A New technique of Initial Boundary Value Problems Using Homotopy Analysis Method

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Abstract. In this paper, a new homotopy analysis technique which is applying to solve initial boundary value problems of partial differential equations by admitted both the initial and boundary conditions in the recursive relation to obtain a good approximate solution for the problem is proposed. The proposed iterative scheme finds the solution without any discretization, linearization, or restrictive assumptions. Furthermore, we can easily control and adjust the convergence domain and rate of series solutions by the convergence control parameter. The effectiveness of the approach is verified by several examples.

1. Introduction
Many important physical phenomena on the engineering and science fields are frequently modeled by partial differential equations. However, it is difficult to obtain closed-form solutions for them, especially for nonlinear ones. In most cases, only approximate solutions (either analytical ones or numerical ones) can be expected. Numerical solutions got by numerical techniques such as Finite Difference Method, Finite Element Method and so on are only the discrete points approximation. It is difficult to have a comprehensive understanding of the nature through the numerical solutions of the problem. Nevertheless, analytical expressions can display different physical quantities are investigated and the variation tendency, so it can reflect the essence of the problem. Thus, analytic methods to obtain approximate solutions have gained importance in recent years. There are Various techniques for seeking analytical solutions to the partial differential equations such as perturbation techniques [1], Darboux transformation [2], the tanh method [3], homogenous balance method [4], Adomian Decomposition Method (ADM) [5], Variation iteration method (VIM) [6], homotopy perturbation method (HPM) [7] and so on.

Comparing with those methods, the HAM initially proposed by Liao [8] who employed the basic idea of the homotopy in topology in his PH.D. thesis is a general analytic approach to get solutions of partial differential equations and has many advantages [9] as follows, first, it is independent of any small or large quantities such as perturbation techniques. So the HAM can be applied no matter if governing equations and boundary/initial conditions contain small or large quantities or not. Second the HAM does not require specific algorithms and complex calculation such as ADM which has difficulties to calculate the so-called Adomian polynomials and VIM which has difficulties of optimal identification of Lagrange multipliers via the variational theory. Furthermore, the HAM always
provides us with a family of solution expressions in the auxiliary parameter $c_0$, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. We can find a proper value of $c_0$ by the so called $c_0$-curves, as suggested by Liao [10]. Liao [10-11] pointed out that ADM and HPM are special case of the homotopy analysis method. When the convergence control parameter $c_0 = -1$, HAM becomes the so-called HPM. Literature [11] showed that sometimes the approximation series given by HPM are not convergent. So the HAM is a general analytic approach to solve differential equations.

In recent years, this method has been successfully employed to solve many types of problems in science and engineering such as the viscous plows of non-Newtonian fluids[12], the KdV-type equations [13], a new technique of using homotopy analysis method for solving high order nonlinear differential equations [14], solving two point boundary value problems by homotopy analysis method [15] and Fourth-order parabolic partial differential equation with variable coefficients [16], The HAM approach is applied obtaining the solution of the one-phase inverse Stefan problem by using the homotopy analysis method [17], Solving the convection–diffusion equation by means of the optimal q-homotopy analysis method [18]. All of these successful applications verified the validity, effectiveness and flexibility of the HAM.

Yet to solve partial differential equations with initial and boundary conditions, the classical HAM among some other series solution methods, build the recurrence scheme of solution using only one type of the problem conditions: either the initial conditions or the boundary conditions. Our aim here is to propose a new HAM technique that incorporates both types of conditions in the scheme to solve initial-boundary value problems (IBVP) over finite domains.

2. The Classical HAM

Consider the following equation

$$N(u(x,t)) = 0, \quad (x,t) \in \Omega,$$

along with boundary conditions:

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad (x,t) \in \Gamma.$$

Where $N$ is a nonlinear operator, $u(x,t)$ is an unknown function, $B$ is a boundary operator, and $\Gamma$ is the boundary of the domain $\Omega$, $x$ and $t$ denote the spatial and the temporal independent variables, respectively. The operator $N$ can be generally divided into linear and nonlinear parts, say $L$ and $F$. Therefore (1) can be written as

$$L(u) + F(u) = 0$$

By means of generalizing the traditional homotopy method, Liao constructs a homotopy $\varphi(x,t;p): \Omega \times [0,1] \rightarrow R$ which satisfies the so-called zero-order deformation equation

$$(1 - p)L[\varphi(x,t;p) - u_0(x,t)] = pc_0N[\varphi(x,t;p)], \quad (x,t) \in \Omega$$

Where $p \in [0,1]$ is an embedding parameter (called homotopy-parameter), $c_0$ is a nonzero auxiliary parameter which is called convergence-control parameter, $L$ is an auxiliary linear operator, $u_0(x,t)$ which satisfy initial conditions or boundary conditions is an initial guess of
\( u(x,t) , \phi(x,t;p) \) is an unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, from (2.4), when \( p = 0 \) and \( p = 1 \) one has

\[
\phi(x,t;0) = u_0(x,t) \quad ; \quad \phi(x,t;1) = u(x,t);
\]

