On localizations of the characteristic classes of \(\ell\)-adic sheaves and conductor formula in characteristic \(p > 0\)

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Abstract

The Grothendieck-Ogg-Shafarevich formula calculates the \(\ell\)-adic Euler-Poincaré number of an \(\ell\)-adic sheaf on a curve by an invariant produced by the wild ramification of the \(\ell\)-adic sheaf named Swan class. A. Abbes, K. Kato and T. Saito generalize this formula to any dimensional scheme in [KS] and [AS]. In this paper, assuming the strong resolution of singularities we prove a localized version of a formula proved by A. Abbes and T. Saito in [AS] using the characteristic class of an \(\ell\)-adic sheaf. As an application, we prove a conductor formula in equal characteristic.

1 Introduction

The Grothendieck-Ogg-Shafarevich formula calculates the \(\ell\)-adic Euler-Poincaré number of an \(\ell\)-adic sheaf on a curve by an invariant produced by the wild ramification of the \(\ell\)-adic sheaf named Swan class. Generalizations of this formula to the surface case are done by Deligne, Kato and Laumon. Recently this formula is generalized to any dimensional scheme by A. Abbes, K. Kato and T. Saito in [AS] and [KS]. To generalize this formula to any dimensional case, K. Kato and T. Saito defines the Swan class of an \(\ell\)-adic sheaf on any dimensional scheme using alteration and logarithmic blow-up. A. Abbes and T. Saito rediscovered the characteristic class of an \(\ell\)-adic sheaf using the Verdier pairing (SGA5) and studied its properties in [AS]. The characteristic class of an \(\ell\)-adic sheaf on a scheme is a cohomological element in the top étale cohomology group which goes to the \(\ell\)-adic Euler-Poincaré number under the trace map in the case where the scheme is proper. They calculate the characteristic class by the Swan class defined by K. Kato and T. Saito. This formula is a refinement of the results proved by Kato and Saito in [KS]. We call this refinement the Abbes-Saito formula.

In this paper, we prove a localized version of the Abbes-Saito formula, a localized version of the Lefschetz-Verdier trace formula and a refinement of the Kato-Saito conductor formula in equal characteristic.

A localized version of the Abbes-Saito formula is an equality (Theorem 4.1) of the localized characteristic class of a smooth \(\ell\)-adic sheaf and the Swan class in an étale cohomology group with support. We call this equality the localized Abbes-Saito formula. To show the localized
Abbes-Saito formula, we generalize the localized characteristic class of a smooth \( \ell \)-adic sheaf also defined in [AS] to the localized characteristic class of an \( \ell \)-adic sheaf with a cohomological correspondence and prove its compatibility with pull-back. Assuming the strong resolution of singularities, as a direct consequence of this compatibility, we prove the localized Abbes-Saito formula in Theorem 4.1.

We prove the compatibility of the localized characteristic class with proper push-forward in Proposition 5.6. This is a localized version of the Lefschetz-Verdié trace formula, which we call the localized Lefschetz-Verdié trace formula.

As an application of the localized Abbes-Saito formula and the localized Lefschetz-Verdié trace formula, we prove a conductor formula in equal characteristic in Corollary 5.8. The conductor formula calculates the Swan conductor of an \( \ell \)-adic representation which appears when we consider a fibration on a curve by the Swan class of an \( \ell \)-adic sheaf defined by Kato and Saito in [KS]. We call this conductor formula the Kato-Saito conductor formula in characteristic \( p > 0 \). To prove this formula is the main purpose to consider localizations. To refine the Kato-Saito conductor formula, we define a localization as a cohomology class with support on the wild locus in subsection 3.3 which we call the logarithmic localized characteristic class and prove its compatibility with proper push-forward in Theorem 5.2. In [T], we prove a refinement of the formula for a smooth sheaf of rank 1 proved by Abbes-Saito in [AS] using an idea of T. Saito in [S]. As an application of Theorem 5.2 and the result in [T], we prove a refinement of the Kato-Saito conductor formula for a smooth sheaf of rank 1 in Corollary 5.4.

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**Notation.** In this paper, \( k \) denotes a field. Schemes over \( k \) are assumed to be separated and of finite type. For a divisor with simple normal crossings of a smooth scheme over \( k \), we assume that the irreducible components and their intersections are also smooth over \( k \). The letter \( l \) denotes a prime number invertible in \( k \) and \( \Lambda \) denotes a finite commutative \( \mathbb{Z}_l \)-algebra. For a scheme \( X \) over \( k \), \( D_{\text{etf}}(X) \) denotes the derived category of complexes of \( \Lambda \)-modules of finite tor-dimension on the étale site of \( X \) with constructible cohomology. Let \( \mathcal{K}_X \) denote \( Rf^!\Lambda \) where \( f : X \to \text{Spec} k \) is the structure map and let \( \mathbf{D} \) denote the functor \( R\mathcal{H}om(\_ , \mathcal{K}_X) \). For objects \( \mathcal{F} \) and \( \mathcal{G} \) of \( D_{\text{etf}}(X) \) and \( D_{\text{etf}}(Y) \) on schemes \( X \) and \( Y \) over \( k \), \( \mathcal{F} \boxtimes \mathcal{G} \) denotes \( \text{pr}_1^* \mathcal{F} \otimes \text{pr}_2^* \mathcal{G} \) on \( X \times Y \). When we say a scheme \( X \) is of dimension \( d \), we understand that every irreducible component of \( X \) is of dimension \( d \).
2 Review of the characteristic class etc.

2.1 Cohomological correspondence and the evaluation map

We recall the definition and some properties of a cohomological correspondence needed in this paper from [AS, subsection 1.2].

**Definition 2.1.** [AS, Definition 1.2.1] Let $X$ and $Y$ be schemes over $k$ and $\mathcal{F}$ and $\mathcal{G}$ be objects of $D_{\text{eff}}(X)$ and of $D_{\text{eff}}(Y)$ respectively. We call a correspondence between $X$ and $Y$ a scheme $C$ over $k$ and morphisms $c_1 : C \to X$ and $c_2 : C \to Y$ over $k$. We put $c = (c_1, c_2) : C \to X \times Y$ the corresponding morphism. We call a morphism $u : c_{\ast}G \to \mathcal{R}c_{\ast}F$ a cohomological correspondence from $\mathcal{G}$ to $\mathcal{F}$ on $C$.

We identify the cohomological correspondence with a section of

$$H^0_c(X \times Y, R\mathcal{H}om(pr^\ast_2 G, pr^\ast_1 \mathcal{F})) \simeq H^0_c(C, \mathcal{H}om(c_{\ast}G, \mathcal{R}c_{\ast} \mathcal{F})).$$

A typical example of a cohomological correspondence is given as follows. Assume $X$ and $Y$ are smooth of dimension $d$ over $k$ and $c = (c_1, c_2) : C \to X \times Y$ is a closed immersion. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of free $\Lambda$-modules on $X$ and $Y$ respectively and assume that $\mathcal{G}$ is smooth.

Then, the canonical map $c^\ast \mathcal{H}om(pr^\ast_2 \mathcal{G}, pr^\ast_1 \mathcal{F}) \to \mathcal{H}om(c_{\ast}G, c_{\ast} \mathcal{F})$ is an isomorphism and we identify $\mathcal{H}om(c_{\ast}G, c_{\ast} \mathcal{F}) = \Gamma(C, c^\ast \mathcal{H}om(pr^\ast_2 \mathcal{G}, pr^\ast_1 \mathcal{F}))$. Since $pr_1 : X \times Y \to X$ is smooth, we have a canonical isomorphism $pr_1^\ast \mathcal{F}(d)[2d] = Rpr_1^\ast \mathcal{F}$ and we identify $R\mathcal{H}om(pr^\ast_2 \mathcal{G}, pr^\ast_1 \mathcal{F}) = \mathcal{H}om(pr^\ast_2 \mathcal{G}, pr^\ast_1 \mathcal{F})(d)[2d]$. Then the cycle class map $CH_d(C) \to H^0_c(X \times Y, \Lambda(d))$ induces a pairing

$$CH_d(C) \otimes \mathcal{H}om(c_{\ast}G, c_{\ast} \mathcal{F}) \to H^0_c(X \times Y, \Lambda(d)) \otimes \Gamma(C, c^\ast \mathcal{H}om(pr^\ast_2 \mathcal{G}, pr^\ast_1 \mathcal{F}))$$

$$\to H^0_c(X \times Y, R\mathcal{H}om(pr^\ast_2 \mathcal{G}, pr^\ast_1 \mathcal{F})).$$

In other words, the pair $(\Gamma, \gamma)$ of a cycle class $\Gamma \in CH_d(C)$ and a homomorphism $\gamma : c_{\ast}G \to c_{\ast} \mathcal{F}$ defines a cohomological correspondence $u(\Gamma, \gamma)$.

We recall the definition of the push-forward of a cohomological correspondence. We consider the commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & C \\
\downarrow f & & \downarrow g \\
X' & \xleftarrow{h} & C'
\end{array}$$

(2.1)

of schemes over $k$. A canonical isomorphism

$$R(f \times g)_\ast R\mathcal{H}om(pr^\ast_2 \mathcal{G}, pr^\ast_1 \mathcal{F}) \to R\mathcal{H}om(pr^\ast_2 Rf^\ast \mathcal{G}, pr^\ast_1 Rf^\ast \mathcal{F})$$

is defined in [Gr (3.3.1)], using the isomorphism

$$R\mathcal{H}om(pr^\ast_2 \mathcal{G}, pr^\ast_1 \mathcal{F}) \to \mathcal{F} \boxtimes^L DG$$

defined in [Gr (3.1.1)]. In the diagram (2.1), we assume that the vertical arrows are proper. The above diagram defines a commutative diagram

$$\begin{array}{ccc}
C & \xrightarrow{c} & X \times Y \\
\downarrow h & & \downarrow f \times g \\
C' & \xleftarrow{c'} & X' \times Y'
\end{array}$$
Let \( u : c^*_2G \rightarrow Rc^*_1F \) be a cohomological correspondence. We identify \( u \) with a map \( u : \Lambda \rightarrow Rc^1R\text{Hom}(pr^*_2G, R\text{pr}^*_1F) \). Then, it induces a map \( \Lambda \rightarrow Rh_\Lambda \rightarrow Rh_\Lambda Rc^1R\text{Hom}(pr^*_2G, R\text{pr}^*_1F) \) where the first map \( \Lambda \rightarrow Rh_\Lambda \) is the adjunction of the identity. By the assumption that \( f, g \) and \( h \) are proper, the base change map defines a map of functors \( Rh_\Lambda Rc^1 = Rh_1Rc^1 \rightarrow Rc^1R(f \times g)_* \). By composing them with the isomorphism

\[
R(f \times g)_*R\text{Hom}(pr^*_2G, R\text{pr}^*_1F) \rightarrow R\text{Hom}(pr^*_2G, R\text{pr}^*_1F),
\]

we obtain a map

\[
\Lambda \rightarrow Rh_\Lambda Rc^1R\text{Hom}(pr^*_2G, R\text{pr}^*_1F) \rightarrow Rc^1R(f \times g)_*R\text{Hom}(pr^*_2G, R\text{pr}^*_1F) \rightarrow Rc^1R\text{Hom}(pr^*_2G, R\text{pr}^*_1F).
\]

We define the push-forward \( h_*u : c^*_2RG = c^*_2R_gG \rightarrow Rc^1_1Rf_*F \) of \( u \) to be the corresponding cohomological correspondence. The push-forward \( h_*u \) is equal to the composition of the maps

\[
c^*_2R_gG \rightarrow Rh_*c^*_2G \rightarrow Rh_*Rc^1_1F \rightarrow Rc^1_1Rf_*F
\]

where the first and the third maps are the base change maps and the second map is the push-forward \( h_*u \).

We consider the commutative diagram

\[
\begin{array}{ccc}
U & \rightarrow & V \\
\downarrow j_U & & \downarrow j_V \\
X & \rightarrow & Y
\end{array}
\]

of schemes over \( k \) where the vertical arrows are open immersions. Let \( F \) and \( G \) be objects of \( D_{\text{ctf}}(X) \) and \( D_{\text{ctf}}(Y) \) respectively and \( \bar{u} : c^*_2G \rightarrow Rc^*_1F \) be a cohomological correspondence on \( C \). Let \( F_U = j^*_U F \) and \( G_V = j^*_V G \) be the restrictions. We identify \( j^*_C Rc^1_1F = j^*_C Rc^1_1F = Rc^1_1F_U \) by the composite isomorphism. Then, the restriction \( j^*_C \bar{u} \) on \( C \) defines a cohomological correspondence \( u : c^*_2G_V = c^*_2G_V \rightarrow j^*_C Rc^1_1F = Rc^1_1F_U \).

We recall the zero-extension of a cohomological correspondence playing an important role when we define a refined (localized) characteristic class.

**Lemma 2.2.** [AS lemma 1.2.2] Let the notation be as above and let \( j : U \times V \rightarrow X \times Y \) be the product \( j_U \times j_V \). We put \( \mathcal{H} = R\text{Hom}(pr^*_2G_U, R\text{pr}^*_1F_U) \) on \( U \times V \) and \( \mathcal{H} = R\text{Hom}(pr^*_2G, R\text{pr}^*_1F) \) on \( X \times Y \). We identify a cohomological correspondence \( \bar{u} : c^*_2G \rightarrow Rc^*_1F \) with a section \( \bar{u} \) of \( Rc^*_1\mathcal{H} \) and the associated map \( \bar{u} : c^*_2\Lambda \rightarrow \mathcal{H} \). We also identify the restriction \( u = j^*_C \bar{u} : c^*_2G_V \rightarrow Rc^1_1F_U \) with a section \( u \) of \( Rc^1\mathcal{H} \) and the associated map \( u : c^*_2\Lambda \rightarrow \mathcal{H} \). Then, we have the following.

1. The section \( u \) of \( Rc^1\mathcal{H} \) is the image of the restriction of \( \bar{u} \) by the composition isomorphism \( j^*_C Rc^1\mathcal{H} \rightarrow Rc^1j_1^*\mathcal{H} = Rc^1j^*\mathcal{H} = Rc^1\mathcal{H} \).
2. The square

\[
\begin{array}{ccc}
j^*c^*_2\Lambda & \rightarrow & j^*\mathcal{H} \\
\downarrow j^*\bar{u} & & \downarrow j^*\mathcal{H} \\
c^*_2\Lambda & \rightarrow & \mathcal{H}
\end{array}
\]

is commutative.

**Lemma 2.3.** [AS lemma 1.2.3] Assume that the right square in the diagram (2.2) is cartesian. Let \( F \) and \( G \) be objects of \( D_{\text{ctf}}(U) \) and \( D_{\text{ctf}}(V) \) respectively and \( u : c^*_2G \rightarrow Rc^*_1F \) be a cohomological correspondence on \( C \). Then, there exists a unique cohomological correspondence \( \bar{u} : c^*_2j_VG \rightarrow Rc^1_1F_U \) on \( C \) such that \( j^*_C \bar{u} = u \).
Corollary 2.4. [AS Corollary 1.2.4] 1. Assume that the map \( c_2 : C \to V \) is proper and \( C \) is dense in \( \bar{C} \). Then the right square in the diagram (2.2) is cartesian.

