A CLASS OF MULTIDIMENSIONAL NONLINEAR DIFFUSIONS WITH THE FELLER PROPERTY

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ABSTRACT. In this note we consider a family of nonlinear (conditional) expectations that can be understood as a multidimensional diffusion with uncertain drift and certain volatility. Here, the drift is prescribed by a set-valued function that depends on time and path in a Markovian way. We establish the Feller property for the associated sublinear Markovian semigroup and we observe a smoothing effect as our framework carries enough randomness. Furthermore, we link the corresponding value function to a semilinear Kolmogorov equation.

1. Introduction

A nonlinear multidimensional diffusion, or nonlinear multidimensional continuous Markov process, is a family of sublinear expectations \( \{E^x: x \in \mathbb{R}^d\} \) on the Wiener space \( C(\mathbb{R}_+; \mathbb{R}^d) \) with \( E^x \circ X_0^{-1} = \delta_x \) for each \( x \in \mathbb{R}^d \) such that the Markov property
\[
\mathcal{E}^x(X_t(\psi(X_s))) = \mathcal{E}^x(\psi(X_{t+s})), \quad x \in \mathbb{R}, \ s, t \in \mathbb{R}_+, \tag{1.1}
\]
holds. Here, \( \psi \) runs through a collection of suitable test functions and \( X \) denotes the canonical process on \( C(\mathbb{R}_+; \mathbb{R}^d) \). Building upon the seminal work of Peng \([14, 15]\) on the \( G \)-Brownian motion, nonlinear Markov processes have been intensively studied in recent years, both from the perspective of processes under uncertainty \([6, 9, 12]\), as well as sublinear Markov semigroups \([3, 8, 10, 11]\).

Using the techniques from \([13]\), a general framework for constructing nonlinear Markov processes was developed in \([8]\). To be more precise, for given \( x \in \mathbb{R}^d \), the sublinear expectation \( \mathcal{E}^x \) has the form \( \mathcal{E}^x = \sup_{P \in \mathcal{R}(x)} E^P \) with a collection \( \mathcal{R}(x) \) of semimartingale laws on the path space, with initial distribution \( \delta_x \), and whose absolutely continuous characteristics are prescribed by a set-valued map.

As in the theory of (linear) Markov processes, there is a strong link to semigroups. Indeed, the Markov property (1.1) ensures the semigroup property \( T_t T_s = T_{s+t}, s, t \in \mathbb{R}_+, \) where the sublinear operators \( T_t, t \in \mathbb{R}_+ \), are defined by
\[
T_t(\psi)(x) := \mathcal{E}^x(\psi(X_t)) = \sup_{P \in \mathcal{R}(x)} E^P[\psi(X_t)] \tag{1.2}
\]
for suitable functions \( \psi \). Using the general theory of \([5, 13]\), the operators \( T_t, t \in \mathbb{R}_+ \), are well-defined on the cone of upper semianalytic functions. We are interested in the \( C_b \)-Feller property of \( (T_t)_{t \in \mathbb{R}_+} \) i.e., \( T_t(C_b(\mathbb{R}^d; \mathbb{R})) \subset C_b(\mathbb{R}^d; \mathbb{R}) \) for all \( t \in \mathbb{R}_+ \). In general, this property seems to be hard to verify, see \([8, \text{Remark 4.43}], [10, \text{Remark 3.4}] \) and \([11, \text{Remark 5.4}] \) for comments. In our previous paper \([2]\), we established the \( C_b \)-Feller property for a large class of one-dimensional nonlinear diffusions. To the best of our knowledge, in a multidimensional framework, there seems to be no result on the continuity of \( x \mapsto T_t(\psi)(x) \) beyond the Lévy case \([8, 12]\). As already acknowledged in the context of controlled diffusions (see, e.g., \([4, 7]\)), the difficulty stems from the (possible) lack of lower hemicontinuity of the set-valued map \( x \mapsto \mathcal{R}(x) \).

\footnotesize
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In this note, we establish the \textit{Feller property} for a class of multidimensional diffusions with \textit{uncertain} drift and \textit{certain} volatility. By means of a \textit{Feller selection principle}, we establish the $C_0$–Feller property of $(T_t)_{t \in \mathbb{R}_+}$, which constitutes our main contribution. This allows us to identify the value function $[0, T] \times \mathbb{R}^d \ni (t, x) \mapsto \mathcal{E}^x(\psi(X_{T-t}))$ as the unique viscosity solution to a semilinear Kolmogorov type PDE. Finally, let us highlight that we also observe a smoothing effect through $(T_t)_{t > 0}$. Namely, we show that $T_t$, for $t > 0$, maps bounded upper semicontinuous functions to bounded continuous functions. This extends the corresponding observation in [2] to a multidimensional setting.

