Derivations of a family of quantum second Weyl algebras

S Launois and I Oppong

April 27, 2022

Abstract

In view of a well-known theorem of Dixmier, its is natural to consider primitive quotients of $U_q^+(g)$ as quantum analogues of Weyl algebras. In this work, we study primitive quotients of $U_q^+(G_2)$ and compute their Lie algebra of derivations.

1 Introduction

Weyl algebras have been extensively studied in the last 60 years due to their link to Lie theory, differential operators, quantum mechanics, etc. One of the main questions remaining is the famous Dixmier Conjecture that asserts that every endomorphism of a complex Weyl algebra is an automorphism.

Let $K$ be a field and $q$ be a non-zero element of $K$ that is not a root of unity. The aim of this article is to produce quantum analogues of the second Weyl algebra and to compare their properties to those of the second Weyl algebra. There exist in the literature various families of “quantum Weyl algebras”, e.g. the so-called quantum Weyl algebras and generalised Weyl algebras (GWA for short). Most of the time, they are obtained by generators and relations through a deformation of the classical defining relation of the first Weyl algebra: $xy - yx = 1$.

To produce potential quantisations, we take a different approach in this article. Our inspiration comes from a Theorem of Dixmier (see, for instance, [7, Théorème 4.7.9]) that asserts that primitive quotients of enveloping algebras of complex nilpotent Lie algebras are isomorphic to Weyl algebras.

We have at hand a quantum analogue of at least some enveloping algebras of complex nilpotent Lie algebras, namely the positive part $U_q^+(g)$ of a quantised enveloping algebra $U_q^+(g)$ of a complex simple Lie algebra $g$. As a consequence, it is natural to consider primitive quotients of $U_q^+(g)$ as quantum analogues of Weyl algebras. In the $A_2$ and $B_2$ cases, primitive ideals of $U_q^+(g)$ have been classified and it turns out that in the $B_2$ case, some of the resulting primitive quotients provide ‘nice’ quantum analogues of the first Weyl algebra. For instance, they are simple—this is not the case of quantum Weyl algebras—and do not possess non-trivial units—this is not the case of a quantum GWA over a Laurent polynomial ring. (See [15] for details.)

The present article is concerned with the $G_2$ case. More precisely, we identify a family of primitive ideals of $U_q^+(G_2)$ and then proceed in proving that the corresponding primitive quotients have (at least for some choice of the parameters) properties similar to those of the second Weyl algebra. More precisely, the center of $U_q^+(G_2)$ is a polynomial algebra in two variables $K[\Omega_1, \Omega_2]$ and we prove that the quotient algebra

$$A_{\alpha, \beta} := U_q^+(G_2)/\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$$

is simple for all $(\alpha, \beta) \neq (0, 0)$. We then proceed and study these quotient algebras. In particular, we show that $A_{\alpha, \beta}$ has the same (Gelfand-Kirillov) dimension as the second Weyl algebra $A_2(K)$. We also establish that for certain choice of the parameters $\alpha$ and $\beta$, the algebra $A_{\alpha, \beta}$ is a deformation of a quadratic extension of $A_2(K)$ at $q = 1$.

In the final section, we compute the derivations of $A_{\alpha, \beta}$. Our results show that when $\alpha$ and $\beta$ are both non-zero, all derivations of $A_{\alpha, \beta}$ are inner, a property that is well known to hold in $A_2(K)$.

In view of the celebrated Dixmier Conjecture, it would be interesting to describe automorphisms and endomorphisms of $A_{\alpha, \beta}$ when $\alpha$ and $\beta$ are both non-zero. We intend to come back to these questions in the future.
This article is organized as follows. In Section 2, we recall the presentation of $U_q^+(G_2)$ as a so-called quantum nilpotent algebra (QNA for short). This allows the use of two different tools to study the prime and primitive spectra of $U_q^+(G_2)$: the $H$-stratification theory of Goodearl and Letzter, and the Deleting Derivation Theory of Cauchon. We recall both theories in the context of $U_q^+(G_2)$ in Section 2. In Section 3, we use these two theories to establish that $(\Omega_1 - \alpha, \Omega_2 - \beta)$ is a maximal ideal of $U_q^+(G_2)$ when $(\alpha, \beta) \neq (0,0)$.

In Section 4, we focus on comparing $A_{\alpha,\beta}$ with the second Weyl algebra $A_2(\mathbb{K})$. In particular, we show that both have Gelfand-Kirillov dimension equal to 4. Through a direct computation, we also establish that $A_{\alpha,\beta}$ is a quadratic extension of $A_2(\mathbb{K})$ at $q = 1$. In this section we also compute a linear basis for $A_{\alpha,\beta}$.

In the final section, we compute the derivations of $A_{\alpha,\beta}$. Our strategy here is to make use of the following tower of algebras arising from the Deleting Derivation Algorithm:

$$A_{\alpha,\beta} \subset R_6 = R_7 \Sigma_{\delta}^{-1} \subset R_5 = R_6 \Sigma_{\delta}^{-1} \subset R_4 = R_5 \Sigma_{4}^{-1} \subset R_3.$$

The later algebra $R_3$ is a simple quantum torus whose derivations have been described by Osborn and Passman in [20]. We pull back their description to obtain a description of the derivations of $A_{\alpha,\beta}$ through a step-by-step process consisting in “reverting” the Deleting Derivation Algorithm. Our results show that when $\alpha$ or $\beta$ is equal to zero, then the first Hochschild cohomology group of $A_{\alpha,\beta}$ is a 1-dimension vector space, whereas when both $\alpha$ and $\beta$ are non-zero, all derivations are inner.

## 2 The quantum nilpotent algebra $U_q^+(G_2)$ and its primitive ideals

### 2.1 The quantum nilpotent algebra $U_q^+(G_2)$

Let $\mathbb{K}$ be a field and $q$ be a non-zero element of $\mathbb{K}$ that is not a root of unity. The algebra of $U_q^+(G_2)$ is the so-called positive part of the quantum enveloping algebra $U_q(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ of type $G_2$. It is well known, see for instance [2], that this algebra is generated over $\mathbb{K}$ by two indeterminates $E_\alpha$ and $E_\beta$ subject to the following quantum Serre relations:

\[
\begin{align*}
(S1) \quad & E_\alpha^4 E_\beta - \left[\begin{array}{c} 4 \\ 1 \end{array}\right]_q E_\alpha^3 E_\beta E_\alpha + \left[\begin{array}{c} 4 \\ 2 \end{array}\right]_q E_\alpha^2 E_\beta E_\alpha^2 - \left[\begin{array}{c} 4 \\ 1 \end{array}\right]_q E_\alpha E_\beta^3 E_\alpha + E_\beta E_\alpha^4 = 0, \\
(S2) \quad & E_\beta^3 E_\alpha - \left[\begin{array}{c} 2 \\ 1 \end{array}\right]_q E_\beta E_\alpha E_\beta + E_\alpha E_\beta^2 = 0,
\end{align*}
\]

where $\left[\begin{array}{c} n \\ i \end{array}\right]_q$ denotes the quantum binomial coefficients (see [2 I.6.1]).

One can construct a PBW-basis of $U_q^+(G_2)$ using the so-called Lusztig automorphisms of $U_q(\mathfrak{g})$, see for instance [2] I.6.8. In the present case, such a basis was computed by De Graaf in [5]. We will use the convention of that paper, but with $E_1 := E_\alpha$, $E_2 := E_{3\alpha+\beta}$, $E_3 := E_{2\alpha+\beta}$, $E_4 := E_{3\alpha+2\beta}$, $E_5 := E_{\alpha+\beta}$ and $E_6 := E_\beta$.

With these notations, the defining relations of $U_q^+(G_2)$ are as follows:

\[
\begin{align*}
E_2 E_1 &= q^{-3} E_1 E_2, & E_3 E_1 &= q^{-1} E_1 E_3 - (q + q^{-1} + q^{-3}) E_2 \\
E_3 E_2 &= q^{-3} E_2 E_3, & E_4 E_1 &= E_1 E_4 + (1 - q^2) E_3^3 \\
E_4 E_2 &= q^{-3} E_2 E_4 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1} E_3^3, & E_4 E_3 &= q^{-3} E_3 E_4 \\
E_5 E_1 &= q E_1 E_5 - (1 + q^2) E_3, & E_5 E_2 &= E_2 E_5 + (1 - q^2) E_3^3 \\
E_5 E_3 &= q^{-1} E_3 E_5 - (q + q^{-1} + q^{-3}) E_4, & E_5 E_4 &= q^{-3} E_4 E_5 \\
E_6 E_1 &= q^3 E_1 E_6 - q^2 E_5, & E_6 E_2 &= q^3 E_2 E_6 + (q^4 + q^2 - 1) E_4 + (q^2 - q^4) E_3 E_5 \\
E_6 E_3 &= E_3 E_6 + (1 - q^2) E_5^2, & E_6 E_5 &= q^{-3} E_5 E_6 \\
E_6 E_4 &= q^{-3} E_4 E_6 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1} E_3^3,
\end{align*}
\]
and the monomials $E_1^{k_1} \cdots E_6^{k_6}$ $(k_1, \ldots, k_6 \in \mathbb{N})$ form a basis of $U_2^+(G_2)$ over $\mathbb{K}$.

Even better, one may write $U_2^+(G_2)$ as a Quantum Nilpotent Algebra (QNA for short) or Cauchon-Goodearl-Letzter extension in the sense of [16] Definition 3.1, by adjoining the generators $E_i$ in lexicographic order. This means in particular that $U_2^+(G_2)$ can be presented as an iterated Ore extension:

$$U_2^+(G_2) = \mathbb{K}[E_1][E_2; \sigma_2, \delta_2] \cdots [E_6; \sigma_6, \delta_6],$$

where the $\sigma_i$ are automorphisms and the $\delta_i$ are left $\sigma_i$-derivations of the appropriate subalgebras. We would not need the precise definition of a QNA for what follows, but it is worth reminding the reader of the algebraic torus action involved in writing $U_2^+(G_2)$ as a QNA.

The algebraic torus $\mathcal{H} = (\mathbb{K}^\times)^2$ acts by automorphisms on $U_2^+(G_2)$ as follows:

$$h \cdot E_i = h_i E_i \text{ for all } i \in \{1, 6\} \text{ and } h = (h_1, h_6) \in \mathcal{H}.$$

Note that the action of the automorphism $h$ on the generators $E_2, \ldots, E_5$ follows from the above defining relations.

By [2] Theorem II.2.7, the action of $\mathcal{H}$ on $U_2^+(G_2)$ is rational in the sense of [2] Definition II.2.6.

A consequence of the QNA condition is that important tools such as Cauchon’s deleting derivations procedure and the Goodearl-Letzter stratification theory (this is the origin of the CGL terminology, see [16]) are available to study prime and primitive ideals. These ideas will be introduced in following sections. At the moment, we merely note that it is immediate that $U_2^+(G_2)$ is a noetherian domain and that all prime ideals are completely prime (in the case of $U_2^+(G_2)$, it was proved in [21] Section 5). We denote by $F_q$ its skew-field of fractions, i.e. $F_q := \text{Frac}(U_2^+(G_2))$.

### 2.2 Prime ideals in $U_2^+(G_2)$ and $\mathcal{H}$-stratification

A two-sided ideal $I$ of $U_2^+(G_2)$ is said to be $\mathcal{H}$-invariant if $h \cdot I = I$ for all $h \in \mathcal{H}$. An $\mathcal{H}$-prime ideal of $U_2^+(G_2)$ is a proper $\mathcal{H}$-invariant ideal $J$ of $U_2^+(G_2)$ such that if $J$ contains the product of two $\mathcal{H}$-invariant ideals of $U_2^+(G_2)$ then $J$ contains at least one of them. We denote by $\mathcal{H}\text{-Spec}(U_2^+(G_2))$ the set of all $\mathcal{H}$-prime ideals of $U_2^+(G_2)$. Observe that if $P$ is a prime ideal of $U_2^+(G_2)$ then

$$(P : \mathcal{H}) := \bigcap_{h \in \mathcal{H}} h \cdot P \quad (1)$$

is an $\mathcal{H}$-prime ideal of $U_2^+(G_2)$. Indeed, let $J$ be an $\mathcal{H}$-prime ideal of $U_2^+(G_2)$. We denote by $\text{Spec}_J(U_2^+(G_2))$ the $\mathcal{H}$-stratum associated to $J$; that is,

$$\text{Spec}_J(U_2^+(G_2)) = \{ P \in \text{Spec}(U_2^+(G_2)) \mid (P : \mathcal{H}) = J \}. \quad (2)$$

Then the $\mathcal{H}$-strata of $\text{Spec}(U_2^+(G_2))$ form a partition of $\text{Spec}(U_2^+(G_2))$ [2] Chapter II.2; that is,

$$\text{Spec}(U_2^+(G_2)) = \bigcup_{J \in \mathcal{H}\text{-Spec}(U_2^+(G_2))} \text{Spec}_J(U_2^+(G_2)). \quad (3)$$

This partition is the so-called $\mathcal{H}$-stratification of $\text{Spec}(U_2^+(G_2))$.

It follows from the work of Goodearl and Letzter [3] that every $\mathcal{H}$-prime ideal of $U_2^+(G_2)$ is completely prime, so $\mathcal{H}\text{-Spec}(U_2^+(G_2))$ coincides with the set of $\mathcal{H}$-invariant completely prime ideals of $U_2^+(G_2)$. Moreover there are precisely $|W|$ $\mathcal{H}$-prime ideals in $U_2^+(G_2)$, where $W$ denotes the Weyl group of type $G_2$ (see [18] Remark 6.2.2). As a consequence, the $\mathcal{H}$-stratification of $\text{Spec}(U_2^+(G_2))$ is finite and so the full strength of the $\mathcal{H}$-stratification theory of Goodearl and Letzter is available to study $\text{Spec}(U_2^+(G_2))$.

For each $\mathcal{H}$-prime ideal $J$ of $U_2^+(G_2)$, the space $\text{Spec}_J(U_2^+(G_2))$ is homeomorphic to the prime spectrum $\text{Spec}(\mathbb{K}[z_1^{\pm 1}, \ldots, z_6^{\pm 1}])$ of a commutative Laurent polynomial ring whose dimension depends on $J$.

These dimensions were computed in [11][23]. Finally, let us mention that the primitive ideals of $A$ are precisely the prime ideals that are maximal in their $\mathcal{H}$-strata [2] Theorem II.8.4].
In this article, we will mainly focus on one specific $\mathcal{H}$-stratum. Since $U_+^q(G_2)$ is a domain, 0 (technically, $(0)$) is clearly an $\mathcal{H}$-invariant completely prime ideal of $U_+^q(G_2)$, and so an $\mathcal{H}$-prime, and we will focus on computing its stratum, the so-called 0-stratum. The motivation here is twofold: first, in the $B_2$ case, we obtain “new” quantum deformation of the first Weyl algebra as $U_+^q(B_2)/P$, where $P$ is a primitive ideal from the 0-stratum of $\text{Spec}(U_+^q(B_2))$ [15]. Next, in the present case, we would like to construct algebras of GK dimension 4 as explained in the introduction. Since Tauvel’s height formula holds in $U_+^q(G_2)$ [8], we need to quotient $U_+^q(G_2)$ by a primitive ideals of height 2. Given that the $\mathcal{H}$-spectrum of $U_+^q(G_2)$ is homeomorphic to the Weyl group of type $G_2$, such primitive ideals can only be found in the 0-stratum and the strata associated to one of the two height 1 $\mathcal{H}$-primes. In this article, we mainly present results for the 0-stratum, but we will also indicate results obtained for the primitive quotients coming from the height 1 $\mathcal{H}$-prime strata.

2.3 Deleting derivations algorithms in $U_+^q(G_2)$

As $U_+^q(G_2)$ is a QNA, we can apply Cauchon’s Deleting Derivation Algorithm to study its prime spectrum.

Recall first that $U_+^q(G_2)$ is an iterated Ore extension of the form:

$$U_+^q(G_2) = \mathbb{K}[E_1][E_2;\sigma_2][E_3;\sigma_3,\delta_3][E_4;\sigma_4,\delta_4][E_5;\sigma_5,\delta_5][E_6;\sigma_6,\delta_6];$$

where, $\sigma_2$ denotes the automorphism of $\mathbb{K}[E_1]$ defined by:

$$\sigma_2(E_1) = q^{-3}E_1,$$

$\sigma_3$ denotes the automorphism of $\mathbb{K}[E_1][E_2;\sigma_2]$ defined by:

$$\sigma_3(E_1) = q^{-1}E_1 \quad \sigma_3(E_2) = q^{-3}E_2,$$

$\delta_3$ denotes the $\sigma_3$-derivation of $\mathbb{K}[E_1][E_2;\sigma_2]$ defined by:

$$\delta_3(E_1) = -(q + q^{-1} + q^{-3})E_2 \quad \delta_3(E_2) = 0,$$

$\sigma_4$ denotes the automorphism of $\mathbb{K}[E_1] \cdots [E_3;\sigma_3,\delta_3]$ defined by:

$$\sigma_4(E_1) = E_1 \quad \sigma_4(E_2) = q^{-3}E_2 \quad \sigma_4(E_3) = q^{-3}E_3,$$

$\delta_4$ denotes the $\sigma_4$-derivation of $\mathbb{K}[E_1] \cdots [E_3;\sigma_3,\delta_3]$ defined by:

$$\delta_4(E_1) = (1 - q^2)E_3^2 \quad \delta_4(E_2) = \frac{-q^4 + 2q^2 - 1}{q^4 + q^2 + 1}E_3^2 \quad \delta_4(E_3) = 0,$$

$\sigma_5$ denotes the automorphism of $\mathbb{K}[E_1] \cdots [E_4;\sigma_4,\delta_4]$ defined by:

$$\sigma_5(E_1) = qE_1 \quad \sigma_5(E_2) = E_2 \quad \sigma_5(E_3) = q^{-1}E_3 \quad \sigma_5(E_4) = q^{-3}E_4,$$

$\delta_5$ denotes the $\sigma_5$-derivation of $\mathbb{K}[E_1] \cdots [E_4;\sigma_4,\delta_4]$ defined by:

$$\delta_5(E_1) = -(1 + q^2)E_3 \quad \delta_5(E_2) = (1 - q^2)E_3^2 \quad \delta_5(E_3) = -(q + q^{-1} + q^{-3})E_4 \quad \delta_5(E_4) = 0,$$

$\sigma_6$ denotes the automorphism of $\mathbb{K}[E_1] \cdots [E_5;\sigma_5,\delta_5]$ defined by:

$$\sigma_6(E_1) = q^3E_1 \quad \sigma_6(E_2) = q^3E_2 \quad \sigma_6(E_3) = E_3 \quad \sigma_6(E_4) = q^{-3}E_4 \quad \sigma_6(E_5) = q^{-3}E_5,$$

and $\delta_6$ denotes the $\sigma_6$-derivation of $\mathbb{K}[E_1] \cdots [E_5;\sigma_5,\delta_5]$ defined by:

$$\delta_6(E_1) = -q^3E_5 \quad \delta_6(E_2) = (q^2 - q^4)E_3E_5 + (q^4 + q^2 - 1)E_4 \quad \delta_6(E_3) = (1 - q^2)E_5^2 \quad \delta_6(E_4) = \frac{-q^4 + 2q^2 - 1}{q^4 + q^2 + 1}E_5^2 \quad \delta_6(E_5) = 0.$$
The deleting derivations algorithm (DDA for short) constructs by a decreasing induction a family \( \{E_{1,j}, \ldots, E_{n,j}\} \) of elements of the division ring of fractions \( F_q = \text{Fract}(U_q^+(G_2)) \) of \( U_q^+(G_2) \) for each \( 2 \leq j \leq 7 \). The precise definition of these elements in the general context of QNAs can be found in \[4\]. In the present case, direct computation leads to:

\[
\begin{align*}
E_{1,6} &= E_1 + rE_3E_6^{-1} \\
E_{2,6} &= E_2 + tE_3E_6^{-1} + uE_4E_6^{-1} + nE_5E_6^{-2} \\
E_{3,6} &= E_3 + sE_3^2E_6^{-1} \\
E_{4,6} &= E_4 + bE_3E_6^{-1} \\
E_{1,5} &= E_{1,6} + hE_3E_6^{-1} + gE_4E_5^{-2} \\
E_{2,5} &= E_{2,6} + fE_3^2E_5^{-1} + pE_3E_6E_5^{-2} + eE_4^2E_5^{-3} \\
E_{3,5} &= E_{3,6} + aE_4E_5^{-1} \\
E_{1,4} &= E_{1,5} + sE_3^2E_4^{-1} \\
E_{2,4} &= E_{2,5} + bE_3E_4^{-1} \\
E_{1,3} &= E_{1,4} + aE_3E_3^{-1} \\
T_1 &:= E_{1,2} = E_{1,3} \\
T_2 &:= E_{2,2} = E_{2,3} = E_{2,4} \\
T_3 &:= E_{3,2} = E_{3,3} = E_{3,4} = E_{3,5} \\
T_4 &:= E_{4,2} = E_{4,3} = E_{4,4} = E_{4,5} = E_{4,6} \\
T_5 &:= E_{5,2} = E_{5,3} = E_{5,4} = E_{5,5} = E_{5,6} = E_5 \\
T_6 &:= E_{6,2} = E_{6,3} = E_{6,4} = E_{6,5} = E_{6,6} = E_6,
\end{align*}
\]

where the parameters \( a, b, c, f, g, h, n, p, r, s, t, u \) are all defined in Appendix \[B\].

In the following, we set \( A := U_q^+(G_2) \) and we denote by \( A^{(j)} \) the subalgebra of \( F_q \) generated by \( E_{1,j}, \ldots, E_{n,j} \). The following results were proved by Cauchon [4, Théorème 3.2.1 and Lemme 4.2.1]. For \( 2 \leq j \leq 7 \), we have:

1. when \( j = 7 \), \( (E_{1,7}, \ldots, E_{n,7}) = (E_1, \ldots, E_6) \), so that \( A^{(7)} = A = U_q^+(G_2) \);

2. \( A^{(j)} \) is isomorphic to an iterated Ore extension of the form

\[ \mathbb{K}[y_1] \cdots [y_{j-1}; \sigma_{j-1}, \delta_{j-1}] [y_j; \tau_j] \cdots [y_6; \tau_6] \]

by an isomorphism that sends \( E_{i,j} \) to \( y_i \) (1 \leq i \leq 6), where \( \tau_j, \ldots, \tau_6 \) denote the \( \mathbb{K} \)-linear automorphisms such that \( \tau_i(y_i) = \lambda_i y_i \) (1 \leq i \leq \ell).

3. Assume that \( j \neq 7 \) and set \( \Sigma_j := \{ E_{n,j+1}^n \mid n \in \mathbb{N} \} = \{ E_{n,j}^n \mid n \in \mathbb{N} \} \).

This is a multiplicative system of regular elements of \( A^{(j)} \) and \( A^{(j+1)} \), that satisfies the Ore condition in \( A^{(j)} \) and \( A^{(j+1)} \). Moreover we have

\[ A^{(j)}\Sigma_j^{-1} = A^{(j+1)}\Sigma_j^{-1}. \]

It follows from these results that \( A^{(j)} \) is a noetherian domain, for all \( 2 \leq j \leq 7 \).

As in [4], we use the following notation.

**Notation 2.1.** We set \( \overline{A} := A^{(2)} \) and \( T_i := E_{i,2} \) for all \( 1 \leq i \leq 6 \).

It follows from [4, Proposition 3.2.1] that \( \overline{A} \) is a quantum affine space in the indeterminates \( T_1, \ldots, T_6 \) and so can be presented as an iterated Ore extension in the \( T_i \)'s with no skew-derivations: it is for this
reason that Cauchon used the expression “effacement des dérivations”. More precisely, let \( M = (\mu_{i,j}) \in M_6(\mathbb{K}^*) \) be the multiplicatively antisymmetric matrix defined as follows:

\[
M = \begin{bmatrix}
0 & 3 & 1 & 0 & -1 & -3 \\
-3 & 0 & 3 & 3 & 0 & -3 \\
-1 & -3 & 0 & 3 & 1 & 0 \\
0 & -3 & -3 & 0 & 3 & 3 \\
1 & 0 & -1 & -3 & 0 & 3 \\
3 & 3 & 0 & -3 & -3 & 0 
\end{bmatrix}.
\]

Then we have

\[
\mathcal{A} = \mathbb{K}_q[M[T_1, \ldots, T_6] = \mathcal{O}_q(\mathbb{K}^6),
\]

where \( \mathbb{K}_q[M[T_1, \ldots, T_6] = \mathcal{O}_q(\mathbb{K}^6) \) denotes the \( \mathbb{K} \)-algebra generated by \( T_1, \ldots, T_6 \) with relations \( T_iT_j = q^{\mu_{i,j}} T_jT_i \) for all \( i,j \).

2.4 Canonical embedding

Since \( A = U_q^+ (G_2) \) is a QNA, one can use Cauchon’s DDA in order to relate the prime spectrum of \( A \) to the prime spectrum of the associated quantum affine space \( \overline{A} \). More precisely, the DDA allows the construction of embeddings

\[
\psi_j : \text{Spec}(A^{(j+1)}) \rightarrow \text{Spec}(A^{(j)}) \quad (2 \leq j \leq 6).
\]

Recall from [4, Section 4.3] that these embeddings are defined as follows.

