Vector-valued general Dirichlet series

by

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Abstract. With early contributions due to, among others, Besicovitch, Bohr, Bohnenblust, Hardy, Hille, Riesz, Neder and Landau, the last 20 years show a substantial revival of systematic research on ordinary Dirichlet series $\sum a_n n^{-s}$, and more recently even on general Dirichlet series $\sum a_n e^{-\lambda n^s}$. This involves the intertwining of classical work with modern functional analysis, harmonic analysis, infinite-dimensional holomorphy and probability theory as well as analytic number theory. The main goal of this article is to start a systematic study of a variety of fundamental aspects of vector-valued general Dirichlet series $\sum a_n e^{-\lambda n^s}$, where the coefficients are in an arbitrary Banach space $X$.

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1. Introduction. Given a frequency $\lambda = (\lambda_n)$, i.e. a strictly increasing sequence of non-negative real numbers, a $\lambda$-Dirichlet series is a (formal) series of the form $D = \sum a_n e^{-\lambda_n s}$, where $s$ is a complex variable and the sequence $(a_n)$ (of Dirichlet coefficients) belongs to $\mathbb{C}$.

In contrast to the theory of general Dirichlet series $D = \sum a_n e^{-\lambda_n s}$, the theory of ordinary Dirichlet series $\sum a_n n^{-s}$ saw a remarkable renaissance within the last two decades which in particular led to the solution of some long-standing problems. One of many fruitful lines of research is the analysis of functional-analytic aspects of vector-valued ordinary Dirichlet series, so series $\sum a_n n^{-s}$ with coefficients $a_n$ in a given normed space $X$. The list [18], [19], [20], [21], [22], [24], [27], [32], [33] of recent articles documents this activity; let us also mention that some of the results proved in these articles are collected in the recent monograph [25, Chapter 26].

Motivated by this line of research the main goal of this article is to start a systematic study of a variety of aspects of vector-valued general Dirichlet series $D = \sum a_n e^{-\lambda_n s}$ with coefficients $a_n$ in a given Banach space $X$. Two different challenges combine in this setting: the behaviour of vector-valued general Dirichlet series depends not only on the structure of the frequency $\lambda$ but also on the geometric structure of the Banach space $X$.

Regarding the frequency, many important tools developed within the study of ordinary Dirichlet series (un)fortunately fail for general Dirichlet series. In other words, making the jump from the frequency $(\log n)$ to arbitrary frequencies $\lambda$ reveals challenging consequences. Much of the ordinary theory relies on ‘Bohr’s theorem’, a fact which in particular implies that for each ordinary Dirichlet series the abscissas of uniform convergence and boundedness coincide. However, for general Dirichlet series, the validity of Bohr’s theorem depends very much on the ‘structure’ of the frequency $\lambda$.

Further due to the fundamental theorem of arithmetic, each natural number $n$ has its prime number decomposition $n = p^\alpha$, where $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and $p = (2, 3, \ldots)$ stands for the sequence of primes, and so the frequency $(\log n)$ can be written as a linear combination of $(\log p_j)$ with natural coefficients. This allows us to translate Dirichlet series $\sum a_n n^{-s}$ into power series of infinitely many variables $\sum_\alpha a_{p^\alpha} z_1^{\alpha_1} z_2^{\alpha_2} \ldots$ in a natural way providing an intimate link between the theory of ordinary Dirichlet series and the theory of
holomorphic functions and polynomials on polydisks. Known as Bohr’s transform, this procedure enables powerful tools to enter the game. However, it is no longer entirely available for arbitrary frequencies.

Regarding vector-valued phenomena, whereas some scalar-valued results can be translated directly to the vector-valued setting, others depend on the geometric structure of the Banach space. Furthermore, some geometric restrictions vary depending on the frequency $\lambda$.

To illustrate all this, in the rest of this introduction we describe four aspects of ordinary as well as general Dirichlet series which are going to guide us in establishing a systematic theory of general Dirichlet series with coefficients in Banach spaces. Sections 3, 4 and 5 are devoted to each of these aspects—but in Section 2 we start with some more preliminaries followed by the definition of several important classes of general Dirichlet series.

**Aspect I.** Let $D_\infty(\lambda, X)$ denote the space of all $\lambda$-Dirichlet series $D = \sum a_ne^{-\lambda ns}$ with coefficients in $X$ which converge and define a bounded, and then necessarily holomorphic, function on the open right half-plane $\{\text{Re} > 0\}$ (endowed with the supremum norm on $\{\text{Re} > 0\}$).

The countable product $\mathbb{T}^\infty$ of the torus $\mathbb{T}$ forms a natural compact abelian group, where the Haar measure is given by the normalized Lebesgue measure. Recall the definition of the Hardy space $H_\infty(\mathbb{T}^\infty, X)$: it is the closed subspace of all $f \in L_\infty(\mathbb{T}^\infty, X)$ such that $\hat{f}(\alpha) = 0$ if $\alpha \in \mathbb{Z}_N \setminus \mathbb{N}_0(N)$. Using Bohr’s transform, these Banach spaces provide us with a notion of ‘Hardy space’ for ordinary Dirichlet series which we denote by $H_\infty((\log n), X)$ (see Section 2.4 for the precise definition); whenever $X = \mathbb{C}$ we write $H_\infty((\log n))$.

One of the celebrated results in the ordinary scalar theory is a result of Hedenmalm, Lindqvist and Seip [41] which shows

\[(1.1) \quad D_\infty((\log n)) = H_\infty((\log n)).\]

This equality can be proved via the intermediate space $H_\infty^+(\log n)$. To understand this, define $H_\infty^+(\log n, X)$ as the Banach space of all $X$-valued ordinary Dirichlet series $D = \sum a_n n^{-s}$ such that all the translates $D_\sigma := \sum a_n n^{-\sigma n^{-s}}$, $\sigma > 0$, form a uniformly bounded set of $H_\infty((\log n), X)$, equipped with the norm

\[\|D\|_\infty^+ := \sup_{\sigma > 0} \|D_\sigma\|_\infty.\]

As a particular case of ideas from [26] (elaborated in [25] Theorems 24.8 and 24.17) we have

\[(1.2) \quad H_\infty((\log n)) = H_\infty^+(\log n) = D_\infty((\log n)).\]

Both identities are proved separately in order to show (1.1), while the original
proof from [41] takes a detour through infinite-dimensional holomorphy using a result of Cole and Gamelin [23] (see also (1.6)).

As it turns out, studying vector-valued general Dirichlet series provides the right setting to shed some light on why the proof is done in two steps. Regarding vector-valued ordinary Dirichlet series, in [26] (see also [25, Theorem 24.8]) it is actually shown that

\[ H_{\infty}((\log n), X) = H_{\infty}^+((\log n), X) \]  

if and only if the Banach space \( X \) has the so-called analytic Radon–Nikodym property ARNP introduced in [16] (see also [25, Chapter 23] for the definition). On the other hand, by [26] (see also [25, Theorem 24.17])

\[ H_{\infty}^+((\log n), X) = D_{\infty}((\log n), X), \]

regardless of the geometry of \( X \).

As shown in Section 2.4, the definitions of the spaces \( H_{\infty}((\log n), X) \) and \( H_{\infty}^+((\log n), X) \) can be extended to arbitrary frequencies \( \lambda \), yielding \( H_{\infty}(\lambda, X) \) and \( H_{\infty}^+(\lambda, X) \). In fact, a standard weak compactness argument shows that for scalar generalized Dirichlet series we always have

\[ H_{\infty}(\lambda) = H_{\infty}^+(\lambda) \]

(see also Proposition 2.5), whereas by [29, Theorem 5.1] the equality

\[ H_{\infty}^+(\lambda) = D_{\infty}(\lambda) \]

holds if and only if \( \lambda \) satisfies Bohr’s theorem. Two questions arise.

(A) For which frequencies \( \lambda \) and Banach spaces \( X \) does \( H_{\infty}(\lambda, X) = H_{\infty}^+(\lambda, X) \) hold?

(B) For which frequencies \( \lambda \) and Banach spaces \( X \) does \( H_{\infty}^+(\lambda, X) = D_{\infty}(\lambda, X) \) hold?

The answer to these questions is at the core of Sections 3 and 4 respectively.

Aspect II. As mentioned before, ordinary Dirichlet series can be identified with holomorphic functions on polydiscs. More precisely, denote by \( H_{\infty}(B_{c_0}, X) \) the Banach space of all holomorphic (Fréchet differentiable) functions \( g : B_{c_0} \to X \) endowed with the sup norm. There is a unique isometric linear bijection

\[ D_{\infty}((\log n), X) = H_{\infty}(B_{c_0}, X) \]

which preserves the Dirichlet and monomial coefficients (see [23] for the scalar case, [26] for the vector-valued case, and also [25, Theorem 24.17]).
Regarding $1 \leq p < \infty$, define the Banach space $H_p(\ell_2 \cap B_{c_0}, X)$ of all holomorphic functions $g : \ell_2 \cap B_{c_0} \to X$ for which

$$\|g\|_{H_p(\ell_2 \cap B_{c_0})} = \sup_{n \in \mathbb{N}} \sup_{0 < r < 1} \left( \int_{\mathbb{T}} \|g(rw_1, \ldots, rw_n, 0, 0, \ldots)\|_X^p \, d(w_1, \ldots, w_n) \right)^{1/p} < \infty.$$ 

In [5] Bayart developed an $H_p$-theory of ordinary Dirichlet series for $1 \leq p \leq \infty$. This was later extended to $\lambda$-Dirichlet series in [28] through Fourier analysis on groups. Providing a vector-valued definition is then straightforward and gives rise to the spaces $\mathcal{H}_p(\lambda, X)$ and $\mathcal{H}_p^+(\lambda, X)$, which are properly defined in Section 2.

In the scalar case $X = \mathbb{C}$, it is proved in [7] that for $1 \leq p < \infty$ there is a unique isometric equality

$$(1.7) \quad \mathcal{H}_p((\log n)) = H_p(\ell_2 \cap B_{c_0})$$

identifying Fourier and monomial coefficients (see also [25, Theorem 13.2]). However, if we consider the vector-valued case and general frequencies, then such identities are no longer available. The spaces $\mathcal{H}_p^+(\lambda, X)$ partly compensate for this loss.

The main goal of Section 3 is to study for which frequencies $\lambda$ and Banach spaces $X$ we have

$$(1.8) \quad \mathcal{H}_p(\lambda, X) = \mathcal{H}_p^+(\lambda, X),$$

which generalizes question (A) to any $1 \leq p \leq \infty$. As in the case $p = \infty$ in (1.3), for vector-valued ordinary Dirichlet series we have

$$(1.9) \quad \mathcal{H}_p((\log n), X) = \mathcal{H}_p^+(((\log n), X)$$

if and only if the Banach space $X$ has ARNP (see [26] and [25, Theorem 24.8]). However, in the general framework the validity of (1.8) depends not only on the geometry of the Banach space but also on the frequency $\lambda$. More precisely, as shown in Theorem 3.6 and Remark 3.7, for any frequency $\lambda$ we have

$X$ has ARNP $\implies \mathcal{H}_p(\lambda, X) = \mathcal{H}_p^+(\lambda, X)$ for all $1 \leq p \leq \infty \implies c_0 \not\subset X.$

Furthermore, both extremes may characterize the coincidence of the spaces for certain frequencies.

**Aspect III.** Suppose that $D = \sum a_ne^{-\lambda_ns}$ (scalar coefficients) converges somewhere and that its limit function extends to a bounded and holomorphic function $f$ on $[\text{Re} > 0]$. Then a prominent problem from the beginning of the 20th century was to determine the class of $\lambda$’s for which under this assumption all $\lambda$-Dirichlet series converge uniformly on $[\text{Re} > \varepsilon]$ for every $\varepsilon > 0$. 
We say that $\lambda$ satisfies Bohr’s theorem if the answer to the preceding problem is affirmative. In Section 4.2 we are going to repeat concrete conditions on $\lambda$ under which Bohr’s theorem holds, but we also give examples which show that not every frequency has this property.

Anyway, this notion seems to be at the heart of every serious study of the space $D_{\infty}(\lambda)$—an opinion which is supported by the equivalence of the following three statements (a result from [29, Theorem 5.1]):

- $\lambda$ satisfies Bohr’s theorem.
- $D_{\infty}(\lambda)$ is complete.
- $D_{\infty}(\lambda) = H_{\infty}(\lambda)$.

The second statement allows one to apply (principles of) functional analysis to $D_{\infty}(\lambda)$, whereas the third statement links its study intimately with Fourier analysis. The proof seems highly non-trivial since it needs, among other tools, a variant of the Carleson theorem (on pointwise convergence of Fourier series in $H_{2}(T)$) for the Hilbert space $H_{2}(\lambda)$ of $\lambda$-Dirichlet series from [29, Theorem 2.1].

In Section 4 the main aim is to find reasonable extensions of the above equivalences within the vector-valued setting. As it turns out, this phenomenon depends only on the frequency once we isolate (1.5) from (1.4). In Theorem 4.7 it is shown among other equivalences that

$$D_{\infty}(\lambda, X) = H_{\infty}^{+}(\lambda, X)$$

if and only if $\lambda$ satisfies Bohr’s theorem, which answers question (B).

In particular we relate Dirichlet series in $D_{\infty}(\lambda, X)$ (and their relatives) to the theory of almost periodic functions on half-planes.

**Aspect IV.** Many solutions of problems which have appeared in the modern theory of ordinary/general Dirichlet series come from appropriate inequalities adapted to these problems—in particular, maximal inequalities (as in the preceding topic). One of the most prominent examples is the quantitative version of Bohr’s theorem which states that for every Dirichlet series $D \in D_{\infty}((\log n))$ and every $N \geq 2$ we have

$$\left\| \sum_{n=1}^{N} a_n n^{-s} \right\|_{\infty} \leq C \log(N) \|D\|_{\infty},$$

where $C > 0$ is a universal constant. Recently, various improvements of different technical versions of this inequality have been proved. In [58], a version of (1.10) is proved which applies to general Dirichlet series under no condition on the frequency. For certain conditions on $\lambda$ isolated by Bohr and Landau this leads to Bohr type estimates like (1.10). More generally, Carleson–Hunt-like maximal inequalities for (Riesz) summation of general Dirichlet series within the scale of $H_{p}(\lambda)$-spaces have been studied in [29].
and [30]. Here it is essential to distinguish carefully between the cases \( p = 1 \) and \( 1 < p \leq \infty \).

In Section 5 we deal with our third goal, of extending a couple of fundamental maximal inequalities for general scalar Dirichlet series to our vector-valued setting. In some situations this just means to apply a Hahn–Banach argument, but there are situations where this is in fact a delicate problem since the underlying geometry of \( X \) becomes essential.

2. Classes of general Dirichlet series. Recall from the above the notion of a general Dirichlet series \( \sum a_n e^{-\lambda_n s} \) with coefficients in a Banach space \( X \). Finite sums of the form \( D = \sum_{n=1}^{N} a_n e^{-\lambda_n s} \) are called Dirichlet polynomials, and we denote by \( \mathcal{D}(\lambda, X) \) the space of all \( X \)-valued \( \lambda \)-Dirichlet series; we write \( \mathcal{D}(\lambda) \) whenever \( X = \mathbb{C} \). It is well-known that if \( D = \sum a_n e^{-\lambda_n s} \) converges at \( s_0 \in \mathbb{C} \), then it also converges for all \( s \in \mathbb{C} \) with \( \text{Re} s > \text{Re} s_0 \), and the limit function \( f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \) defines a holomorphic function on \( [\text{Re} s > \sigma_c(D)] \) with values in \( X \), where

\[
\sigma_c(D) = \inf \{ \sigma \in \mathbb{R} \mid D \text{ converges on } [\text{Re} s > \sigma] \}
\]

determines the so-called abscissa of convergence. The abscissas of absolute and uniform convergence \( \sigma_a(D) \) and \( \sigma_u(D) \) are defined accordingly. These abscissas define the largest half-plane where the Dirichlet series converges in each sense, and the limit function \( f : [\text{Re} s > \sigma_c(D)] \to X \) is holomorphic. However, for our purposes we need summation methods for Dirichlet series \( \sum a_n e^{-\lambda_n s} \) which are more general than just taking limits of partial sums \( \sum_{n=1}^{N} a_n e^{-\lambda_n s} \).

2.1. Riesz means. Fixing a frequency \( \lambda \), some \( k \geq 0 \), and \( D = \sum a_n e^{-\lambda_n s} \in \mathcal{D}(\lambda, X) \), the (first) \( (\lambda, k) \)-Riesz mean of \( D \) of length \( x > 0 \) is the Dirichlet polynomial

\[
R_{x}^{\lambda,k}(D) := \sum_{\lambda_n < x} a_n (1 - \lambda_n/x)^k e^{-\lambda_n s}.
\]

We define three abscissas of \( D \):

\[
\sigma_c^{\lambda,k}(D) = \inf \{ \sigma \in \mathbb{R} \mid (R_{x}^{\lambda,k}(D))_{x \geq 0} \text{ converges on } [\text{Re} s > \sigma] \},
\]

\[
\sigma_a^{\lambda,k}(D) = \inf \{ \sigma \in \mathbb{R} \mid (R_{x}^{\lambda,k}(D))_{x \geq 0} \text{ converges absolutely on } [\text{Re} s > \sigma] \},
\]

\[
\sigma_u^{\lambda,k}(D) = \inf \{ \sigma \in \mathbb{R} \mid (R_{x}^{\lambda,k}(D))_{x \geq 0} \text{ converges uniformly on } [\text{Re} s > \sigma] \}.
\]

By definition \( \sigma_c^{\lambda,k}(D) \leq \sigma_u^{\lambda,k}(D) \leq \sigma_a^{\lambda,k}(D) \), and in general all these abscissas are distinct. For historical reasons we call the following formulas Bohr–Cahen formulas for Riesz summation (see [40] and also [30], [58]).
Proposition 2.1. Let $D \in \mathcal{D}(\lambda, X)$. Then

\[
\begin{align*}
(c) & \quad \sigma_c^{\lambda,k}(D) \leq \limsup_{x \to \infty} \frac{\log(\| \sum_{\lambda n < x} a_n(1 - \lambda n/x)^k \|_X)}{x}, \\
(a) & \quad \sigma_a^{\lambda,k}(D) \leq \limsup_{x \to \infty} \frac{\log(\sum_{\lambda n < x} \|a_n\|_X (1 - \lambda n/x)^k)}{x}, \\
(u) & \quad \sigma_u^{\lambda,k}(D) \leq \limsup_{x \to \infty} \frac{\log(\sup_{t \in \mathbb{R}} \| R_x^{\lambda,k}(D)(it) \|_X)}{x},
\end{align*}
\]

where in each case equality holds if the left hand side is non-negative.

