Stability of viscous shock wave for compressible Navier-Stokes equations with free boundary

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Abstract: A free boundary problem for the one-dimensional compressible Navier-Stokes equations is investigated. The asymptotic stability of the viscous shock wave is established under some smallness conditions. The proof is given by an elementary energy estimate.

1 Introduction

We consider the system of viscous and heat conductive fluid in the Eulerian coordinate

\[
\begin{align*}
\rho_t + (\rho u)_\tilde{x} &= 0, \\
(\rho u)_t + (\rho u^2 + p)_\tilde{x} &= \mu u_{\tilde{x}\tilde{x}}, \\
\left[\rho (e + \frac{u^2}{2})\right]_t + \left[\rho u (e + \frac{u^2}{2}) + pu\right]_{\tilde{x}} &= \kappa \theta_{\tilde{x}\tilde{x}} + (\mu uu_{\tilde{x}})_{\tilde{x}},
\end{align*}
\]

(1.1)

where \(u(\tilde{x}, t)\) is the velocity, \(\rho(\tilde{x}, t) > 0\) is the density, \(\theta(\tilde{x}, t)\) is the absolute temperature, \(p = p(\rho, \theta)\) is the pressure, \(e = e(\rho, \theta)\) is the internal energy, \(\mu > 0\) is the viscosity constant, \(\kappa > 0\) is the coefficient of heat conduction. Here we consider the perfect gas case, that is

\[
p = R\theta \rho, \quad e = \frac{R\theta}{\gamma - 1} + \text{const.},
\]

(1.2)

where \(\gamma > 1\) is the adiabatic constant and \(R > 0\) the gas constant.

There has been a large literature on the asymptotic behaviors of the solutions to the system (1.1). However, most results are concerned with the initial value problem. We refer to [7]-[10], [12]-[13] and references therein. Recently the initial boundary value problem (IBVP) attracts an increasing interest because it has more physically meanings and of course produces new mathematical difficulties due to the boundary effect. We refer to [4], [11], [14], [15] for \(2 \times 2\) case and
[5], [6], [17] for $3 \times 3$ case. However, there is few result on the asymptotic stability of the viscous shock wave to IBVP of the full compressible Navier-Stokes equation (1.1) due to various difficulties. Therefore, the asymptotic stability of the viscous shock wave to IBVP for (1.1) is our main purpose of the present paper. We shall consider a free boundary problem of the full compressible Navier-Stokes equations whose boundary conditions read

$$\left\{ \begin{array}{l}
(p - \mu u_x)\big|_{\tilde{x}=\tilde{x}(t)} = p_0,
\theta\big|_{\tilde{x}=\tilde{x}(t)} = \theta_-, \\
d\tilde{x}(t) = \tilde{u}(\tilde{x}(t), t), \quad \tilde{x}(0) = 0, \quad t > 0,
\end{array} \right. \quad (1.3)$$

and initial data

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0)(x) \to (\rho_+, u_+, \theta_+) \text{ as } \tilde{x} \to +\infty, \quad (1.4)$$

where $p_0 > 0, \theta_- > 0, \rho_+ > 0, \theta_+, u_+$ are prescribed constants. Here the boundary condition (1.3) means the gas is attached at the boundary $\tilde{x} = \tilde{x}(t)$ to the atmosphere with pressure $p_0$(see [15]). We of course assume the initial data satisfy the boundary condition as compatibility condition.

Since the boundary condition (1.3) means the particles always stay on the free boundary $\tilde{x} = \tilde{x}(t)$, if we use the Lagrangian coordinates, then the free boundary becomes a fixed boundary. Thus we transform the Eulerian coordinates $(x, t)$ by

$$x = \int_{\tilde{x}(t)}^{\tilde{x}} \rho(y, t) dy, \quad t = t,$$

and then change the free boundary value problem (1.1)-(1.4) into

$$\left\{ \begin{array}{l}
v_t - u_x = 0, \quad x > 0, \quad t > 0, \\
u_t + p_x = \mu \left( \frac{u_x}{v} \right)_x, \quad x > 0, \quad t > 0, \\
\left( e + \frac{u^2}{2} \right)_t + (pu)_x = \left( \frac{\theta}{v} + \mu \frac{u u_x}{v} \right)_x, \quad x > 0, \quad t > 0, \\
(p - \mu u_x)|_{x=0} = p_0, \\
\theta|_{x=0} = \theta_-,
\end{array} \right. \quad (1.5)$$

where $v = \frac{1}{\rho}$ is the specific volume. Since the domain we consider here in the Lagrange coordinates is $\{x > 0, \ t > 0\}$, we only need to consider the stability of the 3-viscous shock wave.

Before formulating our main result, we briefly recall some results of the shock wave for the inviscid system of (1.1). That is, we consider the system (1.5) without viscosity

$$\left\{ \begin{array}{l}
v_t - u_x = 0, \\
u_t + p_x = 0, \\
\left( e + \frac{u^2}{2} \right)_t + (pu)_x = 0,
\end{array} \right. \quad (1.6)$$

with $v = \frac{1}{\rho}$.
with the Riemann initial data
\[ (v_0, u_0, \theta_0)(x) = \begin{cases} 
(v_-, u_-, \theta_-), & x > 0, \\
(v_+, u_+, \theta_+), & x < 0.
\end{cases} \] (1.7)

It is well known (for example, see [16]) that the Riemann problem (1.6)-(1.7) admits a 3-shock wave if and only if the two states \((v_\pm, u_\pm, \theta_\pm)\) satisfy the so-called Rankine-Hugoniot condition
\[
\begin{align*}
-s(v_+ - v_-) - (u_+ - u_-) &= 0, \\
-s(u_+ - u_-) + (p_+ - p_-) &= 0, \\
-s\left((e_+ + \frac{u_+^2}{2}) - (e_- + \frac{u_-^2}{2})\right) + (p_+ u_+ - p_- u_-) &= 0,
\end{align*}
\] (1.8)
and the Lax’s entropy condition
\[ 0 < \lambda_3^+ < s < \lambda_3^-, \] (1.9)
where \(p_\pm = p(v_\pm, \theta_\pm), e_\pm = e(v_\pm, \theta_\pm)\) and \(\lambda_3 = \sqrt{\gamma R \theta_\pm v_\pm}\) is the third eigenvalue of the inviscid system (1.6). And the shock speed \(s\) is uniquely determined by \((v_\pm, u_\pm, \theta_\pm)\) with (1.8). If the right state \((v_+, u_+, \theta_+)\) is given, it is easy to know that there exists a 3-shock curve \(S_3(v_+, u_+, \theta_+)\) starting from \((v_+, u_+, \theta_+)\).

For any point \((v, u, \theta)\) \(\in S_3(v_+, u_+, \theta_+)\), there exists a unique 3-shock wave connecting \((v, u, \theta)\) with \((v_+, u_+, \theta_+)\). Our assumptions on the boundary values are

(A1). Let \((v_+, u_+, \theta_+)\) and \(\theta_-\) be given, there exist unique \(v_-, u_-\) such that \((v_-, u_-, \theta_-) \in S_3(v_+, u_+, \theta_+)\).

(A2). \(p_0 = \frac{R \theta_-}{v_-} := p_-\).

Remark1. The assumption (A1) is natural.

Remark2. The condition (A2) means that we only consider the stability of a single viscous shock wave.

It is known that the system (1.5) admits smooth travelling wave solution with shock profile \((V, U, \Theta)(x - st)\) under the conditions (1.8) and (1.9) (see [1]). Such travelling wave has been shown nonlinear stable for the initial value problem, see [7] and [9]. A natural question is whether the travelling wave is stable or not for the initial boundary value problem. In this paper, we give a positive answer for the free boundary problem (1.1)-(1.4) or (1.5). Our main result is, roughly speaking, as follows. The precise statement is given in theorem 2.1 below.

