SOFICITY AND VARIATIONS ON HIGMAN’S GROUP

MARTIN KASSABOV, VIVIAN KUPERBERG, AND TIMOTHY RILEY

Abstract. A group is sofic when every finite subset can be well approximated in a finite symmetric group. No example of a non-sofic group is known. Higman’s group, which is a circular amalgamation of four copies of the Baumslag–Solitar group, is a candidate. Here we contribute to the discussion of the problem of its soficity in two ways.

We construct variations on Higman’s group replacing the Baumslag–Solitar group by other groups \( G \). We give an elementary condition on \( G \), enjoyed for example by \( \mathbb{Z} \wr \mathbb{Z} \) and the integral Heisenberg group, under which the resulting group is sofic.

We then use soficity to deduce that there exist permutations of \( \mathbb{Z}/n\mathbb{Z} \) that are seemingly pathological in that they have order dividing four and yet locally they behave like exponential functions over most of their domains. Our approach is based on that of Helfgott and Juschenko, who recently showed the soficity of Higman’s group would imply some existence of some similarly pathological functions. Our results call into question their suggestion that this might be a step towards proving the existence of a non-sofic group.

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1. Our results

The word sofic, derived from the Hebrew for finite, was applied to a group by Weiss in [24] when every finite subset can be well approximated in a finite symmetric group or, equivalently, when the group is a subgroup of a metric ultraproduct of finite symmetric groups. The focus of this article is the outstanding open question about soficity, posed by Gromov in his 1999 paper [13]: is every group sofic? We will give more background on soficity in Section 2.

It is not known whether Higman’s group

\[ H_4 = \langle a, b, c, d \mid b^2 = b^2, c^2 = c^2, d^2 = d^2, a^2 = a^2 \rangle \]

is sofic. This group can be constructed as follows. First amalgamate two copies of the Baumslag–Solitar group \( \text{BS}(1, 2) = \langle a, b \mid b^a = b^2 \rangle \) to give \( \langle a, b, c \mid b^a = b^2, c^b = c^2 \rangle \). By properties of the free products with amalgamation, its subgroup \( \langle a, c \rangle \) is free of rank 2. Amalgamate with a second copy \( \langle c, d, a \mid d^2 = d^2, a^d = a^2 \rangle \) along the common \( \langle a, c \rangle \) subgroup to give \( H_4 \).

Again, properties of free products with amalgamation tell us that the subgroups \( \langle a, b \rangle \), \( \langle b, c \rangle \), \( \langle c, d \rangle \), and \( \langle a, d \rangle \) are copies of \( \text{BS}(1, 2) \), and that \( \langle a, c \rangle \) is free of rank 2. In particular, \( H_4 \) is not amenable, since it contains a non-abelian free subgroup. And \( H_4 \) is not residually finite, because it has no finite quotients [17]. These properties make \( H_4 \) a candidate for a non-sofic group. The case is made all the more compelling because \( H_4 \) fails to have a property slightly more restrictive than soficity: Thom proved in [23] that it does not embed
into a metric ultraproduct of finite groups with a commutator-contractive invariant length function.

The building blocks for our variations on Higman’s group (explored in more detail in Section 3) are a group $G$, subgroups $A$ and $B$, an isomorphism $\phi : B \to A$, and a $k \in \mathbb{N}$. For $1 \leq i \leq k$, let $G_i$ be copies of $G$, let $A_i, B_i \leq G_i$ be copies of its subgroups $A$ and $B$, and let $\phi_i : B_i \to A_{i+1}$ (indices mod $k$) be the map naturally induced by $\phi$. We define

$$\overline{\text{Hig}}_k(G, \phi) := \langle G_1, \ldots, G_k \mid b_i = \phi(b_i) \text{ for all } i \text{ and all } b_i \in B_i \rangle,$$

which is $k$ copies of $G$ assembled in a cyclic analog of a free product with amalgamation. If $G = \text{BS}(1, 2) = \langle a, b \mid b^d = b^2 \rangle$ and $\phi : \langle b \rangle \to \langle a \rangle$ maps $b \mapsto a$, then $\overline{\text{Hig}}_4(G, \phi) = H_4$.

Next we define

$$\text{Hig}_k(G, \phi) := \langle G, t \mid t^k = 1, b^i = \phi(b); \forall b \in B \rangle,$$

a semi-direct product of $\overline{\text{Hig}}_k(G, \phi)$ with a cyclic group of order $k$. The index of $\text{Hig}_k(G, \phi)$ in $\text{Hig}_k(G, \phi)$ is $k$, so one is sofic if and only if the other is; see [21].

(Monod has generalized Higman’s construction in a different direction in [20].)

In contrast to $H_4$, we can often prove soficity for these groups. Indeed, in many cases they are residually solvable, and so sofic. We will prove in Section 4:

**Theorem 1.1.** Suppose $G$ is a residually solvable group and $\phi$ is an isomorphism $B \to A$ between subgroups $A, B \leq G$. Suppose there exists a group homomorphism $\pi : G \to A \times B$ such that $\pi(a) = (a, 1)$ for all $a \in A$ and $\pi(b) = (1, b)$ for all $b \in B$. Then $\text{Hig}_k(G, \phi)$ and $\overline{\text{Hig}}_k(G, \phi)$ are residually solvable for all $k \geq 4$.

Examples of $G$ admitting such a $\pi$ include $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$, the three-dimensional integral Heisenberg group $\mathcal{H} = \langle a, b \mid [a, [a, b]] = [b, [a, b]] = 1 \rangle$, $\mathbb{Z} / \mathbb{Z} = \langle a \rangle \wr \langle b \rangle$, and the free metabelian group on two generators $a$ and $b$ (all with $A = \langle a \rangle$, $B = \langle b \rangle$, and $\phi : b \mapsto a$). There is no such $\pi$ for $\text{BS}(1, 2) = \langle a, b \mid d^k = a^2 \rangle$. See Examples 4.1 for details.

With a view to showing that $H_4$ is not sofic, Helfgott and Juschenko proved:

**Theorem 1.2** (Helfgott–Juschenko [16]). If Higman’s group $H_4$ is sofic, then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$, such that for all odd $n > N$ there exists $f \in \text{Sym}(\mathbb{Z}/n\mathbb{Z})$ of order dividing 4 with $f(x + 1) = 2f(x)$ for at least $(1 - \epsilon)n$ elements $x \in \mathbb{Z}/n\mathbb{Z}$.

The $f$ of Helfgott and Juschenko’s theorem behave locally like an exponential function over most of $\mathbb{Z}/n\mathbb{Z}$ but nevertheless are permutations of order dividing four. They gave a heuristic argument as to why such $f$ are unlikely to exist, based on the assumption that these two properties are independent (an intuition that they backed up with comparisons to prominent conjectures in analytic number theory). In Section 7 we give further analysis as to why one might have expected such $f$ not to exist.

Since Helfgott and Juschenko’s paper first appeared (as a preprint on the arXiv in December 2015) doubt has been cast on this intuition by the following two very similar theorems.

**Theorem 1.3**. For all $\epsilon > 0$ and all $k \geq 3$, there exists $N \in \mathbb{N}$ such that for all coprime integers $m$ and $n$ with $n > N$ and $\ln \ln n < m < \ln n$, there exists $f \in \text{Sym}(\mathbb{Z}/n\mathbb{Z})$ of order dividing $k$ with $f(x + 1) = mf(x)$ for at least $(1 - \epsilon)n$ values of $x \in \mathbb{Z}/n\mathbb{Z}$.
Theorem 1.4 (Helfgott–Juschenko [16], also Glebsky [11]). For all \( m > 2 \) and all \( \varepsilon > 0 \), there exists \( C \) such that for all \( n > C \) coprime to \( m \), there exists \( f \in \text{Sym}(\mathbb{Z}/n\mathbb{Z}) \) of order dividing 4 with \( f(x + 1) = mf(x) \) for at least \( (1 - \varepsilon)n \) values of \( x \in \mathbb{Z}/n\mathbb{Z} \).

Theorems 1.3 and 1.4 both run counter to Helfgott and Juschenko’s heuristics. (However, neither theorem addresses the case \( m = 2 \) directly, so the existence of the functions \( f \) of Helfgott and Juschenko’s theorem remains open.)

We will prove Theorem 1.3 in Section 6.4. It will be apparent there that we could replace \( \ln \ln n \) and \( \ln n \) with other functions.

Theorems 1.2–1.4 all arise from a relationship between soficity and the existence of particular permutations of \( \mathbb{Z}/n\mathbb{Z} \) set out in Theorem 1.5 below, which is a generalization of a result of Helfgott and Juschenko [16]. In the case of Theorem 1.2, soficity of \( H_4 \) is a hypothesis. For Theorem 1.3, we use the soficity of \( \text{Hig}_4(\mathbb{Z} \wr \mathbb{Z}) \) established as a consequence of Theorem 1.1. Theorem 1.4 uses a theorem of Glebsky [11] which says that for \( m \geq 3 \), \( \text{Hig}_4(\text{BS}(1, m)) \) has sofic quotients into which \( \text{BS}(1, m) \) embeds.

Theorem 1.5. Suppose \( G \) is a group, \( \phi \) is an isomorphism \( B \rightarrow A \) between subgroups \( A, B \leq G \), and \( k \geq 1 \) is an integer. The following two conditions are equivalent.

1. \( \text{Hig}_k(G, \phi) \) has a sofic quotient \( Q \) such that the composition \( G \rightarrow \text{Hig}_k(G, \phi) \rightarrow Q \) is injective.
2. Sofic approximations of \( G \) exist for which there are permutations of order dividing \( k \) that almost conjugate the action of \( A \) to the action of \( B \).

If \( G \) is amenable, then these are also equivalent to:

3. For all sofic approximations of \( G \) into sufficiently large symmetric groups, there are permutations of order dividing \( k \) which almost conjugate the action of \( A \) to the action of \( B \).

We will present a precise version of this theorem in Section 5.

The natural map \( G \rightarrow \text{Hig}_k(G, \phi) \) employed in (1) can fail to be injective. Indeed, it is rarely injective when \( k = 1 \) or 2. The case \( k = 3 \) is delicate. As for when \( k \geq 4 \), in Lemma 3.3 we will give sufficient conditions for injectivity and in Example 3.2 will show that injectivity can fail.

We will prove Theorem 1.5 in Section 5, building on the arguments in [16]. The equivalence between Conditions (1) and (2) is analogous to that between the two definitions of soficity outlined at the start of this article—see Proposition 2.2. The idea behind the implication (2) \( \Rightarrow \) (1) is that the sofic approximations together with the almost-conjugating functions can be assembled into a homomorphism from \( \text{Hig}_k(G, \phi) \) to an ultraproduct of finite symmetric groups with image \( Q \). For the implication (1) \( \Rightarrow \) (2), we obtain the requisite sofic approximation of \( S \subseteq G \) and the almost-conjugating permutation from a sofic approximation for the image in \( Q \) of a suitably constructed finite subset \( S' \subseteq \text{Hig}_k(G, \phi) \) with \( S \cup \{t\} \subseteq S' \).

The equivalence of (3) is significantly more complicated. The additional assumption that the group \( G \) is amenable gives better control of the sofic approximations. The key result is a theorem which is due to Helfgott and Juschenko [16] in the form we will use and has
origins in Elek and Szabo \cite{19} and Kerr and Li \cite{19}. It spells out a manner in which any two sofic approximations of an amenable group are almost conjugate.

In Section 6 we give applications of Theorem 1.5. We look at \( G = \mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle \) and 
\( \phi : b \mapsto a \), which we view as an introductory example—in this case, \( \text{Hig}_\mathcal{A}(G, \phi) \) will be a right-angled Artin group. We review the case of \( G = BS(1, m) = \langle a, b \mid a^b = a^m \rangle \) addressed by Helfgott and Juschenko and by Glebsky, where soficity of \( \text{Hig}_\mathcal{A}(G, \phi) \) remains unknown for \( m \geq 2 \). We present our most novel applications which are when \( G \) is the 3-dimensional integral Heisenberg group \( H \), or \( \mathbb{Z} \wr \mathbb{Z} \), or the free metabelian group \( M \) on two generators.

In these cases, \( \text{Hig}_\mathcal{A}(G, \phi) \) will be sofic by Theorem 1.1. We explain how the \( G = \mathbb{Z} \wr \mathbb{Z} \) case leads to Theorem 1.3.

We do not know how to construct functions \( f \) explicitly satisfying the conditions of Theorems 1.3 or 1.4. In principle one could follow the constructions in the proofs, however this would require constructing several Følner sets for \( G \) and switching between sofic approximations several times. (In the case of Theorem 1.3, where the quotients could be taken to be the metabelian groups of Proposition 4.6, sofic approximations could be constructed explicitly; for Theorem 1.4 the quotients are residually nilpotent and constructing explicit sofic approximations is again possible, but significantly more difficult.) It seems unlikely that this will lead to an enlightening description of \( f \).

By the same token, we do not know how \( C \) and \( N \) depend on \( \varepsilon \) in Theorems 1.2–1.4. One could obtain explicit estimates from our proofs, but they will be very weak. We give some examples in Remarks 6.3, 6.6, and 6.15. Sufficiently strong estimates (which may well not exist) could have important applications, including a proof that Higman’s group \( H_4 \) is sofic.

2. Soficity

The normalized Hamming distance \( d \) on the symmetric group \( \text{Sym}(n) \) is

\[
d(\rho, \sigma) = \frac{1}{n} | \{ 1 \leq i \leq n \mid \rho(i) \neq \sigma(i) \} |.
\]

This metric is invariant under both the left and right action of \( \text{Sym}(n) \)—i.e.,

\[
d(\rho, \sigma) = d(\tau \rho \tau', \tau \sigma \tau')
\]

for all \( \rho, \sigma, \tau, \tau' \in \text{Sym}(n) \). It follows that:

**Lemma 2.1.** For \( \sigma, \tau, \mu, \sigma_1, \ldots, \sigma_m \in \text{Sym}(n) \),

\[
(i) \ d(\text{id}, \sigma_1 \cdots \sigma_m) \leq \sum_{i=1}^{m} d(\text{id}, \sigma_i),
(ii) \ d(\tau^{-1}\sigma \tau, \text{id}) = d(\sigma, \text{id}),
(iii) \ d(\tau^{-1}\sigma \tau, \mu^{-1}\sigma \mu) \leq 2d(\tau, \mu).
\]

For \( n \in \mathbb{N}, \delta > 0 \), and \( S \) a finite subset of a group \( G \), an \((S, \delta, n)\)-approximation is a map

\( \psi : G \to \text{Sym}(n) \) such that

- \( d(\psi(g), \psi(h)), \psi(gh)) < \delta \) for all \( g, h \in S \) such that \( gh \in S \), and
- \( d(\psi(g), \text{id}) > 1 - \delta \) for all \( g \in S \setminus \{e\} \).
This implies that for each finite set $S$, is a technical convenience. Its values on $G \setminus S$ are irrelevant to the definition.)

