Local unitary group stabilizers and entanglement for multiqubit symmetric states

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Abstract We refine recent local unitary entanglement classification for symmetric pure states of \( n \) qubits (that is, states invariant under permutations of qubits) using local unitary stabilizer subgroups and Majorana configurations. Stabilizer subgroups carry more entanglement distinguishing power than do the stabilizer subalgebras used in our previous work. We extend to mixed states recent results about local operations on pure symmetric states by showing that if two symmetric density operators are equivalent by a local unitary operation, then they are equivalent via a local unitary operation that is the same in each qubit. A geometric consequence, used in our entanglement classification, is that two symmetric pure states are local unitary equivalent if and only if their Majorana configurations can be interchanged by a rotation of the Bloch sphere.

Keywords symmetric state · local unitary · stabilizer subgroup · Majorana

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1 Introduction

The question of when a given multiparty state can be converted to another by local operations and measurements of subsystems is crucial in quantum information science \(^1\). The fact that entangled states play a role as resources in computation...
and communication protocols motivates problems of measurement and classification of entanglement. In general, these are difficult problems, already rich for the case of pure states of \(n\)-qubits, where the number of real parameters necessary for classifying entanglement types grows exponentially in \(n\).

A promising special case for the general problem of entanglement measurement and classification is that of the symmetric states, that is, states of composite systems that are invariant under permutation of the subsystems. Symmetric states admit simplified analyses, and they are of interest in their own right. Examples of recent work in which permutation invariance has made possible results where the general case remains intractable include: geometric measure of entanglement \([2,3,4]\), classification of states equivalent under stochastic local operations and classical communication (SLOCC) \([6,7]\), and our own work on classification of states equivalent under local unitary (LU) transformations \([8]\).

The main result of this paper (Theorem 3 below) is a classification of LU equivalence classes of \(n\)-qubit symmetric states that refines our own previous work \([8]\), which is based on the following idea. Suppose states \(\rho, \rho'\) are local unitary equivalent via some LU transformation \(U\), that is, we have \(\rho' = U\rho U^\dagger\). If a local unitary operator \(V\) stabilizes \(\rho\), then \(UVU^\dagger\) stabilizes \(\rho'\). The consequence is that stabilizer subgroups of locally equivalent states are isomorphic via conjugation. Thus the isomorphism class of the stabilizer is an LU invariant. This inspires a two-stage classification program.

1. Classify LU stabilizer subgroup conjugacy classes.
2. Classify LU classes of states for each of the stabilizer classes from stage 1.

In previous work \([8,9,10,11,12,13]\), we have exploited the Lie algebra structure of the tangent space of infinitesimal LU transformations, which is a linearization of the stabilizer subgroup, to achieve results in both of these stages for various classes of states. A strength of this method is that linear Lie algebra computations are more tractable than the corresponding nonlinear Lie group computations. The drawback is that a Lie algebra detects only the connected component at the identity element of the corresponding Lie group. A stabilizer subalgebra does not “see” the discrete part of the stabilizer subgroup. For example, the Lie stabilizer subalgebra is the zero vector space for most stabilizer states, that is, states stabilized by the full \(n\)-qubit Pauli group. Group level information is necessary to capture the local unitary stabilizer properties of such states.

In Theorems 1 and 2 of \([8]\), we classify four infinite families and 1 discrete family (that is, the zero vector space) of stabilizer subalgebras for pure symmetric states, and identify LU classes of pure symmetric states that have those stabilizers. In this paper, we advance this classification by separating the four infinite families of subalgebras into six infinite families of groups, inequivalent under isomorphism by local unitary conjugation. The discrete stabilizer subalgebra corresponds to finite subgroups of \(SO(3)\) (rotations of real 3-dimensional Euclidean space): these are the infinite family of cyclic groups; the infinite family of dihedral groups; and the five finite symmetry groups of the Platonic solids.

