From a dichotomy for images to Haagerup’s inequality

Iosif Pinelis*
August 2, 2009

1 Images of subsets: filling/containment dichotomy

Let $X$ be a compact topological space, and let $D$ be a subset of $X$, with the closure $\overline{D}$ and “boundary” $\partial D := \overline{D} \setminus D$; note that, if the set $D$ is open, then $\partial D$ will be the boundary of $D$ in the usual sense. Let $Y$ be a Hausdorff topological space. Let $f : \overline{D} \to Y$ be a continuous map such that $f(D)$ is open.

Dichotomy Principle (DP). Let $E$ be any connected subset of the complement $f(\partial D)^c$ of $f(\partial D)$ to $Y$. Then either $E \subseteq f(D)$ or $f(\overline{D}) \subseteq E^c$; that is, the image $f(D)$ of $D$ under $f$ either fills the set $E$ or is contained in $E^c$.

The proof is almost trivial. Observe first that $E \subseteq f(D) \cup f(\overline{D})^c$ (since $E \subseteq f(\partial D)^c$). Next, $f(D)$ is open (by assumption) and $f(\overline{D})^c$ is open as well (since $f(\overline{D})$ is compact). Therefore and because $E$ is connected, either $E \cap f(D) = E$ (that is, $E \subseteq f(D)$) or $E \cap f(\overline{D})^c = E$ (that is, $f(\overline{D}) \subseteq E^c$).

The DP can be rewritten in the following “containment” form.

Containment Principle (CP). One has $f(\overline{D}) \subseteq \bigcup_{y \in f(\overline{D})} E_y^c$, where $E_y$ denotes the connected component of $y$ in $f(\partial D)^c$.

Indeed, take any $y \in f(\overline{D})^c$. Then $y \in f(\partial D)^c$ and $y \in E_y \setminus f(D)$, whence $E_y \not\subseteq f(D)$. So, by the DP, $f(\overline{D}) \subseteq E_y^c$. Thus, the DP implies the CP.

Vice versa, suppose now that the CP holds. Let $E$ be any connected subset of $f(\partial D)^c$. Suppose that the first alternative, $E \subseteq f(D)$, in the DP is false. Then there exists some $y \in E \setminus f(D)$, so that $y \in f(\overline{D})^c$ (since $y \in E \subseteq f(\partial D)^c$). Hence, by the CP, $f(\overline{D}) \subseteq E_y^c \subseteq E^c$.

The following, tripartite corollary of the DP may be viewed as an abstract, topological generalization of the Jordan Filling Principle for $Y = \overline{C}$ presented in the next section.

*Supported by NSF grant DMS-0805946
Quasi-Jordan Filling Principle (QJFP). Suppose that \( D \neq \emptyset \) and let \( E \) and \( F \) stand for some connected subsets of \( Y \). Then one has the following.

(I) If \( f(D) \subseteq E \subseteq f(\partial D)^c \), then \( f(D) = E \).

(II) If \( E \subseteq f(\partial D)^c \subseteq E \cup F \), \( f(D) \subseteq F^c \), and \( f(\partial D) \subseteq F \), then \( f(D) = E \) (moreover, it follows that \( E \neq \emptyset \), \( E \cap F = \emptyset \), and \( f(D) \subseteq f(\partial D)^c \) — that is, \( f \) does not take on \( D \) any of the values it takes on \( \partial D \)).

(III) If \( f(\partial D)^c = E \cup F \), \( F \nsubseteq f(D) \), and \( f(\partial D) \subseteq F \), then \( f(D) = E \) (moreover, it follows that \( E \neq \emptyset \), \( F \neq \emptyset \), \( E \cap F = \emptyset \), and \( f \) does not take on \( D \) any of the values it takes on \( \partial D \)).

Proof.
(I). The conditions \( D \neq \emptyset \) and \( f(D) \subseteq E \) imply \( f(D) \nsubseteq E^c \) and hence \( f(\partial D) \nsubseteq E^c \). So, by the DP, \( E \subseteq f(D) \subseteq E \), which proves part (I) of the QJFP.

