AN ANALOGUE OF ABEL’S THEOREM

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Abstract. This work makes a parallel construction for curves on threefolds to a “current-theoretic” proof of Abel’s theorem giving the rational equivalence of divisors $P$ and $Q$ on a Riemann surface when $Q - P$ is (equivalent to) zero in the Jacobian variety of the Riemann surface. The parallel construction is made for homologous “sub-canonical” curves $P$ and $Q$ on a general class of threefolds. If $P$ and $Q$ are algebraically equivalent and $Q - P$ is zero in the (intermediate) Jacobian of a threefold, the construction “almost” gives rational equivalence.

1. Introduction

This work establishes a parallel between
1) a proof of the classical theorem of Abel deriving rational equivalence classes of divisors $P$ and $Q$ of degree $d$ on a Riemann surface $X$ from equality of their images in the Jacobian variety $J(X)$,
and
2) a construction for certain cohomologous curves $P$ and $Q$ on a threefold $X$ for which the image of $Q - P$ in the (intermediate) Jacobian $J(X)$ is zero.

This paper was motivated by work of Richard Thomas (see §). Indeed the point of view is in large measure due to him. To explain the analogy with Abel’s theorem, one must recast the classical proof of that theorem in the language of forms with values in distributions, that is, currents (see §). In that language, the classical proof of Abel’s theorem goes something like this.

Take two effective divisors $P$ and $Q$ of degree $d$ on a Riemann surface $X$. Consider a one-current

$$\Gamma$$

such that

$$Q - P = \partial (\Gamma)$$

that is, the operator which assigns to a $C^\infty$-one-form on $X$ its integral over some path $\Gamma$ from $P$ to $Q$. This current is the pull-back of a current on the multiplicative group $\mathbb{C}^*$ of complex numbers as follows:

For the unique topological line bundle $L_\infty$ of degree $d$ on $X$, there are two complex structures, one giving

$$\mathcal{O}(P)$$

and the other giving

$$\mathcal{O}(Q)$$.

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Thus the holomorphic sections $s_P$ and $s_Q$ with respect to these complex structures give two different $C^\infty$-sections of $L_\infty$ which we will continue to call $s_P$ and $s_Q$. The quotient 

$$g := \frac{s_Q}{s_P}$$

then gives a $C^\infty$-map 

$$X - (|P| \cup |Q|) \rightarrow \mathbb{C}^*.$$ 

The current 

$$\Gamma$$

is just the pull-back of the current on $\mathbb{C}^*$ given by the positive real axis, that is 

$$\Gamma = g^* ((0, +\infty)).$$

Furthermore, on the compactification $\mathbb{P}^1$ of $\mathbb{C}^*$ we have the cohomological relation 

$$d\log z \sim 2\pi i \cdot (0, +\infty)$$

of currents. Pulling back via $g^*$, we have the equality 

$$\int_\Gamma = \frac{1}{2\pi i} \int_X \alpha \wedge$$ 

as functionals on $H^{1,0}(X)$, where 

$$\alpha := d\log g.$$

Also we note that the $(0,1)$-component of $d\log z$, that is, the linear operator obtained by integrating $(1,0)$-forms against $d\log z$, is actually bounded, that is, is given by integrating against a $C^\infty$-form $\alpha_{0,1}$ on $X$ of type $(0, 1)$.

Now if $P$ is Jacobian-equivalent to $Q$, the $(0,1)$-summand of the current $\alpha_{0,1}$ (appropriately normalized by the $(0,1)$-summand of an integral cocycle) is 

$$\partial b$$

for some $C^\infty$-function $b$. So in this case, 

$$\psi := g^* (d\log z) - db$$

is a form of type $(1, 0)$ which is $d$-closed on $X - (|P| \cup |Q|)$. So $\psi$ is $\bar{\partial}$-closed and of type $(1, 0)$ and therefore meromorphic, having poles with residues which are integral multiples of $2\pi i$. We complete the proof of Abel’s theorem by remarking that 

$$f = e^f \psi$$

is the rational function giving the rational equivalence of $P$ and $Q$.

After first giving in detail the distribution-theoretic proof of classical Abel's theorem, we will make an analogous construction in the case in which $P$ and $Q$ are certain types of effective algebraic one-cycles on a threefold $X$. We first produce a rank-2 vector bundle $E_\infty$ whose first Chern class is trivial and whose second Chern class is represented by $P$ or by $Q$. We note that a choice of a metric $\mu$ on $E_\infty$ is equivalent to giving $E_\infty$ the structure of a quaternionic line bundle. Then we mimic, for the quaternionic line bundle $E_\infty$, all but the “trivial” last step of the proof of Abel’s theorem for divisors on curves. Namely for each metric $\mu$ on $E_\infty$ we use Chern-Simons theory to produce a three-form $\alpha_{\mu, PQ}$ on $X$ giving the normal function $Q - P$ and such that:

1) If $P$ and $Q$ are algebraically equivalent, then $\alpha_{\mu, PQ}^{(1,2)+(0,3)}$ is $\bar{\partial}$-closed as a form on $X$. 

2) If in addition $P$ and $Q$ are Abel-Jacobi equivalent, then $\alpha_{\mu,PQ}^{(1,2)+(0,3)}$ (normalized by the $(1,2) + (0,3)$ summand of an integral cycle) is $\mathcal{O}$-exact as a form on $X$; so there is an associated form $\psi_\mu$ on $X$ of type $(3,0) + (2,1)$ such that

$$\psi_\mu - \alpha_{\mu,PQ} \in dA_X^2.$$ 

Recall that $P$ and $Q$ are algebraically equivalent if they are homologous on some (possibly reducible) divisor on $X$.

Finally, for the quaternionic line bundle $E_\infty$ we will use Chern-Simons theory to produce a 3-current $\alpha_{PQ}$ giving the normal function $Q - P$ such that $\alpha_{PQ}$ is $d$-closed on $X'$ := $X - (|P| \cup |Q|)$ and:

1) If $P$ and $Q$ are algebraically equivalent, then $\alpha_{PQ}^{(1,2)+(0,3)}$ is $\mathcal{O}$-closed as a current on $X$.

