Complete population inversion by a phase jump: an exactly soluble model

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New Journal of Physics 9 (2007) 58
Received 13 December 2006
Published 16 March 2007
Online at http://www.njp.org/
doi:10.1088/1367-2630/9/3/058

Abstract. An exact analytic solution to the time-dependent Schrödinger equation is presented for a two-state quantum system coherently driven by a nonresonant pulsed external field. The pulse has a hyperbolic-secant shape, with a sign jump (i.e. a phase step of $\pi$) at its maximum, which nullifies the overall pulse area. It is shown that off resonance, for detunings larger than the pulse bandwidth, the transition probability approaches unity as the field intensity increases. The population inversion is robust against small-to-moderate variations in the detuning and the intensity, a feature reminiscent of adiabatic passage. The population inversion, however, is not of adiabatic nature; it is induced by $\delta$-function-shaped nonadiabatic coupling with an area of nearly $\pi$. An estimate of the required experimental resources shows that implementation with shaped femtosecond pulses is feasible.
1. Introduction

The two-state system is a fundamental ingredient in quantum mechanics, which can be found in every area of quantum physics, from nuclear magnetic resonance, coherent atomic excitation and atomic collisions to chemical, solid-state and nuclear physics, neutrino oscillations, and quantum information. Moreover, the two-state problem is a basic tool for treating quantum dynamics of systems with multiple states and complicated linkage patterns, single-particle as well as many-particle, which can often be understood only by reduction to effective two-state problems.

Two-state quantum dynamics has been extensively studied. On exact resonance, when the frequency of the driving field is equal to the Bohr transition frequency, the Schrödinger equation is solved exactly, for any time dependence of the coupling $\Omega(t)$ (the Rabi frequency), and the transition probability depends on the pulse area $A = \int_{-\infty}^{\infty} \Omega(t) \, dt$ only, $P = \sin^2 \left( A / 2 \right)$. Resonant pulses of precise area have been used for a long time in a variety of fields, most notably in nuclear magnetic resonance [1] and coherent atomic excitation [2]. Such pulses are now a common tool in quantum information processing [3]. Of particular use are the $\pi$ pulses, which produce complete population inversion (CPI) between the two states, and half-$\pi$ pulses that create an equal coherent superposition of the two states.

There are several exactly soluble non-resonant two-state models, usually in terms of some special function, including the Rabi [4], Landau–Zener [5], Rosen–Zener [6], Allen–Eberly [7, 8], Bambini–Berman [9], Demkov–Kunike [10], Demkov [11], Nikitin [12], and Carroll–Hioe [13] models. Methods for approximate solutions are also available, such as the perturbation theory, the adiabatic approximation, the Magnus approximation, and the Dykhne–Davis–Pechukas [14] approximation. The latter, in particular, is a very useful tool for deriving very accurate approximations for various cases of interest, e.g. transform-limited and chirped Gaussian pulses [15] or nonlinear level crossing [16].

Adiabatic evolution is of particular interest, because, when accompanied by energy level crossing, it leads to CPI—usually referred to as rapid adiabatic passage [7, 17]. Noncrossing
energies produce no excitation at the end of adiabatic evolution—complete population return, which is, for instance, the key to the explanation of the observed absence of power broadening in coherent pulsed excitation [18].

Despite its apparent simplicity, the two-state problem conceals a variety of surprising features, textbook examples being Rabi oscillations [4], Autler–Townes splitting [19], and the Bloch–Siegert shift [20]. Further unexpected features emerge when the pulse shape is asymmetric [21], when the field is a train of pulses [22], when the field is modulated in amplitude [23] or frequency [24], and when it is bichromatic [25] or polychromatic [26].

In this paper, I shall present an analytically exactly soluble two-state model, in which the detuning is constant and the coupling is pulse-shaped, with a sign jump at the time of its maximum. This pulse has a zero area, and hence on resonance it produces no excitation in the end of the interaction. Surprisingly, off resonance this field produces CPI for a sufficiently large pulse area. Moreover, this CPI is robust against variations in the experimental parameters, which is reminiscent of adiabatic passage. The CPI mechanism, however, is not adiabatic passage, but it is induced by a $\delta$-function-shaped interaction (nonadiabatic coupling) in the adiabatic basis.