2. Assume that the right square in the diagram (2.2) is cartesian. Let \( F \) and \( G \) be objects of \( D_{\text{eff}}(U) \) and of \( D_{\text{eff}}(V) \) respectively and \( u : c_2^*G \to Rc_1^1F \) be a cohomological correspondence on \( C \). Then, \( \bar{u} = j_{C1}u : c_2^*j_{V1}G \to Rc_1^1j_{U1}F \) is the unique cohomological correspondence on \( C \) such that \( j_{C2}^*\bar{u} = u \).

We call \( j_{C1}u : c_2^*j_{V1}G \to Rc_1^1j_{U1}F \) the zero-extension of \( u \).

We define the pull-back of a cohomological correspondence. Let \( f : X' \to X \) and \( g : Y' \to Y \) be morphisms of smooth schemes over \( k \). We assume \( \dim X = \dim X' \) and \( \dim Y = \dim Y' \). Then the canonical map \( f^* \to Rc^1 \) and the isomorphism \( Rc^1 \mathcal{R} \text{Hom}(pr^*_2G, Rp^*_1\mathcal{F}) \to \mathcal{R} \text{Hom}(c_2^*G, Rc_1^1\mathcal{F}) \) induce a map

\[
(f \times g)^* \mathcal{R} \text{Hom}(pr^*_2G, Rp^*_1\mathcal{F}) \to (Rf \times g)^* \mathcal{R} \text{Hom}(pr^*_2G, Rp^*_1\mathcal{F}) \to \mathcal{R} \text{Hom}(pr^*_2g^*G, Rp^*_1Rf^*\mathcal{F}).
\]

With the isomorphism \( \mathcal{R} \text{Hom}(pr^*_2G, Rp^*_1\mathcal{F}) \to \mathcal{F} \boxtimes \mathcal{D} \mathcal{G} \), the above composition is identified with the composition

\[
(f \times g)^* (\mathcal{F} \boxtimes \mathcal{D} \mathcal{G}) \to f^* \mathcal{F} \boxtimes g^* \mathcal{D} \mathcal{G} \to Rf^* \mathcal{F} \boxtimes Rg^* \mathcal{D} \mathcal{G} \to Rf^* \mathcal{F} \boxtimes Rg^* \mathcal{D} \mathcal{G}.
\]

Let \( c = (c_1, c_2) : C \to X \times Y \) be a correspondence and \( u : c_2^*G \to Rc_1^1F \) be a cohomological correspondence on \( C \). We identify \( u \) with a map \( u : c_1\Lambda \to \mathcal{R} \text{Hom}(pr^*_2G, Rp^*_1\mathcal{F}) \) as above. We define a correspondence \( c' = (c'_1, c'_2) : C' \to X' \times Y' \) by the cartesian diagram

\[
\begin{array}{ccc}
C' & \xrightarrow{c'} & X' \times Y' \\
\downarrow h & & \downarrow f \times g \\
C & \xrightarrow{c} & X \times Y.
\end{array}
\]

By the proper base change theorem, the base change map \((f \times g)^*c_1\Lambda \to c'_1\Lambda\) is an isomorphism. Hence the map \( u : c_1\Lambda \to \mathcal{R} \text{Hom}(pr^*_2G, Rp^*_1\mathcal{F}) \) induces a map

\[
c'_1\Lambda \approx (f \times g)^*c_1\Lambda \to (f \times g)^* \mathcal{R} \text{Hom}(pr^*_2G, Rp^*_1\mathcal{F}) \to \mathcal{R} \text{Hom}(pr^*_2g^*G, Rp^*_1Rf^*\mathcal{F}).
\]

The composition defines a cohomological correspondence \((f \times g)^*u : c'_2^*g^*G \to Rc_1^1Rf^1\mathcal{F} = Rc_1^1f^*\mathcal{F} \). We call \((f \times g)^*u\) the pull-back of \( u \) by \( f \times g \).

We recall an evaluation map from [AS subsection 2.1]. Let \( X \) be a scheme over \( k \) and \( \delta : X = \Delta_X \to X \times X \) be the diagonal map. Let \( \mathcal{F} \) be an object of \( D_{\text{eff}}(X) \) and let \( 1 = u(X, 1) \) the cohomological correspondence defined by the identity of \( \mathcal{F} \) on the diagonal \( X \). An isomorphism

\[
\mathcal{H} = \mathcal{R} \text{Hom}(pr^*_2\mathcal{F}, Rp^*_1\mathcal{F}) \to \mathcal{F} \boxtimes \mathcal{D} \mathcal{F}
\]

induces an isomorphism

\[
\delta^*\mathcal{H} \to \mathcal{F} \boxtimes \mathcal{D} \mathcal{F}.
\]

Thus the evaluation map

\[
\mathcal{F} \boxtimes \mathcal{D} \mathcal{F} \to K_X
\]

induces a map

\[
e : \delta^*\mathcal{H} \to K_X.
\]  

(2.3)
We call this map the evaluation map of $\mathcal{F}$. We define another evaluation map. Let $X$ be a scheme over $k$ and $j : U \rightarrow X$ be an open immersion over $k$. Let $\delta_X : X \rightarrow X \times X$ and $\delta_U : U \rightarrow U \times U$ denote the diagonal maps. Let $\mathcal{F}$ be an object of $D_{\text{eff}}(U)$. We put $\mathcal{H} = R\mathcal{H}om(pr_2^*\mathcal{F}, pr_1^*\mathcal{F})$ on $U \times U$ and $\mathcal{H} = R\mathcal{H}om(pr_2^*j_!\mathcal{F}, pr_1^*j_!\mathcal{F})$ on $X \times X$ respectively. Since we have $\mathcal{H} = (j \times 1)_!R(1 \times j)_!\mathcal{H}$ by the Kunneth formula, we obtain a canonical isomorphism $j_!\delta_X^*\mathcal{H} \simeq \delta_U^*\mathcal{H}$. Thus the evaluation map of $\mathcal{F}$ on $U$ defined as above

$$e : \delta_U^*\mathcal{H} \rightarrow \mathcal{K}_U$$

induces an evaluation map

$$j e : \delta^*\mathcal{H} \rightarrow j_!\mathcal{K}_U.$$ (2.4)

### 2.2 Characteristic class of a $\Lambda$-sheaf with a cohomological correspondence

We briefly recall the definition of the (refined) characteristic class of a $\Lambda$-sheaf (c.f. [AS, Definition 2.1.8]). Let $X$ be a scheme over $k$, $U$ an open subscheme, $j : U \rightarrow X$ the open immersion and $\delta : X = \Delta_X \rightarrow X \times X$ the diagonal map. Let $C$ be a closed subscheme of $U \times U$, $C$ the closure of $C$ in $X \times X$, and $c : C \rightarrow U \times U$ and $\bar{c} : C \rightarrow X \times X$ the closed immersions. Let $j_C : C \rightarrow \bar{C}$ denote the open immersion. We assume that $C = (X \times U) \cap \bar{C}$. Let $\mathcal{F}$ be an object of $D_{\text{eff}}(U)$. We put $\mathcal{H} := R\mathcal{H}om(pr_2^*j_!\mathcal{F}, pr_1^*j_!\mathcal{F})$ on $X \times X$. Let $u$ be a cohomological correspondence of $\mathcal{F}$ on $C$. We have the zero-extension $j_C!u$ of $u$ by Corollary 2.4. We identify the cohomological correspondence $j_C!u$ with a section

$$j_C!u \in H^0_C(X \times X, \mathcal{H}).$$

The pull-back by $\delta$ and the evaluation map (2.4) induce

$$j_C!u \in H^0_C(X \times X, \mathcal{H}) \rightarrow H^0_C(X, \delta^*\mathcal{H}) \rightarrow H^0_{C\cap X}(X, j_!\mathcal{K}_U).$$

The image of $j_C!u$ under the composite defines a cohomology class in $H^0_{C\cap X}(X, j_!\mathcal{K}_U)$. We denote it by $C(j_C!\mathcal{F}, C, j_C!u)$ and call it the refined characteristic class of $j_C!\mathcal{F}$ with a cohomological correspondence $j_C!u$ on $C$. We define the characteristic class $C(j_C!\mathcal{F}, C, j_C!u) \in H^0_{C\cap X}(X, \mathcal{K}_X)$ to be the image of $C(j_C!\mathcal{F}, C, j_C!u)$ under the canonical map $H^0_{C\cap X}(X, j_!\mathcal{K}_U) \rightarrow H^0_{C\cap X}(X, \mathcal{K}_X)$.

If $\bar{C}$ is the diagonal $\delta(X) \subset X \times X$ and $u : \mathcal{F} \rightarrow \mathcal{F}$ is an endomorphism, we drop $\bar{C}$ from the notation and simply write $C(j_C!\mathcal{F}, j_!u) \in H^0(X, j_!\mathcal{K}_U)$ for the refined characteristic class and $C(j_C!\mathcal{F}, j_!u) \in H^0(X, \mathcal{K}_X)$ for the characteristic class respectively. Further, if $u$ is the identity, we simply write $C(j_C!\mathcal{F})$ (resp. $C(j_C!\mathcal{F})$) and call it the refined characteristic class of $j_C!\mathcal{F}$. (resp. the characteristic class of $j_C!\mathcal{F}$.)

### 3 Refined localized characteristic class

#### 3.1 Refined localized characteristic class

We will define a localized version of the (refined) characteristic class of a $\Lambda$-sheaf with a cohomological correspondence. Let $X$ be a scheme over a field $k$ and $U \subseteq X$ an open dense subscheme smooth of dimension $d$ over $k$, $S := X \setminus U$ the complement, $j : U \rightarrow X$ the open immersion, and $\delta_U : U \rightarrow U \times U$ and $\delta_X : X \rightarrow X \times X$ the diagonal closed immersions. Let
Let $\mathcal{F}$ be a smooth $\Lambda$-sheaf on $U$. We put $\mathcal{H} := \mathcal{RHom}(\text{pr}_2^*\mathcal{F}, \text{pr}_1^*\mathcal{F})$ on $X \times X$. The canonical map $\Lambda \to Rg_C^*\Lambda$ induces a map

$$H^0_C(X \times X, \mathcal{H}) \to H^0_C(X \times X, \mathcal{H} \otimes Rg_C^*\Lambda).$$

**Lemma 3.1.** The canonical map $H^0_{C\cap C}(X \times X, \mathcal{H} \otimes Rg_C^*\Lambda) \to H^0_C(X \times X, \mathcal{H} \otimes Rg_C^*\Lambda)$ is an isomorphism.

**Proof.** We put $\mathcal{H}_U := \mathcal{RHom}(\text{pr}_2^*\mathcal{F}, \text{pr}_1^*\mathcal{F})$ on $U \times U$. By the localization sequence, it is sufficient to prove that $H^i_C(U \times U, \mathcal{H}_U \otimes Rg_C, \Lambda) = 0$ for all $i$ where $g_C : U \times U \setminus C \to U \times U$ denotes the open immersion. Since $\mathcal{F}$ is a smooth sheaf on $U$, the canonical map $\mathcal{H}_U \otimes Rg_C, \Lambda \to Rg_C, g_C^*\mathcal{H}_U$ is an isomorphism by the projection formula. Therefore we obtain isomorphisms $H^0_C(U \times U, \mathcal{H}_U \otimes Rg_C, \Lambda) \simeq H^0(C, R^jC^!Rg_C, g_C^*\mathcal{H}_U) \simeq 0$ where $c : C \to U \times U$ is the closed immersion. Hence the assertion follows. \hfill $\Box$

The pull-back by $\delta_X$ and the evaluation map (2.4) induce a map

$$e \cdot \delta : H^0_{C\cap C}(X \times X, \mathcal{H} \otimes Rg_C^*\Lambda) \to H^0_{C\cap S}(X, j^!K_U \otimes \delta_X^*Rg_C^*\Lambda).$$

We have obtained the maps

$$H^0_C(X \times X, \mathcal{H} \otimes Rg_C^*\Lambda) \overset{\text{can.}}{\longrightarrow} H^0_{C\cap C}(X \times X, \mathcal{H} \otimes Rg_C^*\Lambda) \overset{e \cdot \delta}{\longrightarrow} H^0_{C\cap S}(X, j^!K_U \otimes \delta_X^*Rg_C^*\Lambda).$$

(3.1)

We write

$$\text{loc}_{X,C,F} : H^0_C(X \times X, \mathcal{H}) \to H^0_{C\cap C}(X \times X, \mathcal{H} \otimes Rg_C^*\Lambda)$$

(3.2)

for the composite $H^0_C(X \times X, \mathcal{H}) \to H^0_C(X \times X, \mathcal{H} \otimes Rg_C^*\Lambda) \simeq H^0_{C\cap C}(X \times X, \mathcal{H} \otimes Rg_C^*\Lambda)$.

**Definition 3.2.** Let $u$ be a cohomological correspondence of $\mathcal{F}$ on $C$ and $j_{C\cap}$ the zero-extension of the cohomological correspondence $u$ recalled in subsection 2.1. The image of the element $\text{loc}_{X,C,F}(j_{C\cap}u) \in H^0_{C\cap C}(X \times X, \mathcal{H} \otimes Rg_C^*\Lambda)$ by the map $e \cdot \delta_X^*$ defines a cohomology class in $H^0_{C\cap S}(X, j^!K_U \otimes \delta_X^*Rg_C^*\Lambda)$ and denotes

$$C^0_{S,j^!(\mathcal{F}, \mathcal{H}), j_{C\cap}u} \in H^0_{C\cap S}(X, j^!K_U \otimes \delta_X^*Rg_C^*\Lambda).$$

We call this element the refined localized characteristic class of $j^!(\mathcal{F}, \mathcal{H})$ with a cohomological correspondence $j_{C\cap}$ on $C$. If $C = \delta_U(U)$ is the diagonal and $u : \mathcal{F} \to \mathcal{F}$ is an endomorphism, we drop $C$ from the notation and we write $C^0_{S,j^!(\mathcal{F}, \mathcal{H}), j_{\delta}u} \in H^0_{S}(X, j^!K_U \otimes \delta_X^*Rg_X\Lambda)$. Further if $u$ is the identity, we simply write $C^0_{S,j^!(\mathcal{F}, \mathcal{H})}$.

