Complete subdomain model for radial-flux slotted PM machines with toroidal windings accounting for the iron-part

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Abstract. Most subdomain models do not solve the magnetic vector potential in the iron part of the machine, this paper proposes a complete subdomain model for (PM) machines with toroidal windings and mainly solves the magnetic vector potential in iron part. The proposed model can accurately predict the field distribution under load or no-load conditions. The electromagnetic parameters of the motor (including magnetic flux density, back-emf, electromagnetic torque, and cogging torque) are calculated using the proposed analytical model, and validated by a FEM model. This paper presents a technique to handle with the stator of machines with outer teeth and the very specific techniques to solve the magnetic field in the iron part.

1. Introduction
Permanent magnet machines with toroidal windings are widely used in high speed applications [1]. High speed permanent magnet machines (HSPMMs) are usually designed with 2 poles to minimize the electrical frequency at a given rotational speed. Toroidal windings can shorten the machine's end windings significantly, which in turn, will reduce overall copper loss and enhance machines’ stiffness [2]. Most toroidal windings are coiled on slotless stators to reduce the eddy current loss of the rotor and to smooth the dynamic performance [3], while other applications choose multi-slot structure to promote output power [4] and to enhance heat dissipation [5].

Accuracy and efficiency of a mathematical model for an electrical machine often takes responsibility for the quality of its design, analysis, and optimization. Methodologies for a mathematical model can be roughly categorized as numerical method and analytical approach. Numerical methods have better accuracy and are capable of handling machines with complex geometry, non-homogeneous and non-isotropic materials, and saturation effect [6], while analytical approaches owns better efficiency [7] and give more physical insights. Therefore, an initial design is usually assisted by analytical model and refined by numerical method [8].

Subdomain technique is generally applied to model a slotted machine [9], the subdomain technique divides the solution area into several subdomains, the differential equation is solved in each subdomains and linked by boundary conditions. Most subdomain models do not count in the iron part of the machine, for it needs technique to handle the Laplace or Poisson equations with non-homogeneous Dirichlet or Neumann boundary conditions (BCs).

This paper presents a complementary subdomain model for HSPMMs, which mainly focuses on the analytical solution for the iron part of the machines. The model can accurately predict the magnetic field distribution under load or no load conditions. Electromagnetic parameters such as flux linkage, back-emf, electromagnetic torque, and cogging torque are calculated respectively. All the results are validated by a Finite Element Method (FEM) model.
2. Analytical model

2.1. Geometry and assumptions
A typical toroidally-wound multi-slot HSPMM is shown in Fig.1. With the case of a toroidally wound topology, the outer windings are not accounted for the effective conductors, and the main field barely passes the outer teeth due to magnetic short circuit phenomenon, based on these two facts, the computational regions are confined within the dashed area as shown in the left part of Fig.2. The whole region is divided into nine subdomains as depicted in the left part of Fig.2, Namely, the rotor shaft subdomain, the PM-ring subdomain, the shielding cylinder subdomain, the air-gap subdomain, the slot-opening subdomain, the slot subdomain, the addendum subdomain, the dedendum subdomain, and the tooth-yoke subdomain. Note that the relative permeability of the shielding cylinder and the air-gap is nearly the same, so the air-gap subdomain contains the shielding cylinder and the air-gap in this paper.

\[
\theta_q = -\frac{\beta}{2} + \frac{2\pi}{Q} (q-1), \text{ with } q = 1, 2, \ldots, Q \tag{1}
\]

To make the analytical solution available and concise, followings assumptions are made,
1) The relative permeability of the stator iron and the rotor shaft are assumed to be infinite, meanwhile, both parts do not saturate.
2) The relative permeability of all other parts (the winding, the shielding cylinder and the magnets) are assumed to be unit.
3) No end effects, so the current density vector has only z-axis component and magnetic flux
density has no \( z \)-axis component.

4) No eddy currents in all parts.

Based on these assumptions, a simplified 2-D analytical model can be established for the 3-D machine. The solution area is divided into two main parts as depicted in the right part of Fig.2, namely, non-iron part (including the PM-ring subdomain, the air-gap subdomain, the \( q \)-th slot-opening subdomain, and the \( q \)-th slot subdomain) and iron part (including the rotor shaft subdomain, the addendum subdomain, the dedendum subdomain, and the tooth-yoke subdomain).

2.2. General solution of magnetic vector potential distribution in non-iron part

Since the solution of the magnetic vector potential of the machines’ non-iron part has been studied thoroughly in [7], [9]–[11], the derivation would just be rough for this part. The control equations for each subdomain in non-iron part can be characterized as

\[
\begin{align*}
\Delta A_s^\phi & = 0 & \text{in regions of air-gap and slot-opening} \\
\Delta A_s^z & = -\nabla \times B_{rem}^z & \text{in region of PM-ring} \\
\Delta A_s^z & = -\mu_0 J_{eq}^z & \text{in region of slots}
\end{align*}
\]

(2)

where, \( A_s^\phi \) is the magnetic vector potential in domain \( s \), \( \mu_0 \) is relative permeability of air, \( J_{eq}^z \) is \( z \)-axis component of the surface current density in slots, \( B_{rem}^z \) is remanence in domain \( s \). Combined with the BCs in each region, the solutions are listed as Eq.\((A.2)\) to Eq.\((A.5)\) in appendix, the very detailed solution process is not presented in this paper for it has been developed very particularly in [10], [12].

