Development and Analysis of Deterministic Privacy-Preserving Policies Using Non-Stochastic Information Theory

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Abstract—A non-stochastic privacy metric using non-stochastic information theory is developed. Particularly, minimax information is used to construct a measure of information leakage, which is inversely proportional to the measure of privacy. Anyone can submit a query to a trusted agent with access to a non-stochastic uncertain private dataset. Optimal deterministic privacy-preserving policies for responding to the submitted query are computed by maximizing the measure of privacy subject to a constraint on the worst-case quality of the response (i.e., the worst-case difference between the response by the agent and the output of the query computed on the private dataset). The optimal privacy-preserving policy is proved to be a piecewise constant function in the form of a quantization operator applied on the output of the submitted query. The measure of privacy is also used to analyze the performance of \( k \)-anonymity methodology (a popular deterministic mechanism for privacy-preserving release of datasets using suppression and generalization techniques), proving that it is in fact not privacy-preserving.

Index Terms—Non-stochastic Information Theory, Minimax Information, Privacy, Piecewise Constant Function, Quantization.

I. INTRODUCTION

Advances in communication and computation engineering have enabled the use of big data analysis for answering societal challenges. These advances have motivated incorporation of new tools for collection and analysis of datasets, and reporting data-driven insights. The erosion of privacy caused by the adoption of such new tools has resulted in adoption of new rules by governments, such as the General Data Protection Regulation (GDPR) in the European Union, for protecting citizens, customers, and their data.

Anonymization is most often used as a method of choice by governments or companies alike for releasing private datasets to the broader public for analysis. Although popularly adopted, anonymization has been proved to be insufficient for privacy preservation [1–3]. Therefore, systematic methods for privacy preservation in a provable manner should be developed.

Differential privacy and its variants, such local differential privacy and probabilistic differential privacy, are parts of a category of methodologies with provable privacy guarantees [4–9]. These methods, in summary, rely on the use of randomized policies, such as additive noise, to ensure that the statistics of the reported outputs do not change noticeably by variations in an individual entry of the dataset. This property ensures that an adversary cannot reverse-engineer the differentially-private outputs to accurately estimate an individual private entry of the dataset, even in the presence of side information. Various studies have been devoted to finding “optimal” noise distribution in differential privacy [10–12]; however, off-the-shelf mechanisms, such as the additive Laplace and Gaussian noise with scales proportional to the sensitivity of the submitted query with respect to individual entries of the dataset, are often used to ensure differential privacy [5]. Note that the use of randomized policies for privacy protection in itself is not particularly new [13] but, prior to differential privacy, provable guarantees were often missing.

Another methodology for privacy protection is the use of information theoretic metrics dating back to the pioneering work in [14] on secrecy. In secrecy problem, a sender wishes to devise an encoding scheme to create a secure channel for communicating with a receiver while hiding her data from an eavesdropper (similar to the setup of encryption). The privacy problem with the emphasis on masking or equivocating of information from the intended primary receiver (rather than an eavesdropper) or a secondary receiver with as much information as the primary receiver have been studied in [15–18]. Information-theoretic guarantees have been also provided on the amount of leaked private information when utilizing the differential privacy [19, 20]. Furthermore, entropy, mutual information, Kulback-Leiber divergence, and Fisher information have been repeatedly used as measures of privacy in [21–27].

A common thread or assumption among all these methodologies is that they utilize randomization for safeguarding privacy. In fact, the definition of differential privacy assumes the use of randomized functions and information theoretic tools used so far have been based on randomized random variables. However, many popular heuristic-based privacy-preserving methods, such as \( k \)-anonymity [28, 29] and \( \ell \)-diversity [30], are deterministic.

The popularity of non-stochastic methods is caused by perhaps multiple factors. First, undesirable properties of differentially-private additive noise, especially the Laplace noise, might make it less appealing [31, 32]. Another reason could be the simplicity of implementing deterministic policies, in the sense of not requiring a working knowledge of random variables and their generation by laymen. Finally, and somewhat more troubling, generation of unreasonable and unrealistic outputs by the use of randomness (e.g., inducing panic within the over-analyzing financial sector by reporting noisy outputs that point to the lack of liquidity in a bank or an entire sector—perhaps the reason behind low popularity of differential privacy within the financial sectors).

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See https://data.gov.au for an example of government initiative. Many other examples can be found in https://www.kaggle.com.
So far, deterministic privacy-preserving policies are generated in an ad hoc manner and are often vulnerable to attacks (e.g., k-anonymity has been proved to be vulnerable to attacks, such as homogeneity attack [30]). This is because there is no good measure of privacy that works for deterministic policies on deterministic datasets. Therefore, one cannot prove (in some sense) privacy guarantees of the methods (even if weak or limited in scope or practice). The popularity of non-stochastic privacy-preserving policies justifies requiring a metric for their analysis and comparison (irrespective of their inherent philosophical weaknesses in comparison to stochastic policies).

Motivated by this observation, in this paper, we develop a deterministic privacy metric based on non-stochastic information theory. Traditional information theory, starting with Shannon’s seminal work in [33], usually assumes that data (source) and communication channels are stochastic in nature. This has been proved to be extremely powerful in modelling and analysing communication systems; see, e.g., [34] and references there-in. However, the notion of information within the traditional information theory literature, such as mutual information, is not useful for analysing non-stochastic uncertain variables and deterministic privacy-preserving policies. However, there is also a parallel less-studied (within tertiary education) theory of non-stochastic information theory [35–42], which has been recently used within engineering [40–43].

In this paper, non-stochastic measures of information, such as minimax information, from the non-stochastic information theory literature are used to develop a measure of privacy. Anyone can submit a query to a trusted agent with access to a non-stochastic uncertain private dataset. Optimal deterministic privacy-preserving policies for responding to the submitted query are sought by maximizing the measure of privacy subject to a constraint on the worst-case quality of the response (i.e., the worst-case difference between the response by the agent and the output of the query computed on the private dataset). The optimal privacy-preserving policy is in fact a quantization operator applied on the output of the submitted query computed based on the private dataset. The developed measure of privacy is utilized to analyze the performance of k-anonymity, proving that it is not privacy-preserving, which was previously shown using adverserial attacks in [50].

