On the gravitational stability of a compressed slab of gas in the background of weakly interacting massive particles

David Tsiklauri
Physics Department, Tbilisi State University, 3 Chavchavadze Ave., Tbilisi 380028, Georgia; email: dtsiklau@resonan.ge

ABSTRACT

Linear stability of an isothermal, pressure-bounded, self-gravitating gas slab which is gravitationally coupled with the background weakly interacting massive particles (WIMPs) is investigated. Analytic dispersion relations describing such a configuration are derived. Two novel, distinct oscillatory modes are found. Astrophysical implications of the results are discussed.

Subject headings: dark matter — gravitation — instabilities — ISM: clouds — ISM: kinematics and dynamics — stars: formation

1. Introduction

It is widely accepted that the formation of stars occurs through gravitational collapse of the interstellar gas clouds. One of the possible mechanisms of triggering the star and/or stellar cluster formation process is a high-velocity (supersonic) cloud-cloud collisions. In such a collision a dense gaseous slab is formed with two plane-parallel shock fronts propagating away from the interface (Usami et al. 1995). Then the slab grows in mass becoming unstable against gravitational instability which causes its fragmentation. Newly formed the gaseous clumps, in turn, collapse further and evolve into stars and/or stellar clusters.

The structure of a self-gravitating, isothermal shock pressure-bounded slab of gas which usually is formed in the cloud-cloud collisions was investigated by Ledoux (1951) and later by Elmegreen & Elmegreen (1978) among others. Lubow & Pringle (1993), LP93 thereafter, have also derived an analytic dispersion relation for the linear perturbations and investigated, in detail, the full unstable branch in such a slab of gas.

Possible existence of WIMP matter, one of the form of dark matter which itself is a dominant mass component of the universe, is strongly motivated both by standard models of particle physics and cosmology. It has been established that the mass density associated with the luminous matter (stars, hydrogen clouds, x-ray gas in clusters, etc.) cannot account for the observed dynamics on galactic scales and above (Trimble, 1987), therefore, revealing the existence of large amounts of dark matter in this universe. It has been argued that the dark matter could be anything from novel weakly interacting elementary particles (massive neutrinos, neutralinos) to baryonic matter in some invisible form (brown dwarfs, primordial black holes, cold molecular gas, etc.).

Recently, it has been shown (Tsiklauri 1998) that classical Jeans theory of the stability of self-gravitating interstellar gas clouds is significantly modified by taking into account the presence of background weakly interacting massive particles. It has been found (Tsiklauri 1998) that the presence of WIMP matter yields
an unavoidable reduction of the Jeans length, the Jeans mass and the Jeans time (time-scale of the collapse via gravitational instability).

Resuming aforesaid, and in the light of recent results of Tsiklauri (1998), a revision of a conventional theory (e.g. LP93) of the stability of self-gravitating, isothermal shock pressure-bounded slab of gas against linear perturbations in the case of presence of WIMP matter, or generally speaking any type of microscopic non-baryonic dark matter that is coupled with the slab only via gravitational interaction, seems to be of a considerable importance for full understanding of the star and/or stellar cluster formation process.

2. The model

Equations that govern dynamics of two self-gravitating fluids (a slab of gas and WIMP matter) inter-coupled only via gravitational interaction can be written in the following way:

\[ \frac{\partial \rho_i}{\partial t} + \nabla (\rho_i \vec{V}_i) = 0, \quad (1) \]

\[ \frac{\partial \vec{V}_i}{\partial t} + (\vec{V}_i \cdot \nabla) \vec{V}_i = -\nabla \phi - \frac{\nabla P_i}{\rho_i}, \quad (2) \]

\[ P_i = K_i \rho_i^{\gamma_i}, \quad (3) \]

\[ \Delta \phi = 4\pi G \sum_i \rho_i, \quad (4) \]

where notation is standard: \( \vec{V}_i, P_i, \rho_i \) and \( \phi \) denote velocity, pressure, mass density and gravitational potential of the fluids. The subscript \( i = G, W \) denotes two components, baryonic gas and WIMP matter respectively. Now, writing every physical quantity, for brevity commonly denoted by \( \vec{f}(x, z) \), in a form of \( \vec{f}(x, z) = \vec{f}_0 + \vec{f}'(z) \exp[i(\omega t - kx)] \) and doing usual linearization of the Eqs.(1)–(4) we can obtain following a set of coupled differential equations for the perturbations

\[ \frac{d^2 P'_G}{dz^2} + \left( \frac{\omega^2}{C_{sG}^2} - k^2 + \frac{4\pi G \rho^G_0}{C_{sG}^2} \right) P'_G = -\frac{4\pi G \rho^W_0}{C_{sW}^2} P'_W, \quad (5) \]

