VIRTUAL CLASPER ON LONG VIRTUAL KNOTS

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Abstract. A $C_n$-move is a family of local moves on knots and links, which gives a topological characterization of finite type invariants of knots. We extend the $C_n$-move to (long) virtual knots by using the lower central series of the pure virtual braid, and call it an $L_n$-move. We then prove that for long virtual knots an $L_n$-equivalence generated by $L_n$-moves is equal to $n$-equivalence, which is an equivalence relation on (long) virtual knots defined by Goussarov-Polyak-Viro. Moreover we directly prove that two long virtual knots are not distinguished by any finite type invariants of degree $n−1$ if they are $L_n$-equivalent, for any positive integer $n$.

1. Introduction

The theory of finite type invariants of knots and links was introduced by Vassiliev [20] and Goussarov [4, 5] and developed by Birman-Lin [2]. Goussarov [6, 7] and Habiro [9, 10] independently introduced theories of surgery along embedded graphs in 3-manifolds, called $Y$-graphs or variation axes by Goussarov, and claspers by Habiro. An $n$-variation equivalence (called $n$-equivalence in [7]) or $C_n$-equivalence for links is generated by $n$-variations [7] or $C_n$-moves [10], respectively. Goussarov proved in [7] that for string links and knots in $S^3$, the $n$-variation (or $C_n$-) equivalence coincides with the Goussarov-Ohyama $n$-equivalence [4, 15]. Stanford proved in [18] that two links are not distinguished by any finite type invariant of degree $n$ if one is obtained from the other by inserting an element of the $(n + 1)$-th lower central series subgroup of the pure braid group. Goussarov [7] and Habiro [9, 10] independently proved that two knots are not distinguished by any finite type invariant of degree $n$ if and only if they are related by a finite sequence of $C_n$-moves and ambient isotopies. Moreover Stanford [19] translated Habiro’s result for $C_n$-moves into the pure braid setting.

On the other hand, a (long) virtual knot is defined by a (long) knot diagram with virtual crossings module Reidemeiser moves, introduced by Kauffman [11]. Goussarov-Polyak-Viro [8] showed that the (long) virtual knot can be redefined as Gauss diagram and also gave the theory of finite type invariants on Gauss diagrams. They also defined an $n$-equivalence on (long) virtual knots and notioned that the value of a finite type invariant of degree less than or equal to $n$ depended only the $n$-equivalence class.

In this paper, we extend a $C_n$-move to (long) virtual knots, called an $L_n$-move, by using Stanford’s method. The $L_n$-moves generate the $L_n$-equivalence on (long) virtual knots. We prove that $L_n$-equivalence coincides with $n$-equivalence on long virtual knots. Moreover, we directly prove that, for any non-negative integer $n$, if two long virtual knots are $L_n$-equivalent, then they are not distinguished by any finite type invariants of degree $n−1$. Their extensions and results are also establish on virtual string links.

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2. Gauss diagram

A Gauss diagram on the interval is an oriented interval with several oriented chords having disjoint end points and equipped with sign as in Figure 1. Here, we call the chord an arrow.

![Figure 1. A Gauss diagram](image1)

Reidemeister moves among Gauss diagrams are the following three moves in Figure 2. First Reidemeister move (RI) is in the top row. Second Reidemeister move (RII) is in the second row. Third Reidemeister move (RIII) is in the remain two rows.

![Figure 2. The Reidemeister moves](image2)

**Definition 2.1.** Two Gauss diagrams $D$ and $D'$ are said to be equivalent if $D$ and $D'$ are related by Reidemeister moves. By $D \sim D'$ we mean that $D$ and $D'$ are equivalent. We define a long virtual knot to be the equivalence class of a Gauss diagram $D$, which is denoted by $[D]$. Similarly, the equivalence class of Gauss diagram on circle ($k$ intervals) is virtual knot ($k$-component virtual string, respectively).

3. Finite type invariant of virtual knot

Goussarov Polyak and Viro defined a finite type invariant for (long) virtual knots in [8]. Similar way to classical knots, we can define Vassiliev-Goussarov filtration on $\mathbb{Z}$-module generated by the set of (long) virtual knots.
Definition 3.1. Let $\mathcal{LVK}$ be the set of long virtual knots. For each $n \geq 0$, let $\mathcal{SLVK}^n$ denote the set of equivalence classes of Gauss diagrams with $n$ dashed arrows equipped with sign with fixing dashed arrows. We construct a map $\varphi : \mathcal{SLVK}^n \to \mathcal{LVK}$ as follows. Let $D$ be a Gauss diagram with $n$ dashed arrows. Let $a_1, \ldots, a_n$ be the dashed arrows of $D$. For $\epsilon_1, \ldots, \epsilon_n$ in $\{\pm 1\}$, let $D_{\epsilon_1, \ldots, \epsilon_n}$ denote the Gauss diagram obtained from $D$ by replacing each dashed arrow $a_i$ with an arrow if $\epsilon_i = 1$ and removing each dashed arrow $a_i$ if $\epsilon_i = -1$. We then define

$$\varphi([D]) = \sum_{\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_n [D_{\epsilon_1, \ldots, \epsilon_n}].$$

Let $f$ be an invariant of $\mathcal{LVK}$ with values in an abelian group $A$. We extend it to $\mathcal{SLVK}$ by linearly. Then $f$ is said to be a finite type invariant of degree $n$ if $f \circ \varphi$ vanishes for any long virtual knot with more than $n$ dashed arrows.