Let \( P \in \text{Spec}(A^{(j+1)}) \). Then

\[
\psi_j(P) = \begin{cases}
PS_j^{-1} \cap A^{(j)} & \text{if } E_{j,j+1} = T_j \notin P \\
g_j^{-1}(P/\langle E_{j,j+1} \rangle) & \text{if } E_{j,j+1} \in P
\end{cases}
\]

where \( g_j \) denotes the surjective homomorphism

\[
g_j : A^{(j)} \rightarrow A^{(j+1)}/\langle E_{j,j+1} \rangle
\]

de fined by

\[
g_j(E_{i,j}) := E_{i,j+1} + \langle E_{j,j+1} \rangle
\]

(for more details, see [4, Lemme 4.3.2]). It was proved by Cauchon [4, Proposition 4.3.1] that \( \psi_j \) induces an increasing homeomorphism from the topological space

\[
\{ P \in \text{Spec}(A^{(j+1)}) \mid E_{j,j+1} \notin P \}
\]

onto

\[
\{ Q \in \text{Spec}(A^{(j)}) \mid E_{j,j} \notin Q \}
\]

whose inverse is also an increasing homeomorphism. Also, \( \psi_j \) induces an increasing homeomorphism from

\[
\{ P \in \text{Spec}(A^{(j+1)}) \mid E_{j,j+1} \in P \}
\]

onto its image by \( \psi_j \) whose inverse similarly is an increasing homeomorphism. Note however that, in general, \( \psi_j \) is not an homeomorphism from \( \text{Spec}(A^{(j+1)}) \) onto its image.

Composing these embeddings, we get an embedding

\[
\psi := \psi_2 \circ \cdots \circ \psi_6 : \text{Spec}(A) \rightarrow \text{Spec}(\overline{A}),
\]

which is called the canonical embedding from \( \text{Spec}(A) \) into \( \text{Spec}(\overline{A}) \).

The canonical embedding \( \psi \) is \( \mathcal{H} \)-equivariant so that \( \varphi(\mathcal{H} - \text{Spec}(A)) \subseteq \mathcal{H} - \text{Spec}(\overline{A}) \). Interestingly, the set \( \mathcal{H} - \text{Spec}(\overline{A}) \) has been described by Cauchon as follows. For any subset \( C \) of \( \{1, \ldots, 6\} \), let \( K_C \) denote the \( \mathcal{H} \)-prime ideal of \( \overline{A} \) generated by the \( T_i \) with \( i \in C \), that is

\[
K_C = \langle T_i \mid i \in C \rangle.
\]
It follows from [3, Proposition 5.5.1] that
\[ \mathcal{H} - \text{Spec}(\overline{A}) = \{ K_C \mid C \subseteq \{1, \ldots, 6\} \}, \]
so that
\[ \psi(\mathcal{H} - \text{Spec}(A)) \subseteq \{ K_C \mid C \subseteq \{1, \ldots, 6\} \}. \]

3 Primitive ideals of $U_q^+(G_2)$ in the 0-stratum

The aim of this section is to give explicit generating sets for the primitive ideals of $U_q^+(G_2)$ that belong to the 0-stratum. They are intimately related to the centre of $U_q^+(G_2)$ and so we start this section by making explicit the centre of $U_q^+(G_2)$ and related algebras.

3.1 Centre of $U_q^+(G_2)$

Recall that $\overline{A} := A^{(2)} = K_{\Lambda}[T_1, \ldots, T_6]$ is a quantum affine space. Set $\Omega_1 := T_1 T_3 T_5$ and $\Omega_2 := T_2 T_4 T_6$. One can easily verify that $\Omega_1$ and $\Omega_2$ are central elements of $\overline{A}$ by checking they commute with all $T_i$s.

We now want to successively pull $\Omega_1$ and $\Omega_2$ from the quantum affine space $\overline{A}$ into the algebra $A$ using the data of DDA of $A$ discussed above. Direct computation shows that
\[
\Omega_1 := T_1 T_3 T_5 = E_{1,4} E_{3,4} E_{5,4} + a E_{2,4} E_{5,4} = E_{1,5} E_{3,5} E_{5,5} + a E_{2,5} E_{5,5}
\]
\[
= E_{1,6} E_{3,6} E_{5,6} + a E_{2,6} E_{5,6} + a' E_{3,6}^2 = E_{1} E_{3} E_{5} + a E_{1} E_{4} + a E_{2} E_{5} + a' E_{3}^2,
\]
and
\[
\Omega_2 := T_2 T_4 T_6 = E_{2,5} E_{4,5} E_{6,5} + b E_{3,5} E_{6,5} = E_{2,6} E_{4,6} E_{6,6} + b E_{3,6} E_{6,6}
\]
\[
= E_{2} E_{4} E_{6} + b E_{2} E_{5}^3 + b E_{3} E_{6} + b' E_{2}^2 E_{5}^2 + c E_{3} E_{4} E_{5} + d E_{1}^2,
\]
where the parameters $a, b, a', b', c, d'$ can be found in Appendix 3. Note, $\Omega_1$ and $\Omega_2$ are central elements of $A^{(j)}$ for each $2 \leq j \leq 7$ since $\text{Fract}(A^{(j)}) = \text{Fract}(\overline{A})$.

We now want to show that the centre of $A$ and other related algebras is a polynomial ring generated by $\Omega_1$ and $\Omega_2$ over $\mathbb{K}$. The following discussions will lead us to the proof.

Set $S_j := \{ \lambda T_j \mid i_{j+1} \cdots T_6^n \mid i_j, \ldots, i_6 \in \mathbb{N} \text{ and } \lambda \in \mathbb{K}^* \}$ for each $2 \leq j \leq 6$. One can observe that $S_j$ is a multiplicative system of non-zero divisors of $A^{(j)} = \mathbb{K}(E_{i,j} \mid \text{ for all } i = 1, \ldots, 6)$. Furthermore, the elements $T_j, \ldots, T_6$ are all normal in $A^{(j)}$. Hence, $S_j$ is an Ore set in $A^{(j)}$. We can therefore localize $A^{(j)}$ at $S_j$ as follows:
\[ R_j := A^{(j)} S_j^{-1}. \]
Recall that $\Sigma_j := \{ T_j^n \mid n \in \mathbb{N} \}$ is an Ore set in both $A^{(j)}$ and $A^{(j+1)}$ for each $2 \leq j \leq 6$, and that
\[ A^{(j)} \Sigma_j^{-1} = A^{(j+1)} \Sigma_j^{-1}. \]
For all $2 \leq j \leq 6$, we have that:
\[ R_j = A^{(j)} S_j^{-1} = (A^{(j)} \Sigma_j^{-1}) S_{1+j}^{-1} = (A^{(j+1)} \Sigma_j^{-1}) S_{1+j}^{-1} = (A^{(j+1)} S_{j+1}^{-1}) \Sigma_j^{-1} = R_{j+1} \Sigma_j^{-1}. \]

Note, $R_7 := A$.

Again, one can also observe that $T_1$ is normal in $R_2$. As a result, we can form the localization $R_1 := R_2[T_1^{-1}]$. The algebra $R_1$ is the quantum torus associated to the quantum affine space $\overline{A}$. As a
result, $R_1 = \mathbb{K}_q^{\mathcal{M}}[T_1^{\pm 1}, \ldots, T_5^{\pm 1}]$, where $T_i T_j = q^{\mu_{ij}} T_j T_i$ for all $1 \leq i, j \leq 6$ and $\mu_{ij} \in M$. Similar to [17 §31], we construct the following tower of algebras:

$$A = R_7 \subset R_6 = R_7 \Sigma_6^{-1} \subset R_5 = R_6 \Sigma_5^{-1} \subset R_4 = R_5 \Sigma_4^{-1} \subset R_3 = R_4 \Sigma_3^{-1} \subset R_2 = R_3 \Sigma_2^{-1} \subset R_1.$$  \hfill (8)

Note, the family $(E_{ij}^{k_1} \cdots E_{ij}^{k_5})$, where $k_i \in \mathbb{N}$ if $i < j$ and $k_i \in \mathbb{Z}$ otherwise is a PBW-basis of $R_j$ for all $1 \leq i, j \leq 7$. In particular, the family $(T_1^{k_1} T_2^{k_2} T_3^{k_3} T_4^{k_4} T_5^{k_5} T_6^{k_6})_{k_i \cdots k_6 \in \mathbb{Z}}$ is a basis of $R_1$.

**Lemma 3.1.**
1. $Z(R_1) = \mathbb{K}[\Omega_1^\pm, \Omega_2^\pm]$.
2. $Z(R_3) = \mathbb{K}[\Omega_1, \Omega_2]$.
3. $Z(\mathcal{E}) = \mathbb{K}[\Omega_1, \Omega_2]$.

**Proof.**
1. It follows from [9 1.3] that $Z(R_1)$ is a commutative Laurent polynomial ring generated by certain monomials in the $T_i$s. A direct computation proves the result.

2. Clearly, $\mathbb{K}[\Omega_1, \Omega_2] \subseteq Z(R_3)$. For the reverse inclusion, let $y \in Z(R_3)$. Then, $y$ can be written in terms of the basis of $R_3$ (recall, $T_i = E_{i,3}$) as:

$$y = \sum_{(i, \cdots, n) \in \mathbb{N}^2 \times \mathbb{Z}} a_{i, \cdots, n} T_1^{i} T_2^{j} T_3^{k} T_4^{l} T_5^{m} T_6^{n}.$$  \hfill \hfill (9)

Using the fact that $T_1, \cdots, T_6$ are all normal elements in $R_3$ and $y T_i = T_i y$ for all $i$, one easily conclude that $i = k = m$ and $j = l = n$ for all monomials appearing in $y$. Since $i, j \geq 0$, we have that $y = \sum_{(i, j) \in \mathbb{N}^2} q^i a_{i, j} T_1^{i} T_2^{j} T_3^{k} T_4^{l} T_5^{m} T_6^{n} = \sum_{(i, j) \in \mathbb{N}^2} q^i a_{i, j} \Omega_1^{i} \Omega_2^{j}$. This implies that $y \in \mathbb{K}[\Omega_1, \Omega_2]$ as expected.

3. Similar to the previous case, and so left to the reader.  \hfill \square

**Lemma 3.2.** $Z(A) = \mathbb{K}[\Omega_1, \Omega_2]$.

**Proof.** Since $R_i$ is a localization of $R_{i+1}$, it follows that $Z(R_{i+1}) \subseteq Z(R_i)$. From [8], we have that $Z(A) \subseteq Z(R_3)$. Observe that $\mathbb{K}[\Omega_1, \Omega_2] \subseteq Z(A) \subseteq Z(R_3) = \mathbb{K}[\Omega_1, \Omega_2]$. Hence, $Z(A) = \mathbb{K}[\Omega_1, \Omega_2]$.  \hfill \square

**Remark 3.3.** Since $Z(A) = Z(R_3) = \mathbb{K}[\Omega_1, \Omega_2]$ and $Z(R_{i+1}) \subseteq Z(R_i)$, it follows from [8] that $Z(A) = Z(R_6) = Z(R_5) = Z(R_4) = Z(R_3) = \mathbb{K}[\Omega_1, \Omega_2]$. One can also deduce from Lemma 3.1 that $Z(R_2) = \mathbb{K}[\Omega_1, \Omega_2]$.

**Remark 3.4.** The centre of the positive part of a quantised enveloping algebra of a simple Lie algebra has been described by Caldero in [3] but we will need Remark 3.3 later on.

### 3.2 $\Omega_1$ and $\Omega_2$ generate completely prime ideals of $U_q^+(G_2)$

The aim of this paragraph is to show that $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$ are (completely) prime. We will make use of DDA to establish these facts. Note that we could also have used the results of [11] to obtain these results. However, we will need some of the intermediate steps obtained here to compute the derivations of certain primitive quotients of $U_q^+(G_2)$ in the final section.

From Section 2.4 we know that there is a bijection between \{ $P \in \text{Spec}(A^{(j+1)}) \mid P \cap \Sigma_j = \emptyset$ \} and \{ $Q \in \text{Spec}(A^{(j)}) \mid Q \cap \Sigma_j = \emptyset$ \} via $P = Q \Sigma_j^{-1} \cap A^{(j+1)}$. Note, $(T_1)$ and $(T_2)$ are prime ideals of the quantum affine space $\overline{A}$, since each of the factor algebras $\overline{A}/(T_1)$ and $\overline{A}/(T_2)$ is isomorphic to a quantum affine space of rank 5 which is well known to be a domain.

The following result and its proof show that $(T_1)$ belongs to the image $\text{Im}(\psi)$ of the canonical embedding $\psi$ and that $\langle \Omega_1 \rangle$ is the completely prime ideal of $A$ such that $\psi(\langle \Omega_1 \rangle) = (T_1)$.

**Lemma 3.5.** $\langle \Omega_1 \rangle \in \text{Spec}(A)$. 

---
Proof. We will prove this result in several steps by showing that:

1. \( \langle T_1 \rangle_{\text{A}(3)} \in \text{Spec}(A^{(3)}) \).
2. \( \langle E_{1,4} T_3 + a T_2 \rangle = \langle T_1 \rangle_{\text{A}(3)}[T_3^{-1}] \cap A^{(4)} \), hence \( Q_1 := \langle E_{1,4} T_3 + a T_2 \rangle \in \text{Spec}(A^{(4)}) \).
3. \( \langle E_{1,5} T_3 + a E_{2,5} \rangle = Q_1 [T_4^{-1}] \cap A^{(5)} \), hence \( Q_2 := \langle E_{1,5} T_3 + a E_{2,5} \rangle \in \text{Spec}(A^{(5)}) \).
4. \( \langle \Omega_1 \rangle_{\text{A}(6)} = Q_2 [T_5^{-1}] \cap A^{(6)} \), hence \( \langle \Omega_1 \rangle_{\text{A}(6)} \in \text{Spec}(A^{(6)}) \).
5. \( \langle \Omega_1 \rangle_A = \langle \Omega_1 \rangle_{\text{A}(6)} [T_6^{-1}] \cap A \), hence \( \langle \Omega_1 \rangle_A \in \text{Spec}(A) \).

We now proceed to prove the above claims.

1. One can easily verify that \( A^{(3)}/\langle T_1 \rangle \) is isomorphic to a quantum affine space of rank 5, which is a domain, hence \( \langle T_1 \rangle \) is a prime ideal in \( A^{(3)} \).

2. Note, \( T_1 = E_{1,4} + a T_2 T_3^{-1} \). We want to show that \( \langle E_{1,4} T_3 + a T_2 \rangle = \langle T_1 \rangle_{\text{A}(3)}[T_3^{-1}] \cap A^{(4)} \). Observe that \( \langle E_{1,4} T_3 + a T_2 \rangle \subseteq \langle T_1 \rangle_{\text{A}(3)}[T_3^{-1}] \cap A^{(4)} \). We established the reverse inclusion. Let \( y \in \langle T_1 \rangle_{\text{A}(3)}[T_3^{-1}] \cap A^{(4)} \). Then, \( y \in \langle T_1 \rangle_{\text{A}(3)}[T_3^{-1}] \). Therefore, there exists \( i \in \mathbb{N} \) such that \( y T_3^i = T_1 y \). This implies that \( y T_3^i = T_1 v \), for some \( v \in A^{(3)} \). Since \( A^{(3)}[T_3^{-1}] = A^{(4)}[T_3^{-1}] \), there exists \( j \in \mathbb{N} \) such that \( v T_3^j = v' \), for some \( v' \in A^{(4)} \). It follows that \( y T_3^{i+j} = T_1 v T_3^{j} = T_1 v' = (E_{1,4} + a T_2 T_3^{-1}) v' = (E_{1,4} T_3 + a T_2) T_3^{-1} v' \).

The multiplicative system generated by \( T_3 \) satisfies the Ore condition in \( A^{(4)} \), hence, there exists \( k \in \mathbb{N} \) and \( v'' \in A^{(4)} \) such that \( T_3^{-1} v'' = v'' T_3^{-k} \). One can therefore write \( y T_3^{i+j} = (E_{1,4} T_3 + a T_2) v'' T_3^{-k} \). This implies that \( y T_3^i = \Omega'_4 v'' \), where \( \Omega'_4 := E_{1,4} T_3 + a T_2 \) and \( \delta = i + j \). Set \( S := \{ s \in \mathbb{N} \mid \exists r'' \in A^{(4)} : y T_3^s = \Omega'_4 r'' \} \). Note, \( S \neq \emptyset \), since \( \delta \in S \). Let \( s = s_0 \) be the minimum element of \( S \) such that \( y T_3^{s_0} = \Omega'_4 r'' \). We want to show that \( s_0 = 0 \). Remember, \( \Omega'_4 T_5 = \Omega_1 A^{(4)} \). Since \( \Omega_1 \) is central in \( A^{(4)} \), and \( T_5 \) is normal in \( A^{(4)} \), we must have \( \Omega'_4 \) to be a normal element in \( A^{(4)} \), otherwise, there will be a contradiction. Therefore, there exists \( w \in A^{(4)} \) such that \( y T_3^{s_0} = \Omega'_4 v'' = w \Omega'_4 \). Now, \( A^{(4)} \) can be viewed as a free left \( \mathbb{K}_E \langle E_{1,4}, T_2, T_4, T_5, T_6 \rangle \)-module with basis \( \langle T_3^\xi \rangle_{\xi \in \mathbb{N}} \). One can therefore write \( y = \sum_{\xi=0}^{n} \alpha_\xi T_3^\xi \) and \( w = \sum_{\xi=0}^{n} \beta_\xi T_3^\xi \), where \( \alpha_\xi, \beta_\xi \in \mathbb{K}_E \langle E_{1,4}, T_2, T_4, T_5, T_6 \rangle \). This implies that \( \sum_{\xi=0}^{n} \alpha_\xi T_3^{s_0+\xi} = \sum_{\xi=0}^{n} \beta_\xi T_3^{s_0+\xi} \Omega'_4 = \sum_{\xi=0}^{n} q^\xi \beta_\xi \Omega'_4 T_3^\xi \) (note, \( T_3 \Omega'_4 = q^{-1} \Omega_1 T_3 \)). Given that \( \Omega'_4 = E_{1,4} T_3 + a T_2 \), we have that \( \sum_{\xi=0}^{n} \alpha_\xi T_3^{s_0+\xi} = \sum_{\xi=0}^{n} q^\xi \beta_\xi E_{1,4} T_3^{1+\xi} + \sum_{\xi=0}^{n} q^\xi \alpha_\xi T_3^\xi \). Suppose that \( s_0 > 0 \). Then, identifying the constant coefficients, we have \( \alpha_0 T_3^0 = 0 \). As a result, \( \beta_0 = 0 \), since \( q^\xi a T_2 \neq 0 \). Hence, \( w \) can be written as \( w = \sum_{\xi=1}^{n} \beta_\xi T_3^\xi \). Returning to \( y T_3^{s_0} = \Omega'_4 r'' \), we have that \( y T_3^{s_0} = \sum_{\xi=1}^{n} \beta_\xi T_3^\xi \Omega'_4 = \sum_{\xi=1}^{n} \beta_\xi \Omega'_4 T_3^\xi = \Omega'_4 \sum_{\xi=1}^{n} q^\xi \beta_\xi T_3^\xi \). This implies that \( y T_3^{s_0-1} = \Omega'_4 w' \), where \( w' = \sum_{\xi=1}^{n} q^\xi \beta_\xi T_3^{\xi-1} \in A^{(4)} \), with \( \beta_\xi \in \mathbb{K}_E \langle E_{1,4}, T_2, T_4, T_5, T_6 \rangle \). Consequently, \( s_0 - 1 \in S \), a contradiction! Therefore, \( s_0 = 0 \) and \( y = \Omega'_4 r'' \in \langle \Omega_4 \rangle = \langle E_{1,4} T_3 + a T_2 \rangle \). Hence, \( \langle T_1 \rangle_{\text{A}(3)}[T_3^{-1}] \cap A^{(4)} \subseteq \langle E_{1,4} T_3 + a T_2 \rangle \) as desired.

The following steps are proved in a similar manner to Step 2. They are left to the reader who might want to check details in [19].

Using similar techniques, one can prove that \( \langle T_2 \rangle \in \text{Im}(\psi) \) and that \( \langle \Omega_2 \rangle \) is the completely prime ideal of \( A \) such that \( \psi(\langle \Omega_2 \rangle) = \langle T_2 \rangle \). Again, we refer the interested reader to [19] for details. We record these facts in the following lemma.

Lemma 3.6. \( \langle T_2 \rangle \in \text{Im}(\psi) \) and \( \langle \Omega_2 \rangle \) is the completely prime ideal of \( A \) such that \( \psi(\langle \Omega_2 \rangle) = \langle T_2 \rangle \).

Since \( \mathcal{H}\text{-Spec}(U_q^+(G_2)) \) is homeomorphic to the Weyl group \( W \) of type \( G_2 \) by [22], there are only 2 \( \mathcal{H} \)-primes in \( U_q^+(G_2) \) of height 1. Since \( \Omega_1 \) and \( \Omega_2 \) are central, the prime ideals that they generate have height less than or equal to 1, and so equal to 1. As an immediate consequence, we get the following result.

Lemma 3.7. 1. \( \langle \Omega_1 \rangle \) and \( \langle \Omega_2 \rangle \) are the only height one \( \mathcal{H} \)-invariant prime ideals of \( A \).

2. Every non-zero \( \mathcal{H} \)-invariant prime ideal of \( A \) contains either \( \langle \Omega_1 \rangle \) or \( \langle \Omega_2 \rangle \).
3.3 Description of the 0-stratum and beyond

In this section, we will often assume that our base field $\mathbb{K}$ is algebraically closed. This assumption is actually not necessary for the main result of this section, Theorem 3.22, but makes the description of the 0-stratum easier to present.

This section focuses on finding the height two maximal ideals of $A = U_q^+(G_2)$. Note first that such ideals can only belong to the $H$-stratum of an $H$-prime of height less than or equal to 1 (since $H$-$Spec(A)$ is isomorphic to $W$). It follows from the previous sections that we need to compute the $H$-strata of 3 $H$-primes: $0$, $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$. We start with the 0-stratum.

The strategy is similar to Propositions 2.3 and 2.4. Note, in this subsection, all ideals in $A$ will simply be written as $\langle \Theta \rangle$, where $\Theta \in A$. However, if we want to refer to an ideal in any other algebra, say $R$, then that ideal will be written as $\langle \Theta \rangle_R$, where in this case, $\Theta \in R$.

**Proposition 3.8.** Assume $\mathbb{K}$ is algebraically closed. Let $P$ be the set of those unitary irreducible polynomials $P(\Omega_1, \Omega_2) \in \mathbb{K}[\Omega_1, \Omega_2]$ with $P(\Omega_1, \Omega_2) \neq \Omega_1$ and $P(\Omega_1, \Omega_2) \neq \Omega_2$. Then, $Spec(0)(A) = \{\{0\} \cup \{P(\Omega_1, \Omega_2) \mid P(\Omega_1, \Omega_2) \in P\} \cup \{\Omega_1 - \alpha, \Omega_2 - \beta, \Omega_1 - \alpha, \Omega_2 - \beta \mid \alpha, \beta \in \mathbb{K}^*\}$.

**Proof.** We claim that $Spec(0)(A) = \{Q \in Spec(A) \mid \Omega_1, \Omega_2 \not\in Q\}$. To establish this claim, let us assume that this is not the case. That is, suppose there exists $Q \in Spec(0)(A)$ such that $\Omega_1, \Omega_2 \not\in Q$; then the product $\Omega_1 \Omega_2$ which is an $H$-eigenvector belongs to $Q$. Consequently, $\Omega_1 \Omega_2 \in \bigcap_{h \in H} h \cdot Q = \{0\}$, a contradiction. Hence, we have shown that $Spec(0)(A) \subseteq \{Q \in Spec(A) \mid \Omega_1, \Omega_2 \not\in Q\}$. Conversely, suppose that $Q \in Spec(A)$ such that $\Omega_1, \Omega_2 \not\in Q$, then $\bigcap_{h \in H} h \cdot Q$ is an $H$-invariant prime ideal of $A$, which contains neither $\Omega_1$ nor $\Omega_2$. Obviously, the only possibility for $\bigcap_{h \in H} h \cdot Q = \{0\}$ since every non-zero $H$-invariant prime ideal contains at least $\Omega_1$ or $\Omega_2$. Thus, $\bigcap_{h \in H} h \cdot Q = \{0\}$, hence, $Q \in Spec(0)(A)$. Therefore, $\{Q \in Spec(A) \mid \Omega_1, \Omega_2 \not\in Q\} \subseteq Spec(0)(A)$. This confirms our claim.

Since $\Omega_1, \Omega_2 \in \langle A \rangle$, we have that the set $\{\Omega_1^i \Omega_2^j \mid i, j \in \mathbb{N}\}$ is a right denominator set in the noetherian domain $A$. One can now localize $A$ as $R := A[\Omega_1^{-1}, \Omega_2^{-1}]$. Let $Q \in Spec(0)(A)$, the map $\phi : Q \rightarrow Q[\Omega_1^{-1}, \Omega_2^{-1}]$ is an increasing bijection from $Spec(0)(A)$ onto $Spec(R)$.

Since $\Omega_1$ and $\Omega_2$ are $H$-eigenvectors, and $H$ acts on $A$, we have that $H$ also acts on $R$. Since every $H$-prime ideal of $A$ contains $\Omega_1$ or $\Omega_2$, one can easily check that $R$ is $H$-simple (in the sense that the only $H$-invariant ideal of $R$ is the 0 ideal).

We proceed to describe $Spec(R)$ and $Spec(0)(A)$. We deduce from [2, Exercise II.3.1] that the action of $H$ on $R$ is rational. This rational action coupled with $R$ being $H$-simple implies that the extension and contraction maps provide a mutually inverse bijection between $Spec(R)$ and $Spec(Z(R))$. From Lemma 3.2, $Z(R) = \mathbb{K}[\Omega_1, \Omega_2]$, and $Z(R) = \mathbb{K}[\Omega_1^{-1}, \Omega_2^{-1}]$. Since $\mathbb{K}$ is algebraically closed, we have that $Spec(Z(R)) = \{\{0\} \cup \{P(\Omega_1, \Omega_2) | P(\Omega_1, \Omega_2) \not\in P\} \cup \{\Omega_1 - \alpha, \Omega_2 - \beta \mid \alpha, \beta \in \mathbb{K}^*\}$. Since there is an inverse bijection between $Spec(R)$ and $Spec(Z(R))$, and also $R$ is $H$-simple, one can recover $Spec(R)$ from $Spec(Z(R))$ as follows: $Spec(R) = \{\{0\} \cup \{P(\Omega_1, \Omega_2) | P(\Omega_1, \Omega_2) \not\in P\} \cup \{\Omega_1 - \alpha, \Omega_2 - \beta | \alpha, \beta \in \mathbb{K}^*\}$.

