Enhancement of superconductivity by Anderson localization

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Motivation / Superconductor–Insulator transition

recent review: Gantmakher & Dolgopolov (2010)
Motivation / Experiments on LaAlO$_3$/SrTiO$_3$ interface

Phase diagram of the LaAlO$_3$/SrTiO$_3$ interface  Caviglia et al. (2008)

Giant background dielectric constant: Coulomb interaction strongly screened
Anderson Theorem:

- nonmagnetic impurities do not affect s-wave superconductors
- Cooper-instability is the same for diffusive electrons:

\[
T_c^{BCS} \sim \omega_D \exp(-2/\lambda_{e-ph})
\]
Anderson transition

quasi-1D, 2D: metallic $\rightarrow$ localized crossover with increasing $L$

d $> 2$: metal-insulator transition
Motivation / Anderson ’59 vs Anderson ’58

Anderson Theorem vs Anderson Localization?

Anderson Theorem vs Anderson Localization?
Motivation / Theory: enhancement of $T_c$ – attraction only

- Superconductivity at 3D Anderson metal-insulator transition (no Coulomb repulsion)

- **Enhancement** of $T_c$ as compared to BCS result $T_c^{BCS} \propto \exp(-2/\lambda)$

$$T_c \propto \lambda^{d/|\Delta_2|}$$

Feigelman, Ioffe, Kravtsov, Yuzbashyan (2007); Feigelman, Ioffe, Kravtsov, Cuevas (2010)

where $\Delta_2 < 0$ – multifractal exponent for inverse participation ratio
Multifractality

Wegner (1980)

- critical wave function:

Evers, Mildenberger, Mirlin

- enhanced correlations in matrix elements $\sim \psi^4$ of Cooper attraction
- stronger attraction $\Rightarrow$ enhancement of $T_c$

Feigelman, Ioffe, Kravtsov, Yuzbashyan (2007)
Motivation / Theory: suppression of $T_c$ due to Coulomb repulsion

- RG theory for disorder and interactions
  
  Finkelstein (1983, 1987); Castellani, Di Castro, Lee, Ma (1984)

- Experiments on Mo-Ge films, Graybeal & Beasley (1984)

- 2D: $T_c$ vanishes at the critical resistance

  $$R_{\Box} \sim \left( \ln \frac{1}{T_c^{BCS_T}} \right)^{-2}$$

  Finkelstein (1987)
Motivation / Questions to answer

Can suppression of $T_c$ due to Coulomb repulsion and enhancement of $T_c$ due to multifractality be described in a unified way?

Does weak multifractality enhance $T_c$ in 2D systems?

Does the enhancement of $T_c$ hold if one takes into account short-ranged repulsion in particle-hole channels?
The problem / Microscopic Hamiltonian \( H = H_0 + H_{\text{dis}} + H_{\text{int}} \)

- **Free electrons:**
  \[
  H_0 = \int d^d \mathbf{r} \overline{\psi}_\sigma(\mathbf{r}) \left[ -\frac{\nabla^2}{2m} \right] \psi_\sigma(\mathbf{r})
  \]
  where \( \sigma = \pm 1 \) is spin projection

- **Scattering off random potential:**
  \[
  H_{\text{dis}} = \int d^d \mathbf{r} \overline{\psi}_\sigma(\mathbf{r}) V(\mathbf{r}) \psi_\sigma(\mathbf{r})
  \]
  Gaussian white-noise distribution:
  \[
  \langle V(\mathbf{r}) \rangle = 0, \quad \langle V(\mathbf{r}_1) V(\mathbf{r}_2) \rangle = \frac{1}{2\pi \nu_0 \tau} \delta(\mathbf{r}_1 - \mathbf{r}_2)
  \]
  \( \nu_0 \) – thermodynamic density of states
The problem / Microscopic Hamiltonian $H = H_0 + H_{\text{dis}} + H_{\text{int}}$

- Electron-electron interaction:

  $$H_{\text{int}} = \frac{1}{2} \int d^d r_1 d^d r_2 \overline{\psi}_\sigma(r_1) \psi_\sigma(r_1) \ U(r_1 - r_2) \overline{\psi}'_\sigma(r_2) \psi'_\sigma(r_2)$$

