Fixed points of endomorphisms on two-dimensional complex tori

Thomas Bauer, Thorsten Herrig

August 26, 2015

Abstract. In this paper we investigate fixed-point numbers of endomorphisms on complex tori. Specifically, motivated by the asymptotic perspective that has turned out in recent years to be so fruitful in Algebraic Geometry, we study how the number of fixed points behaves when the endomorphism is iterated. Our first result shows that the fixed-points function of an endomorphism on a two-dimensional complex torus can have only three different kinds of behaviours, and we characterize these behaviours in terms of the analytic eigenvalues. Our second result focuses on simple abelian surfaces and provides criteria for the fixed-points behaviour in terms of the possible types of endomorphism algebras.

Introduction

Given a holomorphic map $f : X \to X$ on a complex variety $X$, one of the natural questions about $f$ is how many fixed points it has. This number may, as expected, vary a lot between different endomorphisms, but it is a recurring theme in Algebraic Geometry that one hopes for much more regularity when adopting an asymptotic perspective. Examples for the fruitfulness of this approach are questions about base loci [ELMNP], growth of higher cohomology [DKL], syzygies [EL] and Betti numbers [EEL]. Concerning the question of fixed points, a natural asymptotic point of view consists in considering large iterates $f^n$ of a given map. Specifically, denoting by $\#\text{Fix}(f)$ the number of fixed points of a map $f$, the question becomes:

What is the asymptotic behaviour of the fixed-points function

$$n \mapsto \#\text{Fix}(f^n)$$

where $f^n = f \circ \ldots \circ f$ denotes $n$-th iterate of $f$.

The growth of the fixed-points function is also of interest in purely analytic contexts (e.g. [SS]). In the present paper we consider it when $f$ is a holomorphic map on a complex torus. As is customary (cf. [BL]) we set $\#\text{Fix}(f) = 0$, if the fixed-points set is infinite, i.e., if $f$ fixes an analytic subvariety of positive dimension.

The second author was supported by Studienstiftung des deutschen Volkes.

Keywords: abelian variety, endomorphism, fixed point.

Mathematics Subject Classification (2010): 14A10, 14K22, 14J50.
Consider for instance the multiplication map \( m_X : X \to X, x \mapsto mx \), on a complex torus \( X \) of dimension \( g \), for a given integer \( m \geq 2 \). Its fixed points are the \((m-1)\)-torsion points, and hence the fixed-points number
\[
\# \text{Fix}((m_X)^n) = (m^n - 1)^{2g}
\]
grows exponentially with \( n \). It is natural to wonder whether this is typical for endomorphisms on complex tori, and what other behaviour, if any, might occur. For two-dimensional complex tori we provide a complete answer:

**Theorem 1.** Let \( X \) be a two-dimensional complex torus and let \( f : X \to X \) be a non-zero endomorphism. Then the fixed-points function \( n \mapsto \# \text{Fix}(f^n) \) has one of the following three behaviours:

- **(B1)** It grows exponentially in \( n \), i.e., there are real constants \( A, B > 1 \) and an integer \( N \) such that for all \( n \geq N \),
  \[
  A^n \leq \# \text{Fix}(f^n) \leq B^n.
  \]
  In this case both eigenvalues of \( f \) (i.e., of its analytic representation \( \rho_a(f) \in M_2(\mathbb{C}) \)) are of absolute value \( \neq 1 \).

- **(B2)** It is a periodic function. In this case the non-zero eigenvalues of \( f \) are roots of unity, and they are contained in the set of \( k \)-th roots of unity where \( k \in \{1, \ldots, 6, 8, 10, 12\} \).

- **(B3)** It is of the form
  \[
  \# \text{Fix}(f^n) = \begin{cases} 
  0, & \text{if } n \equiv 0 \pmod{r} \\
  h(n), & \text{otherwise}
  \end{cases}
  \]
  where \( r \geq 2 \) is an integer and \( h \) is an exponentially growing function. In this case one of the eigenvalues of \( f \) is of absolute value \( > 1 \) and the other is a root of unity.

All three behaviours occur already in the projective case, i.e., on abelian surfaces.

The fact that the eigenvalues govern the behaviour of the fixed-points function is a consequence of the Holomorphic Lefschetz Fixed-Point Formula (see Prop. [11]). The main point of the theorem is that the stated cases are the only ones that can occur.