2.3. General solution of magnetic vector potential distribution in iron part

The iron part of the machine is divided into four subdomains, which contains the rotor shaft subdomain \( (s = \text{shaft}) \), the \( q \)-th addendum subdomain \( (s = 5qt) \), the \( q \)-th dedendum subdomain \( (s = 6qt) \), and the tooth-yoke subdomain \( (s = so) \). Superposition principle technique to solve the Laplace equations with nonhomogeneous Dirichlet and Neumann BCs are introduced in this part specifically.

For solving magnetic vector of the iron part, it is necessary to obtain the results of the non-iron part primarily, for these results involves the BCs for equations of the iron part. All the coefficients involved in this part are developed in appendix from Eq.\((A.6)\) to Eq.\((A.31)\).

2.3.1. Rotor shaft subdomain \( (s = \text{shaft}) \)

The magnetic vector potential of the rotor shaft subdomain yield

\[
\frac{\partial^2 A_{\text{shaft}}^\phi}{\partial r^2} + \frac{1}{r} \frac{\partial A_{\text{shaft}}^\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_{\text{shaft}}^\phi}{\partial \phi^2} = 0
\]

(3)

on surface \( R_1 \), the continuity of magnetic vector potential can be applied as the BC,

\[
A_{\text{shaft}}^\phi (R_1, \phi) = A_{\text{em}}^\phi (R_1, \phi)
\]

(4)

The general solution of Eq.\((3)\) accounting Eq.\((4)\) can be written as Eq.\((5)\), where \( A_n^{(1)} \) and \( B_n^{(1)} \) can be obtained by applying Fourier series.

\[
A_{\text{shaft}}^\phi (r, \phi) = \sum_{n=1}^{\infty} \left( \frac{r}{R_1} \right)^n \left( A_{n}^{(1)} \cos n\phi + B_{n}^{(1)} \sin n\phi \right)
\]

(5)

\[
A_n^{(1)} = \frac{1}{\pi} \int_0^{2\pi} A_{n}^{\text{em}} (R_1, \phi) \cos n\phi \cdot d\phi
\]

(6)

\[
B_n^{(1)} = \frac{1}{\pi} \int_0^{2\pi} A_{n}^{\text{em}} (R_1, \phi) \sin n\phi \cdot d\phi
\]

(7)

Finally, the magnetic vector potential for the rotor shaft subdomain can be written as \((8)\), where the coefficients \( A_{n}^{(2)} \) and \( B_{n}^{(2)} \) are exactly the coefficients of those in the PM-ring subdomain defined in \((A.1)\).
\[ A_{\phi}^{a} (r, \phi) = -B_{m} \frac{E(R_{s}, R_{i}, 1)}{P(R_{s}, R_{i}, 1)} r \cos \phi + \sum_{n=0}^{\infty} \left( \frac{r}{R_{s}} \right)^{n} \left( A_{n}^{(2)} \frac{2}{P(R_{s}, R_{i}, n)} \cos n\phi + B_{n}^{(2)} \frac{2}{P(R_{s}, R_{i}, n)} \sin n\phi \right) \]  

\[ (8) \]

2.3.2. \( q \)-th addendum subdomain \( (s = 5q) \)

The \( q \)-th addendum subdomain is restricted by azimuthal range of \( \theta_{q} + \beta \) to \( \theta_{q+1} \), and radial range of \( R_{4} \) to \( R_{5} \) as shown in Fig.3(a). The magnetic vector potential of the rotor shaft subdomain yields

\[ \frac{\partial^2 A_{\phi}^{q}}{\partial r^2} + \frac{1}{r} \frac{\partial A_{\phi}^{q}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_{\phi}^{q}}{\partial \phi^2} = 0 \]

\[ (9) \]

As depicted distinctly in Fig.3, the \( q \)-th addendum subdomain possesses non-homogeneous Dirichlet BCs, which are listed as

\[ A_{\phi}^{q} (r, \theta_{q} + \beta) = A_{\phi}^{q} (r, \theta_{q} + \beta) \]

\[ A_{\phi}^{q} (r, \theta_{q+1}) = A_{\phi}^{q} (r, \theta_{q+1}) \]

\[ A_{\phi}^{q} (R_{s}, \phi) = A_{\phi}^{q} (R_{s}, \phi) \]

\[ A_{\phi}^{q} (R_{s}, \phi) = f^{\phi q} (R_{s}, \phi) = \begin{cases} A_{\phi}^{q} (R_{s}, \phi) & \forall \phi \in [\theta_{q} + \beta, \theta_{q+1}] \\ A_{\phi}^{q} (R_{s}, \phi) & \forall \phi \in [\theta_{q+1}, \theta_{q+1}] \\ A_{\phi}^{q} (R_{s}, \phi) & \forall \phi \in [\theta_{q+1}, \theta_{q+1}] \end{cases} \]

\[ (10) \]

where for writing concise, \( \theta_{q3} \) and \( \theta_{q4} \) are defined as

\[ \theta_{q3} = \theta_{q} + \frac{1}{2}(\alpha + \beta) \]

\[ \theta_{q4} = \theta_{q+1} + \frac{1}{2}(\beta - \alpha) \]

\[ (11) \]

