Non-Abelian Landau-Ginzburg Theory of Ferromagnetic Superconductivity and Photon-Spinon Mixing

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We propose an effective theory of non-Abelian superconductivity, an SU(2)xU(1) extension of the Abelian Landau-Ginzburg theory, which could be viewed as an effective theory of ferromagnetic superconductivity made of spin-up and spin-down doublet Cooper pair. Just like the Abelian Landau-Ginzburg theory it has the U(1) electromagnetic interaction, but the new ingredient is the non-Abelian SU(2) gauge interaction between the spin doublet Cooper pair. A remarkable feature of the theory is the mixing between the photon and the diagonal part of the SU(2) gauge boson. After the mixing it has massless gauge boson (the massless non-Abelian spinon) and massive gauge boson (the massive photon), in addition to the massive off-diagonal gauge bosons (the massive non-Abelian spinons) which induces the spin-flip interaction between the spin up and down components of the Cooper pair. So, unlike the ordinary Landau-Ginzburg theory it has a long range interaction mediated by the massless non-Abelian spinon, which could be responsible for the long range magnetic order and spin waves observed in ferromagnetic superconductors. The theory is characterized by three scales. In addition to the correlation length fixed by the mass of the Higgs field it has two different penetration lengths, the one fixed by the mass of the photon (which generates the well known Meissner effect) and the other fixed by the mass of the off-diagonal spinons (which determines the range of the spin flip interaction). The non-Abelian structure of the theory naturally accommodates new topological objects, the non-Abrikosov quantized spin vortex (as well as the well known Abrikosov vortex) and non-Abelian spin monopole. We discuss the physical implications of the non-Abelian Landau-Ginzburg theory.

Keywords: non-Abelian ferromagnetic superconductivity, two-gap ferromagnetic superconductors, non-Abelian spinon, photon-spinon mixing, long range magnetic order, massive photon, massless spinon, massive charged spinon, non-Abelian spinon vortex, quantized spin flux, non-Abelian spinon monopole

I. INTRODUCTION

The superconductivity in condensed matters has played a fundamental role in the the progress of physics. The physics of the superconductivity is very complicated. But the Landau-Ginzburg theory, as the effective theory of the Abelian superconductivity, has helped us to understand the mechanism of the superconductivity intuitively very much [1]. It has explained the Meissner effect with the supercurrent generated by the Cooper pair in terms of the two scales, the correlation length of the electron pairs and the penetration length of themassive photon. Moreover, it has successfully demonstrated the existence of the quantized magnetic vortex in type II superconductors in terms of the Abrikosov vortex [2, 3].

On the other hand, the advent of two-gap and/or spin-triplet Cooper pair superconductors motivates us to think of the possibility of a non-Abelian superconductivity [4, 7]. This is because the two-gap and/or triplet Cooper pair could naturally be related to non-Abelian stucture. In this circumstance an effective theory of a real non-Abelian gauge theory of superconductivity which could replace the Abelian Landau-Ginzburg theory is needed.

A theory of two-gap superconductivity which has the Abelian (i.e., electromagnetic) gauge interaction but has the global SU(2) symmetry has been discussed before [8–10]. But a genuine non-Abelian gauge theory of two-component superconductivity which has the full SU(2) gauge interaction that can be viewed as the non-Abelian generalization of the Landau-Ginzburg theory appears to be missing. The purpose of this paper is to propose a genuine effective theory of non-Abelian Landau-Ginzburg theory of superconductivity, and discuss the physical contents of such theory.

Consider a ferromagnetic two-component superconductor made of spin-up and spin-down Cooper pairs. Since the Cooper pairs carry the electric charge, they must have the U(1) electromagnetic interaction as in ordinary superconductors. As we have remarked, we could construct a theory of non-Abelian superconductivity with this electromagnetic interaction alone, treating the doublet Cooper pair as a global SU(2) spin doublet [8, 10]. But we could also introduce the gauge interaction with the SU(2) gauge bosons, treating the SU(2) symmetry of as a local (i.e., gauge) symmetry. In this case the di-
agonal gauge boson couples to the spin of the Cooper pair and the off diagonal ones induce the spin flip interaction on the doublet Cooper pair. In this sense the gauge bosons might be called “the non-Abelian gauge spinons” or simply the g spinons, and the diagonal and off-diagonal gauge bosons “the diagonal spinon” and “the off-diagonal spinons”.

It should be mentioned that these “spinons” are only a temporary name that we adopt here for simplicity, in the absence of a better name. Our “spinon” could be related to the conventional spinon in the spin-charge separation of the Cooper pair, but this is not clear at the moment. So in principle this triplet “spinon” should be viewed different from the conventional spinon quasiparticle that we have in the spin-charge separation of electron.

Admittedly, at the moment it is completely unclear if this type of spin gauge interaction exists or not. Probably such gauge interaction may not exist in reality. At the best, the idea of the spin gauge interaction is hypothetical, possible only theoretically. But this does not prevent us to consider such possibility. In fact logically this is a natural generalization of the global SU(2) symmetry we have in the two-gap superconductor.

We emphasize that the above spin gauge interaction is, as far as we understand, the first spin-spin interaction mediated by the messenger bosons. So far the spin-spin interactions in physics have always been treated as the instantaneous action at a distance which has no messenger particle. But this appears to be very strange, considering the fact that in modern physics all interactions are mediated by messenger particles. Consider two spins separated apart which has the spin-spin interaction, for example. To have the interaction, two spins must have a way of communication which could tell the spin state of the other. If so, we need a mediator between the two spins. From this point of view the above spinon gauge interaction may not be so strange after all.

With this understanding we discuss how to construct an effective theory of the two-gap superconductor which can be viewed as a SU(2)xU(1) generalization of the Abelian Landau-Ginzburg theory in the following. As we will see the non-Abelian Landau-Ginzburg theory has very interesting new features. Unlike the Abelian Landau-Ginzburg theory, the theory is characterized by three scales, the correlation length of the electron pairs set by the mass of the Higgs scalar and two penetration lengths of the electromagnetic and spinon fields set by the masses of two massive gauge bosons.

More importantly, in this theory the mixing between the photon and diagonal spinon takes place, which creates a new massless gauge boson and a massive neutral gauge boson. So, unlike the Abelian Landau-Ginzburg theory, the non-Abelian theory has a long range interaction mediated by the massless gauge boson. This is completely unexpected. And the physical content of the theory crucially depends on how we interpret the massless the gauge boson. If we identify the massless particle as the photon, the massive gauge boson becomes “the massive spinon”. However, if we interpret the massive gauge boson as the massive photon, the massless field becomes “the massless spinon” which generates a long range spin-spin interaction.

Fortunately, we can determine which interpretation is correct by experiment. This is because the two interpretations predict different physics. For example, they predict different mass ratio between the neutral gauge boson and charged spinons, so that measuring the mass ratio by experiment we can decide which interpretation is correct.

In this paper we prefer to interpret the massive gauge boson as the massive photon for two reasons. First, this makes the non-Abelian superconductors more like the ordinary one because in this case the massive photon generates the same Meissner effect that we have in ordinary superconductors. Second, the massless spinon can be interpreted to generate the long range magnetic order and spin waves known to exist in the non-Abelian ferromagnetic superconductors [6, 7].

