FRACTAL LOCAL GALAXY NUMBER COUNTS MAY IMPLY STRONG BIAS

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Abstract
Exact Tolman solutions are used to analyse the implications if the galactic number count has a fractal form out to a distance of about 150 Mpc in a universe which is homogeneous on the large scale. It is concluded that such a model requires either a non-linear Hubble law or a very low density if galaxies trace the total matter distribution.

1 Introduction
This paper is based on joint work with Roy Maartens and Neil Humphreys. More details are given in three joint papers. There is substantial evidence to support a claim that the number of galaxies within a volume defined by a distance \( y \) has a fractal distribution,

\[ N(y) = Ay^\nu \]

where \( 0 < \nu < 3 \) is the fractal index and \( A \) is a constant. How large \( y \) can be is a matter of dispute but there is some agreement that for \( y < 150 \) Mpc the formula \( N(y) = Ay^\nu \) holds with \( \nu \) between 1.5 and 2.

Unfortunately at this stage in the history of cosmology there is no convincing theoretical model which predicts such a distribution. Some work has been done recently but it is only a beginning. This is a problem because it is notoriously difficult to derive a convincing theoretical model from data alone. Statistical procedures are excellent if we know the model and wish to determine the parameters. However they become suspect when we have to derive the model and the parameters from the same data. Accumulation of data and improved statistics help but they do not remove unease and doubt, e.g. is the distribution really multi-fractal or is the data set incomplete? We can expect the debate to continue.

Our contribution has been to provide a different approach to the problem. We assume; (a) a fractal form for the number counts holds to a distance (observer area distance) 150 Mpc from us, (b) space-time is spherically symmetric about us, (c) Einstein’s field equations hold, (d) the matter distribution can be modelled as dust on a suitable averaging scale, (e) beyond 150 Mpc the universe can be modelled by a
Friedmann-Lemaître-Robertson-Walker (FLRW) geometry, (f) the solutions are regular, and there is no shell crossing or surface density layers and (g) the Darmois conditions hold at matching surfaces. A consequence of these assumptions is that we can use a Tolman model within 150 Mpc. Consistent with the picture we take seriously that the observations are on the observer’s past light cone. This is often not done by astronomers as pointed out by Ellis et al. and Laix and Starkman, and theorists do not always do it either, but it has significant effects, for instance on the measurements of the density power spectrum as pointed out by Laix and Starkman. Even in the spatially homogeneous FLRW models it has a non-trivial effect leading to an in-homogeneous number count formula.

Observational coordinates provide a powerful tool for analysis on the past light cone of an observer and hence for discussions of real observations. They are especially well suited to spherically symmetric space-times. In these coordinates the null geodesic generators of the observer’s past light cone are trivially integrated whereas they cannot be integrated exactly for Tolman models in (3 + 1) coordinates. The reverse situation holds for the field equations. It follows that, in general, the transformation between observational and (3 + 1) coordinates cannot be integrated exactly in simple functions. Thus, as could be anticipated, the two coordinate systems give different viewpoints on the geometry and the physics; the (3 + 1) coordinate approach provides the physical interpretations and the observational coordinates relate to astronomical observations. In this work we needed both viewpoints and therefore found it necessary to develop results which enable us to flip between the two systems.

2 Notation and Formulae

In this section a very brief sketch will be given of the cosmological model and the notation. For details see Humphreys et al. In observational coordinates the spherically symmetric model takes the form:

\[ ds^2 = -A(w,y)^2 dw^2 + 2A(w,y)B(w,y)dy + C(w,y)^2 d\Omega^2 \]  

(2)

where \( \{w = \text{const}\} \) label the past light cones on the observer’s world line \( \{y = 0\} \), \( y \) is a comoving radial distance coordinate down the null geodesic \( \{w = \text{constant}, (\theta, \phi) = \text{const}\} \) and \( d\Omega^2 = d\theta^2 + \sin^2 \phi d\phi^2 \).

The dust velocity is given by \( u^i = A^{-1} \delta^i_w \). The FLRW models are given by choosing \( A = B \). In (3 + 1) coordinates the model is given by,

\[ ds^2 = -dt^2 + \left[ \frac{\partial R(r,t)}{\partial r} \right]^2 \frac{dr^2}{1 - kf(r)^2} + R(r,t)^2 d\Omega^2 \]

(3)
where $f$ is arbitrary and relates to the total energy of the system. The geometry of the $\{t = \text{const}\}$ hyper-surfaces is parabolic if $k = 0$, hyperbolic for $k = -1$ and elliptic for $k = 1$. The dust velocity is, $u^i = \delta^i_0$.

