STRICT HOMOTOPY INVARIANCE VIA COMPACTIFIED HOMOTOPIES
AND CORRESPONDENCES, AND FIBRES OF ESSENTIALLY SMOOTH
SCHEMES OVER ONE-DIMENSIONAL BASE SCHEMES.

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Abstract. We develop the technique of compactified correspondences and homotopies over
one-dimensional base schemes, and illuminate the perfectness and the inverting of characteristic
assumptions from the celebrating Voevodsky’s strict homotopy invariance theorem and its framed
correspondences generalisation over an arbitrary base field. The assumption in this crucial theo-
rem for Voevodsky’s motives theory was kept from the origins of the study, and came later into
more modern theory of framed motives by Garkusha-Panin. Applying the technique, we obtain
also analogs of Gersten and Nisnevich conjectures for Cousin complexes of generalised motivic
cohomotopies over a field, and acyclicity of Cousin complexes on generic fibres of essentially
smooth local schemes over one-dimensional base schemes.

Contents

1. Introduction. 2
1.1. Strict homotopy invariance 2
1.2. Key construction and applications 3
1.3. Proof strategy and overview 4
1.4. Conventions 4
2. Framed correspondences and presheaves 6
2.1. Correspondences categories and presheaves with transfers 6
2.2. Linearised correspondences, and Nisnevich sheafification 7
3. Topology over a quotient-space $X/(X-Z)$ 8
3.1. Topology Nis$_X$ 8
3.2. Framed transfers for cohomologies 8
4. Cohomologies out of infinity on $\AA^1_{V}$ 10
4.1. Excision isomorphisms 11
4.2. Coverings and cohomologies 13
5. Finite support injectivity criterion 14
6. Compactified correspondences over one-dimensional base schemes 15
6.1. Compactified correspondences 15
6.2. Universal endo-correspondence 16
6.3. Contracting correspondence of dimension one 17
6.4. Finite correspondence homotopy 19
6.5. Vanishing of cohomologies 21
7. Strict homotopy invariance 21
8. Cousin complex 22
Appendix A. Relative dimension over one-dimensional base 23
Appendix B. Sections of line bundles 23
Appendix C. Framed correspondences and homotopies 24
References 25

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1. Introduction.

The philosophy of motivic $\mathbb{A}^1$-homotopy theory [36] suggests to consider the affine line $\mathbb{A}^1_k \in \text{Sm}_k$ over a base field $k$ by analogy to the topological interval $[0, 1] \in \text{Top}$ in algebraic topology. The $p$-torsion, where $p$ is the characteristic of the base field $k$, is a mysterious part of the study. Because of finite étale coverings of degrees $p^i$, in distinct to $[0, 1] \in \text{Top}$, the affine line $\mathbb{A}^1_k$ is not one-connected. At the same time, for reasons that look formally independent, various technical obstructions appear inside proofs of complicated structural results on the motivic categories. One well-know example is the theorem [51] regarding Voevodsky’s motives category $\text{DM}(k)$. The proof of the strict homotopy invariance theorem [49, Theorem 5.6], which is a milestone technical result, was based on an upgrade of powerful algebro geometric ideas from proofs of Gersten conjecture in [7, 39] combined with a new instrument of finite correspondences. Nevertheless, the approach required either perfectness assumption on $k$, or inverting of $p$ in the coefficient ring, see Section 1.1.1. Our article develops a modern technique of compactified homotopies and compactified $d$-dimensional correspondences over one-dimensional base schemes. This allows to illuminate the assumption, because the obstruction was hidden, from some viewpoint, at infinity of $\mathbb{A}^1_k$, $X \in \text{Sm}_k$, $x \in X$.

Covering framed correspondences context, we have generalised [24, Theorem 1.1] and immediately extended all results of Garkusha-Panin’s framed motives theory [23], which applies the approach of Voevodsky’s motives theory on $\text{DM}(k)$ to the stable motivic homotopy category $\text{SH}(k)$ [17, 26, 32, 36, 47]. See Sections 1.2 and 1.3 for details and other applications.

**Theorem (Theorem [7, 1]).** Let $F$ be an $\mathbb{A}^1$-invariant quasi-stable framed linear presheaf over a field $k$, and $F_{\text{Nis}}$ denote the associated Nisnevich sheaf, and similarly $F_{\text{zar}}$ for Zariski topology. Then for any $X \in \text{Sm}_k$, the canonical projection induces isomorphisms on Nisnevich cohomologies

$$H_{\text{Nis}}^n(\mathbb{A}^1_k \times X, F_{\text{Nis}}) \cong H_{\text{Nis}}^n(X, F_{\text{Nis}}), \quad n \in \mathbb{Z}$$

(1.1)

and similarly for Zariski cohomologies, and for each $n \in \mathbb{Z}$, there are canonical isomorphisms

$$H_{\text{zar}}^n(X, F_{\text{zar}}) \cong H_{\text{Nis}}^n(X, F_{\text{Nis}}), \quad n \in \mathbb{Z}.$$  

(1.2)

1.1. Strict homotopy invariance. In the topological setting, homotopy classes of something tautologically form a discrete set. In the motivic homotopy theory, this property is called strict homotopy invariance and relates to the fundamental complexity formed by the combination of $\mathbb{A}^1$-homotopy and Nisnevich localisations in the construction of the $\mathbb{A}^1$-motivic localisation. Under certain assumptions on the motivic spaces various strict homotopy invariance theorems reduce infinite iterative composition of $\mathbb{A}^1$-invariantisations and Nisnevich localisations to the single iteration. The theorem discussed in the article regards structure of transfers, which gives a universal computational framework for motivic localisation functors and hom-groups, see [23, 24, 49, 50].

[24, Theorem 5.6] is the result for $F$ with $\text{Cor}(k)$-transfers over a perfect base field $k$. Suslin proved in [45, 46] the result with $\mathbb{Z}[1/p]$-coefficients for non-perfect base fields. The argument is a reduction to the perfect base field case known early for algebraic K-theory as was explained in [46]. The framed motives theory by Garkusha and Panin [24] provides a computation for the stable motivic homotopy types in $\text{SH}(k)$ based on the unpublished notes [51] by Voevodsky. Notes 51 introduced the framed correspondences as an ingredient that would allow to study $\text{SH}(k)$ in parallel to $\text{DM}(k)$ [51]. Large part of framed motives theory [1, 14, 15, 20, 22, 23, 24, 37] upgrades the respective results for $\text{Cor}(k)$-correspondences, and upgrading the original Voevodsky’s reasoning Garkusha and Panin in [24] covered initially the case of an infinite perfect base field of odd characteristic, and additional arguments in [14, 15, 19] covered the generality of arbitrary perfect fields. Note that the resolution of singularities over $k$ required for the cancellation theorem in [51] was illuminated by Voevodsky in [43], and due to the upgrade of the latter argument for framed correspondences in [1], the strict homotopy invariance theorem [24, Theorem 1.1] is the only reason of assumptions on the base field for results of [23, 25, 26, and 19]. Here is the summary of early known facts: (1) $n = 0$ by [24, 49]; (2) for $F$ being an inverse image of presheaves over perfect fields; (3) for presheaves of $\mathbb{Z}[1/p]$-modules with $\text{Cor}(k)$-transfers [46]; (4) it was known the equivalence of the natural isomorphism (1.1) with the one for Zariski topology [17, 19, 45], though isomorphisms $H_{\text{zar}}^n(X, F_{\text{zar}}) \cong H_{\text{Nis}}^n(X, F_{\text{Nis}})$ were not.
The general case for anyone Zariski or Nisnevich topology was an entire mystery, and our own opinion on whether the claim holds was several times changed during years of intensive work and periodical looking for a disproof before the argument was found. Our proof is elementary. The only external ingredients are étale excision and sheafification theorems [15, Theorem 3.10], [24, Proposition 16.2],

1.2. Key construction and applications. While the article subject belongs to the classical area of $\mathbb{A}^1$-homotopy theory over base fields, the developed instrument is applicable simultaneously in the following two modern directions: (1) studies over positive-dimensional base schemes oriented on a computational results for the lowest non-trivial stable motivic homotopy groups and cohomologies of Cousin or Gersten complexes, see [2, 3, 5], and (2) studies of $\square$-homotopy theories and categories [2, 6, 29, 30, 31]. Note that (1) the perfectness assumption excludes such natural class of base schemes as $k^1 \times B$ for any non-zero characteristic scheme $B$, (2) $\square$-homotopy analog of $\text{DM}_{et}(k, \mathbb{Z}/p)$ is non-trivial, which additionally increases the interest with respect to the $p$-torsion in Nisnevich motivic categories.

To achieve Theorem [7,3] we consider $k^1$ as a base scheme with the compactification $B = \mathbb{P}^1_k$ and the point $z = \infty$ and prove the following injectivity theorem, which is called Proposition because of the technical form, while it is the main result of the article in the appropriate sense.

Proposition (Proposition 6.15). Let $B$ be a one-dimensional scheme, $z \in B^{(1)}$, $\eta \in B^{(0)}$. Let $X \in \text{Sm}_B$, $x \in X \times_B z$. Denote by $U = X_x$ the local scheme at $x$, and define $U_\eta = U \times_B \eta$. Then for any closed subscheme $Z$ in $X_\eta = X \times_B \eta$, there are framed correspondences $c \in ZF(N(U_\eta \times k^1, X_\eta)$, $c' \in ZF(N(U_\eta \times k^1, X_\eta - Z)$ such that

(1) $c \circ i_0 = \sigma^N_0 \circ \text{can}_u$, $c \circ i_0 = j \circ c'$, where $i_0, i_1 : U_\eta \rightarrow U_\eta \times k^1$ are the zero and unit sections, $\text{can}_u : U_\eta \rightarrow X_\eta$ is the canonical morphism, $j : X_\eta - Z \rightarrow X_\eta$ is the open immersion,

(2) $c^{-1}(Z)$ is finite over $U_\eta$, see (1,0) for $c^{-1}(Z)$.

A direct application of Part (1) of Proposition 6.15 is the result, which follows by the argument for the Gersten conjecture in [49].

Theorem (Theorem 8.8). Under the assumptions of Proposition 6.15, for any $\text{SH}(B)$-representable cohomology theory $E^*$, the Cousin complex [2]

$$0 \rightarrow E^0(U_\eta) \rightarrow E^0(U_\eta^{(0)}) \rightarrow \bigoplus_{y \in U_\eta^{(c)}} E^0_y(U_\eta) \rightarrow \cdots$$

(1.3)

is acyclic.

As shown in [2, §1.2, 1.3.3] the above theorem has further applications: (1) the acyclicity of the Cousin complex

$$0 \rightarrow E^0(U) \rightarrow E^0(U^{(0)}) \rightarrow \bigoplus_{x \in U^{(1)}} E^1_x(U) \rightarrow \bigoplus_{x \in U^{(c)}} E^2_x(U) \rightarrow \cdots$$

above the terms $E^1_x(U)$ [4] and (2) isomorphisms

$$\pi_{i+j}(\Sigma^\infty_+ Y)(U) \cong 0, \quad Y \in \text{Sm}_B, \quad i < -1, \quad j \in \mathbb{Z},$$

which is a Zariski local improvement of the stable connectivity theorem [3, 43, 44] generalising results for $\text{SH}(k)$ from [34, 35] by Morel with respect to [4] by Ayoub.

Part (2) of Proposition 6.15 has no role for Theorem 8.8 and has a crucial role for Theorem 7.1. Part (2) provides a compactification of the $\mathbb{A}^1$-homotopy from Part (1). In detail, the claim means that the compactification of the homotopy sends the closed point $(\infty,x) \in \mathbb{P}^1 \times X_x$ to the complement of the closure of $Z$ in the compactification of $X_\eta$ over $B$. The key role in the construction belongs to certain projective compactifications of $B$-schemes over the one-dimensional local irreducible base scheme $B$ giving rise to compactified homotopies of so called focused compactified framed correspondences, see Definition 6.2. A compactification of an $\mathbb{A}^1$-homotopy above means

\[\text{Proposition 16.2,}\]

\[\text{over a DVR, an alternative proof is provided by [38].}\]
an extension of the data defining the homotopy along the embedding of \( \mathbb{A}^1 \times U \) to \( \mathbb{P}^1 \times \overline{U} \) for a given compactification \( U \to \overline{U} \). We construct it as a focused compactified one-dimensional framed correspondence equipped with a special morphism to the pair \((\mathbb{A}^1 \times U, \mathbb{P}^1 \times \overline{U})\). A compactification of a \( d \)-dimensional framed correspondence includes two parts, we call “compactification” and “focusing”. The role of “compactification” is similar to such instruments used over base fields as Quillen’s trick from \([40]\) or standard triples in \([49]\). The “focusing” is a positive-dimensional base schemes specific part; an example could be given by the base scheme \( B \) itself considered as a compactification of its generic point \( \eta \), or a smooth projective \( B \)-scheme as a compactification of the generic fibre.

1.2.1. Directions of further applications. Being equipped by the construction with compactifications in the above senses, both of which are important in the \( \square \)-homotopy motivic studies, our homotopies and correspondences form an instrument to obtain certain analogs of Theorems 7.1 and 8.8 for \( \square \)-homotopy invariant theories. We mean a respective generalisations of results in \([41, 42]\) on reciprocity sheaves, which are supposed to be a natural further development of this work.

1.3. Proof strategy and overview. The proof strategy of graded isomorphism \( H^*_{\text{Nis}}(\mathbb{A}_k^1 \times X, F_{\text{Nis}}) \cong F_{\text{Nis}}(X) \) in \([24, 43]\) includes the following parts: (1) an isomorphism

\[
H^*_{\text{Nis}}(\text{Spec } K \times \mathbb{A}^1, F) \cong F(\text{Spec } K)
\]

for any field extension \( K/k \) and \( (\mathbb{A}_k^1 \times X, F_{\text{Nis}}) \) to \( (\mathbb{A}_k^1, F_{\text{Nis}}) \) (1.4).

The proof uses two ingredients: a moving homotopy provided by Proposition 6.15 applied over a “compactified base scheme” \((\mathbb{A}_k^1, F_{\text{Nis}})\), and the partial strict homotopy invariance provided by the above result on cohomologies “out of infinity”. Proposition 6.15 is proven by the technique of compactifications discussed above. The construction of the homotopy requires that the base field contains three different elements used to control values of the compactified homotopy at 0, 1, and \( \infty \). The claim for any base field follows by the argument from \([14, 19]\).

The argument for strict homotopy invariance theorem in Section 7 summarises results of Sections 1 to 6. Section 8 deduces the result on the Cousin complexes. Recollection of framed correspondences and some used homotopies is started in Section 2 and extended in Appendix A. Appendix A formulates the used result on the relative dimension of irreducible schemes over one-dimensional schemes with non-empty generic fibre. Appendix B collects results on line bundles.

1.4. Conventions.

1.4.1. Categories of schemes, relative dimension. \( \text{Sch}_S, \text{Aff}_S, \text{Sm}_S, \text{SmAff}_S, \text{Et}_S \) are the categories of schemes, affine schemes, smooth schemes, smooth affine schemes, and étale schemes over a base scheme \( S \). We say a scheme \( X \) has pure dimension \( d \) and write \( \dim X \), if each irreducible component of \( X \) has dimension \( d \). We say that \( X \in \text{Sch}_S \) is equidimensional of pure relative dimension \( d \) over \( S \), and write \( \dim_S X = d \) if for each \( z \in S \), \( \dim X \times_S z = d \). We say that a closed subscheme \( X' \) of \( X \) has positive relative codimension, and write \( \text{codim}_{X/S}(X') > 0 \), if \( \text{codim}(X' \times_X z)/(X \times_S z) > 0 \) for each \( z \in S \). \( T_{X/S} \) denotes the tangent sheaf of a scheme \( X \) over a scheme \( S \). Usually in Section 2, Section 4, Section 6 the base schemes are denoted \( S, B \) and \( V \) respectively.
1.4.2. Functions, sections, subschemes. Let \( F(X) = \Gamma(X, F) \) be the set of sections of a presheaf \( F \) on \( X \). We use notation \( \mathcal{O}(X) = \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X) \), where \( \mathcal{O}_X \) the sheaf of regular functions on the small Zariski site over \( X \).

