Low-Temperature Series for the Correlation Length in $d = 3$ Ising Model

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Abstract

We extend low-temperature series for the second moment of the correlation function in $d = 3$ simple-cubic Ising model from $u^{15}$ to $u^{26}$ using finite-lattice method, and combining with the series for the susceptibility we obtain the low-temperature series for the second-moment correlation length to $u^{23}$. An analysis of the obtained series by inhomogeneous differential approximants gives critical exponents $2\nu' + \gamma' \approx 2.55$ and $2\nu' \approx 1.27$. 

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1 Introduction

The low-temperature series of $d = 3$ Ising model or equivalently strong coupling series of $d = 3 \ Z_2$ lattice gauge theory had long been shorter than high-temperature series. Recently they have been extended to higher orders [1, 2, 3, 4] using finite-lattice method [3, 4, 5]. In this method free energy density in the infinite volume limit, for instance, is given by a linear combination of the free energy on appropriate finite-size lattices. The algorithm to give the coefficients of the linear combination is so simple. In the standard graphical method, it is rather difficult to list up all the diagrams completely that contribute to the relevant order of the series. The finite-lattice method avoids this problem involved in the standard graphical method and enables us to obtain longer series. The method was applied in $d = 3$ Ising model to get the low-temperature series of the true inverse correlation length (which is equivalent to the mass gap in lattice gauge theory) [1], free energy [2], magnetization and zero-field susceptibility [3], and surface tension (which is equivalent to the string tension in lattice gauge theory) [4].

In this paper we apply the method to calculate the low-temperature expansion series for the second moment of the correlation function $\mu_2$ in $d = 3$ simple-cubic Ising model to order $u^{26}$, extending the previous result of order $u^{15}$ by Tarko and Fisher [5]. It gives the low-temperature series for the second-moment correlation length squared $\Lambda_2 = \xi_{1/2}^2$ [6] to order $u^{23}$, when combined with the low-temperature series of the susceptibility. This is longer by five terms than the low-temperature series for true correlation length squared $\Lambda'_2$ that was derived from the true inverse correlation-length given in Ref. [1].

In the next section we present the algorithm to obtain low-temperature expansion series for $\mu_2$ using finite-lattice method. In section 3 the expansion series for $\mu_2$ and $\Lambda_2$ is given. The low-temperature series for true correlation length squared $\Lambda'_2$ is also listed for comparison. The result of series analysis by inhomogeneous differential approximants is described in section 4.

2 Algorithm of low-temperature expansion

The second-moment correlation length squared is defined by

$$\Lambda_2 = \frac{\mu_2}{2d\mu_0},$$  (1)
where \( \mu_2 \) is the second moment of the correlation function

\[
\mu_2 = \lim_{V \to \infty} \frac{1}{V} \sum_{i,j} (r_i - r_j)^2 \langle s_i s_j \rangle_c,
\]

(2)

with \( V \) the lattice volume and \( r_i = (x_i, y_i, z_i) \) the coordinate of the lattice site \( i \), and \( \mu_0 \) is the zero-th moment of the correlation function or the susceptibility and \( d \) is the dimensionality of the lattice. Here in this paper we take the lattice spacing \( a = 1 \).

The algorithm to calculate the low-temperature expansion of the second moment \( \mu_2 \) is the following. We consider the partition function

\[
Z(\beta, h, \eta, \gamma_1, \gamma_2, \gamma_3) = \sum_{\{s_i\}} \exp (-\mathcal{H}),
\]

(3)

with Hamiltonian

\[
\mathcal{H} = \beta \sum_{\langle ij \rangle} s_i s_j + \sum_i (h + \gamma_1 x_i + \gamma_2 y_i + \gamma_3 z_i + \eta r_i^2) s_i,
\]

