CONFORMAL IMAGES OF CARLESON CURVES

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Abstract. We show that if $\gamma$ is a curve in the unit disk, then arclength on $\gamma$ is a Carleson measure iff the image of $\gamma$ has finite length under every conformal map of the disk onto a bounded domain with a rectifiable boundary.

In this note we characterize curves in $\mathbb{D}$ for which arclength is a Carleson measure, in terms of conformal maps onto rectifiable domains, answering a question asked by Percy Deift (personal communication) arising from his work on Riemann-Hilbert problems. The question seems natural and the proof follows from standard techniques, but I have not been able to locate this result in the literature.

Recall that a positive measure $\mu$ on the open unit disk, $\mathbb{D}$, is called a Carleson measure if

$$\|\mu\|_C = \sup_{|z|=1, r>0} \frac{\mu(D(z, r))}{r} < \infty.$$ 

The left hand side is called the Carleson norm of the measure.

Theorem 1. If $\gamma$ is a curve in the unit disk, then arclength on $\gamma$ is a Carleson measure iff the image of $\gamma$ has finite length under every conformal map onto a bounded domain with rectifiable boundary.

Proof. One direction is an easy consequence of known facts. If $f$ is a conformal map onto a rectifiable domain, then the F. and M. Riesz theorem (e.g., Theorem VI.1.2 of [2]) says that its derivative is in the Hardy space $H^1$. For a Jordan domain, the $H^1$ norm of $f'$ is the length of the image’s boundary. If the boundary is not a Jordan curve then we may replace “length” by “1-dimensional Hausdorff measure” (also denoted by $\ell$) and get $\ell(\partial \Omega) \leq \|f'\|_{H^1} \leq 2\ell(\partial \Omega)$. For any $H^p$ function $g$ on the
unit disk
\[ \int |g|^p d\mu \leq C_p \|\mu\|_C \|g\|_{H^p}, \]

(e.g., Theorem II.3.9 of [1]) where \( \| \cdot \|_{H^p} \) is the Hardy space norm. Thus taking \( g = f' \) we see that
\[ \ell(f(\gamma)) = \int_{\gamma} |f'| ds \leq C_1 \|\mu\|_C \cdot \ell(\partial f(D)), \]

where \( \mu \) denotes arclength measure on \( \gamma \).

The converse requires more work. Theorem II.3.9 of [1] implies that if \( \mu \) is not Carleson, then there is a \( g \in H^1 \) so that \( \int |g| d\mu = \infty \). By the usual factorization theorems for Hardy spaces (e.g., Corollary II.5.7 of [1]), we can assume \( g \) never vanishes in \( \mathbb{D} \), but this is not quite enough to deduce that \( g = h' \) for some conformal map \( h \). Instead, we will explicitly construct a conformal map \( h \) onto a rectifiable domain so that \( \int |h'| d\mu = \infty \).

Our conformal map \( h \) will be built as a limit of compositions from a collection of conformal maps defined as follows. Suppose \( 0 < a < 1 \) and let \( \Omega_{a,\epsilon} = \mathbb{D} \cup D(1 + a, (1 + \epsilon)a) \) be the overlapping union of the unit disk \( \mathbb{D} \) and a smaller disk centered outside of \( \mathbb{D} \). See Figure 1. The conformal map \( \mathbb{D} \to \Omega_{a,\epsilon} \) is a composition of Möbius transformations and power functions, but we will not need the explicit formula. We will only use the following facts.

**Lemma 2.** There is a constant \( 0 < c < 1 \) so that given any \( 0 < a < 1 \) and \( 0 < \delta < 1/2 \), there exists an \( 0 < r_a < 1 \) so that the following holds. For any \( 0 < r < r_a \) there is an \( \epsilon > 0 \) and a conformal map \( f : \mathbb{D} \to \Omega_{a,\epsilon} \) such that:

1. \( f(0) = 0 \) and \( f \) is symmetric with respect to \( \mathbb{R} \),
2. \( f(1 - r) = 1 + a \),
3. \( |f'| \geq ca/r \) on \( D(1, r) \),
4. \( f \) has a conformal extension across \( \mathbb{T} \),
5. \( |f(z) - z| < \delta \) and \( |f'(z) - 1| < \delta \) on \( \mathbb{D} \setminus D(1, \delta) \).