It follows that $Spec(0)(A) = \{\{0\} \cap A \cup \{\Omega_1, \Omega_2 \mid \Omega_1 - \alpha, \Omega_2 - \beta, \alpha, \beta \in \mathbb{K}^*\}$.

Undoubtedly, $\langle 0 \rangle \cap A = \{0\}$. We now have to show that $\langle P(\Omega_1, \Omega_2) \rangle \cap A = \langle P(\Omega_1, \Omega_2) \rangle, \forall P(\Omega_1, \Omega_2) \in P$, and $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \cap A = \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle, \forall \alpha, \beta \in \mathbb{K}^*$ to complete the proof.

Fix $P(\Omega_1, \Omega_2) \in P$. Observe that $\langle P(\Omega_1, \Omega_2) \rangle \subseteq \langle P(\Omega_1, \Omega_2) \rangle_R \cap A$. To show the reverse inclusion, let $y \in \langle P(\Omega_1, \Omega_2) \rangle_R \cap A$. This implies that $y \in \langle P(\Omega_1, \Omega_2) \rangle$, where $d \in R$, since $y \in \langle P(\Omega_1, \Omega_2) \rangle$. Also, $d \in R$ implies that there exist $i, j \in \mathbb{N}$ such that $d = a_{\Omega_1}^{-i} \Omega_2^{-j}$, where $a \in A$. Therefore, $y = a_{\Omega_1}^{-i} \Omega_2^{-j} \langle P(\Omega_1, \Omega_2) \rangle$, which implies that $y \Omega_1^i \Omega_2^j = a P(\Omega_1, \Omega_2)$. Choose $(i, j) \in \mathbb{N}^2$ minimal (in the lexicographic order on $\mathbb{N}^2$) such that the equality holds. Without loss of generality, suppose that $i > 0$, then $a P(\Omega_1, \Omega_2) \in \langle \Omega_1 \rangle$. Given that $\langle \Omega_1 \rangle$ is a completely prime ideal, it implies that $a \in \langle \Omega_1 \rangle$ or $P(\Omega_1, \Omega_2) \in \langle \Omega_1 \rangle$. Since $P(\Omega_1, \Omega_2) \in P$, it implies that $P(\Omega_1, \Omega_2) \not\in \langle \Omega_1 \rangle$, hence $a \in \langle \Omega_1 \rangle$. This further implies that $a = t\Omega_1$, where $t \in A$. Returning to $y \Omega_1^i \Omega_2^j = a P(\Omega_1, \Omega_2)$, we have that $y \Omega_1^i \Omega_2^j = t \Omega_1 P(\Omega_1, \Omega_2)$. Therefore, $y \Omega_1^{-i} \Omega_2^{-j} = t P(\Omega_1, \Omega_2)$. This clearly contradicts the minimality of $(i, j)$, hence $(i, j) = (0, 0)$, and $y = a P(\Omega_1, \Omega_2) \in \langle P(\Omega_1, \Omega_2) \rangle$. Consequently, $\langle P(\Omega_1, \Omega_2) \rangle \cap A = \langle P(\Omega_1, \Omega_2) \rangle$ for all $P(\Omega_1, \Omega_2) \in P$ as desired.

Similarly, we show that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \cap A = \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle, \forall \alpha, \beta \in \mathbb{K}^*$. Fix $\alpha, \beta \in \mathbb{K}^*$. Observe that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \subseteq \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \cap A$. We establish the reverse inclusion. Let $y \in \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle_R \cap A.$
Since \( y \in \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle_R \), there exist \( i, j \in \mathbb{N} \) such that \( y \Omega_1^i \Omega_2^j = m(\Omega_1 - \alpha) + n(\Omega_2 - \beta) \), where \( m, n \in A \). Choose \((i,j) \in \mathbb{N}^2\) minimal (in the lexicographic order on \( \mathbb{N}^2 \)) such that the equality holds. Without loss of generality, suppose that \( i > 0 \) and let \( f : A \rightarrow A/\langle \Omega_2 - \beta \rangle \) be a canonical surjection. We have that \( f(y)f(\Omega_1)^i f(\Omega_2)^j = f(m)f(\Omega_1 - \alpha) \). It follows that \( f(m)f(\Omega_1 - \alpha) \in \langle f(\Omega_1) \rangle \) and so \( \alpha f(m) \in \langle f(\Omega_1) \rangle \). Observe that \( f(\Omega_1 - \alpha) \notin \langle f(\Omega_1) \rangle \), hence \( f(m) \in \langle f(\Omega_1) \rangle \). As \( \alpha \neq 0 \), we obtain the existence of \( \lambda \in A \) such that \( f(m) = f(\lambda)f(\Omega_1) \). Consequently, \( f(y)f(\Omega_1)^i f(\Omega_2)^j = f(\lambda)f(\Omega_1)f(\Omega_1 - \alpha) \). Since \( f(\Omega_1) \neq 0 \), it implies that \( f(y)f(\Omega_1)^{-1} f(\Omega_2)^j = f(\lambda)f(\Omega_1 - \alpha) \). Therefore, \( y \Omega_1^{-1} \Omega_2^j = \lambda(\Omega_1 - \alpha) + \lambda' (\Omega_2 - \beta) \) for some \( \lambda' \in A \). This contradicts the minimality of \((i, j)\). Hence, \((i, j) = (0, 0)\) and so \( y = m(\Omega_1 - \alpha) + n(\Omega_2 - \beta) \in \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \). In conclusion, \( \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle_R \cap A = \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle, \forall \alpha, \beta \in \mathbb{K}^* \). □

Using similar techniques, we obtain the following description for the \( H \)-strata of \( \langle \Omega_1 \rangle \) and \( \langle \Omega_2 \rangle \).

**Proposition 3.9.** Assume \( \mathbb{K} \) is algebraically closed.

1. \( \text{Spec}_{\langle \Omega_1 \rangle}(A) = \{ \langle \Omega_1 \rangle \} \cup \{ \langle \Omega_1, \Omega_2 - \beta \rangle \mid \beta \in \mathbb{K}^* \} \).
2. \( \text{Spec}_{\langle \Omega_2 \rangle}(A) = \{ \langle \Omega_2 \rangle \} \cup \{ \langle \Omega_1 - \alpha, \Omega_2 \rangle \mid \alpha \in \mathbb{K}^* \} \).

Since maximal ideals in their stratum are primitive for a QNA, we obtain the following result.

**Corollary 3.10.** Assume \( \mathbb{K} \) is algebraically closed and let \( (\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\} \). The ideal \( \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \) of \( A \) is primitive.

**Remark 3.11.** The statement of the above corollary is still valid without the assumption that \( \mathbb{K} \) is algebraically closed. The proof is actually similar as we only use this assumption to get a full description of the strata we were interested in.

We can actually prove a stronger result.

**Theorem 3.12.** Let \( (\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\} \). The prime ideal \( \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \) of \( A \) is maximal.

**Proof.** Let \( (\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\} \). Suppose that there exists a maximal ideal \( I \) of \( A \) such that \( \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \subset I \subset A \). Let \( J \) be the \( H \)-invariant prime ideal in \( A \) such that \( I \in \text{Spec}_J(A) \).

We claim that \( J \) cannot be \( \langle 0 \rangle, \langle \Omega_1 \rangle \) or \( \langle \Omega_2 \rangle \). For instance, if \( \alpha, \beta \neq 0 \), then \( J \) cannot be equal to \( \langle 0 \rangle \) since in this case \( \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \) is maximal in the 0-stratum. Moreover, \( J \neq \langle \Omega_1 \rangle \) as otherwise \( I \) would contain \( \alpha = \Omega_1 - (\Omega_1 - \alpha) \), a contradiction. The other cases are similar and left to the reader.

This means that \( J \) is an \( H \)-prime of height at least equal to 2. As the poset of \( H \)-primes is isomorphic to \( W \), this forces \( J \) to contain both \( \Omega_1 \) and \( \Omega_2 \). Moreover, since \( J \subset I \), it implies that \( \Omega_1, \Omega_2 \in I \). Given that \( \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \subset I \), we have that \( \Omega_1 - \alpha, \Omega_2 - \beta \in I \). It follows that \( \alpha, \beta \in I \), hence \( I = A \), a contradiction! This confirms that \( \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \) is a maximal ideal in \( A \). □

4. **Simple quotients of** \( U_q^+(G_2) \) **and their relation to the second Weyl algebra**

Now that we have found primitive ideals of \( A = U_q^+(G_2) \), we are going to study the corresponding simple quotient algebras. In view of Dixmier’s theorem, we consider these simple quotients as deformations of a Weyl algebra (of appropriate dimension), and so we compare their properties with some known properties of the Weyl algebras. In this section, we prove that the Gelfand-Kirillov dimension of \( A_{\alpha, \beta} \) is 4 and consequently prove that the height of the maximal ideal \( \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \) is 2 as expected. Then we focus on describing a linear basis of \( A_{\alpha, \beta} \); we use this basis in the following section to study the derivations of \( A_{\alpha, \beta} \). Finally, we show that with appropriate choices of \( \alpha \) and \( \beta \), the algebra \( A_{\alpha, \beta} \) is a quadratic extension of the second Weyl algebra \( A_2(\mathbb{K}) \) at \( q = 1 \).

Recall from Theorem 3.12 that \( \Omega_1 - \alpha \) and \( \Omega_2 - \beta \), where \((\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\} \), generate a maximal ideal of \( A \). As a result, the corresponding quotient

\[ A_{\alpha, \beta} := \frac{A}{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle} \]
is a simple noetherian domain. Denote the canonical images of $E_i$ in $A_{\alpha,\beta}$ by $e_i := E_i + (\Omega_1 - \alpha, \Omega_2 - \beta)$ for all $1 \leq i \leq 6$. The algebra $A_{\alpha,\beta}$ satisfies the following relations:

$$
e_2 e_1 = q^{-3} e_1 e_2 \quad \quad e_3 e_1 = q^{-1} e_1 e_3 - (q + q^{-1} + q^{-3}) e_2 \quad \quad e_4 e_1 = e_1 e_4 + (1 - q^2) e_3^2$$

$$e_4 e_2 = q^{-3} e_2 e_4 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1} e_3^3 \quad \quad e_4 e_3 = q^{-3} e_3 e_4$$

$$e_5 e_1 = q e_1 e_5 - (1 + q^2) e_3 \quad \quad e_5 e_2 = e_2 e_5 + (1 - q^2) e_3^2 \quad \quad e_5 e_3 = q^{-1} e_3 e_5 - (q + q^{-1} + q^{-3}) e_4$$

$$e_6 e_1 = q^3 e_1 e_6 - q^3 e_5 \quad \quad e_6 e_2 = q^3 e_2 e_6 + (q^4 + q^2 - 1) e_4 + (q^2 - q^4) e_3 e_5$$

$$e_6 e_3 = e_3 e_6 + (1 - q^2) e_5^2 \quad \quad e_6 e_4 = q^{-3} e_4 e_6 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1} e_5^3$$

$$e_6 e_5 = q^{-3} e_5 e_6,$$

and

$$e_1 e_3 e_5 + a e_1 e_4 + a e_2 e_5 + a' e_4^2 = \alpha, \quad (9)$$

$$e_2 e_4 e_6 + b e_2 e_3^3 + b e_3 e_6 + b' e_5 e_6^2 + c' e_3 e_4 e_5 + d' e_1^2 = \beta. \quad (10)$$

The following additional relations of $A_{\alpha,\beta}$ in the lemma below will be helpful when computing linear basis for $A_{\alpha,\beta}$. Note, we put constant coefficients of monomials in a square bracket $[\, \,]$ in order to distinguish them from monomials easily. These constants are defined in Appendix H.

**Lemma 4.1.**

1. $c_4^1 = c_1 \alpha + [c_2] e_2 e_5 + [c_2] e_1 e_4 + [c_2] e_1 e_3 e_5$.

2. $c_4^2 = b_1 \beta + [b_2] e_2 e_4 e_6 + [b_3] e_2 e_3^2 + [b_4] a e_3 e_6 + [b_5] e_2 e_3 e_5 e_6 + [b_6] e_1 e_3 e_4 e_6$

$$+ [b_7] a e_1 e_5 e_6 + [b_8] e_1 e_2 e_5^2 e_6 + [b_9] e_1 e_4 e_6 e_6 + [b_{10}] e_1^2 e_3 e_2^2 e_6 + [b_{11}] a e_5 e_6^2$$

$$+ [b_{12}] e_1 e_3 e_5^3 + [b_{13}] e_3 e_4 e_5 + [b_{14}] e_1 e_4 e_5^2$$.  

3. $c_3^2 e_4 = [c_1] e_4 + [q^{-3} c_2] e_2 e_4 e_5 + [c_2 b_4] a e_3 e_6 + [b_{15}] e_1 e_3 e_4 e_5 + [b_{16}] e_1 e_3 e_5 e_6 + [b_{17}] a e_2 e_4 e_6$

$$+ [c_{20}] a e_3 e_2 e_5 + [c_{21}] a e_2 e_3 e_5 e_6 + [b_{20}] e_1^2 e_3 e_4 e_6 + [c_{20}] a e_3 e_2^2 e_6 + [c_{21}] a e_2 e_3 e_5 e_6$

$$+ [c_{22}] a e_3 e_2 e_5 + [c_{23}] a e_2 e_3 e_5 e_6 + [c_{24}] a e_3 e_2 ^2 e_6 + [c_{25}] a e_2 e_3 e_5 e_6$$.  

4. $c_3 e_2^2 = [b_{15}] e_3 + [k_1] e_2 e_3 e_5 e_6 + [k_2] e_2 e_3 e_6^2 + [k_3] a e_3 e_5 e_6 + [k_4] a e_2 e_3 e_5 e_6 + [k_5] a e_1 e_4 e_6$

$$+ [k_6] a e_1 e_3 e_5 e_6 + [k_7] a e_2 e_5 e_6 + [k_8] a e_1 e_3 e_5 e_6 + [k_9] a e_1 e_2 e_4 e_5 e_6 + [k_{10}] a e_3 e_2 e_3 e_5 e_6 + [k_{11}] a e_1 e_4 e_6$

$$+ [k_{12}] a e_3 e_2^2 e_6 + [k_{13}] a e_2 e_3 e_5 e_6 + [k_{14}] a e_3 e_2 e_5 e_6 + [k_{15}] a e_2 e_3 e_5 e_6 + [k_{16}] a e_2 e_3 e_5 e_6 + [k_{17}] a e_2 e_3 e_5 e_6 + [k_{18}] a e_2 e_3 e_5 e_6 + [k_{19}] a e_3 e_2 e_5 e_6$$.  

$$+ [k_{20}] a e_3 e_2 e_5^2 + [k_{21}] a e_2 e_3 e_5 e_6 + [k_{22}] a e_2 e_3 e_5 e_6 + [k_{23}] a e_3 e_2 e_5 e_6 + [k_{24}] a e_3 e_2 e_5 e_6$$.  

$$+ [k_{25}] a e_3 e_2 e_5 + [k_{26}] a e_2 e_3 e_5 e_6 + [k_{27}] a e_3 e_2 e_5 e_6 + [k_{28}] e_2 e_4 e_5 + [k_{29}] e_1 e_3 e_4 e_5^2 + [k_{30}] e_1 e_2 e_5^2 + [k_{31}] a e_1 e_2 e_3 e_5 e_6$.  

**Proof.** This is proved by brute-force computation, left to the reader. \(\square\)
4.1 Gelfand-Kirillov dimension of $A_{\alpha,\beta}$

We refer the reader to [13] for background on the Gelfand-Kirillov dimension (GKdim for short).

Assume first that $\alpha, \beta \neq 0$. Recall from Section 3.11 that $R_1 = \mathbb{K}_q\mu[T_1^\pm 1, \ldots, T_6^\pm 1]$ is the quantum torus associated to the quantum affine space $\overline{A} = A^{(2)}$. Also, $\Omega_1 = T_1 T_3 T_5$ and $\Omega_2 = T_2 T_4 T_6$ in $\overline{A}$. It follows from [2] Theorem 5.4.1] that there exists an Ore set $S_{\alpha,\beta}$ in $A_{\alpha,\beta}$ such that $A_{\alpha,\beta}S_{\alpha,\beta}^{-1} \cong R_1/(T_1 T_3 T_5 - \alpha, T_2 T_4 T_6 - \beta)$.

Now, set

$$\mathcal{A}_{\alpha, \beta} := \frac{R_1}{\langle T_1 T_3 T_5 - \alpha, T_2 T_4 T_6 - \beta \rangle}.$$ 

Let $t_i := T_i + \langle T_1 T_3 T_5 - \alpha, T_2 T_4 T_6 - \beta \rangle$ denote the canonical images of the generators $T_i$ of $R_1$ in $\mathcal{A}_{\alpha, \beta}$. The algebra $\mathcal{A}_{\alpha, \beta}$ is generated by $t_i^{\pm 1}, \ldots, t_6^{\pm 1}$ subject to the following relations:

$$t_i t_j = q^{\mu_{ij}} t_j t_i, \quad t_1 = \alpha t_1^{-1} t_3^{-1}, \quad t_2 = \beta t_6^{-1} t_4^{-1},$$

for all $1 \leq i, j \leq 6$ and $M = (\mu_{ij})$ (the skew-symmetric matrix in Section 3.11). Observe that $\mathcal{A}_{\alpha, \beta} \cong \mathbb{K}_q^N[t_3^\pm 1, t_4^\pm 1, t_5^\pm 1, t_6^\pm 1]$, where the skew-symmetric matrix $N$ can easily be deduced from $M$ (by deleting the first two rows and columns) as follows:

$$N := \begin{bmatrix} 0 & 3 & 1 & 0 \\ -3 & 0 & 3 & 3 \\ -1 & -3 & 0 & 3 \\ 0 & -3 & -3 & 0 \end{bmatrix}.$$ 

Secondly, suppose that $\alpha = 0$ and $\beta \neq 0$.

Then, $A_{\alpha,\beta}S_{0,\beta}^{-1} \cong R_{0,\beta} = R_1/(T_1, T_2 T_4 T_6 - \beta)$. The algebra $\mathcal{A}_{0, \beta}$ is generated by $t_i^{\pm 1}, \ldots, t_6^{\pm 1}$ subject to the relations

$$t_i t_j = q^{\mu_{ij}} t_j t_i, \quad t_1 = 0, \quad t_2 = \beta t_6^{-1} t_4^{-1},$$

for all $1 \leq i, j \leq 6$ and $\mu_{ij} \in M$. We also have that $\mathcal{A}_{0,\beta} \cong \mathbb{K}_q^N[t_3^\pm 1, t_4^\pm 1, t_5^\pm 1, t_6^\pm 1]$.

Finally, when $\alpha \neq 0$ and $\beta = 0$, then one can also verify that $\mathcal{A}_{\alpha,0} \cong \mathbb{K}_q^N[t_3^\pm 1, t_4^\pm 1, t_5^\pm 1, t_6^\pm 1]$.

As a result of the above discussion, in all cases, we have that $A_{\alpha,\beta}S_{\alpha,\beta}^{-1} \cong \mathcal{A}_{\alpha, \beta} \cong \mathbb{K}_q^N[t_3^\pm 1, t_4^\pm 1, t_5^\pm 1, t_6^\pm 1]$.

With a slight abuse of notation, we write $A_{\alpha,\beta}S_{\alpha,\beta}^{-1} = \mathcal{A}_{\alpha, \beta} = \mathbb{K}_q^N[t_3^\pm 1, t_4^\pm 1, t_5^\pm 1, t_6^\pm 1]$ for all $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\}$. It follows from [8] Theorem 6.3] that GKdim($A_{\alpha,\beta}$) = GKdim($A_{\alpha,\beta}S_{\alpha,\beta}^{-1}$) = GKdim($\mathcal{A}_{\alpha, \beta}$) = 4. Since Taucel’s height formula holds in $A = U_q^+(G_2)$ [8], we have that GKdim($A$) = ht($\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$) + GKdim($A_{\alpha,\beta}$). Since GKdim($A$) = 6, we conclude that ht($\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$) = 2 for all $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\}$.

**Proposition 4.2.** GKdim($A_{\alpha,\beta}$) = 4 for all $(\alpha, \beta) \neq (0, 0)$.

4.2 Linear basis for $A_{\alpha,\beta}$

Set $A_{\beta} := A/\langle \Omega_2 - \beta \rangle$, where $\beta \in \mathbb{K}$. Now, denote the canonical images of $E_i$ by $\hat{e}_i := E_i + \langle \Omega_2 - \beta \rangle$ in $A_{\beta}$. Clearly, $A_{\alpha,\beta} \cong A_{\beta}/\langle \Omega_1 - \alpha \rangle$. As a result, one can identify $A_{\alpha,\beta}$ with $A_{\beta}/\langle \Omega_1 - \alpha \rangle$. Moreover, the algebra $A_{\beta}$ satisfies the relations of $A = U_q^+(G_2)$ and

$$\hat{e}_2 \hat{e}_4 \hat{e}_6 + b \hat{e}_2 \hat{e}_3^3 + b \hat{e}_3^3 \hat{e}_6 + b' \hat{e}_3^2 \hat{e}_5^2 + c' \hat{e}_3 \hat{e}_4 \hat{e}_5 + d' \hat{e}_4^2 = \beta.$$ 

(11)

From Propositions 3.8 and 3.9, one can conclude that $\langle \Omega_2 - \beta \rangle$ is a completely prime ideal (since it is a prime ideal) of $A$ for all $\beta \in \mathbb{K}$. Hence, the algebra $A_{\beta}$ is a noetherian domain.

We are now going to find a linear basis for $A_{\alpha,\beta}$, where $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\}$. Since $A_{\alpha,\beta}$ is identified with $A_{\beta}/\langle \Omega_1 - \alpha \rangle$, we will first and foremost find a basis for $A_{\beta}$, and then proceed to find a basis for $A_{\alpha,\beta}$. Note, the relations in Lemma A.1 are also valid in $A_{\beta}$ and $A_{\alpha,\beta}$, and are going to be very useful in this section.
Proposition 4.3. The set \( \mathcal{G} = \left\{ \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6^m \mid i, j, k, l, m \in \mathbb{N} \text{ and } \xi = 0, 1 \right\} \) is a \( K \)-basis of \( A_\beta \).

Proof. Since the family \((\Pi_{s=1}^{6} E_{s, i})_{i, \xi, \varepsilon} \subseteq K\) is a PBW-basis of \( A \over \mathbb{K} \), it follows that the family \((\Pi_{s=1}^{6} \hat{e}_s E_{s, i})_{i, \xi, \varepsilon} \subseteq \mathbb{K}\) is a spanning set of \( A_\beta \over \mathbb{K} \). We want to show that \( \mathcal{G} \) spans \( A_\beta \). We do this by showing that \( \Pi_{s=1}^{6} \hat{e}_s E_{s, i} \) can be written as a finite linear combination of the elements of \( \mathcal{G} \) for all \( i_1, \ldots, i_6 \in \mathbb{N} \) by an induction on \( i_4 \). The result is obvious when \( i_4 = 0 \) or 1. For \( i_4 \geq 1 \), assume that

\[
\Pi_{s=1}^{6} \hat{e}_s E_{s, i} = \sum_{(\xi, \omega) \in I} a_{(\xi, \omega)} \hat{e}_{1}^{i_1} \hat{e}_{2}^{i_2} \hat{e}_{3}^{i_3} \hat{e}_{4}^{i_4, \xi} \hat{e}_{5}^{l} \hat{e}_{6}^{m},
\]

where \( \omega := (i, j, k, l, m) \in \mathbb{N}^5 \) and \( a_{(\xi, \omega)} \) are all scalars. Note, \( I \) is a finite subset of \( \{0, 1\} \times \mathbb{N}^5 \). It follows from the commutation relations of \( A_\beta \) (see Lemma [4.1])

\[
\hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6 = q^* \Pi_{s=1}^{6} \hat{e}_s E_{s, i} = q^* d_1 [i_6] \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6^i 3, 5, 6, 6, 1, 6, 1, 6, 0, 1.
\]

From the inductive hypothesis, \( \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6 = \sum_{(\xi, \omega) \in I} a_{(\xi, \omega)} \hat{e}_{1}^{i_1} \hat{e}_{2}^{i_2} \hat{e}_{3}^{i_3} \hat{e}_{4}^{i_4, \xi} \hat{e}_{5}^{l} \hat{e}_{6}^{m}. \) Hence, we proceed to show that \( \Pi_{s=1}^{6} \hat{e}_s E_{s, i} \) is also in the span of \( \mathcal{G} \). From the inductive hypothesis, we have

\[
\Pi_{s=1}^{6} \hat{e}_s E_{s, i} = \sum_{(\xi, \omega) \in I} a_{(\xi, \omega)} \hat{e}_{1}^{i_1} \hat{e}_{2}^{i_2} \hat{e}_{3}^{i_3} \hat{e}_{4}^{i_4, \xi} \hat{e}_{5}^{l} \hat{e}_{6}^{m}.
\]

Using the commutation relations in Lemma [4.1] we have that

\[
\Pi_{s=1}^{6} \hat{e}_s E_{s, i} = \sum_{(\xi, \omega) \in I} q^* a_{(\xi, \omega)} \hat{e}_{1}^{i_1} \hat{e}_{2}^{i_2} \hat{e}_{3}^{i_3} \hat{e}_{4}^{i_4, \xi} \hat{e}_{5}^{l} \hat{e}_{6}^{m} + \sum_{(\xi, \omega) \in I} q^* d_1 [m] a_{(\xi, \omega)} \hat{e}_{1}^{i_1} \hat{e}_{2}^{i_2} \hat{e}_{3}^{i_3} \hat{e}_{4}^{i_4, \xi} \hat{e}_{5}^{l} \hat{e}_{6}^{m}.
\]

All the terms in the above expression belong to the span of \( \mathcal{G} \) except \( \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6^m = \sum_{(\xi, \omega) \in I} a_{(\xi, \omega)} \hat{e}_{1}^{i_1} \hat{e}_{2}^{i_2} \hat{e}_{3}^{i_3} \hat{e}_{4}^{i_4, \xi} \hat{e}_{5}^{l} \hat{e}_{6}^{m}, \) From (11), we have that

\[
\hat{e}_4^2 = \beta_0 \hat{e}_4 \hat{e}_6 + b \hat{e}_3 \hat{e}_4 \hat{e}_6 + b \hat{e}_3 \hat{e}_4 \hat{e}_6 + b \hat{e}_3 \hat{e}_4 \hat{e}_6 + b \hat{e}_3 \hat{e}_4 \hat{e}_6 - \beta \beta_0,
\]

where \( \beta_0 = -1/d' \). Substituting (12) into \( \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6^m \), one can easily verify that

\[
\hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6^m \in \text{Span}(\mathcal{G}).
\]

Therefore, \( \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6^m \) can be written as a finite linear combination of the elements of \( \mathcal{G} \) over \( \mathbb{K} \) for all \( i_1, \ldots, i_6 \in \mathbb{N} \). By the principle of mathematical induction, \( \mathcal{G} \) is a spanning set of \( A_\beta \) over \( \mathbb{K} \).

