Doubly Supersymmetric Geometric Approach for Heterotic String: from Generalized Action Principle to Exactly Solvable Nonlinear Equations

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Abstract

The previously proposed generalized action principle approach to supersymmetric extended objects is considered in some details for the case of heterotic string in $D = 3, 4, 6$ and $10$ space–time dimensions. The proof of the 'off–shell' superdiffeomorphism invariance of the generalized action is presented. The doubly supersymmetric geometric approach to heterotic string is constructed on the basis of generalized action principle (instead of the geometrodynamical condition, used for this previously).

It is demonstrated that $D = 3$ heterotic string is described by $n = (1, 0)$ supersymmetric generalization of the nonlinear Liouville equation.
Introduction

This talk contains some results of the investigations, which were performed in collaboration with Dmitrij P. Sorokin and Dmitrij V. Volkov and devoted to the development of twistor–like approach for superstring and supermembrane theories.

These investigations are the natural continuation of ones from Ref. [1], where generalized action principle for supersymmetric extended objects was proposed, as well as of ones from Ref. [3], where the doubly supersymmetric geometric approach for superstrings and supermembranes was built on the basis of the so–called geometrodynamic condition (see below).

The main original motivation for the development of different versions of a twistor–like approach for superparticles and supersymmetric extended objects [3]–[21] was the construction of an adequate basis for future attempts to attack the covariant quantization problem, which solution seems to be necessary for deeper understanding of the quantum theory of supersymmetric extended objects [1].

In a Lorentz–harmonic twistor–like (component) formulation of refs. [17]–[21] the $\kappa$–symmetry was represented in an irreducible but rather complicated form.

The twistor–like approach based on a superfield formulation of super–p–branes in world superspace [3]–[16] allowed one to replace the $\kappa$–symmetry by more fundamental local world supersymmetry and thereby to solve the problem of the infinite reducibility of the former.

At the same time some basic problems have not been solved satisfactory in the known versions of the approach both from the aesthetic and practical point of view. For instance, for constructing the superfield action one should use superfield Lagrange multipliers. Though some of their components can be identified (on the mass shell) with the momentum density and the tension of the super–p–brane, in general, the geometrical and physical meaning of the Lagrange multipliers is obscure. Moreover, in a version suitable for the description of D=10, 11 objects [10, 12, 14, 15, 16] their presence in the action gives rise to some new symmetries which turn out to be infinite reducible themselves, so that the problem which we fought in the conventional Green–Schwarz formulation reappeared in a new form in the twistor–like formulation. Another point concerning the Lagrange multipliers is that in the superfield formulation of D=10 type II superstrings [14] and

\[1\] In the standard formulations of the superparticle [22], superstrings [23, 24] and supermembranes [25] the $\kappa$–symmetry [26] play a significant role [24], however its meaning and origin had been unclear [24]. Moreover, it has an infinitely reducible form and its generator can not be splitted covariantly from the fermionic second class constraints in the Hamiltonian formalism. The later properties hamper a covariant quantization of superstrings.
a $D=11$, $N=1$ supermembrane \cite{15} Lagrange multipliers become propagative redundant degrees of freedom which may spoil the theory at the quantum level.

All this has forced us to revise the twistor–like superfield approach on the basis of more geometrically grounded reasons \cite{2,1} and to apply the generalized action principle of the rheonomic approach \cite{27} to superstrings and supermembrane \cite{1}.

The generalized action principle, first of all, gives the possibility to reproduce the superfield equations of motion for superstrings and other super–$p$–branes just in the form suitable for the development of the doubly supersymmetric geometrical approach.

And, hence, it open a new possibility for a natural application of the doubly supersymmetric twistor–like approach to the studying of some (quasi–) classical problems of superstring and supermembrane theories. Among such problems are ones related to a coupling of the super–$p$–branes to the natural background (super)fields, including the investigation of $T$–duality (see, for example, \cite{28} and refs. therein) in terms of cotangent bundle \cite{29}, and investigation of nonlinear equations of motion of super–$p$–branes with $p \geq 2$.

In this talk we will consider in details generalized action for $D=3,4,6$ and $10$ heterotic string \cite{3}, describe its variation and derivation of the superfield equations of motion. Then we will construct the doubly supersymmetric geometric approach \cite{2} for $D=3,4,6$ and $10$ heterotic string, analyze its equations in the simplest case $D=3$ and prove, that they can be reduced to $n = (1,0)$ supersymmetric generalization of nonlinear Liouville equation.

1 The generalized action principle for heterotic superstrings

The basic concepts and properties of generalized action for super–$p$–branes can be found in Ref. \cite{1}. All of them have the counterparts in the rheonomic approach \cite{27} developed for supergravity (see \cite{30} -- \cite{31} and refs. in \cite{31}). However the super–$p$–brane case is much more simple since for constructing the action only the simplest geometrical objects (i.e. vielbeins, and not connection and curvature) are involved.

So, let us begin from the prescription suitable for the construction of the generalized action.

--- step 1 --- we shall find the component superstring action which is written (or can be

\footnote{For simplicity, we will not consider heterotic fermions inputs here, so 'heterotic string' means here a closed string with $N=1$ target space supersymmetry.}
rewritten) in terms of differential forms without use of the Hodge operation $\ast$.

- step 2 – we shall replace in this action all the fields by superfields and ordinary world-sheet $\mathcal{M}_0^2$

$$\mathcal{M}_0^2 = \{ (\xi^m, \eta^q); \eta^q = 0 \}$$

(1)

by an arbitrary two–dimensional bosonic surface $\mathcal{M}^2$

$$\mathcal{M}^2 = \{ (\xi^m, \eta^q); \eta^q = \eta^q(\xi) \}$$

(2)

in world–sheet superspace $\Sigma^{(2|D−2)}$

$$\Sigma^{(2|D−2)} = \{ (\xi^m, \eta^q) \}, \quad m = 0, 1 \quad q = 1, \ldots, (D−2)$$

(3)

Fortunately, such component action exists. It is known as the action of the so–called twistor–like Lorentz harmonic formulation [19]–[21]. So, rewriting it as a product of differential 1–forms [2] and then doing with it the step 2, we get the generalized action described below.

1.1 The action functional

So, the action for $D = 3, 4, 6$ and 10 heterotic string is

$$S = \int_{\mathcal{M}^2} L_2$$

(4)

where $\mathcal{M}^2$ is an arbitrary surface (3) in world sheet superspace (3), Lagrangian 2–form

$$L_2 = -\frac{1}{2} \left( E^{++} e^{--} - E^{--} e^{++} + e^{--} e^{++} \right) - idX^m d\Theta^m \Theta,$$

(5)

is constructed out of world sheet superspace bosonic vielbein 1–forms

$$e^a(\xi, \eta) \equiv (e^{++}(\xi, \eta), e^{--}(\xi, \eta))$$

(6)

and two vielbein 1–forms of a flat target superspace

$$E^{±±} \equiv \Pi^{m±} m^{±},$$

(7)

by use of exterior product of the forms without any application of the Hodge operation.

