Distance Measures for Geometric Graphs

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Abstract

A geometric graph is a combinatorial graph, endowed with a geometry that is inherited from its embedding in a Euclidean space. Formulation of a meaningful measure of (dis-)similarity in both the combinatorial and geometric structures of two such geometric graphs is a challenging problem in pattern recognition. We study two notions of distance measures for geometric graphs, called the geometric edit distance (GED) and geometric graph distance (GGD). While the former is based on the idea of editing one graph to transform it into the other graph, the latter is inspired by inexact matching of the graphs. For decades, both notions have been lending themselves well as measures of similarity between attributed graphs. If used without any modification, however, they fail to provide a meaningful distance measure for geometric graphs—even cease to be a metric. We have curated their associated cost functions for the context of geometric graphs. Alongside studying the metric properties of GED and GGD, we investigate how the two notions compare. We further our understanding of the computational aspects of GGD by showing that the distance is \text{NP}-hard to compute, even if the graphs are planar and arbitrary cost coefficients are allowed.

1 Introduction

Graphs have been a widely accepted object for providing structural representation of patterns involving relational properties. The framework of representing complex and repetitive patterns using graphical structures can facilitate their description, manipulation, and recognition. While hierarchical patterns are commonly reduced to a string\cite{6} or a tree representation\cite{7}, non-hierarchical patterns generally require a graph representation. One of the most important aspects of such representation is that the problem of pattern recognition becomes the problem of quantifying (dis-)similarity between a query graph and a model or prototype graph. The problem of defining a relevant distance measure for a class of graphs has been looked into for almost five decades now and has a myriad of applications including chemical structure matching\cite{15}, fingerprint matching\cite{12}, face identification\cite{10}, and symbol recognition\cite{11}. All these applications demand a reliable and efficient means of comparing two graphs. A meaningful graph distance measure is expected to yield a small distance implying similarity, and a large distance revealing disparity.

Depending on the class of graphs of interest and the area of application, several methods have been proposed. If the use case requires a perfect matching of two graphs, then the problem of graph isomorphism can be considered\cite{5}; whereas, subgraph isomorphism can by applied for a perfect matching of parts of two graphs. These techniques are not, however, lenient with (sometimes minor) local and structural deformations of the two graphs. To address this issue, several alternative distance measures have been studied. We particularly investigate edit distance\cite{9,14} and inexact matching distance\cite{3}. The former makes use of elementary edit transformations (such as deletion, insertion, relabeling of vertices and edges), while the latter is based

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on partially matching two graphs through an inexact matching relation (Definition 11). And, the distance is defined as the minimum cost of transforming or matching one graph to the other. Although these distance measures have been battle-proven for attributed graphs (i.e., combinatorial graphs with finite label sets), the formulations seem inadequate in providing meaningful similarity measures for geometric graphs.

A geometric graph belongs to a special class of attributed graphs having an embedding into a Euclidean space \( \mathbb{R}^d \), where the vertex and edge labels are inferred from the Euclidean locations of the vertices and Euclidean lengths of the edges, respectively. In the last decade, there has been a gain in practical applications involving comparison of geometric graphs. Examples include road-network or map comparison [1], detection of chemical structures using their spatial bonding geometry, etc. In addition, large datasets like [13] are being curated by pattern recognition and machine learning communities.

Despite a rich literature on the matching of attributed graphs and a fair count of algorithms benchmarked by both the database community and the pattern recognition community, most of the frameworks become untenable for matching geometric graphs. They remain oblivious to the spatial geometry such graphs are endowed with, consequently giving rise to very artificial measures of similarity for geometric graphs. This is not surprising at all—geometric graphs are a special class of labeled graphs after all! For a geometric graph, the significant differences include:

(i) Edge relabeling is not an independent edit operation, but vertex labels dictate the incident edge labels.

(ii) Vertex relabeling amounts to its translation to a different location in the ambient space, and additionally incurs the cost of relabeling of all its adjacent edges.

1.1 Our Contribution

We study two distance measures, the geometric edit distance (GED) and geometric graph distance (GGD), in order to provide a meaningful measure of similarity between two geometric graphs. For attributed graphs the corresponding distance measures are equivalent as shown in [2, Proposition 1]. In contrast, we show in Section 2.3 they are not equivalent for geometric graphs. In addition to bounding each distance measure by a constant factor of the other in Proposition 18, we provide polynomial-time computable bounds on them.

We mention here the contribution of [4] for introducing GGD as well as discussing different definitions of edit distance in the context of geometric graphs. The authors also prove certain complexity results for GGD, which we improve upon in this paper. One of the major contributions of our study is to further our understanding of the computational complexity of GGD. In [4], the authors show that computing GGD is \( \mathcal{N}\mathcal{P} \)-hard for non-planar graphs, when arbitrary cost coefficients \( C_V, C_E \) (as defined in Definition 13) are allowed. For planar graphs, \( \mathcal{N}\mathcal{P} \)-hardness is proved under a very strict condition that \( C_V \ll C_E \). We show in Proposition 21 that computing the GGD is \( \mathcal{N}\mathcal{P} \)-hard, even if the graphs are planar and arbitrary \( C_V, C_E \) are allowed.

The paper is organized in the following way. In Section 2.1 and Section 2.2, we formally define the two distances GGD and GED, respectively, and explore some of their important properties. We then compare the two distances in Section 2.3. Finally, Section 3 is devoted to our findings on the computational complexity of the GGD.

2 Two Distances for Geometric Graphs

A geometric graph is a combinatorial graph that is also embedded in a Euclidean space. We begin with the formal definition.

**Definition 1** (Geometric Graph). A (finite) combinatorial graph \( G = (V^G, E^G) \) is called a geometric graph of \( \mathbb{R}^d \) if the vertex set \( V^G \subset \mathbb{R}^d \) and the Euclidean straight-line segments \( \{ab \mid (a,b) \in E^G\} \) intersect (possibly) at their endpoints.
We denote the set of all geometric graphs of $\mathbb{R}^d$ by $G(\mathbb{R}^d)$, and the subset of geometric graphs without any isolated vertex by $G_0(\mathbb{R}^d)$. Two geometric graphs $G = (V^G, E^G)$ and $H = (V^H, E^H)$ are said to be equal, written $G = H$, if and only if $V^G = V^H$ and $E^G = E^H$. We make no distinction between a geometric graph $G = (V^G, E^G)$ and its geometric realization as a subset of $\mathbb{R}^d$; an edge $(u, v) \in E^G$ can be identified as the line-segment $\overrightarrow{uv}$ in $\mathbb{R}^d$, and its length by the Euclidean length $|\overrightarrow{uv}|$. We denote by $\text{Vol}(G)$ the sum of the edge lengths of $G$.