Extending our previous stabilizer subalgebra classification for symmetric states to a stabilizer subgroup classification makes use of the Majorana representation for pure symmetric states: Given a collection \(|\psi_1\rangle, \ldots, |\psi_n\rangle\) of 1-qubit states, we
can symmetrize to form the state
\[ |\psi\rangle = \alpha \sum_{\pi} |\psi_{\pi(1)}\rangle |\psi_{\pi(2)}\rangle \cdots |\psi_{\pi(n)}\rangle \]
where \(\pi\) ranges over all \(n!\) permutations of the \(n\)-qubits, and \(\alpha\) is a normalization factor. It is a fact (see [3]) that any symmetric pure state can be written as such a symmetrization, and further, that the set of \(n\) 1-qubit states whose symmetrization is \(|\psi\rangle\) is unique up to phase factors. Thus the set of symmetric pure states is in one-to-one correspondence with configurations of multisets (one or more of the 1-qubit states may be repeated) of \(n\) of points on the Bloch sphere.

Using the fact that a rotation of the Bloch sphere corresponds to unitary operation on 1-qubit states, it is a simple observation that a rotation of the Majorana configuration of points representing a state \(|\psi\rangle\) results in an LU equivalent state \(|\psi'\rangle = V^{\otimes n} |\psi\rangle\), where \(V\) is the \(2 \times 2\) unitary operator corresponding to the given rotation of the sphere. Not obvious, but true nonetheless, is that given any LU operation \(U = U_1 \otimes U_2 \otimes \cdots \otimes U_n\) that transforms a symmetric state \(|\psi\rangle\) to another symmetric state \(|\psi'\rangle\), there is a 1-qubit operation \(V\) such that \(|\psi'\rangle = V^{\otimes n} |\psi\rangle\). This was proved by Mathonet et al. [14] for SLOCC operations on pure symmetric states. We show in Theorem 1 below that this holds more generally for LU operations on mixed symmetric states. A consequence (Theorem 2) is that \(\rho, \rho'\) are LU equivalent if and only if their Majorana configurations can be interchanged by a rotation of the Bloch sphere. This fact is used in the proof of the main result (Theorem 3).

2 Preliminaries

We take \(n\)-qubit state space to be the set of \(2^n \times 2^n\) density matrices (positive semidefinite matrices with trace 1). Pure states are represented by density matrices of rank 1. We write \(|D^{(k)}\rangle\) to denote the Dicke state with \(k\) excitations.

We take the local unitary group to be \(PU(2)^n\), where \(PU(2) = U(2)/\{\lambda \text{Id}: \lambda \in \mathbb{C} | |\lambda| = 1\}\), called the projective unitary group, is the set of projective equivalence classes of matrices in \(U(2)\). That is, matrices \(g, h\) in \(U(2)\) represent the same element in \(PU(2)\) if and only if \(g = \lambda h\) for complex number \(\lambda\). The projective unitary group \(PU(2)\) is isomorphic to the group \(SO(3)\) of rotations of 3-dimensional Euclidean space via
\[ \lambda \exp(-i\theta/2v \cdot \sigma) \leftrightarrow \text{rotation by } \theta \text{ radians about axis } v \]
where \(\lambda\) is a norm 1 complex number, \(\theta\) is a real number, \(v = (v_1, v_2, v_3)\) is a unit vector in \(\mathbb{R}^3\), and \(\sigma = (\sigma_x, \sigma_y, \sigma_z)\) is the vector of Pauli matrices (see [1] Ex. 4.5.).

We will denote elements of \(PU(2)\) and the local unitary group \(PU(2)^n\) by their representatives \(g\) in \(U(2)\) and \(U = (g_1, \ldots, g_n)\) in \(U(2)^n\), and will write \(g \equiv h\) or \(U \equiv V\) to denote equality in \(PU(2)\) and \(PU(2)^n\), and will use the equals sign to indicate equality in \(U(2)\) and \(U(2)^n\). Similarly, we will write \(|\psi\rangle \equiv |\phi\rangle\) to indicate that two state vectors are equal up to phase.

