Perturbative Confinement in a 4-d Lorentzian Complex Structure Dependent YM-like Model

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ABSTRACT
I continue the study of a renormalizable four-dimensional generally covariant Yang-Mills-like action, which depends on the Lorentzian complex structure of spacetime and not its metric. The field equations and their integrability conditions are written down explicitly. The model is studied with the presence of two static external sources in the trivial cylindrical complex structure. The energy of two static “colored” sources is found to increase linearly with respect to their distance, providing an explicit proof of their perturbative confinement.

In the present model, confinement is not a consequence of the non-Abelian character of the gauge group, but it is implied by the complex structure dependence of the model.

1 INTRODUCTION
There is a strong experimental evidence that hadrons contain fundamental constituents (the quarks) which cannot exist free. The spectrum of a linear attractive potential between the quarks fits quite well the observed heavy hadrons. In the context of the Standard Model, current strong interactions are incorporated through an SU(3) Yang-Mills (YM) field which couples with colored quarks. It is well known that the static potential of a YM field is Coulomb like \( \frac{1}{r} \). It is generally believed that the non-abelian YM Lagrangian somehow generates a linear potential, while no explicit theoretical proof has yet been presented. This belief is supported by the asymptotic freedom in the ultraviolet limit and some computer calculations on the lattice. I will not continue on the current hadronic phenomenology, the achievements of Quantum Chromodynamics and its failures. The purpose of the present work is to provide an explicit counter example to the general belief that in four dimensions a linear perturbative potential is generated only in the context of Lagrangians with higher order derivatives.

The present model emerged from my attempt to transfer in four dimensions the characteristic property of the two-dimensional string action

\[
I_S = \frac{1}{2} \int d^2 \xi \sqrt{-\gamma} \, \gamma^{\alpha\beta} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \eta_{\mu\nu}
\]

(1.1)
to depend on the complex structure of the two-dimensional surface and not its metric \( \gamma_{\alpha\beta} \). It is well known that in the structure coordinates \((z^0, z^0)\) the string
action takes the form

\[ I_S = \int d^2z \left( \partial_0 X^\mu (\partial_0 X^\nu) \eta_{\mu\nu} \right) \quad (1.2) \]

which does not depend on the metric of the 2-dimensional surface.

The null tetrad form of the present model action\[9,10\] is

\[ I_G = \int d^4x \sqrt{-g} \left\{ (\ell^\mu m^\rho F_{j\mu\rho}) (n^\nu \overline{m}^\sigma F_{j\nu\sigma}) + (\ell^\mu \overline{m}^\rho F_{j\mu\rho}) (n^\nu m^\sigma F_{j\nu\sigma}) \right\} \]

\[ F_{j\mu\nu} = \partial_\mu A_{j\nu} - \partial_\nu A_{j\mu} - \gamma f_{jik} A_{i\mu} A_{k\nu} \quad (1.3) \]

where \( A_{j\mu} \) is a gauge field and \((\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)\) is a null tetrad, which defines an integrable complex structure\[3,4\]. The metric \( g_{\mu\nu} \) and the complex structure tensor \( J^\nu_\mu \) take the form

\[ g_{\mu\nu} = \ell_\mu n_\nu + n_\mu \ell_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu \]

\[ J^\nu_\mu = i(\ell_\mu n^\nu - n_\mu \ell^\nu - m_\mu \overline{m}^\nu + \overline{m}_\mu m^\nu) \quad (1.4) \]

The integrability condition of this complex structure implies the Frobenius integrability conditions of the pairs \((\ell_\mu, m_\mu)\) and \((n_\mu, \overline{m}_\mu)\). That is

\[ (\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu \ell_\nu) = 0 \quad , \quad (n^\mu m^\nu - n^\nu m^\mu)(\partial_\mu n_\nu) = 0 \]

\[ (n^\mu \overline{m}^\nu - \overline{m}^\nu n^\mu)(\partial_\mu n_\nu) = 0 \quad , \quad (n^\mu m^\nu - n^\nu m^\mu)(\partial_\mu m_\nu) = 0 \]

which restricts the spacetime to have two geodetic and shear free congruences.

Then Frobenius theorem states that there are four complex functions \((z^\alpha, \overline{z}^\alpha)\), \(\alpha = 0, 1\), such that

\[ dz^\alpha = f_\alpha \ell_\mu dx^\mu + h_\alpha m_\mu dx^\mu \quad , \quad d\overline{z}^\alpha = f_\overline{\alpha} n_\mu dx^\mu + h_\overline{\alpha} \overline{m}_\mu dx^\mu \quad (1.6) \]

These four functions are the structure coordinates of the (integrable) complex structure. Recall that in the euclidean manifolds the complex structure is defined as a real tensor. But in the present case of Lorentzian spacetimes the coordinates \(z^\alpha\) are not complex conjugate of \(\overline{z}^\alpha\), because \(J^\nu_\mu\) is no longer a real tensor.

