Form factors in relativistic quantum mechanics:
Is there a favored approach? Why?

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Abstract

Form factors of a simple system have been calculated in various forms of relativistic quantum mechanics, using a single-particle current. Their comparison has shown large discrepancies. The comparison is extended here to instant- and front-form calculations in unusual momentum configurations as well as to a point-form approach inspired from the Dirac’s one (based on a hyperboloid surface). It is found that these new results depend on the momentum transfer, $Q$, through its ratio to the total mass, $Q/M$, (closely related to the Breit-frame velocity of the system). They evidence features similar for a part to those shown by an earlier “point-form” implementation (based on hyperplanes perpendicular to the velocity of the initial and final states). It thus appears that the standard instant- and front-form calculations, which generally do well compared either to experiment or to predictions of a theoretical model, rather represent exceptional cases. An argument explaining the success of these last approaches is presented and discussed. It is based on transformations of currents under Poincaré space-time translations, going beyond the energy-momentum conservation property which results from the Lagrangian invariance under them. Depending on the approach, analytic or approximate numerical methods are proposed to correct form factors for missing constraints then expected.

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1 Introduction

Relativistic quantum mechanics (RQM) can be approached by different forms, originally classified by Dirac [1]. When calculating properties of hadrons like form factors, using a single-particle current, front and instant ones are often considered as more appropriate. In the case of the pion charge form factor for instance, these approaches [2, 3, 4, 5, 6] provide results relatively close to experiment [7, 8, 9]. A similar statement holds in the case of theoretical models involving spinless constituents [10]. It is not always stressed however that these calculations correspond to a particular momentum configuration, \( q^+ = 0 \) in the former case and Breit frame in the latter.

Recently, an implementation of the point-form approach [11] has been employed for the calculation of form factors of hadronic systems such as the pion [12, 13, 14], the deuteron [15] and the nucleon [16]. In the pion case, which has been revisited [13, 14], the approach produces a huge discrepancy with experiment. A similar statement could be made in the case of theoretical models involving a two-body system with equal-mass constituents [10, 17], especially when the total mass of the system, \( M \), becomes small in comparison with the sum of the constituent masses, \( 2m \). The discrepancy can be traced back to the dependence of form factors on the momentum transfer \( Q \) through the velocity of the system in the Breit frame for instance, which involves the ratio \( Q/(2M) \). This ratio, which is the only parameter entering the boost transformation in the point-form approach, has striking consequences. The charge radius varies like the inverse of the mass of the system with the result that the larger the binding, the larger the radius, which is somewhat counter-intuitive. Moreover, at high momentum transfer, the power-law behavior of the form factor evidences a suppression by as many powers of \( M/(2m) \) as there are powers of \( Q \). The appearance in the point-form approach of the total mass of the system, \( M \), and its essential role, especially when it goes to zero, were emphasized in various works [18, 17, 13, 19]. At the same time, it was noticed that this mass was playing no role in the boost transformation required for calculating form factors in the standard instant- and front-form approaches (as far as a single-particle current is concerned).

When the first “point-form” result evidencing a sizeable discrepancy appeared [17], it could be thought that this was a peculiarity of the approach [11], suggesting a specific problem. It was thus found that form factors so obtained systematically evidence wrong power-law behavior at high \( Q^2 \). This problem could be solved by introducing the simplest two-body currents [20]. These ones could not however remove other important drawbacks evidenced by the approach.

Interestingly, results showing similarities with the above “point-form” results have recently appeared in other approaches, as a by-product of studies aiming to look at the frame dependence of form factors and the role of two-body currents. Thus, an instant-form calculation of the form factor for a strongly bound system showed a fast fall-off when going away from the Breit frame [10]. In a front-form approach, Simula examined results for the form factor of a pseudo-scalar meson with the pion mass [21]. In the case \( q^+ \neq 0 \), a large drop-off of the single-particle contribution was observed while the relation of the effect to the dependence of the form factor on the ratio \( Q/(2M) \) was emphasized. A similar effect was also found in field-theory motivated approaches, with \( q^+ \neq 0 \) [22, 23].
No relation to a dependence of the corresponding contribution on the ratio $Q/(2M)$ was made in these last cases but, in view of the results, there is not much doubt on the origin of the effect. Thus, the striking behavior of the single-particle current contribution to form factors of strongly bound systems in the earlier “point-form” approach is far to be an isolated fact.

The above observation has been obtained in different schemes however. Its general character needs to be confirmed and specified by dedicated studies involving the same inputs as much as possible. Though we believe that there is some relationship between effects in field-theory and RQM approaches, the last ones have their own rules and a full correspondence is not guaranteed. It is therefore appropriate to extend earlier comparisons of form factors obtained in different forms of relativistic quantum mechanics [10]. Given that form factors calculated in the front and instant forms are not Lorentz invariant, one can thus look at them for non-standard momentum configurations. The comparison can provide some information on violations of Lorentz invariance but this supposes that no other symmetry is significantly violated at the same time. One can also consider a point-form approach more in the spirit of the Dirac’s one, based on a hyperboloid surface [21]. Apart from the fact that such an approach has never been used, it can provide information on the specific character of results obtained in the earlier “point form” implementation. As noticed by Sokolov [25], this one implies a hyperplane perpendicular to the velocity of the system and therefore differs from the Dirac’s point form. Though these features are not so clearly expressed, they also stem from an earlier work by Bakamjian [26] where it is shown that “an instant form of relativistic quantum mechanics can be constructed which displays the symmetry properties inherently present in the point form”. The two approaches (hyperplane and hyperboloid based) have in common that only the generators $P^\mu$ of the Poincaré algebra contain the interaction. To distinguish them, we use quotation marks when referring to the first one.

Ultimately, one would like to get a sufficiently large insight on form factors in different forms and different momentum configurations so that it can provide a clue as to why calculations based on a single-particle current do relatively well in some approaches while they cannot in other ones. For these last cases, a major role is played by two-body currents which, in principle, should ensure that form factors be independent of the form and frame under consideration. Evidently, a comparison to the predictions of an underlying field-theory model, as done in Refs. [17, 10], can tell about the efficiency of an approach. However, apart from the fact that this is not always possible, we believe that a sensible argument, if any, should be found within the formalism that is employed. In RQM approaches, the description of the initial and final states entering the calculation of form factors fulfills Poincaré covariance properties by construction of the corresponding algebra. This involves (homogeneous) Lorentz transformations (boosts and rotations) generated by the algebra operators, $M^\mu_\nu$, and space-time translations generated by the 4-momentum operators, $P^\mu$. However, at the interaction vertex of the external probe with the constituents of the system under consideration, Poincaré covariance is generally violated. Depending on the approach, only part of the expected properties is fulfilled. Those related to Lorentz transformations, which can be easily checked by moving or rotating the system, are currently emphasized. On the contrary, due to the absence of a similar check, the properties related to space-time translations, beyond the global 4-momentum
conservation that results from the Lagrangian invariance under these transformations, are essentially unexplored. It is our intent in this paper to show that these properties can play an important role in discriminating various results.

The study is done for the ground state of a two-body system with equal-mass constituents, in a scalar-particle model. This offers the advantage of minimizing specific difficulties pertinent to the description of a more realistic system like a hadronic one, due to the non-zero spin of the constituents or to a complicated dynamics which one would generally like to learn about. Beside results of approaches currently considered in the literature, we consider new ones which involve, on the one hand, extensions of instant- and front-form approaches to a “parallel” momentum configuration most often ignored and, on the other hand, a point-form approach inspired from Dirac’s one [21]. In the last case, the calculation of form factors supposes some elaboration. They represent a straightforward generalization of those obtained in the front form with an arbitrary orientation of the front. Demonstrating their Lorentz invariance is more tedious however. Together with earlier results, the new ones turn out to be important in revealing both the respective merit of different approaches and the role of properties related to space-time translations.

The plan of the paper is as follows. After reminding some generalities relative to the ingredients entering the calculation of form factors in relativistic quantum mechanics, we successively consider in the second section expressions of form factors in front and instant forms for unusual momentum configurations, and in a point form closer to Dirac’s one. The section is ended by the consideration of form factors in a field-theory model, which in some sense play here the role of an “experiment”. Results are presented in the third section. They involve two form factors (Lorentz vector and scalar) while attention is given to both the low- and high-$Q^2$ behavior, in relation respectively with the radius and power law expectations. A discussion of the results in the light of transformation properties of currents under Poincaré space-time translations is given in the fourth section. The conclusion follows in the fifth section. An appendix contains many details relevant to the derivation of form factors in the “parallel” momentum configuration (with $P \to \infty$) and in the Dirac’s inspired point form. A part is devoted to corrections that allow one to get a Lorentz-invariant scalar form factor at $Q^2 = 0$ in the instant form.

2 Expression of form factors in different approaches

For the ground state of the system considered here, made of scalar particles, there are two form factors, $F_1(Q^2)$ and $F_0(Q^2)$, corresponding to a vector and a scalar probe respectively. Their general definition may be found in Ref. [10] while a schematic representation of the contribution in the single-particle approximation is given in Fig. 1. Considering both of them can provide a better insight on their properties. Their determination in relativistic quantum mechanics implies two ingredients: the relation of the constituent momenta to the total momentum and the solution of a mass operator. These ingredients and the corresponding form factors are successively considered in what follows.
Figure 1: Photon absorption on a two-body system: kinematics relative to a RQM approach (particles on-mass shell: \( e_p = \sqrt{m^2 + p^2} \)). Our convention assumes \( P_f^\mu = P_i^\mu + q^\mu \)

### 2.1 Wave functions

In all cases we are considering, the relation of the constituent momenta, \( \vec{p}_1 \) and \( \vec{p}_2 \), to the total momentum, \( \vec{P} \), can be cast into a unique form:

\[
\vec{p}_1 + \vec{p}_2 - \vec{P} = \frac{\vec{\xi}}{\xi_0} (e_1 + e_2 - E_P),
\]

where the 4-vector, \( \vec{\xi}^\mu \), characterizes each approach. Following the work underlying the Bakamjian-Thomas construction of the Poincaré algebra in the instant form \[27\], it is appropriate to introduce a Lorentz-type transformation that allows one to express the constituent momenta in terms of an internal variable, \( \vec{k} \), which enters the mass operator, and the total momentum, \( \vec{P} \). This transformation, which preserves the on-mass shell character of constituents while fulfilling Eq. (1), is given by:

\[
\begin{align*}
\vec{p}_{1,2} &= \pm \vec{k} \pm \vec{w} \frac{\vec{w} \cdot \vec{k}}{w^0 + 1} + \vec{w} e_k, \\
&\quad e_{1,2} = w^0 e_k \pm \vec{w} \cdot \vec{k},
\end{align*}
\]

where the \( \vec{k} \) vector is defined up to a rotation\(^1\) while the components of the 4-vector \( w^\mu \), \( \vec{w} \) and \( w^0 = \sqrt{1 + (\vec{w})^2} \), are given by:

\[
w^\mu = \frac{P^\mu}{2 e_k} + \frac{\xi^\mu}{2 e_k} \frac{4 e_k^2 - M^2}{\sqrt{(\xi \cdot P)^2 + (4 e_k^2 - M^2) \xi^2} + \xi \cdot P}.\]

The above details, pertinent to the Bakamjian-Thomas construction of the Poincaré algebra, can often be skipped for practical purposes. With this respect, relations of interest, which in particular are independent of the orientation of the \( \vec{k} \)-vector, are the following ones:

\[
\begin{align*}
(p_1 + p_2)^2 &= 4 e_k^2, \\
(p_1 + p_2)^\mu &= P^\mu + \xi^\mu \frac{4 e_k^2 - M^2}{\sqrt{(\xi \cdot P)^2 + (4 e_k^2 - M^2) \xi^2} + \xi \cdot P}.
\end{align*}
\]

\(^1\)This indetermination has no effect on the calculation of form factors. It could however affect, for instance, the comparison of integral expressions aiming to the calculation of the same quantity in different frames when given by an integral over the \( \vec{k} \) variable.
It is noticed that the 4-vector, $\xi^\mu$, appearing in the above expressions, is always associated with a factor $(4e_k^2 - M^2)$, which is nothing but an interaction term. The appearance of this one is a consequence of relying on a unique hypersurface to describe the physics, independent of the system under consideration. It is also seen that the expression is independent of the scale of the 4-vector $\xi^\mu$. Thus, up to an irrelevant scale, the 4-vector $\xi^\mu$, which reflects the symmetry properties of the hypersurface underlying each approach, is given as follows:

- **instant form**
  $$\xi^0 = 1, \quad \vec{\xi} = 0,$$
  (6)

- **front form**
  $$\xi^0 = 1, \quad \vec{\xi} = \hat{n},$$
  (7)
  where $\hat{n}$ is a unit vector with a fixed direction ($\xi^2 = 0$),

- **Dirac’s inspired point form** [24]
  $$\xi^0 = u^0 = 1, \quad \vec{\xi} = \hat{u},$$
  (8)
  where $\hat{u}$ is a unit vector that points to any direction ($\xi^2 = 0$).

