A cut-free proof system for a predicate extension of the logic of provability

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Abstract
In this paper, we introduce a proof system $\text{NQGL}$ for a Kripke complete predicate extension of the logic $\text{GL}$, that is, the logic of provability, which is defined by $\text{K}$ and the L"ob formula $\square(\square p \supset p) \supset \square p$. $\text{NQGL}$ is a modal extension of Gentzen’s sequent calculus $\text{LK}$. Although the propositional fragment of $\text{NQGL}$ axiomatizes $\text{GL}$, it does not have the L"ob formula as its axiom. Instead, it has a non-compact rule, that is, a derivation rule with countably many premises. We show that $\text{NQGL}$ enjoys cut admissibility and is complete with respect to the class of Kripke frames such that for each world, the supremum of the length of the paths from the world is finite.

1 Introduction
In this paper, we introduce a cut-free proof system for a Kripke complete predicate extension of $\text{GL}$, where $\text{GL}$ is a propositional normal modal logic defined by $\text{K}$ and the L"ob formula

$$\square(\square p \supset p) \supset \square p.$$ (1)

$\text{GL}$ is well-known as the logic of provability, in the sense that a propositional modal formula $\phi$ is in $\text{GL}$ if and only if $f(\phi)$ is provable in the Peano arithmetic $\text{PA}$ for every arithmetical interpretation $f$ (e.g. [3]).

A Kripke frame $(W, R)$ is said to be conversely well-founded, if there exists no countably infinite list $(w_i)_{i \in \mathbb{N}}$ of elements of $W$ which satisfies $(w_i, w_{i+1}) \in R$ for any $i \in \mathbb{N}$, and is said to be of bounded length, if for any $w \in W$ the supremum of the length of the lists $w_0, w_1, \ldots, w_n$ which satisfy $(w_i, w_{i+1}) \in R$ and $w_0 = w$ is finite. We write $\mathfrak{T}$, $\mathfrak{B}$, and $\mathfrak{CM}$ for the classes of transitive Kripke frames which are finite and irreflexive, of bounded length, and conversely well-founded, respectively. For any class $C$ of Kripke frames, we write $\text{MP}(C)$ and $\text{MQ}(C)$ for the sets of propositional modal formulas and predicate modal formulas which are valid in $C$, respectively. It is known (e.g. [3]) that

$$\text{GL} = \text{MP}(\mathfrak{T}) = \text{MP}(\mathfrak{CM}).$$
Therefore,
\[
\text{GL} = \text{MP}(\mathcal{B}\mathcal{L}).
\] (2)

However, the situation in predicate extensions of GL is not so clear. Let QGL be the smallest predicate normal modal logic which includes GL as its propositional fragment. Let QPL(PA) be the set of predicate modal formulas defined by
\[
\text{QPL}(\text{PA}) = \{ \phi \mid \text{PA} \vdash f(\phi) \text{ for every interpretation } f \}.
\]

It is shown in [7] that QGL \(\subseteq\) MQ(\(\mathfrak{FJ}\)) and QGL is incomplete with respect to any classes of Kripke frames. It is also proved in [7] that QGL \(\nsubseteq\) QPL(PA), that is, QGL is arithmetically incomplete, and QPL(PA) \(\nsubseteq\) MQ(\(\mathfrak{FJ}\)). Subsequently, [4] shows that if a closed predicate modal formula \(\phi\) is not valid in a finite irreflexive Kripke model with finite domains then there exists an interpretation \(f\) such that PA \(\not\vdash f(\phi)\). To summarize these results, we have the following:

\[
\begin{array}{c}
\text{QPL}(\text{PA}) \subseteq \\
\text{MQ}(\mathfrak{FJ} \text{ with finite domains}) \\
\cup \mathfrak{S} \\
\text{QGL} \\
\end{array}
\]

On the other hand, [12] introduces a logic QGL\(^b\), a predicate extension of GL, in which all occurrences of individual variables in a scope of a modal operator are considered to be bound, and
\[
\Box \phi \rightarrow \Box \forall x \phi
\]
is an axiom schema. It is proved in [12] that QGL\(^b\) is both arithmetically complete and Kripke complete with respect to \(\mathfrak{FJ}\), under the above restriction in the construction of formulas.

