A Flexible Multi-Facility Capacity Expansion Problem
with Risk Aversion

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Abstract

This paper studies flexible multi-facility capacity expansion with risk aversion. In this setting, the decision maker can periodically expand the capacity of facilities given observations of uncertain demand. We model this situation as a multi-stage stochastic programming problem. We express risk aversion in this problem through conditional value-at-risk (CVaR), and we formulate a mean-CVaR objective. To solve the multi-stage problem, we optimize over decision rules. In particular, we approximate the full policy space of the problem with a tractable family of if-then policies. Subsequently, a decomposition algorithm is proposed to optimize the decision rule. This algorithm decomposes the model over scenarios and it updates solutions via the subgradients of the recourse function. We demonstrate that this algorithm can quickly converge to high-performance policies. To illustrate the practical effectiveness of this method, a case study on the waste-to-energy system in Singapore is presented. These simulation results show that by adjusting the weight factor of the objective function, decision makers are able to trade off between a risk-averse policy that has a higher expected cost but a lower value-at-risk, and a risk-neutral policy that has a lower expected cost but a higher value-at-risk risk.

Key words: Capacity expansion problem, real options, risk aversion, multi-stage stochastic programming, decision rules.
1 Introduction

The capacity expansion problem aims to determine the capacity plan—that is, the optimal amount and timing of capacity acquisition—to address growth of demand. This problem has been widely studied for a variety of systems, such as semiconductor manufacturing (Geng et al., 2009), airport facilities (Sun and Schonfeld, 2015), and waste-to-energy systems (Cardin and Hu, 2015). This problem is challenging because future demand is uncertain, but capital expenditure for capacity is high and these investments are usually irreversible (for example, the capacity of airport facilities, such as highway links and ports, is hard to decrease once established (Sun and Schonfeld, 2015)).

The traditional ‘inflexible’ method determines the capacity plan at the beginning before demand is observed, and then this plan is implemented regardless of the realizations of future demand. However, this method may suffer unexpected costs if demands fail to grow as expected.

To cope with this issue, increasing attention has been paid to designing ‘flexible’ systems in capacity expansion problems (a.k.a. real options analysis). A flexible system has the ability to adjust its capacity dynamically as demands are observed. We can then expand the capacity (i.e. exercise the option) if demand surges, and do nothing when demand remain steady. In the multi-facility capacity expansion problem (MCEP), a flexible system has the option not only to adjust capacity, but also to switch service between facilities. For example, if one facility runs out of capacity, we can either expand the capacity of this facility, or allocate excess demand to an adjacent facility. It has been verified by many industrial case studies that flexibility can improve the expected lifecycle performance of engineering systems by 10% to 30% compared with inflexible methods (Neufville and Scholtes, 2011; Cardin et al., 2017a).

Evaluating the economic performance of a flexible MCEP is a dynamic optimization problem with uncertainty. This problem needs to find the optimal expansion policy, which is a mapping from the historical data to the capacity decisions. However, the optimal policy is usually hard to characterize for practically-sized MCEP with existing methods. In particular, dynamic programming (DP)-based algorithms suffer the curse of dimensionality (in the action space) when the number of facilities is large. Stochastic programming algorithms can also be inefficient because of the non-convexity caused by discrete capacity decisions.
To solve MCEPs with discrete capacity, if–then decision rules have been proposed to approximate the policy space (Cardin et al., 2017a,b; Zhang and Cardin, 2017). An if–then rule states that if the capacity gap of a facility exceeds a threshold then its capacity is expanded to a certain level, and the capacity is unchanged otherwise. In this framework, we want to find the best if–then decision rule. This type of rule mimics the behavior of human decision makers so it is intuitive from a managerial standpoint. In numerical terms, the decision rule-based method not only provides high-performance solutions for MECPs, it is also more scalable than DP or stochastic programming (Zhao et al., 2018).

Classical MCEP models suppose that the decision makers are risk-neutral; in other words, they maximize the expected reward. However, decision makers often have their own attitudes about risk. We employ the popular conditional value-at-risk (CVaR) to measure risk in our model. The motivation for this choice is two-fold. First, CVaR is a coherent risk measure and has strong decision-theoretic support. Second, CVaR is convex and enjoys significant computational advantages compared to other risk-aware objectives (Rockafellar and Uryasev, 2000).

Our present paper investigates a risk-averse MCEP with a mean-CVaR objective. Our specific contributions are summarized below:

1. We formulate the objective of the MCEP as a weighted sum of the expected cost and the CVaR of cost. We solve this risk-averse problem by using an if–then decision rule to approximate the policy space. To the best knowledge of the authors, no studies have yet used decision rules for risk-averse MCEPs.

2. We prove that the value of the risk-averse MCEP is always smaller than or equal to the risk-neutral MCEP, implying that decision makers may prefer to pay less for flexibility if they are risk-averse.

3. We optimize over decision rules (with respect to the mean-CVaR objective) by using a customized decomposition algorithm. This algorithm can be viewed as an improved version of the one from (Zhao et al., 2018). We design subgradient cuts to update the parameters of the decision rule, which is not only more time-efficient but can also provide problem insight compared with the algorithm in (Zhao et al., 2018).
The remainder of this paper is organized as follows. Section 2 summarizes the relevant literature. In Section 3, the risk-neutral MCEP model is presented, and then its risk-averse counterpart is discussed. If–then decision rules are introduced in Section 4. In Section 5, we present the risk-averse MCEP with decision rules, and solve it with our subgradient-based decomposition algorithm. A numerical study on a multi-facility waste-to-energy (WTE) system is elaborated in Section 6. Finally, the strengths and limitations of the proposed method and the opportunities for future research are summarized. All proofs may be found in the Appendix.

2 Literature Review

2.1 Flexible Capacity Expansion Problems

Capacity expansion problems have been widely studied since the seminal paper by (Manne, 1961). Manne (1961) investigated the trade-off between the discount factor and the economies of scale in capacity expansion problems with deterministic/stochastic demand. Many variations on this original model have been studied. Comprehensive reviews may be found in (Luss, 1982; Van Mieghem, 2003; Martínez-Costa et al., 2014).

In the framework of real options analysis, capacity expansion decisions are viewed as a series of options that can be exercised over time (Dixit and Pindyck, 1994). The main advantage of this framework comes from its ‘wait-and-see’ nature; capacity decisions can be exercised or deferred based on the realizations of uncertainty. Eberly and Van Mieghem (1997) studied a multi-factor capacity investment problem and characterized the structure of the optimal policy as an ISD (invest, stay put, and disinvest) policy. Kouvelis and Tian (2014) studied a flexible capacity investment problem and investigated the value of the postponed capacity commitment option for uncertain demand. Our work differs from these models because we deal with discrete capacity expansion decisions. In capacity expansion problems with discrete capacity, Huang and Ahmed (2009) derived an analytical bound for the value of the multi-stage problem compared to the two-stage problem, but this result requires linear expansion costs. Cardin and Hu (2015) and Zhao et al. (2018) studied MCEPs with nonlinear expansion costs and used power functions to model economies of scale.
2.2 Risk Measures

The objectives in the aforementioned papers are all risk-neutral. To capture the risk preferences of real decision makers, a variety of risk measures have been reported in the literature including: utility functions, mean-variance, value-at-risk (VaR), and CVaR. For example, Hugonnier and Morellec (2007) extended the standard real options analysis by introducing a utility function to address risk preferences. Birge (2000) incorporated utility functions into a general linear capacity planning model, and formulated the problem as a multi-stage stochastic programming problem. Compared to utility functions, CVaR is more intuitive and easier to specify. Decision makers can express their risk preferences by directly adjusting the percentile terms of gains or losses (Krokhmal et al., 2002), rather than choosing a utility function. In addition, CVaR is a coherent risk measure and thus has strong decision-theoretic support (Artzner et al., 1999). Furthermore, CVaR can be formulated as a convex optimization problem and it is thus more tractable (Rockafellar and Uryasev, 2000) compared to other risk-aware objectives. Maceira et al. (2015) applied CVaR to a multi-stage power generation planning problem, and solved it by combining a scenario-tree based method with stochastic dual dynamic programming. Applications of CVaR in capacity planning problems can be found in (Szolgayová et al., 2011; Delgado and Claro, 2013).

2.3 Solution Methods

Early work on MCEP modeled the problem as a Markov decision process (MDP) and solved it with (exact) dynamic programming (Wu and Chuang, 2010, 2012) or approximate dynamic programming (Zhao et al., 2017). These methods are subject to the curse of dimensionality; more specifically, the size of the action space of the MDPs grows exponentially in the number of facilities. Alternative solution methods for risk-averse MDP can be found in (Ruszczyński, 2010; Haskell and Jain, 2015), but these methods can be inefficient when the actions are discrete and high-dimensional.

In an alternative stream, scenario tree-based multi-stage stochastic programming has been widely applied to MCEP (Huang and Ahmed, 2009; Taghavi and Huang, 2016). In this method, the evolution of uncertain parameters is modeled as a scenario-tree, and the model is solved with decomposing by fixing the allocation plan (Huang and Ahmed, 2009), by a Benders decomposition-
based heuristic (Taghavi and Huang, 2016), or by Lagrangian relaxation (Taghavi and Huang, 2018). Nevertheless, the size of the scenario tree grows exponentially when the number of stages or the dimension of uncertain parameters increases.

To address this open problem, decision rule-based multi-stage stochastic programming was proposed. This method approximates the policy space with parameterized decision rules. Then, the focus is on optimizing the parameters of the decision rule rather than optimizing the policy itself. Well-known decision rules include linear and piecewise linear (Georghiou et al., 2015), but these rules may not be applicable to MCEP as the system is usually modular—i.e., the capacity is discrete. To solve MCEPs with discrete capacity, researchers have investigated if–then decision rules and proposed customized decomposition algorithms to optimize the parameters (Cardin et al., 2017a; Zhao et al., 2018). The solution technique in our present paper is similar to the branch-and-cut based decomposition (BAC-D) algorithm proposed in (Zhao et al., 2018), but our study differs from the previous literature in the following two respects. First, in our paper, we construct cuts for the master problem with the subgradients of the recourse function. These subgradients can be computed by solving some small-scale linear programs (LPs). Compared to the BAC-D algorithm that constructs cuts by solving large-scale LPs, our method is much less time-consuming and it can also provide more interpretation for the resulting policy. Second, both BAC-D and the subgradient method cannot ensure convergence to the global optimum, but we discuss how to improve the best-found solution by using a multi-cut version of our algorithm.

3 Model Formulation

For our multi-stage capacity expansion problem, we introduce a set of facilities \( \mathcal{N} \triangleq \{1, \ldots, N\} \), a set of customers \( \mathcal{I} \triangleq \{1, \ldots, I\} \), and a finite planning horizon \( \mathcal{T} \triangleq \{1, \ldots, T\} \). In each period, the demand generated by the customers is allocated to facilities subject to available capacity. Given observations of demand, the decision maker can expand the capacity at the end of each period. The objective is to maximize the net present value (ENPV) of the system by optimizing the capacity expansion policy. In this paper, we focus on capacity planning at the strategic level, so the following assumptions are made: 1) there is no temporary shutdown of the facilities and the contraction of
Table 1: Notations for the multi-stage MCEP

| Symbol | Description |
|--------|-------------|
| \(\mathcal{I}\) | Set of customers, \(i \in \mathcal{I}\) and \(|\mathcal{I}| = I\) |
| \(\mathcal{N}\) | Set of facilities, \(n \in \mathcal{N}\) and \(|\mathcal{N}| = N\) |
| \(\mathcal{T}\) | Set of time periods, \(t \in \mathcal{T}\) and \(|\mathcal{T}| = T\) |
| \(\mathcal{L}\) | Set of line segments of the piecewise expansion cost, \(l \in \mathcal{L}\) and \(|\mathcal{L}| = L\) |
| \(\Xi_t\) | Sample space of the uncertain demands in time \(t \in \mathcal{T}\) |
| \(\xi_{it}\) | Amount of demand generated from customer \(i \in \mathcal{I}\) in time \(t \in \mathcal{T}\); its vector form is \(\xi_t = (\xi_{1t}, \ldots, \xi_{It})\) such that \(\xi_t \in \Xi_t\) |
| \(K_{\max}^n\) | The maximum capacity of facility \(n \in \mathcal{N}\) that can be installed; its vector form is \(K_{\max}^n = (K_{\max}^1, \ldots, K_{\max}^N)\) |
| \(\mathbb{K}\) | The feasible set of capacity \(\mathbb{K} \triangleq \{K \in \mathbb{Z}_+^N | K \leq K_{\max}\}\) |
| \(K_{nt}\) | Capacity of facility \(n \in \mathcal{N}\) in time \(t \in \mathcal{T} \cup \{0\}\); its vector form is \(K_t = (K_{1t}, \ldots, K_{Nt})\) |
| \(K_t\) | Policies that from the historical demands \(\xi_{[t]}\) to the capacity in time \(t \in \mathcal{T}\) |
| \(\mathcal{K}\) | Set of policies that are feasible to MCEP |
| \(\gamma\) | Discount factor, \(0 < \gamma < 1\) |
| \(r_{int}\) | Unit revenue from satisfying customer \(i \in \mathcal{I}\) with facility \(n \in \mathcal{N}\) in time \(t \in \mathcal{T}\) |
| \(b_{it}\) | Unit penalty cost for unsatisfied customer \(i \in \mathcal{I}\) in time \(t \in \mathcal{T}\) |
| \(p_{nt}\) | Slope of the \(l^{th}\) line segment of the expansion costs corresponding to facility \(n \in \mathcal{N}\) in time \(t\) |
| \(q_{nt}\) | Intercept of the \(l^{th}\) line segment of the expansion costs corresponding to facility \(n \in \mathcal{N}\) in time \(t\) |
| \(z_{int}\) | Amount of demands allocated from customer \(i \in \mathcal{I}\) to facility \(n \in \mathcal{N}\) in time \(t \in \mathcal{T}\) |

capacity is not allowed; 2) the expansion lead time is negligible. The notation for our model is summarized in Table 1.

