On the Geometry of Lightlike Submanifolds in Metallic Semi-Riemannian Manifolds

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Abstract

In the present paper, we introduce screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds with its subclasses, namely screen transversal anti-invariant, radical screen transversal and isotropic screen transversal lightlike submanifolds, and give an example. We show that there do not exist co-isotropic and totally screen transversal type of screen transversal anti-invariant lightlike submanifolds of a metallic semi-Riemannian manifold. We investigate the geometry of distributions involved in the definition of such submanifolds and the conditions for the induced connection to be a metric connection. Furthermore, we give a necessary and sufficient condition for an isotropic screen transversal lightlike submanifold to be totally geodesic.

1 Introduction

In Riemannian geometry, it is well known that the induced metric on a submanifold of a Riemannian manifold is always a Riemannian one. But in semi-Riemannian manifolds the induced metric by the semi-Riemannian metric on the ambient manifold is not necessarily non-degenerate. This case leads to provide an interesting type of submanifolds called lightlike submanifolds. Because of degeneracy of the induced metric on lightlike submanifolds, the tools which are used to investigate the geometry of submanifolds in Riemannian case are not applicable in semi-Riemannian case and so the classical theory fails while defining any induced object on a lightlike submanifold. The main difficulties
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arise from the fact that the intersection of the normal bundle and the tangent bundle of a lightlike submanifold is nonzero. In order to resolve the difficulties that arise during studying lightlike submanifolds, K. Duggal, A. Bejancu [9] introduced a non-degenerate distribution called screen distribution to construct a lightlike transversal vector bundle which does not intersect to its lightlike tangent bundle. It is well-known that a suitable choice of screen distribution gives rise to many substantial characterizations in lightlike geometry. Many authors have studied the geometry of lightlike submanifolds in different manifolds (see [13, 33, 11, 12, 25, 2, 1, 3]). For a more comprehensive reading, we refer [9, 10] and the references therein.

Different kinds of geometric structures (such as almost product, almost contact, almost paracontact etc.) allow to get rich results while studying on submanifolds. Recently, Riemannian manifolds with metallic structures are one of the most studied topics in differential geometry.

In 2002, as a generalization of the Golden mean, metallic means family was introduced by V. W. de Spinadel [29], which contains the Silver mean, the Bronze mean, the Copper mean and the Nickel mean etc. The positive solution of the equation given by

\[x^2 - px - q = 0,\]

for some positive integer \(p\) and \(q\), is called a \((p, q)\)-metallic number [27, 28] and it has the form

\[\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.\]

For \(p = q = 1\) and \(p = 2, q = 1\), it is well-known that we have the Golden mean \(\phi = \frac{1+\sqrt{5}}{2}\) and Silver mean \(\sigma_{2,1} = 1 + \sqrt{2}\), respectively. The metallic mean family plays an important role to establish a relationship between mathematics and architecture. For example, Golden mean and Silver mean can be seen in the sacred art of Egypt, Turkey, India, China and other ancient civilizations [31].

Polynomial structures on manifolds were introduced by S. I. Goldberg, K. Yano and N. C. Petridis in (18 and 19). C. E. Hretcanu and M. Crasmareanu defined Golden structure as a particular case of polynomial structure [6, 7, 8] and some generalizations of this, called metallic structure [17]. Being inspired by the metallic mean, the notion of metallic manifold \(\tilde{N}\) was defined in [23] by a \((1,1)\)-tensor field \(\tilde{J}\) on \(\tilde{N}\), which satisfies \(\tilde{J}^2 = p\tilde{J} + qI\), where \(I\) is the identity operator on \(\Gamma(T\tilde{N})\) and \(p, q\) are fixed positive integer numbers. Moreover, if \((\tilde{N}, \tilde{g})\) is a Riemannian manifold endowed with a metallic structure \(\tilde{J}\) such that the Riemannian metric \(\tilde{g}\) is \(\tilde{J}\)-compatible, i.e., \(\tilde{g}(\tilde{J}V, W) = \tilde{g}(V, \tilde{J}W)\), for any \(V, W \in \Gamma(T\tilde{N})\), then \((\tilde{N}, \tilde{g}, \tilde{J})\) is called metallic Riemannian structure and \((\tilde{N}, \tilde{g}, \tilde{J})\) is a metallic Riemannian manifold. Metallic structure on the ambient Riemannian manifold provides important geometrical results on the submanifolds, since it is an important
tool while investigating the geometry of submanifolds. Invariant, anti-invariant, semi-invariant, slant, semi-slant and hemi-slant submanifolds of a metallic Riemannian manifold were studied in [22, 21, 5, 20] and the authors obtained important characterizations on submanifolds of metallic Riemannian manifolds. One of the most important subclass of metallic Riemannian manifolds consists of the Golden Riemannian manifolds. In recent years, many authors have studied Golden Riemannian manifolds and their submanifolds (see [32, 14, 24, 15]). N. Poyraz Önen and E. Yaşar [26] initiated the study of lightlike geometry in Golden semi-Riemannian manifolds, by investigating lightlike hypersurfaces of Golden semi-Riemannian manifolds. B. E. Acet introduced lightlike hypersurfaces of a metallic semi-Riemannian manifold [4]. Transversal lightlike submanifolds in metallic semi-Riemannian manifolds were firstly studied by F. E. Erdoğan [16].

In the present paper, we introduce screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds with its subclasses, namely screen transversal anti-invariant, radical screen transversal and isotropic screen transversal lightlike submanifolds, and give an example. We show that there do not exist co-isotropic and totally screen transversal type of screen transversal anti-invariant lightlike submanifolds of a metallic semi-Riemannian manifold. We investigate the geometry of distributions involved in the definition of such submanifolds and find necessary and sufficient conditions for the induced connection to be a metric connection. Furthermore, we give a necessary and sufficient condition for an isotropic screen transversal lightlike submanifold to be totally geodesic.

