Mathematical modeling and numerical analysis of reinforced composite beams

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Abstract. This paper is devoted to modeling the properties of composite materials and reinforced composite beams. Mathematical relations describing the nonlinear elastic three-point bending of isotropic and reinforced beams with account of different strength and stiffness behavior in tension and compression are obtained. An algorithm for numerical solution of corresponding boundary-value problems is proposed and implemented. Results of numerical analysis were compared to acquired data for polymer matrix and structural carbon fiber reinforced plastics.

1. Introduction
Composites are the most promising modern materials. High-duty structures used in aviation and space industry, car manufacturing and building sector require new materials as well as ways to improve their characteristics. Applying computer modeling techniques significantly reduces both the time and cost of investigations aimed at searching optimal parameters of composite structures [1–3]. Mathematical modeling provides an opportunity for comprehensive analysis of both composite materials and composite structures. It has become an effective tool for solving important applied problems.

To build a mathematical model of composite materials, including those made of carbon fibers, one relies upon the experimental data acquired in mechanical testing. Wide application of digital testing machines has brought such experiments to higher level of quality. By measuring a large number of parameters with a high discretization frequency, modern testing machines allow for high amount of information on material deformation and failure to be obtained within a single experiment. Therefore, data processing has become an important step for mathematical modeling of carbon fiber reinforced plastics (CFRP) and CFRP structures. This is exceptionally important because of quite specific behavior of CFRPs and of their components: fibers and matrices.

One of the features of such materials is their different strength and stiffness behavior in tension and compression combined with nonlinearities of stress-strain curves. Multiple studies for epoxy matrices showed that their ultimate strains in tension were much lower than in compression: approximately 4\% versus 20\% and more [4]. Moreover, under tension and compression the deformation behavior of epoxy matrices significantly differs. The corresponding stress-strain curves have different stiffness (secant modulus) at the same values of strain. The similar
difference can be observed for CFRPs. In [5], it was shown that in tension tests of carbon fiber specimens with reinforcement angles less than 20° the stiffness grows together with strains (stiffening) whereas for epoxy matrices softening is observed. This phenomenon was explained by the properties of carbon fibers.

Contrasting behavior in tension and compression, stiffening, softening and other nonlinearities are forcing researchers to build and use special mathematical models and computing algorithms. Mathematical models taking into account the above-mentioned properties of materials were proposed and studied theoretically by S.P. Timoshenko [6] and S.A. Ambartsumyan [7,8] in the mid-20th century. Later R.M. Jones had experimentally, theoretically and numerically studied the non-linear behavior of several fiber reinforced composites. Main focus of the research was on the difference in stiffness and strength behavior under tension and compression [9]. After Ambartsumyan’s and Jones’ researches a lot of studies were dedicated to this problem. Most of them were dealing with linear bi-modulus models of materials or 3D finite elements. In [10] Ambartsumyan with a coauthor suggested a theoretical approach to modeling of multi-modulus nonlinear elastic beams under bending, but still without calculations.

A comprehensive approach to modeling and simulation of nonlinear elastic deformation of polymer matrices and different CFRPs was presented in [11]. This article deals with different strength and stiffness behavior of the materials in tension and compression exemplified by a case of three-point bending. This approach implements a full cycle of model development and validation, which comprises the following stages: carrying out tests and acquiring experimental data, data preprocessing and building stress-strain curves, analytical approximation of acquired curves, mathematical modeling and numerical simulation of deformation processes, comparative analysis of results of numerical modeling to acquired data.

Currently, Russia is implementing a program for the development of the Arctic, requiring new effective ways of building roads, ice crossings, runways, dams and piers. Ice can be used as a material for the construction of such objects. Ice is a free unlimited natural resource, it suits to the extreme natural and climatic conditions of the Arctic: long low temperatures, significant annual temperature differences, strong wind loads, high humidity in the sea zone, intense solar radiation during the polar day. In [12,13] the analysis of the current state of Arctic material science and promising fields of application of Arctic materials is given. The development of Arctic material science is facilitated by the high geopolitical interest of various states to the development of the Arctic – a possible field of competition for world powers in the future. It should be noted that ice is a very “whimsical” material whose behavior under the influence of external forces is hardly predictable, therefore, ice-based designs require strengthening. The use of geosynthetic materials (geogrids) as reinforcing materials of the ice matrix opens wide possibilities for increasing the strength, stiffness and bearing capacity of ice composite structures.