The rest of the paper is organized as follows. Section II provides a summary of non-stochastic information theory. The problem formulation is also presented in this section. In Section III, a piecewise constant function, in the form of a quantization operator, is proved to be an optimal privacy-preserving policy. The privacy of k-anonymity is analyzed using the proposed non-stochastic privacy metrics in Section IV. Finally, Section V concludes the paper and presents future directions for work.

II. NON-STOCHASTIC INFORMATION THEORY

A. Uncertain Variables

Consider sample space \( \Omega \). Each element \( \omega \in \Omega \) is referred to as a sample. The sample space is the source of the uncertainty. Any mapping \( X : \Omega \to \mathbb{X} \) defines an uncertain variable. A realization of such a variable is \( X(\omega) \). However, for the ease of presentation, \( X(\omega) \) is replaced by \( X \) when the dependence of the uncertain variable to the sample is evident from the context. Up to this point, the difference between uncertain variables and random variables is the absence of a measure on the space \( \Omega \). Throughout this paper, we assume that all uncertain variables are real-valued, i.e., \( \mathbb{X} \subseteq \mathbb{R}^{n_x} \) for some \( n_x \in \mathbb{N} \). Marginal range of \( X \) is defined as

\[
[X] := \{ X(\omega) : \omega \in \Omega \} \subseteq \mathbb{X}.
\]

Joint range of two uncertain variables \( X : \Omega \to \mathbb{X} \) and \( Y : \Omega \to \mathbb{Y} \) is

\[
[X,Y] := \{ (X(\omega), Y(\omega)) : \omega \in \Omega \} \subseteq \mathbb{X} \times \mathbb{Y}.
\]

Finally, conditional range of \( X \) (conditioned on the observation of another uncertain variable \( Y(\omega) = y \)) is given by

\[
[X|y] := \{ X(\omega) : \exists \omega \in \Omega \ \exists Y(\omega) = y \} \subseteq [X].
\]

The family of all conditional ranges is denoted by

\[
[X|Y] := \{ [X|y] : y \in [Y] \} \subseteq 2^{[X]}.
\]

This should not be mistaken with the union of all such conditional ranges given by \( \bigcup_{y \in [Y]} [X|y] = [X] \). In fact, regarding the union, we can prove that \( \bigcup_{y \in [Y]} [X|y] \times \{ y \} = [X,Y] \).

Definition II.1 (Unrelatedness). Uncertain variables \( X_i, i = 1,\ldots,n, \) are unrelated if \( [X_1,\ldots,X_n] = [X_1] \times \cdots \times [X_n] \). Further, they are conditionally unrelated (conditional on \( Y \)) if \( [X_1,\ldots,X_n|y] = [X_1|y] \times \cdots \times [X_n|y] \) for all \( y \in [Y] \).

For two uncertain variables, this definition is equivalent to stating that \( X_1 \) and \( X_2 \) are unrelated if \( [X_1|x_2] = [X_1] \), \( \forall x_2 \in [X_2] \), and vice versa. Again, for two uncertain variables, this definition is equivalent as saying that \( X_1 \) and \( X_2 \) are conditionally unrelated (conditional on \( Y \)) if \([X_1, x_2, y] = [X_1|y] \times \{ x_2 \} = [X_2, y] \). Finally, for uncertain variables \( X \) and \( Y_i, i = 1,\ldots,n, \) it can be seen that \( [X|y_1,\ldots,y_n] \subseteq \bigcap_{i=1}^n [X|y_i] \), where the inequality is achieved if \( Y_i = 1,\ldots,n, \) are unrelated conditional on \( X \).

B. Non-stochastic Entropy and Information

The non-stochastic entropy of uncertain variable \( X \) can be defined as

\[
h_0(X) := \log(\mu([X])) \in \mathbb{R}, \tag{1}
\]

where \( \mathbb{R} \) is the extended real line \( \mathbb{R} \cup \{ \pm \infty \} \) and the logarithm can be taken in any basis. In line with the differential entropy for random variables, the logarithm is in the natural basis throughout the rest of the paper. The non-stochastic entropy in (1) is sometimes referred to as Rényi differential 0-entropy [38].

Remark II.1 (\( \varepsilon \)-entropy). This notion of Rényi differential 0-entropy is intimately related to the \( \varepsilon \)-entropy [39] defined as

\[
h_\varepsilon(X) := \log(N_{\varepsilon}([X])),
\]

where \( N_{\varepsilon}(\cdot) \) is the smallest number of sets of diameter \( 2\varepsilon \) that their union covers \([X]\), referred to as the minimal \( \varepsilon \)-covering. The inequality \( \varepsilon^m N_{\varepsilon}([X]) \leq (\log(N_{\varepsilon}([X])) \leq h_0(X) \leq \log(\mu([X])) \) holds for all non-negative \( \varepsilon \).
\( \mu(\|X\|) \leq (2e)^m N_e(\|X\|) \) implies that \( 0 \leq h(X) - |h_e(X) + m \log(e)| \leq m \log(2). \) This implies that these two notions of entropy are similar.

Similarly, the non-stochastic relative (or conditional) entropy of uncertain variable \( X \) conditioned on uncertain variable \( Y \) can be defined as

\[
h_0(X|Y) := \text{ess sup}_{y \in [Y]} \log(\mu(\|X\|)) - \text{ess inf}_{y \in [Y]} \log\left( \frac{\mu(\|Y\|)}{\mu(\|X\|)} \right),
\]

where, for any real-valued function \( f : \mathcal{X} \to \mathbb{R} \) for some \( \mathcal{X} \subseteq \mathbb{R}^m \), the essential supremum is defined as

\[
\text{ess sup}_{x \in \mathcal{X}} f(x) := \inf\{b \in \mathbb{R} : \mu(\{x \in \mathcal{X} : f(x) > b\}) = 0\}.
\]

Based on the definition of entropy, the non-stochastic information between two uncertain variables \( X \) and \( Y \) can also be defined as

\[
I_0(X;Y) := h_0(X) - h_0(X|Y) = \text{ess inf}_{y \in [Y]} \log\left( \frac{\mu(\|X\|)}{\mu(\|X\|)} \right),
\]

Note that Kolmogorov had defined ‘combinatorial’ conditional entropy using \( \log(\mu(\|X\|)) \) and the measure of information gain was defined as \( \mu(\|X\|)/\mu(\|Y\|) \) in [36]. These quantities are only defined for an observed value of uncertain variable \( Y = y \); however, the definition in (3) relies on the worst-case ratio.