2) If in addition $P$ and $Q$ are Abel-Jacobi equivalent, then $\alpha_{PQ}^{(1,2)+(0,3)}$ (normalized by the $(1,2) + (0,3)$ summand of an integral cycle) is $\mathcal{O}$-exact as a current on $X$; so there is a canonically associated current $\psi$ on $X$ of type $(3,0) + (2,1)$ such that

$$\psi - \alpha_{PQ} \in d\{2 - currents on X\}.$$

Furthermore

$$\psi|_{X'},$$

is a $d$-closed 3-current, that is, its integral against the coboundary of compactly supported 2-forms on $X'$ is zero.

To understand a potential significance of $\psi$, suppose that $\psi|_{X'}$ turns out to be a 3-form and the algebraic equivalence of $P$ and $Q$ is given by an algebraic family

$$S \xrightarrow{s} X \xleftarrow{r} C$$

such that

1) $C$ is a smooth irreducible curve,

2) $S$ is a smooth surface proper and flat over $C$,

3) for two points $p$ and $q$ in $C$ and the corresponding fibers $S_p$ and $S_q$ of $r$, we have a rational equivalence

$$s_* (S_q) - s_* (S_p) \equiv Q - P,$$

4) $$s^{-1}(|P| \cup |Q|) \subseteq S_p \cup S_q.$$

Then

$$r_* s^* \psi$$

is a $d$-closed form of type $(1,0)$ on $C' := C - \{p,q\}$ and

$$f = e^i r_* s^* \psi$$

would therefore be a meromorphic function on $C$ giving the rational equivalence of $P$ and $Q$. 

2. Classical theorem of Abel

Let \( L_\infty \) denote the unique \( C^\infty \) complex–line bundle on \( X \) with 
\[
c_1 (L_\infty) = d.
\]
If \( P \) is an effective divisor of degree \( d \) on \( X \), then there is a holomorphic structure 
\[
\overline{\partial}_P : A^0_X (L_\infty) \to A^{0,1}_X (L_\infty)
\]
on \( L_\infty \) and a section \( s_P \) of \( L_\infty \) such that 
\[
\overline{\partial}_P (s_P) = 0
\]
\[
div (s_P) = P.
\]
Given any metric \( \mu \) on \( L_\infty \), there is a unique metric-(1, 0) connection \( D_{\mu,P} \) on \( L_\infty \) such that 
\[
D_{0,1}^{0,1}_{\mu,P} = \overline{\partial}_P.
\]
Alternatively, if we restrict our consideration to \( X - |P| \), there is a unique connection \( D_P \) on 
\[
L_\infty|_{X-|P|}
\]
such that \( s_P \) is flat. Again for this connection 
\[
D_{0,1}^{0,1}_P = \overline{\partial}_P.
\]
If \( Q \) is another effective divisor of degree \( d \), we consider both \( s_P \) and \( s_Q \) as 
\( C^\infty \)-sections of the \( C^\infty \)-line bundle \( L_\infty \) and define as above the map 
\[
g = \frac{s_Q}{s_P} : X - (|P| \cup |Q|) \to \mathbb{C}^*
\]
where we assume that \( g \) is meromorphic in a small analytic neighborhood of \((|P| \cup |Q|)\). We compute 
\[
0 = D_Q (g \cdot s_P) = dg \cdot s_P + g \cdot D_Q s_P \\
D_Q s_P = -g^{-1} \cdot dg \cdot s_P.
\]
So, for 
\[
\alpha_{PQ} = D_Q - D_P,
\]
we have 
\[
\alpha \cdot s_P = (D_Q - D_P) s_P \\
= D_Q s_P \\
= -g^{-1} \cdot dg \cdot s_Q
\]
so that 
\[
\alpha_{PQ} = -g^{-1} \cdot dg \\
\alpha_{PQ}^{(0,1)} = \overline{\partial}_Q - \overline{\partial}_P \\
= -g^{-1} \cdot \overline{dg}.
\]
Now via residue and the fact that 
\[
\lim_{r \to 0} r \log r = 0,
\]
we have cohomologous currents
\[
2\pi i \int_0^\infty \sim \int_X d\theta \wedge 
\sim \int_X (z^{-1} \cdot dz) \wedge.
\]
on $\mathbb{C}^* = \mathbb{P}^1$. We therefore obtain the equality
\[
(1) \quad \int_P^Q \eta = \frac{1}{2\pi i} \int_X \alpha_{PQ} \wedge \eta
\]
for $\eta \in H^{1,0}(X)$ by pulling back the above cohomology between currents via $g$.
Also, since
\[
D_{P,0}^{0,1} = D_{\mu,P}^{0,1}, \\
D_{Q,0}^{0,1} = D_{\mu,Q}^{0,1}
\]
we have for
\[
\alpha_{\mu,PQ} := D_{\mu,Q} - D_{\mu,P}
\]
that
\[
\alpha_{\mu,PQ}^{0,1} = \alpha_{PQ}^{0,1}
\]
so that the $(0,1)$-summand of $-g^{-1} \cdot dg$ is bounded.
The equality (1) shows that if, for some
\[
\varepsilon \in H_1(X;\mathbb{Z}),
\]
one has
\[
\int_P^Q \varepsilon = \int_\varepsilon : H^{1,0}(X) \to \mathbb{C},
\]
then the deRham class
\[
A = A^{1,0} (\text{with poles}) + A^{0,1} (\text{bounded})
\]
such that
\[
\{ A^{0,1} \} \in 2\pi i \cdot H^1(X;\mathbb{Z}) + H^{1,0}(X).
\]
So the Poincaré dual of $\varepsilon$ is a deRham class
\[
\{ \xi \} \in H^1(X;\mathbb{Z})
\]
such that
\[
A^{0,1} - \xi^{0,1} = \overline{\partial} \gamma
\]
for some $C^\infty$-function
\[
\gamma \in A^0_X.
\]
Then the form
\[
\psi := g^{-1} \cdot dg + \xi + d\gamma
\]
on $X - (|P| \cup |Q|)$ is $d$-closed and of type $(1,0)$ and therefore meromorphic on $X$
with poles at the divisor $Q - P$.
As mentioned above, the “second part” of of the proof of Abel’s theorem consists
in defining the meromorphic function
\[
f = e^{f \psi}
\]
and noticing that
\[
\text{div} (f) = Q - P.
\]
Thus
\[ f : X \to \mathbb{P}^1 \]
gives the rational equivalence between \( P \) and \( Q \).

