Adjunctions in Quantaloid-enriched Categories

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A Dissertation Submitted to
Sichuan University
for the Degree of Doctor of Philosophy
in Mathematics

(English Version)

March 2014, Chengdu, China
Abstract

Adjunction is a fundamental notion and an extremely useful tool in category theory. This dissertation is devoted to a study of adjunctions concerning categories enriched over a quantaloid \( Q \) (or \( Q \)-categories for short), with the following types of adjunctions involved:

1. adjoint functors between \( Q \)-categories;
2. adjoint distributors between \( Q \)-categories;
3. adjoint functors between categories consisting of \( Q \)-categories.

A quantaloid is a “simplified” bicategory in which all hom-sets are sup-lattices. So, a theory of categories enriched over a quantaloid has been developed within the general framework, initiated by Betti and Walters, of categories enriched over a bicategory.

For a small quantaloid \( Q \) and a distributor between \( Q \)-categories, two adjunctions between the \( Q \)-categories of contravariant and covariant presheaves are presented. These adjunctions respectively extend the fundamental construction of Isbell adjunctions and Kan extensions in category theory, so, they will be called the Isbell adjunction and Kan adjunction, respectively. The functoriality of these constructions is the central topic of this dissertation.

In order to achieve this, infomorphisms between distributors are introduced to organize distributors (as objects) into a category. Then we proceed as follows.

First, the Isbell adjunction and Kan adjunction associated with each distributor \( \phi : A \leftrightarrow B \) between \( Q \)-categories respectively give rise to a monad \( \phi^* \circ \phi^! \) on \( PA \) and a monad \( \phi_* \circ \phi^* \) on \( PB \), where \( PA \) and \( PB \) denote respectively the
\(Q\)-categories of contravariant presheaves on \(A\) and \(B\). A \(Q\)-category \(A\) together with a monad on \(\mathcal{P}A\) is called a \(Q\)-closure spaces due to its resemblance to closure spaces in topology. It is proved that the correspondences

\[
\phi \mapsto (A, \phi^! \circ \phi^?)
\]

and

\[
\phi \mapsto (B, \phi_* \circ \phi^*)
\]

are respectively (covariant) functorial and contravariant functorial from the category of distributors and infomorphisms to the category of \(Q\)-closure spaces.

Second, for each \(Q\)-closure space \((A, C)\), the fixed objects of \(C : \mathcal{P}A \to \mathcal{P}A\) is a complete \(Q\)-category. It is shown that the assignments of a distributor \(\phi\) to the fixed points of \(\phi^! \circ \phi^?\) and \(\phi_* \circ \phi^*\) are respectively (covariant) functorial and contravariant functorial from the category of distributors and infomorphisms to that of skeletal complete \(Q\)-categories and left adjoint functors.

As consequences of the functoriality of the above processes, three factorizations of the free cocompletion functor of \(Q\)-categories are presented.

Finally, as applications, the theory of formal concept analysis and that of rough sets are extended to theories based on fuzzy relations between fuzzy sets.

**Keywords:** Quantaloid, \(Q\)-category, \(Q\)-distributor, \(Q\)-closure space, Isbell adjunction, Kan adjunction.
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Chapter 1

Introduction

In this dissertation, we present a study of adjunctions concerning categories enriched over a quantaloid, with special attention to Isbell adjunctions and Kan adjunctions.

A quantaloid [Ros96, Stu05a] is a category enriched over the symmetric monoidal closed category $\text{Sup}$ consisting of sup-lattices and sup-preserving maps. Since a quantaloid $Q$ is a closed and locally complete bicategory, following the framework of R. Betti [BC82] and R. F. C. Walters [Wal81], one can develop a theory of categories enriched over $Q$ (or $Q$-categories for short). It should be stressed that since a quantaloid is a “simplified” bicategory in the sense that each hom-set is partially ordered (which is indeed a sup-lattice), the coherence issues will not be a concern for $Q$-categories. For an overview of the theory of $Q$-categories, we refer to [Hey10, HST11, Stu03, Stu05a, Stu05b, Stu06, Stu13b].

1.1 Adjoint functors

“Adjoint functors arise everywhere”, as S. Mac Lane said in his classic textbook *Categories for the Working Mathematician* [ML98], the notion of adjoint functors introduced by D. M. Kan [Kan58] is now a core concept in category theory.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A pair of functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ forms an adjunction, written $F \dashv G : \mathcal{A} \to \mathcal{B}$, if there are natural transformations

$$\eta : 1_{\mathcal{A}} \to G \circ F \quad \text{and} \quad \epsilon : F \circ G \to 1_{\mathcal{B}}$$
Chapter 1. Introduction

satisfying the triangle identities \cite{Awo10, ML98}

\[
\begin{align*}
F & \xrightarrow{\eta} F \\
F \circ G \circ F & \xrightarrow{\varepsilon_F} F \\
G \circ F \circ G & \xrightarrow{\eta_G} G
\end{align*}
\]

where \(1_A\) and \(1_B\) are respectively the identity functors on \(A\) and \(B\). We present below two important examples of adjoint functors in category theory — Isbell adjunctions and Kan extensions.

1. Let \(A\) be a small category. The Isbell adjunction (or Isbell conjugacy) refers to the adjunction

\[
\begin{align*}
\text{ub} \dashv \text{lb} : \text{Set}^{\text{A}^{\text{op}}} \rightarrow (\text{Set}^{\text{A}})^{\text{op}}
\end{align*}
\]

arising from the Yoneda embedding \(Y : A \rightarrow \text{Set}^{\text{A}^{\text{op}}}\) and the co-Yoneda embedding \(Y^{\dagger} : A \rightarrow (\text{Set}^{\text{A}})^{\text{op}}\) given by

\[
\text{ub}(F) = \text{Set}^{\text{A}^{\text{op}}}(F, Y-) \quad \text{and} \quad \text{lb}(G) = (\text{Set}^{\text{A}})^{\text{op}}(Y^{\dagger} -, G).
\]

2. Let \(K : A \rightarrow \mathbb{C}\) be a functor, with \(A\) small. Composing with \(K\) induces a functor

\[
- \circ K : \text{Set}^{\text{C}^{\text{op}}} \rightarrow \text{Set}^{\text{A}^{\text{op}}}
\]

between the categories of presheaves on \(C\) and \(A\). The functor \(- \circ K\) has a left adjoint \(\text{Lan}_K\) and a right adjoint \(\text{Ran}_K\). For each presheaf \(F : \text{A}^{\text{op}} \rightarrow \text{Set}\), \(\text{Lan}_K F\) and \(\text{Ran}_K F\) are respectively the left and right Kan extension of \(F\) along \(K\).

Isbell adjunctions and Kan extensions have also been considered for categories enriched over a symmetric monoidal closed category \cite{Bor94b, DL07, Dub70, Kel82, KS05, Law73, Law86, Ric14}, and will be outlined in Chapter 2.

1.2 Adjoint morphisms in a bicategory

The notion of adjoint functors is a special case of adjoint morphisms (or 1-cells) in a 2-category, or more generally, that in a bicategory.
Let $\mathcal{B}$ be a bicategory with the associator $\alpha$, left unitor $\lambda$ and right unitor $\rho$ (will be introduced in Section 2.4). A pair of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow X$ forms an adjunction, written $f \dashv g : X \nrightarrow Y$, if there are 2-cells

$$\eta : 1_X \rightarrow g \circ f \quad \text{and} \quad \epsilon : f \circ g \rightarrow 1_Y$$

such that the compositions of 2-cells

$$f \xrightarrow{\rho_{XY}^{-1}} f \circ 1_X \xrightarrow{f \eta} f \circ (g \circ f) \xrightarrow{\alpha_{XYXY}} (f \circ g) \circ f \xrightarrow{\epsilon \circ f} 1_Y \circ f \xrightarrow{\lambda_{XY}} f$$

and

$$g \xrightarrow{\lambda_{YX}^{-1}} 1_X \circ g \xrightarrow{\eta \circ g} (g \circ f) \circ g \xrightarrow{\alpha_{YXYX}^{-1}} g \circ (f \circ g) \xrightarrow{g \circ \epsilon} g \circ 1_Y \xrightarrow{\rho_{YX}} g$$

are identities [Gra74], where $1_X$ and $1_Y$ are respectively the image of the unit functor on $\mathcal{B}(X, X)$ and $\mathcal{B}(Y, Y)$.

In particular, if $\mathcal{B}$ is a 2-category, then associators and unitors are identities, and $1_X$ and $1_Y$ are respectively the identity morphism on $X$ and $Y$. In this case, a pair of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow X$ forms an adjunction $f \dashv g : X \nrightarrow Y$, if there are 2-cells

$$\eta : 1_X \rightarrow g \circ f \quad \text{and} \quad \epsilon : f \circ g \rightarrow 1_Y$$

satisfying the triangle identities

$$f \xrightarrow{\epsilon \circ f} \xrightarrow{\eta \circ f} f \circ g \circ f \xrightarrow{f \eta} f \quad \text{and} \quad g \xrightarrow{g \circ \epsilon} \xrightarrow{\eta \circ g} g \circ f \circ g \xrightarrow{g \circ \epsilon} g.$$

Adjoint functors are exactly adjoint morphisms in the 2-category $\mathbf{Cat}$ consisting of categories, functors and natural transformations.

## 1.3 Adjunctions in $Q$-categories

There are two basic types of adjunctions between categories enriched over a quantaloid $Q$, i.e., adjoint $Q$-functors between $Q$-categories (corresponding to Section
and adjoint $Q$-distributors as adjoint morphisms in the 2-category of $Q$-categories and $Q$-distributors (corresponding to Section 1.2).

A pair of $Q$-functors $F : A \rightarrow B$ and $G : B \rightarrow A$ forms an adjunction, written $F \dashv G : A \rightarrow B$, if

$$1_A \leq G \circ F \quad \text{and} \quad F \circ G \leq 1_B,$$

where $1_A$ and $1_B$ are respectively the identity $Q$-functors on $A$ and $B$. Adjoint $Q$-functors can be viewed as adjoint morphisms in the locally ordered 2-category $Q\text{-Cat}$ of $Q$-categories and $Q$-functors.

Since a quantaloid $Q$ is itself a locally ordered 2-category, there are adjoint morphisms $f \dashv g : X \rightarrow Y$ in $Q$ given by

$$1_X \leq g \circ f \quad \text{and} \quad f \circ g \leq 1_Y.$$

In our case, $Q$-categories and $Q$-distributors constitute a quantaloid $Q\text{-Dist}$. Adjoint morphisms in $Q\text{-Dist}$ give rise to the notion of adjoint $Q$-distributors. Specifically, a pair of $Q$-distributors $\phi : A \rightarrow B$ and $\psi : B \rightarrow A$ forms an adjunction, written $\phi \dashv \psi : A \rightarrow B$, if

$$A \leq \psi \circ \phi \quad \text{and} \quad \phi \circ \psi \leq B,$$

where $A$ and $B$ are respectively the identity $Q$-distributors on $A$ and $B$.

The following constructions exhibit the abundance of adjoint $Q$-functors and adjoint $Q$-distributors.

1. Each $Q$-functor $F : A \rightarrow B$ induces a pair of adjoint $Q$-distributors

$$F \uparrow \dashv F \downarrow : A \rightarrow B,$$

called respectively the graph and the cograph of $F$.

2. If $Q$ is a small quantaloid, then each $Q$-distributor $\phi : A \rightarrow B$ induces two pairs of adjoint $Q$-functors

$$\phi \uparrow \dashv \phi \downarrow : \mathcal{P}A \rightarrow \mathcal{P}B$$

(1.1)
1.4. Functoriality of the Isbell adjunction and Kan adjunction

and

\[ \phi^* \dashv \phi_* : \mathcal{P}B \rightarrow \mathcal{P}A \]  \hspace{1cm} (1.2)

between the \( Q \)-categories of contravariant presheaves and covariant presheaves.

These adjunctions will be the core subject of our discussion.

It should be pointed out that the adjunctions (1.1) and (1.2) extend the construction of Isbell adjunctions and Kan extensions presented in Section 1.1:

1. The \( Q \)-category \( \mathcal{P}A \) of contravariant presheaves and the \( Q \)-category \( \mathcal{P}^\dagger A \) of covariant presheaves are the counterparts of \( \text{Set}^A \) and \( (\text{Set}^A)^{\text{op}} \), respectively.

2. If \( \phi \) is the identity \( Q \)-distributor on \( A \), then the adjunction \( \phi^* \dashv \phi_* \) reduces to the Isbell adjunction

\[ A \xrightarrow{\phi} (-) \dashv (-) \xleftarrow{\phi} \mathcal{P}A \rightarrow \mathcal{P}^\dagger A \]

presented in [Stu05a].

3. Given a \( Q \)-functor \( F : A \rightarrow B \), consider the graph \( F^\natural : A \rightarrow \text{Graph} \) and the cograph \( F^\sharp : B \rightarrow \text{Cograph} \). Then it holds that (Theorem 6.2.4)

\[ (F^\natural)^* \dashv (F^\sharp)_* = F^{\rightarrow} = (F^\natural)^* \dashv (F^\sharp)_* \]

where \( F^{\rightarrow} : \mathcal{P}B \rightarrow \mathcal{P}A \) is the counterpart of the functor \( - \circ F \) for \( Q \)-categories.

Therefore, adjunctions of the forms (1.1) and (1.2) are called Isbell adjunctions and Kan adjunctions, respectively.

1.4 Functoriality of the Isbell adjunction and Kan adjunction

The primary goal of this dissertation is to discuss the functoriality of the constructions of the Isbell adjunction (1.1) and Kan adjunction (1.2) in the premise that \( Q \) is a small quantaloid.
For each $Q$-distributor $\phi : A \leftrightarrow B$, the related Isbell adjunction (1.1) and Kan adjunction (1.2) give rise to a monad $\phi^! \circ \phi_!$ on $\mathcal{P}A$ and a monad $\phi_* \circ \phi^*$ on $\mathcal{P}B$, respectively. In order to establish functoriality of the correspondences

\[(\phi : A \leftrightarrow B) \mapsto (A, \phi^! \circ \phi_!) \tag{1.3}\]

and

\[(\phi : A \leftrightarrow B) \mapsto (B, \phi_* \circ \phi^*) \tag{1.4}\]

we introduce infomorphisms between $Q$-distributors to construct a category with $Q$-distributors as objects.

Given $Q$-distributors $\phi : A \leftrightarrow B$ and $\psi : A' \leftrightarrow B'$, an infomorphism $(F, G) : \phi \leftrightarrow \psi$ is a pair of $Q$-functors $F : A \to A'$ and $G : B' \to B$ such that

$$\phi(-, G-) = \psi(F-, -).$$

$Q$-distributors and infomorphisms constitute a category $Q$-Info.

A $Q$-category $\mathcal{A}$ together with a monad $C : \mathcal{P}\mathcal{A} \to \mathcal{P}\mathcal{A}$ is called a $Q$-closure space, and lax maps between monads on $\mathcal{P}\mathcal{A}$ are called continuous $Q$-functors, due to their resemblance to closure spaces in topology (will be introduced in Chapter 4). $Q$-closure space and continuous $Q$-functors constitute a category $Q$-Cls.

It is shown that, the correspondence (1.3) gives rise to a right adjoint (co-variant) functor from $Q$-Info to $Q$-Cls, and the correspondence (1.4) defines a contravariant functor (which is a right adjoint if $Q$ is a Girard quantaloid) from $Q$-Info to $Q$-Cls.

Furthermore, for each $Q$-closure space $(\mathcal{A}, C)$, the fixed points of $C : \mathcal{P}\mathcal{A} \to \mathcal{P}\mathcal{A}$ (or equivalently, all the algebras if we consider $C$ as a monad) is a complete $Q$-category. We show that the correspondence

$$(\mathcal{A}, C) \mapsto C(\mathcal{P}\mathcal{A})$$

gives rise to a left adjoint functor from $Q$-Cls to the category $Q$-CCat of skeletal complete $Q$-categories and left adjoint $Q$-functors.
Therefore, the assignments of a $\mathcal{Q}$-distributor $\phi$ to the fixed points of $\phi^\downarrow \circ \phi^\uparrow$ and $\phi_* \circ \phi^*$ respectively give rise to a (covariant) functor

$$\mathcal{M} : \mathcal{Q}\text{-Info} \rightarrow \mathcal{Q}\text{-CCat}$$

and a contravariant functor

$$\mathcal{K} : (\mathcal{Q}\text{-Info})^{\text{op}} \rightarrow \mathcal{Q}\text{-CCat}.$$  

As consequences of the functoriality of the above processes, we present three factorizations of the free cocompletion functor $\mathcal{P}$ of $\mathcal{Q}$-categories:

1. $\mathcal{P}$ factors through the category $\mathcal{Q}\text{-Cls}$ of $\mathcal{Q}$-closure spaces;

2. $\mathcal{P}$ factors through the functor $\mathcal{M}$ induced by Isbell adjunctions;

3. $\mathcal{P}$ factors through the functor $\mathcal{K}$ induced by Kan adjunctions.