A filter $\mathcal{F}$ on a set $I$ is a nonempty set of subsets of $I$ such that $\emptyset \notin \mathcal{F}$; for all $U, V \in \mathcal{U}$, $U \cap V \in \mathcal{F}$; and if $U \in \mathcal{F}$ and $U \subseteq V$, then $V \in \mathcal{F}$. An ultrafilter $\mathcal{U}$ on $I$ is a maximal filter; equivalently, for all $U \subseteq I$, either $U \in \mathcal{U}$ or $(I \setminus U) \in \mathcal{U}$.

Suppose $\mathcal{U}$ is an ultrafilter on a set $I$. To each $i \in I$ associate some $n_i \in \mathbb{N}$. For $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ in the direct product $\prod_{i \in I} \text{Sym}(n_i)$, we write $x \approx_{\mathcal{U}} y$ when $\{i \in I \mid d(x_i, y_i) < \delta\} \in \mathcal{U}$ for all $\delta > 0$. Let $\text{id} = (\text{id}_{n_i})_{i \in I}$. Define $\mathcal{N} := \{x \in \prod_{i \in I} \text{Sym}(n_i) \mid x \approx_{\mathcal{U}} \text{id}\}$, which is called the normal subgroup of infinitesimals. Define the (metric) ultraproduct $\prod_{\mathcal{U}} \text{Sym}(n_i) := (\prod_{i \in I} \text{Sym}(n_i))/\mathcal{N}$. See [21] for further background.

A group $G$ is sofic when it satisfies either of the conditions of the following proposition.

**Proposition 2.2.** For a group $G$, the following are equivalent.

1. The group $G$ is isomorphic to a subgroup of some metric ultraproduct of finite symmetric groups—that is, there exist an ultrafilter $\mathcal{U}$ on a set $I$, natural numbers $\{n_i\}_{i \in I}$, and an injective homomorphism $G \hookrightarrow \prod_{\mathcal{U}} \text{Sym}(n_i)$.

2. For all finite subsets $S \subseteq G$ and all $\delta > 0$, there exists an $(S, \delta, n)$-approximation for some $n$.

**Proof.** Here is a sketch. Details are in [8, 21].

For (1) $\Rightarrow$ (2), a homomorphic embedding $G \hookrightarrow \prod_{\mathcal{U}} \text{Sym}(n_i)$ can be lifted (non-uniquely) to a map $\psi = (\psi_i) : G \rightarrow \prod_{i \in I} \text{Sym}(n_i)$, where $\psi_i : G \rightarrow \text{Sym}(n_i)$. However, $\psi$ may fail to be a group homomorphism. For all $a, b \in G$, $\psi(a)\psi(b)\psi(ab)^{-1}$ is an infinitesimal.

This implies that for each finite set $S$ and each $\delta > 0$, the set of $i$ such that $\psi_i$ is an $(S, \delta, n_i)$-approximation is in the ultrafilter $\mathcal{U}$, and so is not empty. The second condition of the approximation is not immediately satisfied—one only gets that $d(\psi_i(g), \text{id}) > \delta$ for $g \in S \setminus \{e\}$. An ‘amplification trick’ improves this to $1 - \delta$.

For (2) $\Rightarrow$ (1), let $I = \{(S, \delta) \mid \text{ finite } S \subseteq G, \delta > 0\}$. For $(S, \delta) \in I$, define

$$(S, \delta) := \{(S', \delta') \in I \mid \text{ finite } S' \supseteq S, \delta' \leq \delta\}.$$

The family $\mathcal{F}$ of all subsets $(S, \delta)$ of $I$ where $(S, \delta) \in I$ enjoys the finite intersection property since $\bigcap_{i=1}^k (S_i, \delta_i) = \left(\bigcup_{i=1}^k S_i, \max_{i=1}^k \delta_i\right)$. So there is an ultrafilter $\mathcal{U}$ on $I$ with $\mathcal{F} \subseteq \mathcal{U}$. For all $i = (S, \delta) \in I$, let $\psi_i : G \rightarrow \text{Sym}(n_i)$ be an $(S, \delta, n_i)$-approximation. These maps combine in $g \mapsto (\psi_i(g))_{i \in I}$ to induce a monomorphism $G \hookrightarrow \prod_{\mathcal{U}} \text{Sym}(n_i)$: it is a homomorphism because for all $g, h \in G$,

$$(\psi_i(g)\psi_i(h)\psi_i(gh)^{-1})_{i \in I} \in \mathcal{N}$$

since for all $\delta > 0$, $\{i \in I \mid d(\psi_i(g)\psi_i(h), \psi_i(gh)) < \delta\} \in \mathcal{U}$ as it is a superset of $((g, h, gh), \delta) \in \mathcal{U}$; and it is injective because likewise for $\delta > 0$ and $g \in G \setminus \{e\}$, the set

$$\{i \in I \mid d(\psi_i(g), \text{id}_{n_i}) > 1 - \delta\} \in \mathcal{U}$$

and so $(\psi_i(g))_{i \in I} \notin \mathcal{N}$. \qed
Given that the formulation (2) of soficity is in terms of finite subsets of \( G \), it is immediate that a group is sofic if and only if its finitely generated subgroups are sofic. Also, subgroups of finite symmetric groups are sofic, so all finite groups are sofic. This generalizes as follows. A group \( G \) is \textit{residually} \( P \) if for every \( x \in G \setminus \{ e \} \), there is some quotient \( \varphi_x : G \to H_x \) such that \( \varphi_x(x) \) is not trivial and \( H_x \) satisfies condition \( P \). Residually finite groups are sofic: if \( S \subseteq G \) is finite and \( \varphi_x : G \to H_x \) are as per the definition with \( H_x \) finite, then \( \varphi := \bigoplus_{x \in S \setminus \{ e \}} \varphi_x \) is a faithful map to a finite group and composes with a map to some \( \text{Sym}(n) \) to give an \((S, 0, n)\)-sofic approximation. More generally, residually sofic groups are sofic.

Amenable groups are also sofic. A group \( G \) is \textit{amenable} when it satisfies the Følner condition: for all finite subsets \( S \subseteq G \) and for all \( \varepsilon > 0 \), there is a finite subset \( \Phi \subseteq G \) such that for each \( g \in S \), \(|g\Phi\Delta\Phi| < \varepsilon|\Phi|\) (where \( \Delta \) denotes symmetric difference: \( A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B) \)). For every \( g \in S \), the map \( \Phi \to \Phi \) given by \( x \mapsto gx \) is well-defined on all but \( \varepsilon|\Phi| \) elements of \( \Phi \). Extend to the rest of \( \Phi \) arbitrarily so that the map is a bijection, and then each element of \( g \) corresponds to an element of the symmetric group \( \text{Sym}(|\Phi|) \). The function identifying each \( g \) with the corresponding map gives an \((S, 2\varepsilon, |\Phi|)\)-approximation of \( G \). More details are in [21].

It then follows that residually amenable and, in particular, residually solvable groups are sofic (a fact we will use for Corollary 4.5).

The class of sofic groups enjoys various closure properties. These are all sofic: graph products (e.g. free or direct products) of sofic groups [4], amalgamated free products or HNN extensions of sofic groups over amenable groups [5, 9, 22], wreath products of sofic groups [15], groups with finite index sofic subgroups, limits of sofic groups in the space of marked groups (but not all finitely generated sofic groups are limits of amenable groups [6]), locally sofic groups (e.g. groups locally embeddable into sofic groups or direct limits of sofic groups). If \( G \) has a sofic normal subgroup \( N \) such that \( G/N \) is amenable, then \( G \) is sofic. Whether the same conclusion can be drawn when \( N \) is amenable and \( G/N \) is sofic, is open.

Soficity relates to a number of outstanding open problems. In 1973 Gottschalk defined a group \( G \) to be \textit{surjunctive} when for every finite set \( S \) and for \( S^G \) the set of functions \( G \to S \), every continuous \( G \)-equivariant injective function \( f : S^G \to S^G \) is also surjective. Gottschalk conjectured that all groups are surjunctive. A group is hyperlinear when every finite subset can be well approximated in a unitary group with the normalized Hilbert–Schmidt norm. Connes’ Embedding Conjecture states that every group is hyperlinear. Kaplansky’s Direct Finiteness Conjecture is that if \( G \) is a group and \( K \) is a field and if \( a, b \in K[G] \) satisfy \( ab = 1 \), then \( ba = 1 \). Sofic groups are surjunctive [13, 24], are hyperlinear (since finite permutation groups embed in unitary groups), and satisfy the Direct Finiteness Conjecture [8]. The most recent progress is the construction by De Chiara, Glebsky, Lubotzky, and Thom of groups that do not satisfy an alternate version of the hyperlinear condition where the Hilbert–Schmidt norm is not normalized [7].

For further background, we refer to the surveys [2, 21].

3. Variations on Higman’s group

Our notation is \( b^a = a^{-1}ba \) and \([a, b] = a^{-1}b^{-1}ab\).
As explained in Section 1, for a group $G$, subgroups $A$ and $B$, an isomorphism $\phi : B \to A$, and a $k \in \mathbb{N}$, we define $G_k$, where $1 \leq i \leq k$, to be copies of $G$ and $A_i, B_i \leq G_i$ to be copies of its subgroups $A$ and $B$. Then $\phi$ induces an isomorphism $\phi_i : B_i \to A_{i+1}$ and we define

$$\text{Hig}_k(G, \phi) := \langle G_1, \ldots, G_k | b_i = \phi_i(b_i) \text{ for all } i \text{ and all } b_i \in B_i \rangle.$$ 

Thus, $\text{Hig}_k(G, \phi)$ is the quotient of the free product of $k$ copies of $G$ in which $B$ in the $i$-th is identified with $A$ in the $(i+1)$-st for $i = 0, \ldots, k−1$ (indices modulo $k$).

By construction there are maps $t_1, \ldots, t_k$ from the group $G$ to $\text{Hig}_k(G, \phi)$. We regard $t := t_1$ as the natural map $G \to \text{Hig}_k(G, \phi)$. We will often work in settings where these maps are injective, and then for simplicity we will suppress them and consider $G$ as a subgroup of $\text{Hig}_k(G, \phi)$ via $t$.

For example, if $G = \langle a_1, a_2 | R \rangle$ is a 2-generator group such that $a_1$ and $a_2$ have the same order, then $\text{Hig}_k(G, \phi)$, where $\phi : a_2 \mapsto a_1$, is the cyclically presented group

$$\langle a_1, \ldots, a_k | \sigma^i(r) ; r \in R, i = 0, \ldots, k−1 \rangle,$$

where $\sigma$ cycles the indices of the letters of $r$.

The semi-direct product of $\text{Hig}_k(G, \phi)$ with the cyclic group $C_k$ of order $k$ in which a generator $t$ of $C_k$ conjugates $G_i$ to $G_{i+1}$ (indices mod $k$) is

$$\text{Hig}_k(G, \phi) = \langle G, t | t^k = 1, b^t = \phi(b) ; \forall b \in B \rangle.$$ 

Then $\text{Hig}_k(G, \phi)$ is the normal closure of $t(G)$ in $\text{Hig}_k(G, \phi)$ and is the kernel of $\text{Hig}_k(G, \phi) \to C_k$.

In the case when $G$ is a group generated by two elements $a, b \in G$ of the same order, with $A = \langle a \rangle, B = \langle b \rangle$, and $\phi : B \to A$ given by $\phi(b) = a$, we will write $\text{Hig}_k(G, \phi)$ and $\text{Hig}_k(G, \phi)$ in place of $\text{Hig}_k(G, \phi)$ and $\text{Hig}_k(G, \phi)$.

The cases $k = 1, 2$ are degenerate:

**Lemma 3.1.** $\text{Hig}_1(G, \phi)$ is a quotient of $G$. If $G$ is generated by the subgroups $A$ and $B$, then $\text{Hig}_2(G, \phi)$ is a quotient of $G$.

For large $k$ one expects $G$ generally to embed in $\text{Hig}_k(G, \phi)$, but this can fail:

**Example 3.2.** When $G = B = \mathbb{Z}$, $A = 2\mathbb{Z}$, and $\phi$ is multiplication by 2, $\text{Hig}_k(G, \phi)$ is finite for all $k$, and so $t : G \not\hookrightarrow \text{Hig}_k(G, \phi)$.

When $k \geq 4$, here is a sufficient condition:

**Lemma 3.3.** If $A \cap B = \{1\}$ and $k \geq 4$, then $G$ and $A \ast A$ both embed in $\text{Hig}_k(G, \phi)$. In particular, if $G \neq \{1\}$, then $\text{Hig}_k(G, \phi)$ is not amenable.

**Proof.** Let $J = G_1 \ast_{\phi_1} G_2 \ast_{\phi_2} \cdots \ast_{\phi_{k−2}} G_{k−2}$, and let $K = G_{k−1} \ast_{\phi_{k−1}} G_k$. Since $A_1 \cap B_1 = A_1 \cap A_2 = \{1\}$ and $A_2 \cap B_2 = B_1 \cap B_2 = \{1\}$, the subgroup generated by $A_1$ and $B_2$ in $G_1 \ast_{\phi_1} G_2$ is $A_1 \ast B_2$. Inductively, the same holds for the subgroup generated by $A_1$ and $B_{k−2}$ in $J$, and similarly for the subgroup generated by $A_{k−1}$ and $B_k$ in $K$. Then $\text{Hig}_k(G, \phi)$ is the amalgamated free product of $J$ and $K$ along the subgroup $\langle A_1, B_{k−2} \rangle = A_1 \ast B_{k−2}$, which is identified with $B_k \ast A_{k−1}$ via identifying $A_1$ with $B_k$ and $B_{k−2}$ with $A_{k−1}$. Thus $A \ast A$ embeds...
in \( \text{Hig}_k(G, \phi) \) since \( A \ast A \cong A_1 \ast B_{k-2} \leq G \). Meanwhile \( G_1 \leq J \), so \( G_1 \leq \text{Hig}_k(G, \phi) \) as well, and the canonical map \( G \to \text{Hig}_k(G, \phi) \) is injective.

