Deep Differential System Stability
Learning advanced computations from examples

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Abstract

Can advanced mathematical computations be learned from examples? Using transformers over large generated datasets, we train models to learn properties of differential systems, such as local stability, behavior at infinity and controllability. We achieve near perfect estimates of qualitative characteristics of the systems, and good approximations of numerical quantities, demonstrating that neural networks can learn advanced theorems and complex computations without built-in mathematical knowledge.

1 Introduction

Mathematicians solve problems by relying on rules, correct derivations and proven methods of computation that are guaranteed to lead to a correct solution. Over time, they have developed a rich set of computational techniques that can be applied to many problems, and were said to be “unreasonably effective” (Wigner [44]). Most of those computational techniques are not intuitive, they have to be derived from theory and applied by trained scientists or built into software libraries. They are the building blocks of every advanced computation.

Many recent studies showed that deep learning models can learn complex rules from large datasets, only from examples. In natural language processing, models learn to output grammatically correct sentences without prior knowledge of grammar and syntax [34], or to automatically map one language into another [8, 39]. In mathematics, deep learning models have been trained to perform logical inference [10], SAT solving [37], basic arithmetic [16] and symbolic integration [22].

In this paper, we investigate whether deep learning models can be trained to perform complex computations and to deduce the qualitative behavior of mathematical objects, without built-in mathematical knowledge. We consider three questions of higher mathematics: the local stability and controllability of differential systems, and the existence and behavior at infinity of solutions of partial differential equations. All three problems have been widely researched and have many applications outside of pure mathematics. They have known solutions that rely on advanced symbolic and computational techniques, from formal differentiation, Fourier transform, geometrical full-rank conditions, to function evaluation, matrix inversion, and computation of complex eigenvalues. Surprisingly, we find that neural networks can solve these problems with a very high accuracy, by simply looking at instances of problems and their solutions, while being totally unaware of the underlying theory. These results are unintuitive given the advanced numerical techniques required by the theory and the difficulty of neural networks to perform simple arithmetic tasks [36, 41, 47], suggesting that the model might be using a different approach than the known theory to correctly predict the output.

After reviewing prior applications of deep learning to differential equations and symbolic computation, we introduce the three problems we consider, describe how we generate datasets, and detail how we train our models. Finally, we present our experiments and discuss their results.

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2 Related work

Research on the application of neural networks to differential equations has mainly focused on two aspects. A first line of research investigates the relation between residual neural networks \[12\] and the different discretization schemes of Ordinary Differential Equations (ODEs). In particular, an ODE of the form
\[\frac{dy}{dt} = f(y(t), t)\]
can be approximated by the Euler method with
\[y(t+h) = y(t) + h \times f(y(t), t)\]
which can be seen as a particular instantiation of a residual network: the chain of layers in a residual neural network represents an approximation of the ODE solution with the Euler method. Chen et al. \[6\] used this observation to build a memory efficient architecture where the network depth can be adapted by simply modifying the discretization scheme (with a potentially infinite number of layers). Subsequent work by Lu et al. \[25\] analyzed the connection between neural architectures and numerical differential equations. In particular, they showed that other categories of neural networks such as PolyNets \[48\] or FractalNets \[23\] could also be interpreted as different discretization schemes of ODEs, although these networks were not designed with differential equations in mind.

Although the majority of Partial Differential Equations (PDEs) do not have any explicit solution, their solutions can be approximated using numerical methods. These methods are particularly efficient for low-dimension problems, but they are based on the discretization of the input domain, which does not scale with the dimensionality of the problem. A second line of research studies the ability of neural networks to approximate the solution of PDEs. The idea relies on the universal approximation theorem, that states that any continuous function can be approximated by a neural network with one hidden layer \[8, 15, 14, 31, 32\]. In particular, if a PDE does not have an explicit solution, then its solution can be approximated with a neural network \[20, 21, 24, 35, 38\]. Unlike numerical methods, this approach is not as sensitive to the problem dimension.

Lample and Charton \[22\] proposed several approaches to generate arbitrarily large datasets of functions with their integrals, and differential equations with their solutions. They found that a transformer model \[42\] trained on millions of examples could outperform state-of-the-art symbolic frameworks such as Mathematica or MATLAB \[45, 27\] on a particular subset of equations. Their model was used to guess solutions, while verification (arguably a simpler task) was left to a symbolic framework \[29\]. Arabshahi et al. \[1, 2\] proposed to use neural networks to verify the solution of differential equations, and found that Tree-LSTMs \[40\] were better than sequential LSTMs \[13\] at generalizing beyond the training distribution. Other approaches investigated the capacity of neural networks to perform arithmetic operations \[16, 36, 41\] or to run short computer programs \[47\]. More recently, Saxton et al. \[36\] found that neural networks were good at solving arithmetic problems or at performing operations such as differentiation or polynomial expansion, but struggled on tasks like prime number decomposition or on primality tests that can typically not be guessed and require a significant number of steps to compute. Unlike problems like prime number factorization, the problems we address in this paper cannot be solved by simple algorithmic computations.

3 Differential systems and their stability

A differential system of degree \(n\) is a system of \(n\) equations of \(n\) variables \(x_1(t), \ldots, x_n(t)\),
\[
\frac{dx_i(t)}{dt} = f_i(x_1(t), x_2(t), \ldots, x_n(t)), \quad \text{for} \quad i = 1 \ldots n
\]
or, in vector form, with \(x \in \mathbb{R}^n\) and \(f : \mathbb{R}^n \to \mathbb{R}^n\),
\[
\frac{dx(t)}{dt} = f(x(t))
\]
Many problems can be set as differential systems. Special cases include n-th order ordinary differential equations (letting \(x_1 = y, x_2 = y', \ldots, x_n = y^{(n-1)}\)), systems of coupled differential equations, and some particular partial differential equations (separable equations or equations with characteristics). Differential systems are one of the most studied areas of mathematical sciences.
They are found in physics, mechanics, chemistry, biology, and economics as well as in pure mathematics. Most differential systems have no explicit solution. Therefore, mathematicians have studied the properties of their solutions, and first and foremost their stability, a notion of paramount importance in many engineering applications. Studies on stability began in the 18th century with Euler, Lagrange, and then Dirichlet, with significant contributions by Poincaré and Lyapunov at the turn of the 20th century. This will be the first problem we investigate.

3.1 Local stability

Let \( x_e \in \mathbb{R}^n \) be an equilibrium point, that is, \( f(x_e) = 0 \). If all solutions \( x(t) \) converge to \( x_e \) when their initial positions \( x(0) \) at \( t = 0 \) are close enough, the equilibrium is said to be locally stable (see Appendix B for a proper mathematical definition). This problem is well known, if \( f \) is differentiable in \( x_e \), an answer is provided by the Spectral Mapping Theorem (SMT) [7, Theorem 10.10]:

**Theorem 3.1.** Let \( J(f)(x_e) \) be the Jacobian matrix of \( f \) in \( x_e \) (the matrix of its partial derivatives relative to its variables). Let \( \lambda \) be the largest real part of its (complex) eigenvalues. If \( \lambda \) is positive, \( x_e \) is an unstable equilibrium. If \( \lambda \) is negative, then \( x_e \) is a locally stable equilibrium.