2.1. Analogue for one-cycles on threefolds. In the remainder of this paper we present the analogue of Abel’s theorem in the case in which \( P \) and \( Q \) are cohomologous (sums of) smooth sub-canonical curves on a threefold \( X \) and \( E_{\infty} \) is a \( C^\infty \) rank-2 vector bundle on \( X \) whose first Chern class is trivial and whose second Chern class is represented by \( P \) (and so also by \( Q \)).

3. Quaternionic connections

3.1. Serre’s construction. Let \( X \) be a smooth projective threefold with
\[ H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0 \]
and that \( P \) is a smooth (but possibly reducible) Riemann surface lying inside \( X \). We wish to consider situations in which
\[ (2) \quad \omega_X\big|_P = \omega_P, \]
that is, the line bundle
\[ \omega_X^{-1} \otimes \omega_P = \omega_X^{-1} \otimes \text{ext}^2_X(\mathcal{O}_P, \omega_X) = \omega_X^{-1} \otimes \text{ext}^2_X(\mathcal{O}_P, \mathcal{O}_P) \]
on \( P \) is the trivial bundle. Of course this is often not the case. However if we assume, more generally, that there exists an effective divisor
\[ \sum_j x_j \]
on \( P \) representing the line bundle
\[ \omega_X^{-1} \otimes \omega_P \]
with \( x_j \) distinct, and denote by
\[ (3) \quad \tilde{X} \to X \]
the blow-up of \( X \) at the points \( x_j \) containing the proper transform \( \tilde{P} \) of \( P \), then
\[ \omega_{\tilde{X}}\big|_{\tilde{P}} = \omega_{\tilde{P}}. \]

Assuming \( (2) \) from now on, we obtain
\[ (4) \quad \mathcal{O}_P = \text{ext}^2_X(\mathcal{O}_P, \mathcal{O}_X) = \text{ext}^1_X(\mathcal{I}_P, \mathcal{O}_X). \]

Since
\[ H^1(\text{hom}_X(\mathcal{I}_P, \mathcal{O}_X)) = H^1(\mathcal{O}_X) = 0, \]
we have
\[ \text{Ext}^1(\mathcal{I}_P, \mathcal{O}_X) = H^0(\text{ext}^1_X(\mathcal{I}_P, \mathcal{O}_X)) \]
and the nowhere-vanishing section of \( \text{ext}^1_X(\mathcal{I}_P, \mathcal{O}_X) \) gives, via the Serre construction, an exact sequence of vector bundles
\[ (5) \quad 0 \to \mathcal{O}_X \to E \to \mathcal{I}_P \to 0. \]
Also \( E \) has a distinguished global section
\[ s_P \]
vanishing exactly at $P$. By the irreducibility of $P$,

$$H^1(I_P) = 0.$$ 

Thus

$$H^1(E) = 0$$

and so $s$ deforms with each deformation of $(E, X)$. Also, since

$$\det(E) = O_X$$

we have

(6) \[ E' = E. \]

Tensor the surjection

(7) \[ E' \to I_P \]

induced by $s$ with $O_P$ to obtain a surjection of rank-2 bundles on $P$ which is therefore an isomorphism. Dualize to obtain an isomorphism

$$N_{P_X} \to E|_P.$$ 

Thus we have the isomorphism

(8) \[ H^\cdot\left(N_{P_X}\right) = H^\cdot\left(I_P \otimes E \to E\right). \]

On the other hand, (5) and (6) give the exact sequence

(9) \[ 0 \to E \to E' \otimes E \to I_P \otimes E \to 0 \]

from which comes

(10) \[ H^\cdot\left(E' \otimes E\right) \to H^\cdot\left(I_P \otimes E\right). \]

Together (8) and (10) relate the deformation functor of $E$ to that of $P$.

### 3.2. Quaternionic line bundles.

Now

$$\det(E) = O_X$$

allowing us to choose a non-vanishing holomorphic section

$$1 \in H^0\left(\det(E)\right).$$

We next chose a hermitian metric $\mu$ on $E$. We have an associated structure of a quaternionic line bundle on $E$ by defining

$$(j \cdot s')$$

for any (locally defined) $C^\infty$-section $s'$ of $E$ as the unique element such that we have an equality of linear operators

$$\frac{s \wedge (j \cdot s')}{1} = \mu(s, s')$$

on sections $s$ of $E$. Thus

(11) \[ (j \cdot s) \perp s \]

$$s \wedge (j \cdot s) = \|s\|^2 \cdot 1.$$ 

Since

$$s \wedge (j \cdot i \cdot s') = \mu(s, i \cdot s') = -i \cdot \mu(s, s') = s \wedge (-i \cdot j \cdot s'),$$
we have a well-defined (left) action on $E$ by the group $\mathbb{H}^*$ of non-zero quaternions and the action of $\mathbf{j}$ is conjugate linear. Since

\[
\mathbf{j} \cdot (\mathbf{j} \cdot s) = - (\mathbf{j} \cdot \mathbf{j} \cdot s) = (\mathbf{j} \cdot s) \wedge (\mathbf{j} \cdot s) = ||\mathbf{j} \cdot s||^2 \cdot 1
\]

$\mathbf{j} \cdot s$ acts as an isometry. Also, for any two sections $s$ and $s'$, we have that

\[
\mathbf{j} \cdot s \wedge \mathbf{j} \cdot s' = \mu(\mathbf{j} \cdot s, s') = \mu(s', \mathbf{j} \cdot s) = s' \wedge \mathbf{j} \cdot s = s \wedge s'.
\]