(4)

for a three-dimensional rectangular lattice \( \Lambda_0 \) with a volume \( V = L_x \times L_y \times L_z \). We take the fixed boundary condition that all the spins outside \( \Lambda_0 \) are aligned, for instance, to be \( \{s_i = +1\} \). Then the second moment \( \mu_2 \) is given by

\[
\mu_2 = \lim_{V \to \infty} \frac{2}{V} \left( \frac{\partial^2}{\partial h \partial \eta} - \frac{\partial^2}{\partial \gamma_1^2} - \frac{\partial^2}{\partial \gamma_2^2} - \frac{\partial^2}{\partial \gamma_3^2} \right) \ln Z(\beta, h, \eta, \gamma_1, \gamma_2, \gamma_3)|_{h=\eta=\gamma_1=\gamma_2=\gamma_3=0}.
\]

(5)

Let us consider the set \( \{\Lambda\} \) of all three-dimensional rectangular sub-lattices of \( \Lambda_0 \) (\( \Lambda \subseteq \Lambda_0 \)) with the volume \( l_x \times l_y \times l_z \) and define \( H \) of \( \Lambda \) as

\[
H(\Lambda) = 2 \left( \frac{\partial^2}{\partial h \partial \eta} - \frac{\partial^2}{\partial \gamma_1^2} - \frac{\partial^2}{\partial \gamma_2^2} - \frac{\partial^2}{\partial \gamma_3^2} \right) \ln Z(\Lambda)|_{h=\eta=\gamma_1=\gamma_2=\gamma_3=0}.
\]

(6)

where \( Z(\Lambda) \) is the partition function for \( \Lambda \) with the fixed boundary condition that all the spins outside \( \Lambda \) is aligned, and define \( W \) of \( \Lambda \) recursively as

\[
W(\Lambda) = H(\Lambda) - \sum_{\Lambda' \subset \Lambda} W(\Lambda').
\]

(7)

Note that \( H(\Lambda) \) and \( W(\Lambda) \) depend not on the position but on the size \( l_x, l_y, l_z \) of \( \Lambda \). We know

\[
H(\Lambda_0) = \sum_{\Lambda \subseteq \Lambda_0} W(\Lambda).
\]

(8)
Taking the infinite volume limit we obtain

\[ \mu_2 = \lim_{V \to \infty} \frac{1}{V} H(\Lambda_0) \]

\[ = \lim_{L_x, L_y, L_z \to \infty} \frac{1}{L_x L_y L_z} \sum_{l_x, l_y, l_z} (L_x - l_x + 1)(L_y - l_y + 1) \times (L_z - l_z + 1) W(l_x, l_y, l_z) \]

\[ = \sum_{l_x, l_y, l_z} W(l_x, l_y, l_z). \]

(9)

We can prove that the Taylor expansion of \( W(\Lambda) \) with respect to \( u = \exp(-4\beta) \) includes the contribution from all the clusters of polymers in standard cluster expansion that can be embedded into the rectangular lattice \( \Lambda \) but cannot be embedded into any of its rectangular sub-lattice \( \Lambda'(\subset \Lambda) \). The series expansion of \( W(\Lambda) \) starts from the order of \( u^n \) with \( n = 2(l_x + l_y + l_z) - 3 \). So we should take all the finite-size rectangular lattices that satisfy \( 2(l_x + l_y + l_z) - 3 \leq N \) for the summation in eq. (9) to obtain the expansion series to order \( u^N \).

In practice for the calculation of \( \frac{\partial^2}{\partial h^2} \ln Z(l_x, l_y, l_z)|_{h=\eta=\gamma_1=\gamma_2=\gamma_3=0} \), for instance, we have only to calculate the partition function \( Z(\beta, h, \eta; \gamma_1 = \gamma_2 = \gamma_3 = 0) \) to order \( h\eta u^N \).

