Thermodynamics and phase transitions of black holes in contact with a gravitating heat bath

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Abstract
We study the thermodynamics of a shell of self-gravitating radiation, bounded by two spherical surfaces. This system provides a consistent model for a gravitating thermal reservoir for different solutions to vacuum Einstein equations in the shell’s interior. The latter include black holes and flat space, hence, this model allows for the study of black hole phase transitions. Following the analysis of Anastopoulos C and Savvidou N (2012 *Class. Quantum Grav.* 29 025004), we show that the inclusion of appropriate entropy terms to the space-time boundaries (including the Bekenstein–Hawking entropy for black hole horizons) leads to a consistent thermodynamic description. The system is characterized by four phases, two black hole phases distinguished by the size of the horizon, a flat space phase and one phase that describes naked singularities. We undertake a detailed analysis of black-hole phase transitions, the non-concave entropy function, the properties of temperature at infinity, and system’s heat capacity.

Keywords: entropy, radiation, shell, self-gravitating, singularity

Supplementary material for this article is available online
(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Motivation

Ever since Bekenstein’s proposal of black hole entropy [1] and Hawking’s derivation of black hole radiation [2], black holes are understood as thermodynamic objects. According to the

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generalized second law of thermodynamics (GSL) [3], black hole entropy adds up with matter entropy, and their sum never decreases with time.

The GSL implies that there exists a larger thermodynamic space that contains black holes and self-gravitating systems, i.e. systems of ordinary matter in which the gravitational self-interaction contributes significantly to their thermodynamic properties. Therefore, we expect phase transitions between black holes and self-gravitating systems.

In this work, we undertake the analysis of such phase transitions in a simple system, that consists of a shell of self-gravitating radiation. The geometry inside the shell is either Minkowski space, or a black hole or a singular solution. The different geometries correspond to different thermodynamic phases for the system, while the shell acts as a thermal bath for these phases. We analyze the thermodynamic properties of the system and the associated phase transitions.

There is substantial work on black-hole to black-hole phase transitions, first on Kerr–Newman black holes [4–6], and later on asymptotically anti-de Sitter (AdS) black holes [7–10]. Phase transitions in self-gravitating systems have also been extensively studied, see, for example, [11–14]. However, there is much less work on phase transitions between black holes and self-gravitating systems. The most well known case is the Hawking–Page phase transition between black holes and radiation [15], albeit in asymptotically AdS spacetimes. In asymptotically flat spacetimes, phase transitions have been studied by comparing the entropy of a Schwarzschild black hole in a box with the entropy of (non-gravitating) radiation in the box [4, 5, 16–18]. Backreaction from the Hawking radiation can also be included in the thermodynamical description [19].

Understanding phase transitions between black holes and self-gravitating systems is by itself important. Progress in this direction could also provide significant insights to quantum gravity theories [20], for example, on the relation between black hole hair and quantum effects, or on black-hole information loss.

A different context for this work is the physics of non-extensive thermodynamic systems. Non-extensivity arises whenever the range of interactions of the system is larger than the size of the system: this is possible, either with short-range forces in small systems [21] or with long-range forces [22, 23], including gravity [24, 25]. The model examined here is novel and informative for non-extensive thermodynamics. It involves full general relativity (rather than Newtonian approximations) and it brings together black holes and self-gravitating systems.

1.2. Background

An analysis of black hole phase transitions should start with the simplest self-gravitating system in general relativity, namely, static, spherically symmetric solutions to Einstein’s equations. These are described by the Tolman–Oppenheimer–Volkoff (TOV) equation. One might expect that for a given type of matter, there exists a region of the parameter space that corresponds to black hole horizons coexisting with matter. This turns out to not be the case: no horizons are encountered when integrating the TOV equation from the boundary inwards [26, 27]. In fact, there are only two types of solutions, regular ones—i.e. everywhere locally Minkowskian—and singular ones [27]. The latter contain a strongly repulsive naked singularity at the center, where the geometry is locally isomorphic to negative-mass Schwarzschild spacetime.

The situation is different when we consider a shell rather than a ball of matter. Suppose we place an interior boundary at \( r = r_0 \) so that the geometry for \( r < r_0 \) is a solution to the vacuum Einstein equations. Then, the associated thermodynamic state space contains regular solutions, singular solutions and black hole solutions. Hence, we can undertake a thermodynamic analysis that includes black hole phase transitions.
In this paper, we study a shell of radiation, bounded between two ideal reflecting and non-thermally conducting surfaces at $r = r_0$ and at $r = R$ (for past studies of self-gravitating radiation, see, references [28–33]). The shell defines a concrete model of a thermal reservoir in contact with a black hole that substantially improves past ones, e.g. [18], which employ the rather extreme idealization of switching off the gravitational interaction of the reservoir.

Our analysis employs the formal structure of equilibrium thermodynamics, as described by Callen [34]. As yet there exists no set of thermodynamic axioms applicable to non-extensive systems, even if some axiomatic approaches to thermodynamics [35–37] are amenable to generalization—for work in this direction see, reference [38–40]. The key point in Callen’s formulation of thermodynamics is that an isolated thermodynamic system is described in terms of a set of macroscopic constraints that determine its thermodynamic state space $Q$. The values of unconstrained variables correspond to global maxima of the entropy function. This statement is the maximum entropy principle (MEP). Eventually, all thermodynamic information is contained in the fundamental thermodynamic function, i.e. the entropy function $S : Q \rightarrow \mathbb{R}^+$. In the present system, the thermodynamic state space $Q$ is three dimensional, it is determined by the shell radii $r_0$ and $R$ and by the Arnowitt–Deser–Misner (ADM) mass $M$.

