Anti-parity-time topologically undefined state

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Abstract

Researches on the topological edge state in the photonic lattice are attracting considerable attention. Here, we report the studies on a particular state for which the topological invariant is undefined. We constructed an anti-parity-time-symmetric photonic lattice by using the perturbation method. Light distributes only in the wide waveguides with equal magnitude for the state with undefined winding numbers. Further studies show that the equal intensity transmission is unaffected except for the defect site. Our work provides a new way to study the topological state and the equally divided light transmission and might be applicable in optical circuits and optical interconnect.

1. Introduction

Topological photonics, as one of the most remarkable topics, is inspired from the discovery of the quantum Hall effects and the topological insulators in the condensed matter [1, 2]. In recent years, a variety of discrete photonic systems have been proved to realize the topological phase [3], such as the Floquet topological insulators with spatial modulation based on the array of laser-written optical waveguides [4] and a network of surface plasmon rings [5], topological photonics in meta-waveguides [6] and topological insulators [7, 8]. Based on these studies, a large number of phenomena have emerged, such as the topologically protected edge states and corner states in photonic crystals [9], slow light [10, 11], nonlinear optical isolation [12]. Clearly, a common feature of these studies is that the systems must be Hermitian, which means the topological invariant of a system with open boundary condition could be judged by the bulk-boundary correspondence [13, 14].

In the non-Hermitian systems where there exists energy exchange with the ambient environment, the fractional winding number could be observed, and the usual bulk-boundary correspondence is invalid [15, 16]. The global Berry phase [17], the non-Bloch winding number [18, 19], and the Kronecker index [20] could be employed to calculate the winding numbers [21]. These findings promote the development of topology research in the non-Hermitian system [3, 22–26]. The propagation of light in these systems will be affected by external conditions, leading to many novel phenomena, such as lasering generation in topological edge states of a polariton micropillar array [3], Fermi nodal disk in magnetic plasma [22], skin effects [23, 24], boundary modes [24], exceptional non-Hermitian topological edge mode [25], and anomalous topological edge state [26] and so on.

However, there exists a particular state with an undefined topological invariant in all of these studies. For the two-band model, once the closed parametric curve of the coefficient vector $d(k)$ of bulk momentum-space Hamiltonian $\hat{H}(k)$ is tangent to the origin, the winding number is undefined [27], which means that such a topological undefined state is a critical state between the topological non-trivial and...
topological trivial ones. But in recent years, most of the studies are focused on the non-trivial states while the researches on the topological undefined states are still absent.

In this letter, we propose a one-dimensional anti-parity-time-symmetric (anti-PT) photonic lattice with perturbations. By eliminating the amplitudes of the perturbation sites adiabatically, we obtained an effective Hamiltonian with anti-PT symmetry. The system is proved theoretically to be topologically non-trivial when the coupling constants $\kappa_1 < \kappa_2$; while there exists a topologically undefined state when $\kappa_1 = \kappa_2$. It is also found that the introduction of the defect will strengthen the edge state for the topologically non-trivial case while it will not affect the equal intensity transmission for the topologically undefined state. Our study enriches the physics of topological photonics and paves a new way to achieve the topological edge state and undefined state, which might have potential applications in the fields of optical circuits and optical communication.

2. Model and analysis

We construct a one-dimensional photonic lattice with perturbations, as shown in figure 1(a). $\rho_n$, $a_n$, and $b_n$ are the light amplitudes of the perturbation site, ordinary waveguides $A_n$ and $B_n$, respectively. Assuming that the refractive index of the perturbation site is much smaller than those on sites $A_n$ and $B_n$, $\rho_n \ll \rho_{a,b}$, therefore the modal amplitudes in the ordinary waveguides are more substantial than in the perturbations [31, 32]. Accordingly, the ordinary waveguide will not be affected by the next neighbor perturbation. Under the above conditions, we can obtain the Hamiltonian of the system, $H'$, which is described as

$$H' = \begin{pmatrix}
\beta_a & \epsilon & \delta & 0 \\
\epsilon & \beta_b & 0 & \delta \\
\delta & \epsilon & \beta_{\rho,1} & 0 \\
\epsilon & \delta & 0 & \beta_{\rho,2}
\end{pmatrix},$$

(1)

where $\delta$ and $\epsilon$ are the nearest and next-nearest coupling constants between the perturbations and the ordinary waveguides. $\epsilon$ is the coupling constant between ordinary waveguides. $\beta$ is the propagation constant. One can describe the evolution of this system by the Schrödinger-like equation [33] $i\hbar/dt = H'\psi'$ and the wave functions of $\psi' = [a_n, b_n, \rho_1, \rho_2]^{T}$.

