Delay sober up drunkers: Control of diffusion in random walkers

Hiroyasu Ando¹, Kohta Takehara², Miki U. Kobayashi³

¹Faculty of Engineering, Information and Systems, University of Tsukuba, 1-1-1 Ten-noudai, Tsukuba, 305-8573 Japan,
²Department of Business Administration, Graduate School of Social Science, Tokyo Metropolitan University, 1-4-1 Marunouchi, Chiyoda-ku, Tokyo, 100-0005, Japan and
³Faculty of Economics, Rissho University, 4-2-16 Osaki, Shinagawa-ku, Tokyo 141-8602, Japan

(Dated: March 19, 2018)

Time delay in general leads to instability in some systems, while a specific feedback with delay can control fluctuated motion in nonlinear deterministic systems to a stable state. In this paper, we consider a non-stationary stochastic process, i.e., a random walk and observe its diffusion phenomenon with time delayed feedback. Surprisingly, the diffusion coefficient decreases with increasing the delay time. We analytically illustrate this suppression of diffusion by using stochastic delay differential equations and justify the feasibility of this suppression by applying the time-delay feedback to a molecular dynamics model.

PACS numbers: 05.40.-a, 05.40.Fb

Introduction. — Diffusion phenomena are ubiquitous in nature, e.g. in the Brownian motion [1], biological membranes [2, 3], fluid systems [4], engineering systems [5], material science [6], and so on. These diffusion phenomena are sources of noise and are unavoidable in experimental systems [7, 8]. Noise can prevent the precise manipulation of small scale systems. However, it can also be helpful in engineering a system by using physical phenomena such as Logical Stochastic Resonance, where a certain amount of noise is required for logical operation [9]. In both cases, it is important to control noise in terms of variance of stochastic processes, because noise is normally adjusted according to its strength. If the variance of noise is controlled, it would be useful for various kinds of applications.

When we consider systems with noise from a theoretical viewpoint, mathematical models are convenient, because they can numerically simulate an experiment. A mathematical model of the Brownian motion is a typical example of a noisy system, which can be modeled by a simple stochastic description, like the Langevin equation. For practical purposes, it is important to consider how diffusion can be suppressed in the Brownian motion model. The most commonly used approach for suppressing diffusion is to decrease the temperature of the whole system, resulting in a suppression of thermal fluctuations, namely the relation $D \sim T$, where $D$ is the diffusion coefficient and $T$ is temperature. However, the temperature is decreased for all elements in the system and is not selective for specific targets. Therefore, such a control is expensive. We introduce an alternative method that can control the diffusion processes of one particle in the Brownian motion.

In terms of cooling nano-microscopic systems, a mathematical model of feedback cooling in electromechanical oscillator [11] and a generalized model of Langevin dynamics with non-Markovian feedback [12] have been explored. In these systems, time delay in feedback loop is inevitable due to a time lag between the sensor and the manipulator. Furthermore, in nonlinear systems, a time-delayed feedback control (DFC) method has been proposed to stabilize regular motion in fluctuated dynamics [13]. In general, time delay leads to system instability. However, DFC proves that time delay can stabilize deterministic systems. On the other hand, in stochastic systems, e.g. a random walk model with time delay, destabilization has been observed as a result of delay [14]. Thus, we consider whether DFC can stabilize non-stationary stochastic systems.

In this Letter, we focus on stochastic diffusion of random walks, corresponding to the Brownian motion in discrete time, and apply the DFC method to control the extent of this diffusion. Firstly, we observe diffusion processes in a one-dimensional (1D) random walk model, controlled by the DFC method and analyze the control in terms of stochastic delay differential equations. Secondly, we apply the method to a molecular dynamics model of two flocculated particles with thermal fluctuations and that interact via the Lennard-Jones (LJ) potential. We numerically confirm that it is possible to suppress diffusion in the molecular dynamics model by the DFC method. It is noticed that, in conventional control theory, a single path is controlled to a certain state. In contrast, the purpose of this study was to statistically control stochastic processes.

Delayed feedback control of random walk. — In general, nonlinear functions with neutral fixed points show noise-induced diffusion phenomena. If these functions are linearized around the neutral fixed points, the following random walk model can be derived.

\[ x_{n+1} = x_n + D\xi_n, \]  

where $x_n$ is a state variable for discrete time $n$, $\xi_n$ is a $m$-dimensional random variable following the normal distribution $N(0, 1)$ for each entry, and $D$ is an amplitude vector of the noise. Without loss of generality, we consider a 1D case, where $m = 1$.