2.1 Geometric Edit Distance (GED)

Given two geometric graphs $G, H \in G(\mathbb{R}^d)$, we transform $G$ into $H$ by applying a sequence of edit operations. The allowed edit operations and their costs are i) inserting (and deleting) a vertex costs nothing, ii) inserting (and deleting) an edge costs $C_E$ times its length, and iii) translating a vertex costs $C_V$ times the displacement of the vertex plus $C_E$ times the total change in the length of all its incident edges. The operations and their costs are summarized in Table 1. Throughout the paper, we assume that the cost coefficients $C_V$ and $C_E$ are positive constants. In order to denote a deleted vertex and a deleted edge, we introduce the dummy vertex $e_V$ and the dummy edge $e_E$, respectively. While computing edit costs, we follow the convention that $|e_E| = 0$, $|a - e_V| = 0$ for any $a \in \mathbb{R}^d$, and $(u, v) = e_E$ if either $u = e_V$ or $v = e_V$. For each operation $o$ listed in Table 1 note that its inverse, denoted $o^{-1}$, is also an edit operation with the same cost.

| Operation                              | Cost                                      |
|----------------------------------------|-------------------------------------------|
| delete (isolated) vertex $u$           | 0                                         |
| insert vertex $u \in \mathbb{R}^d$    | 0                                         |
| add edge $e$ between existing vertices | $C_E|e|$                                  |
| delete edge $e$                        | $C_E|e|$                                  |
| translate a vertex at $u \in \mathbb{R}^d$ to vertex at $v \in \mathbb{R}^d$ | $C_V|u - v| + \sum_{(s, u) \in E} C_E|\overrightarrow{su} - \overrightarrow{sv}|$ |

Table 1: Allowed edit operations on a geometric graph and associated costs

**Definition 2 (Edit Path).** Given two geometric graphs $G, H \in G(\mathbb{R}^d)$, an edit path $P$ from $G$ to $H$ is a (finite) sequence of edit operations $\{o_i\}_{i=1}^k$ that satisfies the following:

(a) $(o_k \circ \ldots \circ o_2 \circ o_1)(G) = H$, i.e., $P(G) = H$, and

(b) $o_{i+1}$ is a legal edit operation on $(o_1 \circ \ldots \circ o_2 \circ o_1)(G)$ for any $1 \leq i \leq k - 1$.

Note that we do not require for an intermediate edit operation to yield a geometric graph. The set of all edit paths between $G, H \in G(\mathbb{R}^d)$ is denoted by $\mathcal{P}(G, H)$. For an edit path $P = \{o_i\}_{i=1}^k$, the edit path $\{o_{i-1}\}_{i=0}^k$ from $H$ to $G$ is called its inverse path, and is denoted by $P^{-1}$. For any vertex $u \in V^G$ (resp. edge $e \in E^G$), we denote by $P(u)$ (resp. $P(e)$) the end result after its evolution under $P$. If $P$ deletes the vertex $u$ (resp. edge $e$), we write $P(v) = e_V$ (resp. $P(e) = e_E$). The cost, $\text{Cost}(P)$, of an edit path $P$ is defined to be the total cost of the individual edits.

**Definition 3 (Cost of Edit Paths).** The cost of an edit path $P \in \mathcal{P}(G, H)$, denoted $\text{Cost}(P)$, is the sum of the cost of the individual edits, i.e.,

$$\text{Cost}(P) \stackrel{\text{def}}{=} \sum_{o_i \in P} \text{Cost}(o_i).$$

It is not difficult to note that $\text{Cost}(P) = \text{Cost}(P^{-1})$. Then, GED($G, H$) is defined as cost of the least expensive edit path.
Definition 4 (Geometric Edit Distance). For geometric graphs $G, H \in \mathcal{G}(\mathbb{R}^d)$, their geometric edit distance, denoted $\text{GED}(G, H)$, is defined to be the infimum cost of the edit paths, i.e.,

$$\text{GED}(G, H) \overset{\text{def}}{=} \inf_{P \in \mathcal{P}(G, H)} \text{Cost}(P).$$

In Proposition 10 we prove that GED is, in fact, a metric on the space of geometric graphs without any isolated vertex. As also observed in [4], the following example demonstrates that the distance may not be attained by an edit path, unless an infinite number of edits are allowed: Consider $G, H \in \mathcal{G}(\mathbb{R}^2)$, where $G$ has only one edge $(u_1, u_2)$ and $H$ has only one edge $(v_1, v_2)$ as shown in Figure 1. For any fixed $k \geq 1$, consider the edit path $P_k = \{o_i\}_{i=1}^{2k}$, where $o_i$ translates the left vertex of $G$ up by a distance $1/k$ and then $o_{i+1}$ moves the right vertex by the same distance for any odd $i$. So, for any $i$

$$\text{Cost}(o_i) = C_V \frac{1}{k} + C_E \left[ \sqrt{(1/k)^2 + 1^2} - 1 \right] = C_V \frac{1}{k} + C_E \frac{1}{\sqrt{k^2 + 1} + 1},$$

and therefore

$$\text{GED}(G, H) \leq \text{Cost}(P_k) = \sum_{i=1}^{2k} \text{Cost}(o_i) = 2C_V + C_E \frac{2}{\sqrt{k^2 + 1} + 1} \xrightarrow{k \to \infty} 2C_V.$$

Now, if we assume that $C_E > C_V$, then any edit path with an edge deletion costs more than $2C_V$ from (2). Therefore, $\text{GED}(G, H) = 2C_V$. However, there is no edit path that attains this cost.

In Definition 3 the cost of an edit path $P$ is defined as the aggregated cost from the individual edits involved in $P$. Another perspective of the cost of $P$ is the total amount paid by $P$ for the evolution of each vertex and edge of $G$ and $H$. We make this notion more precise by tracking the evolution of vertices and edges through their orbit.

Definition 5 (Orbit of a Vertex). Let $P \in \mathcal{P}(G, H)$ be an edit path and $u$ a vertex of $G$. The orbit of $u$ under $P = \{o_i\}_{i=1}^k$ is the sequence of vertices $\{u_i\}_{i=0}^k$, where $u_0 = u$ and $u_i = (o_i \circ o_{i-1} \circ \ldots \circ o_1)(u)$ for $i \geq 1$. And, the cost of the orbit, denoted $\text{Cost}_P(u)$, is defined by

$$\text{Cost}_P(u) \overset{\text{def}}{=} C_V \sum_{i=1}^k |u_i - u_{i-1}|.$$

The $i$th summand above is positive only if $o_i$ is a translation of the vertex. Using the triangle inequality, we can immediately note the following fact.

