Abstract: We point out that results of Cappell-Shaneson, Hambleton-Kreck-Teichner, and Stolz imply the existence of an inequivalent smooth structure on any closed smooth non-orientable 4-manifold with fundamental group of order two that admits a $Pin^+$-structure, as well as examples of several such structures. The study of the smooth structure on the universal covers yields existence results of orientation-reversing exotic free involutions.

1. Introduction and main result

The first example of an inequivalent smooth structure in dimension four was constructed on the topological type of the real projective 4-space in [3], where a collection of manifolds that are simply homotopy equivalent but not smoothly s-cobordant to $\mathbb{R}P^4$ was constructed (cf. Section 2.2). A corollary of this result is the existence of an orientation-reversing $DIFF$ exotic free involution on a homotopy 4-sphere [3, Theorem p. 61]. The first example of such an involution on $S^4$ was constructed in [4]. It was shown in [7] that the universal cover of the exotic non-orientable 4-manifold in [3] is diffeomorphic to $S^4$.

These results can be interpreted in terms of existence of an orientation-reversing $DIFF$ free exotic involution on the connected sum of an even number of copies of $S^2 \times S^2$. That is, the connected sum of $k - 1$ copies of $S^2 \times S^2$ with the exotic $\mathbb{R}P^4$ (see Section 2.1) is homeomorphic but not diffeomorphic to the connected sum $(k - 1)(S^2 \times S^2) \# \mathbb{R}P^4$ for $k \in \mathbb{N}$ [3]. The universal cover of these manifolds is diffeomorphic to a connected sum of an even number of copies of $S^2 \times S^2$.

Coupling the homeomorphism classification of [8] and the cut-and-paste construction of [3] with the spectral invariant and computations in [12, 10] yields the following result regarding the existence of inequivalent smooth structures on non-orientable 4-manifolds. Two smooth structures are said to be inequivalent if the corresponding smooth manifolds are homeomorphic but not diffeomorphic. We denote the connected sum of two manifolds $M_1$ and $M_2$ by $M_1 \# M_2$, the circle sum along the loop that represents the generator of the fundamental group (see Section 2.1) by $M_1 \#_{S^1} M_2$, and the real Hopf bundle over $\mathbb{R}P^2$ by $\gamma$.

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Theorem 1. Every closed smooth non-orientable 4-manifold with fundamental group of order two that admits a $\text{Pin}^+$-structure has an inequivalent smooth structure. Explicitly, there exists a manifold homeomorphic, but not diffeomorphic to 

$$S(2\gamma \oplus \mathbb{R})\#(k-1)(S^2 \times S^2) \text{ and } \#_{S^1} r \cdot (\mathbb{RP}^4)\#(k-1)(S^2 \times S^2)$$

for $1 \leq r \leq 4$ and any $k \in \mathbb{N}$.

If $r \geq 2$, then there exist at least four pairwise inequivalent smooth structures. All smooth structures are discerned by the $\eta$-invariant.

If $r \neq 4$, existence of an inequivalent smooth structure follows from [12, Theorem 7.4] (cf. Theorem 11). More than two inequivalent smooth structures on closed non-orientable 4-manifolds were not previously known to exist. Handlebodies of exotic manifolds of Theorem 1 are constructed in Section 4.

The universal covers of manifolds that correspond to inequivalent smooth structures on $\# S^1 r \cdot (\mathbb{RP}^4)\#(k-1)(S^2 \times S^2)$ (see Section 2.1, Section 3.1) are standard, i.e., they are diffeomorphic to connected sums of $n - 1$ copies of $S^2 \times S^2$ for $n \in \mathbb{N}$ (see Section 5). The following theorem concerns the exotic nature of involutions on 4-manifolds. We say that an involution is $\text{DIFF exotic}$ if the same smooth manifold is the universal cover for different smooth structures on the same homeomorphism class of the quotient space. We say that an involution is $\text{TOP exotic}$ if the same topological manifold is the universal cover for different homeomorphism classes of orbit space.