Let \(\mathbb{K} \triangleq \{K \in \mathbb{Z}_+^N | K \leq K_{\max}\}\) be the finite set of possible capacity levels, where \(K_{\max} = (K^1_{\max}, \ldots, K^N_{\max})\) is the vector of maximum possible capacity levels. Denote \(K_t \triangleq (K_{1t}, \ldots, K_{Nt}) \in \mathbb{K}\) as the vector of the installed capacity at the end of time period \(t \in \mathcal{T} \cup \{0\}\), and \(K_{[t]} \triangleq (K_0, K_1, \ldots, K_t)\) as the history of installed capacity levels up to time \(t\). Denote \(\Delta K_{nt} \triangleq K_{nt} - K_{n(t-1)}\) and \(\Delta K_t \triangleq (\Delta K_{1t}, \ldots, \Delta K_{Nt})\) as the change in capacity at time \(t \in \mathcal{T}\). Without loss of generality, we assume that no capacity is installed at the beginning so that \(\Delta K_0 = K_0\).

Denote \(\Xi_t \subset \mathbb{R}^I\) as the sample space for the customer demand in time \(t \in \mathcal{T} \cup \{0\}\), and \(\xi_t \triangleq (\xi_{1t}, \ldots, \xi_{It})\) as the realized demand such that \(\xi_t \in \Xi_t\). Without loss of generality, we assume that the demand at \(t = 0\), i.e. \(\xi_0 \in \Xi_0\), is known. We further denote \(\xi_{[t]} \triangleq (\xi_0, \xi_1, \ldots, \xi_t)\) as the history of demand up to time \(t\), \(\Xi \triangleq \times_{t=0}^{T} \Xi_t\) to be the set of all possible demand realizations, and \(\xi \triangleq \xi_{[T]}\).
3.1 Policies for Flexible MCEPs

In the flexible MCEP, capacity decisions are made sequentially based on observations of demand. Specifically, $K_t$ is a mapping from historical demand $\xi[t]$ to capacity expansion decisions, for all $t \in T$. We denote the policies for the flexible MCEP as

$$K_t : \Xi_0 \times \cdots \times \Xi_t \mapsto \mathbb{K}, \quad \forall t = 0, \ldots, T.$$ 

We require that $K_t$ for all $t \in T \cup \{0\}$ be non-anticipative; the capacity decision $K_t = K_t(\xi[t])$ in time $t$ may only depend on $\xi[t]$ (it does not have access to future information).

We further denote $K_{\lfloor t \rfloor} \triangleq (K_0, \ldots, K_{\lfloor t \rfloor})$ as the policies up to time $t$. As contraction of capacity is not allowed, the set of feasible policies for the flexible MCEP is

$$\bar{K} \triangleq \{ (K_0, \ldots, K_T) | K_{t-1}(\xi_{t-1}) \leq K_t(\xi_t), \forall t \in T, \xi \in \Xi \}.$$ 

3.2 Profits and Costs

Denote $\Pi_t (K_{t-1}, \xi_t)$ as the profit given the realized demand $\xi_t$ and the installed capacity $K_{t-1}$. Denote $z_{\text{int}}$ as the demand allocated from customer $i \in I$ to facility $n \in N$ in time $t \in T$. Then, $\Pi_t (K_{t-1}, \xi_t)$ is given by the value of the following linear program:

$$\Pi_t (K_{t-1}, \xi_t) \triangleq \max_z \sum_{i \in I} \sum_{n \in N} r_{int} z_{int} - \sum_{i \in I} b_{it} \left( \xi_{it} - \sum_{n \in N} z_{int} \right)$$

subject to:

$$\sum_{n \in N} z_{int} \leq \xi_{it}, \quad \forall i \in I,$$

$$\sum_{i \in I} z_{int} \leq K_{n(t-1)}, \quad \forall n \in N,$$

$$z_{int} \geq 0, \quad \forall i \in I, n \in N,$$

where $r_{int}$ is the unit revenue for satisfying customer $i$’s demand with facility $n$, and $b_{it}$ is the unit penalty for unsatisfied demand of customer $i$. We do not enforce the constraint that all demands must be met.

Denote $c_t (\Delta K_t)$ for all $t \in T \cup \{0\}$ as the capacity expansion cost given $\Delta K_t$, we assume $c_t(\cdot)$ is piecewise linear. Denote $L \triangleq \{1, \ldots, L\}$ as a set of indices for $L$ line segments, and denote
\( \left( a_n, \ldots, a_n(L+1) \right) \) as a set of breakpoints for the expansion costs for facility \( n \in \mathcal{N} \) such that \( a_n = 0 \) and \( K_n^\text{max} < a_n(L+1) \). Let \( p_{nl} \) and \( q_{nl} \) be the slope and intercept of the \( l^{th} \) line segment of the expansion costs for facility \( n \) in time \( t \). The cost function is then:

\[
c_t (\Delta K_t) \triangleq \left\{ \sum_{n \in \mathcal{N}} c_{nt} (\Delta K_{nt}) \right\}_{\Delta K_{nt} = p_{nl} \cdot \Delta K_{nt} + q_{nl}, \; \forall l \in \mathcal{L}},
\]

\( \forall t \in T \cup \{0\} . \) \hspace{1cm} (2)

Note that \( c_t (\cdot) \) can be \textit{concave} in \( \Delta K_t \), as the expansion costs may enjoy the economies of scale. Eq. (2) can represent/approximate a variety of concave cost functions; for example, the fixed-charge function and the power function (Van Mieghem, 2003).

**Example 1.** The fixed-charge function addresses economies of scale—that is, the system incurs a fixed cost whenever capacity is expanded, and the remaining costs increase linearly. We set the number of line segments to be two and take \([a_n, a_n(2)] = (0, 1)\) and \([a_n, a_n(3)] = (1, K_n^\text{max} + 1)\), and \( p_2 < p_1 \) and \( q_2 > q_1 = 0 \).

**Example 2.** The power cost function can be represented by \((\Delta K_{nt})^v\) where \( 0 \leq v < 1 \) is the factor for the economies of scale. It can be approximated by piecewise linear functions; in addition, the relative gap between the original cost function and the approximate one is smaller than 1.5\% if we take the line segments and breakpoints to be \((a_1, a_2, a_3, \ldots, a_{L+1}) = (0, 2^0, 2^1, \ldots, 2^{L-1})\) for \( L \) satisfying \( 2^{L-2} \leq \max \{ K_1^\text{max}, \ldots, K_N^\text{max} \} \leq 2^{L-1} \) (Zhao et al., 2018).

**Remark 1.** Essentially, Eq. (2) can formulate arbitrary proper cost functions. As \( \Delta K_{nt} \) is finite \((i.e. \; \Delta K_{nt} \in \{0, 1, \ldots, K_n^\text{max}\})\), we can set \( L = K_n^\text{max} + 1 \) so that each line segment corresponds to a specific expansion cost at point \( \Delta K_{nt} \).

We denote the discount factor as \( 0 < \gamma \leq 1 \). Given policies \( K_{[T]} \in \bar{\mathcal{K}} \) and the profit/cost structure described above, the cumulative future costs from stages \( t = 0 \) to \( t = T \) for a particular \( \xi \in \Xi \) are

\[
Q \left( K_{[T]}, \xi \right) \triangleq \left\{ c_0 (K_0) + \sum_{t=1}^T \gamma^t \left( c_t (K_t - K_{t-1}) - \Pi_t (K_{t-1}, \xi_t) \right) \right\}_{K_t = K_t (\xi_{[t]}), \; \forall t \in T \cup \{0\}, \; \xi \in \Xi}.
\]

The overall objective of the MCEP is to minimize \( Q \left( K_{[T]}, \xi \right) \) (or equivalently, to maximize \( -Q \left( K_{[T]}, \xi \right) \)).
3.3 A Risk-Averse Flexible MCEP

In this subsection, we first formally present the risk-neutral flexible MCEP and then its risk-averse counterpart. If the decision maker is risk-neutral, the objective of the flexible MCEP is to find the expansion policy that maximizes the ENPV:

$$
\text{ENPV}_{\text{flex}} \triangleq \max_{K \in K} \mathbb{E} \left[ -Q \left( K_{[T]}, \xi \right) \right].
$$

The costs in the above model can be especially high for some particular realizations of $\xi$, especially when the variances of demand is high. We incorporate CVaR into the objective as a remedy, the definition of CVaR is next.

**Definition 1.** (Sarykalin et al., 2008) Denote $X$ as a continuous random variable and $F_X(y) = P\{X \leq y\}$ for all $y \in \mathbb{R}$ as its cumulative distribution function.

(i) The VaR of $X$ at confidence level $\alpha \in (0, 1)$ is

$$
\text{VaR}_\alpha(X) \triangleq \inf \{ y | F_X(y) \geq \alpha \}.
$$

(ii) The CVaR of $X$ at confidence level $\alpha \in (0, 1)$ is

$$
\text{CVaR}_\alpha(X) \triangleq \int_{-\infty}^{\infty} y dF_X^{\alpha}(y),
$$

where

$$
F_X^{\alpha}(y) \triangleq \begin{cases} 
0, & \text{when } y < \text{VaR}_\alpha(X), \\
\frac{F_X(y) - \alpha}{1 - \alpha}, & \text{when } y \geq \text{VaR}_\alpha(X). 
\end{cases}
$$

For a continuous random variable $X$ and a confidence level $\alpha \in (0, 1)$, $\text{CVaR}_\alpha(X)$ is the conditional expectation of $X$ greater than or equal to $\text{VaR}_\alpha(X)$. This coincides the definition of “expected shortfall” (Acerbi, 2002). If $\alpha \to 1$, $\text{CVaR}_\alpha(X)$ approaches the worst-case cost; whereas if $\alpha \to 0$, $\text{CVaR}_\alpha(X)$ approaches the expectation of $X$.

We introduce a mean-CVaR objective for the MCEP by introducing a weight factor $0 \leq \beta \leq 1$
(Shapiro, 2011) to obtain:

$$\max_{K[T] \in K} \beta \mathbb{E}\left[ -Q(K[T], \xi) \right] + (1 - \beta) \left[ -\text{CVaR}_\alpha \left( Q(K[T], \xi) \right) \right].$$

(4)

In this formulation, decision makers can compromise between risk-neutral and risk-averse policies by adjusting the weight factor $\beta$. If we choose $\beta = 1$, we recover the original risk-neutral model; conversely, if $\beta = 0$, we minimize $\text{CVaR}_\alpha \left( Q(K[T], \xi) \right)$.