(5)

Respectively. Thus, as \( p \) increases from 0 to 1, the solution \( \phi(x,t;p) \) varies from the initial guess \( u_0(x,t) \) to the solution \( u(x,t) \). Expanding \( \phi(x,t;p) \) in Maclaurin series with respect to \( p \), one has

\[
\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m
\]

(6)

Where

\[
u_m(x,t) = \left. \frac{\partial^m \phi(x,t;p)}{\partial p^m} \right|_{p=0}
\]

(7)

Here the series (6) is called homotopy-series and Eq.(7) is called the \( m \)-th-order homotopy-derivative of \( \phi \). If the auxiliary linear operator, the initial guess and the convergence-control parameter \( c_0 \) are so properly chosen, the homotopy-series (6) converges at \( p = 1 \), then using the relationship \( \phi(x,t;1) = u(x,t) \), one has the so-called homotopy-series solution

\[
u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)
\]

(8)

Which must be one of solutions of original equation, as proved by Liao\(^{10}\). As \( c_0 = -1 \), Eq.(4) becomes

\[
(1 - p)L[\phi(x,t;p) - u_0(x,t)] + pN[\phi(x,t;p)] = 0, \quad (x,t) \in \Omega
\]

(9)

Which is used in the homotopy perturbation method.

Let us define the vector

\[
\tilde{u}_m = \{u_0(x,t), u_1(x,t), \ldots, u_m(x,t)\}
\]

(10)

Substituting (6) into (4) and equating the terms with identical powers of \( p \), we have the so-called \( m \)-th-order deformation equation

\[
L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = c_0 R(\tilde{u}_{m-1}(x,t))
\]

(11)

Where

\[
R(\tilde{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(x,t;p)]}{\partial p^{m-1}} \right|_{p=0}
\]

(12)

And

\[
\chi_m
\]

3
\[ \chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m \geq 2.
\end{cases} \quad (13) \]

It should be emphasized that \( u_m(x,t) \) for \( m \geq 1 \) is governed by the linear equation (11) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica. The solutions \( u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \) is approximated by the truncated series

\[ U_n(x,t) = \sum_{i=0}^{n} u_i(x,t) \quad (14) \]

### 3. The HAM technique for IBVP over finite domains

In this section, we propose the HAM technique for solving IBVP over finite domains. Suppose (1) can be written

\[ Qu + Mu + F = 0 \quad (15) \]

Associated with initial and boundary conditions where \( Q \) denotes the highest-order partial derivative with respect to \( t \), \( M \) denotes the highest-order partial derivative with respect to \( x \) and \( F \) is a function of \( x,t,u \), and its temporal and spatial partial derivatives of order less than the order of \( Q \) and \( M \), respectively. Then, to include both the initial and boundary conditions in the solution, we construct the two homotopies:

\[ (1 - p)Q[\varphi(x,t;p) - u_{01}(x,t)] = c_0 p N[\varphi(x,t;p)]; \quad (16) \]

\[ (1 - p)M[\varphi(x,t;p) - u_{02}(x,t)] = c_0 p N[\varphi(x,t;p)]; \quad (17) \]

\( u_{01}(x,t) \) satisfies initial conditions and \( u_{02}(x,t) \) satisfies boundary conditions respectively. for each homotopy the corresponding powers of \( p \) are compared to obtain two systems of partial differential equations with the prescribed conditions. We assume the solutions of problem (15) in the form

\[ u = \sum_{i=0}^{\infty} u_i \quad (18) \]

Where \( u_i \) is given by

\[ u_i = \frac{u_{i_1} + u_{i_2}}{2} \quad i = 0, 1, \ldots \quad (19) \]

Where \( u_{i_1} \) and \( u_{i_2} \) are solutions of the \( i \)th equations in the PDE systems obtained from the homotopies (16) and (17), respectively.
4. Numerical Implementation

In this section, two numerical examples are presented to validate the proposed solution scheme. The results are calculated using the symbolic software maple.

Example 4.1 Consider the wave equation:

\[
\begin{align*}
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sin 3\pi x, & 0 < x < 4, t > 0 \\
u(0, t) = u(4, t) = 0 \\
u(x, 0) = \sin(\pi x), \quad \frac{\partial u}{\partial t}(0, 0) = \sin 2\pi x
\end{cases}
\end{align*}
\]

(20)

According to the homotopies (16) and (17), the two following systems of PDEs are obtained

\[
\begin{align*}
\frac{\partial^2 u_0}{\partial t^2} &= 0, \quad u_0(x, 0) = \sin \pi x, \quad \frac{\partial u_0}{\partial t}(x, 0) = \sin 2\pi x, \\
(1 + c_0) \frac{\partial^2 u_{m-1}}{\partial t^2} - c_0 \left( \frac{\partial^2 u_{m-1}}{\partial x^2} + \sin 3\pi x \right), & m = 1 \\
(1 + c_0) \frac{\partial^2 u_{m-1}}{\partial t^2} - c_0 \frac{\partial^2 u_{m-1}}{\partial x^2}, & m > 1 \\
u_m(x, 0) = 0, \quad \frac{\partial u_m}{\partial t}(x, 0) = 0, & (m \geq 1)
\end{align*}
\]

