WAVE BREAKING AND GLOBAL EXISTENCE FOR THE PERIODIC ROTATION-CAMASSA-HOLM SYSTEM

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Abstract. The rotation-two-component Camassa-Holm system with the effect of the Coriolis force in the rotating fluid is a model in the equatorial water waves. In this paper we consider its periodic Cauchy problem. The precise blow-up scenarios of strong solutions and several conditions on the initial data that produce blow-up of the induced solutions are described in detail. Finally, a sufficient condition for global solutions is established.

1. Introduction. In this paper, we are concerned with the behavior of solutions to the periodic problem for the following rotation-two-component Camassa-Holm (R2CH) system

\[
\begin{align*}
u_t - u_{xxt} - Au_x + 3uu_x &= \sigma(2u_x u_{xx} + uu_{xxx}) - \mu u_{xxx} \\
\rho_t + u\rho_x &= -\rho u_x,
\end{align*}
\]

where \(A\) characterizes a linear underlying shear flow, the real dimensionless constant \(\sigma\) provides the competition, or balance, in fluid convection between nonlinear steepening and amplification due to stretching, \(\mu\) is a nondimensional parameter and \(\Omega\) characterizes the constant rotational speed of the Earth. The system was introduced by Fan-Gao-Liu [19] from the \(f\)-plane governing equations for equatorial geophysical water waves which admit a constant underlying current, where \(u(t,x)\) is the fluid velocity in the \(x\)-direction, \(\rho(t,x)\) is related to the free surface elevation from equilibrium.

The R2CH system (1.1) was derived by applying the approach of Ivanov’s asymptotic perturbation analysis for the governing equations of two-dimensional rotational gravity water waves [27], which is particularly appealing in comparison to other models that have been studied in the past due to the following reasons. First, the model may be the first approach in incorporating the effect of a current into the asymptotic model to the \(f\)-plane equatorial geophysical governing equation although there have been lot of works concerning the asymptotic models to the geophysical governing equation [19] [25] [6] [18] [15] [24]. Second, the R2CH system is another significant modification model in incorporating the geophysical effects into the generalized Dullin-Gottwald-Holm (DGH) system and two-component Camassa-Holm

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system \[19, 27, 14, 61\]. Finally, it should be interesting to see the effects of the constant Equatorial Undercurrent and the Earth’s rotational speed on the evolution of water waves by mathematical analysis \[19\].

The R2CH system \((1.1)\) has significant relationship with several models describing the motion of waves at the free surface of shallow water under the influence of gravity. For instance, if we consider the system in \((1.1)\) without effect of the Earths rotation, i.e., \(\Omega = 0\), it becomes the generalized DGH system

\[
\begin{align*}
    u_t - u_{xxt} - Au_x + 3uu_x = & \sigma(2u_xu_{xx} + uu_{xxx}) - \mu u_{xxx} - \rho \rho_x, \\
    \rho_t + u\rho_x = & -\rho u_x,
\end{align*}
\]

(1.2)

which was first derived by Han-Guo-Gao \[23\] in the shallow-water regime following Ivanov’s approach \[27\]. And blow-up mechanism of the generalized DGH system was analyzed in detail in the same paper. Let \(\mu = 0\), system \(\eqref{1.2}\) reduces to the generalized Camassa-Holm (CH) system \[4\]. Wave-breaking phenomena and existence of global solutions for its Cauchy problem in non-periodic and periodic setting were established in \[4\] and \[5\], respectively. See also for \[14, 17, 18, 20, 21, 22, 33\] and references therein.

When \(\sigma = 1\), and \(\mu = 0\), \(\eqref{1.2}\) recovers the standard two-component integrable Camassa-Holm system \[14, 31\]

\[
\begin{align*}
    u_t - u_{xxt} - Au_x + 3uu_x + \rho \rho_x = & 2u_xu_{xx} + uu_{xxx}, \\
    \rho_t + u\rho_x = & -\rho u_x.
\end{align*}
\]

(1.3)

Moreover, in the case \(\rho = 0\), \(\eqref{1.2}\) becomes the DGH equation \[16\] and \(\eqref{1.3}\) becomes the Camassa-Holm (CH) equation \[3\]. Well-posedness and blow up phenomena for the CH equation have been studied extensively. Indeed, the local well-posedness of the periodic Camassa-Holm equation with initial data were proved and the global strong solutions for certain class of initial data were also studied \[7, 9, 10\]. Existence and uniqueness results for classical solution of the periodic CH equation were established in \[30\]. The blow up phenomena of the periodic CH equation were investigated in a number of papers (see \[3, 7, 9, 10, 11, 8, 12, 29\] and references therein). After wave breaking the solutions can be continued as either global conservative or global dissipative solutions \[1, 2, 26\].

Consider that system \((1.1)\) is a generalization of system \((1.2)\) with the rotation of Earth-these effects feature significantly for such large scale phenomena as currents, and the Coriolis force has introduced a higher order nonlinear term into the generalized two-component CH system, which has interesting implications for the fluid motion, particular in the relation to the wave breaking phenomena and the permanent waves. Fan-Gao-Liu \[19\] investigated the effects of the Coriolis force caused by the Earth’s rotation and nonlocal nonlinearities on blow-up criteria and wave-breaking phenomena. Furthermore, conditions which guarantee the permanent waves were also obtained by using a method of the Lyapunov function when \(\sigma = 1, \mu = 0\).

The goal of the present paper is to derive some conditions on the initial data for the periodic Cauchy problem of system \((1.1)\) that guarantees the formulation of singularities in the resulting solution in finite time. The first step is to establish a wave-breaking criterion. The theory of transport equations implies that the solution \((u, \rho)\) will not blow up as long as the slope of the velocity, i.e. \(u_x\) remain bounded, while the solution blows up in finite time when the slope \(u_x\) is unbounded from blow. Then we try to find conditions of the initial data which can guarantee the
wave breaking in finite time when \( \sigma = 1, \mu = 0 \). It is shown that the initial velocity has to decrease much faster at some point in the presence of the Earth’s rotation and the positive vorticity in order to ensure the occurrence of the wave-breaking phenomena. Furthermore, we also give a blow-up result if \( u_0 \) is odd and \( \rho_0 \) is even when \( \sigma = 1, \mu = 0 \).