The difference between the present action and the ordinary Yang-Mills action becomes more clear in the following covariant form of the action.

\[ I_G = -\frac{1}{8} \int d^4x \sqrt{-g} \left( 2g^{\mu\nu} g^{\rho\sigma} - J^\mu_\nu J^{\rho\sigma} - \overline{J}^{\mu\nu} \overline{J}^{\rho\sigma} \right) F_{j\mu\rho} F_{j\nu\sigma} \quad (1.7) \]

where \(g_{\mu\nu}\) is a metric derived from the null tetrad and \(J^\nu_\mu\) is the tensor \(1.4\) of the integrable complex structure.

The integrability of the Lorentzian complex structure is essential, because only for these spacetimes the action can take a metric independent form, which
assures its renormalizability\[11\]. When we transcribe it in its structure coordinates, it takes the following form

\[ I_G = \int d^4z \ F_{j01} F_{\bar{0}1j} + \text{com. conj.} \]  
(1.8)

\[ F_{j\alpha} = \partial_\alpha A_{j\beta} - \partial_\beta A_{j\alpha} - \gamma f_{j\beta\gamma} A_{j\alpha} A_{k\beta} \]

which is metric independent, analogous to the form (1.2) of the string action. Therefore we have to implement the integrability conditions (1.5) using Lagrange multipliers

\[ I_C = \int d^4x \sqrt{-g} \{ \phi_0 (\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu \ell_\nu) + \phi_1 (\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu m_\nu) + \phi_0 (n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\partial_\mu n_\nu) + \phi_1 (n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\partial_\mu \overline{m}_\nu) + c.\text{conj.} \} \]  
(1.9)

The complete action \( I = I_G + I_C \) is self-consistent and the usual quantization techniques may be used\[6\],\[7\],\[8\]. The renormalizability is the great value of the present model, because if supersymmetry is not found in the current experiments, superstrings have to be abandoned. The characteristic properties of the present model appear to be very appealing, to provide a pathway to a ”theory of everything”.

The model uses the Newman-Penrose null tetrad formalism, but the essential calculations of the present paper will be trivial. In section II the complete field equations and their integrability conditions are explicitly written down. Their forms are quite complicated for readers non familiar with the Newman-Penrose formalism. The unfamiliar readers may skip it. But these equations permit the familiar reader to understand the emergence of an ”energy-momentum” tensor as integrability conditions of the field equations. In fact this tensor coincides with that found considering the YM field in an external time independent null tetrad.

An indication of perturbative ”gluonic” confinement has been presented in my previous works\[9\],\[10\] through the computation of the static ”gluonic” potential in the trivial spherical complex structure determined by the following (spherical) null tetrad in spherical coordinates \((t, r, \theta, \varphi)\)

\[ \ell_\mu = (1, -1, 0, 0) \]
\[ n_\mu = \frac{1}{2} (1, 1, 0, 0) \]
\[ m_\mu = \frac{1}{\sqrt{2}} (0, 0, 1, i \sin \theta) \]  
(1.10)

with its contravariant coordinates

\[ \ell_\nu = (1, 1, 0, 0) \]
\[ n_\nu = \frac{1}{2} (1, -1, 0, 0) \]
\[ m_\nu = \frac{1}{r \sqrt{2}} (0, 0, 1, \frac{i}{\sin \theta}) \]  
(1.11)

If we expand the gauge field into the null tetrad

\[ A_{j\mu} = B_{j1} \ell_\mu + B_{j2} n_\mu + \overline{B}_{j3} m_\mu + B_{j4} \overline{m}_\mu \]  
(1.12)
we find the gauge field components $B_{j1}$, $B_{j2}$, $B_j$. In the present null tetrad, the conjugate momenta of $B_{j1}$, $B_{j2}$ vanish, i.e. $P_{j1} = 0 = P_{j2}$. Therefore we must assume $B_{j1} = 0 = B_{j2}$. Assuming the convenient gauge condition

$$m^\nu \partial_\nu (r \sin \theta m^\mu A_{j\mu}) + m^\nu \partial_\nu (r \sin \theta m^\mu A_{j\mu}) = 0 \quad (1.13)$$

the field equation takes the form

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} \right) (r m^\mu A_{j\mu}) = [\text{source}] \quad (1.14)$$

which apparently implies a linear "gluonic" potential for the field variable $(r m^\mu A_{j\mu})$. It is not enough. I have also to show that this variable diagonalizes the energy of the model.