(21)

\[
\begin{align*}
\frac{\partial^2 u_0}{\partial x^2} &= 0, \quad u_0(0, t) = 0, \quad u_0(4, t) = 0, \\
(1 + c_0) \frac{\partial^2 u_{m-1}}{\partial x^2} - c_0 \left( \frac{\partial^2 u_{m-1}}{\partial t^2} - \sin 3\pi x \right), & m = 1 \\
(1 + c_0) \frac{\partial^2 u_{m-1}}{\partial x^2} - c_0 \frac{\partial^2 u_{m-1}}{\partial t^2}, & m > 1 \\
u_m(0, t) = 0, \quad u_m(4, t) = 0, & (m \geq 1)
\end{align*}
\]

(22)

Solving (21) and (22), we can get any order homotopy solutions for problem (20). Fig.1. shows the \(c_0\)-curve for 3th-order HAM approximation of \(u(0.2, 0.6)\) of the problem(20). Fig.2. gives the comparison at \(t=0.6\) between the NHAM 3rd-order approximate solutions of the problem(20) and the exact solution \(u(x, t) = \cos \pi x \sin \pi x + \frac{1}{2\pi} \sin 2\pi x \sin 2\pi x + \left(-\frac{1}{9\pi^2} \cos 3\pi x + \frac{1}{9} \right) \sin 3\pi x\). From Fig.1. and Fig.2., we can find that when \(c_0 \in [-0.25, -0.3]\), the NHAM3rd-order approximate solutions meet the exact solution closely.

Example 4.2 Consider the heat-like equation:
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \frac{1}{2} x^2 \frac{\partial^2 u}{\partial x^2} &= 0 \quad 0 < x < 1, \ t > 0 \\
u(0, t) &= 0, \quad u(1, t) = e^t \\
u(x, 0) &= x^2 
\end{aligned}
\]

(23)

According to the homotopies (16) and (17), the two following systems of PDEs are obtained

\[
\begin{aligned}
\frac{\partial u_0}{\partial t} &= 0, \quad u_0(x, 0) = x^2, \\
\frac{\partial u_m}{\partial t} &= (1 + c_0) \frac{\partial u_{m-1}}{\partial t} - c_0 \cdot \frac{x^2 \partial^2 u_{m-1}}{2 \partial x^2}, \\
u_m(x, 0) &= 0, \quad (m \geq 1)
\end{aligned}
\]

(24)

\[
\begin{aligned}
\frac{\partial^2 u_0}{\partial x^2} &= 0, \quad u_0(0, t) = 0, \quad u_0(1, t) = e^t, \\
\frac{\partial^2 u_m}{\partial x^2} &= (1 + c_0) \frac{\partial^2 u_{m-1}}{\partial x^2} - c_0 \cdot \frac{2 \partial u_{m-1}}{x^2 \partial t}, \\
u_m(0, t) &= 0, \quad u_m(1, t) = 0, \quad (m \geq 1)
\end{aligned}
\]

(25)

Solving (24) and (25), we can get any order homotopy solutions for problem (23). Fig.3. depicts the \(c_0\) -curve for 5th-order HAM approximation of \(u(0.2,0.5)\) of the problem (23). Fig.4. gives the comparison at \(t=0.5\) between the NHAM 5\(^{th}\)-order approximate solution and the exact solution \(u(x,t)=x^2e^t\) of the problem (23). From Fig.3. and Fig.4., we can find that when \(c_0 \in [-0.5, -0.2]\), the NHAM5\(^{th}\)-order approximate solutions meet the exact solution closely. Especially, when \(c_0 = -0.3\), the NHAM5\(^{th}\)-order approximate solution curve and the exact solution curve are coincident completely.

\[
\text{Figure 1. The } c_0 \text{-curve for 3th-order NHAM approximation of } u(0.2,0.6)
\]
Figure 2. Comparison at t=0.6 between the NHAM3rd-order approximate solution (at $c_0 = -0.35, -0.2995, -1$) and the exact solution of problem (23).

Figure 3. The $c_0$-curve for 5th-order NHAM approximation of $u(0.2,0.5)$.

Figure 4. Comparison at t=0.6 between the NHAM5rd-order approximate solution (at $c_0 = -0.3, -0.6, -0.25$) and the exact solution of problem (26).

5. Conclusion
In this paper, a new analytical technique based on the HAM to solve IBVP over finite domains has been proposed. The advantage of this technique is to include both the initial and boundary conditions in the recursive relation, so that we can obtain a good approximate solution for the problems. Otherwise, the results obtained in the numerical examples show that the auxiliary parameter $c_0$ play an important role on the solution series. According to the $c_0$-curve, we can find a so proper value of $c_0$ that we can get exact results of the problem only by a few truncated series. Furthermore, the results obtained in the numerical examples show that, as a special case of the HAM when $c_0 = -1$, the HPM cannot ensure the convergence of solution series and might give useless results.

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