An Efficient Algorithm for Fair Multi-Agent Multi-Armed Bandit with Low Regret

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1 Introduction

The multi-armed bandit (MAB) problem poses an identical set of choices (or actions, arms) at each time step to the decision-maker. When the decision maker pulls a choice, she receives a reward drawn from a distribution associated with that choice. The goal is to select arms to maximize the total reward in expectation, or to minimize their expected regret with respect to the optimal strategy (or policy). The MAB problem lends itself to many applications including allocation of resources in a financial portfolio (Hoffman et al. 2011; Shen et al. 2015; Huo and Fu 2017), selection of both dosages (Bastani and Bayati 2020) and treatments (Durand et al. 2018) in clinical healthcare, and a broad range of recommender systems (Zhou et al. 2017; Bouniffouf, Bouzeghoub, and Gańcarski 2012, 2013).

In many scenarios, making a decision impacts not only one but multiple agents. For instance, consider a policy-maker making decisions that might influence various groups of their constituents (Zhu and Walker 2018), or a recommendation engine that picks hyperparameters for their system to serve many users. The multi-agent multi-armed bandit problem (MA-MAB), proposed in Hossain, Micha, and Shah (2021), is a variant of the MAB problem in which there are $N$ agents and $K$ arms, and in every round $t$, the algorithm selects the same arm $a$ for each agent and receive $N$ rewards realizations drawn from unknown distributions $\{D_{j,a}\}_{j=1}^N$ with unknown means $\{\mu_{j,a}^*\}_{j=1}^N$. The reward distributions of the same arm for different agents need not be the same, and hence an optimal arm for one agent might be sub-optimal for the others. As such, the goal of picking the optimal arm in the standard MAB problem is no longer meaningful, and a different objective is needed.

In this setting, one intuitive objective would be to optimize the expected rewards over all agents. This reduces the problem to the classic MAB problem and can be solved accordingly (Slivkins 2019). In practice, this objective has a key weakness: individual rewards may be neglected in order to improve the reward of the collective. Consider an extremely divided case with two arms where just over half of agents receive reward 1 from the first arm and reward 0 from the second arm, and the other agents receive the opposite rewards. The optimal strategy in this case is to always choose the first arm: half of the agents receive reward 1, and almost half of the agents receive reward 0. While this formulation is well-studied by reduction, it unfortunately does not serve to enforce fairness among the agents. A fairer choice would be to pull each arm with roughly the same probability, which only slightly reduces the individual utilities. In other words, the agents would receive similar rewards with each other if the algorithm learns a distribution $\pi$ over the arms and converges to pulling each arm almost uniformly. Thus, the optimal strategy for a MA-MAB problem corresponds not to picking the optimal arm as in the classic MAB problem, but a probability distribution over the arms.

In (Hossain, Micha, and Shah 2021), the authors proposed an objective motivated by studies on fairness in social choice: the Nash social welfare (NSW). This is a classical notion going back to (Nash 1950) and (Kaneko and Nakamura 1979). For $n$ agents with expected utility $u_1, u_2, \ldots, u_n$, the Nash product $\prod_i u_i$ is sandwiched between $(\min_i u_i)^n \leq \prod_i u_i \leq \left(\frac{\sum_i u_i}{n}\right)^n$, serving as a compromise between an egalitarian approach (optimizing the minimum reward across agents) and a utilitarian approach (optimizing the sum of rewards across agents). In the context of the MA-MAB problem, the optimal fair strategy corresponds to the distribution that maximizes the cumulative Nash product of the agents’ expected rewards. Formally, $\pi^* = \arg \max_{\pi \in \Delta^K} \prod_j [N] \left(\sum_{a \in [K]} \pi_a \mu_j^*, a\right)$, where $\Delta^K$ is the...
the probability of selecting each arm should be similar if the two arms have a similar quality distribution. (Gillen et al. 2018) studies fairness in the linear contextual bandits setting where there are individual fairness constraints imposed by an unknown similarity metric. (Patil et al. 2020) proposes a fair MAB variant that seeks to optimize the cumulative reward while also ensures that, at any round, each arm is pulled at least a specified fraction of times.

Our objective of finding a probability distribution over arms to optimize the Nash welfare objective can also be cast as the continuum-armed bandit problem where the Nash welfare function is the objective. (Kleinberg, Slivkins, and Upfal 2019) designs an algorithm with a regret bound of $O(\sqrt{T \log T})$, where $\gamma$, defined as the zooming dimension, would be $O(K)$ for the MA-MAB problem. The resulting bound would be no better than $O(T^{2/3})$ and approaches $O(T)$ as $K$ increases. It is also important to note that there is a long line of work on bandit convex optimization (Hazar and Levy 2014; Bubeck and Eldan 2016; Bubeck, Lee, and Eldan 2017; Chen, Zhang, and Karbasi 2019). One can apply the approaches to optimize the log of the Nash welfare function. However, the regret bound of the new objective does not translate to that of the original objective.