Definition 3.2. Denote by $J_n$ the subgroup of $\mathcal{SLVK}$ generated by the set consisting of the element $\varphi([D])$, where $[D]$ is in $\mathcal{SLVK}^n$. It is easy to see that the $J_n$’s form a descending filtration of two-sided ideals of the monoid ring $\mathcal{SLVK}$ under the composition:

$$\mathcal{LVK} = J_0 \supset J_1 \supset J_2 \supset \cdots,$$

which we call the Vassiliev-Goussarov filtration on $\mathcal{LVK}$. Here for Gauss diagrams (or virtual knots) $D$ and $D'$ $(K$ and $K')$, we denote by $D \cdot D'$ $(K \cdot K')$ their composition.

Later, we will redefine $J_n$ by using claspers.

Remark 3.3. Let $A$ be an abelian group and $n$ a positive integer. The following two conditions are equivalent. A map is an $A$-valued finite type invariant of degree $n$ on $\mathcal{LVK}$ and the map is a homomorphism of $\mathcal{SLVK}$ into $A$ which vanishes on $J_{n+1}$.

Definition 3.4. For $n \geq 0$, two long virtual knots $K$ and $K'$ are said to be $V_n$-equivalent if $K$ and $K'$ are not distinguished by any finite type invariants of degree $n$ with values in any abelian group, equivalently, $K \sim K' \in J_{n+1}$.

4. Definition of $L_n$-equivalence

By using the pure virtual braid group, we introduce a new equivalence relation on Gauss diagrams, called $L_n$-equivalence. Because the pure braid group is a subgroup of the pure virtual braid group (see [13]), this is an extension of $C_n$-equivalence. We then give properties of the set of $L_n$-equivalence classes.

Definition 4.1 ([12]). A pure virtual braid group $PV_k$ on $k$ strands is a group represented by the following group representation.

$$PV_k = \langle \mu_{ij} \mid 1 \leq i, j \leq k, i \neq j \rangle \quad \begin{cases} \mu_{ij} \mu_{ij} \mu_{ij} = \mu_{ij} \mu_{ij} \mu_{ij} & \text{(for all distinct } i, j, l) \\ \mu_{ij} \mu_{lm} = \mu_{lm} \mu_{ij} & \{i, j\} \cap \{l, m\} = \emptyset \end{cases}$$

Here, the element of the pure braid group is represented by a diagram as in Figure 3, where $\mu'_{ij}$ is correspondence with a horizontal arrow equipped with sign $\epsilon$ from the $i$-th strand to the $j$-th strand, and we determine that the orientation of the strand is from top to bottom. For example, the diagram in Figure 3 is correspondence with $\mu_{12} \mu_{31}^{-1} \mu_{23} \mu_{12}^{-1} \in PV_3$.

Let $h \in PV_k$ and $h' \in PV_{k'}$. We denote the composition and tensor product of two elements of the pure virtual braid group as $h \cdot h' \in PV_{k+k'} \in PV_{k+k'}$ for any $k$, $k'$, respectively. By $\Gamma_n(G)$ we mean the $n$-th lower central subgroup of the group $G$, that is, $\Gamma_1(G) = G$ and $\Gamma_n(G) = \Gamma_{n-1}(G), G$, which is the commutator of $\Gamma_{n-1}(G)$ and $G$, that is $< a, b > a \in \Gamma_{n-1}(G), b \in G >$ where $[a, b] = aba^{-1}b^{-1}$. 

Figure 3, where
**Definition 4.2.** Two Gauss diagrams $D$ and $D'$ are related by an $L_n$-move if there are a positive integer $k$, an element $h$ in the $n$-th lower central subgroup $\Gamma_n(PV_k)$ of the pure virtual braid group $PV_k$ on $k$ strands and not in $\Gamma_{n+1}(PV_k)$, and an embedding $e$ of $k$ strands such that $D^{(h,e)} = D'$, where $D^{(h,e)}$ is obtained from $D$ by attaching $h$ by an embedding $e$ of $k$ strands of $h$ in the interval of $D$ except for the end points of all arrows of $D$ as in Figure 4. By $D \xrightarrow{L_n} D'$ we mean that $D'$ is obtained from $D$ by $L_n$-move. In particular, we write $D^{(h,e)} \xrightarrow{L_n} D'$ if $D' = D^{(h,e)}$.

**Figure 3.** An element of pure virtual braid group

**Figure 4.** A clasper for a Gauss diagram

We call a pair $(h, e)$ for $D$ a clasper for $D$. We define that a clasper $(h, e)$ is of degree $n$ if $h \in \Gamma_n(PV_k)$ and $h \notin \Gamma_{n+1}(PV_k)$, where $k$ is a positive integer, and denote the degree of the clasper $(h, e)$ by $\deg((h, e))$. Hereinafter, we omit the number $k$ of strands if it is not important. In particular, we call a pair $(h, e)$ a tree clasper for $D$ if $h \in \Gamma_n(PV_k)$ is an $n$-th commutator $[a_1, [a_2, \cdots [a_n, a_{n+1}] \cdots]$ where $a_i \in PV_k$ and a forest clasper otherwise. Two claspers for $D$ are disjoint if the embeddings of all strands of claspers are disjoint in the interval of $D$. For disjoint claspers $(h_1, e_1)$ and $(h_2, e_2)$ for $D$, $D^{((h_1, e_1),(h_2, e_2))}$ means $(D^{(h_1, e_1)})^{(h_2, e_2)}$ or equivalently $(D^{(h_2, e_2)})^{(h_1, e_1)}$.