  - short-ranged repulsion with BCS-type attraction ($\lambda > 0$)

    $$U(R) = u_0 \frac{a^{2\alpha}}{[a^2 + R^2]^{\alpha}} - \frac{\lambda}{\nu_0} \delta(R), \quad \alpha > 2d, \quad u_0 > 0$$

  - Coulomb (long-ranged) repulsion with BCS-type attraction ($\lambda > 0$)

    $$U(R) = \frac{e^2}{\epsilon R} - \frac{\lambda}{\nu_0} \delta(R)$$
The problem / Interaction Hamiltonian at small momentum transfer

- Particle-hole channel:

\[
H_{\text{int}}^{p-h} = \frac{1}{2\nu_0} \int_{q l \lesssim 1} \frac{d^d q}{(2\pi)^d} \sum_{a=0}^{3} F_a(q) m^a(q) m^a(-q)
\]

where \( m^a(q) = \int \frac{d^d k}{(2\pi)^d} \bar{\psi}(k + q) \sigma_a \psi(k) \)

\[
F_0(q) = F_s, \quad F_1(q) = F_2(q) = F_3(q) = F_t
\]

- Particle-particle channel:

\[
H_{\text{int}}^{p-p} = -\frac{F_c}{\nu_0} \int_{q l \lesssim 1} \frac{d^d q}{(2\pi)^d} \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \bar{\psi}_\sigma(k_1) \bar{\psi}_\sigma(-k_1 + q) \psi_{-\sigma}(k_2 + q) \psi_\sigma(-k_2)
\]
Field theory: non-linear sigma-model

\[
S[Q] = \frac{\pi \nu}{4} \int d^{d}r \text{ Tr } [-D(\nabla Q)^{2} - 2i \omega \Lambda Q], \quad Q^{2}(r) = 1
\]

Wegner (1979)

supersymmetry: Efetov (1982)

sigma-model manifold \( \mathcal{M} = \{ \mathcal{M}_{B} \times \mathcal{M}_{F} \} \)

“dressed” by anticommuting variables

\( \mathcal{M}_{B} \) — non-compact, \( \mathcal{M}_{F} \) — compact

with electron-electron interaction: fermionic replicas or Keldysh

Finkelstein (1983), Kamenev, Andreev (1999)
Field theory: interacting non-linear sigma-model

Finkelstein (1983)

\[
S = S_0 + S_{\text{int}}^{(s)} + S_{\text{int}}^{(t)} + S_{\text{int}}^{(c)},
\]

\[
S_0 = -\frac{g}{32} \int d\mathbf{r} \, \text{Tr}(\nabla Q)^2 + 4\pi Tz \int d\mathbf{r} \, \text{Tr} \eta(Q - \Lambda),
\]

\[
S_{\text{int}}^{(s)} = -\frac{\pi T}{4} \Gamma_s \sum_{\alpha, n} \sum_{p=0,3} \int d\mathbf{r} \, \text{Tr} \left[ I^\alpha_n t_p Q \right] \text{Tr} \left[ I^\alpha_n t_p Q \right],
\]

\[
S_{\text{int}}^{(t)} = -\frac{\pi T}{4} \Gamma_t \sum_{\alpha, n} \sum_{p=0,3} \sum_{j=1}^3 \int d\mathbf{r} \, \text{Tr} \left[ I^\alpha_n t_{pj} Q \right] \text{Tr} \left[ I^\alpha_n t_{pj} Q \right],
\]

\[
S_{\text{int}}^{(c)} = -\frac{\pi T}{2} \Gamma_c \sum_{\alpha, n} \sum_{p=0,3} (-1)^p \int d\mathbf{r} \, \text{Tr} \left[ I^\alpha_n t_p Q I^\alpha_n t_p Q \right],
\]

\[
\Lambda_{\alpha \beta}^{nm} = \text{sgn} \, n \delta_{nm} \delta^{\alpha \beta} t_{00}, \quad \eta_{nm}^{\alpha \beta} = n \delta_{nm} \delta^{\alpha \beta} t_{00}, \quad (I^\alpha_k)_{nm} = \delta_{n-m,k} \delta^{\alpha \beta} \delta^{\alpha \gamma} t_{00}
\]

\[
Q^2 = 1, \quad \text{Tr} \, Q = 0, \quad Q^\dagger = C^T Q^T C, \quad C = it_{12}, \quad C^T = -C
\]

n, m – Matsubara, \( \alpha, \beta \) – replicas, p – particle-hole, j – spin; \( t_{pj} = \tau_p \otimes s_j \)
The problem / Interaction parameters at \( q \to 0 \)