The paper is organized as follows. In Section II we construct the SU(2)xU(1) gauge theory of two-gap ferromagnetic superconductivity which can be viewed as the non-Abelian extension of the Abelian Landau-Ginzburg theory. In Section III we discuss the Abelian decomposition and photon-spinon mixing which produces a massless gauge boson mediating a long range interaction in the theory. In Section IV we present two logically possible interpretations of the theory, and argue that the massless gauge boson should be interpreted as the massless spinon which is responsible for the long range ferromagnetic order and spin interaction. In Section V we discuss the topological objects of the theory, in particular the non-Abrikosov spinon vortex which carries the quantized spin flux and non-Abelian spin monopole dressed by Higgs and massive non-Abelian spinons. In Section VI we compare the theory with the Weinberg-Salam model in high energy physics. Finally, in Section VII we discuss the physical implications of the theory.

II. NON-ABELIAN TWO-COMPONENT LANDAU-GINSBURG THEORY

Before we discuss the non-Abelian Landau-Ginzburg theory, we review the Abelian Landau-Ginzburg theory which describes the ordinary superconductors first. Consider the Landau-Ginzburg Lagrangian made of the complex scalar field \( \phi \) which represents the Cooper pair and the electromagnetic field \( A_\mu \),

\[
\mathcal{L} = -(D_\mu \phi)^*(D_\mu \phi) - V(\phi^* \phi) - \frac{1}{4} F_{\mu \nu}^2,
\]

\[
D_\mu = \partial_\mu - igA_\mu.
\]

(1)
This has the equation of motion

\[ D^2 \phi = \frac{dV}{d\phi}, \]
\[ \partial_\mu F_{\mu\nu} = j_\nu = ig \left[ (D_\mu \phi)^\dagger \phi - \phi^\dagger (D_\mu \phi) \right]. \]  
(2)

With

\[ \phi = \frac{1}{\sqrt{2}} \rho \exp(-i\theta), \]
\[ V(\phi^* \phi) = \frac{\lambda}{2} \left( \phi^* \phi - \frac{\mu^2}{\lambda} \right)^2 = \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2, \]
\[ \rho_0 = \sqrt{2\mu^2/\lambda}, \]

we can express the Lagrangian by

\[ L = -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2 - \frac{1}{4} X_{\mu\nu}^2 - \frac{1}{2} g^2 \rho^2 X_{\mu\nu}^2, \]
\[ X_\mu = A_\mu + \frac{1}{g} \partial_\mu \theta, \quad X_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu. \]
(4)

where \( \rho \) and \( X_\mu \) represent the Higgs scalar of the Cooper pair and the massive photon.

The Lagrangian is characterized by two scales, the Higgs mass \( \sqrt{2}\mu \) which provides the coherence length of the Cooper pair and the photon mass \( \sqrt{2/\lambda}g\mu \) which provides the penetration length of the magnetic field. When the coherence length is smaller than the penetration length (i.e., when \( \sqrt{\lambda} \leq g \)) it describes type II superconductor, but the coherence length becomes larger, it describes the type I superconductor.

The Lagrangian (4) teaches us an important lesson. It describes a theory of massive photon \( X_\mu \) interacting with a neutral scalar field \( \rho \), which acquires the mass by the Higgs mechanism. In the popular view this mass generation of the photon absorbing the phase field \( \theta \) of the complex scalar field \( \phi \) is interpreted as the mass generation by ‘spontaneous symmetry breaking’ of the U(1) gauge symmetry. Our discussion above, however, tells that the mass generation need not be related to any symmetry breaking, spontaneous or not. In fact, we have derived (4) with the simple reparametrization of the fields \( \phi \) and \( A_\mu \), which assures that (4) is mathematically identical to (1). This tells that (4) inherits all mathematical properties of (1), in particular the U(1) gauge symmetry. From this we can conclude that the Higgs mechanism can be explained without any symmetry breaking, spontaneous or not.

To generalize the above Abelian Landau-Ginzburg theory and construct a genuine non-Abelian Landau-Ginzburg theory, we let \( \phi = (\phi_1, \phi_2) \) be the doublet made of the spin-up and spin-down Cooper pairs, and consider the following Lagrangian

\[ L = -|D_\mu \phi|^2 - V(\phi) - \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} \bar{G}_{\mu\nu}^2, \]
\[ D_\mu \phi = (\partial_\mu - ig A_\mu - ig' \sigma \cdot \vec{B}_\mu) \phi = (D_\mu - i g A_\mu) \phi, \]
\[ V(\phi) = \lambda \left( \frac{1}{2} |\phi|^2 - \frac{\mu^2}{\lambda} \right)^2. \]
(5)

Here \( A_\mu \) and \( \vec{B}_\mu \) are the ordinary electromagnetic U(1) gauge potential and the new SU(2) gauge potential which describe the photon and the SU(2) spinons, \( F_{\mu\nu} \) and \( G_{\mu\nu} \) are the corresponding field strengths, \( g \) and \( g' \) are the coupling constants, and \( V(\phi) \) is the self-interaction potential of the Cooper pairs which we assume to have the above quartic form in this paper for simplicity. For the Cooper pairs we have \( g = 2e \), and later we will show how we could fix \( g' \). But for the moment we leave \( g \) and \( g' \) arbitrary.

One might have the following questions on the above Lagrangian. First, why do we introduce the gauge interaction to the doublet \( \phi \)? Of course, we do not have to introduce such interaction, treating the SU(2) symmetry as a global symmetry. In fact, with this we still have very interesting non-Abelian superconductivity. But this case has already been discussed before \cite{[S][T]}. And, if we wish to introduce an interaction to the Cooper pair, the gauge interaction which localizes the global symmetry stands out as the simplest and most natural interaction from theoretical point of view. This is why we have the gauge interaction.

Of course, we do not have to view \( \phi \) as the spin doublet. In this paper we treat it as a spin doublet just to be specific, in which case \( \vec{B}_\mu \) which mediates the interaction between the doublet can be identified as the non-Abelian spinon. But we emphasize that in principle \( \phi \) could be any doublet, and \( \vec{B}_\mu \) could be any gauge boson which mediates the interaction between the doublet.

Obviously the Lagrangian (5) is a simplest non-Abelian generalization of (1). It is made of the spin doublet Cooper pair \( \phi \), massless U(1) gauge boson \( A_\mu \), and three massless SU(2) spinons \( \vec{B}_\mu \). The U(1) interaction is the familiar electromagnetic interaction of the Cooper pairs that we have in the Abelian Landau-Ginzburg theory, and the SU(2) gauge interaction describes the non-Abelian spinon interaction between the spin doublet Cooper pairs. So the only new thing here absent in the Abelian Landau-Ginzburg Lagrangian is the non-Abelian spinon gauge interaction.

We can express the complex doublet \( \phi \) with the scalar field \( \rho \) and the unit doublet \( \xi \) by

\[ \phi = \frac{1}{\sqrt{2}} \rho \xi, \quad (\xi^\dagger \xi = 1). \]
(6)
and have
\[ \mathcal{L} = -\frac{1}{2}(\partial_{\mu}\rho)^2 - \frac{\rho^2}{2}|D_{\mu}\xi|^2 - \lambda(\rho^2 - \rho_0^2)^2 - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}G_{\mu\nu}^2, \]
(7)
where \( \rho_0 = \sqrt{2}\mu/\lambda \) is the magnitude of the vacuum expectation value of the complex doublet field.