**Remark 2.** The objective in Eq. (4) can be understood as a weighted sum of two “expected shortfall” measures (since the expectation is an expected shortfall for $\alpha = 0$). Expected shortfall is a spectral risk measure (Acerbi, 2002), so the objective of Eq. (4) is essentially a special case of the spectral risk measure (which is also coherent).

Based on (Rockafellar and Uryasev, 2000, Theorem 2), we can introduce an auxiliary variable $u \in \mathbb{R}$ and calculate $\text{CVaR}_\alpha \left( Q(K[T], \xi) \right)$ by solving the following optimization problem:

$$\text{CVaR}_\alpha \left( Q(K[T], \xi) \right) = \inf_{u \in \mathbb{R}} \left\{ u + \frac{1}{1 - \alpha} \mathbb{E} \left[ Q(K[T], \xi) - u \right]_+ \right\},$$

$$= -\sup_{u \in \mathbb{R}} \left\{ -u - \frac{1}{1 - \alpha} \mathbb{E} \left[ Q(K[T], \xi) - u \right]_+ \right\},$$

where $[\cdot]_+$ denotes $\max \{\cdot, 0\}$. Problem (4) is then equivalent to

$$\text{ENPV}_\alpha (\beta) \triangleq \max_{u \in \mathbb{R}, K[T] \in K} - (1 - \beta) u - \mathbb{E} \left[ \beta Q(K[T], \xi) + \frac{1 - \beta}{1 - \alpha} \left[ Q(K[T], \xi) - u \right]_+ \right],$$

(5)

using the variational form of CVaR. Like Problem (3), Problem (5) is a multi-stage stochastic programming problem. If $\beta = 1$, Problem (3) is equivalent to Problem (5). In addition, we have the following result since $\text{CVaR}_\alpha (X) \geq \mathbb{E} [X]$ for all $\alpha \in (0, 1)$ for any continuous random variable $X$.

**Proposition 1.** Given Problems (3) and (5):

(i) $\text{ENPV}_\alpha (\beta)$ is non-decreasing in $\beta$ given any $\alpha \in (0, 1)$.

(ii) $\text{ENPV}_{\text{flex}} \geq \text{ENPV}_\alpha (\beta)$ for any $\beta \in [0, 1], \alpha \in (0, 1)$. 

11
Proposition 1 states that \( \text{ENPV}_\alpha (\beta) \) will not exceed \( \text{ENPV}_{\text{flex}} \) given appropriately chosen \( \alpha \) and \( \beta \). Furthermore, as \( \beta \) decreases, decision makers put more emphases on minimizing CVaR rather than minimizing expected cost, and so \( \text{ENPV}_\alpha (\beta) \) decreases. Our interpretation is that, as the decision maker becomes more risk-averse, it will tend to underestimate system performance.

From another perspective, we often want to estimate the value of flexibility over an inflexible benchmark problem (i.e. the system has no capacity adjustment options, detailed discussions on this problem appear in Section 6). Problems (3) and (5) have the same level of flexibility (both have capacity adjustment options and facility switching options), and so the willingness of the decision maker to enable flexibility may decrease as risk-aversion increases.

4 Decision Rules

Problem (5) is a multi-stage stochastic programming problem with a non-convex objective. It is widely believed that multi-stage stochastic programming is in general “computationally intractable already when medium-accuracy solutions are sought” (Shapiro and Nemirovski, 2005).

To develop a tractable solution strategy, we approximate \( \bar{K} \) with decision rules. That is, we restrict the policy space to a class of parameterized functions \( \tilde{K} (\Theta) \), where \( \Theta \subset \mathbb{R}^{\text{dim}(\Theta)} \) is some admissible set of the parameters. Then, we can optimize the parameters \( \theta \in \Theta \) which determine the decision rule \( \tilde{K} (\Theta) \), instead of optimizing over all non-anticipative policies in \( \bar{K} \). Of course, \( \tilde{K} (\Theta) \subset \bar{K} \), and so a decision rule may not be optimal for the original problem. However, decision rules offer significant computational advantages as well as managerial insight. In particular, they are far more accessible to non-experts in practice.

4.1 If–Then Decision Rules

We focus on if–then decision rules to approximate the policy space \( \tilde{K} \). The motivation for our choice of if–then decision rules is threefold. First, expansion decisions are binary by their very nature—the capacity is either expanded or not. Also, the output of the decision rule should be integral since capacity is discrete, so a nonlinear decision rule is required. Second, some optimal if–then policies for capacity expansion problems have been reported in the literature. For example, the invest-stay
put-disinvest policy from (Eberly and Van Mieghem, 1997) and the (s, S) policy from (Angelus et al., 2000) are both optimal for their respective problems. Therefore, it is reasonable to speculate that an optimal if-then decision rule should at least have good performance from the perspective of Problem (5). Third, if-then decision rules mimic the behavior of human beings and are more intuitive in practical implementation (Cardin et al., 2017a).

An if–then decision rule in a single facility setting is stated as: if the capacity gap of the facility exceeds a threshold, then we expand capacity up to a certain level. However, in multi-facility problems the dimensions of capacity levels and the demands may not be equal, and so we need to transform one to make it comparable to the other in order to calculate the capacity gap. We take $W \in \mathbb{R}^{I \times N}$ as a preset weight matrix such that $W^T \xi \in \mathbb{R}^N$. Then, we may compute weighted capacity gaps $\left\lfloor W^T \xi \right\rfloor - K_{t-1}$, where $\lfloor \cdot \rfloor$ denotes rounding to the nearest integer. These weighted capacity gaps are the trigger condition for our if–then rules.

The design of the weight matrix $W$ is case-specific. From a managerial point of view, the entry of the weight matrix $W_{in}$ can be interpreted as the profit coefficient of customer $i$ with respect to facility $n$. In other words, if the per unit demand from customer $i$ is more profitable for facility $n$, then we should tend to allocate more demand from customer $i$ to facility $n$ and $W_{in}$ should be larger.

**Example 3.** If the numbers of facilities and customers are the same (i.e. $I = N$) then there is a bijective map from the facility/customer to its most profitable customer/facility counterpart, and the weight matrix can be chosen as an $N \times N$ identity matrix. In this case, the weighted capacity gap of a facility is calculated in terms of subtracting the current capacity from the most profitable demand.

**Example 4.** For more general cases, $W$ can be designed by calculating and weighting the profit coefficients (i.e. $r_{int} + b_{it}$) of the allocation model $\Pi_t (\cdot)$. We refer interested readers to (Zhao et al., 2018) for further discussion of this choice.

Denote $\theta_{1,n}$ as the capacity adjustment parameter and $\theta_{2,nt}$ as the threshold parameter, and define the parameter vectors $\theta_1 \triangleq (\theta_{1,1}, \ldots, \theta_{1,N})$ and $\theta_2 \triangleq (\theta_{2,11}, \ldots, \theta_{2,NT})$. The admissible sets
for parameters $\theta_1$ and $\theta_2$ are

$$
\Theta_1 \triangleq \left\{ \theta_1 \in \mathbb{Z}^N | 0 \leq \theta_1 \leq \theta_{1}^{\max} \right\} \quad \text{and} \quad \Theta_2 \triangleq \left\{ \theta_2 \in \mathbb{R}^{N \times T} | 0 \leq \theta_2 \leq \theta_{2}^{\max} \right\},
$$

respectively, and we further define $\theta \triangleq (\theta_1, \theta_2)$ and $\Theta \triangleq \Theta_1 \times \Theta_2$ for succinctness. Then, we may define our if–then decision rule as follows: for all $n \in \mathcal{N}$, $t \in \mathcal{T}$, $\xi \in \Xi$, $K_{t-1} \in \mathbb{K}$,

$$
\tilde{K}_{nt} \left( K_{n(t-1)} ; \xi ; \theta \right) \triangleq \begin{cases} 
\sum_{i \in \mathcal{I}} W_{in} \xi_{it} & \text{if } \sum_{i \in \mathcal{I}} W_{in} \xi_{it} - K_{n(t-1)} \geq \theta_{2,nt} \text{ and } \sum_{i \in \mathcal{I}} W_{in} \xi_{it} + \theta_{1,n} \leq K_{n}^{\max}, \\
K_{n(t-1)} & \text{otherwise}.
\end{cases}
$$

(6)

This decision rule states that if the weighted capacity gap of facility $n$ (i.e. $\sum_{i \in \mathcal{I}} W_{in} \xi_{it} - K_{n(t-1)}$) exceeds the threshold $\theta_{2,nt}$ and the expanded capacity does not exceed the maximum capacity, then we expand the capacity of facility $n$ up to level $\sum_{i \in \mathcal{I}} W_{in} \xi_{it} + \theta_{1,n}$. Otherwise, the capacity is unchanged. Note that the parameters $\theta_{2,nt}$ are continuous and allowed to vary with time (the purpose of which is to enable a better approximation of the true optimal policy). In addition, $\theta_{1,n}$ is integral so the decision rule automatically yields integral expansion decisions. Now with Policy (6), we want to optimize the scenario-independent parameters $\theta_1$ and $\theta_2$ with respect to the mean-CVaR.

### 4.2 Risk-Averse MCEP with Decision Rule

Denote $\tilde{K}_{[T]} (\theta) \triangleq \left( \tilde{K}_0, \tilde{K}_1 (\cdot ; \theta), \ldots, \tilde{K}_T (\cdot ; \theta) \right)$ for $\theta \in \Theta$ as the vector of parameterized if–then decision rules encoded by Policy (6). Note that $\mathcal{K}_0 : \Xi_0 \mapsto \mathbb{K}$ is the policy for the initial capacity decision $K_0$, and is independent of $\theta$. Then, given Policy (6), Problem (5) can be approximated with the following model

$$
\max_{\theta \in \Theta, u \in \mathbb{R}} - (1 - \beta) u - \mathbb{E} \left[ \beta Q \left( \tilde{K}_{[T]} (\theta) , \xi \right) + \frac{1 - \beta}{1 - \alpha} \left[ Q \left( \tilde{K}_{[T]} (\theta) , \xi \right) - u \right]_+ \right],
$$

(7)
which optimizes over policies of the form (6). We now want to optimize the policy parameters $\theta$, rather than the policy $K \in \mathcal{K}$, which is more tractable.

4.3 MILP Transformations of the Decision Rule-based Model

To transform Problem (7) into a computationally workable form, we apply sample average approximation and linearization to transform Problem (7) into an MILP.

Denote $S \triangleq \{1, \ldots, S\}$ as a set of scenario indices, and $\{\xi^1, \ldots, \xi^S\}$ as a set of sample paths generated via Monte-Carlo simulation of customer demand, where $\xi^s \triangleq (\xi^s_1, \ldots, \xi^s_T)$ for all $s \in S$. Now, we introduce a set of binary auxiliary variables $\delta^s_t \triangleq (\delta^s_{1t}, \ldots, \delta^s_{Nt})$ and transform the if-then decision rule $\tilde{K}_t(K_{t-1}, \xi^s_t; \theta)$ into constraints. Let $\delta^s_{nt}$ be a binary variable such that for all $t \in T$, $s \in S$, the capacity of facility $n$ is expanded to $\lfloor W^\top \xi^s_t \rfloor + \theta_{1n}$ if $\delta^s_{nt} = 1$; otherwise, the capacity is unchanged. This effect is achieved by the following Big-M constraints:

$$
\begin{align*}
    W^\top \xi^s_t - K^s_t - \theta_{2t} &\geq (M\mathbb{I}_{N \times N})^\top (\delta^s_t - \mathbf{1}_N), \quad \forall t \in T, s \in S, \quad (8a) \\
    W^\top \xi^s_t - K^s_t - \theta_{2t} &< (M\mathbb{I}_{N \times N})^\top \delta^s_t, \quad \forall t \in T, s \in S, \quad (8b) \\
    K^s_t &\leq \lfloor W^\top \xi^s_t \rfloor + \theta_{1} + (M\mathbb{I}_{N \times N})^\top (\mathbf{1}_N - \delta^s_t), \quad \forall t \in T, s \in S, \quad (8c) \\
    K^s_t &\geq \lfloor W^\top \xi^s_t \rfloor + \theta_{1} + (M\mathbb{I}_{N \times N})^\top (\delta^s_t - \mathbf{1}_N), \quad \forall t \in T, s \in S, \quad (8d) \\
    K^s_t - K^s_{t-1} &\leq (M\mathbb{I}_{N \times N})^\top \delta^s_t, \quad \forall t \in T, s \in S, \quad (8e) \\
    K^s_t &\leq K_{\text{max}}, \quad \forall t \in T, s \in S, \quad (8f) \\
    \delta^s_t &\in \{0,1\}^N, \quad \forall t \in T, s \in S, \quad (8g)
\end{align*}
$$

where $M \gg 0$ is a large constant, $\mathbb{I}_{N \times N}$ is an $N \times N$ identity matrix, and $\mathbf{1}_N$ denotes an $N$-dimensional vector of ones. In addition, the right-hand sides of Policy (6) are integral so that Constraints (8) map from continuous capacity levels to discrete capacity expansion decisions.

