Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds.

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1 Introduction

This paper proves the following conjecture of Boyer and Zhang: If a small hyperbolic knot in a homotopy sphere has a non-trivial cyclic surgery slope $r$, then it has an incompressible surface with non-integer boundary slope strictly between $r - 1$ and $r + 1$. I state this result below as Theorem 4.1, after giving the background needed to understand it. Corollary 1.1 of Theorem 4.1 is that any small knot which has only integer boundary slopes has Property P. The proof of Theorem 4.1 also gives information about the diameter of the set of boundary slopes of a hyperbolic knot.

The proof of Theorem 4.1 uses a new theorem about the $\text{PSL}_2\mathbb{C}$ character variety of the exterior, $M$, of the knot. This result, which should be of independent interest, is given below as Theorem 3.1. It says that for certain components of the character variety of $M$, the map on character varieties induced by $\partial M \hookrightarrow M$ is a birational isomorphism onto its image. The proof of Theorem 3.1 depends on a fancy version of Mostow rigidity due to Gromov, Thurston, and Goldman. The connection between Theorem 3.1 and Theorem 4.1 is the techniques introduced by Culler and Shalen which connect the topology of $M$ with its $\text{PSL}_2\mathbb{C}$ character variety.

I will begin with the background needed for Theorem 4.1. Let $K$ be a knot in a compact, closed, 3-manifold $\Sigma$, that is, a tame embedding $S^1 \hookrightarrow \Sigma$. The exterior of $K$, $M$, is $\Sigma$ minus an open regular neighborhood of $K$. So $M$ is a compact 3-manifold whose boundary is a torus. Suppose $\gamma$ is a simple closed curve in $\partial M$. We can create a closed manifold $M_{\gamma}$ from $M$ by taking a solid torus and gluing its boundary to $\partial M$ in such a way that $\gamma$ bounds a disc in the solid torus ($M_{\gamma}$ depends only on the isotopy class of $\gamma$). The new manifold $M_{\gamma}$ is called a Dehn filling of $M$ or a Dehn surgery on

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Recently, many people have studied what kinds of manifolds arise when you do this, especially in the case where $K$ is a knot in $S^3$ (see the surveys [Gor,Lue]).

From now on, let $\Sigma$ be a homotopy sphere. An interesting question is: Are there any non-trivial Dehn surgeries on $K$ such that the resulting manifold is a homotopy sphere? Or more generally, where the resulting manifold has cyclic fundamental group? A knot is said to have Property P if no non-trivial Dehn surgery on it yields a homotopy sphere. By the Cyclic Surgery Theorem [CGLS], for fixed $K$ there is at most one $\gamma$ other than the meridian for which $M_\gamma$ is a homotopy sphere. Gordon and Luecke [GL] have shown that non-trivial Dehn surgery on a knot in $S^3$ never yields $S^3$, and so the Poincaré Conjecture would imply that every knot has Property P. On the other hand, there are plenty of examples of knots with non-trivial cyclic surgeries. Fintushel and Stern [FS] discovered that the $(-2,3,7)$ pretzel knot has two non-trivial surgeries where the resulting manifold is a lens space (see also [BMZ]). More examples of knots with non-trivial cyclic surgeries are given in [Lue,BZ2]. This famed pretzel knot was also the first example found to have a non-integer boundary slope (I will define what boundary slopes are in a moment). Boyer and Zhang conjectured that there was a general connection between cyclic surgeries and non-integer boundary slopes.

A properly embedded surface $(F,\partial F) \hookrightarrow (M,\partial M)$ will be called incompressible if $\pi_1(F) \to \pi_1(M)$ is injective and not a 2-sphere bounding a ball. An incompressible surface $F$ which is not boundary parallel is called essential. Isotopy classes of oriented simple closed curves in the torus $\partial M$ are in bijective correspondence with primitive elements of $H_1(\partial M,\mathbb{Z})$; unoriented isotopy classes with pairs of primitive elements $\{\gamma,-\gamma\}$. If $(\alpha,\beta)$ is a basis for $H_1(\partial M,\mathbb{Z})$, the slope of $\gamma=a\alpha+b\beta$ with respect to this basis is $\frac{a}{b} \in \mathbb{Q} \cup \{\infty\}$. Note that the slope of an unoriented isotopy class is well defined, and gives a bijection between unoriented isotopy classes of simple closed curves and $\mathbb{Q} \cup \{\infty\}$. If $\partial F$ is non-empty, it consists of disjoint simple closed curves in $\partial M$. These curves must all be parallel, and so correspond to the same pair of primitive elements $\{\gamma,-\gamma\}$ in $H_1(\partial M,\mathbb{Z})$. These are called the boundary classes of $F$, and their slope the boundary slope of $F$. Hatcher [Hat] has shown that the number of boundary slopes is always finite. As $M$ is the exterior of a knot in a homotopy sphere, fix a meridian-longitude basis $(\mu,\lambda)$, for $H_1(\partial M)$; all slopes will be with respect to this basis. A slope is integral if it is in $\mathbb{Z}$. Many knots have only integral boundary slopes, e.g. all 2-bridge knots [HT]. If the Dehn filling $M_\gamma$ has cyclic fundamental group, then the slope of $\gamma$ is called a cyclic surgery slope. A manifold is small if it does not contain a closed, non-boundary parallel, incompressible surface. A knot is small if its exterior is. A knot is hyperbolic if the interior of $M$ admits a complete hyperbolic metric of finite volume.

Finally, here is the theorem which was conjectured by Boyer and Zhang in [BZ2], and improves their partial results in this direction:

**Theorem 4.1** Suppose $K$ is a small hyperbolic knot in a homotopy sphere which has a non-trivial cyclic surgery slope $r$. Then there is an essential surface in the exterior of $K$ whose boundary slope is non-integer and lies in $(r-1,r+1)$. 
Note that the Cyclic Surgery Theorem [CGLS] shows that a cyclic slope for a knot in a homotopy sphere is always integral. The $(-2, 3, 7)$ pretzel satisfies the hypotheses of this theorem, which explains why it has a non-integral boundary slope. Theorem 4.1 remains true for some knot exteriors in manifolds with non-trivial cyclic fundamental group, see Section 4. A corollary of the theorem is a condition which implies that a knot has Property P:

**Corollary 1.1** If a small knot in a homotopy sphere has only integral boundary slopes, then it has Property P.

This follows as a small non-hyperbolic knot is a torus knot, and surgeries on these have been classified [Mos].

I should mention that there are small hyperbolic knots which have non-integral boundary slopes but no cyclic surgeries. Boyer and Zhang gave examples in [BZ2], and one can also use the proof of Proposition 2.2 of [HO] to give examples of small Montesinos knots with non-integral boundary slopes of absolute value less than 2. So it is not possible to use Corollary 1.1 to show that all small knots have Property P.

The ideas of the proof of Theorem 4.1 can also be used to give information about the diameter, $d$, of the set of boundary slopes. In [CS3], Culler and Shalen showed that for any knot the diameter $d \geq 2$. For a hyperbolic knot, I show that if $d = 2$ then the greatest and least slopes are not integral (see Section 5 for a detailed statement).

Let me change course and state the theorem about character varieties that underlies the proof of Theorem 4.1, and which I hope will be of use in other situations. Let $M$ be a finite-volume hyperbolic 3-manifold with one cusp. Let $X(M)$ denote the $PSL_2\mathbb{C}$ character variety of $M$. Basically, $X(M)$ is the set of representations of $\pi_1(M)$ into $PSL_2\mathbb{C}$ mod conjugacy, and is an algebraic variety over $\mathbb{C}$ (for details, see Section 2). Culler and Shalen introduced a way of getting essential surfaces from $X(M)$, which has been very useful in proving theorems about Dehn surgery. I will postpone explaining how this works until the next section, but this is what connects the next theorem with the preceding. Let $X_0$ denote an irreducible component of $X(M)$ which contains the conjugacy class of a discrete faithful representation. Since $M$ has one cusp, the (complex) dimension of $X_0$ is 1. The inclusion $i: \partial M \hookrightarrow M$ induces a map on character varieties $i^*: X_0 \to X(\partial M)$. I will prove:

**Theorem 3.1** The map $i^*: X_0 \to X(\partial M)$ is a birational isomorphism onto its image.

The conclusion of this theorem does not always hold for other components of $X(M)$. The key to the proof of Theorem 3.1 is:

**Volume Rigidity Theorem 6.1 (Gromov-Thurston-Goldman)** Suppose $N$ is a compact, closed, hyperbolic 3-manifold. If $\rho: \pi_1(N) \to PSL_2\mathbb{C}$ is a representation with $\text{vol}(\rho) = \text{vol}(N)$, then $\rho$ is discrete and faithful.

If $N$ is a closed manifold, the volume of a representation $\rho: \pi_1(N) \to PSL_2\mathbb{C}$ is defined as follows. Choose any smooth equivariant map $f: \tilde{N} \to \mathbb{H}^3$. The form $f^*(\text{Vol}_{\mathbb{H}^3})$ descends to a form on $N$. The volume $\text{vol}(\rho)$ is the absolute value of the integral of this
The volume is independent of $f$ as any two such maps are equivariantly homotopic (see Section 2.5). Goldman [Gol] noticed that one can prove Theorem 6.1 in essentially the same way as the Strict Version of Gromov’s Theorem given by Thurston in his lecture notes [Thu]. As this section of Thurston’s notes remains unpublished, I will include a proof of Theorem 6.1. (Theorem 6.1 is now known to hold for all connected semisimple Lie groups, where in the definition of volume $\mathbb{H}^3$ is replaced by the appropriate symmetric space. The cases not done in [Gol] are covered by [Cor1] and [Cor2]).

The Volume Rigidity Theorem is connected to Theorem 3.1 by a theorem which is a reinterpretation of some of the results of [CCGLS]. It shows, roughly, that the volume of a representation $\rho$ depends only on the image of its conjugacy class under the map $i^*: X(M) \to X(\partial M)$ (for a precise statement, see Section 2.6).

Let me end the introduction with an outline of the proof of Theorem 4.1. The proof goes by contradiction, and so suppose that $K$ has a cyclic surgery slope but no non-integer boundary slopes. A simple algebraic argument determines exactly what the image of $i^*: X_0 \to X(\partial M)$ is. Theorem 3.1 says that $X_0$ is essentially the same as $i^*(X_0)$, and so $X_0$ is now known. Reading off from $X_0$ information about the number of boundary components of a certain essential surface contained in $M$ leads to a contradiction.

The rest of this paper is organized as follows: In Section 2 I will review the facts about character varieties that I will need later, and also prove some needed lemmas. Sections 3 and 4 are devoted to the proofs of Theorems 3.1 and 4.1 respectively. Section 5 discusses the proof of Theorem 4.1 in the context of the norm introduced in [CGLS], and gives an application to the question of the diameter of the set of boundary slopes. Finally, Section 6 gives a proof of the Volume Rigidity Theorem.

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2 Character varieties

2.1 Basics

In this section, I will review facts about character varieties that will be needed later. The basic references are the first chapter of [CGLS] and [CS1], as well as the expository article [Sha]. The case of $\text{PSL}_2\mathbb{C}^-$, as opposed to $\text{SL}_2\mathbb{C}^-$, character varieties, is treated in [BZ3].

If $M$ is a topological space with finitely generated fundamental group, denote by $R(M)$ the set of representations of $\pi_1(M)$ into $\text{PSL}_2\mathbb{C}$. This set has a natural structure as an affine complex algebraic variety. Now $\text{PSL}_2\mathbb{C}$ acts on $R(M)$ by conjugation of representations. Let $X(M)$ denote the quotient space (strictly speaking, the algebro-geometric quotient), and $t: R(M) \to X(M)$ the quotient map. The affine variety $X(M)$
is called the character variety of $R(M)$, as two representations in $R(M)$ map to the same point of $X(M)$ if and only if they have the same character. If two representations have the same character and one of them is irreducible, then they are conjugate. Moreover, for an irreducible component $X$ of $X(M)$ which contains the character of an irreducible representation, characters of reducible representations form a subvariety of strictly smaller dimension. So usually just think of $X(M)$ as representations mod conjugacy. A character is called discrete, faithful, or whatever if all representations with that character are discrete, faithful, or whatever. For each $\gamma$ in $\pi_1(M)$ there is a regular function $f_\gamma$ on $X(M)$ such that if $\chi$ in $X(M)$ is the character of a representation $\rho$ then $f_\gamma(\chi) = \text{tr}(\rho(\gamma))^2 - 4$. This is well defined as the trace of an element of $\text{PSL}_2\mathbb{C}$ is defined up to sign. A map between spaces $f: M \rightarrow N$ gives a map $f^*: X(N) \rightarrow X(M)$ via composition with $f_*: \pi_1(M) \rightarrow \pi_1(N)$. In particular, $i: \partial M \rightarrow M$ induces a map $i^*: X(M) \rightarrow X(\partial M)$.

We can also consider $\text{SL}_2\mathbb{C}$- rather than $\text{PSL}_2\mathbb{C}$- representations, and everything works the same way. Let $\tilde{R}(M)$ denote the variety of representations of $\pi_1(M)$ into $\text{SL}_2\mathbb{C}$, and $\tilde{\chi}(M)$ the associated character variety. In this case, there are also regular functions $\text{tr}_\gamma$ on $\tilde{\chi}(M)$ defined by $\text{tr}_\gamma(\chi) = \text{tr}(\rho(\gamma))$. The natural map $\tilde{\chi}(M) \rightarrow X(M)$ is finite to 1, though it may not be onto, as not all representations into $\text{PSL}_2\mathbb{C}$ lift to ones into $\text{SL}_2\mathbb{C}$.

When $M$ is a finite-volume hyperbolic manifold with one cusp, I will denote by $X_0$ a component of $X(M)$ which contains the character of a discrete faithful representation (note $X_0$ may not be unique, see Section 2.7). By [Cul], a torsion free discrete faithful representation lifts to $\text{SL}_2\mathbb{C}$ and there is a component $\tilde{X}_0$ of $\tilde{\chi}(M)$ which covers $X_0$. Said another way, every representation in $X_0$ lifts to one into $\text{SL}_2(\mathbb{C})$.