For simple abelian surfaces there are only three non-trivial types of possible endomorphism algebras, and it is desirable to know about the fixed-points behaviour in terms of these types. Our second result contains this information:

**Theorem 2.** Let \( X \) be a simple abelian surface. Then the fixed-points function of any non-zero endomorphism \( f \in \text{End}(X) \) is either exponential (B1) or periodic (B2), but has never behaviour (B3). Specifically, we have:

- **(a)** Suppose that \( X \) has real multiplication, i.e., \( \text{End}_\mathbb{Q}(X) = \mathbb{Q}(\sqrt{d}) \) for a square-free integer \( d > 0 \). Then \( \# \text{Fix}(f^n) \) is periodic if \( f = \pm \text{id}_X \), and it grows exponentially otherwise.
(b) Suppose that \( X \) has indefinite quaternion multiplication, i.e., \( \text{End}_Q(X) \) is of the form \( Q + iQ + jQ + ijQ \), where \( i^2 = \alpha \in Q \setminus \{0\}, j^2 = \beta \in Q \setminus \{0\} \) with \( ij = -ji \) and \( \alpha > 0, \alpha \geq \beta \). Write \( f \in \text{End}(X) \) as \( f = a + bi + cj + dij \) with \( a, b, c, d \in Q \). Then \# Fix\((f^n)\) is periodic if \(|a + \sqrt{2}\alpha + c^2\beta^2 - d^2\alpha\beta| = 1\), and it grows exponentially otherwise.

(c) Suppose that \( X \) has complex multiplication, and let \( \sigma : \text{End}(X) \hookrightarrow \mathbb{C} \) be an embedding. Then \( f \) has periodic fixed-point behaviour if \(|\sigma(f)| = 1\), and it has exponential fixed-points growth otherwise.

We give a more detailed description of the three types in Sect. 2 and we provide a list of the finitely many eigenvalues that occur in endomorphisms with periodic fixed-points behaviour (see Prop. 2.1).

Fixed points of endomorphisms on complex tori and abelian varieties have been studied previously by Birkenhake and Lange [BL] with a focus on the classification of fixed-point free automorphisms. The question of fixed point numbers for iterates of endomorphisms on abelian varieties was first addressed in the preprint [Rin]; however, Theorem 1.2 in [Rin] is unfortunately erroneous.

1. Fixed points and eigenvalues

Let \( X \) be a complex torus of dimension \( g \) and let \( f : X \to X \) be a holomorphic map. The map \( f \) is a translate \( f = h + a \) of a group endomorphism \( h \in \text{End}(X) \) by some \( a \in X \), and we have (see [BL])

\[
\# \text{Fix}(f) = \# \text{Fix}(h).
\]

So, as far as fixed point numbers are concerned, it is enough to consider endomorphisms. The following proposition allows one to determine fixed point numbers from the eigenvalues of the analytic representation.

**Proposition 1.1.** Let \( f : X \to X \) be an endomorphism of a \( g \)-dimensional complex torus, and let \( \lambda_1, \ldots, \lambda_g \) be the eigenvalues of its analytic representation (counted with algebraic multiplicities). Then we have for every integer \( n \geq 1 \),

\[
\# \text{Fix}(f^n) = \left| \prod_{i=1}^{g} (1 - \lambda_i^n) \right|^2.
\]

**Proof.** Thanks to the Holomorphic Lefschetz Fixed-Point Formula [BH 13.1.2], the fixed point number can be computed from the analytic representation \( \rho_a(f) \in M_g(\mathbb{C}) \),

\[
\# \text{Fix}(f^n) = |\det(1 - \rho_a(f)^n)|^2.
\]

As the eigenvalues of \( \rho_a(f)^n \) are \( \lambda_1^n, \ldots, \lambda_g^n \), we get the asserted formula. \( \square \)

The proposition shows that the fixed-points function \( n \mapsto \# \text{Fix}(f^n) \) is governed by the size of the eigenvalues. If, for instance, it were to happen that all eigenvalues of \( f \) are of absolute value bigger than 1, then clearly \# Fix\((f^n)\) grows exponentially. (This is the case for the multiplication maps mentioned in the introduction.) The following examples show, however, that eigenvalues of absolute value equal to 1 as well as less than 1 occur, too.
**Example 1.2.** (Eigenvalues of absolute value 1). Take an elliptic curve $E$ and consider the complex torus $X = E \times E$. The endomorphism

$$f : X \to X, \quad (x, y) \mapsto (x - y, x)$$

has the eigenvalues $\frac{1 + \sqrt{-3}}{2}$ and $\frac{1 - \sqrt{-3}}{2}$. Using Prop. 1.1 we find that the fixed-points function of $f$ is periodic:

$$\# \text{Fix}(f^n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{6} \\ 1, & \text{if } n \equiv 1 \text{ or } n \equiv 5 \pmod{6} \\ 9, & \text{if } n \equiv 2 \text{ or } n \equiv 4 \pmod{6} \\ 16, & \text{if } n \equiv 3 \pmod{6}. \end{cases}$$

