Levinson’s Theorem for the Klein-Gordon Equation
in Two Dimensions

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Abstract

The two-dimensional Levinson theorem for the Klein-Gordon equation with a cylindrically symmetric potential $V(r)$ is established. It is shown that $N_m \pi = \pi (n_m^+ - n_m^-) = [\delta_m (M) + \beta_1] - [\delta_m (-M) + \beta_2]$, where $N_m$ denotes the difference between the number of bound states of the particle $n_m^+$ and the ones of antiparticle $n_m^-$ with a fixed angular momentum $m$, and the $\delta_m$ is named phase shifts. The constants $\beta_1$ and $\beta_2$ are introduced to symbol the critical cases where the half bound states occur at $E = \pm M$.

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I. Introduction

The Levinson theorem[3], an important theorem in scattering theory, established the relation between the total number of bound states and the phase shifts at zero momentum. During the past half century, the Levinson theorem has been proved by several authors with different methods, and generalized to different fields [4-11]. Rough speaking, there are three main methods used to prove the Levinson theorem. One [3] is based on the elaborate analysis of the Jost function first introduced by Jost. The second is relied on the Green function method [7]. The third method is used to demonstrate the Levinson theorem by the Sturm-Liouville theorem [8-10]. This simple, intuitive method is readily to be generalized and has been verified by the proofs of many physical problems [8-10,22-24]. Furthermore, some obstacles and ambiguities, which may occur in other two methods, disappear in the third method. However, it is found in the later proof that the Sturm-Liouville theorem can’t be directly used to prove the Levinson theorem for the Klein-Gordon equation, but a modified method which is similar to the Sturm-Liouville theorem will be applied to prove the Levinson theorem. Consequently, such a generalization may be useful for the method of bosonization method which has been widely utilized in the literature[26].

The Klein-Gordon equation, which describes the motion of a relativistic scalar particle, is a second-order differential equation with respect to both space and time. When there exists a potential as the fourth component of the vector field, the energy eigenvalues are not necessarily real and the eigenfunctions satisfy the orthogonal relations with a weight factor[1-2] such that a parameter $\epsilon$ which is not always real and positive appears in the normalized relation with a weight factor. As pointed out by Pauli and Snyder at al[1-2], after boson quantization, that those amplitudes with real and positive $\epsilon$ describe particles, but those with real and negative $\epsilon$ antiparticles.

Recalling in the three-dimensional spaces, two main methods are used to set up the Levinson theorem for the Klein-Gordon equation. One is relied on some formulae which are valid for the cases without complex energies[7]. The other, which is similar to that of Sturm-Liouville theorem, is applied to arrive at the Levinson theorem for the Klein-Gordon equation[9]. This result is correct for the cases both without complex
energies and with complex energies.

The reasons why we write this paper are that, on the one hand the Levinson theorem in two dimensions has been studied in experiment [19] as well as in theory [20-24] in virtue of the wide interest in lower-dimensional field theories and other modern physics [12-18], on the other hand the Levinson theorem for the Klein-Gordon equation in two dimensions has never been appeared in the literature. In our previous works[22-24], some surprised results are obtained from the nonrelativistic and relativistic particle as well as the non-local interactions in two dimensions. We attempt to set up the Levinson theorem for the Klein-Gordon equation in two dimensions. What new results will be appeared?

This paper is organized as follows. In Sec. II, we review the properties of the Klein-Gordon equation, especially those related with the parameter $\epsilon$. In Sec. III, it is proved that the difference between the numbers of bound states of particle and the ones of antiparticle only relies on the changes of the logarithmic derivatives of the wave functions at $E = \pm M$ as the potential $V(r)$ changes from zero to the given value. In Sec. IV, it is also turned out that these changes are closely connected with the phase shifts at $E = \pm M$ which then results in the establishment of the two-dimensional Levinson theorem for the Klein-Gordon equation.