Theorem 2. There exist orientation-reversing $\text{DIFF}$ and $\text{TOP}$ exotic free involutions on the connected sum

$$(n - 1)(S^2 \times S^2)\# S^4,$$

Therefore, a closed smooth simply connected 4-manifold with zero signature and zero second Stiefel-Whitney class is homeomorphic to a 4-manifold that admits both kinds of orientation-reversing involutions.

The case of odd $n$ of Theorem 2 was proven in [4, 3, 7, 2]. The claim on existence of $\text{TOP}$ exotic involutions in Theorem 2 follows by considering connected sums of copies of $S^2 \times S^2$ with either $\mathbb{RP}^2 \times S^2, S(2\gamma \oplus \mathbb{R}), \# S^1 r \cdot \mathbb{RP}^4$, or the respective $\ast$-partner homotopy-equivalent manifold with non-trivial Kirby-Siebenmann invariant (cf. [3, 11, 8]).

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2. Two topological constructions and an invariant

2.1. Circle sums. The basic surgical construction used in this paper is the following glueing procedure along codimension three submanifolds. Along with connected sums, it is used to realize every homeomorphism class considered in Theorem 1.

Definition 3. (cf. [3, 8]) Let $M_1, M_2$ be non-orientable closed smooth 4-manifolds with fundamental group of order two such that both admit a $\text{Pin}^+$-structure. Denote the non-trivial $D^3$-bundle over $S^1$ by $D^3 \times S^1$, and fix a $\text{Pin}^+$-structure on it.
Let $i_i : D^3 \times S^1 \hookrightarrow M_i$ for $i = 1, 2$ be smooth embeddings that represent a non-trivial orientation reversing element of order two in the group $\pi_1(M_i)$ such that $i_1$ preserves the $Pin^+$-structure and $i_2$ reverses it. Define by

$$M_1 \#_S M_2 := (M_1 - i_1(D^3 \times S^1)) \cup (M_2 - i_2(D^3 \times S^1))$$

the circle sum of $M_1$ and $M_2$. The manifold $M_1 \#_S M_2$ admits a $Pin^+$-structure $\phi_1 \#_S \phi_2$.

The Seifert-van Kampen theorem implies $\pi_1(M_1 \#_S M_2) \cong \mathbb{Z}/2\mathbb{Z}$. The choices of $Pin^+$-structures are parametrized by $H^1(M_1 \#_S M_2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ [12, Corollary 6.4], therefore $\pm \phi := \pm(\phi_1 \#_S \phi_2)$ are the only $Pin^+$-structures. Regarding the bordism class of the resulting manifold, we have the following result.

**Proposition 5.** Let $(M_i, \phi_i)$ be a closed non-orientable manifold with $Pin^+$-structure $\phi_i$ for $i = 1, 2$. The circle sum $M_1 \#_S M_2$ and the connected sum $M_1 \# M_2$ are $Pin^+$-bordant to the disjoint union $(M_1, \phi_1) \sqcup (M_2, \phi_2)$.

**Proof.** The case of connected sums is proven in [12, Section 7]. To prove the claim for circle sums, we use an argument in [12, p. 160]. Denote by $\rho_i \subset M_i$ for $i = 1, 2$ the loops involved in the construction of the circle sum of Definition 3, and let $D(\rho_i)$ be its tubular neighborhood. In particular, $D(\rho_1) = D(\rho_2)$, and we denote both by $D(\rho)$. The claimed bordism is

$$M_1 \times [-1, 1] \cup_{D(\rho)} D(\rho) \times [-1, 1] \cup_{D(\rho)} M_2 \times [-1, 1].$$

\[\Box\]

### 2.2. Mapping tori, exotic $\mathbb{R}P^4$s and their universal covers.

Circle sums with mapping tori were used in [3] to unveil inequivalent smooth structures on $\mathbb{R}P^4$. We now recall the construction in [3], since it is basic for the results of Section 3.2. Take the element in the group $GL(3; \mathbb{Z})$ given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & u & v \end{pmatrix}$$

that satisfies $det A = -1$ and $det(I - A^2) = \pm 1$. Considering the 3-torus as $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, the choice of matrix yields a diffeomorphism $\varphi_A : T^3 \to T^3$ with induced map $(\varphi_A)^* = A$ on $H_1(T^3; \mathbb{Z}) = \mathbb{Z}^3$. The mapping torus $M_A$ of $\varphi_A$, i.e., the quotient of $T^3 \times [0, 1]$ under the identification $(x, 0) \sim (\varphi_A(x), 1)$, is a $T^3$-bundle over $S^1$.

