ALTERNATING SUMS OVER $\pi$-SUBGROUPS

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Abstract. Dade’s conjecture predicts that if $p$ is a prime, then the number of irreducible characters of a finite group of a given $p$-defect is determined by local subgroups. In this paper we replace $p$ by a set of primes $\pi$ and prove a $\pi$-version of Dade’s conjecture for $\pi$-separable groups. This extends the (known) $p$-solvable case of the original conjecture and relates to a $\pi$-version of Alperin’s weight conjecture previously established by the authors.

1. Introduction

One of the most general local-global counting conjecture for irreducible complex characters of finite groups is due to E. C. Dade [D]. For a finite group $G$, a prime $p$ and an integer $d > 0$, the conjecture asserts that the number of irreducible characters of $G$ of $p$-defect $d$ can be computed by an alternating sum over chains of $p$-subgroups. (In this paper, we only deal with the group-wise ordinary conjecture; see [N, Conjecture 9.25].) Dade [D] already showed that his conjecture implies Alperin’s weight conjecture. The first author has proved that McKay’s conjecture is also a consequence of Dade’s conjecture (see [N, Theorem 9.27]). Dade’s conjecture is known to be true for $p$-solvable groups by work of G. R. Robinson [R] (see also Turull [T17]), and a reduction of it to simple groups has been recently conducted by B. Späth [Sp].

In previous work by Isaacs–Navarro [IN] and the present authors [NS], we have replaced $p$ by a set of primes $\pi$ in order to prove variants of Alperin’s weight conjecture for $\pi$-separable groups. In this paper, we are interested in chains of $\pi$-subgroups and alternating sums: that is, we look for a $\pi$-version of Dade’s conjecture and for possible applications. For instance: if $C(G)$ is the set of chains of $\pi$-subgroups of $G$, $G_C$ is the stabilizer in $G$ of the chain $C$, and $k(G_C)$ is the number of conjugacy classes of $G_C$, does the number

$$\mu_\pi(G) = \sum_{C \in C(G)} (-1)^{|C|} \frac{|G_C|}{|G|} k(G_C)$$



\textit{2010 Mathematics Subject Classification.} Primary 20C15; Secondary 20C20.
\textit{Key words and phrases.} Dade’s Conjecture, Alternating Sums, $\pi$-subgroups.

The research of the first author supported by Ministerio de Ciencia e Innovación PID2019-103854GB-I00 and FEDER funds. The second author thanks the German Research Foundation (projects SA 2864/1-2 and SA 2864/3-1).
If $\pi = \{p\}$, then the Alperin weight conjecture (with the Knörr–Robinson [KR] reformulation) asserts that $\mu_p(G)$ is the number of $p$-defect zero characters of $G$. In particular, this is the case for $p$-solvable groups. If $G = \text{PSL}_2(11)$ and $\pi = \{2, 3\}$, say, then $\mu_2(G) = 0$, while $G$ has 2 irreducible characters with $\pi$-defect zero for every $p \in \pi$. So, whatever the meaning of $\mu_\pi(G)$ is, certainly it is not the number of $\pi$-defect zero characters of $G$.

For an integer $d \geq 1$, we let $k_d(G)$ to be the number of irreducible characters $\chi \in \text{Irr}(G)$ such that $|G|_\pi = d\chi(1)_\pi$, where $n_\pi = \prod_{p \in \pi} n_p$, and $n_p$ is the largest power of $p$ dividing the positive integer $n$. (Notice that this deviates slightly from the usual notation for $\pi = \{p\}$.)

Our main result is a natural generalization of Dade’s conjecture for $p$-solvable groups:

**Theorem A.** Let $G$ be a $\pi$-separable group, and let $d > 1$. Then

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C| |G_C| k_d(G_C)} = 0.$$ 

Unlike the original conjecture in the case $\pi = \{p\}$, we cannot restrict ourselves to so-called normal chains in Theorem A (see [N], Theorem 9.16). In fact, $G = S_3$ with $\pi = \{2, 3\}$ is already a counterexample. (This is related to the fact that $\pi$-subgroups are not in general nilpotent!) For this reason, the known proofs of the $p$-solvable case cited above cannot be carried over to $\pi$. We will obtain Theorem A as a special case of a more general projective statement with respect to normal subgroups.