1.3. Results

Our results are the following.

(a) The space $Z$ of solutions to the TOV equation is larger than the thermodynamic state space $Q$. The equilibrium values of the additional degrees of freedom must be determined by the MEP. We show that if the entropy function involves only contributions from radiation, then it is unbounded and the MEP cannot be implemented. This is an analogue of the well known gravothermal catastrophe [42, 43].

(b) To resolve this, we must add entropy terms associated to the internal boundaries of the system, i.e. the horizons and the singularities that appear in the interior region. For horizons, we use the Bekenstein–Hawking entropy. For the repulsive singularity, the associated entropy is obtained by Wald’s Noether charge for spacetime boundaries [41], modulo a multiplicative constant. The latter is determined uniquely by the requirement of thermodynamic consistency [31], i.e. that the MEP can be implemented.

(c) We construct the entropy function on the thermodynamic state space $Q$. $Q$ splits into four regions, corresponding to four distinct phases. Phase $F$ corresponds to a shell in locally Minkowski spacetime, phase $B_I$ describes a black hole solution with practically no radiation in the shell, phase $B_{II}$ describes a black hole coexisting with radiation, and phase $S$ corresponds to singular solutions.

(d) Phase transitions between the $F$ phase and the $B_I$ and $B_{II}$ phases are first-order. The latent heat in all transitions from the $F$ phase to the black hole phases is negative, i.e. heat must be removed from a self-gravitating system in order to form a black hole. The $B_I$–$B_{II}$ and the $F$–$S$ transitions are continuous. There is also one triple point for the $F$, $B_I$ and $B_{II}$ phases. There is no coexistence curve between the black hole phases and the $S$ phase.

(e) We analyse the behavior of other thermodynamic observables, focusing on the temperature at infinity and the heat capacity.

The plan of this paper is the following. In section 2 we analyse the geometry of the shell of gravitating radiation, and all types of solutions. In section 3, we show that the inclusion of entropy contributions from boundary terms allows for a thermodynamically consistent description of the system. In section 4, we implement the MEP, we identify the four phases that...
characterize the system, and we analyze phase transitions and other thermodynamic properties. In section 5, we summarize and discuss our results.

2. Spacetime geometry for a shell of self-gravitating radiation

2.1. Constitutive equations

The system under study is a spherical shell of self-gravitating radiation in thermal equilibrium. The thermodynamic equations for radiation are

$$\rho = 3P = bT^4, \quad s = \frac{4}{3}b^{1/4}r^{3/4},$$

where $\rho$ is the energy density, $P$ is the pressure, $T$ is the temperature, $s$ is the entropy density and $b = \pi^2/15$ in geometrized units ($c = G = \hbar = k_B = 1$).

The spacetime metric is static and spherically symmetric. In the usual coordinates $(t, r, \theta, \phi)$,

$$ds^2 = -L(r)^2 dt^2 + \frac{dr^2}{1 - 2m(r)/r} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

where $L(r)$ is the lapse function and $m(r)$ is the mass function.

The radiation is contained between two reflecting spherical boundaries, an external boundary at $r = R$, and an internal one at $r = r_0 < R$. The spherical boundaries are assumed to be ideal totally reflecting surfaces of negligible mass. This idealization is common in the study of thermodynamic systems, including the treatment of self-gravitating radiation [28]. Ordinary metals are good approximations to such surfaces, if the temperature at the boundary is significantly smaller than the metal’s plasma frequency.

For $r > R$, the solution is Schwarzschild with ADM mass $M$, i.e.

$$L(r) = \sqrt{1 - \frac{2M}{r}}, \quad m(r) = M.$$ (3)

For $r \in [r_0, R]$, the geometry is determined by the TOV equation for radiation,

$$\frac{dm}{dr} = 4\pi r^2 \rho, \quad \frac{d\rho}{dr} = \frac{-4\rho(m + \frac{4}{3}\pi r^3 \rho)}{1 - \frac{2m}{r}}.$$ (4)

For fixed ADM mass $M$, we solve the TOV equation from the boundary $r = R$ inwards. To this end, in addition to $M$, we must specify the density $\rho_R := \rho(R)$ or equivalently the temperature $T_R := T(R)$.

Given $\rho(r)$ the lapse function $L(r)$ is specified by Tolman’s law, $L(r)T(r) = T_\infty$, where $T_\infty$ is the temperature at infinity. It implies that

$$L(r) = \frac{T_\infty}{T(r)} = \sqrt{1 - \frac{2M}{R} \left(\frac{\rho_R}{\rho(r)}\right)^{1/4}}.$$ (5)

The solution for $r < r_0$ is Schwarzschild with ADM mass $m_0 := m(r_0)$. As already mentioned in section 1.2, it has been proven [26, 27] that the integration of the TOV equations from the boundary inwards never encounters a horizon, so $2m_0 < r_0$ always. If $m_0 > 0$, there is a Schwarzschild horizon at $r = 2m_0$. If $m_0 = 0$, the geometry for $r < r_0$ is Minkowskian. If $m_0 < 0$, the geometry for $r < r_0$ is Schwarzschild with negative mass, i.e. there is a naked curvature singularity at the center.
2.2. Solution curves

Here, we consider the solution curves obtained when solving the TOV equations from the boundary inwards.