Next we show that \( \mathcal{G} \) is a linearly independent set. Suppose that

\[
\sum_{(\xi, \omega) \in I} a_{(\xi, \omega)} \hat{e}_{1}^{i_1} \hat{e}_{2}^{i_2} \hat{e}_{3}^{i_3} \hat{e}_{4}^{i_4, \xi} \hat{e}_{5}^{l} \hat{e}_{6}^{m} = 0.
\]

Since \( A_\beta = A/\langle \Omega_2 - \beta \rangle \), it implies that

\[
\sum_{(\xi, \omega) \in I} a_{(\xi, \omega)} E_{1}^{i_1} E_{2}^{j_1} E_{3}^{k_1} E_{4}^{l_1} E_{5}^{m_1} = (\Omega_2 - \beta) \nu,
\]

with \( \nu \in A \). Write \( \nu = \sum_{(i, \ldots, n) \in J} b_{(i, \ldots, n)} E_{1}^{i_1} E_{2}^{j_1} E_{3}^{k_1} E_{4}^{l_1} E_{5}^{m_1}, \) where \( J \) is a finite subset of \( \mathbb{N}^6 \) and \( b_{(i, \ldots, n)} \) are all scalars. It follows that

\[
\sum_{(\xi, \omega) \in I} a_{(\xi, \omega)} E_{1}^{i_1} E_{2}^{j_1} E_{3}^{k_1} E_{4}^{l_1} E_{5}^{m_1} = \sum_{(i, \ldots, n) \in J} b_{(i, \ldots, n)} E_{1}^{i_1} E_{2}^{j_1} E_{3}^{k_1} E_{4}^{l_1} E_{5}^{m_1} (\Omega_2 - \beta) E_{4}^{m_1} E_{6}^{n_1}.
\]

Before we continue the proof, the following needs to be noted.
Let \((i', j', k', l', m', n')\), \((i, j, k, l, m, n)\) ∈ \(\mathbb{N}^6\). We say that \((i, j, k, l, m, n) < (i', j', k', l', m', n')\) if \([l < l']\) or \([l = l' \land i < i']\) or \([l = l' \land i = i' \land j < j']\) or \([l = l' \land i = i' \land j = j' \land k < k']\) or \([l = l' \land i = i' \land j = j' \land k = k' \land m < m']\) or \([l = l' \land i = i' \land j = j' \land k = k' \land m = m' \land n < n']\).

Note, the purpose of the square bracket \([\cdot]\) is to differentiate the types.

From Section 3.1 we have that \(\Omega_2 = E_2E_4E_6 + bE_2E_3^3 + bE_3^3E_6 + b'E_4^2E_5^2 + c'E_3E_4E_5 + d'E_4^2\) in \(A = U_7^6(G_2)\).

\[
\sum_{(\xi, \nu) \in I} a_{(\xi, \nu)} E_1^3 E_2^3 E_3^5 E_4^5 E_5^6 E_6^6 = \sum_{(i, \ldots, n) \in J} b_{(i, \ldots, n)} E_1^3 E_2^3 E_3^5 E_4^5 E_5^6 E_6^6 = \sum_{(i, \ldots, n) \in J} d'b_{(i, \ldots, n)} E_1^3 E_2^3 E_3^5 E_4^5 E_5^6 E_6^6 + \text{LT}_{<4},
\]

where \(\text{LT}_{<4}\) contains lower order terms with respect to \(<_4\) (as in \(\clubsuit\)). Moreover, \(\text{LT}_{<4}\) vanishes when \(b_{(i, \ldots, n)} = 0\) for all \((i, \ldots, n) \in J\) (one can easily confirm this by fully expanding the right hand side of \(\clubsuit\)).

Now, suppose that there exists \((i, j, k, l, m, n) \in J\) such that \(b_{(i,j,k,l,m,n)} \neq 0\). Let \((i', j', k', l', m', n')\) be the greatest element of \(J\) with respect to \(<_4\) (defined in \(\clubsuit\) above) such that \(b_{(i',j',k',l',m',n')} \neq 0\). Note, the family \((E_1^3 E_2^3 E_3^5 E_4^5 E_5^6 E_6^6)_{(i, \ldots, n) \in \mathbb{N}^6}\) is a basis of \(A\). Since \(\text{LT}_{<4}\) contains lower order terms, identifying the coefficients of \(E_1'^j E_2'^j E_3'^j E_4'^j E_5'^j E_6'^j E_7'^j E_8'^j\) in the above equality, we have that \(d'b_{(i', \ldots, n')} = 0\). Since \(b_{(i', j', k', l', m', n')} \neq 0\), it follows that \(d' = q^3(2/(q^2 - 1)) = 0\), a contradiction (see Appendix [3] for the expression of \(d'\)). As a result, \(b_{(i,j,k,l,m,n)} = 0\) for all \((i, j, k, l, m, n) \in J\). Therefore, \(\sum_{(\xi, \nu) \in I} a_{(\xi, \nu)} E_1^3 E_2^3 E_3^5 E_4^5 E_5^6 E_6^6 = 0\). Since \((E_1^3 E_2^3 E_3^5 E_4^5 E_5^6 E_6^6)_{(\xi, \nu) \in \mathbb{N}^6}\) is a basis of \(A\), it follows that \(a_{(\xi, \nu)} = 0\) for all \((\xi, \nu) \in I\). In conclusion, \(\mathcal{G}\) is a linearly independent set and hence forms a basis of \(A_\beta\) as desired.

**Proposition 4.4.** Let \((\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\}\). The set \(\mathcal{B} = \{e_1^i e_2^j e_3^k e_4^l e_5^m e_6^n | i, j, k, l \in \mathbb{N} \land e_1, e_2 \in \{0, 1\}\}\) is a \(\mathbb{K}\)-basis of \(A_\beta\).

**Proof.** Since the set \(\mathcal{G} = \{e_1^i e_2^j e_3^k e_4^l e_5^m e_6^n | i_1, i_2, i_3, i_4, i_5, i_6 \in \mathbb{N} \land \xi = 0, 1\}\) is a \(\mathbb{K}\)-basis of \(A_\beta\) (Proposition 4.3), and \(A_{\alpha, \beta}\) is identified with \(A_\beta / (\Omega_1 - \alpha)\), it follows that

\[
(e_1^i e_2^j e_3^k e_4^l e_5^m e_6^n)_{i_1, i_2, i_3, i_4, i_5, i_6} \in \mathbb{K}, \xi \in \{0, 1\}
\]

is a spanning set of \(A_{\alpha, \beta}\) over \(\mathbb{K}\). We want to show that \(\mathcal{B}\) spans \(A_{\alpha, \beta}\) by showing that \(e_1^i e_2^j e_3^k e_4^l e_5^m e_6^n\) can be written as a finite linear combination of the elements of \(A_\beta\) over \(\mathbb{K}\) for all \(i_1, i_2, i_3, i_4, i_5, i_6 \in \mathbb{N}\) and \(\xi = 0, 1\). By Proposition 4.3, it is sufficient to do this by an induction on \(i_3\). The result is obvious when \(i_3 = 0\) or \(1\). For \(i_3 \geq 1\), suppose that

\[
e_1^{i_1} e_2^{i_2} e_3^{i_3+1} e_4^{i_4} e_5^{i_5} e_6^{i_6} = \sum_{(e_1, e_2, e_3) \in \mathcal{G}} a_{(e_1, e_2, e_3)} e_1^{i_1} e_2^{i_2} e_3^{i_3+1} e_4^{i_4} e_5^{i_5} e_6^{i_6},
\]

where \(\underline{t} = (i, j, k, l) \in \mathbb{N}^4\). \(J\) is a finite subset of \(\{0, 1\}^2 \times \mathbb{N}^4\), and \((a_{(e_1, e_2, e_3)})_{(e_1, e_2, e_3) \in \mathcal{G}}\) is a family of scalars. Using the commutation relations in \(A_{\alpha, \beta}\) (Lemma [A.1]), we have that:

\[
e_1^{i_1} e_2^{i_2} e_3^{i_3+1} e_4^{i_4} e_5^{i_5} e_6^{i_6} = q^* e_3^{i_3} e_4^{i_4} e_5^{i_5} e_6^{i_6} - q^* d_{[i_1]} e_1^{i_1-1} e_2^{i_2} e_3^{i_3} e_4^{i_4} e_5^{i_5} e_6^{i_6}.
\]

From the inductive hypothesis, \(e_1^{i_1-1} e_2^{i_2} e_3^{i_3} e_4^{i_4} e_5^{i_5} e_6^{i_6} \in \text{Span}(B)\) for all \(i_1 > 0\) (note: \(d_{[0]} = 0\)). As a result, we proceed to show that \(e_3 e_1^{i_1} e_2^{i_2} e_3^{i_3} e_4^{i_4} e_5^{i_5} e_6^{i_6}\) is also in the span of \(B\). It follows from the inductive hypothesis that

\[
e_3 e_1^{i_1} e_2^{i_2} e_3^{i_3} e_4^{i_4} e_5^{i_5} e_6^{i_6} = \sum_{(e_1, e_2, e_3) \in \mathcal{G}} a_{(e_1, e_2, e_3)} e_1^{i_1} e_2^{i_2} e_3^{i_3+1} e_4^{i_4} e_5^{i_5} e_6^{i_6} = \sum_{(e_1, e_2, e_3) \in \mathcal{G}} a_{(e_1, e_2, e_3)} e_1^{j_1} e_2^{j_2} e_3^{j_3+1} e_4^{j_4} e_5^{j_5} e_6^{j_6}
\]
Clearly, the monomial $e_1^{i-1}e_2^{1+j}e_3^i e_4^{e_5} e_6$ belongs to the span of $B$ for all $(e_1, e_2, \omega) \in I$ (with $i > 0$). Again, the monomial $e_1^i e_2^j e_3^{i+1} e_4^{j+k} e_5^l e_6$ belongs to the span of $B$ for all $(e_1, e_2, \omega) \in I$; with $(e_1, e_2) = (0,0), (0,1)$. For $(e_1, e_2) = (1,0), (1,1)$, we must show that $e_1^i e_2^j e_3^{i+1} e_4^{j+k} e_5^l e_6$ belong to the span of $B$. From Lemma 4.1, one can write $e_1^i e_2^j e_3^{i+1} e_4^{j+k} e_5^l e_6$ as finite linear combinations of the monomials of $B$ over $\mathbb{K}$. Hence, each $e_1^i e_2^j e_3^{i+1} e_4^{j+k} e_5^l e_6$ belongs to the span of $B$ for all $(e_1, e_2, \omega) \in I$; with $(e_1, e_2) = (1,0), (1,1)$. We have therefore established that $e_3 e_1^i e_2^j e_3^{i+1} e_4^{j+k} e_5^l e_6 \in \text{Span}(B)$. Consequently, each $e_1^i e_2^j e_3^{i+1} e_4^{j+k} e_5^l e_6$ belongs to the span of $B$. By the principle of mathematical induction, $B$ is a spanning set of $A_{\alpha, \beta}$ over $\mathbb{K}$ as expected.

Finally, we show that $B$ is a linearly independent set. Suppose that

$$\sum_{(e_1, e_2, \omega) \in I} a_{(e_1, e_2, \omega)} e_1^i e_2^j e_3^i e_4^{e_5} e_6 = 0.$$ 

In $A_{\beta}$, we have that

$$\sum_{(e_1, e_2, \omega) \in I} a_{(e_1, e_2, \omega)} e_1^i e_2^j e_3^i e_4^{e_5} e_6 = (\hat{\Omega}_1 - \alpha) \nu,$$

with $\nu \in A_{\beta}$. One can write $\nu$ in terms of the basis $\mathcal{S}$ of $A_{\beta}$ (Proposition 3.3) as:

$$\nu = \sum_{w \in J_1} b_w e_1^i e_2^j e_3^i e_4^{e_5} e_6 \sum_{w \in J_2} c_w e_1^i e_2^j e_3^i e_4^{e_5} e_6,$$

where $b_w$ and $c_w$ are all scalars and $w := (i, j, k, l, m)$. Hence,

$$\sum_{(e_1, e_2, \omega) \in I} a_{(e_1, e_2, \omega)} e_1^i e_2^j e_3^i e_4^{e_5} e_6 = \sum_{w \in J_1} b_w e_1^i e_2^j e_3^i e_4^{e_5} e_6 (\hat{\Omega}_1 - \alpha) e_3^l e_6^m$$

$$+ \sum_{w \in J_2} c_w e_1^i e_2^j e_3^i e_4^{e_5} e_6 (\hat{\Omega}_1 - \alpha) e_3^l e_6^m.$$

Note, $\hat{\Omega}_1 = \hat{e}_1 \hat{e}_3 \hat{e}_5 + a \hat{e}_1 \hat{e}_4 + a \hat{e}_2 \hat{e}_5 + a \hat{e}_3^2$. Using (12) and the relation $e_3^k e_1 = q^{-k} e_1 e_3^k + d_2 k e_2 e_3^{k-1}$ (see Lemma A.1), one can verify that the above equality can be written in terms of the basis of $A_{\beta}$ (Propositions 3.3) as:

$$\sum_{(e_1, e_2, \omega) \in I} a_{(e_1, e_2, \omega)} e_1^i e_2^j e_3^i e_4^{e_5} e_6 = \sum_{w \in J_1} b_w a \hat{e}_1^i e_2^j e_3^i e_4^{e_5} e_6 e_6^m$$

$$+ \sum_{w \in J_1} q^* b_w a \hat{e}_1^i e_2^j e_3^i e_4^{e_5} e_6 e_6^m$$

$$+ \sum_{w \in J_2} q^* c_w a \hat{e}_1^i e_2^j e_3^i e_4^{e_5} e_6 e_6^m$$

$$+ \sum_{w \in J_2} c_w a \hat{e}_1^i e_2^j e_3^i e_4^{e_5} e_6 e_6^m + \mathcal{Y},$$

where $\mathcal{Y}$ is defined as follows:

$$\mathcal{Y} = \sum_{w \in J_1} q^* b_w a \hat{e}_1^i e_2^j e_3^i e_4^{e_5} e_6 e_6^m + \sum_{w \in J_1} q^* b_w d_2 k e_1^i e_2^j e_3^i e_4^{e_5} e_6 e_6^m$$

$$- \sum_{w \in J_1} q^* b_w a \hat{e}_1^i e_2^j e_3^i e_4^{e_5} e_6 e_6^m + \sum_{w \in J_2} q^* b_w a \hat{e}_1^i e_2^j e_3^i e_4^{e_5} e_6 e_6^m + \sum_{w \in J_1} q^* b_w a \hat{e}_1^i e_2^j e_3^i e_4^{e_5} e_6 e_6^m.$$
Before we continue the proof, the following point needs to be noted.

\[ \bigvee \text{Let } (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6) \text{, } (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \mathbb{N}^6. \text{ We say that } (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) <_3 (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6) \text{ if } \left[ \begin{array}{c} \vartheta_3 > \xi_3 \\ \vartheta_5 > \xi_5 \\ \vartheta_7 > \xi_7 \end{array} \right] \text{ or } \left[ \begin{array}{c} \vartheta_3 < \xi_3 \\ \vartheta_5 < \xi_5 \\ \vartheta_7 < \xi_7 \end{array} \right]. \]

Observe that \( \Upsilon \) contains lower order terms with respect to \( <_3 \) (defined in \( \bullet \) above) in each monomial type (note, there are two different types of monomials in the basis of \( \Lambda \): one with \( \mathcal{B} \) and the other without \( \mathcal{B} \)). Now, suppose that there exists \((i, j, k, l, m) \in J_1 \) and \((i, j, k, l, m) \in J_2 \) such that \( b_{(i, j, k, l, m)} \neq 0 \) and \( c_{(i, j, k, l, m)} \neq 0 \). Let \((v_1, v_2, v_3, v_4, v_5, v_6) \) and \((w_1, w_2, w_3, w_4, w_5, w_6) \) be the greatest elements of \( J_1 \) and \( J_2 \) respectively with respect to \( <_3 \) such that \( b_{(v_1, v_2, v_3, v_4, v_5, v_6)} \) and \( c_{(w_1, w_2, w_3, w_4, w_5, w_6)} \) are non-zero. Since \( \mathcal{S} \) is a linear basis for \( \Lambda \), and \( \Upsilon \) contains lower order terms with respect to \( <_3 \), we have the following: if \( v_3 - v_2 < 2 \), then identifying the coefficients of \( \xi_1^2 \xi_2^2 \xi_3^2 \xi_4^2 \xi_5^2 \xi_6^2 \) in \( \mathbb{I} \), we have that \( b_{(v_1, v_2, v_3, v_4, v_5, v_6)} <_3 0 \). But \( b_{(v_1, v_2, v_3, v_4, v_5, v_6)} \neq 0 \), hence \( a' = q^2/(q^2 - 1) = 0 \), a contradiction (see Appendix \( \mathbb{I} \) for the expression of \( a' \)). Finally, if \( w_3 - w_2 > 2 \), then identifying the coefficient of \( \xi_1^2 \xi_2^2 \xi_3^2 \xi_4^2 \xi_5^2 \xi_6^2 \), we have that \( a'c_{(w_1, w_2, w_3, w_4, w_5, w_6)} = 0 \). But \( c_{(w_1, w_2, w_3, w_4, w_5, w_6)} \neq 0 \), hence \( a' = 0 \), a contradiction! This implies that either all \( b_{(i, j, k, l, m)} \) or all \( c_{(i, j, k, l, m)} \) are zero. Without loss of generality, suppose that there exists \((i, j, k, l, m) \in J_2 \) such that \( c_{(i, j, k, l, m)} \) is not zero. Then, \( b_{(i, j, k, l, m)} \) are all zero. Let \((w_1, w_2, w_3, w_4, w_5, w_6) \) be the greatest element of \( J_2 \) such that \( c_{(w_1, w_2, w_3, w_4, w_5, w_6)} \neq 0 \). Identifying the coefficients of \( \xi_1^2 \xi_2^2 \xi_3^2 \xi_4^2 \xi_5^2 \xi_6^2 \) in the above equality, we have that \( a'c_{(w_1, w_2, w_3, w_4, w_5, w_6)} = 0 \). Since \( c_{(w_1, w_2, w_3, w_4, w_5, w_6)} \neq 0 \), it follows that \( a' = 0 \), a contradiction! We can therefore conclude that \( b_{(i, j, k, l, m)} \) and \( c_{(i, j, k, l, m)} \) are all zero. Consequently, \( \sum_{(\epsilon_1, \epsilon_2, \epsilon_3) \in I} a_{(\epsilon_1, \epsilon_2, \epsilon_3)} \xi_1^i \xi_2^j \xi_3^k \xi_4^{i} \xi_5^{j} \xi_6^{k} = 0 \). Since \( \xi_{(i, j, k, l, m)} \) is a basis of \( \Lambda \), it implies that \( a_{(\epsilon_1, \epsilon_2, \epsilon_3)} = 0 \) for all \((\epsilon_1, \epsilon_2, \epsilon_3) \in I \). Therefore, \( \mathbb{B} \) is a linearly independent set.

We note for future use the following immediate consequence of Proposition \( \mathbb{I} \).

**Corollary 4.5.** Let \( v = (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2 \), I represent a finite subset of \( \{0, 1\} \times \mathbb{N}^2 \times \mathbb{Z}^2 \) and \((a_{(\epsilon_1, \epsilon_2, \epsilon_3)} \in I \) be a family of scalars. If

\[ \sum_{(\epsilon_1, \epsilon_2, \epsilon_3) \in I} a_{(\epsilon_1, \epsilon_2, \epsilon_3)} \xi_1^i \xi_2^j \xi_3^k \xi_4^{i} \xi_5^{j} \xi_6^{k} = 0, \]

then \( a_{(\epsilon_1, \epsilon_2, \epsilon_3)} = 0 \) for all \((\epsilon_1, \epsilon_2, \epsilon_3) \in I \).
Remark 4.6. Given the basis of $A_{\alpha, \beta}$, we have computed the group of units of $A_{\alpha, \beta}$, however, we do not include the details in this manuscript due to the voluminous computations involved. We only summarise our findings below. Set

$$h_1 := e_3 e_5 + a e_4 \quad \text{and} \quad h_2 := (q^{-3} - q^{-9}) e_2 e_4 - (q^{-4} - 2q^2 + 1)/(q^4 + q^2 + 1) e_3^3.$$

**Theorem 4.7.** Let $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$ and $\mathcal{U}(A_{\alpha, \beta})$ denote the group of units of $A_{\alpha, \beta}$. We have that:

$$\mathcal{U}(A_{\alpha, \beta}) = \begin{cases} \{\lambda h_1^i \mid \lambda \in \mathbb{K}^*, \; i \in \mathbb{Z}\} & \text{if } \alpha = 0 \\ \{\lambda h_2^i \mid \lambda \in \mathbb{K}^*, \; i \in \mathbb{Z}\} & \text{if } \beta = 0 \\ \mathbb{K}^* & \text{otherwise.} \end{cases}$$

### 4.3 $A_{\alpha, \beta}$ as a $q$-deformation of a quadratic extension of $A_2(\mathbb{K})$

Recall that $\text{GKdim} A_{\alpha, \beta} = 4$ and so we should compare $A_{\alpha, \beta}$ to the second Weyl algebra. In this section, we prove that, for a suitable choice of $\alpha$ and $\beta$, the simple algebra $A_{\alpha, \beta}$ is a $q$-deformation of (a quadratic extension of) $A_2(\mathbb{K})$.

Recall that $A_2(\mathbb{K})$ is generated by $x_1, x_2, y_1$ and $y_2$ subject to the relations:

$$\begin{align*}
y y_2 & = y_2 y_1, & x_2 y_1 & = y_1 x_2, & x_1 x_2 & = x_2 x_1, & x_1 y_1 - y_1 x_1 & = 1, \\
y_1 y_2 & = y_2 y_1, & x_1 y_2 & = y_2 x_1, & x_2 y_1 & = y_1 x_2, & x_2 y_2 - y_2 x_2 & = 1.
\end{align*}$$

Given the relations of $A_{\alpha, \beta}$ at the onset of this section, we have that $A_{1, \frac{1}{\sqrt{q^6 - 1}}}$ satisfies the following relations:

\[ e_2 e_1 = q^{-3} e_1 e_2, \quad e_3 e_2 = q^{-3} e_2 e_3, \quad e_4 e_2 = q^{-3} e_2 e_4 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1} e_3, \quad e_4 e_3 = q^{-3} e_3 e_4, \]

\[ e_5 e_1 = q e_1 e_5 - (q + q^{-1} + q^{-3}) e_3, \quad e_5 e_2 = q e_2 e_5 + (q + q^{-1}) e_3, \quad e_5 e_3 = q^{-1} e_3 e_5 - (q + q^{-1} + q^{-3}) e_4, \quad e_5 e_4 = q^{-3} e_4 e_5, \]

\[ e_6 e_1 = q^3 e_1 e_6 - q^3 e_5, \quad e_6 e_2 = q^3 e_2 e_6 + (q^4 + q^2 - 1) e_4 + (q^2 - q^4) e_3 e_5, \quad e_6 e_3 = e_3 e_6 + (1 - q^2) e_5^2, \quad e_6 e_4 = q^{-3} e_4 e_6 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1} e_3^2, \]

\[ e_6 e_5 = q^{-3} e_5 e_6, \]

and

\[ (q^{-2} - 1) e_1 e_3 e_5 + (q^2 + 1 + q^{-2}) e_1 e_4 + (q^2 + 1 + q^{-2}) e_2 e_5 - q^4 e_3^2 = q^{-2} - 1, \]

\[ (q^6 - 1) e_2 e_4 e_6 + \frac{2q^{-1} - q^{-3} - q e_2 e_3 + 2q^{-1} - q^{-3} - q e_3 e_6}{q^4 + q^2 + 1} e_3^2 + \frac{q^2 e_3 e_4 e_5 + q^2 e_2 e_6}{q^4 + q^2 + 1} e_3^2 = \frac{1}{9}. \]

Note, we have made the necessary substitutions for $a, \; a', b, \; b', c$ and $d'$ from Appendix [B]

Set $F := \mathbb{K}[z^{\pm 1}]$. One can define a $F[(z^4 + z^2 + 1)^{-1}]$-algebra $A_2$ generated by $e_1, \cdots, e_6$ subject to the following relations:

\[ e_2 e_1 = z^{-3} e_1 e_2, \quad e_3 e_1 = z^{-1} e_1 e_3 - (z + z^{-1} + z^{-3}) e_2, \quad e_4 e_1 = e_1 e_4 + (1 - z^2) e_3^2, \]

\[ e_4 e_2 = z^{-3} e_2 e_4 - \frac{z^4 - 2z^2 + 1}{z^4 + z^2 + 1} e_3^3, \quad e_4 e_3 = z^{-3} e_3 e_4. \]
Lemma 4.8. \( e_i \in Z(A_1) \) and it is also invertible.