**Definition 4.3.** An $L_n$-equivalence is an equivalence relation on Gauss diagrams generated by the $L_n$-moves and Reidemeister moves. By $D \xRightarrow{L_n} D'$ we mean that $D$ and $D'$ are $L_n$-equivalent.

**Proposition 4.4.** The $L_n$-equivalence is an equivalence relation on Gauss diagrams.

**Proof.** First of all, we show the reflexive relation. For any $k \geq 1$ and $n \geq 1$, the identity element $1 \in \Gamma_n(PV_k)$ and $D^{(1,e)} = D$ for any embedding $e$. Therefore $D \xRightarrow{L_n} D$. Secondly, we show the symmetric relation. Let $D' = D^{(h,e)}$ where $h \in \Gamma_n(PV_k)$. Then there is an embedding $e'$ of $h^{-1} \in \Gamma_n(PV_k)$ such that $D^{(h^{-1},e')} = D^{(h,h^{-1},e)}$. 

Since the Gauss diagram $D^{(h, h^{-1}, e)}$ is $D$ up to a sequence of second Reidemeister moves, we have that $D' \overset{L_2}{\sim} D$. Finally, the case of transitive relation is obvious. □

**Proposition 4.5.** If $1 \leq n \leq n'$, then an $L_n$-move is achieved by an $L_{n'}$-move. Therefore $L_{n'}$-equivalence implies $L_n$-equivalence.

**Proof.** By the property of the lower central series, $\Gamma_n(PV_k) < \Gamma_{n-1}(PV_k)$. □

**Proposition 4.6.** Two Gauss diagrams $D$ and $D'$ are $L_n$-equivalent if and only if there exists a clasper $(h, e)$ of degree $n$ such that $D^{(h, e)}$ equals to $D'$ up to a sequence of Reidemeister moves.

**Proof.** A necessary condition is obvious. To prove a sufficient condition, we will show the following three statements (1), (2) and (3). (1) If $D_2$ is obtained from $D_1$ by a first (second or third, respectively) Reidemeister move and then an $L_n$-move $(h_1, e_1)$ (or $(h_2, e_2)$ or $(h_3, e_3)$, respectively) $(n' \geq n)$, then there is an $L_{n'}$-move $(h'_1, e'_1)$ or $(h'_2, e'_2)$ or $(h'_3, e'_3)$, respectively) and a sequence of Reidemeister moves such that $D_2$ is obtained from $D_1$ by the $L_{n'}$-move $(h'_1, e'_1)$ or $(h'_2, e'_2)$ or $(h'_3, e'_3)$, respectively) and then the sequence of Reidemeister moves. (2) If $D_2$ is obtained from $D_1$ by an $L_n$-move $(h, e)$ and then another $L_n$-move $(h', e')$, then there is an $L_{n'}$-move $(h'', e'')$ $(n' \geq n)$ and two sequences of second Reidemeister moves such that $D_2$ is obtained from $D_1$ by one sequence of the second Reidemeister moves and then the $L_{n'}$-move $(h'', e'')$ and then the other sequence of the second Reidemeister moves. (3) For any clasper $(h, e)$ of degree more than or equal to $n$ for $D$, there exists a clasper $(h', e')$ of degree $n$ for $D$ such that $D^{(h, e)} = D^{(h', e')}$, because any $h \in \Gamma_{n+1}$ is represented by the product of elements in $\Gamma_n$. By (1), (2) and (3), if $D$ and $D'$ are $L_n$-equivalent, there is an $L_n$-move and a sequence of Reidemeister moves such that $D'$ is obtained from $D$ by the $L_n$-move and then the sequence of Reidemeister moves.

We show (1). We consider the case of the first Reidemeister move $RI$. In Figure 5 these Gauss diagrams are identical except in a local place of $RI$ represented by this figure. By gray line we mean a clasper. Given a clasper $(h_1, e_1)$, we can move the ends of chords of clasper out the arrow derived from $RI$ by a sequence of second Reidemeister moves. We denote the obtained clasper by $(h'_1, e'_1)$ (See Figure 5). Moreover similar considerations apply to the other first Reidemeister move.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Change of an $L_n$-move and a first Reidemeister move}
\end{figure}
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Similar way to $RI$, in the case of $RII$ and $RIII$, we give claspers $(h'_2, e'_2)$ and $(h'_3, e'_3)$ as in Figure 6 and 7 which are one of $RII$ and $RIII$. Here, in Figure 5 for simplicity we draw only one strand is embedding in each interval between endpoints of arrows derived from $RIII$.