### 2.2 Associated actions on trees

Culler and Shalen discovered a way to construct essential surfaces from $X(M)$ or $\tilde{\chi}(M)$. In this and the next section, I will explain their method for an irreducible curve $X$ in $\tilde{\chi}(M)$ which contains the character of an irreducible representation. Let $Y$ be a smooth projective model for $X$ and $p: Y \rightarrow X$ the associated birational isomorphism. The finite set of points $Y \setminus p^{-1}(X)$ are called ideal points. Culler and Shalen showed how to associate to each ideal point an action of $\pi_1(M)$ on a simplicial tree as follows: Choose a curve $R \subset \tilde{R}(M)$ so that the Zariski closure of $t(R)$ is $X$ (I will say more about how to choose $R$ later in Lemma 2.2). If $\mathbb{C}(X)$ denotes the field of rational functions on $X$, then $t$ induces an inclusion of fields $\mathbb{C}(X) \hookrightarrow \mathbb{C}(R)$. The point $y$ determines a valuation $v$ on $\mathbb{C}(X)$ where $v(f)$ is the order of $f$ at $y$ (zeros getting positive valuation). There is a valuation $w$ on $\mathbb{C}(R)$ which extends $v$ in the sense that there is a positive integer $d$ so that $w(f) = d \cdot v(f)$ for all $f \in \mathbb{C}(X)$. Associated to the field $\mathbb{C}(R)$ and the valuation $w$ is its Bruhat-Tits tree, which I will denote $T_y$. This is a simplicial tree on which $\text{SL}_2\mathbb{C}(R)$ acts by isometries. The translation length of an element $A$ of $\text{SL}_2\mathbb{C}(R)$ acting on $T_y$ is $\min(0, -2w(\text{tr}(A)))$ (Proposition II.3.15 of [MS]). There is a tautological
representation $P: \pi_1(M) \to \text{SL}_2 \mathbb{C}[R]$, where $\mathbb{C}[R] \subset \mathbb{C}(R)$ is the ring of regular functions, defined by $P(\gamma) = A$ where $A$ is the matrix of regular functions so that at any representation $\rho \in R$, $A(\rho) = \rho(\gamma)$. Composing $P$ and the action of $\text{SL}_2 \mathbb{C}(R)$ on $T_y$ gives the promised action of $\pi_1(M)$ on a tree.

In order to prove Theorem 4.1 it will be necessary to know the relationship between the valuations $v$ and $w$ exactly. I will show:

**Proposition 2.2** For each ideal point $y$ in $Y$ there is a simplicial tree $T_y$ on which $\pi_1(M)$ acts so that the translation length of $\gamma \in \pi_1(M)$ acting on $T_y$ is twice the degree of pole of $\text{tr}_\gamma$ at $y$.

Because of the formula for translation length above, to prove Proposition 2.2 it suffices to show we can choose $R$ so that $d = 1$ (you may wish to skip ahead until this becomes crucial in the proof of Theorem 4.1). Let $Z$ be a smooth projective model for $R$, and $\tilde{t}: Z \to Y$ be the map induced by $t: R \to X$. The map $\tilde{t}$ is a holomorphic branched covering of Riemann surfaces. If there is a $z \in \tilde{t}^{-1}(y)$ such that $\tilde{t}$ is not branched at $z$, take $w$ to be the valuation determined by order at $z$, and then $d = 1$. Thus Proposition 2.2 will be proved by:

**Lemma 2.2** Let $y$ be a fixed ideal point of $X$. Then there is a curve $R \subset \tilde{R}(M)$ such that the induced map $\tilde{t}: Z \to Y$ has the property that for any $z \in \tilde{t}^{-1}(y)$, $\tilde{t}$ is not branched at $z$.

**Proof** By Corollary 1.4.5 of [CS1] there is an $\alpha \in \pi_1(M)$ so that $\text{tr}_\alpha$ has a pole at $y$. Since $\text{tr}_\alpha$ is non-constant there is an irreducible character $\chi$ in $X$ such that $\text{tr}_\alpha(\chi) \neq 2$. Let $\rho$ be a representation with character $\chi$. By Proposition 1.5.1 of [CS1] there is a $\beta \in \pi_1(M)$ such that $\rho$ restricted to the subgroup $\langle \alpha, \beta \rangle$ generated by $\alpha$ and $\beta$ is irreducible, or equivalently $\text{tr}_\rho(\langle \alpha, \beta \rangle) \neq 2$. Considering the regular function $\text{tr}_{\langle \alpha, \beta \rangle}$, we see that all but finitely many characters $\chi \in X$ are irreducible when restricted to $\langle \alpha, \beta \rangle$.

There are two cases, depending on whether $\text{tr}_\beta$ is constant or not. Suppose $\text{tr}_\beta$ is non-constant. Then there is a representation $\rho_0$ whose character is in $X$ so that $\rho_0$ restricted to $\langle \alpha, \beta \rangle$ is irreducible and $\rho_0(\alpha)$ and $\rho_0(\beta)$ are hyperbolic elements of $\text{SL}_2 \mathbb{C}$. Then $\rho_0$ restricted to $\langle \alpha, \beta^n \alpha \beta^{-n} \rangle$ is irreducible for large $n$. This is because the fixed point set of $\rho_0(\beta^n \alpha \beta^{-n})$ is $\rho_0(\beta^n)(\text{fix}(\alpha))$, which is disjoint from $\text{fix}(\alpha)$ for large $n$. So there is a conjugate $\gamma$ of $\alpha$ such that $\rho_0$ restricted to $\langle \alpha, \gamma \rangle$ is irreducible. Again, all but finitely many characters of $X$ are irreducible when restricted to $\langle \alpha, \gamma \rangle$. Let $V$ be the subvariety of $\tilde{R}(M)$ consisting of representations $\rho$ such that

$$\rho(\alpha) = \begin{pmatrix} a & 1 \\ 0 & 1/a \end{pmatrix}, \quad \text{and} \quad \rho(\gamma) = \begin{pmatrix} a & 0 \\ s & 1/a \end{pmatrix},$$

for some $a, s \in \mathbb{C}$, with $a \neq 0$. Any representation $\rho$ which is irreducible when restricted to $\langle \alpha, \gamma \rangle$ and where $\rho(\alpha)$ is not parabolic is conjugate to exactly two representations in $V$. For a conjugate of such a $\rho$ lying in $V$, there are two choices for $a$, and once $a$ is fixed, $s$ is determined by $\text{tr}(\rho(\alpha \gamma))$. Any two conjugates of $\rho$ which agree on $\langle \alpha, \gamma \rangle$
are actually equal, as the stabilizer under conjugation of any irreducible representation is \( \{1, -1\} \). Thus all but finitely many points of \( X \) are in \( t(V) \), so we can choose an irreducible curve \( R \subset V \) so that \( t(R) \) is dense in \( X \). The map \( t: R \to X \) is generically either 1-to-1 or 2-to-1. In the former case, we are done. In the latter case \( R = V \) and it is enough to show that \( \tilde{t}^{-1}(y) \subset Z \) consists of two points. Fix \( z \) in \( \tilde{t}^{-1}(y) \). We can define a regular function \( a: R \to \mathbb{C} \) by Eqn. (1). Since \( tr_\alpha \) has a pole at \( y \), \( a \) must have either a pole or a zero at any point of \( \tilde{t}^{-1}(y) \). Choose a sequence of representations \( \rho_j \) converging to \( y \). From Eqn. (1), we have \( \rho_j(\alpha) = \begin{pmatrix} a_j & 1 \\ 0 & 1/a_j \end{pmatrix} \). Since generically a point of \( X \) has two inverse images in \( R \), for all but finitely many \( j \) there are representations \( \psi_j \in R \) which are conjugate to, but not equal to, \( \rho_j \). Hence \( \psi_j(\alpha) = \begin{pmatrix} 1/a_j & 1 \\ 0 & a_j \end{pmatrix} \) because as mentioned above, conjugates which agree on \( \langle \alpha, \gamma \rangle \) are equal. Then the \( \psi_j \) converge to a point \( z' \) in \( \tilde{t}^{-1}(y) \). But \( z \) is not \( z' \) since if \( a \) has a zero at \( z \) then \( a \) has a pole at \( z' \) and vice versa. So \( \tilde{t}^{-1}(y) \) consists of two points and we are done.

If \( tr_\beta \) is constant, fix \( \lambda \in \mathbb{C} \) so that \( \lambda + 1/\lambda = tr_\beta \). Define \( V \) to be the subset of \( R \) given by:

\[
\rho(\alpha) = \begin{pmatrix} a & s \\ 0 & 1/a \end{pmatrix}, \quad \text{and} \quad \rho(\beta) = \begin{pmatrix} \lambda & 0 \\ 1 & 1/\lambda \end{pmatrix}
\]

where \( a, s \in \mathbb{C} \) and proceed as before. \( \square \)

2.3 Associated surfaces

There is a dual surface to any action of \( \pi_1(M) \) on a tree \( T \) as follows. Let \( \tilde{M} \) be the universal cover of \( M \). Choose an equivariant map \( f: \tilde{M} \to T \) which is transverse to the midpoints of the edges of \( T \). If \( \tilde{S} \) is the inverse image under \( f \) of the midpoints of the edges of \( T \), then \( \tilde{S} \) is an equivariant family of surfaces in \( \tilde{M} \) which descends to a surface \( S \) in \( M \). In Section 1.3 of [CGLS] it is shown how any such \( f \) can be modified so that \( S \) becomes essential. If \( y \) is an ideal point, this construction gives an essential surface dual to the action on \( T_y \). This surface, \( S \), is said to be associated to \( y \). Note that \( S \) need not be connected or unique up to isotopy.

Suppose \( M \) has torus boundary, and \( \partial M \) is essential in \( M \). Suppose \( y \) is an ideal point where for some peripheral element \( \gamma \in \pi_1(\partial M) \), \( tr_\gamma \) has a pole (there may be ideal points where this does not happen). In this case, there is exactly one slope \( \{\alpha, -\alpha\} \) such that \( tr_\alpha \) is finite at \( y \) (again, see [CGLS] for details). Then \( \alpha \) fixes a point of \( T_y \). If \( S \) is an essential surface associated to \( T_y \), then some component of \( S \) has non-empty boundary with boundary classes \( \{\alpha, -\alpha\} \). Let \( \beta \) be such that \( (\alpha, \beta) \) is a basis for \( \pi_1(\partial M) \). I will need:

**Proposition 2.3 (Culler and Shalen)** The surface associated to an ideal point \( y \) can be chosen so that the number of boundary components is equal to twice the order of pole of \( tr_\beta \) at \( y \).
Proof

This is essentially part of Section 5.6 of [CCGLS]. By Proposition 2.2, twice the order of pole of \( \text{tr}_\beta \) at \( y \) is the same as the translation length, \( l(\beta) \), of \( \beta \) acting on \( T_y \). First, I claim \( |\partial S| \geq l(\beta) \), where \( |\partial S| \) is the number of components of \( \partial S \). Let \( p: \tilde{M} \to M \) be the covering map. Pick an arc \( a \) in \( \tilde{M} \) which is a lift of a loop in \( M \) representing \( \beta \) and which intersects \( p^{-1}(S) \) in \( |\partial S| \) points. Hence the image of \( a \) in \( T_y \) intersects the midpoints of \( |\partial S| \) edges of \( T_y \). One of the endpoints of the image of \( a \) in \( T_y \) is sent to the other under the action of \( \beta \). Therefore the translation length of \( \beta \) acting on \( T_y \) is at most \( |\partial S| \).

Now I will produce a surface associated to \( y \) with at most \( l(\beta) \) boundary components. Choose a connected component of \( p^{-1}(\partial M) \), \( C \), which we identify with \( \mathbb{R}^2 \) so that \( \alpha \) acts on \( C \) by unit translation in the first coordinate, and \( \beta \) acts by unit translation in the second coordinate. The abelian subgroup \( \pi_1(\partial M) \) leaves invariant a unique line \( L \) in \( T_y \), and acts on \( L \) via translations. An element \( \gamma = a\alpha + b\beta \in \pi_1(\partial M) \) acts on \( L \) by translation by \( b \cdot l(\beta) \). Choose a map \( f: C \to L \) which is equivariant under the action of \( \pi_1(\partial M) \) as follows. Let \( Y \) be the second coordinate axis. First, project onto the second coordinate to get a map from \( C \) to \( Y \). Then compose this with a linear map from \( Y \) to \( L \) that expands by a factor of \( l(\beta) \). There is a unique extension of \( f \) to all of \( \pi_1(\partial M) \) which is equivariant under \( \pi_1(\partial M) \). Since \( T_y \) is contractible, we can extend \( f \) to an equivariant map of all of \( \tilde{M} \) to \( T_y \). The dual surface \( S \) has \( |\partial S| = l(\beta) \). Changing \( f \) so that \( S \) becomes essential does not increase the number of boundary components, so there is a surface associated to \( y \) with \( |\partial S| \leq l(\beta) \). As we also have \( |\partial S| \geq l(\beta) \), \( S \) must have exactly \( l(\beta) \) boundary components.

\( \square \)

2.4 Associated plane curves

The authors of [CCGLS] introduced a plane curve associated to the character variety. This gives a nice set of coordinates for the computations in the proof of Theorem 4.1. Let \( M \) be a compact 3-manifold with torus boundary. Let \( \Delta \) be the set of diagonal representations of \( \pi_1(\partial M) \) into \( \text{SL}_2\mathbb{C} \). For any \( \gamma \in \pi_1(\partial M) \) there is a well-defined eigenvalue function \( \xi_\gamma: \Delta \to \mathbb{C}^* \) which takes a representation \( \rho \) to the upper left hand entry of \( \rho(\gamma) \). Fixing a basis \( (\alpha, \beta) \) for \( \pi_1(\partial M) \), the pair of eigenvalue functions \( (\xi_\alpha, \xi_\beta) \) gives coordinates on \( \Delta \), and allows us to identify it with \( \mathbb{C}^* \times \mathbb{C}^* \). Said another way, we can identify a point \( (a, b) \in \mathbb{C}^* \times \mathbb{C}^* \) with the representation \( \rho \) such that:

\[ \rho(\alpha) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \quad \rho(\gamma) = \begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix}. \]

I will say that \( (a, b) = (\xi_\alpha, \xi_\beta) \) are the eigenvalue coordinates corresponding to the basis \( (\alpha, \beta) \).

There is a natural map \( t: \Delta \to \tilde{X}(\partial M) \) which is onto and generically 2-to-1. If \( Y \) is a one-dimensional subvariety of \( \tilde{X}(\partial M) \), we can take the closure of \( t^{-1}(Y) \subset \Delta \subset \mathbb{C} \mathbb{P}^2 \) to get a plane curve \( D \) which is a double cover of \( Y \). If \( \tilde{X} \) is an irreducible component of \( \tilde{X}(M) \) such that \( i^*(\tilde{X}) \subset \tilde{X}(\partial M) \) is one-dimensional, we can associate
the plane curve \( D(\tilde{X}) \equiv t^{-1}(i^*(\tilde{X})) \) to \( \tilde{X} \). The union of \( D(\tilde{X}) \) over all components \( \tilde{X} \) of \( X(M) \) with \( i^*(\tilde{X}) \) one-dimensional is called the associated plane curve and denoted \( D_M \). Note for a component \( \tilde{X}_0 \) of \( X(M) \) which contains the character of a discrete faithful representation, \( i^*(\tilde{X}_0) \) is one-dimensional by Proposition 1.1.1 of [CGLS].

