A CAPACITY-BASED CONDITION FOR EXISTENCE OF SOLUTIONS TO FRACTIONAL ELLIPTIC EQUATIONS WITH FIRST-ORDER TERMS AND MEASURES

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Abstract. In this manuscript, we appeal to Potential Theory to provide a sufficient condition for existence of distributional solutions to fractional elliptic problems with non-linear first-order terms and measure data \( \omega \):

\[
\begin{cases}
  (-\Delta)^s u = |\nabla u|^q + \omega \quad \text{in } \mathbb{R}^n, \quad s \in (1/2, 1) \\
  u > 0 \quad \text{in } \mathbb{R}^n \\
  \lim_{|x| \to \infty} u(x) = 0,
\end{cases}
\]

under suitable assumptions on \( q \) and \( \omega \). Roughly speaking, the condition for existence states that if the measure data is locally controlled by the Riesz fractional capacity, then there is a global solution for the equation. We also show that if a positive solution exists, necessarily the measure \( \omega \) will be absolutely continuous with respect to the associated Riesz capacity, which gives a partial reciprocal of the main result of this work. Finally, estimates of \( u \) in terms of \( \omega \) are also given in different function spaces.

1. Introduction

We study the solvability of the following fractional elliptic problem

\[
\begin{cases}
  (-\Delta)^s u = |\nabla u|^q + \omega \quad \text{in } \mathbb{R}^n \\
  u > 0 \quad \text{in } \mathbb{R}^n \\
  \lim_{|x| \to \infty} u(x) = 0,
\end{cases}
\]

where \( \frac{1}{2} < s < 1, \ n > 2s \) and \( (-\Delta)^s \) is the classic fractional Laplacian operator of order \( 2s \). Here \( \omega \) will be a non-negative Radon measure with compact support in \( \mathbb{R}^n \). We consider the super-critical case

\[ q > p^* = \frac{n}{n - 2s + 1}, \quad p + q = pq. \]

This assumption on \( q \) is motivated from the fact that in the local-case, as we shall detail below, no solutions exist for sub-critical \( q \) unless \( \omega \equiv 0 \). For \( W^{1,q}(\mathbb{R}^n) \)-solutions, a similar conclusion is obtained in our framework (we refer the reader to Remark 2.6 for details). Moreover, the super-critical case allows us to obtain basic estimates on potentials as will be clarify in the proof of the main results.

We highlight that non-local type operators arise naturally in continuum mechanics, image processing, crystal dislocation, Non-linear Dynamics (Geophysical Flows), Potentials and capacity, PDE’s with measures, non-linear gradient terms.

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phase transition phenomena, population dynamics, non-local optimal control and game theory ([5], [6], [10], [11], [12], [15], [17], [18], [19], [21] and the references therein). Indeed, models like (2.5) may be understood as a Kardar-Parisi-Zhang stationary problem (models of growing interfaces) driving by fractional diffusion (see [21] for the model in the local setting and [1] in the nonlocal stage). In the works [29] and [30] the description of anomalous diffusion via fractional dynamics is investigated and various fractional partial differential equations are derived from Lévy random walk models, extending Brownian motion models in a natural way. Finally, fractional type operators are also encompassed in mathematical modeling of financial markets, since Lévy type processes with jumps take place as more accurate models of stock pricing (cf. [4] and [16] for some illustrative examples).

In this work we provide a sufficient condition for existence of global solutions to (2.5) based in a relation between \( \omega \) and a fractional capacity. We also derive a representation formula for the solution \( u \) in terms of Riesz potentials and, as a result, we obtain pointwise and norm estimates and behaviour at infinity of \( u \). Finally we demonstrate a necessary condition for the existence of solutions to the problem (1.1). We show that if problem (1.1) has a positive \( W^{1,q}(\mathbb{R}^n) \)-solution, then \( \omega \) does not charge sets of Riesz capacity zero, which means \( \omega \) will necessarily be absolutely continuous with respect to the corresponding Riesz capacity. This result gives us a partial reciprocal of the main theorem. We refer the reader to the next section for further details.

The current approach of the problem has been inspired by the enlightening results from [20]. In that paper, the authors provide a criteria for existence of solutions to equations of the form

\[
- \Delta u = |\nabla u|^q + \omega
\]

in \( \mathbb{R}^n \). They also prove the same characterization in bounded domains, but in this case \( q \geq 2 \) is needed for sufficient conditions. The criteria of solvability is given explicitly in terms of pointwise behaviour of the corresponding Riesz potentials as well as in geometric capacitary terms. We also note that in a parallel theory of equations of the form

\[
-\Delta u = |u|^q + \omega
\]

the role played by the Riesz capacity of order \( (2,q') \) is analogous to the results in [20] (see for instance [3] and [22]). As a consequence of the results in [20] and the potential estimates from [28] (see also [2] Section 7.2 and [27] Section 11.5) it follows that no solution exists to (1.2) when \( 1 < q \leq n/(n-1) \) unless \( \omega \equiv 0 \).

A different approach to characterize existence of solutions to local elliptic equations in terms of capacities was stated in [8]. There, it was proved that a measure is absolutely continuous with respect to \( (1,p) \) Riesz capacity if and only if it belongs to \( L^1(\Omega) + W^{-1,q'}(\Omega) \). This characterization allows to prove that a solution exists if and only if the measure data is absolutely continuous with respect to the Riesz capacity. Extensions of these results may be found, for instance, in [7].

We point out that the problem of existence of solutions to fractional elliptic problems with first-order terms like (2.5) in bounded domains (with boundary data) has been considered in [14], [15] and [1]. We highlight that our approach is also comparable to [1] Section 5 where a sufficient condition in terms of fractional capacity...
is also obtained for bounded domains and highly integrable sources. For fractional diffusion problems with non-homogeneous boundary conditions we refer to [13].

The paper is organized as follows. In Section 2 we provide the basic notation and definitions and also state the main result of the paper. Consequences of the main result, such as representation formula for the solution, pointwise and norm estimates and behaviour at infinite, are also provided. In Section 3 we give full details on the proof of Theorem 2.4. Finally, Section 4 is dedicated to show the necessary condition for existence of solutions to problem (1.1) and also discuss the converse of the main theorem which is currently an open problem.

2. Preliminaries and main results

The Fourier definition of the fractional Laplace operator \((-\Delta)^s\) is given by

\[
(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(u)(\xi))(x)
\]

where \(u \in \mathcal{S}(\mathbb{R}^n)\), and \(\mathcal{S}(\mathbb{R}^n)\) is the Schwartz class of smooth real-valued rapidly decreasing functions. We recall the following integral formulation of \((-\Delta)^s\) for functions in \(\mathcal{S}(\mathbb{R}^n)\)

\[
(-\Delta)^s u(x) := a(n, s) \text{P. V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy
\]

where

\[
a(n, s) = \frac{2^{2s} \Gamma\left(\frac{n}{2} + s\right)}{\pi^{n/2} \Gamma(1 - s)}
\]

is a normalization constant to recover (2.1). We refer to [23] and [33] for extensions to Hölder function spaces.

Following the approach in [33], we consider the following spaces

\[
\mathcal{S}_s(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) : \forall \beta \in \mathbb{N}_0^n, \sup_{x \in \mathbb{R}^n} (1 + |x|^{n+2s}) |D^\beta f(x)| < \infty \right\}
\]

with the family of seminorms

\[
[f]_{\mathcal{S}_s(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} (1 + |x|^{n+2s}) |D^\beta f(x)|
\]

and the weighted Lebesgue space

\[
L_s(\mathbb{R}^n) := \left\{ u \in L_{1,\text{loc}}^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} \, dx < \infty \right\}.
\]

We denote by \(\mathcal{S}_s'\) the topological dual of \(\mathcal{S}_s\). It is easy to see that \(L_s(\mathbb{R}^n) \subset \mathcal{S}_s'(\mathbb{R}^n)\) and if \(\varphi \in \mathcal{S}(\mathbb{R}^n)\), then \((-\Delta)^s \varphi \in \mathcal{S}_s(\mathbb{R}^n)\) (see [9]). Then, for \(u \in L_s(\mathbb{R}^n)\), we can define \((-\Delta)^s u\) in sense of tempered distributions as

\[
\langle (-\Delta)^s u, \varphi \rangle = \langle u, (-\Delta)^s \varphi \rangle = \int_{\mathbb{R}^n} u(x)(-\Delta)^s \varphi(x) \, dx
\]

for \(\varphi \in \mathcal{S}(\mathbb{R}^n)\). (See [26]).
We shall consider weak solution of \((\ref{2.5})\) in the following sense. We start with the case without first-order terms.

**Definition 2.1.** Consider the equation
\[
(-\Delta)^s u = \omega.
\]
Then, \(u \in L_s(\mathbb{R}^n)\) is a weak (distributional) solution of \((\ref{2.2})\) if
\[
\int_{\mathbb{R}^n} u(x)(-\Delta)^s \varphi(x) \, dx = \int_{\mathbb{R}^n} \varphi(x) \, d\omega(x)
\]
for all \(\varphi \in S(\mathbb{R}^n)\). If, instead of \(\omega\) we have an integrable function \(f\), we replace \(d\omega(x)\) by \(f(x) \, dx\).

