On the synthesis of control policies from noisy example datasets: a probabilistic approach

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Abstract: In this note we consider the problem of synthesizing optimal control policies for a system from noisy datasets. We present a novel algorithm that takes as input the available dataset and, based on these inputs, computes an optimal policy for possibly stochastic and nonlinear systems that also satisfies actuation constraints. The algorithm relies on solid theoretical foundations, which have their key roots into a probabilistic interpretation of dynamical systems. The effectiveness of our approach is illustrated by considering an autonomous car use case. For such use case, we make use of our algorithm to synthesize a control policy from noisy data allowing the car to merge onto an intersection, while satisfying additional constraints on the variance of the car speed.

1. INTRODUCTION

A framework that is becoming particularly appealing to design control algorithms is that of devising the control policy from examples (or demonstrations), see e.g. Hanawal et al. (2019); Wabersich and Zeilinger (2018) and references therein. At their roots these control techniques, which are gaining considerable attention under the label of Inverse Reinforcement Learning (IRL), rely on Inverse Optimal Control and Optimization (IRL). Today, IRL/control is recognized as an appealing framework to learn policies from successes/prescriptions learning Xu and Paschalidis (2019).

There is then no surprise that, over the years, a number of techniques have been developed to address the problem of devising control policies from demonstrations, mainly in the context of Markov Decision Processes (MDPs) Sutton and Barto (1998). Results include Ratliff et al. (2009), which leverages a linear programming approach, Ratliff et al. (2006) which relies on a maximum margin approach, Ziebart et al. (2008) that makes use of the maximum entropy principle and Ramachandran and Amir (2007) that formalizes the problem via Bayesian statistics.

In this context, the main contributions of this extended abstract can be summarized as follows. First, we introduce an approach to synthesize control policies from datasets which is based on the Fully Probabilistic Design (FPD) Karner (1996); Karner and Guy (2006); Herzallah (2015); Pegueroles and Russo (2019); Kran and Kroupa (2012). This approach formalizes the control problem as an optimization problem where the Kullback-Leibler divergence is minimized subject to constraints on the control variables. By relying on the FPD, one of the main advantages of our approach is that policies can be synthesized from noisy data without requiring that the system is a MDP. Moreover, by embedding actuation constraints into the problem formulation and by solving the resulting optimization, we can export the policy that has been learned on other systems that have different actuation capabilities.

As an additional contribution, we devise from our theoretical results an algorithmic procedure. The key reference applications over which the algorithm was tested involved an autonomous driving use case and full results are presented here.

2. MATHEMATICAL PRELIMINARIES

2.1 Notation

Sets, as well as operators, are denoted with calligraphic characters, while vector quantities are denoted in bold. Let \( n \) be a positive integer and consider the measurable space \((\mathcal{Z}, \mathcal{F}_z)\), with \( \mathcal{Z} \subseteq \mathbb{R}^n \), and with \( \mathcal{F}_z \) being a \( \sigma \)-algebra on \( \mathcal{Z} \). Then, the random variable on \((\mathcal{Z}, \mathcal{F}_z)\) is denoted by \( Z \) and its realization is denoted by \( z \). The pdf of \( Z \) is denoted by \( f(z) \) (or, equivalently, by \( f_Z(z) \) and its support is denoted by \( S(f) \). We recall that, given a function \( h(.) \), the expectation of \( h(z) \), i.e. \( E_f[h(Z)] \) is defined as \( E_f[h(Z)] := \int_{\mathcal{F}_z} h(z) f(z) dz \). For notational convenience, whenever it is clear from the context, we omit the domain of integration as well as the subscript in the expectation. The conditional probability density function (cpdf) of \( Z \) with respect to the random variable \( Y \) is denoted by \( f_X|Y(x) \) and sometimes we will use the shorthand notation \( f_Z \). Given \( Z \subseteq \mathbb{R}^n \), its indicator function is denoted by \( 1_Z(z) \) and \( 1_Z(z) = 1, \forall z \in Z \) and...
2.2 The Kullback-Leibler divergence

The control problem considered in this abstract will be stated (see Section 2.4) in terms of the Kullback-Leibler (KL, Kullback and Leibler (1951)) divergence:

**Definition 1.** (Kullback-Leibler(KL) divergence) Consider two pdfs, \( f_1(z) \) and \( f_2(z) \), with \( f_1(z) \) being absolutely continuous with respect to \( f_2(z) \). Then, the KL-divergence of \( f_1(z) \) with respect to \( f_2(z) \) is

\[
\mathcal{D}_{KL}(f_1 \| f_2) := \int_{S(f_1)} f_1 \ln \left( \frac{f_1}{f_2} \right) \, dz. \tag{1}
\]

Intuitively, \( \mathcal{D}_{KL}(f_1 \| f_2) \) measures how well \( f_1(z) \) approximates \( f_2(z) \). We also recall that \( \mathcal{D}_{KL}(f_1 \| f_2) \) exists if \( S(f_1) \subseteq S(f_2) \). We assume that the KL-divergence for the pdfs of our interest exists.