There is an alternate construction of \( D_M \) given in [CL] that we will need for the next section. Let \( \tilde{R}_U(M) \) be the subset of \( \tilde{R}(M) \) consisting of representations whose restriction to the subgroup \( \pi_1(\partial M) \) is upper-triangular. Then there is a map \( i^*: \tilde{R}_U(M) \to \Delta \) which sends \( \rho \) to the pair consisting of the upper left hand entries of \( \rho(\alpha) \) and \( \rho(\beta) \). It is not hard to see that the union of the \( i^*(\tilde{R}) \) over all components \( \tilde{R} \subset \tilde{R}_U(M) \) with \( i^*(\tilde{R}) \) one-dimensional is exactly \( D_M \).

### 2.5 Volume of a representation

Let \( N \) be a closed 3-manifold. The volume, \( \text{vol}(\rho) \), of a representation \( \rho: \pi_1(N) \to \text{PSL}_2\mathbb{C} \) is defined as follows. Choose any smooth equivariant map \( f: \bar{N} \to \mathbb{H}^3 \). The form \( f^*(\text{Vol}_{\mathbb{H}^3}) \) on \( \bar{N} \) descends to a form on \( N \). The volume \( \text{vol}(\rho) \) is the absolute value of the integral of this form over \( N \). Since any two such maps are equivariantly homotopic, the volume is independent of \( f \). If \( f \) and \( g \) are two such maps one can use the straight line homotopy \( H \) defined by \( H(p, t) = tf(p) + (1-t)g(p) \), where this linear combination is along the geodesic joining \( f(p) \) to \( g(p) \).

The purpose of this section is to define the volume of a representation for a compact 3-manifold \( M \) whose boundary is a torus. The definition given in the closed case does not work here. As \( \partial M \) is non-empty, the value of the integral used to define the volume depends on the choice of equivariant map.

For convenience, I will assume throughout that the torus \( \partial M \) is incompressible in \( M \). Let \( \bar{M} \) be the universal cover of \( M \) with \( \pi: \bar{M} \to M \) the covering map. Let \( \tilde{M} \) denote the quotient space of \( \bar{M} \) obtained by collapsing each component plane of \( \pi^{-1}(\partial M) \) to a point. The points of \( \tilde{M} \) coming from the collapsed components of \( \pi^{-1}(\partial M) \) will be denoted \( \partial \tilde{M} \); the set \( \tilde{M} \setminus \partial \tilde{M} \) will be denoted \( \text{int}(\tilde{M}) \). The action of \( \pi_1(M) \) on \( \tilde{M} \) induces an action on \( \tilde{M} \). This action is free on \( \text{int}(\tilde{M}) \), but each point of \( \partial \tilde{M} \) is stabilized by some peripheral subgroup of \( \pi_1(M) \). Let \( \mathbb{H}^3 \) denote the union of \( \mathbb{H}^3 \) with the sphere at infinity \( S^2_{\infty} \). The idea for defining the volume of a representation \( \rho: \pi_1(M) \to \text{PSL}_2\mathbb{C} \) is to consider equivariant maps \( f: \tilde{M} \to \mathbb{H}^3 \) which send \( \partial \tilde{M} \) into \( S^2_{\infty} \) and send \( \text{int}(\tilde{M}) \) into \( \mathbb{H}^3 \), and then proceed as in the closed case.

Before I can define the precise types of maps to be considered, I need to define some notation. Let \( T \) be a torus and fix a product structure \( T \times [0, \infty] \) on a neighborhood of \( \partial M \) such that \( \partial M = T \times \{\infty\} \) (the reason for this choice of closed interval will become clear in a moment). This gives an equivariant product structure on \( \pi^{-1}(T \times [0, \infty]) \) in \( \tilde{M} \). This product structure induces an equivariant cone structure on a neighborhood of \( \partial \tilde{M} \) in \( \tilde{M} \). If \( \nu \) is in \( \partial \tilde{M} \), I will denote the component of this neighborhood containing \( \nu \) by \( N_\nu \), and the cone structure on \( N_\nu \) by as \( \nu \times [0, \infty] \), where \( \nu \times \nu \) is a plane that covers
$T \times \{0\}$ and $P_v \times \{\infty\}$ is really just the cone point $v$. The peripheral subgroup of $\pi_1(M)$ which fixes $v$ will be denoted $\text{Stab}(v)$. Note that $\text{Stab}(v)$ preserves $N_v$.

Let $\rho: \pi_1(M) \to \text{PSL}_2\mathbb{C}$ be a representation. A pseudo-developing map for $\rho$ is a smooth equivariant map $f: M \to \mathbb{H}^3$ which sends $\partial M$ into $S^2_\infty\mathbb{C}$ and $\text{int}(M)$ into $\mathbb{H}^3$ and which satisfies the following additional condition. For each $v$ in $\partial M$, I require that $f$ maps each ray $\{p\} \times [0, \infty]$ in the cone neighborhood $N_v$ to a geodesic ray, and that $f$ parameterizes this ray by arc length with respect to cone parameter in $[0, \infty]$ (by continuity, this ray has endpoint $f(v)$ in $S^2_\infty\mathbb{C}$). Note that $f(v)$ must be a fixed point of $\rho(\text{Stab}(v))$. As an example, if $\rho$ is the holonomy representation for a complete hyperbolic structure on $\text{int}(M)$, then a developing map for this structure extends to a pseudo-developing map of $\bar{M}$ by sending each $v$ in $\partial M$ to the unique point in $S^2_\infty\mathbb{C}$ fixed by $\rho(\text{Stab}(v))$. Another example is if we have a decomposition of $M$ into ideal tetrahedra; an equivariant map which is piecewise straight with respect to this triangulation is a pseudo-developing map. Every $\rho$ has a pseudo-developing map. To construct one, pick a $v$ in $\partial M$ and define $f(v)$ to be some point fixed by $\rho(\text{Stab}(v))$. Extend $f$ to $N_v$ by picking any $\text{Stab}(v)$ equivariant map of $P \times \{0\}$ into $\mathbb{H}^3$ and then extending geodesically along the cone structure of $N_v$. There is a unique equivariant extension of $f$ to the cone neighborhood of $\partial M$. As $\mathbb{H}^3$ is contractible, $f$ extends to a pseudo-developing map for $\rho$.

I will define the volume of a pseudo-developing map $f$ in the same way as in the closed case. The pull back $f^*(\text{Vol}_{\mathbb{H}^3})$ descends to a form on $M$. The absolute value of the integral of this form over $M$ is defined to be $\text{vol}(f)$. To see that this integral is well defined, pick a $v \in \partial M$ and choose a fundamental domain $F = D \times [0, \infty]$ for the action of $\text{Stab}(v)$ on $N_v$. We need that the integral of $f^*(\text{Vol}_{\mathbb{H}^3})$ over $F$ is defined and finite. Pick a horoball in $\mathbb{H}^3$ centered at $f(v)$ which contains $f(F)$. Project $f(D \times \{0\})$ out to the corresponding horosphere $S$ along geodesic rays with limit $f(v)$. Call this projection $\theta$. Let $\text{Area}_S$ be the area form on $S$. Let $A = \int_{D \times \{0\}} |\theta^*\text{Area}_S|$, the total unoriented area of $\theta(D \times \{0\})$. As $f$ restricted to $\{p\} \times [0, \infty]$ is a geodesic parameterized by arc length, the same computation as computing the volume of a cusp shows that the integral of $|f^*(\text{Vol}_{\mathbb{H}^3})|$ over $F$ is bounded by $A/2$. Thus the integral of the absolute value of $f^*(\text{Vol}_{\mathbb{H}^3})$ over $M$ is finite, and so $\text{vol}(M)$ is well defined. Note that for the two examples of pseudo-developing maps given in the last paragraph, the volume of such a map is the volume that you would expect. It would also be possible to define the volume of $f$ without taking the absolute value of the given integral. All the lemmas in this section remain true with this altered definition; this fact will be needed in Section 2.6.

I would now like to say that the volume of a pseudo-developing map is independent of the choice of map using the same argument as in the closed case. However, there is a problem. Suppose $f$ is a pseudo-developing map for a representation $\rho$ such that $\rho(\pi_1(\partial M))$ has exactly two distinct fixed points. Consider a $v \in \partial M$. Let $p$ and $q$ in $S^2_\infty\mathbb{C}$ be the two fixed points of $\rho(\text{Stab}(v))$. Then $f(v)$ is either $p$ or $q$, say $p$. We can construct a pseudo-developing map $g$ for $\rho$ with $g(v) = q$. Then $f$ and $g$ are not homotopic through equivariant maps from $\bar{M}$ to $\mathbb{H}^3$ which send $\partial M$ into $S^2_\infty\mathbb{C}$. I see no a priori way to compare the volumes of these two maps. However, we have:
Lemma 2.5.1 If \( f \) and \( g \) are two pseudo-developing maps for \( \rho \) which agree on \( \partial \overline{M} \), then \( \text{vol}(f) = \text{vol}(g) \).

Proof Construct a homotopy \( H: \overline{M} \times [0,1] \rightarrow \mathbb{H}^3 \) between \( h \) and \( g \) through pseudo-developing maps as follows. Let \( v \) be in \( \partial M \). Consider the cone \( N_v = P_v \times [0,\infty] \). Begin to construct \( H \) by setting \( H(v, t) = f(v) = g(v) \) for all \( t \). Extend \( H \) over \( P_v \times \{0\} \) by any \( \text{Stab}(v) \) equivariant homotopy between the restrictions of \( f \) and \( g \) to \( P_v \times \{0\} \). As before, cone along geodesic rays ending at \( f(v) \) to extend \( H \) over \( N_v \). We can now extend \( H \) to the desired homotopy.

To see that \( \text{vol}(f) = \text{vol}(g) \) consider \( M_t = M \setminus T \times (0,\infty] \), where \( T \times [0,\infty] \) is our collar on \( \partial M \). For \( t \in [0,\infty] \), define \( V_t(f) = \int_{M_t} f^*(\text{Vol}_{\mathbb{H}^3}) \) and let \( V_t(g) \) be the corresponding quantity for \( g \). Note \( |V_f(\infty)| = \text{vol}(f) \) and similarly for \( g \). The form \( \text{Vol}_{\mathbb{H}^3} \) is closed, so by Stokes Theorem:

\[
0 = \int_{M_t \times [0,1]} dH^*(\text{Vol}_{\mathbb{H}^3}) = V_f(t) - V_g(t) + \int_{(\partial M_t) \times [0,1]} H^*(\text{Vol}_{\mathbb{H}^3})
\]

Because \( H \) was constructed by geodesically coning over the restriction of \( H \) to \( P_v \times \{0\} \), the integral \( \int_{(\partial M_t) \times [0,1]} H^*(\text{Vol}_{\mathbb{H}^3}) \) goes to zero exponentially with \( t \). Therefore \( \text{vol}(f) = \text{vol}(g) \). \( \square \)

If \( \rho \) is a representation such that the volume of any two pseudo-developing maps agree, I will say that the volume of \( \rho \) is defined and set \( \text{vol}(\rho) = \text{vol}(f) \) for any pseudo-developing map \( f \). If \( \rho \) is a representation where \( \rho(\pi_1(\partial M)) \) contains a non-trivial parabolic, then \( \rho(\pi_1(\partial M)) \) has a unique fixed point in \( S_\infty^2 \). By the lemma, any two pseudo-developing maps for \( \rho \) have the same volume, and so the volume is defined for such \( \rho \). I will show:

Lemma 2.5.2 If \( \rho \) is a representation whose character lies in an irreducible component of \( X(M) \) which contains the character of a discrete faithful representation then \( \text{vol}(\rho) \) is defined.

which follows immediately from:

Lemma 2.5.3 Suppose \( \rho_t \), with \( t \in [0,1] \) is a smooth one parameter family of representations of \( \pi_1(M) \) into \( \text{PSL}_2\mathbb{C} \). If the volume of \( \rho_0 \) is defined then so is the volume of \( \rho_1 \).

Proof I will need the results of Section 4.5 of [CCGLS]. There, they construct a particularly nice form of pseudo-developing map as follows. Let \( N \) denote \( \overline{M} \) with \( \partial M \) collapsed to a point. Decompose \( N \) as a simplicial complex so that each simplex has at most one vertex at the collapsed \( \partial M \). This induces an equivariant simplicial decomposition of \( \overline{M} \). Pick a preferred vertex \( v_\infty \) of \( \partial M \) and a fixed point \( p \) of \( \rho(\text{Stab}(v_\infty)) \). For a representation \( \rho \) we can construct a pseudo-developing map \( f \) for \( \rho \) so that \( f \) is piecewise straight on the simplices of \( \overline{M} \) and which sends \( v_\infty \) to \( p \). Simply pick an equivariant map of the zero skeleton of \( \overline{M} \) sending \( v_\infty \) to \( p \) and extend linearly along the simplices. This
gives a pseudo-developing map because each simplex has at most one vertex in \( \partial \tilde{M} \) and so the only points sent to \( S^2_\infty \) are those in \( \partial M \).

Pick two pseudo-developing maps \( f_1 \) and \( g_1 \) for \( \rho_1 \). We can build a smooth family of maps \( f_t \), for \( t \in [0, 1] \) where \( f_t \) is a pseudo-developing map for \( \rho_t \) of the special type just discussed. Let \( g_t \) be a similar family of maps for \( g_1 \). In Section 4.5 of [CCGLS] it is shown that work of Hodgson [Hod] implies that the derivative of \( \text{vol}(f_t) \) (or \( \text{vol}(g_t) \)) depends only on the restriction of \( \rho_t \) to \( \pi_1(\partial M) \). This implies that the derivatives of \( \text{vol}(f_t) \) and \( \text{vol}(g_t) \) are the same. As \( \text{vol}(f_0) = \text{vol}(g_0) = \text{vol}(\rho_0) \), we must have \( \text{vol}(f_1) = \text{vol}(g_1) \). So \( \text{vol}(\rho_1) \) is well defined. \( \square \)

For the proof of Theorem 3.1, I will need:

**Lemma 2.5.4** Suppose \( \rho \) is a representation of \( \pi_1(M) \) which factors through the fundamental group of a Dehn filling \( M_\gamma \) of \( M \). Then the volume of \( \rho \) with respect to \( M \) is defined, and is equal to the volume of \( \rho \) with respect to the closed manifold \( M_\gamma \).

