On the Modular Dynamics of Financial Market Networks

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The financial market is a complex dynamical system composed of a large variety of intricate relationships between several entities, such as banks, corporations and institutions. At the heart of the system lies the stock exchange mechanism, which establishes a time-evolving network of transactions among companies and individuals. Such network can be inferred through correlations between time series of companies stock prices, allowing the overall system to be characterized by techniques borrowed from network science. Here we study the presence of communities in the inferred stock market network, and show that the knowledge about the communities alone can provide a nearly complete representation of the system topology. This is done by defining a simple random model sharing only the sizes and interconnectivity between communities observed in the time-evolving stock market network. We show that many topological characteristics of the inferred networks are preserved in the modeled networks. In particular, we find that in periods of instability, such as during a financial crisis, the network strays away from a state of well-defined community structure to a much more uniform topological organization. This behavior is found to be strongly related to the recently formalized von Neumann entropy for graphs. In addition, the community structure of the market network is used to define a community-based model of crisis periods observed in the market. We show that such model provides a good representation of topological variations taking place in the market during the crises. The general approach used in this work can be extended to other systems.

I. INTRODUCTION

The quantitative analysis of real complex systems is mostly focused on the study of time series, since the full knowledge of the underlying equations of a given dynamic system is hard or usually impossible to obtain from static data [1–4]. This situation is frequently found in the analysis of a number of applicable domains, including financial markets [5], physiological data [6], climate modeling [7] and several other systems [7].

Many tools based on nonlinear dynamics (e.g., fractal dimensions, Lyapunov exponents and recurrence properties) have been developed to analyze time series originating from complex systems [7]. However, besides such conventionally studied approaches, recently, concepts of network science have been applied to this problem too. Generally, most available methods map time series into the network domain, so that a set of network measurements can be used to calculate the statistical properties of the system. These methods can be quite different and include cycle networks [8], transition networks [9], k-nearest-neighbors [10], visibility-graphs [11] and also recurrence networks [12, 13] which is a natural extension of the traditional recurrence analysis of dynamical systems [14, 15]. Ref. [13] provides a brief review of the analysis of time series using complex networks techniques, also comparing the approaches and pointing to possible pitfalls and limitations of their applications.

The aforementioned methods analyze a single time series using complex networks. However, one could also tackle complex systems consisting of sets of time series, rather than just a single sample. Examples of such systems include the stock market in which each asset has a time evolving market price [16–20]; temporal spatio-grid climate data, where each grid point has a time series associated to a given climate variable (e.g., temperature, wind velocity, etc.) [21–24]; and also grid temporal data collected from electroencephalograms (EEG) or functional magnetic resonance images (fMRI) [25, 26]. Each of these systems are represented by discrete elements whose physical interactions can be inferred from the corresponding time series.

Recently, considerable effort has been expended on analyzing such systems using complex networks techniques [27]. The motivation for using the network approach in this case is natural, since these systems are composed of discrete parts forming a complex pattern of interactions. Network techniques usually establish the connections (interactions) by quantifying the statistical similarities between each discrete element, yielding a so-called functional network [25]. Moreover, the complex network approach to complex systems also provides an effective and comprehensive set of visualization tools, which allows important insights on the relation between the network structure and the underlying system dynam-
The benefits of network visualization can be realized, for instance, in the study of biological [28-31], social [32-33] and transportation networks [34], among others [33-35].

Naturally, these functional networks can evolve with time, and techniques can be developed for network construction in order to analyze the underlying dynamic system. In such an approach, the main objective is the detection of extreme events that significantly modify the network structure, which can in turn be related to critical events in the underlying complex system. For instance, in the temporal analysis of climate networks one frequently occurring problem is detecting extreme events caused by El Niño-Southern Oscillation (ENSO) [23, 24, 36-38] or the occurrences of Monsoons [39, 40]. In financial market networks, it is the effect of financial instabilities in the cluster organization of stocks that is of interest [16-20, 41]. In each case, the occurrence of extreme events is inferred from the detection of anomalies in the time series originating from the network evolution.

In this paper, we compare the set of time series obtained from the evolution of the inferred network with those obtained from a model of the evolution process. In particular, we focus on the emergence of community structures as a control feature for inducing extreme events or crises in the network.

