A Statistical Framework for Replicability

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Abstract: We introduce a novel statistical framework to study replicability which simultaneously offers overall Type-I error control, an assessment of compatibility and a combined confidence region. The approach is based on a recently proposed reverse-Bayes method for the analysis of replication success. We show how the method can be recalibrated to obtain a family of combination tests for two studies with exact overall Type-I error control. The approach avoids the double dichotomization for significance of the two-trials rule and has larger project power to detect existing effects. It gives rise to a \( p \)-value function which can be used to compute a confidence region for the underlying true effect. If the effect estimates are compatible, the resulting confidence interval is similar to the meta-analytic one, but in the presence of conflict, the confidence region splits into two disjoint intervals. The proposed approach is applied to data from the Experimental Economics Replication Project.

Key Words: Combination test; Confidence region; Conflict; Replicability; Sceptical \( p \)-value; \( P \)-value function; Type-I error control

1. Introduction

Replication plays a key role to build confidence in the scientific merit of published results and the so-called replication crisis has led to increased interest in replication studies over the last decade (Royal Netherlands Academy of Arts and Science, 2018; National Academies of Sciences, Engineering, and Medicine, 2019). These developments eventually culminated in large-scale replication projects that were conducted in various fields (Open Science Collaboration, 2015; Camerer et al., 2016, 2018; Errington et al., 2021). Deciding whether a replication is successful is, however, not a straightforward task, and different statistical methods are currently being used. For example, the
Reproducibility Project: Cancer Biology (Errington et al., 2021), an 8-year effort to replicate experiments from high-impact cancer biology paper, has used no less than seven different methods to assess replicability, including significance of both the original and replication study, compatibility of the original and replication effect estimates, and computation of a meta-analytic combined effect estimate with confidence interval.

However, all these methods have been developed for different purposes so have their limitations in the replication setting. For example, a meta-analytic combined effect estimate treats original and replication study as exchangeable, which is often questionable if the original study has not been conducted to the same standards as the replication study. The $Q$-test from meta-analysis investigates whether there is evidence for heterogeneity between original and replication studies, but does not take into account the direction nor the significance of the effect estimates. Finally, a bright-line threshold for significance makes it pointless to replicate original experiments that are just beyond that threshold, even if they constitute scientifically interesting and relevant claims of new discoveries.

The problems in the application of standard statistical methods to assess replicability have led to a new proposal for the statistical assessment of replication studies (Held, 2020a). The method combines a reverse-Bayes approach (see Held et al., 2022a, for a recent review) with a prior-predictive check for conflict and gives rise to a quantitative measure of replication success, the skeptical $p$-value. The skeptical $p$-value depends on the two study-specific $p$-values, but also on the variance ratio $c$ of the squared standard errors of the original and replication effect estimates. The method therefore treats the original and replication study not as exchangeable and specifically penalizes shrinkage of the replication effect estimate, compared to the original one. The effect size perspective has been explored further (Held et al., 2022b) to propose a modification that allows original studies with a “trend to significance” to be successful at replication, but only if the effect estimate at replication is larger than at original.
An alternative reverse-Bayes formulation based on Bayesian hypothesis testing has recently also been developed (Pawel and Held, 2022).

In this paper we study the skeptical $p$-value (Held, 2020a) from a frequentist perspective, aiming to achieve exact overall Type-I error (TIE) control for any value of $c > 0$, not necessarily linked to the variance ratio. Declaring a replication as successful if both the original and replication study are significant at level $\alpha$ is known as the two-trials rule in drug development (Senn, 2007) and has overall TIE rate of $\alpha^2$. The two-trials rule suggests to distinguish between linear and squared TIE control and is identified as a limiting case of the proposed framework for $c \to 0$. The case $c = 1$ corresponds to the harmonic mean $\chi^2$-test (Held, 2020b) where exact TIE control is also possible. We use this insight to refine the skeptical $p$-value to obtain exact overall TIE control at level $\alpha^2$ for any value of $c > 0$. This is achieved by deriving the required null distribution which gives rise to a new family of combination tests for two studies with larger project power than the two-trials rule. The approach can be used to compute confidence regions based on the corresponding $p$-value function and is particularly attractive in the reverse-Bayes setting where the variance ratio $c$ usually reduces to the relative sample size of the replication study compared to the original one. This perspective is further explored with additional power calculations, a study of the required minimum relative effect size for replication success, and an application to data from the Experimental Economics Replication Project.

**A novel criterion for replicability**

Let $\hat{\theta}_i$ denote the estimate of the unknown effect size $\theta$ and $\sigma_i$ the corresponding standard error from the original and replication study, $i \in \{o, r\}$. The squared standard errors are usually inversely proportional to the sample sizes $n_o$ and $n_r$, i.e. $\sigma^2_o = \kappa^2 / n_o$ and $\sigma^2_r = \kappa^2 / n_r$ for some unit variance $\kappa^2$. As in standard meta-analysis we assume
that the $\hat{\theta}_i$’s are independent and follow a normal distribution with mean $\theta$ and variance $\sigma_i^2$. Let $z_i = \hat{\theta}_i / \sigma_i$ denote the test statistic for the null hypothesis $H_0$: $\theta = 0$ and $p_i = 1 - \Phi(z_i)$ the corresponding one-sided $p$-value for the alternative $H_1$: $\theta > 0$, so $z_i = \Phi^{-1}(1 - p_i)$, here $\Phi(.)$ denotes the standard normal cumulative distribution function.

The general criterion for replication success is defined as follows. Replicability is achieved if
\[
\left( \frac{z_o^2}{z_u^2} - 1 \right)_+ \left( \frac{z_r^2}{z_u^2} - 1 \right)_+ \geq c
\]
holds, here $x_+ = \max\{0, x\}$, $c > 0$ is a fixed constant and $z_u > 0$ is a suitably chosen threshold. Note that a necessary but not sufficient condition for (1) to hold is $\min\{|z_o|, |z_r|\} > z_u$, as otherwise the left-hand side of (1) is zero.

In the two-sided formulation (1) is sufficient for replication success, irrespectively of the signs of the estimates $\hat{\theta}_o$ and $\hat{\theta}_r$. The one-sided formulation has the additional requirement that the two estimates are both in the same pre-specified (positive) direction. The necessary requirement $\min\{|z_o|, |z_r|\} > z_u = \Phi^{-1}(1 - \alpha_u)$ then translates to $\max\{p_o, p_r\} < \alpha_u$, so $\alpha_u$ serves as an upper threshold for the one-sided study-specific $p$-values $p_o$ and $p_r$.

The requirement (1) can be motivated from a recent proposal to define replication success with a two-step procedure (Held, 2020a): First, a significant original study at level $\alpha$ is challenged with a skeptical normal prior with mean zero and variance chosen such that the resulting posterior is no longer significant (Matthews, 2018). Secondly, the conflict between the replication study result and the skeptical prior is quantified with a prior-predictive tail probability $p_{Box}$ (Box, 1980). Replication success at level $\alpha$ is achieved if $p_{Box} \leq \alpha$. This definition turns out to be equivalent to the requirement (1) with $z_u = z_\alpha = \Phi^{-1}(1 - \alpha)$ and $c = \sigma_o^2 / \sigma_r^2$, the variance ratio original to replication. We will use this specific choice of $c$ in the replication setting but treat $c$ for the moment as a free parameter not necessarily related to the standard errors $\sigma_o$ and $\sigma_r$ of the two
For fixed $z_o$, $z_r$ and $c$ we are often interested in the smallest possible value of $z_u^2$ where (1) holds and denote this value as $z^*_u \in (0, \min\{z_o^2, z_r^2\})$, defined as the smallest positive root of
\[ (z_o^2 / z^*_u - 1) (z_r^2 / z^*_u - 1) = c. \] (2)

Any $|z_S| \geq z_u$ will hence lead to replication success, so the threshold $z_u$ in (1) can now also be interpreted as a critical value for the test statistic $z_S = \sqrt{z^*_u}$.