3 Expansion series

The low-temperature series of \( \mu_2 \) have been obtained to order \( u^{26} \) using rectangular lattices with cross-section up to \( 4 \times 5 \). Bhanot’s algorithm of calculating the exact partition function for finite-size lattices can be applied to the partition function (3), in which the necessary memory and CPU time are proportional to \( N \times 2^{l_x \times l_y} \) and \( N \times 2^{l_x \times l_y} \times l_x \times l_y \times l_z \), respectively. The calculation was performed on FACOM-VP2600 at Kyoto University Data Processing Center and HITAC-S820 at KEK, both of which have about 0.5 Gbyte of main memory and 1 – 2 Gbyte of extended storage. Total CPU time necessary was about 3 hours.

The low-temperature series obtained is listed in table 1, where the coefficients \( \{m_n\} \) are defined by

\[ \mu_2 = \sum_n m_n u^n \]

(10)
Table 1: The low-temperature expansion coefficients \( \{m_n\} \) for the second moment of the correlation function \( \mu_2 \), \( \{\lambda_n\} \) for the second-moment correlation length squared \( \Lambda_2 \), and \( \{\lambda'_n\} \) for the true correlation length squared \( \Lambda'_2 \) in \( d = 3 \) simple-cubic Ising model.

| \( n \) | \( m_n \) | \( \lambda_n \) | \( \lambda'_n \) |
|-------|---------|---------------|-------------|
| 0     | 0       | 0             | 0           |
| 1     | 0       | 0             | 0           |
| 2     | 0       | 1             | 1           |
| 3     | 0       | -1            | -1          |
| 4     | 0       | 10            | 10          |
| 5     | 24      | -14           | -14         |
| 6     | -24     | 85            | 93          |
| 7     | 528     | -169          | -201        |
| 8     | -960    | 884           | 972         |
| 9     | 8496    | -2390         | -2510       |
| 10    | -21312  | 10212         | 10618       |
| 11    | 125904  | -30594        | -31250      |
| 12    | -380016 | 116134        | 118792      |
| 13    | 1813416 | -368934       | -378902     |
| 14    | -6046440| 1337519       | 1377207     |
| 15    | 25675200| -4435616      | -4547052    |
| 16    | -90096000| 15764526     | 16140138    |
| 17    | 358481304| -53464296    | -54602714   |
| 18    | -1289158128| 187665313   |             |
| 19    | 4943015520| -643021360   |             |
| 20    | -17962232976| 2242649168   |             |
| 21    | 67393016880| -7729951680  |             |
| 22    | -245697661872| 26894409824  |             |
| 23    | 909676085232| -93043627527|             |
| 24    | -3315864327216|            |             |
| 25    | 12172005334848|            |             |
| 26    | -44293518847536|            |             |
and \{\lambda_n\} by

\[ \Lambda_2 = \sum_n \lambda_n u^n. \]  \hspace{1cm} (11)

The latter has been derived from the expansion series for \mu_2 obtained here and the low-temperature series of the susceptibility obtained by Guttmann and Enting \[3\] to order \(u^{24}\) ( They obtained the series to order \(u^{26}\) and we have checked using the finite-lattice method that their coefficients are correct to order \(u^{24}\)). Among the series coefficients obtained here, those for \mu_2 to order \(u^{15}\) and those for \Lambda_2 to order \(u^{12}\) coincide to those obtained by Tarko and Fisher \[8\] and we have added new 11 terms.

We also list the low-temperature series for true correlation length squared \(\Lambda'_2\) defined by \[8\]

\[ \Lambda'_2 = \frac{1}{2[\cosh (m) - 1]}, \]  \hspace{1cm} (12)

where \(m\) is the true inverse correlation-length;

\[ m = - \lim_{L \to \infty} \frac{1}{L} \log \langle O(L)O(0) \rangle, \]  \hspace{1cm} (13)

where \(O(\ell) = \sum_{\{i \mid z_i=\ell\}} s_i\) is the summation of all the spin variables in \(z = \ell\) plane. The coefficients \{\lambda'_n\} are defined by

\[ \Lambda'_2 = \sum_n \lambda'_n u^n, \]  \hspace{1cm} (14)

and are derived from the true inverse correlation-length \(m\) given in Ref. \[1\]. We note that, although the coefficients \{\lambda_n\} differ from \{\lambda'_n\} for \(n \geq 6\), their ratios \{\lambda'_n/\lambda_n\} are within a range of 1.02 to 1.03 for \(11 \leq n \leq 17\).