Lemma 6 (Cost of Vertex Orbit). For a vertex $u \in V^G$ and $P \in \mathcal{P}(G, H)$, we have

$$\text{Cost}_P(u) \geq C_V |u - P(u)|.$$
Figure 2: Two graphs $G, H \in \mathcal{G}(\mathbb{R}^2)$ have been shown on the left and right, respectively. In the middle, the evolution of $G$ under an edit path $P = \{o_1, o_2, o_3, o_4\}$ is demonstrated. The edit $o_1$ deletes the edge $(u_1, u_2)$, then $o_2$ translates $u_2$ to $v_3$, after that $o_4$ translates $u_3$ to $v_2$, and finally $o_4$ inserts the edge $(v_1, v_2)$. The orbit of the vertex $u_2$ is $\{u_2, u_2, v_3, v_3, v_3\}$, whereas the orbit of $(u_2, u_3)$ is $\{(u_2, u_3), (u_2, u_3), (v_3, u_3), (v_3, v_2), (v_3, v_2)\}$.

We similarly define the orbit of an edge and its cost.

**Definition 7 (Orbit of an Edge).** Let $P \in \mathcal{P}(G, H)$ be an edit path and $e$ an edge of $G$. The orbit of $e$ under $P = \{o_i\}_{i=1}^k$ is the sequence of edges $\{e_i\}_{i=0}^k$, where $e_0 = e$ and $e_i = (o_i \circ o_{i-1} \circ \ldots \circ o_1)(e)$ for $i \geq 1$. And, the cost of the orbit, denoted $\text{Cost}_P(e)$, is defined by

$$
\text{Cost}_P(e) \overset{\text{def}}{=} C_E \sum_{i=1}^k |e_i| - |e_{i-1}|.
$$

We note that deletion of the edge or translation of an incident vertex are the only edit operations in $P$ that can potentially contribute to a positive summand in the cost function above. Again, the triangle inequality implies the following lemma.

**Lemma 8 (Cost of Edge Orbit).** For an edge $e \in E^G$ and $P \in \mathcal{P}(G, H)$, we have

$$
\text{Cost}_P(e) \geq C_E |e| - |P(e)|.
$$

In particular, $\text{Cost}_P(e) \geq |e|$ if $P$ eventually deletes $e$, i.e., $P(e) = \epsilon_E$.

For examples of vertex and edge orbits see Figure 2. In order to describe $\text{Cost}(P)$ in terms of the costs of individual orbits, we note that $\text{Cost}(P)$ accounts for the costs of the orbits of:

(a) vertices $u \in V^G$ that end up as a vertex of $H$, i.e., $P(u) \neq \epsilon_V$

(b) vertices $u \in V^G$ with $P(u) = \epsilon_V$

(c) vertices $v \in V^H$ that have been inserted, i.e., $P^{-1}(v) = \epsilon_V$

(d) edges $e \in E^G$ that end up as an edge of $H$, i.e., $P(e) \neq \epsilon_E$

(e) edges $e \in E^G$ with $P(e) = \epsilon_E$

(f) edges $f \in E^H$ that have been inserted, i.e., $P^{-1}(f) = \epsilon_E$

(g) vertices and edges that have been inserted at some point and have also been deleted eventually.
Moreover, we observe that two vertex (resp. edge) orbits \( \{x_i\} \) and \( \{y_i\} \) intersect at the \( i_0 \)th position only if \( x_i = y_i = \epsilon_V \) (resp. \( x_i = y_i = \epsilon_E \)) for all \( i \geq i_0 \). As a consequence, the positive summands in the costs of two orbits are necessarily distinct. Accumulating the costs for all orbits of type (a)--(f), we can, therefore, write

\[
\text{Cost}(P) \geq \sum_{u \in V^G} \text{Cost}_P(u) + \sum_{v \in V^H} \text{Cost}_{P^{-1}}(v) \\
+ \sum_{e \in E^G} \text{Cost}_P(e) + \sum_{f \in E^H} \text{Cost}_{P^{-1}}(f).
\]

Equation (1) together with Lemma 6 and Lemma 8 readily imply the following useful result.

**Lemma 9.** For any edit path \( P \in \mathcal{P}(G, H) \), it holds that

\[
\text{Cost}(P) \geq \sum_{u \in V^G} C_V |u - P(u)| + \sum_{e \in E^G} |e| - |P(e)| + \sum_{f \in E^H} |e| + \sum_{f \in E^H} |e|.
\]

**Proposition 10 (GED is a Metric).** The GED defines a metric on \( \mathcal{G}_0(\mathbb{R}^d) \), the space of geometric graphs without any isolated vertex.

**Proof.** Non-negativity. Since the cost of edit paths are non-negative, Definition 4 implies that GED \((G, H)\) is non-negative for any \( G, H \in \mathcal{G}_0(\mathbb{R}^d) \).

Separability. If \( G(H) = 0 \), we claim that \( G = H \), i.e., \( V^G = V^H \) and \( E^G = E^H \). In order to show that \( V^G = V^H \), it suffices to show that the Hausdorff distance \( r := d_H(V^G, V^H) \) between the vertex sets is zero. Fix

\[
2\xi = \begin{cases} 
C_E \min\{l^G, l^H\}, & \text{if } r = 0 \\
\min\{C_V r, C_E l^G, C_E l^H\}, & \text{if } r \neq 0
\end{cases}
\]

where \( l^G \) and \( l^H \) denote the smallest edge lengths of \( G \) and \( H \), respectively. Since \( \xi > 0 \), the definition of GED implies that there is an edit path \( P \in \mathcal{P}(G, H) \) with \( \text{Cost}(P) \leq \xi \). Consequently, each of the four summands in (2) is no larger than \( \xi \). We immediately see that there is no edge \( e \in E^G \) such that \( P(e) = \epsilon_E \). Otherwise, the third summand in (2) would be at least

\[
C_E |e| \geq C_E l^G \geq 2\xi > \xi,
\]

leading to a contradiction. The last inequality above is due to the observation that \( \xi > 0 \). Similarly using the fourth summand in (2), we conclude there is no edge \( f \in E^H \) such that \( P^{-1}(f) = \epsilon_E \). In other words, \( P \) does not delete any edge of \( G \) or \( H \), i.e., \( |E^G| = |E^H| \). As a result, we can further say that no vertex of \( G \) can be removed and no vertex of \( H \) can be inserted, since the input graphs do not have any isolated vertices. Since \( H = P(G) \), the graphs \( G \) and \( H \) must be isomorphic. Lastly, we show that \( V^G = V^H \), i.e., \( r = 0 \). If not, i.e., \( r \neq 0 \) and \( u_0 \in V^G \) such that all the vertices of \( H \) are at least \( r \) distance away from it, then

\[
C_V |u_0 - P(u_0)| \geq C_V r \geq 2\xi > \xi.
\]