It is evident that the Hahn–Banach theorem plays a fundamental role when extending results on scalar-valued $\lambda$-Dirichlet series to vector-valued $\lambda$-Dirichlet series. For the following proposition, for a given vector-valued $\lambda$-Dirichlet series $D = \sum a_n e^{-\lambda n}s \in \mathcal{D}(\lambda, X)$ and $x^* \in X^*$, we define the scalar $\lambda$-Dirichlet series

\[x^* \circ D := \sum x^*(a_n)e^{-\lambda n}s \in \mathcal{D}(\lambda).\]

Proposition 2.2. Let $D \in \mathcal{D}(\lambda, X)$, $k \geq 0$ and let $\iota = c, u$. Then

\[\sigma_\iota^{\lambda,k}(D) = \sup_{x^* \in X^*} \sigma_\iota^{\lambda,k}(x^* \circ D).\]

Before we prove this, let us mention that Proposition 2.2 does not hold for the abscissa of absolute convergence. For instance for $X = \ell^1$ and $a_n = e_n$ (the $n$th unit vector), the ordinary Dirichlet series $D = \sum e_n e^{-\lambda n}s$ satisfies $\sigma_a(D) = 1$ and $\sigma_u(x^*(D)) = 0$ for all $x^* \in X^*$ (this example is taken from [13, Example 3.1]).

Proof of Proposition 2.2. The inequality $\geq$ is obvious. For the other inequality we assume that

\[\sup_{x^* \in X^*} \sigma_\iota^{\lambda,k}(x^* \circ D) < \infty\]

(otherwise the claim is trivial), and let $\sigma > \sup_{x^* \in X^*} \sigma_\iota^{\lambda,k}(x^* \circ D)$. If $\iota = c$, then the net $(R_x^{\lambda,k}(D)(\sigma))_{x \geq 0}$ is weakly bounded in $X$, and so by Mackey’s theorem norm bounded. Now Proposition 2.1(c) implies that $(R_x^{\lambda,k}(D))_{x \geq 0}$ converges on $[\text{Re} > \sigma]$ as $x \to \infty$ and so $\sigma_c^{\lambda,k}(D) \leq \sigma$. If $\iota = u$, then the set

\[A(\sigma) := \left\{ \sum_{\lambda n < x} a_n e^{-\lambda n}\sigma(1 - \lambda n/x)^k e^{-\lambda n}it \mid t \in \mathbb{R}, x \in \mathbb{N} \right\}\]

is weakly bounded in $X$. Again by Mackey’s theorem, $A(\sigma)$ is norm bounded in $X$, and the Bohr–Cahen formula for $\sigma_u^{\lambda,k}$ (Proposition 2.1(u)) implies that $\sigma_u^{\lambda,k}(D) \leq \sigma$. $\blacksquare$
2.2. The spaces $D_\infty(\lambda, X)$. Recall from the introduction the definition of $D_\infty(\lambda, X)$ and $D_\infty(\lambda, \mathbb{C}) = D_\infty(\lambda)$. We endow $D_\infty(\lambda, X)$ with the supremum norm on $[\text{Re} > 0]$. In order to see that this is a norm, we need to show that every $D \in D_\infty(\lambda, X)$ with limit function vanishing on $[\text{Re} > 0]$ is in fact zero, i.e. all its coefficients are zero. But a standard application of the Hahn–Banach theorem to [58, Corollary 3.9] shows that for every $D = \sum a_n e^{-\lambda_n s} \in D_\infty(\lambda, X)$ with limit function $f : [\text{Re} > 0] \to \mathbb{C}$ and for all $n$ and $\sigma > 0$,

$$a_n = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(\sigma + it)e^{(\sigma + it)\lambda_n} \, dt.$$  

In particular,

$$\sup_{n \in \mathbb{N}} \|a_n\|_X \leq \|D\|_\infty.$$  

But $(D_\infty(\lambda, X), \|\cdot\|_\infty)$ may not be complete (see [58, Theorem 5.2] for examples); its completeness will be carefully studied in Section 4.3.

**Theorem 2.3.** Let $\lambda$ be an arbitrary frequency, and $D = \sum a_n e^{-\lambda_n s} \in D_\infty(\lambda, X)$ with limit function $f$. Then for every $k > 0$ and $\varepsilon > 0$ we have

$$f = \lim_{x \to \infty} \sum_{\lambda_n < x} a_n (1 - \lambda_n/x)^k e^{-\lambda_n s} \text{ uniformly on } [\text{Re} > \varepsilon].$$

**Proof.** By [58, Proposition 3.4] we know that $\sigma_u^{\lambda,k}(x^* \circ D) \leq 0$ for every $x^* \in X^*$, so Proposition 2.2 yields $\sigma_u^{\lambda,k}(D) \leq 0$. ■

**Theorem 2.4.** Let $D = \sum a_n e^{-\lambda_n s} \in D(\lambda, X)$. Then the following are equivalent:

(i) $D \in D_\infty(\lambda, X)$.
(ii) $x^* \circ D \in D_\infty(\lambda)$ for all $x^* \in X^*$ and $\sup_{\|x^*\| = 1} \|x^* \circ D\|_\infty < \infty$.

Moreover, in this case $\|D\|_\infty = \sup_{\|x^*\| = 1} \|x^* \circ D\|_\infty$.

**Proof.** Obviously, (i) implies (ii). Conversely, for every $x^* \in X^*$ by the first assumption of (ii) we have $\sigma_u^{\lambda,0}(x^* \circ D) \leq 0$, and hence $\sigma_u^{\lambda,0}(D) \leq 0$ by Proposition 2.2. This implies that $D$ converges on $[\text{Re} > 0]$. By the second assumption of (ii) the limit function of $D$ is bounded on $[\text{Re} > 0]$. ■

2.3. The spaces $H_\infty^\lambda([\text{Re} > 0], X)$. In the following we define almost periodicity for functions on the real line or half-planes with values in a Banach spaces; all our definitions are straightforward extensions of the well-known definitions for complex-valued functions (see e.g. [9]).

Given a Banach space $X$, a continuous function $g : \mathbb{R} \to X$ is said to be uniformly almost periodic (see [9] pp. 1–2) for $X = \mathbb{C}$ if for every $\varepsilon > 0$
there is a number \( l > 0 \) such that for all intervals \( I \subset \mathbb{R} \) with \( |I| = l \) there is \( \tau \in I \) such that
\[
\sup_{x \in \mathbb{R}} \|g(x + \tau) - g(x)\|_X < \varepsilon.
\]
Let now \( F : [\operatorname{Re} > 0] \to X \) be a bounded holomorphic function such that for all \( \sigma > 0 \) the restriction \( t \mapsto F(\sigma + it) \) to the vertical line \([\operatorname{Re} = \sigma]\) is uniformly almost periodic. For fixed \( x \in \mathbb{R} \) and \( \sigma > 0 \), the limit
\[
a_x(F) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} F(\sigma + it) e^{(\sigma + it)x} \, dt
\]
exists and is independent of the choice of \( \sigma \); we call it the \( x \)th Bohr coefficient of \( F \). All this follows from the scalar case discussed in [9, p. 147]) and a straightforward application of the Hahn–Banach theorem. For all \( \sigma > 0 \) and \( x \in \mathbb{R} \) we have
\[
|a_x(F)| \leq e^{\sigma x} \|F\|_{\infty},
\]
and hence \( |a_x(F)| \leq \|F\|_{\infty} \) for \( x \in \mathbb{R} \) and \( a_x(F) = 0 \) for \( x < 0 \). Moreover, at most countably many Bohr coefficients are non-zero, and \( F \) vanishes whenever its Bohr coefficients vanish (see [9, pp. 148, 18]).

Note that typical examples of such functions are finite polynomials \( F(z) := \sum_{n=1}^{N} a_n e^{-\lambda_n z} \) with coefficients \( 0 \neq a_n \in X \) and frequencies \( \lambda_n \geq 0 \), and then the \( a_n \) are precisely the (non-zero) Bohr coefficients of \( F \).

The following definition will be important. Given a frequency \( \lambda \) and a Banach space \( X \), the space \( \mathcal{H}_\infty^\lambda([\operatorname{Re} > 0], X) \) consists of all bounded holomorphic functions \( F : [\operatorname{Re} > 0] \to X \) which are uniformly almost periodic on all vertical lines \([\operatorname{Re} = \sigma] \), \( \sigma > 0 \), and for which the Bohr coefficients \( a_x(F) \) vanish whenever \( x \notin \{\lambda_n \mid n \in \mathbb{N}\} \).

When endowed with the norm \( \|F\|_{\infty} := \sup_{z \in [\operatorname{Re} > 0]} \|F(z)\|_X \), the linear space \( \mathcal{H}_\infty^\lambda([\operatorname{Re} > 0], X) \) becomes a Banach space, and we call this class of spaces Besicovitch spaces.

2.4. The spaces \( \mathcal{H}_p(\lambda, X) \). From [28] (see also [31]) we recall the definition of and some basic facts about Dirichlet groups and we refer to [57] for background on Fourier analysis on groups. Let \( G \) be a compact abelian group and \( \beta : (\mathbb{R}, +) \to G \) a homomorphism of groups. The pair \((G, \beta)\) is called a Dirichlet group if \( \beta \) is continuous and has dense range. In this case the dual map \( \hat{\beta} : \hat{G} \to \mathbb{R} \) is injective, where we identify \( \mathbb{R} = (\hat{\mathbb{R}}, +) \) (note that we do not assume \( \beta \) to be injective).

Consequently, the characters \( e^{-ix \cdot} : \mathbb{R} \to \mathbb{T}, \ x \in \hat{\beta}(\hat{G}) \), are precisely those which define a unique \( h_x \in \hat{G} \) such that \( h_x \circ \beta = e^{-ix \cdot} \). In particular,
\[
\hat{G} = \{h_x \mid x \in \hat{\beta}(\hat{G})\}.
\]
From [28, Section 3.1] we know that every $L_1(\mathbb{R})$-function may be interpreted as a bounded regular Borel measure on $G$. In particular, for every $u > 0$ the Poisson kernel

$$P_u(t) := \frac{1}{\pi} \frac{u}{u^2 + t^2}, \quad t \in \mathbb{R},$$

defines a measure $p_u$ on $G$, which we call the Poisson measure on $G$. We have $\|p_u\| \leq \|P_u\|_{L_1(\mathbb{R})} = 1$ and

$$\widehat{p}_u(h_x) = \widehat{P}_u(x) = e^{-u|x|} \quad \text{for all } u > 0 \text{ and } x \in \widehat{\beta}(\widehat{G}).$$

Now, given a frequency $\lambda$, we call a Dirichlet group $(G, \beta)$ a $\lambda$-Dirichlet group whenever $\lambda \in \widehat{\beta}(\widehat{G})$, or equivalently for every $e^{-i\lambda_n} \in (\mathbb{R}, +)$ there is (a unique) $h_{\lambda_n} \in \widehat{G}$ with $h_{\lambda_n} \circ \beta = e^{-i\lambda_n}$. 

Note that for every $\lambda$ there exists a $\lambda$-Dirichlet group $(G, \beta)$ (not unique). To see a very first example, take the Bohr compactification $\mathbb{R}$ together with the mapping

$$\beta_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto [x \mapsto e^{-itx}].$$

Then $\beta_{\mathbb{R}}$ is continuous and has dense range (see e.g. [55, Theorem 1.5.4, p. 24] or [28, Example 3.6]), and so $(\mathbb{R}, \beta_{\mathbb{R}})$ is a $\lambda$-Dirichlet group for all $\lambda$’s. We refer to [28] for more ‘universal’ examples of Dirichlet groups. Looking at the frequency $\lambda = (n) = (0, 1, 2, \ldots)$, the group $G = \mathbb{T}$ with

$$\beta_\mathbb{T} : \mathbb{R} \to \mathbb{T}, \quad \beta_\mathbb{T}(t) = e^{-it},$$

is a $\lambda$-Dirichlet group, and the so-called Kronecker flow

$$\beta_{\mathbb{T}^\infty} : \mathbb{R} \to \mathbb{T}^\infty, \quad t \mapsto p^{-it} = (2^{-it}, 3^{-it}, 5^{-it}, \ldots),$$

turns the infinite-dimensional torus $\mathbb{T}^\infty$ into a $\lambda$-Dirichlet group for $\lambda = (\log n)$. We note that, identifying $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T}^\infty = \mathbb{Z}^{(N)}$ (all finite sequences of integers), in the first case $h_n(z) = z^n$ for $z \in \mathbb{T}, n \in \mathbb{Z}$, and in the second case $h_{\sum \alpha_j \log p_j}(z) = z^\alpha$ for $z \in \mathbb{T}^\infty, \alpha \in \mathbb{Z}^{(N)}$.

We finish with another crucial tool given by the following fact from [30, Lemma 3.10]: For any $\lambda$-Dirichlet group $(G, \beta)$ and $k > 0$ there is a constant $C = C(k) > 0$ such that for all $x > 0$ there is a measure $\mu_x \in M(G)$ which satisfies $\|\mu_x\| \leq C$ and for all $n$,

$$\widehat{\mu_x}(h_{\lambda_n}) = \begin{cases} (1 - \lambda_n/x)^k, & \lambda_n < x, \\ 0, & \lambda_n \geq x. \end{cases} \tag{2.1}$$

In the following we are going to extend several results from [26] from the ordinary to the general case. But many arguments in the ordinary case rely on the good properties of the Poisson kernel

$$p_N : \mathbb{D}^N \times \mathbb{T}^N \to \mathbb{C}, \quad p_N(z, w) = \sum_{\alpha \in \mathbb{Z}^N} w^{-\alpha} |z|^\alpha \left( \frac{z}{|z|} \right)^\alpha. \tag{2.2}$$

In our much more general setting of general Dirichlet series and Dirichlet
groups, this fundamental tool is not available—but in many cases the measures from (2.1) will be an appropriate substitute.

Let us turn to Hardy spaces of $\lambda$-Dirichlet series. Fix some $\lambda$-Dirichlet group $(G, \beta)$, a Banach space $X$, and $1 \leq p \leq \infty$. We denote by $H^\lambda_p(G, X)$ the Hardy space of all functions $f \in L^p(G, X)$ (the Banach space of all $X$-valued $p$-Bochner integrable functions on $G$) having Fourier transform
\[
\hat{f}(\gamma) = \int_G f(\omega)\overline{\gamma}(\omega)\,dm(\omega), \quad \gamma \in \hat{G},
\]
supported on $\{h_{\lambda n} \mid n \in \mathbb{N}\} \subset \hat{G}$. Being a closed subspace of $L^p(G, X)$, it is clearly a Banach space.

With the spaces $H^\lambda_p(G, X)$ at hand we define $H^\lambda_p$’s of $\lambda$-Dirichlet series in a natural way. Let $H^\lambda_p(\lambda, X)$ be the class of all $\lambda$-Dirichlet series $D = \sum a_n e^{-\lambda_n s}$ for which there is some $f \in H^\lambda_p(G, X)$ such that $a_n = \hat{f}(h_{\lambda n})$ for all $n$. In this case the function $f$ is unique, and with the norm
\[
\|D\|_p := \|f\|_p
\]
the linear space $H^\lambda_p(\lambda, X)$ is obviously a Banach space. So (by definition) the Bohr map
\[
(2.3) \quad \mathcal{B}: H^\lambda_p(G, X) \to H^\lambda_p(\lambda, X), \quad f \mapsto \sum \hat{f}(h_{\lambda n})e^{-\lambda_n s},
\]
is an onto isometry. A fundamental fact (extend the proof of [28, Theorem 3.24] word for word to the vector-valued case) is that the definition of $H^\lambda_p(\lambda, X)$ is independent of the chosen $\lambda$-Dirichlet group $(G, \beta)$.

Our two basic examples of frequencies, $\lambda = (n)$ and $\lambda = (\log n)$, lead to well-known examples:
\[
(2.4) \quad H_p(\mathbb{T}, X) = H_p^{(n)}(\mathbb{T}, X) \quad \text{and} \quad H_p(\mathbb{T}^\infty, X) = H_p^{(\log n)}(\mathbb{T}^\infty, X).
\]
In particular, $f \in H_p^{(n)}(\mathbb{T}, X)$ if and only if $f \in L_p(\mathbb{T}, X)$ and $\hat{f}(n) = 0$ for any $n \in \mathbb{Z}$ with $n < 0$, and $f \in H_p^{(\log n)}(\mathbb{T}^\infty, X)$ if and only if $f \in L_p(\mathbb{T}^\infty, X)$ and $\hat{f}(\alpha) = 0$ for any finite sequence $\alpha = (\alpha_k)$ of integers with $\alpha_k < 0$ for some $k$ (where as usual $\hat{f}(\alpha) := \hat{f}(h_{\log p^k})$). Consequently, if we turn to ordinary Dirichlet series, then the Banach spaces
\[
H_p(X) = H_p((\log n), X)
\]
are precisely Bayart’s Hardy spaces of ordinary $X$-valued Dirichlet series from [5] (see also [25] and [55]).

We use ideas from the proof of [28, Theorem 3.26] to give the following internal description of $H_p(\lambda, X)$, $1 \leq p < \infty$: The limit
\[
(2.5) \quad \|D\|_p := \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^T \left\| \sum_{n=1}^N a_n e^{-\lambda_n it} \right\|^p dt \right)^{1/p}
\]
exists, since we integrate an almost periodic function on \( \mathbb{R} \), and it defines a norm on the space \( \text{Pol}(\lambda, X) \) of all \( X \)-valued \( \lambda \)-Dirichlet polynomials. Then \( \mathcal{H}_p(\lambda, X) \) is the completion of \( (\text{Pol}(\lambda, X), \| \cdot \|_p) \); here the density of \( \text{Pol}(\lambda, X) \) in \( \mathcal{H}_p(\lambda, X) \) follows by an analysis of the arguments in the scalar case, given in [28, Proposition 3.14]. For completeness we sketch the proof: Fix some Dirichlet group \((G, \beta)\) and \( A \subset \hat{G} \). Denote by \( C_A(G, X) \) the Banach space of all continuous functions \( f : G \to X \) with Fourier transform supported on \( A \), and by \( \text{Pol}_A(G, X) \) the set of all \( X \)-valued polynomials of the form \( \sum_{\gamma \in A} x_\gamma \gamma^\ast \). Once we prove that \( \text{Pol}_A(G, X) \) is dense in \( C_A(G, X) \) with respect to the sup norm, the claim follows as in [28, Proposition 3.14] with \( A = \{ h_{\lambda_n} \mid n \in \mathbb{N} \} \). Arguing towards a contradiction, assume that there exists \( g \in C_A(G, X) \setminus \text{Pol}_A(G, X) \). Then by the Hahn–Banach theorem there is \( \varphi \in C(G, X)^\ast \) with \( \varphi(g) \neq 0 \) that vanishes on \( \text{Pol}_A(G, X) \). A direct calculation shows that the continuous function \( \varphi \ast g(x) := \varphi(g(x + \cdot)) : G \to X \) satisfies \( \hat{\varphi} \ast \hat{g}(\gamma) = \varphi(\hat{g}(\gamma)\gamma) \) for every \( \gamma \in \hat{G} \). Then we easily deduce that \( \hat{\varphi} \ast \hat{g} = 0 \), and consequently \( \varphi \ast g = 0 \) (here again the Hahn–Banach theorem is needed). In particular, \( 0 = \varphi \ast g(0) = \varphi(g) \neq 0 \), a contradiction.

