Abstract. The paper is dedicated to studying the problem of Poisson stability (in particular stationarity, periodicity, quasi-periodicity, Bohr almost periodicity, Bohr almost automorphy, Birkhoff recurrence, almost recurrence in the sense of Bebutov, Levitan almost periodicity, pseudo-periodicity, pseudo-recurrence, Poisson stability) of solutions for semi-linear stochastic equation

\[ dx(t) = (Ax(t) + f(t, x(t)))dt + g(t, x(t))dW(t) \quad (\ast) \]

with exponentially stable linear operator \( A \) and Poisson stable in time coefficients \( f \) and \( g \). We prove that if the functions \( f \) and \( g \) are appropriately “small”, then equation (\( \ast \)) admits at least one solution which has the same character of recurrence as the functions \( f \) and \( g \).

1. Introduction

A continuous function \( \varphi \) defined on real line \( \mathbb{R} \) with values in a metric space \( (X, \rho) \) is said to be Poisson stable [43, 44, 45, 49] in the positive (respectively, negative) direction if there is a sequence \( \{t_n\} \subset \mathbb{R} \) with \( t_n \to +\infty \) (respectively, \( t_n \to -\infty \)) such that \( \varphi(t + t_n) \to \varphi(t) \) uniformly with respect to \( t \) on every compact interval \([-l, l] (l > 0)\) as \( n \to \infty \). If \( \varphi \) is Poisson stable in both directions, then it is called Poisson stable.

One considers [43, 44, 45, 49] the following classes of Poisson stable functions: stationary (respectively, periodic, quasi-periodic [8, 9], Bohr almost periodic [10, 11, 12, 13, 14], almost automorphic [6, 7, 52], Birkhoff recurrent [5], Levitan almost periodic [35, 36], almost recurrent in the sense of Bebutov [3, 49], pseudo-periodic [14, p.32], pseudo-recurrent [42, 43, 49], Poisson stable [43, 49]) functions, among others.

In his works [43, 44, 45, 46, 47], B. A. Shcherbakov systematically studied the problem of existence of Poisson stable solutions of the equation

\[ x' = f(t, x), \quad x \in \mathcal{B} \]

with the right-hand side \( f \) Poisson stable in \( t \in \mathbb{R} \) uniformly with respect to \( x \) on every compact subset from \( \mathcal{B} \), where \( \mathcal{B} \) is a Banach space.
To study this problem, B. A. Shcherbakov established a method (principle) of comparability of functions by character of their recurrence. Using his method B. A. Shcherbakov studied different classes of equations of the form (1.1) for which he gave conditions of existence at least one (or exactly one) solution with the same character of recurrence as right-hand side \( f \). He named this type of solution comparable (respectively, uniformly comparable) solution of equation (1.1).

Later the works of B. A. Shcherbakov were extended and generalized by many authors: I. Bronshtein [15, ChIV], T. Caraballo and D. Cheban [16 17 18 19], D. Cheban [20 21], D. Cheban and C. Mammana [23], D. Cheban and B. Schmalfuss [24], and others.

In this paper, we try to extend and generalize Shcherbakov’s ideas and methods to study the Poisson stability of solutions for stochastic differential equations

\[
dx(t) = f(t, x(t))dt + g(t, x(t))dW(t),
\]

where \( f \) and \( g \) are Poisson stable functions in \( t \).

Note that this problem was studied before only for periodic, Bohr almost periodic and Bochner almost automorphic equations: see, e.g. [25 27 22 34 33] for periodic equations, [14 27 31 36 39 38 51] for Bohr almost periodic equations and [25 30 35 33] for Bochner almost automorphic equations, and references therein. It should be pointed out that either Bohr almost periodic or Bochner almost automorphic solutions can be only in distribution sense instead of in square-mean sense, see [33 38] for details. We consider in our present work the general problem of Poisson stability for all classes listed above.

This paper is organized as follows.

In the second section we collect some known notions and facts. Namely we present the definitions of all important classes of Poisson stable functions and their basic properties. We also give a short survey of Shcherbakov’s results on comparability of functions by character of their recurrence.

The third section is dedicated to studying Poisson stable solutions for the linear equation

\[
dx(t) = (Ax(t) + f(t))dt + g(t)dW(t)
\]

with exponentially stable linear operator \( A \) (generally unbounded). The main result of this section (Theorem 3.6) states that equation (1.2) with bounded coefficients \( f \) and \( g \) admits a unique bounded solution \( \varphi \) which has the same character of recurrence in distribution as \( f \) and \( g \).

In the fourth section we study the problem of Poisson stability for the semi-linear equation

\[
dx(t) = (Ax(t) + F(t, x(t)))dt + G(t, x(t))dW(t).
\]

We prove (Theorem 4.6) that the equation (1.3) has a unique bounded solution \( \xi \) which has the same character of recurrence as the functions \( F \) and \( G \).

The fifth section is dedicated to studying the dissipativity (Theorem 5.2) and the convergence (Theorem 5.4) for equation (1.3).

In the last section, we give some applications of our theoretical results.
2. Preliminaries

2.1. The space $C(\mathbb{R}, X)$. Let $(X, \rho)$ be a complete metric space. Denote by $C(\mathbb{R}, X)$ the space of all continuous functions $\varphi : \mathbb{R} \to X$ equipped with the distance

$$d(\varphi, \psi) := \sup_{L > 0} \min_{|t| \leq L} \{\max_{|t| \leq L} \rho(\varphi(t), \psi(t)), L^{-1}\}.$$ 

The space $(C(\mathbb{R}, X), d)$ is a complete metric space (see, for example, [43, ChI], [45, 49]). Throughout the paper, convergence in $C(\mathbb{R}, X)$ means the convergence with respect to this metric $d$ if not specified otherwise.

**Lemma 2.1.** ([43, ChI], [45, 49]) The following statements hold:

1. $d(\varphi, \psi) = \varepsilon$ if and only if $\max_{|t| \leq \varepsilon^{-1}} \rho(\varphi(t), \psi(t)) = \varepsilon$;
2. $d(\varphi, \psi) < \varepsilon$ if and only if $\max_{|t| \leq \varepsilon^{-1}} \rho(\varphi(t), \psi(t)) < \varepsilon$;
3. $d(\varphi, \psi) > \varepsilon$ if and only if $\max_{|t| \leq \varepsilon^{-1}} \rho(\varphi(t), \psi(t)) > \varepsilon$.

**Remark 2.2.** 1. The distance $d$ generates on $C(\mathbb{R}, X)$ the compact-open topology.
   2. The following statements are equivalent:
      1. $d(\varphi_n, \varphi) \to 0$ as $n \to \infty$;
      2. $\lim_{n \to \infty} \max_{|t| \leq L} \rho(\varphi_n(t), \varphi(t)) = 0$ for each $L > 0$;
      3. there exists a sequence $l_n \to +\infty$ such that $\lim_{n \to \infty} \max_{|t| \leq l_n} \rho(\varphi_n(t), \varphi(t)) = 0$.

2.2. Poisson stable functions. Let us recall the types of Poisson stable functions to be studied in this paper; we refer the reader to [11, 43, 45, 49] for further details and the relations among these types of functions.

**Definition 2.3.** A function $\varphi \in C(\mathbb{R}, X)$ is called stationary (respectively, $\tau$-periodic) if $\varphi(t) = \varphi(0)$ (respectively, $\varphi(t + \tau) = \varphi(t)$) for all $t \in \mathbb{R}$.

**Definition 2.4.** Let $\varepsilon > 0$. A number $\tau \in \mathbb{R}$ is called $\varepsilon$-almost period of the function $\varphi$ if $\rho(\varphi(t + \tau), \varphi(t)) < \varepsilon$ for all $t \in \mathbb{R}$. Denote by $T(\varphi, \varepsilon)$ the set of $\varepsilon$-almost periods of $\varphi$.

**Definition 2.5.** A function $\varphi \in C(\mathbb{R}, X)$ is said to be Bohr almost periodic if the set of $\varepsilon$-almost periods of $\varphi$ is relatively dense for each $\varepsilon > 0$, i.e. for each $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that $T(\varphi, \varepsilon) \cap [a, a + l] \neq \emptyset$ for all $a \in \mathbb{R}$.

**Definition 2.6.** A function $\varphi \in C(\mathbb{R}, X)$ is said to be pseudo-periodic in the positive (respectively, negative) direction if for each $\varepsilon > 0$ and $l > 0$ there exists an $\varepsilon$-almost period $\tau > l$ (respectively, $\tau < -l$) of the function $\varphi$. The function $\varphi$ is called pseudo-periodic if it is pseudo-periodic in both directions.

**Definition 2.7.** For given $\varphi \in C(\mathbb{R}, X)$, denote by $\varphi^h$ the $h$-translation of $\varphi$, i.e. $\varphi^h(t) = \varphi(h + t)$ for $t \in \mathbb{R}$. The hull of $\varphi$, denoted by $H(\varphi)$, is the set of all the limits of $\varphi^n$ in $C(\mathbb{R}, X)$, i.e.

$$H(\varphi) := \{\psi \in C(\mathbb{R}, X) : \psi = \lim_{n \to \infty} \varphi^n \text{ for some sequence } \{h_n\} \subset \mathbb{R}\}.$$
It is well-known (see, e.g. [22]) that the mapping \( \sigma : \mathbb{R} \times C(\mathbb{R}, X) \to C(\mathbb{R}, X) \) defined by \( \sigma(h, \varphi) = \varphi^h \) is a dynamical system, i.e. \( \sigma(0, \varphi) = \varphi, \sigma(h_1 + h_2, \varphi) = \sigma(h_2, \sigma(h_1, \varphi)) \) and the mapping \( \sigma \) is continuous. In particular, the mapping \( \sigma \) restricted to \( \mathbb{R} \times H(\varphi) \) is a dynamical system.

**Remark 2.8.** A function \( \varphi \in C(\mathbb{R}, X) \) is pseudo-periodic in the positive (respectively, negative) direction if and only if there is a sequence \( t_n \to +\infty \) (respectively, \( t_n \to -\infty \)) such that \( \varphi^{t_n} \) converges to \( \varphi \) uniformly in \( t \in \mathbb{R} \) as \( n \to \infty \).

**Definition 2.9.** A number \( \tau \in \mathbb{R} \) is said to be \( \varepsilon \)-shift for \( \varphi \in C(\mathbb{R}, X) \) if \( d(\varphi^\tau, \varphi) < \varepsilon \).

**Definition 2.10.** A function \( \varphi \in C(\mathbb{R}, X) \) is called almost recurrent (in the sense of Bebutov) if for every \( \varepsilon > 0 \) the set \( \{ \tau : d(\varphi^\tau, \varphi) < \varepsilon \} \) is relatively dense.

**Definition 2.11.** A function \( \varphi \in C(\mathbb{R}, X) \) is called Lagrange stable if \( \{ \varphi^h : h \in \mathbb{R} \} \) is a relatively compact subset of \( C(\mathbb{R}, X) \).

**Lemma 2.12.** ([33] ChI) Let \( \varphi \in C(\mathbb{R}, X) \), then the following statements are equivalent:

(i) the function \( \varphi \) is Lagrange stable;

(ii) the function \( \varphi \) is uniformly continuous on \( \mathbb{R} \) and its image \( \varphi(\mathbb{R}) \) is a relatively compact subset of \( X \).

**Definition 2.13.** A function \( \varphi \in C(\mathbb{R}, X) \) is called Birkhoff recurrent if it is almost recurrent and Lagrange stable.

**Definition 2.14.** A function \( \varphi \in C(\mathbb{R}, X) \) is called Poisson stable in the positive (respectively, negative) direction if for every \( \varepsilon > 0 \) and \( l > 0 \) there exists \( \tau > l \) (respectively, \( \tau < -l \)) such that \( d(\varphi^\tau, \varphi) < \varepsilon \). The function \( \varphi \) is called Poisson stable if it is Poisson stable in both directions.

In what follows, we denote as well \( Y \) a complete metric space.

**Definition 2.15.** A function \( \varphi \in C(\mathbb{R}, X) \) is called Levitian almost periodic if there exists a Bohr almost periodic function \( \psi \in C(\mathbb{R}, Y) \) such that for any \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that \( d(\varphi^\tau, \varphi) < \varepsilon \) for all \( \tau \in T(\psi, \delta) \), recalling that \( T(\psi, \delta) \) denotes the set of \( \delta \)-almost periods of \( \psi \).

