Emergent Einstein Universe under Deconstruction

--- Self-Consistent Geometry Induced in Theory Space ---

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We study self-consistent static solutions for an Einstein universe in a graph-based induced gravity. The one-loop quantum action is computed at finite temperature. In particular, we demonstrate specific results for the models based on cycle graphs.

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\section{Introduction}

We do not know yet any consistent quantum theory of gravity. While the success of the combined model of three fundamental interactions described using gauge theories was achieved many years ago, only gravitational interaction is left unbound today. An eccentric idea that the gravitational interaction can be induced by quantum fluctuations of matter fields was introduced almost forty years ago. There are many versions of such ‘induced gravity’ (or ‘emergent gravity’).\textsuperscript{***} In most cases, explicit values for fundamental constants cannot be calculated because of the cutoff dependence, even if higher-loop contributions are neglected.

In our previous paper,\textsuperscript{2)} it is shown that calculable models of induced gravity can be constructed on the basis of knowledge of spectral graph theory.\textsuperscript{3)} The Newton constant and the cosmological constant can be calculated at one-loop level in flat-space limit. On the basis of this work, we consider self-consistent equations for a ‘graph theory space’ in the present paper. We demonstrate specific results for models based on cycle graphs, which represent the original moose diagram in dimensional deconstruction.\textsuperscript{4)}

The present paper is organized as follows. In §2, we will begin by reviewing background matters, i.e., the basic ideas behind the present work and techniques used in the following sections. Our previous work is briefly summarized in §3, for convenience. In §4, it is shown that the heat kernel method is used to evaluate the self-consistent solutions in specific models. The results for the models are shown in §5. We provide a summary with some future perspectives in the last section.

We use the metric signature \((- + ++\)) throughout this paper.

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§2. Background of our work

2.1. Induced gravity

Induced gravity or emergent gravity has been studied by many authors.\textsuperscript{1)} The idea of induced gravity is, “Gravity emerges from the quantum effect of matter fields”. In terms of the heat kernel method,\textsuperscript{5)} the one-loop effective action can schematically be expressed as

\begin{equation}
-\frac{1}{2} \int \frac{dt}{t} \sum_i \text{str} \exp \left[ -(\Box + M_i^2) t \right].
\end{equation}

Here, $\Box$ is the d’Alembertian and $M_i$ is the mass of the $i$-th field. They appear in the wave equation of the field.\textsuperscript{*}) The super-trace (str) in (2.1) is considered to be the trace including a sign dependent on the statistics of the field. In curved four-dimensional space-time, the super-trace part including the d’Alembertian becomes

\begin{equation}
\text{str} \exp \left[ -(\Box) t \right] = \sqrt{\left| \det g \right|} \frac{t^{-2}}{(4\pi)^{2}} (a_0 + a_1 t + \cdots),
\end{equation}

where $g$ is the determinant of the metric and the coefficients depend on the background fields. In four-dimensional space-time, the values of coefficients are $a_0 = 1$ and $a_1 = R/6$ for minimal scalar fields, $a_0 = -4$ and $a_1 = R/3$ for Dirac fields, $a_0 = 3$ and $a_1 = -R/2$ for massive vector fields, where $R$ is the scalar curvature of the space-time. The coefficient $a_1$ leads to the Einstein-Hilbert term, while $a_0$ yields the cosmological term.

In Kaluza-Klein (KK) theories, inducing the Einstein-Hilbert term was also studied.\textsuperscript{6)}

2.2. Dimensional deconstruction

The concept of dimensional deconstruction (DD)\textsuperscript{4)} is equivalent to considering a higher-dimensional theory with discretized extra dimensions at a low-energy scale.