This amazing novel feature of two-state systems has been deduced in [27] on the basis of numerical simulations for several pulse shapes, supplemented by approximate analytic results in the adiabatic limit. The present exactly soluble analytic model has the advantage of being, of course, applicable to any values of the interaction parameters. It therefore allows one to derive accurate prescriptions of the resources required to achieve CPI. Another motivation is the conceptual significance of the two-state problem, for which exact analytical solutions are highly valuable. Finally, this model is experimentally feasible by using femtosecond pulse-shaping technology [28].

2. The step-sech model

2.1. The model

The time evolution of a coherently driven two-state quantum system is described by two coupled ordinary differential equations for the probability amplitudes $c_1(t)$ and $c_2(t)$ of states $\psi_1$ and $\psi_2$,

$$i \frac{d}{dt} c_1(t) = \frac{i}{2} \Omega(t) e^{-iD(t)} c_2(t), \quad (1a)$$

$$i \frac{d}{dt} c_2(t) = \frac{i}{2} \Omega(t) e^{iD(t)} c_1(t), \quad (1b)$$

where $D = \int_0^t \Delta(t') \, dt'$. Equations (1) are derived from the Schrödinger equation within the conventional rotating-wave approximation (RWA) [2]. For laser-driven atomic or molecular transitions, $\Delta = \omega_0 - \omega$ is the frequency detuning between the laser carrier frequency $\omega$ and the Bohr transition frequency $\omega_0$, and $\Omega(t) = -\mathbf{d} \cdot \mathbf{E}(t)/\hbar$ is the Rabi frequency, where $\mathbf{d}$ is the transition dipole moment and $\mathbf{E}(t)$ is the laser electric-field envelope.
I shall derive the solution to equation (1) for a step-sech model, with the coupling and the detuning given by

$$\Omega(t) = \begin{cases} \Omega_0 \sech(t/T) & (t < 0) \\ -\Omega_0 \sech(t/T) & (t \geq 0) \end{cases} \quad \text{(2a)}$$

$$\Delta(t) = \text{const.} \quad \text{(2b)}$$

Without loss of generality the constant frequencies $\Omega_0$ and $\Delta$ and the pulse width $T$ will be assumed to be positive.

The coupling $\Omega(t)$ is displayed in figure 1. Its temporal area is $A = \int_{-\infty}^{+\infty} \Omega(t) \, dt = 0$ (zero-area pulse); hence the transition probability on resonance is zero,

$$P = \sin^2(A/2) = 0 \quad \text{(for } \Delta = 0).$$

Off resonance, $P$ is nonzero and its exact value is derived below.

### 2.2. Exact analytic solution

The step-sech model (2) resembles the well-known Rosen–Zener model [6], where the coupling $\Omega(t)$ is a bell-shaped sech function at all times, without the step at $t = 0$. I shall therefore follow the derivation of [6] up to time $t = 0$, where the sign step will be dealt with.

The first step is to decouple equation (1) by differentiating the equation for $\dot{c}_1$ (with the overdot denoting a time derivative) and replacing $c_2$ and $\dot{c}_2$, found from equation (1); this gives

$$\ddot{c}_1 - \left( \frac{\Omega}{\Omega_0} - i\Delta \right) \dot{\dot{c}}_1 + \frac{1}{4} \Omega^2 c_1 = 0.$$

