Functional determinant of the massive Laplace operator and the multiplicative anomaly

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Abstract

After a brief survey of zeta function regularization issues and of the related multiplicative anomaly, illustrated with a couple of basic examples, namely the harmonic oscillator and quantum field theory at finite temperature, an application of these methods to the computation of functional determinants corresponding to massive Laplacians on spheres in arbitrary dimensions is presented. Explicit formulas are provided for the Laplace operator on spheres in $N = 1, 2, 3, 4$ dimensions and for ‘vector’ and ‘tensor’ Laplacians on the unitary sphere $S^4$.

Keywords: functional determinant, multiplicative anomaly, arbitrary dimensional spheres

1. Introduction

In quantum field theory (QFT), the Euclidean partition function plays a very important role. The full propagator and all other $n$-point correlation functions can be computed by means of it. Moreover, this tool can be extended without problem to curved space-time [1]. As a formalism this is extremely beautiful but it must be noticed that in relativistic quantum field theories an infinite number of degrees of freedom is involved and, as a consequence, ultraviolet divergences will be present, thus rendering regularization and renormalization compulsory.

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In the one-loop approximation, and in the external field approximation too, one may describe (scalar) quantum fields by means of a (Euclidean) path integral and express the Euclidean partition function as a function of functional determinants associated with the differential operators involved. In this way, the partition function reads

\[ Z \simeq (\det L)^{-1/2}, \tag{1.1} \]

with \( L \) being an elliptic self-adjoint non-negative operator, the small fluctuation operator. We thus see that the computation of Euclidean one-loop partition functions reduces to the computation of functional determinants.

As already mentioned, functional determinants are divergent quantities which need to be regularized. For a long period this was performed in the physics literature case by case, by adding ‘reasonable’ correcting terms to formulas such as \( \det(AB) = (\det A)(\det B) \), which did look suspect (see, e.g. [2], among very many references)—and are, in fact, generically wrong. No wonder, since those \( \det \)'s are in no way regular determinants, but \( \text{regularized ones}, \) which do not satisfy the usual properties of \( \det \)'s, in particular, the multiplicative property\(^5\).

Soon after this was clearly understood, some seminal works appeared [3–5] (see also [6], and [7] and references therein) where the already existing rigorous, simple, and also very beautiful mathematical formulation of the so-called Wodzicki’s or residue calculus for pseudo-differential operators was put forward and made explicit for use in theoretical physics. However, many practitioner physicists are still now unaware of these fundamental methods. In particular, the so-called multiplicative anomaly or defect of the zeta-regularized determinant (a well-established definition stemming from Atiyah, Ray and Singer [8]) is a perfectly-under-control quantity which can be given by a very simple formula in terms of the Wodzicki residue, which is in its turn the only true trace (up to a multiplicative constant) one can define on the whole class of pseudo-differential operators—and extends, in a unique way, the Dixmier trace and the Adler–Manin one, which are just particular cases of it. This is, in a word, standard theory since the early 90s, at the very least\(^6\). Concerning the spherical case in which we will be particularly interested in this paper, a series of notorious contributions in this respect (see [9, 10]) allowed for the explicit and systematic calculation of Casimir energies and all of the heat-kernel coefficients, which was a long-standing, very hard problem.

We recall that in gauge theory the small fluctuation operator \( L \) is singular due to gauge invariance, and therefore a gauge fixing term and the ensuing ghost contributions will necessarily appear.

The one-loop quantum partition function \( Z[L], S_0 \) being the classical action,

\[ Z[L] \simeq e^{-S_0} \int d[\eta] e^{-\frac{1}{2} \int d^4 \eta \partial^2 \eta}, \tag{1.2} \]

reduces to a Gaussian functional integral and, as is well known, it can be computed in terms of the real eigenvalues, \( \lambda_n \), of the operator, namely \( L\phi_n = \lambda_n \phi_n \). Since \( \phi = \sum_n c_n \phi_n \), the formal functional measure \( d[\phi] \) reads (\( \mu \) is an arbitrary renormalization parameter)

\[ d[\phi] = \prod_n \frac{dc_n}{\sqrt{\mu}}. \tag{1.3} \]

\(^5\) The same happens with regularized traces which, to begin with, are nonlinear, generically.

\(^6\) There is nothing mysterious, uncontrolled, or even difficult, in this matter, contrary to the impression one may get from the many papers around carrying out ad hoc calculations for each particular situation.
As a consequence, the one-loop quantum ‘prefactor’ is

$$Z_1[L] = \prod_n \frac{1}{\sqrt{\mu}} \int_{-\infty}^{\infty} dc_n e^{-c_n^2} = \left[ \det \left( \mu^{-2} L \right) \right]^{1/2}$$

(1.4)

and the one-loop Euclidean effective action reads

$$\Gamma_\epsilon = -\log Z = S_0 + \frac{1}{2} \log \left( \det \mu^{-2} L \right).$$

(1.5)

