Super Gravitons Interacting with the Super Virasoro Group

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ABSTRACT

We describe actions that correspond to the interaction of the Super Virasoro algebra with supergravitons. These new field theories introduce a superfield that corresponds to dual elements of the super Virasoro algebra. Such elements already appear as background fields in the geometric action associated with two dimensional Polyakov gravity. The actions derived here supply dynamics to these otherwise background fields. We are also able to extend the definition of these field theories to higher dimensions. We explicitly exhibit the 2, 3 and 4 dimensional cases. Remarkably, the fundamental prepotentials describing these dual elements of the super Virasoro algebra, in each model, agrees with the known prepotentials of the corresponding supergravity theory. These theories might be important in the quantization of the super Virasoro group, supergravity and in AdS and super AdS gravity.

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1 Introduction

The Virasoro algebra and the Super Virasoro algebra are the underpinnings of string theories and superstring theories. Representations of these algebras are used in constructing string field theories, conformal field theories and low dimensional gravitational theories. Recent work has developed an action principle for the dual representation of the Virasoro algebra [1, 2] which introduces a rank two tensor, $D_{AB}$. One possible interpretation of this field is as a “covariant” background graviton field. In two dimensions it turns out that, one of the components of this rank two diffeomorphism tensor appears as the background quadratic differential present in computations of the two dimensional gravitational anomaly [3]. Its presence determines the symplectic structure on the coadjoint orbit of the Virasoro group [4, 5] and is a signature that the diffeomorphism fields serve as classical gravitational fields in two dimensions. In other words the constant quadratic differentials $b^\pm$ that are often seen in the literature (for example [15]), are precisely the $D_{\pm\pm}$ components of the diffeomorphism field in light-cone coordinates. The aforementioned action then, supplies dynamics to this field yielding a Virasoro inspired definition of two dimensional classical gravity. The field equations of the action yield constraint equations that are precisely the isotropy equations of coadjoint orbits of the Virasoro group where these orbits are associated with the two dimensional gravitational anomalies. Furthermore the fields $D_{\pm\pm}$ serves as Lagrange multipliers for the Gauss’ Law generators that generate “+” and “−” independent coordinate transformations respectively.

The action constructed in [1, 2] is such that the field equations for the “space-time” components of $D_{AB}$ in two dimensions become constraints on the initial data. Explicitly, in Minkowski space the field equations for the $D_{01}$ component dimensions leads to the isotropy equation for the one remaining dynamical degree of freedom, $D_{11}$. To see this we, consider the action

$$S_{\text{diff}} = \int d^n x \sqrt{g} \left( X^{LMR} D^A R X_{MLA} + 2 X^{LMR} D_{LA} X^A RM \right)$$

$$- \int d^n x \sqrt{g} \left( \frac{\beta}{2} X^{AB} B \nabla L \nabla M X^{LM} + \frac{\beta}{2} X^{BGA} B X_{BGA} \right),$$

where $X^{MNR} = \nabla^R D^{MN}$. Throughout this report we will reserve the capital Latin letters for space-time indices and small Greek letters for spinor indices. In $n$ dimensions, the field $D_{AB}$ has dimensions $M^{\frac{n-1}{2}}$, $\beta$ has dimensions $M^{2}$, and $q$ has dimensions $M^{\frac{n-3}{2}}$. $\beta$ may be interpreted as the string tension for a two dimensional theory whereas $q$ is determined by the central extension. Variation with respect to the space-time component $D_{00}$ and setting $D_{N0} = 0$ leads to the equation,

$$X^{lm0} \partial_l D^{lm} - \partial_m (X^{m0} D_{0l}) - \partial_l (X^{m0} D_{ml}) - q \partial_l \partial_m \partial_l X^{lm0} = 0.$$

(2)