**Figure 1.** The step-sech pulse shape.
The next step is to change the independent variable from $t$ to $z(t) = [1 + \tanh(t/T)]/2$. Note that $z(-\infty) = 0$, $z(0) = \frac{1}{2}$, and $z(+\infty) = 1$. Using $d/dt = \dot{z}(d/dz)$, $d^2/dt^2 = \ddot{z}(d^2/dz^2)$, $\dot{z} = 2z(1 - z)/T$, $\ddot{z} = 4z(1 - z)(1 - 2z)/T^2$, $\Omega^2 = 4\Omega_0^2 z(1 - z)$, $\Omega/\Omega = (1 - 2z)/T$, equation (4) is transformed into

$$z(1 - z)\frac{d^2C_1}{dz^2} + \left(\frac{1}{2} + \frac{1}{2}i\Delta T - z\right)\frac{dC_1}{dz} + \frac{1}{4}\Omega_0^2 T^2 C_1 = 0,$$

where $C_n(z) = c_n(t(z))$. This equation has the same form as the Gauss hypergeometric equation and its solution can be expressed in terms of the Gauss hypergeometric function $F(\alpha, \beta; \gamma; z)$ [29] as

$$C_1(z) = AF(\alpha, -\alpha; \gamma; z) + Bz^{1-\gamma}F(1 + \alpha - \gamma, 1 - \alpha - \gamma; 2 - \gamma; z),$$

where $A$ and $B$ are integration constants and

$$\alpha = \frac{1}{2}\Omega_0 T, \quad \gamma = \frac{1}{2} + i\delta, \quad \delta = \frac{1}{2}\Delta T.$$

By using equations (A.2a), (A.2b) and the relation $e^{i\Delta t} = z^{\gamma - 1/2}(1 - z)^{1/2 - \gamma}$, one obtains

$$C_2(z) = i(1 - z)^{1-\gamma}\left[-\alpha\frac{\alpha}{\gamma} z^\gamma F(1 + \alpha, 1 - \alpha; 1 + \gamma; z)\right.$$

$$+ B\frac{1 - \gamma}{\alpha} F(1 + \alpha - \gamma, 1 - \alpha - \gamma; 1 - \gamma; z)\bigg].$$

The constants $A$ and $B$ are determined from the initial conditions $C_1(0)$ and $C_2(0)$. By using equation (A.2d) one finds $A = C_1(0)$ and $B = -i\alpha C_2(0)/(1 - \gamma)$. Hence for $t < 0$ ($0 \leq z \leq \frac{1}{2}$) the elements of the propagator $U(z, 0)$, defined by $C(z) = U(z, 0)C(0)$ with $C(z) = [C_1(z), C_2(z)]^T$, are given by

$$U_{11}(z, 0) = U_{22}(z, 0)^* = F(\alpha, -\alpha; \gamma; z),$$

$$U_{21}(z, 0) = -U_{12}(z, 0)^* = -i\frac{\alpha}{\gamma} (1 - z)^{1-\gamma} F(1 + \alpha, 1 - \alpha; 1 + \gamma; z).$$

By using equations (A.2f) and (A.2g), the propagator from $t \to -\infty$ ($z = 0$) to time $t = 0$ ($z = \frac{1}{2}$) is expressed as

$$U(\frac{1}{2}, 0) = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix},$$

where the Cayley–Klein parameters are

$$a = F(\alpha, -\alpha; \gamma; \frac{1}{2}) = \pi^{1/2}2^{-\gamma} \Gamma(\gamma) (\xi + \eta),$$

$$N = \pi^{1/2}2^{-\gamma} \Gamma(\gamma) (\xi - \eta) = \frac{\omega}{\Omega},$$

$$\xi = \frac{\omega}{\Omega} + \frac{\omega}{\Omega} = \pi^{1/2}2^{-\gamma} \Gamma(\gamma) \left(\frac{\omega}{\Omega} + \frac{\omega}{\Omega}\right),$$

$$\eta = \frac{\omega}{\Omega} - \frac{\omega}{\Omega} = \pi^{1/2}2^{-\gamma} \Gamma(\gamma) \left(\frac{\omega}{\Omega} - \frac{\omega}{\Omega}\right).$$

