Gravitomagnetism and Non–commutative Geometry

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Abstract

Similarity between the gravitoelectromagnetism and the electromagnetism is discussed. We show that the gravitomagnetic field (similar to the magnetic field) can be equivalent to the non–commutative effect of the momentum sector of the phase space when one maintains only the first order of the non–commutative parameters. This is performed through two approaches. In one approach, by employing the Feynman proof, the existence of a Lorentz–like force in the gravitoelectromagnetism is indicated. The appearance of such a force is subjected to the slow motion and the weak field approximations for stationary fields. The analogy between this Lorentz–like force and the motion equation of a test particle in a non–commutative space leads to the mentioned equivalency. In fact, this equivalency is achieved by the comparison of the two motion equations. In the other and quietly independent approach, we demonstrate that a gravitomagnetic background can be treated as a Dirac constraint. That is, the gravitoelectromagnetic field can be regarded as a constrained system from the sense of the Dirac theory. Indeed, the application of the Dirac formalism for the gravitoelectromagnetic field reveals that the phase space coordinates have non–commutative structure from the view of the Dirac bracket. Particularly, the gravitomagnetic field as a weak field induces the non–trivial Dirac bracket of the momentum sector which displays the non–commutativity.

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1 Introduction

The analogy between gravitation and electromagnetism originates in the similarity between the Coulomb law of electricity and the Newton law of gravitation. Actually, the analogous idea of the electric theory and the Newtonian gravitational theory, which inspires a Maxwell–type of gravitational theory, is dated back to the second half of the nineteenth century [1]–[4]. The motivation that the motion of a mass can generate a field analogous to the magnetic field is emerged from the fact that the magnetic field is produced by the electric current. Indeed, the mass current produces a field called gravitomagnetic (GM) field. The introduction of GM field as an analog to the magnetic field comes from the need for knowing the force exerted by a moving body based on the intriguing interplay between geometry and dynamics, as emphasized by Sciama [5]. Of course, such an analogy is necessarily incomplete, for instance, unlike the electric charge, the mass charges are not invariant nor additive [6]. Thus, a more natural framework where this topic can be developed, is the theory of general relativity. The first general investigation of the GM field within this theory is due to Thirring [7]. In fact, the introduction of GM field is unavoidable when one brings the Newtonian

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gravitational theory and the Lorentz invariance together in a consistent framework. This effect is
the usual Maxwellian feature which has Machian root, see, e.g., Ref. [8]. Dynamical equations for a
weak gravitational field, similar to the Maxwell equations, has been deduced by the parameterized
post Newtonian formalism [9]. Also, more attention has been made in the analogy between general
relativity and electromagnetism for slowly motion in weak gravitational fields [10].

There are other dissimilarities in this analogy. For example, a negative mass charge has not
been detected yet and, the like mass charges attract rather than repelling each others. Also, there
are issues which partially are stated as the weak version of the equivalence principle, that is, the
gravitational field is coupled to everything and/or all forms of energy act as sources of the gravitational
field. On the other hand, there is another (outward) type of similarity between the GM and magnetic
fields. This similarity is related to the effect of non–commutative (NC) parameters on the motion
equation of a particle in a NC space.

The canonical coordinates and momenta, as independent variables, have equal status and can be
designated as the physical quantities. They are both needed to describe the motion of the system
in the Hamiltonian formulation, and works have been performed in devising schemes that result
in entirely symmetric equations. That is, the distinction between these two classes of variables is
basically one of nomenclature. In this aspect, the NC geometry has a genuine base and can be viewed
as a manifestation of this respect, in which it presents a newer and perhaps deeper insight into the
physical contents of the nature (phenomena). Indeed, when the geometry of the phase space is NC,
the motion equation contains terms including the NC parameters which appear as additional force
terms with respect to the usual space. Also, when in the usual space, an electromagnetic field is
present, the Lorentzian force gives the motion equation of a moving charged particle. Under special
conditions, by comparing these motion equations (in the NC and usual spaces), one can deduce that
the effect of the NC parameters on the motion equation is equivalent to the effect of the magnetic field
in the usual space. This equivalency has been demonstrated in the literature, e.g., in the classical [14]
and in the quantum [15] perspectives.

In this work, we purpose to investigate the same equivalency for the GM field and the NC ef-
facts, and we will show that this is possible if a Lorentz–like force exists for the gravitoelectro-
magnetic (GEM) field. Actually, such possibility can be demonstrated by employing the Feynman
proof (FP) for the gravitational field. In 1948, Feynman gave a proof2 of the Maxwell equations by
assuming only the Newton law of motion and the commutation relation between the position and the
velocity of a single non–relativistic particle. Interestingly, his proof leads to the relativistic result,
that is the existence of a Lorentz–like force for the motion of a particle. Also, a more important
result of the proof is that, the corresponding electric and magnetic like fields satisfy the homogenous
Maxwell equations. A glance at the proof illustrates that, it is not restricted only to the electromagnetic
interactions, but also includes other interactions, among them, the gravitational one. Of course,
the implications of the FP to the GM interaction is subjected to special conditions [17].