Given morphism \( f : Z \to X \),
\[
\text{Cl}_X(Z) = \text{Cl}_{f,X}(Z) = \text{Cl}_X(f(Z)) = \overline{f(Z)}
\]
(1.5)
is the closure of the image; we skip \( f \) above, when it is defined by the context or \( Z \) is a constructible subscheme of \( X \).

Given a scheme \( X \), we write
\[
i_0 : X \simeq X \times \{0\} \to X \times \mathbb{A}^1, \quad i_1 : X \simeq X \times \{1\} \to X \times \mathbb{A}^1
\]
for the zero and unit sections.

Given a correspondence that defines a span of schemes \( X \xrightarrow{p} S \to Y \) for a closed subscheme \( Z \in Y \), \( c^{-1}(Z) \) is the closure of the image of \( S \times_Y X \) along \( p \), i.e.
\[
c^{-1}(Z) = \text{Cl}_X(p(S \times_Y X)).
\]
(1.6)

For a closed subscheme \( Z \subset X \), we denote by \( \mathcal{I}_{Z/X} = I_X(Z) \) the sheaf of ideals in \( \mathcal{O}_X \) formed by functions that vanish on \( Z \), and \( I_{Z/X} = I_X(Z) \subset \mathcal{O}(X) \) denotes the global sections. The vanishing locus of a sheaf of ideals \( I \subset \mathcal{O}(X) \) is denoted by \( Z(I) \). The vanishing locus of a section \( s \in \Gamma(X, V) \) of a vector bundle \( V \) is \( Z(s) \). We write \( s|_Z = r^*(s) \) for the inverse image of \( s \in \Gamma(X, F) \) along a given constructible injective morphism \( r : \tilde{Z} \to X \).

We denote by \( X^{(c)} \) the set of points and by \( X^{(c)} \) the set of closed subschemes of codimension \( c \) in a scheme \( X \). We use notation \( \mathcal{O}_{X,z} \) for the local ring of \( X \) at a point \( x \in X \), \( \mathcal{O}^h_{X,z} \) for the henselisation, and
\[
X_z = \text{Spec} \mathcal{O}_{X,z}, \quad X^h_z = \text{Spec} \mathcal{O}^h_{X,z},
\]
for schemes and canonical morphisms. Similarly, \( \mathcal{O}_{X,Z} \) and \( \mathcal{O}^h_{X,Z} \) are the localisation and the henselisation of \( \mathcal{O}_X \) at \( \mathcal{I}_{Z/X} \) for a closed subscheme \( Z \in X \), \( X_Z = \text{Spec} \mathcal{O}_{X,Z}, X^h_Z = \text{Spec} \mathcal{O}^h_{X,Z} \).

For \( X \in \text{Sch}_B \), \( \nu \in B \), we use notation
\[
X_\nu = X \times_B \nu.
\]

1.4.3. Cohomologies with respect to a topology. Given morphism \( \tilde{X} \to X \), there are the Čech complex \( \check{C}(X, F) \) and the extended Čech complex \( \check{C}^{\infty}(X, F) \):
\[
\ldots \to 0 \to F(\tilde{X}) \to \cdots \to F(\tilde{X} \times (t-1)) \to F(\tilde{X} \times t) \to \cdots,
\]
\[
\ldots \to 0 \to F(X) \to \cdots \to F(\tilde{X}) \to \cdots \to F(\tilde{X} \times (t-1)) \to F(\tilde{X} \times t) \to \cdots,
\]
where \( F(\tilde{X}) \) is located in the zeroth deg. For a Grothendieck topology \( \tau \) on a category \( C \), and a presheaf \( F \) on \( C \), we denote by \( F_{\tau} \) the sheafification, and use notation \( H^*_\tau(X, F) = H^*_\tau(X, F_{\tau}) \).

Definition 1.7. We call by reduced cohomologies the following groups
\[
\begin{align*}
H^i_{\tau}(X, F) &= 0, \ i < -1, \\
H^i_{\tau}(X, F) &= \ker(F(\tilde{X}) \to F_{\tau}(X)), \\
H^i_{\tau}(X, F) &= \ker(F(\tilde{X}) \to F_{\tau}(X)), \\
H^i_{\tau}(X, F) &= H^i_{\tau}(X, F_{\tau}), \ i > 0.
\end{align*}
\]
(1.8)

Remark 1.9. Note that the following conditions are equivalent
\[
H^0_{\tau}(X, F) = 0 \iff H^0_{\tau}(X, F) \cong F(X) \iff \begin{cases} H^0_{\tau}(X, F_{\tau}) \cong F(X), \\
H^i_{\tau}(X, F_{\tau}) = 0, \ i > 0. \end{cases}
\]

1.4.4. Sites, and sheaves. Given a Grothendieck topology \( \tau \) on a category \( C \), we denote the site by \( C^\tau \). \( \text{Pre} (S), \text{Sh}_n (S) \) and \( Z\text{Pre} (S) \) and \( Z\text{Sh}_n (S) \) denote the categories of presheaves and \( \tau \)-sheaves and additive presheaves and \( \tau \)-sheaves of abelian groups on \( \text{Sch}_S \). Given \( F \in \text{Pre}(S) \), and an \( X \in \text{Sch}_S \), denote by \( F_X \in \text{Pre}(X) \) the inverse image of \( F \) along the structure morphism \( X \to S \). We let us write \( H^*_\tau(X, F) \) for \( H^*_\tau(X, F_X) \) for a topology \( \tau \) on \( \text{Sch}_X \).
1.4.5. *Remark for the Nisnevich topology.* Also we use notation Nis$^X$ for the Nisnevich topology on ${\text{Sch}}_S$ over a scheme $X$. We let us write $F_{\text{Nis}}$ instead of $F_{\text{Nis}_X}$ for a presheaf $F$ on ${\text{Sch}}_S$. We use the notation Nis$^X_Z$ for the subtopology of Nis$^X$ that is trivial over the open subscheme $X - Z$, for a closed subscheme $Z$, and also use the notation Nis$^X$ and Nis$^X_Z$ for certain subtopologies of Nis$^X$ and Nis$^X_Z$, see Definition 1.5.

1.4.6. *Category of open pairs and linear additivisation.*

**Definition 1.10.** For any category $\mathcal{C}$ denote by $\mathcal{C}^{\text{pair}}$ that category with objects being morphisms of $\mathcal{C}$ and morphisms given by commutative squares. Given a functor of categories $\text{Sm}_S \rightarrow \mathcal{C}$, consider the subcategory of $\mathcal{C}^{\text{pair}}$ spanned by open immersions and denote by $\mathcal{C}^{\text{pair}}_{\text{open}}$ the quotient category with morphisms

$$
\mathcal{C}^{\text{pair}}(\{(X_1, U_1), (X_2, U_2)\}) = \text{Cofib}(\mathcal{C}(X_1, U_2) \to \mathcal{C}(X_1, U_1))
$$

where $c: \alpha \to (j_2 \circ \alpha, \alpha \circ j_1)$, $j_1: U_1 \to X_1$, $j_2: U_2 \to X_2$.

**Definition 1.11.** Given a category $\mathcal{C}$ enriched over pointed sets, the *linear additivisation* is the universal colimit functor $\mathcal{C} \to \mathcal{ZC}^{\oplus}$ to an additive category $\mathcal{ZC}^{\oplus}$ enriched over abelian groups.

2. **Framed correspondences and presheaves**

The section recollects some definitions from [23, 24, 50] and recalls the notion of first-order framed correspondences from [8], which is between framed correspondences and normally framed correspondences introduced independently in [19, 21].

**Remark 2.1.** First-order framed correspondences allow us to make to some technical simplification, see Proposition 3.12 though the original framed correspondences from [23] would be enough for our strategy.

**Remark 2.2.** The definition of correspondences of pairs here is slightly different from the one in [24].

Throughout the section $S$ denotes a base scheme.

2.1. **Correspondences categories and presheaves with transfers.** We recall the following notions from [23, 50] and [8].

**Definition 2.3.** (1) An *explicit framed correspondence of level $n$ from $X$ to $Y$ in $\text{Sch}_S$* is data $(Z, V, \varphi, g)$ given by an étale neighbourhood $V$ of a closed subscheme $Z$ in $\mathbb{A}_X^n$ finite over $X$, $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathcal{O}_V(\mathcal{V})^{\oplus}$, such that $Z(\varphi)$ in $X$ equals $Z$, and a regular map $g: V \to Y$.

(2) A *first-order framed correspondence of level $n$ from $X$ to $Y$* is data $(Z, \tau, g)$ given by a closed subscheme $Z$ in $\mathbb{A}_X^n$, a trivialisation of the conormal sheaf of $Z \tau: \mathcal{I}_Z/\mathcal{I}_Z^2 \simeq \mathcal{O}_Z^n$, and a regular map $g: \mathcal{I}_Z^2/\mathcal{I}_Z^2 \to Y$. Note that, the pair $(\tau, g)$ defines the morphism of schemes

$$
(\tau, g): Z(\mathcal{I}_Z^2/\mathcal{I}_Z^2) \to \mathbb{A}_X^n \times Y
$$

such that $Z(\mathcal{I}_Z^2/\mathcal{I}_Z^2 \times \mathbb{A}_X^n) (0 \times Y) \simeq Z$. Denote by $\text{Fr}_{\alpha}^{\text{nth}}(X, Y)$ the set of such $(Z, \tau, g)$ pointed at the element with $Z = \emptyset$.

**Definition 2.5.** (1) Denote by $\text{Fr}_n(X, Y)$ the set of equivalence classes with respect to the equivalence relation $(Z, V_1, \varphi_1, g_1) \sim (Z, V_2, \varphi_2, g_2)$ given by common shrinkings $V$ of $V_1$ and $V_2$ such that the inverse images of $(\varphi_1, g_1)$ and $(\varphi_2, g_2)$ are equal. Define the category $\text{Fr}_n(X, Y)$ with objects being the same as in $\text{Sm}_S$, morphisms given by pointed sets $\text{Fr}_n(X, Y) = \bigvee_n \text{Fr}_n(X, Y)$ pointed at the elements with $Z = \emptyset$, see [23, Definition 2.3].

(2) Define the **pointed category** $\text{Fr}_n^{\text{nth}}(S)$ with morphisms $\bigvee_n \text{Fr}_n^{\text{nth}}(X, Y)$, and composite morphism for $(Z_1, \tau_1, g_1) \in \text{Fr}_n^{\text{nth}}(X_0, X_1)$ and $(Z_2, \tau_2, g_2) \in \text{Fr}_n^{\text{nth}}(X_1, X_2)$ given by $(Z_{12}, \tau_{12}, g_{12}) \in \text{Fr}_{n_1 + n_2}^{\text{nth}}(X_0, X_2)$, where $Z_{12} = Z_1 \times_X Z_2$, $(\tau_{12}, g_{12}): Z(\mathcal{I}_Z^2/\mathcal{I}_Z^2) \to Z(\mathcal{I}_Z^2/\mathcal{I}_Z^2) \to X_2$, the first morphism is induced by $(\tau_2, g_2)$, and the second one is induced by $(\tau_1, g_1)$.

(3) Using Definition 1.10 we define categories $\text{Sm}_S^{\text{pair}}$ and $\text{Fr}_n^{\text{pair}}(S)$. Denote $\text{Fr}_n^{\text{nth}}(\text{Sm}_S^{\text{pair}}) = (\text{Fr}_n^{\text{nth}}(S))^{\text{pair}}$. 
Remark 2.6. $Fr^n_{th}(X,U),(Y,V))$ is the pointed set that is the quotient of the set of the elements $(Z,\tau,g) \in Fr^n_{th}(X,Y)$ such that

$$((X \setminus U) \times_X Z)_{red} \supset (Z \times_Y (Y \setminus V))_{red},$$

with respect to the subset of elements $(Z,\tau,g) \in Fr^n_{th}(X,Y)$ such that $\emptyset = Z \times_Y (Y \setminus V)$.

Example 2.7. $\sigma_X = (0 \times X,t,g) \in Fr_1(X,X)$, where $t$ is the coordinate function on $A_X^1$, and $g$ is the projection to $X$. For a matrix $G \in GL_n(X)$, we define

$$\sigma^G_{\tau} = (0 \times V,G(t_1,\ldots,t_n),pr),$$

where $pr: A^n \to V$ is the canonical projection. Denote $\sigma^D f = f \circ \sigma^D \in Fr_n(X,Y)$ for a given $S$-schemes $X$ and $Y$.

A framed presheaf is a functor $Fr_+(S)^{op} \to Set$. A framed presheaf is quasi-stable, if the morphisms $\sigma^\tau_X$ are auto-morphisms of $F(X)$ for all $X$. There is the functor $Fr_+(S) \to Fr^n_{th}(S)$ defined by the natural map

$$Fr_n(X,Y) \to Fr^n_{th}(X,Y): (Z,V,\varphi,g) \mapsto (Z,\tau,g)_{|Z}, \quad (2.8)$$

where $\tau$ is induced by $\varphi|_{Z(\mathbb{A}^2_X(Z))}$, and similar definitions apply to first-order framed presheaves.

Lemma 2.9. Any $A^1$-invariant quasi-stable framed presheaf $F$ on $Aff_S$ for an affine scheme $S$ is canonically equipped with the structure of a quasi-stable first-order framed presheaf.

Proof. Let $c, c' \in Fr_n(X,Y)$ be elements that images in $Fr^n_{th}(X,Y)$ are equal. Denote $Z_{[2]} = Z(\mathcal{I}^2(Z))$. Denote by $g Z_{[2]}$ the morphism $g|_{Z_{[2]}} = g'|_{Z_{[2]}}: Z_{[2]} \to Y$. Consider the map

$$\tilde{g}: E \to Y,$$

that exists by [32, Theorem 1.8] see also [38, [4, Lemma 3.11], since the canonical morphism $E \to (A^1 \times A^n)^{th}_{A^1 \times Z}$ is an affine henseleian pair. Consider

$$h = (A^1 \times Z, \lambda \varphi + (1 - \lambda)\varphi', \tilde{g}) \in Fr_n(A^1 \times X,Y).$$

Then $h \circ i_0 = c, h \circ i_1 = c'$, and since $F$ is $A^1$-invariant, it follows that $c' = c^*$. Thus the induced morphisms $c^*, c'^*: F(Y) \to F(X)$ are equal. Hence the claim of the lemma follows, because the map $(2.8)$ is surjective and preserves correspondences of the form $\sigma_X, X \in Sm_S$.

2.2. Linearised correspondences, and Nisnevich sheafification.

Definition 2.11. (1) Let $ZF_n(S)$ denote the linear additivisations of the category $Fr_+(S)$, see Definition [11, F] $ZF_+(S)$ is denoted by $ZF_n(X,Y) = coker(ZF_n(X \times A^1, Y) \xrightarrow{i_{0,-}} ZF_n(X,Y))$. Similarly for $Fr^n_{th}(S)$.

Proposition 2.12. Consider the inverse and direct image functors $\gamma^*: ZSh(S) \to ZSh(ZF_+(S)), \gamma_*: ZSh(ZF_+(S)) \to ZSh(S)$, see Section [13.4]. Then for any $E \in ZSh(S)$ and $F \in ZSh(ZF_+(S))$, $Ext^1_{ZSh(S)}(E,\gamma_*F) \simeq Ext^1_{ZSh(ZF_+(S))}(\gamma^*E,F)$.

