fermions;
- a fixed physical volume large enough for finite size effects to be negligible for single particle masses;
- four (small) lattice spacings to extrapolate to the continuum limit ($a = 0$);
- five valence quark masses, spanning the range $m_{\pi}/m_{\rho} = 0.4 - 0.75$, to extrapolate to the physical light quark masses;
- large ensembles of gauge configurations so that the final errors, after extrapolation, are of size $1 - 3\%$.

Some parameters of their simulations are collected in Table 2. I focus mainly on the CP-PACS results for the spectrum since their computational power outstrips that of other groups by an order of magnitude.

This calculation supersedes that done using the GF11 machine and presented in 1994. The improvement, due to more than an order of magnitude increase in computer power, is evident from the final results, shown in Figs. 2 and 3. These figures give the masses of pseudoscalar and vector mesons and octet and decuplet baryons. To understand them it is useful to review how the parameters of lattice simulations are fixed in quenched simulations. First, for each lattice spacing one extrapolates in the average light quark mass to the value at which the physical $m_{\pi}/m_{\rho}$ ratio is obtained. The resulting rho mass is then used to determine the lattice spacing (one extracts $m_{\rho}a$ from the lattice, so $a = (m_{\rho}a)/m_{\rho}^{\text{phys}}$). At this point all masses can be converted to units of GeV (although one should keep in mind that one is really determining the ratio of the masses to $m_{\rho}$). Next, one interpolates in the lattice strange quark

| $\beta = 6/g^2$ | $a$ (fm) | size | $La$ (fm) | Configs. |
|-----------------|---------|------|-----------|----------|
| 5.90            | 0.10    | $32^3 \times 56$ | 3.3       | 800      |
| 6.10            | 0.078   | $40^3 \times 70$ | 3.1       | 600      |
| 6.25            | 0.064   | $48^3 \times 84$ | 3.1       | 420      |
| 6.47            | 0.05    | $64^3 \times 112$ | 3.2       | 150      |

Table 2: Parameters of CP-PACS quenched simulations.

Note that isospin breaking effects are being ignored.

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Figure 1: Approaching QCD.
mass until a particular strange particle has its physical mass—in the present cases the $K$ and $\phi$ are used. And, finally, one extrapolates in the lattice spacing to $a = 0$. The results of this procedure are shown in the Figures—the pion and rho masses are not shown since they are input parameters.

I draw two important lessons from these results. First and foremost, the quenched spectrum disagrees with experiment! This is an important check on the calculations, since we know that we are not calculating in the correct theory. The most significant disagreement is for the hyperfine splitting, $m_{K^*} - m_K$, which is $\sim 10\%$ too small for either choice of $m_s$. This is a $5 - 6\sigma$ discrepancy. The nucleon mass is $7\%$ too low (independent of $m_s$), although this is less significant ($2.5\sigma$). Finally, the splittings within the octet and decuplet are too small, although this discrepancy is greatly reduced if $m_s$ is fixed using $m_{\phi}$. However, the Gell-Mann–Okubo formula for octet baryons and the equal-spacing rule for decuplet baryons are both well satisfied for either value of $m_s$.

The second lesson is that quenched QCD yields a spectrum which is reasonably close ($5 - 10\%$) to that expected from QCD itself. In other words, although quenching is a drastic approximation, it gives us a model in which the gross properties of light hadrons are numerically close to those in QCD. This is important because we want to have an idea of the accuracy of quenched results for other quantities, such as the hadronic matrix elements discussed below. One should keep in mind, however, that there is no universal size to quenching errors.

Extrapolations play a key rôle in the CP-PACS analysis, and I want to spend a little time discussing their reliability. I focus on the chiral extrapolation here, deferring discussion of the continuum extrapolation until the next section. To do the chiral extrapolation requires theoretical input as to the functional form, and this is provided by chiral perturbation theory ($\chi$PT). For the quenched theory one must use the quenched variant, $Q\chi$PT, and this yields the following predictions for the chiral expansions of pion and nucleon masses:

\begin{align}
m^2_\pi &= 2A m_q \left( 1 - \delta \log(2A m_q/\Lambda^2_\chi) + O(m_q) \right) \quad (6) \\
m_N &= M_0 - \delta C_1 m_\pi + C_2 m_\pi^2 + C_3 m_\pi^3 + O(m_\pi^4) \quad (7)
\end{align}

In these expressions, $\delta$, $A$, $\Lambda_\chi$, $M_0$ and the $C_i$ are constants which are related to the coefficients appearing in the quenched chiral Lagrangian. The result for $m_\rho$ has the same form as that for $m_N$ but with different constants. The important features of these expressions are as follows.

- Terms involving the constant $\delta$ are artifacts of
quenching—they are not present in the chiral expansions in QCD. They arise from loops containing the $\eta'$, which, in the quenched theory, remains degenerate with the pion. They give the dominant corrections to the leading order term as $m_q \to 0$. In particular, the $\delta \log m_q$ term in $f_\pi$ diverges in the chiral limit, although the contribution to $m_\pi^2$ itself vanishes in this limit.

- Terms not involving $\delta$ are present in both quenched and unquenched theories, although the detailed expressions for the coefficients differ.
- Particle masses are predicted to be non-analytic functions of the quark mass—the non-analyticity is proportional to $m_q \log m_q$ in $f_\pi^3$ and to $m_\pi \sim m_q^{1/2}$ and $m_\rho \sim m_q^{3/2}$ in $f_\pi^2$. 

In the past, most analyses have extrapolated using low order polynomials in $m_q$, i.e. ignoring possible non-analytic terms. These parameterizations have adequately represented the data. The small statistical errors and long chiral level-arm of the CP-PACS calculation make possible a test of the more complicated forms predicted Q$\chi$PT.

The most dramatic effect predicted by Q$\chi$PT is for the pion mass: the quantity $y = m_\pi^2/(m_1 + m_2)$ should diverge logarithmically at small $m_q$. (See Ref. [4] for the full expression for non-degenerate quarks.) This is to be compared to QCD, where $y$ linearly approaches a constant in the chiral limit. Evidence for this effect has previously been seen with staggered fermions (and is reviewed in Ref. [31]), but not with Wilson fermions. The CP-PACS results are shown in Fig. 4. The predicted rise at low quark masses is seen for all four lattice spacings, with the fits yielding consistent values of $\delta \sim 0.10 \pm 0.02$. This rise has also been seen by the FNAL group [32] who are able to work with very small quark masses by using a modification of the quenched approximation, and by the QCDSF group using improved Wilson fermions [33]. The values for $\delta$ are all roughly consistent with each other, and with those obtained using other quantities [30].

The most economical explanation of these results is that they are quenched artifacts due to $\eta'$ loops, as predicted by Q$\chi$PT. This interpretation is supported by the fact that fits to the predicted chiral behavior of vector mesons, baryons and pseudoscalar decay constants (including those for particles composed of non-degenerate quarks) represent the CP-PACS data well [34]. It is reassuring that the lattice results are good enough to pick out physical effects coming from loops of light particles, and thus to make contact with the analytical predictions of Q$\chi$PT. It also gives credibility to attempts to use Q$\chi$PT to estimate the size of quenching errors in other quantities [34].

On the other hand, one cannot rule out that the rise in $y$ is due to a collusion between terms analytic in $m_q$ of linear, quadratic and cubic order. CP-PACS have investigated this possibility by fitting all the masses to polynomials of the lowest order required to give a good fit. The resulting light hadron masses differ at finite lattice spacing from those obtained using the Q$\chi$PT forms, but it turns out that they agree within errors after extrapolation to the continuum limit (see Fig. 4 below). In particular, the hyperfine splitting is still significantly below the experimental value. Thus the fact that quenched QCD disagrees with experiment appears to be robust.