For example, let us consider an $SU(m)^N$ gauge theory. The lagrangian density for vector fields is

\begin{equation}
\mathcal{L} = -\frac{1}{2e^2} \text{Tr} \sum_{k=1}^{N} F_{\mu\nu}^2 - \frac{2}{e^2} \text{Tr} \sum_{k=1}^{N} (D_{\mu} U_{k,k+1})^2,
\end{equation}

where $F_{k}^{\mu\nu} = \left[ \partial^{\mu} - i A_{k}^{\mu}, \partial^{\nu} - i A_{k}^{\nu} \right]$ is the field strength of $SU(m)_k$ and $\mu, \nu = 0, 1, 2, 3$, while $e$ is a common gauge coupling constant. We should read $A_{N+k}^{\mu} = A_{k}^{\mu}$, etc. $U_{k,k+1}$, called ‘a link field’, is transformed as

\begin{equation}
U_{k,k+1} \rightarrow L_k U_{k,k+1} L_{k+1}^\dagger, \quad (L_k \in SU(m)_k)
\end{equation}

under $SU(m)_k$. The covariant derivative is defined as

\begin{equation}
D_{\mu} U_{k,k+1} \equiv \partial_{\mu} U_{k,k+1} - i A_{k}^{\mu} U_{k,k+1} + i U_{k,k+1} A_{k+1}^{\mu}.
\end{equation}

\textsuperscript{*}) Precisely speaking, the “d’Alembertian” comes from the second functional derivative of the action with respect to the field. Thus the “d’Alembertian” for spinor and vector fields includes the contribution from curvature of space-time in addition to the differential operator.
A polygon with \( N \) edges, called a ‘moose’ diagram, is used to describe this theory, in other words, to characterize the transformation (2.4). This diagram consists of ‘sites’ and ‘links’, which are usually represented by open circles and single directed lines attached to these circles, respectively. A gauge transformation is assigned to each site and four-dimensional fields are assigned to sites and links, in general. The geometrical figure built up from sites and links (and also sometimes faces) is sometimes called ‘theory space’.4)

Turning back to our example, we assume that the absolute value of each link field \( |U_{k,k+1}| \) takes the same scale, \( f \). Then \( U_{k,k+1} \) is expressed as

\[
U_{k,k+1} = f \exp(i\chi_k/f).
\] (2.6)

We then find that the kinetic terms of \( U_{k,k+1} \) go over to a mass-matrix for the gauge fields. Since the gauge boson (mass)\(^2\) matrix for \( N = 5 \) turns out to be

\[
f^2 \begin{pmatrix}
2 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & -1 & 2
\end{pmatrix},
\] (2.7)

we obtain the gauge boson mass spectrum

\[
M_p^2 = 4f^2 \sin^2 \left( \frac{\pi p}{N} \right), \quad (p \in \mathbb{Z})
\] (2.8)

by diagonalizing (2.7).

For \( |p| \ll N \), the mass spectrum becomes

\[
M_p \simeq \frac{2\pi|p|f}{N}.
\] (2.9)

If we define

\[
b = \frac{1}{f}, \quad \ell = Nb,
\] (2.10)

then the masses (2.9) can be written as

\[
M_p \simeq \frac{2\pi|p|}{\ell}.
\] (2.11)

This is precisely the KK spectrum for a five-dimensional gauge boson compactified on a circle of circumference \( \ell \). In the limit of a large number of sites, \( N \to \infty \), DD leads to a five-dimensional theory, where the extra space is a circle.

2.3. Spectral graph theory

Sites can be, in general, more complicatedly connected by links. The theory space does not necessarily have a continuum limit and the diagram would have complicated connections or links. Such a diagram is called a graph. The \( N \)-sided polygon is identified as a simple graph, a cycle graph \( C_N \).
We identify the moose diagram of the theory space as a graph consisting of vertices and edges, which correspond to sites and links, respectively. Therefore, DD can be generalized to field theory on a graph.\(^7\)

A graph \(G\) consists of a vertex set and an edge set where an edge is a pair of distinct vertices of \(G\). The degree of a vertex \(v\), denoted by \(\text{deg}(v)\), is the number of edges incident with \(v\).

We will argue about the orientation of an edge. The graph with directed edges is dubbed as a directed graph. An oriented edge \(e = [u, v]\) connects the origin \(u = o(e)\) and the terminus \(v = t(e)\).