Therefore, in this approach by using the FP, we first derive the motion equation of a particle in
the presence of the GM field and then, compare this equation with the equation motion in the NC
space in order to illustrate the mentioned equivalency. Note that, in this approach, we deal with two
different spaces, where in one of them (the usual space) an external GM field is present, and in the
other one (the NC space) there is not. Incidentally, the equations of motion must be written in the
canonical (Hamiltonian) form, as the functions3 in the NC frame, are defined on a relevant phase
space. Also, we consider the non–commutativity in the both sectors of the phase space, namely the
space sector and the momentum sector.

On the other hand, there is another straightforward approach [18] which shows, from the
geometrical point of view, that the phase space of the electromagnetic field has NC structure. In

1However, there are some interpretation on the cosmological constant (associated with dark energy) that acts effect-
ively as a repulsive (gravity) force, particularly on the large scale. In this issue, see, e.g., Refs. [11, 12, 13].

2This proof was never published by Feynman [16].

3The functions that appear in the definition of the NC product.
other words, unlike the previous approach that is based on the comparison of the motion equations, the NC structure of the phase space can be shown directly. Furthermore, for a classical charged particle in an electromagnetic field, one can display [18], under the special condition and by a new definition of the Poisson bracket, that the phase space variables no longer commute in the sense of the new bracket. Indeed, it has been demonstrated that the new bracket describes the motion equations with respect to the new Hamiltonian [18]. This approach is based on the Dirac theory [19] which is the extension of the Hamiltonian formalism for a dynamical system including constraints. Of course, it has been indicated that, in the presence of an external background magnetic field, the NC coordinates can naturally be introduced [20], that is, the (quantum) commutator of the phase space coordinates are no longer the Poisson commutator.

We show that the Dirac theory can also be applied for the GEM field provided that the particle velocity, say \( v \), is sufficiently slow such that one can neglect the kinetic term in the Lagrangian. It is in this limit that the gravitational system can be regarded as a constrained system, and therefore, the Dirac bracket reveals the NC structure of the phase space.

This work is organized as follows. In the next section, we describe the main ideas of the NC geometry needed for the work, and give a brief formulation of the classical mechanics in this frame. In Section 3, the FP and its application for the GM field is introduced. The main purpose of this work, i.e. the equivalency between the GM field and the NC effects, is performed in Section 4. In Section 5, we provide a classical formulation of the gravitoelectromagnetism based on the Dirac theory. The conclusions are presented in the last section.

2 Non–commutative Classical Mechanics

The NC geometry has played an increasingly important role, more notably, in the attempts to understand the space–time structure at very small distances. The NC structure idea at small length scale first introduced by Snyder [21]. He applied the NC structure concept for discrete space–time coordinates instead of the continuum ones. From the quantum theory point of view, the coordinates operators of a NC space–time do not satisfy the usual commutation relations, i.e. the usual \( [\hat{x}^\mu, \hat{x}^\nu] = 0 \), and instead obey non–trivial ones. The commutation relations are defined with respect to the notion of * (or NC) product\(^5\) such that for any two different operators, e.g. \( A \) and \( B \), one has, in general, \( A \ast B \neq B \ast A \).

The NC concept is not limited only to the space–time operators, and can be extended to the phase space classical variables as well. In this respect, the versions of classical mechanics, e.g. Ref. [14], and ordinary quantum mechanics, e.g. Ref. [15], have been studied in terms of the NC geometry.

It is well–known that the passage from the commutative to NC frame is simply achieved by replacing the ordinary product by the * product, as has been performed in the literature, see, e.g., Refs. [20]–[24]. Our approach to the NC formalism is also based on this passage. That is, for considering a physical theory in the NC frame, one should take its classical version defined on a commutating frame and then, replaces the usual product by the NC product. In the classical physics, the non–commutativity can be described by this product, in where the product law is defined between two arbitrary (infinitely differentiable) functions, e.g. \( f \) and \( g \), of the phase space variables (that is, \( \zeta^a = (x^i, p^j) \) for \( i, j = 1, \cdots, n \)) as [20]

\[
(f \ast g)(\zeta) = \exp \left[ \frac{1}{2} \alpha^{ab}_{\ 0} \partial_0 \partial_1 \right] f(\zeta_1)g(\zeta_2) |_{\zeta_1 = \zeta_2 = \zeta}, \tag{1}
\]

where \( a, b = 1, 2, \cdots, 2n \) and \( 2n \) is the dimension of the phase space. The real matrix \( \alpha_{ab} \) is the generalized symplectic structure and can be written as

\[
\alpha_{ab} = \begin{pmatrix}
\theta_{ij} & \delta_{ij} + \sigma_{ij} \\
-\delta_{ij} - \sigma_{ij} & \beta_{ij}
\end{pmatrix} \tag{2}
\]

\(^4\)That is, the ratio \( v^2/c^2 \) is ignored against the unity.