On short scales \( \gamma_{s,t,c} = \Gamma_{s,t,c}/z \) are related to microscopic parameters:

- **Particle-hole channel**

\[
\gamma_s = -\frac{F_s}{1 + F_s}, \quad \gamma_t = -\frac{F_t}{1 + F_t}
\]

- **Cooper channel** (provided \( \omega_D \tau \gg 1 \))

\[
\gamma_c = \frac{1}{\ln T_c^{BCS \tau}}
\]

**Coulomb repulsion:** \( \gamma_s = -1 \)
2D electrons (orth. symmetry class) / RG equations from $\sigma$-model

\[
\begin{align*}
\frac{dt}{dy} &= t^2 \left[ 1 + f(\gamma_s) + 3f(\gamma_t) - \gamma_c \right] \\
\frac{d\gamma_s}{dy} &= -\frac{t}{2} \left[ 1 + \gamma_s \right] \left[ \gamma_s + 3\gamma_t + 2\gamma_c \right] \\
\frac{d\gamma_t}{dy} &= -\frac{t}{2} \left[ 1 + \gamma_t \right] \left[ \gamma_s - \gamma_t - 2\gamma_c (1 + 2\gamma_t) \right] \\
\frac{d\gamma_c}{dy} &= -\frac{t}{2} \left[ \gamma_s - 3\gamma_t + \gamma_c (\gamma_s + 3\gamma_t) \right] - 2\gamma_c^2
\end{align*}
\]

Finkelstein (1984); Castellani, Di Castro, Lee, Ma (1984)

Castellani, DiCastro, Forgacs, Sorella (1984); Ma, Fradkin (1986); Finkelstein (1984)

where $y = \ln \frac{L}{l}$ and $f(x) = 1 - (1 + x^{-1}) \ln(1 + x)$

- lowest order in disorder, $t = 2/\pi g$, $g$ is conductivity in units $e^2/h$
- exact in $\gamma_s$ and $\gamma_t$
- lowest order in $\gamma_c$
2D electrons (orth. symmetry class) / weak short-range interaction

\[
\begin{align*}
\frac{dt}{dy} &= t^2 \left( 1 - \left[ \gamma_s + 3 \gamma_t + 2 \gamma_c \right]/2 \right) \\
\frac{d}{dy} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix} &= -\frac{t}{2} \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & -2 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \gamma_c^2 \end{pmatrix}
\end{align*}
\]

- Weak interaction, \( |\gamma_s|, |\gamma_t|, |\gamma_c| \ll 1 \)
- Weak disorder, \( t \ll 1 \).
- Initial values \( \gamma_s(0) = \gamma_{s0} < 0, \gamma_t(0) = \gamma_{t0} > 0, \gamma_c(0) = \gamma_{c0} < 0, t(0) = t_0 \)
2D electrons (orth. symmetry class) / weak short-range interaction

- Moderately strong disorder: $|\gamma_{c0}| \ll t_0 \ll 1$

$$\frac{dt}{dy} = t^2, \quad \frac{d}{dy} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix} = -\frac{t}{2} \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & -2 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix}$$

Eigenvalues and eigenvectors

$$\lambda = 2t \quad : \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} ; \quad \lambda' = -t \quad : \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

- attraction to the (BCS) line $-\gamma_s = \gamma_t = \gamma_c \equiv \gamma$

- Anomalous dimensions of operators with two $Q$'s in NL$\sigma$M:
  $$\Delta_2 = -\lambda + \ldots \quad \text{and} \quad \mu_2 = -\lambda' + \ldots$$

- $\Delta_2 < 0$ is due to multifractality of electron wave functions without interactions
Projected RG equations:

\[
\frac{dt}{dy} = t^2, \quad \frac{d\gamma}{dy} = 2t\gamma - 2\gamma^2 / 3
\]

\[t(0) = t_0, \quad \gamma(0) = \gamma_0 = (-\gamma_{s0} + 3\gamma_{t0} + 2\gamma_{c0})/6, \quad |\gamma_0| \ll t_0 \ll 1\]