Let $X$ be a closed smooth non-orientable 4-manifold with fundamental group of order two, let $\rho \subset X$ be the loop representing an element in the group $\pi_1(X)$ that reverses orientation. Denote its tubular neighborhood by $D(\rho)$. Let $M_0$ be the tubular neighborhood of the loop $(x_0 \times [0, 1]/ \sim) \subset M_A$, where $\varphi_A(x_0) = x_0$ for $x_0 \in T^3$. Construct the circle sum

$$X_A := X \#_S M_A = (X - D(\rho)) \cup_{\partial D(\rho)} (M_A - M_0)$$

of Definition 3 by glueing together $X - D(\rho)$ and $M_0$ along their common boundary.

Following a suggestion in [6], the known inequivalent smooth structures on $\mathbb{R}P^4$ [3, 4] were detected using an spectral invariant in [12, 10] (see Section 2.3). For the
remaining part of the paper, we will denote by $Q$ an exotic real projective 4-space whose universal cover is known to be diffeomorphic to the 4-sphere $\mathbb{R}P^4$. 

2.3. The $\eta$-invariant: discerning inequivalent smooth structures. Let $M$ be a 4-manifold with a Riemannian metric $g$, and a $Pin^+$-structure. The fundamental invariant of this paper is the $\eta$-invariant, $\eta(M, g, \phi)$. We refer the reader to [6, 12] for details. It is proven in [12, Proposition 4.3] that $\eta(M, g, \phi) \mod 2\mathbb{Z}$ is a $Pin^+$-bordism invariant, i.e., it depends on the $Pin^+$-bordism class of $(M, \phi)$ but not on the choice of Riemannian metric. Thus, we drop $g$ from our notation, and we denote it by

$$\eta(M, \phi).$$

The spectral invariant $\eta(M, \phi) \mod 2\mathbb{Z}$ completely determines $Pin^+$-bordism classes $[M, \phi] \in \Omega^{Pin^+} \cong \mathbb{Z}/16\mathbb{Z}$, where addition in this group is the circle sum $\# S^1$ of Definition 3 [9, 12] and $\mathbb{R}P^4$ is the generator. In particular, it was proven in [12, Theorem A] that this invariant tells apart the smooth structure of the exotic $\mathbb{R}P^4$'s [3, 4] from the standard one.

**Theorem 9.** [6, Theorem 3.3], [12, Theorem A], [10, Theorem A].

$$\eta(\mathbb{R}P^4, \phi) = \pm 1/8 \mod 2\mathbb{Z} \text{ and } \eta(Q, \phi') = \pm 7/8 \mod 2\mathbb{Z}$$

for all $Pin^+$-structures $\phi, \phi'$ on $\mathbb{R}P^4$ and $Q$ respectively.

The change of the $\eta$-invariant under the surgical procedure of Section 2.1 is

$$\eta(M \# S^1 M, \phi') = \eta(M, \phi) + \eta(MA, \phi_A) = \eta(M, \phi) + 1$$

for the corresponding $Pin^+$-structures [12, Proposition 7.3], which yields the following general result.

**Theorem 11.** [12, Theorem 7.4]. Let $M$ be a closed smooth non-orientable 4-manifold that admits a $Pin^+$-structure $\phi$. Suppose $H^1(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and $\eta(M, \phi) \neq \pm 1/2 \mod 2\mathbb{Z}$. Then,

$$\eta(M \# S^1 MA, \phi') \neq \eta(M, \phi)$$

for all $Pin^+$-structures on $M \# S^1 MA$.

Particularly, the $\eta$-invariant of circle sums and connected sums can be computed in terms of the $\eta$-invariants of the pieces using the following proposition. Its proof follows from Proposition 5 (cf. [12, Paragraph preceding Theorem 7.4]).

**Proposition 13.** The $\eta$-invariant is additive modulo $2\mathbb{Z}$ with respect to circle sums, as well as with respect to connected sums.