4 Analysis of the series

In our preliminary analysis, we estimate the critical exponents of \mu_2 and \Lambda_2 by inhomogeneous differential approximants \[11\], in which the approximants to a function \(f(u)\) satisfy

\[ Q_M(u)f'(u) + P_L(u)f(u) + R_N(u) = O(u^{L+M+N+2}). \]  \hspace{1cm} (15)

The approximants are equivalent to Dlog Padé approximants when \(R_N(u) = 0\).

We first give the results of the analysis for \mu_2. We plot in fig. \[\] its critical exponent \(2\nu' + \gamma'\) against the critical point \(u_c\) consisting of the data given by all the approximants
Figure 1: Unbiased estimates of $2\nu' + \gamma'$ versus $u_c$ for $\mu_2$; the data includes the estimates from all the approximants with $N = \phi, 0, 1, \cdots, 6$ and $L + M + N = 19$

with $L + M + N = 19$ and $N = \phi, 0, 1, \cdots, 6$. We find a liner correlation between the estimates for the critical point and the exponent. Linear fitting of these data gives $2\nu' + \gamma' = 98.1648u_c - 37.8980$. Recent Monte Carlo renormalization-group analysis of $d = 3$ Ising model gives a precise estimate of critical point $u_c = 0.412051 \pm 0.000006$ [12], and the series analysis of the high-temperature expansions for susceptibility, spontaneous magnetization and specific heat gives $u_c$ which is consistent with this [13]. Using this value of the critical point, we can read from fig. [1] that $2\nu' + \gamma' = 2.5509 \pm 0.010$, where the error comes from the statistical error in the linear fitting. The data given by the approximants with $N > 6$ also fit to almost the same line, but their deviation from the line is larger.

We plot in fig. [3] the estimate of $2\nu' + \gamma'$ obtained by fitting the data from all the approximants for each fixed $L + M + N$ with $N = \phi, 0, 1, \cdots, 6$. We note that the estimate appears stable for $L + M + N \geq 16$. Fitting the data from all the approximants with $16 \leq L + M + N \leq 19$ and $N = \phi, 0, 1, \cdots, 6$ gives

$$2\nu' + \gamma' = 2.545 \pm 0.012.$$ (16)
Figure 2: Estimate of $2\nu' + \gamma'$ by fitting the data from all the unbiased approximants with $N = \phi, 0, 1, \cdots, 6$ for each fixed $L + M + N$; the estimates denoted by (a) and (b) are from the high-temperature series and (c) from the $\epsilon$-expansion, respectively.
Figure 3: Histogram of the number of biased estimates of $2\nu' + \gamma'$ for $\mu_2$; estimates are obtained from the approximants with $N = \phi, 0, 1, \cdots, 6$ and $16 \leq L + M + N \leq 19.$
We also obtain biased estimates by replacing $Q_M(u)$ with $(u - u_c)Q_M(u)$ in eq. (15). The estimates of $2\nu' + \gamma'$ show a good accumulation around 2.55. We show in fig. 3 a histogram of the number of estimates given by the biased approximants with $16 \leq L + N \leq 19$ and $N = \phi, 0, 1, \cdots, 6$. The average and one standard deviation of the data that satisfy $|2\nu' + \gamma' - 2.55| < 0.05$ are

$$2\nu' + \gamma' = 2.550 \pm 0.018. \quad (17)$$

![Figure 4](image)

**Figure 4:** Unbiased estimates of $2\nu'$ versus $u_c$ for $\Lambda_2$; the data includes the estimates from all the approximants with $N = \phi$ and $17 \leq M + L \leq 20$.