Proof. Since \( e_4 e_i = e_i e_4 \) for all \( 1 \leq i \leq 6 \), we have that \( e_4 \in Z(A_1) \). Again, from \( e_4^2 = 1/9 \), we have that \( e_4(9e_4) = (9e_4)e_4 = 1 \). Hence \( e_4 \) is invertible with \( e_4^{-1} = 9e_4 \).

Given that \( e_4^{-1} = 9e_4 \) and \( e_4 \in Z(A_1) \), it follows from the relation \( e_4^2 - 3e_1 e_4 - 3e_2 e_5 = 0 \) that \( e_1 = 3e_2 e_4 - 9e_2 e_4 e_5 \). Therefore, \( A_1 \) can be generated by only \( e_2, \ldots, e_6 \). All these generators commute except that

\[
e_6 e_2 = e_2 e_6 + e_4 \quad \text{and} \quad e_5 e_3 = e_3 e_5 - 3e_4.
\]

Since \( e_4 \) is invertible, one can also verify that \( 9e_2 e_4, 3e_3 e_4, e_4, e_5 \) and \( e_6 \) generate \( A_1 \).

Let \( R \) be an algebra generated by \( f_2, f_3, f_4, f_5, f_6 \) subject to the following defining relations:

\[
\begin{align*}
f_3 f_2 &= f_2 f_3 & f_4 f_2 &= f_2 f_4 & f_4 f_3 &= f_3 f_4 \\
f_5 f_2 &= f_2 f_5 & f_5 f_4 &= f_4 f_5 & f_6 f_3 &= f_3 f_6 \\
f_6 f_4 &= f_4 f_6 & f_6 f_5 &= f_5 f_6 & f_1^2 &= 1/9 \\
f_6 f_2 &= f_2 f_6 + 1 & f_5 f_3 &= f_3 f_5 - 1.
\end{align*}
\]

Proposition 4.9. \( R \cong A_1 \).

Proof. One can easily check that we define a homomorphism \( \phi : R \rightarrow A_1 \) via

\[
\begin{align*}
\phi(f_2) &= 9e_2 e_4 & \phi(f_3) &= 3e_3 e_4 & \phi(f_4) &= e_4 & \phi(f_5) &= e_5 & \phi(f_6) &= e_6.
\end{align*}
\]

Recall, \( e_4^2 = 1/9 \). To check that \( \phi \) is indeed a homomorphism, we just need to check its compatibility with the defining relations of \( R \). We check this on the relation \( f_6 f_2 - f_2 f_6 = 1 \) and \( f_3 f_5 - f_5 f_3 = 1 \), and leave the remaining ones for the reader to verify. We do that as follows: \( \phi(f_6) \phi(f_2) - \phi(f_2) \phi(f_6) = 9e_6 e_2 e_4 - 9e_2 e_4 e_6 = 9e_4(e_6 e_2 - e_2 e_6) = 9 e_4^2 = 9(1/9) = 1 \) as needed. Also, \( \phi(f_3) \phi(f_5) - \phi(f_5) \phi(f_3) = 3e_3 e_4 e_5 - 3e_5 e_3 e_4 = 3e_4(e_3 e_5 - e_5 e_3) = 3e_4(3e_4) = 9e_4^2 = 1 \).
Conversely, one can check that we define a homomorphism \( \varphi : A_1 \to R \) via
\[
\varphi(e_1) = 3f_3^2f_4 - f_2f_5 \\
\varphi(e_4) = f_4 \\
\varphi(e_2) = f_2f_4 \\
\varphi(e_5) = f_5 \\
\varphi(e_3) = 3f_3f_4 \\
\varphi(e_6) = f_6.
\]
We check this on the relation \( e_3^2 - 3e_2e_4 - 3e_2e_5 = 0 \), and leave the remaining ones for the reader to verify. We do that as follows: \( \varphi(e_3)^2 - 3\varphi(e_1)\varphi(e_4) - 3\varphi(e_2)\varphi(e_5) = (3f_3f_4)^2 - 3(3f_3^2f_4 - f_2f_5)f_4 - 3f_2f_4f_5 = 9f_6^2f_3^2/2 - 9f_2^2f_4^2 + 3f_2f_4f_5 - 3f_2f_4f_5 = 0 \) as expected.
To conclude we just observe that \( \phi \) and \( \varphi \) are inverse of each other. \( \square \)

The corollary below can easily be deduced from the above proposition.

**Corollary 4.10.** Set \( \mathbb{F} := \mathbb{K}[f_4]/(f_3^2 - 1/9) \), we have that \( R \cong A_2(\mathbb{F}) \), where \( A_2(\mathbb{F}) \) is the second Weyl algebra over the ring \( \mathbb{F} \).

**Remark 4.11.** Observe that the subalgebra \( B \) of \( R \) generated by \( f_2, f_3, f_5, f_6 \) is isomorphic to \( A_2(\mathbb{K}) \) and \( R \cong B[f_4] \cong A_2(\mathbb{K})[f_4] \). Therefore, \( R \) is a quadratic extension of \( A_2(\mathbb{K}) \). Note, \( A_{1, \frac{1}{9(q^2-1)}} \) is a \( q \)-deformation of \( A_1 \cong R \cong A_2(\mathbb{F}) \cong A_2(\mathbb{K})[f_4] \).

## 5 Derivations of the simple quotients of \( U_q^+(G_2) \)

In this section, we compute the derivations of the algebra \( A_{\alpha, \beta} \) using DDA that allows to embed \( A_{\alpha, \beta} \) into a suitable quantum torus. Derivations of quantum tori are known, thanks to the work of Osborn and Passman [20]. In our cases, such derivations are always the sum of an inner derivation and a scalar derivation (of the quantum torus). Since \( A_{\alpha, \beta} \) can be embedded into a quantum torus, we first extend every derivation of \( A_{\alpha, \beta} \) to a derivation of such quantum torus, and then pull back their description as a derivation of the quantum torus to a description of their action on the generators of \( A_{\alpha, \beta} \) by “reverting” DDA process. We conclude that every derivation of \( A_{\alpha, \beta} \) is inner when \( \alpha \) and \( \beta \) are both non-zero. However, when either \( \alpha \) or \( \beta \) is zero, we conclude that every derivation of \( A_{\alpha, \beta} \) is the sum of an inner and a scalar derivation. In fact, the first Hochschild cohomology group of \( A_{\alpha, \beta} \) is of dimension 0 when \( \alpha \) and \( \beta \) are both non-zero and 1 when either \( \alpha \) or \( \beta \) is zero.

### 5.1 Preliminaries and strategy

Let \( 2 \leq j \leq 7 \) and \( (\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\} \). Set
\[
A_{\alpha, \beta}^{(j)} := \frac{A^{(j)}}{(\Omega_1 - \alpha, \Omega_2 - \beta)},
\]
where \( A^{(j)} \) is defined in Section 3.1 and, \( \Omega_1 \) and \( \Omega_2 \) are the generators of the center of \( A^{(j)} \), see Remark 3.3. Note in particular that \( A_{\alpha, \beta}^{(7)} = A_{\alpha, \beta} \). For each \( 2 \leq j \leq 7 \), denote the canonical images of the generators \( E_{i,j} \) of \( A_{\alpha, \beta}^{(j)} \) by \( e_{i,j} \) for all \( 1 \leq i \leq 6 \).

As usual we denote by \( t_i \) the canonical image of \( T_i \) in \( A_{\alpha, \beta}^{(2)} \) for each \( 1 \leq i \leq 6 \). For each \( 3 \leq j \leq 6 \), define \( S_j := \left\{ \lambda_{i,j}^{(j)} t_1^{i_1} \cdots t_6^{i_6} \mid i_1, \ldots, i_6 \in \mathbb{N} \text{ and } \lambda \in \mathbb{K}^* \right\} \). One can observe that \( S_j \) is a multiplicative system of non-zero divisors (or regular elements) of \( A_{\alpha, \beta}^{(j)} \). Furthermore; \( t_1, \cdots, t_6 \) are all normal elements of \( A_{\alpha, \beta}^{(j)} \) and so \( S_j \) is an Ore set in \( A_{\alpha, \beta}^{(j)} \). One can localize \( A_{\alpha, \beta}^{(j)} \) at \( S_j \) as follows:
\[
\mathcal{R}_j := A_{\alpha, \beta}^{(j)} S_j^{-1}.
\]

Let \( 3 \leq j \leq 6 \), and set \( \Sigma_j := \left\{ t_j^k \mid k \in \mathbb{N} \right\} \). By [4] Lemma 5.3.2, \( \Sigma_j \) is an Ore set in both \( A_{\alpha, \beta}^{(j)} \) and \( A_{\alpha, \beta}^{(j+1)} \), and
\[
A_{\alpha, \beta}^{(j)} S_j^{-1} = A_{\alpha, \beta}^{(j+1)} \Sigma_j^{-1}.
\]
As a consequence, similar to (7), we have that
\[ R_j = R_{j+1} \Sigma_j^{-1}, \]  
for all \( 2 \leq j \leq 6 \). By convention, \( R_7 := A_{\alpha, \beta} \). We also construct the following tower of algebras in a manner similar to (8):
\[ R_7 = A_{\alpha, \beta} \subseteq R_6 = R_7 \Sigma_6^{-1} \subseteq R_5 = R_6 \Sigma_5^{-1} \subseteq R_4 = R_5 \Sigma_4^{-1} \subseteq R_3. \]

Note, \( R_3 = A^{(3)}_3 S_3^{-1} = R_3 \Sigma_3^{-1} \) is the quantum torus \( \mathcal{A}_{\alpha, \beta} = \mathbb{K}_{q^N} [t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}] \) studied in Section 4.3.

Our strategy to compute the derivations of \( R_7 \) is to extend these derivations to derivations of the quantum torus \( R_3 \). Then we can use the description of the derivations of a quantum torus obtained by Osborn and Passman in [20]. Once this is done, we will have a “nice” description but involving elements of \( R_3 \) and we will then use the fact that these derivations fix (globally) all \( R_i \) to obtain a description only involving elements of \( R_7 \). This is a step by step process requiring knowing linear bases for \( R_i \). We find such bases in the next section.

Before doing so, we note from [4, Lemme 5.3.2] that DDA theory predicts the following relations between the elements \( e_{i,j} \):
\[
\begin{align*}
e_{1,6} &= e_1^{\pm 1} r e_5 e_6^{\pm 1} \\
e_{2,6} &= e_2 + t e_3 e_5 e_6^{\pm 1} + u e_4 e_6^{\pm 1} + n e_3^{\pm 1} e_6 \\
e_{3,6} &= e_3 + s e_5 e_6^{\pm 1} \\
e_{4,6} &= e_4 + b e_5 e_6^{\pm 1} \\
e_{1,5} &= e_{1,6} + h e_3 e_5 e_6^{\pm 1} + g e_4 e_6^{\pm 1} \\
e_{2,5} &= e_{2,6} + f e_3 e_5 e_6^{\pm 1} + p e_3 e_4 e_6^{\pm 1} + e e_3^{\pm 1} e_6 \\
e_{3,5} &= e_{3,6} + a e_4 e_5 e_6^{\pm 1} \\
e_{1,4} &= e_{1,5} + s e_3 e_4 e_5^{\pm 1} \\
e_{2,4} &= e_{2,5} + b e_3 e_4 e_5^{\pm 1} \\
e_{1,3} &= e_{1,4} + a e_{2,4} e_3 e_5^{\pm 1} \\
t_1 &:= e_{1,2} = e_{1,3} \\
t_2 &:= e_{2,2} = e_{2,3} = e_{2,4} \\
t_3 &:= e_{3,2} = e_{3,3} = e_{3,4} = e_{3,5} \\
t_4 &:= e_{4,2} = e_{4,3} = e_{4,4} = e_{4,5} = e_{4,6} \\
t_5 &:= e_{5,2} = e_{5,3} = e_{5,4} = e_{5,5} = e_{5,6} = e_5 \\
t_6 &:= e_{6,2} = e_{6,3} = e_{6,4} = e_{6,5} = e_{6,6} = e_6,
\end{align*}
\]

where, as usual, the necessary parameters can be found in Appendix B.

We also note that we have complete control over the centers of the algebras \( R_i \).

**Lemma 5.1.** Let \( Z(R_i) \) denote the center of \( R_i \), then \( Z(R_i) = \mathbb{K} \) for each \( 3 \leq i \leq 7 \).

**Proof.** One can easily verify that \( Z(R_3) = \mathbb{K} \). Note, \( R_7 = A_{\alpha, \beta} \). Since \( R_i \) is a localization of \( R_{i+1} \) (see [13]), we have that \( \mathbb{K} \subseteq Z(R_7) \subseteq Z(R_6) \subseteq Z(R_5) \subseteq Z(R_4) \subseteq Z(R_3) = \mathbb{K} \). Therefore, \( Z(R_7) = Z(R_6) = Z(R_5) = Z(R_4) = Z(R_3) = \mathbb{K} \). \( \square \)

### 5.2 Linear bases for \( R_3, R_4 \) and \( R_5 \)

Let \((\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}\). We aim to find a basis of \( R_j \) for each \( j = 3, 4, 5 \). Since \( R_3 = \mathcal{A}_{\alpha, \beta} \), the set \( \{ t_3^{i_1} t_4^{i_2} t_5^{i_3} | \, i, j, k, l \in \mathbb{Z} \} \) is a \( \mathbb{K} \)-basis of \( R_3 \).

For simplicity, we set
\[
\begin{align*}
f_1 &= e_{1,4} & F_1 &= E_{1,4}
\end{align*}
\]

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is a family of scalars. Given that \( \Omega \in F \) and \( \nu \sum_{J_k} \) of \( J \). Consequently, \( \nu \).

**Proof.** We begin by showing that \( \mathcal{S}_4 \) is a spanning set for \( A_4^{(4)} \). It is sufficient to do this by showing that \( \hat{f}_1^{k_1}k_2\hat{f}_3k_4\hat{f}_5k_6 \) can be written as a finite linear combination of the elements of \( \mathcal{S}_4 \) for all \( (k_1, \ldots, k_6) \in \mathbb{N}^3 \times \mathbb{Z}^3 \). This can easily be done through an induction on \( k_2 \) using the fact that \( \hat{f}_2 = \beta \hat{f}_6^{-1} \hat{f}_4^{-1} \) (note that, if \( \beta = 0 \), then \( \hat{f}_2 = 0 \)).

We now prove that \( \mathcal{S}_4 \) is a linearly independent set. Suppose that

\[
\sum_{i \in I} a_i \hat{f}_1^{i_1}t_3^{i_3}t_4^{i_4}t_5^{i_5}t_6^{i_6} = 0.
\]

This implies that

\[
\sum_{i \in I} a_i F_1^{i_1}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6} (\Omega_2 - \beta) \nu,
\]

for some \( \nu \in A_4^{(4)} \). Write \( \nu = \sum_{j \in J} b_j \) and \( b_j \) is a family of scalars. Given that \( \Omega_2 = T_2T_4T_6 \), it follows from the above equality that

\[
\sum_{j \in J} a_j F_1^{i_1}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6} = \sum_{j \in J} q^* b_j F_1^{i_1}T_2^{j_2}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6} - \sum_{j \in J} \beta b_j F_1^{i_1}T_2^{j_2}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6}.
\]

We denote by \(<_2\) the total order on \( \mathbb{Z}^6 \) defined by \((i_1, i_2, i_3, i_4, i_5, i_6) <_2 (j_1, j_2, j_3, j_4, j_5, j_6)\) if \([w_2 > i_2]\) or \([w_2 = i_2, w_1 > i_1]\) or \([w_2 = i_2, w_1 = i_1, w_3 > i_3]\) or \(\cdots\) or \([w_1 = i_1, w_6 \geq t_6, l = 2, 3, 4, 5]\).

Suppose that there exists \((i_1, \cdots, i_6) \in J\) such that \(a_{(i_1, \cdots, i_6)} \neq 0\). Let \((w_1, \cdots, w_6) \in J\) be the greatest element of \( J \) with respect to \(<_2\). Thus \(b_{(w_1, \cdots, w_6)} \neq 0\). Note, \((F_1^{i_1}T_2^{j_2}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6})_{(i_1, \cdots, i_6)J_{(i_1, \cdots, i_6)}}\) is a basis of \( A_4^{(4)} \). Identifying the coefficients of \( F_1^{i_1}T_2^{j_2}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6+1} \), we have that \(b_{(w_1, \cdots, w_6)} = 0\). This is a contradiction to our assumption, hence \(b_{(i_1, \cdots, i_6)} = 0\) for all \((i_1, \cdots, i_6) \in J\). This implies that

\[
\sum_{j \in J} a_j F_1^{i_1}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6} = 0.
\]

Consequently, \(a_i = 0\) for all \(i \in I\). Therefore, \( \mathcal{S}_4 \) is a linearly independent set. 

\( \square \)
In $\mathcal{R}_4 = A^{(4)}_{n,\beta} S_4^{-1}$, we have the following two relations: $f_1 t_3 t_5 + a t_2 t_5 = \alpha$ and $t_2 t_4 t_6 = \beta$. This implies that $f_1 t_3 = \alpha t_5^{-1} - a t_5^{-1} t_2$. Putting these two relations together, we have that

$$f_1 t_3 = \alpha t_5^{-1} - a \beta t_6^{-1} t_4^{-1}. \quad (17)$$

Note, we will usually identify $\mathcal{R}_4$ with $A^{(4)}_{n,\beta} S_4^{-1}/(\hat{\Omega}_1 - \alpha)$.

**Proposition 5.3.** The set $\mathcal{B}_4 = \left\{ f^{i_1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6} \mid i_1, i_3, i_4, i_5, i_6 \in \mathbb{N} \right\}$ is a $K$-basis of $\mathcal{R}_4$.

**Proof.** Since $\left( f^{k_1}_{1} t_{i_3}^{k_3} t_{i_4}^{k_4} t_{i_5}^{k_5} t_{i_6}^{k_6} \right)_{(k_1, k_3, \ldots, k_6) \in \mathbb{N}^2 \times \mathbb{Z}^3}$ is a basis of $A^{(4)}_{n,\beta} S_4^{-1}$ (Proposition 5.2), the set

$$f^{k_1}_{1} t_{i_3}^{k_3} t_{i_4}^{k_4} t_{i_5}^{k_5} t_{i_6}^{k_6} \text{ spans } \mathcal{R}_4.$$ We show that $\mathcal{B}_4$ is a spanning set of $\mathcal{R}_4$ by showing that $f^{k_1}_{1} t_{i_3}^{k_3} t_{i_4}^{k_4} t_{i_5}^{k_5} t_{i_6}^{k_6}$ can be written as a finite linear combination of the elements of $\mathcal{B}_4$ for all $(k_1, k_3, \ldots, k_6) \in \mathbb{N}^2 \times \mathbb{Z}^3$. By Proposition 5.2, it is sufficient to do this by induction on $k_1$. The result is clear when $k_1 = 0$. Assume that the statement is true for $k_1 \geq 0$. That is,

$$f^{k_1}_{1} t_{i_3}^{k_3} t_{i_4}^{k_4} t_{i_5}^{k_5} t_{i_6}^{k_6} = \sum_{j \in I_1} a_j f^{i_1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6} + \sum_{j \in I_2} b_j f^{i_3}_{1} t_{i_4}^{i_4} t_{i_5}^{i_5} t_{i_6}^{i_6},$$

where $i = (i_1, i_4, i_5, i_6) \in I_1 \subset \mathbb{N} \times \mathbb{Z}^3$ and $j = (i_3, i_4, i_5, i_6) \in I_2 \subset \mathbb{N} \times \mathbb{Z}^3$. Note, $a_j$ and $b_j$ are all scalars. It follows that

$$f^{k_1+1}_{1} t_{i_3}^{k_3} t_{i_4}^{k_4} t_{i_5}^{k_5} t_{i_6}^{k_6} = f^{k_1}_{1} f^{i_1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6} + \sum_{j \in I_2} b_j f^{i_3}_{1} t_{i_4}^{i_4} t_{i_5}^{i_5} t_{i_6}^{i_6}.$$

Clearly, the monomial $f^{i_1+1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6} \in \text{Span}(\mathcal{B}_4)$. We have to also show that $f^{i_3}_{1} t_{i_4}^{i_4} t_{i_5}^{i_5} t_{i_6}^{i_6} \in \text{Span}(\mathcal{B}_4)$ for all $i_3 \in \mathbb{N}$ and $i_4, i_5, i_6 \in \mathbb{Z}$. This can easily be achieved by an induction on $i_4$. Therefore, by the principle of mathematical induction, $\mathcal{B}_4$ is a spanning set of $\mathcal{R}_4$ over $K$.

We prove that $\mathcal{B}_4$ is a linearly independent set. Suppose that

$$\sum_{j \in I_1} a_j f^{i_1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6} + \sum_{j \in I_2} b_j f^{i_3}_{1} t_{i_4}^{i_4} t_{i_5}^{i_5} t_{i_6}^{i_6} = 0.$$

It follows that there exists $\nu \in A^{(4)}_{n,\beta} S_4^{-1}$ such that

$$\sum_{j \in I_1} a_j f^{i_1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6} + \sum_{j \in I_2} b_j f^{i_3}_{1} t_{i_4}^{i_4} t_{i_5}^{i_5} t_{i_6}^{i_6} = (\hat{\Omega}_1 - \alpha) \nu.$$

Write $\nu = \sum_{j \in J} c_j f^{i_1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6}$, where $J = (i_1, i_3, i_4, i_5, i_6) \in J \subset \mathbb{N}^2 \times \mathbb{Z}^3$ and $c_j \in K$. Note, $\hat{t}_2 = \beta t_6^{-1} t_4^{-1}$. We have that $\hat{\Omega}_1 = \hat{f}_1 f^{i_1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6} + a \beta t_6^{-1} t_4^{-1} t_5$. Therefore,

$$\sum_{j \in I_1} c_j f^{i_1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6} + \sum_{j \in I_2} b_j f^{i_3}_{1} t_{i_4}^{i_4} t_{i_5}^{i_5} t_{i_6}^{i_6} = \sum_{j \in J} q^* c_j f^{i_1+1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6} + \sum_{j \in J} q^* \beta c_j f^{i_1}_{1} t_{i_2}^{i_3+1} t_{i_4}^{i_5} t_{i_6}^{i_6} + \sum_{j \in J} q^* \beta c_j f^{i_1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5+1} t_{i_6}^{i_6} + \sum_{j \in J} q^* \beta c_j f^{i_1+1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6} + \sum_{j \in J} q^* \beta c_j f^{i_1}_{1} t_{i_2}^{i_3} t_{i_4}^{i_5} t_{i_6}^{i_6}.

Suppose that there exists $(i_1, i_3, i_4, i_5, i_6) \in J$ such that $c_{(i_1, i_3, i_4, i_5, i_6)} \neq 0$.

Let $(w_1, w_2, w_3, w_4, w_5, w_6) \in J$ be the greatest element (in the lexicographic order on $\mathbb{N}^2 \times \mathbb{Z}^3$) of $J$. 23
such that \( c(w_1,w_3,w_4,w_5,w_6) \neq 0 \). Since 
\[
\left( \hat{k}_1 ^ {w_1} \hat{k}_3 ^ {w_3} \hat{k}_4 ^ {w_4} \hat{k}_5 ^ {w_5} \hat{k}_6 ^ {w_6} \right) \text{ is a basis of } A^{(4)}S^{-1}_4,
\]
it implies that the coefficients of 
\[
\hat{w}_1 ^ {w_1} \hat{w}_3 ^ {w_3} \hat{w}_4 ^ {w_4} \hat{w}_5 ^ {w_5} \hat{w}_6 ^ {w_6}
\]
in the above equality can be identified as: 
\[
q^*c(w_1,w_3,w_4,w_5,w_6) = 0.
\]
Hence, 
\[
c_{(i_1,i_3,i_4,i_5,i_6)} = 0
\]
for all \((i_1,i_3,i_4,i_5,i_6) \in J\). This further implies that 
\[
\sum _{j \in I_1} a_j \hat{t}_1 ^ {i_1} \hat{t}_4 ^ {i_4} \hat{t}_3 ^ {i_3} \hat{t}_5 ^ {i_5} \hat{t}_6 ^ {i_6} + \sum _{j \in I_2} b_j \hat{t}_1 ^ {i_1} \hat{t}_4 ^ {i_4} \hat{t}_5 ^ {i_5} \hat{t}_6 ^ {i_6} = 0.
\]
It follows from the previous proposition that \( a_\perp \) and \( b_\perp \) are all zero. In conclusion, \( B_4 \) is a linearly independent set.