Spectral graph theory is the mathematical study of a graph by investigating various properties on eigenvalues, and eigenvectors of matrices associated with it.\(^3\)\(^,\)\(^7\)

Now we introduce various matrices that are naturally associated with a graph.\(^3\)\(^,\)\(^7\)

The incidence matrix \(E(G)\) is defined as

\[
(E)_{ve} = \begin{cases} 
1 & \text{if } v = o(e) \\
-1 & \text{if } v = t(e) \\
0 & \text{otherwise}
\end{cases}
\]  

(2.12)

The adjacency matrix \(A(G)\) is defined as

\[
(A)_{vv'} = \begin{cases} 
1 & \text{if } v \text{ is adjacent to } v' \\
0 & \text{otherwise}
\end{cases}
\]  

(2.13)

The degree matrix \(D(G)\) is defined as

\[
(D)_{vv'} = \begin{cases} 
\text{deg}(v) & \text{if } v = v' \\
0 & \text{otherwise}
\end{cases}
\]  

(2.14)

Note that \(\text{Tr } A = 0\) and \(\text{Tr } A^2 = \text{Tr } D\).

The graph Laplacian (or combinatorial Laplacian) \(\Delta(G)\) is defined as

\[
(\Delta)_{vv'} = (D - A)_{vv'} = \begin{cases} 
\text{deg}(v) & \text{if } v = v' \\
-1 & \text{if } v \text{ is adjacent to } v' \\
0 & \text{otherwise}
\end{cases}
\]  

(2.15)

The most important fact is

\[
\Delta = EE^T,
\]  

(2.16)

where \(E^T\) is the transposed matrix of \(E\). The Laplacian matrix is symmetric, so its eigenvalues are non-negative. Note also that \(\text{Tr } \Delta = \text{Tr } D\) and \(\text{Tr } \Delta^2 = \text{Tr } D^2 + \text{Tr } D\).

For a concrete example, we consider a cycle graph, which is equivalent to a popular moose diagram in DD, usually expressed as a polygon. The cycle graph with \(p\) vertices is denoted as \(C_p\). For \(C_5\), see Fig. 1.

The incidence matrix for \(C_5\) can be written, if the orientation of edges is ‘cyclic’,
as

\[ E(C_5) = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1
\end{pmatrix}. \]

(2.17)

The adjacency matrix for \( C_5 \) takes the form

\[ A(C_5) = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{pmatrix}. \]

(2.18)

The degree matrix for \( C_5 \) takes the form

\[ D(C_5) = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0
\end{pmatrix}. \]

(2.19)

The Laplacian matrix for \( C_5 \) is given by

\[ \Delta(C_5) = \begin{pmatrix}
2 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & -1 & 2
\end{pmatrix}. \]

(2.20)

Up to the dimensionful coefficient \( f^2 \), this matrix is identified with the gauge boson \((mass)^2\) matrix (2.7). The structure of the theory can be generalized to that based on other complicated graphs.

Indeed, any theory space can be associated with a graph and the \((mass)^2\) matrix for a field on a graph can be expressed using the graph Laplacian through the Green’s theorem for a graph

\[ (d\phi_1, d\phi_2) = (\phi_1, \Delta\phi_2), \]

(2.21)

where \( \phi \) is a function that assigns a real value \( \phi(v) \) to each vertex \( v \) of \( G \) and \( (\phi_1, \phi_2) \) denotes the inner product \( \sum_v \phi_1(v)\phi_2(v) \). The difference defined on each edge \( e \) is

\[ (d\phi)(e) \equiv \phi(t(e)) - \phi(o(e)), \]

(2.22)
where \( t(e) \) is the terminus of \( e \) and \( o(e) \) is the origin of \( e \). \((d\phi_1, d\phi_2)\) is defined as \( \sum_e d\phi_1(e)d\phi_2(e) \). For example, a mass term of scalar fields can be constructed as \( f^2(d\phi, d\phi) = f^2(\phi, \Delta\phi) \).

The mass term for fermion fields can be written using the incidence matrix \( E \). For example, the lagrangian density of fermion fields can be written as

\[
- \langle \bar{\psi}_R, D\bar{\psi}_R \rangle - \langle \bar{\psi}_L, D\bar{\psi}_L \rangle - f[\langle \bar{\psi}_L, E^T\psi_R \rangle + \text{h.c.}],
\]

(2.23)

where the subscripts \( L \) and \( R \) denote left-handed and right-handed fermions, respectively. Namely, the left-handed fermions are assigned to the edges while the right-handed ones are assigned to the vertices. The \((\text{mass})^2\) matrix for \( \psi_R \) is expressed as \( f^2E E^T = f^2\Delta \) while that for \( \psi_L \) is \( f^2E^T E \equiv f^2\tilde{\Delta} \). The matrices \( \Delta \) and \( \tilde{\Delta} \) have the same spectrum up to zero modes. Thus the mass spectrum of fermions governed by the lagrangian (2.23) is also given by the eigenvalues of the graph Laplacian (2.16). For details, see Ref. 7.