If \( A \neq \{1\} \) then the subgroup \( A \ast A \) prevents \( \text{Hig}_k(G, \phi) \) from being amenable. If \( A = \{1\} \), then \( \text{Hig}_k(G) \) is a free product.

\( \Box \)

The case \( k = 3 \) is trickier. Sometimes \( G \) does not embed in \( \text{Hig}_3(G, \phi) \) because the latter group is very small—for example, \( \text{Hig}_3(\text{BS}(1, 2)) = \{1\} \)—but it is also possible that \( G \) embeds in \( \text{Hig}_3(G, \phi) \), which is the case for most other examples considered in this paper.

\( \Box \)

### 4. Soficity via residual solvability

Here we prove Theorem 1.1 by an approach which is similar to our proof of Lemma 3.3: it is based on viewing the amalgamated products as a combination of a free product and a semidirect product.

We have that \( G \) is residually solvable and has subgroups \( A \) and \( B \) for which there is an isomorphism \( \phi : B \to A \), and that there exists a group homomorphism \( \pi : G \to A \times B \) such that \( \pi(a) = (a, 1) \) for all \( a \in A \) and \( \pi(b) = (1, b) \) for all \( b \in B \). We aim to show that \( \text{Hig}_k(G, \phi) \) and \( \text{Hig}_k(G, \phi) \) are residually solvable for all \( k \geq 4 \).

Define \( G_A = \pi^{-1}(1, \ast) \) or, equivalently, \( G_A = \text{ker}(\phi_A \circ \pi) \), where \( \phi_A \) is the projection \( A \times B \to A \). So \( G_A \) is a normal subgroup of \( G \) and \( G/G_A \cong A \). The hypothesis that \( \pi(a) = (a, 1) \) for all \( a \in A \) implies that \( A \) is a complement of \( G_A \) in \( G \), and so \( G \) can be expressed as a semidirect product \( G = A \ltimes G_A \). And \( B \leq G_A \) because \( \pi(b) = (1, b) \) for all \( b \in B \). Likewise, \( G = B \rtimes G_B \) with \( A \leq G_B \).

As (3)–(5) of the following examples show, the hypotheses of Theorem 1.1 do not imply that \( A \) and \( B \) commute. Rather, they imply that \( [A, B] \leq G_A \cap G_B = \ker \pi \).

#### Examples 4.1

In each case take \( A = \langle a \rangle = \mathbb{Z} \) and \( B = \langle b \rangle = \mathbb{Z} \):

1. \( G = \mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle \). Take \( \pi \) to be the identity. The semi-direct products are direct products \( \mathbb{Z} \times \mathbb{Z} \).

2. \( G = \text{BS}(1, m) = \langle a, b \mid a^m = b^2 \rangle \). In this case there is no map \( \pi \) for \( m \neq 1 \) because \( [a, b] = a^{m-1} \), and so cannot be in \( \ker \pi \).

3. \( G = \mathcal{H} = \langle a, b \mid [a, [a, b]] = [b, [a, b]] = 1 \rangle \), the three-dimensional integral Heisenberg group. Take \( \pi \) to be the map onto \( \mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle \) quotienting by the center \( \langle [a, b] \rangle \) of \( \mathcal{H} \). Then \( G_A = \langle b, [a, b] \rangle \cong \mathbb{Z}^2 \) and \( G_B = \langle a, [a, b] \rangle \cong \mathbb{Z}^2 \).

4. \( G = \mathbb{Z} \rtimes \mathbb{Z} = \langle a, b \mid [a^i, a^j] = 1 \text{ for all } i, j \rangle \), which is \( \mathbb{Z} \ltimes \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} = \langle b \rangle \ltimes \bigoplus_{i \in \mathbb{Z}} \langle a \rangle \) where \( a_i = a^i \) and \( b \) acts so as to map \( a_i \mapsto a_{i+1} \). Again, take \( \pi \) to be the abelianization map onto \( \mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle \). Then \( G_B \) is the kernel of the map \( \mathbb{Z} \rtimes \mathbb{Z} \to \langle b \rangle \) given by quotienting by \( a \), which is \( \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} = \bigoplus_{i \in \mathbb{Z}} \langle a_i \rangle \). And \( G_A \) is the kernel of the map \( \mathbb{Z} \rtimes \mathbb{Z} \to \langle a \rangle \) given by quotienting by \( b \), which is \( \langle b \rangle \ltimes \bigoplus_{i \in \mathbb{Z}} \langle a_i^{-1} a_{i+1} \rangle \) and is isomorphic to \( G \).

5. \( G = \mathcal{M} = \langle a, b \mid [a, b], [a, b]^{e^j} = 1 \text{ for all } i, j \in \mathbb{Z} \rangle \), the free metabelian group on two generators. Again, take \( \pi \) to be the abelianization map onto \( \mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle \).
We will use the following description of amalgamated products over subgroups which have a normal complement.

**Lemma 4.2.** Suppose $G_1$ and $G_2$ are groups having subgroups $H_1$ and $H_2$ respectively with normal complements—i.e., $G_1 = H_1 \ltimes N_1$ and $G_2 = H_2 \ltimes N_2$ for some $N_1$ and $N_2$. For any isomorphism $\phi : H_1 \rightarrow H_2$, the amalgamated product $G_1 \ast_\phi G_2$ can be expressed as a semidirect product $H_1 \ltimes (N_1 \ast N_2)$, where the action of $H_1$ on $N_2$ comes from that of $H_2$ via the isomorphism $\phi$.

**Proof.** An arbitrary element of $G_1 \ast_\phi G_2$ can be represented as a product

$$w = x_1y_1x_2y_2 \ldots x_ry_r,$$

where $x_1, \ldots, x_r \in G_1$ and $y_1, \ldots, y_r \in G_2$. Express $x_1$ as $m_1h_1$ where $m_1 \in G_1$ and $h_1 \in h$. Since we are working in the amalgamated product, we can move $h_1$ to $G_2$ and write

$$w = m_1(\phi(h_1)y_1)x_2y_2 \ldots x_ry_r.$$

The element $\phi(h_1)y_1$ in $G_2$ can then be expressed as $n_1g_1$ where $n_1 \in N_2$ and $g_1 \in H_2$. Continuing this process, moving elements from $H_1$ or $H_2$ to the right, expresses $w$ as

$$(1) \quad w = m_1n_1m_2n_2 \ldots m_rn_rh,$$

where $m_1, \ldots, m_r \in N_1 \subseteq G_1$, $n_1, \ldots, n_r \in N_2 \subseteq G_2$, and $h \in H_1$. The product $m_1n_1 \ldots m_rn_r$ can be considered as an element in $N_1 \ast N_2$. Such elements form a normal subgroup in $G_1 \ast_\phi G_2$, with quotient $H_1$. All that remains to check is that the action of $H_1$ on the free product is the one described. □

**Corollary 4.3.** Suppose $G_1$ and $G_2$ are residually solvable groups satisfying the conditions of Lemma 4.2. Then the amalgamated product $G_1 \ast_\phi G_2$ is residually solvable.

**Proof.** Free products of residually solvable groups are residually solvable, but semidirect products of residually solvable groups can fail to be residually solvable. Nevertheless we will see that the semidirect products of Lemma 4.2 are residually solvable.

Let $w = m_1n_1m_2n_2 \ldots m_rn_rh$ be a non-trivial element in $G_1 \ast_\phi G_2$ as per (1), where all $m_i \in G_1$ and $n_i \in G_2$ are non-identity, with the possible exceptions of $m_i$ and $n_i$, with $h \in H_1$. If $h \neq 1$, then there is a solvable quotient $H_1 \rightarrow H$ of $H_1$ where $h$ survives (since subgroups of residually solvable groups are residually solvable), which leads to a quotient $G_1 \ast_\phi G_2$, where $w$ has a nontrivial image. Therefore it suffices to consider the case $h = 1$.

Take $k$ such that for $i = 1, 2$, $G_i \rightarrow \overline{G}_i := G_i/G_i^{(k)}$ are quotients of $G_i$ by some derived subgroup such that all the $m_i$ and $n_i$ have nontrivial images in $\overline{G}_1$ and $\overline{G}_2$. Let $H_1$, $N_1$ and $N_2$ denote the (necessarily solvable) images of $H_1$, $N_1$ and $N_2$, respectively, in $\overline{G}_1$ and $\overline{G}_2$. We can view $w$ as element in the free product $\overline{G}_A \ast \overline{G}_B$. Therefore, by the argument that free products of solvable groups are residually solvable (see e.g. [14]), there exists a quotient $\overline{N}_1 \ast \overline{N}_2$ of $\overline{N}_1 \ast \overline{N}_2$ by one of its derived subgroups where $w$ is non-trivial. Since this quotient is characteristic, it has a natural action of $H_1$ which extends the actions of $H_1$ on $\overline{N}_1$ and on $\overline{N}_2$. This allows us to map

$$G_1 \ast_\phi G_2 = H_1 \ltimes (N_1 \ast N_2) \rightarrow H_1 \ltimes (\overline{N}_1 \ast \overline{N}_2) \rightarrow H_1 \ltimes \overline{N}_1 \ast \overline{N}_2,$$

where $H_1 \ltimes \overline{N}_1 \ast \overline{N}_2$ is a solvable quotient of $G \ast_\phi G$ in which $w$ has a nontrivial image. □
Lemma 4.4. Suppose $G$ is a group satisfying the conditions in Theorem 1.1. Then the amalgamated product $G *_\phi G$ can be written as a semidirect product $(A*B)\rtimes H$ for some normal subgroup $H$, and therefore there is a projection $G *_\phi G \to A*B$.

Proof. Annihilating the first factor in $B \rtimes (G_A * G_B)$ and then using the maps $G_A \to B$ and $G_B \to A$ induced by $\pi$, maps $G *_\phi G$ to $A*B$. This map is clearly surjective with some kernel $H$ and restricts to the identity on $A*B$ (viewed as a subgroup of $G_A * G_B$ via $B \leq G_A$ and $A \leq G_B$), so splits $G *_\phi G$ into a semidirect product. □

Proof of Theorem 1.1. Applying Corollary 4.3 and Lemma 4.4 repeatedly, we find that if $k \geq 4$, then the groups $J := G_1 *_{\phi_1} G_2 *_{\phi_2} \cdots *_{\phi_{k-2}} G_{k-2}$ and $K := G_{k-1} *_{\phi_{k-1}} G_k$ (in the notation of Section 3) are both residually solvable, and both contain $A*B$ in such a way that they both split over this group as semidirect products, and $\text{Hig}_k(G, \phi) = J *_{A*B} K$. So the hypotheses of Lemma 4.2 are met and $\text{Hig}_k(G, \phi)$ is residually finite by a final application of Corollary 4.3.

Finally, $\text{Hig}_k(G, \phi) = \text{Hig}_k(G, \phi) \rtimes C_2$, so is also residually solvable. (Semidirect products $H \rtimes A$ of residually solvable groups $H$ and solvable groups $A$ are residually solvable.) □

Theorem 1.1 may also hold when ‘residually solvable’ is replaced with ‘residually nilpotent’ or ‘residually finite’; however, our proof would need further ideas and the given theorem suffices for our application:

Corollary 4.5. When $G$ is $\mathbb{Z}^2$, $\mathcal{H} \sqcup \mathbb{Z}$, or $\mathcal{M}$ as per Examples 4.1, $\text{Hig}_k(G)$ is residually solvable, and so sofic, for all $k \geq 4$.

Finally, we remark on an alternative route:

Proposition 4.6. Suppose there exists a homomorphism $\pi : G \to A \times B$ as per Theorem 1.1. Then for all $k \geq 3$, there are homomorphisms $\mu : \text{Hig}_k(G, \phi) \to C_k \rtimes G^k$ and $\overline{\mu} : \text{Hig}_k(G, \phi) \to G^k$. Moreover, the restrictions of $\mu$ and $\overline{\mu}$ to any copy $G_i$ of $G$ inside $\text{Hig}_k(G)$ are injective.

Proof. Let $\pi_A : G \to A$ (respectively, $\pi_B : G \to B$) be the composition of $\pi$ with projection onto $A$ (respectively, $B$). Define the homomorphism $\overline{\mu} : \text{Hig}_k(G, \phi) \to G^k$, given by

$$\overline{\mu}(\iota_l(g)) = (1, \ldots, 1, \phi^{-1}(\pi_A(g)), g, \phi(\pi_B(g)), 1, \ldots, 1),$$

where $g$ is an arbitrary element of $G$, and $\iota_l(g)$ is the element in $\text{Hig}_k(G)$ corresponding to $g$ sitting in the $l$-th copy of $G$. The elements $\phi(\pi_B(g))$, $g$, and $\phi^{-1}(\pi_A(g))$ are sitting in coordinates $l-1$, $l$, and $l+1$. Clearly $\overline{\mu}$ is well defined on each copy $G_i$ appearing in the presentation of $\text{Hig}_k(G)$, so it suffices to verify that $\overline{\mu}$ identifies the $l$-th copy of $B$ with the $l+1$-st copy of $A$. By definition we have

$$\overline{\mu}(\iota_l(b)) = (1, \ldots, 1, \phi^{-1}(\pi_A(b)), b, \phi(\pi_B(b)), 1, \ldots, 1) = (1, \ldots, 1, b, \phi(b), 1, \ldots, 1)$$

$$\text{and thus } \overline{\mu}(\iota_l(b)) = \overline{\mu}(\iota_{l+1}(b)), \text{i.e., } \overline{\mu} \text{ extends to the group } \text{Hig}_k(G). \text{ By construction, the restriction of } \overline{\mu} \text{ on each copy of } G \text{ is injective. (Unless we are in a degenerate case, the maps } \mu \text{ and } \overline{\mu} \text{ are not surjective.)} \square$$
This is weaker than Theorem 1.1 in that it does not tell us that \( \text{Hig}_k(G, \phi) \) is sofic or residually solvable. But this proposition would suffice for our applications in Section 6 because it tells us that when \( G \) is sofic, \( \text{Hig}_k(G, \phi) \) has a sofic quotient into which \( G \) injects (condition (1) of Theorem 1.5). Moreover, it does so in a manner that makes sofic approximations of that quotient easy to construct explicitly from sofic approximations of \( G \).