In $1 + 1$ this corresponds to the isotropy equation found on the coadjoint orbit where $D$ corresponds to the quadratic differential. In other words, this field equation becomes
\[ \xi D' + 2\xi'D + q\xi''' = 0, \] where the adjoint element \( \xi \) corresponds to the conjugate momentum, \( X^{110} \), of \( D \equiv D_{11} \). We interpret this as a Gauss Law constraint equation associated with the residual time-independent coordinate transformations on the Cauchy data. Indeed in 1+1 dimensions this Gauss’ Law constraint is the generator of time-independent spatial translations \([1]\). This constraint arises because the space components of conjugate momentum, \( X^{AB} = \sqrt{g}\nabla_0 D^{AB} \), transforms under a time independent spatial translation as

\[
\delta_\xi X^{ab} = \xi^c \partial_c X^{ab} - X^{cb} \partial_c \xi^a - X^{ac} \partial_c \xi^b + (\partial_c \xi^c) X^{cb},
\]

where \( \xi^a \) is the space component a time-independent vector field. In the 1+1 dimensional case the transformation law for the “space-space” component of \( X^{AB} \) reduces to

\[
\delta_\xi X^{11} = \xi (X^{11})' - (\xi')' X^{11},
\]

which is the transformation law for adjoint elements of the Virasoro algebra (modulo central extensions). Similarly the field \( D_{11} \) transforms as a quadratic differential. The constraint guarantees that the conjugate momentum which lives in the adjoint will not transport the Cauchy data into spurious directions. Eq.(2) is the higher dimensional extension of this constraint. For our purposes it is only necessary that the one dimensional reduction of Eq.(2) reduce to the isotropy equation of Virasoro coadjoint orbits.

Although this report is not interested in geometric action on the coadjoint orbits of the super Virasoro group, it is worth noting the appearance of the diffeomorphism field in those cases. In the computation of the two dimensional gravitational anomaly on a cylinder with coordinates \([2]\) \( (\theta, \tau) \), the field \( D \) appears in the Polyakov effective action as

\[
S = \int d^2x D(\theta) \frac{\partial s}{\partial \theta} + \frac{C_\mu}{4\pi} \int d^2x \left[ \frac{\partial^2 s}{(\partial \theta s)^2} \partial_\tau \partial_\theta s - \frac{(\partial^2 s)^2}{(\partial \theta s)^3} \right],
\]

and its value governs the symmetries of the symplectic structure for the field \( s(\theta, t) \). This is to be compared to the geometric action for the WZNW model on a cylinder in the presence of a background gauge potential \( A_\mu \) after gauge fixing,

\[
S = \int d^2x \text{Tr} \left\{ A_\theta g^{-1} \partial_\tau g \right\} + k\mu \int d^2x \text{Tr} \left\{ \left( \frac{\partial g^{-1}}{\partial \tau} \right) \frac{\partial g}{\partial \theta} \right\}
- k\mu \int d^3x \text{Tr} \left\{ g^{-1} \frac{\partial g}{\partial \lambda} \left[ \left( \frac{\partial g^{-1}}{\partial \tau} \right) \frac{\partial g}{\partial \theta} - \left( \frac{\partial g^{-1}}{\partial \theta} \right) \frac{\partial g}{\partial \tau} \right] \right\}.
\]

In this work our efforts will be focused on writing an action that will complete the picture for the superdiffeomorphism fields that appear in the geometric action for the super Virasoro group.

\[ ^4 \text{In this chapter, we use } \theta \text{ to describe an angular variable.} \]
2 Construction of a Super Diffeomorphism Interaction

2.1 Procedural Outline

In [2] a principle to determine interactions with matter fields was developed that was based on the interaction of the diffeomorphism field with itself. There the structure of the interaction Lagrangian of the diffeomorphism field was of the form

\[ L_{\text{int}} = X^{LMR} Y_{LMR}, \]  

(7)

where \( X^{LMR} \) acts as the “covariantized” conjugate momentum and \( Y_{LMR} \) is the “covariantized” Lie derivative of the diff field \( D_{ij} \), where the small Latin indices will denote “space” indices and “0” will denote the time index. The action in [1] and later refine in [2] was based on the observation that the isotropy equations for the coadjoint orbits can be interpreted as constraints arising from the time-independent coordinate freedom that still persists after gauge fixing. The details of this are explained in references [1, 2] but we can highlight the salient features of the construction of the action. We present an annotated outline of the construction of the action in what follows. The construction of the covariant interaction Lagrangian went in the following stages.