In order to control the random walk in $x_n$, we intro-
duce a time-delayed feedback control method proposed by Pyragas [13] that is defined by the expression:

\[ x_{n+1} = x_n + D\xi_n - K(x_n - x_{n-\tau}), \]  

(2)

In eq. (2), the DFC is applied to the model (1), where \( K \) is feedback gain with a value of \( 0 \leq K < 1 \) and \( \tau \) is the delay time. Any nonlinear function with noise can be linearized around its stable fixed point and described in the form of (2) as follows.

If we consider the system:

\[ x_{n+1} = f(x_n, a), \]  

(3)

where the map \( f \) is nonlinear and has a stable or neutrally stable fixed point. \( a \) is a parameter. We assume that the fixed point is denoted by \( \tilde{x} \) and linearize \( f \) around \( \tilde{x} \). This results in:

\[ x_{n+1} = f(\tilde{x}) + f'(\tilde{x})(x_n - \tilde{x}), \]  

(4)

which can be described as:

\[ x_{n+1} = f'(\tilde{x})x_n + (1 - f'(\tilde{x}))\tilde{x}, \]  

(5)

where we use the relation \( \tilde{x} = f(\tilde{x}) \). For this equation, if we denote \( K = 1 - f'(\tilde{x}) \):

\[ x_{n+1} = (1 - K)x_n + K\tilde{x}. \]  

(6)

Regarding this linear map, we add the noise term and rewrite (6) as:

\[ x_{n+1} = x_n + D\xi_n - K(x_n - \tilde{x}). \]  

(7)

The dynamics of the system (7) is simple such that a trajectory fluctuates around the fixed point \( \tilde{x} \). It is possible to suppress diffusion of a random walk by the form of (7), in which the diffusion coefficient is zero. On the other hand, in order to control a non-zero diffusion coefficient, we add the control term \( K(x_n - \tilde{x}) \). Consequently, the system is in the form of DFC, namely eq. (2).

It should be noted that this discussion is relevant for systems with only stable fixed points and thus, with no diffusion observed. By the procedure above, such stable systems are converted to systems showing diffusion because of the neutral fixed points.

Figure 1 shows the time series of the (2) system dynamics. It is observed that a longer \( \tau \) suppresses the diffusion of a random walk. This result is counterintuitive, since long time delay should make a system unstable. However, in the system (2), the opposite is observed. Thus, we quantify the diffusion decay with respect to \( \tau \). In Figure 2, the diffusion coefficients \( D \) defined by \( \lim_{T \to \infty} \langle (X_T - X_0)^2 \rangle / T \), is shown as a function of \( \tau \). The result is averaged over 1000 realizations. We takes \( T_{\infty} = 100000 \). It is evident that \( D \) decays by \( \tau^{-2} \). The decay order is illustrated in the following.

We consider a continuous time (2) system:

\[ \dot{y}_t = D\xi_t - K(y_t - y_{t-\tau}), \]  

(8)

where \( y \) is a real variable and \( t \) and \( t - l \) are continuous time points. For this system, let \( \Delta = l/\tau \), \( \tau \in \{2, 3, \ldots\} \). Then, we define \( t_n \) as \( n\Delta \), that is \( n\Delta \), where \( n \in \{-\tau, -\tau + 1, \ldots, 0, 1, \ldots\} \). If we define \( Y_n = y_{t_n} \) and \( \xi_n = \xi_{t_n} \), eq. (8) can be rewritten as

\[ Y_{n+1} = Y_n + D\Delta\xi_n - \Delta K(Y_n - Y_{n-\tau}). \]  

This form is equivalent to eq. (2).

We consider the continuous time system (8) using stochastic delay differential equations. In the standard setting of stochastic calculus, eq. (8) is now redefined as

\[ dy_t = K(y_{t-\tau} - y_t)dt + Dd\xi_t; \quad t \geq 0 \]  

(9)

where \( \{w_t\}_{0 \leq t \leq T} \) is a standard Wiener process on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\), and \( y_s \) for \( s \in [\tau, 0] \) is assumed to be a given \( \mathcal{F}_0 \)-measurable random variable \( Z_s \). It should be noted that \( \mathcal{F}_t \) contains all information for the realization of \( w_t \) and \( y_s \) for \( s \leq t \). Thus, given \( \mathcal{F}_t \), all of these variables can be all treated as constants. In the following, the method for determining the diffusion coefficient \( D \) is briefly described. For details,
see Supplemental Materials.

Equation (9) has the unique solution of

\[
y_t = y^* + D \int_0^t y_{t-s} \, dw_s; 0 \leq t,
\]

(10)

where

\[
y_t^* := y_0^* Z_0 + K \int_{-\tau}^0 y_{t-s} Z_s \, ds
\]

\[
y_0^* := \sum_{n=0}^{\lceil t/\tau \rceil} \left( \frac{K^n}{n!} \right) (t - n\tau)^n \exp(-K(t - n\tau)).
\]

(11)

(See [16]). It should be noted that \( y_0^* \) is a deterministic, rather than a stochastic process.