Now, we can establish a non-stochastic version of the Fano’s inequality in the information theory. Let the uncertain variable \( \hat{X}(y) \) denotes an estimate of an uncertain variable \( X \) based on uncertain variable \( Y \) for measurement \( Y = y \). In this paper, we only consider unbiased estimators, defined below.

**Assumption II.1** (Unbiased Estimator). An estimator \( \hat{X} : \mathcal{Y} \to \mathcal{X} \) is unbiased if \( \hat{X}(y) \in [X|y] \).

This essentially means that the estimate is consistent with the received measurement, i.e., \( X, \hat{X}(y) \in [X|y] \). A measure of the quality of the estimate can be defined as

\[
d_{\text{max}}(X, \hat{X}(Y)) := \text{ess sup}_{y \in [Y]} \text{ess sup}_{x \in [X|y]} \|x - \hat{X}(y)\|_2.
\]

This measure captures the largest worst-case distance between uncertain variable \( X \) and its estimate. Before stating the following theorem, a notation needs to be defined. Let \( \Gamma : z \mapsto \int_0^\infty x^{z-1} \exp(-x)dx \) be the Gamma function (extension of factorial to real numbers).

**Theorem II.1.** Consider \( X \) and \( Y = f(X) \) are uncertain variables for some function \( f : \mathcal{X} \to \mathcal{Y} \). Assume that \( [X|Y] \) is a Borel set for all \( y \in [Y] \). Then,

\[
\frac{2\pi^{n_e/2} - 1}{\pi^n/2} \exp(h_0(X|Y)) \leq d_{\text{max}}(X, \hat{X}(Y)).
\]

**Proof.** Further note that

\[
\text{ess sup}_{y \in [Y]} \text{ess sup}_{x \in [X|y]} \|x - \hat{X}(y)\|_2 \geq \text{ess sup}_{y \in [Y]} \text{ess sup}_{x \in [X|y]} \|x - \hat{X}(y)\|_2 \geq \text{ess sup}_{y \in [Y]} \text{diam}([X|y]),
\]

where the last inequality follows from the fact that \( \text{ess sup}_{x \in [X|y]} \|x - \hat{X}(y)\|_2 \) is the radius of a ball that encompasses \( [X|y] \) and is centred at \( \hat{X}(y) \in [X|y] \) (see Assumption II.1) and the smallest such radius is always larger than equal to half of the diameter. Therefore,

\[
\text{ess sup}_{y \in [Y]} \text{ess sup}_{x \in [X|y]} \|x - \hat{X}(y)\|_2 \geq \frac{1}{2} \frac{2\pi^{n_e/2} - 1}{\pi^n/2} \exp(h_0(X|Y)) \leq d_{\text{min}}(X, \hat{X}(Y)).
\]

This completes the proof. \( \square \)

**Example II.1.** The notions of non-stochastic information and relative entropy are not useful for measuring privacy leakage, at least in the problem of privacy preservation. This is because it considers the worst-case \( \mu([X|y]) \), while privacy wants to ensure that all \( \mu([X|y]) \) are large. To see this, consider the following example:

\[
f(X) := \begin{cases} X, & 0 \leq X < 1/2, \\ 1, & \text{otherwise}, \end{cases}
\]

where \( X \) is an uncertain variable with \( \|X\| = [0,1] \). It is easy to show that \( \text{ess sup}_{y \in [Y]} \mu([X|y]) = \log(1/2) \), which is large in comparison to \( h_0(X) = 1 \). Construct an estimator of the form

\[
\hat{X}(Y) := \begin{cases} Y, & 0 \leq Y < 1/2, \\ 3/4, & \text{otherwise}, \end{cases}
\]

Observe that \( d_{\text{max}}(X, \hat{X}(f(X))) = 1/4 \), which attains the lower bound in Theorem II.1 (recall that \( \Gamma(3/2) = \sqrt{\pi}/2 \)), proving that \( \hat{X}(\cdot) \) is optimal in the sense of minimizing \( d_{\text{max}}(X, \hat{X}(f(X))) \). The function \( f(\cdot) \) is clearly not privacy-preserving as \( f(X) = X \) for many inputs! In fact, \( \inf_{y \in [Y]} \mu([X|y]) = 0 \).

Therefore, a notion of relative disarray can be defined:

\[
d_0(X|Y) := \text{ess sup}_{y \in [Y]} \log(\mu([X|y])).
\]

Following this, non-stochastic information leakage can be defined as

\[
L_0(X;Y) := h_0(X) - d_0(X|Y).
\]

Another useful measure of the quality of an estimator is

\[
d_{\text{min}}(X, \hat{X}(Y)) := \text{ess inf}_{y \in [Y]} \text{ess sup}_{x \in [X|y]} \|x - \hat{X}(y)\|_2.
\]

This measure captures the smallest worst-case distance between uncertain variable \( X \) and its estimate. If \( d_{\text{min}}(X, \hat{X}(Y)) \) is small, it means that there exists some values for uncertain variable \( X \) for which the privacy is not preserved in the sense that an adversary can reconstruct \( X \) for those values accurately based on \( Y \).