2.5. The spaces \( \mathcal{H}^+_p(\lambda, X) \). We already remarked in the introduction that when we pass from ordinary Dirichlet series to general ones, we pay the price of losing holomorphic functions in infinite-dimensional polydiscs. Here we define a class of spaces that can compensate for this loss.

Recall that there is an isometric coefficient-preserving equality

\[ H_p(\mathbb{T}, X) = H_p(\mathbb{D}, X) \]

if and only if \( X \) has ARNP (see e.g. [25, Theorem 23.6]). This says that \( X \) has ARNP if and only if \( f \in H_p(\mathbb{T}, X) \) if and only if \( f \ast p(r, \cdot) \in H_p(\mathbb{T}, X) \) for all \( 0 < r < 1 \), and in this case

\[ \| f \|_p = \sup_{0 < r < 1} \| f \ast p(r, \cdot) \|_p \]

(recall from (2.2) the definition of the Poisson kernel \( p = p_1 \)). If we translate this in terms of scalar \( (n) \)-Dirichlet series, then it reads: \( D \in H_p((n)) \) if and only if all translates \( D_\sigma = \sum a_n e^{-n\sigma} e^{-ns} \) are in \( H_p((n)) \), \( 0 < \sigma < \infty \), and in this case

\[ \| D \|_p = \sup_{0 < \sigma < \infty} \| D_\sigma \|_p. \]

Let now \( \lambda \) be an arbitrary frequency, \( X \) a Banach space, and \( 1 \leq p \leq \infty \). Inspired by the ‘ordinary definition’ from [26] (see also [25, Chapter 24.3]), we define \( \mathcal{H}_p^+(\lambda, X) \) as the space of all (formal) \( X \)-valued \( \lambda \)-Dirichlet series \( D \) such that the translate \( D_\sigma = \sum a_n e^{-\sigma \lambda_n} e^{-\lambda_n s} \) is in \( \mathcal{H}_p(\lambda, X) \) for all \( \sigma > 0 \) and \( \| D \|_p^+ := \sup_{\sigma > 0} \| D_\sigma \|_p < \infty \). It is worth mentioning that \( \| D_\sigma \|_p / \| D \|_p^+ \) as \( \sigma \to 0 \), whenever \( 1 \leq p < \infty \). The proof is analogous to that of [26].
Proposition 2.3 and is sketched in Lemma 3.8. We will see that this space takes over the role holomorphic functions in infinite dimensions play in the ordinary world.

**Proposition 2.5.** For every $1 \leq p \leq \infty$ and frequency $\lambda$ the isometric equality $\mathcal{H}_p(\lambda) = \mathcal{H}_p^+(\lambda)$ holds.

*Proof.* Although parts of this result are proved in [28], we prefer to sketch the argument. For $1 \leq p < \infty$ the embedding $\mathcal{H}_p(\lambda) \subset \mathcal{H}_p^+(\lambda)$ is isometric—this is proved in [28, Theorem 4.7]. The proof extends to $p = \infty$ since for each $D \in \mathcal{H}_\infty(\lambda)$, we have $\|D\|_\infty^+ = \lim_{p \to \infty} \|D\|_p^+$ (alternatively, see the proof of the more general result of Corollary 3.4). Conversely, for $1 < p \leq \infty$ the inclusion $\mathcal{H}_p^+(\lambda) \subset \mathcal{H}_p(\lambda)$ follows from a standard weak-compactness argument in $L_q(G)^*$, where $(G, \beta)$ is an appropriate $\lambda$-Dirichlet group and $1/p + 1/q = 1$ (compare the proof for the ordinary case given in [25, proof of Theorem 11.21]). The case $p = 1$ is proved in [28, Theorem 4.7]; the proof uses [28, Lemma 4.9] which has a further assumption on $\lambda$, but an easy analysis shows that this assumption is in fact not needed. Alternatively, see the proof of the more general result of Theorem 3.6. \[\] We finish with a characterization of the Dirichlet series $\mathcal{H}_p^+(\lambda, X)$ in terms of Riesz means, an important tool in many of the forthcoming proofs.

**Proposition 2.6.** Let $k > 0$, $1 \leq p \leq \infty$, and $D = \sum a_n e^{-\lambda_n s} \in \mathcal{D}(\lambda, X)$. Then $D \in \mathcal{H}_p^+(\lambda, X)$ if and only if

$$M_{k,p}(D) := \sup_{x>0} \left\| \sum_{\lambda_n < x} a_n (1 - \lambda_n / x)^k e^{-\lambda_n s} \right\|_p < \infty,$$

and in this case for all $\sigma > 0$,

$$D_\sigma = \lim_{x \to \infty} \sum_{\lambda_n < x} a_n (1 - \lambda_n / x)^k e^{-\sigma \lambda_n} e^{-\lambda_n s}$$

with convergence in $\mathcal{H}_p(\lambda, X)$. Moreover, for every $k > 0$ there is a constant $C_k > 0$ such that for every $1 \leq p \leq \infty$ and $D \in \mathcal{H}_p^+(\lambda, X)$,

$$\|D\|_p^+ \leq M_{k,p}(D) \leq C_k \|D\|_p^+;$$

here for $0 < k \leq 1$ the choice $C_k = C/k$ with some absolute constant $C$ is possible.

*Proof.* Assume first that $M_{k,p}(D) < \infty$, and define $E = \sum (a_n e^{-\lambda_n s}) e^{-\lambda_n z} \in \mathcal{D}(\lambda, \mathcal{H}_p(\lambda, X))$. Then the Bohr–Cahen formula for $\sigma e^{\lambda k}$ from Proposition 2.1 implies that $\sigma e^{\lambda k}(E) \leq 0$, that is, for every $\sigma > 0$ the limit

$$\lim_{x \to \infty} \sum_{\lambda_n < x} (a_n e^{-\lambda_n s})(1 - \lambda_n / x)^k e^{-\sigma \lambda_n}$$
exists in $H_p(\lambda, X)$, and the limit is given by $D_\sigma$. In particular, by convolution with the Poisson measure we obtain, for every $\sigma > 0$,
\[
\|D_\sigma\|_p = \lim_{x \to \infty} \left\| \sum_{\lambda_n < x} (a_n e^{-\lambda_n s})(1 - \lambda_n/x)^k e^{-\sigma \lambda_n e^{-\lambda_n s}} \right\|_p \\
= \lim_{x \to \infty} \left\| \left( \sum_{\lambda_n < x} (a_n e^{-\lambda_n s})(1 - \lambda_n/x)^k h_{\lambda_n} \right) \ast p_\sigma \right\|_p \\
\leq \lim_{x \to \infty} \left\| \sum_{\lambda_n < x} (a_n e^{-\lambda_n s})(1 - \lambda_n/x)^k e^{-\lambda_n s} \right\|_p \leq M_{k,p}(D),
\]
and so $\|D\|_p^+ \leq M_{k,p}(D)$.

Suppose conversely that $D \in H^+_p(\lambda, X)$. Then for every $\sigma > 0$ we have $D_\sigma \in H_p(\lambda, X)$ with $\|D_\sigma\|_p \leq \|D\|_p^+$. Hence, if $f_\sigma \in H^\lambda_p(G, X)$ is associated with $D_\sigma$ (where $(G, \beta)$ is an appropriate $\lambda$-Dirichlet group), then by (2.1), for every $x$,
\[
\left\| \sum_{\lambda_n < x} a_n e^{-\sigma \lambda_n (1 - \lambda_n/x)^k e^{-\lambda_n s}} \right\|_p = \left\| \sum_{\lambda_n < x} a_n e^{-\sigma \lambda_n (1 - \lambda_n/x)^k h_{\lambda_n}} \right\|_p \\
= \left\| f_\sigma \ast \mu_x \right\|_p \leq \|f_\sigma\|_p \|\mu_x\| = \|D_\sigma\|_p \|\mu_x\| \leq C_k \|D\|_p^+,
\]
where $C_k$ only depends on $k$ and for $0 < k \leq 1$ the choice $C_k = C/k$ with an absolute constant $C$ is possible (see [30, Lemma 3.10, Proposition 3.2 and Theorem 2.1]). If now in the preceding estimate $\sigma \to 0$, then we get $M_{k,p}(D) \leq C_k \|D\|_p^+$ as desired.

Let us revisit the case $\lambda = (n)$ with the $(n)$-Dirichlet group $(\mathbb{T}, \beta_\mathbb{T})$. Then, given $f \in H_\infty(\mathbb{T})$ and $\sigma > 0$, we have
\[
f \ast p_\sigma(z) = \sum_{n=0}^\infty \hat{f}(n)e^{-\sigma n}z^n
\]
uniformly on $\mathbb{T}$, and in particular $f \ast p_\sigma$ is continuous on $\mathbb{T}$. The following corollary extends this observation to general $\lambda$’s.

**Corollary 2.7.** Let $(G, \beta)$ be a $\lambda$-Dirichlet group and $f \in H^\lambda_\infty(G, X)$. Then for every $k, \sigma > 0$ we have, uniformly on $G$,
\[
f \ast p_\sigma = \lim_{x \to \infty} \sum_{\lambda_n < x} \hat{f}(h_{\lambda_n})(1 - \lambda_n/x)^k e^{-\sigma \lambda_n} h_{\lambda_n}.
\]
In particular, the function $f \ast p_\sigma$ is continuous for every $\sigma > 0$.

**3. General Dirichlet series vs. operators.** As mentioned in the introduction, the main goal of this section is to study the coincidence of the spaces $H_p(\lambda, X)$ and $H^+_p(\lambda, X)$. As an application we obtain a generalization of the brothers Riesz theorem and conditions for $H_p(\lambda, X^*)$ to be a dual
space. A useful tool of independent interest will be to identify $\mathcal{H}^+_p(\lambda, X)$ with a suitable space of cone summing operators.

### 3.1. Cone summing operators

Let $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$. To simplify, from now on we fix a $\lambda$-Dirichlet group $(G, \beta)$ with Haar measure $m$. Then for $1 \leq q < \infty$ we write $E_q(G) = L^q(G)$ and $E_q(G) = C(G)$ for $q = \infty$. Moreover, let $X$ be some Banach space.

Recall that a (bounded linear) operator $T : E_q(G) \to X$ is cone summing (written $T \in \Pi_{cone}(E_q(G), X)$) if there is a constant $C > 0$ such that for every choice of finitely many $f_1, \ldots, f_N \in E_q(G)$,

$$\sum_{k=1}^N \|T(f_k)\|_X \leq C \left\| \sum_{k=1}^N |f_k| \right\|_{E_q(G)},$$

and in this case $\pi_{cone}(T) := \inf C$ is called the cone summing norm. To see more examples recall the following classical results:

\begin{align}
\Pi_{cone}(L_1(G), X) &= \mathcal{L}(L_1(G), X), \\
\Pi_{cone}(C(G), X) &= \Pi_1(C(G), X),
\end{align}

where $\mathcal{L}(L_1(G), X)$ denotes all bounded operators from $L_1(G)$ into $X$, and $\Pi_1(C(G), X)$ all summing operators from $C(G)$ into $X$. For information on cone summing operators in the context of Dirichlet series see [25, 24.4] and [26].

**Remark 3.1.** From [25, Proposition 24.16] we deduce that the canonical inclusion

$$L_p(G, X) \hookrightarrow \Pi_{cone}(E_q(G), X), \quad f \mapsto \left[ g \mapsto \int g f \, dm \right],$$

defines an into isometry.

Given $T \in \Pi_{cone}(E_q(G), X)$, we call

$$\hat{T} : \hat{G} \to X, \quad \gamma \mapsto T(\overline{\gamma}),$$

the Fourier transform of $T$. With this definition we see by density of the polynomials in $E_q(G)$ that $T$ is uniquely determined by its Fourier coefficients, that is, $T = 0$ whenever $\hat{T} = 0$.

**Remark 3.2.** Let $T \in \Pi_{cone}(E_q(G), X)$ and $\mu \in M(G)$. Then

$$(T \ast \mu)(g) := T(g \ast \mu) : E_q(G) \to X$$

belongs to $\Pi_{cone}(E_q(G), X)$ with $\pi_{cone}(T \ast \mu) \leq \pi_{cone}(T) \|\mu\|$ and $\widehat{T \ast \mu} = \widehat{T} \mu$.

**Proof.** The proofs of the first two statements are straightforward from the definition of cone summing operators. For the Fourier coefficients we
have
\[
\hat{T} \ast \mu(\gamma) = (T \ast \mu)(\gamma) = T \left( \int_G \gamma(\cdot + x) \, d\mu(x) \right)
\]
\[
= T \left( \int_G \gamma(\cdot) \gamma(x) \, d\mu(x) \right) = T \left( \gamma(\cdot) \int_G \gamma(x) \, d\mu(x) \right) = \hat{\mu}(\gamma) \hat{T}(\gamma).
\]

For \(1 \leq p \leq \infty\) we denote by \(\Pi^\lambda_{\text{cone}}(E_q(G), X)\) the Banach space of all \(T \in \Pi_{\text{cone}}(E_q(G), X)\) such that \(\hat{T}(\gamma) \neq 0\) implies \(\gamma = h_{\lambda_n}\) for some \(n\). Our proof is inspired by the proof of the ordinary case, but it is substantially different since, among other things, no Poisson kernel is available in this case of general Dirichlet series.

\[\begin{align*}
\text{Proof of Theorem 3.3.} & \quad \text{Let } \lambda \text{ be arbitrary, } (G, \beta) \text{ a } \lambda\text{-Dirichlet group and } X \text{ a Banach space. Then for every } 1 \leq p, q \leq \infty \text{ with } 1/p + 1/q = 1 \text{ the map} \\
\mathcal{T}: \Pi^\lambda_{\text{cone}}(E_q(G), X) & \to \mathcal{H}_p^+(\lambda, X), \\
T & \mapsto \sum \hat{T}(h_{\lambda_n}) e^{-\lambda_n s},
\end{align*}\]

is an onto isometry.

Our proof is inspired by the proof of the ordinary case, but it is substantially different since, among other things, no Poisson kernel is available in the case of general Dirichlet series.

\[\begin{align*}
\text{Proof of Theorem 3.3.} & \quad \text{Let } T \in \Pi^\lambda_{\text{cone}}(E_q(G), X), \text{ and } \mu_x \text{ the measure from (2.1) with } k = 1 \text{ and } x > 0. \text{ We define } D = \sum a_n e^{-\lambda_n s}, \text{ where } a_n := \hat{T}(h_{\lambda_n}) \text{ for all } n. \text{ Then by Remarks 3.1 and 3.2 we have} \\
\left\| \sum_{\lambda_n < x} a_n (1 - \lambda_n/x) e^{-\lambda_n s} \right\|_p & = \pi_{\text{cone}}(T \ast \mu_x) \leq C_1 \pi_{\text{cone}}(T),
\end{align*}\]

and Proposition 2.6 yields \(D = \sum a_n e^{-\lambda_n s} \in \mathcal{H}_p^+(\lambda, X)\) with \(\|D\|_p^+ \leq C_1 \pi_{\text{cone}}(T)\). Actually, it can be shown that \(C_1 = 1\) by noticing that \(\mu_x\) can be obtained from a Fejér kernel in \(\mathbb{R}\) just as the Poisson measure \(p_u\) was obtained from \(P_u\). However, we show that \(\|D\|_p^+ \leq \pi_{\text{cone}}(T)\) using the Poisson measure directly at the end of the proof.

Suppose conversely that \(D = \sum a_n e^{-\lambda_n s} \in \mathcal{H}_p^+(\lambda, X)\) and let \(f_\sigma \in H^\lambda_p(G, X)\) correspond to \(D_\sigma = \sum a_n e^{-\lambda_n \sigma} e^{-\lambda_n s}, \sigma > 0\). We define an operator \(T: \text{Pol}(G) \to X\) by

\[
\hat{T}(h_x) = \begin{cases} 
  a_n & \text{if } x = \lambda_n \text{ for some } n, \\
  0 & \text{else}.
\end{cases}
\]
Then we claim that there is \( g \in E_q(G)^* \) with \( \|g\|_{E_q(G)^*} \leq \|D\|_p^+ \) such that \( \|T(P)\|_X \leq g(|P|) \) for all \( P \in \text{Pol}(G) \). This will complete the proof, since by density of the polynomials and the Pietsch type domination theorem for cone summing operators (see e.g. [25, Proposition 24.12]) \( T \) is cone summing with \( \pi_{\text{cone}}(T) \leq \|g\|_{E_q(G)^*} \), and so \( T(T) = D \) and \( \pi_{\text{cone}}(T) \leq \|D\|_p^+ \). Indeed, the desired \( g \in E_q(G)^* \) exists: Since for all \( x \in \mathbb{R} \),

\[
T(h_x)e^{-|x|\sigma} = \hat{f}_\sigma(x) = \int_{\sigma} f(\omega) \, h_x(\omega) \, d\omega,
\]

we obtain

\[
T(P) = \int_{\sigma} \left\{ f(\omega) \sum c_x e^{x|\sigma|} h_x(\omega) \right\} d\omega
\]

for all \( P = \sum c_x h_x \in \text{Pol}(G) \). Hence

\[
\|T(P)\|_X \leq \int_{\sigma} \left\{ \|f(\omega)\|_X \sum c_x e^{x|\sigma|} h_x(\omega) \right\} d\omega = g\left( \sum c_x h_x \right).
\]

Since the net \( (\|f(\cdot)\|_X) \) is weak-star bounded, there is a sequence \( \sigma_j \to 0 \) and \( g \in E_q(G)^* \) such that \( \|f(\cdot)\|_X \to g \) in the weak-star topology and \( \|g\| \leq \|D\|_p^+ \). This implies that

\[
\|T(P)\|_X \leq \lim_{j \to \infty} \int_{\sigma_j} \left\{ \|f(\omega)\|_X \sum c_x e^{x|\sigma|} h_x(\omega) \right\} d\omega = g\left( \sum c_x h_x \right),
\]

and so \( \|T(P)\|_X \leq g(|P|) \) for all \( P \in \text{Pol}(G) \) as desired. As mentioned above, it remains to show that \( \|D\|_p^+ \leq \pi_{\text{cone}}(T) \). Indeed, for all \( \sigma > 0 \) let \( T_\sigma \in II^\lambda_{\text{cone}}(E_q(G), X) \) be the operator associated to \( D_\sigma \in \mathcal{H}_p(\lambda, X) \). From Remark 3.2 we see that \( T_\sigma = T \ast p_\sigma \), and hence

\[
\|D_\sigma\|_p \leq \pi_{\text{cone}}(T_\sigma) \leq \pi_{\text{cone}}(T)\|p_\sigma\|_1 \leq \pi_{\text{cone}}(T). \quad \blacksquare
\]

From the into isometry in Remark 3.1 we immediately see that the inclusion

\[
\mathcal{H}_p(\lambda, X) = H_p^\lambda(G, X) \subset II^\lambda_{\text{cone}}(E_q(G), X)
\]

is isometric as well. Hence Theorem 3.3 immediately implies the following.