**Example 1.3.** (Eigenvalues of absolute value < 1). Examples of this kind have been constructed by McMullen [McM, Sect. 4] in order to exhibit degree 6 Salem numbers (see also [Res]). Specifically, McMullen shows that for every integer $a \geq 0$, the polynomial

$$P(t) = t^4 + at^2 + t + 1$$

occurs as the characteristic polynomial of the rational representation of an endomorphism on a two-dimensional complex torus. One checks that the zeros of $P(t)$ appear in conjugate pairs $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ with $|\alpha| < |\beta|$. As the constant term of $P$ equals 1, it follows that $|\alpha| < 1$.

In fact, a closer analysis of the preceding example shows that arbitrarily small eigenvalues occur:

**Proposition 1.4.** For every $\varepsilon > 0$ there exists a two-dimensional complex torus with an endomorphism that has a non-zero eigenvalue of absolute value less than $\varepsilon$.

**Proof.** We draw on the complex tori constructed in Example 1.3. An application of Rouché’s theorem shows that for all sufficiently large values of $a$ there exists a root of the polynomial

$$t^4 + at^2 + t + 1$$

in the disk of radius $\varepsilon$ around the origin. This proves the proposition. \qed

The following example shows that eigenvalues of absolute value < 1 occur also in the projective case:

**Example 1.5.** Let $E$ be an elliptic curve, and consider the abelian surface $X = E \times E$. Every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ defines an automorphism $f$ of $X$. Its analytic characteristic polynomial is $t^2 - (a + d)t + 1$, and from this one checks that $f$ has an eigenvalue arbitrarily close to 0 when $a + d$ is sufficiently large.

With a little more effort the same behaviour can even be found among simple abelian surfaces: There is a two-dimensional family of principally polarized abelian surfaces $X$ with endomorphism ring $\text{End}(X) = \mathbb{Z}[\sqrt{2}]$ (cf. [Bir, Prop. 2.1]). On such a surface consider the endomorphism $f = -1 + \sqrt{2}$. Its analytic characteristic polynomial is $t^2 + 2t - 1$, and hence it has $-1 + \sqrt{2}$ as an eigenvalue. Therefore $f^n$ has an eigenvalue arbitrarily close to 0 when $n$ is sufficiently large.
In view of Prop. 1.1, and considering the preceding examples, it becomes apparent that the issue, in the general case, is to understand what kind of eigenvalues may occur in endomorphisms of complex tori. We focus from now on on the surface case, where we show as a first step:

**Proposition 1.6.** Let $X$ be a two-dimensional complex torus and $f : X \to X$ an endomorphism. If $f$ has a non-zero eigenvalue of absolute value $< 1$, then it also has an eigenvalue of absolute value $> 1$.

**Proof.** The characteristic polynomial of the analytic representation $\rho_a(f)$ is

$$P^a_f(t) = \det(tI_2 - \rho_a(f)) = (t - \lambda_1)(t - \lambda_2).$$

Since the rational representation $\rho_r(f)$ is the direct sum of $\rho_a(f)$ and its conjugate, the characteristic polynomial of $\rho_r(f)$ is given by

$$P^r_f(t) = P^a_f(t) \cdot \overline{P^a_f}(t) = (t - \lambda_1)(t - \lambda_2)(t - \overline{\lambda_1})(t - \overline{\lambda_2}).$$

Assume now by way of contradiction that $0 < \lambda_1 < 1$ and $\lambda_2 \leq 1$.

Consider first the case that $\lambda_2 = 0$. Then $\lambda_1\overline{\lambda_1}$ appears as a coefficient in $P^r_f(t)$, and it must therefore be an integer. So $|\lambda_1| \geq 1$, a contradiction.

Next, suppose $\lambda_2 > 0$. We have

$$\det(\rho_r(f)) = \lambda_1\lambda_2\overline{\lambda_1}\overline{\lambda_2} = |\lambda_1|^2|\lambda_2|^2.$$

and this implies $0 < |\det(\rho_r(f))| < 1$, which again is impossible as this number is an integer. □

The next statement will be crucial for dealing with the case of eigenvalues of absolute value 1.

**Proposition 1.7.** Let $X$ be a two-dimensional complex torus and $f : X \to X$ an endomorphism. If $\lambda$ is an eigenvalue of $f$ with $|\lambda| = 1$, then $\lambda$ is a root of unity.