The importance of understanding the chiral extrapolations has also been stressed by the MILC collaboration, who have done extensive studies using staggered fermions [33].

My overall conclusion concerning chiral fits from present data is that, while it is clear that the lowest order polynomial fits are inadequate once one goes below $m_\pi/m_\rho < 0.5$, a complete resolution of the issue probably requires smaller masses than those used by CP-PACS. This is part of the reason why I chose $m_\pi/m_\rho = 0.3$ (rather than the minimal value $m_\pi/m_\rho = 0.4$ used by CP-PACS) as the number to aim at in Table 1. Actually, sorting out the chiral extrapolation may be easier in the unquenched theory, since there are no “enhanced” logarithms. On the other hand, such logarithms are present in partially quenched QCD [34] and give rise to the artifacts in the lower shaded region in Fig. 4.
While the CP-PACS results are very impressive, they do rely on certain assumptions about extrapolations. It is clearly important to check them using at least one other fermion discretization, for which the systematics will differ. The MILC results with staggered fermions, while not as extensive (e.g. they do not consider baryons composed on non-degenerate quarks), do provide a partial check. They quote the extrapolated value $m_N/m_\rho = 1.254 \pm 0.018 \pm 0.028$, consistent with the experimental value 1.22. This is higher than the CP-PACS result of 1.143 $\pm$ 0.033, but only by about 2.5$\sigma$. It would be very interesting to see results from staggered fermions for the $K-K^*$ hyperfine splitting. There are also a number of large scale calculations underway using improved Wilson fermions, while not as extensive (e.g. they do not consider baryons composed on non-degenerate quarks), but these do not yet have a range of lattice spacings or statistical accuracy comparable to that of the CP-PACS collaboration.

4 The improvement program

The discussion of sec. 2 highlighted the importance of reducing discretization errors. This is essential if we are to have unquenched simulations of QCD in the near future, and it has been the focus of much research in the last few years. In this section I report on the progress in this direction.

With an unimproved action, such as that for Wilson fermions, the discretization errors start at $O(a)$

$$M^\text{lat}_{\text{Had}}(a) = M^\text{cont}_{\text{Had}} [1 + a\Lambda_1 + (a\Lambda_2)^2 + \cdots] \tag{8}$$

The size of the corrections is set by non-perturbative physics, so we expect $\Lambda_1 \sim \Lambda_2 \sim \Lambda_{\text{QCD}}$. Examples of continuum extrapolations from the CP-PACS data are shown in Fig. 5. The fits are to a linear form ($\Lambda_2 = 0$), and give values for $\Lambda_1$ in the range $0 - 300 \text{ MeV}$. Note that a linear fit is reasonable for such small lattice spacings. If one introduces a quadratic term of the expected size the extrapolated values change by less than 1$\sigma$.

It is useful to estimate the minimal lattice spacing ($a_{\text{min}}$) needed to extrapolate to the continuum with a given accuracy. I take the accuracy required after extrapolation to be 1%. To approximate the extrapolation procedure I assume that $M_{\text{had}}$ and $dM_{\text{had}}/da$ can be determined accurately at $a_{\text{min}}$, and used to extrapolate linearly to zero. For the quadratic function in Eq. 8 the resulting error is $-(a_{\text{min}}\Lambda_2)^2$. Setting this to be 1%, and taking the fairly conservative value $\Lambda_2 = 400 \text{ MeV}$, I find $a_{\text{min}} = 0.05 \text{ fm}$. This estimate is consistent with what CP-PACS finds in practice.

Now assume that the calculation has been improved so that $\Lambda_1 = 0$. The functional form is then quadratic + cubic. Proceeding in the same way, except this time extrapolating linearly in $a^2$, the error is $-(a_{\text{min}}\Lambda_3)^3/2$.

Figure 5: CP-PACS continuum extrapolations, with $m_\rho$ fixed using $m_K$, and the scale in GeV set by $m_\rho$. Solid (open) points are from chiral extrapolations based on Q$\chi$PT (polynomial) fits. Experimental values are shown by diamonds at $a = 0$.

Setting this to be 1%, and taking $\Lambda_3 = 400 \text{ MeV}$, I find $a_{\text{min}} = 0.14 \text{ fm}$. To be conservative I assume that there are some quantities with larger discretization errors (and there are known examples where this happens, such as $B_K$ with staggered fermions), and so reduce the minimal lattice spacing to 0.1 fm. Even so, this doubling of the minimal lattice spacing compared to the unimproved case makes a huge difference in our ability to simulate QCD, as discussed in sec. 2.

How does one go about improving the results? Most of the work so far has followed the Symanzik program of improvement order by order in $a$. Symanzik showed that, without improvement, the long-distance ($p \ll 1/a$) modes of the lattice theory can be described by an effective continuum Lagrangian containing all operators of dimension 5 allowed by the lattice symmetries. This is analogous to the more familiar effective theories such as the Fermi theory of the weak interactions. For QCD this...
appears at dimension 6 and give rise to errors of any physical quantity. Note that there are no gluonic operators of dimension 5—they first appear at dimension 6 and give rise to errors of $O(a^2)$. Thus for the first stage of improvement for QCD the gluon action does not need to be improved (although it may still be advantageous to do so).

The key, then, is to have a method for determining $c_{SW}$. This can be done in perturbation theory, order by order in $g^2$, and has been carried out in practice up to one-loop order. This leads to what are called “perturbatively improved Wilson fermions”, which have discretization errors of size $a^2g^2$ and $a^2$. It is clearly preferable, however, to have a non-perturbative determination. A method for doing so has been developed and applied by the ALPHA collaboration in the last few years, both for the quenched and two-flavor unquenched theories. I will not discuss the method since it was reviewed last year by Martin Lüscher. What is new this year are the results of extensive numerical tests of “non-perturbative $O(a)$ improvement” as it is called. In other words, with $c_{SW}$ determined non-perturbatively, do hadron masses behave as in Eq. (8) with $\Lambda_1 = 0$? The answer is yes!

4.1 Numerical tests of improvement

Studies involving several lattice spacings have been done by the SCRI group, the QCDSF collaboration, and the APE-TOV collaboration, and the ALPHA collaboration. I will show one plot, Fig. 6, which nicely captures the main features of all the results.

To test improvement one need not extrapolate to physical light quark masses—in fact it is better to use quarks with roughly the strange quark mass which can be simulated directly and for which the statistical errors are small. This is also advantageous because it avoids so-called “exceptional” configurations discussed below. The plot shows results for the vector meson mass $m_\rho$ versus $a^2$ at a fixed value of the pseudoscalar to vector meson mass ratio. The lattice spacing is determined using the “string tension” $\sigma$, which is the coefficient of the linear term in the potential between a heavy quark and antiquark in the quenched theory. All we need to know about $\sigma$ is that it is a gluonic quantity and thus approaches the continuum limit with errors of $O(a^2)$. Any linear dependence on $a$ is due to discretization errors in the fermion action. The absolute scale is only approximately defined since this is a quenched calculation, but the arrow in the figure labelled “$\beta = 6.0$” roughly corresponds to $a = 0.1$ fm.