The above functional determinant is ill-defined but it can be expressed in the formal way

$$\left( \log \det L \right) = -\left( \int_0^\infty dt \ t^{-1} \ Tr e^{-tL} \right).$$

(1.6)

For large $t$ one faces no problem, since $L$ is non-negative; for small $t$ the heat kernel expansion in the regular smooth case and for $D = 4$ reads (see, for example, [11, 12])

$$\text{Tr} e^{-tL} \approx \sum_{r=0}^{\infty} A_r t^{-r-2}.$$}

(1.7)

It follows that the formal functional determinant is divergent at $t = 0$, and one needs an ulterior regularization. A simple and useful way to proceed is the use of the dimensional one [13], which in our formulation amounts to the replacement

$$t^{-1} \rightarrow \frac{t^{-1}}{\Gamma(1 + \epsilon)}.$$}

(1.8)

The related regularized functional determinant, with $\epsilon$ sufficiently large, is thus

$$\log \det \left( \epsilon L \right) = -\int_0^\infty dt \ t^{\epsilon - 1} \ Tr e^{-tL} = -\frac{\zeta(\epsilon, L)}{\epsilon},$$

(1.9)

where the generalized zeta function associated with $L$, defined for $\text{Res} > 2$ by

$$\zeta(s, L) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \ Tr e^{-tL},$$

(1.10)

has been introduced. In order to be able to handle the cutoff one makes use of the celebrated theorem by Seeley [14]: ‘If $L$ is an elliptic differential operator, defined on a smooth and compact manifold, the analytic continuation of $\zeta(s, L)$ to the whole complex plane $s$ is regular at $s = 0$.’ This provides the zeta-function determinant [15–18], in the form

$$\log \det L = -\zeta'(0, L).$$

(1.11)

Making use of dimensional regularization, one then arrives at

$$\log \det L(\epsilon) = -\frac{1}{\epsilon} \zeta'(0, L) - \zeta'(0, L) + O(\epsilon).$$

(1.12)

The computable Seeley–de Witt coefficient $A_2 = \zeta'(0, L)$ controls the ultraviolet divergence, while $\zeta'(0, A)$ gives a finite contribution which is, in general, difficult to evaluate, since it is non-local. For examples of exact evaluation see, for instance, [12] and references therein.

The main aim of the paper is to compute in a closed form the functional determinants related to massive Laplacians on spheres by making use of the non-multiplicative property of zeta-function regularized functional determinants. There are several physical motivations for considering massive Laplacians on spheres.
First, we recall the recent approach presented in [19, 20], where the Euclidean partition function associated with a de Sitter black hole is computed making use of the related quasinormal modes. In this paper, concrete examples of the very interesting claim [21] that black hole quasinormal modes determine the one-loop determinants will be discussed.

A second example is related to the computation of the one-loop effective action associated with modified gravitational models on the four dimensional sphere (Euclidean version of the de Sitter space-time) with applications to inflation and dark energy issue [22, 23].

Furthermore, other physical applications are mentioned in the recent paper by Dowker, [24], where a multiplicative anomaly interpretation is also advocated (see also previous references quoted there). A direct computation appeared also in [25].

The paper is organized as follows. In section 2 we will recall some details of the multiplicative anomaly issue. In sections 3 and 4, the factorization technique will be applied to two well known examples: the one-dimensional harmonic oscillator and QFT at finite temperature. In section 5 the general case of massive Laplacians on spheres will be studied and section 6 will be devoted to conclusions. Finally, appendix A contains explicit expressions for the Laplace operator on spheres in $N = 1, 2, 3, 4$ dimensions, and in appendix B the expressions for ‘vector’ and ‘tensor’ Laplacians on the unitary sphere $S^4$ are explicitly computed.

2. Multiplicative anomaly

The multiplicative anomaly issue arises most naturally in our approach. Quite often one has to deal with products of operators, mainly for convenience, in order to drastically simplify calculations and then the crucial point arises, that zeta-function regularized determinants do not satisfy the multiplicative property, in other words:

$$\ln \det (AB) \neq \ln \det A + \ln \det B. \quad (2.1)$$

In fact, generically, there exists the so-called multiplicative anomaly contribution, defined as the difference

$$a(A, B) = \ln \det(AB) - \ln \det(A) - \ln \det(B). \quad (2.2)$$

Here it is left understood that the determinants of the two operators, $A$ and $B$ (which do not exist in a strict sense, being both divergent) are defined by means of the corresponding zeta functions. This anomaly was discovered by several authors who had detected the problem independently and came up with particular solutions for each case (sometimes with erroneous results). Wodzicki was the first to give its name to the multiplicative anomaly and to construct a final theory for the whole class of pseudo-differential operators (PDO, the ones that appear in all physical applications) by discovering the only trace (up to trivial multiplication) which exists for the whole class.$^7$ This trace is now called the Wodzicki residue and the anomaly can be expressed in terms of it by a very simple general formula, as we will now see.