**Proof** Let \( C \) be the solid torus added to \( M \) to make \( M_\gamma \). Let \( \tilde{M}_\gamma \) be the universal cover of \( M_\gamma \), with \( \phi \) the covering map. Pick an equivariant map \( f \) from \( \tilde{M}_\gamma \) to \( \mathbb{H}^3 \). Let \( V = \tilde{M}_\gamma \setminus \phi^{-1}({\text{int}}(C)) \). Let \( W \) be \( \tilde{M} \) minus the open cone neighborhood of \( \partial \tilde{M} \). Adjusting collars, I will view \( V \) as a quotient of \( W \). The restriction of \( f \) to \( V \) is a \( \pi_1(M_\gamma) \)-equivariant map which induces a \( \pi_1(M) \)-equivariant map \( F : W \to \mathbb{H}^3 \). Extend \( F \) over \( \partial \tilde{M} \) by any \( \pi_1(M) \)-equivariant map. By coning along geodesics we can extend \( F \) over the cone neighborhood of \( \partial \tilde{M} \) to a pseudo developing map for \( \rho \).

To compare volumes, choose \( f : \tilde{M}_\gamma \to \mathbb{H}^3 \) so that \( f({\phi^{-1}}(C)) \) is one-dimensional (this is possible because there is map from \( M_\gamma \) to itself which is the identity outside a neighborhood of \( C \) and which collapses \( C \) to a core curve in \( C \)). Thus the volume of \( \rho \) with respect to \( M_\gamma \) is the integral of \( f^*(\text{Vol}_{\mathbb{H}^3}) \) over \( M_\gamma \setminus C \). The image of the cone neighborhood of \( \partial \tilde{M} \) under \( F \) is at most two-dimensional, and so \( \text{vol}(f) \) is the integral of \( F^*(\text{Vol}_{\mathbb{H}^3}) \) over \( M \) minus the collar on \( \partial \tilde{M} \). Because \( F \) is a lift of \( f \) these two integrals have the same value. Thus \( \text{vol}(f) \) is equal to the volume of \( \rho \) with respect to \( M_\gamma \).

Since the restriction of \( F \) to \( \partial \tilde{M} \) was an arbitrary equivariant map, the volume of \( \rho \) with respect to \( M \) is defined and is equal to the volume of \( \rho \) with respect to \( M_\gamma \). \( \square \)

### 2.6 Volume form

In this section \( M \) will be a hyperbolic 3-manifold with one cusp. In Section 2.5, I discussed the volume of a representation of \( \pi_1(M) \) into \( \text{PSL}_2 \mathbb{C} \). As volume is invariant under conjugation, there is a map \( \text{vol} \) from the irreducible characters of \( X(M) \) to \( \mathbb{R}^+ \) where \( \text{vol}(\chi) \) is the volume of any representation with character \( \chi \). The next proposition is a reinterpretation of the results of Sections 4.4-4.5 of [CCGLS], and will be one of the keys to the proof of Theorem 3.1. A normalization of a curve \( X \) is a smooth curve \( Y \) together with a regular birational map \( f : Y \to X \). A normalization of \( X \) can
be constructed by taking a smooth projective model $Y'$ with birational isomorphism $f: Y' \to X$ and letting $Y = f^{-1}(X)$.

**Theorem 2.6** Let $X_0$ be an irreducible component of $X(M)$ which contains the character of a discrete faithful representation. Let $Y$ be a normalization of $i^*(X_0)$ where $i^*: X(M) \to X(\partial M)$ is the map induced by $i: \partial M \hookrightarrow M$. Then the map $\text{vol}$ on the irreducible characters of $X_0$ factors through a map $Y \to \mathbb{R}^+$.

That is, there is a map $\text{vol}: Y \to \mathbb{R}^+$ such that the diagram

\[
\begin{array}{ccc}
\mathbb{R}^+ & \xrightarrow{\text{vol}} & X_0 \\
\downarrow{\text{vol}} & & \downarrow{i^*(X_0)} \\
Y & \xrightarrow{f} & \mathbb{R}^+
\end{array}
\]

commutes. In fact, $\text{vol}$ will be the absolute value of a generically smooth function from $Y$ to $\mathbb{R}$.

**Proof** I will use the notation of Section 2.4. In Section 4 of [CCGLS], the authors define a real-valued differential form $\eta$ on $\Delta$ by

$$\eta = \log |a| \ d \arg(b) - \log |b| \ d \arg(a).$$

which measures the change in volume in the following sense. Let $\tilde{R}_0$ denote the subset of $\tilde{R}_H(M)$ which maps to $X_0$. In the language of Section 2.5, we can define a map $V: \tilde{R}_0 \to \mathbb{R}$ by setting $V(\rho)$ to be the integral of $f^*(\text{Vol}_{\mathbb{R}^3})$ over $M$ where $f$ is any pseudo-developing map for $\rho$. Thus $V(\rho) = \pm \text{vol}(\rho)$, and the results of Section 2.5 show that $V$ is well defined. Work of Hodgson, see Section 4.5 of [CCGLS], shows that $V$ is smooth and that $dV$ is the pull-back of $-\frac{1}{2} \eta$ along $i^*: \tilde{R}_0 \to D_M$. By either Section 4.4 or 4.5 of [CCGLS], the form $\eta$ is exact on $D_M$, in the sense that it is exact (where defined) on any smooth projective model of $D_M$. Let $D_M'$ be subset of $D_M$ on which $\eta$ is defined. Therefore, $V$ factors through a map from a normalization of $D_M'$ to $\mathbb{R}$. More precisely, let $\tilde{D}_M$ be a normalization of $D_M'$, and $f: D_M' \to \tilde{D}_M$ a birational isomorphism. Then there is a smooth function $\nu: \tilde{D}_M \to \mathbb{R}$ such that $\nu \circ f \circ i^* = V$. Note that the form $\eta$ is invariant under the transformations which quotient $\Delta$ down to $X(\partial M)$ (these transformations are $(a, b) \mapsto (1/a, 1/b)$, $(a, b) \mapsto (-a, b)$, etc.). Therefore $\nu$ descends to a function $\nu'$ on a normalization of $t(D_M')$. As any character in $X_0$ lifts to one in $\text{SL}_2\mathbb{C}$ and $\eta$ is defined on all of $\Delta$, the curve $i^*(X_0)$ is contained in $t(D_M')$. So the function $\nu'$ is defined on a normalization of $i^*(X_0)$. If $\chi \in X_0$ is the character of a representation $\rho \in \tilde{R}_0$, we have $|\nu'(i^*(\chi))| = |V(\rho)| = \text{vol}(\rho) = \text{vol}(\chi)$. So $|\nu'|$ is the required function. \qed
2.7 Discrete faithful representations

It is important to remember that if \( \rho \) and \( \rho' \): \( \pi_1(M) \to \text{PSL}_2\mathbb{C} \) are holonomy representations of a one-cusped finite-volume hyperbolic 3-manifold \( M \), then \( \rho \) and \( \rho' \) need not be conjugate in \( \text{PSL}_2\mathbb{C} \). They are conjugate in \( O(3,1) \) by Mostow rigidity, and it’s not difficult to see that \( \rho' \) is conjugate in \( \text{PSL}_2\mathbb{C} \) to either \( \rho \) or to the complex conjugate \( \bar{\rho} \) of \( \rho \). By complex conjugate, I mean \( \bar{\rho}(\gamma) \) is the matrix whose entries are the complex conjugates of those of \( \rho(\gamma) \), for all \( \gamma \in \pi_1(M) \). I will need the following little lemma, which the reader may wish to skip until it is used in the proof of Theorem 3.1. It is just an application of Lemma 6.1 of [CL], and describes the behavior of the map \( i^*: X(M) \to X(\partial M) \) near the two discrete faithful characters. Let \( p \) be the point of \( X(\partial M) \) where the trace of any element of \( \pi_1(\partial M) \) is \( \pm 2 \). If \( \chi \) is a discrete faithful character, then \( i^*(\chi) = p \). The lemma is:

Lemma 2.7 Let \( M \) be a finite-volume hyperbolic 3-manifold with one cusp. Then the two discrete faithful characters have neighborhoods in \( X(M) \) whose images under \( i^* \) are distinct branches of \( i^*(X_0) \) through \( p \).

Proof Let \( (\alpha, \beta) \) be a basis of \( \pi_1(\partial M) \). If \( \rho_0 \) is a discrete faithful representation in \( \text{PSL}_2\mathbb{C} \) we can conjugate it by an element of \( \text{PSL}_2\mathbb{C} \) so that

\[
\rho_0(\alpha) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_0(\beta) = \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.
\]

The cusp shape of \( M \) with respect to this basis is \( a \) (Eqn. (2) uniquely determines \( a \)). The cusp shape \( a \) is always non-real, and, by changing the basis of \( \pi_1(\partial M) \) if necessary, we can assume \( a \) is not pure imaginary. Let \( \chi_0 \in X(M) \) be the character of \( \rho_0 \). As in Lemma 6.1 of [CL] it is not hard to show that \( \lim_{\chi \to \chi_0} f_\beta/f_\alpha = a^2 \) (the key is to note that \( \alpha \) and \( \beta \) commute). From Eqn. (2) we see that the cusp shape of \( \bar{\rho}_0 \) is \( \bar{a} \). So, \( \lim_{\chi \to \chi_0} f_\beta/f_\alpha = a^2 \neq a^2 \). Since \( f_\alpha \) and \( f_\beta \) depend only on the image of a character in \( X(\partial M) \), neighborhoods of \( \chi_0 \) and \( \bar{\chi}_0 \) must go to distinct branches of \( i^*(X(M)) \) through \( p \). \( \square \)

3 Degrees of maps of character curves

Theorem 3.1 Let \( M \) be a finite-volume hyperbolic 3-manifold with one cusp. Let \( X_0 \) be a component of the \( \text{PSL}_2\mathbb{C} \) character variety of \( M \) which contains the character of a discrete faithful representation. The inclusion \( i: \partial M \hookrightarrow M \) induces a regular map \( i^*: X_0 \to X(\partial M) \). This map is a birational isomorphism onto its image.

Proof Fix a discrete faithful character \( \chi_{\text{df}} \) in \( X_0 \). By Proposition 1.1.1 of [CGLS], \( X_0 \) has complex dimension 1, and \( i^* \) is non-constant, so \( i^*(X_0) \) also has dimension 1. As \( X_0 \) is irreducible, so is \( i^*(X_0) \). The map \( i^*: X_0 \to i^*(X_0) \) is a regular map of irreducible algebraic curves, and so has a degree which is the number of points in \( (i^*)^{-1}(p) \) for
generic $p \in i^*(X_0)$ (here, generic means except for a finite number of points of $i^*(X_0)$). A degree-1 map is always a birational isomorphism. Thus it suffices to show that there are infinitely many points $p$ in $i^*(X_0)$ where $(i^*)^{-1}(p)$ consists of a single point.

We construct $p_j$, $j \in \mathbb{N}$, so that $(i^*)^{-1}(p_j)$ consists of a single point as follows. By Thurston’s Hyperbolic Dehn Surgery Theorem, all but finitely many Dehn fillings of $M$ are hyperbolic (see [Thu] and [NZ], or [BP]). Choose an infinite sequence of distinct Dehn fillings so that the resulting manifolds $M_{\gamma_1}, M_{\gamma_2}, \ldots$ are all hyperbolic. Moreover, Thurston’s theorem says that we can choose the $\gamma_j$’s so that the hyperbolic structures of the $M_{\gamma_j}$ converge to the hyperbolic structure of $M$. In particular, there are holonomy characters $\chi_j$ of the $M_{\gamma_j}$ which converge to $\chi^\text{df}$. Additionally, we can assume for each $j$ that the core of the solid torus attached to $M$ to form $M_{\gamma_j}$ is a geodesic in $M_{\gamma_j}$. By Corollary 3.28 in [Por], $\chi^\text{df}$ is a smooth point of $X(M)$, and since the $\chi_j$ converge to $\chi^\text{df}$, infinitely many $\chi_j$ lie in $X_0$ (alternatively, for our purposes, we could just change $X_0$ if necessary).

By Proposition 2.6 the volume of a character depends only on its image in $X(\partial M)$, or more precisely in the smooth projective model of $i^*(X_0)$. This is key to the proof.

Consider one of the holonomy characters $\chi_j$ which comes from Dehn-filling $M$ along the curve $\gamma_j$ in $\partial M$. From now on, consider the map $i^*: X_0 \to i^*(X_0)$ as a map to a smooth projective model of $i^*(X_0)$, though I will not change notation. Let $p_j = i^*(\chi_j)$. Suppose $\chi$ is a point in $(i^*)^{-1}(p_j)$ besides $\chi_j$. Now if $\rho_j$ is a representation corresponding to $\chi_j$ then $\rho_j(\gamma_j) = I$. Let $\beta$ be a curve in $\partial M$ which forms a basis with $\gamma_j$ of $\pi_1(\partial M)$. Then $\beta$ is homotopic in $\partial M$ to the core of the solid torus attached to $M$ to form $M_{\gamma_j}$, and so $\beta$ is homotopic to a closed geodesic in $M_{\gamma_j}$. So $\rho_j(\beta)$ is a hyperbolic element of $\text{PSL}_2\mathbb{C}$. In particular, $\text{tr}(\rho_j(\beta)) \neq \pm 2$. Now if $\rho$ is a representation whose character is $\chi$ then $\text{tr}(\rho(\gamma_j)) = \text{tr}(\rho_j(\gamma_j)) = \pm 2$ and $\text{tr}(\rho(\beta)) = \text{tr}(\rho_j(\beta)) \neq \pm 2$. Since $\rho(\beta)$ and $\rho(\gamma_j)$ commute, we must have $\rho(\gamma_j) = I$. Hence $\rho$ is also a representation of $\pi_1(M_{\gamma_j})$.

Since $\chi_j$ and $\chi$ map to the same point in $i^*(X_0)$, they have the same volume. Moreover by Lemma 2.5.4, for any representation $\psi: \pi_1(M) \to \text{PSL}_2\mathbb{C}$ which factors through $\pi_1(M_{\gamma_j})$, the volume of $\psi$ does not depend on whether it is computed with respect to $M_{\gamma_j}$ or $M$. Now by Volume Rigidity for $M_{\gamma_j}$, the representation $\rho: \pi_1(M_{\gamma_j}) \to \text{PSL}_2\mathbb{C}$ must be discrete and faithful. Hence, as discussed in Section 2.7, $\rho$ is conjugate to $\rho_j$ or $\tilde{\rho}_j$. I claim that for large $j$, $i^*(\rho_j) \neq i^*(\tilde{\rho}_j)$. We know that the $\chi_j$ converge to $\chi^\text{df}$ and so the $\tilde{\chi}_j$ converge to $\tilde{\chi}^\text{df}$ by Lemma 2.7 we know neighborhoods of $\chi^\text{df}$ and $\tilde{\chi}^\text{df}$ go to distinct branches of $i^*(X(M))$. So for large $j$, $i^*(\chi_j) \neq i^*(\tilde{\chi}_j)$. Hence for large $j$, $(i^*)^{-1}(p_j)$ consists only of $\chi_j$. So the map $i^*: X_0 \to i^*(X_0)$ has degree 1.

It is easy to deduce the corresponding result for $\text{SL}_2\mathbb{C}$ character varieties.