In [6], a sequent system for GL is introduced, of which modal rule is
\[
\frac{\Box \Gamma, \Box \phi \rightarrow \phi}{\Box \Gamma \rightarrow \Box \phi}.
\] (4)

A proof of the cut-elimination theorem of the system is given in [11] by a syntactic method, and a semantic proof of it is given in [2]. It is also proved in [2] that the simple predicate extension of the system does not admit cut-elimination. While a sequent of the above sequent system is defined to be a pair of sets of formulas, [5] gives a translation of the argument in [11] to a sequent system built from multisets. A cut-free proof system for QGL\(^b\) is introduced in [8].

Though none of MQ(\(\mathfrak{C}\mathfrak{M}\)), MQ(\(\mathcal{B}\mathcal{L}\)), nor MQ(\(\mathfrak{FJ}\)) are arithmetically complete as described in [4], it could be of some interest as a problem of pure modal
logic to give a cut-free proof system for a Kripke complete predicate extension of GL without any restriction in the construction of formulas. In this paper, we introduce a proof system NQGL, which is a modal extension of Gentzen’s sequent calculus LK for predicate logic, and show the admissibility of the cut-rule and Kripke completeness with respect to BL. From the Kripke completeness, it follows by [2] that the propositional fragment of NQGL axiomatizes GL, but NQGL does not include [1] nor [4] as an axiom schema or a derivation rule, respectively. Instead, it has a non-compact rule, that is, a derivation rule with countably many premises. In [3] and [9], a general theory for model existence theorem for propositional modal logic with non-compact rules is given, also, in [10], for their predicate extension with Barcan formula

\[ BF = \forall x \Box \phi \supset \Box \forall x \phi. \]

It follows immediately as a corollary of the main theorem of [10], that the system defined by NQGL and BF is Kripke complete with respect to BL with constant domains. However, it is shown in [7] that BF is not PA-valid. Therefore, we do not add BF.

The outline of the paper is the following: In Section 2 we give basic definitions for syntax and semantics. In Section 3 we introduce the system NQGL. In Section 4 the notions of finitely consistent pairs and saturated pairs are introduced. In Section 5 we show Kripke completeness of NQGL with respect to BL, as well as the admissibility of the cut-rule.

2 Preliminaries

The language we consider consists of the following symbols:

1. a countable set \( V \) of variables;
2. \( \top \) and \( \bot \);
3. logical connectives: \&, \neg, \supset;
4. quantifier: \( \forall \);
5. for each \( n \in \mathbb{N} \), countably many predicate symbols \( P, Q, R, \cdots \) of arity \( n \);
6. modal operator \( \Box \).

The set \( \Phi(\mathcal{V}) \) of formulas over \( \mathcal{V} \) is the smallest set which satisfies:

1. \( \top \) and \( \bot \) are in \( \Phi(\mathcal{V}) \);
2. if \( P \) is a predicate symbol of arity \( n \) and \( x_1, \ldots, x_n \) are variables in \( \mathcal{V} \) then \( P(x_1, \ldots, x_n) \) is in \( \Phi(\mathcal{V}) \);
3. if \( \phi \) and \( \psi \) are in \( \Phi(\mathcal{V}) \) then \( (\phi \land \psi) \) and \( (\phi \supset \psi) \) are in \( \Phi(\mathcal{V}) \);
4. if \( \phi \in \Phi(V) \) then \((\neg \phi)\) and \((\Box \phi)\) are in \(\Phi(V)\);

5. if \( \phi \in \Phi(V) \) and \( x \in V \) then \((\forall x \phi)\) ∈ \(\Phi(V)\).

As usual, \( \vee \) and \( \exists \) are the duals of \( \land \) and \( \forall \), respectively. The symbol \( \Diamond \) is an abbreviation of \( \neg \Box \neg \cdot \), and for each \( n \in \mathbb{N} \), \( \Box^n \) denote \( n \)-times applications of \( \Box \) and \( \Diamond \), respectively. For each set \( S \) of formulas, we write \( \Box S \) and \( \Box^{-1} S \) for the sets

\[
\Box S = \{ \Box \phi \mid \phi \in S \}, \quad \Box^{-1} S = \{ \phi \mid \Box \phi \in S \}
\]

of formulas, respectively. For each formula \( \phi \), we write \( \text{Var}(\phi) \) for the set of variables which have some free or bound occurrences in \( \phi \). For each set \( S \) of formulas, \( \text{Var}(S) \) denotes the set \( \bigcup_{\phi \in S} \text{Var}(\phi) \). For each subset \( U \) of \( V \),

\[
\Phi(U) = \{ \phi \in \Phi(V) \mid \text{Var}(\phi) \subseteq U \}.
\]