### 3 Preliminaries

Define $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$. In the multi-agent multi-armed bandit problem, we have a set of agents $[N]$ and a set of arms $[K]$ for $N, K \in \mathbb{N}$. For each agent $j \in [N]$ and arm $a \in [K]$, we have a reward distribution $D_{j,a}$ with mean $\mu_{j,a}$ and support in $[0, 1]$. Define $\mu^* = (\mu_{j,a})_{j \in [N], a \in [K]}$ as the mean reward matrix. At each round $t$, $a_t$ denotes the selected arm of round $t$ and $r_{j,a_t,t} \sim D_{j,a_t}$, the realization of the reward distribution associated with arm $a_t$ of each agent $j$.

For reward matrix $\mu = (\mu_{j,a})_{j \in [N], a \in [K]} \in [0, 1]^{N \times K}$ and policy $\pi \in \Delta^K$, we denote the Nash Social Welfare function by the product over the expected reward of each agent, $\text{NSW}(\pi, \mu) = \prod_{j \in [N]} \left(\sum_{a \in [K]} \pi_a \mu_{j,a}\right)$. For a time horizon $T$, our goal is to choose policies $\pi_t \in \Delta^K$ at each round $t \in [T]$ to minimize the cumulative regret $R_T = \sum_{t \in [T]} \text{NSW}(\pi^*, \mu^*) - \sum_{t \in [T]} \text{NSW}(\pi_t, \mu^*)$. We use $\pi_{a,t}$ to denote the probability of selecting arm $a$ under policy $\pi_t$.

At each round $t$, our algorithm samples an arm $a_t$ from $\pi_t$ and pulls the same arm for every agent. Let $N_{a,t} = \sum_{\tau \leq t} 1\{a_\tau = a\}$ denote the number of times arm $a$ has been sampled before round $t$, and $\hat{\mu}_{j,a,t} = \frac{1}{N_{a,t}} \sum_{\tau \leq t} r_{j,a_\tau,t} 1\{a_\tau = a\}$ the corresponding empirical mean of agent $j$’s reward on arm $a$. Our main algorithm maintains a confidence bound for the mean reward matrix $\mu^*$ at every round. Let $\hat{\mu}_{j,a,t} = (\hat{\mu}_{j,a,t})_{j \in [N], a \in [K]}$ denote the estimated mean reward matrix and $\hat{w}_{j,a,t} = (\hat{w}_{j,a,t})_{j \in [N], a \in [K]}$ the confidence bound matrix for round $t$.

### 4 Algorithms

Our UCB algorithm, Algorithm 1, first selects each arm once in order to obtain an initial estimate $\hat{\mu}$ for $\mu$. Then,
Algorithm 1: Fair multi-agent UCB algorithm

1: input: $K, N, T, \delta$
2: for $t = 1$ to $T$ do
3: if $t < K$ then
4: $\pi_t \leftarrow$ policy that puts probability 1 on arm $t$
5: else
6: $\forall j, a, \mu_{j,a,t} = \frac{1}{N_{a,t}} \sum_{\tau=1}^{t-1} r_{j,a,\tau} 1\{a_\tau = a\}$
7: $\forall j, a, U_{j,a,t} = \min(\mu_{j,a,t} + w_{j,a,t}, 1)$
8: $\pi_t \leftarrow \arg\max_{\pi \in \Delta_K} \text{NSW} (\pi, U_t)$
9: end if
10: Sample $a_t$ from $\pi_t$
11: Observe rewards $\{r_{j,a_t,\tau}\}_{j \in N}$
12: $N_{a_t,t+1} \leftarrow N_{a_t,t} + 1$
13: end for

it computes the upper confidence bound estimate $U_t = \{U_{j,a,t}\}_{j \in [N], a \in [K]}$ of the true mean for each arm $a$ of $N$ agents and finds a policy $\pi_t$ that optimizes the Nash social welfare function given $U_t$. Due to the log-concavity of Nash social welfare, we use standard convex optimization tools to optimize $\log(\text{NSW}(\pi, U_t))$ and simultaneously optimize $\text{NSW}(\pi, U_t)$.

This approach differs from the UCB algorithm in Hossain, Micha, and Shah (2021) in two aspects. First, the optimization step in their UCB algorithm uses an additive regularization term in the objective rather than on the estimate $\hat{\mu}$, and is therefore not log-concave. Second, our confidence interval is defined in terms of the empirical mean and as a result our confidence interval is about a factor of $\sqrt{1-\mu_{j,a,t}}$ tighter than that of Hossain, Micha, and Shah (2021). As our proof involves bounding the regret by the Lipschitz continuity of the Nash social welfare, our confidence interval allows for a careful analysis of the algorithm.