The remainder of the paper is organized as follows. In Section 2, some preliminary estimates and results are recalled and presented. Section 3 is devoted to the proofs of our main wave-breaking results, i.e., Theorem 3.1-Theorem 3.3. In Section 4, we provide a sufficient condition for global solutions.

**Notation.** Throughout this paper, we identify all spaces of periodic functions with function spaces over the unit circle \( \mathbb{S} \), i.e., \( \mathbb{S} = \mathbb{R}/\mathbb{Z} \). The norm of the Sobolev space \( H^s(\mathbb{S}) \), \( s \in \mathbb{R} \), by \( \| \cdot \|_{H^s} \). Since all space of functions are over \( \mathbb{S} \), for simplicity, we drop \( \mathbb{S} \) in our notations of function spaces if there is no ambiguity.

### 2. Preliminaries

Let \( G(x) := \frac{\cosh(x - \frac{1}{2})}{2 \sinh(\frac{1}{2})}, x \in \mathbb{R} \). Then \( (1 - \partial_t^2)^{-1} f = G \ast f \) for all \( f \in L^2(\mathbb{S}) \). Our system (1.1) can be written in the following transport type:

\[
\begin{align*}
\begin{cases}
    u_t + (\sigma u - \mu) u_x = -\partial_x G \ast ((\mu - A)u + \frac{3 - \sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1 - 2\Omega A}{2} \rho^2 - \Omega \rho^2 u) + \Omega G \ast (\rho^2 u_x), & t > 0, \quad x \in \mathbb{R}, \\
    \rho_t + u \rho_x = -\rho u_x, & t > 0, \quad x \in \mathbb{R}, \\
    u(t, x + 1) = u(t, x), & \rho(t, x + 1) = \rho(t, x), \quad t \geq 0, \quad x \in \mathbb{R}, \\
    u(0, x) = u_0(x), & \rho(0, x) = \rho_0(x), \quad x \in \mathbb{R}.
\end{cases}
\end{align*}
\]

(2.1)

We begin by present the local well-posedness result for the periodic Cauchy problem of system (1.1). Concerning the R2CH system (1.1) is suitable for applying Kato’s theory [28], one may follow the similar argument as in [19, 13] to obtain the following theorem.

**Theorem 2.1.** Given \( z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \) with \( s > 3/2 \). Then there exists a maximal time \( T = T(z_0) > 0 \) and unique solution \( z = (u, \rho) \) to system (2.1) such that

\[
z = z(\cdot, z_0) \in C([0, T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0, T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})).
\]

Moreover, the solution depends continuously on the initial data, i.e., the mapping \( z_0 \to z(\cdot, z_0) \) is continuous from a neighborhood of the initial data \( z_0 \) in \( H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \) into \( C([0, T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0, T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})) \).

Consider the following associated Lagrangian scales of (2.1), namely,

\[
\begin{align*}
\begin{cases}
    \frac{d q(t, x)}{dt} = u(t, q(t, x)), & t \in [0, T) \\
    q(0, x) = x, & x \in \mathbb{S},
\end{cases}
\end{align*}
\]

(2.2)

where \( u \in C^1([0, T); H^{s-1}) \) is the first component of the solution \( (u, \rho) \) to system (2.1).

**Lemma 2.1.** Let \( (u, \rho) \) be the solution of system (2.1) with initial data \( (u_0, \rho_0) \in H^s \times H^{s-1}, s > 3/2 \) and \( T > 0 \) is the maximal time of existence. Then equation
Let quantities which will play an important role in all analysis of the solutions.

The above lemma indicates that \( q(t, \cdot) \) is an diffeomorphism of the line for each \( t \in [0, T) \). Hence, the \( L^\infty \)-norm of any function \( v(t, \cdot) \in L^\infty(\mathbb{S}), t \in [0, T) \) is preserved under the family of diffeomorphisms \( q(t, \cdot) \) with \( t \in [0, T) \), that is,

\[
\|v(t, \cdot)\|_{L^\infty(\mathbb{S})} = \|v(t, q(t, \cdot))\|_{L^\infty(\mathbb{S})}, \quad t \in [0, T).
\]

Similarly,

\[
\inf_{x \in \mathbb{S}} v(t, x) = \inf_{x \in \mathbb{S}} v(t, q(t, x)), \quad \sup_{x \in \mathbb{S}} v(t, x) = \sup_{x \in \mathbb{S}} v(t, q(t, x)), \quad t \in [0, T).
\]

Now we briefly give the some needed results to pursue our goal.

**Lemma 2.2.** [18] Let \((u, \rho)\) be the solution of system (2.1) with initial data \((u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2 \) and \( T > 0 \) the maximal time of existence. Then we have

\[
\rho(t, q(t, x))q_\tau(t, x) = \rho_0(x), \quad (t, x) \in [0, T) \times \mathbb{R}.
\]

Moreover, if there exists \( x_0 \in \mathbb{S} \) such that \( \rho_0(x_0) = 0 \), then \( \rho(t, q(t, x_0)) = 0 \) for all \( t \in [0, T) \).

Using this result and performing the same argument as in [21], we can establish the following blow-up criterion.

**Theorem 2.2.** Let \((u, \rho)\) be the solution of system (2.1) with initial data \((u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2 \) and \( T > 0 \) the maximal time of existence. Then

\[
T < \infty \Rightarrow \int_0^T \|u_\tau(\tau)\|_{L^\infty} \, d\tau = \infty.
\]

**Lemma 2.3.** [32] (i) For every \( f \in H^1(\mathbb{S}) \), we have

\[
\max_{x \in [0,1]} f^2(x) = \frac{e+1}{2(e-1)} \|f\|_{H^1(\mathbb{S})}^2,
\]

where the constant \( \frac{e+1}{2(e-1)} \) is sharp.

(ii) For every \( f \in H^3(\mathbb{S}) \), we have

\[
\max_{x \in [0,1]} f^2(x) \leq \|f\|_{H^3(\mathbb{S})}^2,
\]

with the best possible constant \( c \) lying within the range \((1, \frac{13}{12}]\). Moreover, the best constant \( c \) is \( \frac{e+1}{2(e-1)} \).

At last, we present some conserved properties. Firstly, we give three conserved quantities which will play an important role in all analysis of the solutions.