A solution of equation (4) is uniquely specified by the boundary data \((R, M, T_R)\). In terms of the dimensionless variables \(\xi := \log(r/R), u := 2m(r)/r\) and \(v := 4\pi r^2 \rho(r)\) the TOV equations become

\[
\frac{du}{d\xi} = 2v - u, \quad \frac{dv}{d\xi} = \frac{2v(1 - 2u - \frac{2}{3}v)}{1 - u}.
\]

Equation (6) are simpler than equation (4), because they define an autonomous two-dimensional dynamical system. They need only two positive numbers \((u_R, v_R)\) for boundary data, the outer boundary corresponding to \(\xi = 0\). The center corresponds to \(\xi \to -\infty\).

This simplification is possible only in linear equations of state, because they introduce no scale into the TOV equations. Hence, the solutions are invariant under the transformation \(r \to \lambda r, m \to \lambda^2 m, \rho \to \lambda^{-2} \rho\), for any \(\lambda > 0\).

In order to describe the behavior of the solution curves, we identify two straight lines on the \(u - v\) plane (figure 1) namely \(\epsilon_1: 2v - u = 0\), at which all solution curves satisfy \(du/d\xi = 0\) and \(\epsilon_2: 1 - 2u - \frac{2}{3}v = 0\), at which all solution curves satisfy \(dv/d\xi = 0\).

The two curves intersect at the point \(K = (\frac{3}{7}, \frac{1}{14})\). \(K\) and \(O = (0, 0)\) are the equilibrium points of the dynamical system (6).

A typical solution curve \(C: u = u(\xi), v = v(\xi)\) satisfies the following properties [27]: there is a point \(r_1 < R\), such that \(u(r) = 0\) and \(u(r) < 0\) for all \(r < r_1\); there is a point \(r_2 < r_1\), such that \(\frac{du}{dr}(r_2) = 0\) and \(\frac{du}{dr} > 0\) for all \(r < r_2\); finally \(\lim_{r \to -\infty}(u, v) = (-\infty, 0)\) and \(\lim_{r \to -\infty}(u, v) = K\).

The only non-trivial exception is the curve \(\Gamma\) of regular solutions, i.e. solutions that satisfy \(m(0) = 0\). On \(\Gamma\), \(\lim_{r \to -\infty}(u, v) = O\). The variables \(v\) and \(u\) attain maximum values on \(\Gamma\) at the points \(P\) and \(Q\) respectively, where \((u_P, v_P) \approx (0.3861, 0.3416)\) and \((u_Q, v_Q) \approx (0.4926, 0.2463)\).

The following solution curves are degenerate cases. (i) The points \(K\) and \(O = (0, 0)\) are equilibrium points, so each defines a distinct solution curve. (ii) A point on the \(u\) axis \((v_R = 0, u_R \neq 0)\), evolves with \(u(\xi) = u_R e^{-\xi}, v(\xi) = 0\), and it encounters an event horizon. This corresponds to a Schwarzschild black hole without matter.
2.3. Shell configurations

The solution curves uniquely determine the spacetime geometry associated to a shell of radiation—for a past treatment of this system, see, reference [33]. We select $R, M, T_R$, and we follow the solution curve until we encounter $r_0$. The segment of the solution curve between $r_0$ and $R$ determines the shell’s geometry, for $r > R$, and $r < r_0$ the geometry is Schwarzschild with mass $M$ and $m_0$ respectively.

Hence, the space $Z$ of equilibrium configurations for a shell of self-gravitating radiation is four-dimensional. It can be described by the coordinates $(R, r_0, M, T_R)$ or equivalently by the coordinates $(R, \xi_0, u_R, v_R)$, where $\xi_0 := \log(r_0/R)$.

The solutions are of three types, depending on $m_0 := m(r_0)$.

- $m_0 = 0$: $F$-type (flat) solution for $r < r_0$.
- $m_0 > 0$: $B$-type (black-hole), it contains a Schwarzschild horizon at $r = 2m_0$.
- $m_0 < 0$: $S$-type (naked singularity), it contains the negative-mass Schwarzschild singularity at $r = 0$.

$F$-type solutions form a set of measure zero in $Z$ that acts as a the boundary between the subset of $B$-type and $S$-type solutions. The curve of the $F$-type solutions on the $u_R - v_R$ plane (the $F$-curve for brevity) depends only on $\xi_0$, because of the scaling symmetry.

In figure 2, the $F$-curve is plotted for different $\xi_0$. $B$-type solutions lie in the region between the $F$-curve and the $u_R$ axis; the remainder corresponds to $S$-type solutions. As $\xi_0$ decreases, so does the area of the $B$ phase. At $\xi_0 \to -\infty$, $B$-type solutions disappear and the $F$-curve coincides with the line $\Gamma$ of figure 1.

For fixed $\xi_0$, the $F$-curve has two distinctive points: the point $u_{\text{max}}(\xi_0)$ that corresponds to the maximum value of $u_R$; the final point $u_f(\xi_0) \neq 0$, where the curve intersects the horizontal axis ($v_R = 0$).

In figure 2, we see that both $u_{\text{max}}(\xi_0)$ and $u_f(\xi_0)$ decrease with decreasing $\xi_0$—see, figure 3. As $\xi_0 \to -\infty$, $u_{\text{max}} \to u_Q$, i.e. it corresponds to the Oppenheimer–Volkoff limit for a sphere of radiation [28]. In the same limit, $u_f$ vanishes as $e^{\xi_0}$. The dependence of $u_{\text{max}}$ and of $u_f$ on $\xi_0$ is plotted in figure 3.

From figure 2, we can analyse how the three types of solution are distributed in the one-dimensional submanifolds (fibers) $V_{(u_R, \xi_0, R)}$ of constant $(u_R, \xi_0, R)$. For given $\xi_0$, each fiber corresponds to a line $u_R = \text{constant}$ in the plots of figure 2. We characterize the fibers in terms of their intersection with the $F$-curve. There are three types of behavior, by which we characterize the fibers as being of type I, II, and III.
**Figure 3.** Plot of $u_{\text{max}}$ and $u_{f}$ against $\xi_0$. Notice that $\lim_{\xi_0 \to -\infty} u_{\text{max}} = u_{Q}$.