We show (2). Given a Gauss diagram $D_1$ and a clasper $(h, e)$ for $D_1$, we can transform $D_1$ to $D_1^{(h, h^{-1}, e)}$ by a sequence of the second Reidemeister moves. Let
$h \in \Gamma_n(PV_k)$ and $h' \in \Gamma_n(PV_{k'})$. There is an embedding $\bar{e}$ of $h \otimes h'$ in $\Gamma_n(PV_{k+k'})$ such that $(D^h(h^{-1}, e))(h \otimes h', \bar{e}) = (D_1^h(h^{-1}, h', \bar{e}), \bar{e}_{h'})$. Moreover, $(D_1^h(h^{-1}, h', \bar{e})) \sim (D_2^h(h', \bar{e}_{h'})) = D_2$ up to a sequence of the second Reidemeister moves. We set $(h'', e'') = (h \otimes h', \bar{e})$. $\Box$

Remark 4.7. It is obvious that Proposition 4.6 is equivalent to the following statement. There exists the union $H$ of disjoint claspers of degree $n$ such that $D^H$ equals to $D'_{\text{up}}$ up to a sequence of the Reidemeister moves.

Remark 4.8. In [14], Meilhan and Yasuhara also extended the concept of the clasper to welded knots, which is a quotient of virtual knot.

Lemma 4.9. Let $n \geq 1$. Let $D$ be a Gauss diagram and $(h, e)$ a clasper of degree $n$ for $D$. Then for any Gauss diagram $D'$ which is equivalent to $D$ there is a clasper $(h', e')$ of degree $n$ for $D'$ such that $D^h$ is equivalent to $D^{h'}$.

Proof. Since $D' \sim D$ and $D \overset{L2}{\sim} D^{(h, e)}$, we have that $D' \overset{L2}{\sim} D^{(h, e)}$. It is from Proposition 4.6 that there is a clasper $(h', e')$ of degree $n$ for $D'$ such that $D^{(h, e)} \sim D^{(h', e')}$.

Remark 4.10. We can show Lemma 4.9 directly. If $D'$ be obtained from $D$ by Reidemeister move RI, RI or RIII, then given a clasper $(h, e)$ for $D$ we can construct a clasper $(h', e')$ such that $D^h \sim D^{(h', e')}$ by similar method of Figure 5, 6 and 7 in the proof of Proposition 4.6.
Definition 4.11. A Gauss diagram $D$ is $L_n$-trivial if $D$ is $L_n$-equivalent to the trivial Gauss diagram $D_0$.

The next proposition is well-known fact of group theory.

Proposition 4.12. Let $G$ be a group. Let $x$ and $y$ be elements in the $n$-th and $n'$-th lower central subgroup of $G$, respectively. Then the commutator $[x, y]$ of $x$ and $y$ is in $(n + n')$-th lower central subgroup of $G$.

Lemma 4.13. Let $D$ be a Gauss diagram. Let $n_1, n_2 \geq 1$. Let $(h_1, e_1)$ be a clasper of degree $n_1$ and $(h_2, e_2)$ a clasper of degree $n_2$ for $D$, where they are disjoint. Let $s_i$ be the $i$-th strand of $h_1$ and $t_j$ the $j$-th one of $h_2$. Suppose that there is no end point of arrows and no embedding of another strands of claspsers on the interval between embeddings $e_1(s_i)$ and $e_2(t_j)$. Then, these embeddings may replace each other up to $L_{n_1 + n_2}$-equivalence as in Figure 8. Let $(h_1, e_1')$ and $(h_2, e_2')$ be claspsers of degree $n_1$ and $n_2$ obtained from $(h_1, e_1)$ and $(h_2, e_2)$ by replacing $e_1(s_i)$ and $e_2(t_j)$ as in Figure 8. Then, there exists a clasper $(h, e)$ of degree $n_1 + n_2$ such that $(D((h_1, e_1'), (h_2, e_2'))) (h, e)$ is equivalent to $D((h_1, e_1), (h_2, e_2))$.

We call the transformation between two claspsers a sliding.

Proof. For $h_1 \in \Gamma_{n_1}(PV_{k_1})$ and $h_2 \in \Gamma_{n_2}(PV_{k_2})$, we construct $h \in \Gamma_{n_1 + n_2}(PV_{k_1 + k_2 - 1})$ and its embedding $e$. Let $h_1$ be an element of pure virtual braid $\Gamma_{n_1}(PV_{k_1 + k_2 - 1})$ from $h_1$ by adding $j - 1$ strands before 1st strand of $h_1$ and $k_2 - j$ strands after 1st strand of $h_1$. Let $h_2$ be an element of $\Gamma_{n_2}(PV_{k_1 + k_2 - 1})$ from $h_2$ by adding $i - 1$ strands between $(j - 1)$-th and $j$-th strand of $h_2$ and, $k_1 - i$ strands between $j$-th and $(j + 1)$-th strand of $h_2$. Then the $i$-th strand of $h_1$ and $j$-th strand of $h_2$ are the same order in $h_1$ and $h_2$.