In the precise complex structure (null tetrad) the energy of the YM field is

$$E = \int_0^T g^0 \sqrt{-g} d^3x =$$

$$= \int drd\theta d\varphi \sin \theta \partial_t (r m^\mu A_{j\mu}) \partial_t (r m^\mu A_{j\mu}) + \partial_r (r m^\mu A_{j\mu}) \partial_r (r m^\mu A_{j\mu})$$

$$\quad (1.15)$$

Notice that the dynamical variable of the gauge field is $(r m^\mu A_{j\mu})$, because this form diagonalizes the energy. From its above field equation, we see that this dynamical variable apparently gives a linear classical static potential.

In section III, I will review the trivial computation of the energy of two electric charges. This trivial computation is reviewed, in order to reveal its similarities and differences with the energy of two "gluonic" charges, which is subsequently computed in the same section.

### 2 FIELD EQUATIONS AND INTEGRABILITY CONDITIONS

Variation of the action with respect to the gauge field $A_{j\mu}$ gives the field equations

$$D_\mu \left\{ \sqrt{-g} [ (\ell^\mu m^\nu - \ell^\nu m^\mu) (n^\rho m^\sigma F_{j\rho\sigma}) + (n^\rho m^\nu - n^\nu m^\rho) (\ell^\sigma m^\tau F_{j\rho\sigma}) + (\ell^\sigma m^\tau - \ell^\tau m^\sigma) (n^\rho m^\sigma F_{j\rho\sigma})] + (n^\rho m^\nu - n^\nu m^\rho) (\ell^\sigma m^\tau F_{j\rho\sigma}) ] \right\} = 0 \quad (2.1)$$

where $D_\mu = \delta_\mu^\lambda \partial_\lambda + \gamma f_{\mu j} A_{k\mu}$ is the gauge symmetry covariant derivative and $\gamma$ the coupling constant. Multiplying with the null tetrad, these equations take
the form
\[ m^\nu D_\nu (\overline{m}F_j) + \overline{m}^\mu D_\mu (\ell mF_j) + (\ell mF_j)[(\nabla_\mu m^\nu) + (nm\partial_\ell)] + \\
+ (\ell mF_j)[(\nabla_\mu \overline{m}^\nu) + (n\overline{m}\partial_\ell)] = 0 \]

\[ m^\nu D_\nu (n\overline{m}F_j) + \overline{m}^\mu D_\mu (nmF_j) + (n\overline{m}F_j)[(\nabla_\mu m^\nu) + (\ell m\partial n)] + \\
+ (nmF_j)[(\nabla_\mu \overline{m}^\nu) + (\ell \overline{m}\partial n)] = 0 \] (2.2)

\[ \ell^\nu D_\nu (nmF_j) + n^\mu D_\mu (\ell mF_j) + (nmF_j)[(\nabla_\mu \ell^\nu) + (\ell \overline{m}\partial n) + \\
+ (\ell mF_j)[(\nabla_\mu n^\nu) + (\ell \overline{m}\partial m)] = 0 \]

Variation of the action with respect to the Lagrange multipliers \( \phi_0, \phi_1, \phi_{\bar{0}}, \phi_{\bar{1}} \) imply the complex structure integrability conditions on the tetrad. Variation of the action with respect to the tetrad gives PDEs on the Lagrange multipliers. In order to preserve the relations between the covariant and contravariant forms of the tetrad we will use the identities
\[ \delta e_\mu^\nu = e_\mu^\lambda \left[ -n^\mu \delta \ell_\lambda - \ell^\nu \delta n_\lambda + m^\mu \delta \overline{m}_\lambda \right] \\
\delta \sqrt{-g} = \sqrt{-g} \left[ n^\lambda \delta \ell_\lambda + \ell^\lambda \delta n_\lambda - \overline{m}_\lambda \delta m_\lambda - m^\lambda \delta \overline{m}_\lambda \right] \] (2.3)

where we denote \( (e_\mu^0 = e_\mu = m_\mu) \) and \( (e_\mu^1 = n_\mu, e_\mu^{\bar{1}} = \overline{m}_\mu) \). Variation with respect to \( \ell_\lambda \) gives the PDEs
\[ 2\ell^\lambda (nmF_j)(n\overline{m}F_j) + m^\lambda (\ell nF_j)(n\overline{m}F_j) + \overline{m}^\lambda (\ell nF_j)(nmF_j) = \\
= -\nabla_\mu \left[ \phi_0 (\ell^\mu n^\lambda - \ell^\lambda m^\mu) \right] - \nabla_\mu \left[ \phi_0 (\ell^\mu \overline{m}^\lambda - \ell^\lambda \overline{m}^\mu) \right] - \\
- \ell^\lambda \left[ \phi_0 (nm\partial_\ell) + \phi_{\bar{0}} (n\overline{m}\partial_\ell) \right] - m^\lambda \left[ \phi_0 (\ell n\partial n) + \phi_{\bar{1}} (\ell \overline{m}\partial n) \right] \] (2.4)

which take the tetrad form
\[ m^\nu \partial_\nu \phi_0 + \overline{m}^\mu \partial_\mu \phi_0 + \phi_0 [(\nabla_\mu m^\nu) + (nm\partial_\ell)] + \\
+ \phi_0 [(\nabla_\mu \overline{m}^\nu) + (\ell \overline{m}\partial n)] = 0 \] (2.5)