The methodology outlined in our paper is described in detail in Fig. 1. Using a financial market data set, we obtain a time-evolving network (step a), which is inferred from the Pearson correlation coefficient between company daily closing stock values. A set of time series of topological measurements is obtained from the evolving network (step b), and used to characterize the evolution of network structure (step c). Using the widely documented crisis periods (step d), we evaluate the extent of which topological feature correctly reflects the changes in network structure during different crises (step e).

In order to model the evolution of the stock market network, a set of community properties is obtained (step f). These properties are used as input for two different models of the time-evolving network, namely the instantaneous model and the crisis simulation model. Starting with the instantaneous model, at each time step, a random network with same average degree and community structure as the corresponding inferred network is created, generating a time-evolving community network (step g). Finally, once the time series of topological measurements are extracted (steps h and i), the network evolution derived from the random model can be compared with the time series of topological measurements of the inferred network (step j).

The final part of our analysis is devoted to the comparison of the second random model considered, namely the crisis simulation model, with the inferred network evolution. More precisely, using information about the community structure of the inferred network and the documented market crisis periods, we simulate the expected community evolution of the stock market network (step k). The topological properties of the simulated networks during the crisis periods (steps l and m) are obtained, and the time series of the topological measurements found in step (e) are compared with those retrieved from the real data (step n).

Our suggested models for the stock market networks are simple stochastic blockmodels and based solely on the community structure of the time evolving inferred network. As far as we known, our approach is different from other approaches on network community models, as most of them are focused on testing community detection algorithms [12, 14]. While more sophisticated community models could also have been used, they would incorporate not only the community structure but also other characteristics of the original system, such as degree distribution and average path length. Therefore it could be difficult to disentangle the effects of these properties on the results. Also, because our model is stochastic, it allows us to naturally derive the dynamical model used to simulate stock market crisis. It is important to highlight that the purpose of the models are not to predict the behavior of the financial market networks, but to understand how informative is the community structure when traditional topological features are used to analyze such networks.
We validate our framework by analyzing functional networks constructed through statistical similarities between stocks traded at the New York Stock Exchange (NYSE). We show that the financial crashes are characterized by the presence of well-defined stock clusters, whereas outside these critical periods the network topology is predominantly homogeneous, i.e., without a clear formation of communities between the stocks.

This paper is organized as follows. In Sec. II we specify how the time evolving network of the financial market is inferred, and describe some basic community changes observed in the network during a single crisis period. In Sec. III we present the instantaneous community model, which is used to highlight the relevance of the community structure for the financial market. Sec. IV is devoted to the crisis simulation model. Finally, in Sec. V we present the conclusions of the study.

II. THE TIME-EVOLVING STOCK MARKET NETWORK

The stock market database consists of daily prices of 3799 stocks traded on the New York Stock Exchange. The stock prices were obtained from the Yahoo! financial database (http://finance.yahoo.com). We selected $N = 348$ stocks from this set, which are the stocks for which there are historical data from January 1986 to February 2011. For these stock, we obtained 6008 closures prices per stock over the trading period.

In the financial market networks analyzed in this paper, the nodes correspond to stocks and the links quantify the statistical similarity between the time series associated to the stock closure prices evolution. In particular, in order to quantify the similarity between two time series, we adopt the Pearson correlation coefficient

$$\rho_{ij} = \frac{(Y_i Y_j) - (Y_i) (Y_j)}{\sqrt{(Y_i^2) - (Y_i)^2} \sqrt{(Y_j^2) - (Y_j)^2}},$$

where $Y_i(t)$ is the logarithm of return, i.e., $Y_i(t) = \ln P_i(t) - \ln P_i(t - 1)$ is the price return, where $P_i(t)$ is the closing price of the $i$-th stock at day $t$. By calculating Eq. 1 between all pairs of stocks we obtain a fully connected weighted graph in which the link weights are given by $\rho_{ij}$. However, in order to analyze the network structure according to the stronger connections, we discard links whose weights are below a threshold $\epsilon$. This leads to a network defined by the adjacency matrix $A_{ij} = \Theta(\rho_{ij} - \epsilon) - \delta_{ij}$, where $\Theta(\cdot)$ is the Heaviside function and $\delta_{ij}$ the Kronecker delta.