The multiplicative anomaly can be evaluated by the Wodzicki formula (a discussion can be found in [26] and references therein). In the simple but important case in which $A$ and $B$ are two operators of the same order $a = b$ such a formula becomes

$^7$ And which extends, in a unique way, the celebrated Dixmier trace and the Adler–Manin residue (for a more detailed description see, for instance, [6] and references therein).
where the non-commutative residue \( \text{res} Q \) related to a classical pseudo-differential operator \( Q \) of order zero is defined as the coefficient of the logarithmic term in \( t \) of the following expansion

\[
\text{Tr} \left( Q e^{-itH} \right) = \sum_j c_j t^{(j-N)/2} - \frac{\text{res} Q}{2} \ln t + O(t \ln t),
\]

\( H \) being an elliptic non-negative operator of second order (usually the Laplacian is taken), which precise form is irrelevant for the evaluation of \( \text{res} Q \).

As already anticipated in this paper we will mainly consider the shift of the Laplace operator (a massive Laplacian) defined on arbitrary dimensional spheres, and in order to compute the related regularized functional determinants, we shall make use of a product factorization in terms of two first-order operators, \( L = AB \), and compute the related functional determinant by the rule

\[
\ln \det(L) = \ln \det(A) + \ln \det(B) + a(A, B).
\]

In all the cases here considered, the evaluation of functional determinants for the first order operators \( A \) and \( B \) is easier than the evaluation of a functional determinant of the second-order Laplace-like operator \( L \). Of course there is a price to pay which consists in the computation of the multiplicative anomaly \( a(A, B) \). This can be evaluated in most physical applications and that is why the anomaly is so important and useful.

3. Application: path integral for harmonic oscillator

As a first example of the advantages of the factorization technique, let us consider the one-dimensional harmonic oscillator. Setting \( \hbar = 1 \), we can formally write the related Euclidean propagator as

\[
K_T := \int [dq] \ e^{-I_E[q]},
\]

with the Euclidean action given by

\[
I_E[q] = \int_0^T dt \left( \frac{1}{2} \ddot{q}^2(t) + \frac{\alpha^2 q^2(t)}{2} \right).
\]

Here, \([dq]\) represents the formal functional measure, the boundary conditions necessary to give a meaning to a formal path integral.

As is well known, the propagator (3.1) can be re-written in the form

\[
K_T(A) = \langle \hat{q}, T \mid q_0, 0 \rangle.
\]

As usual, one may formally proceed by splitting \( q \) into a ‘classical’ part, \( q_{cl} \), and a quantum fluctuation \( \hat{q} \), i.e.

\[
q(t) := q_{cl}(t) + \hat{q}(t).
\]

Here \( q_{cl} \) solves the classical equations of motion obtained by \( \delta I_E = 0 \) with boundary conditions \( q(0) = 0, \ q(T) = q \). Thus, from (3.4), it turns out that also the quantum fluctuations have to satisfy the boundary conditions \( \hat{q}(0) = 0 = \hat{q}(T) \).
The Euclidean action (3.2) becomes:

\[ I_E[q] = I_E[q_\text{cl}] + \frac{1}{2} \int_0^T dt \ 2 \dot{q}^2 \left[ -\frac{d^2}{dt^2} + \omega^2 \right] \dot{q} \tag{3.5} \]

while the classical action reads

\[ I_E[q_\text{cl}] = \int_0^T dt \left( \frac{1}{2} q_\text{cl}^2 + \frac{\omega^2}{2} q_\text{cl}^2 \right) \tag{3.6} \]

and it can be easily evaluated.

The propagator assumes the form of a Gaussian integral in the quantum fluctuation variables:

\[ \langle q, T | q_0, 0 \rangle = \exp \left( - I_E[q_\text{cl}] \right) \int [dq] \exp \left\{ - \frac{1}{2} \int_0^T dt \ 2 \dot{q}^2 \left[ -\frac{d^2}{dt^2} + \omega^2 \right] \dot{q} \right\} \tag{3.7} \]

As a consequence, one has to give a meaning to the formal Gaussian path integral. To this aim, let us denote by

\[ L := -\frac{d^2}{dt^2} + \omega^2 = L_0 + \omega^2 \tag{3.8} \]

the second-order differential operator in \( L_2(0, T) \), defined in the dense domain \( D(L) := \{ f, Lf \in L_2(0, T) \} \), with Dirichlet boundary conditions \( f(0) = f(T) = 0 \). The eigenfunctions are \( \sin(\pi t/T) \) and the spectrum reads

\[ \sigma(L) := \lambda_n := \left( \frac{n\pi}{T} \right)^2 + \omega^2, \ n = 1, 2, 3,... \tag{3.9} \]

We conclude this section with the final form of the propagator of the harmonic oscillator obtained by performing the Gaussian integral (3.7), that is

\[ \langle q, T | q_0, 0 \rangle = N \sqrt{\frac{1}{\det L}} \exp \left( - I_E[q_\text{cl}] \right), \tag{3.10} \]

where \( N \) is a normalization constant which can be fixed by requiring that \( \langle q, T = 0 | q_0, 0 \rangle = \delta(q - q_0) \).