To simplify the equations, one can eliminate the amplitudes of perturbation sites $\rho_n$ adiabatically [34],

$$\rho_1 \approx -a_n \delta - b_n \epsilon, \quad \rho_2 \approx -b_n \delta - a_n \epsilon.$$

(2)

After setting the propagation constants of the perturbation sites as $\beta_{\rho,1} = \delta \epsilon/(c - \kappa)$, $\beta_{\rho,2} = \delta \epsilon/(c + \kappa)$ and those for the ordinary waveguides as $\beta_a = (c \delta + \Delta \epsilon - \beta \kappa)/\epsilon$ and $\beta_b = (c \delta - \Delta \epsilon + \beta \kappa)/\epsilon$, one can obtain the effective Hamiltonian

$$H = \begin{pmatrix}
\Delta & \kappa \\
-\kappa & -\Delta
\end{pmatrix},$$

(3)

where $\Delta$ and $\kappa$ are the corresponding effective propagation constant and coupling constant, respectively. The simplified system is shown in figure 1(b). For this case, a wide waveguide $A_n$ (red circle) and a narrow waveguide $B_n$ (blue circle) consist a unit cell. The effective Hamiltonian satisfies $PTH + HPT = 0$, where $P = \sigma_x$, and $T = IK$ are the parity and time operators, respectively. $\sigma_{x,y}$ are the Pauli matrices, $I$ is the identity operator, $K$ is the elementwise complex conjugation in the effective Hamiltonian. Apparently, the

Figure 1. Schematic of photonic lattices with perturbations. (a) Perturbations used to realize anti-PT symmetry. (b) Effective model. (c) Periodic anti-PT structure. $\rho_n$, the perturbation sites (the gray sites); $A_n$ (the red sites) and $B_n$ (the blue sites) are wide and narrow waveguides, respectively. $\kappa_{1,2} > 0$ are the coupling constants.
Figure 2. Bulk band structure with different $\kappa$. Here $\kappa_1 = 1$, $\Delta = 2$.

effective Hamiltonian of such a system satisfies anti-PT symmetry but is not chirally symmetric, $\Gamma H^{-1} \neq -H$, where $\Gamma = \sigma_z$.

As shown in the aforementioned analysis, we have proposed an anti-PT symmetric optical coupler. Based on this, we employ such a configuration to construct a topological Su–Schrieffer–Heeger (SSH) model [35, 36] (see figure 1(c)). The Bloch Hamiltonian of such a periodic system is

$$H = -id_x \sigma_x + id_y \sigma_y + \Delta \sigma_z,$$

$$d_x = \kappa_2 \sin k,$n
$$d_y = \kappa_1 + \kappa_2 \cos k,$n

(4)

where $\kappa_1$ and $\kappa_2$ are the coupling constants inside and outside one cell. One can learn that the Hamiltonian satisfies $\{ H, \tau \} = 0$, where $\tau = \sigma_x K$, in other words, the system has non-Hermitian particle–hole symmetry, which guarantees that the eigenvalues always appear in pair $\pm E(k)$. The corresponding eigenvalues of the bulk Hamiltonian read

$$E(k) = \pm \sqrt{\Delta^2 - d_x^2 - d_y^2},$$

(5)