Variation with respect to \( n_\lambda \) gives the PDEs
\[ 2n^\lambda (\ell mF_j)(\ell \overline{m}F_j) - m^\lambda (\ell nF_j)(\ell \overline{m}F_j) - \overline{m}^\lambda (\ell nF_j)(\ell mF_j) = \\
= -n^\lambda \left[ \phi_0 (n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu) \right] + \nabla_\mu \left[ \phi_0 (n^\mu m^\nu - n^\nu m^\mu) \right] - \\
- \ell^\lambda \left[ \phi_0 (\ell \overline{m}\partial n) + \phi_{\bar{0}} (\ell m\partial n) \right] + \overline{m}^\lambda \left[ \phi_0 (\ell \overline{m}\partial m) + \phi_{\bar{1}} (\ell n\partial m) \right] \] (2.6)

which take the tetrad form
\[ \overline{m}^\mu \partial_\mu \phi_0 + m^\nu \partial_\nu \phi_0 + \phi_0 [(\nabla_\mu \overline{m}^\nu) + (n\overline{m}\partial_\ell) - (\ell \overline{m}\partial n)] + \\
+ \phi_0 [(\nabla_\mu m^\nu) + (nm\partial_\ell) - (\ell m\partial n)] = 0 \] (2.7)

\[ n^\nu \partial_\nu \phi_0 + \phi_0 [(\nabla_\mu n^\nu) + (n \overline{m}\partial m) - (\ell n \partial n)] - \phi_0 [(\ell \overline{m}\partial m) - (\ell n \partial n)] = 0 \]
Variation with respect to $m_\lambda$ gives the PDEs

$$\ell^\lambda (m m F_j)(n m F_j) - n^\lambda (m m F_j)(\ell m F_j) - 2m^\lambda (\ell m F_j)(n m F_j) =$$

\begin{equation}
= -\nabla_\mu \left[ \phi_1 (\ell^\mu m^\lambda - \ell^\lambda m^\mu) \right] - \nabla_\mu \left[ \phi_0 (n^\mu m^\lambda - n^\lambda m^\mu) \right] - \nabla_\mu \left[ \phi_0 (n^\mu m^\lambda - n^\lambda m^\mu) \right] - \nabla_\mu \left[ \phi_1 (n^\mu m^\lambda - n^\lambda m^\mu) \right] + m^\lambda \left[ \phi_1 (\ell m \phi_0) + \phi_0 (n m \phi_1) \right] + m^\lambda \left[ \phi_1 (\ell m \phi_0) + \phi_0 (n m \phi_1) \right]
\end{equation}

which take the tetradd form

$$m^\mu \partial_\mu \phi_1 + \phi_1 [(\nabla_\mu m^\mu) + (n m m) - (n m \phi_0)] - \phi_0 (n m m) - m^\mu \partial_\mu \phi_1 + \phi_1 [(\nabla_\mu m^\mu) + (n m m) - (n m \phi_0)] - \phi_0 (n m m) = 0 \quad \text{(2.8)}$$

In order to simplify the relations, I made the bracket notations like $(n m \partial \ell) \equiv (n^\mu m^\nu - n^\nu m^\mu) \partial_\mu \ell_\nu$ for the spin coefficients and like $(n m F_j) \equiv n^\mu m^\nu F_{j\mu\nu}$ for the gauge field components.

On the other hand the $e^a_\mu$ field equations imply the four conserved currents

$$\nabla_\lambda \left[ 2(n m F_j)(n m F_j) + \phi_0 (n m m) + \phi_0 (n m m) \right] +$$

\begin{equation}
+m^\lambda [(n F_j)(n m F_j) + \phi_0 (n m m) + \phi_1 (n m m)] +
+m^\lambda [(n F_j)(n m F_j) + \phi_0 (n m m) + \phi_1 (n m m)] = 0 \quad \text{(2.9)}
\end{equation}

These last relations combined with the tetradd integrability conditions imply relations between the surface geometric quantities and the gauge field invariants.

