A reconstruction of the multipreference closure

Laura Giordano¹ and Valentina Gliozzi²

¹ DISIT - Università del Piemonte Orientale, Alessandria, Italy, laura.giordano@uniupo.it
² Dipartimento di Informatica, Università di Torino, Italy, valentina.gliozzi@unito.it

Abstract. The paper describes a preferential approach for dealing with exceptions in KLM preferential logics, based on the rational closure. It is well known that the rational closure does not allow an independent handling of the inheritance of different defeasible properties of concepts. Several solutions have been proposed to face this problem and the lexicographic closure is the most notable one. In this work, we consider an alternative closure construction, called the Multi Preference closure (MP-closure), that has been first considered for reasoning with exceptions in DLs. Here, we reconstruct the notion of MP-closure in the propositional case and we show that it is a natural variant of Lehmann’s lexicographic closure. Abandoning Maximal Entropy (an alternative route already considered but not explored by Lehmann) leads to a construction which exploits a different lexicographic ordering w.r.t. the lexicographic closure, and determines a preferential consequence relation rather than a rational consequence relation. We show that, building on the MP-closure semantics, rationality can be recovered, at least from the semantic point of view, resulting in a rational consequence relation which is stronger than the rational closure, but incomparable with the lexicographic closure. We also show that the MP-closure is stronger than the Relevant Closure.

1 Introduction

Kraus, Lehmann and Magidor in [22,23] and Lehmann and Magidor in [23] investigate the properties that a notion of plausible inference from a conditional knowledge base should satisfy (KLM properties, for short). These properties led to the definition of the notions of preferential and rational consequence relation, as well as to the definition of the rational closure of a conditional knowledge base [23]. Although not all non-monotonic formalisms in the literature satisfy KLM properties, and although the adequacy of these properties has been and still is subject of debate (see, for instance, [3]), the rational closure construction (which is a polynomial construction) recently has been considered for defeasible reasoning in description logics [9,11,19,7], which are the formalisms at the basis of OWL ontologies [26].

While the rational closure provides a simple and efficient approach for reasoning with exceptions, it is well known that “it does not provide for inheritance of generic properties to exceptional subclasses” [24]. This problem was called by Pearl [27] “the blocking of property inheritance problem”, and it is an instance of the “drowning problem” in [1].

To overcome this weakness of the rational closure, Lehmann in [24] introduced the notion of lexicographic closure, as a “uniform way of constructing a rational superset of
the rational closure” thus strengthening the rational closure but still defining a rational consequence relation.

In this paper, we consider another closure construction, that we call multi-preference closure (MP-closure), as it was first proposed in the context of description logics as a construction to soundly approximate the multipreferential semantics [21], a strengthening of the rational closure. Here, we consider the MP-closure in the propositional setting, and reconstruct its semantics, showing that it is a natural (weaker) variant of Lehmann’s lexicographic closure which simply uses a different lexicographic ordering. Following the pattern in [24], in the following, we will present a characterization of the closure both in terms of maxiconsistent sets and of a model-theoretic construction. In both cases, the characterization exploits a lexicographic ordering which compares tuples of sets of defaults rather than tuples of numbers (i.e., the number of defaults in the sets), as in the lexicographic ordering used by the lexicographic closure.

The MP-closure construction departs from lexicographic closure in the choice that, in case of contradictory defaults with the same rank, one tries to satisfy as many defaults as possible (where the number of defaults matters, rather than the defaults themselves). This choice was adopted by the lexicographic closure in agreement with the Maximal Entropy approach [25]. Abandoning the Maximal Entropy approach and following instead Poole’s proposal [28] (an alternative route that was also considered but not explored by Lehmann) leads to a construction which defines a preferential consequence relation rather than a rational consequence relation. Rational Monotonicity is not satisfied when reasoning under the MP-closure, but a more cautious notion of entailment is obtained with respect to reasoning under the lexicographic closure.

We believe that the MP-closure defines an interesting notion of entailment per se that may be reasonable in specific contexts, for instance, when reasoning about multiple inheritance in ontologies, where the lexicographic closure appears to be too bold. Nevertheless, we will also see that from a semantic point of view rationality can be easily regained by defining a rational consequence relation, starting from the MP-closure semantics, which is a superset of the MP-closure, while is incomparable with the lexicographic closure.

We conclude the paper establishing some relationships with the multi-preference semantics in [21] and with the Relevant Closure, a notion of closure proposed by Casini et al. [6] as a weaker alternative to the lexicographic closure. We show that the Relevant Closure is also weaker than the MP-closure.

The paper is organized as follows. In Section 2, we recall the definition of the rational closure and its semantics and, in Section 3, the definition of the lexicographic closure and discuss some examples which motivate the interest in investigating an alternative notion of closure, abandoning the Maximal Entropy approach. In Section 4, we reformulate the MP-closure construction in the propositional setting in terms of maxiconsistent sets and, then, we study its model-theoretic semantics, its properties and its relations with the lexicographic closure. In Section 5, a rational consequence relation is defined, which is a superset of the MP-closure but neither stronger nor weaker than the lexicographic closure. The relationships with the Relevant Closure and the multipreference semantics are also investigated. Section 6 concludes the paper.
2 The rational closure

In this section we recall the definition of the rational closure by Lehmann and Magidor [23] and its semantics, that we will exploit to define the semantics of the MP-closure.

Let the language $\mathcal{L}$ be defined from a set of propositional variables $\text{ATM}$, the boolean connectives and the conditional operator $\models$. Following the presentation in [19], as in [12,4] and with a minor deviation from the original presentation in [23], here we consider the conditionals $A \models B$ as formulas belonging to the object language.

The formulas of $\mathcal{L}$ are defined as follows: if $A$ is a propositional formula, $A \in \mathcal{L}$; if $A$ and $B$ are propositional formulas, $A \models B \in \mathcal{L}$; if $F$ is a boolean combination of formulas of $\mathcal{L}$, then $F \in \mathcal{L}$. A knowledge base $K$ is a set of conditional assertions $A \models B$. In the following, we will restrict our attention to finite knowledge bases over a finite language.

The semantics of conditional KBs is defined by considering a set of worlds $\mathcal{W}$ equipped with a preference relation $\prec$. Intuitively the meaning of $x \prec y$ is that $x$ is more typical/more normal/less exceptional than $y$. We say that a conditional $A \models B$ is true in a model if $B$ holds in all most normal worlds where $A$ is true, i.e. in all $\prec$-minimal worlds satisfying $A$. In [23] Lehmann and Magidor introduce ranked models as a family of preferential models [22].

**Definition 1 (Preferential models and ranked models).** A preferential model is a triple $\mathcal{M} = \langle \mathcal{W}, \prec, v \rangle$ where:

- $\mathcal{W}$ is a non-empty set of worlds;
- $\prec$ is an irreflexive, transitive relation on $\mathcal{W}$ satisfying the Smoothness condition defined below;
- $v$ is a function $v : \mathcal{W} \mapsto 2^{\text{ATM}}$, which assigns to every world $w$ the set of atoms holding in that world. If $F$ is a boolean combination of formulas, its truth conditions $(\mathcal{M}, w \models F)$ are defined as for propositional logic. Let $A$ be a propositional formula; we define $\text{Min}^A_\mathcal{M}(A) = \{ w \in \mathcal{W} | \mathcal{M}, w \models A \text{ and } \forall w', w' \prec w \text{ implies } \mathcal{M}, w' \not\models A \}$. Moreover:

$$\mathcal{M}, w \models A \models B \text{ iff for all } w', \text{ if } w' \in \text{Min}^A_\mathcal{M}(A) \text{ then } \mathcal{M}, w' \models B.$$

At this point we can define the Smoothness condition: if $\mathcal{M}, w \models A$, then either $w \in \text{Min}^A_\mathcal{M}(A)$ or there is $w' \in \text{Min}^A_\mathcal{M}(A)$ such that $w' \prec w$.

A ranked model is a preferential model $\mathcal{M} = \langle \mathcal{W}, \prec, v \rangle$ for which the relation $\prec$ is modular: for all $x, y, z$, if $x \prec y$ then either $x \prec z \text{ or } z \prec y$.

Validity and satisfiability of a formula are defined as usual. We say that a formula $F$ is *satisfiable* in the preferential (rational) semantics if there is a preferential (ranked) model $\mathcal{M} = \langle \mathcal{W}, \prec, v \rangle$ and a world $w \in \mathcal{W}$ such that $\mathcal{M}, w \models F$. We say that a formula $F$ is valid in a preferential (ranked) model $\mathcal{M} = \langle \mathcal{W}, \prec, v \rangle$, and we write $\mathcal{M} \models F$, if, for all $w \in \mathcal{W}$, it holds that $\mathcal{M}, w \models F$. We say that a formula $F$ is valid in the preferential (rational) semantics if it is valid in all preferential (ranked) models, i.e. if, for all preferential (ranked) models $\mathcal{M} = \langle \mathcal{W}, \prec, v \rangle$, it holds that $\mathcal{M} \models F$.

Given a set of formulas $K$ of $\mathcal{L}$ and a model $\mathcal{M} = \langle \mathcal{W}, \prec, v \rangle$, we say that $\mathcal{M}$ is a model of $K$, written $\mathcal{M} \models K$, if for every $F \in K$ and every $w \in \mathcal{W}$, we have that
Laura Giordano and Valentina Gliozzi

\[ \mathcal{M}, w \models F. \]  
\[ K \text{ preferentially entails a formula } F, \text{ written } K \models_\mathbf{P} F \text{ if } F \text{ is valid in all preferential models of } K. \]

\[ K \text{ rationally entails a formula } F, \text{ written } K \models_\mathbf{R} F \text{ if } F \text{ is valid in all rational models of } K. \]

As a consequence of Theorems 6.8 and 6.9 in [17], if a set of formulas \( K \) is satisfiable in a ranked model, then it is satisfiable in a finite ranked model. In the following, we will restrict our consideration to ranked models with a finite set of worlds.

Given a (finite) ranked model \( \mathcal{M} = \langle \mathcal{W}, <, v \rangle \), we can define the rank of a world in \( \mathcal{M} \).

**Definition 2 (Rank \( k_\mathcal{M}(w) \) of a world in \( \mathcal{M} \)).** Given a (finite) ranked model \( \mathcal{M} = \langle \mathcal{W}, <, v \rangle \), the rank \( k_\mathcal{M} \) of a world \( w \in \mathcal{W} \), written \( k_\mathcal{M}(w) \), is the length of the longest chain \( w_0 \prec \cdots \prec w \) from \( w \) to a minimal \( w_0 \) (i.e. there is no \( w' \) such that \( w' \prec w_0 \)).

Hence, the preference relation \( < \) of a ranked model \( \mathcal{M} \) defines a ranking function \( k_\mathcal{M} : \mathcal{W} \rightarrow \mathbb{N} \) (this just is a special case of the general result in [23] where there is no restriction to finite models). Observe that, according to [23], here we might have forgotten the smoothness condition, which is satisfied in any well-founded model and, in particular, in any finite model. Notice also that Definition 2 makes sense even if the relation \( < \) is not modular and that, for a modular relation on a finite set, all maximal chains\(^3\) from an element \( w \) to a minimal \( w_0 \) have the same length.

The previous definition defines from \( < \) a rank function \( k_\mathcal{M} : \mathcal{W} \rightarrow \mathbb{N} \). The opposite is also possible and \( < \) can be defined from a ranking function \( k_\mathcal{M} \) by letting \( x < y \) if and only if \( k_\mathcal{M}(x) < k_\mathcal{M}(y) \) (this is similarly stated in [23], where a ranking function \( r \) over a possibly infinite set is considered, since there is no restriction to finite models).

The rank of a formula \( F \) in a model \( \mathcal{M} \) depends on the rank of the worlds satisfying the formula.

**Definition 3 (Rank of a formula in a model).** The rank \( k_\mathcal{M}(F) \) of a formula \( F \) in a model \( \mathcal{M} \) is \( i = \min \{k_\mathcal{M}(w) : \mathcal{M}, w \models F \} \). If there is no \( w \) such that \( \mathcal{M}, w \models F \), then \( F \) has no rank in \( \mathcal{M} \).

The previous definition defines from \( < \) a rank function \( k_\mathcal{M} : \mathcal{W} \rightarrow \mathbb{N} \). The opposite is also possible and in general in ranked models the rank function \( k_\mathcal{M} \) and \( < \) can be defined from each other by letting \( x < y \) if and only if \( k_\mathcal{M}(x) < k_\mathcal{M}(y) \) (this is similarly stated by [23] where a rank function \( k \) over a possibly infinite set is used, since there is no restriction to finite models).

Lehmann and Magidor proved that, for a knowledge base \( K \) which is a set of positive conditional assertions of the form \( A \vdash B \) the rational (ranked) entailment is equivalent to the preferential entailment. Also, the rational entailment does not define a rational consequence relation (i.e., a consequence relation which also satisfies the property of Rational Monotonicity). A possible formulation of Rational Monotonicity is the following:

\[ (RM) \quad \alpha \vdash \gamma \text{ and } \alpha \not\vdash \neg \beta \text{ then } \alpha \wedge \beta \vdash \gamma \]

\(^3\)A chain \( w_0 < w_1 < \cdots < w_n \) is maximal if there is no element \( w' \) such that for some \( i = 0, \ldots, n-1 \) it holds \( w_i < w' < w_{i+1} \).
i.e., if $\alpha \vdash \gamma$ belongs to the consequence relation and $\alpha \vdash \neg \beta$ does not, then $\alpha \land \beta \vdash \gamma$ must belong as well to the consequence relation.

In order to strengthen rational entailment, Lehmann and Magidor in [23] introduce the notion of rational closure, which provides a solution to both the problems above and can be seen as the “minimal” (in some sense) rational consequence completing a set of conditionals. In the following we recall the definition of the rational closure.

**Definition 4 (Exceptionality of formulas).** Let $K$ be a knowledge base (i.e. a finite set of positive conditional assertions) and $A$ a propositional formula. $A$ is said to be exceptional for $K$ if and only if $K \models \top \vdash \neg A$. A conditional formula $A \vdash B$ is exceptional for $K$ if its antecedent $A$ is exceptional for $K$. The set of conditional formulas of $K$ which are exceptional for $K$ will be denoted as $E(K)$.

It is possible to define a non increasing sequence of subsets of $K$, $C_0 \supseteq C_1, C_1 \supseteq C_2, \ldots$ by letting $C_0 = K$ and, for $i > 0$, $C_i$ the set of conditionals of $C_{i-1}$ exceptional for $C_{i-1}$, i.e. $C_i = E(C_{i-1})$. Observe that, being $K$ finite, there is an $n \geq 0$ such that $C_n = \emptyset$ or for all $m > n$, $C_m = C_n$. The sets $C_i$ are used to define the rank of a formula, as in the next definition. Notice that if there is an $m$ such that $C_m = C_{m+1}$, then for all $k > m$, it will hold that $C_m = C_k$ (indeed $E(C_m) = E(C_{m+1}) = \cdots = E(C_k)$).

**Definition 5 (Rank of a formula).** A propositional formula $A$ has rank $i$ (for $K$), written $\text{rank}(A) = i$, if and only if $i$ is the least natural number for which $A$ is not exceptional for $C_i$. If $A$ is exceptional for all $C_i$ then $A$ has no rank, and we let $\text{rank}(A) = \infty$.

A conditional $A \vdash B$ has rank equal to $\text{rank}(A)$, and $C_i \setminus C_{i-1}$ is the set of conditionals (defaults) in $K$ having rank $i$.

**Example 1.** Let $K$ be the knowledge base containing the conditionals:

1. $\text{Student} \vdash \neg \text{Pay\_Taxes}$
2. $\text{Student} \vdash \text{Young}$
3. $\text{Employee} \land \text{Student} \vdash \text{Pay\_Taxes}$

stating that normally students do not pay taxes and are young, while employed students normally are students and pay taxes. It is possible to see that, from the definition of exceptionality above:

$$C_0 = K$$
$$C_1 = \{\text{Employee} \land \text{Student} \vdash \text{Pay\_Taxes}\}.$$ 

In particular, $\text{rank}(\text{Student}) = 0$, as $\text{Student}$ is non-exceptional for $C_0$, while $\text{rank}(\text{Employee} \land \text{Student}) = 1$, as $\text{Employee} \land \text{Student}$ is exceptional w.r.t. the property that students typically are not taxpayers. Thus, the third conditional describing the properties of employed students has rank 1 and is more specific than the conditionals describing the properties of students, which have rank 0.
Rational closure builds on the notion of exceptionality. Roughly speaking a conditional $A \vdash B$ is in the rational closure of $K$ if $A \land B$ is less exceptional than $A \land \neg B$. We recall the construction of the rational closure for admissible knowledge bases in [23], remembering that we are considering finite knowledge base, and any finite knowledge base is admissible.

**Definition 6 (Rational closure).** Let $K$ be a (finite) knowledge base. The rational closure of $K$ is defined as:

$$
\overline{K} = \{ A \vdash B \mid \text{either rank}(A) < \text{rank}(A \land \neg B) \text{ or rank}(A) = \infty \}
$$

where $A$ and $B$ are propositions in the language of $K$.

Referring to Example 1, $\text{Student} \land \text{Italian} \vdash \neg \text{Pay}_\text{Taxes}$ is in the rational closure of $K$, as $\text{rank(}\text{Student} \land \text{Italian}) = 0 < \text{rank(}\text{Student} \land \text{Italian} \land \neg \text{Pay}_\text{Taxes}) = 1$. Similarly, $\text{Employee} \land \text{Student} \land \text{Italian} \vdash \text{Pay}_\text{Taxes}$ is in $\overline{K}$.

Lehmann and Magidor in [23] develop a model theoretic semantics for the rational closure, by a canonical model construction. In [19] it was shown that a semantic characterization of the rational closure can also be given in terms of *minimal canonical ranked models*. In such models the rank of worlds is minimized to make each world as normal as possible. This is expressed by the following definitions corresponding to the fixed interpretations minimal semantics, $\text{FIMS}$, in [19], where only models with the same set of worlds $W$ and valuation function $V$ are comparable.

