Antiferromagnetic Operators in N=4 Supersymmetric Yang-Mills Theory

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Abstract

The spectrum of operators in the $su(2)$ sector of $\mathcal{N} = 4$ SYM is bounded because the number of operators is finite. According to the AdS/CFT correspondence, the string spectrum in this sector should be also bounded. In this paper the upper bound on the scaling dimension is calculated in the limit of the large R-charge using Bethe ansatz.
The AdS/CFT correspondence [1] predicts that the spectrum of local gauge-invariant operators in $\mathcal{N} = 4$ SYM theory and the string spectrum in $AdS_5 \times S^5$ are identical [2, 3]. One puzzling moment in this identification is that the set of operators superficially looks more discrete than the set of string states. This point is best illustrated by considering a particular set of operators

$$\mathcal{O} = \text{tr}(Z^{J_1} W^{J_2} + \text{permutations}),$$

where $Z$ and $W$ are two complex scalar fields from $\mathcal{N} = 4$ supermultiplet. This set of operators is closed under renormalization [4], but mixing of the operators among themselves is non-trivial and is best described by mapping to a quantum spin chain of length $L = J_1 + J_2$ [5]. Each occurrence of $Z$ in an operator represents spin up and an occurrence of $W$ represents spin down. Cyclically symmetric distributions of spins on a one-dimensional lattice of length $L$ are then in the one-to-one correspondence with all possible orderings of the fields $Z$ and $W$ under the trace. The planar dilatation operator, whose eigenvalues are large-$N$ scaling dimensions of operators (1), can be identified with the spin-chain Hamiltonian [5, 6, 7, 8]. The ferromagnetic ground state of the spin chain corresponds to the chiral primary operator $\text{tr}Z^L$ with zero anomalous dimension. The excited states (magnons), described by a collection of spin flips $Z \rightarrow W$ that propagate along the lattice with momentum $p = 2\pi n/L$, correspond to operators with parametrically small anomalous dimensions (BMN operators [9]). The contribution of a single magnon to the anomalous dimension is determined by the dispersion relation $\varepsilon = (\lambda/2\pi^2)\sin^2(p/2) + O(\lambda^2)$, where $\lambda = g_Y^2 N$ is the 't Hooft coupling of the SYM.

It is possible to identify the dual of the $su(2)$ sector (1) in the classical string theory [10, 11, 12, 13], although not without subtleties [14]. The magnons correspond to transverse fluctuations of the point-like string orbiting around a big circle of $S^5$ [15]. The dispersion relation for the string modes is $\varepsilon = \sqrt{1 + \lambda p^2/4\pi^2} - 1$. At weak coupling and at small momenta $p \ll 1$ the spin-chain and the string dispersion relations match. If we believe in the AdS/CFT correspondence, quantum corrections on the string side and higher-loop effects in the SYM should eliminate the difference completely and should produce a common exact dispersion relation, e.g. [8, 16, 17]

$$\varepsilon = \sqrt{1 + \frac{\lambda}{\pi^2} \frac{p^2}{2} } - 1,$$  

(2)
which reduces to the magnon energy at $\lambda \ll 1$ and to the energy of a classical string mode at $p \ll 1$. There is a small subtlety here, however. The momentum of a magnon is confined to a single Brillouin zone: $0 \leq p < 2\pi$ which together with the momentum quantization makes the total number of states finite. After all, there is a finite number of operators \( 1 \) with $L$ fixed\(^1\). On the contrary, the momentum of the classical string oscillations is unbounded. Of course, one can trust the semiclassical approximation only for sufficiently low world-sheet momenta\(^2\) $p \ll 1$. We currently do not know what happens when the string is quantized. The quantization can somehow hide states with the large momentum.

Since the number of operators is finite, their scaling dimensions are bounded above and can vary in a finite interval

$$L \leq \Delta \leq \Delta_{\text{max}}.$$  

(3)

As long as the $su(2)$ sector can be identified in the quantum string theory, the total number of states in it must be also finite and energies of the string states must satisfy the same bound.

The aim of this note is to calculate $\Delta_{\text{max}}$ in the thermodynamic limit of $L \to \infty$. The state with the largest possible energy is the antiferromagnetic (AF) vacuum of the spin chain. The numbers of up and down spins in the AF vacuum are equal: $J_1 = J_2 = L/2$. One motivation to study this state is that classical string solutions with the same quantum numbers have been recently constructed [23, 24]. Another motivation comes from the study of the spin chain that describes one-loop anomalous dimensions in large-$N$ QCD [25, 26, 27, 28, 29, 30]. The true ground state there is AF [28, 30]. There is much less control over higher-loop corrections in QCD, not to say over the dual string theory. It would be thus interesting to study the AF state in the SYM setting where loop corrections and the string dual are much better understood.