The solving strategy for differential equation with non-homogeneous BCs is firstly utilizing superposition principle to homogenize the BCs [13], and then solving the equations by applying those with homogeneous techniques. Auxiliary functions \( u^{s \phi q} (r, \phi) \) and \( v^{s \phi q} (r, \phi) \) are set for calculation convenience, suppose that

\[ A_{\phi}^{s} (r, \phi) = u^{s \phi q} (r, \phi) + v^{s \phi q} (r, \phi) \]

\[ (12) \]

Let \( v^{s \phi q} (r, \phi) \) formulated as [14]

\[ v^{s \phi q} (r, \phi) = \frac{\phi}{\phi^{s \phi q}} \left[ A_{\phi}^{q} (r, \theta_{q+1}) - A_{\phi}^{q} (r, \theta_{q} + \beta) \right] + \frac{1}{\phi^{s \phi q}} \left[ A_{\phi}^{q} (r, \theta_{q} + \beta) \theta_{q+1} - A_{\phi}^{q} (r, \theta_{q+1}) \theta_{q+1} \right] \]

\[ (13) \]

Where, \( \phi^{s \phi q} \) is the angular range of the \( q \)-th addendum.

\[ \phi^{s \phi q} = \theta_{q+1} - (\theta_{q} + \beta) = \frac{2\pi}{Q} - \beta \]

\[ (14) \]

Obviously, \( v^{s \phi q} (r, \phi) \) possesses the attributions of

\[ v^{s \phi q} (r, \theta_{q} + \beta) = A_{\phi}^{s} (r, \theta_{q} + \beta) \]

\[ v^{s \phi q} (r, \theta_{q+1}) = A_{\phi}^{q} (r, \theta_{q+1}) \]

\[ \Delta v^{s \phi q} (r, \phi) = 0 \]

\[ (15) \]

Considering of Eq.(9) to Eq.(15), the property of \( u^{s \phi q} (r, \phi) \) can be obtained by applying several subtraction operations
\[\Delta u^{s\varphi}(r, \varphi) = 0\]
\[u^{s\varphi}(r, \theta_q + \beta) = 0\]
\[u^{s\varphi}(r, \theta_{q+1}) = 0\]  \hspace{1cm} (16)
\[u^{s\varphi}(R_q, \varphi) = A_q^{s\varphi \varphi}(R_q, \varphi) - v^{s\varphi}(R_q, \varphi)\]
\[u^{s\varphi}(R_q, \varphi) = f^{s\varphi}(R_q, \varphi) - v^{s\varphi}(R_q, \varphi)\]

Hence the general solution of \(u^{s\varphi}(r, \varphi)\) can be expressed as
\[u^{s\varphi}(r, \varphi) = \sum_{k=1}^{\infty} \left[ \frac{A_k^{s\varphi}}{E(r, R_k, \vec{k} \pi / \phi^{s\varphi})} - \frac{B_k^{s\varphi}}{E(r, R_k, \vec{k} \pi / \phi^{s\varphi})} \right] \sin \left( \frac{\vec{k} \pi}{\phi^{s\varphi}} (\varphi - (\theta_q + \beta)) \right)\]  \hspace{1cm} (17)

Where, coefficients \(A_k^{s\varphi}\) and \(B_k^{s\varphi}\) can be calculated by
\[A_k^{s\varphi} = \frac{2}{\vec{k} \pi \phi^{s\varphi}} \int_{\theta_q}^{\theta_{q+1}} \left[ A_k^{s\varphi \varphi}(R_k, \varphi) - v^{s\varphi}(R_k, \varphi) \right] \sin \left[ \frac{\vec{k} \pi}{\phi^{s\varphi}} (\varphi - (\theta_q + \beta)) \right] d\varphi\]
\[B_k^{s\varphi} = \frac{2}{\vec{k} \pi \phi^{s\varphi}} \int_{\theta_q}^{\theta_{q+1}} \left[ f^{s\varphi}(R_k, \varphi) - v^{s\varphi}(R_k, \varphi) \right] \sin \left[ \frac{\vec{k} \pi}{\phi^{s\varphi}} (\varphi - (\theta_q + \beta)) \right] d\varphi\]  \hspace{1cm} (18)

Fig. 3 \(q\)-th addendum (a) and dedendum (b) subdomain with their BCs.