We assume that $C$ is smooth purely of dimension $d$ over $k$ in the following. We have a distinguished triangle

$$c_*R^j\Lambda \to \Lambda \to Rg_C^*\Lambda \to .$$

Since $C$ is smooth over $k$, the cycle class $[C]$ defines an isomorphism $\Lambda(-d)[-2d] \simeq R^j\Lambda$ by the purity theorem. Therefore we acquire a distinguished triangle $c_*\Lambda(-d)[-2d] \to \Lambda \to .$
$Rg_{C_*}\Lambda \longrightarrow$. Applying the functor $j!(K_U \otimes \delta_U^* (-))$ to this triangle, we obtain a distinguished triangle

$$j!(K_U \otimes \delta_U^* c_4 \Lambda ) (-d)[-2d] \longrightarrow j!K_U \longrightarrow j!(K_U \otimes \delta_U^* Rg_{C_*}\Lambda ) \longrightarrow .$$

The canonical isomorphism on the right term $j!K_U \otimes \delta_X^* Rg_{C_*}\Lambda \simeq j!(K_U \otimes \delta_U^* Rg_{C_*}\Lambda )$ and the isomorphism $\Lambda_U(d)[2d] \simeq K_U$ induce a distinguished triangle

$$i_{1,*}j'!\Lambda_{U\cap C} \longrightarrow j!K_U \longrightarrow j!K_U \otimes \delta_X^* Rg_{C_*}\Lambda \longrightarrow$$

where $j': C \cap U \longrightarrow \overline{C} \cap X$ is the open immersion and $i_1 : \overline{C} \cap X \longrightarrow X$ is the closed immersion and hence a long exact sequence

$$H^0_C(C \cap X, j!*\Lambda_{U\cap C}) \longrightarrow H^0_C(C \cap X, j!K_U) \longrightarrow H^0_C(C \cap X, j!K_U \otimes \delta_X^* Rg_{C_*}\Lambda ) \longrightarrow \cdots \quad (3.3)$$

**Lemma 3.3.** Let the notation be as above. Further we assume that $X$ is smooth over $k$. If $C = \delta_U(U)$ is the diagonal, the difference $C^S_{S,1}(j!*\mathcal{F}, j!*\mathcal{U}) - \text{Tr}(U) \cdot C^S_0(j!*\Lambda_U) \in H^0_S(X, j!*\mathcal{K}_U \otimes \delta_X^* Rg_{X_*}\Lambda)$ is in the image of the injection $H^0_S(X, j!*\mathcal{K}_U) \longrightarrow H^0_S(X, j!*\mathcal{K}_U \otimes \delta_X^* Rg_{X_*}\Lambda)$ where $\text{Tr}(U)$ denotes the image of $u \in H^0(U \times U, \mathcal{H}_U) \simeq \text{End}_U(\mathcal{F})$ under the trace map $\text{End}_U(\mathcal{F}) \longrightarrow \Lambda$.

**Proof.** This is proved in the same way as in [AS, Section 5, Lemma 5.2.4.1]. □

**Definition 3.4.** Let the notation and the assumption be as in Lemma 3.3. We call the element in $H^0_S(X, j!*\mathcal{K}_U)$ lifting the difference $C^S_{S,1}(j!*\mathcal{F}, j!*\mathcal{U}) - \text{Tr}(U) \cdot C^S_0(j!*\Lambda_U) \in H^0_S(X, j!*\mathcal{K}_U \otimes \delta_X^* Rg_{X_*}\Lambda)$ by Lemma 3.3 the refined localized characteristic class of $j!*\mathcal{F}$ and denote it by $C^S_{S,1}(j!*\mathcal{F}, j!*\mathcal{U})$. We call the image of the element $C^S_{S,1}(j!*\mathcal{F}, j!*\mathcal{U})$ under the canonical map $H^0_S(X, j!*\mathcal{K}_U) \longrightarrow H^0_S(X, X_{\ast})$ the localized characteristic class of $j!*\mathcal{F}$ and denote it by $C^S_0(j!*\mathcal{F}, j!*\mathcal{U})$.

**Remark 3.5.** In the case where $X$ is smooth and $C = \delta_U(U)$, by the exact sequence similar to (3.3) and the purity theorem, we have an isomorphism $H^0_S(X, X_{\ast}) \simeq H^0_S(X, X_{\ast} \otimes \delta_X^* Rg_{X_*}\Lambda)$. Hence we obtain the localized characteristic class $C^S_0(j!*\mathcal{F})$ in $H^0_S(X, X_{\ast})$ without taking the difference. (c.f. [AS, Definition 5.2.1.]) The class $C^S_{S,1}(j!*\mathcal{F}, j!*\mathcal{U}) \in H^0_S(X, j!*\mathcal{K}_U)$ in Definition 3.4 goes to the difference $C^S_0(j!*\mathcal{F}, j!*\mathcal{U}) - \text{Tr}(U) \cdot C^S_0(j!*\Lambda) \in H^0_S(X, X_{\ast})$ by the canonical map $H^0_S(X, j!*\mathcal{K}_U) \longrightarrow H^0_S(X, X_{\ast})$.

### 3.2 Logarithmic localized characteristic class

In this subsection, for a $\Lambda$-sheaf, we will define a cohomology class with support on its wild locus by killing its tame ramifications, which we call the logarithmic localized characteristic class. We defined the localized characteristic class as a cohomology class with support on the boundary locus in subsection 3.1. To kill the tame ramifications, we use logarithmic blow-up. For a smooth sheaf of rank 1, we introduce a more elementary definition of the logarithmic localized characteristic class in [T, Definition 2.4].

Let $X$ be a smooth scheme of dimension $d$ over $k$, $U \subset X$ an open subscheme. We assume that the complement $X \setminus U = \bigcup_{i \in I} D_i$ is a divisor with simple normal crossings. Let

$$(X \times X) \subset (X \times X)'$$

denote the log product and the log blow-up with respect to divisors $\{D_i\}_{i \in I}$ defined in [AS, subsection 2.2] and [KS, subsection 1.1].
Lemma 3.6. Let the notation be as above. We consider the following cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & (X \times X)' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\delta} & X \times X
\end{array}
\]

where \( f : (X \times X)' \longrightarrow X \times X \) is the projection and \( \delta : X \longrightarrow X \times X \) is the diagonal closed immersion. Then \( X' \) is the union of the diagonal \( X \) and \((\mathbb{P}^1)^{\log} \)-bundles for a subset \( \phi \neq J \subset I \) over \( D_J \) where \( D_J \) is the intersection of \( \{ D_i \}_{i \in J} \) in \( X \).

Proof. For \( i \in I \), we define \( (X \times X)'_i \) to be the blow up of \( X \times X \) along the closed subscheme \( D_i \times D_i \subset X \times X \). Let \( X'_i \) denote the inverse image of the diagonal \( X \) by the projection \( (X \times X)'_i \longrightarrow X \times X \) for \( i \in I \). By the definition of the log blow-up, \( X'_i \) is the union of the diagonal \( X \subset (X \times X)'_i \) and a \( \mathbb{P}^1 \)-bundle over \( D_i \). Since \( (X \times X)'_i \) is the fiber product of \( (X \times X)'_i \) \( \cap \) \( (i \in I) \) over \( X \times X \), \( X' \) is the fiber product of the schemes \( \{ X'_i \}_{i \in I} \) \( \cap \) \( i \in I \) over \( X \). Therefore \( X' \) is the union of the diagonal \( X \) and \((\mathbb{P}^1)^{\log} \)-bundles for \( \phi \neq J \subset I \) over \( D_J \) where \( D_J \) is the intersection of \( \{ D_i \}_{i \in J} \) in \( X \). Hence the assertion follows. \( \square \)

We consider the following situation. Let \( X \) be a scheme of dimension \( d \) over \( k \). Let \( U \subset V \) be open subschemes of \( X \), and \( S = X \setminus V \) and \( D \subset S = X \setminus U \) the complements respectively. We assume that \( V \) is smooth over \( k \), \( D \) is a Cartier divisor and \( D \cap V \subset V \) is a divisor with simple normal crossings. Let \( j : U \longrightarrow X \) and \( j' : U \rightarrow V \) denote the open immersions. Let \( U' \) be the complement of \( D := \bigcup_{i \in I} D_i \) in \( X \). We have \( U = V \cap U' \). Let \( (X \times X) \cap (X \times X)' \) and \( (V \times V) \subset (V \times V)' \) denote the log products and the log blow-ups with respect to \( \{ D_i \}_{i \in I} \) and \( \{ D_i \cap V \}_{i \in I} \) respectively.

We consider the following commutative diagram

\[
\begin{array}{ccc}
(V \times V)' & \xrightarrow{j_1} & (U \times V)' \\
\downarrow f \downarrow & & \downarrow k_1 \downarrow \\
V \times V & \xleftarrow{f_1} & U \times V \xleftarrow{j} U \times U
\end{array}
\]

where \( (U \times V)' \subset (V \times V)' \) is the open subscheme which is the complement of the union of the proper transforms of \( (D_i \cap V) \times V \) for all \( i \in I \). (c.f. [AS Section 2.2],)

We consider the cartesian diagram

\[
\begin{array}{ccc}
(X \times X)' & \xrightarrow{j_1} & (V \times X)' \\
\downarrow f \downarrow & & \downarrow k_1 \downarrow \\
X \times X & \xleftarrow{f_1} & V \times X \xleftarrow{k} V \times V
\end{array}
\]

where the horizontal arrows are the open immersions and the vertical arrows are the projections.

Let \( F \) be a smooth \( \Lambda \)-sheaf on \( U \) which is tamely ramified along \( V \setminus U = V \cap D \). We put \( \mathcal{H}_0 := \mathcal{H}(pr^*_2 \mathcal{F}, pr^*_1 \mathcal{F}) \) on \( U \times U \) and \( \tilde{\mathcal{H}} := R\mathcal{H}(pr^*_2 j_! \mathcal{F}, Rpr^*_1 j_! \mathcal{F}) \) on \( X \times X \) respectively.

We recall a construction defined in loc. cit. We put \( \tilde{\mathcal{H}}_V := R\mathcal{H}(pr^*_2 j_! \mathcal{F}, Rpr^*_1 j_! \mathcal{F}) \) on \( V \times V \), \( \tilde{\mathcal{H}}_V := (\tilde{j}_* \mathcal{H}_0)(d)[2d] \) on \( (V \times V)' \) and \( \tilde{\mathcal{H}}_V := \tilde{j}_1 \tilde{R}k_1(\tilde{j}_* \mathcal{H}_0)(d)[2d] \) on \( (V \times V)' \) respectively.
There exists a unique map
\[ f_V^* \mathcal{H}_V \rightarrow \mathcal{H}'_V \] (3.5)
inducing the canonical isomorphism \( R\text{Hom}(\text{pr}_2^*\mathcal{F}, R\text{pr}_1^*\mathcal{F}) \rightarrow \mathcal{H}_0(d)[2d] \) on \( U \times U \) by [S] the proof of Proposition 3.1.1.1.

We put \( \mathcal{H}' := j_{11}^! Rk_{1*}^! \mathcal{H}_V \) on \((X \times X)\)''. We define a map
\[ f^* \mathcal{H} \rightarrow \mathcal{H}' \] (3.6)
to be the composite of the following maps
\[ f^* \mathcal{H} \simeq f^* j_{11} Rk_{1*} \mathcal{H}_V \simeq j_{11}^! f_V^* \mathcal{H}_V \rightarrow j_{11}^! Rk_{1*} f_V^* \mathcal{H}_V \rightarrow j_{11}^! Rk_{1*}^! \mathcal{H}_V = \mathcal{H}' \]
where the first map is induced by the Kunneth formula and the second and third maps are induced by the base change maps \( f^* j_{11} \simeq j_{11}^! f_V^* \) and \( f_V^* Rk_{1*} \rightarrow Rk_{1*}^! f_V^* \) and the fourth map is induced by the map (3.5).

**Lemma 3.7.** Let the notation be as above. Then the adjunction of the map \( \mathcal{H} \rightarrow Rf_* \mathcal{H}' \) is an isomorphism.

**Proof.** By [AS] lemma 2.2.4, the adjunction \( \mathcal{H}_V \rightarrow Rf_{V*} \mathcal{H}'_V \) is an isomorphism. Since \( f \) is proper, the assertion follows from the cartesian diagram (3.4) and the definition of \( \mathcal{H}' \).

We consider the following cartesian diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{i'} & (X \times X)' \\
\downarrow f' & & \downarrow g' \\
X & \xrightarrow{g_X} & X \times X \setminus \delta_X(X)
\end{array}
\]
where \( f : (X \times X)' \rightarrow X \times X \) is the projection, \( i' : X' \rightarrow (X \times X)' \) is the closed immersion and \( g' : (X \times X)' \setminus X' \rightarrow (X \times X)' \) is the open immersion. Let \( \delta' : X \rightarrow (X \times X)', \delta_V : V \rightarrow (V \times V) \) and \( \delta_V' : V \rightarrow (V \times V)' \) be the logarithmic diagonal closed immersions induced by the universality of blow-up. Let \( V' \) denote the intersection \( X' \cap (V \times V)' \) in \((X \times X)'\). Let \( g_V' : (V \times V)' \setminus V' \rightarrow (V \times V)' \) be the open immersion.

We define an evaluation map. The composite of the canonical isomorphism \( \delta'^* \mathcal{H}' \simeq j_{V!} \delta_V^* ((j_* \mathcal{H}_0)(d)[2d]) \) and an evaluation map \( j_{V!} e : j_{V!} \delta_V^* ((j_* \mathcal{H}_0)(d)[2d]) \rightarrow j_{V!} \Lambda_V(d)[2d] = j_{V!} \mathcal{K}_V([AS] (2.9)) \) induces an evaluation map
\[ e' : \delta'^* \mathcal{H}' \rightarrow j_{V!} \mathcal{K}_V. \] (3.7)
The map (3.4) induces the pull-back
\[ f^* : H^0_X(X \times X, \mathcal{H}) \rightarrow H^0_{X'}((X \times X)', \mathcal{H}'). \] (3.8)
The canonical map \( \Lambda \rightarrow Rg'_* \Lambda \) induces a map
\[ H^0_{X'}((X \times X)', \mathcal{H}') \rightarrow H^0_{X'}((X \times X)', \mathcal{H}' \otimes Rg'_* \Lambda). \]

**Lemma 3.8.** Let the notation be as above. Then the canonical map
\[ H^0_{X \setminus V'}((X \times X)', \mathcal{H}' \otimes Rg'_* \Lambda) \rightarrow H^0_{X'}((X \times X)', \mathcal{H}' \otimes Rg'_* \Lambda) \]
is injective.
Proof. By the localization sequence, it suffices to prove \( H^{-1}_{\nu}(V \times V)', \tilde{\mathcal{H}}' \otimes Rg'_\nu, \Lambda) = 0. \) In the following, we may assume that \( V = X \) and \( \mathcal{F} \) is tamely ramified along the boundary \( \partial X_1 \cup U = D \) which is a divisor with simple normal crossings. We will prove the vanishing \( H^{-1}_{\nu}(X \times X)', \tilde{\mathcal{H}}' \otimes Rg'_\nu, \Lambda) = 0. \) Let \( P \) denote the closed subscheme \( X \setminus U \) of \( (X \times X)' \). By a similar argument to the proof of Lemma \( 5.1 \), we obtain an isomorphism \( H^{-1}_{\nu}(X \times X)', \tilde{\mathcal{H}}' \otimes Rg'_\nu, \Lambda) \simeq H^{-1}_{\nu}(X \times X)', \tilde{\mathcal{H}}' \otimes Rg'_\nu, \Lambda) \). We have a distinguished triangle

\[ i'_*(i'' \tilde{\mathcal{H}}' \otimes R\iota'_! \Lambda) \to \tilde{\mathcal{H}}' \to \tilde{\mathcal{H}}' \otimes Rg'_\nu \Lambda \to \]

and hence a long exact sequence

\[ H^{-1}_{\nu}(X \times X)', \tilde{\mathcal{H}}' \to H^{-1}_{\nu}(X \times X)', \tilde{\mathcal{H}}' \otimes Rg'_\nu, \Lambda) \to H^0(X', i'' \tilde{\mathcal{H}}' \otimes R\iota'_! \Lambda) \to \]

\[ \cdots . \]

Because we have \( H^i(X \times X)', \tilde{\mathcal{H}}' \simeq H_{\nu}(X \times X, \tilde{\mathcal{H}}) \simeq 0 \) for all \( i \) by the isomorphism \( Rf_\nu, \tilde{\mathcal{H}}' \simeq \mathcal{H} \) proved in \( \text{[AS, Lemma 2.2.4]} \), we obtain an isomorphism \( H^{-1}_{\nu}(X \times X)', \tilde{\mathcal{H}}' \otimes Rg'_\nu, \Lambda) \simeq H^0_{\nu}(X', i'' \tilde{\mathcal{H}}' \otimes R\iota'_! \Lambda) \) by the above long exact sequence.