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\[ b = -i \frac{\alpha}{2y} F(1 + \alpha, 1 - \alpha; 1 + \gamma; \frac{1}{2}) = -i \pi^{1/2} 2^{-\gamma} \Gamma(\gamma) (\xi - \eta), \quad (11b) \]

with

\[ \xi = \left[ \Gamma\left(\frac{3}{4}, \frac{1}{2} \alpha + \frac{i}{2} \delta\right) \Gamma\left(\frac{3}{4} - \frac{1}{2} \alpha + \frac{1}{2} i \delta\right) \right]^{-1}, \quad (12a) \]

\[ \eta = \left[ \Gamma\left(\frac{3}{4}, \frac{1}{2} \alpha + \frac{i}{2} \delta\right) \Gamma\left(\frac{1}{4} - \frac{1}{2} \alpha + \frac{1}{2} i \delta\right) \right]^{-1}. \quad (12b) \]

Using the symmetry of equation (1) and taking into account that the only change for \( t > 0 \) is the sign of \( \Omega(t) \), see equation (2), it is a simple matter to show that the propagator for \( 0 \leq t < \infty \) (\( \frac{1}{2} \leq z \leq 1 \)) reads

\[ U(1, \frac{1}{2}) = \begin{bmatrix} a & -b \\ b^* & a^* \end{bmatrix}. \quad (13) \]

The full propagator is \( U(1, 0) = U(1, \frac{1}{2}) U(\frac{1}{2}, 0) \), or

\[ U(1, 0) = \begin{bmatrix} a^2 - b^2 & -2\text{Re}(ab^*) \\ 2\text{Re}(ab^*) & (a^2 - b^2)^* \end{bmatrix}. \quad (14) \]

### 2.3. Transition probability

The transition probability is

\[ P = [2\text{Re}(ab^*)]^2 = [2\pi^2 \text{sech}(\pi \delta) \text{Im}(\xi \eta^*)]^2, \]

where equation (11a), (11b), and (A.3b) have been used. Using equation (A.3a) one finds after simple algebra

\[ P = \{ \text{sech} \pi \delta \text{Im}[e^{i\phi} \cos(\pi(\alpha + i \delta))] \}^2, \quad (15) \]

where

\[ \phi = 2 \arg \left[ \Gamma\left(\frac{1}{4} - \frac{1}{2} \alpha - \frac{1}{2} i \delta\right) \Gamma\left(\frac{1}{4} + \frac{1}{2} \alpha + \frac{1}{2} i \delta\right) \right]. \quad (16) \]

Recalling equation (7) one finally obtains

\[ P = \left[ \text{tanh} \left(\frac{1}{2} \pi \Delta T \right) \sin \left(\frac{1}{2} \pi \Omega_0 T \right) \cos \phi - \cos \left(\frac{1}{2} \pi \Omega_0 T \right) \sin \phi \right]^2. \quad (17) \]

It is readily verified that on resonance (\( \Delta = 0 \)), \( \phi = 0 \) and hence equation (17) gives the correct limit (3), \( P = 0 \).
3. CPI

3.1. Conditions for CPI

When $|\alpha + i\delta|$ is large one can use equation (A.3a), (A.4b) and $\tanh x \sim 1 - e^{-2x}$ to obtain

$$\phi = 2 \arg \left\{ \sin \left[ \pi \left( \frac{1}{4} - \frac{1}{2} \alpha + \frac{1}{2} i \delta \right) \right] \frac{\Gamma \left( \frac{1}{2} + \frac{1}{2} \alpha + \frac{1}{2} i \delta \right)}{\Gamma \left( \frac{1}{2} + \frac{1}{2} \alpha + \frac{1}{2} i \delta \right)} \right\}$$

$$\sim \frac{\pi}{2} + \pi \alpha - \arctan \frac{\delta}{\alpha} - 2e^{-\pi \delta} \cos \pi \alpha + O(e^{-2\pi \delta}, |\alpha + i\delta|^{-2}), \quad (18)$$

and hence

$$P \sim \frac{\Omega_0^2}{\Omega_0^2 + \Delta^2} \left[ 1 - \frac{2\Delta}{\Omega_0} e^{-\pi \Delta / T} \cos \left( \frac{1}{2} \pi \Omega_0 T \right) + O(e^{-2\pi \delta}, |\alpha + i\delta|^{-2}) \right]^2. \quad (19)$$