**Definition 2.2.** We say that \(u \in L_s(\mathbb{R}^n) \cap W^{1,q}_{\text{loc}}(\mathbb{R}^n)\) is a weak solution of the equation \((\ref{2.5})\) if for all \(\varphi \in S(\mathbb{R}^n)\),
\[
\int_{\mathbb{R}^n} \left| \nabla u(x) \right|^q \varphi(x) \, dx + \int_{\mathbb{R}^n} \varphi(y) \, d\omega(y).
\]

Next we define the Riesz potential \(I_\alpha = (-\Delta)^{-\frac{\alpha}{2}}\) on \(\mathbb{R}^n\) of order \(\alpha\) for \(0 < \alpha < n\) as follows
\[
I_\alpha(g)(x) = c(n, \alpha) \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}} \, dy
\]
for \(g \in L^1_{\text{loc}}(\mathbb{R}^n)\) such that
\[
\int_{|y| \geq 1} \frac{|g(y)|}{|y|^{n-\alpha}} \, dy < \infty.
\]
The constant \(c(n, \alpha)\) is defined by
\[
c(n, \alpha) = \pi^{-\frac{n}{2}} 2^{-\alpha} \Gamma \left( \frac{n-\alpha}{2} \right) \Gamma \left( \frac{\alpha}{2} \right)^{-1}.
\]
For a non-negative Radon measure \(\omega\), we define the Riesz potential
\[
I_\alpha(\omega)(x) = c(n, \alpha) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} \, d\omega(y).
\]
The kernel
\[
I_\alpha(x) = \frac{c(n, \alpha)}{|x|^{n-\alpha}}
\]
is called the Riesz kernel. In this way, we see that
\[
I_\alpha(g)(x) = (I_\alpha * g)(x)
\]
and
\[
I_\alpha(\omega)(x) = (I_\alpha * \omega)(x).
\]

In order to state our main result, we first give the definition of Riesz capacity that will be employed along the work.
Definition 2.3. For $0 < \alpha < n$ and $1 < q < \infty$, the Riesz capacity $\text{cap}_{\alpha,q}(E)$ of a measurable set $E \subset \mathbb{R}^n$ is defined by

\begin{equation}
\text{cap}_{\alpha,q}(E) = \inf \left\{ \|u\|^q_{L^q(\mathbb{R}^n)} : \mathcal{I}_\alpha(u) \geq \mathcal{X}_E, u \in L^q_+(\mathbb{R}^n) \right\}.
\end{equation}

The main result of this work is the following theorem.

Theorem 2.4. Let $\frac{1}{2} < s < 1$, $q > p^*$. Suppose that there exists $C = C(n,q,s) > 0$, such that

\begin{equation}
\omega(E) \leq C \text{cap}_{2s-1,q}(E)
\end{equation}

for all compact sets $E \subset \mathbb{R}^n$, then the equation

\begin{equation}
(-\Delta)^s u = |\nabla u|^q + \omega \quad \text{in } \mathbb{R}^n
\end{equation}

has a non-negative weak solution $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^n)$. Moreover, $u$ has the representation

\begin{equation}
u = \mathcal{I}(|\nabla u|^q + \omega)
\end{equation}

We point out that (2.4) is equivalent, by Theorem 2.1 from [28], to

\begin{equation}
I_{2s-1}(\mathcal{I}_{2s-1}(\omega))^q \leq C_1 \mathcal{I}_{2s-1}(\omega), \quad \text{a.e. in } \mathbb{R}^n
\end{equation}

for some $C_1 > 0$ depending on $n$, $q$ and $s$. This relation, involving the non-linear potential $\mathcal{I}_{2s-1}(\mathcal{I}_{2s-1}(\omega)^q)$, will be frequently used in the proof of Theorem 2.4.

The following result constitutes a partial converse of Theorem 2.4. For the feasibility of the $W^{1,q}(\mathbb{R}^n)$-regularity of solutions, we refer the reader to Proposition 2.7.

Theorem 2.5. Suppose that (2.5) has a solution $u \in W^{1,q}(\mathbb{R}^n)$. Then the measure $\omega$ does not charge sets of Riesz capacity zero.

Remark 2.6. In other words, Theorem 2.5 says that a necessary condition for existence of solutions is that $\omega$ is absolutely continuous with respect to the Riesz capacity. A direct consequence of this fact is that no global solution $u$ in $W^{1,q}(\mathbb{R}^n)$ exists if $q \leq p^*$ (see [2, Proposition 2.6.1]).

In the rest of this section, we shall provide some consequences of Theorem 2.4.

Proposition 2.7. The solution from Theorem 2.4 satisfies

\begin{equation}
\mathcal{I}_{2s}(\omega) \leq u \leq C \mathcal{I}_{2s}(\omega)
\end{equation}

and

\begin{equation}
|\nabla u| \leq C \mathcal{I}_{2s-1}(\omega).
\end{equation}

As a result, if $q > n/(n-2s)$, we have $u \in W^{1,q}(\mathbb{R}^n)$.

Proof. Observe that the estimate (2.8) is obtained as follows. From (3.24), we have

\begin{equation}
u \leq C \left( \mathcal{I}_{2s}(\omega) + \mathcal{I}_{2s} \left[ |\mathcal{I}_{2s}(\omega)|^q \right] \right)
\end{equation}

Now, applying $\mathcal{I}$ in both sides of (2.7) yields

\begin{equation}
\mathcal{I}_{2s} \left( |\mathcal{I}_{2s-1}(\omega)|^q \right) \leq C \mathcal{I}_{2s}(\omega)
\end{equation}
Plugging this inequality into (2.10) gives (2.8). The lower bound in (2.8) is a consequence of (2.6). The estimate for the gradient (2.9) follows from (3.25). Next, we prove the final assertion. Observe $I_{2s}(\omega) \in L^q(\mathbb{R}^n)$ for $q > n/(n - 2s)$. Indeed, for any $R > 0$ so that supp$(\omega) \subset B_R$ we have

$$
\int_{B_R} I_{2s}(\omega)^q dx \leq C(n, s, q, R) \int_{B_R} I_{2s-1}(\omega)^q dx < \infty \quad \text{by Lemma 3.7}
$$

and, on the other hand, for large $R$ and $q > n/(n - 2s)$ it follows by Minkowski’s inequality

$$
\int_{\mathbb{R}^n \setminus B_R} I_{2s}(\omega)^q dx \leq C \left( \int_{\text{supp } \omega} \left[ \int_{\mathbb{R}^n \setminus B_R} \frac{dx}{|x-y|^{q(n-2s)}} \right]^{1/q} d\omega(y) \right)^q < \infty
$$

Consequently by (2.8) $u \in L^q(\mathbb{R}^n)$. Moreover, by Remark 3.2 and (2.9), it follows that $\nabla u \in L^q(\mathbb{R}^n)$. Hence $u \in W^{1,q}(\mathbb{R}^n)$.

The next corollary shows that $u$ from (2.6) is actually a solution of (1.1).

**Corollary 2.8.** We have the following for the solution $u$ from Theorem 2.4:

(i) $u$ vanishes at infinite

$$
\lim_{|x| \to \infty} u(x) = 0;
$$

(ii) $u$ is positive everywhere.

**Proof.** Take $R > 0$ so that supp$(\omega) \subset B_R$. Hence for $|x| > R$ we have by (2.8)

$$
u(x) \leq C I_{2s}(\omega) = C \int_{B_R} \frac{d\omega(y)}{|x-y|^{n-2s}} \leq C \frac{\omega(B_R)}{(|x|-R)^{n-2s}}.
$$

This proves (i). For (ii), observe that for $x \in \mathbb{R}^n$ and $r > 0$ so that supp$(\omega) \subset B_r(x)$ the lower bound in (2.8) implies

$$
u(x) \geq c(n, 2s) \int_{B_r(x)} \frac{d\omega(y)}{|x-y|^{n-2s}} \geq c(n, 2s) r^{2s-n} \omega(\text{supp}(\omega)) > 0.
$$

Observe that from Theorem 2.4 $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^n)$ and thus $u$ is finite almost everywhere. The following consequence of Proposition 2.7 gives finiteness everywhere for $q > 2s$.

