Fermionic current induced by magnetic flux in compactified cosmic string spacetime

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Abstract

In this paper, we investigate the fermionic current densities induced by a magnetic flux running along the idealized cosmic string in a four-dimensional spacetime, admitting that the coordinate along the string’s axis is compactified. In order to develop this investigation we construct the complete set of fermionic mode functions obeying a general quasiperiodicity condition along the compactified dimension. The vacuum expectation value of the azimuthal current density is decomposed into two parts. The first one corresponds to the uncompactified cosmic string geometry and the second one is the correction induced by the compactification. For the first part we provide a closed expression which includes various special cases previously discussed in the literature. The second part is an odd periodic function of the magnetic flux along the string axis with the period equal to the flux quantum and it is an even function of the magnetic flux enclosed by the string axis. The compactification of the cosmic string axis in combination with the quasiperiodicity condition leads to the nonzero axial current density. The latter is an even periodic function of the magnetic flux along the string axis and an odd periodic function of the magnetic flux enclosed by the string axis. The axial current density vanishes for untwisted and twisted fields in the absence of the magnetic flux enclosed by the string axis. The asymptotic behavior of the vacuum fermionic current is investigated near the string and at large distances from it. In particular, the topological part of the azimuthal current and the axial current are finite on the string’s axis.

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1 Introduction

In quantum field theory, because of quantum nature of the vacuum state, its properties depend crucially on both the local geometry and topology of the background spacetime. In particular, the nontrivial topology leads to the change of the vacuum energy. This is the topological Casimir effect which has been investigated for large number of geometries (see, for instance,
In higher-dimensional models with compact extra dimensions, the dependence of the vacuum energy on the length scale of the compact subspace provides a stabilization mechanism for moduli fields. In addition, the topological Casimir effect can be a source of dark energy driving the accelerating expansion of the Universe at recent epoch \([2]\). In the present paper we consider an exactly solvable problem for a charged massive fermionic field with two types of the topological vacuum polarization. The first one is generated by a planar angle deficit in the cosmic string geometry and the second one is induced by the compactification of the string axis. Recently, the calculations of topological Casimir densities associated with a quantum scalar field in compactified cosmic string spacetime, has been developed in \([3]\).

The cosmic strings are among the most interesting topological defects which may have been created by phase transitions in the early Universe \([4]\). Though the observation data on the cosmic microwave background have ruled out cosmic strings as primary source for primordial density perturbations, they are still candidate for the generation of a number of interesting physical effects such as gamma ray bursts \([5]\), gravitational waves \([6]\) and high energy cosmic rays \([7]\). Recently, cosmic strings have attracted renewed interest, partly because a variant of their formation mechanism is proposed in the framework of brane inflation \([8]-[10]\).

The geometry of the spacetime produced by an idealized infinite straight cosmic has a conical structure. It is locally flat except on the top of the string where it has a delta shaped curvature tensor. This nontrivial structure raises a number of interesting physical effects. One of these concerns the effect of a string on the properties of quantum vacuum. Explicit calculations for vacuum expectation values of the energy-momentum tensor associated with various fields in the vicinity of a string have been done in \([11]-[25]\). Moreover, considering the presence of a magnetic flux running along the cosmic strings, there appear additional contributions to the corresponding vacuum polarization effects associated with charged fields \([26]-[30]\). The presence of a magnetic flux along the cosmic string induces vacuum current densities, as well. This phenomenon has been investigated for massless scalar field in \([32]\) and recently for massive one in \([33]\). There the authors have shown that the induced vacuum current densities along the azimuthal direction appear if the ratio of the magnetic flux by the quantum one has a nonzero fractional part. The analysis of induced fermionic currents in higher-dimensional cosmic string spacetime in the presence of a magnetic flux have been developed in \([34]\). Moreover, induced fermionic current by a magnetic flux in \((2 + 1)\)-dimensional conical spacetime and in presence of a circular boundary has also been analyzed in \([35]\) (for combined effects of topology and boundaries on the quantum vacuum in the geometry of a cosmic string see \([36]\)).

Another type of topological quantum effects takes place in models which present compact spatial dimensions. The presence of compact dimensions is an important feature of most high-energy theories of fundamental physics, including supergravity and superstring theories. An interesting application of the field theoretical models with compact dimensions recently appeared in nanophysics. The long-wavelength description of the electronic states in graphene can be formulated in terms of the Dirac-like theory in three-dimensional spacetime with the Fermi velocity playing the role of speed of light (see, e.g., \([37]\)). Single-walled carbon nanotubes are generated by rolling up a graphene sheet to form a cylinder and the background spacetime for the corresponding Dirac-like theory has topology \(R^2 \times S^1\). In this paper we shall analyze the influence of compactification of the spatial dimension along the cosmic string’s axis, on the vacuum expectation value (VEV) of the fermionic current induced by a magnetic flux running along this axis and by the magnetic flux enclosed by the string axis. This VEV is among the most important local characteristics of the quantum vacuum. In addition to describing the physical structure of a charged quantum field at a given point, the current acts as the source in the Maxwell equations and plays an important role in modeling a self-consistent dynamics

\(^{1}\)Recently the fluxes by gauge fields play an important role in higher-dimensional models including braneworld scenarios (see, for example, \([31]\)).
involving the electromagnetic field.

We have organized the paper as follows. In Section 2 we present the background geometry associated with the spacetime under consideration and construct the complete set of normalized positive- and negative-energy fermionic wave-functions obeying quasiperiodic boundary condition with an arbitrary phase along the string axis. In Section 3 by using the mode-summation method, we evaluate the renormalized fermionic vacuum current density induced by a magnetic flux running along the string’s axis. As we shall see, the charge density and the radial current vanish. The azimuthal current density is decomposed into two parts: the first one corresponding to the geometry of a cosmic string without compactification and the second one being induced by the compactification. As a consequence of the string axis compactification, there appears a non-vanishing axial current. This is a purely topological effect and is investigated in Section 4 for the general case of periodicity conditions along the compact dimension. In addition, we assume the presence of a constant gauge field with the non-vanishing component along the string axis. The most relevant conclusions of the paper are summarized in Section 5. Throughout the paper we use the units with $G = h = c = 1$.

2 Fermionic wave-functions

The main objective of this section is to obtain the complete set of normalized fermionic wave-functions in a four-dimensional cosmic string spacetime considering that these fields obey a quasiperiodicity condition along the string’s axis. This set is needed for the calculation of the VEVs of the fermionic current densities by using the mode-summation approach.

The background geometry of the spacetime corresponding to a cosmic string along the $z$-axis, can be given, by using cylindrical coordinates, through the line element below:

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2,$$

where the coordinates take values in the ranges $r \geq 0$, $0 \leq \phi \leq \phi_0 = 2\pi/q$, $-\infty < t < +\infty$. The parameter $q$, which codify the planar angle deficit, is related to the mass per unit length of the string, $\mu_0$, by $q^{-1} = 1 - 4\mu_0$. Additionally we shall assume that the direction along the $z$-axis is compactified to a circle with the length $L$: $0 \leq z \leq L$.

The quantum dynamic of a massive charged spinor field in curved spacetime and in the presence of an electromagnetic four-vector potential, $A_\mu$, is governed by the Dirac equation

$$i\gamma^\mu(\nabla_\mu + ieA_\mu)\psi - m\psi = 0 \ , \ \nabla_\mu = \partial_\mu + \Gamma_\mu \ ,$$

where $\gamma^\mu$ are the Dirac matrices in curved spacetime and $\Gamma_\mu$ is the spin connection. Both matrices are given in terms of the flat spacetime Dirac matrices, $\gamma^{(a)}$, by the relations,

$$\gamma^\mu = e^\mu_{(a)}\gamma^{(a)} \ , \ \Gamma_\mu = \frac{1}{4}\gamma^{(a)}\gamma^{(b)}e^\nu_{(a)e^{(b)})\nu;\mu} \ .$$

In (2.3), $e^\mu_{(a)}$ represents the tetrad basis satisfying the relation $e^\mu_{(a)}e^{\nu}_{(b)}\eta^{ab} = g^{\mu\nu}$, with $\eta^{ab}$ being the Minkowski spacetime metric tensor. We assume that along the compact dimension the field obeys the quasiperiodicity condition

$$\psi(t, r, \phi, z + L) = e^{2\pi i\beta}\psi(t, r, \phi, z) \ ,$$

with a constant phase $\beta$, $0 \leq \beta \leq 1$. The special cases $\beta = 0$ and $\beta = 1/2$ correspond to the periodic and antiperiodic boundary conditions (untwisted and twisted fields respectively).