2.3 Formulation of the Control Problem

Let: (i) \( \mathcal{I} := \{i\}_{i=0}^{n-1}, \mathcal{I}_0 := \mathcal{I} \cup \{\emptyset\} \) be the time horizon over which the system is observed; (ii) \( x_k \in \mathbb{R}^d \) and \( u_k \in \mathbb{R}^{d_u} \) be, respectively, the system state and input at time \( t_k \in \mathcal{T} \); (iii) \( d_k := (x_k, u_k) \) be the data collected from the system at time \( t_k \in \mathcal{T} \) and \( d^k \) the data collected from \( t_0 \in \mathcal{T} \) up to time \( t_k \in \mathcal{T} \) (\( t_k > t_0 \)). Then, as shown in e.g. Peterka (1981), the system behavior can be described via the joint pdf of the observed data, say \( f(d^u) \). Then, as shown in the same paper, the application of the chain rule for probability density functions leads to the following factorization for \( f(d^u) \):

\[
f(d^u) = \prod_{i \in \mathcal{I}} f(x_k | u_k, x_{k-1}) f(u_k | x_{k-1}) f(x_0). \tag{2}
\]

Throughout this abstract we will refer to (2) as the probabilistic description of the closed loop system, or simply say that (2) is our closed loop system.

**Remark 1.** The cpdf \( f(x_k | u_k, x_{k-1}) \) describes the system behavior at time \( t_k \), given the previous state and the input at time \( t_k \). In turn, the input is also generated from the cpdf of a randomized control algorithm \( f(u_k | x_{k-1}) \), which indeed returns the input given the previous system state. We also note that initial conditions are embedded in the probabilistic system description through the prior \( f(x_0) \).

In the following we will use the shorthand notations \( f_k^u := f(x_k | u_k, x_{k-1}) \), \( f_k^u := f(x_k | u_k, x_{k-1}) \), \( f_k^u := f(x_k | u_k, x_{k-1}) \), \( f^u := f(d^u) \) and \( f^u := f(d^u) \) so that (2) can be written in the more compact form

\[
f^u = \prod_{i \in \mathcal{I}} f_k^u f_k^u f_k^u = f^u f_0, \quad f^u := \prod_{i \in \mathcal{I}} f_k^u f_k^u. \tag{3}
\]

2.4 The control problem

Our goal is to synthesize the control pdf \( f(u_k | x_{k-1}) \) so that the behavior illustrated via an example dataset, say \( d^u \), can be tracked by system (3) subject to its actuation constraints. As in Kárný (1996); Quinn et al. (2016); Piegler and Russo (2019); Kárný and Guy (2006); Herzallah (2015) the behavior illustrated in the example dataset can be specified through the reference pdf \( g(d^u) \) extracted from the dataset (as e.g. its empirical distribution). Following the chain rule for pdfs we have

\[
g(d^u) := \prod_{i \in \mathcal{I}} g(x_k | u_k, x_{k-1}) g(u_k | x_{k-1}) g(x_0). \tag{4}
\]

Again, by setting \( \tilde{g}_k^u := g(x_k | u_k, x_{k-1}) \), \( \tilde{g}_k^u := g(u_k | x_{k-1}) \), \( g_0 := g(x_0) \) and \( g^u := g(d^u) \) we get:

\[
\tilde{g}^u := \prod_{i \in \mathcal{I}} \tilde{g}_k^u \tilde{g}_k^u g_0 = \tilde{g}^u g_0, \tag{5}
\]

where \( \tilde{g}^u := \prod_{i \in \mathcal{I}} \tilde{g}_k^u \tilde{g}_k^u g_0 \).

Our tracking problem can then be recast as the problem of designing \( f(u_k | x_{k-1}) \) so that \( \tilde{g}^u \) approximates \( \tilde{g}^u \). This leads to the following formalization:

**Problem 1.** Determine the sequence of cpdfs, say \( \{\tilde{f}_k^u\}_{k \in \mathcal{I}} \), solving the nonlinear program

\[
\begin{align*}
\min_{\tilde{f}_k^u} & \quad \mathcal{D}_{KL}(f^u || g^u) \\
\text{s.t.} & \quad \mathbb{E}_{\tilde{f}_k^u} \left[ h_{u,k} \right] = \tilde{H}_{u,k}, \quad k \in \mathcal{I}
\end{align*}
\tag{6}
\]