(II). Assume now that the conditions of part (II) of the QJFP hold. Verify first the last conclusion of part (II) — that \( f(D) \subseteq f(\partial D)^c \); indeed, if that conclusion were false, then one would have \( \emptyset \neq f(D) \cap f(\partial D) \subseteq f(D) \cap F \), which would contradict the conditions that \( f(D) \) is open and \( f(D) \subseteq F^c \). So, \( f(D) \subseteq f(\partial D)^c \cap F^c \subseteq (E \cup F) \cap F^c = E \setminus F \). Therefore and by (I), \( E = f(D) \subseteq E \setminus F \). This in turn yields \( E \cap F = \emptyset \). Also, the conditions \( D \neq \emptyset \) and \( E = f(D) \) imply \( E \neq \emptyset \). This completes the proof of part (II).

(III). Part (III) follows from part (II). Indeed, the condition \( f(\partial D)^c = E \cup F \) implies \( F \subseteq f(\partial D)^c \); hence, by the DP, the condition \( F \nsubseteq f(D) \) yields \( f(D) \subseteq F^c \), so that all the conditions of part (II) hold. Also, the condition \( F \nsubseteq f(D) \) implies \( F \neq \emptyset \). □

In the above proof, we deduced QJFP(II) from QJFP(I), and QJFP(III) from QJFP(II). So, one may say that QJFP(I) is the most general of the three parts of the QJFP, while QJFP(III) is the most special one.

In the case when \( f: \Omega \to \Omega' \) is a proper holomorphic map, where \( \Omega \) and \( \Omega' \) are open connected subsets of \( \mathbb{C}^n \), Rudin [7, Proposition 15.1.5] shows that the "filling" conclusion \( f(\Omega) = \Omega' \) holds (\( f \) is said to be proper if \( f^{-1}(K) \) is compact in \( \Omega \) for any compact \( K \subseteq \Omega' \)). Rudin [7, Theorem 15.1.6] also shows that, for a holomorphic map \( f: \Omega \to \Omega' \) to be locally proper and hence open, it is enough that the set \( f^{-1}(w) \) be compact (or, equivalently, finite) for every \( w \in \Omega' \).

The QJFP (especially its parts (II) and (III)) will be quite useful in certain contexts, such as the proof of the JFP in the next section. However, at this point let us just present a simple, almost trivial illustration of how the QJFP can be applied:

If \( D = X \neq \emptyset \) and \( Y \) is connected, then \( f(D) = Y \).

This follows immediately by invoking the QJFP(II) with \( E = Y \) and \( F = \emptyset \).

Perhaps surprisingly, the purely topological (and almost trivial) dichotomy principle (DP) turns out to be convenient and useful in the applications to various interesting inequalities, even in the special case when the map \( f \) is holomorphic.
2 Special cases and applications

2.1 Case \( Y = \overline{\mathbb{C}} \)

In this subsection, let us assume that the general conditions stated in the first paragraph of Section 1 hold. In addition, assume here that \( Y = \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \), the Riemann sphere, whereas \( X \) may still be any compact topological space.

However, when it is the case that \( X = \mathbb{C} \), \( D \) is a domain (that is, an open connected set), and the map \( f : D \to \overline{\mathbb{C}} \) is non-constant and holomorphic on \( D \), then, by the open map theorem (cf. e.g. [1] Theorem 5.77 or [2] VII.1.3), the condition that \( f(D) \) be open will be satisfied. (Here we shall say that \( f \) is holomorphic on \( D \) if for any point \( z_0 \in D \) there exist Möbius transformations \( M_1 \) and \( M_2 \) of \( \overline{\mathbb{C}} \) such that \( M_1(z_0) \) is finite (that is, is in \( \mathbb{C} \)) and the function \( M_2 \circ f \circ M_1^{-1} \) is finite and differentiable (in the complex-variable sense) in a neighborhood of \( M_1(z_0) \). Clearly, if \( D \subseteq \mathbb{C} \) and \( f(D) \subseteq \mathbb{C} \), then this extended notion of a holomorphic function is equivalent to the more usual one.)

**Finite Containment Principle (FCP).** If \( f \) is finite on \( \overline{D} \), then \( f(D) \subseteq E_\infty \).

This follows immediately from the CP.

**Jordan Filling Principle (JFP).** Suppose that \( f \) is finite on \( \overline{D} \) and \( f(\partial D) = J \), where \( J \) is the image (in \( \overline{\mathbb{C}} \)) of a Jordan curve. Then \( f(D) = \mathcal{I}(J) \), where \( \mathcal{I}(J) \) denotes the inside of \( J \), that is, the bounded connected component in \( \mathbb{C} \) of \( \mathbb{C} \setminus J \).