- Fibers of type I are defined by $u_R \leq u_{f}(\xi_0)$. The line $u_R = \text{constant}$ intersects the $F$-curve only once, at some point $v_1$. For $u_R < v_1$, the fiber contains $B$-type solutions, and for $u_R > v_1$, it contains $S$-type solutions.

- Fibers of type II are defined by $u_{f}(\xi_0) < u_R < u_{\text{max}}(\xi_0)$. The line $u_R = \text{constant}$ intersects the $F$-curve at least twice. Hence, these fibers involve two $F$-type solutions, at $v_R = v_1$ and $v_R = v_2$, such that all solutions with $v \notin [v_1, v_2]$ are $S$-type. In the interval $(v_1, v_2)$ the solutions are either $B$-type, or there exist alternating regions of $B$-type and $S$-type solutions. The latter is the case for large negative values of $\xi_0$ where the $F$-curve develops a spiraling shape, hence, it intersects the line $u_R = \text{constant}$ more than twice.

- Fibers of type III are defined by $u_R > u_{\text{max}}(\xi_0)$. All points in these fibers correspond to $S$-type solutions.

### 3. Thermodynamic consistency and maximum entropy principle

#### 3.1. The fundamental representation

We proceed with a study of the thermodynamics of the shell system. We will employ the fundamental representation (i.e. the entropy representation) of thermodynamic systems: the relevant thermodynamic potential is the entropy $S$, expressed as a function of the total energy $M$ and of the variables $r_0$ and $R$ that describe the area of the bounding surfaces.

In extensive systems, the choice of representation does not affect physical predictions. Extensivity together with the second law of thermodynamics imply that $S$ is a concave function of the extensive variables [34]. The Legendre transform for concave functions fully preserves their information. Hence, all representations connected by a Legendre transform describe the same physics.

In a non-extensive system, the entropy $S$ needs not be a concave function. Then, the Legendre transform does not preserve all thermodynamically relevant information, and other thermodynamic representations are not equivalent to the fundamental one. Hence, for isolated gravitating systems, we must use the fundamental representation.

The shell of radiation considered here acts as a reservoir for the geometry in the interior region $r < r_0$. However, radiation is strongly affected and it strongly affects the interior region, so that a split between a system and a reservoir makes no sense. In gravitational systems, we must treat system and reservoir as a single isolated system, and for this reason, we must always work with the fundamental representation.
3.2. The maximum-entropy principle

First, we consider the shell system with gravity switched off. In the entropy representation, the entropy $S$ is a function of $R$, $r_0$, and $M$,

$$S(R, r_0, M) = \frac{4}{3} b^{1/4} V^{1/4}(R, r_0) M^{3/4},$$

(7)

where $V(R, r_0) = \frac{4}{3} \pi (R^3 - r_0^3)$ is the volume of the shell. The thermodynamic state space $Q = \{(R, r_0, M)\}$ is three-dimensional.

The thermodynamic state space remains the same when gravity is switched on. Since the volume is not fixed by area the bounding surfaces in general relativity, the dependence of $S$ on $R$ and $r_0$ is nontrivial. However, the set $Z$ of solutions to the TOV equation is four dimensional. Hence, the independent thermodynamic variables do not fix uniquely the solution.

In compact stars ($r_0 = 0$), this problem is usually addressed by an additional assumption, namely, regularity at the center $m(0) = 0$. Regular solutions form a set of measure zero in the set of all solutions. Almost all solutions have $m(0) < 0$, and there are no solutions with $m(0) > 0$. Solutions with $m(0) < 0$ have a curvature singularity at the center.

The regularity condition is inadequate because it does not cover the whole thermodynamic state space, as there are no solutions with $m(0) = 0$, if $M > M_{OV}$, where $M_{OV}$ is the Oppenheimer–Volkoff limit. Furthermore, there is no good physical reason to a priori exclude singular solutions from all considerations, for example, they could appear in a quantum or in a statistical sum of geometries as virtual solutions [31]. Furthermore, singular solutions in this system cause no problem to causality and predictability: the spacetime has no inextensible geodesics, it is bounded-acceleration complete, and it is conformal to a globally hyperbolic spacetime with boundary [27].

The shell system studied here demonstrates more forcefully the inadequacy of the regularity condition. Even if one wants to a priori exclude $S$-type solutions—presumably because of the naked singularity—, there is no justification in excluding $B$-type solutions.

We will select the equilibrium configurations by employing the MEP. The MEP asserts that the values assumed by any unconstrained parameter maximize the entropy over the manifold of constrained states [34].

In the entropy representation, the total mass $M$ and the shell radii $R$ and $r_0$ are assumed to be constrained. As shown in section 2.3, the manifold $Z$ is foliated by surfaces of constant $(M, R, r_0)$, or equivalently, of constant $(R, u_R, \xi_0)$. Each fiber $V(R, u_R, \xi_0)$ of the foliation is parameterized by $v_R$. We construct an entropy function $S_Z$ on $Z$, $S_Z(R, u_R, \xi_0, v_R)$. The MEP asserts that the equilibrium state for $M, R, r_0$ is obtained by maximizing the entropy functional along the associated fiber,

$$S_{eq}(M, R, r_0) = \max_{v_R} S_Z(R, u_R, \xi_0, v_R).$$

(8)

If there is no global maximum of $S_Z$ on a fiber, the MEP fails to apply. This is the case in some gravitating systems, known as the gravothermal catastrophe [42].