Note that the UCB algorithm in Hossain, Micha, and Shah (2021) is horizon-independent. Although Algorithm 1 requires the value of the time horizon $T$ as input, it can be easily modified to be horizon-independent. One approach is to modify the confidence interval $w_{j,a,t}$ so it becomes a function of the current time step, $t$, rather than the time horizon, $T$. Specifically, both $\ln(4NKT^2/\delta)$ terms of $w_{j,a,t}$ in Algorithm 1 would become $\ln(8NKt^2/\delta)$. Lemma 4.2 and Theorem 4.5 can be easily adapted to the new confidence interval. The horizon-independent variant of our algorithm would have the same regret bound of $\tilde{O} \left( NK + \sqrt{NKT} \right)$.

**Overall approach:** Our objective is to bound the regret $\sum_{t \in [T]} \text{NSW}(\pi^*, \mu^*) - \text{NSW}(\pi_t, \mu^*)$. Observe that $\text{NSW} (\cdot, \cdot)$ is monotone in the second argument so $\text{NSW}(\pi^*, U_t) \geq \text{NSW}(\pi^*, \mu^*)$. By the optimality of $\pi_t$, we also have $\text{NSW}(\pi_t, U_t) \geq \text{NSW}(\pi^*, U_t)$. Thus, we can reduce the problem to bounding $\sum_{t \in [T]} \text{NSW}(\pi_t, U_t) - \text{NSW}(\pi_t, \mu^*)$.

The key idea to bound the regret in a single round $t$ is to look at the expected reward of each agent $j$ if the mean reward was $U_t$. Formally, let $g_{j,t} = \sum_{a \in [K]} \pi_{a,t}(1 - U_{j,a,t}) = 1 - \sum_{a \in [K]} \pi_{a,t}U_{j,a,t}$. If there are a lot of agents with large $g_{j,t}$, then the Nash product $\text{NSW}(\pi_t, U_t) = \prod_j (1 - g_{j,t})$ is small and the regret is therefore small. More precisely, we consider two cases depending on whether there exists a $p \geq 0$ such that the set of agents $\{j \in [N] : g_{j,t} \geq 2^{-p}\}$ is of size at least $3 \cdot 2^p \ln(T)$. If such a $p$ exists, then

$$\text{NSW}(\pi_t, U_t) \leq (1 - 2^{-p})^{3 \cdot 2^p \ln T} \leq \frac{1}{T^3}$$

As a result, the regret of round $t$ is negligible.

We now need to bound the regret of rounds where no such $p$ exists. The key idea is to show that when no $p$ exists, the upper confidence bounds are on average very close to the true means. For intuition, suppose the following similar statement holds. Let $g'_{j,t} = \sum_{a \in [K]} \pi_{a,t}(1 - \mu_{j,a,t})$ and for all $p \geq 0$, the set of agents $\{j \in [N] : g'_{j,t} \geq 2^{-p}\}$ is of size at most $3 \cdot 2^p \ln(T)$. Given this condition we can bound

$$\sum_{j \in [N]} g'_{j,t} \leq \int_0^1 \text{number of agents } j \text{ s.t. } g'_{j,t} \geq x dx \leq 1 + 6 \ln T \log N$$

Notice that our estimation error $w_{j,a,t}$ is a function of $1 - \mu_{j,a,t}$ and we showed that $1 - \mu_{j,a,t}$ is small on average. Thus, by making a careful averaging argument, we can show that the upper bound $U_t$ is close to $\mu^*$ on average. The regret bound then follows from the smoothness of the function $\text{NSW}(\pi_t, \cdot)$. The actual proof has to overcome additional technical challenges due to the difference between the desired $g'_{j,t}$ and the actual $g_{j,t}$. 

| Algorithm       | Regret Bound                        | Efficient | Reference                                    |
|-----------------|-------------------------------------|-----------|----------------------------------------------|
| Explore-First   | $O \left( \sqrt{NKT^2 \cdot \min\{N,K\}} \right)$ | ✓         | (Hossain, Micha, and Shah 2021), Theorem 1   |
| $\epsilon$-Greedy | $O \left( \sqrt{NKT^2 \cdot \min\{N,K\}} \right)$ | ✓         | (Hossain, Micha, and Shah 2021), Theorem 2   |
| UCB             | $O \left( K \sqrt{NT \cdot \min\{N,K\}} \right)$ | ×         | (Hossain, Micha, and Shah 2021), Theorem 3   |
| Algorithm 1     | $O \left( NK + \sqrt{NKT} \right)$ | ✓         | Theorem 4.5                                 |
| Algorithm 2     | $O \left( N^2K + \sqrt{KT} \right)$ | ×         | Theorem 4.6                                 |

Table 1: Fair algorithms for the MA-MAB problem.
Analysis
We seek to bound $\sum_{t \in [T]} \text{NSW}(\pi_t, U_t) - \text{NSW}(\pi_t, \mu^*)$ using the smoothness of the NSW objective. We include the missing proof of the Lipschitz-continuity property and the other lemmas in the appendix.