In this paper, let $l(G)$ be the number of conjugacy classes of $\pi'$-elements in $G$. Recall that $\chi \in \text{Irr}(G)$ has $\pi$-defect zero if $\chi(1)_p = |G|_p$ for all $p \in \pi$. The number of those characters is $k_1(G)$, using the notation above.

**Corollary B.** Let $G$ be a $\pi$-separable group. Then

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C| |G_C| k(G_C)} = \sum_{C \in \mathcal{C}(G)} (-1)^{|C| |G_C| l(G_C)}$$

is the number of $\pi$-defect zero characters of $G$.

2. Proofs

We fix a set of primes $\pi$ for the rest of the paper. If $G$ is a finite group, we consider chains $C$ of $\pi$-subgroups in $G$ of the form $1 = P_0 < P_1 < \ldots < P_n$ where $n = 0$ is allowed (the trivial chain). Let $|C| = n$ and let

$$G_C = N_G(P_0) \cap \ldots \cap N_G(P_n)$$

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1Theorem A was proposed as a conjecture in the second author’s Oberwolfach talk in 2019.
be the stabilizer of $C$ in $G$. The set of all such chains of $G$ is denoted by $\mathcal{C}(G)$.

For a normal subgroup $N$ of $G$ and $\theta \in \text{Irr}(N)$, let $k_\theta(G|\theta)$ be the number of irreducible characters $\chi$ of $G$ lying over $\theta$ with $|G|_\pi = d\chi(1)_\pi$. We denote by $G_\theta$ the stabilizer of $\theta$ in $G$. By the Clifford correspondence, notice that

$$k_\theta(G|\theta) = k_\theta(G_\theta|\theta).$$

We start with the following.

**Lemma 2.1.** Let $G$ be a finite group, and let $f$ be a real-valued function on the set of subgroups of $G$. If $O_\pi(G) > 1$, then

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|}|G_C|f(G_C) = 0.$$

**Proof.** Let $C : 1 = P_0 < \ldots < P_n$ be a chain in $\mathcal{C}(G)$. If $N = O_\pi(G) \not\subseteq P_n$, we obtain $C^* \in \mathcal{C}(G)$ from $C$ by adding $NP_n$ at the end which is still a $\pi$-group. Otherwise let $N \subseteq P_k$ and $N \not\subseteq P_{k-1}$. If $P_{k-1}N = P_k$, then we delete $P_k$, otherwise we add $P_{k-1}N$ between $P_{k-1}$ and $P_k$. It is easy to see that in all cases $|C^*| = |C| \pm 1$, $(C^*)_* = C$ and $G_C = G_{C^*}$. Hence, the map $C \mapsto C^*$ is a bijection on $\mathcal{C}(G)$ such that

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|}|G_C|f(G_C) = \sum_{C \in \mathcal{C}(G)} (-1)^{|C^*|}|G_{C^*}|f(G_{C^*}) = -\sum_{C \in \mathcal{C}(G)} (-1)^{|C|}|G_C|f(G_C) = 0. \quad \Box$$

It is obvious that $G$ acts by conjugation on $\mathcal{C}(G)$. The set of $G$-orbits is denoted by $\mathcal{C}(G)/G$ in the following. If $K \lhd G$, notice that $G$ also acts on $\mathcal{C}(G/K)$.

**Lemma 2.2.** Let $G$ be a finite group with a normal $\pi'$-subgroup $K$. Let $\overline{H} := HK/K$ for $H \leq G$.

(a) The map $\mathcal{C}(G) \mapsto \mathcal{C}(\overline{G})$ given by

$$C : P_0 < \ldots < P_n \mapsto \overline{C} : P_0 < \ldots < P_n$$

induces a bijection $\mathcal{C}(G)/G \rightarrow \mathcal{C}(\overline{G})/\overline{G}$.

(b) For $\overline{C} \in \mathcal{C}(\overline{G})$, we have that $\overline{C_{\overline{C}}} = G_{\overline{C}}/K = G_{C/K}$.

