Weyl metrics and the generating conjecture

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By means of Ernst complex potential formalism it is shown, that previously studied static axisymmetric Einstein-Maxwell fields obtained through the application of the Horský-Mitskievitch generating conjecture represent a combination of Kinnersley’s transformations [W. Kinnersley: J. Math. Phys. 14 (1973) 651]. New theoretical background for the conjecture is suggested and commented.

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1 Introduction

The complexity of Einstein and coupled Einstein-Maxwell (EM) equations in general relativity led many authors to invent a lot of generating techniques that enable us to obtain new families of EM solutions from those already known (see e.g. [1], §30.1–§30.6 for a compact overview) instead of solving the field equations. Most of these techniques require or employ spacetime symmetries in some way which allows to simplify the problem to a certain extent. One of such methods demanding the presence of at least one Killing vector for the seed metric was proposed in [2]. Though the application of the Horský-Mitskievitch (HM) conjecture have resulted in finding several classes of new EM fields [3, 4, 5, 6, 7, 8, 9], the conjecture itself has not been proved so far. Our objective is to contribute to a deeper understanding of the HM conjecture and to explain its possible connections with other more thoroughly explored generating methods.

The paper is organized as follows. In Section 2 we summarize basic facts about complex potential formalism developed by Ernst [10, 11], Section 3 is devoted to particular examples of Harrison transformations applied to the EM fields generated

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by means of HM conjecture from some classes of the Weyl vacuum metrics, in Section 4 we compare both approaches and discuss the common features of these generating techniques.

Throughout the text geometrized units in which $c = 1$, $G = 1$ are used. The metric signature $- + ++$ and the indexing conventions follow [12].

2 Symmetries of stationary Einstein-Maxwell fields

Though the EM equations describing the coupling of the electromagnetic field with gravity are very complicated, it has been demonstrated they are endowed with hidden symmetries. The groups of corresponding transformations for stationary EM spacetimes were systematically described by Kinnersley and his coworkers (see [13, 14, 15] and references cited therein).

Following Kinnersley, in this paper we also make use of complex potential formalism designed by Ernst [10, 11] for an effective description of stationary axially symmetric EM fields, namely the Kerr and Kerr-Newman solutions. Afterwards it was generalized for stationary spacetimes by Ernst [16] and independently by Israel and Wilson [17]. Let us recall the main ideas and necessary formulae. A general stationary line-element may be written in the form

$$ds^2 = -f (dt - \omega_j dx^j)^2 + \frac{1}{f} h_{jk} dx^j dx^k, \quad j, k = 1, 2, 3$$

where $\{x^j\}$ represent some spacelike coordinates and metric functions $f$, $w_j$, $h_{jk}$ do not depend on $t$. Denoting the covariant derivative with respect to the 3-dimensional metric $h_{jk}$ as $\nabla$, one can define a twist vector

$$\chi = f^2 \nabla \times \omega + i (\Phi^* \nabla \Phi - \Phi \nabla \Phi^*), \quad i = \sqrt{-1}$$

satisfying the equation

$$\nabla \times \chi = 0,$$

which implies the existence of a scalar “twist potential” $\psi$, such that

$$\chi = \nabla \psi.$$  \hspace{1cm} (2)

If we introduce a complex scalar potential $\Phi$ describing the electromagnetic field (see below) and another complex scalar $\mathcal{E}$ by the relation

$$\mathcal{E} = f - |\Phi|^2 + i \psi,$$

then the system of coupled EM equations may be replaced by two complex 3-dimensional Ernst equations for Ernst potentials $\mathcal{E}$, $\Phi$

$$f \Delta \mathcal{E} = (\nabla \mathcal{E} + 2 \Phi^* \nabla \Phi) \cdot \nabla \mathcal{E},$$

$$f \Delta \Phi = (\nabla \mathcal{E} + 2 \Phi^* \nabla \Phi) \cdot \nabla \Phi.$$  \hspace{1cm} (4)
Table 1. Ernst potentials for various types of stationary EM fields

| Spacetime                      | E  | Φ  |
|--------------------------------|----|----|
| Stationary EM fields           | Complex | Complex |
| Static electrovac fields       | Real     | Real     |
| Static magnetovac fields       | Real     | Imaginary |
| Stationary vacuum field        | Complex             | 0      |
| Static vacuum field            | Real       | 0       |
| Conform-stationary EM fields   | 0          | Complex  |