Lemma 4.1. (Lemma 3, (Hossain, Micha, and Shah 2021))
Given a policy $\pi \in \Delta^k$ and reward matrices $\mu^t, \mu^2 \in [0, 1]^{N \times K}$, we have

$$|\text{NSW}(\pi, \mu^1) - \text{NSW}(\pi, \mu^2)| \leq \sum_{j \in [N]} \sum_{a \in [K]} \pi_{a,j} |\mu_{j,a}^1 - \mu_{j,a}^2|$$

Lemma 4.1 implies that a Lipschitz-continuity analysis for bounding $\sum_{t \in [T]} \text{NSW}(\pi_t, U_t) - \text{NSW}(\pi_t, \mu^*)$ would benefit from a tight confidence bound on the means of the rewards. The following lemma proves a confidence interval that is about factor of $\sqrt{T - \hat{\mu}_{j,a,t}}$ tighter than that of Hossain, Micha, and Shah (2021).

Lemma 4.2. For any $\delta \in (0, 1)$, with probability at least $1 - \delta/2$, $\forall t > K, a \in [K], j \in [N], \mu_{j,a}^* - \hat{\mu}_{j,a,t} \leq \sqrt{\frac{12(1 - \mu_{j,a}^*)\ln(4NK\delta)}{N_{a,t}}} + \frac{12\ln(4NK\delta)}{N_{a,t}} = w_{j,a,t}$.

Our confidence bound $w_{j,a,t}$ in Lemma 4.2 has a $\tilde{O}\left(\sqrt{\frac{1 - \mu_{j,a}^*}{N_{a,t}}}\right)$ term and a $\tilde{O}\left(\frac{1}{N_{a,t}}\right)$ term. Using Young’s inequality, we can bound the first term by $\tilde{O}\left((1 - \hat{\mu}_{j,a,t}) + \frac{1}{N_{a,t}}\right)$, and thus we have

$$w_{j,a,t} \in \tilde{O}\left(1 - \hat{\mu}_{j,a,t} + \frac{1}{N_{a,t}}\right). \quad (1)$$

As mentioned above, if there are a lot of agents with large $g_{j,t}$ at round $t$, then the regret will be negligible. Thus, our main goal is to analyze the regret for rounds $t$’s whose are not enough such agents. The following lemma formalizes the intuition and bounds the sum of expected empirical reward over all agents by the confidence intervals $w_{t}$. In other words, it bounds the $(1 - \mu_{j,a,t})$ term of Equation 1 over all actions $a$ and agents $j$.

Lemma 4.3. Define $g_{j,t} = \sum_{a \in [K]} \pi_{a,t}(1 - U_{j,a,t})$, and $S(t, p) = \{j \in [N] : g_{j,t} \geq 2^{-p}\}$. If $|S(t, p)| < 2^p$, then for all $p \geq 0$, then

$$\sum_{j \in [N]} \sum_{a \in [K]} \pi_{a,t}(1 - \hat{\mu}_{j,a,t}) \leq 1 + 6\ln T \log N + \sum_{j \in [N]} \sum_{a \in [K]} \pi_{a,t}w_{j,a,t}. \quad (2)$$

Lemma 4.4 bounds the error incurred by the $\frac{1}{N_{a,t}}$ term of Equation 1. We carefully analyze the martingale sequence to obtain a tighter bound than that of black-box approaches, i.e. in Lemma 4.4 we bound the martingale sequence to be $O(K \ln T / K)$ while Azuma-Hoeffding inequality would give us $O(\sqrt{T})$.

Lemma 4.4. With probability $1 - \delta/2$,

$$\sum_{t \in [T]} \sum_{a \in [K]} \pi_{a,t}/N_{a,t} \leq 2K \left(\ln \frac{T}{K} + 1\right) + \ln(2/\delta).$$

With Lemma 4.3 and Lemma 4.4, we are ready to bound $\sum_{t \in [T]} \text{NSW}(\pi_t, U_t) - \text{NSW}(\pi_t, \mu^*)$ by a Lipschitz-continuity analysis of the Nash social welfare function.

Theorem 4.5. Suppose $\forall j, a, t, r_{j,a,t} \in [0, 1]$ and $w_{j,a,t} = \sqrt{\frac{12(1 - \mu_{j,a,t})\ln(4NK\delta)}{N_{a,t}}} + \frac{12\ln(4NK\delta)}{N_{a,t}}$, for any $\delta \in (0, 1)$, the regret of the Fair multi-agent UCB algorithm (Algorithm 1) is $R_T = \tilde{O}\left(\sqrt{NK^T} + N\delta\right)$ with probability at least $1 - \delta$.

