Wormholes with asymptotic Lifshitz scaling in Hořava gravity

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Abstract. We study static spherically symmetric solutions of the nonprojectable Hořava theory with and without cosmological constant. The solutions we find are two-side wormholes and (single-side) naked singularities. Interestingly, in the case of negative cosmological constant we find that in the exterior side the wormhole acquires an asymptotic scaling between space and time equal to the scaling of the Lifshitz solution, which was previously found to be a vacuum solution of the same theory. This result leads us to pose the question whether in the case of negative cosmological constant the asymptotic anisotropic Lifshitz scaling is a generic feature of the vacuum field equations rather than the asymptotic AdS-like scaling.

1. Introduction

Hořava theory has been proposed as an UV completion of general relativity with the property of being power-counting renormalizable. The foliation-preserving-diffeomorphisms gauge symmetry enables the introduction of higher order spatial derivatives (higher spatial curvature) in the potential without need of introducing higher order time derivatives. This mechanism, in principle, could avoid adding ghosts while rendering the theory renormalizable (careful analysis of the propagating modes is needed in any case).

In this work we study the problem of finding solutions, under a combination of analytical and numerical approaches, of the large-distance effective action of the nonprojectable Hořava theory [1, 2]. We impose three conditions on the set of solutions: staticity, spherical symmetry and vanishing of the shift vector. Unlike General Relativity (GR), in Hořava theory spherical symmetry does not imply staticity on the vacuum solutions. Moreover the asymptotic behavior of the solutions does not need to be the same dictated by GR. Thus, we impose staticity and spherically symmetry independently. Since the cosmological constant represents the lowest-order term in the Lagrangian, it has important consequences on the form of the solutions of the large-distance effective action. We study the cases of negative and vanishing cosmological constant, which can be engulfed into a unique mathematical approach, as we shall see.

In [3] solutions of the Einstein-aether theory without cosmological constant under conditions analog to the ones we impose here were found. This theory is dual to the large-distance effective action of the Hořava theory if the aether vector is hypersurface orthogonal. Those authors found a wormhole-like geometry. In [4] the same problem was focussed directly on the Hořava theory. In [5] a symmetric-wormhole ansatz was studied in Hořava theory.
In this work we shall perform further developments including a negative cosmological constant. Part of our analysis is based on the introduction of a new radial coordinate that helps to clarify the geometry of the solutions. We shall obtain two kinds of solutions: a wormhole-like geometry formed by two sides joined by a throat and a single-side solution with a naked singularity at the origin. We shall pay special attention to the asymptotic behavior of the solutions. We shall see that in the case of the wormhole the asymptotia of its two sides is different, the wormhole is not symmetric. When the cosmological constant vanishes one of the sides is asymptotically flat whereas the other one is asymptotically singular. The asymptotically singular side can be regarded as a kind of internal space. When the cosmological constant is negative, there is again an internal side with singular asymptotia and an asymptotically regular side. It comes out the question on whether the regular side follows the AdS asymptotia or other kind of geometry instead.

2. Finding the solutions

The theory is written in ADM variables. The most general effective action for large distances with cosmological constant is 

\[ S = \int dt d^3x \sqrt{g} N \left( K_{ij} K^{ij} - \lambda K^2 - 2\Lambda + \xi R + \alpha a_i a^i \right), \]

where \( K_{ij} = \frac{1}{2N} \left( g_{ij} - 2\nabla_i (N_j) \right) \), \( K \) is its trace, \( R \) is the spatial Ricci scalar, \( a_i = \partial_i \ln N \) and \( \lambda \), \( \Lambda \), \( \xi \) and \( \alpha \) are independent coupling constants. Under the conditions we are going to use we may set \( \xi = 1 \) without loss of generality. When the value of the coupling constant \( \lambda \) is out of the kinetic conformal point, \( \lambda \neq 1/3 \), the theory propagates three degrees of freedom: the two tensorial modes that are also present in GR plus one extra scalar mode [2, 6, 7]. Sometimes this is believed to be the general situation and several studies are devoted to the physics of the extra mode. However, when \( \lambda = 1/3 \), which is the kinetic conformal point, the theory acquires two extra second-class constraints that eliminate the extra mode completely for all ranges of energy [8]. Another exception that is relevant for large-distance dynamics is the model studied in [9].