**Corollary 3.4.** Let \( \lambda \) be a frequency and \( X \) a Banach space. Then for every \( 1 \leq p \leq \infty \), isometrically \( \mathcal{H}_p(\lambda, X) \subset \mathcal{H}_p^+(\lambda, X) \).

**3.2. Nth abschnitte.** Hilbert’s criterion from [25, Theorem 15.26, p. 372] states that for every family \((c_\alpha)_{\alpha \in \mathbb{N}_0^{(N)}}\) in a Banach space \( X \) there is a function \( f \in H_\infty(B_{c_0}, X) \) such that \( c_\alpha = \partial f(0)/\alpha! \) for every \( \alpha \in \mathbb{N}_0^{(N)} \) if and only if

\[
\sup_{N \in \mathbb{N}} \sup_{z \in \mathbb{D}^N} \left\| \sum_{\alpha \in N_0^N} c(z)^N \alpha \right\|_X < \infty.
\]
In view of (1.6) this result translates to ordinary Dirichlet series. In fact, an extension of this result to the framework of vector-valued $\lambda$-Dirichlet series is possible, which requires no assumption on the frequency $\lambda$.

Therefore, every frequency $\lambda$ admits another real sequence $B = (b_k)$ such that for every $n$ there are finitely many unique rationals $q_k^n$ for which $\lambda_n = \sum q_k^n b_k$. We call the matrix $R = (q_k^n)$ Bohr matrix, and we write $\lambda = (R, B)$ whenever $\lambda$ decomposes with respect to $B$ with associated Bohr matrix $R$.

Given a formal Dirichlet series $D = \sum a_n e^{-\lambda_n s}$ and a decomposition $\lambda = (R, B)$, the $N$th abschnitt $D|_N$ of $D$ is the sum $\sum a_n e^{-\lambda_n s}$, where only those $a_n$ differ from 0 for which $\lambda_n$ is a linear combination of the first $b_1, \ldots, b_N$.

**Theorem 3.5.** Let $\lambda$ be any frequency with decomposition $\lambda = (R, B)$ and $X$ a Banach space. Let $1 \leq p \leq \infty$ and $D = \sum a_n e^{-\lambda_n s} \in D(\lambda, X)$. Then the following are equivalent:

1. $D \in \mathcal{H}_p^{+}(\lambda, X)$,
2. $D|_N \in \mathcal{H}_p^{+}(\lambda, X)$ for all $N$ and $\sup_{N \in \mathbb{N}} \|D|_N\|_p^+ < \infty$.

Moreover, in this case

\[ \|D\|_p^+ = \sup_{N \in \mathbb{N}} \|D|_N\|_p^+ < \infty. \]  

(3.3)

We mention that in [29, Theorem 5.9] this theorem for $X = \mathbb{C}$ is proved under the assumption that Bohr’s theorem holds for $\lambda$. We now present a proof that avoids this assumption.

**Proof of Theorem 3.5.** First assume that (1) holds. Then, following the argument from [29, Remark 4.21] together with Remark 3.2 for all $N$, we have $D|_N \in \mathcal{H}_p^{+}(\lambda, X)$ with $\|D|_N\|_p^+ \leq \|D\|_p^+$. Suppose that (2) holds, and let $\mu_x$ be the measure from (2.1) with $x > 0$ and $k = 1$. Moreover, let $T|_N \in \Pi_{\text{cone}}^\lambda(E_q(G), X)$ correspond to $D|_N$ (Theorem 3.3). Then for fixed $x$ and $N$ large enough, by Remarks 3.1 and 3.2 we have

\[ \left\| \sum_{\lambda_n < x} a_n (1 - \lambda_n / x) e^{-\lambda_n s} \right\|_p = \pi_{\text{cone}}(T|_N * \mu_x) \leq \sup_M \pi_{\text{cone}}(T|M), \]

which is finite by assumption and Theorem 3.3. Hence Proposition 2.6 implies $D \in \mathcal{H}_p^{+}(\lambda, X)$. It remains to check (3.3). So let $T \in \Pi_{\text{cone}}^\lambda(E_q(G), X)$ correspond to $D$ (Theorem 3.3). Then for every choice of finitely many polynomials $g_k \in E_q(G)$ we have, for large $M$,

\[ \sum_{k=1}^n \|T(g_k)\|_X = \sum_{k=1}^n \|T|M(g_k)\|_X \leq \sup_N \pi_{\text{cone}}(T|_N) \left\| \sum_{k=1}^n |g_k| \right\|_q, \]

which by density shows that $\|D\|_p^+ \leq \sup_N \|D|_N\|_p^+$. 

**3.3. Coincidence.** Recall that by Corollary 3.4 for every $1 \leq p \leq \infty$ and every frequency $\lambda$ we have the isometric inclusion $\mathcal{H}_p(\lambda, X) \subset \mathcal{H}_p^{+}(\lambda, X)$. 

However as stated in (1.9), for the vector-valued case with \( \lambda = (\log n) \) equality is attained if and only if \( X \) has ARNP. Likewise, the same is true for the frequency \( \lambda = (n) \). As mentioned before, \( X \) has ARNP if and only if \( H_p(T, X) = H_p(D, X) \), which we can rewrite as \( H_p((n), X) = H_p^+(n), X) \).

Indeed, as the following theorem proves, ARNP is sufficient for every frequency \( \lambda \).

**Theorem 3.6.** Let \( \lambda \) be an arbitrary frequency and \( X \) a Banach space with ARNP. Then for all \( 1 \leq p \leq \infty \) we have

\[
H_p^+(\lambda, X) = H_p(\lambda, X).
\]

The proof needs some preparation, and is given at the end of this section.

Recall that a Banach space \( X \) with ARNP never contains an isomorphic copy of \( c_0 \). We also point out that Banach lattices \( X \) have this property if and only if \( c_0 \) is not isomorphically contained in \( X \). This is actually a necessary condition for the coincidence result.

**Remark 3.7.** If \( H_p(\lambda, X) = H_p^+(\lambda, X) \) for some \( 1 \leq p \leq \infty \), then \( X \) contains no isomorphic copy of \( c_0 \).

**Proof.** Assume by way of contradiction that there is an isomorphic embedding \( c_0 \hookrightarrow X \), and choose moreover a \( \lambda \)-Dirichlet group \( (G, \beta) \) such that \( H_p(\lambda, X) = H_p^+(\lambda, X) \). Define \( f^N = \sum_{n=1}^N e_n h_{\lambda_n} \), where the \( e_n \) denote the unit vectors in \( c_0 \). Then for every \( \sigma > 0 \) and \( M > N \) we have \( \| f^M_\sigma - f^N_\sigma \|_p = e^{-\lambda_{N+1}\sigma} \). Consequently, \( (f^N_\sigma) \) is Cauchy in \( H_p^+(\lambda, X) \), with limit \( f_\sigma \) say, and

\[
\| f_\sigma \|_p = \lim_{N \to \infty} \| f^N_\sigma \|_p \leq 1.
\]

Hence \( D = \sum e_n e^{-\lambda_n s} \in H_p^+(\lambda, X) \), and so \( D \in H_p(\lambda, X) = H_p^+(\lambda, X) \).

Consequently, by the Riemann–Lebesgue lemma \( (e_n) \) is a zero sequence in \( X \) and so in \( c_0 \), a contradiction.

Our proof of Theorem 3.6 follows the same strategy as in [26]. As an important ingredient we use Lemma 5.4 from [26] which states that for every bounded and holomorphic function \( F: [\text{Re} > 0] \to X \) the horizontal limit

\[
\lim_{\varepsilon \to 0} F(\varepsilon + it)
\]

exists for Lebesgue almost all \( t \in \mathbb{R} \) whenever \( X \) has ARNP (see also [25, Lemma 11.22]). The following lemma is an analogue of [26, Proposition 2.3] (also proved in [25, Proposition 11.20]).

**Lemma 3.8.** For \( 1 \leq p < \infty \) and \( D \in H_p^+(\lambda, X) \) the function

\[
F: [\text{Re} > 0] \to H_p(\lambda, X), \quad z \mapsto D_z = \sum a_n(D) e^{-\lambda_n z} e^{-\lambda_n s},
\]

is defined, continuous on \([\text{Re} \geq 0]\) and holomorphic on \([\text{Re} > 0]\). Moreover,

(1) \( \| D_{\varepsilon+it} \|_p = \| D_{\varepsilon} \|_p \) for all \( t \in \mathbb{R} \) and \( \varepsilon \geq 0 \),
(2) sup_{Re \geq 0} \|D_z\|_p = \|D\|_p^+
(3) the function $\varepsilon \mapsto \|D_\varepsilon\|_p$ is decreasing,
(4) if $D \in \mathcal{H}_p(\lambda, X)$, then $\lim_{\varepsilon \to 0} D_\varepsilon = D$ in $\mathcal{H}_p(\lambda, X)$.

For the proof of (1) take a $\lambda$-Dirichlet group $(G, \lambda)$, and note that by the translation invariance of the Haar measure $m$ on $G$, for every polynomial $D = \sum_{n=1}^N a_ne^{-\lambda ns}$ we have

$$\|Dt\|_p^p = \int_G \left\| \sum_{n=1}^N a_ne^{-it\lambda_n h_{\lambda_n}(\omega)} \right\|^p d\omega = \int_G \left\| \sum_{n=1}^N a_n h_{\lambda_n}(\omega \beta(t)) \right\|^p d\omega = \|D\|_p^p.$$

Now all claims follow as in the ordinary case.

The third ingredient for the proof of Theorem 3.6 reduces the case $p = \infty$ to finite $p$’s.

**Lemma 3.9.** Let $X$ be a Banach space and $\lambda$ a frequency. If $\mathcal{H}_p^+(\lambda, X) = \mathcal{H}_p(\lambda, X)$ for some $1 \leq p < \infty$, then $\mathcal{H}_\infty^+(\lambda, X) = \mathcal{H}_\infty(\lambda, X)$.

**Proof.** By Corollary 3.4 we only have to check one inclusion, so let $D \in \mathcal{H}_\infty^+(\lambda, X)$. Then $D \in \mathcal{H}_p^+(\lambda, X)$, and so there is $f \in H_p^\lambda(G, X)$ such that $\hat{f}(h_{\lambda_n}) = a_n(D)$ for all $n$ (for an appropriate $\lambda$-Dirichlet group $(G, \beta)$, e.g. the Bohr compactification).

We show now that in fact $f \in H_\infty^\lambda(G, X)$. Indeed, by Proposition 2.5 we know that $x^* \circ D \in \mathcal{H}_\infty(\lambda)$ with $\|x^* \circ D\|_\infty \leq \|D\|_\infty^+$. Hence, comparing Fourier and Dirichlet coefficients we see that $x^* \circ f \in H_\infty^\lambda(G)$ for every $x^* \in X^*$ with $\|x^* \circ f\|_\infty \leq \|D\|_\infty^+$. Now recall that, since $f$ is measurable, it is separable-valued, i.e. there is a zero set $E \subset G$ and separable subspace $X_0$ such that $f(G \setminus E) \subset X_0$. An application of the Hahn–Banach theorem shows that there is a sequence $(x^*_n) \subset X_0^*$ such that

$$\|x_0\| = \sup_n \|x^*_n(x_0)\| \quad \text{for every } x_0 \in X_0.$$

Hence, for every $n$ there is a zero set $E_n \subset G$ such that $|x^*_n(f(\omega))| \leq \|D\|_\infty^+$ for all $\omega \notin E_N \cup E$. Collecting the countably many zero sets we deduce from (3.5) that $\|f\|_\infty \leq \|D\|_\infty^+$.  

**Proof of Theorem 3.6** By Lemma 3.9 it suffices to prove the claim for $1 \leq p < \infty$. Choose a $\lambda$-Dirichlet group $(G, \beta)$ such that $\mathcal{H}_p(\lambda, X) = H_p^\lambda(G, X)$. Since by assumption $X$ has ARNP, the space $L^p(G, X)$ has ARNP, and hence so does $\mathcal{H}_p(\lambda, X)$ as a closed isometric subspace (see 38 and also 25 Proposition 23.20]). Now, given $D \in \mathcal{H}_p^+(\lambda, X)$, by Lemma 3.8 and (3.4) there is $t \in \mathbb{R}$ such that $\lim_{\varepsilon \to 0} D_{\varepsilon+it}$ exists in $\mathcal{H}_p(\lambda, X)$. Comparing Dirichlet coefficients we see that this limit is exactly $D_{it}$, and so by translation invariance $D \in \mathcal{H}_p(\lambda, X)$. 

3.4. \( \mathbb{Q} \)-linear independence I. Theorem 3.6 and Remark 3.7 show that for a given frequency \( \lambda \) and a Banach space \( X \) we have

\[ X \text{ has ARNP } \Rightarrow \mathcal{H}_p(\lambda, X) = \mathcal{H}_p^+(\lambda, X) \]

for all \( 1 \leq p \leq \infty \) \( \Rightarrow c_0 \not\subset X \).

As mentioned before, for the case of classical Fourier series \( \lambda = (n) \) and ordinary Dirichlet series \( \lambda = (\log n) \), it is known that \( X \) has ARNP if and only if \( \mathcal{H}_p(\lambda, X) = \mathcal{H}_p^+(\lambda, X) \). To complete the picture we show that for \( \mathbb{Q} \)-linearly independent frequencies \( \lambda \) we have \( \mathcal{H}_p(\lambda, X) = \mathcal{H}_p^+(\lambda, X) \) if and only if \( c_0 \) is not isomorphic to a subspace of \( X \). To this end we identify \( \mathcal{H}_p(\lambda, X) \) and \( \mathcal{H}_p^+(\lambda, X) \) with two sequence spaces of random convergence on the probability space \( T^\infty \). For a Banach space \( X \) we define

\[
\text{RAD}(X) = \left\{ (x_n)_{n \in \mathbb{N}} \subseteq X \left| \sup_{N \in \mathbb{N}} \left\| \sum_{n=1}^{N} x_n z_n \right\|_{L_2(T^\infty, X)} < \infty \right. \right\},
\]

\[
\text{Rad}(X) = \left\{ (x_n)_{n \in \mathbb{N}} \subseteq X \left| \sum_{n=1}^{N} x_n z_n \text{ converges in } X \text{ a.e. on } T^\infty \right. \right\}.
\]

A straightforward computation shows that \( \text{RAD}(X) \) is a Banach space under the norm provided in its definition. On the other hand, in [60, Theorem 3.1(b)] it is shown that \( \text{Rad}(X) \) can be identified with the closure in \( L_2(T^\infty, X) \) of

\[
\text{span} \left\{ \sum_{n=1}^{N} x_n z_n \left| N \in \mathbb{N}, x_n \in X, z_n \in \mathbb{C} \right. \right\}.
\]

This endows \( \text{Rad}(X) \) with a Banach space structure.

As a consequence of [60, Theorem 6.1] we see that \( \text{Rad}(X) = \text{RAD}(X) \) if and only if \( c_0 \) is not isomorphic to a subspace of \( X \) (apply the equivalence \((a) \Leftrightarrow (c)\) from [60, Theorem 6.1] for \( \xi_k : T^\infty \to L_2(T^\infty, X) \) given by \( \xi_k(w) = w_k(z_k x_k) \)). This together with the characterization provided in the following theorem settles the issue.

**Theorem 3.10.** Let \( \lambda \) be \( \mathbb{Q} \)-linearly independent and \( X \) a Banach space. Then for all \( 1 \leq p < \infty \) we have isomorphically

\[ \text{Rad}(X) \simeq \mathcal{H}_p(\lambda, X) \subset \mathcal{H}_p^+(\lambda, X) \simeq \text{RAD}(X). \]

In particular for every \( 1 \leq p < \infty \) we have \( \mathcal{H}_p(\lambda, X) = \mathcal{H}_p^+(\lambda, X) \) if and only if \( c_0 \) is not isomorphic to a subspace of \( X \).

We will need the Kahane–Khinchin inequality (see e.g. [25] (14.18) and Theorem 25.9]) to compare the \( p \)th moments of random sums: For every \( 1 \leq p < \infty \) there are constants \( A_p, B_p > 0 \) such that for every Banach space \( X \) and every choice of vectors \( (x_n)_{n=1}^{N} \subseteq X \) we have
(3.6)  \[ A_p^{-1} \left\| \sum_{n=1}^{N} x_n z_n \right\|_{L_p(T^\infty, X)} \leq \left\| \sum_{n=1}^{N} x_n z_n \right\|_{L_2(T^\infty, X)} \leq B_p \left\| \sum_{n=1}^{N} x_n z_n \right\|_{L_p(T^\infty, X)}. \]

Proof of Theorem 3.10. Notice that \( T^\infty \) (with the canonical morphism \( \beta \)) is a Dirichlet group for \( \lambda \). As a direct consequence of the Kahane–Khinchin inequality (3.6) and the density of finite sums \( \sum_{n=1}^{N} x_n z_n \) in \( \text{Rad}(X) \) and \( H_p^\lambda(T^\infty, X) \) we have

\[ \text{Rad}(X) \simeq H_p^\lambda(T^\infty, X) = H_p(\lambda, X). \]

For the second equality notice that another use of the Kahane–Khinchin inequality and of Corollary 3.4 shows that for any sequence \( (x_n) \subseteq X \) we have

\[ \left\| \sum_{n=1}^{N} x_n z_n \right\|_{\text{Rad}(X)} = \sup_{N \in \mathbb{N}} \left\| \sum_{n=1}^{N} x_n z_n \right\|_2 = \sup_{N \in \mathbb{N}, \sigma > 0} \left\| \sum_{n=1}^{N} x_n e^{-\lambda_n \sigma} z_n \right\|_2 \]
\[ \simeq \sup_{N \in \mathbb{N}, \sigma > 0} \left\| \sum_{n=1}^{N} x_n e^{-\lambda_n \sigma} z_n \right\|_p = \sup_{N \in \mathbb{N}} \|D|N\|_{H_p^+}(\lambda, X). \]

This together with Theorem 3.5 proves the equality.