A **Kripke frame** is a pair \((W, R)\), where \( W \) is a non-empty set and \( R \) is a binary relation on \( W \). A **system of domains** over a frame \( F = (W, R) \) is a family \( D = (D_w)_{w \in W} \) of non-empty sets such that for all \( w_1 \) and \( w_2 \) in \( W \),

\[
(w_1, w_2) \in R \Rightarrow D_{w_1} \subseteq D_{w_2}.
\]

A **predicate Kripke frame** over \( F = (W, R) \) is a triple \((W, R, D)\), where \( D \) is a system of domains over \( F \). A **Kripke model** is a four tuple \((W, R, D, I)\), where \((W, R, D)\) is a predicate Kripke frame and \( I \) is a mapping called an **interpretation** which maps each pair \((w, P)\), where \( w \) is a member of \( W \) and \( P \) is an \( n \)-ary predicate symbol, to an \( n \)-ary relation \( I(w, P) \subseteq (D_w)^n \) over \( D_w \). The relation \( \models \) among a Kripke model \( M = (W, R, D, I) \), a world \( w \in W \), and a closed formula \( \phi \) is defined inductively as follows:

1. \( M, w \models \top, M, w \not\models \bot; \)

2. for any predicate \( P \) of arity \( n \),
   \[
   M, w \models P(d_1, \ldots, d_n) \iff (d_1, \ldots, d_n) \in I(w, P); \]

3. \( M, w \models \phi \land \psi \iff M, w \models \phi \) and \( M, w \models \psi; \)

4. \( M, w \models \phi \lor \psi \iff M, w \not\models \phi \) or \( M, w \models \psi; \)

5. \( M, w \models \neg \phi \iff M, w \not\models \phi; \)

6. \( M, w \models \forall x \phi \iff M, w \models \phi[d/x] \) for any \( d \in D_w; \)

7. \( M, w \models \Box \phi \iff (w, w') \in R \) implies \( M, w' \models \phi \) for any \( w' \) in \( W \).

Validity of a non-closed formula is defined by the validity of the universal closure of it. Let \( \phi \) be a formula. If every world \( w \) in a Kripke model \( M \) satisfies \( M, w \models \phi \), we write \( M \models \phi \). If every Kripke model \( M \) over a frame \( F \) satisfies \( M \models \phi \), we write \( F \models \phi \). If every \( F \) in a class \( C \) of Kripke frames satisfies \( F \models \phi \), we write \( C \models \phi \). The following lemma holds immediately:

**Lemma 2.1.** For any Kripke model \( M = (W, R, D, I) \), the underlying frame \((W, R)\) is of bounded length if and only if for any \( w \in W \) there exists some \( n \in \mathbb{N} \) such that \( M, w \models \neg \Box^n \top \).
3 Non-compact proof system for predicate extension of the logic of provability

In this section, we introduce a proof system NQGL for a predicate extension of GL. The proof system NQGL is a variant of Gentzen-style sequent calculus. A sequent \( \Gamma \rightarrow \Delta \) is defined to be a pair of finite sets \( \Gamma \) and \( \Delta \) of formulas. The axiom schemata of NQGL are \( p \rightarrow p \), \( \rightarrow \top \), \( \bot \rightarrow \), and the derivation rules of NQGL are the following:

Set

\[
\frac{\Gamma \rightarrow \Delta}{\Gamma' \rightarrow \Delta'} \quad (\text{where } \Gamma \subseteq \Gamma' \text{ and } \Delta \subseteq \Delta')
\]

Cut

\[
\frac{\Gamma \rightarrow \Delta, \phi \quad \phi, \Lambda \rightarrow \Xi}{\Gamma, \Lambda \rightarrow \Delta, \Xi}
\]

Conjunction

\[
\frac{\Gamma \rightarrow \Delta, \phi \quad \Gamma \rightarrow \Delta, \psi}{\Gamma \rightarrow \Delta, \phi \land \psi}
\]

\[
\frac{\phi, \Gamma \rightarrow \Delta \quad \psi, \Gamma \rightarrow \Delta}{\phi \land \psi, \Gamma \rightarrow \Delta}
\]