The complete basis of the superspace cotangent to a world supersurface (supervielbein) contains besides (5) also $(D−2)$ fermionic 1–forms $e^{+q}(\xi, \eta)$:

$$e^A = (e^{++}, e^{--}, e^{+q}), \quad q = 1, \ldots, (D−2)$$

(8)

We will stress, that the standard Green–Schwarz superstring action [23, 24] can not be used for this. Indeed, even its bosonic limit, which is the Polyakov’s string action, has the form $\int dx^m \ast dx^m$.
but they are not involved into the action explicitly.

Thus $\xi$--directions have a privilege over $\eta$--directions.

However, the external differential $d$ should be expended in the complete $e^A$ basis

$$d = e^a \nabla_a + e^{op} \nabla_{op},$$

where $\nabla_{\pm}$, $\nabla_{+q}$ are covariant derivatives for world--sheet scalar superfields.

The forms

$$\Pi^m = dX^m - id\Theta\Gamma^m\Theta, \quad d\Theta\mu,$$

involved into (7) and (3), are the pullbacks onto the world sheet superspace $\Sigma^{(2|D-2)}$ of the basic supercovariant forms [32] of flat target superspace.

$u_{m}^{++}(\xi, \eta), u_{m}^{--}(\xi, \eta)$, involved in Eq.(7), are the light--like vector components of a local frame (supervielbein)

$$E_{\pm} \equiv (E^\pm; E^\alpha) \equiv (E^{++}, E^{--}, E^i; E^{0q}, E^{0q}),$$

$$E^{++} \equiv \Pi^m u_{m}^{++}, \quad E^i \equiv \Pi^m u_{m}^i, \quad E^\alpha \equiv d\Theta\mu u_{\mu}^\alpha,$$

in target superspace. Together with the $(D-2)$ components $u^i_m(\xi, \eta)$ they are naturally [17, 18, 19, 20, 21, 2] composed of the spinor moving frame matrix components (Lorentz harmonics or generalized Newman– Penrose dyades)

$$v_{\mu}^\alpha = (v^+_{\mu q}, v^-_{\mu q}) \quad  \in \quad Spin(1, D-1)$$

as follows

$$u_{m}^{++}\Gamma^m_{\mu\nu} = 2v^+_{\mu q}v^+_{\nu q}, \quad u_{m}^{--}\Gamma^m_{\mu\nu} = 2v^-_{\mu q}v^-_{\nu q},$$

$$u_{m}^{i}\Gamma^m_{\mu\nu} = (v^+_{\mu q}v^-_{\nu q} + v^-_{\mu q}v^+_{\nu q})\gamma^i_{q\bar{q}},$$

$$u_{m}^{++}\Gamma^m_{\mu\nu} = 2v^+_{\mu q}v^+_{\nu q}, \quad u_{m}^{--}\Gamma^m_{\mu\nu} = 2v^-_{\mu q}v^-_{\nu q},$$

$$u_{m}^{i}\Gamma^m_{\mu\nu} = -(v^-_{\mu q}v^+_{\nu q} + v^+_{\mu q}v^-_{\nu q})\gamma^i_{q\bar{q}},$$

In [13] we have presented the considered expressions for the case of target superspace with $D = 10$, where the $SO(1, 1) \otimes SO(8)$ invariant representation for the (chiral) gamma matrices $\Gamma_{\mu}$ has the form

$$\Gamma^{++}_{\alpha\beta} = \begin{pmatrix} 2\delta_{qp} & 0 \\ 0 & 0 \end{pmatrix} = \Gamma^{--}_{\alpha\beta},$$

$$\Gamma^{--}_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 2\delta_{qp} \end{pmatrix} = \Gamma^{++}_{\alpha\beta},$$

$$\Gamma^i_{\alpha\beta} = \begin{pmatrix} 0 & \gamma^i_{q\bar{q}} \\ \bar{\gamma}^i_{p\bar{q}} & 0 \end{pmatrix} = -\Gamma^i_{\alpha\beta}. $$

(14)
(\gamma_{i\dot{q}}\dot{p} are \sigma\text{-matrices for }SO(8)\text{ group, }\tilde{\gamma}_{i\dot{q}}^i \equiv \gamma_{i\dot{p}q},\text{ and the inverse spinor moving frame matrix}

\begin{equation}
\varepsilon_\mu = (v_{-\mu}^+, v_{+\mu}^-) \in Spin(1, D-1)
\end{equation}

can not be expressed in terms of the variables \([12]\) in a simple manner.

For example, for the simplest \(D = 3\) case the expressions \([13]\), relating vector and spinor harmonics, have the form

\begin{equation}
\begin{aligned}
 u_{m+}^+ \Gamma_{m+}^{\mu\nu} &= 2v^+_{\mu}v^+_{\nu}, & v_{m-}^- \Gamma_{m-}^{\mu\nu} &= 2v^-_{\mu}v^-_{\nu}, \\
 u_{m\bot}^\bot \Gamma_{m\bot}^{\mu\nu} &= v^+_{\mu}v^-_{\nu} + v^-_{\mu}v^+_{\nu},
\end{aligned}
\end{equation}

with spinor harmonics \(v^\pm_{\mu}\) being bosonic spinors restricted by the normalization conditions

\begin{equation}
\begin{aligned}
 v^+_{\mu}v^-_{\nu} &= 1,
\end{aligned}
\end{equation}

(named the harmonicity conditions \([34, 35, 17]\)) only.

Note that eqs. \([13]\) or \([16]\) result in the orthonormality relations for the composed moving frame vectors

\begin{equation}
\begin{aligned}
 u_{\mu\nu}^\sigma u_{\sigma}^{\mu\nu} = \eta_{\mu\nu}^{ab} = diag(1, -1, \ldots, -1)
\end{aligned}
\end{equation}

which, in particular, include the light–likeness conditions for \(u^{\pm\pm}\):

\begin{equation}
\begin{aligned}
 u_{m+}^+ u_{m-}^- &= 0 = u_{m-}^- u_{m+}^+, & u_{m+}^+ u_{m-}^- &= 0, \\
 u_{m+}^+ u_{m-}^- &= 2, & u^i_{m} u^j_{m} &= -\delta^{ij},
\end{aligned}
\end{equation}

More details about harmonics can be founded in Refs. \([34, 35, 17]\) - \([21, 22]\).

To get the superfield equations of motion from the generalized action \((4), (5)\), both the coefficients of the forms and the bosonic submanifold are varied.

The variation of the action over \(\mathcal{M}_2\) is amount to superdiffeomorphism transformations on the world supersurface. This allows one to extend the superfield equations from \(\mathcal{M}_2\) to the whole supersurface.

Then we will stress, that intrinsic geometry of the world supersurface is not \textit{a priori} restricted by any superfield constraints, and the embedding of the world supersurface into the target superspace is not \textit{a priori} specified by any condition such as a geometrodynamical condition \([3] - [16]\) (see eq. \((39)\), the latter playing the crucial role in the twistor–like superfield approach). All the constraints and the geometrodynamical condition are obtained as equations from the generalized action.
This guarantees that the equations of motion, which are the differential form equations, can be extended to the whole world superspace and that variations of the integration surface do not give new independent equations to those which are get by variations of fields.