**Proof.** We may assume that $\lambda \neq \pm 1$, as otherwise there is nothing to prove. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $f$. Suppose to begin with that $|\lambda_1| = |\lambda_2| = 1$. The characteristic polynomial $P^r_f$ of the rational representation of $f$ is then a monic polynomial over the integers, all of whose roots are of absolute value 1. It follows that all of its roots are then roots of unity (see [Coh], Prop. 3.3.9), and we are done in this case.

It remains to consider the case that $|\lambda_1| = 1$ and $|\lambda_2| \neq 1$. By Lemma 1.6, we know that then necessarily $|\lambda_2| > 1$ or $\lambda_2 = 0$. Let $h$ be the minimal polynomial of $\lambda_1$ over $\mathbb{Q}$ (\(\lambda_1\) is a root of $P^r_f$, and hence an algebraic integer.) According to Lemma 1.8 below, $h$ is a symmetric integer polynomial of degree 2 or 4, whose roots appear in reciprocal pairs. So there are two cases:

**Case 1:** $\deg h = 2$. In that case, the second root of $h$ is $\overline{\lambda_1}$. So all roots of $h$ are of absolute value 1. And as above, we conclude that $\lambda_1$ is a root of unity.

**Case 2:** $\deg h = 4$. Then $h$ coincides with $P^r_f$, so its roots are $\lambda_1, \overline{\lambda_1}, \lambda_2, \overline{\lambda_2}$. But because of $|\lambda_2| > |\lambda_1| = 1$, there is no way that the roots can occur in reciprocal pairs – so this case does not happen. □
We very much assume that the following elementary algebraic lemma is well-known. For lack of a reference we include a proof.

**Lemma 1.8.** Let $a \in \mathbb{C}$ be an algebraic integer of absolute value 1 and different from $\pm 1$. Then its minimal polynomial is a polynomial over $\mathbb{Z}$ of even degree with symmetric coefficients, whose roots occur in reciprocal pairs.

**Proof.** By definition there is a monic polynomial $g$ over $\mathbb{Z}$ with $g(a) = 0$. As $g$ is a multiple of the minimal polynomial $h$ of $a$, it follows from Gauß’ Lemma that $h$ is integral as well.

We now prove the symmetry statement. As $a$ is of absolute value 1, we know that $1/a = \overline{a}$ appears as a root of $h$ as well. On the other hand, we have

$$h(1/a) = h(a) = 0,$$

hence $\overline{a}$ is a root of the integral polynomial $t^n h(1/t)$, which is also of degree $n$. Therefore $t^n h(1/t) = c \cdot h(t)$ for some $c \in \mathbb{Q}$. Setting $t = 1$, we get $h(1) = c \cdot h(1)$. Since $a$ is irrational, the degree of $h$ is at least 2, and hence $h(1) \neq 0$. We conclude that $c = 1$, i.e.,

$$t^n h(1/t) = h(t),$$

and this shows that the roots of $h$ occur in reciprocal pairs, and hence that its degree is even.

We can now give the

**Proof of Theorem 1.** Let $f : X \to X$ be an endomorphism of a two-dimensional complex torus, and let $\lambda_1$ and $\lambda_2$ be the eigenvalues of its analytic representation (counted with algebraic multiplicities). After reordering we assume $|\lambda_1| \leq |\lambda_2|$. We have then by Prop. 1.1 for every integer $n \geq 1$,

$$\# \text{Fix}(f^n) = |(1 - \lambda_1^n)(1 - \lambda_2^n)|^2.$$

Suppose first that $|\lambda_1| > 1$. Then by our setup we have $|\lambda_2| > 1$ as well, and hence the fixed-points function $\# \text{Fix}(f^n)$ grows exponentially. So we are in Case (B1) of the theorem.

Suppose next that $0 < |\lambda_1| < 1$. It follows from Prop. 1.6 that then $|\lambda_2| > 1$, and hence the function $\# \text{Fix}(f^n)$ grows exponentially again.

Suppose now that $|\lambda_1| = 1$. Proposition 1.7 tells us that $\lambda_1$ is then a root of unity. If $|\lambda_2| = 1$, then the same is true for $\lambda_2$ and hence $\# \text{Fix}(f^n)$ is a periodic function, so we are in Case (B2) of the theorem. And if $|\lambda_2| > 1$, then the fixed-points function has the form described in Case (B3) of the theorem. In either case, the roots of unity are of algebraic degree $\leq 4$, since they appear as roots of the rational characteristic polynomial $P_f(f)$. They are therefore $k$-th roots of unity, where $k \in \{1, \ldots, 6, 8, 10, 12\}$.

Finally, suppose that $\lambda_1 = 0$. Then we have behaviour (B1) if $|\lambda_2| > 1$ and behaviour (B2) if $|\lambda_2| = 1$.