In what follows, we will show that if $S_Z(R, u_R, \xi_0, v_R)$ involves only a contribution $S_{rad}$ from radiation, the shell system is not thermodynamically consistent. In contrast, an appropriate gravity contribution $S_{gr}$ makes the system consistent. The gravitational entropy $S_{gr}$ is a Noether charge associated to the spacetime boundaries in the region $r < r_0$. For $B$-type solutions, $S_{gr}$ is the Bekenstein–Hawking entropy. For the $S$-type solutions, the working expression for $S_{gr}$ is the same with the one identified in reference [31].
3.3. Radiation entropy

The radiation entropy $S_{\text{rad}}$ of a solution to the TOV equation is given by

$$S_{\text{rad}} = 4\pi\int_{R0}^{R} \frac{r^2 s}{\sqrt{1 - 2m(r)}} \, dr = \frac{4}{3} (4\pi b)^{1/4} \int_{R0}^{R} \frac{r^{1/2} v^{3/4}}{\sqrt{1 - u}} \, dr. \quad (9)$$

Solutions to the TOV equation satisfy \[28\]

$$r^{1/2} v^{3/4} = \frac{d}{dr} \left( \frac{v + \frac{1}{2} u}{6v^{1/4}\sqrt{1 - u}} \right), \quad (10)$$

from which we obtain

$$S_{\text{rad}}(R, \xi_0, u_R, v_R) = \frac{2}{9} (4\pi b)^{1/4} \left( \frac{v_R + \frac{1}{2} u_R}{v_R^{1/4}\sqrt{1 - u_R}} - \frac{v_0 + \frac{1}{2} u_0}{v_0^{1/4}\sqrt{1 - u_0}} e^{3\xi_0/2} \right) R^{3/2}, \quad (11)$$

where $u_0 = u(\xi_0)$ and $v_0 = v(\xi_0)$. The simple dependence of $S_{\text{rad}}$ on $R$ is due to the scaling symmetry.

The key point here is that $S_{\text{rad}}$ has no global maximum since $\lim_{v \to \infty} S_{\text{rad}} = \infty$ for any $(R, \xi_0, u_R)$. Furthermore, $\lim_{v \to 0} S_{\text{rad}} = \infty$ for all $R$ whenever $\xi_0 < \log u_R$. These limits can easily be seen in numerical calculation. Figure 4 shows the results of such a calculation, $S_{\text{rad}}/R^{3/2}$ is plotted as function of $v_R$ for fixed $u_R$ and $\xi_0$.

To conclude, the MEP cannot be applied in the shell system if $S_{\text{rad}}$ is the only contribution to the system’s entropy. There is no global maximum of entropy in the fibers of constant $(R, u_R, \xi_0)$.

3.4. Entropy from spacetime boundaries

Ever since Bekenstein’s and Hawking’s work on black hole thermodynamics, we know that entropy can be meaningfully assigned to gravitational degrees of freedom. Wald showed that black hole entropy can be expressed in terms of the Noether charge $Q(\xi)$ of spacetime diffeomorphisms \[41\], as

$$S = \frac{Q(\xi)}{T_{\infty}}. \quad (12)$$
Figure 5. Left: we plot $S_{\text{rad}} + S_{\text{sing}} R^3/2$ as a function of $v_R$ for $\xi_0 = -4$ and for different values of $u_R$: (a) $u_R = 0.44$ (b) $u_R = 0.47$, and (c) $u_R = 0.50$. Case (a) has four $F$-type solutions, corresponding to the local maxima of the entropy function. Case (b) has two $F$-type solutions, again corresponding to local maxima of entropy. Case (c) has no $F$-type solution, the entropy maximum corresponds to a $S$-type solution. Right: $S_{\text{rad}} + S_{\text{sing}} R^3/2$ (blue) and $u_0$ (orange) are plotted as function of $v_R$ for $\xi_0 = -4$ and $u_R = 0.44$. This plot demonstrates clearly the one-to-one correspondence between maxima of entropy and $F$-type solutions ($u_0 = 0$).

The Noether charge $Q(\xi)$ is defined in terms of the time-like Killing vector $\xi = \frac{\partial}{\partial t}$, normalized so that $\xi^\mu \xi_\mu = -1$ at infinity, and evaluated on the horizon, viewed as a boundary of the surfaces of constant $t$:

$$ Q(\xi) = \frac{\lambda}{4\pi} \oint_{\partial \Sigma} d\sigma_{\mu\nu} \nabla^\mu \xi^\nu, \quad (13) $$

where $\lambda$ is an arbitrary multiplicative constant.

For positive-mass Schwarzschild spacetime, $Q(\xi) = 2\lambda M$, when evaluated at the horizon. Since $T_\infty = 8\pi M$, the Bekenstein–Hawking entropy $S_{\text{BH}} = 4\pi M^2$ is obtained for $\lambda = \frac{1}{2}$. We will use the Bekenstein–Hawking entropy for the entropy of the horizon in $B$-type solutions: $S_{\text{grav}} = 4\pi m_0^2$.

