New perspectives on the TOV equilibrium from a dual null approach

Alan Maciel,1,* Morgan Le Delliou,2,† and José P. Mimoso3,*

1Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, Avenida dos Estados 5001, CEP 09210-580, Santo André, São Paulo, Brazil.
2Institute of Theoretical Physics, Physics Department, Lanzhou University, No.222, South Tianshui Road, Lanzhou, Gansu 730000, P R China
3Departamento de Física and Instituto de Astrofísica e Ciências do Espaço, Faculdade de Ciências da Universidade de Lisboa, Campo Grande, Ed. C8 1749-016 Lisboa, Portugal

We apply the dual null formalism to spacetimes with two dimensional spherical, planar and hyperbolic symmetries with a perfect fluid as the source. We also assume a Killing vector field orthogonal to the surfaces of symmetry, which gives us static solutions, in the timelike Killing field case, and homogeneous dynamical solutions in the case the Killing field is spacelike. In order to treat equally all the aforementioned cases, we discuss the definition of a quasi-local energy for the spacetimes with planar and hyperbolic foliations, since the Hawking-Hayward definition only applies to compact foliations. After this procedure, we are able to translate our geometrical formalism to the fluid dynamics language in a unified way, to find the generalized TOV equation, for the three cases when the solution is static, and to obtain the evolution equation, for the homogeneous spacetime cases.

I. INTRODUCTION

The interpretation of General Relativity (GR) as general relativistic fluid dynamics allows both the remarkable departure from classical Newtonian gravity in two opposite scale limits, and the tackling of general relativistic gravitational problems, regardless of the scale of their domain.

On the one hand, these opposite departures from Newtonian gravity manifest themselves in the small distance scales where the sources have a high density, and on the other hand, in the large scales where the effect of mass distribution is accumulated (even though the mass density might be low). The former is, for instance, well illustrated by the Schwarzschild solutions [1], which revealed the existence of Black Holes [now at last observationally confirmed, see 2], while the latter appears in the standard cosmological solutions [3], which introduced an expanding universe. Interestingly these limits also seem opposite in the sense that the former gravitational solutions are static and the latter clearly dynamical.

Perhaps because of this opposition, they have been addressed using GR in somewhat different perspectives. While the Schwarzschild solutions are associated to bound gravitational systems in static equilibrium, and correspond to matter distributions surrounded by vacuum, the cosmological solutions stem from the consideration of an unbound, spatial distribution of matter.

The latter can be understood as an example of a general relativistic fluid dynamics. This viewpoint also applies when one considers the equilibrium of spherical stars with the outer Schwarzschild solution, but otherwise is not adopted when characterising static configurations which just depend on some symmetries and boundary conditions. This is remarkably done by invoking Birkhoff theorem [4], which is, on the one hand, usually assumed to be restricted to the spherically symmetric case [although this is a misled assumption; see, e.g. 5], and on the other hand, understood as exempting one from considering the actual distribution of matter. However, this viewpoint runs into a manifest exception when one describes spherical stars in equilibrium in which the matter distribution must satisfy the Tolman-Oppenheimer-Volkoff (TOV) equation [6, 7]. The TOV equation gives a first approximation to describe virtually any body in the sky large enough such that its dynamics is dominated by gravity and stationary enough to enable us to assume them to be in static equilibrium, such as planets and stars. Due to their evident relevance, many solutions for this type of configurations have been found [8–20], and more specifically, different formalisms [21–24] and solutions generating techniques have been developed [25, 26]. Extensions of the TOV equation have also been investigated in the framework of modified gravity theories [27–33]. Yet a unified characterization of the underlying features of the TOV equation has attracted little attention, and this is what concerns us in the present work.

Focusing on GR fluid dynamics [for a review of the pioneering work on this viewpoint see for instance 34–36, and references therein], we can trace it back to Hawking and Penrose’s singularity theorems [37], and it eventually enables one to tackle “small scale” problems with the same tools are those applicable to “large scale” ones. In this context, an approach using the properties of light
cones is most likely to reveal the structures of spacetime both at small and large scales.