§3. Our previous work

We have constructed models of one-loop finite induced gravity using several graphs in Ref. 2). With knowledge in spectral graph theory, we can easily find that the UV divergent terms are concerned with the graph Laplacian in DD or the theory on a graph.

We have observed important relations: The quadratic divergence is proportional to the trace of \(((\text{mass})^2\text{matrix})\) and the logarithmic divergence is proportional to the trace of \(((\text{mass})^2\text{matrix})^2\) matrix. For the cancellation of the quartic divergence, we choose the particle content so that the bosonic and fermionic degrees of freedom should be the same. This is the same consequence required from supersymmetry.

Therefore, the UV divergences can be controlled using the graph Laplacian and we can construct the models of one-loop finite induced gravity from the graph. In the model of Ref. 2), the one-loop finite Newton’s constant is induced and the positive-definite cosmological constant can also be obtained.

In flat-space limit, the one-loop vacuum energy has been calculated for field theory associated with \( CN^2 \). In the theory, the \((\text{mass})^2\) matrix is expressed using the graph Laplacian of \( CN \). The finite parts from various spin fields are:

For a scalar field, \( V_0 = -\frac{3}{4\pi^2} \left( \frac{f}{N} \right)^4 Z(5, N) \),

(3.1)

For a vector field, \( V_0 = -\frac{9}{4\pi^2} \left( \frac{f}{N} \right)^4 Z(5, N) \),

(3.2)

For a Dirac field, \( V_0 = +\frac{3}{\pi^2} \left( \frac{f}{N} \right)^4 Z(5, N) \),

(3.3)

where ‘a’ field denotes a lagrangian for a set of four-dimensional fields assigned on a
The function $Z(5, N)$ is defined as

$$Z(5, N) \equiv \sum_{q=1}^{\infty} \frac{1}{q (q^2 - \frac{1}{N^2}) (q^2 - \frac{4}{N^2})}. \quad (3.4)$$

The inverse of the Newton constant $G$ has also been computed for theory associated with $C_N$. The finite parts from various spin fields are:

For a scalar field, $(16\pi G)^{-1} = \frac{1}{48\pi^2} \left( \frac{f}{N} \right)^2 Z(3, N), \quad (3.5)$

For a vector field, $(16\pi G)^{-1} = -\frac{1}{16\pi^2} \left( \frac{f}{N} \right)^2 Z(3, N), \quad (3.6)$

For a Dirac field, $(16\pi G)^{-1} = \frac{1}{24\pi^2} \left( \frac{f}{N} \right)^2 Z(3, N), \quad (3.7)$

where $Z(3, N)$ is defined as

$$Z(3, N) \equiv \sum_{q=1}^{\infty} \frac{1}{q (q^2 - \frac{1}{N^2})}. \quad (3.8)$$

Note that $Z(5, \infty) = \zeta_R(5)$ and $Z(3, \infty) = \zeta_R(3)$, where $\zeta_R(z)$ is Riemann’s zeta function.

In the next section, we study self-consistent cosmological solutions for an Einstein universe in the graph-based induced gravity model.

§4. Self-consistent Einstein universe

The static homogeneous, closed space is often called an Einstein universe. The solution for the static hot universe of positively curved space can be deduced from Einstein equations with a cosmological constant and homogeneous matter in thermal equilibrium at finite temperature.

The back reaction problem was studied for various spin fields in such a static curved space at finite temperature. Einstein equations were solved self-consistently in Refs. 8) and 9). Self-consistent solutions were also investigated in higher-dimensional theory with compactification.10) The present work may be similar to Ref. 10), because we should consider the excited states, which come from selected fields.

The metric of the spherically symmetric, static homogeneous universe is given by

$$ds^2 = -dt^2 + a^2[d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2)], \quad (4.1)$$

where $a$ is the radius of the spatial part of the universe $S^3$ and $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. The scalar curvature of this space-time is given by $R = 6/a^2$.