(c) Let $f$ be a real-valued function on the set of subgroups of $G$ such that $f(H) = f(H^g)$ for all $H \leq G$ and $g \in G$. Then

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|}|G_C|f(G_CK) = \sum_{\overline{C} \in \mathcal{C}(\overline{G})} (-1)^{|\overline{C}|}|G_{\overline{C}}|f(G_{\overline{C}}).$$

**Proof.** First, we notice that the map $\mathcal{C}(G) \rightarrow \mathcal{C}(\overline{G})$ given by $C \mapsto \overline{C}$ is surjective. Indeed, suppose that $\overline{C} : 1 = U_0/K < \ldots < U_n/K$ is a chain of $\pi$-subgroups of $G/K$. By the Schur–Zassenhaus theorem, we have that $U_n = KP_n$ for some $\pi$-subgroup $P_n$. 
of $G$. Then $U_i = K(U_i \cap P_n)$, and therefore the chain $1 = U_0 \cap P_n < \ldots < P_n$ maps to $\overline{C}$.

If chains $C : P_0 < \ldots < P_n$ and $D : Q_0 < \ldots < Q_n$ are conjugate in $G$, then $\overline{C}$ and $\overline{D}$ are obviously conjugate in $\overline{G}$. Suppose conversely that $\overline{C}$ and $\overline{D}$ are $\overline{G}$-conjugate. Without loss of generality, we may assume that $P_i K = Q_i K$ for $i = 0, \ldots, n$. Again by the Schur–Zassenhaus theorem (this time relying on the Feit–Thompson theorem), $P_n$ is conjugate to $Q_n$ by some $x \in K$. We still have $P_i^x K = Q_i K$ for $i = 0, \ldots, n$. Since $P_i^x, Q_i \leq Q_n$ it follows that $P_i^x = Q_i$ for $i = 0, \ldots, n$. Hence, $C$ and $D$ are $G$-conjugate. This proves (a).

Suppose that $P_1 < P_2$ are $\pi$-subgroups of $G$. We claim that

$$\overline{N}(P_1) K \cap \overline{N}(P_2) K = (\overline{N}(P_1) \cap \overline{N}(P_2)) K.$$

If $x \in \overline{N}(P_1) K \cap \overline{N}(P_2) K$, then $P_2^x = P_2^k$ for some $k \in K$. Therefore $x k^{-1} \in \overline{N}(P_2) \cap \overline{N}(P_1) K$. Since $P_1 K \cap P_2 = P_1$, we have that $x k^{-1} \in \overline{N}(P_1)$, and therefore $x \in (\overline{N}(P_1) \cap \overline{N}(P_2)) K$. This proves the claim.

Suppose now that $C : P_0 < \ldots < P_n$ is a chain of $\pi$-subgroups of $G$. By the Frattini argument, $\overline{N}(P_1) = \overline{N}(P_1)$ and therefore $\overline{G} = \overline{G}$, using the last paragraph. Regarding the action of $G$ on $\overline{C}(G)$ we also have $G_{\overline{C}}/K = \overline{G}$. Finally, we prove (c). The $G$-orbit of $C$ has size $|G : G_C|$, while the $\overline{G}$-orbit of $\overline{C}$ has size $|\overline{G} : \overline{G}| = |G : G_{\overline{C}}|$. Let $C_1, \ldots, C_k$ be representatives for $\overline{C}(G)/G$, so that $\overline{C}_1, \ldots, \overline{C}_k$ are representatives for $\overline{C}(G)/\overline{G}$. Then

$$\sum_{C \in \overline{C}(G)} (-1)^{|C|} |G_C| f(G_C K) = |G| \sum_{i=1}^k (-1)^{|C_i|} f(G_{C_i} K) = \sum_{\overline{C} \in \overline{C}(G)} (-1)^{|\overline{C}|} |G_{\overline{C}}| f(G_{\overline{C}}). \quad \square$$

The deep part of our results comes from the “above Glauberman–Isaacs correspondence” theory. If $A$ is a solvable finite group, acting coprimely on $G$, recall that Glauberman discovered a natural bijection $^*$ from $\text{Irr}_A(G)$, the set of $A$-invariant irreducible characters of $G$, and $\text{Irr}(\text{C}_G(A))$, the irreducible characters of the fixed-point subgroup. The case where $A$ is a $p$-group is fundamental in the local/global counting conjectures. If $A$ is not solvable, an important case in this paper, then $G$ has odd order by the Feit-Thompson theorem. In this case, Isaacs [I] proved that there is also a natural bijection $\text{Irr}_A(G) \to \text{Irr}(\text{C}_G(A))$. T. R. Wolf [Wo] proved that both correspondences agree in the intersection of their hypotheses.