It only remains to check the case \( p = \infty \) of the last statement. This is a direct consequence of Lemma 3.9 and Remark 3.7.

3.5. Brothers Riesz theorem. Let us discuss the special case \( p = 1 \) in Theorem 3.3. For every \( \lambda \)-Dirichlet group \((G, \beta)\) and every Banach space \( X \),

\[ \Pi^\lambda_{\text{cone}}(C(G), X) = H_1^+(\lambda, X), \quad T \mapsto \sum \hat{T}(h_{\lambda_n}) e^{-\lambda_n s}, \]

is an onto isometry. Denote by \( M(G, X) \) the Banach space of all regular \( X \)-valued Borel measures on \( G \) of bounded variation, and by \( M_\lambda(G, X) \) its subspace of \( \lambda \)-analytic measures \( \mu \), i.e. \( \hat{\mu}(\gamma) \not= 0 \) only if \( \gamma = h_{\lambda_n} \) for some \( n \). A well-known result (see e.g. [36, Chapter VI]) is that isometrically

\[ M(G, X) = \Pi_1(C(G), X), \quad \mu \mapsto \left[ g \mapsto \int g \, d\mu \right], \]

where as above \( \Pi_1 \) denotes the summing operators. As a consequence of what we have achieved so far, we state the following brothers Riesz type theorem for general \( X \)-valued \( \lambda \)-Dirichlet series.

**Theorem 3.11.** Let \( \lambda \) be arbitrary, \((G, \beta)\) a \( \lambda \)-Dirichlet group, and \( X \) a Banach space. Then the mapping

\[ M_\lambda(G, X) = H_1^+(\lambda, X), \quad \mu \mapsto \sum \hat{\mu}(h_{\lambda_n}) e^{-\lambda_n s}, \]

defines an onto isometry preserving the Fourier and Dirichlet coefficients. Moreover,
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(1) $M_\lambda(G, X) = \mathcal{H}_1(\lambda, X)$ whenever $X$ has ARNP,

and under the assumption that $\lambda$ is $\mathbb{Q}$-linearly independent,

(2) $M_\lambda(G, X) = \mathcal{H}_1(\lambda, X)$ if and only if $X$ contains no isomorphic copy of $c_0$.

Note that (3.2) and Theorem 3.3 immediately give the first result. Then both statements (1) and (2) follow from Theorems 3.6 and 3.10. The scalar ordinary case is due to [45] (for an alternative proof see also [1]). In the ordinary vector-valued case, (1) was proved in [26] (see also [25, Section 26.6]). The identity $M_\lambda(G) = \mathcal{H}_1(\lambda)$, where $X = \mathbb{C}$ and $\lambda$ is arbitrary, is found in [28, Theorem 4.25]; there the proof uses as a crucial ingredient a result of Doss from [37, Theorem 4] for locally compact and connected groups with ordered duals. Note that the proof given here is entirely performed within the realm of Dirichlet series.

3.6. Preduals of $\mathcal{H}_p(\lambda, X^*)$. As proved e.g. in [25, Proposition 24.16], for every $1 \leq p \leq \infty$ and every Banach space $X$ we have isometrically

(3.7) $\Pi_{\text{cone}}(E_q(G), X^*) = E_q(G, X)^*, \quad T \mapsto [f \otimes x \mapsto \langle Tf, x \rangle]$.

Hence, fixing a frequency $\lambda$ and denoting by $E_q(G, X)^*_\lambda$ the weak*-closed subspace of all functionals $\varphi \in E_q(G, X)^*$ such that $\varphi(\overline{\mathfrak{h}_t} \otimes x) = 0$ for all $x \in X$ whenever $t \notin \{\lambda_n \mid n \in \mathbb{N}\}$, by (3.7) and Theorem 3.3 we have the isometric equalities

$$
\mathcal{H}_p^+(\lambda, X^*) = \Pi_{\text{cone}}^\lambda(E_q(G), X^*) = E_q(G, X)^*_\lambda,
$$

where again the Dirichlet and Fourier coefficients are preserved. In the ordinary case the first part of the following theorem was proved in [26, Theorem 7.3] (compare also [25, Theorem 24.15]); our proof follows similar lines.

**Theorem 3.12.** Let $\lambda$ be a frequency, $X$ a Banach space and $1 \leq p \leq \infty$. Then $\mathcal{H}_p(\lambda, X^*)$ has a predual if and only if

$$
\mathcal{H}_p^+(\lambda, X^*) = \mathcal{H}_p(\lambda, X^*).
$$

In particular, this is the case

(1) whenever $X^*$ has ARNP,

and, assuming that $\lambda$ is $\mathbb{Q}$-linearly independent,

(2) if and only if $X^*$ does not contain an isomorphic copy of $c_0$.

In contrast to the ordinary case, there are Banach spaces $X$ such that $\mathcal{H}_p(\lambda, X^*)$ has a predual but $X^*$ fails ARNP. Indeed, it suffices to find a dual space failing ARNP that contains no isomorphic copy of $c_0$, since then for $\mathbb{Q}$-linearly independent frequencies (2) holds.

We show that if $A$ is the disc algebra, then $A^*$ has the desired properties. It does not contain $c_0$, since $A^*$ has cotype 2 (see [15, Corollary 2.11]). We
now check that $A^*$ fails ARNP. As shown in [52, p. 11], the F. and M. Riesz theorem implies that

$$A^* \simeq L_1(\mathbb{T})/H^1_0(\mathbb{T}) \oplus M_s(\mathbb{T}),$$

where $M_s(\mathbb{T})$ denotes the Banach space of all singular measures on $\mathbb{T}$. Since the quotient $L_1(\mathbb{T})/H^1_0(\mathbb{T})$ does not have ARNP (see [54, Remark 4.35]) and this property is inherited by closed subspaces, we deduce that $A^*$ also fails ARNP. In conclusion, for $\mathbb{Q}$-linearly independent frequencies $\lambda$, Theorem 3.12 implies that $\mathcal{H}_p(\lambda, A^*)$ has a predual but $A^*$ fails ARNP.

4. General Dirichlet series vs. holomorphic functions. The main goal of this section is to extend the equivalences of Bohr’s theorem to the vector-valued setting (see Aspect III). Surprisingly, this depends only on the frequency $\lambda$ and not on the geometry of the Banach space $X$ once we replace $\mathcal{H}_\infty(\lambda, X)$ with $\mathcal{H}^+_{\infty}(\lambda, X)$ as mentioned in the introduction.

4.1. Besicovitch spaces. Recall that the Banach space $\mathcal{H}_\infty^\lambda([\text{Re} > 0], X)$ consists of all bounded holomorphic functions $F: [\text{Re} > 0] \to X$ which are almost periodic on all abscissas and for which the Bohr coefficients $a_x(F)$ vanish whenever $x \notin \{\lambda_n \mid n \in \mathbb{N}\}$.

In the scalar case $X = \mathbb{C}$ we know from [30, Theorem 2.16] that there is an onto isometry

$$\mathcal{H}_\infty^\lambda[\text{Re} > 0] = \mathcal{H}_\infty(\lambda)$$

(4.1)

preserving the Bohr and Dirichlet coefficients. We extend this result to the $X$-valued case.

THEOREM 4.1. For all frequencies $\lambda$ and Banach spaces $X$ there is an onto isometry

$$\mathcal{H}_\infty^\lambda([\text{Re} > 0], X) = \mathcal{H}^+_{\infty}(\lambda, X), \quad F \mapsto D,$n

such that $a_{\lambda_n}(F) = a_n(D)$ for all $n$. In particular, the inclusion

$$\mathcal{D}_{\infty}(\lambda, X) \hookrightarrow \mathcal{H}^+_{\infty}(\lambda, X)$$

is isometric.

Before going into the proof, we easily deduce from Corollary 2.7 the following approximation theorem for almost periodic functions.

COROLLARY 4.2. Let $\lambda$ be a frequency, $X$ a Banach space, and $k > 0$. Then for every $F \in \mathcal{H}_\infty^\lambda([\text{Re} > 0], X)$ and $\sigma > 0$ the restriction

$$F_{\sigma} : \mathbb{R} \to X, \quad F_{\sigma}(t) = F(\sigma + it),$$

is, as $x \to \infty$, the uniform limit of the polynomials

$$\sum_{\lambda_n < x} a_{\lambda_n}(F)(1 - \lambda_n/x)^k e^{-\lambda_n(\sigma + it)}.$$
A useful tool for the proof of Theorem 4.1 is the following extension of Theorem 2.4 from $\mathcal{D}_\infty(\lambda, X)$ to $\mathcal{H}^+_{\infty}(\lambda, X)$.

**Theorem 4.3.** Let $D \in \mathcal{D}(\lambda, X)$. Then the following are equivalent:

1. $D \in \mathcal{H}^+_{\infty}(\lambda, X)$.
2. $x^* \circ D \in \mathcal{H}_\infty(\lambda)$ for all $x^* \in X^*$.

Moreover, in this case

$$\|D\|_{\infty}^+ = \sup_{x^* \in B_{X^*}} \|x^* \circ D\|_{\mathcal{H}_\infty(\lambda)}. \quad (4.2)$$

**Proof.** Suppose that $D \in \mathcal{H}^+_{\infty}(\lambda, X)$. Then $x^* \circ D \in \mathcal{H}^+_{\infty}(\lambda)$ with $\|x^* \circ D\|_{\infty}^+ \leq \|D\|_{\infty}^+$ for every $x^* \in X^*$ with $\|x^*\| \leq 1$. Now the claim follows, since $\mathcal{H}^+_{\infty}(\lambda) = \mathcal{H}_\infty(\lambda)$ by Proposition 2.5. Assume conversely that $D$ satisfies (2).

By a closed graph argument we have

$$\sup_{x^* \in B_{X^*}} \|x^* \circ D\|_{\mathcal{H}_\infty(\lambda)} = C < \infty.$$  

Moreover fixing a $\lambda$-Dirichlet group $G$, for every $x^* \in B_{X^*}$ there is a function $f_{x^*} \in \mathcal{H}_{\infty}^\lambda(G)$ such that $\widehat{f}_{x^*}(\lambda, X) = x^*(a_n(D))$ for all $n$ and

$$\|f_{x^*}\|_{\infty} = \|x^* \circ D\|_{\mathcal{H}_\infty(\lambda)}.$$  

Now consider the linear operator

$$T : L_1(G) \to X^{**}, \quad g \mapsto \left[ x^* \mapsto \int_G g(\omega) f_{x^*}(\omega) \, d\omega \right].$$

Then $T$ is bounded with $\|T\| \leq C$, and so $T \in \Pi_{\text{cone}}(L_1(G), X^{**})$ by (3.1). Since for all $x^* \in X^*$ and all $x > 0$,

$$\widehat{T}(h_x)(x^*) = T(h_x)(x^*) = \widehat{f}_{x^*}(h_x),$$

we have $T \in \Pi_{\text{cone}}^\lambda(L_1(G), X^{**})$ with

$$\widehat{T}(h_{\lambda n})(x^*) = x^*(a_n(D)) \quad \text{for all } n \text{ and } x^* \in X^*.$$  

But then $\widehat{T}(h_{\lambda n}) = a_n(D) \in X$ for all $n$, and hence $T \in \Pi_{\text{cone}}^\lambda(L_1(G), X)$ by the density of the polynomials. Finally, Theorem 3.3 finishes the proof. \hfill \blacksquare

**Proof of Theorem 4.1.** Let $F \in \mathcal{H}_\infty^\lambda([\text{Re} > 0], X)$ and $D = \sum a_n e^{-\lambda_n s}$ be defined by $a_n = a_n(F)$. Then $x^* \circ F \in \mathcal{H}_\infty^\lambda([\text{Re} > 0]$, and so by (4.1) and comparing coefficients we have $x^* \circ D \in \mathcal{H}_\infty(\lambda)$ for all $x^*$. Now Theorem 4.3 implies $D \in \mathcal{H}^+_{\infty}(\lambda, X)$. Conversely, take some $D = \sum a_n e^{\lambda_n s} \in \mathcal{H}^+_{\infty}(\lambda, X)$, i.e. for each $\sigma > 0$ there is some $f_{\sigma} \in \mathcal{H}_\infty^\lambda(G, X)$ such that $f_{\sigma}(h_{\lambda n}) = a_n e^{-\sigma \lambda_n}$ for all $n$ and $\|f_{\sigma}\| = \|D_{\sigma}\|_{\infty}$ (where $(G, \beta)$ is an appropriate $\lambda$-Dirichlet group). Using the measures from (2.1) with $k = 1$, for all $x > 0$ we have
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\[ \sup_{t \in \mathbb{R}} \left| \sum_{\lambda_n < x} x^*(a_n)e^{-\lambda_n \sigma}(1 - \lambda_n/x)e^{-it\lambda_n} \right| = \|(x^* \circ f_\sigma) \ast \mu_x\|_\infty \leq \|f_\sigma\|_\infty \|\mu_x\| \leq C_1 \|D\|_\infty. \]

We conclude by Proposition 2.1 that \( \sigma_u^\lambda(D) \leq 0 \), so that the function

\[ F : [\text{Re} > 0] \rightarrow X, \quad F(s) := \lim_{x \rightarrow \infty} \sum_{\lambda_n < x} a_n(1 - \lambda_n/x)e^{-\lambda_n s}, \]

belongs to \( \mathcal{H}_\infty^\lambda([\text{Re} > 0], X) \) and has \( a_n \) as its \( \lambda_n \)th Bohr coefficient. Indeed, this function is bounded, since for every \( \sigma > 0 \),

\[ \sup_{t \in \mathbb{R}} \|F(\sigma + it)\|_X \leq \lim_{x \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \sum_{\lambda_n < x} a_n(1 - \lambda_n/x)e^{-\sigma \lambda_n e^{-i\lambda_n t}} \right|_X \leq \|f_\sigma\|_\infty C_1 \leq C_1 \|D\|_\infty^+, \]

which finishes the proof. ■

4.2. Bohr’s theorem. Suppose that \( D = \sum a_n e^{-\lambda_n s} \) converges somewhere and that its limit function extends to a bounded holomorphic function \( f \) on \([\text{Re} > 0]\). Then as already mentioned in Aspect III in the introduction, a prominent problem from the beginning of the 20th century was to determine the class of \( \lambda \)'s for which under this assumption all \( \lambda \)-Dirichlet series converge uniformly on \([\text{Re} > \epsilon]\) for every \( \epsilon > 0 \). We then say that \( \lambda \) satisfies Bohr’s theorem. See [58, Theorem 5.2] for examples of \( \lambda \)'s which fail to satisfy Bohr’s theorem.

Considering \( X \)-valued Dirichlet series one may ask if it makes sense to define \( \lambda \) satisfies Bohr’s theorem for the Banach space \( X \) to mean that every Dirichlet series \( D = \sum a_n e^{-\lambda_n s} \) with coefficients in \( X \), which converges somewhere and has a limit function extending to a bounded holomorphic function \( f \) on \([\text{Re} > 0]\) with values in \( X \), converges uniformly on \([\text{Re} > \epsilon]\) for every \( \epsilon > 0 \). In this context the space \( \mathcal{D}_\infty^{\text{ext}}(\lambda, X) \) of all somewhere convergent \( D \in \mathcal{D}(\lambda, X) \) that allow a bounded holomorphic extension \( f \) to \([\text{Re} > 0]\) is natural. Actually, as a consequence of Proposition 2.1 we see that the Banach space \( X \) does not affect satisfying Bohr’s theorem.

**Proposition 4.4.** Let \( \lambda \) be a frequency and \( X \) a non-trivial Banach space. Then \( \lambda \) satisfies Bohr’s theorem if and only if \( \lambda \) satisfies Bohr’s theorem for \( X \).

**Proof.** Assume that \( \lambda \) satisfies Bohr’s theorem, and let \( D \in \mathcal{D}_\infty^{\text{ext}}(\lambda, X) \). Then by assumption \( \sigma_u(x^* \circ D) \leq 0 \) for every \( x^* \in X^* \), which by Proposition 2.1 implies that \( \sigma_u(D) \leq 0 \) as desired. The converse implication is trivial. ■

Given a frequency \( \lambda \) and a Banach space \( X \), to obtain quantitative versions of Bohr’s theorem means to estimate the norm of the partial sum
operator

\[ S_N : D^\text{ext}_\infty(\lambda, X) \to D_\infty(\lambda, X), \quad D = \sum_{n=1}^{N} a_n e^{-\lambda_n s} \mapsto \sum_{n=1}^{N} a_n e^{-\lambda_n s}. \]

For the scalar case \( X = \mathbb{C} \) we deduce from [58, Theorem 3.2] the following estimate which does not assume any condition on \( \lambda \): There is a universal constant \( C > 0 \) such that for every \( D = \sum a_n e^{-\lambda_n s} \in H_\infty(\lambda) \) and \( 0 < k \leq 1 \),

\[ \left\| \sum_{n=1}^{N} a_n(D) e^{-\lambda_n s} \right\|_\infty \leq C \frac{k}{k} \left( \frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right) \| D \|_\infty. \]

Applying the Hahn–Banach theorem and (4.2) we obtain an extension of this result to \( H^+_\infty(\lambda, X) \).

**Theorem 4.5.** Let \( \lambda \) be an arbitrary frequency and \( X \) a Banach space. Then for all \( D \in H^+_\infty(\lambda, X) \), \( 0 < k \leq 1 \) and \( N \in \mathbb{N} \) we have

\[ \left\| \sum_{n=1}^{N} a_n(D) e^{-\lambda_n s} \right\|_\infty \leq C \frac{k}{k} \left( \frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right) \| D \|_\infty^+, \]

where \( C > 0 \) is a universal constant.

Several sufficient conditions on \( \lambda \) that guarantee Bohr’s theorem are known. Originally, Bohr [11] introduced the following condition, which we call Bohr’s condition (BC):

\[ \exists l > 0 \forall \delta > 0 \exists C > 0 \forall n \in \mathbb{N}: \quad \lambda_{n+1} - \lambda_n \geq C e^{-(l+\delta)\lambda_n}. \]

Note that \( \lambda = (\log n) \) has (BC) with \( l = 1 \). Secondly, there is a weaker condition than (BC), namely Landau’s condition (LC) from [49]:

\[ \forall \delta > 0 \exists C > 0 \forall n \in \mathbb{N}: \quad \lambda_{n+1} - \lambda_n \geq C e^{-\delta \lambda_n}. \]

To see an example that has (LC) and fails (BC), take \( \lambda = (\sqrt{\log n}) \). Assuming (LC), the choice \( k_N = e^{-\delta \lambda_N} \) in Theorem 4.5 leads to

\[ \| S_N \|_\infty \leq C e^{\delta \lambda_N}, \]

which is in fact the vector-valued quantitative variant of Bohr’s theorem under (LC). Assuming (BC) the choice \( k_N = \lambda^{-1}_N \) (here \( N \geq 2 \), since \( \lambda_1 = 0 \) is possible) yields

\[ \| S_N \| \leq C \lambda_N; \]

in the ordinary scalar case \( \lambda = (\log n) \) this was first proved in [4] (see also [55, Theorem 6.2.2] and [25, Theorem 1.13 and (24.14)])).

