CHIRAL HODGE COHOMOLOGY AND MATHIEU MOONSHINE

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ABSTRACT. We construct a filtration of chiral Hodge cohomology of a K3 surface $X$, such that its associated graded object is a unitary representation of the $\mathbb{N}=4$ vertex algebra with central charge 6 and its subspace of primitive vectors has the property: its equivariant character for a symplectic automorphisms $g$ of $X$ is agree with the McKay-Thompson series for $g$ in Mathieu moonshine.

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1. INTRODUCTION

In 2010, Eguchi, Ooguri and Tachikawa [9] observed that when the elliptic genus of a K3 surface, the Jacobi form $2\phi_{0,1}(z;\tau)$ of weight 0 and index 1, is decomposed into a sum of the characters of the N = 4 superconformal algebra with central charge $c = 6$,

$$2\phi_{0,1}(z;\tau) = -2ch_{1,\frac{1}{2}}(z;\tau) + 20ch_{1,0}(z;\tau) + 2\sum_{n=1}^{\infty} A_n ch_{1,n+\frac{1}{2},\frac{1}{2}}(z;\tau),$$  \hspace{1cm} (1.1)

the first few coefficients $A_n$ are the sums of the dimensions of the irreducible representations of the largest Mathieu group $M_{24}$. Let

$$\Sigma(q) = q^{-\frac{1}{8}}(-2 + 2\sum_{n=1}^{\infty} A_n q^n).$$  \hspace{1cm} (1.2)

It is a mock modular form of weight $\frac{1}{2}$. They conjectured that there exist a graded $M_{24}$-module $K = \sum_{n=0}^{\infty} K_n q^{n-1/8}$ with graded dimension $\Sigma(q)$. It is Mathieu analogue to the modular function $J(q)$ in the famous monstrous moonshine\[1\]\[7\]\[25\]: the expansion coefficients of $J(q)$

$$J(q) = \frac{1}{q} + 196884q + 21493760q^2 + \cdots$$

could be naturally decomposed into the sums of the dimensions of the irreducible representations of the largest sporadic group–the Fischer-Griess monster.

Subsequently the analogues to McKay-Thompson series in monstrous moonshine were proposed in several works\[4\]\[5\]\[8\]\[13\]\[14\]. The McKay-Thompson series for $g$ in $M_{24}$ are

$$\Sigma_g(q) = q^{-\frac{1}{8}}\sum_{n=0}^{\infty} q^n \text{Trace}_{K_n} g = \frac{e(g)}{24}\Sigma_e(q) - \frac{f_g}{\eta(q)^3},$$  \hspace{1cm} (1.3)

where $\Sigma_e(q) = \Sigma(q)$, $e(g)$ is the character of the 24-dimensional permutation representation of $M_{24}$, the series $f_g$ is a certain explicit modular form of weight 2 for some subgroup $\Gamma^0(N_g)$ of $SL(2,\mathbb{Z})$ and $\eta$ is the Dedekind eta function. $\Sigma_g(q)$ satisfies the Rademacher sum property, which is equivariant to the principal modulus property of the MonstrousMcKay-Thompson series. Terry Gannon \[12\] has proven that these McKay-Thompson series indeed determine a $M_{24}$-module:

**Theorem 1.1 (Gannon).** The McKay-Thompson series as in \[8\]\[13\] determine a virtual graded $M_{24}$-module $K = \sum_{n=0}^{\infty} K_n q^{n-1/8}$. For $n \geq 1$, the $K_n$ are honest (and not only virtual) $M_{24}$-representations.
But the proof does not explain any connection to geometry or physics, and a concrete construction of $K$ remains unknown.

There is a deep relation between the K3 surfaces and the Mathieu group $M_{24}$. Mukai has classified the finite symplectic automorphism groups of K3 surfaces in $[15][17]$, which are all isomorphic to subgroups of the Mathieu group $M_{23}$ of a particular type. $M_{23}$ is isomorphic to a one-point stabilizer for the permutation action of $M_{24}$ on 24 elements.

If $g$ is a symplectic automorphism of a K3 surface, the functions $\Sigma_g(q)$ admit a geometric interpretation in terms of K3 surfaces. Thomas Creutzig and Gerald Höhn in $[6]$ showed that

**Theorem 1.2 (Creutzig-Höhn).** For a non-trivial finite symplectic automorphism $g$ acting on a K3 surface $X$, the equivariant elliptic genus and the twining character determined by the McKay-Thompson series of Mathieu moonshine agree, i.e. one has

$$\text{Ell}_{X,g}(z; \tau) = \frac{e(g)}{12} \phi_{0,1} + f_g \phi_{-2,1}.$$  

They also showed that the complex elliptic genus of a K3 surface can be given the structure of a virtual $M_{24}$-module which is compatible with the $H$-module structure for all possible groups $H$ of symplectic automorphisms of K3 surfaces under restriction.

In this work, we will construct a graded vector space from the chiral Hodge cohomology of the K3 surface with graded dimension $\Sigma(q) + 2q^{-\frac{1}{2}}$.