II. the Klein-Gordon Equation

Throughout this paper the natural units $\hbar = c = 1$ are employed. Consider a relativistic scalar particle satisfying the Klein-Gordon equation

$$\left(-\nabla^2 + M^2\right)\psi(x) = [E - V(x)]^2\psi(x),$$

where the potential $V(x)$ is the fourth component of a vector field and the $M, E$ denote the mass and the energy of the particle, respectively. In order to simplify the discussion, we only research that the potential is static and cylindrical symmetric one

$$V(x) = V(r),$$

and its asymptotic behavior is written

$$r|V(r)| \to 0 \quad \text{when} \quad r \to 0,$$  \hspace{1cm} (3a)
and
\[ V(r) = 0 \quad \text{when} \quad r > r_0. \] (3b)

Equation (3a) is required to make the wave function single value at the origin, and (3b) is called the cutoff potential for the sake of the simplicity of discussion, i.e., it is vanishing beyond a sufficiently large radius \( r_0 \). It is proved that, following the method [22-23], the results obtained in this paper will not change the essence of the proof if the potential vanishes faster than \( r^{-2} \) at infinity.

Introduce a parameter \( \lambda \) for the potential \( V(r) \)
\[ V(r, \lambda) = \lambda V(r), \] (4)
which shows that the potential \( V(r, \lambda) \) changes from zero to the given potential \( V(r) \) when \( \lambda \) increases from zero to one.

Due to the symmetry of the potential, let
\[ \psi(x, \lambda) = r^{-1/2} R_m(r, \lambda) e^{\pm im\phi}, \quad m = 0, 1, 2, \ldots. \] (5)

where the radial wave equation \( R_m(r, \lambda) \) satisfies the radial equation
\[ \frac{\partial^2 R_m(r, \lambda)}{\partial r^2} + \left\{ (E^2 - M^2) - (2EV - V^2) - \frac{m^2 - 1/4}{r^2} \right\} R_m(r, \lambda) = 0. \] (6)

Denote by \( R_{m1}(r, \lambda) \) the solution to Eq.(6) for the energy \( E_1 \)
\[ \frac{\partial^2 R_{m1}(r, \lambda)}{\partial r^2} + \left\{ (E_1^2 - M^2) - (2E_1V - V^2) - \frac{m^2 - 1/4}{r^2} \right\} R_{m1}(r, \lambda) = 0. \] (7)

Multiplying Eq.(6) and Eq.(7) by \( R_{m1}(r, \lambda) \) and \( R_m(r, \lambda) \), respectively, and calculating their difference, we have
\[ \frac{\partial}{\partial r} \{ R_m(r, \lambda) R_{m1}^*(r, \lambda) - R_{m1}(r, \lambda) R_m^*(r, \lambda) \} = -(E_1^* - E) R_{m1}(r, \lambda) \cdot (E_1^* + E - 2V) R_m(r, \lambda), \] (8)

where the primes denote the derivative of the radial wave function with respect to the variable \( r \). As we know, the energy eigenvalues are not necessarily real for some potential \( V(r) \) which origins from the Klein paradox. Integrating (8) over the whole space and noting that \( R_m(r, \lambda) R_{m1}^*(r, \lambda) - R_{m1}(r, \lambda) R_m^*(r, \lambda) \) vanishes both at the origin.
and at infinity for the physically admissible solutions with the different energies \( E \) and \( E_1 \), we get the weighted orthogonality relation of the radial wave function

\[
(E_1^* - E) \int_0^\infty R_{m1}^*(r, \lambda)(E_1^* + E - 2V)R_m(r, \lambda)dr = 0.
\]

As a matter of fact, we always able to obtain the real solutions for the real energies. However, it is easy to see from Eq. (9) that the normalized relation for the solutions with real energies are not always positive on account of the weight factor \((E_1^* + E - 2V)\):

\[
\int_0^\infty R_{m1}(r, \lambda)(E_1 + E - 2V)R_m(r, \lambda)dr = \begin{cases} 
\epsilon_E \delta(E_1 - E), & |E| > M, \\
\epsilon_E \delta E_1, & |E| < M.
\end{cases}
\]