If \( \lambda \) is negative, close to \( x_e \), the solutions will converge to this equilibrium exponentially fast with decay rate \( |\lambda| \). Predicting \( \lambda \) at a point \( x_e \) for a given differential system \( f \) is our first problem. Therefore, to measure the stability of a differential system at a point \( x_e \), we need to:

1. differentiate each function with respect to each variable, obtain the formal Jacobian \( J(x) \)

\[
f(x) = \left( \begin{array}{c}
\cos(x_2) - 1 - \sin(x_1) \\
x_1^2 - \sqrt{1 + x_2}
\end{array} \right), \quad J(x) = \left( \begin{array}{cc}
-\cos(x_1) & -\sin(x_2) \\
2x_1 & -(2\sqrt{1 + x_2})^{-1}
\end{array} \right)
\]

2. evaluate \( J(x_e) \), the Jacobian in \( x_e \) (a real or complex matrix)

\[
x_e = (0.1, ... 0.1) \in \mathbb{R}^n, \quad J(x_e) = \left( \begin{array}{cc}
-\cos(0.1) & -\sin(0.1) \\
0.2 & -(2\sqrt{1 + 0.1})^{-1}
\end{array} \right)
\]

3. calculate the eigenvalues \( \lambda_i, i = 1...n \) of \( J(x_e) \)

\[
\lambda_1 = -1.031, \quad \lambda_2 = -0.441
\]

4. return \( \lambda = -\max(\text{Real}(\lambda_i)) \) the speed of convergence of the system

\[
\lambda = 0.441 > 0 \rightarrow \text{locally stable with decay rate 0.441}
\]

3.2 Control theory

One of the lessons of the spectral mapping theorem is that instability is very common. In fact, unstable systems are plenty in nature (Lagrange points, epidemics, satellite orbits, etc.), and the idea of trying to control them through external variables comes naturally. This is the controllability problem. Control theory has a lot of practical applications, including space launch and the landing on the moon, the US Navy automated pilot, or recently autonomous vehicles [5, 33, 11]. Formally, we are given a system

\[
\frac{dx}{dt} = f(x(t), u(t)),
\]

where \( x \in \mathbb{R}^n \) is the state of the system. We want to find a function \( u(t) \in \mathbb{R}^p \), the control action, such that, beginning from a position \( x_0 \) at \( t = 0 \), we can reach a position \( x_1 \) at \( t = T \) (see Appendix B). The first rigorous mathematical analysis of this problem was given by Maxwell [28], but a turning point was reached in 1963, when Kalman gave a precise condition for a linear system [17], later adapted to nonlinear system [7] as follows:

**Theorem 3.2 (Kalman condition).** Let \( A = \partial_x f(x_e, u_e) \) and \( B = \partial_u f(x_e, u_e) \), if

\[
\text{Span}\{A^iBu : u \in \mathbb{R}^m, i \in \{0, ..., n-1\}\} = \mathbb{R}^n,
\]

then the system is locally controllable around \( x = x_e, u = u_e \).
When this condition holds, a solution to the control problem that makes the system locally stable in $x_e$ is $u(t) = u_e + K(x(t) - x_e)$ (c.f. [7], [19], [26] and appendix C for key steps of the proof), where $K$ is the $m \times n$ control feedback matrix:

$$K = -B^{tr} \left( e^{-AT} \left[ \int_0^T e^{-At} BB^{tr} e^{-A^{tr}t} dt \right] e^{-A^{tr}T} \right)^{-1}. \quad (3)$$

In the non-autonomous case, where $f = f(x, u, t)$ (and $A$ and $B$) depends on $t$, (2) becomes:

$$\text{Span}\{D_i u : u \in \mathbb{R}^m, i \in \{0, ..., 2n - 1\} = \mathbb{R}^n\}, \quad (4)$$

where $D_0(t) = B(t)$ and $D_{i+1}(t) = D_i(t) - A(t)D_i(t)$. All these theorems make use of advanced mathematical results, such as the Cayley-Hamilton theorem, or LaSalle invariance principle. Learning them by predicting controllability and computing the control feedback matrix $K$ is our second problem. To measure whether the system is controllable at a point $x_e$, we need to:

1. differentiate the system with respect to its internal variables, obtain the Jacobian $A(x, u)$
2. differentiate the system with respect to its control variables, obtain the matrix $B(x, u)$
3. evaluate $A$ and $B$ in $(x_e, u_e)$
4. calculate the controllability matrix $C$ with (2) (resp. (3) in the non-autonomous case)
5. calculate the rank $d$ of $C$, if $d = n$, the system is controllable
6. (optionally) if $d = n$, compute the control feedback matrix $K$ with (3)

In: $f(x, u) = \left( \sin(x_1^2) + \log(1 + x_2^2) + \frac{\tan(x_1)}{1 + x_2^2} \right)$, $x_e = [0.1]$, $u_e = 1$, Out: $\begin{cases} n - d = 0 \\ \text{System is controllable} \\ K = (-22.8 \quad 44.0) \end{cases}$

A step by step derivation of this example is given in Section A of the appendix.

### 3.3 Stability of partial differential equations using Fourier Transform

Partial Differential Equations (PDEs) naturally appear when studying continuous phenomena (e.g. sound, electromagnetism, gravitation). Over such problems, ordinary differential systems are not sufficient. Like differential systems, PDEs seldom have explicit solutions, and studying their stability has many practical applications. It is also a much more difficult subject, where few general theorems exist. We consider linear PDEs of the form

$$\partial_t u(t, x) + \sum_{|\alpha| \leq k} a_{\alpha} \partial_x^\alpha u(t, x) = 0, \quad (5)$$

where $t, x \in \mathbb{R}^n$, and $u(t, x)$ are time, position, and state. $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ is a multi-index and $a_{\alpha}$ are constants. Famous examples of such problems include the heat equation, transport equations or Schrodinger equation [3]. We want to determine whether a solution $u(t, x)$ of (5) exists for a given an initial condition $u(0, x) = u_0$, and if it tends to zero as $t \to +\infty$. This is mathematically answered (see appendix [C] and [3], [4] for similar arguments) by:

**Proposition 3.1.** Given $u_0 \in S'(\mathbb{R}^n)$, the space of tempered distribution, there exists a solution $u \in S'(\mathbb{R}^n)$ if there exists a constant $C$ such that

$$\forall \xi \in \mathbb{R}^n, \quad u_0(\xi) = 0 \quad \text{or} \quad \text{Real}(f(\xi)) > C, \quad (6)$$

where $\overline{u_0}$ is the Fourier transform of $u_0$ and $f(\xi)$ is the Fourier polynomial associated with the differential operator $D_x = \sum_{|\alpha| \leq k} a_{\alpha} \partial_x^\alpha$. In addition, if $C > 0$, this solution $u(t, x)$ goes to zero when $t \to +\infty$.