2 Preliminaries

A submanifold $\tilde{N}^m$ immersed in a semi-Riemannian manifold $(\tilde{N}^{m+k}, \tilde{g})$ is called a lightlike submanifold if it admits a degenerate metric $g$ induced from $\tilde{g}$, whose radical distribution $\text{Rad}(T\tilde{N})$ is of rank $r$, where $1 \leq r \leq m$. Then $\text{Rad}(T\tilde{N}) = T\tilde{N} \cap T\tilde{N}^\perp$, where

$$T\tilde{N}^\perp = \bigcup_{x \in \tilde{N}} \left\{ u \in T_x \tilde{N} \mid \tilde{g}(u, v) = 0, \forall v \in T_x \tilde{N} \right\}.$$  

Let $S(T\tilde{N})$ be a screen distribution which is a semi-Riemannian complementary distribution of $\text{Rad}(T\tilde{N})$ in $T\tilde{N}$, i.e., $T\tilde{N} = \text{Rad}(T\tilde{N}) \oplus_{\text{ort}} S(T\tilde{N})$.

We consider a screen transversal vector bundle $S(T\tilde{N}^\perp)$, which is a semi-Riemannian complementary vector bundle of $\text{Rad}(T\tilde{N})$ in $T\tilde{N}^\perp$. For any local basis $\{\xi_i\}$ of $\text{Rad}(T\tilde{N})$, there exists a lightlike transversal vector bundle $ltr(T\tilde{N})$ locally spanned by $\{N_i\}$ [9]. Let $\text{tr}(T\tilde{N})$ be complementary (but not orthogonal) vector bundle to $T\tilde{N}$ in $T\tilde{N}^\perp |_{\tilde{N}}$. Then we have

$$\text{tr}(T\tilde{N}) = ltr(T\tilde{N}) \oplus_{\text{ort}} S(T\tilde{N}^\perp),$$

$$T\tilde{N} |_{\tilde{N}} = S(T\tilde{N}) \oplus_{\text{ort}} [\text{Rad}(T\tilde{N}) \oplus ltr(T\tilde{N})] \oplus_{\text{ort}} S(T\tilde{N}^\perp).$$
Although $S(T\tilde{N})$ is not unique, it is canonically isomorphic to the factor vector bundle $T\tilde{N}/\text{Rad}(T\tilde{N})$.

Note that the lightlike second fundamental forms of a lightlike submanifold $\tilde{N}$ do not depend on $S(T\tilde{N})$, $S(T\tilde{N}^\perp)$ and $\text{ltr}(T\tilde{N})$.

We say that a submanifold $(\tilde{N}, g, S(T\tilde{N}))$ of $\tilde{N}$ is

Case 1: $r$-lightlike if $r < \min\{m, k\}$;

Case 2: Co-isotropic if $r = k < m$; $S(T\tilde{N}^\perp) = \{0\}$;

Case 3: Isotropic if $r = m < k$; $S(T\tilde{N}) = \{0\}$;

Case 4: Totally lightlike if $r = k = m$; $S(T\tilde{N}) = \{0\} = S(T\tilde{N}^\perp)$.

The Gauss and Weingarten equations are given by

\begin{align*}
\tilde{\nabla}_W U &= \nabla_W U + h(W, U), \quad \forall W, U \in \Gamma(T\tilde{N}), \\
\tilde{\nabla}_W V &= -A_V W + \nabla^t_W V, \quad \forall W \in \Gamma(T\tilde{N}), V \in \Gamma(\text{ltr}(T\tilde{N})),
\end{align*}

where \{\nabla_W U, A_V W\} and \{h(W, U), \nabla^t_W V\} belong to $\Gamma(T\tilde{N})$ and $\Gamma(\text{ltr}(T\tilde{N}))$, respectively. Here, $\nabla$ and $\nabla^t$ denote linear connections on $\tilde{N}$ and the vector bundle $\text{ltr}(T\tilde{N})$, respectively. Also, for any $W, U \in \Gamma(T\tilde{N})$, $N \in \Gamma(\text{ltr}(T\tilde{N}))$ and $Z \in \Gamma(S(T\tilde{N}^\perp))$, we have

\begin{align*}
\tilde{\nabla}_W U &= \nabla_W U + h^\ell(W, U) + h^s(W, U), \\
\tilde{\nabla}_W N &= -A_N W + \nabla^s_W N + D^s(W, N), \\
\tilde{\nabla}_W Z &= -A_Z W + \nabla^s_W Z + D^\ell(W, Z).
\end{align*}

Let $P$ denotes the projection of $T\tilde{N}$ on $S(T\tilde{N})$. Since $\tilde{\nabla}$ is a metric connection, then by using (2), (4)-(6) we get

\begin{align*}
\check{\nabla}(h^s(W, U), Z) + \check{\nabla}(U, D^\ell(W, Z)) &= \check{\nabla}(A_Z W, U), \\
\check{\nabla}(D^s(W, N), Z) &= \check{\nabla}(N, A_Z W).
\end{align*}

From the decomposition of the tangent bundle of a lightlike submanifold, we write

\begin{align*}
\nabla_W PU &= \nabla^*_W PU + h^*(W, PU), \\
\nabla_W \xi &= -A^*_\xi W + \nabla^*_W \xi,
\end{align*}

for $W, U \in \Gamma(T\tilde{N})$ and $\xi \in \Gamma(\text{Rad}(T\tilde{N}))$, which imply

\begin{align*}
g(h^\ell(W, PU), \xi) &= g(A^*_\xi W, PU), \\
g(h^s(W, PU), N) &= g(A_N W, PU), \\
g(h^\ell(W, \xi), \xi) &= 0, \quad A^*_\xi \xi = 0.
\end{align*}
In general, the induced connection $\nabla$ on $\tilde{N}$ is not a metric connection. Since $\tilde{\nabla}$ is a metric connection, by using (4) we get

$$\nabla_W g(U, V) = g(h^\ell(W, U), V) + g(h^\ell(W, V), U).$$

However, we note that $\nabla^*$ is a metric connection on $S(T\tilde{N})$.