**Definition 7 (Minimal ranked models).** Let $\mathcal{M} = (W, <, v)$ and $\mathcal{M'} = (W', <', v')$ be two ranked models. $\mathcal{M}$ is preferred to $\mathcal{M'}$ with respect to the fixed interpretations minimal semantics (and we write $\mathcal{M} <_{\text{FIMS}} \mathcal{M'}$) if: $W = W'$, $v = v'$ and

- for all $x \in W$, $k_\mathcal{M}(x) \leq k_{\mathcal{M}'}(x)$ and
- there exists $x' \in W$ such that $k_\mathcal{M}(x') < k_{\mathcal{M}'}(x')$.

Given a knowledge base $K$, we say that $\mathcal{M}$ is a minimal model of $K$ with respect to $<_{\text{FIMS}}$ if $\mathcal{M}$ is a model of $K$ and there is no $\mathcal{M}'$ such that $\mathcal{M}'$ is a model of $K$ and $\mathcal{M}' <_{\text{FIMS}} \mathcal{M}$.

In [19] it was also shown that, a notion of canonical model is needed when reasoning about the (relative) rank of the propositions in a model of $K$: it is important to have them true in some world of the model, whenever they are consistent with the knowledge base.

Given a knowledge base $K$ and a query $Q$, let $\text{ATM}_{K,Q}$ be the set of all the propositional variables of $\text{ATM}$ occurring in $K$ or in the query $Q$, and let $\mathcal{L}_{K,Q}$ be the restriction of the language $\mathcal{L}$ to the propositional variables in $\text{ATM}_{K,Q}$.

A truth assignment $v_0 : \text{ATM}_{K,Q} \rightarrow \{\text{true}, \text{false}\}$ is compatible with $K$, if there is no propositional formula $A \in \mathcal{L}_{K,Q}$ such that $v_0(A) = \text{true}$ and $K \models A \vdash \bot$ (where $v_0$ is extended as usual to arbitrary propositional formulas over the language $\mathcal{L}_{K,Q}$).

**Definition 8 (Canonical models).** A model $\mathcal{M} = (W, <, v)$ satisfying a knowledge base $K$ is said to be canonical if it contains (at least) a world associated with each truth assignment compatible with $K$, that is to say: if $v_0$ is compatible with $K$, then there exists a world $w$ in $W$ such that, for all propositional formulas $B \in \mathcal{L}_{K,Q}$, $\mathcal{M}, w \models B$ if and only if $v_0(B) = \text{true}$. 
**Definition 9 (Minimal canonical ranked models).** $M$ is a minimal canonical ranked model of $K$, if it is a canonical ranked model of $K$ and it is minimal with respect $<_{FIMS}$ (see Definition 7) among the canonical ranked models of $K$.

We define a notion of minimal entailment w.r.t. minimal canonical ranked models of $K$. $K$ minimally entails a formula $F$, and we write $K \models_{\text{min}} F$, if $F$ is true in all the minimal canonical ranked models of $K$.

It has been shown that, for any satisfiable knowledge base, a finite minimal canonical ranked model exists (see [19], Theorem 1), and that minimal canonical ranked models are an adequate semantic counterpart of rational closure. The correspondence between minimal canonical ranked models and rational closure is established by the following theorem.

**Theorem 1 ([19]).** Let $K$ be a knowledge base and $M \in \text{Min}_{RC}(K)$ be a minimal canonical ranked model of $K$. For all conditionals $A \not\rightarrow B \in \mathcal{L}$:

$$M \models A \not\rightarrow B \text{ if and only if } A \not\rightarrow B \in \overline{K},$$

where $\overline{K}$ is the rational closure of $K$.

Furthermore, when $\text{rank}(A)$ is finite, the rank $k_M(A)$ of a proposition $A$ in any minimal canonical ranked model of $K$ is equal to the rank $\text{rank}(A)$ assigned by the rational closure construction. Otherwise, $\text{rank}(A) = \infty$ and proposition $A$ is not satisfiable in any ranked model of $K$ (in any ranked model of $K$, $A$ has no rank).

Observe that, by Theorem 1, the set of conditionals minimally entailed from $K$ coincide with the set of conditionals true in any (arbitrarily chosen) minimal canonical ranked model $M$ of $K$. In the following, we will restrict our consideration to the finite minimal canonical models of the knowledge base $K$ (which, as said above, always exist when $K$ is consistent), and we denote their set by $\text{Min}_{RC}(K)$.

**Example 2.** Considering again the knowledge base in Example 1, we can see that conditional assertions $\text{Student} \wedge \text{Italian} \not\rightarrow \neg \text{Pay Taxes}$ and $\text{Employee} \wedge \text{Student} \not\rightarrow \text{Young}$ are satisfied in all the minimal canonical models of $K$. For the first conditional, in all the minimal canonical models of $K$, $\text{Student} \wedge \text{Italian}$ has rank 0, while $\text{Student} \wedge \text{Italian} \wedge \neg \text{Pay Taxes}$ has rank 1. Thus, in all the minimal canonical models of $K$ each typical Italian student must be an instance of $\neg \text{Pay Taxes}$. Similarly for the second conditional assertion.

Instead, the conditional $\text{Employee} \wedge \text{Student} \not\rightarrow \text{Young}$ is not minimally entailed from $K$ and, hence, it does not belong to the rational closure of $K$. Indeed, the proposition $\text{Employee} \wedge \text{Student}$ is exceptional for $E_0$, as it violates the property of students that normally they do not pay taxes and, then, in all models $M \in \text{Min}_{RC}(K)$, $k_M(\text{Employee} \wedge \text{Student}) = 1$. Furthermore, both $k_M(\text{Employee} \wedge \text{Student} \wedge \neg \text{Young}) = 1$ and $k_M(\text{Employee} \wedge \text{Student} \wedge \neg \text{Young}) = 1$, hence nothing can be concluded about the typical employed students being young or not. Employed students do not "inherit" any of the more general defeasible properties of students, not even the property that students are normally young. In general, the rational closure “does not provide for inheritance of generic properties to exceptional subclasses” [24].
In particular, the rational closure does not satisfy (among others) the desirable condition called by Lehmann the presumption of typicality. By Rational Monotony, if the rational closure of a KB contains \( \alpha \vdash \beta \) then it must contain either \( \alpha \land \gamma \vdash \beta \) or \( \alpha \vdash \neg \gamma \). But which one? Lehmann suggests that “in the absence of convincing reason to accept the latter, we should accept the former”. This and other desirable conditions led to the definition of the lexicographic closure as a “uniform way of constructing a rational superset of the rational closure” [24], thus strengthening the rational closure but still providing a rational consequence relation.

3 From the Lexicographic closure to the MP-closure

To overcome the weakness of rational closure, Lehmann introduced the notion of lexicographic closure [24], which strengthens the rational closure by allowing, roughly speaking, a class to inherit as many as possible of the defeasible properties of more general classes, giving preference to the more specific properties. In the example above, the property of students being young should be inherited by employed students, as it is consistent with all other default properties of employed students (i.e., with default 4) and, by “presumption of independence” [24], even if typicality is lost with respect to one consequent (the property that typically students are not taxpayers), we may still presume typicality of employed students with respect to other typical properties of students, such as the property of being young.

Let us recap the definition of the lexicographic closure in [24]. In order to compare alternative sets of defaults, in [24] a seriousness ordering \( \prec \) among sets of defaults is defined by associating with each set of defaults \( D \subseteq K \) a tuple of numbers \( \langle n_0, n_1, \ldots, n_k \rangle_D \), where \( k \) is the order of \( K \), i.e. the least finite \( i \) such that \( C_i - C_{i-1} = \emptyset \) (i.e. there is no defaults with finite rank \( k \) or rank higher than \( k \), but there is at least one default with rank \( k - 1 \)). The tuple is constructed considering the ranks of defaults in the rational closure. \( n_0 \) is the number of defaults in \( D \) with rank \( \infty \), and, for \( 1 \leq i \leq k \), \( n_i \) is the number of defaults in \( D \) with rank \( k - i \).

For instance, in the example Example 1 above, the set of defaults \( D = \{ \text{Student} \vdash \neg \text{Young}, \text{Employee} \land \text{Student} \vdash \neg \text{Pay Taxes} \} \) (that we will denote, synthetically as \( D = \{2, 3\} \)) is associated with the tuple \( \langle 0, 1, 1 \rangle_D \) meaning that \( D \) contains: no default with rank \( \infty \), one default with rank 1 (default 3) and one default with rank 0 (default 2).

A modular order \( \prec \) among sets of defaults is obtained from the natural lexicographic order over the tuples \( \langle n_0, n_1, \ldots, n_k \rangle_D \). This order gives preference to those sets of defaults containing more specific defaults. Notice that the numbers \( n_i \) in the tuple are in decreasing order w.r.t. the rank of the defaults, and the highest is the rank, the more specific is the default.

Lehmann defines a notion of basis for a formula \( A \) in a knowledge base \( K \). A basis for \( A \) is a set \( D \) of defaults in \( K \) such that \( A \) is consistent with \( D \), the material counterpart of \( D \), and \( D \) is maximal w.r.t. the seriousness ordering \( \prec \) for this property.\(^4\)

In the example above, the set of defaults \( D = \{2, 3\} \) forms a basis for \( \text{Employee} \land \text{Student} \), as its materialization \( \hat{D} = \{\text{Student} \rightarrow \text{Young}, \text{Employee} \rightarrow \text{Pay Taxes}\} \)

\(^4\) The material counterpart of \( D \), \( \hat{D} \), is the set containing a material implication \( A \rightarrow B \), for each conditional \( A \vdash B \) in \( D \).
is consistent (in the propositional calculus) with \( \text{Employee} \land \text{Student} \), and \( D \) is maximal w.r.t. the seriousness ordering among the sets having this property. \( D \) is actually the unique basis for \( \text{Employee} \land \text{Student} \).

A conditional \( A \rightarrow B \) is in \( K' \), the lexicographic closure of \( K \), if \( \hat{D} \cup A \models B \), for any basis \( D \) for \( A \). In the example, \( \text{Employee} \land \text{Student} \rightarrow \text{Young} \) belongs to the lexicographic closure of \( K \), as \( \hat{D} \cup \{(\text{Employee} \land \text{Student})\} \models \text{Young} \), for the unique basis \( D \) for \( \text{Employee} \land \text{Student} \). This is what is expected, as the property of typical students of being young is inherited by employed students by presumption of independence.

In the following we will consider two variants of the knowledge base in Example 1 to illustrate the lexicographic closure and, later, to describe its common points and differences with the MP-closure.

**Example 3.** Let \( K' \) be the knowledge base containing the conditionals:

1. \( \text{Student} \rightarrow \neg \text{Pay\_Taxes} \)
2. \( \text{Student} \rightarrow \text{Bright} \)
3. \( \text{Employee} \rightarrow \neg \text{Pay\_Taxes} \)
4. \( \text{Employee} \land \text{Student} \rightarrow \text{Busy} \)

Here, Students and Employee have a conflicting property: students normally do not pay taxes, while employees normally do pay taxes. Furthermore, students are normally bright and employed students are normally busy.

According to the rational closure, the formulas \( \text{Student} \) and \( \text{Employee} \) have both rank 0, while the formula \( \text{Employee} \land \text{Student} \) has rank 1. Therefore, conditionals 1, 2, 3 have rank 0, while conditional 4 has rank 1. It is easy to see that the conditionals \( \text{Employee} \land \text{Student} \rightarrow \neg \text{Pay\_Taxes} \) and \( \text{Employee} \land \text{Student} \rightarrow \neg \text{Pay\_Taxes} \) do not belong to the rational closure of \( K' \). The same can be said about the conditional \( \text{Employee} \land \text{Student} \rightarrow \text{Brigh} \), which also is not in the rational closure of \( K' \), although we would like to conclude it, as the property of typical student of being bright is not conflicting with other properties of typical employees and of typical employed students.

In this example, there are two bases for \( \text{Employee} \land \text{Student} \): \( D = \{1, 2, 4\} \) and \( B = \{2, 3, 4\} \). They represent two alternative scenarios, the first one in which typical employed students inherit from typical students the property of not paying taxes, and the second one in which typical employed students inherit from typical employees the property of paying taxes. It is easy to see that \( D = \{1, 2, 4\} \) and \( B = \{2, 3, 4\} \) are not comparable with each other, i.e. none of them is more serious than the other (that is, \( D \not\prec B \) and \( B \not\prec D \)), as the tuples \( (0, 1, 2)_D \) and \( (0, 1, 2)_B \), associated with \( D \) and \( B \) (respectively), are not comparable in the lexicographic order.

Both the bases contain the default that normally students are bright and, as intended, this property extends to employed students. It is easy to see that \( \text{Employee} \land \text{Student} \rightarrow \text{Brigh} \) is in the lexicographic closure of \( K' \). Instead, the lexicographic closure neither contains the conditional \( \text{Employee} \land \text{Student} \rightarrow \neg \text{Pay\_Taxes} \) nor the conditional \( \text{Employee} \land \text{Student} \rightarrow \neg \text{Pay\_Taxes} \), as each of them is false in one of the two bases (they are conflicting).
The following variant of Example 3 has a single basis and may suggest that the lexicographic closure is sometimes too bold.

**Example 4.** Let the knowledge base $K''$ contain the following conditionals:

1. $Student \not\succ \neg Pay\_Taxes$
2. $Student \not\succ Young$
3. $Employee \not\succ \neg Young \land Pay\_Taxes$
4. $Employee \land Student \not\succ Busy$

Again, defaults 1, 2 and 3 have rank 0 in the rational closure, while default 4 has rank 1. As a difference with the previous example, the lexicographic closure has a single basis, $D = \{1, 2, 4\}$. Indeed, of the two sets of defaults $D = \{1, 2, 4\}$ and $B = \{3, 4\}$, whose materializations are both consistent with $Employee \land Student$, have the associated tuples $\langle 0, 1, 2 \rangle_D$ and $\langle 0, 1, 1 \rangle_B$ and, therefore, $B$ is less serious than $D (B \prec D)$. As a consequence, there is a single basis $D$ for $Employee \land Student$, and we can conclude that typical employed students are not only busy, but (like typical students) they are also young and do not pay taxes. The conditional

$$Employee \land Student \not\succ Young \land \neg Pay\_Taxes$$

is in the lexicographic closure of $K''$, as $\bar{D} \cup \{Employee \land Student\} \models Young \land \neg Pay\_Taxes$, and $D$ is the only basis for $K''$.

The result above is in line with the choice of the lexicographic closure that, in the case of contradictory defaults with the same rank, as many as possible defaults should be satisfied, a choice taken in agreement with the Maximal Entropy approach \[25\]. However, the reason to accept that typical employed students are not young and pay taxes (rather than the converse) may be questioned and, in this last example, the lexicographic closure appears to be too bold. Indeed, the conclusion that normally employed students are young and do not pay taxes, i.e. conditional (1) here follows from the accidental fact that the properties of Employees are expressed by a single default, while the properties of Students are expressed by two defaults. Notice that, if we replace default 3 with the two defaults $T(Employee) \subseteq \neg Young$ and $T(Employee) \subseteq Pay\_Taxes$, there would be two bases in the lexicographic closure, and one would not be allowed to conclude any more that typical employed students are young and do not pay taxes. As observed by Lehmann, the lexicographic closure construction is “extremely sensitive to the way defaults are presented” and “the way defaults are presented is important” \[24\].

In the following section, we will consider a different notion of closure, the MP-closure, that departs from Maximal Entropy assumption of the lexicographic closure and, for instance, in Example 4 it considers both the sets of defaults $D$ and $B$ to be maximally serious, and it does not conclude conditional (1). Although also the MP-closure is somewhat syntax dependent, in this case, differently from the lexicographic closure, it treats in the same way the two different formulations of the knowledge base $K''$ above. We will show that the MP-closure is stronger than the rational closure but weaker than the lexicographic closure. Abandoning the Maximal Entropy assumption leads to a construction which defines a preferential consequence relation, rather than a rational consequence relation, and which defines a more cautious notion of entailment.
(with respect to the lexicographic closure), that does not satisfy the property of Rational Monotonicity. We will see later that, however, rationality can be recovered, at least from the semantic point of view, by considering a rational extension of the MP-closure, which provides another (different) solution to the technical problem, risen by Lehmann, of defining a rational consequence relation which is a rational superset of the rational closure.

Following the pattern in [24], in Section 4 we present both a characterization of the MP-closure in terms of maxiconsistent sets and a model-theoretic construction. In Section 5.1 we will then exploit the semantic construction to define a rational consequence relation which is a superset of the MP-closure and is neither stronger nor weaker than the lexicographical closure.

4 The MP-closure revisited

The multipreference closure (MP-closure, for short), was preliminarily introduced in the technical report [16] as a construction which soundly approximates the multipreference semantics proposed by Gliozzi [20, 21] for the description logic \( \mathcal{ALC} \) with typicality, thus defining a refinement of the rational closure of \( \mathcal{ALC} \). This semantics was originally proposed for separately reason about the inheritance of different properties and, hence, to provide a solution to the drowning problem related to the rational closure.

We believe that the interest of the MP-closure construction goes beyond description logics and that its definition and semantics can be reconstructed and significantly simplified in the context of propositional logic. The MP-closure can be regarded as the natural variant of the lexicographic closure, if we are ready to abandon the Maximal Entropy approach (as illustrated by Example 4 in the previous section), an alternative route already considered but not explored by Lehmann in [24]. In this section we reformulate the MP-closure construction from [16] in the propositional setting and, then, we focus on its semantics, its properties and its relations with the lexicographic closure. Further relationships, and in particular the relationships with the Relevant closure, will be investigated in Section 5.

4.1 The MP-closure construction

Given a finite knowledge base \( K \), and a formula \( A \) whose rank \( \text{rank}(A) \) in the rational closure of \( K \) is finite, we let \( k \geq 0 \) be the maximum finite rank for a conditional assertion (default) in the rational closure of \( K \). In the following, we exploit the MP-closure construction to define the plausible consequences of a formula \( A \). Observe that, for any formula \( A \) with infinite rank, i.e. such that \( \text{rank}(F) = \infty \), the conditional \( A \vdash C \) is in the rational closure of \( K \), for any \( C \).