Learning this proposition and predicting, given an input $D_x$ and $u_0$, whether a solution $u$ exists, if so, whether it vanishes at infinite time, will be our third and last problem.

To predict whether our PDE has a solution under given initial conditions, and determine its behavior at infinity, we need to:
To train our models, we generate datasets of problems and solutions. Since all problems have
software [43, 29], using the techniques described in Section 3. For both the stability and controlla-
4 Datasets and models
n
Generating a system amounts to sampling
stability, q

Since many operators or their derivatives are undefined at zero, choosing the origin as
result in a lot of undefined Jacobians, slowing generation and biasing the dataset by reducing the
number of significant digits. A derivation of the size of the problem space is provided in appendix D.

Datasets for local stability include systems with
Local stability

functions

Under these conditions, more than 95% of systems are controllable. This makes learning control-
property
iable) yields high accuracy (95%). To mitigate this, we build a balanced dataset by oversampling non-controllable solutions to achieve 50/50 balance between controllable and non-controllable systems. For feedback matrix prediction, we restrict generation to controllable systems. For the non-autonomous case, we generate systems with 2 or 3 equations. Overall, we generate 3 datasets with more than 50 million examples each: a balanced dataset for predicting autonomous controllability, a dataset of controllable systems with feedback matrices, and a dataset for non-autonomous controllability.

**Stability of partial differential equations using Fourier Transform** We generate a differential operator and an initial condition $u_0$. The operator is a polynomial in $\partial_x$. $u_0$ is the product of $n$ functions $f(a_jx_j)$ with known Fourier transforms, and $d$ operators $\exp(ib_kx_k)$, with $0 \leq d \leq 2n$ and $a_j, b_k \in \{-100, \ldots, 100\}$. We calculate 1) the existence of solutions, 2) their behavior when $t \to +\infty$, and 3) the set of frequencies, and express these three values as a sequence of two Booleans followed by floating point decimals. We create a dataset of over 50 million examples.

**Models and evaluation** In all experiments, we use a transformer architecture [42] with 8 attention heads. We train our models with the Adam optimizer [18], a learning rate of $10^{-4}$, and follow the learning rate scheduler of Vaswani et al. [42]. We vary the dimension from 64 to 1024, and the number of layers from 1 to 8. We use mini-batches composed of 1024 systems, and train our models on 8 V100 GPUs with float16 operations to speed up training and reduce memory usage.

At the end of each epoch, evaluation is carried out on a held-out validation set of 10000 examples. We ensure that test examples are never seen during training (given the size of the problem space, this never happens in practice). To evaluate a model output, we either compare it with the reference solution, or use a problem-specific metric.

5 Experiments

5.1 Local stability

In these experiments, the model is given functions $f : \mathbb{R}^n \to \mathbb{R}^n$, where $n \in \{2, \ldots, 6\}$. It is trained to predict $\lambda$, the largest real part of the eigenvalues of their Jacobians at $x_0 = [0.01]$, corresponding to the convergence speed to the equilibrium. We consider that predictions are correct when they fall within 10% of the ground truth.

An 8-layer transformer model of 1024 dimensions correctly predicts $\lambda$ in 86.6% of cases. Performances range from 96.3% for systems of degree 2, to 77.3% for systems of degree 6. Overall, increasing the dimension and adding layers improves the performance. Table 1 summarizes results for different sets of parameters.

Prediction of $\lambda$ to better accuracy can be achieved by training models on data rounded to 2, 3 or 4 significant digits, and measuring the number of exact predictions on the test sample. Overall, we predict $\lambda$ with two significant digits in 59.2% of test cases. Table 2 summarizes the results for different precisions (for transformers with 6 layers and a dimensionality of 512).

| Degree | Degree 2 | Degree 3 | Degree 4 | Degree 5 | Degree 6 | Overall |
|--------|----------|----------|----------|----------|----------|---------|
| 4 layers, dim 512 | 88.0 | 74.3 | 63.8 | 54.2 | 45.0 | 65.1 |
| 6 layers, dim 512 | 93.6 | 85.5 | 77.4 | 71.5 | 64.9 | 78.6 |
| 8 layers, dim 512 | 95.3 | 88.4 | 83.4 | 79.2 | 72.4 | 83.8 |
| 4 layers, dim 1024 | 91.2 | 80.1 | 71.6 | 61.8 | 54.4 | 71.9 |
| 6 layers, dim 1024 | 95.7 | 89.0 | 83.4 | 78.4 | 72.6 | 83.8 |
| 8 layers, dim 1024 | 96.3 | 90.4 | 86.2 | 82.7 | 77.3 | 86.6 |

| Degree | Degree 2 | Degree 3 | Degree 4 | Degree 5 | Degree 6 | Overall |
|--------|----------|----------|----------|----------|----------|---------|
| 2 digits | 83.5 | 68.6 | 55.6 | 48.3 | 40.0 | 59.2 |
| 3 digits | 75.3 | 53.2 | 39.4 | 33.4 | 26.8 | 45.7 |
| 4 digits | 62.0 | 35.9 | 25.0 | 19.0 | 14.0 | 31.3 |
5.2 Control theory

In these experiments, the model is given functions \( f : \mathbb{R}^{n+p} \rightarrow \mathbb{R}^n \) (\( \mathbb{R}^{n+p+1} \rightarrow \mathbb{R}^n \) in the non-autonomous case) and is trained on a qualitative task, predicting controllability in \( x_e \), and a numerical task, calculating feedback matrices for the control variables.

When predicting controllability in the autonomous case, transformers with 512 dimensions and 6 layers achieve 97.4\% accuracy. Performance decreases with the size of the system, from 99.1\% for 3 equations to 95.2\% for 6 equations. Surprisingly, even models with a dimensionality of 64 and only 1 or 2 layers achieve a reasonable performance, above 80\%.

Table 3: Accuracy of autonomous control task over a balanced sample of systems with 3 to 6 equations.

| Dimension       | 64    | 128   | 256   | 512   |
|-----------------|-------|-------|-------|-------|
| 1 layers        | 81.0  | 85.5  | 88.3  | 90.4  |
| 2 layers        | 82.7  | 88.0  | 93.9  | 95.5  |
| 4 layers        | 84.1  | 89.2  | 95.6  | 96.9  |
| 6 layers        | 84.2  | 90.7  | 96.3  | 97.4  |

In the non-autonomous case, transformers with 512 dimensions and 6 layers achieve 99.7\% accuracy, but using smaller models makes little difference in accuracy. Overall, this task is solved with a very high accuracy, even by very small models.