**Theorem II.2.** Consider \( X \) and \( Y = f(X) \) are uncertain variables for some function \( f : \mathcal{X} \to \mathcal{Y} \). Assume that \( [X|y] \) is a Borel set for all \( y \in [Y] \). Then,

\[
\frac{2\pi^{n_e/2} - 1}{\pi^n/2} \exp(d_0(X|Y)) \leq d_{\text{min}}(X, \hat{X}(Y)).
\]
Proof. The proof follows the same line of reasoning as in the proof of Theorem II.1. Note that,

\[
\frac{1}{2} \text{diam}(\mathcal{Y}[x|y]) \geq \frac{1}{2} \text{diam}(\mathcal{Y}[x|y]) \geq \text{ess sup}_{y \in \mathcal{Y}} \text{ess sup}_{x \in \mathcal{X}[y]} \| x - \hat{X}(y) \|_2
\]

This completes the proof. \( \square \)

Example II.1 (Cont’d). In this example, \( d_0(X; f(X)) = -\infty \) (by the convention that \( \log(0) = \lim_{\alpha \to 0} \log(\alpha) = -\infty \)) and \( L_0(X; f(X)) = +\infty \). Hence, non-stochastic information leakage \( L_0(X; f(X)) \) can accurately capture the fact that \( f(X) \) is not privacy preserving. In addition, we can show that \( d_{\min}(X; \hat{X}(Y)) = 0 \), which proves that again \( \hat{X}(Y) \) is optimal in the sense of the cost function \( d_{\min}(X; \hat{X}(Y)) \) (as the lower bound in Theorem II.2 is achieved).

In general, the non-stochastic information and non-stochastic information leakage are not symmetrical, that is, \( I_0(X; Y) \neq I_0(Y; X) \) and \( L_0(X; Y) \neq L_0(Y; X) \) (contrary to mutual information in the information theory literature). Therefore, non-stochastic information transmission was proposed in [38], defined as

\[
T_0(X; Y) := h_0(X) + h_0(Y) - h_0(X, Y).
\] (8)

This new measure of information is symmetric, that is, \( T_0(X; Y) = T_0(Y; X) \). Although being symmetric in general, utilization of this measure is not possible (because \( [\mathcal{Y}] \) can be a discrete set \( \mu([\mathcal{Y}]) = 0 \) and thus \( h_0(X, Y) = 0 \) in all such cases). Another symmetric measure of information is the minimax information. In order to define this measure of information, we need to first define the notion of taxicab connectivity.

Definition II.2 (Taxicab Connectivity).
- \((x, y), (x', y') \in [X, Y] \) are taxicab\(^2\) connected if there exists a sequence of points \( \{x_i, y_i\}_{i=1}^n \subseteq [X, Y] \) such that \( (x_1, y_1) = (x, y) \), \( (x_n, y_n) = (x', y') \), and either \( x_i = x_{i-1} \) or \( y_i = y_{i-1} \) for all \( i \in \{2, \ldots, n\} \);
- \( \mathcal{A} \subseteq [X, Y] \) is taxicab connected if all points in \( [X, Y] \) are taxicab connected;
- \( \mathcal{A}, \mathcal{B} \subseteq [X, Y] \) are taxicab isolated if there does not exist any points \( \mathcal{A} \) and \( \mathcal{B} \) that are taxicab connected;
- A taxicab partition of \( [X, Y] \) is a set of sets \( \mathcal{G}(X, Y) := \{\mathcal{A}_i\}_{i=1}^n \) such that \( [X, Y] \subseteq \bigcup_{i=1}^n \mathcal{Y}_i \), any \( \mathcal{A}_i, \mathcal{A}_j \) are taxicab isolated if \( j \neq i \), and \( \mathcal{A}_i \) is taxicab connected for all \( i \).

There exists a unique taxicab partition for any \( [X, Y] \) [38].

Minimax information can be defined as

\[
I_* (X; Y) := \text{ess inf}_{z \in [\mathcal{Z}]} \mu([X|z])
\]

where \( \mathcal{G}(X, Y) \) denotes the unique taxicab partition of \([X|Y]\). It has been proved that \( |\mathcal{G}(X, Y)| = |\mathcal{G}(Y, X)| \) and thus \( L_0(X; Y) = L_0(Y; X) \) resulting in a symmetric notion of information [38].

Example II.1 (Cont’d). In this example, \( L_0(X; f(X)) = +\infty \). This instantly shows that \( f(X) \) is not privacy preserving.

C. Problem Formulation

In what follows, we assume that that a private dataset \( X \) is available to secure trusted agent. Anyone may submit a query of the form \( f(\cdot) \), i.e., it can request that a trusted agent compute and provide the response \( f(X) \).

Definition II.3 (Measure of Privacy). Let \( f'(\cdot) \) be a reporting function and define uncertain variable \( Y \) based on uncertain variable \( X \) such that \( Y = f'(X) \). Then, the measure of privacy for the reporting function \( f' \) is

\[
\Psi_1 (f') := \frac{1}{L_0(X; Y)};
\] (10a)

\[
\Psi_2 (f') := \frac{1}{I_* (X; Y)}.
\] (10b)

The inverse relationship between the measures of privacy and information in [10] is because information leakage reduces the privacy guarantee. We can prove a useful and intuitive property for the aforementioned measures of privacy that after releasing an output it is not possible to gain more information from the data by additional manipulations.

Theorem II.3 (Post Processing). \( \Psi_1 (g \circ f) \geq \Psi_1 (f) \).

Proof. Let uncertain variable \( Y \) and \( Z \) be defined as \( Y(\omega) := f(X(\omega)) \) and \( Z(\omega) := g(Y(\omega)) \) for all \( \omega \in \Omega \). The data processing inequality in [38] implies that \( I_*(X; Z) \leq I_*(X; Y) \). Therefore, \( \Psi_2 \) can be only increased by post processing. For the other measure of privacy note that

\[
d_0(Z|X) = \text{ess inf}_{z \in [\mathcal{Z}]} \mu([X|z])
\]

\[
= \text{ess inf}_{z \in [\mathcal{Z}]} \left( \bigcup_{y' \in [\mathcal{Y}]} [X|y'] \right)
\]

\[
\geq \text{ess inf}_{z \in [\mathcal{Z}]} \text{ess inf}_{y \in [\mathcal{Y}]} \mu([X|y])
\]

\[
\geq \text{ess inf}_{y \in [\mathcal{Y}]} \mu([X|y])
\]

\[
\geq d_0(X|Y).
\]

Hence, \( \Psi_1 \) can also be only increased by post processing. This concludes the proof. \( \square \)

The best policy for preserving privacy, maximizing the measure of privacy, is to ensure that \( X \) and \( f(X) \) are unrelated (making \( \Psi_1 (f) = 0 \)). This is, of course, without any value as all the information regarding \( X \) would be lost and the utility of the report (in every possible sense) is zero. Therefore, there is a need for balancing utility and privacy.