Proof. Let $I' \subseteq [T]$ denote the set of all rounds $t$ where there exists $p \geq 0$ such that $|S(t, p)| \geq 2^p \ln T$. Let $I = [K] \cup I'$. We have

$$\sum_{t \in I} \text{NSW}(\pi_t, U_t) - \text{NSW}(\pi_t, \mu^*)$$

$$\leq \sum_{t \in I} \text{NSW}(\pi_t, U_t) + K$$

$$\leq \sum_{t \in I'} \sum_{j \in S(t, p)} \sum_{a \in [K]} \pi_{a,t}|U_{j,a,t} - \mu_{j,a,t}^*|$$

$$\leq 2\sum_{t \in I} \sum_{j \in S(t, p)} \sum_{a \in [K]} \pi_{a,t}\sqrt{\frac{12(1 - \mu_{j,a,t})\ln(4NK\delta)}{N_{a,t}}} + \frac{24\ln(4NK\delta)}{N_{a,t}}. \quad (3)$$

The first inequality follows from Lemma 4.1. The last inequality follows from Lemma 4.2 and the definition of $U_{j,a,t}$. By Lemma 4.4, we can bound the error incurred by the linear term of the confidence interval in Equation 3,

$$\sum_{t \in I} \sum_{j \in [N]} \sum_{a \in [K]} \pi_{a,t}\sqrt{\frac{24\ln(4NK\delta)}{N_{a,t}}}$$

$$\leq \sum_{t \in [T]} \sum_{j \in [N]} \sum_{a \in [K]} \pi_{a,t}\sqrt{\frac{24\ln(4NK\delta)}{N_{a,t}}}$$

$$= \sum_{j \in [N]} \sum_{t \in [T]} \sum_{a \in [K]} \pi_{a,t}\sqrt{\frac{24\ln(4NK\delta)}{N_{a,t}}}$$

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We are done after bounding the remaining term of Equation 3. For brevity, we bound it without the log term,

\[
\sum_{t \in T} \sum_{j \in [N]} \sum_{a \in [K]} \pi_{a,t} \left( 1 - \frac{1}{N_{a,t}} \right) \leq \sum_{t \in T} \sum_{j \in [N]} \sum_{a \in [K]} \pi_{a,t} \left( \frac{q(1 - \mu_{j,a,t})}{2} + \frac{1}{2q \cdot N_{a,t}} \right).
\]

The inequality follows from Young's inequality for \( q \geq 0 \). Applying Lemma 4.3 to the first term and Lemma 4.4 to the second term of Equation 5, we have:

\[
\sum_{t \in T} \sum_{j \in [N]} \sum_{a \in [K]} \pi_{a,t} \left( 1 - \frac{1}{N_{a,t}} \right) \leq \frac{q}{2} \sum_{t \in T} \left( 1 + 6 \ln T + \sum_{j \in [N]} \sum_{a \in [K]} \pi_{a,t} w_{j,a,t} \right) + \frac{N}{2q} \left( 2K \left( \ln \frac{T}{K} + 1 \right) + \ln(2/\delta) \right)
\]

\[
\leq \frac{q}{2} \sum_{t \in T} \sum_{j \in [N]} \sum_{a \in [K]} \sqrt{\frac{12(1 - \mu_{j,a,t}) \ln(4NK/T)}{N_{a,t}}} + \frac{N}{2q} \left( 2K \left( \ln \frac{T}{K} + 1 \right) + \ln(2/\delta) \right) + 6q \cdot T \cdot \ln T.
\]

where we use the fact that \( 6 \ln T \geq 6 \ln 2 \geq 1 \). Suppose \( q \in (0, 1] \), applying Lemma 4.4 to the linear term of the confidence interval, we have

\[
\sum_{t \in T} \sum_{j \in [N]} \sum_{a \in [K]} \pi_{a,t} \left( 1 - \frac{1}{N_{a,t}} \right) \leq \frac{q}{2} \sum_{t \in T} \sum_{j \in [N]} \sum_{a \in [K]} \sqrt{\frac{12(1 - \mu_{j,a,t}) \ln(4NK/T)}{N_{a,t}}} + \frac{N}{2q} \left( 2K \left( \ln \frac{T}{K} + 1 \right) + \ln(2/\delta) \right) + 6q \cdot T \cdot \ln T.
\]

Setting \( q = \frac{\sqrt{KN}}{(\sqrt{KN} + T)\sqrt{12 \ln(4NK/T)}} \leq 1 \) and re-arranging the terms, we have

\[
\left( \ln(4NK/T) \right)^{3/2} \left( 1 + \frac{\sqrt{T}}{\sqrt{KN}} \right) 2NK \left( \ln \frac{T}{K} + 1 \right)
\]

\[
+ 6T \sqrt{KN} + (\sqrt{KN} + T).
\]

From Equations 2, 3, 4, and 6, we have

\[
\sum_{t \in [T]} \text{NSW}(\pi_t, U_t) - \text{NSW}(\pi_t, \mu^*) = O \left( \sqrt{NK} + N \right) \cdot \text{polylog}(NK/T).
\]