The positive solution of the equation

$$x^2 - px - q = 0,$$

for fixed two positive integers $p$ and $q$, is called a member of metallic means family ([27]-[31]). These numbers, given by

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2},$$

are called $(p, q)$-metallic numbers.

**Definition 2.1.** [23] A polynomial structure on a manifold $\tilde{N}$ is called a metallic structure if it is determined by an $(1, 1)$-tensor field $\tilde{J}$ which satisfies

$$\tilde{J}^2 = p\tilde{J} + qI,$$

where $I$ is the identity map on $\Gamma(T\tilde{N})$ and $p, q$ are positive integers. Also, if

$$\tilde{g}(\tilde{J}W, U) = \tilde{g}(W, \tilde{J}U)$$

holds for every $U, W \in \Gamma(T\tilde{N})$, then the semi-Riemannian metric $\tilde{g}$ is called $\tilde{J}$-compatible. In this case, $(\tilde{N}, \tilde{g}, \tilde{J})$ is called a metallic semi-Riemannian manifold. Furthermore, a metallic semi-Riemannian structure $\tilde{J}$ is called a locally metallic structure if $\tilde{J}$ is parallel with respect to the Levi-Civita connection $\tilde{\nabla}$, that is

$$\tilde{\nabla}_W \tilde{J}U = \tilde{J}\tilde{\nabla}_W U.$$

If $\tilde{J}$ is a metallic structure, then (17) is equivalent to [23]

$$\tilde{g}(\tilde{J}W, \tilde{J}U) = p\tilde{g}(\tilde{J}W, U) + q\tilde{g}(W, U),$$

for any $W, U \in \Gamma(T\tilde{N})$.

It is known, from [19], that a polynomial structure on a manifold $\tilde{N}$ defined by a smooth tensor field of type $(1, 1)$ induces a generalized almost product structure $F$, i.e., $F^2 = I$, on $\tilde{N}$ with number of distributions of $F$ equal to the number of distinct irreducible factors of the structure polynomial over the real field while the projectors are expressed as polynomials in $F$. 

Proposition 2.2. [23] Every almost product structure $F$ induces two metallic structures on $\tilde{N}$ given as follows:

\[ \tilde{J}_1 = \frac{p}{2} I + \left( \frac{2\sigma_{p,q} - p}{2} \right) F, \quad \tilde{J}_2 = \frac{p}{2} I - \left( \frac{2\sigma_{p,q} - p}{2} \right) F. \]

Conversely, every metallic structure $\tilde{J}$ on $\tilde{N}$ induces two almost product structures on this manifold:

\[ F = \pm \left( \frac{2}{2\sigma_{p,q} - p} \tilde{j} - \frac{p}{2\sigma_{p,q} - p} I \right). \]

3 Screen Transversal Lightlike Submanifolds of Metallic Semi-Riemannian Manifolds

In this section, before introducing a screen transversal lightlike submanifold of metallic semi-Riemannian manifolds, we begin with the following.

Lemma 3.1. Let $\tilde{N}$ be a lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$ with a vector subbundle $\text{ltr}(T\tilde{N})$ of the screen transversal vector bundle. Then we have

\[ \tilde{J}\text{Rad}(T\tilde{N}) \cap \tilde{J}\text{ltr}(T\tilde{N}) = \{0\}. \]

Proof. Assume that $\text{ltr}(T\tilde{N})$ is invariant with respect to $\tilde{J}$, i.e., $\tilde{J}\text{ltr}(T\tilde{N}) = \text{ltr}(T\tilde{N})$.

From the definition of a lightlike submanifold, we have

\[ \tilde{g}(N, \xi) = 1, \]

for $\xi \in \Gamma(\text{Rad}(T\tilde{N}))$ and $N \in \Gamma(\text{ltr}(T\tilde{N}))$. Also from (19), we find that

\[ \tilde{g}(\tilde{J}N, \tilde{J}\xi) = p + q. \]

However, since $\tilde{J}N \in \Gamma(\text{ltr}(T\tilde{N}))$, then by hypothesis, we get $\tilde{g}(\tilde{J}N, \tilde{J}\xi) = 0$, which is a contradiction. So $\tilde{J}N$ can not belong to $\Gamma(\text{ltr}(T\tilde{N}))$.

Now, let us consider $\tilde{J}N \in \Gamma(S(T\tilde{N}))$. Then we obtain $\tilde{g}(\tilde{J}N, \tilde{J}\xi) = 0$, which contradicts (20). When we assume $\tilde{J}N \in \Gamma(\text{Rad}(T\tilde{N}))$, we get the same contradiction. Thus, $\tilde{J}N$ does not belong to $S(T\tilde{N})$ as well as to $\text{Rad}(T\tilde{N})$. Then, from the decomposition of a lightlike submanifold, we conclude that $\tilde{J}N \in \Gamma(S(T\tilde{N}^\perp))$. This completes the proof.