**Basis for \( R_5 \).** We will identify \( R_5 \) with \( A_\alpha^{(5)}S^{-1}_5 / (\hat{\Omega}_2 - \beta) \), where \( A_\alpha^{(5)}S^{-1}_5 = \frac{A^{(5)}S^{-1}_5}{\langle \Omega_1 - \alpha \rangle} \). Note, the canonical images of \( E_{i,j} \) (resp. \( T_i \)) in \( A_\alpha^{(5)}S^{-1}_5 \) will be denoted by \( \hat{e}_{i,j} \) (resp. \( \hat{t}_i \)). We now find a basis for \( A_\alpha^{(5)}S^{-1}_5 \). Recall that \( \Omega_1 = Z_1T_3T_5 + aZ_2T_5 \) and \( \Omega_2 = Z_2T_4T_6 + bT_3^3T_6 \) in \( A^{(5)} \) (remember, \( Z_1 := E_{1,5} \) and \( Z_2 := E_{2,5} \)). Since \( 2zt_6 + bt_3^3t_6 = \beta \) and \( \hat{\alpha} \hat{t}_3 \hat{t}_5 + a\hat{\alpha} \hat{t}_5 = \alpha \) in \( R_5 \) and \( A_\alpha^{(5)}S^{-1}_5 \) respectively, we have the relation 
\[
\hat{z}_2 = \frac{1}{a} (\hat{\alpha} \hat{t}_5 - \hat{z}_1 \hat{t}_3)
\]
in \( A_\alpha^{(5)}S^{-1}_5 \) and, in \( R_5 \), we have the following two relations:
\[
\hat{z}_2 = \frac{1}{a} (\alpha \hat{t}_5 - \hat{z}_1 \hat{t}_3) ,
\]
\[
\hat{t}_3^2 = \frac{1}{b} (\beta \hat{t}_6 - \hat{z}_2 \hat{t}_4) = \frac{\beta}{b} \hat{t}_6^2 - \frac{q^3 \alpha}{ab} \hat{t}_4 \hat{t}_5^2 + \frac{1}{ab} \hat{z}_1 \hat{t}_3 \hat{t}_4.
\]
**Proposition 5.4.** The set \( S_5 = \{ \hat{z}_1 \hat{t}_3 \hat{t}_4 \hat{t}_5 \hat{t}_6 | (i_1,i_3,i_4,i_5,i_6) \in \mathbb{N}^3 \times \mathbb{Z}^2 \} \) is a \( \mathbb{K} \)-basis of \( A_\alpha^{(5)}S^{-1}_5 \), where \( \alpha \in \mathbb{K} \).
**Proof.** The proof is similar to that of Proposition 5.2 and is so left to the reader. Details can be found in [19].

**Proposition 5.5.** The set \( B_5 = \{ z_1 \hat{t}_3 \hat{t}_4 \hat{t}_5 \hat{t}_6 | (\xi,i_1,i_4,i_5,i_6) \in \{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2 \} \) is a \( \mathbb{K} \)-basis of \( R_5 \).
**Proof.** The proof is similar to that of Proposition 5.3 and so is left to the reader. Details can be found in [19].

We note for future reference the following immediate corollary.

**Corollary 5.6.** Let \( I \) be a finite subset of \( \{0,1,2\} \times \mathbb{N} \times \mathbb{Z}^2 \) and \( (a_{(i,j)} \xi \hat{t}_3 \hat{t}_4 \hat{t}_5 \hat{t}_6 | \in I \) a family of scalars. If 
\[
\sum _{(i,j) \in I} a_{(i,j)} \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \xi_6 = 0,
\]
then \( a_{(i,j)} = 0 \) for all \((i,j) \in I \).

**Remark 5.7.** We were not successful in finding a basis for \( R_6 \). However, this has no effect on our main results in this section. Since \( R_7 = A_{\alpha,\beta} \), we already have a basis for \( R_7 \) (Proposition 3.4).

### 5.3 Derivations of \( A_{\alpha,\beta} \)

We are now going to study the derivations of \( A_{\alpha,\beta} \). We will only treat the case when both \( \alpha \) and \( \beta \) are non-zero, and mention results when either \( \alpha \) or \( \beta \) is zero without details.

Throughout this subsection, we assume that \( \alpha \) and \( \beta \) are non-zero. Let \( \text{Der}(A_{\alpha,\beta}) \) denote the \( \mathbb{K} \)-derivations of \( A_{\alpha,\beta} \) and \( D \in \text{Der}(A_{\alpha,\beta}) \). Via localization, \( D \) extends uniquely to a derivation of
Proof. 1. Set $R$ obtained by deleting the first row and first column of each of the series of algebras in (16). Therefore, $R$ where

Furthermore, since $R$ where

$x \in R$, and $\delta$ is a scalar derivation of $R$ defined as $\delta(t_i) = \lambda_i t_i$ for each $i = 3, 4, 5, 6$. Note, $\lambda_i \in Z(R_3) = \mathbb{K}$. Also, $ad_p$ is an inner derivation of $R$ defined as $ad_p(L) = xL - Lx$ for all $L \in R_3$.

We aim to describe $D$ as a derivation of $A_{x, \beta} = R_7$. We do this in several steps.

Before starting the process we note the following relations that will be used in this section. They all follow from \cite[Lemma 5.3.2]{1}.

**Remark 5.8.** Recall the notations:

$$
\begin{align*}
  f_1 & := e_{1,4} & F_1 & := E_{1,4} \\
  z_1 & := e_{1,5} & Z_1 & := E_{1,5} \\
  z_2 & := e_{2,5} & Z_2 & := E_{2,5}.
\end{align*}
$$

Then

$$
\begin{align*}
  f_1 &= t_1 - at_2t_3^{-1} & e_{3,6} &= t_3 - at_4t_5^{-1} \\
  z_1 &= f_1 - st_3^2t_4^{-1} & e_1 &= e_{1,6} - rt_5t_6^{-1} \\
  z_2 &= t_2 - bt_3^2t_4^{-1} & e_3 &= e_{3,6} - st_5^2t_6^{-1} \\
  e_{1,6} &= z_1 - he_{3,6}t_5^{-1} - gt_4t_5^{-2} & e_4 &= t_4 - bt_5^2t_6^{-1}.
\end{align*}
$$

We first describe $D$ as a derivation of $R_4$.

**Lemma 5.9.** 1. $x \in R_4$.

2. $\lambda_5 = \lambda_4 + \lambda_6$, $\delta(f_1) = -(\lambda_3 + \lambda_5)f_1$ and $\delta(t_2) = -\lambda_5 t_2$.

3. Set $\lambda_1 := -(\lambda_3 + \lambda_5)$ and $\lambda_2 := -\lambda_5$. Then, $D(e_{\kappa,4}) = ad_{x}(e_{\kappa,4}) + \lambda_{\kappa} e_{\kappa,4}$ for all $\kappa \in \{1, \cdots, 6\}$.

**Proof.** 1. Set $Q_4 := K[\mu][t_{3}^{\pm 1}, t_{5}^{\pm 1}, t_{6}^{\pm 1}]$, where $\mu$ is the skew-symmetric sub-matrix of $N$ (see Section 4.1) obtained by deleting the first row and first column of $N$. Observe that $Q_4$ is a subalgebra of both $R_3$ and $R_4$ with central element $z := t_4t_5^{-1}t_6$.

Furthermore, since $R_3$ is a quantum torus, we can present it as a free left $Q_4$-module with basis $(t_{3}^{s})_{s \in \mathbb{Z}}$.

With this presentation, $x \in R_3$ can be written as

$$x = \sum_{s \in \mathbb{Z}} y_{s} t_{3}^{s},$$

where $y_{s} \in Q_4$. Set

$$x_{+} := \sum_{s \geq 0} y_{s} t_{3}^{s} \text{ and } x_{-} := \sum_{s < 0} y_{s} t_{3}^{s}.$$ 

Clearly, $x = x_{+} + x_{-}$. Obviously, $x_{+} \in R_4$, hence we aim to also show that $x_{-}$ belongs to $R_4$ by following a pattern similar to \cite[Proposition 7.1.2]{2}. As $D$ is a derivation of $R_4$, we have that $D(z^j) \in R_4$ for all $j \in \mathbb{N}_{\geq 1}$. Now $D(z^j) = ad_{x}(z^j) + \delta(z^j) = ad_{x_{+}}(z^j) + ad_{x_{-}}(z^j) + \delta(z^j)$. Observe that $ad_{x_{+}}(z^j) \in R_4$; since $x_{+}, z^j \in R_4$. Also, $\delta(z) = \delta(t_4t_5^{-1}t_6) = (\lambda_4 - \lambda_5 + \lambda_6)t_4t_5^{-1}t_6 = (\lambda_4 - \lambda_5 + \lambda_6)z$, where $\lambda_4, \lambda_5, \lambda_6 \in \mathbb{K}$. It follows that $\delta(z^j) = j(\lambda_4 - \lambda_5 + \lambda_6)z^j \in R_4$. We can therefore conclude that each $ad_{x_{-}}(z^j)$ belongs to $R_4$ since $D(z^j), ad_{x_{+}}(z^j), \delta(z^j) \in R_4$. We have:

$$ad_{x_{-}}(z^j) = x_{-} z^j - z^{j} x_{-} = \sum_{s=-n}^{-1} y_{s} t_{3}^{s} z^{j} - \sum_{s=-n}^{-1} y_{s} z^{j} t_{3}^{s}.$$ 

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One can verify that \( zt_3 = q^{-2}t_3z \). Therefore,

\[
\text{ad}_{x_j}(z^j) = \sum_{s=1}^{-n} (1 - q^{-2js}) y_s t_3^j z, \quad \text{hence,} \quad \text{ad}_{x_j}(z^j)z^{-j} = \sum_{s=1}^{-n} (1 - q^{-2js}) y_s t_3^s.
\]

Set \( \nu_j := \text{ad}_{x_j}(z^j)z^{-j} \in \mathcal{R}_4 \). It follows that

\[
\nu_j = \sum_{s=1}^{-n} (1 - q^{-2js}) y_s t_3^s,
\]

for each \( j \in \{1, \cdots, n\} \). One can therefore write the above equality as a matrix equation as follows:

\[
\begin{pmatrix}
(1 - q^2) & (1 - q^4) & (1 - q^6) & \cdots & (1 - q^{2n}) \\
(1 - q^4) & (1 - q^8) & (1 - q^{12}) & \cdots & (1 - q^{4n}) \\
(1 - q^6) & (1 - q^{12}) & (1 - q^{18}) & \cdots & (1 - q^{6n}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(1 - q^{2n}) & (1 - q^{4n}) & (1 - q^{6n}) & \cdots & (1 - q^{2n})
\end{pmatrix}
\begin{pmatrix}
y_{-1}t_3^{-1} \\ y_{-2}t_3^{-2} \\ y_{-3}t_3^{-3} \\ \vdots \\ y_{-n}t_3^{-n}
\end{pmatrix}
= \begin{pmatrix}
\nu_1 \\ \nu_2 \\ \nu_3 \\ \vdots \\ \nu_n
\end{pmatrix}.
\]

We already know that each \( \nu_j \) belongs to \( \mathcal{R}_4 \). We want to show that \( y_s t_3^s \) also belongs to \( \mathcal{R}_4 \) for each \( s \in \{-1, \cdots, -n\} \). To establish this, it is sufficient to show that the coefficient matrix of the above matrix equation is invertible. Let \( U \) represent this matrix. Thus,

\[
U = \begin{pmatrix}
(1 - q^2) & (1 - q^4) & (1 - q^6) & \cdots & (1 - q^{2n}) \\
(1 - q^4) & (1 - q^8) & (1 - q^{12}) & \cdots & (1 - q^{4n}) \\
(1 - q^6) & (1 - q^{12}) & (1 - q^{18}) & \cdots & (1 - q^{6n}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(1 - q^{2n}) & (1 - q^{4n}) & (1 - q^{6n}) & \cdots & (1 - q^{2n})
\end{pmatrix},
\]

Apply row operations: \(-r_{n-1} + r_n \rightarrow r_n, \cdots, -r_2 + r_3 \rightarrow r_3, -r_1 + r_2 \rightarrow r_2\) to \( U \) to obtain:

\[
U' = \begin{pmatrix}
l_1 & l_2 & l_3 & \cdots & l_n \\
l_1q^2 & l_1q^4 & l_3q^6 & \cdots & l_nq^{2n} \\
l_1q^4 & l_2q^8 & l_3q^{12} & \cdots & l_nq^{4n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
l_1q^{2n} & l_1q^{4n} & l_3q^{6n} & \cdots & l_nq^{2n}
\end{pmatrix},
\]

where \( l_i := 1 - q^{2i}; i \in \{1, 2, \cdots, n\} \). Clearly, \( U' \) is similar to a Vandermonde matrix (since the terms in each column form a geometric sequence) which is well known to be invertible when all parameters are pairwise distinct (this is the case here as \( q \) is not a root of unity). This further implies that \( U \) is invertible. So each \( y_s t_3^s \) is a linear combination of the \( \nu_j \in \mathcal{R}_4 \). As a result, \( y_s t_3^s \in \mathcal{R}_4 \) for all \( s \in \{-1, \cdots, -n\} \). Consequently, \( x_+ = \sum_{s=-1}^{-n} y_s t_3^s \in \mathcal{R}_4 \) as desired.

2. Recall that \( \delta(t_\kappa) = \lambda_\kappa t_\kappa \) for all \( \kappa \in \{3, 4, 5, 6\} \) and \( \lambda_\kappa \in \mathbb{K} \). From Remark 5.3, we have that \( f_1 = t_1 - at_2t_3^{-1} \). Recall from Section 4.1 that \( t_1 = \alpha t_5^{-1}t_3^{-1} \) and \( t_2^{-1} = \beta t_6^{-1}t_4^{-1} \) in \( \mathcal{R}_3 = \mathscr{A}_{a, \beta} \). As a result, \( f_1 = \alpha t_5^{-1}t_3^{-1} - a\beta t_6^{-1}t_4^{-1}t_3^{-1} \). Hence,

\[
\delta(f_1) = - (\lambda_5 + \lambda_3) \alpha t_5^{-1}t_3^{-1} - (\lambda_6 + \lambda_4 + \lambda_3) a\beta t_6^{-1}t_4^{-1}t_3^{-1}.
\]

From Proposition 5.3, the set \( \mathcal{B}_3 = \{ f_1 i_1 t_4^{i_5} t_5^{i_6}, t_3 i_4 t_4^{i_5} t_6^{i_6} | i_1, i_3 \in \mathbb{N} \text{ and } i_4, i_5, i_6 \in \mathbb{Z} \} \) is a \( \mathbb{K} \)-basis of \( \mathcal{R}_4 \). Since \( t_4, t_5 \) and \( t_6 \) \( q \)-commute with \( f_1 \) and \( t_3 \), one can also write \( \delta(f_1) \in \mathcal{R}_4 \) in terms of \( \mathcal{B}_4 \) as follows:

\[
\delta(f_1) = \sum_{r>0} a_r f_1^r + \sum_{s \geq 0} b_st_3^s.
\]
where $a_r$ and $b_s$ belong to $Q_q = \mathbb{K}[t_{4}^{\pm 1}, t_{5}^{\pm 1}, t_{6}^{\pm 1}]$.

$$
f_1' = (\alpha t_5^{-1}t_3 - a\beta t_6^{-1}t_4^{-1}t_3^{-1})' = \sum_{i=0}^{r} \binom{r}{i} q^i \cdot (\alpha t_5^{-1}t_3^{-1})^i(-a\beta t_6^{-1}t_4^{-1}t_3^{-1})^{r-i}.
$$

$$
= \sum_{i=0}^{r} \binom{r}{i} q^i \cdot \alpha^i(-a\beta)^{r-i}q^{6(i-1)+\frac{1}{2}((r-i)-1)+3(i-r)}t_5^{-i}t_6^{-i}t_4^{-i}t_3^{-r} = c_r t_3^{-r}, \quad (22)
$$

where

$$
c_r = \sum_{i=0}^{r} \binom{r}{i} q^i \cdot \alpha^i(-a\beta)^{r-i}t_5^{-i}t_6^{-i}t_4^{-i}t_3^{-r} \in Q_q \setminus \{0\}, \quad (23)
$$

Substitute (22) into (21) to obtain:

$$
\delta(f_1) = \sum_{r>0} a_r c_r t_3^{-r} + \sum_{s \geq 0} b_s t_3^s.
$$

One can rewrite (20) as

$$
\delta(f_1) = dt_3^{-1}, \quad (25)
$$

where $d = -(\lambda_5 + \lambda_3)\alpha t_5^{-1} + (\lambda_6 + \lambda_4 + \lambda_3) a\beta t_6^{-1}t_4^{-1} \in Q_q$. Comparing (24) to (25) shows that $b_s = 0$ for all $s \geq 0$, and $a_r c_r = 0$ for all $r \neq 1$. Therefore $\delta(f_1) = a_1 c_1 t_3^{-1}$. Moreover, from (23), $c_1 = -(\alpha t_6^{-1}t_4^{-1} + t_6^{-1}t_4^{-1})$. Hence,

$$
\delta(f_1) = a_1 c_1 t_3^{-1} = a_1(\alpha t_6^{-1}t_4^{-1} + t_5^{-1})t_3^{-1} = a_1\alpha t_5^{-1}t_3^{-1} - a_1\beta t_6^{-1}t_4^{-1}t_3^{-1}. \quad (26)
$$

Comparing (20) to (24) reveals that $a_1 = -(\lambda_5 + \lambda_3) = -(\lambda_6 + \lambda_4 + \lambda_3)$. Consequently, $\lambda_5 = \lambda_6 + \lambda_4$. Hence, $\delta(f_1) = -(\lambda_5 + \lambda_3)\alpha t_5^{-1}t_3^{-1} + (\lambda_5 + \lambda_3) a\beta t_6^{-1}t_4^{-1}t_3^{-1} = -(\lambda_5 + \lambda_3)f_1$. Finally, since $t_2 = \beta t_6^{-1}t_4^{-1} \in R_5$, it follows that $\delta(t_2) = -(\lambda_5 + \lambda_4)\beta t_6^{-1}t_4^{-1} = -(\lambda_5 + \lambda_4)t_2 = -t_2$.

3. Set $\lambda_1 := -(\lambda_3 + \lambda_5)$ and $\lambda_2 := -\lambda_5$. It follows from points (1) and (2) that $D(e_{\kappa, 4}) = ad_x(e_{\kappa, 4}) + \delta(e_{\kappa, 4}) = ad_x(e_{\kappa, 4}) + \lambda_4 e_{\kappa, 4}$ for all $\kappa \in \{1, \cdots , 6\}$. In conclusion, $D = ad_x + \delta$, with $x \in R_4$. \qed

We proceed to describe $D$ as a derivation of $R_5$.

**Lemma 5.10.** 1. $x \in R_5$.

2. $\lambda_4 = 3\lambda_3 + \lambda_5$, $\lambda_6 = -3\lambda_3$, $\delta(z_1) = -(\lambda_3 + \lambda_5)z_1$ and $\delta(z_2) = -\lambda_5z_2$.

3. Set $\lambda_1 := -(\lambda_3 + \lambda_5)$, $\lambda_2 := -\lambda_5$ and $\lambda_6 := -3\lambda_3$. Then, $D(e_{\kappa, 5}) = ad_x(e_{\kappa, 5}) + \lambda_4 e_{\kappa, 5}$ for all $\kappa \in \{1, \cdots , 6\}$.

**Proof.** In this proof, we denote $\nu := (i, j, k, l) \in \mathbb{N} \times \mathbb{Z}^3$.

1. We already know that $x \in R_4 = R_5^{[t_{4}^{-1}]}$. Given the basis $B_5$ of $R_5$ (Proposition 5.5), $x$ can be written as $x = \sum_{(\xi, \omega) \in I} a_{(\xi, \omega)} z_1^{i}t_3^{j}t_4^{k}t_5^{l}$, where $I$ is a finite subset of $\{0, 1, 2\} \times \mathbb{N} \times \mathbb{Z}^3$ and $a_{(\xi, \omega)}$ are scalars.

Write $x = x_+ + x_-$, where

$$
x_+ = \sum_{(\xi, \omega) \in I, j \geq 0} a_{(\xi, \omega)} z_1^{i}t_3^{j}t_4^{k}t_5^{l} \quad \text{and} \quad x_- = \sum_{(\xi, \omega) \in I, j < 0} a_{(\xi, \omega)} z_1^{i}t_3^{j}t_4^{k}t_5^{l}.
$$

Suppose that there exists a minimum $j_0 < 0$ such that $a_{(\xi, i, j_0, k, l)} \neq 0$ for some $(\xi, i, j_0, k, l) \in I$ and $a_{(\xi, i, j, k, l)} = 0$ for all $(\xi, i, j_0, k, l) \in I$ with $j < j_0$. Given this assumption, write

$$
x_- = \sum_{(\xi, \omega) \in I, j_0 \leq j \leq -1} a_{(\xi, \omega)} z_1^{i}t_3^{j}t_4^{k}t_5^{l}.
$$
Now, \( D(t_6) = ad_{x_+}(t_6) + ad_{x_+}(t_6) + \delta(t_6) \in \mathcal{R}_5 \). This implies that \( ad_{x_+}(t_6) \in \mathcal{R}_5 \), since \( ad_{x_+}(t_6) + \delta(t_6) = ad_{x_+}(t_6) + \lambda_6 t_6 \in \mathcal{R}_5 \). We aim to show that \( x_- = 0 \). Since \( t_6 \) is normal in \( \mathcal{R}_5 \), one can easily verify that

\[
ad_{x_-}(t_6) = \sum_{\langle \xi, \nu \rangle \in I \atop j_0,j \leq -1} \left( q^{3(i-j-k)} - 1 \right) a_{\langle \xi, \nu \rangle} z_1^j t_3^j t_4^k t_5^l t_6^e.
\]

Set \( w := (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2 \). One can equally write \( ad_{x_-}(t_6) \in \mathcal{R}_5 \) in terms of the basis \( \mathcal{B}_5 \) of \( \mathcal{R}_5 \) (Proposition 5.5) as:

\[
ad_{x_-}(t_6) = \sum_{\langle \xi, \nu \rangle \in J \atop j_0 \leq -1} b_{\langle \xi, \nu \rangle} z_1^j t_3^j t_4^k t_5^l t_6^e,
\]

where \( J \) is a finite subset of \( \{0, 1, 2\} \times \mathbb{N}^2 \times \mathbb{Z}^2 \) and \( b_{\langle \xi, \nu \rangle} \) are all scalars. It follows that

\[
\sum_{\langle \xi, \nu \rangle \in I \atop j_0 \leq -1} \left( q^{3(i-j-k)} - 1 \right) a_{\langle \xi, \nu \rangle} z_1^j t_3^j t_4^k t_5^l t_6^e = \sum_{\langle \xi, \nu \rangle \in J \atop j_0 \leq -1} b_{\langle \xi, \nu \rangle} z_1^j t_3^j t_4^k t_5^l t_6^e.
\]

As \( \mathcal{B}_5 \) is a basis for \( \mathcal{R}_5 \), we deduce from Corollary 5.4 that \( \left( z_1^j t_3^j t_4^k t_5^l t_6^e \right)_{\langle \xi, \nu \rangle \in \mathbb{N}^2 \times \mathbb{Z}^2} \) is a basis for \( \mathcal{R}_5[t_4^{-1}] \). Now, at \( j = j_0 \), denote \( \nu = (i, j, k, l) \) by \( \nu_0 := (i, j_0, k, l) \). Since \( \nu_0 \in \mathbb{N} \times \mathbb{Z}^2 \) (with \( j_0 < 0 \)) and \( w := (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2 \) (with \( j \geq 0 \)), it follows from the above equality that, at \( w_0 \), we must have

\[
\left( q^{3(i-j_0-k)} - 1 \right) a_{\langle \xi, \nu \rangle} = 0.
\]

From our initial assumption, the coefficients \( a_{\langle \xi, \nu \rangle} \) are all not zero, therefore \( q^{3(i-j_0-k)} - 1 = 0 \). This implies that

\[
k = i - j_0,
\]

for some \( \langle \xi, \nu_0 \rangle \in I \).