For $S$-type solutions, the singularity at $r = 0$ defines a timelike boundary [27]. For this boundary, $Q(\xi) = 2\lambda m_0 \kappa$, suggesting an entropy associated to singularity equal to

$$ S_{\text{sing}} = \frac{2\lambda m_0 \kappa}{T_\infty} = \lambda(4\pi b)^{1/4} \frac{u_0}{v_0^{1/4}} \sqrt{1 - u_0} e^{3\xi_0/2} R^{3/2}. \quad (14) $$

The only way we have found to specify $\lambda$ is through the requirement of thermodynamic consistency. In [31], it was shown that the only way to implement the MEP for a sphere of self-gravitating radiation is by assigning entropy to the singularity with $\lambda = 2$. The same holds in the shell system studied here. We found numerically that the only value of $\lambda$ that provides a consistent implementation of the MEP is $\lambda = 2$. The reader may see this in the plots of the supplementary material (https://stacks.iop.org/CQG/38/195026/mmedia).

For $\lambda = 2$, $S_{\text{sing}}$ is negative, and vanishes for $m_0 = 0$, thereby enhancing the stability of $F$-type solutions. We found numerically that all $F$-type solutions at a fiber of constant $(u_R, R, \xi_0)$ correspond to local maxima of $S_{\text{rad}} + S_{\text{sing}}$, with respect to $v_R$. $S$-type maxima are only possible in fibers with no $F$-type solutions, i.e. for $u_R > u_{\text{max}}(\xi_0)$. This behavior is demonstrated in figure 5. This result is identical with the one of reference [31] for balls of radiation. It also persists for other equations of state.
Figure 6. Entropy maxima and phase transition in fibers of type I for $R = 10^{38}$ and $\xi_0 = -1$.

We conclude that the entropy function is

$$S_Z = \begin{cases} S_{\text{rad}} + S_{\text{BH}}, & u_0 \geq 0 \\ S_{\text{rad}} + S_{\text{sing}}, & u_0 < 0 \end{cases}.$$  \hspace{1cm} (15)

We note that both $S_{\text{rad}}(R, \xi_0, u_R, v_R)$ and $S_{\text{sing}}(R, \xi_0, u_R, v_R)$ can be expressed as $f(\xi_0, u_R, v_R) R^{3/2}$, i.e. they scale with $R^{3/2}$. In contrast,

$$S_{\text{BH}} = 4\pi m_0^2 = \pi (u_0 e^{\xi_0})^2 R^2$$  \hspace{1cm} (16)

scales with $R^2$. The black hole contribution breaks the scaling invariance of the entropy that originates from the scale independence of the equation of state.

4. Phase transitions and other thermodynamic properties

4.1. The four phases of the system

Next, we implement the MEP on each fiber of constant $(R, u_R, \xi_0)$. There are three different scenarios, one for each type of fiber, see, section 2.3.

Type I fibers: if $u_R < u_f(\xi_0)$, then the fiber contains a single $F$-type solution, say at $v_R = v_1$, that is a local maximum. Solutions for $v_R > v_1$ are $S$-type, and they have all smaller entropy than the $F$-type solution. There is no local maximum of entropy for $S$-type solutions.

For $u_R < v_1$, solutions are $B$-type. The maximum value of entropy for $B$-type solutions occurs typically at very small $v_R$, often numerically indistinguishable from $v_R = 0$, i.e. a black hole with very little radiation in the shell, in the sense that $M - m(r_o) \ll M$. We will refer to this type of black hole, as a $B_1$-type solution.

Hence, the entropy along a type I fiber contains one $F$-type local entropy maximum $S_F$, and one $B_1$-type local maximum $S_{\text{BH}}$. The global maximum is the highest of the two local maxima.

Typical plots of the entropy along a fiber are given in figure 6. The behavior is characteristic of a first-order phase transition. As the location of the fiber changes, so does the relative height of the two maxima. If $S_{\text{BH}} > S_F$, the equilibrium phase is $B_1$-type; if $S_F > S_{\text{BH}}$ the equilibrium phase is $F$-type. The submanifold of $Q$ where $S_{\text{BH}} = S_F$ is the coexistence curve for the $B$–$F$ transition.

Type II fibers: type II fibers are characterized by several $F$-type local maxima and at least one $B$-type local maximum. The main difference from fiber I is that the $B$-type local maxima lie at an intermediate value between two $F$-type maxima—see, figure 7.
Figure 7. Entropy maxima and phase transition in fibers of type II for $R = 10^{38}$ and $\xi_0 = -1$.

Table 1. The three types of fiber and the four phases.

| Fiber type | Definition | Types of solution | Phases |
|------------|------------|-------------------|--------|
| I          | $u_R < u_f(\xi_0)$ | S, F, B          | F, B_I |
| II         | $u_f(\xi_0) < u_R < u_{\text{max}}(\xi_0)$ | S, F, B          | F, B_{II} |
| III        | $u_R > u_{\text{max}}(\xi_0)$ | S            | S      |

In these $B$-type solutions a significant fraction of mass is in the form of radiation in the shell, and the black hole is often very small, in the sense that $m_0 \ll M$. We will denote these solutions as $B_{II}$.

Type III fibers: type III fibers contain only $S$-type solutions. Obviously, the entropy-maximizing solutions are $S$-type.

The results of our implementation of the MEP in the shell system are summarized in table 1.

4.2. Phase diagrams and coexistence curves

In figure 8, we show how the submanifolds of constant $R$ are partitioned into the four phases, for different values of $R$. We remark the following.

(a) The phases are separated by four coexistence curves. Two coexistence curves coincide with the submanifolds that separate fibers of different types. In particular, the surface $u_R = u_f(\xi_0)$ separates between fibers of type I and fibers of type II, and the surface $u_R = u_{\text{max}}(\xi_0)$ separates between fibers of type II and fibers of type III. *Phase transitions across these surfaces are continuous*, because the value of $v_R$ at maximum entropy is continuous. Hence, the temperature at infinity $T_\infty$ is also continuous. The former surface describes the $F$–$S$ phase transition, and the second surface describes the $B_I$–$B_{II}$ phase transition.