**Lemma 2.4.** Let \((u, \rho)\) be the solution of system (2.1) with initial data \((u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2 \) and \( T \) the maximal time of existence. Then for all \( t \in (0, T) \), we have

\[
E_1(u_0, \rho_0) = \int_\mathbb{S} \rho \, dx = \int_\mathbb{S} \rho_0 \, dx, \quad E_2(u_0, \rho_0) = \int_\mathbb{S} (u - \Omega \rho)^2 \, dx = \int_\mathbb{S} (u_0 - \Omega \rho_0)^2 \, dx,
\]

\[
\tilde{E}(u_0, \rho_0) = \int_\mathbb{S} (u^2 + u_\tau^2 + (1 - 2\Omega A) \rho^2) \, dx = \int_\mathbb{S} (u_0^2 + u_{\tau,0}^2 + (1 - 2\Omega A) \rho_0^2) \, dx.
\]
Proof. Integrating the second equation of system (1.1) by parts, in view of the periodicity of $u$ and $\rho$, we have
\[
\frac{d}{dt} \int_S \rho(t, x) \, dx = - \int_S (u \rho)_x \, dx = 0
\]
which implies
\[
E_1(u_0, \rho_0) = \int_S \rho \, dx = \int_S \rho_0 \, dx.
\]

For the proof of the other two conserved quantities, integrating the first equation of system (1.1) by parts, in view of the periodicity of $u$ and $\rho$, we have
\[
\frac{d}{dt} \int_S u(t, x) \, dx = \int_S u_{xxt} \, dx + \int_S u_x \, dx - \int_S 3u u_x \, dx - \mu \int_S u_{xxx} \, dx + 2\Omega \int_S \rho(u) \, dx + \sigma \int_S (2u_x u_x + uu_{xxx}) \, dx
\]
\[
= -\Omega \int_S u_x \rho^2 \, dx.
\]

(2.4)

Multiplying the first equation of system (1.1) by $2u$ and integrating by parts, in view of the periodicity of $u$ and $\rho$, we have
\[
\frac{d}{dt} \int_S (u^2(t, x) + u_x^2(t, x)) \, dx = (1 - 2\Omega A) \int_S u_x \rho^2 \, dx
\]

(2.5)

Multiplying the second equation of system (1.1) by $2\rho$ and integrating by parts, in view of the periodicity of $u$ and $\rho$, we get
\[
\frac{d}{dt} \int_S \rho^2(t, x) \, dx = -\int_S u_x \rho^2 \, dx.
\]

(2.6)

Combining (2.4)-(2.6), we obtain
\[
\frac{d}{dt} \int_S (u(t, x) - \Omega \rho^2(t, x)) \, dx = 0,
\]
\[
\frac{d}{dt} \int_S (u^2(t, x) + u_x^2(t, x) + (1 - 2\Omega A)\rho^2(t, x)) \, dx = 0,
\]
which completes the proof of the lemma.

By the conservation laws stated in Lemma 2.4 and Lemma 2.3 (i), we have the following useful corollary.

Corollary 2.1. Let $(u, \rho)$ be the solution of system (1.1) with initial data $(u_0, \rho_0) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$ and $T > 0$ be the maximal time of existence. Then for all $t \in [0, T)$, we have
\[
\|u(t)\|^2_{L^\infty(S)} \leq \frac{e + 1}{2(e - 1)} \|u(t, \cdot)\|^2_{H^s(S)} \leq \frac{e + 1}{2(e - 1)} \bar{E}(u_0, \rho_0).
\]

Furthermore, let $1 - 2\Omega A > 0$, the following relations hold:
\[
\|u(t)\|^2_{L^2(S)} \leq \bar{E}(u_0, \rho_0), \quad \|u_x(t)\|^2_{L^2(S)} \leq \bar{E}(u_0, \rho_0),
\]
\[
\|\rho(t)\|^2_{L^2(S)} \leq \frac{1}{1 - 2\Omega A} \bar{E}(u_0, \rho_0).
\]
3. Wave-breaking phenomena. In this section, we establish a blow up criterion and also derive some sufficient conditions for the breaking of waves for the initial-value problem \([2.1]\). First, we give the wave-breaking criterion.

**Theorem 3.1.** Assume that \(1 - 2\Omega A > 0\). Let \((u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})\) with \(s > 3/2\), and \(T > 0\) be the maximal time of existence of solution \((u, \rho)\) to system \([2.1]\) with initial data \((u_0, \rho_0)\). Then the corresponding solution \((u, \rho)\) blows up in finite time \(T < \infty\) if and only if

\[
\liminf_{t \uparrow T^-} \{ \sup_{x \in \mathbb{S}} |u_x(t, x)| \} = +\infty. \tag{3.1}
\]

Furthermore, if \(\sigma = 1\) and \(\mu = 0\), then the the corresponding solution \((u, \rho)\) blows up in finite time \(T < \infty\) if and only if

\[
\liminf_{t \uparrow T^-} \{ \inf_{x \in \mathbb{S}} u_x(t, x) \} = -\infty. \tag{3.2}
\]

**Proof.** By Theorem \([2.1]\) and a simple density argument, we need only to consider this theorem for \(s \geq 3\). We may also assume \(u_0 \neq 0\), otherwise it is trivial.

We first try to prove the blow-up criterion \([3.1]\). If \([3.1]\) holds, the Sobolev embedding theorem \(H^s \hookrightarrow L^\infty, s > 1/2\) implies that the corresponding solution blows up in finite time. Conversely, assume that \(T < \infty\) and \([3.1]\) is not valid. Then there is some positive number \(M_0 > 0\), such that

\[
|u_x| < M_0, \quad \forall (t, x) \in [0, T) \times \mathbb{S}.
\]

Therefore, it follows from Theorem \([2.2]\) that the maximal existence time \(T = \infty\), which contradicts the assumption that \(T < \infty\).