Definition 3.2. Let $\tilde{N}$ be a lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. If

\[ \tilde{J}\text{Rad}(T\tilde{N}) \subset S(T\tilde{N}^\perp), \]

then $\tilde{N}$ is called a screen transversal lightlike submanifold of a metallic semi-Riemannian manifold.
Example 3.3. Let $\tilde{N} = \mathbb{R}^5_2, \tilde{g}, \tilde{J}$ be the five-dimensional semi-Euclidean space with the semi-Euclidean metric $\tilde{g}$ of sign $(-,-,+,+,+)$. If we take

$$\tilde{J}(x_1, x_2, x_3, x_4, x_5) = ((p - \sigma)x_1, (p - \sigma)x_2, \sigma x_3, \sigma x_4, \sigma x_5),$$

where $(x_1, x_2, x_3, x_4, x_5)$ is the standard coordinate system of $\mathbb{R}^5_2$, then one can easily see that $\tilde{J}$ is a metallic structure on $\mathbb{R}^5_2$. Let $\dot{N}$ be a submanifold in $\tilde{N}$ defined by $x_3 = 0, x_5 = x_1 + x_2$.

Then we get $T\dot{N} = Sp\{W_1, W_2, W_3\}$, for

$$W_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, \quad W_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}, \quad W_3 = \frac{\partial}{\partial x_4}.$$ 

It is easy to check that $\dot{N}$ is a lightlike submanifold. Therefore,

$$\text{Rad}(T\dot{N}) = Sp\{W_1 = \xi\},$$

$$\text{ltr}(T\dot{N}) = Sp\left\{N = \frac{1}{2} \left( -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5} \right) \right\},$$

$$S(T\dot{N}) = Sp\{W_4\},$$

and we have

$$\tilde{J}\xi = (p - \sigma) \frac{\partial}{\partial x_1} + \sigma \frac{\partial}{\partial x_5} \in S(T\dot{N}^\perp),$$

$$\tilde{J}N = \frac{1}{2} \left( -(p - \sigma) \frac{\partial}{\partial x_1} + \sigma \frac{\partial}{\partial x_5} \right) \in S(T\dot{N}^\perp).$$

Thus, $\dot{N}$ is a radical screen transversal lightlike submanifold of $\tilde{N}$.

Definition 3.4. Let $\dot{N}$ be a screen transversal lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$.

1. If $\tilde{J}S(T\dot{N}) \subset S(T\dot{N}^\perp)$, then we say that $\dot{N}$ is a screen transversal anti-invariant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$.

2. If $\tilde{J}S(T\dot{N}) = S(T\dot{N})$, then we say that $\dot{N}$ is a radical screen transversal lightlike submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$.

Remark 3.5. Let $\dot{N}$ be a lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Considering the definition of a lightlike submanifold, we note the followings [10]:

...
(i) the radical distribution $Rad(T\tilde{N})$ is integrable (resp., defines a totally geodesic foliation) if and only if
\begin{equation}
 g([W,U], Z) = 0, \quad (\text{resp., } \tilde{g}(\nabla_W U, Z) = 0),
\end{equation}
for $W, U \in \Gamma(Rad(T\tilde{N}))$ and $Z \in \Gamma(S(T\tilde{N}))$.

(ii) the screen distribution $S(T\tilde{N})$ is integrable (resp., defines a totally geodesic foliation) if and only if
\begin{equation}
 g([W,U], N) = 0, \quad (\text{resp., } \tilde{g}(\nabla_W U, N) = 0),
\end{equation}
for $W, U \in \Gamma(S(T\tilde{N}))$ and $N \in \Gamma(ltr(T\tilde{N}))$.

### 3.1 Screen Transversal Anti-Invariant Submanifolds

Let $\tilde{N}$ be a screen transversal anti-invariant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then we have the following decomposition:

$$S(T\tilde{N}^\perp) = \{\tilde{J}Rad(T\tilde{N}) \oplus \tilde{J}ltr(T\tilde{N}) \oplus \tilde{J}S(T\tilde{N})\} \oplus \text{ort } D_o,$$

where, $D_o$ is a non-degenerate distribution orthogonal complement to $\tilde{J}Rad(T\tilde{N}) \oplus \tilde{J}ltr(T\tilde{N}) \oplus \tilde{J}S(T\tilde{N})$.

**Proposition 3.6.** Let $\tilde{N}$ be a screen transversal anti-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then the distribution $D_o$ is invariant with respect to $\tilde{J}$.

**Proof.** Using (17), we obtain
\begin{align*}
\tilde{g}(\tilde{J}W, \xi) &= \tilde{g}(W, \tilde{J}\xi) = 0,
\end{align*}
which implies that $\tilde{J}U$ does not belong to $\Gamma(ltr(T\tilde{N}))$. Since we have
\begin{align*}
\tilde{g}(\tilde{J}W, N) &= \tilde{g}(W, \tilde{J}N) = 0, \\
\tilde{g}(\tilde{J}W, \tilde{J}\xi) &= \tilde{g}(W, \tilde{J}\xi) + \tilde{g}(W, \xi) = 0, \\
\tilde{g}(\tilde{J}W, \tilde{J}N) &= 0, \\
\tilde{g}(\tilde{J}W, U) &= \tilde{g}(W, \tilde{J}U) = 0, \\
\tilde{g}(\tilde{J}W, \tilde{J}U) &= 0
\end{align*}
for $W \in \Gamma(D_o)$, $\xi \in \Gamma(Rad(T\tilde{N}))$, $N \in \Gamma(ltr(T\tilde{N}))$ and $U \in \Gamma(S(T\tilde{N}))$, then we complete the proof. \(\square\)
Proposition 3.7. Let $\tilde{N}$ be a screen transversal anti-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then there do not exist co-isotropic and totally screen transversal type of such lightlike submanifolds.

Proof. If $\tilde{N}$ is a co-isotropic or totally screen transversal lightlike submanifold, then we have

$$S(T\tilde{N}^\perp) = \{0\}.$$ 

Therefore, from Definition 3.2, the proof is trivial.