We note that the program constraints can be equivalently written as

\[
\int_{S(f_k^u)} h_{u,k} \, d\tilde{f}_k^u = \tilde{H}_{u,k} \quad \mathbb{E}_{\tilde{f}_k^u} \left[ h_{u,k} \right] = \tilde{H}_{u,k}, \quad k \in \mathcal{I}
\]

Finally, the constraints of the program are time-varying and the number of constraints can change over time (the number of constraints at time \( t_k \) is denoted by \( c_{u,k} \)). Indeed, in the constraints of (5): (i) \( \tilde{H}_{u,k} \) is a (column) vector of coefficients, i.e. \( \tilde{H}_{u,k} := \left[ H_{u,0,k}, H_{u,T,k}^T \right]^T \) and \( h_{u,k} \) is \( \begin{cases} h_{u,0,k}, & H_{u,0,k} \\ h_{u,T,k}, & H_{u,T,k} \end{cases} \), \( h_{u,0,k}, \tilde{H}_{u,k} : S(\tilde{f}_k^u) \mapsto \mathbb{R}^c_{u,k} \); (ii) \( \tilde{H}_{u,k} \in \mathbb{R}^{c_{u,k}} \) and \( h_{u,k} \in S(\tilde{f}_k^u) \); (iii) \( H_{u,0,k} := 1 \) and \( h_{u,0,k} \) ensure that the solution of the program is a cpdf.

3. TECHNICAL RESULTS

We now introduce the main technical result (i.e. Theorem 1) behind the algorithm of Section 4. A sketch of the proofs of the technical lemmas (i.e. Lemma 1 and Lemma 2) are given in the appendix.

**Lemma 1.** Let \( Z \) be a random variable on the measurable space \( (\mathcal{Z}, \mathcal{F}_Z) \), \( f := f_Z(z), g := g_Z(z) \) be two probability distributions over \( (\mathcal{Z}, \mathcal{F}_Z) \), \( \alpha : \mathcal{Z} \mapsto \mathbb{R}_+^n \) be a nonnegative function of \( Z \), integrable under the measure given by \( f_Z(z) \). Given this set-up, assume that \( f_Z(z) \) satisfies the following set of algebraically independent constraints

\[
\int f_Z(z) \, dH(z) = \tilde{H}, \tag{6}
\]

where: (i) \( \tilde{H}(z) := [h_{00}(z), h_T(z)] \), with \( h_{00}(z) := 1 \in S(Z(z)) \) and \( h : \mathcal{Z} \rightarrow \mathbb{R}^c \) being a measurable map; (ii) \( \tilde{H}(z) := [H_0(z), H_T(z)] \) with \( H_0 := 1 \) and \( H \in \mathbb{R} \) being a vector of constants. Then, the solution of the constrained optimization problem

\[
\begin{align*}
\min_{f_Z} & \quad \mathcal{D}_{KL}(f \| g) + \int f_Z(z) \, d\alpha(z) \\
\text{s.t.} & \quad \text{constraints in (6)}
\end{align*}
\tag{7}
\]

is the pdf

\[
f_Z^\ast(z) = \frac{g(z) e^{-(\alpha(z)+\lambda^*(z)h(z))}}{e^{+\lambda^*(z)h(z)}}, \tag{8}
\]

where \( \lambda^* \) and \( \lambda^* := [\lambda^*_1, \ldots, \lambda^*_n]^T \) are the Lagrange multipliers associated to the constraints. Moreover, the
corresponding minimum of the cost function $\mathcal{J}(f) := \mathcal{D}_{\text{KL}}(f||g) + \int f(x) \alpha(x) \, dx$ is
\[
\mathcal{J}^* := \mathcal{J}(f^*) = - (1 + \lambda^*_i + \langle \lambda^*, \mathbf{H} \rangle).
\]

**Proof:** See the appendix \(\Box\)

Note that, in Lemma 1, the optimal solution $f^*_Z(z)$ depends on the Lagrange multipliers (LMs) $\lambda^*_i$ and $\lambda^*$. While $\lambda^*_i$ can be obtained by integration, all the other LMs need to be computed numerically. With the next result, we propose a strategy for finding the LMs $\lambda^*$. In particular, our key idea is to recast the problem of finding the solutions of non-linear equations as a minimization problem. In general, the approach can be also used to fit the parameters of a pdf so that it meets a set of pre-specified constraints (for example, to find pdfs that satisfy the Maximum Entropy principle Guilleminot and Soize (2013)).