**Remark 4.7.** If we remove the defining relator \( t^k = 1 \) from our presentation for \( \text{Hig}_k(G, \phi) \), then \( t \) becomes the stable letter of the HNN-extension \( \langle G, t \mid b^t = \phi(b) \forall b \in B \rangle \), which is more straightforward to understand in the context of soficity. For example, the instance where \( G = \langle a, b \mid b^a = b^2 \rangle \) and \( \phi : b \mapsto a \) is Baumslag’s one-relator group \( \langle b, t \mid b^t = b^2 \rangle \). If \( G \) is solvable, then \( \langle G, t \mid b^t = \phi(b) \forall b \in B \rangle \) is sofic: Collins and Dykema [5, Corollary 3.6] show that an HNN-extension of a sofic group \( G \) relative to an injective group homomorphism \( \theta : H \to G \), for \( H \leq G \) monolabelably amenable, is sofic. If \( G \) is solvable, then so is its subgroup \( B \). Solvable groups are monolabelably amenable, thereby implying \( \langle G, t \mid b^t = \phi(b) \forall b \in B \rangle \) is sofic.

### 5. SOFIC QUOTIENTS AND ALMOST CONJUGATION

This section is devoted to proving Theorem 1.5 relating soficity in the context of \( \text{Hig}_k(G, \phi) \) to seemingly pathological permutations \( f \). These \( f \) come from permutations approximating \( t \in \text{Hig}_k(G, \phi) \). They will have order dividing \( k \) since \( t^k = 1 \) and, for all \( b \in B \), will ‘almost conjugate’ permutations approximating \( b \) to permutations approximating \( \phi(b) \) since \( b^t = \phi(b) \) in \( \text{Hig}_k(G, \phi) \). When \( G \) is amenable and we have explicit sofic approximations for \( G \), the permutations approximating \( b \) and \( \phi(b) \) in \( \text{Hig}_k(G, \phi) \) essentially have to be those sofic approximations. In examples, the ‘almost conjugate’ conclusion will then amount to a local recurrence such as \( f(x + 1) = mf(x) \) holding for most values of \( x \).

We make Theorem 1.5 precise as:

**Theorem 5.1.** Suppose \( G \) is a group, \( \phi \) is an isomorphism \( B \to A \) between subgroups \( A, B \subseteq G \), and \( k \geq 1 \) is an integer. The following two conditions are equivalent.

1. \( \text{Hig}_k(G, \phi) \) has a sofic quotient \( Q \) such that the composition \( G \xrightarrow{i} \text{Hig}_k(G, \phi) \to Q \) is injective.
2. For all finite subsets \( S \subseteq G \) and all \( \delta, \varepsilon > 0 \), there exists an \((S, \delta, n)\)-approximation \( \psi \) of \( G \) and a permutation \( f \in \text{Sym}(n) \) of order dividing \( k \) such that for all \( b \in S \cap \phi^{-1}(A \cap S) \),
   \[
   d(\psi(b) \circ f, f \circ \psi(\phi(b))) < \varepsilon.
   \]

If \( G \) is amenable, then these are also equivalent to:

3. For all finite sets \( S \subseteq G \) and all \( \varepsilon > 0 \), there exist a finite set \( S' \subseteq G \) with \( S \subseteq S' \) and \( \delta > 0 \) and an integer \( N \) such that if \( \psi \) is an \((S', \delta, n)\)-sofic approximation of \( G \) with \( n > N \), then there exists a permutation \( f \in \text{Sym}(n) \) of order dividing \( k \) such that for all \( b \in S \cap \phi^{-1}(A \cap S) \),
   \[
   d(\psi(b) \circ f, f \circ \psi(\phi(b))) < \varepsilon.
   \]

**Proof of Theorem 5.1.** (1) \( \Rightarrow \) (2). We have that there is a sofic quotient \( Q \) such that the composition of the natural map \( G \xrightarrow{i} \text{Hig}_k(G, \phi) \) with the quotient map \( \pi : \text{Hig}_k(G, \phi) \to Q \) is injective. In particular, the map \( i \) is injective.
Suppose $S \subseteq G$ is a finite subset and $\varepsilon, \delta > 0$. We seek an $n$ and an $(S, \delta, n)$-approximation $\psi$ of $G$ together with a permutation $f \in \text{Sym}(n)$ of order dividing $k$ such that $d(\psi(\phi(b)) \circ f, f \circ \psi(b)) < \varepsilon$ for all $b \in S \cap \phi^{-1}(A \cap S)$.

Let
\[ S' = \{\text{id}, t, \ldots, t^{k-1}\} \cup \iota(S) \cup \iota(S \cap \phi^{-1}(A \cap S))t \subseteq \text{Hig}_k(G, \phi). \]

Let $\delta' = \min[\delta, \varepsilon]/6k$. Then $\pi(S')$ is a finite subset of the sofic group $Q$, so there exists an $n \in \mathbb{N}$ and an $(\pi(S'), \delta', n)$-approximation $\pi : \text{Sym}(n) \to \text{Sym}(n)$. Via $\pi$ this gives a map $\psi' : \text{Hig}_k(G, \phi) \to \text{Sym}(n)$ which enjoys the first defining property of an $(S', \delta', n)$-approximation, but may fail the second as it could map some elements of $S'$ to the identity. Since $G$ naturally maps into $\text{Hig}_k(G, \phi)$ and $\delta' < \delta$, the composition $\psi$ of $\iota$ and $\psi'$ is an $(S, \delta, n)$-approximation of $G$, as required.

We will obtain the requisite permutation $f \in \text{Sym}(n)$ by the action of $t$ under $\psi'$. First set $\tilde{f} = \psi'(t)$. The order of this permutation may fail to divide $k$ since $\psi'$ is not necessarily a homomorphism. However,
\[ d(\tilde{f}^k, \text{id}) = d(\psi'(t)^k, \text{id}) \leq d(\psi'(t)^k, \psi'(t^k)) + d(\psi'(t^k), \text{id}) < (k - 1)\delta' + \delta' = k\delta', \]
where the second inequality holds because $t^k = \text{id}$ and $t^i \in S'$ for all $i$. Therefore the set of points which are not part of a cycle of length dividing $k$ under the action of $\tilde{f}$ is correspondingly small and we can find a permutation $f$ of order dividing $k$ such that $d(f, \tilde{f}) < k\delta'$.

Suppose $b \in S \cap \phi^{-1}(A \cap S)$. It remains to show that
\[ d(\psi(b) \circ f, f \circ \psi(b)) \leq \varepsilon. \]
As $d(f, \tilde{f}) \leq k\delta'$, Lemma 2.1 (iii) yields
\[ d(\tilde{f}^{-1} \circ \psi(b) \circ f, f, \tilde{f}^{-1} \circ \psi(b) \circ \tilde{f}) < 2k\delta'. \]
By definition of $\tilde{f}$,
\[ \tilde{f}^{-1} \circ \psi(b) \circ \tilde{f} = \psi'(t)^{-1} \circ \psi'(\iota(b)) \circ \psi'(t). \]
Now, as $t, t^{-1}, \text{id} \in S'$ and $\psi'$ is an $(S', \delta', n)$-approximation,
\[ d\left( \psi'(t)^{-1} \circ \psi'(\iota(b)) \circ \psi'(t), \psi'(t^{-1}) \circ \psi'(\iota(b)) \circ \psi'(t) \right) = d\left( \psi'(t)^{-1}, \psi'(t^{-1}) \right) \leq 2\delta'. \]
And, likewise, as $t^{-1}, b, t, bt, t^{-1}bt \in S'$ and $\phi(\iota(b)) = t^{-1}t(b)t$ in $\text{Hig}_k(G, \phi)$,
\[ d(\psi'(t^{-1}) \circ \psi'(b(b))) \circ \psi'(t), \psi'(t^{-1}bt)) = d(\psi'(t^{-1}) \circ \psi'(b(b)) \circ \psi'(t), \psi'(t^{-1}b(b)t)) \leq 2\delta'. \]
In combination, (2)–(5) yield the first inequality of:
\[ d(\psi(b) \circ f, f \circ \psi(b)) = d(\tilde{f}^{-1} \circ \psi(b) \circ f, f, \psi(\phi(b))) \leq (2k + 4)\delta' \leq 6k\delta' \leq \varepsilon. \]
\[ \Box \]

The following lemma will provide the heart of our proof that (2) \Rightarrow (1). Since $\text{Hig}_k(G, \phi)$ is generated by $\iota(G)$ and $t$, we can choose a section $\sigma : \text{Hig}_k(G, \phi) \to \{G, t\}^*$ for the evaluation map $\{G, t\}^* \to \text{Hig}_k(G, \phi)$—that is, for every $g \in \text{Hig}_k(G, \phi)$ we choose a way of expressing $g$ as a product $\sigma(g) = t(g_1)t^{j_1} \cdots t(g_r)t^{j_r}$ of elements of $G$ and powers of $t$. 

Given a map \( \psi : G \rightarrow \text{Sym}(n) \) (not necessarily a homomorphism) and a permutation \( f \in \text{Sym}(n) \), define a map \( \psi^f : \text{Hig}_k(G, \phi) \rightarrow \text{Sym}(n) \) by

\[
\psi^f(g) := \psi(g_1)^{f_1} \cdots \psi(g_r)^{f_r},
\]

where \( \sigma(g) = \psi(g_1)^{\phi(1)} \cdots \psi(g_r)^{\phi(r)} \). The lemma will tell us that if \( \psi \) and \( f \) are suitably compatible then \( \psi^f \) is close to a homomorphism.

**Lemma 5.2.** For all finite subsets \( S \subseteq \text{Hig}_k(G, \phi) \) and \( \overline{S} \subseteq G \) such that \( \overline{S} \subseteq S \) and all \( \delta > 0 \), there exists a finite set \( S_0 \subseteq G \) with \( \overline{S}_0 \subseteq S_0 \) and an \( \varepsilon > 0 \) satisfying the following.

Suppose \( \psi : G \rightarrow \text{Sym}(n) \) is an \((S_0, \varepsilon, n)\)-approximation and \( f \in \text{Sym}(n) \) is a permutation of order dividing \( k \) such that for all \( b \in S_0 \cap B \)

\[
d(\psi(b)), f \circ \psi(b)) < \varepsilon.
\]

Then for all \( s_1, s_2 \in S \) for which \( s_1s_2 \in S \),

\[
d(\psi^f(s_1), \psi^f(s_2)) < \varepsilon
\]

and for all \( g \in \overline{S} \)

\[
d(\psi^f(g)), \psi(g)) < \varepsilon.
\]

**Proof.** Since \( S \) is finite there exists an integer \( m \) and a finite subset \( S' \subseteq G \) containing \( \overline{S} \) such that \( \sigma(S) \subseteq \{\sigma(S'), t\}^m \). Then \( S \) sits inside the subgroup \( \Gamma = \langle \sigma(S'), t \rangle \) of \( \text{Hig}_k(G, \phi) \).

As \( \Gamma \) is finitely generated, there exists a finitely presented group \( \Gamma' = \langle S', t \mid R' \rangle \) which projects onto \( \Gamma \)—that is, the composition \( S' \hookrightarrow \Gamma' \twoheadrightarrow \Gamma \) is the identity.

By construction, every relation in \( R' \) is also satisfied in \( \text{Hig}_k(G, \phi) \), and so can be deduced from the defining relations in the presentation of \( \text{Hig}_k(G, \phi) \). These defining relations come in three types: relations in \( G \), the relation \( t^k = 1 \), and relations of the form \( \sigma(b)^{\phi(b)} = \sigma(t) \) for some \( b \in B \). We can enlarge the set \( S' \) to another finite subset \( S'' \subseteq G \) by gathering all elements in \( G \) needed to deduce all the relations \( r \in R' \), so as to view \( S \) as a subset of a finitely presented group

\[
\Gamma'' = \langle S'', t \mid \hat{r}, R'', b', \phi(b)^{-1} \rangle
\]

where \( R'' \) is a finite set of relations satisfied in the subgroup \( \langle S'' \rangle \) of \( G \), and \( B'' \) is a finite subset of \( B \). Let \( N \geq k \) be a number such that every defining relation in \( R'' \) has length at most \( N \) in the generating set \( S'' \) and all elements in \( B'' \) and \( \phi(B'') \) can be expressed as words in \( S'' \) of length at most \( N - 1 \). By construction, there exists a constant \( M \) such that each relator in \( \Gamma'' \) of the form \( s_1^{-1}s_1{s_2}^{-1}s_3 \) for \( s_1, s_2, s_3 \in S \) or of the form \( g^{-1}\sigma(g) \) for \( g \in \overline{S} \) can be written as product of at most \( M \) conjugates of the defining relators in the above presentation.