**Remark 2.16.**

(i) Every Bohr almost periodic function is Levitian almost periodic.

(ii) The function \( \varphi \in C(\mathbb{R}, \mathbb{R}) \) defined by equality \( \varphi(t) = \frac{1}{2 + \cos t + \cos \sqrt{2}t} \) is Levitian almost periodic, but it is not Bohr almost periodic ([37] ChIV].

**Definition 2.17.** A function \( \varphi \in C(\mathbb{R}, X) \) is said to be Bohr almost automorphic if it is Levitian almost periodic and Lagrange stable.

**Remark 2.18.**

(i) The function \( \varphi \in C(\mathbb{R}, X) \) is Bohr almost automorphic if and only if for any sequence \( \{t_n\} \subset \mathbb{R} \) there are a subsequence \( \{t_n\} \) and some function \( \psi : \mathbb{R} \to X \) such that

\[
\varphi(t + t_n) \to \psi(t) \quad \text{and} \quad \psi(t - t_n) \to \varphi(t)
\]

uniformly in \( t \) on every compact subset from \( \mathbb{R} \). Some authors call this later equivalent version “compact almost automorphy”.
(ii) In [52] Veech introduced a bit weaker version of Bohr almost automorphy as follows: the function \( \varphi \in C(\mathbb{R}, X) \) is called Bohr almost automorphic if it is Levitan almost periodic and \( \varphi(\mathbb{R}) \) is relatively compact. In what follows, we mean our version when we mention Bohr almost automorphy.

(iii) A function \( \varphi \in C(\mathbb{R}, X) \) is said to be Bohr almost automorphic (see [4] [7] for details) if from every sequence \( \{t'_n\} \subset \mathbb{R} \) we can extract a subsequence \( \{t_n\} \) such that the relations (2.1) take place pointwise for \( t \in \mathbb{R} \).

(iv) It is natural to consider almost automorphy in the sense of Bohr since the solutions of differential equations satisfy this stronger property; see [4N] for details.

**Lemma 2.19.** Suppose that the function \( \varphi \in C(\mathbb{R}, X) \) is uniformly continuous on \( \mathbb{R} \) and almost automorphic in the sense of Bohr. Then it is almost automorphic in the sense of Bohr.

**Proof.** Suppose that the function \( \varphi \in C(\mathbb{R}, X) \) is almost automorphic in Bohr’s sense and \( \{t'_n\} \) is an arbitrary sequence from \( \mathbb{R} \), then there exists a subsequence \( \{t_n\} \) of \( \{t'_n\} \) such that the relations (2.1) hold for every \( t \in \mathbb{R} \). If, additionally, the function \( \varphi \) is uniformly continuous on \( \mathbb{R} \), then the convergence in the relations (2.1) are uniform with respect to \( t \) on every compact interval \([-l,l] \) \((l > 0)\). In fact, if we suppose that this is not true, then there exist \( \varepsilon_0 > 0 \), \( l_0 > 0 \) and a subsequence \( \{t_{n_k}\} \subseteq \{t_n\} \) such that at least one of the inequalities

\[
\max_{|t| \leq l_0} \rho(\varphi(t + t_{n_k}), \psi(t)) \geq \varepsilon_0
\]

and

\[
\max_{|t| \leq l_0} \rho(\psi(t - t_{n_k}), \varphi(t)) \geq \varepsilon_0
\]

takes place.

Since the function \( \varphi \) is almost automorphic in the sense of Bohcer, the closure \( \overline{\varphi(\mathbb{R})} \) of its image is a compact subset of \( X \). Since \( \varphi \) is uniformly continuous, it is Lagrange stable. Consequently, without loss of generality, we may suppose that the sequence \( \varphi^{t_{n_k}} \) converges in the space \( C(\mathbb{R}, X) \). Thus the function \( \psi \), figuring in the relations (2.1), belongs to \( H(\varphi) \) and \( H(\psi) \subseteq H(\varphi) \). It’s clear that the function \( \psi \) is also Lagrange stable and, consequently, without loss of generality we may suppose that the sequence \( \{\varphi^{t_{n_k}}\} \) converges to \( \psi \) and \( \{\psi^{-t_{n_k}}\} \) converges to \( \varphi \) in the space \( C(\mathbb{R}, X) \). The last fact contradicts to relations (2.2) and (2.3). This contradiction proves our statement. \( \square \)

**Remark 2.20.** The function \( \varphi(t) = \sin\left(\frac{t}{2 + \cos t + \cos \sqrt{2}t}\right) \) is

(i) almost automorphic in the sense of Bohr [2, Example 3.1];

(ii) Levitan almost periodic, but it is not Bohr almost automorphic, because \( \varphi \) is not uniformly continuous on \( \mathbb{R} \) [52, Ch.V, pp.212–213].

**Definition 2.21.** A function \( \varphi \in C(\mathbb{R}, X) \) is called quasi-periodic with the spectrum of frequencies \( \nu_1, \nu_2, \ldots, \nu_k \) if the following conditions are fulfilled:

(i) the numbers \( \nu_1, \nu_2, \ldots, \nu_k \) are rationally independent;

(ii) there exists a continuous function \( \Phi : \mathbb{R}^k \to X \) such that \( \Phi(t_1 + 2\pi, t_2 + 2\pi, \ldots, t_k + 2\pi) = \Phi(t_1, t_2, \ldots, t_k) \) for all \( (t_1, t_2, \ldots, t_k) \in \mathbb{R}^k \);

(iii) \( \varphi(t) = \Phi(\nu_1 t, \nu_2 t, \ldots, \nu_k t) \) for \( t \in \mathbb{R} \).
Let $\varphi \in C(\mathbb{R}, X)$. Denote by $\mathcal{N}_\varphi$ (respectively, $\mathcal{M}_\varphi$) the family of all sequences \( \{t_n\} \subseteq \mathbb{R} \) such that $\varphi^{t_n} \to \varphi$ (respectively, $\{\varphi^{t_n}\}$ converges) in $C(\mathbb{R}, X)$ as $n \to \infty$.

By $\mathcal{N}_\psi^u$ (respectively, $\mathcal{M}_\psi^u$) we denote the family of sequences $\{t_n\} \subseteq \mathcal{N}_\varphi$ such that $\varphi^{t_n}$ converges to $\varphi$ (respectively, $\varphi^{t_n}$ converges) uniformly in $t \in \mathbb{R}$ as $n \to \infty$.

**Remark 2.22.**

(i) The function $\varphi \in C(\mathbb{R}, X)$ is pseudo-periodic in the positive (respectively, negative) direction if and only if there is a sequence $\{t_n\} \subseteq \mathcal{M}_\varphi^u$ such that $t_n \to +\infty$ (respectively, $t_n \to -\infty$) as $n \to \infty$.

(ii) Let $\varphi \in C(\mathbb{R}, X)$, $\psi \in C(\mathbb{R}, Y)$ and $\mathcal{M}_\psi^u \subseteq \mathcal{M}_\varphi^u$. If the function $\psi$ is pseudo-periodic in the positive (respectively, negative) direction, then so is $\varphi$.

**Definition 2.23.** A function $\varphi \in C(\mathbb{R}, X)$ is called pseudo-recurrent if for any $\varepsilon > 0$ and $l \in \mathbb{R}$ there exists $L \geq l$ such that for any $\tau_0 \in \mathbb{R}$ we can find a number $\tau \in [l, L]$ satisfying

$$
\sup_{|t| \leq 1/\varepsilon} \rho(\varphi(t + \tau_0 + \tau), \varphi(t + \tau_0)) \leq \varepsilon.
$$

**Remark 2.24.** ([42, 43, 45, 49])

(i) Every Birkhoff recurrent function is pseudo-recurrent, but the inverse statement is not true in general.

(ii) If the function $\varphi \in C(\mathbb{R}, X)$ is pseudo-recurrent, then every function $\psi \in H(\varphi)$ is pseudo-recurrent.

(iii) If the function $\varphi \in C(\mathbb{R}, X)$ is Lagrange stable and every function $\psi \in H(\varphi)$ is Poisson stable, then $\varphi$ is pseudo-recurrent.

Finally, we remark that a Lagrange stable function is not Poisson stable in general, but all other types of functions introduced above are Poisson stable.

### 2.3. Shcherbakov’s comparability method by character of recurrence

**Definition 2.25.** A function $\varphi \in C(\mathbb{R}, X)$ is said to be comparable (respectively, uniformly comparable) by character of recurrence with $\psi \in C(\mathbb{R}, Y)$ if $\mathcal{M}_\psi \subseteq \mathcal{M}_\varphi$ (respectively, $\mathcal{M}_\psi^u \subseteq \mathcal{M}_\varphi^u$).

**Theorem 2.26.** ([43 ChII], [44]) The following statements hold:

(i) $\mathcal{M}_\psi \subseteq \mathcal{M}_\varphi$ implies $\mathcal{N}_\psi \subseteq \mathcal{N}_\varphi$, and hence uniform comparability implies comparability.

(ii) Let $\varphi \in C(\mathbb{R}, X)$ be comparable by character of recurrence with $\psi \in C(\mathbb{R}, Y)$. If the function $\psi$ is stationary (respectively, $\tau$-periodic, Levi-tan almost periodic, almost recurrent, Poisson stable), then so is $\varphi$.

(iii) Let $\varphi \in C(\mathbb{R}, X)$ be uniformly comparable by character of recurrence with $\psi \in C(\mathbb{R}, Y)$. If the function $\psi$ is quasi-periodic with the spectrum of frequencies $\nu_1, \nu_2, \ldots, \nu_k$ (respectively, Bohr almost periodic, Bohr almost automorphic, Birkhoff recurrent, Lagrange stable), then so is $\varphi$.

(iv) Let $\varphi \in C(\mathbb{R}, X)$ be uniformly comparable by character of recurrence with $\psi \in C(\mathbb{R}, Y)$ and $\psi$ be Lagrange stable. If $\psi$ is pseudo-periodic (respectively, pseudo-recurrent), then so is $\varphi$.

**Lemma 2.27.** Let $\varphi \in C(\mathbb{R}, X)$, $\psi \in C(\mathbb{R}, Y)$ and $\mathcal{M}_\psi^u \subseteq \mathcal{M}_\varphi^u$. Then the following statements hold:

(i) $\mathcal{N}_\psi \subseteq \mathcal{N}_\varphi$;

(ii) If the function $\psi$ is Bohr almost periodic, then so is $\varphi$. 
subset $Q$ and $F$ equality is a complete metric space. \[\sigma \]

Let $\hat{\tau} = \hat{\tau}(\cdot, \cdot)$ be a function such that the mapping $\hat{\tau}(\cdot, \cdot)$ makes $d$ uniformly bounded on every bounded subset from $\mathbb{R}_+$, i.e., $\hat{\tau}(\cdot, \cdot)$ is uniformly comparable by character of recurrence with $\psi$ and $\hat{\tau}$ such that $\sup_{(t, x)} \hat{\tau}(t, x) < \infty$. Since $\hat{\tau}(t, x)$ is a subsequence of $\{\hat{\tau}(t, x)\}$, we have $\hat{\tau} = \hat{\tau}$, i.e., $\{t_n\} \in M$.

Let $\psi \in C(\mathbb{R}, Y)$ be Bohr almost periodic, then $\mathfrak{B} = \mathfrak{B}_\psi$ (see, for example, [43, ChII]). Consequently, we have $\mathfrak{B} = \mathfrak{B}_\psi \subseteq \mathfrak{B}_\phi \subseteq \mathfrak{B}_\phi$. This means that the function $\phi$ is uniformly comparable by character of recurrence with $\psi$ and $\phi$ such that $\sup_{(t, x)} \phi(t, x) < \infty$. According to Theorem 2.20, the function $\phi$ is Bohr almost periodic.

2.4. The function space $BUC$.

Definition 2.28. A function $F : \mathbb{R} \times X \to X$ is called continuous at $t_0 \in \mathbb{R}$ uniformly with respect to (w.r.t.) $x \in Q$ if for any $\varepsilon > 0$ there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $|t - t_0| < \delta$ implies $\rho(F(t, x)) < \varepsilon$. The function $F : \mathbb{R} \times X \to X$ is called continuous on $\mathbb{R}$ uniformly w.r.t. $x \in Q$ if it is continuous at every point $t_0 \in \mathbb{R}$ uniformly w.r.t. $x \in Q$.