Implication

\[
\frac{\phi, \Gamma \rightarrow \Delta, \psi}{\Gamma \rightarrow \Delta, \phi \supset \psi}
\]

\[
\frac{\Gamma \rightarrow \Delta, \phi \quad \psi, \Lambda \rightarrow \Xi}{\phi \supset \psi, \Gamma, \Lambda \rightarrow \Delta, \Xi}
\]

Negation

\[
\frac{\phi, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg \phi}
\]

\[
\frac{\Gamma \rightarrow \Delta, \phi}{\neg \phi, \Gamma \rightarrow \Delta}
\]

For all

\[
\frac{\Gamma \rightarrow \Delta, \phi[y/x]}{\Gamma \rightarrow \Delta, \forall x \phi}
\]

\[
\frac{\phi[z/x], \Gamma \rightarrow \Delta}{\forall x \phi, \Gamma \rightarrow \Delta}
\]

Here, \( y \) is a variable in \( V \) which does not occur in any formulas in the lower sequent, and \( z \) is any variable in \( V \).

Box

\[
\frac{\Box \Gamma, \Delta \rightarrow \phi}{\Box \Gamma, \Box \Delta \rightarrow \Box \phi}
\]

Boundedness of length

\[
\frac{\Gamma \rightarrow \Delta, \Diamond^n \top}{\Gamma \rightarrow \Delta} \quad (\text{for any } n \in \mathbb{N})
\]

Here, the set of upper sequents is countably infinite.
For any sequent $\Gamma \rightarrow \Delta$, we write $\vdash_{\text{NQGL}} \Gamma \rightarrow \Delta$ if it is derivable in NQGL. A formula $\phi$ is said to be derivable in NQGL, if $\vdash_{\text{NQGL}} \phi$. If this is the case, we write $\vdash_{\text{NQGL}} \phi$. It is easy to see that the rule Box is equivalent to $\Box p \supset \Box \Box p$ plus standard necessitation rule.

\[
\Gamma \rightarrow \phi \\
\Box \Gamma \rightarrow \Box \phi.
\]

The rule Boundedness of length denotes that

\[
\bigwedge_{n \in \mathbb{N}} \Diamond^n 1 = 0
\]

(5)

holds in the Lindenbaum algebra of the logic defined by NQGL. Note that if a Boolean algebra with operators satisfies (5), the following equation holds in it, either:

\[
\bigwedge_{n \in \mathbb{N}} \Box \Diamond^n 1 = \Box 0.
\]

**Theorem 3.1.** (Soundness of NQGL). If $\vdash_{\text{NQGL}} \phi$, then $\mathfrak{B} \models \phi$, for any formula $\phi$.

### 4 Finitely consistent pairs and saturated pairs

In this section, we introduce some notions which are used to show the Kripke completeness and the admissibility of the cut-rule. We write NQGL$^-$ for the cut-free fragment of NQGL, and $\vdash_{\text{NQGL}}^\neg \Gamma \rightarrow \Delta$ if a sequent $\Gamma \rightarrow \Delta$ is derivable in NQGL$^-$.

**Definition 4.1.** A pair $(S, T)$ of sets of formulas is said to be finitely consistent if for any finite sets $S' \subseteq S$ and $T' \subseteq T$,

$\forall_{\text{NQGL}} - S' \rightarrow T'$.

**Definition 4.2.** Let $\mathcal{U}$ be a set of variables. A finitely consistent pair $(S, T)$ of subsets of $\Phi(\mathcal{U})$ is said to be $\mathcal{U}$-saturated, if the following conditions are satisfied:

1. If $\phi_1 \land \phi_2 \in S$, then $\phi_1, \phi_2 \in S$, and if $\phi_1 \land \phi_2 \in T$, then either $\phi_1 \in T$ or $\phi_2 \in T$.
2. If $\phi_1 \supset \phi_2 \in S$, then either $\phi_1 \in T$ or $\phi_2 \in S$, and if $\phi_1 \supset \phi_2 \in T$, then $\phi_1 \in S$ and $\phi_2 \in T$.
3. If $\neg \phi \in S$, then $\phi \in T$, and if $\neg \phi \in T$, then $\phi \in S$.
4. If $\forall x \phi \in S$, then $\phi[z/x] \in S$ for all $z \in \mathcal{U}$, and if $\forall x \phi \in T$, then $\phi[z/x] \in T$ for some $z \in \mathcal{U}$.
Definition 4.3. A finitely consistent pair \((S, T)\) of formulas is called a GL-pair, if \(\Box \neg \Diamond^n \top \in S\) for some \(n \in \mathbb{N}\).