\[ \frac{d^2 P'_W}{dz^2} + \left( \frac{\omega^2}{C_{sW}^2} - k^2 + \frac{4\pi G \rho^W_0}{C_{sW}^2} \right) P'_W = -\frac{4\pi G \rho^G_0}{C_{sG}^2} P'_G, \quad (6) \]

\[ \frac{d^2 \phi'}{dz^2} - k^2 \phi' = 4\pi G \left( \frac{P'_G}{C_{sG}^2} + \frac{P'_W}{C_{sW}^2} \right). \quad (7) \]

Here, \( C_{sG} \) and \( C_{sW} \) denote speeds of sound for the baryonic gas and WIMP matter respectively. It is worthwhile to note that Eqs.(5) and (6) resemble closely to the equations describing coupled pendulums which arise in the non-modal study of linear perturbations in shear flows (cf. Chagelishvili, Rogava & Tsiklauri 1996, 1997). The fundamental (normal) oscillatory modes of the Eqs.(5) and (6) are given by the following expression:

\[ q^2_{\pm} = \frac{1}{2} \left[ q^2_G + q^2_W \pm \sqrt{(q^2_G - q^2_W)^2 + 4\alpha_G \alpha_W} \right], \quad (8) \]

where

\[ q^2_G = \frac{\omega^2}{C_{sG}^2} - k^2 + \frac{4\pi G \rho^G_0}{C_{sG}^2}, \quad (9) \]
and

\[ q_w^2 = \left[ \frac{\omega^2}{C_{sw}^2} - k^2 + \frac{4\pi G \rho_0}{C_{sw}^2} \right], \tag{10} \]

are the eigenfrequencies of the system. Also, \( \alpha_G = 4\pi G \rho_0/C_{sw}^2 \) and \( \alpha_W = 4\pi G \rho_0/C_{sw}^2 \).

A general solution of the Eqs.(5)–(7) can be readily found. Following LP93 we consider only symmetric modes \( dP'_G/dz = dP'_W/dz = 0 \) at \( z = 0 \). There are two distinct, general solutions of the Eqs.(5)–(7) depending on the sign of \( q^2 \).

For the \( q^2 > 0 \) we obtain

\[ P'_G = A_\pm \cos(q_- z) + A_+ \cos(q_+ z), \tag{11} \]

\[ P'_W = \frac{q^2 - q_G^2}{\alpha_W} A_\pm \cos(q_- z) + \frac{q^2 - q_G^2}{\alpha_W} A_+ \cos(q_+ z), \tag{12} \]

\[ \phi' = C \cosh(kz) - \left( \beta_G + \frac{q^2 - q_G^2}{\alpha_W} \beta_W \right) \frac{A_-}{k^2 + q_-^2} \cos(q_- z) - \]

\[ \left( \beta_G + \frac{q^2 - q_G^2}{\alpha_W} \beta_W \right) \frac{A_+}{k^2 + q_+^2} \cos(q_+ z), \tag{13} \]

here we introduced notation \( \beta_G = 4\pi G/C_{sw}^2 \) and \( \beta_W = 4\pi G/C_{sw}^2 \).

For the \( q^2 < 0 \) we obtain

\[ P'_G = A_\pm \cosh(\sqrt{-q^2} z) + A_+ \cos(q_+ z), \tag{14} \]

\[ P'_W = -\frac{q^2 - q_G^2}{\alpha_W} A_\pm \cosh(\sqrt{-q^2} z) + \frac{q^2 - q_G^2}{\alpha_W} A_+ \cos(q_+ z), \tag{15} \]

\[ \phi' = C \cosh(kz) - \left( \beta_G + \frac{-q^2 - q_G^2}{\alpha_W} \beta_W \right) \frac{A_-}{k^2 - q_-^2} \cosh(\sqrt{-q^2} z) - \]

\[ \left( \beta_G + \frac{q^2 - q_G^2}{\alpha_W} \beta_W \right) \frac{A_+}{k^2 + q_+^2} \cos(q_+ z), \tag{16} \]

Note that in the case of presence of the background WIMP matter there are two fundamental (normal) modes \( q_{\pm} \) (see Eq.(8)). Whereas, in the case with no WIMP matter there is only one oscillatory mode, namely \( q_G \) (see Eq.(32) in LP93). This is the presence of WIMP matter that causes appearance of a novel unstable mode described by Eqs.(14)–(16).