Proof. Since the category $ZSh(S)$ is generated via colimits by the representable sheaves $Z(U), U \in Sm_S$, it is enough to prove the claim for $E = Z(U)$. Since $\gamma^* E = ZF_+(U)_{Nis}$, the claim follows from [24, Proposition 16.2].
Definition 2.13. Let $U, X \in \text{Sch}_X$, and $c \in \text{Fr}_c(U, X)$. For any closed subscheme $Z$ in $X$, and $Y = c^{-1}(Z)$, see (1.9), for any $F \in Z\text{Sh}(ZF_*(S))$, define the canonical morphism on reduced cohomologies with support $c_* : H^i_Z(X, F) \to H^i_Y(U, F)$ as the composite morphism

$$H^i_Z(X, F) = \text{Ext}^i_{Z\text{Sh}(S)}(Z(X)/Z(X - Z), F) \cong \text{Ext}^i_{Z\text{Sh}(ZF_*(S))}(ZF(X)/ZF(X - Z), F) \to \text{Ext}^i_{Z\text{Sh}(ZF_*(S))}(ZF(U)/ZF(U - Y), F) \cong \text{Ext}^i_{Z\text{Sh}(S)}(Z(U)/Z(U - Y), F) = H^i_Y(U, F).$$

3. Topology over a quotient-space $X/(X - Z)$
sect:FrCorPresh

3.1. Topology $\text{Nis}_X^Z$. Recall that $\text{Nis}_X^Z$ denotes the Nisnevich topology on $\text{Sch}_X$ over a scheme $X$.

Definition 3.1. Let $Z$ be a closed subscheme of $X$. Define the topology $\text{Nis}_X^Z$ on $\text{Sch}_X$ as the strongest subtopology of $\text{Nis}_X$ that restriction to $\text{Sch}_{X - Z}$ is trivial.

Lemma 3.2. (1) An étale morphism $\tilde{U} \to U$ is a $\text{Nis}_Z^Z$-covering if there is a dashed arrow in the triangle

$$\xymatrix{ \tilde{U} \ar[dr] \ar[r] & U \ar[d] \cr V \ar[r] & U }$$

for any scheme $V$ of the forms

$$V = U^h_v, \quad v \in U, \quad V = U \times X (X - Z). \quad (3.3)$$

(2) The topology $\text{Nis}_Z^Z$ has enough set of points given by the schemes $\text{3.3}$ for all $U \in \text{Sch}_X$.

Proof. It follows from Definition 3.1 that a $\text{Nis}_Z^Z$-covering $\tilde{U} \to U$ is a Nisnevich coverings such that there is a dashed arrow in the diagram

$$\xymatrix{ \tilde{U} \ar[d] \cr U \times X (X - Z) \ar[r] & U. } \quad (3.4)$$

Then both points (1) and (2) follow. □

3.2. Framed transfers for cohomologies. Recall that $F_T = p^*(F)$ is the inverse image of $F \in \text{Pre}(\text{Sch}_X)$ along a given morphism $p : T \to X$, and $H^*_T(T, F) = H^*_p(T, F_T)$, see Section 1.4.4

Lemma 3.5. For any $T \in \text{Sch}_X$ finite over $X$ and any additive presheaf of abelian groups $F \in Z\text{Pre}(X)$,

$$H^i_{\text{Nis}_Z^Z}(X, p_*(F_T)) \cong H^i_{\text{Nis}_Z^Z}(T, F)$$

for all $i \in \mathbb{Z}$, where $p : T \to X$ is the canonical morphism.

Proof. Consider the spectral sequence $H^i_{\text{Nis}_Z^Z}(X, Rp_!(F_T)) \Rightarrow H^{i+j}_{\text{Nis}_Z^Z}(T, F)$, where $Rp_!(F_T)$ is the $\text{Nis}_Z^Z$-sheafification of the presheaf $H^j_{\text{Nis}_Z^Z}(\_ X T, F)$ on $\text{Sch}_X$. It follows by Lemma 3.2 (2) that $H^j_{\text{Nis}_Z^Z}(T \times_X U^h_v, F) = 0$ for any $U \in \text{Sch}_X$, $v \in U$, because

$$T \times_X U^h_v = \coprod_{y \in T \times_X v} (T \times_X U)^h_y$$
since $T$ is finite over $X$. On the other hand, $H^j_{Nis^X_Z}(T \times_X (X - Z), F) = 0$ since $Nis^X_Z$ is trivial on $\text{Sch}_{X-Z}$. Thus by Lemma 3.2(2),
\[
\begin{cases}
Rp^*_j(p^*(F)) = 0 & \text{for } j > 0, \\
Rp^*_j(p^*(F)) = p_*(p^*(F)) & \text{for } j = 0.
\end{cases}
\]

Lemma 3.6. For a closed subscheme $Z$ in a scheme $X$, for any quasi-stable linear (first-order) framed presheaf $F$ over $X$, the presheaf $\text{Sch}_X \to \text{Ab}; \ T \mapsto H^*_\text{Nis}^X_Z(X, p_*(F_T))$, $p: T \to X$, has a canonical structure of a quasi-stable (first-order) framed presheaf, such that for a pair of closed subschemes $Z_1 \subset Z_2$ in a scheme $X$, the canonical natural transformations
\[
H^*_\text{Nis}^X_{Z_2}(X, p_*(F_{T_2})) \to H^*_\text{Nis}^X_{Z_1}(X, p_*(F_{T_1})) \to H^*_\text{Nis}(T, F)
\]
are morphisms of framed presheaves. Similarly for the reduced cohomologies (3.8).

Proof. For any $c \in \text{Fr}_n(U_1, U_2)$ over $X$ and a morphism $\tilde{X} \to X$, the base change along a covering $\tilde{X}^{\times l} \to X$ for each $l \in \mathbb{Z}_{\geq 0}$ induces $c^l \in \text{Fr}_n(U_1 \times X \tilde{X}^{\times l}, U_2 \times X \tilde{X}^{\times l})$. For any $F \in Z\text{Pre}(Z\text{Fr}_n(X))$, correspondences $c^l$ induce the morphism of Čech complexes with respect to the coverings of $U_1$ and $U_2$. This defines a structure of a quasi-stable linear (first-order) framed presheaf on cohomologies natural with respect to morphisms (3.7). The claim for $\text{Fr}^{1th}_+(S)$ and reduced cohomologies follows similarly.

For a morphism of schemes $T \to X$, and a closed subscheme $Z$ in $X$, the topology $Nis^X_Z$ restricted on $\text{Sch}_T$ equals $Nis^T_{Z \times X Z}$. So for each $F \in Z\text{Pre}(X)$, there is a canonical isomorphism
\[
H^*_{Nis^T_{Z \times X Z}}(T, F) \cong H^*_{Nis^X_Z}(T, F).
\]

Definition 3.9. Let $c = (T, \tau, g) \in \text{Fr}^{1th}_n((X, U), (Y, V))$ for $(X, U), (Y, V) \in \text{Sm}_S^{\text{pair}}$ over a base scheme $S$. Let $W = Y \setminus V$, $Z = X \setminus U$. Define the homomorphism
\[
c^* : H^*_{\text{Nis}^Y_W}(Y, F_Y) := H^*_{\text{Nis}^W}(Y, F_Y) \to H^*_{\text{Nis}^X_Z}(X, F_X) =: H^*_{\text{Nis}^X_Z}(X, F_X)
\]
as the composite homomorphism
\[
\begin{array}{c}
H^*_{\text{Nis}^Y_W}(Y, F_Y) \\
\downarrow g^* \\
\downarrow \text{Lm}[3, 6] \\
H^*_{\text{Nis}^X_Z}(X, F_X)
\end{array}
\]
where $H^*(-) = H^*_\nu(-, F_\nu)$, and the second morphism in the sequence is provided by (3.8) and the equality $c^{-1}(W) = Z$. The same applies to $c^* : H^*_{\text{Nis}^X_Z}(X, F) \to H^*_{\text{Nis}^Y_W}(Y, F)$.

Let $\text{Fr}^{1th}_{+(p)}(S)$ denote the subcategory in $\text{Fr}^{1th}_+(S)$ formed by morphisms such that $g \cong \text{id}_{T_2}$, where $T_2 = Z(T^2(T))$, in (2.4). For any $c = (T, \tau, g) \in \text{Fr}^{1th}_n(X, Y)$, there is a decomposition
\[
c = g \circ p_!, \quad g \in \text{Fr}_0(T_{[2]}), \quad c_p = (T, \tau, \text{id}_{T_{[2]}}) \in \text{Fr}_n^{1th}(X, T_{[2]}),
\]
and similarly for $\text{Fr}^{1th}_{+(p)}(S)$.

Proposition 3.12. For any quasi-stable first-order framed linear presheaf $F$ over a base scheme $S$, the presheaf $(X, X - Z) \mapsto H^*_{\text{Nis}^X_Z}(X, F)$ has the canonical structure of the quasi-stable first-order framed presheaf. The same applies to the presheaf $(X, X - Z) \mapsto H^*_{\text{Nis}^X_Z}(X, F)$ and also to $\Lambda^1$-invariant framed presheaves over a base scheme $S$. 
Proof. To prove the claim we need to show isomorphisms $(c_2 \circ c_1)^* \cong c_1^* \circ c_2^*$. Because of (3.11) it is enough to consider four cases:

1. For $c_1, c_2 \in \text{Fr}_{+}^\text{pair}(S)$, the claim follows by Lemma 3.6.
2. For $c_1 = (T, \tau, \text{id}_{Z(Z^2(T))})$, $c_2 = g$, the claim holds by Definition 3.9.
3. For $c_1 = f \in \text{Sch}_S^\text{pair}((X, X - Z), (X', X' - Z'))$, $c_2 = (Y, \tau, \text{id}_{Y}) \in \text{Fr}_{+}^\text{pair}((X', X' - Z'), (Y, Y - W))$, there is the commutative diagram:

\[
\begin{array}{ccc}
H^*_\text{Nis}_X(T, F) & \to & H^*_\text{Nis}_Y(Y, F) \\
\downarrow \cong & & \downarrow \cong \\
H^*_\text{Nis}_X'(T, F) & \to & H^*_\text{Nis}_Y'(Y, F) \\
\downarrow \cong & & \downarrow \cong \\
H^*_\text{Nis}_X(X, p_1^T(FT)) & \to & H^*_\text{Nis}_Y'(X, p_1^Y(FY)) \\
\downarrow \cong & & \downarrow \cong \\
H^*_\text{Nis}_X(X, F) & \to & H^*_\text{Nis}_X'(X', F),
\end{array}
\]

where $T = Y \times X$, $p^T : T \to S$, $p^Y : Y \to S$.

The case of reduced cohomologies (1.8) is similar. The claim for $\mathbb{A}^1$-invariant framed presheaves follows by (2.8) and Lemma 2.9.

Lemma 3.13. Under the assumptions of Proposition 3.12 the homomorphism

\[ H^*_\text{Nis}_X(X, F) \to H^*_\text{Nis}_X(X, F) \]

defines the homomorphism of presheaves on $\text{Fr}_{+}^\text{pair}(S)$.

Proof. The claim follows because natural transformations in (3.10) commute with the morphisms $H^*_\nu(-, F_\nu) \to H^*_\text{Nis}_X(-, F_\text{Nis})$ for $\nu$ as in Definition 3.9.

4. Cohomologies out of infinity on $\mathbb{A}^1$. 

In this section, we show triviality of “cohomologies out of infinity” on the relative affine line $\mathbb{A}^1_V$ over a local henselian scheme $V$, where “cohomologies out of infinity” means “the universal cohomologies trivial at infinity” defined as cohomologies with respect to the topology $\text{Fin}_V$, see Example 4.2.

Definition 4.1. Let $X$ be a scheme over a scheme $S$. Denote by $\text{Prop}_S^X$ the minimal topology on $\text{Sh}_X$ that contains topologies $\text{Nis}_Z^X$ for all closed subschemes $Z$ in $X$ proper over $S$, see Definition 3.11.

Example 4.2. Let $S$ be a scheme. Define $\text{Fin}_S = \text{Prop}_{S}^{\mathbb{A}^1 \times S}$, which is the minimal topology on $\text{Sh}_{\mathbb{A}^1 \times S}$ that contains topologies $\text{Nis}_Z^{\mathbb{A}^1 \times S}$ for all closed subschemes $Z$ in $\mathbb{A}^1 \times S$ finite over $S$.

Remark 4.3. The basic principles of the arguments in this section up to “cohomologies out of infinity” were introduced in 49 and reworked with the use of open pairs categories suggested in 11 and Serre’s theorem 24, III, Corollary 10.7 suggested to the author by I. Panin.

The argument strategy for excision isomorphisms used here came from 12, 13, and constructions of correspondences from 22.
4.1. Excision isomorphisms.

**Proposition 4.4 (Étale excision)** Let $X$ be an étale neighbourhood of a closed subscheme $Z$ in an essentially smooth local henselian scheme $X$ over a field $k$. Then for any quasi-stable framed linear presheaf $F$, the following square is Cartesian and cocartesian

\[
\begin{array}{ccc}
F(X - Z) & \longrightarrow & F(\widetilde{X}) \\
\downarrow & & \downarrow \\
F(X) & \longrightarrow & F(X).
\end{array}
\]

**Proposition 4.5. (Zariski excision with finite support)** Let $V$ be a local scheme. Let $Z_0, Z_1$ be closed subschemes in $\mathbb{A}^1_V$, such that $Z_0 \cap Z_1 = \emptyset$, and $Z_0$ is finite over $V$. Then for any quasi-stable framed linear presheaf $G$ of $\mathcal{H}_{\text{pair}}(V) \to \text{Ab}$, the morphism

\[
j^* : G(\mathbb{A}^1_V, \mathbb{A}^1_V - Z_0) \to G(\mathbb{A}^1_V, \mathbb{A}^1_V - Z_1 - Z_0),
\]

where $g = \infty$ is induced by the canonical open immersion of pairs

\[
j : (\mathbb{A}^1_V - Z_1, \mathbb{A}^1_V - Z_1 - Z_0) \to G(\mathbb{A}^1_V, \mathbb{A}^1_V - Z_0),
\]

is an isomorphism. See Definition 2.3, Definition 1.11, Section 2.2, and Definition 1.10 for the category $\mathcal{H}_{\text{pair}}^*(V)$.

**Proof.** Denote by $\mathcal{O}(1)$ the canonical ample bundle on $\mathbb{P}^1$ and let $t_{\infty}$ denote the section that vanishing locus is $\infty$. Denote $\mathbb{A}^1_V = \mathbb{A}^1 \times V = \mathbb{P}^1 \setminus \{\mathbb{A}^1\}$, and let $Z_1$ be the closure of $Z_1$ in $\mathbb{P}^1_V$. Since $Z_0$ is finite over $V$ it is a closed subscheme in $\mathbb{P}^1_V$. So $Z_0 \cap Z_1 = Z_0 \cap Z_1 = \emptyset$. Denote by $\delta \in \Gamma(\mathbb{A}^1_V \times V, \mathbb{P}^1_V, \mathcal{O}(1))$ a section that vanishing locus is the graph of the canonical immersion $\mathbb{A}^1_V \to \mathbb{P}^1_V$. By Lemma 3.1 for any $l \in Z$, there are invertible sections

\[
\gamma_{l, \infty} \in \Gamma(\infty \times V \cup Z_1, \mathcal{O}(l)), \gamma_{Z_0} \in \Gamma(Z_0, \mathcal{O}(l - 1)).
\]

We are going to show that (4.6) has left and right inverse morphisms.