Table 4: Accuracy for non-autonomous control over systems with 2 to 3 equations.

| Dimension       | 64    | 128   | 256   | 512   |
|-----------------|-------|-------|-------|-------|
| 1 layer         | 97.9  | 98.3  | 98.5  | 98.9  |
| 2 layers        | 98.4  | 98.9  | 99.3  | 99.5  |
| 4 layers        | 98.6  | 99.1  | 99.4  | 99.6  |
| 6 layers        | 98.7  | 99.1  | 99.5  | 99.7  |

On the feedback matrix prediction task, we train transformers with 6 layers and a dimensionality of 512. We use two metrics to evaluate the performance: 1) prediction within 10\% of all coefficients in the target matrix, and 2) verifying that the model outputs a correct feedback matrix \( K \), i.e. that all eigenvalues in \( A + BK \) have negative real parts. The second metric makes more mathematical sense, as it checks that the model actually solves the control problem (like a differential equation, a feedback control problem can have many different solutions).

Using the first metric, 15.8\% of feedback matrices are predicted with less than 10\% error. Accuracy is 50.0\% for systems with 3 equations, but drops fast as systems becomes larger. These results, although low, are well above chance level (less than 0.0001\%). However, with the second metric (i.e. the metric that actually matters mathematically), we achieve 66.5\% accuracy, a much better result. Accuracy decreases with system size, but even degree 6 systems, with \( 1 \times 6 \) to \( 3 \times 6 \) feedback matrices, are correctly predicted 41.5\% of the time. Therefore, while the model fails to approximate \( K \) to a satisfactory level, it does learn to predict correct solutions to the control problem to a high accuracy. This result is very surprising, as it suggests that a mathematical property characterizing feedback matrices might have been learned.

Table 5: Prediction of feedback matrices - Approximation vs. correct mathematical feedback.

|                | Degree 3 | Degree 4 | Degree 5 | Degree 6 | Overall |
|----------------|----------|----------|----------|----------|---------|
| Prediction within 10\% | 50.0     | 9.3      | 2.1      | 0.4      | 15.8    |
| Correct feedback matrix | 87.5     | 77.4     | 58.0     | 41.5     | 66.5    |

5.3 Partial differential equations and Fourier Transform

In these experiments, the model is given a differential operator \( D_x \) and an initial condition \( u_0 \), and is trained to predict if a solution to \( \partial_t u + D_x u = 0 \) exists and, if so, whether it converges to 0 when \( t \to +\infty \). The space dimension (i.e. dimension of \( x \)) is between 2 and 6.
In a first series of experiments the model is only trained to predict the existence and behavior of solutions, encoded as a sequence of two Booleans. Overall accuracy is 98.4%. In a second series, we introduce an auxiliary task by adding to the output the frequency bounds $F$ of $u_0$. We observe that this auxiliary task significantly contributes to the stability of the model with respect to hyperparameters. In particular, without the auxiliary task, the model is very sensitive to the learning rate scheduling and often fails to converge to something better than random guessing. However, in case of convergence, the model reaches the same overall accuracy, with and without auxiliary task. Table 6 details the results for different space dimensions.

### 5.4 Discussion

We studied five problems of advanced mathematics from widely researched areas of mathematical analysis. In three of them, we predict qualitative and theoretical features of differential systems. In two, we perform numerical computations. According to mathematical theory, solving these problems requires a combination of advanced techniques, symbolic and numerical, that seem unlikely to be learnable from examples. Yet, our model achieves more than 95% accuracy on all qualitative tasks, and between 65 and 85% on numerical computations. Such high performances over difficult mathematical tasks may come as a surprise.

One way to generate datasets of problems with their solutions consists in sampling the solution first, and deriving an associated problem. For instance, pairs of functions with their integrals can be generated by sampling then differentiating random functions. Such backward approach might result in a biased dataset [46]. In this paper, datasets for all considered tasks are generated using a forward approach, by directly sampling from the input space. As a result, potential biases caused by backward generative methods do not apply here. Besides, we studied problems from three different fields, that use different mathematical techniques, and different generators, suggesting that the high results we obtain do not come from some specific aspect of a particular problem.

An objection traditionally raised is that the model might memorize a very large number of cases, and interpolate between them. This is unlikely. First, because the size of our problem space is too large to be memorized (for all considered problems, we did not get a single duplicate over 50 million generated examples). Second, because in some of our problems such as non-autonomous control, even a model with one layer and 64 dimensions obtains a high accuracy, and such a small model would never be able to memorize that many examples. Third, because for some of our problems (e.g. local stability), we know from mathematical theory that solutions (e.g. the real values of eigenvalues) cannot be obtained by simple interpolation.

### 6 Conclusion

In this paper, we show that by training transformers over generated datasets of mathematical problems, advanced and complex computations can be learned, and qualitative and numerical tasks performed with high accuracy. Our models have no built-in mathematical knowledge, and learn from examples only. It seems that our models have learned to solve these problems, but this does not mean they learned the techniques we use to compute their solutions. Problems such as non-autonomous control involve long and complex chains of computations. Yet, even very small models (one layer transformers with 64 dimensions) achieve high accuracy.

Most probably, our models learn shortcuts that allow them to solve specific problems, without having to learn or understand their theoretical background. Such a situation is common in everyday life. Most of us learn and use language without understanding its rules. On many practical subjects, we have tacit knowledge and know more than we can tell (Polanyi and Sen [33]). This may be the way neural networks learn advanced mathematics. Understanding what these shortcuts are and how neural networks discover them is a subject for future research.
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A Examples of computations

A.1 Step by step example: autonomous control

To measure whether the system

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= \sin(x_1^2) + \log(1 + x_2) + \frac{\text{atan}(ux_1)}{1 + x_2} \\
\frac{dx_2(t)}{dt} &= x_2 - e^{x_1x_2},
\end{align*}
\]

is controllable at a point \(x_e\), with asymptotic control \(u_e\), we need to

1. differentiate the system with respect to its internal variables, obtain the Jacobian \(A(x, u)\)

\[
A(x, u) = \begin{pmatrix}
2x_1 \cos(x_1^2) + \frac{u(1+x_2)^{-1}}{1+u^2x_1^2} & (1 + x_2)^{-1} - \frac{\text{atan}(ux_1)}{(1+x_2)^2} \\
-x_2 e^{x_1x_2} & 1 - x_1 e^{x_1x_2}
\end{pmatrix}
\]