Given a subset \( D \) of the conditional assertions in \( K \) (a set of defaults), we let \( D_i \) be the set of defaults in \( D \) with finite rank \( i \leq k \), and \( D_\infty \) be the set of defaults in \( D \) with rank \( \infty \). The tuple \( \langle D_\infty, D_k, \ldots, D_1, D_0 \rangle_D \), associated with \( D \), defines a partition of \( D \), according to the ranks of the defaults in the rational closure of \( K \).

We define a preference relation \( \prec_{MP} \) among sets of defaults, by comparing the tuples associated to these sets according to the natural lexicographic order on such tuples,
defined inductively as follows. Given two tuples \( (X_1, \ldots, X_1) \) and \( (X_1', \ldots, X_1') \) of sets of defaults in \( K \), we let:

\[
(\langle X_1, \ldots, X_1 \rangle) \preceq (\langle X_1', \ldots, X_1' \rangle) \text{ iff } X_1 \subset X_1'
\]
\[
(\langle X_n, \ldots, X_1 \rangle) \preceq (\langle X_n', \ldots, X_1' \rangle) \text{ iff } X_n \subset X_n'
\]

As the (strict) subset inclusion relation \( \subset \) among sets is a strict partial order, the lexicographic order \( \preceq \) on the tuples of sets of defaults is a strict partial order as well. This lexicographic order provides a new seriousness ordering among sets of defaults.

**Definition 10 (MP-seriousness ordering).** \( D \prec^{MP} B \) \( (D \text{ is less serious than } B \text{ w.r.t. the } MP \text{-seriousness ordering}) \) iff

\[
(\langle D_\infty, D_k, \ldots, D_1, D_0 \rangle)_D \preceq (\langle B_\infty, B_k, \ldots, B_1, B_0 \rangle)_B.
\]

Notice that the relation \( \prec^{MP} \) defines a seriousness ordering among sets of defaults, which is different from the seriousness ordering used by the lexicographic closure, where the corresponding tuple associated with \( D \) would be \( \langle | D_\infty |, | D_k |, \ldots, | D_1 |, | D_0 | \rangle \) \( (\text{see Section 3}) \), i.e. the cardinality of the sets \( D_i \) matters, rather than the defaults in \( D_i \). As the lexicographic order \( \preceq \) on the tuples of sets of defaults is a strict partial order, \( \prec^{MP} \) is a strict partial order as well, although it is not necessarily modular.

The difference of the seriousness ordering between lexicographic closure and the MP-closure has an impact on the kind of conclusions one draws in the two cases, as we will see below. Let us first give a characterization of the MP-closure in terms of bases.

**Definition 11 (MP-basis).** Given a finite knowledge base \( K \), and a formula \( A \) with finite rank, a set of defaults \( D \subseteq K \) is a basis for \( A \) if \( A \) is consistent with \( \hat{D} \) \( (\text{the material counterpart of } D) \) and \( D \) is maximal w.r.t. the MP-seriousness ordering for this property.

Notice that the definition of a basis is exactly the same as in the lexicographic closure \([24]\), but for the fact that it uses a different lexicographic ordering.

**Definition 12 (MP-closure).** A default \( A \vdash B \) is in \( MP(K) \), the MP-closure of a knowledge base \( K \), if for all the MP-bases \( D \) for \( A \):

\[
\hat{D} \cup \{ A \} \models B,
\]

where \( \models \) is logical consequence in the propositional calculus and \( \hat{D} \) is the materialization of \( D \).

Consider again Example \([4]\) the two sets of defaults \( D = \{1, 2, 4\} \) and \( B = \{3, 4\} \) are now incomparable using the \( \prec^{MP} \) preference relation, as the tuples associated to the sets \( D \) and \( B \) are respectively: \( \langle \emptyset, \{4\}, \{1, 2\} \rangle_D \) and \( \langle \emptyset, \{4\}, \{3\} \rangle_B \) and neither \( \langle \emptyset, \{4\}, \{1, 2\} \rangle_D \preceq \langle \emptyset, \{4\}, \{3\} \rangle_B \) nor \( \langle \emptyset, \{4\}, \{3\} \rangle_B \preceq \langle \emptyset, \{4\}, \{1, 2\} \rangle_D \) (instead, as we have seen above, in the lexicographic closure, \( D \) is more serious than \( B \)). Thus, there are two MP-bases for \( Employee \land Student \), namely \( D = \{1, 2, 4\} \)
and \( E = \{3, 4\} \). Therefore, neither \( \text{Employee} \land \text{Student} \not\models \text{Young} \) nor \( \text{Employee} \land \text{Student} \not\models \lnot \text{Young} \) are in the MP-closure of \( K'' \). In this example, the MP-closure is less bold than the lexicographic closure, which, as we have seen, includes the default \( \text{Employee} \land \text{Student} \not\models \text{Young} \), as \( D \) is the only basis for \( \text{Employee} \land \text{Student} \) in the lexicographic closure.

Concerning Examples 1 and 3 above, it is easy to see that in both of them the MP-closure has the same bases as the lexicographic closure, as well as the same consequences.

It can be proved that the MP-closure is stronger than the rational closure, but weaker than the lexicographic closure. We prove the second result, while postponing the proof of the first one after the introduction of the semantics of the MP-closure in Section 4.2. We prove that \( \prec^{\text{MP}} \) is coarser than \( \prec \).

**Proposition 1.** \( \prec^{\text{MP}} \) is coarser than \( \prec \), that is, for all the sets of defaults \( D \) and \( B \), if \( D \prec^{\text{MP}} B \) then \( D \prec B \).

**Proof.** Given a knowledge base \( K \) and two sets of conditionals \( D, B \subseteq K \), let us assume that \( D \prec^{\text{MP}} B \). As \( D \) is less serious than \( B \) in the MP-ordering, it must be that:

\[
\langle D_\infty, D_k, \ldots, D_1, D_0 \rangle_D \ll \langle B_\infty, B_k, \ldots, B_1, B_0 \rangle_B.
\]

Consider the highest \( j \), with \( 0 \leq j \leq k \), such that \( D_j \neq B_j \). For such a \( j \), it must be that \( D_j \subset B_j \), while \( D_r = B_r \), for all \( r \) such that \( k \geq r > j \).

Let us now consider the two tuples of numbers

\[
\langle n_\infty, n_k, \ldots, n_1, n_0 \rangle_D \quad \text{and} \quad \langle m_\infty, m_k, \ldots, m_1, m_0 \rangle_B
\]

associated with \( D \) and \( B \), respectively, in the lexicographic closure construction. Notice that, \( n_i = |D_i| \), for all \( i \), and \( m_i = |B_i| \), for all \( i \). Furthermore, \( D_\infty (= B_\infty) \) is the set of all the conditionals with rank \( \infty \) in the rational closure and, hence, \( n_\infty = m_\infty \). For all \( r \) such that \( k \geq r > j \), as \( D_r = B_r \), it must be that \( n_r = m_r \). Also, from \( D_j \subset B_j \), we get \( n_j < m_j \). Thus, using the lexicographic ordering on numbers:

\[
\langle n_\infty, n_k, \ldots, n_1, n_0 \rangle_D \ll \langle m_\infty, m_k, \ldots, m_1, m_0 \rangle_B
\]

and, therefore, \( D \prec B \). \( \Box \)

As a consequence of this result, it is easy to prove the following corollaries.

**Corollary 1.** Let \( K \) be a knowledge base, \( A \) a formula and \( D \subseteq K \) a set of defaults. If \( D \) is a basis for \( A \) in the lexicographic closure, then \( D \) is a basis for \( K \) in the MP-closure.

**Proof.** Let \( D \) be a basis for \( A \) in the lexicographic closure, i.e. \( A \) is consistent with \( \check{D} \) and \( D \) is maximal w.r.t. \( \prec \)-seriousness ordering for this property.

We show that \( B \) is also a basis in the MP-closure. If not, there is a set of defaults \( B \subseteq K \) such that \( A \) is consistent with \( \check{B} \) and \( D \prec^{\text{MP}} B \). But, then, by Proposition 1 \( D \prec B \), and \( D \) is not a maximal w.r.t. \( \prec \) among the sets of default whose materialization is consistent with \( A \), which contradicts the hypothesis that \( D \) is a basis for \( A \) in the lexicographic closure. \( \Box \)
Corollary 2. Let $K$ be a knowledge base and $A$ a formula. If $A \not\vdash C$ is in the MP-closure of $K$, then $A \not\vdash C$ is in the lexicographic closure of $K$.

Proof. If $A \not\vdash C$ is in the MP-closure of $K$, then, in all the bases $D$ for $A$ in the MP-closure, $D \cup \{A\} \models C$. Let $D$ be any basis for $A$ in the lexicographic closure of $K$. As, by Corollary 1, all the bases for $A$ in lexicographic closure of $K$ are also MP-bases for $A$, $\tilde{D} \cup \{A\} \models C$. Hence, for all the bases $D$ for $A$ in the lexicographic closure of $K$, $\tilde{D} \cup \{A\} \models C$, and $A \not\vdash C$ is in the lexicographic closure of $K$. $\blacksquare$

To conclude this section, we show that the MP-closure does not define a rational consequence relation. Let $\mathcal{MP}_K$ be the set of conditionals in the MP-closure of $K$. The following counterexample shows that $\mathcal{MP}_K$ does not satisfy the property of Rational Monotonicity, and is a reformulation of Lehmann’s musician example[24].

Example 5. The following knowledge base $K$

1. $\text{Student} \not\vdash \text{Merry}$
2. $\text{Student} \not\vdash \text{Young}$
3. $\text{Adult} \not\vdash \text{Serious}$
4. $\text{Student} \land \text{Adult} \not\vdash (\neg \text{Young} \land \neg \text{Merry}) \lor \neg \text{Serious}$

The conditionals 1, 2 and 3 have rank 0 in the rational closure, while conditional 4 has rank 1. There are two bases for $\text{Student} \land \text{Adult}$ in the MP-closure of $K$, $D = \{1, 2, 4\}$ and $B = \{3, 4\}$, and the conditional $\text{Student} \land \text{Adult} \not\vdash \text{Young} \leftrightarrow \text{Merry}$ is in the MP-closure of $K$ (in $\mathcal{MP}_K$). Instead, the conditional $\text{Student} \land \text{Adult} \not\vdash \text{Young}$ is not in $\mathcal{MP}_K$ (as Young does not hold in the basis $B$), that is, $\text{Student} \land \text{Adult} \not\vdash \text{Young}$. By the property of Rational Monotonicity, the conditional

$$\text{Student} \land \text{Adult} \land \neg \text{Young} \not\vdash \text{Young} \leftrightarrow \text{Merry}$$

should be in $\mathcal{MP}_K$. Instead, the last conditional is not in the MP-closure of $K$. In fact, there are two bases for $\text{Student} \land \text{Adult} \land \neg \text{Young}$, namely $D' = \{1, 4\}$ and $B' = \{3, 4\}$, and the formula $\text{Young} \leftrightarrow \text{Merry}$ does not hold in the first basis, as $D' \cup \{\text{Student} \land \text{Adult} \land \neg \text{Young}\} \models \neg \text{Young} \land \text{Merry}$.

Notice that the example above is not a counterexample to Rational Monotonicity for the lexicographic closure, which in known to define a rational consequence relation. In fact, $D = \{1, 2, 4\}$ is the only basis for $\text{Student} \land \text{Adult}$ in the lexicographic closure of $K$ and, hence, the conditional $\text{Student} \land \text{Adult} \not\vdash \text{Young}$ is in the lexicographic closure of $K$.

4.2 A semantic characterization for the MP-closure

A semantics for the MP-closure is defined in [14] building on the preferential semantics for rational closure of $\mathcal{ALC} + T_n$, introducing a notion of refined, bi-preference interpretation, which contains two preference relations, let us call them $<$ and $<'$: the first one plays the role of the preference relation in a model of the RC in $\mathcal{ALC} + T_n$, while the second one $<'$ is built from $<$ exploiting a specificity criterium, and represents a refinement of $<$. 
In this section we define a simpler semantic characterization of the MP-closure of a propositional knowledge base, starting from the propositional models of the rational closure. This simplified setting, that corresponds to the one considered by Lehmann in his semantic characterization of the lexicographic closure \([24]\), also allows an easy comparison among the two semantics.

Given a finite satisfiable knowledge base \(K\), in the following we define the semantics of the MP-closure by means of some preferential models of \(K\) (that we call MP-models) and, then, we prove a characterization result. To this purpose, we introduce a functor \(\mathcal{F}\) associating a preferential interpretation \(\mathcal{N}\) to each finite minimal canonical ranked model \(\mathcal{M} \in \text{Min}_{RC}(K)\) characterizing the rational closure of \(K\) according to Theorem 1. As we will see, \(\mathcal{N}\) is a model of the MP-closure.

**Definition 13 (Functor \(\mathcal{F}\)).** Given a minimal canonical ranked interpretation \(\mathcal{M} = \langle W, <, v \rangle\) in \(\text{Min}_{RC}(K)\), we let \(\mathcal{F}(\mathcal{M}) = \mathcal{N}\) such that: \(\mathcal{N} = \langle W, <', v \rangle\) and

\[
x <' y \iff V(x) <' MP V(y)
\]

where, for \(z \in W\), \(V(z)\) is the set of defaults in \(K\) which are violated by \(z\) (i.e., the set of conditionals \(A \not\vdash C \in K\) such that \(\mathcal{M}, z \models A \wedge \neg C\).

As \(<' MP\), introduced in Definition 10, is a strict partial order, it is easy to see that \(<'\) in the definition above is a strict partial order as well. Indeed, \(<'\) is irreflexive: if \(x <' y\) then \(V(x) <' MP V(y)\) and, by irreflexivity of \(<' MP, V(y) \not< MP V(x)\). Hence, \(y <' x\) does not hold. Also, \(<'\) is transitive: if \(x <' y\) and \(y <' z\), then, by definition of \(<'\), \(V(x) <' MP V(y)\) and \(V(y) <' MP V(z)\). From the transitivity of \(<' MP, V(x) <' MP V(z)\). Therefore, \(x <' z\).

Hence, \(\mathcal{N} = \langle W, <', v \rangle\), in Definition 13 is a preferential interpretation. Propositions 2 and 4 below will show that \(\leq <'\), i.e. the preference relation \(<'\) is finer than the modular preference relation \(<\), and that \(\mathcal{N}\) is a model of \(K\). Thus, \(\mathcal{N}\) is a preferential model of \(K\) which is the refinement of the model \(\mathcal{M}\) of the rational closure of \(K\) (in the sense that the preference relation in \(\mathcal{N}\) is finer than the preference relation in \(\mathcal{M}\).

**Proposition 2.** For all \(\mathcal{M} = \langle W, <, v \rangle \in \text{Min}_{RC}(K)\) and \(\mathcal{N} = \langle W, <', v \rangle\) such that \(\mathcal{N} = \mathcal{F}(\mathcal{M})\), it holds that \(\leq <'\), i.e. the preference relation \(<'\) is finer than \(<\).

**Proof.** We show that, for all \(x, y \in W\), \(x < y\) implies \(x <' y\).

If \(x < y\) in \(\mathcal{M}\), then for some \(j, h, k_M(x) = j < h = k_M(y)\). As \(\mathcal{M}\) is a ranked model of \(K\), by Proposition 2 in [19], \(\mathcal{M}, x \models A \rightarrow C\) for all the conditionals in \(A \not\vdash C \in C_j\) (i.e., for all the conditionals \(A \not\vdash C\) with rank \(r \geq j\)). In particular, letting \(V_r(x)\) the defaults with rank \(r\) violated by \(x\), we have: \(V_r(x) = \emptyset\), for all \(r \geq j\). The tuple associated with \(V(x)\) has the form

\[
\langle C_\infty, \emptyset, \ldots, \emptyset, V_{j-1}(x), \ldots, V_0(x)\rangle_{V(x)},
\]

where \(C_\infty\) is the set of the conditionals in \(K\) with rank \(\infty\). Such conditionals must be in \(V(x)\) as they cannot be satisfied in any model of \(K\).
As \( k_M(y) = h > j \), there must be some conditional \( A \vdash B \) with \( \text{rank}(A) = h - 1 \geq j \), which is falsified by \( y \). Hence, \( \mathcal{V}_{h-1}(y) \) is nonempty. The tuple associated with \( V(y) \) has the form

\[
(C_\infty, \emptyset, \ldots, \emptyset, V_{h-1}(y), \ldots, V_j(y), V_{j-1}(y), \ldots, V_0(y))_V(y),
\]

with \( V_{h-1}(y) \neq \emptyset \) and \( h - 1 \geq j \). Then, the tuple associated with \( V(x) \) is less serious (in the MP-seriousness ordering) than the tuple associated to \( V(y) \). Hence, \( V(x) \prec_{MP} V(y) \). Therefore, by (2), \( x \prec y \) holds, and we can then conclude that, \( x \prec \prec y \). \( \square \)

To prove that a preferential interpretation \( \mathcal{N} \) such that \( \mathcal{N} = \mathcal{F}(\mathcal{M}) \), for some \( \mathcal{M} \in \text{Min}_{RC}(K) \), is a model of \( K \), we exploit the following general property of preferential interpretations.

**Proposition 3.** Let \( \mathcal{N}' = (\mathcal{W}, <', v) \) and \( \mathcal{N}'' = (\mathcal{W}, <'', v) \) be two preferential interpretations such that \( <' \sqsubseteq <'' \). For all conditionals \( A \vdash C \), if \( \mathcal{N}' \models A \vdash C \) then \( \mathcal{N}'' \models A \vdash C \).

**Proof.** Let \( \mathcal{N}' \models A \vdash C \), i.e., for all \( w \in \mathcal{W} \), if \( w \in \text{Min}_{<'}(\mathcal{N}) \) then \( \mathcal{N}', w \models C \). We prove that \( \mathcal{N}'' \models A \vdash C \).