3.1. Evaluation of functional determinant of massive Laplace operator

The functional determinant of the one-dimensional Laplace operator \( L \) may be evaluated by several different techniques. A general one consists in making use of binomial expansion, and expressing the final result as an infinite series of Riemann zeta functions. Another approach is the use of the Gelfand–Yaglom–Levit–Smilanski–Forman theorem [27–29] that gives the ratio of two functional determinants associated with ordinary differential operators \( L \) and \( L_0 \) in the form

\[ \frac{\det L}{\det L_0} = \frac{Y(T)}{Y_0(T)} \tag{3.11} \]

where \( Y(T) \) and \( Y_0(T) \) are the solution of the (Cauchy) problem

\[ LY = 0, \quad Y(0) = 0, \quad \dot{Y}(0) = 1, \quad Y(T) = \frac{\sinh \omega T}{\omega} \tag{3.12} \]
and
\[ L_0 Y_0 = 0, \quad Y_0(0) = 0, \quad \dot{Y}_0(0) = 1, \quad Y_0(T) = T. \]  
(3.13)

This leads to the well known result
\[ \frac{\det L}{\det L_0} = \frac{\sinh \omega T}{\omega T}. \]  
(3.14)

We will now show that this regularized determinant can be most conveniently computed with the factorization technique illustrated in previous section.

To this aim let us first factorize the operator \( L \) as
\[ L = K^\dagger K = (P + i\omega)(P - i\omega), \quad K = P - i\omega, \]  
(3.15)

where the self-adjoint operator \( P = \sqrt{-\frac{d^2}{d\tau^2}} \) is defined via the spectral theorem of \( L_0 \) and its spectrum reads \( \sigma(P) = \frac{\pi n}{T}, n = 1,2,3,... \). As a consequence
\[ \det L = \det \left( K^\dagger K \right), \]  
(3.16)

and
\[ \ln \det L = \ln \det K^\dagger + \ln \det K + a\left(K^\dagger, K\right). \]  
(3.17)

Making use of equations (2.3) and (2.4), a direct calculation leads to \( a(K^\dagger, K) = 0 \) (see also section 5), and thus we have
\[ \ln \det L = -\zeta'\left(0 \mid K^\dagger\right) - \zeta'\left(0 \mid K\right). \]  
(3.18)

Since \( \sigma(K^\dagger) = \frac{\pi n}{T} + i\omega, \sigma(K) = \frac{\pi n}{T} - i\omega (n = 1,2,3,...) \), one easily has
\[ \zeta\left(s \mid K^\dagger\right)(\omega) = \left(\frac{\pi}{T}\right)^s \left[ \zeta\left(s, \frac{i\omega T}{\pi}\right) - \left(\frac{i\omega T}{\pi}\right)^s \right]. \]  
(3.19)

where \( \zeta(s, q) \) is the Hurwitz zeta function defined by
\[ \zeta(s, q) = \sum_{n=0}^{\infty} (n + q)^{-s}, \quad \text{Re} \ s > 1, \quad q \neq -n. \]  
(3.20)

The latter expression can be analytical extended to the whole complex \( s \)-plane and \( q \)-plane. The following properties are valid:
\[ \zeta(0, q) = \frac{1}{2} - q, \quad \zeta'(0, q) = \ln \Gamma(q) - \frac{1}{2} \ln 2\pi. \]  
(3.21)

Making use of (3.18), (3.19) and (3.21), a direct calculation yields
\[ \det L = 2 \sinh \omega T \omega. \]  
(3.22)

In the limit \( \omega \to 0 \) one obtains
\[ \det L_0 = 2T, \]  
(3.23)
in complete agreement with the G–Y–L–S–F theorem result (3.14).
4. Application: QFT at finite temperature

As a second example let us consider a free charged boson field at finite temperature $\beta = 1/T$, and chemical potential $\mu$. The related grand canonical partition function reads

$$Z_{\beta, \mu} = \int d\phi e^{\frac{1}{\beta} \int d^4x \mathcal{L} \phi \partial_x \phi}$$ (4.1)

with $A_{ij} = \left( L_\tau + L_3 - \mu^2 \right) \delta_{ij} + 2 \mu \psi \sqrt{L_\tau}$. $L_\tau = -\Delta_3 + m^2$, $\Delta_3$ being the Laplace operator on $\mathbb{R}^3$ (it has a continuous spectrum $\vec{k}^2$) and $L_\tau = -\partial_\tau^2$ (it has a discrete spectrum, with Matsubara frequencies $\omega_n^2 = 4\pi^2 / \beta^2$). Thus, the grand canonical partition function can be written as (see, for example, [30] and references therein)

$$\ln Z_{\beta, \mu} = -\ln \det |A_{ik}|. \quad (4.2)$$

Now the algebraic determinant of $A$, $|A|$, can be evaluated through the factorization