2. Main Result

2.1. The Setting. Fix a dimension $d \in \mathbb{N}$ and define $\Omega$ to be the space of continuous functions $\mathbb{R}^d \rightarrow \mathbb{R}^d$ endowed with the local uniform topology. The canonical process on $\Omega$ is denoted by $X$, i.e., $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \in \mathbb{R}_+$. It is well-known that $\mathcal{F} := \mathcal{B}(\Omega) = \sigma(X_t, t \in \mathbb{R}_+)$. We define $\mathbf{F} := (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ as the canonical filtration generated by $X$, i.e., $\mathcal{F}_t := \sigma(X_s, s \in [0, t])$ for $t \in \mathbb{R}_+$. Notice that we do not make the filtration $\mathcal{F}$ right-continuous. The set of probability measures on $(\Omega, \mathcal{F})$ is denoted by $\mathfrak{P}(\Omega)$ and endowed with the usual topology of convergence in distribution. We denote the space of symmetric positive semidefinite real-valued $d \times d$ matrices by $\mathcal{S}_+^d$. Let $F$ be a metrizable space and let $b: F \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $a: \mathbb{R}^d \rightarrow \mathcal{S}_+^d$ be two Borel functions.

Standing Assumption 2.1.

(i) $F$ is compact.
(ii) $b$ and $a$ are continuous.
(iii) There exists a constant $C > 0$ such that, for all $f \in F$ and $x, \xi \in \mathbb{R}^d$,

$$||b(f, x)|| \leq C, \quad \frac{||\xi||^2}{C} \leq \langle \xi, a(x)\xi \rangle \leq C||\xi||^2.$$ 

Remark 2.2. While it is possible so substantially weaken part (iii) of Standing Assumption 2.1, we assume it for the sake of clarity.

We define the correspondence, i.e., the set-valued map, $\Theta: [0, \infty[, \mathbb{R}^d \rightarrow \mathcal{S}_+^d$ by

$$\Theta(t, \omega) := \{(b(f, \omega(t)), a(\omega(t))): f \in F\} \subset \mathbb{R}^d \times \mathcal{S}_+^d.$$ 

The correspondence $\Theta$ has \textit{Markovian structure}, i.e., for every $(t, \omega) \in [0, \infty[)$, the set $\Theta(t, \omega)$ depends on $(t, \omega)$ only through the value $\omega(t)$.

Remark 2.3. Thanks to [1, Lemma 2.11], the graph of $\Theta$ is measurable.

We denote the set of laws of continuous semimartingales by $\mathfrak{P}_{\text{sem}} \subset \mathfrak{P}(\Omega)$. For $P \in \mathfrak{P}_{\text{sem}}$, we denote the semimartingale characteristics of the coordinate process $X$ by $(B^P, C^P)$, and we set

$$\mathfrak{P}_{\text{sem}}^{\text{ac}} := \{P \in \mathfrak{P}_{\text{sem}}: P\text{-a.s. } (B^P, C^P) \ll \lambda\},$$

where $\lambda$ denotes the Lebesgue measure. For $x \in \mathbb{R}^d$, we further define

$$\mathcal{R}(x) := \{P \in \mathfrak{P}_{\text{sem}}^{\text{ac}}: P \circ X_{-1}^{-1} = \delta_x, (\lambda \otimes P)\text{-a.e. } (dB^P/d\lambda, dC^P/d\lambda) \in \Theta\}.$$ 

2.2. Nonlinear Diffusions and Semigroups. For each $x \in \mathbb{R}^d$, we define the sublinear operator $\mathcal{E}^x$ on the convex cone of bounded upper semianalytic functions $\psi: \Omega \rightarrow \mathbb{R}$ by

$$\mathcal{E}^x(\psi) := \sup_{P \in \mathcal{R}(x)} E^P[\psi].$$

For every $x \in \mathbb{R}^d$, we have by construction that $\mathcal{E}^x(\psi(X_0)) = \phi(x)$ for every bounded upper semianalytic function $\psi: \mathbb{R}^d \rightarrow \mathbb{R}$.