It is possible to classify various types of stationary EM fields according to the values taken by the complex Ernst potentials $E$ and $\Phi$ as it is summarized in Table 1. For the sake of simplicity and without loss of generality we may further consider a timelike electromagnetic vector potential $A = A_t dt$ corresponding to an electric field. In this case the Ernst potential $\Phi = A_t$ reads straightforwardly, while in magnetic case $\Phi$ gets imaginary values and has to be found as a solution of partial differential equations (see e.g. [11, 9] for examples). Moreover, we can turn electric field into magnetic and vice versa via duality rotation in the complex plane of the potential $\Phi$.

Kinnersley [13, 14] proved, that the EM equations in the presence of one non-null Killing vector possess covariance under an 8-parameter group of transformation isomorphic to $SU(2,1)$. Those eight parameters can be combined into three complex parameters $a, b, c$ and two real parameters $\alpha, \beta$, so that from a given solution it is possible to generate a five-parameter family of solutions; the change of Ernst potentials under the symmetry transformations is summarized in Table 2. Let us remind that the spacelike metric coefficients $h_{ik}$ in (1) remain unchanged.

The gauge transformations (5) and (6) are of course not interesting from a physical point of view, as they do not lead to a new metric. The duality rotation (7) for a complex unit parameter $|b| = 1$ changes the type of electromagnetic field not altering the metric line element. The Ehlers transformation (8) reverts static fields into stationary ones and its more general Kinnersley’s form in Table 2 admits the presence of the electromagnetic field unlike its original Ehlers formulation [14]. Finally, the Harrison transformation (9) may add an electromagnetic field with $\Phi \neq 0$ to a vacuum seed metric for which $\Phi = 0$. It is namely this “charging” transformation we would like to concentrate on in comparison with the HM generating conjecture.

We can see, that the existence of a single non-null Killing vector field endows the EM field with a remarkable amount of symmetry and internal structure described above. Naturally, the situation becomes considerably simpler in the presence of two commuting Killing vectors as in the frequently studied case of stationary axisymmetric fields. Most examples described in Section 3 belong to this class. The
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Table 2. Kinnersley transformations: $a, b, c \in C, \alpha, \beta \in R$

| I: Electromagnetic gauge | $\mathcal{E} \rightarrow \mathcal{E} - 2a^*\Phi - aa^*$, $\Phi \rightarrow \Phi + a$, $f \rightarrow f$, $\mathbf{x} \rightarrow \mathbf{x}$ (5) |
|-------------------------|---------------------------------------------------------------|
| II: Gravitational gauge  | $\mathcal{E} \rightarrow \mathcal{E} + i\alpha$, $\Phi \rightarrow \Phi$, $f \rightarrow f$, $\mathbf{x} \rightarrow \mathbf{x}$ (6) |
| III:                    | $|b| = 1$ duality rotation, $|b| \neq 1$ scaling or conformal transf. |
|                         | $\mathcal{E} \rightarrow (bb^*)^{-1} \mathcal{E}$, $\Phi \rightarrow (b^*b^{-2}) \Phi$, $ds^2 \rightarrow (bb^*)^{-1} ds^2$ (7) |
| IV: Ehlers transformation| $\mathcal{E} \rightarrow \frac{\mathcal{E}}{1 + i\beta \mathcal{E}}$, $\Phi \rightarrow \frac{\Phi}{1 + i\beta \mathcal{E}}$ (8) |
| V: Harrison transformation| $\mathcal{E} \rightarrow \frac{\mathcal{E}}{1 - 2c^*\Phi - cc^* \mathcal{E}}$, $\Phi \rightarrow \frac{\Phi + c\mathcal{E}}{1 - 2c^*\Phi - cc^* \mathcal{E}}$ (9) |

Effectiveness of the complex potential approach was systematically demonstrated in a series of papers by Hauser and Ernst and completed by their proof of a generalized Geroch conjecture [18]. Thus, all vacuum spinning mass solutions could be generated from any one such solution (even Minkowski space) by means of an infinite sequence of transformations associated with the internal symmetries and with the choice of the basic Killing vector fields.