**Corollary 3.2** Let $M$ be a finite-volume hyperbolic 3-manifold with one cusp. Let $\tilde{X}_0$ be a component of the $\text{SL}_2\mathbb{C}$ character variety of $M$ which contains the character of a discrete faithful representation. The inclusion $i: \partial M \hookrightarrow M$ induces a map $i^*: \tilde{X}_0 \to \tilde{X}(\partial M)$. This map has degree onto its image at most $|H^1(M, \mathbb{Z}_2)|/2$ where $|\cdot|$ denotes
number of elements. In particular, if $H^1(M,\mathbb{Z}_2) = \mathbb{Z}_2$ then $i^*$ is a birational isomorphism onto its image.

**Proof** If a representation $\rho$ into $\text{PSL}_2\mathbb{C}$ lifts to a representation $\tilde{\rho}$ into $\text{SL}_2\mathbb{C}$ then there are $|H^1(M,\mathbb{Z}_2)|$ distinct lifts which are constructed like this: If $\epsilon \in H^1(M,\mathbb{Z}_2)$ is thought of as a homomorphism $\epsilon: \pi_1(M) \to \mathbb{Z}_2 = \{1, -1\} \subset \text{SL}_2\mathbb{C}$ then we can construct another lift of $\rho$ by $\phi(g) = \epsilon(g)\tilde{\rho}(g)$. By Poincaré duality and the long exact sequence for the pair $(M, \partial M)$, the image $H_1(\partial M, \mathbb{Z}_2) \to H_1(M, \mathbb{Z}_2)$ is one-dimensional. So if $\chi$ is a $\text{PSL}_2\mathbb{C}$ character which lifts to an $\text{SL}_2\mathbb{C}$ character, then the $|H^1(M,\mathbb{Z}_2)|$ distinct lifts map to precisely two points in $\tilde{X}(\partial M)$, unless the traces of $\pi_1(\partial M)$ are all zero. Since all the traces are not zero generically on $\tilde{X}_0$, the last theorem shows that the map $i^*: \tilde{X}_0 \to \tilde{X}(\partial M)$ has degree at most $|H^1(M,\mathbb{Z}_2)|/2$. □

**Remark** In the first example of [Dun] the map $i^*$ has degree 4 on $\tilde{X}_0$ and the first homology is $\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$. For components of $X(M)$ which do not contain a discrete faithful character, the map $i^*: X(M) \to X(\partial M)$ may not have degree 1. For instance, there are sometimes components of $X(M)$ which have dimension greater than 1 whose image under $i^*$ is one-dimensional (see Theorem 8.2 and 10.1 of [CL]). Even if we consider an irreducible component of $X(M)$ of dimension 1 whose image under $i^*$ is one-dimensional, $i^*$ may still fail to have degree 1. This happens with the exterior of the knot $7_4$.

## 4 Cyclic Surgery Slopes and Boundary Slopes

Let $K$ be a knot in a homotopy sphere $\Sigma$ with exterior $M$. Fix a meridian $\mu$ and longitude $\lambda$ in $H_1(\partial M, \mathbb{Z})$. The slope of $\gamma \in H_1(\partial M, \mathbb{Z})$ with respect to the basis $(\mu, \lambda)$ will be denoted $r_\gamma$. In this section I will prove:

**Theorem 4.1** Suppose $K$ is a small hyperbolic knot in a homotopy sphere. Suppose there is a $\beta \in H_1(\partial M, \mathbb{Z})$ with $\pi_1(M_\beta)$ cyclic and $\beta \neq \pm \mu$. Then there is an essential surface in the exterior of $K$ whose boundary slope is non-integral and lies in $(r_\beta - 1, r_\beta + 1)$.

By the Cyclic Surgery Theorem [CGLS], $r_\beta$ is always an integer. Theorem 4.1 is true more generally for a small hyperbolic knot in a manifold $\Sigma$ with cyclic fundamental group whose exterior satisfies $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$. In this setting, take $\lambda$ to be an arbitrary element of $H_1(\partial M, \mathbb{Z})$ such that $(\mu, \lambda)$ is a basis (there is not always a natural choice when $\pi_1(\Sigma)$ is non-trivial). Note the condition on $H^1(M, \mathbb{Z}_2)$ holds if $\pi_1(\Sigma)$ has odd order. Theorem 4.1, including this more general case, follows easily from:

**Theorem 4.2** Let $M$ be a one-cusped finite-volume hyperbolic 3-manifold with $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$. Suppose $(\mu, \beta)$ is a basis for $H_1(\partial M, \mathbb{Z})$ where $\pi_1(M_\beta)$ is cyclic. For $\gamma \in H_1(\partial M, \mathbb{Z})$, denote by $s_\gamma$ the slope with respect to the basis $(\mu, \beta)$. Then there is a boundary class $\gamma$ such that $|s_\gamma| < 1$. 
Note that hypotheses of Theorem 4.2 do not restrict $\pi_1(M_\mu)$, nor is it assumed that $M$ is small. I will now prove Theorem 4.1 assuming Theorem 4.2.

**Proof (of Theorem 4.1)** By the Cyclic Surgery Theorem, the meridian $\mu$ and the given class $\beta$ form a basis for $H_1(\partial M, \mathbb{Z})$. Let $\gamma$ be the boundary class of $M$ given by Theorem 4.2. Note $r_\gamma = s_\gamma + r_\beta$, and recall that $r_\beta$ is an integer. Since we are assuming $M$ is small, Theorem 2.0.3 of [CGLS] shows that $\beta$ is not a boundary class. Therefore $r_\gamma$ is a non-integral boundary slope in $(r_\beta - 1, r_\beta + 1)$.

I will now prove Theorem 4.2:

**Proof (of Theorem 4.2)** I will use $\text{SL}_2 \mathbb{C}$ character varieties here, so fix a component $\tilde{X}_0$ of $\tilde{X}(M)$ which contains the character of a discrete faithful representation. I will break the proof into three lemmas. The first is:

**Lemma 4.3** If $K$ is a counterexample to the theorem, then the function $f_\mu/f_\beta$ is constant on $\tilde{X}_0$.

So suppose $f_\mu/f_\beta$ is constant. From this, it is possible to determine $i^*(\tilde{X}_0)$ precisely. To state the answer, I will use the associated plane curve and the notation of Section 2.4. Let $D_0$ be an irreducible component of $t^{-1}(i^*(\tilde{X}_0))$. Let $(m, b)$ be the eigenvalue coordinates on $\Delta$ corresponding to the basis $(\mu, \beta)$. The second lemma is:

**Lemma 4.4** If $f_\mu/f_\beta$ is constant on $\tilde{X}_0$, there is a constant $C \neq \pm 1$ such that $D_0$ is exactly the set of zeros of the irreducible polynomial

$$P(m, b) \equiv bm^2 - b - Cb^2m + Cm.$$  \hspace{1cm} (3)

The proof of the theorem is finished with:

**Lemma 4.5** The polynomial $P$ of the last lemma can not define $D_0$ for a hyperbolic manifold $M$ with $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$.

I will now prove these lemmas. The first follows immediately from the following result, which is more general in that it makes no assumption about $H^1(M, \mathbb{Z}_2)$:

**Lemma 4.6** Let $M$ be a one-cusped finite-volume hyperbolic 3-manifold. Suppose $(\mu, \beta)$ is a basis for $H_1(\partial M, \mathbb{Z})$ where $\pi_1(M_\beta)$ is cyclic. Either there is a boundary class $\gamma$ such that $|s_\gamma| < 1$, or the function $f_\mu/f_\beta$ is constant on $\tilde{X}_0$.

**Proof (Lemma 4.6)** Set $g = f_\mu/f_\beta$. Let $Y$ be a smooth projective model of $\tilde{X}_0$. If $g$ is constant on $Y$, we are done. Otherwise, I will produce an essential surface with boundary class $\gamma$ where $|s_\gamma| < 1$. As $s_\beta = 0$, if $\beta$ is a boundary class take $\gamma = \beta$. So assume $\beta$ is not a boundary class. Let $y \in Y$ be a pole of $g$. Suppose $f_\beta$ has a zero at $y$. For any rational function $h$ on $Y$, let $Z_y(h)$ denote the order of zero at $y$, where $Z_y(h) = 0$ if $h$ does not have a zero at $y$. As $\beta$ is not a boundary class and $\pi_1(M_\beta)$ is cyclic, Proposition 1.1.3 of [CGLS] shows that the function $f_\mu$ also has a zero at $y$ and
\(Z_y(f_\mu) \geq Z_y(f_\beta)\). But then \(g\) does not have a pole at \(y\), a contradiction. So \(f_\beta\) must not have a zero at \(y\). Thus \(f_\mu\) must have a pole at \(y\), and so \(y\) must be an ideal point where the associated surfaces have non-empty boundary (see Section 2.3). Let \(\gamma\) be a boundary class associated to \(y\). By the proof of Lemma 1.4.1 of [CGLS], we have:

\[
|s_\gamma| = \frac{\Pi_y(f_\beta)}{\Pi_y(f_\mu)},
\]

where \(\Pi_y\) denotes the order of pole at \(y\) or is 0 if there is no pole. Since \(\Pi_y(g) = \Pi_y(f_\mu) - \Pi_y(f_\beta) > 0\), we have \(|s_\gamma| < 1\). □

Now I will prove Lemma 4.4, which determines the equation defining \(D_0\), assuming that \(f_\mu/f_\beta\) is constant.

Proof (Lemma 4.4) Let \(C'\) be the constant such that \(f_\mu/f_\beta = C'\) on \(D_0\). With our coordinates \((m, b)\) on \(\Delta\), we have

\[
f_\mu = \left(\frac{m + 1}{m}\right)^2 - 4 = \left(\frac{m - 1}{m}\right)^2 \quad \text{and} \quad f_\beta = \left(\frac{b - 1}{b}\right)^2.
\]

So on \(D_0\) the equation

\[
\left(\frac{m - 1}{m}\right)^2 = C' \left(\frac{b - 1}{b}\right)^2
\]

holds. Since \(D_0\) is irreducible, for some square root \(C\) of \(C'\) the equation

\[
\left(\frac{m - 1}{m}\right) = C \left(\frac{b - 1}{b}\right)
\]

holds on \(D_0\). Turning this into a polynomial condition, we have

\[
P(m, b) \equiv bm^2 - b - Cb^2 m + Cm = 0
\]

on \(D_0\).

Next, I will show \(C \neq \pm 1\). If \(C = \pm 1\), \(P = (bm + C)(m -Cb)\). Then one of the eigenvalue functions \(\xi_{\mu+\beta}\) or \(\xi_{\mu-\beta}\) would be constant on \(D_0\), which is impossible by Proposition 2 of [CS2].

If \(C \neq \pm 1\), it is an elementary exercise to check that \(P\) is irreducible. Thus \(D_0\) must be exactly the zero set of \(P\). □

Finally, I will prove Lemma 4.5, showing that the polynomial \(P\) of Lemma 4.4 cannot be the defining polynomial for \(D_0\) because \(H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2\).

Proof (Lemma 4.5) As before, let \(Y\) be a smooth projective model of \(\tilde{X}_0\). Since \(H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2\), Corollary 3.2 shows that the map \(i^\ast: \tilde{X}_0 \to i^\ast(\tilde{X}_0)\) is a birational isomorphism. So we can also think of \(Y\) as a smooth projective model of \(i^\ast(\tilde{X}_0)\). Thus we have a rational map \(t: D_0 \to Y\) induced by \(t: D_0 \to i^\ast(\tilde{X}_0)\). Note \(P(0, 0) = 0\), and so \((0, 0) \in D_0\).
This is a smooth point of $D_0$ and the equation for the tangent line through $(0,0)$ is $Cm - b = 0$. It follows that the eigenvalue functions $m = \xi_\mu$ and $b = \xi_\beta$ restricted to $D_0$ have simple zeros at $(0,0)$. So the functions $\text{tr}_\mu = m + 1/m$ and $\text{tr}_\beta = b + 1/b$ on $D_0$ have simple poles at $(0,0)$. Let $y = t((0,0))$. Then $y \in Y$ is an ideal point of $\tilde{X}_0$ and both $\text{tr}_\mu$ and $\text{tr}_\beta$ have simple poles at $y$. Looking again at the line tangent to $D_0$ though $(0,0)$, we see that $\mu - \beta$ is a boundary class associated to $y$ (see Section 2.3). Also, the eigenvalue of $\mu - \beta$ at $(0,0)$ is $C$. By the theorem in Section 5.7 of [CCGLS] the value of $\xi_{\mu-\beta}$ at $y$ is a root of unity whose order divides the number of boundary components of any surface associated to $y$. So since $C \neq \pm 1$, any surface associated to $y$ has at least three boundary components. We now get our contradiction by showing that there is a surface associated to $y$ with only two boundary components. Let $T_y$ be the tree associated to $y$. By Proposition 2.2, the translation length of $\mu$ on $T_y$ is twice the order of pole of $\text{tr}_\mu$ at $y$. From the dual role of $Y$ as the smooth projective model of both $i^*(\tilde{X}_0)$ and $\tilde{X}_0$, we have calculated that the order of pole of $\text{tr}_\mu$ at $y$ is 1, and so the translation length of $\mu$ is 2. But then Proposition 2.3 shows there is a surface associated to $y$ with two boundary components, a contradiction. So $P$ can not be the equation defining $D_0$. 

Since we have proved Lemmas 4.3, 4.4, and 4.5, we have proven the theorem. 

Remark Note that the proof used $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$ in a fundamental way. If the degree of $\tilde{X}_0 \to i^*(\tilde{X}_0)$ were $d$, we could only have concluded that there was a surface associated to $y$ having as few as $2d$ boundary components. The first example in [Dun] has an ideal point whose image in $i^*(\tilde{X}_0)$ looks exactly like in the proof, in the sense that $\text{tr}_\mu$ has a simple pole at the corresponding point of $D_0$. But there, $H_1(M, \mathbb{Z})$ is $\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$ and the associated surface has four boundary components. This illustrates the way that Corollary 3.2 shows there is a surprising connection between the size of $H^1(M, \mathbb{Z}_2)$ and the character variety of $M$.