The eigenvalues of the bulk Hamiltonian are demonstrated in figure 2, one can notice that the two bands are closing to each other with the increasing $\kappa_1$, and a Dirac point is formed at $k = 0$ when $\kappa_1 = 1$ (see figure 2(b1)). It is also shown in the inset in figure 2(b2) that the circle is tangent to the origin in the inset, this indicates that the winding number of the system is undefined. Further increasing the magnitude of the inner coupling constant $\kappa_1$, a flatband starts to emerge in the center of the Brillouin zone as a result of the protection of non-Hermitian particle–hole symmetry (see figure 2(c1)). Such a flatband covers a region of $-\sqrt{d_x^2 + d_y^2} < \Delta < \sqrt{d_x^2 + d_y^2}$ in which $E_1$ and $E_2$ are also known as the exceptional points [37, 38]. According to the definition of winding number, $W = \frac{1}{2} \int_{0}^{2\pi} dk \left< \psi_+ | \partial_k | \psi_- \right>$, we further calculated the winding number $W$ of such a system, where $|\psi_\pm \rangle$ is the eigenfunction of the Bloch Hamiltonian $H$ [35]. It is shown $W = 1$ corresponding to a topological non-trivial state when $\kappa_1 < \kappa_2$, and $W = \frac{1}{2}$ corresponding to an undefined state when $\kappa_1 = \kappa_2$.

The open boundary spectra are given in figure 3. Two non-zero edge modes occur when $\kappa_2 - \Delta < \kappa_1 < \Delta - \kappa_2$ in figures 3(a) and (b), suggesting the ordinary bulk-boundary correspondence is valid in our non-Hermitian anti-PT symmetric system. As shown in figure 3(c), two new non-zero defect modes form (the red lines between $m_1$ and $m_2$, $m'_1$ and $m'_2$) when a defect is introduced in the left-most edge by replacing the coupling constant $\kappa_1$ with $\kappa_1 + \kappa_d$ to test the robustness of the system. Clearly, the
Figure 3. Band structure as a function of $\kappa_1$ in the SSH model with open boundary. Here, the total number of the waveguide is $N = 30, \kappa_2 = 1, \kappa_3 = 0.5, \Delta = 2$. (a) and (b) are the real and imaginary parts of eigenvalues. (c) and (d) are spectra with a defect at the left-most site.

defect modes do not break the topological edge modes and the region of $\kappa_1$ is also unchanged (figure 3(d)), proving that the edge modes are robust to the defect due to topological protection.

To further investigate the energy distribution for the topological state, the stationary Schrödinger-like equation

\[
\begin{align*}
(E - \Delta) a_n &= \kappa_1 b_n + \kappa_2 b_{n-1}, \\
(E + \Delta) b_n &= -\kappa_1 a_n - \kappa_2 a_{n+1}.
\end{align*}
\]  

(6)
is used. As the band analysis predicts that the left edge mode has an eigenvalue of $E_{\text{left}} = \Delta$, with the boundary conditions

\[
\begin{align*}
(E - \Delta) a_0 &= \kappa_1 b_0, \\
(E + \Delta) b_0 &= -\kappa_1 a_0 - \kappa_2 a_1.
\end{align*}
\]  

(7)

One can learn that

\[
|a_n|^2 = \left( \frac{\kappa_1}{\kappa_2} \right)^2 |a_0|^2.
\]  

(8)

This indicates that the left incidence $a_0$ can induce a particular energy distribution $a_n$ and the energy only exists at the wide waveguide, site $A_n$. Peculiarly, a stable edge mode forms when $\kappa_1 < \kappa_2$, which agrees with the topological edge state. When $\kappa_1 = \kappa_2$, it is seen from equation (8) $|a_n|^2 = |a_0|^2$, indicating that the light energy distributes equally in all the wide waveguides $A_n$ for the case with undefined winding number (see figure 2(b2)). Figuratively speaking, the topological state has been ‘copied’ to all the waveguides $A_n$ for this special state.

It can be seen from the previous chapter that when there is a defect at the left-most incident waveguide, it may have a significant impact on the topological state and the undefined state. Therefore, in order to clearly understand the effect of the defect on the system, we further consider the defect in the first cell, one can get the following relations

\[
\begin{align*}
(E - \Delta) a_0 &= (\kappa_1 + \kappa_3) b_0, \\
(E + \Delta) b_0 &= -(\kappa_1 - \kappa_3) a_0 - \kappa_2 a_1.
\end{align*}
\]  

(9)

and obtain similar results with defects $\kappa_3$,

\[
|a_n|^2 = \left( \frac{\kappa_1}{\kappa_2} \right)^2 \left( \frac{\kappa_1 - \kappa_3}{\kappa_1} \right)^2 |a_0|^2.
\]  

(10)
Figure 4. Numerical calculations of output energy distribution in both (a) topological nontrivial ($\kappa_1 = 0.5$) and (b) undefined cases ($\kappa_1 = 1$). (c) and (d) add the defect at the edge with different $\kappa_1$ and same $\kappa_\delta$.