2.3.3. \(q\)-th dedendum subdomain (s = 6qt)
The \(q\)-th dedendum subdomain is restricted by azimuthal range of \(\theta_{q3}\) to \(\theta_{q4}\), and radial range of \(R_5\) to \(R_6\) as depicted in Fig. 3(b). The angular range of the \(q\)-th dedendum is defined as
\[\phi^{s\varphi} = \theta_{q4} - \theta_{q3} = \frac{2\pi}{Q} - \alpha\]  \hspace{1cm} (19)

The Laplace equation of this subdomain can be reduced to
\[\frac{\partial^2 A^{s\varphi}}{\partial r^2} + \frac{1}{r} \frac{\partial A^{s\varphi}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A^{s\varphi}}{\partial \varphi^2} = 0\]  \hspace{1cm} (20)

The \(q\)-th dedendum subdomain possesses non-homogeneous Neumann BCs, which are listed as...
\begin{align*}
A_0^{(m)}(r, \theta) &= A_0^{(m)}(r, \theta) \\
A_0^{(m)}(r, \theta_\phi) &= A_0^{(m+1)}(r, \theta_\phi)
\end{align*}
\begin{align*}
\frac{\partial A_0^{(m)}}{\partial r}
\bigg|_{r=R_\phi} &= \frac{\partial A_0^{(m)}}{\partial r}
\bigg|_{r=R_\phi} \\
\frac{\partial A_0^{(m)}}{\partial \phi}
\bigg|_{\theta=\theta_\phi} &= \frac{\partial A_0^{(m)}}{\partial \phi}
\bigg|_{\theta=\theta_\phi}
\end{align*}
(21)

Similar solving strategy to the addendum subdomain is applied to handle the BCs. Suppose that
\begin{equation}
A_0^{(m)}(r, \phi) = v_0^{(m)}(r, \phi) + u_0^{(m)}(r, \phi)
\end{equation}
(22)

\begin{equation}
v_0^{(m)}(r, \phi) = \frac{\phi}{\phi_{\theta}} \left[ A_0^{(m+1)}(r, \theta_\phi) - A_0^{(m)}(r, \theta_\phi) \right] + \frac{1}{\phi_{\theta}} \left[ \theta_\phi A_0^{(m)}(r, \theta_\phi) - \theta_\phi A_0^{(m+1)}(r, \theta_\phi) \right]
\end{equation}
(23)

The line current density in this paper is deemed to be zero, combining Eq.(22) with Eq.(23), we can easily find that
\begin{align*}
\Delta u_0^{(m)} &= 0 \\
u_0^{(m)}(r, \theta_\phi) &= 0 \\
u_0^{(m)}(r, \theta_\phi) &= 0
\end{align*}
(24)

\begin{align*}
\frac{\partial u_0^{(m)}}{\partial r}
\bigg|_{r=R_\phi} &= \frac{\partial A_0^{(m)}}{\partial r}
\bigg|_{r=R_\phi} - \frac{\partial v_0^{(m)}}{\partial r}
\bigg|_{r=R_\phi} \\
\frac{\partial u_0^{(m)}}{\partial \phi}
\bigg|_{\theta=\theta_\phi} &= \frac{\partial A_0^{(m)}}{\partial \phi}
\bigg|_{\theta=\theta_\phi} - \frac{\partial v_0^{(m)}}{\partial \phi}
\bigg|_{\theta=\theta_\phi}
\end{align*}

Hence the general solution of \( \Delta u_0^{(m)} = 0 \) is written as
\begin{equation}
u_0^{(m)}(r, \phi) = \sum_{n=1}^{\infty} A_n^{(m)}(r, \phi) \frac{\phi}{\phi_{\theta}} \frac{R_\phi}{E(R_\phi, R_\phi, \bar{m}\pi, \phi_{\theta})} - B_n^{(m)}(r, \phi) \frac{R_\phi}{E(R_\phi, R_\phi, \bar{m}\pi, \phi_{\theta})} \left[ \frac{\bar{m}\pi}{\phi_{\theta}} (\varphi - \theta_1) \right]
\end{equation}
(25)

where coefficients \( A_n^{(m)} \) and \( B_n^{(m)} \) can be calculated by
\begin{align*}
A_n^{(m)} &= \frac{2}{\phi_{\theta}} \int_{\theta_\phi}^{\theta_1} \frac{\partial A_n^{(m)}}{\partial \phi}
\left. \right|_{\theta=\theta_\phi} \sin \left[ \frac{\bar{m}\pi}{\phi_{\theta}} (\varphi - \theta) \right] d\varphi \\
B_n^{(m)} &= \frac{2}{\phi_{\theta}} \int_{\theta_\phi}^{\theta_1} \frac{\partial A_n^{(m)}}{\partial \phi}
\left. \right|_{\theta=\theta_\phi} \sin \left[ \frac{\bar{m}\pi}{\phi_{\theta}} (\varphi - \theta) \right] d\varphi
\end{align*}
(26)

2.3.4. Tooth-yoke subdomain \((s = so)\)

The annular region of the tooth-yoke subdomain is \( R_6 \) to \( R_7 \), the Laplace equation for this subdomain can be reduced to
\begin{equation}
\frac{\partial^2 A_0^{(m)}}{\partial r^2} + \frac{1}{r} \frac{\partial A_0^{(m)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_0^{(m)}}{\partial \phi^2} = 0
\end{equation}
(27)

and its BCs on the boundaries as
\begin{align*}
A_0^{(m)}(R_6, \phi) &= f^{(m)}(R_6, \phi) \\
A_0^{(m)}(R_7, \phi) &= 0
\end{align*}
(28)