For a fixed non-zero local section $s_0$ of $E$, we have the framing $(s_0, \mathbf{j} \cdot s_0)$ as a complex vector bundle, and, for any section $s$, we have the formula

\[
s = \frac{\mu(s, s_0)}{||s_0||^2} \cdot s_0 + \frac{\mu(s, \mathbf{j} \cdot s_0)}{||s_0||^2} \cdot (\mathbf{j} \cdot s_0)
\]

(12)

\[
= \frac{s \wedge (\mathbf{j} \cdot s_0)}{||s_0||^2} \cdot s_0 - \frac{s \wedge s_0}{||s_0||^2} \cdot (\mathbf{j} \cdot s_0)
\]

(13)

with respect to the unitary basis

\[
\left( \frac{s_0}{||s_0||}, \frac{\mathbf{j} \cdot s_0}{||s_0||} \right).
\]

Now suppose that $D_{\mu,P}$ denotes the metric $(1,0)$-connection on $E$ with respect to the metric $\mu$. Then we have that

\[
d(\mu(s, s')) = \mu(D_{\mu,P}(s), s') + \mu(s, D_{\mu,P}(s'))
\]

so that

\[
d(s \wedge \mathbf{j} \cdot s') = D_{\mu,P}(s) \wedge \mathbf{j} \cdot s' + s \wedge \mathbf{j} \cdot D_{\mu,P}(s')
\]

from which follows that

\[
d(s \wedge s') = D_{\mu,P}(s) \wedge s' - s \wedge \mathbf{j} \cdot D_{\mu,P}(\mathbf{j} \cdot s')
\]

Since

\[
d(s \wedge s') = D_{\mu,P}(s) \wedge s' + s \wedge D_{\mu,P}(s')
\]

it follows that $\mathbf{j}$ commutes with $D_{\mu,P}$ so that $D_{\mu,P}$ is a quaternionic connection. Since $-\mathbf{j} \cdot a \cdot \mathbf{j} = \overline{a}$,

\[
-\mathbf{j} \cdot D_{\mu,P}^{1,0} \cdot \mathbf{j} = D_{\mu,P}^{0,1}
\]

\[
-\mathbf{j} \cdot D_{\mu,P}^{0,1} \cdot \mathbf{j} = D_{\mu,P}^{1,0}
\]

Thus

\[
D_{\mu,P}^{1,0} = -\mathbf{j} \cdot \overline{\nabla} \cdot \mathbf{j}.
\]

Now let $s = s_P$, the holomorphic section of $E$. Our first goal is to understand. If we write

\[
D_{\mu,P}(s_P) = D_{\mu,P}^{1,0}(s_P) = -\mathbf{j} \cdot \overline{\nabla} \cdot \mathbf{j} \cdot s_P = \alpha_P \cdot s_P + \beta_P \cdot \mathbf{j} \cdot s_P
\]
for forms $\alpha_P$ and $\beta_P$ of type $(1,0)$, then
\[
D_{\mu,P}(j \cdot s_P) = j \cdot D_{\mu,P} s_P = j \cdot (\alpha_P \cdot s_P + \beta_P \cdot j \cdot s_P) = -\beta_P \cdot s_P + \alpha_P \cdot j \cdot s_P.
\]
And since
\[
(D_{\mu,P}(j \cdot s_P))^{0,1} \wedge s_P = \partial (j \cdot s_P) \wedge s_P = -\partial (\|s_P\|^2)
\]
we can write
\[
(14) \quad D_{\mu,P} \left( \begin{array}{c} s_P \\ j \cdot s_P \end{array} \right) = \left( \begin{array}{cc} \partial \log (\|s_P\|^2) & \beta_P \\ -\beta_P & \partial \log (\|s_P\|^2) \end{array} \right) \left( \begin{array}{c} s_P \\ j \cdot s_P \end{array} \right).
\]
With respect to the basis $(s_P, j \cdot s_P)$, the curvature of the connection $D_{\mu,P}$ then becomes
\[
\left( \begin{array}{cc} \partial \log (\|s_P\|^2) & \beta_P \\ -\beta_P & \partial \log (\|s_P\|^2) \end{array} \right) + \left( \begin{array}{c} \beta_P \wedge \beta_P \\ \beta_P \wedge (\partial - \overline{\partial}) \log (\|s_P\|^2) \end{array} \right).
\]
Since this matrix must be of type $(1,1)$ we conclude
\[
(15) \quad \partial \beta_P = \beta_P \wedge \partial \log (\|s_P\|^2)
\]
and the curvature matrix becomes
\[
(16) \quad R_{\mu,P} = \left( \begin{array}{cc} \partial \log (\|s_P\|^2) - \beta_P \wedge \beta_P \\ -\partial \overline{\beta_P} - \overline{\beta_P} \wedge \partial \log (\|s_P\|^2) \end{array} \right) + \left( \begin{array}{c} \beta_P \wedge \beta_P \\ \beta_P \wedge (\partial - \overline{\partial}) \log (\|s_P\|^2) \end{array} \right).
\]
3.3. Chern-Simons functional. Following [DT] define the holomorphic Chern-Simons functional as follows. For any two connections $D_0, D_1$ on a $C^\infty$-vector bundle $E$, form the connection
\[
\hat{D} := D_0 + t (D_1 - D_0) + dt
\]
on
\[
X \times [0,1]
\]
and let
\[
\hat{R} = \hat{D}^2
\]
denote its curvature form. Let $A_X$ denote the $C$-deRham complex in $X$ and let $F$ denote the Hodge filtration on $A_X$.