This follows immediately from the QJFP(III) (on letting \( E := \mathcal{I}(J) \) and \( F := E_\infty \)).

The JFP may be compared with the following result, based on the argument principle (cf. e.g. [1] Corollary 9.16 and Exercise 9.17):

**Darboux-Picard Theorem (DPT).** Assume that \( X = \mathbb{C} \), \( D \) is a domain, and the function \( f \) is non-constant and holomorphic on \( D \). Let \( D \subseteq \mathbb{C} \) be the inside of the image of a Jordan curve, and suppose that \( f \) is finite on \( \overline{D} \) and one-to-one on \( \partial D \). Then \( f \) is one-to-one on \( \overline{D} \), and \( f(D) \) is the inside of \( f(\partial D) \).

A partial extension of the DPT to \( X = Y = \mathbb{C}^n \) was given by Chen [3], where, in addition to the injectivity of \( f \) on \( \overline{D} \), it was proved only that \( f(D) \) is a subset of the inside (rather than exactly the inside) of \( f(\partial D) \).

One can see that, in contrast with the DPT, in the JFP we do not require that \( D \) be a domain, or that \( f \) be one-to-one on the boundary \( \partial D \), or that \( \partial D \) be the image of a Jordan curve (or any other curve), or even that the space \( X \) be \( \mathbb{C} \) or \( \overline{\mathbb{C}} \). On the other hand, the conclusion of the JFP is somewhat weaker than that of the DPT, in that the former is, naturally, missing the injectivity of \( f \) on \( \overline{D} \).

Of course, the QJFP is significantly more general than the JFP, even when \( X = Y = \overline{\mathbb{C}} \) and \( f \) is holomorphic on \( D \).

**Example 1.** Let \( X = Y = \overline{\mathbb{C}} \), \( D = \overline{\mathbb{C}} \setminus \{0, \infty\} \), \( f(z) = z + 1/z \) for \( z \in D \), and \( f(0) = f(\infty) = \infty \), so that \( \partial D = \{0, \infty\} \), \( f(\partial D) = \{\infty\} \), and \( f(\partial D)^c = \mathbb{C} \). Thus, \( f(\partial D) \) is not the image of a Jordan curve; so, the JFP is not applicable here, and therefore the DPT is not applicable either. However, one can easily apply the QJFP(II) (with \( E := \mathbb{C} \) and \( F := \{\infty\} \)), to conclude that \( f(D) = \mathbb{C} \). Of course, in this very simple situation the same conclusion can be obtained directly, by solving a quadratic equation.
The following, less trivial example may be viewed as a toy model for the setting to be considered in Subsection 2.2.

**Example 2.** Let \( X = \overline{\mathbb{C}} \) and \( D = \{ z \in \mathbb{C} : \Re z > 0, \Im z > 0 \} \), so that \( \partial D = \{ \infty \} \cup \Gamma_1 \cup \Gamma_2 \), where \( \Gamma_1 := \{ z \in \mathbb{C} : \Re z \geq 0, \Im z = 0 \} \) and \( \Gamma_2 := \{ z \in \mathbb{C} : \Re z = 0, \Im z > 0 \} \). Let next \( f(z) = \frac{2z}{z-1} \) for \( z \in \overline{D} \setminus \{1, \infty\} \), \( f(1) = \infty \), and \( f(\infty) = 0 \). Then \( f(\partial D) = f(\Gamma_1) \cup f(\Gamma_2) \), \( f(\Gamma_1) = \{ \infty \} \cup \{ w \in \mathbb{C} : \Im w = 0 \} \), and \( f(\Gamma_2) = \{ w \in \mathbb{C} : \Re w = 0, -1 \leq \Re w < 0 \} \). Let now \( H_+ := \{ w \in \mathbb{C} : \Re w \geq 0 \}, F := f(\Gamma_2) \cup H_+ \) and \( E := f(\partial D)^c \setminus F = \mathbb{C} \setminus f(\Gamma_2) \setminus H_+ \). Then one can verify that the condition \( f(D) \subseteq \Delta^\circ \) of the QJFP(II) holds. Indeed, note first that for \( z \in D \) one has \( f(z) = \frac{2(z^2-1)}{z(z-1)} \), whence \( \Re f(z) < 0 \), so that \( f(D) \subseteq H_+^c \). Also, for \( z \in D \) one has \( f(z) = \frac{2|z|^2}{z^2-1} \), whence \( \Re f(z) = 0 \) iff \( |z| = 1 \), in which case \( |f(z)| = |\frac{2}{z-1}| > 1 \), so that \( f(D) \subseteq f(\Gamma_2)^c \). Thus, the condition \( f(D) \subseteq \Delta^\circ \) is verified. The other conditions of the QJFP(II) are even easier to check. Therefore, \( f(D) = E \).