The results in the plot are from several actions involving different degrees of improvement. The most interesting comparison is between unimproved Wilson fermions (crosses and squares) and non-perturbatively improved Wilson fermions (circles). The former have errors of $O(a)$ while the latter should have $O(a^2)$ errors. The solid lines are the result of fits to these hypotheses with the constraint that the continuum value be the same. The fit is reasonable, showing that non-perturbative improvement works. The quantitative success is also striking. For example, at $a \approx 0.18$ fm the...
discretization errors in hadron masses are reduced from 20 – 30% with Wilson fermions to 1 – 3% with non-perturbative improvement.

If this data alone indicated the efficacy of improvement in general then one would be able to work at larger minimal lattice spacings than my conservative estimate of $a_{\text{min}} = 0.1$ fm. The success is not uniform, however. Heitger et al., using the non-perturbatively $O(a)$ improved action, finds that, at $a = 0.1$ fm, the discretization errors are only a few percent in most quantities, but that a quantity related to $f_\rho$ differs by 28% from its continuum value.

The remaining points in the figure illustrate other interesting results. One can attempt to improve the gauge action so as to reduce the $a^2$ errors (which are large in certain glueball masses as we will see below). This improvement has only been implemented perturbatively, reducing the errors to $\alpha_s a^2$. One can then non-perturbatively improve the fermions on these new background gauge fields. This should remove the $O(a)$ errors, but one should not expect much reduction in the $a^2$ errors compared to the usual gauge action, because one has not reduced the $a^2$ errors coming from the fermions. This is what is found, as shown by the diamonds in the figure, with the dashed line being the fit. In this case there is no advantage gained from improving the gluon action.

The set of four points with the “burst” symbol (and no fit line) show the result of perturbatively improving the fermion action. Here one expects that there should be an $O(a)$ term, although reduced from that for unimproved Wilson fermions by a factor of $\alpha_s$. Qualitatively this is what is found. A fit to just an $a^2$ dependence (a linear fit on the plot) does not extrapolate close to the correct continuum value. Note also that the errors at $a \approx 0.2$ fm are substantially larger than for non-perturbatively improved fermions.

I conclude that, for almost all of the quantities studied so far, Symanzik improvement not only removes the leading correction, but also reduces the size of the discretization error substantially for the lattice spacings of interest. This is a very encouraging result.

Staggered fermions provide an interesting exception to this rosy picture, as shown by the two uppermost crosses in the figure. Staggered fermions are already improved—the errors start at $O(a^2)$—but it appears that the coefficient $A_2$ is very large. This is perhaps due to the fact that the effective lattice spacing is twice as large. Attempts to improve staggered fermions are just beginning—and will be needed if they are to present a competitive alternative to improved Wilson fermions.

What I have discussed so far is improvement of the S-matrix: masses and scattering amplitudes. We would also like to improve hadronic matrix elements, such as $f_B$. This requires improvement of operators as well as the action, e.g. for $f_B$ one needs to improve the axial current. This is not an easy task, since for each operator there are new improvement constants, and one wants to have a non-perturbative method for determining all of these. So far, methods have been proposed for all bilinear operators, and have been implemented in several cases. Extension to more complicated operators should be possible.

What is needed now is to extend the improved calculations towards the chiral limit. In particular, one would like to do the improved analogue of the CP-PACS calculation, including chiral and continuum extrapolations, in order to check that the results agree with the CP-PACS benchmark, and to determine what is the minimal lattice spacing needed. In particular, can we use $a_{\text{min}} = 0.1$ or larger, as required to approach QCD in the near future? A number of calculations are underway (as reviewed by Kenway). An unfortunate obstacle is the presence of exceptional configurations. These are configurations on which the operator $D_{\text{lat}} + m$ has a zero eigenvalue for a physical quark mass, and thus the propagator diverges at this mass. In the continuum these zero eigenvalues move to $m_q = 0$ and are the expected zero-modes associated with topologically non-trivial configurations. They are present at positive $m_q$ on the lattice because of explicit chiral symmetry breaking. Formally, this makes the quenched approximation ill-defined at small quark masses with Wilson fermions. It turns out that this problem is exacerbated by non-perturbative improvement, and makes it difficult to study the effect of improvement on the quenched light hadron spectrum for lattice spacings larger than 0.1 fm. This obstacle may be avoided using alternate forms of the improved action. It is important to realize, however, that exceptional configurations are quenched artifacts, and will not present a problem in unquenched QCD because they are suppressed by the fermion determinant. Thus they are a temporary obstacle.

### 4.2 Other approaches to improvement

I have concentrated so far on removing as completely as possible the corrections linear in $a$. Other approaches to improvement are also being pursued. Lepage has argued for pushing Symanzik improvement to higher order, using tadpole-improved perturbation theory to fix the additional improvement coefficients. Tadpole-improved perturbation theory has proven successful at predicting the dominant lattice corrections in many cases. The hope is to work on coarser lattices, perhaps as coarse as $a = 0.25 – 0.4$ fm, which implies $1/a = 0.5 – 0.8$ GeV. For example, Alford et al. argue that errors in hadron masses could be reduced to a few percent even at $a = 0.25$ fm if one combines tadpole estimates of the higher
order improvement coefficients with a non-perturbative determination of the leading improvement coefficient $c_{SW}$. Their numerical results, using the D234c action, for which the tree-level errors begin at $a^4$, support this claim.

I think that this approach is very interesting, but I am uncomfortable at the reliance on perturbation theory at such large lattice spacings. At $a = 0.4$ fm, for example, the relevant coupling constant is $\alpha_s(\pi/a) \approx 0.4$. This introduces two problems. First, one is essentially giving up on doing an extrapolation in $a$, since several terms can contribute. Instead, one aims to reduce the errors to a small enough level that no extrapolation is needed. Second, to estimate the residual errors requires, at the least, calculating the full one-loop contributions to improvement coefficients so as to determine their effect on the final results. It will be difficult to carry out such calculations given the complexity of the improved action. My major concern is for the calculation of matrix elements, for which the perturbative uncertainty enters at leading order. For this reason I am skeptical that one will be able, in general, to push much beyond $a_{\text{min}} = 0.1$ fm and obtain precise results for matrix elements. Hopefully I will be proven wrong.

Another approach to improvement is to determine a good approximation to the “perfect action”—i.e. an action without any discretization errors. This has been done at the classical level, but it has proven more difficult to find a truncation appropriate for numerical simulations which maintains the good properties of the full action. For a recent review see Ref. 55.

4.3 Glueball masses in SU(3) Yang-Mills theory

The major motivation for improvement is to make simulations of unquenched QCD tractable. Before discussing the status of such simulations, however, I want to give two examples from the pure gluon theory for which improvement has allowed substantial progress.

If one removes not only the sea quarks but also the valence quarks from QCD one is left with a pure gauge theory with group SU(3). Unlike quenched QCD, this is a physical quantum theory with a unitary S-matrix. It is an interesting laboratory for lattice methods because, like QCD, its low-energy dynamics are non-perturbative. The spectrum consists of interacting glueballs which, if massive enough, decay into multi-glueball states. What I illustrate here is the progress that has been made in studying this theory through the use of improved actions. I show in Fig. 7 the latest results, due to Morningstar and Peardon 57. They have obtained the masses of the lightest 15 glueballs with small errors, including the extrapolation to the continuum limit. On the LH scale, the masses are expressed in terms of a distance scale, $r_0$, which is determined from the quenched heavy quark potential 58 (and is thus defined in the pure gauge theory). I stress that these results are definitive numbers for dimensionless ratios in the pure gauge theory—there are no remaining approximations to be dealt with. The effects of virtual glueball pairs are fully included.