1.5 Applications in fuzzy set theory

In Chapter 7, we present some applications of quantaloid-enriched categories in fuzzy set theory. We demonstrate that Isbell adjunctions and Kan adjunctions are closely related to the theories of formal concept analysis [CR04, DP02, GSW05, GW99] and rough set theory [Paw82, Pol02, Yao04] in computer science. The functoriality of Isbell adjunctions and Kan adjunctions generalizes the functoriality of concept lattices based on formal concept analysis and rough set theory [SZ13]. Furthermore, the theories of formal concept analysis and rough set theory on fuzzy sets are established.

Recall that a preorder $\leq$ on a (crisp) set $A$ is a reflexive and transitive relation on $A$ in the sense that

1. $\forall x \in A, x \leq x;$

2. $\forall x, y, z \in A, y \leq z \text{ and } x \leq y \text{ implies } x \leq z.$
A preordered set \((A, \leq)\) is exactly a category enriched over the quantaloid \(2\), the two-element Boolean algebra.

A formal context \((A, B, R)\) consists of (crisp) sets \(A, B\) and a relation \(R \subseteq A \times B\) (or equivalently, a \(2\)-distributor between discrete \(2\)-categories). Formal contexts provide a common framework for formal concept analysis and rough set theory. For each formal context \((A, B, R)\), there exists a contravariant Galois connection \((R^\uparrow, R^\downarrow)\) and a covariant Galois connection \((R_\exists, R_\forall)\) between the powersets of \(A\) and \(B\). These two Galois connections play fundamental roles in formal concept analysis and rough set theory, respectively, and they are exactly the Isbell adjunction and Kan adjunction in \(2\)-categories.

Classical preorders are extended to many-valued preorders by replacing \(2\) with a more complicated quantaloid:

1. Let \(Q\) be a unital quantale, i.e., a one-object quantaloid. A fuzzy relation on a (crisp) set \(A\) is a map \(A \times A \longrightarrow Q\), and the notion of reflexivity and transitivity can be extended to fuzzy relations. Then a (crisp) set \(A\) equipped with a reflexive and transitive fuzzy relation on \(A\) is a \(Q\)-preordered set \([\text{Bel04, LZ06, LZ09, SZ13, Zad71}]\), which is exactly a category enriched over \(Q\).

2. Let \(Q\) be a divisible unital quantale. A fuzzy set (or a \(Q\)-subset \([\text{Gog67, Zad65}]\)) is a map \(A : A_0 \longrightarrow Q\) from a (crisp) set \(A_0\) to \(Q\), where the value \(Ax\) is interpreted as the membership degree of \(x\) in \(A_0\). A \(Q\)-preorder on a fuzzy set \([\text{PZ12, TLZ12}]\) is then characterized as a category enriched over a quantaloid \(Q\) induced by \(Q\) (see Proposition \([7.2.2]\) for the definition of \(Q\)), which can also be interpreted elementarily as a reflexive and transitive fuzzy relation on the fuzzy set \(A\).

Thus, by translating Isbell adjunctions and Kan adjunctions in categories enriched over a certain quantaloid \(Q\), the theories of formal concept analysis and rough set theory can be established on fuzzy relations between (crisp) sets, and further on fuzzy relation between fuzzy sets. As consequences of the results in
Chapter 5 and 6 for formal concept analysis and rough set theory built in each one of these frameworks, the processes of generating concept lattices from formal contexts are functorial.
Chapter 2

Preliminaries

We assume that the readers are familiar with the basic notions of classic category theory \cite{Awo10, BW95, Bor94a, LR03, ML98}, including category, functor, natural transformation, Yoneda embedding, limit, colimit and adjunction. In this chapter, we review the theory of categories enriched over a monoidal category $\mathcal{V}$, and then introduce Isbell adjunctions and Kan extensions in $\mathcal{V}$-categories. Finally, we present the basic concepts of categories enriched over a bicategory as a foundation for the next chapter.

2.1 Categories enriched over a monoidal category

In this section, we recall some basic concepts of categories enriched over a monoidal category. Some necessary coherence diagrams are omitted here, and we refer to \cite{AHS90, Bor94b, Kel82, Law73, ML98} for the detailed definitions.

Definition 2.1.1. A monoidal category $\mathcal{V}$ is a category equipped with a bifunctor

\[ \otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V} \]

as the tensor product, which is associative in the sense that there is a natural and coherent isomorphism

\[ a_{XYZ} : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z \]
for each triple $X, Y, Z$ of objects in $\mathcal{V}$, and has a unit object $I$ of $\mathcal{V}$ in the sense that there are natural and coherent isomorphisms

$$l_X : I \otimes X \cong X \quad \text{and} \quad r_X : X \otimes I \cong X$$

for each object $X$ of $\mathcal{V}$.

**Definition 2.1.2.** A monoidal category $\mathcal{V}$ is *symmetric* if there is a natural and coherent isomorphism

$$s_{XY} : X \otimes Y \cong Y \otimes X$$

for each pair $X, Y$ of objects in $\mathcal{V}$.

**Definition 2.1.3.** A monoidal category $\mathcal{V}$ is *closed* if both the functors

$$- \otimes X : \mathcal{V} \to \mathcal{V} \quad \text{and} \quad X \otimes - : \mathcal{V} \to \mathcal{V}$$

have a right adjoint.

**Example 2.1.4.** (1) The category $\textbf{Set}$ of small sets and functions is a symmetric monoidal closed category with cartesian products of sets playing as the tensor products.

(2) [Ros90] A unital quantale is a complete lattice $Q$ equipped with a binary operation $\&$ such that

(i) $(Q, \&)$ is a monoid;

(ii) $a \&(\bigvee b_i) = \bigvee (a \& b_i)$ and $(\bigvee b_i)\& a = \bigvee (b_i \& a)$ for all $a, b_i \in Q$.

A unital quantale $(Q, \&)$ is a monoidal closed category with $\&$ being the tensor product operation. Furthermore, if $(Q, \&)$ is a commutative monoid, then $(Q, \&)$ is a commutative unital quantale, which is a symmetric monoidal closed category.

(3) [JTS84, KW10, Pit88] A sup-lattice is a partially ordered set $X$ in which every subset $A \subseteq X$ has a join (or supremum, least upper bound) $\bigvee A$. A sup-lattice necessarily admits a meet (or infimum, greatest lower bound)
2.1. Categories enriched over a monoidal category

For every subset $A \subseteq X$, a sup-preserving map $f : X \rightarrow Y$ between sup-lattices is a map $f : X \rightarrow Y$ such that $f(\bigvee A) = \bigvee f(A)$ for every subset $A \subseteq X$. Sup-lattices and sup-preserving maps constitute a symmetric monoidal closed category $\text{Sup}$. The tensor product $X \otimes Y$ of two sup-lattices $X$ and $Y$ is constructed as a quotient of the free sup-lattice on $X \times Y$, i.e., the powerset $\mathcal{P}(X \times Y)$, obtained from the equivalence relation generated by

$$
\left( \bigvee_{i} x_{i}, y \right) \sim \bigvee_{i} (x_{i}, y) \quad \text{and} \quad (x, \bigvee_{j} y_{j}) \sim \bigvee_{j} (x, y_{j}).
$$

(4) [Bor94b, HS97] The category $\text{Ab}$ of abelian groups is a symmetric monoidal closed category equipped with the usual tensor products of abelian groups.

(5) [HS97] The category $\text{Ch}_R$ of chain complexes of $R$-modules over a commutative ring $R$ is a symmetric monoidal closed category with the usual tensor products of chain complexes.

(6) The category $\text{Cat}$ of small categories and functors is a symmetric monoidal closed category with the products of categories playing as the tensor products.

**Definition 2.1.5.** Let $\mathcal{V}$ be a monoidal category. A $\mathcal{V}$-category (or category enriched over $\mathcal{V}$) $\mathcal{A}$ consists of a class $\mathcal{A}_0$ as the objects, a hom-object $\mathcal{A}(x, y)$ of $\mathcal{V}$ for all $x, y \in \mathcal{A}_0$, a morphism

$$
c_{xyz} : \mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \rightarrow \mathcal{A}(x, z)
$$

in $\mathcal{V}$ for all $x, y, z \in \mathcal{A}_0$ as the composition, and a morphism

$$
u_{x} : I \rightarrow \mathcal{A}(x, x)
$$

in $\mathcal{V}$ for all $x \in \mathcal{A}_0$ as the unit, such that the diagrams expressing the associativity
and unity laws

\[
\begin{align*}
&\mathcal{A}(z, w) \otimes (\mathcal{A}(y, z) \otimes \mathcal{A}(x, y)) \\
\text{and} &\quad (\mathcal{A}(z, w) \otimes \mathcal{A}(y, z)) \otimes \mathcal{A}(x, y) \\
&\mathcal{A}(z, w) \otimes \mathcal{A}(x, z) \\
&\mathcal{A}(y, w) \otimes \mathcal{A}(x, y) \\
&\mathcal{A}(x, y) \\
&\mathcal{A}(x, y) \otimes I \\
&\mathcal{A}(x, y) \otimes \mathcal{A}(x, x) \\
&\mathcal{A}(x, y) \otimes I
\end{align*}
\]

are commutative for all \(x, y, z, w \in A_0\).

**Example 2.1.6.** (1) **Set**-categories are just the ordinary locally small categories.

(2) \[\text{GMdZ10, SZ13}\] A category enriched over a unital quantale \(Q\) is a \(Q\)-valued preordered set.

(3) \[\text{Ros96, Stu05a}\] A **Sup**-category is a quantaloid.

(4) \[\text{HS97}\] An **Ab**-category is a ringoid. An **Ab**-category becomes an additive category if it admits finite (co)products.

(5) \[\text{Ber11}\] Let \(\text{Ch}_*(\mathcal{A})\) be the category of chain complexes of \(R\)-modules over a commutative ring \(R\). A category enriched over \(\text{Ch}_*(\mathcal{A})\) is a differential graded category, or dg category for short.
A \textbf{Cat}-category is a (strict) 2-category.

If \( \mathcal{V} \) is a symmetric monoidal closed category, then \( \mathcal{V} \) is itself a \( \mathcal{V} \)-category, with \( \mathcal{V}(X, -) \) being the right adjoint of \( - \otimes X \) for each object \( X \in \mathcal{V} \).

If \( \mathcal{V} \) is a symmetric monoidal category, then each \( \mathcal{V} \)-category \( A \) has a dual \( \mathcal{V} \)-category \( A^{op} \) given by the same objects and

\[
A^{op}(x, y) = A(y, x)
\]

for all \( x, y \in A_0 \).

**Definition 2.1.7.** Let \( \mathcal{V} \) be a monoidal category. A \( \mathcal{V} \)-functor \( F : A \rightarrow B \) between \( \mathcal{V} \)-categories consists of a map \( F : A_0 \rightarrow B_0 \) and a morphism

\[
F_{xx'} : A(x, x') \rightarrow B(F x, F x')
\]

in \( \mathcal{V} \) for all \( x, x' \in A_0 \), such that the diagrams

\[
\begin{array}{ccc}
A(x', x''') \otimes A(x, x') & \xrightarrow{(c_{A})_{x', x''}} & A(x, x'') \\
F_{x', x''} \otimes F_{xx'} & & F_{xx''} \\
\mathcal{B}(F x', F x'') \otimes \mathcal{B}(F x, F x') & \xrightarrow{(c_{\mathcal{B}})_{F x', F x', F x''}} & \mathcal{B}(F x, F x'')
\end{array}
\]

and

\[
\begin{array}{ccc}
I & \xrightarrow{(u_A)_x} & A(x, x) \\
(u_B)_F & & F_{xx} \\
\mathcal{B}(F x, F x) & & 
\end{array}
\]

are commutative for all \( x, x', x'' \in A_0 \).

A \( \mathcal{V} \)-functor \( F : A \rightarrow B \) is **fully faithful** if \( F_{xx'} \) is an isomorphism in \( \mathcal{V} \) for all \( x, x' \in A_0 \).

**Definition 2.1.8.** Let \( \mathcal{V} \) be a monoidal category. A \( \mathcal{V} \)-natural transformation \( \alpha : F \Rightarrow G \) between \( \mathcal{V} \)-functors \( F, G : A \rightarrow B \) is given by a morphism

\[
\alpha_x : I \rightarrow \mathcal{B}(F x, G x)
\]
in \( V \) for each \( x \in A_0 \), such that the diagram

\[
\begin{array}{ccc}
I \otimes A(x, x') & \xrightarrow{\alpha \otimes G_{x'}} & B(Fx, Gx) \otimes B(Gx, Gx') \\
\downarrow^{l_{k,x}'-1} & & \downarrow^{c_{Fx,Gx,Gx'}} \\
A(x, x') & \xrightarrow{c_{Fx,Gx,Gx'}} & B(Fx, Gx') \\
\end{array}
\]

\[
\begin{array}{ccc}
A(x', x) & \xrightarrow{F_{x'} \otimes \alpha_{x'}} & B(Fx', Fx') \otimes B(Fx', Gx') \\
\end{array}
\]

is commutative for all \( x, x' \in A_0 \).

If \( V \) is a symmetric monoidal category, then the category \( V\text{-Cat} \) of small \( V \)-categories and \( V \)-functors is itself a symmetric monoidal category, with the tensor product \( A \otimes B \) of \( V \)-categories given by

\[
(A \otimes B)_0 = A_0 \times B_0
\]

and

\[
(A \otimes B)((x, y), (x', y')) = A(x, x') \otimes B(y, y')
\]

for all \( x, x' \in A_0 \) and \( y, y' \in B_0 \).

**Definition 2.1.9.** Let \( V \) be a complete symmetric monoidal closed category and \( A \) a small \( V \)-category. The *end* of a \( V \)-functor \( F : A^{\text{op}} \otimes A \rightarrow V \) consists of an object \( \int_{a \in A_0} F(a, a) \) of \( V \) and a universal \( V \)-natural family

\[
\alpha_x : \int_{a \in A_0} F(a, a) \rightarrow F(x, x)
\]

in the sense that any other \( V \)-natural family \( \beta_x : X \rightarrow F(x, x) \) factors uniquely through \( \alpha_x \) via a morphism \( f : X \rightarrow \int_{a \in A_0} F(a, a) \) in \( V \).

If \( V \) is a complete symmetric monoidal closed category, then the \( V \)-functors between a small \( V \)-category \( A \) and a \( V \)-category \( B \) constitute a \( V \)-category \( B^A \) with

\[
B^A(F, G) = \int_{x \in A_0} B(Fx, Gx)
\]
for all \( \mathcal{V} \)-functors \( F, G : \mathcal{A} \to \mathcal{B} \). The object \( \mathcal{B}^\mathcal{A}(F, G) \in \mathcal{V} \) is called the \textit{object of} \( \mathcal{V} \)-natural transformations from \( F \) to \( G \).

\textbf{Definition 2.1.10.} Let \( \mathcal{V} \) be a symmetric monoidal closed category and \( F : \mathcal{A} \to \mathcal{B} \) a \( \mathcal{V} \)-functor.

\begin{enumerate}
\item The \textit{limit of} \( F \) \textit{weighted by} a \( \mathcal{V} \)-functor \( G : \mathcal{A} \to \mathcal{V} \) is an object \( \operatorname{lim}_G F \in \mathcal{B}_0 \) together with a \( \mathcal{V} \)-natural isomorphism
\[ \mathcal{B}(b, \operatorname{lim}_G F) \cong \mathcal{V}(G(b), \mathcal{B}(F -)). \]
\end{enumerate}

A \( \mathcal{V} \)-category \( \mathcal{B} \) is \textit{complete} if \( \operatorname{lim}_G F \) exists for each \( \mathcal{V} \)-functor \( F : \mathcal{A} \to \mathcal{B} \) and \( G : \mathcal{A} \to \mathcal{V} \) with the domain \( \mathcal{A} \) small.

\begin{enumerate}
\item The \textit{colimit of} \( F \) \textit{weighted by} a \( \mathcal{V} \)-functor \( G : \mathcal{A}^{\text{op}} \to \mathcal{V} \) is an object \( \operatorname{colim}_G F \in \mathcal{B}_0 \) together with a \( \mathcal{V} \)-natural isomorphism
\[ \mathcal{B}(\operatorname{colim}_G F, b) \cong \mathcal{V}^{\text{op}}(G(F -), b). \]
\end{enumerate}

A \( \mathcal{V} \)-category \( \mathcal{B} \) is \textit{cocomplete} if \( \operatorname{colim}_G F \) exists for each \( \mathcal{V} \)-functor \( F : \mathcal{A} \to \mathcal{B} \) and \( G : \mathcal{A}^{\text{op}} \to \mathcal{V} \) with the domain \( \mathcal{A} \) small.

