Affine cellularity of BLN-algebras
With an appendix by Hiraku Nakajima

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Abstract

We show that the BLN-algebra, which was introduced by McGerty, is affine cellular in the sense of Koenig and Xi. In fact, we establish a stronger property, namely that the affine cell ideals are generated by idempotents. This particularly implies that the global dimension of the BLN-algebra is finite. For affine type $A$, thus we obtain that the affine $q$-Schur algebra $U_{D,n,n}$, when $D < n$, is affine cellular and has finite global dimension.

Keywords: Affine cellular algebras; Modified quantum affine algebras; Extremal weight modules; BLN-algebras; Affine $q$-Schur algebras

1 Introduction

Graham and Lehrer [GL] introduced the notion of cellularity to give a unified approach to study the classification of the irreducible representations of a finite dimensional algebra. Hecke algebras of finite type have been shown to be cellular as well as various finite dimensional diagram algebras, such as Temperley-Lieb algebras, Brauer algebras, Birman-Murakami-Wenzl algebras and their cyclotomic analogs.

Let $U$ be a quantized enveloping algebra and $\hat{U}$ be its modified form defined by Lusztig. For any quantum algebra $U$ of finite type, Lusztig [L1] defined a finite dimensional algebra $\hat{U}/\hat{U}[P]$ associated to some saturated set $P$, which was proved to be isomorphic to a ‘generalized $q$-Schur algebra’ in [Do]. Furthermore, Doty [Do] showed that generalized $q$-Schur algebras are cellular algebras in the sense of Graham and Lehrer and their canonical basis, inherited from $\hat{U}$, is a cellular basis.

Koenig and Xi [KX] recently defined the notion of affine cellularity to generalize the notion of cellular algebras to algebras that need not be finite dimensional over a noetherian domain $k$. Extended affine Hecke algebras of type $A$ and affine Temperley-Lieb algebras were proved to be affine cellular in [KX]. Further examples of affine cellular algebras include affine Hecke algebras of rank two with generic parameters ([GM]), KLR algebras of finite type ([KL]) and affine quantum Schur algebras ([C]).

The cells attached to the Kazhdan-Lusztig basis of Hecke algebras associated to Coxeter groups are very useful in understanding the structures and representations of Coxeter groups and their Hecke algebras. The theory of cells attached to the canonical basis of the modified quantum algebra $\hat{U}$, which has been developed by Lusztig, is also
very useful in studying the algebra itself. Lusztig [L2, Sect. 4] described completely the structure of cells in $\tilde{U}$ in the case that $U$ is of finite type. Moreover, Lusztig [L2, Sect. 5] also gave a series of conjectures on the cell structure in the case of the level-zero modified quantum affine algebra $\tilde{\mathfrak{U}}$. These conjectures have been proved by Beck and Nakajima in [BN] (see also [Mc1] for type $A_n^{(1)}$).

Specifically speaking, Beck and Nakajima [BN] studied the canonical basis or global crystal basis $B$ of the level-zero modified quantum affine algebra $\tilde{\mathfrak{U}}$ defined by Lusztig in [L1] or Kashiwara in [Kas1], and obtained an explicit description of the structure of two-sided cells of $\tilde{\mathfrak{U}}$ and the asymptotic algebra of $\tilde{\mathfrak{U}}$ at $q = 0$.

Based on the work of Beck and Nakajima on cell structure in level-zero modified quantum affine algebras, McGerty [Mc2] introduced the definition of a BLN-algebra (for Beck, Lusztig and Nakajima) $\tilde{\mathfrak{U}}/\tilde{\mathfrak{U}}[P]$ associated to a level-zero modified quantum affine algebra $\tilde{\mathfrak{U}}$ and a saturated set $P$. BLN-algebras can be considered as affine analogs of generalized $q$-Schur algebras.

In this paper, we will show that BLN-algebras are affine cellular algebras in the sense of Koenig and Xi. In fact, we establish a stronger property, namely that each affine cell ideal is generated by an idempotent. Applying [KX, Theorem 4.4], we then obtain that the BLN-algebra has finite global dimension and its derived category admits a stratification whose strata are the derived categories of the representation ring of products of general linear groups. For affine type $A$, thus we obtain that the affine $q$-Schur algebras $\mathcal{U}_{D,n,n}$, when $D < n$, are affine cellular algebras and also have the favorable homological properties stated as above. Our proof relies heavily on Beck and Nakajima’s explicit description of two-sided cells and the asymptotic algebra.

This paper is organized as follows. In Section 2, we recall Koenig and Xi’s results on affine cellular algebras. In Section 3, we review many concepts and results, such as extremal weight modules, level-zero modified quantum affine algebras, BLN-algebras and Beck and Nakajima’s explicit description of the cell structures and the asymptotic algebra. In Section 4, we prove our main results, Theorem 4.4, 4.6 and 4.7.

2 Affine cellular algebras

In this section, we recall Koenig and Xi’s ([KX]) definition and results of affine cellular algebras. Throughout, we assume that $k$ is a noetherian domain.

For two $k$-modules $V$ and $W$, let $\tau$ be the switch map: $V \otimes W \to W \otimes V$ defined by $\tau(v \otimes w) = w \otimes v$ for $v \in V$ and $w \in W$. A $k$-linear anti-automorphism $i$ of $A$ which satisfies $i^2 = id_A$ will be called a $k$-involution on $A$. A commutative $k$-algebra $B$ is called an affine $k$-algebra if it is a quotient of a polynomial ring $k[x_1, \ldots, x_r]$ in finitely many variables $x_1, \ldots, x_r$ by some ideal $I$.

**Definition 2.1.** (See [KX, Definition 2.1].) Let $A$ be a unitary $k$-algebra with a $k$-involution $i$. A two-sided ideal $J$ in $A$ is called an affine cell ideal if and only if the following data are given and the following conditions are satisfied:

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(1) \( i(J) = J. \)

(2) There exist a free \( k \)-module \( V \) of finite rank and an affine \( k \)-algebra \( B \) with a \( k \)-involution \( \sigma \) such that \( \Delta := V \otimes_k B \) is an \( A-B \)-bimodule, where the right \( B \)-module structure is induced by the right regular \( B \)-module \( B_B \).

(3) There is an \( A-A \)-bimodule isomorphism \( \alpha : J \rightarrow \Delta \otimes_B \Delta' \), where \( \Delta' := B \otimes_k V \) is a \( B-A \)-bimodule with the left \( B \)-module induced by the left regular \( B \)-module \( B_B \) and with the right \( A \)-module structure defined by \( (b \otimes v) \tau := \tau(i(a)(v \otimes b)) \) for \( a \in A \), \( b \in B \) and \( v \in V \), such that the following diagram is commutative:

\[
\begin{array}{ccc}
J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \\
i & \downarrow & \\
J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta'.
\end{array}
\]

The algebra \( A \) together with the \( k \)-involution \( i \) is called affine cellular if and only if there is a \( k \)-module decomposition \( A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n \) (for some \( n \)) with \( i(J'_k) = J'_k \) for \( 1 \leq k \leq n \), such that, setting \( J_l := \bigoplus_{k=1}^l J'_k \), we have a chain of two-sided ideals of \( A \):

\[
0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A
\]

so that each \( J'_l = J_l/J_{l-1} \) \((1 \leq l \leq n)\) is an affine cell ideal of \( A/J_{l-1} \) (with respect to the involution induced by \( i \) on the quotient).

Given a free \( k \)-module \( V \) of finite rank, an affine \( k \)-algebra \( B \) and a \( k \)-bilinear form \( \rho : V \otimes_k V \rightarrow B \), we define an associative algebra \( A(V, B, \rho) \) as follows: \( A(V, B, \rho) := V \otimes_k B \otimes_k V \) as a \( k \)-module, and the multiplication on \( A(V, B, \rho) \) is defined by

\[
(u_1 \otimes_k b_1 \otimes_k v_1)(u_2 \otimes_k b_2 \otimes_k v_2) := u_1 \otimes_k b_1 \rho(v_1, u_2) b_2 \otimes_k v_2
\]

for all \( u_1, u_2, v_1, v_2 \in V \) and \( b_1, b_2 \in B \). Moreover, if \( B \) admits a \( k \)-involution \( \sigma \) satisfying \( \sigma \rho(v_1, v_2) = \rho(v_2, v_1) \), then \( A(V, B, \rho) \) admits a \( k \)-involution \( \sigma \) which sends \( u \otimes b \otimes v \) to \( v \otimes \sigma(b) \otimes u \) for all \( u, v \in V \) and \( b \in B \).