Let \((v_+, u_+, \theta_+)\) and \(\theta_-\) be given and the assumptions (A1) and (A2) hold, then the 3-viscous shock wave connecting \((v_-, u_-, \theta_-)\) with \((v_+, u_+, \theta_+)\) is asymptotically stable.
The plan of this paper is as follows. After stating the notations, in section 2, we give some properties of the viscous shock wave and the main Theorem 2.1. In Section 3, we reformulate the original problem to a new initial boundary value problem. The proof of the Theorem 2.1 is given in section 4 by the elementary energy method. In section 5, we prove the local existence of the solution by the iteration method.

**Notation:** Throughout this paper, several positive generic constants which are independent of $T, \beta$ and $\alpha$ are denoted by $C$ without confusions. For function spaces, $H^l(\mathbb{R}^+)$ denotes the $l$-th order Sobolev space with its norm

$$
\|f\|_{l} = \left(\sum_{j=0}^{l} \|\partial_j^j f\|^2\right)^{\frac{1}{2}}, \text{ when } \|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^+)}.
$$

## 2 Preliminaries and Main Result

We first recall some properties of the 3-viscous shock wave. The shock profile $(V,U,\Theta)(x,\xi) = x - st$, is determined by

$$
\begin{aligned}
-sV' - U' &= 0, \\
-sU' + P' &= \mu \left(\frac{U'}{V}\right)', \\
-s \left( E + \frac{U^2}{2} \right) + (PU)' &= \left( \kappa \frac{\Theta'}{V} + \mu \frac{UU'}{V} \right)', \\
(V,U,\Theta)(\pm\infty) &= (v_\pm, u_\pm, \theta_\pm),
\end{aligned}
$$

where $P = R\Theta/V$, $E = R\Theta/\gamma + const.$, $(v_\pm, u_\pm, \theta_\pm)$ satisfy R-H condition (1.8) and entropy condition (1.9) and $s$ is determined by (1.8). Integrating (2.1) on $(-\infty, \xi)$ gives

$$
\begin{aligned}
\frac{s\mu V_\xi}{V} &= - \left[ P + s^2 \left( V - \frac{b_1}{s^2} \right) \right], \\
\frac{\kappa \Theta_\xi}{sV} &= - \left[ E - \frac{s^2 \left( V - \frac{b_1}{s^2} \right)^2}{2} + \frac{b_1^2}{2s^2} - b_2 \right], \\
U &= -(sV + a),
\end{aligned}
$$

where $p_\pm = R\theta_\pm/v_\pm$, $e_\pm = R\theta_\pm/\gamma - 1 + const.$, $a = -(sv_\pm + u_\pm)$, $b_1 = p_\pm + s^2 v_\pm$ and $b_2 = e_\pm + p_\pm v_\pm + s^2 v_\pm^2/2$. From [1] and [9], we have the following proposition:

**Proposition 2.1.** Assume that the two states $(v_\pm, u_\pm, \theta_\pm)$ satisfy the conditions (1.8) and (1.9), then there exists a unique shock profile $(V,U,\Theta)(\xi)$, up to a shift, of system (2.1). Moreover, there are positive constants $c_1$ and $c_2$ independent
of $\gamma > 1$ such that for $\xi \in \mathbb{R},$

$$
\begin{aligned}
&\left\{ 
\begin{array}{l}
  sV_\xi = -U_\xi > 0, \quad s\Theta_\xi < 0, \quad \left( |V - v_\pm|, |U - u_\pm| \right) \leq c_1 \|e^{-c_2 d|\xi|} \|,
  |\Theta - \theta_\pm| \leq c_1 (\gamma - 1) \|e^{-c_2 d|\xi|} \|,
  \left( |V_\xi|, |V_{\Theta\xi}|, |\Theta_{\Theta\xi}| \right) \leq c_1 d^2 \|e^{-c_2 d|\xi|} \|,
  |\Theta_\xi| \leq c_1 (\gamma - 1) d^2 e^{-c_2 d|\xi|},
  \frac{\Theta_\xi}{V_\xi} \leq c_1 (\gamma - 1),
\end{array}
\right.
\end{aligned}
$$

(2.3)

where $d = v_+ - v_-.$

As pointed out by Liu [7], a generic perturbation of viscous shock wave produces not only a shift $\alpha$ but also diffusion waves, which decay to zero with a rate $(1 + t)^{-\frac{\gamma}{4}},$ for the Cauchy problem. That is the solution of the compressible Navier-Stokes equations asymptotically tends to the translated travelling wave $(V, U, \Theta)(x - st + \alpha).$ The shift $\alpha$ is explicitly determined by the initial value. Similar to the Cauchy problem, the shift $\alpha$ is also expected for IBVP. For a kind of initial boundary value problem, in which the velocity is zero on the boundary, Matsumura and Mei [11] developed a new way to determine the shift $\alpha.$ A byproduct of [11] showed that, unlike the Cauchy problem, there is no diffusion wave for IBVP due to the boundary effect. This new idea has been used by many authors to treat the initial boundary value problem of the system (1.5) or other related systems (see [4], [14], [15]). In the spirit of [11], we calculate the shift $\alpha$ for the IBVP (1.5).

We consider the situation where the initial data $(v_0, u_0, \theta_0)$ are given in a neighborhood of $(V, U, \Theta)(x - \beta)$ for some large constant $\beta > 0.$ That is, we require the viscous shock wave is far from the boundary initially. Here we can not directly apply the idea of [11] to compute the shift $\alpha$ since the velocity $u(0, t)$ on the boundary is not given, while in [11], the velocity is zero on the boundary and the conservation of the mass (1.5) is then used to determine the shift $\alpha.$ Instead of (1.5)1, we use the conservation of momentum (1.5)2 to determine the shift $\alpha$ because $\rho - \mu \frac{U}{V}$ is given on the boundary for the IBVP (1.5). From (1.5)2 and (2.1)2, we have

$$
(u - U)_t = -[p(v, \theta) - P(V, \Theta)]_x + \mu \left( \frac{u_x}{v} \right)_x - \mu \left( \frac{U_x}{V} \right)_x,
$$

(2.4)

where $(V, U) = (V, U)(x - st + \alpha - \beta).$ Integrating (2.4) over $[0, \infty)$ with respect to $x$ and using (2.1) and (A2) yield

$$
\begin{aligned}
\frac{d}{dt} \int_0^{+\infty} [u(x, t) - U(x - st + \alpha - \beta)] dx &= p_+ - P(V, \Theta)(-st + \alpha - \beta) + \mu \frac{U'}{V}(-st + \alpha - \beta) \\
&= -s(U(-st + \alpha - \beta) - u_-).
\end{aligned}
$$

(2.5)
Integrating (2.5) with respect to \( t \), we have
\[
\int_{0}^{\infty} [u(x, t) - U(x - st + \alpha - \beta)]dx \\
= \int_{0}^{\infty} [u_0 - U(x + \alpha - \beta)]dx - \int_{0}^{t} s(U(-s\tau + \alpha - \beta) - u_-)d\tau.
\]  
(2.6)

We define
\[
I(\alpha) := \int_{0}^{\infty} [u_0 - U(x + \alpha - \beta)]dx \\
- \int_{0}^{\infty} s(U(-st + \alpha - \beta) - u_-)dt.
\]  
(2.7)

It follows that
\[
I'(\alpha) = -\int_{0}^{\infty} U'(x + \alpha - \beta)dx - s\int_{0}^{\infty} U'(-s\tau + \alpha - \beta)d\tau \\
= u_- - u_+.
\]  
(2.8)

Expectation \( \lim_{t \to \infty} \int_{0}^{\infty} [u(x, t) - U(x - st + \alpha - \beta)]dx = I(\alpha) = 0 \)

\[
\alpha = \frac{1}{u_+ - u_-} I(0),
\]  
(2.9)

and
\[
\int_{0}^{\infty} [u(x, t) - U(x - st + \alpha - \beta)]dx \\
= s \int_{t}^{\infty} [U(-s\tau + \alpha - \beta) - u_-]d\tau \leq c_1 e^{-c_2 |t - \alpha - \beta|} \text{ as } t \to +\infty.
\]  
(2.10)

Therefore the shift \( \alpha \) is uniquely determined by the initial value.