Part 1) Firstly, we construct the left inverse. By [27, III, Corollary 10.7] for large enough

\[
l \in Z,
\]

there exist sections

\[
s_1 \in \Gamma(\mathbb{A}^1 \times \mathbb{P}^1_V, \mathcal{O}(l)), \quad s_0 \in \Gamma(\mathbb{A}^1 \times \mathbb{P}^1_V, \mathcal{O}(l - 1)),
\]

\[
s_{1|A^1 \times Z_0} = \gamma_{Z_0} \delta|_{\mathbb{A}^1 \times Z_0}, \quad s_0|_{\mathbb{A}^1 \times Z_0} = \gamma_{Z_0},
\]

\[
s_{1|A^1 \times (\infty \times V \cup Z_1)} = \gamma_{l, \infty}, \quad s_0|_{A^1 \times (\infty \times V)} = \gamma_{l, \infty} \delta^{-1}|_{A^1 \times (\infty \times V)},
\]

\[
s_0|_{\mathbb{A}^1} = t_{\infty}.
\]

Define the section $s = s_1 \lambda + s_0 (1 - \lambda)$ and morphisms in $\mathcal{H}_{\text{pair}}^*(V)$

\[
c = (Z(s), \mathbb{A}^1 \times \mathbb{A}^1_V \times V \mathbb{A}^1_V, s/t_{\infty}, g), \quad c' = (Z(s), 0 \times \mathbb{A}^1_V \times V \mathbb{A}^1_V, 0, s_0/t_{\infty}, g_0)
\]

\[
c_1 = (Z(s_1), 1 \times \mathbb{A}^1_V \times V \mathbb{A}^1_V, s_1, g_1), \quad \tilde{c}_0 \in \mathcal{H}_{\text{pair}}^*(0 \times (\mathbb{A}^1_V, \mathbb{A}^1_V - Z_0), (\mathbb{A}^1_V, \mathbb{A}^1_V - Z_0), (\mathbb{A}^1_V, \mathbb{A}^1_V - Z_0)),
\]

where $g, g_0$ and $\tilde{g}_1$ are given by the projections to the second multiplicand. Define

\[
id^r = (Z(\delta), 0 \times \mathbb{A}^1_V \times V \mathbb{A}^1_V, s_0/t_{\infty}, g_0)
\]

$g_0$ is induced by the projection to the second multiplicand. Then the following equalities hold

\[
c \circ i_1 = j \circ \tilde{c}_0, c \circ i_0 = \text{id} + e \circ c'_0
\]

in $\mathcal{H}_{\text{pair}}^*(V)$, where

\[
i_0, i_1 : (\mathbb{A}^1_V, \mathbb{A}^1_V - Z_0) \to (\mathbb{A}^1 \times (\mathbb{A}^1_V, \mathbb{A}^1_V - Z_0),
\]

\[
c : (\mathbb{A}^1_V - Z_0, \mathbb{A}^1_V - Z_0) \to (\mathbb{A}^1_V, \mathbb{A}^1_V - Z_0).
\]
Since $G$ is $A^1$-invariant the framed correspondence $\id^\tau$ induces the same endomorphism on $G(A^1_1, A^1_1 - Z_0)$ as $\sigma(A^1_1, A^1_1 - Z_0)$ because of the $A^1$-homotopy

$$(Z(\delta), (s_0 \lambda + \delta t_{l-1}^\delta)/(l_{\infty}^l, g_0 \circ \pr)) \in FV^\text{pair}_{1} (A^1 \times (A^1_1, A^1_1 - Z_0), (A^1_1, A^1_1 - Z_0)), \quad (4.7)$$

where $\pr: A^1 \times (A^1_1, A^1_1 - Z_0) \to (A^1_1, A^1_1 - Z_0)$. Since $G$ is additive, we have $i_0^* c^* = (\id^\tau)^* + (e_0^*)^*$. The morphism $c^*$ is trivial morphism in the category of pairs $FV^\text{pair}(V)$. Hence

$$c^* \tau^* = (\id^\tau)^* + (e_0^*)^* = \sigma^*(A^1_1, A^1_1 - Z_0).$$

Since $G$ is quasi-stable, the morphism (4.8) has the left inverse.

Part 2) Next, we construct the right inverse to (4.8). Define $V_1 = A^1_1 - Z_1$. By [27, III, Corollary 10.7], for large enough $l \in Z$, there exist sections

$$s_1 \in \Gamma(A^1 \times F_{V}, O(I)), \quad s_0 \in \Gamma(V_1 \times F_{V}, O(l - 1)).$$

Define sections and framed correspondences of open pairs over $V$

$$c_1 = (Z(s'), 0 \times V_1 \times V, s_1/(l_{\infty}^l, g)) \in FV^\text{pair}_{1}(A^1 \times (V_1, V_1 - Z_0), (V_1, V_1 - Z_0)), \quad c_0 = (Z(s), 0 \times V_1 \times V, s_0/(l_{\infty}^l, g_0)) \in FV^\text{pair}_{1}(0 \times (V_1, V_1 - Z_0), (V_1 - Z_0, V_1 - Z_0)), \quad \tilde{c}_1 = (Z(s_1), 1 \times A^1_1 \times V, s_1/(l_{\infty}^l, g)) \in FV^\text{pair}_{1}(1 \times (A^1_1, A^1_1 - Z_0), (V_1, V_1 - Z_0)),$$

where $g$, $g_0$ and $\tilde{g}_1$ are the morphisms to the right side multiplicands $V_1$ and $V_1 - Z_0$. Define

$$\id^\tau = (Z(\delta), 0 \times V \times V, s_0, g_0) \in FV^\text{pair}_{1}(V_1, V_1 - Z_0),$$

where $g_0$ is the morphism to the right side $V_1$. Then $c \circ i_0 = j \circ \tilde{c}_1$ and $c \circ i_0 = \id^\tau + c_0^*$ in $ZF^\text{pair}_{1}(V)$,

$$i_0, i_1: (V_1, V_1 - Z_0) \to A^1_1 \times (V_1, V_1 - Z_0), \quad c: (V_1 - Z_0, V_1 - Z_0) \to (V_1, V_1 - Z_0).$$

The framed correspondences $\id^\tau$ and $\sigma(A^1_1, A^1_1 - Z_0)$ induce the same endomorphism on $G(V_1, V_1 - Z_0)$ since $G$ is $A^1$-invariant, and because of the $A^1$-homotopy

$$(Z(\delta), (s_0 \lambda + \delta t_{l-1}^\delta)/(l_{\infty}^l, g_0 \circ \pr)) \in FV^\text{pair}_{1}(A^1 \times (V_1, V_1 - Z_0), (V_1, V_1 - Z_0)),$$

where $\pr: A^1 \times (V_1, V_1 - Z_0) \to (V_1, V_1 - Z_0)$. Since $G$ is additive, we have $i_0^* c^* = (\id^\tau)^* + (e_0^*)^*$. The object $(V_1 - Z_0, V_1 - Z_0)$ is trivial in the category of pairs, so the morphism $c_0^*$ is trivial, and the induced morphism $(c^*_0)^*$ is trivial too. Thus

$$j^* c_1^* = (\id^\tau)^* + (e_0^*)^* = (\id^\tau)^* = \sigma^*(V_1 - V_1, V_1 - Z_0),$$

where the right side equality follows because of an $A^1$-homotopy of framed correspondences similar to (4.7). Since $G$ is quasi-stable, (4.8) is an automorphism, and consequently, (4.6) has a right inverse.

**Corollary 4.9.** Let $Z \subset A^1_1$ be finite over a local scheme $V$, and $X' \subset X$ be a pair of open neighbourhoods of $Z$. Then for any $A^1$-invariant quasi-stable framed linear presheaf $G: ZF^\text{pair}_{1}(V) \to \Ab$, the canonical morphism

$$G(X', X' - Z) \to G(X, X - Z),$$

is an isomorphism, see Definition 2.3, Definition 11.1, Section 2.2 and Definition 11.10.
Proof. By Proposition 4.13, the claim holds for \( X = \mathbb{A}^1_V \). For an arbitrary open neighbourhood \( X \) of \( Z \) the claim follows because of the composition \( G(X', X' - Z) \simeq G(\mathbb{A}^1_V, \mathbb{A}^1_V - Z_0) \simeq G(X, X - Z) \). \( \square \)

**Corollary 4.10.** Under the assumptions of Corollary 4.9, for any \( \mathbb{A}^1 \)-invariant quasi-stable framed linear presheaf \( F \) on \( \text{Sm}_V \), the following square is Cartesian and coCartesian

\[
\begin{array}{ccc}
F(X' - Z) & \longrightarrow & F(X') \\
\downarrow & & \downarrow \\
F(X - Z) & \longrightarrow & F(X).
\end{array}
\]

**Proof.** Consider the framed presheaves on \( ZP^\text{pair}_V(V) \) given by

\[
(X, U) \mapsto \ker(F(X) \to F(U)), \quad (X, U) \mapsto \text{coker}(F(X) \to F(U))
\]  
(4.11)

By Corollary 4.9 since the morphism \( j^* \) induced by the canonical morphism

\[
j: (\mathbb{A}^1_V - Z_1, \mathbb{A}^1_V - Z_1 - Z_0) \to (\mathbb{A}^1_V, \mathbb{A}^1_V - Z_0)
\]

is an isomorphism for any of presheaves in (4.11). So the claim follows. \( \square \)

**Corollary 4.12.** Let \( X \) be an open subscheme of \( \mathbb{A}^1_V \) over a local scheme \( V \). Let \( Z \subset X \) be finite over \( V \) with the unique closed point \( z \). Then for any \( \mathbb{A}^1 \)-invariant quasi-stable framed linear presheaf \( F \) over \( V \), the square

\[
\begin{array}{ccc}
F(X_z - Z) & \longrightarrow & F(X_z) \\
\downarrow & & \downarrow \\
F(X - Z) & \longrightarrow & F(X)
\end{array}
\]

is Cartesian and coCartesian, where \( X_z \) denotes the local scheme of \( X \) at \( z \).

**Proof.** The claim follows from Corollary 4.10. \( \square \)

### 4.2. Coverings and cohomologies.

**Theorem 4.13.** Let \( V \) be a local henselian scheme, and \( F \) be an \( \mathbb{A}^1 \)-invariant quasi-stable framed linear presheaf over \( V \). Let \( X \) be an open subscheme of \( V \times \mathbb{A}^1 \). Then

\[
H^\cdot_{\text{Nis}_V}(X, F) = 0
\]

for any closed subscheme \( Z \) in \( X \) finite over \( V \).

**Proof.** Since \( Z \) is finite over a local henselian scheme, the connected components of \( Z \) are local henselian schemes. We prove the claim by induction on the number of the connected components of \( Z \). If \( Z = \emptyset \), the claim is a tautology, since the topology \( \text{Nis}_V \) is trivial. Let \( Z = Z_0 \sqcup Z_1 \), where \( Z_0 \) is non-empty local henselian. Let \( z \) be the closed point of \( Z_0 \).

Consider the long exact sequence

\[
\cdots \to H^{n-1}_{\text{Nis}_V}(X - Z, F) \to H^n_{\text{Nis}_V}(X, F) \to H^n_{\text{Nis}_V}(X - Z_0, F) \oplus H^n_{\text{Nis}_V}(X - Z_1, F) \to H^n_{\text{Nis}_V}(X - Z, F) \to \cdots
\]  
(4.14)

Since any \( \text{Nis}_V \)-covering over \( X - Z \) admits a left inverse morphism, \( H^{n-1}_{\text{Nis}_V}(X - Z, F) = 0 \). Next, \( H^n_{\text{Nis}_V}(X - Z_0, F) \cong H^n_{\text{Nis}_V}(X - Z_0, F) = 0 \), because \( (X - Z_0) \cap Z = Z_1 \), and by the inductive assumption applied to the topology \( \text{Nis}_{V_1} \). Further,

\[
H^n_{\text{Nis}_V}(X - Z_1, F) \cong H^n_{\text{Nis}_{V_0}}(X - Z_1, F) \cong H^n_{\text{Nis}_{V_0}}(X - Z_1, F) \cong H^n_{\text{Nis}_{V_0}}(X - Z_1, F) \cong H^n_{\text{Nis}_{V_0}}(X - Z_1, F) \cong 0.
\]
Here (0) holds since \((X - Z_1) \cap Z = Z_0\), (1) holds since \(H^0_{\text{Nis}_{Z_0}}(X - Z_1 - Z_0, F) \cong 0\) because \(X - Z_1 - Z_0\) is a Nisnevich point, (2) and (3) follow by Corollary 4.12 and Proposition 4.3 respectively both applied with substitutions \(Z = Z_0\) because each Nisnevich covering of \(X - Z_1\) and \(X_2\) has a shrinking given by one Nisnevich square, and (4), (5) hold because \(X^h_{\text{S}}\) and \(X^h_{\text{S}} - Z_0\) are Nisnevich points. 

**Definition 4.15.** Let \(S\) be a scheme. Denote by \(N\text{is}\) the topology on \(\text{Sch}_S\) that is the weakest topology such that the functor \(\text{Et}_S \to \text{Sch}_S\) is continuous.

**Corollary 4.16.** Let \(S\) be a scheme. Consider the topology \(N\text{is}\cup \text{Fin}_S\) on \(\text{Sm}_{S,\mathbb{A}^1}\), see Definition 4.12 and Example 4.2. Then the projection \(S \times \mathbb{A}^1 \to S\) induces a natural isomorphism

\[
H^*_{\text{Nis}_{S\cup \text{Fin}}}(S \times \mathbb{A}^1, F) \cong H^*_S(S, F)
\]

for any \(\mathbb{A}^1\)-invariant quasi-stable framed linear presheaf \(F\) on \(\text{Sm}_S\).

**Proof.** Consider the presheaves on the small étale site over \(S\)

\[
U \mapsto h^q_{\text{Nis}}(U) = H^q_{\text{Nis}}(U \times \mathbb{A}^1, F), \quad U \mapsto h^q_{\text{nfin}}(U) = H^q_{\text{nfin}}(U \times \mathbb{A}^1, F).
\]

The canonical morphism \(h^q_{\text{Nis}} \to h^q_{\text{nfin}}\) is a Nisnevich local isomorphism by Theorem 4.13 applied to the schemes \(X = \mathbb{A}^1 \times U^h_v\) for all \(v \in U\), \(U \in \text{Et}_S\). Then the induced morphism of spectral sequences

\[
H^p_{\text{Nis}}(S, h^q_{\text{Nis}}) \Rightarrow H^p+q_{\text{Nis}}(S \times \mathbb{A}^1, F), \quad H^p_{\text{Nis}}(S, h^q_{\text{nfin}}) \Rightarrow H^p+q_{\text{nfin}}(S \times \mathbb{A}^1, F)
\]

is an isomorphism. □

5. Finite support injectivity criterion

In this section, we formulate an enough criterion for the injectivity of the restriction homomorphism

\[
H^*_S(X, F_{\text{Nis}}) \hookrightarrow H^*_S(X^{(0)}, F_{\text{Nis}})
\]

for an \(\mathbb{A}^1\)-invariant quasi-stable framed linear presheaf \(F\).

**Proposition 5.1.** Let \(S\) be a scheme over a field \(k\), \(X \in \text{Sm}_k\), and \(Z\) be a closed subscheme in \(X\). Suppose there is a framed correspondence \(c \in \text{Fr}_X(S \times \mathbb{A}^1, X)\) such that \(c^{-1}(Z)\) is finite over \(S\). Then the morphisms

\[
c_0, c_1: H^1_{\text{Nis}_{X}}(X, F) \to H^1_{\text{Nis}}(S, F), \quad i \geq 0,
\]

where \(c_0 = c \circ i_0, c_1 = c \circ i_1\), where \(i_0: S \times 0 \to S \times \mathbb{A}^1, i_1: S \times 1 \to S \times \mathbb{A}^1\), are equal for any \(\mathbb{A}^1\)-invariant quasi-stable framed linear presheaf \(F\).