If both of the orientations of $s_i$ and $t_j$ are compatible or not with the orientation of the interval of $D$, the product $\tilde{h}_1 \cdot \tilde{h}_2$ has a natural embedding $e$ induced by $e_1$ and $e_2$. Let $h = [\tilde{h}_2, \tilde{h}_1]$. Then $h \in \Gamma_{n_1 + n_2}(PV_{k_1 + k_2 - 1})$ by Proposition 4.12. Since $h \cdot \tilde{h}_1 \cdot \tilde{h}_2 = \tilde{h}_2 \cdot \tilde{h}_1$, we may replace $s_i$ and $t_j$ each other and leave other embeddings up to $L_{n_1 + n_2}$-equivalence. If only one of the orientations of $s_i$ and $t_j$ is compatible with that of the interval of $D$, to adjust $s_i$ and $t_j$ the orientation we set $h = [\tilde{h}_2, \tilde{h}_1]$, where $\tilde{h}_2$ is the mirror image of $h_2$ for a horizontal line and its embedding $e$ is induced by $e_1$ and $e_2$. \qed

Proposition 4.14. Let $n, n' \geq 1$. Let $D$ be an $L_n$-trivial Gauss diagram and $D'$ be an $L_{n'}$-trivial one. Then the Gauss diagram $D \cdot D'$ is $L_{n + n'}$-equivalent to $D' \cdot D$.

Proof. By assumption, there are two claspsers $(h, e)$ and $(h', e')$ with $h \in \Gamma_n(PV)$ and $h' \in \Gamma_{n'}(PV)$ such that $D_0^{(h, e)} \sim D$ and $D_0^{(h', e')} \sim D'$. Then by Lemma 4.13 we have $D \cdot D' \sim D_0^{(h, e)} \cdot D_0^{(h', e')} \sim L_{n + n'} D_0^{(h', e')} \cdot D_0^{(h, e)} \sim D' \cdot D$. \qed
Lemma 4.17. For any $L_n$-trivial Gauss diagram $D$, there is an $L_n$-trivial Gauss diagram $D'$ such that both $D \cdot D'$ and $D' \cdot D$ are $L_{2n}$-trivial.

Proof. By assumption, there is a clasper $(h,e) \in \Gamma_n(PV_k)$ such that $D_0^{(h,e)} \sim D$. We define $D' = D_0^{(h^{-1},e)}$. Then by Lemma 4.13 we have $D \cdot D' \sim D_0^{(h,e)}$. $D_0^{(h^{-1},e)} \sim D_0$.

Notation 4.16. The set $LVK_n$ of equivalence classes of Gauss diagrams has a monoid structure under the composition. For $n \geq 1$, let $LVK_n$ denote the submonoid of $LVK_n$ consisting of the equivalence classes of Gauss diagrams which are $L_n$-trivial. There is a descending filtration of monoids $LVK = LVK_1 \supset LVK_2 \supset LVK_3 \supset \cdots$.

For $l \geq n$, $LVK_{n}/L_l$ denotes the quotient of $LVK_n$ by $L_l$-equivalence. It is easy to see that the monoid structure on $LVK_n$ induces that of $LVK_n/L_l$. There is a filtration on $LVK_n/L_l$ of finite length $LVK/L_l = LVK_1/L_l \supset LVK_2/L_l \supset LVK_3/L_l \supset \cdots \supset LVK_l/L_l = \{1\}$.

Lemma 4.17. For $n \geq 1$, the monoid $LVK_n/L_l$ ($1 \leq l \leq 2n$) is an abelian group.

Proof. By Proposition 4.15, for any $K \in LVK_n$ there exists $K' \in LVK_n$ such that both $K \cdot K'$ and $K \cdot K'$ are trivial up to $L_{2n}$-equivalence, and the monoid $LVK_n/L_{2n}$ is a group. By Proposition 4.13 for any $K_1, K_2 \in LVK_n$, $K_1 \cdot K_2 = K_2 \cdot K_1$ up to $L_{2n}$-equivalence, and the group $LVK_n/L_{2n}$ is abelian.

Proposition 4.18. Let $1 \leq n \leq l$. We then have as follows.

1. The monoid $LVK_n/L_l$ is a group.
2. $[LVK_n/L_l, LVK_{n'}/L_l] \subset LVK_{n+n'}/L_l$ for $n, n' \geq 1$ with $n + n' \leq l$.
3. The group $LVK_n/L_l$ is nilpotent.

Proof. (1) We fix $l$ and prove it by induction on $n$. If $n = l$, it is obvious. Assume that $LVK_{n+1}/L_l$ is a group for some $n$ with $1 \leq n \leq l$. We then have a shot exact sequence of monoids:

$$1 \to LVK_{n+1}/L_l \to LVK_n/L_l \to LVK_{n+1}/L_l \to 1$$

Here, $LVK_{n+1}/L_l$ and $LVK_n/L_{n+1}$ are groups by the assumption of induction and Lemma 4.17. Therefore $LVK_n/L_l$ is also a group.

(2) It is from Proposition 4.13 $a \in LVK_n/L_l$ and $b \in LVK_{n'}/L_l$ commute up to $L_{n+n'}$-equivalence. Here $[a, b] \sim_{L_{n+n'}} 1$. Therefore $[a, b] \subset LVK_{n+n'}/L_l$.

(3) From (2), it is easy to check.

5. $L_n$-equivalence and $n$-equivalence

In this section, we prove that $L_n$-equivalence coincides with $n$-equivalence defined by Goussarov-Polyak-Viro [8].