Note that the nonlinear term $[Q(\tilde{K}_{[T]}(\theta), \xi^s) - u]_+$ in the objective of Problem (7). To linearize it, we examine its epigraph by introducing auxiliary variables $\eta^s \in \mathbb{R}$ for all $s \in S$, such that $\eta^s \geq Q(\tilde{K}_{[T]}(\theta), \xi^s) - u$ and $\eta^s \geq 0$. Problem (7) is subsequently approximated via the aforementioned
Minimize \( \min_{K_0,u,\theta} \beta c_0(K_0) + (1 - \beta) u + \frac{1}{S} \sum_{s=1}^{S} \left[ \beta \sum_{t=1}^{T} \gamma^t \left( c_t \left( K_t^s - K_{t-1}^s \right) - \Pi_t \left( K_{t-1}^s, \xi_t^s \right) \right) + \frac{1 - \beta}{1 - \alpha} \eta^s \right] \) \tag{9a}

subject to \( (8a) \) to \( (8g) \) \hspace{1cm} \forall t \in T, s \in S, \tag{9b}

\[ \eta^s \geq c_0(K_0) + \sum_{t=1}^{T} \gamma^t \left( c_t \left( K_t^s - K_{t-1}^s \right) - \Pi_t \left( K_{t-1}^s, \xi_t^s \right) \right) - u, \forall s \in S, \tag{9c} \]

\[ \eta^s \in \mathbb{R}_+, \quad K_t^s \in \mathbb{R}^N, \quad \forall t \in T, s \in S, \tag{9d} \]

\[ K_0 = K_0, \quad \forall s \in S, \tag{9e} \]

\[ K_0 \in \mathbb{K}, \quad u \in \mathbb{R}, \quad \theta \in \Theta. \tag{9f} \]

Problem \( (9) \) now appears as an MILP. In this MILP, Constraint \( (9e) \) appears to simplify notation.

### 5 Subgradient-based Decomposition Algorithm

In this section, we propose a decomposition algorithm for Problem \( (9) \). We first reformulate Problem \( (9) \) as a “two-stage” stochastic programming problem. Then, making use of this structure, we decompose the “second-stage” over scenarios, compute the sub-gradients of the recourse function, and then update the parameters of the decision rule. This decomposition algorithm may not converge to the global optimum, so we later introduce a multi-cut procedure to improve upon the best-found solution of this algorithm.

#### 5.1 Two-Stage Decomposition

Problem \( (9) \) can be treated as a “two-stage” stochastic programming problem. The first-stage decisions consist of the parameters \( \theta \), the initial capacity \( K_0 \), and the auxiliary variable \( u \) from the variational form of CVaR. These decisions are all scenario-independent (i.e. here-and-now decisions):

\[
\min_{K_0,u,\theta} \beta c_0(K_0) + (1 - \beta) u + \frac{1}{S} \sum_{s=1}^{S} R^s(K_0, u, \theta), \quad \text{s.t. (9f)} \tag{10}
\]
where $R^s(K_0, u, \theta)$ is the recourse function of the second-stage problem on scenario $s \in S$. The recourse function here actually returns the multi-period revenue over time periods $t = 1, \ldots, T$. It determines the capacity plan $(K^s_1, \ldots, K^s_T)$ and the auxiliary variables $\eta^s$, which are scenario-dependent (i.e. wait-and-see decisions):

$$R^s(K_0, u, \theta) \triangleq \min_{\eta^s, \delta^s, K^s_{[T]}} \left\{ \beta \sum_{t=1}^T \gamma^t \left( c_t(K^s_t - K^s_{t-1}) - \Pi_t(K^s_{t-1}, \xi^s_t) \right) + \frac{1 - \beta}{1 - \alpha} \eta^s, \text{ s.t. (9b)–(9e)} \right\}. \quad (11)$$

This two-stage decomposition, however, is difficult to solve because the recourse function $R^s(K_0, u, \theta)$ is nonconvex due to the integer wait-and-see variables (i.e. $\delta^s_t$) and the nonconvex costs $c_t(\cdot)$.

Fortunately, this problem has appealing structure. Once the initial capacity $K_0$, the parameters $\theta$ of Policy (6), and $u$ are all fixed, we can determine $K^s_{[T]} \triangleq (K^s_1, \ldots, K^s_T)$ via Policy (6) given the demand vector $\xi^s$ on scenario $s \in S$. Once $K^s_{[T]}$ is known, the nonconvex costs $c_t(\cdot)$ for all $t \in T$ can all be determined by Eq. (2). Therefore, Problem (11) can be solved, and the subgradients of the recourse functions $R^s(K_0, u, \theta)$ with respect to $(K_0, u, \theta_1)$ can be computed.

We remark that the subgradients of $R^s(K_0, u, \theta)$ with respect to $\theta_2$ may not be attainable because of the trigger conditions in Policy (6). For counter-example, consider $\theta_{2,nt} = 10$ and a simple if–then rule: if $\xi_{it} \geq \theta_{2,nt}$, then expand the capacity of facility $n$ to 10. In this case, the dependence of the recourse function on $\theta_{2,nt}$ is implicit since the rule is triggered for any $\xi_{it} > 10$, and the subgradient with respect to $\theta_{2,nt}$ is not available.

To address this difficulty, we take a primal decomposition and update $(K_0, u, \theta_1)$ and $\theta_2$ separately. To update $(K_0, u, \theta_1)$, we approximate the recourse functions by using cut generation. Specifically, we solve the epigraph formulation of Problem (10) in each iteration $m$:

$$\min_{K_0, u, \theta_1, y} \beta c_0(K_0) + (1 - \beta) u + y \quad (12a)$$

s.t. $y \geq \left( \phi^{m'} \right)^\top (K_0, u, \theta_1) + \phi^{m'}_0, \quad \forall m' = 1, \ldots, m - 1, \quad (12b)$

$$K_0 \in \mathbb{K}, \quad u \in \mathbb{R}, \quad \theta_1 \in \Theta_1, \quad y \in \mathbb{R}, \quad (12c)$$

where Eq. (12b) contains the cuts generated up to iteration $m - 1$. Essentially, we try to use Eq. (12b) to approximate the recourse functions from below. Problem (12) is an MILP with $N + N \times L$ integer variables, which can be directly solved via commercial solvers.
Our algorithm contains three major steps. Denote \((K_0^m, u^m, \theta_1^m)\) as the optimal solution of Problem (12).

- **Step 1**: Fix \((K_0^m, u^m, \theta_1^m)\) and compute \(\theta_2^m\) by stochastic approximation. We construct and solve single-scenario problems, and then iteratively average the resulting optimal \(\theta_2\).

- **Step 2**: Once \((K_0^m, u^m, \theta_1^m, \theta_2^m)\) are fixed, we compute subgradients of the recourse functions \(R^s(\cdot)\) for all \(s \in S\) at \((K_0^m, u^m, \theta_1^m)\), and construct subgradient cuts.

- **Step 3**: We add the subgradient cut to Eq. (12b), and compute \((K_0^{m+1}, u^{m+1}, \theta_1^{m+1})\) by solving Problem (12). The algorithm then proceeds iteratively until our termination conditions are met.

The overall framework of our algorithm is somewhat similar to the one in (Zhao et al., 2018). One of the main differences is that in the second step, we solve small-scale LPs and compute the subgradients analytically. This way is more efficient than solving large-scale LPs. Figure 1 gives a flow chart of our algorithm.

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**Figure 1**: Flow chart of the subgradient-based decomposition algorithm.
5.2 Subgradient-based Decomposition Algorithm

Step 1: Update $\theta_2$ via Stochastic Approximation

Suppose we have fixed the first-stage decisions $(K^m_0, u^m, \theta_1^m)$ in the $m^{th}$ iteration, and let $k \geq 0$ be a counter for the iterations of stochastic approximation. We assign equal probability to each scenario in $S$, and randomly select one scenario $\{\xi_{s_k}^1, \ldots, \xi_{s_k}^T\}$ for $s_k \in S$ without replacement. Then, we construct the following single-scenario problem given $(K^m_0, u^m, \theta_1^m)$ and $\xi_k$:

$$\begin{align*}
\min_{K[T], \eta, \theta_2} & \quad \beta \sum_{t=1}^{T} \gamma^t (c_t (K_t - K_{t-1}) - \Pi_t (K_{t-1}, \xi_{s_k}^t)) + \frac{1 - \beta}{1 - \alpha} \eta + c_\theta \sum_{t \in T} \sum_{n \in N} \theta_{2,nt} \\
\text{s.t.} & \quad K_t = \tilde{K}_t (K_{t-1}, \xi_{s_k}^t; (\theta_1^m, \theta_2)), \quad \forall t \in T, \\
& \quad \eta \geq c_0 (K^m_0) + \sum_{t=1}^{T} \gamma^t (c_t (K_t - K_{t-1}) - \Pi_t (K_{t-1}, \xi_{s_k}^t)) - u^m, \\
& \quad \eta \in \mathbb{R}_+, \quad \theta_2 \in \Theta_2, \quad K_0 = K^m_0, \quad K[T] \in \mathbb{R}^{N \times (T+1)},
\end{align*}$$

(13)

where $c_\theta$ is a small positive constant intended to regularize $\theta_2$ (this way the optimal $\theta_{2,nt}$ in Problem (13) is not a large number when facility $n$ is not expanded in time $t$). The objective of the above problem is to find the optimal $\theta_2$ such that the cumulative costs given sample path $\{\xi_{s_k}^1, \ldots, \xi_{s_k}^T\}$ are minimized. Since $(K^m_0, u^m, \theta_1^m)$ are fixed and Policy (6) yields integral decisions, the $K[T]$ in Problem (13) are continuous variables.

Remark 3. Note that in Problem (13), we have binary auxiliary variables in $c_t (\cdot)$ and we need to transform $\tilde{K}_t (\cdot)$ into Big-M constraints, so that Problem (13) is an MILP with $N \times (L + 1) \times T$ binary variables. For modestly-sized problem instances, this MILP can be directly solved by commercial solvers. For example, in our upcoming numerical study, our problem has $N = 5$, $L = 5$, and $T = 15$, and 450 binary variables. In this case, Problem (13) can be solved by CPLEX within seconds.

Let $\hat{\theta}_2^{m,k} \triangleq \left(\hat{\theta}_2^{m,k,1}, \ldots, \hat{\theta}_2^{m,k,N_T}\right)$ be the optimal $\theta_2$ of Problem (13) given sample path $\{\xi_{s_k}^1, \ldots, \xi_{s_k}^T\}$. Then, we update $\theta_2$ by the following rule:

$$\hat{\theta}_2^{m,k+1} = \hat{\theta}_2^{m,k} + \sigma_k (\hat{\theta}_2^{m,k} - \bar{\theta}_2^{m,k}), \quad \forall k \geq 0,$$

(14)
where \{\sigma_k\}_{k \geq 0} is a sequence of learning rates that satisfies \(\sum_{k=1}^{\infty} \sigma_k = \infty\) and \(\sum_{k=1}^{\infty} \sigma_k^2 = \infty\). Given a preset precision and a minimum number of iterations \(\bar{k}\), we terminate the stochastic approximation update when \(\|\bar{\theta}_2^{m,k+1} - \bar{\theta}_2^{m,k}\|_\infty \leq \epsilon\) or \(k \geq \bar{k}\), where \(\|\cdot\|_\infty\) denotes the sup-norm. The above update rule is essentially the stochastic approximation algorithm of (Robbins and Monro, 1951).

The advantage is that we can derive an approximate \(\theta_2\) by evaluating a portion of, rather than all of, the scenarios in \(S\).