**Corollary 2.9.** Suppose that $q > 2s$. Then, $u < \infty$ everywhere in $\mathbb{R}^n$.

**Proof.** By (2.8) and the Cavalieri’s representation of Riesz potentials (see for instance Section 2.2 in [32]), we have for $x \in \mathbb{R}^n$

$$
u(x) \leq C I_{2s}(\omega)(x) = C \int_{\text{supp } \omega} \frac{\omega(B_r(x))}{r^{n-2s+1}} dr = C \int_0^{R(x)} \frac{\omega(B_r(x))}{r^{n-2s+1}} dr,
$$

where
where $R(x) > 0$ satisfies $\text{supp} \, \omega \subset B(x, R(x))$. We now prove that

$$\omega(B_r(x)) \leq C_2(n, s, q)r^{n-(2s-1)p}. \tag{2.13}$$

For $0 < \alpha < n$, take

$$g = \frac{2^{n-\alpha}}{c(n, \alpha)\omega_n r^\alpha} \mathcal{X}_{B_r}(x) \in L^p_+(\mathbb{R}^n).$$

Now, let $z \in B_r(x)$, then

$$\mathcal{I}_\alpha g(z) = c(n, \alpha) \int_{\mathbb{R}^n} \frac{1}{|z-y|^{n-\alpha}} \cdot \frac{2^{n-\alpha}}{c(n, \alpha)\omega_n r^\alpha} \mathcal{X}_{B_r}(y) \, dy = \frac{2^{n-\alpha}}{\omega_n r^\alpha} \int_{B_r(x)} \frac{1}{|z-y|^{n-\alpha}} \, dy = 1.$$

Hence $\mathcal{I}_\alpha g \geq \mathcal{X}_{B_r}(x)$. Therefore,

$$\text{cap}_{\alpha,p}(B_r(x)) \leq \|g\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} \left( \frac{2^{n-\alpha}}{c(n, \alpha)\omega_n r^\alpha} \right)^p \mathcal{X}_B(y)^p \, dy = \left( \frac{2^{n-\alpha}}{c(n, \alpha)} \right)^p \omega_n^{1-p} r^{-\alpha p}.$$

Thus

$$\text{cap}_{\alpha,p}(B) \leq Cr^{n-\alpha p}, \quad C = \left( \frac{2^{n-\alpha}}{c(n, \alpha)} \right)^p \omega_n^{1-p}.$$

Letting $\alpha = 2s$ and recalling (2.4), we obtain (2.13). Plugging this into (2.12) gives that $u(x)$ is finite (observe that $q > 2s$ implies $p \in (1, 2s/(2s-1)$ and so the last integral in (2.12) is finite).

The following results show how the regularity of the source is transferred into the regularity of the solution. We start with measure data and then we analyse the regularity of solutions for integrable sources.

To state estimates of the solution in terms of measures, we first recall the definition of the Marcinkiewicz spaces.

**Definition 2.10.** Let $\Omega \subset \mathbb{R}^n$ be a domain and $\mu$ be a positive Borel measure in $\Omega$. For $\kappa > 1$, $\kappa' = \kappa/(\kappa-1)$ we define the Marcinkiewicz space $M^\kappa(\Omega, d\mu)$ of exponent $\kappa$ or weak $L^\kappa$-space, as

$$M^\kappa(\Omega, d\mu) := \{ v \in L^1_{\text{loc}}(\Omega, d\mu) : \| v \|_{M^\kappa(\Omega, d\mu)} < \infty \}$$

where

$$\| v \|_{M^\kappa(\Omega, d\mu)} := \inf \left\{ c \in [0, \infty] : \int_E |v| \, d\mu \leq c \left( \int_E d\mu \right)^{\frac{1}{\kappa'}} \right\},$$

for all Borel $E \subset \Omega$.

The Marcinkiewicz type estimate of the solution $u$ is the following (recall $p^* = n/(n-2s+1)$).
Proposition 2.11. There exists $C = C(n, s) > 0$ so that
\begin{equation}
\|u\|_{W^{n/(n-2s), (n-2s)n/(n-2s+1)}(\mathbb{R}^n, d\mu)} + \|\nabla u\|_{M^p(\mathbb{R}^n, d\mu)} \leq C\|\omega\|_{M^b(\mathbb{R}^n)},
\end{equation}
where
\begin{equation}
d\mu(x) := \frac{dx}{1 + |x|^{n+2s}}
\end{equation}
and
\begin{equation}
\|\omega\|_{M^b(\mathbb{R}^n)} = \omega(\mathbb{R}^n)
\end{equation}
is a norm in the space of bounded Radon measures $M^b(\mathbb{R}^n)$ of $\mathbb{R}^n$.

Proof. We follow closely the proof of [14, Proposition 2.2]. We prove the estimates for $\nabla u$. By similar arguments, the control for $\|u\|_{W^{n/(n-2s), (n-2s)n/(n-2s+1)}(\mathbb{R}^n, d\mu)}$ is obtained. Observe that, in view of (2.9), it is enough to prove that there is $C > 0$ with
\begin{equation}
\|I_{2s-1}(\omega)\|_{M^{n/(n-2s+1)}(\mathbb{R}^n, d\mu)} \leq C\|\omega\|_{M^b(\mathbb{R}^n)}.
\end{equation}
For $y \in \mathbb{R}^n$ and $\lambda > 0$, define
\begin{align*}
A_\lambda(y) := \left\{ x \in \mathbb{R}^n \setminus \{y\} : I_{2s-1}(x - y) = \frac{c(n, 2s)}{|x - y|^{n-2s+1}} > \lambda \right\}, \\
m_\lambda(y) := \int_{A_\lambda(y)} \frac{1}{1 + |x|^{n+2s}} dx.
\end{align*}
Since
\begin{equation}
A_\lambda \subset B_r(y),
\end{equation}
with
\begin{equation}
r = \left[ \frac{c(n, 2s)}{\lambda} \right]^{1/(n-2s+1)},
\end{equation}
we have for some $C > 0$
\begin{equation}
m_\lambda(y) \leq C\lambda^{-p^*}.
\end{equation}
Let now $E$ be a Borel set. Then
\begin{equation}
\int_E I_{2s-1}(x - y)d\mu(x) \leq \lambda \int_E d\mu(x) + \int_{A_\lambda(y)} I_{2s-1}(x - y)d\mu(x)
\end{equation}
and
\begin{equation}
\int_{A_\lambda(y)} I_{2s-1}(x - y)d\mu(x) = \lambda m_\lambda(y) + \int_\lambda^\infty m_s(y)ds \leq C\lambda^{1-p^*},
\end{equation}
for some $C > 0$. Hence
\begin{equation}
\int_E I_{2s-1}(x - y)d\mu(x) \leq \lambda \int_E d\mu(x) + C\lambda^{1-p^*}.
\end{equation}
Choosing $\lambda = (\int_E d\mu)^{-1/p^*}$, we obtain
\begin{equation}
\int_E I_{2s-1}(x - y)d\mu(x) \leq C \left( \int_E d\mu \right)^{\frac{p^*-1}{p^*}},
\end{equation}
for a universal constant $C > 0$ and all $y \in \mathbb{R}^n$. As a result
\begin{equation}
\int_E I_{2s-1}(\omega)(x)d\mu(x) \leq C\|\omega\|_{M^b(\mathbb{R}^n)} \left( \int_E d\mu \right)^{\frac{p^*-1}{p^*}}.
\end{equation}
This ends the proof of the proposition. \qed
In the next proposition, we provide a complete scheme of Lebesgue, Sobolev and Hölder regularity of $u$ in terms of the regularity of the source.

**Proposition 2.12.** Let $\omega = f dx$, where $f \in L^m_\text{loc} (\mathbb{R}^n)$ vanishes outside a compact set. Then for the solution $u$ from Theorem 2.4 we get:

(i) if $m = 1$, then there is $C = C(s, n) > 0$ so that for any $\lambda > 0$

$$| \{ x \in \mathbb{R}^n : u(x) > \lambda \} | \leq C \left( \frac{\| f \|_{L^1(\mathbb{R}^n)}}{\lambda} \right)^{m^*}, \quad m^* = 1 - \frac{2s}{n};$$

(ii) if $m > 1$ and $2sm < n$, there is a constant $C = C(r, s, n, m) > 0$ so that

$$\| u \|_{L^r(\mathbb{R}^n)} \leq C \| f \|_{L^m(\mathbb{R}^n)}, \quad \text{for all } r \in \left( \frac{n}{n - 2s}, m^* \right]$$

where

$$m^* = \frac{nm}{n - 2sm};$$

(iii) for $m > 1$ and $(2s - 1)m < n$, for some $C = C(r, s, n, m) > 0$ we have

$$\| \nabla u \|_{L^r(\mathbb{R}^n)} \leq C \| f \|_{L^m(\mathbb{R}^n)}, \quad \text{for all } r \in \left( \frac{n}{n - 2s + 1}, m^* \right]$$

where

$$m^* = \frac{nm}{n - (2s - 1)m};$$

(iv) if $m = n/(2s)$, then

$$u \in L^r(\mathbb{R}^n) \quad \text{for all } r \in \left( \frac{n}{n - 2s}, \infty \right);$$

(v) in the case $m = n/(2s - 1)$,

$$u \in W^{1,r}(\mathbb{R}^n) \quad \text{for all } r \in \left( \frac{n}{n - 2s + 1}, \infty \right);$$