If both effect estimates go in the same direction, the transformation $p_S = 1 - \Phi(|z_S|)$ defines the (one-sided) skeptical $p$-value in its original formulation, otherwise it is $p_S = \Phi(|z_S|)$ (Held, 2020a). A two-sided $p$-value $\tilde{p}_S = 2 \{1 - \Phi(|z_S|)\}$ can also be considered, but is subject to the “replication paradox” (Ly et al., 2019) where replication success can occur even if the effect estimates $\hat{\theta}_o$ and $\hat{\theta}_r$ are in opposite directions.

**Overall Type-I error control**

Let $\alpha \in (0,1)$ be fixed. We say a $p$-value $p$ has linear TIE control if
\[ \Pr(p \leq \alpha \mid H_0) \leq \alpha \text{ for all } \alpha \in (0,1). \] (3)

A $p$-value that fulfills (3) is called valid (Casella and Berger, 2002). If (3) holds with equality for all $\alpha$ then $p$ has exact linear TIE control. A $p$-value with exact linear TIE control has a uniform distribution under the null.

The standard replication setting involves two studies, where it is useful to introduce $p$-values with squared TIE control. If
\[ \Pr(p_S \leq \alpha \mid H_0) \leq \alpha^2 \text{ holds for all } \alpha \in (0,1), \] (4)

then we say $p_S$ has squared TIE control. A $p$-value $p_S$ with squared TIE control also
has linear T1E control because $\alpha^2 < \alpha$ holds for all $\alpha \in (0, 1)$.

If (4) holds with equality for all $\alpha$ then $p_S$ has exact squared T1E control. A p-value with exact squared T1E control has - by definition - a triangular null distribution (i.e. a beta $\text{Be}(2,1)$ distribution) on the unit interval. With a change-of-variables it can be shown that $p_S = \sqrt{\tilde{p}}$ has exact squared T1E control, if $p$ has exact linear T1E control.

On the other hand, if $p_S$ has exact squared T1E control then $p = p_S^2$ has exact linear T1E control.

The ‘nominal’ skeptical p-value $p_S$ has the property $p_S > \max\{p_o, p_r\}$ (Held, 2020a, Sec. 3.1), so $p_S$ has squared T1E control for every value of $c > 0$. This does, however, not hold exactly and the nominal skeptical p-value has a much smaller T1E rate than the two-trials rule. A perspective on the relative effect size (Held et al., 2022b) leads to the modification $p_S = 1 - \Phi(\sqrt{\phi}|z_S|)$ using the golden ratio $\phi = (\sqrt{5} + 1)/2$. This is less stringent than the original formulation but still ensures that borderline significant original studies ($p_o \approx \alpha$) can only lead to replication success if the replication effect estimate is larger than the original one. The T1E rate of the ‘golden’ skeptical p-value is always smaller than $\alpha^2$ where

$$\alpha_S = 1 - \Phi(z_\alpha / \sqrt{\phi})$$

(5)

and still has squared T1E control if $c \geq 1$ and $\alpha \leq 0.058$ (Held et al., 2022b, Section 3.2).

In the following we will consider both one- and two-sided p-values $p$ and $\tilde{p}$, respectively $p_S$ and $\tilde{p}_S$. If they are based on two studies, then $p = \tilde{p}/4$, if both effect estimates are in the same (pre-defined) direction. This translates to $p_S = \tilde{p}_S/2$ and it is natural to set $p_S = 1 - \tilde{p}_S/2$ if the effect estimates disagree.
The two-trials rule

The two-trials rule requires significance of both studies at level $\alpha$ so corresponds to $z_o^2 \geq z_a^2$ and $z_r^2 \geq z_a^2$, where $z_a = \Phi^{-1}(1 - \alpha)$ is the critical value. This can be identified as a limiting case of (1) with $z_u = z_a$ and $c \downarrow 0$, where $z_S^2$ in (2) converges to $\min\{z_o^2, z_r^2\}$ (Held, 2020a, eq. (11)).

The largest value of $\alpha$ (respectively the smallest value of $z_a$) such that the two-trials rule is fulfilled is $p_S = \max\{p_o, p_r\}$ and the two-trials rule then translates to $p_S \leq \alpha$. Under the null hypothesis both $p_o$ and $p_r$ are uniformly distributed and it is straightforward to show that the maximum of two independent uniform random variables follows a triangular $\text{Be}(2,1)$ distribution. The variable $p_S$ is therefore a $p$-value with exact squared T1E control, so $p = p_S^2 = \max\{p_o^2, p_r^2\}$ has exact linear T1E control.

We may also consider the distribution of $Y = z_S^2 = \min\{z_o^2, z_r^2\}$ under the null, where $X_o = z_o^2$ and $X_r = z_r^2$ are independent $\chi^2(1)$ random variables. The variable $Y$ has cumulative distribution function (cdf)

$$F_0(y) = 1 - 4 (1 - \Phi(\sqrt{y}))^2$$

for $y \geq 0$, (6)

which can be shown using the fact that the cdf of the minimum $Y = \min\{X_1, X_2\}$ of two iid random variables $X_1$ and $X_2$ with cdf $F_X(x)$ has cdf $F_Y(y) = 1 - (1 - F_X(y))^2$, here $F_X(x) = 2\Phi(\sqrt{x}) - 1$ is the cdf of the $\chi^2(1)$ distribution for $x \geq 0$. Furthermore, the expectation of $Y$ can be shown to be $E(Y) = 1 - 2/\pi \approx 0.36$, see Supporting Information (SI). Now $z_S^2$ doesn’t take into account the direction of the effect estimates, so the $p$-value

$$\hat{p} = 1 - F_0(y = z_S^2) = 4 (1 - \Phi(\min\{|z_o|, |z_r|\}))^2,$$
is two-sided with exact linear T1E control. The one-sided \( p \)-value is therefore 
\[
p = \frac{\hat{p}}{4} = \left( \max\{p_o, p_r\} \right)^2 = \max\{p_o^2, p_r^2\},
\]
if both effect estimates are in the correct direction and the relationship to the one-sided \( p \)-value \( p_S = \max\{p_o, p_r\} \) with exact squared T1E control is simply \( p_S = \sqrt{p} \).

This perspective suggests a strategy for exact T1E control for any \( c > 0 \): If we are able to derive the null distribution function \( F_c(.) \) of \( z^2_S \) in (2) then we can use the transformation \( p = (1 - F_c(z^2_S))/4 \) to obtain a one-sided \( p \)-value with exact linear T1E control. The “controlled” skeptical \( p \)-value then is \( p_S = \sqrt{p} \). We first consider the special case \( c = 1 \), where the null distribution of \( z^2_S \) is particularly simple.