2. Structural models of reinforced composite materials

For most of composite materials models we can write the relations between average stresses $\sigma_{\alpha\beta}$, $\tau_{\alpha3}$ and strains $e_{\alpha\beta}$, $\gamma_{\alpha3}$ (generalized Hooke’s Law):

$$
\sigma_{\alpha\alpha} = a_{\alpha\alpha} e_{\alpha\alpha} + a_{\alpha\beta} e_{\beta\beta} + a_{\alpha3} \cdot 2 e_{\alpha\beta} - a_{\alpha\Theta} \Theta,
$$

$$
\sigma_{\alpha\beta} = a_{\alpha\beta} e_{\alpha\alpha} + a_{\beta\beta} e_{\beta\beta} + a_{\beta3} \cdot 2 e_{\alpha\beta} - a_{\beta\Theta} \Theta,
$$

$$
\gamma_{\alpha3} = q_{\alpha\alpha} \tau_{\alpha3} + q_{\alpha\beta} \tau_{\beta3},
$$

where $\Theta$ is the increase of temperature. Relations (1) are called the thermoelasticity relations, or, when no temperature influence is considered, they are simply elasticity relations.

The structural model of fiber reinforced composite described in [14–17] has become a foundation for a large number of current researches. Now it is widely used while simulating the
behaviour of composite structures. The model is based on the following assumptions: the stress-strain state into isotropic elastic fibres and into entire volume of isotropic ideally elastic matrix is homogeneous; fibers and matrix are deformed jointly along the direction of reinforcement; stresses in fibers and in matrix corresponding to other directions are equal.

For computing the effective elastic moduli of unidirectional fiber reinforced composite the Reuss-Voigt average was used giving the following formulae

\[ E_1 = \omega_f E_f + \omega_m E_m, \quad E_2 = \frac{E_f E_m}{\omega_f E_m + \omega_m E_f}, \]

\[ \nu_{12} = \omega_f \nu_f + \omega_m \nu_m, \quad G = \frac{G_f G_m}{\omega_f G_m + \omega_m G_f}, \]

where all the terms having squared Poisson coefficients are neglected.

Herewith \( E_1, E_2 \) are effective moduli along and across the direction of reinforcement, \( G \) is effective share modulus, \( \nu_{12} \) is effective Poisson coefficient in the plain of layer; \( E, \nu, \omega \) with “\( f \)” and “\( m \)” indices are elastic moduli, Poisson coefficients and volume fractions of matrix and fibres correspondingly, hereby \( \omega_m + \omega_f = 1 \).

On the ground of symmetry of compliance tensor one has

\[ \nu_{21} = \nu_{12} E_2 E_1^{-1}. \]

Formulae for effective coefficients of thermal expansion have the following form

\[ \alpha_1 = \omega_f \alpha_f + \omega_m \alpha_m, \quad \alpha_2 = \frac{\omega_f \alpha_f E_f + \omega_m \alpha_m E_m}{\omega_f E_f + \omega_m E_m}. \]

In description of the model it is noted that among formulae for effective moduli, those obtained using Reuss averaging (in particular formulae for \( G \)) lead to the worst results. Estimations for \( G \) obtained using variational method are also obtained, and it is shown that lower boundary

\[ G = \frac{(1 + \omega_f) G_f + \omega_m G_m}{(1 + \omega_f) G_m + \omega_m G_f} G_m \]

gives more accurate approximation than (2) does. Hereinafter share moduli of matrix and fibers are

\[ G_m = \frac{E_m}{2(1 + \nu_m)}, \quad G_f = \frac{E_f}{2(1 + \nu_f)}. \]

Components of effective stiffness tensor for unidirectionally reinforced layer in case of state of plane stress have the following form:

\[ A_{aaaa} = \frac{E_\alpha}{1 - \nu_{12} \nu_{21}}, \quad A_{1122} = \frac{\nu_{21} E_1}{1 - \nu_{12} \nu_{21}}, \quad A_{1212} = G. \]

Unwritten expressions can be obtained using symmetry rule or vanish. Hereinafter we assume \( \alpha, \beta = 1, 2 \) and \( \alpha \neq \beta \).

The coefficients in relations (1) for example are defined by the formulas given in [2,14,15].
3. Mathematical model and numerical analysis of reinforced beam deformation

Three-point bending test has been one of the standard techniques to determine physical and mechanical characteristics of materials. The beam’s stress-strain state is characterized by the following values determined on the reference surface: the shear force $Q(x)$, the bending moment $M(x)$, the longitudinal force $N(x)$, and by the longitudinal displacement and deflection ($u(x)$, $w(x)$ respectively). The corresponding equilibrium equations are written as follows:

$$\frac{dN}{dx} = 0, \quad \frac{dQ}{dx} = 0, \quad \frac{dM}{dx} = Q.$$  \hfill (6)

The reactions $R_A$ and $R_B$ can be determined by considering force equilibrium $R_A = R_B = P/2$. The force $P$ is applied to the center of the beam. The bending moments at the support points are equal to zero: $M_A = M_B = 0$. The solution of the system of equation (6) can be expressed as follows:

$$N = 0, \quad Q(x) = \begin{cases} P/2, & 0 \leq x \leq l/2, \\ -P/2, & l/2 \leq x \leq l, \end{cases}$$ \hfill (7)

$$M(x) = \begin{cases} Px/2, & 0 \leq x \leq l/2, \\ -P(x - l)/2, & l/2 \leq x \leq l. \end{cases}$$ \hfill (7)