The above equations, (2) and (3), can be generalized to an arbitrary hyperplane with orientation $\xi^\mu = \lambda^\mu$ and $\lambda^2 = 1$. They, in particular, allow one to recover the boost transformation introduced in an earlier “point-form” approach [11, 26] by taking $\xi^\mu \propto P^\mu$.

The corresponding expression of the $w^\mu$ four vector, $w^\mu = P^\mu/M$, can be obtained from the other cases by neglecting interaction effects ($2e_k \rightarrow M$). Missing the consistency requirement that underlies them, the calculation of form factors in this approach necessarily implies two hyperplanes determined by the different momenta of the initial and final states.

![Figure 2: Photon absorption on a two-body system: Feynman diagram with corresponding kinematics.](image)

The second main ingredient entering the calculation of form factors concerns a mass operator and its solution, which could be written as:

$$(M^2 - 4e_k^2) \phi_0(k) = \int \frac{d\vec{k}'}{(2\pi)^3} \frac{1}{\sqrt{e_k}} V_{int}(k, \vec{k}') \frac{1}{\sqrt{e_{k'}}} \phi_0(\vec{k}').$$

(9)

A particular case corresponds to the Feynman diagram shown in Fig. 2. The strong interaction vertices appearing in this diagram can be considered as resulting from the exchange
of an infinite-mass boson. Due to this property, little uncertainty on the determination of the mass operator and its solution is expected. Actually, an exact solution can be found \[10\]. It only involves the masses of the constituents, \( m \), and of the system, \( M \). It is given by:

\[
\phi_0(\vec{k}) = \phi_0(k) \propto \frac{1}{\sqrt{\epsilon_k}} \frac{1}{4 \epsilon_k^2 - M^2}.
\]

Another extreme case of interest corresponds to the exchange of a zero-mass boson (Wick-Cutkosky model). Due to a hidden symmetry, the solution of the Bethe-Salpeter equation can be obtained relatively easily. As shown in Refs. \[10, 28\], a reasonable mass operator can be determined.

For practical purposes, we take for the constituent masses and the total mass of the system under consideration values appropriate to the pion, \( m = 0.3 \) GeV and \( M = 0.14 \) GeV. For the zero-mass boson exchange case, we rely on a mass operator of the form given in Eq. (56) of Ref. \[10\] with a coupling constant determined by fitting the total mass. Its high \( k \) behavior is essential in getting the appropriate power-law behavior of form factors. The following normalization of the ground-state wave function of interest here is assumed:

\[
\int \frac{d\vec{k}}{(2\pi)^3} \phi_0^2(k) = 1.
\]

2.2 Form factors in hyperplane-based approaches

In order to calculate form factors, the current to be used has to be specified. This was done in Ref. \[10\] where their expressions in various approaches involving hyperplanes (instant, front and earlier point form) have been given. Written in terms of different variables, it was not always obvious how similar they were looking like.

In the case of the charge form factor, \( F_1(Q^2) \), where the similarity is the most striking, it is found that the different form factors can be cast into a common form given by:

\[
F_1(Q^2) = \frac{1}{(2\pi)^3} \int \frac{d\vec{p}}{2 e_p} \left( \frac{(p_f + p)^2 (p_i + p)^2}{(p_f - p)^2} \right)^{1/4} \phi_0 \left( \frac{(p_f - p)^2}{2} \right) \phi_0 \left( \frac{(p_i - p)^2}{2} \right) \times \frac{\xi_f \cdot (p_f + p) \xi_i \cdot (p_i + p)}{(2 \xi_f \cdot p_f) (2 \xi_i \cdot p_i)} \frac{2 (p_f + p_i) \cdot (\xi_f + \xi_i)}{(p_f + p_i + 2 p) \cdot (\xi_f + \xi_i)}.
\]

The relation of the 4-momenta \( p_i^\mu \) or \( p_f^\mu \) to the 4-momentum of the spectator particle \( p^\mu \), the total momenta \( P_i^\mu \) or \( P_f^\mu \) and the 4-vectors \( \xi_i^\mu \) or \( \xi_f^\mu \) is given by Eq. \[11\] and subsequent ones. The argument of the wave function, \( ((p_1 - p_2)/2)^2 \) is related to the internal \( \vec{k} \) variable by the relation \( ((p_1 - p_2)/2)^2 = -\vec{k}^2 \), so that \( \phi_0(((p_1 - p_2)/2)^2) \) in Eq. \[12\] stands for \( \phi_0(k = ((- (p_1 - p_2)/2)^2)^{1/2}) \). The 4-vectors \( \xi_i^\mu \) and \( \xi_f^\mu \) are identical for the instant and front forms and are given by Eqs. \[13\] and \[17\] respectively. In the front-form case, with the condition \( q^+ = 0 \), standard expressions in terms of the \( x \) and \( k_\perp \) variables can be recovered by making a change of variables (the demonstration is similar to that one given in Appendix \[A\] for a different momentum configuration). For the earlier “point form”, the above 4-vectors are proportional to the velocity of the initial and final states.
and are taken as $\xi^\mu_i = P^\mu_i/M$, $\xi^\mu_f = P^\mu_f/M$. As expected, Eq. (12) is invariant under a change of the scale of the 4-vectors, $\xi^\mu$. It can also be shown that the following equality holds:

$$F_1(Q^2 = 0) = \int \frac{d\vec{k}}{(2\pi)^3} \phi_0^2(k) = 1,$$

(13)

independently of the velocity of the system. The specific form of the last factor at the second line in Eq. (12) is especially relevant to obtain this result in the instant form.

In the case of the scalar form factor, $F_0(Q^2)$, the expression of the different form factors reads:

$$F_0(Q^2) = \frac{1}{(2\pi)^3} \int \frac{d\vec{p}}{2e_p} \left( (p_f + p)(p_i + p)^2 \right)^{1/4} \phi_0 \left( \frac{(p_f - P_i)^2}{2} \right) \phi_0 \left( \frac{(p_i - P_f)^2}{2} \right) \frac{\xi_f \cdot (p_f + p) \cdot \xi_i \cdot (p_i + p)}{(2\xi_f \cdot p_f)(2\xi_i \cdot p_i)} \ cf_0,$$

(14)

where the definitions relative to momenta are the same as for $F_1(Q^2)$. The coefficient, $c_{f_0}$, is introduced to make the value of the scalar form factor at $Q^2 = 0$, $F_0(0)$, independent of the velocity of the system, as expected from a minimal Lorentz invariance requirement. With this respect, only the instant-form approach raises a problem. The method allowing to determine $c_{f_0}$ is described in the Appendix B. Its expression reads:

$$c_{f_0} = 1 + \frac{g(k_i) + g(k_f)}{2} \frac{e_p \left( (e_f + e_p)(e_i + e_p) - E_f E_i \right)}{(e_f + e_p)(e_i + e_p)(e_f + e_i)},$$

(15)

where the function $g(k)$ can be obtained from a quadrature, see Eq. (11). In the case of a scalar-particle model together with a zero-range interaction, it is found that $g(k) = 1$. The above factor then allows one to reproduce the “exact” scalar form factor, $F_0(Q^2)$, at all $Q^2$. In the other extreme corresponding to a zero-mass exchange interaction (Wick-Cutkosky model), $g(k)$ takes a value close to 1/3. In all the other approaches, we assume $c_{f_0} = 1$. Such a result is actually obtained from an extension of Eq. (15) to an arbitrary hyperplane with orientation given by a 4-vector $\xi^\mu$:

$$c_{f_0} = 1 + \frac{g(k_i) + g(k_f)}{2} \frac{\xi \cdot p \left( \xi \cdot (p_f + p) \xi \cdot (p_i + p) - \xi \cdot P_f \xi \cdot P_i \right)}{\xi \cdot (p_f + p) \xi \cdot (p_i + p) \xi \cdot (P_f + P_i)}.$$

(16)

For the front form (using Eq. (11)) or the instant form with the “parallel” momentum configuration and $\tilde{P} \to \infty$ (see Eq. (10)), it is found that the coefficient of the $g(k)$ factor in the above equation vanishes.

It is noticed that the last factor in Eq. (12) characterizes the charge form factor. The numerator contains the factor $(p_i + p_f)^\mu$ which is part of the photon coupling to scalar particles while the denominator corresponds to the factor $(P_i + P_f)^\mu$ that has to be factored out from the matrix element of the current. Its particular form ensures that the ratio $F_1(0)/F_0(0)$ predicted by simple theoretical models is approximately or exactly recovered [10]. The last factor in Eq. (12) is evidently absent for the scalar form factors where it is replaced by the factor $c_{f_0}$ when necessary (instant form), ensuring that the form factor at $Q^2 = 0$ be Lorentz invariant as already explained. This amounts to account for some
two-body currents. We would finally like to remark that the above factor in Eq. \((12)\) is responsible for getting the asymptotic ratio, \(F_1(Q^2)/F_0(Q^2)\) \((Q^2 \to \infty) = 2\) (taking into account the definitions adopted for the form factors \([10]\)). This agrees with expectations from the underlying field-theory model.

### 2.3 Limit for a parallel-momentum configuration (and large momenta)

It is known that the expression of a form factor calculated in the instant form with the momentum configuration \(\vec{q} \perp (\vec{P}_i + \vec{P}_f)\) \((E_i = E_f)\) and \(|\vec{P}_i + \vec{P}_f| \to \infty\) is close or even identical to that one obtained in the standard front-form approach \((q^+ = 0,\ \text{also denoted “perpendicular” in the following})\). The choice of the currents ensures the identity in the present case. Another limit of interest corresponds to take the parallel configuration, \(\vec{q} \parallel (\vec{P}_i + \vec{P}_f)\) \((E_i \neq E_f)\) and \(|\vec{P}_i + \vec{P}_f| \to \infty\) (denoted “parallel” in the following). Apart from a few recent works, this limit is rarely considered. Form factors so obtained however evidence a feature that, in our opinion, casts a completely new insight on earlier “point-form” results. We give here the expression of the form factors in this limit while some details about the derivation are given in Appendix \([A]\). This is conveniently done using the variables employed in the standard front-form approach, \(x\) and \(k_\perp\). The noticeable point is that the dependence on the momentum transfer, \(Q\), appears through the quantity \(v = \sqrt{Q^2/(Q^2 + 4M^2)}\), which is nothing but the velocity of the system in the Breit frame and, most important, is the same as in point-form approaches. The most symmetrical expressions read:

\[
F_1(Q^2) = \frac{1}{(2\pi)^3} \int_0^{1-v} dx \frac{(1-x)}{2x} \frac{1-v^2}{(1-x)^2 - v^2} \int d^2k_\perp \tilde{\phi}(k_i^2) \tilde{\phi}(k_f^2),
\]

\[
F_0(Q^2) = \frac{1}{(2\pi)^3} \int_0^{1-v} dx \frac{1}{4x} \frac{1-v^2}{(1-x)^2 - v^2} \int d^2k_\perp \tilde{\phi}(k_i^2) \tilde{\phi}(k_f^2),
\]

with \(\tilde{\phi}(k) = \sqrt{\epsilon_k} \phi_0(k)\),

\[
k_i^2 = k_\perp^2 + (m^2 + k_\perp^2) \frac{(1-2x-v)^2}{4x(1-x-v)},
\]

\[
k_f^2 = k_\perp^2 + (m^2 + k_\perp^2) \frac{(1-2x+v)^2}{4x(1-x+v)}.
\]

Interestingly, these expressions are identical to those obtained in the front form with the momentum configuration \(\vec{q} \parallel \vec{n}\) (denoted “parallel” in the following), which can be obtained from the original ones, Eqs. \((12, 14)\), by performing a change of variable (see second part of Appendix \([A]\)). It is reminded that a similar identity generally holds for the “perpendicular” momentum configuration.