From the Lagrangian (7) we have the equation of motion
\[ \partial^{2}\rho - |D_{\mu}\xi|^2 \rho = \lambda(\rho^2 - \rho_0^2)\rho, \]
\[ D^2\xi + 2\frac{\partial_{\rho}\rho}{\rho}D_{\mu}\xi + |D_{\mu}\xi|^2\xi = 0, \]
\[ \partial_{\mu}F_{\mu\nu} = j_{\nu} = -\frac{\mu}{2}\rho^2[(D_{\nu}\xi)\xi - \xi^\dagger(D_{\nu}\xi)], \]
\[ \partial_{\mu}G_{\mu\nu} = \kappa_{\nu} = -\frac{\mu}{4}\rho^2[(D_{\nu}\xi)\xi - \xi^\dagger\partial(D_{\nu}\xi)], \]
(8)
where \( j_{\mu} \) and \( \kappa_{\mu} \) are the U(1) and SU(2) supercurrents.

III. ABELIAN DECOMPOSITION AND PHOTON-SPINON MIXING

To discuss the physical meaning of the above Lagrangian we should understand the skeleton structure of the Lagrangian first. For this we need the Abelian decomposition of the Lagrangian. All non-Abelian gauge theory has the Abelian (diagonal) part, but it has generally been believed that the separation of the Abelian part from the non-Abelian part is not possible because the two parts are intimately connected by gauge transformation. This is not true, however, and the Abelian decomposition tells us how to separate them gauge independently [11, 12].

Consider the SU(2) gauge field \( \tilde{B}_{\mu} \) first. To make the Abelian decomposition we choose an arbitrary direction \( \hat{n} \) in SU(2) space to be the Abelian direction at each space-time point, and impose the magnetic symmetry on the gauge potential \( \tilde{B}_{\mu} \),
\[ D_{\mu}\hat{n} = 0 \hspace{1cm} (\hat{n}^2 = 1). \]
(9)
From this we have
\[ \tilde{B}_{\mu} \rightarrow \hat{B}_{\mu} = \tilde{B}_{\mu} + \tilde{C}_{\mu}, \]
\[ \tilde{B}_{\mu} = B_{\mu}\hat{n} \hspace{1cm} (B_{\mu} = \hat{n} \cdot \tilde{B}_{\mu}), \hspace{1cm} \tilde{C}_{\mu} = -\frac{1}{g}\hat{n} \times \partial_{\mu}\hat{n}. \]
(10)
This is the Abelian projection which projects out the restricted potential \( B_{\mu} \) which describes the Abelian subdynamics of the non-Abelian gauge theory [11, 12]. Notice that the restricted potential is precisely the potential which leaves \( \hat{n} \) invariant under parallel transport (which makes \( \hat{n} \) a covariant constant). Remarkably it has a dual structure, made of two potentials \( \hat{B}_{\mu} \) and \( \hat{C}_{\mu} \).

With this we obtain the gauge independent Abelian decomposition of \( \hat{B}_{\mu} \), adding the valence part \( \hat{W}_{\mu} \) which was excluded by the isometry. Introducing a right-handed orthonormal SU(2) basis \( (\hat{n}_1, \hat{n}_2, \hat{n}_3 = \hat{n}) \), we can express \( \hat{B}_{\mu} \) by [11, 12]
\[ \hat{B}_{\mu} = \hat{B}_{\mu} + \hat{W}_{\mu}, \hspace{1cm} \hat{W}_{\mu} = \hat{W}_{\mu}^1\hat{n}_1 + \hat{W}_{\mu}^2\hat{n}_2. \]
(11)
Under the (infinitesimal) gauge transformation
\[ \delta\hat{B}_{\mu} = \frac{1}{g}D_{\mu}\hat{\alpha}, \hspace{1cm} \delta\hat{n} = -\hat{\alpha} \times \hat{n}, \]
(12)
we have
\[ \delta\hat{B}_{\mu} = \frac{1}{g}\hat{n} \cdot \partial_{\mu}\hat{\alpha}, \hspace{1cm} \delta\hat{W}_{\mu} = -\hat{\alpha} \times \hat{W}_{\mu}. \]
(13)
This tells that \( \hat{B}_{\mu} \) by itself describes an SU(2) connection which enjoys the full SU(2) gauge degrees of freedom. Furthermore the valence potential \( \hat{W}_{\mu} \) forms a gauge covariant vector field. But what is really remarkable is that this decomposition is gauge independent. Once \( \hat{n} \) is chosen, the decomposition follows automatically, regardless of the choice of gauge.

The restricted field strength \( \hat{G}_{\mu\nu} \) inherits the dual structure of \( \hat{B}_{\mu} \), which can also be described by two Abelian potentials \( B_{\mu} \) and \( C_{\mu} \),
\[ \hat{G}_{\mu\nu} = \partial_{\mu}\hat{B}_{\nu} - \partial_{\nu}\hat{B}_{\mu} + g\hat{B}_{\mu} \times \hat{B}_{\nu} = G_{\mu\nu}^1\hat{n}, \]
\[ G_{\mu\nu} = G_{\mu\nu}^1 + H_{\mu\nu} = \partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu}, \]
\[ H_{\mu\nu} = -\frac{1}{g}\hat{n} \cdot (\partial_{\mu}\hat{n} \times \partial_{\nu}\hat{n}) = \partial_{\mu}C_{\nu} - \partial_{\nu}C_{\mu}, \]
\[ C_{\mu} = -\frac{1}{g}\hat{n}_1 \cdot \partial_{\mu}\hat{n}_2. \]
\[ B_{\mu} = B_{\mu} + C_{\mu}. \]
(14)
Notice that the potential \( C_{\mu} \) for \( H_{\mu\nu} \) is determined uniquely up to the U(1) gauge freedom which leaves \( \hat{n} \) invariant.

To understand the meaning of the dual structure of \( \hat{B}_{\mu} \) and \( \hat{G}_{\mu\nu} \), notice that \( \hat{B}_{\mu} \) describes the potential parallel to the Abelian direction \( \hat{n} \). So it describes the non-topological Maxwell part of \( \hat{B}_{\mu} \). To understand the meaning of \( \hat{C}_{\mu} \), let
\[ \xi = \begin{pmatrix} \sin\alpha/2 \exp(-i\beta) \\ -\cos\alpha/2 \end{pmatrix}, \]
\[ \hat{n} = -\xi^\dagger\hat{\xi} = \begin{pmatrix} \sin\alpha \cos\beta \\ \sin\alpha \sin\beta \cos\alpha \end{pmatrix}. \]
(15)
With this we have

\[ \tilde{C}_\mu = -\frac{1}{g'} \tilde{n} \times \partial_\mu \tilde{n} = \frac{1}{g'} (\tilde{n}_1 \sin \alpha \partial_\mu \beta - \tilde{n}_2 \partial_\mu \alpha), \]

\[ \tilde{n}_1 = \begin{pmatrix} \cos \alpha \cos \beta \\ \cos \alpha \sin \beta \\ -\sin \alpha \end{pmatrix}, \quad \tilde{n}_2 = \begin{pmatrix} -\sin \beta \\ \cos \beta \\ 0 \end{pmatrix}, \]

\[ C_\mu = -\frac{1}{g'} (1 - \cos \alpha) \partial_\mu \beta. \quad (16) \]

This tells that, when \( \tilde{n} = \tilde{r} \), the potential \( \tilde{C}_\mu \) describes the Wu-Yang monopole and \( C_\mu \) describes the Dirac monopole \[13, 15\]. So \( \tilde{C}_\mu \) describes the topological potential \( \tilde{C}_\mu \) which describes the non-Abelian monopole. This justifies us to call \( B_\mu \) and \( C_\mu \) the electric and magnetic potential.