\textbf{Definition 2.1.11.} A pair of \( \mathcal{V} \)-functors \( F : \mathcal{A} \to \mathcal{B} \) and \( G : \mathcal{B} \to \mathcal{A} \) between \( \mathcal{V} \)-categories is called an \textit{adjunction}, written \( F \dashv G : \mathcal{A} \to \mathcal{B} \), if for every pair \( x \in \mathcal{A}_0 \) and \( y \in \mathcal{B}_0 \), there is an isomorphism in \( \mathcal{V} \)
\[ \mathcal{B}(Fx, y) \cong \mathcal{A}(x, Gy) \]
\( \mathcal{V} \)-natural in \( x \) and \( y \). In this case, \( F \) is called a \textit{left adjoint} of \( G \) and \( G \) a \textit{right adjoint} of \( F \).

\section*{2.2 Isbell adjunctions in \( \mathcal{V} \)-categories}

In this section, we assume that \( \mathcal{V} \) is a complete symmetric monoidal closed category. In this case, \( \mathcal{V} \) is itself a complete \( \mathcal{V} \)-category.
For each small \( \mathcal{V} \)-category \( A \), there is a \( \mathcal{V} \)-functor \( Y : A \rightarrow \mathcal{V}^{A^{\mathrm{op}}} \) given by
\[
Y_x = A(-, x) \quad \text{and} \quad Y_{xx'} = \mathcal{V}^{A^{\mathrm{op}}}(A(-, x), A(-, x'))
\]
and a \( \mathcal{V} \)-functor \( Y^\dagger : A \rightarrow \mathcal{(V^A)^{\mathrm{op}}} \) given by
\[
Y^\dagger x = A(x, -) \quad \text{and} \quad Y^\dagger_{xx'} = \mathcal{(V^A)^{\mathrm{op}}}(A(x, -), A(x', -)).
\]
The following lemma implies that both \( Y \) and \( Y^\dagger \) are fully faithful \( \mathcal{V} \)-functors, and are called respectively the \textit{Yoneda embedding} and the \textit{co-Yoneda embedding}.

**Lemma 2.2.1 (Yoneda).** \cite{Bor94b, Kel82} Let \( A \) be a small \( \mathcal{V} \)-category. For each \( \mathcal{V} \)-functor \( F : A \rightarrow \mathcal{V} \) and \( x \in A_0 \), there is an isomorphism in \( \mathcal{V} \)
\[
\mathcal{V}^{A}(A(x, -), F) \cong Fx.
\]

For each small \( \mathcal{V} \)-category \( A \), the Yoneda embedding
\[
Y : A \rightarrow \mathcal{V}^{A^{\mathrm{op}}}
\]
and the co-Yoneda embedding
\[
Y^\dagger : A \rightarrow \mathcal{(V^A)^{\mathrm{op}}}
\]
induce a pair of \( \mathcal{V} \)-functors
\[
\begin{align*}
\mathrm{ub} : \mathcal{V}^{A^{\mathrm{op}}} & \rightarrow \mathcal{(V^A)^{\mathrm{op}}} \\
\mathrm{lb} : \mathcal{(V^A)^{\mathrm{op}}} & \rightarrow \mathcal{V}^{A^{\mathrm{op}}}
\end{align*}
\]
given by
\[
\begin{align*}
\mathrm{ub}(F) = \mathcal{V}^{A^{\mathrm{op}}}(F, Y-) & \quad \text{and} \quad \mathrm{lb}(G) = \mathcal{(V^A)^{\mathrm{op}}}(Y^\dagger-, G).
\end{align*}
\]

**Proposition 2.2.2 (Isbell).** \cite{DL07, KS05, Law86} \( \mathrm{ub} \dashv \mathrm{lb} : \mathcal{V}^{A^{\mathrm{op}}} \rightarrow \mathcal{(V^A)^{\mathrm{op}}} \).

This adjunction is known as the \textit{Isbell adjunction} (or \textit{Isbell conjugacy}) in category theory.
2.3 Kan extensions of \( \mathcal{V} \)-functors

In this section, we also assume that \( \mathcal{V} \) is a complete symmetric monoidal closed category.

**Definition 2.3.1.** Let \( K : A \to C \) be a \( \mathcal{V} \)-functor and \( B \) a \( \mathcal{V} \)-category.

1. The **left Kan extension** of a \( \mathcal{V} \)-functor \( F : A \to B \) along \( K : A \to C \), if it exists, is a \( \mathcal{V} \)-functor
   \[
   \text{Lan}_K F : C \to B
   \]
   together with a \( \mathcal{V} \)-natural isomorphism
   \[
   B^C(\text{Lan}_K F, S) \cong B^A(F \circ K)
   \]
   for any other \( \mathcal{V} \)-functor \( S : C \to B \).

2. The **right Kan extension** of a \( \mathcal{V} \)-functor \( F : A \to B \) along \( K : A \to C \), if it exists, is a \( \mathcal{V} \)-functor
   \[
   \text{Ran}_K F : C \to B
   \]
   together with a \( \mathcal{V} \)-natural isomorphism
   \[
   B^C(S, \text{Ran}_K F) \cong B^A(S \circ K, F)
   \]
   for any other \( \mathcal{V} \)-functor \( S : C \to B \).

If the left Kan extension \( \text{Lan}_K F \) of a \( \mathcal{V} \)-functor \( F : A \to B \) along \( K : A \to C \) exists, then there is a *universal* \( \mathcal{V} \)-natural transformation \( \eta : F \Rightarrow (\text{Lan}_K F) \circ K \) in the sense that for any other \( \mathcal{V} \)-functor \( S : C \to A \) and \( \mathcal{V} \)-natural transformation \( \gamma : F \Rightarrow S \circ K \), \( \gamma \) factors uniquely through \( \eta \).

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow{\gamma} & & \downarrow{=} \\
C & \xrightarrow{S} & B
\end{array}
\quad \xrightarrow{K} \quad
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow{\eta} & \xleftarrow{\text{Lan}_K F} & \downarrow{\exists} \\
C & \xrightarrow{S} & B
\end{array}
\]

Dually, the right Kan extension \( \text{Ran}_K F \) of a \( \mathcal{V} \)-functor \( F : A \to B \) is equipped with a *universal* \( \mathcal{V} \)-natural transformation \( \varepsilon : (\text{Ran}_K T) \circ K \Rightarrow T \) such that for
any other \( \mathcal{V} \)-functor \( S : \mathcal{C} \rightarrow \mathcal{A} \) and \( \mathcal{V} \)-natural transformation \( \sigma : S \circ K \Rightarrow F \), \( \sigma \) factors uniquely through \( \epsilon \).

Given a \( \mathcal{V} \)-functor \( K : \mathcal{A} \rightarrow \mathcal{C} \) and another \( \mathcal{V} \)-category \( \mathcal{B} \), composing with \( K \) yields a \( \mathcal{V} \)-functor

\[
K^* : \mathcal{B}^C \rightarrow \mathcal{B}^A,
\]

which sends a \( \mathcal{V} \)-functor \( S : \mathcal{C} \rightarrow \mathcal{B} \) to \( S \circ K : \mathcal{A} \rightarrow \mathcal{B} \).

If each \( \mathcal{V} \)-functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) has a left Kan extension \( \text{Lan}_K F : \mathcal{C} \rightarrow \mathcal{B} \) along \( K \), then we obtain an adjunction \[ \text{Bor94b}\]

\[
\text{Lan}_K \dashv K^* : \mathcal{B}^A \rightarrow \mathcal{B}^C.
\]

Dually, if each \( \mathcal{V} \)-functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) has a right Kan extension \( \text{Ran}_K F : \mathcal{C} \rightarrow \mathcal{B} \) along \( K \), then we obtain an adjunction

\[
K^* \dashv \text{Ran}_K : \mathcal{B}^C \rightarrow \mathcal{B}^A.
\]

Given a \( \mathcal{V} \)-functor \( F : \mathcal{A} \rightarrow \mathcal{B} \), if the codomain \( \mathcal{B} \) admits certain weighted colimits, then the left Kan extension of \( F \) along some \( \mathcal{V} \)-functor \( K : \mathcal{A} \rightarrow \mathcal{C} \) can be constructed pointwise for each object \( c \in \mathcal{C}_0 \). Dually, if the codomain \( \mathcal{B} \) admits certain weighted limits, then the right Kan extension of \( F \) along \( K \) can be constructed pointwise.

**Proposition 2.3.2.** \[\text{Kel82}\] Let \( F : \mathcal{A} \rightarrow \mathcal{B} \) and \( K : \mathcal{A} \rightarrow \mathcal{C} \) be \( \mathcal{V} \)-functors.
2.4 Categories enriched over a bicategory

(1) The left Kan extension of $F$ along $K$ can be computed by

$$(\text{Lan}_K F)c = \text{colim}_{(K-,c)} F$$

if the weighted colimit exists for each $c \in C_0$.

(2) The right Kan extension of $F$ along $K$ can be computed by

$$(\text{Ran}_K F)c = \text{lim}_{c(K-,)} F$$

if the weighted limit exists for each $c \in C_0$.

Thus, if $A$ is small, $B$ is complete and cocomplete, then every $V$-functor $F : A \to B$ has a (pointwise) left Kan extension and a (pointwise) right Kan extension along any $V$-functor $K : A \to C$. In particular, we have the following corollary.

**Corollary 2.3.3.** [Law73] Let $V$ be a complete and cocomplete symmetric monoidal closed category. For each $V$-functor $K : A \to C$ with $A$ small, the $V$-functor “composing with $K$”

$$K^* : V^C \to V^A$$

has a left adjoint $\text{Lan}_K$ and a right adjoint $\text{Ran}_K$.

2.4 Categories enriched over a bicategory

A $\text{Cat}$-category is called a (strict) 2-category. However, we are particularly interested in a weak version of 2-categories — categories “weakly enriched” over $\text{Cat}$, in which the associativity and unity laws of enriched categories hold only commutative up to natural and coherent isomorphism. This is what we call a bicategory [Bén67, CKWW08, CW87, Lac10, Lei98].

**Definition 2.4.1.** [Bén67, ML98] A bicategory $B$ consists of the following data:

(1) a class $B_0$ of objects $X, Y, Z, \ldots$, also called 0-cells;

(2) for every pair $X, Y$ of 0-cells, a (small) category $B(X, Y)$, whose objects are called morphisms or 1-cells and whose morphisms are called 2-morphisms or 2-cells;
(3) for every triple \(X, Y, Z\) of 0-cells, a horizontal composition functor,

\[
\circ : \mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) \to \mathcal{B}(X, Z);
\]

(4) for every 0-cell \(X\), a unit functor

\[
u_X : 1 \to \mathcal{B}(X, X)
\]

from the terminal category 1, which picks out a 1-cell \(1_X \in \mathcal{B}(X, X)\);

(5) for every quadruple \(X, Y, Z, W\) of 0-cells, an associativity natural isomorphism

\[
\mathcal{B}(Z, W) \times \mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) \xrightarrow{\circ_{YZW} \times 1_{\mathcal{B}(X, Y)}} \mathcal{B}(X, W) \times \mathcal{B}(X, Y)
\]

\[
\begin{array}{ccc}
\mathcal{B}(Z, W) \times \mathcal{B}(X, Z) & \xrightarrow{\circ_{XZW}} & \mathcal{B}(X, W) \\
\mathcal{B}(Z, W) \times \mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) & \xrightarrow{\circ_{XYZW}} & \mathcal{B}(X, W) \\
\end{array}
\]

between the two functors from \(\mathcal{B}(Z, W) \times \mathcal{B}(Y, Z) \times \mathcal{B}(X, Y)\) to \(\mathcal{B}(X, W)\) arising from the horizontal composition functor \(\circ\), called the associator;

(6) for every pair \(X, Y\) of 0-cells, a left unit natural isomorphism

\[
1 \times \mathcal{B}(X, Y) \xrightarrow{u_Y \times 1_{\mathcal{B}(X, Y)}} \mathcal{B}(Y, Y) \times \mathcal{B}(X, Y)
\]

\[
\begin{array}{ccc}
1 \times \mathcal{B}(X, Y) & \xrightarrow{u_Y \times 1_{\mathcal{B}(X, Y)}} & \mathcal{B}(Y, Y) \times \mathcal{B}(X, Y) \\
\mathcal{B}(X, Y) \xrightarrow{\lambda_{XY}} & & \mathcal{B}(X, Y) \\
\end{array}
\]

and a right unit natural isomorphism

\[
\mathcal{B}(X, Y) \times 1 \xrightarrow{1_{\mathcal{B}(X,Y)} \times u_X} \mathcal{B}(X, Y) \times \mathcal{B}(X, X)
\]

\[
\begin{array}{ccc}
\mathcal{B}(X, Y) \times 1 & \xrightarrow{1_{\mathcal{B}(X,Y)} \times u_X} & \mathcal{B}(X, Y) \times \mathcal{B}(X, X) \\
\mathcal{B}(X, Y) \xrightarrow{\rho_{XY}} & & \mathcal{B}(X, Y) \\
\end{array}
\]

called respectively the left unitor and the right unitor.
(7) the associativity coherence diagram (pentagon identity)

\[
\begin{array}{c}
\alpha_{YXZW} \\
\alpha_{XZWV}
\end{array}
\]

is commutative for every quintuple of 0-cells \(X, Y, Z, W, V \in \mathcal{B}_0\) and every quadruple of 1-cells \(f \in \mathcal{B}(X, Y), g \in \mathcal{B}(Y, Z), h \in \mathcal{B}(Z, W), k \in \mathcal{B}(W, V)\);

(8) the unit coherence diagram (triangle identity)

\[
\begin{array}{c}
g \circ (1_Y \circ f) \\
g \circ f
\end{array}
\]

is commutative for every triple of 0-cells \(X, Y, Z \in \mathcal{B}_0\) and every pair of 1-cells \(f \in \mathcal{B}(X, Y), g \in \mathcal{B}(Y, Z)\).

Example 2.4.2. (1) Each (strict) 2-category is a bicategory in which the associators and the unitors are identities.

(2) A monoidal category is a bicategory with only one object.

(3) [Ros96] A quantaloid is a bicategory (a 2-category, indeed) in which there is at most one 2-cell between each pair of 1-cells.

As stated in the above example, a bicategory can be viewed as a many-object monoidal category. Thus, the theory of categories enriched over a monoidal category can be promoted further to categories enriched over a bicategory [BC82, BCSW83, GP97, Lei02, Wal81].
Definition 2.4.3. Let \( \mathcal{B} \) be a bicategory. A \( \mathcal{B} \)-typed set is a set \( A \) together with a mapping \( t : A \to B_0 \) sending each elements \( x \in A \) to its type \( tx \in B_0 \).

Definition 2.4.4. [BC82, BCSW83, Wal81] Let \( \mathcal{B} \) be a bicategory. A (small) \( \mathcal{B} \)-category (or category enriched over \( \mathcal{B} \)) \( A \) consists of the following data:

1. a \( \mathcal{B} \)-typed set \( A_0 \) of objects;
2. for every pair \( x, y \in A_0 \), a 1-cell \( A(x, y) \in \mathcal{B}(tx, ty) \) as the hom-arrow;
3. for every triple \( x, y, z \in A_0 \), a 2-cell in \( \mathcal{B}(tx, tz) \)
   \[
   c_{xyz} : A(y, z) \circ A(x, y) \to A(x, z)
   \]
   as the composition;
4. for every \( x \in A_0 \), a 2-cell in \( \mathcal{B}(tx, tx) \)
   \[
   u_x : 1_{tx} \to A(x, x)
   \]
   as the unit;
5. the diagram expressing the associativity law
   \[
   \begin{array}{ccc}
   A(z, w) \circ (A(y, z) \circ A(x, y)) & \to & A(z, w) \circ A(x, z) \\
   \downarrow_{c_{tyw} \circ A(x, y)} & & \downarrow_{c_{txz}} \\
   (A(z, w) \circ A(y, z)) \circ A(x, y) & \to & A(x, w) \\
   \end{array}
   \]
   is commutative for every quadruple \( x, y, z, w \in A_0 \);
(6) the diagrams expressing the unity law

\[
\begin{array}{c}
1_{ty} \circ A(x, y) \\
\downarrow \lambda_{tx, ty} \\
A(x, y)
\end{array}
\]

\[
\begin{array}{c}
A(x, y) \circ 1_{tx} \\
\downarrow \rho_{tx, ty} \\
A(x, y)
\end{array}
\]

are commutative for every pair \(x, y \in A_0\).