**4.3. Equivalence.** Given a frequency \( \lambda \) and a Banach space \( X \), we say that \( \lambda \) satisfies Bayart’s Montel theorem for \( X \) whenever the following statement holds: Every sequence \( (D^N) \) of Dirichlet series \( D^N = \sum a_n^N e^{-\lambda_n s} \in D_\infty(\lambda, X) \) admits a subsequence \( (N_k) \) and \( D \in D_\infty(\lambda, X) \) such that \( (D^{N_k}) \)
converges to $D$ uniformly on $[\text{Re} > \varepsilon]$ for every $\varepsilon > 0$ as $k \to \infty$ provided $(D^N)$ satisfies the following two conditions:

(a) There is a subsequence $(N_k)_k$ such that $\lim_{k \to \infty} a_{n_k}^N$ exists for all $n$.
(b) $(D^N)$ is bounded in $D_\infty(\lambda)$.

If $X = \mathbb{C}$, then we just say that $\lambda$ satisfies Bayart’s Montel theorem. In this case the first assumption (a) on $(D^N)$ by compactness is superfluous, since by (b) we have $|a_{n_k}^N| \leq \sup_N \|D^N\|_\infty < \infty$ for all $n$. In [5] Lemma 18 Bayart proves that $\lambda = (\log n)$ has this property.

Mainly collecting results from [29] and [30], we see that in the case of scalar-valued general Dirichlet series, several conditions considered so far in fact generate the same classes of frequencies.

**Theorem 4.6.** Let $\lambda$ be a frequency. Then the following are equivalent:

1. $\lambda$ satisfies Bohr’s theorem.
2. $D_\infty(\lambda)$ is complete.
3. $D_\infty(\lambda) = H_\infty(\lambda)$.
4. $D_\infty(\lambda) = H_\infty^+(\lambda, \text{Re} > 0)$.
5. $\lambda$ satisfies Bayart’s Montel theorem.

The first four equivalences are known from [29, Theorem 5.1] and [30, Theorem 2.16], and looking at [29, Theorem 5.8] we see that each of them implies (5). We close the cycle by including the proof of $(5) \Rightarrow (2)$.

**Proof of $(5) \Rightarrow (2)$**. We check that (5) implies that $D_\infty(\lambda) \subset H_\infty^+(\lambda, \text{Re} > 0)$ is closed, and so (2) follows. Indeed, let $(D^N)$ be a sequence in $D_\infty(\lambda)$ that converges to a function $F \in H_\infty^+(\lambda, \text{Re} > 0)$ uniformly on $[\text{Re} > 0]$. Assuming (5), there is a subsequence $(N_k)$ and $D \in D_\infty(\lambda)$ such that $D^{N_k}$ converges to $D$ on $[\text{Re} > \varepsilon]$ for every $\varepsilon > 0$ as $k \to \infty$. This implies that $F = D$ with $a_{\lambda_n}(F) = a_n(D)$ for all $n$, which finishes the proof.

The following vector-valued extension of Theorem 4.6 is the main contribution of this section.

**Theorem 4.7.** Let $\lambda$ be a frequency and $X$ a non-trivial Banach space. Then the following are equivalent:

1. $\lambda$ satisfies Bohr’s theorem for $X$.
2. $D_\infty(\lambda, X)$ is complete.
3. $D_\infty(\lambda, X) = H_\infty^+(\lambda, X)$.
4. $D_\infty(\lambda, X) = H_\infty^+(\lambda, \text{Re} > 0, X)$.
5. $\lambda$ satisfies Bayart’s Montel theorem for $X$.

Before turning to the proof of these equivalences, we add another remark.

**Remark 4.8.** The equality (4) is equivalent to the fact that every Dirichlet series $D = \sum a_{\lambda_n}(F)e^{-\lambda_n s}$ generated by a function $F \in H_\infty^+(\lambda, \text{Re} > 0, X)$...
converges on $[\text{Re} > 0]$. Then, moreover, by Theorem 4.1 statement (3) holds if and only if every Dirichlet series $D = \sum a_n e^{-\lambda_n s} \in \mathcal{H}_\lambda^+(\lambda, X)$ converges on $[\text{Re} > 0]$.

Proof of Remark 4.8. Clearly, if (4) holds, then every $D = \sum a_n (F) e^{-\lambda_n s}$ generated by some $F \in \mathcal{H}_\lambda^+([\text{Re} > 0], X)$ belongs to $\mathcal{D}_\lambda(\lambda, X)$, so converges on $[\text{Re} > 0]$. Assume conversely that every $D = \sum a_n (F) e^{-\lambda_n s}$ generated by some $F \in \mathcal{H}_\lambda^+([\text{Re} > 0], X)$ converges on $[\text{Re} > 0]$. Then by [30, Proposition 1.2] (see also [40, Chapter V]) the Dirichlet series $D$ is $(\lambda, 1)$-summable at every $s \in [\text{Re} > 0]$, that is,

$$D(s) = \lim_{x \to \infty} \sum_{\lambda_n < x} a_n (1 - \lambda_n/x) e^{-\lambda_n s}.$$ 

By Corollary 4.2 this limit for all $s \in [\text{Re} > 0]$ coincides with $F(s)$, which implies that $D \in \mathcal{D}_\lambda(\lambda, X)$. ■

Starting the proof of Theorem 4.7, we note first that by Proposition 4.4 statement (1) holds if and only if $\lambda$ satisfies Bohr’s theorem. We will see that in view of Theorems 4.1 and 4.6 the proof of Theorem 4.7 becomes evident once we prove the following two results.

**Proposition 4.9.** Let $\lambda$ be a frequency and $X$ a non-trivial Banach space. Then $\mathcal{D}_\lambda(\lambda) = \mathcal{H}_\lambda(\lambda)$ if and only if $\mathcal{D}_\lambda(\lambda, X) = \mathcal{H}_\lambda^+(\lambda, X)$. Moreover, if $X$ has ARNP, then each of these statements is equivalent to $\mathcal{D}_\lambda(\lambda, X) = \mathcal{H}_\lambda(\lambda, X)$.

**Proof.** This is an immediate consequence of Theorems 2.4 and 4.3 ■

**Proposition 4.10.** Let $\lambda$ be a frequency and $X$ a non-trivial Banach space. Then $\mathcal{D}_\lambda(\lambda)$ is complete if and only if $\mathcal{D}_\lambda(\lambda, X)$ is complete.

**Proof.** Completeness of $\mathcal{D}_\lambda(\lambda, X)$ implies completeness of $\mathcal{D}_\lambda(\lambda)$, since the second space can be viewed as a closed subspace of the first one.

Assume conversely that $\mathcal{D}_\lambda(\lambda)$ is complete. If $(D^N) \subset \mathcal{D}_\lambda(\lambda, X)$ is Cauchy, then $(a_n^N)_{N}$ (where $a_n^N$ denotes the $n$th coefficient of $D^N$) is Cauchy in $X$ and $(x^* \circ D^N)$ is Cauchy in $\mathcal{D}_\lambda(\lambda)$ for all $x^* \in X^*$. Define $a_n := \lim_{N \to \infty} a_n^N$ for $n \in \mathbb{N}$ and $D := \sum a_n e^{-\lambda_n s} \in \mathcal{D}(\lambda, X)$. Then $x^* \circ D = \lim_{N \to \infty} x^* \circ D^N \in \mathcal{D}_\lambda(\lambda)$ with $\|D\|_\infty = \sup_{\|x^*\|=1} \|x^* \circ D\|_\infty < \infty$, and by Theorem 2.4 we see that $D \in \mathcal{D}_\lambda(\lambda, X)$.

It remains to check that $\lim_{N \to \infty} D^N = D$ in $\mathcal{D}_\lambda(\lambda, X)$. Indeed, for $\varepsilon > 0$ take $N_0$ such that $\|D^N - D^M\|_{\infty} \leq \varepsilon$ for all $M, N \geq N_0$. Fix now $x^* \in X^*$ with $\|x^*\|=1$, and take $M \geq N_0$ such that $\|x^*(D - D^M)\|_{\infty} \leq \varepsilon$. Then for all $N \geq N_0$,

$$\|x^* \circ (D - D^N)\|_{\infty} \leq \|x^* \circ (D - D^M)\|_{\infty} + \|D^M - D^N\|_{\infty} \leq 2\varepsilon,$$

and so $\|D - D^N\|_{\infty} = \sup_{\|x^*\|=1} \|D - D^N\|_{\infty} \leq 2\varepsilon$. ■
Finally, we collect all partial results to prove Theorem 4.7.

**Proof of Theorem 4.7.** Propositions 4.4, 4.9 and 4.10 show the equivalence of (1)–(3), and Theorem 4.1 clearly proves that (3) and (4) are equivalent. Clearly, if (5) holds, then \( \lambda \) satisfies Bayart’s Montel theorem (for \( \mathbb{C} \)), and then we know from Theorem 4.6 that \( \lambda \) satisfies Bohr’s theorem (for \( \mathbb{C} \)), and hence by Proposition 4.4 also for \( X \). Finally, (1) \( \Rightarrow \) (5) follows by a word-for-word extension of [29, proof of Theorem 5.8] (this proof uses a straightforward vector-valued extension of the Bohr–Cahen formula from [58, Proposition 2.4] as well as [29, Lemma 5.2]).

Let us again come back to a characterization through \( N \)th abschnitte as in Theorem 3.5, this time for \( D_\infty(\lambda, X) \).

**Corollary 4.11.** Let \( \lambda \) satisfy Bohr’s theorem, \( D = \sum a_ne^{-\lambda ns} \) be a formal \( \lambda \)-Dirichlet series and \( X \) be any Banach space. Then the following are equivalent:

1. \( D \in D_\infty(\lambda, X) \).
2. \( D|_N \in D_\infty \) and \( \sup_{N \in \mathbb{N}} \|D|_N\|_\infty < \infty \).

Moreover in this case, \( \|D\|_\infty = \sup_{N \in \mathbb{N}} \|D|_N\|_\infty < \infty \).

**Proof.** This follows immediately from Theorems 3.5 and 4.7.

We finish with another equivalence for Bohr’s theorem—this time in terms of a concrete inequality. Roughly speaking, one might expect that the question of whether or not a given frequency \( \lambda \) satisfies Bohr’s theorem may be decided within \( \lambda \)-polynomials. Indeed, yet another consequence of Theorem 4.7 confirms this intuition.

**Theorem 4.12.** Let \( \lambda \) be an arbitrary frequency and \( X \) a Banach space. Then Bohr’s theorem holds for \( \lambda \) if and only if for every \( \sigma > 0 \) there is a constant \( C = C(\sigma) \) and \( M_0 = M_0(\sigma) \in \mathbb{N} \) such that for every \( M \geq M_0 \) and every sequence \((a_n) \subset X \) we have

\[
(4.3) \quad \sup_{N \leq M} \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{N} a_ne^{-\lambda_n it} \right\|_X \leq Ce^{\lambda M \sigma} \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{M} a_ne^{-\lambda_n it} \right\|_X .
\]

**Proof.** Assume that Bohr’s theorem holds for \( \lambda \). Then by Theorem 4.7 we know that \( D_\infty(\lambda, X) \) is complete. Hence an application of the uniform boundedness principle shows that for every \( \sigma > 0 \) there is a constant \( C = C(\sigma) > 0 \) such that for every \( D \in D_\infty(\lambda, X) \) we have

\[
\sup_{N} \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{N} a_n(D)e^{-\lambda_n(\sigma+it)} \right\|_X \leq C(\sigma) \|D\|_\infty
\]
In particular, for every sequence \((a_n)\) in \(X\),

\[
\sup_{N \leq M} \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{N} a_n e^{-\lambda_n (\sigma + it)} \right\|_X \leq C(\sigma) \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{M} a_n e^{-\lambda_n it} \right\|_X.
\]

Let us now verify (4.3) with \(M_0 = 1\). For some fixed \((a_n)\) in \(X\) we define

\[S_x(s) := \sum_{\lambda_n < x} a_n e^{-\lambda_n s}.\]

Then, using Abel summation, for every \(0 < y \leq x\) and \(t \in \mathbb{R}\) we have

\[S_y(it) = e^{-y\sigma} S_y(\sigma + it) - \sigma \int_0^y \left( e^{-\sigma a} S_a(\sigma + it) \right) da.
\]

Taking norms and applying (4.4) we obtain

\[\| S_y(it) \|_X \leq e^{-y\sigma} C(\sigma) \| S_x \|_\infty + C(\sigma) \| S_x \|_\infty e^{2\sigma x} \leq 2C(\sigma) e^{2\sigma x} \| S_x \|_\infty,
\]

which implies (4.3).

Assume conversely that (4.3) holds with a constant \(C(\sigma)\), and let \(D = \sum a_n e^{-\lambda_n s} \in D_\infty^\text{ext}(\lambda)\). We claim that \(\sigma_u(D) \leq 0\). First we show that (4.3) implies that for every \(M \geq M_0(\sigma)\),

\[
\sup_{N \leq M} \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{N} a_n e^{-\lambda_n (\sigma + it)} \right\|_X \leq C_1(\sigma) \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{M} a_n e^{-\lambda_n it} \right\|_X.
\]

Indeed, keeping the definition of \(S_x\), again by Abel summation we have

\[S_y(2\sigma + it) = e^{-2y\sigma} S_y(it) + 2\sigma \int_0^y \left( e^{-2\sigma a} S_a(\sigma + it) \right) da,
\]

and so by (4.3), for every \(M_0(\sigma) \leq y \leq x\),

\[|S_y(2\sigma + it)| \leq \| S_x \|_\infty \left( e^{-y\sigma} C(\sigma) + 2\sigma \int_0^y \left( e^{-\sigma a} da \right) \right) \leq \| S_x \|_\infty C(\sigma) \left( 1 + 2\sigma \int_0^\infty \left( e^{-\sigma a} da \right) \right).
\]

From this (4.5) follows.

Now let \(\sigma, \varepsilon > 0\). Applying (4.5) to the Dirichlet polynomial \(R_{x, \varepsilon}^1(D_\varepsilon)\) we obtain, for every \(N\) with \(M_0(\sigma) \leq \lambda_N \leq x\),

\[
\left\| \sum_{n=1}^{N-1} a_n (1 - \lambda_n / x) e^{-(\sigma + \varepsilon + s)\lambda_n} \right\|_\infty
\]

\[= \| S_{\lambda_N} (R_{x, \varepsilon}^1(D_\varepsilon + s)) \|_\infty \leq C(\sigma) \| R_{x, \varepsilon}^1(D_\varepsilon) \|_\infty.
\]

Letting \(x \to \infty\) we deduce (by Theorem 2.3) for every \(N\) with \(\lambda_N \geq M_0(\sigma)\)
that
\[ \left\| \sum_{n=1}^{N-1} a_n e^{-\sigma \varepsilon +s} \lambda_n \right\|_\infty \leq C(\sigma) \| D_\varepsilon \|_\infty \leq C(\sigma) \| D \|_\infty. \]

By Proposition 2.1 we infer that \( \sigma_u(D) \leq \sigma + \varepsilon \) for every \( \sigma, \varepsilon > 0 \), which yields the conclusion. □

4.4. Bohr’s strips. Given a frequency \( \lambda \) and a Banach space \( X \), we define
\[ L(\lambda, X) := \sup_{D \in D(\lambda, X)} (\sigma_a(D) - \sigma_c(D)), \]
and abbreviate \( L(\lambda) = L(\lambda, \mathbb{C}) \). Then straightforward arguments show that
\[ L(\lambda) = L(\lambda, X) = \sigma_c \left( \sum e^{-\lambda_n s} \right) = \sigma_a \left( \sum e^{-\lambda_n s} \right), \]
and (with a less obvious argument) Bohr proved in [12, §3, Hilfsätze 2 and 3] that
\[ L(\lambda) = \limsup_{N \to \infty} \frac{\log N}{\lambda N}. \]
Define also
\[ S(\lambda, X) := \sup_{D \in D(\lambda, X)} (\sigma_a(D) - \sigma_u(D)) \]
(again we write \( S(\lambda) = S(\lambda, \mathbb{C}) \)), and note that under Bohr’s theorem for \( \lambda \) (see also Proposition 4.4) we have
\[ S(\lambda, X) = \sup_{D \in D_\infty(\lambda, X)} \sigma_a(D). \]
Then in the ordinary case
\[ S((\log n), X) = 1 - \frac{1}{\cot(X)}, \]
where \( \cot(X) \) denotes the optimal cotype of \( X \). More precisely, for finite-dimensional \( X \) we have
\[ S((\log n), X) = 1/2, \]
which is a celebrated theorem of Bohnenblust and Hille [10], and for infinite-dimensional \( X \) the result was proved in [24] (see also [25, Theorem 26.4]).

For any frequency \( \lambda \) and any finite-dimensional Banach space \( X \), by the Cauchy–Schwarz inequality we deduce that
\[ S(\lambda, X) \leq L(\lambda)/2. \]
But this estimate is far from being an equality: For the scalar case \( X = \mathbb{C} \) a results of Neder [51] shows that for every \( x > 0 \) and \( 0 \leq y \leq x/2 \) there is a frequency \( \lambda \) such that \( S(\lambda, \mathbb{C}) = y \) and \( L(\lambda) = x \). Regarding the case \( x = \infty \),
for $\mathbb{Q}$-linearly independent frequencies $\lambda$ we have $S(\lambda, \mathbb{C}) = 0$ (see Theorem 4.7) although $L(\lambda) = \infty$ for suitable choices of $\lambda$. Hence it seems that for scalar-valued general Dirichlet series it probably only makes sense to ask for the exact value of $S(\lambda, \mathbb{C})$ for concrete (families of) frequencies. The game seems to change drastically if we consider $X$-valued $\lambda$-Dirichlet series for an infinite-dimensional Banach space $X$.

**Proposition 4.13.** Let $\lambda$ be any frequency satisfying Bohr’s theorem, and $X$ an infinite-dimensional Banach space. Then

\begin{equation}
L(\lambda) \left(1 - \frac{1}{\cot(X)}\right) \leq S(\lambda, X).
\end{equation}

Moreover, if $\lambda$ is $\mathbb{Q}$-linearly independent, then equality holds in (4.10).