In the discussion below we admit the existence of a constant gauge field with the vector potential

$$A_\mu = (0, 0, A_\phi, A_z) \ .$$
The component \( A_\phi \) is related to an infinitesimal thin magnetic flux, \( \Phi_2 \), running along the string by \( A_\phi = -q\Phi_2/(2\pi) \). Similarly, the axial component \( A_z \) can be given in terms of the magnetic flux \( \Phi_3 \) enclosed by the \( z \)-axis by the relation \( A_z = -\Phi_3/L \). Though the magnetic field strength corresponding to \( (2.5) \) vanishes, the nontrivial topology of the background geometry leads to Aharonov-Bohm-like effects on the VEVs of physical observables.

In order to find the complete set of mode functions in the problem under consideration, we shall use the standard representation of the flat space Dirac matrices:

\[
\gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{(a)} = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \quad a = 1, 2, 3,
\]

with \( \sigma_1, \sigma_2, \sigma_3 \) being the Pauli matrices. We take the tetrad basis as follows:

\[
e_{(a)}^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(q\phi) & -\sin(q\phi)/r & 0 \\ 0 & \sin(q\phi) & \cos(q\phi)/r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

(2.7)

where the index \( a \) identifies the rows of the matrix. With this choice, the gamma matrices take the form

\[
\gamma^0 = \gamma^{(0)}, \quad \gamma^l = \begin{pmatrix} 0 & \sigma^l \\ -\sigma^l & 0 \end{pmatrix},
\]

(2.8)

where we have introduced the \( 2 \times 2 \) matrices for \( l = (r, \phi, z) \):

\[
\sigma^r = \begin{pmatrix} 0 & e^{-iq\phi} \\ e^{iq\phi} & 0 \end{pmatrix}, \quad \sigma^\phi = -\frac{i}{r} \begin{pmatrix} 0 & e^{-iq\phi} \\ -e^{iq\phi} & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(2.9)

For the spin connection and the combination appearing in the Dirac equation we find

\[
\Gamma_\mu = \frac{1 - q^2}{2r} \gamma^{(1)} \gamma^{(2)} \delta^\mu_\phi, \quad \gamma^\mu \Gamma_\mu = \frac{1 - q^2}{2r} \gamma^r,
\]

(2.10)

and the Dirac equation takes the form

\[
\left( \gamma^\mu \left( \partial_\mu + ieA_\mu \right) + \frac{1 - q^2}{2r} \gamma^r + im \right) \psi = 0.
\]

(2.11)

For positive energy solutions, assuming the time-dependence of the eigenfunctions in the form \( e^{-iEt} \) and decomposing the spinor \( \psi \) into the upper and lower components, denoted by \( \psi^+ \) and \( \psi^- \), respectively, we find the equations

\[
\left( \sigma^l \left( \partial_l + ieA_l \right) + \frac{1 - q^2}{2r} \sigma^r \right) \psi^+ - i \left( E + m \right) \psi^- = 0,
\]

\[
\left( \sigma^l \left( \partial_l + ieA_l \right) + \frac{1 - q^2}{2r} \sigma^r \right) \psi^- - i \left( E - m \right) \psi^+ = 0.
\]

(2.12)

Substituting the function \( \psi^- \) from the first equation into the second one, we obtain the second order differential equation for the spinor \( \psi^+ \):

\[
\left[ \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \left( \partial_\phi + ieA_2 - \frac{1 - q^2}{2} \sigma^z \right)^2 + (\partial_z + ieA_3)^2 + E^2 - m^2 \right] \psi^+ = 0.
\]

(2.13)

The same equation is obtained for the spinor \( \psi^- \).
In order to look for solution for \((2.13)\), we use the the ansatz below, compatible with the cylindrical symmetry of the physical system:

\[
\psi^+ = e^{ikz} \begin{pmatrix} C_1 R_1(r)e^{iqn_1\phi} \\ C_2 R_2(r)e^{iqn_2\phi} \end{pmatrix},
\]

\((2.14)\)

with \(n_l\), for \(l = 1, 2\), being integer numbers and \(C_1\) and \(C_2\) are two arbitrary constants. Substituting this function into \((2.13)\), we can see that the solutions of the equations for the radial functions, regular on the string, are expressed in terms of the Bessel function of the first kind: \(R_l(r) = J_{|\nu_l|}(\lambda r)\), where the order is given below,

\[
\nu_1 = q(n_1 + 1/2) + eA_2 - 1/2, \quad \nu_2 = q(n_2 - 1/2) + eA_2 + 1/2,
\]

\((2.15)\)

and the variable \(\lambda\) is related with the energy by,

\[
\lambda = \sqrt{E^2 - \tilde{k}_+^2 - m^2},
\]

\((2.16)\)

where \(\tilde{k}_+ = k + eA_3\). We may see that \(|\nu_2| = |\nu_1| + \epsilon_{\nu_1}\), where \(\epsilon_{\nu_1}\) is equal to +1 for \(\nu_1 \geq 0\) and -1 for \(\nu_1 < 0\).

Having the upper two-component spinor, we can find the components of the lower one, \(\psi^-\), by using the first equation in \((2.12)\). From this equation we find the following relations,

\[
\psi^- = e^{ikz} \begin{pmatrix} B_1 R_1(r)e^{iqn_1\phi} \\ B_2 R_2(r)e^{iqn_2\phi} \end{pmatrix},
\]

\((2.17)\)

with

\[
n_2 = n_1 + 1.
\]

\((2.18)\)

The coefficients \(B_1\) and \(B_2\) are related with \(C_1\) and \(C_2\) by,

\[
B_1 = \frac{1}{E + m} \left( C_1 \tilde{k}_+ - iC_2 \epsilon_{\nu_1} \lambda \right), \\
B_2 = \frac{1}{E + m} \left( iC_1 \lambda \epsilon_{\nu_1} - C_2 \tilde{k}_+ \right).
\]

\((2.19)\)

Finally we can write the positive energy solution in its complete form:

\[
\psi = \begin{pmatrix} C_1 J_{|\nu_1|}(\lambda r) \\ C_2 J_{|\nu_2|}(\lambda r)e^{iq\phi} \end{pmatrix} e^{ikz+iqn\phi-iEt},
\]

\((2.20)\)

where we have defined \(n_1 = n\). We can see that \((2.20)\) is an eigenfunction of the total angular momentum along the cosmic string:

\[
\tilde{J}_3 \psi = \left( -i\partial_\phi + \frac{q}{2} \gamma^{(1)}(2) \right) \psi = q_j \psi,
\]

\((2.21)\)

where

\[
j = n + 1/2, \quad j = \pm1/2, \pm3/2, \ldots.
\]

\((2.22)\)

The fermionic wave-function above contains four coefficients and there are two equations relating them. The normalization condition on the functions provides an extra equation. Consequently, one of the coefficients remains arbitrary. In order to determine this coefficient some additional condition should be imposed on the coefficients. The necessity for this condition is
related to the fact that the quantum numbers \((\lambda, k, j)\) do not specify the fermionic wave-function uniquely and some additional quantum number is required.