An equivalent description of this construction is as follows: given \( V, B, \rho \) as above, we define the generalized matrix algebra \( (M_n(B), \rho) \) over \( B \) with respect to \( \rho \) as follows: it equals the ordinary matrix algebra \( M_n(B) \) of \( n \times n \) matrices over \( B \) as a \( k \)-space, but the multiplication is deformed as follows:

\[
\widetilde{x} \cdot \widetilde{y} = x \Psi y
\]

for all \( x, y \in M_n(B) \), where \( \widetilde{x} \) and \( \widetilde{y} \) are elements of \( (M_n(B), \rho) \) corresponding to \( x \) and \( y \), respectively, and \( \Psi \) is the matrix describing the bilinear form \( \rho \) with respect to some basis of \( V \). Moreover, if \( B \) admits a \( k \)-involution \( \sigma \) satisfying \( \sigma \rho(v_1, v_2) = \rho(v_2, v_1) \), then \( (M_n(B), \rho) \) admits a \( k \)-involution \( \kappa \) which sends \( E_{jl}(b) \) to \( E_{lj}(\sigma(b)) \), where \( E_{jl}(b) \) denotes a square matrix whose \((j, l)\)-entry is \( b \in B \) and all the other entries are zero.

From the above discussion, we can easily get the following proposition about the description of affine cell ideals, which we will use in Section 4.
Proposition 2.2. (See [KX, Proposition 2.2].) Let $k$ be a noetherian domain, $A$ a unitary $k$-algebra with a $k$-involution $i$. A two-sided ideal $J$ in $A$ is an affine cell ideal if and only if $i(J) = J$, $J$ is isomorphic to some generalized matrix algebra $(M_n(B), \rho)$ for some affine $k$-algebra $B$ with a $k$-involution $\sigma$, a free $k$-module $V$ of finite rank and a $k$-bilinear form $\rho: V \otimes_k V \to B$. Under this isomorphism, if a basis element $a$ of $J$ corresponds to $E_{jl}(b')$ for some $b' \in B$, then $i(a)$ corresponds to $E_{lj}(\sigma(b'))$.

The following theorem plays an important role in investigating homological properties of affine cellular algebras.

Theorem 2.3. (See [KX, Theorem 4.4].) Let $A$ be an affine cellular algebra with a cell chain $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ such that $J_l/J_{l-1} = V_l \otimes_k B_l \otimes_k V_l$ as in Definition 2.1. Suppose that each $B_l$ satisfies $\text{rad}(B_l) = 0$. Moreover, suppose that each $J_l/J_{l-1}$ is idempotent and contains a non-zero idempotent element in $A/J_{l-1}$. Then:

1. The parameter set of simple $A$-modules equals the parameter set of simple modules of the asymptotic algebra, that is, a finite union of affine spaces (one for each $B_l$).
2. The unbounded derived category $D(A\text{-Mod})$ of $A$ admits a stratification, that is, an iterated recollement whose strata are the derived categories of the various affine $k$-algebras $B_l$.
3. The global dimension $\text{gldim}(A)$ is finite if and only if $\text{gldim}(B_l)$ is finite for all $l$.

Remark 2.4. Koenig [Koe, Page 531] called an affine cellular algebra with the assumptions stated as in Theorem 2.3 an affine quasi-hereditary algebra, since it implies the crucial homological properties analogous to known results about quasi-hereditary algebras and highest weight categories; see also [Kle] for the graded version of affine quasi-heredity.

3 BLN-algebras

In this section, we will recall the definition of BLN-algebras, and also Beck and Nakajima’s ([BN]) results on the structure of cells and the asymptotic algebra of the level-zero modified quantum affine algebra $\tilde{\mathbf{U}}$.

3.1 Preliminaries

In this subsection, we briefly review some necessary notions, such as quantum affine algebras, modified quantum affine algebras, global crystal bases and extremal weight modules. We refer to [Kac], [Kas1], [Kas2] and [L1] for more details.

Let us first introduce some notations.

1. $\hat{g}$: an affine Kac-Moody Lie algebra and $A = (a_{ij})$: the associated generalized Cartan matrix of affine type.
(2) $I$: a finite index set, which is identified with the set $\{0, 1, \ldots, n\}$.
(3) $\hat{t}$: the Cartan subalgebra of $\widehat{\mathfrak{g}}$ and $\hat{t}^*$: the dual of $\hat{t}$.
(4) $\{h_i\}_{i \in I} \subset \hat{t}$: the set of simple coroots and $\{\alpha_i\}_{i \in I} \subset \hat{t}^*$: the set of simple roots of $\widehat{\mathfrak{g}}$, respectively, with $\langle h_i, \alpha_j \rangle = \delta_{ij}$.
(5) $\Lambda_i$: the fundamental weights defined by $\langle h_i, \Lambda_j \rangle = \delta_{ij}$ for $i, j \in I$.
(6) $P$: the weight lattice such that $\Lambda_i, \Lambda_i \in P$ and $P^\vee = \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$: the dual weight lattice such that $h_i \in P^\vee$.
(7) $Q = \sum_{i \in I} \mathbb{Z}\alpha_i$: the root lattice and $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}h_i \subset P^\vee$: the coroot lattice.
(8) $\delta \in Q$: the unique element satisfying $\{\lambda \in Q; \langle h_i, \lambda \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta$.
(9) $c \in Q^\vee$: the unique element satisfying $\{h \in Q^\vee; \langle h, \alpha_i \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}c$.
(10) $W$: the affine Weyl group generated by the simple reflections $s_i \ (i \in I)$; $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$ for any $\lambda \in \hat{t}^*$.
(11) $(\cdot, \cdot)$: a $W$-invariant non-degenerate symmetric bilinear form on $\hat{t}^*$ satisfying $\langle h_i, \lambda \rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $i \in I$, $\lambda \in \hat{t}^*$, and $(\delta, \lambda) = \langle c, \lambda \rangle$ for any $\lambda \in \hat{t}^*$.
(12) $P^0$: the set of level-zero weights, that is, $P^0 := \{\lambda \in P; (\alpha_i, \lambda) = 0\}$.
(13) $P^0_{cl} := \text{cl}(P^0)$, where $\text{cl}: \hat{t}^* \to \hat{t}^*/\mathbb{Q}\delta$ is the canonical projection.
(14) $I_0 := I \setminus \{0\} = \{1, 2, \ldots, n\}$, where $0 \in I$ is chosen as the leftmost vertex in [Kac, pages 54, 55] except that we take $\alpha_0$ the longest simple root in $A_{2n}^{(2)}$ case.
(15) $\varpi_i := \Lambda_i - \langle c, \alpha_i \rangle \Lambda_0 \in P^0$ for $1 \leq i \leq n$ when $\widehat{\mathfrak{g}} \neq A_{2n}^{(2)}$.
(16) $\varpi_n := 2\Lambda_n - \Lambda_0 \in P^0$ and $\varpi_i := \Lambda_i - \Lambda_0 \in P^0$ for $1 \leq i \leq n-1$ when $\widehat{\mathfrak{g}} = A_{2n}^{(2)}$.
(17) $P_+ := (\sum_{i \in I} \mathbb{N}I_i) \oplus \mathbb{Z}\delta$, $P^0_+ := \sum_{i \in I_0} \mathbb{N}\varpi_i$ and $P^0_{cl,+} := \sum_{i \in I_0} \mathbb{N}\text{cl}(\varpi_i)$.

Let $d$ be the smallest positive integer such that $(\alpha_i, \alpha_i)/2 = \frac{1}{d} \mathbb{Z}$ for any $i \in I$. Let $q$ be an indeterminate, and let $q_s = q^{1/d}$. Set $K := \mathbb{Q}(q_s)$.

**Definition 3.1.** The quantum affine algebra $U = U_q(\widehat{\mathfrak{g}})$ associated to $\widehat{\mathfrak{g}}$ is a unital associative $K$-algebra generated by the symbols $e_i, f_i \ (i \in I)$ and $q^h \ (h \in \frac{1}{d}P^\vee)$ with the following relations:

\[
q^0 = 1, \quad q^h q^{h'} = q^{h+h'}, \\
q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i, \\
\sum_{k=0}^{\frac{1}{1-a_{ij}}-1} (-1)^k e_i^{(k)} e_j q^{(1-a_{ij})-k} = \sum_{k=0}^{\frac{1}{1-a_{ij}}-1} (-1)^k f_i^{(k)} f_j q^{(1-a_{ij})-k} = 0 \quad \text{for } i \neq j,
\]

where $q_i = q^{(\alpha_i, \alpha_i)/2}, t_i = q^{(\alpha_i, \alpha_i) h_i/2}$, and

\[
[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]_q^! = \prod_{r=1}^{k} [r]_q, \quad e_i^{(k)} = \frac{e_i^k}{[k]_q^!}, \quad f_i^{(k)} = \frac{f_i^k}{[k]_q^!} \quad \text{for } i \in I \text{ and } k \in \mathbb{Z}_{\geq 0}.
\]

Let $A = \mathbb{Z}[q_s, q_s^{-1}]$. We denote by $A U$ the $A$-subalgebra of $U$ generated by the elements $e_i^{(k)}, f_i^{(k)}$, and $q^h$ for $i \in I$, $k \in \mathbb{Z}_{\geq 0}$, and $h \in \frac{1}{d}P^\vee$, and we set $U'$ to be the subalgebra of $U$ generated by the elements $e_i, f_i$, and $q^h$ for $i \in I$ and $h \in \frac{1}{d}Q^\vee$. 