3. Smooth structures on non-orientable $Pin^+$ 4-manifolds

In this section we prove Theorem [1]. Constructions of inequivalent smooth structures on non-orientable 4-manifolds can be found in [3, 4, 9, 12].
3.1. Smooth structures on circle sums. Define
\[ S_r := \#_{S^1} r \cdot \mathbb{R}P^4 \]

The classification of exotic manifolds are needed to prove Theorem 2, define smooth structure. We will show that circle sums of copies of similar computation yields the values Theorem 9 and Proposition 13 imply that \( r \eta \) have cases \( 1 \leq r \leq 4 \). Since explicit constructions of exotic manifolds are needed to prove Theorem 2 we define
\[ E_r := Q\#_{S^1} ((r - 1) \cdot \mathbb{R}P^4) \]

We now distinguish \( S_r \) and \( E_r \) using specific values for their corresponding spectral invariants. The smooth structures remain inequivalent under arbitrarily many connected sums with \( S^2 \times S^2 \). Several inequivalent smooth structures are also given.

**Proposition 16.** Let \( k \in \mathbb{N} \). For all respective \( Pin^+ \)-structures \( \phi_{k-1}, \phi'_{k-1} \), we have
\[ \eta(S_r \# ((k - 1)(S^2 \times S^2), \phi_{k-1}) \neq \eta(E_r \# ((k - 1)(S^2 \times S^2), \phi'_{k-1}). \]

If \( r \geq 2 \), then \( S_r \# ((k - 1)(S^2 \times S^2) \) admits at least four pairwise inequivalent smooth structures that are distinguished by the \( \eta \)-invariant.

**Proof.** Theorem 2 and Proposition 13 imply that \( \eta(S_r, \pm \phi_r) = \pm r/8 \mod 2Z \). A similar computation yields the values
\[ \eta(Q\#_{S^1} \mathbb{R}P^4, \pm \phi'_2, 1) = \pm 1 \mod 2Z \]

for all respective \( Pin^+ \)-structures \( \pm \phi'_2, 1 \). An application of Theorem 11 and the surgery classification of Section 2.2 to \( S_2 \) and \( Q\#_{S^1} \mathbb{R}P^4 \) yield the values
\[ \eta(S_2 \#_{S^1} M_A, \pm \phi'_A) = \pm 3/4 \mod 2Z \text{ and } \eta((Q\#_{S^1} \mathbb{R}P^4)\#_{S^1} A, \pm \phi'_3) = 0 \mod 2Z \]

for all respective \( Pin^+ \)-structures \( \pm \phi'_A, \pm \phi'_3 \). Appealing to the topological classification in [3], we conclude on the existence of at least four pairwise inequivalent smooth structures on \( S_2 \). Four pairwise inequivalent smooth structures on \( S_3 \) are given by
\[ \{S_3, S_3 \#_{S^1} M_A, Q\#_{S^1} S_2, (Q\#_{S^1} S_2)\#_{S^1} M_A\}. \]

Five pairwise inequivalent smooth structures on \( S_4 \) are given by
\[ \{S_4, Q\#_{S^1} S_3, (Q\#_{S^1} S_3)\#_{S^1} M_A, \#_{S^1} 2 \cdot Q\#_{S^1} S_2, (\#_{S^1} 2 \cdot Q\#_{S^1} S_2)\#_{S^1} M_A\} \]

Since by Theorem 11 and Proposition 13 the values of the \( \eta \)-invariant are different for both \( Pin^+ \)-structures, the smooth structures are inequivalent. The \( \eta \)-invariant of \((k - 1)(S^2 \times S^2) \) is zero for any \( k \in \mathbb{N} \), and spin structure \( \phi_{k-1} \), since \( \eta(k - 1)(S^2 \times S^2, \phi_{k-1}) = 1/16 \cdot \sigma((k - 1)(S^2 \times S^2)) = 0 \mod 2Z \) by Proposition 13 and [12, Corollary 5.2]. Thus, the smooth structures remain inequivalent under connected sums with arbitrarily many copies of \( S^2 \times S^2 \).