However, the JFP is not applicable here (and therefore the DPT is not applicable either), because \( f(\partial D) \) cannot be the image of a simple closed curve in \( \overline{\mathbb{C}} \); indeed, \( f(\Gamma_1) \) is a proper closed subset of \( f(\partial D) \), and yet the set \( \overline{\mathbb{C}} \setminus S \) is not connected – cf. e.g. \([1, \text{Exercise 4.39}]\). One may also note the following. Suppose that \( z \) traces out \( \Gamma_1 \) from \( \infty \) to 1 to 0, and then traces out \( \Gamma_2 \) from 0 to \( \infty \); at that, \( f(z) \) will first trace out the positive real semi-axis from 0 to \( \infty \), then will jump to \( -\infty \) and trace out the negative real semi-axis from \( -\infty \) to 0, then will trace out the vertical segment \( f(\Gamma_2) \) from 0 down to \( -i \), and finally will trace out \( f(\Gamma_2) \) back from \( -i \) to 0. (Of course, the “jump” from \( \infty \) to \( -\infty \) is not really a jump on the Riemann sphere \( \overline{\mathbb{C}} \).) Thus, \( f \) is not one-to-one on \( \partial D \). This example is illustrated below.

\[
\begin{array}{c}
0 & \Gamma_1 & \Gamma_2 \\hline
f(\Gamma_2) & f(\Gamma_1) \\hline
\end{array}
\]

Consider now applications of the dichotomy principle to maximum and minimum modulus principles (again for any compact \( X \)). For any \( r \in [0, \infty) \), let \( B_r := \{ w \in \overline{\mathbb{C}} : |w| < r \} \) and \( \overline{B_r} := \{ w \in \overline{\mathbb{C}} : |w| \leq r \} \); one may note that the closure \( \overline{B_r} \) of \( B_r \) coincides with \( \overline{B_r} \) unless \( r = 0 \), in which latter case \( \overline{B_0} = \emptyset \) and \( \overline{B_r} = \{0\} \). Let also \( M := \sup \{|f|(\partial D)| \) and \( m := \inf \{|f|(\partial D)| \). 

**Finite Maximum Modulus Principle (FinMaxMP).** If \( f \) is finite (on \( \overline{D} \)) then 
\[
\max \{|f|(\overline{D})| = \sup \{|f|(\partial D)|,
\]
Indeed, if \( M = \infty \) then the FinMaxMP is trivial. Assume now that \( M < \infty \) and let \( E := \overline{B_M} \). Then \( E \) is a connected subset of \( f(\partial D)^c \) and \( E \not\subseteq f(D) \), since \( \infty \in E \setminus f(D) \). So, by the DP, \( f(\overline{D}) \subseteq E^c = \overline{B_M} \).
More generally, the DP (with \( E = B_M^c \)) immediately yields

Maximum Modulus Principle (MaxMP). Either

\[
    f(D) \supseteq B_M^c, \text{ that is, } f \text{ takes on } D \text{ all the values that are } > M \text{ in modulus}; \quad (\mathcal{F}_{\max})
\]

or \( f(D) \subseteq B_M^c, \text{ that is, all the values that } f \text{ takes on } D \text{ are } \leq M \text{ in modulus}. \quad (\mathcal{C}_{\max})
\]

Note that the “containment” alternative \((\mathcal{C}_{\max})\) can be rewritten as \(\max |f(D)| = \sup |f|(|\partial D|); cf. the FinMaxMP.\)