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There exist unique \( \mathbb{Q}(q_s) \)-algebra anti-automorphisms \(*\), \(#\), and \(\tau\) of \(U\) satisfying
\[
    e_i^* = e_i, \quad f_i^* = f_i, \quad (q^h)^* = q^{-h},
\]
\[
    e_i^# = f_i, \quad f_i^# = e_i, \quad (q^h)^# = q^h.
\]
\[
    \tau(e_i) = q_i^{-1}t_i^{-1}f_i, \quad \tau(f_i) = q_i^{-1}t_ie_i, \quad \tau(q^h) = q^h.
\]

There exists a unique \(\mathbb{Q}\)-algebra automorphism \(-\) of \(U\) satisfying
\[
    \overline{e_i} = q_i s^{-1}, \quad \overline{f_i} = e_i, \quad \overline{q^h} = q^{-h}.
\]

Given any \(\zeta \in \mathbb{Q}\) and an element \(x \in U\), we say that \(x \in U_\zeta\) if \(x\) satisfies \(q^h x q^{-h} = q^{(h,\zeta)} x\) for any \(h \in P^\vee\). We define the modified quantum affine algebra \(\hat{U}\) by
\[
    \hat{U} = \bigoplus_{\lambda \in P} I_\lambda, \quad \text{where}\ I_\lambda = U / \sum_{h \in P^\vee} U(q^h - q^{(h,\lambda)}).
\]

Let \(a_\lambda\) be the image of 1 under the natural projection \(p : U \to I_\lambda\). Then the multiplication on \(\hat{U}\) is defined by
\[
a_{\lambda x} = xa_{\lambda - \zeta} \quad \text{for} \ \zeta \in \mathbb{Q} \text{ and } x \in U_\zeta, \quad a_\lambda a_\mu = \delta_{\lambda \mu} a_\lambda.
\]

Let \(A\hat{U}\) be the \(A\)-subalgebra of \(\hat{U}\) generated by \(e_i^{(n)}a_\lambda, f_i^{(n)}a_\lambda\) for \(i \in I, n \in \mathbb{Z}_{\geq 0}\), and \(\lambda \in P\). Then \(A\hat{U}\) is an integral form of \(\hat{U}\).

Next we briefly recall some background on crystal bases.

Let \(M\) be a \(U\)-module. Given any \(\lambda \in P\) and an element \(u \in M\), we say that \(u \in M_\lambda\) if \(u\) satisfies \(q^h u = q^{(h,\lambda)} u\) for any \(h \in P^\vee\). We say that \(M\) is integrable if \(M\) satisfies the following conditions:

1. \(M\) has a weight space decomposition \(M = \bigoplus_{\lambda \in P} M_\lambda\).
2. For any \(i, e_i\) and \(f_i\) are locally nilpotent on \(M\).

Let \(A\) (resp. \(\overline{A}\)) be the subring of \(\mathbb{Q}(q_s)\) consisting of rational functions regular at \(q_s = 0\) (resp. \(q_s = \infty\)). Let \(\hat{e}_i\) and \(\hat{f}_i\) (\(i \in I\)) be the Kashiwara operators.

**Definition 3.2.** Let \(M\) be an integrable \(U\)-module. A pair \((\mathcal{L}, \mathcal{B})\) is called a crystal basis of \(M\) if it satisfies the following conditions:

1. \(\mathcal{L}\) is a free \(A\)-submodule of \(M\) such that \(M \cong \mathbb{Q}(q_s) \otimes_A \mathcal{L}\);
2. \(\mathcal{B}\) is a \(\mathbb{Q}\)-basis of \(\mathcal{L}/q_s\mathcal{L}\);
3. \(\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda\), where \(\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda\) for any \(\lambda \in P\);
4. \(\hat{e}_i \mathcal{L} \subset \mathcal{L}, \hat{f}_i \mathcal{L} \subset \mathcal{L}\) for any \(i \in I\);
5. we have \(\hat{e}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}\) and \(\hat{f}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}\) for any \(i \in I\);
6. \(\mathcal{B} = \bigsqcup_{\lambda \in P} \mathcal{B}_\lambda\), where \(\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}/q_s\mathcal{L}_\lambda)\) for any \(\lambda \in P\);
7. for any \(b, b' \in \mathcal{B}\) and \(i \in I\), we have \(b' = \hat{f}_i b\) if and only if \(b = \hat{e}_i b'\).

Let \(M\) be an integrable \(U\)-module with a crystal basis \((\mathcal{L}, \mathcal{B})\). Let \(-\) be a bar involution of \(M\) defined by \(\overline{ux} = \overline{x}u\) for any \(x \in U\) and \(u \in M\). We denote by \(\mathcal{A}M\) a \(\mathcal{A}\)-submodule of \(M\) such that
\[
    \overline{\mathcal{A}M} = \mathcal{A}M, \quad u - \overline{u} \in (q_s - 1)\mathcal{A}M \quad \text{for any } u \in \mathcal{A}M.
\]
Set \( E := \mathcal{L} \cap \overline{\mathcal{L}} \cap \mathcal{A}M \). We say that \((\mathcal{L}, \overline{\mathcal{L}}, \mathcal{A}M)\) is a balanced triple of \( M \) if each of \( \mathcal{L}, \overline{\mathcal{L}} \), and \( \mathcal{A}M \) generates \( M \) as a \( K \)-vector space, and moreover, \( E \rightarrow \mathcal{L}/q_s\mathcal{L} \) is an isomorphism.

In such a case, if we denote by \( G \) the inverse isomorphism, that is,

\[
G : \mathcal{L}/q_s\mathcal{L} \xrightarrow{\sim} E := \mathcal{L} \cap \overline{\mathcal{L}} \cap \mathcal{A}M,
\]

then \( \{ G(b); b \in \mathcal{B} \} \) forms an \( \mathcal{A} \)-basis of \( \mathcal{L} \) and an \( \mathcal{A} \)-basis of \( \mathcal{A}M \). We call this basis a global crystal basis of \( M \). We have \( G(b) = G(b) \) for any \( b \in \mathcal{B} \).

\( \mathcal{U} \) has a crystal basis \((\mathcal{L}(\mathcal{U}), \mathcal{B}(\mathcal{U}))\) and a balanced triple \((\mathcal{L}(\mathcal{U}), \overline{\mathcal{L}(\mathcal{U})}, \mathcal{A}\mathcal{U})\). Thus, \( \mathcal{U} \) has a global crystal basis \( \{ G(b); b \in \mathcal{B}(\mathcal{U}) \} \). Furthermore, the same proof of [Kas1, Theorem 4.3.2] works equally well to show that the global crystal basis is invariant under \#.

Finally let us briefly recall the definition of an (integrable) extremal weight module; see [Kas1, §8] for more details (where it is denoted by \( V_{\text{max}}(\lambda) \)).

Let \( M \) be an integrable \( \mathcal{U} \)-module. A vector \( u \in M_\lambda \) is called an extremal vector if there exists a family of vectors \( \{ u_w \}_{w \in W} \) satisfying the following properties:

1. \( u_w = u \) for \( w = e; \)
2. if \( \langle h_i, w\lambda \rangle \geq 0 \), then \( e_i u_w = 0 \) and \( f_i^{(\langle h_i, w\lambda \rangle)} u_w = u_{s_i w}; \)
3. if \( \langle h_i, w\lambda \rangle \leq 0 \), then \( f_i u_w = 0 \) and \( e_i^{(\langle h_i, w\lambda \rangle)} u_w = u_{s_i w}. \)

Similarly, we can define an extremal vector in a regular (normal) crystal \( \mathcal{B} \) by replacing \( e_i \) and \( f_i \) with \( \tilde{e}_i \) and \( \tilde{f}_i \), respectively.

**Definition 3.3.** For each \( \lambda \in P \), the extremal weight module \( V(\lambda) \) is the \( \mathcal{U} \)-module generated by \( u_\lambda \) with the defining relation such that \( u_\lambda \) is an extremal vector of weight \( \lambda \). It can also be represented as

\[
V(\lambda) = \mathcal{U}u_\lambda / J_\lambda, \quad \text{where} \quad J_\lambda = \bigoplus_{b \in \mathcal{B}(\mathcal{U}u_\lambda) \setminus \mathcal{B}(\lambda)} \mathbb{Q}(q)G(b),
\]

where \( \mathcal{B}(\lambda) = \{ b \in \mathcal{B}(\mathcal{U}u_\lambda); b^* \text{ is extremal} \} \).