Remark 2.29. If $Q$ is a compact subset of $X$ and $F : \mathbb{R} \times X \to X$ is a continuous function, then $F$ is continuous on $\mathbb{R}$ uniformly w.r.t. $x \in Q$.

Denote by $BUC(\mathbb{R} \times X, X)$ the set of all functions $F : \mathbb{R} \times X \to X$ possessing the following properties:

(i) continuous in $t$ uniformly w.r.t. $x$ on every bounded subset $Q \subset X$;
(ii) bounded on every bounded subset from $\mathbb{R} \times X$.

For $F, G \in BUC(\mathbb{R} \times X, X)$ and $\{Q_n\}$ a sequence of bounded subsets from $X$ such that $Q_n \subset Q_{n+1}$ for any $n \in \mathbb{N}$ and $X = \bigcup_{n \geq 1} Q_n$, denote

$$d(F, G) := \sum_{n=1}^{\infty} \frac{1}{2^n + 1} \frac{d_n(F, G)}{1 + d_n(F, G)},$$

where $d_n(F, G) := \sup_{(t, x) \in Q_n} \rho(F(t, x), G(t, x))$. Then it is immediate to check that $d$ makes $BUC(\mathbb{R} \times X, X)$ a complete metric space and $d(F_k, F) \to 0$ if and only if $F_n(t, x) \to F(t, x)$ uniformly w.r.t. $(t, x) \in \mathbb{R} \times X$.

For given $F \in BUC(\mathbb{R} \times X, X)$ and $\tau \in \mathbb{R}$, denote by $F^\tau$ the translation of $F$, i.e., $F^\tau(t, x) := F(t + \tau, x)$ for $(t, x) \in \mathbb{R} \times X$, and the hull of $F$ by $H(F) := \{F^\tau : \tau \in \mathbb{R}\}$ with the closure being taken under the metric $d$ given by (2.4). It is immediate to check that the mapping $\sigma : \mathbb{R} \times BUC(\mathbb{R} \times X, X) \to BUC(\mathbb{R} \times X, X)$ defined by $\sigma(\tau, F) := F^\tau$ is a dynamical system, i.e., $\sigma(0, F) = F$, $\sigma(\tau_1 + \tau_2, F) = \sigma(\tau_2, \sigma(\tau_1, F))$ and the mapping $\sigma$ is continuous. See [23, §1.1] for details.

Denote by $BC(X, X)$ the set of all continuous and bounded on every bounded subset $Q \subset X$ functions $F : X \to X$ and let

$$d_{BC}(F, G) := \sum_{n=1}^{\infty} \frac{1}{2^n + 1} \frac{d_n(F, G)}{1 + d_n(F, G)},$$

for any $F, G \in BC(X, X)$, where $d_n(F, G) := \sup_{x \in Q_n} \rho(F(x), G(x))$. Then $BC(X, X)$ is a complete metric space.

Let now $F \in BUC(\mathbb{R} \times X, X)$ and $F : \mathbb{R} \to BC(X, X)$ a mapping defined by equality $F(t) := F(t, \cdot)$.
Remark 2.30. It is not difficult to check that:

(i) \( \mathcal{M}_F = \mathcal{M}_F^p \) for any \( F \in \text{BUC} (\mathbb{R} \times X, X) \);
(ii) \( \mathcal{M}_F^p = \mathcal{M}_F^p \) for any \( F \in \text{BUC} (\mathbb{R} \times X, X) \).

Here \( \mathcal{M}_F \) is the set of all sequences \( \{ t_n \} \) such that \( F^{t_n} \) converges in the space \( \text{BUC}(\mathbb{R} \times X, X) \) and \( \mathcal{M}_F^p \) is the set of all sequences \( \{ t_n \} \) such that \( F^{t_n+t} \) converges in the space \( \text{BUC}(\mathbb{R} \times X, X) \) uniformly w.r.t. \( t \in \mathbb{R} \).

3. Linear Equations

Let \( \mathfrak{B} \) be a Banach space with the norm \( | \cdot |_{\mathfrak{B}} \). Consider the linear nonhomogeneous equation

\[
\dot{x} = Ax + f(t)
\]
on the space \( \mathfrak{B} \), where \( f \in C(\mathbb{R}, \mathfrak{B}) \) and \( A \) is an infinitesimal generator which generates a \( C_0 \)-semigroup \( \{ U(t) \}_{t \geq 0} \) acting on \( \mathfrak{B} \).

Definition 3.1. A semigroup of operators \( \{ U(t) \}_{t \geq 0} \) is said to be exponentially stable, if there are positive numbers \( \mathcal{N}, \nu > 0 \) such that \( ||U(t)|| \leq \mathcal{N} e^{-\nu t} \) for any \( t \geq 0 \).

Denote by \( C_b(\mathbb{R}, \mathfrak{B}) \) the Banach space of all continuous and bounded mappings \( \varphi : \mathbb{R} \to \mathfrak{B} \) equipped with the norm \( ||\varphi||_{\infty} := \text{sup} \{ ||\varphi(t)||_{\mathfrak{B}} : t \in \mathbb{R} \} \).

Let \( (H, | \cdot |_H) \) be a real separable Hilbert space, \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, and \( L^2(\mathbb{P}, H) \) be the space of \( H \)-valued random variables \( x \) such that

\[
\mathbb{E}[|x|^2] := \int_{\Omega} |x|^2 d\mathbb{P} < \infty.
\]

Then \( L^2(\mathbb{P}, H) \) is a Hilbert space equipped with the norm

\[
||x||_2 := \left( \int_{\Omega} |x|^2 d\mathbb{P} \right)^{1/2}.
\]

For \( f \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \), the space of bounded continuous mappings from \( \mathbb{R} \) to \( L^2(\mathbb{P}, H) \), we denote \( ||f||_{\infty} := \text{sup} \{ ||f(t)||_2 \} \).

Consider the following semi-linear stochastic differential equation

\[
dx(t) = (Ax(t) + f(t, x(t)))dt + g(t, x(t))dW(t),
\]

where \( A \) is an infinitesimal generator which generates a \( C_0 \)-semigroup \( \{ U(t) \}_{t \geq 0} \), \( f, g : \mathbb{R} \times H \to H \) and \( W(t) \) is a two-sided standard one-dimensional Brownian motion defined on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). We set \( \mathcal{F}_t := \sigma \{ W(u) : u \leq t \} \).

Definition 3.2. Recall that an \( \mathcal{F}_t \)-adapted processes \( \{ x(t) \}_{t \in \mathbb{R}} \) is said to be a mild solution of equation (3.2) if it satisfies the stochastic integral equation

\[
x(t) = U(t-t_0)x(t_0) + \int_{t_0}^t U(t-s)f(s, x(s))ds + \int_{t_0}^t U(t-s)g(s, x(s))dW(s),
\]

for all \( t \geq t_0 \) and each \( t_0 \in \mathbb{R} \).

Remark 3.3. If \( \varphi \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \), then for any \( \psi \in H(\varphi) \) we have \( ||\psi(t)||_2 \leq ||\varphi||_{\infty} \) for every \( t \in \mathbb{R} \).
Let $\mathcal{P}(H)$ be the space of all Borel probability measures on $H$ endowed with the $eta$ metric:

$$\beta(\mu, \nu) := \sup \left\{ \left| \int f \, d\mu - \int f \, d\nu \right| : ||f||_{BL} \leq 1 \right\}, \quad \text{for } \mu, \nu \in \mathcal{P}(H),$$

where $f$ are bounded Lipschitz continuous real-valued functions on $H$ with the norms

$$||f||_{BL} = \text{Lip}(f) + ||f||_{\infty}, \quad \text{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}, \quad ||f||_{\infty} = \sup_{x \in H} |f(x)|.$$

Recall that a sequence $\{\mu_n\} \subset \mathcal{P}(H)$ is said to weakly converge to $\mu$ if $\int f \, d\mu_n \to \int f \, d\mu$ for all $f \in C_b(H)$, where $C_b(H)$ is the space of all bounded continuous real-valued functions on $H$. It is well-known that $(\mathcal{P}(H), \beta)$ is a separable complete metric space and that a sequence $\{\mu_n\}$ weakly converges to $\mu$ if and only if $\beta(\mu_n, \mu) \to 0$ as $n \to \infty$.

**Definition 3.4.** A sequence of random variables $\{x_n\}$ is said to converge in distribution to the random variable $x$ if the corresponding laws $\{\mu_n\}$ of $\{x_n\}$ weakly converge to the law $\mu$ of $x$, i.e. $\beta(\mu_n, \mu) \to 0$.

**Definition 3.5.** Let $\{\varphi(t)\}_{t \in \mathbb{R}}$ be a mild solution of equation (3.2). Then $\varphi$ is called comparable (respectively, uniformly comparable) in distribution if $\mathfrak{N}_{(f, g)} \subseteq \hat{\mathfrak{N}}_{\varphi}$ (respectively, $\mathfrak{N}_{(f, g)} \subseteq \hat{\mathfrak{M}}_{\varphi}$), where $\hat{\mathfrak{N}}_{\varphi}$ (respectively, $\mathfrak{M}_{\varphi}$) means the set of all sequences $\{t_n\} \subset \mathbb{R}$ such that the sequence $\{\varphi(\cdot + t_n)\}$ converges to $\varphi(\cdot)$ (respectively, $\{\varphi(\cdot + t_n)\}$ converges) in distribution uniformly on any compact interval.

In this section, we consider the following linear stochastic differential equation

$$dx(t) = (Ax(t) + f(t))\,dt + g(t)\,dW(t),$$

where $A$ and $W$ are the same as in (3.2), and $f, g \in C(\mathbb{R}, L^2(\mathbb{P}, H))$ are $\mathcal{F}_t$-adapted.

**Theorem 3.6.** Consider the equation (3.3). Suppose that the semigroup $\{U(t)\}_{t \geq 0}$ acting on $H$ is exponentially stable, then the following statements hold:

(i) for every $f, g \in C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ there exists a unique solution $\varphi \in C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ of equation (3.3) given by the formula

$$\varphi(t) = \int_{-\infty}^{t} U(t - \tau) f(\tau) \,d\tau + \int_{-\infty}^{t} U(t - \tau) g(\tau) \,dW(\tau);$$

(ii) the Green’s operator $G$ defined by

$$G(f, g)(t) := \int_{-\infty}^{t} U(t - \tau) f(\tau) \,d\tau + \int_{-\infty}^{t} U(t - \tau) g(\tau) \,dW(\tau)$$

is a bounded operator defined on $C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \times C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ with values in $C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ and

$$||G(f, g)||_{\infty} \leq \frac{N}{\nu} \left( 2||f||_{\infty}^2 + \nu||g||_{\infty}^2 \right)^{1/2}$$

for all $f, g \in C_b(\mathbb{R}, L^2(\mathbb{P}, H))$;

(iii) if $f, g \in C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ and $l > L > 0$, then

$$\max_{|\tau| \leq L} \mathbb{E}|\varphi(t)|^2 \leq \frac{N^2}{\nu^2} \left( 2\max_{|\tau| \leq l} \mathbb{E}|f(\tau)|^2 + \nu \max_{|\tau| \leq l} \mathbb{E}|g(\tau)|^2 \right).$$
For the first term, by the Cauchy-Schwarz inequality we have
\[
\frac{N^2}{\nu^2} \left( 2e^{-\nu(t-L)} \|f\|_\infty^2 + \nu e^{-2\nu(t-L)} \|g\|_\infty^2 \right);
\]
(iv) if \( f, g \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \), then the unique \( L^2 \)-bounded solution \( \varphi \) of equation (3.3) is uniformly comparable in distribution;
(v) \( M_{u}(f,g) \subseteq M_{\varphi} \), where \( M_{\varphi} \) is the set of all sequences \( \{ t_n \} \) such that the sequence \( \{ \varphi(t + t_n) \} \) converges in distribution uniformly in \( t \in \mathbb{R} \).

**Proof.** (i)-(ii). It is straightforward to verify that the function \( \varphi \) given by (3.4) is a solution of the equation (3.3). If \( \psi \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \) is also a solution, then \( u(t) = \varphi(t) - \psi(t) \) satisfies the equation
\[
x'(t) = Ax(t).
\]
But under the exponential stability condition of \( \{U(t)\}_{t \geq 0} \), this equation has only trivial solution in \( C \).