With \( \tilde{r} \) we have

\[ \tilde{G}_{\mu\nu} = \tilde{G}_{\mu\nu} + \tilde{D}_\mu \tilde{W}_\nu - \tilde{D}_\nu \tilde{W}_\mu + g' \tilde{W}_\mu \times \tilde{W}_\nu, \]

\[ \tilde{D}_\mu = \partial_\mu + g' \tilde{B}_\mu \times, \quad (17) \]

so that the SU(2) gauge theory is decomposed to the restricted part and the valence part gauge independently,

\[ \mathcal{L}_{SU(2)} = -\frac{1}{4} \tilde{G}_{\mu\nu}^2 = -\frac{1}{4} G_{\mu\nu}^2 - \frac{1}{4} (\tilde{D}_\mu \tilde{W}_\nu - \tilde{D}_\nu \tilde{W}_\mu)^2 \]

\[ -\frac{g'}{2} G_{\mu\nu} \cdot (\tilde{W}_\mu \times \tilde{W}_\nu) - \frac{g'^2}{4} (\tilde{W}_\mu \times \tilde{W}_\nu)^2. \quad (18) \]

This is the Abelian decomposition of the SU(2) gauge theory known as the Cho decomposition, Cho-Duan-Ge (CDG) decomposition, or Cho-Faddeev-Niemi (CFN) decomposition \[16, 19\].

The Abelian decomposition reveals important hidden structures of non-Abelian gauge theory. First, it tells that we can construct the restricted gauge theory with the restricted potential \( \tilde{B}_\mu \) alone,

\[ \mathcal{L}_R = -\frac{1}{4} \tilde{G}_{\mu\nu}^2, \quad (19) \]

which is much simpler than the Yang-Mills theory but has the full non-Abelian gauge invariance. Second, it tells that the non-Abelian gauge theory can be viewed as the restricted gauge theory which has the valence potential \( \tilde{W}_\mu \) as a gauge covariant source. This means that we can always remove the valence part (if we like) in non-Abelian gauge theory self consistently, without compromising the full non-Abelian gauge invariance.

Moreover, the Abelian decomposition allows us to put \( \tilde{n} = \tilde{r} \) into the Abelian form gauge independently \[11, 12\]. Indeed with

\[ W_\mu = \frac{1}{\sqrt{2}} (W_\mu^1 + i W_\mu^2), \quad (20) \]

we have

\[ \mathcal{L}_{SU(2)} = -\frac{1}{4} G_{\mu\nu}^2 - \frac{1}{2} |D_\mu W_\nu - D_\nu W_\mu|^2 \]

\[ + ig' G'_\mu W_\mu^* W_\nu + \frac{g'^2}{4} (W_\mu^* W_\nu - W_\nu^* W_\mu)^2, \]

\[ D_\mu' = \partial_\mu + i g' B'_\mu. \quad (21) \]

One might wonder how the non-Abelian structure disappears in this Abelianization. Actually the non-Abelian structure has not disappeared but hidden. To see this notice that the potential \( B'_\mu \) in the Abelian formalism is dual, given by the sum of the electric and magnetic potentials \( B_\mu \) and \( C_\mu \). Clearly \( C_\mu \) represents the topological degrees of the non-Abelian symmetry which does not exist in the naive Abelianization that one obtains by fixing the gauge, choosing \( \tilde{n} = (0, 0, 1) \) \[11, 12\]. And it plays the crucial role to retain the full non-Abelian gauge symmetry in the Abelianized Lagrangian.

With the Abelian decomposition we have

\[ D_\mu \xi = \left[ -\frac{g}{2} A_\mu - i g' \left( B'_\mu n + \tilde{W}_\mu \right) \cdot \tilde{\sigma} \right] \xi, \]

\[ |D_\mu \xi|^2 = \frac{1}{8} (-g A_\mu + g' B'_\mu)^2 + \frac{g'^2}{4} \tilde{W}_\mu^2. \quad (22) \]

Using this we can remove the Cooper pair doublet completely from \( \tilde{r} \) and “abelianize” it gauge independently \[11, 12\].

\[ \mathcal{L} = -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2 \]

\[ -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} C_{\mu\nu}^2 - \frac{1}{2} |D'_\mu W_\nu - D'_\nu W_\mu|^2 \]

\[ -\frac{g^2}{8} ((-g A_\mu + g' B'_\mu)^2 + 2 g^2 W_\mu^\ast W_\mu) \]

\[ + ig' G'_\mu W_\mu^* W_\nu + \frac{g'^2}{4} (W_\mu^* W_\nu - W_\nu^* W_\mu)^2, \]

\[ D'_\mu = \partial_\mu + i g' B'_\mu. \quad (23) \]

This tells that the Lagrangian is made of two Abelian gauge fields \( A_\mu \) and \( B'_\mu \). Notice, however, that the two Abelian gauge fields in the Lagrangian are not mass eigenstates. To express them in terms of mass eigenstates, we introduce the mixing with

\[ \begin{pmatrix} \tilde{A}_\mu \\ \tilde{Z}_\mu \end{pmatrix} = \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} g' & g \\ -g & g' \end{pmatrix} \begin{pmatrix} A_\mu \\ B'_\mu \end{pmatrix} \]

\[ = \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix} \begin{pmatrix} A_\mu \\ B'_\mu \end{pmatrix}, \quad (24) \]

where \( \omega \) is the mixing angle. With this we can express
the Lagrangian \([5]\) in the following form
\[
\mathcal{L} = -\frac{1}{2}(\partial_{\mu}\rho)^2 - \frac{\lambda}{8}(\rho^2 - \rho_0^2)^2 - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}Z_{\mu\nu}^2
- \frac{1}{2}i(\bar{D}_\mu + ig g'\rho \bar{Z}_\mu)W_\nu - (\bar{D}_\nu + ig g'\rho Z_\mu)W_\mu)]^2
- \frac{\rho^2}{4}(g^2W^*_\mu W_\mu + \frac{g^2}{2}Z^2_\mu)
+ i\bar{e}(\bar{F}_{\mu\nu} + \frac{g g'}{g}Z_{\mu\nu})W^*_\nu W_\nu
+ \frac{g^2}{4}(W^*_\mu W_\mu - W^*_\nu W_\nu)^2,
\]
where
\[
\bar{F}_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \quad Z_{\mu\nu} = \partial_{\mu}Z_{\nu} - \partial_{\nu}Z_{\mu},
\]
\[
\bar{D}_\mu = \partial_{\mu} + ig A_{\mu}, \quad \bar{e} = \frac{gg'}{\sqrt{g^2 + g'^2}} = g'\sin\omega = g\cos\omega.
\]
This is the Abelian decomposition, or the Abelianization, of the non-Abelian Landau-Ginzburg Lagrangian \([5]\).

The Abelianized Lagrangian tells that the non-Abelian Landau-Ginzburg theory is made of Higgs scalar \(\rho\), massless gauge boson \(\bar{A}_\mu\), massive neutral gauge boson \(Z_\mu\), and massive complex spinon \(W_\mu\) whose masses are given by
\[
M_H = \sqrt{\lambda}\rho_0, \quad M_Z = (\sqrt{g^2 + g'^2}/2)\rho_0, \quad M_W = (g'/2)\rho_0.
\]
So, unlike the Abelian Landau-Ginzburg theory, it has three mass scales. Moreover, the interaction of the theory becomes simpler in terms of these mass eigenstates.

As importantly, it tells that (just as in the Abelian Landau-Ginzburg theory) here again we have the mass generation (i.e., the Higgs mechanism) for \(W_\mu\) and \(Z_\mu\) without any symmetry breaking, spontaneous or not. The popular interpretation of the Higgs mechanism is that the gauge bosons acquire the mass by the spontaneous symmetry breaking of SU(2)xU(1) down to the unbroken \(\bar{U}(1)\) through the Higgs vacuum. And this unbroken \(\bar{U}(1)\) is supposed to generates the long range interaction. But here we have derived the Lagrangian \([25]\) without any symmetry breaking. All we did in this Abelianization is the reparameterization of the fields which does not involve any symmetry breaking.