1. Contract the conjugate momentum of the field with the variation of the field:

\[ L_{0}^{th} = X^{ij0}(\xi^l \partial_l D_{ij} + D_{lj} \partial_i \xi^l + D_{il} \partial_j \xi^l) \] .

(8)

Here we note that this is akin to the pairing of adjoint elements with coadjoint elements in the geometric actions [4, 5, 6]. This analogy follows since the conjugate momentum \( X^{ij0} \) transforms as an adjoint element (see Eqs.(3-4)), while the Lie derivative of \( D_{ij} \), the term in the parenthesis, transforms as a coadjoint element in one dimension.

2. Replace the fields \( \xi^i \) with a space-time component of the field, \( D_0^a \):

\[ L_{1st} = X^{LM0}(D_0^A \nabla_A D_{LM} + D_{AM} \nabla_L D_0^A + D_{LA} \nabla_M D_0^A) \] .

(9)

This is an important ingredient in the construction in it states that some of the components of the diffeomorphism field, transform in the adjoint representation. Here it is clear that \( D_0^a \) will transform as a adjoint element with respect to time independent spatial translation. Such components serve as Lagrange multipliers for the constraint equation discussed earlier. (One possible gauge fixing condition for the coordinate transformations is to choose coordinates were \( \partial_0 D_{a0} = 0 \).) Later in the superfield actions, we will use a superfield \( F_0^A \), that has to leading order in its component fields \( D_0^a \).

3. Extend the time directions to covariant directions:

\[ L_{\text{int}} = X^{LMR}(D_R^A \nabla_A D_{LM} + D_{AM} \nabla_L D_R^A + D_{LA} \nabla_M D_R^A) \] .

(10)
Here we restore the full general coordinate covariance of the Lagrangian. The importance of this will be seen in the supersymmetric cases later when we restore the vector indices to the superfield, i.e \( F_0^M \rightarrow F_A^M \).

In [10] similar thinking was used to discuss Lagrangians in which the Cauchy data carries a representation of the Virasoro algebra. Perhaps one of the most illuminating comments that we can make is to note that this interaction Lagrangian is to be thought of as the analog of the first term that appears in either Eq.(5) or Eq.(6). In this work we will use this principle in order to define a new class of interactions between target space supergravity superfields and the moduli of Diff \( S^1 \) gauge fields. These classes of interaction are based on the coadjoint representation of the super Virasoro algebra.

### 2.2 Superfields and the Super Virasoro Algebra

The super Virasoro algebra contains the bosonic Virasoro generators \( L_m, m \in \mathbb{Z} \) and fermionic generators \( G_\mu, \mu \in \mathbb{Z} \) or \( \mathbb{Z} + \frac{1}{2} \), and a central charge \( \hat{c} \). It reads

\[
[L_m, L_n] = (m - n) L_{m+n} + \frac{1}{8} \hat{c} (m^3 - m) \delta_{m+n,0} I ,
\]

\[
[L_m, G_\mu] = \left( \frac{1}{2} m - \mu \right) G_{m+\mu} ,
\]

\[
\{ G_\mu, G_\nu \} = -i \frac{4}{2} L_{\mu+\nu} - i \frac{1}{2} \hat{c} \left( \mu^2 - \frac{1}{4} \right) \delta_{\mu+\nu,0} I .
\]

A generic element valued in the adjoint of this algebra takes the form

\[
\hat{A} = \sum_{m=-\infty}^{\infty} A^m L_m + \sum_{\mu=-\infty}^{\infty} A^\mu G_\mu + \frac{1}{8} a \hat{c} I .
\]

