Casimir effect for the sphere revisited

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In a recent work Brevik et al. have offered formal proofs of two results which figure prominently in calculations of the Casimir pressure on a sphere. It is shown by means of simple counterexamples that each of those proofs is necessarily incorrect.

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I INTRODUCTION

The Casimir energy of a conducting sphere has been considered in a number of papers in the past several years. In particular this author [1] reexamined the well known calculation of Milton, DeRaad, and Schwinger [2] and raised certain concerns about the means used in that work to carry out those calculations. These may be conveniently referred to as the pressure-stress relation and the claimed validity of a certain contour rotation.

Subsequent to the publication of ref.1, Brevik, et al. [3] have stated their strong objections to the conclusions reached in that work and have offered what they claim to be proofs of two results which support the conclusions of ref. 2. As is shown in the present work neither of those proofs can be correct since counterexamples can be shown to exist. In II the pressure-stress issue is discussed, it being shown that a proof based merely on the assumptions of refs. 2 and 3 is logically impossible. Section III similarly provides a simple counterexample to the contour rotation result of refs. 2 and 3.

Before proceeding to these two main results it may be appropriate to comment briefly on a third issue discussed in ref. 1 having to do with the issue of boundary conditions. As is well known any calculation based on quantum field theory must utilize so-called causal boundary conditions (i.e., positive frequencies in the future and negative frequencies in the past). Reference 2, however, is based (by its own claim) on outgoing wave boundary conditions, which according to standard usage would imply functional dependence of the form $\omega(r - t)$. This is quite different from the result identified as outgoing wave boundary conditions as given by Eq.(1) of ref. 3. It is therefore clear that this issue has its origin in imprecise language rather than any significant issue of physics.

II THE PRESSURE-STRESS CONNECTION

In ref. 2 and a number of subsequent related applications the Casimir pressure on a sphere is computed by evaluating the discontinuity of the $T^{rr}$ component of the stress tensor, an approach which has a considerable amount of intuitive appeal. What is not clear, however, is whether this calculational method has a rigorous basis. What is shown here (the claim of ref. 3 notwithstanding) is that one cannot hope to establish the legitimacy of this approach making recourse only to the appropriate divergence relation for the stress tensor.

The proof of ref. 3 argues from the result

$$\nabla_\alpha T^{\alpha\beta} = -F^{\beta\lambda} J_\lambda = -f^\beta$$

where $f^\beta$ is identified as the four-vector force density and is nonzero only within the conducting shell [4]. It is then inferred that this force is necessarily identical to the Casimir force, thereby leading to the conclusion that the Casimir pressure is indeed given by the discontinuity of $T^{rr}$. It is easy to see that this is nothing more than proof by definition–namely, that because $f^\beta$ has certain properties characteristic of a force, it must be the Casimir force.

To display the contradiction, one makes reference to the corresponding stress tensor for a spin-zero field $\phi$ which has the form

$$T^{\mu\nu} = T_1^{\mu\nu} + T_2^{\mu\nu}$$

where
\[ T^{\mu\nu}_{1} = \phi^\mu \phi^\nu - \frac{1}{2} g^{\mu\nu} \phi_\alpha^2 \]

\[ T^{\mu\nu}_{2} = \frac{1}{6} [g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu] \phi^2 \]

and \( \phi_\mu = -\partial_\mu \phi \). Note that \( T^{\mu\nu} \) is the so-called “improved” (i.e., traceless) stress tensor, while \( T^{\mu\nu}_{1} \) is the canonical (i.e., “unimproved”) version. One can now include a coupling to a scalar current \( J \) (the analogue of \( J^\mu \) of Eq.(1)) to obtain

\[ \nabla_\alpha T^{\alpha\beta} = -\phi^\beta J = -f^\beta \]

which therefore (in accord with the approach of ref. 3) must imply that the rhs of (3) is the Casimir force for the scalar case. However, it is important to note that the same relation holds for the unimproved tensor \( T^{\alpha\beta}_1 \) as well, and it consequently follows that the Casimir pressure can be obtained from the discontinuity of either \( T^{\alpha\beta}_1 \) or \( T^{\alpha\beta} \). Unless these have the same discontinuity, one must conclude that the result fails in (at least) one of the two cases in which the proof of ref. 3 is extended to the scalar case.