Theorem 4.4. Let \(\mathcal{U}\) be a cofinite subset of \(\mathcal{V}\). Suppose \((S, T)\) is a finitely consistent pair of subsets of \(\Phi(\mathcal{U})\). Then, there exists a cofinite subset \(\mathcal{U}'\) of \(\mathcal{V}\) and a \(\mathcal{U}'\)-saturated pair \((S', T')\) such that \(\mathcal{U} \subseteq \mathcal{U}'\), \(S \subseteq S'\), and \(T \subseteq T'\).

Proof. Take a cofinite subset \(\mathcal{W}\) of \(\mathcal{V}\) such that \(\mathcal{U}\) is a cofinite subset of \(\mathcal{W}\).

Let \((\phi_n)_{n \in \mathbb{N}}\) be a sequence of formulas of \(\Phi(\mathcal{W})\) such that each formula of \(\Phi(\mathcal{W})\) occurs infinitely many times in it. For example, if \((\gamma_n)_{n \in \mathbb{N}}\) is an enumeration of all formulas of \(\Phi(\mathcal{W})\), \((\phi_n)_{n \in \mathbb{N}}\) could be

\[
\gamma_0, \; \gamma_0, \gamma_1, \; \gamma_0, \gamma_1, \gamma_2, \; \gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots
\]

Define lists \((\mathcal{U}_n)_{n \in \mathbb{N}}\) and \(((S_n, T_n))_{n \in \mathbb{N}}\) which satisfies the following:

1. for every \(n \in \mathbb{N}\), \(\mathcal{U}_n\) is a cofinite subset of \(\mathcal{W}\) and \(\mathcal{U}_n \subseteq \mathcal{U}_{n+1}\);
2. for every \(n \in \mathbb{N}\), \((S_n, T_n)\) is a finitely consistent pair of subsets of \(\Phi(\mathcal{U}_n)\), \(S_n \subseteq S_{n+1}\), and \(T_n \subseteq T_{n+1}\).

First, let \(\mathcal{U}_0 = \mathcal{U}\) and \((S_0, T_0) = (S, T)\). Suppose \(\mathcal{U}_i\) and \((S_i, T_i)\) are defined for every \(i \leq n\):

- Case \(\phi_n = \psi_1 \land \psi_2\): \(U_{n+1} = U_n\). If \(\psi_1 \land \psi_2 \in S_n\), then \(S_{n+1} = S_n \cup \{\psi_1, \psi_2\}\) and \(T_{n+1} = T_n\). If \(\psi_1 \land \psi_2 \in T_n\), then \(S_{n+1} = S_n\) and define \(T_{n+1}\) by \(T_{n+1} = T_n \cup \{\psi_1\}\) or \(T_{n+1} = T_n \cup \{\psi_2\}\), so that \((S_{n+1}, T_{n+1})\) is finitely consistent.

- Case \(\phi_n = \psi_1 \lor \psi_2\): \(U_{n+1} = U_n\). If \(\psi_1 \lor \psi_2 \in S_n\), then define \(S_{n+1}\) and \(T_{n+1}\) by \(S_{n+1} = S_n\) and \(T_{n+1} = T_n \cup \{\psi_1\}\), or \(S_{n+1} = S_n \cup \{\psi_2\}\) and \(T_{n+1} = T_n\), so that \((S_{n+1}, T_{n+1})\) is finitely consistent. If \(\psi_1 \lor \psi_2 \in T_n\), then \(S_{n+1} = S_n \cup \{\psi_1\}\) and \(T_{n+1} = T_n \cup \{\psi_2\}\).

- Case \(\phi_n = \neg \psi\): \(U_{n+1} = U_n\). If \(\neg \psi \in S_n\), then \(S_{n+1} = S_n\) and \(T_{n+1} = T_n \cup \{\psi\}\). If \(\neg \psi \in T_n\), then \(S_{n+1} = S_n \cup \{\psi\}\) and \(T_{n+1} = T_n\).

- Case \(\phi_n = \forall x \psi\): If \(\forall x \psi \in S_n\), then \(U_{n+1} = U_n\), \(S_{n+1} = S_n \cup \{\psi[z/x]\} \mid z \in U_n\), and \(T_{n+1} = T_n\). If \(\forall x \psi \in T_n\), then \(U_{n+1} = U_n \cup \{z\}\), where \(z \in \mathcal{W} \setminus U_n\), \(S_{n+1} = S_n\), and \(T_{n+1} = T_n \cup \{\psi[z/x]\}\).