In fact, \([5]\) and \([25]\) are mathematically identical, so the Lagrangian \([25]\) retains all mathematical properties of the original Lagrangian. In particular it retains the full SU(2)xU(1) gauge symmetry of \([5]\), in spite of the appearance. Indeed, in \([25]\) the symmetry is not explicit but hidden, and we can easily confirm the existence of the symmetry \([20]\). This shows that the mass generation (i.e., the Higgs mechanism) actually can take place without any spontaneous symmetry breaking.

\[\text{IV. PHYSICAL INTERPRETATION OF THE NON-ABELIAN LANDAU-GINZBURG THEORY}\]

In spite of the fact that the two Lagrangians \([5]\) and \([25]\) are mathematically identical, they are completely different from the physical point of view. The Abelianized Lagrangian reveals important features which are not evident in \([5]\). The most remarkable feature of the non-Abelian Landau-Ginzburg theory is that it has the massless gauge boson \(A_\mu\) after the mixing \([24]\), which means that it has a long range interaction mediated by the massless gauge boson. This is completely unexpected, because this type of long range interaction is absent in the Abelian Landau-Ginzburg theory. This is remarkable.

To understand the physical implication of this, remember that the U(1) gauge boson \(A_\mu\) is introduced to describe the photon which couples to the electromagnetic charge \(g = 2e\) of the Cooper pairs. But the mixing tells that this \(A_\mu\) can not be viewed a physical state, and thus can not be identified as the photon. In stead, we have the new massless gauge boson \(\bar{A}_\mu\). How can we interpret this?

In this circumstance we have two logically possible choices. The first is that we interpret \(\bar{A}_\mu\) as the real electromagnetic potential which describes the photon. On the surface this looks natural since the photon is the only known massless gauge boson in nature. In this case \(Z_\mu\) should be identified as the massive neutral spinon. But we have to take this interpretation with a grain of salt. This is because this tells that, unlike the Abelian superconductors, the non-Abelian superconductor keeps photon massless. This implies that the photon has nothing to do with the non-Abelian Meissner effect.

The other choice is that, just as in ordinary superconductors we interpret the massive \(Z_\mu\) as the massive photon. In this case the massive photon generates the same Meissner effect we have in ordinary superconductors in the non-Abelian Landau-Ginzburg theory. But in this case \(A_\mu\) should be interpreted as the massless spinon which generates a long range force on the spins. This is remarkable.

A priori, it is hard to tell which interpretation is correct. But depending on which interpretation we choose, the physics of the two-gap superconductor becomes remarkably different. If we choose the first interpretation, the coupling constant in front of \(A_\mu\) should be \(e\), so that we must have
\[
\bar{e} = \frac{gg'}{\sqrt{g^2 + g'^2}} = e,
\]
So, with \(g = 2e\) we can fix the mixing angle and \(g'\) uniquely. From \([26]\) we have
\[
g' = \frac{2}{\sqrt{3}}e, \quad \tan\omega = \sqrt{3} \quad (\omega = \frac{\pi}{3}).
\]
This is remarkable, because \( g' \) was introduced as a free coupling constant at the beginning.

In this case the SU(2) spinons \( \vec{B}_\mu \) transforms to the neutral spinon \( Z_\mu \) and the complex spinon \( W_\mu \) whose masses are given by

\[
M_W = \frac{g'}{2} \rho_0 = \frac{1}{\sqrt{3}} e \rho_0, \\
M_Z = \frac{g'^2 + g^2}{2} \rho_0 = \frac{8}{\sqrt{3}} e \rho_0 = 8M_W.
\]  

(30)

So the Lagrangian \([5]\) can be interpreted to describe a non-Abelian superconductivity in which two massive \( Z_\mu \) spinon and \( W_\mu \) spinon, whose masses describe the penetration lengths of the diagonal and off diagonal spin interactions. And \([30]\) tells that the penetration length of the diagonal spin interaction is less than the penetration length of the off diagonal spin interaction by the factor 1/8.

If we adopt the second interpretation, however, the coupling constant in front of \( Z_\mu \) in \([25]\) should be identified by the real electric charge,

\[
e = \frac{g'}{g} \bar{e} = \frac{g'^2}{\sqrt{g^2 + g'^2}},
\]

so that (with \( g = 2e \)) we have

\[
g' = \frac{\sqrt{1 + \sqrt{17}}}{\sqrt{2}} e, \quad \tan \omega = \frac{2\sqrt{2}}{1 + \sqrt{17}}, \\
\bar{e} = \frac{2\sqrt{2}}{\sqrt{1 + \sqrt{17}}} e = \tan \omega \times e.
\]

(32)

From this we have

\[
M_W = \frac{g'}{2} \rho_0 = \frac{\sqrt{1 + \sqrt{17}}}{2\sqrt{2}} e \rho_0, \\
M_Z = \frac{g'^2 + g^2}{2} \rho_0 = \frac{\sqrt{9 + \sqrt{17}}}{2\sqrt{2}} e \rho_0 \simeq 1.6M_W.
\]

(33)

This should be compared with \([30]\). In this case the massive photon \( Z_\mu \) generates the well known Meissner effect, and the penetration length of the off diagonal spin interaction is more than the penetration length of the magnetic field by the factor 1.6. More importantly, the massless spinon generates a long range spin interaction in the non-Abelian superconductor.

One can ask if there is any way to tell which interpretation is realistic. Fortunately, we could answer the question by experiment, because the two interpretations predict different physics. For example, they predict different mass ratio between \( M_W \) and \( M_Z \), so that measuring the two masses experimentally we could tell which interpretation is realistic. Another difference is the Meissner effect. In the first interpretation the Meissner effect comes from the spinons, so that it is different from the Meissner effect in ordinary superconductors. But in the second interpretation it comes from the massive photon and off diagonal spinon. In this case the magnetic levitation, the well known property of the Abelian superconductors, becomes possible in the non-Abelian superconductor. So, checking this magnetic levitation experimentally, we could tell which interpretation is correct.

Having said this, we prefer the second interpretation for two reasons. First, this interpretation makes the non-Abelian superconductor more like the ordinary ones. This must be clear because in this case the photon becomes massive and generates the well known Meissner effect \([9]\). Second, the massless spinon is precisely what we need to explain the long range magnetic order and spin waves observed in ferromagnetic superconductors \([\text{?}]\). But, of course, we need more experimental evidences to make sure that this interpretation is correct.

Independent of which interpretation we choose, the \( W \) spinon has (not only the spin but also) the electric charge, because it couples to both \( \vec{A}_\mu \) and \( Z_\mu \). So we could also call the \( W \) spinon the charged spinon. Moreover, it must be clear that the non-Abelian Landau-Ginzburg theory has two penetration lengths set by the masses of \( Z_\mu \) and \( W_\mu \). So the off diagonal spinon \( W_\mu \) can be viewed to generate the non-Abelian Meissner effect (the screening of the spin flip interaction) which does not exist in the ordinary superconductors.