The dual null formalism offers a description of the spacetime based on the properties of the optical flow. The latter is characterised by two linearly independent null congruences which are orthogonal to some codimension-two foliation of the spacetime. This approach shares some of the convenience of the choice of dual null coordinates, but it has the significant advantage of being a coordinate free formalism as well as to reveal, by construction, the causal structure of the spacetime in a natural way. It has been originally introduced to study general relativistic problems associated with the behaviour of dynamical black holes [38, 39], but it also is most convenient to apply to the analysis of other diverse questions. For instance, it has been considered in connection with the definition of energy in more general geometries [40], the gravitational collapse of fluids [41], and even the definition of generalized horizons in modified gravity [42]. The dual null formalism has been useful as well to explicit the "linear" behavior of gravity for sources that satisfy the hypotheses of the Birkhoff theorem [43].

Here, we apply the dual null formalism to analyse, in a unified way, the spacetimes which admit a codimension-two foliation with constant curvature leaves. This comprises the spherical, planar and hyperbolic symmetries, sourced by a perfect fluid. Aside from the Killing vector fields that are tangent to those surfaces of symmetry, we assume the existence of an additional symmetry generated by a Killing vector field orthogonal to those surfaces at each event. A particular case of this setup, where the symmetry is spherical and the Killing vector is timelike, corresponds to the spherically symmetric perfect fluid in hydrostatic equilibrium, which leads us to the well-known TOV equation. As we will show this celebrated equation arises most naturally in the dual-null framework which, moreover, allows its generalizations for the planar and hyperbolic cases. When the metric is characterised by a spacelike Killing vector, we have spatially homogeneous spacetimes that, as we will show, correspond to some of the Bianchi spacetimes.

We proceed as follows: we start by giving a short introduction to the dual null formalism, using from the onset the symmetries assumed in our class of problems in order to simplify our expressions, in Sec. II. We then prove a proposition that states that the Killing vector two-expansion always vanishes, Sec. III. From this property we will show that one derives either the equation of hydrostatic equilibrium, when the Killing is timelike, Sec. IV A, or the evolution equation when the Killing is spacelike, Sec. IV B. In order to interpret the geometrical quantities that appear, and to establish their underlying physical content, we discuss the mass-energy definition in these spacetimes, Sec. IV. This will lead us to put forward a generalization of the Misner-Sharp/Hawking-Hayward definitions.

II. MAIN ASSUMPTIONS AND DEFINITIONS

We consider metrics that have a codimension-two maximally symmetric foliation, and can be written as

$$ds^2 = N_{ab} dx^a dx^b + Y^2 (x^c) (d\theta^2 + S_\epsilon^2 d\phi^2),$$

(1)

where

$$S_\epsilon = \begin{cases} 
\sin \theta, & \text{for } \epsilon = 1 \\
1, & \text{for } \epsilon = 0 \\
\sinh \theta, & \text{for } \epsilon = -1
\end{cases},$$

and where we divide the tangent space \( \mathcal{T} \) at each event in two orthogonal subspaces \( \mathcal{T} = N \oplus S \). Here \( S \) is the subspace generated by the orbits of the \( (\theta, \phi) \) and \( N \), the subspace of \( \mathcal{T} \) orthogonal to \( S \). The \( x^a \) coordinates are chosen orthogonal to \( S \), which gives the metric in the warped sum form of Eq. (1).

We denote \( \gamma_{ab} = Y^2 \gamma_{ab} \) the induced metric in each leaf of the foliation. Evidently, \( \gamma_{ab} := \delta^a_{\theta} \delta^b_{\theta} + S_\epsilon^2 \delta^a_{\phi} \delta^b_{\phi} \) has constant curvature and does not depend on the coordinates \( x^c \) which identify each leaf \( \Sigma_x \), defined as the locus spanned by the orbits of \( \theta \) and \( \phi \) for fixed \( x^c \). We define an orthonormal two dimensional basis \( (u^a, e^a) \) for \( N \), whose induced metric is \( N_{ab} \), according to Eq. (1). This basis satisfy

$$-n^an_a = e^ae_a = 1, \quad n^ae_a = n^as_{ab} = e^as_{ab} = 0. \quad (2)$$

We may also define a dual null basis for the same subspace from \( n^a \) and \( e^a \) by