**Proof.** The claim follows because of the commutative diagram

\[
\begin{array}{ccc}
H^*_{\text{Nis}_{X}}(X, F) & \xleftarrow{\cong} & H^*_{\text{Nis}_{S \times 0}^{c^{-1}(Z)}}(S \times 0, F) \\
& & H^*_{\text{Nis}_{S \times \mathbb{A}^1}^{c^{-1}(Z)}}(S \times \mathbb{A}^1, F) \xrightarrow{\cong} H^*_{\text{Nis}_{S \times 1}^{c^{-1}(Z)}}(S \times 1, F) \\
\end{array}
\]

see Definition 6.11 for the topologies in the second row, Definition 4.13 for \(N\text{is}\), and Example 4.2 for \(\text{Fin}_S\); the arrows between the first two rows are given by Definition 2.13 the arrows from the second row to the third one are inverse images for the morphisms of sites. The commutativity of the diagram is provided by Proposition 3.12 and Lemma 3.13. The isomorphisms in the bottom row hold by Corollary 4.16. □
Corollary 5.2. Let $S$ be a scheme over a field $k$, and $X \in \text{Sm}_k$, and $Z$ be a closed subscheme in $X$. Suppose that

(1) there is a framed correspondence $c \in \mathcal{K}_X(S \times \mathbb{A}^1, X)$ given by the difference of explicit framed correspondences $c = c^+ - c^-$ such that schemes $(c^+)^{-1}(Z)$ and $(c^-)^{-1}(Z)$ are finite over $S$.

(2) there is $\tilde{c}_1 \in \mathcal{Z}_X(S, X - Z)$ such that $c_1 = j \circ \tilde{c}_1$, where $c_1 = c \circ i_1$, $i_1 : S \to S \times \mathbb{A}^1$ is the unit section, $j : X - Z \to X$ is the canonical immersion.

Then for any $\mathbb{A}^1$-invariant quasi-stable framed linear presheaf $F$ on $\text{Sm}_k$ the morphisms

$$c_0^* : H^i_{\mathbb{N}is}^< (X, F) \to H^i_{\mathbb{N}is} (X, F)$$

induced by $c_0$ vanish for all $i \geq 0$.

Proof. The claim follows since the homomorphism $j^* : H^i_{\mathbb{N}is}^< (X, F) \to H^i_{\mathbb{N}is} (X - Z, F)$ given by the inverse image vanishes, and $c_0^* = \tilde{c}_1^* j^*$ by Proposition 5.4.

Corollary 5.3. Let $X \in \text{Sm}_k$. If the assumption of Corollary 5.2 holds for any $Z$ in $X$ of positive codimension, then

$$H^i_{\mathbb{N}is} (X, F) \cong 0 \quad \forall i \geq 0$$

for any $\mathbb{A}^1$-invariant quasi-stable framed linear presheaf $F$ on $\text{Sm}_k$.

Proof. Recall that $H^i_{\mathbb{N}is} (X, F_{\mathbb{N}is}) = H^i_{\mathbb{N}is}^< (X, F_{\mathbb{N}is}^x)$. Let $X^{(0)}$ denote the scheme that is the union of generic points of $X$.

Consider the topology $\mathbb{N}is^{X/Y}_{(0)} := \bigcup_Z \mathbb{N}is^X_Z$, where $Z$ runs over all closed subschemes $Z$ in $X$ of positive dimension. Then $H^i_{\mathbb{N}is}^{X/Y}_{(0)} (X, F_{\mathbb{N}is}^{X/Y}_{(0)}) \cong \mathrm{lim}_Z H^i_{\mathbb{N}is}^< (X, F_{\mathbb{N}is}^x)$. By Corollary 5.2 the canonical morphisms $H^i_{\mathbb{N}is}^< (X, F_{\mathbb{N}is}^x) \to H^i_{\mathbb{N}is}^< (X, F_{\mathbb{N}is}^x)$ are trivial for all closed subschemes $Z$ of positive codimension.

Hence the canonical morphism

$$H^i_{\mathbb{N}is}^{X/Y}_{(0)} (X, F_{\mathbb{N}is}^x) \to H^i_{\mathbb{N}is}^< (X, F_{\mathbb{N}is}^x) (5.4)$$

is trivial for each $i \in Z$.

On the other side, $\mathbb{N}is^{X/Y}_{(0)}$ has enough set of points that is the union of sets of Nisnevich points and schemes of the form $U \times_X X^{(0)}$, $U \in \text{Sch}_X$. Consider the restrictions of $\mathbb{N}is^{X/Y}_{(0)}$ and $\mathbb{N}is^X$ on $\text{Et}_X$, and the morphism of sites given by the embedding of the topologies $(\mathbb{N}is^{X/Y}_{(0)})|_{\text{Et}_X} \to (\mathbb{N}is^X)|_{\text{Et}_X}$. For any $U \in \text{Et}_X$, the scheme $U \times_X X^{(0)}$ has Krull dimension zero. Hence for any additive presheaf $F$, there is the isomorphism of presheaves $H^*_\mathbb{N}is^{X/Y}_{(0)} (-, F_{\mathbb{N}is}^{X/Y}_{(0)}) \simeq H^*_\mathbb{N}is^x (-, F_{\mathbb{N}is}^x)$ on $\text{Et}_X$. So the canonical morphism (5.4) is an isomorphism.

Since (5.4) is trivial and is an isomorphism, $H^i_{\mathbb{N}is}^< (X, F) \cong 0$. \hfill $\square$

6. Compactified correspondences over one-dimensional base schemes

In this section, we prove triviality of cohomologies on the generic fibres of essentially smooth local schemes over one-dimensional base schemes applying Corollary 5.3 to a framed $\mathbb{A}^1$-homotopy “with finite supports” constructed in Proposition 6.15, which moves cohomology classes to the trivial ones. For this purpose we develop the technique of compactified correspondences over one-dimensional base schemes.

6.1. Compactified correspondences.

Definition 6.1. A $d$-dimensional framed correspondence $\Phi$ over a scheme $U$ from an affine $U$-scheme $X$ to a $U$-scheme $Y$ is a set of data $(S, \varphi, g)$, where

(1) $S$ is an $X$-scheme of relative dimension $d$ over $X$ equipped with a closed embedding $S \to \mathbb{A}^1_X$.

(2) $\varphi \in \mathcal{O}((\mathbb{A}^1_X)^d)^{\mathbb{N} - d}$ is a set of regular functions such that $S = Z(\varphi)$,

(3) $g : (\mathbb{A}^1_X)^d \to Y$ is a morphism of $U$-schemes.
Definition 6.2. Let $B$ a local irreducible scheme, $\dim B = 1$, $z \in B^{(1)}$, $\eta \in B^{(0)}$. Let $V \in \text{Sch}_B$, $V_\eta = V \times_B \eta$, $T \in \text{Sch}_\eta$. Assume we are given with

1. $\overline{X} \in \text{Sch}_\eta$, $\overline{X}$ is projective, $\dim_V X = d$, see Section 1.3.1, an ample bundle $O(1)$ on $\overline{X}$, a section $t_\infty \in \Gamma(\overline{X}, O(1))$ such that the closed subscheme $X_\infty = Z(t_\infty)$ in $\overline{X}$ has positive relative codimension over $V$;
2. a $d$-dimensional framed correspondence $\Phi = (X_\infty, \varphi, g)$ from $V_\eta$ to $T$ over $\eta$,
3. an isomorphism $X_\eta \cong (\overline{X} - X_\infty) \times_B \eta$.

The set of data $(\overline{X}, X_\infty, \Phi)$ is called $V$-focused compactified $d$-dimensional $\eta$-framed correspondence from $V_\eta$ to $T$.

Definition 6.3. Let $B$ a local irreducible one-dimensional scheme, $z \in B^{(1)}$, $\eta \in B^{(0)}$. Let $V \in \text{Sch}_B$, $T \in \text{Sch}_\eta$, and $V_\eta = V \times_B \eta$. Let $r: V_\eta \to T$ be a morphism of schemes. Let $(\overline{X}, X_\infty, \Phi)$ be $V$-focused compactified $d$-dimensional $\eta$-framed correspondence from $V_\eta$ to $T$, $\Phi = (X_\infty, \varphi, g)$.

We say that $(\overline{X}, X_\infty, \Phi)$ $V$-smoothly contains $r$ if there is a closed subscheme $\Gamma$ in $X = \overline{X} - X_\infty$ such that

1. the canonical projection $X \to V$ is smooth over $\Gamma$ and induces an isomorphism $\gamma: \Gamma \cong V$,
2. $g \circ \gamma^{-1}_{\eta} = r$, where $\gamma_{\eta}: \Gamma \times_B \eta \cong V_\eta$.

6.2. Universal endo-correspondence. An appropriate compactification of a smooth scheme $X$ over $B$ provides a focused compactified framed correspondence of dimension $d = \dim_X X$ from the local scheme of $X \in \text{Sm}_B$ at $x \in X$ to $X$ itself as shown in the following lemma.

Lemma 6.4. Let $B$ be a one-dimensional irreducible local scheme, $z \in B^{(1)}$, $\eta \in B^{(0)}$. Let $X \in \text{Sm}_B$, $x \in X_\eta = X \times_B \eta$. Let $d = \dim^0_B X$ be the relative dimension of $X$ at $x$ over $B$. Let $Z$ be a closed subscheme of positive codimension in $X_\eta = X \times_B \eta$.

Then there is a $d$-dimensional $X_\eta$-focused compactified $\eta$-framed correspondence $(\overline{X}, S_\infty, \Phi)$ from $(X_\eta)_\eta = X_\eta \times_B \eta$ to $X_\eta = X \times_\eta$, where $\Phi = (S_\eta, \varphi, \text{val})$, $S_\eta = S \times_B \eta$, $S = S - S_\infty$, such that

1. $(\overline{X}, S_\infty, \Phi)$ $X_\eta$-smoothly contains $\gamma_\eta: (X_\eta)_\eta \to X_\eta$;
2. the scheme $\overline{Z} = \text{Cl}_\eta(\text{val}^{-1}(Z))$, see Lemma 6.2, is of positive relative codimension,
3. $\text{codim}_{\overline{X} \times_\eta x} (\mathcal{O}_{\overline{X}}(\overline{Z} \cap S_\infty) \times_B \eta \times_\eta x) \geq 2$.

Proof. Suppose $d = 0$, then $Z = \emptyset$, and the claim is trivial. Suppose $d > 0$.

Step 0) We shrink $X$ to an irreducible smooth affine Zariski neighbourhood of $x$ of dimension $d$ such that $T_{X/B} \cong k^d_X$. Note that $X_\eta$ is dense in $X$ since $X \in \text{Sm}_B$. We consider a closed immersion $X \to \mathfrak{A}^{N'}_B$, $N' \in Z$. Let $\overline{X} = \text{Cl}_{\mathfrak{A}^{N'}_B}(X)$,

$$\overline{X}_\eta = \overline{X} \times_B \eta, \quad \overline{X}_\eta = \overline{X} \times_B \eta.$$

Then $\overline{X}$ is irreducible projective $B$-scheme, $\overline{X}_\eta$ is a dense open subscheme. Since $\dim B = 1$, the scheme $\overline{X}$ is equidimensional over $B$ by Corollary 4.3. Consider $X_{\infty, \eta} = X_\eta \times_\eta X_\eta$ and $X_\infty = \text{Cl}_\eta(X_{\infty, \eta})$. Note that

$$X_\infty \times_B \eta = X_{\infty, \eta}, \quad X_\infty \subset \overline{X} \setminus X.$$

Since $\dim B = 1$, by Corollary 4.3, $X_{\infty, \eta}$ is equidimensional over $B$, and $\dim_x (X_\infty \times_B z) = \dim_y (X_\infty \times_B y) < \dim_\eta X_\eta = d$. So $\text{codim}_{\overline{X} \times_\eta x} (X_\infty) < d$.

Step 1) Since $\dim_x (X_\infty \times_B z \cup x) = \dim_x X_\eta$, it follows that $X_\eta - ((X_\infty \times_B z) \cup x)$ is dense in $\overline{X}_\eta$. Let $F \subset \overline{X}_\eta - ((X_\infty \times_B z) \cup x)$ be a finite set of closed points that intersects non-emptyly every irreducible component of $\overline{X}_\eta$. Since $\overline{X}$ is a projective scheme over $B$, there is an ample bundle $O(1)$ on $\overline{X}$ and $t_\infty \in \Gamma(\overline{X}, O(1))$ such that

$$t_\infty|t_\infty|_{\overline{X}} = 0, \quad t_{\infty}|_{F} \in \Gamma(F, O(1)^x).$$

Then $Z(t_\infty) \supset X_{\infty, \eta}$, $\dim_B Z(t_\infty) = \dim_B X - 1$. Denote $X'_{\infty, \eta} = Z(t_\infty)$, $X' = \overline{X} - X'_{\infty, \eta}$. 
Step 2) Since $X_\infty \supset X_S$, and $x \in X'$, there is the canonical open immersion
\[ j: X'_2 = X' \times B \eta \to X_2 \]
and $X'$ is smooth at $x$ over $B$. Note that $\text{codim}_{X/B}(X'_2) > 0$. Since $T_{X_2/\eta} \cong T_{\overline{X}_2/\eta}$, we get $X'_2 \in \text{SmAff}_\eta$, $T_{X'_2/\eta} \simeq T_{\overline{X}_2/\eta}$. Then by Lemma C.5 there is a $d$-dimensional framed correspondence $(X'_2)^{fr} = (X'_2, f_{d+1}, \ldots, f_N, \tilde{j})$ where $\tilde{j}: (h^N_{X'_2} \to X_2)$ is such that $\tilde{j}|_{X'_2} = j$, see [32, Theorem 1.8], see also [18, 10, Lemma 3.11].

So
\[ (X, X_{\infty}^{fr}, (X'_2)^{fr}) \]
is a $d$-dimensional $X_x$-focused compactified $\eta$-framed correspondence from $\eta$ to $X_2$. The base change along the map $X_x \to B$ gives the $d$-dimensional $X_x$-focused compactified $\eta$-framed correspondence $(\overline{S}, \overline{S}_\infty, \Phi)$ from $X_x$ to $X$, where $\overline{S} = \overline{X} \times B X_z$, $\overline{S}_\infty = X_{\infty}^{fr} \times B X_z$, and $\Phi$ is the base change of $(X'_2)^{fr}$.

Step 3) The composite morphism $X_x \to X \to \overline{X}$ induces $\Delta: X_x \to \overline{S}$. Since $X \in \text{Sm}_{\eta}$ by assumption, the morphism $\overline{S} \to X_x$ is smooth over $\Delta(X_x)$. Thus $(\overline{S}, \overline{S}_\infty, \Phi)$ $X_x$-smoothly contains the canonical morphism $(X_x)_\eta \to X'_2$. This proves claim (1).

Step 4) We have $j^{-1}(Z) = \overline{Z} \cap X'_2$, $\overline{\text{Cl}}_{\overline{S}}(j^{-1}(Z)) \times B \eta = \text{Cl}_{X'_2}(j^{-1}(Z))$.

\[ \overline{Z} = \text{Cl}_{\overline{S}}(\text{val}^{-1}(Z)) \subset j^{-1}(\overline{Z}) \times B X_z, \quad (6.5) \]

where $\overline{j^{-1}(Z)} = \text{Cl}_{\overline{S}}(j^{-1}(Z))$. Since $\text{codim}_{X'_2}(j^{-1}(Z)) = \text{codim}_{X_2}(Z) > 0$, $\text{codim}_{X'_2}(j^{-1}(Z)) > 0$. Thus by Corollary A.3

\[ \text{codim}_{\overline{S}/X_z}(\overline{Z}) = \text{codim}_{X/B}(j^{-1}(Z)) \geq 0. \] (6.3)

This proves claim (2).

Step 5) Since $\text{codim}_{X'_2}(j^{-1}(Z) \cap X'_2) \times B \eta = \text{codim}_{X'_2}(j^{-1}(Z) \cap X_{\infty, \eta}) \leq 2$, by Corollary A.3

\[ \text{codim}_{X'_2}(\text{Cl}_{X'_2}(j^{-1}(Z) \cap X'_2) \times B \eta) \times B z) \leq 2. \]

Then claim (3) follows because by (6.3)

\[ \overline{Z} \cap S_\infty \subset (j^{-1}(Z) \cap X_{\infty}^{fr}) \times B X_z. \]

\[ \mathbf{\square} \]

6.3. Contracting correspondence of dimension one. In Proposition [6, 14] we construct a one-dimensional correspondence that is a “curve”-homotopy moving cohomology classes on $(X_x)_\eta$ to the trivial one. Proposition 6.6 is used to deduce Proposition 6.14 from Lemma 6.4.