(vi) if $m > n/(2s - 1)$, then

$$\nabla u \in C^{0,\gamma}(\mathbb{R}^n), \quad \gamma = 2s - 1 - \frac{n}{m}.$$  

**Proof.** The estimates (i) – (iii) are direct consequences of Proposition 2.7 and well-known $L^p$ embeddings of the Riesz potential (see for instance [34, Chapter V] and [2, Chapter 3]). Also, (iv) and (v) follow from (ii) and (iii), respectively, together with the assumption that $f$ has compact support. Finally, for (vi) we first observe that from (2.6)

$$\nabla u = \mathcal{I}_{2s-1}(|\nabla u|^q + f) \quad \text{a.e. } \mathbb{R}^n.$$  

Moreover, $\nabla u \in L^{qm}(\mathbb{R}^n)$ since by (v), $u \in W^{1,r}(\mathbb{R}^n)$ for all $r > p^*$ and $qm > p^*$. Therefore, appealing to [31] Theorem 2.2, Sec. 4.2, there is a constant $M > 0$ so that

$$|\nabla u(x) - \nabla u(y)| \leq M |x - y|^{2s-1-n/m} \| \nabla u \|_{L^q(\mathbb{R}^n)} a.e. \text{ in } \mathbb{R}^n.$$  

By [2] Theorem 3.2.1, we also may obtain the following exponential summability for $u$ and $\nabla u$ which account, for instance, to local integrability:
• if \( m = n/2s \) and \( \text{supp } f \subset B_R \), there is a constant \( A = A(n, m) \) such that
\[
\int_{B_R} \exp \left( A_0 u^{m'} \right) \, dx \leq A R^n, \quad A_0 = \frac{n}{c(n, 2s)^{m'} \omega_{n-1}};
\]
• in the case \( m = n/(2s - 1) \) and \( \text{supp } f \subset B_R \), we have for the same \( A \) as before that
\[
\int_{B_R} \exp \left( A_0 |\nabla u|^{m'} \right) \, dx \leq A R^n, \quad A_0 = \frac{n}{c(n, 2s - 1)^{m'} \omega_{n-1}}.
\]

Remark 2.13. The interested reader may compare the above regularity results to the related findings for bounded domains and no first-order terms presented in [25, Theorem 15-16] and [1, Lemma 2.15].

3. Proof of Theorem 2.4

The structure of the proof consists of the following steps

(I) the starting point will be to consider \( u_0 = I_{2s}(\omega) \) and prove that
\[ (-\Delta)^s u_0 = \omega \]

in the sense of Definition 2.1. This will be done in Lemma 3.1 and Proposition 3.3.

(II) The next step is to consider first-order terms. Indeed, we show in Lemmas 3.6-3.7 and Propositions 3.9-3.10 that \( v = I_{2s}(|\nabla u_0|^q) \) solves
\[ (-\Delta)^s v = |\nabla u_0|^q. \]

(III) The final step, developed at the end of the section, is to define by recursion the sequence
\[ u_{k+1} = I_{2s}(\omega) + I_{2s}(|\nabla u_k|^q), \quad k \geq 0, \]

and prove that \( u_k \) converges in the right topology to a solution of (2.5).

3.1. Step (I).

Lemma 3.1. Let \( \omega \) be a nonnegative Radon measure with compact support in \( \mathbb{R}^n \). Then, for \( n > 2s \), \( I_{2s}(\omega) \in L^1_{\text{loc}}(\mathbb{R}^n) \).

Proof. First note that \( I_{2s}(\omega) \in L^1_{\text{loc}}(\mathbb{R}^n) \). Let \( R > 0 \) such that \( \text{supp} (\omega) \subset B_R \). For simplicity take \( R = 1 \). Then,
\[
\int_{\mathbb{R}^n} \frac{|I_{2s}(\omega)(x)|}{1 + |x|^{n+2s}} \, dx = \int_{B_1} \left( \int_{\mathbb{R}^n} \frac{c(n, 2s)}{(1 + |x|^{n+2s})|x - y|^{n-2s}} \, dx \right) \, d\omega(y).
\]

Now we split the last integral as follows (we omit the constant for simplicity)
\[
\int_{B_1} \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^{n+2s})|x - y|^{n-2s}} \, dx \right) \, d\omega(y) = \int_{B_1} \left( \int_{B_2} \frac{1}{(1 + |x|^{n+2s})|x - y|^{n-2s}} \, dx \right) \, d\omega(y)
\] \[ + \int_{B_1} \left( \int_{\mathbb{R}^n \setminus B_2} \frac{1}{(1 + |x|^{n+2s})|x - y|^{n-2s}} \, dx \right) \, d\omega(y).
\]

Then, for the first integral we have
\[
\int_{B_1} \left( \int_{B_2} \frac{1}{|x|^{n+2s} |x-y|^{n-2s}} \, dx \right) \, d\omega(y) \leq \int_{B_1} \left( \int_{B_2} \frac{1}{|x-y|^{n-2s}} \, dx \right) \, d\omega(y) \\
\leq \int_{B_1} \left( \int_{B_2+|y|} |z|^{2s-n} \, dz \right) \, d\omega(y) \\
= \omega_n \int_{B_1} \frac{(2 + |y|)^2s}{2s} \, d\omega(y) \\
< \infty.
\]

For the second integral we have
\[
\int_{B_1} \left( \int_{\mathbb{R}^n \setminus B_2} \frac{1}{|x|^{n+2s} |x-y|^{n-2s}} \, dx \right) \, d\omega(y) \leq \int_{B_1} \left( \int_{\mathbb{R}^n \setminus B_2} (|x| - |y|)^{2s-n} |x|^{-n-2s} \, dx \right) \, d\omega(y) \\
< \infty.
\]

**Remark 3.2.** Reproducing the proof of Lemma 3.1 with \(2s - 1\) instead of \(2s\), one obtains \(I_{2s-1}(\omega) \in L_s(\mathbb{R}^n)\) as well.

For the proof of Proposition 3.5 is necessary the following lemma (Proposition 2.4 of [9]).

**Lemma 3.3.** Let \(n > 2s\) and \(f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)\) with \(\mathcal{F}^{-1}(f) \in S_s(\mathbb{R}^n)\). Then
\begin{equation}
\int_{\mathbb{R}^n} I_{2s}(x) \mathcal{F}^{-1}(f)(x) \, dx = \int_{\mathbb{R}^n} |x|^{-2s} f(x) \, dx.
\end{equation}

**Remark 3.4.** A closer look at the proof of [9] Proposition 2.4 reveals that the assumption \(\mathcal{F}^{-1}(f) \in S_s(\mathbb{R}^n)\) may be replaced by
i. \(\int_{\mathbb{R}^n} I_{2s}(x) \mathcal{F}^{-1}(f)(x) \, dx < \infty\) and
ii. \(\mathcal{F}^{-1}(f) \in L^2(\mathbb{R}^n)\).

We will use this observation in Proposition 3.10.

**Proposition 3.5.** For \(n > 2s\), \(I_{2s}(\omega)\) is a weak solution of
\begin{equation}
(-\Delta)^s u = \omega \quad \text{in } \mathbb{R}^n.
\end{equation}

**Proof.** We will prove that Definition 2.1 is satisfied. We already know that \(I_{2s}(\omega) \in L_s(\mathbb{R}^n)\) by Lemma 3.1. Let \(\varphi_0 \in S(\mathbb{R}^n)\). Put
\[\varphi(x) = |x|^{2s} \mathcal{F}^{-1}(\varphi_0)(x)\]
and take \(\psi\) such that
\[\varphi(x) = \mathcal{F}(\psi)(-x)\].

Then, \(\psi(x) = \mathcal{F}^{-1}(\varphi)(-x)\). Thus, since \(\varphi_0 \in S(\mathbb{R}^n), \mathcal{F}^{-1}(\varphi) \in S_s(\mathbb{R}^n)\) and therefore \(\psi \in S_s(\mathbb{R}^n)\).
In what follows, we shall employ Lemma 3.3. We first prove that $\psi \ast \omega \in S_s(\mathbb{R}^n)$. Since $D^\alpha \psi \in S_s(\mathbb{R}^n)$ and $D^\alpha (\psi \ast \omega) = (D^\alpha \psi) \ast \omega$ we just need to verify that

$$|\psi \ast \omega(x)| \leq C(1 + |x|^{n+2s})^{-1} \quad x \in \mathbb{R}^n. \quad (3.3)$$

Take $R > 0$ such that supp($\omega$) $\subset B_R$. Then, there is $r > 0$ such that

$$\frac{|x|^{n+2s}}{1 + (|x| - R)^{n+2s}} \rightarrow 1 \quad \text{for } |x| \rightarrow \infty. \quad (3.4)$$

for $|x| > r$. Let $r_0 > \max\{2R, r\}$. Then, for $x \in B_{r_0}$

$$\int_{B_R} \frac{1 + |x|^{n+2s}}{1 + |x - y|^{n+2s}} d\omega(y) \leq (1 + r_0^{n+2s}) \omega(B_R) < \infty$$

and for $x \in \mathbb{R}^n \setminus B_{r_0}$,

$$\int_{B_R} \frac{1 + |x|^{n+2s}}{1 + |x - y|^{n+2s}} d\omega(y) \leq \omega(B_R) + \int_{B_R} \frac{|x|^{n+2s}}{1 + |x - y|^{n+2s}} d\omega(y) < \infty$$

by (3.4) and the choice of $r_0$. Thus, we have proved (3.3) and we conclude that $\psi \ast \omega \in S_s(\mathbb{R}^n)$.