Next we give the result of the analysis for $\Lambda_2$. The estimates of its critical exponent $2\nu'$ by inhomogeneous differential approximants show less convergent results, except for the case of $N = \phi$, that is, Dlog Padé approximants. In fig. 4 we show the plot of the exponent $2\nu'$ against the critical point $u_c$ obtained from Dlog Padé approximants of $\Lambda_2$ with $17 \leq L + M \leq 20$. Linear fitting of these data gives $2\nu' = 43.360u_c - 16.595$. We can read from fig. 4 using $u_c = 0.412051$ that

$$2\nu' = 1.272 \pm 0.004. \quad (18)$$

We also list in table 2 biased estimates of the critical exponent $2\nu'$ by Dlog Padé approximants, and plot in fig. 5 the estimate of $2\nu'$ by Dlog Padé approximants for each $L + M$, where we have excluded the data that do not satisfy $|2\nu' - 1.27| < 0.05$. All the Dlog Padé approximants with $17 \leq L + M \leq 20$ and $|2\nu' - 1.27| < 0.05$ give

$$2\nu' = 1.276 \pm 0.012. \quad (19)$$
Table 2: Biased estimates of $2\nu'$ from $\Lambda_2$; estimates are obtained by Dlog Padé approximants with $K \equiv M + L = 17, 18, 19, 20$.

| $[L/M]$ | $K = 17$ | $K = 18$ | $K = 19$ | $K = 20$ |
|---------|---------|---------|---------|---------|
| $[K - 1/1]$ | 0.4495 | -0.4837 | 0.2279 | 0.1687 |
| $[K - 2/2]$ | -0.4030 | 0.8627 | 2.1319 | 0.8284 |
| $[K - 3/3]$ | 1.2749 | 1.2699 | 1.2794 | 1.2625 |
| $[K - 4/4]$ | 1.2751 | 1.2737 | 1.2731 | 1.2579 |
| $[K - 5/5]$ | 1.2709 | 1.2702 | 1.2731 | 1.2736 |
| $[K - 6/6]$ | 1.2707 | 1.2665 | 1.2708 | 1.2480 |
| $[K - 7/7]$ | 1.4254 | 1.2698 | 1.2707 | 1.2716 |
| $[K - 8/8]$ | 1.2782 | 1.2766 | 1.2815 | 1.2861 |
| $[K - 9/9]$ | 1.2740 | 1.2761 | 1.2778 | 1.2914 |
| $[K - 10/10]$ | 1.2655 | 1.2769 | 1.2735 | 1.2807 |
| $[K - 11/11]$ | 1.2727 | 1.2714 | 1.2488 | 1.2968 |
| $[K - 12/12]$ | 1.2738 | 1.2704 | 1.2725 | 1.2827 |
| $[K - 13/13]$ | 1.2688 | 1.2722 | 1.2735 | 1.2745 |
| $[K - 14/14]$ | 1.2692 | 1.2704 | 1.2637 | 1.2753 |
| $[K - 15/15]$ | 1.4272 | 1.2737 | 1.2675 | 1.2722 |
| $[K - 16/16]$ | 1.3291 | 1.2950 | 1.2889 | 1.3058 |
| $[K - 17/17]$ | 1.2247 | 1.2879 | 1.2923 |   |
| $[K - 18/18]$ |         | 1.3850 | 1.3023 |   |
| $[K - 19/19]$ |         |         | 1.2142 |   |

Finally we give the result of the analysis for $\Lambda'_2$. The approximants for $\Lambda'_2$ with $N = 0, 1, 2$ give better converging estimates of $2\nu'$ than the other approximants with $N \geq 3$, but the average of estimates given by these approximants is too small ($2\nu' \approx 1.20$) and inconsistent with the result of the analysis for $\Lambda_2$. (If we use the approximants with $N \geq 3$, we obtain smaller estimates of $2\nu' \approx 1.17$.) We cannot say that this is due to the fact that the series for $\Lambda'_2$ is shorter than those for $\Lambda_2$. The analysis of the series for $\Lambda_2$ to order $u^{17}$, which is the maximum order for $\Lambda'_2$, gives more convergent and consistent estimates of $2\nu' = 1.28 \pm 0.02$.