Before we leave this section one might wonder if we can simplify the above non-Abelian Landau-Ginzburg theory and obtain the Abelian Landau-Ginzburg theory. This is possible. To show this, notice that we can always switch off \( W_\mu \) in the Lagrangian \([25]\) whenever necessary. In this case, it reduces to

\[
\mathcal{L}_{AZ} = -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2 - \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} Z_{\mu\nu}^2 - \frac{g'^2 + g^2}{8} \rho^2 Z_{\mu\nu}^2.
\]

(34)

Moreover, with \( F_{\mu\nu} = 0 \) we have

\[
\mathcal{L}_Z = -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2 - \frac{1}{4} Z_{\mu\nu}^2 - \frac{g'^2 + g^2}{8} \rho^2 Z_{\mu\nu}^2.
\]

(35)

This is nothing but the Abelian Landau-Ginzburg Lagrangian in which the massive neutral gauge boson \( Z_\mu \) plays the role of the massive photon.

This suggests that \([5]\) is a logical extension of Abelian Landau-Ginzburg theory to a non-Abelian theory. Moreover, this suggests that there is non-Abelian extension of the Abelian Landau-Ginzburg theory which is simpler than \([5]\).
V. TOPOLOGICAL OBJECTS IN NON-ABELIAN SUPERCONDUCTOR

It is well known that the Abelian Landau-Ginzburg theory has the topological Abrikosov-Nielsen-Olesen (ANO) vortex which carries the quantized magnetic flux. This implies that we could also have similar vortex in the non-Abelian superconductor. In fact, the SU(2)xU(1) gauge symmetry of the non-Abelian Landau-Ginsburg theory has more topology so that it allows more topological objects than the Abelian Landau-Ginzburg theory. For example it has two \( \pi_1(S^1) \) topology coming from the U(1) and the Abelian subgroup of SU(2), which allows two different vortices. Moreover, it has \( \pi_2(S^2) \) topology which allows the monopole. This is because \( \xi \) in (5) can be viewed as a \( CP(1) \) field. In the following we show that the theory has not only different types of magnetic vortices but also the non-Abelian monopole.

To see how the topological vortex comes about, we first review the Abrikosov vortex in type II superconductors. Consider the Abelian Landau-Ginzburg theory \[ \text{AN} \] and let us choose the vortex ansatz in the cylindrical coordinates \((r, \varphi, z)\),

\[
\phi = \frac{1}{\sqrt{2}} \rho(r) \exp \left( - i n \varphi \right),
\]

\[
A_\mu = \frac{n}{g} A(r) \partial_\mu \varphi,
\]

where \( n \) is the integer which represents the winding number of \( \pi_1(S^1) \) of the phase angle \( \varphi \) of the charged scalar field \( \phi \). With this we have the equation of motion

\[
\dot{\rho} + \frac{1}{r} \rho - \frac{n^2}{r^2} (A - 1)^2 \rho = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho,
\]

\[
\ddot{A} - \frac{1}{r} \dot{A} - g^2 \rho^2 (A - 1) = 0.
\]

This has a singular solution

\[
\rho = \rho_0, \quad A = 1,
\]

which carries the quantized magnetic flux

\[
\Phi = \oint_{r = \infty} A_\mu dx^\mu = \frac{2 \pi n}{g}.
\]

This confirms that the ansatz is able to describe two types of string.

Moreover, we can regularize this solution. Imposing the boundary condition

\[
\rho(0) = 0, \quad \rho(\infty) = \rho_0, \quad A(0) = 0, \quad A(\infty) = 1,
\]

we obtain the Abrikosov vortex which has the exponential damping asymptotically,

\[
\rho \simeq \rho_0 - \sqrt{\frac{\pi}{2 M_\rho}} \exp(-M_\rho r), \quad M_\rho = \sqrt{2} \mu
\]

\[
A \simeq 1 + \sqrt{\frac{\pi \rho}{2 M_\gamma}} \exp(-M_\gamma r), \quad M_\gamma = g \rho_0.
\]

which carries the quantized magnetic flux \( 2 \pi n/g \).

The Abrikosov vortex clearly shows how the Meissner effect works. When the coherence length of the Higgs field is smaller than the penetration length of the magnetic field, the magnetic field (i.e., the magnetic vortex) can penetrate the superconductor, but asymptotically the magnetic field is confined by the supercurrent made of the Cooper pair.

Now, we discuss two different vortices, the \( \bar{A} \) vortex and \( Z \) vortex, in the non-Abelian superconductor. To do this, we consider the following string ansatz in the cylindrical coordinates \((r, \varphi, z)\),

\[
\phi = \frac{1}{\sqrt{2}} \rho(r) \xi, \quad \xi = \frac{1}{\sqrt{2}} \left( - \exp(-i n \varphi) \right),
\]

\[
A_\mu = \frac{m}{g} A(r) \partial_\mu \varphi,
\]

\[
B_\mu = \frac{n}{g} (B(r) + 1) \partial_\mu \varphi \hat{n}
\]

\[
+ \frac{1}{g^2} \left( f(r) - 1 \right) \hat{\n} \times \partial_\mu \hat{n},
\]

where \( m \) and \( n \) are integers which represents the winding numbers of the \( \pi_1(S^1) \) topology of U(1) and U(1) subgroup of SU(2).

In terms of the physical field, we can express the ansatz by

\[
\rho = \rho(r),
\]

\[
\bar{A}_\mu = \bar{e} \left( \frac{m}{g^2} A + \frac{n}{g^2} B \right) \partial_\mu \varphi,
\]

\[
W_\mu = -\frac{n}{g^2} f \exp(in \varphi) \partial_\mu \varphi,
\]

\[
Z_\mu = -\frac{1}{g^2} \left( m A - n B \right) \partial_\mu \varphi.
\]

So, when \( mA = nB \) the ansatz describes the \( \bar{A} \) vortex

\[
\bar{A}_\mu = \frac{n}{e} B \partial_\mu \varphi, \quad Z_\mu = 0.
\]

But when \( mA/g^2 + nB/g^2 = 0 \), the ansatz describes the \( Z \) vortex

\[
\bar{A}_\mu = 0, \quad Z_\mu = -m \sqrt{g^2 + g^2} A \partial_\mu \varphi.
\]

This confirms that the ansatz is able to describe two types of string.

Consider the \( \bar{A} \) vortex first. With \( B = -1 \) the equation of motion \( [\bar{S}] \) reduces to

\[
\ddot{\rho} + \frac{\dot{\rho}^2}{r} - \frac{n^2 f^2}{4} \rho = \frac{\lambda}{2} \rho^2 - \rho_0^2,
\]

\[
\frac{n}{g} \left( f - \frac{f^2}{4} + \frac{1}{4} g^2 \rho \right) = 0.
\]
Clearly this has the naked singular electromagnetic string solution given by
\[ \rho = \rho_0, \quad f = 0, \]
\[ \bar{A}_\mu = -\frac{n}{e} \partial_\mu \varphi, \tag{47} \]
which has the quantized spin flux generated by \( \bar{A}_\mu \),
\[ \Phi = \int_{r=\infty} \bar{A}_\mu dx^\mu = -\frac{2\pi n}{e}. \tag{48} \]
along the z-axis.

We can solve with the boundary condition
\[ \rho(0) = 0, \quad \rho(\infty) = \rho_0, \]
\[ f(0) = 1, \quad f(\infty) = 0, \tag{49} \]
and find the singular \( \bar{A}_\mu \) vortex dressed by the Higgs and W spinon. The Higgs and W dressing of the solution is shown in Fig. 1. At the origin \( \rho \) and \( f \) can be expressed by
\[ \rho \simeq r^\delta (a_1 + a_2 r + ...), \quad \delta = |n|/2, \]
\[ f \simeq 1 + b_1 r^2 + ... . \tag{50} \]
Asymptotically they have the exponential damping set by the Higgs and W spinon mass
\[ \rho \simeq \rho_0 - \sqrt{\frac{\pi}{2 M_H r}} \exp(-M_H r) + ..., \]
\[ f \simeq \sqrt{\frac{\pi r}{2 M_W}} \exp(-M_W r) + .... \tag{51} \]
Clearly this string singularity is topological, whose quantized spin flux represents the non-trivial winding number of \( \pi_1(S^3) \). We can call this the spin vortex.