It remains to show that all three behaviours actually occur. Case (B1) happens for the multiplication map $x \mapsto mx$ on every complex torus, as soon as $|m| \geq 2$. Example 1.2 is an instance of (B2), and Example 1.9 below shows that (B3) occurs. In all three cases there are projective examples. \qed


Example 1.9. (One eigenvalue of absolute value > 1, the other a root of unity.) Consider the elliptic curve $E$ with complex multiplication in $\mathbb{Z}[i]$, and take the abelian surface $X = E \times E$. The endomorphism

$$X \rightarrow X, \quad (x, y) \mapsto (ix, 2iy)$$

has the eigenvalues $i$ and $2i$, and hence the fixed-points function has the behaviour described in Case (B3) of Theorem 1:

$$\# \text{Fix}(f^n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{4} \\ h(x) & \text{otherwise} \end{cases}$$

where the function $h$ grows exponentially, $h(n) \sim 2^{2n}$.

2. Fixed points on simple abelian surfaces

In this section we will explicitly determine the endomorphisms on simple abelian surfaces whose fixed-points function grows exponentially, thus proving Theorem 2 stated in the introduction.

We will make use of the fixed-point formula for abelian varieties [CAV, 13.1.4], which for an endomorphism $f$ of a simple abelian surface $X$ takes the following form: Let $D = \text{End}_\mathbb{Q}(X)$, $K = \text{center}(D)$, $e = [K : \mathbb{Q}]$, $d^2 = [D : K]$, and let $N : D \rightarrow \mathbb{Q}$ be the reduced norm map. Then for $f \in \text{End}(X)$,

$$\# \text{Fix}(f) = \left( N(1 - f) \right)^{\frac{1}{de}}.$$  

(The formula expresses the fact that the characteristic polynomial of the rational representation $\rho_r(f)$ coincides with the map $N^{4/de}$.) Since on a simple abelian surface every non-zero endomorphism $f$ is an isogeny, both eigenvalues of $f$ are non-zero.

We employ now a strategy as in [BL], i.e., we use Albert’s classification and deal with the possible types of simple abelian surfaces separately.

**Type 0: Integer multiplication.** Suppose that $\text{End}(X) = \mathbb{Z}$. In that case every endomorphism $f$ is a multiplication map $x \mapsto mx$ for some $m \in \mathbb{Z}$. So $f$ has exponential fixed-points growth if and only if $|m| > 1$.

**Type 1: Real multiplication.** Suppose that $X$ has real multiplication, i.e., that $\text{End}_\mathbb{Q}(X) = \mathbb{Q}(\sqrt{d})$ for some square-free integer $d > 0$. Every endomorphism $f \in \text{End}(X)$ is then of the form $f = a + b\omega$ with $a, b \in \mathbb{Z}$, where

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2, 3 \pmod{4} \\ \frac{1}{2}(1 + \sqrt{d}) & \text{if } d \equiv 1 \pmod{4} \end{cases}$$

With respect to suitable coordinates on $\mathbb{C}^2$, the analytic representation $\rho_a : \text{End}_\mathbb{Q}(X) = \mathbb{Q}(\sqrt{d}) \rightarrow M_2(\mathbb{C})$ is given by

$$1 \mapsto I_2 \quad \text{and} \quad \sqrt{d} \mapsto \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix}$$

(see [Rup]). So the eigenvalues of $\rho_a(f) = \rho_a(a + b\omega)$ are
• $a \pm b\sqrt{d}$, if $d \equiv 2, 3 \pmod{4}$,
• $a + b\frac{1}{2}(1 \pm \sqrt{d})$, if $d \equiv 1 \pmod{4}$

If $b = 0$, then $f$ is multiplication by $a$, and hence it has exponential fixed-points growth if and only if $|a| > 1$. And if $b \neq 0$, then both eigenvalues are of absolute value $\neq 1$. Using Prop. 1.6 we see that then $f$ has exponential fixed-points growth.