In order to specify the second constant we impose the condition
\[
C_1/B_1 = -C_2/B_2 .
\]  
(2.23)

By taking into account (2.19), we can write,
\[
C_2 = sC_1 , \quad B_1 = -sB_2 = \frac{\tilde{k}_+ - is\epsilon\nu_1\lambda}{E + m}C_1 , \quad s = \pm 1 .
\]  
(2.24)

With the condition (2.23), the fermionic mode functions are uniquely specified by the set of quantum numbers \(\sigma = (\lambda, k, j, s)\). The eigenvalues of the quantum number \(k\) are determined from the periodicity condition (2.4):
\[
k = k_l^{(+)} = 2\pi(l + \beta)/L , \quad l = 0, \pm 1, \pm 2, \ldots .
\]  
(2.25)

On the basis of all these considerations, the positive-energy fermionic wave function is written in the form
\[
\psi_{\sigma}^{(+)}(x) = C_{\sigma}^{(+)} \left( \begin{array}{c} J_{\beta_1}(\lambda r) \\ sJ_{\beta_j+\epsilon_j}(\lambda r)e^{iq\phi} \\ \frac{s}{E_+ + m} J_{\beta_2}(\lambda r) \\ -s\frac{s}{E_+ + m} J_{\beta_j+\epsilon_j}(\lambda r)e^{iq\phi} \end{array} \right) e^{ik_l^{(+)}z + iq(j-1/2)\phi - iE_+ t} ,
\]  
(2.26)

where \(\epsilon_j = 1\) for \(j > -\alpha\) and \(\epsilon_j = -1\) for \(j < -\alpha\), and
\[
\beta_j = q|j + \alpha| - \epsilon_j/2.
\]  
(2.27)

The energy is expressed in terms of \(\lambda\) and \(\tilde{k}_+\) by the relation
\[
E_+ = \sqrt{\lambda^2 + \tilde{k}_+^2 + m^2} ,
\]  
(2.28)

where \(\tilde{k}_+ = 2\pi(l + \tilde{\beta})/L\), with
\[
\tilde{\beta} = \beta + eA_3 L/(2\pi) = \beta - \Phi_3/\Phi_0,
\]  
(2.29)

and with \(\Phi_0 = 2\pi/e\) being the flux quantum. In (2.27) we have defined
\[
\alpha = eA_2/q = -\Phi_2/\Phi_0,
\]  
(2.30)

The constant \(C_{\sigma}^{(+)}\) is found from the normalization condition
\[
\int d^3x \sqrt{\gamma} (\psi_{\sigma}^{(+)}(x)\bar{\psi}_{\sigma'}^{(+)}(x) = \delta_{\sigma\sigma'} ,
\]  
(2.31)

where \(\gamma\) is the determinant of the spatial metric tensor. The delta symbol on the right-hand side is understood as the Dirac delta function for continuous quantum numbers \((\lambda)\) and the Kronecker delta for discrete ones \((k, j, s)\). From (2.31) one finds
\[
|C_{\sigma}^{(+)}|^2 = \frac{q\lambda(E_+ + m)}{8\pi LE_+} .
\]  
(2.32)

The physical results will depend on the phases in the periodicity condition (2.4) and on the component of the gauge potential along the axis of the string in the form of the combination
The negative-energy fermionic mode-function can be obtained in a similar way. The corresponding result is given by the expression:

\[
\psi^{(-)}(x) = C^{(-)} \begin{pmatrix}
J_{\beta_j + \epsilon_j}(\lambda r) e^{-iq\phi} \\
\frac{-k_- - is_\epsilon \lambda}{E_- - m} J_{\beta_j + \epsilon_j}(\lambda r) e^{-iq\phi} \\
\frac{k_- + is_\epsilon \lambda}{E_- - m} J_{\beta_j}(\lambda r)
\end{pmatrix} e^{iE_- t - i(q(j-1)/2)\phi - ik^{-1}_j z} ,
\]  

with \( k^{-1}_j = 2\pi(l - \beta)/L, l = 0, \pm 1, \pm 2, \ldots, \tilde{k}_- = 2\pi(l - \tilde{\beta})/L, \) and

\[
E_- = \sqrt{\lambda^2 + \tilde{k}_-^2 + m^2}.
\]  

The normalization constant is determined by the relation

\[
|C^{(-)}|^2 = \frac{q\lambda(E_- - m)}{8\pi LE_-}.
\]  

The wave-functions obtained in this section can be used for the investigation of vacuum fermionic current densities induced by the presence of the magnetic flux and also by the compactification along the string’s axis.

In the discussion above we have imposed the regularity condition on the fermionic wave-functions at the cone apex. As it is well known, the theory of von Neumann deficiency indices leads to a one-parameter family of allowed boundary conditions in the background of an Aharonov-Bohm gauge field \[38\]. In addition to the regular modes, these boundary conditions, in general, allow normalizable irregular modes. The VEV of the fermionic current density for general boundary conditions on the cone apex is evaluated in a way similar to that described below. The contribution of the regular modes is the same for all boundary conditions and the results differ by the parts related to the irregular modes. A special case of boundary conditions has been discussed in \[39\], where the Atiyah-Patodi-Singer type nonlocal boundary condition is imposed at a finite radius, which is then taken to zero. Similar approach, with the MIT bag boundary condition, has been used in Refs. \[35\] – \[40\] for a two-dimensional conical space with a circular boundary. Note that in recent investigation of the induced fermionic current for a massless Dirac field in (2+1) dimensions, carried out in \[41\], the authors impose the regularity condition. It was shown that the corresponding result coincides with the result for a finite radius solenoid, assuming that an electron cannot penetrate the region of nonzero magnetic field.

### 3 Fermionic current

The VEV of the fermionic current density, \( j^\mu = e\bar{\psi}\gamma^\mu\psi \), can be evaluated by using the mode sum formula,

\[
\langle j^\mu(x) \rangle = e \sum_\sigma \psi^{(-)}_\sigma(x) \gamma^\mu \psi^{(-)}_\sigma(x) ,
\]  

where we are using the compact notation defined below,

\[
\sum_\sigma = \int_0^\infty d\lambda \sum_{l=\infty}^{+\infty} \sum_{s=\pm 1}^{4\infty} \sum_{j=\pm 1/2}^{4\infty} .
\]
This VEV is a periodic function of the fluxes $\Phi_2$ and $\Phi_3$ with the period equal to the flux quantum. In particular, if we write the parameter $\alpha$ in (2.30) in the form

$$\alpha = n_0 + \alpha_0, \quad |\alpha_0| < 1/2,$$

(3.3)

where $n_0$ is an integer number, the VEV of the current density will depend on $\alpha_0$ only. Note that, for the boundary condition at the cone apex used in [35], there are no square integrable irregular modes for $|\alpha_0| \leq (1 - 1/q)/2$.

### 3.1 Charge density and radial current

Let us start the calculation of charge density,

$$\rho(x) = \langle j^0(x) \rangle = e \sum_\sigma \psi_\sigma^{(-)} \psi_\sigma^{(-)}.$$

(3.4)

Substituting (2.33) and (2.35) into (3.4), we obtain

$$\rho(x) = eq \frac{4\pi L}{e} \sum_\sigma \lambda \left[ J_{2\beta_j}(\lambda r) + J_{2\beta_j + \epsilon_j}(\lambda r) \right].$$

(3.5)

Of course, this expression is divergent. In order to regularize it we introduce a cutoff function $e^{-\eta k^2}$, with the cutoff parameter $\eta > 0$. At the end of the calculation we take the limit $\eta \to 0$. With the cutoff function, the integral can be evaluated using the result from [42]. So, the regularized contribution due to the integral over $\lambda$ gives us:

$$\int_0^\infty d\lambda \lambda e^{-\eta \lambda^2} \left[ J_{2\beta_j}(\lambda r) + J_{2\beta_j + \epsilon_j}(\lambda r) \right] = \frac{1}{2\eta} e^{-r^2/(2\eta)} \left[ I_{\beta_j}(r^2/(2\eta)) + I_{\beta_j + \epsilon_j}(r^2/(2\eta)) \right],$$

(3.6)

with $I_{\nu}(z)$ being the modified Bessel function. As a result, the regularized charge density reads:

$$\rho_{\text{reg}}(x, \eta) = \frac{eq}{4\pi \eta L} \sum_{l=-\infty}^{+\infty} e^{-\eta k_{l}^2} \left[ I(q, -\alpha_0, r^2/(2\eta)) + I(q, \alpha_0, r^2/(2\eta)) \right].$$

(3.7)

