THE INVERSE SPECTRAL THEORY FOR THE WARD EQUATION
AND FOR THE 2+1 CHIRAL MODEL

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ABSTRACT We solve the Cauchy problem of the Ward model in light-cone coordinates using the inverse spectral (scattering) method. In particular we show that the solution can be constructed by solving a $2 \times 2$ local matrix Riemann-Hilbert problem which is uniquely defined in terms of the initial data. These results are also directly applicable to the 2 + 1 Chiral model.

AMS (MOS) Subject Classification 35Q15, 35Q51

1 INTRODUCTION

We study the Cauchy problem for the Ward model in light-cone coordinates:

\begin{align*}
Q_{xt} &= Q_{yy} + [Q_y, Q_x], \quad x, y \in \mathbb{R}, \quad t \geq 0, \\
Q(x, y, 0) &= Q_0(x, y),
\end{align*}

(1)

(2)

where $[ , ]$ denotes the usual matrix commutator, $Q(x, y, t)$ is a traceless $2 \times 2$ anti-Hermitian matrix and $Q_0(x, y)$ is a $2 \times 2$ anti-Hermitian traceless matrix decaying sufficiently fast as $x^2 + y^2 \to \infty$.

We shall solve this problem using the so-called inverse spectral (scattering) method. This method is based on the fact that equation (1) is the compatibility condition of the

\textsuperscript{1}To Appear in Communications in Applied Analysis
following Lax pair,
\[
\mu_y - k\mu_x - Q_x \mu = 0, \quad (3)
\]
\[
\mu_t - k^2 \mu_x - (kQ_x + Q_y) \mu = 0, \quad k \in C, \quad (4)
\]
where \( \mu(x, y, t, k) \) is a \( 2 \times 2 \) matrix. The transformation
\[
x, y, t \rightarrow \frac{t - y}{2}, x, \frac{t + y}{2}, \quad (5)
\]
maps equation (1) to the Ward model \( \mathbb{I} \) in laboratory coordinates. The Cauchy problem in this model is defined by
\[
Q_{tt} - Q_{xx} - Q_{yy} = [Q_x, Q_t] - [Q_x, Q_y], \quad x, y \in R, \quad t \geq 0, \quad (6)
\]
\[
Q(x, y, 0) = Q_1(x, y), \quad Q_t(x, y, 0) = Q_2(x, y). \quad (7)
\]
This problem can be solved by using the fact that equation (3) possess the following Lax pair
\[
(k - \frac{1}{k})\mu_x - 2\mu_y - (Q_x + \frac{Q_t - Q_y}{k}) \mu = 0, \quad (8)
\]
\[
2\mu_t - (k + \frac{1}{k})\mu_x - (-Q_x + \frac{Q_t - Q_y}{k}) \mu = 0, \quad k \in C. \quad (9)
\]
The Cauchy problem (3), (4), was studied in \( \mathbb{I} \) using the Lax pair (8), (9).

Here we study the Cauchy problem (1), (2), using the Lax pair (3), (4). We also make some remarks about the Cauchy problem (6), (7).

We note that the transformations
\[
Q_y \doteq -J^{-1}J_t, \quad Q_x \doteq -J^{-1}J_y, \quad (10)
\]
and
\[
Q_x \doteq -(J^{-1}J_t + J^{-1}J_y), \quad Q_t - Q_y \doteq -J^{-1}J_x, \quad (11)
\]
map equations (1) and (3) to equations
\[
(J^{-1}J_y)_y = (J^{-1}J_t)_x, \quad (12)
\]
and
\[
(J^{-1}J_x)_x = (J^{-1}J_{(t+y)})_{(t-y)}, \quad (13)
\]
respectively. Thus, our results are directly applicable to the solutions of the Cauchy problem for equations (12) and (13). These equations are the 2+1 integrable chiral equations in light-cone and laboratory coordinates, respectively.