Definition II.4 (Measure of Quality). The measure of quality for the reporting function \( f' \) for the query \( f \) is

\[
\Omega(f') := \frac{1}{\text{ess sup}_{x \in [\mathcal{X}]} \| f(x) - f'(x) \|_2^2}.
\] (11)
With these definition ready, the optimal privacy preserving policy can be computed by solving the optimization problem in
\[ P_\gamma: \max_{f' \in F} \mathcal{Q}_i(f'), \] (12a)
\[ \text{s.t.} \quad \mathcal{Q}(f') \geq \gamma, \] (12b)
where \( F \) denotes the set of functions over which the cost function is optimized, i.e., the set of functions that we are interested in implementing as potential privacy-preserving policies.

**Proposition III.1.** Assume that \([Y] \subseteq \mathbb{R}\). For any \( f' = g \circ f \), we get
\[ I_*(X; g(f(X))) \leq I_*(f(X); g(f(X))), \] (13a)
\[ \mathcal{Q}(f') = 1/\text{ess sup}_{y \in [Y]} \| y - g(y) \|_2. \] (13b)

**Proof.** Let uncertain variable \( Y \) and \( Z \) be defined as \( Y(\omega) := f(X(\omega)) \) and \( Z(\omega) := g(f(X(\omega))) \) for all \( \omega \in \Omega \). The data processing inequality \([33]\) shows that \( I_*(X; Z) \leq \min(I_*(X; Y), I_*(Z; Y)) \). This concludes the proof for (13a).

The proof for (13b) follows from the definition. \( \square \)

When restricting the search for privacy-preserving policies over the set of policies \( F := \{ f' \mid \exists g \in G : f' = g \circ f \} \) for some set \( G \), the optimization problem in (12) with privacy measure (10b) can be relaxed into:
\[ P'_\gamma: \min_{g \in G} I_*(Y; g(Y)), \] (14a)
\[ \text{s.t.} \quad \text{ess sup}_{y \in [Y]} \| y - g(y) \|_2 \leq 1/\gamma, \] (14b)

Note that such a relaxation is not possible for the privacy measure (10a) because this measure of information is not symmetric and thus the data processing inequality does not hold for it in both directions.

### III. Privacy-Preserving Policies

Before stating the results of the paper, we need to define the set of piecewise constant functions. Over the real line \( \mathbb{R} \), a mapping \( g: [y, \overline{y}] \rightarrow [y, \overline{y}] \) is a piecewise constant function if there exist \( y = a_1 \leq a_2 \leq \cdots \leq a_{q+1} = \overline{y} \) and \( b_1 \leq b_2 \leq \cdots \leq b_q \) for some arbitrary number \( q \in \mathbb{N} \) such that \( g(y) = b_i \) for all \( y \in [a_i, a_{i+1}) \) except for \( i = q \) in which case \( g(y) = b_q \) for all \( y \in [a_q, a_{q+1}) \). The ordered sets \( (a_i)_{i=1}^{q+1} \) and \( (b_i)_{i=1}^q \) are referred to as the parameters of the piecewise constant function. Let \( \mathcal{Q}[\{y, \overline{y}\}] \) denote the set of all piecewise constant functions. For more general functions \( g: \mathcal{X} \rightarrow \mathbb{R} \) is a piecewise constant function if there exist sets \( \{\mathcal{X}_i\}_{i=1}^q \) such that \( \mathcal{X} \subseteq \bigcup_{i=1}^q \mathcal{X}_i, \mathcal{X}_i \cap \mathcal{X}_j = \emptyset \) if \( i \neq j \), and \( g(x) = b_i \) if \( b_i \in \mathcal{X}_i \). The ordered sets \( \{\mathcal{X}_i\}_{i=1}^q \) and \( (b_i)_{i=1}^q \) are referred to as the parameters of the piecewise constant function. Let \( \mathcal{Q}(\mathcal{X}) \) denote the set of all piecewise constant functions. When \( \mathcal{X} \) is obvious from the context, \( \mathcal{Q} \) is used instead of \( \mathcal{Q}(\mathcal{X}) \). The set of piecewise constant functions is dense in \( L^p \) for all \( p \in [1, +\infty) \) \([35]\). In the next theorem, we show that that searching over the set of piecewise constant functions is enough for finding the solution of (12).

**Theorem III.1** (Solution Class). The solution of (12) for the privacy metric in (10b) over the set of piecewise differentiable functions is a piecewise constant function.

**Proof.** Let \( x \in [X] \) be any point such that \( \nabla f'(x) \neq 0 \). Then there exists a direction \( d \) such that \( d^T \nabla f'(x) > 0 \). Assume that \( d^T \nabla f'(x) > 0 \); the proof for the other case is identical and is thus omitted. By piecewise continuity of the derivatives, it can be deduced that there exists a small enough neighborhood around \( x \) of the form \( \| x - \tilde{x} \| \leq \varepsilon \| d \| \) inside which \( d^T \nabla f'(x) > 0 \). Therefore, for all \( w \in (-\varepsilon, \varepsilon) \), \( f'(x + wd) \) is increasing and takes a unique value. We prove that no two distinct points in \( \{(x + wd, f'(x + wd)) : w \in (-\varepsilon, \varepsilon)\} \) are taxicable connected. Assume that this not the case. Therefore, there exists \((x, y), (x', y') \in \{(x + wd, f'(x + wd)) : w \in (-\varepsilon, \varepsilon)\} \subseteq [X, Y] \) that are taxicable connected. This implies that there exists a sequence of points \( \{(x_i, y_i)\}_{i=1}^n \subseteq [X, Y] \) such that \((x, y) \neq (x', y'), (x_1, y_1) = (x, y), (x_n, y_n) = (x', y') \), and either \( x_i = x_{i-1} \) or \( y_i = y_{i-1} \) for all \( i \in \{2, \ldots, n\} \). Because \( f \) is a function (i.e., \( y_i = f'(x_i) = f'(x_{i-1}) = y_{i-1} \) if \( x_i = x_{i-1} \)), we can eliminate all transitions such that \( x_i = x_{i-1} \) (as it would also implies that \( y_i = y_{i-1} \)). Therefore, we can construct a subsequence of points \( \{(x_i, y_i)\}_{i=1}^n \subseteq \{(x_i, y_i)\}_{i=1}^n \subseteq [X, Y] \) must satisfy that \( x_1, y_1 = (x, y), (x_n, y_n) = (x', y') \), and \( y_i = y_{i-1} \) for all \( i \in \{2, \ldots, n\} \). This implies that \( y'_n = y_n = y_{n-1} = \cdots = y_2 = y_1 = y \). This is in contradiction with the assumption that \( (x, y) \neq (x', y') \) because it must either be that \( y' \neq y \); note that if \( x_1 \neq x_2 \) in \( \{(x + wd, f'(x + wd)) : w \in (-\varepsilon, \varepsilon)\} \), it must also hold that \( y_1 \neq y_2 \). Noting that two distinct points in \( \{(x + wd, f'(x + wd)) : w \in (-\varepsilon, \varepsilon)\} \) are taxicable connected, there needs to be as many taxicable partitions. This implies that \( \mathcal{S}(X, Y) = \infty \). The other category of functions is all functions for which \( \nabla f'(x) = 0 \) (where defined) for all \( x \). The only functions that satisfy this condition are piecewise constant functions. For piecewise constants \( \mathcal{S}(X, Y) = q < \infty \) with \( q \) denoting the number of disjoint sets \( \{\mathcal{X}_i\}_{i=1}^q \).