**Lemma 2.** Let: (i) $Z \subseteq \mathbb{R}^n_z$ and $\mathbf{\hat{\Theta}} \subseteq \mathbb{R}^n_z$; (ii) $\hat{f}_1 : Z \mapsto \hat{f}_1(z)$ be a positive and integrable function on $Z$; (iii) $\hat{f}_2 : (Z \times \hat{\Theta}) \mapsto \hat{f}_2(z)(\mathbf{e}(-\hat{w}_{\hat{z}} \hat{h}(z)))$, where $\hat{h} = [\hat{h}_1(z), \ldots, \hat{h}_c(z)]^T : Z \mapsto \mathbb{R}^c_z$. Consider the set of algebraically independent equations
\[
\int_Z \hat{f}_2(z, \hat{\bar{\theta}}) \hat{h}_i(z) \, dz = \hat{H}_i, \hspace{1cm} i = 1, \ldots, c_z, \tag{10}
\]
where $\hat{H} := [\hat{H}_1, \ldots, \hat{H}_c]^T \in \mathbb{R}^c_z$. Then, the unique solution, say $\hat{\bar{\theta}}^*$, of the minimization problem
\[
\min_{\hat{\bar{\theta}}} \mathcal{J}(\hat{\bar{\theta}}), \tag{11}
\]
with $J(\hat{\bar{\theta}}) = \langle \hat{\bar{\theta}}, \hat{H} \rangle + \int_Z \hat{f}_2(z, \hat{\bar{\theta}}) \, dz$ is also a solution of (10).

**Proof:** See the Appendix \(\Box\)

The main result behind the algorithm of Section 4, the proof of which leverages the above technical lemmas, is presented next.

**Theorem 1.** The solution, $(\hat{f}_z^*)^* = f^*(u_k|x_{k-1})$, of the control Problem 1 is
\[
(\hat{f}_z^*)^* = \hat{g}_n e^{-\omega(u_k, x_{k-1}) + \langle \lambda^*_{u,k}, h_{u,k}(u_k) \rangle} / \alpha_{k, n, u}, \tag{12}
\]
where:

(1) $\hat{w}(\cdot, \cdot)$ is generated via backward recursion. In particular,
\[
\hat{w}(u_k, x_{k-1}) = \hat{\alpha}(u_k, x_{k-1}) + \hat{\beta}(u_k, x_{k-1}), \tag{13}
\]
and
\[
\hat{\alpha}(u_k, x_{k-1}) := \mathcal{D}_{\text{KL}}(\hat{f}_x^*||\hat{g}_x^*) \tag{14}
\]
\[
\hat{\beta}(u_k, x_{k-1}) := -\mathbb{E}_{\hat{f}_x^*} \ln \hat{\gamma}(x_k)
\]
with terminal conditions $\hat{\beta}(u_n, x_{n-1}) = 0$ and $\hat{\alpha}(u_n, x_{n-1}) = \mathcal{D}_{\text{KL}}(\hat{f}_x^*||\hat{g}_x^*)$.

(2) $\hat{\gamma}(\cdot)$ is defined as
\[
\ln \hat{\gamma}(x_{k-1}) := \sum_{i=0}^{c_n} \ln (\hat{\gamma}_{u,i,k}(x_k)) \tag{15}
\]
and $\hat{\gamma}_{u,i,k}(\cdot)$ are given by
\[
\hat{\gamma}_{u,0,k}(x_{k-1}) = \exp \{\lambda^*_{u,0,k} + 1\} \tag{16}
\]
and
\[
\hat{\gamma}_{u,i,k}(x_{k-1}) := \exp \{\lambda^*_{u,i,k} H_{u,i,k}\} \tag{17}
\]
with $\hat{\gamma}_{u,0,n}(x_{n-1}) = 1$, i.e. $\lambda^*_{u,0,n} = 0$, and $\lambda^*_{u,i,n} = 0$, $i = 1, \ldots, n$.

(3) $\lambda^*_{u,0,k}$ and $\lambda^*_{u,k} = [\lambda^*_{u,1,k}, \ldots, \lambda^*_{u,c_n,k}]$ are the Lagrange multipliers associated to the constraints at time $t_k$. In particular,
\[
\lambda^*_{u,0,k} = \ln \left\{ \int \left[ e^{-\omega(u_k, x_{k-1}) + \langle \lambda^*_{u,k}, h_{u,k}(u_k) \rangle} \right] \, du_k \right\} - 1,
\]
while all the other LMs can be obtained numerically (via e.g. Lemma 2).

Moreover, the corresponding minimum is given by:
\[
B_k^* := -\mathbb{E}_{\hat{f}_x^*} \left[ \sum_{i=0}^{c_n} \ln (\hat{\gamma}_{u,i,k}(x_{k-1})) \right] \tag{18}
\]

We report below a sketch of the proof for Theorem 1.