To state our main theorem, we suppose that for some \( \beta > 0 \)

\[
(v_0(x) - V(x - \beta), u_0(x) - U(x - \beta), \theta_0(x) - \Theta(x - \beta)) \in H^1 \cap L^1.
\]  
(2.11)

Let
\[
(\Phi_0, \Psi_0)(x) = -\int_{\infty}^{0} \left[ v_0(y) - V(y - \beta), u_0(y) - U(y - \beta) \right] dy,
\]
\[
\tilde{W}_0(x) = -\int_{x}^{\infty} \left[ (e_0 + \frac{u_0^2}{2})(y) - (E + \frac{U^2}{2})(y - \beta) \right] dy.
\]  
(2.12)

Assume that
\[
(\Phi_0, \Psi_0, \tilde{W}_0) \in L^2.
\]  
(2.13)

Our main result is
**Theorem 2.1.** Suppose that the assumptions (A1) and (A2) hold. Let \((V, U, \Theta)(\xi)\) be the travelling wave solution satisfying (2.1). Assume that \(1 < \gamma \leq 2\) and (2.11-2.13) hold, then there exists positive constants \(\delta_0\) and \(\varepsilon_0\) such that if
\[(\gamma - 1)d \leq \delta_0,\]and
\[||\Phi_0, \Psi_0, \frac{W_0}{\sqrt{\gamma - 1}}||_2 + e^{-c_2d\beta} \leq \varepsilon_0,\]then the system (1.5) has a unique global solution \((v, u, \theta)(x, t)\) satisfying
\[v(x, t) - V(x - st + \alpha - \beta) \in C([0, \infty), H^1) \cap L^2(0, \infty; H^1),\]
\[u(x, t) - U(x - st + \alpha - \beta) \in C([0, \infty), H^1) \cap L^2(0, \infty; H^2),\]
\[\theta(x, t) - \Theta(x - st + \alpha - \beta) \in C([0, \infty), H^1) \cap L^2(0, \infty; H^2),\]
and
\[\sup_{x \in \mathbb{R}^+} |(v, u, \theta)(x, t) - (V, U, \Theta)(x - st + \alpha - \beta)| \to 0, \text{ as } t \to +\infty,\]
where \(\alpha = \alpha(\beta)\) is determined by (2.9).

### 3 Reformulation of the Original Problem

Let
\[(v, u, \theta)(x, t) = (V, U, \Theta)(x - st + \alpha - \beta) + (\phi, \psi, w)(x, t),\]
then we rewrite the system (1.5) as
\[
\begin{aligned}
\phi_t - \psi_x &= 0, \\
\psi_t + R \left( \frac{\Theta + w}{V + \phi} - \frac{\Theta}{V} \right)_x &= \mu \left[ \frac{\psi_x}{V + \phi} + \left( \frac{1}{V + \phi} - \frac{1}{V} \right) U_x \right], \\
\left( \frac{R}{\gamma - 1} w + \frac{\psi^2}{2} + U\psi \right)_t + R \left[ \frac{\Theta + w}{V + \phi} \psi + \left( \frac{\Theta + w}{V + \phi} - \frac{\Theta}{V} U \right) \right]_x &= \kappa \left[ \frac{w_x}{V + \phi} + \left( \frac{1}{V + \phi} - \frac{1}{V} \right) \Theta x \right] \\
& \quad + \mu \left[ \frac{\psi \psi_x + U \psi_x + U_x \psi}{V + \phi} + \left( \frac{1}{V + \phi} - \frac{1}{V} \right) U U_x \right], \\
w_{x=0} &= \theta_0 - \Theta(-st + \alpha - \beta), \quad \left( \frac{R \theta_0}{V + \phi} - \mu U_x + \psi_x \right)_{x=0} = p_-, \\
(\phi, \psi, w)|_{x=0} &= (\phi_0, \psi_0, w_0)(x).
\end{aligned}
\]

We define
\[(\Phi, \Psi)(x, t) = -\int_{-\infty}^{+\infty} (\phi, \psi)(y, t) dy,\]
\[W(x, t) = -\int_{-\infty}^{+\infty} \left( e + \frac{u^2}{2} \right) (y, t) - \left( E + \frac{U^2}{2} \right) (y - st + \alpha - \beta) dy.\]
Then we have
\[
(\phi, \psi, w) = \left( \Phi_x, \Psi_x, \frac{\gamma - 1}{R} [W_x - \left( \frac{1}{2} \Psi_x^2 + U \Psi \right)] \right).
\] (3.4)

Integrating (3.2) with respect to \( x \) yields
\[
\begin{align*}
\Phi_t - \Psi_x &= 0, \\
\Psi_t + R \left( \Theta + w \right) &= \mu \Psi_{xx} + \left( \frac{\mu}{V + \Phi_x} - \frac{\mu}{V} \right) U_x, \\
W_t + R \left( \frac{\Theta + w}{V + \Phi_x} - \frac{\Theta}{V} \right) U_x + R \Theta + \frac{\mu}{V} \Psi_x &= \kappa w_x + \left( \frac{\kappa}{V + \Phi_x} - \frac{\kappa}{V} \right) \Theta_x \\
&\quad + \frac{\kappa}{V + \Phi_x} (\Psi_x \Psi_{xx} + U_x \Psi_x + U \Psi_{xx}) + \left( \frac{\mu}{V + \Phi_x} - \frac{\mu}{V} \right) U U_x.
\end{align*}
\] (3.5)

Introduce a new variable
\[
\tilde{W} = \frac{\gamma - 1}{R} (W - U \Psi),
\] (3.6)

then we write \( w \) in the form
\[
w = \tilde{W}_x + \frac{\gamma - 1}{R} \left( U_x \Psi - \frac{\Psi_x^2}{2} \right),
\] (3.7)

and transform the system (3.5) into
\[
\begin{align*}
\Phi_t - \Psi_x &= 0, \\
\Psi_t - \frac{b_1 - s^2 V}{V} \Phi_x + \frac{R}{V} \tilde{W}_x - \frac{\mu}{V} \Psi_{xx} + \frac{\gamma - 1}{V} U_x \Psi &= F_1, \\
\frac{R}{\gamma - 1} \tilde{W}_t + (b_1 - s^2 V) \Psi_x - \frac{\kappa}{V} \left( \tilde{W}_x + \frac{\gamma - 1}{R} U_x \Psi \right)_x &= F_2,
\end{align*}
\] (3.8)

where \( F_1 \) and \( F_2 \) are nonlinear terms with respect to \((\Phi, \Psi, \tilde{W})\), that is
\[
\begin{align*}
F_1 &= \frac{\gamma - 1}{2V} \frac{1}{\psi^2} - \frac{\psi}{V(\psi + \phi)} \left\{(b_1 - s^2 V) \phi - R \psi + \mu \psi_x \right\}, \\
F_2 &= -\frac{\kappa (\gamma - 1)}{RV} \psi \psi_x + \frac{\psi}{V + \phi} \left\{(b_1 - s^2 V) \phi - R \psi + \mu \psi_x \right\}.
\end{align*}
\] (3.9)