These values of $2\nu' + \gamma'$ and $2\nu'$ are to be compared with the result $2\nu + \gamma = 2.504(2)$, $2.497(4)$ and $2\nu = 1.264(2), 1.260(3)$ of the high-temperature series [14, 15], $2\nu + \gamma = 2.501(4)$ and $2\nu = 1.262(3)$ of the $\epsilon$-expansion [16], and $2\nu = 1.246(6)$ of the Monte Carlo renormalization-group analysis [12]. The values of $2\nu + \gamma$ cited here are estimated from the values of $\nu$ and $\gamma$ given in the respective references. Our results of $2\nu'$ from $\Lambda_2$ [e.g., eqs. (18) and (19)] are not inconsistent with these estimates considering the error bounds. On the other hand, our results of $2\nu' + \gamma'$ from $\mu_2$ [eqs. (16) and (17)] are 2 per cent larger than the estimates
Figure 5: Estimate of $2\nu'$ by biased Dlog Padé approximants for each fixed $L + M$; the estimates denoted by (a) and (b) are from the high-temperature series, (c) from the $\epsilon$-expansion, and (d) from the Monte Carlo renormalization-group analysis, respectively.

from the high-temperature series and $\epsilon$-expansion and they are not within error limits. We cannot, however, conclude that there is a violation of scaling relation $2\nu + \gamma = 2\nu' + \gamma'$, considering the fact that our analysis here does not include the possibility of confluent singularity. To take into account the confluent singularity, we tried an analysis of Roskies-transformed series [17] with the confluent exponent 0.5, but the result was less convergent. It might suggest us to investigate inhomogeneous second-order differential approximants [18], which is another method to include the confluent singularity. Longer series might also solve the discrepancy, although the estimate from $\mu_2$ appears so stable when we change $L + M + N$ from 16 to 19 in our analysis of inhomogeneous differential approximants as mentioned above (See fig. 2).

We find the unphysical but dominant singularity at $u = u_1 = -0.2858 \pm 0.0003$ with a critical exponent

$$2\nu' + \gamma'\text{(unphysical)} = 1.892 \pm 0.034,$$

from $\mu_2$ and at $u = u_1 = -0.2858 \pm 0.0006$ with a critical exponent

$$2\nu'\text{(unphysical)} = 0.811 \pm 0.039,$$

from $\Lambda_2$. These values of the critical point are consistent with $u_1 = -0.2853(3)$ from the susceptibility [3], and the critical exponents are so sensitive to the value $u_1$ of the critical point if we would try a biased estimation, in which the change of the position of the critical point by 0.0005 reduces the critical exponent about 0.08 for $\mu_2$ and about 0.03 for $\Lambda_2$. 

11
5 Summary

We have calculated low-temperature series for the second moment of the correlation function in $d = 3$ simple-cubic Ising model to order $u^{26}$ by finite-lattice method, from which we have obtained the low-temperature series for the second-moment correlation length to order $u^{23}$ using the known low-temperature series for the susceptibility. The obtained series is 11 terms longer than those calculated previously. Preliminary analysis of the series by inhomogeneous differential approximants gives critical exponents $2\nu' + \gamma' \approx 2.55$ and $2\nu' \approx 1.27$. The latter is not inconsistent with the result from high-temperature series and $\epsilon$-expansion, but there is a discrepancy by 2 per cent between the former and the critical exponent from high-temperature series and $\epsilon$-expansion. It appears that further analysis of the series should be done including the possibility of the confluent singularity.

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