The parameter \( \epsilon_E \), which depends on the particular radial wave function \( R_m(r, \lambda) \), may be either positive, negative or vanishing. Normalized factors of the solutions can’t change the sign of \( \epsilon \). Generally speaking, if the solution \( R_m(r, \lambda) \) with a complex energy \( E \) is complex, then \( R_m^* \) is also a solution with complex energy \( E^* \) and a complex \( \epsilon_E \) appears for a pair of the complex solutions. It is evident after bose quantization that those \( R_m(r, \lambda) \) with positive \( \epsilon_E \) describes particles and those with negative \( \epsilon_E \) antiparticles. In the case zero \( \epsilon_E \), the solution can be regarded as a pair of particle and antiparticle bound states. The Hamiltonian and charge operator can’t be written as the diagonal forms for the solutions with complex energy \( \epsilon_E \), therefore they describe neither particles nor antiparticles. In this paper, we only count the number of bound states with the real positive and negative nonvanishing \( \epsilon_E \) is named particle and antiparticle bound states, respectively.

Since we are always able to arrive at the real solution for the real energy, we can now solve Eq.(6) in two regions and match two solutions at \( r_0 \). Actually, the solutions in the region \([0, r_0]\) with \( R_m(0) = 0 \) can be arrived at in principle. We only need one matching condition at \( r_0 \) for the logarithmic derivative of the radial wave function

\[
A_m(E, \lambda) \equiv \left\{ \frac{1}{R_m(r, \lambda)} \frac{\partial R_m(r, \lambda)}{\partial r} \right\}_{r=r_0^-} = \left\{ \frac{1}{R_m(r, \lambda)} \frac{\partial R_m(r, \lambda)}{\partial r} \right\}_{r=r_0^+} \equiv B_m(E).
\]

Only one solution is convergent at the origin because of the condition (3a). For example, for the free particle \( (\lambda = 0) \), the solution to Eq. (6) at the region \([0, r_0]\) is
proportional to the Bessel function $J_m(x)$:

$$R_m(r, 0) = \begin{cases} \sqrt{\frac{\pi kr}{2}} J_m(kr), & \text{when } |E| > M \text{ and } k = \sqrt{E^2 - M^2} \\ e^{-im\pi/2} \sqrt{\frac{\pi kr}{2}} J_m(ikr), & \text{when } |E| < M \text{ and } \kappa = \sqrt{M^2 - E^2} \end{cases}$$

(12)

The solution $R_m(r, 0)$ given in Eq. (12) is a real function. A constant factor on the radial wave function $R_m(r, 0)$ is not important.

In the region $[r_0, \infty)$, we have $V(r) = 0$. For $|E| > M$, there are two oscillatory solutions to Eq. (6). Their combination can always satisfy the matching condition (11), so that there is a continuous spectrum for $|E| > M$.

$$R_m(r, \lambda) = \sqrt{\frac{\pi kr}{2}} \left\{ \cos \eta_m(k, \lambda) J_m(kr) - \sin \eta_m(k, \lambda) N_m(kr) \right\} \sim \cos \left( k r - \frac{m \pi}{2} - \frac{\pi}{4} + \eta_m(k, \lambda) \right), \quad \text{when } r \rightarrow \infty. \quad (13)$$

where $N_m(kr)$ is the Neumann function.

However, there is only one convergent solution in the region $[r_0, \infty)$ for $|E| \leq M$ the matching condition (11) is not always satisfied.

$$R_m(r, \lambda) = e^{i(m+1)\pi/2} \sqrt{\frac{\pi kr}{2}} H_m^{(1)}(ikr) \sim e^{-kr}, \quad \text{when } r \rightarrow \infty. \quad (14)$$

where $H_m^{(1)}(x)$ is the Hankel function of the first kind. When the condition (11) is satisfied, a bound state appears at this energy. It means that there is a discrete spectrum for $|E| \leq M$.