Let $X$ be a complex manifold. In $[18][19]$, Malikov, Schectman and Vaintrob introduced a sheaf of vertex algebras $\Omega^e_X$, called chiral de Rham algebra. The sheaf has a $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ grading

$$\Omega^e_X = \bigoplus_{k=0}^{\infty} \bigoplus_{p} \Omega^e_X[k, p]$$

by fermionic number $p$ and conformal weight $k$. And the weight zero piece coincides with the ordinary de Rham sheaf. This construction has substantial applications to mirror symmetry and is related to stringy invariants of $X$ such as the elliptic genus $[2]$. According to $[16]$, if $X$ is a Calabi-Yau manifold, its cohomology, called *chiral Hodge cohomology* $H^\ast(X, \Omega^e_X)$, can be identified with the infinite-volume limit of the half-twisted sigma model defined by E. Witten.
If $X$ is a hyperKähler manifold, we will show that $H^i(X, \Omega^\text{ch}_X)$ has a filtration \{ $H^i_k(X)$ \}, such that its associated graded object $\mathcal{H}^i(X, \Omega^\text{ch}_X) = \oplus H^i_k(X)/H^i_{k+1}(X)$ is a unitary representation of the N=4 vertex algebra with central charge $3 \dim X$ (Theorem 3.2).

If $X$ is a K3 surface, let $\mathcal{A}^1_{n,2}(X)$ be the space of the primitive vectors with conformal weight $n$ in the unitary representation $\mathcal{H}^1(X, \Omega^\text{ch}_X)$ of the $N = 4$ vertex algebra with central charge 6. Let

$$\mathcal{A}_X(q) = \sum_{n=1}^{\infty} \mathcal{A}^1_{n,2}(X)q^{n-\frac{1}{8}}.$$ 

In this paper, we will show

**Theorem 1.3.** The graded dimension of $\mathcal{A}_X(q)$ is $\Sigma(q) + 2q^{\frac{1}{8}}$. For a finite symplectic automorphism $g$ acting on a K3 surface $X$,

$$\Sigma_g(q) + 2q^{\frac{1}{8}} = \sum_{n=1}^{\infty} q^{n-\frac{1}{8}} \text{Trace}_{\mathcal{A}^1_{n,2}(X)} g = \text{Trace}_{\mathcal{A}_X(q)} g.$$ 

**2. Chiral de Rham complex**

The chiral de Rham algebra [18][19] is a sheaf of vertex algebras $\Omega^\text{ch}_X$ defined on any complex manifold $X$. Let $\Omega_N$ be the tensor product of $N$ copies of the $\beta\gamma - bc$ system. It has $2N$ even generators $\beta^1(z), \cdots, \beta^N(z), \gamma^1(z), \cdots, \gamma^N(z)$ and $2N$ odd generators $b^1(z), \cdots, b^N(z), c^1(z), \cdots, c^N(z)$. Their nontrivial OPEs are

$$\beta^i(z)\gamma^j(w) \sim \frac{\delta^i_j}{z - w}, \quad b^i(z)c^j(w) \sim \frac{\delta^i_j}{z - w}.$$ 

Let $(U, \gamma^1, \cdots, \gamma^N)$ be a complex coordinate system of an $N$ dimensional complex manifold $X$, $\mathbb{C}[\gamma^1, \cdots, \gamma^N] \subset \mathcal{O}(U)$ can be regarded as a subspace of $\Omega_N$ by identifying $\gamma^i$ with $\gamma^i(z) \in \Omega_N$. As a linear space, $\Omega_N$ has a $\mathbb{C}[\gamma^1, \cdots, \gamma^N]$ module structure (it is not compatible with the vertex algebra structure). $\Omega^\text{ch}_X(U)$ is the localization of $\Omega_N$ on $U$,

$$\Omega^\text{ch}_X(U) = \Omega_N \otimes_{\mathbb{C}[\gamma^1, \cdots, \gamma^N]} \mathcal{O}(U).$$

Then $\Omega^\text{ch}_X(U)$ is the vertex algebra generated by $\beta^i(z), b^i(z), c^i(z)$ and $f(z), f \in \mathcal{O}(U)$. These generators satisfy the nontrivial OPEs

$$\beta^i(z)f(w) \sim \frac{\delta f}{\partial \gamma^i}(z) \frac{1}{z - w}, \quad b^i(z)c^j(w) \sim \frac{\delta^i_j}{z - w}.$$
as well as the normally ordered relations

\[
:f(z)g(z) := fg(z), \text{ for } f, g \in \mathcal{O}(U).
\]

\(\Omega^{ch}\) is spanned by the elements

\[
\partial^{k_1} \beta^{i_1} \ldots \partial^{k_s} \beta^{i_s} \partial^{l_1} b^{j_1} \ldots \partial^{l_t} b^{j_t} \partial^{m_1} c^{r_1} \ldots \partial^{m_u} c^{r_u} \partial^{n_1} \gamma^{s_1} \ldots \partial^{n_v} \gamma^{s_v} f(\gamma) :,
\]

\(f(\gamma) \in \mathcal{O}(U), k_1 \geq k_2 \geq \cdots \geq k_s, l_1 \geq \cdots \geq l_t, m_1 \geq \cdots \geq m_u, n_1 \geq \cdots \geq n_v > 0.\)

\(\Omega^{ch}\) is a free \(\mathcal{O}(U)\) module (which is not compatible with the vertex algebra structure) with basis

\[
\partial^{k_1} \beta^{i_1} \ldots \partial^{k_s} \beta^{i_s} \partial^{l_1} b^{j_1} \ldots \partial^{l_t} b^{j_t} \partial^{m_1} c^{r_1} \ldots \partial^{m_u} c^{r_u} \partial^{n_1} \gamma^{s_1} \ldots \partial^{n_v} \gamma^{s_v} :,
\]