3 Examples of Kinnersley’s transformations

In this section we demonstrate, that some EM fields we derived via HM generating conjecture represent either Harrison transformation or a combination of Kinnersley’s transformations (5)–(9) of the corresponding seed metrics. We are going to concentrate on electro- and magnetovacuum solutions obtained from seed
axially symmetric vacuum gravitational fields [8, 9]. For the static EM fields the twist potential $\psi$ in (2) equals zero and the potential $E$ takes real values only (see Table 1). According to (3) the Ernst potentials read as

$$E_{\text{seed}} = f_{\text{seed}}, \quad \Phi_{\text{seed}} = 0,$$

for the seed metrics and

$$E_{\text{charged}} = f_{\text{charged}} - |\Phi_{\text{charged}}|^2, \quad \Phi_{\text{charged}} \neq 0,$$

for the charged solutions provided they remain static. Moreover, for studied electrovacuum solutions with a timelike vector potential we can just put

$$\Phi = A_t.$$

Despite of relative simplicity, the Weyl solutions class includes many astrophysically interesting solutions that might be relevant e.g. for the description of gravitating discs around black holes [19, 20].

All the solutions found by the authors employing HM generating conjecture revealed that for a particular class of charged solutions the metric coefficients were modified in the same way, no matter whether we start with the Levi-Civita [8] or the Darmois-Vorhees-Zipoy (also $\gamma$) [9] metrics. Let us show, that this modification represents the Harrison transformation (9) in fact.

Let us start with the $\gamma$-metric with an electric field, the line element and four-potential of which in the Weyl-Lewis-Papapetrou cylindrical coordinates read as

$$ds^2 = -\frac{f_1(r, z)}{f(r, z)}dt^2 + \frac{f(r, z)^2}{f_1(r, z)} \left[ f_2(r, z) \left( dr^2 + dz^2 \right) + r^2 d\varphi^2 \right],$$

$$A = q \frac{f_1(r, z)}{f(r, z)} dt,$$

where

$$f(r, z) = 1 - q^2 f_1(r, z),$$

$$f_1(r, z) = \left( \frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m} \right)^{\gamma}, \quad f_2(r, z) = \left[ \frac{(R_1 + R_2 - 2m)(R_1 + R_2 + 2m)}{4R_1R_2} \right]^{\gamma^2},$$

$$R_1 = \sqrt{r^2 + (z - m)^2}, \quad R_2 = \sqrt{r^2 + (z + m)^2}.$$

Setting $q = 0$ we obtain the seed $\gamma$-metric with $E_{\text{seed}} = f_1(r, z)$. Extracting the Ernst potentials according to (10) and (12) one come to

$$E = \frac{f_1(r, z)}{f(r, z)} = \frac{f_1(r, z)}{1 - q^2 f_1(r, z)}, \quad \Phi = q E = \frac{q f_1(r, z)}{1 - q^2 f_1(r, z)},$$

which coincides with (9) for a real parameter $c = q$. 

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In case of solutions with magnetic field we cannot just use the relation \( \Phi \) for \( f \), moreover, from the Table \( \text{I} \) we know, that \( \Phi \) takes imaginary values. Rewriting the magnetovacuum metric line element

\[
\begin{align*}
\mathrm{d}s^2 = & -f(r,z)^2f_1(r,z)\mathrm{d}t^2 + \\
& + \frac{1}{f_1(r,z)} \left[ f(r,z)^2f_2(r,z) \left( \mathrm{d}r^2 + \mathrm{d}z^2 \right) + \frac{r^2}{f_1(r,z)^2} \mathrm{d}\varphi^2 \right]
\end{align*}
\]