\(^5\)This product is usually called the Weyl–Moyal product.
The constant antisymmetric components $\theta_{ij}$ and $\beta_{ij}$ are called the NC parameters of the space and momentum sectors, respectively, which can be written in terms of the Levi–Civita antisymmetric tensor, namely

$$\theta_{ij} = \varepsilon_{ijk} \theta_k$$ and $$\beta_{ij} = \varepsilon_{ijk} \beta_k. \quad (3)$$

The real parameters $\theta_k$ and $\beta_k$ are usually assumed to be very small, and hence, we consider them up to the first order. The third parameter $\sigma_{ij}$ can be written as combination (product) of the other two parameters and hence, can be ignored up to the first order. Obviously, the defined product (1) is associative, but not commutative, hence the modified Poisson bracket can be written as

$$\{f, g\} = f * g - g * f. \quad (4)$$

Thus, it is easy to show that

$$\{x_i, x_j\} = \theta_{ij}, \quad \{x_i, p_j\} = \delta_{ij} + \sigma_{ij} \quad \text{and} \quad \{p_i, p_j\} = \beta_{ij}. \quad (5)$$

A simple way to study a physical theory within the NC geometry is the replacement of the Moyal product with the ordinary multiplication, when one considers the following non–canonical transformation on the classical phase space.

$$x'_i = x_i - \frac{1}{2} \theta_{ij} p^j \quad \text{and} \quad p'_i = p_i + \frac{1}{2} \beta_{ij} x^j. \quad (6)$$

By which, the usual Poisson brackets of the primed variables give

$$\{x'_i, x'_j\} = \theta_{ij}, \quad \{x'_i, p'_j\} = \delta_{ij} + \sigma_{ij} \quad \text{and} \quad \{p'_i, p'_j\} = \beta_{ij}, \quad (7)$$

with $\sigma_{ij} = -\theta_{k(i(\beta_{j)l})} \delta^{kl}/4$. The commutation relations (7) are the same as (5), and consequently, for introducing non–commutativity, it is more convenient to work with the Poisson brackets (7) than the modified brackets (5). It is important to note that, the relations represented by equations (5) are defined in the spirit of the Moyal product given above, however, in the relations defined by (6) and used in (7), the variables $(x_i, p_j)$ obey the usual Poisson bracket. Hence, the two sets of the deformed and ordinary Poisson brackets must be considered as distinct.

We assume that one has a symplectic structure consistent with the commutation rules (7) and then, obtain the corresponding equations of motion, where it can be understood as the motion equation of a particle in the NC space. Also, it is assumed that the functional form of the Hamiltonian in the commutative and NC cases are in the same form, i.e.

$$H(x', p') = \frac{p'^2}{2m} + V(x'), \quad (8)$$

where the coordinates $x'_i$ and the momenta $p'_j$ ($i = 1, 2, 3$) yield brackets (7). The Hamiltonian equations are

$$\dot{x}'_i = \{x'_i, H\} \quad \text{and} \quad \dot{p}'_i = \{p'_i, H\}, \quad (9)$$

which easily govern the NC dynamics, up to the first order, as

$$\dot{x}'_i = \frac{p'_i}{m} + \theta_{ij} \frac{\partial V}{\partial x'_j} + \frac{\sigma_{ij} p^j}{m} \approx \frac{p'_i}{m} + \theta_{ij} \frac{\partial V}{\partial x'_j},$$

$$\dot{p}'_i = -\frac{\partial V}{\partial x^n} + \frac{\beta_{ij} p^j}{m} - \frac{\sigma_{ij} \partial V}{\partial x'_j} \approx -\frac{\partial V}{\partial x^n} + \frac{\beta_{ij} p^j}{m}. \quad (10)$$

6The variables of the classical phase space obey the usual Poisson brackets, namely $\{x_i, x_j\} = 0 = \{p_i, p_j\}$ and $\{x_i, p_j\} = \delta_{ij}$. 

4
By eliminating the momentum variables, one gets

\[ m\ddot{x}'_i \simeq -\frac{\partial V}{\partial x}'_i + \left[ m\theta_{ij} \left( \frac{\partial^2 V}{\partial x'^k \partial x'_j} \right) + \beta_{ik} \right] \dot{x}'^k. \] (11)

The first term in the right hand side (11) is the usual conservative force. The second and third terms, which depend on the velocity, are arisen from the non–commutativity and can be interpreted as additional forces. It means that, the NC effects can be equivalent with imposing the additional forces on the usual (commutative) space.

In the case of a free particle, as the simplest example, equation (11) reads

\[ m\ddot{x}'_i \simeq \beta_{ik} \dot{x}'^k = \varepsilon_{ikj} \dot{x}'^j \beta^j = (\mathbf{v}' \times \beta)_i, \]

where we take \( \beta = (\beta^i) \). This equation is similar to the motion equation of a classical moving charged particle \( q \) in the presence of a magnetic field \( \mathbf{B}_m \), i.e. \( m\ddot{x}_i = q(\mathbf{v} \times \mathbf{B}_m)_i \) in the SI units. Thus, the effect of the NC parameters \( \beta^j \) in the NC space is similar to the presence of a magnetic field in the usual space. The comparison between these two equations, gives the magnitude of the NC parameter corresponding to the magnetic field, namely

\[ \mathbf{B}_m = \beta/q. \] (12)

Indeed, equation (12) represents the equivalency of the NC effects and the magnetic force field.