The transition probability (19) can be represented as a sum of two terms, smooth $P$ and oscillatory $\tilde{P}$,

$$P = \mathcal{P} + \tilde{P}, \quad (20a)$$

$$\mathcal{P} = \frac{\Omega_0^2}{\Omega_0^2 + \Delta^2}, \quad (20b)$$

$$\tilde{P} = -\frac{4\Omega_0 \Delta}{\Omega_0^2 + \Delta^2} e^{-\pi \Delta / T} \cos \left( \frac{1}{2} \pi \Omega_0 T \right). \quad (20c)$$

In the limit of large coupling ($\Omega_0 \gg \Delta$) one finds $\mathcal{P} \to 1$, $\tilde{P} \to 0$, and hence complete inversion, $P \to 1$.

There are two conditions for CPI. **Firstly**, the detuning should be sufficiently large compared to the pulse bandwidth $1/T$, to fulfil the condition $e^{-\pi \delta} = e^{-\pi \Delta / T} \lesssim \varepsilon$, where $\varepsilon$ is a small positive number. **Secondly**, the peak Rabi frequency should be sufficiently larger than the detuning, to fulfil the condition $\mathcal{P} \gtrsim 1 - \varepsilon$. The two CPI conditions read

$$\Delta \gtrsim \frac{2 \ln(1/\varepsilon)}{\pi T}, \quad (21a)$$

$$\Omega_0 \gtrsim \Delta \sqrt{\frac{1}{\varepsilon} - 1}. \quad (21b)$$
Figure 2. The transition probability versus the peak Rabi frequency $\Omega_0$ for detuning $\Delta = 2/T$. The solid curve shows the exact probability (17), the dots the approximation (19), and the dashed curve the average probability (20b).

For $\varepsilon = 0.5, 0.2$ or 0.1 (implying respectively $P \gtrsim 0.5, 0.8$ or 0.9) the CPI conditions are

$$0.44/T \lesssim \Delta \lesssim \Omega_0 \quad (P \gtrsim 0.5), \quad (22a)$$

$$1.02/T \lesssim \Delta \lesssim \frac{1}{2}\Omega_0 \quad (P \gtrsim 0.8), \quad (22b)$$

$$1.47/T \lesssim \Delta \lesssim \frac{1}{3}\Omega_0 \quad (P \gtrsim 0.9). \quad (22c)$$

3.2. Examples

Figure 2 shows the transition probability as a function of the peak Rabi frequency $\Omega_0$ for detuning $\Delta = 2/T$, which satisfies the high-inversion condition (21a). As predicted by equations (22), the average probability $P$ reaches the value of 0.5 when $\Omega_0 \approx \Delta$, and 0.9 when $\Omega_0 \approx 3\Delta$. As $\Omega_0$ increases further, the transition probability approaches unity.

Figure 3 shows the transition probability as a function of the detuning $\Delta$ for a sufficiently large peak Rabi frequency, $\Omega_0 = 30/T$, to allow for a region of high population inversion. As predicted by equations (22), the average probability $P$ increases above 0.9 when $\Delta \gtrsim 1/T$, and then, as the detuning increases, $P$ decreases to 0.9 when $\Delta \approx \frac{1}{3}\Omega_0 = 10/T$, and to 0.5 when $\Delta \approx \Omega_0 = 30/T$.

Figure 4 shows the transition probability in a contour plot as a function of the detuning $\Delta$ and the peak Rabi frequency $\Omega_0$. Two symmetric linearly expanding regions of high population
Figure 3. The transition probability versus the detuning $\Delta$ for a peak Rabi frequency $\Omega_0 = 30/T$. The solid curve shows the exact probability (17), and the dashed curve the average probability (20b).