$$n^a = \frac{1}{2} (n^a + e^a), \quad l^a = \frac{1}{2} (n^a - e^a),$$

$$k^a = k^a + l^a, \quad e^a = k^a - l^a. \quad (3)$$

which satisfies

$$k^aal_a = l^al_a = 0, \quad k^al_a = -\frac{1}{2}. \quad (4)$$

The metric \( g_{ab} \) can be written as

$$g_{ab} = \frac{2}{k^al_c} k_{(a}l_{b)} + s_{ab}. \quad (5)$$

We associate the null expansion for each null vector, defined as

$$\Theta_k = \frac{1}{2} \delta^{ab} \mathcal{L}_k s_{ab} = \frac{1}{2} Y^{-2} \gamma^{ab} \mathcal{L}_k Y^2 \gamma_{ab} = \frac{2}{Y} k^a \partial_a Y. \quad (6)$$

We may extend the definition of null expansion to timelike and spacelike vectors in \( N \), calling it the two-expansion, since it measures the rate of variation of area, as in the null case. We may define the mean curvature form \( K_a = \partial_a \ln Y^2 \), such that, we obtain for the two-expansion \( \Theta_{(a)} \) of any vector \( a^a \) in \( N \)

$$\Theta_{(a)} = u^a K_a. \quad (7)$$
We describe our spacetimes with the behaviour of the null expansion, by writing the Einstein equations, $G_{ab} = 8\pi T_{ab}$ in terms of expansions, i.e. the Raychaudhuri equations \cite{[37, 38, 44, 45]}. They read

\begin{align}
\mathcal{L}_k \Theta_{(k)} &= \nu_k \Theta_{(k)} - \frac{\Theta^2_{(k)}}{2} - 8\pi T_{ab} k^a k^b, \quad \text{(8a)}
\mathcal{L}_l \Theta_{(l)} &= \nu_l \Theta_{(l)} - \frac{\Theta^2_{(l)}}{2} - 8\pi T_{ab} k^a k^b, \quad \text{(8b)}
\mathcal{L}_k \Theta_{(l)} + \mathcal{L}_l \Theta_{(k)} &= -\Theta_{(l)} \nu_k - \Theta_{(k)} \nu_l - 2 \Theta_{(k)} \Theta_{(l)} + \epsilon_n \frac{2 k^l a}{Y^2} + 16 \pi T_{ab} k^a k^b, \quad \text{(8c)}
\end{align}

where we included the inaffinities $\nu_k$ and $\nu_l$, defined as

\begin{equation}
\nu_k = \frac{1}{k^a k^b} \nu_a \nu_b, \quad \nu_l = \frac{1}{k^a k^b} \nu_a \nu_b. \quad (9)
\end{equation}

In this work we adapt our vector basis to a fluid source, such that $n^a$ gives its flow. Therefore, it will be useful to relate our quantities to this flow. The flow $n^a$ is, by construction, always orthogonal to the surfaces of symmetry and will be characterized by two quantities

\begin{equation}
A = e^a n_a = e^a n^b \nabla_b n_a, \quad B = e^a n'_a = e^b \nabla_b n_a. \quad (10)
\end{equation}

The scalar $A$ gives us the acceleration of the flow, a positive sign means that the acceleration is outwards, in the spherical, compact case. The scalar $B$ gives the change of direction of $n^a$ as we travel along $e^a$. It is the $-e$ component of the extrinsic curvature $K_{ab}$ of the 3-space orthogonal to this flow, since

\begin{equation}
K_{ab} = \frac{1}{2} L_n h_{ab}, \quad (11)
\end{equation}

where $h_{ab} = g_{ab} + n_a n_b$. We may also write

\begin{equation}
h_{ab} = e_a e_b + Y^2 \gamma_{ab}, \quad (12)
\end{equation}

which gives

\begin{equation}
K_{ab} = B e_a e_b + \frac{\Theta_{(n)}}{2} Y^2 \gamma_{ab}. \quad (13)
\end{equation}

The trace of Eq. (13) gives us the flow volumetric expansion $\Theta_3 = \nabla_a n^a = K^a_a$ as

\begin{equation}
\Theta_3 = B + \Theta_{(n)}. \quad (14)
\end{equation}

In order to relate our quantities with the flow scalars, we compute the shear scalar $\sigma$, by taking the traceless part of $K_{ab}$. We obtain

\begin{equation}
\sigma = \frac{\Theta_{(n)}}{6} - \frac{B}{3}, \quad (15)
\end{equation}

which implies

\begin{equation}
\frac{\Theta_3}{3} + \sigma = \frac{\Theta_{(n)}}{2}, \quad (16)
\end{equation}

in agreement with the result obtained in Ref. [41].