As mentioned above, in the case with the real energy solutions, integrating the Eq. (6) in two regions $[0, r_0]$ and $[r_0, \infty)$, respectively, and taking the limit $E_1 \rightarrow E$, we will obtain the following equations in terms of the boundary condition that $R_m(0) = 0$ and $R_m(\infty) = 0$ for $|E| < M$

$$\frac{\partial A_m(E, \lambda)}{\partial E} = \frac{\partial}{\partial E} \left( \frac{1}{R_m(r, \lambda)} \frac{\partial R_m(r, \lambda)}{\partial r} \right)_{r=r_0^-} = - R_m(r_0, \lambda)^{-2} \int_{r_0}^{\infty} R_m(r, \lambda)^2 \left[ E - V(r) \right] dr < 0. \quad (15a)$$

and

$$\frac{d B_m(E)}{dE} = \frac{\partial}{\partial E} \left( \frac{1}{R_m(r, \lambda)} \frac{\partial R_m(r, \lambda)}{\partial r} \right)_{r=r_0^+} = R_m(r_0, \lambda)^{-2} \int_{r_0}^{\infty} R_m(r, \lambda)^2 2 E \left[ E - V(r) \right] dr > 0. \quad (15b)$$
which demonstrates from Eq. (15) that $A_m(E, \lambda)$ is no longer monotonic with respect to energy, but $B_m(E)$ is still monotonic with respect to energy if the energy doesn’t change sign.

From the matching condition (11) we have

$$\tan \eta_m(k, \lambda) = \frac{J_m(kr_0)}{N_m(kr_0)} \cdot \frac{A_m(E, \lambda) - kJ'_m(kr_0)/J_m(kr_0) - 1/(2r_0)}{A_m(E, \lambda) - kN'_m(kr_0)/N_m(kr_0) - 1/(2r_0)},$$

(16)

$$\eta_m(k) \equiv \eta_m(k, 1).$$

(17)

where the prime denotes the derivative of the Bessel function, the Neumann function, and later the Hankel function with respect to their argument. However, it is not true for $|E| < M$ because of no adjustable phase shift $\delta_m(E)$. Once the matching condition is satisfied, we will get the discrete bound states.

The phase shift $\eta_m(k, \lambda)$ is determined from (16) up to a multiple of $\pi$ due to the period of the tangent function. In this paper, for the free particle ($V(r) = 0$), the definition of phase shift $\eta_m(k, 0)$ is defined to be zero, i.e

$$\eta_m(k, 0) = 0, \quad \text{where} \quad \lambda = 0,$$

(18)

which is same as our previous definition[8-9,22-24].

It is shown from Eq. (10) that scattering states---$|E| > M$---are normalized as the Dirac $\delta$ function, and that the main contribution to the integration Eq. (10) comes from the radial wave functions in the region $[r_0, \infty)$ where there is no potential. For this reason we obtain

$$\epsilon_E = \pi \sqrt{E^2 - M^2} \cdot \frac{E}{|E|}, \quad |E| > M.$$ 

(19)

All the scattering states with positive energy ($E > M$) describe particles and those with negative energy ($E < -M$) describe antiparticles. It is easy to see that this conclusion is not true for the critical case $E = \pm M$ except for $S$ waves where there is a half bound state at $E = \pm M$. The situations which $\epsilon_E$ with $E = \pm M$ and $m > 1$ may be positive, negative or vanishing are relied on the potential.