We now show the boundedness of \( \varphi \). Note that \( \varphi(t) = p(t) + q(t) \) for \( t \in \mathbb{R} \), where
\[
p(t) := \int_{-\infty}^{t} U(t-\tau)f(\tau)d\tau
\]
and
\[
q(t) := \int_{-\infty}^{t} U(t-\tau)g(\tau)dW(\tau).
\]
For the first term, by the Cauchy-Schwarz inequality we have
\[
\mathbb{E}|p(t)|^2 = \mathbb{E} \left[ \int_{-\infty}^{t} |U(t-\tau)f(\tau)|^2 d\tau \right] \leq \mathbb{E} \left[ \int_{-\infty}^{t} |U(t-\tau)||f(\tau)|^2 d\tau \right] \leq \mathbb{E} \left[ \int_{-\infty}^{t} Ne^{-\nu(t-\tau)}|f(\tau)|^2 d\tau \right] \leq \frac{N^2}{\nu^2} \|f\|_\infty^2.
\]
For the second term, using Itô’s isometry property we get
\[
\mathbb{E}|q(t)|^2 = \mathbb{E} \left[ \int_{-\infty}^{t} |U(t-\tau)g(\tau)|^2 d\tau \right] = \mathbb{E} \left[ \int_{-\infty}^{t} |U(t-\tau)g(\tau)|^2 d\tau \right] \leq \int_{-\infty}^{t} \mathbb{E}|g(\tau)|^2 d\tau \leq \frac{N^2}{2\nu} \|g\|_\infty^2.
\]
From (3.7) and (3.8) we have
\[
\mathbb{E}|\varphi(t)|^2 \leq 2(\mathbb{E}|p(t)|^2 + \mathbb{E}|q(t)|^2) \leq \frac{N^2}{\nu^2} \left( 2\|f\|_\infty^2 + \nu \|g\|_\infty^2 \right),
\]
and consequently
\[
\|G(f,g)\|_\infty = \|\varphi\|_\infty \leq \frac{N}{\nu} \left( 2\|f\|_\infty^2 + \nu \|g\|_\infty^2 \right)^{1/2}.
\]
(iii). Let \( L > 0, t \in [-L,L], l > L \) and \( f, g \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \), then from (3.7), (3.8) we have
\[
\mathbb{E}|\psi(t)|^2 \leq 2(\mathbb{E}|p(t)|^2 + \mathbb{E}|q(t)|^2)
\]
(3.9)
\[
\leq 2N^2 \left( \frac{1}{\nu} \int_{-\infty}^{t} e^{-\nu(t-\tau)} \mathbb{E}|f(\tau)|^2 d\tau + \int_{-\infty}^{t} e^{-2\nu(t-\tau)} \mathbb{E}|g(\tau)|^2 d\tau \right).
\]
Note that
\[
\int_{-\infty}^{t} e^{-\nu(t-\tau)} \mathbb{E}|f(\tau)|^2 d\tau = \int_{-t}^{t} e^{-\nu(t-\tau)} \mathbb{E}|f(\tau)|^2 d\tau + \int_{-\infty}^{-t} e^{-\nu(t-\tau)} \mathbb{E}|f(\tau)|^2 d\tau \leq \frac{1}{\nu} \max_{|\tau| \leq t} \mathbb{E}|f(\tau)|^2 + \frac{1}{\nu} e^{-\nu(t+\nu)} |f|^2.
\]
(3.10)
Reasoning as above we have
\[
\int_{-\infty}^{t} e^{-2\nu(t-\tau)} \mathbb{E}|g(\tau)|^2 d\tau \leq \frac{1}{2\nu} \max_{|\tau| \leq t} \mathbb{E}|g(\tau)|^2 + \frac{1}{2\nu} e^{-2\nu(t+\nu)} |g|^2.
\]
(3.11)
From (3.9), (3.10) we obtain
\[
\max_{|t| \leq L} \mathbb{E}|\psi(t)|^2 \leq \frac{N^2}{\nu^2} \left( 2 \max_{|\tau| \leq t} \mathbb{E}|f(\tau)|^2 + \nu \max_{|\tau| \leq t} \mathbb{E}|g(\tau)|^2 \right)
\]
\[
+ \frac{N^2}{\nu^2} \left( 2 e^{-\nu(l-L)} ||f||_\infty^2 + \nu e^{-\nu(l-L)} ||g||_\infty^2 \right).
\]
Thus inequality (3.6) is established.
(iv). Let now \( \{t_n\} \in \mathfrak{M}(f,g) \), then there exists \((\tilde{f}, \tilde{g}) \in H(f,g)\) such that \( f^{t_n} \to \tilde{f} \) and \( g^{t_n} \to \tilde{g} \) in the space \( C(\mathbb{R}, L^2(\mathbb{P}, H)) \) as \( n \to \infty \); that is, for any \( L > 0 \) we have
\[
\max_{|t| \leq L} \mathbb{E}|f(t + t_n) - \tilde{f}(t)|^2 \to 0 \quad \text{and} \quad \max_{|t| \leq L} \mathbb{E}|g(t + t_n) - \tilde{g}(t)|^2 \to 0
\]
as \( n \to \infty \).
Denote by \( h^1_n(t) := f^{t_n}(t) - \tilde{f}(t) \) and \( h^2_n(t) := g^{t_n}(t) - \tilde{g}(t) \) for any \( t \in \mathbb{R}, \)
\( \varphi_n := \mathcal{G}(f^{t_n}, g^{t_n}), \phi := \mathcal{G}(\tilde{f}, \tilde{g}) \) and \( \psi_n := \varphi_n - \phi \). It is easy to check that \( \psi_n = \mathcal{G}(h^1_n, h^2_n) \), \( h^1_n \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \) \( (i = 1, 2) \) and by Remark 3.3 we have \( ||h^1_n||_\infty \leq 2 ||f||_\infty \) (respectively, \( ||h^2_n||_\infty \leq 2 ||g||_\infty \)). Let now \( \{t_n\} \) be a sequence of positive numbers such that \( t_n \to \infty \) as \( n \to \infty \). According to inequality (3.6) we obtain
\[
\max_{|t| \leq L} \mathbb{E}|\psi_n(t)|^2 \leq \frac{N^2}{\nu^2} \left( 2 \max_{|\tau| \leq t_n} \mathbb{E}|h^1_n(\tau)|^2 + \nu \max_{|\tau| \leq t_n} \mathbb{E}|h^2_n(\tau)|^2 \right)
\]
\[
+ \frac{N^2}{\nu^2} \left( 2 e^{-\nu((l_n-L))} ||h^1_n||^2_\infty + \nu e^{-\nu((l_n-L))} ||h^2_n||^2_\infty \right).
\]
(3.12)
Passing to limit in (3.12) as \( n \to \infty \) and taking into consideration Remark 2.2 (iii) we have
\[
\lim_{n \to \infty} \max_{|t| \leq L} \mathbb{E}|\psi_n(t)|^2 = 0
\]
for any \( L > 0 \); that is, \( \varphi_n \to \phi \) in the space \( C(\mathbb{R}, L^2(\mathbb{P}, H)) \) as \( n \to \infty \).
Since \( L^2 \) convergence implies convergence in distribution, we have \( \varphi_n(t) \to \phi(t) \) in distribution uniformly in \( t \in [-L,L] \) for all \( L > 0 \). On the other hand we have
\[
\varphi(t + t_n) = \int_{-\infty}^{t} U(t - \tau) f(\tau + t_n) d\tau + \int_{-\infty}^{t} U(t - \tau) g(\tau + t_n) d\tilde{W}_n(\tau),
\]
where \( \tilde{W}_n(\tau) := W(\tau + t_n) - W(t_n) \) is a shifted Brownian motion. So \( \varphi_n(t) \) and \( \varphi(t+t_n) \) share the same distribution on \( H \), and hence \( \varphi(t+t_n) \to \tilde{\varphi}(t) \) in distribution uniformly in \( t \in [-L, L] \) for all \( L > 0 \). Thus we have \( \{t_n\} \in \mathcal{M}_{\varphi} \). That is, \( \varphi \) is uniformly comparable in distribution.

(\( \psi \)). Let \( \{t_n\} \in \mathcal{M}_{(f,g)} \); then there exists \((\tilde{f}, \tilde{g}) \in H(f,g) \) such that \( f^{t_n} \to \tilde{f} \) and \( g^{t_n} \to \tilde{g} \) uniformly in \( t \in \mathbb{R} \) as \( n \to \infty \), that is,
\[
\max_{t \in \mathbb{R}} \mathbb{E}|f(t + t_n) - \tilde{f}(t)|^2 \to 0 \quad \text{and} \quad \max_{t \in \mathbb{R}} \mathbb{E}|g(t + t_n) - \tilde{g}(t)|^2 \to 0
\]
as \( n \to \infty \). As above we denote by \( h_n^1(t) := f^{t_n}(t) - \tilde{f}(t) \) and \( h_n^2(t) := g^{t_n}(t) - \tilde{g}(t) \) for \( t \in \mathbb{R} \), \( \varphi_n := \mathbb{G}(f^{t_n}, g^{t_n}) \), \( \tilde{\varphi} := \mathbb{G}(\tilde{f}, \tilde{g}) \) and \( \psi_n := \varphi_n - \tilde{\varphi} \).

According to inequality (4.5) we obtain
\[
\|\psi_n\|_{\infty} \leq \frac{N}{2^{\nu}} (2\|h_n^1\|_{\infty}^2 + \nu\|h_n^2\|_{\infty}^2)^{1/2}.
\]

Passing to limit in (4.13) we obtain \( \varphi_n \to \tilde{\varphi} \) uniformly on \( \mathbb{R} \) in \( L^2 \)-norm as \( n \to \infty \). Since \( \varphi_n(t) \) and \( \varphi(t + t_n) \) have the same distributions, \( \varphi(t + t_n) \to \tilde{\varphi}(t) \) in distribution uniformly in \( t \in \mathbb{R} \). Thus we have \( \{t_n\} \in \mathcal{M}_{\varphi} \). The proof is complete. \( \square \)

**Corollary 3.7.** Under the conditions of Theorem 3.6, if the functions \( f,g \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \) are jointly stationary (respectively, \( \tau \)-periodic, quasi-periodic with the spectrum of frequencies \( \nu_1, \ldots, \nu_k \), Bohr almost periodic, Bohr almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable), then equation (3.3) has a unique solution \( \varphi \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \) which is stationary (respectively, \( \tau \)-periodic, quasi-periodic with the spectrum of frequencies \( \nu_1, \ldots, \nu_k \), Bohr almost periodic, Bohr almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in distribution. If the functions \( f,g \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \) are jointly Lagrange stable and jointly pseudo-periodic (respectively, pseudo-recurrent), then equation (3.3) has a unique solution \( \varphi \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \) which is pseudo-periodic (respectively, pseudo-recurrent) in distribution.

**Proof.** This statement follows from Theorems 2.26 and 3.6. \( \square \)

### 4. Semi-linear equations

Let us consider the stochastic differential equation
\[
dx(t) = (Ax(t) + F(t, x(t)))dt + G(t, x(t))dW(t),
\]
where \( F,G \in C(\mathbb{R} \times H, H) \).

**Definition 4.1.** We say that the functions \( F \) and \( G \) satisfy the condition

(C1) if there exists a number \( A_0 \geq 0 \) such that \( |F(t,0)|, |G(t,0)| \leq A_0 \) for any \( t \in \mathbb{R} \);

(C2) if there exists a number \( \mathcal{L} \geq 0 \) such that \( \text{Lip}(F), \text{Lip}(G) \leq \mathcal{L} \), where
\[
\text{Lip}(F) := \sup \left\{ \frac{|F(t,x_t) - F(t,x_2)|}{|x_1 - x_2|} : x_1 \neq x_2, \; t \in \mathbb{R} \right\};
\]

(C3) if \( F \) and \( G \) are continuous in \( t \) uniformly w.r.t. \( x \) on each bounded subset \( Q \subset H \).
Remark 4.2. (i) If $F$ and $G$ satisfy (C1)-(C2) with the constants $A_0$ and $L$, then every pair of functions $(\tilde{F}, \tilde{G})$ in $H(F,G) := \{(F\tau, G\tau) : \tau \in \mathbb{R}\}$, the hull of $(F,G)$, also possess the same property with the same constants.