Figure 4. Contour plot of the transition probability (17) versus the detuning $\Delta$ and the peak Rabi frequency $\Omega_0$.

inversion are identified, one for $\Delta < 0$ and another for $\Delta > 0$. The symmetry stems from the invariance of the Schrödinger equation with respect to the sign of $\Delta$. As predicted by equation (22), the transition probability $P$ is small around resonance ($\Delta \lesssim 1/T$), being exactly zero on exact resonance, equation (3), due to the zero pulse area. The outer borders of the high-inversion regions change linearly, as predicted by conditions (22).

3.3. Discussion

Figures 2, 3, and particularly 4 are reminiscent of adiabatic passage, with broad regions of high transition probability. However, the underlying physical mechanism is not adiabatic passage. As has been discussed in [27], the CPI mechanism is a transition in the adiabatic basis induced by the $\delta$-function behaviour of the nonadiabatic coupling, which derives from the discontinuity of
the Rabi frequency $\Omega(t)$ at the sign jump at $t = 0$. Because the area of the nonadiabatic coupling around this discontinuity is $\arctan(\Omega_0/\Delta)$, and because for a $\delta$-function coupling the energy splitting is irrelevant, the ensuing transition probability is

$$P \approx \sin^2 \arctan \frac{\Omega_0}{\Delta} = \frac{\Omega_0^2}{\Omega_0^2 + \Delta^2},$$

(23)

in complete agreement with the behaviour of the average probability (20b), derived from the exact probability (17). It follows that the average probability (23) should apply universally to any antisymmetric pulse shape, with a sign jump at its maximum. The oscillatory component (20c) derives from interfering transitions in the symmetric tails in the nonadiabatic coupling; it is therefore expected to be shape-specific.

Pulse shapes that change sign smoothly, of the types considered in [27], should typically deliver lower transition probability than shapes with abrupt sign jump, because then the nonadiabatic coupling only approximates $\delta$-function behaviour. The implication is that larger peak Rabi frequencies are needed to simulate this behaviour and make the transition probability approach unity [27].

4. Conclusions

In this paper, an exactly soluble two-state model has been presented. The coupling is a pulse-shaped function, with a sign jump at its maximum. The exact analytic solution is expressible in terms of elementary functions and the phase of a product of $\Gamma$-functions. When the detuning is larger than the pulse bandwidth, and the peak Rabi frequency is sufficiently larger than the detuning, a nearly complete population inversion occurs. The CPI profile has the robustness of adiabatic passage. The underlying physical mechanism, however, is not adiabatic passage, but a $\delta$-function interaction in the adiabatic basis.

The sign jump in the zero-area pulse (2a) can be realized most easily with pulse-shaping femtosecond technology [28]. The latter also allows a virtually perfect sech time dependence to be produced. As follows from equation (22c) the required pulse area for 90% population inversion can be as low as $5\pi$, which is feasible experimentally [30].

Acknowledgments

This work has been supported by the EU ToK project CAMEL (grant no. MTKD-CT-2004-014427) and the EU RTN project EMALI (grant no. MRTN-CT-2006-035369).

Appendix A. Relevant properties of the Gauss hypergeometric function and the Euler $\Gamma$-function

The Gauss hypergeometric equation is [29],

$$z(1 - z)w'' + [\gamma - (\alpha + \beta + 1)z]w' - \alpha\beta w = 0.$$  

(A.1)
Two independent solutions are \( F(\alpha, \beta; \gamma; z) \) and \( z^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma; 2 - \gamma; z) \). The Gauss function has the following properties [29]:

\[
\frac{d}{dz} F(\alpha, \beta; \gamma; z) = \frac{\alpha \beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; z), \quad (A.2a)
\]

\[
\frac{d}{dz} \left[ z^{1-\gamma} F(\alpha, \beta; \gamma; z) \right] = (\gamma - 1) z^{\gamma-2} F(\alpha, \beta; \gamma - 1; z), \quad (A.2b)
\]

\[
F(\alpha, \beta; \gamma; z) = (1 - z)^{\gamma-a-\beta} F(\gamma - \alpha, \gamma - \beta; \gamma; z), \quad (A.2c)
\]

\[
F(\alpha, \beta; \gamma; 0) = 1, \quad (A.2d)
\]