**The harmonic mean \( \chi^2 \)-test**

The harmonic mean \( \chi^2 \)-test (Held, 2020b) arises as a special case of (1) for \( c = 1 \). The solution of (2) then is
\[
z^2_S = z^2_{H}/2
\]
where \( z^2_{H} = 2/(1/z^2_o + 1/z^2_r) \) is the harmonic mean of the squared test statistics \( z^2_o \) and \( z^2_r \). It follows that \( z^2_S \) has a gamma \( \text{Ga}(1/2, 2) \) distribution if \( z^2_o \) and \( z^2_r \) are independent \( \chi^2(1) \) (Pillai and Meng, 2016, eq. (2.3)). The cdf \( F_{c=1}(y) \) of \( Y = z^2_S \) is thus readily available and a two-sided \( p \)-value with exact linear T1E control can be calculated:
\[
\hat{p} = 1 - F_{c=1}(y = z^2_S = z^2_{H}/2).
\]
Division by 4 gives the corresponding one-sided \( p \)-value \( p = \hat{p}/4 \) and the square root \( p_S = \sqrt{p} \) is a one-sided \( p \)-value with exact squared T1E control. This implies that thresholding \( p_S \) at \( \alpha \) will have a T1E rate of \( \alpha^2 \). The corresponding value of \( z_u \) in (1) is (Held, 2020b, Section 2.1)
\[
z_u = \Phi^{-1}(1 - 2\alpha^2)/2.
\]
For $\alpha = 0.025$ we obtain $z_u = 1.51$, which corresponds to $\alpha_u = 0.065$. The threshold $\alpha_u$ has two useful interpretations. First, it serves as a threshold for the nominal skeptical $p$-value to achieve exact T1E control at level $\alpha^2$. Secondly, a necessary but not sufficient requirement for replication success is that both $p_o$ and $p_r$ are smaller than $\alpha_u$. The two interpretations also hold for $c \neq 1$.

We note that the harmonic mean $\chi^2$-test can be extended to combine the results from $n$ studies and can also include weights (Held, 2020b).

**Exact Type-I error control for $c \neq 1$**

For $c > 0$ and $c \neq 1$ there is a unique solution of (2) that fulfills the requirement $0 \leq z^2_S \leq \min\{z^2_o, z^2_r\}$:

$$z^2_S = \frac{z^2_A}{c-1} \left\{ \sqrt{1 + (c-1)z^2_H/z^2_A} - 1 \right\}, \quad (9)$$

here $z^2_H$ is the harmonic and $z^2_A$ the arithmetic mean of the squared test statistics $z^2_o$ and $z^2_r$. Note that (9) is always non-negative and also works for $c = 0$ where we obtain $z^2_S = \min\{z^2_o, z^2_r\}$.

In what follows we derive a new “controlled” version of the skeptical $p$-value with exact squared T1E control for any $c \neq 1$, defined as the square root of a $p$-value that is a function of (9) and has exact linear T1E control. To obtain the required transformation of $z^2_S$, consider the probabilistic version of equation (2), where the random variable $Y = z^2_S$ depends on the two independent random variables $z^2_o$ and $z^2_r$. Under the null hypothesis of no effect, $z^2_o$ and $z^2_r$ are independent $\chi^2(1)$-distributed. Then $z^2_A$ and $z^2_H/z^2_A$ in (9) are also independent (Grimmett and Stirzaker, 2001, Section 4.7) which facilitates the computation of the cdf $F_c(y) = \Pr(Y \leq y \mid c)$ of $Y$ for every value of $c > 0$, $c \neq 1$:

$$F_c(y) = 1 - \frac{1}{\pi} \int_0^1 \frac{\exp\{-g(y,t,c)\}}{\sqrt{t(1-t)}} \, dt$$

(10)
where
\[
g(y, t, c) = \frac{(c - 1)y}{\sqrt{1 + (c - 1)t - 1}}.
\]
Details of the derivation of (10) are given in SI. Computation of \( F_c(y) \) is straightforward with standard numerical integration techniques and so a two-sided \( p \)-value with exact linear T1E control can be calculated for every \( c \neq 1 \):
\[
\tilde{p} = 1 - F_c(z_2^2)
\]
where \( z_2^2 \) is defined in (9). Division by 4 gives the corresponding one-sided \( p \)-value \( p = \tilde{p}/4 \) with exact linear T1E control and \( p_S = \sqrt{p} \) gives the one-sided “controlled” skeptical \( p \)-value with exact squared T1E control. The skeptical \( p \)-value \( p_S \) needs to be thresholded at \( \alpha \) to achieve overall T1E control at \( \alpha^2 \). Equivalently, we can use the adaptive threshold \( \alpha_u \) (as it depends on \( c \)) for the nominal skeptical \( p \)-value to ensure T1E control at level \( \alpha^2 \).

The case \( c \to \infty \)

For \( c \to \infty \) the two-sided \( p \)-value (11) converges to
\[
\tilde{p}_\infty \equiv \lim_{c \to \infty} \tilde{p} = \frac{1}{\pi} \int_0^1 \frac{\exp\left(-z_2^2 \sqrt{t}/\sqrt{t(1-t)}\right)}{\sqrt{t(1-t)}} \, dt,
\]
where \( z_G^2 = \sqrt{z_o^2 z_r^2} = |z_o z_r| \) is the geometric mean of the squared test statistics \( z_o^2 \) and \( z_r^2 \) (proof to be found in SI). Note that \( \tilde{p}_\infty = 1 \) if either \( z_o = 0 \) or \( z_r = 0 \), as \( f(x) = 1/\{\pi \sqrt{x(1-x)}\} \) is the density of a \( X \sim \text{Be}(1/2, 1/2) \) random variable and thus integrates to 1. Furthermore, (12) is a valid (two-sided) \( p \)-value with exact linear T1E control, i.e. \( \tilde{p}_\infty \) is uniformly distributed if \( z_o^2 \) and \( z_r^2 \) are i.i.d. \( \chi^2(1) \).

Equation (12) has some similarities to Fisher’s and Stouffer’s method for combining \( p \)-values from two studies: Fisher’s method is based on the product of the \( p \)-values,
Stouffer’s method is based on the sum of the \( z \)-values, whereas (12) is based on the product of the \( z \)-values.

**Statistical applications**

**A new family of combination tests**

The framework (1) (in the one-sided formulation) can now be used to define a family of combination tests for two studies with exact T1E control, indexed by the parameter \( c \). Specifically, we can compute for fixed \( c \) the required threshold \( \alpha_u \geq \alpha \) for the nominal skeptical \( p \)-value to achieve exact overall T1E control at level \( \alpha^2 \). The threshold \( \alpha_u \) is adaptive, as it depends on \( c \), and is a generalization of the two-trials rule with T1E rate \( \alpha^2 \) where \( \alpha_u = \alpha \). In contrast, the “nominal” and “golden” skeptical \( p \)-value are based on the (non-adaptive) threshold \( \alpha \) respectively \( \alpha_g \) from (5). We can also fix \( \alpha \) and the threshold \( \alpha_u \) and compute the value of \( c \) that achieves T1E rate \( \alpha^2 \).

The special case of the harmonic mean \( \chi^2 \)-test (\( c = 1 \)) gives \( \alpha_u = 0.065 \) for \( \alpha = 0.025 \). In practice one might want to have smaller values of \( \alpha_u \), for example \( \alpha_u = 0.035 \) where \( c = 0.1 \). Importantly, \( c \) respectively \( \alpha_u \) must be chosen before the two \( p \)-values are observed. Choosing \( \alpha_u \) after the first \( p \)-value \( p_o \) has been observed (ensuring \( \alpha_u > p_o \) may not control the T1E anymore.

Computation of the constant \( \alpha_u \) is done with numerical techniques. Briefly, the overall T1E rate for a given value of \( c \) and \( \alpha_u \) can be computed with numerical integration (Held et al., 2022b, Section 3.2). Root-finding methods are then used to find the value of \( \alpha_u \) that gives a T1E rate of \( \alpha^2 \), see the inset plot in Figure 1. One could also fix the threshold \( \alpha_u \) and compute the corresponding value of \( c \).