Here the expression for the regularized charge density is given in terms of the series

$$I(q, \alpha_0, z) = \sum_j I_{\beta_j}(z) = \sum_{n=0}^\infty [I_{q(n+\alpha_0+1/2)-1/2}(z) + I_{q(n-\alpha_0+1/2)+1/2}(z)],$$

(3.8)

and

$$\sum_j I_{\beta_j + \epsilon_j}(z) = I(q, -\alpha_0, z).$$

(3.9)

Here and below

$$\sum_j = \sum_{j=\pm 1/2, \ldots}.$$

(3.10)

An equivalent representation for the charge density can be obtained by using the integral representation below derived in [35]:

$$I(q, \alpha_0, z) = \frac{e^z}{q} - \frac{1}{\pi} \int_0^\infty dy \frac{e^{-z \cosh y f(q, \alpha_0, y)}}{\cosh(qy) - \cos(q\pi)}$$

$$+ \frac{2}{q} \sum_{k=1}^{p} (-1)^k \cos[2\pi k(\alpha_0 - 1/2q)] e^z \cos(2\pi k/q),$$

(3.11)
with \( 2p < q < 2p + 2 \) and with the notation

\[
\begin{align*}
    f(q, \alpha_0, y) &= \cos [q \pi (1/2 - \alpha_0)] \cosh [(q \alpha_0 + q/2 - 1/2) y] \\
    &\quad - \cos [q \pi (1/2 + \alpha_0)] \cosh [(q \alpha_0 - q/2 - 1/2) y]. \tag{3.12}
\end{align*}
\]

In the case \( q = 2p \), the term

\[
- (-1)^{q/2} e^{-z} \frac{e^{-z}}{q} \sin(\pi q \alpha_0), \tag{3.13}
\]

should be added to the right-hand side of Eq. (3.11). For \( 1 \leq q < 2 \), the last term on the right-hand side of Eq. (3.11) is absent.

Note that for integer values of \( q \) and for \( \alpha_0 = 1/2 - n + \frac{1}{2} q \), one has \( f(q, \alpha_0, y) = 0 \). From the condition \( |\alpha_0| < 1/2 \) we find

\[
0 \leq n < q - 1/2. \tag{3.15}
\]

In this case we can see that the functions \( I(q, \pm \alpha_0, z) \) are presented in equivalent forms:

\[
I(q, \alpha_0, z) = 1 + \frac{q}{2} \sum_{k=0}^{q-1} \cos(2\pi k (n + 1)/q) e^{z \cos(2\pi k/q)}, \tag{3.16}
\]

For a general \( q \), by making use of the formula (3.11), the regularized charge density is presented in the form

\[
\rho_{\text{reg}}(x, \eta) = \frac{ee^{-z}}{2\pi \eta L} \sum_{l=-\infty}^{+\infty} e^{-\eta k_l (-)^l} \left[ e^{z} + \frac{q}{\pi} \int_{0}^{\infty} dy \frac{\sinh (y/2) e^{-z} \cosh y h(q, \alpha_0, y)}{\cosh(qy)} - \cos(q\pi) \right] \\
+ 2 \sum_{k=1}^{p} (-1)^k c_k \cos (2\pi k \alpha_0) e^{z \cos(2\pi k/q)} \tag{3.17}
\]

with \( z = r^2/(2\eta) \). In this representation we have introduced the notations

\[
c_k = \cos (\pi k/q), \tag{3.18}
\]

and

\[
h(q, \alpha_0, y) = \cos [q \pi (1/2 + \alpha_0)] \sinh [q (1/2 - \alpha_0) y] \\
+ \cos [q \pi (1/2 - \alpha_0)] \sinh [q (1/2 + \alpha_0) y]. \tag{3.19}
\]

The first term in the square brackets of (3.17) corresponds to the charge density for \( \alpha_0 = 0 \) and \( q = 1 \). The renormalized value for this part vanishes (see Ref. [43] for a general case of spatial topology \( R^p \times (S^1)^q \) and Ref. [44] for the corresponding current densities in de Sitter spacetime). The other contributions contain \( e^{-r^2 \cosh^2(y/2)/\eta} \) and \( e^{-r^2 \sin^2(\pi k/q)/\eta} \), inside the integral and summation respectively; hence in the limit \( \eta \to 0 \) these terms vanish for \( r > 0 \). So, we conclude that the renormalized value for the charge density is zero, i.e., there is no induced charge density.
As to the VEV of the radial current, it is given by the expression

\[ \langle j^r(x) \rangle = e \sum_{\sigma} \psi_\sigma^\dagger(-)^2 \gamma^r \gamma^0 \psi_\sigma(-) . \]  

(3.20)

Substituting (2.33) and the Dirac matrices given in (2.8) and (2.9) in the right-hand side of (3.20), we can easily see that there appears a cancellation between all terms. Consequently there is also no induced radial current density.

3.2 Azimuthal current

The VEV of the azimuthal current is given by

\[ \langle j^\phi(x) \rangle = e \sum_{\sigma} \psi_\sigma^\dagger(-)^2 \gamma^0 \gamma^\phi \psi_\sigma(-) . \]  

(3.21)

Substituting the expression for the negative-energy solution for the fermionic field (2.33) and the corresponding expressions for the Dirac matrices in this background, given in (2.8) and (2.9), into the expression inside the summations of (3.21), after the redefinition \( l \to -l \), we obtain:

\[ \langle j^\phi \rangle = -eq \frac{2\pi \epsilon_j J_{\beta_j}(\lambda r) J_{\beta_j + \epsilon_j}(\lambda r)}{[2\pi(l - \tilde{\beta})/L]^2 + \lambda^2 + m^2} . \]  

(3.22)

We assume the presence of a cutoff function without writing it explicitly. The specific form of this function is not needed in the further discussion.

The summation over the quantum number \( s \) in (3.22) provides a factor 2. In order to develop the summation over \( l \), we shall apply the Abel-Plana summation formula in the form (for generalizations of the Abel-Plana formula see [43])

\[ \sum_{l=-\infty}^{\infty} g(l + \tilde{\beta}) f(|l + \tilde{\beta}|) = \int_0^\infty du \left[ g(u) + g(-u) \right] f(u) \]

\[ + i \int_0^\infty du \left[ f(iu) - f(-iu) \right] \sum_{\lambda = \pm 1} \frac{g(i\lambda u)}{e^{2\pi u + i\lambda \tilde{\beta}}} - 1 , \]  

(3.23)

taking \( g(u) = 1 \) and

\[ f(u) = \frac{1}{\sqrt{(2\pi u/L)^2 + \lambda^2 + m^2}} . \]  

(3.24)

As a result, the induced azimuthal current is decomposed as,

\[ \langle j^\phi \rangle = \langle j^\phi \rangle_s + \langle j^\phi \rangle_c , \]  

(3.25)

where the term \( \langle j^\phi \rangle_s \) is due to the contribution of the first integral in the right-hand side of (3.23) and corresponds to the axial current density in the geometry of a cosmic string without compactification. The part \( \langle j^\phi \rangle_c \) is induced by the compactification of the string along its axis. As we shall see the latter vanishes in the limit \( L \to \infty \).