Strain distribution for the beam’s thickness can be obtained from the Kirchhoff-Love kinematic hypotheses:

$$\varepsilon(x, z) = \varepsilon(x) + z\kappa(x),$$ \hfill (8)

$$\varepsilon(x) = \frac{du}{dx}, \quad \kappa(x) = -\frac{d^2w}{dx^2},$$ \hfill (9)

where $\varepsilon(x, z)$ is the strain in the beam; $\varepsilon(x)$ is the median surface strain; and $\kappa(x)$ denotes changes in the median surface curvature. As mentioned above, the beam undergoes tension and compression strain, whose interface will be marked as $z_1$. In this case for the section area $-h \leq z \leq z_1$ the strain will be negative, and for $z_1 \leq z \leq h$ positive. At the interface of these two states the strains $\varepsilon$ vanish, so the interface itself is determined as follows:

$$z_1 = -\frac{e}{\kappa}, \quad -h \leq z_1 \leq h.$$ \hfill (10)

The constitutive equation can be expressed as:

$$\sigma^\pm (x, z) = f^\pm (\varepsilon),$$ \hfill (11)

where the superscript “+” refers to the areas with positive strains and “−” to the area with negative ones; $f(\varepsilon)$ denotes the approximation selected for the stress-strain curve (a linear function, a polynomial, or a combination of linear and power-law functions).

The longitudinal force $N$ and the bending moment $M$ in the beam cross section are determined by the equations:

$$N = b \left( \int_{-h}^{z_1} \sigma^- dz + \int_{z_1}^{h} \sigma^+ dz \right),$$ \hfill (12)

$$M = b \left( \int_{-h}^{z_1} \sigma^- zdz + \int_{z_1}^{h} \sigma^+ zdz \right).$$ \hfill (12)

Using equation (9) and the beam’s fixing conditions, a system of equations can be derived:

$$\frac{d^2w_1}{dx^2} = -\kappa_1, \quad \frac{d^2w_2}{dx^2} = -\kappa_2,$$
\[ w_1(0) = w_2(l) = 0, \quad w_1(l/2) = w_2(l/2), \]
\[ \frac{dw_1(l/2)}{dx} = \frac{dw_2(l/2)}{dx}. \]

The solution of these equations can be obtained using the methods of solving boundary-value problems for systems of ordinary differential equations. For that purpose the discrete orthogonalization method [18–21] and modified collocation and least squares collocation method [22–24] were applied.

Figure 1. Experimental (solid curves) and dependencies of beam-deflection and load obtained in simulation: linear approximation (1); quadratic approximation by a polynomial of the second degree (2); cubic approximation (3); linear and power-law approximation (4); a–c are specimens 1–3 respectively; d is the solution to a three-point bending problem without account for the different strength and stiffness behaviour in tension (curve 1) and compression (curve 2). The solid line shows the results of mechanical tests.

In figure 1, one can see the simulation results for beam three-point bending, obtained through different approaches to approximation of the constitutive equations, and their comparison with the experimental data.

Applying the linear dependencies to tension and compression has not resulted in adequate approximation even for 30% of curve. Using more complex than quadratic approximation laws at first led to a significant deviation from the experimental curve, and then to divergence of
the Newton type iteration process. This is explained by the fact that in tension tests, due to specimens fragility, the strain range for the polymer matrix specimens was limited to 2%, while in the bending tests the strains in tension zone reached 4–5%.

Thus, to solve the bending problem the tension curve was extrapolated into the domain of high strains. The extrapolations obtained using a polynomial of the third degree, and by linear and power-law function reached the maximum too quickly and then started to decrease, which is against the physics behind the deformation process. A similar effect was observed when calculating the bending of the carbon-fiber specimens cut out along direction of reinforcement filler.

The calculations using quadratic approximation and extrapolation of tension curves and approximation of compression curves within a short (up to 6%) segment have turned out to be best for qualitative and quantitative description of the nonlinear character of polymer matrix bending. In the case of the specimen cut out perpendicular to direction of its reinforcement all the approximations have shown the results close to experiment. At the same time for the test with maximum load the best option has still been application of quadratic approximations.

Conclusions
A test program for VSE-1212 polymer matrix and VKU-28 CFRPs under different load conditions has been developed and implemented in [11] and convenient mathematical tool to process big experimental data and eliminate the most significant artifacts has been developed. A number of approaches to approximation of the stress-strain curves obtained in the course of experiments have been suggested. Comparative analysis to assess the effectiveness of the developed techniques for obtaining analytical approximations of the stress-strain curves for different kinds of tests has been performed.

Mathematical models for nonlinear flexural deformation of CFRPs and polymer matrices with account for their different strength and stiffness behavior in tension and compression have been built. A satisfactory match with the results of mechanical tests has been obtained. The study has proved that the nonlinear properties of polymer matrices and carbon fibers should be taken into account when calculating and designing real structures.

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