**Definition III.1** (Uniform Quantizer). A uniform quantizer is a scalar piecewise constant function with parameters \( (a_i)_{i=1}^{q+1} \) and \( (b_i)_{i=1}^q \) such that \( a_{i+1} - a_i = a_{j+1} - a_j \) and \( b_j = (a_j + a_{j+1})/2 \) for all \( 1 \leq i, j \leq q \). A uniform quantizer can be equivalently represented by the range \( [a_1, a_{q+1}] \) and the number of bins \( q \).

We start by solving the relaxed problem in (14) for scalar cases in the next theorem.

**Theorem III.2** (Relaxed Policy). Assume that \([Y] = [y, \overline{y}] \subseteq \mathbb{R}\). The solution of (14) over \( F = \mathcal{Q} \circ \{f\} \) is a uniform quantizer, equi-dividing \([Y]\) into \( \lfloor (\overline{y} - y)/2 \rfloor \) bins.

**Proof.** Note that, for any \( f' \in F \),
\[ \frac{1}{\mathcal{Q}(f')} = \text{ess sup}_{x \in [X]} \| f(x) - g(f(x)) \| = \text{ess sup}_{y \in [Y]} \| y - g(y) \| = \max_{1 \leq i \leq q} (|b_i - a_i|, |b_i - a_{i+1}|), \]
where \( g \) is any function in \( Q \). Furthermore, \( I_*(Y ; g(Y)) = q \).

The problem (14) can be rewritten as

\[
\begin{align*}
\min_{(a_i)_{i=1}^{q+1}, (b_i)_{i=1}^q} q, \\
\text{s.t. } \max_{1 \leq i \leq q} (|b_i - a_i|, |b_i - a_{i+1}|) \leq \frac{1}{\gamma}, \\
a_{q+1} = \gamma, \quad a_1 = y.
\end{align*}
\]

By selecting \( b_i = (a_i + a_{i+1})/2, \) max\((|b_i - a_i|, |b_i - a_{i+1}|)\) can be made as small as possible. Thus, this problem can be rewritten as

\[
\begin{align*}
\min_{(a_i)_{i=1}^{q+1}, (b_i)_{i=1}^q} q, \\
\text{s.t. } \max_{1 \leq i \leq q} \frac{1}{2} |a_{i+1} - a_i| \leq \frac{1}{\gamma}, \\
\sum_{i=1}^q |a_{i+1} - a_i| = \gamma - y. \tag{15a}
\end{align*}
\]

It is easy to show that \( q < \gamma(\gamma - y)/2 \), the problem is not feasible. This is because

\[
\sum_{i=1}^q |a_{i+1} - a_i| \leq q \max_{1 \leq i \leq q} |a_{i+1} - a_i| \leq q^2/\gamma < \gamma - y.
\]

Therefore, a lower bound on the solution of (15) is then the smallest integer that is larger than \( \gamma(\gamma - y)/2 \), i.e., \( \lceil \gamma(\gamma - y)/2 \rceil \). The uniform quantizer in the statement of theorem achieves the lower bound.

Now, we can consider the general problem in (12) for scalar queries over the set of piecewise continuous functions.

**Theorem III.3 (Optimal Policy).** Assume that \([X] \subseteq \mathbb{R}\).

The solution of (12) for privacy measures in (10a) over \( F = Q([X]) \) is given by

\[
\begin{align*}
&b_i^* \in \arg \min_{b_i \in \mathcal{X}_i^*} |f(x) - b_i|, \tag{16a} \\
&\{ \mathcal{X}_i^* \}_{i=1}^q \subseteq \arg \min_{\{ \mathcal{X}_i \}_{i=1}^q} \min_{\mathcal{X}_i \subseteq \mathbb{R}, \mathcal{X}_i \supseteq [X]} \mu(\mathcal{X}_i), \tag{16b} \\
&\text{s.t. } \max_{1 \leq i \leq q} \text{rad}(\mathcal{X}_i) \leq \frac{1}{\gamma}. \tag{16c}
\end{align*}
\]

For privacy measures in (10b) over \( F = Q([X]) \) is given by

\[
\begin{align*}
&b_i^* \in \arg \min_{b_i \in \mathcal{X}_i^*} |f(x) - b_i|, \tag{17a} \\
&\{ \mathcal{X}_i^* \}_{i=1}^q \subseteq \arg \max_{\{ \mathcal{X}_i \}_{i=1}^q} \min_{\mathcal{X}_i \subseteq \mathbb{R}, \mathcal{X}_i \supseteq [X]} \mu(\mathcal{X}_i), \tag{17b} \\
&\text{s.t. } \max_{1 \leq i \leq q} \text{rad}(\mathcal{X}_i) \leq \frac{1}{\gamma}. \tag{17c}
\end{align*}
\]

