On the Chromatic Number of $\mathbb{R}^n$ for Small Values of $n$

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Abstract

The lower bound for the chromatic number of $\mathbb{R}^n$ is improved for $n = 6, 7, 10, 11, 12, 13$ and $14$.

1 Introduction.

The chromatic number of $n$-dimensional Euclidean space, denoted $\chi(\mathbb{R}^n)$, is the minimum number of colors that can be assigned to the points of $\mathbb{R}^n$ so that no two points at distance one receive the same color. In this note, we establish new lower bounds for $\chi(\mathbb{R}^n)$ for several small values of $n$.

In [10], a table of lower bounds for the $\chi(\mathbb{R}^n)$ was given. Besides the new bounds given in that paper, we are aware of only one other improvement [6]. Based on this improvements, we give a modified table below. The table also indicates the new bounds given here.

2 $\chi(\mathbb{R}^6)$

We construct a graph $G_{175}$ of order 175 with chromatic number 12. The vertices of the graph are a set of points in $\mathbb{R}^6$ generated by 11 special points. The coordinates of each of these special points is permuted in all $6! = 720$ possible ways to obtain the full set of 175 vertices. The graph will be constructed as an...
Table 1: Lower bounds on $\chi(\mathbb{R}^n)$ for small $n$.

$r$-distance graph for $r = \sqrt{8}$. The coordinates for each point can be divided by $r$ to obtain a unit distance graph.

The following table lists the aforementioned 11 special points along with the number of distinct points generated by permuting their coordinates. For each of these 11 points, $v_i$, let $V_i$ denote the set of points obtained by permuting the coordinates, and let $n_i = |V_i|$, as shown in the table.

| Point | Coordinates | $n_i$ |
|-------|-------------|------|
| $v_1$ | 0 0 0 0 0 0 0 0 0 | 1 |
| $v_2$ | 2 2 0 0 0 0 | 15 |
| $v_3$ | 2 2 2 2 2 0 0 | 15 |
| $v_4$ | $\sqrt{3}$ 1 1 1 1 1 1 | 6 |
| $v_5$ | $\sqrt{3}$ 1 1 1 1 1 | -1 30 |
| $v_6$ | $-\sqrt{3}$ 1 1 1 1 1 | 6 |
| $v_7$ | $-\sqrt{3}$ 1 1 1 1 1 | -1 30 |
| $v_8$ | $2 + \sqrt{3}$ 1 1 1 1 1 1 | 6 |
| $v_9$ | $2 + \sqrt{3}$ 1 1 1 1 | -1 30 |
| $v_{10}$ | $2 - \sqrt{3}$ 1 1 1 1 1 | 6 |
| $v_{11}$ | $2 - \sqrt{3}$ 1 1 1 1 | -1 30 |

Table 2: The 11 points that generate the vertices of $G_{175}$.

Observe that the subgraphs induced by $V_2$ and by $V_3$ are each isomorphic to the line graph of $K_6$. In the case of $V_2$ this is because two points in $V_2$
are adjacent if their dot product is 4. This occurs when there is exactly one coordinate position where both points have a 2. In the case of $V_3$, two points are adjacent if there is exactly one coordinate position where both points have a zero. Next define an isomorphism $\phi : V_2 \rightarrow V_3$ by letting

$$\phi(u_1, u_2, u_3, u_4, u_5, u_6) = (2 - u_1, 2 - u_2, 2 - u_3, 2 - u_4, 2 - u_5, 2 - u_6).$$

Then the edges joining $V_2$ and $V_3$ are given as follows. A vertex $x \in V_2$ is adjacent to a vertex $y \in V_3$ whenever $y$ is not adjacent to (or equal to) $\phi(x)$. This gives the subgraph $H$ of the $\sqrt{8}$-distance graph induced by $V_2 \cup V_3$.