The scaling dimensions of the operators \( 1 \) are eigenvalues of the dilata-

\(^1\)An asymptotic upper bound on the number of operators is $2^L$. One can make a better estimate with the help of the Polya theory [18, 19].

\(^2\)All explicit calculations on the string side have so far been done only for such low-momentum states and in fact have been insensitive to the difference between $p^2/4$ and $\sin^2(p/2)$ in the dispersion relation [20] (field-theory calculations, however, can be pushed beyond the leading order in $p^2$ [21, 22]). I am grateful to S. Frolov for this remark.
tion operator

\[ D = \sum_{l=1}^{L} \left[ 1 + \frac{\lambda}{16\pi^2} (1 - \sigma_l \cdot \sigma_{l+1}) - \left( \frac{\lambda}{16\pi^2} \right)^2 (3 - 4\sigma_l \cdot \sigma_{l+1} + \sigma_l \cdot \sigma_{l+2}) \right. \]

\[ + \left. \left( \frac{\lambda}{16\pi^2} \right)^3 (20 - 29\sigma_l \cdot \sigma_{l+1} + 10\sigma_l \cdot \sigma_{l+2} - \sigma_l \cdot \sigma_{l+3} \right. \]

\[ - \sigma_l \cdot \sigma_{l+2} \sigma_{l+1} \cdot \sigma_{l+3} + \sigma_l \cdot \sigma_{l+3} \sigma_{l+1} \cdot \sigma_{l+2} + O(\lambda^4) \]  

At one loop this is the Hamiltonian of the Heisenberg spin chain [5], which is an integrable model. It is truly remarkable that the condition of integrability and the requirement of the BMN scaling imposed on the dispersion relation uniquely fix the dilatation operator up to \( O(\lambda L) \) [6, 8]. An alternative algebraic derivation of [31, 32, 17] and explicit three-loop calculations of anomalous dimensions [33, 34] confirm the validity of these assumptions.

Integrability of the dilatation operator allows one to calculate its spectrum with the help of the Bethe ansatz. By extending the Bethe-ansatz solution of the Heisenberg model [37, 38] to higher orders of perturbation theory, Beisert, Dippel and Staudacher proposed the following all-loop Bethe equations [8]  

\[ \left( \frac{x(u_j + i/2)}{x(u_j - i/2)} \right)^L = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}, \]  

where \( u_j (j = 1, \ldots, J_2) \) are rapidities of elementary excitations and

\[ x(u) = \frac{u}{2} + \frac{1}{2} \sqrt{u^2 - \frac{\lambda}{4\pi^2}}. \]

The scaling dimension is given by

\[ \Delta = L + \frac{i\lambda}{8\pi^2} \sum_{j=1}^{J_2} \left( \frac{1}{x(u_j + \frac{i}{2})} - \frac{1}{x(u_j - \frac{i}{2})} \right). \]  

\(^3\)Strictly speaking, the BMN scaling has been tested only up to three loops. In the plane-wave matrix theory, which is described by a very similar Hamiltonian, the BMN scaling is violated at \( O(\lambda^4) \) [35]. However, the plane-wave matrix theory is not exactly integrable [36].

\(^4\)These equations can be systematically derived in perturbation theory by applying coordinate Bethe ansatz to [41, 39].

\(^5\)This is (2) in the rapidity parametrization: \( e^{ip} = x(u + i/2)/x(u - i/2) \).
The trace cyclicity of the SYM operators requires the wave function to be periodic. This imposes an extra condition

\[ \prod_{j=1}^{J_2} \frac{x(u_j + \frac{i}{2})}{x(u_j - \frac{i}{2})} = 1, \]  

which makes the total momentum an integer multiple of $2\pi$.

These equations are asymptotic in a certain sense [8] and compute the eigenvalues of (4) up to the $O(\lambda^L)$ accuracy. The wrapping interactions [8, 40], which start to contribute at this order of perturbation theory, may invalidate or modify the asymptotic Bethe ansatz. These corrections, however, are exponentially small in the thermodynamic limit of $L \to \infty$, at least if the 't Hooft coupling is not very large.