From the two metrics the subtle relations between the two systems can be demonstrated. On a fixed light cone $\{w = w_0\}$ the observer area distance (angular diameter distance) $D$ in terms of redshifts is given by,

$$D(y(z)) = \begin{cases} 
C(w_0, y(z)) \\
R(r(z), T(r(z)))
\end{cases}$$

where $t = T(r)$ is the equation of the past light cone of the observer. In general the function $T$ cannot be determined as an exact function and so numerical methods have to be employed and thus to determine $R$ we need to know both $r$ and $t$ separately. Mustapha et al. circumvent this by restricting attention to a single light cone, which cuts out the dynamics, and they use time $t$ which is not an observable. These are not appropriate here.

In the metric $[6]$ the hyper-surfaces $\{w = \text{const}\}$ are null but that is not sufficient to make them the past null cones of the observer. This requires central conditions which develops earlier results of Temple $[18]$. As $y \to 0$ the following asymptotic behaviour is required,

$$A(w, y) = A(w, 0) + O(y) \quad (4)$$
$$B(w, y) = B(w, 0) + O(y) \quad (5)$$
$$C(w, y) = B(w, 0)y + O(y^2) \quad (6)$$

with $A(w, 0) \neq 0$ and $B(w, 0) \neq 0$. These conditions imply that near enough to the observer the space-time appears Minkowskian. This imposes restrictions on the asymptotic behaviour of the number count formula as $y \to 0$. In observational coordinates the number count formula is given by $[11]$,

$$N(y) = 4\pi \int_0^y n(w_0, x)B(w_0, x)C(w_0, x)^2 dx \quad (7)$$

where $n(w_0, y)$ is the number density of sources. The dust density is

$$\rho(w_0, y) = n(w_0, y)m \quad (8)$$

where $m$ is the average mass of the sources (galaxies). We ignore selection and evolution effects although they could be incorporated without much difficulty if we knew their functional form. Also we assume no bias. Dark matter could be included via $m$, if unbiased. From the energy conservation equation, the central conditions and the formulae for $N$ and $\rho$ it follows that as $y \to 0$,

$$\rho = 4\pi \rho_0 \frac{B(w_0, 0)^3}{B(w_0)^3} + O(y) \quad (9)$$
and
\[ N(y) = \left( \frac{4\pi\rho_0}{3m} \right) y^3 + O(y^4) \]  
(10)
where \( \rho_0 = mn(w_0, 0) \).

These strictly mathematical results mean that the number count formula cannot be fractal for \( y \) near zero.

3 Sketch to Illustrate the Relations Between the Variables

From the two formulae for the dust velocity we obtain,
\[ A = \frac{\partial t}{\partial w} \]
(11)
\[ B = -\frac{\partial t}{\partial y} \]
(12)
and these can be used to write the field equation for \( A \) in the form
\[ A = \frac{\dot{C}}{\left[ (F^2 - 1) + \left( \frac{2mN_*}{C} \right) \right]^{1/2}} \]
(13)
where \( \dot{C} = \partial C/\partial w \), \( N_* = FN' \), \( F \) is an arbitrary function of integration and a prime denotes the partial derivative with respect to \( y \). We can use (11) to write (13) in the form,
\[ dt = \frac{dC}{\left[ (F^2 - 1) + \left( \frac{2mN_*}{C} \right) \right]^{1/2}} \]
(14)
along the fluid flow field. Along this flow the well known Tolman solution is
\[ dt = \frac{dR}{\left[ -kf + \frac{2M}{R} \right]^{1/2}} \]
(15)
which led us to identify,
\[ F(y)^2 = 1 - kf(y)^2 \]
(16)
\[ M(y) = mN_*(y) \]
(17)
\[ = m \int_0^y (1 - kf(y)^2)^{1/2} \frac{dN}{dx} \]
(18)
Clearly for \( k = 0 \), \( M(y) = mN(y) \).