The lemma be proven by an explicit calculation of \( f \), or by applying symmetry and distortion properties of conformal maps (e.g., Koebe’s \( \frac{1}{4} \)-theorem). The idea for (2) is that the hyperbolic distance between 0 and \( a \) is a continuous function of \( \epsilon \) and it goes to \( \infty \) as \( \epsilon \) goes to zero. For a given \( a, \epsilon \) we can choose \( r \) so the image is \( > 1 + a \),
Figure 1. The top picture shows the domain $\Omega_{a, \epsilon}$ which is a small disk attached to the unit disk. A properly placed Carleson region is expanded by this map to a size comparable to the added "bubble" and $|f'|$ is comparable to the ratio the diameters of the region and its image. By composing maps of this form, we get build a sequence of domains that look like the lower picture, except that in the proof the sizes of the "bubbles" shrink much more dramatically.

but the image tends to 1 as $\epsilon \searrow 0$, so there is an intermediate choice of $\epsilon$ where $r$ maps to $1 + a$. By replacing $f(z)$ by $f(sz)$ for $s$ very close to 1, we can assume $f$ has a conformal extension across $\mathbb{T}$ and the previous conditions still hold. We leave the details to the reader.

By conjugating $f$ with a rotation of $\mathbb{D}$ (i.e., replace $f(z)$ by $f(\lambda z) / \lambda$, $|\lambda| = 1$), we can clearly make $|f'|$ large on any sufficiently small Carleson disk, not just those centered at 1.

Let $\mu$ denote arclength measure on a curve $\gamma$ and suppose this is not a Carleson measure. Then there must be sequence of disks centered at points $\{x_n\}$ on the unit circle and radii $\rho_n \to 0$ so that

$$\mu(D(x_n, \rho_n)) \geq n\rho_n.$$

Fix one such disk $D = D(x, r)$ and let $W_t = D \cap \{|z| < t\}$. Since $D \cap \mathbb{D}$ is the union of the $W_t$’s as $t \nearrow 1$, we can choose a $t$ so that $\mu(W_t) \geq \frac{1}{2}\mu(D)$. For each disk in our sequence, make such a choice and inductively define a subsequence of sets $\{W_n\}$.
so that \( \mu(W_n) \geq nd_n \) and \( d_{n+1} 2^{-n} \cdot \text{dist}(W_n, \mathbb{T}) \), where \( d_n = \text{diam}(W_n) \) (Euclidean diameter). We now proceed by induction to construct a sequence of conformal maps \( \{ h_j \} \) on \( \mathbb{D} \) that map our non-Carleson curve \( \gamma \) to curves with longer and longer length. The limiting map \( h \) will map \( \gamma \) to a curve of infinite length.

Start with \( a = \delta = 1/2 \) and let \( r_a \) be as in the lemma. Choose \( k_1 \) so large that the region \( W_{k_1} \subset D(x_{k_1}, \rho_{k_1}) \) has diameter less than \( r_a \). By the lemma, we can choose a point \( a_1 = a \cdot x_{k_1} \) outside \( \mathbb{D} \), an \( \epsilon_1 > 0 \), and a conformal map \( f_1 : \mathbb{D} \to \Omega_{a_1, \epsilon_1} \) so that \( |f_1'| \geq ca_1 / \rho_{k_1} \) on \( W_{k_1} \), and \( f_1 \) extends to be analytic on \( \{|z| < 1 + s_1\} \) for some positive \( s_1 \). Let \( h_1 = f_1 \).

In general, assume we have used the lemma to choose conformal maps \( f_1, \ldots, f_{n-1} \) and that they and all have a conformal extension to \( \{|z| < 1 + s_{n-1}\} \) for some positive \( s_{n-1} \). Let \( h_{n-1} = f_1 \circ \cdots \circ f_{n-1} \). Let \( M_{n-1} = \max |h_{n-1}'| \) over the closed unit disk (since \( h_{n-1} \) has a holomorphic extension across the boundary, this maximum is certainly finite). Similarly, let \( m_n = \min |h_{n-1}'| > 0 \). Choose \( 0 < a_n < s_{n-1} \) and \( \epsilon_n > 0 \) so small that \( a_n M_{n-1} \leq 2^{-n} \) and so that the conformal map \( f_n \) given by the lemma satisfies both

\[
|f_n(z) - z| \leq s_{n-1}/2, \quad \text{and} \quad |f_n' - 1| \leq 2^{-n},
\]

on \( \mathbb{D} \setminus D(1, s_{n-1}) \). Moreover, \( |f_n'| \geq c/(a_n \rho_{k_n}) \) on \( D(1, r_n) \), where \( r_n = r_{a_n} \) as given by the lemma.

Now choose \( k_n \) so large that the region \( W_{k_n} \) satisfies:

6. \( \text{diam}(W_{k_n}) < r_{a_n} \) (\( r_a \) as given by the lemma),

7. The minimum and maximum of \( |h_{n-1}'| \) over \( W_{k_n} \) differ by at most a factor of 2 (this is possible by the distortion theorem for conformal maps if \( \text{diam}(W_{k_n}) \) is small enough).