The parts of the generic elements that do not lie in the center of the algebra also may be used to introduce the concept of a superfield \( A(z, \zeta) \)

\[
A(z, \zeta) = \sum_{m=-\infty}^{\infty} (A^m z^{m+1}) + 2 \sum_{\mu=-\infty}^{\infty} (A^\mu z^{\mu+\frac{1}{2}}) ,
\]

and the generic element of the algebra written in Eq.(14) has an equivalent representation as a doublet \((A(Z), a)\) with \( Z = \{z, \zeta\} \). We can then derive the following general commutator for two adjoint elements

\[
[[A,B]] = ((\partial A)B - A\partial B - i \frac{1}{2} (\mathcal{D}A)(\mathcal{D}B), \oint dZ (\partial^2 DA) B) ,
\]

where \( A \) and \( B \) are adjoint elements and

\[
\mathcal{D} = \frac{\partial}{\partial \zeta} + i \zeta \frac{\partial}{\partial \zeta} , \quad dZ = \frac{dz}{2\pi i} d\zeta .
\]

\[^5\]Here we use the word “moduli” to refer to the parts of the gauge fields with vanishing field strengths.

\[^6\]The quantity denoted by \( \zeta \) here in Eq.(13) is simply a 1D Grassmann coordinate. In a later section we use \( \theta \) to denote the space-time Grassmann supercoordinate variable.
This result is found by first using the representation in Eq.(12), and calculating the usual bracket using the algebra defined by Eq.(11) \[6, 7\]. (Note that this definition of \(D\) implies \(D^2 = i2\partial_z\) which we use below to derive Eq.(22).)

The action on coadjoint elements can be constructed in a similar way \[4, 7\] and we find for the action of a super field \(F\) on a coadjoint vector \(B^*\) is

\[
\delta_F B^* = -FD^2 B^* - \frac{1}{2} DFDB^* - \frac{3}{2} D^2 F B^* + q D^5 F ,
\]

where \(F\) has the decomposition \(F = \xi + i\zeta \epsilon\) and \(B^* = (u + i\zeta D, b^*) = (u, D, b^*)\). Now the isotropy equation for the coadjoint element \(B^*\) is given by setting Eq.(16) to zero. In terms of component fields this becomes the two coupled equations

\[
-\xi \partial D - \frac{1}{2} \epsilon \partial u - \frac{1}{2} \partial \epsilon u + b^* \partial^2 \xi - 2 \partial \xi D = 0 \quad (17)
\]

\[
-\xi \partial u - \frac{1}{2} \epsilon D - \frac{3}{2} \partial \xi u + q \partial^2 \epsilon = 0 \quad (18)
\]

with \(\partial = \partial_z\).

We can make sense out of these transformation rules in higher dimensions (modulo the central extension). We replace the 1D Grassmann variable \(\zeta\) to a single D-dimensional Majorana spinor \(\theta^\alpha\) and the supersymmetric covariant derivative operator to

\[
D\mu = \partial_\mu - \frac{i}{2} \gamma^{\mu N}\theta^\nu \partial_N .
\]

With this

\[
\{D_\mu, D_\nu\} = -i \gamma^{\mu N}_\nu \frac{\partial}{\partial z^N} ,
\]

where \(\gamma^{\mu N}_\nu\) is determined by the space-time dimension. With these \(\gamma^{\mu N}_\nu\)’s we also introduce \(\gamma^{\alpha \beta}\) such that

\[
\gamma^{A \alpha \beta} B^{\alpha \beta} = \frac{1}{2} \delta^{\lambda \mu}_\alpha \eta^{AB} + \frac{1}{2} \Sigma^{AB \lambda} ,
\]

where \(\Sigma^{AB \lambda}_\alpha\) is anti-symmetric in its space-time indices. Then for two dimensions, say, \(F\) is promoted to the vector superfield \(F^M = (\xi^M + \theta^\alpha \gamma^{M \beta}_\alpha \epsilon^\beta)\) and \(B^*\) is promoted to the \(\frac{3}{2}\) spin superfield \(B_{\mu M} = (\Upsilon_{\mu M} + \theta^\alpha \gamma^{N M}_\alpha D_{M N} + \theta^\alpha \theta^\beta \gamma^{N M}_\alpha A_{\beta | M N})\). The coadjoint element \(D\) that appears in \(B^* = (u + \zeta D, b^*)\) is the “space-space” component of \(D_{M N}\) when the dimension is two, viz. \(D \equiv D_{11}\). Keeping track of the indices we may write Eq.(16) as