One might wonder if the string singularity of the vortex can be removed by the gauge transformation
\[ \bar{A}_\mu \to \bar{A}'_\mu = \bar{A}_\mu + \frac{n}{e} \partial_\mu \varphi = 0, \]
\[ W_\mu \to W'_\mu = \exp(-im\varphi)W_\mu = -\frac{n}{g} \frac{f}{\sqrt{2}} \partial_\mu \varphi . \tag{52} \]
This, of course, is a singular gauge transformation which changes the \( \pi_1(S^3) \) topology of the string. But mathematically there is nothing wrong with this gauge transformation, in the sense that it keeps \( \bar{A}_\mu' \) and \( W'_\mu \) as a qualified solution after the gauge transformation. This tells that the theory has a regular vortex solution made of Higgs and W spinon described by Fig. 1 which does not carry any \( \bar{A}_\mu \) singularity. This is the W vortex.

On might wonder how such a solution is possible. The reason is that, in the absence of \( \bar{A}_\mu \) and \( Z_\mu \), the Lagrangian \( \langle 25 \rangle \) reduces to the Landau-Ginzburg theory with the gauge potential \( W'_\mu \) when \( W_\mu \) becomes \( W'_\mu \). So it must have the Abrikosov vortex solution, which is exactly the solution discussed above. In fact we can easily see that the equation \( \langle 46 \rangle \) is identical to the equation for the Abrikosov vortex. This tells that the singular \( \bar{A}_\mu \) vortex solution dressed by the Higgs and W spinon which has the quantized flux \( \langle 48 \rangle \) is nothing but the W vortex which has the topological singular \( \bar{A}_\mu \) string at the core.

Now, consider the \( Z \) vortex. With \( f = 0 \) the equation of reduces to
\[ \dot{\rho} + \frac{\dot{\rho}}{r} - \frac{m^2}{4} \frac{Z^2}{r^2} \rho = \frac{\lambda}{2} \rho (\rho^2 - \rho_0^2), \]
\[ \dot{Z} - \frac{\dot{Z}}{r} - \frac{g^2 + g'^2}{4} \rho^2 Z = 0. \tag{53} \]
where we have put
\[ Z_\mu = \frac{m}{\sqrt{g^2 + g'^2}} Z \partial_\mu \varphi = -m \sqrt{\frac{g^2 + g'^2}{g^2}} A \partial_\mu \varphi, \]
\[ Z = -\frac{g^2 + g'^2}{g^2} A. \tag{54} \]
Obviously this is mathematically identical to the equation \( \langle 40 \rangle \) which describes the well known Abrikosov vortex \( \langle 23 \rangle \).

The above exercise tells that there are two types of quantized magnetic vortices in the non-Abelian superconductors. This is because the SU(2)xU(1) gauge symmetry in the non-Abelian superconductor has two different \( \pi_1(S^3) \) topology.

Now, we can also show that the non-Abelian superconductor admits the monopole. This is not surprizing, given the fact that QED has the Dirac monopole and SU(2) gauge theory has the Wu-Yang monopole \( \langle 19 \rangle \langle 15 \rangle \). To show this we choose the following monopole ansatz in
the ansatz (55) can be expressed by

\[ \rho = \rho(r), \quad \xi = i \left( \sin(\theta/2) e^{-i\varphi} \right), \]

\[ A_\mu = -\frac{1}{g} (1 - \cos \theta) \partial_\mu \varphi, \]

\[ \bar{B}_\mu = \frac{1}{g'} (f(r) - 1) \hat{r} \times \partial_\mu \hat{r}. \] (55)

Notice that \( A_\mu \) describes the Dirac-type Abelian monopole and \( \bar{B}_\mu \) describes the 'tHooft-Polyakov type monopole \[13, 20\]. So the ansatz is a hybrid between Dirac and 'tHooft-Polyakov. In terms of the physical fields the ansatz (55) can be expressed by

\[ \bar{A}_\mu = -\frac{1}{\bar{e}} (1 - \cos \theta) \partial_\mu \varphi, \quad Z_\mu = 0, \]

\[ W_\mu = \frac{i}{g'} \frac{f(r)}{\sqrt{2}} e^{i\varphi} (\partial_\mu \theta + i \sin \theta \partial_\mu \varphi). \] (56)

This clearly shows that the ansatz is for a real monopole.

The ansatz reduces the equations of motion to

\[ \ddot{\rho} + \frac{2}{\rho} \dot{\rho} - \frac{f^2}{2 r^2} \rho = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho, \]

\[ \ddot{f} - \frac{f^2 - 1}{r^2} f - \frac{g'^2}{4} \rho^2 f = 0. \] (57)

Obviously this has a trivial solution

\[ \rho = \rho_0 = \sqrt{2 \mu^2 / \bar{e}}, \quad f = 0, \]

\[ \bar{A}_\mu = -\frac{1}{\bar{e}} (1 - \cos \theta) \partial_\mu \varphi, \] (58)

which describes the Dirac type point monopole whose monopole charge is given by \( 4\pi/\bar{e} \), not \( 2\pi/\bar{e} \). Remarkably, this monopole naturally admits a non-trivial dressing of Higgs and W spinon. Indeed we can integrate (57) with the boundary condition

\[ \rho(0) = 0, \quad \rho(\infty) = \rho_0, \quad f(0) = 1, \quad f(\infty) = 0. \] (59)

and find the dressed monopole solution shown in Fig. 2.

So, mathematically this monopole can be viewed as a hybrid between the Dirac monopole and the 'tHooft-Polyakov monopole. But physically we could interpret it the monopole made of the massless spinon field which carries the spin charge \( 4\pi/\bar{e} \), which has nontrivial dressing of the Higgs and massive charged spinons.

VI. COMPARISON WITH WEINBERG-SALAM MODEL OF ELECTROWEAK THEORY

One might have noticed that the non-Abelian Landau-Ginzburg theory discussed in this paper is mathematically very similar to the Weinberg-Salam theory (known as the standard model) in high energy physics which unifies the electromagnetic and weak interactions \[21\]. To understand this notice that the (bosonic part) Weinberg-Salam Lagrangian is given by

\[ \mathcal{L}_{WS} = -|D_\mu \phi|^2 - \frac{\lambda}{2} (|\phi|^2 - \mu^2)^2 - \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} G_{\mu\nu}^2, \]

\[ D_\mu \phi = (\partial_\mu - i\frac{g}{2} g' \cdot \bar{A}_\mu - i\frac{g'}{2} B_\mu) \phi \]

\[ = D_\mu \phi - i\frac{g'}{2} B_\mu \phi, \] (60)

where \( \phi \) is the Higgs doublet, \( \bar{A}_\mu, \bar{F}_{\mu\nu} \) and \( G_{\mu\nu} \) are the gauge fields of SU(2) and the hypercharge U(1), and \( D_\mu \) is the covariant derivative of SU(2). Now, it must be clear that, if we replace \( \bar{A}_\mu \) to \( \bar{B}_\mu \), \( B_\mu \) to \( A_\mu \), and \( g \) to \( g' \) and vise versa, this Lagrangian becomes our Lagrangian shown in \[5\]. So the two Lagrangians become mathematically identical.