The analysis of the time evolution of the inferred financial market network is illustrated in Fig. 2. First we set a time window of length $\Delta t = 30$ days inside which the correlations between stocks are calculated creating the network. Next, by sliding the window by amounts $\delta t$ we are then able to generate a sequence of networks corresponding to the evolution of the market. More precisely, the first network $N_1$ is constructed by calculating the correlations of the time series at $t^{(1)}_1 = 1$ and $t^{(1)}_2 = 29$, the second network $N_2$ is constructed considering data between instants $t^{(2)}_1 = 2$ and $t^{(2)}_2 = 30$, and the $n$-th network encompasses the prices between $t^{(n)}_1 = 1 + (n - 1)\delta t$ and $t^{(n)}_2 = t^{(n)}_1 + \Delta t$.

In order to better understand the evolution of the financial market network, it is useful to first visualize how its structure is organized near critical points. A useful and clearly defined event that can be used as a reference for the effect of financial instabilities in the network structure is the famous Black Monday crisis, which occurred in October 19, 1987 [20, 45]. After employing the methodology described above for network construction, in Fig. 3 we show the time evolution of the von Neumann entropy $S$ [10] of the network. The von Neumann entropy has previously been used to quantify changes in the evolving structure of citation networks [47] and can detect critical events due to changes in network topology. In this figure, we also show network visualizations corresponding to four different instants of time. Each node color corresponds to a different community detected by the method presented in [45]. Furthermore, in order to observe community evolution, the community membership of each node is calculated only in network $A$ of Fig. 3 so that the community assignment is kept fixed in the following networks in the time series.

From the figure, it is clear that before the crisis the modular structure is mainly composed of two predominant communities. As the network approaches the crisis the network structure changes drastically, and the community structure substantially vanishes. Only a highly connected cluster at the center of the network remains. At this epoch, most stocks are disconnected, meaning that the prices evolve without strong correlations. Note that during the crisis, the network exhibits a more homogeneous structure, as suggested by the lower values of the von Neumann entropy $S$ [17, 19]. Similar repercussions of the Black Monday were reported in the temporal analysis of the Minimum Spanning Tree (MST) constructed from the NYSE data [20, 45]. Here it is observed that the asset tree shrinks, i.e., the normalized tree length decreases, reducing the topological distances between the stocks. This result also agrees with other findings on the structural organization of financial market networks [41, 50, 54]. It is also interesting to note in Fig. 3 that, throughout the period considered, the communities preserve most of their membership composition, even in very long periods from the crash, when the network becomes reconnected again.

In order to quantitatively investigate the relationship between a financial crisis and the von Neumann entropy, we analyze a set of well known crisis periods. These periods are marked alongside the curve of the von Neumann entropy in Fig. 4. For each considered crisis, we observe a decline of the von Neumann entropy around the time span of the crisis. This indicates that the von Neumann entropy captures topological network properties related
FIG. 2: Diagram illustrating the method to construct the financial market networks. The network is constructed by calculating the correlations between the stocks returns $Y_i$ ($i = 1, 2, ..., N$) inside a time window of length $\Delta t$. Next, by shifting this time window by amounts $\delta t$ until the end of the database is reached, we obtain the network evolution.

FIG. 3: Community structure of the NYSE networks in distinct times during and around the Black Monday crisis [20, 45]. Here the community structure is calculated only for network A, the membership of each node is kept fixed in the following visualizations.
As discussed above, the community structure of the financial market network displays strong variations during crisis periods. Thus, a natural measurement that can capture such variations is the modularity. For each time step, we apply the multilevel community detection algorithm to the networks generated using the methodology described in Sec. I. We then use the community membership of the nodes to calculate what we refer to as the dynamical modularity. The term dynamical community comes from the fact that at each time step the labels of the nodes are being reassigned. In Fig. 5 we show in blue the evolution of the dynamical modularity for the entire time series. We see that this measurement contains spikes, that is, increased values of dynamical modularity during short periods, while there is no long-time variation of the modularity. Although the dynamical modularity is a useful indicator of the overall association between stock prices, it provides no information about the actual changes occurring inside the communities. For example, the membership of the nodes can change along time steps without modifying the dynamical modularity.