### 2.4 Form factor in Dirac’s point form case

The need for developing a point-form approach more in the spirit of the Dirac’s one \([24]\) is due for a part to the drawbacks evidenced by an earlier implementation to reproduce
form factors calculated in a very simple theoretical model \[17\]. As mentioned elsewhere \[18\], this last approach implies hyperplanes perpendicular to the velocity of the initial and final states while the Dirac’s one is based on a hyperboloid surface, which is at the same time unique and independent of the system under consideration. The question therefore arises of whether the new approach can improve the calculation of form factors. In Ref. \[24\], it is shown that the correct power-law behavior of form factors could be recovered in a simple case. We here consider the calculation of these form factors on a more general ground, which is done for the first time. We give and explain the expression of form factors in this subsection while details are given in Appendix C.

Consistently with the absence of a direction on a hyperboloid, it is expected that the sum of the constituent momenta points isotropically to any direction in the c.m. case. This direction, which is to some extent a new degree of freedom and turns out to be conserved, is here represented by a unit vector \(\hat{u}\). It is therefore expected that the expression of form factors involves an integration over this orientation. This represents the main new feature evidenced by the form factors considered here. The integration over \(\hat{u}\) is not arbitrary however. It includes some weight which ensures Lorentz invariance and is obtained from considering the expression of the norm for instance. A minimal expression thus takes the form:

\[
F(Q^2) = \int \frac{d\hat{u}}{4\pi} \left( \frac{M}{P \cdot u} \right)^2 \cdots ,
\]

(18)

where \(P^\mu\) is the 4-momentum of the initial or final state. How factors entering Eq. \[18\] change under a Lorentz transformation while preserving the invariance of the full expression is described in Appendix C. Two points in the demonstration are to be noticed. The property, \((u^0)^2 - \vec{u}^2 = 0\), remains unchanged, allowing one to define a new unit vector, \(\hat{u}' = \vec{u}/u^0\). The change in the scale of the 4-vector, \(u^\mu\), which occurs in the transformation, is compensated by a modification in the other factors so that we can choose \(u^0 = 1\), while the change in the orientation, \(\hat{u} \to \hat{u}'\), can be absorbed into the integration over \(\hat{u}\). As for the dots at the r.h.s. of Eq. \[18\], they represent other factors like wave functions and matrix elements of the current. The important point is that these factors be formally Lorentz invariant and invariant under a change of scale of the 4-vector, \(u^\mu\). For this part, one can thus use Eqs. \[12 \quad 14\], which offers the advantage to introduce no bias in the comparison that will be made later on with other forms. Taking into account the symmetry between initial and final states, the expressions for the charge and scalar form factors thus read:

\[
F_1(Q^2) = \int \frac{d\hat{u}}{4\pi} \left( \frac{M}{P_i \cdot u} \right)^2 \frac{1}{(2\pi)^3} \int \frac{d\vec{p}}{2e_p} \phi_0 \left( \frac{(p_f - p)^2}{2} \right) \phi_0 \left( \frac{(p_i - P_i)^2}{2} \right) \times \left( (p_f + p)^2 (p_i + p)^2 \right)^{1/4} \frac{u \cdot (p_f + p)}{(2u \cdot p_f)} \frac{u \cdot (p_i + p)}{(2u \cdot p_i)} \frac{2 (p_f + p_i) \cdot u}{(p_f + p_i + 2p) \cdot u}
\]

\[
F_0(Q^2) = \int \frac{d\hat{u}}{4\pi} \left( \frac{M}{P_i \cdot u} \right)^2 \frac{1}{(2\pi)^3} \int \frac{d\vec{p}}{2e_p} \phi_0 \left( \frac{(p_f - p)^2}{2} \right) \phi_0 \left( \frac{(p_i - P_i)^2}{2} \right) \times \left( (p_f + p)^2 (p_i + p)^2 \right)^{1/4} \frac{u \cdot (p_f + p)}{(2u \cdot p_f)} \frac{u \cdot (p_i + p)}{(2u \cdot p_i)} .
\]

(19)

In these expressions, the relation of the struck-particle momenta, \(p_i^\mu\), \(p_f^\mu\), are given in terms of the spectator one, \(\vec{p}^\mu\), by Eqs. \[13 \quad 8\].
2.5 “Experiment”

Among theoretical models, two of them, already mentioned, are of special interest. Corresponding to two opposite extreme cases, they involve scalar particles interacting by exchanging an infinite-mass boson (interpretation of strong interaction vertices in the triangle Feynman diagram, Fig. 2) and a zero-mass one (Wick-Cutkosky model). In both cases, the Bethe-Salpeter amplitude is given under a form which is analytical for the essential factors. The calculation of form factors, which implies a Wick rotation, can thus be performed without much difficulty. Expressions of charge and scalar form factors so obtained for the ground state have been given in Eq. (48) of Ref. [10] for the triangle Feynman diagram and the appendix of Ref. [20] for the Wick-Cutkosky model. We here notice that these form factors behave asymptotically as $Q^{-2}$ and $Q^{-4}$ respectively, up to log terms (see Alabiso and Schierholz for more general predictions in relation with the behavior of the interaction at high-momentum transfer [29]). In the former case, an analytic expression is available in the limit $M \to 0$ (see Ref. [10], Eq. (50)). In the latter one, only an approximate expression can be obtained in the same limit. The asymptotic behavior can nevertheless be determined exactly. As it offers some interest (beside providing a useful numerical check), it is given here:

\[
F_1(Q^2)_{Q^2 \to \infty} = 2 F_0(Q^2)_{Q^2 \to \infty} = 540 \left( \frac{m^2}{Q^2} \right)^2 \left( \frac{\sqrt{4 m^2 + Q^2}}{2 Q} \log \left[ \frac{\sqrt{4 m^2 + Q^2} + Q}{\sqrt{4 m^2 + Q^2} - Q} \right] - 2 \right.
\]
\[
+ \left. \frac{m^2}{Q^2} \log^2 \left[ \frac{\sqrt{4 m^2 + Q^2} + Q}{\sqrt{4 m^2 + Q^2} - Q} \right] \right).
\]

(20)

Departures to the exact result, located at low $Q^2$, do not exceed those observed at $Q^2 = 0$ (3/2 and 3/4 instead of 1 and 5/4 for $F_1(0)$ and $F_0(0)$ respectively). Departures due to the non-zero mass of $M$ amount to 10-20% over the full range of $Q^2$. We also stress that the present “Bethe-Salpeter” results, contrary to those sometimes referred to in the literature under the same name [30], do not involve any approximation like an instantaneous one for the interaction. They are fully relativistic, verifying expected properties under both Lorentz transformations and space-time translations. Moreover, they satisfy current conservation when applicable.

To a large extent, the predictions of the above models play the role of measurements. In comparison with a real physical problem, the comparison of these “measurements” with results obtained in the frame of relativistic quantum mechanics offers many advantages. The physics is quite simple (one-boson exchange, no crossed diagram). Intrinsic form factors, if any, cancel in the comparison. Moreover, there is no spin complication. The comparison can thus be particularly useful to test minimal ingredients entering relativistic quantum mechanics. In spite of this simplicity, reproducing the predictions for the asymptotic behavior and especially the log terms is not trivial. They therefore represent a severe test for RQM approaches, both for the mass operator and the currents. They could be quite relevant when looking at a more realistic problem like the pion charge form factor. As a side remark, the comparison can tell about different effects which have been mentioned in the literature in relation with the implementation of relativity. The first one, which is of direct relevance for the present work, concerns the “point-form” approach.
at high $Q^2$. It was found that the dependence of form factors on $Q^2$ was affected by an extra factor as follows [15]:

$$Q^2 \rightarrow Q^2 \left(1 + \frac{Q^2}{4M^2}\right).$$

(21)

It provides an asymptotic dependence $Q^{-2n}$ where other approaches would give $Q^{-n}$, hence a faster drop off. The second effect concerns Lorentz-contraction. In order to take it into account, it was proposed to modify non-relativistic form factors by changing the argument as follows: $Q^2 \rightarrow Q^2/(1 + Q^2/(4M^2))$, which leads to constant form factors at high $Q^2$. Amazingly, this recipe involves the same relativistic correction factor, $(1 + Q^2/(4M^2))$, as in Eq. (21), but at the denominator instead of the numerator. It has been however mentioned that the recipe was incorrect. Some analysis of what it is missing has been described in Ref. [10].

### 3 Form factors: results

![Diagram](image.png)

Figure 3: Charge form factor in various forms of relativistic quantum mechanics and an infinite-mass boson-exchange model: left for low $Q^2$ and right for high $Q^2$ (the last one is multiplied by a factor $Q^2$ to compensate an expected $Q^{-2}$ behavior). The “exact” results (our “experiment”) are represented by diamonds.

In looking at form factors, two domains are of special interest. At low $Q^2$, they are sensitive to the radius (charge or else depending on the probe) while at high $Q^2$, they generally evidence power-law behaviors that can be compared to expectations. We therefore present accordingly the results in two figures for each form factor, up to $Q^2 = 0.2 \text{ (GeV/c)}^2$ and $Q^2 = 100 \text{ (GeV/c)}^2$. Moreover, in this second case, we show the form factor multiplied by the inverse of its expected asymptotic behavior, respectively $Q^2$ and $Q^4$ for the infinite-mass and zero-mass boson exchange. Thus, these last results should evidence a plateau at high $Q^2$, up to possible $\log^2(Q)$ corrections.
Figure 4: Same as in Fig. 3 for the scalar form factor.

Form factors for the infinite-mass boson exchange case, \( F_1(Q^2) \) and \( F_0(Q^2) \), are presented in Figs. 3 and 4 respectively, while those for the zero-mass boson exchange are shown in Figs. 5 and 6. Beside the “exact” result (our “experiment”) represented by data points, each figure contains six curves:
- the front-form form factor in the “perpendicular” momentum configuration \( q^+ = 0 \) (F.F. (perp.)),
- the instant-form form factor in the Breit frame (I.F. (Breit frame)),
- the front-form form factor in a “parallel” momentum configuration \( \vec{q} \parallel (\vec{P}_i + \vec{P}_f) \parallel \vec{n} \) (F.F. (parallel)),
- the instant-form form factor in a “parallel” momentum configuration \( \vec{q} \parallel (\vec{P}_i + \vec{P}_f) \) with \( |\vec{P}_i + \vec{P}_f| \to \infty \) which coincides with the previous one (I.F. (parallel)),
- the form factor calculated in a Dirac’s inspired point form (D.P.F.),
- and the form factor calculated in an earlier implementation of the point-form (“P.F.”).

Examination of Figs. 3-6 shows that the various curves clearly fall into two sets, those that are close to the “experiment” and the other ones that are far apart. This occurs both at low and high \( Q^2 \). The first set comprises standard instant- and front-form calculations while the second one includes the same approaches with non-standard momentum configurations as well as point-form results that stem from a Lorentz covariant approach. Looking for an argument that can discriminate between different curves, we notice that the first set corresponds to approaches where the boost implementation is essentially independent of the mass of the system\(^2\) while form factors in the second set all depend on \( Q \) though the ratio \( Q/(2M) \). As noticed elsewhere, this feature has the consequence that the charge (or Lorentz-scalar) squared radius scales like \( 1/M^2 \), a feature which has so surprising effects that one can suspect that the underlying formalism misses an important

\(^2\)Instant form in the Breit frame and more generally with \( E_i = E_f \) \( (\vec{q} = \vec{P}_f - \vec{P}_i) \perp (\vec{P}_i + \vec{P}_f) \), front form with \( E_f - E_i - (\vec{P}_f - \vec{P}_i) \cdot \vec{n} = 0 \).
Figure 5: Charge form factor in various forms of relativistic quantum mechanics and a zero-mass boson-exchange model: left for low $Q^2$ and right for high $Q^2$ (the last one is multiplied by a factor $Q^4$ to compensate an expected $Q^{-4}$ behavior).

property. Of course, this can be repaired by appropriate two-body currents.