**III. The Number of Bound States**

In our previous works, the Levinson theorem for the nonrelativistic and relativistic particles are set up under the help of Sturm-Liouville theorem. For the Sturm-Liouville
problem, the fundamental trick is the definition of a phase angle which is monotonic with respect to the energy [25]. Although this method is very simple, intuitive and easy to be generalized, from the Eq. (6), it is the weight factor \((E_1^* + E - 2V)\) that makes the Sturm-Liouville theorem not be used for the the Klein-Gordon equation. Nevertheless, a modified method is applied to prove the Levinson theorem for the Klein-Gordon equation. From the difference between the Eq.(15a) and Eq. (15b), we arrive at

\[
\frac{d B_m(E)}{dE} - \frac{\partial A_m(E, \lambda)}{\partial E} \equiv B'_m(E) - A'_m(E, \lambda) = \frac{1}{R_m(r_0)\epsilon_E},
\]

where here and hereafter the primes denote the derivative with respect to the energy.

From Eq. (14), we get

\[
B_m(E) = i k H_m^{(1)}(i k r_0) \left(\frac{1}{H_m^{(1)}(i k r_0)}\right)^{'} - \frac{1}{2r_0} = \begin{cases} 
\frac{(-m + 1/2)/r_0 \equiv \rho_m}{-\kappa \sim -\infty} \quad \text{when } k_1 \rightarrow 0 \\
-\kappa \sim \infty \quad \text{when } k_1 \rightarrow \infty.
\end{cases}
\]

The logarithmic derivative given in Eq. (21) does not depend on \(\lambda\). On the other hand, when \(\lambda = 0\) we obtain from Eq. (12)

\[
A_m(E, 0) = i k J'_m(i k r_0) \left(\frac{1}{J_m(i k r_0)}\right)^{'} - \frac{1}{2r_0} = \begin{cases} 
\frac{(m + 1/2)/r_0}{\kappa \sim \infty} \quad \text{when } k_1 \rightarrow 0 \\
\kappa \sim \infty \quad \text{when } k_1 \rightarrow \infty.
\end{cases}
\]

It is evident from the Eqs. (21) and (22) that both \(B_m(E)\) and \(A_m(E, 0)\) are continuous curves with respect to energy which don’t intersect each other; i.e. the matching condition (11) is not satisfied if \(|E| \leq M\) and \(\lambda = 0\). No bound states appear when there is no potential.

As \(\lambda\) changes from the zero to the given potential, \(B_m(E)\) don’t change, but \(A_m(E, \lambda)\) changes continuously except the points where \(R_m(r_0) = 0\) and \(A_m(E, \lambda)\) tends to infinity. Generally speaking, \(A_m(E, \lambda)\) is continuous except those finite points and intersects with the curve \(B_m(E)\) several times for \(|E| \leq M\). The bound state will appear only if the intersection happens. The points of the intersection determine the number of the bound states. It is shown from Eq. (20) that the relative slopes at the points of intersection decide whether the bound states describe particle or antiparticles.

When the potential \(V(r)\) change with the \(\lambda\), the number of intersection points will change, too. This only origins from the following two sources. Firstly, the intersection points move inward or outward at \(E = \pm M\). Secondly, the curve \(A_m(E, \lambda)\) intersects

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with the curve $B_m(E)$ or departs from it through the tangency point. For the second case, a pair of particle and antiparticle bound state will be created or annihilated at the same time, but the difference of the number of the particle $n^+_m$ and antiparticle bound state $n^-_m$ don’t change. That’s to say, the change of the whole bound states $N_m$ which expresses that the difference of the particle bound state and the antiparticle state only depends on the intersection points moving in or out at $E = \pm M$ where the critical cases occur. Hence, we only discuss this case. There are four cases when $A_m(M, \lambda) = B_m(M)$ when $A_m(M, \lambda)$ decreases across the value $B_m(M) = (-m + 1/2)/r_0$ at $E = M$

$$ (1) A'_m(M, \lambda) < B'_m(M), $$

$$ (2) A^{(n)}_m(M, \lambda) = B^{(n)}_m(M), \quad (-1)^n A^{(n+1)}_m(M, \lambda) < (-1)^n B^{(n+1)}_m(M), $$

$$ (3) A^{(n)}_m(M, \lambda) = B^{(n)}_m(M), \quad (-1)^n A^{(n+1)}_m(M, \lambda) > (-1)^n B^{(n+1)}_m(M), $$