In order to simplify the rigorous aspects of our formalism we first assume that $Q_0(x, y)$ is a Schwartz function which is small in the following sense

$$\int_{R^2} |\hat{Q}_0(\xi, y)|\,d\xi\,dy \ll 1, \tag{14}$$

where $\hat{Q}_0$ is the fourier transformation of $Q_0$ in the $x$ variable. This assumption excludes soliton solutions. We then indicate how the formalism can be extended in the case that the above assumption is violated. In the case that $Q_0$ is sufficient small, the inverse spectral method yields a solution of the Ward model in light-cone coordinates through the following construction.

**Theorem 1** Let $Q_0(x, y), x, y \in R$ be a $2 \times 2$ anti-Hermitian traceless matrix which is a Schwartz function and which is small in the sense of equation (14).

(i) Given $Q_0(x, y)$, define $\mu^+(x, y, k), k \in C^+ = \{k \in C : \text{Im}k \geq 0\}$ and $\mu^-(x, y, k), k \in C^- = \{k \in C : \text{Im}k \leq 0\}$ as the $2 \times 2$ matrix valued functions which are the unique solutions of the linear integral equations

$$\mu^+(x, y, k) = I + \frac{1}{4\pi} \left( \int_{-\infty}^{0} dp \int_{-\infty}^{y} dy' - \int_{0}^{\infty} dp \int_{y}^{\infty} dy' \right) \int_{R^2} dx' e^{ip(x-x'+ky-y')} Q_{0x'}(x', y') \mu^+(x', y', k), \tag{15}$$

and

$$\mu^-(x, y, k) = I + \frac{1}{4\pi} \left( \int_{-\infty}^{0} dp \int_{-\infty}^{y} dy' - \int_{0}^{\infty} dp \int_{y}^{\infty} dy' \right) \int_{R^2} dx' e^{ip(x-x'+ky-y')} Q_{0x'}(x', y') \mu^-(x', y', k), \tag{16}$$

where $I$ denotes the $2 \times 2$ unit matrix.

(ii) Given $\mu^\pm$ define the $2 \times 2$ matrix $S(x + ky, k), x, y, k \in R,$ by

$$I - S = \left( I - \frac{1}{4\pi} \int_{-\infty}^{0} dp e^{ip(x+ky)} \int_{R^2} dx' dy' e^{-ip(x'+ky')} Q_{0x'}(x', y') \mu^+(x', y', k) \right)^{-1} \times \left( I - \frac{1}{4\pi} \int_{0}^{\infty} dp e^{ip(x+ky)} \int_{R^2} dx' dy' e^{-ip(x'+ky')} Q_{0x'}(x', y') \mu^-(x', y', k) \right). \tag{17}$$

(iv) Given $S(x + ky, k)$ define the sectionally holomorphic function $M(x, y, t, k) = M^+(x, y, t, k)$ for $k \in C^+, M(x, y, t, k) = M^-(x, y, t, k)$ for $k \in C^-$ as the unique solution
of the following $2 \times 2$ Riemann-Hilbert problem

$$M^-(x, y, t, k) = M^+(x, y, t, k) \left( I - S(x + ky + k^2 t, k) \right), \quad k \in \mathbb{R}, \quad (18)$$

$$\det M = 1, \quad (19)$$

$$M = I + O(1/k), \quad k \to \infty. \quad (20)$$

(v) Given $M(x, y, t, k)$ define $Q$ as

$$Q_x(x, y, t) = \frac{1}{2i\pi} \int_R dk \, M^+(x, y, t, k) S(x, y, t, k). \quad (21)$$

Then $Q$ solves equation (1) and $Q(x, y, 0) = Q_0(x, y)$.

The rest of the paper is organized as follows. In section 2 we derive the main theorem. In section 3 we show how the relevant formalism can be extended to include soliton solutions. In section 4 we briefly discuss the Cauchy problem for the Ward model in laboratory coordinates.