\( V(\lambda) \) has a crystal basis \((\mathcal{L}(\lambda), \mathcal{B}(\lambda))\) and a balanced triple \((\mathcal{L}(\lambda), \overline{\mathcal{L}(\lambda)}, \mathcal{A}V(\lambda))\). Thus, \( V(\lambda) \) has a global crystal basis \( \{ G(b) \text{ mod } J_\lambda; b \in \mathcal{B}(\lambda) \} \).

Following [Kas2, §5.2], let us denote by \( u_\nu \) the unique global crystal basis in \( V(\varpi_i)_\nu \) for any \( \nu \in W\varpi_i \). Let \( z_i \) be a \( \mathcal{U}' \)-linear automorphism of \( V(\varpi_i) \) of weight \( d_i \delta \), which sends \( u_{\varpi_i} \) to \( u_{\varpi_i + d_i \delta} \), where \( d_i = \max\{1, (\alpha_i, \alpha_i)/2\} \). Then the finite-dimensional fundamental \( \mathcal{U}' \)-module (of level zero) \( W(\varpi_i) \) is defined by

\[
W(\varpi_i) = V(\varpi_i)/(z_i - 1)V(\varpi_i).
\]

For each \( \lambda = \sum_{i \in I_0} \lambda_i \varpi_i \in P^0_+ \), we set

\[
W(\lambda) = \bigotimes_{i \in I_0} W(\varpi_i)^{\otimes \lambda_i}.
\]

It has a global crystal basis \( \{ G(b); b \in \mathcal{B}_W(\lambda) \} \). Note that \( \mathcal{B}_W(\lambda) \) is a finite set.
For each $\lambda = \sum_{i \in I_0} \lambda_i \varpi_i \in P^0_+$, let $\tilde{u}_\lambda = \bigotimes_i \psi_i^{\otimes \lambda_i}$. For each $i$ and each $\mu = 1, \ldots, \lambda_i$, we have a $U'$-linear automorphism $\tilde{z}_{i,\mu}$ of weight $\delta$ following [Kas2, §4.2]. We set

$$\tilde{\mathcal{V}}(\lambda) = U[z_{i,\mu}^{\pm 1}]_{1 \leq i \leq n, 1 \leq \mu \leq \lambda_i} \tilde{u}_\lambda.$$ 

Since $\tilde{\mathcal{V}}(\lambda)$ has the extremal vector $\tilde{u}_\lambda$ of weight $\lambda$, there is a unique $U$-linear morphism

$$\Psi_\lambda : \mathcal{V}(\lambda) \to \tilde{\mathcal{V}}(\lambda),$$

which sends $u_\lambda$ to $\tilde{u}_\lambda$.

For each $\lambda = \sum_{i \in I_0} \lambda_i \varpi_i \in P^0_+$, let $G_\lambda = \prod_{i \in I_0} GL_{\lambda_i}(\mathbb{C})$. Let Irr $G_\lambda$ be the set of irreducible representations of $G_\lambda$, which can be identified with the ring of polynomials in the variables $\{z_{i,\mu}^{\pm 1}\}_{1 \leq i \leq n, 1 \leq \mu \leq \lambda_i}$. For each $s \in \text{Irr} G_\lambda$, we set $s(z)$ to be the polynomial corresponding to $s$, and define $S \in G(\mathcal{B}(\lambda))$ by

$$\Psi_\lambda(S u_\lambda) = s(z) \tilde{u}_\lambda.$$

For each $s \in \text{Irr} G_\lambda$, we denote by $\sigma(s)$ the dual representation of $s$.

### 3.2 BLN-algebras

In this subsection, we will recall the definition of BLN-algebras and an important bilinear form on $V(\lambda)$.

We define the level-zero modified quantum affine algebra $\tilde{U}$ by $\tilde{U} = \bigoplus_{\lambda \in P^0_{cl}} U a_\lambda$. Let $(\tilde{\mathcal{L}}, \tilde{\mathcal{B}})$ be its crystal basis. We set

$$\tilde{\mathcal{A}} \tilde{U} = \bigoplus_{\lambda \in P^0_{cl}} \mathcal{A} U a_\lambda.$$ 

For each element $\beta \in \tilde{\mathcal{B}}$, we denote by $G(\beta)$ the corresponding element of the global crystal basis of $\tilde{U}$.

From now on we always assume that $\lambda \in P^0_{cl+}$. Let $\leq$ be the dominance order on the set $P^0_{cl}$ with respect to its basis $\{\text{cl}(\varpi_i)\}_{i \in I_0}$.

Given an element $u \in \tilde{U}$, we say that $u \in \tilde{U}[\geq \lambda]$ (resp. $\tilde{U}[\geq \lambda]$) if $u$ acts by zero on $V(\lambda')$ for any $\lambda' \neq \lambda$ (resp. $\lambda' \neq \lambda$). Let $\tilde{\mathcal{U}}[\lambda] = \tilde{U}[\geq \lambda]/\tilde{U}[\geq \lambda]$. Let $\mathcal{A} \tilde{\mathcal{U}}[\geq \lambda] = \mathcal{A} \tilde{U} \cap \tilde{U}[\geq \lambda]$, and let $\mathcal{A} \tilde{\mathcal{U}}[\geq \lambda] = \mathcal{A} \tilde{U} \cap \tilde{U}[\geq \lambda]$. We set

$$\mathcal{A} \tilde{\mathcal{U}}[\lambda] = \mathcal{A} \tilde{U}[\geq \lambda]/\mathcal{A} \tilde{U}[\geq \lambda].$$

We denote by $\tilde{\mathcal{L}}[\lambda]$ the $\mathcal{A}$-submodule of $\tilde{U}[\lambda]$ consisting of elements whose images under the map $\tilde{U} \to V(\zeta) \otimes V(-\zeta')$, which is given by $x \mapsto x(u_\zeta \otimes u_{-\zeta'})$, are contained in $\mathcal{L}(\zeta) \otimes \mathcal{L}(-\zeta')$ for any $\zeta, \zeta' \in P_+$. Given an element $\beta \in \tilde{\mathcal{B}}$, we say that $\beta \in \tilde{\mathcal{B}}[\lambda]$ if $G(\beta) \in \tilde{U}[\geq \lambda]$, and moreover, $G(\beta)$ acts nontrivially on $V(\lambda)$.

The following theorem says that $\tilde{U}[\lambda]$ has a global crystal basis.

**Theorem 3.4.** (See [BN, Theorem 6.29].) $\tilde{U}[\lambda]$ has a crystal basis $(\tilde{\mathcal{L}}[\lambda], \tilde{\mathcal{B}}[\lambda])$, and a balanced triple $(\tilde{\mathcal{L}}[\lambda], \tilde{\mathcal{L}}[\lambda], \mathcal{A} \tilde{U}[\lambda])$. Furthermore, $\{G(\beta) \bmod \tilde{U}[\geq \lambda]; \beta \in \tilde{\mathcal{B}}[\lambda]\}$ is its global crystal basis.
The following proposition gives a characterization of $\widetilde{B}[\lambda]$.

**Proposition 3.5.** (See [BN, Proposition 6.23].) For each $\lambda \in P^0_{cl,+}$, we have an isomorphism of bi-crystals

$$\widetilde{B}[\lambda] \cong B_W(\lambda) \times \text{Irr } G_\lambda \times B_W(\lambda).$$

From this proposition, we will identify $\widetilde{B}[\lambda]$ with $B_W(\lambda) \times \text{Irr } G_\lambda \times B_W(\lambda)$ hereafter.

Now we give the definition of BLN-algebras (for Beck, Lusztig and Nakajima). Given a subset $P \subset P^0_{cl,+}$, we say that $P$ is saturated if $P^0_{cl,+} \setminus P$ is finite, and if for all $\lambda \in P^0_{cl,+}$ such that there is a $\mu \in P$ with $\mu \leq \lambda$ we have $\lambda \in P$.

**Definition 3.6.** (See [Mc2, Definition 6.2].) Given a saturated set $P \subset P^0_{cl,+}$, let $\widetilde{U}[P] = \sum_{\lambda \in P} \widetilde{U}[\lambda]$ and set $\widetilde{U}_P = \widetilde{U}/\widetilde{U}[P]$. This $\mathbb{Q}(q_s)$-algebra $\widetilde{U}_P$ is called the BLN-algebra attached to the set $P$.

Let $\mathcal{A}\widetilde{U}[P] = \mathcal{A}\widetilde{U} \cap \widetilde{U}[P] = \sum_{\lambda \in P} \mathcal{A}\widetilde{U}[\lambda]$ and set $\mathcal{A}\widetilde{U}_P = \mathcal{A}\widetilde{U}/\mathcal{A}\widetilde{U}[P]$. Then $\mathcal{A}\widetilde{U}_P$ is an integral form of $\widetilde{U}_P$, i.e. the map $\mathbb{Q}(q_s) \otimes \mathcal{A}\widetilde{U}_P \to \widetilde{U}_P$ is an isomorphism, since the two-sided ideal $\widetilde{U}[P]$ is based, that is, it is spanned by a subset of its basis. Furthermore, # leaves $\widetilde{U}[\lambda]$ invariant ([BN, Proposition 6.9]). In particular, # induces an involutive anti-automorphism on $\mathcal{A}\widetilde{U}[\lambda]$, and on $\mathcal{A}\widetilde{U}_P$.