\[
\delta_F B_{\mu M} = F^N \partial_N B_{\mu M} + \partial_M F^N B_{\mu N} + \frac{1}{2} (\partial_N F^N) B_{\mu M} + i (D_\lambda F^N \gamma^{\lambda \nu}) D_\nu B_{\mu M} .
\]

This is seen as the Lie derivative with respect to \(F^M\) on the space-time index in the first three summands followed by a supersymmetry transformation on \(B_{\mu M}\) with \(\epsilon^\nu \equiv (D_\lambda F^N \gamma^{\lambda \nu})\). This combination of a Lie derivative and supersymmetry transformation is a natural extension of the isotropy equation to dimensions higher than two. Note that the superfield \(B_{\mu M}\)
carries a density weight of \( \frac{1}{2} \). This is consistent with the one dimensional isotropy equation.

Setting Eq.(22) to zero and rewriting in terms of the component fields we have

\[
\theta^\alpha A_{\alpha\mu}(\xi^M_D D_{AB} + \partial_B \xi^M D_{CM} + \partial_M \xi^M_D D_{AB} + \frac{1}{2} \eta_{MA} \eta^{CN} \partial_N \xi^M D_{CB} - \frac{1}{2} \partial_A \xi^C D_{CB}) \\
+ \theta^\alpha A_{\alpha\rho}(\epsilon^\rho \partial_A \Upsilon_{\mu B} + \partial_B \epsilon^\rho \Upsilon_{\mu M} + \frac{1}{2} \partial_A \epsilon^\rho \Upsilon_{\mu B} - \frac{1}{2} n \epsilon^\rho \Upsilon_{\mu B}) \\
+ \xi^A \partial_A \Upsilon_{\mu B} + \partial_B \xi^M \Upsilon_{\mu M} + \frac{1}{2} \partial_M \xi^M \Upsilon_{\mu A} + i n \epsilon^\rho \Upsilon_{\mu B} + O(\theta^2 & \Sigma \text{ terms}) = 0,
\]

where \( n \) in the above is the dimension. Evaluating this in one dimension leads to the Eqs.(17,18).

3 Actions for the Super Virasoro Group

3.1 2D and 3D Majorana Spinors

We are now in a position to construct an action from the procedure outlined above. To begin with we consider the 2D or 3D Majorana action since this requires no modification to the superfield \( B_{\mu N} \). First we define \( F_N^A \) as the covariant extension of \( F^N \) that appears in Eq.(22). As stated earlier in the second part of the procedure of Section(2.1), this should be defined in terms of the superfield \( B_{\mu N} \) since it contains the diffeomorphism field and its superpartners. The fully covariant superfield that satisfies the requirements is

\[
F_N^A = E^\frac{1}{2} \gamma^\alpha_{\alpha\beta} D_A B^\beta_{\mu N}.
\]

Clearly the leading term of \( F_0^N = D_0^N \) which replaces the vector field \( \xi^i \). Thus \( F_0^N \) serves as the super symmetric extension of Eq.(9) and the covariant extension analogous to Eq.(10). Next we defined the covariant superfield \( B_{A\mu M} \) as,

\[
B_{A\mu M} \equiv F_A^N \nabla_N B_{\mu M} + \nabla_M F_A^N B_{\mu N} + \frac{1}{2} (\nabla_N F_A^N) B_{\mu M} - \frac{1}{8} \nabla_C F^A_{D\lambda M} \Sigma^{CD\lambda}_{\mu} B_{\Lambda M} \\
+ i (D_\lambda F_A^N \gamma^\lambda_{\alpha\nu}) D_\nu B_{\mu M} + q D_\mu \nabla_N \nabla_M F_A^N,
\]