The above example motivates the following issue. As the GM field is the analogous of the magnetic field in the gravitoelectromagnetism, there would also be the corresponding equivalency of the GM field and the NC parameters. In the explicit words, one expects that the NC effects can be interpreted as the impose of the GM field in the usual space. Equivalently, it is interesting to find out what is the relevance of the gravitomagnetism from the NC geometry effects. In the next two sections, we probe this modification, though first, in Section 3, we investigate the Lorentz–like behavior of the GM field.

### 3 Gravitomagnetism and Feynman Proof

In the absence of the electromagnetic interactions, the geodesic equation for particles position is obviously valid. For instance, in the particular case of a rotating sphere, when the space–time metric is given by the Kerr metric in the Boyer–Lindquist coordinates [26], the gravitational Lorentz force can easily be obtained by using the geodesic equation [27] \( \ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \). In other words, the existence of a Lorentz–like force law is also possible for the gravitational interactions under the special conditions. Indeed, Feynman gave the proof of the Maxwell equations independent of the electromagnetic theory, just by assuming the Newtonian law of motion and the commutation relation between the position and the velocity for a single non–relativistic particle. The details of this approach have been presented in the Dyson work [28]. It is important to emphasize that the proof reproduces only the homogeneous Maxwell equations (i.e. the two free source equations), which are compatible with the Galilean relativity [29].

Let us make a concise review on the FP without mathematical rigorous proof of the theorem, though for a complete demonstration, see, e.g., Refs. [16, 30, 31]. Incidentally, we consider the FP of the Maxwell equations in the classical form and follow the Dyson approach [28].

The proof takes the classical and quantum concepts for a single non–relativistic particle, and leads to the Lorentz–like force law and the homogeneous Maxwell equations. The result seems somehow strange, for by starting with a classical equation (the Newtonian law), one will end up with the relativistic one. The basic assumptions of the proof consist of

- The Newtonian second law,
- The Galilean relativity,
The commutation relations between the position and the velocity of a particle.

Consider a particle whose the position $x_i$ and the velocity $\dot{x}_i$ satisfy the Newtonian second law

$$m \ddot{x}_i = F_i(x, \dot{x}, t), \quad (13)$$

and also obey the commutation relations

$$[x_i, x_j] = 0 \quad \text{and} \quad m[x_i, \dot{x}_j] = \delta_{ij}. \quad (14)$$

Hence, it can be shown that there exist an $E(x, t)$ field and a $B(x, t)$ field such that

$$F_i = E_i + \varepsilon_{ijk} \dot{x}_j B^k, \quad (15)$$

with

$$\nabla \cdot B = 0 \quad \text{and} \quad \nabla \times E + \frac{\partial B}{\partial t} = 0, \quad (16)$$

in the SI units. As it is obvious, this theorem not only includes the electromagnetic phenomena, but also discusses more general cases, even it can be extended to the case of non–Abelian gauge fields, both in the Newtonian [32] as well as in the relativistic dynamics [16] and the dynamical equations of spinning particles [31]. Also, it can be shown that the FP is applicable to the GEM phenomena in the case of stationary fields [17].

4 Gravitomagnetic Field and Non–commutative Effects

In the above discussion, we consider a moving mass particle in the gravitational field when the stationary weak field and the slow motion approximations are satisfied. Thus, equation (15) can be viewed as the motion equation of the particle in the gravitational field, and hence, the first and second terms in the right hand side of (15) exhibit the gravitoelectric and the GM forces, respectively. That is, equation (15) can be re–written in the form

$$m \ddot{x}_i = -\frac{\partial V}{\partial x^i} + m \varepsilon_{ijk} \dot{x}^j B^k_{gm}, \quad (17)$$

where $E_i = -\partial V/\partial x^i$ (with $V$ as the Newtonian weak potential), and $B^i_{gm} = B^i/m$ is the corresponding GM field.

On the other hand, in the case of stationary fields, potentials are linear in terms of the coordinates, and if one substitutes such a potential in the NC motion equation (11), the second term will be dropped, and one attains

$$m \ddot{x}_i = -\frac{\partial V}{\partial x^i} + \varepsilon_{ijk} \beta^k \dot{x}^j. \quad (18)$$

Though we have dropped the primes, but we should remind that this equation describes the equation of motion of a particle in the NC space. By comparing the second terms on the right hand sides of equations (17) and (18), one is led to the following statement:

*When the geometry of the phase space is NC, the force term, that comes from the non–commutativity of the momentum sector, is equivalent to the exertion of the GM force in the usual space.*

Also, if one formally equates these terms, one will get

$$B_{gm} = \beta/m, \quad (19)$$

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7 In the case of classical mechanics, and when the equations of motion are canonical, the commutation relations can be considered as the usual Poisson brackets [16, 30].