We can endow the extremal weight module $V(\lambda)$ with a natural bilinear form following Nakajima.

**Proposition 3.7.** (See [N, Proposition 4.1].) There exists a unique bilinear form $(\cdot, \cdot)_\lambda$ on the extremal weight module $V(\lambda)$ satisfying

$$(u, G(b))_\lambda = \begin{cases} 1 & \text{if } G(b) = u, \\ 0 & \text{otherwise}, \end{cases}$$

$$(yu, v)_\lambda = (u, \tau(y)v)_\lambda$$

for $y \in U$, and $u, v \in V(\lambda)$.

We can use this bilinear form to give a characterization of the global crystal basis of $V(\lambda)$, which generalizes [VV, Theorem A] from the fundamental representation $V(\varpi_i)$ to arbitrary $V(\lambda)$.

**Theorem 3.8.** (See [N, Theorem 3].)

1. $\{G(b); b \in B(\lambda)\}$ is an almost orthonormal basis with respect to $(\cdot, \cdot)_\lambda$, that is,

$$(G(b), G(b'))_\lambda \equiv \delta_{bb'} \mod q_s \mathbb{Z}[q_s].$$

2. We have

$$\{\pm G(b); b \in B(\lambda)\} = \{u \in \mathcal{A}V(\lambda); \tau = u \text{ and } (u, u)_\lambda \equiv 1 \mod q_s \mathbb{Z}[q_s]\}.$$
Lemma 3.9. Let $\beta_1 = (b_1, s_1, b_1'), \beta_2 = (b_2, s_2, b_2') \in \tilde{B}[\lambda]$. We have

$$G(b_1, s_1, b_1')G(b_2, s_2, b_2')$$

$$= q^{-\alpha(\beta_1)} \sum_{s_1' \in \text{Irr}_G} (G(b_2)S_2u_{\lambda}, G(b_1)S_1'G(b_2)' \pmod{\tilde{U}[\gamma]})$$

$$= q^{-\alpha(\beta_1)} \sum_{s_1, s_1' \in \text{Irr}_G} c_{s_1, s_1'}^{s_1'} (G(b_2)S_2u_{\lambda}, G(b_1)S_1'G(b_2)' \pmod{\tilde{U}[\gamma]})$$

where $c_{s_1, s_1'}^{s_1'} \in \mathbb{N}$ is the multiplicity of $s_1''$ in the tensor product $s_1 \otimes s_1'$.

3.3 The structure of $\tilde{U}[\lambda]_0$

Let us first review the definition of cells in $\tilde{U}$ with respect to the global crystal basis $G(\tilde{B}) = \{G(\beta); \beta \in \tilde{B}\}$. Let $c_{\beta, \beta'}^{\beta''}(q_*) \in A$ be the structure constants of $\tilde{B}$ defined by

$$G(\beta)G(\beta') = \sum_{\beta''} c_{\beta, \beta'}^{\beta''}(q_*)G(\beta'').$$

For $\beta, \beta' \in \tilde{B}$, we say that $\beta \preceq_L \beta'$ (resp. $\beta \preceq_R \beta'$) if there exists a chain $\beta_0 = \beta, \beta_1, \ldots, \beta_k = \beta'$ of elements in $\tilde{B}$ such that for each $0 \leq t \leq k-1$, we have $c_{\gamma_t, \beta_{t+1}}^{\gamma_t} \neq 0$ (resp. $c_{\gamma_t, \beta_{t+1}}^{\gamma_t} = 0$) for some $\gamma_t \in \tilde{B}$. We write $\beta \preceq_L \beta'$ if there exists a chain $\beta_0 = \beta, \beta_1, \ldots, \beta_k = \beta'$ of elements in $\tilde{B}$ such that for each $1 \leq t \leq k$, we have $\beta_{t-1} \preceq_L \beta_t$ or $\beta_{t-1} \preceq_R \beta_t$. We say that $\beta \sim_L \beta'$ (resp. $\beta \sim_R \beta'$) if and only if $\beta \preceq_L \beta'$ and $\beta' \preceq_L \beta$ (resp. $\beta \preceq_R \beta'$ and $\beta' \preceq_R \beta$). The equivalence classes of $\sim_L$ (resp. $\sim_R$) are called left cells (resp. right cells). We say that $\beta \preceq_{LR} \beta'$ if and only if $\beta \preceq_L \beta'$ and $\beta' \preceq_{LR} \beta$. The equivalence classes of $\sim_{LR}$ are called two-sided cells.

In fact, we have $\tilde{B} = \bigsqcup_{\lambda \in P_{cl,+}^0} \tilde{B}[\lambda]$ (BN, Lemma 6.25). Thus, the following proposition gives a complete description of two-sided cells, left and right cells of $\tilde{B}$.

Proposition 3.10. (See [BN, Proposition 6.30].)

1. For each $\lambda \in P_{cl,+}^0$, $\tilde{B}[\lambda]$ is a two-sided cell of $\tilde{B}$.
2. For each $b_2 \in \mathcal{B}_W(\lambda)$, $\{(b_1, s, b_2) \in \tilde{B}[\lambda]; s \in \text{Irr } G, b_1 \in \mathcal{B}_W(\lambda)\}$ is a left cell.
3. For each $b_2 \in \mathcal{B}_W(\lambda)$, $\{(b_1, s, b_2) \in \tilde{B}[\lambda]; s \in \text{Irr } G, b_2 \in \mathcal{B}_W(\lambda)\}$ is a right cell.

Corollary 3.11. (1) There is a one-to-one correspondence $\theta$ between $P_{cl,+}^0$ and the set of two-sided cells of $\tilde{B}$.

2. Under the above correspondence, we have $\theta(\mu) \preceq_{LR} \theta(\lambda) \Rightarrow \lambda \leq \mu$.

Proof. (2) follows from Lemma 3.9. \hfill \Box

Recall the involutive anti-automorphism $\#$ on $\mathcal{A}\tilde{U}[\lambda]$. The following lemma describes explicitly how $\#$ acts on the global crystal basis. In particular, this implies that $\#$ interchanges the left cells and right cells in a two-sided cell $\tilde{B}[\lambda]$. 

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Lemma 3.12. (See [BN, Lemma 6.34].) For any \((b_1, s, b_2) \in \tilde{B}[\lambda]\), we have
\[ G(b_1, s, b_2)^\# = G(b_2, \sigma(s), b_1). \]

Next let us recall the definition of the asymptotic ring \(\tilde{U}[\lambda]_0\). It is equipped with a \(\mathbb{Z}\)-basis \(\{t_\beta; \beta \in \tilde{B}[\lambda]\}\) and the multiplication in \(\tilde{U}[\lambda]_0\) is defined by
\[ t_\beta t_\gamma = \sum_{\gamma' \in \tilde{B}[\lambda]} \gamma_{\beta, \gamma'}^{\gamma'} t_{\gamma'} \quad \text{for all } \beta, \gamma' \in \tilde{B}[\lambda], \]
where \(\gamma_{\beta, \gamma'}^{\gamma'} \in \mathbb{Z}\) is defined by \(q^{\alpha(\beta)} c_{\beta, \gamma'}^{\gamma'} \equiv \gamma_{\beta, \gamma'}^{\gamma'} \mod q_s \mathbb{Z}[q_s].\)

For each \(\lambda \in P_{\text{cl.}+}^0\), let
\[ D_{\tilde{B}[\lambda]} = \{(b, 1, b); b \in B_W(\lambda)\}. \]

From Lemma 3.12, it is clear that \(G(\beta)^\# = G(\beta)\) for \(\beta \in D_{\tilde{B}[\lambda]}\). Furthermore, each left cell and each right cell in a two-sided cell \(\tilde{B}[\lambda]\) contain exactly one element in \(D_{\tilde{B}[\lambda]}\). Note that the \(\mathbb{Z}\)-ring \(\tilde{U}[\lambda]_0\) has an identity element, that is, \(1 = \sum_{b \in B_W(\lambda)} t_{(b, 1, b)}\).

In the following lemma, we collect some properties of the structure constants \(c_{\beta, \gamma'}^{\gamma''}\) of \(\tilde{U}\) and \(\gamma_{\beta, \gamma'}^{\gamma''}\) of \(U[\lambda]_0\), which play a crucial role in Section 4.

Lemma 3.13. For each \(\lambda \in P_{\text{cl.}+}^0\), let \(B[\lambda]\) be a two-sided cell.