We now make some remarks about related work. A method for solving the Cauchy problem for decaying initial data for integrable evolution equations in one spatial variable was discovered in 1967 [3]. This method which we refer to as the inverse spectral method, reduces the solution of the Cauchy problem to the solution of an inverse scattering problem for an associated linear eigenvalue equation (namely for the $x$-part of the associated Lax pair). Such an integrable evolution equation in one spatial dimension is the chiral equation; the associated $x$-part of the Lax pair is

$$\mu_x = -\frac{Q_x}{k} \mu, \quad (22)$$

where the eigenfunction $\mu(x, t, k)$ is a $2 \times 2$ matrix, $k$ is the spectral parameter and $Q(x, t)$ is a solution of the chiral equation.

Each integrable evolution equation in one spatial dimension has several two spatial dimensional integrable generalizations. An integrable generalization of the chiral equation is (1). A method for solving the Cauchy problem for decaying initial data for integrable evolution equations in two spatial variables appeared in the early 1980 (see reviews [4],[5]). For some equations such as the Kadomtsev-Petviashvili I equation, this method is based
on a nonlocal Riemann-Hilbert problem, while for other equations such as the Kadomtsev-Petviashvili II equation, this method is based on a certain generalization of the Riemann-Hilbert problem called the $\bar{\partial}$ (DBAR) problem.

It is interesting that although equation (1) is an equation in two spatial variables, the Cauchy problem can be solved by a local Riemann-Hilbert problem. This is a consequence of the fact that the equation (3) is a first order ODE in the variable $x - ky$.

For integrable equations, there exist several different methods for constructing exact solutions. Such exact solutions for the Ward model in laboratory coordinates have been constructed in [1, 2]. In particular, Ward constructed soliton solutions using the so-called dressing method [3]. These solutions are obtained by assuming that $M(x, y, t, k)$ has simple poles. In this case the corresponding solitons interact trivially, that is they pass through each other without any phase-shift. Recently, new soliton [4, 5] and soliton-antisoliton solutions [6] were derived, by assuming that $M(x, y, t, k)$ has double or higher order poles. The corresponding lumps interact nontrivially, namely they exhibit $\pi/N$ scattering between $N$ initial solitons.

The formalism presented in section 3 can also be used to obtain exact soliton solutions. In particular, it is shown in section 3 that if the assumption (14) is violated then $M(x, y, t, k)$ still satisfies the Riemann-Hilbert problem (18) but now it is generally a meromorphic as opposed to a holomorphic function of $k$. The solitonic part of the solution $Q(x, y, t)$ is generated by the poles of $M$. The main advantage of this approach is that it can be used to establish the generic role played by the soliton solutions. Namely, it is well known [9] that the long time behaviour of the solution of a local Riemann-Hilbert problem of the type (18) where $M$ is a meromorphic function of $k$, is dominated by the associated poles. Thus the long time behaviour of $Q(x, y, t)$ with arbitrary decaying initial data $Q_0(x, y)$ is given by the multisoliton solution.

2 THE CAUCHY PROBLEM WITHOUT SOLITONS

In this section we prove Theorem 1.

We first consider the direct problem, ie, we show that the spectral data $S(x + ky, k)$
are well defined in terms of the initial data \( Q_0(x, y) \). Replacing \( Q(x, y, t) \) by \( Q_0(x, y) \) in equation (3) we find
\[
\frac{\partial \mu(x, y, k)}{\partial y} - k \frac{\partial \mu(x, y, k)}{\partial x} - Q_0(x, y) \mu(x, y, k) = 0. \tag{23}
\]
Let \( \hat{\mu}(p, y, k) \) denote the \( x \)-Fourier transform of \( \mu(x, y, k) \). Then equation (23) gives
\[
\hat{\mu}_y - ipk\hat{\mu} - \int_R dx e^{-ipx} Q_0(x, y) \mu(x, y, k) = 0. \tag{24}
\]
Equations (15) and (16) are integrable forms of equation (24) with different initial values. Under the small norm assumption (14), equations (15) and (16) are uniquely solvable in the space of bounded continuous functions \( f(x, y) \) such that \( f - I \) has a finite \( L_1 \) norm.