Figure 1 compares the success region of the proposed test for different values of \( c \). All methods control the T1E rate at \( 0.025^2 = 0.000625 \), so the area under each curve is equal to this value. The case \( c = \infty \) is based on the one-sided \( p \)-value \( \tilde{p}_{\infty}/4 \) from
The two-trials rule success region is the squared gray area below the black line and corresponds to $\alpha_u = 0.025$ where $c = 0$. The inset plot gives the upper threshold $\alpha_u$ as a function of $c$. For $c = \infty$, the upper threshold is 0.5, indicating that both effect estimates still have to be in the pre-specified direction. In contrast, both Fisher’s and Stouffer’s method can flag success even if one effect estimate is in the wrong direction (Held, 2020b).

An important operating characteristic in the replication setting is the project power (Maca et al., 2002), i.e. the success probability over both studies under the alternative hypothesis. The two studies are assumed to have the same sample size, so the distribution of both $z_o$ and $z_r$ is

\[ N(z_\alpha + z_\beta, 1), \tag{13} \]

where $1 - \beta$ is the individual power of each study to detect the true effect size at level $\alpha$ with a standard $z$-test (Matthews, 2006, Section 3.3). The project power of the proposed test can then be calculated numerically (Held et al., 2022b, Section 3.3). Table 1 reports this quantity for different values of $c$ and the individual power $1 - \beta$ at $\alpha = 0.025$. The project power is simply $(1 - \beta)^2$ for $c = 0$ (corresponding to the two-trials rule) and increases with increasing $c$.

| Individual power (%) | Project power (%) |
|----------------------|-------------------|
| $c = 0$              | $c = 0.01$        | $c = 0.1$       | $c = 1$         | $c = 10$        | $c = \infty$    |
| 80                   | 64                | 65              | 67              | 71              | 74              | 75              |
| 90                   | 81                | 82              | 84              | 87              | 89              | 90              |
| 95                   | 90                | 91              | 92              | 94              | 96              | 96              |

Table 1.: Project power of the combination test at $\alpha = 0.025$ for two equally sized studies with individual power given in the first column.
Figure 1.: Comparison of the combination test for different values of $c$. Below each line is the success region depending on the $p$-values $p_o$ (x-axis) and $p_r$ (y-axis). All methods control the T1E rate at $\alpha_2^2 = 0.025^2 = 0.000625$, so the area under each curve is equal to this value. The y-axis indicates the different values for the upper threshold $\alpha_u$ in color. The thresholds $\alpha_u$ for the cases $c = 1$ and $c = 10$ are 0.065 and 0.14, see the inset plot. The two-trials rule success region is the squared gray area below the black line with $\alpha_u = \alpha = 0.025$ where $c = 0$. 


**P-value function and confidence regions**

Up to now we have considered the test statistic $z_0$ and $z_r$ for the null hypothesis $H_0$: $\theta = 0, i \in \{o, r\}$. We now extend this and consider the generalized test statistic

$$z_i(\mu) = \frac{\hat{\theta}_i - \mu}{\sigma_i}$$

for the null hypothesis $H_0$: $\theta = \mu, i \in \{o, r\}$. The values $z_o(\mu)$ and $z_r(\mu)$ are then used to compute $z_A^2(\mu)$, $z_H^2(\mu)$ and finally $z_S^2(\mu)$ in (7) resp. (9) and so we can calculate a (two-sided) $p$-value function (Fraser, 2019)

$$\tilde{p}(\mu) = 1 - F_c(z_S^2(\mu))$$

as a function of the null value $\mu$ for every value of $c$. A confidence region at any level $\gamma$ can then be defined as the set of $\mu$ values where $\tilde{p}(\mu) \geq 1 - \gamma$. Two examples are given in Figure 5.

The confidence region can be computed with numerical root-finding techniques. However, the $p$-value function (15) is in general bimodal, with peaks at $\mu = \hat{\theta}_i, i \in \{o, r\}$. Indeed, (14) will be zero for $\mu = \hat{\theta}_i, i \in \{o, r\}$, and therefore $z_H^2(\mu)$ will be zero. Then $z_S^2(\mu)$ in (7) resp. (9) will also be zero and therefore $\tilde{p}(\mu = \hat{\theta}_i)$ will be 1 for $i = o$ and $i = r$. The $p$-value function in (15) will be smaller one elsewhere. The confidence region can therefore be a union of two disjoint intervals rather than just one interval, if there is conflict between the original and replication study effect estimates. This phenomenon is more likely to occur for smaller confidence levels $\gamma$. Taking the limit $\gamma \to 0$ shows that the proposed framework does not give one, but two point estimates equal to the effect estimates from the two studies. Whether or not the confidence region splits into two intervals depends on the minimum $\min \tilde{p} = \min_{\hat{\theta}_o < \mu < \hat{\theta}_r} \tilde{p}(\mu)$ of the $p$-value function between the two effect estimates $\hat{\theta}_o$ and $\hat{\theta}_r$. A split will occur in
case \( \min \tilde{p} < 1 - \gamma \).

If \( c \) represents the variance ratio \( c = \sigma_o^2 / \sigma_r^2 \) and we have \( c = 1 \), i.e. \( \sigma_o^2 = \sigma_r^2 = \sigma^2 \), the minimum \( \min \tilde{p} \) of the \( p \)-value function \( \tilde{p}(\mu) \) over the interval \( (\hat{\theta}_o, \hat{\theta}_r) \) is equal to the \( p \)-value from the \( Q \)-statistic. Indeed, the minimum then occurs at \( \mu = (\hat{\theta}_o + \hat{\theta}_r) / 2 \) and the squared test statistic (14) can be written as

\[
  z_i^2 = \frac{(\hat{\theta}_i - \mu)^2}{\sigma^2} = \frac{(\hat{\theta}_o - \hat{\theta}_r)^2}{4\sigma^2} \quad \text{for} \ i \in \{o, r\},
\]

and so

\[
  z_S^2(\mu) = \frac{1}{1/z_o^2 + 1/z_r^2} = \frac{1}{2} \frac{(\hat{\theta}_o - \hat{\theta}_r)^2}{4\sigma^2} = \frac{Q}{4}
\]

where \( Q \) is the \( Q \)-statistic. Now \( Q \) has a \( \chi^2(1) \)-distribution under the null and \( z_S^2(\mu) \) has a \( \text{Ga}(1/2, 2) \) null distribution, so \( 4 z_S^2(\mu) \) also has a \( \text{Ga}(1/2, 1/2) \) \( \chi^2(1) \) null distribution. For \( c \neq 1 \), \( \min \tilde{p} \) will be close, but not equal to the \( p \)-value from the \( Q \)-test.

**Application to replication studies**

We now consider the reverse-Bayes setting where we will use the variance ratio \( c = \sigma_o^2 / \sigma_r^2 \) and assume that it can be re-written as \( c = n_r / n_o \). This means that \( c \) now comes directly from the data and \( \alpha_u \) is a function of \( c \) and \( \alpha \) (as shown in Figure 1, inset plot), ensuring exact TIE control at level \( \alpha^2 \).

**Conditional and predictive power**

The power to achieve replication success, given the results from the original study, is needed for calculation of the replication sample size \( n_r = c n_o \). Conditional power assumes that the effect estimate from the original study is the true effect, whereas predictive power takes the uncertainty of the original effect estimate into account (Held,
Figure 2 (first row) shows conditional and predictive power of the controlled and golden skeptical $p$-value at $\alpha = 0.025$ and compares it to the two-trials rule for $z_o = 2$ (left) and $z_o = 4$ (right). For $z_o = 2$, the power of the controlled skeptical $p$-value is smaller compared to the two-trials rule while it is the other way round for $z_o = 4$. For each method, conditional power is larger than predictive if it is above 50%, and smaller otherwise. It is noteworthy that for a borderline significant result ($z_o = 2$) it is difficult to achieve large power values with the golden skeptical $p$-value, even for large relative sample size $c$. This is not the case for the controlled skeptical $p$-value, where the power curves are only slightly lower than for the two-trials rule. The differences between golden and controlled are less pronounced for $z_o = 4$.