**Definition 3.1.** The holomorphic Chern-Simons current $CS_{D_0}(D_1)$ is the current of type $(1,2) + (0,3)$ given by the functional
\[
CS_{D_0}(D_1) : F^2 A_X^3 \to \mathbb{C}
\]
\[
\tau \mapsto \int_{X \times [0,1]} \tau \wedge tr (\hat{R} \wedge \hat{R}).
\]
(This is an extension of the standard definition of holomorphic Chern-Simons functional which refers to the restriction of the above current to $F^3 H^3(X)$.)
Since $tr\left(\tilde{R} \wedge \tilde{R}\right)$ pulls back to an exact form on the associated principal bundle of frames of $E$ (see §3 of [CS]), one has for three connections $D_0, D_1, D_2$ that
\begin{equation}
CS_{D_0}(D_1) + CS_{D_1}(D_2) = CS_{D_0}(D_2).
\end{equation}

Let
\[A := D_1 - D_0\]
We compute $\tilde{R}$ as follows
\[(D_0 + tA + dt) \circ (D_0 + tA + dt) = R_0 + t \cdot D_0A + t^2 \cdot A \wedge A + dt \wedge A\]
so that
\[\tau \wedge \tilde{R} \wedge \tilde{R} = \tau \wedge dt \wedge 2A \wedge (R_0 + t \cdot D_0A + t^2 \cdot A \wedge A)\]
and finally
\begin{equation}
CS_{D_0}(D_1)(\tau) = \int_X \tau \wedge tr\left(A \wedge \left(R_0 + D_0A + \frac{2}{3}A \wedge A\right)\right).
\end{equation}

Suppose that $D_0^{0,1}$ and $D_1^{0,1}$ both give complex structures on $E$, that is
\[R_0^{0,2} = \left(D_0^{0,1}\right)^2 = 0 = \left(D_1^{0,1}\right)^2.\]
Then
\[0 = \left(D_0^{0,1} + A^{0,1}\right)^2 = D_0^{0,1}A^{0,1} + A^{0,1} \wedge A^{0,1}.\]
Thus in this case
\begin{equation}
tr\left(A \wedge \left(R_0 + D_0A + \frac{2}{3}A \wedge A\right)\right)^{0,3} = -\frac{1}{3}tr\left(A^{0,1} \wedge^3\right).
\end{equation}

On the other hand, suppose that $D_0$ and $D_1$ are both flat. Then $R_0 = 0$ and
\[0 = (D_0 + A)^2 = D_0A + A \wedge A.\]
so that (13) becomes
\begin{equation}
CS_{D_0}(D_1)(\tau) = -\frac{1}{3} \int_X \tau \wedge tr(A \wedge A \wedge A).
\end{equation}

3.4. Comparing connections via Chern-Simons theory. Let $D_P$ denote the unique $\mathbb{H}$-connection on the restriction $E'$ of $E$ to
\[X' := X - |P|\]
such that
\[D_P(s_P) = 0.\]
($D_P$ is of course flat.) We first apply the Chern-Simons theory to the two connections
\[D_0 = D_P, \quad D_1 = D_{0,P}^{1,0} + D_{\mu,P}^{0,1} =: D'_P\]
\[D_0 = D_P, \quad D_1 = D_{0,P}^{1,0} + D_{\mu,P}^{0,1} =: D'_P\]
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where, as above, $D_{\mu,P}$ is the metric $(1,0)$-connection associated to $\mu$ and the complex structure on $E$. Thus $D'_p$ is a $(1,0)$-connection for the complex structure on $E$. Let

$$A_p := D'_p - D_p.$$  

Then by (14)

$$A_p \left( \frac{p}{j \cdot s_p} \right) = \left( \begin{array}{cc} 0 & 0 \\ -\beta_p \cdot \overline{\beta_p} \overline{\log \|s_p\|^2} & 0 \end{array} \right) \left( \frac{p}{j \cdot s_p} \right).$$

Also, in terms of the basis $(s_p, j \cdot s_p)$ we can compute the curvature

(21) $R'_p = (D'_p)^2 = (D_p + A_p)^2$

(22) $= \left( \begin{array}{cc} 0 & 0 \\ -d\beta_p \cdot \overline{\beta_p} \overline{\log \|s_p\|^2} & 0 \end{array} \right)$

using (13).

By (18) and (?), for any $(3,0) + (2,1)$-form $\tau$, the Chern-Simons functional $CS_{D_p} (D'_p) (\tau)$ is given by the expression

(23) $\int_{X'} \text{tr} \left( \overline{\partial} \log \|s_p\|^2 \wedge \overline{\partial} \partial \|s_p\|^2 \right) \wedge \tau$

By (??) and Stokes theorem, if $\tau$ is $d$-closed, this reduces to

(24) $CS_{D_p} (D'_p) (\tau) = \int_{\partial X'} \left( \overline{\partial} \log \|s_p\|^2 \wedge \overline{\partial} \partial \|s_p\|^2 \right) \wedge \tau$

where $\int_{\partial X'}$ is computed as

$$\lim_{\varepsilon \to 0} \int_{B_\varepsilon} \left( \log \|s_p\|^2 \wedge \overline{\partial} \partial \|s_p\|^2 \right) \wedge \tau,$$

where $B_\varepsilon$ is a tubular neighborhood of $P$ in $X$, of radius $\varepsilon$ in some fixed metric. The integrand is bounded by

$$\text{const.} \cdot \frac{(\log(\varepsilon))^2}{\varepsilon^2}.$$  

So the above integral is bounded by a constant multiple of $\varepsilon \cdot (\log(\varepsilon))^2$, which tends to zero as $\varepsilon \to 0$. Thus

$$CS_{D_p} (D'_p) \sim 0.$$  

Finally since

$$D_{\mu,P} - D'_p = A^{1,0},$$
is of type \((1, 0)\) we again use (18) to conclude by type that for any form \(\tau\) of type 
\(\left((3, 0) + (2, 1)\right)\),

\[
CS_{D_P'} (D_{\mu, P}) (\tau) = \int_X \tau \wedge \text{tr} \left(A^{1, 0} \wedge \left(R'_P + dA^{1, 0} + \frac{2}{3} A^{1, 0} \wedge A^{1, 0}\right)\right)
\]

since \(R'_P\) is of type \((1, 1)\). So by the additivity formula (17) we conclude

\[
CS_{D_P} (D_{\mu, P}) \sim 0,
\]

that is, the \((1, 2) + (0, 3)\) current \(CS_{D_P} (D_{\mu, P})\) is \(d\)-exact in the sense of currents.