Now we turn to prove the case in \([3.2]\). Using the identity \(-\partial_x^2 G \ast f = f - G \ast f\) for any \(f \in L^2\). Differentiating the first equation in \([2.1]\) with respect to \(x\), we have

\[
u_t + \rho u_x = -\frac{1}{2} u_x^2 + \frac{1}{2} - \frac{2\Omega A}{2} \rho^2 - \Omega u \rho^2 + A \partial_x^2 G \ast u + u^2
- G \ast \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} - \frac{2\Omega A}{2} \rho^2 - \Omega (\rho^2 u)\right) + \Omega \partial_x G \ast (\rho^2 u_x).
\tag{3.3}
\]

Given \(x \in \mathbb{S}\), define

\[
M(t) = u_x(t, q(t, x)), \quad \gamma(t) = \rho(t, q(t, x)), \quad t \in [0, T),
\]

where \(q(t, x)\) is defined by \([2.2]\). Then along the trajectory \(q(t, x)\), the above equation and the second equation of \([2.1]\) become

\[
\begin{aligned}
M'(t) &= -\frac{1}{2} M(t)^2 + \frac{1}{2} - \frac{2\Omega A}{2} \gamma(t)^2 + f(t, q(t, x)), \\
\gamma'(t) &= -\gamma M,'
\end{aligned} \tag{3.4}
\]

for \(t \in [0, T)\), where \('\) denotes the derivative with respect to \(t\) and \(f(t, q(t, x))\) is given by

\[
\begin{aligned}
f &= u^2 - \Omega u \rho^2 + A \partial_x^2 G \ast u - G \ast \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} - \frac{2\Omega A}{2} \rho^2\right) \\
&\quad + \Omega G \ast (\rho^2 u) + \Omega \partial_x G \ast (\rho^2 u_x).
\end{aligned}
\]

By the definition of \(M(t)\), assume that \(T < \infty\) and \([3.2]\) is not valid. Then there is some positive number \(M_1 > 0\), such that

\[
\inf_{x \in \mathbb{S}} u_x(t, x) \geq -M_1, \quad \forall t \in [0, T).
\]
Then from the second equation of (3.4), we obtain that for \( x \in \mathbb{S} \),
\[
|\rho(t, q(t, x))| = |\gamma(t)| = |\gamma(0)|e^{-\int_0^t M(\tau) d\tau} \leq \|\rho_0\|_{L^\infty} e^{M_1 t},
\]
i.e.,
\[
\|\rho(t, \cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{M_1 t}.
\]
which completes the proof of the lemma.

Now we will derive the upper bound for \( f \) for later use in getting the wave-breaking result. Using that \( \partial_x^2 G \ast u = \partial_x G \ast \partial_x u \), we have
\[
f = u^2 - \Omega u \rho^2 + A \partial_x^2 G \ast u - G \ast \left( u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \Omega \cdot 2 \rho^2 \right) + \Omega \partial_x G \ast (\rho^2 u_x)
\]
\[
\leq u^2 + \Omega |u \rho^2| + |A| |\partial_x G \ast \partial_x u| + |G \ast (u^2 + \frac{1}{2} u_x^2)| + \Omega |G \ast (\rho^2 u)|
\]
\[
+ \Omega |\partial_x G \ast (\rho^2 u_x)|,
\]
for any \( x \in \mathbb{S} \) and \( t \in [0, T) \). Consider that \( G = \frac{\cosh(x - |x| - \frac{1}{2})}{2 \sinh(\frac{1}{2})} \), applying Young’s inequality and Corollary [2.1], we have
\[
u^2 \leq \|u\|^2_{L^\infty(S)} \leq \frac{e + 1}{2(e - 1)} \|u(t, \cdot)\|^2_{H^1(S)} \leq \frac{e + 1}{2(e - 1)} E(u_0, \rho_0),
\]
(3.5)
and
\[
\Omega |u \rho^2| \leq \Omega \|\rho\|^2_{L^\infty} \|u\|_{L^\infty} \leq \frac{\Omega^2}{4} \|\rho\|^4_{L^\infty} + \|u\|^2_{L^\infty} \leq \frac{\Omega^2}{4} \|\rho\|^4_{L^\infty} + \frac{e + 1}{2(e - 1)} \|u\|^2_{H^1}
\]
\[
\leq \frac{\Omega^2}{4} \|\rho\|^4_{L^\infty} + \frac{e + 1}{2(e - 1)} E(u_0, \rho_0),
\]
(3.6)
\[
|A| \|G_x \ast u_x| \leq |A| \|G_x\|_{L^2} \|u_x\|_{L^2} = |A| \frac{\sqrt{\sinh 1 - 1}}{2 \sqrt{2} \sinh \frac{1}{2}} \|u_x\|_{L^2}
\]
\[
\leq \frac{\sinh 1 - 1}{8 \sinh 1} A^2 + \frac{\cosh \frac{1}{2}}{2 \sinh \frac{1}{2}} \|u_x\|_{L^2}^2,
\]
(3.7)
\[
|G \ast (u^2 + \frac{1}{2} u_x^2)| \leq \|G\|_{L^\infty} \|u^2 + \frac{1}{2} u_x^2\|_{L^1} = \cosh \frac{1}{2} \|u^2 + \frac{1}{2} u_x^2\|_{L^1}
\]
\[
\leq \frac{\cosh \frac{1}{2}}{2 \sinh \frac{1}{2}} \|u\|_{L^2}^2 + \frac{\cosh \frac{1}{2}}{4 \sinh \frac{1}{2}} \|u_x\|_{L^2}^2,
\]
\[
\Omega |G \ast (\rho^2 u)| \leq \Omega \|G\|_{L^\infty} \|\rho^2 u\|_{L^1} = \frac{\Omega \cosh \frac{1}{2}}{2 \sinh \frac{1}{2}} \|\rho^2 u\|_{L^1}
\]
\[
\leq \frac{\Omega \cosh \frac{1}{2}}{2 \sinh \frac{1}{2}} \|u\|_{L^\infty} \|\rho\|_{L^2}^2,
\]
\[
\Omega |G_x \ast (\rho^2 u_x)| \leq \Omega \|G_x\|_{L^\infty} \|\rho^2 u_x\|_{L^1} = \frac{\Omega}{2} \|\rho\|_{L^\infty} \|\rho u_x\|_{L^1}
\]
\[
\leq \frac{\Omega^2}{4} \|\rho\|_{L^\infty}^2 + \frac{1}{4} \|\rho u_x\|_{L^1}^2.
\]
Therefore, we obtain the upper bound for $f$

$$f \leq C_1^2 + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^4 e^{4M_1 t} + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^2 e^{2M_1 t},$$

where $C_1$ denotes a positive constant that depends only on $A$, $\Omega$ and $\tilde{E}(u_0, \rho_0)$, where we use the following relations

$$
\|u\|_{L^2}^2 \leq \tilde{E}(u_0, \rho_0), \quad \|u_x\|_{L^2}^2 \leq \tilde{E}(u_0, \rho_0), \quad \|\rho\|_{L^2}^2 \leq \frac{1}{2} |\tilde{E}(u_0, \rho_0)|.
$$