5 The norm and the diameter of the set of boundary slopes

Lemma 4.6 has an interpretation in terms of the norm on $H_1(\partial M, \mathbb{R})$ defined in [CGLS]. Let $V$ denote $H_1(\partial M, \mathbb{R})$ and $L \subset V$ the lattice $H_1(\partial M, \mathbb{Z})$. This norm on $V$ has the property that for $\gamma \in L$, the number $||\gamma||$ is the degree of $f_\gamma$ on $\tilde{X}_0$. Let $r$ be the minimum of $||\gamma||$ over all non-zero $\gamma \in L$. Let $B$ denote the closed ball of radius $r$ about $0$ in $V$. The ball $B$ is a finite-sided convex polygon which is invariant under $v \mapsto -v$. Boundary classes associated to ideal points of $\tilde{X}_0$ correspond bijectively with vertices of $B$, where a boundary class $\gamma$ corresponds to a vertex $v$ where $v = a\gamma$ for a positive rational number $a$ (this is not quite explicit in [CGLS], but see Lemma 6.1 of [BZ1]). An element $\gamma \in L$ such that $\pi_1(M, \gamma)$ is cyclic and which is not a boundary class has minimal norm, that is, $||\gamma|| = r$ and $\gamma \in B$. In this language, a slight modification of the proof of Lemma 4.6 gives:
Lemma 5.1 Suppose \( (\mu, \beta) \) is a basis of \( L \) such that \( \pi_1(M_\mu) \) is cyclic, \( \mu \) is not a boundary slope, and \( \beta \) has minimal norm. Then either there is a boundary class \( \gamma \), associated to an ideal point of \( \tilde{X}_0 \), which satisfies \(|s_\gamma| < 1\) or the function \( f_\mu / f_\beta \) is constant on \( \tilde{X}_0 \).

Proof The needed modification to the proof of Lemma 4.6 is at the step that shows that if \( f_\beta \) has a zero at \( y \) then \( f_\mu \) also has a zero at \( y \) and \( Z_y(f_\mu) \geq Z_y(f_\beta) \). Here, by Proposition 1.1.3 of [CGLS], we know if \( f_\mu \) has a zero at \( y \) then \( Z_y(f_\mu) \leq Z_y(f_\beta) \). The total number of zeros of \( f_\mu \) is equal to the total number of zeros of \( f_\beta \) since \( ||\mu|| = ||\beta|| \).

Thus the sets of zeros (including multiplicity) of \( f_\mu \) and \( f_\beta \) are the same, and we can apply the rest of the proof of Lemma 4.6 unchanged.

I will now give an application to the following question. Consider a knot in a homotopy sphere with reducible exterior \( M \). The set of boundary slopes is finite [Hat], and so has a well-defined diameter \( d \) as a subset of \( \mathbb{Q} \cup \{\infty\} \) (I will use the convention that \( d = \infty \) if \( \infty \) is a boundary slope). Hatcher and Thurston [HT] asked in the case of knots in \( S^3 \) whether \( d \) is always greater than 2. In [CS3], Culler and Shalen showed that for any knot, the diameter \( d \geq 2 \). Another application of the proof of Theorem 4.1 is to show:

Theorem 5.2 If \( d = 2 \) for a hyperbolic knot in a homotopy sphere, the greatest and least boundary slopes are not integers.

Proof Consider such a knot with \( d = 2 \). In [CS3, Proof of Main Theorem] it is shown that for a suitable choice of meridian-longitude basis \((\mu, \lambda)\) for \( L \), \( B \) is as in Fig. 1. Since this is not quite explicit there, let me elaborate. In the notation of [CS3], let \( v_1 \) and \( v_2 \) be the vertices of \( B \) which are the ends of the edge containing \( \mu \). Then \( \mu = tv_1 + (1 - t)v_2 \) for some \( t \in [0, 1] \), and they show that the diameter of the set of boundary slopes is bounded below by \( 1/(2t(t - 1)) \geq 2 \). If the diameter of the set of boundary slopes is 2, we must have \( t = 1/2 \). In this case, the area of the parallelogram with vertices \( \{v_1, v_2, -v_1, -v_2\} \) is 4 and so \( B \) is equal to this parallelogram. Since \( t = 1/2 \), \( \mu \) lies in the middle of an edge of \( B \). Combined with the fact that the area of \( B \) is 4, the sides of \( B \) must be segments of vertical lines \( \lambda = \pm 1 \). Since \( 0 \) is always a boundary slope, after possibly changing the signs of \( \mu \) and \( \lambda \), \( B \) must be as in Fig. 1.

From the correspondence between boundary classes associated to ideal points of \( \tilde{X}_0 \) and vertices of \( B \), there are only two boundary slopes associated to ideal points of \( \tilde{X}_0 \), namely, \(-p/q \) and \( 2 - p/q \) where \( 0 \leq p \leq q \) and \( \gcd(p, q) = 1 \).

Showing that the greatest and least slopes are not integers is equivalent to showing \( q > 1 \). Suppose \( q = 1 \). There are two cases: \( p = 0 \) and \( p = 1 \). If \( p = 1 \), the two boundary slopes are 1 and \(-1 \) with respect to the basis \((\mu, \lambda)\). Since \( \lambda \) is in \( B \) it has minimal norm, and as \( \mu \) is not a boundary class (since \( d \) is finite), Lemma 5.1 shows that \( f_\mu / f_\lambda \) must be constant. We can now apply Lemmas 4.4 and 4.5 to get a contradiction. If \( p = 0 \), we apply the same argument with \( \lambda \) replaced by \( \gamma = \mu + \beta \).

It is possible to prove more:
Property P, degrees of maps of character curves, and volume rigidity

Theorem 5.3 Suppose the diameter of the set of boundary slopes of a hyperbolic knot in a homotopy sphere is 2. Let \(0 \leq p \leq q\) be such that the greatest and least slopes are \(2 - \frac{p}{q}\) and \(-\frac{p}{q}\) respectively. Then \(p\) is even, \(q\) odd, and \(q > 1\).

The key to this version is a lemma which is a consequence of Theorem 3.1. Consider the commutative diagram:

\[
\begin{array}{ccc}
\tilde{X}_0 & \xrightarrow{i^*} & i^*(\tilde{X}_0) \subset \tilde{X}(\partial M) \\
\pi_M \downarrow & & \downarrow \pi_{\partial M} \\
X_0 & \xrightarrow{i^*} & i^*(X_0) \subset X(\partial M)
\end{array}
\]

The map \(\pi_{\partial M}\) from \(\tilde{X}(\partial M)\) to \(X(\partial M)\) has degree 4, so one might expect that the restriction of \(\pi_{\partial M}\) to \(i^*(\tilde{X}_0)\) could also have degree 4. But I will show the degree of this restriction can be no more than 2. An involution \(f\) of \(i^*(\tilde{X}_0)\) which has the property that \(\pi_{\partial M} \circ f = \pi_{\partial M}\) will be called a symmetry of \(i^*(\tilde{X}_0)\). If the restriction of \(\pi_{\partial M}\) to \(i^*(\tilde{X}_0)\) has degree 4, the group of symmetries would have order 4. One possible symmetry, \(\tau\), is the restriction of the involution of \(\tilde{X}(\partial M)\) whose action in coordinates is \((\text{tr}_\mu, \text{tr}_\lambda, \text{tr}_{\mu+\lambda}) \mapsto (-\text{tr}_\mu, \text{tr}_\lambda, -\text{tr}_{\mu+\lambda})\). I will show:

Lemma 5.4 For the exterior of a hyperbolic knot in a homotopy sphere, the only possible symmetry of \(i^*(X_0)\) is \(\tau\).

Proof The discussion in the proof of Corollary 3.2 shows that \(\pi_M\) has degree 1 or 2, depending on whether \((\pi_M)^{-1}(X_0)\) is irreducible. By Theorem 3.1 and Corollary 3.2, the horizontal maps in Eqn. (6) have degree 1. So the degrees of \(\pi_M\) and \(\pi_{\partial M}\) are equal. If these degrees are 1, there are no symmetries. If the degrees are 2, \(i^*(\tilde{X}_0)\) has one symmetry, which I claim is \(\tau\). In this case, there is an involution \(\tau'\) of \(\tilde{X}_0\) for which \(\pi_M \circ \tau' = \pi_M\), namely, multiplication of characters by the unique non-trivial homomorphism \(\pi_1(M) \to \mathbb{Z}_2 = \{I, -I\} \subset \text{SL}_2\mathbb{C}\). Then \(\tau'\) induces the symmetry \(\tau\) of \(i^*(\tilde{X})\).

\(\square\)
Remark For the exterior of a knot in a homotopy sphere, it is conceivable that $i^*(X_0)$ might not have the symmetry $\tau$. Suppose $(\pi_M)^{-1}(X_0)$ consists of two irreducible components. While $i^*((\pi_M)^{-1}(X_0))$ would be invariant under $\tau$, the curve $i^*(\tilde{X}_0)$ might not be. The first example of [Dun] is a one-cusped hyperbolic manifold where $(\pi_M)^{-1}(X_0)$ splits into multiple components. Applying $i^*$ to one component yields a curve which is not invariant under the analogue of $\tau$. However, this manifold is not exterior of a knot in a homotopy sphere.

From Lemma 5.4 we can prove Theorem 5.3:

Proof (Theorem 5.3)

Let $p$ and $q$ be as in the statement. Theorem 5.3 shows that $q > 1$, and I will assume this throughout. Let $\gamma = \mu + \lambda$. Consider the function

$$g = \frac{f_\gamma}{f_\mu} \text{ on } \tilde{X}_0,$$

noting that $q - p > 0$. The proof follows the same basic plan as that of Theorem 4.1. The analogue of Lemma 4.3 is:

Lemma 5.5 If $d = 2$, then $g$ is constant on $\tilde{X}_0$.

Let $(m, l)$ be the eigenvalue coordinates on $\Delta$ corresponding to $(\mu, \lambda)$. The analogue of Lemma 4.4 is:

Lemma 5.6 If $g$ is constant there is a $C \in \mathbb{C}$ so that $D_0$ is exactly the zeros of

$$P(m, l) \equiv m^p (l^2 - 1)^p (l^2 m^2 - 1)^{q-p} - Cl^q (m^2 - 1)^q.$$  (7)

The final step, which is the one that differs most from the proof of Theorem 4.1, is to use Lemma 5.4 to show the following analogue of Lemma 4.5:

Lemma 5.7 If the polynomial $P$ of the last lemma defines $D_0$ for the exterior of a knot in a homotopy sphere, then $p$ is even and $q$ is odd.

I will begin with:

Proof (Lemma 5.5) This is a refinement of the proofs of Lemmas 4.6 and 5.1. Suppose $g$ is not constant. Let $y \in Y$ be a pole of $g$, where $Y$ is a smooth projective model of $\tilde{X}_0$. We know that $\pi_1(M_\mu)$ is a cyclic and that $\mu$ is not a boundary class. From Fig. 1 we know $\mu$, $\lambda$, and $\gamma$ all have minimal norm. Therefore, as in the proof of Lemma 5.1, the sets of zeros (including multiplicity) of $f_\mu$, $f_\lambda$, and $f_\gamma$ are all the same. But then $g$ would not have a pole at $y$, a contradiction. Thus at least one of $f_\lambda$ or $f_\gamma$ has a pole at $y$. So $y$ is an ideal point.

By Lemma 1.4.1 of [CGLS] there is a linear functional $l$ on $H_1(\partial M, \mathbb{Z})$ such that the order of pole of $f_\alpha$, $\Pi_y f_\alpha$, is $|l(\alpha)|$ for all $\alpha$ in $H_1(\partial M, \mathbb{Z})$. The slope associated to $y$ is either $-p/q$ or $2 - p/q$. In the first case, $\Pi_y f_{-p\mu + q\lambda} = 0$ and so there is a $d \in \mathbb{Z}$ such
that \( l(a\mu + b\lambda) = d(qa + pb) \). Then \( \Pi_y f_\mu = |d|q, \Pi_y f_\lambda = |d|p, \) and \( \Pi_y f_\gamma = |d|(p+q) \). So

\[
\Pi_y g = -q\Pi_y f_\mu + p\Pi_y f_\lambda + (q-p)\Pi_y f_\gamma = 0.
\]

If the slope associated to \( y \) is \( 2-p/q = (2q-p)/q \) then \( l(a\mu + b\lambda) = d(qa-(2q-p)b) \). Then \( \Pi_y f_\mu = |d|q, \Pi_y f_\lambda = |d|(2q-p), \) and \( \Pi_y f_\gamma = |d|(q-p) \); hence \( \Pi_y g = 0 \).

So \( g \) has no poles and must be constant.

Now we deduce the equation defining \( D_0 \).

**Proof (Lemma 5.6)**

Just as in the proof of Lemma 4.4, there is a constant \( C \) so that the following equation holds on \( D_0 \):

\[
\left( l - \frac{1}{l} \right)^p \left( ml - \frac{1}{ml} \right)^{q-p} = C \left( \frac{m}{m} \right)^q,
\]

This is equivalent to the polynomial \( P \) given in the statement of Lemma 5.6 being zero.

Now I will show that \( D_0 \) is exactly the zeros of \( P \). We need to take the point of view of [CCGLS] in which information about ideal points of \( X_0 \) is deduced from the Newton polygon of the equation defining \( D_0 \). The Newton polygon of a polynomial \( Q \) in variables \( x \) and \( y \) is the convex hull in \( \mathbb{R}^2 \) of:

\[
\{(i,j) \in \mathbb{Z}^2 \mid \text{the coefficient of } x^i y^j \text{ in } Q \text{ is nonzero}\}.
\]

The Newton polygon, \( N \), of \( P \) is shown in Fig. 2.

I claim that \( \{P = 0\} \) is exactly \( D_0 \). Suppose \( Q \) were a proper factor of \( P \) which defines \( D_0 \). Let \( N' \) be the Newton polygon of \( Q \). In [CCGLS] it is shown that the slopes of the sides of \( N' \) are precisely the boundary slopes of surfaces associated to ideal points of \( X_0 \). Thus the slopes of the sides of \( N' \) are \( -p/q \) and \( 2-p/q \). So we know that \( N' \) is a parallelogram whose sides are parallel to those of \( N \). The idea is that \( N \) is the smallest such parallelogram and therefore \( N = N' \) and \( P = Q \). Let \( \text{diam}_l(N') \) denote the diameter of the projection of \( N' \) onto the axis corresponding to the exponent of \( l \).

If we think of \( Q \) as the a polynomial in a single variable \( l \) over \( \mathbb{C}[m] \), then the degree of this polynomial, \( \text{deg}_l(Q) \), is equal to \( \text{diam}_l(N') \). Since \( Q \) is a factor of \( P \), we know \( \text{deg}_l Q \leq \text{deg}_l P \), and so \( \text{diam}_l(N') \leq \text{diam}_l(N) = 2q \). Projecting onto the other axis, we also have \( \text{diam}_m(N') \leq \text{diam}_m(N) = 2q \). Moreover, if both diameters of \( N' \) and \( N \) agree, we have \( P = Q \). Now consider a side \( S \) of \( N' \) with slope \( -p/q \). Because the endpoints of \( S \) are in \( \mathbb{Z}^2 \), we must have \( \text{diam}_m(S) \geq q \) and \( \text{diam}_m(S) \geq p \). Similarly, a side \( T \) of \( N' \) with slope \( 2-p/q \) must have \( \text{diam}_m(T) \geq q \) and \( \text{diam}_m(T) \geq 2q-p \). Thus \( \text{diam}_l(N') \geq 2q = \text{diam}_l(N) \) and \( \text{diam}_m(N') \geq 2q = \text{diam}_m(N) \). So \( P = Q \) as desired. So \( D_0 \) must be exactly the zero set of \( P \).