The calculation of the induced azimuthal current in the geometry of a straight cosmic string has been developed before by many authors considering massless field. For massive field the expression is provided in a closed form for the special case where \( q \) is an integer and \( \alpha_0 \) given by (3.14) with \( n = 0 \) [34]; however, to our knowledge, a closed expression for the induced azimuthal current considering general values of the parameters is missed. So, in order to fill this blank, we
In order to provide a more workable expression, we use the identity
\[ \lambda \]
Substituting this identity into (3.26), the next step is to develop the integral over the variable \( q \). For \( q < 1 \), we decided to include this calculation in the present paper. Combining (3.22) and (3.23), we get the representation
\[ \langle j^{\phi} \rangle_s = -\frac{eq}{\pi^2r^4} \int_0^\infty \int_0^\infty \frac{\epsilon_j J_{\beta_j}(\lambda r)J_{\beta_j+\epsilon_j}(\lambda r)}{\sqrt{m^2 + k^2 + \lambda^2}} \, d\lambda^2 \, dk \sum_j \epsilon_j. \]  
(3.26)

In order to provide a more workable expression, we use the identity
\[ \frac{1}{\sqrt{m^2 + k^2 + \lambda^2}} = \frac{2}{\sqrt{\pi}} \int_0^\infty dt \, e^{-(m^2+k^2+\lambda^2)t^2}. \]  
(3.27)

Substituting this identity into (3.26), the next step is to develop the integral over the variable \( \lambda \). With the help of [42], we can write,
\[ \int_0^\infty d\lambda^2 e^{-\lambda^2 t^2} J_{\beta_j}(\lambda r)J_{\beta_j+\epsilon_j}(\lambda r) = \frac{e^{-r^2/(2t^2)}}{4t^4} r \epsilon_j [I_{\beta_j}(r^2/(2t^2)) - I_{\beta_j+\epsilon_j}(r^2/(2t^2))]. \]  
(3.28)

Introducing a new variable \( y = r^2/(2t^2) \), we obtain
\[ \langle j^{\phi} \rangle_s = \frac{eq}{2\pi^2r^4} \int_0^\infty dy \, y \, e^{-y-m^2 r^2/(2y)} [\mathcal{I}(q,-\alpha_0,y) - \mathcal{I}(q,\alpha_0,y)], \]  
(3.29)

where \( \mathcal{I}(q,\alpha_0,y) \) is defined in (3.8). From the above expression, we see that the induced azimuthal current is an odd function of \( \alpha_0 \).

By using the formula (3.11), after the integration over \( y \), the expression (3.29) is presented in the form
\[ \langle j^{\phi} \rangle_s = -\frac{em^2}{\pi^2 r^4} \left[ \sum_{k=1}^{p} \frac{(-1)^k}{s_k} \sin(2\pi k \alpha_0) K_2(2mr s_k) \right. \right. \]
\[ + \left. \left. \frac{q}{\pi} \int_0^\infty dy \, g(q,\alpha_0,2y) K_2(2mr \cosh y) \right] \]  
(3.30)

where \( K_\nu(x) \) is the Macdonald function. In (3.30), we have introduced the notations
\[ s_k = \sin(\pi k/q), \]  
(3.31)

and
\[ g(q,\alpha_0,y) = \cos[q\pi(1/2 + \alpha_0)] \cosh[q(1/2 - \alpha_0)y] \]
\[ - \cos[q\pi(1/2 - \alpha_0)] \cosh[q(1/2 + \alpha_0)y]. \]  
(3.32)

The azimuthal current density \( \langle j^{\phi} \rangle_s \) vanishes in the absence of the magnetic flux along the string \( (\alpha_0 = 0) \). In the special case \( q = 1 \), the first term in the square brackets of (3.30) is absent and from this formula we obtain the current density in the Minkowski bulk induced by the magnetic flux.

At large distances from the string, \( mr \gg 1 \), and for \( q \gg 1 \) the dominant contribution to (3.30) comes from the term with \( k = 1 \). To the leading order one gets
\[ \langle j^{\phi} \rangle_s \approx \frac{em^4 \sin(2\pi \alpha_0)}{2(\pi s_1)^{3/2} (mr)^{5/2}} e^{-2mr s_1}. \]  
(3.33)

For \( q < 2 \) and \( mr \gg 1 \), the azimuthal current density is suppressed by the factor \( e^{-2mr} \). Near the string, \( mr \ll 1 \), the leading term in \( \langle j^{\phi} \rangle_s \) behaves as \( 1/r^4 \). This term does not depend on
the mass and coincides with \( \langle j^\phi \rangle_s \) for a massless field. The latter is easily obtained from (3.30) by taking into account that \( K_\nu(x) \sim 2^{\nu-1}\Gamma(\nu)x^{-\nu} \) for \( x \to 0 \).

For the special case (3.14) the integral term in (3.30) vanishes and one finds:

\[
\langle j^\phi \rangle_s = \frac{em^2}{2\pi^2 r^2} \sum_{k=1}^{q-1} \sin \left( \frac{\pi k}{q} \frac{2n + 1}{s_k} \right) K_2(2mr s_k) . \tag{3.34}
\]

For a massless field and for \( n = 0 \), it can be seen that (3.34) becomes,

\[
\langle j^\phi \rangle_s = \frac{e(q^2 - 1)}{12\pi^2 r^4} . \tag{3.35}
\]

In fig. 1 we display the dependence of \( r^4 \langle j^\phi \rangle_s / e \) on \( \alpha_0 \) in the case of a massless fermionic field for separate values of the parameter \( q \) (numbers near the curves). As it is seen, the current density increases with increasing \( q \).

Figure 1: Azimuthal current in the geometry of a straight cosmic string, \( r^4 \langle j^\phi \rangle_s / e \), as a function of the parameter \( \alpha_0 \) in the case of a massless fermionic field for different values of \( q \) (numbers near the curves).

Now let us develop the calculation of the contribution to the azimuthal current induced by the compactification of the string along its axis. This part comes from the second integral in the right-hand side of the summation formula (3.23) and is presented in the form:

\[
\langle j^\phi \rangle_c = \frac{eq}{\pi^2 r} \sum_j \epsilon_j \int_0^\infty d\lambda \lambda^2 J_{\beta_j}(\lambda r) J_{\beta_j + \epsilon_j}(\lambda r) \int_0^\infty \frac{dk}{\sqrt{k^2 - \lambda^2 - m^2}} \left( \frac{1}{e^{LK + 2\pi i \beta} - 1} + \frac{1}{e^{LK - 2\pi i \beta} - 1} \right) . \tag{3.36}
\]

To continue the calculation, we shall use the series expansion, \( (e^u - 1)^{-1} = \sum_{l=1}^\infty e^{-lu} \) in the above expression. Taking this expansion, the integral over \( k \) can be developed with the help of [12]. After some minor steps, we arrive at,

\[
\langle j^\phi \rangle_c = \frac{2eq}{\pi^2 r} \sum_{l=1}^\infty \cos(2\pi \sqrt{\beta}) \sum_j \epsilon_j \int_0^\infty d\lambda \lambda^2 J_{\beta_j}(\lambda r) J_{\beta_j + \epsilon_j}(\lambda r) K_0(1L \sqrt{m^2 + \lambda^2}) . \tag{3.37}
\]
At this point we shall use the integral representation below for the Macdonald function [42]:

\[ K_\nu(x) = \frac{1}{2} \left( \frac{x}{2} \right)^\nu \int_0^{\infty} dt \frac{e^{-t-x^2/(4t)}}{t^{\nu+1}} \].

(3.38)

So, we obtain the representation

\[
\langle j^\phi \rangle_c = -\frac{eq}{\pi^2 r^4} \sum_{l=1}^{\infty} \cos(2\pi l \tilde{\beta}) \sum_j e_j \int_0^{\infty} dt \frac{e^{-t-L^2 \rho^2 m^2/4t} + \eta(\alpha, y)}{t}
\times \int_0^{\infty} d\lambda \lambda^2 J_{\beta卓越}(\lambda r)J_{\beta卓越+\epsilon_j}(\lambda r) e^{-L^2 \lambda^2/4t}.
\]

(3.39)

The integral over \( \lambda \) can be developed by using a similar integral as is written in [3,28]. Defining a new variable \( y = 2r^2 t/(L^2 L^2) \), and after some simplifications, we arrive at,

\[
\langle j^\phi \rangle_c = \frac{eq}{\pi^2 r^4} \sum_{l=1}^{\infty} \cos(2\pi l \tilde{\beta}) \int_0^{\infty} dy \ y^2 e^{-y|1+L^2 L^2/(2r^2)|-m^2 r^2/(2y)}
\times \left[ \mathcal{I}(q, -\alpha_0, y) - \mathcal{I}(q, \alpha_0, y) \right].
\]

(3.40)

We can see that this contribution remains the same for the replacement of \( \tilde{\beta} \) by \( 1 - \tilde{\beta} \).