**Type 2: Indefinite quaternion multiplication.** Suppose that $X$ has indefinite quaternion multiplication, i.e., there are $\alpha, \beta \in \mathbb{Q} \setminus \{0\}$ with $\alpha \geq \beta$ and $\alpha > 0$ such that $\text{End}_\mathbb{Q}(X)$ is isomorphic to the quaternion algebra $(\mathbb{Q}(\sqrt{\alpha}), \mathbb{Q}(\sqrt{\beta}))$, i.e., $\text{End}_\mathbb{Q}(X) = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + ij\mathbb{Q}$, where $i$ and $j$ satisfy the relations $i^2 = \alpha$, $j^2 = \beta$ and $ij = -ji$. Using the splitting field $\mathbb{Q}(\sqrt{\alpha})$ of $\text{End}_\mathbb{Q}(X)$, one has an isomorphism

$$\psi : \text{End}_\mathbb{Q}(X) \otimes \mathbb{Q}(\sqrt{\alpha}) \to M_2(\mathbb{Q}(\sqrt{\alpha}))$$

given by

$$i \otimes 1 \mapsto \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix} \quad \text{and} \quad j \otimes 1 \mapsto \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}.$$

For an element $f \in \text{End}(X)$, written as $f = a + bi + cj + dij$ with $a, b, c, d \in \mathbb{Q}$, we have

$$\psi(f) = \begin{pmatrix} a + b\sqrt{\alpha} & c\beta + d\beta\sqrt{\alpha} \\ c - d\sqrt{\alpha} & a - b\sqrt{\alpha} \end{pmatrix}.$$

The reduced norm of $f$ over $\mathbb{Q}$ is given by $N(f) = \det(\psi(f)) = a^2 - b^2\alpha - c^2\beta + d^2\alpha\beta$.

After diagonalization of $\psi(f)$ the norm of $1 - f^n$ can be written as

$$N(1 - f^n) = \det \begin{pmatrix} 1 - (a + \sqrt{b^2\alpha + c^2\beta - d^2\alpha\beta})^n & 0 \\ 0 & 1 - (a - \sqrt{b^2\alpha + c^2\beta - d^2\alpha\beta})^n \end{pmatrix} = (1 - t_1^n)(1 - t_2^n),$$

where $t_i = a \pm \sqrt{b^2\alpha + c^2\beta - d^2\alpha\beta}$. So we have $\# \text{Fix}(f^n) = ((1 - t_1^n)(1 - t_2^n))^2$.

Consider now the reduced characteristic polynomial of $f$,

$$\chi_f(t) = t^2 - \text{tr}(f)t + N(f) = t^2 - \text{tr}(\psi(f))t + \det(\psi(f)).$$

Since $f$ is contained in an order, it is an integral element, and hence $\chi_f$ has integer coefficients. For every integer $m$ we have

$$\chi_f(m)^2 = (m - t_1)(m - t_2)^2 = N(m - f)^2 = \det(\mathbb{I}_4 - \rho_r(f - (m - 1))) = \det(\mathbb{I}_4 - \rho_r(f) + (m - 1)\mathbb{I}_4) = \det(m\mathbb{I}_4 - \rho_r(f)) = P_f^r(m)$$

and this implies that $\chi_f(t)^2 = P_f^r(t)$ as polynomials in $t$. Therefore, if we denote by $\lambda_1$ and $\lambda_2$ the analytic eigenvalues of $f$, then $t_1 \in \{\lambda_1, \overline{\lambda}_1\}$ and $t_2 \in \{\lambda_2, \overline{\lambda}_2\}$.

• If $|t_1| > 1$ and $|t_2| > 1$, then $\# \text{Fix}(f^n)$ grows asymptotically with $|t_1t_2|^{2n}$. 

If $|t_1| < 1$, then Prop. L.4 tells us that $|\lambda_2| > 1$. Therefore the number of fixed-points grows exponentially in this case.

If $|t_1| = |t_2| = 1$, then the possible real values for $t_1$ and $t_2$ are $\pm 1$. If $t_1$ and $t_2$ are complex, then they are roots of unity, because they are roots of the integral polynomial $\chi_f$. The function $n \mapsto \# \text{Fix}(f^n)$ is then periodic.

We show now that the case $|t_1| > 1$ and $|t_2| = 1$ does not occur. Assume the contrary. Since $t_1$ and $t_2$ are the roots of a rational polynomial of degree 2, we know that $t_2$ has to be $\pm 1$. As $t_1 \neq t_2$, at least one of the coefficients $b, c, d$ is non-zero (since otherwise $a = t_1 = t_2$). But then $t_2 = \pm 1$ implies that $f \neq 1$ is a non-zero endomorphism whose norm is zero, and hence the quaternion algebra cannot be a division algebra, a contradiction.

Type 3: Complex multiplication. Suppose that $X$ has complex multiplication, i.e., that $\text{End}_Q(X)$ is isomorphic to an imaginary quadratic extension $K$ of a real quadratic number field $Q(\sqrt{d})$, where $d$ is a positive square-free integer.