**Proof:** For notational convenience, we use the shorthand notation $(\mathbf{E}_{u_k})$ to denote the constraints of Problem 1 at time $t_k$. We also denote by $(\mathbf{E}_{u,k})$ the set of constraints over the whole time horizon $\mathcal{I}$ and $(\mathbf{E}_{u,k})_{k=1}^{n-1}$ to denote the constraints up to time $t_{n-1}$.

First, we note that Problem 1 can be rewritten as follows as follows:
\[
\min \{f_{k,n} \mid \mathbf{E}_{u,k} \} \tag{19}
\]
\[
\text{s.t.: } \mathcal{D}_{\text{KL}}(f_n^||g_n^) \tag{20a}
\]
\[
\mathcal{A}(x_{n-1}) = \mathcal{D}_{\text{KL}}(\hat{f}_x^*||\hat{g}_x^*). \tag{20c}
\]

That is, Problem 1 can be split in two stages, where the second stage in (19) corresponds to the last time-instant of the time-horizon $\mathcal{I}$. Now, the minimization problem
\[
\min \{f_{k,n} \mid \mathbf{E}_{u,k} \} \tag{21a}
\]
\[
\text{s.t.: } \mathcal{A}(x_{n-1}) \tag{21b}
\]
\[
\mathcal{A}(x_{n-1}) = \mathcal{D}_{\text{KL}}(\hat{f}_x^*||\hat{g}_x^*). \tag{21c}
\]

for any fixed $x_{n-1}$. Note now that, by definition of Kullback-Leibler divergence,
\[
\mathcal{A}(x_{n-1}) = \mathcal{D}_{\text{KL}}(\hat{f}_x^*||\hat{g}_x^*) + \int \hat{f}_x^* \hat{\alpha}(u_n, x_{n-1}) \, du_n.
\]
and therefore Lemma 1 can be applied. Thus, the solution of the above problem is the pdf
\[
(f_u^n)^* = \frac{\tilde{g}_u^n e^{-\{\tilde{\omega}(u_n,x_{n-1}) + (\tilde{\lambda}_{u,n}^*, b_{u,n}(u_n))\}}}{e^{1 + \tilde{\lambda}_{u,0,n}}},
\]
where $\lambda_{u,0,n}$ and $\lambda_{u,n}^*$ are the LMs at the last time instant, $t_n$. Moreover, by integrating the above expression it can be shown that:
\[
\lambda_{u,0,n}^* + 1 = \int \tilde{g}_u^n e^{-\{\tilde{\omega}(u_n,x_{n-1}) + (\lambda_{u,n}^*, b_{u,n}(u_n))\}} \, du_n = \tilde{\gamma}_{u,0,n} (x_{n-1}).
\]

Also, following Lemma 1, the minimum of the problem is given by:
\[
\tilde{A}^*(x_{n-1}) = -\left[1 + \lambda_{u,0,n}^* + \langle \lambda_{u,n}^*, H_{u,n} \rangle \right]
\]
and thus, the corresponding minimum value for $B_n$ is:
\[
B_n^* = -E_{f_{\tilde{x}}^{-1}} \left[1 + \lambda_{u,0,n}^* + \langle \lambda_{u,n}^*, H_{u,n} \rangle \right]
= -E_{f_{\tilde{x}}^{-1}} \left[\sum_{i=0}^{c_u} \ln (\tilde{\gamma}_{u,i,n} (x_{n-1}))\right]
\]
where we have used the definition: in eq.(16) and (17) for $\tilde{\gamma}_{u,i,n}$, $i = 0, \ldots, c_u$.

Now, by means of (23) and (19), Problem 1 becomes
\[
\min \left\{ \frac{n}{k} \right\}_{k=1}^{n} = -E_{f_{\tilde{x}}^{-1}} \left[1 + \lambda_{u,0,n}^* + \langle \lambda_{u,n}^*, H_{u,n} \rangle \right] + B_n^*
\]
\[
\text{s.t.: } \{E_{u,k}\}_{k=1}^{n-1}
\]
Again, the problem can be split, this time with the last stage corresponding to the time instant $t_{n-1}$. Namely, we get:
\[
\min \left\{ \frac{n}{k} \right\}_{k=1}^{n-2} = -E_{f_{\tilde{x}}^{-1}} \left[1 + \lambda_{u,0,n}^* + \langle \lambda_{u,n}^*, H_{u,n} \rangle \right] + B_n^*
\]
\[
\text{s.t.: } \{E_{u,k}\}_{k=1}^{n-2}
\]
\[
+ \min \left\{ \frac{n}{k} \right\}_{k=1}^{n-2} = -E_{f_{\tilde{x}}^{-1}} \left[1 + \lambda_{u,0,n}^* + \langle \lambda_{u,n}^*, H_{u,n} \rangle \right] + B_n^*
\]
\[
\text{s.t.: } \{E_{u,n-1}\}
\]