1. Let \(\beta_2, \beta_4 \in B[\lambda]\), and let \(\beta_1, \beta_3 \in \tilde{B}\). We have
\[ \sum_{\beta \in \tilde{B}[\lambda]} c_{\beta_1, \beta_2}^{\beta} (q_s) \gamma_{\beta, \beta_3}^{\beta_3} = \sum_{\beta \in \tilde{B}[\lambda]} c_{\beta_1, \beta}^{\beta_3} (q_s) \gamma_{\beta_2, \beta_3}^{\beta}. \]
\[ \sum_{\beta \in \tilde{B}[\lambda]} \gamma_{\beta_1, \beta_2}^{\beta} c_{\beta, \beta_3}^{\beta_3} (q_s) = \sum_{\beta \in \tilde{B}[\lambda]} \gamma_{\beta_1, \beta}^{\beta_4} c_{\beta_2, \beta_4}^{\beta_3} (q_s). \]

2. Let \(\beta, \beta', \beta''\) be in a two-sided cell. If \(\gamma_{\beta, \gamma'}^{\gamma''} \neq 0\), then \(\beta \sim_L \beta'^\#, \beta'' \sim_R \beta\) and \(\beta'' \sim_L \beta'.\)

3. Suppose that \(\beta_1, \beta_2 \in \tilde{B}[\lambda]\). Then \(\sum_{\beta \in D_{\tilde{B}[\lambda]}} \gamma_{\beta, \beta_1}^{\beta_1} = 1\), and \(\gamma_{\beta, \beta_1}^{\beta_2} = 0\) for any \(\beta \in D_{\tilde{B}[\lambda]}\) if \(\beta_1 \neq \beta_2\).

Proof. (1) It follows from [BN, Lemma 6.41] and [L2, Proposition 1.9(a)].

(2) It follows from [BN, (6.45)].

(3) It follows from the fact that \(\sum_{\delta \in D_{\tilde{B}[\lambda]}} t_\delta\) is the identity of \(\tilde{U}[\lambda]_0\). □

Since \(B_W(\lambda)\) is a finite set, we will use \(\{1, 2, \ldots, n_\lambda\}\) to label these elements in it, where \(n_\lambda = |B_W(\lambda)|\). From now on we always use this fixed label. By definition, the corresponding elements of \(D_{\tilde{B}[\lambda]}\) will be labeled by the same numbers. From Proposition 3.10(2), the left cell corresponding to \(b' \in B_W(\lambda)\) will be denoted by \(L_j\), if \(j\) labels \(b'\). Then \(L_1, L_2, \ldots, L_{n_\lambda}\) are a list of all left cells in \(\tilde{B}[\lambda]\), and \(R_j := L_j^\#, 1 \leq j \leq n_\lambda,\) are a list of all right cells in \(\tilde{B}[\lambda]\). Let \(A_{jl} = R_j \cap L_l\) for \(1 \leq j, l \leq n_\lambda\). Then \(A_{jl}^\# = A_{lj}\), and \(\tilde{B}[\lambda]\) is a disjoint union of all \(A_{jl}\). Let \(B_\lambda = R(G_\lambda)\) be the representation ring of
the algebraic group $G_\lambda$, which is a product of general linear groups $GL_\lambda$. By [FH, Exercise 23.36(d)], we have

$$R(GL_\lambda) \cong \mathbb{Z}[E_1, \ldots, E_{\lambda_i-1}, E_{\lambda_i}, E_{\lambda_i}^{-1}],$$

where $E_k$ ($1 \leq k \leq \lambda_i$) are the elementary symmetric functions of $x_1, \ldots, x_{\lambda_i}$. So $B_\lambda$ is an affine commutative $\mathbb{Z}$-algebra ([KX, Theorem 5.3]). $B_\lambda$ has a basis $\text{Irr} G_\lambda$ so that $\sigma$ can be extended to the entire algebra $B_\lambda$ as an involutive anti-automorphism.

The following theorem gives an explicit description of the asymptotic ring $\tilde{U}[\lambda]_0$.

**Theorem 3.14.** (See [BN, Theorem 6.44(i)].)

The asymptotic ring $\tilde{U}[\lambda]_0$ is isomorphic to an $n_\lambda \times n_\lambda$ matrix algebra over $B_\lambda$.

This isomorphism is given by $t_\beta \mapsto E_{jl}(s)$ for $\beta = (j, s, l) \in A_{jl}$.

**Proof.** It follows from Lemma 3.13(2) that the given map is a ring homomorphism. □

Thus each element in $\tilde{U}[\lambda]_0$ is a matrix over $B_\lambda$. So we identify $t_\beta$ with $E_{jl}(s)$ for $\beta = (j, s, l) \in A_{jl}$. We can also label the basis element $G(\beta)$ with $\beta \in \tilde{B}[\lambda]$, that is, we write $E_{jl}(s)$ for $G(\beta)$ with $\beta = (j, s, l) \in A_{jl}$.

## 4 Affine cellularity of BLN-algebras

Let $\tilde{B}[\lambda]$ be a two-sided cell. We denote by $\mathcal{A}\tilde{U}^{\leq}[\lambda]$ (resp. $\mathcal{A}\tilde{U}^{<}[\lambda]$) the free $\mathcal{A}$-submodule of $\mathcal{A}\tilde{U}$ generated by all $G(\beta)$ with $\beta \leq_{LR} \beta'$ (resp. $\beta <_{LR} \beta'$) for some $\beta' \in \tilde{B}[\lambda]$. Then both $\mathcal{A}\tilde{U}^{\leq}[\lambda]$ and $\mathcal{A}\tilde{U}^{<}[\lambda]$ are two-sided ideals in $\mathcal{A}\tilde{U}$. We denote by $\mathcal{A}\tilde{U}[\lambda]$ the quotient $\mathcal{A}\tilde{U}^{\leq}[\lambda]/\mathcal{A}\tilde{U}^{<}[\lambda]$. Thus $\mathcal{A}\tilde{U}[\lambda]$ has an $\mathcal{A}$-basis $\{(G(\beta)); \beta \in \tilde{B}[\lambda]\}$ and the multiplication in $\mathcal{A}\tilde{U}[\lambda]$ is defined by

$$[G(\beta)][G(\beta')] = \sum_{\beta'' \in \tilde{B}[\lambda]} c_{\beta, \beta''}^{\beta'}(q_s)[G(\beta'')] \quad \text{for all } \beta, \beta' \in \tilde{B}[\lambda].$$

**Lemma 4.1.** $\tilde{U}[\lambda]_0$ is a $\tilde{U}[\lambda]_0 - \mathcal{A}\tilde{U}[\lambda]$-bimodule via

$$t_\beta \circ [G(\beta')] = \sum_{\beta'' \in \tilde{B}[\lambda]} c_{\beta, \beta''}^{\beta'}(q_s)t_{\beta''}. $$

**Proof.** It follows from Lemma 3.13(1). □

**Lemma 4.2.** In $\tilde{U}[\lambda]_0$, for all $\beta, \beta' \in \tilde{B}[\lambda]$ we have

$$t_\beta \circ [G(\beta')] = t_\beta\left(\left(\sum_{\delta \in \mathcal{D}_{\tilde{B}[\lambda]}} t_\delta\right) \circ \left(\sum_{\delta \in \mathcal{D}_{\tilde{B}[\lambda]}} [G(\delta)]\right)\right)t_{\beta'}. $$

**Proof.** Since $\sum_{\delta \in \mathcal{D}_{\tilde{B}[\lambda]}} t_\delta$ is the identity of $\tilde{U}[\lambda]_0$, there are, by Lemma 4.1, the following equalities:

$$t_\beta\left(\left(\sum_{\delta \in \mathcal{D}_{\tilde{B}[\lambda]}} t_\delta\right) \circ \left(\sum_{\delta \in \mathcal{D}_{\tilde{B}[\lambda]}} [G(\delta)]\right)\right)t_{\beta'} = \left(t_\beta \circ \left(\sum_{\delta \in \mathcal{D}_{\tilde{B}[\lambda]}} [G(\delta)]\right)\right)t_{\beta'} = \sum_{\alpha, \beta'' \in \tilde{B}[\lambda]} c_{\alpha, \beta''}^{\alpha, \beta} t_{\beta''}. $$

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Now, we use Lemma 3.13(1) to replace $\sum_\alpha c_\beta,\delta t^\delta_{\alpha,\beta'}$ by $\sum_\alpha c_\beta,\alpha t^\alpha_{\delta,\beta'}$, and we get
\[
t_\beta \left( \left( \sum_{\delta \in D_{\tilde{B}[\lambda]}} t_\delta \right) \circ \left( \sum_{\delta \in D_{\tilde{B}[\lambda]}} [G(\delta)] \right) \right) t_{\beta'} = \sum_{\delta \in D_{\tilde{B}[\lambda]}} c_\beta,\alpha t^\alpha_{\delta,\beta'} = \sum_{\alpha,\beta'} c_\beta,\alpha \left( \sum_{\delta \in D_{\tilde{B}[\lambda]}} \tau^\alpha_{\delta,\beta'} \right) t_{\beta'}.
\]

It follows from Lemma 3.13(3) that
\[
t_\beta \left( \left( \sum_{\delta \in D_{\tilde{B}[\lambda]}} t_\delta \right) \circ \left( \sum_{\delta \in D_{\tilde{B}[\lambda]}} [G(\delta)] \right) \right) t_{\beta'} = \sum_{\beta'' \in \tilde{G}[\lambda]} c_\beta,\beta'' t_{\beta''} = t_\beta \circ [G(\beta')]. \quad \square
\]

When Lemma 4.2 gets translated into matrix language, the left-hand side of the equation expresses the multiplication $E_{jl}(\beta') E_{pq}(s')$ in $\tilde{A} \tilde{U}[\lambda]$ for $\beta = (j, s, l) \in A_{jl}$ and $\beta' = (p', s', q) \in A_{pq}$, and the right-hand side is just the product $E_{jl}(\beta') E_{pq}(s')$ in the usual matrix algebra $\tilde{U}[\lambda]_0$, where $\Psi_{\tilde{B}[\lambda]}$ is the matrix representing $(\sum_{\delta \in D_{\tilde{B}[\lambda]}} t_\delta) \circ ([G(\delta)])$.