The general solutions of Eq.(27) can be expressed as
\begin{equation}
A_0^{(m)}(r, \phi) = A_0^{(m)} + B_0^{(m)} \ln r + \sum_{n=1}^{\infty} \left[ A_n^{(m)} \frac{E(r, R_n, R_6)}{E(r, R_n, r)} \cos n\varphi + B_n^{(m)} \frac{E(r, R_n, R_6)}{E(r, R_n, r)} \sin n\varphi \right]
\end{equation}
(29)
where, coefficients $A_n^{(0)}$, $B_n^{(0)}$, $A_n^{(m)}$, and $B_n^{(m)}$ can be obtained by

$$
A_n^{(0)} + B_n^{(0)} \ln R_b = \frac{1}{2\pi} \int_0^{2\pi} f^{(0)}(R_b, \varphi) d\varphi
$$

and

$$
A_n^{(m)} + B_n^{(m)} \ln R_b = 0
$$

$$
A_n^{(0)} = \frac{1}{\pi} \int_0^{2\pi} f^{(0)}(R_b, \varphi) \cos n\varphi d\varphi
$$

$$
B_n^{(0)} = \frac{1}{\pi} \int_0^{2\pi} f^{(0)}(R_b, \varphi) \sin n\varphi d\varphi
$$

(30)

3. Analytical prediction and FEM validation

A 2-D FEM model is presented for the validation of the analytical prediction. Electromagnetic performances including flux density, back-emf, electromagnetic torque, and cogging torque are computed using the analytical model and validated by the FEM model. Parameters of the studied machine are given in Table I in appendix.

3.1. Flux density distribution

The flux density distribution of the middle of the air-gap, the middle of the slot-openings, the middle of the slots, and the middle of the tooth-yoke are calculated under load conditions separately. The moment of $t=0$ is chosen for analysis, the peak value of the phase current is set as 20 Amperes. Results are demonstrated in Fig.4 and Fig.5. Excellent agreement is achieved between analytical prediction and FEM results. Thus the technique of handling the outer teeth in this paper is reasonable for approximation.

3.2. Back-emf and electromagnetic torque

Back-emf results are obtained via flux linkage results, to obtain flux linkage results in a certain mechanical angular velocity $\omega_m$, the computation should be applied every moment when the rotor changes its position, which can be implemented by changing the angular position of each slot-opening $\theta_q$, more specifically

$$
\theta_q(t) = -\frac{\beta}{2} + \frac{2\pi}{Q} (q - 1) - \omega_m t \quad \text{with } q = 1, 2, \ldots, Q \text{ and } \omega_m t \in [0, 2\pi]
$$

(31)

and the flux linkage of phase $x$ can be solved as [10]
\[ \psi_x = \sum_{q,s} C(q) \frac{N_{\text{turn}} L}{A_{\text{slot}}} \int_{A_{\text{slot}}}^{} A_s^x r dr d\varphi \]  

(32)

where, \( N_{\text{turn}} \) is the number of turns of each phase in series per slot, \( L \) is the length of the machine. \( A_{\text{slot}} \) is the slot area. \( C(q) \) indicates the direction of the phase windings (+1 for positive winding, -1 for negative winding). The back-emf and the electromagnetic torque can be computed by

\[
    e_x = -\frac{d\psi_x}{dt} = -\omega_n \frac{d\psi_x}{d\varphi} \\
    T_{\text{em}} = \frac{1}{\omega_n} \sum_x e_x i_x 
\]

(33)

The results of back-emf of phase U and electromagnetic torque are shown in Fig.6(a). The analytical solutions perform great consistency with FEM results. Noting that the electromagnetic torque data of FEM doesn’t reflect smooth enough compare with those obtained by analytical model, this exactly reflects that the FEM model is sensitive to the mesh quality.

![Fig.6 (a) Back-emf of phase U & electromagnetic torque versus time. (b) Cogging torque under different \( \beta/\alpha \) values. (c) Instantaneous \((t=0)\) magnetic flux lines under load condition \((\beta/\alpha=0.267)\). (d) Instantaneous \((t=0)\) magnetic vector potential under load condition \((\beta/\alpha=0.267)\).](image)

3.3. Cogging torque

The cogging torque of this specific machine is a quantity with an angular period of \( 2\pi/Q \) [15]. The computations are taken in every increments of angular position \( s \) of rotor, the detail of the implementation is similar to the flux linkage calculation, let

\[
    \theta_x (\Delta \varphi) = -\frac{\beta}{2} + \frac{2\pi}{Q} (q-1) - \Delta \varphi \quad \text{with} \quad q = 1, 2, \ldots, Q \quad \text{and} \quad \Delta \varphi \in [0, 2\pi/Q] \]  

(34)

where \( \Delta \varphi \) denotes the increment of the angular position. The cogging torque is obtained by applying the Maxwell stress tensor theory

\[
    T_{\text{cog}} = \frac{L R^2}{\mu_0} \int_{\Delta \varphi}^{2\pi} B_{\text{air-gap}}^x (R_{\text{gap}}, \varphi) B_{\text{air-gap}}^y (R_{\text{gap}}, \varphi) d\varphi 
\]

(35)

where \( R_{\text{gap}} \) is the radius of the integral path in the air-gap subdomain. The results under different \( \beta/\alpha \) values are presented in Fig.6(b). The cogging torque value of the studied machine is quite small mainly due to the large air-gap length. Some details should be pointed out that it costs quite long time (approximately 13 times as long as the analytical method) to calculate the results correctly for the considerable amount of the mesh quantity.