$$ (4) A'_m(M, \lambda) > B'(M). $$

Where here and hereafter $n$ is positive integer. For the first two situation, a interaction point moves inward from $E > M$ to $E < M$, which results in the appearance of new particle bound state. However, for the last two cases, the interaction point moves outward from $E < M$ to $E > M$, which causes the disappearance of an antiparticle bound state. The converse process occurs when $A_m(M, \lambda)$ increases to cross the value $B_m(M)$, i.e. the number of bound states $N_m$ increases by one only if each time $A_m(M, \lambda)$ decreases to cross the value $B_m(M)$ at $E = M$. Conversely, each time $A_m(M, \lambda)$ increases across the value $B_m(M)$ at $E = M$, $N_m$ decreases by one.

On the other hand, there are also four cases when $A_m(-M, \lambda) = B_m(-M)$:

$$ (1') A'_m(-M, \lambda) > B'_m(-M), $$

$$ (2') A^{(n)}_m(-M, \lambda) = B^{(n)}_m(-M), A^{(n+1)}_m(-M, \lambda) > B^{(n+1)}_m(-M), $$

$$ (3') A^{(n)}_m(-M, \lambda) = B^{(n)}_m(-M), A^{(n+1)}_m(M, \lambda) < B^{(n+1)}_m(M), $$

$$ (4') A'_m(-M, \lambda) < B'_m(-M). $$

If $A_m(-M, \lambda)$ decreases across the value $B_m(-M)$ as $\lambda$ increases, for the first two cases a interaction moves inward from the $E < -M$ to $E > -M$ point which describes
an antiparticle. But for the last two cases an interaction point moves outward from $E > -M$ to $E < -M$ which describes a particle. The number of bound states $N_m$ decreases by one only if each time $A_m(-M, \lambda)$ increases across the value $B(-M)$. The opposite process occurs when $A_m(-M, \lambda)$ increases across the value $B_m(-M)$.

We denote by $N_m(\pm M)$ the difference between the number of times $A(\pm M, \lambda)$ decreasing across the value $B(\pm M)$ and the number of the times that $A(\pm M, \lambda)$ increasing across that value. Hence, we obtain

$$N_m \equiv n^+_m - n^-_m = n_m(+M) - n_m(-M). \quad (25)$$

**IV. The Phase Shifts**

As we know, the solutions in the region $[r_0, \infty)$ for the scattering states have been given by Eq. (13). The phase shift $\eta_m(0, \lambda)$ is the limit of the phase shift $\eta_m(k, \lambda)$ as $k$ tends to zero. Hence, what we are interested in is the phase shift $\eta_m(k, \lambda)$ at a sufficiently small momentum $k$, $k \ll 1/r_0$. For the small momentum we obtain from the matching condition (11)

$$\tan \eta_m(k, \lambda) \sim \begin{cases} 
-\pi (kr_0)^{2m} 
& \frac{2^{2m}m!(m-1)!}{A_m(0, \lambda) - c^2k^2 - \rho_m (1 - (kr_0)^2)/(m-1)(2m-1)} \quad \text{when } m \geq 2 \\
\frac{-\pi (kr_0)^2}{4\pi} 
& \frac{A_m(0, \lambda) - \rho_1 (1 + 2(kr_0)^2 \log(kr_0))}{A_m(0, \lambda) - c^2k^2 - \rho_0 (1 - (kr_0)^2)} \quad \text{when } m = 1 \\
& \frac{2\log(kr_0)}{A_m(0, \lambda) - c^2k^2 - \rho_0 \left(1 + \frac{2}{\log(kr_0)}\right)} \quad \text{when } m = 0.
\end{cases} \quad (26)$$

In addition to the leading terms, we include in (26) some next leading terms, which is useful only for the critical case where the leading terms are canceled with each other. and

$$\frac{\partial \eta_m(k, \lambda)}{\partial A_m(E, \lambda)} \bigg|_k = \frac{-8r_0 \cos^2 \eta_m(k, \lambda)}{\pi \left(2r_0 A_m(E, \lambda) N_m(kr_0) - 2kr_0 N'_m(kr_0) - N_m(kr_0)\right)^2} \leq 0, \quad (27)$$
which shows that the phase shift is monotonic with respect to the logarithmic derivative $A_m(E, \lambda)$ as $\lambda$ increases.