**Proposition 6.6.** Let $B$ be a local scheme, $z$ be the closed point. Let $Z$ be a closed subscheme, $U = B - Z$. Let $X$ be a projective $B$-scheme, and $O(1)$ denote an ample bundle on $X$.

Let $Y \to X$ and $D \to X$ be closed immersions, $F = C_X(F_U)$, $F_U = (Y \cap D) \times B U$. Let $\Delta: B \to X$ be a $B$-morphism. Suppose that

(a1) $\overline{X}$ is equidimensional of pure relative dimension $d$ over $B$;

(a2) $\text{codim}_{X \times_B Z}(Y \times_B Z) > 0$ and $\text{codim}_{X \times_B D}(D \times_B Z) > 0$;

(a3) $\text{codim}_{X \times_B Z}(F \times_B Z) \geq 2$;

(a4) $\overline{X}$ is $B$-smooth over the subscheme $(\Delta(B), \Delta(B) \cap D = \emptyset)$. Then for large enough $l_i \in \mathbb{Z}$, there are sections $s_i \in \Gamma(\overline{X}, O(l_i))$, $i = 2, \ldots, d$, such that

(c1) the vanishing locus $\mathcal{C} = Z(s_2, \ldots, s_d)$ is of pure relative dimension one over $B$;

(c2) $\Delta(B) \subset C$, and the morphism $\overline{C} \to B$ is smooth over $\Delta(B)$;

(c3) $\overline{C} \cap D$ and $\overline{C} \cap Y$ are finite over $B$;

(c4) $(\overline{C} \cap Y \cap D) \times B U = \emptyset$.

**Proof.** Consider the closed subscheme $\Delta(B)$ in $\overline{X}$ and $\Delta(B) \cap = Z(T^2(\Delta(B)))$. Since the morphism $\overline{X} \to B$ is smooth over $\Delta(B)$, and $\Delta(B)$ is local, $O(1)|_{\Delta(B)} \simeq O(\Delta(B))$, $N_{\Delta(B)/\overline{X}} \simeq 1_{\Delta(B)}$. Hence for any $d$, there are sections $\gamma_1, \ldots, \gamma_d \in \Gamma(\Delta(B) \cap \overline{X}, O(d))$ such that $Z(\gamma_1, \ldots, \gamma_d) = \Delta(B)$. 

By induction for \(i = \ldots, 2\), we construct sections \(s_i \in \Gamma(X, \mathcal{O}(l_i))\) and define schemes \(X_i, Y_i, D_i, F_i, S_i\),

\[ s_i \in \Gamma(X, \mathcal{O}(l_i)), \quad s_i|_{\Delta(B)|z} = \gamma_i, \quad s_i|_{S_i} \in \mathcal{O}(l_i)^x, \quad (6.7) \]

\(\overline{X}_n := X_n, \overline{X}_{i-1} = Z(s_i) \subset X_n\). \(Y_i = Y \cap \overline{X}_i\), \(D_i = D \cap \overline{X}_i\), \(F_i \subset Y \times D \cap \overline{X}_i \times_B U\), \(F_i = \text{Cl}_{\overline{X}_i}(F_i, U)\). \(S_i \subset \overline{X}_i \times_B z\) is a finite set of closed points such that \(S_i \cap N \neq \emptyset\) for any irreducible component \(N\) of anyone of schemes \(\overline{X}_i \times_B z\), \(\overline{X}_i \times_B z\), \(Y_i \times_B z\), \(D_i \times_B z\), \(F_i \times_B z\).

The middle equality in (6.7) implies that \(\Delta(B) \subset \overline{X}_i\), and the morphism \(\overline{X}_i \to B\) is smooth over \(\Delta(B)\). The right side condition in (6.7) implies that

\[ \text{dim}_B \overline{X}_i = i, \]

\[ \text{codim}_{\overline{X}_i \times_B z}(Y_i \times_B z) \geq 1, \]

\[ \text{codim}_{\overline{X}_i \times_B z}(D_i \times_B z) \geq 1, \]

\[ \text{codim}_{\overline{X}_i \times_B z}(F_i \times_B z) \geq 2, \]

So the scheme \(\overline{C} = Z(s_2, \ldots, s_d) = X_1\) satisfies claim (c2). Claim (c1) holds by (6.8). Claim (c3) follows by (6.9), and (6.10). Then \(F_1\) is a finite over \(B\), and since by (6.11) \(F_1 \times_B z = \emptyset\), it follows by Nakayama’s Lemma that \(F_1 = \emptyset\). Whence

\[ (Y \cap D \cap \overline{C}) \times_B U = F_1 \times_B U = \emptyset. \]

So claim (c4) holds.

**Lemma 6.12.** Let \(B\) be a one-dimensional irreducible local scheme, \(z \in B^{(1)}, \eta \in B^{(0)}\). Suppose there is a \(d\)-dimensional \(X\)-focused compactified \(\eta\)-focused correspondence \((\overline{S}, S_\infty, \Phi)\) from the \(\eta\)-scheme \(U_\eta = (X_\eta, z) = X \times_B \eta\) to the \(\eta\)-scheme \(X_\eta = X \times_B \eta\), where \(\Phi = (S_\eta, \varphi, \text{val}_{S_\eta})\), \(S_\eta = (\overline{S} - S_\infty) \times_B \eta\), such that

(a1) \((\overline{S}, S_\infty, \Phi)\) smoothly contains the canonical morphism \((X_\eta, z) \to X_\eta, z\).

(a2) \(\text{codim}_{X_\eta}(\overline{Z}) > 0, \text{where } \overline{Z} = \text{Cl}_{\overline{S}}(Z), Z = \text{val}_{S}^{-1}(Z), \text{and} \)

\[ \text{codim}_{\overline{S}_{\times X_{\eta}}}((\text{Cl}_{\overline{S}}(\overline{Z} \cap \overline{S_\infty})) \times_B \eta) \times_{X_\eta} x \geq 2. \]

Then there is a one-dimensional \(X\)-focused compactified \(\eta\)-focused correspondence \((\overline{C}, C_\infty, \Psi)\) from \(U_\eta = (X_\eta, z) \to X_\eta, \Psi = (C_\eta, \psi, \text{val}_{C_{\eta}}), C_\eta = (\overline{C} - C_\infty) \times_B \eta\), such that

(c1) \((\overline{C}, C_\infty, \Psi)\) \(X\)-smoothly contains \((X_\eta, z) \to X_\eta, z\).

(c2) \(\text{val}_{C_{\eta}}^{-1}(Z) = \text{Cl}_{\overline{C}}(\text{val}_{C_{\eta}}^{-1}(Z)), \text{see } (6.5)\) is finite over \(X_\eta\), and

\[ (\text{val}_{C_{\eta}}^{-1}(Z) \cap C_{\infty}) \times_B \eta = \emptyset. \]

**Proof.** Let \(O(1)\) be an ample line bundle on the projective \(B\)-scheme \(\overline{S}\), and then \(t_\infty\) be a global section such that \(Z(t_\infty) = S_\infty\) provided by Definition 6.2. We are going to apply Proposition 6.6 to the scheme \(\overline{S} := \overline{S}\), the subschemes

\[ D := S_{\infty}, \quad Y := \overline{Z}, \quad \Delta(B) := \Delta(X_\eta). \]

where \(B := X_\eta, Z := X_\times B \times_B \eta\). Assumption (a1) in Proposition 6.6 holds by Definition 6.2 (a4) in Proposition 6.6 holds by (a1) in Lemma 6.12 and Definition 6.3 (a2) in Proposition 6.6 follows from (a2) in Lemma 6.12 and Definition 6.2 (a3) in Proposition 6.6 holds by (6.13). Then by Proposition 6.6 there is a vector of sections \((s_2, \ldots, s_d)\), where \(s_i \in \Gamma(\overline{S}, \mathcal{O}(l))\) for a large enough \(l\), such that

(p1) \(\text{dim}_B \overline{C} = 1, \text{where } \overline{C} = Z(s_2, \ldots, s_d) \subset \overline{S}\),

(p2) \(\Delta(X_\eta) \subset \overline{C}\), and the morphism \(\overline{C} \to X_\eta\) is smooth over \(\Delta(X_\eta)\),

(p3) \(\overline{Z} \cap \overline{C}\) is finite over \(X_\eta\), \(C_\infty = S_\infty \cap \overline{C}\) is finite over \(X_\eta, (\overline{Z} \cap C_\infty) \times_B \eta = \emptyset.\)
By assumption we are given with a closed embedding $S_\mathcal{B} \to \mathcal{H}_\mathcal{U}_\mathcal{B}$, a morphism $\text{val}_\mathcal{B}: (\mathcal{H}_\mathcal{U}_\mathcal{B})^h \to X_\mathcal{B}$, and $\varphi \in \mathcal{O}(\mathcal{H}_\mathcal{U}_\mathcal{B})^h$ such that $Z(\varphi) = S_\mathcal{B}$. By property (p1) there are canonical embeddings $\mathcal{C} \to \mathcal{S}, C_\mathcal{B} \to S_\mathcal{B}$ where $C_\mathcal{B} = C \times_B \eta, C = \mathcal{C} - C_\infty$. Define $\text{val}_{C_\mathcal{B}}: (\mathcal{H}_\mathcal{U}_\mathcal{B})^h \to X_\mathcal{B}$ as the composite morphism

$$(\mathcal{H}_\mathcal{U}_\mathcal{B})^h = \text{val}_{C_\mathcal{B}}: (\mathcal{H}_\mathcal{U}_\mathcal{B})^h \to X_\mathcal{B}.$$ Choose liftings $\varphi_j \in \mathcal{O}(\mathcal{H}_\mathcal{U}_\mathcal{B})^h$ of functions $s_j/\eta^\infty \in \mathcal{O}(S_\mathcal{B}), j = 2, \ldots, d$, and denote by the same symbols their inverse images on $(\mathcal{H}_\mathcal{U}_\mathcal{B})^h$. Then $\Psi = (C_\mathcal{B}, \varphi_2, \ldots, \varphi_d, \varphi, \text{val}_{C_\mathcal{B}})$ is a one-dimensional framed correspondence from $U_\mathcal{B}$ to $X_\mathcal{B}$. Moreover, $(\mathcal{C}, C_\infty, \Psi)$ is an $X_\mathcal{B}$-focused compactified $\eta$-framed correspondence because $\mathcal{C}$ is of pure relative dimension one and $C_\infty$ is finite over $X_\mathcal{B}$ by property (p1) and the second part of (p3) in the list above respectively.

Claim (c1) of the lemma follows by property (p2). Claim (c2) follows by property (p3), because $\text{val}_C^{-1}(Z) \subset \mathcal{C} \cap \mathcal{C}$ is finite over $X_\mathcal{B}$, and the fiber of $\text{val}_C^{-1}(Z) \cap C_\infty \subset \mathcal{C} \cap C_\infty$ over $\eta$ is empty. □

**Proposition 6.14.** Let $B$ be a one-dimensional irreducible local scheme, $z \in B^{(1)}, \eta \in B^{(0)}$. Let $X \in \text{Sm}_B, x \in X_{\mathcal{Z}} = X \times_B z$. Let $d = \dim_B X$. Let $Z$ be a closed subscheme of positive codimension in $X_{\mathcal{Z}} = X \times_B \eta$.

Then there is a one-dimensional $X_{\mathcal{Z}}$-focused compactified $\eta$-framed correspondence $(\overline{\mathcal{C}}, C_\infty, \Psi)$ from $U_{\mathcal{Z}} = (X_{\mathcal{Z}})_{\mathcal{Z}}$ to $X_{\mathcal{Z}}$ where $\Psi = (C_{\mathcal{Z}}, \psi, \text{val}_C), C_{\mathcal{Z}} = (\overline{\mathcal{C}} - C_\infty) \times_B \eta$, such that

1. $X_{\mathcal{Z}}$-smoothly contains the canonical morphism $(X_{\mathcal{Z}})_{\eta} \to X_{\mathcal{Z}}$.
2. The closure $\overline{\text{val}_C}(Z)$ of $\text{val}_C^{-1}(Z)$ in $\overline{\mathcal{C}}$ is finite over $X_{\mathcal{Z}}$, and $(\text{val}_C^{-1}(Z) \cap C_\infty) \times_B \eta = \emptyset$.

Proof. The claim follows by Lemma 6.3 and Lemma 6.12. □

**6.4. Finite correspondence homotopy.**

**Proposition 6.15.** Let $B$ be one-dimensional scheme, $z \in B^{(1)}, \eta \in B^{(0)}$. Let $X \in \text{Sm}_B, x \in X_{\mathcal{Z}} = X \times_B z$.

Define $U = X_\mathcal{Z}$ and $U_{\mathcal{Z}} = U \times_B \eta$. Then for any closed subscheme $Z$ in $X_{\mathcal{Z}} = X \times_B \eta$, there are framed correspondences $c \in \text{ZF}_N(U_{\mathcal{Z}} \times A^1, X_{\mathcal{Z}}), c' \in \text{ZF}_N(U_{\mathcal{Z}} \times A^1, X_{\mathcal{Z}} - Z)$ such that

1. $c \circ i_0 = \sigma^N \text{can}_{\mathcal{Z}}, c \circ i_0 = j \circ c'$, where $i_0, i_1: U_{\mathcal{Z}} \to U_{\mathcal{Z}} \times A^1$ are the zero and unit sections, $\text{can}_{\mathcal{Z}}: U_{\mathcal{Z}} \to X_{\mathcal{Z}}, j: X_{\mathcal{Z}} - Z \to X_{\mathcal{Z}}$ are the canonical morphisms,
2. $c^{-1}(Z)$ is finite over $U_{\mathcal{Z}}$, see (1.6) for $c^{-1}(Z)$.

Proof. Using the base change along the morphism $(\mathcal{P})_{\mathcal{Z}} \to B$, we reduce the claim to local irreducible base scheme $B$. By Proposition 6.13 there is a one-dimensional $X_{\mathcal{Z}}$-focused compactified framed correspondence $(\overline{\mathcal{C}}, C_\infty, \Psi)$ from the $\eta$-scheme $U_{\mathcal{Z}}$ to the $\eta$-scheme $X_{\mathcal{Z}}$, where $\Psi = (C_{\mathcal{Z}}, \varphi, \text{val}), C_{\mathcal{Z}} = (\overline{\mathcal{C}} - C_\infty) \times_B \eta$, such that

1. $(\overline{\mathcal{C}}, C_\infty, \Psi)$ smoothly contains $\text{can}_{\mathcal{Z}}$,
2. the closure $\overline{Z}$ of $Z = \text{val}^{-1}(Z)$ in $\overline{\mathcal{C}}$ is finite over $X_{\mathcal{Z}}$, and $(\overline{\mathcal{C}} \cap C_\infty) \times_B \eta = \emptyset$.

We are going to construct framed correspondences $\tilde{c} \in \text{Fr}_N(U_{\mathcal{Z}} \times A^1, X_{\mathcal{Z}}), c_0^+, c_1 \in \text{Fr}_N(U_{\mathcal{Z}} \times A^1, X_{\mathcal{Z}} - Z), N \in \mathbb{Z}$, such that

1. $\tilde{c} \circ i_0 = \sigma^N \text{can}_{\mathcal{Z}} + c_0^+, [\tilde{c} \circ i_1] = [j \circ c_1]$ in $\text{ZF}_N(U_{\mathcal{Z}}, X_{\mathcal{Z}})$ see Definition 2.11,
2. $\tilde{c}^{-1}(Z)$ is finite over $U_{\mathcal{Z}}$.

Then $c = \tilde{c} - (c_0^+ \circ \text{pr})$, where $\text{pr}: U_{\mathcal{Z}} \times A^1 \to U_{\mathcal{Z}}$ is the canonical projection, satisfies properties (1)-(2) in the lemma above.