(ii) If $F$ and $G$ satisfy the conditions (C1)-(C3), then $F,G \in \text{BUC}(\mathbb{R} \times H,H)$ and $H(F,G) \subset \text{BUC}(\mathbb{R} \times H,H) \times \text{BUC}(\mathbb{R} \times H,H)$.

(iii) When we consider stochastic ordinary differential equations, i.e. $F,G \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$, the condition (C3) naturally holds as pointed out in Remark 2.29; but for stochastic partial differential equations, we need to check condition (C3) carefully.

Lemma 4.3. Let $u,f \in C_b(\mathbb{R}, \mathbb{R}^+)$ and $\nu > \alpha \geq 0$, then the following statements hold:

(i) if

$$u(t) \leq \int_{-\infty}^{t} e^{-\nu(t-\tau)}(\alpha u(\tau) + f(\tau))d\tau$$

for any $t \in \mathbb{R}$, then

$$u(t) \leq \int_{-\infty}^{t} e^{-k(t-\tau)}f(\tau)d\tau,$$

where $k := \nu - \alpha$;  

(ii) if $l > L > 0$, then

$$\max_{|t| \leq L} u(t) \leq \frac{e^{kl}e^{-kl}}{k} \sup_{t \in \mathbb{R}} f(t) + \frac{1-e^{-kL}e^{-kl}}{k} \max_{|t| \leq l} f(t).$$

Proof. (i). Consider the equation

$$(4.2) \quad v(t) = \int_{-\infty}^{t} e^{-\nu(t-\tau)}(\alpha v(\tau) + f(\tau))d\tau.$$ 

Note that the linear operator $A : C_b(\mathbb{R}, \mathbb{R}) \to C_b(\mathbb{R}, \mathbb{R})$ defined by

$$(A\varphi)(t) := \int_{-\infty}^{t} e^{-\nu(t-\tau)}\alpha\varphi(\tau)d\tau$$

is a contraction, where $C_b(\mathbb{R}, \mathbb{R})$ is equipped with the norm $||\varphi||_\infty := \sup\{|\varphi(t)| : t \in \mathbb{R}\}$. In fact, it is immediate to check that $||A|| \leq \frac{\nu}{\alpha} < 1$, $C_b(\mathbb{R}, \mathbb{R}^+)$ is a cone in the space $C_b(\mathbb{R}, \mathbb{R})$ and $A(C_b(\mathbb{R}, \mathbb{R}^+)) \subseteq C_b(\mathbb{R}, \mathbb{R}^+)$, and thus the operator $\Phi : C_b(\mathbb{R}, \mathbb{R}) \to C_b(\mathbb{R}, \mathbb{R})$, defined by

$$(\Phi\phi)(t) := \int_{-\infty}^{t} e^{-\nu(t-\tau)}(\alpha\phi(\tau) + f(\tau))d\tau, \quad \text{for} \ t \in \mathbb{R}$$

is a contraction and consequently the equation (4.2) has a unique solution on the space $C_b(\mathbb{R}, \mathbb{R})$.

Note that the unique bounded solution $v(t)$ of equation (4.2) is a solution of the equation

$$v'(t) = -kv(t) + f(t)$$

and consequently it is given by

$$v(t) = \int_{-\infty}^{t} e^{-k(t-\tau)}f(\tau)d\tau.$$
Define an operator $S$ equation Proof. Since the semigroup (4.1) then $x, y$ have for $t \in \mathbb{R}$.

(ii). Let now $l > L > 0$ and $t \in [-L, L]$, then we have

$$\int_{-\infty}^{t} e^{-k(t-\tau)} f(\tau)d\tau = \int_{-\infty}^{-l} e^{-k(t-\tau)} f(\tau)d\tau + \int_{-l}^{t} e^{-k(t-\tau)} f(\tau)d\tau$$

$$\leq \sup_{t \in \mathbb{R}} f(t) \cdot \frac{e^{-k(t+l)}}{k} + \max_{v \leq |t|} f(t) \cdot \frac{1 - e^{-k(t+l)}}{k}.$$ 

Consequently,

$$\max_{|t| \leq L} u(t) \leq \max_{|t| \leq L} v(t) \leq \frac{e^{kL}e^{-kt}}{k} \sup_{t \in \mathbb{R}} f(t) + \frac{1 - e^{-kL}e^{-kt}}{k} \max_{v \leq |t|} f(t).$$

$\square$

Proposition 4.4. Consider the equation (4.1). Suppose that the following conditions hold:

(i) the semigroup $\{U(t)\}_{t \geq 0}$ acting on the space $H$ is exponentially stable;

(ii) $F, G \in C(\mathbb{R} \times H, H)$;

(iii) the functions $F$ and $G$ satisfy the conditions (C1) and (C2).

For $p > 2$, denote

$$c_p := \left[ \frac{p(p-1)}{2} \right] \cdot \left[ \frac{p}{p-1} \right]^{p/2}. $$

If

$$\theta_p := 2^{p-1}N^p L_p \left[ \left( \frac{2(p-1)}{\nu p} \right)^{p-1} + c_p \left( \frac{p-2}{\nu p} \right)^{p/2-1} \right] \cdot \frac{2}{\nu p} < 1,$$

then (4.1) admits a unique bounded solution in $C_b(\mathbb{R}, L^p(\mathbb{P}, H))$.

Proof. Since the semigroup $U(t)$ is exponentially stable, it can be checked that $x_0 \in C_b(\mathbb{R}, L^p(\mathbb{P}, H))$ is a mild solution of (4.1) if and only if it satisfies the integral equation

$$x_0(t) = \int_{-\infty}^{t} U(t-\tau)F(\tau, x_0(\tau))d\tau + \int_{-\infty}^{t} U(t-\tau)G(\tau, x_0(\tau))dW(\tau).$$

Define an operator $S$ on $C_b(\mathbb{R}, L^p(\mathbb{P}, H))$ by

$$(Sx)(t) := \int_{-\infty}^{t} U(t-\tau)F(\tau, x(\tau))d\tau + \int_{-\infty}^{t} U(t-\tau)G(\tau, x(\tau))dW(\tau).$$

Since $F, G$ satisfy the conditions (C1) and (C2), it is not hard to check that $S$ maps $C_b(\mathbb{R}, L^p(\mathbb{P}, H))$ into itself.

By the proof of [25, Theorem 4.36], we have for any $s < t$

$$\mathbb{E} \left| \int_{s}^{t} f(\tau)dW(\tau) \right|^p \leq c_p \left( \mathbb{E} \left| \int_{s}^{t} |f(\tau)|^2d\tau \right|^2 \right)^{p/2}. $$

So by Hölder’s inequality with exponents $(p, \frac{p}{p-1})$ and $(\frac{p}{2}, \frac{p}{p-2})$ respectively, we have for $x, y \in C_b(\mathbb{R}, L^p(\mathbb{P}, H))$ and $t \in \mathbb{R}$

$$\mathbb{E}|(Sx)(t) - (Sy)(t)|^p$$
Remark 4.5. Note that the contraction constant \( \theta_p \) is continuous in \( p \) when \( p > 2 \). Furthermore, \( c_p = 1 \) when \( p = 2 \) in Proposition 4.4, so we have

\[
\theta_p = \frac{2N^2\mathcal{L}^2}{\nu^2} + \frac{2N^2\mathcal{L}^2}{\nu}.
\]

Theorem 4.6. Consider the equation (4.1). Suppose that the following conditions hold:

(i) the semigroup \( \{U(t)\}_{t \geq 0} \) acting on the space \( H \) is exponentially stable;
(ii) \( F, G \in C(\mathbb{R} \times H, H) \);
(iii) the functions \( F \) and \( G \) satisfy the conditions (C1) and (C2).

Then the following statements hold:

(i) If \( \mathcal{L} < \frac{\nu}{N\sqrt{2+\nu}} \), then equation (4.1) has a unique solution \( \xi \in C(\mathbb{R}, B[0, r]) \), where

\[
r = \frac{NA_0\sqrt{2+\nu}}{\nu - N\mathcal{L}\sqrt{2+\nu}}
\]

and

\[B[0, r] := \{ x \in L^2(\mathbb{P}, H) : ||x||_2 \leq r \} .\]
such that \( \Phi(\xi) = \xi \) so \( \Phi \) is a contraction. Consequently, there exists a unique function \( \Phi(\xi) = \xi \) such that \( \xi(t + t_n) \) converges in distribution uniformly in \( t \in \mathbb{R} \).

Proof. (i). Note that \( C(\mathbb{R}, B[0, r]) \) is a complete metric space. Define an operator

\[
\Phi : C(\mathbb{R}, B[0, r]) \to C(\mathbb{R}, B[0, r])
\]

as follows. If \( \phi \in C(\mathbb{R}, B[0, r]) \), then we put \( h_1(t) := F(t, \phi(t)) \) and \( h_2(t) := G(t, \phi(t)) \) for any \( t \in \mathbb{R} \). Since the function \( F \) satisfies conditions (C1) and (C2), we have

\[
\|h_1(t)\|_2 = \|F(t, \phi(t))\|_2 \leq \|F(t, 0)\|_2 + L\|\phi(t)\|_2 \leq A_0 + Lr
\]

for any \( t \in \mathbb{R} \). Analogically we have

\[
\|h_2(t)\|_2 \leq A_0 + L\|\phi(t)\|_2 \leq A_0 + Lr
\]

for \( t \in \mathbb{R} \). According to Theorem 3.6 the equation

\[
dz(t) = (A_z(t) + h_1(t))dt + h_2(t)dW(t)
\]

has a unique solution \( \psi \in C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \). Besides, it obeys the estimate

\[
\|\psi\|_\infty \leq \frac{N}{\nu}(2\|h_1\|_\infty^2 + \nu\|h_2\|_\infty^2)^{1/2}.
\]

From (4.1), (4.10) and (4.13) we have

\[
\|\psi\|_\infty \leq \frac{N}{\nu}(2(A_0 + Lr)^2 + \nu(A_0 + Lr)^2)^{1/2} = \frac{N\sqrt{2 + \nu}}{\nu}(A_0 + Lr) = r.
\]

So \( \psi \in C(\mathbb{R}, B[0, r]) \). Let \( \Phi(\phi) := \psi \). It follows from the above argument that \( \Phi \) is well defined.

Let us show that the operator \( \Phi \) is a contraction. In fact, it is easy to note that the function \( \psi_1 - \psi_2 = \Phi(\phi_1) - \Phi(\phi_2) \) is the unique solution from \( C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \) of the equation

\[
du(t) = (Au(t) + F(t, \phi_1(t)) - F(t, \phi_2(t)))dt + (G(t, \phi_1(t)) - G(t, \phi_2(t)))dW(t).
\]

By Theorem 3.6 we have the following estimate

\[
\|\Phi(\phi_1) - \Phi(\phi_2)\|_\infty \leq \frac{N^2}{\nu^2} \left( 2\sup_{t \in \mathbb{R}} \mathbb{E}|F(t, \phi_1(t)) - F(t, \phi_2(t))|^2 + \nu \sup_{t \in \mathbb{R}} \mathbb{E}|G(t, \phi_1(t)) - G(t, \phi_2(t))|^2 \right)
\]

\[
\leq \frac{N^2\mathcal{L}^2(2 + \nu)}{\nu^2} ||\phi_1 - \phi_2||_\infty^2 =: \theta_2 ||\phi_1 - \phi_2||_\infty^2.
\]

By the assumption on \( \mathcal{L} \) we have

\[
\theta_2 = \frac{N^2\mathcal{L}^2(2 + \nu)}{\nu^2} < \frac{N^2(2 + \nu)}{\nu^2} \cdot \frac{\nu^2}{N^2(2 + \nu)} = 1,
\]

so \( \Phi \) is a contraction. Consequently, there exists a unique function \( \xi \in C(\mathbb{R}, B[0, r]) \) such that \( \Phi(\xi) = \xi \).
(ii)-(a). Let \( \{ t_n \} \in \mathfrak{M}_r(F,G) \). Then there exists \( (\hat{F}, \hat{G}) \in H(F,G) \) such that for any \( r > 0 \)

\[
(4.8) \quad \sup_{t \in \mathbb{R}, |x| \leq r} |F(t + t_n, x) - \hat{F}(t, x)| \to 0
\]

and

\[
(4.9) \quad \sup_{t \in \mathbb{R}, |x| \leq r} |G(t + t_n, x) - \hat{G}(t, x)| \to 0
\]

as \( n \to \infty \). Consider equations

\[
(4.10) \quad dx(t) = (Ax(t) + F(t, x(t)))dt + G(t, x(t))dW(t) \quad (n \in \mathbb{N})
\]

and

\[
(4.11) \quad dx(t) = (Ax(t) + \hat{F}(t, x(t)))dt + \hat{G}(t, x(t))dW(t).
\]