$$|A_{ik}| = \det (K_+ K_-), \quad (4.3)$$

where $K_\pm = L_3 + (\sqrt{L_\tau} \pm i\mu)^2$. However, it is easy to see that another convenient factorization exists [30], namely

$$|A_{ik}| = \det (L_+ L_-), \quad (4.4)$$

with $L_\pm = L_\tau + (\sqrt{L_\tau} \pm \mu)^2$ (again, $A = L_+ L_- = K_+ K_-), and in both cases one is dealing with the product of two $\Psi$DOs, the couple $L_+ \text{ and } L_-$ being also formally self-adjoint. This is a very interesting situation. To wit, the partition function can be written in both the forms

$$\ln Z_{\beta, \mu} = -\ln \det K_+ - \ln \det K_- + a \left( K_+ K_- \right),$$

$$= -\ln \det L_+ - \ln \det L_- + a \left( L_+ L_- \right). \quad (4.5)$$

The evaluation of the multiplicative anomalies which appear in both expressions above can be performed by making use of Wodzicki’s formula and, indeed, complete agreement is found for the two expressions of the partition function. Moreover, it is quite easy to realize that if one neglects the multiplicative anomaly contribution, one arrives at a sound mathematical inconsistency: the results obtained in the two different factorizations are quite different [30].

5. Functional determinants of massive Laplacians on spheres of arbitrary dimension

The multiplicative anomaly plays a relevant role in the case of the evaluation of functional determinants of massive Laplacians on arbitrary dimensional spheres. Its specific importance there has also been recently pointed out in references [19, 20] and by Dowker [2, 24]. Our approach in the present paper is however quite different from the ones advocated in [19, 20, 24], in the sense that we will compute the massive determinant by suitable factorization, similar to the one used in the previous section, and compute the associated multiplicative anomaly by making use of Wodzicki’s formula.

To start, we recall the eigenvalues and relative degeneration of the Laplace operator acting on a scalar function in an $N$-dimensional sphere $S^N$, and we introduce some useful notation, as follows

\[\text{We should again observe that this has caused a substantial number of errors in the literature.}\]
$\Delta_N$, scalar Laplacian on $S^N$;

$$\lambda^N_n = n(n + 2\nu_N) = (\alpha^N_n)^2 - \nu^2_N, \text{ eigenvalues;}$$

$$d^N_n = \sum_{k=0}^{N-1} c_k^N (\alpha^N_n)^k, \text{ degeneration;}$$

$$\Omega_N = \frac{2\pi^{(N+1)/2}}{\Gamma((N+1)/2)}, \text{ volume (hyper-surface) of } S^N;$$

$$\nu_N = \frac{N - 1}{2}, \quad \alpha^N_n = n + \nu_N, \quad (5.1)$$

where $n$ runs from 0 to $\infty$ and $(\alpha^N_n)^2$ are the eigenvalues of the operator $\hat{L}_N = -\Delta_N + \nu^2_N$. This is a positive operator and its square root has eigenvalues $\alpha^N_n$ with degeneration $d^N_n$.

The degeneration of the eigenvalues assumes the explicit form:

$$d^N_n = \begin{cases} d^0_n = 1, & d^1_n = 2, \\ d^2_n = 2(n + \nu_2), & d^2_n = 2(n + \nu_2), \end{cases}$$

$$d^N_n = \frac{2}{(N-1)!} \prod_{k=0}^{(N-3)/2} \left[ (n + \nu_N)^2 - k^2 \right], \quad \text{for odd } n \geq 3,$$

$$d^N_n = \frac{2(N + \nu_N)}{(N-1)!} \prod_{k=0}^{(N-4)/2} \left[ (n + \nu_N)^2 - (k + 1/2)^2 \right], \quad \text{for even } n \geq 4,$$

(5.2)

and this permits to compute the coefficients $c_k^N$ in equation (5.1).

From now on, for simplicity, the index $N$ will be left understood (recall all quantities depend on $N$). Thus, we will write $L, \nu, c_k, \ldots$ in place of $L_N, \nu_N, c^N_k, \ldots$, etc.

We are now ready to compute the determinant of the Laplace-like operator. For the special operator $\hat{L}$, with eigenvalues $\alpha_n$, we can also compute the corresponding zeta function in terms of a finite sum of Hurwitz zeta functions, but this will not be the case for a Laplacian with arbitrary mass.

For $\text{Re } s$ sufficiently large (this depends on the dimension), we have

$$\zeta(s | \hat{L}) = \sum_{k=0}^{\infty} \sum_{n=0}^{N-1} c_k \alpha_n^{-2s+k} = \sum_{k=0}^{N-1} c_k (n + \nu)^{-2s+k} = \sum_{k=0}^{N-1} c_k \zeta(2s - k, \nu), \quad (5.3)$$

where $\zeta(s, q)$ is the Hurwitz zeta function. Note that, for even $N$, $c_{2k} = 0$, while for odd $N$, $c_{2k+1} = 0$ and so the sum is performed on either odd or even $k < N$.