By (3.3)-(3.4), the initial values satisfy
\[
\begin{align*}
\Phi(x, 0) &= -\int_{x}^{+\infty} \left[ v_0(y) - V(y + \alpha - \beta) \right] dy \\
&= \tilde{\Phi}_0(x) + \int_{x}^{+\infty} \left[ V(y + \alpha - \beta) - V(y - \beta) \right] dy \\
&= \tilde{\Phi}_0(x) + \int_{0}^{x} \left[ v_+ - V(x + \varsigma - \beta) \right] d\varsigma =: \tilde{\Phi}_0(x).
\end{align*}
\] (3.10)
\[ \Psi(x,0) = - \int_{x}^{+\infty} [u_0(y) - U(y + \alpha - \beta)]dy \]
\[ = \tilde{\Psi}_0(x) + \int_{0}^{x} [u_+ - U(x + \zeta - \beta)]d\zeta =: \Psi_0(x). \]  
(3.11)

\[ W(x,0) = - \int_{x}^{+\infty} \left[ \frac{R\vartheta_0}{\gamma - 1} + \frac{u_0^2}{2} \right](y) - \left( \frac{R\Theta}{\gamma - 1} + \frac{U^2}{2} \right)(y + \alpha - \beta) \]  
\[ = \tilde{W}_0(x) + \int_{x}^{+\infty} \left[ \left( \frac{R\Theta}{\gamma - 1} + \frac{U^2}{2} \right)(y + \alpha - \beta) - \left( \frac{R\Theta}{\gamma - 1} + \frac{U^2}{2} \right)(y) \right]dy \]  
\[ = \tilde{W}_0(x) + \int_{0}^{x} \frac{R}{\gamma - 1} [\theta_+ - \Theta(x + \zeta - \beta)] + \frac{1}{2} [u_+^2 - U^2(x + \zeta - \beta)]d\zeta \]  
\[ =: W_0(x). \]  
(3.12)

\[ \tilde{W}(x,0) = \frac{\gamma - 1}{R} [W_0(x) - U(x + \alpha - \beta)\Psi_0(x)] =: \tilde{W}_0(x). \]  
(3.13)

Furthermore, by the same way as in [11], we have

**Lemma 3.1.** Under the assumptions (2.11) and (2.13), \((\Phi_0, \Psi_0, \tilde{W}_0) \in H^2 \) and the shift
\[ \alpha \to 0 \quad \text{as} \quad \| (\Phi_0, \Psi_0, \tilde{W}_0) \|_2 \to 0 \quad \text{and} \quad \beta \to +\infty. \]  
(3.14)

**Lemma 3.2.** Under the assumptions (2.11) and (2.13), the initial perturbations \((\Phi_0, \Psi_0, \tilde{W}_0) \in H^2 \) and satisfies
\[ \| (\Phi_0, \Psi_0, \tilde{W}_0) \| \to 0 \quad \text{as} \quad \| (\Phi_0, \Psi_0, \tilde{W}_0) \| \to 0 \quad \text{and} \quad \beta \to +\infty. \]  
(3.15)

By (3.3), (3.5) and (2.5), the boundary values satisfy
\[ \Psi(0,t) = - \int_{0}^{t} + \infty \psi(y,t)dy \]
\[ = - s \int_{t}^{+\infty} [U(-sr + \alpha - \beta) - u_-]d\tau := A(t), \]  
(3.16)

\[ \tilde{W}_x(0,t) - \frac{\gamma - 1}{2R} \Psi_0^0(0,t) = \omega(0,t) - U_x(-st + \alpha - \beta)A(t) := B(t). \]  
(3.17)

For any \( T > 0 \), we define the solution space of the problem (3.5), with the initial values (3.10), (3.11), (3.13) and the boundary values (3.16), (3.17) by
\[ X_{m,M}(0,T) = \left\{ (\Phi, \Psi, \tilde{W}) : (\Phi, \Psi, \tilde{W}) \in C(0,T; H^2); \right\} \]
\[ \Phi_x \in L^2(0,T; H^1); \quad (\Psi \tilde{W}_x) \in L^2(0,T; H^2); \quad \sup_{t \in [0,T]} \| (\Phi, \Psi, W)(t) \|_2 \leq M; \quad \inf_{x,t} (\Phi + \Psi) \geq m \} \]  
(3.18)

where \( T, M, m \) are the positive constants.
4 Proof of Theorem 2.1

In this section, we give the proof of the Theorem 2.1. Without loss of generality, we may restrict \( \beta > 1 \) and \( |\alpha| < 1 \). First we state the local existence result for the IBVP (3.8), (3.10)-(3.13) and (3.16)-(3.17), whose proof is given in section 5.

**Proposition 4.1.** (Local Existence) There exists a positive constant \( b \) such that if \( \| (\Phi_0, \Psi_0, \hat{W}_0) \|_2 \leq M \), and if \( \inf_{x,t} (V + \Phi_0 x) \geq m > 0 \), then there exists a positive constant \( T_0 = T_0(m, M) \) such that the system (3.8), with the initial values (3.10), (3.11), (3.13) and the boundary values (3.16), (3.17), has a unique solution \((\Phi, \Psi, \hat{W}) \in X_{\frac{1}{m}, bM}(0, T_0)\).

Denote that
\[
N(T) = \sup_{\tau \in [0, T]} (\| \Phi(\tau) \|_2 + \| \Psi(\tau) \|_2 + \| W(\tau) \|_2),
\]
\[
N_0 = \| \Phi_0 \|_2 + \| \Psi_0 \|_2 + \| W_0 \|_2.
\]

**Proposition 4.2.** (A Priori Estimates) Let \((\Phi, \Psi, W) \in X_{\frac{1}{m}, b}(0, T)\) be a solution of the problem (3.5) and \( 1 < \gamma \leq 2 \). Then there exist positive constants \( \delta_1, \varepsilon_1 \) and \( C \), which are independent of \( T \), such that if \((\gamma - 1)d \leq \delta_1 \) and \( N_0 + \varepsilon + \beta^{-1} \leq \varepsilon_1 \), then the following estimate holds for \( t \in [0, T] \)
\[
\| (\Phi, \Psi, \hat{W}) (t) \|_2^2 + \| (\phi, \psi, \frac{w}{(\gamma - 1)^2}) (t) \|_2^2 + \int_0^t \| (\psi, w) \|_2^2 + \| \phi \|_2^2 d\tau \leq C \left( N_0 + e^{-\beta \varepsilon_1} \right).
\]

(4.1)

With the local existence Proposition 4.1 in hand, for the proof of the Theorem 2.1 by the standard continuum process, it is sufficient to prove the a priori estimate Proposition 4.2. In order to prove the Proposition 4.2, we first give some Lemmas. The following Lemma is about the boundary estimates.