By monotonicity of the Nash-social welfare function and the optimization step in the algorithm, we have \( \text{NSW}(\pi_t, U_t) \geq \text{NSW}(\pi^*, U_t) \geq \text{NSW}(\pi^*, \mu^*) \). Thus,

\[
\sum_{t \in [T]} \text{NSW}(\pi_t, \mu^*) - \sum_{t \in [T]} \text{NSW}(\pi_t, \mu^*) = O \left( \sqrt{NK} + N \right) \cdot \text{polylog}(NK/T).
\]

\[\blacksquare\]

**An Inefficient Algorithm with Improved Regret**

Algorithm 2: Fair multi-agent UCB algorithm with high start-up cost

1: **input:** \( K, N, T, \delta \)
2: for \( t = 1 \) to \( T \) do
3: if \( t \leq 180N^2 \ln(6NTK/\delta) \ln T \) then
4: \( \pi_t \leftarrow \text{policy that puts probability 1 on arm } t \mod (K+1) \)
5: else
6: \( \forall j, a, \mu_{j,a,t} = \frac{1}{N_{a,t}} \sum_{a=1}^{K} r_{j,a,r} \textbf{1}_{\{a_r = a\}} \)
7: \( S_\pi = \{ \pi \in \Delta K : \sum_{a \in [K], j \in [N]} \pi_a (1 - \mu_{j,a,t}) \leq 1 + 2 \ln T \} \)
8: if \( S_\pi \neq \emptyset \) then
9: \( \pi_t \leftarrow \text{argmax}_{\pi \in S_\pi} (\text{NSW}(\pi, \mu_t) + \pi \cdot \eta_t) \)
10: else
11: \( \pi_t \leftarrow \text{policy that puts probability 1 on a random arm } a \in [K] \)
12: end if
13: end if
14: Sample \( a_t \) from \( \pi_t \)
15: Observe rewards \( \{r_{j,a,t} : j \in N\} \)
16: \( N_{a_t+1} \leftarrow N_{a_t} + 1 \)
17: end for

Algorithm 2 is able to obtain a tighter regret bound in terms of \( T \) by first pulling each arm \( O(N^2) \) times, then selecting \( \pi_t \) to optimize the Nash Social Welfare on \( \mu_t \) plus an additive term \( \pi \cdot \eta_t \), where \( \eta_t = \{\eta_{a,t} : a \in [K]\} \) is a vector. The additive term gives an upper bound on \( \text{NSW}(\pi, \mu^*) - \text{NSW}(\pi, \mu_t) \), and by the optimization step we obtain that for all \( t > O(N^2 K) \),

\[ \text{NSW}(\pi^*, \mu^*) \leq \text{NSW}(\pi^*, \mu_t) + \sum_{a \in [K]} \pi_a^* \cdot \eta_{a,t} \]
We test Algorithm 1 and the UCB algorithm from (Hossain, Micha, and Shah 2021) which drops the leading factor $Q$ and we have a known efficient implementation. Additionally, similar to the UCB algorithm of (Hossain, Micha, and Shah 2021) we can address this issue to some degree by allowing the gradient ascent to run substantially longer: for Algorithm 1 the gradient ascent terminates after the objective changes by less than $2 \cdot 10^{-4}$ after 20 iterations, and for (Hossain, Micha, and Shah 2021) the ascent terminates after changing less than $10^{-6}$ after 30 iterations. By tightening the termination conditions we make it more difficult for the ascent to hang on a suboptimal position in (Hossain, Micha, and Shah 2021) at the tradeoff of longer runtime. Even at this point, we still see instability in the regret graph over time with the non-concave optimization. We include the algorithm none-the-less as it is the only other existing algorithm for fair multi-agent multi-armed bandits with regret on the order of $O(\sqrt{T})$.

Table 2 shows the average regret of the two algorithms after 200,000 iterations and 500,000 iterations over the 5 independent instances for each size, which are the same set of instances for both algorithms. Figure 1 shows cumulative regret graphs of the algorithms’ instances for each setting of $N$ and $K$. In all cases, Algorithm 1 outperforms the previous best algorithm as $T$ becomes large. Once the $\sqrt{T}$ terms in the regret bounds become dominant, the $\sqrt{K \cdot \min N, K}$ factor saved in Algorithm 1 over the previous algorithm becomes apparent. This is especially true as $N$ and $K$ are large. For the small case where $(N, K) = (4, 2)$, both regret curves take on the shape $\sqrt{T}$, with a growing separation between the two as $T$ increases. However, it is worth noting that (Hossain, Micha, and Shah 2021) does not make any changes to $\mu$ in the optimization step, so their algorithm is able to begin improving immediately. Algorithm 1 adds the confidence bound to $\hat{\mu}$ and then caps all elements in $\hat{\mu}$.

\[ \pi_i \sim \mathcal{B}(\mu_i) \]

\[
\eta_{a,t} = O(\sqrt{K/T}) \sum_{j \in [K]} (1 - \hat{\mu}_{j,a,t}) + O(1/\sqrt{N}) \sum_{j \in [N]} w_{j,a,t} + O(1/\sqrt{N}) \sum_{j \in [N]} w_{j,a,t}
\]

for any $\delta \in (0, 1)$, the regret of Algorithm 2 is $R_T = O(\sqrt{KT} + N^2 K)$ with probability at least $1 - \delta$.