To conclude the existence of a homeomorphism, we use an argument in [3] Section 3, [12, Section 7]. Fix an \( r \). The manifolds constructed are homotopy equivalent to \( S_r \), and [3, Theorem 3.1] implies that there exists a simple homotopy equivalence; the Whitehead group \( Wh(Z/2Z) \) is trivial (see [13, Proof of Theorem 13 A.1]). The surgery exact sequence
\[ \rightarrow L_5(Z\pi_1(S_r), \omega_1(S_r)) \rightarrow S^{Top}(S_r) \rightarrow [S_r, G/TOP] \rightarrow \]
Theorem 10.3, Theorem 10.5] implies the existence of an $s$-cobordism between any two of these manifolds. Since the surgery group vanishes [13, Theorem 13.1], it follows from [5] that these manifolds are homeomorphic. One can alternatively appeal to the topological classification in [8, Theorem 3] to conclude the existence of the homeomorphism.

3.2. Smooth structures on an $S^2$-bundle over $\mathbb{RP}^2$. The sphere bundle of the Whitney sum of two copies of the real Hopf bundle and the trivial bundle is the boundary of a five dimensional disk bundle that has a $Pin^+$-structure. Therefore, $S(2\gamma \oplus \mathbb{R})$ represents the zero element in $\Omega^4_{Pin^+}$. Consider the decomposition

$$\mathbb{RP}^4 = (D^2 \times \mathbb{RP}^2) \cup (D^3 \times S^1)$$

as the union of the twisted 2-disk bundle over the real projective plane and the twisted 3-disk bundle over the circle. The $S^2$-bundle over $\mathbb{RP}^2$ is the double

$$S(2\gamma \oplus \mathbb{R}) = (D^2 \times \mathbb{RP}^2) \cup id (D^2 \times \mathbb{RP}^2).$$

Denote by $\mathbb{RP}^4$ the real projective 4-space endowed with the $Pin^+$-structure $-\phi$; $\mathbb{RP}^4$ represents the bordism class $[\mathbb{RP}^4, -\phi] = -[\mathbb{RP}^4, \phi] \in \Omega^4_{Pin^+}$. The double is the circle sum $S(2\gamma \oplus \mathbb{R}) = \mathbb{RP}^4 \#_1 \mathbb{RP}^4$, whose $\eta$-invariant is zero for both $Pin^+$-structures. Theorem [11] implies that $S(2\gamma \oplus \mathbb{R}) \#_1 M_A$ is not diffeomorphic to $S(2\gamma \oplus \mathbb{R})$. We proceed to argue the existence of at least two more inequivalent smooth structure. Consider the manifold

$$F := (Q - D^3 \times S^1) \cup id (\mathbb{RP}^4 - D^3 \times S^1),$$

i.e., $Q \#_1 \mathbb{RP}^4$ with

$$\eta(F, \phi_F) = \eta(Q, \phi_Q) - \eta(\mathbb{RP}^4, g, \phi) = 3/4$$

modulo $2\mathbb{Z}$ by Theorem [9] and Proposition [13]. Thus, $\eta(F, \pm \phi_F) = \pm 3/4$ mod $2\mathbb{Z}$. Theorem [11] now implies that a fourth inequivalent smooth structure is $F \#_1 M_A$, where $M_A$ is the mapping torus of Section 2.1. An argument similar to the one used in the proof of Proposition [16] allows us to conclude the following result.

Proposition 25. There exist at least four pairwise non-diffeomorphic manifolds that are all homeomorphic to

$$S(2\gamma \oplus \mathbb{R}) \# (k - 1)(S^2 \times S^2)$$

for every $k \in \mathbb{N}$.

4. Non-orientable handlebodies

The handlebody structure of an exotic $\mathbb{RP}^4$ was analyzed in [1], where it is shown that it decomposes as the circle sum of an exotic 2-disk bundle over $\mathbb{RP}^2$ and $D^3 \times S^1$. The construction of exotic manifolds of Theorem [1] can be expressed as blowing up an $\mathbb{RP}^2$ as in [1] Section 0, thus we construct their handlebodies building greatly upon [1]. The handlebody of $S(2\gamma \oplus \mathbb{R}) = D^2 \times \mathbb{RP}^2 \cup id D^2 \times \mathbb{RP}^2$ is given in Figure 1. The same figure with the $p_2$- and $q_2$-framed 2-handle removed yields a handlebody of $D^2 \times \mathbb{RP}^2$, provided $p_i + q_i$ is odd [1] Section 0.