Since we have \( X' = \delta'(X) \cup P \) and \( D = \delta'(X) \cap P \), we acquire the following long exact sequence by the excision

\[ H^0(D, i^*_D \tilde{\mathcal{H}}' \otimes R\iota'_D \Lambda) \to H^0_{\nu}(X, i'' \tilde{\mathcal{H}}' \otimes R\iota'! \Lambda) \to H^0(P, i^*_P \tilde{\mathcal{H}}' \otimes R\iota'_P \Lambda) \to H^0_{\nu}(X', i'' \tilde{\mathcal{H}}' \otimes R\iota'_! \Lambda) \]

\[ \to H^1(D, i^*_D \tilde{\mathcal{H}}' \otimes R\iota'_D \Lambda) \to \cdots \text{ where } i_P : P \to (X \times X)' \text{ and } i_D : D \to (X \times X)' \text{ are the closed immersions. By the purity theorem and [S Corollary 2.21(3)], we obtain isomorphisms } H^0(D, i^*_D \tilde{\mathcal{H}}' \otimes R\iota'_D \Lambda) \simeq H^0_{\nu}(X, i'' \tilde{\mathcal{H}}' \otimes R\iota'! \Lambda) \text{ and } i^*_P : P \to (X \times X)' \text{ is the extension by zero of a smooth sheaf on the complement } X \setminus D, \text{ for } i \in I \text{ in the same way as in [AS, the proof of Lemma 2.2.4]. Since } (X \times X)' \text{ is the fiber product of } (X \times X)'_i \text{ over } X \times X \text{ and } P_j \text{ is the fiber product of } P_i \subset (X \times X)'_i \text{ for } i \in J \text{ and the diagonal } X \subset (X \times X)'_i \text{ for } i \in I \setminus J \text{ over } X \text{ where } P_i \text{ is the } \mathbb{P}^1 \text{-bundle over } D_i. \text{ Hence, by the Kunneth formula, it is reduced to the case where } D \text{ is a smooth divisor. By the cartesian diagram}

\[
\begin{array}{ccc}
P & \to & (X \times X)' \\
\downarrow i_P & & \downarrow f \\
P_j & \to & f_j \\
D & \to & X \times X,
\end{array}
\]

we obtain \( RF_{\nu}(\tilde{\mathcal{H}}'|_P) \simeq (RF_{\nu}(\tilde{\mathcal{H}}'))|_P \) by the proper base change theorem. By the isomorphism \( RF_{\nu}(\tilde{\mathcal{H}}') \simeq \mathcal{H} \) proved in loc. cit., the assertion follows.

We have a cohomological correspondence \( j_!1 = \text{id}_{j_!\mathcal{F}} \in H^0_{\nu}(X \times X, \tilde{\mathcal{H}}) \) and its pull-back \( f^*\text{id}_{j_!\mathcal{F}} \in H^0_{\nu}(X \times X), \tilde{\mathcal{H}}) \) by \( \text{[SS]}. \)

Lemma 3.9. Let the notation be as above. There exists a unique element \( (f^*\text{id}_{j_!\mathcal{F}})' \) in \( H^0_{\nu}(X \times X), \tilde{\mathcal{H}} \otimes Rg'_\nu, \Lambda) \) which is sent to \( f^*\text{id}_{j_!\mathcal{F}} \) by the canonical map \( H^0_{\nu}(X \times X), \tilde{\mathcal{H}} \otimes Rg'_\nu, \Lambda) \to H^0_{\nu}(X \times X), \tilde{\mathcal{H}} \otimes Rg'_\nu, \Lambda) \).
Definition 3.11. Let the notation and the assumption be as in Lemma 3.10. We call the vanishing of the localized characteristic class, i.e. equality $e \cup |V| \in H^0_V((V \times V), \mathcal{H}_V \otimes Rg_{r^*} \lambda)$, the image of the class $\delta^* \in H^0_S((X, j, \mathcal{K}_S \otimes \delta^* Rg_{r^*} \lambda)$ by $\Sigma$ Proposition 3.11.2]. We consider the following commutative diagram

$$
eq \rightarrow \quad e \cup [V] \in H^0_V((V \times V); \mathcal{H}_V) \quad \rightarrow \quad H^0_V((V \times V), \mathcal{H}_V \otimes Rg_{r^*} \lambda)$$

where $\mathcal{g}_V : (V \times V) \otimes (V \times V)$ is the open immersion and the horizontal arrows are induced by the canonical map $\lambda \rightarrow Rg_{r^*} \lambda$. Since we have $H^2_0((V \times V), Rg_{r^*} \lambda) = 0$, we acquire an equality $e \cup |V| = 0$ in $H^0_V((V \times V), \mathcal{H}_V \otimes Rg_{r^*} \lambda) = H^0_V((V \times V), \mathcal{H}_V \otimes Rg_{r^*} \lambda)$ by the above commutative diagram. Hence we obtain the vanishing $f^* \text{id}_{j, F} = 0$ in $H^0_V((V \times V), \mathcal{H}_V \otimes Rg_{r^*} \lambda)$.

Lemma 3.10. Let the notation be as above. Further we assume that $X$ is smooth over $k$. Then the canonical map induced by the map $\lambda \rightarrow \delta^* Rg_{r^*} \lambda$

$$H^0_S((X, j, \mathcal{K}_S \otimes \delta^* Rg_{r^*} \lambda) \rightarrow H^0_S((X, X) \otimes \delta^* Rg_{r^*} \lambda)$$

is an isomorphism.

Proof. This is proved in $[T, \text{Lemma 2.3}$].

The pull-back by $\delta^*$ and the evaluation map $\Sigma$ induce a map

$$e' \cdot \delta^* : H^0_S((X \times X), \mathcal{H}_V \otimes Rg_{r^*} \lambda) \rightarrow H^0_S((X, j, \mathcal{K}_S \otimes \delta^* Rg_{r^*} \lambda)$$

We have obtained the following maps

$$H^0_S((X \times X), \mathcal{H}_V) \xrightarrow{\text{can.}} H^0_S((X \times X), \mathcal{H}_V \otimes Rg_{r^*} \lambda)$$

$$f^* \xrightarrow{\Sigma} H^0_S((X \times X), \mathcal{H}_V \otimes Rg_{r^*} \lambda)$$

$$\text{id}_{j, F} \in H^0(X, X, \mathcal{H}_V)$$

$$H^0_S((X \times X), \mathcal{H}_V \otimes Rg_{r^*} \lambda) \xrightarrow{e' \cdot \delta^*} H^0_S((X, j, \mathcal{K}_S \otimes \delta^* Rg_{r^*} \lambda)$$

By the map $e' \cdot \delta^*$ and Lemmas $\Sigma$ and $\Sigma$, we obtain a class $e' \cdot \delta^* (f^* \text{id}_{j, F})'$ in $H^0_S((X, j, \mathcal{K}_S \otimes \delta^* Rg_{r^*} \lambda)$. We put $C^\log_{S, 0}(j, F) := e' \cdot \delta^* (f^* \text{id}_{j, F})'$. We denote by $C^\log_{S, 0}(j, F) \in H^0_S((X, X) \otimes \delta^* Rg_{r^*} \lambda)$, the image of the class $C^\log_{S, 0}(j, \mathcal{F})$ under the canonical map $H^0_S((X, j, \mathcal{K}_S \otimes \delta^* Rg_{r^*} \lambda) \rightarrow H^0_S((X, X) \otimes \delta^* Rg_{r^*} \lambda) \simeq H^0_S((X, X, \mathcal{K}_S \otimes \delta^* Rg_{r^*} \lambda)$.

Definition 3.11. Let the notation and the assumption be as in Lemma $\Sigma$. We call the class $C^\log_{S, 0}(j, \mathcal{F})$ in $H^0_S((X, \mathcal{K}_X)$ the logarithmic localized characteristic class of $j, \mathcal{F}$. We call the element $C^\log_{S, 0}(j, \mathcal{F})$ in $H^0_S((X, j, \mathcal{K}_S \otimes \delta^* Rg_{r^*} \lambda)$ the refined logarithmic localized characteristic class of $j, \mathcal{F}$. We put the difference $C^\log_{S, 0}(j, \mathcal{F}) := C^\log_{S, 0}(j, \mathcal{F}) - \text{rk}(\mathcal{F}) C^\log_{S, 0}(j, \lambda) \in H^0_S((X, \mathcal{K}_X)$.

Let the notation be as above. In the following, we assume that $V = X, X \setminus D = \emptyset$ is a divisor with simple normal crossings and that $\mathcal{F}$ is tamely ramified along the boundary $D$. We will prove the vanishing of the localized characteristic class, i.e. $C^\log_{S, 0}(j, \mathcal{F}) = 0$. This vanishing plays a key role in the proof of the localized Abbes-Saito formula.
Remark 3.12. Let the notation be as in Definition 3.11. We expect that the logarithmic localized characteristic class $C^S\log(\delta)_{ij}(F)$ is sent to the localized characteristic class $C^0(\delta)_{ij}(F)$ by the canonical map $H^0_S(X, \mathcal{K}_X) \to H^0_S \cup D(X, \mathcal{K}_X)$. If we admit this, the vanishing $C^0_S(\delta)_{ij}(F) = 0$ will follow from Definition 3.11 by putting $X = V, S = \emptyset$. However, we do not know a proof. We give a proof of the vanishing $C^0_S(\delta)_{ij}(F) = 0$ in the following.

We write $H^0'_{\mathcal{F}}$ and $e_{\mathcal{F}} \in \Gamma(X, j_* \mathcal{H}_0|_X)$ for the sheaf $H^0'$ and the unique section $e \in \Gamma(X, j_* \mathcal{H}_0|_X)$ lifting the identity $\Gamma(U, \mathcal{H}_0|_U)$ to emphasize that they are associated to the sheaf $\mathcal{F}$.

The canonical map $\Lambda \to f^*Rg_X\Lambda$ induces a map $H^0_X((X \times X)', \mathcal{H}'_{\mathcal{F}}) \to H^0_X((X \times X)', \mathcal{H}'_{\mathcal{F}} \otimes f^*Rg_X\Lambda)$. By the same argument as the proof of Lemma 3.13, the canonical map $H^0_X((X \times X)', \mathcal{H}'_{\mathcal{F}} \otimes f^*Rg_X\Lambda) \to H^0_X((X \times X)', \mathcal{H}'_{\mathcal{F}} \otimes f^*Rg_X\Lambda)$ is an isomorphism. The image of $f^*\text{id}_{\mathcal{F}}$ under the composite $H^0_X((X \times X)', \mathcal{H}'_{\mathcal{F}} \otimes f^*Rg_X\Lambda) \to H^0_X((X \times X)', \mathcal{H}'_{\mathcal{F}} \otimes f^*Rg_X\Lambda)$ denotes $(f^*\text{id}_{\mathcal{F}})^{\text{loc}} \in H^0_X((X \times X)', \mathcal{H}'_{\mathcal{F}} \otimes f^*Rg_X\Lambda)$ under this map.

Lemma 3.13. Let the notation be as above. Then we have the following vanishing

$$C^0_S(\delta)_{ij}(F) = 0$$

in $H^0_b(X, \mathcal{K}_X)$.

Proof. We prove an equality $C^0_S(\delta)_{ij}(F) = e' \cdot \delta^* (f^*\text{id}_{\mathcal{F}})^{\text{loc}}$ in $H^0_b(X, \mathcal{K}_X)$. This follows from Definition 3.11, Remark 3.12 and the following commutative diagram

$$
\begin{array}{ccc}
\text{id}_{\mathcal{F}} & \in & H^0_X((X \times X), \mathcal{H}_{\mathcal{F}}) \\
f^* & & f^* \\
\text{id}_{\mathcal{F}} & \in & H^0_X((X \times X)', \mathcal{H}'_{\mathcal{F}} \otimes f^*Rg_X\Lambda) \\
\text{can} & & \text{can} \\
H^0_X((X \times X), \mathcal{H}_{\mathcal{F}} \otimes Rg_X\Lambda) & & H^0_X((X \times X)', \mathcal{H}'_{\mathcal{F}} \otimes f^*Rg_X\Lambda) \\
f^* & & f^* \\
\text{can} & & \text{can} \\
H^0_X((X \times X), \mathcal{H}_{\mathcal{F}} \otimes Rg_X\Lambda) & & H^0_X((X \times X)', \mathcal{H}'_{\mathcal{F}} \otimes f^*Rg_X\Lambda) \\
ed^* & & ed^* \\
H^0_X((X \times X), \mathcal{H}_{\mathcal{F}} \otimes Rg_X\Lambda) & & H^0_X((X \times X), \mathcal{H}_{\mathcal{F}} \otimes Rg_X\Lambda) \\
f^* & & f^* \\
ed^* & & ed^* \\
H^0_X((X \times X), \mathcal{H}_{\mathcal{F}} \otimes Rg_X\Lambda) & & H^0_X((X \times X), \mathcal{H}_{\mathcal{F}} \otimes Rg_X\Lambda)
\end{array}
$$

where the vertical arrows are induced by the map $\text{can} f^*H \to H_{\mathcal{F}}$. By this diagram and $f^*\text{id}_{\mathcal{F}} = e_{\mathcal{F}} \cup [X]$ by [3, Proposition 3.1.1.2], we acquire equalities $C^0_S(\delta)_{ij}(F) = e' \cdot \delta^* (f^*\text{id}_{\mathcal{F}})^{\text{loc}} = e'(e_{\mathcal{F}}) \cdot \delta^* [X] = \text{rk}(\mathcal{F}) \cdot \delta^* [X]$ and $C^0_S(\delta)_{ij}(F) = e'(e_{\mathcal{F}}) \cdot \delta^* [X] = \delta^* [X]$. Hence the assertion follows.