Now I will show that if \( P \) defines \( D_0 \) for the exterior of a knot in a homotopy sphere, then \( p \) is even and \( q \) is odd. By Lemma 5.4 the only allowed symmetry is the one whose action on \( D_M \) sends \((m,l)\) to \((-m,l)\). If \( q \) is even, \( p \) and \( q - p \) are odd. Note in this
case that Eqn. (8) and hence \( P \) are invariant under \((m, l) \mapsto (m, -l)\). But then there is a symmetry of \( i^*(\tilde{X}_0) \) other than \( \tau \), which is impossible. Therefore \( q \) is odd, and we already know \( q \neq 1 \). If \( p \) and \( q \) are both odd, then Eqn. (8) is invariant under \((m, l) \mapsto (-m, -l)\), so this case is ruled out as well. So \( p \) is even and \( q \) is odd. \( \Box \)

Remarks Let \( Q \) denote the polynomial in Eqn. (7) with \( p = 1 \), \( q = 2 \), and \( C = 1 \). The polynomial \( Q \) actually occurs as the defining equation for \( D_0 \) for \( N \), the sister of the exterior of the figure-8 knot. In this case, \( N_\mu \) has fundamental group \( \mathbb{Z}_{10} \). The results of [CS3], more generally, say that \( d \geq 2 \) for any knot in a manifold with cyclic fundamental group whose exterior is irreducible and not cabled. It turns out that for \( N \), the diameter of the set of all boundary slopes is exactly 2, and this shows that the estimate of Culler and Shalen is sharp in this more general context (see Example 1.4 of [CS3]). It was the observation that equation defining \( D_N \) was \( \{ Q = 0 \} \) that led me to discover this example.

If one cares about the diameter of the set of strict boundary slopes, then the situation stays the same except that \( \lambda \) is replaced by some random class \( \nu \) with integer slope. In this case, you can not rule out \( p \) and \( q \) both being odd since \( \nu \) may generate \( H_1(M, \mathbb{Z}_2) \).

This is also why you can not use the argument about symmetries to prove Theorem 4.1.

6 Proof of volume rigidity

This section provides a proof of:

Theorem 6.1 (Gromov-Thurston-Goldman) Suppose \( M \) is a compact hyperbolic 3-manifold. If \( \rho_1: \pi_1(M) \to \text{PSL}_2\mathbb{C} \) is a representation with \( \text{vol}(\rho_1) = \text{vol}(M) \), then \( \rho_1 \) is discrete and faithful.

Proof The proof is essentially the same as that of Thurston’s strict version of Mostow’s Theorem [Thu, Theorem 6.4]. The only modification is that since \( \mathbb{H}^3/\rho_1 \) may be nasty, rather than a compact manifold, it is necessary to do some things equivariantly. I will follow [Thu], with some details coming from Toledo’s paper [Tol] using the same technique. The same proof works in higher dimensions with the aid of [HM], but I stick to the 3-dimensional case for simplicity. I assume some familiarity with Gromov’s proof of Mostow’s Theorem (for a nice account, see [Mun] or the very through [Rat, Chapter 11]).

Let \( \rho_0 \) be a discrete faithful representation for \( M \). Pick a smooth equivariant map \( f \) from \( \mathbb{H}^3 \) acted on by \( \rho_0 \) to \( \mathbb{H}^3 \) acted on by \( \rho_1 \). If the integral \( \int_M f^*(\text{Vol}_{\mathbb{H}^3}) \) is negative, choose an orientation reversing isometry \( r \) of \( \mathbb{H}^3 \) and replace \( \rho_1 \) by \( r \circ \rho_1 \circ r^{-1} \) and \( f \) by \( r \circ f \) so that the integral is positive. Then the hypothesis on \( \text{vol}(\rho_1) \) gives:

\[
\int_M \text{Vol}_{\mathbb{H}^3} = \text{vol}(M) = \text{vol}(\rho_1) = \int_M f^*(\text{Vol}_{\mathbb{H}^3}).
\]  \hspace{1cm} (9)

By a tetrahedron, I will mean a simplex in \( \mathbb{H}^3 \) with totally geodesic faces. I will divide the proof into the following three claims:
Claim 1 The map \( f \) extends to an equivariant measurable map \( \bar{f} \) from \( S^2_\infty = \partial \mathbb{H}^3 \) to itself.

Claim 2 The map \( \bar{f} \) takes vertices of almost all regular ideal tetrahedra to vertices of regular ideal tetrahedra.

Claim 3 Because of Claim 2, \( \bar{f} \) is essentially a Möbius transformation. That is, there is a Möbius transformation \( F \) so that \( F = \bar{f} \) almost everywhere.

The proof is then completed by noting that since \( F \) is equivariant, it conjugates the action of \( \rho_0 \) on \( S^2_\infty \) to the action of \( \rho_1 \) on \( S^2_\infty \). As \( \rho_0 \) and \( \rho_1 \) are conjugate, \( \rho_1 \) is discrete and faithful.

Let me start in on the proof of Claim 1. The key is that the regular ideal tetrahedron is the unique tetrahedron of maximal volume [Rat, Theorem 10.4.7]. The idea behind Claim 1 is that \( f \) must take the vertices of a non-ideal tetrahedron of near-maximal volume to points which span a tetrahedron of near-maximal volume or else \( f \) would be volume shrinking and Eqn. (9) would be violated.

The proof will use measure homology, which I will briefly describe (for details see Section 4 of [Mun] or Section 11.5 of [Rat]). Let \( C^1(\Delta_k, M) \) be the space of \( C^1 \) maps of the standard \( k \)-simplex \( \Delta_k \) into \( M \), supplied with the \( C^1 \) topology. Let \( \mathcal{C}_k(M, \mathbb{R}) \) be the set of real-valued Borel measures of bounded total variation on \( C^1(\Delta_k, M) \). There is a natural boundary operator from \( \mathcal{C}_k(M, \mathbb{R}) \) to \( \mathcal{C}_{k-1}(M, \mathbb{R}) \), and the homology of the resulting complex is called the measure homology of \( M \). There is an inclusion of the usual \( C^1 \) singular chain complex \( C_*(M, \mathbb{R}) \) into \( \mathcal{C}_*(M, \mathbb{R}) \) which sends a singular simplex (an element of \( C^1(\Delta_k, M) \)) to the associated Dirac measure. This inclusion induces an isomorphism on homology. The pairing between \( C_k(M, \mathbb{R}) \) and smooth differential \( k \)-forms extends to \( \mathcal{C}_k(M, \mathbb{R}) \).

Let \( G = \text{PSL}_2 \mathbb{C} \) and \( \Gamma \subset G \) be \( \pi_1(M) \) acting via \( \rho_0 \). The unit tangent bundle to \( M \) is \( X = \Gamma \backslash G \), and let \( \mu \) be the Haar measure on \( G \) such that \( \mu(X) = \text{vol}(M) \). Let \( D \) be a polyhedron minus some faces which is a fundamental domain for \( M \) in \( \mathbb{H}^3 \) (each orbit of \( \mathbb{H}^3 \) under \( \Gamma \) has exactly one representative in \( D \)).

Let \( \sigma \subset \mathbb{H}^3 \) be a non-ideal tetrahedron. Let \( \text{smear} \sigma \) be the measure cycle in \( \mathcal{C}_3(\mathbb{H}^3, \mathbb{R}) \) consisting of all translates of \( \sigma \) with first vertex in \( D \), uniformly weighted by \( 1/\text{vol}(\sigma) \); that is, if \( \eta \) is a differential 3-form:

\[
\langle \eta, \text{smear} \sigma \rangle = \int_{g \in X} \left( \frac{1}{\text{vol}(\sigma)} \int_{g \sigma} \eta \right) d\mu.
\]

Here, I abuse notation and denote \( D \times \text{SO}(3) \subset G \), a fundamental domain for the unit tangent bundle \( X \), by \( X \). Now if \( \sigma^- \) denotes \( \sigma \) with the opposite orientation, let \( z_\sigma = 1/2(\text{smear} \sigma - \text{smear} \sigma^-) \). If \( \pi : \mathbb{H}^3 \to M \) is the covering map, we have \( \pi_*(z_\sigma) \) is closed. Since measure homology is the same as standard homology, \( z_\sigma \) represents some multiple of the fundamental class. As \( \langle \text{Vol}_{\mathbb{H}^3}, \text{smear} \sigma \rangle = \text{vol}(M) \), the cycle \( z_\sigma \)
represents the fundamental class. Now \( \text{vol}(\rho_1) = \langle f^*\text{Vol}_{\mathbb{H}^3}, z_\sigma \rangle = \langle f^*\text{Vol}_{\mathbb{H}^3}, \text{smear } \sigma \rangle \), and so

\[
\text{vol}(\rho_1) = \int_X \left( \frac{1}{\text{vol}(\sigma)} \int_{g_\sigma} f^*\text{Vol}_{\mathbb{H}^3} \right) \, d\mu.
\] (10)

There is a chain map \( \text{Str} \) from \( C_*(\mathbb{H}^3, \mathbb{R}) \) to itself which replaces a singular simplex by a geodesic simplex with the same vertices (see Section 4 of [Mun] or Section 11.5 of [Rat] for a definition of \( \text{Str} \) and its properties). I claim that we can replace \( \int_{g_\sigma} f^*\text{Vol}_{\mathbb{H}^3} \) in Eqn. (10) by \( \int_{\text{Str}(f(g_\sigma))} \text{Vol}_{\mathbb{H}^3} = \text{vol}(\text{Str}(f(g_\sigma))) \). The idea is this: Consider the analogous question for singular simplicial homology, i.e. let \( z \) be a lift to \( C_3(\mathbb{H}^3, \mathbb{Z}) \) of a cycle in \( C_3(M, \mathbb{Z}) \). The chain \( z \) is now a finite linear combination of singular simplices. While \( \partial z \) is not zero, you can choose elements of \( \Gamma \) that pair up the simplices of \( \partial z \) that reflect the fact that \( \pi(\partial z) = 0 \) in \( C_2(M, \mathbb{Z}) \). The difference between \( f_*(z) \) and \( \text{Str} f_*(z) \) depends on \( f_*(\partial z) \). The way elements of \( \Gamma \) pair up the simplices of \( \partial z \) shows that we can replace \( \int_{g_\sigma} f^*\text{Vol}_{\mathbb{H}^3} \) by \( \int_{\text{Str}(f(g_\sigma))} \text{Vol}_{\mathbb{H}^3} \) in (10).

Formally, there is a chain homotopy, \( H \), from \( \text{Str} \) to the identity which is invariant under isometries. Now

\[
\langle f^*\text{Vol}_{\mathbb{H}^3}, z_\sigma \rangle = \langle \text{Vol}_{\mathbb{H}^3}, f_*(z_\sigma) \rangle = \langle \text{Vol}_{\mathbb{H}^3}, \text{Str}(f_*(z_\sigma)) - H(f_*(\partial z_\sigma)) \rangle
\]

and so to show the claim it is enough to show \( \langle \text{Vol}_{\mathbb{H}^3}, H(f_*(\partial z_\sigma)) \rangle = 0 \). Equivalently, define a cochain \( c \) by \( c(\tau) = \langle \text{Vol}_{\mathbb{H}^3}, H(f_*(\tau)) \rangle \); we want \( c(\partial z_\sigma) = 0 \). Since \( f \) is equivariant and \( H \) commutes with isometries, \( c \) descends to a cochain \( c_M \) on \( M \). As \( \pi_*(\partial z_\sigma) = 0 \), \( c_M(\pi_*(\partial z_\sigma)) = c(\partial z_\sigma) = 0 \), as desired. This proves the claim. Hence we have from Eqn. (10):

\[
\text{vol}(\sigma)\text{vol}(M) = \int_X \text{vol}(\text{Str}(f(g_\sigma))) \, d\mu.
\] (11)

where the volume of \( \text{Str}(f(g_\sigma)) \) is signed volume. This is the formula which will guarantee that \( f \) do not shrink large tetrahedra very much.

Fix a geodesic ray \( r \) with endpoint \( b \in D \). Let \( \sigma_i \) be a regular tetrahedron all of whose sides have length \( i \) with one vertex \( b \) and an edge lying on \( r \). The idea is to use the expanding sequences \( \{g \cdot \sigma_i\} \) for \( g \in X \) to approximate what would happen to an ideal tetrahedron. Let \( v_3 \) denote the volume of a regular ideal tetrahedron, which is the unique tetrahedron of maximal volume. The next lemma is a quantitative version of the statement “\( f \) does not shrink volume.”

**Lemma 6.2** Let \( \epsilon_i = v_3 - \text{vol}(\sigma_i) \), and let \( Y = \{ g \in X : \text{vol}(\text{Str}(f(g \cdot \sigma_i))) < \text{vol}(\sigma_i) - i^2 \epsilon_i \} \). Then \( \mu(Y) < \mu(X)/i^2 \).

**Proof** This is Lemma 2.3 of [Tol]. Since \( \text{vol}(\text{Str}(f(g \cdot \sigma))) < v_3 = \text{vol}(\sigma_i) + \epsilon_i \), Eqn. (11) gives us

\[
\text{vol}(\sigma_i)\text{vol}(M) = \int_X \text{vol}(\text{Str}(f(g \cdot \sigma))) \, d\mu
\]
\[ \int_Y \text{vol}(\text{Str}(f(g \cdot \sigma_i))) \, d\mu + \int_{X \setminus Y} \text{vol}(\text{Str}(f(g \cdot \sigma_i))) \, d\mu \]

\[ < (\text{vol}(\sigma_i) - i^2 \epsilon_i)\mu(Y) + \nu_3 \mu(X \setminus Y) \]

Since

\[ (\text{vol}(\sigma_i) - i^2 \epsilon_i)\mu(Y) + \nu_3 \mu(X \setminus Y) = \text{vol}(\sigma_i)\text{vol}(M) + \epsilon_i(\mu(X \setminus Y) - i^2 \mu(Y)) \]

we have

\[ \text{vol}(\sigma_i)\text{vol}(M) < \text{vol}(\sigma_i)\text{vol}(M) + \epsilon_i(\mu(X \setminus Y) - i^2 \mu(Y)). \]

Therefore \(0 < \mu(X \setminus Y) - i^2 \mu(Y)\), and so \(\mu(Y) < \mu(X)/i^2\). \(\square\)

The next lemma shows that for large \(i\), \(\sigma_i\) is a very good approximation of a regular ideal tetrahedron.