4. Analogue of the first part of Abel’s theorem

4.1. Changing the curve \(P\). Let \(X\) and \(P\) be as in the previous section. Suppose now we have another irreducible smooth curve \(Q \subseteq X\) with

\[
\omega^{-1}_X \otimes \omega_Q = O_Q.
\]

We suppose there is a fixed \(C^\infty\)-vector bundle, which we call \(E_\infty\) such that, as \(C^\infty\)-vector bundles, we have isomorphisms

\[
E_P \leftrightarrow E_\infty \leftrightarrow E_Q.
\]

We use a fixed hermitian structure \(\mu\) on \(E_\infty\) and a fixed section \(1\) of \(\text{det}(E_\infty)\) as above to make \(E_\infty\), and therefore also \(E_P\) and \(E_Q\), into quaternionic line bundles such that the above correspondences are \(C^\infty\)-isomorphisms of quaternionic line bundles. Again considering \(s_P\) and \(s_Q\) as sections of the quaternionic line bundle \(E_\infty\) we have

\[
s_Q = g \cdot s_P
\]

where, for \(X' = X - (|P| \cup |Q|)\),

\[
g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : X' \to \mathbb{H}^*.
\]

That is

\[
s_Q = a \cdot s_P + b \cdot (j \cdot s_P).
\]

Then

\[
\begin{aligned}
   j \cdot s_Q &= j \cdot a \cdot s_P + j \cdot b \cdot (j \cdot s_P) \\
   &= -\bar{b} \cdot s_P + \bar{a} \cdot (j \cdot s_P)
\end{aligned}
\]

so that the matrix \(g\) gives the expression for the \(\mathbb{C}\)-basis \((s_Q, j \cdot s_Q)\) in terms of the \(\mathbb{C}\)-basis \((s_P, j \cdot s_P)\), that is,

\[
\begin{pmatrix} s_Q \\ j \cdot s_Q \end{pmatrix} = g \cdot \begin{pmatrix} s_P \\ j \cdot s_P \end{pmatrix}.
\]

We are interested in comparing two connections on \(E'\), namely the \(\mathbb{H}\)-connection \(D_P\) with flat section \(s_P\) and the \(\mathbb{H}\)-connection \(D_Q\) with flat section \(s_Q\). Again we compute

\[
\begin{aligned}
   0 &= D_Q (g \cdot s_P) = dg \cdot s_P + g \cdot D_Q s_P \\
   D_Q s_P &= -g^{-1} \cdot dg \cdot s_P.
\end{aligned}
\]

So

\[
(D_Q - D_P)(s_P).
\]
Since $\mathbb{H}$-connections are determined by their values on a single section, then in terms of the basis $(s_P, j \cdot s_P)$, we have

$$A_{PQ} := D_Q - D_P = -g^{-1} \cdot dg.$$

Since $D_P$ and $D_Q$ are both flat,

$$0 = (D_P + A_{PQ})^2 = dA_{PQ} + A_{PQ} \wedge A_{PQ}.$$ 

Write

$$h = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = r \cdot \begin{bmatrix} u & v \\ -\bar{v} & \bar{u} \end{bmatrix}$$

with $u\bar{u} + v\bar{v} = 1$. So for

$$\kappa = \begin{bmatrix} u & v \\ -\bar{v} & \bar{u} \end{bmatrix} = u + v \cdot j$$

we have

$$\kappa^{-1} = \bar{v} - v \cdot j$$

since

$$(u + v \cdot j) \cdot (\bar{v} - v \cdot j) = u\bar{v} + v \cdot j \cdot \bar{v} - uv \cdot j \cdot v \cdot j$$

$$= u\bar{v} + uv \cdot j - uv \cdot j + v\bar{v}$$

$$= 1.$$ 

The form

$$h^{-1} \cdot dh = d\log r + \bar{\mathbb{H}} d\kappa$$

$$= r^{-1} dr + (\bar{v} - v \cdot j) (du + dv \cdot j)$$

$$= r^{-1} dr + (\bar{u}du + v\bar{d}v) + (\bar{v}dv - v\bar{d}u) \cdot j$$

is a left-invariant 1-form on

$$\mathbb{H}^* = (0, +\infty) \times SU(2).$$

Since $u\bar{u} + v\bar{v} = 1$, the form

$$\bar{u}du + v\bar{d}v = -(ud\bar{u} + v\bar{d}v)$$

is purely imaginary.

Using the rule

$$(A + B \cdot j) \wedge (C + D \cdot j) = (A \wedge C - B \wedge \bar{D}) + (A \wedge D + B \wedge \bar{C}) \cdot j$$

we have that, if $\bar{A} = -A$,

$$(A + B \cdot j) \wedge (A + B \cdot j) = -B \wedge \bar{B} - 2 (A \wedge B) \cdot j$$

and

$$(A + B \cdot j)^\wedge = 3 \bar{A} \wedge B \wedge \bar{B}.$$ 

We compute

$$(h^{-1} dh)^\wedge = r^{-1} dr \wedge ((\bar{u}du + v\bar{d}v) + (\bar{v}dv - v\bar{d}u) \cdot j)^\wedge$$

$$+ ((\bar{u}du + v\bar{d}v) + (\bar{v}dv - v\bar{d}u) \cdot j)^\wedge$$

and

$$(\bar{u}du + v\bar{d}v) \wedge (\bar{v}dv - v\bar{d}u) = d\bar{u}dv.$$
So
\[
\left( (\overline{u}dU + v\overline{v}) + (\overline{v}dU - v\overline{u}) \cdot \mathbf{j} \right)^3 = 3d\overline{u}dU \wedge (u\overline{v} - \overline{v}u) = 3(u\overline{u}dUd\overline{v} - \overline{v}u\overline{d}Ud\overline{v}).
\]