In the following, we calculate the localized characteristic class by the localized Chern class using Lemma 3.13 in the tamely ramified case. We will not use the results in the following sections. We recall the definition of the localized Chern class from [KS, Section 3.4]. Let $X$ be a scheme of finite type over $k$ and $Z \subset X$ be a closed subscheme. Let $\mathcal{E}$ and $\mathcal{F}$ be locally free $\mathcal{O}_X$-modules of rank $d$ and $f : \mathcal{E} \to \mathcal{F}$ be an $\mathcal{O}_X$-linear map. We assume that $f : \mathcal{E} \to \mathcal{F}$ is an isomorphism on $X \setminus Z$. We consider the complex $K = [\mathcal{E} \to \mathcal{F}]$ of $\mathcal{O}_X$-modules by putting $\mathcal{F}$ on degree 0.
Then, the localized Chern class $c^X_j(\mathcal{K}) - 1$ is defined as an element of $\text{CH}^* (Z \rightarrow X)$ in [Ful Chapter 18.1]. We define an element $c(\mathcal{F} - \mathcal{E})^X_j = (c_i(\mathcal{F} - \mathcal{E})^X_j)_{i>0}$ of $\text{CH}^* (Z \rightarrow X)$ by

$$c(\mathcal{F} - \mathcal{E})^X_j = c(\mathcal{E}) \cap (c^X_j(\mathcal{K}) - 1).$$

In other words, we put $c_i(\mathcal{F} - \mathcal{E})^X_j = \sum_{j=0}^{\text{min}(d,i-1)} c_j(\mathcal{E}) \cap c_{i-j}^X(\mathcal{K})$ for $i > 0$. The image of $c(\mathcal{F} - \mathcal{E})^X_j$ in $\text{CH}^* (X)$ is the difference $c(\mathcal{F}) - c(\mathcal{E})$ of Chern classes.

**Lemma 3.14.** Let $X$ be a smooth scheme over $k$ of dimension $d$, $U$ an open dense subscheme and $D$ the complement $X \setminus U$. We assume that $D$ is a divisor with simple normal crossings of $X$. Let $j : U \rightarrow X$ be the open immersion. Let $\{D_i\}_{i \in I}$ be the irreducible components of $D$. For a subset $J \subseteq I$, we put $D_J := \bigcap_{i \in J} D_i$ and $B_J := \bigcup_{i \notin J} (D_i \cap D_j)$. Let $j_J : D_J - B_J \rightarrow D_J$ be the open immersion.

1. Then, we have

$$C^0_D(j_U^* \Lambda_U) = -\sum_{r = 1}^{\text{min}(d,n)} \Sigma_{|J| = r} C(j_J^* \Lambda_{D_J - B_J})$$

in $H^0_D(X, \mathcal{K}_X)$ where $|I| = n$.

2. We have

$$c_d(\Omega^1_{X/k}(\log D) - \Omega^1_{X/k})^X \cap [X] = -\sum_{r = 1}^{\text{min}(d,n)} \Sigma_{|J| = r} (-1)^r c_d(\Omega^1_{D_J/k}(\log B_J)) \cap [D_J]$$

in $\text{CH}_0(D)$.

**Proof.** The assertion 1 is easy. We omit a proof. We prove 2. We have an exact sequence

$$0 \rightarrow \Omega^1_{X/k} \rightarrow \Omega^1_{X/k}(\log D) \rightarrow \bigoplus_{i \in I} \mathcal{O}_{D_i} \rightarrow 0.$$ 

We put $\mathcal{K} := \bigoplus_{i \in I} \mathcal{O}_{D_i}$. The above sequence induces equalities $c(\Omega^1_{X/k}(\log D) - \Omega^1_{X/k})^X \cap [X] = c(\Omega^1_{X/k}) \cap (c^X_j(\mathcal{K}) - 1) \cap [X] = c(\Omega^1_{X/k}) \cap c^X_j(\mathcal{K}) \cap (1 - c^X_j(\mathcal{K})^{-1}) \cap [X]$ and $(1 - c^X_j(\mathcal{K})^{-1}) \cap [X] = -\sum_{r = 1}^{\text{min}(n,d)} \Sigma_{r \leq |J|, r | J} (-1)^r [D_J]$. Therefore we obtain an equality $c(\Omega^1_{X/k}(\log D) - \Omega^1_{X/k})^X \cap [X] = -\sum_{r = 1}^{\text{min}(n,d)} (-1)^r \sum_{r \leq |J|, r | J} c_{\Omega^1_{X/k}}(J) \cap [D_J]$. On the other hand, the following equality holds $c(\Omega^1_{X/k}(\log D)) = c(\Omega^1_{X/k}(\log D)) \cap c^X_j(\mathcal{K})^{-1}$. Hence we acquire $c(\Omega^1_{X/k}(\log D) - \Omega^1_{X/k})^X \cap [X] = -\sum_{r = 1}^{\text{min}(n,d)} (-1)^r \sum_{r \leq |J|, r | J} c_{\Omega^1_{X/k}(\log D)) \cap [D_J]$. The assertion follows from an equality $c(\Omega^1_{X/k}(\log D)) \cap [D_J] = (\Omega^1_{D_J/k}(\log B_J)) \cap [D_J]$.

**Corollary 3.15.** Let the notation be as in Lemma 3.14 and $\mathcal{F}$ be a smooth $\Lambda$-sheaf on $U$ which is tamely ramified along $D$. Then, we have

$$C^0_D(j_U^* \mathcal{F}) = (-1)^d \cdot \text{rk}(\mathcal{F}) \cdot c_d(\Omega^1_{X/k}(\log D) - \Omega^1_{X/k})^X \cap [X]$$

in $H^0_D(X, \mathcal{K}_X)$.

**Proof.** By Lemma 3.13 and the assumption that $\mathcal{F}$ is tamely ramified along $D$, we obtain an equality $C^0_D(j_U^* \mathcal{F}) - \text{rk}(\mathcal{F}) \cdot C^0_D(j_U^* \Lambda_U) = 0$. Therefore the assertion is reduced to an equality $C^0_D(j_U^* \Lambda_U) = (-1)^d \cdot c_d(\Omega^1_{X/k}(\log D) - \Omega^1_{X/k})^X \cap [X]$.

By Lemma 3.14, the following equality holds:

$$C^0_D(j_U^* \Lambda_U) = -\sum_{r = 1}^{\text{min}(n,d)} (-1)^r c_{\Omega^1_{D_J/k}(\log B_J)) \cap [D_J].$$

Hence the assertion follows from Lemma 3.13.
3.3 Pull-back

In this subsection, we will prove the compatibility of the refined localized characteristic class with pull-back. Let $X$ and $Y$ be schemes over $k$, $U \subseteq X$ and $V \subseteq Y$ open dense subschemes smooth of dimension $d$ over $k$, and $S := X \setminus U$ and $T := Y \setminus V$ the complements respectively. We consider a cartesian diagram

\[
\begin{array}{c}
V 
\begin{array}{c}
\downarrow j_V \\
Y 
\end{array}
\begin{array}{c}
\downarrow j \\
T 
\end{array}
\begin{array}{c}
\downarrow f \\
U 
\end{array}
\begin{array}{c}
\downarrow j \\
X 
\end{array}
\begin{array}{c}
\downarrow f \\
S 
\end{array}
\end{array}
\]

where $f : Y \to X$ is a proper morphism and $f : V \to U$ is a finite flat morphism.

Let $C \subset U \times U$ be a closed subscheme purely of dimension $d$. Let $\overline{C}$ be the closure of $C$ in $X \times X$, $C' \subset V \times V$ the inverse image of $C \subset U \times U$ by $f \times f : V \times V \to U \times U$ and $C''$ the closure of $C'$ in $Y \times Y$. We also assume $C = \overline{C} \cap (X \times U)$. Let $j_C : C \to \overline{C}$ and $j_{C'} : C' \to C'$ denote the open immersions. We consider the following cartesian diagram

\[
Y \times Y \setminus \overline{C'} \xrightarrow{g'} Y \times Y
\]
\[
\begin{array}{c}
X \times X \setminus \overline{C} 
\begin{array}{c}
\downarrow g \\
X \times X 
\end{array}
\end{array}
\]

where $g : X \times X \setminus \overline{C} \to X \times X$ and $g' : Y \times Y \setminus \overline{C'} \to Y \times Y$ are the open immersions.

Let $\mathcal{F}$ be a smooth $\Lambda$-sheaf on $U$ and $u$ a cohomological correspondence on $C$. We put $\mathcal{F}_V = f^* \mathcal{F}$ on $V$, $\mathcal{H} := R\mathcal{H}om(pr_2^* j_V \mathcal{F}, pr_1^* j_V \mathcal{F})$ on $X \times X$ and $\mathcal{H}' := R\mathcal{H}om(pr_2^* j_V \mathcal{F}, pr_1^* j_V \mathcal{F})$ on $Y \times Y$ respectively.

We define a map

\[
\tilde{f}^* (j_K U \otimes \delta_X R\mathcal{g}_* \Lambda) \to j_{V'} \mathcal{K}_V \otimes \delta_Y R\mathcal{g}'_* \Lambda
\]  
(3.9)

to be the composition of the following maps

\[
\tilde{f}^* (j_K U \otimes \delta_X R\mathcal{g}_* \Lambda) \to \tilde{f}^* j_K U \otimes \delta_Y R\mathcal{g}'_* \Lambda \to j_{V'} \mathcal{K}_V \otimes \delta_Y R\mathcal{g}'_* \Lambda
\]

where the first map is induced by the base change map $\tilde{f}^* \mathcal{F} \to \mathcal{F}$ and the second map is induced by an isomorphism $f^* \mathcal{K}_U \simeq \mathcal{K}_V$ by the assumption that $U, V$ are smooth schemes of the same dimension. The map (3.9) induces the pull-back

\[
\tilde{f}^* : H^0_{\mathcal{C}' \cap \mathcal{S}}(X, j_K U \otimes \delta_X R\mathcal{g}_* \Lambda) \to H^0_{\mathcal{C}' \cap \mathcal{T}}(Y, j_{V'} \mathcal{K}_V \otimes \delta_Y R\mathcal{g}'_* \Lambda).
\]  
(3.10)

Proposition 3.16. (Pull-back) Let the notation be as above. Then we have an equality

\[
C^\mathcal{F}_{\mathcal{T}, 1}(j_{V'} \mathcal{F}, \mathcal{C}', j_{C'}(f \times f)^* u) = \tilde{f}^* C^\mathcal{F}_{\mathcal{S}, 1}(j_{C} \mathcal{F}, \mathcal{C}, j_{C} u)
\]
in $H^0_{\mathcal{C}' \cap \mathcal{T}}(Y, j_{V'} \mathcal{K}_V \otimes \delta_Y R\mathcal{g}'_* \Lambda)$.

Proof. We consider the commutative diagram

\[
\begin{array}{c}
H^0_{\mathcal{C}}(Y \times Y, \mathcal{H}) 
\begin{array}{c}
\downarrow (f \times f) \\
H^0_{\mathcal{C} \cap \mathcal{C}'}(X \times X, \mathcal{H} \otimes R\mathcal{g}_* \Lambda) 
\end{array}
\begin{array}{c}
\downarrow (f \times f) \\
H^0_{\mathcal{C} \cap \mathcal{T}}(Y, j_{V'} \mathcal{K}_V \otimes \delta_Y R\mathcal{g}'_* \Lambda) 
\end{array}
\begin{array}{c}
\downarrow \tilde{f}^* \\
H^0_{\mathcal{C}}(X \times X, \mathcal{H}) 
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \tilde{f}^* \\
H^0_{\mathcal{C}}(X \times X, \mathcal{H}) 
\end{array}
\begin{array}{c}
\downarrow \tilde{f}^* \\
H^0_{\mathcal{C} \cap \mathcal{S}}(X, j_K U \otimes \delta_X R\mathcal{g}_* \Lambda) 
\end{array}
\begin{array}{c}
\downarrow \tilde{f}^* \\
H^0_{\mathcal{C} \cap \mathcal{T}}(Y, j_{V'} \mathcal{K}_V \otimes \delta_Y R\mathcal{g}'_* \Lambda) 
\end{array}
\end{array}
\]

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The assertion follows from this commutative diagram, \( j_{C^1}(f \times f)^* u = (\tilde{f} \times \tilde{f})^* (j_{C^1} u) \) by [AS] in the proof of Proposition 2.1.9] and Definition 3.2.

We keep the same notation as above. Let \( \delta_U : U \longrightarrow U \times U \) denote the diagonal map. In the following, we consider the case where \( C \) is the diagonal \( \delta_U(U) \). Further, we assume that \( f : V \longrightarrow U \) is a finite Galois étale morphism of Galois group \( G \). Let \( u \) be an endomorphism of \( \mathcal{F} \). Given \( \sigma \in G \), let \( \Gamma_{\sigma} \subset V \times V \) be the graph of \( \sigma : V \longrightarrow V \). Then we have \( \Gamma_{\sigma} \cap V \times V = \bigsqcup_{\sigma \in G} \Gamma_{\sigma} \), since \( f \) is a finite Galois étale morphism. Let \( \bar{\Gamma}_{\sigma} \) be the closure of \( \Gamma_{\sigma} \) in \( V \times Y \). Let \( j_{\sigma} : \Gamma_{\sigma} \longrightarrow \bar{\Gamma}_{\sigma} \) denote the open immersion. For \( \sigma \in G \), let \( \sigma^* : \sigma^* f^* \mathcal{F} \longrightarrow f^* \mathcal{F} \) be the canonical map. We consider the composite \( f^*(u) \circ \sigma^* : \sigma^* f^* \mathcal{F} \longrightarrow f^* \mathcal{F} \) as a cohomological correspondence of \( f^* \mathcal{F} \) on the graph \( \Gamma_{\sigma} \subset V \times V \). We have the pull-back

\[
\tilde{f}^* : H^0_{\Gamma_{\sigma} \cap V}(X, j_{\sigma} \mathcal{K}_U) \longrightarrow H^0_{\Gamma_{\sigma} \cap V}(Y, j_{\sigma} \mathcal{K}_V).
\]

By this isomorphism, we obtain a class

\[
C^{00}_{\mathcal{F}, j_{\sigma}}(\tilde{f}^*(u) \circ \sigma^*) := C^{00}_{\mathcal{F}, j_{\sigma}}(\tilde{f}^*(u) \circ \sigma^*) \in H^0_{\Gamma_{\sigma} \cap V}(Y, j_{\sigma} \mathcal{K}_V) \cong H^0_{\Gamma_{\sigma} \cap V}(Y, j_{\sigma} \mathcal{K}_V \otimes \delta_{\bar{\Gamma}_{\sigma}} R\tilde{g}_{\sigma}^* \Lambda).
\]

By the canonical map \( H^0_{\Gamma_{\sigma} \cap V}(Y, j_{\sigma} \mathcal{K}_V) \longrightarrow H^0_{\Gamma_{\sigma} \cap V}(Y, j_{\sigma} \mathcal{K}_V \otimes \delta_{\bar{\Gamma}_{\sigma}} R\tilde{g}_{\sigma}^* \Lambda) \), the class \( C^{00}_{\mathcal{F}, j_{\sigma}}(\tilde{f}^*(u) \circ \sigma^*) \) is equal to the refined characteristic class \( C^{00}_{\mathcal{F}, j_{\sigma}}(\tilde{f}^*(u) \circ \sigma^*) \) recalled in subsection 2.2 by Definition 3.2.