On the other hand $v_R$ is discontinuous along the $F$–$B_I$ and $F$–$B_{II}$ phase transitions, so these transitions are of first order.

There is one triple point (actually a curve in the thermodynamic state space $Q$) for the $F$–$B_I$–$B_{II}$ phases. Both $u_R$ and $\xi_0$ on the triple point decrease with $R$.

(b) There is no coexistence curve between the $S$ phase and either of the $B_I$ and the $B_{II}$ phases. The $S$ phase has only a coexistence curve with the $F$ phase. This leads to a rather ‘bizarre’ behavior, of a thin strip of $F$-phase being intermediate between the $B_I$ and the $S$ phase. However, this is mathematically necessary, since one cannot go from positive values of $m_0$ to negative values of $m_0$ without crossing the surface $m_0 = 0$. In absence of this strip,
Figure 8. Phase diagrams for different values of $R$. In plots (a), (b) and (c) we use a logarithmic scale for the $u_R$ axis. In plot (d), we use a linear scale for the $u_R$ axis, the plot is practically insensitive to $R$, since the differences at small $u_R$ cannot be distinguished.

The $F$ phase is always at lower energy than the black hole phases, in accordance with the Page–Hawking phase transition [15] or the heuristic discussions of black hole formation in a box [4, 17].

The strip of $F$-phase may be removed, if the Bekenstein–Hawking expression for entropy changes at small masses. A small-mass black hole emits more energy in Hawking radiation, and in presence of the box the black hole would have to coexist with its Hawking radiation. If the Hawking radiation contributes negatively to the energy, then it would be possible to have $m_0 < 0$ even in presence of the horizon. It would then be possible to pass from the $S$ phase to the $B$ phase without crossing from the $F$ phase. However, such a modification is conjectural. At the current state of knowledge, it can only be implemented by an ad-hoc insertion of an additional parameter in the Bekenstein–Hawking formula.

(c) In linear scale for $u_R$, the $F$-phase can only be distinguished in the thin strip interpolating between the $S$ and $B_{II}$ phases. We need a logarithmic scale for $u_R$ in order to see the intuitively obvious result that the $F$-phase dominates at small masses.

The $B$-phases dominate at high $R$ and they are suppressed at small $R$. This is obvious since the horizon contribution to the entropy grows faster than any other contribution with the scale of the system. However, the $B$-phases disappear as $\xi_0 \to -\infty$ for all $R$, reflecting the fact that there are no horizons in a ball of self-gravitating radiation.

(d) $B_I$ solutions have smaller ADM mass than $B_{II}$ solutions for the same $\xi_0$ and $R$. However, the area of the horizon (determined by $m_0$) in $B_I$ is typically larger, because a large part of the mass of $B_{II}$ consists of radiation in the shell.
Figure 9. Plot of $S_{eq}/R^{3/2}$ as a function of $u_R$ for $R = 10^{38}$ and (a) $\xi_0 = -1$, (b) $\xi_0 = -2$, (c) $\xi_0 = -5$. For $\xi_0 = -1$ auxiliary graphs zoom in specific ranges of $u_R$ where the transitions $F\rightarrow B_1$, $B_1\rightarrow B_{II}$ and $B_{II}\rightarrow F\rightarrow S$ take place, respectively.

4.3. The entropy function in equilibrium

After the implementation of the MEP, we evaluate the entropy function $S_{eq}(R, u_R, \xi_0)$ on $Q$, using equation (8). Characteristic plots of $S_{eq}$ as a function of $u_R$, for fixed $\xi_0$ and $R$ are given in figure 9.

We see that the entropy function is, in general, a non-concave function of $u_R$, and hence, of $M$. By construction, $S$ is continuous across phase transitions.

We note that the entropy is not an increasing function of $u_R$, and that it is bounded from above for fixed $R$ and $\xi_0$. This maximum satisfies Bekenstein’s bound [44, 45], $S < 2\pi MR$, or equivalently $S/\mu R < \pi R^2$, for all macroscopic values of $R$.

4.4. Temperature and heat capacity

The maximization of the MEP also allows us to identify $v_R$ as a function on $Q$ for the equilibrium solutions. Then, the temperature at infinity is

$$T_\infty(R, \xi_0, u_R) = \sqrt{1 - u_R \left( \frac{b v_R(R, \xi_0, u_R)}{4\pi R^2} \right)^{-1/4}}.$$  \hspace{1cm} (17)

The temperature $T_R$ cannot be identified with a partial derivative of the entropy function. This is only possible for solutions with a simply connected boundary [40], which includes regular solutions to the TOV equation [28].

Plots of $T_R$ as a function of $u_R$, for fixed $\xi_0$ and $R$ are given in figure 10. The temperature $T_R$ is not an increasing function of the total energy $M$ for fixed $R$ and $r_0$, and it has finite jumps at first-order phase transitions.

The natural definition of heat capacity $C$ for fix boundaries is [30]

$$C := \left( \frac{\partial M}{\partial T_\infty} \right)_{R, r_0} = \frac{R}{2} \left( \frac{\partial T_\infty}{\partial u_R} \right)_{R, \xi_0}^{-1}. \hspace{1cm} (18)$$

Clearly, the heat capacity is negative in any region of the thermodynamic state space $Q$ where $T_R$ decreases with $u_R$—see, figure 10. At the local maxima of $T_R$, $\left( \frac{\partial T_\infty}{\partial u_R} \right)_{R, \xi_0} = 0$, hence, $C$ diverges and changes sign. At the points of the first-order phase transition, $C$ exhibits discontinuities. Like the ball of self-gravitating radiation [30], the present system is also characterized by alternating regions of positive and negative heat capacities throughout the thermodynamic state space.
Figure 10. Plot of $T_R$ as a function of $u_R$ for fixed $R = 10^{38}$ and different values of $\xi_0$: (a) $\xi_0 = -1$, (b) $\xi_0 = -2$, (c) $\xi_0 = -5$. The insets zoom near the phase transition points for curve (a) and demonstrate a finite discontinuity for $T_R$.