This means that, formally there is exactly one to one correspondence between the two Lagrangians \[5\] and \[60\]. The mixing angle \( \omega \) in \[24\] corresponds to the Weinberg angle, the massive Z boson and W spinon correspond to the Z boson and W boson in the standard model.

Moreover, the \( \bar{A} \) and Z vortices discussed above correspond to the electromagnetic and Z string in the standard model \[22, 23\]. And the above monopole corresponds exactly to the electroweak monopole known as the Cho-Maison monopole in the standard model \[24, 27\].

The similarity, however, stops here. From the physical point of view the two Lagrangians describe completely different physics. The standard model which unifies the electromagnetic interaction with the weak interaction is a fundamental theory of nature. So the coupling constants \( g \) and \( g' \) in the standard model represent the fundamental constants of nature, which determine the Weinberg angle. In particular, \( \bar{e} \) was identified as the real electromagnetic coupling \( e \) in the standard model. Moreover, the Higgs particle, W boson, and Z bosons are the elementary particles of physics. And the Higgs vacuum \( \rho_0 \),

![FIG. 2. The singular monopole solution with W-boson and Higgs scalar dressing in two-gap superconductor.](https://example.com/fig2.png)
sets the electroweak scale of the order of 100 GeV.

On the other hand our Lagrangian (5) here is an effective Lagrangian which is proposed to describe a non-Abelian extention of the Landau-Ginzburg theory of superconductivity, not a fundamental interaction of nature. So here the couplings $g$ and $g'$ (and thus the mixing angle $\omega$) are completely fixed by $e$. Moreover, here the Higgs field $\rho$ and the gauge bosons $A_\mu$, $Z_\mu$, and $W_\mu$ have completely different meaning. They appear as the emergent particles, not as the fundamental particles. Most importantly, the Higgs vacuum $\rho_0$ here represents the correlation length of the electron pairs in superconductors, which is supposed to be of the order of a few meV. This tells that the two Lagrangians describe a totally different physics, in spite of the fact that mathematically they are identical.

In particular, two theories have totally different mass scales. In the non-Abelian Landau-Ginzburg theory the mass of the Higgs scalar and gauge bosons are of the order of few meV. But in the electroweak theory the mass becomes of the order of 100 GeV, different by the factor $10^{14}$.

The same logic applies to the monopole. The electroweak monopole is a fundamental particle which exists in the standard model. So, when discovered, the monopole will be viewed as the first absolutely stable topological elementary particle in the history of physics. And it has the mass of the order of 10 TeV. For this reason MoEDAL and ATLAS at LHC are actively searching for the electroweak monopole \cite{28, 29}.

On the other hand the above monopole in two-gap superconductors may not be viewed as an elementary particle. It has the mass of the order of a fraction of eV. Moreover, it may not be absolutely stable, even though it is topological. This is because it is made of emergent fields. To clarify this point, consider the Abrikosov vortex. We can create it applying magnetic field on superconductor. But it is not fundamental nor stable, although it is topological. When we switch off the magnetic field, it disappears. The monopole here should be similar. We could possibly create it imposing the monopole topology by brute force with an external magnetic field, but when we switch off the magnetic field, it probably will disappear.

But perhaps the most important difference between the two theories could be the character of the massless gauge boson $A_\mu$. In the standard model this describes the real photon. But in this non-Abelian superconductivity this could likely describe the massless spinon which could induce the long range ferromagnetic order. Moreover, in the standard model the massive neutral gauge boson becomes the $Z$ boson, but here it could turn out to be the massive photon. So the two theories become totally different.

What is really remarkable is that, mathematically the same theory can describe totally different physics. On the other hand we emphasize that, while the standard model is a fundamental theory of nature, the above model described by (5) is a phenomenological model proposed to describe real non-Abelian condensed matters. As such, it can not be exact.

VII. DISCUSSIONS

In this paper we have shown how we could generalize the Abelian Langau-Ginzburg theory to a non-Abelian Landau-Ginzburg theory of two-gap ferromagnetic superconductor made of spin-up and spin-down Cooper pairs, which has the $SU(2)xU(1)$ gauge symmetry where the $SU(2)$ gauge interaction is mediated by three non-Abelian gauge spinons. This theory can be viewed a minimum non-Abelian extension of the Abelian Landau-Ginzburg theory, and has many interesting new features. The long range non-Abelian spin interaction mediated by the massless spinon and the non-Abelian Meissner effect generated by the massive spinon are the main new features. The existence of new topological objects, the quantized non-Abrikosov vortices and the non-Abelian monopole, are another example.

A remarkable feature of the theory is the mixing between the $U(1)$ gauge bosons and the diagonal part of $SU(2)$ spinon. After the mixing the theory has the massless spinon and the massive charged spinon, in addition to the massive photon we have in ordinary superconductors. The existence of the massless spinon which mediates a long range spin interaction is completely unexpected, because this type of massless particle is absent in ordinary (Abelian) superconductors.

This tells that the two component ferromagnetic superconductor retains the well known superconductivity that we have in the Abelian superconductors, and has the same Meissner effect generated by the massive photon. The new things here are the followings. First, the appearance of the massless spinon which generates a long range spin interaction, which could explain the long range magnetic order and spin waves observed in the ferromagnetic superconductors. Second, the appearance of the massive charged spinons which generates the spin flip interaction to the Cooper pair. But this interaction is not long range, and has a finite penetration length set by the mass of the spinon.

This means that the ferromagnetic superconductor has two penetration lengths, the one set by the photon mass which confines the magnetic field and the other set by the mass of the charged spinon which confines the spin flip interaction.

There are more new things in this two gap ferromagnetic superconductor. Since the extended $SU(2)xU(1)$ gauge symmetry admits more topology than the $U(1)$...
gauge symmetry in the Abelian superconductor, the non-Abelian superconductor admit more topological objects. In particular, it has the singular spinon string dressed by Higgs and massive spinon which has quantized spin flux, in addition to the well known Abrikosov quantized magnetic vortex we have in the Abelian superconductor. Moreover, we have the singular spinon monopole dressed by the Higgs and the massive spinon fields.

We can easily generalize the theory to non-Abelian superconductors made of the spin triplet Cooper pairs. In detail, the spin triplet superconductivity is different from the spin doublet superconductivity, but the generic features remain the same. The details of the non-Abelian superconductivity and the effective theory of superconducting made of the spin triplet Cooper pairs will be discussed in a separate paper [30].

Obviously the non-Abelian superconductivity proposed in this paper is hypothetical, and the above discussions are purely theoretical. So, we definitely need the experimental check on this, in particular, the existence of the long range spinon gauge field. But independent of this, the above discussion raises a fundamental question. Can we describe the spin-spin interaction by a gauge interaction, mediated by the gauge spinons? So far the spin multiplets in physics have always been regarded as global multiplets which have no gauge interaction. For the first time in this paper we have introduced a gauge interaction to the spin doublet Cooper pair. In what extent could this non-Abelian spin gauge interaction justified? Certainly this is a fundamental question in physics which need to be studied further.

Author Contribution Statement: The first author made the overall plan of the paper and the second author provided the physical interpretation of the non-Abelian spinon gauge interaction.

Data Availability Statement: All data generated during this study are included in this published article.

ACKNOWLEDGEMENT

The work is supported in part by the National Research Foundation of Korea funded by the Ministry of Science and Technology (Grant 2022-R1A2C1006999) and by Center for Quantum Spacetime, Sogang University, Korea.

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