To demonstrate such a failure one makes recourse to the case of the spherical shell \( a < r < R \) and seeks solutions which satisfy the boundary condition \( \partial_r r \phi = 0 \) at the boundaries \( r = a, R \). The issue to be resolved is whether the Casimir pressure

\[ p = -\frac{1}{4\pi a^2} \frac{\partial E}{\partial a} \]

where \( E \) is the Casimir energy

\[ E = \int d\mathbf{x} T^{00}(\mathbf{x}) \]

is given by the negative of \( T^{rr}(r = a) \) for each allowed mode of the system [5]. Since the aim here is to provide a demonstration using the simplest possible choice of \( \phi \), it is convenient to take the spherically symmetric free field solution

\[ \phi = A \frac{\cos[\omega_n (r - a)]}{\omega_n r}. \]

The eigenmodes \( \omega_n \) are readily seen from the boundary conditions to be given by \( \omega_n = \frac{n\pi}{R-a} \) while \( A \) is determined from the condition

\[ 2\omega_n A^2 \int d\mathbf{x} \phi^2 = 1. \]

Using \( T^{\mu\nu}_1 \) and \( T^{\mu\nu}_2 \) successively one finds without difficulty that the two “Casimir energies” \( E_1 \) and \( E_2 \) are given by

\[ \frac{1}{4\pi} E_1 = \frac{1}{2} \omega_n + \frac{1}{2n\pi} \left( \frac{1}{a} - \frac{1}{R} \right) \]

and

\[ \frac{1}{4\pi} E_2 = -\frac{1}{3n\pi} \left( \frac{1}{a} - \frac{1}{R} \right). \]

One has the corresponding results for \( T^{rr}(r = a) \)

\[ T^{rr}_1(r = a) = \frac{n\pi}{2a^2(R-a)^2} + \frac{1}{2n\pi a^4} \]
and
\[ T_2^{rr}(r = a) = -\frac{2}{3n\pi a^4}. \]

Using these results it is straightforward to show that for both \( T_1^{\mu\nu} \) and \( T_2^{\mu\nu} \) the pressure-stress relation fails, but that it does hold for the sum (i.e., for the “improved” case).

On the basis of these results one can conclude that the proof of the pressure-stress relation claimed in [3] is not in fact valid. In general it simply cannot be possible to prove such a relation solely on the basis of arguments deriving from the divergence of the stress tensor. It may, of course, be possible to do so by including as well the requirement that the stress tensor be traceless. However, that property was not invoked in [3] and thus the result quoted there is necessarily incorrect.

For purposes of clarity it is certainly appropriate to remark here that the claimed proof of ref.3 deals only with the case of the electromagnetic field. Yet, having said that, it is striking that (as shown here) virtually identical techniques imply a contradiction when applied to the scalar field case. In addition it should not be overlooked that section III of ref.3 presents a scalar field Casimir calculation which implicitly assumes the pressure-stress relation despite the demonstrated failure of the proof in that case.

### III CONTOUR ROTATION

A significant problem encountered in ref. [2] (and in subsequent related works) is the evaluation of integrals with rapidly oscillating integrands. It is argued there that this can generally be accomplished by a contour rotation which results in a much more manageable integrand. The mathematical basis for that rotation is given in [2] and repeated virtually verbatim in [3]. It is shown here by an explicit calculation that the mathematical steps in that rotation must necessarily be incorrect.