This is a contradiction, because the first term in (2) exceeds \( \xi \). So, \( r = 0 \). Therefore, \( G = H \).
Symmetry. Each elementary edit operation can be reversed at exactly the same cost. Given an edit path $P \in \mathcal{P}(G,H)$, we can reverse the operations to get an edit path $P^{-1} \in \mathcal{P}(H,G)$ with $\text{Cost}(P) = \text{Cost}(P^{-1})$. By Definition \[\text{Definition 4},\] for an arbitrary $\xi > 0$ there exists $P \in \mathcal{P}(G,H)$ such that $\text{Cost}(P) \leq \text{GED}(G,H) + \xi$. On the other hand,

$$\text{GED}(H,G) \leq \text{Cost}(P^{-1}) = \text{Cost}(P) \leq \text{GED}(G,H) + \xi.$$  

Since $\xi$ is arbitrary, this implies $\text{GED}(H,G) \leq \text{GED}(G,H)$. By a similar argument, one can also show $\text{GED}(H,G) \geq \text{GED}(G,H)$. Together, they imply $\text{GED}(H,G) = \text{GED}(G,H)$.

Triangle Inequality. Fix an arbitrary $\xi > 0$ and $G, H, I \in \mathcal{G}_0(\mathbb{R}^d)$. By Definition \[\text{Definition 4},\] there must exist edit paths $P_1 \in \mathcal{P}(G,H)$ and $P_2 \in \mathcal{P}(H,I)$ such that $\text{Cost}(P_1) \leq \text{GED}(G,H) + \xi/2$ and $\text{Cost}(P_2) \leq \text{GED}(H,I) + \xi/2$. If we define $P$ to be the concatenation of the edit operations from $P_1$ and $P_2$ in the same order, then $P \in \mathcal{P}(G,I)$. Moreover, $\text{Cost}(P) = \text{Cost}(P_1) + \text{Cost}(P_2)$. Now,

$$\text{GED}(G,I) \leq \text{Cost}(P), \text{ from the Definition of GED}$$

$$= \text{Cost}(P_1) + \text{Cost}(P_2)$$

$$\leq \left[ \text{GED}(G,H) + \frac{\xi}{2} \right] + \left[ \text{GED}(H,I) + \frac{\xi}{2} \right]$$

$$= \text{GED}(G,H) + \text{GED}(H,I) + \xi.$$  

Since the choice of $\xi$ is arbitrary, we get $\text{GED}(G,I) \leq \text{GED}(G,H) + \text{GED}(H,I)$.

2.2 Geometric Graph Distance (GGD)

The definition of GED is very intuitive but not at all suited for computational purposes. Firstly, there could be infinitely many locations a vertex is allowed to be translated to. Secondly, there are infinitely many edit operations—each elementary edit operation can be reversed at exactly the same cost. Given an edit path $P \in \mathcal{P}(G,H)$, we can reverse the operations to get an edit path $P^{-1} \in \mathcal{P}(H,G)$ with $\text{Cost}(P) = \text{Cost}(P^{-1})$. By Definition \[\text{Definition 4},\] for an arbitrary $\xi > 0$ there exists $P \in \mathcal{P}(G,H)$ such that $\text{Cost}(P) \leq \text{GED}(G,H) + \xi$. On the other hand,

$$\text{GED}(H,G) \leq \text{Cost}(P^{-1}) = \text{Cost}(P) \leq \text{GED}(G,H) + \xi.$$  

Since $\xi$ is arbitrary, this implies $\text{GED}(H,G) \leq \text{GED}(G,H)$. By a similar argument, one can also show $\text{GED}(H,G) \geq \text{GED}(G,H)$. Together, they imply $\text{GED}(H,G) = \text{GED}(G,H)$.
Definition 12 (Cost of a Matching). Let \( G, H \in \mathcal{G}(\mathbb{R}^d) \) be geometric graphs and \( \pi \in \Pi(G, H) \) an inexact matching. The cost of \( \pi \), denoted \( \text{Cost}(\pi) \), is defined as

\[
\text{Cost}(\pi) = \sum_{u \in V^G} C_V |u - \pi(u)| + \sum_{e \in E^G} C_E |e| - |\pi(e)| + \sum_{e \in E^G} C_E |e| + \sum_{f \in E^H} C_E |f|.
\]

(3)

vertex translations  
edge translations  
edge deletions  
edge deletions

Definition 13 (GGD). For geometric graphs \( G, H \in \mathcal{G}(\mathbb{R}^d) \), their geometric graph distance \( \text{GGD}(G, H) \), is defined as the minimum cost of an inexact matching, i.e.,

\[
\text{GGD}(G, H) = \min_{\pi \in \Pi(G, H)} \text{Cost}(\pi).
\]

The minimum cost matching between two graphs along with their GGD has been illustrated in Figure 1.

Lemma 14. Let \( G, H \in \mathcal{G}(\mathbb{R}^d) \) be geometric graphs. For any \( \pi \in \Pi(G, H) \), we have

\[
\text{Cost}(\pi) \geq \sum_{u \in V^G} C_V |u - \pi(u)| + \sum_{e \in E^G} C_E |e| - |\pi(e)| + \min \left\{ \sum_{e \in E^G} C_E |e|, \sum_{f \in E^H} C_E |f| \right\}.
\]

Proof. Without any loss of generality, we assume that

\[
\sum_{e \in E^G} C_E |e| \leq \sum_{f \in E^H} C_E |f|.
\]

(4)