The update rule by Eq. (14) may underestimate \(\theta_2\). Suppose the stochastic approximation algorithm terminates after \(k^*\) iterations. Denote \(\hat{K}^{m,k}_{nt}\) for all \(n \in N, t \in T\) to be the optimal capacity decisions for Problem (13) given the selected scenario in the \(k^{th}\) iteration, and let \(\hat{\delta}^{m,k}_{nt} = 1\) if \(\hat{K}^{m,k}_{nt} > \hat{K}^{m,k}_{n(t-1)}\) and \(\hat{\delta}^{m,k}_{nt} = 0\) otherwise. Then, if for all of the selected scenarios, facility \(n\) is not expanded in time \(t\) (i.e. \(\hat{\delta}^{m,k'}_{nt} = 0\) for all \(k' = 1, \ldots, k^* - 1\)), the approximate \(\bar{\theta}^{m,k^*}_{2,nt}\) may be an underestimate. This is because \(\bar{\theta}^{m,k^*}_{2,nt} \leq \max\{\hat{\theta}^{m,1}_{2,nt}, \ldots, \hat{\theta}^{m,k^*-1}_{2,nt}\}\) if it is updated via Eq. (14). To fix this issue, we take \(M_\theta \gg 0\) to be a large number such that \(M_\theta \geq \hat{\theta}^{m,k'}_{2,nt}\) for all \(k' = 1, \ldots, k^* - 1\).

Then, the optimal \(\theta_2\) given \((K^0_0, u^m, \theta^m_1)\) can be computed by

\[
\theta^{m}_{2,nt} = \begin{cases} 
M_\theta & \text{if } k^*-1 \sum_{k'=1}^{k^*-1} \delta^{m,k'}_{nt} = 0, \\
\bar{\theta}^{m,k^*}_{2,nt} & \text{otherwise},
\end{cases} 
\quad \forall n \in N, t \in T.
\]

**Step 2: Calculating the Subgradients of the Recourse Function**

Once \(\theta^{m}_{2}\) is computed, we can fix \((K^0_0, u^m, \theta^m_1, \theta^m_2)\) and determine the integer wait-and-see variables in Problem (11), and then calculate the subgradients of the recourse functions with respect to \((K^0_0, u^m, \theta^m_1)\). According to Eq. (11), we need to compute the subgradients of \(c_t(\cdot)\) and \(\Pi_t(\cdot)\) in order to compute the subgradient of \(R^s(K^0_0, u^m, \theta^m)\). Therefore, in this step, we first derive the closed-form of the capacity decisions \(K_t\) with respect to \((K^0_0, u^m, \theta^m_1)\). Subsequently, we calculate the subgradients \(\partial c_t(\cdot)\) and \(\partial \Pi_t(\cdot)\).

First, recall that \(\delta^{s}_{nt}\) is the binary variable in Eq. (8) which indicates whether the capacity decision of facility \(n\) is triggered in time \(t\). Given \((K^0_0, u^m, \theta^m_1)\) in the \(m^{th}\) iteration and scenario \(\{\xi^s_1, \ldots, \xi^s_T\}\) for \(s \in S\), the expansion decisions \(\delta^{m,s}_{nt}\) and \(K^{m,s}_{nt}\) can be determined sequentially from
$t = 1$ to $T$ by:

$$
\delta_{nt}^{m,s} = \begin{cases} 
1, & \text{if } \sum_{i \in I} W_{int} \xi_{sit}^- - K_{n(t-1)}^{m,s} \geq \theta_{2,nt}^m \text{ and } \sum_{i \in I} W_{int} \xi_{sit}^+ + \theta_{1,nt}^m \leq K_{n}^{m,s}, \\
0, & \text{otherwise,}
\end{cases}
$$

and

$$
K_{nt}^{m,s} = \delta_{nt}^{m,s} \left( \sum_{i \in I} W_{int} \xi_{sit}^+ + \theta_{1}^m \right) + (1 - \delta_{nt}^{m,s}) K_{n(t-1)}^{m,s}, \quad \forall t \in \mathcal{T},
$$

where we take $K_{n0}^{m,s} = K_0^m$ to simplify notation. We further denote

$$
h_{K,nt}^{m,s} \triangleq \prod_{t_0=1}^{t} (1 - \delta_{nt_0}^{m,s}), \quad \forall n \in \mathcal{N}, t \in \mathcal{T} \cup \{0\}, s \in \mathcal{S},
$$

$$
h_{\theta,nt}^{m,s} \triangleq \sum_{t_0=1}^{t} \prod_{t_1=t_0+1}^{t} (1 - \delta_{nt_1}^{m,s}) \delta_{nt_0}^{m,s}, \quad \forall n \in \mathcal{N}, t \in \mathcal{T} \cup \{0\}, s \in \mathcal{S},
$$

$$
h_{0,nt}^{m,s} \triangleq \sum_{t_0=1}^{t} \prod_{t_1=t_0+1}^{t} (1 - \delta_{nt_1}^{m,s}) \delta_{nt_0}^{m,s} \left( \sum_{i \in I} W_{int} \xi_{sit_0}^- \right), \quad \forall n \in \mathcal{N}, t \in \mathcal{T} \cup \{0\}, s \in \mathcal{S}.
$$

For simplicity, we take $\prod_{t_1=t_0}^{t_1} x_t = 1$ and $\sum_{t_1=t_0}^{t} x_t = 0$ for any $x_t$ if $t_0 > t_1$. Therefore, for $t = 0$, we have $h_{K,0}^{m,s} = 1$ and $h_{\theta,0}^{m,s} = h_{0,0}^{m,s} = 0$. The closed form of $K_{nt}^{m,s}$ with respect to $(K_0^m, u^m, \theta_1^m)$ follows.

**Lemma 1.** Given $(K_0^m, u^m, \theta_1^m)$, scenario $s \in \mathcal{S}$, and Eq. (15), we have

$$
K_{nt}^{m,s} = h_{K,nt}^{m,s} K_{n0}^m + h_{\theta,nt}^{m,s} \theta_{1,n}^m + h_{0,nt}^{m,s}, \quad \forall n \in \mathcal{N}, t \in \mathcal{T} \cup \{0\}.
$$

Next, we compute the subgradients of the expansion costs. Note that the expansion costs $c_t(\Delta K_t^{m,s})$ for all $t \in \mathcal{T}$ are deterministic since $K_{nt}^{m,s}$ are known. Let $\Delta K_{nt}^{m,s} = K_{nt}^{m,s} - K_{n(t-1)}^{m,s}$ for all $t \in \mathcal{T}$ and $\Delta K_{n0}^{m,s} = K_{n0}^m$ for all $s \in \mathcal{S}$, and define

$$
\tau_{nt}^{m,s} \triangleq \begin{cases} 
1, & \text{if } \Delta K_{nt}^{m,s} \in [a_{nl}, a_{n(l+1)}), \\
0, & \text{otherwise,}
\end{cases}
$$

In the above, $\tau_{nlt}^{m,s} = 1$ implies that $\Delta K_{nt}^{m,s}$ lies in the $l^{th}$ line segment of the expansion cost in Eq.
The slope and the intercept of the expansion cost for $\Delta K_{nt}^{m,s}$ are
\[
g_{nt}^{m,s} \triangleq \sum_{l=1}^{L} p_{nl} t_{nl}^{m,s}, \quad \text{and} \quad g_{0,nt}^{m,s} \triangleq \sum_{l=1}^{L} q_{nl} t_{nl}^{m,s}, \quad \forall n \in N, t \in T \cup \{0\}.
\]
Then, the subgradients of $c_t(\cdot)$ are as follows.

**Lemma 2.** Given $(K_{0}^{m,s}, u^{m}, \theta_{1})$, scenario $s \in S$, and Eq. (16), a subgradient of $c_t(\cdot)$ for all $t \in T \cup \{0\}$ at $(K_{0}^{m,s}, u^{m}, \theta_{1})$ is
\[
(\partial c_t^{m,s})^\top (K_{0}^{m,s}, u^{m}, \theta_{1}) + \sum_{n \in N} \left( g_{nt}^{m,s} h_{0,nt}^{m,s} - g_{nt}^{m,s} h_{0,n(t-1)}^{m,s} + g_{0,nt}^{m,s} \right),
\]
where
\[
\partial c_t^{m,s} = \begin{pmatrix}
g_{nt}^{m,s} h_{K,nt}^{m,s} - g_{nt}^{m,s} h_{K,n(t-1)}^{m,s} \quad \text{n} \in N \\
0 \\
g_{nt}^{m,s} h_{0,nt}^{m,s} - g_{nt}^{m,s} h_{0,n(t-1)}^{m,s} \quad \text{n} \in N
\end{pmatrix}, \quad \forall t \in T, \quad \text{and} \quad \partial c_0^{m,s} = \begin{pmatrix}
\left( g_{nt}^{m,s} h_{K,n}^{m,s} \right)_{n \in N} \\
0 \\
\left( g_{nt}^{m,s} h_{0,n}^{m,s} \right)_{n \in N}
\end{pmatrix}.
\]

We now compute the subgradients of the profit. As $\Pi_t(K_{t-1}, \xi_t)$ is given by an LP, the subgradient of $\Pi_t(K_{t-1}, \xi_t)$ with respect to $(K_{0}^{m,s}, u^{m}, \theta_{1})$ can be computed from the dual to Problem (1). Let $(\mu_{nt}^{m,s})_{n \in N, t \in T}$ and $(\psi_{it}^{m,s})_{n \in I, t \in T}$ be the optimal dual variables with respect to the capacity and demand constraints of $\Pi_t(\cdot)$, respectively.

**Lemma 3.** Given $(K_{0}^{m,s}, u^{m}, \theta_{1})$, scenario $s \in S$, and Eq. (16), a subgradient of $\Pi_t(K_{n(t-1)}^{m,s}, \xi_{st})$ for all $t \in T$ at $(K_{0}^{m,s}, u^{m}, \theta_{1})$ is
\[
(\partial \Pi_t^{m,s})^\top (K_{0}^{m,s}, u^{m}, \theta_{1}) + \sum_{i \in I} b_{it} \xi_{st} - \sum_{i \in I} \psi_{it}^{m,s} \xi_{st},
\]
where
\[
\partial \Pi_t^{m,s} = \begin{pmatrix}
\left( \mu_{nt}^{m,s} h_{K,n(t-1)}^{m,s} \right)_{n \in N} \\
0 \\
\left( \mu_{nt}^{m,s} h_{0,n(t-1)}^{m,s} \right)_{n \in N}
\end{pmatrix}, \quad \forall t \in T.
\]

Given Lemmas 1–3, the subgradients of the recourse functions can be computed via Proposition 2.
Proposition 2. Given \((K_0^m, u^m, \theta^m)\) and scenario \(s \in S\), a subgradient in \(\partial R^s(\cdot)\) with respect to \((K_0^m, u^m, \theta^m)\) is given by:

(i) If \(c_0(K_0^m) + \sum_{t=1}^{T} \gamma^t (c_t(\Delta K_t^s) - \Pi_t(K_{t-1}^s, \xi_t^s)) - u^m \geq 0\), we have

\[
\partial R^m_{m,s} = \beta \sum_{t=1}^{T} \gamma^t \left( g_{nt}^m (h_{K,nt}^m - h_{K,n(t-1)}^m) - \mu_{nt}^m h_{K,n(t-1)}^m \right), \quad \forall n \in N,
\]

\[
\partial u^m R^m_{m,s} = 0,
\]

\[
\partial \theta^m_{1,n} R^m_{m,s} = \beta \sum_{t=1}^{T} \gamma^t \left( g_{nt}^m (h_{\theta,nt}^m - h_{\theta,n(t-1)}^m) - \mu_{nt}^m h_{\theta,n(t-1)}^m \right), \quad \forall n \in N,
\]

and

\[
R^0_{m,s} = \beta \sum_{t=1}^{T} \gamma^t \left( g_{nt}^m (h_{0,nt}^m - h_{0,n(t-1)}^m) + g_{0,t}^m + \sum_{i \in I} b_i s_i \right) + \sum_{i \in I} b_i s_i - \mu_{nt}^m h_{0,nt}^m.
\]