The local unitary group element \(U = (g_1, \ldots, g_n)\) acts on the density matrix \(\rho\) by
\[ \rho \rightarrow U\rho U^\dagger = (g_1 \otimes \cdots \otimes g_n) \rho (g_1 \otimes \cdots \otimes g_n)^\dagger. \]
We denote by $\text{Stab}_\rho$ the local unitary stabilizer subgroup for $\rho$

$$\text{Stab}_\rho = \{ U \in PU(2)^n ; U\rho U^\dagger = \rho \}.$$ 

3 Main Results

**Theorem 1** Let $\rho, \rho'$ be $n$-qubit symmetric states, pure or mixed, with $n \geq 3$. Then $\rho, \rho'$ are LU equivalent if and only if there exists an element $g$ in $U(2)$ such that

$$\rho' = (g \otimes n) \rho (g \otimes n)^\dagger.$$ 

**Proof of Theorem 1** We only need to prove the “only if” direction. Let $\rho' = U\rho U^\dagger$, where

$$U = (g_1, g_2, \ldots, g_n) = \prod_{j=1}^n g_j^{(j)}$$

is an LU transformation. Suppose there exist $k, \ell$ with $g_k \not\equiv g_{\ell}$. Transposing the $k$-th and $\ell$-th coordinates of $U$, let

$$V = g_{\ell}^{(k)} g_k^{(\ell)} \prod_{j \not= k, \ell} g_j^{(j)}$$

By symmetry, we have $\rho' = V \rho V^\dagger$, and therefore we have $\rho = (V^\dagger U) \rho (U^\dagger V)$. Let $h = g_{\ell}^\dagger g_k$ so that we have

$$V^\dagger U = h^{(k)} (h^\dagger)^{(\ell)}.$$ 

and choose $u$ in $U(2)$ to diagonalize $h$, so that we have

$$uhu^\dagger \equiv \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$$

for some $t$. Let us call this diagonal matrix $d$, and let $\tau = (u \otimes n) \rho (u \otimes n)^\dagger$, so that we have

$$\tau = d^{(k)} (d^\dagger)^{(\ell)} \tau (d^\dagger)^{(k)} d^{(\ell)}.$$ 

Let $\tau = \sum_{I,J} c_{IJ} |I\rangle \langle J|$ be the expansion of $\tau$ in the computational basis, where $I = i_1 \ldots i_n$, $J = j_1 \ldots j_n$ denote binary strings of length $n$ and the $c_{IJ}$ are complex coefficients. We claim that if $c_{IJ} \not= 0$, then $J = I$ or $J = I^c$, where $I^c$ denotes the string obtained by taking the mod 2 complement of each bit in the string $I$. Suppose, on the other hand, that there exists a pair $I, J$ such that $c_{IJ} \not= 0$ and $J \not= I$ and $J \not= I^c$. Then there exist two qubit labels $k, \ell$ such that $j_k \not= i_k$ and $j_{k^c} \not= (i_k)^c$. Without loss of generality, suppose $(i_k i_{k^c}) = 00$ and $(j_k j_{k^c}) = 01$. Since

$$d^{(k)} (d^\dagger)^{(\ell)} |00\rangle \langle 01| (d^\dagger)^{(k)} d^{(\ell)} = e^{-2it} |00\rangle \langle 01|$$

we must have $t = m\pi$ for some integer $m$. Then $d \equiv Id$, and so $h \equiv Id$, and therefore $g_k \equiv g_{\ell}$, contradicting our assumption. We conclude that

$$\tau = a |I\rangle \langle I| + b |I^c\rangle \langle I^c| + \bar{b} |I^c\rangle \langle I| + (1-a) |I^c\rangle \langle I^c|$$

(1)
for some coefficients $a, b$ and some bit string $I$.