Quite similarly, the DP (with \( E = B_m^c \)) yields

Minimum Modulus Principle (MinMP). Either

\[
    f(D) \supseteq B_m^c, \text{ that is, } f \text{ takes on } D \text{ all the values that are } < m \text{ in modulus}; \quad (\mathcal{F}_{\min})
\]

or \( f(D) \subseteq B_m^c, \text{ that is, all the values that } f \text{ takes on } D \text{ are } \geq m \text{ in modulus}. \quad (\mathcal{C}_{\min})
\]

Note that the “containment” alternative \((\mathcal{C}_{\min})\) can be rewritten as \(\min |f(D)| = \inf |f|(|\partial D|).\)

Note also that each of the two alternatives in the MaxMP and in the MinMP (and thus in the DP) actualizes. Indeed, take the trivial example of \( f(z) = z \) for all \( z \in D \), where \( D \) is either \( B_1 \) or \( B_1^c \).

Various versions of the maximum and minimum modulus principles (for non-constant finite holomorphic functions on domains in \( X = \mathbb{C} \)) may be found e.g. in [4]. The FinMaxMP presented above corresponds to the second of the three maximum principles given in [4, pages 124–125].

Our MinMP can be compared with the minimum modulus principle stated (for non-constant finite holomorphic functions on bounded domains \( D \)) in Exercise 1 on page 125 of [4], which latter has the alternative \( f(D) \ni 0 \) instead of \( f(D) \ni B_m^c \); let us refer to that statement in [4] as the 0-MinMP. This somewhat less informative principle, 0-MinMP, is enough to obtain immediately the main theorem of algebra. Indeed, let \( R \in (0, \infty) \) be such that \( m := \min |z| = R |f(z)| > |f(0)| \), where \( f \) is a given polynomial of degree \( \geq 1 \). Then the polynomial \( f \) takes on the value 0 in \( D := B_R \), since the alternative \((|f| \geq m \text{ on } D)\) cannot take place.

One may note that (again in the case when \( f \) is a non-constant holomorphic function on \( D \) and \( D \) is a domain) it is not hard to deduce the general MinMP from the 0-MinMP. Indeed, fix any \( w_\ast \in B_m \). Let \( g \) be a Möbius transformation of \( \mathbb{C} \) leaving each of sets \( B_m, \partial B_m, \) and \( B_m^c \) invariant, and such that \( g(w_\ast) = 0 \). Let \( h := g \circ f \). Then \( \min |h|(|\partial D|) = m \).

So, by the 0-MinMP, either \( |h| \geq m \) or \( h(D) \ni 0 \); that is, either \( |f| \geq m \) or \( f(D) \ni w_\ast \).

However, our MinMP is more informative and directly derived.

### 2.2 Haagerup’s inequality

Haagerup’s inequalities [5] provide exact upper and lower bounds on the absolute power moments of normalized linear combinations of independent Rademacher random variables. Unfortunately, the proof given in [5] is very long and difficult. Nazarov and Podkorytov [6] discovered a short and ingenious way to prove Haagerup’s result. A seemingly insipid but actually crucial point in their proof is the following.
Argument Containment Proposition (ACP). The domain

\[ D := \{ z \in \mathbb{C} : 0 < \Re z < \frac{\pi}{2}, \Im z > 0 \} \]

is mapped into the set

\[ \Delta_p := \{ w \in \mathbb{C} : -\frac{2\pi}{p} < \arg w \leq 0 \} \]

by the function \( f \) defined by the formula

\[ f(z) := z^{-p} - (\pi - z)^{-p} + (\pi + z)^{-p} - (2\pi - z)^{-p} + (2\pi + z)^{-p} - \cdots, \quad (2.1) \]

where \( 1 < p < 2 \) and the principal branch of the power function is used, so that \( z^{-p} > 0 \) for any \( z > 0 \); as usual, the values of the argument function \( \arg \) are assumed to be in the interval \( (-\pi, \pi] \); let us also assume that \( \arg 0 = 0 \).

The authors of [6] note (on page 259) that for all \( z \in D \) the points \( z^{-p}, (\pi + z)^{-p}, (2\pi + z)^{-p}, \ldots \) are in \( \Delta_p \). Then, to conclude that \( f(z) \in \Delta_p \) for \( z \in D \), they proceed to claim that the points \(- (\pi - z)^{-p}, -(2\pi - z)^{-p}, \ldots \) are also in \( \Delta_p \); however, this claim is obviously false: if a point \( z \in D \) is close (say) to \( \frac{\pi}{4} \), then the arguments of the points \(- (\pi - z)^{-p}, -(2\pi - z)^{-p}, \ldots \) are close to \( -\pi \notin [-\frac{2\pi}{p}, 0] \).