\hspace{1cm} (14)

into form

\[
\begin{align*}
\mathrm{d}s^2 = & \frac{r^2}{f(r,z)f_1(r,z)} \mathrm{d}\varphi^2 + f(r,z)^2f_1(r,z) \left[ \frac{f_2(r,z)}{f_1(r,z)^2} \left( \mathrm{d}r^2 + \mathrm{d}z^2 \right) - \mathrm{d}t^2 \right]
\end{align*}
\]

with

\[ A = \frac{q}{f(r,z)f_1(r,z)} \mathrm{d}\varphi, \quad f(r,z) = 1 + q^2 \frac{r^2}{f_1(r,z)}, \]

we realize, that a formal transformation interchanging \( t \) and \( \phi \) coordinates makes the situation mathematically equivalent to the electrovacuum solution \( \text{II} \) with \( \mathcal{E}_{\text{seed}} = -r^2/f_1(r,z) \). Indeed, the both Ernst potentials of the “charged” solution

\[
\begin{align*}
\mathcal{E} = & -\frac{r^2}{f_1(r,z)} \frac{1}{1 - q^2}, \\
\Phi = & -\frac{q}{f_1(r,z)} \frac{r^2}{1 - q^2} \left[ 1 - \frac{r^2}{f_1(r,z)} \right]
\end{align*}
\]

again fulfil the Harrison transformation equations \( \text{II} \).

The same could be gradually accomplished for all the solutions generated from the Levi-Civita seed metric in \( \text{II} \) – they represent its Harrison transformation \( \text{II} \). Naturally, one would like to check the spacetimes studied in \( \text{II} \) to support the HM conjecture. The first example – the charged Schwarzschild and Kerr, i.e. the Reissner-Nordström and Kerr-Newman solutions respectively – has been proved to be a combination of Kinnersley’s transformations in fact by Ernst \( \text{II} \). The second example in \( \text{II} \) – the charged Taub solution

\[
\begin{align*}
\mathrm{d}s^2 = & -\frac{a + bx}{x^2} \mathrm{d}t^2 + \frac{x^2}{a + bx} \mathrm{d}x^2 + x^2 \left[ \mathrm{d}y^2 + \mathrm{d}z^2 \right], \\
A \sim & \frac{b}{x} \mathrm{d}t
\end{align*}
\]

\hspace{1cm} (15)

reduces to the Taub solution for \( a = 0 \). Thus, according to \( \text{II} \) \( \mathcal{E}_{\text{seed}} = b/x \). The Harrison transformation \( \text{II} \) then gives

\[
\begin{align*}
\Phi = & \frac{cb/x}{1 - c^2b/x}, \\
\mathcal{E} = & \frac{b/x}{1 - c^2b/x}, \\
f = & \text{Re} \mathcal{E} + |\Phi|^2 = \frac{bx}{(x - bc^2)^2}.
\end{align*}
\]

Introducing a new spacelike coordinate \( \xi \) by the relations

\[ x = \xi + bc^2, \quad a = b^2 c^2 \]
we get the metric (15).

We have explicitly demonstrated that the HM conjecture in case of some static fields provides results equivalent to Harrison transformation. The common issues of both methods are discussed in the following section.

4 The generating conjecture in a new context

Having been proposed in [2], the HM conjecture was reformulated to meet more suitably practical generation of new EM fields. Originally, the conjecture says, that having a seed vacuum gravitational field with at least one Killing vector, then it makes sense to search for an EM spacetime for which the fourpotential of the electromagnetic field is proportional (up to a constant factor) to the Killing covector of the seed vacuum metric. When the parameter connected with the electromagnetic field of the self-consistent problem is set equal to zero, one comes back to the seed solution. After several successful application of the conjecture it was generalized by Cataldo et al. [4] in the sense, that the electromagnetic fourpotential need not be just a constant multiple of the seed metric Killing covector, but that it is possible to multiply by a suitable function. Finally, the conjecture was also used in a few cases when the seed spacetimes were non-vacuum solutions of the Einstein equations [3, 5]. Thus the key idea is that the electromagnetic field tensor \( \mathcal{F} \) is in some sense connected with so called Papapetrou fields [21] – exterior derivatives of corresponding Killing fields.