Next we claim that we may assume $I = 0 \cdots 0$ or $I = 1 \cdots 1$. Suppose contrary that there are two qubit positions $k, \ell$ such that $i_k \neq i_\ell$. Choose any third qubit position $r$ (this is where we use the hypothesis that $n \geq 3$). We must have $i_r = i_k$ or $i_r = i_\ell$. Without loss of generality, suppose $i_r = i_k$. Now transpose qubits $\ell, r$. This produces a state $\bar{\tau}$ with nonzero coefficient for the term $|I\rangle \langle I'|$, where $i'_k = i'_\ell$. But this contradicts the fact that $\bar{\tau} = \tau$ because $\tau$ is symmetric. This establishes the claim.

Next we claim that we may take $b$ to be real and nonnegative in $[1]$. If $b$ is not real, let $\phi = \arg(b)/n$ if $I = 0 \cdots 0$ and let $\phi = -\arg(b)/n$ if $I = 1 \cdots 1$. Replacing $\tau$ by $(\text{diag}(1,e^{i\phi}))^\otimes n$ $\tau$ $(\text{diag}(1,e^{-i\phi}))^\otimes n$ establishes the claim.

Now apply the preceding argument to $\rho'$ to construct a sequence of LU transformations that are the same in each qubit to obtain

$$\tau' = a'|I\rangle \langle I'| + b'|I'\rangle \langle I'| + b'(1 - a') |(1')\rangle \langle (1')| + (1 - a') |(1')\rangle \langle (1')|$$

for some real and nonnegative coefficients $a', b'$ and some bit string $I' = 0 \cdots 0$ or $I' = 1 \cdots 1$. Comparing 1-qubit reduced density matrices for $\tau, \tau'$ yields $a = a'$ or $a = 1 - a'$. If the latter, replace $\tau'$ by $(X, \ldots, X) \tau' (X, \ldots, X)$. Finally, comparing eigenvalues of $\tau, \tau'$, we conclude that $b = b'$. Thus we have constructed a chain of symmetric local unitary operations that transform $\rho$ to $\rho'$, as desired. This concludes the proof of Theorem 1.

**Theorem 2** Let $|\psi\rangle, |\psi'\rangle$ be $n$-qubit symmetric states with Majorana configurations $C_\psi, C_{\psi'}$. Then $|\psi\rangle, |\psi'\rangle$ are local unitary equivalent if and only if there exists an element $g$ in $U(2)$ such that

$$C_{\psi'} = gC_\psi.$$

**Proof of Theorem 2** Let $|\psi\rangle, |\psi'\rangle$ be symmetric states with Majorana configurations $C_\psi = \{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ and $C_{\psi'} = \{|\psi'_1\rangle, \ldots, |\psi'_n\rangle\}$. If there is a rotation of the Bloch sphere given by $g$ in $U(2)$ that takes $C_\psi$ to $C_{\psi'}$, then (possibly after renumbering) we have $g|\psi_j\rangle = |\psi'_j\rangle$ for $1 \leq j \leq n$, and hence $|\psi'\rangle \equiv g^\otimes n |\psi\rangle$. Conversely, if $|\psi'\rangle = U|\psi\rangle$ for some local unitary $U$, then by Theorem 1, there is a $g$ in $U(2)$ such that $|\psi'\rangle = g^\otimes n |\psi\rangle$. We can interpret this $g$ as a rotation of the Bloch sphere, and it is clear that we have $C_{\psi'} = gC_\psi$.

**Theorem 3** Let $\rho$ be an $n$-qubit symmetric pure state whose local unitary stabilizer $\text{Stab}_\rho$ is infinite. Then one of the following holds.

(i) The state $\rho$ is LU equivalent to the product state $\tau = |\psi\rangle \langle \psi|$, where $|\psi\rangle = |0\cdots 0\rangle$ and $\text{Stab}_\rho$ is isomorphic to $U(1)^n$, where $(e^{it_1}, \ldots, e^{it_n})$ in $U(1)^n$ corresponds to

$$(\exp(-it_1Z/2), \ldots, \exp(-it_nZ/2))$$

in $\text{Stab}_\tau$. There is one $LU$ equivalence class of this type.