- Otherwise, \(U_{n+1} = U_n\) and \((S_n, T_n) = (S_{n+1}, T_{n+1})\).

It is clear that the conditions 1 and 2 are satisfied. Now, Let

\[
\mathcal{U}' = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n, \; S' = \bigcup_{n \in \mathbb{N}} S_n, \; T' = \bigcup_{n \in \mathbb{N}} T_n.
\]

Since each formula in \(\Phi(\mathcal{W})\) occurs infinitely many times in the list \((\phi_n)_{n \in \mathbb{N}}\), \(\mathcal{U}'\) and \((S', T')\) satisfy the first part of the 4th condition of Definition 4.2. It is easy to check the other conditions are fulfilled. \(\square\)
**Theorem 4.5.** Let $U$ be a coinfinite subset of $V$ and $(S, T)$ a $U$-consistent GL-pair. If $\Box \phi \in T$, there exists a coinfinite subset $U'$ of $V$ and a $U'$-saturated GL-pair $(S', T')$ such that $U \subseteq U'$, $\phi \in T'$, and $\Box^{-1}S \cup \Box^{-1}S \subseteq S'$.

**Proof.** Since $(S, T)$ is finitely consistent, so is $(\Box^{-1}S \cup \Box^{-1}S, \{\phi\})$. Since $(S, T)$ is a GL-pair, $\Box \neg \neg \top \in \Box^{-1}S$ for some $n \in \mathbb{N}$. Now, by Theorem 4.4, there exists a coinfinite subset $U'$ of $V$ and $U'$-saturated pair $(S', T')$ such that $U \subseteq U'$, $\phi \in T'$, and $\Box^{-1}S \cup \Box^{-1}S \subseteq S'$.

5 Kripke completeness of $\text{NQGL}^-$

In this section, we show that the cut-free fragment $\text{NQGL}^-$ of $\text{NQGL}$ is Kripke complete with respect to $\text{BL}$. The admissibility of the cut-rule follows from the completeness theorem and Theorem 3.1.

**Theorem 5.1.** If $\not\vdash_{\text{NQGL}^-} \Gamma \rightarrow \Delta$, there exists a coinfinite subset $U$ of $V$ and a $U$-saturated GL-pair $(S, T)$ such that $\Gamma \subseteq S$ and $\Delta \subseteq T$.

**Proof.** By the rule of boundedness, there exists $n \in \mathbb{N}$ such that $\not\vdash_{\text{NQGL}^-} \Box \neg \neg \top \rightarrow \Gamma \rightarrow \Delta$.

Apply Theorem 4.4 to $\text{Var}(\Gamma \cup \Delta)$ and $(\{\Box \neg \neg \top\} \cup \Gamma, \Delta)$.

**Theorem 5.2.** (Kripke completeness of $\text{NQGL}^-$). A formula $\phi$ is derivable in $\text{NQGL}^-$ if and only if $\text{BL} \models \phi$.

**Proof.** We only show the if-part. Define a model $M = (W, R, D, I)$ as follows:

- $W$ is the set of all triples $(U, S, T)$, where $U$ is a coinfinite subset of $V$ and $(S, T)$ is a $U$-saturated GL-pair.
- For any $(U, S, T)$ and $(U', S', T')$ in $W$,
  $$(U, S, T), (U', S', T') \in R \iff U \subseteq U' \text{ and } \Box^{-1}S \cup \Box^{-1}S \subseteq S'. $$
- For any $(U, S, T) \in W$, $D_{(U, S, T)} = U$.
- For any $(U, S, T) \in W$ and any predicate symbol $P$ of arity $n$,
  $$I((U, S, T), P) = \{(x_1, \ldots, x_n) \in V^n \mid P(x_1, \ldots, x_n) \in S\}.$$ By definition of $R$, the frame $(W, R)$ is transitive. We claim that for any formula $\phi$ and $(U, S, T) \in W$,$$
\phi \in S \Rightarrow M, (U, S, T) \models \phi, \quad \phi \in T \Rightarrow M, (U, S, T) \not\models \phi.$$