**Remark 4.14.** Bayart [6] shows that (4.10) in fact holds without any assumption on $\lambda$ (replacing the closed graph argument we use by a gliding hump argument).

**Proof of Proposition 4.13.** First we show that $S(\lambda, X)$ equals the infimum of all $\sigma \in \mathbb{R}$ for which there is a constant $C > 0$ such that for every $N$ and sequence $(a_n) \subset X$ we have

\begin{equation}
\sum_{n=1}^{N} \|a_n\|_X e^{-\lambda_n \sigma} \leq C \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{N} a_n e^{-\lambda_n \sqrt{t}} \right\|_X.
\end{equation}

Indeed, if $\sigma > S(\lambda, X)$, then a closed graph argument (here we use completeness of $D_\infty(\lambda, X)$ which is guaranteed by assuming Bohr’s theorem and Theorem 4.7) gives (4.11).

Conversely, denote by $A$ the infimum above and take $\sigma > A$. Fix $\varepsilon > 0$. Then by the definition of $A$ we have, for every $D = \sum a_n e^{-\lambda_n s}$,

\begin{equation}
\lim_{N \to \infty} \sum_{n=1}^{N} \|a_n\|_X e^{-\lambda_n (\sigma + \varepsilon)} \leq C \lim_{N \to \infty} \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{N} a_n e^{-\lambda_n (\varepsilon + it)} \right\|_X = \|D\varepsilon\|_\infty,
\end{equation}

where the last equality follows from Bohr’s theorem. Hence $\sigma + \varepsilon \geq S(\lambda, X)$ and so altogether we obtain $A = S(\lambda, X)$. Write $1/\cot(X)' = 1 - 1/\cot(X)$. Then a direct calculation shows

\begin{equation}
L(\lambda) \left(1 - \frac{1}{\cot(X)}\right) = L(\cot(X)'\lambda) = \inf \left\{ \sigma \in \mathbb{R} \mid (e^{-\lambda_n \sigma}) \in \ell_{\cot(X)'}) \right\} =: B.
\end{equation}

Let us show that $B \leq A$, and assume without loss of generality that $A < \infty$. Take some $A < \sigma$, i.e. there is $C > 0$ such that for every $N$ and sequence $(a_n) \subset X$ we have

\begin{equation}
\sum_{n=1}^{N} \|a_n\|_X e^{-\lambda_n \sigma} \leq C \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{N} a_n e^{-\lambda_n it} \right\|_X.
\end{equation}
We show that \((e^{-\lambda n\sigma}) \in \ell_{\cot(X)^\prime}\), that is, \(B \leq \sigma\). Since \(X\) is infinite-dimensional, there are \(x_1, \ldots, x_N \in X\) such that for all \(u = (u_1, \ldots, u_N) \in \mathbb{C}^N\),

\[
\frac{1}{2} \|u\|_\infty \leq \left\| \sum_{n=1}^{N} x_n u_n \right\|_X \leq \|u\|_{\cot(X)}
\]

(see [50] and also [35, Theorem 14.5]). Now let \(w_1, \ldots, w_N \in \mathbb{C}\). Then, applying (4.12) and (4.13) with \(a_n = e_n w_n\) and \(u = (w_1, \ldots, w_N)\), we deduce by the choice of \(\sigma\) that

\[
\sum_{n=1}^{N} |e^{-\lambda n\sigma} w_n| \leq 2 \sum_{n=1}^{N} \|x_n w_n\|_X e^{-\lambda n\sigma} \leq 2C(\sigma) \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{N} x_n w_n e^{-\lambda n t} \right\|_X
\]

\[
\leq 2C(\sigma) \left( \sum_{n=1}^{N} |w_n e^{-i\lambda n t}|_{\cot(X)} \right) \frac{1}{\cot(X)} = 2C(\sigma) \left( \sum_{n=1}^{N} |w_n|_{\cot(X)} \right) \frac{1}{\cot(X)}.
\]

Consequently, by duality \((e^{-\lambda n\sigma}) \in \ell_{\cot(X)^\prime}\), which finally implies \(B \leq A\).

It remains to verify that \(B \geq A\) whenever \(\lambda\) is \(\mathbb{Q}\)-linearly independent. Note that \(T^\infty\) with the mapping

\[
\beta: \mathbb{R} \to T^\infty, \quad t \mapsto (e^{-it\lambda})
\]

forms a \(\lambda\)-Dirichlet group. We assume that \(B < \infty\) and take \(\sigma > B\). Then there is some \(0 < \varepsilon < 1\) such that \(\sigma > L(\lambda) \left( 1 - \frac{1}{\cot(X) + \varepsilon} \right)\). Defining \(q := \cot(X) + \varepsilon\) we obtain, for every sequence \((a_n) \subset X\),

\[
\sum_{n=1}^{N} \|a_n\|_X e^{-\lambda n\sigma} \leq C(\sigma) \left( \sum_{n=1}^{N} \|a_n\|_X^q \right)^{1/q} \leq C(X, \sigma) \left( \int_{T^\infty} \left\| \sum_{n=1}^{N} a_n z_n \right\|_X^q dz \right)^{1/q}
\]

\[
\leq C(X, \sigma) \sup_{z \in T^\infty} \left\| \sum_{n=1}^{N} a_n z_n \right\|_X = \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{N} a_n e^{-it\lambda} \right\|_X,
\]

where the first inequality follows from Hölder’s inequality. This implies that \(B \geq A\) and finishes the proof. ■

The following proposition gives some evidence that the estimate in (4.10) (as in the particular case (4.9)) might be an equality.

**Proposition 4.15.** Given a frequency \(\lambda\) satisfying Bohr’s theorem and an infinite-dimensional Banach space \(X\) of type 2, we have

\[
S(\lambda, X) = L(\lambda) \left( 1 - \frac{1}{\cot(X)} \right).
\]

Since \(\cot(\ell_r) = \max\{2, r\}\), \(1 \leq r \leq \infty\), the following consequence is immediate.
Corollary 4.16. For every frequency \( \lambda \) satisfying Bohr’s theorem and \( 2 \leq r \leq \infty \) we have

\[
S(\lambda, \ell_r) = L(\lambda)(1 - 1/r).
\]

We precede the proof of Proposition 4.15 with some more preliminaries. For \( 1 \leq p \leq \infty \) define, in analogy to (4.8),

\[
S_p(\lambda, X) = \sup_{D \in \mathcal{H}_p(\lambda, X)} \sigma_a(D),
\]

and again \( S_p(\lambda) = S_p(\lambda, \mathbb{C}) \). Observe that an application of the Cauchy–Schwarz inequality shows that

\[
S_2(\lambda) = L(\lambda)/2.
\]

We mention in passing that it would be very interesting to know for which \( p \)'s and for which frequencies \( \lambda \) with \( L(\lambda) < \infty \) this equality also holds true for \( S_p(\lambda) \); note that if \( S_p(\lambda) = \infty \), then \( L(\lambda) = \infty \), but the converse seems unclear. Anyway, in view of Neder’s result mentioned above, the behavior of \( S_p(\lambda) \) might not be as chaotic as that of \( S(\lambda) \).

Recall here that in the ordinary case \( \lambda = (\log n) \) we have \( S_p((\log n)) = S_2((\log n)) = 1/2, 1 \leq p \leq \infty \), since by [5] the following ‘hypercontractivity property’ holds: For every \( \sigma > 0 \) and \( 1 \leq p, q < \infty \) the translation operator

\[
T_\sigma : \mathcal{H}_p((\log n)) \to \mathcal{H}_q((\log n)), \quad \sum a_n n^{-s} \mapsto \sum \frac{a_n}{n^\sigma} n^{-s},
\]

is bounded. We know from [6] that the analogous result for arbitrary frequencies \( \lambda \) fails—even under (BC) (which implies that \( L(\lambda) \) is finite).

We need the following alternative descriptions of \( S_p(\lambda, X) \) (the proof of which follows by a standard closed graph argument and the Bohr–Cahen formulas for \( \sigma_a \) and \( \sigma_u \); analyse e.g. the proof of [25] Proposition 9.5]). Assume that \( \lambda \) satisfies Bohr’s theorem, \( X \) is some Banach space, and \( 1 \leq p \leq \infty \). Then \( S_p(\lambda, X) \) equals the infimum of all \( \sigma > 0 \) for which there is a constant \( c_\sigma > 0 \) such that for all finite Dirichlet polynomials \( D = \sum_{\lambda_n \leq x} a_n e^{-\lambda_n s}, \)

\[
\sum_{\lambda_n \leq x} \|a_n\| e^{-\lambda_n \sigma} \leq c_\sigma \|D\|_p;
\]

alternatively we may replace this estimate by

\[
\sum_{\lambda_n \leq x} \|a_n\| \leq c_\sigma e^{x \sigma} \|D\|_p.
\]

The last ingredient for the proof of Proposition 4.15 is given by the following result, which can be extracted from the proof of [20, Theorem 4.1].
Lemma 4.17. Let $X$ be a Banach space of type 2 and $(G, \beta)$ a $\lambda$-Dirichlet group. Then there is a constant $C > 0$ such that for every choice of finitely many $a_1, \ldots, a_m \in X$ we have

$$
E \left\| \sum_{n=1}^{m} \varepsilon_n a_n \right\|_X^2 \leq C \int_G \left\| \sum_{n=1}^{m} a_n h_{\lambda n}(\omega) \right\|_X^2 \, d\omega,
$$

where the $\varepsilon_n$ are independent Bernoulli variables.

For the sake of completeness we sketch the proof.

Proof of Lemma 4.17. Denote $f = \sum_{n=1}^{m} a_n h_{\lambda n}$ and let $T : \ell^n_2 \to X$ be the operator defined by $T(e_n) = a_n$. Since $X$ has type 2, we have

$$
\left( E \left\| \sum_{n=1}^{m} \varepsilon_n a_n \right\|_X^2 \right)^{1/2} \ll \left( \int_G \left\| \sum_{n=1}^{m} \gamma_n a_n \right\|_X^2 \right)^{1/2} \ll \pi_2(T^*)
$$

(see e.g. [59, (4.2) and Theorem 12.2]). For every $x^* \in X^*$ observe that

$$
\| x^*(f) \|_{L^2(G)} = \| (x^*(a_n)) \|_{\ell^m_2} = \| T^*(x^*) \|_{\ell^m_2}.
$$

Therefore, given a finite collection of vectors $x_k^* \in X^*$, we have

$$
\sum_k \| T^*(x_k^*) \|_{\ell^m_2} = \sum_k \| x_k^*(f) \|_{L^2(G)}^2 = \int_G \sum_k \| x_k^*(f(\omega)) \|_X^2 \, d\omega
$$

$$
\leq \int_G \| f(\omega) \|_X^2 \sup_{x^{**} \in B_{X^{**}}} \sum_k | x^{**}(x_k^*) |^2 \, d\omega
$$

$$
= \| f \|_{L^2(G,X)}^2 \sup_{x^{**} \in B_{X^{**}}} \sum_k | x^{**}(x_k^*) |^2.
$$

From the definition of the 2-summing norm we deduce that $\pi_2(T^*) \leq \| f \|_{L^2(G,X)}$, which concludes the argument. $

$\textbullet$

Having Lemma 4.17 at hand, in order to prove Proposition 4.15 we check that for any frequency $\lambda$ and any Banach space $X$ the equality from (4.14) holds true provided (4.20) is satisfied. Therefore, it is natural to ask for which Banach spaces and frequencies inequality (4.20) holds. By [20, Theorem 4.1], (4.20) is equivalent to type 2 for the case of ordinary Dirichlet series (i.e. $\lambda = (\log n)$). Also, combining [20, Proposition 3.1] and [20, Theorem 4.1] or looking carefully at the proof of [2, Theorem 1.5], we find that (4.20) is equivalent to type 2 whenever $\lambda = (n)$ (so for the case of classical Fourier series). On the other hand, there are frequencies $\lambda$ for which (4.20) is satisfied for every Banach space $X$: $\mathbb{Q}$-linearly independent frequencies and lacunary frequencies (see [53, Theorem 2.1]). As a consequence, in these cases we have $S(\lambda, X) = L(\lambda)(1 - 1/cot(X))$ regardless of the Banach space $X$; see again the second statement in Proposition 4.13.
Proof of Proposition 4.15. Fix some $\lambda$-Dirichlet group $(G, \beta)$. Since $X$ has type 2, we have $\cot(X) < \infty$. Then for $\cot(X) < q < \infty$ and every sequence $(a_n)$ in $X$ each $x$ we have

$$\sum_{\lambda_n \leq x} \|a_n\|_X e^{-\frac{L(\lambda)+\varepsilon}{q} \lambda_n} \leq \left( \sum_{\lambda_n \leq x} e^{-(L(\lambda)+\varepsilon)\lambda_n} \right)^{1/q} \left( \sum_{\lambda_n \leq x} \|a_n\|_X^q \right)^{1/q}.$$  

Applying the fact that $X$ has cotype $q$ and (4.20), we obtain

$$\sum_{\lambda_n \leq x} \|a_n\|_X e^{-\frac{L(\lambda)+\varepsilon}{q} \lambda_n} \leq C \left( \sum_{\lambda_n \leq x} \varepsilon_n a_n \right)^{2/q} \leq C \left( \int_{G} \left\| \sum_{\lambda_n \leq x} a_n h_{\lambda_n}(\omega) \right\|_X^2 d\omega \right)^{1/2}.$$  

By (4.18) we deduce that

$$S(\lambda, X) \leq S_2(\lambda, X) \leq \frac{L(\lambda)}{q} + \varepsilon$$  

for every $\varepsilon > 0$ and every $q > \cot(X)$, which concludes the argument. ■

4.5. $Q$-linear independence II. In the case of a $Q$-linearly independent frequency $\lambda$ we know from Theorem 3.10 that the equality $H^w_\infty(\lambda, X) = H^w_\infty(\lambda, X)$ holds if and only if $c_0$ is not isomorphically contained in $X$. Moreover, for this class of $\lambda$’s and for the scalar case $X = \mathbb{C}$, in [58, Theorem 4.7] it is shown that isometrically

(4.21)  

$$D^w_\infty(\lambda) = \ell_1, \quad \sum a_n e^{-\lambda_n s} \mapsto (a_n).$$  

Hence, by Proposition 4.10 and Theorem 4.7 we have $D^w_\infty(\lambda, X) = H^w_\infty(\lambda, X)$ for every Banach space $X$. Altogether we see that for $Q$-linearly independent frequencies $\lambda$ the equality $D^w_\infty(\lambda, X) = H^w_\infty(\lambda, X)$ holds if and only if $c_0$ is not isomorphically contained in $X$.

In this section we provide another approach to this result by extending the equality $D^w_\infty(\lambda) = \ell_1$ to its vector-valued analog. We write $\ell^w_1(X)$ for the Banach space of all weakly summable sequences $(x_n)$ in $X$, where the norm is given by

$$w((x_n)) = \sup_{x^* \in B_{X^*}} \sum_{n=1}^{\infty} |x^*(x_n)|,$$

and $\ell^{w,0}_1(X)$ for its closed subspace of $\ell^w_1(X)$ consisting of all sequences $(x_n) \in \ell^w_1(X)$ such that

$$\lim_{N \to \infty} w((x_n)_{n \geq N}) = 0.$$  

Recall from [34, Theorem V.8] that $\ell^{w,0}_1(X) = \ell^w_1(X)$ if and only if $c_0$ is not isomorphically contained in $X$.
Theorem 4.18. If the frequency \( \lambda \) is \( \mathbb{Q} \)-linearly independent, then for every Banach space \( X \) we have isometrically
\[
\ell_{1}^{w,0}(X) = \mathcal{H}_{\infty}(\lambda, X) \subset \mathcal{D}_{\infty}(\lambda, X) = \ell_{1}^{w}(X).
\]
In particular, \( \mathcal{D}_{\infty}(\lambda, X) = \mathcal{H}_{\infty}(\lambda, X) \) isometrically if and only if \( X \) does not contain an isomorphic copy of \( c_{0} \).

Proof. We start with the second equality in (4.22). Let \( D \in \mathcal{D}_{\infty}(\lambda, X) \) with Dirichlet coefficients \( (a_{n}) \). Then by (4.21) and Theorem 2.4
\[
w((a_{n})) = \left\| \sup_{x^{*} \in B_{X^{*}}} \sum_{n=1}^{\infty} |x^{*}(a_{n})| \right\|_{\infty} = \left\| x^{*} \circ D \right\|_{\infty} = \left\| D \right\|_{\infty} < \infty.
\]
Conversely, the same arguments give \( \ell_{1}^{w}(X) \subset \mathcal{D}_{\infty}(X) \).

Now we verify the first equality in (4.22). Fix \( (a_{n}) \in \ell_{1}^{w,0}(X) \) and a \( \lambda \)-Dirichlet group \( (G, \beta) \). Then for all \( M \geq N \) we have (using the duality of \( \ell_{1} \) and \( \ell_{\infty} \))
\[
\left\| \sum_{n=N}^{M} a_{n} h_{\lambda n} \right\|_{\infty} = \sup_{\omega \in G} \left\| \sum_{n=N}^{M} a_{n} h_{\lambda n}(\omega) \right\|_{X} \leq \sup_{b \in B_{t_{\infty}}} \left\| \sum_{n=N}^{M} a_{n} b_{n} \right\|_{X}
\]
\[
\leq \sup_{x^{*} \in B_{X^{*}}} \sum_{n=N}^{\infty} |x^{*}(a_{n})| = w((a_{n})_{n \geq N}),
\]
which vanishes as \( N \to \infty \). This shows that \( (\sum_{n=1}^{M} a_{n} h_{\lambda n})_{M} \) is a Cauchy sequence in \( H_{\infty}^{\lambda}(G, X) \), and therefore there is a limit \( f \in H_{\infty}^{\lambda}(G, X) \) with \( \|f\|_{\infty} \leq w((a_{n})) \) and \( \hat{f}(h_{\lambda n}) = a_{n} \) for all \( n \). Conversely, let \( f \in H_{\infty}^{\lambda}(G, X) \) and define, for \( \omega, \eta \in G \),
\[
F(\omega) = [\eta \mapsto f(\eta \omega)].
\]
Then we straightforwardly see that \( F \in L_{1}(G, H_{\infty}^{\lambda}(G, X)) \). Moreover, the Fourier coefficients are given by \( \hat{F}(h_{x}) = \hat{f}(h_{x}) h_{x} \), since
\[
\hat{F}(h_{x})(\eta) = \left( \int_{G} F(\omega) \overline{h_{x}(\omega)} d\omega \right)(\eta) = \int_{G} f(\eta \omega) \overline{h_{x}(\omega)} d\omega = \hat{f}(h_{x}) h_{x}(\eta).
\]
So \( F \in H_{1}^{\lambda}(G, H_{\infty}^{\lambda}(G, X)) \), and then Theorem 5.2 (to be proved in the final section) implies that for almost all \( \omega \in G \) we have
\[
\sum_{n=1}^{\infty} \hat{f}(h_{\lambda n}) h_{\lambda n} h_{\lambda n}(\omega) = \sum_{n=1}^{\infty} \hat{F}(h_{\lambda n}) h_{\lambda n}(\omega) = F(\omega)
\]
with convergence in \( H_{\infty}^{\lambda}(G, X) \). Hence, using the inclusion in (4.22), which is a consequence of Corollary 3.4 and Theorem 4.7, there is some \( \omega \in G \) such
that
\[
\lim_{N \to \infty} w((\hat{f}(h\lambda_n))_{n>N}) = \lim_{N \to \infty} \left\| \sum_{n=1}^{N} \hat{f}(h\lambda_n)h\lambda_n - f \right\|_{H_{\infty}^\lambda(G,X)}
\]
\[
= \lim_{N \to \infty} \left\| \sum_{n=1}^{N} \hat{f}(h\lambda_n)(\omega)h\lambda_n - f(\omega) \right\|_{H_{\infty}^\lambda(G,X)}
\]
\[
= \lim_{N \to \infty} \left\| \sum_{n=1}^{N} \hat{F}(h\lambda_n)(\omega) - F(\omega) \right\|_{H_{\infty}^\lambda(G,X)} = 0,
\]
which is what we aimed for.