**Theorem 2.3.** Let $G$ be a finite group with a normal $\pi'$-subgroup $K$. Let $C \in \mathcal{C}(G)$ with last subgroup $P_C = P_{|C|}$. Let $\tau \in \text{Irr}(K)$ be $P_C$-invariant, and let $\tau^* \in \text{Irr}(\text{C}_K(P_C))$ be its Glauberman–Isaacs correspondent. Then

$$k_d(G_C K | \tau) = k_d(G_C | \tau^*)$$

for every integer $d$. 
Proof. Let $U = K(P_C G_C)$. Notice that $G_C \cap K = C_K(P_C)$. Also, $KP_C \vartriangleleft U$. Thus $U = K N_U(P_C)$, by the Frattini argument and the Schur–Zassenhaus theorem. Also, $N_U(P_C) = N_G(P_C) \cap (P_C G_C) K = (P_C G_C) N_K(P_C) = P_C G_C C_K(P_C)$.

Since $G_C$ normalizes $P_C$, we have that $G_C$ commutes with the $P_C$-Glauberman–Isaacs correspondence. In particular,

$$(G_C)_{\tau^*} = (G_C)_{\tau}.$$ 

Hence, by using the Clifford correspondence, we may assume that $\tau$ is $G_C$-invariant (and therefore $U$-invariant) and that $\tau^*$ is $G_C$-invariant too.

Now, we claim that the character triples $(U, K, \tau)$ and $(N_U(P_C), C_K(P_C), \tau^*)$ are isomorphic. If $P_C$ is solvable, this is a well-known fact which follows from the Dade–Puig theory. (A comprehensive proof is given in [108].) If $P_C$ is not solvable, then $|K|$ is odd, by the Feit–Thompson theorem. Then the claim follows from the theory developed by Isaacs in [1]. (A proof is given in the last paragraphs of [1].)

Since the character triples $(U, K, \tau)$ and $(N_U(P_C), C_K(P_C), \tau^*)$ are isomorphic, it follows from the definition that the sub-triples $(G_C K, K, \tau)$ and $(G_C, C_K(P_C), \tau^*)$ are isomorphic too. This yields a bijection $\text{Irr}(G_C K \mid \tau) \to \text{Irr}(G_C \mid \tau^*)$, $\chi \mapsto \chi^*$ such that $\chi(1)/\tau(1) = \chi^*(1)/\tau^*(1)$ (see [N, p. 87]). In particular, $k_d(G_C K \mid \tau) = k_d(G_C \mid \tau^*)$ (if $d_{\pi} \neq d$, both numbers are 0).

Theorem A is the special case $N = 1$ of the following projective version.

**Theorem 2.4.** Let $G$ be a $\pi$-separable group with a normal $\pi'$-subgroup $N$. Let $\theta \in \text{Irr}(N)$ be $G$-invariant and $d > 1$. Then

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|} |G_C| k_d(G_C N \mid \theta) = 0.$$ 

Proof. We may assume that $d_{\pi} = d$. We argue by induction on $|G : N|$. Let $G = G' \mid N$. By Lemma 2.2 we may sum over $C \in \mathcal{C}(G)$ by replacing $G_C$ with $G_{\overline{C}}$. Recall that a character triple isomorphism $(G, N, \theta) \to (G^*, N^*, \theta^*)$ induces an isomorphism $G \cong G^*/N^*$ and a bijection $\text{Irr}(G \mid \theta) \to \text{Irr}(G^* \mid \theta^*)$, $\chi \mapsto \chi^*$ such that $\chi(1)/\theta(1) = \chi^*(1)/\theta^*(1)$. Thus, $k_d(G_{\overline{C}} \mid \theta) = k_d(G_{\overline{C}} \mid \theta^*)$ and the numbers $|G_{\overline{C}}|, |G_{\overline{C}}^*|$ differ only by a factor independent of $C$. This allows us to replace $N$ by $N^*$. Using [N, Corollary 5.9], we assume that $N$ is a central $\pi'$-subgroup in the following. Now using Lemma 2.2 in the opposite direction, we sum over $C \in \mathcal{C}(G)$ again and note that $N \subseteq G_C$. Thus, it suffices to show that

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|} |G_C| k_d(G_C \mid \theta) = 0.$$ 