It is shown from Eqs. (26) and (27) that they are not different from those of Schrödinger equation. Therefore, we may simply discuss this problem by the same method. Each time $A_m(\pm M, \lambda)$ decreases across the value $B_m(\pm M)$ as the potential changes from the zero to the given potential, the phase shift $\delta_m(\pm M, \lambda)$ increases by $\pi$. Conversely, the phase shift $\delta_m(\pm M, \lambda)$ decreases by $\pi$ if $A_m(\pm M, \lambda)$ increases across the value $B_m(\pm M)$.

As $\lambda$ increases from zero to one, i.e. the potential changes from the zero to the given value, we have

$$\delta_m(\pm M) \equiv \delta_m(\pm M, 1) = n_m(\pm M)\pi, \quad (28)$$

Thus, we draw a conclusion that the Levinson theorem for the Klein-Gordon equation if $A(\pm M, 1) \neq B(\pm M)$

$$N_m \pi = \delta_m(M) - \delta_m(-M). \quad (29)$$

We now discuss the critical cases

$$A_m(M, 1) = B_m(M) \quad \text{and} \quad A_m(-M, 1) = B_m(-M), \quad (30)$$

where the potential changes from the zero to the given potential $V(r)$. Similar to the discussion[22-24], the phase shift $\delta_m(\pm M, \lambda)$ increases by $\pi$ for $m > 1$ or an additional $\pi$ for the $P$ waves if $A_m(\pm M, \lambda)$ decreases from near and larger than the value $B_m(M)$ to smaller than that value when the potential changes from the zero to the given potential. Conversely, $\delta_m(\pm M, \lambda)$ doesn’t decrease by $\pi$ or an additional $\pi$ if $A_m(\pm M, \lambda)$ increases across the value $B_m(M)$ as the potential changes to the given potential $V(r)$. On the other hand, the states for $m = 0, 1$ are called a half bound state which is defined as its wave function is finite but not square integrable. Furthermore, the half bound state is not a bound state. For $M > 1$ states in the critical situations, there is a bound state but its $\epsilon_E$ may be either positive, negative or vanishing, which depends on the different cases (23) and (24). We consider the state with zero $\epsilon_E$ as a pair of particle and antiparticle bound states.
Introduce two parameters $\beta_1, \beta_2$ to describe the appearance or disappearance of the bound states at the critical cases. $\beta_1 = 0$ for the noncritical case $A(M, 1) \neq B_m(M)$, and $\beta_2 = 0$ for the case $A_m(-M, 1) \neq (-M)$.

(1) If $A(M, 1) = B_m(M)$, $\beta_1 = 0$ for the cases (23a) or (23c) with $m > 1$; $\beta_1 = -1$ for the cases (23b) or (23d) with $m > 1$; and $\beta_1 = -1$ for the case (23a) with $m = 1$.

(2) If $A_m(-M, 1) = B_m(-M)$, $\beta_1 = 0$ for the cases (24a) or (24c) with $m > 1$; $\beta_1 = -1$ for the cases (24b) or (24d) with $m > 1$; and $\beta_1 = -1$ for the case (24a) with $m = 1$.

where $\lambda$ is substituted by one. Then, the Levinson theorem for the Klein-Gordon equation with the cylindrical symmetric potential $V(r)$ satisfying the asymptotic behavior (3)

$$N_m \pi = \pi \left( n^+_m - N^-_m \right) = \left[ \delta_m(M) + \beta_1 \right] - \left[ \delta_m(-M) + \beta_2 \right]. \quad (31)$$

According to the above discussion, it is easy to find, compared with the case in the three-dimensional spaces, that the phase shifts for the critical states changes by an additional $\pi$ not by $\pi/2$. This conclusion is same as the relativistic and nonrelativistic particles.