We defer the proof of this theorem and the details of the constants and log terms in the $O$ and $\hat{O}$ of $\eta$ to the appendix. At a high level, this bound comes from using a tighter bound in place of Lemma 4.2, where we bound $\sum_{j \in [N]} |\mu_{j,a} - \hat{\mu}_{j,a,t}|$ instead of $|\mu_{j,a}^* - \hat{\mu}_{j,a,t}|$. We also analyze the regret at each time step $t$ using

\[
NSW(\pi, \hat{\mu}_t) = \prod_{j \in [N]} \left( \mathbb{E}_{a \sim \pi} \mu_{j,a}^* + \mathbb{E}_{a \sim \pi} (\hat{\mu}_{j,a,t} - \mu_{j,a}^*) \right).
\]

Since $NSW(\pi, \mu^*) = \prod_{j \in [N]} \mathbb{E}_{a \sim \pi} \mu_{j,a}^*$, we can bound $NSW(\pi, \mu^*) - NSW(\pi, \hat{\mu}_t)$ by

\[
\sum_{m=1}^{N} \sum_{B \subseteq [N]: |B| = m} \sum_{j \in B} \frac{\mathbb{E}_{a \sim \pi} (\hat{\mu}_{j,a,t} - \mu_{j,a})}{\mathbb{E}_{a \sim \pi} \mu_{j,a}^*},
\]

which drops the leading factor $\prod_{j} \mathbb{E}_{a \sim \pi} \mu_{j,a}^* \leq 1$ from the bound. Note that here the regret is bounded by $\mathbb{E}_{a \sim \pi} (\hat{\mu}_{j,a,t} - \mu_{j,a})$ as opposed to $\mathbb{E}_{a \sim \pi} |\hat{\mu}_{j,a,t} - \mu_{j,a}^*|$ in Lemma 4.1. Analyzing the terms with $m = 1$ and $m \geq 2$ separately allows us to derive the bound $\eta_{a,t}$ in our algorithm.

The algorithm obtains a bound in $T$ which matches the known lower bound $O(\sqrt{KT})$ up to logarithmic terms, at the tradeoff of a high initial cost for pulling each arm $O(N^2)$ times. Additionally, similar to the UCB algorithm of (Hossain, Micha, and Shah 2021) the algorithm uses an additive regularization term in its optimization step and therefore does not have a known efficient implementation.

5 Experiments

We test Algorithm 1 and the UCB algorithm from (Hossain, Micha, and Shah 2021) with both $\alpha_t = N$ and $\alpha_t = \sqrt{12NK \log(NKT)}$, although we exclude the second $\alpha_t$ from the results because it was outperformed by the other algorithms in every test. We test three pairs of $(N, K)$: a small size $(4, 2)$, a medium size $(20, 4)$, and a large size $(80, 8)$. For each pair $(N, K)$ we test the algorithms on 5 values of $\mu^*$ chosen randomly from 1 minus an exponential distribution with mean 0.04, and rounded up to 0.1 in extreme cases. When an action $a \in [K]$ is taken, we draw the rewards for each agent $j \in [N]$ from a Bernoulli distribution with $p = \mu_{j,a}^*$. For the optimization step, we round the empirical mean up to $10^{-3}$ since this is a divisor in gradient computations. We also optimized the additive terms in both algorithms using a constant factor found through empirical binary search: the additive term $w_{j,a,t}$ in Algorithm 1 is scaled by 0.5, and the additive term in the optimization step of (Hossain, Micha, and Shah 2021) is scaled by 0.8.1

In each algorithm we compute $\pi$ using the projected gradient ascent. For Algorithm 1, we take advantage of the log-concavity of the Nash Social Welfare function and the monotonicity of the logarithm and optimize the log of the objective in the gradient ascent. The UCB algorithm from (Hossain, Micha, and Shah 2021) computes the policy $\pi_t$ as $\arg\max_{\pi} NSW(\pi, \hat{\mu}_t) + \alpha_t \sum_{a \in [K]} \left( \pi_a \cdot \sqrt{|\log(NKT)|} \right)$, which is no longer a log-concave objective due to the linear terms (Hossain, Micha, and Shah 2021). We can address this issue to some degree by allowing the gradient ascent to run substantially longer: for Algorithm 1 the gradient ascent terminates after the objective changes by less than $2 \cdot 10^{-4}$ after 20 iterations, and for (Hossain, Micha, and Shah 2021) the ascent terminates after changing less than $10^{-6}$ after 30 iterations. By tightening the termination conditions we make it more difficult for the ascent to hang on a suboptimal position in (Hossain, Micha, and Shah 2021) at the tradeoff of longer runtime. Even at this point, we still see instability in the regret graph over time with the non-concave optimization. We include the algorithm none-the-less as it is the only other existing algorithm for fair multi-agent multi-armed bandits with regret on the order of $O(\sqrt{T})$.