Equations (15) and (16) can also be written in the form
\[
\mu^\pm(x, y, k) = I + \int_{R^2} dx' dy' G^\pm(x - x', y - y', k) Q_{ox'}(x', y') \mu^\pm(x', y', k), \tag{25}
\]
where
\[
G^\pm(x, y, k) = \frac{i}{4\pi^2} \int_{R^2} dp dl \frac{e^{i(px + ly)}}{kp - l} I, \quad k \in C^\pm. \tag{26}
\]
We note that \( G^\pm \) can be evaluated in closed form,
\[
G^\pm(x, y, k) = \pm \frac{i}{2\pi ky} \frac{\delta(y)}{2k} \left( \theta(x) - \theta(-x) \right), \tag{27}
\]
where \( \delta(y) \) and \( \theta(x) \) denote the Dirac and the Heaviside functions, respectively. Indeed, writing \( 1/k = (k_R - ik_I)/|k|^2 \) and using
\[
\int_R dx \frac{e^{ipx}}{x + a + ib} = 2\pi i \text{sgn}(x) \theta(-xb) e^{ip(a+ib)}, \quad a, b, p \in R, \tag{28}
\]
we find
\[
G^\pm(x, y, k) = -\frac{\text{sgn}x}{2\pi k} \int_R dl e^{ily} \theta(xlk_I). \tag{29}
\]
Recall that \( G^+ \) corresponds to \( k_I \geq 0 \); then in this case \( (\text{sgn}x) \theta(xlk_I) = \theta(x)\theta(l) - \theta(-x)\theta(-l) \), and the above equation becomes
\[
G^+ = -\frac{1}{2\pi k} \left( \theta(x) \int_0^\infty dl e^{ily} - \theta(-x) \int_{-\infty}^0 dl e^{ily} \right) \\
= \frac{i}{2\pi k} \left( \frac{\theta(x)}{y + i0} + \frac{\theta(-x)}{y - i0} \right). \tag{30}
\]
Using
\[ \frac{1}{y \pm i0} = \frac{1}{y} \mp i\pi\delta(y), \] (31)
we find the expression for \( G^+ \) given by (27). Similarly for \( G^- \).

Using equation (27) it is straightforward to compute the large \( k \) behaviour of \( \mu^\pm \):
\[ \mu^\pm(x, y, k) = I \pm \frac{i}{2\pi k} \int_{\mathbb{R}^2} dx' dy' \frac{Q_{0x'}(x', y')}{y-y'} + \frac{1}{2k} \left( \int_{-\infty}^{\infty} - \int_{x}^{\infty} \right) dx' Q_{0x'}(x', y) + O\left( \frac{1}{k^2} \right), \] (32)
for \( k \to \infty \). Thus
\[ \mu = I - \frac{Q_0(x, y)}{k} + O\left( \frac{1}{k^2} \right), \quad k \to \infty. \] (33)

Taking the complex conjugate of equation (15), letting \( p \to -p \) and using the fact that
\[ Q_{011} = -Q_{011} = Q_{022}, \quad Q_{012} = -Q_{021}, \] (34)
we find
\[ \overline{\mu_{11}(x, y, k)} = \mu_{22}(x, y, k), \quad \overline{\mu_{21}(x, y, k)} = -\mu_{12}(x, y, k), \]
\[ \overline{\mu_{12}(x, y, k)} = -\mu_{21}(x, y, k), \quad \overline{\mu_{22}(x, y, k)} = \mu_{11}(x, y, k). \] (35)