Now we define

$$P(t) = M(t) - \|u_{0x}\|_{L^\infty} - 2C_1 - \Omega \|\rho_0\|_{L^\infty}^2 e^{2M_1 t} - (\Omega + \sqrt{1 - 2\Omega A})\|\rho_0\|_{L^\infty} e^{M_1 t}.$$

Observe $P(t)$ is a $C^1$-differentiable function in $[0, T)$ and satisfies

$$P(0) = M(0) - \|u_{0x}\|_{L^\infty} - 2C_1 - \Omega \|\rho_0\|_{L^\infty}^2 - (\Omega + \sqrt{1 - 2\Omega A})\|\rho_0\|_{L^\infty} \leq u_{0x}(x) - \|u_{0x}\|_{L^\infty} \leq 0.$$

We will show that

$$P(t) = M(t) - \|u_{0x}\|_{L^\infty} - 2C_1 - \Omega \|\rho_0\|_{L^\infty}^2 e^{2M_1 t} - (\Omega + \sqrt{1 - 2\Omega A})\|\rho_0\|_{L^\infty} e^{M_1 t}.$$

If not, then suppose there is a $t_0 \in [0, T)$ such that $P(t_0) > 0$. Define

$$t_1 = \max\{t \leq t_0, P(t) = 0\}.$$

Then $P(t_1) = 0$ and $P'(t_1) \leq 0$, or equivalently,

$$M(t_1) = \|u_{0x}\|_{L^\infty} + 2C_1 + \Omega \|\rho_0\|_{L^\infty}^2 e^{2M_1 t_1} + (\Omega + \sqrt{1 - 2\Omega A})\|\rho_0\|_{L^\infty} e^{M_1 t_1},$$

$$M'(t_1) = 2M_1 \Omega \|\rho_0\|_{L^\infty}^2 e^{2M_1 t_1} + M_1 (\Omega + \sqrt{1 - 2\Omega A})\|\rho_0\|_{L^\infty} e^{M_1 t_1} \geq 0. \quad (3.8)$$

On the other hand, we have

$$M'(t_1)$$

$$= -\frac{1}{2} M^2(t_1) + \frac{1}{2} 2\Omega A \gamma^2(t_1) + f(t, q(t_1, x))$$

$$\leq -\frac{1}{2} \left( \|u_{0x}\|_{L^\infty}^2 + 2C_1 + \Omega \|\rho_0\|_{L^\infty}^2 e^{2M_1 t_1} + (\Omega + \sqrt{1 - 2\Omega A})\|\rho_0\|_{L^\infty} e^{M_1 t_1} \right)^2$$

$$+ \frac{1}{2} 2\Omega A \|\rho_0\|_{L^\infty}^2 e^{2M_1 t_1} + C_1^2 + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^4 e^{4M_1 t_1} + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^2 e^{2M_1 t_1}$$

$$\leq 0,$$

which is a contradiction to (3.8). This verifies the estimate in (3.7). Therefore, it implies for arbitrary $x \in S$,

$$\sup_{x \in S} u_x(t, x) \leq \|u_{0x}\|_{L^\infty} + 2C_1 + \Omega \|\rho_0\|_{L^\infty}^2 e^{2M_1 t} + (\Omega + \sqrt{1 - 2\Omega A})\|\rho_0\|_{L^\infty} e^{M_1 t}. $$

Recall that

$$\inf_{x \in S} u_x(t, x) \geq -M_1, \quad \forall t \in [0, T),$$

we have $|u_x| < \infty$. This contradicts our assumption $T < \infty$, which completes the proof of Theorem 3.1.

Now we are in position to state the following that provide some cases that wave breaks in finite time.
Theorem 3.2. Assume that $1 - 2\Omega A > 0$, $\sigma = 1$ and $\mu = 0$. Let $(u_0, \rho_0) \in H^s(S) \times H^{s-1}(S)$ with $s > 3/2$, and $T > 0$ be the maximal time of existence of solution $(u, \rho)$ to system (2.1) with initial data $(u_0, \rho_0)$. Assume there exists a $x_0$ such that $\rho_0(x_0) = 0$ and $u_{0,x}(x_0) < -C_3 - \frac{2\Omega}{1 - 2\Omega A} C_2$. Then the corresponding solution $(u, \rho)$ blows up in finite time in the following sense: there exists a $T_1$ with

$$0 < T_1 < \frac{1}{C_3} \ln \left( \frac{(1 - 2\Omega A)u_{0,x}(x_0) + \Omega \partial_{0,x}G \ast \rho^2(x_0) + \Omega C_2 - (1 - 2\Omega A)C_3}{(1 - 2\Omega A)u_{0,x}(x_0) + \Omega \partial_{0,x}G \ast \rho^2(x_0) + \Omega C_2 + (1 - 2\Omega A)C_3} \right),$$

(3.9)

such that

$$\liminf_{t \uparrow T_1} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty,$$

where

$$C_2 = \frac{\cosh \frac{1}{2}}{2 \sinh \frac{1}{2}} E(u_0, \rho_0),$$

$$C_3 = \left( 2\Omega C_2^2 + 8C_2 + \frac{\sinh 1 - 1 - A^2}{4 \sinh 1} \right)^{\frac{1}{2}}.$$

The proof of Theorem 3.2 relies on the following crucial lemma.