Using the inaffinities of the null basis vectors, $A$ and $B$ can be expressed as

\begin{equation}
A = \nu_k - \nu_l, \quad B = \nu_k + \nu_l. \quad (17)
\end{equation}

\section{Orthogonal Killing Vector}

We now assume that our metric has a Killing vector orthogonal to maximally symmetric surfaces. We note the hypersurface orthogonal Killing vector field $\chi^a$. It satisfies the Killing equation,

\begin{equation}
\mathcal{L}_\chi g_{ab} = 0. \quad (18)
\end{equation}

\textbf{Proposition III.1.} If a spacetime is described by a metric of the form (1) and admit an orthogonal Killing vector $\chi^a \in \mathcal{N}$, then $\Theta_3 = 0$.

\textit{Proof.} We may write, from Eq. (6), $\Theta_3 = \frac{1}{2} s_{ab} \mathcal{L}_\chi s_{ab}$

\begin{equation}
g_{ab} = N_{ab} + Y^2 \gamma_{ab}. \quad (19)
\end{equation}

Then

\begin{equation}
0 = Y^{-2} \gamma_{ab} \mathcal{L}_\chi g_{ab} = Y^{-2} \gamma_{ab} \mathcal{L}_\chi N_{ab} + 2 \Theta_3 = -Y^{-2} N_{ab} \mathcal{L}_\chi \gamma_{ab} + 2 \Theta_3. \quad (20)
\end{equation}

However

\begin{equation}
\mathcal{L}_\chi \gamma_{ab} = 0, \quad (21)
\end{equation}

since $\gamma^a$ does not admit components in $S$ and $\gamma_{ab}$ doesn’t depend on coordinates along $\mathcal{N}$. Therefore, Eq. (20) implies that $\Theta_3 = 0$.

Consequently, if there is an extra symmetry with orbits orthogonal to those of the maximally symmetric leaves of the foliation, the two-expansion of its generator vanishes. This also implies that if $dY$ is spacelike, then $\gamma_a$ is timelike and vice-versa. If $dY$ is null, the Killing vector will also be null.

\section{Mass-Energy}

In order to interpret our spacetimes properly, we have to understand their mass-energy content. There is a widely known mass-energy definition suitable to the spherically symmetric case, the Misner-Sharp mass-energy \cite{[46, 47]}, defined regardless of asymptotic assumptions. However, as we also intend to analyze nonspherical spacetimes in this work, it leads us to a more general mass-energy definition such as the Hawking-Hayward one \cite{[40, 48]}. The HH mass-energy gives the mass-energy content inside a closed compact surface in terms of an integral over that surface, in a manner similar to the Gauss law in Newtonian gravity. This quasi-isothermal mass-energy has been explored in different contexts, such as seen in Refs. [49–51].

In the case where the $\Sigma_{xx^*}$ are spheres, which are compact, we can compute the Hawking-Hayward mass-energy enclosed by $\Sigma$ (we drop the $x^*$ index for short)

\begin{equation}
\mathcal{M} = \frac{1}{2} \left[ \left( \int_{\Sigma} \mathcal{L}_\chi g_{ab} d^2 x \right)^2 - \int_{\Sigma} \mathcal{L}_\chi g_{ab} d^2 x \right],
\end{equation}

...
as
\[
M = \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_{\Sigma} [\mathcal{R} - \frac{1}{k^a l_a} \left( \Theta_k(\Theta_l) - \frac{1}{2} \sigma_k(a) \sigma_l(a) - 2\omega_k \omega_l \right)] d\Sigma
\]
(22)
where \( \mathcal{R} \) is the two-dimensional Ricci scalar and \( A \) is the area of \( \Sigma \), \( \sigma_k(a) \) and \( \sigma_l(a) \) are the two-dimensional shear tensors along \( \Sigma \), associated with the \( k \) and \( l \) congruences, respectively, and \( \omega_k \) is the twist vector given by the projection on \( \Sigma \) of the commutator of the null basis vectors. We have included a factor of \(-\frac{1}{k^a l_a}\) in the optical scalars part of the mass-energy, compared with the formula present in Ref. [40], in order to take account of our different normalization of the null normals.