Step 1) Firstly, assume the residue field $\mathcal{O}(x)$ has at least three elements. According to Definitions 6.1 to 6.3, we are given with

1. the projective equidimensional scheme $\overline{\mathcal{C}}$ over $X_\mathcal{Z}$ of relative dimension one, a closed subscheme $C_\mathcal{Z}$ finite over $X_\mathcal{Z}$, a closed embedding $C_\mathcal{Z} \to \mathcal{H}_\mathcal{U}_\mathcal{B}$ for some $N \in \mathbb{Z}$;
(d2) functions $e_i \in \mathcal{O}(E)$, $i = 2, \ldots, N$, and a regular map val: $E \to X_{\overline{\eta}}$, such that $C_{\overline{\eta}} = Z(e_2, \ldots, e_N)$, where $E$ is the hypersurface $(\overline{\Delta})_{\overline{\eta}}$.

(d3) a map $\Delta: X_{\overline{\eta}} \to C$, such that $\text{val} \circ \Delta = \text{can}: X_{\overline{\eta}} \to X$ is the canonical map, and the morphism $C \to X_{\overline{\eta}}$ is smooth over $\Delta(X_{\overline{\eta}})$.

Here $C_{\overline{\eta}} = C \times_B \eta$, and denote $C_{\infty, \overline{\eta}} = C_{\infty} \times_B \eta$, and note that $Z = \text{val}^{-1}(Z) = C \times_B \eta$. Then $C_{\overline{\eta}}$ is a projective scheme of relative dimension one over $U_{\overline{\eta}}$, and $C_{\infty, \overline{\eta}}$ and $Z$ are closed subschemes finite over $U_{\overline{\eta}}$.

We let us write $\Delta \subset C$ for the subscheme $\Delta(X_{\overline{\eta}})$. By (d3) the morphism $C \to X_{\overline{\eta}}$ is smooth over $\Delta$. It follows that there is a line bundle $\mathcal{L}(\Delta)$ on $C$ with a section $\delta$ such that $Z(\delta) = \Delta$. We write $\mathcal{L}(\Delta)$ for $\mathcal{L}(\Delta)|_{C_{\overline{\eta}}}$, since $Z$, $\Delta$ and $C_{\infty, \overline{\eta}}$ are finite over $X_{\overline{\eta}}$, by Lemma [B.1], the restrictions $\mathcal{O}(1)|_{Z \cup \Delta}$, $\mathcal{L}(\Delta)|_{Z \cup \Delta}$, $\mathcal{O}(1)|_{C_{\infty, \overline{\eta}}}$ are trivial. Since $\#O_z(z) > 2$, by Lemma [B.2] for any $l \in \mathbb{Z}_{\geq 0}$, there are invertible sections $\gamma_\infty, \gamma_0, \gamma_1 \in \mathcal{O}(\mathcal{L}(\Delta)|_{X_{\infty}}, \mathcal{O}(l)^\times)$, $\gamma_\infty = \gamma_1 - \gamma_0^l \delta|_{Z \cup \Delta}$, and let $\beta_0^l \in \Gamma(C_{\infty, \overline{\eta}}, \mathcal{O}(1))$ be an invertible section.

Since $Z \cap C_{\infty, \overline{\eta}} = \emptyset$ by (a2), by Serre’s theorem [27, III, Corollary 10.7], for a large enough $l \in \mathbb{Z}$, there are sections

$$s_0^l + \gamma_0, \quad s_1 + s_\infty, \quad s_\lambda = s_0(1 - \lambda) + s_1 \lambda.$$

Define

$$s_0 = \delta s_0^l, \quad s_1 = s_0 + s_\infty, \quad s_\lambda = s_0(1 - \lambda) + s_1 \lambda.$$

Let $g: (\tilde{\mathcal{A}_{\overline{U}_{\overline{X}}}}_{\mathcal{O}(s_\lambda)})^b \to X_{\overline{\eta}}$ be a lifting of the morphism

$$Z(s_\lambda) \to C_{\overline{\eta}} \times \mathbb{A}^1 \to C_{\overline{\eta}} \to X_{\overline{\eta}}$$

along the canonical closed immersion, see [32, Theorem I.8], see also [18, 10, Lemma 3.11], and $e_i \in \mathcal{O}(\tilde{\mathcal{A}_{\overline{U}_{\overline{X}}}}_{\mathcal{O}(s_\lambda)})$ be a lifting of $s_\lambda/\epsilon_{\infty}^l \in \mathcal{O}(C_{\overline{\eta}} \times \mathbb{A}^1)$. Define

$$s_i^l = (Z(s_\lambda), e_1, e_2, \ldots, e_N, g) \in \mathcal{O}(U_{\overline{U}} \times \mathbb{A}^1, X_{\overline{\eta}}),$$

$$c_0^l = (Z(s_0^l), e_1, e_2, \ldots, e_N, g, \mathcal{O}(U_{\overline{U}} \times \mathbb{A}^1, X_{\overline{\eta}} - Z),$$

$$c_1 = (Z(s_1), s_1/l, e_2, \ldots, e_N, \mathcal{O}(U_{\overline{U}} \times \mathbb{A}^1, X_{\overline{\eta}} - Z)).$$

where $\epsilon_{\infty}^l: (\tilde{\mathcal{A}_{\overline{U}_{\overline{X}}}}_{\mathcal{O}(s_\lambda)})^b \to X_{\overline{\eta}} - Z$, $i = 0, 1$, are induced by $g$, because sections $s_i^l|_Z = \gamma_0^l$ and $s_1|_Z = \gamma_1$ are invertible. Then since the section $s_0^l|_{\alpha} = \gamma_0^l|_{\alpha}$ is invertible, $\epsilon \circ i_0 = \epsilon_{\infty}^l + j \circ c_0^l$, where $c_0^l = (\tilde{\Delta}(U_{\overline{U}}), e_1, e_2, \ldots, e_N, g) = \sigma \epsilon_{\infty}^l \text{can}_{\overline{\eta}}$, and $D \in \text{GL}_N(U_{\overline{U}})$ is defined by the differentials of functions $e_i$. We redenote $c := \epsilon \circ \sigma \epsilon_{\infty}^l \text{can}_{\overline{\eta}}$, and similarly for $c_0, c_0^l, c_1$, then by Lemma [B.3] since $X_{\overline{\eta}}$ is local, we conclude $[c \circ i_0]| = [\text{can}_{\overline{\eta}} + j \circ c_0^l]$, $\epsilon \circ i_1 = j \circ c_1$.

Since the section $s_1|_{C_{\infty, \overline{\eta}} \times \mathbb{P}^1} = \delta|_{C_{\infty, \overline{\eta}} \times \mathbb{P}^1}$ is invertible, $Z(s_\lambda) \subset C_{\overline{\eta}} \times \mathbb{P}^1$, and the morphism $p: Z(s_\lambda) \to U_{\overline{U}} \times \mathbb{A}^1$ is finite. Hence the scheme

$$c^{-1}(Z) = p(g_{\overline{Z}(s_\lambda)}(Z)) = p(Z(s_\lambda)|_{Z \times \mathbb{A}^1})$$

is finite over $U_{\overline{U}}$ since the section $s_\infty|_Z = \gamma_\infty$ is invertible.

Step 2) For an arbitrary residue field $k = O_z(z)$, let $q_0, q_1 \in \mathbb{Z}_{>0}$, $(q_0, q_1) = 1$, $(q_0, \text{char} k) = 1$, $(q_2, \text{char} k) = 1$. Define polynomials

$$f_i = t^{q_i - 1} + t^{q_i - 2} + \ldots + 1 = (t^{q_i - 1})/(t - 1) \in R[t], R = O_B(B).$$

Then $f_0$ and $f_1$ are separable over the closed point $z$. Denote by $B_i = Z(f_i), i = 0, 1$, then the projection to $\mathbb{A}_B^1 \to B$ induces finite étale morphisms $p_i: B_i \to B$, and there are étale neighbourhoods $V_i$ of $B_i$ with retractions $r_i: V_i \to B_i$. Define $e_i = (Z(f_i), V_i, f_i, r_i) \in \text{Fr}_1(B, B_i)$.
and \( c = \sum_{i=0}^{1} (-1)^i (p_i \times_B \text{id}_{U_i}) \circ e_B^i \circ (e_i \times_B \text{id}_{U_i \times A^1}) \circ (b_i)_e \in \text{ZF}_{2+N}(U_n \times A^1, X) \), where \( e_B^i \in \text{ZF}_N(U_n \times B, A^1, X) \) from (Step 1). Then

\[
c \circ i_0 = \sum_{i=0}^{1} (-1)^i (p_i \times_B \text{id}_{U_i}) \circ e_B^i \circ (e_i \times_B \text{id}_{U_i}) \circ (b_i)_e \]

\[
\text{Step 1} \quad \sum_{i=0}^{1} (-1)^i (p_i \times_B \text{id}_{X_i}) \circ c_B^i \circ (e_i \times_B \text{id}_{U_i}) \circ (b_i)_e \\
= \sigma_N \circ \left( \sum_{i=0}^{1} (-1)^i (p_i \circ e_i \circ (b_i)_e) \times_B \text{id}_{U_i} \right) \\
\text{Lemma } C.1 \quad \sigma_N \circ \left( \sum_{i=0}^{1} (-1)^i ((q_i)_e \circ (b_i)_e) \times_B \text{id}_{U_i} \right) \\
= \sigma^2 \circ \left( (\sigma^2 \times_B \text{id}_{U_i}) \right) \\
= \sigma^2 + N \circ \left( (\text{id}_{U_i}) \right) \in \text{ZF}_{2+N}(U_n \times B, X).
\]

6.5. Vanishing of cohomologies.

**Theorem 6.16.** Let \( X \) be a smooth scheme over a one-dimensional irreducible base scheme \( B \). Let \( x \in X \) be a point over a closed point \( z \in B \). Let \( F \) be an \( A^1 \)-invariant quasi-stable framed linear presheaf over the generic point \( \eta \in B \). Then \( H^n_{\text{Nis}}(X \times_B \eta, F) \cong 0 \).

**Proof.** The claim follows by the combination of Corollary 5.3 with Proposition 6.15.

7. Strict homotopy invariance

We prove the strict homotopy invariance theorem combining Theorem 6.16 and Theorem 4.13.

**Theorem 7.1.** Let \( F \) be a quasi-stable homotopy invariant framed linear presheaf over \( k \).

1. The presheaves \( H^*_\text{Nis}(F) : \text{Sm}_k \to \text{Ab}, X \mapsto H^*_{\text{Nis}}(X, F) \), are \( A^1 \)-invariant for all \( i \geq 0 \).

2. There are canonical isomorphisms \( H^*_\text{Nis}(X, F_{\text{Nis}}) \cong H^*_\text{zar}(X, F_{\text{zar}}) \) for all \( X \in \text{Sm}_k, i \geq 0 \).

**Proof.** (1) The claim is equivalent to isomorphisms

\[
\begin{align*}
H^0_{\text{Nis}}(V \times A^1, F) & \cong F(V \times A^1), \\
H^i_{\text{Nis}}(V \times A^1, F) & = 0, & i > 0,
\end{align*}
\]

for all local henselian essentially smooth schemes \( V \) over \( k \). The morphism of sites given by the embedding of the topology \( \text{Fin}_S \) on \( \text{Sm}_{A^1 \times V} \) into the Nisnevich topology leads to the spectral sequence

\[
H^p_{\text{Fin}^V}(A^1 \times V, H^q_{\text{Nis}}(F)_{\text{Fin}^V}) \Rightarrow H^r_{\text{Nis}}(A^1 \times V, F_{\text{Nis}}), r = p + q.
\]

where \( H^*_{\text{Nis}}(F)_{\text{Fin}^V} \) denotes the sheafification with respect to \( \text{Fin}^V \) of the presheaf \( H^*_{\text{Nis}}(-, F_{\text{Nis}}) \).

By Lemma 3.2 the topology \( \text{Fin}^V \) has enough set of points given by schemes of the forms \( U^h_v \), for \( v \in U \), and \( U \times_{p_1 \times V} (P^1 \times V)_{(\infty, z)} \), where \( z \) is the closed point of \( V \). Since the schemes \( U^h_v \) are Nisnevich points, \( H^*_{\text{Nis}}(U^h_v, F) \cong 0 \). By Theorem 6.16

\[
H^*_{\text{Nis}}(U \times_{p_1 \times V} (P^1 \times V)_{(\infty, z)}, F) \cong 0.
\]

Combining the latter two isomorphisms, we conclude \( H^*_{\text{Nis}}(F)_{\text{Fin}^V} \cong 0 \), and consequently, \( H^*_{\text{Nis}}(F)_{\text{Fin}^V} \cong F_{\text{Fin}^V} \). Thus (7.2) implies

\[
H^*_{\text{Fin}^V}(A^1 \times V, F_{\text{Fin}^V}) \cong H^0_{\text{Nis}}(A^1 \times V, F_{\text{Nis}});
\]

and the claim, since \( H^*_{\text{Fin}^V}(A^1 \times V, F) \cong 0 \) by Theorem 4.13.

(2) We repeat the argument from 13. By point (1) and 24. Theorem 3.11] the canonical morphism \( H^*_{\text{Nis}}(U, F_{\text{Nis}}) \to H^*_{\text{Nis}}(\eta, F_{\text{Nis}}) \) is injective for any essentially smooth local scheme \( U \) with the generic point \( \eta \). Hence the presheaf \( H^*_{\text{Nis}}(-, F_{\text{Nis}}) \) is Zariski locally trivial. Then because of the spectral sequence \( H^*_\text{zar}(X, H^*_{\text{Nis}}(-, F_{\text{Nis}})) \Rightarrow H^*_{\text{Nis}}(X, F_{\text{Nis}}) \) the claim follows.
8. Cousin complex

Let \( B \) be a one-dimensional local irreducible noetherian scheme. Let \( E \in \text{SH}(B) \). Then by Voevodsky's Lemma, see [24, Construction 3.1], \( E \) defines the functor \( E : \text{Fr}_+(B) \to \text{SH} \) such that for any \( X, X_0, X_1 \in \text{Sm}_B \), the morphisms

\[
F(X) \xrightarrow{\sigma^X} F(X), \quad F(X_0 \amalg X_1) \to F(X_0) \oplus F(X_1), \quad F(X) \to F(X \times \Delta^1)
\]

are isomorphisms in \( \text{SH} \). Consequently, since the stable homotopy category \( \text{SH} \) is additive, \( F \) defines the functor

\[
F : ZF_+(B) \to \text{SH}.
\]  

(8.1)

**Definition 8.2.** For \( X \in \text{Sm}_B \), and a closed subscheme \( Z \) in \( X \), denote

\[
E^l_Z(X) = \pi_{-l} \text{coCone}(E(X) \to E(X - Z)), l \in \mathbb{Z},
\]

and define functors

\[
ZF_+(B)^{\text{pair}} \to \text{Ab}; \quad ((X - Z) \hookrightarrow X) \to E^l_Z(X).
\]  

(8.3)

**Lemma 8.4.** Presheaves \( E^l \) are \( \Delta^1 \)-invariant.

*Proof.* The claim follows because (8.1) is \( \Delta^1 \)-invariant. \qed

**Lemma 8.5.** Let \( U \) be essentially smooth local scheme over \( B \), and \( U_\eta = U \times_B \eta, \) where \( \eta \) is the generic point of \( B \). Then for any closed subscheme \( W \) of codimension \( r \) in \( U_\eta \), the morphism

\[
E^l_Y(X \times_B \eta) \to \lim_{W \in U_\eta[r-1]} E^l_W(U_\eta)
\]

is trivial for any \( l \in \mathbb{Z} \), where \( X = U, Y = W, \) and \( W' \) runs over the filtered set of closed subschemes of codimension \( r - 1 \).