(iia) The state $\rho$ is LU equivalent to the GHZ state $\tau = |\psi\rangle \langle \psi|$, where $|\psi\rangle = (1/\sqrt{2})(|0\cdots 0\rangle + |1\cdots 1\rangle)$ for some $n \geq 3$ and $\text{Stab}_\rho$ is isomorphic to $U(1)^{n-1} \ltimes \mathbb{Z}_2$, where $(e^{it_1}, \ldots, e^{it_{n-1}}, b)$ in $U(1)^{n-1} \ltimes \mathbb{Z}_2$ corresponds to

$$(\exp(-it_1Z/2), \ldots, \exp(-it_{n-1}Z/2), \exp(i \sum_k t_k Z/2)(X, \ldots, X)^b)$$

in $\text{Stab}_\tau$. There is one $LU$ equivalence class of this type.
(iib) The state $\rho$ is LU equivalent to the generalized GHZ state $\tau = |\psi\rangle \langle \psi|$, where $|\psi\rangle = a|0\cdots0\rangle + b|1\cdots1\rangle$ for some $n \geq 3$ with $|a| \neq |b|$, and $\text{Stab}_\rho$ is isomorphic to $U(1)^{n-1}$, where $(e^{it_1}, \ldots, e^{it_{n-1}})$ in $U(1)^{n-1}$ corresponds to
\[\left(\exp(-it_1Z/2), \ldots, \exp(-it_{n-1}Z/2), \exp\left(-i \sum_k t_k\right)Z/2\right)\]
in $\text{Stab}_\tau$. We may take $a$ and $b$ to both be positive and real with $a > b$. The LU equivalence classes of this type are parameterized by the interval $0 < t < 1$ by $a = \cos \frac{\pi}{4}t$, $b = \sin \frac{\pi}{4}t$.

(iii) The state $\psi$ is LU equivalent to the singlet state $\tau = |\psi\rangle \langle \psi|$, where $|\psi\rangle = |01\rangle - |10\rangle$ and $\text{Stab}_\rho$ is isomorphic to $PU(2)$, where $g$ in $PU(2)$ corresponds to $(g, g)$ in $\text{Stab}_\tau$. There is one LU equivalence class of this type.

(iva) The state $\psi$ is LU equivalent to the Dicke state $\tau = |\psi\rangle \langle \psi|$, where $|\psi\rangle = D_n^{(1/2)}$ for some even $n \geq 4$, and $\text{Stab}_\rho$ is isomorphic to $U(1) \times Z_2$, where $(e^{it}, b)$ in $U(1) \times Z_2$ corresponds to
\[\left(\exp(-itZ/2), \ldots, \exp(-itZ/2)\right) \cdot (X, \ldots, X)^b\]
in $\text{Stab}_\tau$. There are one LU equivalence class of this type.

(ivb) The state $\psi$ is LU equivalent to the Dicke state $\tau = |\psi\rangle \langle \psi|$, where $|\psi\rangle = D_k^n$ for some $n \geq 3$ and some $k$ in the range $0 < k < n$ and $k \neq n/2$, and $\text{Stab}_\rho$ is isomorphic to $U(1)$, where $(e^{it})$ in $U(1)$ corresponds to
\[\left(\exp(-itZ/2), \ldots, \exp(-itZ/2)\right)\]
in $\text{Stab}_\tau$. There are $\lfloor n/2 \rfloor$ LU equivalence classes of this type, with representatives
\[\left|D_n^{(1)}\right>, \left|D_n^{(2)}\right>, \ldots, \left|D_n^{(\lfloor n/2 \rfloor-1)}\right>\].

**Proof of Theorem 3.** We show in [8] that an arbitrary pure symmetric state $\rho$ is LU equivalent to one of the states $\tau$ listed in (i)–(ivb). Theorem 1 of that paper identifies 4 families of nonzero stabilizer Lie subalgebras, which exponentiate to stabilizer subgroup elements of the forms given in (i)–(ivb). In each case, it is easy to see that the given correspondences are one-to-one. To establish the claimed isomorphisms, it remains to be shown that the groups

(i) $U(1)^n$
(ii) $U(1)^{n-1} \times Z_2$
(iii) $PU(2)$
(iva) $U(1) \times Z_2$
(ivb) $U(1)$
given in (i)–(ivb) above map surjectively onto the full stabilizer subgroups of the corresponding states given in (i)–(ivb).