\(k_1 \geq k_2 \geq \cdots \geq k_s, l_1 \geq \cdots \geq l_t, m_1 \geq \cdots \geq m_u, n_1 \geq \cdots \geq n_v > 0.\)

Let \(\tilde{\gamma}^1, \ldots \tilde{\gamma}^N\) be another set of coordinates on \(U\), with

\[
\tilde{\gamma}^i = f^i(\gamma^1, \ldots \gamma^N), \quad \gamma^i = g^i(\tilde{\gamma}^1, \ldots \tilde{\gamma}^N).
\]

The coordinate transfer equations for the generators of \(\Omega^{ch}\) are

\[
\partial \tilde{\gamma}^i(z) := \frac{\partial f^i}{\partial \gamma^j}(z) \partial \gamma^j(z) :,
\]

\[
\tilde{b}^j(z) := \frac{\partial g^j}{\partial \gamma^i}(g(\gamma)) b^i :,
\]

\[
\tilde{c}^i(z) := \frac{\partial f^i}{\partial \gamma^j}(z) c^j(z) :,
\]

\[
\tilde{\beta}^j(z) := \frac{\partial g^j}{\partial \gamma^i}(g(\gamma))(z) \beta^i(z) + \cdots + \frac{\partial}{\partial \gamma^k}(\frac{\partial g^j}{\partial \gamma^i}(g(\gamma)))(z) c^k(z) b^j(z) :.
\]

**Filtration.** Let \(\Omega^{ch}_{X,k}(U)\) be the subspace of \(\Omega^{ch}(U)\), which is spanned by elements of (2.1) with \(v - s \geq k\).

\[
\cdots \subset \Omega^{ch}_{X,k+1}(U) \subset \Omega^{ch}_{X,k}(U) \subset \Omega^{ch}_{X,k-1}(U) \subset \cdots
\]

is a filtration of \(\Omega^{ch}(U)\). The filtration is compatible with the grading of fermionic number and conformal weight. And restricted on \(\Omega^{ch}_{X}[k,p]\), the filtration is finite. We have

**Lemma 2.1.** For any \(A \in \Omega^{ch}_{X,k}(U), A' \in \Omega^{ch}_{X,l}(U), A_{(n)}A' \in \Omega^{ch}_{X,k+l}(U)\). For any holomorphic function \(f\) on \(U\), \(f(n)A \in \Omega^{ch}_{X,k+1}(U)\), for \(n \geq 0\).
Proof. We can assume \( A \) and \( A' \) are elements in the form of (2.1). \( A(n)A' \) is obtained from \( A \) and \( A' \) by contracting some \( \beta' \)'s with \( \gamma' \)'s and some \( b' \)'s with \( c' \)'s and acting \( \partial' \)'s on some \( \beta' \)'s, \( \gamma' \)'s, \( b' \)'s and \( c' \)'s in \( A \) and \( A' \). One \( \partial'\gamma, s > 0 \) contracts with one \( \beta \) and one \( \beta \) contracts with one \( \partial'\gamma, s > 0 \) or a holomorphic function. Thus through the contraction, the number of \( \partial'\gamma, s > 0 \) minus the number of \( \beta' \)'s in \( A \) and \( A' \) will not decrease. The number of \( \beta' \)'s will not change and the number of \( \partial'\gamma, s > 0 \) will not decrease by acting \( \partial' \)'s on \( \beta' \)'s, \( \gamma' \)'s, \( b' \)'s and \( c' \)'s. So \( A(n)A' \in \Omega_{X,k,l}(U) \). If \( A = f \) and \( n \geq 0 \) there is some \( \beta \) contracts with \( f \), so \( f(n)A \in \Omega_{X,k,l}(U) \). □

Let

\[
gr(\Omega^ch_X(U)) = \bigoplus_k gr(\Omega^ch_{X,k}(U)), \quad gr(\Omega^ch_{X,k}(U)) = \Omega^ch_{X,k}(U)/\Omega^ch_{X,k+1}(U).
\]

Let \( p_k : \Omega^ch_{X,k}(U) \to gr(\Omega^ch_{X,k}(U)) \) be the projection. \( gr(\Omega^ch_X(U)) \) is a vertex algebra with the circle product

\[
-(n)_- : gr(\Omega^ch_{X,k}(U)) \times gr(\Omega^ch_{X,l}(U)) \to gr(\Omega^ch_{X,k+l}(U))
\]

given by

\[
p_k(A(a)p_l(A') = p_{k+l}(A(a)A').
\]

By Lemma 2.1 the circle product is well defined on \( gr(\Omega^ch_X(U)) \) and for any \( f \in \mathcal{O}(U), A \in gr(\Omega^ch_X(U)), f(n)A = 0 \). So \( gr(\Omega^ch_X(U)) \) is a free \( \mathcal{O}(U) \) module under the wick product.