**Corollary 3.17.** Let the notation be as above. Further we assume that \( X, Y \) are smooth over \( k \). Then, we have an equality

\[
\tilde{f}^* C^{00}_{\mathcal{F}, j_{\sigma}}(j_{\sigma} \mathcal{F}, j_{\sigma} u) = \sum_{\sigma \in G} C^{00}_{\mathcal{F}, j_{\sigma}}(j_{\sigma} \mathcal{F}, j_{\sigma}(f^*(u) \circ \sigma^*))
\]

in \( H^0_{\Gamma_{\sigma} \cap V}(Y, j_{\sigma} \mathcal{K}_V) \).

**Proof.** By Proposition 3.10 and \((f \times f)^* u = \sum_{\sigma \in G} f^*(u) \circ \sigma^* \) by [AS] the proof of Corollary 2.1.11], we obtain an equality

\[
\tilde{f}^* C^{00}_{\mathcal{F}, j_{\sigma}}(j_{\sigma} \mathcal{F}, j_{\sigma} u) = \sum_{\sigma \in G} C^{00}_{\mathcal{F}, j_{\sigma}}(j_{\sigma} \mathcal{F}, j_{\sigma}(f^*(u) \circ \sigma^*))
\]

in \( H^0_{\Gamma_{\sigma} \cap V}(Y, j_{\sigma} \mathcal{K}_V \otimes \delta_{\bar{\Gamma}_{\sigma}} R\tilde{g}_{\sigma}^* \Lambda) \). Since the canonical map \( H^0_{\Gamma_{\sigma} \cap V}(Y, j_{\sigma} \mathcal{K}_V) \longrightarrow H^0_{\Gamma_{\sigma} \cap V}(Y, j_{\sigma} \mathcal{K}_V \otimes \delta_{\bar{\Gamma}_{\sigma}} R\tilde{g}_{\sigma}^* \Lambda) \) is injective, the assertion follows from Definition 3.2.

4 Proof of the localized Abbes-Saito formula

In this section, we give a proof of the localized Abbes-Saito formula assuming the strong resolution of singularities. Let \( X \) be a smooth scheme of dimension \( d \) over a perfect field \( k \) and \( j : U \longrightarrow X \) be an open immersion with dense image. Let \( S \) denote the complement \( X \setminus U \). We assume that \( I \) denotes a prime number invertible in \( k \) and \( E \) denotes a finite extension of \( \mathbb{Q}_I \). Let \( \mathcal{F} \) be a smooth \( E \)-sheaf on \( U \). Let \( E_0 \) denote \( E \cap \mathbb{Q}(\mu_\infty) \). The naive Swan class \( Sw_{\text{naive}}(\mathcal{F}) \in CH_0(S)_{E_0} \) is defined in [KS] Definition 4.2.2] and recalled in [AS] subsection 3.2.
Theorem 4.1. (the localized Abbes-Saito formula) Let the notation and the assumption be as above. Further, we assume the strong resolution of singularities. Then we have

\[ C_{S}^{00}(j, F) = -\text{cl}(\text{Sw}^{\text{naive}}(F)) \]

in \( H^0_S(X, K_X) \) where \( \text{cl} : CH_0(S)_{E_0} \to H^0_S(X, K_X) \) denotes the cycle class map.

Proof. Let \( O \) be the integer ring of \( E \) and \( \lambda \) the maximal ideal of \( O \). For a constructible \( E \)-sheaf \( F \) on \( X \), \( F_O \) denotes an \( O \)-lattice and \( F_n \) denotes the reduction \( F_O \otimes_O O/\lambda^n \). We put \( F = F_1 \).

We take the following cartesian diagram

\[
\begin{array}{ccc}
V & \xrightarrow{j_Y} & Y \\
\downarrow f & & \downarrow f \\
U & \xrightarrow{j} & X & \xleftarrow{S} & D
\end{array}
\]

where \( f \) is a finite Galois étale morphism of Galois group \( G \) that trivializes the reduction \( \tilde{f} \) and \( \tilde{f} : Y \to X \) is a proper morphism. Since we assume the strong resolution of singularities, we may assume that \( Y \) is smooth over \( k \) and \( D = \bigcup_{i \in I} D_i \subset Y \) is a divisor with simple normal crossings. We put \( F_V := f^*F \) on \( V \).

By Corollary 3.17 we have

\[ \tilde{f}^* C_{S}^{00}(j_Y F) = \sum_{\sigma \in G} C_{S}^{00}(j_{V!, F_V}, \tilde{\Gamma}_{\sigma}, j_{\sigma!}\sigma^*) \quad (4.1) \]

in \( H^0_{\Gamma_{\sigma} \cap Y}(Y, K_Y) \). Since we assume the strong resolution of singularities, the condition in [AS, Theorem 3.3.1] is satisfied. Therefore we obtain an equality

\[ C(j_{V!, F_V}, \tilde{\Gamma}_{\sigma}, j_{\sigma!}\sigma^*) = -s_{V/U}(\sigma)\text{Tr}^{Br}(\sigma : \hat{M}) \]

in \( H^0_{\Gamma_{\sigma} \cap Y}(Y, K_Y) \) for \( \sigma \neq 1 \) by loc. cit. Since we have \( \tilde{\Gamma}_{\sigma} \cap Y = \tilde{\Gamma}_{\sigma} \cap D \) for \( \sigma \neq 1 \), the canonical map \( H^0_{\Gamma_{\sigma} \cap Y}(Y, K_Y) \to H^0_{\Gamma_{\sigma} \cap Y}(Y, K_Y) \) is an isomorphism. By this isomorphism and Definition 3.2 we understand the following equalities

\[ C_{D}(j_{V!, F_V}, \tilde{\Gamma}_{\sigma}, j_{\sigma!}\sigma^*) = C(j_{V!, F_V}, \tilde{\Gamma}_{\sigma}, j_{\sigma!}\sigma^*) = -s_{V/U}(\sigma)\text{Tr}^{Br}(\sigma : \hat{M}) \quad (4.2) \]

in \( H^0_{\Gamma_{\sigma} \cap Y}(Y, K_Y) = H^0_{\Gamma_{\sigma} \cap D}(Y, K_Y) \) where \( C_{D}(j_{V!, F_V}, \tilde{\Gamma}_{\sigma}, j_{\sigma!}\sigma^*) \) denotes the image of the class \( C_{D}(j_{V!, F_V}, \tilde{\Gamma}_{\sigma}, j_{\sigma!}\sigma^*) \) under the canonical map \( H^0_{\Gamma_{\sigma} \cap D}(Y, j_{V!, K_V}) \to H^0_{\Gamma_{\sigma} \cap D}(Y, K_Y) \). By (4.1), Lemma 3.13 (Here we use the strong resolution of singularities.) and (4.2), we acquire equalities

\[ \tilde{f}^* C_{S}^{00}(j_Y F) = \sum_{\sigma \in G} C_{D}(j_{V!, F_V}, \tilde{\Gamma}_{\sigma}, j_{\sigma!}\sigma^*) = \sum_{\sigma \in G, \sigma \neq 1} -s_{V/U}(\sigma)(\text{Tr}^{Br}(\sigma : \hat{M}) - \text{rk}(F)) \]

in \( H^0_B(Y, K_Y) \). Since we have \( \sum_{\sigma \in G} s_{V/U}(\sigma) = 0 \), the following equality holds

\[ \tilde{f}^* C_{S}^{00}(j_Y F) = \sum_{\sigma \in G} -s_{V/U}(\sigma)\text{Tr}^{Br}(\sigma : \hat{M}). \]

Applying the functor \( \tilde{f}_* \), we obtain

\[ |G| \cdot C_{S}^{00}(j_Y F) = \sum_{\sigma \in G} -\tilde{f}_* s_{V/U}(\sigma)\text{Tr}^{Br}(\sigma : \hat{M}) = -|G| \cdot \text{Sw}^{\text{naive}}(F). \]

Hence we have proved the required assertion. \( \square \)
Remark 4.2. Let the notation be as in Definition 3.11. We expect that the following equality $C^{\log,00}_S(j_!\mathcal{F}) = -\text{cl}(\text{Sw}^{\text{naive}}(\mathcal{F}))$ holds in $H^0(X, \mathcal{K}_X)$. However we do not know a proof. In the case where $\mathcal{F}$ is a sheaf of rank 1 which is clean with respect to the boundary, we prove this equality in [11 Corollary 3.10].

Remark 4.3. Without assuming that $X$ is smooth over $k$, we will define the localized characteristic class $C^0_S(j_!\mathcal{F}) \in H^0_2(X, \mathcal{K}_X)$ in Definition 5.10 and prove the equality $C^0_S(j_!\mathcal{F}) = -\text{cl}(\text{Sw}^{\text{naive}}(\mathcal{F}))$ in Corollary 5.11.

5 Kato-Saito conductor formula in characteristic $p > 0$

In this section, we will prove the compatibility of the (logarithmic) localized characteristic class with proper push-forward. This is a localized version of the Lefschetz-Verdier trace formula. As a corollary, we will prove the Kato-Saito conductor formula in characteristic $p > 0$. Originally the Kato-Saito conductor formula calculates the Swan conductor of a Galois representation which appears when we consider an $\ell$-adic sheaf on a proper smooth curve over a discrete valuation field by the 0-cycle class (Kato 0-cycle class defined in [K2] for a sheaf of rank 1 or Swan class) on the boundary which is produced by the wild ramification of the $\ell$-adic sheaf.

We prove the compatibility of the logarithmic localized characteristic class of a smooth $\Lambda$-sheaf with proper push-forward. Let the notation be as in Lemma 3.10. We write $\phi$ for the projection $f : (X \times X)' \longrightarrow X \times X$ in this section. Moreover let $Z$ be a smooth scheme of dimension $e$, $W$ an open subscheme of $Z$. Let $\delta_Z : Z \longrightarrow Z \times Z$ and $\delta_W : W \longrightarrow W \times W$ be the diagonal closed immersions, and $g_Z : Z \times Z \backslash \delta_Z(Z) \longrightarrow Z \times Z$ and $g_W : W \times W \backslash \delta_W(W) \longrightarrow W \times W$ the open immersions. We consider a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{f'} & V \\
\downarrow{j''} & & \downarrow{jv} \\
X & \xleftarrow{j'} & S \\
\downarrow{f} & \downarrow{jw} & \downarrow{j} \\
W & \xleftarrow{w''} & Z \\
\end{array}
\]

where the squares are cartesian, $f : V \longrightarrow W$ is a proper smooth morphism and $\tilde{f} : X \longrightarrow Z$ is a proper morphism.

We consider the following cartesian diagram

\[
\begin{array}{c}
X'' \longrightarrow (X \times X)' \xleftarrow{g''} (X \times X)' \backslash X'' \\
\downarrow{\phi} & & \downarrow{\phi} \\
X \times Z X \xlongrightarrow{f \times f} X \times X \xleftarrow{g \times g} X \times X \backslash X \times Z X \\
\downarrow{\delta_Z} & & \downarrow{\delta_Z} \\
Z \times Z \xlongrightarrow{g_Z} Z \times Z \backslash \delta_Z(Z). \\
\end{array}
\]

where $g'' : (X \times X)' \backslash X'' \longrightarrow (X \times X)'$ is the open immersion. Let $V''$ be the intersection $X'' \cap (V \times V)'$ in $(X \times X)'$, and $i'' : V'' \longrightarrow (V \times V)'$ the closed immersion. Let $\tilde{h} : (X \times X)' \longrightarrow Z \times Z$ denote the projection. We have $X' \subset X''$ and $V''$ is a smooth scheme of codimension $e$ in $(V \times V)'$ since the projection $h : (V \times V)' \longrightarrow W \times W$ is a smooth morphism. We consider
the cartesian diagram

\[
\begin{array}{ccc}
V'' \xrightarrow{\iota''} (V \times V)' & \xrightarrow{\delta''} (V \times V) \setminus V'' \\
h_W \downarrow & & \downarrow h \\
W \xrightarrow{\delta_W} W \times W & \xrightarrow{g_W} W \times W \setminus W
\end{array}
\] (5.1)

where \( g_{\nu} \) and \( g_W \) are the open immersions, and \( h \) and \( h_W \) are the projections.

We define a map

\[
Rf_* (j_{V*} K_V \otimes \delta^* Rg_A) \to j_{W*} K_W \otimes \delta^2 Rg_A. \tag{5.2}
\]

By the smooth base change theorem and the projection formula, we acquire isomorphisms

\[
j_{W*} Rf_* (K_V \otimes \delta^* Rg'_A) \simeq j_{W*} Rf_* (K_V \otimes f^* \delta^* Rg_{W*}) \simeq j_{W*} Rf_* (K_V \otimes \delta^2 Rg_A).
\]

Therefore we obtain an isomorphism

\[
Rf_* j_{V*} K_V \otimes \delta^2 Rg_A \simeq Rf_* (j_{V*} K_V \otimes \delta^* Rg''_A). \tag{5.3}
\]

We define the map \( \tag{5.2} \) to be the composite of the following maps

\[
Rf_* (j_{V*} K_V \otimes \delta^* Rg_A) \to Rf_* (j_{V*} K_V \otimes \delta^* Rg''_A) \to Rf_* j_{V*} K_V \otimes \delta^2 Rg_A \to j_{W*} K_W \otimes \delta^2 Rg_A
\]

where the first map is induced by the canonical map \( Rg'_A \to Rg''_A \) and the second isomorphism is \( \tag{5.3} \) and the third map is induced by the proper push-forward \( Rf_* K_V \to K_W \). Then the map \( \tag{5.2} \) induces the proper push-forward

\[
\bar{f}_* : H^n_{\mathcal{S}} ((X, j_{V*} K_V \otimes \delta^* Rg_A)) \to H^n_{\mathcal{S}} (Z, j_{W*} K_W \otimes \delta^2 Rg_A). \tag{5.4}
\]

**Lemma 5.1.** The canonical map

\[
H^n_{\mathcal{X} \setminus \mathcal{W}} ((X \times X)', \mathcal{H}' \otimes Rg''_A) \to H^n_{\mathcal{X} \setminus \mathcal{W}} ((X \times X)', \mathcal{H}' \otimes Rg''_A)
\]

is an isomorphism.