Below we give the proof for (iia) and (iva). The other proofs are both similar and easier. The proofs that the homomorphisms in (iia) and (iva) are onto share the following outline. We consider an arbitrary element \( U = (g_1, \ldots, g_n) \) in \( \text{Stab}_\rho \), and we wish to show that \( U \) is in the image of the given homomorphism. First, we show that it suffices to show that either all \( g_k \) are diagonal, or all \( g_k \) are antidiagonal. Then we consider two cases. The first case is where \( g_k \equiv g_\ell \) for all \( k, \ell \), so that \( U \) has the form \( U \equiv (g_1, \ldots, g_1) \) for some \( g_1 \) in \( U(2) \). The second case with where there exists a pair of qubits \( k, \ell \) such that \( g_k \not\equiv g_\ell \). We show that both cases lead to the conclusion that either all the \( g_k \) are diagonal, or all the \( g_k \) are antidiagonal. By the earlier reduction, this completes the proof.

**Proof of surjectivity in Theorem 3 (iia).** Let \( \rho = |\psi\rangle \langle \psi| \) be the \( n \)-qubit GHZ state, where \( |\psi\rangle = |0\cdots0\rangle + |1\cdots1\rangle \) for some \( n \geq 3 \), and let \( U = (g_1, \ldots, g_n) \) be an element of \( \text{Stab}_\rho \). We aim to prove that \( U \) can be written in the form

\[
\left( \exp(-it_1 Z/2), \ldots, \exp(-it_{n-1} Z/2), \exp(i \left( \sum_k t_k \right) Z/2) \right) \cdot (X, \ldots, X)^b
\]

for some real \( t_1, \ldots, t_{n-1} \) and some \( b = 0, 1 \).

We begin with the claim that it suffices to show that either \( g_k \) is diagonal for all \( k \), or \( g_k \) is antidiagonal for all \( k \). Indeed, if every \( g_k \) is diagonal, say \( g_k \equiv e^{it_k i Z} \), then from

\[
U |0\cdots0\rangle \langle 1\cdots1| U^\dagger = \exp(2i(\sum t_k)) |0\cdots0\rangle \langle 1\cdots1|
\]

we conclude that \( \sum t_k \) is an integer multiple of \( \pi \), and so \( \sum t_k \) may be taken to be zero (because we are working projectively; we have \( e^{in\pi Z} = -\text{Id} \equiv \text{Id} \)). Hence \( U \) is the image of the element \( (e^{-it_1/2}, \ldots, e^{-it_{n-1}/2}) \) in \( U(1)^{n-1} \). If every \( g_k \) is antidiagonal, then we can write

\[
g_k \equiv \begin{bmatrix} 0 & e^{it_k} \\ e^{-it_k} & 0 \end{bmatrix} = \begin{bmatrix} e^{it_k} & 0 \\ 0 & e^{-it_k} \end{bmatrix} X. \tag{2}
\]

Then from

\[
U |0\cdots0\rangle \langle 1\cdots1| U^\dagger = \exp(-2i(\sum t_k)) |1\cdots1\rangle \langle 0\cdots0|
\]

we have \( \sum t_k \) is \( \pi \) times and integer, \( U \) is projectively equivalent to the image of \( (e^{-it_1/2}, \ldots, e^{-it_{n-1}/2}) \) in \( U(1)^{n-1} \times \mathbb{Z}_2 \). This establishes the claim.

Next we show that either all \( g_k \) are diagonal, or all \( g_k \) are antidiagonal.