Let \( B^i = p_{-1}(\beta^i), A^i = p_{1}(\partial\gamma^i), b^i = p_0(b^i), c^i = p_0(c^i) \). We have

\[
(2.5) \quad B_{(0)} = 0
\]
on \( gr(\Omega^ch_X(U)) \) since \( \beta_0 \) maps \( \Omega^ch_{X,k}(U) \) to \( \Omega^ch_{X,k+1}(U) \). The nontrivial OPEs of these elements are

\[
B^i(z)A^j(w) \sim \frac{\delta^i_j}{(z-w)^2}, \quad b^i(z)c^j(w) \sim \frac{\delta^i_j}{z-w}.
\]

Let \( W \) be the vertex algebra generated by these elements, then \( gr(\Omega^ch_X(U)) = W \otimes \mathcal{O}(U) \).

From Lemma 2.1 and the coordinate transfer equations (2.4), we have

**Lemma 2.2.** The filtration \( \{\Omega^ch_{X,k}(U)\} \) is preserved under coordination transfer.

Thus the vertex algebra sheaf \( \Omega^ch_X \) has a filtration \( \{\Omega^ch_{X,k}\} \). Its associated graded object

\[
gr(\Omega^ch_X) = \bigoplus \Omega^ch_{X,k}/\Omega^ch_{X,k+1}, \quad \text{with } \Omega^ch_{X,k}/\Omega^ch_{X,k+1}(U) = gr(\Omega^ch_{X,k})(U)
\]
is a sheaf of vertex algebra. Under the coordinate transfer (2.3),

\[ \tilde{A}^i = \frac{\partial f^i}{\partial \gamma^j} A^j : , \]

\[ \tilde{b}^i = \frac{\partial g^j}{\partial \tilde{\gamma}^i} (g(\gamma)) b^j : , \]

\[ \tilde{c}^i = \frac{\partial f^i}{\partial \gamma^j} c^j : , \]

\[ \tilde{B}^i = \frac{\partial g^j}{\partial \tilde{\gamma}^i} (g(\gamma)) B^j : . \]

(2.6)

So \( gr(\Omega^\text{ch}_X) \) is the sheaf of the sections of holomorphic vector bundle

\[ V = \bigotimes_{n=1}^{\infty} (S_{q^n}(T) \otimes S_{q^n}(T^*) \otimes \wedge_{y-1} q^n T \otimes \wedge_{yq^n-1} T^*) = \sum_{k,p} V[k,p] y^p q^k \]

with \( gr(\Omega^\text{ch}_X)[k,p] \) is the sheaf of the sections of \( V[k,p] \). \( V \) is a holomorphic vector bundle of vertex algebra and its fibre is isomorphic to \( W \).

**Holomorphic sections.** If \( X \) is a Calabi-Yau manifold, there are four global sections \( Q(z), L(z), J(z) \) and \( G(z) \) on \( X \), which generate an \( N = 2 \) super conformal vertex algebra with the central charge \( c = 3 \dim X \). Locally,

\[ Q(z) = \sum_{i=1}^{N} : \beta^i(z) c^i(z) : , \quad L(z) = \sum_{i=1}^{N} ( : \beta^i(z) \partial \gamma^i(z) : - : b^i(z) \partial c^i(z) : ) , \]

\[ J(z) = - \sum_{i=1}^{N} : b^i(z) c^i(z) : , \quad G(z) = \sum_{i=1}^{N} : b^i(z) \partial \gamma^i(z) : . \]

\( L_{(1)} \) and \( J_{(0)} \) give \( \Omega^\text{ch}_X \) the the grading of conformal weights \( k \) and fermionic numbers \( p \), respectively.

If \( X \) has a nowhere vanishing holomorphic volume form \( \omega_0 \). Let \(( U, \gamma) \) be a coordinate system of \( X \) such that \( \omega_0 = d\gamma^1 \ldots d\gamma^N \). There are two global sections \( D(z) \) and \( E(z) \) of \( \Omega^\text{ch}_X \), which can be constructed from \( \omega_0 \). Locally, \( D(z) \) and \( E(z) \) can be represented by

\[ D(z) = : b^1(z) b^2(z) \cdots b^N(z) : , \quad E(z) = : c^1(z) c^2(z) \cdots c^N(z) : . \]

Let \( B(z) = Q(z)(0) D(z), C(z) = G(z)(0) E(z) \). These eight sections \( Q, J, L, G, E, D, B, C \) generate a vertex algebra \( \mathcal{V}_0 \). If \( N = 3 \), it is Odake’s algebra\[^{[21]}\].

Let

\[ \tilde{Q} = p_{-1}(Q), \quad \tilde{L} = p_0(L), \quad \tilde{J} = p_0(J), \quad \tilde{G} = p_1(G), \]
These eight elements are holomorphic sections of \( gr(\Omega_X^{ch}) \). They generate a vertex algebra which is isomorphic to \( \mathcal{V}_0 \).

If \( X \) is a hyperKähler manifold with holomorphic symplectic form \( \omega_1 \), let \( \omega^{-1}_1 \) be the inverse bivector of \( \omega_1 \). Locally, it is given by \( \omega_1 = \omega_{ij} d\gamma^i \wedge d\gamma^j \) and \( \omega^{-1}_1 = \omega_{ij} \frac{\partial}{\partial \gamma^i} \frac{\partial}{\partial \gamma^j} \). Let

\[
E_1(z) = \sum_{ij} \omega_{ij} c^i(z) c^j(z), \quad D_1(z) = \sum_{ij} \omega_{ij} b^i(z) b^j(z);
\]

\[
B_1(z) = Q(0) D_1(z), \quad C_1 = G(z)(0) E_1(z).
\]

\( Q, J, L, G, E_1, D_1, B_1, C_1 \) are global sections of \( \Omega_X^{ch} \) and they generate a copy of \( N = 4 \) super conformal vertex algebra \( \mathcal{V}_1 \) of central charge \( c = 3 \dim X \). \( \mathcal{V}_0 \) is a sub vertex algebra of \( \mathcal{V}_1 \).