**Proof.** It suffices to show that \( H^n_{\mathcal{X} \setminus \mathcal{W}} ((V \times V)', \mathcal{H}' \otimes Rg''_A) = 0 \) for all \( i \) by the localization sequence. By the proper base change theorem and the cartesian diagram \( \tag{5.1} \), we acquire an isomorphism \( H^n_{\mathcal{U} \setminus \mathcal{V}} ((V \times V)', \mathcal{H}' \otimes Rg''_A) \simeq H^n(W, R\delta^i_W R\mathcal{H}' \otimes Rg''_A) \). We put \( \mathcal{H}_W := R\text{Hom}(pr^*_2 Rf_U, \mathcal{F}, pr^*_1 Rf_U) \) on \( W \times W \). We write \( \phi \) for the projection \( f_V : (V \times V)' \to V \times V \) in subsection 3.3. The isomorphism \( R\phi_* \mathcal{H}' \simeq \mathcal{H}_W \) by \cite{AS} Lemma 2.2.4 and the Kunneth formula induce an isomorphism \( R\phi_* \mathcal{H}' \simeq \mathcal{H}_W \). Since we have an isomorphism \( h^* Rg_{W*} \simeq Rg''_{W*} \) by the smooth base change theorem, we acquire an isomorphism \( Rh_* (\mathcal{H}' \otimes Rg''_A) \simeq \mathcal{H}_W \otimes Rg_{W*} \) by the projection formula. Since \( Rf_U \mathcal{F} \) is a smooth sheaf on \( W \) for all \( q \), we obtain the following vanishing \( H^i(W, R\delta^i_W R\mathcal{H}' \otimes Rg''_A) \simeq H^i(W, R\delta^i_W (\mathcal{H}_W \otimes Rg_{W*})) = 0 \) again by the projection formula. Hence we have proved the required assertion. \( \square \)

**Theorem 5.2.** (localized Lefschetz-Verdier trace formula) Let the notation and the assumption be as above. Then we have an equality

\[
\bar{f}_* C^0_{\mathcal{S}^g \otimes 0} (j; \mathcal{F}) = C^0_T (j_{W*} Rf_U, \mathcal{F})
\]

in \( H^n_{\mathcal{S}} (Z, K_Z) \).
Proof. We prove the assertion by a similar method to the one in [Gr Théorème 4.4]. We put
\( \mathcal{H}_Z := R\text{Hom}(pr^*_Z jw_1 Rf_{1!}, \mathcal{F}) \) on \( Z \times Z \). By Lemma \([5, 7]\) and the Kunneth formula, we have an isomorphism \( Rh_*, \mathcal{H}' \simeq \mathcal{H}_Z \). We consider the following commutative diagram

\[
\begin{array}{ccc}
\phi^* \text{id}_{\mathcal{I}, \mathcal{F}} \in H^0_\mathcal{I}_X((X \times X)', \mathcal{H}') & \xrightarrow{\text{can.}} & H^0_{X'}((X \times X)', \mathcal{H}' \otimes Rg'_\Lambda) \\
\downarrow \& \downarrow \& \\
H^0_{X''}((X \times X)', \mathcal{H}') & \xrightarrow{\text{can.}} & H^0_{X''}((X \times X)', \mathcal{H}' \otimes Rg''_\Lambda)
\end{array}
\]

(5.5)

We denote by \( e^* \delta^* \phi^* (\text{id}_{\mathcal{I}_{X}', \mathcal{F}})'' \) the image of the element \( \phi^* \text{id}_{\mathcal{I}, \mathcal{F}} \) in \( H^0_{X''}((X \times X)', \mathcal{H}') \) by the composite of the maps in the lower line in the above diagram. By the above commutative diagram, Lemma \([5, 9]\) and Definition \([5, 11]\) we obtain an equality

\[
C^0_{S}(j, \mathcal{F}) = e^* \delta^* \phi^* (\text{id}_{\mathcal{I}_{X}', \mathcal{F}})''
\]

(5.6)

in \( H^0_\mathcal{I}_X((X, j_! K \mathcal{V} \otimes \delta^* Rg'_\Lambda) \) where we denote by the same letter \( C^0_{S}(j, \mathcal{F}) \) the image of \( \phi^* \text{id}_{\mathcal{I}_{X}', \mathcal{F}} \).

We consider the following commutative diagram

\[
\begin{array}{ccc}
H^0_{X''}((X \times X)', \mathcal{H}') & \xrightarrow{(0)} & H^0_{X'' \setminus V''}((X \times X)', \mathcal{H}' \otimes Rg''_\Lambda) \\
\downarrow \& \downarrow \& \\
H^0_\mathcal{I}_X((Z \times Z), \mathcal{H}_Z) & \xrightarrow{(1)} & H^0_{\mathcal{I}_X \setminus W}(Z \times Z, R\text{Hom}(\mathcal{H}' \otimes Rg'_\Lambda)) \\
\downarrow \& \downarrow \& \\
H^0_\mathcal{I}_X((Z \times Z), \mathcal{H}_Z) & \xrightarrow{(2)} & H^0_\mathcal{I}_X((Z \times Z), \mathcal{H}_Z \otimes Rg_\Lambda) \\
\downarrow \& \downarrow \& \\
H^0_\mathcal{I}_X((Z \times Z), \mathcal{H}_Z \otimes Rg_\Lambda) & \xrightarrow{(3)} & H^0_\mathcal{I}_X((Z \times Z), \mathcal{H}_Z \otimes Rg_\Lambda).
\end{array}
\]

(5.7)

We explain the maps and the commutativities in the above diagram. The commutativities except for the bottom one follow from definitions of the maps immediately.

(0): This map is the composite of the first two maps in the lower line in the diagram \([5, 5]\).

(1): The adjoint map \( h^* \mathcal{H}_Z \longrightarrow \mathcal{H}' \) of the isomorphism \( \mathcal{H}_Z \longrightarrow Rh_*, \mathcal{H}' \) and the canonical map \( \Lambda \longrightarrow Rg'_\Lambda \) induce a map \( h^* \mathcal{H}_Z \longrightarrow \mathcal{H}' \otimes Rg'_\Lambda \). The adjoint map \( \mathcal{H}_Z \longrightarrow Rh_*(\mathcal{H}' \otimes Rg'_\Lambda) \) of this map induces a map \( H^0_{\mathcal{I}_X}(Z \times Z, \mathcal{H}_Z) \longrightarrow H^0_{\mathcal{I}_X}(Z \times Z, Rh_* (\mathcal{H}' \otimes Rg'_\Lambda)) \). By Lemma \([5, 1]\) the canonical map \( H^0_{\mathcal{I}_X \setminus W}(Z \times Z, Rh_* (\mathcal{H}' \otimes Rg'_\Lambda)) \longrightarrow H^0_{\mathcal{I}_X}(Z \times Z, Rh_* (\mathcal{H}' \otimes Rg'_\Lambda)) \) is an isomorphism. The map (1) is the composition of these maps.

(1)'': The base change map \( \delta_Z Rh_* \longrightarrow R\tilde{f}_* \delta^* \) induces a map \( \delta_Z Rh_* \longrightarrow \mathcal{H}' \otimes Rg'_\Lambda \). The evaluation map \([5, 3]\) induces a map \( R\tilde{f}_* (\delta^* \mathcal{H}' \otimes \delta^* Rg'_\Lambda) \longrightarrow R\tilde{f}_* (j_! K \mathcal{V} \otimes \delta^* Rg'_\Lambda) \). We define (1)' to be the composite of these two maps. By these definitions, the commutativities in the first line in the diagram are clear.
(2): This map is the map loc_{Z,W,Rf_U,F} : H^0_Z(Z \times Z, \hat{H}_Z) \to H^0_Z(Z \times Z, \hat{H}_Z \otimes Rg_Z, \Lambda) \simeq H^0_{Z,W}(Z \times Z, \hat{H}_Z \otimes Rg_Z, \Lambda). (c.f. [3.2].)

(2)' : The isomorphism \( R\hat{h}_*\hat{H}' \simeq \hat{H}_Z \) and the base change map \( \delta^*_Z R\hat{h}_* \to Rf'_* \delta'^* \) induce a map \( \delta^*_Z \hat{H}_Z \otimes \delta^*_Z Rg_Z, \Lambda \to Rf'_* \delta'^* \hat{H}' \otimes \delta^*_Z Rg_Z, \Lambda. \) The evaluation map (3.7) induces a map \( Rf'_* \delta'^* \hat{H}' \otimes \delta^*_Z Rg_Z, \Lambda \to Rf'_* j_V K_V \otimes Rg_Z, \Lambda. \) We define (2)' to be the composite of these maps.

(2)'' : The map \( \tilde{h}^* \hat{H}_Z \to \hat{H}' \) and the base change map \( \tilde{h}^* Rg_Z, \Lambda \to Rg'_* \Lambda \) induce a map \( \tilde{h}^* (\hat{H}_Z \otimes \delta^*_Z Rg_Z, \Lambda) \to \hat{H}' \otimes Rg''_* \Lambda. \) The map (2)'' is induced by the adjoint of this map. By these descriptions and the definition of the map (5.3), the commutativities in the second line in the diagram (5.7) follow.

(3)' : This map is induced by the pull-back by \( \delta_Z \) and the usual evaluation map \( \delta^*_Z \hat{H}_Z \to j_{W'} K_{W'} \) for \( Rf_U, F \). The right bottom commutativity is a consequence of the compatibility of evaluation maps with proper push-forward which is proved in [3.4 Theorème 4.4. (4.4.4)].

We consider the following commutative diagram

\[
\begin{array}{ccc}
\phi^* \quad & \quad & \phi^* \\
\downarrow \quad & \quad & \downarrow \quad & \quad & \downarrow \quad & \quad & \downarrow \\
\tilde{h}_* \quad & \quad & \tilde{h}_* \\
\end{array}
\]

\[
\begin{array}{ccc}
\phi^* \quad & \quad & \phi^* \\
\downarrow \quad & \quad & \downarrow \quad & \quad & \downarrow \quad & \quad & \downarrow \\
H^0_X(X \times X, \hat{H}') & \simeq & H^0_X(X \times X, \hat{H}) \ni \text{id}_{j,F} \\
\end{array}
\]

id_{jW,Rf_U,F} \in H^0_Z(Z \times Z, \hat{H}_Z).

By this diagram, an equality \((\bar{f} \times \bar{f}) \text{id}_{j,F} = \text{id}_{jW,Rf_U,F}\) which is a consequence of the compatibility of the cohomological correspondence with proper push-forward and \( \phi_* \phi^* = \text{id} \), we obtain an equality

\[
\tilde{h}_* \phi^* \text{id}_{j,F} = \text{id}_{jW,Rf_U,F}.
\]

We consider the following commutative diagram

\[
\begin{array}{ccc}
H^0_S(X, j_{V!} K_V \otimes \delta'^* Rg'_* \Lambda) & \xrightarrow{\text{can}} & H^0_S(X, K_X \otimes \delta'^* Rg'_* \Lambda) \\
\downarrow \quad & \quad & \downarrow \\
H^0_T(Z, j_{W!} K_W \otimes \delta^*_Z Rg_Z, \Lambda) & \xrightarrow{\text{can}} & H^0_T(Z, K_Z \otimes \delta^*_Z Rg_Z, \Lambda) \\
\end{array}
\]

where the left vertical arrow is the proper push-forward (5.4) and the right vertical arrow \( f_* : H^0_S(X, K_X) \to H^0_T(Z, K_Z) \) is the usual proper push-forward. Clearly the composite \( H^0_S(X, j_{V!} K_V \otimes \delta'^* Rg'_* \Lambda) \to H^0_T(Z, j_{W!} K_W \otimes \delta^*_Z Rg_Z, \Lambda) \) of the right vertical arrows in the diagram (5.7) is equal to the map \( \bar{f}_* : H^0_S(X, j_{V!} K_V \otimes \delta'^* Rg'_* \Lambda) \to H^0_T(Z, j_{W!} K_W \otimes \delta^*_Z Rg_Z, \Lambda) \) in the diagram (5.9). The localized characteristic class \( C^0_T(j_{W!} Rf_U, F) \) is the image of the cohomological correspondence id_{jW,Rf_U,F} by the composite of the maps (2) and (3)' in the diagram (5.7) by Definition 5.2. Hence the assertion follows from the equalities (5.6) and (5.3), and the commutative diagrams (5.7) and (5.9).

**Corollary 5.3.** Let the notation and the assumption be as in Theorem 5.4. Then we have an equality

\[
\bar{f}_* C^0_S(j, F) = C^0_T(j_{W!} Rf_U, F) - \text{rk}(F) \cdot C^0_T(j_{W!} Rf_U, \Lambda_U)
\]

in \( H^0_T(Z, K_Z) \).

**Proof.** By Theorem 5.2 we have equalities \( \bar{f}_* C^0_S(j, F) = C^0_T(j_{W!} Rf_U, F) \) and \( \bar{f}_* C^0_S(j, \Lambda_U) = C^0_T(j_{W!} Rf_U, \Lambda_U). \) Hence the assertion follows from an equality \( \text{rk}(Rf_U, F) = \text{rk}(F) \cdot \text{rk}(Rf_U, \Lambda_U). \)
Corollary 5.4. Let the notation be as in Theorem 4.1. Further we assume that \( k \) is a perfect field, that \( D \cup S \) is a divisor with simple normal crossings, that \( \dim Z \leq 2 \) and that \( \mathcal{F} \) is a smooth \( E \)-sheaf of rank 1 which is clean with respect to the boundary where \( E \) is a finite extension of \( \mathbb{Q}_l \) and \( l \) is invertible in \( k \). Then we have

\[
-\bar{f}_*c_{\mathcal{F}} = Sw^{\text{naive}}(Rf_{U!*}\mathcal{F}) - \text{rk}(\mathcal{F}) \cdot Sw^{\text{naive}}(Rf_{U!*}\Lambda_U)
\]

in \( H^0_\mathbb{F}(Z, K_Z) \) where \( c_{\mathcal{F}} \in CH_0(S) \) is the Kato 0-cycle class defined by K. Kato in \([K1]\) and \([K2]\), and recalled in \([AS, \text{Section 4}]\).

Proof. We prove an equality \(-c_{\mathcal{F}} = C_{\mathcal{S}}^{\log, 00}(j_!\mathcal{F}) \) in \( H^0_\mathbb{F}(X, K_X) \) in \([T, \text{Corollary 3.10}]\). By this equality, Theorem 4.1 and Corollary 5.3 the assertion follows.