For the further evaluation of the topological part we use the representation [3,11] for the functions \( \mathcal{I}(q, \pm \alpha_0, y) \). After the integration over \( y \) we come to the expression

\[
\langle j^\phi \rangle_c = -\frac{8em^2}{\pi^2 L^2} \sum_{l=1}^{\infty} \cos(2\pi l \tilde{\beta}) \left[ \sum_{k=1}^{p} (-1)^k s_k \sin(2\pi k \alpha_0) \frac{K_2(mL \sqrt{\rho_k^2 + l^2})}{\rho_k^2 + l^2} + \frac{q}{\pi} \int_0^\infty dy \ \frac{\cosh(y) g(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} \right]
\]

(3.41)

where we have defined \( \rho_k = 2r s_k L \), \( \eta(y) = \frac{2r \cosh y}{L} \).

As we see, the part in the current density induced by the compactification of the string axis is an odd function of the magnetic flux along the string and it is an even function of the parameter \( \tilde{\beta} \). In particular, in the case of an untwisted fermionic field, \( \langle j^\phi \rangle_c \) is an even function of the magnetic flux enclosed by the string axis. In the absence of the magnetic flux along the string axis one has \( \langle j^\phi \rangle_c = 0 \).

The topological part of the current density is finite on the string and from (3.41) we get

\[
\langle j^\phi \rangle_c|_{r=0} = -\frac{8em^2}{\pi^2 L^2} \sum_{l=1}^{\infty} \cos(2\pi l \tilde{\beta}) \frac{K_2(mL)}{l^2}
\times \left[ \sum_{k=1}^{p} (-1)^k s_k \sin(2\pi k \alpha_0) + \frac{q}{\pi} \int_0^\infty dy \ \frac{\cosh(y) g(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} \right].
\]

(3.43)

Hence, near the string the total current is dominated by the part \( \langle j^\phi \rangle_c \). For large values of the length of the compact dimension, \( mL \gg 1 \), assuming that \( nr \) is fixed, the main contribution comes from the \( l = 1 \) term and to the leading order we find

\[
\langle j^\phi \rangle_c \approx -\frac{2^{5/2}em^3}{\pi^{3/2} L^{5/2}} \cos(2\pi \tilde{\beta}) e^{-mL} \sum_{k=1}^{p} (-1)^k s_k \sin(2\pi k \alpha_0)
+ \frac{q}{\pi} \int_0^\infty dy \ \frac{\cosh(y) g(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)}.
\]

(3.44)
In this limit the dominant contribution to the total current density comes from the term \( \langle j^\phi \rangle_s \).

In the special case (3.14) the formula (3.41) is simplified to

\[
\langle j^\phi \rangle_c = \frac{4em^2}{\pi^2L^2} \sum_{l=1}^{\infty} \cos(2\pi l \tilde{\beta}) \sum_{k=1}^{q-1} s_k \sin\left(\pi k \frac{2n+1}{q}\right) \frac{K_2(mL \sqrt{\rho_k^2 + l^2})}{\rho_k^2 + l^2},
\]

(3.45)

where \( n \) is restricted by the condition (3.15).

For a massless field, combining (3.41) with (3.30), the expression for the total azimuthal current density takes the form

\[
\langle j^\phi \rangle = -\frac{16e}{\pi^2L^4} \left[ \sum_{k=1}^{p} (-1)^k s_k \sin(2\pi k\alpha_0) C(\tilde{\beta}, \rho_k) + \frac{q}{\pi} \int_0^{\infty} dy \frac{y(y, \alpha_0, 2y) \cosh y}{\cosh(2qy) - \cos(q\pi)} C(\tilde{\beta}, \eta(y)) \right],
\]

(3.46)

where we have defined

\[
C(\tilde{\beta}, x) = \sum_{l=0}^{\infty} \frac{\cos(2\pi l \tilde{\beta})}{(L^2 + x^2)^{\frac{3}{2}}},
\]

(3.47)

and the prime on the sum means that the term \( l = 0 \) should be taken with the coefficient \( 1/2 \).

In (3.46), the part with \( l = 0 \) in (3.47) coincides with \( \langle j^\phi \rangle_s \) and the remaining part corresponds to \( \langle j^\phi \rangle_c \). Note that for the series (3.47) one has

\[
C(\tilde{\beta}, x) = \frac{\pi^2 \cosh(2\pi \tilde{\beta} x)}{4x^2 \sinh^2(\pi x)} + \frac{\cosh[\pi(1 - 2\tilde{\beta})x] + 2\tilde{\beta}x \sinh[\pi(1 - 2\tilde{\beta})x]}{4x^3 \sinh(\pi x)},
\]

(3.48)

where \( 0 \leq \tilde{\beta} \leq 1 \).

For \( r \ll L \) and for a massless field, for the topological part to the leading order one has

\[
\langle j^\phi \rangle_c \approx \frac{16\pi^2 e}{3L^4} \left[ \tilde{\beta}^2(1 - \tilde{\beta})^2 - \frac{1}{30} \right] \left[ \sum_{k=1}^{p} (-1)^k s_k \sin(2\pi k\alpha_0) \right] + \frac{q}{\pi} \int_0^{\infty} dy \frac{\cosh(y) y(y, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)},
\]

(3.49)

where we have assumed that \( 0 \leq \tilde{\beta} \leq 1 \). Note that in this limit \( \langle j^\phi \rangle_c / \langle j^\phi \rangle_s \sim (r/L)^4 \), and the total current density is dominated by the part \( \langle j^\phi \rangle_s \). In the opposite limit, \( r \gg L \), we use the asymptotic expression for the function (3.48) for \( x \gg 1 \) and for \( 0 < \tilde{\beta} < 1 \): \( C(\tilde{\beta}, x) \approx \pi^2 e^{-2\pi \sigma x}/(2x^2) \), where \( \sigma = \min(\tilde{\beta}, 1 - \tilde{\beta}) \). For \( \tilde{\beta} = 0 \) one has the asymptotic \( C(0, x) \approx \pi/(4x^2) \). From these expressions we conclude that at large distances from the string, \( r \gg L \), the azimuthal current density is exponentially suppressed by the factor \( \exp[-4\pi \sigma x/(\pi q/L)] \) for \( q \geq 2 \) and for \( 0 < \tilde{\beta} < 1 \). For \( q < 2 \) the suppression is by the factor \( \exp[-4\pi \sigma r/L] \). For \( \tilde{\beta} = 0 \) the current density decays as power-law: \( \langle j^\phi \rangle \sim (L/r)^3 \). Note that in the latter case the total current density is dominated by the topological part: \( \langle j^\phi \rangle_c / \langle j^\phi \rangle_s \sim r/L \). Hence, at distances larger than the length of the compactification the behavior of the azimuthal current density depends crucially on whether \( \tilde{\beta} = 0 \) or not. For \( \tilde{\beta} \neq 0 \) the compactification of the string along its axis leads to the suppression of the current density, whereas for \( \tilde{\beta} = 0 \) the current density is increased by the compactification. This feature is illustrated in figure 2 where we display the total azimuthal current density for a massless fermionic field as a function of \( r/L \). The graphs are plotted for
Figure 2: The total azimuthal current density, $L^4 \langle j_\phi \rangle / e$, for a massless fermionic field as a function of $r/L$ for $q = 2.5$ and $\alpha_0 = 0.4$. The numbers near the curves are the values of the parameter $\tilde{\beta}$ and the dashed curve corresponds to the current density for the geometry of a straight cosmic string, $L^4 \langle j_\phi \rangle_s / e$.

$q = 2.5$, $\alpha_0 = 0.4$ and the numbers near the curves correspond to the values of the parameter $\tilde{\beta}$. The dashed curve presents the current density for the geometry of a straight cosmic string ($L^4 \langle j_\phi \rangle_s / e$).

In fig. 3 we plot the topological part of the azimuthal current density, $L^4 \langle j_\phi \rangle_c / e$, as a function of $\alpha_0$ and $\tilde{\beta}$ in the geometry of a cosmic string with the parameter $q = 2.5$ and for $r/L = 1$.

4 Axial current

In this section we shall present the evaluation of the induced fermionic current along the string’s axis. As we shall see, this new current is a consequence of the quasiperiodicity condition imposed on the fermionic field and also of the magnetic flux enclosed by the compact dimension.