(ii) If \(c_0(K_0^m) + \sum_{t=1}^{T} \gamma^t (c_t(\Delta K_t^s) - \Pi_t(K_{t-1}^s, \xi_t^s)) - u^m < 0\), we have

\[
\partial R^m_{m,s} = \left( \beta + \frac{1 - \beta}{1 - \alpha} \right) \sum_{t=1}^{T} \gamma^t \left( g_{nt}^m (h_{K,nt}^m - h_{K,n(t-1)}^m) - \mu_{nt}^m h_{K,n(t-1)}^m \right) + \frac{1 - \beta}{1 - \alpha} g_{n0}^m, \quad \forall n \in N,
\]

\[
\partial u^m R^m_{m,s} = \frac{\beta - 1}{1 - \alpha},
\]

\[
\partial \theta^m_{1,n} R^m_{m,s} = \left( \beta + \frac{1 - \beta}{1 - \alpha} \right) \sum_{t=1}^{T} \gamma^t \left( g_{nt}^m (h_{\theta,nt}^m - h_{\theta,n(t-1)}^m) - \mu_{nt}^m h_{\theta,n(t-1)}^m \right), \quad \forall n \in N,
\]

and

\[
R^0_{m,s} = \left( \beta + \frac{1 - \beta}{1 - \alpha} \right) \sum_{t=1}^{T} \gamma^t \left( \sum_{n \in N} g_{nt}^m (h_{0,nt}^m - h_{0,n(t-1)}^m) + g_{0,t}^m \right)
\]

\[
+ \sum_{i \in I} b_i s_i \right) + \frac{1 - \beta}{1 - \alpha} \sum_{n \in N} g_{n0}^m,
\]

such that

\[
R^s(K_0^m, u^m, \theta^m) = \begin{pmatrix} \partial K_0^m R^m_{m,s} \\ \partial u^m R^m_{m,s} \\ \partial \theta^m_{1,n} R^m_{m,s} \end{pmatrix}^T \begin{pmatrix} K_0^m \\ u^m \\ \theta^m \end{pmatrix} + R^0_{m,s}.
\]
Eq. (17) indicates that $R_s(K_0^m, u^m, \theta^m)$, the value of the recourse function at $(K_0^m, u^m, \theta^m)$, can be recovered by a linear function with respect to $(K_0^m, u^m, \theta^m)$ if $\partial R_{m,s}^m$ and $R_0^m$ are given. Denote

$$\phi^m \triangleq \mathbb{E}_{s \in S} \left[ \begin{array}{c} \partial_{K_0^m} R_s \\
\partial_{u^m} R_s \\
\partial_{\theta_1^m} R_s \end{array} \right] \quad \text{and} \quad \phi_0^m \triangleq \mathbb{E}_{s \in S} [R_0^m],$$

then a subgradient cut is given by

$$y \geq (\phi_m^m) ^\top (K_0, u, \theta_1) + \phi_0^m. \quad (18)$$

We see that if $(K_0, u, \theta_1) = (K_0^m, u^m, \theta_1^m)$, the cut recovers $\mathbb{E}[R_{m,s} (K_0^m, u^m, \theta^m)]$, otherwise it returns the recourse along the computed subgradient. Then, we can update $(K_0, u, \theta_1)$ by adding the subgradient cut, Eq. (18), into Problem (12).

**Step 3: Update $(K_0, u, \theta_1)$ by Solving the First-Stage Problem**

Suppose we have a set of subgradient cuts computed from Step 2 up to iteration $m$. We add the subgradient cuts to Problem (12), and solve the problem in iteration $m + 1$. Once solved, we denote its optimal solution as $(K_0^{m+1}, u^{m+1}, \theta_1^{m+1})$ and go to Step 1. Problem (12) is a small-scale MILP, which has $N + N \times L$ integer variables, and can be efficiently solved with commercial solvers.

The decomposition algorithm terminates when the objective value of Problem (10), computed from the best-found solution $(K_0, u, \theta)$, is close enough to the objective value derived from Problem (12) or when a preset number of iterations is reached. Algorithm 1 overviews the details of the entire procedure.

### 5.3 Improving the Best-Found Solution

The solution $(K_0, u, \theta_1)$ obtained in Step 3 is not necessarily globally optimal. In Problem (12), the epigraph of the recourse function is successively approximated from below by the subgradient cuts, i.e. Eq. (18). If the recourse function is convex, this procedure is the same as Benders decomposition, and the global optimum can be found since the subgradient gives a global underestimator.
for convex recourse functions (Benders, 1962). However, this is not the case for $R_s(K_0, u, \theta)$ which is non-convex due to the integer wait-and-see variables and the non-convex cost function $c_t(\cdot)$. In this case, the subgradient cuts may cut off part of the epigraph. Therefore, we wish to further improve upon the best-found solution.

We use the multi-cut method proposed in (Zhao et al., 2019) to improve the solution given by Algorithm 1. First, we introduce another type of cut, i.e. the integer optimality cut (Laporte and Louveaux, 1993), simultaneously with the subgradient cuts in Problem (12) to update $(K_0, u, \theta)$. The integer optimality cut is valid, i.e. it is a global underestimator of the epigraph, but it is conservative when updating the first-stage decisions. Second, we run the decomposition algorithm until the termination condition is triggered. Then, we stop adding subgradient cuts and remove one of the subgradient cuts which has the minimum slack. This procedure relaxes the approximation of the epigraph and helps the algorithm avoid getting stuck at a local solution.

## 6 Case Study: Capacity Planning for a Waste-to-Energy System

Our case study about a multi-facility WTE system in Singapore is adapted from (Cardin and Hu, 2015; Zhao et al., 2018). All data in this section are taken from the real case study.
The system has five candidate sites in different parts of Singapore. The WTE facility disposes of food waste collected from each sector by using an innovative anaerobic digestion technique which transforms the food waste into electricity. Undisposed waste will be subjected to further treatment via landfill, incurring greater disposal costs (i.e. penalties). The profit of the system comes from selling the electricity, and the costs consist of unit disposal costs, penalty costs, transportation costs, and capacity expansion costs. For simplicity, we omit annual fixed costs incurred once the facility is installed, which differs from the models in (Cardin and Hu, 2015; Zhao et al., 2018). However, the proposed method can still solve the problem with fixed costs if we use a multi-cut version, i.e. subgradient cuts with integer optimality cuts, in Algorithm 1.

The generation of the food waste from each sector (i.e. the demand of each customer) is assumed to be standard geometric Brownian motion (GBM) (Cardin and Hu, 2015), represented by

\[
\xi_{it} = (\bar{\mu} + \bar{\sigma}\omega_t)\xi_{i(t-1)}, \quad \forall t \in \mathcal{T},
\]

where \(\xi_{it}\) is the waste amount generated from sector \(i\) in time \(t\), \(\bar{\mu}\) is the percentage drift, \(\bar{\sigma}\) is the percentage volatility, and \(\omega_t\) is a standard normal random variable. In the numerical study, we assume that \(\bar{\mu}\) is 4% and \(\bar{\sigma}\) is 16%, and the initial waste vector at the beginning is \(\xi_0 = [498, 518, 293, 460, 382]\) (unit: tonnes per day).

The WTE facilities are assumed to be modular; that is, one unit of capacity can dispose of 100 tonnes of food waste per day, and the capacity limit is \(K^{\text{max}} = [16, 10, 10, 10, 10]\). The expansion cost of facility \(n\) is given by a power function:

\[
\bar{c}_{nt}(\Delta K_{nt}) = 7 \times 10^6 \times (\Delta K_{nt})^{0.9} \times 100, \quad \forall n \in \mathcal{N}, t \in \mathcal{T}.
\]

We linearize the expansion costs by setting breakpoints at \((a_1, a_2, \ldots, a_7) = (0, 2^0, 2^1, \ldots, 2^4)\) and derive the cost function presented in Eq. (2). The penalty for undisposed food waste is \(b_{it} = 77\).
per tonne, and the profit matrix is

\[
(r_{in})_{i \in I, n \in N} = \begin{bmatrix}
59.9 & 42.9 & 30.4 & 34.9 & 21.9 \\
42.9 & 59.9 & 40.9 & 52.1 & 38.9 \\
30.4 & 40.9 & 59.9 & 49.8 & 34.4 \\
34.9 & 52.1 & 49.9 & 59.9 & 44.4 \\
21.9 & 38.9 & 34.4 & 44.4 & 59.9 \\
\end{bmatrix},
\]

where \( r_{in} \) denotes the net profit from disposing every tonne of food waste from sector \( i \) by facility \( n \).

Since the system has a one-to-one correspondence between a sector and its most profitable facility, the allocation matrix \( W \) is a \( 5 \times 5 \) identity matrix. Therefore, the decision rule is

\[
K_{nt}(\theta) = \begin{cases}
\lfloor \xi_{it}/100 \rfloor + \theta_{1,n}, & \text{if } \lfloor \xi_{it}/100 \rfloor - K_{n(t-1)} \geq \theta_{2,nt}, \\
K_{n(t-1)}, & \text{otherwise}.
\end{cases}
\]

We set the discount factor \( \gamma = 0.926 \).

### 6.1 A Baseline Design and the Value of Flexibility

Problem (5) is a multi-stage stochastic programming problem and it is very difficult to solve. The solution provided by the decision-rule based method may not be globally optimal. Therefore, we need an attainable baseline design as a lower bound, to see how the proposed method improves upon the economic performance.

A baseline is a restriction of Problem (3): the decision maker fixes the capacity plan at the beginning, and has no options to adjust the capacity in the future. Mathematically, the capacity decisions \((K_0, K_1, \ldots, K_T)\) of the baseline model, once determined, do not change with the realizations of \( \xi \) over time. We call this baseline MCEP the \textit{inflexible} model. This inflexible model can be solved by the Benders decomposition: we fix the capacity decisions, i.e. \((K_0, K_1, \ldots, K_T)\), solve the allocation problems (i.e. \( \Pi_t(\cdot) \) for all \( t \in T \)), and then iteratively update \((K_0, K_1, \ldots, K_T)\) via the generated Benders cuts. The allocation problems are all LPs, and we have finite number of capacity decisions, so the optimal solution of the inflexible model will be reached within a finite number of iterations (Benders, 1962).
Table 2: Simulation results with $\alpha = 0.95$ (unit: million S$).

| Criterion          | Flexible policies with different $\beta$ | Inflexible policy |
|--------------------|-----------------------------------------|-------------------|
|                    | 0.01 0.25 0.5 0.75 0.99                 |                   |
| Min                | 125.1 135.5 135.0 135.5 130.1           | −291.8            |
| 5$^{th}$-percentile| 224.3 221.4 220.5 218.7 217.1           | 146.4             |
| ENPV               | 284.8 287.4 288.7 290.3 291.3           | 237.7             |
| 95$^{th}$-percentile| 344.3 354.3 361.1 365.1 366.2           | 306.4             |
| Max                | 492.5 516.6 523.0 539.9 542.3           | 342.4             |
| VoF                | 47.1 49.7 51.0 52.6 53.6                | -                 |

The flexible and inflexible models are solved separately by generating 4,000 demand scenarios via Monte-Carlo simulation. Then, to compare, we conduct out-of-sample tests to evaluate the economic performance of each design by using an identical sample set. The out-of-sample test consists of 12,000 scenarios that are generated via Monte-Carlo simulation. The rationale underlying the out-of-sample test is to eliminate the bias introduced by using different sample sets.

Given the baseline design, we can calculate the difference between the economic performance of the flexible multi-stage model and its inflexible counterpart, i.e. the value of flexibility (VoF).

6.2 The Proposed Method Captures Decision-Maker Risk-Preferences

In our simulations, we set $\alpha = 0.95$ and vary $\beta$ from 0.01 to 0.99. When $\alpha = 0.95$, we are minimizing expected costs that are greater than or equal to the 95$^{th}$-percentile. Equivalently, the objective of Problem (5) maximizes the expected value less than the 5$^{th}$-percentile. The simulation results from different policies are presented in Table 2. All cases are run three times and the values are averaged. The inflexible policy comes from the baseline design, where ENPV is 237.7 million, the 5$^{th}$-percentile of the NPV is 146.4 million, and the worst-case NPV is −291.8 million. The flexible policies from Problem (5) are computed for varying $\beta$. If we set $\beta = 0.99$, the ENPV is 291.3 million, the 5$^{th}$-percentile is 217.1 million, and the worst-case scenario is 130.1 million. We see that the flexible policy dominates the inflexible one in terms of the five metrics presented in Table 2. In particular, the worst-case scenario is improved significantly. The worst-case NPV for the inflexible policy is negative, while the worst-case NPV of the flexible policies ranges from 125.1 million to 130.1 million.
These results demonstrate that the 5th-percentile of the NPVs decreases with $\beta$ and the ENPV increases with $\beta$ (see Figure 2). If we change $\beta$ from 0.99 to 0.01, the 5th-percentile rises from 217.1 million to 224.3 million and the ENPV declines from 291.3 million to 284.8 million. In addition, the VoF decreases from 53.6 million to 47.1 million, which means that the value gained from flexibility decreases as the weight factor decreases. One possible explanation is because the particular flexible capacity-expansion policy analyzed here focuses on improving upside potential. This means that risk-averse decision-makers may gain less value from being flexible. The result from Figure 2 also verifies the conclusion of Proposition 1 numerically.