For each two-sided cell $\tilde{B}[\lambda]$, we can define the following $\mathcal{A}$-linear map $\Phi : \tilde{A} \tilde{U} \rightarrow \mathcal{A} \otimes \tilde{U}[\lambda]_0$, due to Lusztig, by
\[
\Phi(G(\beta)) = \sum_{\delta \in D_{\tilde{B}[\lambda]}; \beta' \in \tilde{G}[\lambda]} c_\beta,\beta'(q_\delta) t_{\beta'} \quad \text{for any } \beta \in \tilde{B},
\]
which is well-defined since $D_{\tilde{B}[\lambda]}$ is a finite set and for fixed $\beta, \delta$ there are only finitely many $c_\beta,\beta'(q_\delta) \neq 0$.

The algebra homomorphism $\Phi$ induces an algebra homomorphism $\Psi_{\tilde{B}[\lambda]}$ from $\tilde{A} \tilde{U}[\lambda]$ to $\mathcal{A} \otimes \tilde{U}[\lambda]_0$, which is given by
\[
[G(\beta)] \mapsto \sum_{\delta \in D_{\tilde{B}[\lambda]}; \beta' \in \tilde{G}[\lambda]} c_\beta,\beta'(q_\delta) t_{\beta'} = t_\beta \circ \left( \sum_{\delta \in D_{\tilde{B}[\lambda]}} [G(\delta)] \right) = t_\beta \left( \left( \sum_{\delta \in D_{\tilde{B}[\lambda]}} t_\delta \right) \circ \left( \sum_{\delta \in D_{\tilde{B}[\lambda]}} [G(\delta)] \right) \right).
\]

In terms of matrix language, we see that for $\beta = (j, s, l) \in A_{jl}$,
\[
\Phi_{\tilde{B}[\lambda]} : E_{jl}(\beta) \mapsto E_{jl}(\beta) \cdot \Phi_{\tilde{B}[\lambda]}.
\]

From Lemma 3.12, there is an $\mathcal{A}$-involution $\#$ on $\tilde{A} \tilde{U}$, which sends $G(\beta)$ to $G(\beta\#)$ for any $\beta \in \tilde{B}$. This induces an $\mathcal{A}$-involution $\#$ on $\tilde{A} \tilde{U}[\lambda]$, which is given by $[G(\beta)] \mapsto [G(\beta\#)]$ for any $\beta \in \tilde{B}[\lambda]$.

Summarizing, we get the following result.

**Proposition 4.3.** Let $\tilde{B}[\lambda]$ be a two-sided cell in $\tilde{B}$. Then there is a matrix $\Psi_{\tilde{B}[\lambda]}$ in $\tilde{U}[\lambda]_0$ such that $\tilde{A} \tilde{U}[\lambda]$ can be identified with the generalized matrix algebra $(M_{n_{\lambda}}(B_{\lambda}), \Psi_{\tilde{B}[\lambda]})$. Moreover, if $[G(\beta)]$ is identified with $E_{jl}(\beta)$ for $\beta = (j, s, l) \in \tilde{B}[\lambda]$, then we obtain that $[G(\beta\#)]$ is identified with $\tilde{E}_{jl}(\sigma(s))$. The multiplication in $(M_{n_{\lambda}}(B_{\lambda}), \Psi_{\tilde{B}[\lambda]})$ is given by
\[
\tilde{E}_{jl}(s) \cdot \tilde{E}_{pq}(s') = E_{jl}(s) \Psi_{\tilde{B}[\lambda]} E_{pq}(s').
\]

In addition, the homomorphism $\Phi_{\tilde{B}[\lambda]}$ defined by Lusztig from $\tilde{A} \tilde{U}[\lambda]$ to $\mathcal{A} \otimes \tilde{U}[\lambda]_0$ can be identified with the map from $(M_{n_{\lambda}}(B_{\lambda}), \Psi_{\tilde{B}[\lambda]})$ to $M_{n_{\lambda}}(B_{\lambda})$ by multiplying $\Psi_{\tilde{B}[\lambda]}$ from the right.
From Corollary 3.11(2), we can easily get a linear order on the set of two-sided cells such that $\tilde{B}[\lambda] \preceq_{LR} \tilde{B}[\mu]$ implies that $\lambda \leq \mu$, where $\preceq$ means that we choose the opposite dominance order, that is, $\lambda \leq \mu$ if and only if $\mu \leq \lambda$. We denote by $\lambda \prec \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$.

For each $\lambda \in P_{cl,+}^0$, we define $\mathcal{C}'_\lambda$ to be the $\mathcal{A}$-submodule generated by all $G(\beta)$ with $\beta \in \tilde{B}[\lambda]$, and set $\mathcal{C}_\lambda = \bigoplus_{\mu \leq \lambda} \mathcal{C}'_\mu$. Then $\mathcal{C}'_\lambda$ is invariant under the involution $\#$, $\mathcal{C}_\lambda$ is a two-sided ideal in $\mathcal{A}\tilde{U}$ with $\mathcal{C}_\lambda / \mathcal{C}_{<\lambda} = \mathcal{A}\tilde{U}[\tilde{B}[\lambda]]$, where $\mathcal{C}_{<\lambda} = \bigoplus_{\mu < \lambda} \mathcal{C}'_\mu$, and the infinite chain
\[
\cdots \subset \cdots \subset \mathcal{C}_\lambda \subset \cdots \subset \bigoplus_{\lambda \in P_{cl,+}^0} \mathcal{C}'_\lambda = \mathcal{A}\tilde{U}
\]
is a cell chain for $\mathcal{A}\tilde{U}$ by Proposition 2.2 and Proposition 4.3. Thus we have proved the following theorem.

**Theorem 4.4.** Let $\mathcal{A} = \mathbb{Z}[q_s, q_s^{-1}]$. Then the level-zero modified quantum affine algebra $\mathcal{A}\tilde{U}$ is a generalized affine cellular $\mathbb{Z}$-algebra with respect to the $\mathcal{A}$-involution $\# : G(\beta) \mapsto G(\beta^\#)$ for any $\beta \in \tilde{B}$, where ‘generalized’ means that we allow the cell chain in Definition 2.1 to be infinite.

In the following lemma, we prove that $\mathcal{A}\tilde{U}[\lambda]$, as a two-sided ideal (affine cell ideal) of $\mathcal{A}\tilde{U} / \mathcal{A}\tilde{U}[>\lambda]$, satisfies the assumptions in Theorem 2.3.

**Lemma 4.5.** For each $\lambda \in P_{cl,+}^0$, $\mathcal{A}\tilde{U}[\lambda]$, as a two-sided ideal of $\mathcal{A}\tilde{U} / \mathcal{A}\tilde{U}[>\lambda]$, is idempotent, and moreover, is generated by a non-zero idempotent element.