### Project power

Computation of the project power over both studies assumes that the distribution of $z_o$ is as in (13). However, the sample size $n_r$ of the replication study now depends on $c$ via $n_r = c n_o$, so the distribution of $z_r$ is $z_r \sim N(\sqrt{c}(z_\alpha + z_\beta), 1)$. The power of the original study to detect the assumed effect $\theta$ is $1 - \beta$, and the power of the replication study also depends on $c$.

Figure 2 (second row) shows the project power as a function of the relative sample size $c$ for $\alpha = 0.025$ and $\beta = 0.1$. For the controlled skeptical $p$-value, the project power is smaller than with the golden level for $c < 0.85$, but larger otherwise and increases to values close to 98.3% for $c = 10$, where the golden level reaches only 93.7% project power. The project power based on the two-trials rule is always smaller and converges to 90% for large $c$.

If we assume that a replication study will only be conducted if the original study is significant ($p_o \leq 0.025$), then the project power of both the golden and controlled skeptical $p$-value converges to 90% for large $c$, but remains larger than with the two-trials rule for smaller values of $c$. 

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Figure 2.: First row: Conditional (solid) and predictive (dashed) power of the controlled and golden skeptical $p$-value and the two-trials rule (2TR) for $z_0 = 2$ (left) and $z_0 = 4$ (right). Second row: Project power as a function of the relative sample size $c$ based on all original studies (left) or the significant ones ($p_o < \alpha = 0.025$) only. Results are given for the golden and controlled skeptical $p$-value and compared with the two-trials rule (2TR).

**Minimum relative effect size**

The one-sided assessment of replication success allows to rearrange Equation (1) to a condition based on the relative effect size $d = \hat{\theta}_r / \hat{\theta}_o$ (Held et al., 2022b). Figure 3 displays the minimum relative effect size for replication success with the controlled skeptical $p$-value and the two-trials rule for different relative sample sizes $c$. As com-
pared to the two-trials rule, the controlled skeptical $p$-value has a smaller minimum relative effect size for more convincing original studies, and a larger one for less convincing ones. The success region of the controlled skeptical $p$-value does not have the strong cut-off of the two-trials rule at $\alpha = 0.025$. The largest value of $p_o$ where replication success is possible is the threshold $\alpha_u$, which increases with increasing relative sample size $c$ and is displayed in color in the top axis of Figure 3.

![Figure 3.](image)

**Figure 3.** Minimum relative effect size to achieve replication success with the two-trials rule and the controlled skeptical $p$-value for selected values of the relative sample size $c$ at $\alpha = 0.025$.

**Application to Experimental Economics Replication Project**

We now illustrate the proposed methodology using all 18 studies from the Experimental Economics replication project (Camerer et al., 2016). The different effect estim-
Figure 4: Comparison of the golden (x-axis) and controlled (y-axis) skeptical $p$-value. The vertical and horizontal lines are at the standard 0.025 threshold.

Parameters were all transformed to correlation coefficients, where Fisher’s $z$-transformation achieves asymptotically normal effect estimates $\hat{\theta}_i$ with standard errors $\sigma_i = 1/\sqrt{n_i} - 3$. Figure 4 compares the golden (x-axis) and the controlled (y-axis) skeptical $p$-value. The color of the dots indicates the relative effect size $d$, the size represents the relative sample size $c$. There is good agreement between the two $p$-values (with Kendall’s rank correlation coefficient $\tau = 0.93$) as both increase with decreasing relative effect size. For comparison, the correlation of the controlled skeptical $p$-value with $p_S = \max\{p_o, p_r\}$ from the two-trials rule is $\tau = 0.91$.

The controlled skeptical $p$-value tends to be slightly smaller than the golden one but there are 3 exceptions. One is the study by Kessler & Roth (Kessler and Roth, 2012).
where controlled $p_S = 0.003$ while golden $p_S = 0.001$. This study has a highly significant original study ($z_0 = 9.0$) and by far the smallest relative sample size among all 18 studies ($c = 0.16$). For such small values of $c$, the adaptive threshold $\alpha_u$ will be smaller than the (non-adaptive) threshold $\alpha_g = 0.062$ from (5) (compare the inset plot in Figure 1), so the controlled skeptical $p$-value will be larger than the golden one.

The study originally conducted by Ambrus & Greiner (Ambrus and Greiner, 2012), where the controlled approach leads to success at level $\alpha = 0.025$ ($p_S = 0.024$) whereas the golden approach doesn’t ($p_S = 0.049$) is of particular interest. Figure 5 (left) displays the forest plot (with 95% confidence interval for the original and replication effect, as well as the skeptical and the meta-analytic 95% confidence interval for the combined effect) and the $p$-value function for this study. The forest plot shows that the original study was not significant and there was some shrinkage of effect size so the skeptical $p$-value at the golden level could not be successful by definition. In contrast, the controlled skeptical $p$-value flags replication success at $\alpha = 0.025$. Note that one-sided controlled $p_S = 0.024$ translates into two-sided $\hat{p} = (2 \cdot 0.024)^2 = 0.002$ at $\mu = 0$, as indicated in the title of the $p$-value function plot. There is no evidence for conflict between original and replication ($p$-value from $Q$-test = 0.65, min $p = 0.66$). The 95% ‘skeptical’ confidence interval is [0.10, 0.49], its upper limit slightly larger compared to the (fixed-effect) meta-analytic one ([0.10, 0.41]). The skeptical confidence interval at level $1 - (2\alpha)^2 = 99.75\%$ is [0.002, 0.580] and does - by definition (as controlled $p_S < \alpha$) - not include zero.

As a third example we consider the original study by de Clippel et al. (de Clippel et al., 2014). Forest plot and $p$-value function are shown in Figure 5 (right). This is an example where there is clear replication success ($p_S < 0.0001$), but also conflict between original and replication study (with the replication effect estimate considerably larger than the original one) and so the skeptical confidence region splits into two intervals. In contrast, the meta-analytic pooled confidence interval is barely supported
Figure 5.: Forest plot (with 95% confidence interval for the original and replication effect, as well as the skeptical and the meta-analytic 95% confidence interval/region for the combined effect) and p-value function for the Ambrus & Greiner (Ambrus and Greiner, 2012) (left) and the de Clippel et al. (de Clippel et al., 2014) (right) study.

by the effect estimates from both studies. The p-value from the Q-test is 0.002 and the minimum of the p-value function also has this value, since the variance ratio is $c \approx 1$.

**Discussion**

We have described a novel statistical framework for the assessment of replicability. It offers exact overall Type-I error control for the assessment of replication success, a confidence region for the combined treatment effect, and an assessment of conflict between the two studies. All these aspects can be derived from the p-value function based on the criterion (1), which we are able to compute based on the null distribution.
of the quantity $z_S^2$ in (2).

The framework stems from a recently proposed reverse-Bayes approach to assess replication success (Held, 2020a), where the parameter $c$ is the variance ratio $\sigma_o^2/\sigma_r^2$, so the order of the two studies matters. The skeptical $p$-value $p_S$ can now be recalibrated to have exact squared Type-I error control. This is achieved by an adaptive level $\alpha_u$ which depends on the variance ratio $c$. The transformed $p$-value $p = p_S^2$ then has exact linear T1E control and the corresponding $p$-value function produces a confidence region fully compatible with the skeptical $p$-value. The framework thus addresses an important point raised by Diggle (Diggle, 2020) about the need to accompany the sceptical $p$-value with suitable estimation procedures to assess the relevance of the observed effects (Stahel, 2021).