We show the claim only for the cases of $\phi = P(x_1, \ldots, x_n)$, $\forall x \psi(x)$, and $\Box \psi$:
• Case $\phi = P(x_1, \ldots, x_n)$: By definitions of $I$ and $|=,$

$$P(x_1, \ldots, x_n) \in S \iff (x_1, \ldots, x_n) \in I((U, S, T), P)$$

$$\iff M, (U, S, T) \models P(x_1, \ldots, x_n).$$

Since $(S, T)$ is finitely consistent,

$$P(x_1, \ldots, x_n) \in T \Rightarrow P(x_1, \ldots, x_n) \notin S$$

$$\iff (x_1, \ldots, x_n) \notin I((U, S, T), P)$$

$$\iff M, (U, S, T) \not\models P(x_1, \ldots, x_n).$$

• Case $\phi = \forall x \psi$: If $\forall x \psi(x) \in S,$ then $\psi(z) \in S$ for any $z \in U,$ since $(S, T)$ is $U$-saturated. Hence, by induction hypothesis, $M, (U, S, T) \models \psi(z)$ for any $u \in D(U, S, T).$ If $\forall x \psi(x) \in T,$ then, $\psi(z) \in T$ for some $z \in U,$ since $(S, T)$ is $U$-saturated. By induction hypothesis, $M, (U, S, T) \not\models \psi(z)$ for some $z \in D(U, S, T).$

• Case $\phi = \Box \psi$: Suppose $\Box \psi \in S$ and $((U, S, T), (U', S', T')) \in R.$ Then, $\psi \in S'$ by definition of $R.$ By induction hypothesis, $M, (U', S', T') \models \psi.$ Suppose $\Box \psi \in T.$ Then, by Theorem [1.3] there exists a co-finite subset $U'$ of $V$ and a $U'$-saturated $GL$-pair $(S', T')$ such that $U \subseteq U',$ $\phi \in T',$ and $\Box^{-1}S \cup \Box^{-1} \subseteq S'.$ Then, $(U', S', T') \in W,$ $((U, S, T), (U', S', T')) \in R,$ and, by induction hypothesis, $M, (U', S', T') \not\models \psi.$

This complete the proof of the claim. By using the claim and Lemma 2.1 $(W, R) \in \mathfrak{B}L.$ Now, suppose $\not\models_{\mathfrak{B}L} \Gamma \rightarrow \Delta.$ Then, by Theorem 5.1 there exists $(U, S, T) \in W$ such that $\Gamma \subseteq S$ and $\Delta \subseteq T.$ Hence, $M, (U, S, T) \not\models \Gamma \rightarrow \Delta.$

References

[1] S. Artemov and G. Dzhaparidze. Finite Kripke models and predicate logics of provability. The Journal of Symbolic Logic, 55:1090–1098, 1990.

[2] A. Avron. On modal systems having arithmetical interpretations. The Journal of Symbolic Logic, 49:935–942, 1984.

[3] G. Boolos. The logic of provability. Cambridge University Press, 1993.

[4] R. Goldblatt. Mathematics of Modality, volume 43 of CSLI Lecture Notes. CSLI Publications, 1993.

[5] R. Goré and R. Ramanayake. Valentini’s cut-elimination for provability logic resolved. In C. Areces and R. Goldblatt, editors, Advances in Modal Logic, volume 7, pages 67–86. CSLI Publications, 2008.

[6] D. Leivant. On the proof theory of modal logic for arithmetic provability. The Journal of Symbolic Logic, 46:531–538, 1981.
[7] F. Montagna. The predicate modal logic of provability. *Notre Dame Journal of Formal Logic*, 25:179–189, 1984.

[8] Y. Schwarz and G. Tourlakis. On the proof-theory of a first-order extension of GL. *Logic and Logical Philosophy*, 23:329–363, 2014.

[9] K. Segerberg. A model existence theorem in infinitary propositional modal logic. *Journal of Philosophical Logic*, 23:337–367, 1994.

[10] Y. Tanaka. Model existence in non-compact modal logic. *Studia Logica*, 67:61–73, 2001.

[11] S. Valentini. The modal logic of provability. *Journal of Philosophical Logic*, 12:471–476, 1983.

[12] R. E. Yavorsky. On arithmetical completeness of first order logics of provability. In F. Wolter, H. Wansin, and M. Zakharyaschev, editors, *Advances in Modal Logic*, volume 3, pages 1–16. CSLI Publications, 2001.