Moreover, observe that the integrability of $\psi \ast \omega$ implies $\mathcal{F}(\psi \ast \omega) \in C^\infty(\mathbb{R}^n)$. To verify the integrability of $\mathcal{F}(\psi \ast \omega)$ note that it easily follows

$$\mathcal{F}(\psi \ast \omega)(x) = \mathcal{F}(\psi)(x) \mathcal{F}(\omega)(x) \quad \text{for all } x.$$
\[
\int_{\mathbb{R}^n} |\mathcal{F}(\psi * \omega)(x)| \, dx = \int_{\mathbb{R}^n} |\mathcal{F}(\psi)(x)||\mathcal{F}(\omega)(x)| \, dx \\
\leq \omega(B_R) \int_{\mathbb{R}^n} |\mathcal{F}(\psi)(x)| \, dx \\
= \omega(B_R) \int_{\mathbb{R}^n} |x|^{2s} |\mathcal{F}(\varphi_0)(-x)| \, dx \\
= \omega(B_R) \left[ \int_{B_1} |x|^{2s} |\mathcal{F}(\varphi_0)(-x)| \, dx + \int_{\mathbb{R}^n \setminus B_1} |x|^{2s} |\mathcal{F}(\varphi_0)(-x)| \, dx \right] \\
\leq C + \int_{\mathbb{R}^n \setminus B_1} |x|^{2s}(1 + |x|)^{-n-2} \, dx < \infty.
\]

We can apply now Lemma 3.3 to \( f = \mathcal{F}(\psi * \omega) \) to obtain

\[
\int_{\mathbb{R}^n} I_{2s}(x)(\psi * \omega)(x) \, dx = \int_{\mathbb{R}^n} I_{2s}(x)\mathcal{F}^{-1}(\mathcal{F}(\psi * \omega))(x) \, dx \\
= \int_{\mathbb{R}^n} |x|^{-2s} \mathcal{F}(\omega)(x)\mathcal{F}(\psi)(x) \, dx.
\]

(3.5)

Now,

\[
\int_{\mathbb{R}^n} I_{2s}(x)(\psi * \omega)(x) \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I_{2s}(x)\psi(x - y) \, d\omega(y) \, dx \\
= \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} I_{2s}(x)\mathcal{F}^{-1}(\varphi)(y - x) \, dx \right] \, d\omega(y).
\]

(3.6)

We make the change of variable \( z = x - y \) in the last integral and obtain

\[
\int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} I_{2s}(y + z)\mathcal{F}^{-1}(\varphi)(-z) \, dz \right] \, d\omega(y) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\varphi)(-z) \left[ \int_{\mathbb{R}^n} I_{2s}(y + z) \, d\omega(y) \right] \, dz \\
= \int_{\mathbb{R}^n} I_{2s}(\omega)(-z)\mathcal{F}^{-1}(\varphi)(-z) \, dz \\
= -\int_{\mathbb{R}^n} I_{2s}(\omega)(x)\mathcal{F}^{-1}(\varphi)(x) \, dx.
\]

(3.7)

Thus from (3.5), (3.6) and (3.7) we get

\[
\int_{\mathbb{R}^n} I_{2s}(\omega)(x)\mathcal{F}^{-1}(\varphi)(x) \, dx = -\int_{\mathbb{R}^n} |x|^{-2s} \mathcal{F}(\omega)(x)\mathcal{F}(\psi)(x) \, dx = -\int_{\mathbb{R}^n} |x|^{-2s} \mathcal{F}(\omega)(x)\varphi(-x) \, dx.
\]

(3.8)

Then
\[
\int_{\mathbb{R}^n} \mathcal{I}_{2s}(\omega)(x)(-\Delta)^s(\varphi_0)(x) \, dx = -\int_{\mathbb{R}^n} |x|^{-2s} \mathcal{F}(\omega)(x)\varphi(-x) \, dx \tag{3.8}
\]

by \(3.8\)

\[
= \int_{\mathbb{R}^n} |x|^{-2s} \mathcal{F}(\omega)(-x)\varphi(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} |x|^{-2s} \mathcal{F}^{-1}(\omega)(x)|x|^{2s} \mathcal{F}(\varphi_0)(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} \omega(y) \right] \mathcal{F}(\varphi_0)(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} \mathcal{F}(\varphi_0)(x) \, dx \right] d\omega(y)
\]

3.2. Step (II).

**Lemma 3.6.** Let \(u_0 = \mathcal{I}_{2s}(\omega)\). There exists \(C_0 = C_0(n, s)\) such that

\[
|\nabla u_0| \leq C_0 \mathcal{I}_{2s-1}(\omega).
\]

**Proof.** We will show that

\[
\frac{\partial u_0}{\partial x_i}(x) = c(n, 2s) \int_{\mathbb{R}^n} \frac{(x_i - y_i)}{|x - y|^{n-2s+2}} \, d\omega(y) \quad i = 1, \ldots, n.
\]

in the weak sense. Let \(\varphi \in C_0^{\infty}(\mathbb{R}^n)\) and let \(R > 0\) such that \(\text{supp}(\varphi) \subset B_R\). We want to show

\[
\int_{\mathbb{R}^n} u_0(x) \frac{\partial \varphi}{\partial x_i}(x) \, dx = -\int_{\mathbb{R}^n} \left[ c(n, 2s) \int_{\mathbb{R}^n} \frac{(x_i - y_i)}{|x - y|^{n-2s+2}} \, d\omega(y) \right] \varphi(x) \, dx.
\]

Now, by Fubini’s Theorem we have that

\[
\int_{\mathbb{R}^n} \left[ \int_{\text{supp}(\omega)} \frac{c(n, 2s)}{|x - y|^{n-2s+2}} \frac{\partial \varphi}{\partial x_i}(x) \, d\omega(y) \right] dx
\]

\[
= \int_{\text{supp}(\omega)} \left[ \int_{B_R} \frac{c(n, 2s)}{|x - y|^{n-2s+2}} \frac{\partial \varphi}{\partial x_i}(x) \, dx \right] d\omega(y)
\]

and

\[
-\int_{\mathbb{R}^n} \left[ c(n, 2s) \int_{\text{supp}(\omega)} \frac{(x_i - y_i)}{|x - y|^{n-2s+2}} \varphi(x) \, d\omega(y) \right] dx
\]

\[
= -c(n, 2s) \int_{\text{supp}(\omega)} \left[ \int_{B_R} \frac{(x_i - y_i)}{|x - y|^{n-2s+2}} \varphi(x) \, dx \right] d\omega(y).
\]
For $r > 0$ small and $y \in \text{supp}(\omega)$ fixed, integration by parts gives

\[(3.12)\]
\[
\int_{B_R \setminus B_r(y)} \frac{1}{|x - y|^{n - 2s}} \frac{\partial \varphi}{\partial x_i}(x) \, dx = \int_{\partial B_r(y)} \frac{\varphi(x) \eta_i}{|x - y|^{n - 2s + 1}} \, dS_x - \int_{B_R \setminus B_r(y)} \frac{(x_i - y_i)}{|x - y|^{n - 2s + 2}} \varphi(x) \, dx.
\]

Here we have used the fact that $\varphi = 0$ in $\partial B_R$. The vector $\eta = (\eta_1, ..., \eta_n)$ is the exterior normal unit vector to $\partial B_r(y)$, so

$$\eta_i = \frac{(x_i - y_i)}{|x - y|}.$$ 

Now, for all $y \in \text{supp}(\omega)$

$$\left| \int_{\partial B_r(y)} \frac{1}{|x - y|^{n - 2s}} \varphi(x) \eta_i \, dS_x \right| = \left| \int_{\partial B_r(y)} \frac{(x_i - y_i)}{|x - y|^{n - 2s + 1}} \varphi(x) \, dS_x \right|$$

\[\leq \int_{\partial B_r(y)} \frac{1}{|x - y|^{n - 2s}} |\varphi(x)| \, dS_x \]

\[= r^{2s - n} \int_{\partial B_r(y)} |\varphi(x)| \, dS_x \]

\[= r^{2s - 1 - n} \int_{\partial B_r(y)} |\varphi(x)| \, dS_x.
\]