Define \( S_0 = \langle S'' \rangle^N \) and \( \varepsilon = \delta/8MN \). Suppose that \( \psi \) is an \((S_0, \varepsilon, n)\)-approximation of \( G \) and \( f \in \text{Sym}(n) \) is a permutation of order dividing \( k \) such that \( d(\psi(b)), f \circ \psi(b)) < \varepsilon \) for all \( b \in S_0 \cap \phi^{-1}(A \cap S) \). Extend \( \psi|_{S''} \) to a homomorphism \( \psi \) from the free group generated by \( S'' \) and \( t \) to \( \text{Sym}(n) \) by mapping \( t \) to the permutation \( f \). Defining relations \( r = \hat{r} \) or \( r \in R'' \) or \( r = b', \phi(b)^{-1} \) in our presentation of \( \Gamma'' \) have lengths at most \( k, N \) and \( 2N \), respectively, and so \( d(\psi(r), \text{id}) \leq 2N\varepsilon \) by Lemma 2.1 (i). It then follows from Lemma 2.1 (i) and (ii) that for all relators \( r' \) of the form \( s_1^{-1}s_1{s_2}^{-1}s_3 \) or \( g^{-1}\sigma(g) \), we have

\[
d(\psi(r'), \text{id}) < 2MN\varepsilon < \delta/4.
\]
For the relators of the first type this gives us that \( d(\tilde{\psi}(s_3^{-1}s_1s_2), \text{id}) < \frac{\delta}{4} \). For those of second type we get both (7) for all \( g \in \overline{S} \), and

\[
(9) \quad d(\tilde{\psi}(s_i), \psi^f(s_i)) < \frac{\delta}{4}
\]

for \( i = 1, 2, 3 \). Then (8) applied to \( r' = s_3^{-1}s_1s_2 \) and (9) give

\[
d(\psi^f(s_3), \psi^f(s_1)) = d(\psi^f(s_3)^{-1}\psi^f(s_1), \text{id}) < d(\tilde{\psi}(s_3)^{-1}\tilde{\psi}(s_1), \text{id}) + \frac{3\delta}{4} = d(\tilde{\psi}(s_3^{-1}s_1s_2), \text{id}) + \frac{3\delta}{4} < \delta,
\]

which yields inequality (6).

\( \square \)

**Proof of Theorem 5.1, (2) \( \Rightarrow \) (1).** This proof is similar to that of (2) \( \Rightarrow \) (1) of Proposition 2.2. Define

\[
I := \{ (S, \overline{S}, \delta) \mid \text{finite } S \subseteq \text{Hig}_k(G, \phi), \text{finite } \overline{S} \subseteq G \text{ with } \iota(\overline{S}) \subseteq S, \delta > 0 \}.
\]

For \( (S, \overline{S}, \delta) \in I \), define

\[
\overline{(S, \overline{S}, \delta)} \equiv \{ (S', \overline{S'}, \delta') \in I \mid S' \supseteq S, \overline{S'} \supseteq \overline{S}, \delta' \leq \delta \}.
\]

As in our proof of Proposition 2.2, the family \( \mathcal{F} \) of all subsets \( (S, \overline{S}, \delta) \) enjoys the finite intersection property, and so there is an ultrafilter \( \mathcal{U} \) on \( I \) with \( \mathcal{F} \subseteq \mathcal{U} \).

Suppose \( i = (S, \overline{S}, \delta) \in I \). Let \( S_0 \subseteq G \) and \( \varepsilon > 0 \) be as per Lemma 5.2. Let \( \psi_i \) be an \((S_0, \varepsilon, n_i)\)-approximation of \( G \) and \( f_i \in \text{Sym}(n_i) \) a permutation as per condition (2). Together \( \psi_i \) and \( f_i \) define maps \( \psi_i^f : \text{Hig}_k(G, \phi) \rightarrow \text{Sym}(n_i) \) and Lemma 5.2 tells us that these \( \psi_i^f \) enjoy conditions (6) and (7).

If \( g \in \overline{S} \subseteq S_0 \), then (7) gives us that \( d(\psi_i^f(\iota(g)), \psi(g)) < \delta \). If, additionally, \( g \neq e \), then \( d(\psi_i(\iota(g)), \text{id}) > 1 - \varepsilon \) because \( \psi_i \) is an \((S_0, \varepsilon, n_i)\)-approximation. Together these give

\[
(10) \quad d(\psi_i^f(\iota(g)), \text{id}) > 1 - \delta - \varepsilon
\]

for all \( g \in \overline{S} \setminus \{ e \} \).

The \( \{ \psi_i^f \}_{i \in I} \) combine to induce a map

\[
\Psi^f : \text{Hig}_k(G, \phi) \rightarrow \prod_{\mathcal{U}} \text{Sym}(n_i).
\]

This is a group homomorphism because of condition (6). Its image \( Q = \Psi^f(\text{Hig}_k(G, \phi)) \) is a sofic quotient of \( \text{Hig}_k(G, \phi) \). In general, \( \Psi^f \) might not be injective, but (10) tells us that the composition \( G \xrightarrow{\iota} \text{Hig}_k(G, \phi) \xrightarrow{\Psi^f} Q \) is injective. In both cases, the details are similar to our derivations of corresponding statements in our proof of Proposition 2.2.

\( \square \)

**Proof of Theorem 5.1, (3) \( \Rightarrow \) (2).** This implication is immediate since \( G \) is sofic.

\( \square \)
The remaining implication (2) ⇒ (3) is significantly more complicated and uses that for an amenable group, any two approximations into the same Sym(n) are almost conjugate. This result is due to Helfgott and Juschenko in the form given but, as they explain, has origins in Elek and Szabo [9], builds on a lemma from Kerr and Li [19], and is also comparable to Arzhantseva and Păunescu [1]. Helfgott and Juschenko’s proof is a delicate analysis of the interplay between sofic approximations and the Følner characterization of amenability.

**Theorem 5.3** (Helfgott–Juschenko [16]). Suppose G is an amenable group, ε > 0, and S is a finite subset of G. Then there is a finite subset S′ ⊆ G with S ⊆ S′ and constants N ∈ ℤ+, δ > 0 such that for any two (S′, δ, n)-approximations ρ1, ρ2 of G with n ≥ N, there exists τ ∈ Sym(n) such that, for every s ∈ S,

\[ d(τ^{-1} ◦ ρ_1(s) ◦ τ, ρ_2(s)) < ε. \]

We will also use the following lemma which essentially says that the n in the definition of an (S, δ, n)-approximation is irrelevant provided it is sufficiently large.

**Lemma 5.4.** Suppose n = qm + r where m, n, q, r are non-negative integers with m, n ≥ 1 and q = [n/m]. If α : S → Sym(m) is an (S, η, m)-approximation of a finite subset S of a group, then composing

\[ \left( α, \ldots, α, 1 \right) : S → Sym(m) × ⋯ × Sym(m) × Sym(r) \]

with the diagonal embedding into Sym(n) gives an (S, η + 1/(q + 1), n)-approximation β.

**Proof.** If s1, s2, s1s2 ∈ S, then

\[ d(β(s_1)β(s_2), β(s_1s_2)) < \frac{1}{n} \left( \frac{qm + \cdots + qm}{mn} \right) = \frac{qmη}{qm + r} < \eta < \eta + \frac{1}{q + 1}. \]

As for the second condition on approximations, suppose s ∈ S \ {e}. Then

\[ d(β(s), id) > \frac{1}{n} \left( \frac{q}{m(1 - η) + \cdots + m(1 - η)} \right) = \frac{1 - η}{n} > \frac{1 - η - \frac{r}{n}}{q + 1}, \]

with the final inequality coming from combining 1 − η < η and r/n < 1/(q + 1), the latter of which holds because r < m implies that qr + r < n.

**Proof of Theorem 5.1 (2) ⇒ (3).** We are given a finite set S ⊆ G and some ε > 0.

We aim to show that for a suitable finite set S′ ⊆ G with S ⊆ S′ and suitable δ > 0 and N, every (S′, δ, n)-sofic approximation ψ of G with n > N admits some ψ ∈ Sym(n) of order dividing k almost conjugating the action of A under ψ to the action of B under ψ. The idea will be to apply condition (2) and Lemma 5.4 to obtain some approximation |f| of G together with a permutation f of order k which will almost conjugate the action of A to the action of B. A priori ψ and |f| will be unrelated, but in fact by Theorem 5.3 will essentially be conjugate. We will apply this conjugation to f to obtain the requisite permutation f.

Here are the details. Let ε = ε/3. By Theorem 5.3 there exists a finite subset S′ ⊆ G with S′ ⋃ φ(S ⋂ B) ⊆ S′ and δ > 0 and N₀ ∈ ℤ+ such that any two (S′, δ, n)-approximations ρ₁
and $\rho_2$ of $G$ with $n \geq N_0$ are \textit{almost conjugate} in that there exists $\tau \in \text{Sym}(n)$ such that for all $s \in S \cup \phi(S \cap B)$,
\begin{equation}
\label{eq:conjugation}
    d(\tau^{-1} \circ \rho_1(s) \circ \tau, \rho_2(s)) < \bar{\epsilon}.
\end{equation}
Let $\delta = \min\{\bar{\delta}, \bar{\epsilon}\}$.

By Condition (2), there exists an $(S', \delta/2, m)$-approximation $\psi'$ of $G$ together with a permutation $f \in \text{Sym}(m)$ of order dividing $k$ such that for all $b \in S' \cap \phi^{-1}(A \cap S')$,
\begin{equation}
\label{eq:approximation}
    d(\psi(b) \circ f, f \circ \psi(\phi(b))) < \bar{\epsilon}.
\end{equation}

Let $N = \max\{N_0, 2m/\delta\}$. With $S'$ and $\delta$ as defined above, suppose $\psi$ is an $(S', \delta, n)$-approximation of $G$ with $n > N$.

Via Lemma 5.4, we can use $\psi'$ to construct another $(S', \delta, n)$-approximation $\overline{\psi}$ of $G$ and an associated permutation $\overline{f}$ which almost conjugates the action of $A$ to $B$ with the same error $\bar{\epsilon}$:
\begin{equation}
\label{eq:conjugation_bar}
    d(\overline{\psi}(b) \circ \overline{f}, \overline{f} \circ \overline{\psi}(\phi(b))) < \bar{\epsilon}
\end{equation}
for all $b \in S' \cap B$. This is possible because
\[
    \frac{\delta}{2} + \frac{1}{[n/m] + 1} \leq \frac{\delta}{2} + \frac{1}{[2/\delta] + 1} < \frac{\delta}{2} + \frac{\delta}{2} = \delta
\]
and given how $\overline{\psi}$ is assembled from copies of $\psi$ and the identity (and correspondingly $\overline{f}$ from copies of $f$ and the identity), the error $\bar{\epsilon}$ of (13) does not increase and $\overline{f}$, like $f$, has order dividing $k$.

By Theorem 5.3 there is a permutation $\tau \in \text{Sym}(n)$ which almost conjugates $\psi$ to $\overline{\psi}$—i.e.,
\begin{equation}
\label{eq:conjugation_bar_2}
    d\left(\tau^{-1} \circ \psi(s) \circ \tau, \overline{\psi}(s)\right) < \bar{\epsilon}
\end{equation}
for all $s \in S'$.

Define $f = \tau \circ \overline{f} \circ \tau^{-1}$, which is a permutation of order dividing $k$ since $\overline{f}$ has order dividing $k$. Suppose $b \in S \cap \phi^{-1}(A \cap S)$. We will complete our proof by showing that
\[
    d(\psi(b) \circ f, f \circ \psi(\phi(b))) < \epsilon.
\]

By definition of $f$,
\begin{equation}
\label{eq:permutation}
    f^{-1} \circ \psi(b) \circ f = \tau \circ \overline{f}^{-1} \circ \tau^{-1} \circ \psi(b) \circ \tau \circ \overline{f} \circ \tau^{-1}.
\end{equation}

Since $b \in S \subseteq S'$, by (14),
\begin{equation}
\label{eq:approximation_bar}
    d\left(\tau \circ \overline{f}^{-1} \circ \tau^{-1} \circ \psi(b) \circ \tau \circ \overline{f} \circ \tau^{-1}, \tau \circ \overline{f}^{-1} \circ \tau^{-1} \circ \overline{\psi}(b) \circ \overline{f} \circ \tau^{-1}\right) < \bar{\epsilon}.
\end{equation}

By (13), $d\left(\overline{f}^{-1} \circ \overline{\psi}(b) \circ \overline{f}, \overline{\psi}(\phi(b))\right) < \bar{\epsilon}$, and therefore
\begin{equation}
\label{eq:approximation_bar_2}
    d\left(\tau \circ \overline{f}^{-1} \circ \tau^{-1} \circ \overline{\psi}(b) \circ \overline{f} \circ \tau^{-1}, \tau \circ \overline{\psi}(\phi(b)) \circ \tau^{-1}\right) < \bar{\epsilon}.
\end{equation}

Since $\phi(b) \in S \cap B \subseteq S'$, by (14) again,
\begin{equation}
\label{eq:approximation_bar_3}
    d\left(\tau \circ \overline{\psi}(\phi(b)) \circ \tau^{-1}, \psi(\phi(b))\right) < \bar{\epsilon}.
\end{equation}

Together, (15)–(18) yield the first inequality of:
\[
    d(\psi(b) \circ f, f \circ \psi(\phi(b))) = d\left(f^{-1} \circ \psi(b) \circ f, \psi(\phi(b))\right) < 3\bar{\epsilon} = \epsilon.
\]
which completes the proof. □

6. Applications of Theorem 5.1

In this section we will examine the groups $\mathbb{Z}^2$, the 3-dimensional integral Heisenberg group $\mathcal{H}$, BS(1, $m$), $\mathbb{Z} \wr \mathbb{Z}$, and the 2-generator metabelian group in the context of Theorem 1.5 (or, in its precise form, Theorem 5.1). Each of these groups is amenable. We will exhibit families of maps witnessing to their soficity, and will then explain what Theorem 5.1 allows us to conclude about the existence of seemingly pathological permutations. In particular, we will explain how the case of BS(1, $m$) yields Theorems 1.2 and 1.4, and how $\mathbb{Z} \wr \mathbb{Z}$ yields Theorem 1.3.

We begin with $\mathbb{Z}^2$, which we view as an introductory example.

6.1. $\mathbb{Z}^2$. We present $\mathbb{Z}^2$ as $\langle a, b \mid ab = ba \rangle$, so $\phi : \mathbb{Z} \to \mathbb{Z}$, given by $b \mapsto a$, is the map defining Hig$_k(\mathbb{Z}^2)$.

To obtain a family of functions witnessing to the soficity of $\mathbb{Z}^2$, we identify Sym(n) with Sym($\mathbb{Z}/n\mathbb{Z}$) and then for $p, q \in \mathbb{N}$, define $\psi_{n, p, q} : G \to$ Sym(n) by

\[
\psi_{n, p, q}(a) : x \mapsto x + p, \quad \text{and} \quad \psi_{n, p, q}(b) : x \mapsto x + q.
\]

Lemma 6.1. For any finite set $S \subseteq \mathbb{Z}^2$ and any $\delta > 0$, there exists a constant $C$ such that $\psi_{n, p, q}$ is an $(S, \delta, n)$-approximation of $\mathbb{Z}^2$ provided that $p > Cq$ and $n > Cp$.