From (3), we have

\[
\text{Cost}(\pi) = \sum_{u \in V^G} C_V |u - \pi(u)| + \sum_{e \in E^G} C_E |e| - |\pi(e)| + \sum_{e \in E^G} C_E |e| + \sum_{f \in E^H} C_E |f|
\]

\[
= \sum_{u \in V^G} C_V |u - \pi(u)| + \sum_{e \in E^G} C_E |\pi(e) - |e|| + \sum_{f \in E^H} C_E |f| - \sum_{e \in E^G} C_E |e| + 2 \sum_{e \in E^G} C_E |e|
\]

\[
= \sum_{u \in V^G} C_V |u - \pi(u)| + \sum_{e \in E^G} C_E |\pi(e) - |e|| + \sum_{f \in E^H} C_E |f| - \sum_{e \in E^G} C_E |e| + 2 \sum_{e \in E^G} C_E |e|
\]

\[
+ 2 \sum_{e \in E^G} C_E |e|, \text{ from (4)}
\]

\[
\geq \sum_{u \in V^G} C_V |u - \pi(u)| + \sum_{e \in E^G} C_E (|\pi(e)| - |e|) + \sum_{f \in E^H} C_E |f| - \sum_{e \in E^G} C_E |e|
\]

\[
+ 2 \sum_{e \in E^G} C_E |e|, \text{ by the triangle inequality}
\]

\[
\geq \sum_{u \in V^G} C_V |u - \pi(u)| + \sum_{e \in E^G} C_E (|\pi(e)| - |e|) + \sum_{f \in E^H} C_E |f| - \sum_{e \in E^G} C_E |e|
\]

\[
\geq \sum_{u \in V^G} C_V |u - \pi(u)| + \sum_{e \in E^G} C_E (|\pi(e)| - |e|) + \sum_{f \in E^H} C_E |f| - \sum_{e \in E^G} C_E |e|
\]
Proof. For any arbitrary matching \(\pi \in \Pi(G, H)\), from Lemma 14 we get

\[ C_E|\text{Vol}(G) - \text{Vol}(H)| \leq \text{Cost}(P). \]

Since \(\pi\) is arbitrary, we conclude \(C_E|\text{Vol}(G) - \text{Vol}(H)| \leq \text{GGD}(G, H)\).

For the second inequality, we choose the trivial matching \(\pi_0 \in \Pi(G, H)\), where \(\pi_0(u) = \pi_0^{-1}(v) = e_V\) for all \(u \in V^G\) and \(v \in V^H\). So,

\[ \text{GGD}(G, H) \leq \text{Cost}(\pi) = C_E[\text{Vol}(G) + \text{Vol}(H)]. \]

This proves the result. \(\square\)

The follow proposition provides a lower and upper bound for the GGD that are computable in polynomial time.

**Proposition 15** (Bounding the GGD). For geometric graphs \(G, H \in \mathcal{G}(\mathbb{R}^d)\), we have

\[ C_E|\text{Vol}(G) - \text{Vol}(H)| \leq \text{GGD}(G, H) \leq C_E|\text{Vol}(G) + \text{Vol}(H)|. \]

Proof. For any arbitrary matching \(\pi \in \Pi(G, H)\), from Lemma 14 we get

\[ C_E|\text{Vol}(G) - \text{Vol}(H)| \leq \text{Cost}(P). \]

Since \(\pi\) is arbitrary, we conclude \(C_E|\text{Vol}(G) - \text{Vol}(H)| \leq \text{GGD}(G, H)\).

For the second inequality, we choose the trivial matching \(\pi_0 \in \Pi(G, H)\), where \(\pi_0(u) = \pi_0^{-1}(v) = e_V\) for all \(u \in V^G\) and \(v \in V^H\). So,

\[ \text{GGD}(G, H) \leq \text{Cost}(\pi) = C_E[\text{Vol}(G) + \text{Vol}(H)]. \]

\(\square\)

As also shown in [4], the GGD is also a metric. We present a proof here, using our notation, for the sake of completion.

**Proposition 16** (GGD is a Metric). The GGD defines a metric on \(\mathcal{G}_0(\mathbb{R}^d)\), the space of geometric graphs without any isolated vertex.

Proof. **Non-negativity.** Since the cost of any matching in \(\Pi(G, H)\) is non-negative, Definition 13 implies that GGD\((G, H)\) is non-negative for any \(G, H \in \mathcal{G}_0(\mathbb{R}^d)\).

**Separability.** If GGD\((G, H) = 0\), then there is \(\pi \in \Pi(G, H)\) with \(\text{Cost}(\pi) = 0\). So, all the four summands in 3 are identically zero. In particular, the third and fourth summands imply that no edge has been deleted from \(G\) or \(H\) by \(\pi\), i.e., \(|E^G| = |E^H|\). Since the graphs do not have any isolated vertex, this implies that \(\pi(u) \neq e_V, \pi(v) \neq e_V\) for all \(u \in V^G\) and \(v \in V^H\). As a result, \(|E^G| = |E^H|\). Moreover, the first summand of 3 implies that \(\pi(u) = u\) for all \(u \in V^G\). Therefore, \(G = H\).
**Symmetry.** We conclude that \( \text{GGD}(G, H) = \text{GGD}(H, G) \) due to the fact that any matching in \( \Pi(G, H) \) induces a matching in \( \Pi(H, G) \) with exactly the same cost and vice versa.

**Triangle Inequality.** For the triangle inequality, let us assume that \( \text{Cost}(\pi_1) = \text{GGD}(G, H) \) and \( \text{Cost}(\pi_2) = \text{GGD}(H, I) \) for some \( \pi_1 \in \Pi(G, H) \) and \( \pi_2 \in \Pi(H, I) \). For any \( u \in V^G \) and \( v \in V^I \), define \( \pi \in \Pi(G, I) \) such that:

\[
\pi(u) = \begin{cases} 
\pi_2 \circ \pi_1(u), & \text{if } \pi_1(u) \neq \epsilon \\
\epsilon_V, & \text{otherwise}
\end{cases}
\]

and

\[
\pi^{-1}(v) = \begin{cases} 
\pi_1^{-1} \circ \pi_2^{-1}(v), & \text{if } \pi_2^{-1}(u) \neq \epsilon \\
\epsilon_V, & \text{otherwise}
\end{cases}
\]

Using the triangle inequality, it can be easily seen from (3) that \( \text{Cost}(\pi) \leq \text{Cost}(\pi_1) + \text{Cost}(\pi_2) \). So,

\[
\text{GED}(G, I) \leq \text{Cost}(\pi), \text{ from the Definition of GED}
\]

\[
\leq \text{Cost}(\pi_1) + \text{Cost}(\pi_2)
\]

\[
= \text{GGD}(G, H) + \text{GGD}(H, I).
\]

Therefore, we get \( \text{GGD}(G, I) \leq \text{GGD}(G, H) + \text{GGD}(H, I) \) as desired. \( \square \)

### 2.3 Comparing GED and GGD

As we now have the two notions of distances under our belts, the question of how they compare arises naturally. We have already pointed out that the analogous notions for attributed graphs yield equivalent distances. To our surprise, they are not generally equal for geometric graphs, as the following proposition demonstrates.