Here we present a summary of the approach for solving the field equations under the symmetry conditions we are interested in. The details can be seen in [10]. The static spherically symmetric spacetime metric with vanishing shift vector, \( N_i = 0 \), can be written in time×spherical coordinates as

\[ ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2. \]

From the action (1) we derive the field equations and then evaluate them on this ansatz. We get that the system is well posed in the sense the number of independent equations matches with the number of unknowns (\( N(r) \) and \( f(r) \)). From the resulting field equations the following quadratic equation can be obtained

\[ \left( \sqrt{\frac{1}{\rho N}} + \frac{r \sqrt{\frac{f'}{N'}}}{\rho N} \right)^2 - \left( \frac{\beta r \sqrt{\frac{f'}{N'}}}{\rho N} \right)^2 = 1, \]

where the prime denotes differentiation with respect to \( r \), \( \rho \equiv 1 - \Lambda r^2 \) and \( \beta \equiv \sqrt{1 - \alpha/2} \). We find that for any \( \chi \in (-\infty, +\infty) \) the solution to this quadratic equation can be written as (modulo coordinate transformations)

\[ \frac{\sqrt{\rho}}{\rho} + \frac{r \sqrt{\frac{f'}{N'}}}{\rho N} = \cosh \chi, \quad \frac{\beta r \sqrt{\frac{f'}{N'}}}{\rho N} = \sinh \chi. \]
A further combination of the field equations with this solution, regarding $r$, $N$ and $f$ as functions of $\chi$, yields the following differential equation for $r(\chi)$:

$$\frac{dr}{d\chi} = \left(\frac{1}{\beta} - \frac{\cosh \chi}{\sinh \chi}\right) \frac{r \rho^2}{H},$$

where $H$ is a factor that depends on $\chi$ and $r(\chi)$. The solution $r(\chi)$ of (5) can be interpreted as a coordinate transformation from $\chi$ to $r$. By doing further combinations a differential equation for $N(\chi)$ can be obtained and the radial component $g_{\chi\chi}$ can be algebraically determined in terms of $r(\chi)$. Thus, the search for solutions can be closely casted as a problem of solving first-order differential equations in $\chi$, including the function $r(\chi)$.

When $\Lambda = 0$ the $H$ factor becomes equal to one. In this case (5) can be explicitly solved for $r(\chi)$ [11]. We shall return to this case later on.

Next we perform the numerical integration of Eq.(5) when $\Lambda < 0$ providing several initial data $r(\chi_0)$ for $\chi_0 \in (-\infty, +\infty)$. We find that there are three classes of curves. In Fig. 1 we show the plot of a representative curve for each class.

In all of the three classes the solutions stop at some finite values of $\chi$ where the numerical integration diverges ($dr/d\chi$ diverges). We have classified these points into two kinds of divergence points according to whether $r(\chi)$ remains finite or not there. The point at which also $r(\chi)$ diverges, denoted by $\chi_s$, is fixed for all solutions at the value $\tanh \chi_s = 1/2\beta$. It corresponds to the spatial infinity. We can show analytically that in the $\chi$ coordinate the spatial infinity is reached only at $\chi = \chi_s$ and at $\chi = +\infty$.

The other points of divergence at which $r(\chi)$ remains finite vary their location on each solution. It turns out that there is always a pair of different-class solutions that tend to the same divergence point of this kind (from above and below of the corresponding value of $r$). Thus, we can join such two solutions to form a single extended solution. In Fig. 1 we show how this joining works in the red and blue curves which are different solutions in $\chi$ and stop at the same joining point. We have checked numerically that the joining is continuous in the metric components and their derivatives as well as in the square-curvature scalar (Kretschmann scalar). Therefore, the divergence points where $r(\chi)$ remains finite are nothing but a failure of the $\chi$ coordinate and are the junction points of the extended solution.

Thus, we finally get that there are two kinds of vacuum solutions that satisfy the conditions we have imposed. The most prominent geometric feature of the joined solution is that it has a minimum value of $r(\chi)$ located at $\tanh \tilde{\chi} = \beta$, and it is valid at both sides of the minimum. This means that the solution has a throat at $\tilde{\chi}$. The solution is regular at the throat and, unlike $r$, the coordinate $\chi$ is useful to parameterize the space around the throat in a single chart. The two sides of the throat extend themselves from the throat to spatial infinity. One of the sides...
is completely regular both in its inner points and at its infinite boundary. The other side is regular internally but exhibits an essential singularity at infinity. We regard this side as a kind of internal space. The wormhole is not symmetric between its two sides. We remark that the coordinate transformation from $r$ to $\chi$ was purely spatial, thus there is no place for a horizon in this solution. We have that, except for the infinite boundary of the internal side, the solution is regular and free of horizons; any physical particle can pass through the throat from one side to the other one in both directions, except, again, if it reaches the singular boundary of the internal side. We have checked this regularity numerically with the metric components and their derivatives. We have also checked regularity of the solution at the throat, at the joining point and at the exterior asymptotic boundary by evaluating the square-curvature scalar. We have also checked that the square-curvature scalar diverges at the interior asymptotic boundary.

The other solution is composed of a single spatial side, extends itself from the origin to spatial infinity and at the origin exhibits a naked singularity. All these features have been checked with the metric components and the square-curvature scalar.