Definition 2.4. Let $\mathcal{H}$ be a convex cone of functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ containing all constant functions. A family of sublinear operators $T_t: \mathcal{H} \rightarrow \mathcal{H}, \ t \in \mathbb{R}_+$, is called a sublinear Markovian semigroup on $\mathcal{H}$ if it satisfies the following properties:

(a) \textit{Feller property} for $T_t$.
(b) $T_tP \ll T_sP$, for all $t > s$.
(c) $\mathbf{E}^{T_t}_{\Psi}(\psi) = \mathbf{E}^{T_s}_{\Psi}(\psi) \quad \text{for all } t > s$.
(d) $\mathbf{E}^{T_t}_{\Psi}(\psi) = \mathbf{E}^{T_s}_{\Psi}(\psi)$, for all $t > s$.
(e) $T_tP \ll T_sP$, for all $t > s$.
(i) \((T_t)_{t \in \mathbb{R}_+}\) has the semigroup property, i.e., \(T_sT_t = T_{s+t}\) for all \(s, t \in \mathbb{R}_+\) and \(T_0 = \text{id}\),

(ii) \(T_t\) is monotone for each \(t \in \mathbb{R}_+\), i.e., \(f, g \in \mathcal{H}\) with \(f \leq g\) implies \(T_tf \leq T_tg\),

(iii) \(T_t\) preserves constants for each \(t \in \mathbb{R}_+\), i.e., \(T_t(c) = c\) for each \(c \in \mathbb{R}\).

The following proposition should be compared to [8, Remark 4.33] and [2, Proposition 2.9]. For brevity, we omit a detailed proof.

**Proposition 2.5.** The family of operators \((T_t)_{t \in \mathbb{R}_+}\) given by

\[
T_t(\psi)(x) := \mathcal{E}^x(\psi(X_t)), \quad t \in \mathbb{R}_+, \ x \in \mathbb{R}^d,
\]

defines a sublinear Markovian semigroup on the set of bounded upper semianalytic functions.

### 2.3. The Feller Property

In the following theorem, which is the main result of this note, we show that \((T_t)_{t \in \mathbb{R}_+}\) is a nonlinear semigroup on the space \(C_b(\mathbb{R}^d; \mathbb{R})\) of bounded continuous functions from \(\mathbb{R}^d\) into \(\mathbb{R}\). In fact, we show a bit more, namely the existence of a strong Feller selection.

**Condition 2.6.** The set \(\{b(f, x) : f \in F\} \subset \mathbb{R}^d\) is convex for every \(x \in \mathbb{R}^d\).

**Theorem 2.7.** Suppose that Condition 2.6 holds. Then, for every \(t > 0\) and any bounded upper semicontinuous function \(\psi : \mathbb{R}^d \to \mathbb{R}\), the map \(x \mapsto T_t(\psi)(x)\) is continuous. In particular, \((T_t)_{t \in \mathbb{R}_+}\) has the \(C_b\)-Feller property, i.e., it is a nonlinear semigroup on \(C_b(\mathbb{R}^d; \mathbb{R})\).

**Remark 2.8.** Theorem 2.7 shows that \((T_t)_{t \in \mathbb{R}_+}\) has a weak version of the strong Feller property, i.e., that regularity is gained through \(T_t\), with \(t > 0\), since bounded upper semicontinuous functions are mapped to bounded continuous functions.

For a one-dimensional continuous nonlinear framework with uncertain volatility, i.e., with \(F\)-dependent diffusion coefficient \(a\), a version of Theorem 2.7 was proved in our previous paper [2]. Theorem 2.7 seems to be the first multidimensional result beyond the Lévy case ([8, 12]).

### 2.4. An Application to Semilinear PDEs

We fix a finite time horizon \(T > 0\). For \((t, x, \phi) \in \mathbb{R}_+ \times \mathbb{R}^d \times C^{2,3}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R})\), we define

\[
G(t, x, \phi) := \sup \{ (b(f, x), \partial_x \phi(t, x)) : f \in F \} + \frac{1}{2} \text{tr} \left[ a(x) \partial^2_{x} \phi(t, x) \right].
\]

Recall that a function \(u : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) is said to be a weak sense viscosity subsolution to the semilinear PDE

\[
\begin{aligned}
\partial_t v(t, x) + G(t, x, v) &= 0, \quad \text{for } (t, x) \in [0, T) \times \mathbb{R}^d, \\
v(T, x) &= \psi(x), \quad \text{for } x \in \mathbb{R}^d,
\end{aligned}
\]

where \(\psi \in C_b(\mathbb{R}^d; \mathbb{R})\), if the following two properties hold:

(a) \(u(T, \cdot) \leq \psi\);

(b) \(\partial_t \phi(t, x) + G(t, x, \phi) \geq 0\) for all \(\phi \in C^{2,3}([0, T] \times \mathbb{R}^d; \mathbb{R})\) such that \(\phi \geq u\) and \(\phi(t, x) = u(t, x)\) for some \((t, x) \in [0, T) \times \mathbb{R}^d\).