2. differentiate the system with respect to its control variables, obtain a matrix \(B(x, u)\)

\[
B(x, u) = \begin{pmatrix}
x_1 ((1 + u^2 x_1^2)(1 + x_2))^{-1} \\
0
\end{pmatrix}
\]

3. evaluate \(A\) and \(B\) in \(x_e = [0.5]\), \(u_e = 1\)

\[
A(x_e, u_e) = \begin{pmatrix}
1.50 & 0.46 \\
-0.64 & 0.36
\end{pmatrix}, \quad B(x_e, u_e) = \begin{pmatrix}
0.27 \\
0
\end{pmatrix}
\]

4. calculate the controllability matrix given by (2).

\[
C = [B, AB]((x_e, u_e)) = \begin{pmatrix}
(0.27) & (1.50 & 0.46 & 0.27) \\
0 & -0.64 & 0.36 & 0
\end{pmatrix} = \begin{pmatrix}
0.27 & 0.40 \\
0 & -0.17
\end{pmatrix}
\]

5. output \(n - d\), with \(d\) the rank of the controllability matrix, the system is controllable if \(n - d = 0\)

\(n - \text{rank}(C) = 2 - 2 = 0\) : System is controllable in \((x_e = [0.5], u_e = 1)\)

6. (optionally) if \(n - d = 0\), compute the control feedback matrix \(K\) as in (3)

\[
K = (-22.8 \quad 44.0).
\]
### A.2 Examples of inputs and outputs

#### A.2.1 Local stability

| System | Speed of convergence at $x_c = [0.01]$ |
|--------|-------------------------------------|
| $\frac{d}{dt}x_0 = -\frac{x_1}{\arctan(8x_0x_2)} + \frac{0.01}{\arctan(0.0008)}$ | |
| $\frac{d}{dt}x_1 = -\cos(9x_0) + \cos(0.09)$ | $-1250$ |
| $\frac{d}{dt}x_2 = x_0 - \sqrt{x_1 + x_2} - 0.01 + 0.1\sqrt{2}$ | |
| $\frac{d}{dt}x_0 = -\frac{2x_2}{x_0 - 2x_2(x_1 - 5)} + 0.182$ | |
| $\frac{d}{dt}x_1 = (x_1 + (x_2 - e^{x_1}) (\tan(x_0) + 3)) (\log(3) + i\pi)$ | $-0.445$ |
| $\frac{d}{dt}x_2 = \arcsin\left(\frac{x_0}{x_0 + 2}\right) - \arcsin(0.06 + 0.01i\pi)$ | |
| $\frac{d}{dt}x_0 = e^{x_1}e^{-\sin(x_0 - x_2)} - 1.01e^{-\sin(0.01 - x_2)}$ | |
| $\frac{d}{dt}x_1 = 0.06 - 6x_1$ | 6.0 (locally stable) |
| $\frac{d}{dt}x_2 = -201 + \frac{x_0 + 2}{x_3 + x_2}$ | |
| $\frac{d}{dt}x_0 = e^{x_1}e^{-\sin(x_0 - x_1)} - 9.9 \cdot 10^{-5}$ | |
| $\frac{d}{dt}x_1 = 7.75 \cdot 10^{-4} - \frac{e^{x_2} \arctan(\arctan(x_1))}{4e^{x_2} + 9}$ | $-0.0384$ |
| $\frac{d}{dt}x_2 = (x_1 - \arcsin(9)) e^{-\frac{0.01}{\log(3) + i\pi}} - (0.01 - \arcsin(9)) e^{-\frac{0.01}{\log(3) + i\pi}}$ | |
| $\frac{d}{dt}x_0 = \frac{-x_0(7 - \sqrt{7})}{9} - x_1 + 0.0178 - 0.00111\sqrt{7}\sqrt{7}$ | |
| $\frac{d}{dt}x_1 = -0.000379 + e^{-\frac{0.63}{\cos((x_0 - \sin(x_0)) + x_0) + x_0}}$ | $3.52 \cdot 10^{-11}$ (locally stable) |
| $\frac{d}{dt}x_2 = -x_0 - x_1 + \arcsin\left(\frac{\cos(x_0) + x_0}{x_0}\right)$ | $1.55 + 1.32i$ |
### A.2.2 Controllability: autonomous systems

| Autonomous system | Dimension of uncontrollable space at $x_e = [0.5, u_e = [0.5]$ |
|-------------------|-------------------------------------------------------------------|
| $\begin{align*}
\frac{dx_0}{dt} &= -\sin \left( \frac{x_0}{9} - \frac{4\tan(\cos(10))}{9} \right) \\
\frac{dx_1}{dt} &= u - x_2 + \log \left( 10 + \frac{\tan(x_1)}{u+x_0} \right) - 2.36 \\
\frac{dx_2}{dt} &= 2x_1 + x_2 - 1.5
\end{align*}$ | 0 (controllable) |
| $\begin{align*}
\frac{dx_0}{dt} &= u - \sin(x_0) - 0.5 + \frac{\pi}{6} \\
\frac{dx_1}{dt} &= x_0 - x_1 + 2x_2 + \tan(x_0) - 1.46 \\
\frac{dx_2}{dt} &= \frac{5x_2}{\cos(x_2)} - 2.85
\end{align*}$ | 1 |
| $\begin{align*}
\frac{dx_0}{dt} &= 6u + 6x_0 - \frac{6x_1}{x_0} \\
\frac{dx_1}{dt} &= 0.75 + x_1^2 - \cos(u - x_2) \\
\frac{dx_2}{dt} &= -x_2^2 + x_0 + \log(e^{x_2}) - 0.75
\end{align*}$ | 2 |
| $\begin{align*}
\frac{dx_0}{dt} &= +x_0 \left( \cos \left( \frac{u}{x_0+2x_2} \right) + \sin(u) \right) \\
\frac{dx_1}{dt} &= \frac{\pi x_1}{4(x_2+4)} - \frac{\pi}{36} \\
\frac{dx_2}{dt} &= 2.5 - 108e^{0.5} - 12x_0x_2 + x_1 + 108e^u
\end{align*}$ | 0 (controllable) |
| $\begin{align*}
\frac{dx_0}{dt} &= -10 \sin \left( \frac{3x_0}{\log(5)} - 22 \right) - 6.54 \\
\frac{dx_1}{dt} &= \sin \left( 9 + \frac{x_1-4}{8x_2} \right) - 1 \\
\frac{dx_2}{dt} &= 4\tan \left( \frac{4x_0}{u} \right) - 4\tan(4)
\end{align*}$ | 1 |
### A.2.3 Controllability: non-autonomous systems