Since the functions \((F^{t_n}(t), G^{t_n}(t)) (n \in \mathbb{N})\) and \((\hat{F}, \hat{G})\) satisfy conditions (C1) and (C2) (see Remark 1.2, by the first part of the theorem equation (4.10) (respectively, equation (4.11)) has a unique solution \( \xi_n \in C(\mathbb{R}, B[0, r]) \) (respectively, \( \xi \in C(\mathbb{R}, B[0, r]) \)). We will show that \( \{\xi_n(t)\} \) converges, in \( L^2 \) norm, to \( \xi(t) \) uniformly in \( t \in \mathbb{R} \). To this end we note that \( \xi_n \ (n \in \mathbb{N}) \) is the unique solution from \( C(\mathbb{R}, B[0, r]) \) of equation

\[
dx(t) = (Ax(t) + h_n(t))dt + g_n(t)dW(t) \quad (n \in \mathbb{N}),
\]

where \((h_n(t), g_n(t)) := (F(t, \xi_n(t)), G(t, \xi_n(t))) \) for \( t \in \mathbb{R} \) and \( n \in \mathbb{N} \) and, respectively, \( \xi \) is the unique solution from \( C(\mathbb{R}, B[0, r]) \) of the equation

\[
dx(t) = (Ax(t) + \bar{h}(t))dt + \bar{g}(t)dW(t) \quad (n \in \mathbb{N}),
\]

where \((\bar{h}(t), \bar{g}(t)) := (\hat{F}(t, \xi(t)), \hat{G}(t, \xi(t))) \) for \( t \in \mathbb{R} \). It is easy to check that \( \phi_n := \xi_n - \xi \) is the unique solution from \( C(\mathbb{R}, B[0, 2r]) \) of the equation

\[
(4.12) \quad dx(t) = (Ax(t) + h_n(t) - \bar{h}(t))dt + (g_n(t) - \bar{g}(t))dW(t) \quad (n \in \mathbb{N}),
\]

where \( h_n - \bar{h}, g_n - \bar{g} \in C_b(\mathbb{R}, L^2(\mathbb{R}, H)) \). In virtue of Theorem 3.6 (item (ii)) we have

\[
(4.13) \quad ||\phi_n||_{L^2}^2 \leq \frac{N^2}{\nu^2} (2||h_n - \bar{h}||_{L^2}^2 + \nu||g_n - \bar{g}||_{L^2}^2).
\]

Taking into consideration that the functions \((F^{t_n}, G^{t_n}) (n \in \mathbb{N})\) and \((\hat{F}, \hat{G})\) satisfy conditions (C1) and (C2), and \( \xi_n, \xi \in C(\mathbb{R}, B[0, r]) \) \((n \in \mathbb{N})\) we have

\[
E|h_n(\tau) - \bar{h}(\tau)|^2 = E|F^{t_n}(\tau, \xi_n(\tau)) - F^{t_n}(\tau, \bar{\xi}(\tau)) + F^{t_n}(\tau, \tilde{\xi}(\tau)) - \hat{F}(\tau, \tilde{\xi}(\tau))|^2 \leq 2(E|F^{t_n}(\tau, \xi_n(\tau)) - F^{t_n}(\tau, \bar{\xi}(\tau))|^2 + E|F^{t_n}(\tau, \tilde{\xi}(\tau)) - \hat{F}(\tau, \tilde{\xi}(\tau))|^2) \leq 2(L^2 E||\xi_n(\tau) - \bar{\xi}(\tau)||^2 + \sup_{\tau \in \mathbb{R}} E|F^{t_n}(\tau, \tilde{\xi}(\tau)) - \hat{F}(\tau, \tilde{\xi}(\tau))|^2) \leq 2(L^2||\phi_n||_{L^2}^2 + \sup_{\tau \in \mathbb{R}} E\phi_n^2(\tau, \bar{\tau})),
\]

where

\[
a_n,\tau := |F^{t_n}(\tau, \tilde{\xi}(\tau)) - \hat{F}(\tau, \tilde{\xi}(\tau))|.
\]

Using the same arguments we have

\[
E|g_n(\tau) - \bar{g}(\tau)|^2 \leq 2(L^2||\phi_n||_{L^2}^2 + \sup_{\tau \in \mathbb{R}} E\phi_n^2(\tau, \bar{\tau})),
\]
with 
\[ b_{n,\tau} := |G^{t_n}(\tau, \tilde{\xi}(\tau)) - \tilde{G}(\tau, \tilde{\xi}(\tau))|, \]

From (4.13)-(4.16) we obtain
\[ ||\phi_n||^2_\infty \leq \frac{N^2}{\nu^2} \left[ 4(L^2||\phi_n||^2_\infty + \sup_{\tau \in \mathbb{R}}\mathbb{E}a^2_{n,\tau}) + 2\nu(L^2||\phi_n||^2_\infty + \sup_{\tau \in \mathbb{R}}\mathbb{E}b^2_{n,\tau}) \right]. \]

Consequently,
\[ (4.16) \left( 1 - \frac{2N^2L^2}{\nu^2}(2 + \nu) \right)||\phi_n||^2_\infty \leq \frac{4N^2}{\nu^2} \sup_{\tau \in \mathbb{R}}\mathbb{E}a^2_{n,\tau} + \frac{2N^2}{\nu} \sup_{\tau \in \mathbb{R}}\mathbb{E}b^2_{n,\tau}. \]

By our assumption on \( L \), the coefficients of \( ||\phi_n||^2_\infty \) is positive.

We note by (4.7) and Remark 4.2 that, for \( p > 2 \), the contradiction constant \( \theta_2 \) for the equation (4.11) is
\[ \theta_2 = \frac{2N^2L^2}{\nu^2} + \frac{N^2L^2}{\nu}. \]

Comparing to Remark 4.5 we have
\[ \lim_{p \to 2^+} \theta_p = \theta_2 + \frac{N^2L^2}{\nu}. \]

We also note that \( \lim_{p \to 2^+} \theta_p < 1 \) if and only if
\[ (4.17) \mathcal{L} < \frac{\nu}{N\sqrt{2(1 + \nu)}}. \]

which is satisfied by our assumption on \( L \). So it follows from Proposition 4.4 that (4.11) admits a unique \( L^p \)-bounded solution for some \( p > 2 \). This \( L^p \)-bounded solution is exactly the unique \( L^2 \)-bounded solution \( \tilde{\xi} \) of (4.11). So the family
\[ \{||\tilde{\xi}(\tau)||^2 : \tau \in \mathbb{R}\} \]

is uniformly integrable, and hence by conditions (C1) and (C2) the families
\[ \{a^2_{n,\tau} : n \in \mathbb{N}, \tau \in \mathbb{R}\} \quad \text{and} \quad \{b^2_{n,\tau} : n \in \mathbb{N}, \tau \in \mathbb{R}\} \]

are uniformly integrable. This together with (4.8) and (4.9) implies: taking limit in (4.10), we obtain the required result, i.e. \( \xi_n(t) \to \tilde{\xi}(t) \) in distribution uniformly on \( \mathbb{R} \). On the other hand, \( \xi(t + t_n) \) satisfies the equation
\[ \xi(t + t_n) = \int_{-\infty}^t U(t - \tau) F(\tau + t_n, \xi(\tau + t_n))d\tau + \int_{-\infty}^t U(t - \tau) G(\tau + t_n, \xi(\tau + t_n))d\tilde{W}_n(\tau), \]

with \( \tilde{W}_n(t) = W(t + t_n) - W(t_n) \). Note that \( \tilde{W}_n(\cdot) \) is also a standard Brownian motion with the same distribution as \( W(\cdot) \), so \( \xi_n(t) \) and \( \xi(t + t_n) \) share the same distribution on \( H \). This implies \( \xi(t + t_n) \to \tilde{\xi}(t) \) in distribution uniformly in \( t \in \mathbb{R} \). Thus we have \( \{t_n\} \in \mathcal{M}_F \).

(ii)-(b). Let \( \{t_n\} \in \mathcal{M}_{F,G} \). Then there exists \( (\tilde{F}, \tilde{G}) \in H(F,G) \) such that for any \( r, l > 0 \)
\[ (4.18) \sup_{|s| \leq l, |x| \leq r} |F(t + t_n, x) - \tilde{F}(t, x)| \to 0 \]
and
\begin{equation}
(4.19) \quad \sup_{|t| \leq L, |x| \leq \tau} |G(t + t_n, x) - \tilde{G}(t, x)| \to 0
\end{equation}
as $n \to \infty$. Like what we did in the proof of (ii)-(a): let $\xi_n$ and $\tilde{\xi}$ be the unique bounded solutions of the shift equation and the limit equation respectively, and still denote $\phi_n = \xi_n - \tilde{\xi}$. To finish the proof, it suffices to show $\phi_n \to 0$ in the space $C(\mathbb{R}, L^2(\mathbb{R}, H))$, i.e. \( \lim_{n \to \infty} \max_{|t| \leq L} \mathbb{E}|\phi_n(t)|^2 = 0 \) for any $L > 0$.

Since $\phi_n$ is the unique bounded solution of equation (4.12), by the Cauchy-Schwarz inequality and Itô’s isometry property we have
\begin{equation}
(4.20) \quad \mathbb{E}|\phi_n(t)|^2 \leq 2N^2 \left( \frac{1}{\nu} \int_{-\infty}^{t} e^{-\nu(t-\tau)} \mathbb{E}|h_n(\tau) - \tilde{h}(\tau)|^2 d\tau \right) + \int_{-\infty}^{t} e^{-2\nu(t-\tau)} \mathbb{E}|g_n(\tau) - \tilde{g}(\tau)|^2 d\tau.
\end{equation}

By (4.14) we have
\begin{equation}
(4.21) \quad \mathbb{E}|h_n(\tau) - \tilde{h}(\tau)|^2 \leq 2(L^2 \mathbb{E}|\phi_n(\tau)|^2 + \mathbb{E} a_{n,\tau}^2).
\end{equation}

Similar to (4.13), we have
\begin{align*}
\mathbb{E}|h_n(\tau) - \tilde{h}(\tau)|^2 &= \mathbb{E}|F^{t_n}(\tau, \xi_n(\tau)) - \tilde{F}(\tau, \tilde{\xi}(\tau))|^2 \\
& \leq 2(\mathbb{E}|F^{t_n}(\tau, \xi_n(\tau))|^2 + \mathbb{E}|\tilde{F}(\tau, \tilde{\xi}(\tau))|^2) \\
& \leq 4(A_0 + L\nu)^2
\end{align*}
for any $\tau \in \mathbb{R}$ and, consequently,
\begin{equation}
(4.22) \quad ||h - \tilde{h}||_\infty^2 \leq 4(A_0 + L\nu)^2
\end{equation}
for any $n \in \mathbb{N}$.

Using the same arguments as above we have
\begin{equation}
(4.23) \quad \mathbb{E}|g_n(\tau) - \tilde{g}(\tau)|^2 \leq 2(L^2 \mathbb{E}|\phi_n(\tau)|^2 + \mathbb{E} b_{n,\tau}^2)
\end{equation}
and
\begin{equation}
(4.24) \quad ||g - \tilde{g}||_\infty^2 \leq 4(A_0 + L\nu)^2
\end{equation}
for any $n \in \mathbb{N}$.