Therefore, we also characterize the community evolution of the financial market by using a fixed modularity defined as follows. The community structure found on the first time step defines the membership of the nodes for all time steps, and the modularity is calculated based on this membership. In Fig. 5 we show in orange the evolution of the fixed modularity. It is likely that, over a long time, stocks may evolve into different communities, so fixed modularity tends to decrease with time. It is clear that spikes are less pronounced than when using the dynamical modularity. Also, there is a clear transition of the fixed modularity around the year 2002, which means that there was a noticeable change of the financial market at this epoch. We will discuss this transition on the following section. Nevertheless, an obvious problem with the fixed modularity is that there is nothing to distinguish the starting point of the time series, where the community composition and node membership was calculated. This problem can be solved by defining a lagged modularity, in which the memberships are calculated at time $t-t_\Delta$ and used to determine the modularity at time $t$. If $t_\Delta = 0$, we simply recover the dynamical modularity. For large $t_\Delta$ the values tend to the fixed modularity, since when $t-t_\Delta < 0$ we consider the memberships calculated in the first time step. To some extent, the value $t_\Delta$ is related to the memory of the market, that is, the extent to which its structure changed during the interval $t_\Delta$. In Fig. 5 we show the lagged modularity for $t_\Delta = 100$ business days. In Fig. S1 of the supplementary material we present the lagged modularity for different values of $t_\Delta$. In a manner similar to the fixed modularity, the spikes of the lagged modularity are also less pronounced than when using the dynamical modularity. This means that although the market contains pronounced communities during the spike periods, the composition of the communities are varying with time.
III. INSTANTANEOUS COMMUNITY MODEL

In the stock exchange system, companies are typically organized into several sectors (i.e., industry, services, banking, etc.) which may possess very distinctive relationship patterns and time series behavior. The inferred networks are also expected to exhibit a modular organization based on node sectors. In the previous section, we showed qualitatively that the community structure on the inferred networks may be related to periods of crisis (Fig. 3).

To further investigate the importance of the community structure in this system, we devise a simple stochastic model to generate networks that preserve only information about the community structure of the inferred networks. Starting from the set of inferred networks obtained from the NYSE dataset, for each time step $t$ we obtain the community structure $C(t)$ by employing the multilevel community detection method proposed in [48].

From the resulting partitioning of the nodes, we obtain a set of community mixing matrices $\Pi(t)$. The elements in $\Pi_{\alpha\beta}(t)$ correspond to the number of connections between each pair of communities ($\alpha, \beta$) at time $t$, where $\alpha \in C(t)$ and $\beta \in C(t)$. Additionally, the number of nodes within each community $\alpha$, $|\alpha|$, is also considered. We define our model in terms of a sequence of stochastic mixing matrices, $\pi(t)$, that can be expressed by a normalization of $\Pi(t)$:

$$\pi_{\alpha\beta}(t) = \begin{cases} \frac{2\Pi_{\alpha\beta}(t)}{|\alpha|(|\alpha|-1)} & \text{if } \alpha = \beta, \\ \frac{\Pi_{\alpha\beta}(t)}{|\alpha||\beta|} & \text{if } \alpha \neq \beta. \end{cases}$$

The parameter $\pi(t)$ corresponds to the same probability of connection between groups on the simplest blockmodel [56]. However, here we took a different approach to the generative model, we apply a wiring process much more similar to the way that simple Erdős Rényi networks are constructed. For each instant, a model network is initially generated as a fully disconnected graph with $\sum_{\alpha \in C(t)} |\alpha|$ nodes. Each node is assigned to a community $\alpha \in C(t)$ in agreement with the given community sizes, $|\alpha|$. Next, pairs of nodes $(i,j)$, with community memberships $i \in \alpha$ and $j \in \beta$, are connected according to the probability $\pi_{\alpha\beta}(t)$.

The resulting networks contain communities formed by uniform random graphs (similar to the Erdős Rényi model), and the only characteristic they share is that they evolve with time as the communities at each time step. We refer to this model as the instantaneous market model.

The remainder of this section is devoted to comparing the topological properties obtained from our model with those for the inferred networks. We start by measuring six topological characteristics [57], namely dynamical modularity [55], average shortest path length [58], average betweenness [59], degree assortativity [60], transitivity [61] and von Neumann entropy [46]. The dynamical modularity, shown in Fig. 6(a), is measured to provide a confirmation that our model is indeed capturing the community structure of the inferred networks. The Pearson correlation coefficient between the dynamical modularity of the inferred and artificial networks for the entire series is $\rho_{1987 \rightarrow 2011} = 0.97$. As noted before, there is an apparent transition of the fixed modularity around the year 2002. Therefore, we also divide the whole series into two intervals, the period 1987 → 2002 and 2002 → 2011.

We calculate the Pearson coefficient for each period separately, and show the results in the plots for each respective measurement. We note that the correlation of dynamical modularity for the period 2002 → 2011 is lower than in the previous period.