**Case a.** Suppose that \( g_k \equiv g_\ell \) for all \( k, \ell \). Then \( U \) has the form \( U \equiv (h, \ldots, h) \) for some \( h \) in \( U(2) \). As in the proof of Theorem 2 we can read \( h \), and therefore \( U \), as a rigid motion of the Bloch sphere, and conclude that \( U \) takes the Majorana configuration for the GHZ state into itself. Thus \( U \) is a symmetry of the regular \( n \)-gon in the equatorial plane, and is therefore either a rotation about the \( Z \)-axis, or a 180-degree-rotation about the \( X \)-axis followed by a rotation about the \( Z \)-axis. Thus \( h \) is either diagonal or antidiagonal.
antidiagonal. From this we have that $h$, $d$, $n$ take the Majorana configuration for $|\psi\rangle$ for some $e$ of the Bloch sphere that must be of the form $g$ so we conclude that all Bloch sphere coming from either a diagonal or antidiagonal matrix as in case a, $d$ matrix the Majorana configuration of $|\tau\rangle$ $k, \ell$ that $g$ leads to the contradiction that $g_k \equiv g_t$, so we conclude that $\tau = |\psi'\rangle \langle \psi'|$ where $|\psi'\rangle = a |0\cdots0\rangle + b |1\cdots1\rangle$ is an LU-equivalent GHZ state with $|a| = |b|$. The Majorana configuration for $\tau$ is a regular $n$-gon in the equatorial plane, so we may conclude that $u$ is a rigid motion of the Bloch sphere that must be of the form $e^{i\phi Z}$ or $e^{i\phi Z} X$, so $u$ is diagonal or antidiagonal. From this we have that $h$ is diagonal, so $g_k = g_t d_{tk}$ for some diagonal matrix $d_{tk}$. It follows that $U$ is of the form $U \equiv (g_1, g_1 d_{12}, \ldots, g_1 d_{1n}) = (g_1, \ldots, g_1)(1, \mathrm{Id}, d_{12}, \ldots, d_{1n})$.

Since the action of $(\mathrm{Id}, d_{12}, \ldots, d_{1n})$ is a rotation about the $Z$-axis, and $U$ takes the Majorana configuration of $\tau$ to itself, it must be that $g_1$ is a rigid motion of the Bloch sphere coming from either a diagonal or antidiagonal matrix as in case a, so we conclude that all $g_k$ are diagonal or all $g_k$ are antidiagonal, as desired.

Proof of surjectivity in Theorem A (iv). Let $\rho$ be the Dicke state $\rho = |\psi\rangle \langle \psi|$, where $|\psi\rangle = \left| D_n^{(n/2)} \right\rangle$ for some even $n \geq 4$, and let $U = (e^{it}, g_1, \ldots, g_n)$ be an element of Stab$_\rho$. We aim to prove that $U$ can be written in the form $(\exp(-itZ/2), \ldots, \exp(-itZ/2)) \cdot (X, \ldots, X)^b$ for real $t$ and some $b = 0, 1$.

We begin with the claim that it suffices to show that either $g_k$ is diagonal for all $k$, or $g_k$ is antidiagonal for all $k$. Suppose that all $g_k$ are diagonal, say $g_k \equiv e^{it_k Z}$. Choose two qubit labels $k, \ell$, choose a weight $n/2$ multindex $I = i_1 i_2 \ldots i_n$ such that $i_k = 0, i_\ell = 1$, and let $J = j_1 j_2 \ldots j_n$ denote the multindex that is formed by complementing the $k$-th and $\ell$-th bits of $I$. Then from $U |I\rangle \langle J| U^\dagger = \exp(2it_k - t_\ell) |I\rangle \langle J|$ we conclude that $t_k - t_\ell$ is an integer multiple of $\pi$. This holds for all $k, \ell$, so we have $g_k \equiv g_1$ for all $k$, so that $U \equiv (g_1, \ldots, g_1)$. Thus $U$ is the image of $(e^{-it_1/2}, 0)$ in $U(1) \times Z_2$. If every $g_k$ is antidiagonal, then again we may write $g_k$ in the form of equation 2. Considering the action of $U$ on $|I\rangle \langle J|$ above, the same argument goes through with minor changes, and we have that $U$ is the image of $(e^{-it_1/2}, 1)$ in $U(1) \times Z_2$. This establishes the claim.