Then, we can express \( D \) in terms of \( y_0^* \) as:

\[
D = \lim_{T \to \infty} \frac{D^2}{T} \int_0^T (y_0^*)^2 \, ds.
\]

(12)

Moreover, we examined how the function \( y_0^* \) behaves in our numerical computations of the equation (11), and found that it seemed converging to a constant of \( \tilde{y} \propto t^{-1} \), as described in the Supplemental Materials and in [17]. As a consequence, \( D \) is given by:

\[
D = (D\tilde{y})^2.
\]

In addition, it should be noted that this \( \tilde{y} \) and \( D \) decreases with increasing \( \tau \), which is consistent with Figure 2.

Here, the characteristics of our model should be stressed. In [16], the equivalent conditions to stationary solutions of \( y = \{y_i\} \) are given, which are not met in our system (10). In the case of the stationary \( y \), its diffusion coefficient \( D \) goes to zero as \( T \to \infty \). Conversely, \( D \) seems to explode when the solution \( y \) is non-stationary. In this sense, our model with the non-zero \( D \) lies at the boundary between a stationary and non-stationary state.

This can be seen in another intuitive way. If a limiting process of our model (9) is considered with \( \tau \to 0 \):

\[
dy_t = D \, dw_t.
\]

It should be noted that this is just a scaled random walk, which still includes \( D = D \). On the other hand, \( D = 0 \), under another limiting process with \( \tau \to \infty \):

\[
dy_t = K(y_{-\infty} - y_t) \, dt + D \, dw_t
\]

where \( y_{-\infty} := \lim_{t \to -\infty} y_t \) is given as an initial condition and as a constant target control level.

As derived before, in our model, the diffusion coefficient is given by \( D = D^2 \times (\frac{1}{1+K\tau})^2 \). Thus, any level of \( D \) between \( 0, D \) can be achieved with an appropriate choice of \( K \) and \( \tau \).

**Application to Molecular Dynamics.** — In the following, we apply the control scheme which is shown in the above to a molecular dynamics model.

We consider the dynamics of two particles with noise, and attractive and dissipative forces in a 1D space:

\[
\begin{align*}
\frac{dx_i}{dt} &= v_i, \\
\frac{dv_i}{dt} &= P_i(\{x_j\}_{j=1}^m) + D_i(\{v_j\}_{j=1}^m) + R_i\xi_i \quad (14)
\end{align*}
\]

where \( x_i \) and \( v_i \) are the position and velocity of the \( i \)-th particle \( (i = 1, 2) \), and \( \xi \) is noise from the Gaussian distribution with intensity, \( R \). For simplicity, we assume that the mass of the \( i \)-th particles, \( m_i = 1 \). The first term in eq. (14) is a conservative force derived as a derivative of the potential \( U \) written as:

\[
P_i = -\frac{dU(\{x_j\}_{j=1}^m)}{dx_i},
\]

\[
U(\{x_j\}_{j=1}^m) = \sum_{i=1}^2 \sum_{j=i+1}^m \tilde{V}(x_{ij}).
\]

(15)

(16)

where \( \tilde{V} \) is a central force potential, and \( x_{ij} = x_i - x_j \).

In this paper, we use the Lennard-Jones potential as the central potential as given by:

\[
\tilde{V}(x) = \epsilon \left( \frac{\sigma}{x} \right)^{12} - \frac{\epsilon}{2} \left( \frac{\sigma}{x} \right)^6.
\]

(17)

The second term in eq. (14) represents a dissipative force:

\[
D_i = -\gamma v_i,
\]

(18)

where \( \gamma \) is a coefficient of drag. We set \( \sigma = 2^{-1/6} \) so as that the LJ potential \( \tilde{V}(x) \) has a minimum at \( x = 1 \) and \( \epsilon = 1 \).

We calculate dynamics of two particles with initial conditions \( x_1 = 0.8, v_1 = 0.0, x_2 = 5.8, \) and \( v_2 = 0.0 \). In this...
case, the particles flocculate because of the attracting force from the LJ potential, and move together diffusively (see Fig. 3).