Now, let us consider the operator $L = \hat{L} + \alpha^2$ ($\alpha$ is an arbitrary constant), and the two pseudo-differential operators $D_\pm$, such that

$$L = D_+ D_-, \quad D_\pm = \sqrt{\hat{L}} \pm i\alpha. \quad (5.4)$$

One has

$$\log \det L = \log \det D_+ + \log \det D_- + a(D_+, D_-), \quad (5.5)$$

where $a(D_+, D_-)$ is the multiplicative anomaly. In order to compute it, we make use of equations (2.3) and (2.4), choosing $H = \hat{L}$ in (2.4), the spectral theorem gives
\[
\text{Tr}\left(\log \frac{L + ia}{\sqrt{L} - ia}\right) e^{-L} = \sum_{n=0}^{\infty} e^{-i\alpha^2 f(a_n)}
\]
\[
= \sum_j c_j f^{(j-N)/2} - \frac{\text{res} \left[ \log \frac{\sqrt{L} + ia}{\sqrt{L} - ia} \right]^2}{2} \ln t + O(t \ln t),
\] (5.6)

where
\[
f(a_n) = d_n^N(a_n) \left( \log \frac{a_n + ia}{a_n - ia} \right)^2.
\] (5.7)

In this way
\[
\text{res } Q = -\text{Res } (f(z), z = \infty),
\] (5.8)

\[
\text{Res } (f, z) \text{ being the ordinary Cauchy residue. As a result, the multiplicative anomaly reads}
\]
\[
a(D_+, D_-) = -\frac{\text{Res } (f(z), z = \infty)}{4}. (5.9)
\]

In odd dimensions, \(d_n^N\) is an even polynomial in \(a_n\) and so the multiplicative anomaly is trivially vanishing, while in even dimensions the multiplicative anomaly is a polynomial of order \(N\).

Using (1.11) for the regularized definition of the determinant of the operator we get, in the present case,
\[
\log \det L = \log \det (D_+ D_-) = -\zeta'(0)D_+ - \zeta'(0)D_- + a(D_+, D_-). (5.10)
\]

For \(\text{Re } s\) sufficiently large (the actual value depends on the dimension), we obtain
\[
\zeta(s \mid D_+) = \sum_{n=0}^{\infty} \sum_{k=0}^{N-1} c_k \alpha_n^k (\alpha_n \pm ia)^{-s}
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{N-1} \sum_{j=0}^{k} c_k b_{kj} (\alpha_n \pm ia)^j (\mp ia)^{k-j} (\alpha_n \pm ia)^{-s}
\]
\[
= \sum_{k=0}^{N-1} \sum_{j=0}^{k} c_k b_{kj} (\mp ia)^{k-j} \zeta(s-j, \nu \pm ia), (5.11)
\]

where \(b_{kj}\) are binomial coefficients. Then, it finally follows that
\[
\log \det L = a(D_+, D_-)
\]
\[
= \sum_{k=0}^{N-1} \sum_{j=0}^{k} c_k b_{kj} (ia)^{k-j} \left[ \zeta'(-j, \nu - ia) + (-1)^{k-j} \zeta'(-j, \nu + ia) \right]. (5.12)
\]

This general expression yields the logarithm of the determinant of the massive Laplace operator on a hypersphere in any number of dimensions in terms of a finite sum of Hurwitz zeta functions. In appendix A we write explicit formulae for dimensions \(N = 1, 2, 3, 4\), while in appendix B we extend the computation to Laplace–Beltrami operators acting on vector and tensor fields. In these last cases, we limit ourselves to the physical dimension \(N = 4\).
Finally, with regard to the comparison with other works, our results agree with similar results obtained by Dowker in [2, 24].

6. Conclusions

At the beginning of this paper we have presented a brief survey of zeta function regularization and, in particular, of the related multiplicative anomaly issue, which has been done in section 2. Our point being that, as can be easily checked in the physical literature on the subject, even if those issues have long been well and rigorously established in the mathematical community dealing with physical applications, this is still not the case among physicists. It is worthwhile to summarize the main points again and explain them with the help of a couple of basic but very useful examples, namely the one-dimensional harmonic oscillator and QFT at finite temperature, as we did in sections 3 and 4, respectively.

A major new contribution has been the computation, carried out in section 5, of the general case corresponding to massive Laplacians on spheres in arbitrary dimensions. A general formula which yields the determinant of the massive Laplace operator on a hypersphere in any number of dimensions in terms of a finite sum of Hurwitz zeta functions has been obtained. Further, explicit expressions for the Laplace operator on spheres in \( N = 1, 2, 3, 4 \) dimensions have been given, in appendix A, and those for ‘vector’ and ‘tensor’ Laplacians on the unitary sphere \( S^4 \), in appendix B. These last are, in no way, straightforward cases, and the fact that they could be treated in such a simple and general way by using the zeta function procedure is an excellent proof of the power of this method.

The results concerning the scalar Laplacian agree with analogous results obtained in [2, 24].