A closer examination at Figs. 3-4, which correspond to an interaction model with an infinite-mass boson exchange, shows that some results coincide with “experiment”. These results include the standard front-form results and some instant-form ones. The agreement in the first case is not totally surprising. On the one hand, the “zero-range” nature of the interaction does not leave much freedom on the solution of the mass operator. On the other hand, in a field-theory approach, it is known that corrections due to $Z$-type diagrams are suppressed in the case of scalar particles and $q^+ = 0$ [31]. This result cannot be applied to relativistic quantum mechanics but, taking into account that a $Z$-type diagram has often a contact term as a counterpart in this formalism, it makes the absence of correction in this case plausible. The agreement in the instant form for the scalar form factor is more surprising. As mentioned in Sect. 2, the current was including a correction ensuring that $F_0(Q^2 = 0)$ be Lorentz invariant (factor $c f_0$ in Eq. (14)). It turns out that this constraint entails the identity of the form factor $F_0(Q^2)$ with the “experiment” at all $Q^2$. In the case of the charge form factor, $F_1(Q^2 = 0)$ was in any case Lorentz invariant, requiring no correction. We notice however that a correction to the current could have been introduced in Eq. (12), preserving the Lorentz invariance of $F_1(Q^2 = 0)$, while providing identity with the “experiment” [10].

A last remark about the infinite-mass boson results concerns the asymptotic behavior of form factors. Most of them scale like $Q^{-2}$. This is clearly seen for some of the results but not so clear for the “experiment” and the standard instant- and front-form results. In these cases, the slight increase (after multiplying form factors by $Q^2$) is due to non-trivial $\log^2(Q)$ corrections whose reproduction is a stringent test of the implementation of relativity. The only exception concerns the “point-form” implementation that produces form factors scaling like $Q^{-4}$. This behavior can be traced back to the observation that
results in Eq. (21). As this feature is not evidenced by the Dirac’s point-form, we believe that it is a specific feature of the approach. Most probably, it is due to the fact that, contrary to all other ones, it implies different surfaces in the description of initial and final states.

The general pattern of results presented in Figs. 5, 6, which correspond to a zero-mass boson exchange, is similar to that of Figs. 3, 4. Significant differences are nevertheless worthwhile to be mentioned. Considering first the “good” results, those for the standard instant and front forms are close to the “experiment” but none is identical. Due to the long range of the underlying interaction model, some uncertainty is expected in the derivation of the mass operator. We however notice that the “good” result is largely due to the choice of the high momentum behavior of the interaction $V(\vec{k}, \vec{k}')$ entering the mass operator. As noticed by Alabiso and Schierholz [29], the behavior of form factors at high $Q^2$ is closely related to the interaction one at high $k$, which has thus to fulfill well determined conditions. This result is important as it provides minimal guidelines for further work concerning the pion form factor for instance. At low $Q^2$, one can notice some discrepancy for the form factor $F_0(Q^2)$, which, contrary to the charge form factor, $F_1(Q^2)$, is not protected by some charge conservation. This points to the missing contribution of relatively standard two-body currents. It is nevertheless interesting that the two-body currents implied by the introduction of the factor, $c f_0$, in Eq. (14), make the standard instant- and front-form ones equal to each other at $Q^2 = 0$, while decreasing their difference at non-zero values of $Q^2$. The smaller discrepancy, which does not exceed 6% (instead of 30%) points to a partial restoration of Lorentz invariance. Considering now the “bad” results, it is found that the sensitivity to the approach under consideration is much larger than for the zero-range interaction model. This increased sensitivity is in relation with the expected asymptotic behavior $Q^{-4}$, which implies a ratio of instant- and front-form results with standard and non-standard momentum configurations of the order $(2 m/M)^4$, instead of $(2 m/M)^2$ previously (up to log terms). As for the “point-form” approach, the asymptotic behavior is $Q^{-8}$, as expected from the above $Q^{-4}$ behavior together with the modification
A few comments have already been made about the results obtained in a Dirac motivated point-form. As such results are presented for the first time, it is appropriate to discuss them separately a little more. It is first noticed that its Lorentz covariance does not ensure it provides “good” results. While it does better than an earlier “point-form” approach, especially with respect to the asymptotic behavior of form factors that now evidence the right $Q^2$ power law, it suffers from the fact that their dependence on the momentum transfer involves the ratio $Q/2M$, implying obvious drawbacks in the limit $M \to 0$. Actually, it turns out that these undesirable features are shared by instant- and front-form approaches with unusual momentum configurations, which suggests that the problems raised in the above limit have a more general character, independent of the intrinsic Lorentz covariance of the point-form approach. A second observation concerns how these point-form results compare quantitatively with other ones. Representing a weighted average of contributions that involve in particular the standard and non-standard front-form approaches, it is not surprising that the new point-form results fall in between. Correcting the misleading impression produced by the logarithmic scale in the right panels of Figs. 3-6, these results are however closer to the later ones (non-standard) than to the former ones (standard momentum configuration). Throughout this paper, we considered a strongly bound system. Apart from the fact that some of the inputs correspond to a physical system (the pion), an extreme case like the one we considered offers the advantage of better emphasizing the peculiarities pertinent to the formalism. Looking at a weakly bound system would not have been so instructive.

Results for form factors presented in this section show that they strongly depend on the underlying formalism when only the single-particle current is considered. Though no detailed study was made here, it appears that the largest discrepancies with “experiment” can be interpreted as if the momentum transfer was effectively larger than the physical one. The factor could be of the order $2\bar{\epsilon}_k/M$ in the instant and front forms with “parallel” momentum configuration as well as in the Dirac’s inspired point form ($\bar{\epsilon}_k$ is some average value for the internal kinetic energy). An extra factor, $(1 + Q^2/4M^2)^{1/2}$ should be considered for the “point form”. The discrepancy between the form factors calculated in the standard instant- and front-form approaches does not exceed a few percent’s. This roughly summarizes the main features of numerical results presented in this section.

4 Discussion and relationship to Poincaré space-time translation invariance

In view of the results presented in the previous section, the question arises of whether there is a way to discriminate results from a simple argument and, possibly, to remove the main discrepancies. We first notice that a Lorentz-covariant approach like the point form, which a priori represents a sensible feature, does not guarantee to get a “good” result. Such situations often occur in physics. In a region of intrinsically deformed nuclei for instance, the binding energy of a spherical nucleus ($J = 0$) is better obtained by using a mean field which breaks the spherical symmetry. Observing that the “good”
results are obtained in the following cases, instant form with $E_f = E_i$ (this goes beyond the standard Breit frame mostly mentioned in the present work) and front form with $\xi \cdot (P_i - P_f) = q^+ = 0$, a more important criterion could be the conservation of the 4-momentum at the interaction vertex of the external probe with the struck constituent. This condition, which is fulfilled in field-theory models, can only be verified approximately in relativistic quantum mechanics. The best that one can require is that this condition be fulfilled on the average, $< (p_i + q - p_f)^\mu > = 0$ (which is a much weaker constraint than $(p_i + q - p_f)^\mu = 0$). As the conservation of the 4-momentum stems from Poincaré space-time translation invariance, one can also infer that an equivalent criterion involves the conditions that make this result possible. With this respect, it was noticed by Coester that the momentum $p$ in the point-form kinematics does not generate translations consistent with the dynamics [32].

Quite generally, Poincaré covariance implies that a 4-vector (or a scalar) current transforms under space-time translations as follows:

$$e^{iP^a J^\nu(x) (S(x))} e^{-iP^a} = J^\nu(x + a) (S(x + a)),$$

where $P^\mu$ is the 4-momentum operator of the Poincaré algebra that generates space-time translations. In a particular case, this equation reads:

$$J^\nu(x) (S(x)) = e^{iP^a J^\nu(0)} (S(0)) e^{-iP^a}.$$  

Matrix elements of this relation between states with 4-momentum $P^\mu_i$ and $P^\mu_f$ can be considered. By construction of the Poincaré algebra pertinent to a RQM approach, these states are eigenstates of the 4-momentum $P^\mu$ with eigenvalues $P^\mu_i$ and $P^\mu_f$. This allows one to factorize the $x$ dependence of the matrix elements as $\exp(i (P_i - P_f) \cdot x)$. The integration over $x$ of this factor together with the factor $\exp(i q \cdot x)$ describing the external probe then provides the well known energy-momentum conservation relation $(P_i + q - P_f)^\mu = 0$. Quite generally, fulfilling the above relations requires the consideration of many-body components in the current $J^\nu(x)$ (or $S(x)$), beside the one-body component most often retained in the calculations. Covariant transformations of currents under space-time translations in RQM approaches can therefore imply further constraints beyond the usual energy-momentum conservation relation that is made possible by these covariance properties and is supposed to hold in any case.

Further equations that could be more amenable to some check are obtained by considering an expansion of Eq. (22) for small space-time translations. In the simplest case, they read [33]:

$$[P^\mu, J^\nu(x)] = -i \partial^\mu J^\nu(x), \quad [P^\mu, S(x)] = -i \partial^\mu S(x).$$  

While some information about the many-body components entering the current $J^\nu(x)$ (or $S(x)$) can be obtained from a parallel study within field theory, the above constraints are, in first place, the proper way to introduce them in the RQM formalism used here, where they reduce to two-body ones. In their absence, one can at least demand that the matrix elements of the r.h.s. and l.h.s. of the above equations be equal. For form factors considered in the present work, the commutator of the momentum operator
at the l.h.s. can be transformed into a matrix element involving the momentum transfer \( q^\mu \), using the conservation of the overall momentum. The equality requirement with the r.h.s. then implies that the momentum transferred to the system of interest and that one transferred to the struck constituent be the same on the average, \( \langle q^\mu \rangle = \langle (p_f - p_i)^\mu \rangle \). This is precisely the condition that is suggested for getting “good” form factors. Looking at the other results, it appears that they all violate this equality. In the instant form with the “parallel” momentum configuration, the equality is verified for the spatial components but not for the time one. In the point form, due to the presence of the interaction in the momentum, it is \textit{a priori} violated in all components.

The discussion can be extended to any number of commutators or derivatives in Eqs. (24). Among them, the relations involving the following double commutators:

\[
\left[ P_\mu, \left[ P^\mu, J^\nu (x) \text{ or } S(x) \right] \right] = -\partial_\mu \partial^\mu J^\nu (x) \text{ or } S(x),
\]

and its matrix elements at \( x = 0 \):

\[
\langle |q^2 J^\nu (0) \text{ or } S(0)| \rangle = \langle |(p_i - p_f)^2 J^\nu (0) \text{ or } S(0)| \rangle,
\]

are particularly relevant here. In getting the l.h.s., we made use of the energy-momentum conservation. This relation allows one to replace by \( q^2 \) the product \((P_f - P_i)^2\) which results from applying the momentum operator in Eq. (25) on the initial and final states. The appearance at the l.h.s. of the scalar quantity, \( q^2 = -Q^2 \), on which form factors considered in this work exclusively depend, provides a basis for a quantitative discussion better than \( q^\mu \). As for the quantity, \((p_i - p_f)^2\), appearing at the r.h.s. of the last equation, we notice at this point that the use of Eq. (5) allows one to write it as:

\[
(p_i - p_f)^2 = \left((P_i - P_f)^\mu + \xi^\mu (\Delta_i - \Delta_f)\right)^2 = (P_i - P_f)^2 + 2 \xi \cdot (P_i - P_f) (\Delta_i - \Delta_f) + \xi^2 (\Delta_i - \Delta_f)^2,
\]

where

\[
\Delta_{(i,f)} = \frac{4 e_k^2 - M^2}{\sqrt{(\xi \cdot P_{(i,f)})^2 + (4 e_k^2 - M^2) \xi^2}}.
\]

As the examination of the last quantity shows, the discrepancy with the quantity \( q^2 \) appearing at the l.h.s of Eq. (26) involves interaction effects. We stress that the factor \((p_i - p_f)^2\) depends on derivatives of the current around \( x = 0 \), implying therefore current properties going beyond those required to obtain the energy-momentum conservation relation.