*Proof.* For any scheme \( X \in \text{Sm}_B \), point \( x \in X \) and closed subscheme \( Y \) of \( X_\eta = X \times_B \eta \) such that \( U = X_x, W = Y \times_X U \) by Proposition [6.15] there are

\[
c' \in ZF_N(U_\eta, X_\eta - Y), \quad c = (Z, V, \varphi, g) \in ZF(U_\eta \times \Delta^1, X_\eta)
\]

such that \( c \circ i_0 = \sigma_N^U \text{can}, c \circ i_1 = j \circ c' \), see Section [13.2] where \( c \text{can} : U_\eta \to X_\eta, j : X_\eta - Y \to X_\eta \) are the canonical morphisms. Denote by \( \tilde{W} \) the image closure along the composite morphism \( Y \times_X Z \to Z \to U_\eta \). Then codim \( \tilde{W} \geq r - 1 \). Since for each \( l \in \mathbb{Z} \), by Lemma 8.4 there is the equality of homomorphisms

\[
(j \circ c')^* = \text{can}^*: E^l_Y(X_\eta) \to E^l_{\tilde{W}}(U_\eta).
\]

So since \( (j \circ c')^* = 0 \), it follows that \( \text{can}^* = 0 \), and morphism (8.6) is trivial for all \( X \) and \( Y \) as above. Hence (8.6) is trivial for \( X = U, Y = W \). \qed

**Corollary 8.7.** Under the assumptions of Lemma 8.5, the sequence

\[
0 \to \lim_{W' \in U_\eta[r-1]} E^l_W(U_\eta) \to \lim_{W' \in U_\eta[r-1]} E^l_W(U_\eta - W) \to E^{l+1}_W(U_\eta) \to 0
\]

is exact, for any \( l \in \mathbb{Z} \) and \( W \in U_\eta[r] \).

*Proof.* The claim follows from Lemma 8.5 because of the long exact sequence

\[
\cdots \to \lim_{W' \in U_\eta[r-1]} E^l_W(U_\eta) \to \lim_{W' \in U_\eta[r-1]} E^l_W(U_\eta - W) \to E^{l+1}_W(U_\eta) \to \cdots.
\]  

(8.5)

**Theorem 8.8.** The Cousin complex (1.3) is acyclic for \( U_\eta = U \times_B \eta \) for any essentially smooth local scheme \( U \) over \( B \).
Proof. The claim follows because Corollary 8.7 provides short exact sequences
\[
0 \to \lim_{W' \in \mathcal{U}_\mathcal{Z}^{(r-1)}} E_{W'}^{d+r-1}(U_\mathcal{Z}) \to \bigoplus_{y \in \mathcal{U}_\mathcal{Z}^{(r-1)}} E_{y}^{d+r-1}(U_\mathcal{Z}) \to \bigoplus_{W' \in \mathcal{U}_\mathcal{Z}^{(r)}} E_{W'}^{d+r}(U_\mathcal{Z}) \to 0.
\]

Example 8.9. For a base field \(k\), the Cousin complex \([13]\) is exact for \(U_\mathcal{Z} = U - Z(f)\) for an essentially smooth local scheme \(U\) over \(k\) and \(f \in \mathcal{O}(U)\) such that \(Z(f)\) is smooth.

Appendix A. Relative dimension over one-dimensional base.

Proposition A.1. Let \(X\) be an affine scheme such that \(\dim X' \geq 1\) for each irreducible component \(X'\) of \(X\). Then there is a function \(f \in \mathcal{O}_X(X)\) such that \(\text{codim}_X(Z(f) \cap X') = 1\) for each irreducible component \(X'\).

Proof. Since for each irreducible component \(O_X(X')\) is not a field, it follows that there is a closed subscheme \(F\) of \(X\) that intersects non-emptily and does not contain each irreducible component \(X'\). Let \(X_2\) be the closed subscheme in \(X\) that is the union of pairwise intersections of the irreducible components of \(X\). Then \(X_2\) and \(X_2 \cap F\) do not contain any irreducible component \(X'\). Hence for any \(X'\), there is a function \(f_{X'} \in O_X(X')\) such that \(f_{X'} \neq 0\), \(f_{X'}|_{(X_2 \cap F) \cap X'} = 0\), and combining the functions \(f_{X'}\) for all \(X'\) together, we get the function \(X\) such that \(Z(f) \cap X' \neq \emptyset\), and \(f|_{X'} \neq 0\) for any irreducible component \(X'\) of \(X\). Hence \(Z(f)\) is of positive codimension in \(X\) and intersect non-empty each \(X'\). Since \(\text{codim}_X(Z(f) \cap X') \leq 1\) for any function \(f\), \(\text{codim}_X(Z(f)) = 1\).

Proposition A.2. Let \(B\) be a one-dimensional irreducible local scheme, \(z \in B^{(1)}\), \(\eta \in B^{(0)}\). Given an irreducible scheme \(X\) over \(B\) such that the schemes \(X_\mathcal{Z} = X \times_B \mathcal{Z}\) and \(X_\eta = X \times_B \eta\) are non-empty, then \(X\) is equidimensional over \(B\), i.e. \(\dim X_\mathcal{Z} = \dim X_\eta\).

Proof. Since any scheme has a Zariski covering by affine schemes, we can assume that \(X\) is affine, and consequently, \(X_\eta\) is of such type. If \(\dim X_\mathcal{Z} = 0\), then \(X\) is quasi-finite over \(B\), and \(\dim X_\eta = 0\) as well. Suppose \(\dim X_\mathcal{Z} = d \geq 1\), and the claim is proven for all \(X\) such that \(\dim X_\mathcal{Z} < d\).

By Proposition A.1 there is a regular function \(f_\mathcal{Z} \in \mathcal{O}_X(X_\mathcal{Z})\) such that \(\text{codim}_X Z(f_\mathcal{Z}) = 1\). Since \(X\) is affine, there is a lifting \(f \in \mathcal{O}_X(X)\) of \(f_\mathcal{Z}\). Then since \(X\) is irreducible, it follows that \(\text{codim}_X Z(f) = 1\). Moreover, since \(X\) is irreducible, it follows that \(X_\eta\) is dense in \(X\), and \(X_\eta\) is irreducible too. Hence \(f_\mathcal{Z} := f|_{X_\eta} \neq 0\), and it follows that \(\text{codim}_X_\eta Z(f_\mathcal{Z}) = 1\). Let \(X_1\) denote the irreducible component of \(Z(f)\). By the above \(\text{codim}_X(X_1 \times_B \mathcal{Z}) = 1\) and \(\text{codim}_X(X_1 \times_B \eta) = 1\). Then \(X_1\) is an irreducible affine scheme over \(B\) such that \(X_1 \times_B \mathcal{Z}\) and \(X_1 \times_B \eta\) are non-empty. By the inductive assumption \(\dim(X_1 \times_B \mathcal{Z}) = \dim(X_1 \times_B \eta)\). Hence \(\dim X_\mathcal{Z} = \dim X_\eta\).

Corollary A.3. Let \(B\) be a one-dimensional irreducible local scheme, \(z \in B^{(1)}\), \(\eta \in B^{(0)}\). Given a projective scheme \(X\) over \(B\) such that \(\text{Cl}_X(X_\mathcal{Z}) = X\), where \(X_\mathcal{Z} = X \times_B \mathcal{Z}\), the scheme \(X\) is equidimensional over \(B\), i.e. \(\dim X_\mathcal{Z} = \dim X_\eta\).

Proof. Without loss of generality, we can assume that \(X\) is irreducible and non-empty. Suppose that \(X \times_B \mathcal{Z} = \emptyset\), then the morphism \(p: X \to B\) passes throw \(\eta\), and since \(p\) is proper, and \(\eta\) equals \(p(\mathcal{Z})\), it follows that \(\eta\) is closed subscheme in \(B\), that contradicts to that \(\dim B = 1\). So \(X \times_B \mathcal{Z} \neq \emptyset\), and the claim follows by Proposition A.2.

Appendix B. Sections of line bundles

Lemma B.1. Any line bundle \(\mathcal{L}\) on a semi-local affine scheme \(Z\) is trivial. In particular, the claim holds for any finite scheme \(Z\) over a local scheme \(X\).

Proof. Since \(Z\) is affine, the restriction homomorphism \(\Gamma(Z, \mathcal{L}) \to \Gamma(F, \mathcal{L})\) is surjective, where \(F\) is the set of closed points of \(Z\). Since \(F\) is a union of spectra of fields there is an invertible section \(f \in \Gamma(F, \mathcal{L})\). Then any section \(s \in \Gamma(Z, \mathcal{L})\) such that \(s|_F = f\) is invertible.
Lemma C.2. Let $X$ be a local scheme, and $Z$ be a finite scheme over $X$. Assume the residue field at the closed point of $X$ has at least three elements. Let $L_1$, $L_0$ be linear bundles on $Z$. Then for any section $\delta \in \Gamma(Z, L_0)$, there are invertible sections $s_1, s_\infty \in \Gamma(Z, L_1)$, $s_0^\delta \in \Gamma(Z, L_1 \otimes L_0^{-1})$, $s_1 = s_\infty + \delta s_0^\delta$.

Proof. Since $Z$ is finite over a local scheme, $Z$ is semi-local and affine. Denote by $F$ the union of closed points of $Z$.

Suppose $Z = F$. If $\delta = 0$, we choose any invertible sections $s_0^+$ and $s_\infty$, and put $s_1 = s_\infty$. If $\delta$ is invertible, since by assumption $\# k \geq 3$, there are invertible sections $s_0, s_1 \in \Gamma(F, L_1)$ such that the section $s_\infty := s_1 - s_0$ is invertible. Put $s_0^\delta = s_0 \delta^{-1}$. Since for any $\delta$, if $Z = F$, then $Z = Z(\delta) \otimes (Z - Z(\Delta))$, the claim is proven.

Thus by the above, for any $Z$ as in the lemma, there are invertible sections $f_1$, $f_\infty \in \Gamma(F, L_1)$, $f_0^\delta \in \Gamma(F, L_1 \otimes L_0^{-1})$, $f_1 = f_\infty + \delta f_0^\delta$. Since $L_0$ and $L_1 \otimes L_0^{-1}$ are coherent sheaves on the affine scheme, there are sections $s_0 \in \Gamma(Z, L_0)$, $s_1 \in \Gamma(Z, L_1 \otimes L_0^{-1})$ such that $s_\infty |_F = f_\infty$, $s_0^\delta |_F = f_0^\delta$. Define $s_1 = s_\infty + \delta s_0^+$, then $s_1 |_F = f_1$. Since $f_0, f_1, f_\infty$ are invertible $s_0, s_1, s_\infty$ are invertible. □

Appendix C. Framed correspondences and homotopies

Lemma C.1. Given a scheme $U$ over a base scheme $S$ and monic polynomial $f$ in $\mathcal{O}(U)[t]$ of degree $1$, there is the equality $[(Z(f), f, r)] = t e \in \mathcal{F}_1(U, U)$, where $r : \mathbb{A}_1^1 \rightarrow U$ is the canonical projection.

Proof. The equality follows by the sequence

\[
[(Z(f), f, r)] = [(Z((t-1)^{l-1}), (t-(t-1)^{l-1}, r))] + [(Z(t), (t-1)^{l-1}, r)]
\]

the second because of homotopies

\[
(Z((t-1)^{l-1}) \times \mathbb{A}_1^1, \tilde{g}_0, r) \in \mathcal{F}_1(U \times \mathbb{A}_1^1, U), \tilde{g}_0 = f(t - \lambda) + t(l - 1)^{l-1} l \in \mathcal{O}(U)[\lambda][t],
\]

and the third equality holds, since $[(Z((t-1)^{l-1}, (t-(t-1)^{l-1}, r))] = [(l-1)2]$ by the inductive assumption, and because $(Z(t), (t-1)^{l-1}, r) \in (l-1)^{l-1}$.

Lemma C.2. $(l_1)z \circ (l_2)z = \sigma(l_1, l_2)z \in \mathcal{F}_2(U, U)$ for any scheme $U$ and integers $l_1, l_2$.

Proof. By Lemma C.1 $[(l_1)z \circ ((-1)^l)] = \sigma((-1)^l \cdot (l_1)z]$. Note that $(-1)^l \cdot (l_1)z] = (-1)^{l+1} \cdot (l_1)z]$, because for even $l_1$, we have $((-1)^l \cdot (l_1)z] = (-1)^{l-1} \cdot (l_1)z]$, and for odd $l_1$, $(-1)^l = (-1)^{l+1}$. Thus $[(l_1)z \circ \sum_{j=0}^{l-1} (-1)^j \cdot (l_1)z] = [\sigma(\sum_{j=0}^{l-1} ((-1)^j \cdot (l_1)z)] = [(l_2)z]$. □

Denote by $E_n(V)$ the subgroup of elementary matrices in $\text{GL}_n(V)$.

Lemma C.3. For a scheme $V$, If the matrices $G_1, G_2 \in \text{GL}_n(V)$ have the same class in $E_n(V) \otimes \text{GL}_n(V) / E_n(V)$, then $[\sigma_{G_1}^{G_2}] = [\sigma_{G_2}^{G_1}] \in \mathcal{F}_n(V)$.

Proof. The claim follows because $E_n(V)$ is generated by elementary transvections $e_{\nu, t}(l) \in \mathcal{V}(V), v \in \Gamma(V, \mathcal{O}^\mathcal{V}_n), l \in \Gamma(V, \mathcal{O}^\mathcal{V}_n)$, and for each matrix $G \in \text{GL}_n(V)$, each transvection $e_{\nu, t}(l)$ defines $\mathbb{A}^1$-homotopies $\sigma_{G, e_{\nu, t}(l)}^G, \sigma_{G, e_{\nu, t}(l)}^{G, G} \in \mathcal{F}_n(V \times \mathbb{A}^1, V)$, where $\mathcal{O}(V \times \mathbb{A}^1) = \mathcal{O}(V)[\lambda]$. □

Lemma C.4. $[(\lambda_1)z \circ (\lambda_2)z] = [\sigma(\lambda_1, \lambda_2)z] \in \mathcal{F}_2(U)$, for any scheme $U$ and $\lambda_1, \lambda_2 \in \mathcal{O}^\mathcal{V}(U)$. 

24 SI THEOREM VIA COMPACTIFIED HOMOTOPIES
Proof. Consider the diagonal matrices $T_1 = (\lambda_1, \lambda_2)$ and $T_2 = (1, \lambda_1 \lambda_2)$ in $\text{GL}_2(U)$, and the rotation $r_{1,2} = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$, and elementary transvections $e_{2,1}(\alpha)$ for $\alpha \in O_U(U)$ in $E_2(U)$. Since

$$r_{1,2} e_{2,1}(-\lambda_1^{-1}) T_1 e_{2,1}(\lambda_2) e_{2,1}(-\lambda_2^{-1}) = T_2,$$

the claim follows from Lemma C.3 because $(\lambda_1) \circ (\lambda_2) = c r_1$, and $\sigma[(\lambda_1 \lambda_2)] = c r_2$ in sense of notation from Lemma C.5. □

Lemma C.5. Let $X \in S_{M_K}$ be a smooth affine scheme, $T_U(X) \cong \mathbb{A}_K^d$. There is a $d$-dimensional framed correspondence $(X)^{fr} = (S, \varphi, g)$ from $U$ to $X$ such that $g|_S : S \to X$ is an isomorphism, see Definition B.4.

Proof. Since the tangent bundle of $X$ is trivial then there is a closed immersion $X \to \mathbb{A}^N_X$ such that the normal bundle $N_{X/\mathbb{A}^N_X}$ is trivial. Then there is a vector of regular functions $f_{d+1}, \ldots, f_N \in O(\mathbb{A}^N_X)$, $d = \dim_X X_U$, such that $Z(f_{d+1}, \ldots, f_N) = X \cap \mathbb{A}^N_X$ for some $X$. By [22, Theorem I.8], see also [18, 10, Lemma 3.11], the canonical isomorphism $g : S \to X$ admits a lifting $g : (\mathbb{A}^N_X)^N_X \to X$. Thus we get the $d$-dimensional framed correspondence $(X)^{fr} = (X, f_{d+1}, \ldots, f_N, g)$. □

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26

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