**Lemma 4.3.** For \( 0 \leq t \leq T \), the following inequalities hold:
\[
\int_0^t (\Phi \Psi) |_{x=0} d\tau, \int_0^t (\Psi \Psi_x) |_{x=0} d\tau, \int_0^t (\hat{W} \Psi) |_{x=0} d\tau, \int_0^t (\psi w) |_{x=0} d\tau \leq Ce^{-\beta d},
\]
\[
\int_0^t (\hat{W}_x \hat{W}) |_{x=0} d\tau \leq Ce^{-\beta d} + C N(T) \int_0^t (\| \Psi_x \|^2 + \| \Psi_{xx} \|^2) d\tau,
\]
\[
\int_0^t (\phi \psi) |_{x=0} d\tau, \int_0^t (\psi_x \psi) |_{x=0} d\tau, \int_0^t (\psi_{x} \psi_{x}) |_{x=0} d\tau \leq C(e^{-\beta d} + \| \phi_0 \|_1),
\]
\[
\int_0^t (w w_x) |_{x=0} d\tau, \int_0^t (w_x w_{x}) |_{x=0} d\tau \leq (\gamma - 1)d \int_0^t \| w_{xx} \|^2 (\tau) d\tau + Ce^{-\beta d}.
\]
Proof. Since $s > 0$, and $\beta \gg 1, |\alpha| < 1, we have from (2.3) and (3.16) that

$$|\Psi(0,t)| = |A(t)| \leq Ce^{-\epsilon\beta}\epsilon^{-\epsilon\beta}.$$ 

Thus,

$$\int_0^t (\Phi\Psi)_{x=0} d\tau \leq Cd^{-1}N(T)e^{-\epsilon\beta} \leq Ce^{-\epsilon\beta}.$$ 

Similarly, we can estimate the term $\int_0^t (\Psi x)_{x=0} d\tau, \int_0^t (\hat{W}\Psi)_{x=0} d\tau$. Also,

$$\int_0^t (\psi w)_{x=0} d\tau \leq N(T) \int_0^t (\theta)_{x=0} d\tau \leq \hat{W}(0) = w(0) - \gamma - 1 R(U_{xx} \Psi(0,t) - \frac{\Psi^2(0,t)}{2}),$$

so we have from (2.3) that

$$\int_0^t (\hat{W}\hat{W})_{x=0} d\tau \leq Ce^{-\epsilon\beta} + CN(T) \int_0^t (\|\Psi\|^2 + \|\Psi x\|^2) d\tau.$$ 

By using the free boundary condition in (1.5), one has

$$\frac{R\theta - \epsilon v(0,t)}{v} = \frac{R\theta}{v},$$

and then

$$v(0,t) - v_- = (v_0 - v_--)e^{-\frac{p_0}{\mu}t} = (V(\alpha - \beta) - v_- + \phi(0))e^{-\frac{p_0}{\mu}t} \leq C(e^{-\epsilon\beta} + \|\phi\|_1)e^{-\frac{p_0}{\mu}t} \quad (4.2)$$

By using (2.3) and (4.2), we obtain

$$|\phi(0,t)| = |v(0,t) - V(-st + \alpha - \beta)| \leq |v_0(t) - v_-| + \|V(-st + \alpha - \beta) - v_-\| \leq C(e^{-\epsilon\beta} + \|\phi(0\|_1)e^{-\frac{p_0}{\mu}t} + Ce^{-\epsilon\beta}e^{-\epsilon\beta},$$

and

$$|\psi(0,t)| = \frac{|v_0(t) - v_0(0)|e^{-\frac{p_0}{\mu}t} - U_{xx}(-st + \alpha - \beta)| \leq C(e^{-\epsilon\beta} + \|\phi(0\|_1)e^{-\frac{p_0}{\mu}t} + Ce^{-\epsilon\beta}e^{-\epsilon\beta}.$$

Then we get at once

$$\int_0^t \phi_{x}_t \frac{x=0} \leq CN(T)(e^{-\epsilon\beta} + \|\phi\|_1) \leq C(e^{-\epsilon\beta} + \|\phi\|_1),$$

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Finally, and

\[ \int_0^t (\psi_x \psi_r)_{x=0} d\tau = \int_0^t (\psi_x(0, \tau) \psi(0, \tau))_{x=0} d\tau - \int_0^t (\psi_x \psi)_{x=0} d\tau \]

\[ = \psi_x(0, \tau) \psi(0, \tau) \big|_t^0 = - \int_0^t (\psi_x \psi)_{x=0} d\tau \]

\[ \leq CN(T)(e^{-cd} + \|\phi_0\|_1) \leq C(e^{-cd} + \|\phi_0\|_1). \]

Finally, \[ \int_0^t (w w_x)_{x=0} d\tau \leq C(\gamma - 1)e^{-cd} \int_0^t e^{-cd\tau} \|w_x\|^2 \|w_{xx}\| \frac{1}{2} d\tau \]

\[ \leq (\gamma - 1)\int_0^t \|w_{xx}\|^2 (\tau) d\tau + C(\gamma - 1)e^{-cd} \int_0^t \|w_x\|^2 \frac{1}{2}(\tau) e^{-\frac{cd}{2}} e^{\frac{1}{2}cd\tau} d\tau \]

\[ \leq (\gamma - 1)\int_0^t \|w_{xx}\|^2 (\tau) d\tau + C(\gamma - 1)e^{-cd}[N(T)] \frac{1}{2} \int_0^t e^{-\frac{cd}{2} \tau} d\tau \]

\[ \leq (\gamma - 1)\int_0^t \|w_{xx}\|^2 (\tau) d\tau + Ce^{-cd}. \]

and

\[ \int_0^t (w_x w_r)_{x=0} d\tau \leq C(\gamma - 1)d^2e^{-cd} \int_0^t e^{-cd\tau} \|w_x\|^2 \|w_{xx}\| \frac{1}{2} d\tau \]

\[ \leq (\gamma - 1)d \int_0^t \|w_{xx}\|^2 (\tau) d\tau + C(\gamma - 1)d^2e^{-cd} \int_0^t \|w_x\|^2 \frac{1}{2}(\tau) e^{-\frac{cd}{2}} e^{\frac{1}{2}cd\tau} d\tau \]

\[ \leq (\gamma - 1)d \int_0^t \|w_{xx}\|^2 (\tau) d\tau + C(\gamma - 1)d^2e^{-cd}[N(T)] \frac{1}{2} \int_0^t e^{-\frac{cd}{2} \tau} d\tau \]

\[ \leq (\gamma - 1)d \int_0^t \|w_{xx}\|^2 (\tau) d\tau + Ce^{-cd}. \]

We complete the proof of the lemma 4.3.

**Lemma 4.4.** For \((\gamma - 1)d \leq \delta_0\) small enough, then

\[ \|(\Phi, \Psi, \frac{\partial \tilde{W}}{(\gamma - 1)^{\frac{1}{2}}})(t)\|^2 + \int_0^t \|V_2\|^2(\Psi, \frac{\partial \tilde{W}}{(\gamma - 1)^{\frac{1}{2}}})(\tau)\|^2 d\tau \]

\[ + \int_0^t \|\Psi, \overline{W_x}\|^2 (\tau) d\tau - C(\gamma - 1)d \int_0^t \|\Phi_x(\tau)\|^2 d\tau \]

\[ \leq C \left\{ \|(\Phi_0, \Psi_0, \frac{\partial \tilde{W}_0}{(\gamma - 1)^{\frac{1}{2}}})\|^2 + \int_0^t \int_0^\infty |\Psi| |F_1| + |\tilde{W}| |F_2| dx d\tau \right\} + CN(T) \int_0^t \|\Psi_{xx}\|^2 d\tau + Ce^{-cd}. \]