where the \( \Sigma \) term only contributes to the 3D case. One can see that this is the superfield analogue of the term in parenthesis in Eq.(8) when the subscript \( A = 0 \) above. Note that \( E \) is the superdeterminant. We have also included a term that will reproduce the contribution from the central extension when evaluated in one spatial dimension. For the final part we note that \( \nabla^A B^{\Lambda M} \) will serve as the “covariantized” conjugate momentum. A suitable kinetic term for the Lagrangian would be

\[
\int d^2x d\theta^\mu d\theta^\nu \beta (\nabla^A B_{\mu N})(\nabla_A B_{\nu M}) \eta^{NM},
\]

where \( \beta \) is the string tension found in Eq.(1).
With this we can write the action in two dimensions as

\[
S = - \int d^2 x d\theta^\mu d\theta^\nu \beta (\nabla^A B_{\mu N})(\nabla^A B_{\nu M}) \eta^{NM} \\
- \int d^2 x d^2 \theta \frac{1}{2} \epsilon^{\mu\nu} (\nabla^A B_{\nu B}) B_{\mu M} \eta^{BM} ,
\]

(27)
or

\[
S = - \int d^2 x d^2 \theta \beta \epsilon^{\mu\nu} (\nabla^A B_{\mu N})(\nabla_A B_{\nu M}) \eta^{NM} \\
- \int d^2 x d^2 \theta \frac{1}{2} \epsilon^{\mu\nu} (\nabla^A B_{\nu B}) B_{\mu M} \eta^{BM} .
\]

(28)

Here we have not introduced the determinant $E^{-1}$ in the action since our $B_{\nu M}$ field is a tensor density of weight $1/2$. This action yields the equations of motion and constraint equations for the 2D super diffeomorphism field. The constraint equations from the bosonic field equations of $D_{01}$ are the analogs of the bosonic part of the isotropy equations found on the coadjoint orbits of the super Virasoro algebra. The field equations of the $\Upsilon_{\mu 0}$ and the auxiliary field $A_{\alpha MN}$ yield the fermionic constraints on the orbit.

### 3.2 2D Chiral Spinors

The procedure to build the other supersymmetric actions mimics the previous example. One should be careful to choose $F^M_N$ so that it agrees with Eq.(9) in the bosonic limit. Furthermore the kinetic terms and “covariantized” conjugate momenta must respect the Grassmann integration. Actions with an odd number of fermions will differ in form from Eq.(28) since a “quadratic” action would not be able to recover the proper bosonic limit (Eq.(1)). The 2D chiral case epitomizes these concerns.

Since the 2D Majorana spinors can be eigenstates of $\gamma^3$ we can write spinors in terms of the one component eigenstates of $\gamma^3$, $\bar{\theta}^\mu$ and $\theta^\mu$ corresponding to the “+” and “-” eigenvalues. We write the “+” eigenstate superfield

\[
\bar{B}_{\mu M} = (\bar{\Upsilon}_{\mu M} + \bar{\theta}^\nu \gamma^N_{\nu \mu} D_{NM}) ,
\]

(29)

and again define $F_{AB}$ through,

\[
\bar{F}^N_A = \gamma^\alpha_{A \beta} \bar{D}^\alpha_{\beta} \bar{B}^N .
\]

(30)

Although there is only one spinor component we have purposely preserved the index structure of the gamma matrices for comparison to higher dimensions. Since we are only integrating over one component we must insure that the bosonic sector preserves the isotropy equations for orbits of the Virasoro algebra. The action appropriate for chiral fermions in 2D is

\[
S = - \int d^2 x d\bar{\theta}^\mu \beta \gamma^B_{\mu \lambda} \nabla^A \bar{F}_{BM}(\nabla^A \bar{B}^{\lambda M}) \\
- \int d^2 x d\bar{\theta}^\mu \frac{1}{2} \gamma^B_{\mu \lambda} (\nabla^A \bar{F}_{BM}) \bar{B}^\lambda_M + h.c. .
\]

(31)
This action, called the affirmative action, gives precisely the one dimensional isotropy equations for the super Virasoro sector without having to introduce auxiliary fields. It is this action that governs the classical dynamics of the quadratic differentials that appear on the 2D superstring world sheet. The vacuum expectation value of this field determine the sub-algebra that will be preserved on the world sheet and hence govern the vacuum symmetry of the superstring. We are investigating its importance for heterotic string theories.