**Proposition 17.** Given any \( D > 0 \), there exist graphs \( G, H \in \mathcal{G}(\mathbb{R}) \) such that

\[
\text{GGD}(G, H) = D \text{ and GED}(G, H) = \left(1 + \frac{C_E}{C_V}\right) D.
\]

In particular, \( \text{GGD}(G, H) < \text{GED}(G, H) \).

**Proof.** We take two graphs \( G, H \in \mathcal{G}(\mathbb{R}) \) as shown in Figure 3. In each graph, the two vertices are separated by a distance \( L \), whereas the second graph is a copy of the first but shifted by \( x \). We also choose

\[
x = \frac{D}{2C_V} \text{ and } L = \left(1 + \frac{2C_V}{C_E}\right) x.
\]

To see that \( \text{GGD}(G, H) = D \), we consider the matching \( \pi(u_i) = v_i \) for \( i = 1, 2 \). The cost of the matching is

![Figure 3: The graphs G (top) and H (bottom) are embedded in the real line, where \( u_2 - u_1 = v_2 - v_1 = L \) and \( v_2 - u_2 = v_1 - u_1 = x \).](image-url)
\[ \text{Cost} (\pi) = C_V \sum_{i=1}^{2} |u_i - v_i| = C_V \sum_{i=1}^{2} x = 2C_Vx = D. \]

It is worth noting here that a matching \( \pi' \) that is not bijective on the vertex sets has cost

\[ \text{Cost} (\pi') \geq C_E L > C_E \times \frac{2C_V}{C_E}x = D = \text{Cost} (\pi). \]

Since \( L > x \), the cost of \( \pi \) is also (strictly) smaller that \( 2C_V L \), which is the cost of the other possible bijective matching. So, we have \( \text{GGD}(G, H) = D \).

In order to compute \( \text{GED}(G, H) \), we consider the edit path \( P_0 \) that moves the vertex \( u_1 \) to \( v_1 \), then moves \( u_2 \) to \( v_2 \). The cost of \( P_0 \) is

\[ 2C_Vx + 2C_Ex = 2C_Vx \left(1 + \frac{C_E}{C_V}\right) = \left(1 + \frac{C_E}{C_V}\right)D. \]

We now claim that the cost of any edit path \( P \) is at least \((1 + C_E/C_V)D\). Consider the following two cases:

**Case I.** If \( P(u_1, u_2) \neq \epsilon_E \), then from (2), we have

\[ \text{Cost}(P) \geq 2C_EL = 2(C_E + 2C_V)x = 2(C_E + 2C_V) \frac{D}{2C_V} = (2 + C_E/C_V)D > (1 + C_E/C_V)D. \]

**Case II.** For this case, we assume that \( P(u_1, u_2) = \epsilon_E \). So, \( P \) contains only vertex translations. Let \( O = \{o_i\}_{i=1}^{k} \) be the subsequence of \( P \) containing only those translations that do not flip the order of the endpoints of the incident edge. Due to the position of \( G \) and \( H \), it is evident that \( O \) is non-empty. Moreover, the vertices must travel at least \( x \) distance each under \( O \). When an endpoint \( u \) is moved to a location \( w \in \mathbb{R} \) by such an \( o_i \), the associated cost of translating the edge becomes \( C_E |w - u| \). Therefore, the cost

\[ \text{Cost}(P) \geq \text{Cost}(O) \geq 2C_Vx + 2C_E2x = 2(C_E + C_V) \frac{D}{2C_V} = (1 + C_E/C_V)D. \]

Considering the above the cases, we conclude that \( \text{GED}(G, H) = (1 + C_E/C_V)D \). \( \square \)

More generally, we prove that following result to compare the two distances.

**Proposition 18.** For any two geometric graphs \( G, H \in \mathcal{G}(\mathbb{R}^d) \), we have

\[ \text{GGD}(G, H) \leq \text{GED}(G, H) \leq \left(1 + \Delta \frac{C_E}{C_V}\right) \text{GGD}(G, H), \]

where \( \Delta \) denotes the maximum degree of the graphs \( G, H \).

**Proof.** Take an arbitrary edit path \( P \in \mathcal{P}(G, H) \). Let us define a matching \( \pi_P \in \Pi(G, H) \) such that

\[ \pi_P \overset{\text{def}}{=} \{(u, P(u)) \mid u \in V^G\} \cup \{(P^{-1}(v), v) \mid v \in V^H\}. \]

This definition of \( \pi_P \) implies that \( P(u) = \pi_P(u) \) for all \( u \in V^G \), \( P(e) = \pi_P(e) \) for all \( e \in E^G \), and \( P^{-1}(f) = \pi_P^{-1}(f) \) for all \( f \in E^H \). From (3) and Lemma 9 it follows that \( \text{Cost} (\pi_P) \leq \text{Cost} (P) \). The definition of \( \text{GGD}(G, H) \) then implies that

\[ \text{GGD}(G, H) \leq \text{Cost}(\pi_P) \leq \text{Cost}(P). \]

Since \( P \) is chosen arbitrarily, the definition of \( \text{GED}(G, H) \) then implies the first inequality.

For the second inequality, we take an arbitrary \( \pi \in \Pi(G, H) \). From \( \pi \), we define an edit path \( P_{\pi} \) to be the sequence \( (D_E, D_V, T_V, I_V, I_E) \) of edit operations, where
(i) \( D_E \) is a sequence of deletions of edges \( e \in E^G \) with \( \pi(e) = \epsilon_E \).

(ii) \( D_V \) is a sequence of deletions of vertices \( u \in V^G \) with \( \pi(u) = \epsilon_V \).

(iii) \( T_V \) is a sequence of translations of vertices \( u \in V^G \) with \( \pi(u) \neq \epsilon_V \) to \( \pi(u) \).

(iv) \( I_V \) is a sequence of insertions of vertices \( v \in V^H \) with \( \pi^{-1}(v) = \epsilon_V \), and

(v) \( I_E \) is a sequence of insertions of edges \( f \in E^H \) with \( \pi^{-1}(f) = \epsilon_E \).