**Definition 2.4.5.** [BC82] A \(B\)-functor \(F : A \to B\) between \(B\)-categories consists of the following data:

1. a type-preserving map \(F : A_0 \to B_0\) in the sense that \(\forall x \in A_0, tx = t(Fx)\);
2. for every pair \(x, x' \in A_0\), a 2-cell in \(B(tx, tx')\)

\[
F_{xx'} : A(x, x') \to B(Fx,Fx')
\]

3. the diagram expressing the composition law

\[
\begin{array}{c}
A(x', x'') \circ A(x, x') \\
\downarrow F_{x'x''} \circ F_{xx'} \\
B(Fx', Fx'') \circ B(Fx, Fx')
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{(ca)_{xx,x''}} A(x, x'') \\
\downarrow F_{xx''} \\
B(Fx', Fx'') \circ B(Fx, Fx')
\end{array}
\]

is commutative for every triple \(x, x', x'' \in A_0\);

4. the diagram expressing the unity law

\[
\begin{array}{c}
1_{tx} \\
\downarrow (u_{A})_{tx} \\
A(x, x)
\end{array}
\]

\[
\begin{array}{c}
A(x, x) \\
\downarrow (u_{B})_{Fx} \\
B(Fx, Fx)
\end{array}
\]
is commutative for every $x \in A_0$.

**Definition 2.4.6.** [BCS2] A $B$-distributor (also called $B$-bimodule or $B$-profunctor) $\phi : A \leftrightarrow B$ between $B$-categories consists of the following data:

1. for every pair $x \in A_0$ and $y \in B_0$, a 1-cell in $\phi(x, y) \in B(tx, ty)$;

2. for every triple $x \in A_0$ and $y, y' \in B_0$, a 2-cell in $B(tx, ty)$

   \[
   \phi_{x'y'y} : B(y', y) \circ \phi(x, y') \to \phi(x, y);
   \]

3. for every triple $x, x' \in A_0$ and $y \in B_0$, a 2-cell in $B(tx, ty)$

   \[
   \phi_{xx'y} : \phi(x', y) \circ A(x, x') \to \phi(x, y);
   \]

4. the diagrams expressing the associativity law

\[
\begin{array}{ccc}
B(y'', y) \circ B(y', y'') \circ \phi(x, y') & \xrightarrow{B(y'', y) \circ \phi_{x'y'y''}} & B(y'', y) \circ \phi(x, y'') \\
\downarrow{(c_\phi)_{y'y''y'} \circ \phi(x, y')} & & \downarrow{\phi_{x'y'y'}} \\
B(y', y) \circ \phi(x, y') & \xrightarrow{\phi_{x'y'y}} & \phi(x, y)
\end{array}
\]

\[
\begin{array}{ccc}
\phi(x'', y) \circ A(x', x'') \circ A(x, x') & \xrightarrow{\phi_{x''y''y} \circ A(x, x')} & \phi(x', y) \circ A(x, x') \\
\downarrow{\phi(x'', y) \circ (c_\phi)_{xx'y''}} & & \downarrow{\phi_{xx'y'}} \\
\phi(x'', y) \circ A(x, x'') & \xrightarrow{\phi_{xx'y'}} & \phi(x, y)
\end{array}
\]

\[
\begin{array}{ccc}
B(y', y) \circ \phi(x', y') \circ A(x, x') & \xrightarrow{B(y', y) \circ \phi_{xx'y'}} & B(y', y) \circ \phi(x, y') \\
\downarrow{\phi_{x'y'y} \circ A(x, x')} & & \downarrow{\phi_{xy'y}} \\
\phi(x', y) \circ A(x, x') & \xrightarrow{\phi_{xx'y'}} & \phi(x, y)
\end{array}
\]

are commutative for every sextuple $x, x', x'' \in A_0$ and $y, y', y'' \in B_0$;
(6) the diagrams expressing the unity law

\[
\begin{align*}
1_{ty} \circ \phi(x, y) & \xrightarrow{(u_{xy})_y \circ \phi(x, y)} B(y, y) \circ \phi(x, y) \\
& \xrightarrow{\lambda_{ty, ty}} \phi(x, y) \\
& \xrightarrow{\phi_{xyy}} B(y, y) \circ \phi(x, y)
\end{align*}
\]

\[
\begin{align*}
\phi(x, y) \circ 1_{tx} & \xrightarrow{\phi(x, y) \circ (u_{tx})_x} \phi(x, y) \circ \Delta(x, x) \\
& \xrightarrow{\rho_{tx, ty}} \phi(x, y) \\
& \xrightarrow{\phi_{xxy}} \phi(x, y)
\end{align*}
\]

are commutative for every pair \(x \in A_0\) and \(y \in B_0\).

It is easily seen from Definition 2.4.5 that \(B\)-functors can be composed in an obvious way. Thus, \(B\)-categories and \(B\)-functors constitute a category \(B\text{-Cat}\).

So, it is natural to ask that whether \(B\)-distributors have reasonable compositions, and give rise to a category \(B\text{-Dist}\) consisting of \(B\)-categories and \(B\)-distributors.

A straightforward idea comes from the composition of distributors between categories enriched over a symmetric monoidal closed category. Explicitly, the composition \(\psi \circ \phi : A \leftrightarrow C\) of \(B\)-distributors \(\phi : A \leftrightarrow B\) and \(\psi : B \leftrightarrow C\) is given by the following coequalizer diagram

\[
\begin{align*}
\prod_{y, y' \in B_0} \psi(y', z) \circ B(y, y') \circ \phi(x, y) & \xrightarrow{\prod_{y \in B_0} \psi(y, z) \circ \phi(x, y)} \prod_{y \in B_0} (\psi \circ \phi)(x, z)
\end{align*}
\]

for all \(x \in A_0\) and \(z \in C_0\). In particular, the small category \(B(tx, tz)\) admits all set-indexed coproducts if each pair of \(B\)-distributors \(\phi : A \leftrightarrow B\) and \(\psi : B \leftrightarrow C\) can be composed. This implies that \(B(tx, tz)\) is a (cocomplete) preordered set.

Therefore, the category \(B\text{-Dist}\) only makes sense when \(B\) is (cocomplete) locally ordered.

Throughout this dissertation, the composition of distributors is essential for developing our results. Therefore, we consider categories enriched over a quan-
taloid $\mathcal{Q}$, which is a (cocompletely) locally ordered bicategory, instead of categories enriched over a general bicategory $\mathcal{B}$. 
Chapter 3
Quantaloids and $\mathcal{Q}$-categories

In this chapter, we go over the theory of quantaloids and $\mathcal{Q}$-categories, and fix the notations that will be used in the sequel. We refer to [Ros96] for the theory of quantaloids, and [Hey10, HS11, Stu05a, Stu06] for the theory of categories enriched over a quantaloid.

3.1 Quantaloids

Definition 3.1.1. A quantaloid $\mathcal{Q}$ is a category enriched over the symmetric monoidal closed category $\text{Sup}$.

Explicitly, a quantaloid $\mathcal{Q}$ is a category with a class of objects $\mathcal{Q}_0$ such that $\mathcal{Q}(X,Y)$ is a sup-lattice for all $X, Y \in \mathcal{Q}_0$, and the composition $\circ$ of morphisms preserves joins in both variables, i.e.,

$$g \circ \left( \bigvee_i f_i \right) = \bigvee_i (g \circ f_i) \quad \text{and} \quad \left( \bigvee_j g_j \right) \circ f = \bigvee_j (g_j \circ f) \quad (3.1)$$

for all $f, f_i \in \mathcal{Q}(X,Y)$ and $g, g_j \in \mathcal{Q}(Y,Z)$. We denote the top and the bottom element of the sup-lattice $\mathcal{Q}(X,Y)$ by $\top_{X,Y}$ and $\bot_{X,Y}$ respectively, and the identity arrow on $X \in \mathcal{Q}_0$ by $1_X$.

Remark 3.1.2. Quantaloids are the horizontal categorification or oidification [BD98] of quantales. The suffix -oid of the word quantaloid is short for oidified, which means generalizing a certain type of one-object categories to such type of
categories with more than one object. In other words, a quantaloid is a quantale with many objects, while a quantale is a one-object quantaloid. There are other similar examples:

1. The oidification of groups are groupoids, i.e., categories with every morphism invertible.
2. The oidification of rings are ringoids, i.e., categories enriched over the category of abelian groups.
3. The oidification of monoids are just ordinary categories.
4. The oidification of monoidal categories are bicategories.

Throughout this dissertation, $Q$ always denotes a quantaloid, while $Q_0$ and $Q_1$ stand for its class of objects and its class of morphisms, respectively.

**Definition 3.1.3.** For $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : X \rightarrow Z$ in a quantaloid $Q$, define the left implication $h \uparrow f : Y \rightarrow Z$ and the right implication $g \downarrow h : X \rightarrow Y$ by

$$h \uparrow f = \bigvee \{g' : Y \rightarrow Z \mid g' \circ f \leq h\}$$

and

$$g \downarrow h = \bigvee \{f' : X \rightarrow Y \mid g \circ f' \leq h\}.$$ 

In other words, $h \uparrow f$ is the largest arrow in $Q(Y, Z)$ that satisfies $(h \uparrow f) \circ f \leq h$, and $g \downarrow h$ is the largest arrow in $Q(X, Y)$ that satisfies $g \circ (g \downarrow h) \leq h$.

It is easy to see that for each $f \in Q(X, Y)$ and $g \in Q(Y, Z)$, functions

$$- \uparrow f : Q(X, Z) \rightarrow Q(Y, Z) : h \mapsto h \uparrow f,$$
3.1. Quantaloids

\[ g \downarrow \_ : Q(X, Z) \rightarrow Q(X, Y) : h \mapsto g \downarrow h \]

are respective right adjoints of

\[ - \circ f : Q(Y, Z) \rightarrow Q(X, Z) : g \mapsto g \circ f, \]

\[ g \circ - : Q(X, Y) \rightarrow Q(X, Z) : f \mapsto g \circ f. \]

Example 3.1.4. Some basic examples of quantaloids are listed below.

1. A unital quantale is a one-object quantaloid, as stated in Remark 3.1.2. In particular, the two-element Boolean algebra \(2\) is a quantaloid.

2. The category \(\text{Sup}\) is itself a quantaloid, in which sup-preserving maps are endowed with the pointwise order.

3. \([\text{Ros}96]\) The category \(\text{Rel}\) of sets and (binary) relations is a quantaloid, in which

   - the local order between relations is given by the inclusion;
   - for relations \(R \subseteq A \times B\) and \(S \subseteq B \times C\) between sets, the composition \(S \circ R \subseteq A \times C\) is given by
     \[
     S \circ R = \{ (x, z) \mid \exists y \in B, (x, y) \in R \text{ and } (y, z) \in S \};
     \]
   - the identity relation on a set \(A\) is the diagonal relation
     \[
     \Delta_A = \{ (x, x) \mid x \in A \};
     \]
   - for relations \(R \subseteq A \times B, S \subseteq B \times C\) and \(T \subseteq A \times C\), the left implication \(T \nleftarrow R \subseteq B \times C\) and the right implication \(S \nrightarrow T \subseteq A \times B\) are given by
     \[
     T \nleftarrow R = \{ (y, z) \mid \forall x \in A, (x, y) \in R \rightarrow (x, z) \in T \}
     \]
     and
     \[
     S \nrightarrow T = \{ (x, y) \mid \forall z \in C, (y, z) \in S \rightarrow (x, z) \in T \}.\]
In general, each Grothendieck topos \( \mathcal{E} \) gives rise to a quantaloid \( \text{Rel}(\mathcal{E}) \), in which

- the objects are the same as \( \mathcal{E} \);
- A morphism \( \phi \in \text{Rel}(\mathcal{E})(X,Y) \) is a relation from \( X \) to \( Y \), i.e., a subobject \( \phi : R \rightrightarrows X \times Y \);
- the local order is given by the inclusion of subobjects;
- for relations \( \phi : R \rightrightarrows X \times Y \) and \( \psi : S \rightrightarrows Y \times Z \), the composition \( \eta : T \rightrightarrows X \times Z \) is given by the pullback and image factorization in \( \mathcal{E} \):

\[
\begin{array}{c}
X \times Z \\
\downarrow \eta \\
T \\
\downarrow \\
R \times Y S \\
\downarrow \phi \downarrow \psi \\
X \times X Y \times Y \times Z Z \\
\end{array}
\]

- the identity relation on an object \( X \) is the diagonal subobject

\( \Delta_X : X \rightrightarrows X \times X \)

given by the universal property of the product \( X \times X \):

\[
\begin{array}{c}
X \\
\downarrow \Delta_X \\
X \times X \\
\end{array}
\]

In the following two propositions, we list some useful formulas for calculating the compositions and implications of arrows in a quantaloid. They can be obtained by routine calculation and we omit the proof.
3.1. Quantaloids

Proposition 3.1.5. In a quantaloid $Q$, the following properties hold for all $Q$-arrows $f, g, h, f_i, g_i, h_i$ whenever the compositions and implications make sense:

1. $g \circ f \leq h \iff g \leq h \uparrow f \iff f \leq g \downarrow h$.

2. $\bigwedge_i h_i \uparrow f = \bigwedge_i (h_i \uparrow f), g \downarrow \bigwedge_i h_i = \bigwedge_i (g \downarrow h_i)$.

3. $h \uparrow \bigvee_i f_i = \bigwedge_i (h \uparrow f_i), \bigvee_i g_i \downarrow h = \bigwedge_i (g_i \downarrow h)$.

4. $(h \uparrow g) \circ (g \uparrow f) \leq h \uparrow f, (f \downarrow g) \circ (g \downarrow h) \leq f \downarrow h$.

5. $(h \uparrow f) \uparrow g = h \uparrow (g \circ f), f \downarrow (g \downarrow h) = (g \circ f) \downarrow h$.

6. $(g \downarrow h) \uparrow f = g \downarrow (h \uparrow f)$.

7. $(h \uparrow f) \circ f \leq h, g \circ (g \downarrow h) \leq h$.

8. $h \circ (g \uparrow f) \leq (h \circ g) \uparrow f, (g \downarrow h) \circ f \leq g \downarrow (h \circ f)$.

Proposition 3.1.6. In a quantaloid $Q$, the following properties hold for all $Q$-arrows $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : X \rightarrow Z$:

1. $f \uparrow 1_X = 1_Y \downarrow f = f$.

2. $\bot_{Y,Z} \circ f = g \circ \bot_{X,Y} = \bot_{X,Z}$.

3. $h \uparrow \bot_{X,Y} = \top_{Y,Z}, \bot_{Y,Z} \downarrow h = \top_{X,Y}$.

4. $\top_{X,Z} \uparrow f = \top_{Y,Z}, g \downarrow \top_{X,Z} = \top_{X,Y}$.

Definition 3.1.7. An adjunction in a quantaloid $Q$, $f \vdash g : X \rightarrow Y$ in symbols, is a pair of $Q$-arrows $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that

$$1_X \leq g \circ f \quad \text{and} \quad f \circ g \leq 1_Y.$$ 

In this case, $f$ is a left adjoint of $g$ and $g$ is a right adjoint of $f$.

The following two propositions can be derived by straightforward calculation.
Proposition 3.1.8. If \( f \dashv g : X \rightarrow Y \) in a quantaloid \( Q \), then
\[
f \circ g \circ f = f \quad \text{and} \quad g \circ f \circ g = g.
\]

Proposition 3.1.9. If \( f \dashv g : X \rightarrow Y \) in a quantaloid \( Q \), then the following identities hold for all \( Q \)-arrows \( h, h' \) whenever the compositions and implications make sense:

1. \( h \circ f = h \upharpoonright g, g \circ h = f \downharpoonleft h. \)
2. \( (f \circ h) \downharpoonleft h' = h \downharpoonleft (g \circ h'), (h' \circ f) \upharpoonright h = h' \upharpoonright (h \circ g). \)
3. \( (h \downharpoonleft h') \circ f = h \downharpoonleft (h' \circ f), g \circ (h' \upharpoonright h) = (g \circ h') \upharpoonright h. \)
4. \( g \circ (h \downharpoonleft h') = (h \circ f) \downharpoonleft h', (h' \upharpoonright h) \circ f = h' \upharpoonright (g \circ h). \)

The identities in Proposition 3.1.9 will be frequently applied to adjunctions of the form \( F_\sharp \vdash F^\natural : \mathbb{A} \rightarrow \mathbb{B} \), the graph and cograph of a \( Q \)-functor \( F : \mathbb{A} \rightarrow \mathbb{B} \) (will be defined in Section 3.3).