The two-trials rule is a limiting case of the formulation for $c \to 0$, where $p_S$ is the maximum of the two study-specific $p$-values. We could also derive the limiting form of $p_S$ for $c \to \infty$. The success region of our formulation (both in terms of the two $p$-values and the relative sample size) can be viewed as a smoothed version of the two-trials rule, avoiding its “double dichotomization” and offering larger project power. However, T1E control through the adaptive level comes at a certain price: the explicit penalization of small relative effect sizes in the previous (non-adaptive) nominal or golden versions of the sceptical $p$-value is lost and replication success may occur even for large shrinkage of the effect estimate, as long as the relative sample size $c$ is large enough. Our conclusion is that T1E control and penalization of small effect sizes are two competing goals that cannot be achieved by a single criterion. It would be interesting to extend the recently proposed dual-criterion for replication studies (Rosenkranz, 2021), which simultaneously requires significance and relevance, to the sceptical $p$-value.

Nevertheless, there is good agreement between the controlled version and the golden version in our application to data from the Experimental Economics replication pro-
ject. Furthermore, the controlled version addresses concerns about the “stubbornness” (Ly and Wagenmakers, 2020) of the original (non-adaptive) formulation, if the original study is not particularly convincing and there is shrinkage in effect size, as exemplified by the reanalysis of the Ambrus & Greiner replication study.

This raises the question whether the controlled version is appropriate in the presence of heterogeneity between the two studies, with different underlying true effects. In addition, the original effect estimate is likely to be biased if only significant studies have been selected for replication. The present formulation assumes that the underlying effects are the same but could be extended to incorporate heterogeneity and bias if reasonable assumptions can be made about the size of it (Pawel and Held, 2020).

The paper has focused on the analysis, but we plan to investigate the design of replication studies in future work. Available methodology (Held, 2020a, Section 4.2) then needs to be extended to the proposed adaptive level. Since the adaptive level \( \alpha_u \) depends on the relative sample size \( c \), sample size calculation does not have a closed-form expression. Root-finding algorithms are therefore required to find the value of \( c \) which corresponds to the desired (conditional or predictive) power.

**Data and Software Availability** The data used in Section 1 is originally from https://osf.io/pnwuz/ and available in the R-package ReplicationSuccess, available from CRAN (Pawel and Held, 2020, supplement S1). The code used in this work can be accessed at https://gitlab.uzh.ch/charlotte.micheloud/framework-for-replicability.

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A. The null distribution of $Y = z^2_S$

Throughout we assume that $z_o$ and $z_r$ are independent standard normal variables, so $X_1 = z^2_o$ and $X_2 = z^2_r$ are i.i.d. $\sim \chi^2(1) \equiv \text{Ga}(1/2, 1/2)$.

A.1. The case $c < 1$

Consider $Y = z^2_S$ to be equal to the smallest positive root of the equation

$$\left( \frac{X_1}{Y} - 1 \right) \left( \frac{X_2}{Y} - 1 \right) = c. \quad (16)$$

After some algebra, we find for $c < 1$ the solution

$$Y = \frac{1}{2(1-c)} \left( X_1 + X_2 - \sqrt{(X_1 + X_2)^2 - 4(1-c)X_1X_2} \right)$$

$$= \frac{X_1 + X_2}{2(1-c)} \left( 1 - \sqrt{1 - 4(1-c) \frac{X_1X_2}{(X_1 + X_2)^2}} \right)$$

$$= \frac{S}{2(1-c)} \left( 1 - \sqrt{1 - 4(1-c)R(1-R)} \right)$$

$$= \frac{S}{2(1-c)} \left( 1 - \sqrt{1 - (1-c)B} \right)$$

where

$$S = X_1 + X_2, \quad R = \frac{X_1}{X_1 + X_2} \quad \text{and} \quad B = 4R(1-R).$$

A well-known result (Grimmett and Stirzaker, 2001, Section 4.7, Exercise 14) says that if $X_1$ and $X_2$ are independent such that $X_i \sim \text{Ga}(\alpha_i, \beta), i = 1, 2$, then

$$S = X_1 + X_2 \sim \text{Ga}(\alpha_1 + \alpha_2, \beta) \quad \perp \perp \quad R = \frac{X_1}{X_1 + X_2} \sim \text{Be}(\alpha_1, \alpha_2).$$
We have $X_i \sim \text{Ga}(1/2, 1/2)$ and therefore $S \sim \text{Ga}(1, 1/2) \overset{d}{=} \text{Exp}(1/2)$ and $R \sim \text{Be}(1/2, 1/2)$ are independent. The $\text{Be}(1/2, 1/2)$ is also known as the Arcsine(0, 1) distribution on the support $[0, 1]$ (Rogozin, 2001). The general arcsine distribution $\text{Arcsine}(a, a+b)$ on the support $[a, a+b]$, $a \in \mathbb{R}$ and $b > 0$, is obtained by the linear transformation $a+bX$ with $X \sim \text{Arcsine}(0, 1)$. If $b < 0$, then $a+bX \sim \text{Arcsine}(a+b, a)$. The arcsine distribution has the property

$$X \sim \text{Arcsine}(-1,1) \implies X^2 \sim \text{Arcsine}(0,1)$$

which implies that $B = 4R(1-R) = 1 - (1-2R)^2 \sim \text{Be}(1/2, 1/2)$ holds with $S$ and $B$ independent.

It is clear that $F_c(y) = 0$ for $y \leq 0$. For $y > 0$, we now obtain

$$F_c(y) = \Pr(Y \leq y) = \Pr \left( S \left( 1 - \sqrt{1 - (1-c)B} \right) \leq 2(1-c)y \right) = \int_0^1 \Pr \left( S \leq \frac{2(1-c)y}{1 - \sqrt{1 - (1-c)t}} \right) \frac{\Gamma(1)}{\Gamma(1/2)} t^{-1/2} (1-t)^{-1/2} dt = 1 - \frac{1}{\pi} \int_0^1 \exp \left( -\frac{(1-c)y}{1 - \sqrt{1 + (1-c)t}} \right) \frac{1}{\sqrt{t}(1-t)} dt \quad (17)$$

using independence of $S$ and $B$ and the fact that the cdf of $\text{Exp}(1/2)$ is given by $t \mapsto 1 - \exp(-t/2), t > 0$ and that $1/\pi \int_0^1 1/\sqrt{r(r-1)} dr = 1$ (the density of $\text{Be}(1/2, 1/2)$ integrates to 1).

**A.2. The case $c > 1$**

The solution of equation (16) is in this case

$$Y = \frac{S}{2(c-1)} \left( \sqrt{1 + (c-1)B} - 1 \right)$$
with S and B defined as before. Thus, for \( y > 0 \),

\[
F_c(y) = 1 - \frac{1}{\pi} \int_0^1 \exp \left( -\frac{(c-1)y}{\sqrt{1 + (c-1)t - 1}} \right) \frac{1}{\sqrt{t(1-t)}} \, dt. \tag{18}
\]

Figure 6 compares \( F_c(y) \) for different values of \( c \), including the special cases \( c = 0 \) and \( c = 1 \).

![Cumulative distribution function](image)

Figure 6.: Cumulative distribution function (cdf) \( F_c(y) \) of \( Y = z_S^2 \) (main plot) and expectation of \( Y \) (inset plot) for different values of \( c \).