Each of the above sequences (i)-(v) is unique up to the ordering of its operations. Also in \( P_\pi \), the edges are deleted in \( D_E \) before deleting their endpoints in \( D_V \), and the edges are inserted in \( I_E \) only after inserting their endpoints in \( I_V \). Consequently, \( P_\pi \) defines a legal edit path between \( G \) and \( H \), i.e., \( P_\pi \in \mathcal{P}(G,H) \). We claim that

\[
\text{Cost}(P_\pi) \leq \left( 1 + \Delta \frac{C_E}{C_V} \right) \text{Cost}(\pi).
\]

To prove the claim, we note that \( P_\pi \) does not insert any vertex or edge that has been later deleted. As a result, the item (g) above \( \square \) has a zero cost. So, \( \square \) is, in fact, an equality:

\[
\text{Cost}(P_\pi) = \sum_{u \in V^G} \text{Cost}_{P_\pi}(u) + \sum_{e \in E^G} \text{Cost}_{P_\pi}(e) + \sum_{v \in V^H} \text{Cost}_{P_\pi}(v) + \sum_{f \in E^H} \text{Cost}_{P_\pi}(f).
\]

Moreover, a deleted (resp. inserted) vertex has never been translated, yielding a zero cost for its orbit. So, the second and the third summands are identically zero. We can then write

\[
\text{Cost}(P_\pi) = \sum_{u \in V^G} \text{Cost}_{P_\pi}(u) + \sum_{e \in E^G} \text{Cost}_{P_\pi}(e) + \sum_{f \in E^H} \text{Cost}_{P_\pi}(f).
\]

In order to get upper bound on the last term, we observe for any edge \( e = (u_1, u_2) \in E^G \) with \( \pi(e) \neq \epsilon_E \) that its orbit under \( T_V \) is \( \{(u_1, u_2), (u_1, \pi(u_2)), (\pi(u_1), \pi(u_2))\} \). The cost of the orbit of each \( e \) then is

\[
C_E \left( ||u_1 - \pi(u_2)|| - |u_1 - u_2| + ||\pi(u_1) - \pi(u_2)|| - |u_1 - \pi(u_2)| \right) \leq C_E (|u_2 - \pi(u_2)| + |u_1 - \pi(u_1)|).
\]

So,

\[
\text{Cost}(P_\pi) \leq \text{Cost}(\pi) + \sum_{e \in E^G} \text{Cost}_{P_\pi}(e).
\]
\[
\leq \text{Cost}(\pi) + \sum_{e=(u_1, u_2) \in E^G, \pi(e) \neq e} C_E(|u_2 - \pi(u_2)| + |u_1 - \pi(u_1)|)
\leq \text{Cost}(\pi) + \Delta \sum_{u \in V} C_E|u - \pi(u)|
\leq \text{Cost}(\pi) + \Delta \frac{C_E}{C_V} \sum_{u \in V} C_V|u - \pi(u)|
\leq \text{Cost}(\pi) + \Delta \frac{C_E}{C_V} \text{Cost}(\pi) = \left(1 + \Delta \frac{C_E}{C_V}\right) \text{Cost}(\pi).
\]

By the definition GED, it is implies that \(\text{GED}(G, H) \leq \left(1 + \Delta \frac{C_E}{C_V}\right) \text{Cost}(\pi)\). Since \(\pi\) is chosen arbitrarily, we then conclude from the definition of GGD that \(\text{GGD}(G, H) \leq \left(1 + \Delta \frac{C_E}{C_V}\right) \text{GGD}(G, H)\).

We remark that the configuration in Figure 1 and Proposition 17 show that the bounds presented in Proposition 18 are, in fact, tight.

### 3 Computational Complexity

In this section, we discuss the computational aspects of the GGD. The computation is algorithmically feasible, since the there are only a finite number of matchings between two graphs. However, it has been already shown in [4] that the distance is generally hard to compute. We define the decision problem as follows.

**Definition 19 (PROBLEM GGD).** Given geometric graphs \(G, H \in \mathcal{G}(\mathbb{R}^d)\) and \(\tau \geq 0\), is there a matching \(\pi \in \Pi(G, H)\) such that \(\text{Cost}(\pi) \leq \tau\)?

In [4], the authors show that PROBLEM GGD is \(\mathcal{NP}\)-hard for non-planar graphs. For planar graphs, however, its \(\mathcal{NP}\)-hardness is proved under the very strict condition that \(C_V << C_E\). In both cases, the problem instances seem non-practical. In Proposition 21 we prove a stronger result that the problem is \(\mathcal{NP}\)-hard, even if the graphs are planar and arbitrary \(C_V, C_E\) are allowed. Our reduction is from the well-known 3-PARTITION problem.

**Definition 20 (Problem 3-PARTITION).** Given positive integers \(N > 1, B\) and a multiset of positive integers \(S = \{a_1, a_2, \ldots, a_{3N}\}\) so that \(\frac{1}{4} < a_i < \frac{1}{2}\) and \(\sum_{i=1}^{3N} a_i = NB\), does there exist a partition of \(S\) into \(N\) multisets \(S_1, S_2, \ldots, S_N\) such that \(|S_i| = 3\) and \(\sum_{a \in S_i} a = B\) for all \(1 \leq i \leq N\)?

The problem is known to be strongly \(\mathcal{NP}\)-complete [8]. We reduce an instance \(I := (N, B, S)\) of 3-PARTITION to an instance of PROBLEM GGD.

**Proposition 21 (Hardness of PROBLEM GGD).** The PROBLEM GGD is \(\mathcal{NP}\)-hard to decide. This result holds even if

(i) the input graphs are embedded in \(\mathbb{R}^2\), and

(ii) the cost coefficients \(C_E, C_V\) are arbitrary.

**Proof.** Given an instance \(I := (N, B, S)\) of 3-PARTITION, we construct two planar graphs \(G, H\) such that the existence of a 3-PARTITION of \(S\) implies \(\text{GGD}(G, H) \leq \tau\), otherwise \(\text{GGD}(G, H) > \tau\).

We now describe the construction of \(G\) and \(H\). Each of them will have a certain number of connected components, which we call blobs. A blob of size \(k\) is a connected block of \(k\) vertices \(\{u_1, u_2, \ldots, u_k\}\) in the
upper row and \( k \) vertices \( \{l_1, l_2, \ldots, l_k\} \) in the lower row. The two rows are separated by distance \( L \), and the consecutive vertices in each row are equidistant. The choice of \( L \) will be made explicit later on. Except for \( u_1 \), each vertex \( u_j \) in the upper row is connected to \( l_{j-1} \) and \( l_j \) in the bottom row, making the blob path-connected. The configuration of such a typical blob and its shorthand are depicted in Figure 4.