**Proof.** Let

\[ k(V) = (b_1 - s^2 V)^{-1}. \]
Multiplying the first equation of (3.8) by $\Phi$, the second equation of (3.8) by $k(V)V\Psi$ and the third equation of (3.8) by $Rk(V)^2\hat{W}$, respectively, summing them up, we have

$$
E_1(\Phi, \Psi, \hat{W})_t + E_2(\Psi, \Psi_x + E_3(\hat{W}, \hat{W}_x) + G(\Psi, \hat{W}, \Phi_x, \hat{W}_x) + \left\{ \mu k(V)\Psi\Psi_x - \Phi\Psi - \frac{Rk(V)^2}{V}(\hat{W}_x + \frac{\gamma - 1}{R}U_x\Psi)\hat{W} + Rk(V)\hat{W}\Psi \right\}_x
$$

$$
= k(V)V\Psi F_1 + Rk(V)^2\hat{W}F_2,
$$

(4.4)

where

$$
E_1(\Phi, \Psi, \hat{W}) = \frac{1}{2}\left( \Phi^2 + k(V)V\Psi^2 + \frac{R^2}{\gamma - 1}k(V)^2\hat{W}^2 \right),
$$

$$
E_2(\Psi, \Psi_x) = \left[ \frac{s}{2}(k(V)V)_x + (\gamma - 1)k(V)U_x \right] \Psi + \mu k(V)\Psi_x + \mu k(V)_x\Psi_x,
$$

$$
E_3(\hat{W}, \hat{W}_x) = \frac{sR^2}{\gamma - 1}k(V)k(V)_x\hat{W}^2 + \kappa R\frac{k(V)^2}{V}\hat{W}_x^2,
$$

$$
G(\Psi, \hat{W}, \Phi_x, \hat{W}_x) = \kappa R\left( \frac{k(V)^2}{V} \right)_x \hat{W}\left( \hat{W}_x + \frac{\gamma - 1}{R}U_x\Psi \right)
$$

$$
+ \kappa R\frac{k(V)^2}{V^2}\Psi\Theta_x\hat{W} + \kappa(\gamma - 1)\frac{k(V)^2}{V}U_x\Psi\hat{W}_x,
$$

Since

$$
p_\gamma \leq k(V)^{-1} = b_1 - s^2V \leq p_\gamma,
$$

one has

$$
c\left( \Phi^2 + \Psi^2 + \frac{\hat{W}^2}{\gamma - 1} \right) \leq E_1 \leq C\left( \Phi^2 + \Psi^2 + \frac{\hat{W}^2}{\gamma - 1} \right),
$$

(4.6)

$$
E_3 \geq c\left( |V_x|\frac{\hat{W}^2}{\gamma - 1} + \hat{W}_x^2 \right),
$$

(4.7)

and for $\forall \alpha_1 > 0$, there $\exists$ a constant $C_{\alpha_1}$ such that

$$
|G| \leq \alpha_1\left( |V_x|\frac{\hat{W}^2}{\gamma - 1} + \hat{W}_x^2 \right) + C_{\alpha_1}(\gamma - 1)d \left[ |V_x|\left( \Psi^2 + \frac{\hat{W}^2}{\gamma - 1} \right) + \Phi_x^2 \right].
$$

(4.8)

By using the method in [9], for $\gamma \in (1, 2]$ and suitably small $(\gamma - 1)d > 0$, one has

$$
\inf_{x > 0} \frac{k(V)V_x + (\gamma - 1)k(V)U_x}{V_x} > 0,
$$

$$
\sup_{x > 0} \frac{\mu k(V)_x^2}{V_x} - 4 \frac{k(V)_x}{V_x} > 0,
$$

(4.9)

(4.10)

Combining with the boundary estimates in Lemma 4.3, (4.3) is obtained.
Lemma 4.5. There is a constant $C$ such that
\[
\|\phi(t)\|^2 + \int_0^t \|\phi(\tau)\|^2 d\tau \\
- C\{\|\Psi(t)\|^2 + \int_0^t \|V_x \frac{d}{d\tau} \Psi(t)\|^2 + \|\psi_x \W_x(t)\|^2 d\tau\} \leq C\left\{\|\Psi(0)\|^2 + \|\phi(0)\|^2 + \int_0^t \int_0^{+\infty} |\psi_x| |F_1(x) dx| dx\right\}.
\]

Proof. Multiplying (3.8) by $V \Psi_x - V_x \Psi$, (3.8) by $-V \Phi_x$, then applying $\partial_x$ to (3.8) and multiplying the resulting equation by $\mu \Phi_x$, calculating all their sums, we get
\[
\left(\frac{\mu \Phi_x^2}{2} - V \Phi_x \Psi\right)_t + (V \Psi_x \Psi_x)_x + (b_1 - s^2 V) \Phi_x^2 \\
= V_x \Psi_x + V \Phi_x^2 + \left[R \W_x - s(\gamma - 1) V_x \Psi - V \Phi_x - VF_1\right] \Phi_x.
\]
Integrating (4.12) over $[0, +\infty) \times [0, t]$ with respect to $x, t$ and using the boundary estimates Lemma 4.2, we obtain Lemma 4.5.

From (3.7), we easily have
\[
\int_0^t \|w(\tau)\|^2 d\tau - C \left\{\int_0^t \|V_x \frac{d}{d\tau} \Psi(t)\|^2 + \|\W_x(t)\|^2 d\tau\right\} \leq C \int_0^t |\psi^2 w| dx d\tau.
\]
Now we rewrite (3.2) in the form
\[
\left\begin{array}{l}
\phi_t - \psi_x = 0, \\
\psi_t - \frac{b_1 - s^2 V}{V} \phi_x + \frac{R}{V} w_x - \left(\frac{\mu}{V} \psi_x\right)_x - \left(\frac{b_1 - s^2 V}{V}\right) x \phi + \left(\frac{R}{V}\right)_x w = f_1 \\
\frac{R}{\gamma - 1} w + (b_1 - s^2 V) \psi_x - \left(\frac{\kappa}{V} \theta_x\right)_x + \left(\frac{\kappa}{V^2} \theta_x \phi\right)_x \\
- \frac{1}{V} \{((b_1 - s^2 V) \phi - Rw + \mu \psi_x) U_x = f_2,
\end{array}\right.
\]
where $f_1$ and $f_2$ are nonlinear terms with respect to $(\phi, \psi, w)$
\[
f_1 = -\left\{\frac{\phi}{V(V + \phi)} [(b_1 - s^2 V) \phi - Rw + \mu \psi_x]\right\}_x,
\]
\[
f_2 = \left\{\frac{1}{V + \phi}\{((b_1 - s^2 V) \phi - Rw + \mu \psi_x)(\psi_x - \frac{1}{V} U_x \phi) \\
- \left(\frac{\kappa \phi}{V(V + \phi)} (w_x - \frac{1}{V} \Theta_x \phi)\right)_x.\right\}
\]
The following Lemma is the estimates of \((\phi, \psi, \omega)\) and \((\phi_x, \psi_x, \omega_x)\).