In a similar manner, \( D(t_3) = ad_{x_+}(t_3) + ad_{x_-}(t_3) + \delta(t_3) \in \mathcal{R}_5 \). This implies that \( ad_{x_-}(t_3) \in \mathcal{R}_5 \), since \( ad_{x_+}(t_3) + \delta(t_3) = ad_{x_+}(t_3) + \lambda_3 t_3 \in \mathcal{R}_5 \). We have that

\[
ad_{x_-}(t_3) = \sum_{\langle \xi, \nu \rangle \in I \atop j_0 \leq -1} a_{\langle \xi, \nu \rangle} z_1^j t_3^j t_4^k t_5^l t_6^e - \sum_{\langle \xi, \nu \rangle \in I \atop j_0 \leq -1} a_{\langle \xi, \nu \rangle} z_1^j t_3^j t_4^k t_5^l t_6^e.
\]

One can deduce from Lemma A.1 3a) that

\[
t_3 z_1^j = q^{-i} z_1^j t_3 + d_2[i] z_1^{i-1} z_2,
\]

where \( d_2[i] = q^{-i} d_2[1] \left[ \frac{1 - q^{-2i}}{1 - q^{-2}} \right] \), \( d_2[1] = -(q + q^{-1} + q^{-3}) \) and \( d_2[0] = 0 \). Therefore, the above expression for \( ad_{x_-}(t_3) \) can be expressed as:

\[
ad_{x_-}(t_3) = \sum_{\langle 0, \nu \rangle \in I \atop j_0 \leq -1} f[i, j, k] a_{\langle 0, \nu \rangle} z_1^j t_3^j t_4^k t_5^l t_6^e + \sum_{\langle 1, \nu \rangle \in I \atop j_0 \leq -1} f[i, j, k] a_{\langle 1, \nu \rangle} z_1^j t_3^j t_4^k t_5^l t_6^e + \sum_{\langle 2, \nu \rangle \in I \atop j_0 \leq -1} f[i, j, k] a_{\langle 2, \nu \rangle} z_1^j t_3^j t_4^k t_5^l t_6^e - \sum_{\langle 3, \nu \rangle \in I \atop j_0 \leq -1} a_{\langle \xi, \nu \rangle} z_1^{-1} z_2 t_3^j t_4^k t_5^l t_6^e,
\]

where \( f[i, j, k] := q^{-(k+3j)} - q^{-i} \). Recall from (18) and (19) that

\[
z_2 = \frac{1}{a} (\alpha t_3^{-1} - z_1 t_3) \quad \text{and} \quad t_3^2 = \frac{\beta}{b} t_4^{-1} - \frac{q^3}{ab} t_4 t_5^{-1} + \frac{1}{ab} z_1 t_3 t_4,
\]

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where \(a\) and \(b\) are non-zero scalars (Appendix B). Using these two expressions, one can write \(\text{ad}_{x^-}(t_3)\) in terms of the basis of \(\mathcal{R}_5\) as:

\[
\text{ad}_{x^-}(t_3) = \mathcal{K} + \sum_{(0,\omega) \in I} g[i,j_0,k]a_{(0,\omega)} z_i^1 t_3^1 t_4^j t_5^k t_6^l + \sum_{(1,\omega) \in I} g[i,j_0,k]a_{(1,\omega)} z_i^2 t_3^1 t_4^j t_5^k t_6^l
\]

\[+ \sum_{(2,\omega) \in I} \frac{q^*\beta}{b} a_{(2,\omega)} g[i,j_0,k] z_i^1 t_3^j t_4^k t_5^l t_6^l - \sum_{(\xi,\omega) \in I} \frac{q^*\alpha}{a} d_2[i] a_{(\xi,\omega)} z_i^1 t_3^j t_4^k t_5^l t_6^l\]

\[= \sum_{(0,\omega) \in I} 1/b \left( q^*\beta g[i,j_0,k]a_{(0,i,j_0,k,l+1)} + (q^*\alpha bd_2[i+1]/a) a_{(0,i+1,j_0,k,l+1)} \right) z_i^1 t_3^j t_4^k t_5^l t_6^l
\]

\[+ \sum_{(1,\omega) \in I} (g[i,j_0,k]a_{(0,i,j_0,k,l)} + (q^*\alpha d_2[i+1]/a) a_{(1,i+1,j_0,k,l)}) z_i^1 t_3^j t_4^k t_5^l t_6^l
\]

\[+ \sum_{(2,\omega) \in I} (g[i,j_0,k]a_{(1,i,j_0,k,l)} + (q^*\alpha d_2[i+1]/a) a_{(2,i+1,j_0,k,l)}) z_i^1 t_3^j t_4^k t_5^l t_6^l + \mathcal{K}, \quad (28)
\]

where \(g[i,j_0,k] := q^{-(k+3j_0)} - q^{-i} + d_2[i]/a\) and

\(\mathcal{K} \in \text{Span} \left( \mathcal{B}_5 \setminus \{ z_i^1 t_3^j t_4^k t_5^l t_6^l \mid (\xi, i, j_0, k, l) \in \{0,1,2\} \times \mathbb{N} \times \mathbb{Z}^2 \} \right).

One can also write \(\text{ad}_{x^-}(t_3) \in \mathcal{R}_5\) in terms of the basis \(\mathcal{B}_5\) of \(\mathcal{R}_5\) (Proposition 5.5) as:

\[
\text{ad}_{x^-}(t_3) = \sum_{\xi,\omega \in I} b_{(\xi,\omega)} z_i^1 t_3^j t_4^k t_5^l t_6^l, \quad (29)
\]

where \(J\) is a finite subset of \(\{0,1,2\} \times \mathbb{N} \times \mathbb{Z}^2\), and \(b_{(\xi,\omega)} \in \mathbb{K}\). Recall: \(w = (i,j,k,l) \in \mathbb{N}^2 \times \mathbb{Z}^2\). Now, (28), and (29) imply that

\[
\sum_{(\xi,\omega) \in J} b_{(\xi,\omega)} z_i^1 t_3^j t_4^k t_5^l t_6^l = \sum_{(0,\omega) \in I} 1/b \left( q^*\beta g[i,j_0,k]a_{(0,i,j_0,k,l+1)} + (q^*\alpha bd_2[i+1]/a) a_{(0,i+1,j_0,k,l+1)} \right) z_i^1 t_3^j t_4^k t_5^l t_6^l
\]

\[+ \sum_{(1,\omega) \in I} (g[i,j_0,k]a_{(0,i,j_0,k,l)} + (q^*\alpha d_2[i+1]/a) a_{(1,i+1,j_0,k,l)}) z_i^1 t_3^j t_4^k t_5^l t_6^l
\]

\[+ \sum_{(2,\omega) \in I} (g[i,j_0,k]a_{(1,i,j_0,k,l)} + (q^*\alpha d_2[i+1]/a) a_{(2,i+1,j_0,k,l)}) z_i^1 t_3^j t_4^k t_5^l t_6^l + \mathcal{K}.
\]

We have already established that \(\{ z_i^1 t_3^j t_4^k t_5^l t_6^l \}_{i \in \mathbb{N}; j,k,l \in \mathbb{Z}; \xi \in \{0,1,2\}} \) is a basis for \(\mathcal{R}_5[t_4^{-1}]\). Given that \(\nu_0 = (i,j_0,k,l) \in \mathbb{N} \times \mathbb{Z}^3\) (with \(j_0 < 0\)) and \(w = (i,j,k,l) \in \mathbb{N}^2 \times \mathbb{Z}^2\) (with \(j \geq 0\)), it follows that

\[
q^*\beta g[i,j_0,k]a_{(2,i,j_0,k,l+1)} + (q^*\alpha bd_2[i+1]/a) a_{(0,i+1,j_0,k,l+1)} = 0. \quad (30)
\]

\[
g[i,j_0,k]a_{(0,i,j_0,k,l)} + (q^*\alpha d_2[i+1]/a) a_{(1,i+1,j_0,k,l)} = 0. \quad (31)
\]

\[
g[i,j_0,k]a_{(1,i,j_0,k,l)} + (q^*\alpha d_2[i+1]/a) a_{(2,i+1,j_0,k,l)} = 0. \quad (32)
\]

Suppose that there exists \((\xi, i, j_0, k, l) \in I\) such that \(g[i,j_0,k] = 0\). Then,

\[
g[i,j_0,k] = q^{-(k+3j_0)} - q^{-i} + d_2[i]/a = 0.
\]

Note, \(d_2[i] = d_2[1]q^{1-i} \left( \frac{1 - q^{2i}}{1 - q^{-2}} \right)\), where \(d_2[1] = -(q + q^{-1} + q^{-3})\) and \(d_2[0] = 0\). Again, recall from Appendix B that \(a = (q^2 + 1 + q^{-2})/(q^{-2} - 1) = \frac{qd_2[1]}{1 - q^2}\). Given these expressions for \(d_2[i]\) and \(a\), we have that

\[
g[i,j_0,k] = q^{-(k+3j_0)} - q^{-i} + d_2[i]/a = q^{-3j_0-k} - q^{-3i} = 0.
\]

Since \(q\) is not a root of unity, we get

\[
k = 3(i - j_0). \quad (33)
\]
Comparing (33) to (27) shows that \( i - j_0 = 0 \) which implies that \( i = j_0 < 0 \), a contradiction (note, \( i \geq 0 \)). Therefore, \( g[i, j_0, k] \neq 0 \) for all \( (\xi, i, j, k, l) \in I \).

Now, observe that if there exists \( \xi \in \{0, 1, 2\} \) such that \( a_{(\xi, i, j_0, k, l)} = 0 \) for all \((i, j_0, k, l) \in \mathbb{N} \times \mathbb{Z}^3 \), then one can easily deduce from equations (30), (31) and (32) that \( a_{(\xi, i, j_0, k, l)} = 0 \) for all \((\xi, i, j_0, k, l) \in I \).

This will contradict our initial assumption. Therefore, there exists some \((i, j_0, k, l) \in \mathbb{N} \times \mathbb{Z}^3 \) such that \( a_{(\xi, i, j_0, k, l)} \neq 0 \) for each \( \xi \in \{0, 1, 2\} \). Without loss of generality, let \((u, j_0, v, w)\) be the greatest element in the lexicographic order on \( \mathbb{N} \times \mathbb{Z}^3 \) such that \( a_{(0, u, j_0, v, w)} \neq 0 \) and \( a_{(0, i, j_0, k, l)} = 0 \) for all \( i > u \).

From (31), at \((i, j_0, k, l) = (u, j_0, v, w)\), we have:

\[
g[u, j_0, v]a_{(0, u, j_0, v, w)} + (q \cdot ad_2[u + 1]/a)a_{(1, u + 1, j_0, v + 1, w)} = 0.
\]

From (32), at \((i, j_0, k, l) = (u + 1, j_0, v + 1, w)\), we have:

\[
g[u + 1, j_0, v + 1]a_{(1, u + 1, j_0, v + 1, w)} + (q \cdot ad_2[u + 2]/a)a_{(2, u + 2, j_0, v + 2, w)} = 0.
\]

Finally, from (30), at \((i, j_0, k, l) = (u + 2, j_0, v + 2, w - 1)\), we have:

\[
q \cdot b g[u + 2, j_0, v + 2]a_{(2, u + 2, j_0, v + 2, w)} + (q \cdot abd_2[u + 3]/a)a_{(0, u + 3, j_0, v + 3, w - 1)} = 0.
\]

Note: \( a, b, \alpha, \beta, q \neq 0; g[i, j_0, k] \neq 0 \) for all \((\xi, i, j_0, k, l) \in I \); and \( d_2[i] \neq 0 \) for \( i > 0 \). Since \( u + 3 > u \), it follows from the above list of equations (starting from the last one) that

\[
a_{(0, u + 3, j_0, v + 3, w - 1)} = 0 \Rightarrow a_{(2, u + 2, j_0, v + 2, w)} = 0 \Rightarrow a_{(1, u + 1, j_0, v + 1, w)} = 0 \Rightarrow a_{(0, u, j_0, v, w)} = 0,
\]

a contradiction! Hence, \( a_{(0, i, j_0, k, l)} = 0 \) for all \((i, j_0, k, l) \in \mathbb{N} \times \mathbb{Z}^3 \). From (30), (31) and (32), one can easily conclude that \( a_{(\xi, i, j_0, k, l)} = 0 \) for all \((\xi, i, j_0, k, l) \in I \). This contradicts our initial assumption, hence \( x_\infty = 0 \). Consequently, \( x = x_\infty \in \mathcal{R}_5 \) as desired.

2. From Remark 5.8, we have \( z_2 = t_2 - bt_3 t_4^{-1} \). Since \( \delta(t_\infty) = \lambda_\kappa t_\kappa, \kappa \in \{2, \ldots, 6\} \), with \( \lambda_2 := -\lambda_5 \) (see Lemma 5.9), it follows that

\[
\delta(z_2) = -\lambda_5 t_2 - b(3\lambda_3 - \lambda_4)t_3^3 t_4^{-1} = -\lambda_5 z_2 - b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3 t_4^{-1}.
\]

Furthermore,

\[
D(z_2) = ad_x(z_2) + \delta(z_2) = ad_x(z_2) - \lambda_5 z_2 - b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3 t_4^{-1} \in \mathcal{R}_5.
\]

Hence \( b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3 t_4^{-1} \in \mathcal{R}_5 \), since \( ad_x(z_2) - \lambda_5 z_2 \in \mathcal{R}_5 \). This implies that \( b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3 \in \mathcal{R}_5 t_4 \) (note, from Appendix 11 \( b \neq 0 \)). Set \( w := 3\lambda_3 - \lambda_4 + \lambda_5 \). Suppose that \( w \neq 0 \). From (19), we have:

\[
t_3^3 = \frac{\beta}{b} t_6^{-1} - \frac{\alpha}{ab} t_4 t_5^{-1} + \frac{1}{ab} z_1 t_3 t_4.
\]

It follows that

\[
wbt_3^3 = w\beta t_6^{-1} - \frac{\alpha}{ab} t_4 t_5^{-1} + \frac{w}{a} z_1 t_3 t_4 \in \mathcal{R}_5 t_4.
\]

Since \( t_3^3, t_4 t_5^{-1} \) and \( z_1 t_3 t_4 \) are all elements of \( \mathcal{R}_5 t_4 \), it implies that \( t_6^{-1} \in \mathcal{R}_5 t_4 \). Hence, \( 1 \in \mathcal{R}_5 t_4 \).

Using the basis \( \mathcal{B}_5 \) of \( \mathcal{R}_5 \) (Proposition 5.5), this leads to a contradiction. Therefore, \( w = 0 \). That is, \( 3\lambda_3 - \lambda_4 + \lambda_5 = 0 \), and so \( \lambda_4 = 3\lambda_3 + \lambda_5 \). This further implies that \( \delta(z_2) = -\lambda_5 z_2 \) as desired.

Again, from Lemma 5.9 we have that \( \delta(f_1) = -(\lambda_3 + \lambda_5)f_1 \). Recall from Remark 5.8 that \( z_1 = f_1 - st_3^3 t_4^{-1} \). It follows that

\[
\delta(z_1) = -(\lambda_3 + \lambda_5)f_1 - s(2\lambda_3 - \lambda_4)t_3^3 t_4^{-1} = -(\lambda_3 + \lambda_5)z_1 - s(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3 t_4^{-1} = -(\lambda_3 + \lambda_5)z_1.
\]

Finally, we know that \( \delta(t_6) = \lambda_6 t_6 \). This implies that \( \delta(t_6^{-1}) = -\lambda_6 t_6^{-1} \). From (19), we have that

\[
t_3^3 = \frac{\beta}{b} t_6^{-1} - \frac{\alpha}{ab} t_4 t_5^{-1} + \frac{1}{ab} z_1 t_3 t_4,
\]
where \( a \) and \( b \) are non-zero scalars (Appendix [B]). This implies that

\[
t_6^{-1} = \frac{b}{\beta} t_3^3 + \frac{q^3 \alpha}{\alpha_3 \beta} t_4 t_5^{-1} - \frac{1}{a \beta} z_1 t_3 t_4.
\]

Given that \( \delta(z_1) = -(\lambda_3 + \lambda_5)z_1 \), \( \delta(t_3) = \lambda_3 t_3 \), \( \delta(t_4) = (3\lambda_3 + \lambda_5)t_4 \) and \( \delta(t_5) = \lambda_5 t_5 \), applying \( \delta \) to the above relation gives

\[
-\lambda_6 t_6^{-1} = 3\lambda_3 \left( \frac{b}{\beta} t_3^3 + \frac{q^3 \alpha}{\alpha_3 \beta} t_4 t_5^{-1} - \frac{1}{a \beta} z_1 t_3 t_4 \right).
\]

It follows that \( \lambda_6 = -3\lambda_3 \) as desired.

3. Set \( \lambda_1 := -(\lambda_3 + \lambda_5) \) and \( \lambda_2 := -\lambda_5 \). It follows from points (1) and (2) that \( D(e_{\kappa,5}) = ad_x(e_{\kappa,5}) + \delta(e_{\kappa,5}) = ad_x(e_{\kappa,5}) + \lambda_\kappa e_{\kappa,5} \) for all \( \kappa \in \{1, \cdots, 6\} \). In conclusion, \( D = ad_x + \delta \) with \( x \in R_5 \).

We are now ready to describe \( D \) as a derivation of \( A_{\alpha,\beta} \).

**Lemma 5.11.**

1. \( x \in A_{\alpha,\beta} \).

2. \( \delta(e_\kappa) = 0 \) for all \( \kappa \in \{1, \cdots, 6\} \).

3. \( D = ad_x \).

**Proof.** In this proof, we denote \( \nu := (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2 \). Also, recall from DDA of \( A_{\alpha,\beta} \) at the beginning of this section that \( t_5 = e_5 \) and \( t_6 = e_6 \).

1. Given the basis \( B \) of \( A_{\alpha,\beta} \) (Proposition [1.4]), one can write \( x \in R_5 = A_{\alpha,\beta}[t_5^{-1}, t_6^{-1}] \) as:

\[
x = \sum_{(e_1, e_2, \nu) \in I} a_{(e_1, e_2, \nu)} e_1^i e_2^j e_3^k e_4^l t_5^{-i} t_6^{-l},
\]

where \( I \) is a finite subset of \( \{0,1\}^2 \times \mathbb{N}^2 \times \mathbb{Z}^2 \), and \( a_{(e_1, e_2, \nu)} \) are scalars. Write \( x = x_- + x_+ \), where

\[
x_+ = \sum_{(e_1, e_2, \nu) \in I} a_{(e_1, e_2, \nu)} e_1^i e_2^j e_3^k e_4^l t_5^{-i} t_6^{-l},
\]

and

\[
x_- = \sum_{(e_1, e_2, \nu) \in I} a_{(e_1, e_2, \nu)} e_1^i e_2^j e_3^k e_4^l t_5^{-i} t_6^{-l}.
\]

Suppose that there exists a minimum negative integer \( k_0 \) or \( l_0 \) such that \( a_{(e_1, e_2, i, j, k_0, l_0)} \neq 0 \) or \( a_{(e_1, e_2, i, j, k_0, l_0)} \neq 0 \) for some \((e_1, e_2, i, j, k_0, l_0) \in I \), and \( a_{(e_1, e_2, i, j, k_0, l_0)} = 0 \) whenever \( k \leq k_0 \) or \( l \leq l_0 \). Write

\[
x_- = \sum_{(e_1, e_2, \nu) \in I} a_{(e_1, e_2, \nu)} e_1^i e_2^j e_3^k e_4^l t_5^{-i} t_6^{-l}.
\]

Now, \( D(e_3) = ad_x(e_3) + ad_x(-e_3) + \delta(e_3) \in A_{\alpha,\beta} \). From Remark [5.8] we have that \( e_3 = e_{3,6} - st_5^{-1} t_6^{-1} \) and \( e_{3,6} = t_3 - at_4 t_5^{-1} \). Putting these two together gives

\[
e_3 = t_3 - at_4 t_5^{-1} - st_5^2 t_6^{-1}.
\]

Again, from Remark [5.8] we also have that \( t_4 = e_4 + bt_5^2 t_6^{-1} \). Note, \( \delta(t_\kappa) = \lambda_\kappa x_\kappa, \kappa \in \{3, 4, 5, 6\} \). Now,

\[
\delta(e_3) = \lambda_3 e_3 - (\lambda_4 - \lambda_5) t_4 t_5^{-1} - s(2\lambda_5 - \lambda_6) t_5^3 t_6^{-1} - \lambda_3 e_3 + a(\lambda_4 - \lambda_4) t_4 t_5^{-1} + s(\lambda_4 - 2\lambda_5) t_5^2 t_6^{-1} = \lambda_3 e_3 + a(\lambda_3 - \lambda_4 + \lambda_5) t_4 t_5^{-1} + s(\lambda_4 - 2\lambda_5) t_5^2 t_6^{-1} = \lambda_3 e_3 + a(\lambda_3 + \lambda_4 + \lambda_5) (e_4 + bt_5^2 t_6^{-1}) t_5^{-1} + s(\lambda_4 - 2\lambda_5) t_5^2 t_6^{-1} = \lambda_3 e_3 + a_1 e_4 t_5^{-1} + a_2 t_5^2 t_6^{-1},
\]

(34)
where $\alpha_1 = a(\lambda_3 - \lambda_4 + \lambda_5)$ and $\alpha_2 = s(\lambda_3 - 2\lambda_5 + \lambda_6) + q^{-3}ab(\lambda_3 - \lambda_4 + \lambda_5)$. Therefore, $D(e_3) = \text{ad}_{x_+}(e_3) + \text{ad}_{x_-}(e_3) + \lambda_3 e_3 + \alpha_1 e_4 t_5^{-1} + \alpha_2 t_6^{-1} \in A_{\alpha, \beta}$. It follows that $D(e_3) t_5 t_6 = \text{ad}_{x_+}(e_3) t_5 t_6 + \text{ad}_{x_-}(e_3) t_5 t_6 + \lambda_3 e_3 t_5 t_6 + \alpha_1 e_4 t_6 + q^5 \alpha_2 e_3 \in A_{\alpha, \beta}$. Hence, $\text{ad}_{x_+}(e_3) t_5 t_6 \in A_{\alpha, \beta}$, since $\text{ad}_{x_+}(e_3) t_5 t_6 + \lambda_3 e_3 t_5 t_6 + \alpha_1 e_4 t_6 + q^5 \alpha_2 e_3 \in A_{\alpha, \beta}$.

Now,

$$\text{ad}_{x_-}(e_3) = \sum_{(e_1, e_2, \omega) \in I} a_{(e_1, e_2, \omega)} e_1 e_2 e_3 e_4 e_5 t_6 e_3 - \sum_{(e_1, e_2, \omega) \in I} a_{(e_1, e_2, \omega)} e_3 e_1 e_3 e_4 e_5 t_6 e_3. \quad (35)$$

Using Lemma A.1, we have the following:

$$t_5^k e_3 = q^{-k} e_3 t_5^{-1} + d_2[k] e_4 t_5^{-1} + d_3[k] t_5^{-1} + q^{-3} d_2[k] e_3 e_4 e_5 t_5^{-1} + q^{-3} d_2[k] e_3 e_4 e_5 t_5^{-1} + q^{-3} d_2[k] e_3 e_4 e_5 t_5^{-1}, \quad (36)$$

$$e_3 e_1 e_3 = q^{-3} e_1 e_2 e_3 + d_2[l] e_4 e_5 t_5^{-1} + q^{-3} d_2[l] e_4 e_5 t_5^{-1} + q^{-3} d_2[l] e_4 e_5 t_5^{-1}. \quad (37)$$

Substitute (36) and (37) into (35), simplify and multiply (on the right) by $t_5 t_6$ to obtain

$$\text{ad}_{x_-}(e_3) t_5 t_6 = \sum_{(e_1, e_2, \omega) \in I} a_{(e_1, e_2, \omega)} \left( g[i, j, e_2, l] e_1 e_2 e_3 + q^{-3} d_2[k] e_3 e_4 e_5 t_5^{-1} + q^{-3} d_2[k] e_3 e_4 e_5 t_5^{-1} + q^{-3} d_2[k] e_3 e_4 e_5 t_5^{-1} \right), \quad (38)$$

where $g[i, j, e_2, l] := q^{-k} e_2 + q^{-3} e_3 - q^{-3} e_3 - 3_l$.

Assume that there exists $l < 0$ such that $a_{(e_1, e_2, \omega)} \neq 0$. It follows from our initial assumption that $a_{(e_1, e_2, i, j, k, l)} = 0$. Now, at $l = l_0$, denote $\omega = (i, j, k, l)$ by $\omega_0 := (i, j, k, l_0)$. From (35), we have that

$$\text{ad}_{x_-}(e_3) t_5 t_6 = \sum_{(e_1, e_2, \omega) \in I} q^{-3(l_0 - 1)} a_{(e_1, e_2, \omega)} d_3[l_0] e_1 e_3 e_3 e_4 e_5 t_5^{-1} t_6 + \mathcal{J}_1,$n

where $\mathcal{J}_1 \in \text{Span} \left( \mathcal{B} \setminus \{ e_1 e_2 e_3 e_4 e_5 t_5^{-1} \mid e_1, e_2 \in \{0, 1\}, k \in \mathbb{Z} \text{ and } i, j \in \mathbb{N} \} \right)$.

Set $\omega := (i, j, k, l) \in \mathbb{N}^4$. One can also write $\text{ad}_{x_-}(e_3) t_5 t_6 \in A_{\alpha, \beta}$ in terms of the basis $\mathcal{B}$ of $A_{\alpha, \beta}$ (Proposition A.4) as:

$$\text{ad}_{x_-}(e_3) t_5 t_6 = \sum_{(e_1, e_2, \omega) \in J} b_{(e_1, e_2, \omega)} e_1 e_2 e_3 e_4 e_5 t_5^{-1}, \quad (39)$$

where $J$ is a finite subset of $\{0, 1\}^2 \times \mathbb{N}^4$, and $b_{(e_1, e_2, \omega)} \in K$. It follows that

$$\sum_{(e_1, e_2, \omega) \in J} b_{(e_1, e_2, \omega)} e_1 e_2 e_3 e_4 e_5 t_5^{-1} = \sum_{(e_1, e_2, \omega) \in I} q^{-3(l_0 - 1)} a_{(e_1, e_2, \omega)} d_3[l_0] e_1 e_3 e_3 e_4 e_5 t_5^{-1} t_6 + \mathcal{J}_1.$$

Since $\mathcal{B}$ is a basis for $A_{\alpha, \beta}$, we deduce from Corollary A.3 that $(e_1 e_2 e_3 e_4 e_5 t_5^{-1} \omega \in \{0, 1\}^2 \times \mathbb{N}^2 \times \mathbb{Z}^2)$ is also a basis for $A_{\alpha, \beta}[-1, -1, t_6^{-1}]$. Since $\omega_0 = (i, j, k, l_0) \in \mathbb{N}^2 \times \mathbb{Z}^2$ (with $l_0 < 0$) and $\omega_0 = (i, j, k, l) \in \mathbb{N}^4$ (with $l \geq 0$) in the above equality, we must have

$$q^{-3(l_0 - 1)} a_{(e_1, e_2, \omega)} d_3[l_0] = 0.$$

Given that $q^{-3(l_0 - 1)} d_3[l_0] \neq 0$, it follows that $a_{(e_1, e_2, i, j, k, l_0)} = 0$. This is a contradiction. Therefore, $l \geq 0$ (i.e. there is no negative exponent for $t_6$).

Since $l \geq 0$, it follows from our initial assumption that there exists $k = k_0 < 0$ such that $a_{(e_1, e_2, i, j, k_0, l)} \neq 0$. The rest of the proof will show that this assumption cannot also hold.