Note that the independence number $\alpha(L(K_6)) = 3$. However, in $H$ an independent set of size three consisting of vertices from $V_2$ dominates every vertex in $V_3$. So $\alpha(H) = 4$. In fact, it can be seen that every maximum independent set can be obtained from the following independent set by an appropriate permutation of coordinates

$$
\begin{align*}
(2, & 2, 0, 0, 0) \\
(0, & 0, 2, 2, 0, 0) \\
(2, & 0, 2, 0, 2, 2) \\
(2, & 0, 0, 2, 2, 2)
\end{align*}
$$

Thus we have the following lemma.

**Lemma 2.1.** The graph $H$ has chromatic number 8.

The proof of the following theorem can be completed by a short computer search that makes strong use of this lemma.

**Theorem 2.1.** The graph $G$ has chromatic number 12 and can be represented as a unit distance graph in $\mathbb{R}^6$.

## 3 $\chi(\mathbb{R}^7)$

**Theorem 3.1.** $\chi(\mathbb{Q}^7) \geq 16$.

**Proof.** Consider the following fourteen sets in $\mathbb{Q}^7$:

$$
\begin{align*}
S_{123} &= [\pm 2, \pm 2, \pm 2, 0, 0, 0, 0], & T_{123} &= [0, 0, 0, \pm 1, \pm 1, \pm 1, \pm 1], \\
S_{145} &= [\pm 2, 0, 0, \pm 2, \pm 2, 0, 0], & T_{145} &= [0, \pm 1, \pm 1, 0, \pm 1, \pm 1, \pm 1], \\
S_{167} &= [\pm 2, 0, 0, 0, \pm 2, \pm 2], & T_{167} &= [0, \pm 1, \pm 1, \pm 1, 0, 0], \\
S_{247} &= [0, \pm 2, 0, \pm 2, 0, 0, 0], & T_{247} &= [\pm 1, 0, \pm 1, \pm 1, 0, \pm 1, \pm 1], \\
S_{256} &= [0, \pm 2, 0, 0, \pm 2, \pm 2], & T_{256} &= [\pm 1, 0, \pm 1, \pm 1, 0, \pm 1], \\
S_{346} &= [0, 0, \pm 2, \pm 2, 0, \pm 2, 0], & T_{346} &= [\pm 1, \pm 1, 0, 0, \pm 1, \pm 1, \pm 1], \\
S_{357} &= [0, 0, \pm 2, \pm 2, 0, \pm 2], & T_{357} &= [\pm 1, \pm 1, 0, \pm 1, 0, \pm 1, 0].
\end{align*}
$$
Denote
\[
S = S_{123} \cup S_{145} \cup S_{167} \cup S_{247} \cup S_{256} \cup S_{346} \cup S_{357},
\]
\[
T = T_{123} \cup T_{145} \cup T_{167} \cup T_{247} \cup T_{256} \cup T_{346} \cup T_{357}.
\]

Let \(G\) be the graph whose vertices are the points in \(S \cup T\). Two vertices are adjacent if and only if their distance is 4. It can be checked that \(G\) has 168 vertices and 4396 edges. We are going to prove that \(\chi(G) = 16\).

Let \(H\) be the subgraph of \(G\) induced by the points in \(S\). One can verify that \(H\) is a graph of order \(|V(H)| = 56\), size \(|E(H)| = 756\), independence number \(\alpha(H) = 4\) and chromatic number \(\chi(H) = |V(H)|/\alpha(H) = 14\).

Similarly, let \(K\) be the subgraph of \(G\) induced by the points in \(T\). One can verify that \(K\) is a matching of order \(|V(K)| = 112\), size \(|E(K)| = 56\), independence number \(\alpha(K) = 56\) and chromatic number \(\chi(K) = |V(K)|/\alpha(K) = 2\). It follows that \(\chi(G) \leq \chi(H) + \chi(K) = 14 + 2 = 16\).