\[
F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \quad [\text{Re}(\gamma - \alpha - \beta) > 0], \quad (A.2e)
\]

\[
F(\alpha, -\alpha; \beta; \frac{1}{2}) = \pi^{1/2} 2^{-\gamma} \Gamma(\gamma) \left[ \frac{1}{\Gamma((\alpha + \gamma)/2) \Gamma((\gamma - \alpha + 1)/2)} \right.
\]

\[
+ \frac{1}{\Gamma((\alpha + \gamma + 1)/2) \Gamma((\gamma - \alpha)/2)}, \quad (A.2f)
\]

\[
F(1 + \alpha, 1 - \alpha; 1 + \gamma; \frac{1}{2}) = \frac{\pi^{1/2} 2^{1-\gamma}}{\alpha} \Gamma(\gamma + 1) \left[ \frac{1}{\Gamma((\alpha + \gamma)/2) \Gamma((\gamma - \alpha + 1)/2)} \right.
\]

\[
- \frac{1}{\Gamma((\alpha + \gamma + 1)/2) \Gamma((\gamma - \alpha)/2} \right]. \quad (A.2g)
\]

The \( \Gamma \)-function obeys the reflection formulae [29]

\[
\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin \pi z}, \quad (A.3a)
\]

\[
\Gamma \left( \frac{1}{2} + i \delta \right) \Gamma \left( \frac{1}{2} - i \delta \right) = \frac{\pi}{\cosh \pi \delta}, \quad (A.3b)
\]

and has the asymptotics expansions [29]

\[
\ln \Gamma(z) \sim \frac{1}{2} \ln 2 \pi + \left( z - \frac{1}{2} \right) \ln z - z + \frac{1}{12z} + O(|z|^{-3}) \quad (|\arg z| < \pi), \quad (A.4a)
\]

\[
\frac{\Gamma(z + a)}{\Gamma(z + b)} \sim z^{a-b} \left[ 1 + \frac{(a + b - 1)(a - b)}{2z} + O(|z|^{-2}) \right] \quad [|\arg(a + z)| < \pi]. \quad (A.4b)
\]
References

[1] Abraham A 1961 *The Principles of Nuclear Magnetism* (Oxford: Clarendon)
Slichter C P 1990 *Principles of Magnetic Resonance* (Berlin: Springer)

[2] Shore B W 1990 *The Theory of Coherent Atomic Excitation* (New York: Wiley)

[3] Nielsen M A and Chuang I L 1990 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[4] Rabi I I 1937 *Phys. Rev.* 51 652

[5] Landau L D 1932 *Proc. R. Soc. Lond.* A 137 696

[6] Rosen N and Zener C 1932 *Phys. Rev.* 40 502

[7] Allen L and Eberly J H 1987 *Optical Resonance and Two-Level Atoms* (New York: Dover)

[8] Hioe F T 1984 *Phys. Rev.* A 30 2100

[9] Bambini A and Berman P R 1981 *Phys. Rev.* A 23 2496

[10] Demkov Y N and Kunike M 1969 Vestn. Leningr. Univ. Fiz. Khim. 16 39

Zakrzewski J 1985 *Phys. Rev.* A 32 3748

[11] Demkov Y N 1964 *Sov. Phys.—JETP* 18 138

Vitanov N V 1993 *J. Phys. B: At. Mol. Opt. Phys.* 26 L53

Vitanov N V 1993 *J. Phys. B: At. Mol. Opt. Phys.* 26 2085 (erratum)