Consider an endomorphism $f \in \text{End}(X)$ and let $\lambda_1$ and $\lambda_2$ be the eigenvalues of its analytic representation $\rho_\alpha(f)$. By the Cayley-Hamilton theorem, $f$ is annihilated by its rational characteristic polynomial $P_f$. As the isomorphism $\sigma : \text{End}_Q(X) \to K$ fixes $Q$, the complex number $\tilde{f} = \sigma(f)$ is then a zero of $P_f$ as well. This implies that it is contained in the set $\{\lambda_1, \lambda_2, \overline{\lambda}_1, \overline{\lambda}_2\}$. So, after renumbering, we have $\tilde{f} = \lambda_1$ or $\tilde{f} = \overline{\lambda}_1$. We distinguish now cases according to $\tilde{f}$:

- If $|\tilde{f}| < 1$, then we know by Prop. L.6 that $|\lambda_2| > 1$. Therefore in this case the number of fixed points grows exponentially, which is behaviour (B1).

- If $|\tilde{f}| = 1$, then by Prop. L.7 we know that $\lambda_1$, and hence $\tilde{f}$, is a root of unity. The fixed-points function $n \mapsto \# \text{Fix}(f^n) = N(1 - f^n)$ is then periodic (B2).

- Let $|\tilde{f}| > 1$ and $\tilde{f} \in \mathbb{R}$. Then $\tilde{f}$ is of the form $a + b\sqrt{d}$, hence we have behaviour (B1) as in the case of real multiplication.

- Let $|\tilde{f}| > 1$ and $\tilde{f} \notin \mathbb{R}$. So $|\lambda_1| > 1$. Our aim is to show that then $|\lambda_2| \neq 1$, which implies behaviour (B1). Assuming by way of contradiction that $|\lambda_2| = 1$, note first that the proof of Prop. L.7 shows that the minimal polynomial of $\lambda_1$ must be of degree 2. Therefore $\lambda_1$ can be written as $a + b\sqrt{-c}$ with $a, b \in \mathbb{Q}$ and $c$ a square-free positive integer. For $t \in \mathbb{Z}$ we compute the norm of $t - \tilde{f}$,

$$N_{K/Q}(t - \tilde{f}) = N_{Q(\sqrt{-c})/Q}(N_{K/Q(\sqrt{-c})}(t - \tilde{f}))$$

$$= N_{Q(\sqrt{-c})/Q}((t - \tilde{f})^2)$$

$$= ((t - a)^2 + b^2c)^2.$$

On the other hand, $N_{K/Q}(t - \tilde{f})$ coincides as a polynomial in $t$ with $P_f(t)$, and therefore $\lambda_2$ equals $\tilde{f}$ or its conjugate. It cannot be of absolute value 1 then, and so we arrive at a contradiction.

We provide a complete list of the possible eigenvalues that occur for endomorphisms with periodic fixed-points behaviour in quaternion and complex multiplication.
Proposition 2.1.  a) If $X$ is a simple abelian surface with quaternion multiplication and $f$ a non-zero endomorphism on $X$ with periodic fixed-points behaviour, then all eigenvalues of $f$ are roots of unity of algebraic degree $\leq 2$, i.e., they are contained in the following set:

| Degree | Eigenvalues |
|--------|-------------|
| 1      | $\pm 1$    |
| 2      | $\pm i$, $\pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}$ |

Conversely, each of these numbers occurs as an eigenvalue of an endomorphism on some simple abelian surface with quaternion multiplication.

b) If $X$ is a simple abelian surface with complex multiplication and $f$ a non-zero endomorphism on $X$ with periodic fixed-points behaviour, then all eigenvalues of $f$ are roots of unity of algebraic degree $\leq 4$, i.e., they are contained in the following set:

| Degree | Eigenvalues |
|--------|-------------|
| 1      | $\pm 1$    |
| 2      | $\pm i$, $\pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}$ |
| 3      | $\pm \frac{1}{2} \pm \frac{\sqrt{5}}{2}, \pm \frac{\sqrt{2}}{2} \pm \frac{\sqrt{4}}{2}$ |
| 4      | $\pm (\frac{1}{2} + \frac{\sqrt{5}}{2}) \pm i \sqrt{\frac{3}{8} - \frac{\sqrt{5}}{8}}, \pm (\frac{1}{2} - \frac{\sqrt{5}}{2}) \pm i \sqrt{\frac{3}{8} + \frac{\sqrt{5}}{8}}$ |

Conversely, each of these numbers occurs as an eigenvalue of an endomorphism on some simple abelian surface with complex multiplication.

Proof. One direction is clear by now: If $f$ has periodic fixed-points behaviour, then we know that all eigenvalues are roots of unity. In the quaternion case they are roots of $f^2$ and therefore of degree $\leq 2$, and in the complex multiplication case they are roots of $P_f^2$ and therefore of degree $\leq 4$.