As our next step towards derivation of the dispersion relation for linear perturbations, following LP93, we impose relevant boundary conditions. The first boundary condition is that the Lagrangian pressure perturbation is zero at the slab boundary, namely that

\[ P'_i + \xi_z \frac{dP'_i}{dz} = 0, \tag{17} \]

at \( z = a \) (with \( a \) being slab half-thickness), where \( \xi \) vector is the Lagrangian displacement vector of the perturbation and \( i = W, G \). In this case, the displacement is simply related to \( \vec{V}' \) by \( \xi = -i \vec{V}'/\omega \). Using relation \( \nabla P'_i = -\phi'_i \nabla \phi = -4\pi G \rho_0^i (\rho_G^i + \rho_W^i) \) for the unperturbed physical quantities, Eq. (17), at \( z = a \), reduces to the two following conditions

\[ \frac{P'_i}{\rho_i^0} = \xi_z 4\pi G (\rho_G^0 + \rho_W^0) a, \tag{18} \]
The other boundary condition is the one for $\phi'$ (LP93):

$$\frac{d\phi'}{dz} - \frac{d\phi'}{dz} = 4\pi G(\rho_G^0 + \rho_W^0)\xi_z,$$

evaluated at $z = a$, where $\phi'$ is the perturbed gravitational potential in the region $|z| > a$ and is given by

$$\phi'_e = \phi'(a) \exp[-k(|z| - a)],$$

where it has been assumed that $k > 0$. Thus, we obtain

$$\frac{d\phi'}{dz} \bigg|_{z=a} = -k\phi'(a) - 4\pi G(\rho_G^0 + \rho_W^0)\xi_z.$$

(19)

2.1. The case when $q^2 > 0$

Applying boundary conditions (18) and (19) to the general solutions in the case when $q^2 > 0$ [Eqs.(11)-(13)] yields

$$\frac{A_+}{C} = \frac{1}{\rho_G^0 ak} \alpha \left[ \frac{D_+q_+}{k} \sin(q_+a) - \frac{D_-q_-}{k} \sin(q_-a) + \frac{D_+q_+}{k} \cos(q_+a) - \frac{D_-q_-}{k} \cos(q_-a) \right]^{-1}.

(20)

Here we have introduced following notation:

$$D_+ = \left( \beta_G + \frac{q^2 - q_G^2}{\alpha W} \beta_W \right) \frac{1}{k^2 + q^2_+},$$

$$D_- = \left( \beta_G + \frac{q^2 - q_G^2}{\alpha W} \beta_W \right) \frac{1}{k^2 + q^2_-},$$

and

$$\chi = \frac{\cos(q_+a)}{\cos(q_-a)} \frac{1}{\delta(q_+ - q_G^2)/\alpha W} \frac{1}{\delta(q_- - q_G^2)/\alpha W},$$

$$\delta = \frac{\rho_G^0}{\rho_W^0}.$$

Another condition which relates $A_+$ and $C$ can be obtained using $z$-component of the Eq.(2) for the perturbations in baryonic gas and $\xi = -i\tilde{V}/\omega$ (LP93). Doing this yields

$$\frac{A_+}{C} = \frac{\rho_G^0 ak \sinh(ka)}{\sinh(ka)} \left[ \frac{\omega^2 \cos(q_+a)}{4\pi G(\rho_G^0 + \rho_W^0)} - \frac{\chi \omega^2 \cos(q_-a)}{4\pi G(\rho_G^0 + \rho_W^0)} - \frac{\chi q_-a(1 - \rho_G^0 D_-)\sin(q_-a) + q_+a(1 - \rho_G^0 D_+)\sin(q_+a)}{4\pi G(\rho_G^0 + \rho_W^0)} \right]^{-1}.

(21)

Now, combining Eqs.(20) and (21) allows us to obtain the dispersion relation for the perturbations for the case when $q^2 > 0$

$$\frac{\sinh(ka) + \cosh(ka)}{\sinh(ka)} = -b_1^+ \cos(q_+a) + b_3^+ \cos(q_-a) + b_5 \sin(q_+a) + b_2 \sin(q_-a)

b_1^+ = 1 - D_+\rho_G^0 ak, b_3^+ = -\chi(1 - D_-\rho_G^0 ak), b_5 = D_+\rho_G^0 q_+a, b_2^- = -\chi D_-\rho_G^0 q_-a,

b_2 = \omega^2/[4\pi G(\rho_G^0 + \rho_W^0)], b_5 = -\chi \omega^2/[4\pi G(\rho_G^0 + \rho_W^0)], b_4^+ = q_+a(1 - \rho_G^0 D_+), b_4^- = -\chi q_-a(1 - \rho_G^0 D_-).$$