As we will demonstrate below, the field variations of the action give two kinds of relations:
1) relations between target superspace and world supersurface vielbeins which originate them along one another and are the standard relations of surface embedding theory; we have called them ”rheotropic” conditions [1][4];
2) dynamical equations causing the embedding to be minimal.
Only the latter equations put the theory on the mass shell.

The last term in (3) is the Wess–Zumino 2–form [24]. Its coefficient being fixed by the requirement that the action (4), (5) has $(D-2)$–parametric fermionic gauge symmetry, which is the projection of the world sheet superspace supertranslations onto an integration surface $\mathcal{M}^2$.

From the rheonomy point of view [27], the theory is off the mass shell superdiffeomorphism invariant if for the action (4) to be independent of the surface $\mathcal{M}_2$ (i.e. $d\mathcal{L}_2 = 0$) only the rheotropic relations are required, and the latter do not lead to the equations of motion.

In the next section, after a complete analysis of consequences of rheotropic conditions we will prove the off–shell superdiffeomorphism invariance of the generalized action (4), (5) for heterotic string.

The other evident gauge symmetries of the generalized action (4), (5) are $SO(1,1)$ (identified with world sheet Lorentz group) and $SO(D – 2)$. This results in the possibility to consider the components of the spinor moving frame matrix $v^\mu_{\underline{\alpha}}$ (taking it values in $Spin(1, D – 1)$ (12), (13)), as well as the components of orthogonal ‘vector’ matrix $u^a_{\underline{m}} = (u^a_{\underline{m}}, u^l_{\underline{m}})$ (see (18), (19)) as coordinates parametrizing (noncompact) coset space

\[ \frac{SO(1,D-1)}{SO(1,1) \times SO(D-2)} \] (see [35, 19, 20, 21, 2]).

4’rheo’ is ’current’ and ’tropic’ is ’direction, rotation’ in Greek
1.2 Equations of motion

Varying the action \((\ref{eq:action1}), (\ref{eq:action2})\) over 1–forms \(e^{++}, e^{--}\) and the fields \(X^m\) and \(\Theta^\mu\) we get the following differential form equations

\[
\frac{\delta S}{\delta e^{++}} = 0 \Rightarrow E^{++} \equiv \Pi^m u^{++}_m = e^{++} \tag{20}
\]

\[
\frac{\delta S}{\delta X^m} = \frac{\delta S}{\omega^m(\delta)} = 0 \Rightarrow d(u_m^{++} e^{--} - u_m^{--} e^{++}) - 2i d\Theta \Gamma^m_m d\Theta = 0, \tag{21}
\]

\[
\frac{\delta S}{\delta \Theta^\mu} |_{\omega^m(\delta) = 0} = 0 \Rightarrow d\Theta \Gamma^m_m (u_m^{--} e^{++} - u_m^{++} e^{--} + 2\Pi^m_m) = 0 \tag{22}
\]

To get the rest of the equations we will perform the varying of the action with respect to the harmonic variables. The composed nature of the light–like vectors \(u_m^{++}\) (see \((\ref{eq:lightlike1})\) or \((\ref{eq:lightlike2})\) and similar relations for \(D = 4\) and 6) or, equivalently, the orthonormality conditions \((\ref{eq:orthonormal1}), (\ref{eq:orthonormal2})\) should be taken into account in such varying.

Indeed, due to the orthonormality conditions \((\ref{eq:orthonormal1})\), the matrix

\[
||u_a^m|| \equiv ||u^a_m| | u^i_m|| \equiv ||u^{++}_m| | u^{--}_m|| \in SO(1, D - 1) \tag{23}
\]

takes its values in the vector representation of the Lorentz group, as well the matrix \(v^a_{\mu}\) \((\ref{eq:spinor1})\) takes its values in its spinor representation. So, they both have \(D(D - 1)/2\) degrees of freedom. Henceforth, the dimension of the space tangent to the harmonic (or moving frame) sector should be \(D(D - 1)/2\) too.

The natural basis for this tangent space is given by \(D(D - 1)/2\) Cartan forms

\[
\Omega^{ab} \equiv u^a_m du^{km} \tag{24}
\]

which split naturally into the set of

\(-2(D-2)\) covariant forms being the basis of the \((noncompact)\) coset space \(\frac{SO(1,9)}{SO(1,1) \otimes SO(8)}\)

\[
\Omega^{++} i \equiv u^{++}_m du^{m} i = \frac{1}{4} v^+_{iq} \gamma^i_{qq} dv^{++}_{\mu q}, \tag{25}
\]

\[
\Omega^{--} i \equiv u^{--}_m du^{m} i = \frac{1}{4} v^-_{iq} \gamma^i_{qq} dv^{--}_{\mu q}, \tag{26}
\]

— 1 form having the transformation properties of the \(SO(1,1)\) connection

\[
\Omega^{(0)} \equiv \frac{1}{2} u^{--}_m du^{m} ++ = \frac{1}{4} v^-_{iq} dv^{++}_{\mu q} = \frac{1}{4} v^+_{iq} dv^{--}_{\mu q}, \tag{27}
\]
and, at least,

\[-(D - 2)(D - 3)/2\] forms being the \(SO(D - 2)\) connections

\[
\Omega^{ij} \equiv u_m^i \delta u^m_j = -\frac{1}{4} v_q u^i_{\, jq} \delta v^j_p + \frac{1}{4} v_q u^j_{\, qp} \delta v^i_p ,
\]

(28)

(For definiteness, all the expressions in terms of spinor harmonics are presented for \(D = 10\) case).

For the further analyzes it is essential that, due to (24), \(\Omega^{\hat{a} \hat{b}}\) satisfy identically Maurer–Cartan equations

\[
d\Omega^{\hat{a} \hat{b}} + \Omega_{\hat{a} \hat{c}} \Omega^{\hat{c} \hat{b}} = 0\] (29)

which split naturally into the set of equations for the forms (25)–(28), which are

\[
D\Omega^{\pm+i} \equiv d\Omega^{\pm+i} - \Omega^{\pm+i} \Omega^{(0)} + \Omega^{\pm+j} \Omega^{ji} = 0 \quad (30)
\]

\[
D\Omega^{-i} \equiv d\Omega^{-i} + \Omega^{-i} \Omega^{(0)} + \Omega^{-j} \Omega^{ji} = 0 \quad (31)
\]

\[
F \equiv d\Omega^{(0)} = \frac{1}{2} \Omega^{-i} \Omega^{++i} \quad (32)
\]

\[
R^{ij} \equiv d\Omega^{ij} + \Omega^{ik} \Omega^{kj} = -\Omega^{-[i} \Omega^{++j]} \quad (33)
\]

The admissible variation of the composed vectors \(u_m^\pm\) (as well as the variation of the spinor harmonics \(v\)) can be considered as an element of the cotangent space and, hence, can be decomposed onto the same basis of the forms (taken to be dependent on the variation symbol \(\delta\) instead of the external differential symbol \(d\)). So,

\[
\delta u_m^{\pm \pm} = \pm 1/2 u_m^{\pm \pm} \Omega^{(0)}(\delta) + u_m^i \Omega^{\pm \pm i}(\delta),\] (34)

in spite of is the moving frame vectors considered as composed from the spinor harmonics, or supposed to be fundamental.