Observe

$$\lim_{r \to 0} r^{1 - n} \int_{\partial B_r(y)} |\varphi(x)| \, dS_x = \omega_n |\varphi(y)| \leq C$$

for all $y \in \text{supp}(\omega)$. So

$$\int_{\partial B_r(y)} \frac{1}{|x - y|^{n - 2s}} \varphi(x) \eta_i \, dS_x = o(1)$$

for $r \to 0$ uniformly on $y \in \text{supp}(\omega)$. Regarding the last term in (3.12) we have

$$\int_{B_R \setminus B_r(y)} \frac{(x_i - y_i)}{|x - y|^{n - 2s + 2}} \varphi(x) \, dx \leq C \int_{B_R} \frac{1}{|x - y|^{n - 2s + 1}} \, dx$$

and taking $R_0 > 0$ big enough we obtain that

$$\int_{\text{supp}(\omega)} \int_{B_{R_0}} \frac{1}{|x - y|^{n - 2s + 1}} \, dx \, d\omega(y) = \int_{\text{supp}(\omega)} \int_0^{R_0} \omega_n r^{2s - 1 - 1} r^{n - 1} \, dr \, d\omega(y)$$

\[\leq CR_0^{2s - 1} < \infty.
\]

Hence, by Lebesgue dominated convergence theorem

$$\lim_{r \to 0} \int_{\text{supp}(\omega)} \int_{B_R} \frac{(x_i - y_i)}{|x - y|^{n - 2s + 2}} \varphi(x) \chi_{B_R \setminus B_r(y)} \, dx \, d\omega(y) = \int_{\text{supp}(\omega)} \int_{B_R} \frac{(x_i - y_i)}{|x - y|^{n - 2s + 2}} \varphi(x) \, dx \, d\omega(y).$$

Therefore
\[
\lim_{r \to 0} \left[ \int_{\text{supp}(\omega)} \int_{B_R \setminus B_r(y)} \frac{1}{|x - y|^{n-2s}} \frac{\partial \varphi}{\partial x_i}(x) \, dx \, d\omega(y) \right] = - \int_{\text{supp}(\omega)} \int_{B_R} \frac{(x_i - y_i)}{|x - y|^{n-2s+2}} \varphi(x) \, dx \, d\omega(y).
\]

Similarly
\[
\lim_{r \to 0} \int_{\text{supp}(\omega)} \int_{B_R \setminus B_r(y)} \frac{1}{|x - y|^{n-2s}} \frac{\partial \varphi}{\partial x_i} \, dx \, d\omega(y) = \int_{\text{supp}(\omega)} \int_{B_R} \frac{1}{|x - y|^{n-2s}} \frac{\partial \varphi}{\partial x_i} \, dx \, d\omega(y).
\]

Then, we have proved (3.11) and therefore we have
\[
\nabla u_0(x) = c(n, 2s) \int_{\mathbb{R}^n} \frac{(x - y)}{|x - y|^{n-2s+2}} \, d\omega(y)
\]
in the weak sense.

Lemma 3.7. For \( n > 2s \) and \( q > p^* \), if hypothesis (2.7) is satisfied with a constant \( C_1 = C_1(n, q, s) \), then \( I_{2s-1}(\omega) \in L^q(\mathbb{R}^n) \).

Proof. First of all, by Theorem 2.1 from [28] and the assumption (2.7) we have that
\[
(I_{2s-1}(\omega)(x))^q dx \leq C(n, 2s-1)^q \left[ \int_{B_R} \left( \int_{|x - y|^{q(n-2s+1)}} \frac{1}{|x - y|^{q(n-2s+1)}} \, dx \right)^{1/q} d\omega(y) \right]^q.
\]

Now, since \( q > \frac{n}{n-2s+1} \)
\[
\int_{\mathbb{R}^n \setminus B_{2R}} \frac{1}{|x - y|^{q(n-2s+1)}} \, dx \leq \frac{R^{n+(2s-n-1)q}}{(n-2s+1)q - n}
\]
and so the integral
\[
\left[ \int_{B_R} \left( \int_{\mathbb{R}^n \setminus B_{2R}} \frac{1}{|x - y|^{q(n-2s+1)}} \, dx \right)^{1/q} d\omega(y) \right]^q
\]
is finite.
Remark 3.8. Observe that by Lemma 3.6 and Lemma 3.7, $|\nabla u_0| \in L^q(\mathbb{R}^n)$.

Proposition 3.9. Under the same conditions of Lemma 3.7, $I_{2s}(|\nabla u_0|^q) \in L_s(\mathbb{R}^n)$.

Proof. Using the inequality (3.9), we just need to verify that $I_{2s}(|\nabla u_0|^q) \in L_s(\mathbb{R}^n)$. Hence

\[
I_{2s}(|\nabla u_0|^q)(x) = \int_{\mathbb{R}^n} \frac{c(n, 2s)}{|x-y|^{n-2s}} [I_{2s-1}(\omega)]^q(y) \, dy
\]

\[
= c(n, 2s) \left( \int_{B_1(x)} [I_{2s-1}(\omega)]^q(y) \, dy + \int_{\mathbb{R}^n \setminus B_1(x)} [I_{2s-1}(\omega)]^q(y) \, dy \right)
\]

\[
\leq c(n, 2s) \left( \int_{B_1(x)} [I_{2s-1}(\omega)]^q(y) \, dy + \int_{\mathbb{R}^n \setminus B_1(x)} [I_{2s-1}(\omega)]^q(y) \, dy \right)
\]

\[
= C(n, s) \left[ I_{2s-1}(I_{2s-1}(\omega))^q(x) + \|I_{2s-1}(\omega)\|_{L^q(\mathbb{R}^n)}^q \right]
\]

\[
\leq C(q, n, s) \left[ I_{2s-1}(\omega)(x) + \|I_{2s-1}(\omega)\|_{L^q(\mathbb{R}^n)}^q \right].
\]

Thus, using Remark 3.2 we conclude the proof. ■

Proposition 3.10. For $n > 2s$ and $q > \frac{n}{n-2s+1}$, if there exists $C_1 = C_1(n, q, s) > 0$ such that (2.17) is satisfied, then $I_{2s}(|\nabla u_0|^q)$ is a weak solution of

\[
(-\Delta)^s v = |\nabla u_0|^q \quad \text{in} \quad \mathbb{R}^n.
\]

Proof. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$. We proceed as in Proposition 3.5 naming $\varphi(x) = |x|^{2s} \mathcal{F}(\varphi_0)(x)$ and $\psi(x) = \mathcal{F}(\varphi)(-x)$. Recall that $\psi \in \mathcal{S}_s(\mathbb{R}^n)$.

We will apply Lemma 3.4 as in the proof of Lemma 3.5 to obtain the desired result. According to Remark 3.4, we just need to verify the following statements

i. $\mathcal{F}(\psi * |\nabla u_0|^q) \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$,

ii. $\int_{\mathbb{R}^n} I_{2s}(x)(\psi * |\nabla u_0|^q) \, dx < \infty$, and

iii. $\psi * |\nabla u_0|^q \in L^2(\mathbb{R}^n)$.

First of all, $\mathcal{F}(\psi * |\nabla u_0|^q) \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. The continuity follows from the fact that $\psi * |\nabla u_0|^q \in L^1(\mathbb{R}^n)$ for $\psi \in \mathcal{S}_s(\mathbb{R}^n)$. The integrability can be checked easily using that $|\nabla u_0|^q \in L^1(\mathbb{R}^n)$. Thus, we will concentrate on ii. and iii.

ii.
\[
\int_{\mathbb{R}^n} |x|^{2s-n}(\psi \ast |\nabla u_0|^q)(x) \, dx = \int_{\mathbb{R}^n} \frac{1}{|x|^{n-2s}} \left( \int_{\mathbb{R}^n} \psi(x-y)|\nabla u_0(y)|^q \, dy \right) \, dx
\]
\[
= \int_{\mathbb{R}^n} |\nabla u_0(y)|^q \left( \int_{\mathbb{R}^n} \frac{\psi(x-y)}{|x|^{n-2s}} \, dx \right) \, dy
\]
\[
= \int_{\mathbb{R}^n} |\nabla u_0(y)|^q \left( \int_{B_1} \frac{\psi(x-y)}{|x|^{n-2s}} \, dx + \int_{\mathbb{R}^n \setminus B_1} \frac{\psi(x-y)}{|x|^{n-2s}} \, dx \right) \, dy.
\]

Now, since \( \psi \in \mathcal{S}_s(\mathbb{R}^n) \),
\[
\int_{B_1} \frac{\psi(x-y)}{|x|^{n-2s}} \, dx \leq C \int_{B_1} \frac{1}{|x|^{n-2s}} \, dx = \frac{C \omega_n}{2s}.
\]

On the other hand,
\[
\int_{\mathbb{R}^n \setminus B_1} \frac{\psi(x-y)}{|x|^{n-2s}} \, dx \leq \|\psi\|_{L^1(\mathbb{R}^n)}.
\]

Then,
\[
\int_{\mathbb{R}^n} |x|^{2s-n}(\psi \ast |\nabla u_0|^q)(x) \, dx \leq C(n, s, \psi)\|\nabla u_0\|_{L^q(\mathbb{R}^n)} < \infty.
\]

iii. Follows by Young's inequality.