Proof. Take $C$ sufficiently large that $S \subseteq \{a^i b^j \mid |i| < C/3, |j| < C/3\}$. Since the map $\psi_{n, p, q}$ is a group homomorphism, we only need to show that $d(\psi_{n, p, q}(s), \text{id}) > 1 - \delta$ for all $s \in S \setminus \{1\}$ provided that $p > Cq$ and $n > Cp$. Then for $s = a^i b^j \in S$ we find $\psi_{n, p, q}(s)$ is translation by $\lambda p + \mu q$, which is not divisible by $n$ (unless $\lambda = \mu = 0$), and therefore $d(\text{id}, \psi_{n, p, q}(s)) = 1$. □

The equivalence $(1) \Leftrightarrow (3)$ of Theorem 5.1 tells us that for $k \in \mathbb{N}$, the group Hig$_k(\mathbb{Z}^2)$ has a sofic quotient $Q$ such that the composition $\mathbb{Z}^2 \to$ Hig$_k(\mathbb{Z}^2) \to Q$ is injective if and only if for any $n, p, q$ such that $n/p$ and $p/q$ are sufficiently large, there is a permutation $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ of order $k$ with $d(\psi_{n, p, q}(b) \circ f, \psi_{n, p, q}(a)) < \epsilon$. As $(\psi_{n, p, q}(b) \circ f)(x) = f(x) + q$ and $(f \circ \psi_{n, p, q}(a))(x) = f(x + p)$, the latter condition amounts to $f(x + p) = f(x) + q$ for at least $(1 - \epsilon)n$ elements $x \in \mathbb{Z}/n\mathbb{Z}$.

However, for $k \geq 1$, the group Hig$_k(\mathbb{Z}^2)$ is a right-angled Artin group, so it is linear and thus residually finite (see [18]). Thus Hig$_k(\mathbb{Z}^2)$ and Hig$_k(\mathbb{Z}^2)$ are sofic. (For $k \geq 4$, we reached the same conclusion in Corollary 4.5 via the residual solvability established in Theorem 1.1. For $k \leq 3$ the group is abelian, and thus also sofic.) And for $k \geq 2$, $\mathbb{Z}^2 \hookrightarrow$ Hig$_k(\mathbb{Z}^2)$.

Thus:

Theorem 6.2. Suppose $k \geq 2$ and $\epsilon > 0$. Then there exists $C > 0$ such that for all $n, p, q$ satisfying $n \geq Cq$ and $p \geq Cq$, there exists a permutation $f \in$ Sym($\mathbb{Z}/n\mathbb{Z}$) of order dividing $k$ such that

\[
f(x + p) = f(x) + q
\]

for at least $(1 - \epsilon)n$ elements $x \in \mathbb{Z}/n\mathbb{Z}$. 

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In some cases this is straight-forward. If \( n \) is a prime congruent to 1 modulo \( k \), then there exists \( l \in \mathbb{Z}/n\mathbb{Z} \) such that \( \ell^k = 1 \) and \( q = lp \) in \( \mathbb{Z}/n\mathbb{Z} \), and then \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \) mapping \( x \mapsto lx \) satisfies the given conditions, because \( f(x + p) = lx + lp = lx + q = f(x) + q \) for all \( x \in \mathbb{Z}/n\mathbb{Z} \) and \( f^k = \text{id} \). Indeed, such \( f \) arise from a natural sofic quotient of \( \text{Hig}_k(\mathbb{Z}^2) \)—take the semidirect product of the cyclic group of order \( k \) and the abelianization of \( \text{Hig}_k(G) \). Then \( \text{Hig}_k(\mathbb{Z}^2) \) maps onto \( C_k \rtimes \mathbb{Z}/n\mathbb{Z} \), where the action is by multiplication by \( l \).

But in most cases errors are inevitable. Suppose \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \) satisfies \( f(x + p) = f(x) + q \) for all \( x \in \mathbb{Z}/n\mathbb{Z} \). Then \( f^k(x + q') = f^k(x) + q' \) for all \( x \in \mathbb{Z}/n\mathbb{Z} \) and all \( l \in \mathbb{N} \). So if \( f^k = \text{id} \), then \( n \) divides \( q^k - p^k \).

Whether or not \( n \) divides \( q^k - p^k \), by Theorem 6.2, there exist such functions \( f \) satisfying \( f(x + p) = f(x) + q \) for most \( x \in \mathbb{Z}/n\mathbb{Z} \). Such \( f \) could be constructed explicitly by carefully following the arguments in our proofs of Theorems 5.1 and 5.3 (using that there are Følner sets for \( \mathbb{Z}^2 \) of a very simple form), but doing this in general would be quite technical.

**Remark 6.3.** In such a simple example it is possible to determine the dependance of the constant \( C \) in Theorem 6.2: one can take \( C = O(\varepsilon^{-k}) \).

### 6.2. The Heisenberg group

The Heisenberg group \( \mathcal{H} \) has presentation

\[
\mathcal{H} = \langle a, b \mid [a, [a, b]] = [b, [a, b]] = 1 \rangle.
\]

It is nilpotent and so is amenable and residually finite. Identify \( \text{Sym}(n^2) \) with \( \text{Sym} \left( (\mathbb{Z}/n\mathbb{Z})^2 \right) \).

Define \( \psi_n : \mathcal{H} \to \text{Sym}(n^2) \) for \( n \in \mathbb{N} \) by

\[
\psi_n(a) : (x, y) \mapsto (x, y + 1), \quad \psi_n(b) : (x, y) \mapsto (x + y, y),
\]

which extends to \( \mathcal{H} \) since \( \psi_n(a) \) and \( \psi_n(b) \) satisfy the defining relations of \( \mathcal{H} \). This action of \( \mathcal{H} \) arises the finite quotient \( \mathcal{H}_n := \mathcal{H} / (a^n, b^n) \) acting on cosets of the subgroup \( \langle a \rangle \).

**Lemma 6.4.** For all finite sets \( S \subseteq \mathcal{H} \) and all \( \delta > 0 \), there exists \( C > 0 \) such that \( \psi_n \) is \((S, \delta, n^2)\)-approximation of \( \mathcal{H} \) for all \( n > C \).

**Proof.** Suppose \( S \subseteq \mathcal{H} \) is finite. As in Lemma 6.1, as \( \psi_n \) is a homomorphism, it suffices to check that there exists an integer \( C \) such that when \( n > C \), the permutation \( \psi_n(s) \) is far from the identity for all \( s \in S \setminus \{e\} \). Every element of \( \mathcal{H} \) can be expressed uniquely as \( a^i b^j \bar{a}^k \). So there exists \( N \) such that

\[
S \subseteq \left\{ a^i b^j \bar{a}^k [a, b]^\ell \mid |\ell|, |a|, |b| < N \right\}.
\]

One computes that

\[
\psi_n(a^i b^j \bar{a}^k [a, b]^\ell) : (x, y) \mapsto (x + \ell y - \nu, y + \lambda),
\]

which is a permutation with at most \(|\ell|n \) fixed points (provided that this element is not trivial—i.e., \( n \notdivides \lambda \) or \( n \notdivides \mu \) or \( n \notdivides \nu \)). Therefore if \( n > C := N/\delta \), then \( \psi_n \) is an \((S, \delta, n^2)\)-approximation. \( \square \)

So Theorem 5.1 tells us that for all \( k \in \mathbb{N} \) the group \( \text{Hig}_k(\mathcal{H}) \) has a sofic quotient \( \mathcal{Q} \) such that \( \mathcal{H} \to \text{Hig}_k(\mathcal{H}) \to \mathcal{Q} \) is injective if and only there exist infinitely many \( n \) and functions \( f : (\mathbb{Z}/n\mathbb{Z})^2 \to (\mathbb{Z}/n\mathbb{Z})^2 \) of order dividing \( k \) which conjugate the action of \( b \) to the action of \( a \) under \( \psi_n \) up to error \( \varepsilon \).
For $k \geq 4$, Corollary 4.5 tells us that $\text{Hig}_k(\mathcal{H})$ is sofic. As for the case $k = 2$, we have that $\text{Hig}_2(\mathcal{H}) \cong \mathcal{H}$ is also sofic. And for $k = 3$, it is not hard to construct a surjective map from $\text{Hig}_3(\mathcal{H})$ onto the sofic group $\text{SL}_3(\mathbb{Z})$ such that the composition $\mathcal{H} \to \text{Hig}_3(\mathcal{H}) \to \text{SL}_3(\mathbb{Z})$ is injective. A sofic quotient of $\text{Hig}_3(\mathcal{H})$ into which $\mathcal{H}$ injects could also be obtained via Proposition 4.6. (We do not know whether the group $\text{Hig}_4(\mathcal{H})$ itself is sofic, but see no reason it should not be.)

So applying the condition $d(\psi(b) \circ f, f \circ \psi(\phi(b))) < \varepsilon$ of Theorem 5.1 (3) to $\psi_n$ and to $b$ and $a = \phi(b)$ we get the following.

**Theorem 6.5.** For all $\varepsilon > 0$ and all $k \geq 2$, there exists an integer $C$ such that for all $n > C$ there exists a permutation $f \in \text{Sym}(n^2)$ of order dividing $k$ which, when expressed as $f(x, y) = (f_1(x, y), f_2(x, y))$ so that $f_1, f_2 : \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ are its coordinate functions, satisfies

$$f_1(x, y + 1) = f_1(x, y) + f_2(x, y) \quad \text{and} \quad f_2(x, y + 1) = f_2(x, y)$$

for at least $(1 - \varepsilon)n^2$ pairs $(x, y)$.

**Remark 6.6.** There is no $f$ such that the above equality holds for all pairs $(x, y)$, since $\psi_n(a)$ and $\psi_n(b)$ are not conjugate inside $\text{Sym}(n^2)$—one of them has (a few) fixed points and the other has none.

**Remark 6.7.** It is again possible to estimate the dependance of the constant $C$. One can show that we can take $C = O(\varepsilon^{-3k})$.

**Remark 6.8.** The generators $a$ and $b$ play asymmetric roles in the definition of $\psi_n$: for example, $d(\psi_n(a), \text{id}) = 1$, but $d(\psi_n(b), \text{id}) = 1 - 1/n < 1$. One can instead take the action of $\mathcal{H}_n$ on itself which will lead to a permutation representation $\mathcal{H} \to \text{Sym}(n^3)$ in which the roles of $a$ and $b$ are symmetric. The functions $f$ of the resulting analogue of Theorem 6.5 can be constructed in such a way that the equations are satisfied for all points if and only if there is a nontrivial semisimple element of order dividing $k$ in the group $\text{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

**Remark 6.9.** One can also consider a Higman-like construction from $\mathcal{H}$ in which $\phi$ maps one of the standard generators to a generator of the center: define

$$G = \langle \mathcal{H} = \langle b, c \mid [b, [b, c]] = [c, [b, c]] = 1 \rangle, \quad \phi : b \mapsto a, \quad \phi : c \mapsto c \rangle,$$

where $a := [b, c]$ so that

$$\text{Hig}_k(G, \phi) = \langle b_1, c_1, \ldots, b_k, c_k \mid [b_i, [b_i, c_i]] = [c_i, [b_i, c_i]] = 1, b_i = [b_{i+1}, c_{i+1}] \forall i (\text{mod } k) \rangle.$$

We do not know whether the group $\text{Hig}_k(G, \phi)$ is sofic, since Theorem 1.1 does not apply. However $b_i \mapsto \text{id} + e_{(i+1)}$ and $c_i \mapsto \text{id} - e_{(i+1)}$ (indices mod $k$) defines a homomorphism

$$\text{Hig}_k(G, \phi) \to \text{SL}_k(\mathbb{Z}) \cong \mathbb{Z}_k \subseteq \text{SL}_k(\mathbb{Z}),$$

and for $k \geq 2$ the group $\mathcal{H}$ injects into this quotient (i.e., image) of $\text{Hig}_k(G, \phi)$ which is linear and thus sofic.

As before (but with $a$ and $b$ now changed to $b$ and $c$, respectively) define $\psi_n : \mathcal{H} \to \text{Sym}(n^2)$ for $n \in \mathbb{N}$ by

$$\psi_n(b) : (x, y) \mapsto (x, y + 1), \quad \text{and} \quad \psi_n(c) : (x, y) \mapsto (x + y, y).$$

Then, as $\psi_n([b, c]) : (x, y) \mapsto (x - 1, y)$, applying Theorem 5.1 leads to:
Theorem 6.10. Functions \( f \) exist exactly as per Theorem 6.5, except with the displayed equations replaced by:

\[
f_1(x - 1, y) = f_1(x, y) + f_2(x, y) \quad \text{and} \quad f_2(x - 1, y) = f_2(x, y).
\]

Despite their similarity, we do not see a way to derive one of Theorems 6.5 and 6.10 immediately from the other. Defining \( g(x, y) := f(y, x) \) transforms one set of recurrences to the other, but the condition that the function’s order divides \( k \) is lost.

6.3. The Baumslag–Solitar group \( BS(1, m) \). This is the case addressed by Helfgott and Juschenko in [16]. Here we explain how it fits into our framework and give our own account of how it relates to recent work of Glebsky.

The Baumslag–Solitar group \( BS(1, m) \) has presentation

\[
BS(1, m) = \langle a, b \mid a^m = b^m \rangle.
\]

It is a residually finite solvable group, and so is amenable. If \( m \neq \pm 1 \), then the image of \( a \) in any proper quotient of \( BS(1, m) \) is finite. (Every element can be expressed as \( b^\mu a^\lambda b^{-\lambda} \) for some \( \mu, \lambda \geq 0 \) and \( \nu \in \mathbb{Z} \). The result then follows from consequences of a non-trivial \( b^n a^n b^{-n} \) mapping to the identity and the relation \( a^m = b^m \).)