The conjecture does not specify, in what way the metric tensor of the EM field is modified in comparison with the seed vacuum spacetime, thus it does not provide an exact algorithm, how to generate charged solutions from the seed ones. Unfortunately, in many cases it is extremely difficult (if not even impossible) to solve EM equations without any additional condition, even if we set the electromagnetic fourpotential in accordance with the conjecture.

There is only one condition required by the conjecture: the existence of a Killing vector field. The conjecture does not impose any restriction, whether the Killing vector should be timelike, spacelike, null or whether some Killing vectors should be excluded. The possible connection of the electromagnetic field with spacetime symmetries described by the Killing vectors is also not closely specified.

On the other hand, we have demonstrated in Section 3 that some classes of the EM fields found by means of the conjecture are in fact examples of Kinnersley’s transformations. Similarly, the usage of these transformations demands an existence of a non-null Killing vector. Moreover, the charging Harrison transformation prescribes exactly, how to modify Ernst potentials of the seed metric and thus provides generating algorithm of consequent calculations. And finally, the Harrison transformation is also connected with the concept of symmetry: a non-null Killing vector is needed for the 8-parameter group of transformations described in Section 2.

Conversely, for all classes of vacuum space time admitting a non-null Killing vector the Harrison transformation ensures an existence of a correspondent charged
EM field and supplies a procedure, how to construct it. From this point of view for all solutions generated from the seed Weyl metrics in [8, 9] the HM conjecture in its generalized formulation [4] necessarily had to work and provide new EM fields. Of course, the Harrison transformation is really simple for static metrics in Section 3, where we do not need to take into account the zero twist potential $\psi$ in (3). Anyway, the Kinnersley transformations (5)–(9) explain the validity of the HM conjecture for a wide class of seed metrics.

5 Conclusions

The connection between the HM conjecture and inner symmetries of EM fields described in special case by Kinnersley’s transformations as suggested in the preceding section might give a more solid theoretical background to the conjecture and could lead to its more precise formulation or even explanation. Let us remind, the HM conjecture in connection with complex potentials has been considered by Stephani [22] who explored the original formulation with a fourpotential being a constant multiple of a corresponding Killing vector. He has proved the HM conjecture for some class of EM fields admitting a diverging, geodesic and shearfree null congruence and with a non-radiative Maxwell field.

The possible connection of the HM conjecture with inner symmetries of the EM equations proposed above would connect the conjecture with a set of generating methods elaborated by Ernst and other authors (see e.g. [23,13] and references cited therein) for axisymmetric fields. These methods are based on the solution of the homogeneous Hilbert problem for the axes-accessible Einstein equations (solutions with singularities along the whole axis such as Levi-Civita’s one are excluded) and it was proved [18], that these vacuum fields are deducible through the action of a huge group with infinitesimal generators. It turns out that the axis values of $E$ contain enough information to construct Ernst potentials at off-axis points, the axis mass distribution, angular momentum, electric and magnetic charge distributions. Such axis relation provides a way to identify a corresponding Kinnersley-Chitre transformation to generate a spacetime with prescribed $E$ potential from Minkowski space via this Geroch group. The application of this group covers physically interesting problems such as derivation of the Kerr metric, spinning-mass solutions of arbitrary complexity, the cylindrical gravitational wave and the colliding plane gravitational waves solution. Moreover, the proposed connection of the HM conjecture with Kinnersley’s transformations would suit Stephani’s demand [22] on its invariant formulation.

Naturally, there still remain open problems. It is necessary to check other EM fields generated through the conjecture, especially those with non-static or non-vacuum seed metrics. The Kinnersley transformations does not support the HM conjecture employing null Killing vectors.

We believe that the HM conjecture reflects some hidden principles. The Kinnersley’s transformation may represent a right clue to its better understanding.
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