**Proof.** (a) For each $(b, 1, b') \in \tilde{B}[\lambda]$, we have
\[
G(b, 1, b')G(b, 1, b')
\]
(by Lemma 3.9) $\equiv q^{-a(\beta)} \sum_{s' \in \text{Irr}_{\tilde{G}}^\lambda} (G(b)u_\lambda, G(b')S'u_\lambda)_{\lambda}G(b)S'G(b')^\# \pmod{\tilde{U}[>\lambda]}$
\[\text{(by Theorem 3.8)} \equiv q^{-a(\beta)} \delta_{b,b'}G(b, 1, b') \pmod{\tilde{U}[>\lambda] + q_s\tilde{L}[\lambda]}, \]
where $\beta = (b, 1, b')$. So if we choose $(b, 1, b')$ such that $b = b'$ and $G(b, 1, b') \in \tilde{U}a_\lambda$, then $a(\beta) = 0$, and we see that
\[
G(b, 1, b)G(b, 1, b) \equiv G(b, 1, b) \pmod{q_s\tilde{L}[\lambda]} \in \tilde{U}[\lambda].
\]
Taking $-\lambda$ of both sides and using the $-\lambda$ invariance of the global crystal basis, we get
\[
G(b, 1, b)G(b, 1, b) \equiv G(b, 1, b) \pmod{q_s^{-1}\tilde{L}[\lambda]} \in \tilde{U}[\lambda].
\]
So we get
\[
G(b, 1, b)G(b, 1, b) = G(b, 1, b) \pmod{q_s\tilde{L}[\lambda] \cap q_s^{-1}\tilde{L}[\lambda] \cap \mathcal{A}\tilde{U}[\lambda]}.
\]
Thus, we must have
\[
G(b, 1, b)G(b, 1, b) = G(b, 1, b) \text{ in } \mathcal{A}\tilde{U}[\lambda],
\]
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which means that $\mathcal{A}\tilde{U}[\lambda]$ contains a nonzero idempotent element.

(b) For $(b, s, b'', (b', 1, b') \in \mathcal{B}[\lambda]$, we have

$$G(b, s, b'')(G(b'', 1, b') = q^{-a(\beta)} \sum_{\lambda \in \text{Irr}_G} (G(b')u_\lambda, G(b')S'u_\lambda)\lambda G(b)SS'G(b')# (\mod \tilde{U}[\lambda])$$

$$q^{-a(\beta)}G(b, s, b') (\mod \tilde{U}[\lambda] + q_s\tilde{L}^f[\lambda]),$$

where $\beta = (b, s, b'')$. Now we choose $b'' \in \mathcal{B}_W(\lambda)$ such that $G(b'', 1, b'') \in \tilde{U}u_\lambda$, then for any $(b, s)$, we have $a(\beta) = a(b, s, b') = a(b'', 1, b'') = 0$, and we have

$$G(b, s, b'')G(b'', 1, b') \equiv G(b, s, b') (\mod q_s\tilde{L}^f[\lambda]) \text{ in } \tilde{U}[\lambda].$$

Taking $-$ of both sides and using the fact that $G(\mathcal{B}) = G(\beta)$ for any $\beta \in \tilde{B}[\lambda]$, we get

$$G(b, s, b'')G(b'', 1, b') \equiv G(b, s, b') (\mod q_s\tilde{L}^f[\lambda]) \text{ in } \tilde{U}[\lambda].$$

So we get

$$G(b, s, b'')G(b'', 1, b') - G(b, s, b') \in q_s\tilde{L}^f[\lambda] \cap q_s^{-1}\tilde{L}^f[\lambda] \cap \mathcal{A}\tilde{U}[\lambda].$$

Thus, we must have

$$G(b, s, b') = G(b, s, b'')G(b'', 1, b') \text{ in } \mathcal{A}\tilde{U}[\lambda],$$

which means that the two-sided ideal $\mathcal{A}\tilde{U}[\lambda]$ is an idempotent ideal. In particular, we have $G(b, s, b'') = G(b, s, b')G(b'', 1, b'')$ in $\mathcal{A}\tilde{U}[\lambda]$. From the choice of $b''$ and (a), we can see that $G(b'', 1, b'')$ is idempotent. Thus, we get

$$G(b, s, b') = G(b, s, b'')G(b'', 1, b'')G(b'', 1, b') \text{ in } \mathcal{A}\tilde{U}[\lambda],$$

which means that the two-sided ideal $\mathcal{A}\tilde{U}[\lambda]$ is generated by a non-zero idempotent element.

For each saturated subset $P \subset P_{cl, +}^0$, we have defined the BLN-algebra $\tilde{U}_P$ and its integral form $\mathcal{A}\tilde{U}_P$, then we immediately get the following theorem from Theorem 2.3, Theorem 4.4 and Lemma 4.5.

**Theorem 4.6.** Let $\mathcal{A} = \mathbb{Z}[q_s, q_s^{-1}]$. Then the BLN-algebra $\mathcal{A}\tilde{U}_P$ is an affine cellular $\mathbb{Z}$-algebra with respect to the induced $\mathcal{A}$-involution #. For any noetherian domain $k$, the parameter set of simple $k \otimes_\mathbb{Z} \mathcal{A}\tilde{U}_P$-modules equals the parameter set of simple modules of the asymptotic algebra, and so it is a finite union of affine spaces. Moreover, it has finite global dimension provided that $k$ has that, and its derived category admits a stratification whose strata are the derived categories of the various algebras $B_\lambda$.

For affine type A, the BLN-algebra $\tilde{U}_P$ is just the generalization of the algebra $U_{D,n,n}$ studied in [L3] ([Mc2, Lemma 6.6]), which is exactly the affine $q$-Schur algebra $U_{D,n,n}$ defined in [L3] when $D < n$. It is not difficult to see that $\mathcal{A}U_{D,n,n}$ defined in [L3] also coincides with $\mathcal{A}\tilde{U}_P$ for the same saturated set $P$. Thus we have also obtained the following theorem.
Theorem 4.7. Let $A = \mathbb{Z}[v, v^{-1}]$ ($v$ an indeterminate). Then the affine $q$-Schur algebra $\mathcal{A}\mathfrak{U}_{D,n,n}$ over $A$ is an affine cellular $\mathbb{Z}$-algebra when $D < n$. In such a case, for any noetherian domain $k$, $k \otimes_{\mathbb{Z}} \mathcal{A}\mathfrak{U}_{D,n,n}$ has finite global dimension provided that $k$ has that, and its derived category admits a stratification whose strata are the derived categories of the various algebras $B_{\lambda}$.

Remark 4.8. From Theorem 4.6 and 4.7, we immediately get that BLN-algebras $\mathcal{A}\tilde{\mathfrak{U}}_P$ and affine quantum Schur algebras $\mathcal{A}\mathfrak{U}_{D,n,n}$, when $D < n$, are affine quasi-hereditary algebras.

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References

[BN] J. Beck and H. Nakajima, Crystal bases and two-sided cells of quantum affine algebras, Duke Math. J. 123 (2004) 335-402.
[C] W. Cui, Affine cellularity of affine $q$-Schur algebras, submitted, arXiv: 1405.6705.
[Do] S. Doty, Presenting generalized $q$-Schur algebras, Represent. Theory 7 (2003) 196-213.
[FH] W. Fulton and J. Harris, Representation Theory. A First Course, Grad. Texts in Math., vol. 129, Springer-Verlag, 1991.
[GL] J.J. Graham and G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1996) 1-34.
[GM] J. Guilhot and V. Miemietz, Affine cellularity of affine Hecke algebras of rank two, Math. Z. 271 (2012) 373-397.
[Kac] V. Kac, Infinite Dimensional Lie Algebras, 3rd edition, Cambridge University Press, Cambridge, 1990.
[Kas1] M. Kashiwara, Crystal bases of modified quantized enveloping algebras, Duke Math. J. 73 (1994) 383-413.
[Kas2] M. Kashiwara, On level-zero representations of quantized affine algebras, Duke Math. J. 112 (2002) 117-175.
[Kle] A. Kleshchev, Affine highest weight categories and affine quasihereditary algebras, arXiv: 1405.3328.
[KL] A. Kleshchev and J. Loubert, Affine cellularity of Khovanov-Lauda-Rouquier algebras of finite types, to appear in Int. Math. Res. Not. arXiv: 1310.4467.
[Koe] S. Koenig, A panorama of diagram algebras, in: Trends in representation theory and related topics, EMS Ser. Congr. Rep., pp. 491-540. Eur. Math. Soc., Zürich, 2008.
[KX] S. Koenig and C. Xi, Affine cellular algebras, Adv. Math. 229 (2012) 139-182.
[L1] G. Lusztig, Introduction to Quantum Groups, Progress in Mathematics 110, Birkhäuser, Boston · Basel · Berlin, 1993.

[L2] G. Lusztig, Quantum groups at $v = \infty$, in: Functional Analysis on the Eve of the 21st Century: In Honor of I. M. Gelfand, vol. I, in: Progr. Math., vol 131, Birkhäuser, 1995, pp. 199-221.

[L3] G. Lusztig, Aperiodicity in quantum affine $\mathfrak{g}l_n$, Asian J. Math. 3 (1999) 147-177.

[Mc1] K. McGerty, Cells in quantum affine $\mathfrak{sl}_n$, Int. Math. Res. Not. 24 (2003) 1341-1361.

[Mc2] K. McGerty, Generalized $q$-Schur algebras and quantum Frobenius, Adv. Math. 214 (2007) 116-131.

[N] H. Nakajima, Extremal weight modules of quantum affine algebras, Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math. 40, Math. Soc. Japan, Tokyo, (2004), pp. 343-369.

[VV] M. Varagnolo and E. Vasserot, Canonical bases and quiver varieties, Represent. Theory 7 (2003) 227-258.

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