**Lemma 4.6.** There is a constant \(C\) such that

\[
\begin{align*}
    \|(\phi, \psi, \omega) \frac{w}{(\gamma - 1)^{\frac{3}{2}}}(t)\|^2 + & \int_0^t \|\partial_x(\psi, w)(\tau)\|^2 d\tau - C \int_0^t \|(\phi, \psi, w)(\tau)\|^2 d\tau \\
    - (\gamma - 1) d \int_0^t w_{xx}(\tau) d\tau & \leq C\|(\phi_0, \psi_0, \omega_0) \frac{w_0}{(\gamma - 1)^{\frac{3}{2}}}\|^2 \\
    + C \int_0^t \int_0^{+\infty} (|\psi| f_1 + |w| f_2) dxd\tau + C(e^{-cd\beta} + \|\phi_0\|_1). \\
\end{align*}
\]

\[
\|\phi_x(t)\|^2 + \int_0^t \|\phi_x(\tau)\|^2 d\tau - C\left\{ \|\psi(t)\|^2 + \int_0^t \|\psi, w(\tau)\|^2 d\tau \right\} \\
\leq C \left\{ \|\psi_0\|^2 + \int_0^t \int_0^{+\infty} |\phi_x| f_1 dxd\tau + (e^{-cd\beta} + \|\phi_0\|_1) \right\}.
\]

\[
\|\partial_x(\psi, \omega) \frac{w}{(\gamma - 1)^{\frac{3}{2}}}(t)\|^2 + \int_0^t \|\partial_x(\psi, w)(\tau)\|^2 d\tau - C \int_0^t \|(\phi, \psi, w)(\tau)\|^2 d\tau \\
\leq C\|\partial_x(\psi_0, \omega_0) \frac{w_0}{(\gamma - 1)^{\frac{3}{2}}}\|^2 + C(e^{-cd\beta} + \|\phi_0\|_1) \\
+ C \int_0^t \int_0^{+\infty} (|\psi| f_1 + |w| f_2) dxd\tau.
\]

**Proof.** Multiplying (4.14)_1 by \(\phi\), (4.14)_2 by \(V k(V)\psi\), (4.14)_3 by \(R k^2(V)w\), and adding them and integrating over \(x, t\), we have

\[
\begin{align*}
    \|(\phi, \psi, \omega) \frac{w}{(\gamma - 1)^{\frac{3}{2}}}(t)\|^2 + & \int_0^t \|\partial_x(\psi, w)(\tau)\|^2 d\tau - C \int_0^t \|(\phi, \psi, w)(\tau)\|^2 d\tau \\
    + & \int_0^t \left[ - \phi \psi + R k(V)\psi w - \mu k(V)\psi \psi_x + \frac{\kappa R}{V^2} \Theta_x k^2(V) \phi w - \frac{\kappa R k^2(V)}{V} w w_x \right]_{x=0} d\tau \\
\end{align*}
\]

\[
\leq C\|(\phi_0, \psi_0, \omega_0) \frac{w_0}{(\gamma - 1)^{\frac{3}{2}}}\|^2 + C \int_0^t \int_0^{+\infty} (|\psi| f_1 + |w| f_2) dxd\tau.
\]

Using the boundary estimate in Lemma 4.3, we can get the first inequality in (4.15).

Now we want to get the estimate of \(\|\phi_x\|^2\) in (4.15)_2. Multiplying (4.14)_1 by \(V\psi\), (4.14)_2 by \(-V\phi\), then applying \(\partial_x\) to (4.14)_1 and multiplying the resulting equation by \(\mu\phi_x\), calculating all their sums, we get

\[
\begin{align*}
    \left( \frac{\mu \phi_x^2}{2} - V \phi_x \psi \right)_t + (V \psi \psi_x)_x + (b_1 - s^2 V) \phi_x^2 = V_x \psi \psi_x + V \psi_x^2 + \\
    \left[ R \omega_x + V(\frac{\mu}{V}) \psi_x + V(b_1 - s^2 V) \phi - \frac{R}{V} \right] \omega - V_t \psi - V f_1 \phi_x.
\end{align*}
\]

Thus integrating the equation (4.16) and using the boundary estimate Lemma 4.3, we obtain (4.15)_2.
Combining (4.17) with the estimates (4.3), (4.11), (4.13), (4.15) and using the a priori assumption equality of (4.15). The proof of Lemma 4.6 is complete.

Integrating the above equality and using Lemma 4.3, we can get the third inequality of (4.15). The proof of Lemma 4.6 is complete.

Since

\[ |F_1, F_2| = O(1) \left[ |(\phi, \psi, w)|^2 + |(\phi, \psi)| |(\psi, w_x)| \right], \]

\[ |f_1, f_2| = O(1) \left[ |(\phi, w)|^2 + |(\phi, w)| |(\psi, w_x)| \right] \tag{4.17} + |(\phi_x, \psi_x)| |(\psi, w_x)| + |\phi||(|w_x, w_{xx}|), \]

combining (4.17) with the estimates (4.3), (4.11), (4.13), (4.15) and using the a priori assumption \( N(T) \leq 6 \) sufficiently small, and also letting \( (\gamma - 1)d \) small enough, we can get the following estimate

\[ N^2(T) + \int_0^T \| (\psi, w) \|^2_2 + \| \phi \|^2_2 \, dt \leq \bar{C}(N_0 + e^{-cd\beta}). \]

where the constant \( \bar{C} \) is independent of \( T \). Thus we get the desired a priori estimate (4.1) if we choose \( N_0 \) and \( e^{-cd\beta} \) small enough.

5 The Local Existence

In this section, we prove the local existence result Proposition 4.1 by the iteration method. First we rewrite the equation (3.8) with the initial values (3.10)-(3.13) and the boundary values (3.16)-(3.17) as the following

\[
\left\{ \begin{array}{l}
\Psi_t - \frac{\mu}{V + \Phi_x} \Psi_{xx} = g_1 := g_1(\Psi, \Phi_x, \Psi_x, \widehat{W}_x), \\
\Psi(0, t) = A(t), \\
\Psi(x, 0) = \Psi_0(x),
\end{array} \right. \tag{5.1}
\]

\[
\left\{ \begin{array}{l}
\frac{R}{\gamma - 1} \widehat{W}_t - \frac{\kappa}{V + \Phi_x} (\widehat{W}_x - \frac{\gamma - 1}{2R} \Psi_x^2)_x = g_2 := g_2(\Psi, \Phi_x, \Psi_x, \widehat{W}_x, \Psi_{xx}), \\
\widehat{W}_x(0, t) - \frac{\gamma - 1}{2R} \Psi_x^2(0, t) = B(t), \\
\widehat{W}(x, 0) = \widehat{W}_0(x),
\end{array} \right. \tag{5.2}
\]
and
\[
\Phi(x, t) = \Phi_0(x) + \int_0^t \Psi_s(x, \tau) d\tau, \tag{5.3}
\]
where \(A(t), B(t)\) is given in (3.16), (3.17) respectively, and
\[
g_1 = \frac{b_1 - s^2 V}{V + \Phi_x} \Phi_x - \frac{R}{V + \Phi_x} (\hat{W}_x + \gamma - \frac{1}{R} U_x \Psi - \frac{\gamma - 1}{2R} \Psi_x^2), \tag{5.4}
\]
\[
g_2 = -\frac{b_1 - s^2 V}{V + \Phi_x} \Phi_x \Psi_x + \frac{\kappa (\gamma - 1)}{R (V + \Phi_x)} (U_x \Psi)_x + s U_x \Psi - \frac{\kappa \Theta_x \Phi_x}{V (V + \Phi_x)} + \frac{\mu \Psi_x \Psi_{xx}}{V + \Phi_x} \frac{R \Psi_x}{V + \Phi_x} [\hat{W}_x + \gamma - \frac{1}{R} (U_x \Psi - \Psi_x^2)] \tag{5.5}
\]
To use the iteration method, we approximate the initial values \((\Phi_0, \Psi_0, \hat{W}_0) \in H^2(0, +\infty)\) by \((\Phi_{0k}, \Psi_{0k}, \hat{W}_{0k}) \in H^5(0, +\infty)\) which will be determined later.