In the static space-time, it is known that the effective action can be interpreted as the total free energy of the quantum fields.10) The one-loop part of the effective potential considered here is given by the free energy in the canonical ensemble.
To evaluate the effective action at the one-loop level, we use the heat kernel method. The free energy of a system of bosonic fields can be computed as the integration over the parameter $t$,

$$F_b = -\frac{1}{2\beta} \int_{1/\Lambda^2}^{\infty} \frac{dt}{t} \sum_{n=-\infty}^{\infty} \exp \left[ -\left( \frac{2\pi n}{\beta} \right)^2 t \right] \{ \text{tr exp} \left[ -(-\nabla^2) t \right] \} \sum_i \exp \left[ -M_i^2 t \right], \quad (4.2)$$

while the free energy of a system of fermionic fields can be computed as

$$F_f = \frac{1}{2\beta} \int_{1/\Lambda^2}^{\infty} \frac{dt}{t} \sum_{n=-\infty}^{\infty} \exp \left[ -\left( \frac{2\pi}{\beta} \right)^2 \left( n + \frac{1}{2} \right)^2 t \right] \{ \text{tr exp} \left[ -(-\nabla^2) t \right] \} \sum_i \exp \left[ -M_i^2 t \right]. \quad (4.3)$$

Here, $\beta$ is the inverse of the temperature of the system, that is, $\beta = 1/T$. $\Lambda$ is the mass scale of the UV cutoff. $M_p$ is the mass and $\nabla^2$ is the Laplacian on the $S^3$ (with radius $a$) of the correspondent field.*

The trace part $\{ \text{tr exp} \left[ -(-\nabla^2) t \right] \}$ can be evaluated using the eigenvalues of the Laplacian. Thus, we need the eigenvalues of the Laplacian ($\nabla^2$) on $S^3$ with radius $a$ for various spin fields.8)

For a minimally coupled scalar, the eigenvalues of $-\nabla^2$ are

$$\frac{\ell(\ell + 2)}{a^2}, \quad (4.4)$$

and the degeneracy of each eigenvalue is $(\ell + 1)^2$, where $\ell = 0, 1, 2, 3, \ldots$.

For a vector field, two parts of spectra are known. One is the transverse mode. Its eigenvalues are

$$\frac{\ell^2}{a^2}, \quad (4.5)$$

with the degeneracy $2(\ell^2 - 1)$, where $\ell = 2, 3, \ldots$. Another is the longitudinal mode whose eigenvalues are

$$\frac{\ell(\ell + 2)}{a^2}, \quad (4.6)$$

with the degeneracy $(\ell + 1)^2$, where $\ell = 1, 2, 3, \ldots$. Note that this spectrum is the same as that of a minimally coupled scalar, except for its zero mode.

The Laplacian eigenvalues for a Dirac fermion field, where the Laplacian is interpreted as the square of the Dirac operator, are

$$\frac{(\ell + 1/2)^2}{a^2}, \quad (4.7)$$

and the degeneracy is $4\ell(\ell + 1)$, where $\ell = 1, 2, 3, \ldots$.

The trace part $\{ \text{tr exp} \left[ -(-\nabla^2) t \right] \}$ for each field can be evaluated as follows and has an asymptotic form for small $t$:***

---

* Precisely speaking, the “Laplacian” is defined so that the absolute value of its eigenvalue is the square of the eigenfrequency of the corresponding field.

*** The useful summation formulas are found in Appendix.
that, for a small $t$, where $f$ behaves as $\sim 1/\Lambda^2$ and $\sim 1/\Lambda^2$ for a transverse vector, (4-9)

$$\sum_{\ell=1}^{\infty} (\ell+2)^2 \exp \left[ -\frac{(\ell+2)^2}{a^2} t \right] = 4 \frac{2\pi^2 a^3}{(4\pi t)^{3/2}} \left( 1 - \frac{1}{2a^2} t + \cdots \right), \quad \text{for a Dirac spinor.} \quad (4-10)$$

Now one finds that the trace part $\{ \text{tr} \exp \left[ -(-\nabla^2) t \right] \}$ for each field behaves as $\sim 1/t^{3/2}$ for $t \to 0$, while the part expressed as the sum over $n$ in (4-2) or (4-3) behaves as $\sim 1/t^{1/2}$ for $t \to 0$. Therefore, in generic cases, we recognized that the free energy diverges as $\Lambda^4$ for a large cutoff $\Lambda$.