To obtain further equations, e.g. for \((1 - kf(y)^2)^{1/2}\), in terms of observational data \( D(z) \) and \( N(z) \) requires further use of the field equations,
given by Maartens et al. This results in,

\[ \sqrt{1 - kf(z)^2} = \frac{1 + z}{D(z)} \int_0^z \frac{1}{Q} \left\{ D'(x) + \left[ \frac{D(x)Q(x)^2}{(1 + x)^2} \right]' \right\} dx \quad (19) \]

\[ Q(z) = 1 - \frac{m}{D(z)} \int_0^z \frac{(1 + x)N'(x)}{D(x)} dx. \quad (20) \]

The full set of equations is given by Humphreys et al. The equation (18) can be inverted to give,

\[ N(y) = \frac{1}{m} \int_0^y M'(x)[1 - kf(x)^2]^{-\frac{m}{2}} dx \quad (21) \]

4 An Application of the Formulae

To apply the formulae we only need to consider one past light cone so we can use the area distance \( D \) as our distance measure. We will also limit ourselves to the \( k = 0 \) case which is particularly simple because only one of the functions \( N \) and \( D \) is arbitrary. The other more complicated cases which cannot be handled completely analytically are considered by Humphreys et al.

The dependence of \( N \) and \( D \) is expressed explicitly by the integral equation

\[ 1 + z = \left( 1 - \frac{2mN}{D} \right)^{-1} \exp \left[ - \int \frac{mN'}{D} dD \left( 1 - \frac{2mN}{D} \right)^{-1} dD \right] \quad (22) \]

Once \( N \) is known in terms of \( D \), through for instance a fractal formula, this gives a relatively simple expression for \( z \) in terms of \( D \). Unfortunately the central conditions do not permit such a simple procedure since the form of the function \( N \) has to change. A simple ansatz which combines the limiting behaviour with a fractal number count formula outside the immediate vicinity of \( D = 0 \), is

\[ N(D) = \begin{cases} \frac{3m}{3m} D^3 & \text{for } D \leq D_I \\ \frac{3m}{3m} D_I \left( \frac{D}{D_I} \right)^3 & \text{for } D_I < D \leq D_h \end{cases} \quad (23) \]

where \( D_h \) is the distance at which transition to FLRW geometry occurs. We will assume that in the core \((D \leq D_I)\) and in the fractal region \((D_I < D \leq D_h)\) the space-time is parabolic, i.e., \( k = 0 \). Then the Hubble constant is given by

\[ H_0 = \sqrt{\frac{8}{3} \pi \rho_0} \quad (24) \]
which puts a precise constraint on the central density. It can be shown that matching the number count \( N(y) \) at \( D_h \) relates the four parameters \( H_0, D_I, D_h, \) and \( \nu \). From observational data one could estimate \( H_0, D_h \) and \( \nu \) which would then fix the minimum fractal scale \( D_I \). Unfortunately the critical observations are controversial.

For small \( D \) an explicit form of the behaviour of \( z \) with respect to \( D \) is given by

\[
\frac{dz}{dD} \approx \frac{1}{2} H_0 \left[ (\nu - 1) \left( \frac{D_I}{D} \right)^{(3-\nu)/2} - \nu H_0 D_I \left( \frac{D_I}{D} \right)^{(2-\nu)} \right],
\]

which shows that after initial behaviour which is linear by construction to \( D_I \) the \( z(D) \) graph curves upwards contradicting the linear Hubble Law for scales less than 100 Mpc. This rules out the parabolic models.

It is interesting that for any rational fractal index \( \nu \) (22) can be integrated to give an explicit expression for \( z \) in terms of \( D \). Note that for the simple model of the transition from the core to the fractal region given in (23) the integration is particularly simple but it can become very complicated for more sophisticated transition formulae. In the non-parabolic models it can be shown that the fractal number count forces a very low density for the universe.

5 Conclusion

The talk outlined some of the ideas and methods employed by Humphreys et al. Only some aspects have been discussed to show the mathematical ideas involved. The comprehensive treatment concludes that exact models using fractal counts models out to 150 Mpc either contradict the well established linear Hubble law out to 150 Mpc or they yield a very low density universe. Both conclusions conflict with observations. There are various modifications that could be made to the model but the most compelling is that the luminous matter may not trace the actual distribution of density. In this connection a recent paper by Labini and Durer argues for a fractal baryonic matter distribution with non fractal dark matter. The approach developed here gives an alternative method by which bias could be detected or measured.

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