We now compare the matrix elements of both sides of Eq. (25), numerically, or analytically when possible, while retaining only the one-body component of the current as done most often. Beginning with the standard front-form approach \((\xi^2 = 0, \xi \cdot (P_i - P_f) = 0)\), one can easily check from the expression of \((p_i - p_f)^2\), Eq. (27), that the equality is always fulfilled. In the standard instant-form approach \((E_i = E_f)\), the second term at the r.h.s. of Eq. (27) vanishes and only the last term \( (\propto \xi^2) \) provides some departure. This one is found to amount to 20% at most (at low \( Q^2 \)) and vanishes when the average momentum of the system goes away from the Breit-frame case, \( \vec{P}_i + \vec{P}_f = 0 \), to the limit \( |\vec{P}_i + \vec{P}_f| = \infty \), where the above front-form result is recovered.
Contrary to the above results, those for the instant and front forms in the “parallel” momentum configuration involve a non-zero contribution from the second term at the r.h.s. of Eq. (27) but none from the last term. For the charge form factor, the departure to the expected equality of the two members of Eq. (26) decreases from a factor 45 at low $Q^2$ to 30 at the highest values of $Q^2$ considered in this work. Typically, this factor is of the order $(2 \bar{e}_k/M)^2$, as suggested by Eq. (27). In the Dirac’s point-form inspired approach, the departure, which is also produced by the second term at the r.h.s. of Eq. (27), decreases from a factor 15 to 3, a value which is intermediate between the two front-form ones given above. Finally, for the “point-form” approach, where one has to account for different $\xi^\mu$ in the initial and final states, the departure is found to increase from a factor 30 at low $Q^2$ to about 35000 at $Q^2 = 100 \text{(GeV/c)}^2$, the factor being roughly given by $((2 \bar{e}_k/M) (E_Q/2M))^2$, as can be checked from Eq. (27). To a large extent, the above departures are in accordance with those inferred from the numerical examination of form factors in the previous section. This shows that the discrepancy of the “bad” form factors with the “exact” ones is closely related to a violation of properties related to Poincaré translational covariance of currents. In comparison to the large effects mentioned above, we notice that the violation of Lorentz invariance is only a few 10% in cases where the above properties are approximately fulfilled (see effect for the scalar form factor $F_0(Q^2 = 0)$ or that one resulting from the difference of the standard instant and front-form form factors at any $Q^2$).

As can be expected, the above departures point to the missing contribution of interaction effects. This is easily checked in the most striking cases where the factor multiplying $q^2$ at the r.h.s. of Eq. (26) is given by $4 \bar{e}_k^2/M^2$. As the consideration of various examples shows [20], the numerator has often to be completed by an interaction term as follows, $4 \bar{e}_k^2 \rightarrow (4 \bar{e}_k^2 + 4 m \tilde{V})$. When this is done, the use of the mass equation, Eq. (9), allows one to replace $(4 \bar{e}_k^2 + 4 m \tilde{V})/M^2$ by 1. It is not rare that such a result stems from the restoration of some symmetry properties, in relation with Poincaré translational covariance of currents in the present case. In principle, the missing interaction effects should be accounted for by two-body currents pertinent to the underlying formalism so that to ensure the equivalence of different approaches.

Examining the problem of these two-body currents, we found that they differ from those mostly encountered in nuclear physics for instance. In some limit, they have the form $0/0$, which rather suggests that their role is related to the restoration of a symmetry. Only such currents, with a non-trivial behavior, can correct either for the paradox of a radius going to $\infty$ while the mass of the system goes to 0 (which can be obtained by increasing the attraction), or for the discrepancy with dispersion approaches (which cannot a priori produce such a radius scaling). Actually, these currents involve a slowly converging series in terms of the coupling constant, generally requiring an infinite number of contributions tending individually to zero in the $M \rightarrow 0$ limit. Having a specific role, it is not a surprise if such currents could not be obtained in an earlier work where two-body currents were motivated by current conservation and the high-$Q^2$ behavior of the Born amplitude [20]. Evidently, these conventional currents can play a role but rather at the level of the relatively slight difference between the standard instant- and front-form form factors. This is illustrated by results presented in the previous section for the scalar form factor where this difference was reduced, therefore tending to restore Lorentz invariance.
as far as these approaches are concerned.

Figure 7: Charge form factor in various forms of relativistic quantum mechanics and a zero-mass boson-exchange model: same as in Fig. 5 but with including contributions that partly correct for missing properties related to Poincaré translational covariance of currents.

An alternative but approximate way to account for the missing interaction effects is suggested by the above observation about their role in restoring some symmetry. To remove the effect of the undesirable factor, $4 e^2 k^2 / M^2$, it suffices to compensate for it in the matrix element of the single-particle current. This can be done by multiplying the coefficient of the squared momentum transfer, $Q^2$, appearing in the calculations, by the inverse of the above factor. This represents a schematic way to proceed. In practice, there are some corrections to this factor, especially for the “point-form” approach. An improved implementation of the missing interaction effects along the above lines thus consists in introducing, in place of $4 e^2 k^2 / M^2$, the departure factor resulting from the numerical comparison of both sides of Eq. (26). This can be done analytically in the standard instant-form and in the “point-form” approaches. Equation (26) is then fulfilled exactly (which does not necessarily imply that symmetry properties related to the translational covariance of currents are fully restored). In the other cases, it is done numerically. No change is required for the standard front-form approach which satisfies this equation identically as already mentioned.

The corrected results are presented in Fig. 7 for the charge form factor calculated in the infinite-range interaction model at both low and high $Q^2$. Comparison with the corresponding results presented in Fig. 5 shows a spectacular decrease of the discrepancies. At low $Q^2$, the slope of the form factor, which determines the charge radius, differs from the standard front-form one by 25% at most instead of a factor up to 20 before. At large $Q^2$, the discrepancy now reaches one order of magnitude instead of a few orders, up to 7 in the worse case. Altogether, these results demonstrate that properties related to Poincaré space-time translations are relevant in describing form factors reliably. These
properties go beyond the energy-momentum conservation generally expected from the Lagrangian invariance under these transformations. At the same time, all the drawbacks that appear especially in the limit of a zero-mass system and could point to the violation of some symmetry are removed. Somewhat incidentally, the above approach was used in Ref. [12], allowing the authors to get a reasonable result for the pion charge form factor, but no justification could be given within the underlying formalism. It was also noticed that the same wave function could give rise to quite different form factors, depending on the total mass of the system [20]. This arbitrariness is largely removed by fulfilling the constraints discussed in this section.

There remain some discrepancies. In this respect, we notice that the above method works relatively better when the violation of the equality given by Eq. (26) is larger. Thus, for the case where the remaining discrepancy for the form factor is the largest (D.P.F. at large $Q^2$), the violation of Eq. (26) was of the order of a factor 3 before correction. This indicates that some refinement is necessary. Pursuing along this line in the cases where the violation of Eq. (26) is originally large is questionable however. Using an approach that introduces interaction effects that turn out to be fictitious, as they have to be removed later on in one way or another, is not the best strategy, especially if further corrections have to be considered. Thus, from the present study, only the standard front-form approach ($\xi \cdot (P_i - P_f) = q^+ = 0$) or the the instant-form one ($\xi \cdot (P_i - P_f) = E_i - E_f = 0$) appear as viable when the current is restricted to a single-body one. The last approach includes the standard Breit-frame case ($\vec{P}_i + \vec{P}_f = 0$) but also the case $\vec{P}_i + \vec{P}_f \neq 0$ with $(\vec{P}_i + \vec{P}_f) \perp (\vec{P}_i - \vec{P}_f)$, which allows one to make the relation with the previous front-form case in the limit $|\vec{P}_i + \vec{P}_f| \to \infty$. It has been often thought that these frameworks were more relevant than other ones. The fact that they better fulfill constraints from translational covariance of currents provides a sensible justification. Most likely, the argument based on these constraints can be extended to field-theory type calculations made in the light front [21, 22, 23]. In these ones, the condition $q^+ = 0$ is currently used but does not seem to have received any justification other than providing results close to the exact ones [31].

5 Conclusion

In this paper, we presented charge and scalar form factors corresponding to different forms or different momentum configurations, calculated using a single-particle current. This has been done for two opposite extreme interaction models corresponding to the exchange of an infinite- and a zero-mass boson. In comparison to a previous work [10], the present one includes results for instant and front forms in a “parallel” momentum configuration, which are most often ignored, and a point form inspired from Dirac’s work, which differs from the currently referred one [25, 26]. A method allowing one to restore Lorentz invariance for the scalar form factor at $Q^2 = 0$ is presented. Anticipating on a future work, the constituent mass and the mass of the system are those appropriate

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3It can be shown that predictions for the charge form factor in different forms as well as in a dispersion-relation approach [34] (partly corrected) could be made identical [35]. This result supposes a single-particle-like current slightly different from what is generally assumed.
for the pion. As noticed elsewhere, the small mass of the system in comparison of the constituent one, apart from the fact it corresponds to a physical case, makes differences between various approaches more striking.

At low $Q^2$, the results clearly fall into two sets, close and sometimes identical to the “experiment” for some of them, far apart for the other ones. In the first category, one finds the standard instant- and front-form results while the second one contains the instant- and front-form results for non-standard momentum configurations and the point-form ones. The quantity characterizing these different results is the squared charge or scalar radius. Up to some numerical factor, it is roughly given by the inverse of the squared constituent mass in one case (notice that the binding is here close to the sum of the constituent masses) and by the inverse of the squared mass of the system in the other. This last property results from the dependence of form factors on $Q^2$ through the ratio $Q^2/(2M)^2$. At high $Q^2$, a similar separation into two sets occurs. Again, the mass of the system is the essential parameter [18, 17, 13, 19]. One can guess however that the “point-form” behaves differently. As noticed in Ref. [15], the asymptotic behavior of the corresponding form factors is rather $(Q^-)^2$ where other approaches give $(Q^-)^2$.

Altogether, there is not much doubt about which approach is an efficient one in the sense that most of the contribution is due to a single-particle current. The other ones (instant and front forms with $\xi \cdot (P_i - P_j) \neq 0$ and point forms) require large if not dominant contributions from two-body currents, which is not the best for incorporating further corrections or for discussing physics when a comparison to experiment is done. Interestingly, the relevant values of the spectator-particle momentum in these approaches are considerably increased in comparison to the value of the order of $Q$ in the case where “good” results are obtained. The enhancement, which partly depends on the momentum configurations, currently involves a factor of the order $2m/M$. In the “point form” approach, an extra enhancement factor, $1 + Q^2/(2M)^2$, has to be considered for the struck particle. These approaches, especially the last one, therefore require a significantly improved implementation of relativity. This qualitatively agrees with the expected increased role of two-body currents mentioned above. The important point we want to stress however is that the earlier “point-form” approach is not any more the only one to evidence obvious drawbacks due to the dependence of form factors on $Q^2$ through the ratio $Q^2/(2M)^2$. Actually, we are inclined to believe that the “good” result is an exception while the “bad” result is more likely the rule as soon as calculations are made for an arbitrary momentum configuration.

Looking for an argument that could explain the discrepancy evidenced by the comparison of different approaches in a single-particle current approximation, we found a close relationship between two features. The first one involves the departure of the form factors to the “exact” results (or the standard front-form ones). The second one is concerned with the violation of an equality relating the squared-momentum transferred to the whole system and that one transferred to the constituents, Eq. (26), which stems from Poincaré space-time translation covariance. Absent in the standard front-form case, the violation amounts to 20% at most in the instant-form one. In the other cases, where the effect is more striking and can reach orders of magnitude, the violation is roughly given by a factor $4e_k^2/M^2$ possibly corrected by a factor $1 + Q^2/(4M^2)$ in the “point-form” case. Again, the violation points to the missing contribution of two-body currents.
Examining the explicit derivation of these two-body currents, we find it is rather hopeless as it requires the consideration of terms to all orders in the interaction in the limit of a zero-mass system. Another approach is suggested by the expression of the above discrepancy factor, $4 e^2 k^2 / M^2$, which deviates from the expected value 1 by a term which involves the interaction. It consists in rescaling the factor multiplying $Q^2$ in the expression of the form factor so that the constraints from transformations of the current under Poincaré space-time translations discussed above be fulfilled. This method, which, in some sense, amounts to sum up some of the above two-body contributions, has been applied to form factors considered in this work. It is found to remove the largest discrepancies that some of the approaches were evidencing either with "experiment" or with the standard front-form results that are close to it. In particular, all the drawbacks related to a small ratio of the total mass to the kinetic energy, $M/(2 e_k)$, vanish. We believe that these results, which involve many orders of magnitude effects, demonstrate the relevance of properties related to Poincaré space-time translations, beyond the usual energy-momentum conservation. In comparison, effects due to a violation of Lorentz invariance evidenced in the present work, assuming that the other symmetry approximately holds, only amount to a few 10%. In this respect, we notice that large effects attributed to a violation of the first symmetry, like a frame dependence, could be actually due to a simultaneous violation of the second one.