| Non-autonomous system                                                                 | Local controllability at $x_e = [0.5], u_e = [0.5]$ |
|-------------------------------------------------------------------------------------|------------------------------------------------------|
| $\begin{align*}
  \frac{dx_0}{dt} &= (x_2 - 0.5) e^{-\sin (8)} \\
  \frac{dx_1}{dt} &= e^{t+0.5} - e^{t+x_1} + \frac{\sqrt{x_2}}{x_2} + 1 - 2e \\
  \frac{dx_2}{dt} &= t(x_2 - 0.5) \left( \sin (6) + \sqrt{\tan (8)} \right)
\end{align*}$                      | False                                               |
| $\begin{align*}
  \frac{dx_0}{dt} &= \frac{\tan (\sqrt{25})}{x_0 - 1} - 2 \tan \left( \frac{\sqrt{2}}{2} \right) \\
  \frac{dx_1}{dt} &= - \frac{u}{\sqrt{x_0x_1 + 3}} + x_2 + \log (x_0) \\
  \frac{dx_2}{dt} &= -70t(x_0 - 0.5)
\end{align*}$                      | False                                               |
| $\begin{align*}
  \frac{dx_0}{dt} &= \frac{x_2 + 7}{\sin (x_0e^{\cos (33.0)}) + 7} \\
  \frac{dx_1}{dt} &= - \frac{9x_0 e^{-\sin (\sqrt{\log (1/t)})}}{x_0} \\
  \frac{dx_2}{dt} &= t + \sin (tx_2 + 4)
\end{align*}$                      | False                                               |
| $\begin{align*}
  \frac{dx_0}{dt} &= 0.5 - x_2 + \tan (x_0) - \tan (0.5) \\
  \frac{dx_1}{dt} &= \frac{t}{t} - t \\
  \frac{dx_2}{dt} &= 2.75 - x_0 (u + 4) - x_0
\end{align*}$                      | True                                               |
| $\begin{align*}
  \frac{dx_0}{dt} &= u (u - x_0 - \tan (8)) + 0.5(\tan (8)) \\
  \frac{dx_1}{dt} &= -6(2 - \sqrt{2}) - 12t (4 - \pi) \\
  \frac{dx_2}{dt} &= -7(u - 0.5) - 7 \tan (\log (x_2)) \\
  + 7 \tan (\log (0.5))
\end{align*}$                      | True                                               |

### A.2.4 Stability of partial differential equations using Fourier transform

| PDE $\partial_t u + D_x u = 0$ and initial condition | Existence of a solution, $u \to 0$ at $t \to +\infty$ |
|------------------------------------------------------|-------------------------------------------------------|
| $\begin{align*}
  D_x &= 2\partial_{x_0} \left( 2\partial_{x_0}^2 \partial_{x_2}^4 + 3\partial_{x_1}^3 + 3\partial_{x_1}^2 \partial_{x_2} \right) \\
  u_0 &= \delta_0(-18x_0)\delta_0(-62x_2)e^{89ix_0-8649x_2^2+98ix_1-59ix_2}
\end{align*}$                      | False , False                                        |
| $\begin{align*}
  D_x &= -4\partial_{x_0}^4 - 5\partial_{x_0}^3 - 6\partial_{x_0}^2 \partial_{x_1} \partial_{x_2}^2 + 3\partial_{x_0} \partial_{x_1} - 4\partial_{x_1}^6 \\
  u_0 &= (162x_0x_2)^{-1} \left( e^{-25x_0+96x_2} \sin (54x_0) \sin (3x_2) \right)
\end{align*}$                      | True , False                                         |
| $\begin{align*}
  D_x &= \partial_{x_1} \left( 4\partial_{x_0}^5 \partial_{x_1} + 4\partial_{x_0}^2 \partial_{x_2}^2 - 3\partial_{x_0} \partial_{x_1} - 4\partial_{x_1}^6 \right) \\
  + 2\partial_{x_1}^3 \partial_{x_2} - 4\partial_{x_1}^4 \partial_{x_2} - 2\partial_{x_1} \partial_{x_2}^2 \\
  u_0 &= (33x_0)^{-1} \left( e^{86ix_0-56ix_1-16x_2^2+87ix_2} \sin (33x_0) \right)
\end{align*}$                      | True , False                                         |
| $\begin{align*}
  D_x &= -6\partial_{x_0}^2 \partial_{x_1}^2 + \partial_{x_0}^2 \partial_{x_2}^6 - 9\partial_{x_0} \partial_{x_1}^2 - 9\partial_{x_0}^4 \partial_{x_2}^4 \\
  + 7\partial_{x_0} \partial_{x_2}^6 + 4\partial_{x_0} \partial_{x_2}^5 - 6\partial_{x_1}^6 \\
  u_0 &= \delta_0(88x_1)\partial_{x_0}e^{-2x_0(2312x_0+15i)}
\end{align*}$                      | True , True                                          |
B Mathematical definitions

B.1 Notions of stability

Let us consider a system
\[
\frac{dx(t)}{dt} = f(x(t)).
\] (7)

\(x_e\) is an attractor, if there exists \(\rho > 0\) such that
\[
|x(0) - x_e| < \rho \implies \lim_{t \to +\infty} x(t) = x_e.
\] (8)

But, counter intuitive as it may seem, this is not enough for asymptotic stability to take place.

**Definition B.1.** We say that \(x_e\) is a locally (asymptotically) stable equilibrium if the two following conditions are satisfied:

(i) \(x_e\) is a stable point, i.e. for every \(\varepsilon > 0\), there exists \(\eta > 0\) such that
\[
|x(0) - x_e| < \eta \implies |x(t) - x_e| < \varepsilon, \forall t \geq 0.
\] (9)

(ii) \(x_e\) is an attractor, i.e. there exists \(\rho > 0\) such that
\[
|x(0) - x_e| < \rho \implies \lim_{t \to +\infty} x(t) = x_e.
\] (10)

In fact, the SMT of Subsection 3.1 deals with an even stronger notion of stability, namely the exponential stability defined as follows:

**Definition B.2.** We say that \(x_e\) is an exponentially stable equilibrium if \(x_e\) is locally stable equilibrium and, in addition, there exist \(\rho > 0, \lambda > 0, \) and \(M > 0\) such that
\[
|x(0) - x_e| < \rho \implies |x(t)| \leq Me^{-\lambda t}|x(0)|.
\]

In this definition, \(\lambda\) is called the exponential convergence rate, which is the quantity predicted in our first task. Of course, if \(x_e\) is locally exponentially stable it is in addition locally asymptotically stable.

B.2 Controllability

We give here a proper mathematical definition of controllability. Let us consider a non-autonomous system
\[
\frac{dx(t)}{dt} = f(x(t), u(t), t),
\] (11)
such that \(f(x_e, u_e) = 0\).