Hence, the varying with respect to harmonic variables leads to the equations

\[
\Sigma_\pm(\pm u_m^{\mp \pm} \delta S) \equiv \frac{\delta S}{\Omega^{(0)}(\delta)} = 0 \quad \Rightarrow \quad E^{\pm \pm} e^{--} + E^{--} e^{++} = 0\] (35)

and

\[
u_m^i \delta S \equiv \frac{\delta S}{\Omega^{\pm \pm i}(\delta)} = 0 \quad \Rightarrow \quad E^i e^{\mp \mp} = 0\] (36)

It is easy to see, that Eq. (35) is satisfied identically due to (20). This reflect the local \(SO(1, 1)\) (world–sheet Lorentz) symmetry of the considered action.

So, the only independent equations of motion appearing as a result of the varying with respect to harmonic (or moving frame) variables is (36), which means

\[
E^i \equiv \Pi^m u_m^i = 0\] (37)
Eqs. (37) and (20) are just the bosonic subset of the set of the rheotropic relations for the case of superstring. They cause the light-like bosonic components $E^{\pm\pm}$ of superspace vielbein (11) to become tangent to the world sheet superspace and the rest of them to become orthogonal one.

So, the bosonic part of rheotropic conditions can be rewritten as unique vector 1–form equation

$$\Pi^m = \frac{1}{2}(e^{++}u^{--}_m + e^{--}u^{++}_m)$$

which is the supersymmetric counterpart of the basic relations of the geometric approach to bosonic string theory [33]. But here, following [1], we have derived it from the action principle for superfield case [4].

Moreover, now Eq.(38) has the projection onto the Grassmann directions of the world sheet tangent superspace

$$\Pi^{m}_{\tau q} = 0$$

which is just the Geometrodynamic equation [3]–[16], which was the basis of the previous doubly supersymmetric formulations.

Substituting (38) into the equations (22) we derive the simple two form equation

$$e^{++}(d\Theta\Gamma)_{\mu}u^{--}_{\mu} = 0,$$

which, using for the $u^{--}_{\mu}$ the expression from the first line of Eq. (13) or (16), can be further simplified

$$e^{++}d\Theta\mu v^{--}_{\mu} = 0,$$

(for $D = 3$ the symbol $\dot{q}$ should be omitted). The latter is most suitable for the further analyzes.

Below, after the consideration of the relation with the component twistor–like superstring formulation of Refs. [19, 20, 21] we will study the set of equations (37), (20) (or (38)), (11), (21) for the case of $D = 10$ heterotic superstrings (the results for simpler cases $D = 3, 4$ and $6$ can be derived by reduction).

It will be proved that Eqs. (21) are always dependent on (11), (37) and (20). Moreover, we will prove that the only dynamical equation in the set of the rest relations are the component of (11) appeared as the coefficient for the basic two–form $e^{++} e^{--}$ and having the form

$$\nabla^{--}\Theta\mu v^{--}_{\mu} = 0,$$

5See the first section of [2] for the similar result for the bosonic string. Such component (not superfield) supersymmetric equations had been derived from the action principle in refs. [14, 21, 21], however without discussion of the relation with the geometric approach.
Eq. (42) coincides formally with the equation of motion for the field $\theta$ appearing in the component twistor–like formulation of Refs. [19, 20, 21]. The another equation contained in (41)

$$\nabla + q \Theta \mu v_{\mu} = 0,$$

(43)

can be considered as the fermionic part of the set of the rheotropic conditions and, as it will be proved below, do not lead to any dynamical equations.

This will be done by the investigation of the selfconsistency conditions for these equations.

The presence of the spinor moving frame variables (Lorentz harmonics) gives, from one hand, the possibility to formulate the doubly supersymmetric geometrical approach [2] to heterotic superstring as the result of such investigation, and require, from the other hand, to investigate the Maurer–Cartan equations (29) (or Eqs. (30) – (33) being the counterpart of the Peterson–Codazzi, Gauss and Ricci equations) as the part of the selfconsistency conditions.

Moreover, we will prove that the closure of the Lagrangian 2–form holds when only the equations (37), (20) (or (38)) and the equation (43) are taken into account. This reflect the off–shell diffeomorphism invariance of the discussed action for $D = 3$, 4, 6 and 10 heterotic string in the rheonomic sense.

1.3 Component formulation and local fermionic symmetry

The component formulation [19, 20, 21] of the heterotic superstring is obtained by choosing the surface $\mathcal{M}_2$ to be defined by the condition $\eta^{+q} = 0$ and taking into account only the vector components of (9). In this case the action (4) is just the action of Refs. [19, 20, 21] for the fields

$$X^m|_{\eta=0} = x^m(\xi), \quad \Theta^\mu|_{\eta=0} = \theta^\mu(\xi),$$

$$u^m_a|_{\eta=0} = u^m_a(\xi) \quad \text{and} \quad e^a|_{\eta=0} = e^a(\xi).$$

but rewritten in terms of the differential forms.

For these fields one can get from (37)–(40) the following equations:

$$\Pi^m_{\pm} = e^m_{\pm}(\partial_m x^m - \imath \partial_m \theta \Gamma^m \theta) = u^m_{\mp}(\xi),$$

(44)

$$e^m_{-\pm} \partial_m \theta u_{\mp} = 0,$$

(45)

$$\partial_m(e e^m_{\pm} u_{\pm}^m) - 4i \varepsilon^m \partial_m \theta \Gamma^m \partial_n \theta = 0,$$

(46)

Which are just the string equations of the component twistor–like formulation [19, 20, 21].
Using the component rheotropic equation (44) (which can be transformed into the form representing any of the sets of the vector variables $\Pi^m_m$, $u^{\pm\pm}_m$, or $e^{\pm\pm}_m$ through two others) we can transform equations (45) and (46) into the form of the standard Green–Schwarz formulation [23, 24]

$$\Pi^m_m g^{mn} \partial_n \theta^m \Gamma^m_{\mu \nu} = 0, \tag{47}$$

and

$$\partial_m (\sqrt{-g} g^{mn} \Pi^m_n) - 2i \varepsilon^{mn} \partial_m \theta \Gamma^{mn} \partial_n \theta = 0, \tag{48}$$

where $g_{mn} = e_a^m e_an = \Pi^m_m \Pi^m_n$ is the induced metric on the world sheet.

The component action obtained from (4), (5) by choosing $M^2 = M^2_0$ (1) possess the $\kappa$-symmetry in the following irreducible form [19, 20, 21]

$$\delta \theta^\mu = \kappa^+ q^\mu,$$

$$\omega^m (\delta) = 0, \quad \Rightarrow \quad \delta x^m = i \kappa^+ q^\mu \Gamma^m_{\mu \nu} \theta^\nu,$$

$$\delta e^{++} = -4i (d \Theta v^+_q) \kappa^+ q, \quad \delta e^{--} = 0, \tag{49}$$

$$\delta v^+_m = 2i \kappa^p \gamma^i_{pq} e^m_p \partial_m \theta \gamma^i_{pq} v^+_q,$$

$$\delta v^-_m = -2i \kappa^p \gamma^i_{pq} e^m_p \partial_m \theta v^-_q \gamma^i_{pq} v^+_q. \tag{50}$$

The basic feature of the twistor–like superfield approach is that this transformations (49), (50) are the relic of the world surface superdiffeomorphisms [3], for instance, $\theta^\mu$ and $v^\mu_{\tau \rho}$ are transformed as superpartners.