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Appendix A. Laplace operator on the sphere in \( N = 1, 2, 3, 4 \) dimensions

Consider the operators \( L = -\Delta + \rho^2 \) acting on functions on the sphere in 1, 2, 3 and 4 dimensions. Here we use the notation as in section 5.

\( A.1. N = 1 \)

For eigenvalues and degeneration one trivially has \( \lambda_n = n^2, d_n = 2 \) (excluding the vanishing eigenvalue \( \lambda_0 = 0 \)). The multiplicative anomaly is vanishing and so

\[
\zeta(s|L_1) = 2\zeta(2s), \quad \alpha = \rho, \quad \zeta(0, i\alpha) = \zeta(0, i\alpha). \tag{A.1}
\]

\[
\log \det L_1 = -2[\zeta'(0, -i\alpha) + \zeta'(0, i\alpha)]. \tag{A.2}
\]
A.2. $N = 2$

$$L_2 = L_2 + a^2, \quad \alpha^2 = \rho^2 - \nu^2 = \rho^2 - \frac{1}{4}, \quad (A.3)$$

$$\nu = \frac{1}{2}, \quad d_n = 2(n + \nu), \quad c_1 = 2, \quad c_k = 0 \text{ for } k \neq 1, \quad (A.4)$$

$$\zeta \left( s \mid \hat{L}_2 \right) = 2\zeta(2s - 1, 1/2) = 2(2^{2s-1} - 1) \zeta(2s - 1), \quad (A.5)$$

$$\log \det L_2 = -2\alpha^2 - 2i\alpha \left[ \zeta'(0, 1/2 - i\alpha) - \zeta'(0, 1/2 + i\alpha) \right]$$

$$- 2[\zeta'(-1, 1/2 - i\alpha) + \zeta'(-1, 1/2 + i\alpha)]. \quad (A.6)$$

A.3. $N = 3$

$$L_3 = \hat{L}_3 + a^2, \quad \alpha^2 = \rho^2 - \nu^2 = \rho^2 - 1, \quad (A.7)$$

$$\nu = 1, \quad d_3 = 2(n + 1), \quad c_2 = 1, \quad c_k = 0 \text{ for } k \neq 2, \quad (A.8)$$

$$\zeta \left( s \mid \hat{L}_3 \right) = \zeta(2s - 2, 1) = \zeta(2s - 2), \quad (A.9)$$

$$\log \det L_3 = a^2 \left[ \zeta'(0, 1 - i\alpha) + \zeta'(0, 1 + i\alpha) \right] - 2i\alpha \left[ \zeta'(-1, 1 - i\alpha) - \zeta'(-1, 1 + i\alpha) \right]$$

$$- [\zeta'(-2, 1 - i\alpha) + \zeta'(-2, 1 + i\alpha)]. \quad (A.10)$$

A.4. $N = 4$

Here $\alpha^2 = \rho^2 - 9/4$ and

$$\zeta \left( s \mid \hat{L}_4 \right) = -\frac{1}{12} \zeta(2s - 1, 3/2) + \frac{1}{3} \zeta(2s - 3, 3/2), \quad (A.11)$$

$$\log \det L_4 = \frac{2}{9} \alpha^4 + \frac{1}{12} \alpha^3 + \frac{1}{3} i\alpha \left[ \zeta'(0, 3/2 - i\alpha) - \zeta'(0, 3/2 + i\alpha) \right]$$

$$+ \frac{1}{12} i\alpha \left[ \zeta'(0, 3/2 - i\alpha) - \zeta'(0, 3/2 + i\alpha) \right]$$

$$+ \alpha^2 \left[ \zeta'(-1, 3/2 - i\alpha) + \zeta'(-1, 3/2 + i\alpha) \right]$$

$$+ \frac{1}{12} \left[ \zeta'(-1, 3/2 - i\alpha) + \zeta'(-1, 3/2 + i\alpha) \right]$$

$$- i\alpha \left[ \zeta'(-2, 3/2 - i\alpha) - \zeta'(-2, 3/2 - i\alpha) \right]$$

$$- \frac{1}{3} \left[ \zeta'(-3, 3/2 - i\alpha) + \zeta'(-3, 3/2 + i\alpha) \right]. \quad (A.12)$$

Appendix B. ‘Vector’ and ‘tensor’ Laplacian on the unitary sphere $S^4$

Now we consider the Laplace–Beltrami operators $\Delta^{(v,t)}$ acting on traceless-transverse vector and tensor fields on the unitary sphere $S^4$. In such a case the eigenvalues and the corresponding degenerations read
\[ \Delta^{(v,t)}, \] vector/tensor Laplacian on \( S^4; \]

\[ \lambda_n^{(v,t)} = \left[ n + \nu^{(v,t)} \right]^2 - \gamma^{(v,t)}, \] eigenvalues;

\[ d_n^{(v,t)} = a^{(v,t)} \left[ n + \nu^{(v,t)} \right] + b^{(v,t)} \left[ n + \nu^{(v,t)} \right]^3, \] degeneration;

\[ \nu^{(v)} = \frac{5}{2}, \quad \nu^{(t)} = \frac{7}{2}, \]

\[ \gamma^{(v)} = \frac{13}{4}, \quad \gamma^{(t)} = \frac{17}{4}, \]

\[ a^{(v)} = -\frac{9}{4}, \quad b^{(v)} = 1, \]

\[ a^{(t)} = -\frac{125}{12}, \quad b^{(t)} = \frac{5}{3}, \quad (B.1) \]