Assume that $\tilde{N}$ is a screen transversal anti-invariant submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Let $T_1, T_2, T_3,$ and $T_4$ be the projection morphisms on $\tilde{J}Rad(T\tilde{N}), \tilde{J}S(T\tilde{N}), \tilde{J}ltr(T\tilde{N}),$ and $D_o$, respectively. Then, for $U \in \Gamma(S(T\tilde{N}^\perp))$, we have expression

$$U = T_1 U + T_2 U + T_3 U + T_4 U. \tag{23}$$

If we apply $\tilde{J}$ to (23), then we find

$$\tilde{J}U = \tilde{J}T_1 U + \tilde{J}T_2 U + \tilde{J}T_3 U + \tilde{J}T_4 U. \tag{24}$$

On the other hand, we have

$$\tilde{J}U = BU + CU, \tag{25}$$

for $U \in \Gamma(S(T\tilde{N}^\perp))$, where, $BU$ and $CU$ are the tangent and transversal components of $\tilde{J}U$, respectively.

Also, let $R$ and $R'$ be the projection morphisms of $\tilde{J}T_1 U$ on $Rad(T\tilde{N})$ and $\tilde{J}Rad(T\tilde{N})$, respectively; $S$ and $S'$ be the projection morphisms of $\tilde{J}T_2 U$ on $S(T\tilde{N})$ and $\tilde{J}S(T\tilde{N})$, respectively; $L$ and $L'$ be the projection morphisms of $\tilde{J}T_3 U$ on $ltr(T\tilde{N})$ and $\tilde{J}ltr(T\tilde{N})$, respectively; $D$ and $D'$ be the projection morphisms of $\tilde{J}T_4 U$ on $D_o$ and $\tilde{J}D_o$, respectively. Then, from (24) and (25), we have

$$BU = R\tilde{J}T_1 U + S\tilde{J}T_2 U,$$

$$CU = R'\tilde{J}T_1 U + S'\tilde{J}T_2 U + L\tilde{J}T_3 U + L'\tilde{J}T_3 U + D\tilde{J}T_4 U + D'\tilde{J}T_4 U.$$

If we put $B_1 = R\tilde{J}T_1, B_2 = S\tilde{J}T_2, C_1 = L\tilde{J}T_3,$ and $C_2 = R'\tilde{J}T_1 + S'\tilde{J}T_2 + L\tilde{J}T_3 + L'\tilde{J}T_3 + D\tilde{J}T_4 + D'\tilde{J}T_4$, then we can rewrite (24) as

$$\tilde{J}U = B_1 U + B_2 U + C_1 U + C_2 U. \tag{26}$$
Here there are components of $B_1 U$, $B_2 U$, $C_1 U$, and $C_2 U$ at $\Gamma(Rad(T\tilde{N}))$, $\Gamma(S(T\tilde{N}))$, $\Gamma(ltr(T\tilde{N}))$, and $\Gamma(S(T\tilde{N}^\perp))$, respectively, namely $\tilde{J} U$ belongs to $T\tilde{N}|_{\tilde{N}}$.

It is known that the induced connection on a screen transversal anti-invariant lightlike submanifold immersed in metallic semi-Riemannian manifolds is not a metric connection. The condition under which the induced connection on the submanifold would be a metric connection is given by the following theorem.

**Theorem 3.8.** Let $\tilde{N}$ be a screen transversal anti-invariant lightlike submanifold of a locally metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then the induced connection $\nabla$ on $\tilde{N}$ is a metric connection if and only if

$$B_2 \nabla^*_W \tilde{J} \xi = 0,$$

for $W \in \Gamma(T\tilde{N})$ and $\xi \in \Gamma(Rad(T\tilde{N}))$.

**Proof.** From (18), for $W \in \Gamma(T\tilde{N})$ and $\xi \in \Gamma(Rad(T\tilde{N}))$, we have

$$\tilde{\nabla}_W \tilde{J} \xi = \tilde{J} \tilde{\nabla}_W \xi.$$

If we use (1) and (3), then we get

$$-A_{\tilde{J} \xi} W + \nabla^*_W \tilde{J} \xi + D^l(W, \tilde{J} \xi) = \tilde{J} \left( \nabla_W \xi + h^l(W, \xi) + h^s(W, \xi) \right).$$

Applying $\tilde{J}$ to above equation, we find

$$-\tilde{J} A_{\tilde{J} \xi} W + \tilde{J} \nabla^*_W \tilde{J} \xi + \tilde{J} D^l(W, \tilde{J} \xi) = \tilde{J}^2 \left( \nabla_W \xi + h^l(W, \xi) + h^s(W, \xi) \right).$$

Then from (16), we obtain

$$-\tilde{J} A_{\tilde{J} \xi} W + \tilde{J} \nabla^*_W \tilde{J} \xi + \tilde{J} D^l(W, \tilde{J} \xi) = \begin{pmatrix} p\tilde{J} \nabla_W \xi + p\tilde{J} h^l(W, \xi) \\ +p\tilde{J} h^s(W, \xi) + q\nabla_W \xi \\ +q h^l(W, \xi) + q h^s(W, \xi) \end{pmatrix}.$$ 

If we use (26) in last equation above, we can write

$$\begin{pmatrix} -\tilde{J} A_{\tilde{J} \xi} W + B_1 \nabla^*_W \tilde{J} \xi \\ +B_2 \nabla^*_W \tilde{J} \xi + C_1 \nabla_W \tilde{J} \xi \\ +C_2 \nabla^*_W \tilde{J} \xi + \tilde{J} D^l(W, \tilde{J} \xi) \end{pmatrix} = \begin{pmatrix} p\tilde{J} \nabla_W \xi + p\tilde{J} h^l(W, \xi) \\ +p\tilde{J} h^s(W, \xi) + q\nabla_W \xi \\ +q h^l(W, \xi) + q h^s(W, \xi) \end{pmatrix}.$$ 

By equating the tangent parts of the last equation, we have

$$\frac{1}{q} (B_1 \nabla^*_W \tilde{J} \xi + B_2 \nabla^*_W \tilde{J} \xi) = \nabla_W \xi.$$

Hence, the proof is completed.
Theorem 3.9. Let $\tilde{N}$ be a screen transversal anti-invariant lightlike submanifold of a locally metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then the radical distribution is integrable if and only if

$$\nabla^s_W \tilde{J}V = \nabla^s_V \tilde{J}W,$$

for $V, W \in \Gamma(\text{Rad}(T\tilde{N}))$.