If the mass spectrum of a specific field is determined using a graph, as briefly explained in §2 and in Refs. 2) and 7), the last piece in (4-2) or (4-3) becomes

$$\sum_{p} \exp \left[ -M_p^2 t \right] = \text{Tr} \exp \left[ -f^2 \Delta(G) t \right], \quad (4-12)$$

where $f$ is a mass scale and $\Delta(G)$ is the graph Laplacian for the graph $G$. We found that, for a small $t$, the coefficients of $t^0$, $t^1$, and $t^2$ of (4-12) depend on the number of vertices and edges and the degrees of vertices. Thus without enormous effort, we can choose a particle content and graphs for the mass spectrum of each field to cancel the $\Lambda^4$ and $\Lambda^2$ terms in the total free energy of bosonic and fermionic fields.

When we considered fields only minimally coupled to gravity, we found a unique set of field contents for a finite free energy: one minimally coupled neutral scalar field, one vector field, and one Dirac field. Here, ‘one’ field denotes that there is a theory of the four-dimensional fields on a graph. Note that this ratio is the same as the content of five-dimensional supersymmetric theory including a vector multiplet. This particle (field) content is a condition of necessity for finite theory. We select a graph whose Laplacian gives the mass spectrum for each field to cancel the cutoff-dependent parts in total. We adopt a graph $G_S$ for a scalar field, $G_V$ for a vector, and $G_D$ for a Dirac field, respectively, as a graph on which each field theory is based.

We find a sufficient condition for the cancellation of UV divergences on graphs:

$$\text{Tr} D(G_S) = \text{Tr} D(G_V) = \text{Tr} D(G_D) \quad \text{and} \quad \text{Tr} D^2(G_S) = \text{Tr} D^2(G_V) = \text{Tr} D^2(G_D), \quad (4-13)$$

for the above-mentioned field content.2)}
constant contribution (i.e., independent of \( \beta \) and \( a \)) to \( \beta F_b \) is small-mass-scale cutoff-dependent.

It is known that the fastest way to obtain self-consistent equations is by using the total free energy \( F \). The self-consistent equations can be derived as

\[
\frac{\partial (\beta F)}{\partial \beta} = 0 \quad \text{and} \quad \frac{\partial (\beta F)}{\partial a} = 0 ,
\]

where the first equation corresponds to the 00-component of the Einstein equation with one-loop corrections and the second corresponds to the diagonal component in a spatial direction. Thus the extremal point of \( \beta F(a, \beta) \) provides a solution to the self-consistent equation and the solution is independent of the small-mass-scale cutoff mentioned above.

Now we will consider specific models and will show the self-consistent solutions. We show two models. In our models, four-dimensional fields are defined on graphs consisting of one or several cycle graphs. Here, \( C_N \) denotes a cycle graph with \( N \) vertices, equivalent to an \( N \)-sided polygon. Thus, each degree of vertex is two. Since we do not show each lagrangian explicitly, please consult our papers\(^2\), \(^7\) for details. We mention here only that the mass spectra are given by \( f^2 \Delta(G_S) \), \( f^2 \Delta(G_V) \) and \( f^2 \Delta(G_D) \) as the \((mass)^2\) matrices for minimal scalar fields, vector fields and Dirac fields, respectively.

Model 1 is now described as follows. We consider that scalar fields are assigned on \( G_S = 8C_{N/2} \), vector fields on \( G_V = 4C_N \) and Dirac fermions on \( G_D = 2C_{N/2} + 3C_N \). Each graph has an equal number of vertices and edges, \( 4N \). We find also \( \text{Tr} D(G_S) = \text{Tr} D(G_V) = \text{Tr} D(G_D) = 8N \) and \( \text{Tr} D^2(G_S) = \text{Tr} D^2(G_V) = \text{Tr} D^2(G_D) = 16N \).