To obtain these results two technical tricks were combined: improvement and anisotropic lattices. The parts of the action connecting degrees of freedom in spatial directions were improved, allowing the use of a large spatial lattice spacing: $a_s = 0.2 - 0.4$ fm. The temporal lattice spacing was kept smaller, $a_t/a_t = 3 - 5$, so that the improvement of those parts of the action involving temporal separations was not required. This has two advantages. First, using a smaller $a_t$ allows one to trace out the exponential fall-off of correlators $[C(\tau) \propto \exp(-M\tau)]$ more accurately and thus extract more precise values for glueball masses. Second, if one had to improve the action in the temporal direction this would introduce unphysical “ghost” states which would interfere with the extraction of the masses. The use of a larger $a_s$ not only reduces the computer time required (the calculations have been done on DEC-ALPHA workstations), but also makes it easier to construct operators having a larger overlap with the glueballs.

![Figure 7: Continuum glueball spectrum in SU(3) pure Yang-Mills theory. Boxes indicate 1σ errors. Hollow boxes indicate ambiguous spin interpretation or ambiguous continuum extrapolation.](image-url)
These results are a considerable advance over previous work, which used the unimproved Wilson gauge action, and needed to use lattice spacings as small as \( a = 0.05 \text{ fm} \) in order to extrapolate to the continuum limit. This demonstrates the potential power of improvement: the minimal lattice spacing is increased by a factor of four. In particular, since the improvement was done perturbatively, it is a test case for the adequacy of perturbative as opposed to non-perturbative improvement. The errors of \( O(a^2) \) in the unimproved action are reduced to \( a_4, a_4^2 \) and \( a_4^2 \), and one might be concerned that the continuum extrapolations would be difficult if all three terms were comparable. In practice, it turns out, with one exception, that the \( a_4^2 \) errors are dominant and continuum extrapolations seem straightforward. The exception is the scalar glueball (the lightest state), for which it appears that the \( a_4, a_4^2 \) errors are significant. This leads to a more difficult continuum extrapolation, and explains why the errors for this state in Fig. 7 are larger than those for most of the more massive states, contrary to the usual pattern. I should stress that improvement still works— the \( a_4, a_4^2 \) errors are smaller than the \( a_4^2 \) errors with unimproved Wilson fermions. On the other hand, if one wants to reduce the error on the scalar glueball mass to a few percent using the improved action one would have to work at smaller values of \( a \) than 0.2 fm. This is one of the reasons why I chose the conservative estimate \( a_{\text{min}} = 0.1 \text{ fm} \) for QCD.

What are the implications of these results for QCD? The masses can be converted to physical units by matching, say, the quenched rho mass or the heavy-quark potential to experiment. This is done on the RH scale in Fig. 6. Let me stress, however, that the uncertainty in these predictions will generically be larger than the 5-10\% discrepancies between the quenched light meson and baryon masses and experiment. This is because when we “unquench” glueballs, they have many more decay channels than the light mesons and baryons, and, furthermore, most of the glueballs can mix with quark states. Shifts in the masses will, however, be very state dependent; for example, they will probably be smaller for the quark model exotics with \( J^{PC} = 2^{+-} \) and \( 0^{+-} \) which have no quark states to mix with. Because of the potentially larger quenching errors, I view the results in the Figure as only a rough guide to experimental masses, and identification with observed glueball candidates difficult. I should note that Weingarten and collaborators have studied the effects of mixing and decays within the quenched approximation, and conclude that they are small for the scalar glueball \( J^{PC} = 0^{++} \). These conclusions are, however, controversial.

Irrespective of its phenomenological implications, the numerically determined glueball spectrum provides a benchmark against which to test theoretical models. An interesting example is the prediction that states should be ordered roughly according to the minimal dimension of the operators which create them. This proposal works well for the states in Fig. 5 which are created by operators of dimension 4, 5 and 6. This comparison as well as that to the bag model are described by Kuti.

### 4.4 Heavy hybrids

In pure gauge theory one can introduce an infinitely heavy quark and antiquark pair, and calculate the energy of the induced gluonic flux as a function of the separation \( r \). One can excite the gluonic flux into states classified by their angular momentum along the axis and by other discrete symmetries. There is thus a tower of “potentials” which, up to an overall constant, are physical quantities in the pure-gauge theory.

The interest in these potentials is twofold. First, they give us insight into the physics of confinement. In the standard picture of a flux tube of constant transverse size, all the potentials should grow linearly at large \( r \), with the excitations corresponding to excited transverse modes. At distances of order the transverse size, \( r \approx 0.5 \text{ fm} \), this picture breaks down, and the detailed form of the potentials contains information about how this happens. The second interest is phenomenological. The potentials can be used, in the Born-Oppenheimer approximation, to calculate the spectrum of quarkonium states. The lowest lying potential gives rise to the usual \( QQ \) states, while the excited potentials correspond to hybrid \( QQG \) states. A prediction for the mass of the lightest hybrids in say, the \( b \bar{b} \) system would be very interesting. Of course, such a prediction would be in the quenched approximation, since the effects of virtual quarks (light and heavy) have been ignored.

Recent work by Juge, Kuti and Morningstar has considerably extended our knowledge of these potentials. These authors use the same improved, anisotropic lattice action as in the glueball calculation of Ref. 2, with spatial and temporal lattice spacings in the ranges \( a_s = 0.12 - 0.29 \text{ fm} \) and \( a_t = 0.04 - 0.06 \text{ fm} \) respectively. Compared to previous work using the unimproved action, they are able to calculate more excited potentials and to extend the results out to much greater separations. A subset of the new results are shown in Fig. 8. For clarity, the potentials are shown only out to about 2 fm, although they have been determined out to about 4 fm. The group theoretic symbols indicate different representations of the

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\(^1\)It is possible to make alterations to the action so as to reduce the discretization errors in the scalar glueball mass, and this might lead to an action for which extrapolations could be done using \( a_{\text{min}} \approx 0.2 \text{ fm} \). While this is an interesting approach, it is not a substitute for a systematic improvement program since one must make the alterations on a case-by-case basis.
effective string theory the first four predicted bands are: (I) $\Sigma^+$; (II) $\Sigma^+$, $\Pi^0$, and $\Delta_g$; (IV) $\Sigma^+$, $\Pi^0$, and $\Delta_g$. The two lines for each curve indicate the estimated error. In the effective string theory the first four predicted bands are: (I) $\Sigma^+$; (II) $\Sigma^+$, $\Pi^0$, and $\Delta_g$; (IV) $\Sigma^+$, $\Pi^0$, and $\Delta_g$. Units are $r_0 \approx 0.5$ fm.

What I want to stress is that, as for the glueball spectrum, numerical methods have produced detailed, accurate results for non-perturbative quantities. These results are good enough to test theoretical models. For example, at long distances, an effective string theory description should apply, which predicts that the excited states should fall into bands. This band structure is indeed observed by about 2 fm and is visible in the figure. The string model also predicts that the differences between the levels asymptotically approach multiples of $\pi/r$, but this turns out not to work. This discrepancy may be due to a flaw in the model itself, as discussed by Ref. for a discussion of possibilities. An important point, stressed by Kuti, is that the fact that the ground state potential remains very close to linear all the way down to $r \approx 0.5$ fm does not necessarily provide evidence in favor of the sting picture at these distances. Such “precocious” linearity is predicted by models (e.g. the adiabatic bag model) to occur well before the onset of string-like behavior.