In Fig. 6(b) we show the average shortest path length of the inferred and model networks. This measurement does not change significantly with time, except at epochs where the modularity increases, since a more modular
The degree assortativity of the networks is shown in Fig. 7(b), where again there was a good agreement between the model and the inferred network. The assortativity of a uniformly random network should be close to zero, since there is no preferential connectivity between nodes of similar degree. Therefore, one might conclude that our model should produce non-assortative networks, but the community description of a network is so effective that it also constrains the degree assortativity. For example, if one of the network communities is denser than remaining communities, then the nodes belonging to this community will have large degrees and have a tendency to be more strongly interconnected. This increases the assortativity of the network. The transitivity (shown in Fig. 7(c)) is the characteristic that displays the largest deviation between the inferred networks and the model. This occurs because the transitivity of uniformly random networks tends to zero for large network sizes [25]. Since the inferred networks actually have a non-zero transitivity, the model will usually have a lower transitivity values than for the inferred case. Nevertheless, the transitivity variations between time steps are still well-explained by the simulated networks, since the community structure captured by the model also has a strong influence on the transitivity. The final measurement applied to test the veracity of our model is the von Neumann entropy [46]. The calculated values are shown in Fig. 7(c). Again, the community structure of the inferred network is sufficient to represent the evolution of the von Neumann entropy of the data with reasonable accuracy.

In order to summarize the results obtained for the six measurements presented above, we apply Principal Component Analysis technique [62] to the data. The original 6-dimensional space spanned by the network measurements is projected onto a 2-dimensional one, defined by the first two PCA components. The result is shown in Fig. 8. It is clear that the model can accurately reproduce the distribution of inferred networks for the period 1987 → 2002. Surprisingly, the year 2002 represents a transition of the financial market, where the networks have a fairly a distinct structure from those belonging to the previous period. After this transition, the community structure of the inferred networks does not seem to possess the same amount of information as in the previous period, since the model is no longer able to explain the measured quantities of the networks with the same accuracy.

Finally, as a final example, in Fig. 9 we visually compare the inferred networks with those generated by our model. The inferred networks were sampled every 1000 time steps between 10 May 1991 and 26 March 2007. To construct the visualization we use a force-directed method based on the Fruchterman-Reingold algorithm [83] for which only the network topology is taken into account, hence no information about the communities is considered aside from nodes colors. Again, we observe that the networks constructed by the mixing model are able to qualitatively describe the topology of the inferred financial networks.

IV. CRISIS SIMULATION MODEL

As observed in the previous section, the community structure of financial networks is of fundamental importance to characterize such systems. In addition, the instantaneous model is able to capture many topological features of the inferred networks. Specifically, the von Neumann entropy calculated from the model adheres closely to the values obtained for the inferred networks. As observed in Sec. I this measurement appears to be intrinsically related to periods of financial crisis. Therefore, we use this measurement to evaluate a crisis model of the financial market networks, and was allowed to correctly represent crises occurring in the inferred networks.

We define the crisis model as an extension of the instantaneous community model described in the previous section. However, in contrast with the instantaneous model, the main objective of the crisis model is to simulate the dynamics occurring in the system. Hence the need for both structural and dynamical parameters.

The crisis model is based on a stochastic rewiring process occurring over time on a network initially derived from the instantaneous model. We opted for a rewiring process mainly because it is one of the simplest topological dynamics used to model time evolving networks. Thus, initially a network is generated by employing the instantaneous model with parameters $H_{\text{initial}}$ as the initial mixing matrix and community partitioning set, $C = \{\alpha_0, \alpha_1, \cdots, \alpha_N\}$, which encompass both the community sizes $|\alpha_n|$ and memberships of nodes. To obtain the time evolving network, a sequence of epochs $T^* = (T_0, T_1, \cdots, T_{m+1})$ is needed as an extra parameter. The epochs in $T^*$ indicate marked changes of the system properties, such as when the system enters or leaves a crisis period. From $T^*$ we define the intervals $\Delta T_m$ as the time period between $T_m$ and $T_{m+1}$. For instance, during an interval $\Delta T_m$ the system may be undergoing an intense crisis, while in $\Delta T_{m+1}$ the network can be enduring a weaker crisis period. The model is constrained by the initial community memberships given by $C$, which is maintained fixed during the simulation. However, the stochastic probabilities of rewiring are free to change over the time and are defined by a sequence
FIG. 6: Dynamical modularity, average path length and average betweenness centrality obtained for the NYSE networks and the model. The Pearson correlation coefficient was calculated for three time intervals: \( p_{1987\rightarrow2011} \) comprising the entire time series, \( p_{1987\rightarrow2002} \) and \( p_{2002\rightarrow2011} \) calculated over the respective time intervals.