Proposition 3.1. [19] The first equation of system (2.1) can be rewritten as

$$K_t + (\sigma u - \mu)K_x = \Omega \left[ (A - \mu) + \sigma u \right] \partial_x G \ast \rho^2 - \partial_x G \ast \left[ (\mu - A)u + \frac{3 - \sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1}{2} \rho^2 \right],$$

(3.10)

where $K = u + \Omega G \ast \rho^2$. Furthermore,

$$K_{xt} + (\sigma u - \mu)K_{xx}$$

$$= -\frac{\sigma}{2} (K_x - \Omega \partial_x G \ast \rho^2)^2 + 1 + 2\Omega (\mu - A) - 2\Omega \sigma u \rho^2 + \sigma u \partial_x G \ast \rho^2$$

$$+ (A - \mu) \partial_x^2 G \ast u + \frac{3 - \sigma}{2} u^2 - G \ast \left[ \frac{3 - \sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1}{2} \rho^2 \right].$$

(3.11)

Proof. Similar to the arguments in the beginning of the proof of Theorem 3.1, we have just need to consider $s \geq 3$. First take $\sigma = 1, \mu = 0$ in (3.11), we get

$$K_{xt} + uK_{xx} = -\frac{1}{2} (K_x - \Omega \partial_x G \ast \rho^2)^2 + \frac{1 - 2\Omega A - 2\Omega u}{2} \rho^2 + \Omega u \partial_x G \ast \rho^2$$

$$+ A \partial_x^2 G \ast u + u^2 - G \ast \left( u^2 + \frac{1}{2} u_x^2 + \frac{1 - 2\Omega A}{2} \rho^2 \right).$$

(3.12)

Given $x \in S$, let

$$M_1(t) = K_x(t, q(t, x)), \quad \gamma(t) = \rho(t, q(t, x)), \quad t \in [0, T),$$

(3.13)

where $q(t, x)$ is defined by (2.2). Then we can write the equation of $\rho$ in (2.1) along the trajectory of $q(t, x)$ as

$$\gamma'(t) = -\gamma u_x, \quad t \in [0, T),$$

Taking $x = x_0$, the assumption $\gamma(0) = \rho_0(x_0) = 0$ and the above equation imply

$$\gamma(t) \equiv 0, \quad \forall t \in [0, T).$$

Therefore, it follows from (3.12) that

$$M_1'(t) = -\frac{1}{2} (M_1 - \Omega \partial_x G \ast \rho^2)^2 + f(t, q(t, x_0))$$

(3.14)
We now claim that
\[ f = \Omega u G * \rho^2 + A \delta_x^2 G * u + u^2 - G * \left( u^2 + \frac{1}{2} u_x^2 + \frac{1}{2 \Omega A} \rho^2 \right) \tag{3.15} \]
Since
\[
|\partial_x G * \rho^2| \leq |G * \rho^2| \leq \|G\|_{L^\infty} \|\rho^2\|_{L^1} = \|G\|_{L^\infty} \|\rho\|^2_{L^1},
\]
we deduce
\[
|u G * \rho^2| \leq \|u\|_{L^\infty} \|G * \rho^2\|_{L^1} \leq \sqrt{\frac{e+1}{2(e-1)}} \tilde{E}(u_0, \rho_0) C_2 = C_2^3,
\]
and in view of (3.5)-(3.7) and , we have that
\[
f \leq \Omega C_2^2 + \frac{\sinh 1 - 1}{8 \sinh 1} A^2 + \cosh \frac{1}{2} \|u_x\|^2_{L^2} + \frac{e+1}{2(e-1)} \tilde{E}(u_0, \rho_0) \]
\[ + \cosh \frac{1}{2} \|u\|^2_{L^2} + \cosh \frac{1}{2} \|u_x\|^2_{L^2} + \frac{1}{2} C_2 \]
\[ = \Omega C_2^2 + 4 C_2 + \frac{\sinh 1 - 1}{8 \sinh 1} A^2 \]
\[ := \frac{1}{2} C_3^3. \tag{3.17} \]
From (3.17), we deduce
\[ M_1'(t) = -\frac{1}{2} (M_1 - \Omega \partial_x G * \rho^2)^2 + \frac{1}{2} C_3^2, \quad t \in [0, T). \tag{3.18} \]
If the assumption holds, then
\[ M_1(0) = u_{0,x}(x_0) + \Omega \partial_{0,x} G * \rho^2(x_0) < u_{0,x}(x_0) + \frac{\Omega}{1 - 2\Omega A} C_2 \leq -C_3 - \frac{\Omega}{1 - 2\Omega A} C_2. \]
We now claim that
\[ M_1(t) \leq -C_3 - \frac{\Omega}{1 - 2\Omega A} C_2, \quad \forall t \in [0, T). \tag{3.19} \]
If not, there is a \( t_0 \in [0, T) \) such that \( M_1(t) \leq -C_3 - \frac{\Omega}{1 - 2\Omega A} C_2 \) on \([0, t_0)\), while \( M_1(t_0) = -C_3 - \frac{\Omega}{1 - 2\Omega A} C_2 \). But then we would have by (3.18)
\[ \frac{dM_1(t)}{dt} < 0, \quad a.e. \quad t \in [0, t_0). \]
Being locally Lipshitz, the function \( M_1(t) \) is absolutely continuous on \([0, t_0)\), and thus an integration of the previous inequality would lead us to
\[ M_1(t_0) < M_1(0) < -C_3 - \frac{\Omega}{1 - 2\Omega A} C_2, \]
which contradicts our assumption \( M_1(t_0) = -C_3 - \frac{\Omega}{1 - 2\Omega A} C_2 \). Hence (3.19) holds, implying that \( M_1(t) \) is strictly decrease on \([0, T)\). Then
\[ M_1'(t) \leq -\frac{1}{2} \left( M_1 + \frac{\Omega}{1 - 2\Omega A} C_2 \right)^2 + \frac{1}{2} C_3^2, \quad t \in [0, T). \]
Solving the above inequality gives
\[
\frac{(M_1(0) + \frac{\Omega}{1 - 2\Omega A} C_2 + C_3)}{(M_1(0) + \frac{\Omega}{1 - 2\Omega A} C_2 - C_3)} e^{C_3 t} - 1 \leq \frac{2 C_3}{M_1(t) + \frac{\Omega}{1 - 2\Omega A} C_2 - C_3} \leq 0.
\]
In view of $0 < \frac{(M_1(0) + \frac{\rho_0}{2})C_2 + C_3}{M_1(0) + \frac{\rho_0}{2}C_2 - C_3} < 1$, we obtain that there exists $T_1$ satisfying

$$0 < T_1 < \frac{1}{C_3} \ln \frac{1 - 2\Omega A}{1 - 2\Omega A}u_{0,x}(x_0) + \Omega \partial_x G \ast \rho^2(x_0) + \Omega C_2 - (1 - 2\Omega A)C_3,$$

such that $\lim_{t \to T_1} M_1(t) = -\infty$, i.e., $\lim_{t \to T_1} u_x(t, x) = -\infty$ as a result of the boundedness of $\partial_x G \ast \rho^2$. This completes the proof of the proposition.