3.3. Step (III).

**Proof of Theorem 2.4**. We begin constructing a sequence of functions \( u_{k+1} \) as follows. Let \( u_0 \) be as in Lemma 3.6 and define by recursion
\[
(3.15) \quad u_{k+1} = \mathcal{I}_{2s}( |\nabla u_k|^q ) + \mathcal{I}_{2s}( \omega).
\]

Then by propositions 3.5 and 3.10
\[
(3.16) \quad (-\Delta)^s u_{k+1} = |\nabla u_k|^q + \omega \quad \text{in } \mathbb{R}^n.
\]

We claim that
\[
(3.17) \quad |\nabla u_k| \leq C_2 \mathcal{I}_{2s-1}(\omega),
\]
\[
(3.18) \quad |\nabla u_{k+1} - \nabla u_k| \leq C_3 \delta^k \mathcal{I}_{2s-1}(\omega)
\]
for some \( 0 < \delta < 1 \). Let us begin with (3.17) proceeding by induction. Because of Lemma 3.6 (3.17) holds for \( k = 0 \). Suppose that
\[
(3.19) \quad |\nabla u_k| \leq a_k \mathcal{I}_{2s-1}(\omega).
\]

Then, we see that
\[ |\nabla u_{k+1}| \leq |\nabla (I_{2s}(|\nabla u_k|^q))| + |\nabla (I_{2s}(\omega))| \]
\[ \leq C_0 (I_{2s-1}(|\nabla u_k|^q) + I_{2s-1}(\omega)) \]
\[ \leq C_0 (a_k^q I_{2s-1}(\omega)^q + I_{2s-1}(\omega)) \] by (3.19)
\[ \leq C_0 (a_k^q C_1 I_{2s-1}(\omega) + I_{2s-1}(\omega)) \] by (2.7)
\[ = C_0 (a_k^q C_1 + 1) I_{2s-1}(\omega). \]

Then
\[ |\nabla u_{k+1}| \leq a_{k+1} I_{2s-1}(\omega) \]
with \( a_{k+1} = C_0 (a_k^q C_1 + 1) \). Then, if \( C_1 \leq (q')^{1-q} q^{-1} C_0^{-q} \) we see that
\[ \lim_{k \to \infty} a_k = a \leq C_0 q' \]
where \( a \) is a root of the equation \( x = C_0 (x^q C_1 + 1) \). Hence (3.17) holds with \( C_2 = C_0 q' \). Now we prove (3.18). Assume (2.7) holds with \( C_1 \leq (q')^{1-q} q^{-1} C_0^{-q} \) so that (3.17) is satisfied with \( C_2 = C_0 q' \), where \( C_0 \) is the constant of (3.9).

Now,
\[ u_1 - u_0 = I_{2s}(|\nabla u_0|^q). \]

Then,
\[ |\nabla u_1 - \nabla u_0| \leq C_0 I_{2s-1}(|\nabla u_0|^q) \]
\[ \leq C_0 C_2^0 I_{2s-1}(|I_{2s-1}(\omega)|^q) \]
\[ \leq C_0 C_2^0 C_1 I_{2s-1}(\omega). \]

Therefore
\[ |\nabla u_1 - \nabla u_0| \leq b_0 I_{2s-1}(\omega) \]
with \( b_0 = C_0 C_2^0 C_1 \). Analogously,
\[ u_{k+1} - u_k = I_{2s}(|\nabla u_k|^q - |\nabla u_{k-1}|^q) \]
and
\[ |\nabla u_{k+1} - \nabla u_k| \leq C_0 I_{2s-1}(|\nabla u_k|^q - |\nabla u_{k-1}|^q). \]

Using the inequality \( |r^q - s^q| \leq q|r - s| \max\{r, s\}^{q-1} \) with \( r = |\nabla u_k| \) and \( s = |\nabla u_{k-1}| \) we obtain
\[ |\nabla u_k|^q - |\nabla u_{k-1}|^q | \leq q \| \nabla u_k \| - |\nabla u_{k-1}| \| \max\{|\nabla u_k|, |\nabla u_{k-1}|\}^{q-1} \]
\[ \leq C_2^{q-1} q |\nabla u_k| - |\nabla u_{k-1}| \| I_{2s-1}(\omega)|^{q-1}. \]

Thus by (3.20) and (3.21)
\[ |\nabla u_{k+1} - \nabla u_k| \leq C_0 C_2^{q-1} q I_{2s-1}(|\nabla u_k - \nabla u_{k-1}| \| I_{2s-1}(\omega)|^q). \]
Now suppose $|\nabla u_k - \nabla u_{k-1}| \leq b_k \mathcal{I}_{2s-1}(\omega)$. Then, by (3.22)

$$|\nabla u_{k+1} - \nabla u_k| \leq C_0 C_2^{q-1} q \mathcal{I}_{2s-1} (b_k [\mathcal{I}_{2s-1}(\omega)]^q) \leq C_0 C_2^{q-1} q b_k C_1 \mathcal{I}_{2s-1}(\omega).$$

Then, arguing by induction we see that

$$|\nabla u_{k+1} - \nabla u_k| \leq b_{k+1} \mathcal{I}_{2s-1}(\omega)$$

with $b_{k+1} \leq C_0 C_2^{q-1} q C_1 b_k$. Thus,

$$b_{k+1} \leq \left( C_0 C_2^{q-1} q C_1 \right)^{k+1} b_0$$

with $b_0 = C_0 C_2^{q-1} C_1$. Taking $C_1$ such that $\delta = C_0 C_2^{q-1} q C_1 < 1$, we obtain (3.18) with $C_3 = b_0$.

Now we claim that

$$|u_{k+1} - u_k| \leq C_4 \delta^k \mathcal{I}_{2s} ([\mathcal{I}_{2s-1}(\omega)]^q)$$

with $C_4 > 0$ and $0 < \delta < 1$ depending only on $q, n$ and $s$. Indeed, by (3.21) and (3.18)

$$|u_{k+1} - u_k| \leq \mathcal{I}_{2s} (||\nabla u_k||^q - ||\nabla u_{k-1}||^q) \leq C_2^{q-1} q \mathcal{I}_{2s} (|\nabla u_k - \nabla u_{k-1}| |\mathcal{I}_{2s-1}(\omega)|^{q-1}) \leq C_2^{q-1} q C_3 \delta \mathcal{I}_{2s} ([\mathcal{I}_{2s-1}(\omega)]^q).$$

Then (3.23) holds for $C_4 = C_2^{q-1} q C_3$.

Now suppose (2.7) holds for $C_1$ small enough so that (3.17) and (3.18) are satisfied. Let

$$u(x) = u_0(x) + \sum_{k=0}^{\infty} (u_{k+1}(x) - u_k(x))$$

with $u_k$ defined by (3.15). By (3.23),

$$|u_{k+1}(x) - u_k(x)| \leq C_4 \delta^k \mathcal{I}_{2s} ([\mathcal{I}_{2s-1}(\omega)]^q).$$

Hence, $u(x) = \lim_{k \to \infty} u_k(x)$ and $u \in L_s(\mathbb{R}^n)$ since

$$|u| \leq C (\mathcal{I}_{2s} \omega + \mathcal{I}_{2s} [\mathcal{I}_{2s-1} \omega]^q)$$

On the other side, by (3.18)

$$|\nabla u_{k+1} - \nabla u_k| \leq C_3 \delta^k \mathcal{I}_{2s-1}(\omega),$$

hence

$$|\nabla u| \leq C \mathcal{I}_{2s-1}(\omega).$$

Then, by Lemma 3.7 $|\nabla u| \in L^q(\mathbb{R}^n)$. Therefore, $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^n) \cap L_s(\mathbb{R}^n)$ (observe that $u \in L^q_{\text{loc}}(\mathbb{R}^n)$ by (2.11)).
We now check that Definition 2.2 holds. Let \( \varphi \in S(\mathbb{R}^n) \). We have that \( \nabla u(x) = \lim_{k \to \infty} \nabla u_k(x) = \nabla u_0(x) + \sum_{k=0}^{\infty} (\nabla u_{k-1}(x) - \nabla u_k(x)) \) a.e. in \( \mathbb{R}^n \). Also, \( \nabla u_k|^{q} \leq C I_{2s-1}(\omega)|^{q} \in L^1(\mathbb{R}^n) \) for all \( k \). Then, letting \( k \to \infty \)

\[
\int_{\mathbb{R}^n} \varphi |\nabla u_k|^{q} \, dx \to \int_{\mathbb{R}^n} \varphi |\nabla u|^{q} \, dx.
\]

On the other side observe that, for all \( k \)

\[
|u_k| \leq C [I_{2s}(\omega) + I_{2s}([I_{2s-1}(\omega)]^{q})]
\]

Thus, since \( \varphi \in S(\mathbb{R}^n) \) and \( I_{2s}(\omega), I_{2s}([I_{2s-1}(\omega)]^{q}) \in L_s(\mathbb{R}^n) \), we can apply Lebesgue dominated convergence theorem and obtain

\[
\int_{\mathbb{R}^n} u_{k+1}(-\Delta)^s \varphi \, dx \to \int_{\mathbb{R}^n} u(-\Delta)^s \varphi \, dx
\]

for \( k \to \infty \).