Now, let
\[
B_{n-1} = E_{f_{\tilde{x}}^{-1}} \left[1 + \lambda_{u,0,n}^* + \langle \lambda_{u,n}^*, H_{u,n} \rangle \right] + B_n^*
\]
\[
= -E_{f_{\tilde{x}}^{-1}} \left[1 + \lambda_{u,0,n}^* + \langle \lambda_{u,n}^*, H_{u,n} \rangle \right] - E_{f_{\tilde{x}}^{-1}} \left[\sum_{i=0}^{c_u} \ln (\tilde{\gamma}_{u,i,n} (x_{n-1}))\right],
\]
and note that $B_{n-1}$ can be written as
\[
B_{n-1} = E_{f_{\tilde{x}}^{-1}} \left[A (x_{n-2})\right],
\]
where
\[
A (x_{n-2}) :=
\]
\[
D_{KL} \left(f_{\tilde{x}}^{-1}\right) - E_{f_{\tilde{x}}^{-1}} \left[\sum_{i=0}^{c_u} \ln (\tilde{\gamma}_{u,i,n} (x_{n-1}))\right] .
\]

To obtain the above expression we used the fact that the following identity holds for any function $\varphi$ of $x_{n-1}$:
\[
E_{f_{\tilde{x}}^{-1}} [\varphi (x_{n-1})] = E_{f_{\tilde{x}}^{-1}} \left[ E_{f_{\tilde{x}}^{-1}} [\varphi (x_{n-1})] \right].
\]
Now, $A (x_{n-2})$ can be explicitly written in compact form as
\[
A (x_{n-2}) = \int f_{u,n-1}^{-1} \left\{ \ln \left(\frac{f_{u,n-1}}{g_{u}}\right) + \tilde{\omega} (u_{n-1}, x_{n-2}) \right\} \, du_{n-1},
\]
where $\tilde{\omega} (u_{n-1}, x_{n-2}) := \tilde{\omega} (u_{n-1}, x_{n-2}) + \beta (u_{n-1}, x_{n-2})$ and
\[
\beta (u_{n-1}, x_{n-2}) := D_{KL} \left(f_{\tilde{x}}^{-1}||\tilde{g}_{\tilde{x}}^{-1}\right),
\]
with the corresponding minimum value for $B_{n-1}^*$ given by
\[
B_{n-1}^* = -E_{f_{\tilde{x}}^{-1}} \left[\sum_{i=0}^{c_u} \ln (\tilde{\gamma}_{u,i,n-1} (x_{n-2}))\right]
\]
The proof can then be concluded by observing that at each further backward iteration, the general solution $\left(f_{\tilde{x}}^{k}\right)^*$ has the same shape as $\left(f_{\tilde{x}}^{k-1}\right)^*$, since at each iteration backward the resulting Lagrangian involves the same functions $\tilde{\omega}, \beta, \tilde{\omega}$ evaluated in previous instants. In order to make valid the general solution (21) also for the last step (n) the quantity $\beta (u_n, x_n)$ is set to 0. This last statement is equivalent to setting no constraints on $n + 1$ iterations i.e. to have $\lambda_{u,n+1}^* = 0, \forall i$.

We are now ready to introduce our algorithm translating the above theoretical results into a computational tool.

4. THE ALGORITHM

We developed an algorithmic procedure that, by leveraging the technical results introduced above, outputs the solution $\left(f_{\tilde{x}}^k\right)^*$ to Problem 1. The only inputs that are necessary to the algorithm are $g (d^n)$, extracted from the example dataset and the $f_{\tilde{x}}^k$’s modeling the plant.
Algorithm 1 Pseudo-code

Inputs: $g(d^n)$ and $f^k_X$

Output: $\{\hat f^k_u\}_{k\in\mathbb{Z}}$ solving Problem 1

Initialize

$\gamma_{u,0,n}(x_n) = 1$ $\lambda_{u,0,n} = 0$, $\lambda_{u,i,n} = 0$,

$\tilde{\gamma} = \gamma_{u,0,n}$;

$\tilde{\beta} = 0$;

for $k = n$ to 1 do

By backward recursion

$\hat{\alpha}(u_k, x_{k-1}) \leftarrow \int f(x_k|u_k, x_{k-1}) \frac{f(x_k|u_k, x_{k-1})}{g(x_k|u_k, x_{k-1})} dX_k$

$\hat{\beta}(u_k, x_{k-1}) \leftarrow \int f(x_k|u_k, x_{k-1}) \{- \ln (\hat{\gamma}(x_k))\}$