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**References**

[1] W. Pauli, princeton mimeographed notes (1935).

[2] H. Snyder and J. Weinberg, Phys. Rev. 15 307 (1940); L.I. Schiff, H. Snyder and J. Weinberg, *ibid* 15 315 (1940).

[3] N. Levinson, K. Danske Vidensk. Selsk. Mat-fys. Medd. 25, No. 9 (1949).

[4] R. G. Newton, J. Math. Phys. 1, 319 (1960); *ibid* 18, 1348, 1582 (1977); *Scattering theory of waves and particles*, (Springer-Verlag, New York, 2nd ed., 1982) and references therein.

[5] J. M. Jauch. Helv. Phys. Acta 30, 143 (1957).
[6] A. Martin, Nuovo Cimento 7, 607 (1958).

[7] G. J. Ni, Phys. Energ. Fort. Phys. Nucl. 3, 432 (1979); Z. Q. Ma and G. J. Ni, Phys. Rev. D31, 1482 (1985).

[8] Z. Q. Ma, J. Math. Phys. 26(8), 1995 (1985).

[9] Z. Q. Ma, Phys. Rev. D32, 2203 and 2213 (1985).

[10] Z. R. Iwinski, L. Rosenberg, and L. Spruch, Phys. Rev. 31, 1229 (1985).

[11] N. Poliatzky, Phys. Rev. Lett. 70, 2507 (1993); R. G. Newton, Helv. Phys. Acta 67, 20 (1994); Z. Q. Ma, Phys. Rev. Lett. 76, 3654 (1996).

[12] Z. R. Iwinski, L. Rosenberg, and L. Spruch, Phys. Rev. A 33, 946 (1986); L. Rosenberg, and L. Spruch, Phys. Rev. A 54, 4985 (1996).

[13] R. Blankenbecler and D. Boyanovsky, Physica 18D, 367 (1986).

[14] A. J. Niemi and G. W. Semenoff, Phys. Rev. D32, 471 (1985).

[15] F. Vidal and J. Letourneaux, Phys. Rev. C 45, 418 (1992).

[16] K. A. Kiers, W. van Dijk, J. Math. Phys. 37, 6033 (1996).

[17] M. S. Debianchi, J. Math. Phys., 35, 2719 (1994).

[18] P. A. Martin and M. S. Debianchi, Europhys. Lett. 34, 639 (1996).

[19] M. E. Portnoi and I. Galbraith, Solid State Commun. 103, 325 (1997).

[20] D. Bollé, F. Gesztesy, C. Danneels, and S. F. J. Wilk, Phys. Rev. Lett. 56, 900 (1986).

[21] Q. G. Lin, Phys. Rev. A56, 1938 (1997).

[22] Shi-Hai Dong, Xi-Wen Hou and Zhong-Qi Ma, Levinson’s theorem for the Schrödinger equation in two dimensions, Phys. Rev. A. accepted (will be published in August of 1998).
[23] Shi-Hai Dong, Xi-Wen Hou and Zhong-Qi Ma, The relativistic Levinson’s theorem in two dimensions, preprint, submitted to Phys. Rev. A.

[24] Shi-Hai Dong, Xi-Wen Hou and Zhong-Qi Ma, Levinson’s theorem for the non-local interactions in two dimensions, accepted to J. Phys. A.

[25] C. N. Yang, in Monopoles in Quantum Field Theory, Proceedings of the Monopole Meeting, Trieste, Italy, 1981, ed. by N. S. Craigie, P. Goddard, and W. Nahm (World Scientific, Singapore, 1982), p.237.

[26] C. G. Callan, Jr., Phys. Rev. 26, 2058 (1982); E. Witten, Commun. Math. Phys. 92, 455 (1984).