Finally, complete magnetic flux line and magnetic vector potential are presented in Fig.6(c) and (d) respectively.

4. Conclusion

In this paper, a complete 2-D analytical model for multi-slot HSPMMs with toroidal windings has been developed. The proposed analytical model mainly solves magnetic vector potential in non-iron part. The model counts for the magnetic field under both no-load and load conditions. Flux density distribution, flux linkage, back-emf, electromagnetic torque, and cogging torque are calculated using this analytical model, the results reflect a great consistency with these issues from the FEM model.
This paper provides a technique to handle the slotted machines with toroidal windings, and the method to solve magnetic vector potential in the iron parts. With little modification, the analytical model can be used to predict the field distribution with iron parts that possess finite relative permeability.

Appendix
For writing conciseness, follow notations are defined

\[
P(a,b,c) = (a + b)\gamma + (a - b)\gamma
\]

\[
E(a,b,c) = (a + b)\gamma - (a - b)\gamma
\]

Main parameters of the machine are listed in Table I.

### Table I Machine parameters

| Symbol & Quantity | value | Symbol & Quantity | value |
|-------------------|-------|-------------------|-------|
| \(R_i\) | Radius of the rotor shaft | 9mm | \(E_{\text{L}}\) | Length of the machine | 35mm |
| \(R_o\) | Outer radius of the PM-ring | 14mm | \(\alpha\) | Slot pitch angle | 15° |
| \(R_s\) | Outer radius of the shielding cylinder | 15mm | \(\beta\) | Slot-opening angle | 4° |
| \(R_k\) | Inner radius of the slot-opening | 16mm | \(B_{\text{rem}}\) | Remanence of the Pn-ring | 1.02T |
| \(R_f\) | Outer radius of the slot-opening | 17mm | \(N_{\text{turns}}\) | Number of turns of each phase in series per slot | 2 |
| \(R_t\) | Outer radius of the tooth-yoke | 20mm | \(Q\) | Slot number | 18 |

General solutions of magnetic vector potential for non-iron parts are listed as follows

\[
A_{n}^{\alpha\varphi} (r, \varphi) = -B_{n}^{\alpha\varphi} E(r/R_{n}) \cos \varphi + \sum_{i \neq 0} \left[ A_{n}^{(3)} (r, R_{i}, n) \cos n\varphi + B_{n}^{(3)} (r, R_{i}, n) \sin n\varphi \right]
\]

\[
A_{n}^{\text{air}\varphi} (r, \varphi) = \sum_{i = 1}^{\infty} \left[ A_{n}^{(3)} (r, R_{i}, n) \cos n\varphi + B_{n}^{(3)} (r, R_{i}, n) \sin n\varphi \right]
\]

\[
A_{n}^{\eta} (r, \varphi) = A_{n}^{(5)} + B_{n}^{(5)} \ln r + \cos (k\pi / \beta) (\varphi - \theta_q) \sum_{i = 1}^{\infty} \left[ A_{n}^{(5)} (r, R_{i}, k\pi / \beta) / E(R_{i}, R_{i}, k\pi / \beta) - B_{n}^{(5)} (r, R_{i}, k\pi / \beta) / E(R_{i}, R_{i}, k\pi / \beta) \right]
\]

\[
A_{n}^{\eta} (r, \varphi) = A_{n}^{(6)} + \frac{1}{2} \mu_{0} \frac{\beta_{\varphi}}{r} \left[ R_{n}^{2} \ln r - \frac{1}{2} \right] + \sum_{i = 1}^{\infty} A_{n}^{(6)} \alpha R_{i} \cos (m\pi / \alpha) \left[ \frac{m\pi}{\alpha} \left( \varphi - \theta_q - \frac{1}{2} (\beta - \alpha) \right) \right]
\]

Coefficients for non-iron parts are listed in Table II, where functions involved in Eq.(A.6) to Eq.(A.13) are defined in Eq.(A.14) to Eq.(A.16).