Part of this work was originally motivated by the drawbacks evidenced by a "point-form" approach for the calculation of form factors. An implementation more in the spirit of the Dirac’s point-form has not alleviated much the problems. While these approaches fulfill Lorentz covariance, it sounds from the present work that a more sensible criterion might be Poincaré space-time translation covariance. With this respect, the Lorentz invariance of the form factors in the point-form approach, which represents a priori a desirable property, turns out to be a disadvantage as it implies that there is no frame where the effect of the violation of the other symmetry can be minimized. Different aspects of Poincaré covariance have been discussed at length in the past, especially in relation with rotation symmetry within the front-form dynamics (see for instance Ref. [36]). The relevance of translational-covariance properties for applications of relativistic quantum mechanics has hardly been discussed however. It should probably be given increased attention in the future. In the case of form factors, this should be facilitated by the test and the subsequent correction we proposed. In short, the squared momentum transferred to the constituents should match as much as possible that one transferred to the system under consideration, which, after all, is not surprising.

Acknowledgements
We are very grateful to T. Melde for a question about a possible relation of the strange behavior of form factors in some approaches and a possible violation of space-time translation invariance, which our results were tending to support. The question motivated further search for a more quantitative argument, which is presented in the last part of the paper.
A Derivation of form factors in “parallel” momentum configurations and $\vec{P} \rightarrow \infty$

We here derive successively expressions taken by the instant and front-form form factors in the “parallel” momentum configuration, $\vec{P}_i \parallel \vec{P}_f$, with the further condition $\vec{P} \rightarrow \infty$ in the former case, $\vec{P}_i \parallel \vec{P}_f \parallel \hat{n}$ in the latter.

A.1 Instant-form case

Defining the longitudinal direction as the one carried by the common direction of $\vec{P}_i$ and $\vec{P}_f$, we introduce the parallel and perpendicular components as follows:

$$\bar{P} = \frac{1}{2} (P_i + P_f), \quad (P_i)_\perp = (P_f)_\perp = 0,$$

$$Q_\parallel = (P_f - P_i)_\parallel = 2 y \bar{P}, \quad Q_\parallel^2 = \frac{Q^2 (Q^2 + 4 (M^2 + \bar{P}^2))}{Q^2 + 4 M^2},$$

$$y_{P \rightarrow \infty} = \left( \frac{Q_\parallel}{2 \bar{P}} \right)_{P \rightarrow \infty} = \sqrt{\frac{Q^2}{Q^2 + 4 M^2}} = v_{B.F.} = v,$$

$$p_\parallel = x \bar{P}, \quad (p_i)_\parallel = (1 - x) \bar{P} - \frac{Q_\parallel}{2}, \quad (p_f)_\parallel = (1 - x) \bar{P} + \frac{Q_\parallel}{2}. \quad (29)$$

One has now to consider the limit $\bar{P} \rightarrow \infty$ of the different factors entering Eqs. (12, 14). The results are simply listed below for quantities that multiply the wave functions in the integral displayed there:

$$\left( \frac{d\bar{P}}{e_p} \right)_{\bar{P} \rightarrow \infty} = d^2 k_\perp \frac{dx}{x},$$

$$\left( \frac{4 e_i e_f}{(e_i + e_f + e_p)} \right)_{\bar{P} \rightarrow \infty} = \frac{1 - v^2}{4 \left( (1 - x)^2 - v^2 \right)},$$

$$\left( \frac{2 (e_i + e_f)}{e_i + e_f + 2 e_p} \right)_{\bar{P} \rightarrow \infty} = 2 (1 - x),$$

$$\left( \frac{e_p (e_f + e_p)(e_i + e_p) - E_f E_i}{e_p (e_f + e_p)(e_i + e_p)(e_f + e_i)} \right)_{\bar{P} \rightarrow \infty} = 0. \quad (30)$$

Dealing with the wave function, $\phi_0(k)$, entering the expression of the form factors, Eqs. (12, 14), is more delicate. Its argument can be written in terms of $x, k_\perp^2, \bar{P}$ and $Q_\parallel$. For the initial state, it reads:

$$k_i^2 = k_\perp^2 + \frac{1}{2} \left[ \sqrt{m^2 + k_\perp^2 + \left( (1 - x) \bar{P} - \frac{Q_\parallel}{2} \right)^2} \right] \sqrt{m^2 + k_\perp^2 + (x \bar{P})^2} - \left( (1 - x) \bar{P} - \frac{Q_\parallel}{2} \right) (x \bar{P}). \quad (31)$$
Replacing \( Q_\parallel \) in terms of \( y \) and \( \vec{P} \), and taking the limit \( \vec{P} \rightarrow \infty \), one successively gets:

\[
(k_i^2)_{\vec{P} \rightarrow \infty} = \left(k_i^2 + \frac{(m^2 + k_\perp^2)}{4x} \left(1 - \frac{2x}{1-x-y} \right) \right)_{\vec{P} \rightarrow \infty}
\]

\[
= \left(k_i^2 + \frac{(m^2 + k_\perp^2)}{4x} \left(1 - \frac{2x}{1-x-y} \right) \right)_{\vec{P} \rightarrow \infty}
\]

\[
= k_i^2 + \frac{(m^2 + k_\perp^2)}{4x} \left(1 - \frac{2x}{1-x-y} \right)
\]

(32)

### A.2 Front-form case

In the front-form case, it is appropriate to introduce quantities such as \( \omega \cdot P = \omega^0 (E - \hat{n} \cdot \vec{P}) \), \( \omega \cdot p = \omega^0 (e - \hat{n} \cdot \vec{p}) \), \( \ldots \), where, up to a factor, \( \omega^\mu \) stands for the \( \xi^\mu \) introduced in the text with the condition \( \omega^2 = 0 \). In the “parallel” momentum configuration of interest here, one has \( \vec{P}_i \parallel \vec{P}_f \), while \( \hat{n} \) is taken to be opposite to \( \vec{P}_i + \vec{P}_f \). Defining this direction as the parallel one, one can write \( \omega \cdot P = \omega^0 (E + P_\parallel) \), \( \omega \cdot p = \omega^0 (e_p + p_\parallel) \), \( \ldots \). Proceeding as above and using similar notations as much as possible, one can first write:

\[
\omega \cdot \vec{P} = \frac{1}{2} (\omega \cdot P_i + \omega \cdot P_f), \quad (\omega \cdot P_f - \omega \cdot P_i) = 2y \omega \cdot \vec{P},
\]

\[
\omega \cdot P_i = (1 - y) \omega \cdot \vec{P}, \quad \omega \cdot P_f = (1 + y) \omega \cdot \vec{P}.
\]

(33)

Making now use of relations pertinent to the “parallel” momentum configuration, different expressions of \( y \) are obtained, allowing one to identify this quantity with the Breit frame velocity, \( v \), already mentioned. Some intermediate steps are displayed below:

\[
y = \frac{E_f + P_f - E_i - P_i}{E_f + P_f + E_i + P_i} = \frac{(P_f - P_i) \left(1 + \frac{P_f + P_i}{E_f + E_i}\right)}{(E_f + E_i) \left(1 + \frac{P_f + P_i}{E_f + E_i}\right)} = \frac{P_f - P_i}{E_f + E_i}
\]

\[
= \frac{(E_f + P_f - E_i - P_i)(E_f - P_f + E_i - P_i)}{(E_f + P_f + E_i + P_i)(E_f - P_f + E_i - P_i)}
\]

\[
= \frac{(E_f + E_i)(P_f - P_i) - (E_f - E_i)(P_f + P_i)}{4 M^2 + Q^2}
\]

\[
= \sqrt{\frac{(P_f - P_i)^2 - (E_f - E_i)^2}{4 M^2 + Q^2}} = \sqrt{\frac{Q^2}{4 M^2 + Q^2}} = v,
\]

(34)

where, in order to simplify the notation and in absence of ambiguity, the parallel components of \( \vec{P}_i \) and \( \vec{P}_f \) have been denoted \( P_i \) and \( P_f \).

Introducing now the variable \( x \) defined as:

\[
\omega \cdot p = x \omega \cdot \vec{P} \left(= \omega^0 (e_p - \hat{n} \cdot \vec{p}) = \omega^0 (e_p + p_\parallel)\right),
\]

(35)
and using the equality, $y = v$, one can write the following relations:

$$
\omega \cdot p_i = \omega \cdot P_i - \omega \cdot p = (1 - x - v) \omega \cdot \bar{P},
$$

$$
\omega \cdot p_f = \omega \cdot P_f - \omega \cdot p = (1 - x + v) \omega \cdot \bar{P},
$$

$$
p_{\parallel} = \frac{x^2 (\bar{E} + \bar{P}_{\parallel})^2 - (m^2 + k_\perp^2)}{2 x (\bar{E} + \bar{P}_{\parallel})}, \\
e_p = \frac{x^2 (\bar{E} + \bar{P}_{\parallel})^2 + (m^2 + k_\perp^2)}{2 x (\bar{E} + \bar{P}_{\parallel})},
$$

$$
d\parallel = \frac{e_p}{x}, \\
d\perp = \frac{d^2 k_\perp}{x},
$$

$$
\frac{\omega \cdot (p_f + p)}{2 \omega \cdot p_f} \frac{\omega \cdot (p_i + p)}{2 \omega \cdot p_i} = \frac{1 - v^2}{4 \left( (1 - x)^2 - v^2 \right)}.
$$

$$
2 \frac{(p_f + p_i) \cdot \omega}{(p_f + p_i + 2 p) \cdot \omega} = 2 (1 - x). \tag{36}
$$

The argument of the wave function in the case of the initial state is now given by:

$$
k_i^2 = \frac{-1}{4} (p_i - p)^2 = \frac{k_\perp^2}{2} - \frac{m^2}{2} + \frac{m^2 + k_\perp^2}{4} \left( \frac{\omega \cdot p}{\omega \cdot p_i} + \frac{\omega \cdot p_i}{\omega \cdot p} \right)
$$

$$
= k_\perp^2 + \frac{m^2 + k_\perp^2}{4} \left( \frac{\omega \cdot p}{\omega \cdot p_i} + \frac{\omega \cdot p_i}{\omega \cdot p} - 2 \right)
$$

$$
= k_\perp^2 + \frac{m^2 + k_\perp^2}{4} \frac{(1 - 2x - v)^2}{4 x (1 - x - v)}. \tag{37}
$$

The r.h.s. of last equalities in Eqs. (36, 37) are identical to those in Eqs. (30, 32) but they have been obtained without taking the limit \( \bar{P} \to \infty \). This in in accordance with the statement often made about the “perpendicular” momentum configuration that taking the above limit in an instant-form approach or working on an hyperplane with the limit \( |\vec{\xi}/\xi_i| = |\hat{n}| = 1 \) (front-form) are equivalent.

### B Making the instant-form scalar form factor, \( F_0(Q^2 = 0) \), Lorentz invariant

Considering the RQM expression of the charge form factor \( F_1(Q^2) \) in the instant form, it is generally found that its value at \( Q^2 = 0 \) is independent of the momentum of the system. With this respect, the particular form of the last factor in Eq. (12) is essential. This minimal Lorentz invariance property is deeply related to current conservation or to a meaningful definition of the norm. It seems to imply that standard instant- and front-form charge form factors be close to each other, thus pointing to a small violation of Lorentz invariance, a few % at most. In contrast, examination of the scalar form factor, \( F_0(Q^2 = 0) \), generally evidences a sizeable one. Thus, in absence of two-body currents, discrepancies of a few 10% are observed between its value at \( \bar{P} = 0 \) and that other one at \( |\bar{P}| \to \infty \), where it is equal to the standard front-form result. Determining two-body currents that can restore the equality, thus fulfilling a minimal restoration of Lorentz invariance, can be done from examination of diagrams. Apart from the fact that this is not straightforward in a RQM framework and, in any case, does not apply to
a phenomenological approach, the question arises of whether there is a general method allowing one to derive currents fulfilling the above invariance property.