At intermediate distances, $r \approx 1$ fm, the excited states undergo a complicated pattern of level crossings. While some of this behavior can be understood in terms of degeneracies that are predicted for $r = 0$ (Ref. 69), it remains a challenge to understand these levels in detail. This work of interpretation is just beginning—for the moment the important progress is having data good enough to compare with models.

Let me turn now to the predictions for the masses of hybrids. In the Born-Oppenheimer approximation, the lightest $b\bar{b}q$ level is eightfold degenerate. This level contains a $J^{PC} = 1^{-+}$, which could be produced in $e^+e^-$ annihilation, and three exotics with $J^{PC} = 1^{++}$, $0^{+-}$, and $2^{+—}$. This degeneracy is lifted when one includes retardation effects, as can be done using non-relativistic heavy quarks simulated on the lattice. Such calculations have been done by Manke et al. and by Juge et al.

The retardation effects are found to be small—for example, all eight levels remain close in mass. The detailed predictions from the two groups are in slight disagreement. For example, the mass of the $b\bar{b}q$ hybrid, with $J^{PC} = 1^{-+}$, is found to be 11.1(1) GeV by Ref. 58 and 10.8 GeV by Ref. 73, both in the quenched approximation. This is an interesting mass range for hybrids, because, although above the $B\bar{B}$ threshold at 10.56 GeV, it brackets the S-wave+P-wave $B_s^0\bar{B}$ threshold at 11.01 GeV. There are theoretical arguments suggesting that the dominant decay mechanism of a hybrid is to an S- and a P-wave meson. Thus, if the hybrid lies below or close to the $B_s^0\bar{B}$ threshold, it could be a narrow state, and thus observable. Improvements in the quenched results are likely in the next year.

Calculations of $c\bar{c}g$ hybrids have been done by the MILC collaboration, using improved relativistic quarks, and by Juge et al., using non-relativistic quarks. MILC finds 4.3 $^{+0.2}_{-0.1}$ GeV for the $1^{+-}$ state, and the result from Juge et al is consistent with this.

5 Simulations of two-flavor “QCD”

An important step on the way to simulating full QCD is the study of the theory with two flavors of light quarks. As noted in sec. 3, the use of an improved fermion action will likely be essential in order to simulate this theory with quarks light enough to allow a controlled extrapolation to the chiral limit. The hope is that improvement will work well enough to allow a continuum extrapolation to be done using lattice spacings no smaller than 0.1 fm. The successes of improvement in the pure-gauge theory auger well for this program.

Machines sustaining several hundred Gigaflops have been used to do preliminary studies of two-flavor QCD. The extra cost of including dynamical fermions restricts these simulations to larger lattice spacings ($a > 0.1$ fm), smaller volumes ($L < 2.4$ fm), and larger quark masses.
String breaking has already been observed in finite temperature QCD. Here one has the option of using the correlation between Polyakov lines to measure the potential and these have a good overlap with the broken-string state as well.

6 Hadronic matrix elements

A major reason for simulating lattice QCD is to extract results for the hadronic matrix elements of various operators. We would like to have both the numerical values of these matrix elements and reliable estimates of the uncertainties in these values. In this section I report on results for those quantities for which the calculations are mature enough to include a continuum extrapolation within the quenched approximation. These results are collected in Table 3 and I will discuss them in more detail in the following. My aim is to allow the non-specialist to make a judgement on the reliability of the error estimates. For more technical details, I refer the reader to the reviews also contain updates on many other interesting quantities for which the calculations are less developed.

$(m_+/m_\rho > 0.6)$ than the state-of-the-art quenched simulations discussed above. The most important of these restrictions is on the quark masses—reliable extrapolations to physical quark masses are not yet possible. Nevertheless, some interesting results have been obtained and I will highlight a few of these.

Several collaborations have undertaken simulations aimed at studying the spectrum of light hadrons in two-flavor QCD: CP-PACS [87, 88], using a perturbatively $O(a)$ improved fermion action and an improved gauge action determined by renormalization group methods; MILC [89], using staggered fermions (which are automatically $O(a)$ improved); SESAM [90], using unimproved Wilson fermions; and UKQCD [91], using non-perturbatively $O(a)$ improved Wilson fermions. Of these, the CP-PACS and MILC studies are the most extensive, using several lattice spacings and the widest ranges of quark masses.

An interesting first question to study is whether the previously noted discrepancy between the hyperfine splitting in quenched QCD and experiment is reduced in two-flavor QCD. The preliminary CP-PACS results, shown in Fig. 9, do show a significant increase in this splitting which moves the results toward the experimental value. This is encouraging, although much more work is needed to control the chiral and continuum extrapolations.

Other quantities for which a significant effect of unquenching has been observed are the quark masses (discussed in the next section), and the heavy quark potential. The latter should behave differently both at short and at long distances. At short distances, the one-gluon exchange contribution $\propto \alpha_s(r)/r$ should be stronger in the unquenched theory because the coupling runs more slowly due to the dynamical quarks. This effect has been observed with the expected magnitude.

At long distances a more striking phenomenon should occur—the string should break! This should occur at $r \approx 1.5\,\text{fm}$, when it becomes energetically favorable for the flux tube to decay into two sources screened by light (anti)quarks. At this point the potential should bend over and reach a plateau. Surprisingly, this rather basic phenomenon has not yet been seen in simulations. A year ago, as summarized in a review by Güsken [92], results for the potential had been extended out to distances at which the expected flattening should begin. This year, CP-PACS has pushed the calculation of the potential out to somewhat longer distances, yet still finds that the potential keeps on rising [93]. There has been progress, however, in understanding why string-breaking is not observed [94, 95]. The key point is that the operators used to create the gluon flux have very small overlap with the broken-string state. The authors of Refs. [94, 95] find that if they include in their bases operators designed to have substantial overlap with the broken-string state, then they clearly observe string breaking. These studies are done, however, on theories with scalar quarks, and it would be very interesting to repeat them on QCD itself.
Three other quantities, $\alpha_s$, $m_c$ and $m_b$, are also well determined from lattice simulations. I do not discuss them here because there has been little new work on them in the last year. Detailed discussions are given in earlier reviews: for $\alpha_s$ see the particle data tables and references therein; for heavy quark masses see Ref. [24].

6.1 Light quark masses

Quark masses are fundamental parameters in the standard model. In principle, the masses of the up, down and strange quarks are encoded in the masses of the light hadrons, but to determine them requires a non-perturbative method. Lattice calculations will ultimately provide results with all errors under control, but at present they are plagued by the uncertainties due to the quenched or partially quenched approximations. Calculations using QCD sum rules remain for the moment of comparable reliability, and competition between the two “camps” has produced useful reanalyses on both sides.

Quark masses are also interesting for a more practical reason. In the usual parameterization of the standard model prediction for $\epsilon'/\epsilon$ in terms of $B-$parameters, the dominant contributions are inversely proportional to $m_u^2$. If, as lattice calculations suggest, $m_s$ is smaller than expected a few years ago, then the prediction for $\epsilon'/\epsilon$ is likely to increase. Unfortunately, this is not a generic statement because there are delicate cancelations. For recent reviews see Refs. [13, 21].