8 In the non–stationary case, the electric–like and the magnetic–like fields in equation (15) do not satisfy the two Maxwell–like equations [33].
which represents the gravitational counterpart of equation (12). Therefore, the NC effects of the momentum sector is analogous to the presence of the GM field. Another and a more accurate statement of the above result can be expressed as follows:

The effect of NC parameter of the momentum sector for a static mass distribution in a NC space is equivalent to when the same mass distribution is in uniform motion in the usual space.

The following thought experiment, as a simple example, can be instructive and, lends more justification for the subject.

Suppose, in the usual space, a constant distribution of mass surface density \( \sigma \), say, in the \( xy \)-plane, is moving in the \( x \)-direction with constant speed \( u \). With the calculation of the corresponding electromagnetic problem [34], or the direct calculation [35], one can show that the gravitational Lorentzian force on a mass particle \( m \) that moves in the \( x \)-direction is

\[
F = F_{ge} + F_{gm} = -2\pi G m \sigma \hat{z} + 4\pi G m \sigma u_x \hat{z}/c^2,
\]

where \( v_x \) is the \( x \)-component of the particle velocity and \( F_{ge} \) and \( F_{gm} \) are the gravitoelectric and the GM forces, respectively. In accord with equation (17), the GM field can easily be found to be

\[
B_{gm} = 4\pi G \sigma u \hat{y}/c^2.
\]

By equation (19), the NC parameter \( \beta \) can be described in terms of the statical and the kinematical quantities of the subject, namely

\[
\beta = 4\pi G m \sigma u/c^2.
\]

Note that, in the NC space, the mass distribution (in the \( xy \)-plane) is in the rest, but still the same force acts on the particle. This fact is concealed in the dependency of the NC parameter on the velocity of the mass distribution.

5 Gravitomagnetism and Dirac Bracket

Canonical variables of the phase space obey the usual Poisson brackets. However, we have shown that when one considers transformation (6) on the usual phase space, then the Poisson brackets of the new (prime) variables obey the new brackets (7). In geometrical language, the Poisson brackets are mapped into the modified brackets through this transformation. Hence, one obtains the motion equation in the NC space, and can match it with those of the usual space in the presence of a GM field. This leads to the equivalency of the NC effects and the GM force field. Now, in this section, we illustrate that, without referring to the NC space (and hence, without appealing to the coordinates transformation (6)), one can define the new brackets on the phase space of the GEM field, and shows directly that the geometry of the phase space in the scene of these new brackets is actually NC. Of course, defining a new set of brackets can be considered as a typical transformation of the Poisson brackets. From the geometrical aspect, it means that the Poisson structure defined on the manifold is actually replaced by a new algebraic structure known as the Dirac structure (brackets).

Indeed, for constrained dynamical systems, Dirac introduced his brackets instead of the Poisson brackets, and as we will show, the GEM system can be considered as a constrained system. Hence, one just needs to apply the Dirac canonical formalism to proceed the above issue.

Dirac formalism is actually an extension of the Hamiltonian formulation for mechanical systems which are subject to constraints. That is, although the motion equations of systems without constraints are expressible in terms of the Poisson brackets, but for systems involving with constraints, the Hamiltonian equations of motion can be expressed in terms of the Dirac brackets. And, a constrained system is a system whose position and momentum variables obey certain identities (constraints) and

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9Here, the underlying manifold is the phase space.
therefore, the canonical variables are not independent. Of course, such a dependency comes from the form of the Lagrangian function, which is often referred to as the nonstandard or the degenerate Lagrangian. Indeed, Dirac extended a standard technique and used a new Hamiltonian form for the extension of such nonstandard Lagrangians. For a more justification, let us look at the subject from the following point of view.

In the usual formulation of classical mechanics, the passage from the Lagrangian variables of generalized positions \( q_i \) and velocities \( \dot{q}_i \), to the Hamiltonian variables (i.e., generalized positions and momenta \( p_i \)) is possible when, and only when, the velocities can be expressed in terms of the positions and momenta. This is done by the Legendre transformation, i.e. \((q_i, \dot{q}_i) \rightarrow (q_i, p_i)\), where \( p_i = \partial L(q, \dot{q}) / \partial \dot{q}_i \) and \( L(q, \dot{q}) \) is the Lagrangian, see, e.g., Ref. [36]. It is expected that the definition of momenta should not lead to any identities among the positions and momenta alone, for the canonical variables must be independent. But, there are some situations for which the definition of generalized momenta leads to such identities or, in another word, constraints. An explicit example of such situation is when the Lagrangian is linear in generalized velocities. Thus, the transition from the Lagrangian to the Hamiltonian formalism at the classical level is non–trivial. This transition and also the canonical quantization of such constrained systems were worked out by Dirac [19]. Both the Poisson bracket and commutator are representations of a Lie algebra product, hence the corresponding principle, in replacing the classical Poisson bracket by the commutator of the quantum mechanical operators, can work. Dirac introduced a procedure for the Hamiltonian formalism of a constrained system, and considered the new Hamiltonian form for developing the quantum mechanics of such systems. Of course, the classical version of the Dirac theory has been well studied in the literature, see, e.g., Ref. [37]. In below, we give a brief review on account of the Dirac theory in the classical version while avoiding the mathematical rigorous.\(^{10}\)