**Definition B.3.** Let \(\tau > 0\), we say that the nonlinear system (11) is locally controllable at the equilibrium \(x_e\) in time \(\tau\) with asymptotic control \(u_e\) if, for every \(\varepsilon > 0\), there exists \(\eta > 0\) such that, for every \((x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n\) with \(|x_0 - x_e| \leq \eta\) and \(|x_1 - x_e| \leq \eta\) there exists a trajectory \((x, u)\) such that
\[
\begin{align*}
    x(0) &= x_0, & x(\tau) &= x_1 \\
    |u(t) - u_e| &\leq \varepsilon, & \forall t \in [0, \tau].
\end{align*}
\] (12)

An interesting remark is that if the system is autonomous, the local controllability does not depend on the time \(\tau\) considered, which explains that it is not precised in Theorem 3.2.

B.3 Tempered distribution

We start by recalling the multi-index notation: let \(\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n, x \in \mathbb{R}^n, \) and \(f \in C^\infty(\mathbb{R}^n), \) we denote
\[
\begin{align*}
    x^\alpha &= x_1^{\alpha_1} \times \cdots \times x_n^{\alpha_n} \\
    \partial_\alpha^\alpha f &= \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f.
\end{align*}
\] (13)

\(\alpha\) is said to be a multi-index and \(|\alpha| = \sum_{i=1}^n |\alpha_i|\). Then we give the definition of the Schwartz functions:
**Definition B.4.** A function $\phi \in C^\infty$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ if, for any multi-index $\alpha$ and $\beta$,

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta \phi| < +\infty.$$  \hfill (14)

Finally, we define the space of tempered distributions:

**Definition B.5.** A tempered distribution $\phi \in \mathcal{S}'(\mathbb{R}^n)$ is a linear form $u$ on $\mathcal{S}(\mathbb{R}^n)$ such that there exists $p > 0$ and $C > 0$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha|, |\beta| < p} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta \phi|, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$ \hfill (15)

### C Proofs of theorems

#### C.1 Analysis of Problem 2

The proofs of Theorem 3.2, of validity of the feedback matrix given by the expression (3), and of the extension of Theorem 3.2 to the non-autonomous system given by condition (4) can be found in Coron [7]. We give here the key steps of the proof for showing that the matrix $K$ given by (3) is a valid feedback matrix to illustrate the underlying mechanisms:

- **Setting** $V(x(t)) = x(t)^T C^{-1}_T x(t)$, where $x$ is solution to $x'(t) = f(x, u_e + K.(x - x_e))$, and
  $$C_T = \left( e^{-AT} \left[ \int_0^T e^{-At} BB^{tr} e^{-A'\tau} d\tau \right] e^{-A'\tau} \right).$$ \hfill (16)

- **Showing,** using the form of $C_T$, that
  $$\frac{d}{dt}(V(x(t))) = -|B^{tr} C_T^{-1} x(t)|^2 - |B^{tr} e^{-T A'\tau} C_T^{-1} x(t)|^2$$

- **Showing** that, if for any $t \in [0, T]$, $|B^{tr} C_T^{-1} x(t)|^2 = 0$, then for any $i \in \{0, ..., n-1\}$, $x^{tr} C_T^{-1} A^i B = 0$, $\forall t \in [0, T]$.

- **Deducing** from the controllability condition (3), that
  $$x(t)^T C_T^{-1} = 0, \quad \forall t \in [0, T],$$
  and therefore from the invertibility of $C_T^{-1}$,
  $$x(t) = 0, \quad \forall t \in [0, T].$$

- **Concluding** from the previous and LaSalle invariance principle that the system is locally exponentially stable.

#### C.2 Analysis of Problem 3

In this section we prove Proposition 3.1. We study the problem

$$\partial_t u + \sum_{|\alpha| \leq k} a^{\alpha} \partial_x^\alpha u = 0 \text{ on } \mathbb{R}_+ \times \mathbb{R}^n,$$ \hfill (17)

with initial condition

$$u(0, \cdot) = u_0 \in \mathcal{S}'(\mathbb{R}^n),$$ \hfill (18)

and we want to find a solution $u \in C^0([0, T], \mathcal{S}'(\mathbb{R}^n))$.

Denoting $\tilde{u}$ the Fourier transform of $u$ with respect to $x$, the problem is equivalent to

$$\partial_t \tilde{u}(t, \xi) + \sum_{|\alpha| \leq k} a^{\alpha}(i\xi)^\alpha \tilde{u}(t, \xi) = 0,$$ \hfill (19)
with initial condition $\tilde{u}_0 \in \mathcal{S}(\mathbb{R}^n)$. As the only derivative now is with respect to time, we can check that

$$\tilde{u}(t, \xi) = \tilde{u}_0(\xi)e^{-f(\xi)t}, \quad (20)$$

where $f(\xi) = \sum_{|\alpha| \leq k} a_{\alpha}(i\xi)^\alpha$, is a weak solution to (19) belonging to the space $C^0([0, +\infty), \mathcal{D}'(\mathbb{R}^n))$. Indeed, first of all we can check that for any $t \in [0, +\infty)$, $\xi \to \exp(-f(\xi)t)$ is a continuous function and $\tilde{u}_0$ belongs to $\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$, thus $\tilde{u}(t, \cdot)$ belongs to $\mathcal{D}'(\mathbb{R}^n)$. Besides, $t \to e^{-f(\xi)t}$ is a $C^\infty$ function whose derivative in time are of the form $P(\xi)e^{-f(\xi)t}$ where $P(\xi)$ is a polynomial function. $\tilde{u}$ is continuous in time and $\tilde{u} \in C^0([0, +\infty), \mathcal{D}'(\mathbb{R}^n))$. Now we check that it is a weak solution to (19) with initial condition $\tilde{u}_0$. Let $\phi \in C^\infty_c([0, +\infty) \times \mathbb{R}^n)$ the space of smooth functions with compact support, we have

$$-\langle \tilde{u}_0, \partial_t \phi \rangle + \sum_{|\alpha| \leq k} a_{\alpha}(i\xi)^\alpha \langle \tilde{u}, \phi \rangle + \langle \tilde{u}_0, \phi \rangle = -\langle \tilde{u}_0, \partial_t(e^{-f(\xi)t}\phi) \rangle - \langle \tilde{u}_0, f(\xi)e^{-f(\xi)t}\phi \rangle + \langle \tilde{u}_0, e^{-f(\xi)t}f(\xi)\phi \rangle + \langle \tilde{u}_0, \phi \rangle = 0. \quad (21)$$

Hence, $\tilde{u}$ defined by (20) is indeed a weak solution of (19) in $C^0([0, +\infty), \mathcal{D}'(\mathbb{R}^n))$. Now, this does not answer our question as this only tells us that at time $t > 0$, $u(t, \cdot) \in \mathcal{D}'(\mathbb{R}^n)$ which is a less regular space than the space of tempered distribution $\mathcal{S}'(\mathbb{R}^n)$. In other words, at $t = 0$, $\tilde{u} = \tilde{u}_0$ has a higher regularity by being in $\mathcal{S}'(\mathbb{R}^n)$ and we would like to know if equation (19) preserves this regularity. This is more than a regularity issue as, if not, one cannot define a solution $u$ as the inverse Fourier Transform of $\tilde{u}$ because such function might not exist. Assume now that there exists a constant $C$ such that

$$\forall \xi \in \mathbb{R}^n, \; \tilde{u}_0(\xi) = 0 \; \text{or} \; \text{Re}(f(\xi)) > C. \quad (22)$$