### 6.3 Risk-Averse Policy is More Conservative in Expansion

We examine the optimal solutions of Problem (5) for different risk preferences. The optimal initial capacity given $\beta = 0.99$ is $K_0^* = [6, 7, 3, 6, 4]$ and the optimal parameter is $\theta_1^* = [0, 0, 0, 1, 0]$. In contrast, the optimal solution given $\beta = 0.01$ is $K_0^* = [5, 6, 3, 5, 4]$ and $\theta_1^* = [0, 0, 0, 0, 0]$. The policy with $\beta = 0.01$ has a smaller initial capacity, and its expansion is more conservative when the decision rule is triggered. The cumulative density functions of the out-of-sample tests of these two policies are plotted in Figure 3. As can be seen, the policy with $\beta = 0.01$ does not fully exploit
the upside expansion opportunity whereas the one with $\beta = 0.99$ does, but it reduces the downside cost as it is more conservative.

6.4 Flexible Policies have Better Performance under an Inaccurate Demand Model

We also test the robustness of the policies when the demand model is inaccurate. Recall that our demand model is GBM with $\bar{\mu} = 0.04$ and $\bar{\sigma} = 0.16$. We perform out-of-sample tests via a set of samples generated from a GBM with identical $\bar{\mu}$ and $\bar{\sigma}$. To see how the performance changes if we use incorrect training data, we intentionally generate samples via GBMs that have different parameters and perform out-of-sample tests. We generate two sample sets:

- Set A with 12,000 samples generated via GBM ($0.02, 0.08$), which has a lower percentage drift and lower percentage volatility compared to the training data;
- Set B with 12,000 samples generated via GBM ($0.05, 0.16$), which has a higher percentage drift and higher percentage volatility compared to the training data.

The simulation results of the out-of-sample tests are presented in Table 3. We see that for the risk-averse policy (with $\beta = 0.01$), the ENPV computed under the original sampling condition is
Table 3: Simulation results given inaccurate demand models (unit: million S$).

| Policies       | Criteria       | Training samples | Set A          | Set B          |
|---------------|----------------|------------------|----------------|----------------|
|               |                | GBM(0.04, 0.16)  | Results | Variations | Results | Variations |
| Flexible policy | 5th-percentile | 224.3            | 249.4 | +11.2%       | 236.4 | +5.1%       |
| ($\beta = 0.01$) | ENPV           | 284.8            | 278.2 | −2.3%        | 295.9 | +3.1%       |
| Flexible policy | 5th-percentile | 217.1            | 239.7 | +10.4%       | 233.2 | +7.1%       |
| ($\beta = 0.99$) | ENPV           | 291.3            | 279.2 | −4.2%        | 307.9 | +5.5%       |
| Inflexible policy | 5th-percentile | 146.4            | 157.5 | +7.6%        | 147.5 | +0.8%       |
|               | ENPV           | 237.7            | 213.2 | −10.3%       | 242.2 | +1.8%       |

284.8 million, but if we perform out-of-sample tests via the alternative sample Sets A and B, its ENPV decreases 2.3% and increases 3.1%, respectively. The ENPV derived from the risk-neutral flexible policy (with $\beta = 0.99$) decreases 4.2% in Set A and increases 5.5% in Set B. However, the ENPVs of the inflexible policy decrease −10.3% in Set A and increase merely 1.8% in Set B.

Several conclusions follow from Table 3. First, a risk-averse policy may suffer lower loss compared to a risk-neutral policy when the predicted demands (i.e., the training data) are not as large as the true demands (i.e., the out-of-test data). Second, we see that the risk-neutral policy can better exploit upside expansion opportunities if the testing demands are generally larger than the training demands. Third, flexible policies are more robust in comparison with inflexible policies for an incorrect demand model.

7 Conclusion

In this paper, we establish a multi-facility capacity expansion model with a mean-CVaR objective to capture decision maker risk preferences. We verify that the economic performance of the risk-averse model is not higher than that of the risk-neutral model.

To solve the our risk-averse model, we approximate the expansion policy of the multi-stage problem with an if-then decision rule, and then solve the resulting model by a customized subgradient decomposition algorithm. Though our algorithm may not reach the global optimum of the problem, numerical studies show that its improvement over the baseline design (i.e. the inflexible MCEP) is significant, especially when the demand model is inaccurate.
Simulation results also reveal that the decision maker is able to choose a policy with a higher ENPV or one with a higher 5th-percentile of the NPVs by simply adjusting the weight factor of the objective function. In addition, the ENPV decreases as the decision maker becomes more risk-averse, indicating that the decision maker will prefer to pay less for flexibility. The simulation results also illustrate that a risk-averse expansion policy, compared with the risk-neutral one, tends to establish smaller initial capacity at the beginning, and is more conservative in future expansion.

Many opportunities exist for future work. The model proposed in this paper does not account for annual fixed costs. Such costs can take the form of annual land rental costs which are incurred once a facility is established (i.e., has a non-zero capacity). To formulate a problem with fixed costs, we need to introduce extra binary auxiliary variables as the costs are concave with respect to the capacity, so the model becomes more difficult to handle numerically. In another research direction, we may consider robustness against an unknown demand model. Our present numerical experiments suggest that our approach has some intrinsic robustness against demand uncertainty, so we can build on this initial proof of concept.

**Appendix. Proofs of Main Results**

**Proof of Proposition 1:** Firstly, we know from the definition of CVaR that, given a random variable $X$, we have $\text{CVaR}_\alpha (X) \geq \mathbb{E}[X]$ for any $\alpha \in (0, 1)$ (Sarykalin et al., 2008).

(i) Denote $\mathcal{K}_{[T]}^*$ as the optimal solution of Problem (5) given $\beta$. Denote $\Delta \beta > 0$ as a positive increment such that $\beta + \Delta \beta \leq 1$. According to Problem (5), we have

$$\text{ENPV}_\alpha (\beta + \Delta \beta) = \max_{\mathcal{K}_{[T]} \in \mathcal{K}} \left( -\beta \mathbb{E} \left[ Q \left( \mathcal{K}_{[T]}, \xi \right) \right] - (1 - \beta) \text{CVaR}_\alpha \left( Q \left( \mathcal{K}_{[T]}, \xi \right) \right) ight)$$

$$+ \Delta \beta \left( \text{CVaR}_\alpha \left( Q \left( \mathcal{K}_{[T]}^*, \xi \right) \right) - \mathbb{E} \left[ Q \left( \mathcal{K}_{[T]}, \xi \right) \right] \right)$$

$$\geq -\beta \mathbb{E} \left[ Q \left( \mathcal{K}_{[T]}^*, \xi \right) \right] - (1 - \beta) \text{CVaR}_\alpha \left( Q \left( \mathcal{K}_{[T]}^*, \xi \right) \right)$$

$$+ \Delta \beta \left( \text{CVaR}_\alpha \left( Q \left( \mathcal{K}_{[T]}^*, \xi \right) \right) - \mathbb{E} \left[ Q \left( \mathcal{K}_{[T]}^*, \xi \right) \right] \right)$$

$$= \text{ENPV}_\alpha (\beta) + \Delta \beta \left( \text{CVaR}_\alpha \left( Q \left( \mathcal{K}_{[T]}^*, \xi \right) \right) - \mathbb{E} \left[ Q \left( \mathcal{K}_{[T]}^*, \xi \right) \right] \right)$$

The inequality holds as $\mathcal{K}_{[T]}^*$ is the optimal solution for ENPV$_\alpha (\beta)$. Since CVaR$_\alpha \left( Q \left( \mathcal{K}_{[T]}, \xi \right) \right)$
The last equality holds as we have \( n \in \mathbb{N} \), we can conclude that ENPV \( \alpha (\beta) \) is non-decreasing in \( \beta \).

(ii) Since ENPV \( \alpha (1) = \text{ENPV}_{\text{flex}} \) and ENPV \( \alpha (\beta) \) is non-decreasing in \( \beta \), given any \( \alpha \in (0, 1) \), we have ENPV \( \text{inflex} = \text{ENPV} \alpha (1) \geq \text{ENPV} \alpha (\beta) \) for all \( \beta \in [0, 1] \) according to (i). Hence, we can conclude the result.

**Proof of Lemma 1:** We first show that

\[
K_{nt}^{m,s} = \prod_{t_0=1}^{t} (1 - \delta_{nt_0}^{m,s}) K_{nt_0}^{m} + \sum_{t_1=t_0+1}^{t} \prod_{t_0=1}^{t} (1 - \delta_{nt_1}^{m,s}) \delta_{nt_0}^{m,s} \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] + \theta_{1,n}^{m} \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] + \theta_{1,n}^{m}, \quad \forall n \in \mathcal{N}, t \in \mathcal{T} \cup \{0\}.
\]

The result can be proved by induction. For \( t = 1 \), we have

\[
K_{n1}^{m,s} = (1 - \delta_{n1}^{m,s}) K_{n0}^{m} + \delta_{n1}^{m,s} \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{i1}^{s} \right] + \theta_{1,n}^{m},
\]

and the result holds. Suppose the result holds for \( t = t' - 1 \):

\[
K_{n(t' - 1)}^{m,s} = \prod_{t_0=1}^{t'-1} (1 - \delta_{nt_0}^{m,s}) K_{nt_0}^{m} + \sum_{t_1=t_0+1}^{t'-1} \prod_{t_0=1}^{t'-1} (1 - \delta_{nt_1}^{m,s}) \delta_{nt_0}^{m,s} \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] + \theta_{1,n}^{m} \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] + \theta_{1,n}^{m}.
\]

Then, for \( t = t' \) we have

\[
K_{nt'}^{m,s} = (1 - \delta_{nt'}^{m,s}) K_{n(t'-1)}^{m,s} + \delta_{nt'}^{m,s} \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it'}^{s} \right] + \theta_{1,n}^{m}
\]

\[
= \prod_{t_0=1}^{t'} (1 - \delta_{nt_0}^{m,s}) K_{nt_0}^{m} + \sum_{t_1=t_0+1}^{t'} \prod_{t_0=1}^{t'} (1 - \delta_{nt_1}^{m,s}) \delta_{nt_0}^{m,s} \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] + \theta_{1,n}^{m} \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] + \theta_{1,n}^{m}
\]

\[
= \prod_{t_0=1}^{t'} (1 - \delta_{nt_0}^{m,s}) K_{nt_0}^{m} + \sum_{t_1=t_0+1}^{t'} \prod_{t_0=1}^{t'} (1 - \delta_{nt_1}^{m,s}) \delta_{nt_0}^{m,s} \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] \left[ \sum_{i \in \mathcal{I}} W_{in} \xi_{it_0}^{s} \right] + \theta_{1,n}^{m}.
\]

The last equality holds as we have \( \prod_{t_1=t_0+1}^{t'} (1 - \delta_{nt_1}^{m,s}) = \prod_{t_1=t'}^{t+1} (1 - \delta_{nt_1}^{m,s}) = 1 \). Reorganize the above equation and we can conclude the result. \( \square \)
Proof of Lemma 2: According to Eq. (2) and the definitions of $g_{nt}^{m,s}$, we have

$$\partial c_{nt}^{m,s} = \partial \left( \sum_{n \in N} \left[ g_{nt}^{m,s} \cdot \left( K_{nt}^{m,s} - K_{n(t-1)}^{m,s} \right) + g_{0,nt}^{m,s} \right] \right),$$

$$= \partial \left( \sum_{n \in N} \left[ g_{nt}^{m,s} \cdot \left( h_{K,nt}^{m,s} K_{n0}^{m,s} + h_{\theta,nt}^{m,s} \theta_{1,n}^{m,s} + h_{0,nt}^{m,s} - h_{K,n(t-1)}^{m,s} K_{n0}^{m,s} - h_{\theta,n(t-1)}^{m,s} \theta_{1,n}^{m,s} - h_{0,n(t-1)}^{m,s} \right) + g_{0,nt}^{m,s} \right] \right).$$

Then, the following result holds trivially according to Eq. (16) from Lemma 1.