For that we will use the following relations of my spin coefficients and the
ordinary Newman-Penrose ones

\[
\begin{align*}
\alpha &= \pm \left[ (\ell n \bar{\partial} m) + (\ell m \bar{\partial} n) - (n m \bar{\partial} m) - 2(n m \bar{\partial} m) \right] \\
\beta &= \pm \left[ (\ell n \bar{\partial} m) + (\ell m \bar{\partial} n) - (n m \bar{\partial} m) - 2(n m \bar{\partial} m) \right] \\
\gamma &= \pm \left[ (n m \bar{\partial} m) - (n m \bar{\partial} m) - (n m \bar{\partial} m) + 2(n m \bar{\partial} m) \right] \\
\varepsilon &= \pm \left[ (\ell m \bar{\partial} m) - (\ell n \bar{\partial} m) - (n m \bar{\partial} m) + 2(\ell n \bar{\partial} m) \right] \\
\mu &= \pm \left[ (n m \bar{\partial} n) + (n m \bar{\partial} m) + (n m \bar{\partial} m) \right] \\
\pi &= \pm \left[ (\ell m \bar{\partial} m) - (\ell n \bar{\partial} m) - (n m \bar{\partial} m) \right] \\
\rho &= \pm \left[ (\ell n \bar{\partial} m) + (\ell m \bar{\partial} n) - (n m \bar{\partial} m) \right] \\
\tau &= \pm \left[ (n m \bar{\partial} m) + (\ell m \bar{\partial} n) + (\ell n \bar{\partial} m) \right] \\
\kappa &= (\ell m \bar{\partial} m) , \quad \sigma = (\ell m \bar{\partial} m) \\
\nu &= - (n m \bar{\partial} n) , \quad \lambda = - (n m \bar{\partial} m) 
\end{align*}
\] (2.11)

and the inverse relations

\[
\begin{align*}
(\ell n \bar{\partial} m) &= \varepsilon + \pi - \rho - \beta \\
(\ell m \bar{\partial} n) &= \gamma + \tau , \quad (\ell m \bar{\partial} m) = \beta - \tau - \pi \\
(\ell n \bar{\partial} n) &= \gamma + \tau , \quad (\ell n \bar{\partial} n) = \beta - \tau - \pi \\
(\ell m \bar{\partial} n) &= \varepsilon - \pi + \rho , \quad (\ell m \bar{\partial} n) = \pi - \varepsilon - \rho \\
(n m \bar{\partial} m) &= - \lambda , \quad (n m \bar{\partial} m) = \gamma - \pi , \quad (n m \bar{\partial} m) = \beta - \varepsilon\] (2.12)

\[
\begin{align*}
\nabla_\mu \ell^\nu &= \varepsilon + \tau - \rho - \beta \\
\nabla_\mu m^\nu &= \mu + \pi - \gamma - \tau
\end{align*}
\]

which are implied by the following formula[2] of the covariant derivatives of the null tetrad

\[
\begin{align*}
\nabla_\mu \ell_\nu &= (\gamma + \tau) \ell_\mu \ell_\nu - \bar{\pi}_\mu m_\nu - \tau \ell_\mu \bar{m}_\nu + (\varepsilon + \pi) n_\mu \ell_\nu - \\
&\quad - \bar{\kappa}_\mu m_\nu - \bar{\kappa}_\mu m_\nu - (\alpha + \beta) m_\mu \ell_\nu + \sigma m_\nu m_\nu + \\
&\quad + \rho m_\nu m_\nu - (\alpha + \beta) \bar{m}_\mu \ell_\nu + \bar{\pi}_\mu m_\nu + \bar{\sigma}_\mu m_\nu \\
\nabla_\mu n_\nu &= - (\gamma + \tau) \ell_\mu n_\nu + \nu \ell_\mu m_\nu + \bar{\pi}_\mu \bar{m}_\nu - (\varepsilon + \pi) n_\mu n_\nu + \\
&\quad + \pi n_\mu m_\nu + \pi n_\mu \bar{m}_\nu - (\alpha + \beta) m_\mu n_\nu - \lambda m_\mu m_\nu - \\
&\quad - \bar{\pi}_\mu m_\nu - (\alpha + \beta) \bar{m}_\mu n_\nu - \mu m_\mu m_\nu - \bar{\lambda} m_\mu m_\nu \\
\nabla_\mu m_\nu &= \bar{\pi}_\mu \ell_\nu - \tau \ell_\mu n_\nu + (\gamma - \tau) \ell_\mu m_\nu + \bar{\pi}_\mu \bar{m}_\nu - \kappa n_\mu n_\nu + \\
&\quad + (\varepsilon - \beta) n_\mu m_\nu - \bar{\kappa} m_\mu \ell_\nu + \rho m_\mu n_\nu + (\beta - \alpha) m_\mu m_\nu - \\
&\quad - \bar{\lambda} m_\mu \ell_\nu + \sigma m_\mu m_\nu + (\alpha - \beta) \bar{m}_\mu m_\nu
\end{align*}
\] (2.13)