3.3 4D Chiral Spinors

By using the above 3/2 spin superfield we can make contact with an interesting chiral variant of this action in four dimensions. By considering a symmetric superfield \( B_{\mu\alpha\beta} \) field that satisfy
\[
\bar{D}_\alpha B_{\mu\lambda\beta} = 0,
\]
one has a chiral gravitational theory with the action
\[
S = - \int d^4x \, \frac{1}{2} C^{\mu\nu} (\nabla^A B_{\mu\nu\gamma})(\nabla_A B_{\nu\alpha\gamma}) C^{\lambda\alpha} C^{\rho\gamma} - \int d^4x \, \frac{1}{2} \frac{1}{q} C^{\mu\nu} (\nabla^A B_{\nu\alpha\gamma}) B_{\lambda\mu\lambda\rho} C^{\lambda\alpha} C^{\rho\gamma} + h.c. .
\] (32)
The field \( d_{\alpha\beta\chi\delta} = D_\alpha B_{\beta\chi\delta} \) contains the chiral graviton \( D_\alpha B_{\beta\chi\delta} \mid _{\theta=0} \) and a Lorentz \((1,0)\) field, \( D^\alpha B_{\alpha\beta\delta} \mid _{\theta=0} \).

3.4 4D Spinors

More interesting is the full non-chiral four dimensional theory. Here the relevant superfield is a supervector field akin to the field \( U_M \) found in supergravity theories related to the Einstein-Hilbert action. Let us write our superfield as
\[
B_M = h_M + (\theta^\alpha \Upsilon_{\alpha M} + h.c.) + \theta^\alpha \bar{\theta}^\dot{\alpha} D_{\alpha\dot{\alpha}M} + O(\theta^3),
\] (33)
where we use “dot” and “undotted” notation for spinor indices when necessary. The field \( F_M^N \) that is suitable for this case is
\[
F_{\alpha\dot{\alpha}}^M \equiv \bar{D}_\alpha D_{\dot{\alpha}} B^M .
\] (34)
Note again that the \( F_0^M \) component recovers \( D_0^M \) as required in Eq.(31). Now since \( B_N \) is a tensor of rank one and density zero (i.e. no fermionic indices) we can define
\[
B_{\alpha\dot{\alpha}N} \equiv F_{\alpha\dot{\alpha}}^M \nabla_M B_N + B_M \nabla_N F_{\alpha\dot{\alpha}}^M + \frac{1}{2} (D_\mu F_{\alpha\dot{\alpha}}^{\mu\dot{\mu}}) D_\mu B_N + \frac{1}{2} (\bar{D}_\mu F_{\alpha\dot{\alpha}}^{\mu\dot{\mu}}) D_\mu B_N + q \nabla_N \nabla_M F_{\alpha\dot{\alpha}}^M .
\] (35)
With this we write
\[
S = - \int d^4x \, E^{-1/2} d\theta^2 d\bar{\theta}^2 \beta (\nabla^A B_N)(\nabla_A B_M) g^{NM} - \int d^4x \, E^{-1/2} d\theta^2 d\bar{\theta}^2 \frac{1}{q} (\nabla^A B_B) B_{AM} g^{BM} + h.c. .
\] (36)
There is a rather interesting observation to be made about the first line above. Had we begun with the action for 4D, \( N = 1 \) supergravity used the background-quantum field method and gauge fixed the quantum supergravity pre-potential, we would arrive at precisely the result on the first line above. Stated another way, by using solely arguments about the algebraic structure of the coadjoint super Virasoro representation we are lead the the gauge-fixed 4D, \( N = 1 \) superfield supergravity action expressed correctly in term of its prepotential to lowest order in a background field formalism.