Letting \( \xi = x + ky, \eta = x - ky \), equation (23) becomes
\[ 2k \frac{\partial\mu}{\partial\eta} - Q_{0x} \mu = 0. \] (36)
Thus any two solutions of this equation are related by a matrix which is a function of \( x + ky \) and of \( k \). Hence
\[ \mu^-(x, y, k) = \mu^+(x, y, k) \left( I - S(x + ky, k) \right), \quad k \in \mathbb{R}. \] (37)
Equation (37) and the symmetry relations (35), imply that \( I - S \) is a Hermitian matrix.
In particular, the determinant of \( I - S \) is real. The determinant of equation (37) yields
\[ \det \mu^- = \det \mu^+ \det (I - S). \] (38)
Taking the complex conjugate of this equation and using the symmetry relations (35), we find
\[ \det \mu^+ = \det \mu^- \det (I - S). \] (39)
Equations (38) and (39) imply \( \det (I - S) = \pm 1 \). However, equation (33) implies that
\[ \det \mu^\pm = 1 + O(1/k), \quad k \to \infty. \] (40)
Thus equation (38) implies det\((I - S) = 1 + O(1/k)\) as \(k \to \infty\), and since det\((I - S) = \pm 1\) it follows that

\[
\text{det}(I - S) = 1. \tag{41}
\]

Equations (38) and (41) imply

\[
\text{det } \mu^+ = \text{det } \mu^- \tag{42}
\]

Since \(\mu^\pm\) are analytic in \(C^\pm\), equations (10) and (12) define a local Riemann-Hilbert problem [10]. Its unique solution is

\[
\text{det } \mu^+ = \text{det } \mu^- = 1. \tag{43}
\]

Evaluating equation (37) as \(y \to -\infty\) (keeping \(x + ky\) fixed) we find

\[
I - S = \left(\lim_{y \to -\infty} (\mu^+)\right)^{-1} \left(\lim_{y \to -\infty} (\mu^-)\right), \tag{44}
\]

which is equation (17).

We now consider the inverse problem, ie, we show how to construct the solution of the Cauchy problem (1), (2), starting from \(S(x + ky, k)\). Given \(S(x + ky + k^2t, k)\), we define \(M(x, y, t, k)\) as the solution of the Riemann-Hilbert problem (18). In general, if the \(L_2\) norm with respect to \(k\) of \(S\) and of \(\frac{\partial S}{\partial k}\) are sufficiently small, then the problem has a unique solution. However, in our particular case the solution exists without a small norm assumption. This is a consequence of the fact that \(I - S\) is a Hermitian matrix. Using this fact it can be shown (see for example [11]) that the homogeneous problem, ie, the problem

\[
\Phi^- = \Phi^+(I - S), \quad k \in R, \tag{45}
\]

\[
\Phi = O\left(\frac{1}{k}\right), \quad k \to \infty, \tag{46}
\]

has only the zero solution.

Given \(M\), we define \(Q(x, y, t)\) by equation (21). A direct computation shows that if \(M^+\) solves the Riemann-Hilbert problem (18), ie, if \(M^+\) satisfies

\[
M^+(x, y, t, k) = I + \frac{1}{2\pi i} \int_R dk' \frac{M^+(x, y, t, k')S(x + k'y + k'^2t, k')}{k' - (k + i0)}, \quad k \in R, \tag{47}
\]

and if \(Q(x, y, t)\) is defined by equation (21) then \(M^+\) satisfies equations (3) and (4). Hence \(Q\) satisfies equation (1). Furthermore the investigation of the Riemann-Hilbert problem
(18) at $t = 0$, implies that $Q(x, y, 0) = Q_0(x, y)$. Also since $I - S$ is Hermitian, $M^+$ and $M^-$ have the proper symmetry properties (see equations (33)), which in turn imply that $Q(x, y, t)$ is a traceless anti-Hermitian matrix.

**Remark** It is important to note that equation (3) involves $Q_x$ and not $Q$. Because of this fact, equation (32) reduces to equation (33). This is to be contrasted with the Kadomtsev-Petviashvili equation, whose Lax pair involves $Q$. In that case equation (32) simplifies to equation (33) only if $\int R Q(x, y) \, dx = 0$. This is the reason why it is usually assumed that the initial data of the Kadomtsev-Petviashvili equation satisfy the above condition. Without this assumption, the inverse spectral method is more complicated [12].