Proof. From (21), we get

$$0 = g(\tilde{\nabla}_W \tilde{J}V, \tilde{J}Z) - pg(\tilde{\nabla}_W V, \tilde{J}Z) - g(\tilde{\nabla}_V \tilde{J}W, \tilde{J}Z) + pg(\tilde{\nabla}_V W, \tilde{J}Z),$$

for $V, W \in \Gamma(\text{Rad}(T\tilde{N}))$ and $Z \in \Gamma(S(T\tilde{N}))$. Since $\tilde{J}U, \tilde{J}W \in \Gamma(S(T\tilde{N}^\perp))$, then by using (5), we find

$$0 = g(\nabla^s_W \tilde{J}V - \nabla^s_V \tilde{J}W, \tilde{J}Z).$$

Theorem 3.10. Let $\tilde{N}$ be a screen transversal anti-invariant lightlike submanifold of a locally metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. In this case, the screen distribution is integrable if and only if

$$\nabla^s_W \tilde{J}U = \nabla^s_U \tilde{J}W,$$

for $W, U \in \Gamma(S(T\tilde{N}))$.

Proof. By using (22), from (13), (17) and (19), we find

$$0 = g(\tilde{\nabla}_W \tilde{J}U, \tilde{J}N) - pg(\tilde{\nabla}_W U, \tilde{J}N) - g(\tilde{\nabla}_U \tilde{J}W, \tilde{J}N) + pg(\tilde{\nabla}_U W, \tilde{J}N) = g(\nabla^s_W \tilde{J}U, \tilde{J}N) - g(\nabla^s_U \tilde{J}W, \tilde{J}N) - g(ph^s(W, U), \tilde{J}N) + g(ph^s(U, W), \tilde{J}N),$$

for $W, U \in \Gamma(S(T\tilde{N}))$ and $N \in \Gamma(ltr(T\tilde{N}))$. The last equation implies

$$\nabla^s_W \tilde{J}U - \nabla^s_U \tilde{J}W = ph^s(W, U) - ph^s(U, W).$$

Since $h^s$ is symmetric, we get $\nabla^s_W \tilde{J}U = \nabla^s_U \tilde{J}W$.

Theorem 3.11. Let $\tilde{N}$ be a radical screen transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then the radical distribution defines a totally geodesic foliation if and only if there is no component of $\nabla^s_W \tilde{J}U - ph^s(W, U)$ in $S(T\tilde{N})$, for $W, U \in \Gamma(\text{Rad}(T\tilde{N}))$. 

\if\showproof

Proof. Let $\tilde{N}$ be a radical screen transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then the radical distribution defines a totally geodesic foliation if and only if there is no component of $\nabla^s_W \tilde{J}U - ph^s(W, U)$ in $S(T\tilde{N})$, for $W, U \in \Gamma(\text{Rad}(T\tilde{N}))$. 

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Proof. From the second part of (21), for \( W, U \in \Gamma(\text{Rad}(T\tilde{N})) \) and \( Z \in S(T\tilde{N}) \), if we use (11), (18) and (19), we have

\[
\tilde{g}(\tilde{\nabla}_W \tilde{J}U, \tilde{J}Z) - p\tilde{g}(\tilde{\nabla}_W U, \tilde{J}Z) = 0.
\]

Then we find

\[
\tilde{g}(\nabla^s_W \tilde{J}U - ph^s(W, U), \tilde{J}Z) = 0,
\]

by virtue of (6). So, the proof is completed.

Theorem 3.12. Let \( \tilde{N} \) be a screen transversal lightlike submanifold of a locally metallic semi-Riemannian manifold \( (\tilde{N}, \tilde{g}, \tilde{J}) \). Then the screen distribution defines a totally geodesic foliation if and only if there is no component of \( \nabla^s_W \tilde{J}U - ph^s(W, U) \) in \( \tilde{J}\text{ltr}(T\tilde{N}) \), for \( W, U \in \Gamma(S(T\tilde{N})) \).

Proof. By using (22), (4), (19) and (18), we get

\[
\tilde{g}(\tilde{\nabla}_W \tilde{J}U, \tilde{J}N) - pg(\tilde{\nabla}_W U, \tilde{J}N) = 0,
\]

for any \( W, U \in \Gamma(S(T\tilde{N})) \), \( N \in \Gamma(\text{ltr}(T\tilde{N})) \). Since \( \tilde{J}U \in \Gamma(S(T\tilde{N}^\perp)) \), from the (6), we obtain

\[
0 = \tilde{g}(\nabla^s_W \tilde{J}U - ph^s(W, U), \tilde{J}N).
\]

3.2 Radical Screen Transversal Lightlike Submanifolds of Metallic Semi-Riemannian Manifolds

We begin with investigating the integrability conditions of the distributions.

Theorem 3.13. Let \( \tilde{N} \) be a radical screen transversal lightlike submanifold of a locally metallic semi-Riemannian manifold \( (\tilde{N}, \tilde{g}, \tilde{J}) \). In this case, the screen distribution is integrable if and only if there is no component of \( h^s(W, \tilde{J}U) - h^s(U, \tilde{J}W) \) in \( \tilde{J}\text{ltr}(T\tilde{N}) \), for \( W, U \in \Gamma(S(T\tilde{N})) \).