On the other hand
\[
0 = d(h \cdot h^{-1}) = (dh^{-1}) \cdot h + h^{-1} \cdot dh
\]
so that
\[
(28) \quad d(h^{-1} \cdot dh) = -h^{-1} \cdot dh \wedge h^{-1} \cdot dh
\]
and
\[
(29) \quad (h^{-1} \cdot dh)^{\wedge 3} = -d\log r \wedge d(\chi^{-1} \cdot d\chi) + (\chi^{-1} \cdot d\chi)^{\wedge 3}.
\]
So
\[
(30) \quad tr\left((\chi^{-1} \cdot d\chi)^{\wedge 3}\right) = 3(ud\overline{u}dUd\overline{v} - \overline{v}u\overline{d}Ud\overline{v}) + 3(ud\overline{u}dUd\overline{v} - \overline{v}u\overline{d}Ud\overline{v})
\]
is real and \(d\)-closed on \(X'\) as is
\[
tr\left((h^{-1} \cdot dh)^{\wedge 3}\right).
\]
Since the left-invariant form \((30)\) evaluated at \((u, v) = (1, 0)\) is
\[
12d(\text{Im} a) \wedge d(\text{Re} b) \wedge d(\text{Im} b)
\]
we have
\[
\int_{S^3} tr\left((h^{-1} \cdot dh)^{\wedge 3}\right) = 12 \cdot vol(S^3) = 24\pi^2.
\]

We compare these connections on \(E_{\infty}|_{X'}\) via the Chern-Simons functional. Now
\[
A_{PQ} = -g^* (h^{-1} \cdot dh)
\]
for the invariant one-form \(h^{-1} \cdot dh\) on \(\mathbb{H}^*\). Then by \((28)\) and \((29)\) we have the formula
\[
(31) CS_{DP} (DQ) (\tau) = \int_X g^* tr\left((h^{-1} \cdot dh) \wedge d(h^{-1} \cdot dh) - \frac{2}{3} (h^{-1} \cdot dh)^{\wedge 3}\right) \wedge \tau
\]
\[
= \frac{1}{3} \int_X g^* tr\left((h^{-1} \cdot dh)^{\wedge 3}\right) \wedge \tau.
\]
Analogously to the classical Abel theorem on curves, via residue and the fact that
\[
\lim_{r \to 0} r \log r = 0,
\]
we have cohomologous currents
\[
tr\left((h^{-1} \cdot dh)^{\wedge 3}\right) \sim tr(\chi^{-1} \cdot d\chi)^{\wedge 3}
\]
\[
\sim 24\pi^2 \int_{(0, +\infty)} \tau.
\]
on \(\mathbb{H}^* = \mathbb{H} \mathbb{P}^1\). Pulling back via \(g\) we therefore have by \((31)\) that, whenever \(\tau\) is a \(d\)-closed form of type \((3, 0) + (2, 1)\),
\[
(32) \quad CS_{DP} (DQ) (\tau) = \frac{24\pi^2}{3} \int_{D}^{Q} \tau.
\]
Here on the right-hand side we integrate over the 3-chain
\[ g^{-1}((0, \infty)), \]
which indeed bounds \( Q - P \). This is of course completely analogous, for rank-2 vector bundles, to the main step in the above version of the proof of classical Abel’s theorem for line bundles on curves. Thus
\[ CS_{D_P}(D_Q) = 8\pi^2 \int_P^Q \]
is \( d \)-exact in the sense of currents. Since
\[ tr\left((h^{-1} \cdot dh)^{\wedge 3}\right) \]
is \( d \)-closed on \( \mathbb{H}^* \), \( CS_{D_P}(D_Q) \) is the \((1, 2) + (0, 3)\) summand of a \( d \)-closed form on \( X' \), not of a \( d \)-closed current on \( X \). Indeed, by what we have just shown,
\[ \frac{1}{3} \int_X g^* tr\left((h^{-1} \cdot dh)^{\wedge 3}\right) \wedge d\beta = 8\pi^2 \int_P^Q d\beta = 8\pi^2 \left( \int_Q^P \beta - \int_P^Q \beta \right). \]

Finally by (32), (26) and the additivity property (17), we have that
\[ CS_{D_{\mu,P}}(D_{\mu,Q}) \]
is a current coboundary and so, for any \( d \)-closed form \( \tau \) of type \((3, 0) + (2, 1)\) that
\[ (33) \]
\[ CS_{D_{\mu,P}}(D_{\mu,Q})(\tau) = 8\pi^2 \int_Q^P \tau. \]

4.2. \( P \) and \( Q \) Abel-Jacobi equivalent. Let
\[ A_{\mu,PQ} = D_{\mu,Q} - D_{\mu,P} \]
where we recall that
\[ A_{\mu,PQ}^{0,1} = \overline{\partial} Q - \overline{\partial} P \]
\[ A_{\mu,PQ}^{1,0} = -j \cdot (\overline{\partial} Q - \overline{\partial} P) \cdot j. \]

Let
\[ \alpha_{\mu,PQ} := tr\left( A_{\mu,PQ} \wedge \left( R_{\mu,P} + dA_{\mu,PQ} + \frac{2}{3} A_{\mu,PQ} \wedge A_{\mu,PQ} \right) \right) \]
be the \( C^\infty \)-deRham form giving \( CS_{D_{\mu,P}}(D_{\mu,Q}) \) so that
\[ CS_{D_{\mu,P}}(D_{\mu,Q})(\tau) = \int_X \tau \wedge \alpha_{\mu,PQ} = \int_X \tau \wedge \alpha_{\mu,PQ}^{(1,2) + (0,3)}. \]

If \( P \) and \( Q \) are algebraically equivalent, we can arrange that \( \Gamma \) with
\[ \partial \Gamma = Q - P \]
be chosen to lie inside a (possibly reducible) algebraic surface on \( X \). Suppose now that
\[ \tau = \overline{\partial} \beta^{2,0}. \]
Then, since any \((3, 0)\)-form restricts to zero on \(\Gamma\), we conclude by (33) that
\[
\int_X \tau \wedge \alpha^{(1,2)+(0,3)}_{\mu,PQ} = \int_X \overline{\partial} \beta^{2,0} \wedge \alpha_{\mu,PQ} = 8\pi^2 \int_{\Gamma} \overline{\partial} \beta^{2,0} = 8\pi^2 \int_{\Gamma} d\beta^{2,0} = 8\pi^2 \left( \int_Q \beta^{2,0} - \int_P \beta^{2,0} \right) = 0.
\]

Thus:

**Lemma 4.1.** If \(P\) and \(Q\) are algebraically equivalent, \(\alpha^{(1,2)+(0,3)}_{\mu,PQ}\) is \(\overline{\partial}\)-closed.