(22)
2.2. The case when \( q^2_+ < 0 \)

We now apply boundary conditions (18) and (19) to the general solutions in the case when \( q^2_+ < 0 \) [Eqs.(14)–(16)]. Thus we obtain

\[
\frac{A_+}{C} = -[\cosh(ka) + \sinh(ka)] \times \left[ \frac{D_+ q_+}{k} \sin(q_a a) + \chi^u \frac{D_u}{k} \sqrt{-q^2_a} \sinh(\sqrt{-q^2_a} a) + \right.
\]

\[
\left. \left\{ \frac{1}{\rho_G^2 ak} - D_+ \right\} \cos(q_a a) - \chi^u \left\{ \frac{1}{\rho_G^2 ak} - D_u \right\} \cosh(\sqrt{-q^2_a} a) \right]^{-1}
\]

(23)

Here we have introduced following notation:

\[
D_+ = \left( \beta_G + \frac{-q^2 - q^2_G}{\alpha_W} \beta_W \right) \frac{1}{k^2 - q^2_+},
\]

and

\[
\chi^u = \frac{\cos(q_a a)}{\cosh(\sqrt{-q^2_a} a)} \left[ \frac{1 - \delta(q^2_+ - q^2_G)/\alpha_W}{1 - \delta(-q^2_+ - q^2_G)/\alpha_W} \right].
\]

Again using the condition which relates \( A_+ \) and \( C \) that is obtained using z-component of the Eq.(2) for the perturbations in baryonic gas and \( \xi = -i \vec{V}/\omega \), we get

\[
\frac{A_+}{C} = \rho^0_G ak \sinh(ka) \left[ \frac{\omega^2 \cos(q_a a)}{4\pi G(\rho^0_G + \rho^0_W)} - \frac{\chi^u \omega^2 \cosh(\sqrt{-q^2_a} a)}{4\pi G(\rho^0_G + \rho^0_W)} + \right.
\]

\[
\left. \chi^u \sqrt{-q^2_a (1 - \rho^0_G D_+) \sinh(\sqrt{-q^2_a} a)} + q_a (1 - \rho^0_G D_+) \sin(q_a a) \right]^{-1}
\]

(24)

Finally, using Eqs.(23) and (24) we to obtain the dispersion relation for the perturbations in the case when \( q^2_+ < 0 \)

\[
\frac{\sinh(ka) + \cosh(ka)}{\sinh(ka)} = -\frac{b^+_1 \cos(q_a a) + b^-_u \cosh(\sqrt{-q^2_a} a) + b^+_2 \sin(q_a a) + b^-_u \sinh(\sqrt{-q^2_a} a)}{b^+_3 \cos(q_a a) + b^-_3 \cosh(\sqrt{-q^2_a} a) + b^+_4 \sin(q_a a) + b^-_4 \sinh(\sqrt{-q^2_a} a)}.
\]

(25)

where

\[
b^-_{1u} = -\chi^u (1 - D_u \rho^0_G ak), \quad b^+_2 = \chi^u D_u \rho^0_G \sqrt{-q^2_a}, \quad b^-_3 = -\chi^u \omega^2 / [4\pi G(\rho^0_G + \rho^0_W)], \quad b^-_{4u} = \chi^u \sqrt{-q^2_a (1 - \rho^0_G D_u)}.
\]

3. Discussion

In this paper we have derived two dispersions relations for linear perturbations in a pressure-bounded, self-gravitating gas slab which is gravitationally coupled with the background weakly interacting massive particles. In the original paper LP93 authors found that the slab alone is unstable at sufficiently long wavelength and the growth rate is of the order of \( \sqrt{\frac{G}{\rho}} \), with \( \rho \) being mass density of the slab, but at high external pressures the nature of the instability is quite different from the classical Jeans instability.
Moreover, they have developed an analytic model which reproduces numerical results of Elmegreen & Elmegreen (1978). Interesting result of these authors is that the critical wavenumber for the onset of the instability is always of the order of the slab thickness, regardless of the level of self-gravity and the external pressure.

Present analysis revealed that taking into account presence of the putative WIMP dark matter significantly modifies results of LP93 with major changes being appearance of two distinct fundamental (normal) oscillatory modes $q_\pm$ and the existence of an additional unstable mode when $q^2 < 0$. Obtained dispersion relations Eqs.(22) and (25) need detailed investigation in various limiting cases. Especially interesting would be doing analysis of the low-frequency modes, for which LP93 found an unexpected result that the critical wavenumber for the onset of the instability is always of the order of the slab thickness.

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