Assume that \( w \in \text{Min}_{<''}(\mathcal{N}) \). We show that \( \mathcal{N}'', w \models C \).

As \( w \in \text{Min}_{<''}(\mathcal{N}) \), clearly \( w \in \mathcal{W} \) and \( \mathcal{N}'', w \models A \). Since \( \mathcal{N}' \) and \( \mathcal{N}'' \) have the same set of worlds \( \mathcal{W} \) and valuation function \( v \), it follows that \( \mathcal{N}', w \models A \). Furthermore, as the worlds satisfying \( A \) in \( \mathcal{N}' \) and in \( \mathcal{N}'' \) are the same and, from \( <' \sqsubseteq <'' \), it follows that \( \text{Min}_{<''}(\mathcal{N}) \subseteq \text{Min}_{<'}(\mathcal{N}) \).

Therefore \( w \in \text{Min}_{<'}(\mathcal{N}) \) and, \( \mathcal{N}' \models A \vdash C \), then \( \mathcal{N}', w \models C \). Again, as \( \mathcal{N}' \) and \( \mathcal{N}'' \) have the same set of worlds \( \mathcal{W} \) and valuation function \( v \), it follows that \( \mathcal{N}'' \), \( w \models C \), which concludes the proof. \( \square \)

As a consequence of the proposition above, an interpretation \( \mathcal{N} \) such that \( \mathcal{N} = \mathcal{F}(\mathcal{M}) \) is a model of \( K \).

**Proposition 4.** For all \( \mathcal{M} = (\mathcal{W}, <, v) \in \text{Min}_{RC}(K) \) and \( \mathcal{N} = (\mathcal{W}, <', v) \) such that \( \mathcal{N} = \mathcal{F}(\mathcal{M}) \), it holds that \( \mathcal{N} \) is a model of \( K \).

**Proof.** From the hypothesis we know that \( \mathcal{M} \) is a minimal canonical ranked model of \( K \). Hence, \( \mathcal{M} \) satisfies all the conditionals in \( K \), i.e., for all conditionals \( A \vdash B \in K \), \( \mathcal{M} \models A \vdash C \). As \( \mathcal{M} \) and \( \mathcal{N} \) are preferential models and, by Proposition 2, \( <' \sqsubseteq <' \), we can use Proposition 3 to conclude that, for all conditionals \( A \vdash B \in K \), as \( \mathcal{M} \models A \vdash C \), also \( \mathcal{N} \models A \vdash C \). Thus \( \mathcal{N} \) is a model of \( K \). \( \square \)

We can naturally extend the functor \( \mathcal{F} \) to a set of models, and let

\[
\mathcal{F}(\text{Min}_{RC}(K)) = \{ \mathcal{N} \mid \mathcal{N} = \mathcal{F}(\mathcal{M}) \text{ for all } \mathcal{M} \in \text{Min}_{RC}(K) \}.
\]

\( \mathcal{F}(\text{Min}_{RC}(K)) \) is a set of preferential models of \( K \) that we call MP-models of \( K \).

**Definition 14 (MP-models of \( K \)).** Given a knowledge base \( K \), an MP-model of \( K \) is any preferential model \( \mathcal{N} \in \mathcal{F}(\text{Min}_{RC}(K)) \).
We prove that MP-models provide a semantic characterization of the MP-closure.

**Theorem 2 (characterization result for the MP-closure).** Given a satisfiable knowledge base $K$, a conditional $A \vdash C$ is true in all the MP-models of $K$ if and only if $A \vdash C$ belongs to the MP-closure of $K$.

**Proof.** ($\Rightarrow$) By contraposition, let us assume that $A \vdash C$ does not belong to the MP-closure of $K$. We show that there is a model $\mathcal{N}$ in $\mathcal{F}(\text{Min}_{RC}(K))$ such that $A \vdash C$ is not satisfied in $\mathcal{N}$.

If $A \vdash C$ does not belong to the MP-closure of $K$, there must be some basis $D$ for $A$ in $K$ such that $D \cup \{A\} \nvdash C$. Therefore, there is some propositional interpretation satisfying $\bar{D} \land A \land \lnot C$.

Let $\mathcal{M} = \langle W, <, v \rangle$ be any minimal canonical ranked model in $\text{Min}_{RC}(K)$ (whose existence, for a satisfiable knowledge base $K$, was proved in [19]). There must be a world $w \in W$ such that $\mathcal{M}, w \models \bar{D} \land A \land \lnot C$.

Let $\mathcal{N} = \mathcal{F}(\mathcal{M})$, i.e., $\mathcal{N}$ is an MP-model of $K$. By construction, $\mathcal{N} = \langle W, <', v \rangle$ and $x <' y$ iff $V(x) <^\text{MP} V(y)$. We show that $\mathcal{N}$ falsifies $A \vdash C$.

As $W$ and $v$ in $\mathcal{N}$ are the same as those in $\mathcal{M}$, it must be that $\mathcal{N}, w \models D \land A \land \lnot C$. We show that $w$ is a minimal world satisfying $A$ in $\mathcal{N}$, i.e., that $w \in \text{Min}^\text{MP}_\mathcal{N}(A)$. As $w$ falsifies $C$, this is enough to show that $\mathcal{N}$ falsifies $A \vdash C$.

By absurd, if $w$ were not in $\text{Min}^\text{MP}_\mathcal{N}(A)$, there would be an element $z \in \text{Min}^\text{MP}_\mathcal{N}(A)$, such that $z <' w$. We show that this leads to a contradiction with the hypothesis that $D$ is a basis for $A$ in $K$.

Let $B$ be the set of defaults $A' \vdash C'$ in $K$ that are not violated in $z$, i.e., $B = K \setminus V(z)$. As $z <' w$, by definition of $<'$, $V(z) <^\text{MP} V(w)$, that is, there is some $h$ such that $V_h(z) \subset V_h(w)$ and $V_j(z) = V_j(w)$ for all $j$ such that $k \geq j > h$ (and the defaults with rank $\infty$ violated in $z$ and $w$ must be the same). Hence, there is some $E \vdash F \in V_h(w) \setminus V_h(z)$. $E \vdash F$ is a conditional with rank $h$ which is violated by $w$ and not by $z$). Clearly, $E \vdash F \in B$. Instead, $E \vdash F \notin D$, as default $E \vdash F$ is violated in $w$ while, by construction, all the defaults in $D$ are satisfied in $w (\mathcal{N}, w \models \bar{D})$.

To show that $D <^\text{MP} B$, we show that all defaults $G \vdash G'$ with rank $l \geq h$ that belongs to $D$ also belongs to $B$. Let $G \vdash G' \in D$, then $G \rightarrow G' \in \bar{D}$ and hence $\mathcal{N}, w \models G \rightarrow G'$. Thus $G \vdash G' \notin V(w)$. But, then, $G \vdash G' \notin V(z)$, as we know that $V_h(z) \subset V_h(w)$ and, for all $j$ such that $j > h$, $V_j(z) = V_j(w)$. Therefore, by definition of $B$, $G \vdash G' \in B$. Therefore, $D <^\text{MP} B$.

This contradicts the hypothesis. In fact, as $D$ is a basis for $A$ in $K$, there cannot be a set of defaults $B$ such that $B \cup \{A\}$ is consistent and $D <^\text{MP} B$. Therefore, we can conclude that $w \in \text{Min}^\text{MP}_\mathcal{N}(A)$ and that the conditional $A \vdash C$ is false in $\mathcal{N}$.

($\Leftarrow$) By contraposition, let us assume that there is an MP-model $\mathcal{N}$ in $\mathcal{F}(\text{Min}_{RC}(K))$ such that $A \vdash C$ is false in $\mathcal{N}$. Let $\mathcal{N} = \mathcal{F}(\mathcal{M})$ for some $\mathcal{M} = \langle W, <, v \rangle \in \text{Min}_{RC}(K)$. We prove that $A \vdash C$ does not belong to the MP-closure of $K$.

If $A \vdash C$ is false in $\mathcal{N}$, then there is a world $x \in \text{Min}^\text{MP}_\mathcal{N}(A)$ such that $\mathcal{N}, x \models \lnot C$. Let $D$ be the set of conditionals $F \vdash E$ in $K$ which are not violated in $x (D = K \setminus V(w))$, i.e., the conditionals $F \vdash E$ such that $\mathcal{N}, x \models \lnot F \lor E$. We show that $D$ is a basis for $A$ in $K$ and that $D \cup \{A\} \nvdash C$. As a consequence, $A \vdash C$ does not belong to the MP-closure of $K$. 

Let us prove that $D$ is a basis for $A$. First, $\tilde{D} \cup \{A\}$ is consistent. In fact, by construction, $\mathcal{N}, x \models \neg F \lor E$, for all $F \models E$ in $D$. Hence, $\mathcal{N}, x \models D$. Also, $\mathcal{N}, x \models A$, as $x \in \min_{\mathcal{N}}(A)$. Therefore, $\mathcal{N}, x \models \tilde{D} \cup \{A\}$.

We prove that $D$ is maximal w.r.t. $\prec_{\text{MP}}$ among the sets of defaults $B \subseteq K$ such that $\tilde{B} \cup \{A\}$ is consistent. By absurd, suppose that there is a set $B$ of defaults in $K$ such that $\mathcal{N}, x \models D \cup \{A\}$ and $D \prec_{\text{MP}} B$. We show that this leads to a contradiction.

As $\tilde{B} \cup \{A\}$ is consistent, there must be a propositional interpretation satisfying $\tilde{B} \cup \{A\}$. As the model $\mathcal{M} \in \min_{\text{RC}}(K)$ is a canonical model of $K$, there must be a world $w \in \mathcal{W}$ such that $\mathcal{M}, w \models \tilde{B} \cup \{A\}$. The same for $\mathcal{N}$, i.e., $\mathcal{N}, w \models B \cup \{A\}$.

Let us consider the tuples $(D_\infty, D_k, \ldots, D_1, D_0)_D$ and $(B_\infty, B_k, \ldots, B_1, B_0)_B$ associated with the sets $D$ and $B$, respectively. As $D \prec_{\text{MP}} B$, there is some $h$ such that $D_h \subset B_h$ and $D_j = B_j$ for all $j$ such that $k \geq j > h$.

Let $F \models G \in B_h \setminus D_h$.

By construction, $V(x) = K \setminus D$ and, hence, $F \models G \in V_h(x)$. Also, $F \models G \not\in V_h(w)$, as $F \models G \in B$ and $\mathcal{N}, w \models B$. Therefore, $F \models G \in V_h(x) \setminus V_h(w)$.

We show that, for all $j \geq h$, if $V_h(w) \subset V_h(x)$, from this, together with $F \models G \in V_h(x) \setminus V_h(w)$, it follows that $V_h(w) \subset V_h(x)$, and hence that $V(w) \prec_{\text{MP}} V(x)$.

To show that, for all $j \geq h$, if $V_j(w) \subset V_j(x)$, let $E \models E' \subset B_j$. As, for all $j \geq h$, $D_j \subset B_j$, $E \models E' \notin D_j$ (and $E \models E' \notin D$). Hence, $E \models E' \subset V(x)$ and, specifically, $E \models E' \subset V_j(x)$ (as $E \models E'$ has rank $j$).

Then we can conclude that $V(w) \prec_{\text{MP}} V(x)$ and, by definition of $\prec$, that $w \models x$. As $\mathcal{N}, w \models A$, this contradicts the hypothesis that $x$ is minimal among the worlds satisfying $A$ in $\mathcal{N}$, i.e. $x \in \min_{\mathcal{N}}(A)$. The assumption that $D$ is not maximal w.r.t. $\prec_{\text{MP}}$ leads to a contradiction. Thus $D$ is a basis for $A$ in $K$.

To show that $\tilde{D} \cup \{A\} \not\models C$, remember that $\mathcal{N}, x \models \neg C$ and that $\mathcal{N}, x \models \tilde{D} \cup \{A\}$. Therefore, the propositional valuation $v(x)$ satisfies $\tilde{D} \cup \{A\}$ but falsifies $C$.

We have proved that there is a basis $D$ for $A$ in $K$ such that $\tilde{D} \cup \{A\} \not\models C$. It follows that $A \models C$ does not belong to the MP-closure of $K$.

We conclude this section showing that the functor $\mathcal{F}$ in Definition 13 [14] can be reformulated by making explicit the dependency of the preference relation $\prec'$ in $\mathcal{N}$ from the preference relation $\prec$ in $\mathcal{M}$. Observe that, by Proposition 13 in [19], for a model $\mathcal{M} \in \min_{\text{RC}}(K)$, the ranking function $k_\mathcal{M}$ associated with $M$ is such that, for all formulas $A$ in the language of $K$, $k_\mathcal{M}(A) = \text{rank}(A)$, where $\text{rank}(A)$ is the rank of $A$ in the rational closure of $K$. We can therefore give an alternative equivalent definition of the functor $\mathcal{F}$ above replacing $\prec_{\text{MP}}$ with the preference relation $\prec_{k_\mathcal{M}}$ defined below, which makes the dependency of $\prec'$ on $k_\mathcal{M}$ (and thus on $\prec$) explicit.

**Definition 15 (k_\mathcal{M}-seriousness ordering).** Given a finite ranked model $\mathcal{M}$ of $K$ and a set of defaults $D \subseteq K$, we let $k$ be the greatest rank of a world in the finite model $\mathcal{M}$, and we let the tuple associated with $D$ in $\mathcal{M}$ be the following tuple of subsets of $D$

$\langle D_\infty, D_k, \ldots, D_1, D_0 \rangle^\mathcal{M}_D$,

where $D_\infty$ is the set of defaults $A \models C$ in $D$ such that $A$ has no rank in the model $\mathcal{M}$, and, for all finite $i$, $D_i$ is the set of defaults $A \models C$ in $D$ such that $k_\mathcal{M}(A) = i$. 
Given two sets of defaults $D$ and $B$, $D$ is less serious than $B$ w.r.t. the $k_M$-seriousness ordering, written $D \prec k_M B$, if and only if

$$\langle D_\infty, D_k, \ldots, D_1, D_0 \rangle_D^M \ll \langle B_\infty, B_k, \ldots, B_1, B_0 \rangle_B^M,$$

where, in the tuple $\langle D_\infty, D_k, \ldots, D_1, D_0 \rangle_D^M$ and $\langle B_\infty, B_k, \ldots, B_1, B_0 \rangle_B^M$ are, respectively, the tuples associated with $D$ and with $B$ in the model $M$.

Observe that the definition above uses the same lexicographic order $\ll$ on tuples of sets of defaults used in the definition of the MP-seriousness ordering $\prec_{MP}$, which is inductively defined just before Definition 10.

We can then reformulate the functor $F$ as follows:

**Proposition 5.** Given a ranked interpretation $M = (W, <, v) \in \text{Min}_{RC}(K)$ and a preferential interpretation $N = (W, <', v)$, if $N = F(M)$, then

$$x <' y \iff V(x) \prec_{k_M} V(y)$$

**Proof.** The proof is trivial since, as a consequence of Proposition 13 in [19], the rank of a conditional $A \cni B$ in the rational closure of $K$ corresponds to the rank of $A$ in any minimal canonical model of $K$ (and hence in $M$). In particular, when $\text{rank}(A \cni B) = \infty$, $A$ has no rank in $M$. For any set of defaults $D$ (and in particular for $V(x)$ and for $V(y)$), the tuple $\langle D_\infty, D_k, \ldots, D_1, D_0 \rangle_D$ associated with $D$ in the definition of MP-seriousness ordering (Definition 10) exactly corresponds to the tuple $\langle D_\infty, D_k, \ldots, D_1, D_0 \rangle_D^M$ associated with $D$ in $M$ according to the Definition 15 of $k_M$-seriousness ordering, when $M$ is a minimal canonical model of $K$. Therefore, the $k_M$-seriousness ordering $\prec_{k_M}$ and the MP-seriousness ordering $\prec_{MP}$ coincide, when $M \in \text{Min}_{RC}(K)$. Under this condition, $V(x) \prec_{k_M} V(y) \iff V(x) \prec_{MP} V(y)$, and the thesis follows. \qed

This last reformulation of the functor $F$, besides making the dependency of $<'$ on $<$ more evident, also allows to clarify the relationships between MP-models and the notion of bi-preference interpretation in [14]. A quadruple $(W, <, <', v)$, where $<$ is a ranked model and $<'$ is defined by the *if part* of Condition (3), corresponds to a BP-interpretation. Minimization was then used therein to define the minimal BP-models of $K$ characterizing the MP-closure. Here, instead, we have directly constructed the MP-models of a knowledge base $K$ by applying the functor $F$ to the minimal canonical models of the rational closure of $K$ (see Definition 13), without requiring a further minimization step.

### 4.3 Some properties and relation with the lexicographic closure

Observe that there is a strong correspondence among the ordering on the models defined by Condition (2) in Definition 13 and the semantics of the lexicographic closure given by Lehmann [24], where the seriousness ordering $<$ based on the lexicographic order on the tuples of numbers is used to order the propositional models as follows:

$$m < m' \iff V(m) < V(m)$$

(4)
“This modular ordering on models defines a modular preferential model, that, in turn defines a consequence relation”, the lexicographic closure of the knowledge base. It was proven by Lehmann (see [24], Theorem 2) that the relation \( \prec \) on propositional models is finer than \( \ll \), a modular relation defining a model of the rational closure. In Proposition 1 in Section 4.1 we have proved that \( \prec^M \) is coarser than \( \prec \). As a consequence, all the conditionals belonging to the MP-closure of a knowledge base \( K \) also belong to the lexicographic closure of \( K \).

Let \( \mathcal{L}C_K \) be the set of conditionals belonging to the lexicographic closure of \( K \), and \( \mathcal{R}C_K \) be the set of conditionals belonging to the rational closure of \( K \). Remember that \( \mathcal{M}P_K \) is the set of conditionals belonging to the MP-closure of \( K \). We have already proven that \( \mathcal{M}P_K \subseteq \mathcal{L}C_K \). Example 4 shows a knowledge base \( K \) for which the inclusion is strict, i.e., \( \mathcal{M}P_K \subset \mathcal{L}C_K \). In fact, the conditional \( 1 \) belongs to \( \mathcal{L}C_K \) but not to \( \mathcal{L}C_K \). Therefore the converse inclusion (\( \mathcal{L}C_K \subseteq \mathcal{M}P_K \)) does not hold.