of mixing matrices \( \Pi^* = (\Pi(\Delta T_0), \Pi(\Delta T_1), \ldots, \Pi(\Delta T_M)) \) corresponding to the community organization of the inferred network at each considered time interval. In addition, a sequence of dynamical rewiring matrices \( \Xi^* = (\Xi(\Delta T_0), \Xi(\Delta T_1), \ldots, \Xi(\Delta T_M)) \) is also calculated for the inferred network time series. They correspond to the average number of rewirings between each pair of communities for the time intervals in \( T^* \), as given by:

\[
\Xi(\Delta T_m)_{\alpha\beta} = \sum_{t \in \tau(\Delta T_m)} \frac{\Psi_{\alpha\beta}(t)}{|\tau(\Delta T_m)|},
\]

where \( \Psi_{\alpha\beta}(t) \) accounts for the number of edges connecting nodes from community \( \alpha \) to community \( \beta \) that does not exist in the next epoch \( t+1 \). The term \( \tau(\Delta T_m) \) represents the set comprising the time series intervals in the period \( \Delta T_m \).

To summarize, the crisis simulation model uses five characteristics from the real data: the starting community structure \( C \) which remains fixed during the entire simulation; the mixing matrix \( \Pi_{\text{initial}} \) used to construct the initial network corresponding to the epoch \( T_0 \); the sequence \( T^* \), which defines the crisis and stability intervals; the mixing matrices \( \Pi^* \) corresponding to the community connectivity of the networks for each element of time intervals defined by \( T^* \); and the rewiring matrices \( \Xi^* \) indicating the average rewiring occurring with time. It is important to stress that in this model, the memberships of nodes are provided at the start of the sequence and do not change with time. As a result this approach is limited to small time periods where the membership of nodes remains approximately constant over the considered intervals.

For each time step \( t \) in a time interval \( T_m < t < T_{m+1} \) we first normalize the rewiring matrix \( \Xi(\Delta T_m) \) in a sim-
FIG. 7: Assortativity, transitivity and von Neumann entropy calculated for the NYSE networks and the model. In the same fashion as figure [6]

ilar manner as present in Eq. [2]

\[ \xi(\Delta T_m)_{\alpha\beta} = \begin{cases} \frac{\pi(\Delta T_m)_{\alpha\beta}}{|\alpha|(|\beta|-1)} & \text{if } \alpha = \beta \\ \frac{\pi(\Delta T_m)_{\alpha\beta}}{|\alpha||\beta|} & \text{if } \alpha \neq \beta \end{cases} \]  

(4)

The resulting stochastic matrix \( \xi(\Delta T_m) \) corresponds to the probability of rewiring each class of edges between and within communities.

We start by selecting the set of edges, \( R \) to be rewired. This is done by testing each edge against the probability distribution given by \( \xi(\Delta T_m) \). The edges that pass the test are rewired according to the normalized matrix \( \pi(\Delta T_m) \) obtained by applying Eq. [2] to \( \Pi(\Delta T_m) \). Specifically, for each selected edge \((i, j) \in R\), we choose the new community \( \alpha \) to which it will be connected following the probability distribution:

\[ p_{\text{from}}^{\alpha}(\alpha) = \frac{|\alpha|}{|\gamma|} \sum_{\gamma \in C} \frac{\pi(\Delta T_m)_{\alpha\gamma}}{|\gamma|}, \]

(5)

for a given time period \( \Delta T_m \). The community for the other endpoint \( \beta \) is chosen according to the probabilities \( p_{\alpha}^{\beta}(\beta) \) given by the column representing the chosen community \( \alpha \) of \( \pi(\Delta T_m) \), thus:

\[ p_{\alpha}^{\beta}(\beta) = \pi(\Delta T_m)_{\alpha\beta}. \]

(6)

We rewire the edge by uniformly selecting the endpoints among the nodes of the chosen communities \( \alpha \) and \( \beta \). To avoid obtaining a network with self or multiple links we repeat these steps until the resulting edge is not already present on the new network.