However, if we consider world surface superspace diffeomorphisms as the symmetry of the generalized action (4), (5), they will be projected onto an integration surface $M^2$ and, so, are realized nonlinearly.

Namely, the fermionic symmetry of the generalized action (4), (5) is defined by relations (49) with all the fields replaced by superfields taken at a surface $M^2$ (2) and variations of harmonic superfields are defined by relations

$$\delta v^+_m = 1/2 \Omega^{++} (\delta) \gamma^i_{pq} v^+_q,$$

$$\delta v^-_m = 1/2 \Omega^{--} (\delta) v^-_q \gamma^i_{pq},$$

with $\Omega^{++} (\delta)$ being determined by the solution of 1–form equation

$$e^{-} \Omega^{++} (\delta) - e^{++} \Omega^{--} (\delta) + 4id \Theta v^-_q \gamma^i_{pq} v^+_q = 0 \tag{51}$$

In (52) all the forms are pulled back on the surface $M^2$ (2), i.e.

$$e^{-} = d\xi^m e^{-}_m (\xi, \eta(\xi)) + d\eta^q (\xi) e^{-}_q (\xi, \eta(\xi)) = d\xi^m (e^{-}_m (\xi, \eta(\xi)) + \partial_m \eta^q (\xi) e^{-}_q (\xi, \eta(\xi)) \tag{53}$$
For the choice of surface $\mathcal{M}^2 = \mathcal{M}^2$, which corresponds to the component formulation, the solutions of Eq. (52) define just the transformation rules (50) for harmonic variables.

### 2 $D = 10$ heterotic superstring: Doubly supersymmetric geometric approach and the ’off–shell’ superdiffeomorphism invariance of the generalized action.

In this section we will investigate completely the set of the superfield equations following from the generalized action principle (4) for $D = 10$ heterotic string. Such investigation naturally results in the construction of the doubly supersymmetric geometric approach \[3\] for the heterotic string in $D = 10$.

We will prove that the set of the rheotropic equations (37), (20) and (43) do not result in any dynamical equation. This will be done by studying of all the consequences of the rheotropic equations; as a result we will construct the geometric approach based on the rheotropic equations only and prove that they define the nonminimal embedding of the world sheet into the target superspace. We will prove also that the equations (21) are always dependent on the other ones and that the only independent dynamical equation for the case of the heterotic string is (42).

At the end of the section we present also the complete set of the equations of the geometrical approach, which describes the minimal embedding of the heterotic string into the $D = 10$ target superspace. The results for $D = 3, 4 \text{ and } 6$ can be derived by reduction.

Such set of the equations for $D = 3$ heterotic string will be used in the next section for the investigation of the relation between the heterotic string and the supersymmetric extension of the nonlinear Liouville equation.

#### 2.1 The conventional rheotropic conditions and the choice of the world–sheet connections

\[6\]

but based on the generalized action principle instead of the geometrodynamic equation, as it was in \[2\]; it should be stressed, that just for the case of heterotic string, where the Geometrodynamic condition (39) do not lead to the equations of motion and, moreover, the complete superfield form of the equations of motion had not be known, it was not completely understood previously how to formulate the minimal embedding of the heterotic string.
2.2 The geometry of the world sheet superspace

The world–sheet geometry (or the world–sheet supergravity) can be described by the

\[ \text{supervielbein forms:} \quad e^A(d) \equiv (e^{\pm\pm}, e^{+q}), \]

\[ \text{Lorentz \ (or \ SO(1,1)) connection form:} \quad w(d) = e^A(d)w_A, \]

\[ \text{and} \]

\[ \text{SO(D – 2) connection form:} \quad B^{ij}(d) = e^A(d)B_A^{ij}. \]  

(54)

However, only the bosonic vielbein forms \( e^{\pm\pm} \) are involved explicitly into the action \((4),(5)\).

Hence, we can choose arbitrary the Grassmann vielbein form \( e^{+q} \) and the connections of the both types \( w(d) \) and \( B^{ij}(d) \).

The most natural way is to choose them being induced by the embedding.

For the connections this means, that they are chosen to be equal to the pull–backs of the corresponding Cartan forms \((27)\) and \((28)\). Such coincidence can be formulated as

\[ \Omega^{(0)}(D) \equiv \Omega^{(0)}(d) - 2w = 0, \quad \Omega^{ij}(D) \equiv \Omega^{ij}(d) - B^{ij} = 0. \]  

(55)

where \( D \) is differential covariant with respect to both world sheet Lorentz \( (SO(1,1)) \) and natural 'internal' \( SO(D – 2) \) symmetries.

As the result of \((55)\), the covariant world surface derivatives of the spinor and vector moving frame variables acquire the forms

\[ \mathcal{D}v^\mu_+ = \frac{1}{2} \Omega^{++ \mu} v_q^+ \Omega^{++ i} + i(d), \quad \mathcal{D}v^\mu_- = \frac{1}{2} v^{\mu}_q i_{qq}^i \Omega^{-- i} + \mu(d), \]

(56)

\[ \mathcal{D}v^\mu_q = -\frac{1}{2} \Omega^{-- i} i(d) \gamma^{i}_{qq} v^\mu_q, \quad \mathcal{D}v^\mu_q = -\frac{1}{2} v^\mu_q i_{qq}^i \Omega^{++ i} + \mu(d), \]

(57)

and

\[ \mathcal{D}u^\mu_+ = u^\mu_+ \Omega^{++ i} + i(d), \quad \mathcal{D}u^\mu_- = u^\mu_- \Omega^{-- i} - i(d), \quad \mathcal{D}u^\mu_q = u^\mu_+ \Omega^{++ i} + i(d) + u^\mu_- \Omega^{-- i} + i(d), \]

(58)

respectively.

The fermionic vielbein induced by the embedding has the form

\[ e^{+q} = E^{+q} \equiv d\Theta^\mu_{-\mu} v^\mu_+ \]

(59)

Eq. \((59)\) is the evident Grassmann counterpart of the rheotropic conditions \((20)\). However it is not derived as an equation of motion, but chosen using the arbitrariness or symmetry.
of the action [1].

So, it is naturally to refer on it as on the conventional rheotropic condition.

Other rheotropic conditions are (20), (37)

\[ E^{++} \equiv \Pi^{m} u_{m}^{++} = e^{++}, \] (60)

\[ E^{--} \equiv \Pi^{m} u_{m}^{--} = e^{--}, \] (61)

\[ E^{i} \equiv \Pi^{m} u_{m}^{i} = 0, \] (62)

and (13). The latter can be presented in terms of 1–forms as follows

\[ E^{-q} = d\Theta^{q}_{-} = e^{\pm \pm} \psi_{\pm \pm}, \] (63)

### 2.3 The inducing of the torsion constraints and the doubly SUSY geometric approach generation

Let us investigate the selfconsistency conditions for the rheotropic relations (59) – (63).