Referring to (??) and writing the distribution-valued differential
\[
\alpha_{PQ} : = tr \left( A_{PQ} \wedge \left( dA_{PQ} + \frac{2}{3} A_{PQ} \wedge A_{PQ} \right) \right) = \frac{1}{3} tr \left( A_{PQ} \wedge A_{PQ} \wedge A_{PQ} \right),
\]
then
\[
CS_{DP}(DQ)(\tau) = \int_X \tau \wedge \alpha_{PQ} = \int_X \tau \wedge \alpha^{(1,2)+(0,3)}_{PQ}.
\]

Again if \(P\) is algebraically equivalent to \(Q\), by (??), \(\Gamma\) can be chosen so that
\[
\int_X \tau \wedge \alpha^{(1,2)+(0,3)}_{PQ} = 0
\]
whenever \(\tau \in F^2 A_X^3\) is \(\overline{\partial}\)-exact. Thus:

**Lemma 4.2.** If \(P\) is algebraically equivalent to \(Q\), then \(\alpha^{(1,2)+(0,3)}_{PQ}\) is \(\overline{\partial}\)-closed.

The equality (??) shows that, if \(P\) is algebraically equivalent to \(Q\) and
\[
\int_Q^P = \int_\varepsilon : F^2 H^3(X) \to \mathbb{C}
\]
for some
\[
\varepsilon \in H_3(X; \mathbb{Z}),
\]
then we have the containment
\[
\left\{ \alpha^{(1,2)+(0,3)}_{\mu,PQ} \right\} \in H^3(X; \mathbb{Z}) + F^2 H^3(X)
\]
of the Dolbeault class of \(\alpha^{(1,2)+(0,3)}_{\mu,PQ}\). So the Poincaré dual of \(\varepsilon\) is a deRham class
\[
\{ \xi \} \in H^3(X; \mathbb{Z})
\]
with the property that
\[
\alpha^{(1,2)+(0,3)}_{\mu,PQ} - \xi^{(1,2)+(0,3)} = \overline{\partial} \gamma_{\mu}
\]
for some $C^\infty$-form

$$\gamma_{\mu} \in A_X^{(1,1)+(0,2)}.$$  

Then the form

$$\psi_{\mu} := \alpha_{\mu,PQ} - \xi - \overline{\partial} \gamma_{\mu}$$

on $X$ is of type $(3,0) + (2,1)$.

The equality (32) shows that, if $P$ is algebraically equivalent to $Q$ and

$$\int_{\epsilon} F^3 H^3 (X) \rightarrow \mathbb{C}$$

for some

$$\epsilon \in H_3 (X; \mathbb{Z}),$$

then we have as above, taking the deRham dual

$$\{\xi\} \in H^3 (X; \mathbb{Z})$$

of $\epsilon$, we have

$$\int_X (\alpha_{PQ} - \xi) \wedge \tau = 0$$

for any $d$-closed form $\tau$ of type $(3,0) + (2,1)$ and any $\overline{\partial}$-exact form of type $(3,0) + (2,1)$, hence for any $\overline{\partial}$-closed form of type $(3,0) + (2,1)$. This means that the current $(\alpha_{PQ} - \xi)^{(1,2)+(0,3)}$ is $\overline{\partial}$-exact, and so, in particular,

$$\alpha_{PQ}^{0,3} - \epsilon^{0,3} = \overline{\partial} \gamma^{0,2}$$

for some distribution–valued form $\gamma^{0,2}$ of type $(0,2)$. Then the current

$$\alpha_{PQ} - \xi - d\gamma^{0,2}$$

on $X$ is of type $(3,0) + (2,1) + (1,2)$ and is $d$-closed on $X'$. Since $\alpha_{PQ} - \xi - d\gamma$ integrates to zero against any $\overline{\partial}$-closed form of type $(3,0) + (2,1)$ on $X$, the current $(\alpha_{PQ} - \xi - d\gamma)^{1,2}$ is again $\overline{\partial}$-exact, we can find a current $\gamma^{1,1}$ such that the distribution valued-form

$$\psi : \alpha_{PQ} - \xi - d\gamma^{0,2} - d\gamma^{1,1}$$

on $X$ is a current of type $(3,0) + (2,1)$ and is $d$-closed on $X'$.

Of course, in the case of classical Abel’s theorem, $P$ and $Q$ are always algebraically equivalent and $\psi$ gives directly the rational equivalence of $P$ and $Q$. $\psi_{\mu}$ does not enter the picture because $\alpha^{0,1} = \alpha^{0,1}_{\mu}$ for any metric $\mu$ is already smooth. In the threefold case, (23) and (24) imply that

$$\alpha_{PQ}^{0,3} = \alpha_{\mu,PQ}^{0,3}$$

for any metric $\mu$ but in general the currents $\alpha_{PQ}^{1,2}$ and $\alpha_{\mu,PQ}^{1,2}$ are not equal. However it should be true that, as we vary the metric $\mu$ nicely so that it becomes flat on $X'$, we achieve that $\alpha_{\mu,PQ}^{1,2}$ converges to $\alpha_{PQ}^{1,2}$.  

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