**A.3. The expectation of \( Y \)**

For \( c = 1 \) we obtain \( E(Y) = 0.25 \) from \( Y \sim \text{Ga}(1/2, 2) \).
For $c = 0$ we have

\[
E(Y) = \min\{X_1, X_2\} \\
= \frac{1}{2}(X_1 + X_2 - |X_1 - X_2|) \\
= \frac{1}{2}(z_o^2 + z_r^2 - |z_o^2 - z_r^2|) \\
= \frac{1}{2}(z_o^2 + z_r^2 - |z_o - z_r| |z_o + z_r|).
\]

Now Cov$(z_o - z_r, z_o + z_r) = \text{Var}(z_o) - \text{Var}(z_r) = 0$ and hence $z_o - z_r$ and $z_o + z_r$ are independent N$(0, 2)$ variables. This implies that $|z_o - z_r|$ and $|z_o + z_r|$ are also independent and identically distributed according to a half-normal distribution with expectation $2/\sqrt{\pi}$. Thus,

\[
E(Y) = \frac{1}{2}(E(z_o^2) + E(z_r^2) - E(|z_o - z_r|) E(|z_o + z_r|)) \\
= \frac{1}{2}(1 + 1 - (2/\sqrt{\pi})^2) \\
= 1 - 2/\pi \approx 0.36.
\]

For $0 < c < 1$, the random variable $Y$ has cdf (17) with expectation

\[
E(Y) = \int_0^\infty (1 - F_c(y))dy \\
= \frac{1}{\pi} \int_0^1 \int_0^1 \exp \left(-\frac{(1-c)y}{1 - \sqrt{1 - (1-c)t}}\right) \frac{1}{\sqrt{t(1-t)}} dt \ dy \\
= \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{t(1-t)}} \int_0^\infty \exp \left(-\frac{(1-c)y}{1 - \sqrt{1 - (1-c)t}}\right) dy \ dt.
\]

Now

\[
\int_0^\infty \exp \left(-\frac{(1-c)y}{1 - \sqrt{1 - (1-c)t}}\right) dy = \frac{1 - \sqrt{1 - (1-c)t}}{1 - c}
\]
and therefore

\[ E(Y) = \frac{1}{\pi(1 - c)} \int_0^1 \frac{1 - \sqrt{1 - (1 - c)t}}{\sqrt{t(1 - t)}} \, dt. \]

For \( c > 1 \) we obtain with (18)

\[ E(Y) = \frac{1}{\pi(c - 1)} \int_0^1 \frac{\sqrt{1 - (c - 1)t} - 1}{\sqrt{t(1 - t)}} \, dt. \]

The expectation \( E(Y) \) is shown in the inset plot of Figure 6 as a function of \( c \).

**B. The limit of the \( p \)-value as \( c \to \infty \)**

Recall that for \( c > 1 \)

\[ Z = \frac{S}{2(c - 1)} \left( -1 + \sqrt{1 + (c - 1)B} \right) \]

with \( S = X_1 + X_2 \) and \( B = 4R(1 - R) \). The \( p \)-value \( \tilde{p}(c) = 1 - F_c(z_2^2) \) can be factorized as

\[
\tilde{p}(c) = \frac{1}{\pi} \int_0^1 \exp \left( -\frac{(c - 1)Z}{\sqrt{1 + (c - 1)t - 1}} \right) \frac{1}{\sqrt{t(1 - t)}} \, dt
\]

\[
= \frac{1}{\pi} \int_0^\eta \exp \left( -\frac{(c - 1)Z}{\sqrt{1 + (c - 1)t - 1}} \right) \frac{1}{\sqrt{t(1 - t)}} \, dt
\]

\[
+ \frac{1}{\pi} \int_{1-\eta}^{1-\frac{1}{\eta}} \exp \left( -\frac{(c - 1)Z}{\sqrt{1 + (c - 1)t - 1}} \right) \frac{1}{\sqrt{t(1 - t)}} \, dt
\]

\[
+ \frac{1}{\pi} \int_{\eta}^{1-\eta} \exp \left( -\frac{(c - 1)Z}{\sqrt{1 + (c - 1)t - 1}} \right) \frac{1}{\sqrt{t(1 - t)}} \, dt
\]

\[ = A(\eta, c) + B(\eta, c) + C(\eta, c). \]
It is easy to see that
\[
\sup_{c > 1} A(\eta, c) \leq \int_0^\eta \frac{1}{\sqrt{t(1-t)}} dt \searrow 0, \quad \text{as } \eta \searrow 0.
\]

Similarly,
\[
\sup_{c > 1} B(\eta, c) \leq \int_{1-\eta}^1 \frac{1}{\sqrt{t(1-t)}} dt \searrow 0, \quad \text{as } \eta \nearrow 1.
\]

Now, we want to show that
\[
\lim_{c \to \infty} C(\eta, c) = \frac{1}{\pi} \int_{\eta}^{1-\eta} \exp \left( -\frac{S \sqrt{B}}{2 \sqrt{t}} \right) \frac{1}{\sqrt{t(1-t)}} dt.
\]

Let us put
\[
\Delta(x, c) = \frac{1}{\pi} \int_{\eta}^{1-\eta} \left\{ \exp \left( \frac{S \left( -1 + \sqrt{1 + (c-1)B} \right)}{2 \left( \sqrt{1 + (c-1)t} - 1 \right)} \right) - \exp \left( \frac{S \sqrt{B}}{2 \sqrt{t}} \right) \right\} \frac{1}{\sqrt{t(1-t)}} dt.
\]

We show now that \( \lim_{c \to \infty} \Delta(\eta, c) = 0 \). It is enough to show that for a \( \eta > 0 \) small enough,
\[
\lim_{c \to \infty} \sup_{t \in [\eta,1-\eta]} \left| \exp \left( \frac{S \left( -1 + \sqrt{1 + (c-1)B} \right)}{2 \left( \sqrt{1 + (c-1)t} - 1 \right)} \right) - \exp \left( \frac{S \sqrt{B}}{2 \sqrt{t}} \right) \right| = 0.
\]

By the mean-value theorem, \( \exp(y) - \exp(x) = \exp(\theta_{x,y})(y - x) \) for some \( \theta_{x,y} \) between \( x \) and \( y \). Thus, \( |\exp(y) - \exp(x)| \leq \exp(\max(x,y)) |y - x| \). In what follows,
\[
x = \frac{S \sqrt{B}}{2 \sqrt{t}} \quad \text{and} \quad y = \frac{S \left( -1 + \sqrt{1 + (c-1)B} \right)}{2 \left( \sqrt{1 + (c-1)t} - 1 \right)}.
\]
We have

\[ y = S \times \frac{\sqrt{B(c-1)}}{(c-1)t} \left( \frac{\sqrt{1 + \frac{1}{(c-1)B}} - \frac{1}{\sqrt{(c-1)t}}}{\sqrt{1 + \frac{1}{\sqrt{(c-1)t}}}} \right) \]

\[ = S \times \frac{\sqrt{1 + \frac{1}{(c-1)B}} - \frac{1}{\sqrt{(c-1)t}}}{\sqrt{1 + \frac{1}{\sqrt{(c-1)t}}}} \]

\[ = x \times \frac{\sqrt{1 + \frac{1}{(c-1)B}} - \frac{1}{\sqrt{(c-1)t}}}{\sqrt{1 + \frac{1}{(c-1)t}} - \frac{1}{\sqrt{(c-1)t}}} \]