Selfconsistency conditions for eq. (60), after taking into account (62), acquire the form

\[ T^{++} \equiv D e^{++} = -2i d\Theta_{v}^{q} d\Theta_{v}^{q} \]

which, after taking into account the conventional rheotropic condition (59), coincides with the component of the flat torsion of the world sheet superspace

\[ T^{++} \equiv D e^{++} = -2i e^{q} e^{q} \] (64)

The selfconsistency conditions for (61) after taking into account (62) and (63) acquire, respectively, the forms

\[ T^{--} \equiv D e^{--} = -2i d\Theta_{v_{q}} d\Theta_{v_{q}} \]

and

\[ T^{--} \equiv D e^{--} = -4i e^{++} e^{--} \psi_{++}^{q} \psi_{--}^{q}. \] (65)

\[ ^{7} \text{Let us stress that they can be discussed as the result of the gauge fixing for the redefinition ”symmetry”} \]

\[ e^{q} \rightarrow \tilde{e}^{q} = (e^{+p} + e^{\pm \pm} \chi_{\pm \pm}^{+p}) W_{p}^{q} \quad \text{det} W \neq 0 \]

which holds due to the mentioned absence of the Grassmann vielbein in the action [1] in the proper form (i.e. it is present in the external differential decomposition [1] only).

The related redefinition of the derivatives (which follows from \( d \rightarrow d \) and \( e^{\pm \pm} \rightarrow e^{\pm \pm} \)) is

\[ \nabla_{+q} \rightarrow \tilde{\nabla}_{+q} = (W^{-1})_{q}^{p} \nabla_{+p}, \quad \nabla_{\pm \pm} \rightarrow \tilde{\nabla}_{\pm \pm} = \nabla_{\pm \pm} - \chi_{\pm \pm}^{+p} \nabla_{+p}, \]

15
So the torsion constraints of the 'heterotic' supergravity \[10\] are reproduced now as the selfconsistency conditions of the "tangent vector" rheotropic conditions \((60), (61)\).

The selfconsistency conditions for "orthogonal vector" rheotropic relation \((62)\) produce the following conditions for the forms \(\Omega^{++i}\) and \(\Omega^{--i}\)

\[
(\Omega^{++i} + 4ie^{+q}\psi_{-q})e^{-} + (\Omega^{--i} + 4ie^{+q}\psi_{+q})e^{+} = 0, \tag{66}
\]

This means, in particular, that their bosonic components with zero \(SO(1, 1)\) weight coincide and are equal to the main curvatures \(h^i\) of the embedded (super)surface

\[
\Omega^{++i} = \Omega^{--i} = h^i, \tag{67}
\]

(see \[2\] and refs. therein). Eq. \((66)\) contain also the expressions for the spinor components of the Cartan forms through the superfields \(\psi\)

\[
\Omega^{++i}_{+q} = -4i\gamma_{qq}^i \psi_{-q}, \quad \Omega^{--i}_{+q} = -4i\gamma_{qq}^i \psi_{+q}, \tag{68}
\]

So the only components of the forms \(\Omega^{\pm\pm i}\) undetermined by \((66)\) are bosonic ones with the Weyl weights \(\pm 4\), namely \(\Omega^{\pm\pm i}_{\mp\mp}\). Hence

\[
\Omega^{--i} = -4ie^{+q}\gamma_{qq}^i \psi_{-q} + e^{+\pm} \Omega^{--i}_{++} + e^{-\pm} h^i, \tag{69}
\]

\[
\Omega^{++i} = -4ie^{+q}\gamma_{qq}^i \psi_{-q} + e^{-\pm} h^i + e^{+\pm} \Omega^{++i}_{--} \tag{70}
\]

Then, the selfconsistency conditions for the conventional rheotropic equation \((59)\) result in the expression for the spinor torsion \(T^{+q} \equiv De^{+q}\) of the world–sheet superspace

\[
T^{+q} \equiv De^{+q} = -2ie^{\pm e^{+p}\gamma_{pp}^i \gamma_{qq}^i \psi_{-q} \psi_{+q} + e^{+e^{-}\frac{1}{2} (\Omega^{++i}_{--} \gamma_{qq}^i \psi_{+q} - h^i \gamma_{qq}^i \psi_{-q})}, \tag{71}
\]

which means, in particular, that the well known conventional torsion constraint holds

\[
T^{+q}_{+p} = 0. \tag{72}
\]

Hence we reproduce (as the result of the selfconsistency of the rheotropic conditions) the ordinary form of the covariant spinor derivative algebra

\[
\{D_{+q}, D_{+p}\} = 4i\delta_{qp}D_{++} + \text{curvature} \tag{72}
\]

The selfconsistency conditions for the last rheotropic equation \((63)\) produce the restrictions on the \(\psi\) superfields

\[
D_{+q} \psi_{+++} = -\frac{1}{2} \gamma_{qq}^i \Omega^{--i}_{++}, \tag{73}
\]
\[
D_{++}\psi_{-\tilde{q}} = -\frac{1}{2}\gamma_{\tilde{q}\bar{q}}^i h^i \\
D_{--}\psi_{+\tilde{q}} = D_{++}\psi_{-\tilde{q}} + 4i\psi_{+\tilde{p}}\psi_{-\tilde{q}}\psi_{-\tilde{r}} (74)
\]

The only nontrivial component of the Peterson–Codazzi equations is one of \(D\Omega^{++i}\), which are proportional to the basic two form \(e^{-\tilde{e}^q}\):

\[
D_{++}\Omega_{-\tilde{r}}^{++i} = -4i(\gamma_{\tilde{q}\bar{q}}^i D_{--}\psi_{-\tilde{q}} - 2\gamma_{\tilde{q}\bar{r}}^i \gamma_{\tilde{q}\bar{p}}^j \psi_{-\tilde{q}}\psi_{-\tilde{p}}\psi_{-\tilde{r}}) (76)
\]

The Gauss and Ricci equation define the \(SO(1,1)\) and \(SO(8)\) curvatures \(F\) and \(F^{ij}\), which satisfy the Bianchi identities

\[
D_{T+q} = -\frac{1}{2} e^{+q} F + \frac{1}{4} e^{+p} \gamma^{ij}_{\tilde{q}\bar{p}} F^{ij} (77)
\]

\[
D_{T++} = -e^{++} F (78)
\]

\[
D_{T--} = e^{--} F (79)
\]

2.4 Minimal embedding of the heterotic string world sheet superspace.

As it can be seen from eq. (74), the only independent superfield dynamical equation of motion is just eq. (42)

\[
\psi_{-\tilde{q}} \equiv \nabla_{-\tilde{q}} \Theta^\mu_{\mu\tilde{q}} = 0 (80)
\]

which means in particular the vanishing of the main curvatures

\[
h^i = 0 (81)
\]

and, hence, the minimality of the embedding [2].

Projecting equations of motion (21) for \(X\) superfield onto different components of vector moving frame (23), it is easy to see that all of them are satisfied identically due to rheotropic conditions (59)–(62) and Eq.(80).

So, the equations for \(X\) superfield are dependent ones.