We will now exploit the correspondence result in Theorem 3 to show that the conditionals belonging to the rational closure of a knowledge base \( K \) also belong to the MP-closure of \( K \), i.e. that \( \mathcal{R}C_K \subseteq \mathcal{M}P_K \).

**Proposition 6.** Given a knowledge base \( K \), if \( A \vdash C \) is in the rational closure of \( K \), than \( A \vdash C \) is in the MP-closure of \( K \).

**Proof.** Let \( A \vdash C \) be in the rational closure of \( K \), then, by the correspondence Theorem 1 \( \mathcal{M} \models A \vdash C \), for all \( \mathcal{M} \) in \( \text{Min}_{RC}(K) \). To show that \( A \vdash C \) is in the MP-closure of \( K \), we prove that for all \( \mathcal{N} \in \mathcal{F}(\text{Min}_{RC}(K)) \), \( \mathcal{N} \models A \vdash C \).

Let us consider any \( \mathcal{N} \in \mathcal{F}(\text{Min}_{RC}(K)) \). It must be \( \mathcal{N} = \mathcal{F}(\mathcal{M}) \) for some \( \mathcal{M} = \langle W, \prec, v \rangle \) in \( \text{Min}_{RC}(K) \). From the hypothesis we know that that \( \mathcal{M} \models A \vdash C \) and, hence, for all \( w \in \text{Min}^<_{\prec}(A) \), \( \mathcal{M}, w \models C \).

We prove that \( \mathcal{N} \models A \vdash C \), i.e., that for all \( w' \in \text{Min}^<_{\prec}(A) \), \( \mathcal{N}, w' \models C \). From \( w' \in \text{Min}^<_{\prec}(A) \), we know that \( w' \in W \) and that \( \mathcal{N}, w' \models A \). But then \( \mathcal{M}, w' \models A \), as \( W \) and \( v \) are the same in \( \mathcal{M} \) and \( \mathcal{N} \). Furthermore, the set of the worlds satisfying \( A \) in \( \mathcal{M} \) is the same as the set of the worlds satisfying \( A \) in \( \mathcal{N} \). By Proposition 2 \( \ll \subseteq < \) and, hence, \( \text{Min}^<_{\prec}(A) \subseteq \text{Min}^<_<(A) \). Thus, \( w' \in \text{Min}^<_<(A) \). As \( \mathcal{M} \) satisfies \( A \vdash C \), then \( \mathcal{M}, w' \models C \). Again, as \( W \) and \( v \) are the same in \( \mathcal{M} \) and \( \mathcal{N} \), \( \mathcal{N}, w' \models C \). We can then conclude that \( \mathcal{N} \models A \vdash C \). As this is true for all \( \mathcal{N} \in \mathcal{F}(\text{Min}_{RC}(K)) \), this proves the thesis.

The following corollary follows:

**Corollary 3.** \( \mathcal{R}C_K \subseteq \mathcal{M}P_K \subseteq \mathcal{L}C_K \).

To show that, for some knowledge base \( K \), the strict inclusion \( \mathcal{R}C_K \subset \mathcal{M}P_K \) holds, it is enough to observe that, in Example 1, conditional \( \text{Employee} \land \text{Student} \vdash \text{Young} \) is in the MP-closure of \( K \), but not in the rational closure of \( K \).

Observe that, Lehmann in his semantics considers a single model including the propositional interpretations \( m \), ordered according to the preference relation \( \prec \), where, for \( m \) and \( m' \) two propositional interpretations, \( m \prec m' \) is defined by condition (4). Instead, in our semantics we consider a set of models \( \mathcal{F}(\text{Min}_{RC}(K)) \), where the models \( \mathcal{N} \in \mathcal{F}(\text{Min}_{RC}(K)) \) may differ w.r.t. the set of worlds \( W \) and the valuation function.
The property that the preference relation $\prec$ in $\mathcal{F}(\text{Min}_{RC}(K))$, however, is canonical, and it must contain at least one world for each propositional truth assignment (over the language $\mathcal{L}_{K,Q}$) which is compatible with $K$. We will see that each model $\mathcal{N}$ in $\mathcal{F}(\text{Min}_{RC}(K))$ satisfies exactly the same conditionals. This is a consequence of the fact that the relative ordering of two worlds $x, y \in \mathcal{W}$ in any interpretation $\mathcal{N} \in \mathcal{F}(\text{Min}_{RC}(K))$ only depends on the set of defaults violated in $x$ and $y$. In turn, the defaults violated in $x$ and in $y$ only depend on the valuation of the formulas $B \in \mathcal{L}_{K,Q}$ in $x$ and $y$.

The following lemma shows that the preference relation $\prec'$ among worlds in a model $\mathcal{N}$ is determined by the valuation of the propositions in the language $\mathcal{L}_{K,Q}$ of the knowledge base $K$ and of a query $Q$ (a conditional). The lemma will be used to prove the next proposition.

**Lemma 1.** Given two models $\mathcal{N}' = (\mathcal{W}', <', v')$ and $\mathcal{N}'' = (\mathcal{W}'', <'', v'')$, and two worlds $x \in \mathcal{W}'$ and $y \in \mathcal{W}''$, we say that the valuations $v'(x)$ and $v''(y)$ coincide over the language $\mathcal{L}_{K,Q}$, when, for all $B \in \mathcal{L}_{K,Q}$, $\mathcal{N}'_x \models B$ iff $\mathcal{N}''_y \models B$. That is, the propositional formulas of the language $\mathcal{L}_{K,Q}$ have the same truth value in $x$ and in $y$.

**Proof.** Let $x', y' \in \mathcal{W}'$ and $x'', y'' \in \mathcal{W}''$ and assume that the valuations $v'(x')$ and $v''(x'')$ coincide over the language $\mathcal{L}_{K,Q}$, and that the $v'(y')$ and $v''(y'')$ coincide over the language $\mathcal{L}_{K,Q}$. Hence, for all $B \in \mathcal{L}_{K,Q}$,

- $\mathcal{N}'_x \models B$ iff $\mathcal{N}''_y \models B$, and,
- $\mathcal{N}'_y \models B$ iff $\mathcal{N}''_y \models B$.

Then the defaults of $K$ which are violated in $x'$ and in $x''$ are the same ones, i.e. $V(x') = V(x'')$ and, similarly, $V(y') = V(y'')$. As $\mathcal{N} \in \mathcal{F}(\text{Min}_{RC}(K))$, by construction, condition (2) holds and it must be:

$$x' \prec y' \iff V(x') \prec_{MP} V(y').$$

As $\mathcal{N}'' \in \mathcal{F}(\text{Min}_{RC}(K))$, again by condition (2),

$$x'' \prec'' y'' \iff V(x'') \prec_{MP} V(y'').$$

From the two equivalences above, given that $V(x') = V(x'')$ and $V(y') = V(y'')$, it follows that

$$x' \prec y' \iff x'' \prec'' y''.$$

The property that the preference relation $\prec'$ among worlds in each $\mathcal{N} \in \mathcal{F}(\text{Min}_{RC}(K))$ is only determined by the valuation of the propositional formulas of the language $\mathcal{L}_{K,Q}$ at the worlds, can be used to prove that all the models in $\mathcal{F}(\text{Min}_{RC}(K))$ satisfy the same conditional formulas.
Proposition 7. Given two models \( N' = \langle W', <', \nu' \rangle \) and \( N'' = \langle W'', <'', \nu'' \rangle \) such that \( N', N'' \in F(\text{Min}_{RC}(K)) \) and a query \( Q = A \rightarrow C \). Then,

\[
N' \models A \rightarrow C \Rightarrow N'' \models A \rightarrow C.
\]

Proof. (\( \Rightarrow \)) Assume \( N' \models A \rightarrow C \). We want to show that \( N'' \models A \rightarrow C \), i.e., for all \( w \in W'' \), if \( w \in \text{Min}_{RC}(A) \) then \( N'', w \models C \). Suppose, by absurd, that, for some \( w \in W'' \), \( w \in \text{Min}_{RC}(A) \) but \( N, w \not\models C \). Then there is truth assignment \( v_0 : \text{ATM}_{K,Q} \rightarrow \{\text{true}, \text{false}\} \) such that \( N', w \models B \) iff \( v_0(B) = \text{true} \), for all formulas \( B \in L_{K,Q} \). Clearly, \( v_0 \) is compatible with \( K \), as \( N'' \) is a model of \( K \), by Proposition 4.

As \( N' \) is a canonical model of \( K \) (\( N' \in F(M) \) for some canonical model \( M \in \text{Min}_{RC}(K) \)), there must be a world \( w' \in W' \) such that \( N', w' \models B \) iff \( v_0(B) = \text{true} \), for all formulas \( B \in L_{K,Q} \). In particular, \( N', w' \models A \) and \( N', w' \not\models C \). We show that \( w' \in \text{Min}_{RC}(A) \).

If \( w' \not\in \text{Min}_{RC}(A) \), there must be a \( z' \in W'' \) such that \( z' <' w' \) and \( N', z' \models A \). This leads to a contradiction with the assumption that \( w \in \text{Min}_{RC}(A) \). As \( N'' \) is equal to \( F(M'') \) for some minimal canonical model \( M'' \) of \( K \), there must be an element \( z'' \in W'' \), whose valuation corresponds to the same truth assignment over \( L_{K,Q} \) of \( z' \) in \( N' \). In particular, \( z' \) satisfies \( A \) and, hence, \( z'' \) satisfies \( A \) in \( N'' \). By Lemma 1, as \( \nu''(w) \) and \( \nu'(w') \) coincide over the language \( L_{K,Q} \), and \( \nu''(z'') \) and \( \nu'(z') \) coincide over the language \( L_{K,Q} \), then

\[
z' <' w' \iff z'' <'' w.
\]

Then, it must be \( z'' <'' w \), which contradicts the assumption that \( w \in \text{Min}_{RC}(A) \). Hence, it must be that \( w \in \text{min}(A) \). As \( N', w \not\models C \), we can conclude that \( N' \not\models A \rightarrow C \), which contradicts the hypothesis.

As a consequence of this result, all the models in \( F(\text{Min}_{RC}(K)) \) must satisfy the same conditionals. Hence, for a satisfiable knowledge base \( K \) we can take any (arbitrarily chosen) model \( \mathcal{N}_{MP} \) in \( F(\text{Min}_{RC}(K)) \) as the semantic characterization of the MP-closure, thus coming up with a semantics quite similar to the model-theoretic semantics by Lehmann in [24]. In particular, by the characterization Theorem 2,

\[
\mathcal{N}_{MP} \models A \rightarrow C \iff A \vdash C \in \mathcal{M}P_K
\]

Then, the MP-closure is the consequence relation defined by a preferential model, \( \mathcal{N}_{MP} \). As a consequence of the property, proved by Lehmann and Magidor [23], that any preferential model defines a preferential consequence relation, it follows that the MP-closure is a preferential consequence relation.

Corollary 4. The MP-closure is a preferential consequence relation.

We have already seen in Section 4.1 that the MP-closure, instead, is not a rational consequence relation as it violates the property of Rational Monotonicity.
5 Further Issues

In this section we discuss some further issues. Recall that in Section 4.1 we have shown that the MP-closure does not define a rational consequence relation. In Section 5.1 we will show that rationality can be recovered, and we define a semantic construction which, starting from the MP-closure, defines a rational consequence relation, called $\mathcal{M} \mathcal{P}_R^K$. We show that $\mathcal{M} \mathcal{P}_R^K$ is incomparable with $\mathcal{L} \mathcal{C}_K$ the lexicographic closure of $K$. Then, in Section 5.2 we use this semantic construction to establish some relationships with the multi-preference semantics proposed by Gliozzi in [20,21]. In Section 5.3 we compare the MP-closure with the Relevant Closure proposed by Casini et al. [6] for description logics, and we show that the Minimal Relevant Closure (when considered in the propositional calculus) is weaker than the MP-closure.

5.1 Can rationality be recovered?

We have seen that the MP-closure does not define a rational consequence relation and that the models characterizing the MP-closure by Theorem 2 are preferential models, in which the preference relation is not necessarily modular. In particular, the MP-closure does not satisfy the property of rational monotonicity (see Example 5). While the adequacy of the property of Rational Monotonicity might be subject of discussion, in this section we show that a rational consequence relation which is a superset of the MP-closure can be defined for any finite knowledge base $K$ and, in particular, there is a simple way to define a rational consequence relation starting from the semantic characterization of the MP-closure. We will also see that such a rational consequence relation does not coincide with the lexicographic closure of the knowledge base and is incomparable with it.

Given any MP-model of $K$, i.e. a preferential model $\mathcal{N} = \langle \mathcal{W}, <', v \rangle$ in $\mathcal{F}(\text{Min}_{RC}(K))$, we want to minimally extend the preference relation $<'$ to a ranked preference relation $<^R$, so to define a ranked model $\mathcal{N}^R = \langle \mathcal{W}, <^R, v \rangle$ of $K$. Definition 2 in Section 2 provides a simple way to extend the preference relation $<'$ in a preferential interpretation $\mathcal{N}$ to a modular one, by assigning a rank to each world, thus defining a ranked model $\mathcal{N}^R$.

Definition 16. Given $\mathcal{N} = \langle \mathcal{W}, <', v \rangle$ a preferential interpretation, we let $\mathcal{N}^R = \langle \mathcal{W}, <^R, v \rangle$ be the ranked interpretation where: the rank $k_{\mathcal{N}^R}(w)$ of a world $w \in \mathcal{W}$ in $\mathcal{N}^R$ is the length of a longest path $w_0 <' \ldots <' w$ from $w$ to a minimal world $w_0$ (i.e., a world for which there is no $z \in \mathcal{W}$ such that $z <' w_0$). In particular, each minimal world $w_0$ is given rank $k_{\mathcal{N}^R}(w_0) = 0$.

Observe that an iterative construction to “transform a preferential model $\mathcal{W}$ into a ranked model $\mathcal{W}'$ letting all the states of $\mathcal{W}$ sink as low as they can respecting the order of $\mathcal{W}$” [23] was already considered by Lehmann and Magidor in their model theoretic description of the rational closure. Applying the same kind of transformation to a (finite) preferential model $\mathcal{N}$ one can give an alternative (but equivalent) inductive definition of the rank $k_{\mathcal{N}^R}(w)$ of a world $w \in \mathcal{W}$ in $\mathcal{N}^R$.
Given a preferential interpretation \( \mathcal{N} = (\mathcal{W}, <', v) \), let \( U_i \) be the set of the worlds in \( \mathcal{W} \) with rank \( i \), defined inductively as follows:

\[
U_0 = \{ w \mid w \in \mathcal{W} \text{ and there is no } z \in \mathcal{W}, \text{ such that } z <' w \} \\
U_i = \{ w \mid w \in \mathcal{W} - (U_1 \cup \ldots U_{i-1}) \text{ and there is no } z \in \mathcal{W} - (U_1 \cup \ldots U_{i-1}) \text{ such that } z <' w \} 
\]

One can easily prove, by induction on the rank \( i \), that:

**Proposition 8.** Given a preferential interpretation \( \mathcal{N} = (\mathcal{W}, <', v) \), the ranked interpretation \( \mathcal{N}^R = (\mathcal{W}, <^R, v) \) can be defined by letting, for all \( w \in \mathcal{W} \),

\[
k_{\mathcal{N}^R}(w) = i \text{ if } w \in U_i.
\]

We show that, when \( \mathcal{N} \) is an MP-model of \( K \), \( \mathcal{N}^R \) is a model of \( K \) as well. We need the following proposition.

**Proposition 9.** Given a preferential interpretation \( \mathcal{N} = (\mathcal{W}, <', v) \), and a ranked interpretation \( \mathcal{N}^R = (\mathcal{W}, <^R, v) \), defined as above, it holds that \( <' \subseteq <^R \).

**Proof.** The proof is immediate. Just observe that, if \( x <' y \) holds for some \( x, y \in \mathcal{W} \) in \( \mathcal{N} \), the length of the longest path \( w_0 <' \ldots <' y \) from \( y \) to a minimal world \( w_0' \) must be longer than the length of the longest path \( w_0' <' \ldots <' x \) from \( x \) to a minimal world \( w_0'' \). Hence, \( k_{\mathcal{N}^R}(x) < k_{\mathcal{N}^R}(y) \), and then \( x <^R y \).

From Proposition 9 and Proposition 3, it is easy to prove that, when \( \mathcal{N} \) is an MP-model of a knowledge base \( K \), \( \mathcal{N}^R \) is a model of \( K \).

**Proposition 10.** Let \( \mathcal{N} \) be an MP-model of \( K \) and let \( \mathcal{N}^R \) be the ranked interpretation as defined above. \( \mathcal{N}^R \) satisfies a superset of the conditionals satisfied by \( \mathcal{N} \) and is a model of \( K \).

**Proof.** Let \( \mathcal{N} = (\mathcal{W}, <', v) \) be an MP-model of \( K \). As both \( \mathcal{N} \) and \( \mathcal{N}^R \) are preferential interpretations and, by Proposition 9 \( <' \subseteq <^R \), we can conclude, as a consequence of Proposition 3, that all the conditionals \( A \models C \) satisfied by \( \mathcal{N} \) are satisfied by \( \mathcal{N}^R \) as well. In particular, as \( \mathcal{N} \) is a model of \( K \), \( \mathcal{N} \) satisfies all the conditionals in \( K \) and \( \mathcal{N}^R \) too.

We will now consider the rational consequence relation containing all the conditionals satisfied in \( \mathcal{N}^R \). Let us define a new functor \( \mathcal{F}^R \) as follows:

\[
\mathcal{F}^R(\mathcal{M}) = \{ \mathcal{N}^R \mid \mathcal{N} = \mathcal{F}(\mathcal{M}) \}.
\]

In particular, given a model \( \mathcal{M} \in \text{Min}_{RC}(K) \), the functor \( \mathcal{F}^R \) first applies the functor \( \mathcal{F} \) to \( \mathcal{M} \) and then transforms the resulting preferential model \( \mathcal{N} \) into a ranked model \( \mathcal{N}^R \).