Proof of Lemma 3: Let $(\mu_{nt})_{n \in N', t \in T}$ and $(\psi_{it})_{i \in I, t \in T}$ be the dual variables with respect to the capacity and demand constraints of $\Pi_t(\cdot)$ respectively, and the dual problem of $\Pi_t(K_{t-1}^{m,s}, \xi_t^s)$ can then be formulated by

$$\min_{\psi, \mu} \sum_{n \in N} \mu_{nt} K_{n(t-1)}^{m,s} + \sum_{i \in I} \psi_{it} \xi_{it}^s - \sum_{i \in I} b_{it} \xi_{it}^s$$

$$\text{s.t.} \quad -r_{nt} - b_{it} + \psi_{it} + \mu_{nt} \geq 0, \quad \forall i \in I, n \in N,$$

$$\psi_{it} \geq 0, \quad \mu_{nt} \geq 0, \quad \forall i \in I, n \in N.$$ (19)

Since Problem (1) is a linear program and the optimal solution is not empty as $K_{t-1}^{m,s}$ and $\xi_t^s$ are all bounded and nonnegative, the strong duality holds for Problem (1) (Bertsimas and Tsitsiklis, 1997, Theorem 4.4). Therefore, we have

$$\Pi_t(K_{t-1}^{m,s}, \xi_t^s) = \sum_{n \in N} \mu_{nt} K_{n(t-1)}^{m,s} + \sum_{i \in I} \psi_{it} \xi_{it}^s - \sum_{i \in I} b_{it} \xi_{it}^s,$$

where $(\mu_{nt})_{n \in N', t \in T}$ and $(\psi_{it})_{i \in I, t \in T}$ are the optimal solutions of Problem (19) by definitions. Then, according to Lemma 1, Lemma 3 can be proved.

Proof of Proposition 2: We know that $K_{1}^{s}, \ldots, K_{T}^{s}, \eta^s$ can be computed from Policy (6) once
\((K_0^m, u^m, \theta^m)\) and scenario \(s \in S\) are given. According to Eq. (11), we have

\[
\partial R_{m,s} = \partial \left[ \beta \sum_{t=1}^{T} \gamma^t \left( c_t (K_t^s - K_{t-1}^s) - \Pi_t (K_{t-1}^s, \xi_t^s) \right) + \frac{1 - \beta}{1 - \alpha} \eta_t \right],
\]

\[
= \beta \sum_{t=1}^{T} \gamma^t (\partial c_t^{m,s} - \partial \Pi_t^{m,s}) + \frac{1 - \beta}{1 - \alpha} \partial \eta_t^{m,s},
\]

where \(\partial c_t^{m,s}, \partial \Pi_t^{m,s},\) and \(\partial \eta_t^{m,s}\) denote the subgradients of the corresponding functions with respect to \((K_0^m, u^m, \theta_t^m)\). Then, when \(c_0 (K_0) + \sum_{t=1}^{T} \gamma^t (c_t (K_t^s - K_{t-1}^s) - \Pi_t (K_{t-1}^s, \xi_t^s)) - u \geq 0\), we have

\[
\partial \eta_t^{m,s} = \partial c_0^{m,s} + \sum_{t=1}^{T} \gamma^t (\partial c_t^{m,s} - \partial \Pi_t^{m,s}) + \partial u; \text{ therefore}
\]

\[
\partial R_{m,s} = \left( \beta + \frac{1 - \beta}{1 - \alpha} \right) \sum_{t=1}^{T} \gamma^t (\partial c_t^{m,s} - \partial \Pi_t^{m,s}) + \frac{1 - \beta}{1 - \alpha} \left( \partial c_0^{m,s} + \partial u \right). \tag{20}
\]

When \(c_0 (K_0) + \sum_{t=1}^{T} \gamma^t (c_t (K_t^s - K_{t-1}^s) - \Pi_t (K_{t-1}^s, \xi_t^s)) - u < 0\), we have \(\partial \eta_t^{m,s} = 0\), and thus

\[
\partial R_{m,s} = \beta \sum_{t=1}^{T} \gamma^t (\partial c_t^{m,s} - \partial \Pi_t^{m,s}). \tag{21}
\]

Proposition 2 can then be concluded by substituting the results from Lemma 2 and 3 into Eqs. (20)–(21).

\[
\square
\]

References

Acerbi, C. (2002). Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking & Finance*, 26(7):1505–1518.

Angelus, A., Porteus, E., and Wood, S. C. (2000). Optimal sizing and timing of modular capacity expansions. *Graduate School of Business Research Paper*, (1479R2).

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent Measures of Risk. *Mathematical Finance*, 9(3):203–228.

Benders, J. F. (1962). Partitioning procedures for solving mixed-variables programming problems. *Numerische mathematik*, 4(1):238–252.
Bertsimas, D. and Tsitsiklis, J. N. (1997). *Introduction to linear optimization*, volume 6. Athena Scientific Belmont, MA.

Birge, J. R. (2000). Option Methods for Incorporating Risk into Linear Capacity Planning Models. *Manufacturing & Service Operations Management*, 2(1):19–31.

Cardin, M.-A. and Hu, J. (2015). Analyzing the Tradeoffs Between Economies of Scale, Time-Value of Money, and Flexibility in Design Under Uncertainty: Study of Centralized Versus Decentralized Waste-to-Energy Systems. *Journal of Mechanical Design*, 138(1):011401–011401.

Cardin, M.-A., Xie, Q., Ng, T. S., Wang, S., and Hu, J. (2017a). An approach for analyzing and managing flexibility in engineering systems design based on decision rules and multistage stochastic programming. *IISE Transactions*, 49(1):1–12.

Cardin, M.-A., Zhang, S., and Nuttall, W. J. (2017b). Strategic real option and flexibility analysis for nuclear power plants considering uncertainty in electricity demand and public acceptance. *Energy Economics*, 64:226–237.

Delgado, D. and Claro, J. (2013). Transmission network expansion planning under demand uncertainty and risk aversion. *International Journal of Electrical Power & Energy Systems*, 44(1):696–702.

Dixit, A. K. and Pindyck, R. S. (1994). *Investment Under Uncertainty*. Princeton University Press. Google-Books-ID: VahsELa_qC8C.

Eberly, J. C. and Van Mieghem, J. A. (1997). Multi-factor Dynamic Investment under Uncertainty. *Journal of Economic Theory*, 75(2):345–387.

Geng, N., Jiang, Z., and Chen, F. (2009). Stochastic programming based capacity planning for semiconductor wafer fab with uncertain demand and capacity. *European Journal of Operational Research*, 198(3):899–908.

Georghiou, A., Wiesemann, W., and Kuhn, D. (2015). Generalized decision rule approximations for stochastic programming via liftings. *Mathematical Programming*, 152(1-2):301–338.
Haskell, W. and Jain, R. (2015). A Convex Analytic Approach to Risk-Aware Markov Decision Processes. *SIAM Journal on Control and Optimization*, 53(3):1569–1598.

Huang, K. and Ahmed, S. (2009). The Value of Multistage Stochastic Programming in Capacity Planning Under Uncertainty. *Operations Research*, 57(4):893–904.

Hugonnier, J. and Morellec, E. (2007). Real Options and Risk Aversion. SSRN Scholarly Paper ID 422600, Social Science Research Network, Rochester, NY.

Kouvelis, P. and Tian, Z. (2014). Flexible Capacity Investments and Product Mix: Optimal Decisions and Value of Postponement Options. *Production and Operations Management*, 23(5):861–876.

Krokhmal, P., Palmquist, J., and Uryasev, S. (2002). Portfolio optimization with conditional value-at-risk objective and constraints. *Journal of Risk*, 4:43–68.

Laporte, G. and Louveaux, F. V. (1993). The integer L-shaped method for stochastic integer programs with complete recourse. *Operations Research Letters*, 13(3):133–142.

Luss, H. (1982). Operations Research and Capacity Expansion Problems: A Survey. *Operations Research*, 30(5):907–947.

Maceira, M., Marzano, L., Penna, D., Diniz, A., and Justino, T. (2015). Application of CVaR risk aversion approach in the expansion and operation planning and for setting the spot price in the Brazilian hydrothermal interconnected system. *International Journal of Electrical Power & Energy Systems*, 72:126–135.

Manne, A. S. (1961). Capacity Expansion and Probabilistic Growth. *Econometrica*, 29(4):632–649.

Martínez-Costa, C., Mas-Machuca, M., Benedito, E., and Corominas, A. (2014). A review of mathematical programming models for strategic capacity planning in manufacturing. *International Journal of Production Economics*, 153:66–85.

Neufville, R. D. and Scholtes, S. (2011). *Flexibility in Engineering Design*. MIT Press. Google-Books-ID: pKjnnqih3EC.
Robbins, H. and Monro, S. (1951). A Stochastic Approximation Method. *The Annals of Mathematical Statistics*, 22(3):400–407.

Rockafellar, R. T. and Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of Risk*, 2:21–42.

Ruszczyński, A. (2010). Risk-averse dynamic programming for Markov decision processes. *Mathematical Programming*, 125(2):235–261.

Sarykalin, S., Serraino, G., and Uryasev, S. (2008). Value-at-Risk vs. Conditional Value-at-Risk in Risk Management and Optimization. In Chen, Z.-L., Raghavan, S., Gray, P., and Greenberg, H. J., editors, *State-of-the-Art Decision-Making Tools in the Information-Intensive Age*, pages 270–294. INFORMS.

Shapiro, A. (2011). Analysis of stochastic dual dynamic programming method. *European Journal of Operational Research*, 209(1):63–72.

Shapiro, A. and Nemirovski, A. (2005). On complexity of stochastic programming problems. In *Continuous optimization*, pages 111–146. Springer.

Sun, Y. and Schonfeld, P. (2015). Stochastic capacity expansion models for airport facilities. *Transportation Research Part B: Methodological*, 80:1–18.

Szolgayová, J., Fuss, S., Khabarov, N., and Obersteiner, M. (2011). A dynamic CVaR-portfolio approach using real options: an application to energy investments. *European Transactions on Electrical Power*, 21(6):1825–1841.

Taghavi, M. and Huang, K. (2016). A multi-stage stochastic programming approach for network capacity expansion with multiple sources of capacity. *Naval Research Logistics (NRL)*, 63(8):600–614.

Taghavi, M. and Huang, K. (2018). A Lagrangian relaxation approach for stochastic network capacity expansion with budget constraints. *Annals of Operations Research*. 

38
Van Mieghem, J. A. (2003). Commissioned Paper: Capacity Management, Investment, and Hedging: Review and Recent Developments. *Manufacturing & Service Operations Management*, 5(4):269–302.

Wu, C.-H. and Chuang, Y.-T. (2010). An innovative approach for strategic capacity portfolio planning under uncertainties. *European Journal of Operational Research*, 207(2):1002–1013.

Wu, C.-H. and Chuang, Y.-T. (2012). An efficient algorithm for stochastic capacity portfolio planning problems. *Journal of Intelligent Manufacturing*, 23(6):2161–2170.

Zhang, S. and Cardin, M.-A. (2017). Flexibility and real options analysis in emergency medical services systems using decision rules and multi-stage stochastic programming. *Transportation Research Part E: Logistics and Transportation Review*, 107:120–140.

Zhao, S., Haskell, W. B., and Cardin, M.-A. (2017). An Approximate Dynamic Programming Approach for Multi-Facility Capacity Expansion Problem with Flexibility Design. *IIE Annual Conference. Proceedings; Norcross*, pages 440–445.

Zhao, S., Haskell, W. B., and Cardin, M.-A. (2018). Decision rule-based method for flexible multi-facility capacity expansion problem. *IISE Transactions*, 50(7):553–569.

Zhao, S., Haskell, W. B., and Cardin, M.-A. (2019). An Approximate Dynamic Programming Algorithm for Multi-Stage Capacity Investment Problems. *arXiv:1901.05154 [math]*. arXiv: 1901.05154.