Identify \( \text{Sym}(n) \) with \( \text{Sym}(\mathbb{Z}/n\mathbb{Z}) \). For all \( n \in \mathbb{N} \) relatively prime to \( m \), define a map \( \psi_n : BS(1, m) \to \text{Sym}(n) \) by

\[
\psi_n(a) : x \mapsto x + 1, \quad \text{and} \quad \psi_n(b) : x \mapsto m^{-1}x,
\]

which extends to a homomorphism defined on the whole of \( BS(1, m) \) since \( \psi_n(a) \) and \( \psi_n(b) \) satisfy the defining relation of \( BS(1, m) \). This action of \( BS(1, m) \) arises from the quotient \( \text{BS}(1, m)_n := \text{BS}(1, m)/(a^n) \) acting on cosets of the subgroup \( (b) \).

Lemma 6.11. For all finite sets \( S \subseteq BS(1, m) \) and all \( \delta > 0 \), there exists an integer \( C \) such that for all \( n > C \), the map \( \psi_n \) is an \( (S, \delta, n) \)-approximation of \( BS(1, m) \), provided that \( |m| \geq 2 \).

Proof. The group \( BS(1, m) \) can be represented by \( 2 \times 2 \) matrices via

\[
a \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}.
\]

The image of this embedding is

\[
\left\{ \begin{bmatrix} 1 & \lambda m^\mu \\ 0 & m^\mu \end{bmatrix} \bigg| \mu \in \mathbb{Z}, \lambda \in \mathbb{Z} \left[ \frac{1}{m} \right] \right\}.
\]

For every finite set \( S \subseteq BS(1, m) \) there exists a positive integer \( N \) such that \( s \in S \) is sent to

\[
\begin{bmatrix} 1 & \lambda s m^{-N} \\ 0 & m^{\mu_s} \end{bmatrix}
\]

where \( \lambda_s \) and \( \mu_s \) are integers such that \( |\mu_s| \leq N \) and \( |\lambda_s| \leq |m^{2N}| \). One computes that

\[
\psi_n(s) : x \mapsto m^{-\mu_s}(x + \lambda_s m^{-N}).
\]

If this permutation is non-trivial (i.e., \( \mu_s \neq 0 \) or \( n \nmid \lambda_s \)), then it has at most \( m^N \) fixed points. Therefore if \( n > C := \max \left\{ \left\lfloor \frac{m^2}{2} \right\rfloor, |m^{2N}| \right\} \), then \( \psi_n \) is an \( (S, \delta, n) \)-approximation. \( \square \)
Theorem 5.1 now tells us that for all $k \in \mathbb{N}$ the group $\text{Hig}_k(\text{BS}(1, m))$ has a sofic quotient into which $\text{BS}(1, m)$ naturally embeds if and only there exist infinitely many $n$ and functions $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ of order dividing $k$ which conjugate addition to multiplication by $m$ up to error $\varepsilon$—that is, $m^{-1}f(x) = f(x + 1)$ for at least $(1 - \varepsilon)n$ values of $x \in \mathbb{Z}/n\mathbb{Z}$. Define $\tilde{f} \in \text{Sym}(\mathbb{Z}/n\mathbb{Z})$ by $\tilde{f}(x) = -f(-x)$, which has order dividing $k$ if and only if $f$ does. The equation $m^{-1}f(x) = f(x + 1)$ can be re-expressed as $f(x) = mf(x + 1)$, and then as

$$\tilde{f}(x + 1) = -f(-x - 1) = -mf(-x) = m\tilde{f}(x).$$

So $m^{-1}f(x) = f(x + 1)$ is satisfied by at least $(1 - \varepsilon)n$ values of $x \in \mathbb{Z}/n\mathbb{Z}$ if and only if the same is true of $\tilde{f}(x + 1) = mf(x)$. Theorem 1.2 then follows, or, in more detail, we have:

**Theorem 6.12** (Helfgott–Juschenko [16]). The group $\text{Hig}_k(\text{BS}(1, m))$ has a sofic quotient $Q$ such that the composition $\text{BS}(1, m) \to \text{Hig}_k(\text{BS}(1, m)) \to Q$ is injective if and only if for all $\varepsilon > 0$ there exists an integer $C$ such that for all $n > C$ coprime to $m$, there exist a permutation $f \in \text{Sym}(\mathbb{Z}/n\mathbb{Z})$ of order dividing $k$ such that $f(x + 1) = mf(x)$ for at least $(1 - \varepsilon)n$ values of $x$.

We do not know whether $\text{Hig}_k(\text{BS}(1, m))$ is sofic for $k \geq 4$. Both $\text{Hig}_k(\text{BS}(1, m))$ and $\text{Hig}_k(\text{BS}(1, m))$ are finite for $k \leq 3$ (assuming $m \neq \pm 1$), and so cannot have a sofic quotient into which $\text{BS}(1, m)$ injects. A beautiful argument due to Higman [17] shows that $\text{Hig}_4(\text{BS}(1, 2))$ has no finite quotients. Glebsky [11, 12] shows that, by contrast, if $k \geq 4$ and $p$ is a prime dividing $m - 1$, then the groups $\text{Hig}_k(\text{BS}(1, m))$ have many quotients which are finite $p$-groups. His main theorem in [11] amounts to the following. His proof is more combinatorial than the one we give below via Golod–Shafarevich machinery.

**Theorem 6.13** (Glebsky [11]). Suppose $k \geq 4$ and $p$ is a prime dividing $m - 1$. Then the pro-$p$ completion of $\text{Hig}_k(\text{BS}(1, m))$ is infinite. If, moreover, $k$ is even and $m \neq \pm 1$, then $\text{BS}(1, m)$ embeds into this pro-$p$ completion.

**Proof.** The defining relator of $\text{BS}(1, m)$ can be written in the from $[a, b]^{-1}a^{m-1}$ and lies in the $p$-Frattini subgroup of the free group. This implies that the pro-$p$ completion $\hat{G}$ of $\text{Hig}_k(\text{BS}(1, m))$ has a minimal pro-$p$ presentation with $k$ generators and $k$ relations. Such a presentation satisfies the Golod–Shafarevich condition (since $k \leq k^2/4$) and therefore it defines an infinite pro-$p$ group (see, for example, [10]).

Since $a$ has finite order in any proper quotient of $\text{BS}(1, m)$, to prove that $\text{BS}(1, m)$ embeds into this pro-$p$ completion, it suffices to show that the images of the generators $a_1, \ldots, a_k$ of $\text{Hig}_k(\text{BS}(1, m))$ have infinite order—indeed, that and one of them has infinite order. In the case $k = 4$, the defining relations of $\hat{G}$ are similar to the relations of $F_2 \times F_2$, where the first copy of the free group $F_2$ is generated by $a_1$ and $a_3$ and the second copy is generated by $a_2$ and $a_4$. It can be shown that $\hat{G}$ contains the free pro-$p$ groups $\Gamma_1$ and $\Gamma_2$ generated by $\Gamma_1 = \langle a_1, a_3 \rangle$ and $\Gamma_2 = \langle a_2, a_4 \rangle$, and moreover that any element in $\hat{G}$ can be written uniquely as a product of two elements, one from $\Gamma_1$ and one from $\Gamma_2$. This shows that the order of $a_i$ in $\hat{G}$ is infinite and that $\text{BS}(1, m)$ embeds in $\hat{G}$. (When $k > 4$ the group $\hat{G}$ does not have such nice combinatorial description, but there is a quotient of $\hat{G}$, which has similar structure, provided that $k$ is even.)

Theorems 6.12 and 6.13 together imply Theorem 1.4: if $|m| > 2$ and $\varepsilon > 0$, then there exists $C$ such that for all $n > C$ coprime to $m$, there are permutations $g \in \text{Sym}(\mathbb{Z}/n\mathbb{Z})$ with $g^4 = \text{id}$ and with $g(x + 1) = mg(x)$ for at least $(1 - \varepsilon)n$ values of $x \in \mathbb{Z}/n\mathbb{Z}$. 

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Remark 6.14. Theorem 1.4 applies to all integers \( m \neq 0, 2 \), these being the cases where \((m - 1) \neq \pm 1\) and has no prime divisors are \( m = 0, 2 \). The above proof via Theorem 6.13 works for \( m \neq -1, 0, 1, 2 \). For \( m = 1 \), when \( BS(1, 1) = \mathbb{Z} \times \mathbb{Z} \), the analogue is a special case of Theorem 6.2. The case \( m = -1 \) only requires a minor strengthening of Theorem 6.13. The case \( m = 0 \) is degenerate since \( a \) and \( b \) have different orders. Also one can have the order of \( g \) divide any given even integer \( k \geq 4 \), not just 4. We stress that the analogue of Theorem 1.4 is unknown when \( m = 2 \).

Remark 6.15. Estimating the dependance of the constant \( C \) in Theorem 1.4 is quite hard because it involves explicitly constructing the sets \( S' \) in Theorem 5.3. We believe that by carefully tracking all bounds one gets that \( C = O\left(2^{Kx^2}\right) \).

6.4. \( \mathbb{Z} \wr \mathbb{Z} \). The wreath product \( \mathbb{Z} \wr \mathbb{Z} \) has presentation

\[
\mathbb{Z} \wr \mathbb{Z} = \left\{ a, b \mid [a, a^i] = 1 \forall i \in \mathbb{N} \right\}.
\]

It is a residually finite solvable group, and so is amenable. For any \( n \in \mathbb{N} \) and any \( m \) coprime to \( n \), define a homomorphism \( \psi_{n,m} : \mathbb{Z} \wr \mathbb{Z} \to \text{Sym}(\mathbb{Z}/n\mathbb{Z}) \) by

\[
\psi_{n,m}(a) = (x \mapsto x + 1), \quad \psi_{n,m}(b) = (x \mapsto m^{-1}x),
\]

which is well-defined since the permutations \( \psi_{n,m}(a) \) and \( \psi_{n,m}(b) \) satisfy the defining relations of \( \mathbb{Z} \wr \mathbb{Z} \). This action of \( \mathbb{Z} \wr \mathbb{Z} \) arises from the quotient \((\mathbb{Z} \wr \mathbb{Z})_{n,m} = (\mathbb{Z} \wr \mathbb{Z})/(a^n = 1, a^b = a^m)\) acting on cosets of the subgroup \( \langle b \rangle \).

Lemma 6.16. For all finite sets \( S \subseteq \mathbb{Z} \wr \mathbb{Z} \) and all \( \delta > 0 \), there exists \( C > 0 \) with the property that \( \psi_{n,m} \) is an \((S, \delta, n)\)-approximation of \( \mathbb{Z} \wr \mathbb{Z} \) for all coprime \( n, m \) satisfying \( n \nmid t(m) \) for every nonzero polynomial \( t(x) = \sum t_i x^i \in \mathbb{Z}[x] \) whose degree is most \( C \) and whose coefficients all satisfy \(|t| < C \).

If, moreover, \(|m| > 2C + 1 \) and \( n > |m|C^2 \), then all such polynomials satisfy \( n \nmid t(m) \), and so \( \psi_{n,m} \) is necessarily an \((S, \delta, n)\)-approximation of \( \mathbb{Z} \wr \mathbb{Z} \).

Proof. The group \( \mathbb{Z} \wr \mathbb{Z} \) can be represented by the group of matrices

\[
\left\{ \begin{bmatrix} 1 & \bar{t}(x) \\ 0 & x^k \end{bmatrix} \mid k \in \mathbb{Z}, \bar{t}(x) \in \mathbb{Z}[x, x^{-1}] \right\}
\]

via

\[
a \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & x^{-1} \end{bmatrix}.
\]

Suppose \( S \) is a finite subset of \( \mathbb{Z} \wr \mathbb{Z} \). Then there exists an integer \( N \) with the following property. Every \( s \in S \) can be represented by

\[
\begin{bmatrix} 1 & \bar{t}_{(s)}(x) \\ 0 & x^{b(s)} \end{bmatrix}
\]

where

\[
\bar{t}_{(s)}(x) = \sum_{i=-N}^{N} \bar{t}_i x^i \in \mathbb{Z}[x, x^{-1}]
\]
and $|\mu(s)| \leq N$ and $|t_i| \leq N$ for all $i$. One computes that for $s$ as above,

$$\psi_{n,m}(s) : \left( x \mapsto m^{-|\mu(s)|} \left( x + \tilde{t}_i(m) \right) \right).$$

Define $t_{(i)}(x) := x^N \tilde{t}_i(x) \in \mathbb{Z}[x]$. Let $C = \max \left\{ 2N + 1, \left\lfloor \frac{1}{\delta} + 1 \right\rfloor \right\}$, so that $t_{(i)}(x)$ is within the scope of the lemma. Assume that $n, m$ satisfy the condition in the first part of the lemma, so that, in particular, $n \nmid t_{(i)}(m)$.

If $\mu(s) = 0$, then $\psi_{n,m}(s)$ maps $x$ to $x + \tilde{t}_i(m) = x + m^{-N}t_{(i)}(m)$ and so has no fixed points as $m^{-1}$ is coprime to $n$ and $n \nmid t_{(i)}(m)$.

Suppose $\mu(s) \neq 0$. If $n|(m^{\mu(s)} - 1)$, then $\psi_{n,m}(s) : x \mapsto x + \tilde{t}_i(m)$ and so is either the identity (when $n \mid t_{(i)}(m)$, which would contradict $n \nmid t_{(i)}(m)$, and so does not occur) or has no fixed points. If $n \nmid (m^{\mu(s)} - 1)$, then the permutation $\psi_{n,m}(s)$ has at most $\gcd(m^{\mu(s)} - 1, n)$ fixed points. But $\gcd(m^{\mu(s)} - 1, n) \leq \delta n$, else the polynomial $M(x^{\mu(s)} - 1)$ will have $m$ as a root mod $n$ for some $M < 1/\delta$. So $\psi_{n,m}(s)$ has at most $\delta n$ fixed points.

In every case we have $d(\psi_{n,m}(s), \text{id}) > 1 - \delta$. And the almost homomorphism condition is immediate since $\psi_{n,m}$ is, in fact, a homomorphism. So $\psi_{n,m}$ is an $(\delta, \delta, n)$-approximation of $\mathbb{Z} \wr \mathbb{Z}$, as required.