A weak sense viscosity supersolution is obtained by reversing the inequalities. Further, \(u\) is called weak sense viscosity solution if it is a weak sense viscosity sub- and supersolution. Additionally, \(u\) is called viscosity subsolution if it is both, a weak sense viscosity subsolution, and upper semicontinuous. The notions of viscosity supersolution and viscosity solution are defined accordingly.

Almost verbatim as in our previous paper [1], under linear growth and local Hölder assumptions on \(b\) and \(a\), one can prove that the value function

\[
v(t, x) := \sup_{P \in \mathcal{P}(x)} E^P[\psi(X_{T-t})], \quad (t, x) \in [0, T] \times \mathbb{R}^d,
\]

is a weak-sense viscosity solution to (2.1). The strong Feller property from Theorem 2.7 yields additional regularity of \(v\) which can be used to show that \(v\) is a viscosity solution in the classical sense. Under Lipschitz conditions on \(b\) and \(a\), we can even deduce a uniqueness statement.
Condition 2.9 (Lipschitz Continuity in Space). There exists a decomposition $a = \sigma \sigma^*$ and a constant $C > 0$ such that
\[
\|b(f, x) - b(f, y)\| + \|\sigma(x) - \sigma(y)\| \leq C\|x - y\|,
\]
for all $f \in F$ and $x, y \in \mathbb{R}^d$.

Theorem 2.10. Suppose that the Conditions 2.6 and 2.9 hold. Then, the value function $v$ is the unique bounded viscosity solution to the semilinear PDE (2.1).

Proof. We already mentioned that $v$ is a weak sense viscosity solution to (2.1). Furthermore, thanks to Theorem 2.7, it follows verbatim as in the proof of [2, Theorem 2.34] that $v$ is continuous (in both arguments). Finally, the comparison principle [8, Corollary 2.34], in combination with [8, Lemmata 2.4, 2.6] and [8, Remark 2.5], implies uniqueness. \hfill \Box

3. PROOF OF THEOREM 2.7

We call an $\mathbb{R}^d$-valued continuous process $Y = (Y_t)_{t \geq 0}$ a (continuous) semimartingale after a time $t^* \in \mathbb{R}_+$ if the process $Y_{t^*} = (Y_t)_{t \geq 0}$ is a semimartingale for its natural right-continuous filtration. The law of a semimartingale after $t^*$ is said to be a semimartingale law after $t^*$ and the set of them is denoted by $\mathcal{P}_{\text{sem}}(t^*)$. For $P \in \mathcal{P}_{\text{sem}}(t^*)$ we denote the semimartingale characteristics of the shifted coordinate process $X_{t^*}$ by $(B_{t^*}^P, C_{t^*}^P)$, and we set
\[
\mathcal{P}_{\text{sem}}^{ac}(t^*) := \{ P \in \mathcal{P}_{\text{sem}}(t^*) : P\text{-a.s. } (B_{t^*}^P, C_{t^*}^P) \ll \lambda \}.
\]

For $P \in \mathcal{P}_{\text{sem}}^{ac}(t^*)$, we define $\mathcal{K}(t, x) := \{ P \in \mathcal{P}_{\text{sem}}^{ac}(t) : P(X_s = x \text{ for all } s \in [0, t]) = 1, (\lambda \otimes P)\text{-a.e. } (dB_{t^*}^P/d\lambda, dC_{t^*}^P/d\lambda) \in \Theta(\{+, t, X_{t^*}\}) \}$. For a probability measure $P$ on $(\Omega, \mathcal{F})$, a kernel $\Omega \ni \omega \mapsto Q_\omega \in \mathcal{P}(\Omega)$, and a finite stopping time $\tau$, we define the pasting measure
\[
(P \otimes_{\Omega} Q)(A) \triangleq \int \int \mathbf{1}_A(\omega \otimes_{\tau(\omega)} \omega')Q_\omega(d\omega')P(d\omega), \quad A \in \mathcal{F},
\]
where
\[
\omega \otimes_{\tau(\omega)} \omega' := \omega|_{[0, t]} + (\omega(t) + \omega' - \omega'(t))\mathbf{1}_{(t, \infty]}.
\]