On the other hand, model 2 is described as follows. We consider that scalar fields are assigned on \( 16C_{N/4} + 2C_{N/2} \), vector fields on \( 5C_N \) and Dirac fermions on \( 4C_{N/4} + 3C_N + 2C_{N/2} \). Each graph has an equal number of vertices and edges, \( 5N \). We find also \( \text{Tr} D(G_S) = \text{Tr} D(G_V) = \text{Tr} D(G_D) = 10N \) and \( \text{Tr} D^2(G_S) = \text{Tr} D^2(G_V) = \text{Tr} D^2(G_D) = 20N \).

We emphasize again that in each model, Newton’s constant and the cosmological constant are calculable using the criterion of Ref. 2) and are not given by hand as in Refs. 8) and 9).

For the field assigned on \( C_N \), the trace of the kernel part of the mass spectrum is given by

\[
\sum_p \exp(-M_p^2t) = Ne^{-2f^2t} \sum_{q=-\infty}^{\infty} I_{qN}(2f^2t) , \quad (4.15)
\]

where \( f \) is a mass scale in the model.

For \( N \gg 1 \) and \( 2f^2t \gg 1 \), we find

\[
\sum_p \exp(-M_p^2t) \sim \frac{N}{f} \frac{1}{\sqrt{4\pi t}} \sum_{q=-\infty}^{\infty} \exp \left( -\frac{N^2q^2}{4f^2t} \right) , \quad (4.16)
\]
This asymptotic formula holds as much like in the case of the KK spectrum on $S^1$. Because the cancellation of divergence denotes that the region of small $t$ is ineffective, the calculation of the free energy can be approximated as the one in KK theories. The results given in the next section are on models 1 and 2 in the limit of $N \to \infty$.

Thus, the present calculation is much akin to the one in the case of KK theories.

We selected the content of each model so that the induced Newton constant becomes positive. In model 1, the induced vacuum energy vanishes in flat-space limit, while in model 2, that is positive in the limit. Referring to the result of Ref. 2), noted in §3, one can evaluate the one-loop vacuum energy $V_0$ and the inverse of the Newton constant $G$ in flat-space limit and for large $N$ as follows:

\[
\text{Model 1: } V_0 = 0, \quad (16\pi G)^{-1} = \frac{7}{8\pi^2} \left( \frac{f}{N} \right)^2 \zeta_R(3),
\]

\[
\text{Model 2: } V_0 = \frac{279}{4\pi^2} \left( \frac{f}{N} \right)^4 \zeta_R(5), \quad (16\pi G)^{-1} = \frac{133}{16\pi^2} \left( \frac{f}{N} \right)^2 \zeta_R(3).
\]

The feature that $V_0$ and $G$ are merely non-negative in our toy models is essential and we will not go into pursuing models with realistic values for parameters here.

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\text{Fig. 2.} A contour plot of $\beta F$ in the first model, in which scalar assigned on $8C_{N/2}$, vectors on $4C_N$, and Dirac fermions on $2C_{N/2} + 3C_N$. A solution of the self-consistent equation can be found at the maximum point.

* It is safe to say that the ‘approximation’ of the large $N$ limit is valid even for $N \approx 10$.\(^{12)}\)
§5. Results

We show the contour plots for $\beta F$ obtained by numerical calculations, whose extrema provide self-consistent solutions. The horizontal axis indicates the scale factor $a$, while the vertical one indicates the inverse of temperature $\beta = 1/T$. The scale of each axis is in the unit of $N/f$.

There are two regimes introduced in Ref. 8). For large $Ta$, we have the Planck regime, and for small $Ta$, we have the Casimir regime. They are characterized by the leading contributions $F_l$ in the total free energy, $F_l \propto -a^3T^4$ in the Planck regime and $F_l \propto -a^3 \times 1/a^4$ in the Casimir regime.

At high temperature, the energy of the black-body radiation in the Planck distribution overcomes the Casimir effect in the one-loop contribution from quantum fluctuations. At low temperature, finite-temperature piece of quantum correction is negligible, and the Casimir density dependent on the scale factor $a$, as well as zero-temperature-induced gravity, is left.

We show $\beta F$ for the first model in Fig. 2 and for the second in Fig. 3, for large $N$.