Let

\[
\tilde{D}_1 = p_0(D_1), \quad \tilde{E}_1 = p_0(E_1), \quad \tilde{B}_1 = p_{-1}(B_1), \quad \tilde{C}_1 = p_1(C_1).
\]

\( \tilde{Q}, \tilde{J}, \tilde{L}, \tilde{G}, \tilde{E}_1, \tilde{D}_1, \tilde{B}_1, \tilde{C}_1 \) are holomorphic sections of \( gr(\Omega_X^{ch}) \). They generate a copy of \( N = 4 \) vertex algebra which is isomorphic to \( \mathcal{V}_1 \).

### 3. Unitary Representation

Let \( h = (-, -)_X \) be the Ricci flat Kähler metric on a compact Calabi-Yau manifold \( X \) with \( H_{ij} = (\frac{\partial}{\partial \gamma^i}, \frac{\partial}{\partial \gamma^j})_X \). It induces a canonical Hermitian metric \((-, -)\) on the vector bundle \( V \). For any \( a, b \in gr(\Omega_X^{ch})(U) \),

\[
(\mathcal{B}^i_{(n)} a, b) = H_{ij}(a, A^j_{(-n)} b), \quad \text{for any } n \in \mathbb{Z}, n \neq 0, ;
\]

\[
(\mathcal{B}^i_{(n)} a, b) = H_{ij}(a, c^j_{(-n-1)} b), \quad \text{for any } n \in \mathbb{Z}.
\]

Since \( \mathcal{B}_{(0)} = 0 \) (equation \((2.5)\)), we have

\[
(\tilde{Q}_{(n)} a, b) = (a, \tilde{G}_{(-n+1)} b),
\]

\[
(\tilde{J}_{(n)} a, b) = (a, \tilde{J}_{(-n)} b),
\]

\[
(\tilde{L}_{(n)} a, b) = (a, (\tilde{L}_{(-n+2)} - (n-1)\tilde{J}_{(-n+1)}) b),
\]

\[
(\tilde{D}_{(n)} a, b) = (a, (-1)^{\frac{n(n-1)}{2}} \tilde{E}_{(N-2-n)} b), \quad X \text{ has nonvanishing } N \text{ form}
\]

\[
(\tilde{D}_1_{(n)} a, b) = (a, \tilde{E}_{1(-n)} b), \quad X \text{ is a hyperKähler manifold}.
\]
The above equations induce the corresponding equations in the cohomology of \( gr(\Omega^ch_X) \).

**Lemma 3.1.** If \( X \) is a compact Calabi-Yau manifold, the cohomology \( H^i(X, gr(\Omega^ch_X)) \) of the sheaf \( gr(\Omega^ch_X) \) is a unitary representation of the \( N = 2 \) super conformal vertex algebra with

\[
\begin{align*}
\tilde{Q}^*_{(n)} &= \tilde{G}_{(-n+1)}, & \tilde{J}^*_{(n)} &= \tilde{J}_{(-n)}, \\
\tilde{L}^*_{(n)} &= \tilde{L}_{(-n+2)} - (n - 1)\tilde{J}_{(-n+1)}.
\end{align*}
\]

If \( X \) has a nonvanishing holomorphic volume form, \( H^i(X, gr(\Omega^ch_X)) \) is a unitary representation of \( \mathcal{V}_0 \) with

\[
\tilde{D}^*_{(n)} = (-1)^{\frac{N(N-1)}{2}} E_{(N-2-n)}.
\]

If \( X \) is a hyperKähler manifold, \( H^i(X, gr(\Omega^ch_X)) \) is a unitary representation of \( \mathcal{V}_1 \) with

\[
\tilde{D}_1^*_{(n)} = \tilde{E}_1^*_{(-n)}.
\]

**A filtration of chiral Hodge cohomology.** Let

\[
\tau_k : \Omega^ch_{X,k} \to \Omega^ch_X
\]

be the imbedding, which induces the morphism of their cohomology

\[
\tau_k^* : H^i(X, \Omega^ch_{X,k}) \to H^i(X, \Omega^ch_X).
\]

Let \( H_k^i(X) \) be the image of \( \tau_k^* \). \( \{H_k^i(X)\} \) is a filtration of \( H^i(X, \Omega^ch_X) \). Let \( \mathcal{H}^i(X, \Omega^ch_X) = \bigoplus_k H_k^i(X)/H_{k+1}^i(X) \) be its associated graded object.