Our symmetry assumptions imply that the only non-vanishing optical scalar on \( \Sigma \) is the null expansion. Therefore, the Hawking-Hayward mass-energy is reduced to
\[
M = \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_{\Sigma} \left( \mathcal{R} - \frac{1}{k^a l_a} \Theta_k(\Theta_l) \right) d\Sigma
\]
(23)
Since we assume that \( \Sigma \) is maximally symmetric, we have \( \mathcal{R} = \frac{2\epsilon}{Y^2} \). We also have
\[
\Theta_k(\Theta_l) = k^a \partial_a \ln Y^2 \partial_k \ln Y^2 = k^a l_a \partial_a \ln Y^2 \partial_k \ln Y^2 = \frac{k^a l_a}{2} g^{ab} \partial_a \ln Y^2 \partial_b \ln Y^2 = \frac{1}{2} k^a l_a dt \partial_t \ln Y^2 = \frac{k^a l_a}{2} \ln Y^2 ||dY||^2 \Rightarrow
\]
\[
\Theta_k(\Theta_l) = \frac{2}{Y^2} ||dY||^2,
\]
where we used Eq. (5) in the fourth step. For the spherical case \( \epsilon = 1 \) and \( A = 4\pi Y^2 \), we obtain the known interpretation of \( ||dY|| \) in terms of the Misner-Sharp mass-energy, which coincides with the Hawking-Hayward one
\[
M = \frac{Y}{2} \left( 1 - ||dY||^2 \right) \Leftrightarrow ||dY||^2 = 1 - \frac{2M}{Y}.
\]
(25)

In the planar and hyperbolic cases (\( \epsilon = 0 \) and \( \epsilon = -1 \), respectively), the Hawking-Hayward mass is not rigorously defined for our preferred foliation, as it requires a closed compact surface to be the integration domain. Since the area \( A \) is infinite in our case, the expression (23) is not defined. However, we interpret this as the presence of an infinite mass distribution and we will adapt (23) to a definition of a quasilocal mass-energy surface density. First we make the replacement
\[
\frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \rightarrow \frac{Y}{4\pi \kappa}.
\]
(26)
in order to just keep its dimensionality, but eliminating the explicit dependence on the area of \( \Sigma \). Evidently, by setting \( \kappa = 4 \) we recover the Hawking-Hayward mass-energy in the spherical case. We then define the quasilocal mass-energy surface density \( \mu(Y) \) by
\[
M = \frac{\mu(Y)}{4\pi} \int S^2(\Theta) d\theta d\phi,
\]
(27)
and we write
\[
\frac{Y}{4\pi \kappa} \left[ \mathcal{R} - \frac{\Theta_k(\Theta_l)}{k^a l_a} \right] \int_{\Sigma} d\Sigma = \frac{Y}{4\pi \kappa} [2\epsilon - 2||dY||^2] \int S^2_2(\Theta) d\theta d\phi.
\]
(28)
We equate Eqs. (27) and (28) and eliminate the improper area integral on both sides
\[
||dY||^2 = \epsilon - \frac{\kappa \mu(Y)}{2Y}.
\]
(29)
Equation (29) coincides with the known mass function [see 52, Eq. 15.7a] that appears as we integrate the Einstein equations for spacetimes with metric of the form (1) for planar and hyperbolic symmetries. From now on, we will consider Eq. (29) with the choice \( \kappa = 4 \) as the mass-energy definition.

**A. Timelike Killing Vector**

We assume \( \chi^a \chi_a < 0 \). In this case, the spacetime is static and \( n^a \sim \chi^a \). Therefore, from Proposition III.1, \( \Theta_{(a)} = 0 \) everywhere and \( dY \) is spacelike, since it is orthogonal to \( n^a \). If \( \Theta_{(a)} \) vanishes everywhere, this means that the fluid has no radial velocity, therefore we are dealing with a static fluid with a flow parallel to the Killing vector field.

In order to characterize its static equilibrium, we need to compute the derivative of the flow 2-expansion along the flow itself:
\[
\mathcal{L}_n \Theta_{(a)} = 0,
\]
(30)
since \( \Theta_{(a)} = 0 \) everywhere.