The following corollary follows immediately from Proposition 3.1.9(1).

Corollary 3.1.10. If \( f \dashv g : X \rightarrow Y \) in a quantaloid \( Q \), then
\[
g = f \downharpoonleft 1_Y \quad \text{and} \quad f = 1_X \upharpoonright g.
\]

In particular, the left adjoint and right adjoint of a \( Q \)-arrow are unique when they exist.

Definition 3.1.11. Let \( \mathcal{D} = \{ d_X : X \rightarrow X \mid X \in Q_0 \} \) be a family of \( Q \)-arrows in a quantaloid \( Q \).

1. \( \mathcal{D} \) is called a cyclic family if
\[
d_X \upharpoonright f = f \downharpoonleft d_Y \quad \text{(3.2)}
\]

   for all \( f : X \rightarrow Y \) in \( Q \).
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(2) $\mathcal{D}$ is called a dualizing family if

$$
(d_X \not\geq f) \geq d_X = f = d_Y \not\geq (f \geq d_Y) \quad (3.3)
$$

for all $f : X \to Y$ in $\mathcal{Q}$.

**Definition 3.1.12.** A Girard quantaloid is a quantaloid with a cyclic dualizing family $\mathcal{D}$ of $\mathcal{Q}$-arrows.

**Example 3.1.13.** Some examples of Girard quantaloids are listed below.

1. A Girard quantale [Ros90] is a one-object Girard quantaloid.

2. [Ros92] The quantaloid $\text{Rel}$ of sets and relations (see Example 3.1.4(3)) is a Girard quantaloid. A cyclic dualizing family is given by

$$
\mathcal{D} = \{\neg \Delta_A \subseteq A \times A \mid A \text{ is a set}\},
$$

where

$$
\neg \Delta_A = \{(x, x') \in A \times A \mid x \neq x'\}.
$$

3. In Chapter 7, we will construct a quantaloid $\mathcal{Q}$ from a divisible unital quantale $\mathcal{Q}$ (see Proposition 7.2.2). If $(\mathcal{Q}, \&)$ is a Boolean algebra (with $\&$ being $\land$) [Bir48], then the induced quantaloid $\mathcal{Q}$ is a Girard quantaloid.

**Proposition 3.1.14.** If $1_X$ is the top element of $\mathcal{Q}(X, X)$ for all $X \in \mathcal{Q}_0$ and $\mathcal{D} = \{d_X : X \to X \mid X \in \mathcal{Q}_0\}$ is a dualizing family, then $d_X = \perp_{X,X}$ for all $X \in \mathcal{Q}_0$.

**Proof.** For any $f : X \to X$,

$$
f \not\geq d_X = ((d_X \not\geq f) \geq d_X) \not\geq d_X \quad \text{(Equation 8.3)}
= (d_X \not\geq f) \geq (d_X \not\geq d_X) \quad \text{(Proposition 3.1.5(6))}
= (d_X \not\geq f) \geq 1_X \quad \text{(1_X = \top_{X,X})}
= 1_X \quad \text{(Proposition 3.1.6(4))}
$$

hence $d_X \leq f$ and consequently $d_X = \perp_{X,X}$. \qed
Proposition 3.1.15. Suppose that $Q$ has a dualizing family $\mathcal{D} = \{d_X : X \to X \mid X \in Q_0\}$. For $f, f_i : X \to Y$, $g : Y \to Z$, $h : X \to Z$ in $Q_1$:

1. $\bigvee_i (d_X \nleftarrow f_i) = d_X \nleftarrow (\bigwedge_i f_i)$, $\bigvee_i (f_i \nrightarrow d_Y) = (\bigwedge_i f_i) \nrightarrow d_Y$.
2. $g \circ f = d_Z \nleftarrow (f \nrightarrow (g \nrightarrow d_Z)) = ((d_X \nleftarrow f) \nleftarrow g) \nrightarrow d_Y$.
3. $h \nleftarrow f = (d_X \nleftarrow h) \nrightarrow (d_X \nleftarrow f) = d_Z \nleftarrow (f \circ (h \nrightarrow d_Z))$.
4. $g \nrightarrow h = (g \nrightarrow d_Z) \nleftarrow (h \nrightarrow d_Z) = ((d_X \nleftarrow h) \circ g) \nrightarrow d_Y$.
5. $(d_Y \nleftarrow g) \nrightarrow f = g \nleftarrow (f \nrightarrow d_Y)$.

Proof. All the identities can be obtained by routine calculation. We just prove (5) for example. Note that $(d_Y \nleftarrow g) \nrightarrow f \leq g \nleftarrow (f \nrightarrow d_Y)$ follows from

$$((d_Y \nleftarrow g) \nrightarrow f) \circ (f \nrightarrow d_Y) \leq (d_Y \nleftarrow g) \nrightarrow d_Y = g,$$

and the reverse inequality follows from

$$(d_Y \nleftarrow g) \circ (g \nleftarrow (f \nrightarrow d_Y)) \leq d_Y \nleftarrow (f \nrightarrow d_Y) = f.$$ 

3.2 $Q$-categories and $Q$-functors

A quantaloid $Q$ is a locally ordered bicategory, which means that there is at most one 2-cell between each pair of $Q$-arrows, i.e., the order $\leq$. From now on, $Q$ always denotes a small quantaloid, i.e., $Q_0$ and $Q_1$ are sets.

Definition 3.2.1. A $Q$-typed set is a set $A$ equipped with a map $t : A \to Q_0$ sending each elements $x \in A$ to its type $tx \in Q_0$.

Definition 3.2.2. A (small) $Q$-category (or category enriched over $Q$) $A$ consists of a $Q$-typed set $A_0$ and hom-arrow $A(x,y) \in Q(tx,ty)$ for all $x,y \in A_0$, such that

1. $1_{tx} \leq A(x,x)$ for all $x \in A_0$;
(2) \( A(y, z) \circ A(x, y) \leq A(x, z) \) for all \( x, y, z \in A_0 \).

**Definition 3.2.3.** A \( \mathcal{Q} \)-functor \( F : \mathcal{A} \longrightarrow \mathcal{B} \) between \( \mathcal{Q} \)-categories is a map \( F : A_0 \longrightarrow B_0 \) such that

1. \( F \) is *type-preserving* in the sense that \( \forall x \in A_0, tx = t(Fx) \);
2. \( \forall x, x' \in A_0, A(x, x') \leq B(Fx, Fx') \).

A \( \mathcal{Q} \)-functor \( F : \mathcal{A} \longrightarrow \mathcal{B} \) is *fully faithful* if \( A(x, x') = B(Fx, Fx') \) for all \( x, x' \in A_0 \). Bijective fully faithful \( \mathcal{Q} \)-functors are exactly the isomorphisms in the category \( \mathcal{Q}\text{-Cat} \) of \( \mathcal{Q} \)-categories and \( \mathcal{Q} \)-functors.

**Example 3.2.4.** We list here some basic examples of \( \mathcal{Q} \)-categories.

1. For the two-element Boolean algebra \( 2 \), \( 2 \)-categories are preordered sets, and \( 2 \)-functors are order-preserving maps.
2. Each \( \mathcal{Q} \)-typed set \( A \) gives rise to a *discrete* \( \mathcal{Q} \)-category \( \mathcal{A} \) given by \( A_0 = A \) and
   
   \[
   A(x, y) = \begin{cases} 
   1, & x = y; \\
   \bot, & x \neq y.
   \end{cases}
   \]
3. For each \( X \in \mathcal{Q}_0 \), \( *_X \) is a \( \mathcal{Q} \)-category with only one object \( * \) of type \( t* = X \) and hom-arrow \( 1_X \).
4. Let \( \mathcal{A} \) be a \( \mathcal{Q} \)-category, a \( \mathcal{Q} \)-category \( \mathcal{B} \) is a (full) \( \mathcal{Q} \)-subcategory of \( \mathcal{A} \) if \( B_0 \) is a subset of \( A_0 \) and \( B(x, y) = A(x, y) \) for all \( x, y \in B_0 \). In particular, for each \( \mathcal{Q} \)-functor \( F : \mathcal{A} \longrightarrow \mathcal{B} \), the elements \( \{ y \in B_0 : \exists x \in A_0, Fx = y \} \) is a subset of \( B_0 \) and constitute a \( \mathcal{Q} \)-subcategory \( F(A) \) of \( B \).

Given a \( \mathcal{Q} \)-category \( \mathcal{A} \), there is a natural underlying preorder \( \leq \) on \( A_0 \). For \( x, y \in A_0 \),

\[
x \leq y \iff tx = ty = X \quad \text{and} \quad 1_X \leq A(x, y).
\]

For each \( X \in \mathcal{Q}_0 \), the objects in \( \mathcal{A} \) with type \( X \) constitute a subset of the underlying preordered set \( A_0 \) and we denote it by \( A_X \). It is clear that the underlying preordered set \( A_0 \) is the disjoint union of all \( A_X \), i.e., \( x \leq y \) in \( A_0 \) necessarily implies that \( x \) and \( y \) belong to the same \( A_X \).
Proposition 3.2.5. Let $\mathbb{A}$ be a $\mathbb{Q}$-category and $x, y \in \mathbb{A}_0$, then the following conditions are equivalent.

1. $x \leq y$.
2. $\mathbb{A}(y, z) \leq \mathbb{A}(x, z)$ for all $z \in \mathbb{A}_0$.
3. $\mathbb{A}(z, x) \leq \mathbb{A}(z, y)$ for all $z \in \mathbb{A}_0$.

Proof. We prove the equivalence of (1) and (2) for example.

(1) $\implies$ (2): If $tx = ty = X$ and $\mathbb{A}(x, y) \geq 1_X$, then for all $z \in \mathbb{A}_0$,

$$\mathbb{A}(y, z) \leq \mathbb{A}(y, z) \circ \mathbb{A}(x, y) \leq \mathbb{A}(x, z).$$

(2) $\implies$ (1): In particular, $\mathbb{A}(y, y) \leq \mathbb{A}(x, y)$ implies $tx = ty = X$ and $\mathbb{A}(x, y) \geq 1_X$. $\square$

Two objects $x, y$ in $\mathbb{A}$ are isomorphic if $x \leq y$ and $y \leq x$, written $x \cong y$. $\mathbb{A}$ is skeletal if no two different objects in $\mathbb{A}$ are isomorphic. The following proposition follows immediately from Proposition 3.2.5.

Proposition 3.2.6. Let $\mathbb{A}$ be a $\mathbb{Q}$-category and $x, y \in \mathbb{A}_0$, then the following conditions are equivalent.

1. $x \cong y$.
2. $\mathbb{A}(x, z) = \mathbb{A}(y, z)$ for all $z \in \mathbb{A}_0$.
3. $\mathbb{A}(z, x) = \mathbb{A}(z, y)$ for all $z \in \mathbb{A}_0$.

Corollary 3.2.7. For any $\mathbb{Q}$-category $\mathbb{A}$, the following conditions are equivalent.

1. $\mathbb{A}$ is skeletal.
2. For any $x, y \in \mathbb{A}_0$, $x = y$ if and only if $\mathbb{A}(x, z) = \mathbb{A}(y, z)$ for all $z \in \mathbb{A}_0$.
3. For any $x, y \in \mathbb{A}_0$, $x = y$ if and only if $\mathbb{A}(z, x) = \mathbb{A}(z, y)$ for all $z \in \mathbb{A}_0$. 
3.2. Q-categories and Q-functors

The underlying preorders on Q-categories induce an preorder between Q-functors:

\[ F \leq G : A \rightarrow B \iff \forall x \in A_0, Fx \leq Gx \text{ in } B_0 \]

\[ \iff \forall x \in A_0, 1_{tx} \leq B(Fx,Gx). \]

Thus \( \mathcal{Q}\text{-Cat}(A,B) \) becomes a preordered set, in which \( F \leq G \) is an “enriched natural transformation” from \( F \) to \( G \). This makes \( \mathcal{Q}\text{-Cat} \) a (locally ordered) 2-category. We denote \( F \cong G : A \rightarrow B \) if \( F \leq G \) and \( G \leq F \). Furthermore, \( \mathcal{Q}\text{-Cat}(A,B) \) becomes a poset (partially ordered set) if \( B \) is a skeletal \( Q \)-category.

**Proposition 3.2.8.** Let \( F : A \rightarrow B \) be a fully faithful \( Q \)-functor between \( Q \)-categories. Then \( F : A_0 \rightarrow B_0 \) is essentially injective in the sense that \( x \cong x' \) whenever \( Fx = Fx' \).

**Proof.** Suppose \( x, x' \in A_0 \) and \( Fx = Fx' \), then \( A(x,x') = B(Fx,Fx) \geq 1_{tx} \).

Similarly \( A(x',x) \geq 1_{tx} \), hence \( x \cong x' \).

**Definition 3.2.9.** A pair of \( Q \)-functors \( F : A \rightarrow B \) and \( G : B \rightarrow A \) forms an adjunction, written \( F \dashv G : A \Rightarrow B \), if

\[ 1_A \leq G \circ F \quad \text{and} \quad F \circ G \leq 1_B, \]

where \( 1_A \) and \( 1_B \) are respectively the identity \( Q \)-functors on \( A \) and \( B \). In this case, \( F \) is called a left adjoint of \( G \) and \( G \) a right adjoint of \( F \).

Adjoint \( Q \)-functors can be viewed as adjoint morphisms in the 2-category \( \mathcal{Q}\text{-Cat} \). The following proposition is easily derived from the definition.

**Proposition 3.2.10.** If \( F \dashv G : A \Rightarrow B \) in \( \mathcal{Q}\text{-Cat} \), then

\[ F \circ G \circ F \cong F \quad \text{and} \quad G \circ F \circ G \cong G. \]

We present below another useful characterization of adjoint \( Q \)-functors. It also implies that left adjoints and right adjoints of a \( Q \)-functor, when they exist, are essentially unique, i.e., unique up to isomorphism.
Proposition 3.2.11. Let \( F : \mathbb{A} \rightarrow \mathbb{B} \) and \( G : \mathbb{B} \rightarrow \mathbb{A} \) be type-preserving functions between \( \mathcal{Q} \)-categories (not necessarily \( \mathcal{Q} \)-functors). Then \( F \dashv G : \mathbb{A} \rightarrow \mathbb{B} \) if and only if
\[
\mathbb{B}(Fx, y) = \mathbb{A}(x, Gy)
\]
for all \( x \in \mathbb{A} \) and \( y \in \mathbb{B} \).

Proof. Suppose \( F \dashv G : \mathbb{A} \rightarrow \mathbb{B} \), then \( 1_{\mathbb{A}} \leq G \circ F \) and \( F \circ G \leq 1_{\mathbb{B}} \). For all \( x \in \mathbb{A}_0 \) and \( y \in \mathbb{B}_0 \),
\[
\mathbb{B}(Fx, y) \leq \mathbb{A}(GFx, Gy) \leq \mathbb{A}(GFx, Gy) \circ \mathbb{A}(x, GFx) \leq \mathbb{A}(x, Gy).
\]
Similarly \( \mathbb{A}(x, Gy) \leq \mathbb{B}(Fx, y) \). Hence \( \mathbb{B}(Fx, y) = \mathbb{A}(x, Gy) \).

Conversely, first we show that \( F \) and \( G \) are \( \mathcal{Q} \)-functors. For all \( x, x' \in \mathbb{A}_0 \),
\[
\mathbb{A}(x, x') \leq \mathbb{A}(x', GFx') \circ \mathbb{A}(x, x') \leq \mathbb{A}(x, GFx') = \mathbb{B}(Fx, Fx').
\]
Thus \( F \) is a \( \mathcal{Q} \)-functor. Similarly \( G \) is a \( \mathcal{Q} \)-functor.

Second, for all \( x \in \mathbb{A}_0 \) and \( y \in \mathbb{B}_0 \),
\[
\mathbb{A}(x, GFx) = \mathbb{B}(Fx, Fx) \geq 1_{tx} \quad \text{and} \quad \mathbb{B}(FGy, y) = \mathbb{A}(Gy, Gy) \geq 1_{ty}.
\]
Thus \( 1_{\mathbb{A}} \leq G \circ F \) and \( F \circ G \leq 1_{\mathbb{B}} \). \( \square \)

3.3 \( \mathcal{Q} \)-distributors

\( \mathcal{Q} \)-distributors between \( \mathcal{Q} \)-categories generalize relations between sets.