The extraction of masses from lattice calculations is, in principle, straightforward. Ignoring isospin breaking, one has two dials to turn: the bare average light quark mass $\overline{m}$ and the bare strange quark mass $m_s$. One adjusts these until the results for two mass ratios, say $m_\pi/m_u$ and $m_K/m_s$, agree with experiment. One then determines the “physical” quark mass (defined in some renormalization scheme at a fixed scale) by matching the lattice and continuum definitions of quark masses. There are several ways of doing this—but the key point here is that all require knowledge of matching factors relating lattice and continuum operators. For example, one can use the ratio of the divergence of the axial current to the pseudoscalar density to define what is called the “axial Ward identity” mass:

$$2\overline{m}_{\text{cont}} = \frac{\langle 0 | \partial_\mu A_\mu^{A\text{cont}} | H \rangle}{\langle 0 | \mathcal{P}^{\text{cont}} | H \rangle} = \frac{Z_A}{Z_P} \frac{\langle 0 | \partial_\mu A_\mu^{A\text{lat}} | H \rangle}{\langle 0 | \mathcal{P}^{\text{lat}} | H \rangle}.$$  \hspace{1cm} (11)

Here $| H \rangle$ is an arbitrary hadronic state (usually the pion), “cont” and “lat” refer respectively to continuum and lattice regularized operators, and the matrix elements are to be evaluated with the bare quark masses set to their previously determined values. In this method the matching factors one needs are $Z_A$ and $Z_P$.

Let me first discuss the average of the up and down quark masses, $\overline{m}$. The state-of-the-art a year ago was, for the most part, to use one-loop perturbative results for

| Quantity | Quenched Result | Effect of unquenching | Comments |
|----------|-----------------|-----------------------|----------|
| $\overline{m} = 0.5(m_u + m_d)$ | $4 - 5 \text{ MeV}$ | $\approx 40\% \text{ reduction}$ | Non-perturbative matching crucial |
| $m_s$ | $100 - 150 \text{ MeV}$ | $\approx 40\% \text{ reduction}$ | Using $m_K$ and $m_\phi$ gives different results |
| $B_K$ | $0.85 \pm 0.06$ (Ref. [10]) | $\times 1.10 \pm 0.16$ (Ref. [30]) | Unquenched result is $0.94 \pm 0.06 \pm 0.14$ |
| $f_B$ | $165 \pm 20 \text{ MeV}$ (Ref. [83]) | $^{+23}_{-0} \text{ MeV}$ (Ref. [83]) | Ref. [10] quotes $175 \pm 25 \text{ MeV}$ (quenched) |
| $\hat{B}_B$ | $1.4 \pm 0.1$ (Ref. [22]) | See text |
| $f_B \sqrt{\hat{B}_B}$ | $195 \pm 25 \text{ MeV}$ | See text | Ref. [22] quotes $201(42) \text{ MeV}$ (quenched) |
| $\xi = \sqrt{\frac{f_B^2 \hat{B}_B}{f_B^2 \hat{B}_B}}$ | $1.14^{+6}_{-5}$ (Ref. [25]) | See text | Most reliable of all B-meson results |
| $f_{D_s}$ | $220^{+25}_{-20} \text{ MeV}$ (Ref. [23]) | $^{+17}_{-1} \text{ MeV}$ (Ref. [83]) | Compare with experiment |
matching factors, and to do chiral extrapolations with the lowest order polynomial that gave a good fit. After continuum extrapolation of all available data Bhat- 

Tschakary and Gupta concluded that, in the quenched approximation,
\[
\begin{align*}
\overline{m} & = 4.1 \pm 0.1 \text{ MeV} \quad \text{(Wilson)} \\
& = 3.8 \pm 0.1 \text{ MeV} \quad \text{(Perturbatively Improved Wilson)} \\
& = 3.5 \pm 0.1 \text{ MeV} \quad \text{(staggered)}
\end{align*}
\]

for the different fermion discretizations. I emphasize that these results are for an \( \overline{MS} \) scale of \( \mu = 2 \text{ GeV} \) (and not \( 1 \text{ GeV} \) as often used in the sum-rule literature). Results using different discretizations of the fermion action should agree in the continuum limit. The discrepancies between them were attributed to systematic errors associated with uncertainties in the continuum extrapolations and matching factors. Other groups, making different assumptions about these systematics, found somewhat larger values for the quark masses.

The most important improvement in the last year is that matching factors have been determined non-perturbatively. The technology for doing this has been developed over the last decade, and is an interesting example of “applied field theory”. The normalization of the axial current is determined by enforcing Ward identities, while that of the pseudoscalar density (and other divergent operators) is determined using off-shell quark correlation functions. I give several examples of the difference between one-loop perturbative and non-perturbative results in Table 4. The one-loop result works well for \( Z_A \) but not for \( Z_P \), which is not unexpected given the sizes of the one-loop corrections in the two cases. This is an example of the importance of using non-perturbative methods to normalize operators—a point I stressed previously when discussing the improvement program. Another success of non-perturbative normalization is that it brings results for quark masses determined using different methods (vector versus axial Ward Identity) into agreement.

Referring back to Eq. 11, we see that smaller values for \( Z_P \) imply larger quark masses. For example, the new result for staggered fermions, \( \overline{m} = 4.2(3) \text{ MeV} \), is 20% larger than that quoted above. The inclusion of non-perturbative matching factors does not, however, seem to bring the results using different types of fermion into closer agreement. For example, there is, for the first time, a calculation of quark masses using non-perturbatively \( O(a) \) improved Wilson fermions, which gives \( \overline{m} = 4.9(4) \text{ MeV} \). Although this result does not include a continuum extrapolation it is from a small lattice spacing, \( a \approx 0.07 \text{ fm} \), for which the \( O(a^2) \) discretization errors are likely to be small.

Another advance in the last year concerns the chiral fits. As discussed above, there is reasonable evidence in the quenched spectrum for the artifacts predicted by quenched chiral perturbation theory. Fitting including these artifacts results in quark masses that, at finite lattice spacing, are significantly reduced compared to those from polynomial fits. It turns out, however, that, with present statistics, the shift is not significant after continuum extrapolation.

All in all, despite the progress that has been made, we are lacking the comprehensive data sets, including non-perturbative matching factors, which would allow fully controlled chiral and continuum extrapolations. My summary of the present situation is that, in the quenched approximation, \( \overline{m} \) lies in the range \( 4 - 5 \text{ MeV} \). I expect some clarification in the upcoming year, particularly using the matching factors calculated using the Schrödinger functional.

The upwards movement of the quenched results brings them into reasonably comfortable consistency with the bounds derived using QCD sum rules. But what are the quenching errors? Early calculations suggested that quark masses were smaller when quark loops were included, and this has been confirmed by the results from CP-PACS. Their preliminary analysis suggests that two flavors of dynamical quarks reduce the light quark masses by about 40%, which might well bring them into contradiction with the sum rule bounds. It will be very interesting to see how things evolve over the next few years. There must be unexpectedly large systematic errors in either the lattice or the sum rule methods (or both). Suggestions for the latter have been discussed in Ref.

\[^{102}\]In the talk I mistakenly quoted a larger range \( 4 - 6 \text{ MeV} \). The upper end of this range was based on the result from Ref. in the RI, rather than the \( \overline{MS} \), scheme.
The systematic errors in determining $\overline{m}$ occur also in the
determination of the strange quark mass. In addition, as is clear from comparing Figs. 3 and 4, a different value results if one uses the $m_K$ or $m_{\phi}$ to fix the strange mass.\footnote{There is an analogous dependence of $\overline{m}$ on whether it is determined using $m_s/m_{\rho}$ or $m_s/m_N$, but, as can be seen from Fig. 3, the difference is less statistically significant than that for $m_s$. It is conventional to fix $\overline{m}$ using $m_{\rho}$ rather than $m_N$, because the former has a smaller statistical error.} Using $m_K$ leads to $m_s/\overline{m} \approx 25$ if the non-linearities in $m_s^2/m_q$ are small, whereas using $m_{\phi}$ turns out to give $m_s/\overline{m} \approx 30$. Thus I have converted $\overline{m} = 4 - 5$ MeV to $m_s = 100 - 150$ MeV for my estimate of the likely range for the quenched strange quark mass.