In the scalar-particle case and for an interaction model corresponding to the exchange of an infinite-mass boson, the underlying model allowed one to get the explicit expression of the correcting factor to be inserted in the one-body current contribution [10]:

\[ c f_0 = 1 + \frac{e_p}{2 e_i} \frac{(e_i + e_p)^2 - E^2}{(e_i + e_p)^2}. \]  

This one has a typical off-energy shell character and, therefore, can be cast into the form of an interaction term, using the mass equation, Eq. (38). In this simple model, the factor makes the instant-form form factor \( F_0(Q^2 = 0) \) equal to the standard front-form one. Moreover, a simple generalization at \( Q^2 \neq 0 \) was found to preserve the identity [10]. With the idea that an extension of the above correction factor could contribute to restore Lorentz invariance, we looked at its determination.

For our purpose, it is convenient to start from an expression of \( F_0(Q^2 = 0) \) where the momenta of the constituents are expressed in terms of the total momentum \( \vec{P} \) and the internal variable \( \vec{k} \). In the above mentioned simple model, this quantity reads:

\[ F_0(Q^2 = 0) = \int \frac{d\vec{k}}{(2 \pi)^3} \frac{\phi_0^2(k)}{1 + \vec{k} \cdot \vec{P}} \left( 1 + \frac{1}{2} \frac{1 - \vec{P}/4 e_k^2 - M^2}{1 + \vec{k} \cdot \vec{P}/4 e_k^2 + \vec{P}^2} \right). \]  

where \( \phi_0(k) \propto \sqrt{2 e_k} (4 e_k^2 - M^2) \), \( \vec{k} = k/e_k \) and \( \vec{P} = P/\sqrt{4 e_k^2 + \vec{P}^2} \). Despite appearances, it can be shown that \( F_0(Q^2 = 0) \) does not depend on the momentum \( \vec{P} \). A generalization of the above result to any function \( \phi_0(k) \) supposes to modify appropriately the interaction term proportional to \( 4 e_k^2 - M^2 \). The simplest change consists in multiplying this term by a factor \( g(k) \) so that \( F_0(Q^2 = 0) \) now reads:

\[ F_0(Q^2 = 0) = \int \frac{d\vec{k}}{(2 \pi)^3} \frac{\phi_0^2(k)}{1 + \vec{k} \cdot \vec{P}} \left( 1 + \frac{g(k)}{2} \frac{1 - \vec{k} \cdot \vec{P}/4 e_k^2 - M^2}{1 + \vec{k} \cdot \vec{P}/4 e_k^2 + \vec{P}^2} \right). \]  

In order to \( F_0(Q^2 = 0) \) be independent of \( \vec{P} \), it is found that the following relation has to be fulfilled:

\[ \frac{g(k)}{8} e_k (4 e_k^2 - M^2) \phi_0^2(k) = \int_k^\infty dk' k' e_{k'} \phi_0^2(k'), \]  

which allows one to easily get \( g(k) \). As the above result is obtained for the first time and that details may be useful in other cases, we provide here some steps. An integration over the orientation of \( \vec{k} \) is first made in Eq. (40) with the result:

\[ F_0(Q^2 = 0) = \frac{1}{2 \pi^2} \int d\vec{k} k^2 \phi_0^2(k) \times \left( 1 - \frac{g(k)}{2} \frac{4 e_k^2 - M^2}{4 e_k^2 + \vec{P}^2} \right) \frac{1}{2 k \vec{P}} \log \left( \frac{1 + \vec{k} \cdot \vec{P}}{1 - \vec{k} \cdot \vec{P}} \right) + g(k) \frac{e_k^2 (4 e_k^2 - M^2)}{4 e_k^2 + \vec{P}^2 m^2}. \]  

(42)
An integration by parts has now to be done to remove the undesirable log term. This supposes that the corresponding coefficient can be cast into the form of the derivative of a function as follows:

\[ k^2 \phi_0^2(k) \left( 1 - \frac{g(k)}{2} \left( \frac{4 e_k^2 - M^2}{4 e_k^2 + P^2} \right) \right) \frac{1}{2 k \cdot \vec{P}} \]

\[ = - \frac{d}{dk} \left( \frac{\sqrt{4 e_k^2 + P^2}}{2 P} \int_k^\infty dk' k' e_{k'} \phi_0^2(k') \right). \] (43)

By identifying the left- and right-hand sides of the equation, Eq. (41) is obtained. After the integration by parts is performed, the expression of \( F_0(Q^2 = 0) \) reads:

\[ F_0(Q^2 = 0) = \frac{1}{2 \pi^2} \int dk \phi_0^2(k) \frac{g(k)}{8} \left( 4 e_k^2 - M^2 \right) \]

\[ \times \left( \frac{m^2 (4 e_k^2 + P^2) - 4 e_k^2 k^2}{4 e_k^4 + P^2 m^2} + \frac{8 e_k^2 k^2}{4 e_k^4 + P^2 m^2} \right), \] (44)

which simplifies to get the total momentum independent result:

\[ F_0(Q^2 = 0) = \frac{1}{2 \pi^2} \int dk \phi_0^2(k) \frac{g(k)}{8} \left( 4 e_k^2 - M^2 \right). \] (45)

In the particular case of a zero-range interaction, the square of the wave function fulfills the relation \( \phi_0^2(k) \propto (2 e_k (4 e_k^2 - M^2)^2)^{-1} \), from which one gets \( g(k) = 1 \), in agreement with Eq. (38). In the case \( \phi_0^2(k) \propto (2 e_k (4 e_k^2 - M^2)^4)^{-1} \), which is not a bad approximation for a Coulombian type interaction (Wick-Cutkosky model), one gets \( g(k) = 1/3 \). It is noticed that the factor \((1 - \vec{k} \cdot \vec{P})/(1 + \vec{k} \cdot \vec{P})\) in Eq. (39) has been conserved but one can imagine to split it into two parts and find a more general expression of two-body currents. This freedom could be used to make the instant- and front-form scalar form factors closer to each other at any \( Q^2 \), beyond the equality at \( Q^2 = 0 \) which is achieved by the two-body currents determined above. The method to make the instant-form scalar form factor, \( F_0(Q^2 = 0) \), independent of the momentum of the system has a rather general character. It could be applied for instance to the pion system whose constituents have a non-zero spin.

C Details about form factors in Dirac’s point-form

We give here a few details pertinent to the derivation of the expression of form factors appropriate to the implementation of a Dirac’s inspired point form which is considered in this work, among other forms.

C.1 Minimal expression

On the basis of expressions of form factors for a two-body system in other forms, it is expected that, in the case of a single-particle current, the expression in the point-form approach will involve integration over the 3-momenta of the struck and spectator
particles, constrained by relations of these 3-momenta to the total momentum. A minimal expression that evidences Lorentz invariance reads:

\[
F(q^2) \propto \int \frac{d\vec{p}}{2e_p (2\pi)^3} \frac{d\vec{p}_i}{2e_i (2\pi)^3} \frac{d\vec{p}_f}{2e_f (2\pi)^3} \frac{(2\pi)^6}{\pi^2} \times \delta \left( (p_i + p - P_i)^2 \right) \delta \left( (p_i + p - P_i) \cdot (p_f + p - P_f) \right) \delta \left( (p_f + p - P_f)^2 \right)
\times M \sqrt{e_{k_i}} \phi_0(k_i^2) \times M \sqrt{e_{k_f}} \phi_0(k_f^2) \cdots ,
\]

where the dots account for factors pertinent to the current describing the interaction with an external probe. The two functions, \( \delta((p_i + p - P_i)^2) \) and \( \delta((p_f + p - P_f)^2) \), are part of the wave functions for the initial and final states [24]. The middle one, \( \delta((p_i + p - P_i) \cdot (p_f + p - P_f)) \), stems from the integration over the coordinate at the interaction point with the external probe, constrained to be on a hyperboloid. Involving plane waves relative to the struck particle and the probe one, it provides a function, \( \delta((p_f - p_f + q)^2) \), which can be rearranged into the above one taking into account the other \( \delta(\cdots) \) functions and the 4-momentum conservation relation, \( P^\mu_i - P^\mu_f + q^\mu = 0 \). It is noticed that the introduction of a 4-vector at the r.h.s. of Eq. (46), \( p^\mu \) or \( p^\mu_i + p^\mu_f \) for instance, would produce the appearance of a 4-vector at the l.h.s. with the correct transformation properties under a Lorentz transformation, similarly to the earlier “point form”. The structure of the integrand is however quite different (compare with Eq. 8 in Ref. [17] or Eq. 42 in Ref. [10]).

C.2 Removing the \( \delta(\cdots) \) functions

In the following, we transform the above expression into a one where integrations over the various \( \delta(\cdots) \) functions are performed. Introducing vectors \( \vec{u}_i \) and \( \vec{u}_f \), the above expression can first be written:

\[
F(q^2) \propto \int \frac{d\vec{p}}{2e_p (2\pi)^3} \frac{d\vec{p}_i}{2e_i (2\pi)^3} \frac{d\vec{p}_f}{2e_f (2\pi)^3} \times \int d\vec{u}_i d\vec{u}_f \delta(u_i^2 - 1) \delta(1 - \vec{u}_i \cdot \vec{u}_f) \delta(u_f^2 - 1) \frac{(2\pi)^6}{\pi^2} \times \delta \left( \vec{p}_i + \vec{p} - \vec{P}_i - \vec{u}_i (e_i + e_p - E_i) \right) \delta \left( \vec{p}_f + \vec{p} - \vec{P}_f - \vec{u}_f (e_f + e_p - E_f) \right) \times M \sqrt{e_{k_i}} \phi_0(k_i^2) \times M \sqrt{e_{k_f}} \phi_0(k_f^2) \cdots .
\]

(47)

Taking advantage of the fact that the \( \delta(\cdots) \) functions involving the \( \vec{u} \) variable can be transformed for a part into a \( \delta(\vec{u}_i - \vec{u}_f) \) function, the above expression can be successively written after performing various integrations:

\[
F(q^2) \propto \int \frac{d\vec{p}}{2e_p (2\pi)^3} \frac{d\vec{p}_i}{2e_i (2\pi)^3} \frac{d\vec{p}_f}{2e_f (2\pi)^3} \times \int d\vec{u}_i d\vec{u}_f \delta(u_i^2 - 1) \delta(\vec{u}_i - \vec{u}_f) \frac{(2\pi)^6}{\pi} \times \delta \left( \vec{p}_i + \vec{p} - \vec{P}_i - \vec{u}_i (e_i + e_p - E_i) \right) \delta \left( \vec{p}_f + \vec{p} - \vec{P}_f - \vec{u}_f (e_f + e_p - E_f) \right) \times M \sqrt{e_{k_i}} \phi_0(k_i^2) \times M \sqrt{e_{k_f}} \phi_0(k_f^2) \cdots .
\]
\[\propto \int \frac{d\vec{p}}{2e_p (2\pi)^3} \frac{d\vec{p}_i}{2e_i (2\pi)^3} \frac{d\vec{p}_f}{2e_f (2\pi)^3} \ d\vec{u} \delta(u^2-1) \frac{(2\pi)^6}{\pi}\]

\[\times \delta(\vec{p}_i + \vec{p} - \vec{P}_i - \vec{u} (e_i + e_p - E_i)) \delta(\vec{p}_f + \vec{p} - \vec{P}_f - \vec{u} (e_f + e_p - E_f))\]

\[\times M \sqrt{\epsilon_{k_i}} \phi_0(k_i^2) M \sqrt{\epsilon_{k_f}} \phi_0(k_f^2) \cdots\]

\[\propto \int \frac{d\vec{p}}{2e_p (2\pi)^3} \frac{M}{2u \cdot p_i} \frac{2u \cdot p_f}{2u \cdot p_f} d\vec{u} \delta(u^2-1) \sqrt{\epsilon_{k_i}} \phi_0(k_i^2) \sqrt{\epsilon_{k_f}} \phi_0(k_f^2) \cdots\]

\[\propto \int \frac{d\vec{p}}{2e_p (2\pi)^3} \frac{M}{2u \cdot p_i} \frac{M}{4\pi} \sqrt{2\epsilon_{k_i}} \phi_0(k_i^2) \sqrt{2\epsilon_{k_f}} \phi_0(k_f^2) \cdots, \tag{48}\]

where \(u \cdot p_i = u \cdot (P_i - p), u \cdot p_f = u \cdot (P_f - p)\). The Lorentz invariance of the last expressions (dots put apart) is not straightforward though the property stems from the starting point, Eq. (46). In order to generalize the above expression and specify the dots, we show directly on the last expression how Lorentz invariance is fulfilled despite it involves the \(\vec{u}\) variable.