**Theorem 3.3.** Assume that $1 - 2\Omega A > 0$, $\sigma = 1$ and $\mu = 0$. Let $(u_0, \rho_0) \in H^s(S) \times H^{s-1}(S)$ with $s \geq 3/2$, and $T > 0$ be the maximal time of existence of solution $(u, \rho)$ to system (2.1) with initial data $(u_0, \rho_0)$. Assume that $u_0$ is odd, $\rho_0$ is even, $u_{0,x}(0) < 0$ and $\rho_0(0) = 0$. Then the corresponding solution $(u, \rho)$ to the system (2.1) blows up in finite time.

**Proof.** Similar to the proof of Theorem 3.1, it suffices to consider $s \geq 3$. Since $u_0$ is odd and $\rho_0$ is even, the corresponding solution $(u, \rho)$ satisfies $u(t, x)$ is odd and $\rho(t, x)$ is even with respect to $x$ for given $0 < t < T$. Hence $u(t, 0) = 0$, and $\rho_x(t, 0) = 0$. In view of the transport equation of $\rho$ in (2.1), we have

$$\begin{cases}
\rho(t, 0) + \rho(t, 0)u_x(t, 0) = 0, \\
\rho(0, 0) = 0.
\end{cases}$$

Thus $\rho(t, 0) = 0$. Evaluating (3.3) at $(t, 0)$ and denoting $M_2(t) = u_x(t, 0)$, we obtain

$$M_2'(t) + \frac{1}{2}M_2^2(t) = A(\partial_x^2 G \ast u)(t, 0) - G \ast \left( u^2 + \frac{1}{2}u_x^2 + \frac{1 - 2\Omega A}{2}\rho^2 \right)(t, 0) + \Omega G \ast (\rho^2 u)(t, 0) + \Omega \partial_x G \ast (\rho^2 u_x)(t, 0).$$

Note that $u(t, x)$ is odd and $G(x)$ is even, so

$$A(\partial_x^2 G \ast u)(t, 0) = 0, \quad \Omega G \ast (\rho^2 u)(t, 0) = 0, \quad \Omega \partial_x G \ast (\rho^2 u_x)(t, 0) = 0.$$

Therefore,

$$M_2'(t) + \frac{1}{2}M_2^2(t) = -G \ast \left( u^2 + \frac{1}{2}u_x^2 + \frac{1 - 2\Omega A}{2}\rho^2 \right)(t, 0) \leq 0.$$

Hence

$$M_2(t) \leq M_2(0) = u_{0,x}(0) < 0, t \in [0, T),$$

$$-\frac{1}{M_2(t)} + \frac{1}{M_2(0)} \leq M_2(0) \leq -\frac{t}{2},$$

then

$$u_x(t, 0) = M_2(t) \leq \frac{2M_2(0)}{2 + M_2(0)t} \to -\infty, \quad t \to -\frac{2}{M_2(0)},$$

which indicates that the maximal existence time $T \leq -\frac{2}{u_{0,x}(0)}$ and hence it completes the proof of the theorem.

4. **Global existence.** In this section, we turn our attention to existence of the global solution of system (2.1) in the case when $\sigma = 1, \mu = 0$.

**Theorem 4.1.** Let $\sigma = 1, \mu = 0$ and $(u_0, \rho_0) \in H^s(S) \times H^{s-1}(S)$ with $s \geq 3/2$, and $T > 0$ be the maximal time of existence of solution $(u, \rho)$ to system (2.1) with initial data $(u_0, \rho_0)$. If

$$\inf_{x \in S} \rho_0(x) > 0,$$

(4.1)
which is always positive for every $x$ for $t \in [0, T)$, where $f$ is defined in (3.15). The second equation above implies that $\gamma(t)$ and $\gamma(0)$ are of the same sign.

Recalling the assumptions of the theorem, we know $\gamma(0) = \rho_0(x) > 0$ for every $x \in S$. Define the following Lyapunov function by

$$
\omega(t) = \left( 1 - 2\Omega A - 2\Omega \sqrt{\frac{e + 1}{2(e - 1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \gamma(0) \gamma(t) + \frac{\gamma(0)}{\gamma(t)} \gamma'(t)(1 + M_1^2(t))
$$

which is always positive for $t \in [0, T)$ in view of the assumption (4.2). Differentiating $\omega$ and using (4.3), we obtain

$$
\omega'(t) = \left( 1 - 2\Omega A - 2\Omega \sqrt{\frac{e + 1}{2(e - 1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \gamma(0) \gamma'(t) - \frac{\gamma(0)}{\gamma(t)} \gamma'(t)(1 + M_1^2(t))
$$

$$
+ \frac{2\gamma(0)}{\gamma(t)} M_1(t) M_1'(t)
$$

$$
= \frac{2\gamma(0)}{\gamma(t)} M_1(t) \left( f + \frac{\Omega^2}{2} (\partial_x G * \rho^2) \right) + \frac{\Omega \gamma(0)}{\gamma(t)} \partial_x G * \rho^2 (M_1^2(t) - 1)
$$

$$
- 2\Omega \left( u - \sqrt{\frac{e + 1}{2(e - 1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \gamma(0) \gamma(t) M_1(t)
$$

$$
\leq \frac{\gamma(0)}{\gamma(t)} (1 + M_1^2(t)) \left( \frac{1}{2} + \frac{\Omega^2}{2} |\partial_x G * \rho^2|^2 + |f| + \Omega |\partial_x G * \rho^2| \right)
$$

$$
+ \Omega \left( 1 - 2\Omega A - 2\Omega \sqrt{\frac{e + 1}{2(e - 1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \gamma(0) \gamma(t) |\partial_x G * \rho^2|
$$

$$
- 2\Omega \left( u - \sqrt{\frac{e + 1}{2(e - 1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \gamma(0) \gamma(t) M_1(t)
$$