When the cosmological constant vanishes all the analysis we have presented in the coordinate $\chi$ is valid, with the advantage that in this case we can find the solution analytically. The solution of (5) is

$$r(\chi) = \frac{ke^{\chi/\beta}}{\sinh \chi},$$

where $k$ is an integration constant. The metric components can also be obtained analytically [11]. In the full range $\chi \in (-\infty, +\infty)$ these expressions comprise again two solutions that are analog to the ones of the $\Lambda < 0$ case: the wormhole with a throat (covered in $\chi \in (0, +\infty)$) and a single-side space with a naked singularity in the origin (covered in $\chi \in (-\infty, 0)$). The wormhole is very similar to the previous case. It has two nonsymmetric sides; the internal one is asymptotically singular whereas the exterior side is asymptotically flat. The solution with the naked singularity at the origin is also asymptotically flat.

3. Asymptotic Lifshitz scaling

Returning to the $\Lambda < 0$ case, now we present the interesting asymptotic behavior of the solutions. In [12] it was argued that Hořava theory, rather than a relativistic theory, provides the minimal holographic dual for Lifshitz-type field theories with anisotropic scaling between space and time. In that study those authors showed that the Lifshitz spacetime is a vacuum solution of the nonprojectable Hořava theory with negative cosmological constant. This spacetime can be written as

$$ds^2 = -\left(\frac{r}{\ell}\right)^2 dt^2 + \left(\frac{\ell}{r}\right)^2 dr^2 + \left(\frac{r}{\ell}\right)^2 dx^i dx^i.$$

The exponent $z$ determines the degree of anisotropy between the time and the spatial direction $r$. In order to be a solution of the large-distance effective Hořava action its geometrical parameters $z$ and $\ell$ must be related to the dynamical coupling constants $\Lambda$ and $\alpha$ of the theory. The relations between constants those authors found can be written in the form (we set 3+1 spacetime dimensions whereas in [12] it is fixed $\ell = 1$)

$$z = \frac{1}{1 - \alpha}, \quad \ell^2 = \frac{(2 - \alpha)(3 - 2\alpha)}{2\Lambda(1 - \alpha)^2}.$$

We stress that the Lifshitz spacetime is not spherically symmetric. In the coordinates of (7) the surfaces of $t$ and $r$ constant are not spheres. However, we find interesting to compare the Lifshitz spacetime with our solutions. To establish the connection we equate the bulk coordinate $r$ of (7) with our original radial coordinate $r$. The asymptotic behavior of our solutions can be investigated analytically in the original system of field equations where the radius $r$ is the
independent variable. To achieve this we assume that if $N$ and $f$ diverge at $r \to \infty$, then they do so with some dominant powers of $r$. That is, we assume that as $r \to \infty$ these components diverge as $N = (r/\ell)^z$ and $f = (r/\ell)^b$, with $z, b, \ell > 0$. Under this assumption, the asymptotic analysis of the field equations derived from (1) (see [10] for details) shows that necessarily the form of the asymptotic solution in the time-racial sector is dictated by the scaling of the Lifshitz solution; that is, necessarily $b = 2$ and $z$ and $\ell$ are fixed in terms of $\alpha$ and $\Lambda$ exactly as in (8). This is an analytical result. Now, in the case of the wormhole we have that only the exterior side is asymptotically regular. Therefore, in the exterior side there is a place for the asymptotic Lifshitz scaling (8). We have checked on the numerical solutions that effectively this is the case. The same holds for the asymptotic limit of the solution with the naked singularity at the origin.

Our results on the asymptotic behavior and the results of [12] lead us to ask whether this Lifshitz scaling, rather that the relativistic AdS scaling, is a generic feature of this family of vacuum solutions of the Horava theory with a negative cosmological constant. That is, there could be general ansatz for geometries where a bulk coordinate $r$ can be picked up. Then the generic asymptotic behavior could be a Lifshitz scaling in the form $g_{tt} \sim -r^{2z}$ and $g_{rr} \sim r^{-2}$, with $z$ given in terms of $\alpha$ by (8).

4. Conclusions

The vacuum solutions of the large-distance effective action of complete nonprojectable Hořava gravity with cosmological constant $\Lambda \leq 0$ that satisfy the conditions of staticity, spherical symmetry and vanishing of the shift vector are two: a wormhole of two sides and a space with a naked singularity at the origin. We have shown that both the exterior side of the wormhole and the other solution acquire a kind of Lifshitz scaling asymptotically. Surprisingly, the dynamical exponent $z$ (adopting the terminology of statistical physics) and even the coefficient of the “longitudinal” component of the metric coincide exactly with that of the Lifshitz spacetime when this is casted as a vacuum solution of the theory. It would be interesting to elucidate whether in this theory this a general behavior for solutions that possess common structures in the set of isometries and noncompact directions. A related analysis on certain solutions of the Hořava theory with asymptotic Lifshitz scaling has been recently done in 2 + 1 dimensions in [13]. It would be also interesting to study the case with positive cosmological constant, although likely other coordinate systems are more appropriate in this case.

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