Next we show that either all $g_k$ are diagonal, or all $g_k$ are antidiagonal.

Case b. Suppose there exist qubits $k, \ell$ such that $g_k \not\equiv g_\ell$. As in the proof of Theorem A let $V = g_k^{(k)} g_\ell^{(\ell)} \prod_{j \neq k, \ell} g_j^{(j)}$, let $h = g_t g_k$, so that

$$V^\dagger U = h^{(k)} (h^{1})^{(\ell)}$$

is in Stab$_h$. Choose $u$ in $U(2)$ to diagonalize $h$, and let $\tau = u^\dagger u \rho (u^\dagger u)^\dagger$, so that we have $d^{(k)} (d^{1})^{(\ell)}$ in Stab$_\tau$, where $d = e^{itZ} \equiv uhu^\dagger$, for some real $t$. Continuing to follow the proof of Theorem A considering the action of $d^{(k)} (d^{1})^{(\ell)}$ on qubits $k, \ell$, the presence of a standard nonzero coefficient $c_{k,\ell}$ in the expansion of $\tau$ in the computational basis with $J \neq I$ and $J \neq I^c$ in qubits $k, \ell$ leads to the contradiction that $g_k \equiv g_\ell$, so we conclude that $\tau = |\psi''\rangle \langle \psi'|$ where $|\psi''\rangle = a |0\cdots0\rangle + b |1\cdots1\rangle$ is an LU-equivalent GHZ state with $|a| = |b|$. The Majorana configuration for $\tau$ is a regular $n$-gon in the equatorial plane, so we may conclude that $u$ is a rigid motion of the Bloch sphere that must be of the form $e^{i\phi Z}$ or $e^{i\phi Z} X$, so $u$ is diagonal or antidiagonal. From this we have that $h$ is diagonal, so $g_k = g_t d_{tk}$ for some diagonal matrix $d_{tk}$. It follows that $U$ is of the form $U \equiv (g_1, g_1 d_{12}, \ldots, g_1 d_{1n}) = (g_1, \ldots, g_1)(1, \mathrm{Id}, d_{12}, \ldots, d_{1n})$.

Since the action of $(\mathrm{Id}, d_{12}, \ldots, d_{1n})$ is a rotation about the $Z$-axis, and $U$ takes the Majorana configuration of $\tau$ to itself, it must be that $g_1$ is a rigid motion of the Bloch sphere coming from either a diagonal or antidiagonal matrix as in case a, so we conclude that all $g_k$ are diagonal or all $g_k$ are antidiagonal, as desired.
the $Z$-axis, or $h$ is a 180-degree-rotation about the $X$-axis followed by a $Z$-axis rotation. In the first case, $h$ is diagonal. In the second case, $h$ is antidiagonal.

Case b. Suppose there exist qubits $k, \ell$ such that $g_k \neq g_\ell$. By the same argument as for case b in the previous proof of surjectivity for (iia), we conclude that $|\psi\rangle$ is LU equivalent to a state of the form $a|0\cdots0\rangle + b|1\cdots1\rangle$, which is either a product state or a generalized GHZ state. But this violates the known LU classification (Theorem 1 of [8]) for symmetric states. We conclude that case b cannot hold, and this ends the proof. □

4 Conclusion

We have completely classified LU equivalence classes of LU stabilizer subgroups for pure symmetric states. For finite stabilizer subgroups, we have given a complete classification of LU equivalence classes of symmetric states. For each finite stabilizer subgroup, there are an infinite number of LU equivalence classes of symmetric states. Each family is characterized by a Majorana configuration, and the LU equivalent states are precisely those whose Majorana configurations are obtained by rotating the Bloch sphere.

In future work we hope to extend these results to mixed symmetric states. We are encouraged by the success of recent work [7] by Bastin et al., in which they extend to mixed symmetric states their own SLOCC classification [6] for pure symmetric states.

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