**C.3 Lorentz invariance of expressions with integration over \(\vec{u}\)**

Invariance under rotations being straightforward, the Lorentz transformation of interest here involves boosts in some direction represented by a vector \(\vec{V}\). Introducing the notation \(V^0 = (1 + \vec{V}^2)^{-1/2}\), it is defined as:

\[x^0 \to x^0 V^0 - \vec{x} \cdot \vec{V}, \tag{49}\]

\[\vec{x} \to \vec{x} + \frac{\vec{x} \cdot \vec{V}}{V_0 + 1} \vec{V} - x^0 \vec{V}.\]

Under the above transformation, a seemingly Lorentz scalar quantity like \(u \cdot X\) transforms as follows:

\[u \cdot X = X^0 - \hat{u} \cdot \vec{X} \to X^0 V_0 + \vec{X} \cdot \vec{V} - \hat{u} \cdot (\vec{X} + \vec{V} \frac{\vec{V} \cdot \vec{X}}{V_0 + 1} + X^0)\]

\[= (V_0 - \hat{u} \cdot \vec{V}) X^0 - \left(\hat{u} + \vec{V} \frac{\hat{u} \cdot \vec{V}}{V_0 + 1} - 1\right) \cdot \vec{X}\]

\[= (V_0 - \hat{u} \cdot \vec{V}) u^\prime \cdot X, \tag{50}\]

where, at the last line, we introduce a new 4-vector \(u''^\mu\) defined as:

\[u''^0 = 1, \quad \vec{u}'' = \frac{1}{V_0 - \hat{u} \cdot \vec{V}} \left(\hat{u} + \vec{V} \left(\hat{u} \cdot \vec{V} \frac{\hat{u} \cdot \vec{V}}{V_0 + 1} - 1\right)\right) = \hat{u}'', \tag{51}\]

The last equality can be traced back to the relation \(\vec{u}''^2 = \vec{u}^2 = 1\), which is expected from the Lorentz-invariant condition \(u' \cdot u' = u \cdot u = 0\).

In order to determine how the volume integration in Eq. (48), \(d\vec{u}\), changes under the Lorentz transformation, it is useful to invert Eq. (51). One thus gets:

\[\frac{1}{V_0 - \hat{u} \cdot \vec{V}} = V_0 + \hat{u}' \cdot \vec{V},\]

\[\hat{u} = \frac{1}{V_0 + \hat{u}' \cdot \vec{V}} \left(\hat{u}' + \vec{V} \left(\frac{\hat{u}' \cdot \vec{V}}{V_0 + 1} + 1\right)\right), \tag{52}\]
The Jacobian of the transformation, \( \hat{u} \to \hat{u}' \), can also be calculated, taking into account the normalization \( \hat{u}'^2 = \hat{u}^2 = 1 \):

\[
\frac{d\hat{u}}{d\hat{u}'} = \left( \frac{1}{V_0 + \hat{u}' \cdot \hat{V}} \right)^2 = (V_0 - \hat{u} \cdot \hat{V})^2. \tag{53}
\]

In the expression of \( F(q^2) \) given by the last line of Eq. (48), the boost transformation leads to the appearance of a factor \((V_0 - \hat{u} \cdot \hat{V})^2\) (one factor separately for the quantities \( e_i - \hat{u} \cdot \vec{p}_i \) and \( e_f - \hat{u} \cdot \vec{p}_f \) at the denominator). This factor cancels the one from the Jacobian, ensuring the Lorentz invariance of the expression (\( \hat{u} \) is replaced by \( \hat{u}' \), which can be renamed \( \hat{u} \)). From the above, it immediately follows that the Lorentz invariance property of the quantity \( F(q^2) \) will not be affected if the dot part in the integrand at the last line of Eq. (13) contains factors depending on \( u^\mu \) provided that they evidence a seemingly Lorentz-scalar form and are invariant under changing the scale of \( u^\mu \). In such a case, the factors \((V_0 - \hat{u} \cdot \hat{V})\) appearing in the Lorentz transformation, last line of Eq. (50), cancel out. We stress that this simplification, which is essential to demonstrate the above Lorentz-invariance property, is possible because \( u \cdot u = 0 \).

**C.4 Other factors: the dot part**

In considering the part involving dots in Eqs. (46-48), which was unspecified till here, we first look at the case, \( q^\mu = 0 \), where the normalization condition, \( F_1(0) = 1 \), should be recovered. For this quantity, the dots are replaced by:

\[
\ldots = \frac{2 \, u \cdot (p_i + p_f)}{u \cdot (p_i + p_f + 2p)} \tag{54}
\]

which is suggested by the close relationship of the normalization to the charge current density and is unchanged when the scale of \( u^\mu \) is modified. As explained elsewhere [10], the factor \((p_i + p_f)^\mu\) at the numerator could represent the interaction of the photon with the constituents while the quantity at the denominator represents the sum of the momenta of the constituents that has to be factored out in calculating the charge form factor \((p_i + p_f + 2p)^\mu = (P_i + P_f)^\mu\) in absence of interaction.

For our purpose, either equation (46-48) could be used. Starting from Eq. (46) for instance, and taking into account the relation \( P_i^\mu = P_f^\mu \) for \( q^\mu = 0 \), it is first noticed that the product of the three \( \delta(\cdots) \) function in the integrand can be expressed as follows:

\[
\delta((p_i + p - P_i)^2) \delta((p_i + p - P_i) \cdot (p_f + p - P_f)) \delta((p_f + p - P_f)^2) = \delta((p_i + p - P_i)^2) \delta((p_i + p - P_i) \cdot (p_f - P_i)) \delta((p_f - P_f)^2) = \delta((p_i + p - P_i)^2) \pi \frac{e_f}{(p_i + p - P_i) \cdot P_f} \delta(p_i - p_f). \tag{55}
\]

In writing the last line, we employed relations similar to Eqs. (74) and (75) of Ref. [24], taking into account that \( p_i^2 = p_f^2 = m^2 \). The last \( \delta(\cdots) \) function in the above equation allows one to perform the integration over \( \vec{p}_f \) in Eq. (46). The form factor \( F_1(0) \) now
which allows one to make the integration over \( \vec{p} \) is obtained. However, one can imagine to insert extra factors such as used for a non-zero momentum transfer since the correct charge form factor at given by Eq. (54) for the charge form factor, by 1 for the Lorentz-scalar one, could be of the energy conservation is taken, one could get different results: function that could appear in a more complete treatment. Depending on how the limit is varied, in relation with off-energy shell effects. In the case of the electromagnetic interaction for instance, one can consider that the photon is emitted either from the

\[
F_1(0) = \int \frac{1}{(2\pi)^6} \frac{d\vec{p}}{2e_p} \frac{d\vec{P}_i}{2e_i} \frac{4\pi^2}{(p_i + p - P_i) \cdot (p_i + p)} \times \delta \left( (p_i + p - P_i)^2 \right) \left( M \sqrt{e_{k_i}} \phi_0(k_i^2) \right)^2
\]

\[
= \int \frac{1}{(2\pi)^6} \frac{d\vec{p}}{2e_p} \frac{d\vec{P}_i}{2e_i} \frac{e_{k_i} \phi_0^2(k_i^2)}{4e_i^2 - M^2} \frac{8\pi^2}{(p_i + p - P_i)^2} \frac{M^2}{(u \cdot P_i)^2} = 1. \tag{56}
\]

which, apart from notations, can be seen to be identical to Eq. (31) given in Ref. [21]. After making a change of variable described in this last work, one also finds:

\[
F_1(0) = \int \frac{d\vec{k}}{(2\pi)^3} \phi_0^2(k) \int \frac{d\vec{u}}{2\pi} \delta(1 - \vec{u}^2) \left( \frac{M^2}{(u \cdot P_i)^2} \right) = 1. \tag{57}
\]

It is noticed that Eq. (56) can be easily recovered from the second relation of Eq. (48). For \( P_i^\mu = P_f^\mu \), the second \( \delta(\cdots) \) function can be readily transformed into a \( \delta(\vec{p} - \vec{p}_f) \) function, which allows one to make the integration over \( \vec{p}_f \). Accounting for the appropriate factors, the integration over \( \vec{u} \) is easily performed using the other 3-dimensional \( \delta(\cdots) \) function.

At first sight, expressions (46-48) together with the appropriate choice for the dots, given by Eq. (54) for the charge form factor, by 1 for the Lorentz-scalar one, could be used for a non-zero momentum transfer since the correct charge form factor at \( q^2 = 0 \) is obtained. However, one can imagine to insert extra factors such as \( u \cdot P_i / u \cdot P_f \) or \( u \cdot P_f / u \cdot P_i \) in the dots part of Eqs. (46-48) since these ones preserve Lorentz invariance and reduce to 1 at zero momentum transfer, allowing one to fulfill the above limit. While trying to fix this extra factor, we have in mind that the structure for the currents should be close to each other in different forms so that to avoid some bias in comparing their predictions. From considering Eq. (12), a minimal factor, corresponding to the quantity \( \xi_f \cdot (p_f + p) \), \( \xi_i \cdot (p_i + p) \) appearing at its numerator, is given by \( u \cdot P_i u \cdot P_f \). The full factor should be \( u^\mu \)-scale independent and, therefore, this quantity has to be divided by a factor that is bilinear in \( u^\mu \). As the expression of the norm given in Eq. (57) suggests, it could be either \( (u \cdot P_i)^2 \) or \( (u \cdot P_f)^2 \). These possibilities correspond to the expectation that, in the c.m., the integration over \( \vec{u} \) should be made isotropically. Moreover, what is isotropic for the initial state may not be for the final state and vice versa. The ambiguity has probably its origin in a partial treatment of the “time” evolution of the interaction with an external probe. It is illustrated here by the consideration of the following \( \delta(\cdots) \) function that could appear in a more complete treatment. Depending on how the limit of the energy conservation is taken, one could get different results:

\[
\delta(\vec{P}_i - \vec{P}_f + \vec{q} - \vec{u} (E_i - E_f + q^0)) = \delta(\vec{P}_i - \vec{P}_f + \vec{q}) = \frac{E_i}{u \cdot P_i} \delta(\vec{P}_i - \vec{P}_f + \vec{q}) = \frac{E_f}{u \cdot P_f} \delta(\vec{P}_i - \vec{P}_f + \vec{q}). \tag{58}
\]

Of course, physical results should not depend on either expression. Other examples can be encountered, in relation with off-energy shell effects. In the case of the electromagnetic interaction for instance, one can consider that the photon is emitted either from the
initial or final state. Accordingly, one could have relations like $P_f^2 = (P_i + q)^2$ and $P_i^2 = (P_f - q)^2$. In principle, the corresponding contributions should be the same. However, in an incomplete calculation, they may differ. In analogy with this example, the two contributions should be considered on an equal footing. The dot part of Eqs. (46-48) can therefore be replaced as follows:

\[
F_1(Q^2) \rightarrow \ldots = \frac{u \cdot P_i u \cdot P_f}{2} \left( \frac{1}{(u \cdot P_f)^2} + \frac{1}{(u \cdot P_i)^2} \right) \frac{2 u \cdot (p_i + p_f)}{u \cdot (p_i + P_f + 2 p)} ,
\]

\[
F_0(Q^2) \rightarrow \ldots = \frac{u \cdot P_i u \cdot P_f}{2} \left( \frac{1}{(u \cdot P_f)^2} + \frac{1}{(u \cdot P_i)^2} \right) .
\]

(59)

Actually, form factors calculated here turn out to be independent of which term is considered in the above expression. We can therefore omit one of them together with the front factor $1/2$, what is made in the text, Eq. (19). It is noticed that this expression allows one to recover the expected asymptotic behavior of the form factor in the Born-amplitude approximation [20, 24]. In our opinion, this result, which could be used, the other way round, to discriminate among choices for the factor in place of $(u \cdot P_f)^{-2} + (u \cdot P_i)^{-2}$ in Eq. (59), is not fortuitous. The asymptotic Born amplitude is the sum of two terms where one of the initial or final states is on mass-shell while the other one is not. For an off-mass-shell initial state for instance and an on-mass-shell final one, the absence of interaction in this last state discards dependence on $u \cdot P_f$, leaving only $(u \cdot P_i)^{-2}$ as a possible choice. The symmetry between the initial and final states then suggests to take the combination $(u \cdot P_f)^{-2} + (u \cdot P_i)^{-2}$.

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