$\hat{\omega}(u_k, x_{k-1}) \leftarrow \hat{\alpha}(u_k, x_{k-1}) + \hat{\beta}(u_k, x_{k-1})$

$\hat{\nu}(u_k, x_{k-1}) \leftarrow g(u_k|x_{k-1}) \exp \{- \hat{\omega}(u_k, x_{k-1})\}$

$\hat{\gamma}_0(x_{k-1}) \leftarrow \int \hat{\nu}(u_k, x_{k-1}) dU_k$

$\hat{f}(u_k|x_{k-1}) \leftarrow \frac{\hat{\gamma}_0(u_k,x_{k-1})}{\gamma_0(x_{k-1})}$

Use Lemma 2 with $Z := S(f(u_k|x_{k-1}))$, $\hat{f}_1 = f$,

$\tilde{H} := H_{u,k}$, $\tilde{h} := h_{x,k}$, $\lambda_0 := \lambda_{u,0,k}$, $\lambda := \lambda_{u,k}$,

$\tilde{\theta} := \left[\theta_0, \theta^T\right]^T = \left[1 + \lambda_0, \lambda^T\right]^T$ to find the Lagrange multipliers:

$\lambda_{u,k} \equiv \lambda^*$ \leftarrow $\theta^*$

$\lambda_{u,0,k}(x_{k-1}) = \lambda_{0}^* \leftarrow \theta_0^* - 1$.

Compute the policy and prepare variables for the next iteration, $k - 1$:

$\hat{f}^*_u \leftarrow f(u_k|x_{k-1}) e^{\hat{\gamma}(x_{k-1})}$

$\hat{\gamma}_{u,i,k}(x_{k-1}) \leftarrow \exp \{\lambda_{u,i,k} H_{u,i,k}\}$

$\hat{\gamma}_{u,0,k} = \exp \{\theta_0^*\}$

$\hat{\gamma}(x_{k-1}) \leftarrow \exp \left[\sum_{i=1}^N \ln (\hat{\gamma}_{u,i,k}(x_k))\right]$

end for

5. VALIDATION

We used Algorithm 1 to synthesize a control policy (from real data) that would allow an autonomous car to merge on a highway. The scenario considered in our test is described in Fig. 1. Data were collected using the infrastructure of Stillorgan Road, Dublin 4.

Fig. 1. Autonomous driving scenario for Section 5: a car that is trying to merge onto a highway. The figure illustrates the stretch of road where the experiments took place. The area is outside the UCD entrance on Stillorgan Road, Dublin 4.

Fig. 2. Data collected during the experiments: speed, acceleration, jerk as a function of distance (measured from the beginning of the trip, the UCD entrance). The vertical line in each panel denotes the physical location of the junction highlighted in Fig. 1. The panels on the left report all the data collected from 100 trips, while panels on the right report the subset of 20 trips with the lowest jerk.

We used the distance between the the road junction point and the car position as state variable ($x_k = d(t_k)$) and the car longitudinal speed as control variable ($u_k = v(t_k)$). From the dataset, we extracted the 20 trips with the lowest jerk (in red in Fig. 2). We used this reduced dataset as desired behavior for the car. Given this set-up, we were able to compute both $f(d^n)$ and $g(d^n)$ from the complete dataset of 100 trips and the reduced dataset of 20 trips respectively. These pdfs are shown in Fig. 3, together with the corresponding control pdf (rightward panel).

Fig. 3. Pdfs extracted from the datasets of Fig. 2. On the axes, $x$ and $u$ denote the full series of collected distances and speeds.

Finally, we decided to constraint the variance of the acceleration (the control variable) and solved the resulting Problem 1 via Algorithm 1. In particular, to make the problem computationally efficient, we approximated all the above pdfs as Gaussian distributions via the Maximum Entropy Principle. Once this was done, we were able to control the closed loop pdf of the system so that it became
as close as possible to \( g(d^n) \), given the constraint on the variance. In the figure, the control time horizon is 20 iterations, i.e. \( n = 20 \), and the initial condition is \( y_0 = 18 \) meters (physically, this is a traffic light outside the UCD gate). Also, the equality constraint was set to have a variance of the closed-loop system higher than the variance of \( g(d^n) \) (this is why the closed-loop pdf is flatter).

![Image](41x75 to 292x687)

**Fig. 4.** The results obtained using Algorithm 1. For the sake of clarity, the results are illustrated at time \( k = 1 \) and are representative of the other time instants. The optimal control pdf (left panel) and the resulting closed loop pdf (right panel).