Definition 5.1. [8] Let $n \geq 0$. A Gauss diagram $D$ on $k$ strands is said to be $n$-trivial if the Gauss diagram satisfies the following condition. There exist $n + 1$ non-empty disjoint subsets $A_1, A_2, \cdots, A_{n+1}$ of the set of arrows of $D$ such that for any non-empty subfamily of the set $\{A_1, A_2, \cdots, A_{n+1}\}$ the Gauss diagram obtained from $D$ by removing all arrows which belongs to the subfamily is trivial up to a sequence of second Reidemeister moves.

Two Gauss diagrams $D$ and $D'$ are related by $n$-variation if $D'$ is obtained from $D$ by attaching an $(n-1)$-trivial Gauss diagram on several strands to segments of $D$ without endpoints of any arrow. Two Gauss diagrams are said to be $n$-equivalent if they are related by $(n+1)$-variations and Reidemeister moves.
Theorem 5.2. For any \( n \geq 1 \), \( L_n \)-equivalence and \((n-1)\)-equivalence on long virtual knots are equal.

Proof. It is obvious that if two Gauss diagrams are \( L_n \)-equivalent, then they are \((n-1)\)-equivalent. Therefore it suffices to prove that if Gauss diagrams \( D \) and \( D' \) are related by an \( n \)-variation then they are \( L_n \)-equivalent.

Let \( D_t \) be an \((n-1)\)-trivial Gauss diagram such that \( D' \) is obtained from \( D \) by attaching \( D_t \). Let \( A_1, A_2, \cdots, A_n \) be disjoint sets of arrows of \( D \) satisfying the condition in Definition 5.1. By the property of \((n-1)\)-triviality, \( D \) coincides with the Gauss diagram obtained from \( D' \) by removing all arrows in \( A_1 \cup \cdots \cup A_n \) up to second Reidemeister move. Therefore, by the method of (1) in proof of Proposition 4.6, it is sufficient to consider the case that all arrows of \( D_t \) belong to \( A_1 \cup \cdots \cup A_n \). Let \( H \) be the set of tree claspers of degree 1 corresponding to the arrows in \( A_1 \cup \cdots \cup A_n \).

We define a weight for a clasper \( h \), which is a subset of \( \mathbb{N} \), and denote it by \( w(h) \). We consider \( H \) as a set of tree claspers each clasper of which assigns \( i \) as weight if the clasper corresponds with an arrow of \( A_i \). Let \( I \) be a finite subset of \( \mathbb{N} \). Then \( H(I) \) denote the subset of \( H \) each clasper of which has a subset of \( I \) as weight, and \( H_I \) denote the subset of \( H(I) \) each clasper of which has \( I \) as weight. Let \( N = \{1, 2, \cdots, n\} \). We then can regard \( D' \) as \( D^H(N) \). Moreover, by the property of \((n-1)\)-triviality if \( I \) is a proper subset of \( N \), then \( D^H(I) = D \) up to a sequence of second Reidemeister moves. We show the following claim, which proves the theorem.

Claim 5.3. Suppose that \( D \) and \( H \) are as above. Then there exists a set \( H' \) of tree claspers of degree \( \geq n \) with weight \( N \) such that \( D^{H'} \) is equivalent to \( D^H \).

Let us first prove the case that \( D \) is equivalent to the trivial Gauss diagram \( D_0 \). Then by Lemma 4.9, it is sufficient to show that the case that \( D \) is trivial. To prove this claim, we prove the following statement depending on a positive integer \( s \).

(A) There exists a set \( G = \{(h_1, e_1), \ldots, (h_k, e_k)\} \) of tree claspers \( (h_i, e_i), \ldots, (h_k, e_k) \) \((k \geq 0)\) for \( D_0 \) such that for each \( i = 1, \ldots, k \) \( s \leq |w((h_i, e_i))| \leq \deg((h_i, e_i)) \) where \( \cdot \) means the number of a set, and \( D^{G(I)}_0 \sim D^{H(I)}_0 \) for every subset \( I \) of \( N \).

We prove it by induction on \( s \) for \( s = 1, 2, \cdots, n \). For \( s = 1 \), we can set \( G = H \). Under the assumption of the claim, assuming the statement (A) to hold for \( s < n \), we will prove it for \( s + 1 \). Let \( G \) be a set of tree claspers for \( D_0 \) satisfying (A) for \( s \). We take a subset \( I \) of \( N \) such that \( |I| = s \) and \( G_I \) is not empty. Then we shift all tree clasper in \( G_I \) to the ahead with fixing claspers in \( G \setminus G_I \) by sliding of claspers (Lemma 4.13) until all end points of all clasper in \( G_I \) are completely to the ahead of those in \( G \setminus G_I \). We denote the obtained set of tree claspers for \( D_0 \) by \( G' \). Here we define the weight of new tree claspers obtained by sliding in Lemma 4.13 as follows. If two claspers have the weight \( w_1 \) and \( w_2 \), then the new tree clasper \((h, e)\) has the weight \( w_1 \cup w_2 \). We remark that \( D^{G(I)}_0 \sim D^{G'(I)}_0 \) for every \( I \subset N \) and \( s < |w((h, e))| \leq \deg((h, e)) \) for any new tree \((h, e)\) \( G' \).