Definition 3.3.1. A \( \mathcal{Q} \)-distributor (or \( \mathcal{Q} \)-profunctor, \( \mathcal{Q} \)-bimodule) \( \phi : \mathbb{A} \rightleftharpoons \mathbb{B} \) between \( \mathcal{Q} \)-categories is a map \( \mathbb{A}_0 \times \mathbb{B}_0 \rightarrow \mathcal{Q}_1 \) that assigns to each pair \( (x, y) \in \mathbb{A}_0 \times \mathbb{B}_0 \) a \( \mathcal{Q} \)-arrow \( \phi(x, y) \in \mathcal{Q}(tx, ty) \), such that

1. \( \forall x \in \mathbb{A}_0, \forall y, y' \in \mathbb{B}_0, \mathbb{B}(y', y) \circ \phi(x, y') \leq \phi(x, y) \);
2. \( \forall x, x' \in \mathbb{A}_0, \forall y \in \mathbb{B}_0, \phi(x', y) \circ \mathbb{A}(x, x') \leq \phi(x, y) \).

The following proposition is an analogue of Example 3.1.4(3)(4) for \( \mathcal{Q} \)-categories.
Proposition 3.3.2. [Stu05a] $Q$-categories and $Q$-distributors constitute a quantaloid $Q$-$\text{Dist}$ in which

1. the local order is pointwise, i.e., for $Q$-distributors $\phi, \psi : A \rightarrow B$,

$$\phi \leq \psi \iff \forall x \in A_0, \forall y \in B_0, \phi(x,y) \leq \psi(x,y);$$

2. the composition $\psi \circ \phi : A \rightarrow C$ of $Q$-distributors $\phi : A \rightarrow B$ and $\psi : B \rightarrow C$ is given by

$$\forall x \in A_0, \forall z \in C_0, (\psi \circ \phi)(x,z) = \bigvee_{y \in B_0} \psi(y,z) \circ \phi(x,y);$$

3. the identity $Q$-distributor on a $Q$-category $A$ is the hom-arrows of $A$ and will be denoted by $A : A \rightarrow A$;

4. for $Q$-distributors $\phi : A \rightarrow B$, $\psi : B \rightarrow C$ and $\eta : A \rightarrow C$, the left implication $\eta \upharpoonright \phi : B \rightarrow C$ and the right implication $\psi \downharpoonright \eta : A \rightarrow B$ are given by

$$\forall y \in B_0, \forall z \in C_0, (\eta \upharpoonright \phi)(y,z) = \bigwedge_{x \in A_0} \eta(x,z) \upharpoonright \phi(x,y)$$

and

$$\forall x \in A_0, \forall y \in B_0, (\psi \downharpoonright \eta)(x,y) = \bigwedge_{z \in C_0} \psi(y,z) \downharpoonright \eta(x,z).$$

As a special case of Definition 3.1.7, a pair of $Q$-distributors $\phi : A \rightarrow B$ and $\psi : B \rightarrow A$ forms an adjunction $\phi \dashv \psi : A \rightarrow B$ in the quantaloid $Q$-$\text{Dist}$ if

$$A \leq \psi \circ \phi \quad \text{and} \quad \phi \circ \psi \leq B.$$ 

Every $Q$-functor $F : A \rightarrow B$ induces an adjunction $F_\downarrow \dashv F^\uparrow : A \rightarrow B$ in $Q$-$\text{Dist}$ with

$$F_\downarrow(x,y) = B(Fx,y) \quad \text{and} \quad F^\uparrow(y,x) = B(y,Fx)$$

for all $x \in A_0$ and $y \in B_0$. The $Q$-distributors $F_\downarrow : A \rightarrow B$ and $F^\uparrow : B \rightarrow A$ are called the graph and cograph of $F$, respectively.
It follows immediately from Proposition 3.2.5 that for each pair of \( Q \)-functors \( F, G : \mathcal{A} \rightarrow \mathcal{B} \),
\[
F \leq G \iff G^\sharp \leq F^\sharp \iff F^\sharp \leq G^\sharp. \tag{3.4}
\]

**Proposition 3.3.3.** [Stu05a] Let \( F : \mathcal{A} \rightarrow \mathcal{B} \) and \( G : \mathcal{B} \rightarrow \mathcal{C} \) be \( Q \)-functors between \( Q \)-categories.

(1) \( (1_A)^\sharp = (1_A)^\natural = A \).

(2) \( G^\flat \circ F^\sharp = (G \circ F)^\sharp, \ F^\sharp \circ G^\sharp = (G \circ F)^\sharp \).

This proposition gives rise to a functor

\[
(\_)^\sharp : \text{Q-Cat} \rightarrow \text{Q-Dist}
\]
and a contravariant functor

\[
(\_^\natural) : (\text{Q-Cat})^{\text{op}} \rightarrow \text{Q-Dist}.
\]

**Proposition 3.3.4.** [Stu05a] Let \( F : \mathcal{A} \rightarrow \mathcal{B} \) and \( G : \mathcal{B} \rightarrow \mathcal{A} \) be a pair of \( Q \)-functors. The following conditions are equivalent:

(1) \( F \dashv G : \mathcal{A} \rightarrow \mathcal{B} \).

(2) \( F^\sharp \leq G^\sharp \).

(3) \( G^\sharp \dashv F^\sharp : \mathcal{B} \rightarrow \mathcal{A} \) in \( Q \)-\text{Dist}.

(4) \( G^\sharp \dashv F^\sharp : \mathcal{A} \rightarrow \mathcal{B} \) in \( Q \)-\text{Dist}.

*Proof.* The equivalence of (1) and (2) is a reformulation of Proposition 3.2.11. The equivalence of (2), (3) and (4) follows immediately from Corollary 3.1.10. \( \square \)

**Proposition 3.3.5.** Let \( F : \mathcal{A} \rightarrow \mathcal{B} \) be a \( Q \)-functor.

(1) If \( F \) is fully faithful, then \( F^\sharp \circ F^\sharp = \mathcal{A} \).

(2) If \( F \) is essentially surjective in the sense that there is some \( x \in \mathcal{A}_0 \) such that \( Fx \cong y \) in \( \mathcal{B} \) for all \( y \in \mathcal{B}_0 \), then \( F^\sharp \circ F^\sharp = \mathcal{B} \).
3.3. \( \mathbb{Q} \)-distributors

Proof. (1) If \( F \) is fully faithful, then for all \( x, x' \in \mathbb{A}_0 \),
\[
(F^\sharp \circ F^\sharp)(x, x') = \bigvee_{y \in \mathbb{B}_0} \mathbb{B}(y, Fx') \circ \mathbb{B}(Fx, y) = \mathbb{B}(Fx, Fx') = \mathbb{A}(x, x').
\]

(2) If \( F \) is essentially surjective, then for all \( y, y' \in \mathbb{B}_0 \), there is some \( x \in \mathbb{A}_0 \) such that \( Fx \cong y \). Thus
\[
(F_\sharp \circ F^\sharp)(y, y') = \bigvee_{a \in \mathbb{A}_0} \mathbb{B}(Fa, y') \circ \mathbb{B}(y, Fa) \\
\geq \mathbb{B}(Fx, y') \circ \mathbb{B}(y, Fx) \\
= \mathbb{B}(y, y') \circ \mathbb{B}(y, y) \\
\geq \mathbb{B}(y, y').
\]

Since \( F_\sharp \circ F^\sharp \leq \mathbb{B} \) holds trivially, it follows that \( F_\sharp \circ F^\sharp = \mathbb{B} \).

\[\square\]

Definition 3.3.6. (1) A contravariant presheaf on a \( \mathbb{Q} \)-category \( \mathbb{A} \) is a \( \mathbb{Q} \)-distributor \( \mu : \mathbb{A} \to \star_X \) with \( X \in \mathbb{Q}_0 \).

(2) A covariant presheaf on a \( \mathbb{Q} \)-category \( \mathbb{A} \) is a \( \mathbb{Q} \)-distributor \( \mu : \star_X \to \mathbb{A} \) with \( X \in \mathbb{Q}_0 \).

Contravariant presheaves on a \( \mathbb{Q} \)-category \( \mathbb{A} \) constitute a \( \mathbb{Q} \)-category \( \mathcal{P}\mathbb{A}_\mathbb{A} \) in which
\[
t_\mu = X \quad \text{and} \quad \mathcal{P}\mathbb{A}_\mathbb{A}(\mu, \lambda) = \lambda \nLeftarrow \mu \tag{3.5}
\]
for all \( \mu : \mathbb{A} \to \star_X \) and \( \lambda : \mathbb{A} \to \star_Y \) in \( (\mathcal{P}\mathbb{A}_\mathbb{A})_0 \).

Dually, covariant presheaves on \( \mathbb{A} \) constitute a \( \mathbb{Q} \)-category \( \mathcal{P}^\dagger \mathbb{A}_\mathbb{A} \) in which
\[
t_\mu = X \quad \text{and} \quad \mathcal{P}^\dagger \mathbb{A}_\mathbb{A}(\mu, \lambda) = \lambda \nLeftarrow \mu \tag{3.6}
\]
for all \( \mu : \star_X \to \mathbb{A} \) and \( \lambda : \star_Y \to \mathbb{A} \) in \( (\mathcal{P}^\dagger \mathbb{A}_\mathbb{A})_0 \).

In particular, we denote \( \mathcal{P}(\star_X) = \mathcal{P}X \) and \( \mathcal{P}^\dagger(\star_X) = \mathcal{P}^\dagger X \) for each \( X \in \mathbb{Q}_0 \).
Proposition 3.3.7. For each $\mathcal{Q}$-category $\mathcal{A}$, $\mathcal{P}\mathcal{A}$ and $\mathcal{P}^\dagger\mathcal{A}$ are both skeletal $\mathcal{Q}$-categories. In particular, $\mathcal{P}X$ and $\mathcal{P}^\dagger X$ are both skeletal $\mathcal{Q}$-categories for each $X \in \mathcal{Q}_0$.

Remark 3.3.8. For each $\mathcal{Q}$-category $\mathcal{A}$, it follows from the definition that the underlying preorder in $\mathcal{P}\mathcal{A}$ coincides with the local order in $\mathcal{Q}\text{-Dist}$, while the underlying preorder in $\mathcal{P}^\dagger\mathcal{A}$ is the reverse local order in $\mathcal{Q}\text{-Dist}$. That is to say, for all $\mu, \lambda \in \mathcal{P}^\dagger\mathcal{A}$, we have

$$\mu \leq \lambda \text{ in } (\mathcal{P}^\dagger\mathcal{A})_0 \iff \lambda \leq \mu \text{ in } \mathcal{Q}\text{-Dist}.$$  

In order to get rid of the confusion about the symbol $\leq$, from now on we make the convention that the symbol $\leq$ between $\mathcal{Q}$-distributors always denotes the local order in $\mathcal{Q}\text{-Dist}$ if not otherwise specified.

Given a $\mathcal{Q}$-category $\mathcal{A}$ and $a \in \mathcal{A}_0$, write $Y_a$ for the $\mathcal{Q}$-distributor

$$\mathcal{A} \leftrightarrow *_{ta}, \quad x \mapsto \mathcal{A}(x, a);$$  

write $Y^\dagger a$ for the $\mathcal{Q}$-distributor

$$*_{ta} \leftrightarrow \mathcal{A}, \quad x \mapsto \mathcal{A}(a, x).$$

The following lemma implies that both $Y : \mathcal{A} \to \mathcal{P}\mathcal{A}, a \mapsto Y_a$ and $Y^\dagger : \mathcal{A} \to \mathcal{P}^\dagger\mathcal{A}, a \mapsto Y^\dagger a$ are fully faithful $\mathcal{Q}$-functors (hence embeddings if $\mathcal{A}$ is skeletal). Thus, $Y$ and $Y^\dagger$ are called respectively the Yoneda embedding and the co-Yoneda embedding.

Lemma 3.3.9 (Yoneda). [Stu05a] For all $a \in \mathcal{A}_0$, $\mu \in \mathcal{P}\mathcal{A}$ and $\lambda \in \mathcal{P}^\dagger\mathcal{A}$,

$$\mathcal{P}\mathcal{A}(Y_a, \mu) = \mu(a) \quad \text{and} \quad \mathcal{P}^\dagger\mathcal{A}(\lambda, Y^\dagger a) = \lambda(a).$$

For each $\mathcal{Q}$-distributor $\phi : \mathcal{A} \to \mathcal{B}$ and $x \in \mathcal{A}_0, y \in \mathcal{B}_0$, write $\phi(x, -)$ for the $\mathcal{Q}$-distributor $\phi \circ Y^\dagger_a x : *_{tx} \to \mathcal{A} \to \mathcal{B}$; and write $\phi(-, y)$ for the $\mathcal{Q}$-distributor $Y_B y \circ \phi : \mathcal{A} \to \mathcal{B} \to *_{ty}$. Then the Yoneda lemma can be phrased as the
commutativity of the following diagrams:

\[
\begin{align*}
\mathcal{P} A \xrightarrow{\mathcal{P} \Delta(-, \mu)} \ast_{t \mu} \\
\mu & \downarrow \\
A & \xrightarrow{Y_z} & Y_z(-, \mu)
\end{align*}
\]

That is,

\[
\mu = \mathcal{P} A(Y-, \mu) = Y_z(-, \mu)
\]

and

\[
\lambda = \mathcal{P} \uparrow A(\lambda, Y\uparrow -) = (Y\uparrow)^2(\lambda, -).
\]

**Remark 3.3.10.** Given \(Q\)-distributors \(\phi : A \rightharpoonup B\), \(\psi : B \rightharpoonup C\) and \(\eta : A \rightharpoonup C\), one can form \(Q\)-distributors such as \(\phi(x, -), \eta \updownarrow \phi, \eta \downarrow \psi(y, -), \) etc. We list here some basic formulas related to these \(Q\)-distributors that will be used in the sequel.

1. \(\forall x \in A_0, \forall z \in C_0, (\psi \circ \phi)(x, z) = \psi(-, z) \circ \phi(x, -);\)

2. \(\forall y \in A_0, \forall z \in C_0, (\eta \rhd \phi)(y, z) = \eta(-, z) \rhd \phi(-, y);\)

3. \(\forall x \in A_0, \forall y \in B_0, (\psi \downarrow \eta)(x, y) = \psi(y, -) \downarrow \eta(x, -);\)

4. \(\forall x \in A_0, (\psi \circ \phi)(x, -) = \psi \circ \phi(x, -);\)

5. \(\forall z \in C_0, (\psi \circ \phi)(-z) = \psi(-, z) \circ \phi;\)

6. \(\forall y \in B_0, (\eta \rhd \phi)(y, -) = \eta \rhd \phi(-, y);\)

7. \(\forall z \in C_0, (\eta \rhd \phi)(-z) = \eta(-, z) \rhd \phi;\)

8. \(\forall y \in B_0, (\psi \downarrow \eta)(-y) = \psi(y, -) \downarrow \eta;\)

9. \(\forall x \in A_0, (\psi \downarrow \eta)(x, -) = \psi \downarrow \eta(x, -).\)

As an application of Remark 3.3.10, we derive some useful formulas for graphs and cographs of \(Q\)-functors.
Proposition 3.3.11. Let \( F : \mathbb{A} \rightarrow \mathbb{B} \) be a \( \mathcal{Q} \)-functor and \( \phi : \mathbb{B} \rightarrow \mathbb{C} \), \( \psi : \mathbb{C} \rightarrow \mathbb{B} \) be \( \mathcal{Q} \)-distributors. Then

1. \( \phi \circ F_{\sharp} = \phi(F-,-) = \phi \triangleright F^\sharp \);

2. \( F^\sharp \circ \psi = \psi(-,F-) = F_{\flat} \downarrow \psi \).

Proof. We prove (1) for example. Although \( \phi \circ F_{\sharp} = \phi \triangleright F^\sharp \) follows immediately from Proposition 3.1.9(1), we present here a proof that they are respectively equal to \( \phi(F-,-) \) as an illustration for Remark 3.3.10. For each \( x \in \mathbb{A}_0 \) and \( z \in \mathbb{C}_0 \),

\[
(\phi \circ F_{\sharp})(x,z) = \phi(-,z) \circ F_{\sharp}(x,-) = (\phi(-,z) \circ \mathbb{B})(Fx,-) = \phi(-,z)(Fx,-) = \phi(Fx,z) = (\phi \triangleright \mathbb{B})(Fx,z)
\]

(Remark 3.3.10(1))

\[
= \phi(-,z) \triangleright \mathbb{B}(-,Fx) = \phi(-,z) \triangleright F^\sharp(-,x) = (\phi \triangleright F^\sharp)(x,z).
\]

(Remark 3.3.10(2))

Thus \( \phi \circ F_{\sharp} = \phi(F-,-) = \phi \triangleright F^\sharp \).