Let us consider a satisfiable knowledge base \( K \). We have shown in Propositions 7 that all models \( \mathcal{N} \) in \( \mathcal{F}(\text{Min}_{RC}(K)) \) satisfy the same conditionals and, essentially, define the same preference relation among the worlds (which only depends on the truth value assignments to the formulas in \( L_{K,Q} \)). As a consequence, a single model
We show that we arrive to a contradiction with the assumption that

\[ \mathcal{N}_{MP} \models A \vdash C \text{ iff } A \vdash C \in \mathcal{MP}_K \]

Let us now consider the ranked model \( \mathcal{N}^{R}_{MP} \in \mathcal{F}^{R}(\text{Min}_{RC}(K)) \), obtained from \( \mathcal{N}_{MP} \) according to the construction above. As a consequence of the result proven by Lehmann and Magidor [23], that the consequence relation defined by any ranked model is rational, the model \( \mathcal{N}^{R}_{MP} \) defines a rational consequence relation, that we call \( \mathcal{MP}^{R}_K \), where

\[ \mathcal{MP}^{R}_K = \{ A \vdash C \ | \ \mathcal{N}^{R}_{MP} \models A \vdash C \} \]

The definition of the rational consequence relation \( \mathcal{MP}^{R}_K \) does not depend on the choice of the model \( \mathcal{N}_{MP} \) in \( \mathcal{F}(\text{Min}_{RC}(K)) \), as shown by the following proposition.

**Proposition 11.** For all \( \mathcal{N}_1, \mathcal{N}_2 \in \mathcal{F}(\text{Min}_{RC}(K)) \), \( \mathcal{N}_1^{R} \) and \( \mathcal{N}_2^{R} \) satisfy the same conditionals.

**Proof.** We prove that, for any conditional \( A \vdash C \in K \), if \( \mathcal{N}_1^{R} \models A \vdash C \), then \( \mathcal{N}_2^{R} \models A \vdash C \). As \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) are any two models in \( \mathcal{F}(\text{Min}_{RC}(K)) \), the thesis follows.

Let \( \mathcal{N}_1 = (\mathcal{W}_1, <_1, v_1) \) and \( \mathcal{N}_2 = (\mathcal{W}_2, <_2, v_2) \) and assume that \( \mathcal{N}_1^{R} \models A \vdash C \). We show that \( \mathcal{N}_2^{R} \models A \vdash C \), i.e., for all \( w \in \mathcal{W}_2 \), if \( w \in \text{Min}_{<_2}^{R} (A) \), then \( \mathcal{N}_2^{R} \models w \vdash C \).

Suppose, by absurd, that, for some \( w \in \mathcal{W}_2 \), \( w \in \text{Min}_{<_2}^{R} (A) \) and \( \mathcal{N}_2^{R} \models w \not\vdash C \). Then there is a truth assignment \( v_0 : \text{ATM}_{K,Q} \rightarrow \{ \text{true, false} \} \) such that \( \mathcal{N}_2^{R} \models w \models B \) iff \( v_0(B) = \text{true} \), for all formulas \( B \in \mathcal{L}_{K,Q} \). Clearly, \( v_0 \) is compatible with \( K \), as \( \mathcal{N}_2^{R} \) is a model of \( K \), by Proposition [10].

As \( \mathcal{N}_2^{R} \) is a canonical model of \( K \) (\( \mathcal{N}_2 \in \mathcal{F}(\mathcal{M}) \) for some canonical model \( \mathcal{M} \in \text{Min}_{RC}(K) \)), \( \mathcal{N}_2^{R} \) is a canonical model of \( K \) as well, and there must be a world \( w' \in \mathcal{W}_1 \) such that \( \mathcal{N}_1^{R} \models w' \models B \) iff \( v_0(B) = \text{true} \), for all formulas \( B \in \mathcal{L}_{K,Q} \). In particular, \( \mathcal{N}_1^{R} \models w' \models A \) and \( \mathcal{N}_2^{R} \models w' \not\models C \). We show that \( w' \in \text{Min}_{<_1}^{R} (A) \).

If \( w' \not\in \text{Min}_{<_1}^{R} (A) \), there must be a \( z' \in \mathcal{W}_1 \) such that \( z' <_1 w' \) and \( \mathcal{N}_1^{R} \models z' \models A \). We show that we arrive to a contradiction with the assumption that \( w \in \text{Min}_{<_2}^{R} (A) \).

In fact, as \( \mathcal{N}_2 \) is equal to \( \mathcal{F}(\mathcal{M}'') \) for some minimal canonical model \( \mathcal{M}' \) of \( K \), there must be an element \( z'' \in \mathcal{W}_2 \), whose valuation corresponds to the same truth assignment over \( \mathcal{L}_{K,Q} \) of \( z' \) in \( \mathcal{N}_2^{R} \). In particular, as \( z' \) satisfies \( A \) in \( \mathcal{N}_1^{R} \), \( z'' \) satisfies \( A \) in \( \mathcal{N}_2^{R} \). By Lemma [1] as \( v_2(w) \) and \( v_1(w') \) coincide over the language \( \mathcal{L}_{K,Q} \), and \( v_2(z'') \) and \( v_1(z') \) coincide over the language \( \mathcal{L}_{K,Q} \), then

\[ z' <_1 w' \text{ iff } z'' <_2 w. \]

Then, it must be \( z'' <_2 w \), which contradicts the assumption that \( w \in \text{Min}_{<_2}^{R} (A) \).

Hence, it must be that \( w' \in \text{Min}_{<_1}^{R} (A) \). As \( \mathcal{N}_2^{R} \models w' \not\models C \), we can conclude that \( \mathcal{N}_1^{R} \models w' \not\models A \vdash C \), which contradicts the hypothesis. \( \square \)
Next proposition proves that the rational consequence relation $\mathcal{MP}_R$ is a superset of $\mathcal{MP}_K$, the set of conditionals which belong to the MP-closure of $K$.

**Proposition 12.** For a satisfiable knowledge base $K$, $\mathcal{MP}_K \subseteq \mathcal{MP}_R$.

**Proof.** We have seen that $\mathcal{MP}_K$ is the set of the conditional satisfied by an MP-model $\mathcal{N}_{MP}$ and that, by construction, $\mathcal{MP}_R$ is the set of the conditional satisfied by the ranked model $\mathcal{N}_{MP}^R$. As, by Proposition 10, $\mathcal{N}_{MP}^R$ satisfies a superset of the conditionals satisfied by $\mathcal{N}_{MP}$, we can conclude that $\mathcal{MP}_K \subseteq \mathcal{MP}_R$. \hfill $\square$

To see that the converse inclusion $\mathcal{MP}_R \subseteq \mathcal{MP}_K$ does not hold, let us consider again Example 5 showing a knowledge base $K$ such that $\mathcal{MP}_R$ contains a conditional which is not in $\mathcal{MP}_K$.

**Example 6.** Let $K$ be the knowledge base:

1. $\text{Student} \not\vDash \text{Merry}$
2. $\text{Student} \not\vDash \text{Young}$
3. $\text{Adult} \not\vDash \text{Serious}$
4. $\text{Student} \wedge \text{Adult} \not\vDash (\neg \text{Young} \wedge \neg \text{Merry}) \vee \neg \text{Serious}$

It is possible to see that the conditional $\text{Student} \wedge \text{Adult} \not\vDash \text{Young}$ which (as we have seen in Example 5) does not belong to $\mathcal{MP}_K$, instead belongs to $\mathcal{MP}_R$. Indeed, if we consider any preferential model $\mathcal{N}_{MP} = (\mathcal{W}, <^R, v)$ in $\mathcal{F}(\text{Min}_{RC}(K))$, the ranked model $\mathcal{N}_{MP}^R = (\mathcal{W}, <^R, v)$ satisfies $\text{Student} \wedge \text{Adult} \not\vDash \text{Young}$. Let us explain why.

First, there must be a world $w \in \mathcal{W}$ whose propositional valuation is $v(w) = \{\text{Student}, \text{Adult}, \text{Merry}, \text{Young}\}$, as this valuation is compatible with $K$ and the model $\mathcal{N}_{MP}$ is a canonical model. The world $w$ satisfies all the conditionals in $K$ except conditional 3: $\text{Adult} \not\vDash \text{Serious}$. Thus the set $V(w)$ of the conditionals violated by $w$ is associated with the tuple $\langle \emptyset, \emptyset, \{3\}\rangle_{V(w)}$ (conditional 3 has rank 0). All the worlds $z \in \mathcal{W}$ such that $z <^R w$ must have the associated tuple $\langle \emptyset, \emptyset, \emptyset\rangle_{V(z)}$ that is, they violate no conditionals, and they must have rank 0 in $\mathcal{N}_{MP}^R$ (i.e., $k_{\mathcal{N}_{MP}^R}(z) = 0$).

Therefore, $w$ must have rank 1 in $\mathcal{N}_{MP}^R$ (i.e., $k_{\mathcal{N}_{MP}^R}(w) = 1$). One can see that $w$ is the unique minimal world in $\mathcal{N}_{MP}^R$ satisfying $\text{Student} \wedge \text{Adult}$. In fact, the world $x \in \mathcal{W}$ such that $v(x) = \{\text{Student}, \text{Adult}, \text{Serious}\}$ (which falsifies $\text{Young}$) has rank 2 in $\mathcal{N}_{MP}^R$. While in the preferential model $\mathcal{N}_{MP}$ both $x$ and $w$ are minimal worlds satisfying $\text{Student} \wedge \text{Adult}$ (i.e., $x, w \in \text{Min}_{\mathcal{N}_{MP}}(\text{Student} \wedge \text{Adult})$), so that $\mathcal{N}_{MP} \not\models \text{Student} \wedge \text{Adult} \not\vDash \text{Young}$, in the ranked model $\mathcal{N}_{MP}^R$ the world $x$ has rank 2 and is not minimal. Indeed $x$ violates two conditionals (conditionals 1 and 2) and it has a rank higher that the worlds falsifying just one of these two defaults. In particular, a world $y$ such that $v(y) = \{\text{Student}, \text{Adult}, \text{Merry}, \text{Serious}\}$ must be as well in $\mathcal{W}$ and $y <^R x$, as $y$ violates a (strict) subset of the conditionals violated by $x$ (conditional 2, but not 1). World $y$ has rank 1 in $\mathcal{N}_{MP}^R$. Therefore, $x$ is not minimal among the worlds satisfying $\text{Student} \wedge \text{Adult}$ in $\mathcal{N}_{MP}^R$ and it holds that $\text{Min}_{\mathcal{N}_{MP}^R}(\text{Student} \wedge \text{Adult}) = \{w\}$. Thus, $\mathcal{N}_{MP}^R \models \text{Student} \wedge \text{Adult} \not\vDash \text{Young}$.

In this example, the model $\mathcal{N}_{MP}^R$ looks similar to a model of the lexicographic closure (according to the semantics by Lehmann), as the rank of a world in $\mathcal{N}_{MP}^R$, as shown
above, may also depend on the number of defaults it satisfies. One may wonder whether the rational consequence relation $\mathcal{MP}^R_K$ coincides with the lexicographic closure of $K$, $\mathcal{LC}_K$. Next example shows that this is not the case, providing a knowledge base $K$ which falsifies both the inclusions $\mathcal{MP}^R_K \subseteq \mathcal{LC}_K$ and $\mathcal{LC}_K \subseteq \mathcal{MP}^R_K$.

**Example 7.** Let $K$ be the knowledge base:

1. $A \vdash E \land F$
2. $A \vdash E \land F \land E$
3. $A \vdash E \land F \land E \land F$
4. $C \vdash \neg E$
5. $C \vdash \neg F$

The knowledge base $K$ is satisfiable and, in the rational closure construction, all the conditions have rank 0. In the lexicographic closure, $D = \{1, 2, 3\}$ is the only basis for $A \land C$, and the conditional $A \land C \vdash E$ is in $\mathcal{LC}_K$. In the MP-closure, there are two bases for $A \land C$: $D = \{1, 2, 3\}$ and $D' = \{4, 5\}$, and neither $A \land C \vdash E$ nor $A \land C \vdash \neg E$ are in $\mathcal{MP}_K$. We show that the $\mathcal{MP}^R_K$ contains $A \land C \vdash \neg E$.

Consider an MP-model $\mathcal{N} = \langle W, <', v \rangle$ of $K$ and the model $\mathcal{N}^R$. Consistently with the fact that in the MP-closure there are two bases $D$ and $D'$ for $A \land C$, $\mathcal{N}$ contains two minimal worlds $x$ and $y$ satisfying $A \land C$, where $v(x) = \{A, C, E, F\}$ and $v(y) = \{A, C\}$, so that $x$ violates defaults 4 and 5 ($V(x) = \langle \emptyset, \{4, 5\}\rangle$) and $y$ violates defaults 1, 2 and 3 ($V(y) = \langle \emptyset, \{1, 2, 3\}\rangle$).

All worlds $z$ such that $z <' y$ do not violate any default and have rank 0 in $\mathcal{N}^R$, so that $y$ has rank 1 in $\mathcal{N}^R$. In fact, there is no propositional assignment (and no world in $\mathcal{N}$) that may violate a subset of the defaults 1, 2 or 3, but not all of them.

Concerning $x$, there are two propositional assignments, $\{C, E\}$ and $\{C, F\}$ violating a (strict) subset of the defaults violated by $x$. As $\mathcal{N}$ is canonical there are two worlds $z$ and $w$ such that $v(z) = \{C, E\}, v(w) = \{C, F\}$, $z <' x$ and $w <' x$. In fact, $z$ violates default 5 ($V(z) = \langle \emptyset, \{5\}\rangle$) and $w$ violates default 4 ($V(w) = \langle \emptyset, \{4\}\rangle$). Both $z$ and $w$ have rank 1 in $\mathcal{N}^R$, so that $x$ has rank 2.

While $x$ and $y$ are both minimal $A \land C$ worlds in the MP-model $\mathcal{N}$, instead $y <' x$ in $\mathcal{N}^R$, as $x$ has rank 2 while $y$ has rank 1. It can be seen that all the minimal worlds satisfying $A \land C$ in $\mathcal{N}^R$ have the same truth assignment as $y$, and falsify $E$ and $F$. Hence, $A \land C \vdash \neg E$ is satisfied by $\mathcal{N}^R$ and is in $\mathcal{MP}^R_K$. Instead, $A \land C \vdash E$ is not in $\mathcal{MP}^R_K$.

As $A \land C \vdash \neg E$ is not in the lexicographic closure of $K$, we can conclude that $A \land C \vdash \neg E$ is in $\mathcal{MP}^R_K - \mathcal{LC}_K$ while $A \land C \vdash E$ is in $\mathcal{LC}_K - \mathcal{MP}^R_K$.

We have seen previously that $\mathcal{RC}_K \subseteq \mathcal{MP}_K \subseteq \mathcal{LC}_K$, and that, for a satisfiable knowledge base $K$, $\mathcal{MP}_K \subseteq \mathcal{MP}^R_K$. The previous example shows that $\mathcal{MP}^R_K$ and $\mathcal{LC}_K$ are incomparable.

**Corollary 5.** There is some knowledge base $K$, such that neither $\mathcal{MP}^R_K \subseteq \mathcal{LC}_K$, nor $\mathcal{LC}_K \subseteq \mathcal{MP}^R_K$ hold.
We observe that, while the lexicographic closure of $K$ and the MP-closure have a definition in terms of maxiconsistent sets the rational extension of the MP-closure $\mathcal{M}P^K$ has just a semantic definition. We have introduced it to show that there is at least another rational consequence relations, different from the lexicographic closure and incomparable with it, which is still a superset of the rational closure.

To conclude this section we show that, among the ranked modes whose modular preference relation extends $\prec'$, $\mathcal{N}^R$ is the least one, with respect to $\prec_{FIMS}$.

**Proposition 13.** Let $\mathcal{N} = (\mathcal{W}, \prec', \nu)$ in $\mathcal{F}(\text{Min}_{\text{RC}}(K))$. For all the ranked interpretations $\mathcal{N}^\ast = (\mathcal{W}, \prec^\ast, \nu)$ such that $\prec' \subseteq \prec^\ast$ and $\mathcal{N}^\ast \neq \mathcal{N}^R$, it holds $\mathcal{N}^R <_{\text{FIMS}} \mathcal{N}^\ast$.

**Proof.** We prove that, for all $x \in \mathcal{W}$,

$$k_{\mathcal{N}^R}(x) \leq k_{\mathcal{N}^\ast}(x).$$

The proof is by induction on the rank $i$ of $x$ in $\mathcal{N}^R$. Let $k_{\mathcal{N}^R}(x) = i$.

For $i = 0$, condition (6) holds trivially, as $k_{\mathcal{N}^R}(x) \geq 0$.

For $i > 0$, $i$ is the length of the longest paths from $x$ to a minimal world $w_0$. Let $w_0 <' \ldots <' w_{i-1} <' x$ be such a path. Then, $w_{i-1} \in \mathcal{W}$ and $k_{\mathcal{N}^R}(w_{i-1}) = i-1$.

The rank of $w_{i-1}$ cannot be less than $i - 1$ as, otherwise, this would not be a path of length $i$. The rank of $w_{i-1}$ cannot be more than $i - 1$ as, otherwise, there would be a path longer than $i$ from $x$ to some minimal world.

By inductive hypothesis, $k_{\mathcal{N}^R}(w_{i-1}) \leq k_{\mathcal{N}^\ast}(w_{i-1})$. Then, $k_{\mathcal{N}^\ast}(w_{i-1}) \geq i - 1$. As $w_{i-1} <' x$ and $<' \subseteq <^\ast$, then $w_{i-1} <^\ast x$. Therefore, the rank of $x$ in $\mathcal{N}^\ast$ must be higher than the rank of $w_{i-1}$, i.e., $k_{\mathcal{N}^\ast}(x) \geq i$. Thus, $k_{\mathcal{N}^R}(x) \leq k_{\mathcal{N}^\ast}(x)$ and condition (6) follows.