We define \( G \) as the graph with \( 3N \) many blobs \( G_1, G_2, \ldots, G_{3N} \) of size \( a_1, a_2, \ldots, a_{3N} \), respectively, placed side-by-side so that they do not overlap. Now, \( H \) is defined as the graph with \( N \) many blobs \( H_1, H_2, \ldots, H_N \) of size \( B \) each placed side-by-side so that they do not overlap. Now, \( G \) and \( H \) are placed side-by-side in a bounding-box of width \( x \) and height \( L \), where

\[
x = \frac{\tau}{2C_v(N+1)NB}, \quad \text{and} \quad L = \frac{\tau}{2C_E(N+1)}.
\]

We remark that appropriately small inter-vertex and inter-blob distances can always be chosen to fit them in the bounding-box, keeping the length of all the vertical (resp. slanted) edges the same. See Figure 5 for the configuration of the graphs.

Let us first assume that \( \mathcal{I} \) is a YES instance, and that \( \{S_1, S_2, \ldots, S_N\} \) is a partition of \( S \). A (bijective) matching \( \pi \in \Pi(G, H) \) can be defined in the following way. For any \( i \in \{1, 2, \ldots, N\} \), if \( S_i = \{a_{i_1}, a_{i_2}, a_{i_3}\} \) then the upper and lower vertices of the blobs \( G_{i_1}, G_{i_2}, \text{ and } G_{i_3} \) of \( G \) are mapped, consecutively, to the corresponding upper and lower vertices of the \( i \)th blob \( H_i \) of \( H \). We argue that \( \text{Cost}(\pi) \leq \tau \). In light of (3), the cost is the total contribution from the following two types:

(a) There are \((2NB - N)\) many edges in \( G \), whereas there are \((2BN - 3N)\) many in \( H \). So, there are exactly \( 2N \) many vertical edges \( e \) in \( G \) such that \( \pi(e) = e_V \). The resulting cost is at most \( C_E \cdot 2N \cdot L \).
(b) Since no vertex in the upper row is mapped to a vertex in the lower row and vice versa, we have

\[ |u - \pi(u)| \leq x \text{ for all } u \in G. \]

There are \(2(\sum_{i}^{3N} a_i) = 2NB\) many vertices in \(G\), so the total cost for vertex translation is at most \(C_V \cdot x \cdot 2NB\).

As a result, the total cost is

\[ \text{Cost}(\pi) \leq 2C_ENL + 2C_VNBx = 2C_EN \frac{\tau}{2C_E(N + 1)} + 2C_VNB \frac{\tau}{2C_V(N + 1)NB} = \tau. \]

Hence, GGD\((G, H)\) \(\leq \tau\).

For the other direction, we assume that GGD\((G, H)\) \(\leq \tau\), i.e., there is a matching \(\pi \in \Pi(G, H)\) such that \(\text{Cost}(\pi) \leq \tau\). We observe that \(\pi(V^G) \neq \{\epsilon_V\}\). Otherwise, from 3 the cost of \(\pi\) would be

\[ \text{Cost}(\pi) \geq C_E \text{Vol}(G) + C_E \text{Vol}(H) \geq C_E(4NB - 4N)L = 4C_EN(B - 1) \tau \frac{C_V(N + 1)}{2C_E(N + 1)} = \frac{2N(B - 1)\tau}{N + 1} > \tau. \]

The above volume estimates use the fact that there are \((2NB - 3N)\) edges in \(G\) and \((2NB - N)\) edges in \(H\), and the length of each edge is at least \(L\). Also, the last inequality above is strict because \(2N > N + 1\) for any \(N > 1\). Since this is a contradiction, there must be some \(u_0 \in V^G\) with \(\pi(u_0) \neq \epsilon_V\).

Moreover, we claim that \(\pi : V^G \rightarrow V^H\) must be a bijection. Let us assume the contrary, i.e., there is \(u_1 \in V^G\) such that \(\pi(u_1) = \epsilon_V\). Since there is at least one edge (of length at least \(L\)) incident to \(u_1\), we then have from Lemma[14]

\[
\text{Cost}(\pi) \geq C_V|u_0 - \pi(u_0)| + C_E[\text{Vol}(H) - \text{Vol}(G)] + 2C_EL \\
\geq C_V|u_0 - \pi(u_0)| + C_E \times 2N \times L + 2C_EL \\
= C_V|u_0 - \pi(u_0)| + 2C_E(N + 1)L \\
= C_V|u_0 - \pi(u_0)| + 2C_E(N + 1) \tau \frac{C_V(N + 1)}{2C_E(N + 1)} \\
= C_V|u_0 - \pi(u_0)| + \tau.
\]

Since the graphs are non-overlapping, \(u_0 - \pi(u_0) > 0\). Hence, \(\text{Cost}(\pi) > \tau\). This is a contradiction, so \(\pi\) must be a bijection. Finally, we show that \(\pi\) defines a partition of \(S\) by arguing that a blob \(G_r\) of \(G\) cannot split into two blobs \(H_s\) and \(H_t\) of \(H\) when mapped by \(\pi\). If it did, there would be an edge \(e_0\) of \(G_r\) with \(\pi(e_0) = \epsilon_E\), since the blobs \(H_s\) and \(H_t\) are not connected. This would lead to a contradiction using the exact same argument just presented. Therefore, \(\pi\) defines a partition of the blobs of \(G\), so a partition of \(S\). This completes the proof. \(\square\)

### 4 Discussions and Future Work

We have studied two notions for a similarity measure between geometric graphs. In addition to the metric properties of GED and GGD, we also establish tight bounds in order to compare them. Although the distance measures induce equivalent metrics on the space of geometric graphs, it is not clear which one is better performant in practical applications. We have also shown the hardness of computing the GGD even for planar graphs. This naturally provokes the question of the hardness of its polynomial-time approximation. We conjecture that for any \(\alpha > 1\), an \(\alpha\)-approximation is also \(NP\)-hard, i.e, PROBLEM GGD is generally \(APX\)-hard. One can also investigate an alternative version of the GED that is algorithmically feasible to compute. This can probably be achieved by putting the graphs on a (Euclidean) grid and avoiding redundant edit operations in an edit path. It also remains unclear how to adjust the definitions of the proposed distances to incorporate merging and splitting of vertices and edges.
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