The energy density in the anti-ferromagnetic vacuum can be calculated by standard techniques [41, 38]. Taking logarithm of both sides of (5) we get

\[ Lp(u_j) = 2\pi k_j + \sum_{k \neq j} \Phi(u_j - u_k), \]  

where

\[ p(u) = \frac{1}{i} \ln \frac{u + \frac{i}{2} + \sqrt{(u + \frac{i}{2})^2 - \frac{\lambda}{4\pi^2}}}{u - \frac{i}{2} + \sqrt{(u - \frac{i}{2})^2 - \frac{\lambda}{4\pi^2}}} \]  

is the momentum of an elementary excitation and

\[ \Phi(u) = \pi - 2 \arctan u \]  

is the phaseshift due to pairwise scattering. It is assumed that the same branch of the logarithm is used for all momenta in (10). Let us fix the conventions by requiring that $p(u)$ and $\Phi(u)$ change from $2\pi$ at $u \to -\infty$ to 0 at $u \to +\infty$. The arbitrariness in choosing the branch of the logarithm is then entirely encoded in the mode numbers $k_j$.

There is one excitation per mode number in the AF state. It then follows from (9) that $k_j$ can take $L/2$ values from 1 to $L - M = L/2$. Thus all available levels are filled and, assuming that $u_j$ monotonously decreases with $j$, we can set $k_j = j$. After introducing the scaling variable $\xi = j/L$ and taking the thermodynamic limit $L \to \infty$, the Bethe equation (9) can be written as

\[ p(u(\xi)) = 2\pi\xi + \int d\eta \Phi(u(\xi) - u(\eta)). \]
Differentiating in $u$ we get

\[
\frac{i}{2} \left[ \frac{1}{\sqrt{(u + \frac{i}{2})^2 - \frac{\lambda}{4\pi^2}}} - \frac{1}{\sqrt{(u - \frac{i}{2})^2 - \frac{\lambda}{4\pi^2}}} \right] = \pi \rho(u) + \int_{-\infty}^{+\infty} \frac{dv \rho(v)}{(u - v)^2 + 1},
\]

where

\[\rho(u) = -\frac{d\xi}{du}\]

is the density of Bethe roots.

The integral equation (13) can be solved by the Fourier transform:

\[
\rho(u) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{iku} \frac{J_0\left(\frac{\sqrt{\lambda} k}{2\pi}\right)}{2 \cosh \frac{k}{2}},
\]

where $J_0$ is the Bessel function. Plugging this solution into the equation for the energy:

\[
\frac{\Delta_{\text{max}}}{L} = 1 + \frac{i\lambda}{8\pi^2} \int_{-\infty}^{+\infty} du \rho(u) \left( \frac{1}{x(u + \frac{i}{2})} - \frac{1}{x(u - \frac{i}{2})} \right),
\]

we find:

\[
\frac{\Delta_{\text{max}}}{L} = 1 + \frac{\sqrt{\lambda}}{\pi} \int_{0}^{\infty} \frac{dk}{k} J_0\left(\frac{\sqrt{\lambda} k}{2\pi}\right) J_1\left(\frac{\sqrt{\lambda} k}{2\pi}\right) e^k + 1.
\]

This is the main result of this paper.

Expanding in $\lambda$ we get

\[
\frac{\Delta_{\text{max}}}{L} = 1 + 4 \ln 2 \frac{\lambda}{16\pi^2} - 9\zeta(3) \left(\frac{\lambda}{16\pi^2}\right)^2 + 75\zeta(5) \left(\frac{\lambda}{16\pi^2}\right)^3 + \ldots
\]

The second term is the ground-state energy of the Heisenberg AF. The third term can be computed from (4) by first-order perturbation theory, since the two-site spin correlator in the Heisenberg model is known [42]

\[
\langle\text{AF}0|\sigma_l \cdot \sigma_{l+2}|\text{AF}0\rangle = 1 - 16 \ln 2 + 9\zeta(3).
\]

Alternatively, the two-loop correction can be extracted from the exact solution [43] of the Inozemtsev model [44] 6.

The three-loop dilatation operator can be also embedded into the Inozemtsev model, but the embedding is rather non-trivial [7].
Figure 1: The density of Bethe roots \((2\pi \rho / \sqrt{\lambda})\) as a function of the scaling variable \(U = 2\pi u / \sqrt{\lambda}\) at \(\lambda = 7\) (left pane); \(\lambda = 30\) (middle pane); and \(\lambda = 2000\) (right pane).