Let now consider any interpretation $\mathcal{N}^\ast$ such that $<' \subseteq <^\ast$ and $\mathcal{N}^\ast \neq \mathcal{N}^R$. As the two interpretations may only differ in the ranking, it must be that, for some world $w$, $k_{\mathcal{N}^R}(w) \neq k_{\mathcal{N}^\ast}(w)$. Form condition (6) it follows that $k_{\mathcal{N}^R}(w) < k_{\mathcal{N}^\ast}(w)$. As for all the other worlds $w' \in \mathcal{W}$, by (6), $k_{\mathcal{N}^R}(w') \leq k_{\mathcal{N}^\ast}(w')$, then $\mathcal{N}^R <_{\text{FIMS}} \mathcal{N}^\ast$. \hfill \square

### 5.2 Relations with the multipreference semantics

The MP-closure was proposed in [16] as a construction which is a sound approximation of the multipreference semantics [21] for the description logic $ALC$. Here we recall this semantics and compare with it, abstracting away from the peculiarities of description logics and considering only the propositional part (with no ABox, no individual constants, no roles and no universal and existential restrictions). The idea of the multipreference semantics was to define a refinement of the rational closure in which preference with respect to specific aspects was considered. It is formulated in terms of enriched models, which also consider the preference relations $<_{A_i}$ associated with the different aspects, where each relation $<_{A_i}$ refers (only) to the defeasible inclusions of the form $T(D) \sqsubseteq A_i$ (corresponding, here, to conditionals of the form $D \rightsquigarrow A_i$). The idea of having different preference relations, associated to different typicality operators, was already proposed by Gil to define a multipreference formulation of the logic.
\(\mathcal{ALC} + T_{\text{min}}\) with multiple typicality operators, a logic which exploits a different minimal model semantics w.r.t. the rational closure semantics. In [21], instead, a refinement of the rational closure was considered and with a single typicality operator.

The aim of the semantics in [21] is to define a refinement of the ranked models of the rational closure of a knowledge base \(K\) (i.e., the models in \(\text{Min}_{\text{RC}}(K)\)) in which the modular preference relation \(<\) satisfies the following additional condition on the preference relations \(<_{A_i}\):

(a) If \(x <_{A_i} y\), for some \(A_i\), and there is no \(A_j\) such that \(y <_{A_j} x\), then \(x < y\).

while the typicality operator remains just one.

The intended meaning of \(x <_{A_i} y\) is that \(x\) satisfies some default for \(A_i\) which is violated by \(y\). More precisely, \(<_{A_i}\) is the preference relation in a ranked model of a knowledge base \(K_i\) containing only the defaults of the form \(D \vdash A_i \in K\). In the minimal ranked models \(M = \langle W, <_{A_i}, v \rangle\) of \(K_i\) (minimal according to Definition 7), \(x <_{A_i} y\) has precisely the meaning that \(x\) satisfies some default for \(A_i\) which is violated by \(y\). Condition (a) alone, however, is too weak to define models of the rational closure and a specificity condition was added to define enriched models. Here, we refer to the definition of \(S\)-enriched rational models in [16], which is slightly stronger than the one in [21] (although both of them lead to refinements of the rational closure), and reformulate it in the propositional case replacing typicality inclusions \(T(D) \sqsubseteq C\) with conditionals \(D \not\vdash C\).

**Definition 17 (S-Enriched rational models of \(K\)).** \(\mathcal{M} = \langle W, <_{A_1}, \ldots, <_{A_n}, <, v \rangle\) is a strongly enriched model of \(K\) if the following conditions hold:

- \(\langle W, <, v \rangle\) is a ranked model of \(K\) (as in Section 2, Definition 1);
- for all \(C \vdash A_i \in K\), for all \(w \in W\), if \(w \in \text{Min}_{<_{A_i}}^\mathcal{M}(C)\) then \(M, w \models A_i\) and
- the preference relation \(<\) satisfies the conditions (a) above, and the following specificity condition:

\[
x < y \text{ if } (i) \ y \text{ violates some defeasible inclusion satisfied by } x \text{ and } \\
(ii) \text{ for all } C_j \vdash D_j \in K, \text{ which is violated by } x \text{ and not by } y, \\
\text{there is a } C_k \vdash D_k \in K, \text{ which is violated by } y \text{ and not by } x, \\
\text{such that } k_M(C_j) < k_M(C_k).
\]

In (i) and (ii) the ranking function \(k_M\) is the ranking function of model \(\mathcal{M}\) itself and the intended meaning of the specificity condition is that preference should be given to the worlds that falsifies less specific defaults (defaults with lower ranks). The defaults violated by \(x\) are less serious than the defaults violated by \(y\), as formula \(C_k\) is more specific than \(C_j\).

It is easy to see that conditions (i) and (ii) in the specificity condition above (together) are equivalent to the condition

\[
V(x) \prec_{k_M} V(y),
\]

where \(\prec_{k_M}\) is the \(k_M\)-seriousness ordering in Definition 15. As a consequence, one can reformulate \(S\)-enriched models by reformulating the specificity condition as:

\[
x < y \text{ if } V(x) \prec_{k_M} V(y).
\]
A further simplification to the notion of S-enriched models comes from the fact that the semantics in [21,16] considers the minimal S-enriched models, among all the S-enriched models if $K$, which are obtained by first minimizing the $<_{A_i}$ and then minimizing $<$ (as done for the ranked models of the rational closure), in this order, thus giving preference to models with lower ranks. It was proved in [16] (Proposition 1 therein) that, in minimal S-enriched models, the specificity condition is strong enough to enforce condition (a). As a consequence, one can simplify the definition of S-enriched rational models from the beginning, by removing condition (a) as well as the preference relations $<_1, \ldots, <_n$, thus starting from the following simplified notion of enriched model.

**Definition 18 (simplified-enriched models of $K$).** A simplified-enriched model of $K$ is a ranked model $M = \langle \Delta, <, I \rangle$ of $K$ (according to Definition 1 in Section 2) such that the preference relation $<$ satisfies the condition

$$x < y \text{ if } V(x) \prec_k M V(y)$$

With this simplification, the minimal S-enriched models in [16] would correspond to the $<_{FIMS}$-minimal simplified-enriched rational models of $K$, where $<_{FIMS}$-minimality is precisely the same notion of minimality used in the semantic characterization of the rational closure (see Definition 7 in Section 2). Furthermore, minimal simplified enriched models also satisfy property (a), for $x <_{A_i} y$ meaning that $x$ satisfies some default for $A_i$ which is violated by $y$. Thus the multipreference semantics in [16] collapses into a semantics without multiple preferences (a result that actually was implied by Proposition 1 therein).

We prove that $<_{FIMS}$-minimal simplified-enriched models of $K$ include all the $<_{FIMS}$-minimal ranked models $M$ such that $F_R(M) = M$. It is easy to prove that the ranked models of $K$ such that $F_R(M) = M$ are simplified-enriched models of $K$.

**Proposition 14.** Given a ranked model $M = \langle W, <, v \rangle$ of $K$, if $F_R(M) = M$ then $M$ is a simplified-enriched model of $K$.

**Proof.** Let $M$ be a ranked model of $K$ such that $F_R(M) = M$. Then $M = N^R$ for some $N = F(M)$ where, based on the formulation of functor $F$ in Proposition 5 $N = \langle W, <', v \rangle$ and

$$x <' y \text{ iff } V(x) <^{k_M} V(y).$$

As $<' \subseteq <^R$ (by Proposition 9), then:

$$x <^R y \text{ if } V(x) <^{k_M} V(y).$$

But, as $M = F_R(M)$, the relation $<$ in $M$ must be equal to $<^R$ and, therefore:

$$x < y \text{ if } V(x) <^{k_M} V(y).$$

Thus $M$ is a simplified-enriched model of $K$. \hfill $\square$
From the proposition above, the next corollary follows.

**Corollary 6.** If $\mathcal{M}$ is a $<_{FIMS}$-minimal canonical ranked model of $K$ such that $\mathcal{F}^{R}(\mathcal{M}) = \mathcal{M}$ then $\mathcal{M}$ is a $<_{FIMS}$-minimal canonical simplified-enriched model of $K$.

Observe that the converse of Proposition 14 does not hold. However, whether the converse of Corollary 6 holds or not, and whether entailment in the multipreference semantics defines a rational consequence relation is still to be understood and will be subject of future work.

In [16] it was shown (Proposition 2) that the MP-closure is a sound approximation of the multipreference semantics, that is, the typicality inclusions (the conditionals) that follow from the MP-closure hold in all the minimal canonical s-enriched models of the KB. This result, henceforth, extends to the $<_{FIMS}$-minimal canonical ranked model of $K$ such that $\mathcal{F}^{R}(\mathcal{M}) = \mathcal{M}$.

### 5.3 Relations with the Relevant Closure

The relevant closure [6] was developed by Casini et al. as a proposal for defeasible reasoning in description logics to overcome the inferential weakness of rational closure. It is based on the idea of relevance of subsumptions to a query, where relevance is determined based on justifications, minimal sets of sentences responsible for a conflict. Any sentence occurring in some justification is potentially relevant for resolving the conflict.

In a defeasible description logic, the knowledge bases contains a set of defeasible subsumptions $D \sqsubseteq C$, a DBox $\mathcal{D}$, corresponding to a set of conditionals $D \sqsupseteq C$, and a set of classical subsumptions $D \sqsubseteq C$, a TBox $\mathcal{T}$, which have to be satisfied by all the elements of the domain in any model. When evaluating a query $C \sqsubseteq D$, one has to compute the $C$-justifications w.r.t. $\mathcal{D}$, that is, the minimal sets of defaults $J \subseteq D$ making $C$ exceptional (or, supporting $\neg C$). The idea is that, for each $C$-justification $J$, some defeasible subsumption occurring in $J$ is to be removed from $D$ for consistency with $C$, and it is convenient to remove first the defeasible subsumptions with lower ranks in the $C$-justifications. This is done by the Relevant Closure algorithm.

For a given query $C \sqsubseteq D$, the algorithm receives in input the ranking in the rational closure of the defeasible subsumptions in $D$, and a set $R$ of the defeasible subsumptions which are relevant to the query, i.e., the set of the defeasible subsumptions which are eligible for removal during the execution of the Relevant Closure algorithm. The algorithm determines from $D$ a new set of defeasible subsumptions $D'$, by removing from $D$, rank by rank, starting from the lower rank 0, all the subsumptions in $R$ with that rank, until the remaining set of (non-removed) defeasible subsumptions $D'$ is consistent with $\mathcal{T}$ and with $C$. For the pseudocode of this algorithm, we refer to Algorithm 2 in [6].

In the Basic Relevant closure, the set $R$ of relevant defeasible subsumptions is the union $\bigcup J_j$ of all the $C$-justifications $J_j$ w.r.t. $\mathcal{D}$, where, by Corollary 1 in [6], $J_j$ is a $C$-justification if it is an inclusion-minimal subset of $\mathcal{D}$ such that $\mathcal{T} \models J_j \subseteq \neg C$ ($J$ being the materialization of the defeasible subsumptions in $J$ and $\models$ the rational entailment in the preferential description logic). At the end of the iteration phase, a set $D' \subseteq D$ of defeasible subsumptions consistent with $\mathcal{T}$ and with $C$ is obtained, and $D'$
is used, together with \( T \), to check whether or not \( C \subseteq D \) follows from \( D' \) and \( T \) (i.e., whether \( T \models D' \cap C \subseteq D \)).

The Minimal Relevant closure exploits exactly the same algorithm as the basic Relevant closure, but it takes \( \bigcup J_j^{min} \), the union of all sets \( J_j^{min} \) containing the conditionals with lowest rank in each \( C \)-justification \( J_j \), as the set \( R \) of relevant defaults which are eligible for removal (instead of \( \bigcup J_j \)). The Basic Relevant Closure is weaker than the Minimal Relevant Closure, and the Minimal Relevant Closure is weaker than the lexicographic closure \( [6] \).

For a comparison with the MP-closure, let us consider the case when the TBox \( T \) is empty and \( D \) is a set of conditionals, so that the knowledge base \( K \) is just a set of conditionals, as before (i.e., \( K = C \)). In the following, we transpose the definition of the basic and minimal Relevant Closure to the propositional case.

Let \( K \) be a knowledge base and \( C \models D \) a query. A \( C \)-justification w.r.t. \( K \) is an (inclusion) minimal subset \( J \) of \( K \) such that \( \models J \rightarrow \neg A \), that is, \( J \cup \{ A \} \) is inconsistent, where \( J \) is the materialization of the conditionals in \( J \), as in Section \( [3] \) and \( \models \) is logical consequence in the propositional calculus. Let \( \bigcup J_j \) be the union of all the \( C \)-justifications w.r.t. \( K \). The algorithm exploits the ranking of conditionals computed by the rational closure of \( K \).

Given a query \( C \models D \), and \( R = \bigcup J_j \), the Basic Relevant closure algorithm, for each rank \( i \) (in the rational closure of \( K \)) starting from 0, removes from \( K \) all the defaults with rank \( i \) occurring in \( R \), until the remaining set of conditionals \( D' \) is consistent with \( C \) (i.e., \( D' \cup \{ C \} \) is consistent in the propositional calculus (at least one conditional has been removed from any \( C \)-justification \( J_j \)). A conditional \( C \models D \) is in the Basic Relevant Closure of \( K \) if \( \models (\bigwedge D' \land C) \rightarrow D \) (in the propositional calculus).

As before, the Minimal Relevant Closure algorithm differs from the previous one only in that it takes \( R \) as \( \bigcup J_j^{min} \) the union of all sets \( J_j^{min} \), where \( J_j^{min} \) is the set of the conditionals with lowest rank in the \( C \)-justification \( J_j \).

Let us consider again Example \( [4] \), the one in which the lexicographic closure comes to the conclusion that typical employed students, like typical students, are young and do not pay taxes (which appears to be too bold). We can see that neither the Basic Relevant Closure nor the Minimal Relevant Closure conclude \( \text{Employee} \land \text{Student} \models \neg \text{Pay}_m \text{Taxes} \).

Example 8. The knowledge base \( K'' \) contains the conditionals:

1. \( \text{Student} \models \neg \text{Pay}_m \text{Taxes} \)
2. \( \text{Student} \models \text{Young} \)
3. \( \text{Employee} \models \neg \text{Young} \land \text{Pay}_m \text{Taxes} \)
4. \( \text{Employee} \land \text{Student} \models \text{Busy} \)

There are two justifications of the exceptionality of \( \text{Employee} \land \text{Student} \) w.r.t. \( K'' \), namely \( J_1 = \{1, 3\} \) and \( J_2 = \{2, 3\} \), and their union \( \bigcup J_j = \{1, 2, 3\} \), used in the basic relevant closure algorithm, contains only conditionals with rank 0, which are all removed as responsible of the exceptionality of \( \text{Employee} \land \text{Student} \) at the first iteration stage (for rank 0). The set of remaining conditionals is then \( D' = \{4\} = \{ \text{Employee} \land \text{Student} \models \text{Busy} \} \), so that \( \bigwedge D' = \{ \text{Employee} \land \text{Student} \rightarrow \text{Busy} \} \).
and
\[ \not \models (\bigwedge D' \land \text{Employee} \land \text{Student}) \rightarrow (\text{Young} \land \neg \text{Pay \_Taxes}). \]

Therefore, the conditional \( \text{Employee} \land \text{Student} \not\models \text{Young} \land \neg \text{Pay \_Taxes} \) is not in the Basic Relevant closure of \( K'' \). The result is the same for the Minimal Relevant closure, as \( J_1 \) and \( J_2 \) only contain conditionals with rank 0 and, therefore, \( J_1 = J_1^{\text{min}} \) and \( J_2 = J_2^{\text{min}} \).

The Basic and Minimal Relevant closure as well as the MP-closure are all more cautious than the lexicographic closure. The next example shows that the MP-closure is neither equivalent to Basic nor to Minimal Relevant closure.

**Example 9.** Let \( K \) be the knowledge base containing the conditionals:

1. \( \text{Italian} \not\models \text{Residence in Italy} \)
2. \( \text{German} \not\models \text{Residence in Germany} \)
3. \( \text{Residence in Italy} \land \neg \text{Has Residence} \not\models \bot \)
4. \( \text{Residence in Germany} \land \neg \text{Has Residence} \not\models \bot \)
5. \( \text{Residence in Italy} \land \text{Residence in Germany} \not\models \bot \)

Italians normally have a residence in Italy and Germans normally have a residence in Germany. Those who have a residence in Italy have residence. Those who have a residence in Germany have residence. It is not the case that somebody has residence both in Germany and in Italy. Observe that the last three conditionals have infinite rank in the rational closure, and they represent properties which no model of \( K \) can violate.

There is a unique \( \text{Italian} \land \text{German} \)-justification w.r.t. \( K \), namely \( J = \{1, 2, 5\} \). In fact, \( J \) is a minimal set of conditionals such that \( \models \bigwedge J \rightarrow \neg (\text{Italian} \land \text{German}) \). As defaults 1 and 2 in \( J \) have rank 0 in the rational closure, while default 5 has an infinite rank, the basic Relevant closure algorithm first removes conditionals 1 and 2 from \( D \). The resulting set of defaults \( D' = \{3, 4, 5\} \) is consistent with \( \text{German} \not\models \text{Has Residence} \), and nothing else needs to be removed. The conditional \( \text{Italian} \land \text{German} \not\models \text{Has Residence} \) is not in the basic Relevant closure of \( K \) as \( \not\models (\bigwedge D' \land \text{Italian} \land \text{German}) \rightarrow \text{Has Residence} \).

Concerning the Minimal Relevant closure, as defaults 1 and 2 have rank 0, the set of the defaults in \( J \) with lowest rank is \( J^{\text{min}} = \{1, 2\} \). Therefore, also in the Minimal Relevant Closure construction, justifications 1 and 2 are both removed from \( D \), and the conditional \( \text{Italian} \land \text{German} \not\models \text{Has Residence} \) is not in the Minimal Relevant Closure of \( K \).