6.2 The kaon $B$-parameter

Little has changed in the result for $B_K$ since the reviews
by Flynn and myself. Nevertheless, since it is an important number for phenomenology, I want to stress a few points.

The most reliable determination comes from staggered fermions, and the best data by far is that from JLQCD. In a tour de force they have distinguished between discretization errors and errors due to the use of one-loop perturbative expressions for matching factors. The former are proportional to $a^2$ while the latter vary as $[a_s(a)]^2 \propto |\ln a|^{-2}$. Their final result is that $B_K(N\Delta R, \mu = 2\text{GeV}) = 0.628(42)$. The relatively large error is almost entirely due to the uncertainties in fitting the $a$ dependence—their statistical errors are much smaller. This is yet another indication that accurate numbers for matrix elements require non-perturbative determinations of matching factors.

The JLQCD data also contains another warning. Although the calculation is automatically improved, with the discretization errors being of $O(a^2)$, the continuum extrapolation requires accurate data down to $a = 0.05$ fm. This is because the dimensionful coefficients multiplying powers of $a$ turn out to be large. In the notation of Eq. 5\footnote{For a clear discussion of the errors in the relativistic and non-relativistic approaches the errors arise from the truncation of the non-relativistic expansion, and the need (so far at least) to match the non-relativistic theory with QCD using perturbation theory truncated to one-loop order. Although the various groups are making quite sophisticated error estimates for their respective calculations, the uncertainties in the errors themselves are large. This makes it} even though $\Lambda_1 = 0$, $\Lambda_2$ is a few times larger than $\Lambda_{QCD}$, as are the higher order $\Lambda_n$. Thus improvement does not always work as expected.

Converting the JLQCD result to the renormalization group invariant quantity $\hat{B}_K$ (using the assumptions of Ref. 1\footnote{There is an analogous dependence of $\overline{m}$ on whether it is determined using $m_s/m_{\rho}$ or $m_s/m_N$, but, as can be seen from Fig. 3, the difference is less statistically significant than that for $m_s$. It is conventional to fix $\overline{m}$ using $m_{\rho}$ rather than $m_N$, because the former has a smaller statistical error.}) gives the result quoted in Table 3. Fortunately, as noted by Gupta, the result for $\hat{B}_K$ is quite insensitive to whether this conversion is done using the quenched or unquenched formula. This final quenched result differs from that quoted in 1996 because the final JLQCD number is slightly higher than their preliminary result due to the more sophisticated continuum extrapolation.

The estimate of quenching errors has not changed significantly since 1996. Numerical data from unquenched simulations with two and four flavors implies that quark loops increase $B_K$ by a factor of $1.05 \pm 0.15$, where the error estimate is quite conservative. The change from degenerate quarks to physical non-degenerate quarks in the kaon is estimated, using chiral perturbation theory, to increase $B_K$ by another factor of $1.05 \pm 0.05$. Here the error estimate is less conservative. Combining these two factors gives $1.10 \pm 0.16$, where the errors have been combined in quadrature. Multiplying the quenched result by this factor gives the final result quoted in the table. Whether one should prefer this number to the quenched result, supplemented by an estimate of the quenching error, is a matter of taste. I prefer to include the estimates of the effect of unquenching, along with their fairly conservative errors, since they are based on reasonably reliable arguments.

6.3 $B-$meson matrix elements

The study of heavy-light mesons and baryons is now the major focus of that part of the lattice community oriented towards phenomenology. This is certainly appropriate given that $B$-factories promise a wealth of precise data in the near future. Lattice calculations will likely play an important role in the extraction of the underlying parameters of the standard model from these data. Although I report here results only for decay constants and $B-$parameters, I want to stress that fully controlled quenched results for many other quantities are likely to appear soon. Examples include form factors for $B \to D^{(*)}\ell\nu$, $B \to (\rho, \pi)\ell\nu$, $D \to K^{(*)}\ell\nu$ and $B \to K^*\gamma$, and the spectrum of excited hadrons containing $b$ quarks.

For further details see the above-mentioned reviews.\footnote{There is an analogous dependence of $\overline{m}$ on whether it is determined using $m_s/m_{\rho}$ or $m_s/m_N$, but, as can be seen from Fig. 3, the difference is less statistically significant than that for $m_s$. It is conventional to fix $\overline{m}$ using $m_{\rho}$ rather than $m_N$, because the former has a smaller statistical error.}

The most important theoretical issue in heavy quark calculations on the lattice is the accuracy with which one can simulate the $b$ quark. Direct simulation of a $b$-quark with a relativistic lattice action is not possible because discretization errors are large for present lattice spacings since $m_b a \gg 1$. To avoid these errors, one can either use a relativistic action but extrapolate from lighter quarks (typically somewhat heavier than the charm quark), or treat the $b-$quark using a non-relativistic expansion. Both approaches have potentially large systematic errors. In the relativistic approach these are from discretization errors, which can be reduced using improved actions and operators. In the non-relativistic approach the errors arise from the truncation of the non-relativistic expansion, and the need (so far at least) to match the non-relativistic theory with QCD using perturbation theory truncated to one-loop order. Although the various groups are making quite sophisticated error estimates for their respective calculations, the uncertainties in the errors themselves are large. This makes it
from various collaborations, which contain various improvements, but which are preliminary.

The quantities of particular interest to phenomenology are the last three in Table 3: $f_B\sqrt{B_B}$ is needed to extract information on the CKM elements from $B-\bar{B}$ mixing; $\xi$ is needed to determine the constraints from $B_s-\bar{B}_s$ mixing; and $f_D$, can be compared with experimental results. The errors in the quenched results for these quantities are relatively small, and it is now crucial to obtain estimates of the quenching error. For the decay constants themselves, such estimates have been made using numerical simulations by the MILC collaboration, and are included in the table. Reference 106 gives a detailed exposition of how these estimates are made, and notes that they are quite uncertain—the error on the quenching error is itself estimated to be $\sim 50\%$. For the $B$ parameters there is little numerical guidance as to the size of quenching errors. One can make theoretical estimates, based upon comparing chiral loops in quenched and full QCD 113,114. These suggest quenching errors as large as $\sim 10\%$ in the individual $B$ parameters, though somewhat smaller in the ratio $B_{B_s}/B_B$. Given the roughness of these estimates, as well as the uncertainty in the quenching errors for the decay constants, it is probably premature to attempt an estimate of the unquenched results for $f_B\sqrt{B_B}$ and $\xi$. Nevertheless, one requires numbers in order to place bounds on CKM parameters. My best estimates for the unquenched numbers are $f_B\sqrt{B_B} = 210 \pm 40$ MeV and $\xi = 1.14(13)$. To obtain these I have included the MILC estimate for the shift in $f_B$ due to unquenching, but inflated their quenching errors by $50\%$, and taken quenching errors on $B_{B_s}/B_B, f_{B_s}/f_B$, and $B_{B_s}/B_B$ to be $10\%$, $10\%$ and $5\%$ respectively, based on the chiral loop estimates. Draper has given similar unquenched estimates 86.