**Lemma 6.3** Let \(\sigma_i\) be a regular tetrahedron in \(\mathbb{H}^3\) all of whose sides have length \(i\). Then for \(i\) large, \(\epsilon_i = \nu_3 - \text{vol}(\sigma_i)\) decreases exponentially with \(i\).

**Proof** This is Lemma 6.4.1 of [Thu]. Let \(\delta_\infty\) be a fixed regular ideal tetrahedron. Let \(p\) be the center of mass of \(\delta_\infty\) and consider the four rays starting at \(p\) and ending at the vertices of \(\delta_\infty\). The tetrahedron whose vertices are the points on these rays a distance \(t\) from \(p\) is regular, and will be denoted \(\delta_i\) where \(i \in \mathbb{R}^+\) is the length of any of its edges (thus \(\sigma_i\) and \(\delta_i\) are isometric). Any tetrahedron has a natural straight (or barycentric) parameterization coming from the affine structure of the hyperboloid model of \(\mathbb{H}^3\) (see Section 11.4 of [Rat]). Parameterizing \(\delta_i\) in this way, we get a one parameter family of diffeomorphisms \(\varphi_i\) from the standard 3-simplex \(\Delta_3\) to \(\delta_i\). A point \(y = \varphi_i(x)\) on a face of \(\delta_i\) has a velocity which is defined as the derivative of \(\varphi_i(x)\) with respect to \(i\). The normal velocity of \(y\) is the component of velocity normal to the face of \(\delta_i\) containing \(y\).

![Fig. 3. The tetrahedron \(\delta_i\)](image1)

![Fig. 4. Finding the maximum normal velocity](image2)
The derivative \( \frac{d\text{vol}(\delta_i)}{dt} \) is bounded by the area of \( \partial\delta_i \) times the maximum normal velocity of \( \partial\delta_i \). Let \( q \) be the center of mass of one of the faces of \( \delta_i \). From Figs. 3 and 4, we see that the maximum normal velocity of \( \partial\delta_i \) is \( \sin \theta \).

By the law of sines, \( \sin \theta = \frac{\sinh d}{\sinh t} \). Since any triangle in hyperbolic space has area less than \( \pi \), we have:

\[
\text{vol}(\delta_i) < 4\pi \sinh d \sinh t.
\]

Since \( \sinh d \) is bounded, \( \frac{d\text{vol}(\delta_i)}{dt} \) decreases exponentially with \( t \).

By Lemma 6.2, if we fix \( i_0 \) then the set of \( g \in X \) for which \( \text{vol}(\operatorname{Str}(f(g \cdot \sigma_i))) < \text{vol}(\sigma_i) - i^2\epsilon_i \) for some \( i \geq i_0 \) has measure less than \( \mu(X) \sum_{i=i_0}^{\infty} 1/i^2 \). Thus except on a small exceptional set, we have

\[
\text{vol}(\operatorname{Str}(f(g \cdot \sigma_i))) \geq \text{vol}(\sigma_i) - i^2\epsilon_i
\]

for all \( i \geq i_0 \).

By Lemma 6.3, \( i^2\epsilon_i \to 0 \) as \( i \to \infty \). So letting \( i_0 \to \infty \) we have that for almost all \( g \in X \), \( \text{vol}(\operatorname{Str}(f(g \cdot \sigma_i))) \) converges to \( v_3 \). Let \( r(i) \) denote the point of \( r \) at a distance \( i \) from \( b \), the basepoint of \( r \).

Let \( \tau_i \) be a tetrahedron with vertices \( b, r(i+1) \), and the two vertices of \( \sigma_i \) not on \( r \) (see Fig. 5). Note that \( v_3 - \text{vol}(\sigma_i) > v_3 - \text{vol}(\tau_i) \) and so the above arguments show that

![Fig. 5. The tetrahedra \( \sigma_i \) and \( \tau_i \)](image)

![Fig. 6. The tetrahedra \( \text{Str}(f(\sigma_i)) \) and \( \text{Str}(f(\tau_i)) \)](image)

for almost all \( g \in X \), \( \text{vol}(\operatorname{Str}(f(g \cdot \tau_i))) \) converges to \( v_3 \). Suppose \( g \) is such that both \( \text{vol}(\operatorname{Str}(f(g \cdot \sigma_i))) \) and \( \text{vol}(\operatorname{Str}(f(g \cdot \tau_i))) \) converge to \( v_3 \). For notational convenience, take \( g \) to be the identity. I claim \( f(r) \) converges to a point in \( S_\infty^2 \). Since \( f \) is Lipschitz, it is enough to show that the \( f(r(i)) \) converge to a point in \( S_\infty^2 \). Regular ideal tetrahedra are the only tetrahedra of maximal volume, so since \( \text{vol}(\operatorname{Str}(f(\sigma_i))) \) goes to \( v_3 \), we must have the distance from \( f(b) \) to \( f(r(i)) \) going to \( \infty \) as \( i \) goes to \( \infty \). Hence the \( f(r(i)) \)
head out toward $S^2_{\infty}$. To show they converge, as opposed to wandering about willy-nilly, we need to show the visual angle of $f(r(i))$ with respect to $f(b)$ converges. The change in visual angle between $f(r(i))$ and $f(r(i+1))$ is the angle between the lines from $f(b)$ to $f(r(i))$ and from $f(b)$ and $f(r(i+1))$. From Fig. 6 we see this change is less than the sum of the two indicated face angles of $\text{Str}(f(\sigma_i))$ and $\text{Str}(f(\tau_i))$.

The following lemma allows us to estimate these face angles:

**Lemma 6.4** There is a constant $C > 0$ such that if $\sigma$ is a tetrahedron with $\text{vol}(\sigma)$ sufficiently close to $v_3$, then for any face angle $\beta$ of $\sigma$:

$$v_3 - \text{vol}(\sigma) > C\beta^2$$

For large $i$, Eqn. (12) shows $v_3 - \text{vol}(\text{Str}(\sigma_i)) \leq (v_3 - \text{vol}(\sigma_i)) + i^2\epsilon_i = \epsilon_i(1 + i^2)$. From Lemma 6.4 we have that the change in visual angle for large $i$ is less than

$$2\sqrt{\frac{\epsilon_i(1 + i^2)}{C}}.$$

By Lemma 6.3 this is eventually exponentially decreasing with $i$, and so the visual angles of the $f(r(i))$ converge. Hence $f(g \cdot r)$ converges for almost all $g \in X$. Therefore, for almost all geodesic rays $r$ in $\mathbb{H}^3$, $f(r)$ converges to a point in $S^2_{\infty}$.

Moreover, as $f$ is Lipschitz any two rays which are asymptotic have images under $f$ which converge to the same point of $S^2_{\infty}$. Hence we have an extension of $f$, $\tilde{f} : S^2_{\infty} \to S^2_{\infty}$, and you can check that $\tilde{f}$ is measurable. Modulo the proof of Lemma 6.4, we have proven Claim 1.

Let us go back and prove the lemma.

**Proof (Lemma 6.4)** Let $v$ be the vertex of $\sigma$ which is the endpoint of the angle $\beta$. Without changing a neighborhood of $v$, push the other three vertices of $\sigma$ to $S^2_{\infty}$ (this only decreases $v_3 - \text{vol}(\sigma)$). Extend an edge through $v$ which is a side of $\beta$ to $S^2_{\infty}$, as in Fig. 7. Look at the part $P$ added on by doing this. Now $\text{vol}(P) \leq v_3 - \text{vol}(\sigma)$. Consider Fig. 8 in the upper half space model where we are using Euclidean coordinates such that $\text{dist}(x, w) = 1$.

We will estimate the volume of $P$ above the dotted line. By requiring that $\text{vol}(\sigma)$ be large, we can assume the dihedral angles of $\sigma$ are close to $\pi/3$. The indicated cross section is then about an equilateral triangle whose area is bounded below by $C_1d^2$. Note $d = 1 - \cos \beta$ with respect to our Euclidean coordinate system, and so

$$\text{vol}(P) \geq \int_{\sin \beta}^{\infty} \frac{C_1(1 - \cos \beta)^2}{z^3} dz$$

Evaluating the integral we get

$$\text{vol}(P) \geq \frac{C_1(1 - \cos \beta)^2}{2\sin^2 \beta} \geq C_2\beta^2$$

for some constant $C_2$. Thus $v_3 - \text{vol}(\sigma) \geq C_2\beta^2$, as desired.
Next, I will check Claim 2 that \( \bar{f} \) sends the vertices of almost every positively oriented regular ideal tetrahedron to the vertices of a positively oriented regular ideal tetrahedron. Let \( \delta_i \) denote a positively oriented regular tetrahedron with side length \( i \), with center of mass at a fixed point \( b \in D \), and whose vertices lie along four fixed geodesic rays \( r_1, r_2, r_3, r_4 \) emanating from \( b \). Arguing as above, we can show that for almost all \( g \in X \), \( \text{vol} (\text{Str}(f(g \cdot \delta_i))) \) converges to \( v_3 \). Moreover for almost all \( g \) this is true and, in addition, \( f(g \cdot r_j) \) converges to a point \( p_j \in S^2_\infty \) for all \( j \). Since \( \text{vol} (\text{Str}(f(g \cdot \delta_i))) \) converges to \( v_3 \), the \( p_j \) must span a regular ideal tetrahedron. Since this is true for almost all \( g \in X \), the vertices of almost all regular ideal tetrahedra are sent to regular ideal tetrahedra. This proof of Claim 2.

I will now prove Claim 3, that \( \bar{f} \) is essentially a Möbius transformation. The space \( \mathcal{T} \) of regular oriented ideal tetrahedra with labeled vertices is a full measure subset of \( S^2_\infty \times S^2_\infty \times S^2_\infty \). Let \( \mathcal{T}^G \) be the subset of \( \mathcal{T} \) consisting of tetrahedra which \( \bar{f} \) takes to regular oriented ideal tetrahedra. We have just shown that \( \mathcal{T}^G \) has full measure. By Fubini’s Theorem there is a \( v_0 \in S^2_\infty \) such that almost all \( T \in \mathcal{T} \) with first vertex \( v_0 \) are in \( \mathcal{T}^G \). In fact, this is true for almost all \( v_0 \), so we can assume that \( f(v_0) \) is defined (recall that \( \bar{f} \) is only defined by the process of looking at images of geodesic rays for a full measure subset of \( S^2_\infty \)).

Without loss of generality, we can take both \( v_0 \) and \( f(v_0) \) to be the point at infinity in the upper half space model of \( \mathbb{H}^3 \). Tetrahedra in \( \mathcal{T} \) with first vertex at \( \infty \) are equivalent to oriented equilateral triangles in \( \mathbb{C} \) with labeled vertices, which are parameterized by a full measure subset of \( \mathbb{C} \times \mathbb{C} \). It will help the reader to think of \( \mathbb{C} \) as having finite measure when applying Fubini; we will only be concerned with which sets have measure zero,
a property which is invariant under diffeomorphism [Boo, Section VI.1]. For almost all lines \( l \) through \( 0 \), almost all equilateral triangles with the edge between the first and second vertices parallel to \( l \) define tetrahedra which are in \( T^G \). Assume that one such line is the real axis. Let \( S \) denote tetrahedra with first vertex at \( \infty \) and such that the edge between the second and third vertices (the first and second vertices of the corresponding triangle) is parallel to the real axis.

We know that \( S^G \equiv S \cap T^G \) has full measure in \( S \). Let \( \omega \) be the \( \sqrt[3]{-1} \) which has positive imaginary part. Then \( \{0, 1, \omega\} \) is an oriented equilateral triangle. Let \( L_0 \) be all equilateral triangles in the tiling of \( \mathbb{C} \) by the triangle \( \{0, 1, \omega\} \). Let \( L_k \) be the same set of triangles scaled by \( 2^{-k} \). Let \( L = \bigcup_{k \in \mathbb{Z}} L_k \) be this nested family of equitriangular lattices (See Fig. 9).

![Fig. 9. Some of the triangles in the nested family of lattices \( L \)](image)

I claim there is an \( r \in \mathbb{R} \) such that for almost all \( z \in \mathbb{C} \), the entire countable set of triangles \( z + rL \) are in \( T^G \). Consider the submersion \( \pi: \mathbb{C} \times \mathbb{R} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to S \) which sends \( (z, r, k, n, m) \) to the equilateral triangle with vertices

\[
(z + r2^{-k}(n + m\omega), z + r2^{-k}(n + 1 + m\omega), z + r2^{-k}(n + (m + 1)\omega))
\]

in \( z + rL_k \). We will think of \( \mathbb{Z} \) as having a measure where the measure of \( q \) is \( 1/q^2 \). As \( \pi \) is a submersion, \( \pi^{-1}(S^G) \) has full measure. Thus by Fubini, for almost all \( r \) and \( z \), we have \( \pi^{-1}(S^G) \cap \{r\} \times \{z\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) has full measure, that is equal to \( \{r\} \times \{z\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \), as desired. Without loss of generality assume \( r = 1 \) has this property. So for almost all \( z \in \mathbb{C} \) all triangles in \( z + L \) are in \( S^G \). This forces \( \bar{f}(z + L) \) to be family of nested equitriangular lattices (see Fig. 9). For each \( z \) there is a complex number \( h(z) \) such that:

\[
\bar{f} \left( z + 2^{-k}(n + m\omega) \right) = \bar{f}(z) + h(z)2^{-k}(n + m\omega)
\]

for all \( \{n, m, k\} \subset \mathbb{Z} \). I claim the function \( h \) is invariant under the group of translations of the form \( z \mapsto z + 2^{-j}(a + b\omega) \), where \( \{j, a, b\} \subset \mathbb{Z} \). Let \( z' = z + 2^{-j}(a + b\omega) \). We
have
\[ \bar{f}(z') = \bar{f}(z) + h(z)2^{-j}(a + b\omega) \]  
\[ \bar{f}(z') + h(z')2^{-j} = \bar{f}(z) + h(z)2^{-j}(a + 1 + b\omega) = \bar{f}(z' + 2^{-j}) \]  
(13)  
(14)

Subtracting Eqn. (13) from Eqn. (14) we get
\[ h(z') = h(z). \]

Our group of translations is dense, and so acts ergodically. Therefore \( h \) is constant almost everywhere. But then
\[ \bar{f}(z') = \bar{f}(z) + h(z)2^{-j} \]
almost everywhere which implies that \( \bar{f}(z) - h \cdot z \) is invariant under our group of translations. So there is a constant \( c \) such that \( \bar{f}(z) - h \cdot z = c \) almost everywhere and thus \( \bar{f}(z) = c + hz \) almost everywhere.

Thus \( \bar{f} \) is essentially a Möbius transformation. The corresponding Möbius transformation conjugates the actions of \( \rho_0 \) and \( \rho_1 \) on \( S_\infty \) to one another. Thus \( \rho_0 \) and \( \rho_1 \) are conjugate, and \( \rho_1 \) is discrete and faithful. \( \square \)

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