5. Maximal inequalities. A fundamental question in Fourier analysis is under which assumptions the Fourier series of \( f \in H_1^\lambda(G,X) \) represents \( f \),
\[
f \sim \sum \hat{f}(h\lambda_n)h\lambda_n,
\]
in the sense of pointwise convergence, or convergence with respect to some norm. More questions appear if one here replaces ordinary summation by other summation methods. For the scalar case, various results in this direction can be found in [29] and [30], and our aim in this final section is to study their vector-valued counterparts (and related topics).

5.1. Carleson–Hunt theorem. A famous example in the direction of (5.1) for \( X = \mathbb{C} \) is given by the Carleson–Hunt theorem, which states that
\[
f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k
\]
almost everywhere on \( \mathbb{T} \) provided that \( f \in L_p(\mathbb{T}) \) and \( 1 < p \leq \infty \). As proved in [29, Theorem 2.2] (see also [39] and [42] for earlier results in this direction) this result extends to \( H_p^\lambda(G) \), \( 1 < p \leq \infty \), for arbitrary frequencies \( \lambda \) and \( \lambda \)-Dirichlet groups \( (G,\beta) \). The techniques of the proof in [29] extend to \( H_p^\lambda(G,X) \) once we assume that the \( X \)-valued counterpart of (5.2) is valid.

**Theorem 5.1.** Assume that \( X \) is a Banach space for which the Carleson–Hunt maximal inequality holds, i.e. for every \( 1 < p < \infty \) there is some \( C > 0 \) such that for every \( f \in L_p(\mathbb{T},X) \),
\[
\left\| \sup_{N} \left\| \sum_{n=-N}^{N} \hat{f}(n)z^n \right\|_X \right\|_{L_p(\mathbb{T})} \leq C \|f\|_{L_p(\mathbb{T},X)}.
\]
Then for every frequency \( \lambda \), every \( \lambda \)-Dirichlet group \( (G,\beta) \), every \( 1 < p < \infty \) and all \( f \in H_p^\lambda(G,X) \) we have
\[
f = \sum_{n=1}^{\infty} \hat{f}(h\lambda_n)h\lambda_n
\]
almost everywhere on $G$, and

\begin{equation}
\left\| \sup_N \left\| \sum_{n=1}^{N} \hat{f}(h_{\lambda_n})h_{\lambda_n} \right\|_X \right\|_p \leq C \|f\|_p,
\end{equation}

where $C = C(p)$ is a constant which only depends on $p$.

A Banach space for which (5.3) holds must have UMD (see for example [46], where also some sufficient conditions are presented). To our knowledge, the class of Banach spaces for which (5.3) holds is not known. In the case of Banach lattices, (5.3) holds true if and only if $X$ has UMD [56]. The next result shows that for $\mathbb{Q}$-linearly independent frequencies, (5.5) is valid for every Banach space $X$ and $1 \leq p < \infty$.

**Theorem 5.2.** Let $\lambda$ be a $\mathbb{Q}$-linearly independent frequency, $(G, \beta)$ a $\lambda$-Dirichlet group, and $X$ a Banach space. Then for every $1 \leq p < \infty$ and all $f \in H_p^\lambda(G, X)$ we have

\begin{equation}
\left\| \sup_N \left\| \sum_{n=1}^{N} \hat{f}(h_{\lambda_n})h_{\lambda_n} \right\|_X \right\|_p \leq 2 \|f\|_p.
\end{equation}

In particular, almost everywhere on $G$ we have

$$f = \sum_{n=1}^{\infty} \hat{f}(h_{\lambda_n})h_{\lambda_n}.$$ 

**Proof.** Consider the $\lambda$-Dirichlet group induced by $\beta : \mathbb{R} \to \mathbb{T}^\infty$ given by $\beta(t) = (e^{-\lambda_n t})$. We have to prove that for every $f \in H_p^\lambda(\mathbb{T}^\infty, X)$,

$$\left( \int_{\mathbb{T}^\infty} \sup_N \left\| \sum_{k=1}^{N} \hat{f}(e_k)z_k \right\|_X^p dz \right)^{1/p} \leq 2 \|f\|_p.$$ 

This estimate follows from Lévy’s inequality for Banach spaces (see for example [48, Proposition 1.1.1]). This result is usually stated for Bernoulli random variables but in fact it holds for Steinhaus variables $\mathbb{T}^\infty \to \mathbb{T}$, $z \mapsto z_k$, with the same proof. For $1 \leq N \leq M$, we apply Lévy’s inequality to get

$$\left( \int_{\mathbb{T}^\infty} \sup_{1 \leq N \leq M} \left\| \sum_{k=1}^{N} \hat{f}(e_k)z_k \right\|_X^p dz \right)^{1/p} \leq 2 \left( \int_{\mathbb{T}^\infty} \left\| \sum_{k=1}^{M} \hat{f}(e_k)z_k \right\|_X^p dz \right)^{1/p} \leq 2 \|f\|_p,$$

where in the last inequality we use Theorem 3.5 and Corollary 3.4. The result then follows from the monotone convergence theorem taking $M \to \infty$. 

Clearly, Theorem 5.1 does not apply to the case $p = 1$. But under Landau’s condition (LC), this loss can be offset if we replace the function $f \in H_1^\lambda(G, X)$ by its convolution $f \ast p_\sigma$ with the Poisson measure—this even works for every $X$. 

Theorem 5.3. Suppose that $\lambda$ satisfies (LC), $(G, \beta)$ is a $\lambda$-Dirichlet group, and $X$ a Banach space. Then for every $\varepsilon > 0$ there is a constant $C = C(\varepsilon, \lambda)$ such that for all $f \in H^\lambda_1(G, X)$ we have

\[
\left\| \sup_{\sigma \geq \varepsilon} \sup_{N} \left\| \sum_{n=1}^{N} \hat{f}(h_{\lambda n}) e^{-\sigma \lambda_n} h_{\lambda n} \right\|_{L^\infty(G, X)} \right\|_{1, \infty} \leq C \|f\|_{1, \infty},
\]

where $\| \cdot \|_{1, \infty}$ denotes the norm of the weak $L^1$-space $L^1_{1, \infty}(G, X)$. Moreover, for every $f \in H^\lambda_1(G, X)$ there is a null set $N \subset G$ such that for every $\sigma > 0$ and every $\omega \in G \setminus N$,

\[
f * p_\sigma(\omega) = \sum_{n=1}^{\infty} \hat{f}(h_{\lambda n}) e^{-\sigma \lambda_n} h_{\lambda n}(\omega).
\]

The scalar-valued variant of this result can be found in [29, Theorem 3.2], and its proof extends word for word to $H^\lambda_1(G, X)$.

5.2. Almost everywhere convergence of Riesz means. It is known that (5.4) may fail for $p = 1$ as it does for the power series case $\lambda = (n)$. For the scalar case $X = \mathbb{C}$, a substitute for this failure is given by [30, Theorem 2.1 and Corollary 2.2], which states that, given any $k > 0$, for every $f \in H^\lambda_1(G)$ we have

\[
f(\omega) = \lim_{x \to \infty} \sum_{\lambda_n < x} \hat{f}(h_{\lambda n})(1 - \lambda_n/x)^k h_{\lambda n}(\omega)
\]

almost everywhere on $G$; in the language of [30] this means that $f$ is summable by its first $(\lambda, k)$-Riesz means almost everywhere on $G$.

Analysing the proof of this result (more precisely, replacing there the absolute values by the norms), we see that (5.7) extends to $X$-valued functions without any restrictions on $X$ (in contrast to Theorem 5.1).

Theorem 5.4. Let $\lambda$ be a frequency, $k > 0$, $(G, \beta)$ any $\lambda$-Dirichlet group, and $X$ a Banach space. Then

\[
P^\lambda_k f(\omega) = \sup_{x > 0} \left\| \sum_{\lambda_n < x} \hat{f}(h_{\lambda n})(1 - \lambda_n/x)^k h_{\lambda n}(\omega) \right\|_X
\]

defines a bounded operator from $H^\lambda_1(G, X)$ to $L^1_{1, \infty}(G, X)$. In particular, given $f \in H^\lambda_1(G, X)$, for every $k > 0$, almost everywhere on $G$ we have

\[
f = \lim_{x \to \infty} \sum_{\lambda_n < x} \hat{f}(h_{\lambda n})(1 - \lambda_n/x)^k h_{\lambda n}.
\]

5.3. Helson type theorems. Let us translate Theorems 5.1, 5.3, and 5.4 on pointwise summability of the Fourier series of functions $f \in H^\lambda_p(G, X)$ into convergence theorems for so-called vertical limits

\[
D^\omega = \sum \hat{f}(h_{\lambda n}) h_{\lambda n}(\omega) e^{-\lambda_n s}, \quad \omega \in G,
\]
of the corresponding Dirichlet series \( D = \mathcal{B}(f) \in \mathcal{H}_p(\lambda, X) \). The key to translating these results into results on Dirichlet series is given by the vector-valued analogs of [28, Remark 1.3] and [30, Lemma 1.4]. We remark that results of this type were first addressed by Helson in [43], [44].

**Theorem 5.5.** Let \((G, \beta)\) be a \(\lambda\)-Dirichlet group and \(D = \sum a_n e^{-\lambda_n s} \in \mathcal{H}_p(\lambda, X)\), where \(1 \leq p \leq \infty\) and \(X\) is a Banach space.

1. Provided \(1 < p < \infty\) and \(X\) satisfies (5.3), almost all vertical limits \(D^\omega\), \(\omega \in G\), converge almost everywhere on \([\text{Re} = 0]\) and so consequently at every \(s \in [\text{Re} > 0]\).
2. If \(\lambda\) satisfies (LC) and \(p = 1\), then \(D^\omega\) converges on \([\text{Re} > 0]\) for almost every \(\omega \in G\).
3. If \(p = 1\) and \(k > 0\), then almost all \(D^\omega\), \(\omega \in G\), are \((\lambda,k)\)-Riesz summable almost everywhere on \([\text{Re} = 0]\), that is, the limit
   \[
   \lim_{x \to \infty} \sum_{\lambda_n < x} a_n h_{\lambda_n}(\omega)(1 - \lambda_n/x)^k e^{-it\lambda_n}
   \]
   exists for almost all \(t \in \mathbb{R}\). In particular, \(\sigma_c^{\lambda,k}(D^\omega) \leq 0\) for almost every \(\omega\).

**5.4. Riesz projection.** We finally study the boundedness of the vector-valued Riesz projection for Dirichlet groups, or equivalently the boundedness of the vector-valued Hilbert transform on these groups.

Let \(G\) be a compact abelian group and \(P \subset \hat{G}\) such that \(P + P \subset P\), \(P \cup (-P) = \hat{G}\) and \(P \cap (-P) = \{0\}\). Notice that \(P\) (which stands for ‘positive’) defines an order on \(\hat{G}\). In the case of Dirichlet groups we will always consider the order inherited from \(\mathbb{R}\) given by
\[
P = \{h_x \in \hat{G} \mid 0 \leq x \in \hat{\beta}(\hat{G})\}.
\]
A distinction between positive and negative characters allows us to define a Hilbert transform, also known as abstract conjugate function. Indeed, define the Hilbert transform \(T_P\) and the Riesz projection \(R_P\) onto \(X\)-valued trigonometric polynomials on \(G\) by
\[
T_P\left(\sum_{\gamma \in \hat{G}} x_{\gamma} \gamma\right) = -i \sum_{\gamma \in \hat{G}} \text{sg}(\gamma) x_{\gamma} \gamma \quad \text{and} \quad R_P\left(\sum_{\gamma \in \hat{G}} x_{\gamma} \gamma\right) = \sum_{\gamma \in P} x_{\gamma} \gamma,
\]
where \(\text{sg}(\gamma) = \chi_{P}(\gamma) - \chi_{-P}(\gamma)\). A Banach space has the ACF (abstract conjugate function) property if there is \(1 < p < \infty\) and a constant \(C > 0\) such that for every compact abelian group \(G\) with ordered characters, \(T_P\) extends to a bounded operator on \(L_p(G, X)\) with norm bounded by \(C\).

It turns out that UMD is equivalent to ACF for some (equivalently, every) \(1 < p < \infty\) (see [8], [14], [17], and also [3] for a complete picture and an alternative proof of ACF \(\Rightarrow\) UMD).
Theorem 5.6. Let \((G, \beta)\) be a non-trivial Dirichlet group and \(X\) a Banach space. Then \(X\) has UMD if and only if the Riesz projection 
\[ R : L_p(G, X) \to H_p(G, X) \]
is bounded for some (and then for all) \(1 < p < \infty\), where we denote by \(H_p(G, X)\) the space of all \(f \in L_p(G, X)\) such that \(\hat{f}(h_x) \neq 0\) implies \(x \geq 0\) for every \(x\).

Proof. If \(X\) has UMD, then joining [17] and [3, Theorem 2.1] we deduce that \(X\) enjoys the ACF property for every \(1 < p < \infty\). In our setting we have 
\[ P = \{h_x \in \hat{G} \mid x \geq 0\} \]
and so \(T_P\) is bounded. Therefore, the Riesz projection \(R\) is bounded, since 
\[ Rf = \frac{1}{2}(f + iT_Pf + \hat{f}(0)) \]
for every \(f \in L_p(G, X)\).

Conversely, since \((G, \beta)\) is a Dirichlet group, \(\hat{G}\) is a subgroup of \(\mathbb{R}\). Take any non-zero character \(h_x \in \hat{G}\) with \(x > 0\). Then the mapping
\[ \alpha : \mathbb{Z} \hookrightarrow \hat{G}, \quad k \mapsto h_{kx}, \]
where \(kx\) is a product in \(\mathbb{R}\), is an injective homomorphism. So the dual map \(\hat{\alpha} : G \to \mathbb{T}\) is continuous and has dense range. Then 
\[ L_p(T, X) = H_p^\alpha(\mathbb{Z})(G, X), \quad f \mapsto f \circ \hat{\alpha}, \]
is an onto isometry and \(\hat{f} \circ \hat{\alpha} \circ \alpha = \hat{f}\), that is, \(\hat{f}(k) = \hat{f}(\hat{\alpha}(kx))\) for all \(k \in \mathbb{Z}\) (see [28], Proposition 3.17 with \(E = \mathbb{Z}\), where the proof extends to the vector-valued setting). On the other hand, with \(E = \mathbb{N}_0\) we isometrically have
\[ H_p(T, X) = H_p^\alpha(\mathbb{N}_0)(G, X). \]
Now we apply the Riesz projection of \(G\). Let \(f \in L_p(T, X)\) and define \(g := R(f \circ \hat{\alpha})\). Then \(\hat{g}(h_{xk}) = 0\) if \(k < 0\), and \(\hat{g}(h_{xk}) = \hat{f}(k)\) if \(k \geq 0\), since \(x > 0\). Hence \(g \in H_p^\alpha(\mathbb{N}_0)(G, X)\) and so there is a corresponding \(\tilde{f} \in H_p(T, X)\) in the sense of (5.8). In this way we obtain a bounded map 
\[ L_p(T, X) \to H_p(T, X), \quad f \mapsto \tilde{f}, \]
which coincides with the Riesz projection on \(T\). Therefore \(X\) has UMD (see e.g. [47], Corollary 5.2.11, p. 399).

The following corollary complements Theorem 4.5 and its consequences in the reflexive case \(1 < p < \infty\).

Corollary 5.7. Let \(\lambda\) be a frequency. If \(X\) has UMD, then for all \(1 < p < \infty\),
\[ \sup_{N \in \mathbb{N}} \|\pi_p^N : H_p(\lambda, X) \to H_p(\lambda, X), \sum a_n e^{-\lambda_n s} \mapsto \sum_{n=1}^N a_n e^{-\lambda_n s}\| < \infty. \]
In particular, every \(\lambda\)-Dirichlet series from \(H_p(\lambda, X)\) converges in \(H_p(\lambda, X)\).
Proof. Let \((G, \beta)\) be a \(\lambda\)-Dirichlet group; identify \(\mathcal{H}_p(\lambda, X) = H^\lambda_p(G, X)\). Then fixing \(D \in \mathcal{H}_p(\lambda, X)\), for every \(N \in \mathbb{N}\) we have
\[
\pi^N_p(D) = D - e^{-\lambda N + 1s} R(e^{\lambda N + 1s} D),
\]
which leads to
\[
\|\pi^N_p(D)\|_p \leq (1 + \|R\|) \|D\|_p.
\]
Now taking the supremum over \(N\) we find by Theorem 5.6 that
\[
\sup_N \|\pi^N_p\| \leq 1 + \|R\| < \infty.
\]
In order to prove the second statement fix again \(D \in \mathcal{H}_p(\lambda, X)\). For \(\varepsilon > 0\) let \(P \in \mathcal{H}_p(\lambda, X)\) be a Dirichlet polynomial such that \(\|D - P\|_p < \varepsilon\). Choose \(N_0\) such that \(\pi^N_p(P) = P\). Then for every \(N \geq N_0\) we have
\[
\|D - \pi^N_p(D)\|_p \leq \|D - P\|_p + \|P - \pi^N_p(D)\|_p \\
\leq \varepsilon + \|\pi^N_p(D - P)\|_p \leq (2 + \|R\|)\varepsilon.
\]
Hence, the partial sums of \(D\) converge in \(\mathcal{H}_p(\lambda, X)\).

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