Definition 3.3.12. Let \( F : \mathbb{A} \rightarrow \mathbb{B} \) be a \( \mathcal{Q} \)-functor.

1. The contravariant direct image \( \mathcal{Q} \)-functor of \( F \) is a \( \mathcal{Q} \)-functor \( F^\rightarrow : \mathcal{P}\mathbb{A} \rightarrow \mathcal{P}\mathbb{B} \) between the \( \mathcal{Q} \)-categories of contravariant presheaves given by

\[
F^\rightarrow(\mu) = \mu \circ F^\sharp.
\]

2. The contravariant inverse image \( \mathcal{Q} \)-functor of \( F \) is a \( \mathcal{Q} \)-functor \( F^\leftarrow : \mathcal{P}\mathbb{B} \rightarrow \mathcal{P}\mathbb{A} \) between the \( \mathcal{Q} \)-categories of contravariant presheaves given by

\[
F^\leftarrow(\lambda) = \lambda \circ F_{\flat}.
\]
(3) The covariant direct image $Q$-functor of $F$ is a $Q$-functor $F^\rightarrow: \mathcal{P}^\uparrow A \rightarrow \mathcal{P}^\uparrow B$ between the $Q$-categories of covariant presheaves given by

$$F^\rightarrow(\mu) = F^\sharp \circ \mu.$$ 

(4) The covariant inverse image $Q$-functor of $F$ is a $Q$-functor $F^\leftarrow: \mathcal{P}^\uparrow B \rightarrow \mathcal{P}^\uparrow A$ between the $Q$-categories of covariant presheaves given by

$$F^\leftarrow(\lambda) = F^\flat \circ \lambda.$$ 

We illustrate Definition 3.3.12 through the following commutative diagrams:

![Diagrams](image)

Proposition 3.3.13. Let $F: A \rightarrow B$ be a $Q$-functor.

(1) If $F$ is fully faithful, then

(i) $F^\rightarrow: \mathcal{P}A \rightarrow \mathcal{P}B$ and $F^\rightarrow: \mathcal{P}^\uparrow A \rightarrow \mathcal{P}^\uparrow B$ are fully faithful and injective.

(ii) $F^\leftarrow: \mathcal{P}B \rightarrow \mathcal{P}A$ and $F^\leftarrow: \mathcal{P}^\uparrow B \rightarrow \mathcal{P}^\uparrow A$ are surjective.

(2) If $F$ is essentially surjective, then

(i) $F^\rightarrow: \mathcal{P}A \rightarrow \mathcal{P}B$ and $F^\rightarrow: \mathcal{P}^\uparrow A \rightarrow \mathcal{P}^\uparrow B$ are surjective.

(ii) $F^\leftarrow: \mathcal{P}B \rightarrow \mathcal{P}A$ and $F^\leftarrow: \mathcal{P}^\uparrow B \rightarrow \mathcal{P}^\uparrow A$ are fully faithful and injective.

Proof. We prove (1) for example. If $F$ is fully faithful, then it follows from Proposition 3.3.5(1) that $F^\sharp \circ F^\sharp = A$. 

(i) For all $\mu, \mu' \in \mathcal{P}A$,

\[
\mathcal{P}B(F^{\to} (\mu), F^{\to} (\mu')) = F^{\to} (\mu') \not\subseteq F^{\to} (\mu) \quad \text{(Equation (3.5))}
\]

\[
= (\mu' \circ F^\sharp) \not\subseteq (\mu \circ F^\sharp) \quad \text{(Definition 3.3.12)}
\]

\[
= (\mu' \circ F^\sharp \circ F_\sharp) \not\subseteq \mu \quad \text{(Proposition 3.1.9(2))}
\]

\[
= \mu' \not\subseteq \mu \quad \text{(Proposition 3.3.5(1))}
\]

\[
= \mathcal{P}A(\mu, \mu'). \quad \text{(Equation (3.5))}
\]

Thus $F^{\to} : \mathcal{P}A \to \mathcal{P}B$ is fully faithful. Since $\mathcal{P}A$ is skeletal, by Proposition 3.2.8 we obtain that $F^{\to}$ is injective. Similarly $F^{\leftarrow} : \mathcal{P}A \to \mathcal{P}B$ is fully faithful and injective.

(ii) For all $\mu \in \mathcal{P}A$,

\[
F^{\leftarrow} \circ F^{\to} (\mu) = \mu \circ F^\sharp \circ F_\sharp = \mu \circ A = \mu.
\]

Thus $F^{\leftarrow} : \mathcal{P}B \to \mathcal{P}A$ is surjective. Similarly $F^{\leftarrow} : \mathcal{P}B \to \mathcal{P}A$ is surjective.

**Proposition 3.3.14.** For each $\mathcal{Q}$-functor $F : A \to B$,

\[
F^{\to} \dashv F^{\leftarrow} : \mathcal{P}A \to \mathcal{P}B
\]

and

\[
F^{\leftarrow} \dashv F^{\to} : \mathcal{P}B \to \mathcal{P}A.
\]

**Proof.** For all $\mu \in \mathcal{P}A$, $\lambda \in \mathcal{P}B$,

\[
\mathcal{P}B(F^{\to} (\mu), \lambda) = \lambda \not\subseteq (\mu \circ F^\sharp) \quad \text{(Formula (3.5))}
\]

\[
= (\lambda \circ F_\sharp) \not\subseteq \mu \quad \text{(Proposition 3.1.9(2))}
\]

\[
= \mathcal{P}A(\mu, F^{\leftarrow} (\lambda)). \quad \text{(Formula (3.5))}
\]

Thus $F^{\to} \dashv F^{\leftarrow} : \mathcal{P}A \to \mathcal{P}B$. That $F^{\leftarrow} \dashv F^{\to} : \mathcal{P}B \to \mathcal{P}A$ can be deduced similarly.

The following proposition is an immediate consequence of Proposition 3.3.11.

**Proposition 3.3.15.** Let $F : A \to B$ be a $\mathcal{Q}$-functor.
(1) For all $\lambda \in \mathcal{P}B$ and $x \in \mathcal{A}_0$,

$$F^\leftarrow(\lambda)(x) = \lambda(Fx) \in Q(tx, t\lambda).$$

(2) For all $\lambda \in \mathcal{P}^\dagger B$ and $x \in \mathcal{A}_0$,

$$F^\rightarrow(\lambda)(x) = \lambda(Fx) \in Q(t\lambda, tx).$$

We would like to stress that

$$\mu \leq F^\leftarrow \circ F^\rightarrow(\mu) \quad \text{and} \quad F^\rightarrow \circ F^\leftarrow(\lambda) \leq \lambda$$

(3.9)

for all $\mu \in \mathcal{P}A$ and $\lambda \in \mathcal{P}B$; whereas

$$\nu \leq F^\rightarrow \circ F^\leftarrow(\nu) \quad \text{and} \quad F^\leftarrow \circ F^\rightarrow(\gamma) \leq \gamma$$

(3.10)

for all $\nu \in \mathcal{P}^\dagger A$ and $\gamma \in \mathcal{P}^\dagger B$ by Remark 3.3.8.

**Proposition 3.3.16.** Let $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ be $Q$-functors between $Q$-categories.

(1) $G^\rightarrow \circ F^\rightarrow = (G \circ F)^\rightarrow : \mathcal{P}A \rightarrow \mathcal{P}C$.

(2) $F^\leftarrow \circ G^\leftarrow = (G \circ F)^\leftarrow : \mathcal{P}C \rightarrow \mathcal{P}A$.

(3) $G^\rightarrow \circ F^\leftarrow = (G \circ F)^\rightarrow : \mathcal{P}^\dagger A \rightarrow \mathcal{P}^\dagger C$.

(4) $F^\leftarrow \circ G^\rightarrow = (G \circ F)^\leftarrow : \mathcal{P}^\dagger C \rightarrow \mathcal{P}^\dagger A$.

**Proof.** Straightforward calculation by help of Proposition 3.3.3. \qed

This proposition gives rise to two functors and two contravariant functors:

(1) $\mathcal{P} : \mathcal{Q}\text{-Cat} \rightarrow \mathcal{Q}\text{-Cat}$ that sends a $Q$-functor $F : \mathcal{A} \rightarrow \mathcal{B}$ to its contravariant direct image $Q$-functor $F^\leftarrow : \mathcal{P}A \rightarrow \mathcal{P}B$;

(2) $\mathcal{P}^\dagger : \mathcal{Q}\text{-Cat} \rightarrow \mathcal{Q}\text{-Cat}$ that sends a $Q$-functor $F : \mathcal{A} \rightarrow \mathcal{B}$ to its covariant direct image $Q$-functor $F^\rightarrow : \mathcal{P}^\dagger A \rightarrow \mathcal{P}^\dagger B$;
(3) \( \mathcal{P}^{\text{op}} : (\mathcal{Q}\text{-Cat})^{\text{op}} \rightarrow \mathcal{Q}\text{-Cat} \) that sends a \( \mathcal{Q} \)-functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) to its contravariant inverse image \( \mathcal{Q} \)-functor \( F^{-} : \mathcal{P}\mathcal{B} \rightarrow \mathcal{P}\mathcal{A} \); and

(4) \( (\mathcal{P}^{\dagger})^{\text{op}} : (\mathcal{Q}\text{-Cat})^{\text{op}} \rightarrow \mathcal{Q}\text{-Cat} \) that sends a \( \mathcal{Q} \)-functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) to its covariant inverse image \( \mathcal{Q} \)-functor \( F^{\dagger} : \mathcal{P}^{\dagger}\mathcal{B} \rightarrow \mathcal{P}^{\dagger}\mathcal{A} \).

**Proposition 3.3.17.** Let \( \mathcal{Y} \) and \( \mathcal{Y}^{\dagger} \) be functions that assign to each \( \mathcal{Q} \)-category \( \mathcal{A} \), respectively, the Yoneda embedding \( \mathcal{Y}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{P}\mathcal{A} \) and the co-Yoneda embedding \( \mathcal{Y}^{\dagger}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{P}^{\dagger}\mathcal{A} \).

(1) \( \mathcal{Y} = \{ \mathcal{Y}_{\mathcal{A}} \} \) is a natural transformation from the identity functor on \( \mathcal{Q}\text{-Cat} \) to \( \mathcal{P} \).

(2) \( \mathcal{Y}^{\dagger} = \{ \mathcal{Y}^{\dagger}_{\mathcal{A}} \} \) is a natural transformation from the identity functor on \( \mathcal{Q}\text{-Cat} \) to \( \mathcal{P}^{\dagger} \).

**Proof.** (1) We show that the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
\downarrow \mathcal{Y}_{\mathcal{A}} & & \downarrow \mathcal{Y}_{\mathcal{B}} \\
\mathcal{P}\mathcal{A} & \xrightarrow{F^{-}} & \mathcal{P}\mathcal{B}
\end{array}
\]

is commutative for each \( \mathcal{Q} \)-functor \( F : \mathcal{A} \rightarrow \mathcal{B} \). Indeed, for all \( x \in \mathcal{A}_{0} \),

\[
\mathcal{Y}_{\mathcal{B}} \circ Fx = \mathcal{B}(\mathcal{A}, Fx) \quad \text{(Formula (3.7))}
\]

\[
= \mathcal{F}^{\mathcal{B}}(\mathcal{A}, x) \quad \text{(Definition of \( \mathcal{F}^{\mathcal{B}} \))}
\]

\[
= (\mathcal{A} \circ \mathcal{F}^{\mathcal{B}})(\mathcal{A}, x) \quad \text{(Remark 3.3.10(5))}
\]

\[
= \mathcal{A}^{-} \circ \mathcal{Y}_{\mathcal{A}} x. \quad \text{(Definition 3.3.12)}
\]

(2) Similar to (1), one can prove that the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
\downarrow \mathcal{Y}_{\mathcal{A}}^{\dagger} & & \downarrow \mathcal{Y}_{\mathcal{B}}^{\dagger} \\
\mathcal{P}^{\dagger}\mathcal{A} & \xrightarrow{F^{-}} & \mathcal{P}^{\dagger}\mathcal{B}
\end{array}
\]
is commutative for each $\mathbb{Q}$-functor $F : \mathbb{A} \rightarrow \mathbb{B}$.

3.4 Weighted colimits and limits

In order to describe the cocompleteness and completeness of $\mathbb{Q}$-categories, we first define weighted colimits and limits in $\mathbb{Q}$-categories.

**Definition 3.4.1.** Let $F : \mathbb{A} \rightarrow \mathbb{B}$ be a $\mathbb{Q}$-functor.

1. The *colimit* of $F$ weighted by a contravariant presheaf $\mu \in \mathcal{P}_A$ is an object $\text{colim}_\mu F \in \mathbb{B}_0$ (necessarily of type $t\mu$) such that

$$\mathbb{B}(\text{colim}_\mu F, -) = F_\sharp \sqcup \mu.$$ 

A $\mathbb{Q}$-category $\mathbb{B}$ is *cocomplete* if $\text{colim}_\mu F$ exists for each $\mathbb{Q}$-functor $F : \mathbb{A} \rightarrow \mathbb{B}$ and $\mu \in \mathcal{P}_A$.

2. The *limit* of $F$ weighted by a covariant presheaf $\lambda \in \mathcal{P}^\dagger_A$ is an object $\text{lim}_\lambda F \in \mathbb{B}_0$ (necessarily of type $t\lambda$) such that

$$\mathbb{B}(-, \text{lim}_\lambda F) = \lambda \sqcap F_\sharp.$$ 

A $\mathbb{Q}$-category $\mathbb{B}$ is *complete* if $\text{lim}_\lambda F$ exists for each $\mathbb{Q}$-functor $F : \mathbb{A} \rightarrow \mathbb{B}$ and $\lambda \in \mathcal{P}^\dagger_A$.

**Remark 3.4.2.** Let $F : \mathbb{A} \rightarrow \mathbb{B}$ be a $\mathbb{Q}$-functor. The definition of $\text{colim}_\mu F$ for some $\mu \in \mathcal{P}_A$ can be extended to the colimit of $F$ weighted by a $\mathbb{Q}$-distributor $\phi : \mathbb{A} \rightarrow \mathbb{C}$ as in [Stu05a]. Explicitly, a $\mathbb{Q}$-functor $G : \mathbb{C} \rightarrow \mathbb{B}$ is the colimit of $F$ weighted by $\phi : \mathbb{A} \rightarrow \mathbb{C}$, denoted by $\text{colim}_\phi F$, if

$$G_\sharp = F_\sharp \sqcup \phi.$$ (3.11)

If $\mathbb{C}$ is a one-object $\mathbb{Q}$-category, then $\phi$ is a contravariant presheaf and $G$ picks out an object $\text{colim}_\phi F \in \mathbb{B}_0$, as formulated in Definition 3.4.1.
Conversely, if \( G = \text{colim}_\phi F \), then for each \( z \in C_0 \), \( \phi(-, z) : A \to *_{tz} \) is a contravariant presheaf, and
\[
\mathbb{B}(Gz, -) = G_z(z, -) \quad \text{(Definition of } G_z) \quad \text{and} \\
= F_z \vee \phi(-, z). \quad \text{(Remark 3.3.10(6))}
\]
Thus \( Gz = \text{colim}_{\phi(-, z)} F \). This means that the colimit of a \( \mathbb{Q} \)-functor weighted by a \( \mathbb{Q} \)-distributor can be obtained from Definition 3.4.1 via pointwise calculation.

Dually, given a \( \mathbb{Q} \)-functor \( F : A \to B \) and a \( \mathbb{Q} \)-distributor \( \psi : C \to A \), the limit of \( F \) weighted by \( \psi \), denoted by \( \lim_{\psi} F \), is a \( \mathbb{Q} \)-functor \( G : C \to B \) such that
\[
G^\psi = \psi \searrow F^\psi. \tag{3.12}
\]
Similarly, for each \( z \in C_0 \), \( Gz = \lim_{\psi(z, -)} F \).

As illustrated in the following diagrams, \( (\text{colim}_\phi F)_z \) is the largest \( \mathbb{Q} \)-distributor that satisfies
\[
(\text{colim}_\phi F)_z \circ \phi \leq F_z,
\]
and \( (\lim_{\psi} F)^\psi \) is the largest \( \mathbb{Q} \)-distributor that satisfies
\[
\psi \circ (\lim_{\psi} F)^\psi \leq F^\psi.
\]

This description applies to Definition 3.4.1 if \( A \) is replaced by a one-object \( \mathbb{Q} \)-category.