One may then wonder whether the ACP can be saved by simple means such as trying to prove that each of the differences \( z^{-p} - (\pi - z)^{-p}, (\pi + z)^{-p} - (2\pi - z)^{-p}, \ldots \) is in \( \Delta_p \). However, this latter conjecture is false, even if one instead considers partial sums of these differences; e.g., the argument of the sum of the first 100 differences is \( < -\frac{2\pi}{p} (1 + 3.5 \times 10^{-18}) < -\frac{2\pi}{6} \) for \( z = 10^{-30} + 10^{-6}i \) and \( p = 19/10 \). Alternatively, one may try to consider \( f(z) \) as the sum of the terms \( z^{-p}, -(\pi - z)^{-p} + (\pi + z)^{-p}, -(2\pi - z)^{-p} + (2\pi + z)^{-p}, \ldots \); however, this simple trick does not work either, as already the term \( -(\pi - z)^{-p} + (\pi + z)^{-p} \) is outside \( \Delta_p \), e.g. when \( z = \frac{\pi}{6} + 10^{-2}i \) and \( p = 19/10 \).

Fortunately, the ACP can be rather easily proved using the topological dichotomy principle (DP), which – now to disclose the motivation for this paper – was developed specifically to address this containment concern.

Proof of the ACP. Note that \( \partial D = \Gamma_1 \cup \cdots \cup \Gamma_5 \), where (trying to keep up with the corresponding notation in [6])

\[ \Gamma_1 := \{ 0 \}, \]
\[ \Gamma_2 := \{ z \in \mathbb{C} : \Re z = 0, \Im z > 0 \}, \]
\[ \Gamma_3 := \{ \infty \}, \]
\[ \Gamma_4 := \{ z \in \mathbb{C} : \Re z = \frac{\pi}{2}, \Im z \geq 0 \}, \]
\[ \Gamma_5 := \{ z \in \mathbb{C} : 0 < \Re z < \frac{\pi}{2}, \Im z = 0 \}; \]

this is illustrated by the picture below on the left, with a portion of \( D \) near \( \infty \) cut off. Note that \( |f(z) - z^{-p}| \leq (\frac{2\pi}{p})^{-p/2}(1^{-p} + 2^{-p} + \cdots) < \infty \) for all \( z \in D \); so, by dominated convergence, \( f \) can be extended to \( \overline{D} \setminus \{ 0, \infty \} \) by the same formula \( (2.1) \); let also \( f(0) := \infty \) and \( f(\infty) := 0 \), so that \( f \) is continuous on \( \overline{D} \), and non-constant and holomorphic on \( D \).

Thus, \( f = \infty \) on \( \Gamma_1 \) and \( f = 0 \in \Delta_p \) on \( \Gamma_3 \). Let us now prove that the images of \( \Gamma_2, \Gamma_4, \Gamma_5 \) under \( f \) are contained in \( \Delta_p \). (These images, as well as part of the boundary of the angular set \( \Delta_p \), are shown in the picture below on the right, with a portion of \( \Delta_p \) near \( \infty \) cut off.)
For $z \in \Gamma_4$, one has $\Re f(z) = 0$ and $\Im f(z) \leq 0$ (cf. the second displayed formula on page 262 in \[1\]). Thus, $f(\Gamma_4) \subseteq \Delta_p$. At this point one may note that $f\left(\frac{\pi}{2}\right) = f(\infty) = 0$; so, $f(z)$ traces out the vertical segment $f(\Gamma_4)$ on the imaginary axis (at least) twice as $z$ traces out the vertical ray $\Gamma_4$. Therefore, the one-to-one condition of the Darboux-Picard theorem stated in Subsection 2.1 does not hold here. Yet, the dichotomy principle (DP) of Section 1 allows us to proceed and obtain the containment result.

For $z \in \Gamma_5$, one has $\exists f(z) = 0$ and $\Re f(z) > 0$, since $(k\pi + t)^{-p} > (k\pi + t - \pi)^{-p}$ for all $k \geq 0$ and $t \in (0, \frac{\pi}{2})$. Thus, $f(\Gamma_5) \subseteq \Delta_p$.