In this application, we control fluctuations in the system by using the time-delayed feedback control method. Velocities fluctuate randomly around 0 because of the Gaussian noise and are stable at 0 without the noise. Thus, the dynamics of the velocities is the same as the Gaussian noise. This Gaussian noise-like behavior makes the dynamics of the positions analogous to a simple random walk, \( \frac{dx_i}{dt} \sim \xi \). Thus, control inputs are given only according to the dynamics of the positions, which are driven by noise originally from the velocities. The control scheme for the molecular dynamics model is written as:

\[
\frac{dx_i}{dt} = v_i - K(x_i - x_i(t - \tau)), \quad (19) \\
\frac{dv_i}{dt} = P_i(\{x_j\}_{j=1}^2) + D_i(\{v_j\}_{j=1}^2) + R\xi. \quad (20)
\]

Control signals are injected when the particles are flocculated. Figure 4 shows the dynamics of the center positions of two particles without control (green dotted line) and with control with \( \tau = 50 \) (red solid line). The movement of the flocculated particles is diffusive because of the noise. The figure shows that the diffusion of the flocculated particles is suppressed by our control method.

Figure 5 shows the dependence of the \( \mathcal{D} \) with \( \tau \). As is the case with the simple random walk, the \( \mathcal{D} \) decreases with a change in \( \tau \).

**Summary.**—In this work, we have applied time-delayed feedback control to a 1D random walk model. The interaction between noise and time delay is nontrivial. We have counterintuitively observed that the diffusion coefficient decreases with increasing delay time. Thus, we have analytically explained the decay of the diffusion coefficient by solving stochastic delay differential equations derived from the controlled system. In principle, it is possible to control the amount of diffusion from zero to a specific value.

We have applied the proposed method to a molecular dynamics model described by continuous time differential equations. We have numerically demonstrated the control of two flocculated particles with the LJ potential. Consequently, diffusion generated by noise was successfully controlled and decay of the diffusion coefficient was observed in this system.

The proposed method does not change physical properties of the system in comparison with the conventional technique of reducing diffusion, i.e., decreasing temperature, which can change the thermophysical properties. Moreover, the previous analysis of delayed random walks is based on the assumption of a stationarity [14, 15]. In contrast, our method is non-stationary and a conventional analysis by stochastic delay differential equations in [16] cannot be applied. In addition to such theoretical findings, we remark that the method can be applied to nano- microscopic resonators for reducing the thermal noise. In those experimental systems, time delay cannot be avoided or shortened less than some length [11, 12]. However, making longer time delay can be easy so that our method is effective in the systems.

Other types of time-delayed feedback control are also feasible. For example, we have proposed the adaptive time delay system as follows [18]:

\[
x_{n+1} = x_n + D\xi_n - \frac{1}{x_n - \tau}(x_n - x_{n-\tau}).
\]

This form of adaptive time-delayed feedback control shows almost similar performance to the original system (2) in terms of control of the diffusion coefficient. The difference between the original and the adaptive control is in determining \( K \) or determining the initial values for the system. We can apply the adaptive control scheme to real world systems depending on the availability of the control parameters. If we cannot access \( K \), adaptive feedback gain \( 1/x_{n-\tau} \) can be alternatively used.
[1] A. Einstein, *Annalen der Physik.* 17, 549, 1905.
[2] Z. Siwy and A. Fulinski, *Phys. Rev. Lett.* 89, 198103, 2002.
[3] D. Arcizet, B. Meier, E. Sackmann, J. O. Röder, and D. Heinrich, *Phys. Rev. Lett.* 101, 248103, 2008.
[4] B. I. Shraiman and E. D. Siggia, *Nature* 405, 639–646, 2000.
[5] J. J. Collins, C. J. De Luca, *Exp. Brain Res.* 95, 308–318, 1993.
[6] T. Fujita, et al., *Nano Lett.* 14 (3), 1172–1177, 2014.
[7] V. Tabard-Cossa, et al., *Nanotechnology* 18, 305505, 2007.
[8] H. J. Butt and M. Jaschke, *Nanotechnology* 6, 1, 1995.
[9] K. Murali et al., *Phys. Rev. Lett.* 102 (10), 10410, 2009.
[10] M. Igarashi, C.-H. Huang, T. Morie, and S. Samukawa., *App. Phys. Express,* vol. 3, pages 085202-1–3, 2010.
[11] M. Bonaldi et al., *Phys. Rev. Lett* 103, 010601, 2009.
[12] T. Munakata and M. L. Rosinberg, *Phys. Rev. Lett* 112, 180601, 2014.
[13] K. Pyragas, *Phys. lett. A,* vol. 170, pages 421–428, 1992.
[14] T. Ohira, J. G. Milton, *Phys. Rev. E* 52, 3277, 1995.
[15] T. Ohira, T. Yamane, *Phys. Rev. E* 61, 1247, 2000.
[16] U. Küchler, B. Mensch, *Stoch. Stoch. Rep.* 40, 23, 1992.
[17] Kobayashi et al., in preparation.
[18] H. Ando, M. U. Kobayashi, *Proc. of the 4th IFAC Conference on Analysis and Control of Chaotic Systems,* 262, 2015.