**Theorem 3.2.** If \( X \) is a compact Calabi-Yau manifold, \( H^i(X, \Omega^ch_X) \) has a filtration \( \{H_k^i(X)\} \), such that \( \mathcal{H}^i(X, \Omega^ch_X) \) is a unitary representation of the \( N = 2 \) super conformal vertex algebra with central charge \( 3 \dim X \); if \( X \) has a nonvanishing holomorphic volume form, \( \mathcal{H}^i(X, \Omega^ch_X) \) is a unitary representation of \( \mathcal{V}_0 \); if \( X \) is a hyperKähler manifold, \( \mathcal{H}^i(X, \Omega^ch_X) \) is a unitary representation of the \( N=4 \) vertex algebra \( \mathcal{V}_1 \) with central charge \( 3 \dim X \).

**Proof.** Assume \( X \) is a hyperKähler manifold. \( \bigoplus_k \Omega^ch_{X,k} \) has a \( \mathcal{V}_1 \) module structure given by

\[
\begin{align*}
Q^*_{(n)}a, B^*_{(n)}a & \in \Omega^ch_{X,k-1}(U); \\
L^*_{(n)}a, J^*_{(n)}a, D^*_{(n)}a, E^*_{(n)}a & \in \Omega^ch_{X,k}(U); \\
G^*_{(n)}a, C^*_{(n)}a & \in \Omega^ch_{X,k+1}(U).
\end{align*}
\]
The exact sequences of sheaves

\[ 0 \to \Omega^\text{ch}_{X,k+1} \to \Omega^\text{ch}_{X,k} \to \Omega^\text{ch}_{X,k}/\Omega^\text{ch}_{X,k+1} \to 0 \]

gives an exact sequence of \( \mathcal{V}_1 \) modules

\[ 0 \to \bigoplus_k \Omega^\text{ch}_{X,k+1} \to \bigoplus_k \Omega^\text{ch}_{X,k} \to \text{gr}(\Omega^\text{ch}_X) \to 0, \]

Which induces a long exact sequence of \( \mathcal{V}_1 \) modules

\[ \cdots \to \bigoplus_k H^i(X, \Omega^\text{ch}_{X,k+1}) \to \bigoplus_k H^i(X, \Omega^\text{ch}_{X,k}) \to H^i(X, \text{gr}(\Omega^\text{ch}_X)) \to \cdots. \]  

(3.3)

We have an imbedding of \( \mathcal{V}_1 \) modules

\[ \bigoplus_k H^i(X, \Omega^\text{ch}_{X,k+1})/\iota_k^* H^i(X, \Omega^\text{ch}_{X,k+1}) \to H^i(X, \text{gr}(\Omega^\text{ch}_X)). \]

In particular, \( \bigoplus_k H^i(X, \Omega^\text{ch}_{X,k})/\iota_k^* H^i(X, \Omega^\text{ch}_{X,k}) \) is a unitary representation of the vertex algebra \( \mathcal{V}_1 \), since by Lemma 3.1, \( H^i(X, \text{gr}(\Omega^\text{ch}_X)) \) is a unitary representation of \( \mathcal{V}_1 \).

\( \bigoplus_k \Omega^\text{ch}_X \) has a \( \mathcal{V}_1 \) module structure, which is similar to the \( \mathcal{V}_1 \) module structure of \( \bigoplus_k \Omega^\text{ch}_{X,k} \).

Under these \( \mathcal{V}_1 \) module structure,

\[ \bigoplus_k \Omega^\text{ch}_{X,k} \to \bigoplus_k \Omega^\text{ch}_X \]

is a morphism of \( \mathcal{V}_1 \) modules. So

\[ \bigoplus_k H^i(X, \Omega^\text{ch}_{X,k}) \to \bigoplus_k H^i_k(X) \]

is a morphism of \( \mathcal{V}_1 \) module. We get a morphism of \( \mathcal{V}_1 \) module

\[ \tau_* : \bigoplus_k H^i(X, \Omega^\text{ch}_{X,k})/\iota_k^* H^i(X, \Omega^\text{ch}_{X,k+1}) \to \bigoplus_k H^i_k(X)/H^i_{k+1}(X). \]

By the definition of \( H^i_k(X) \), \( \tau_* \) is surjective. \( \bigoplus_k H^i(X, \Omega^\text{ch}_{X,k})/\iota_k^* H^i(X, \Omega^\text{ch}_{X,k}) \) is unitary representation of \( \mathcal{V}_1 \), so \( \bigoplus_k H^i_k(X)/H^i_{k+1}(X) \) is a unitary representation of \( \mathcal{V}_1 \).

The proofs for the other two cases are similar. \( \square \)

For the rest of the paper, we assume \( X \) is a hyperKähler manifold. By Theorem 3.2, \( \mathcal{H}^i(X, \Omega^\text{ch}_X) \) is a direct sum of irreducible representation of \( N = 4 \) vertex algebra with central charge \( 3 \dim X \).
Unitary representation of $V_1$. For an irreducible unitary representation $M_{k,h,l}$ of $N = 4$ vertex algebra $V_1$ with central charge $c = 6k$, the highest weight vector $v$ of $M_{k,h,l}$ is labeled by conformal weight $h$ and the isospin $l$.