\([n + \nu^{(v,t)}]^2\) being the eigenvalues of the operators \( \hat{L}^{(v,t)} = -\Delta^{(v,t)} + \gamma^{(v,t)} \). These are positive operators and their square roots have eigenvalues \( n + \nu^{(v,t)} \) with degenerations \( d_n^{(v,t)} \).

Now we can proceed as in section 5 and define

\[ L^{(v,t)} = -\Delta^{(v,t)} + \rho^2 = L^{(v,t)} + a_0^{(v,t)} = D_+^{(v,t)}D_-^{(v,t)}, \quad (B.2) \]

\[ \log \det L^{(v,t)} = -\zeta \left( 0 \mid D_+^{(v,t)} \right) - \zeta \left( 0 \mid D_-^{(v,t)} \right) + a \left( D_+^{(v,t)}, D_-^{(v,t)} \right) \]

\[ = -\zeta \left( 0 \mid D_+^{(v,t)} \right) - \zeta \left( 0 \mid D_-^{(v,t)} \right) + a \left( D_+^{(v,t)}, D_-^{(v,t)} \right), \quad (B.3) \]

where

\[ a_0^{(v)} = \rho^2 - \gamma^{(v)} = \rho^2 - \frac{13}{4}, \quad \quad a_0^{(t)} = \rho^2 - \gamma^{(t)} = \rho^2 - \frac{17}{4}. \quad (B.4) \]

Also in these cases the anomaly can be easily computed by means of equation \((5.9)\), with appropriate eigenvalues and degeneration.

For simplicity here we assume positive eigenvalues of the operators \( L^{(v,t)} \). Negative or vanishing eigenvalues have to be considered separately (see appendix in \( [22] \)).

The specific computation is similar to the one in example A.4. One gets

\[ \zeta \left( 0 \mid L^{(v,t)} \right) = \frac{1}{4} \rho^4 - \frac{1}{2} \rho^3 - \frac{15}{4}, \quad (B.5) \]

\[ \log \det L^{(v,t)} = \frac{2}{3} \alpha^4 + \frac{9}{4} \alpha^2 + 3\zeta \left( 0, \frac{5}{2} - ia \right) + i \zeta \left( 0, \frac{5}{2} + ia \right) \]

\[ + 3\zeta \left( -1, \frac{5}{2} - ia \right) + 3\zeta \left( -1, \frac{5}{2} + ia \right) - 3\zeta \left( -2, \frac{5}{2} - ia \right) + 3i \zeta \left( -2, \frac{5}{2} + ia \right) \]

\[ + 3i \zeta \left( -2, \frac{5}{2} + ia \right) + 9 \zeta \left( 0, \frac{5}{2} - ia \right) + 9 \zeta \left( 0, \frac{5}{2} + ia \right) - 9 \zeta \left( -1, \frac{5}{2} - ia \right) \]

\[ - \zeta \left( -3, \frac{5}{2} - ia \right) - \zeta \left( -3, \frac{5}{2} + ia \right) + 9 \zeta \left( -1, \frac{5}{2} - ia \right) \]

\[ + \frac{9}{4} \zeta \left( -1, \frac{5}{2} + ia \right). \quad (B.6) \]
\[ \zeta (0 | L^{(i)}) = \frac{5}{12} \rho^4 + \frac{5}{3} \rho^2 - \frac{40}{3}, \quad (B.7) \]

\[ \log \det L^{(i)} = \frac{10}{9} \alpha^4 + \frac{125}{12} \alpha^2 + \frac{5}{3} i \zeta' \left( -\frac{7}{2} - i \alpha \right) \alpha^2 \]

\[ + 5 \zeta' \left( -1, \frac{7}{2} - i \alpha \right) \alpha + \frac{5}{3} i \zeta' \left( -1, \frac{7}{2} + i \alpha \right) \alpha^2 \]

\[ + 5 i \zeta' \left( -2, \frac{7}{2} + i \alpha \right) + \frac{125}{12} i \zeta' \left( -3, \frac{7}{2} + i \alpha \right) + \frac{125}{12} i \zeta' \left( -3, \frac{7}{2} - i \alpha \right) \]

\[ + \frac{125}{12} i \zeta' \left( -1, \frac{7}{2} + i \alpha \right) \alpha. \quad (B.8) \]

It is understood that in all equations above, \( \alpha \) has to be replaced by the appropriate expression (\( \alpha_v \) or \( \alpha = \alpha_t \)).

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