We may write \( \mathcal{L}_n \Theta_{(a)} \) in terms of the null expansions as
\[
\mathcal{L}_n \Theta_{(a)} = \mathcal{L}_k \Theta_{(k)} + \mathcal{L}_l \Theta_{(l)} + \mathcal{L}_c \Theta_{(c)}.
\]
(31)
Substituting the Eqs. (8a), (8b), and (8c), we obtain
\[
\mathcal{L}_n \Theta_{(a)} = - \frac{\Theta_k(\Theta_l)}{2} - \frac{\epsilon}{2} - 8\pi T_{ab} \epsilon_a \epsilon_b + A(\Theta_k - \Theta_l).
\]
(32)
Recall that we are assuming \( \Theta_{(a)} = \Theta_k(\Theta_l) = 0 \), and that \( \Theta_k(\Theta_l) = \Theta_{(c)} \), using Eqs. (7) and (3). We
identify here $\Theta_{(k)} \Theta_{(l)}$ as the mass term, since it equals
\[
\frac{2}{Y^2} \| dY \|^2 = \frac{2}{Y^2} \left( \epsilon - 2\mu(Y) \right).
\]

We assume the source to be a perfect fluid, then the energy momentum tensor
\[
T_{ab} = \rho n_an_b + P (e_a e_b + s_{ab}).
\]

Contracting the conservation of the energy-momentum tensor with $e_3$ [Euler equation in 53]:
\[
e_3 \nabla_a T^{ab} = (\rho + P) n^b e_b + e^a \nabla_a P = 0 \quad \Rightarrow \quad A = -\frac{e^a \partial_a P}{\rho + P},
\]

Since $\Theta_{(n)} = 0$, this implies that $e^a$ is proportional to $\partial Y$. As $e^a$ is normalized, we have $e_a = \frac{1}{\| dY \|} \partial_a Y$.

Imposing $e^a e_a = 1$ we obtain
\[
e^a = \| dY \| (\partial Y)^a,
\]
which gives us
\[
A \Theta_{(n)} = -\| dY \|^2 \frac{2}{Y} \frac{\partial Y P}{\rho + P}.
\]

Therefore, replacing $\| dY \|^2$ by its meaning in terms of mass, $L_n \Theta_{(n)} = 0$ corresponds to
\[
\left( \epsilon - 2\mu(Y) \right) \frac{2}{Y} \frac{\partial Y P}{\rho + P} = -\frac{2}{Y} \frac{\mu(Y)}{Y} - 8\pi P - \frac{\epsilon}{Y^2} \frac{\partial Y}{\rho + P},
\]
or,
\[
\frac{\partial Y P}{\rho + P} = -\left( \frac{\mu(Y)}{Y^2} + 4\pi P Y \right) \left( \epsilon - 2\mu(Y) \right)^{-1},
\]
which is the known TOV equation for space-times with spherical, planar or hyperbolic symmetry depending on the choice of $\epsilon$. This underlines the fact that the TOV equation is a hydrostatic equilibrium equation and not an equation of state, as it is erroneously stated sometimes.

In order to determine $\mu(Y)$ we consider the $L_e \Theta_{(e)}$ Raychaudhuri equation
\[
L_e \Theta_{(e)} = B \Theta_{(n)} - \frac{\Theta_{(e)}^2}{2} + \frac{\epsilon}{4} \left( \Theta_{(n)}^2 - \Theta_{(e)}^2 \right)
\]
\[
+ \frac{\epsilon}{Y^2} - 8\pi T_{ab} n^a n^b,
\]
which, by using $\Theta_{(n)} = 0$, and Eq. (35) lead us to
\[
\| dY \| \partial Y \left( \frac{2}{Y} \| dY \| \right) = -\frac{3}{Y^2} \| dY \|^2 + \frac{\epsilon}{Y^2} - 8\pi \rho.
\]

Substituting Eq. (29) into Eq. (40), we obtain
\[
\partial Y \mu = 4\pi \rho Y^2,
\]
which is the known mass-energy equation in spherical symmetry. We warn that the $4\pi$ factor is an artifact of our definition of mass-energy surface density. This choice is convenient in order to treat all the three cases simultaneously and recover the mass function with the traditional factor. With Eq. (41), the last requirement to solve Eq. (38) is the equation of state of the fluid, $f(\rho, P) = 0$ which should come from physical considerations.