This implies that, for any $t > 0$, $\tilde{u} \in \mathcal{S}'(\mathbb{R}^n)$. Besides, defining for any $p \in \mathbb{N}$,

$$N_p(\phi) = \sum_{|\alpha|, |\beta| < p} \sup_{\xi \in \mathbb{R}^n} |\xi^\alpha \partial_\xi^\beta \phi(\xi)|, \quad (24)$$

then for $t_1, t_2 \in [0, T]$,

$$N_p((e^{-f(\xi)t_1} - e^{-f(\xi)t_2})\phi) = \sum_{|\alpha|, |\beta| < p} \sup_{\xi \in \mathbb{R}^n} |\xi^\alpha P_\beta(\xi, \phi)|, \quad (25)$$

where $P_\beta(\xi, \phi)$ is polynomial with $f(\xi), \phi(\xi)$, and their derivatives of order strictly smaller than $p$. Besides, each term of this polynomial tend to 0 when $t_1$ tends to $t_2$ on supp($\tilde{u}_0$), the set of frequency of $u_0$. Indeed, let $\beta_i$ be a multi-index, $k \in \mathbb{N}$, and $Q_i(\xi)$ be polynomials in $\xi$, where $i \in \{1, \ldots, k\}$.

$$\left|1_{\text{supp}(\tilde{u}_0)} \partial_\xi^{\beta_1} \phi(\xi) \left(\sum_{i=0}^k Q_i(\xi) t_1^i e^{-f(\xi)t_1} - Q_i(\xi) t_2^i e^{-f(\xi)t_2}\right)\right| \leq \max_{i=0}^k \left|\partial_\xi^{\beta_1} \phi(\xi) Q_i(\xi, t)\right|. \quad (26)$$

From (22), the time-dependant terms in the right-hand sides converge to 0 when $t_1$ tends to $t_2$. This implies that $u \in C^0([0, T], \mathcal{S}'(\mathbb{R}^n))$. Finally let us show the property of the behavior at infinity. Assume that $C > 0$, one has, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \tilde{u}(t, \cdot), \phi \rangle = \langle \tilde{u}_0, 1_{\text{supp}(\tilde{u}_0)} e^{-f(\xi)t}\phi \rangle. \quad (27)$$

Let us set $g(\xi) = e^{-f(\xi)t}\phi(\xi)$, one has for two multi-index $\alpha$ and $\beta$

$$|\xi^\alpha \partial_\xi^\beta g(\xi)| \leq |\xi^\alpha Q(\xi)| e^{-f(\xi)t}, \quad (28)$$

where $Q$ is a sum of polynomials, each multiplied by $\phi(\xi)$ or one of its derivatives. Thus $\xi^\alpha Q(\xi)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and therefore, from assumption (22),

$$|\xi^\alpha \partial_\xi^\beta g(\xi)| 1_{\text{supp}(\tilde{u}_0)} \leq \max_{\xi \in \mathbb{R}^n} |\xi^\alpha Q(\xi)| e^{-Ct}, \quad (29)$$
which goes to 0 when \( t \to +\infty \). This imply that \( \tilde{u}(t, \cdot) \to 0 \) in \( \mathcal{S}'(\mathbb{R}^n) \) when \( t \to +\infty \), and hence \( u(t, \cdot) \to 0 \). This ends the proof of Proposition 3.1.

Let us note that one could try to find solutions with lower regularity, where \( u \) is a distribution of \( \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n) \), and satisfies the equation

\[
\partial_t u + \sum_{|\alpha| \leq k} a_\alpha \partial_\alpha u = \delta_{t=0} u_0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^n.
\]

This could be done using for instance Malgrange-Erhenpreis theorem, however, studying the behavior at \( t \to +\infty \) may be harder mathematically, hence this approach was not considered in this paper.

**D Size of the problem space**

Lample and Charton [22] provide the following formula to calculate the number of functions with \( m \) operators:

\[
E_0 = L \\
E_1 = (q_1 + q_2 L)L \\
(m+1)E_m = (q_1 + 2q_2 L)(2m-1)E_{m-1} - q_1 (m-2)E_{m-2}
\]

Where \( L \) is the number of possible leaves (integers or variables), and \( q_1 \) and \( q_2 \) the number of unary and binary operators. In the stability and controllability problems, we have \( q_1 = 9 \), \( q_2 = 4 \) and \( L = 20 + q \), with \( q \) the number of variables.

Replacing, we have, for a function with \( q \) variables and \( m \) operators

\[
E_0(q) = 20 + q \\
E_1(q) = (89 + 4q)(20 + q) \\
(m+1)E_m(q) = (169 + 8q)(2m-1)E_{m-1} - 4(m-2)E_{m-2}
\]

In the stability problem, we sampled systems of \( n \) functions, with \( n \) variables, \( n \) from 2 to 6. Functions have between 3 and 2\( n + 2 \) operators. The number of possible systems is

\[
PS_{st} = \sum_{n=2}^{6} \left( \sum_{m=3}^{2n+2} E_m(n) \right)^n > E_{14}(6)^6 \approx 3.10^{212}
\]

(since \( E_m(n) \) increases exponentially with \( m \) and \( n \), the dominant factor in the sum is the term with largest \( m \) and \( n \))

In the autonomous controllability problem, we generated systems with \( n \) functions (\( n \) between 3 and 6), and \( n + p \) variables (\( p \) between 1 and \( n/2 \)). Functions had between \( n + p \) and \( 2n + 2p + 2 \) operators. The number of systems is

\[
PS_{aut} = \sum_{n=3}^{6} \left( \sum_{p=1}^{n/2} \sum_{m=n+p}^{2(n+p+1)} E_m(n+p) \right)^n > E_{20}(9)^6 \approx 4.10^{310}
\]

For the non-autonomous case, the number of variables in \( n + p + 1 \), \( n \) is between 2 and 3 and \( p = 1 \), therefore

\[
PS_{naut} = \sum_{n=2}^{3} \left( \sum_{m=n+1}^{2(n+2)} E_m(n+2) \right)^n > E_{10}(5)^3 \approx 5.10^{74}
\]

Because expressions with undefined or degenerate jacobians are skipped, the actual problem space size will be smaller by several orders of magnitude. Yet, problem space remains large enough for overfitting by memorizing problems and solutions to be impossible.