8. \( k_n \geq c/(m_n a_n) \).

By the definition of \( W_n \), Condition (8) implies

\[
\mu(W_{k_n}) / \text{diam}(W_{k_n}) \geq k_n \geq c/(m_n a_n)
\]

or

\[
\mu(W_{k_n}) \geq c \cdot \frac{\text{diam}(W_{k_n})}{m_n a_n}.
\]

By conjugating \( f_n \) by an appropriate rotation, we get a function (also called \( f_n \)) so that \( |f_n'| \geq ca_n / \rho_{k_n} \) on \( W_{k_n} \). This implies that the length of \( \sigma \) inside \( W_{k_n} \) is expanded
to approximately unit length under $f_n$. We want to show this is also true for the composition $h_n = h_{n-1} \circ f_n = f_1 \circ \cdots f_{n-1} \circ f_n$ and show these maps have a limit $h$ with the same property.

By construction, the image of each map $f_j$ lies inside a disk where the map $f_{j-1}$ is defined and conformal so the composition is well defined and conformal on $\mathbb{D}$. Since the maps $f_j$ converge uniformly to the identity on compact subsets of $\mathbb{D}$ (as rapidly as we wish), the limiting map $h$ exists and is conformal on $\mathbb{D}$. Next we check that $h(\gamma)$ has infinite length and that $h(\mathbb{T})$ is rectifiable.

On each $W_{kj}$ we have

$$|h'_n| \geq |h'_j|\left(\prod_{m=j+1}^{n} (1 - 2^{-m})\right) \geq c|h'_j|.$$ 

Thus later generations of the construction do not greatly effect the expansion we have already created on earlier regions. Since $h_n \to h$ uniformly on compact sets, we also have $h'_n \to h'$ uniformly on compact sets and hence

$$\int_K |h'|d\mu = \lim_n \int_K |h'_n|d\mu,$$

for any compact $K \subset \mathbb{D}$. In particular,, we can let $K = W_{k1} \cup \cdots \cup W_{kn}$ be a finite union of the sets $W_{kj}$ and note that

$$\int_K |h'_n|d\mu \geq c \sum_{j=1}^{n} \int_{W_{kj}} |h'_j|d\mu \geq \sum_{j=1}^{n} m_{n-1} \cdot \frac{1}{|a_j|\rho_{k_j}} \cdot \rho_{k_j} k_j \geq \sum_{j=1}^{n} 1 \to \infty$$

by our choice of $k_j$ in Condition (8) above. Thus $h(\gamma)$ has infinite length.

Finally, we have to check that $h$ maps $\mathbb{D}$ to a domain with rectifiable boundary. However, the domain $h_n(\mathbb{D})$ is obtained by taking the union of $\mathbb{D}$ with disk of diameter $a_n$ and composing with the map $h_{n-1}$ and then dilating the map very slightly to make sure it has a conformal extension across the unit circle. Adding the disk adds length $O(a_n)$ and composing with $h_{n-1}$ gives a curve which is in the union of $\partial h_{n-1}(\mathbb{D})$ and the image of the small disk. This image has length $O(M_{n-1}a_n) = O(2^{-n})$. Dilating shortens the length of the boundary curve (since $|f'|$ is subharmonic the length of $f(|z| = r)$ is always less than the length of $f(|z| = 1)$ for any conformal map). Thus we can choose $|a_n| \searrow 0$ so rapidly that the length of $\partial h_n(\mathbb{D})$ is uniformly bounded above by some $L < \infty$. 
Next, note that the length of $\partial h(\mathbb{D})$ is equal to

$$\sup_{0<r<1} \int |h'(re^{i\theta})|d\theta.$$ 

On the other hand, for any fixed $r$, $h_n$ converges uniformly to $h$ on the compact set $\{|z|=r\}$ and hence its derivative converges uniformly to $h'$ on this set. Thus for a fixed $0<r<1$,

$$\int |h'(re^{i\theta})|d\theta \leq \sup_n \int |h'_n(re^{i\theta})|d\theta \leq L.$$ 

Taking the sup over $r$ we see $h' \in H^1$ and so $h(\mathbb{T})$ is rectifiable. $\square$

Although Deift’s question concerned curves, we never used this, and we have actually proven that a positive measure $\mu$ on the disk is Carleson iff $\int |f'|d\mu < \infty$ for any conformal map $f$ onto a rectifiable domain.

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References

[1] John B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[2] John B. Garnett and Donald E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008. Reprint of the 2005 original.