This is easily accomplished by considering the \( l = 0 \) contribution [5] to the integral in Eq.(2) of ref. [3]. To within an uninteresting normalization factor one can write that part of Eq.(2) as
\[ f = \frac{ia}{2} \int_C d\omega e^{-i\omega\tau} \left( \frac{ka}{H_{1/2}(ka)} \left[ \frac{H_{1/2}(ka)}{J_{1/2}(ka)} + 1 \right] \right) \]
where \( k = |\omega| \) and \( C \) is a contour just above the real axis for \( \omega > 0 \) and just below the real axis for \( \omega < 0 \). It is claimed in [3] that as a result of contour rotation this can be transformed into
\[ f_E = -\frac{1}{2} \int_{-\infty}^{\infty} dy e^{i\delta y} \left( \frac{K_{1/2}(x)}{K_{1/2}(x)} + \frac{I_{1/2}(x)}{I_{1/2}(x)} + 1 \right) \]
where \( x = |y| \). The content of this claim lies in the fact that, if true, \( f \) and \( f_E \) must be equal in the limit of vanishing cutoff (i.e., in the limit \( \tau, \delta \to 0 \)). It is the advantage of the \( l = 0 \) case that both of the above integrals can be evaluated analytically. The latter is particularly simple, with the result being
\[ f_E = -\frac{\pi^2}{12}. \]

To evaluate \( f \) one notes that the Bessel functions of order \( \frac{1}{2} \) allow it to be written as
\[ f = \frac{i}{2} \int_C dx \exp[i|x| - i\frac{\tau}{a}x] |\frac{x}{\sin x}|. \]
It is convenient to break the integral into a principal value term and a sum over contributions from the poles of the integrand. Upon taking the limits of the integral as \( \pm R \) where \( R = (N + \xi)\pi \) with \( N \) a large integer and \( 0 < \xi < 1 \), the real part of this integral can readily be evaluated. For small \( \tau \) the result is
\[ \Re f = -\frac{\pi^2}{12} + \frac{\pi^2}{2} \cos(N\pi\tau/a)[N(1 - 2\xi) - \xi^2 + \frac{1}{6}]. \]
Since this has no well defined $N \to \infty$ limit, one concludes by this direct calculation that the original integral $f$ does not agree with the result for $f_E$. It is not difficult to show that this occurs precisely because the integrals along the quarter circles of radius $R$ in the first and third quadrants fail to vanish in the $R \to \infty$ limit. At least for the $l = 0$ case it appears that the use of the alternative cutoff $e^{-k|x|}$ with $k > 0$ could formally avoid this difficult issue of contour rotation. However, such a cutoff would require significant modification of the approach of refs. [2] and [3] where the cutoff originates in the underlying Minkowski space formulation of the theory.

**IV CONCLUSION**

In this work two alleged theorems of ref. 3 have been shown by specific counterexamples to be incorrect. The first of these has to do with the pressure-stress relation, and it has been demonstrated here that the proof offered in [3] *must* fail since the spin-zero canonical stress tensor is in specific disagreement with the claimed pressure-stress relation. If such a relation can in fact be established, its proof necessarily must include the property of tracelessness, an aspect which nowhere appears in the proof of ref. 3. Similarly, the contour rotation result supposedly proved in [2] and then again in [3] has been shown to fail for the case of $l = 0$, the only instance for which it appears possible to do an exact calculation.

A final comment has to do with the rather extensive set of remarks made in ref. 3 concerning the experimental side of the Casimir effect. These were apparently intended to rebut the two sentences of ref. 1 (described in [3] as “objectionable”) which remarked that some recent experiments [6] might be viewed as less than compelling evidence. This had to do with the fact that strict plane parallel plate geometry has not been utilized in those experiments, and the correction to a spherical lens has not been rigorously carried out. It remains true that those corrections have not been calculated beyond a reasonable doubt.

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[3] I. Brevik, B. Jensen, and K. A. Milton, Phys. Rev. 64, 088701 2002.
[4] It is appropriate to remark here that the statement in ref. 3 to the effect that ref. 1 assumes that $\nabla_\alpha T^{\alpha\beta} = 0$ everywhere is simply incorrect. Such an assertion would be equivalent to the insupportable claim that the stress tensor has no discontinuity at the surface of the sphere.
[5] Clearly one must require that the claimed relation hold for each mode of the system if the pressure-stress relation is to be viewed as a governing principle rather than as an accident of a particular summation procedure.
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