A handlebody of an exotic copy of $S(2\gamma \oplus \mathbb{R})$ is given in Figure 2, which is constructed by turning the handlebody of $D^2 \times \mathbb{RP}^2$ upside down and adjoining
it to the handlebody of the exotic 2-disk bundle over the real projective plane constructed in Figure 1.33]. Different choice of gluing map between the common boundary of $D^2 \times \mathbb{R}P^2$ and its exotic copy yields a handlebody for the inequivalent smooth structures on the manifolds $\#_{S^1 \cdot \mathbb{R}} \mathbb{R}P^4$.

5. Smooth structures on the universal covers

In this section we show that the smooth structure of the universal covers of the manifolds that were constructed using circle sums is standard, thus concluding the proof of Theorem 2.
Proposition 27. The universal cover \( \pi : \tilde{M}_{n-1} \to \#_n \cdot Q \) is diffeomorphic to the connected sum \((n-1)(S^2 \times S^2) \# S^4\) for \(n \in \mathbb{N}\). The same conclusion holds for hybrid circle sums of copies of \(Q\) and \(\mathbb{R}P^4\).

Proof. It is proven in [3][7][2] that \(\tilde{M}_0\) is diffeomorphic to \(S^4\). Suppose \(n = 2\), and consider \(\pi : \tilde{M}_1 \to Q \# S^1 \cdot Q\). Following notation of [3], let \(\rho\) be the nontrivial loop generating \(\pi_1(Q)\), and denote by \(D(\rho)\) its tubular neighborhood. The circle sum can then be written as

\[
(Q - D(\rho)) \cup (Q - D(\rho)).
\]

The symbol "\(\approx\)" denotes the existence of a diffeomorphism. Using the hypothesis on the manifold \(Q\), we have

\[
\tilde{M}_1 \approx (S^4 - \pi^{-1}(D(\rho))) \cup (S^4 - \pi^{-1}(D(\rho))),
\]

since the corresponding loops inside \(S^4\) are isotopic. We are abusing notation, denoting any covering map by \(\pi\). Decomposing the 4-sphere as

\[
S^4 \approx \partial D^5 \approx \partial(D^3 \times D^2) \approx (D^3 \times S^1) \cup (S^2 \times D^2),
\]

and substituting in (29) we obtain

\[
\tilde{M}_1 \approx (S^2 \times D^2) \cup (S^2 \times D^2) \approx S^2 \times S^2,
\]
as it was claimed.

Assume \(n = 3\), and consider

\[
\tilde{M}_2 \to (Q \# S^1 \cdot Q) \# S^1 \cdot Q.
\]

Decompose the circle sum associatively as

\[
((Q \# S^1 \cdot Q) - D(\rho)) \cup (Q - D(\rho)).
\]

Employing the hypothesis on \(Q\) and [3][7] we have the decomposition

\[
\tilde{M}_2 \approx (S^2 \times S^2 - \pi^{-1}(D(\rho))) \cup (S^4 - \pi^{-1}(D(\rho))).
\]

Write

\[
(S^2 \times S^2 - \pi^{-1}(D(\rho))) \cup (S^2 \times D^2)
\]
as

\[
((S^3 - D^3 \cup D^3)) \cup (D^2 \times S^2 \cup S^2 \times D^2) - \pi^{-1}(D(\rho))) \cup (S^2 \times D^2) \approx
\]

\[
(S^1 \times (S^3 - D^3 \cup D^3)) \cup (D^2 \times (S^2 \cup S^2 \cup S^2)) \approx
\]

\[
(S^2 \times S^2) \# (S^2 \times S^2).
\]

The remaining cases follow from an iteration of the previous arguments to the decomposition

\[
m(S^2 \times S^2) \approx (S^1 \times (S^3 - D^3 \cup \cdots \cup D^3)) \cup (D^2 \times (S^2 \cup \cdots \cup S^2)),
\]

where all gluing diffeomorphisms are the corresponding identity maps.

\[\square\]
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