7 Exactly massless quarks on the lattice

My final topic concerns discretizations of the fermion action which exactly preserve (or almost preserve) the full global chiral symmetries of continuum QCD at non-zero lattice spacing. This is to be contrasted with existing discretizations (Wilson and staggered fermions), for which one cannot separate the chiral and continuum limits—chiral symmetry is only fully restored in the continuum limit. Separating these limits is very attractive from both a formal and a practical point of view. Continuum discussions of chiral symmetry breaking, topological properties, and rigorous inequalities often rely on exact chiral symmetry in the massless limit. These arguments carry over directly onto the lattice if chiral symmetry is maintained. Practical calculations of quantities constrained by chiral

Figure 10: Comparison of results for $f_B$, taken from Draper 88. Statistical and systematic errors are superimposed, the former indicated by the larger tick marks. The quoted average is also shown, together with the results included in determining it. difficult, at least for someone not involved in the calculations, to make a meaningful average of results from the different groups, and so I have chosen to quote averages from the review of Draper 88, if available, or from the review of Flynn and Sachrajda 109. Fortunately, in most cases, the results from different methods are consistent, so the central values are not likely to change much with a different weighting of the different results. I have illustrated this robustness by quoting examples of results from other reviews in the table. For the decays constants, Draper averages results from the following collaborations: GLOK 107, who use non-relativistic heavy quarks; FNAL 108 and JLQCD 109, who use improved Wilson quarks interpreted non-relativistically following Ref. 113, MILC 98, who extrapolate using unimproved Wilson quarks, interpreted following Ref. 113, and APE 110, who use both unimproved and improved Wilson quarks. The consistency between these results is illustrated in Fig. 10, along with Draper’s average. Also shown are various numbers which are not included in the average: old results which do not have as good control over systematic errors, and new results

relativistic approaches see Ref. 106 and Ref. 108 respectively.
symmetry, such as the quark condensate and kaon mixing and decay amplitudes would also be greatly simplified if there were no explicit chiral symmetry breaking in the lattice action. During the last year there has been a flurry of activity on this possibility, and I will attempt to sketch the essence of this work and its implications. For more details see the recent review of Niedermayer and the contribution by Neuberger.

Recall that, formally, chiral symmetry is restored in the massless limit because the Dirac operator anticommutes with $\gamma_5$: $\{ \mathcal{D}, \gamma_5 \} = 0$. When one discretizes the fermion action, and removes the doublers in a reasonable way (local Dirac operator with correct continuum limit), one finds that the anticommutator no longer vanishes

$$\{ \mathcal{D}_{\text{lat}}, \gamma_5 \} = 0(a).$$

This result follows from the famous Nielsen-Ninomiya no-go theorem. Two lines of work have attempted to evade the explicit chiral symmetry breaking on the RHS of Eq. \((12)\). The first was begun by Ginsparg and Wilson in 1982. They noted that the $O(a)$ term in Eq. \((12)\) could, in principle, be made to vanish in physical matrix elements by a suitable choice of fermion action. They formulated a condition, now known as the “Ginsparg-Wilson relation”, the simplest form of which is

$$\{ \mathcal{D}_{\text{lat}}, \gamma_5 \} = a \mathcal{D}_{\text{lat}} \gamma_5 \mathcal{D}_{\text{lat}}$$

If this relation is satisfied, then chiral Ward identities are violated only by contact terms (because the two factors of $\mathcal{D}$ on the rhs of Eq. \((13)\) cancel the propagators joining the lattice artifact to other parts of correlation functions), and thus all physical consequences of these identities remain valid. In other words, the Ginsparg-Wilson relation ensures that there is, in effect, a chiral symmetry. Despite these nice properties, no explicit solutions to the relation were found when gauge interactions are included. The difficulty is clear from an equivalent form of the relation, $\{ \mathcal{D}_{\text{lat}}^{-1}, \gamma_5 \} = a \gamma_5$, which requires the anticommutator of the inverse Dirac operator with $\gamma_5$ to be local. Thus the Ginsparg-Wilson relation lay in obscurity for many years.

The second line of work was the attempt, by many authors over many years, to come up with a discretized version of a chiral gauge theory (or at least a lattice theory which one could show became a chiral gauge theory in the continuum limit). A very interesting approach to this problem was initiated by Kaplan’s domain wall proposal. As explained in Ref. \cite{18} this and other work prompted Narayanan and Neuberger to develop the “overlap” formulation of chiral gauge theories. Whether this proposal works for chiral gauge theories is a matter of some controversy, but it is uncontroversial that the “overlap” provides a formulation of vector gauge theories such as QCD with exactly massless fermions. It evades the no-go theorem by having, in effect, an infinite number of regulator fields. This solution to the problem of exactly massless quarks in vector theories also lay in relative obscurity for the last few years, perhaps because a practical implementation of the overlap formulation seemed difficult.

In the last two years these threads have come together. Neuberger showed how one could integrate out the extra degrees of freedom in the overlap formulation and obtain a relatively simple form for a lattice Dirac operator corresponding to exactly massless quarks. This he called the “overlap-Dirac operator”. The second step was the “rediscovery” of the Ginsparg-Wilson relation: Peter Hasenfratz noted that “classically perfect” fermion actions satisfy the relation. These actions (reviewed in Ref. \cite{9}) thus provide another solution to the problem of discretizing a vector gauge theory with massless quarks. Neuberger then noted that the overlap-Dirac operator also satisfied the Ginsparg-Wilson relation.

Finally, Lüscher clarified the previous work by showing that any lattice Dirac operator satisfying the Ginsparg-Wilson relation has an exact, albeit non-local, chiral symmetry. Furthermore, he showed with a very compact argument how “Ginsparg-Wilson fermions” satisfy an exact lattice version of the continuum Atiyah-Singer index theorem. His work has led to further developments in the formulation of chiral gauge theories.

The end result is that we now have explicit forms of lattice fermion actions with exact chiral symmetry, and a simple and general formalism for understanding them based on the Ginsparg-Wilson relation. This advance has spawned a flood both of theoretical work aimed at determining the consequences of the Ginsparg-Wilson relation in detail, and at finding a practical implementation.

### 7.1 Numerical implementations

My previous sketch left out what may be the most important development from a practical point of view, namely “domain-wall fermions”. Kaplan’s proposal for a chiral gauge theory requires an extra, fifth dimension of infinite extent, in which the chiral theory lives on a four-dimensional domain-wall. Truncating to a finite fifth dimension introduces a second domain-wall inhabited by fermions of opposite chirality, so that overall one has a vector theory. Thus, as stressed by Shamir, domain-walls provide a practical method for studying QCD. The coupling between the left and right-handed fermions is suppressed exponentially by the distance between the two walls, and so violations of chiral symmetry are con-
trollable and can, in principle, be made as small as one wants. One has again succeeded in separating the chiral and continuum limits.

In the last year there have been extensive numerical simulations using domain-wall fermions, and they indicate that this is a very promising approach. In particular, it may be possible to get away with only 10 – 20 sites in the extra dimension. Of course, the simulations slow down by this factor, but this may be more than compensated for by the lack of chiral symmetry breaking, and by the fact that the domain-wall action is automatically $O(a)$ improved. For more details see the recent review by Blum. There have also been investigations of the properties of the overlap-Dirac operator and the feasibility of numerically inverting it. The non-locality makes inversion slow, but the potential gains are great. I expect a lot of development in these directions during the upcoming year.

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