As a difference, the defeasible inclusion \( \text{Italian} \land \text{German} \not\models \text{Has Residence} \) is in the MP-closure of \( K \) as well as in the Lexicographic closure of \( K \), which both have two bases, \( \{1\} \) and \( \{2\} \).

Notice also that, for each justification \( J_j \), the minimal Relevant closure always removes all the defaults in \( J_j^{\text{min}} \). Indeed, the defaults in \( J_j^{\text{min}} \) have all the same rank and they have to be removed all together at the same iteration of the algorithm (at iteration \( i \) if they have rank \( i \)). It cannot be the case that some default in \( J_j^{\text{min}} \) is removed from \( D \) but not all of them. This observation will be useful in the following.
The example above shows that the MP-closure is different from both the Basic Relevant Closure and the Minimal Relevant Closure, and that the MP-closure cannot be weaker than such closures. We prove that the Minimal Relevant Closure is a subset of the MP closure.

**Proposition 15.** Let $K$ be a set of conditionals. If $C \vdash A$ is in the Minimal Relevant Closure of $K$, then $C \vdash A$ is in the MP-closure of $K$.

**Proof.** The proof is by contraposition. If $C \vdash A$ is not in the MP-closure of $K$, then there must be an MP-basis $B$ for $C$ such that $B \cup \{C\} \not\models A$. By definition of MP-basis, $B$ is maximal w.r.t. the MP-seriousness ordering among the subsets of $K$ consistent with $C$.

We show that, when executing the Minimal Relevant Closure algorithm for the goal $C \vdash A$, each conditional $d = E \vdash F \notin B$ is removed from $K$ by the Minimal Relevant Closure algorithm, so that the resulting set of defaults $D'$ must be a subset of the MP-basis $B$. As a consequence, $D' \cup \{C\} \not\models A$ and, by the deduction theorem, $\not\models D' \land C \rightarrow A$. We can then conclude that $C \vdash A$ is not in the Minimal Relevant Closure of $K$.

First observe that, as $B$ is an MP basis for $C$, it is a maximal set of defaults such that $B \cup \{C\}$ is consistent. Then, if $d \notin B, B \cup \{d\} \cup \{C\}$ is inconsistent and there must be some $C$-justification $J$ such that $d \in J$. We show that, in particular, there must be a $C$-justification $J_r$, such that $d \in J_{r_{\text{min}}}$, i.e., $d$ must have the lowest rank among the conditionals in $J_r$.

Suppose, by absurd, that a $C$-justification $J_r$ with $d \in J_{r_{\text{min}}}$ does not exists. Then for all $C$-justifications $J_{s_h}$ such that $d \in J_{s_h}$ (with $h = 1, \ldots, t$), the rank of $d$ is not the lowest among the ranks of the conditionals in $J_{s_h}$, and (for each $h$) there must be another conditional $d_{s_h} \in J_{s_h}$ such that $\text{rank}(d_{s_h}) < \text{rank}(d)$. Observe that the set of conditionals $E = B \cup \{d\}\setminus \{d_{s_1}, \ldots, d_{s_t}\}$ is then more serious than $B$ in the MP-ordering $(B \prec_{\text{MP}} E)$, and $E$ must be consistent with $C$ as, for all the $C$-justifications $J_{s_h}$, a conditional ($d_{s_h}$) is not in $E$. This contradicts the assumption that $B$ (being a basis for $C$) is a maximally MP-serious set of conditionals in $K$ consistent with $C$.

Hence, there must be a $C$-justification $J_{s_h}$ (for some $h$) such that $d \in J_{s_h}^{\text{min}}$. As a consequence, the Minimal Relevant Closure algorithm must remove $d$ from the set of conditionals in $K$ at the iteration stage $i = \text{rank}(d)$. All the conditionals with rank $\text{rank}(d)$ are removed from $K$ at that stage as no-other conditional in $J_{s_h}$ has been removed in advance (there is none in $J_{s_h}$ with rank lower than $i$).

As conditional $d$ is removed at some iteration, $d$ is not in $D'$ the set of defaults resulting from the execution of the Minimal Relevant Closure algorithm. But, as our choice of $d$ is arbitrary, this holds for all $d \notin B$. Therefore, $D' \subseteq B$, that is, the set of conditionals $D'$ computed by the Minimal Relevant Closure algorithm, for the query $C \vdash A$, is a subset of the MP-basis $B$ for $C$.

As mentioned above, we can now conclude that $C \vdash A$ is not in the Minimal Relevant Closure of $K$. As we know that $B \cup \{C\} \not\models A$, then $\not\models B \land \{C\} \rightarrow A$. As $D' \subseteq B$, then $\not\models D' \land \{C\} \rightarrow A$. Thus, $C \vdash A$ is not in the Minimal Relevant Closure of $K$. □
Basic Relevant closure is known to be weaker than Minimal Relevant Closure and, therefore, is also weaker than the MP-closure.

As for the MP-closure and the lexicographic closure, which may have an exponential number of bases (in the size of the knowledge base), computing the Relevant closure may require as well an exponential number of classical entailment checks [6]. Concerning the properties of the relevant closure in [6], it was shown that both Basic Relevant Closure and Minimal Relevant Closure, among the properties of a rational inference relation, do not satisfy Or, Cautious Monotonicity and Rational Monotonicity.

6 Conclusions

In this paper we have studied the notion of MP-closure in the propositional case. The MP-closure was originally introduced for description logics in [16] as an approximation of the multipreference semantics. As the lexicographic closure, the MP-closure builds on the rational closure but it exploits a different seriousness ordering to compare sets of defaults: a different lexicographic order is used which compares tuples of sets of defaults rather than tuples of numbers (the number of the defaults in the sets).

The option of abandoning the Maximal Entropy approach was already considered by Lehmann in his seminal work on the lexicographic closure [24]. Here we have explored this option, presenting a characterization of the MP-closure both in terms of maxiconsistent sets and of a model-theoretic construction. In particular, we have developed a simple preferential semantics for the MP-closure, in terms of a functor \( F \) that maps each minimal canonical model of the knowledge base (characterizing the rational closure) into a preferential model, thus defining, for a knowledge base \( K \), a consequence relation \( \mathcal{MP}_K \) which is a superset of the rational closure of \( K \), \( \mathcal{RC}_K \), but a subset of the lexicographic closure, \( \mathcal{LC}_K \).

\( \mathcal{MP}_K \) is not rational; however, we have seen that rationality can be recovered and, starting from the semantics of the MP-closure, a ranked semantics can be defined such as a rational consequence relation (that we called \( \mathcal{MP}^R_K \)), which is a superset of the MP-closure. We have shown that \( \mathcal{MP}^R_K \) is incomparable with the lexicographic closure, that is \( \mathcal{MP}^R_K \) neither includes \( \mathcal{LC}_K \), nor is included in \( \mathcal{LC}_K \).

In the paper we have compared the MP-closure with the multipreference semantics introduced in [21] and with the Relevant closure [6]. They are both refinements of the rational closure as they build on rational closure ranking to define stronger consequence relations. These formalisms have been defined for description logics, but their definition can be transposed to propositional logic. Concerning the multipreference semantics, which is a semantic strengthening of the rational closure, we have shown that, in the propositional setting, there is an inclusion relation among the \( <_{FILMS} \)-minimal canonical fixed-points of the operator \( F^R \) (used for defining a rational consequence relation extending the MP-closure) and the minimal canonical enriched models in the multipreference semantics. Concerning the Relevant Closure, we have seen that MP-closure is stronger than both the Basic Relevant Closure and the Minimal Basic Relevant Closure.

A first semantic characterization of the MP-closure for the description logic \( ALC \) was developed in [14] using bi-preferential (BP) interpretations, preferential interpretations developed along the lines of the preferential semantics introduced by Kraus,
Lehmann and Magidor [22,23], but containing two preference relations, the first one
\(<_1\) playing the role of the ranked preference relations in the models of the RC, and
the second one \(<_2\) representing a preferential refinement of \(<_1\). Another construction,
developed for DLs, the skeptical closure [15], was shown to be a weaker (polynomial)
variant of the MP-closure in [14], and we refer therein for detailed comparisons.

There are other related approaches that build on the rational closure, and deal with
its limitations. In particular, the logic \(\mathcal{DL}^N\), proposed by Bonatti et al. [2], and the
inheritance-based rational closure by Casini and Straccia [10,11].

\(\mathcal{DL}^N\) captures a form of “inheritance with overriding”: a defeasible inclusion is
inherited by a more specific class if it is not overridden by more specific (conflicting)
properties. \(\mathcal{DL}^N\) is not necessarily defined starting from the ranking given by the rational
closure but, when it does, it provides a possible approach to address the problem
of inheritance blocking in the rational closure. Inference is based on a polynomial algo-
rithm which allows a default property to be inherited. When a defeasible property of a
concept is conflicting with another defeasible property, and none of them is more
specific so to override the other, the concept may have an inconsistent prototype. For
instance, in Example 3 the concept \(\text{Employee} \land \text{Student}\) has an inconsistent proto-
type, as employed students inherit the property of students of non paying taxes and the
property of employee of paying taxes, none is more specific than the other. In such an
example, as we have seen, the MP-closure and the lexicographic closure only conclude
that employed students are busy, and silently ignore the conflicting defaults. In \(\mathcal{DL}^N\)
unresolved conflicts have to be detected and then fixed by modifying the knowledge
base. The logical properties of \(\mathcal{DL}^N\) are studied in [3]. It is shown that, when consid-
ering the internalized KLM postulates, where each inclusion \(NC \subseteq D\) corresponds
to a conditional \(C \supset D\), few of the postulates are satisfied (namely, Reflexivity, Left
Logical Equivalence and Right Weakening) but, when only \(N\)-free knowledge bases are
allowed (i.e., knowledge bases which do not allow normality concepts \(NC\) on the r.h.s.
of conditionals), all the postulates are satisfied, with the partial exception of Cautious
Monotonicity. That is, satisfying KLM properties in \(\mathcal{DL}^N\) comes at the price of re-
nouncing to the full expressiveness of the non monotonic DL (such as, supporting role
restrictions to normal instances).

The inheritance-based rational closure in [10,11], is a closure construction which
is defined by combining the rational closure with defeasible inheritance networks. For
answering a query "if \(A\), normally \(B\)”, it relies on the idea that only the information
related to the connection of \(A\) and \(B\) (and, in particular, only the defeasible inclusions
occurring on the routes connecting \(A\) and \(B\) in the corresponding net) are relevant and
have to be considered in the rational closure construction for answering the query.

The idea of considering subsets of the axioms in the knowledge base has also been
considered by Fernandez Gil in [13], who developed a multi-typicality version of the
typicality logic \(\mathcal{ALC} + T_{min}\) [18], allowing for different typicality operators \(T_i\) in the
knowledge base, where \(\mathcal{ALC} + T_{min}\) is another defeasible description logic based on
a preferential extension of \(\mathcal{ALC}\) with typicality, which, differently from the rational
closure, is not based on a ranked semantics but, nevertheless, also suffers from the
blocking of property inheritance problem.
Other approaches in the literature deal with the problem of inheritance with exceptions. A recent one, by Bozzato et al. in [5] presents an extension of the CKR framework in which defeasible axioms are allowed in the global context and can be overridden by knowledge in a local context. Exceptions have to be justified in terms of semantic consequence. A translation of extended CHRs (with knowledge bases in $SROIQ$-RL) into Datalog programs under the answer set semantics is also defined.

In this paper, we have studied the properties of the MP-closure strengthening the rational closure and we have compared it with other closures extending the rational closure, showing that they have different strengths. We believe that, depending on the application context one approach might be more suitable than another or vice-versa (and, accordingly, we might want to accept the principle of Maximal Entropy or not). There are several aspects which deserve investigation, for instance, how inheritance-based rational closure relates with the other notions of closure; whether the multipreference semantics defines a rational consequence relation, and if the converse of Corollary [6] holds (this would produce an alternative semantic characterization of the multipreference semantics); which are the relationships among the MP-closure, the Relevant Closure and the Lexicographic closure with other preferential approaches, which are not based on the rational closure, but also address the problem of inheritance blocking, such as the proposal in [13]. As in the MP-closure and in the lexicographic closure the number of possible bases for a given formula may be exponential in the number of defaults (and, in the Relevant Closure, an exponential number of justifications is to be computed in the worst case), from the practical point of view, it may be interesting to consider sound approximations of these notions of closures and especially, polynomial, approximations.

Acknowledgement: This research is partially supported by INDAM-GNCS Project 2019 “METALLIC #2: METodi di prova per il ragionamento Automatico per Logiche non-cLassIChe”.

References

1. Salem Benferhat, Didier Dubois, and Henri Prade. Possibilistic logic: From nonmonotonicity to logic programming. In *Symbolic and Quantitative Approaches to Reasoning and Uncertainty, European Conference, ECSQARU’93, Granada, Spain, November 8-10, 1993, Proceedings*, pages 17–24, 1993.
2. P. A. Bonatti, M. Faella, I. Petrova, and L. Sauro. A new semantics for overriding in description logics. *Artif. Intell.*, 222:1–48, 2015.
3. P. A. Bonatti and L. Sauro. On the logical properties of the nonmonotonic description logic $DL^N$. *Artif. Intell.*, 248:85–111, 2017.
4. C. Boutilier. Conditional logics of normality: a modal approach. *Artificial Intelligence*, 68(1):87–154, 1994.
5. L. Bozzato, T. Eiter, and L. Serafini. Enhancing context knowledge repositories with justifiable exceptions. *Artif. Intell.*, 257:72–126, 2018.
6. G. Casini, T. Meyer, K. Moodley, and R. Nortje. Relevant closure: A new form of defeasible reasoning for description logics. In *JELIA 2014, LNCS 8761, pages 92–106*. Springer, 2014.
7. G. Casini, T. Meyer, K. Moodley, U. Sattler, and I.J. Varzinczak. Introducing defeasibility into OWL ontologies. In *The Semantic Web - ISWC 2015 - 14th International Semantic Web*
8. G. Casini, T. Meyer, I. J. Varzinczak, and K. Moodley. Nonmonotonic Reasoning in Description Logics: Rational Closure for the ABox. In DL 2013, 26th International Workshop on Description Logics, volume 1014 of CEUR Workshop Proceedings, pages 600–615. CEUR-WS.org, 2013.
9. G. Casini and U. Straccia. Rational Closure for Defeasible Description Logics. In T. Janhunen and I. Niemelä, editors, Proceedings of the 12th European Conference on Logics in Artificial Intelligence (JELIA 2010), volume 6341 of Lecture Notes in Artificial Intelligence, pages 77–90, Helsinki, Finland, September 2010. Springer.
10. G. Casini and U. Straccia. Defeasible Inheritance-Based Description Logics. In Toby Walsh, editor, Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI 2011), pages 813–818, Barcelona, Spain, July 2011. Morgan Kaufmann.
11. G. Casini and U. Straccia. Defeasible inheritance-based description logics. Journal of Artificial Intelligence Research (JAIR), 48:415–473, 2013.
12. N. Friedman and J. Y. Halpern. Plausibility measures and default reasoning. Journal of the ACM, 48(4):648–685, 2001.
13. Oliver Fernandez Gil. On the Non-Monotonic Description Logic ALC+Tmin. CoRR, abs/1404.6566, 2014.
14. L. Giordano and V. Gliozzi. Reasoning about exceptions in ontologies: from the lexicographic closure to the skeptical closure. CoRR, abs/1807.02879, 2018.
15. L. Giordano and V. Gliozzi. Reasoning about exceptions in ontologies: from the lexicographic closure to the skeptical closure. In Proceedings of the Second Workshop on Logics for Reasoning about Preferences, Uncertainty, and Vagueness, PRUV@IJCAR 2018, Oxford, UK, July 19th, 2018.
16. L. Giordano and V. Gliozzi. Reasoning about multiple aspects in dls: Semantics and closure construction. CoRR, abs/1801.07161, 2018.
17. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Analytic Tableaux Calculi for KLM Logics of Nonmonotonic Reasoning. ACM Transactions on Computational Logics (TOCL), 10(3), 2009.
18. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. A NonMonotonic Description Logic for Reasoning About Typicality. Artificial Intelligence, 195:165–202, 2013.
19. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Semantic characterization of rational closure: From propositional logic to description logics. Artificial Intelligence, 226:1–33, 2015.
20. V. Gliozzi. A minimal model semantics for rational closure. In G. Kern-Isberner and R. Wassermann, editors, NMR 2016 (16th International Workshop on Non-Monotonic Reasoning), Cape Town, South Africa, 2016.
21. V. Gliozzi. Reasoning about multiple aspects in rational closure for dls. In Proc. AI*IA 2016 - XVth International Conference of the Italian Association for Artificial Intelligence, Genova, Italy, November 29 - December 1, 2016, pages 392–405, 2016.
22. S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. Artificial Intelligence, 44(1-2):167–207, 1990.
23. D. Lehmann and M. Magidor. What does a conditional knowledge base entail? Artificial Intelligence, 55(1):1–60, 1992.
24. D. J. Lehmann. Another perspective on default reasoning. Ann. Math. Artif. Intell., 15(1):61–82, 1995.
25. M. Goldszmidt, P. H. Morris, and J. Pearl. A maximum entropy approach to nonmonotonic reasoning. In Proceedings of the 8th National Conference on Artificial Intelligence. Boston, Massachusetts, USA, July 29 - August 3, 1990, 2 Volumes., pages 646–652, 1990.
26. P.F. Patel-Schneider, P.H. Hayes, and I. Horrocks. OWL Web Ontology Language; Semantics and Abstract Syntax. In http://www.w3.org/TR/owl-semantics/, 2002.

27. J. Pearl. System Z: A natural ordering of defaults with tractable applications to nonmonotonic reasoning. In R. Parikh, editor, TARK (3rd Conference on Theoretical Aspects of Reasoning about Knowledge), pages 121–135, Pacific Grove, CA, USA, 1990. Morgan Kaufmann.

28. D. Poole. A logical framework for default reasoning. Artif. Intell., 36(1):27–47, 1988.