The field equations[2,5] become

\[
\begin{align*}
m^\mu \partial_\nu \phi_0 + m^\mu \partial_\nu \bar{\phi}_0 + \phi_0 [3 \beta - 2 \tau + \alpha] + \bar{\phi}_0 [3 \beta - 2 \tau + \alpha] + \\
+ 2(n m F_j)(n m F_j) = 0
\end{align*}
\] (2.14)

\[
\begin{align*}
\ell^\mu \partial_\nu \phi_0 + \phi_0 [3 \varepsilon + \tau - \rho] + \phi_1 [\tau + \pi] + (\ell n F_j)(n m F_j) = 0
\end{align*}
\]
The field equations (2.7) become
\[\overline{m}^\mu \partial_\mu \phi_3 + m^\mu \partial_\mu \phi_3 + \phi_3[3\alpha + 2\pi - \beta] - \phi_3[3\alpha + 2\pi - \beta] - 2(\ell \overline{m} F_j)(\ell m F_j) = 0 \quad (2.15)\]

\[n^\mu \partial_\mu \phi_3 + \phi_3[3\gamma - \pi + \mu] - \phi_3[\pi + \pi] - (\ell n F_j)(\ell m F_j) = 0 \]

The field equations (2.9) become
\[m^\mu \partial_\mu \phi_1 + \phi_1[3\beta - \pi - \tau] + \phi_0[\rho - \pi] - (\ell m F_j)(m m F_j) = 0 \quad (2.16)\]

\[\ell^\mu \partial_\mu \phi_1 + n^\mu \partial_\mu \phi_1 + \phi_1[3\pi - 2\rho - \pi] + \phi_1[-3\pi + 2\pi + \gamma] - 2(\ell m F_j)(n m F_j) = 0 \]

Using the Newman-Penrose spin coefficients, the field equations (2.2) become
\[m^\mu D_\mu (m m F_j) + \overline{m}^\mu D_\mu (\ell m F_j) + (\ell m F_j)[\pi - 2\alpha] + (\ell m F_j)[\pi - 2\alpha] = 0 \]
\[m^\mu D_\mu (n m F_j) + \overline{m}^\mu D_\mu (n m F_j) + (n m F_j)[2\beta - \tau] + (n m F_j)[2\beta - \tau] = 0 \quad (2.17)\]
\[\ell^\mu D_\mu (n m F_j) + n^\mu D_\mu (\ell m F_j) + (n m F_j)[2\pi - \pi] + (\ell m F_j)[\mu - 2\gamma] = 0 \]

Their integrability conditions are satisfied identically.

The integrability condition of the equations (2.15) is
\[m^\mu \partial_\mu [(\ell n F_j)(n m F_j)] + m^\mu \partial_\mu [(\ell n F_j)(n m F_j)] - 2m^\mu \partial_\mu [(n m F_j)(n m F_j)] + (2\beta + \pi - 2\tau)(\ell m F_j)(m m F_j) + (2\beta + \pi - 2\tau)(\ell m F_j)(m m F_j) + \tau + \pi](m m F_j)(m m F_j) + (\ell n F_j)(\ell m F_j) + \tau + \pi](m m F_j)(m m F_j) + 2(\rho + \pi - 2\pi)(m m F_j)(n m F_j) = 0 \quad (2.18)\]

where the tetrad commutation relations\(^2\) are used. The equations (2.16) imply
\[m^\mu \partial_\mu [(\ell n F_j)(\ell m F_j)] + \overline{m}^\mu \partial_\mu [(\ell n F_j)(\ell m F_j)] - 2m^\mu \partial_\mu [(\ell n F_j)(\ell m F_j)] + (-2\pi + 2\pi - \tau)(\ell n F_j)(\ell m F_j) + (\ell n F_j)(\ell m F_j) + (\ell n F_j)(\ell m F_j) + (-2\alpha + 2\pi - \pi)(\ell n F_j)(\ell m F_j) + (\ell n F_j)(\ell m F_j) + \tau + \pi](m m F_j)(\ell m F_j) + (\ell n F_j)(\ell m F_j) + \tau + \pi](m m F_j)(\ell m F_j) + 2(\gamma + 2\pi - \mu - \pi)(\ell m F_j)(\ell m F_j) = 0 \quad (2.19)\]

and the equations (2.17) imply the integrability condition
\[\ell^\mu \partial_\mu [(\ell m F_j)(n m F_j)] + n^\mu \partial_\mu [(\ell m F_j)(n m F_j)] - 2m^\mu \partial_\mu [(\ell m F_j)(n m F_j)] + (2\pi - 2\rho - \pi)(n m F_j)(n m F_j) + (2\beta - 2\pi + \mu)(\ell m F_j)(m m F_j) + (\ell n F_j)(n m F_j) + (\ell n F_j)(n m F_j) + (\ell n F_j)(n m F_j) + 2(2\pi - 2\beta + \tau - \pi)(\ell m F_j)(\ell m F_j) = 0 \quad (2.20)\]
Notice that the curvature terms cancel out in all these integrability conditions.