B. Spacelike Killing Vector

In the spacelike Killing vector case, $dY$ is timelike, the flow $n_a$ is orthogonal to the Killing vector and the unitary base vector $e^a$ is parallel to it.

One dynamical equation is given by $L_e \Theta_{(e)} = 0$, which by Eq. (39) gives:
\[
B \Theta_{(n)} - \frac{\Theta_{(e)}^2}{2} + \frac{\epsilon}{4} \left( \Theta_{(n)}^2 - \Theta_{(e)}^2 \right) + \frac{\epsilon}{Y^2} - 8\pi \rho = 0,
\]

From Proposition III.1, we have $\Theta_{(e)} = 0$. Replacing Eq. (15) in Eq. (39), we obtain
\[
\frac{3}{4} \Theta_{(n)}^2 - 3\sigma \Theta_{(n)} + \frac{\epsilon}{Y^2} = 8\pi \rho,
\]
or, using Eq. (16) we can express Eq. (43) in terms of the volume expansion $\Theta_3$
\[
\frac{\Theta_3^2}{3} - 3\sigma^2 = 8\pi \rho - \frac{\epsilon}{Y^2},
\]
which correspond to the evolution of a homogeneous and anisotropic universe. The isotropic case corresponds to $\sigma = 0$, where we may identify $\Theta_3 = 3H$ and we obtain the usual Friedmann equation.

The $L_n \Theta_{(n)}$ Raychaudhuri equation gives the evolution of $\Theta_{(n)}$. Using Eq. (32)
\[
L_n \Theta_{(n)} = -\frac{\Theta_{(n)}^2}{2} - \Theta_{(e)} \Theta_{(n)} - \frac{\epsilon}{Y^2} - 8\pi T_{ab} e^a e^b \Rightarrow \nabla \left( \frac{3}{4} \Theta_{(n)} - \frac{\epsilon}{Y^2} - 8\pi T_{ab} e^a e^b \right).
\]

Subtracting Eq. (43) from Eq. (45), we obtain
\[
L_n \Theta_{(n)} = -3\sigma \Theta_{(n)} - 8\pi (\rho + P),
\]
which, together with an equation of state relating $\rho$ and $P$ closes our system. By adding half of the Eq. (46) with one third of Eq. (44), we obtain:
\[
L_n \left( \frac{\Theta_3}{3} \right) + \left( \frac{\Theta_3}{3} \right)^2 = -2\sigma \left( \frac{\Theta_3}{3} + \sigma \right) - \frac{4\pi}{3} (\rho + 3P).
\]

Those homogeneous, generally anisotropic spacetimes are a subclass of Bianchi universes, with the case $\epsilon = 0$ corresponding to type I universes.

V. CONCLUSION

In this paper we analyse spacetimes with a two-dimensional maximally symmetric foliation sourced by a perfect fluid. We proved that in those cases, if a Killing vector orthogonal to the leaves exists, its two-expansion
vanishes, which allows us to simplify our dynamical equations in terms of the two-expansion of a unitary vector orthogonal to the Killing field.

When the Killing vector $\chi^a$ is timelike, we find that the flow lines must be tangent to $\chi^a$ and as this is true at all times, the equations describe a hydrostatic equilibrium, governed by a (generalised) TOV equation. When the Killing vector is spacelike, we have instead a spatially homogeneous dynamical spacetime. The result is a subclass of Bianchi universes, with only two distinct shear eigenvalues [54, 55]. The corresponding equation gives the evolution of expansion and shear scalars.

We have discussed the geometric meaning of the mass-energy in such spacetimes and our procedure matches the traditional mass function found in those cases by usual methods of integration of Einstein equations. Our approach relates the mass function to the geometrically defined quasi-local mass-energies of Misner-Sharp and Hawking-Hayward.

Using these we could recover the physical interpretation of the geometrical quantities appearing in the equilibrium/evolution equations, translate the dual null formalism to the more usual relativistic fluid dynamics framework, and state that the TOV equation arises as a particular case of those equations, its generalizations appear automatically by just setting $\epsilon = 0$ or $-1$ accordingly. The same treatment also gives us the evolution equation of Bianchi spacetimes with two distinct shear eigenvectors.

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