For fixed \(k\), we define the sequence \(\{(\Phi_k^{(n)}, \Psi_k^{(n)}, \hat{W}_k^{(n)}(x, t))\}_{n=1}^{\infty}\) by
\[
(\Phi_k^{(0)}, \Psi_k^{(0)}, \hat{W}_k^{(0)}(x, t)) = (\Phi_{0k}, \Psi_{0k}, \hat{W}_{0k}(x)), \tag{5.6}
\]
and if \((\Phi_k^{(n-1)}, \Psi_k^{(n-1)}, \hat{W}_k^{(n-1)})) (x, t)\) is given, then we define \((\Phi_k^{(n)}, \Psi_k^{(n)}, \hat{W}_k^{(n)}(x, t))\) as the following
\[
\left\{\begin{array}{l}
\Psi_k^{(n)}(0, t) = A(t), \\
\Psi_k^{(n)}(x, 0) = \Psi_{0k}(x),
\end{array}\right.
\]
\[
\frac{R}{\gamma - 1} \hat{W}_{kxx}^{(n)} - \frac{\kappa}{V + \Phi_k^{(n-1)}} (\hat{W}_{kx}^{(n)} - \frac{\gamma - 1}{2R} \Psi_{kx}^{(n)}),
\]
\[
\hat{W}_{k}^{(n)}(0, t) - \frac{\gamma - 1}{2R} \Psi_{kx}^{(n)}(0, t) = B(t),
\]
\[
\hat{W}_{k}^{(n)}(x, 0) = \hat{W}_{0k}(x), \tag{5.7}
\]
and
\[
\Phi_k^{(n)}(x, t) = \Phi_{0k}(x) + \int_0^t \Psi_{kx}^{(n)}(x, \tau) d\tau. \tag{5.9}
\]
Now we construct the approximate initial values \((\Phi_{0k}, \Psi_{0k}, \hat{W}_{0k}(x))\). Firstly we choose \(\Phi_{0k} \in H^5\) such that \(\Phi_{0k} \rightarrow \Phi_0\) strongly in \(H^2\) as \(k \rightarrow \infty\). Let
\[
\Phi_0(x) := \Psi_0(x) - A(0)e^{-x^2}. \tag{5.10}
\]
Note that \(A(0) = \Psi_0(0)\). Then we have \(\Phi_0(x) \in H^2_0\). Now we choose \(\Psi_{0k}(x) \in H^3_0 \cap H^5\) such that \(\Psi_{0k} \rightarrow \Psi_0\) strongly in \(H^2\) as \(k \rightarrow \infty\). We construct
\[
\Psi_{0k}(x) := \Psi_{0k}(x) + A(0)e^{-x^2}, \tag{5.10}
\]

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then we have \( \Psi_{0k} \to \Psi_0(x) + A(0)e^{-x^2} = \Psi_0(x) \) strongly in \( H^2 \) as \( k \to \infty \). Moreover, \( \Psi_{0k}(x) \) constructed in (5.10) satisfies the compatibility condition \( \Psi_{0k}(0) = A(0) \) for the approximate equation (5.7). Now we turn to the compatibility condition for the equation (5.8). Let

\[
\overline{W}_0(x) := \overline{W}_0(x) - B(0)x e^{-x^2} - \overline{W}_0(0)e^{-x^2}.
\]

It is obvious that \( \overline{W}_0(x) \in H_0^2 \). So we can choose \( \overline{W}_{0k}(x) \in H_0^2 \cap H^5 \) such that \( \overline{W}_{0k}(x) \to \overline{W}_0(x) \) strongly as \( k \to \infty \). Set

\[
\overline{W}_{0k}(x) := \overline{W}_{0k}(x) + B(0)x e^{-x^2} + \overline{W}_0(0)e^{-x^2}.
\]

Then we have \( \overline{W}_{0k}(x) \to \overline{W}_0(x) + B(0)x e^{-x^2} + \overline{W}_0(0)e^{-x^2} = \overline{W}_0(x) \) strongly in \( H^2 \) as \( k \to \infty \). Note that \( B(0) = \overline{W}_{0k}(0) - \frac{\gamma - 1}{2R} \psi_{0k}^2(0) \). We verify that the approximated initial values \( \Psi_{0k}(x), \overline{W}_{0k}(x) \) satisfy the following compatibility condition for the equation (5.8),

\[
\overline{W}_{0k}(0) - \frac{\gamma - 1}{2R} \psi_{0k}^2(0) = \overline{W}_{0k}(0) + B(0) - \frac{\gamma - 1}{2R} \psi_{0k}^2(0) = B(0).
\]

And it is easy to choose that the above approximation \( (\Phi_{0k}(x), \Psi_{0k}(x), \overline{W}_{0k}(x)) \) satisfies \( \| (\Phi_{0k}, \Psi_{0k}, \overline{W}_{0k}) \|^2 \leq \frac{3}{2} M \) and \( \inf_{x} (V + \Phi_{0k}(x)) \geq \frac{3}{2} m \) for any fixed \( k \).

If \( (\Psi_{0k}^{(n-1)}, \Psi_{0k}^{(n-1)}, \overline{W}_{0k}^{(n-1)}) \in X_{m,b;H_0^3}(t_0, t_0) \cap C(0, t_0; H^5), \) then \( g_1^{(n-1)} \in C(0, t_0; H^4). \) By linear parabolic theory, since \( \Psi_{0k} \in H^5 \), there exists a unique solution to (5.7) satisfying

\[
\Psi_{0k}^{(n)}(0) \subset C(0, t_0; H^5) \cap C^1(0, t_0; H^3) \cap L^2(0, t_0; H^6).
\]

Substituting \( \Psi_{0k}^{(n)} \) into \( g_2^{(n-1)} \), we have that \( g_2^{(n-1)} \in C(0, t_0; H^3) \). Using linear parabolic theory again, we obtain

\[
\overline{W}_{0k}^{(n)} \subset C(0, t_0; H^5) \cap C^1(0, t_0; H^3) \cap L^2(0, T; H^6).
\]

From (5.9), we also have

\[
\Phi_{0k}^{(n)} \subset C(0, t_0; H^5) \cap C^1(0, t_0; H^3) \cap L^2(0, t_0; H^6).
\]

The elementary energy estimates to the equation (5.7)-(5.8) yield that

\[
\| (\Psi_{0k}^{(n)}, \overline{W}_{0k}^{(n)}) \|^2 \leq (bM)^2.
\]

if the time interval \( t_0 = t_0(m, M) \) is suitably small. We omit the detailed calculations for brevity.

Now from (5.9), we can compute that

\[
\| \Phi_{0k}^{(n)} \|^2 \leq (bM)^2,
\]
and
\[
\inf_{x,t \in [0,t_0]} (V + \Phi_{kx}^{(n)}) \geq \frac{1}{2}m.
\]
Therefore we have \((\Psi_k^{(n)}, \Psi_k^{(n)}, \hat{W}_{kx}^{(n)}) \in X_{4m,bM}(0,t_0) \cap C(0,t_0;H^5)\). Since \(\| (\Psi_k^{(0)}, \Psi_k^{(0)}, \hat{W}_{k}^{(0)}) \|_5 \) is uniformly bounded for fixed \(k\), we can show that \((\Psi_k^{(n)}, \Psi_k^{(n)}, \hat{W}_{kx}^{(n)})\) is the Cauchy sequence in \(C(0,t_0;H^4)\). Letting \(n \to \infty\) in (5.7)-(5.9), we get a solution \((\Phi_k, \Psi_k, \hat{W}_k)(x,t)\) of (5.1)-(5.3) with the initial values replaced by \((\Phi_{0k}, \Psi_{0k}, \hat{W}_{0k})(x)\) in the time interval \([0,t_0]\).

In the same way we can show that \((\Phi_k, \Psi_k, \hat{W}_k)(x)\) is a Cauchy sequence in \(C(0,T_0;H^2)\) (taking \(T_0\) smaller than \(t_0\) if necessary). Now letting \(k \to \infty\), we get the desired unique solution \((\Phi, \Psi, \hat{W})(x,t)\) to (5.1)-(5.3) in the time interval \([0,T_0]\).

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