By Lemma 2.1 we may assume that $O_{\pi}(G) = 1$. Let $K = O_{\pi}(G)$. If $K = N$, then $N = G$ by the Hall–Higman 1.2.3 lemma. In this case, the theorem is correct because $d > 1$. So we may assume that $K > N$. Let $P_C$ be the last member of
$C \in \mathcal{C}(G)$. Observe that $G_C \cap K = C_K(P_C)$. Each $\psi \in \text{Irr}(G_C|\theta)$ lies over some $\mu \in \text{Irr}(C_K(P_C)|\theta)$. But $\psi$ lies also over $\mu^\theta$ for every $g \in G_C$. Therefore,

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|} |G_C| |k_d(G_C|\theta)| = \sum_{C \in \mathcal{C}(G)} (-1)^{|C|} |G_C| \left( \sum_{\mu \in \text{Irr}(C_K(P_C)|\theta)} \frac{k_d(G_C|\mu)}{|G_C : G_C|\mu} \right)$$

where $G_{C,\mu} = G_C \cap G_\mu$. According to Theorem 2.3, we replace $\text{Irr}(C_K(P_C)|\theta)$ by $\text{Irr}_{P_C}(K|\theta)$ and $k_d(G_C|\mu)$ by $k_d(G_C|\mu)$. By the Clifford correspondence, $k_d(G_C|\mu) = k_d(G_{C,\mu}|K|\mu)$. Moreover, $\mu \in \text{Irr}_{P_C}(K|\theta)$ implies $P_C \leq G_\mu$. Thus, for a fixed $\mu$ we only need to consider chains in $G_\mu$. Hence,

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|} |G_C| |k_d(G_C|\theta)| = \sum_{\mu \in \text{Irr}(K|\theta)} \left( \sum_{\mu \in \text{Irr}(G_\mu|\theta)} (-1)^{|C|} |G_{C,\mu}| k_d(G_{C,\mu}|\mu) \right).$$

Since $|G_\mu : K| < |G : N|$, the inner sum vanishes for every $\mu$ by induction. Hence, we are done. \hfill \Box

Finally, we come to our second result.

**Proof of Corollary B.** Let $C : P_0 < \ldots < P_n$ in $\mathcal{C}(G)$ such that $n > 0$. Then $P_1 \leq G_C$. Let $\chi \in \text{Irr}(G_C)$ and $\theta \in \text{Irr}(P_1)$ under $\chi$. By Clifford theory, $\chi(1)/\theta(1)$ divides $|G_C/P_1|$ (see [N] Theorem 5.12]). Since $\theta(1) < |P_1|$, it follows that $\chi(1)_\pi < |G_C|_\pi$ and $k_1(G_C) = 0$. Summing over $d \geq 1$ in Theorem A yields

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|} \frac{|G_C|}{|G|} k_d(G_C) = k_1(G).$$

The second equality follows from a straight-forward generalization of the Knörr–Robinson argument. In fact, the proofs of [N] 9.18–9.23] go through word by word (replacing $p$ by $\pi$, of course). \hfill \Box

Given the proof above, we take the opportunity to point out that a theorem of Webb [We] (see also [N] Corollary 9.20]) remains true in the $\pi$-setting:

**Theorem 2.5.** Let $G$ be an arbitrary finite group, and let $\pi$ be a set of primes. Then the generalized character

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|} |G_C| (1_G)^G$$

vanishes on all elements of $G$ whose order is divisible by a prime in $\pi$.

Unlike the case where $\pi = \{p\}$, Alperin’s weight conjecture cannot be deduced from Corollary B. As a matter of fact, for $\pi$-separable groups $G$, we can prove that there is no formula of the form

$$l(G) = \sum_P \alpha_P k_1(N_G(P)/P)$$
where $P$ runs through the $G$-conjugacy classes of $\pi$-subgroups and the coefficients $\alpha_P \geq 0$ depend only on the isomorphism type of $P$.

It is interesting to speculate on variations of Theorem A, that is projective versions of Dade’s conjecture, that might be even true for arbitrary normal subgroups of any finite group $G$, whenever $\pi = \{p\}$. Outside $\pi$-separable groups, we do not know what is the meaning, if any, of the number $\mu_\pi(G)$. In fact, this number can even be negative in groups with a Hall $\pi$-subgroup. We have not attempted a block version of Theorem A. Although $\pi$-block theory is well-developed in $\pi$-separable groups (see [Sl], for instance), Brauer’s block induction does not behave well if Hall $\pi$-subgroups are not nilpotent.

Computations with chains are almost impossible to do by hand. The results of this paper would not have been discovered without the help of [GAP].

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