Now, since \( (-\Delta)^s u_{k+1} = |\nabla u_k|^{q} + \omega \) in \( \mathbb{R}^n \), by Definition 2.2

\[
\int_{\mathbb{R}^n} u_{k+1}(-\Delta)^s \varphi \, dx = \int_{\mathbb{R}^n} \varphi |\nabla u_k|^{q} \, dx + \int_{\mathbb{R}^n} \varphi \, d\omega.
\]

Consequently, letting \( k \to \infty \) in the preceding equality we get the desired conclusion.

Finally, we prove the representation (2.6). By (3.15) is enough to show that, as \( k \to \infty \), \( I_{2s}(|\nabla u_k|^{q}) \to I_{2s}(|\nabla u|^{q}) \).

First, by definition of Riesz potential we have

\[
|I_{2s}(|\nabla u_k|^{q}) - I_{2s}(|\nabla u|^{q})| \leq c(n, 2s) \int_{\mathbb{R}^n} \frac{|\nabla u_k|^{q} - |\nabla u|^{q}|}{|x-y|^{n-2s}} \, dy
\]

Now, by (3.17) and (3.25) we get

\[
\frac{|\nabla u_k|^{q} - |\nabla u|^{q}|}{|x-y|^{n-2s}} \leq qC |I_{2s-1}(\omega)|^{q} \frac{|x-y|^{n-2s}}{|x-y|^{n-2s}} \leq qC \frac{|I_{2s-1}(\omega)|^{q}}{|x-y|^{n-2s}}
\]

In Proposition 2.12, we proved

\[
I_{2s}([I_{2s-1}(\omega)]^{q})(x) \leq C(q, n, s) \left[ I_{2s-1}(\omega)(x) + \|I_{2s-1}(\omega)\|_{L^q(\mathbb{R}^n)}\right]
\]

in \( \mathbb{R}^n \). Therefore,

\[
\int_{\mathbb{R}^n} \frac{qC |I_{2s-1}(\omega)|^{q} |x-y|^{n-2s}}{|x-y|^{n-2s}} \, dy = C(n, q, s) I_{2s}([I_{2s-1}(\omega)]^{q})(x)
\]

is finite, so we can apply Lebesgue’s Theorem to obtain

\[
\lim_{k \to \infty} |I_{2s}(|\nabla u_k|^{q}) - I_{2s}(|\nabla u|^{q})| = c(n, 2s) \int_{\mathbb{R}^n} \lim_{k \to \infty} \frac{|\nabla u_k|^{q} - |\nabla u|^{q}|}{|x-y|^{n-2s}} \, dy = 0
\]

Hence, we get (2.6).
4. Proof of Theorem 2.5

Suppose that (2.3) has a solution \( u \in W^{1,q}(\mathbb{R}^n) \). Let \( u_k \in C_0^\infty(\mathbb{R}^n) \) be a sequence converging to \( u \) in \( W^{1,q}(\mathbb{R}^n) \). Then for non-negative \( \varphi \in C_0^\infty(\mathbb{R}^n) \) with \( \varphi \geq \chi_E \) we have

\[
\omega(E) \leq \int_{\mathbb{R}^n} \varphi d\omega
\leq \int_{\mathbb{R}^n} u(-\Delta)^s \varphi \quad \text{since } u \text{ is a solution}
\]

\[
= \int_{\mathbb{R}^n} u_k(-\Delta)^s \varphi + o(1)
\]

\[
= \int_{\mathbb{R}^n} (-\Delta)^{1/2} u_k(-\Delta)^{(2s-1)/2} \varphi + o(1) \quad \text{by Lemma 2.2 in [14]}
\]

By continuity of the operator \((-\Delta)^{1/2} : W^{1,q}(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)\) (see for instance [?, Theorem 2.1]) and H"{o}lder’s inequality, we derive

\[
\omega(E) \leq \int_{\mathbb{R}^n} (-\Delta)^{1/2} u(-\Delta)^{(2s-1)/2} \varphi
\]

\[
\leq \|(1-\Delta)^{1/2} u\|_{L^q(\mathbb{R}^n)} \|(-\Delta)^{(2s-1)/2} \varphi\|_{L^p(\mathbb{R}^n)}
\]

\[
\leq C \|\nabla u\|_{L^q(\mathbb{R}^n)} \|\varphi\|_{L^p_{2s-1}(\mathbb{R}^n)},
\]

where for \( r \in (1, \infty) \) and \( \alpha > 0 \), \( L^r_{\alpha}(\mathbb{R}^n) \) denotes the Bessel potential space defined as

\[
L^r_{\alpha}(\mathbb{R}^n) := \left\{ v \in L^r(\mathbb{R}^n) : (I - \Delta)^{\alpha/2} v \in L^r(\mathbb{R}^n) \right\}
\]

with the norm

\[
\|v\|_{L^r_{\alpha}(\mathbb{R}^n)} = \|v\|_{L^r(\mathbb{R}^n)} + \|(1-\Delta)^{\alpha/2} v\|_{L^r(\mathbb{R}^n)}.
\]

Since \( \omega \) has compact support and the Riesz capacity of \( E \) is zero, we obtain from [2 Proposition 5.1.4 (b)] that the Bessel capacity of \( E \), denoted by \( C_{2s-1,p}(E) \), is also zero. Then there is a sequence of non-negative and smooth functions \( \varphi_k \in L^p_{2s-1}(\mathbb{R}^n) \) such that

\[
\varphi_k \geq \chi_E, \quad \varphi_k \rightarrow 0 \quad \text{in } L^p_{2s-1}(\mathbb{R}^n).
\]

Applying (4.2) to the sequence \( \varphi_k \) and taking \( k \rightarrow \infty \), we conclude

\[
\omega(E) = 0.
\]

This ends the proof.

Remark 4.1. We now comment on the necessity of (2.4) for existence of solutions. We have not proved the converse of Theorem 2.4, but we exhibit below how some arguments may be applied to find a relation between the measure \( \omega \), the Riesz capacity and the solution \( u \). We consider \( \varphi \) as before and we take \( \varphi^p \) as a test function to derive

\[
\omega(E) \leq \int_{\mathbb{R}^n} u_k(-\Delta)^s \varphi^p - \int_{\mathbb{R}^n} |\nabla u|^q \varphi^p + o(1)
\]

\[
= \int_{\mathbb{R}^n} (-\Delta)^{1/2} u_k(-\Delta)^{(2s-1)/2} \varphi^p - \int_{\mathbb{R}^n} |\nabla u|^q \varphi^p + o(1)
\]
Letting $k \to \infty$ and by Young’s inequality, we get

\[
\omega(E) \leq \int_{\mathbb{R}^n} (-\Delta)^{1/2} u (-\Delta)^{(2s-1)/2} \varphi^p - \int_{\mathbb{R}^n} |\nabla u|^q \varphi^p
\]

\[
\leq C_p \int_{\mathbb{R}^n} |(-\Delta)^{1/2} u|^{p-1} (-\Delta)^{(2s-1)/2} \varphi - \int_{\mathbb{R}^n} |\nabla u|^q \varphi^p
\]

\[
\leq \int_{\mathbb{R}^n} |(-\Delta)^{1/2} u|^q \varphi^p + C_p \int_{\mathbb{R}^n} |(-\Delta)^{(2s-1)/2} \varphi|^p - \int_{\mathbb{R}^n} |\nabla u|^q \varphi^p
\]

\[
= \int_{\mathbb{R}^n} (|(-\Delta)^{1/2} u|^q - |\nabla u|^q) \varphi^p + C_p \|\varphi\|_{L^p_{2s-1}(\mathbb{R}^n)}^p.
\]

Taking the infimum over $\varphi$ in (4.4), we derive

\[
\omega(E) \leq \inf_{\varphi} \int_{\mathbb{R}^n} \left( |(-\Delta)^{1/2} u|^q - |\nabla u|^q \right) \varphi^p + C_p C_{2s-1,p}(E).
\]

Since $\omega$ has compact support, we derive from [2, Proposition 5.1.4 (b)] that there is a constant $C > 0$ so that

\[
\omega(E) \leq \inf_{\varphi} \int_{\mathbb{R}^n} \left( |(-\Delta)^{1/2} u|^q - |\nabla u|^q \right) \varphi^p + C_p C_{2s-1,p}(E).
\]

In the local case, exposed in [20], the first term in (4.5) does not appear and hence (2.4) is also necessary for existence. In the current scenario, we have not obtained the vanishing of that term. Indeed, adapting directly the proof of [20, Lemma 2.1] to our case does not seem to be straightforward and the problem is left as open.

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