In order to simulate crises using the crisis model, we extract parameters of the inferred networks, considering both periods of crisis and stability. The crisis intervals, which correspond to the sequence \( T^* \) in our model, are found by employing the von Neumann entropy of the inferred time-evolving networks and looking for locations with high variance. The matrices in \( \Pi^* \) are obtained directly from the inferred networks at each sampled epoch.
over the intervals $T^*$. The rewiring matrices $\Xi^*$ can be obtained by taking the average over the number of times the edges change during the corresponding periods.

In Fig. 10 we show both comparison of the von Neumann entropy obtained for the inferred and modeled networks, for 5 different crisis periods. Each plot is labeled above with the name of the crisis pertinent to the period analyzed together with the Pearson correlation coefficient between the two time series. The Black Monday crisis, which is a single day market crash, is well-represented by the model around the day of the crash. After the end of the crisis, our model displays an entropy rebound effect that is not observed in the inferred networks. Nevertheless, the correlation between the time series extracted from the inferred networks and predicted model is fairly high. For all other crises, the model represents the variations observed in the time series derived from the topological measurements of the inferred networks with high accuracy. When there is a sharp decrease or increase of von Neumann entropy in the inferred time series, the corresponding modeled networks change accordingly. The main characteristic of the inferred time series that is not captured by the model is the purposeful scale of the crisis. To overcome this problem, we apply a simple renormalization to the results of the model. That is, we rescale the entropy range of the time series generated from the model so that its maximum and minimum entropy values become equal to the maximum and minimum of the inferred time series. Using this entropy renormalization, we see that the model does indeed present good agreement with the original data. Still, the fact that the inferred and modeled time series are on different entropy scales means that the strength of the crises cannot be explained by the change in community structure alone. This may be due to the fact that von Neumann entropy reaches its maximum value on random networks \cite{17} and our proposed model is based on a random rewiring process.

V. CONCLUSIONS

The study of the evolving financial market network is of great interest for many reasons. Besides improving decisions related to industrial development and overall national growth, a reliable model of the evolution of a financial market can provide indicators of an imminent widespread stock value decline, which we refer to as a stock market crisis. In order to infer the underlying financial market network for a group of companies, one widespread technique is to obtain the correlation networks of the stocks exchanged between them \cite{17}. Such inferred networks tend to convey many relevant properties of the actual financial network established by the financial market. In addition, since companies can always be divided into distinct subgroups, related to their respective sector of activity or partnerships, the analysis of the community structure of the financial market is of considerable importance to understanding its dynamics.

The goal of this paper is to show that the community structure of the financial market contains an almost complete description of the dynamics of the system, in the sense that many of its characteristics can be recovered by knowing only the communities over a given time interval. We here demonstrated that by using the mixing matrix obtained from the inferred networks, many network measurements can indeed be recovered by generating a random network following the connectivity pattern indicated by this matrix. Interestingly, after the year 2002 the model becomes more ineffective in representing the inferred networks. This effect remains the topic of future work.

We also considered the fundamental problem of modeling market crises. Using the mixing matrices of the inferred networks together with derivatives, we were able to develop a crisis model for the stock exchange network. Using information about the beginning and duration of a number of well-known crises, we showed that such a crisis model can correctly predict variations in entropy observed during the crises. The extent of the crisis is not correctly represented by the model alone, but we demonstrated that a simple normalization based on the real data can improve the description provided by the model. Further, refinements to the crisis model are needed to improve its performance in simulating real crises. For instance, by considering that the community membership of nodes may change between crises. However, time-evolving networks with community structure are still an open problem in network science \cite{44}. Another interesting development would be study the analytical behavior of the von Neumann entropy on networks presenting community structure.

Our methodology can be applied to any system where
FIG. 9: Network visualizations comparing the inferred networks and those generated by the model. We sampled the networks every 1000 time steps. Colors indicate the community membership of nodes.

FIG. 10: Simulations of real crisis using the dynamical community model for five crisis periods. The curves in blue depict the von Neumann entropy obtained from the inferred networks. The results obtained for the model are displayed in orange and green, where the last is rescaled for better comparison with the inferred networks.

the underlying network can be inferred through similarity measurements. When the system also displays a natural representation based on its constituent communities, we expect the model to provide an accurate description of events occurring in the system.

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FIG. S1: Evolution of the dynamical, fixed and lagged modularity (considering 3 lag values) of the stock market network with time.