### Table II Expressions of the coefficients for the iron parts

\[
A_{n}^{(4)} = \frac{2}{k \pi} \left[ (-1)^{n+1} \left( \Theta_{i}(R_{i}, \theta_{i}) - \Theta_{n}(R_{i}, \theta_{i} + \beta) \right) + \sum_{i = 1}^{\infty} A_{n}^{(2)} \frac{2R_{i}}{n\varphi} E(R_{i}, R_{i}, \beta) - B_{n}^{(2)} \frac{2R_{i}}{n\varphi} E(R_{i}, R_{i}, \beta) \right] \tilde{p}_{n}(n,q,K)
\]

\[
+ \sum_{i = 1}^{\infty} C_{n}^{(2)} \frac{2R_{i}}{n\varphi} E(R_{i}, R_{i}, \beta) - D_{n}^{(2)} \frac{2R_{i}}{n\varphi} E(R_{i}, R_{i}, \beta) \right] \tilde{p}_{n}(n,q,K)
\]
\[ B_{\alpha}^{(\nu)} = \sum_{m \in \mathbb{Z}} A_{\alpha}^{(\nu)} \mathcal{P}(R_{\alpha}, R_{\alpha}, m \pi / \alpha) \mathcal{E}_{\alpha}(m, \bar{m}) + \sum_{m \in \mathbb{Z}} A_{\alpha}^{(\nu)} \mathcal{P}(R_{\alpha}, R_{\alpha}, m \pi / \alpha) \mathcal{E}_{\alpha}(m, \bar{m}) + F_{\alpha}(q, \bar{q}) + F_{\alpha}(q, \bar{q}) - B_{\alpha}^{(\nu)}\]

\[ A_{\alpha}^{(\nu)} = \sum_{m \in \mathbb{Z}} A_{\alpha}^{(\nu)} \mathcal{P}(R_{\alpha}, R_{\alpha}, m \pi / \alpha) \mathcal{E}_{\alpha}(m, \bar{m}) + \sum_{m \in \mathbb{Z}} A_{\alpha}^{(\nu)} \mathcal{P}(R_{\alpha}, R_{\alpha}, m \pi / \alpha) \mathcal{E}_{\alpha}(m, \bar{m}) + F_{\alpha}(q, \bar{q}) + F_{\alpha}(q, \bar{q}) - B_{\alpha}^{(\nu)}\]

\[ B_{\alpha}^{(\nu)} = \frac{2}{R_{\alpha} m \pi} \left[ 1 - (-1)^{\nu} \right] + \sum_{m \in \mathbb{Z}} A_{\alpha}^{(\nu)} \frac{n \mathcal{P}(R_{\alpha}, R_{\alpha}, n)}{R_{\alpha} m \pi} \mathcal{E}_{\alpha}(n, q, \bar{m}) + B_{\alpha}^{(\nu)} \frac{n \mathcal{P}(R_{\alpha}, R_{\alpha}, n)}{R_{\alpha} m \pi} \mathcal{E}_{\alpha}(n, q, \bar{m})\]

\[ A_{\alpha}^{(\nu)} = -B_{\alpha}^{(\nu)} \ln R_{\alpha}\]

\[ B_{\alpha}^{(\nu)} = \left\{ \frac{1}{2 \pi} \sum_{m \in \mathbb{Z}} A_{\alpha}^{(\nu)} \frac{\mathcal{P}(R_{\alpha}, R_{\alpha}, m \pi / \alpha)}{m \pi} \mathcal{E}_{\alpha}(m, \bar{m}) \mathcal{E}_{\alpha}(n, q, \bar{m}) - B_{\alpha}^{(\nu)} \frac{\mathcal{P}(R_{\alpha}, R_{\alpha}, m \pi / \alpha)}{m \pi} \mathcal{E}_{\alpha}(m, \bar{m}) \mathcal{E}_{\alpha}(n, q, \bar{m}) \left[ 1 - (-1)^{\nu} \right] \right\} \]

\[ \sum_{q \in \mathbb{Z}} \frac{\alpha}{2 \pi} A_{\alpha}^{(\nu)} + \frac{1}{2} \mu_{\alpha} \frac{f_{\alpha}}{R_{\alpha} \ln R_{\alpha} - 2 R_{\alpha}} \left[ \frac{1}{2} A_{\alpha}^{(\nu)} (R_{\alpha}, \theta) - A_{\alpha}^{(\nu)} (R_{\alpha}, \theta) \right] \ln \frac{R_{\alpha}}{R_{\alpha}}\]

**Expressions of the coefficients**

\[ A_{\alpha}^{(\nu)} = \frac{1}{2 \pi} \sum_{m \in \mathbb{Z}} A_{\alpha}^{(\nu)} \frac{\mathcal{P}(R_{\alpha}, R_{\alpha}, m \pi / \alpha)}{m \pi} \mathcal{E}_{\alpha}(m, \bar{m}) + \sum_{m \in \mathbb{Z}} A_{\alpha}^{(\nu)} \frac{\mathcal{P}(R_{\alpha}, R_{\alpha}, m \pi / \alpha)}{m \pi} \mathcal{E}_{\alpha}(m, \bar{m}) - A_{\alpha}^{(\nu)} (R_{\alpha}, \theta) \]