Proof. From (22), and then using (4), (17), (18), (19), we find

\[
0 = g(\tilde{\nabla}_W \tilde{J}U, \tilde{J}N) - pg(\tilde{\nabla}_W U, \tilde{J}N) + g(\tilde{\nabla}_U \tilde{J}W, \tilde{J}N) + pg(\tilde{\nabla}_U W, \tilde{J}N)
= g(h^s(W, \tilde{J}U) - h^s(U, \tilde{J}W) - ph^s(W, U) + ph^s(U, W), \tilde{J}N),
\]

for \( W, U \in \Gamma(S(T\tilde{N})) \) and \( N \in \Gamma(\text{ltr}(T\tilde{N})) \). Here, since \( h^s \) is symmetric, then we have

\[
\tilde{g}(h^s(W, \tilde{J}U) - h^s(U, \tilde{J}W), \tilde{J}N) = 0.
\]

Therefore, the proof is completed.
THEOREM 3.14. Let \( \tilde{N} \) be a radical screen transversal lightlike submanifold of a locally metallic semi-Riemannian manifold \((\tilde{N}, \tilde{g}, \tilde{J})\). Then the radical distribution is integrable if and only if either \( A_{jW}U - A_{jU}W = p(A^*_wU - A^*_uW) \) or \( A_{wU} = A_{uW} \) and \( A_{jW}U - A_{jU}W \) belong to \( \Gamma(\text{Rad}(T\tilde{N})) \), for \( W, U \in \Gamma(\text{Rad}(T\tilde{N})) \).

PROOF. From (21), (17), (18) and (19), we have

\[
0 = g(\tilde{\nabla}_W \tilde{J}U, \tilde{J}Z) - g(\tilde{\nabla}_U \tilde{J}W, \tilde{J}Z) - pg(\tilde{\nabla}_W U, \tilde{J}Z) + pg(\tilde{\nabla}_U W, \tilde{J}Z),
\]
for \( W, U \in \Gamma(\text{Rad}(T\tilde{N})) \) and \( Z \in \Gamma(S(T\tilde{N})) \). Since \( \tilde{J}U, \tilde{J}W \in \Gamma(S(T\tilde{N}^\perp)) \) and \( \tilde{J}Z \in \Gamma(S(T\tilde{N})) \), from (4) and (6), we obtain

\[
0 = g(A_{jW}W - A_{jW}U - pA^*_wU + pA^*_uW, \tilde{J}Z),
\]
which completes the proof. \(\square\)

PROPOSITION 3.15. Let \( \tilde{N} \) be a radical screen transversal lightlike submanifold of a locally metallic semi-Riemannian manifold \((\tilde{N}, \tilde{g}, \tilde{J})\). Then the distribution \( D_o \) is invariant with respect to \( \tilde{J} \).

PROOF. For a radical screen transversal lightlike submanifold, we have

\[
S(T\tilde{N}^\perp) = \tilde{J}\text{Rad}(T\tilde{N}) \oplus \tilde{J}\text{ltr}(T\tilde{N}) \oplus_{\text{ort}} D_o,
\]
\[
\tilde{J}S(T\tilde{N}) = S(T\tilde{N}).
\]
Here, for \( W \in D_o \) and \( U \in \Gamma(S(T\tilde{N})) \), if we use (17) and (19), we find

\[
g(\tilde{J}W, \xi) = g(W, \tilde{J}\xi) = 0,
\]
\[
g(\tilde{J}W, \tilde{J}\xi) = 0,
\]
\[
g(\tilde{J}W, N) = g(W, \tilde{J}N) = 0,
\]
\[
g(\tilde{J}W, \tilde{J}N) = 0,
\]
\[
g(\tilde{J}W, U) = g(W, \tilde{J}U) = 0,
\]
\[
g(\tilde{J}W, \tilde{J}U) = 0.
\]
Therefore, we obtain

\[
\tilde{J}D_o \cap \tilde{J}\text{Rad}(T\tilde{N}) = \{0\}, \quad \tilde{J}D_o \cap \tilde{J}\text{ltr}(T\tilde{N}) = \{0\},
\]
\[
\tilde{J}D_o \cap \text{Rad}(T\tilde{N}) = \{0\}, \quad \tilde{J}D_o \cap \text{ltr}(T\tilde{N}) = \{0\},
\]
\[
\tilde{J}D_o \cap \tilde{J}S(T\tilde{N}) = \{0\}, \quad \tilde{J}D_o \cap S(T\tilde{N}) = \{0\},
\]
which give the assertion of the theorem. \(\square\)
Theorem 3.16. Let $\hat{N}$ be a radical screen transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $(\hat{N}, \hat{g}, \hat{J})$. Then the screen distribution defines a totally geodesic foliation if and only if there is no component of $h^s(W, \hat{J}U) - ph^s(W, U)$ in $\hat{J}ltr(T\hat{N})$, for any $W, U \in \Gamma(S(T\hat{N}))$.

Proof. By using (22), (17), (18) and (19), we find
$$0 = g(\hat{\nabla}_W \hat{J}U, \hat{J}N) - pg(\hat{\nabla}_W U, \hat{J}N),$$
where $W, U \in \Gamma(S(T\hat{N}))$ and $N \in \Gamma(ltr(T\hat{N}))$. So, we have
$$0 = g(h^s(W, \hat{J}U) - ph^s(W, U), \hat{J}N).$$
Hence, we get the conclusion. 

Theorem 3.17. Let $\hat{N}$ be a radical screen transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $(\hat{N}, \hat{g}, \hat{J})$. Then the radical distribution defines a totally geodesic foliation if and only if one of the followings hold:

(i) $A_{\hat{J}U}W$ belongs to $\Gamma(\text{Rad}(T\hat{N}))$ and $A^*_UW = 0$,
(ii) $A_{\hat{J}U}W = pA^*_UW$,
(iii) there is no component of $h^s(W, \hat{J}Z)$ in $\hat{J}\text{Rad}(T\hat{N})$,

for any $W, U \in \Gamma(\text{Rad}(T\hat{N}))$ and $Z \in \Gamma(S(T\hat{N}))$.