Table 2 shows the average regret of the two algorithms after 200,000 iterations and 500,000 iterations over the 5 independent instances for each size, which are the same set of instances for both algorithms. Figure 1 shows cumulative regret graphs of the algorithms’ instances for each setting of $N$ and $K$.

In all cases, Algorithm 1 outperforms the previous best algorithm as $T$ becomes large. Once the $\sqrt{T}$ terms in the regret bounds become dominant, the $\sqrt{K \cdot \min N, K}$ factor saved in Algorithm 1 over the previous algorithm becomes apparent. This is especially true as $N$ and $K$ are large. For the small case where $(N, K) = (4, 2)$, both regret curves take on the shape $\sqrt{T}$, with a growing separation between the two as $T$ increases. However, it is worth noting that (Hossain, Micha, and Shah 2021) does not make any changes to $\hat{\mu}$ in the optimization step, so their algorithm is able to begin improving immediately. Algorithm 1 adds the confidence bound to $\hat{\mu}$ and then caps all elements in $\hat{\mu}$.

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1The repository is hosted at https://github.com/MetricJones/Fair-MultiAgent-MultiArmed-Bandits.
Figure 1: Sample Cumulative Regret Graphs for each setting. \((N, K)\) is \((4,2)\) in the top row, \((20,4)\) in the center row, and \((80,8)\) on the bottom row. The regret graphs show up to 20,000 iterations in the first column, 100,000 iterations in the second column, and 500,000 iterations in the third for Algorithm 1 (blue) and the UCB (red) from (Hossain, Micha, and Shah 2021).

| \(N\) | \(K\) | Algorithm 1 \(t = 2 \cdot 10^5\) | (Hossain, Micha, and Shah 2021) \(t = 5 \cdot 10^5\) | NSW(\(\pi^*, \mu^*\)) |
|------|------|-------------------------------|---------------------------------|---------------------|
| 4    | 2    | 1222                          | 1226                            | 0.9428 ± 0.0370     |
| 20   | 4    | 8313                          | 10801                           | 0.6353 ± 0.0358     |
| 80   | 8    | 4521                          | 5230                            | 0.0808 ± 0.0154     |

Table 2: Average cumulative regrets over 10 values of \(\mu^*\), with \(T = 5 \cdot 10^5\)

at 1, so until one of the terms in the upper confidence bound drops below 1 the algorithm will choose a uniform \(\pi\), which accounts for the large linear cumulative regret in the first 7500 iterations before the upper confidence bound is non-trivial and Algorithm 1 begins to outperform (Hossain, Micha, and Shah 2021). For the small size, it seems that Algorithm 1 performs better as long as the number of iterations is at least 25,000.

In the medium case, the linear section in the regret curve of Algorithm 1 still does not outperform (Hossain, Micha, and Shah 2021), but (Hossain, Micha, and Shah 2021) sees a steeper regret curve which narrows the gap for small \(T\) and creates a larger regret gap for large \(T\). In the largest case, where \((N, K) = (80, 8)\), we see that (Hossain, Micha, and Shah 2021) barely ever outperforms Algorithm 1 even at small \(T\) when Algorithm 1’s upper confidence bound is a matrix of 1s. At high values of \(T\) we still see that Algorithm 1 outperform and we observe that the other factors in the \(\sqrt{T}\) terms begin to play a significant role. Algorithm 1’s regret curve still takes on a \(\sqrt{T}\) shape although it is much gentler than the smaller cases, which can be at least partially attributed to the fact that the instantaneous regret at each round is bounded by the optimal value of the Nash social welfare, which is the product of expected rewards. The algorithm from (Hossain, Micha, and Shah 2021) still appears almost linear even at 500,000 iterations, as substantially larger values of \(T\) are required to overcome the factors of \(N\) and \(K\) and see the \(\sqrt{T}\) shape. There is still a substantial difference between the cumulative regrets of the two algorithms as \(T\) increases.

Our experiments support the theoretical gains of our results. We see that at small values of \(T\), under 10,000, our algorithm may be outperformed by (Hossain, Micha, and Shah 2021) for small sizes of \(N\) and \(K\). This effect becomes significantly weaker as \(N\) and \(K\) increase. At sufficiently large \(T\), on the order of 25,000, Algorithm 1 outperforms the previous state-of-the-art with increasing significance and consistency as \(K, N,\) and \(T\) increase. Additionally, as \(T\) increases the regret curves of Algorithm 1 are significantly smoother than those of the previous algorithm due to the efficiency of the optimization step.

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