Hence, the minimal embedding of heterotic superstring world sheet superspace into a flat \(D = 10\) (and 3, 4, 6) target superspace is described in terms of

\(-i\) the matter superfields \(\psi_{+\tilde{q}}\) and \(\Omega_{-\tilde{r}}^{++i}\), which satisfy the equations

\[
D_{--}\psi_{+\tilde{q}} = 0, (82)
\]

\[
D_{+q}\psi_{+\tilde{q}} = -\frac{1}{2}\gamma_{\tilde{q}\bar{q}}^i \Omega^{+++i}, (83)
\]
(where $\Omega_{++}^-i$ is defined just by this condition),
\begin{equation}
\mathcal{D}_{+q}\Omega_{--+}^{++i} = 0,
\end{equation}
and
-ii- vielbeins $e^A \equiv (e^{\pm \pm}, e^{+q})$ (world-sheet "supergravity"), restricted by the torsion constraints
\begin{align}
T^{++} &\equiv D e^{++} = -2ie^{+q} e^{+q} \\
T^{--} &\equiv D e^{--} = 0 \\
T^{+q} &\equiv D e^{+q} = e^{++} e^{--} - \frac{1}{2} \Omega^{++i} \gamma^{i} q_{q} \psi_{++}^{i} + \Omega^{++i} \gamma^{i} q_{q} \psi_{++}^{i} - \Omega^{++i} \gamma^{i} q_{q} \psi_{++}^{i} 
\end{align}
In fact, the matter superfields define the torsion components except for one leaving nonvanishing in the flat limit.

The curvatures are defined by Gauss and Ricci equations and have the forms
\begin{equation}
SO(1,1)
\end{equation}
\begin{equation}
\mathcal{F} \equiv d\Omega^{(0)} = 2ie^{--} e^{q} \gamma_{q} i \Omega^{++i} \psi_{++}^{i} + e^{++} e^{--} \Omega^{++i} \Omega^{++i} 
\end{equation}
and, for $D = 4, 6, 10$,
\begin{equation}
SO(D-2)
\end{equation}
\begin{equation}
F^{ij} = 4ie^{--} e^{q} \Omega^{--[i} \gamma_{q} i \Omega^{++i]} \psi_{+++}^{i} - e^{++} e^{--} \Omega^{++[i} \Omega^{++i]} 
\end{equation}

### 2.5 The action independence on the surface

The conditions of the off–shell superdiffeomorphysm invariance of the action [I] for $D = 10$ (and 3, 4, 6) heterotic string has the form
\begin{equation}
d\mathcal{L}_{2} = \frac{1}{2} [ (E^{++} - e^{++})(T^{--} - 2id\Theta v_{q}^{--} d\Theta v_{q}^{--}) - (E^{--} - e^{--})(T^{++} - 2id\Theta v_{q}^{++} d\Theta v_{q}^{++}) ]
\end{equation}
The first three terms vanish due to the rheotrophic equations (37) and (20). The last term (after the complete decomposition onto the supervielbein forms) can be rewritten as follows
\begin{equation}
-4ie^{++} e^{+q} [2e^{--} \nabla_{--} \Theta v_{q}^{--} + e^{+p} \nabla_{+p} \Theta v_{q}^{--}] \nabla_{+q} \Theta v_{q}^{--}
\end{equation}
It is evident, that the latter expression vanishes due to the "fermionic" rheotrophic equation (43) only.
Hence,

\[ d\mathcal{L}_2 = 0 \]

holds as the result of the rheotropic conditions only.

This means the off–shell superdiffeomorphism invariance of the heterotic string action in the rheonomy sense \([1]\), because it have been proved above that the rheotropic conditions do not lead to the equations of motion.

3 \( D = 3 \) heterotic superstring and \( n = (1, 0) \) supersymmetric generalization of nonlinear Liouville equation

In conclusion, let us analyze the set of geometric approach equations (82) – (88) for the simplest case of \( D = 3, N = 1 \) string (where Eq. (89) is absent).

Eq. (84) and the consequence

\[ \mathcal{D}_+\mathcal{D}_-\psi_+^- = \mathcal{D}_-\mathcal{D}_+\psi_+^- = 0 \]

of Eq. (82) can be used to determine \( SO(1, 1) \) connection

\[ \Omega^{(0)} = \frac{1}{2\Omega^{++}}(e^+\nabla_+ + e^{++}\nabla_{++})\Omega^{++} - \frac{1}{2\mathcal{D}_+\psi_+^+}e^{-+}\nabla_-\mathcal{D}_+\psi_+^- , \quad (91) \]

Then Eq. (88) reproduces the equation

\[ \mathcal{D}_+\nabla_- lg(\Omega^{++}\mathcal{D}_+\psi_+^-) = -4i\Omega^{++}\psi_+^- \quad (92) \]

and its consequence

\[ \mathcal{D}_+\nabla_- lg|\Omega^{++}\mathcal{D}_+\psi_+^-| = -2\Omega^{++}\mathcal{D}_+\psi_+^- \quad (93) \]

Taking into account (91), we can rewrite Eqs. (86) – (87) as follows

\[ d((\Omega^{++})^{1/2}e^{--}) = 0 \quad (94) \]

\[ d((\mathcal{D}_+\psi_+^-)^{1/2}e^{++}) = -2i\bar{e}^+ e^+ \quad (95) \]

\[ d\bar{e}^+ = 0 \quad (96) \]

where

\[ e^+ \equiv (\mathcal{D}_+\psi_+^-)^{1/4}(e^+ - i/8e^{++}\nabla_+ lg|\Omega^{++}\mathcal{D}_+\psi_+^-|) \quad (97) \]

In the derivation of Eq. (96) Eq. (92) should be taken into account.
Relations (94) – (96) coincides with the expressions for the torsion of flat superspace. This reflects the known statement: two dimensional $n = (1, 0)$ supergeometry is always conformally flat [36].

Hence, neglecting possible inputs from a nontrivial world sheet topology, we can represent supervielbein as follows

$$e^{--} = (\Omega^{++})^{-1/2} d\xi^{(--)}, \quad (98)$$

$$e^{++} = (\mathcal{D}_+ \psi_+^-)^{-1/2} w^{(++)} \equiv (\mathcal{D}_+ \psi_+^-)^{-1/2} (d\xi^{(++)} - 2i\eta^+ \eta^+), \quad (99)$$

$$e^+ = (\mathcal{D}_+ \psi_+^-)^{-1/4} (d\eta^+ + i/8 w^{(++)} D_{(+)\gamma} \Omega^{-+} \mathcal{D}_+ \psi_+^-), \quad (100)$$

where

$$w^{(++)} \equiv d\xi^{(++)} - 2i\eta^+ \eta^+,$$

is the basic supersymmetric 1–form [32] of a flat $d = 2, n = (1, 0)$ superspace and

$$D_{(+)\gamma} \equiv \frac{\partial}{\partial \eta^{(+)\gamma}} + 2i\eta^+ \partial_{(++)} \quad (101)$$

is flat covariant derivative of the superspace.

It is convenient to identify variables $\xi^{(\pm\pm)}$, $\eta^{(+)}$ with local coordinates of world sheet superspace. Of course, the gauge with respect to superdiffeomorphisms is fixed by this step. Only $SO(1, 1)$ gauge invariance remains unbroken.

Now we are ready to present geometric approach equations in terms of flat superspace derivatives $\partial_{(\pm\pm)} = \partial/\partial \xi^{(\pm\pm)}$ and $D_{(+)\gamma}$ (101).