Figure 5. Numerical simulation of light propagation incident from the left-most waveguide $A_0$.

Considering the modulated coupling constants should not break the lattice symmetry (see figure 1(c)), the coefficient term of $a_0$ on the right-hand side in equation (9) should satisfy $(\kappa_1 - \kappa_\delta) \geq 0$. Therefore the defect $0 \leq \kappa_\delta \leq \kappa_1$ is deduced. Since the imaginary part of the energy spectrum is zero for both the topological state (figures 2(a1) and (a2)) and undefined state (figures 2(b1) and (b2)), it is easy to find from equation (10) that $|a_n|^2 \leq |a_0|^2$, indicating that the strongest light energy still distributes in the first waveguide $A_0$ and the light field in other waveguides $A_n$ can be controlled by adjusting the coupling constants $\kappa_1$, $\kappa_2$ and $\kappa_\delta$.

To see the characteristics of the system more vividly, we calculated the output energy distribution both numerically and theoretically. As shown in figure 4, the energy will only reside on the sites $A_n$ when the left edge mode is excited for $\kappa_1 < \kappa_2$ (see figure 4(a)). The edge state will be strengthened when the defect is introduced $\kappa_\delta = 0.5$, namely, the system is topologically nontrivial (see figure 4(c)). Interestingly, it is found that all the light intensity has the same magnitudes when $\kappa_1 = \kappa_2$ and $\kappa_\delta = 0$, for which the winding number of the system is undefined (see the inset in figure 2(b2)). When $\kappa_1 = \kappa_2$ and $\kappa_\delta = 0.5$, the equal intensity distribution keeps except for the first waveguide $A_0$, this means that this kind of undefined state is also robust and be immune to the defect (figure 4(d)).
Figure 5 shows the numerical simulation of light propagation in our system. Most of the light energy will be localized on waveguide $A_0$ and propagates along with it for an incident beam with a function of $e^{-\omega_0/\kappa} (\omega_0 = 0.1)$ when $\kappa_1 < \kappa_2$ (see figure 5(a)). When $\kappa_1 = \kappa_2 = 1$, the light will be coupled to the next nearest waveguides and finally propagates in all the sites $A_n (n \in [0, 14])$ with equal intensities (see figure 5(b)). Contrastly, light wave is localized completely at $A_0$ when the defect is introduced ($\kappa_3 = 0.5$) (see figure 5(c)); this kind of strong localization could be attributed to the coaction of topological edge state and the defect mode as shown in figures 3(c) and 4(c). For the case of $\kappa_1 = 1$ and $\kappa_2 = 0.5$ (figure 5(d)), the light will also distribute evenly in all the $A_n$ waveguides except that the first waveguide has larger energy due to the defect. Apparently, the light propagation is in good agreement with the former theoretical and numerical results.

An applicable experimental proposal could be performed with the setup shown in reference [39, 40]. A femtosecond laser pulse $@1064$ nm is focused through a micro-objective lens with 20 times magnification. The wide/site $A_n$ and narrow/site $B_n$ waveguides could be induced in the fused silica glass by moving the optical platform relatively. The width and coupling coefficient of the waveguide can be modulated by controlling the moving velocity and the power of the femtosecond laser. A $632.8$ nm He–Ne laser can be used as the excitation light source to probe the light dynamics.

3. Conclusions and discussion

In summary, we present an anti-PT symmetric topological photonic lattice with perturbations. The topological edge state and undefined state can both be achieved in such an anti-PT photonic system by tuning the coupling constant. When the coupling constant $\kappa_1 < \kappa_2$ most light transmits at the edge site $A_0$, forming a topological state; when $\kappa_1 = \kappa_2$ the light distributes equally on all the wide waveguides $A_n$, corresponding to a winding number undefined state. Further investigation on the impaction of defect shows it might be helpful for the formation of a more localized edge state when the defect is positive, and the topologically undefined state is robust to the defect except for the incident waveguide. Thus, our work opens a gap in the in-depth study of the topologically undefined state, which provides a practical way for light beam equal-splitter and may offer important applications in the fields of optical circuits and optical communications in the future.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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