Let \( G'' \) be a set \( G \setminus G_I \) of tree claspers for the Gauss diagram \( D^{G_I}_0 \). We consider a subset \( J \) of \( N \). If \( J \supseteq I \), then it is clear that \( D^{G(J)}_0 = (D^{G_I}_0)^{(G(J))} \). If not, then the new tree claspers do not contain in \( G'' \). Hence \( (D^{G_I}_0)^{(G(J))} = D^{G(J)}_0 \cdot D^{G_I}_0 \). By the assumption of claim, \( D^{G_I}_0 \) is equivalent to \( D_0 \). Therefore the Gauss diagrams \( D^{G(J)}_0 \) and \( (D^{G_I}_0)^{(G(J))} \) are equivalent for every \( J \subset N \).

Repeating this procedure for \( I \) such that \( |I| = s \), we obtain a set of tree clasers \((h, e)\)'s with \( d((h, e)) \geq |w((h, e))| \geq s + 1 \) for a Gauss diagram which is equivalent to \( D_0 \). By Lemma 4.9, we obtain a set of tree clasers preserving above condition for \( D_0 \), which is the required set satisfying (A) for \( s + 1 \). This proves the claim for the case that \( D \sim D_0 \).
Next we prove the case that $D$ is not equivalent to the trivial one. Since the set of $L_n$-equivalence classes has a group structure, there is an inverse $D^{-1}$ of $D$ up to $L_n$-equivalence. Then $D$ is $L_n$-equivalent to $D \cdot D^{-1} \cdot D$. It is $L_n$-equivalent to $D' \cdot D^{-1} \cdot D$, since $D \cdot D^{-1}$ and $D' \cdot D^{-1}$ are $L_n$-equivalent. Because $D' \cdot D^{-1}$ is $(D \cdot D^{-1})^{H'}$, by Proposition 6.3 there exists a clasper $h$ of degree $\geq n$ such that $(D \cdot D^{-1})^h \sim D_0$. Therefore it follows from the case $D \sim D_0$ that there exists a set $H'$ of claspers of degree $\geq n$ such that $((D \cdot D^{-1})^h)^{H'} \sim ((D \cdot D^{-1})^h)^{H'}$. Hence $D$ is $L_n$-equivalent to $D'$.

Remark 5.4. Even though we change “second Reidemeister moves” into “Reidemeister moves” in the definition of the $n$-trivial in Definition 5.1, we can show Theorem 5.2 similarly. Therefore it is concluded that these two $n$-equivalences coincide.

6. $L_n$-EQUIVALENCE AND $V_n$-EQUIVALENCE

Goussarov-Polyak-Viro \cite{GPV} mentioned that the value of a finite type invariant of degree less than or equal to $n$ depends only on the $n$-equivalence classes. Therefore it follows from Theorem 5.2 that $L_{n+1}$-equivalence implies $V_n$-equivalently, indirectly. In this section, we give this relation directly, by redefining the two-sided ideal $J_n$ of the monoid ring $\mathbb{Z}\mathcal{LVK}$ by using claspers.

Definition 6.1. Let $l \geq 1$. A scheme of size $l$, $H = \{(h_1, e_1), (h_2, e_2), \ldots , (h_l, e_l)\}$, for a Gauss diagram $D$ is the set of disjoint claspers for $D$. Denote an element $[D, H]$ of $\mathbb{Z}\mathcal{LVK}$ by

$$[D, H] = \sum_{G \subseteq H} (-1)^{|G|} [DG],$$

where $G$ runs over all $2^l$ subsets of $H$. The degree of a scheme $H = \{(h_1, e_1), (h_2, e_2), \ldots , (h_l, e_l)\}$ is the sum of the degree of its elements, denoted by $\deg(H)$.

Lemma 6.2. Let $D$ be a Gauss diagram and $H$ a scheme of size $l$ for $D$ of degree $n$. Then for any Gauss diagram $D'$ which is equivalent to $D$ there is a scheme $H'$ of size $l$ for $D'$ of degree $n$ such that $[D, H] = [D', H']$ in $\mathbb{Z}\mathcal{LVK}$.

Proof. If $D'$ is obtained from $D$ by Reidemeister move RI, RII or RIII, we can construct a scheme $H'$ such that $D^h \sim D'^{H'}$ by similar method of Figure 5, 6 and 7 in the proof of Proposition 4.4. From the construction of $H'$, for each $G \subseteq H'$, there is the corresponding $G' \subset H'$ such that $D^{G'} \sim D'^{G'}$. Therefore $[D, H] = [D', H']$ in $\mathbb{Z}\mathcal{LVK}$.

Lemma 6.3. (1) $[D, \emptyset] = [D]$.
(2) $[D, \{(h_1, e_1), (h_2, e_2), \ldots , (h_l, e_l)\}]$ = $[D,(h_1, e_1), \{(h_2, e_2), \ldots , (h_l, e_l)\}] = [D, \{(h_1, e_1), \ldots , (h_l, e_l)\}]$.
(3) $[D, \{(h_{i_1}, e_{i_1}), \ldots , (h_{i_m}, e_{i_m}), (h_{i_{m+1}}, e_{i_{m+1}}), \ldots , (h_{i_l}, e_{i_l})\}]$ = $\sum_{i=1}^m [D^{(h_{i_1}, e_{i_1}), \ldots , (h_{i_{m+1}}, e_{i_{m+1}}), \ldots , (h_{i_l}, e_{i_l})}]$.