Let \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) be a coordinate system on \( 2n \) dimensional Euclidean space \( \mathbb{R}^{2n} \) and let \( U \) be an open subset of \( \mathbb{R}^{2n} \). Then, the Poisson bracket of any two infinitely differentiable functions on \( U \), say \( f \) and \( g \), is defined and denoted by

\[
\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right). \tag{20}
\]

The symplectic form \( \omega \), corresponding to the Poisson bracket, is called the classical symplectic structure and is represented by the \( 2n \times 2n \) matrix

\[
\omega = \begin{pmatrix}
0 & 1_{n \times n} \\
-1_{n \times n} & 0
\end{pmatrix} \tag{21}
\]

such that, the Poisson bracket can be written in the form

\[
\{f, g\} = \pi^{ij} \partial_i f \partial_j g, \tag{22}
\]

where \( \pi^{ki} \omega_{jk} = \delta^i_j \).

For mechanical systems, \( \mathbb{R}^{2n} \) can be considered as \( 2n \) dimensional phase space with the canonical variables \( (q_1, \ldots, q_n \text{ and } p_1, \ldots, p_n) \). On the other hand, in many cases, constraints can be written in terms of the phase space functions. In this manner, let \( \phi_i(q,p) = 0 \ (i = 1, \ldots, m) \) denotes all constraints for the Hamiltonian system. These constraints can be divided into two classes by analyzing the \( m \times m \) antisymmetric matrix of their mutual Poisson brackets, say \( M_{ij} = \{\phi_i, \phi_j\} \). Since \( M_{ij} \) is antisymmetric, its rank \( r \) must be even,\(^{11}\) and after redefining the constraints by taking their linear combinations (known as the Dirac separating constraints algorithm), the top left \( r \times r \) sub–matrix of \( M_{ij} \), which we denote by \( C_{ij} \), is regular (i.e. nonsingular). The constraint functions \( \phi_{r+1}, \ldots, \phi_m \) are called the first–class and \( \phi_1, \ldots, \phi_r \) are called the second–class. Regarding the

\(^{10}\)For a more complete discussion of this issue see, e.g., Refs. [36, 38, 39].

\(^{11}\)The rank of a matrix is the number of its independent row (column).
definition of Dirac bracket, the second–class constraints are only considered, and for them, we can introduce the Dirac bracket of any two phase space functions $f$ and $g$ as

$$\{f, g\}_D = \{f, g\} - \{f, \phi_i\} C^{ij} \{\phi_j, g\},$$

(23)

where the matrix $C^{ij}$ is the inverse of $C_{ij}$. Note that, the functions $\phi_i(q, p)$ are a certain subset (second–class) of all those functions whose vanishing give the constraints. Dirac argued that one can generalize the canonical Hamiltonian $H$ to the new Hamiltonian $H_T$ (the total Hamiltonian) such that, the dynamics is confined by the constrains and is governed by the total Hamiltonian. Namely,

$$\dot{q}_i = \{q_i, H_T\}_D \quad \text{and} \quad \dot{p}_i = \{p_i, H_T\}_D$$

(24)

with

$$H_T = H + \sum_{k=1}^{m} u^k \phi_k,$$

(25)

where $u^k$'s are unknown coefficients and are not necessarily functions of the coordinates and momenta. Recall that, when the constraints are exerted, one gets $H_T = H$. Thus, a constrained Hamiltonian system is defined by a given Lagrangian together with the Dirac bracket corresponding to the set of constraints. The physical phase space of the system is a constraint manifold, which is a sub–manifold of the unconstraint (naive) phase space $\mathbb{R}^{2n}$ spanned by the $p_i$'s and $q_i$'s. An another (short) statement of the above discussion can be asserted as follows.

The flat symplectic (Poisson) structure on the naive phase space induces a non–trivial symplectic structure on the physical phase space. This non–trivial symplectic structure is given by the Dirac bracket and it is just what can show the non–commutativity in the physical phase space.

Let us state the result of an example\(^{12}\) in the electromagnetism for the motion of a charged particle $q$ with mass $m$ in a constant magnetic field $B_0$. The motion is subjected to the following condition which we refer to it as the magnetic condition:

The ratio $qB_0/2mc$ is sufficiently large such that the kinetic term in the corresponding Lagrangian can be neglected.

In this example and under the magnetic condition, the Dirac brackets show the non–commutativity in the phase space of the electromagnetic field. Evidently, this condition induces a strong magnetic field\(^{13}\) while in the corresponding gravitoelectromagnetism, the GM field is a weak field. However, one can still employ the Dirac theory for the GEM field without the magnetic condition, but when the particle, that moves in a constant GM field, is subjected to the GM condition as:

The velocity of particle, $v$, is so small that the ratio $v^2/c^2$ can be neglected against the unity.