$$
\leq \frac{1}{2} \left( 1 + \frac{\Omega^2 C_2^2}{(1 - 2\Omega A)^2} + C_3^2 + \frac{4\Omega C_2}{1 - 2\Omega A} \right) \omega(t)
$$

$$
- 2\Omega \left( u - \sqrt{\frac{e + 1}{2(e - 1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \gamma(0) \gamma(t) M_1(t)
$$

$$
:= C_4 \omega(t) - 2\Omega \gamma(0) \gamma(t) M_1(t) \left( u - \sqrt{\frac{e + 1}{2(e - 1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right),
$$

where we have used the bound (3.16) and (3.17) for $\partial_x G * \rho^2$ and $f$. 
Let us assume on the contrary, i.e., \( u_x(t, q(t, x)) \) becomes unbounded from below within finite time for some fixed \( \xi \in \mathbb{S} \). Denote by \( T_1 \) the first time when wave breaking occurs, then
\[
u_x(t, q(t, \xi)) \to -\infty, \quad \text{as} \quad t \to T_1. \tag{4.5}
\]
Thus there exists a time \( 0 \leq \bar{t} \leq T_1 \), such that
\[
u_x(t, q(t, \xi)) \leq -\frac{\Omega}{1 - 2\Omega A} C_2, \quad \text{for} \quad \bar{t} \leq t < T_1,
\]
where \( C_2 \) is defined in Theorem 3.2. Then we have \( M_1(t, q(t, \xi)) = \left( u_x + \Omega \partial_x G * \rho^2 \right)(t, q(t, \xi)) \leq 0 \) for \( \bar{t} \leq t < T_1 \). Therefore,
\[
M_1(t) \left( u - \sqrt{\frac{e+1}{2(e-1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \geq 0, \quad t \in [\bar{t}, T_1).
\]
Consider \( (4.4) \), it then implies that
\[
\nu'(t) \leq C_4 \nu(t) - 2\Omega \gamma(0) \gamma(t) M_1(t) \left( u - \sqrt{\frac{e+1}{2(e-1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \leq C_4 \nu(t),
\]
and
\[
\nu(t) \leq \nu(\bar{t}) e^{C_4(t - \bar{t})} \leq \nu(\bar{t}) e^{C_4 t_1}, \quad \text{for} \quad \forall \quad t \in [\bar{t}, T_1). \tag{4.6}
\]
For the case when \( 0 \leq t \leq \bar{t} \), there exist a \( \bar{C} \) such that
\[
u_x(t, q(t, \xi)) \geq -\bar{C}, \quad \text{for} \quad 0 \leq t \leq \bar{t},
\]
then the second equation of \( (4.3) \) implies
\[
\gamma(t) = \gamma(0)e^{-\int_0^t u_x(\tau) d\tau} \leq \gamma(0) e^{\bar{C} t} \leq \| \gamma(0) \|_{L^\infty} e^{\bar{C} t}, \quad \forall \quad t \in [0, \bar{t}].
\]
It then follows from \( (4.4) \) that
\[
\nu'(t) \leq C_4 \nu(t) - 2\Omega \frac{\gamma(0)}{\gamma(t)} M_1(t) \left( u - \sqrt{\frac{e+1}{2(e-1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \gamma^2(t)
\]
\[
\leq C_4 \nu(t) + \Omega \frac{\gamma(0)}{\gamma(t)} \left( 1 + M_1^2(t) \right) \left( u - \sqrt{\frac{e+1}{2(e-1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \gamma^2(t)
\]
\[
\leq C_4 \nu(t) + 2\Omega \sqrt{\frac{e+1}{2(e-1)}} \sqrt{\bar{E}(u_0, \rho_0)} \| \gamma(t) \|_{L^\infty}^2 e^{2\bar{C} t} \nu(t)
\]
\[
:= (C_4 + C_5) \nu(t),
\]
which implies that
\[
\nu(\bar{t}) \leq \nu(0) e^{(C_4 + C_5) \bar{t}} \leq \nu(0) e^{(C_4 + C_5) t_1}, \tag{4.7}
\]
with
\[
\nu(0) = \left( 1 - 2\Omega A - 2\Omega \sqrt{\frac{e+1}{2(e-1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \gamma^2(0) + 1 + M_1^2(0)
\]
\[
\leq \left( 1 - 2\Omega A - 2\Omega \sqrt{\frac{e+1}{2(e-1)}} \sqrt{\bar{E}(u_0, \rho_0)} \right) \| \rho_0 \|_{L^\infty}^2 + 1 + 2\| u_{0, x} \|_{L^\infty}^2 \tag{4.8}
\]
\[ + 2 \left( \frac{\Omega}{1 - 2\Omega A} \right)^2 C_2^2 \]

\[ := C_6. \]

In view of (4.6)-(4.8), we can get

\[ \omega(t) \leq C_6 e^{(2C_4 + C_5)T_1}. \]

(4.9)

Recall that \( \gamma(t) \) and \( \gamma(0) \) are all positive, the definition of \( \omega(t) \) implies

\[ \sqrt{1 - 2\Omega A - 2\Omega \sqrt{\frac{e_1 + 1}{2(e_1 - 1)}} \sqrt{\tilde{E}(u_0, \rho_0)}} \gamma(0)|M_1(t)| \leq \omega(t). \]

It is then inferred by applying the inequality (4.9) that

\[ |M_1(t)| = \left| \left( u_x + \Omega \partial_x G * \rho^2 \right)(t, q(t, \xi)) \right| \]

\[ \leq \frac{\omega(t)}{\sqrt{1 - 2\Omega A - 2\Omega \sqrt{\frac{e_1 + 1}{2(e_1 - 1)}} \sqrt{\tilde{E}(u_0, \rho_0)}} \gamma(0)} \]

\[ \leq \frac{1}{\sqrt{1 - 2\Omega A - 2\Omega \sqrt{\frac{e_1 + 1}{2(e_1 - 1)}} \sqrt{\tilde{E}(u_0, \rho_0)}} \inf_{x \in \mathbb{S}} \rho_0(x)} C_6 e^{(2C_4 + C_5)T_1} \]

\[ < \infty, \]

which implying \( |u_x(t, q(t, \xi))| < \infty, \ t \in [\tilde{t}, T_1) \) and then contradicts our assumption (4.5). This completes the proof of Theorem 4.1.

\[ \square \]

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