$$L_{(1)}v = (h - \frac{k}{4})v, \quad J_{(0)}v = (2l + k)v.$$ 

There exist two types of representations in the $N = 4$ vertex algebra\[10\][11], massless (BPS) ($h = \frac{k}{4}, \ l = 0, \frac{1}{2}, 1, \cdots, \frac{k}{2}$) and massive (non-BPS) ($h > \frac{k}{4}, \ l = \frac{1}{2}, 1, \cdots, \frac{k}{2}$) representations. The character of a representation $V$ of the $N = 4$ vertex algebra for Ramond sectors is defined by

$$\text{ch}\tilde{R}_V(z; \tau) = y^{-k} \text{Trace}_V y^{J_{(0)}} q^{L_{(1)}},$$

and

$$\text{ch}\tilde{R}_V(z; \tau) = \text{ch}\tilde{R}_V(z + \frac{1}{2}; \tau) = (-y)^{-k} \text{Trace}_V (-y)^{J_{(0)}} q^{L_{(1)}},$$

where $q = e^{2\pi i \tau}, y = e^{2\pi iz}$. Let $\text{ch}\tilde{R}_k,h,l(z; \tau)$ be the character of the representation $M_{k,h,l}$.

Let $A_{i,n,l}(X)$ be the subspace of $\mathcal{H}_i(X, \Omega^c_X)$, which consist primitive vectors $v$ with

$$\bar{L}_{(1)}v = hv; \quad \bar{L}_{(n)}v = 0, n > 1;$$

$$\bar{J}_{(0)}v = lv; \quad \bar{J}_{(n)}v = 0, n > 0;$$

$$\bar{Q}_{(n)}v = 0, n \geq 0; \quad \bar{G}_{(n)}v = 0, n > 1;$$

$$\bar{E}_{1(n)}v = 0, n \geq -1; \quad \bar{D}_{1(n)}v = 0, n > 1;$$

$$\bar{B}_{1(n)}v = 0, n > 1; \quad \bar{C}_{1(n)}v = 0, n \geq 0.$$ 

By Theorem 3.2,

$$\mathcal{H}_i(X, \Omega^c_X) = (\bigoplus_{l=0}^{k} M_{k,\frac{k}{4}+\frac{l}{2}} \otimes A_{0,k+l}(X)) \bigoplus (\bigoplus_{n=1}^{\infty} \bigoplus_{l=1}^{k} M_{k,n+\frac{k}{4}+\frac{l}{2}} \otimes A_{n,k+l}(X)).$$ 

Let $A_{i,n,l}(X) = \dim A_{i,n,l}(X)$.

$$\text{ch}\tilde{R}_{\mathcal{H}_i(X, \Omega^c_X)}(z; \tau) = \sum_{l=0}^{k} A_{0,l+k} \text{ch}\tilde{R}_{k,\frac{k}{4}+\frac{l}{2}}(z; \tau) + \sum_{n=1}^{\infty} \sum_{l=1}^{k} A_{i,n,l+k} \text{ch}\tilde{R}_{k,n+\frac{k}{4}+\frac{l}{2}}(z; \tau).$$
The complex elliptic genus. For a holomorphic vector bundle $E$ on $X$, its Euler characteristic is $\chi(X, E) = \sum_{i=0}^{N} (-1)^i H^i(X, E)$. The complex elliptic genus of $X$ is

$$\text{Ell}_X(z; \tau) = y^{-\dim X/2} \sum_{i=0}^{N} (-1)^i \text{Trace}_{H^i(X, \Omega^i_X)}(-y)^{J(0)} q^{L(1)}.$$ 

By [3] or by the long exact sequence (3.3), it is equal to the graded dimension of the cohomology of the chiral de Rham algebra of $X$, i.e.

$$\text{Ell}_X(z; \tau) = y^{-\dim X/2} \sum_{i=0}^{N} (-1)^i \text{Trace}_{H^i(X, \Omega^i_X)}(-y)^{J(0)} q^{L(1)}.$$ 

If $g$ is an automorphism of the holomorphic vector bundle $E$, let

$$\chi(g; X, E) = \sum_{i=0}^{N} (-1)^i \text{Trace}_{H^i(X, E)} g.$$ 

For an automorphism $g$ of $X$, by the long exact sequence (3.3), the equivariant elliptic genus of $X$ is

$$\text{Ell}_{X,g}(z; \tau) = \sum_{i=0}^{N} (-1)^i \text{Trace}_{H^i(X, \Omega^i_X)} g(-y)^{J(0)} q^{L(1)}.$$ 

It is equal to

$$\text{Ell}_{X,g}(z; \tau) = (-1)^k \sum_{i=0}^{N} (-1)^i \left( \sum_{l=0}^{k} \text{ch}_{k, l + \frac{1}{2}} R(z; \tau) \text{Trace}_{A_{0,l+k}(X)} g \right)$$

$$+ \sum_{n=1}^{\infty} \sum_{l=1}^{k} \text{ch}_{k, n + \frac{1}{2}} R(z; \tau) \text{Trace}_{A_{n,l+k}(X)} g.$$ 

A complex automorphism $g$ of the hyperKähler manifold $X$ is called symplectic if it preserve the holomorphic symplectic 2-form $\omega_1$, i.e. $g^* \omega_1 = \omega_1$. If $g$ is symplectic automorphism, from the definition of the eight generators of $V_1$, the elements of $V_1$ are $g$ invariant. So $g$ acts on $A_{n,l+k}(X)$. We have