Indeed, this GM condition is a typical statement of the slow motion condition, i.e. the smallness of the particle velocity in contrast to the light velocity. And, it is well–known that the slow motion is one of the conditions that general relativity reveals the electromagnetism\(^{14}\). It looks that the magnetic condition and the GM condition are another dissimilarity between the GM field and the magnetic field, but, in Ref. [18], it has been shown when the magnetic field is weak, the non–trivial Dirac bracket can also be resulted. This apparent limitation of weak magnetic fields is even a desirable feature of the approach of Ref. [18], where it has been proved that by a given set of Dirac constraints, a magnetic field induces the corresponding Dirac brackets. In fact, by extending the usual understanding of the classical phase space and regarding it as a generalized complex manifold, a unified picture is obtained in which the magnetic fields and the Dirac constraints turn out to be equivalent objects. In below,

\(^{12}\)The full illustration of this example can be found in Ref. [40].

\(^{13}\)Though, a considerable amount of charge $q$ or a very small mass $m$ can also make this large ratio, but we do not consider these cases.

\(^{14}\)Obviously, the full conditions are the weak field and the slow motion approximations.
analogous to the electromagnetic example, we illustrate that the latter statement is also true for the GM field.

The Lagrangian for a particle of mass \( m \) and velocity \( \mathbf{v} \) moving in a GEM field is given by [41]

\[
L = -mc^2(1 - v^2/c^2)^{1/2} + m\frac{1 + v^2/c^2}{\sqrt{1 - v^2/c^2}} U - \frac{2m/c}{\sqrt{1 - v^2/c^2}} \mathbf{A} \cdot \mathbf{v},
\]

where \( U \) and \( \mathbf{A} \) are the gravitoelectric and the GM potentials, respectively. By ignoring \( v^2/c^2 \) against the unity, multiplying by \( 1/m \) and discarding a constant, the Lagrangian can be written in the form

\[
L \approx U - \frac{2}{c} \mathbf{A} \cdot \mathbf{v}.
\]

Lagrangian (27) is linear in terms of velocity and therefore, the equations of the generalized momenta can be considered as the Dirac constraints. It is instructive to re–scale the potential as

\[
U \rightarrow \epsilon U
\]

with

\[
\epsilon = \frac{B_{gm}}{c},
\]

where we assume that the particle is confined to the \( xy \)–plane in a constant GM field \( \mathbf{B}_{gm} = B_{gm}\hat{z} \). Therefore, Lagrangian (27) reads

\[
L \approx \epsilon U - \epsilon (x\dot{y} - y\dot{x}),
\]

where the GM vector potential has been considered to be \( \mathbf{A} = -\mathbf{r} \times \mathbf{B}_{gm}/2 \). The conjugate momenta are obtained from Lagrangian (28) as

\[
p_x = \epsilon y \quad \text{and} \quad p_y = -\epsilon x,
\]

and thus, the corresponding Hamiltonian takes the simple form

\[
H = -\epsilon U.
\]

Equations (29) give two primary constraints, namely

\[
\phi_1 = \epsilon y - p_x \quad \text{and} \quad \phi_2 = \epsilon x + p_y.
\]

Therefore, according to definition (25), the total Hamiltonian is

\[
H_T = -\epsilon U + u_1(\epsilon y - p_x) + u_2(\epsilon x + p_y),
\]

and hence, we have

\[
\{\phi_1, H_T\} = \epsilon(2u_2 - \frac{\partial U}{\partial x}) \quad \text{and} \quad \{\phi_2, H_T\} = -\epsilon(2u_1 - \frac{\partial U}{\partial y}).
\]

Note that, for consistency [19], when the constraints are imposed, one must set \( \{\phi_i, H_T\} = 0 \) \((i = 1, 2)\).

The latter equations are regarded as the relations that determine the coefficients \( u_1 \) and \( u_2 \) in (32). Also, it can easily be found that

\[
\{\phi_1, \phi_2\} = -\{\phi_2, \phi_1\} = 2\epsilon,
\]

as expected not to be zero, for our primary constraints are the second–class ones. Thus, one can define the Dirac bracket in terms of these constraints. Let us constitute the matrix \( C_{ij} = \{\phi_i, \phi_j\} \), that is

\[
(C_{ij}) = -2\epsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

whose inverse is

\[
(C^{ij}) = -\frac{1}{2\epsilon} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

\[\text{For this reason, these relations are usually called the consistency conditions.}\]
The inverse matrix elements can be written as\textsuperscript{16} $C^{ij} = -\varepsilon^{ij}/2\epsilon$, hence, the Dirac bracket of any two quantities $f$ and $g$ is
\begin{equation}
\{f, g\}_D = \{f, g\} + \frac{1}{2\epsilon}\varepsilon^{ij}\{f, \phi_i\}\{\phi_j, g\}.
\end{equation}

Now, the non–trivial Dirac bracket of the phase space coordinates can be resulted. However, before this, it will be an instructive and useful example, if one writes the motion equation in terms of the Dirac bracket. For this purpose, equation (37) for $x$ variable gives
\begin{equation}
\dot{x} = \{x, H_T\}_D = u_1\{x, \phi_1\} + \frac{1}{2\epsilon}\{x, \phi_1\}\{\phi_2, H_T\} = u_1\{x, \phi_1\} - \frac{1}{2\epsilon}\{x, \phi_1\}\epsilon(2u_1 - \frac{\partial U}{\partial y}),
\end{equation}
where by using (31) and hence, substituting for $\{x, \phi_1\} = -1$, leads to
\begin{equation}
\dot{x} = -\frac{1}{2}\frac{\partial U}{\partial y}.
\end{equation}