### 3 SOLITON SOLUTIONS

In this section we show how the formalism of section 2 can be modified to include the soliton solutions.

Since the matrix $I - S$ is Hermitian of determinant one, it can be represented as

$$ I - S = \begin{pmatrix} 1 & \overline{\alpha} \\ \alpha & 1 + |\alpha|^2 \end{pmatrix}, $$

where $\alpha$ is an arbitrary function of $(x + ky + k^2 t, k)$.

Then equation (18) becomes

$$ (M_1^- \quad M_2^-) = (M_1^+ \quad M_2^+) \begin{pmatrix} 1 & \overline{\alpha} \\ \alpha & 1 + |\alpha|^2 \end{pmatrix}, $$

where $M_1^+$ and $M_2^-$ are 2-dimensional column vectors which are functions of $(x, y, t, k)$. In particular

$$ M_1^- = M_1^+ + \alpha M_2^+. $$

Equations (15) and (16) are Fredholm integral equations of the second type; thus they may have homogeneous solutions. These homogeneous solutions which correspond to discrete eigenvalues are rather important since they give rise to solitons. We assume
that there exists a finite number of discrete eigenvalues and that they are all simple. Then
Fredholm theory implies that \( M_1^+ \) admits the representation
\[
M_1^+ = m_1^+ + \sum_{l=1}^{N} \frac{\phi_l(x, y, t)}{k - k_l},
\]
where \( m_1^+(x, y, t, k) \) is analytic for \( k \in C^+ \) and the vectors \( \phi_l(x, y, t) \), \( 1 \leq l \leq N \) are homogenous solutions of the first column vector of equation (54). Following the arguments of [13] it can be shown that
\[
\phi_l(x, y, t) = -c_l (x + k_l y + k_l^2 t) M_2^+(x, y, t, k_l), \tag{52}
\]
where \( c_l \) is a scalar function of the argument indicated. Hence equation (51) becomes
\[
M_1^+(x, y, t, k) = m_1^+(x, y, t, k) - \sum_{l=1}^{N} \frac{c_l (x + k_l y + k_l^2 t) M_2^+(x, y, t, k_l)}{k - k_l}. \tag{53}
\]
Substituting this equation into equation (50) solving the resulting Riemann-Hilbert problem we find
\[
M_1^-(x, y, t, k) + \sum_{l=1}^{N} \frac{c_l (x + k_l y + k_l^2 t) M_2^+(x, y, t, k_l)}{k - k_l} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi} \int_R \frac{\alpha(x + k_l y + k_l^2 t) M_2^+(x, y, t, k_l')}{k' - (k - i0)} dk'. \tag{54}
\]
Let
\[
M_2^+(x, y, t, k) = \begin{pmatrix} A(x, y, t, k) \\ B(x, y, t, k) \end{pmatrix}. \tag{55}
\]
In what follows, for simplicity of notion we suppress the \( x, y, t \) dependence. Using the notation (53) together with the symmetry relation (35), equation (54) becomes
\[
\begin{pmatrix} B(k) \\ -A(k) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{l=1}^{N} \frac{c_l}{k - k_l} \begin{pmatrix} A(k_l) \\ B(k_l) \end{pmatrix} - \frac{1}{2i\pi} \int_R \frac{dk' \alpha(k')}{k' - (k - i0)} \begin{pmatrix} A(k') \\ B(k') \end{pmatrix}. \tag{56}
\]
Equation (56) express the solution of the Riemann-Hilbert problem (18) in the case that solitons are included.