### 6. CONCLUSIONS

We presented an approach to the synthesis of policies from examples. The key technical novelty of the results is the inclusion of actuation constraints in the problem formulation. This in turn yields policies that can be exported to different systems having different actuation capabilities. This yields to the following set of conditions:

\[
\frac{\partial \mathcal{L}_{aug} \{ f, \lambda_0, \lambda \}}{\partial \lambda_i} = 0, \quad i = 0, \ldots, c_z, \tag{A.5}
\]

which, in particular, means that the LMs associated to the constraints must satisfy the following condition:

\[
\int g e^{-\left(\alpha(z) + (\lambda, h(z))\right)} \hat{h}_i (z) \, dz = \bar{H}_i, \quad i = 0, \ldots, c_z, \tag{A.6}
\]

which was obtained by replacing the expression of the optimal solution candidate, i.e. (A.4), in the derivatives \( \frac{\partial \mathcal{L}_{aug} \{ f, \lambda_0, \lambda \}}{\partial \lambda_i} \), \( i = 0, \ldots, c_z \).

Now, the above set of equations can be solved via Lemma 2 and here we let \( \lambda_0, \lambda^* \) be values of LMs satisfying the above constraints. By substituting the optimal LMs into the expression of the optimal solution candidates yields:

\[
f^*(z) = g e^{-\left(\alpha(z) + (\lambda^*, h(z))\right)} e^{1+\lambda_0^*}. \tag{A.7}
\]

The proof is then concluded by noticing that \( f^*(z) \) is indeed the optimal solution since the Lagrangian is convex in \( f \). To show convexity, it suffices to consider the second derivative of \( I'(f) \) and to observe that this is always positive definite (indeed \( \frac{\partial^2 I}{\partial z^2} = 2gL_0(z) + \bar{H}_i - \hat{h}_i \lambda \bar{H}_i > 0 \)).

Finally, the optimal value of the cost function is given by:

\[
J \{ f^* \} = \int f^* \left[ \ln \left( \frac{f^*}{g} \right) + \alpha + (\lambda^*, h(z)) \right] \, dz = -\int f^* \left( 1 + \lambda_0^* + (\lambda^*, h(z)) \right) \, dz = (A.7)
\]

and this concludes the proof. \( \square \)
Proof of Lemma 2

We prove this result by showing that: (i) $J(\hat{\theta})$ is strictly convex; (ii) its minimizer must satisfy the set of equations (10).

The proof of statement (ii) comes directly from the evaluation of the first order stationary condition. Indeed, any optimal candidate, say $\hat{\theta}^*$, must satisfy the condition

$$\nabla J \{ \hat{\theta}^* \} = 0.$$  

Now, since the equations in (10) are algebraically independent, we have that:

$$\nabla J \{ \hat{\theta}^* \} = \begin{bmatrix} \mathbf{H}_{1} \mathbf{f}_1(z) e^{-\langle \hat{\theta}^*, \hat{\mathbf{h}}(z) \rangle} & \mathbf{H}_{2} \mathbf{f}_2(z) \end{bmatrix} \Rightarrow \mathbf{H} = \mathbf{f}_2(z) \mathbf{f}_2(\hat{\theta}^*),$$

where we used the definition of $\mathbf{f}_2$ to obtain the second equality. That is, the above condition immediately implies that any candidate minimizer of the optimization problem in (11) must fulfill the set of equations (10).

In order to prove strict convexity (i.e. statement (i)) we compute the Hessian of $J(\hat{\theta})$ and show that this is strictly positive definite in $\hat{\theta}$. Indeed, computing the Hessian yields

$$\nabla^2 J \{ \hat{\theta} \} = \begin{bmatrix} \mathbf{H}_{1} & \mathbf{H}_{2} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1(z) e^{-\langle \hat{\theta}, \hat{\mathbf{h}}(z) \rangle} \mathbf{f}_1(z) & \mathbf{f}_2(z) \mathbf{f}_2(\hat{\theta}) \end{bmatrix},$$

where $\otimes$ denotes the external product between tensors. Now, since the equations in (10) are algebraically independent, we have that:

$$\exists \mathbf{S} \subset \mathcal{Z} \ni \forall \mathbf{\tilde{v}} \in \mathbb{R}^n - \{0\} \Rightarrow \left\langle \nabla^2 J \{ \hat{\theta} \} \mathbf{\tilde{v}}, \mathbf{\tilde{v}} \right\rangle = \int_{\mathcal{Z}} (\mathbf{\hat{h}}(z) \mathbf{\hat{h}}(z))^\top \mathbf{f}_1(z) e^{-\langle \hat{\theta}, \hat{\mathbf{h}}(z) \rangle} dz > 0,$$

and this implies that $\hat{\theta}^*$ is the unique minimizer of the optimization problem, thus concluding the proof. \qed

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