Set $\omega_0 := (i, j, k_0, l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}$. From (35), we have that

$$\text{ad}_{x_-}(e_3) t_5 t_6 = \sum_{(e_1, e_2, \omega) \in I} q^{-3} a_{(e_1, e_2, \omega)} d_2[k_0] e_1 e_2 e_3 e_4 e_5 t_5^{-1} t_6 + V,$$

32
where $V \in \mathcal{J}_2 := \text{Span} \left( B \setminus \{e_1^j e_2^i e_3^e e_4^k e_5^l t^{k_0 l_6} : e_1, e_2 \in \{0, 1\} \text{ and } i, j, l \in \mathbb{N} \right)$. It follows that:

$$
ad_{x_-}(e_3)t_6 = \sum_{(0,0,i,j,k) \in I} q^{-3i}d_{(0,0,i,j,k)}d_2[k_0]e_1^j e_2^i e_4^k t^{k_0 l_6} + \sum_{(1,0,i,j,k) \in I} a_{(1,0,i,j,k)}d_2[k_0]e_1^j e_2^i e_3^e e_4^k t^{k_0 l_6} + 1 \sum_{(1,1,i,j,k) \in I} a_{(1,1,i,j,k)}d_2[k_0]e_1^j e_2^i e_4^k t^{k_0 l_6} + 1 + V. \quad (40)$$

Write the relations in Lemma 4.1(2),(4) as:

$$
e_1^2 = b_1 \beta + b_2 e_2 e_4 e_6 + b_4 \alpha e_3 e_6 + b_6 e_1 e_3 e_4 e_6 + L_1, \quad (41)$$

$$
e_3 e_4^2 = \beta b_1 e_3 + k_1 e_2 e_4 e_6 + k_2 \alpha e_2 e_4 e_6 + k_3 \alpha e_1 e_4 e_6 + k_4 \beta e_4 e_2 e_3 e_6 + k_5 \alpha e_2 e_4 e_6 + k_6 \alpha e_3 e_4 e_6 + k_7 \alpha e_2 e_4 e_6 + k_8 \alpha e_3 e_4 e_6 + L_2, \quad (42)$$

where $L_1$ and $L_2$ are some elements of the left ideal $A_{\alpha, \beta} t_5 \subseteq \mathcal{J}_2$. Substitute (41) and (42) into (40), and simplify to obtain:

$$
ad_{x_-}(e_3)t_6 = \sum_{(0,0,i,j,k,l) \in I} [\lambda_{1,1,\beta}a_{(0,1,i,j,k,l)} + \lambda_{1,2,\alpha}^2a_{(1,1,i,j,k,l)}] e_1^j e_2^i e_3^e e_4^k e_5^l \text{ t}_6$$

$$+ \sum_{(1,0,i,j,k,l) \in I} \lambda_{1,3,\beta}a_{(1,1,i,j,k,l)} e_1^j e_2^i e_3^e e_4^k e_5^l \text{ t}_6$$

$$+ \sum_{(1,1,i,j,k,l) \in I} \lambda_{1,4,\alpha}a_{(1,1,i,j,k,l)} e_1^j e_2^i e_3^e e_4^k e_5^l \text{ t}_6 + V'. \quad (43)$$

where $V' \in \mathcal{J}_2$. Also, $\lambda_{s,t} := \lambda_{s,t}(j,k_0,l)$ are some families of scalars which are non-zero for all $s, t \in \{1, 2, 3, 4\}$ and $i, j, l \in \mathbb{N}$, except $\lambda_{1,4}$ and $\lambda_{2,4}$ which are assumed to be zero since they do not exist in the above expression. Note, although each $\lambda_{s,t}$ depends on $j$, $k_0$, $l$, we have not made this dependency explicit in the above expression since the minimum requirement we need to complete the proof is for all the $\lambda_{s,t}$ existing in the above expression to be non-zero, which we already have.

Observe that (43) and (39) are equal, hence,

$$
\sum_{(e_1,e_2,w) \in I} b_{(e_1,e_2,w)} e_1^j e_2^i e_3^e e_4^k e_5^l \text{ t}_6 = \sum_{(0,0,i,j,k,l) \in I} [\lambda_{1,1,\beta}a_{(0,1,i,j,k,l)} + \lambda_{1,2,\alpha}^2a_{(1,1,i,j,k,l)}] e_1^j e_2^i e_3^e e_4^k e_5^l \text{ t}_6$$

$$+ \sum_{(1,0,i,j,k,l) \in I} \lambda_{1,3,\beta}a_{(1,1,i,j,k,l)} e_1^j e_2^i e_3^e e_4^k e_5^l \text{ t}_6$$

$$+ \sum_{(1,1,i,j,k,l) \in I} \lambda_{1,4,\alpha}a_{(1,1,i,j,k,l)} e_1^j e_2^i e_3^e e_4^k e_5^l \text{ t}_6 + V'.$$  

We have previously established that $\left( e_1^j e_2^i e_3^e e_4^k e_5^l t^{k_0 l_6} \right)_{(e_1,e_2,w) \in \{0,1\} \times \mathbb{N}^2 \times \mathbb{Z}^2}$ is a basis for $A_{\alpha, \beta} t^{l_6}$ (note, in this part of the proof $l \geq 0$). Since $w_0 = (i, j, k_0, l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}$ (with $k_0 < 0$) and $w = (i, j, k, l) \in \mathbb{N}^4$ (with $k \geq 0$) in the above equality, it follows that

$$
\lambda_{1,1,\beta}a_{(0,1,i,j,k_0,l)} + \lambda_{1,2,\alpha}^2a_{(1,1,i,j,k_0,l)} + \lambda_{1,3,\beta}a_{(1,1,i,j,k_0,l)} = 0, \quad (44)
$$

(continued on next page)
From (44) and (45), one can easily deduce that
\[
\begin{align*}
\lambda_2 & \alpha a_{0,1,i,j,k_0,l-2} + \lambda_2 \beta a_{1,1,i,j,k_0,l-1} + \lambda_2 \alpha a_{1,1,i-2,j,k_0,l-3} = 0, \\
\lambda_3 & \alpha a_{0,1,i,j,k_0,l-2} + \lambda_2 \beta a_{1,1,i,j,k_0,l-2} + \lambda_3 \alpha a_{1,1,i-2,j,k_0,l-3} \\
& \quad + \lambda_4 a_{0,0,i,j,k_0,l-1} = 0, \\
\lambda_4 & \alpha a_{0,1,i,j,k_0,l-2} + \lambda_4 a_{1,1,i,j,k_0,l-2} + \lambda_4 a_{1,1,i-3,j,k_0,l-3} + \lambda_4 a_{1,0,i,j,k_0,l-1} = 0.
\end{align*}
\]  
(45)
(46)
(47)

From (44) and (45), one can easily deduce that
\[
\begin{align*}
a_{0,1,i,j,k_0,l} &= -\frac{\alpha^2 \lambda_{1,2}}{\beta \lambda_{1,1}} a_{1,1,i,j,k_0,l-1} - \frac{\lambda_{1,3}}{\lambda_{1,1}} a_{1,1,i-2,j,k_0,l-1}, \\
a_{1,1,i,j,k_0,l} &= -\frac{\alpha \lambda_{2,1}}{\beta \lambda_{2,2}} a_{0,1,i,j,k_0,l-1} - \frac{\alpha \lambda_{3,3}}{\beta \lambda_{2,2}} a_{1,1,i-2,j,k_0,l-2}.
\end{align*}
\]  
(48)
(49)

Note, \(a_{(i_1,i_2,i,j,k_0,l)} := 0\) whenever \(i < 0\) or \(j < 0\) or \(l < 0\) for all \(i_1, i_2 \in \{0,1\}\).

**Claim.** The coefficients \(a_{0,1,i,j,k_0,l}\) and \(a_{1,1,i,j,k_0,l}\) are all zero for all \(l \geq 0\). We now justify the claim by an induction on \(l\). From (48) and (49), the result is obviously true when \(l = 0\). For \(l \geq 0\), assume that \(a_{0,1,i,j,k_0,l} = a_{1,1,i,j,k_0,l} = 0\). Then, it follows from (48) and (49) that
\[
\begin{align*}
a_{0,1,i,j,k_0,l+1} &= -\frac{\alpha^2 \lambda_{1,2}}{\beta \lambda_{1,1}} a_{1,1,i,j,k_0,l} - \frac{\lambda_{1,3}}{\lambda_{1,1}} a_{1,1,i-2,j,k_0,l}, \\
a_{1,1,i,j,k_0,l+1} &= -\frac{\alpha \lambda_{2,1}}{\beta \lambda_{2,2}} a_{0,1,i,j,k_0,l} - \frac{\alpha \lambda_{3,3}}{\beta \lambda_{2,2}} a_{1,1,i-2,j,k_0,l-1}.
\end{align*}
\]

From the inductive hypothesis, \(a_{0,1,i,j,k_0,l} = a_{1,1,i,j,k_0,l} = a_{0,1,i,j,k_0,l-1} = a_{1,1,i,j,k_0,l-1} = 0\). Hence, \(a_{0,1,i,j,k_0,l+1} = a_{1,1,i,j,k_0,l+1} = 0\) for all \(l \geq 0\) as desired. Given that the families \(a_{0,1,i,j,k_0,l}\) and \(a_{1,1,i,j,k_0,l}\) are all zero, it follows from (40) and (47) that \(a_{0,0,0,i,j,k_0,l}\) and \(a_{1,0,i,j,k_0,l}\) are also zero for all \((i, j, k_0, l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}\).

Since \(a_{(i_1,i_2,i,j,k_0,l)}\) are all zero, it contradicts our assumption. Hence, \(x_\ast = 0\). Consequently, \(x = x_+ \in A_{\alpha,\beta}\) as desired.

2. From Remark [5.8] we have \(e_4 = t_4 - bt_5^2 t_6^{-1}\). Again, from Lemma [5.10] we have that \(\lambda_4 = 3\lambda_3 + \lambda_5\) and \(\lambda_6 = -3\lambda_3\). Therefore,
\[
\delta(e_4) = \lambda_4 t_4 - b(3\lambda_3 - \lambda_5) t_5^3 t_6^{-1} - (3\lambda_3 + \lambda_5) t_5 t_6 \in A_{\alpha,\beta},
\]
\[
= (3\lambda_3 + \lambda_5) t_4 - b(3\lambda_3 + \lambda_5) t_5^3 t_6^{-1} - 2b\lambda_3 t_5 t_6 \in A_{\alpha,\beta}.
\]

Moreover, \(D(e_4) = ad_x(e_4) + \delta(e_4) = ad_x(e_4) + (3\lambda_3 + \lambda_5) t_4 - 2b\lambda_3 t_6 \not\in A_{\alpha,\beta}\). It follows that \(b\lambda_3 t_6^2 t_6^{-1} \not\in A_{\alpha,\beta}\), since \(ad_x(e_4) + (3\lambda_3 + \lambda_5) e_4 \in A_{\alpha,\beta}\). Consequently, \(b\lambda_3 t_6^2 \not\in A_{\alpha,\beta}\). Since \(b \neq 0\) (Appendix [13]), we must have \(\lambda_5 = 0\), otherwise, there will be a contradiction using the basis of \(A_{\alpha,\beta}\) (Proposition [3.4]). Therefore, \(\delta(e_4) = 3\lambda_3 e_4\) and \(\delta(t_5) = 0\). We already know from Lemma [5.10] that \(\delta(t_4) = -3\lambda_3 t_6\). From (34), we have that \(\delta(e_3) = \lambda_3 e_3 + (\lambda_3 - \lambda_5) e_4 t_5^{-1} + (s(\lambda_3 - 2\lambda_5 + \lambda_6) + g^{-3}a b (\lambda_3 - \lambda_5 + \lambda_5) t_5^2 t_6^{-1}\). Given that \(\lambda_3 = 3\lambda_3\), \(\lambda_5 = 0\) and \(\lambda_6 = -3\lambda_3\), we have that \(\delta(e_3) = \lambda_3 e_3 - 3a \lambda_3 e_4 t_5^{-1}\) (note, from Appendix [13] one can confirm that \(g^{-3}a b + s = 0\)). Now, \(D(e_3) = ad_x(e_3) + \delta(e_3) = ad_x(e_3) + \lambda_3 e_3 + 2a \lambda_3 e_4 t_5^{-1} \in A_{\alpha,\beta}\). Observe that \(ad_x(e_3) + \lambda_3 e_3 \in A_{\alpha,\beta}\), and so \(2a \lambda_3 e_4 t_5^{-1} \in A_{\alpha,\beta}\). Since \(a \neq 0\), it implies that \(\lambda_3 = 0\), otherwise, there will be a contradiction using the basis of \(A_{\alpha,\beta}\). We now have that \(\delta(e_3) = \delta(e_4) = \delta(e_5) = \delta(e_6) = 0\). We finish the proof by showing that \(\delta(e_1) = \delta(e_2) = 0\). Recall from (10) that
\[
e_2 e_4 e_6 + be_2 e_3^2 + be_3^3 e_6 + b'e_2 e_3^2 + c'e_3 e_4 e_5 + d' e_4 = \beta.
\]

Apply \(\delta\) to this relation to obtain \(\delta(e_2) e_4 e_6 + \delta(e_2) e_3^2 = 0\). This implies that \(\delta(e_2)(e_4 e_6 + be_3^2) = 0\). Since \(e_4 e_6 + be_3^2 \neq 0\), it follows that \(\delta(e_2) = 0\). Similarly, from (9), we have that
\[
e_1 e_3 e_5 + ae_1 e_4 + ae_2 e_5 + a'e_3^2 = \alpha.
\]
Apply \( \delta \) to this relation to obtain \( \delta(e_1)(e_3e_5 + ae_4) = 0 \). Since \( e_3e_5 + ae_4 \neq 0 \), we have that \( \delta(e_1) = 0 \).

In conclusion, \( \delta(e_\kappa) = 0 \) for all \( \kappa \in \{1, \cdots, 6\} \).

3. As a result of (1) and (2), we have that \( D(e_\kappa) = \text{ad}_x(e_\kappa) \). Therefore, \( D = \text{ad}_x \) as desired.

Using similar techniques, one can describe the derivations of \( A_{\alpha,0} \) and \( A_{0,\beta} \). Details can be found in [19]. There are fundamental differences in these two cases. Indeed, there exist in both cases derivations which are not inner. More precisely, one can check that the linear map \( \theta \) of \( A_{\alpha,0} \) defined by

\[
\theta(e_1) = -e_1, \quad \theta(e_2) = -e_2, \quad \theta(e_3) = 0, \quad \theta(e_4) = e_4, \quad \theta(e_5) = e_5, \quad \theta(e_6) = 2e_6
\]

is a \( \mathbb{K} \)-derivation of \( A_{\alpha,0} \).

Similarly, the linear map \( \tilde{\theta} \) of \( A_{0,\beta} \) by

\[
\tilde{\theta}(e_1) = -2e_1, \quad \tilde{\theta}(e_2) = -3e_2, \quad \tilde{\theta}(e_3) = -e_3, \quad \tilde{\theta}(e_4) = 0, \quad \tilde{\theta}(e_5) = e_5, \quad \tilde{\theta}(e_6) = 3e_6
\]

is a \( \mathbb{K} \)-derivation of \( A_{0,\beta} \).

We summarize our main results in the theorem below.

**Theorem 5.12.** Given that \( A_{\alpha,\beta} = U_q^+(G_2)/\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \), with \( (\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\} \), we have the following results:

1. If \( \alpha, \beta \neq 0 \), then every derivation \( D \) of \( A_{\alpha,\beta} \) can uniquely be written as \( D = \text{ad}_x \), where \( x \in A_{\alpha,\beta} \).
2. If \( \alpha \neq 0 \) and \( \beta = 0 \), then every derivation \( D \) of \( A_{\alpha,0} \) can uniquely be written as \( D = \text{ad}_x + \lambda \theta \), where \( \lambda \in \mathbb{K} \) and \( x \in A_{\alpha,0} \).
3. If \( \alpha = 0 \) and \( \beta \neq 0 \), then every derivation \( D \) of \( A_{0,\beta} \) can uniquely be written as \( D = \text{ad}_x + \tilde{\lambda} \tilde{\theta} \), where \( \lambda \in \mathbb{K} \) and \( x \in A_{0,\beta} \).

4. \( HH^1(A_{\alpha,0}) = \mathbb{K}[\theta] \) and \( HH^1(A_{0,\beta}) = \mathbb{K}[\tilde{\theta}] \), where \( [\theta] \) and \( [\tilde{\theta}] \) respectively denote the classes of \( \theta \) and \( \tilde{\theta} \) modulo the space of inner derivations.

5. If \( \alpha, \beta \neq 0 \), then \( HH^1(A_{\alpha,\beta}) = \{[0]\} \), where \([0]\) denotes the class of 0 modulo the space of inner derivations.

The above theorem shows that \( A_{\alpha,\beta} \) when both \( \alpha \) and \( \beta \) are nonzero shares a number of properties with the second Weyl algebra over \( \mathbb{K} \): it is simple, units are reduced to scalars, and all derivations are inner.

It would be interesting to compute the automorphism group of these algebras and verify if all endomorphisms are automorphisms, i.e. an analogue of the celebrated Dixmier Conjecture [6].

In general, the present work and [15] suggest that the primitive quotients of \( U_q^+(g) \) by primitive ideals from the 0-stratum provide algebras that could (should?) be regarded (and studied) as quantum analogue of Weyl algebras.

### A Some general relations of \( U_q^+(G_2) \)

We have the following selected general relations of \( U_q^+(G_2) \).

**Lemma A.1.** For any \( n \in \mathbb{N} \), we have that:

1. \( E_jE_i^* = q^{-3n}E_i^*E_j \) for all \( 1 \leq i, j \leq 6 \), with \( j - i = 1 \).

2. \( E_6E_6^* = q^{-3n}E_6E_6^* + d_1[n]E_6E_6^* + d_1[n]E_6E_6^* - 1 \)
   (c) \( E_4E_2^* = q^{-3n}E_4E_2^* + d_1[n]E_4E_2^* - 1 \)
   (d) \( E_4E_2 = q^{-3n}E_2E_4 + d_1[n]E_3E_4 + d_1[n]E_3E_4^* - 1 \),
   where \( d_1[n] = q^{3(1-n)}d_1[1] \left( \frac{1}{1 - q^{-6n}} \right) \);
   \( d_1[1] = -\frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1} \) and \( d_1[0] := 0 \).

3. \( E_3E_3^* = q^{-n}E_3E_3^* + d_2[n]E_3E_3^* - 1 \)
   (c) \( E_3E_3^* = q^{-n}E_3E_3^* + d_2[n]E_3E_3^* - 1 \)
   (d) \( E_3E_3 = q^{-n}E_3E_3^* + d_2[n]E_4E_3^* - 1 \),
where \( d_2[n] = q^{1-n}d_2[1] \left( \frac{1 - q^{2-n}}{1 - q^{-2}} \right) \); \( d_2[1] = -(q + q^{-1} + q^{-3}) \) and \( d_2[0] := 0 \).

4(a) \( E_6^n E_3 = E_3 E_6^n + d_3[n]E_6^{n-1} \) (b) \( E_3 E_6^n = E_3 E_5^n + d_3[n]E_6^{n-1} E_3 \),

where \( d_3[n] = d_3[1] \left( \frac{1 - q^{-6n}}{1 - q^{-6}} \right) \); \( d_3[1] = 1 - q^2 \) and \( d_3[0] := 0 \).

**Proof.** This is an easy proof by induction, left to the reader. \( \square \)

**B Definition of parameters used throughout**

In this appendix, we define parameters used in this article. Note, for all \( n \in \mathbb{N} \), we have already defined the scalars \( d_2[n] \) in Lemma [A.1] hence, we are not going to repeat them here. Any other scalars not defined here must be defined in/before the context in which it is found.

\[
\begin{align*}
a &= \frac{q^2 + 1 + q^{-2}}{q^2 - 1} \\
g &= \frac{q + q^{-1} + q^{-3}}{(1 - q^{-2})^2} \\
h &= \frac{q + q^{-1}}{q^2 - 1} \\
t &= \frac{q^{-1} - q}{1 - q^{-6}} \\
p &= \frac{q^4 + q^2 + 1}{q^2 - 1} \\
e &= \frac{-(q^7 + q^5 + q^3)}{q^4 - 2q^2 + 1} \\
n &= \frac{q^{12}}{(q^4 + q^2 + 1)^3} \\
k_1 &= q^{-3}b_2 + b_6d_2[1] \\
k_2 &= q^{-3}b_3 + b_{12}d_2[1] \\
k_3 &= b_4c_1 \\
k_4 &= b_4c_2 + q^{-3}b_5c_1 + b_7d_2[1] \\
k_5 &= b_4c_2 + q^{-1}b_6c_1 \\
k_6 &= c_3b_4 + q^{-1}b_7 + q^{-3}b_4b_{13}c_2 \\
k_7 &= q^{-3}c_2b_5 + b_8d_2[1] \\
k_8 &= b_1b_{13}c_2 \\
k_9 &= q^{-4}b_6c_2 + b_9d_2[2] + q^{-3}b_2c_2b_{13} + q^{-3}b_5c_2 \\
k_{10} &= q^{-1}b_5b_8c_2 \\
k_{11} &= b_{13}c_1 \\
k_{12} &= q^{-1}b_6b_7c_2 \\
k_{13} &= q^{-4}b_6c_3 + q^{-2}b_9 + q^{-3}b_6b_{13}c_2 + q^{-1}b_6b_{13}c_2 \\
k_{14} &= q^{-1}b_1b_6c_2 \\
k_{15} &= q^{-1}b_2b_6c_2 \\
b &= -\frac{q^7 - 2q^5 + q^3}{(q^4 + q^2 + 1)(1 - q^{-6})} \\
f &= \frac{1 - q^2}{1 - q^{-2}} \\
s &= \frac{1 - q^2}{1 - q^{-6}} \\
u &= \frac{q + q^{-1} - q^{-3}}{1 - q^{-6}} \\
r &= -\frac{1}{1 - q^{-6}} \\
q'' &= \frac{q^7 - 2q^5 + q^3}{q^4 + q^2 + 1} \\
q' &= -(q^2 + 1 + q^{-2}) \\
a' &= af + hq = \frac{q^6}{q^{-2} - 1} \\
b' &= \frac{q^{13} - q^{11}}{(q^4 + q^2 + 1)^2} \\
d' &= \frac{q^{12}}{q^3 - 1} \\
c' &= -\frac{q^9}{q^4 + q^2 + 1} \\
c_1 &= \frac{1}{a'} \\
c_2 &= -ac_1 \\
c_3 &= -c_1 \\
b_1 &= \frac{1}{d'} \\
b_2 &= b_1b_2c_2(q + q^{-1} + q^{-3}) - b_1 \\
b_3 &= -b'b_1c_2 - bb_1 \\
b_4 &= -b_1b_3 \\
b_5 &= b_1b(c_3(q + q^{-1} + q^{-3}) - q^{-3}c_2) \\
b_6 &= -q^{-1}c_2b_1b \\
b_7 &= -q^{-1}b_1b_{13}c_3 \\
b_8 &= b_9 = -q^{-1}b_1b_2c_3 \\
\end{align*}
\]
\[ k_{16} = q^{-1}b_3b_b c_2 + q^{-2}b_{10}c_2 + q^{-3}b_8b_{13}c_2 \]
\[ k_{17} = q^{-1}b_4b_b c_2 \]
\[ k_{18} = q^{-1}b_5b_b c_2 \]
\[ k_{19} = q^{-1}b_6^2c_2 \]
\[ k_{20} = q^{-1}b_6b_b c_2 \]
\[ k_{21} = q^{-1}b_6b_1c_2 + q^{-2}b_{10}c_1 + q^{-3}b_7b_{13}c_2 \]
\[ k_{22} = q^{-1}b_6b_{10}c_2 \]
\[ k_{23} = q^{-1}b_6b_{12}c_2 + q^{-2}b_{10}c_3 + q^{-3}b_{10}b_{13}c_2 \]
\[ k_{24} = q^{-1}b_6b_{14}c_2 + q^{-2}b_{10}c_2 + q^{-3}b_9b_{13}c_2 \]
\[ k_{25} = q^{-1}b_{12}c_1 + b_{11}b_{13}c_2 \]
\[ k_{26} = q^{-1}b_{12}c_2 + b_{13}b_{14}c_2 \]
\[ k_{27} = q^{-1}b_{12}c_3 + b_{12}b_{13}c_2 \]
\[ k_{28} = b_{13}b_{15} + q^{-1}b_{14} \]
\[ k_{29} = b_{13}b_{15} + q^{-1}b_{14} \]
\[ k_{30} = b_{3}b_{13}c_2 + q^{-1}b_{12}c_2 \]
\[ k_{31} = q^{-3}b_5b_{13}c_2 + q^{-3}b_5c_3 + q^{-4}b_8 + b_{10}d_2[2] \]
\[ b_{10} = q^{-1}c_3b_1b \]
\[ b_{11} = b'c_1b \]
\[ b_{12} = b'c_3b \]
\[ b_{13} = -b'c'' \]
\[ b_{14} = -b'c_2b \]
\[ b_{15} = q^{-3}c_3 + c_2b_{13} \]
\[ b_{22} = q^{-1}b_6b_{10}c_2 \]
\[ b_{24} = q^{-1}b_6b_{14}c_2 + q^{-2}b_{10}c_2 + q^{-3}b_9b_{13}c_2 \]
\[ b_{26} = q^{-1}b_{12}c_2 + b_{13}b_{14}c_2 \]
\[ b_{28} = q^{-3}b_{13}c_2 + b_{14}d_2[1] \]

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S Launois
School of Mathematics, Statistics and Actuarial Science, University of Kent
Canterbury, Kent, CT2 7FS, UK
e-mail: S.Launois@kent.ac.uk

I Oppong
School of Mathematics, Statistics and Actuarial Science, University of Kent
Canterbury, Kent, CT2 7FS, UK
e-mail: isaac oppong@aims.ac.rw