PURE SOLITONS

Soliton solutions correspond to \( \alpha = 0 \). In this case evaluating equation (56) at \( k = \bar{k}_j \) we find
\[
\begin{aligned}
B(k_j) &= 1 - \sum_{l=1}^{N} \frac{c_l}{k_j - k_l} A(k_l), \\
A(k_j) &= \sum_{l=1}^{N} \frac{c_l}{k_j - k_l} B(k_l).
\end{aligned} \tag{57}
\]
The complex conjugate of these equations yields
\[ B(k_j) = 1 - \sum_{l=1}^{N} \frac{\bar{c}_l}{k_j - k_l} A(k_l), \]
\[ A(k_j) = \sum_{l=1}^{N} \frac{\bar{c}_l}{k_l - k_j} B(k_l). \] (58)
Equations (57) and (58) determine \( A(k_l) \) and \( B(k_l), l = 1, \ldots, N. \)

Equation (54) yields
\[ M_{i}^{-1} = \left( \begin{array}{cc} 1 & 0 \\ -\sum_{l=1}^{N} \frac{c_l}{k_j - k_l} \left( A(k_l) \\ B(k_l) \right) \end{array} \right). \] (59)
Thus using equation (53) we find
\[ Q_{11} = -Q_{22} = \sum_{l=1}^{N} c_l A(k_l), \quad Q_{21} = -Q_{12} = \sum_{l=1}^{N} c_l B(k_l). \] (60)

In summary, the \( N \)-soliton solution is given by equations (54), where \( c_l = c_l(x + k_1 y + k_1^2 t) \) and \( A(k_l), B(k_l) \) are the solutions of the equations (57) and (58). In the case of 1-soliton equations (54) and (58) yield
\begin{align*}
A(k_1) &= \frac{1}{1 - \frac{|c_1|^2}{(k_1 - k_1)^2}} \frac{\bar{c}_1}{k_1 - k_1}, \\
B(k_1) &= \frac{1}{1 - \frac{|c_1|^2}{(k_1 - k_1)^2}}.
\end{align*} (61)
Thus
\begin{align*}
Q_{11} &= -Q_{22} = -\frac{1}{1 - \frac{|c_1|^2}{(k_1 - k_1)^2}} \frac{|c_1|^2}{k_1 - k_1}, \\
Q_{21} &= -Q_{12} = \frac{c_1}{1 - \frac{|c_1|^2}{(k_1 - k_1)^2}}.
\end{align*} (62)
Figure 1 represent a snapshot of the solution of equation (1) by taking \( c_1 = x + k_1 y + k_1^2 t \) for \( k_1 = i \) at time \( t = -3. \)

4 THE WARD MODEL IN LABORATORY COORDINATES

Here we study the Cauchy problem (3), (4) using the Lax pair (8), (9). In the case that \( Q_1 \) and \( Q_2 \) are sufficient small, the inverse spectral method yields a solution of the Ward model in laboratory coordinates through the following construction.

**Theorem 2** Let \( Q_1(x, y), Q_2(x, y), x, y \in R \) be \( 2 \times 2 \) anti-Hermitian traceless matrices which are Schwartz functions and which satisfy the small norm conditions
\[ \int_{R^2} |\hat{Q}_j(\xi, y)| d\xi dy \ll 1, \quad j = 1, 2, \] (63)
where $\hat{Q}_j$ is the Fourier transformation of $Q_j$ in the $x$ variable.

(i) Given $Q_1(x, y)$, $Q_2(x, y)$, define $\mu^+(x, y, k)$, $k \in C^+$ as the unique solutions of the linear integral equations

$$\mu^+ = I + \frac{1}{4\pi} \left( \int_0^\infty dp \int_{-\infty}^y dy' - \int_{-\infty}^0 dp \int_{y}^\infty dy' \right) \int_R dx' e^{ip(x-x'+\frac{k^2-1}{2k}(y-y'))} \left( Q_{1x'} + \frac{Q_2 - Q_{1y'}}{k} \right) \mu^+, \quad (64)$$

and

$$\mu^- = I + \frac{1}{4\pi} \left( \int_{-\infty}^0 dp \int_{-\infty}^y dy' - \int_{-\infty}^\infty dp \int_{y}^\infty dy' \right) \int_R dx' e^{ip(x-x'+\frac{k^2-1}{2k}(y-y'))} \left( Q_{1x'} + \frac{Q_2 - Q_{1y'}}{k} \right) \mu^-, \quad (65)$$