For the final part of the lemma, assume $|m| > 2C + 1$ and $n > |m|^{C+1}$. In particular, $|m| \neq 1$. Let $d \leq C$ be the degree of $t(x)$. Then

$$|t(m)| \leq \sum_{i=0}^{d} |t_i| |m|^d \leq C \frac{|m|^{d+1} - 1}{|m| - 1} < |m|^{C+1} < n.$$

If $d = 0$, then $0 \neq |t(m)| = |t_0| < C < n$, so $n \nmid t(m)$, as required. Assume $d \geq 1$. Then

$$|t(m)| = \left| t_0 m^d + \sum_{i=0}^{d-1} t_i m^i \right| \geq |t_0 m^d| - \left| \sum_{i=0}^{d-1} t_i m^i \right| \geq |m|^d - C \frac{|m|^{d-1} - 1}{|m| - 1} \geq |m|^d/2,$$

where the final inequality holds because $|m| > 2C + 1$. As $d \geq 1$, we now have that $0 < |m|/2 \leq |t(m)| < n$, which implies that $n \nmid t(m)$.

So Theorem 5.1 tells us that for all $k \in \mathbb{N}$ the group $\text{Hig}_k(\mathbb{Z} \wr \mathbb{Z})$ has a sofic quotient into which $\mathbb{Z} \wr \mathbb{Z}$ naturally embeds if and only there exist infinitely many $n$ and functions $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ of order dividing $k$ which conjugate addition of 1 to multiplication by $m$ up to error $\varepsilon$. After conjugating by a minus sign (in the manner of replacing $f$ by $f'$ in Section 6.3) this latter condition becomes $f(x + 1) = mf(x)$ for at least $(1 - \varepsilon)n$ values of $x \in \mathbb{Z}/n\mathbb{Z}$. However, for $k \geq 4$ we know by Lemma 3.3 that $\mathbb{Z} \wr \mathbb{Z}$ naturally embeds in $\text{Hig}_4(\mathbb{Z} \wr \mathbb{Z})$ and by Corollary 4.5 that $\text{Hig}_4(\mathbb{Z} \wr \mathbb{Z})$ is sofic. We also know by Proposition 4.6 that $\text{Hig}_3(\mathbb{Z} \wr \mathbb{Z})$ has a sofic quotient into which $\mathbb{Z} \wr \mathbb{Z}$ naturally embeds. So we have that such $n$ and $f$ do exist for $k \geq 3$. Adjusting the constant $C$ of Lemma 6.16 suitably, we have:

**Theorem 6.17.** For all $\varepsilon > 0$ and $k \geq 3$, there exists $C$ such that if $n$ is coprime to $m$ and $|m| > C$ and $n > |m|^\varepsilon$, then there exists $f \in \text{Sym}(\mathbb{Z}/n\mathbb{Z})$ which has order dividing $k$ and the property that $f(x + 1) = mf(x)$ for at least $(1 - \varepsilon)n$ values of $x \in \mathbb{Z}/n\mathbb{Z}$.

This result is stronger but less clean than the version we preferred to present in the introduction as Theorem 1.3.
Proof of Theorem 1.3. Theorem 1.3 states that for all \(e > 0\) and all \(k \geq 3\), there exists \(N \in \mathbb{N}\) such that for all coprime integers \(m\) and \(n\) with \(n > N\) and \(\ln n < m < \ln n\), there exists an \(f\) as per Theorem 6.17. This follows from Theorem 6.17 by taking \(N\) sufficiently large that \(\ln \ln N > C\) and \(N > (\ln N)^C\).

As mentioned above, \(\operatorname{Hig}_k(\mathbb{Z} \wr \mathbb{Z})\) is sofic for all \(k \geq 4\) by Corollary 4.5. For \(k = 1\) and \(k = 2\) the groups \(\operatorname{Hig}_k(\mathbb{Z} \wr \mathbb{Z})\) are sofic since

\[
\operatorname{Hig}_1(\mathbb{Z} \wr \mathbb{Z}) = \mathbb{Z} \wr \mathbb{Z} \cong \mathbb{Z} \quad \text{and} \quad \operatorname{Hig}_2(\mathbb{Z} \wr \mathbb{Z}) \cong \mathcal{H}
\]

(but \(\mathbb{Z} \wr \mathbb{Z}\) does not embed in \(\operatorname{Hig}_1(\mathbb{Z} \wr \mathbb{Z})\) or \(\operatorname{Hig}_2(\mathbb{Z} \wr \mathbb{Z})\)). As in the case of \(\mathcal{H}\), we do not know whether \(\operatorname{Hig}_3(\mathbb{Z} \wr \mathbb{Z})\) is sofic, but we see no reason it should not be.

Remark 6.18. Before proving Theorem 1.1, which implies that \(\operatorname{Hig}_4(\mathbb{Z} \wr \mathbb{Z})\) is sofic, we constructed finite quotients which can be combined to give a residually finite quotient \(Q\) of \(\operatorname{Hig}_4(\mathbb{Z} \wr \mathbb{Z})\) in which \(\mathbb{Z} \wr \mathbb{Z}\) embeds. We find these quotients interesting on their own, and will briefly describe them. Pick a prime \(p\) and two functions \(f, \lambda : \mathbb{F}_p \to \mathbb{F}^*_p\). Then there is an action \(\psi_{p,f,\lambda}\) of \(\operatorname{Hig}_4(\mathbb{Z} \wr \mathbb{Z})\) on \(S = \mathbb{F}_p \times \mathbb{F}_p \times \mathbb{F}_p \times \mathbb{F}_p\) defined by

\[
t : (x, y, z, w) \mapsto (y, z, w, x), \quad a : (x, y, z, w) \mapsto (x + f(w), y, z, w),
\]

and since \(a^p = d\), we then can compute that

\[
a : (x, y, z, w) \mapsto (x + f(w), y, z, w), \quad c : (x, y, z, w) \mapsto (x, y + f(x), z, w),
\]

\[
b : (x, y, z, w) \mapsto (x, y, z + f(y), w), \quad d : (x, y, z, w) \mapsto (x, y + f(w), z, w).
\]

To verify that this is an action, observe that \((a^p)^d\) acts via

\[
(a^p)^d : (x, y, z, w) \mapsto (x + \lambda(z)^{-1}f(w + jf(z)), y, w + jf(z), z, w).
\]

Since this distorts \(x\) and \(y\) by only multiplication or only addition of elements that are not distorted (namely \(z\) and \(w\)), \((a^p)^d\) will commute with \((a^p)^d\). So, \(\psi_{p,f,\lambda}\) is a homomorphism \(\operatorname{Hig}_4(\mathbb{Z} \wr \mathbb{Z}) \to \operatorname{Sym}(\mathbb{P}^n)\). Using a combinatorial description of the elements in \(\mathbb{Z} \wr \mathbb{Z}\) it is possible to show that the restriction of \(\psi_{p,f,\lambda}\) to \(\mathbb{Z} \wr \mathbb{Z}\) is injective.

These actions \(\psi_{p,f,\lambda}\) are quite different from those arising in our proof of Theorem 1.1. We expect that for generic functions \(f\) and \(\lambda\), the image of \(\psi_{p,f,\lambda}\) will either be the full symmetric group, or the alternating group, and so will be very far from (residually) solvable. It is intriguing question whether the actions \(\psi_{p,f,\lambda}\) distinguish all elements in \(\operatorname{Hig}_4(\mathbb{Z} \wr \mathbb{Z})\). We see no reason why this should not the case, but without having an easily understandable combinatorial model of \(\operatorname{Hig}_4(\mathbb{Z} \wr \mathbb{Z})\), it is hard to prove such claim.

6.5. The free metabelian group on two generators. The free metabelian group on two generators \(M\) has a presentation

\[
M = \left\langle a, b \mid [a, b], [a, b]^{[b]} = 1 \forall i, j \in \mathbb{Z} \right\rangle.
\]

It is a residually finite solvable group, so is amenable. For \(n \in \mathbb{N}\) and \(p\) and \(q\) relatively prime to \(n\), define a map \(\psi_{n,p,q} : \mathbb{Z} \wr \mathbb{Z} \to \operatorname{Sym}(\mathbb{Z}/n\mathbb{Z})\) by

\[
\psi_{n,p,q}(a) = \left\{ x \mapsto q^{-1}(x + 1) \right\}, \quad \text{and} \quad \psi_{n,p,q}(b) = \left\{ x \mapsto p^{-1}x \right\}.
\]
which extends to the whole of $M$ since the permutations $\psi_{n,p,q}(a)$ and $\psi_{n,p,q}(b)$ satisfy the defining relations of $M$. We have the analogue of Lemma 6.16, whose proof is practically the same:

**Lemma 6.19.** For all finite sets $S \subseteq M$ and all $\delta > 0$, there exists a constant $C$ such that for all integers $n$, $p$, and $q$ with $n$ coprime to $p$ and $q$ and satisfying $n \not| \ell(p,q)$ for all nonzero polynomials $t(x,y) = \sum t_{ij}x^iy^j \in \mathbb{Z}[x,y]$ with integer coefficients $|t_{ij}| < C$ and total degree at most $C$, the map $\psi_{n,p,q}$ is an $(S, \delta, n)$-approximation of $M$.

If, moreover, $|q| > 2C + 1$, $|p| > |q|^{C+1}$ and $n > |p|^{C+1}$, then all such polynomials satisfy $n \not| \ell(p,q)$ and so $\psi_{n,p,q}$ necessary is an $(S, \delta, n)$-approximation of $M$.

So Theorem 5.1 tells us that for all $k \in \mathbb{N}$ the group $\text{Hig}_k(M)$ has a sofic quotient into which $M$ naturally injects if and only there exist infinitely many $n$ and functions $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ of order dividing $k$ which conjugate the actions of $\psi_{n,p,q}(a)$ and $\psi_{n,p,q}(b)$ up to error at most $\varepsilon$.

For $k \geq 4$ we can use Corollary 4.5 to see that $\text{Hig}_k(M)$ is sofic and Lemma 3.3 to see that $M$ naturally embeds into it. The case $k = 2$ is handled by the observation $\text{Hig}_2(M) \cong M$, and $k = 3$ by Proposition 4.6. (We do not know whether $\text{Hig}_3(M)$ is sofic.) As for the case $k = 1$, the group $\text{Hig}_1(M)$ is also sofic since $\text{Hig}_1(M) = \text{Hig}_1(M) \cong \mathbb{Z}$ but $M$ does not embed in $\text{Hig}_1(M)$.

Theorem 5.1 then allows us to conclude:

**Theorem 6.20.** For all $\varepsilon > 0$ and $k \geq 2$, there exists $C$ such that if $n$ is coprime to $p$ and $q$ and $|q| > C$, $|p| > |q|^C$ and $n > |p|^C$, then there exists $f \in \text{Sym}(\mathbb{Z}/n\mathbb{Z})$ such that $f^k = \text{id}$ and $f(qx + 1) = pf(x)$ for at least $(1 - \varepsilon)n$ values of $x \in \mathbb{Z}/n\mathbb{Z}$.

7. Heuristic

Here we will explain why the existence of the permutations $f \in \text{Sym}(n)$ proved in Theorem 5.1 is surprising. We will focus on instances where $A = \langle a \rangle \cong B = \langle b \rangle \cong \mathbb{Z}$ and $k = 4$ which is the case in most of our examples. (We could generalize to $k \geq 4$ without significantly changing the following argument, but the assumption that $A \cong B \cong \mathbb{Z}$ is essential.)

Denote $a = \psi(a)$ and $b = \psi(b)$. Each permutation $f$ satisfies the global condition $f^4 = \text{id}$; and many local conditions: the condition concerning $d(\psi(b) \circ f, f \circ \psi(b))$ is equivalent to $f^4(\alpha(x)) = \beta(f(x))$ for at least $(1 - \varepsilon)n$ points $x$.

One can estimate the probability that a permutation $f$ chosen uniformly at random from $\text{Sym}(n)$ satisfies that global condition: by considering cycle structure, one counts the number of elements of order dividing 4 in the symmetric group $\text{Sym}(n)$ (see \cite{3}), which leads to

$$P = \text{Prob}(f^4 = \text{id}) \approx \frac{1}{\sqrt{|\text{Sym}(n)|}} \approx n^{-n/4}.$$  

(Here we are only describing the leading term of the expansion of $\log P$.)

It is also quite easy to estimate the probability that a local condition is satisfied. A local condition asserts that $f(\alpha(x))$ is determined by $f(x)$—the probability that this happens at
a given $x$ is approximately $1/n$ (only approximately since $x$ might be a fixed point of $\alpha$ or $f(x)$ might be a fixed point for $\beta$). However, this probability is irrelevant since we want $f(\alpha(x)) = \beta(f(x))$ for the majority of $x$ (for at least $(1 - \varepsilon)n$ points $x$, to be precise). Informally, a small number of local conditions are almost independent from each other, but this is not true if we consider many local conditions.

The number of permutations in $\text{Sym}(n)$ satisfying $f(\alpha(x)) = \beta(f(x))$ for at least $(1 - \varepsilon)n$ points $x$ is at most $n^{2\varepsilon + k}$ where $k$ is the number cycles in the action of $\alpha$—such permutations are determined by the following: the points $x$ where the local condition is not satisfied; values of $f(\alpha(x))$ at each of these points; and the values of $f$ at a single point on each cycle on the action of $\alpha$. (This information specifies a function $f$ but it may not be a permutation.) For all $\varepsilon' > 0$, we have $k < \varepsilon'n$ for large $n$, otherwise a small power of $\alpha$ will be close to the identity permutation, which would contradict the fact that $\psi$ detects the soficity of $G$. Thus, the number of permutations satisfying the majority of the local conditions is at most $K = n^{(2\varepsilon + \varepsilon')n}$.

If the global condition is almost independent from the local conditions, then the expected number of permutation satisfying both, should be around

$$P.K = n^{-n/4}n^{(2\varepsilon + \varepsilon')n} = n^{(2\varepsilon + \varepsilon' - 1/4)n} \ll 1,$$

if $\varepsilon, \varepsilon' < 1/20$. Thus, one should expect that there are no such permutations when $n$ is sufficiently large.

The independence assumption is somewhat justified by the observation that the global condition is independent form each of the local conditions (it is also almost independent from any fixed number of local conditions). Notice that this heuristic does not really depend on the group $G$.

The main weakness of this heuristic is the assumption that the global condition is almost independent from the majority of the local conditions. One can interpret Theorem 5.1 as saying there is a connection between the soficity of the group $\text{Hi}_{\delta}(G)$ and the independence of global versus local conditions.

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