where

\[ \frac{\sqrt{1 + \frac{1}{(c-1)B}} - \frac{1}{\sqrt{(c-1)t}}}{\sqrt{1 + \frac{1}{(c-1)t}} - \frac{1}{\sqrt{(c-1)t}}} \leq \sqrt{1 + \frac{1}{(c-1)\eta} + \frac{1}{\sqrt{(c-1)\eta}}} \]

\[ \leq \sqrt{1 + \frac{1}{(c-1)\eta}} + \frac{1}{\sqrt{(c-1)\eta}} \]

\[ \leq \frac{3}{2} \]

for \( c \) large enough. It follows that

\[ |\exp(y) - \exp(x)| \leq \exp \left( \frac{3}{2} x \right) |y - x| \]

\[ \leq \exp \left( \frac{3S\sqrt{B}}{4\sqrt{\eta}} \right) \left| \frac{\sqrt{1 + \frac{1}{(c-1)B}} - \frac{1}{\sqrt{(c-1)t}}}{\sqrt{1 + \frac{1}{\sqrt{(c-1)t}}} - 1} - x \right| \]

\[ \leq \exp \left( \frac{3S\sqrt{B}}{4\sqrt{\eta}} \right) \frac{S\sqrt{B}}{2\sqrt{\eta}} \left| \frac{\sqrt{1 + \frac{1}{(c-1)B}} - \frac{1}{\sqrt{(c-1)t}}}{\sqrt{1 + \frac{1}{\sqrt{(c-1)t}}} - \frac{1}{\sqrt{(c-1)t}}} - 1 \right| \] (19)
where
\[
\frac{\sqrt{1 + \frac{1}{(c-1)B}} - \frac{1}{\sqrt{(c-1)B}}}{\sqrt{1 + \frac{1}{(c-1)t}} - \frac{1}{\sqrt{(c-1)t}}} - 1 = \frac{\sqrt{1 + \frac{1}{(c-1)\eta}} + \frac{1}{\sqrt{(c-1)\eta}}}{\sqrt{1 + \frac{1}{(c-1)t}} + \frac{1}{\sqrt{(c-1)t}}} - 1
\]

implying that
\[
\frac{\sqrt{1 + \frac{1}{(c-1)(1-\eta)}} + \frac{1}{\sqrt{(c-1)(1-\eta)}}}{\sqrt{1 + \frac{1}{(c-1)B}} + \frac{1}{\sqrt{(c-1)B}}} - 1 \leq \frac{\sqrt{1 + \frac{1}{(c-1)B}} - \frac{1}{\sqrt{(c-1)B}}}{\sqrt{1 + \frac{1}{(c-1)t}} - \frac{1}{\sqrt{(c-1)t}}} - 1 \leq \frac{\sqrt{1 + \frac{1}{(c-1)\eta}} + \frac{1}{\sqrt{(c-1)\eta}}}{\sqrt{1 + \frac{1}{(c-1)B}} + \frac{1}{\sqrt{(c-1)B}}} - 1.
\]

As the terms in both sides of the inequality converge to 0 as \(c \to \infty\) we conclude that
the absolute value of the term in the middle converges to 0 and hence the right side of
the inequality in (19) has also to converge to 0 where the convergence is uniform for
\(t \in [\eta, 1 - \eta]\). It follows that \(\lim_{c \to \infty} \Delta(\eta, c) = 0\) and hence, for \(\eta\) small enough

\[
\lim_{c \to \infty} C(\eta, c) = \frac{1}{\pi} \int_0^{1-\eta} \exp \left(-\frac{S\sqrt{B}}{2\sqrt{t}}\right) \frac{1}{\sqrt{t(1-t)}} dt.
\]

Therefore, and since \(\tilde{p}(c)\) does not depend on \(\eta\)

\[
\lim_{c \to \infty} \tilde{p}(c) = \lim_{\eta \to 0} \lim_{c \to \infty} \tilde{p}(c) = 0 + 0 + \frac{1}{\pi} \int_0^1 \exp \left(-\frac{S\sqrt{B}}{2\sqrt{t}}\right) \frac{1}{\sqrt{t(1-t)}} dt
\]

\[
= \frac{1}{\pi} \int_0^1 \exp \left(-\frac{S\sqrt{B}}{2\sqrt{t}}\right) \frac{1}{\sqrt{t(1-t)}} dt
\]

(20)

which finishes the proof because

\[
W = \frac{S\sqrt{B}}{2} = \sqrt{X_1X_2} = |z_1z_2|.
\]
C. Uniform distribution of the limiting $p$-value

We want to show that the $p$-value from (20) is uniformly distributed on $[0, 1]$. To this aim, we need to determine the density of $S\sqrt{B}/2 = |z_1z_2|$ with $z_1, z_2$ i.i.d $\sim \mathcal{N}(0, 1)$. The density of the variable $V = z_1z_2$ is

$$f_V(v) = \frac{1}{\pi} K_0(|v|)$$

with $K_0$ is second class zero order modified Bessel function (Craig, 1936). Now, the cumulative distribution of $W$ is given for $w > 0$ by $F_W(w) = 2F_V(w) - 1$, implying that density of $W$ is given by

$$f_W(w) = 2f_V(w) = \frac{2}{\pi} K_0(w), \text{ for } w > 0.$$

To show that the $p$-value from (20) is uniformly distributed on $[0, 1]$ it is enough to show that 1 minus the $p$-value from (20) is uniformly distributed on $[0, 1]$. This is equivalent to showing that the density of $W$ is also equal to the derivative of

$$w \mapsto 1 - \frac{1}{\pi} \int_0^1 \exp\left(-\frac{w}{\sqrt{t}}\right) \frac{1}{\sqrt{t}(1-t)} dt.$$

Since this derivative is

$$w \mapsto \frac{1}{\pi} \int_0^1 \exp\left(-\frac{w}{\sqrt{t}}\right) \frac{1}{t\sqrt{1-t}} dt,$$
we need to show that
\[
\frac{1}{\pi} \int_0^1 \exp \left( -\frac{w}{\sqrt{t}} \right) \frac{1}{t \sqrt{1-t}} dt = \frac{2}{\pi} K_0(w)
\]
for all \( w > 0 \) or equivalently
\[
\int_0^1 \exp \left( -\frac{w}{\sqrt{t}} \right) \frac{1}{t \sqrt{1-t}} dt = 2K_0(w).
\] (21)

It is known that for \( n > -1/2 \), the second class modified Bessel function of order \( n \) is for \( x > 0 \)
\[
K_n(x) = \frac{\sqrt{\pi}}{\Gamma(n+1/2)} \left( \frac{1}{2} \right)^n \int_1^\infty \exp(-xt)(t^2 - 1)^{n-1/2} dt.
\]

Thus, for \( n = 0 \),
\[
K_0(x) = \int_1^\infty \frac{\exp(-xt)}{\sqrt{t^2 - 1}} dt.
\]

Thus, to show the identity in (21), it is enough to show that
\[
\int_0^1 \exp \left( -\frac{w}{\sqrt{t}} \right) \frac{1}{t \sqrt{1-t}} dt = 2 \int_1^\infty \frac{\exp(-wt)}{\sqrt{t^2 - 1}} dt
\]
for all \( w > 0 \). Using the change of variable \( u = 1/\sqrt{t} \) we obtain
\[
\int_0^1 \exp \left( -\frac{w}{\sqrt{t}} \right) \frac{1}{t \sqrt{1-t}} dt = \int_1^\infty \exp(-wu)u^2 \frac{1}{\sqrt{1-u^2}} \frac{2}{u^3} du = 2 \int_1^\infty \frac{\exp(-wu)}{\sqrt{u^2 - 1}} du
\]
which completes the proof. \( \Box \)