4.5. Latent heat

Since the transitions $F$–$B_I$ and $B_{II}$–$F$ are first-order, they involve latent heat. In extensive systems, the latent heat $L$ is identified as the difference $\Delta H$ of the enthalpy between the two phases, while the Gibbs free energy is constant. Hence, $L = \Delta H = \Delta(G + TS) = \Delta(TS)$. In the fundamental representation, $S$ is constant along the transition, hence, $L = S\Delta T$.

We employ the analogue of this formula in our system, i.e. we consider the quantity

$$L = S\Delta T_{\infty}$$

(19)

as a candidate for the latent heat in the first order transitions between flat space and the black hole phases. This choice is based primarily on the basis of analogy with extensive thermodynamics. However, it appears plausible that this is the amount of heat (19) we must give the system on the coexistence curve—together with work $\Delta W = -L$, so that $\Delta U = 0$—for the phase change to occur.

In figure 11, we plot the reduced latent heat $L/M$ as a function of $\xi_0$, for fixed $R$. We note the following.

(a) The latent heat is negative for both the $F \rightarrow B_I$ and the $F \rightarrow B_{II}$ transitions. This means that the black hole phases are always low temperature phases compared to flat space, one needs to ‘boil’ a black hole in order to remove the horizon. This result may appear surprising. Except for the narrow strip of $F$ phase before the $S$ phase, the $F$ phase has lower internal energy than the $B$ phases. However, the temperature of the $F$ phase is higher. This is due to the fact that the temperature $T_R$ is not a monotonous function of the energy.

It is easy to show that black holes always have lower temperature than self-gravitating systems along the coexistence curve. Consider, for example, a simplified analysis of black hole phase transitions in the vein of [4]. A box of radius $R$ contains either a black hole of mass $M$ or (non-gravitating) radiation with the same mass $M$. In the former case, the entropy is given by the Bekenstein–Hawking expression $S_1 = 4\pi M^2$, in the latter by $S_2 = \frac{4\pi}{\pi^2}b^{1/4}M^{3/4}R^{3/4}$, where $b$ is the constant in equation (1). The associated temperatures are $T_1 = (8\pi M)^{-1}$ and $T_2 = \frac{1}{4}b^{-1/4}M^{1/4}R^{-3/4}$.

A black hole is entropically favored for $M > M_c$, where the critical mass $M_c = \left(\frac{b}{81}\right)^{1/5}R^{3/5}$. The temperature of the black hole phase at $M_c$ is $T_1(M_c) = \frac{1}{8\pi R} \left(\frac{81}{b}\right)^{1/5}R^{-3/5}$. The temperature of the radiation phase is $T_2(M_c) = \frac{1}{\pi(81)^{1/6}}b^{-1/5}R^{-3/5}$. We compute
Figure 11. The reduced latent heat $L/M$ for transitions $F \to B_I$ (left) and $B_{II} \to F$ (right) as a function of $\xi_0$ and for constant $R = 10^{38}$.

$$\Delta T = T_1(M_c) - T_2(M_c),$$
and we find that, indeed,

$$\Delta T = -\frac{5}{24\pi R^{3/5}} \left( \frac{81}{b} \right)^{1/5} < 0. \quad (20)$$

A plausible conjecture is that negative latent heat refers to the amount of radiation (electromagnetic or gravitational) that must be emitted during gravitational collapse before the system settles as a black hole. To test this conjecture, we must undertake an analysis of gravitational collapse in the context of non-equilibrium thermodynamics.

(b) The reduced latent heat is almost constant for a large range of values of $\xi_0$. It vanishes as $\xi_0 \to -\infty$, well outside the range of the plots in figure 11, because the black hole phases disappear at this limit.

(c) The latent heat is much smaller in transitions involving the $B_{II}$ phase than in transitions with the $B_I$ phase. This is because the horizon in the $B_{II}$ phase is much smaller than the horizon in the $B_I$ phase.

5. Conclusions

We analysed the thermodynamics of a shell of gravitating radiation surrounding a solution to vacuum Einstein’s equation, which may either correspond to flat space, a black hole or a repulsive singularity. The shell can be interpreted as a gravitating heat reservoir. However, the presence of long range forces necessitates an analysis of the total system that consists of the shell and its interior.

We showed that the only way to obtain a consistent thermodynamic description of the system is by assigning a specific expression for entropy to the naked singularity, thus, confirming the proposal of reference [31]. The result is a concrete model for describing phase transitions between black holes and self-gravitating systems that is fully compatible with the rules of thermodynamics.

The methods of this paper can be straightforwardly generalized, for example, to different equations of state, rotating systems and other shell geometries. The self-gravitating shell of radiation can also be used as a generic thermal reservoir in studies of system-reservoir thermodynamics in self-gravitating systems.

Furthermore, our results strongly suggest the importance of a thermodynamic analysis to gravitational collapse. The non-equilibrium evolution of a self-gravitating shell is perhaps the simplest model for analyzing the interplay between horizon formation and thermodynamics.
in a gravitating system. We expect that the solutions studied in this paper will correspond to asymptotic states of a non-equilibrium analysis.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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