First of all, let us note, that Eq.(82) $\mathcal{D}_- \psi_+^- = 0$ can be represented as a flat space chirality condition

$$\partial_{(-)} \Psi_L^{(+)\gamma} = 0 \quad (102)$$

for fermionic superfield

$$\Psi_L^{(+)\gamma} = (\mathcal{D}_+ \psi_+^-)^{-3/4} \psi_+^- \quad (103)$$

Now, if we try to write the expression $\mathcal{D}_+ \psi_+^-$ in terms of superfield $\Psi_L^{(+)\gamma}$ and flat fermionic derivative (101), we get the following identity

$$\left(\Omega^{-+} \mathcal{D}_+ \psi_+^-\right)^{3/4} \equiv D_{(+)\gamma} \left(\left(\Omega^{-+} \mathcal{D}_+ \psi_+^-\right)^{3/4} \Psi_L^{(+)\gamma}\right) \quad (104)$$

Let us introduce the gauge invariant superfield $W$ by

$$\exp\{4W\} = \Omega^{-+} \mathcal{D}_+ \psi_+^- \quad (105)$$

Then the identity (104) gives us the connection between $W$ and $\Psi_L^{(+)\gamma}$

$$D_{(+)\gamma} (\exp\{4W\} \Psi_L^{(+)\gamma}) = e^{4W} \quad \Leftrightarrow \quad D_{(+)\gamma} \Psi_L^{(+)\gamma} = 1 - 3 D_{(+)\gamma} W \Psi_L^{(+)\gamma} \quad (106)$$
Now all the supervielbeins and Cartan forms can be expressed in terms of superfields \( \Psi_{L}^{(+)} \), \( W \) and
\[
L = 1/4lg|D_{+}\psi_{++}+/\Omega_{--}| \tag{107}
\]
(being the compensator for \( SO(1,1) \) gauge transformations) as follows
\[
\Omega_{--} = e^{W+L}(-2w^{(++)}(1 - D_{(+)W}\Psi_{L}^{(+)}) - 4i\eta^{(+)\Psi_{L}^{(+)}}), \tag{108}
\]
\[
\Omega^{++} = d\xi^{(-)}e^{W+L} \tag{109}
\]
\[
\Omega^{(0)} = (d\eta^{(+)D_{(+)} + w^{(++)}\partial_{(++)} - d\xi^{(-)}\partial_{(-)})W - dL, \tag{110}
\]
Eq.\[\text{(108)}\]
\[
d\Omega^{(0)} = 1/2\Omega^{--}\Omega^{++} \tag{111}
\]
for the forms \([108] - [110]\) results in the equation
\[
\partial_{(-)}D_{(+)W} = -ie^{2W}\Psi_{L}^{(+)}, \tag{112}
\]
which, together with the identity \([106]\) and chirality conditions \([102]\), describes completely \( D = 3 \) heterotic superstring in the framework of geometric approach.

Decomposing the superfields in the (finite) power series on the only nilpotent Grassmann coordinate \( \eta^{+} \), we can verify that the constraint \([106]\) has no dynamical consequences and simply expresses the highest components of the superfields \( W \) and \( \Psi_{L}^{(+)} \) through the leading ones
\[
W = w + (2i/3)\eta^{(+)e^{-3w}\partial_{(++)}}(e^{3w}\psi_{L}^{(+)}) \tag{113}
\]
\[
\Psi_{L}^{(+)} = \psi_{L}^{(+)\xi^{(++)}} + \eta^{(+)1 - 2i\partial_{(++)}\psi_{L}^{(+)\psi_{L}^{(+)}}}, \tag{114}
\]
The only nontrivial consequence of the superfield equation \([112]\) is
\[
\partial_{(++)}\partial_{(-)}w = 1/2e^{2w}(1 - 2i/3\partial_{(++)}\psi_{L}^{(+)\psi_{L}^{(+)}}) \tag{115}
\]
which can be reduced to the standard bosonic Liouville equation
\[
\partial_{(++)}\partial_{(-)}\tilde{w} = 1/4exp\{2\tilde{w}\} \tag{116}
\]
by the field redefinition
\[
w = \tilde{w} + (2i/3)\partial_{++}\psi_{L}\psi_{L} \tag{117}
\]
This, from one hand, gives us a reason to conclude, that \( D = 3 \) heterotic string (without heterotic fermions) is described by \( n = (1,0) \) supersymmetric extension of the
nonlinear Liouville equation and, from the other hand, means that this nonlinear system is exactly solvable.

Indeed, it can be reduced to the system of the exactly solvable nonlinear Liouville equation \((116)\) and free field equation

\[
\partial_{(-)}\psi_L^{(+)} = 0
\]

for fermionic field.

It is remarkable, that all equations \((102), (106), (112)\) appears as consequences of zero curvature representation

\[
d\Omega^\beta_{\underline{\gamma}} - \Omega^\gamma_{\underline{\delta}}\Omega^\delta_{\underline{\alpha}} = 0
\]

for \(SL(2,R) (= SO(1,2))\) connection

\[
\Omega^\beta_{\underline{\alpha}} = \frac{1}{2} \begin{pmatrix}
\Omega^{(0)} & \Omega^{-} \\
\Omega^{+} & -\Omega^{(0)}
\end{pmatrix}
\]

with the forms \(\Omega^{\pm}, \Omega^{(0)}\) determined by Eqs. \((108), (109), (110)\).

In conclusion, let us present the Bäcklund transform for \(n = (1,0)\) supersymmetric Liouville system in the superfield form

\[
D^{(+)}W - D^{(+)}L = \frac{-i}{a} \exp\{W + L\}\psi_L^{(+)},
\]

\[
\partial_{(-)}W + \partial_{(-)}L = a \exp\{W - L\}
\]

One of the selfconsistency conditions for \((121)\) gives eq.\((112)\) and another do not contain \(W\) and restrict \(L\) by free superfield equation

\[
\partial_{-}D_{+}L = 0
\]

**Conclusion**

In this talk we have consider in details the previously proposed \[1\] generalized action principle on the simple example of \(D = 10\) (and 3, 4, 6) heterotic (super)string, have proved the off–shell (in rheonomy sense) superdiffeomorphism invariance of the generalized action, and demonstrate that it produce naturally the torsion constraint of the world–sheet geometry, as well as the doubly supersymmetric geometrical approach. The latter had been developed preciously in Ref. \[2\] on the basis of the postulated \textit{a priori} geometrodynamic equation \((39)\) and torsion constraints.

But just for the heterotic string, where the geometrodynamic equation \((39)\) do not leads to any dynamical equation \[12\] and, moreover, the superfield form of these dynamical
equations had not bee known, the description of the minimal embedding of the world sheet superspace was unclear [2].

We have investigated completely the geometric approach equations for $D = 3, N = 1$ superstring and have proved that they can be reduced to $n = (1, 0)$ supersymmetric generalization of the nonlinear Liouville equation.

We have stressed that this system of equations appears in the form of zero curvature representation.

The later property holds for nonlinear equations describing any super–$p$–brane in the doubly supersymmetric geometric approach.

The consequences of this fact are under investigation now.

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