A family $\{P_{(s, x)} : (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\} \subset \mathcal{P}(\Omega)$ is said to be a (time inhomogeneous) strong Markov family if $(t, x) \mapsto P_{(t, x)}$ is Borel and the strong Markov property holds, i.e., for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and every finite stopping time $\tau \geq s$, for $P_{(s, x)}$-a.a. $\omega \in \Omega$
\[
P_{(s, x)}(\cdot | \mathcal{F}_\tau)(\omega) = \omega \otimes_{\tau(\omega)} P_{(\tau(\omega), \omega(\tau(\omega)))}.
\]

The following general strong Markov selection principle can be proved as in Section 5 of [2]. We omit a detailed proof.

Theorem 3.1 (Strong Markov Selection Principle). For every $\psi \in \text{USC}_b(\mathbb{R}; \mathbb{R}^d)$ and every $t > 0$, there exists a strong Markov family $\{P_{(s, x)} : (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ such that, for all $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $P_{(s, x)} \in \mathcal{K}(s, x)$ and
\[
E^{P_{(s, x)}}[\phi(X_t)] = \sup_{P \in \mathcal{K}(s, x)} E^P[\phi(X_t)].
\]
In particular, for all $x \in \mathbb{R}^d$,
\[
T_t(\psi)(x) = E^{P_{(0, x)}}[\psi(X_t)].
\]

We say that a strong Markov family $\{P_{(t, x)} : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ has the strong Feller property if, for every $T > 0$ and every bounded Borel function $\phi : \mathbb{R} \to \mathbb{R}$, the map $[0, T) \times \mathbb{R}^d \ni (s, x) \mapsto E^{P_{(s, x)}}[\phi(X_T)]$ is continuous. The next result is the key observation for the proof of Theorem 2.7.
Theorem 3.2. Let \( \{P_{(s,x)}: (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d\} \) be a strong Markov family such that, for all \((s,x) \in \mathbb{R}_+ \times \mathbb{R}^d\), \(P_{(s,x)} \in \mathcal{K}(s,x)\). Then, it is also a strong Feller family.

Proof. We adapt the argument from [16, Theorem 7.1.9]. Recall from [16] that a probability measure \(Q\) on \((\Omega, \mathcal{F})\) is said to be a solution to the martingale problem for \((0,a)\) starting from \((s,x) \in \mathbb{R}_+ \times \mathbb{R}^d\) if \(Q(X_t = x) \) for all \(t \in [0, s] \) = 1 and the processes

\[
f(X_t) - \int_s^t \frac{1}{2} \text{tr} \left[ a(X_r) \nabla^2 f(X_r) \right] dr: \quad t \geq s, \quad f \in C_{c}^{\infty}(\mathbb{R}^d; \mathbb{R}),
\]

are \(Q\)-martingales. Thanks to [16, Theorem 7.2.1], for every \((s,x) \in \mathbb{R}_+ \times \mathbb{R}^d\), there exists a unique solution \(Q_{(s,x)}\) to the martingale problem for \((0,a)\) starting from \((s,x)\).

By definition of the correspondence \(\mathcal{K}\), we have \(P_{(s,x)} \in \mathcal{Q}_{\text{sem}}^{\text{ac}}(s)\) and we denote the Lebesgue densities of the \(P_{(s,x)}\)-characteristics of the shifted coordinate process \(X_{t+s}\) by \((b_{t+s}^{(s,x)}, a_{t+s}^{(s,x)})\). Notice that \((\mathcal{L} \otimes P_{(s,x)})\text{-a.e.} a_{t+s}^{(s,x)} = a(X_{t+s})\) by the definition of \(\Theta\) and \(\mathcal{K}(s,x)\). We define

\[
Z_t^{(s,x)} := \exp \left( -\int_s^{t \vee s} \langle a^{-1}(X_r)b_r^{(s,x)}, dX_r^{(s,x)} \rangle - \frac{1}{2} \int_s^{t \vee s} (b_r^{(s,x)}, a^{-1}(X_r)b_r^{(s,x)}) dr \right), \quad t \in \mathbb{R}_+,
\]

where

\[
X^{(s,x)} := X - \int_s^{t \vee s} b_r^{(s,x)} dr.
\]