(ii) Given $\mu^\pm$ define the $2 \times 2$ matrix $S(x + \frac{k^2-1}{2k}y, k)$, $x, y, k \in \mathbb{R}$, by

$$I - S = \left( I - \frac{1}{4\pi} \int_{-\infty}^0 dp \int_{-\infty}^\infty dx' dy' e^{ip(x-x'+\frac{k^2-1}{2k}(y-y'))} \left( Q_{1x'} + \frac{Q_2 - Q_{1y'}}{k} \right) \mu^+ \right)^{-1} \times$$

$$\left( I - \frac{1}{4\pi} \int_0^\infty dp \int_{-\infty}^\infty dx' dy' e^{ip(x-x'+\frac{k^2-1}{2k}(y-y'))} \left( Q_{1x'} + \frac{Q_2 - Q_{1y'}}{k} \right) \mu^- \right). \quad (66)$$

(iv) Given $S(x + \frac{k^2-1}{2k}y, k)$ define the sectionally holomorphic function $M(x, y, t, k) = M^+(x, y, t, k)$ for $k \in C^+$, $M(x, y, t, k) = M^-(x, y, t, k)$ for $k \in C^-$ as the unique solution of the following $2 \times 2$ Riemann-Hilbert problem

$$M^-(x, y, t, k) = M^+(x, y, t, k) \left( I - S(x + \frac{k^2-1}{2k}y + \frac{k^2+1}{2k}t, k) \right), \quad k \in \mathbb{R}, \quad (67)$$
\[ \det M = 1, \quad (68) \]
\[ M = I + O(1/k), \quad k \to \infty. \quad (69) \]

(v) Given \( M(x, y, t, k) \), define \( Q \) as
\[ Q_x(x, y, t) = \frac{1}{2i\pi} \int_R dk \, M^+(x, y, t, k) S(x, y, t, k). \quad (70) \]
Then \( Q \) solves equation (6) and \( Q(x, y, 0) = Q_1(x, y), \quad Q_t(x, y, 0) = Q_2(x, y) \).

The proof of Theorem 2 is similar to section 2.

Equations (64) and (65) can also be written in the form
\[ \mu^\pm(x, y, k) = I + \int_{R^2} dx' dy' G(x - x', y - y', k) \left( Q_{1x'} + \frac{Q_2 - Q_{1y'}}{k} \right) \mu^\pm(x', y', k), \quad (71) \]
where
\[ G^\pm(x, y, k) = \frac{1}{(2\pi)^2i} \int_{R^2} dp \, dn \, \frac{e^{i(px + ny)}}{(k - k^{-1})p - 2nI}, \quad k \in C^\pm, \quad (72) \]
or
\[ G^\pm(x, y, k) = \pm \frac{1}{2\pi iky} + \frac{\delta(y)}{2k} (\theta(x) - \theta(-x)). \quad (73) \]
Substituting this equation into equation (71), it is straightforward to compute the large \( k \) behaviour of \( \mu^\pm \),
\[ \mu^\pm = I \pm \frac{i}{2\pi k} \int_{R^2} dx' dy' \left( \frac{Q_{1x'} + \frac{Q_2 - Q_{1y'}}{k}}{y - y'} \right) + \frac{1}{2k} \left( \int_{-\infty}^{x} - \int_{x}^{\infty} \right) dx' \left( Q_{1x'} + \frac{Q_2 - Q_{1y'}}{k} \right) + O\left( \frac{1}{k^2} \right), \quad (74) \]
for \( k \to \infty \). Thus
\[ \mu = I - \frac{Q_1(x, y)}{k} + O\left( \frac{1}{k^2} \right), \quad k \to \infty. \quad (75) \]

The corresponding soliton solutions of equation (6) can be derived following the method of section 3.

5 ACKNOWLEDGMENTS

TI acknowledges support from EU ERBFMBICT950035.
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