Proof. From (21), for $W, U \in \Gamma(\text{Rad}(T\hat{N}))$ and $Z \in \Gamma(S(T\hat{N}))$, we have
$$0 = \hat{g}(\hat{\nabla}_W \hat{J}U, \hat{J}Z) - p\hat{g}(\hat{\nabla}_W U, \hat{J}Z) = \hat{g}(A_{\hat{J}U}W + pA^*_UW, \hat{J}Z),$$
by virtue of (17), (18) and (19), which implies either (i) or (ii). Also, we can write
$$\hat{g}(h^s(W, \hat{J}Z), \hat{J}U) = \hat{g}(A_{\hat{J}U}W, \hat{J}Z) = 0,$$
by virtue of
$$\hat{g}(A_{\hat{J}U}W, \hat{J}Z) = p\hat{g}(\hat{\nabla}_W U, \hat{J}Z) = 0.$$ 
Therefore, the proof is completed. 

Theorem 3.18. Let $\hat{N}$ be a radical screen transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $(\hat{N}, \hat{g}, \hat{J})$. Then the induced connection on $\hat{N}$ is a metric connection if and only if either there is no component of $A_{\hat{J}\xi}W$ in $S(T\hat{N})$ or there is no component of $h^s(U, W)$ in $\hat{J}\text{Rad}(T\hat{N})$, for any $W, U \in \Gamma(S(T\hat{N}))$, $\xi \in \Gamma(\text{Rad}(T\hat{N}))$. 

Proof. Since \((\bar{N}, \bar{g}, \bar{J})\) is a locally metallic semi-Riemannian manifold, then, for \(W, U \in \Gamma(S(T\bar{N}))\) and \(\xi \in \Gamma(\text{Rad}(T\bar{N}))\), we have
\[
g(\bar{\nabla}_W \bar{J}\xi, U) = g(\bar{\nabla}_W \xi, \bar{J}U).
\]
By using (6), (17) and (18), we find
\[
-g(A_{J\xi} W, U) = g(\nabla W \xi, \bar{J}U),
\]
which implies that, either there is no component of \(A_{J\xi} W\) in \(S(T\bar{N})\) or from (7) in last equation, we have
\[
-g(h^s(U, W), \bar{J}\xi) = g(\nabla W \xi, \bar{J}U).
\]
So, we complete the proof. \(\square\)

3.3 Isotropic Screen Transversal Lightlike Submanifolds

In case when \(\bar{N}\) is an isotropic screen transversal lightlike submanifold of a metallic semi-Riemannian manifold \((\bar{N}, \bar{g}, \bar{J})\), from the Definition 3.2 and Proposition 3.6, we can write
\[
T\bar{N} = \text{Rad}(T\bar{N})
\]
and tangent bundle of the main space has the decomposition
\[
T\bar{N} = \{T\bar{N} \oplus \text{ltr}(T\bar{N})\} \oplus_{\text{ort}} \{J\text{Rad}(T\bar{N}) \oplus J\text{ltr}(T\bar{N})\} \oplus_{\text{ort}} D_0.
\]

Theorem 3.19. Let \(\bar{N}\) be an isotropic screen transversal lightlike submanifold of a locally metallic semi-Riemannian manifold \((\bar{N}, \bar{g}, \bar{J})\). In this case, \(\bar{N}\) is totally geodesic if and only if \(D^i(\xi_1, \bar{J}\xi_2) = 0\) and \(D^i(\xi_1, Z) = 0\) and there is no component of \(D^s(\xi_1, N)\) in \(\Gamma(\bar{J}\text{Rad}(T\bar{N}))\), for any \(\xi_1, \xi_2 \in \Gamma(\text{Rad}(T\bar{N})), N \in \Gamma(\text{ltr}(T\bar{N}))\) and \(Z \in \Gamma(D_0)\).

Proof. From (17) and (18), we find
\[
\bar{g}(\bar{\nabla}_{\xi_1} \bar{J}\xi_2, \xi) = \bar{g}(\bar{\nabla}_{\xi_1} \xi_2, \bar{J}\xi).
\]
Then, for \(\bar{J}\xi_2 \in \Gamma(\bar{J}\text{Rad}(T\bar{N})) \subset S(T\bar{N}^\perp)\), from (6), we obtain
\[
(27) \quad \bar{g}(D^i(\xi_1, \bar{J}\xi_2), \xi) = \bar{g}(h^s(\xi_1, \xi_2), \bar{J}\xi).
\]
Similarly, we have
\[
\bar{g}(\bar{\nabla}_{\xi_1} \bar{J}\xi_2, N) = \bar{g}(\bar{\nabla}_{\xi_1} \xi_2, \bar{J}N).
\]
Using (4) and (6), we get
\[
(28) \quad -\bar{g}(D^s(\xi, N), \bar{J}\xi_2) = \bar{g}(h^s(\xi_1, \xi_2), \bar{J}N).
\]
Also, since $\nabla$ is a metric connection, for $Z \in \Gamma(D_0)$, from (1) and (6) again, we have

\begin{align*}
\hat{g}(\tilde{\nabla}_{\xi_1}\xi_2, Z) &= -\hat{g}(\xi_2, \tilde{\nabla}_{\xi_1}Z) \\
\hat{g}(h^s(\xi_1, \xi_2), Z) &= \hat{g}(\xi_2, D^l(\xi_1, Z)).
\end{align*} \tag{29}

(27), (28) and (29) complete the proof. \hfill \Box

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