The above integrability conditions are the null tetrad forms of the following relations implied by the gauge field equations (2.1).

\[ \nabla_{\mu} \{ \Gamma_{\mu\lambda\rho\sigma} F_{j\nu\lambda} F_{j\rho\sigma} - \frac{1}{4} \delta^\mu_\nu (\Gamma_{\tau\lambda\rho\sigma} F_{j\tau\lambda} F_{j\rho\sigma}) \} = -\frac{1}{4} (\nabla_\nu \Gamma_{\tau\lambda\rho\sigma}) F_{j\tau\lambda} F_{j\rho\sigma} \]

\[ \Gamma_{\mu\nu\rho\sigma} = \frac{1}{2} \left[ (\ell^\mu m^\nu - \ell^\nu m^\mu) (n^\rho m^\sigma - n^\sigma m^\rho) + (n^\mu m^\nu - n^\nu m^\mu) (\ell^\rho m^\sigma - \ell^\sigma m^\rho) \right] + c.c. \]  

(2.21)

which takes the following form with ordinary derivatives derivatives

\[ \frac{1}{\sqrt{-g}} \partial_\mu \{ \sqrt{-g} (\Gamma_{\mu\lambda\rho\sigma} F_{j\nu\lambda} F_{j\rho\sigma} - \frac{1}{4} \delta^\mu_\nu (\Gamma_{\tau\lambda\rho\sigma} F_{j\tau\lambda} F_{j\rho\sigma})) \} = -\frac{1}{4} \sqrt{-g} \partial_\nu (\sqrt{-g} \Gamma_{\tau\lambda\rho\sigma}) F_{j\tau\lambda} F_{j\rho\sigma} \]

(2.22)

It is well-known that the translation generators of a generally covariant Lagrangian are first class constraints which must vanish. But notice that the left term of the above equation looks like the energy-momentum tensor of the new action. It is not generally conserved, but if the null tetrad is time-independent, it does provide a conserved energy for the gauge field in an external spacetime complex structure. In fact this energy was used (1.15) and we will use below, in the next section.

3 THE ENERGY OF TWO EXTERNAL ”COLORED” SOURCES

I will first consider the energy of two electric charges located at \( \vec{x}_1 \) and \( \vec{x}_2 \).

The particle and electromagnetic energy momentum tensors are

\[ T^\mu_\nu (p) = \sum_n \frac{d\mathbf{x}_n}{dt} F_{\mu\nu}(\mathbf{x} - \mathbf{x}_n) \]

\[ T^\mu_\nu (e) = -F^\mu\rho F_{\nu\rho} + \frac{1}{4} \delta^\mu_\nu F_{\rho\sigma} F^{\rho\sigma} \]  

(3.1)

We know that the total energy momentum tensor is conserved, because of the following EM field equations and the Lorentz force

\[ \frac{\partial F^{\mu\rho}}{\partial \nu} = J^\rho \]

\[ \frac{d^\mu_n}{dt} = q_n F^\mu_{\nu} \frac{d\mathbf{x}_n}{dt} \]  

(3.2)

where the EM charge current is

\[ J^\mu = \sum_n q_n \delta^3(\vec{x} - \vec{x}_n) \frac{d\mathbf{x}_n}{dt} \]  

(3.3)

The total energy of the two charges is

\[ E = \int d^3x \left\{ m_1 \delta^3(\vec{x} - \vec{x}_1) + m_2 \delta^3(\vec{x} - \vec{x}_2) \right\} + 
+ [(-\partial_t A_0)^2 + \frac{1}{4} (\partial_x A_0)^2] \right\} = 
= m_1 + m_2 + \frac{1}{2} \int d^3x (\partial_t A_0)^2 \]  

(3.4)
From the FE and the static current we find
\[
A_0 = \frac{1}{4\pi} \frac{q_1}{|x - x_1|} + \frac{1}{4\pi} \frac{q_2}{|x - x_2|} \tag{3.5}
\]
Which gives
\[
E = m_1 + m_2 + q_1 q_2 |\rightarrow x - \leftarrow x_1| + \text{self-interaction} \tag{3.6}
\]
Notice that the energy of two opposite charges has an upper bound \(m_1 + m_2\) when their distance goes to infinity.