Remark 5.5. Let the notation be as in Corollary 5.4. If we assume the strong resolution of singularities, the equality

\[
-\bar{f}_*c_{\mathcal{F}} = Sw^{\text{naive}}(Rf_{U!*}\mathcal{F}) - \text{rk}(\mathcal{F}) \cdot Sw^{\text{naive}}(Rf_{U!*}\Lambda_U)
\]

in \( H^0_\mathbb{F}(Z, K_Z) \) holds for any dimensional scheme \( Z \).

We prove the compatibility of the localized characteristic class of a smooth \( \Lambda \)-adic sheaf with a cohomological correspondence with proper push-forward. Let \( X \) and \( Y \) be schemes over \( k \) and \( U \subseteq X \) and \( V \subseteq Y \) open dense subschemes smooth over \( k \) respectively. Let \( j_V : V \to Y \) and \( j : U \to X \) denote the open immersions, and \( T := Y \setminus V \) and \( S := X \setminus U \) the complements respectively. Let \( \delta_X : X \to X \times X, \delta_Y : Y \to Y \times Y, \delta_V : V \to V \times V \) and \( \delta_U : U \to U \times U \) be the diagonal closed immersions. We consider a cartesian diagram

\[
\begin{array}{ccc}
V & \xrightarrow{j_V} & Y \\
\downarrow f & & \downarrow \bar{f} \\
U & \xrightarrow{j} & X \\
\end{array}
\]

where \( \bar{f} : Y \to X \) is a proper morphism and \( f : V \to U \) is a proper smooth morphism.

Let \( C \) and \( C' \) be closed subschemes of \( U \times U \) and \( V \times V \) respectively. Let \( \bar{C} \) be the closure of \( C \) in \( X \times X \) and \( \bar{C}' \) the closure of \( C' \) in \( Y \times Y \) respectively. We assume that \((f \times f)^{-1}(C) \subseteq \bar{C}' \) and \( C = (X \times U) \cap \bar{C} \) and \( C' = (Y \times V) \cap \bar{C}' \). Let \( C'' \) denote the inverse \((f \times f)^{-1}(C) \) and \( \bar{C}'' \) the closure of \( C'' \) in \( Y \times Y \). Let \( j_C : C \to \bar{C} \) and \( j_{C'} : C' \to \bar{C}' \) be the open immersions.

We consider the following cartesian diagram

\[
\begin{array}{ccc}
\bar{C}' & \xrightarrow{\bar{C}''} & Y \times Y \\
\downarrow g_{\bar{C}''} & & \downarrow \bar{g} \\
\bar{C} & \xrightarrow{\bar{g}_{\bar{C}''}} & X \times X \\
\end{array}
\]

where \( g_{\bar{C}} : X \times X \setminus \bar{C} \to X \times X \) and \( g_{\bar{C}''} : Y \times Y \setminus \bar{C}'' \to Y \times Y \) are the open immersions and the squares are cartesian. Similarly \( g_C : U \setminus U \setminus C \to U \times U, g_{C'} : V \setminus V \setminus C' \to V \times V \) and \( g_{C''} : Y \setminus Y \setminus \bar{C}'' \to Y \times Y \) denote the open immersions. Let \( h : C' \to C \) be the projection.

Let \( \mathcal{F} \) be a smooth \( \Lambda \)-sheaf on \( V \). Since \( f \) is a proper smooth morphism, the sheaves \( R^qf_*\mathcal{F} \) are smooth for all \( q \). Let \( u' \) be a cohomological correspondence of \( \mathcal{F} \) on \( C' \). We put \( \mathcal{H} := R\text{Hom}(pr_2^*j_{!}Rf_*\mathcal{F}, Rpr_1^*j_{!}Rf_*\mathcal{F}) \) on \( X \times X \) and \( \mathcal{H}' := R\text{Hom}(pr_2^*j_{!}\mathcal{F}, Rpr_1^*j_{!}f_*\mathcal{F}) \) on \( Y \times Y \) respectively.
We define a map in the same way as (5.10)
\[ Rf_* (jV_! K_V \otimes \delta^*_Y RgC_* \Lambda) \longrightarrow jK_U \otimes \delta_X^* RgC_* \Lambda \]  
(5.10)
to be the composite of the following maps
\[ Rf_* (jV_! K_V \otimes \delta^*_Y RgC_* \Lambda) \longrightarrow Rf_* (jV_! K_V \otimes \delta^*_Y RgC'_* \Lambda) \simeq Rf_* jV! (K_V \otimes \delta^*_Y RgC'_* \Lambda) \]
\[ \simeq Rf_* jV! K_V \otimes \delta^*_X RgC_* \Lambda \longrightarrow jK_U \otimes \delta^*_X RgC_* \Lambda. \]
The first map is induced by the canonical map $RgC'_* \Lambda \longrightarrow RgC'_* \Lambda$. The second isomorphism is induced by the projection formula. The third isomorphism follows from the smooth base change theorem. The fourth map is induced by the proper push-forward $Rf_* jV! K_V \longrightarrow jK_U$.
The map (5.10) induces the proper push-forward
\[ f_* : H^0_{C(\gamma)} (Y, jV_! K_V \otimes \delta^*_Y RgC_* \Lambda) \longrightarrow H^0_{C(\gamma)} (X, jK_U \otimes \delta^*_X RgC_* \Lambda). \]  
(5.11)

**Proposition 5.6.** Let the notation and the assumption be as above. Then we have
\[ f_* C^0_{T'} (jV_! \mathcal{F}, \tilde{C}'_!, jC'_! u') = C^0_{S'_!} (jRf_* \mathcal{F}, \tilde{C}'_!, jC'_! h_* u') \]
in $H^0_{C(\gamma)} (X, jK_U \otimes \delta^*_X RgC_* \Lambda)$.  

**Proof.** We prove this formula in the same way as Theorem 4.8. We omit a proof. \hfill \square

We keep the same notation as above. In the following, we consider the case where $C = \delta_U (U)$ and $C' = \delta_V (V)$ are the diagonals and $u' = \text{id}_F$. We assume that $X, Y$ are smooth over $k$. We have the proper push-forward $f_* : H^0_F (Y, jV_! K_V) \longrightarrow H^0_S (X, jK_U)$.  

**Corollary 5.7.** Let the notation and the assumption be as above.  
1. We have
\[ f_* C^0_{T'} (jV_! \mathcal{F}) = C^0_{S'_!} (jRf_* \mathcal{F}) - \text{rk}(\mathcal{F}) \cdot C^0_{S'_!} (jRf_* \Lambda_V) \]
in $H^0_S (X, jK_U)$.  
2. We keep the same notation as in 1. Then we have an equality
\[ f_* C^0_{T'} (jV_! \mathcal{F}) = C^0_S (jRf_* \mathcal{F}) - \text{rk}(\mathcal{F}) \cdot C^0_S (jRf_* \Lambda_V) \]
in $H^0_S (X, K_X)$.

**Proof.** 1. We consider the commutative diagram
\[
\begin{array}{ccc}
H^0_F (Y, jV_! K_V) & \longrightarrow & H^0_F (Y, jV_! K_V \otimes \delta^*_Y RgY_* \Lambda) \\
\downarrow f_* & & \downarrow f_* \\
H^0_S (X, jK_U) & \longrightarrow & H^0_S (X, jK_U \otimes \delta^*_X RgX_* \Lambda)
\end{array}
\]
where the right vertical arrow is the map (5.11) in the case where $C = \delta_U (U)$ and $C' = \delta_V (V)$ are the diagonals. Since the canonical map $H^0_S (X, jK_U) \longrightarrow H^0_S (X, jK_U \otimes \delta^*_X RgX_* \Lambda)$ is injective, we may regard the equality as an equality in $H^0_S (X, jK_U \otimes \delta^*_X RgX_* \Lambda)$. Hence the assertion follows from Proposition 5.6 and Lemma 5.3.

2. The assertion follows from 1 and Remark 5.3 immediately. \hfill \square
Corollary 5.8. \((\text{Kato-Saito conductor formula in characteristic } p > 0)\) Let the notation be as in Corollary 5.7. We assume that \(k\) is a perfect field, that \(E\) denotes a finite extension of \(\mathbb{Q}_l\) and that \(\mathcal{F}\) is a smooth \(E\)-sheaf and the strong resolution of singularities. Then we have an equality

\[
-f_*\text{Sw}^\text{naive}\left(\mathcal{F}\right) = \text{Sw}^\text{naive}(Rf_*\mathcal{F}) - \text{rk}(\mathcal{F}) \cdot \text{Sw}^\text{naive}(Rf_*E)
\]

in \(H^0_k(X, K_X)\).

\[\text{Proof.}\] This follows from Theorem 4.11 and Corollary 5.7.2. \(\square\)

Let \(X\) be a scheme and \(U \subseteq X\) an open dense subscheme smooth of dimension \(d\) over \(k\). Let \(S\) be the complement \(X \setminus U\). Let \(j : U \rightarrow X\) denote the open immersion.

Let \(\mathcal{F}\) denote a smooth \(\Lambda\)-sheaf on \(U\). Assuming the strong resolution of singularities, we define the localized characteristic class \(C^0_S(j, \mathcal{F}) \in H^0_k(X, K_X)\) and prove the equality \(C^0_S(j, \mathcal{F}) = -\text{cl} \left(\text{Sw}^\text{naive}(\mathcal{F})\right)\) in \(H^0_k(X, K_X)\). Let \(f : X' \rightarrow X\) be a desingularization preserving the open subscheme \(U\) by the assumption of the strong resolution of singularities. Let \(j' : U \rightarrow X'\) denote the open immersion and \(S' = X' \setminus U\). We denote by \(C^0_{X'}(j', \mathcal{F}) \in H^0_k(X', K_{X'})\) the image of the localized characteristic class \(C^0_S(j, \mathcal{F}) \in H^0_k(X, K_X)\) in Remark 5.9 by the proper push-forward \(f_* : H^0_k(X', K_{X'}) \rightarrow H^0_k(X, K_X)\).

Corollary 5.9. Let the notation be as above. We consider two desingularizations \(X' \rightarrow X\) and \(X'' \rightarrow X\) preserving the open subscheme \(U\). Then we have an equality

\[
C^0_{X'}(j, \mathcal{F}) = C^0_{X''}(j, \mathcal{F})
\]

in \(H^0_k(X, K_X)\).

\[\text{Proof.}\] We take a smooth model \(\tilde{X} \rightarrow X' \times_X X''\) preserving \(U\) by the strong resolution of singularities. We consider the following cartesian diagram

\[
\begin{array}{ccc}
U & \xrightarrow{j} & \tilde{X} \\
| \downarrow \text{id} & & \downarrow \pi \\
U & \xrightarrow{j'} & X'
\end{array}
\]

where the right vertical arrow is the canonical projection. Let \(\tilde{S}\) denote the complement \(\tilde{X} \setminus \tilde{U}\). It suffices to show \(\pi_*C^0_S(j, \mathcal{F}) = C^0_{\tilde{S}}(j', \mathcal{F})\) in \(H^0_k(X', K_{X'})\). This equality follows immediately from Corollary 5.7.2. Hence the required assertion follows. \(\square\)

Definition 5.10. Let the notation be as in Corollary 5.8. We take a smooth model \(f : X' \rightarrow X\) preserving \(U\). We put \(C^0_S(j, \mathcal{F}) := C^0_{X'}(j, \mathcal{F}) \in H^0_k(X, K_X)\). This class is independent of a choice of a smooth model \(f : X' \rightarrow X\) by Corollary 5.9. We call it the localized characteristic class of \(j, \mathcal{F}\). Further we put \(C^0_S(j, \mathcal{F}) := C^0_S(j, \mathcal{F}) - \text{rk}(\mathcal{F}) \cdot C^0_S(j, \Lambda) \in H^0_k(X, K_X)\).

Corollary 5.11. Let the notation be as in Theorem 4.11. We do not assume that \(X\) is smooth over \(k\). Then we have

\[
C^0_S(j, \mathcal{F}) = -\text{cl} \left(\text{Sw}^\text{naive}(\mathcal{F})\right)
\]

in \(H^0_k(X, K_X)\) where the left hand side is the localized characteristic class defined in Definition 5.10.
Proof. We take a smooth model $f : X' \to X$ preserving $U$ by the strong resolution of singularities. We denote by $\text{Sw}^{\text{naive}}_{X'}(\mathcal{F}) \in CH_0(X')_{E_0}$ the naive Swan class of $\mathcal{F}$ with respect to $(U, X')$. We have an equality $f_*\text{Sw}^{\text{naive}}_{X'}(\mathcal{F}) = \text{Sw}^{\text{naive}}_{X}(\mathcal{F})$ in $CH_0(S)_{E_0}$. By Definition 5.11 and Theorem 4.1 we acquire the following equalities

$$C_{S}^{00}(j'_!*\mathcal{F}) = -f_*\text{cl}(\text{Sw}^{\text{naive}}_{X'}(\mathcal{F})) = -\text{cl}(\text{Sw}^{\text{naive}}_{X}(\mathcal{F}))$$

in $H_{S}^{0}(X, K_X)$. Hence the assertion follows.

Corollary 5.12. Let the notation be as in Corollary 5.7. Moreover, we assume the strong resolution of singularities and that $f$ is a finite étale morphism. Let $d_{V/U} \in CH_0(S) \otimes \mathbb{Q}$ be the wild discriminant of $V$ over $U$ defined in ([KS, Definition 4.3.1]). We have

$$\tilde{f}_* C_{T}^{00}(j_V!*\mathcal{F}) = C_{S}^{00}(j_*f_*\mathcal{F}) + \text{rk}(\mathcal{F}) \cdot \text{cl}(d_{V/U})$$

in $H_{S}^{0}(X, K_X)$.

Proof. By Corollary 5.7.2, we obtain an equality $\tilde{f}_* C_{T}^{00}(j_V!*\mathcal{F}) = C_{S}^{00}(j_*f_*\mathcal{F}) - \text{rk}(\mathcal{F}) \cdot C_{S}^{00}(j_*f_*\Lambda_V)$. By Theorem 4.1 we acquire $C_{S}^{00}(j_*f_*\Lambda_V) = -\text{cl}(\text{Sw}(f_*\Lambda_V))$. By the definition of the Swan class, we have $\text{Sw}(f_*\Lambda_V) = d_{V/U}^{\log}$. Hence the following equalities hold

$$\tilde{f}_* C_{T}^{00}(j_V!*\mathcal{F}) = C_{S}^{00}(j_*f_*\mathcal{F}) - \text{rk}(\mathcal{F}) \cdot C_{S}^{00}(j_*f_*\Lambda_V)$$

$$= C_{S}^{00}(j_*f_*\mathcal{F}) + \text{rk}(\mathcal{F}) \cdot \text{cl}(d_{V/U}^{\log}).$$

Thus the assertion follows.

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