4 Conclusion

We have constructed an action for the super diffeomorphism field that is consistent with the isotropy equations found on the orbits of the super Virasoro groups. In 2D, 3D and chiral 4D, the fundamental field is a spin \( \frac{3}{2} \) superfield, \( B_{\mu N} \) whereas in 4D non-chiral we require a spin one prepotential \( B_{\alpha \dot{\alpha}} \). This new description of the graviton can augment the description of quantum gravity in low dimensions since this action supplies dynamics to the background field that is often called \( B \) found in the super geometric action \[ S = 2q \int d\tau \oint dZ \left( -\frac{1}{2} \partial_x^2 \bar{z} D \bar{z} \partial_x \bar{z} + \frac{3}{2} \partial_x^2 \bar{z} D \bar{z} \partial_x \bar{z} D \bar{z} - \partial_x^2 D \bar{z} \partial_x \bar{z} D \bar{z} \right. \\
+ \left. \partial_x^2 \bar{z} D \bar{z} \partial_x \bar{z} D \bar{z} - D \partial_x^2 \bar{z} D \bar{z} \partial_x \bar{z} D \bar{z} + \frac{1}{2} (\partial_x^2 \bar{z})^2 D \bar{z} \partial_x \bar{z} + \frac{1}{4} B(z)(\partial_x \bar{z} / \partial_x \bar{z}) \right). \]

Furthermore, just as the bosonic diffeomorphism field theory for \( D_{AB} \) might play an important role in the quantization of the Virasoro group and \( AdS^3 \) gravity so might \( B_{\mu N} \) in the quantization of the super Virasoro Group and supergravity on \( AdS^3 \). In fact in \[ \text{[15]} \] the coadjoint orbits of the Virasoro group are related to the phase space of \( AdS^3 \) gravity. However, in both the bosonic and supersymmetric cases only certain orbits admit Kähler structures and hence have a symplectic structure that can be geometrically quantized \[ \text{[11, 16, 17]} \]. The origin of the obstruction to quantization could be due to the restrictions on the diffeomorphism field \( D_{AB} \) which foliates the dual space into orbits. Each orbit can be described by a vacuum expectation value of \( D_{AB} \) where \( < D_{AB} > = \delta^+_A \delta^+_B b^* \), where \( b^* \) is a constant. In general the value of \( b^* \) determine the coset space and therefore the global symmetry of \( AdS^3 \) gravity. One may think of \( D_{AB} \) as having been “quenched”. However quantum mechanics may only be compatible with certain values of the quenched field. A more ambitious approach to quantization would be to incorporate the interactions of \( D_{AB} \) into the partition function as one does during “annealing.” The annealed theory might admit a consistent quantization scheme that allows transitions from the different coadjoint orbits that by themselves cannot be quantized. This is exactly analogous to the gauged
WZNW models\cite{18, 19, 20, 21}. Also the presence of this field will affect the number of microstates which could influence the entropy of black holes derived from the Virasoro and super Virasoro coadjoint orbits.

As we have seen for the case of spinors possessing four components, the superfield can exist in higher dimensions and extended to several supersymmetries. The number of spinors will determine the spin content of the $B$-superfield prepotential and will interact with metric as a covariant graviton. Another interesting extension to the 1+1 dimensional chiral models would be to use the dual representation of the geometrically realized super Virasoro algebra that are defined in \cite{23, 22}.

Finally, we would like to note a possible additional implication of this work. By the proposed method of this work, we see that there appears a way in which to use the embedding of representations of the super Virasoro algebra to obtain information directly about the pre-potential of supergravity theories in greater than two dimensions. In our work we have found a curious coincidence arising from an embedding procedure that leads to identification between quantities associated with the dual coadjoint elements on the one side and supergravity pre-potentials on the other Eq.(24) and Eq.(34). Should it be possible to extend this construction universally, it is highly suggestive that the representation theory of super Virasoro algebras may be in some way determining the superfield pre-potential structure of superspace supergravity. These issues are under investigation.

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