**Theorem 3.3.** If $X$ is a hyperKähler manifold, $g$ is a symplectic automorphism of $X$,

$$\text{Ell}_{X,g}(z; \tau) = (-1)^k \sum_{i=0}^{N} (-1)^i \left( \sum_{l=0}^{k} \text{ch}_{k, l + \frac{1}{2}} R(z; \tau) \text{Trace}_{A_{0,l+k}(X)} g \right)$$

$$+ \sum_{n=1}^{\infty} \sum_{l=1}^{k} \text{ch}_{k, n + \frac{1}{2}} R(z; \tau) \text{Trace}_{A_{n,l+k}(X)} g.$$ 

In particular

$$\text{Ell}_{X}(z; \tau) = (-1)^k \sum_{i=0}^{N} (-1)^i \left( \sum_{l=0}^{k} A^{i}_{0,l+k} \text{ch}_{k, l + \frac{1}{2}} R(z; \tau) + \sum_{n=1}^{\infty} \sum_{l=1}^{k} A^{i}_{n,l+k} \text{ch}_{k, n + \frac{1}{2}} R(z; \tau) \right).$$
Proof. $H^i_k$ is a filtration of $H^i(X, \Omega^ch_X)$ and it is compatible with the grading of the conformal weight and fermionic number, so

$$\text{Trace}_{H^i(X, \Omega^ch_X)} g(-y)^{J(0)} q^{L(1)} = \text{Trace}_{H^i(X, \Omega^ch_X)} g(-y)^{J(0)} q^{L(1)}.$$  

By equation (3.5),

$$\text{Ell}_{X,g}(z; \tau) = y - \dim X \sum_{i=0}^{n} (-1)^i \text{Trace}_{H^i(X, \Omega^ch_X)} g(-y)^{J(0)} q^{L(1)}.$$  

Thus by equation (3.4) and the fact that elements of $V_1$ is $g$ invariant, we get equation (3.6). $\square$

4. RELATION TO MATHEIU MOONSHINE

In this section we explain the relation between the Mathieu Moonshine and chiral Hodge cohomologe of the K3 surfaces.

K3 surface. Let $X$ be a K3 surface, in [23][24], we showed that $H^0(X, \Omega^ch_X)$ is a simple $N = 4$ vertex algebra. So it is isomorphic to $M_{1,\frac{1}{4},\frac{1}{2}}$. By chiral Poincaré duality[20],

$$H^0(X, \Omega^ch_X) \cong H^2(X, \Omega^ch_X).$$

The elliptic genus of a K3 surface is $2\phi_{0,1},$ So

$$\text{char}^{R}_{H^1(X, \Omega^ch_X)}(z; \tau) = 2\phi_{0,1}(z; \tau) + 2ch^{R}_{1,\frac{1}{4},\frac{1}{2}}(z; \tau).$$

We also have $\dim H^0,1(X) = 0$ and $\dim H^1,1(X) = 20$. Obviously $H^{1,1}(X) = A_{0,1}(X)$. By (3.4),

$$H^1(X, \Omega^ch_X) = M_{1,\frac{1}{4},0} \otimes H^{1,1}(X) \oplus M_{1,n+\frac{1}{4},\frac{1}{2}} \otimes A_{n,2}(X).$$

By Theorem 3.3 we get equation (1.1), the decomposition of the elliptic genus of K3 surface,

$$\text{Ell}_X(z; \tau) = -2ch^{R}_{1,\frac{1}{4},\frac{1}{2}}(z; \tau) + 20ch^{R}_{1,\frac{1}{4},0}(z; \tau) + 2 \sum_{n=1}^{\infty} A_n \text{ch}^{R}_{1,n+\frac{1}{4},\frac{1}{2}}(z; \tau).$$

Here $A_n = \frac{1}{2} A_{n,2}(X)$. If $g$ is a symplectic automorphism of $X$, elements of $H^0(X, \Omega^ch_X)$ and $H^2(X, \Omega^ch_X)$ are $g$ invariant, so

(4.1)

$$\text{Ell}_{X,g}(z; \tau) = -2ch^{R}_{1,\frac{1}{4},\frac{1}{2}}(z; \tau) + ch^{R}_{1,\frac{1}{4},0}(z; \tau) \text{Trace}_{H^{1,1}(X)} g + \sum_{n=1}^{\infty} \text{Trace}_{A_{n,2}(X)} g ch^{R}_{1,n+\frac{1}{4},\frac{1}{2}}(z; \tau).$$
Relation to Mathieu Moonshine. Let
\[ A_X(q) = \sum_{n=1}^{\infty} A_{n,2}(X)q^{n-\frac{1}{2}}. \]

If \( H \) is a symplectic automorphism group of \( X \), then \( A_X(q) \) is a representation of \( H \). The finite symplectic automorphism group of a K3 surface has been classified by Mukai in [17],

**Theorem 4.1 (Mukai).** A finite group \( H \) acting symplectically on a K3 surface is isomorphic to a subgroup of \( M_{23} \) with at least five orbits on the regular permutation representation of the Mathieu group \( M_{24} \) on 24 elements.

If \( g \) is a symplectic automorphism \( g \) acting on a K3 surface \( X \), By Theorem [1.2] equation (1.3) and equation (4.1), we immediately get Theorem [1.3]

By Gannon’s theorem (Theorem 1.1), \( A_X(q) \) is a graded \( M_{24} \) module. One may ask whether it is possible to construct a concrete action of the Mathieu group \( M_{24} \) on \( A_X(q) \)?

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