Two-step renormalization

1. \(t > \gamma\): neglect Cooper instability \((\gamma^2)\);
   enhancement of interaction matrix element due to weak multifractality

2. \(\gamma > t\): neglect disorder-induced term \((t\gamma) \implies\) conventional BCS,
   but with the new “bare” coupling constant \(t\) determined by disorder
   \(\implies\) exponential enhancement of the mean-field \(T_c\)
If $t_0 \ll \sqrt{|\gamma_0|} \ll 1$, superconductor wins:

$$|\gamma(y_*)| \sim 1, \quad t(y_*) \sim t_0^2/|\gamma_0| \ll 1, \quad y_* \sim \frac{1}{t_0}$$

superconductor transition temperature $T_c \propto e^{-2/t_0} \gg T_c^{BCS}$

Enhancement of $T_c$ due to weak multifractality!

If $\sqrt{|\gamma_0|} \ll t_0 \ll 1$, insulator wins:

$$t(y_*) \sim 1, \quad |\gamma(y_*)| \sim |\gamma_0|/t_0^2 \ll 1, \quad y_* \sim \frac{1}{t_0}$$
Sketch of phase diagram

Superconductor-Insulator Transition (SIT)
2D electrons (orth. symmetry class) / weak short-range interaction

- Resistivity $t$ near SIT and dependence of $T_c$ on $t_0$

\[ \gamma_{s0} = -0.005, \quad \gamma_{t0} = 0.005, \quad \gamma_{c0} = -0.04, \]
\[ t_0 = 0.065, 0.075, 0.085, 0.095, 0.10, 0.105, 0.11, 0.12 \text{ (from bottom to top)} \]
2D electrons: BKT transition

- SC transition in 2D is of Berezinskii-Kosterlitz-Thouless type

- Mean-field $T_c$ differs only slightly from $T_c^{BKT}$ for weak disorder

  Beasley, Mooij, Orlando ’79, Halperin, Nelson ’79

- Future work: effect of multifractality on BKT transition
3D electrons near Anderson transition/ weak short-range interaction

RG equations near free electron fixed point \( t = t_c, \gamma = 0 \)

\[
\frac{dt}{dy} = \frac{1}{\nu}(t - t_c) + \eta \gamma, \quad \frac{d\gamma}{dy} = -\Delta_2 \gamma - A \gamma^2, \quad A \sim 1
\]

\( t(0) = t_0 \) and \( \gamma(0) = \gamma_0 < 0 \) at the UV energy scale \( E_0 \sim 1/(\nu_0 l^d) \)

- Correlation length:
  \[
  \xi = |\tilde{t}_0 - t_c|^{-\nu}
  \]

\( \tilde{t} = t - \eta \nu \gamma /(|\Delta_2| \nu - 1) \) and \( \tilde{t}_0 = \tilde{t}(0) \)

- 3D Anderson transition (orth. symmetry class):
  \( \nu = 1.57 \pm 0.02 \) and \( \Delta_2 = -1.7 \pm 0.05 \)
Schematic phase diagram in the interaction–disorder plane and $T_c$

\[ T_c \]

I: $T_c = E_0 |\gamma_0|^{d/|\Delta_2|}$  
II: $T_c = \xi^{-d} E_0 \exp \left( -\frac{d}{a|\gamma_0|\xi|\Delta_2|} \right)$  
III: $T_c = T_c^{BCS}$

$T_c$ for region I agrees with Feigelman, Ioffe, Kravtsov, Yuzbashyan (2007)
Conclusions

- **Strong enhancement** of $T_c$ for 2D electrons (short-range interactions)
- **Strong enhancement** of $T_c$ near (free electron) 3D Anderson transition (short-range interactions)
- In both cases enhancement of superconductivity is due to **multifractality**
- **No Coulomb interaction:** Anderson localization facilitates superconductivity  \( \Rightarrow \) high-$T_c$ superconductivity?
- Anderson ’59 vs Anderson ’58: Anderson wins!