Thanks to [16, Lemma 6.4.1], each \(Z^{(s,x)}\) is a \(P_{(s,x)}\)-martingale and

\[
dQ_{(s,x)} = Z_T^{(s,x)} dP_{(s,x)} \text{ on } \mathcal{F}_T \text{ for all } T \in \mathbb{R}_+.
\]

Let \(\psi: \mathbb{R}^d \to \mathbb{R}\) be a bounded Borel function such that \(|\psi| \leq 1\) and fix a finite time horizon \(T > 0\). Furthermore, take a sequence \((s^n, x^n)_{n=0}^{\infty} \subset [0, T) \times \mathbb{R}^d\) such that \((s^n, x^n) \to (s^0, x^0)\). Choose \(N \in \mathbb{N}\) large enough such that \(t_N := \max(s^0, \sup_{n \geq N} s^n) + \frac{T}{N} < T\) and set

\[
\Psi(s,x) := E^{P_{(s,x)}} [\psi(X_T)], \quad (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d.
\]

For all \(n \geq N\), using the (strong) Markov property of \(\{P_{(s,x)}: (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}\), we obtain

\[
|E^{P_{(s^n,x^n)}}[\psi(X_T)] - E^{P_{(s^0,x^0)}}[\psi(X_T)]| \leq |E^{P_{(s^n,x^n)}}[\Psi(t_N, X_{t_N})] - E^{P_{(s^0,x^0)}}[\Psi(t_N, X_{t_N})]| \leq |E^{P_{(s^n,x^n)}}[Z^{(s^n,x^n)}_{t_N}] - E^{P_{(s^0,x^0)}}[Z^{(s^0,x^0)}_{t_N}]| + E^{P_{(s^n,x^n)}}[[1 - Z^{(s^n,x^n)}_{t_N}]] + E^{P_{(s^0,x^0)}}[[1 - Z^{(s^0,x^0)}_{t_N}]] \quad \text{(3.1)}
\]

For every \(n \in \{0, N, N + 1, \ldots\}\), we obtain that

\[
(E^{P_{(s^n,x^n)}}[[1 - Z^{(s^n,x^n)}_{t_N}]])^2 \leq E^{P_{(s^n,x^n)}}[[1 - Z^{(s^n,x^n)}_{t_N}]^2] = E^{P_{(s^n,x^n)}}[(Z^{(s^n,x^n)}_{t_N})^2] - 1 \leq C(TN^{-s^n}) - 1,
\]

where \(C > 0\) only depends on the constant from part (iii) of Standing Assumption 2.1. Fix \(\varepsilon > 0\) and choose \(N\) large enough such that

\[
t_N - s_n \leq \frac{\log(1 + \varepsilon^2/9)}{C}
\]

for all \(n \in \{0, N, N + 1, \ldots\}\). In this case, we have

\[
E^{P_{(s^n,x^n)}}[[1 - Z^{(s^n,x^n)}_{t_N}]] \leq \frac{\varepsilon}{3}
\]

(3.2)
for all $n \in \{0, N, N+1, \ldots \}$. Due to [16, Theorem 7.2.4], there exists an $M \in \mathbb{N}$, which in particular depends on $\varepsilon$ and $N$, such that, for all $n \geq M$,

$$(3.3) \quad \left| E^{Q^{(n^n,n_n^n)}} \left[ \Psi(t_N, X^{t_N}) \right] - E^{Q^{(n_0^n,n_0^n)}} \left[ \Psi(t_N, X^{t_N}) \right] \right| \leq \varepsilon.$$ 

Notice that we used here that $s_n < t_N$ for all $n \in \mathbb{Z}_+$. Thanks to (3.1), (3.2) and (3.3), for all $n \geq N \vee M$, we conclude that

$$\left| E^{P^{(s^n,x_n^n)}} \left[ \psi(X^T) \right] - E^{P^{(s_0^n,x_0^n)}} \left[ \psi(X^T) \right] \right| \leq \varepsilon.$$ 

This proves the strong Feller property of $\{P^{(s,x)} : (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ and therefore, the proof is complete. \hfill \Box

Finally, we are in the position to prove our main result, Theorem 2.7.

**Proof of Theorem 2.7.** Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be bounded and upper semicontinuous and take $t > 0$. By the Theorems 3.1 and 3.2, there exists a strong Feller family $\{P^{(s,x)} : (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ such that

$$T_t(\psi)(x) = E^{P^{(0,x)}}[\psi(X_t)].$$

The strong Feller property yields the continuity of $x \mapsto T_t(\psi)(x)$. This completes the proof. \hfill \Box

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