**Proof.** Note that, for any \( f' \in F = Q([X]) \), there exists \( \{\mathcal{X}_i, b_i\}_{i=1}^q \) such that \( [X] \subseteq \bigcup_{i=1}^q \mathcal{X}_i, \mathcal{X}_i \cap \mathcal{X}_j = \emptyset \) if \( i \neq j \), and \( f'(x) = b_i \) if \( b_i \in \mathcal{X}_i \). Hence,

\[
\frac{1}{Q(f)} = \text{ess sup}_{x \in [X]} |f(x) - f'(x)| = \max_{1 \leq i \leq q} \text{sup}_{x \in \mathcal{X}_i} |f(x) - b_i|.
\]

Let us consider the privacy measure in (10a). It can be shown that

\[
\begin{align*}
d_0(X|f'(X)) &= \text{ess inf}_{x \in [X]} \log(\mu([X|f'(x)]) \\
&= \text{ess inf}_{1 \leq i \leq q} \mu(\mathcal{X}_i), \\
&\text{s.t. } \max_{1 \leq i \leq q} \text{sup}_{x \in \mathcal{X}_i} |f(x) - b_i| \leq \frac{1}{\gamma}.
\end{align*}
\]

Noting that \( \text{rad}(\mathcal{X}_i) = \max_{1 \leq i \leq q} \text{rad}(\mathcal{X}_i) \), the problem (12) can be rewritten as

\[
\begin{align*}
&\max_{\{ \mathcal{X}_i, b_i\}_{i=1}^q} \min_{1 \leq i \leq q} \mu(\mathcal{X}_i), \\
&\text{s.t. } \max_{1 \leq i \leq q} \text{sup}_{x \in \mathcal{X}_i} |f(x) - b_i| \leq \frac{1}{\gamma}.
\end{align*}
\]

This problem can be rewritten again as

\[
\begin{align*}
&\max_{\{ \mathcal{X}_i, b_i\}_{i=1}^q} \min_{1 \leq i \leq q} \mu(\mathcal{X}_i), \\
&\text{s.t. } \max_{1 \leq i \leq q} \text{sup}_{x \in \mathcal{X}_i} |f(x) - b_i| \leq \frac{1}{\gamma}.
\end{align*}
\]

Noting that \( \text{rad}(\mathcal{X}_i) = \max_{1 \leq i \leq q} \text{rad}(\mathcal{X}_i) \) concludes the proof for the first part. Now, let us consider the privacy measure in (10b). It can be seen that \( I_*(X; f'(X)) = q \).

This is because \( (\mathcal{X}_i \times \{ b_i \})_{i=1}^q \) forms a taxicab partition for \([X], f(X)]\). Hence, the problem (12) can be rewritten as

\[
\begin{align*}
&\max_{\{ \mathcal{X}_i, b_i\}_{i=1}^q} \min_{1 \leq i \leq q} \mu(\mathcal{X}_i), \\
&\text{s.t. } \max_{1 \leq i \leq q} \text{sup}_{x \in \mathcal{X}_i} |f(x) - b_i| \leq \frac{1}{\gamma}.
\end{align*}
\]

This concludes the proof.

For the case where \([X] \subseteq \mathbb{R}\), the results of Theorems III.3 and III.2 [46]. Therefore, there is no loss of generality in designing the quantizer after computing \( f(x) \) rather than designing a general \( f'(x) \). In the next corollary, we show that this holds for more general queries under mild assumptions.
Theorem III.3 to find the optimal privacy-preserving policy

This function can be rewritten as

where $\mathcal{X}$ denotes the quantizer. We have

Clearly, $\mathcal{X} \subseteq \mathcal{Y}'$ and thus $\max_{1 \leq i \leq n} \min_{x \in \mathcal{Y}_{i}} |y - b_{i}| \leq 1/\gamma$. We can define $\mathcal{Y}_i$ such that $\mathcal{Y}_1 = \mathcal{Y}_i'$ and $\mathcal{Y}_i = \mathcal{Y}_i' \setminus (\bigcup_{1 \leq j \leq i - 1} \mathcal{Y}_j)$ for all $i > 1$. Therefore, by selecting $b_i = (a_i + a_{i+1})/2$ minimizes $\sup_{y \in \mathcal{Y}_i} |y - b_i|$. This implies that (18) can be rewritten as the optimization problem in the statement of Theorem III.2.

Example III.1. Let $f$ be a functional that $f^{-1}(y) := \{x|f(x) = y\}$ is a connected set for all $y \in [Y]$. Then, the optimal policy in Theorem III.3 for the privacy metric (10b) is equal to the optimal policy in Theorem III.2.

Proof. The solution of (12) for the privacy measure in (10b) is given by (18). We can define $Y_i' = \{y|\exists x \in X_i : y = f(x)\}$. The inequality constraint in (18) is equivalent to saying that that $\max_{1 \leq i \leq n} \min_{x \in \mathcal{Y}_i} \sup_{y \in \mathcal{Y}_i'} |y - b_{i}| \leq 1/\gamma$. We can define $\mathcal{Y}_i$ such that $\mathcal{Y}_1 = \mathcal{Y}_i'$ and $\mathcal{Y}_i = \mathcal{Y}_i' \setminus (\bigcup_{1 \leq j \leq i - 1} \mathcal{Y}_j)$ for all $i > 1$. If $\mathcal{Y}_i$ is connected, it should take one of the following forms $[a_i, a_{i+1}]$, $[a_i, a_{i+1})$, $(a_i, a_{i+1})$, or $(a_i, a_{i+1})$. Therefore, for the optimal policy, we have $b_i = (a_i + a_{i+1})/2$ minimizes $\sup_{y \in \mathcal{Y}_i} |y - b_i|$. This implies that (18) can be rewritten as the optimization problem in the statement of Theorem III.2. \qed

Corollary III.1. Let $f$ be a function that $f^{-1}(y) := \{x|f(x) = y\}$ is a connected set for all $y \in [Y]$. Then, the optimal policy in Theorem III.3 for the privacy metric (10b) is equal to the optimal policy in Theorem III.2.