\[ B_{\alpha}^{(\nu)} = \frac{1}{2 \pi} \sum_{m \in \mathbb{Z}} A_{\alpha}^{(\nu)} \frac{\mathcal{P}(R_{\alpha}, R_{\alpha}, m \pi / \alpha)}{m \pi} \mathcal{E}_{\alpha}(m, \bar{m}) + \sum_{m \in \mathbb{Z}} A_{\alpha}^{(\nu)} \frac{\mathcal{P}(R_{\alpha}, R_{\alpha}, m \pi / \alpha)}{m \pi} \mathcal{E}_{\alpha}(m, \bar{m}) - B_{\alpha}^{(\nu)} (R_{\alpha}, \theta)\]
\[ \tilde{e}(\bar{m}, \tilde{k}) = \int_{\theta\in\mathbb{R}} \sin \left[ \frac{\pi}{\phi^w} \left( x - (\varphi + \beta) \right) \right] \cdot \sin \left[ \frac{\pi}{\phi^w} \left( x - (\varphi + \beta) \right) \right] \cdot dx \]

\[ \tilde{e}(m, \tilde{k}) = \int_{\theta\in\mathbb{R}} \cos \left[ \frac{\pi}{\alpha} \left( x - (\varphi - \frac{1}{2} (\beta - \alpha)) \right) \right] \cdot \sin \left[ \frac{\pi}{\phi^w} \left( x - (\varphi + \beta) \right) \right] \cdot dx \]

\[ \tilde{e}_x(m, \tilde{k}) = \int_{\theta_x\in\mathbb{R}} \cos \left[ \frac{\pi}{\alpha} \left( x - (\varphi - \frac{1}{2} (\beta - \alpha)) \right) \right] \cdot \sin \left[ \frac{\pi}{\phi^w} \left( x - (\varphi + \beta) \right) \right] \cdot dx \]

\[ e_x(n, q, m) = \int_{\theta_x\in\mathbb{R}} \sin(n x) \cdot \cos \left[ \frac{\pi}{\alpha} \left( x - \frac{1}{2} (\beta - \alpha) \right) \right] \cdot dx \]

\[ e_x(n, q, m) = \int_{\theta_x\in\mathbb{R}} \sin(n x) \cdot \cos \left[ \frac{\pi}{\phi^w} \left( x - \frac{1}{2} (\beta - \alpha) \right) \right] \cdot dx \]

\[ \overline{e}_x(n, q, m) = \int_{\theta_x\in\mathbb{R}} \sin(n x) \cdot \sin \left[ \frac{\pi}{\phi^w} \left( x - \frac{1}{2} (\beta - \alpha) \right) \right] \cdot dx \]

\[ \overline{e}_x(n, q, m) = \int_{\theta_x\in\mathbb{R}} \sin(n x) \cdot \sin \left[ \frac{\pi}{\phi^w} \left( x - \frac{1}{2} (\beta - \alpha) \right) \right] \cdot dx \]

\[ \overline{p}_x(n, q, \tilde{k}) = \int_{\theta_x\in\mathbb{R}} \sin(n x) \cdot \sin \left[ \frac{\pi}{\phi^w} \left( x - (\varphi + \beta) \right) \right] \cdot dx \]

\[ F_i(q, \tilde{k}) = \frac{2}{\pi} \left[ (1) \cos \left( \frac{\pi}{\phi^w} \right) - A^\alpha \right] \]

\[ F_i(q, \tilde{k}) = \frac{-2}{\pi} \left[ A^\alpha + \frac{1}{2} \mu_0 J^{\alpha \beta} \left( R^2 - \frac{R^2}{2} \right) \right] \left[ \cos \left( \frac{\pi}{\phi^w} \frac{\alpha - \beta}{2} \right) - 1 \right] \]

\[ F_i(q, \tilde{k}) = \frac{-2}{\pi} \left[ A^\alpha + \frac{1}{2} \mu_0 J^{\alpha \beta} \left( R^2 - \frac{R^2}{2} \right) \right] \left[ (1) \cos \left( \frac{\pi}{\phi^w} \frac{\alpha - \beta}{2} \right) \right] \]

\[ F_i(q, \tilde{k}) = \frac{2}{\pi} \cos \left( \frac{\pi}{\phi^w} \frac{\alpha - \beta}{2} \right) \left[ A^\alpha \right] \]

\[ G_i(q, n) = \frac{1}{2} \sum_{\varphi=\pi} ^{\infty} \left[ A^\alpha \left( \frac{\varphi}{\phi^w} \right) \right] \cos \left( \frac{\pi}{\alpha} \frac{\varphi}{2} \right) \left[ \sin \left( \frac{\pi}{\phi^w} \frac{\varphi}{2} \right) \right] \left[ \sin \left( \frac{\pi}{\phi^w} \frac{\varphi}{2} \right) \right] \]

\[ G_i(q, n) = \frac{1}{2} \sum_{\varphi=\pi} ^{\infty} \left[ A^\alpha \left( \frac{\varphi}{\phi^w} \right) \right] \cos \left( \frac{\pi}{\alpha} \frac{\varphi}{2} \right) \left[ \sin \left( \frac{\pi}{\phi^w} \frac{\varphi}{2} \right) \right] \left[ \sin \left( \frac{\pi}{\phi^w} \frac{\varphi}{2} \right) \right] \]

(A.14)
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