The VEV of the axial current is given by the expression

$$\langle j^z \rangle = e \sum_\sigma \psi^{(-)}_\sigma \gamma^0 \gamma^z \psi^{(-)}_\sigma.$$  \hspace{1cm} (4.1)

Once more, substituting the expression for the negative-energy mode functions (2.33) and the corresponding expressions for the Dirac matrices in this background, given in (2.8), into the expression in the right-hand side of (4.1), we obtain:

$$\langle j^z \rangle = -\frac{eq}{4\pi L} \sum_\sigma \lambda k_{l}^{(-)} \frac{E_-}{J_{\tilde{\beta}j}(\lambda r) + J_{\tilde{\beta}j+\epsilon_j}(\lambda r)},$$  \hspace{1cm} (4.2)

with $\sum_\sigma$ defined as (3.2). The summation over $s$ provides the factor 2. As to the summation over the quantum number $l$, we redefine $l \rightarrow -l$ (this is equivalent to the replacements $k^{(-)} \rightarrow -k^{(+)}$ and $E_- \rightarrow E_+$ in (4.2)) and then use the formula (3.23), taking $g(u) = 2\pi u/L$ and the expression (3.24) for the function $f(u)$. For this case, the function $g(u)$ is an odd function and the contribution due to the first term on the right-hand side of (3.23) vanishes, remaining only the contribution associated with the second one, which will provide the current
due to the compactification. As a consequence, the axial current density in the geometry of an uncompactified cosmic string vanishes.

The current density induced by the compactification of the string along its axis is written as,

\[
\langle j^z \rangle_c = \frac{iqe}{2\pi^2} \sum_j^\infty \int_0^\infty d\lambda \lambda \left[ J_{j+\epsilon_j}^2(\lambda r) + J_{j+\epsilon_j}^2(\lambda r) \right] \int_0^\infty \frac{dk}{\sqrt{k^2 + \lambda^2 + m^2}} \left( \frac{1}{e^{Lk+2\pi i\tilde{\beta}} - 1} - \frac{1}{e^{Lk-2\pi i\tilde{\beta}} - 1} \right).
\]

As before, the next step is to use the expansion \((e^u - 1)^{-1} = \sum_{l=1}^\infty e^{-lu}\), in the terms inside the bracket, and with the help of (3.38), the integral over \(k\) can be evaluated with the result:

\[
\langle j^z \rangle_c = \frac{qieL}{4\pi^2} \sum_{l=1}^\infty 2\pi l \sin(2\pi l\tilde{\beta}) \int_0^\infty dt \frac{e^{-t^2L^2m^2/(4t)}}{t^2} \int_0^\infty d\lambda \lambda \left( \frac{1}{e^{Lk+2\pi i\tilde{\beta}} - 1} - \frac{1}{e^{Lk-2\pi i\tilde{\beta}} - 1} \right) \int_0^\infty \frac{dk}{\sqrt{k^2 + \lambda^2 + m^2}} \left( \frac{1}{e^{Lk+2\pi i\tilde{\beta}} - 1} - \frac{1}{e^{Lk-2\pi i\tilde{\beta}} - 1} \right) \left( J_{j+\epsilon_j}^2(\lambda r) + J_{j+\epsilon_j}^2(\lambda r) \right)K_1 \left( L\sqrt{\lambda^2 + m^2} \right). \]

As we see the induced axial current is an odd periodic function of \(\tilde{\beta}\) with the period 1. Now using the integral representation for the Macdonald function given in (3.38), we arrive:

\[
\langle j^z \rangle_c = \frac{qieL}{4\pi^2} \sum_{l=1}^\infty 2\pi l \sin(2\pi l\tilde{\beta}) \int_0^\infty dt \frac{e^{-t^2L^2m^2/(4t)}}{t^2} \int_0^\infty d\lambda \lambda \left( J_{j+\epsilon_j}^2(\lambda r) + J_{j+\epsilon_j}^2(\lambda r) \right) \left( \frac{1}{e^{Lk+2\pi i\tilde{\beta}} - 1} - \frac{1}{e^{Lk-2\pi i\tilde{\beta}} - 1} \right) \int_0^\infty \frac{dk}{\sqrt{k^2 + \lambda^2 + m^2}} \left( \frac{1}{e^{Lk+2\pi i\tilde{\beta}} - 1} - \frac{1}{e^{Lk-2\pi i\tilde{\beta}} - 1} \right) \left( J_{j+\epsilon_j}^2(\lambda r) + J_{j+\epsilon_j}^2(\lambda r) \right). \]

The integral over \(\lambda\) presents two terms. The term proportional to \(m^2\) is given in (3.32) and the other term that contains the power \(\lambda^3\) can be evaluated by using a well-known trick. So, we
have
\[
\int_0^\infty d\lambda \; \lambda (\lambda^2 + m^2) \, e^{-a\lambda^2} J^2(\lambda r) = (m^2 - \partial_a) \int_0^\infty d\lambda \; \lambda \, e^{-a\lambda^2} J^2(\lambda r) = \frac{1}{2a^2} \left( m^2 a + 1 + y\partial_y \right) [e^{-y} I_\nu(y)] ,
\]
with \( y = r^2/(2a) \). Adapting the above expression to our calculation we have,
\[
\langle j^z \rangle_c = \frac{16ge}{8\pi^2 L} \sum_{l=1}^\infty \frac{\sin(2\pi l \bar{\beta})}{l^3} \int_0^\infty dt \; e^{-t-l^2 m^2/(4t)}
\[
\times \left( \frac{l^2 L^2 m^2}{4t} + 1 + y\partial_y \right) \{ e^{-y} [I(q, \alpha_0, y) + I(q, -\alpha_0 y, y)] \} ,
\]
where \( y = 2tr^2/(lL)^2 \). As it is seen, the axial current density is an even function of \( \alpha_0 \).

Now we use the formula (3.11) for the function \( I(q, \alpha_0, y) \). After the integration over \( t \) this gives
\[
\langle j^z \rangle_c = \frac{4m^2e}{\pi^2 L} \sum_{l=1}^\infty l \sin(2\pi l \bar{\beta}) \left[ \sum_{k=0}^p (-1)^k c_k \cos(2\pi k\alpha_0) \frac{K_2(mL \sqrt{l^2 + \rho_k^2})}{l^2 + \rho_k^2} \right]
\[
+ \frac{q}{\pi} \int_0^\infty dv \frac{h(q, \alpha_0, 2v) \sinh v}{\cosh(2qv) - \cos(q\pi)} \frac{K_2(mL \sqrt{l^2 + \eta^2(y)})}{l^2 + \eta^2(y)} \right] ,
\]
where \( c_k \) is defined in (3.18), the function \( h(q, \alpha_0, v) \) is given in (3.19), and the prime on the sign of the sum means that the term with \( k = 0 \) should be taken with the coefficient 1/2. This term does not depend on \( \alpha_0 \) and \( q \) and corresponds to the current density in the absence of the string. For this part one has
\[
\langle j^z \rangle_{c}^{(1)} = \frac{2m^2e}{\pi^2 L} \sum_{l=1}^\infty \frac{\sin(2\pi l \bar{\beta})}{l} K_2(lLm) ,
\]
which does not depend on the radial coordinate \( r \). Eq. (4.9) presents the current density in the Minkowski spacetime with the spatial topology \( R^2 \times S^1 \). It is a special case of a general formula given in Ref. (43) for the topology \( R^p \times (S^1)^q \) with arbitrary \( p \) and \( q \). As we see from (4.8), the axial current density vanishes for integer and half-integer values of \( \bar{\beta} \). In particular, this is the case for untwisted and twisted fields in the absence of the magnetic flux enclosed by the string axis \( (\Phi_3 = 0) \).