In the first model, the induced cosmological constant, which is independent of $a$ or $\beta$, vanishes and the solution can be found at the maximum of $\beta F$, corresponding to the Casimir regime.\(^8\)

In the second model, solutions in the Casimir regime and Planck regime are both found.

In both models, no solution corresponding to the minimum of $\beta F$ can be found. Because the instability of a hot closed universe is well known, this result can be
expected. The result tells us that the one-loop quantum effect cannot stabilize the solution.

§6. Summary and outlook

We have studied self-consistent Einstein universe in induced gravity models constructed on the basis of ‘graph theory space’. The solution can be systematically obtained using the knowledge of the graph structure. We find that various patterns of regimes can been seen depending on the model, i.e., the choice of graphs.

As future works, we should discuss the possibility of obtaining a small cosmological constant and a large Planck scale in a model in which scalar fields are assigned on $4G_{(1)}$, vector fields are on $4G_{(2)}$ and Dirac fermions are on $G_{(1)} + 3G_{(2)}$, where the number of vertices in $G_{(1)}$ and $G_{(2)}$ is equal. We should also investigate the model with the time-dependent scale factor, $a(t)$.

In the present analysis, we have constructed models using cycle graphs, but we are also interested in the model of general graphs. For a $k$-regular graph, trace formula\footnote{13} is useful if we have a single mass scale. Field theory on weighted graphs, corresponding to warped spaces in the continuous limit or not, is worth studying.

Concerning the properties of matter fields at finite temperature, how the possible condensation or degenerate matters in a closed space\footnote{14} affects the mechanism of induced gravity is also an interesting subject.

There are other possible exotic ideas. A quasi-continuous mass spectrum is conceivable, and dynamics of graphs such as Hosotani mechanism\footnote{15} are also reasonable to consider. A discrete ‘flux’-like object may be associated with a closed circuit in a graph.

Finally, we expect that the knowledge of spectral graph theory will help produce useful results on deconstructed theories in all directions and open up another possibility of gravity models.

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Appendix A

Formulas on Summations

\begin{align}
\sum_{n=-\infty}^{\infty} \exp \left[ - \left( \frac{2\pi}{\beta} \right)^2 n^2 t \right] &= \frac{\beta}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \exp \left( -\frac{\beta^2 n^2}{4t} \right). \\
\sum_{n=-\infty}^{\infty} \exp \left[ - \left( \frac{2\pi}{\beta} \right)^2 \left( n + \frac{1}{2} \right)^2 t \right] &= \frac{\beta}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} (-1)^n \exp \left( -\frac{\beta^2 n^2}{4t} \right).
\end{align}
\begin{align}
\sum_{\ell=0}^{\infty} (\ell + 1)^2 \exp \left[ -\frac{\ell(\ell + 2)}{a^2} t \right] &= \frac{2\pi^2 a^3}{(4\pi t)^{3/2}} \sum_{\ell=-\infty}^{\infty} \left( 1 - \frac{2\pi^2 a^2 \ell^2}{t} \right) e^{-\frac{\pi^2 \ell^2}{t}}. \quad \text{(A.3)}
\end{align}

\begin{align}
\sum_{\ell=2}^{\infty} 2(\ell^2-1) \exp \left[ -\frac{\ell^2}{a^2} t \right] &= 2 \frac{2\pi^2 a^3}{(4\pi t)^{3/2}} \sum_{\ell=-\infty}^{\infty} \left( 1 - \frac{2t}{a^2} - \frac{2\pi^2 a^2 \ell^2}{t} \right) e^{-\frac{\pi^2 \ell^2}{t}} + 1. \quad \text{(A.4)}
\end{align}

\begin{align}
\sum_{\ell=1}^{\infty} 4\ell (\ell + 1) \exp \left[ -\frac{(\ell + 1/2)^2}{a^2} t \right] &\quad \\
&= 4 \frac{2\pi^2 a^3}{(4\pi t)^{3/2}} \sum_{\ell=-\infty}^{\infty} (-1)^\ell \left( 1 - \frac{t}{2a^2} - \frac{2\pi^2 a^2 \ell^2}{t} \right) e^{-\frac{\pi^2 \ell^2}{t}}. \quad \text{(A.5)}
\end{align}

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