We will now compute the energy of two "gluonic" charges located at \(\vec{x}_1 = (0, 0, d)\) and \(\vec{x}_2 = (0, 0, -d)\). Because of the cylindrical symmetry of the system, we assume that its complex structure may be asymptotically approximated by the cylindrical null tetrad
\[
\ell_{\mu} dx^{\mu} = dt - dz \\
n_{\mu} dx^{\mu} = \frac{1}{2} (dt + dz) \\
m_{\mu} dx^{\mu} = \frac{1}{\sqrt{2}} (dp + i\rho d\varphi) \\
\sqrt{-g} = \rho \tag{3.7}
\]
with its contravariant coordinates
\[
\ell^{\mu} \partial_{\mu} = \partial_t + \partial_z \\
n^{\mu} \partial_{\mu} = \frac{1}{2} (\partial_t - \partial_z) \\
m^{\mu} \partial_{\mu} = \frac{1}{\sqrt{2}} (\partial_p + \frac{i}{\rho} \partial_{\varphi}) \tag{3.8}
\]
The gluonic energy current is
\[
T_{(g)}^\mu \ 0 = \Gamma^{\nu\rho\sigma} F_{0\nu} F_{\rho\sigma} - \frac{1}{4} \delta_0^{\mu} \Gamma^{\nu\rho\sigma} F_{\nu\tau} F_{\rho\sigma} \tag{3.9}
\]
which is conserved for any static complex structure. The conservation of the total energy current is implied by the following gluonic FE and particle energy relation ("Lorentz" force)
\[
\frac{1}{\sqrt{-g}} \partial_{\mu} [\sqrt{-g} \Gamma^{\mu\nu\rho\sigma} F_{\nu\rho\sigma}] = - J^\nu_{(g)} \tag{3.10}
\]
where I assume a \(U(1)\) gauge group for convenience and the "gluonic" current
\[
J^0 = J^1_g = J^2_g = 0 \\
J^3_g = \Phi(\rho) [q_1 \delta(z - d) + q_2 \delta(z + d)] \\
with \ \Phi(\rho) = \text{real} \ \text{and} \ \int d\rho \Phi(\rho) = 1 \tag{3.11}
\]
in \((t, z, \rho, \varphi)\) coordinates. Notice that this precise form of external current satisfies the relation
\[
\partial_{\mu} T_{(g)}^\mu \ 0 = \sqrt{-g} F_{0\nu} J^\nu_{(g)} = 0 \tag{3.12}
\]
and it is found to be confined.

We look for $t, \varphi$-independent solutions with $A_0 = A_1 = A_2 = 0$. Then

$$\frac{\partial}{\partial z} (m^\nu A_\nu) = -i 2 \partial_z (m^\mu A_\mu)$$

$$\partial_2 \partial_\rho \{ \rho [(m^\mu A_\mu) + (\overline{m^\mu A_\mu})] \} = 0$$

We find the solution

$$\frac{\partial}{\partial z} (m^\mu A_\mu) = -\varphi (\rho G(z - d) + q_2 G(z + d))$$

where the Green function $G(z, z')$ satisfies the equation $\partial_2 G(z, z') = \delta (z - z')$. Its form is known to be

$$G(z, z') = \left\{ \begin{array}{ll}
\frac{z' - L}{z' - L}, & -L \leq z \leq z' \\
\frac{z - L}{z - L}, & z' \leq z \leq L
\end{array} \right.$$

where $(-L, L)$ is a box for convenience.

For two opposite charges the energy of the system is

$$E = m_1 + m_2 + \int dz d\rho d\varphi \rho \left[ 2 \mathcal{F}_\varphi(z - d) + \frac{1}{2} \mathcal{F}_\varphi(z + d) \right] =$$

$$= m_1 + m_2 + 2\pi \int d\rho \rho [(m^\mu A_\mu) + (\overline{m^\mu A_\mu})]$$

We find that the interaction energy of the two opposite charges is linearly increasing with their distance. Notice that the present "gluonic" current is not analogous to the EM current. It has only the $\varphi$-component. It rotates around the $z$-axis at the $z$-positions of the particles.

4 DISCUSSION

Confinement of the hadronic constituents (partons) is an experimental fact. It is generally believed that non-Abelian gauge theories are confined, but it has not yet been mathematically proven. In four dimensions up to now, linear perturbative potentials have been derived from Lagrangian models with second order derivatives, which have serious unitarity problems. As far as I know, the present Lagrangian with first order derivatives is the unique four dimensional model which exhibits confinement of external sources. But the characteristic property of the model, its independence on the metric of the spacetime, which assures its renormalizability, imposes severe constraints on the permitted Lagrangian terms.

Despite my efforts, I have not yet found a way to incorporate terms with the Dirac field. Therefore the "external sources" cannot be classical considerations of the well known fermionic currents. It seems that these "external sources" have to emerge from the solitonic sectors of the model.
The model has a quite rich solitonic sector. It is well known that spacetimes with two geodetic and shear free null congruences, which admit an integrable Lorentzian complex structure, exhibit fermionic solitonic properties. A typical example is the Kerr-Newman spacetime which has the fermionic gyromagnetic ratio \( g = 2 \). These fermionic solitons appear without any Dirac field present in the action. Therefore it is natural to consider that the confined "external sources" of the present work may emerge as gauge field excitations of these fermionic solitons.

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