[12] Nikitin E E 1962 *Opt. Spectrosc.* 13 431

Nikitin E E 1970 *Adv. Quantum Chem.* 5 135

Vitanov N V 1994 *J. Phys. B: At. Mol. Opt. Phys.* 27 1791

[13] Carroll C E and Hioe F T 1986 *J. Phys. A: Math. Gen.* 19 3579

[14] Dykhne A M 1960 *Sov. Phys.—JETP* 11 411

Davis J P and Pechukas P 1976 *J. Chem. Phys.* 64 3129

[15] Berman P R, Yan L, Chiam K-H and Sung R 1998 *Phys. Rev.* A 57 79

Vasilev G S and Vitanov N V 2004 *Phys. Rev.* A 70 053407

Vasilev G S and Vitanov N V 2005 *J. Chem. Phys.* 123 174106

[16] Vitanov N V and Suominen K-A 1999 *Phys. Rev.* A 59 4580

[17] Vitanov N V, Halfmann T, Shore B W and Bergmann K 2001 *Annu. Rev. Phys. Chem.* 52 763

[18] Vitanov N V, Shore B W, Yatsenko L P, Böhmer K, Halfmann T, Rickes T and Bergmann K 2001

*Opt. Commun.* 199 117

Halfmann T, Rickes T, Vitanov N V and Bergmann K 2003 *Opt. Commun.* 220 353

[19] Autler S H and Townes C H 1955 *Phys. Rev.* 100 703

[20] Bloch F and Siegent A 1940 *Phys. Rev.* 57 522

[21] Vitanov N V 1994 *J. Phys. B: At. Mol. Opt. Phys.* 27 1351

Vitanov N V and Knight P L 1995 *J. Phys. B: At. Mol. Opt. Phys.* 28 1905

[22] Vitanov N V and Knight P L 1995 *Phys. Rev.* A 52 2245

[23] Agarwal G S and Nayak N 1986 *J. Phys. B: At. Mol. Phys.* 19 3385

Ruyten W M 1989 *Phys. Rev.* A 39 442

Papademetriou S, Van Leeuwen M F and Stroud C R Jr 1996 *Phys. Rev.* A 53 997

Vitanov N V, Yatsenko L P and Bergmann K 2003 *Phys. Rev.* A 68 043401

[24] Agarwal G S and Harshawardhan W 1994 *Phys. Rev.* A 50 R4465

Garraway B M and Vitanov N V 1997 *Phys. Rev.* A 55 4418

Noel M W, Griffith W M and Gallagher T F 1998 *Phys. Rev.* A 58 2265

Cashen M, Rivoire O, Romanenko V, Yatsenko L and Metcalf H 2001 *Phys. Rev.* A 64 063411
[25] Guccione-Gush R and Gush H P 1974 Phys. Rev. A 10 1474
Goreslavskii S P and Krainov V P 1979 Sov. Phys.—JETP 49 13
Zhu Y, Wu Q, Lezama A, Gauthier D J and Mossberg T W 1990 Phys. Rev. A 41 6574
Guérin S, Yatsenko L P and Jauslin H R 2001 Phys. Rev. A 63 031403
Yatsenko L P, Shore B W, Vitanov N V and Bergmann K 2003 Phys. Rev. A 68 043405
Conde A P, Yatsenko L P, Klein J, Oberst M and Halfmann T 2005 Phys. Rev. A 72 053808
[26] Greentree A D, Wei C and Manson N B 1999 Phys. Rev. A 59 4083
[27] Vasilev G S and Vitanov N V 2006 Phys. Rev. A 73 023416
[28] Diels J-C and Rudolph W 1996 Ultrashort Laser Pulse Phenomena: Fundamentals, Techniques, and Applications on a Femtosecond Time Scale (San Diego: Academic)
Wollenhaupt M, Engel V and Baumert T 2005 Annu. Rev. Phys. Chem. 56 25
Brixner T, Pfeifer T, Gerber G, Wollenhaupt M and Baumert T 2005 Femtosecond Laser Spectroscopy ed P Hannaford (New York: Springer) chapter 9
[29] Erdélyi A, Magnus W, Oberhettinger F and Tricomi FG 1953 Higher Transcendental Functions (New York: McGraw-Hill)
Abramowitz M and Stegun I A 1964 Handbook of Mathematical Functions (New York: Dover)
[30] Wollenhaupt M, Assion A, Bazhan O, Horn C, Liese D, Sarpe-Tudoran C, Winter M and Baumert T 2003 Phys. Rev. A 68 015401

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