Extrapolating (17) to the strong coupling we find

\[
\frac{\Delta_{\text{max}}}{L} = \frac{\sqrt{\lambda}}{\pi^2} + \ldots
\]  

(19)

It is not clear how justified is this extrapolation. The string Bethe equations \[16, 45\], which are supposed to describe the spectrum at strong coupling, contain a correction term that definitely contributes in the thermodynamic limit. The order of limits is also important here\(^7\). The derivation of (19) assumes that the limit \(L \to \infty\) is taken before \(\lambda \to \infty\). The consistency of replacing (9) by (12) and (13) requires that the distance between nearby roots is small: \(\Delta u \sim 1 / L \rho(u) \ll 1\). Since \(\rho \sim 1 / \sqrt{\lambda}\) at large \(\lambda\), (19) holds only for \(\lambda \ll L^2\).

It is interesting to see how the density behaves at large \(\lambda\). It will then become clear that the \(\sqrt{\lambda} L\) scaling of \(\Delta_{\text{max}}\) is the robust prediction, to a large degree independent of a particular form of Bethe equations. At one-loop,

\[
\rho(u)|_{\lambda \to 0} = \frac{1}{2 \cosh \pi u},
\]  

(20)

which monotonously decreases with \(|u|\). As \(\lambda\) grows the density develops two peaks which become more and more pronounced (fig. I). The positions of the peaks approach \(u = \pm \sqrt{\lambda \pi} / 2\pi\) at strong coupling and the density asymptotically approaches

\[
\rho(u)|_{\lambda \to \infty} = \frac{\theta \left( \frac{\lambda}{4\pi^2} - u^2 \right)}{2\pi \sqrt{\frac{\lambda}{4\pi^2} - u^2}}.
\]  

(21)

\(^7\)I would like to thank J. Minahan for the discussion of this point.
This is somewhat similar to the rapidity distribution in the conformal super-
coset $O(2m + 2|2m)$ sigma-model [46], where the peaks correspond to special
low-energy modes coined non-movers in [46]. The distribution (21), however,
has a very simple form in the momentum representation. At strong coupling,

$$x \left( u \pm \frac{i}{2} \right) \approx \frac{u}{2} \pm \frac{i}{2} \sqrt{\frac{\lambda}{4\pi^2} - u^2} \quad (22)$$

for $|u| < \sqrt{\lambda}/2\pi$ and

$$u = \frac{\sqrt{\lambda}}{2\pi} \cos \frac{p}{2}. \quad (23)$$

Changing the variables from $u$ to $p$ we find that (21) corresponds to the flat
distribution of momenta in the whole Brillouin zone:

$$\rho(p)|_{\lambda \to \infty} = \frac{1}{4\pi}. \quad (24)$$

In other words, the Bethe equations are solved by $p_n = 4\pi n/L; n = 1, \ldots, L/2$.
This looks like ordinary momentum quantization, but the quantum of momentum is twice as large as allowed by the periodic boundary conditions.
This is because the scattering phase is also non-trivial:

$$\prod_{n=1}^{L/2} \frac{\sqrt{\lambda}}{2\pi} \left( \cos \frac{p}{2} - \cos \frac{2\pi n}{L} \right) + i \approx -e^{ipL/2},$$

and forces the momentum to be quantized in the units of $4\pi/L$.

One can in principle do the same calculation for the string Bethe ansatz
of [16]. The result should not be much different, since the dispersion relation
is the same eq. (2) [48]. At strong coupling it becomes

$$\varepsilon \approx \frac{\sqrt{\lambda}}{2\pi} \sin \frac{p}{2}.$$ 

As a result, the upper bound on the energy is proportional to $\sqrt{\lambda}$. The mo-
mentum distribution is not very important for this conclusion. The concrete
form of the distribution only determines the coefficient of proportionality.

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As a matter of fact, the string Bethe ansatz [16] postulates the same mechanism for
the finiteness of the number of states: the dispersion relation is periodic in $p$, momentum
is thus confined to one Brillouin zone.
It would be extremely interesting to check this prediction directly from the string theory in $AdS_5 \times S^5$.

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Note added: While this paper was being prepared for publication, [47] appeared which also contains the calculation of the energy of the AF state. The authors of [47] in addition established an interesting relationship between the spin chain [4] and the Hubbard model.

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