For $y$ variable, we get
\begin{equation}
\dot{y} = \{y, H_T\}_D = u_2\{y, \phi_2\} - \frac{1}{2\epsilon}\{y, \phi_2\}\epsilon(2u_2 - \frac{\partial U}{\partial x}),
\end{equation}
that as $\{y, \phi_2\} = 1$, gives
\begin{equation}
\dot{y} = \frac{1}{2}\frac{\partial U}{\partial x}.
\end{equation}

With a similar calculation, the equation motions of momenta are
\begin{align*}
\dot{p}_x &= \{p_x, H_T\}_D = \frac{1}{2}\frac{\partial U}{\partial x} \quad \text{and} \quad 
\dot{p}_y &= \{p_y, H_T\}_D = \frac{1}{2}\epsilon\frac{\partial U}{\partial y}.
\end{align*}

However, the usual Hamilton equations give
\begin{align*}
\dot{x} &= 0 = \dot{y},
\end{align*}
which are not the correct results, and also
\begin{align*}
\dot{p}_x &= \epsilon\frac{\partial U}{\partial x} \quad \text{and} \quad 
\dot{p}_y &= \epsilon\frac{\partial U}{\partial y},
\end{align*}
which contradicts the outcomes of the constrain equations (29). These results confirm the necessity of the new Hamiltonian $H_T$.

Let us return to the main purpose of the work. It is easily to show that the Dirac brackets of the phase space coordinates are
\begin{equation}
\{x, y\}_D = \{x, y\} + \frac{1}{2\epsilon}\epsilon^{ij}\{x, \phi_i\}\{\phi_j, y\} = \frac{1}{2\epsilon}\{x, \phi_1\}\{\phi_2, y\} = \frac{1}{2\epsilon}.
\end{equation}

Similarly, we get
\begin{equation}
\{x, p_x\}_D = \frac{1}{2} = \{y, p_y\}_D,
\end{equation}
and
\begin{equation}
\{p_x, p_y\}_D = \frac{\epsilon}{2}.
\end{equation}

Obviously, the Dirac brackets between the phase space coordinates are nonzero, particularly for the momentum sector. It can be resulted that, the geometry of the phase space is NC and the associated parameter of the momentum sector is proportional to the magnitude of the GM field. In other words, the non–commutativity in the momentum sector is equivalent to the presence of the GM force whose weakness implies that the NC parameter of the momentum (space) sector is small (large).

\textsuperscript{16}The notation $\epsilon^{ij}$ is the Levi–Civita symbol in two dimensions.
6 Conclusions

Gravitomagnetism is an old subject in general relativity around which many well written papers exist. However, the effects of accompanying non-commutativity lends some weight to its significance. On the other hand, the generalized uncertainty principle which is somehow what we have considered in this study has gained a considerable amount of momentum in recent years and is thus carries a noticeable importance. However, its use in the framework of gravitomagnetism has been somewhat left on the side. It is thus interesting to study within this framework. Hence, in this work, we have investigated one of the other similarities between the GM and the magnetic fields. The investigation is relevant to the equivalence between the GM field and the NC space effects. For this purpose, we have proceeded through two approaches. The first approach is based on the comparison of the motion equations of a particle in a NC space and the usual space in the presence of a GM field. In the usual space, according to the FP, the motion equation includes a GM term that comes from the motion of the potential source. Indeed, the presence of such term is resulted from the existence of the Lorentz-like force for the GEM field. The existence of such force is provided by the slow motion and the weak field approximations when the GEM fields are in the stationary case. The comparison of the two motion equations shows that, the GM field (like its magnetic analogous) can be equivalent to the NC parameter of the momentum sector. That is, the NC effect can be interpreted as the presence of the GM field in the usual space.

The second approach is based on the fact that the GEM system can be considered as a constraint system in the Dirac formalism. The weakness of the field and sufficiently low velocity conditions permit the GEM system to be described in terms of the degenerate Lagrangian which is characterized by the constraints. Such constraints are divided into two classes, the first– and the second–classes. The first–class constraints are correlated, at least to some extent, with the gauge properties of the system, while the second–class constraints reflect the appearance of non dynamical degrees of freedom in the theory. The dynamics of a constrained system is described by the Dirac bracket which is a generalization of the Poisson bracket in the symplectic mechanics. By the definition of Dirac bracket in terms of the second–class constraints, the dynamics is governed by the total (extended) Hamiltonian instead of the canonical one. With respect to the new (algebraic) symplectic structure, the phase space variables no longer commute and this illustrates the NC structure of the phase space geometry. Particularly, the Dirac bracket of the momentum sector variables is non–trivial and proportional to the magnitude of the GM field. The main conclusion of the second approach can be summarized in the following statement:

The GM field as a typical weak field is equivalent to the Dirac constraint which induces the corresponding Dirac bracket.

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