Let \(M\) be an independent set of \(G\). From the observation above \(|M \cap V(H)| \leq 4\). We say that \(M\) is an independent set of type \(k\) if \(|M \cap V(K)| = k\) for some \(0 \leq k \leq 4\). The following claim can be easily checked

**Claim 3.1.** Let \(M\) be an independent set of type \(k\) in \(G\). Then, the following hold:

\[
\begin{align*}
\text{If } k &= 0, \quad \text{then} \quad |M \cap V(K)| \leq 56. \\
\text{If } k &= 1, \quad \text{then} \quad |M \cap V(K)| \leq 24. \\
\text{If } k &= 2, \quad \text{then} \quad |M \cap V(K)| \leq 24. \\
\text{If } k &= 3, \quad \text{then} \quad |M \cap V(K)| \leq 3. \\
\text{If } k &= 4, \quad \text{then} \quad |M \cap V(K)| \leq 3.
\end{align*}
\]

Suppose that \(\chi(G) \leq 15\). Then the set of vertices of \(G\) can be partitioned into 15 independent sets. Denote by \(m_k\) the number of independent sets of type \(k\) in this partition, \(0 \leq k \leq 4\). Then from Claim 3.1 the following relations hold true:

\[
15 = m_0 + m_1 + m_2 + m_3 + m_4. \\
56 = m_1 + 2m_2 + 3m_3 + 4m_4. \\
112 \leq 56m_0 + 24m_1 + 24m_2 + 3m_3 + 3m_4.
\]

But it is easy to check (either by hand or a short program) that this system has no solutions in nonnegative integers. Thus \(\chi(G) \geq 16\). \(\square\)
4 \( \chi(\mathbb{R}^{10}) \)

The construction for \( \chi(\mathbb{R}^{10}) \) is related to the well-known Frankl-Wilson construction [7] which established an exponential lower bound for \( \chi(\mathbb{R}^{10}) \), and also gives the best constructive lower bound for classical diagonal Ramsey numbers.

The vertices of this graph are identified with the points \((x_1, x_2, \ldots, x_{11})\) in \( \mathbb{R}^{11} \) such that each \( x_i = 0 \) or 1, and

\[
\sum_{i=1}^{11} x_i = 5
\]

There are \( \binom{11}{5} = 462 \) such points. Two points are adjacent if their distance is 2 (and so their Hamming distance is 4). This graph is regular of degree \( \binom{5}{2} \binom{6}{2} = 150 \).

A computation reveals that the independence number of \( G \), denoted as usual by \( \alpha(G) \), is 18. We did this computation two ways. First, our own special program written for graph of this type was used. Second, the result was verified by the mcqd program of Konc and Janežič [9]. Using the fact that \( \chi(G) \geq \frac{n}{\alpha(G)} \), for any graph \( G \) of order \( n \), we find that \( \chi(G) \geq \lceil \frac{462}{18} \rceil = 26 \).

Finally, for each point, the sum of the coordinates is 5, so the points are located on a 10-dimensional hyperplane, so we have the following.

**Theorem 4.1.** \( \chi(\mathbb{R}^{n}) \geq 26 \), for \( n = 10, 11 \).

5 \( \chi(\mathbb{R}^{12}) \)

The construction for \( \mathbb{R}^{12} \) parallels that of \( \mathbb{R}^{10} \). In this case the vertex set consists of all \( 0 - 1 \)-vectors in \( \mathbb{R}^{13} \) with Hamming weight 6. Again two vertices are adjacent if their distance is 2 (Hamming distance 4). So the the graph has order \( \binom{13}{6} = 1716 \) and degree \( \binom{13}{6} \binom{13}{6} \).

For this case, computing the independence number is a much harder computation. But again, both mcqd and our specialized program were able to determine that the independence number is 46. Hence the chromatic number is at least \( \lceil \frac{1716}{46} \rceil = 36 \)

**Theorem 5.1.** \( \chi(\mathbb{R}^{n}) \geq 36 \), for \( n = 12, 13, 14 \).

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