From (4.20), (4.21), (4.23) and taking into account that $e^{-2\nu(t-\tau)} \leq e^{-\nu(t-\tau)}$ ($t \geq \tau$), we obtain
\begin{align*}
\mathbb{E}|\phi_n(t)|^2 &\leq \left( \frac{4N^2 \mathbb{E}|\phi_n(\tau)|^2}{\nu} + 4N^2 \mathbb{E}|\phi_n(\tau)|^2 \right) \int_{-\infty}^{t} e^{-\nu(t-\tau)} \mathbb{E}|\phi_n(\tau)|^2 d\tau + 4N^2 \int_{-\infty}^{t} e^{-\nu(t-\tau)} \left( \frac{1}{\nu} \mathbb{E} a_{n,\tau}^2 + \mathbb{E} b_{n,\tau}^2 \right) d\tau.
\end{align*}

By Lemma 4.3 we have
\begin{align*}
\max_{|t| \leq L} \mathbb{E}|\phi_n(t)|^2 &\leq 4N^2 \frac{e^{kl}}{k} e^{-kt} \sup_{t \in \mathbb{R}} \left( \frac{1}{\nu} \mathbb{E} a_{n,t}^2 + \mathbb{E} b_{n,t}^2 \right) \left( 1 + 4N^2 \max_{|t| \leq L} \left( \frac{1}{\nu} \mathbb{E} a_{n,t}^2 + \mathbb{E} b_{n,t}^2 \right) \right),
\end{align*}
where $k > 0$ is a constant.
where
\[ k := \nu - \frac{4N^2}{\nu} \left( \mathcal{L}^2 + 4N^2 \mathcal{L}^2 \right) > 0 \]
by the assumption \( \mathcal{L} < \frac{\nu}{2N^2 + \nu} \).

Let now \( \{l_n\} \) be a sequence of positive numbers such that \( l_n \to +\infty \) as \( n \to \infty \). According to inequality (4.22), (4.24) and (4.25) we obtain
\[
\max_{|t| \leq L} \mathbb{E}[\phi_n(t)]^2 \leq \frac{16N^2 e^{kL e^{-kl_n}}}{k} \left( \frac{1}{\nu} + 1 \right) (A_0 + Lr)^2
\]
\[ + \frac{4N^2 (1 - e^{-kL e^{-kl_n}})}{k} \max_{|t| \leq l_n} \left( \frac{1}{\nu} \mathbb{E}a^2 + \mathbb{E}b^2 \right) . \]

(4.26)

By Remark 2.2 (iii), passing to limit in (4.26) as \( n \to \infty \) we obtain for any \( L > 0 \)
\[
\lim_{n \to \infty} \max_{|t| \leq L} \mathbb{E}[\phi_n(t)]^2 = 0
\]
by (4.18), (4.19) and the uniform integrability of the families \( \{a_{n, \tau} : n \in \mathbb{N}, \tau \in \mathbb{R}\} \)
and \( \left\{b_{n, \tau}^2 : n \in \mathbb{N}, \tau \in \mathbb{R}\right\} \). That is, \( \xi_n \to \hat{\xi} \) as \( n \to \infty \) in the space \( C(\mathbb{R}, L^2(\mathbb{P}, H)) \).

So we have \( \xi_n(t) \to \hat{\xi}(t) \) in distribution uniformly in \( t \in [-L, L] \) for any \( L > 0 \).
Since \( \xi_n(t) \) and \( \xi(t + t_n) \) share the same distribution, \( \xi(t + t_n) \to \xi(\hat{\xi}) \) in distribution uniformly in \( t \in [-L, L] \) for all \( L > 0 \). Thus we have \( \{t_n\} \in \mathfrak{M}_{\xi} \), and hence \( \xi \) is uniformly comparable in distribution. The theorem is completely proved. □

**Corollary 4.7.** Assume that the conditions of Theorem 4.6 hold.

(i) If the functions \( F \) and \( G \) are jointly stationary (respectively, \( \tau \)-periodic, quasi-periodic with the spectrum of frequencies \( \nu_1, \nu_2, \ldots, \nu_k \), Bohr almost periodic, Bohr almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in \( t \in \mathbb{R} \) uniformly with respect to \( x \in H \) on every bounded subset, then so is the unique bounded solution \( \xi \) of equation (4.1) in distribution.

(ii) If \( F \) and \( G \) are jointly pseudo-periodic (respectively, pseudo-recurrent) and \( F \) and \( G \) are jointly Lagrange stable, in \( t \in \mathbb{R} \) uniformly with respect to \( x \in H \) on every bounded subset, then the unique bounded solution \( \xi \) of (4.1) is pseudo-periodic (respectively, pseudo-recurrent) in distribution.

**Proof.** This statement follows from Theorems 2.26, 4.6 and Remark 2.30 □

5. CONVERGENCE IN SEMI-LINEAR SDEs

In this section we consider the stochastic differential equation
\[
dx(t) = (Ax(t) + F(t, x(t)))dt + G(t, x(t))dW(t),
\]
where \( F, G \in C(\mathbb{R} \times H, H) \) and the linear operator \( A \) is an infinitesimal generator which generates a \( C_0 \)-semigroup \( \{U(t)\}_{t \geq 0} \), which is exponentially stable.

**Definition 5.1.** An \( F_t \)-adapted processes \( \{x(t)\}_{t \geq t_0} \) is said to be a mild solution of equation (5.1) with initial value \( x(t_0) = x_0 \) \( (t_0 \in \mathbb{R}) \) if it satisfies the stochastic integral equation
\[
x(t) = U(t - t_0)x_0 + \int_{t_0}^{t} U(t - s)F(s, x(s))ds
\]
Theorem 5.2. Consider the equation (5.1). Suppose that the following conditions hold:

(i) the semigroup \( \{U(t)\}_{t \geq 0} \) acting on the space \( H \) is exponentially stable;
(ii) \( F, G \in C(\mathbb{R} \times H, H) \) are locally Lipschitz in \( x \in H \);
(iii) there exist two positive constants \( A_0, M \) such that \( |F(t, x)|, |G(t, x)| \leq A_0 + M|x| \) for all \( x \in H \) and \( t \in \mathbb{R} \);
(iv) \( M < \frac{\nu}{N\sqrt{6}(\nu+1)} \).

Then for any initial value \( x_0 \) with \( \mathbb{E}|x_0|^2 < \infty \) we have

\[
\mathbb{E}|x(t; t_0, x_0)|^2 
\leq 3N^2 \left[ \mathbb{E}|x_0|^2 - \frac{2A_0^2(\nu + 1)}{\nu^2 - 6N^2M^2(\nu + 1)} \right] \exp\left\{ -[\nu - 6N^2M^2(1 + 1/\nu)](t - t_0) \right\} 
\]

for any \( t \geq t_0 \), where \( x(t; t_0, x_0) \) denotes the solution of the equation (5.1) passing through \( x_0 \) at the initial moment \( t_0 \).

Proof. Since

\[
x(t; t_0, x_0) = U(t - t_0)x_0 + \int_{t_0}^{t} U(t - s)F(s, x(s; t_0, x_0))ds 
+ \int_{t_0}^{t} U(t - s)G(s, x(s; t_0, x_0))dW(s)
\]

for any \( t \geq t_0 \), by the Cauchy-Schwarz inequality and Itô’s isometry property we have

\[
\mathbb{E}|x(t; t_0, x_0)|^2 = \mathbb{E} \left| U(t - t_0)x_0 + \int_{t_0}^{t} U(t - s)F(s, x(s; t_0, x_0))ds 
+ \int_{t_0}^{t} U(t - s)G(s, x(s; t_0, x_0))dW(s) \right|^2
\]

\[
\leq 3 \left( \mathbb{E}|U(t - t_0)x_0|^2 + \mathbb{E} \left| \int_{t_0}^{t} U(t - s)F(s, x(s; t_0, x_0))ds \right|^2 
+ \mathbb{E} \left| \int_{t_0}^{t} U(t - s)G(s, x(s; t_0, x_0))dW(s) \right|^2 \right)
\]

\[
\leq 3 \left[ N^2e^{-2\nu(t-t_0)} \mathbb{E}|x_0|^2 + \mathbb{E} \left| \int_{t_0}^{t} U(t - s)F(s, x(s; t_0, x_0))ds \right|^2 
+ \mathbb{E} \left| \int_{t_0}^{t} U(t - s)G(s, x(s; t_0, x_0))dW(s) \right|^2 \right]
\]

\[
\leq 3 \left[ N^2e^{-2\nu(t-t_0)} \mathbb{E}|x_0|^2 + \frac{1}{\nu} \int_{t_0}^{t} N^2e^{-\nu(t-s)}\mathbb{E}|F(s, x(s; t_0, x_0))|^2ds \right]
\]
Then it follows from (5.3) that
\[ u(t) := e^{vt}E|x(t; t_0, x_0)|^2, \quad \text{for } t \geq t_0. \]

Denote
\[ (5.3) \quad 3\lambda^2 e^{-vt} \left[ e^{v_{t_0}}E|x_0|^2 + 2(1 + \frac{1}{\nu}) \int_{t_0}^t e^{vs}(A_0^2 + M^2E|x(s; t_0, x_0)|^2)ds \right]. \]

Along with inequality (5.4) we consider the equation
\[ v(t) = 3\lambda^2 e^{v_{t_0}}E|x_0|^2 + \frac{6\lambda^2 A_0^2}{\nu^2}(\nu + 1)(e^{vt} - e^{v_{t_0}}) + 6\lambda^2 M^2(1 + \frac{1}{\nu}) \int_{t_0}^t u(s)ds. \]
that is, \( v(t) \) satisfies the equation
\[ v'(t) = 6\lambda^2 M^2(1 + \frac{1}{\nu})v(t) + 6\lambda^2 A_0^2(1 + \frac{1}{\nu})e^{vt} \]
with initial condition \( v(t_0) = 3\lambda^2 e^{v_{t_0}}E|x_0|^2 \). Solving this equation for \( v(t) \) we get
\[ v(t) = 3\lambda^2 e^{\alpha(t-t_0) + v_{t_0}}E|x_0|^2 + \frac{\beta}{\nu - \alpha}[e^{vt} - e^{\alpha(t-t_0) + v_{t_0}}] \]
\[ = \left( 3\lambda^2 E|x_0|^2 - \frac{\beta}{\nu - \alpha} \right) e^{\alpha(t-t_0) + v_{t_0}} + \frac{\beta}{\nu - \alpha} e^{vt}, \]
where
\[ \alpha := 6\lambda^2 M^2(1 + \frac{1}{\nu}) \quad \text{and} \quad \beta := 6\lambda^2 A_0^2(1 + \frac{1}{\nu}). \]
The comparison principle then implies that \( u(t) \leq v(t) \) for all \( t \geq t_0 \), so it follows from the definition of \( u(t) \) that for \( t \geq t_0 \) we have
\[ E|x(t; t_0, x_0)|^2 \leq \left( 3\lambda^2 E|x_0|^2 - \frac{\beta}{\nu - \alpha} \right) e^{-(\nu-\alpha)(t-t_0)} + \frac{\beta}{\nu - \alpha}, \]
which is just (5.2). The proof is complete. \( \square \)

Note that condition (iv) in Theorem 5.2 implies \( \nu > \alpha = 6\lambda^2 M^2(1 + 1/\nu) \). So we have the following

**Corollary 5.3.** Under the conditions of Theorem 5.2 for arbitrary \( \varepsilon > 0 \) and \( r > 0 \) there exists a positive number \( T(\varepsilon, r) \) such that
\[ E|x(t; t_0, x_0)|^2 \leq \frac{6\lambda^2 A_0^2(\nu + 1)}{\nu^2 - 6\lambda^2 M^2(\nu + 1)} + \varepsilon \]
for all \( ||x_0||_2 \leq r \) and \( t \geq t_0 + T(\varepsilon, r) \). In other words, we have
\[ \limsup_{t \to \infty} E|x(t; t_0, x_0)|^2 \leq \frac{6\lambda^2 A_0^2(\nu + 1)}{\nu^2 - 6\lambda^2 M^2(\nu + 1)} \]
uniformly with respect to \( x_0 \) on every bounded subset of \( L^2(\mathbb{P}, H) \).
Theorem 5.4. Consider the equation (5.1). Suppose that the following conditions hold:

(i) the semigroup \( \{U(t)\}_{t \geq 0} \) acting on the space \( H \) is exponentially stable;

(ii) \( F, G \in C(\mathbb{R} \times H, H) \) are globally Lipschitz in \( x \in H \) and \( \text{Lip}(F), \text{Lip}(G) \leq \mathcal{L} \);

(iii) there exists a positive constant \( A_0 \) such that \( |F(t,0)|, |G(t,0)| \leq A_0 \) for all \( t \in \mathbb{R} \);

(iv) \( \mathcal{L} < \frac{\nu}{N \sqrt{3(\nu+1)}}. \)

Then the following statements hold:

(i) for any \( t \geq t_0 \) and \( x_1, x_2 \in L^2(\mathbb{P}, H) \),

\[
\mathbb{E}|x(t; t_0, x_1) - x(t; t_0, x_2)|^2 \leq 3N^2 \exp \left\{ -[\nu - 3(1 + \frac{1}{\nu})N^2L^2](t - t_0) \right\} \mathbb{E}|x_1 - x_2|^2;
\]

(ii) equation (5.1) has a unique solution \( \varphi \in \mathcal{C}_b(\mathbb{R}, L^2(\mathbb{P}, H)) \) which is globally asymptotically stable and

\[
\mathbb{E}|x(t; t_0, x_0) - \varphi(t)|^2 \leq 3N^2 \exp \left\{ -\left[ \nu - 3(1 + \frac{1}{\nu})N^2L^2 \right] (t - t_0) \right\} \mathbb{E}|x_0 - \varphi(t_0)|^2
\]

for any \( t \geq t_0 \) and \( x_0 \in L^2(\mathbb{P}, H) \).