As for the converse statement in (a): The assertion being obvious for $\pm 1$, we now exhibit quaternion algebras $B_1$ and $B_2$ that are skew-fields, and orders $\mathcal{O}_1 \subset B_1$ and $\mathcal{O}_2 \subset B_2$ containing elements that lead to the required eigenvalues $\pm i$ and $\pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}$ respectively. By Shimura’s theory (cf. [CAV, §9.4]), each of the orders $\mathcal{O}_k$ is contained in the endomorphism ring of some simple abelian surface.

To this end, consider first the quaternion algebra $B_1 = (\frac{3+i}{3})$. It is a skew field and contains the order $\mathcal{O}_1 = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij$. The element $f_1 = i + j + ij$ has reduced characteristic polynomial $\chi_{f_1}(t) = t^2 + 1$, whose roots are $\pm \sqrt{-1}$.

Secondly, consider $B_2 = (-\frac{3,2}{3})$ and the splitting field $L = \mathbb{Q}(i)$, where $i^2 = -3$. The maximal order in $L$ is $S = \mathbb{Z} + \frac{1+i}{2} \mathbb{Z}$, and it follows from this that $S + jS$, with $j^2 = 2$, is an order in $B_2$. The element $f_2 = \frac{1+i}{2} \in S$ has reduced characteristic polynomial $\chi_{f_2}(t) = t^2 - t + 1$, whose roots are $\frac{1}{2} \pm \frac{1}{2} \sqrt{-3}$. And the element $f_3 = \frac{-1+i}{2} \in S$ has reduced characteristic polynomial $\chi_{f_3}(t) = t^2 + t + 1$, whose roots are $\frac{1}{2} \pm i \sqrt{-3}$.

As for the converse statement in (b): Every root of unity $\zeta$ of algebraic degree 4 is an algebraic integer in the CM field $\mathbb{Q}(\zeta)$, and therefore there exists a simple abelian surface $X$ of CM-type, where the root represents an endomorphism $f$ (cf. [CAV, §9.6]). For the roots of unity of degree 2 just consider any real quadratic number field $L$ and take the CM field $L(\zeta)$. $\square$
References

[Bir] Birkenhake, Ch.: Tensor products of ample line bundles on abelian varieties. Manuscripta math. 84 (1994), 21-28

[BL] Birkenhake, C., Lange, H.: Fixed-point free automorphisms of abelian varieties. Geom. Dedicata 51, No.3, 201–213 (1994)

[CAV] Birkenhake, C., Lange, H.: Complex abelian varieties. Springer, 2004.

[Coh] Cohen, H.: Number Theory. Volume I: Tools and Diophantine Equations. Springer, 2007.

[DKL] De Fernex, T., Küronya, A., Lazarsfeld, R.: Higher cohomology of divisors on a projective variety. Math. Ann. 337, No. 2, 443-455 (2007)

[EEL] Ein, L., Erman, D., Lazarsfeld, R.: Asymptotics of random Betti tables. J. Reine Angew. Math. 702, 55-75 (2015)

[EL] Ein, L., Lazarsfeld, R.: Asymptotic syzygies of algebraic varieties. Invent. Math. 190, No. 3, 603-646 (2012)

[ELMNP] Ein, L., Lazarsfeld, R., Mustaţă, M., Nakamaye, M., Popa, M.: Asymptotic invariants of base loci. Ann. Inst. Fourier 56, No. 6, 1701-1734 (2006)

[LB] Lange, H., Birkenhake, Ch.: Complex Abelian Varieties. Grundlehren der math. Wiss. 302, Springer-Verlag, 1992.

[McM] McMullen, C.: Dynamics on K3 surfaces: Salem numbers and Siegel disks. J. reine angew. Math 545, 201–233 (2001)

[Res] Reschke, P.: Salem Numbers and Automorphisms of Complex Surfaces. Math. Res. Lett. 19(2), 475–482 (2012)

[Rin] Ringler, A.: Fixed points of smooth varieties with Kodaira dimension zero. [arXiv:0708.3587] [math.AG]

[Rup] Ruppert, W.: Two-dimensional complex tori with multiplication by $\sqrt{d}$. Arch. Math. 72, 278–281 (1999)

[SS] Shub, M., Sullivan, D.: A remark on the Lefschetz fixed point formula for differentiable maps. Topology 13, 189–191 (1974)

Thomas Bauer, Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, D-35032 Marburg, Germany.

E-mail address: tbauer@mathematik.uni-marburg.de

Thorsten Herrig, Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, D-35032 Marburg, Germany.

E-mail address: herrig106@mathematik.uni-marburg.de