Finally, for $z \in \Gamma_2$ one has $\Re f(z) < 0$ and $\exists f(z) < \Re f(z) \tan(-\frac{\pi}{2})$ (cf. \[6\], the middle of page 261). Thus, $f(\Gamma_2) \subseteq \Delta_p$.

We conclude that $f(\partial D) \subseteq \Delta_p \cup \{\infty\}$. Let now $E := \mathbb{C} \setminus \Delta_p$. Then $E$ is a connected subset of $f(\partial D)^c$. Moreover, every $w \in \mathbb{C}$ with $\exists w > 0$ is in $E$. On the other hand, $\exists w < 0$ for any $w \in f(D)$ (since $\exists[(k\pi + z)^{-p}] < 0$ and $\exists[-(k\pi + z - \pi)^{-p}] < 0$ for any $k \geq 0$ and any $z \in D$). So, $E \nsubseteq f(D)$. Thus, by the dichotomy principle, $f(D) \subseteq E^c = \Delta_p \cup \{\infty\}$. Since $f(D) \subseteq \mathbb{C}$, it follows that $f(D) \subseteq \Delta_p$. In fact, since $\exists w < 0$ for any $w \in f(D)$, one has a little more: $f(D)$ is contained in the interior of $\Delta_p$. 

While the ACP is enough as far as the proof of Haagerup’s inequality is concerned, one might want to prove more. One improvement is easy. Let $A$ and $B$ denote, respectively, the sets of all points in $\mathbb{C}$ strictly above and below $f(\partial D)$. Then the DP with $E = A$ (instead of $E = \mathbb{C} \setminus \Delta_p$ in the above proof of the ACP) yields $f(D) \subseteq A^c = B \cup f(\partial D)$, and the latter set is a proper subset of $\Delta_p$. Now, by the QJFP(II) of Section 1 with $E = B$ and $F = A \cup f(\Gamma_4)$, one could conclude that $f(D) = B$ – provided that one could show that $f(D) \subseteq \mathbb{C} \setminus \Delta_p$ (cf. Example 2 of Subsection 2.1); however, this does not appear easy to do.

The inequality $\exists f(z) < \Re f(z) \tan(-\frac{\pi}{2})$ for $p \in (1, 2)$ and $z = x + iy$ with $x \in (0, \frac{\pi}{2})$ and $y > 0$, implied by the ACP, can be rewritten as

$$\sum_{k=0}^{\infty} \frac{\sin\left(p \arccot \frac{y}{\pi k + x}\right)}{((\pi k + x)^2 + y^2)^{p/2}} > \sum_{k=0}^{\infty} \frac{\sin\left(p \arccot \frac{y}{\pi k + x - \pi}\right)}{((\pi k + x)^2 + y^2)^{p/2}}$$

(*)

(with the values of the function $\arccot$ in the interval $(0, \pi)$). To appreciate the usefulness of the topological dichotomy principle, one may try to prove (*) by other methods, say by methods of the calculus of functions of real variables.

**Acknowledgment.** I am pleased to thank J. Michael Steele for bringing the paper by Nazarov and Podkorytov to my attention.
References

[1] Burckel, R. B. An introduction to classical complex analysis. Vol. 1, vol. 82 of Pure and Applied Mathematics. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979.

[2] Cartan, H. Elementary theory of analytic functions of one or several complex variables. Editions Scientifiques Hermann, Paris, 1963.

[3] Chen, S.-C. On the Darboux-Picard theorem in $\mathbb{C}^n$. Michigan Math. J. 40, 3 (1993), 605–608.

[4] Conway, J. B. Functions of one complex variable. Springer-Verlag, New York, 1973. Graduate Texts in Mathematics, 11.

[5] Haagerup, U. The best constants in the Khintchine inequality. Studia Math. 70, 3 (1981), 231–283 (1982).

[6] Nazarov, F. L., and Podkorytov, A. N. Ball, Haagerup, and distribution functions. In Complex analysis, operators, and related topics, vol. 113 of Oper. Theory Adv. Appl. Birkhäuser, Basel, 2000, pp. 247–267.

[7] Rudin, W. Function theory in the unit ball of $\mathbb{C}^n$, vol. 241 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]. Springer-Verlag, New York, 1980.

Department of Mathematical Sciences
Michigan Technological University
Houghton, MI 49931
ipinelis@mtu.edu