Proof. (i). Denote by \( \omega(t) := x(t; t_0, x_1) - x(t; t_0, x_2) \) for any \( t \geq t_0 \). Since

\[
x(t; t_0, x_i) = U(t-t_0)x_i + \int_{t_0}^{t} U(t-s)F(s, x(s; t_0, x_i))ds \]

\[
+ \int_{t_0}^{t} U(t-s)G(s, x(s; t_0, x_i))dW(s)
\]

for \( i = 1, 2 \), we have

\[
\omega(t) = U(t-t_0)(x_1 - x_2) + \int_{t_0}^{t} U(t-s)[F(s, x(s; t_0, x_1)) - F(s, x(s; t_0, x_2))]ds
\]

\[
+ \int_{t_0}^{t} U(t-s)[G(s, x(s; t_0, x_1)) - G(s, x(s; t_0, x_2))]dW(s).
\]

Consequently,

\[
\mathbb{E}|\omega(t)|^2 \leq 3 \left( \mathbb{E}|U(t-t_0)(x_1 - x_2)|^2 
\right.
\]

\[
+ \mathbb{E} \left| \int_{t_0}^{t} U(t-s)[F(s, x(s; t_0, x_1)) - F(s, x(s; t_0, x_2))]ds \right|^2
\]

\[
+ \mathbb{E} \left| \int_{t_0}^{t} U(t-s)[G(s, x(s; t_0, x_1)) - G(s, x(s; t_0, x_2))]dW(s) \right|^2
\]

\[
\leq 3 \left( N^2 e^{-2\nu(t-t_0)} \mathbb{E}|x_1 - x_2|^2 
\right.
\]

\[
+ N^2 \int_{t_0}^{t} e^{-\nu(t-s)} ds \int_{t_0}^{t} e^{-\nu(t-s)} \mathbb{E}|F(s, x(s; t_0, x_1)) - F(s, x(s; t_0, x_2)|^2 ds
\]

\[
+ N^2 \int_{t_0}^{t} e^{-\nu(t-s)} ds \int_{t_0}^{t} e^{-\nu(t-s)} \mathbb{E}|G(s, x(s; t_0, x_1)) - G(s, x(s; t_0, x_2)|^2 ds
\]

\[
\right).
\]

\[
\mathbb{E}|x(t; t_0, x_0) - \varphi(t)|^2 \leq 3N^2 \exp \left\{ -\left[ \nu - 3(1 + \frac{1}{\nu})N^2L^2 \right] (t - t_0) \right\} \mathbb{E}|x_0 - \varphi(t_0)|^2
\]

for any \( t \geq t_0 \) and \( x_0 \in L^2(\mathbb{P}, H) \).
Consider an ordinary differential equation perturbed by white noise: 

\[ + \int_{t_0}^{t} \mathcal{N}^2 e^{-2\nu(t-s)} \mathbb{E}[G(s, x(s; t_0, x_1)) - G(s, x(s; t_0, x_2))]^2 \, ds \]

\[ \leq 3 \mathcal{N}^2 e^{-2\nu(t-t_0)} \mathbb{E}|x_1 - x_2|^2 \]

\[ + (1 + \frac{1}{\nu}) \mathcal{N}^2 \mathcal{L}^2 \int_{t_0}^{t} e^{-\nu(t-s)} \mathbb{E}|x(s; t_0, x_1) - x(s; t_0, x_2)|^2 \, ds \]

(5.7)

Set \( u(t) := e^{\nu t} \mathbb{E}|\omega(t)|^2 \) for \( t \geq t_0 \), then from (5.7) we get

(5.8) 

\[ u(t) \leq 3 \mathcal{N}^2 e^{\nu t_0} \mathbb{E}|x_1 - x_2|^2 + 3(1 + \frac{1}{\nu}) \mathcal{N}^2 \mathcal{L}^2 \int_{t_0}^{t} u(s) \, ds. \]

Along with inequality (5.8) we consider the equation

\[ v(t) = 3 \mathcal{N}^2 e^{\nu t_0} \mathbb{E}|x_1 - x_2|^2 + 3(1 + \frac{1}{\nu}) \mathcal{N}^2 \mathcal{L}^2 \int_{t_0}^{t} v(s) \, ds. \]

Solving this equation for \( v(t) \) we obtain

\[ v(t) - 3 \mathcal{N}^2 e^{\nu t_0} \mathbb{E}|x_1 - x_2|^2 \exp \left\{ 3(1 + \frac{1}{\nu}) \mathcal{N}^2 \mathcal{L}^2 (t - t_0) \right\}. \]

The comparison principle then implies that \( u(t) \leq v(t) \), i.e.

\[ u(t) \leq 3 \mathcal{N}^2 e^{\nu t_0} \mathbb{E}|x_1 - x_2|^2 \exp \left\{ 3(1 + \frac{1}{\nu}) \mathcal{N}^2 \mathcal{L}^2 (t - t_0) \right\}, \quad \text{for } t \geq t_0 \]

and consequently by the definition of \( u(t) \) we get

\[ \mathbb{E}|x(t; t_0, x_1) - x(t; t_0, x_2)|^2 \leq 3 \mathcal{N}^2 \mathbb{E}|x_1 - x_2|^2 \exp \left\{ - \nu - 3(1 + \frac{1}{\nu}) \mathcal{N}^2 \mathcal{L}^2 (t - t_0) \right\} \]

for any \( t \geq t_0 \).

(ii). By the proof of Theorem 4.6 (see (5.7) and the paragraph following it), equation (5.1) admits a unique bounded solution \( \varphi \in C_0(\mathbb{R}, L^2(\mathbb{P}, H)) \) under the condition (4.17), which is met under the current condition (iv).

To establish inequality (5.6) we note that \( \varphi(t) = x(t; t_0, \varphi(t_0)) \) for any \( t \geq t_0 \). Applying (5.6) we obtain inequality (5.6).

6. Applications

In this section, we illustrate our theoretical results by two examples.

**Example 6.1.** Consider an ordinary differential equation perturbed by white noise:

\[ (6.1) \]

\[ dy = \left( -5y + \cos t + \sin \sqrt{2}t \cdot \frac{y}{y^2 + 1} \right) \, dt + \frac{1}{2} y \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2}t} \right) \, dW \]

\[ = : (A y + f(t, y)) \, dt + g(t, y) \, dW, \]

where \( W \) is a one-dimensional two-sided Brownian motion. It is clear that \( A \) generates an exponentially stable semigroup on \( \mathbb{R} \) with \( \mathcal{N} = 1 \) and \( \nu = 5 \). Note that \( f \) is quasi-periodic in \( t \) and \( g \) is Levitan almost periodic in \( t \), uniformly w.r.t \( y \) on
any bounded subset of $\mathbb{R}$, so $f, g$ are jointly Levitan almost periodic. The Lipschitz constants of $f, g$ satisfy $\max\{\text{Lip}(f), \text{Lip}(g)\} \leq 2/3$, so the conditions of Theorems 4.6, 5.2 and 5.4 are met.

Since the coefficients satisfy both Lipschitz and global linear growth conditions, it follows that the equation (6.1) admits global in time solutions. By Theorem 4.6, (6.1) admits a unique $L^2$-bounded mild solution; furthermore, this unique $L^2$-bounded solution is Levitan almost periodic in distribution by Corollary 4.7. By Theorem 5.4 this Levitan almost periodic in distribution solution is globally asymptotically stable in square-mean sense. By Corollary 5.3 all the solutions of (6.1) with $L^2$-initial value are bounded by a constant after sufficiently long time.

If $f$ remains unchanged but $g(t, y) = y(\sin t + \cos \sqrt{2}t)/4$, then $g$ is quasi-periodic in $t$, uniformly w.r.t $y$ on any bounded subset. In this case $f, g$ are jointly quasi-periodic, so (6.1) admits a quasi-periodic in distribution solution.

Example 6.2. Consider the stochastic heat equation on the interval $[0,1]$ with Dirichlet boundary condition:

\begin{equation}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2} + \frac{(\sin t + \cos \sqrt{3}t) \sin u}{3} + \frac{u}{u^2+1} \cdot \cos \left( \frac{1}{2 + \sin t + \sin \sqrt{2}t} \right) \frac{\partial W}{\partial t} - \frac{\partial^2 u}{\partial \xi^2} + f(t, u) + g(t, u) \frac{\partial W}{\partial t},
\end{equation}

$u(t, 0) = u(t, 1) = 0, \quad t > 0.$

Here $W$ is a one-dimensional two-sided Brownian motion. Let $A$ be the Laplace operator, then $A : D(A) = H^2(0,1) \cap H^1_0(0,1) \to L^2(0,1)$. Denote $H := L^2(0,1)$ and the norm on $H$ by $\| \cdot \|$. Then the stochastic heat equation can be written as an abstract evolution equation

\begin{equation}
\frac{dY(t)}{dt} = (AY(t) + F(t, Y(t)))dt + G(t, Y(t))dW(t)
\end{equation}

on the Hilbert space $H$, where

$Y(t) := u(t, \cdot), \quad F(t, Y(t)) := f(t, u(t, \cdot)), \quad G(t, Y(t)) := g(t, u(t, \cdot)).$

Note that, the operator $A$ has eigenvalues $\{-n^2 \pi^2\}_{n=1}^{\infty}$ and generates a $C^0$-semigroup $T(t)$ on $H$ satisfying $\|T(t)\| \leq e^{-\pi^2 t}$ for $t \geq 0$, i.e. $\mathcal{N} = 1$ and $\nu = \pi^2$. Note that $\max\{\text{Lip}(F), \text{Lip}(G)\} \leq 1$, so it is immediate to verify that conditions (C1)-(C2) hold and the restrictions on Lipschitz constant in Theorems 4.6, 5.2 and 5.4 are satisfied. As pointed out in Remark 4.2 (iii), we need to check condition (C3). Indeed, since $f$ is bounded, for given $\alpha > 0$ we have

\begin{equation}
\sup_{t \in \mathbb{R}, \|u\| \leq M} \int_{[0,1]} |f(t, u(x))|^{2+\alpha} dx < \infty
\end{equation}

for any $M > 0$, i.e. the family $\{||f(t, u(x))||^2 : t \in \mathbb{R}, ||u|| \leq M\}$ of functions of $x$ is uniformly integrable on $[0,1]$. This implies that for $t_n \to t$, by choosing $k$ large enough,

\[ \int_{[0,1]} |f(t_n, u(x)) - f(t, u(x))|^2 dx \]
\[
\int_{[0,1] \cap M_k} |f(t_n, u(x)) - f(t, u(x))|^2 \, dx + \int_{[0,1] \setminus M_k} |f(t_n, u(x)) - f(t, u(x))|^2 \, dx
\]

is sufficiently small, where \( M_k := \{ x \in [0,1] : |u(x)| \leq k \} \). That is, (C3) holds.

Finally note that \( F \) is quasi-periodic in \( t \) and \( G \) is Levitan almost periodic in \( t \), uniformly w.r.t. \( Y \in H \).

By Theorem 4.6, (6.3) (and hence (6.2)) admits a unique \( L^2(\mathbb{P}, H) \)-bounded mild solution, and by Corollary 4.7 this unique bounded solution is Levitan almost periodic in distribution. By Theorem 5.4 this bounded solution is globally asymptotically stable in square-mean sense. By Corollary 5.3 all the solutions of (6.2) with \( L^2 \)-initial value are bounded by a constant after sufficiently long time.

**Remark 6.3.** As pointed out in Remark 4.2 to apply our results for stochastic PDEs, we need to check the condition (C3), which is not easy to check in some situations. We will try to weaken or remove this condition in our future work.

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