The axial current density is finite on the string:
\[
\langle j^z \rangle_{c} \big|_{r=0} = \frac{4m^2e}{\pi^2 L} \sum_{l=1}^\infty \frac{\sin(2\pi l \bar{\beta})}{l} \frac{K_2(mLl)}{l} \left[ \sum_{k=0}^p (-1)^k c_k \cos(2\pi k\alpha_0) \right]
\[
+ \frac{q}{\pi} \int_0^\infty dv \frac{h(q, \alpha_0, 2v) \sinh v}{\cosh(2qv) - \cos(q\pi)} \right] .
\]
For the special case with (3.14) the integral term in (4.8) vanishes and for the current density we get
\[
\langle j^z \rangle_c = \frac{2m^2e}{\pi^2 L} \sum_{l=1}^\infty l \sin(2\pi l \bar{\beta}) \sum_{k=0}^{q-1} c_k \cos \left( \frac{\pi k}{q} \frac{2n + 1}{q} \right) \frac{K_2(mL \sqrt{l^2 + \rho_k^2})}{l^2 + \rho_k^2} .
\]
For a massless field this formula simplifies to

\[ \langle j^z \rangle_c = \frac{4e}{\pi^2 L^3} \sum_{l=1}^{\infty} \frac{\sin(2\pi l \tilde{\beta})}{l} \sum_{k=0}^{q-1} \cos \left( \frac{\pi k}{q} \right) \frac{c_k}{(l^2 + \rho_k^2)^2}. \]  

(4.12)

For a massless field the general formula (4.8) reduces to

\[ \langle j^z \rangle_c = \frac{8e}{\pi^2 L^3} \left[ \sum_{k=0}^{p} (-1)^k c_k \cos (2\pi k \alpha_0) S(\tilde{\beta}, \rho_k) \right. \\
+ \frac{q}{\pi} \int_0^\infty dv \sinh v \left( q(2qv) - \cos(qv) \right) S(\tilde{\beta}, \eta(v)) \left. \right], \]  

(4.13)

where

\[ S(\tilde{\beta}, x) = \sum_{l=1}^{\infty} \frac{l \sin(2\pi l \tilde{\beta})}{(l^2 + x^2)^3}. \]  

(4.14)

The function in (4.14) can be written as

\[ S(\tilde{\beta}, x) = -\frac{1}{4x} \partial_x \sum_{l=1}^{\infty} \frac{l \sin(2\pi l \tilde{\beta})}{(l^2 + x^2)^2}, \]  

(4.15)

and the expression for the series in the right-hand side can be found in [46] (with the sign missprint corrected). In this way one gets:

\[ S(\tilde{\beta}, x) = \frac{\pi^2}{16x^3 \sinh y} \left\{ -\frac{\sinh (2\tilde{\beta}y)}{\sinh y} (2y \coth y + 1) \\
+ 2\tilde{\beta} \left[ 2y \frac{\cosh (2\tilde{\beta}y)}{\sinh y} + 2\tilde{\beta}y \sinh (1 - 2\tilde{\beta}y) + \cosh (1 - 2\tilde{\beta}y) \right] \right\}, \]  

(4.16)

with \(0 \leq \tilde{\beta} \leq 1\) and \(y = \pi x\). For \(r \ll L\), the leading term in the axial current is given by (4.13) with the replacement \(S(\tilde{\beta}, x) \to S(\tilde{\beta}, 0)\), where

\[ S(\tilde{\beta}, 0) = \frac{\pi^5}{45} \tilde{\beta} (1 - \tilde{\beta})(1 - 2\tilde{\beta})(1 + 3\tilde{\beta} - 3\tilde{\beta}^2). \]  

(4.17)

At large distances from the string the axial current density, given by (4.13), is dominated by the \(k = 0\) term. In order to estimate the contribution of the remaining part we note that for \(0 < \tilde{\beta} < 1/2\) and assuming \(\tilde{\beta} x \gg 1\) one has the asymptotic expression \(S(\tilde{\beta}, x) \approx \pi^3 (\tilde{\beta}/x)^2 e^{-2\pi \tilde{\beta} x}/4\). Hence, this contribution is suppressed by the factor \(\exp[-4\pi \tilde{\beta} r \sin(\pi/q)/L]\) for \(q \geq 2\) and by the factor \(\exp(-4\pi \tilde{\beta} r/L)\) for \(q < 2\). The asymptotic behavior for \(1/2 < \tilde{\beta} < 1\) is obtained by using the property that the axial current density changes the sign under the replacement \(\tilde{\beta} \to 1 - \tilde{\beta}\).

In fig. 4 we plot \(L^3 \langle j^z \rangle_c / e\) as a function of \(\alpha_0\) and \(\tilde{\beta}\) in the geometry of a cosmic string with the parameter \(q = 2.5\) and for \(r/L = 0.25\).

5 Conclusion

In this paper we have investigated the influence of the non-trivial spatial topology on the VEV of the fermionic current densities. The combined effects of two types of topology change are considered. The first one is due to the planar angle deficit induced by the cosmic string and the second one is induced by the compactification of the string along its axis. Along the compactified
dimension we have considered a general quasiperiodicity condition with an arbitrary phase $\beta$. As special cases, it includes periodicity and antiperiodicity conditions corresponding to untwisted and twisted fields respectively. As we could observe, all the new contributions to the current induced by the compactification depend crucially on the parameter $\beta$. In addition, we have assumed the presence of a constant gauge field with nonzero azimuthal and axial components. They correspond to a magnetic flux along the string axes and to a magnetic flux enclosed by the compact dimension.

For the evaluation of the VEV for the current density we have used the direct summation method over a complete set of modes. In this method a complete set of fermionic mode functions are employed. The complete set of positive- and negative-energy mode functions is constructed in section 2 and they are given by the expressions (2.26) and (2.33). Given these functions, the VEV of the current density is presented in the form of the mode sum (3.1). We have shown that the charge density and the radial component of the current density vanish. By making use of the Abel-Plana-type summation formula (3.23), the azimuthal current density is decomposed into two parts. The first one corresponds to the current density for the geometry of a straight cosmic string and the second one is induced by the compactification of the string axis. For the first part we have provided a closed form (3.30) for a massive fermionic field valid for general value of the planar angle deficit. It includes various special cases previously discussed in the literature. A simple expression, (3.34), is obtained for a special value (3.14) for the parameter $\alpha_0$ characterizing the magnetic flux along the axis of the string and $q$ being an integer number.

The part in the azimuthal current density induced by the compactification of the string axis is given by the formula (3.41). This part is an odd periodic function of the magnetic flux along the string axis with the period equal to the flux quantum and it is an even function of the parameter $\tilde{\beta}$, defined by (2.29), with the period 1. Unlike to the part $\langle j^\phi \rangle_s$, which diverges on the string as $r^{-4}$, the topological part of the azimuthal current density is finite on the string (see (3.43)). For a massless field, the topological part is further simplified and the total current density is given by the expression (3.46). Near the string the total current density is dominated by the part $\langle j^\phi \rangle_s$. For a massless field, at distances from the string, larger than the length of the compactification, the behavior of the azimuthal current density depends crucially on whether $\tilde{\beta} = 0$ or not. For $\tilde{\beta} \neq 0$ the compactification of the string along its axis leads to the suppression...
of the current density, whereas for $\tilde{\beta} = 0$ the current density is increased by the compactification: $\langle j^0 \rangle_s \sim r/L$.

The compactification of the cosmic string axis, in combination with the quasiperiodicity condition (2.4) and with the component of the gauge field along the string axis, leads to the nonzero VEV of the axial current density. The phases in the periodicity conditions and the axial component of the gauge field are related to each other through a gauge transformation and the physical results depend on the combination (2.29). The VEV of the axial current density is given by the expression (4.8). This VEV has a purely topological origin and vanishes in the geometry of a straight cosmic string. Of course, the latter is a direct consequence of the problem symmetry. The axial current density is a periodic function of the magnetic flux along the string axis with the period equal to the flux quantum. It is an odd periodic function of the parameter $\tilde{\beta}$ with the period 1. The axial current density vanishes for integer and half-integer values of $\tilde{\beta}$. In particular, this is the case for untwisted and twisted fields in the absence of the magnetic flux enclosed by the string axis. The axial current density is finite on the string’s axis. At large distances from the string, it tends to a limiting value which corresponds to the current density in the Minkowski spacetime with the spatial topology $R^2 \times S^1$. For a massless field, the summation over $l$ in (4.8) is done explicitly and the expression for the axial current density takes simpler form (4.13).

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