Proof. It is easy to check.

Definition 6.4. Let $n, l$ be integers with $1 \leq l \leq n$. Let $J_{n,l}$ denote the two-sided ideal of $\mathbb{Z}\mathcal{LVK}$ generated by the elements $[D, H]$ under the composition, where $D$ is any Gauss diagram and $H$ is any scheme of size $l$ for $D$ of degree $n$.

Remark 6.5. By Lemma 6.3(3), $J_{n,l}$ can be generated by schemes $[D, H]$ where $H$ is the set of tree claspers for $D$.

Remark 6.6. The natural homomorphism $i : \mathbb{Z}\mathcal{LVK} \rightarrow \mathbb{Z}(\mathcal{LVK}/L_n)$ induces the ring isomorphism $\mathbb{Z}\mathcal{LVK}/J_{n,1} \approx \mathbb{Z}(\mathcal{LVK}/L_n)$. 
Lemma 6.7. Let $D$ be a Gauss diagram. We then have the following properties.

(1) For any integer $n \geq 1$ $J_{n,n} = J_n$
(2) For any integers $n$, $l$ with $1 \leq l \leq n$, $J_{n,l} \subset J_{n,l+1}$
(3) For any integers $n$, $n'$ with $1 \leq l \leq n$, $J_{n',l} \subset J_{n,l}$

Proof. (1) We show that

$$\varphi([\overbrace{\cdots}^{\nu_1} \overbrace{\cdots}^{\nu_{n-2}} \overbrace{\cdots}^{\nu_n}]) = [\overbrace{\cdots}^{\nu_1} \overbrace{\cdots}^{\nu_{n-2}} \overbrace{\cdots}^{\nu_n}]$$

where the left-hand side of the equation means the image of $n$ dashed arrows by $\varphi$ and the right-hand side of the equation means a Gauss diagram with a scheme of size $n$ of degree $n$, which consists of $n$ claspers of degree 1. If $n = 1$, $\varphi([\overbrace{\cdots}^{\nu_1} \overbrace{\cdots}^{\nu_{n-2}} \overbrace{\cdots}^{\nu_n}]) = [\overbrace{\cdots}^{\nu_1} \overbrace{\cdots}^{\nu_{n-2}} \overbrace{\cdots}^{\nu_n}]$. Assume the formula holds less than or equal to $n$, it is easy to check that the formula holds $n + 1$.

(2) It suffices to show that $J_{n,l} \subset J_{n,l+1}$ for $n$, $l$ with $1 \leq l \leq n-1$. Let $[D, H] \in J_{n,l}$. By assumption, there is a clasper of degree $d$ in $H$, say to $(h_1, e_1)$, where $d \geq 2$. Then $h_1$ can be represented by a pure virtual braid $[h_{1,1}, h_{1,2}] \cdot [h_{2,1}, h_{2,2}] \cdot \cdots \cdot [h_{m,1}, h_{m,2}]$ where $\deg(h_{i,j}) = d_j$ for any $i$ and $j$ and $d_1 + d_2 = d$. Let $h_{i,j} = h_{i,j} \otimes h_{i,j}^{-1} \otimes h_{i,j}^{-1} \otimes \cdots \cdot h_{j,m} \otimes h_{j,m}^{-1}$ for $j = 1, 2$. Then $\deg(h_{i,j}^{-1}) = d_j$ and $D_{(h_{i,j}^{-1})} \sim D_0$, where $e_{i}^{1}$ is induced by $e_1$. Therefore we have

$$[D_0, \{(h_1, e_1)\}] = [D_0^{(h_1, e_1)}] - [D_0]$$
$$= [D_0^{(h_1, e_1)}] - [D_0^{(h_2, e_1)}] + [D_0]$$

Hence

$$[D, H] = [D, \{(h_1, e_1), \cdots, (h_l, e_l)\}]$$
$$= [D, \{(h_1^{e_1}, e_1^{1}), (h_2^{e_1}, e_1^{2}), (h_2, e_2), \cdots, (h_l, e_l)\}]$$

$\in J_{n,l+1}$.

(3) It suffices to show that $J_{n+1,l} \subset J_{n,l}$ for $n$, $l$ with $1 \leq l \leq n$. Let $[D, H] \in J_{n+1,l}$. By assumption, there is a clasper of degree $d$ in $H$, say to $(h_1, e_1)$, where $d \geq 2$. Since $h_1 \in \Gamma_d(PV_{m,k}) \subset \Gamma_{d-1}(PV_k)$ for some $k$, we have $[D, H] \in J_{n,l}$. By Lemma 6.7 we can redefine $J_n$ as the ideal of $\mathbb{Z}[LVK]$ generated by elements $[D, H]$ where $D$ is any Gauss diagram and $H$ is any scheme for $D$ of degree $n$.

Proposition 6.8. For any $n \geq 0$, if $K$ and $K'$ are $L_{n+1}$-equivalent, then $K$ and $K'$ are $V_n$-equivalent.

Proof. By Remark 6.6 and Lemma 6.7 if $K$ and $K'$ are $L_{n+1}$-equivalent, then $K - K' \in J_{n+1,1} \subset J_{n+1,n+1} = J_{n+1}$. It is equivalent to that $K$ and $K'$ are $V_n$-equivalent.

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