Proof. The solution of (12) for the privacy measure in (10b) is given by (18). We can define $Y_i' = \{y|\exists x \in X_i : y = f(x)\}$. The inequality constraint in (18) is equivalent to saying that that $\max_{1 \leq i \leq n} \min_{x \in \mathcal{Y}_i} \sup_{y \in \mathcal{Y}_i'} |y - b_{i}| \leq 1/\gamma$. We can define $\mathcal{Y}_i$ such that $\mathcal{Y}_1 = \mathcal{Y}_i'$ and $\mathcal{Y}_i = \mathcal{Y}_i' \setminus (\bigcup_{1 \leq j \leq i - 1} \mathcal{Y}_j)$ for all $i > 1$. If $\mathcal{Y}_i$ is connected, it should take one of the following forms $[a_i, a_{i+1}]$, $[a_i, a_{i+1})$, $(a_i, a_{i+1})$, or $(a_i, a_{i+1})$. Therefore, by selecting $b_i = (a_i + a_{i+1})/2$ minimizes $\sup_{y \in \mathcal{Y}_i} |y - b_i|$. This implies that (18) can be rewritten as the optimization problem in the statement of Theorem III.2. \qed

IV. RELATIONSHIP TO OTHER NOTIONS OF PRIVACY

In this section, we analyse the privacy merits of $k$-anonymity using the measures of privacy in (10). Consider a dataset $x \in \mathbb{X} \subseteq \mathbb{R}^{n \times m}$ with $n$ rows (entries or individuals) and $m$ columns (attributes). The following argument can easily be extended to other sets and thus, without loss of generality, we focus on real numbers.

Definition IV.1 ($k$-anonymity [28, 29, 47]). A release of data is said to have the $k$-anonymity property if the information for each individual contained in the release cannot be distinguished from at least $k-1$ individuals whose information also appear in the release.

Proposition IV.1. There exists a reporting function $f'(X)$ admitting $k$-anonymity property for which the following holds:

- $d_0(X|f(X)) = 0$ (and thus $L_0(X; f(X)) = h_0(X)$);
- $I_r(X; f(X)) = \infty$.

Proof. Consider the case where $x$ is a dataset that has $k$ identical individuals. Let the first $k$ rows denote the identical individuals. This is without the loss of generality as otherwise the rows can be swapped. Let $f : \mathbb{R}^{n \times m} \to \mathbb{R}^{n \times m}$ be any $k$-anonymous reporting function. Assume that the $i$-th row of $f(x)$ is report corresponding to the $i$-th row of $x$. This is again without the loss of generality as otherwise the output rows can be swapped. Construct $f'$ such that

$$f'(x) = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ [0_{(n-k) \times m} \ I_{n-k}] f(x) \end{bmatrix},$$

where $x_i$ denotes the $i$-th entry of $x$. Now, we can use Theorem III.3 to find the optimal privacy-preserving policy for the case with privacy metric in (10b). Figure 7 illustrates the regions $\{X_i\}_{i=1}^{12}$ for the optimal privacy-preserving policy in Theorem III.3 for $[X] = [-2, 2]^2$, $\gamma = 2$, and linear query $f(x) = 1^T x/2$. For the optimal policy in Figure 7, we have $b_1 = -1.5$, $b_2 = -0.5$, $b_3 = 0.5$, and $b_4 = 1.5$. It is interesting to note that the optimal policy in Figure 7 is in fact equal to (20). Therefore, the relaxation in (14) is without loss of generality in this example. This is because $f$ meets the condition of Corollary III.1. Now, we focus on a nonlinear query of the form $f(x) = x_1^2 + 2x_2^2$. In this case, we have $[Y] = [0, 12]$. Therefore, the optimal policy of the relaxed problem in (14) for $\gamma = 2$ is a uniform quantizer over $[0, 12]$ with 12 bins. Again, we use $g$ denote this quantizer. We can see that

$$f'(x) = i + 0.5, \ i \leq x_1^2 + 2x_2^2 < i + 1, \ \forall i \in \{0, \ldots, 11\}, \quad (21)$$

Again, we can use Theorem III.3 to find the optimal privacy-preserving policy in this case. Figure 7 illustrates the regions $\{X_i\}_{i=1}^{12}$ for the optimal privacy-preserving policy in Theorem III.3 for $[X] = [-2, 2]^2$, $\gamma = 2$, and nonlinear query $f(x) = x_1^2 \text{diag}(1,2)x$. For the optimal policy, we have $b_1 = i - 0.5$ for all $1 \leq i \leq 12$. Similarly, the optimal policy in Figure 7 is equal to (21) and thus, the relaxation in (14) is again without loss of generality. This is because $f$ meets the condition of Corollary III.1.
By construction, \( f' \) is also a \( k \)-anonymous reporting function. However, 
\[
[X | f'(x)] = [X | f(x)] \cap \left\{ \frac{w}{z} \in X \mid w = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \right\},
\]
which shows that \( \mu([X | f'(x)]) = 0 \). Thus, \( d_0(X | f'(X)) = 0 \).
Finally, noting that \( [X | f'(x)] \) must be included in the taxicab partitions for all choices of \( x_1 = \cdots = x_k \), \( \|X | f'(X)\| = +\infty \). This shows that \( I_s(X; f'(X)) = +\infty \).

Proposition IV.1 shows that \( k \)-anonymity is not private. This is because of the homogeneity attack, i.e., attacks that leverage the cases in which all the values for a sensitive value within a set of \( k \) records are identical. In such cases, even though the data has been \( k \)-anonymized, the sensitive value for the set of \( k \) records may be exactly predicted. Such cases are explored to prove Proposition V.1.

V. CONCLUSIONS AND FUTURE WORK

We presented a deterministic privacy metric using non-stochastic information theory is developed. We considered the case where anyone can submit a query to a trusted server with access to a non-stochastic uncertain private data. Optimal privacy-preserving policy is proved to be a quantized version of the output of the submitted query. Finally, we proved that \( k \)-anonymity is not privacy-preserving using the proposed privacy metric. Future work can focus on analysing non-scale queries as well as demonstrating the performance of the method on publicly available datasets.

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