**Example 3.4.3.** Let \( A \) be a \( \mathbb{Q} \)-category and \( \mu \in \mathcal{P}A \), the \( \mathbb{Q} \)-distributor
\[
\text{ub} \mu = A \vee \mu
\]
is in \( \mathcal{P}^1A \), called the upper bounds of \( \mu \). \( \text{ub} : \mathcal{P}A \to \mathcal{P}^1A \) is a \( \mathbb{Q} \)-functor, since for all \( \mu, \lambda \in \mathcal{P}^1A \),
\[
\mathcal{P}A(\mu, \lambda) = \lambda \vee \mu \leq (A \vee \lambda) \searrow (A \vee \mu) = \mathcal{P}^1A(\text{ub} \mu, \text{ub} \lambda).
\]
Note that for all $\lambda \in \mathcal{P}^\dagger A$,

\[
\mathcal{P}^\dagger A(\text{ub } \mu, \lambda) = \lambda \searrow \text{ub } \mu \\
= \lambda \searrow (A \searrow \mu) \\
= (\lambda \searrow A) \searrow \mu \\
= (\mathcal{Y}^\dagger)_\ell (-, \lambda) \searrow \mu,
\]

thus $\text{ub } \mu = \text{colim}_\mu \mathcal{Y}^\dagger$.

Dually, let $\lambda \in \mathcal{P}^\dagger A$, the $Q$-distributor $\text{lb } \lambda = \lambda \searrow A$ is in $\mathcal{P}A$, called the lower bounds of $\lambda$, and gives a $Q$-functor $\text{lb} : \mathcal{P}^\dagger A \rightarrow \mathcal{P}A$.

Similar calculation leads to $\text{lb } \mu = \lim_\mu Y$.

**Proposition 3.4.4** (Isbell). $\text{Stu05a}$ $\text{ub } \dashv \text{lb} : \mathcal{P}A \Rightarrow \mathcal{P}^\dagger A$ in $Q$-$\text{Cat}$.

**Proof.** For all $\mu \in \mathcal{P}A$ and $\lambda \in \mathcal{P}^\dagger A$,

\[
\mathcal{P}^\dagger A(\text{ub } \mu, \lambda) = \lambda \searrow \text{ub } \mu \\
= \lambda \searrow (A \searrow \mu) \\
= (\lambda \searrow A) \searrow \mu \\
= \text{lb } \lambda \searrow \mu \\
= \mathcal{P}A(\mu, \text{lb } \lambda). \quad \text{(Equation 3.5)}
\]

Hence the conclusion holds.

**Proposition 3.4.4** presents the Isbell adjunction

\[
A \searrow (-) \dashv (-) \searrow A : \mathcal{P}A \Rightarrow \mathcal{P}^\dagger A
\]

in $Q$-categories. We will extend the adjunction to a generalized version in Chapter 5 by replacing the identity $Q$-distributor $A$ with a general $Q$-distributor $\phi$. 
Example 3.4.5. Let $\mathbb{A}$ be a $\mathcal{Q}$-category and $\mu \in \mathcal{P} \mathbb{A}$. If $\text{ub} \mu$ is represented by some $a \in \mathbb{A}_0$, i.e.,

$$\text{ub} \mu = \mathbb{A}(a, -),$$

we say that $a$ is a supremum of $\mu$, and denote it by $\text{sup} \mu$. Note that

$$\mathbb{A}(\text{sup} \mu, -) = \mathbb{A}(\text{sup} \mu) = (1_{\mathbb{A}}) \backslash \mu,$$

thus $\text{sup} \mu = \text{colim}_\mu 1_{\mathbb{A}}$.

Dually, let $\lambda \in \mathcal{P}^\dagger \mathbb{A}$, if $\text{lb} \lambda$ is represented by some $b \in \mathbb{A}_0$, i.e.,

$$\text{lb} \lambda = \mathbb{A}(-, b),$$

we say that $b$ is an infimum of $\lambda$, and denote it by $\text{inf} \lambda$. Note that

$$\mathbb{A}(-, \text{inf} \lambda) = \mathbb{A}(-, \text{inf} \lambda) = \lambda \backslash \mathbb{A} = \lambda \backslash (1_{\mathbb{A}})^\dagger,$$

thus $\text{inf} \lambda = \text{lim}_\lambda 1_{\mathbb{A}}$.

Remark 3.4.6. It follows immediately from Definition 3.4.1 and Proposition 3.2.6 that weighted colimits and weighted limits, when they exist, are essentially unique. It should be noted that there is an abuse of notation. Strictly speaking, when we refer to essentially unique objects, such as suprema, we should use the isomorphism $a \cong \text{sup} \mu$, since it is not necessarily unique if the related $\mathcal{Q}$-category is not skeletal. But we still write $a = \text{sup} \mu$, where we mean $a$ is one of the suprema of $\mu$. Proposition 3.2.6 ensures that which object we “pick out” to represent the supremum makes no difference in most cases. However, we must be aware that $a = \text{sup} \mu$ and $a' = \text{sup} \mu$ does not imply $a = a'$.

Proposition 3.4.7. Let $F : \mathbb{A} \to \mathbb{B}$ be a $\mathcal{Q}$-functor.

(1) For each $\mu \in \mathcal{P} \mathbb{A}$,

$$\text{colim}_\mu F = \text{sup} \mathbb{B} F (\mu)$$

whenever the colimit or the supremum exists.
(2) For each \( \lambda \in \mathcal{P}^\dagger \mathbf{A} \),
\[
\lim_{\lambda} F = \inf_{\mathbf{B}} F^\to(\lambda)
\]
whenever the limit or the infimum exists.

Proof. We prove (1) for example.
\[
\mathbb{B}(\sup_{\mathbf{B}} F^\to(\mu),-) = \mathbb{B} \prec (\mu \circ F^\circ)
\]
(Example 3.4.5)
\[
= (\mathbb{B} \circ F_z) \prec \mu
\]
(Definition 3.3.12(1))
\[
= F_z \prec \mu
\]
(Proposition 3.1.9(2))
\[
= \mathbb{B}(\text{colim}_\mu F,-).
\]
(Definition 3.4.1)

Thus \( \text{colim}_\lambda F = \sup_{\mathbf{B}} F^\to(\mu) \).

Let \( \mathbf{A} \) be a cocomplete \( \mathcal{Q} \)-category. Since the supremum of each \( \mu \in \mathcal{P} \mathbf{A} \) is essentially unique, we can pick out a supremum for each \( \mu \in \mathcal{P} \mathbf{A} \). This gives rise to a \( \mathcal{Q} \)-functor \( \text{sup} : \mathcal{P} \mathbf{A} \to \mathbf{A} \), as shown in the following proposition. Similarly, \( \text{inf} : \mathcal{P}^\dagger \mathbf{A} \to \mathbf{A} \) is a \( \mathcal{Q} \)-functor if \( \mathbf{A} \) is a complete \( \mathcal{Q} \)-category.

**Proposition 3.4.8.** Let \( \mathbf{A} \) be a \( \mathcal{Q} \)-category.

1. If \( \mathbf{A} \) is cocomplete, then \( \text{sup} : \mathcal{P} \mathbf{A} \to \mathbf{A} \) is a \( \mathcal{Q} \)-functor and
\[
\text{sup} \circ \mathcal{Y} \cong 1_{\mathbf{A}}.
\]
2. If \( \mathbf{A} \) is complete, then \( \text{inf} : \mathcal{P}^\dagger \mathbf{A} \to \mathbf{A} \) is a \( \mathcal{Q} \)-functor and
\[
\text{inf} \circ \mathcal{Y}^\dagger \cong 1_{\mathbf{A}}.
\]

Proof. We prove (1) for example. For each \( \mu, \lambda \in \mathcal{P} \mathbf{A} \),
\[
\mathcal{P} \mathbf{A}(\mu,\lambda) = \lambda \prec \mu
\]
(Formula 3.5)
\[
\leq (\mathbf{A} \prec \lambda) \setminus (\mathbf{A} \prec \mu)
\]
(Proposition 3.1.5(1)(4))
\[
= \mathbf{A}(\text{sup} \lambda, -) \setminus \mathbf{A}(\text{sup} \mu, -)
\]
(Example 3.4.5)
\[
= \mathbf{A}(\text{sup} \mu, \text{sup} \lambda).
\]
(Remark 3.3.10(3))
Thus sup is a $\mathcal{Q}$-functor. Furthermore, for each $x \in \mathcal{A}_0$,

\[
\mathcal{A}(\text{sup} \circ \mathcal{Y}x, -) = \mathcal{A} \vee \mathcal{Y}x \quad \text{(Example 3.4.5)}
\]

\[
= \mathcal{A} \vee \mathcal{A}(-, x)
\]

\[
= (\mathcal{A} \vee \mathcal{A})(x, -) \quad \text{(Remark 3.3.10(6))}
\]

\[
= \mathcal{A}(x, -).
\]

Thus $\text{sup} \circ \mathcal{Y}x \cong x$, and consequently $\text{sup} \circ \mathcal{Y} \cong 1_A$.

\[\square\]

## 3.5 Tensors and cotensors

We introduce the notions of tensors and cotensors to provide a clear characterization for cocomplete and complete $\mathcal{Q}$-categories.

**Definition 3.5.1.** Let $\mathcal{A}$ be a $\mathcal{Q}$-category.

1. For $x \in \mathcal{A}_0$ and $f \in \mathcal{P}(tx)$, the **tensor** of $f$ and $x$, denoted by $f \otimes x$, is an object in $\mathcal{A}_0$ of type $t(f \otimes x) = tf$ such that

\[
\mathcal{A}(f \otimes x, -) = \mathcal{A}(x, -) \vee f.
\]

A $\mathcal{Q}$-category $\mathcal{A}$ is **tensored** if the tensor $f \otimes x$ exists for all choices of $x$ and $f$.

2. For $x \in \mathcal{A}_0$ and $f \in \mathcal{P}^t(tx)$, the **cotensor** of $f$ and $x$, denoted by $f \multimap x$, is an object in $\mathcal{A}_0$ of type $t(f \multimap x) = tf$ such that

\[
\mathcal{A}(-, f \multimap x) = f \multimap \mathcal{A}(-, x).
\]

A $\mathcal{Q}$-category $\mathcal{A}$ is **cotensored** if the cotensor $f \multimap x$ exists for all choices of $x$ and $f$.

**Example 3.5.2.**

1. For all $x \in \mathcal{A}_0$, $1_{tx} \otimes x \cong x \cong 1_{tx} \multimap x$.

2. If $\mathcal{A}$ is tensored, then each $\mathcal{A}_X$ has a bottom element $\bot_{\mathcal{A}_X}$ and for all $x \in \mathcal{A}_0$,

\[
\bot_{tx,X} \otimes x \cong \bot_{\mathcal{A}_X}.
\]
(3) If $\mathbb{A}$ is cotensored, then each $\mathbb{A}_X$ has a top element $\top_{\mathbb{A}_X}$ and for all $y \in \mathbb{A}_0$, 
$\bot_{X,Y} \to y \cong \top_{\mathbb{A}_X}$.

The following proposition is a direct consequence of Definition 3.5.1.

**Proposition 3.5.3.** Let $\mathbb{A}$ be a $\mathbb{Q}$-category.

(1) If $\mathbb{A}$ is tensored, then for all $x \in \mathbb{A}_0$,

$$(-) \otimes x \dashv \mathbb{A}(x,-) : \mathcal{P}(tx) \to \mathbb{A}.$$  

(2) If $\mathbb{A}$ is cotensored, then for all $x \in \mathbb{A}_0$,

$$\mathbb{A}(-,x) \dashv (-) \to x : \mathbb{A} \to \mathcal{P}^\dagger(tx).$$

**Proposition 3.5.4.** Let $\mathbb{A}$ be a $\mathbb{Q}$-category and $x \in \mathbb{A}_0$, $Y \in \mathbb{Q}_0$.

(1) If $\mathbb{A}$ is tensored, then for each subset $\{y_j\} \subseteq \mathbb{A}_Y$,

$$\mathbb{A}\left(x, \bigwedge_j y_j\right) = \bigwedge_j \mathbb{A}(x,y_j)$$

whenever the meet $\bigwedge_j y_j$ in $\mathbb{A}_Y$ exists, where $\bigwedge$ on the right hand denotes the meet in $\mathbb{Q}(tx,Y)$.

(2) If $\mathbb{A}$ is cotensored, then for each subset $\{y_j\} \subseteq \mathbb{A}_Y$,

$$\mathbb{A}\left(\bigvee_j y_j, x\right) = \bigwedge_j \mathbb{A}(y_j,x)$$

where the join $\bigvee_j y_j$ in $\mathbb{A}_Y$ exists, where $\bigwedge$ on the right hand denotes the meet in $\mathbb{Q}(Y,tx)$.

**Proof.** We prove (1) for example. Suppose the meet $\bigwedge_j y_j$ in $\mathbb{A}_Y$ exists, then for all $j$, by Proposition 3.2.5,

$$\mathbb{A}\left(x, \bigwedge_j y_j\right) \leq \mathbb{A}(x, y_j).$$
If \( f \in Q(tx,Y) \) satisfies \( f \leq \Lambda(x,y_j) \) for all \( j \), then
\[
1_Y \leq \Lambda(x,y_j) \vee f = \Lambda(f \otimes x,y_j)
\]
because \( \Lambda \) is tensored. Thus \( f \otimes x \leq y_j \) for all \( j \) in \( A_Y \), and it follows that \( f \otimes x \leq \bigwedge_j y_j \). Hence
\[
1_Y \leq \Lambda\left(f \otimes x, \bigwedge_j y_j \right) = \Lambda(x, \bigwedge_j y_j) \not\leq f,
\]
and consequently \( f \leq \Lambda(x, \bigwedge_j y_j) \). Therefore, \( \Lambda(x, \bigwedge_j y_j) = \bigwedge_j \Lambda(x,y_j) \). \( \Box \)

**Proposition 3.5.5.** Let \( \Lambda \) be a \( Q \)-category.

1. For \( x \in \Lambda_0 \) and \( f \in P(tx) \),
\[
f \otimes x = \sup(f \circ \Upsilon x)
\]
whenever the tensor or the supremum exists.

2. For \( x \in \Lambda_0 \) and \( g \in P^i(tx) \),
\[
g \mapsto x = \inf(\Upsilon^i x \circ g)
\]
whenever the cotensor or the infimum exists.

**Proof.** We prove (1) for example. Note that \( f \circ \Upsilon x \in P\Lambda \) because \( f \) is viewed as a \( Q \)-distributor \( *_{tx} \mapsto *_{tf} \) between one-object \( Q \)-categories. Then
\[
\Lambda(f \otimes x,-) = \Lambda(x,-) \vee f \quad \text{(Definition 3.5.1)}
\]
\[
= (\Lambda \vee \Lambda(-,x)) \vee f \quad \text{(Remark 3.3.10)}
\]
\[
= \Lambda \vee (f \circ \Upsilon x) \quad \text{(Proposition 3.1.5(5))}
\]
\[
= \Lambda(\sup(f \circ \Upsilon x),-) \quad \text{(Example 3.4.5)}
\]
Thus \( f \otimes x = \sup(f \circ \Upsilon x) \). \( \Box \)

**Example 3.5.6.** Let \( \Lambda \) be a \( Q \)-category.
(1) \( \mathcal{P} \mathbb{A} \) is a tensored and cotensored \( \mathbb{Q} \)-category in which
\[ f \otimes \mu = f \circ \mu, \quad g \rightarrow \mu = g \downarrow \mu \]
for all \( \mu \in \mathcal{P} \mathbb{A} \) and \( f \in \mathcal{P}(t\mu), \ g \in \mathcal{P}^\dagger(t\mu) \).

(2) \( \mathcal{P}^\dagger \mathbb{A} \) is a tensored and cotensored \( \mathbb{Q} \)-category in which
\[ f \otimes \lambda = \lambda \vee f, \quad g \rightarrow \lambda = \lambda \circ g \]
for all \( \lambda \in \mathcal{P}^\dagger \mathbb{A} \) and \( f \in \mathcal{P}(t\lambda), \ g \in \mathcal{P}(t\lambda) \).

**Definition 3.5.7.** A \( \mathbb{Q} \)-category \( \mathbb{A} \) is order-complete if each \( \mathbb{A}_X \) admits all joins (or equivalently, all meets) in the underlying preorder.

**Definition 3.5.8.** Let \( \mathbb{A} \) be a \( \mathbb{Q} \)-category and \( X \in \mathbb{Q}_0 \).

(1) The conical colimit of a subset \( \{ x_i \} \subseteq \mathbb{A}_X \) is the supremum of the join
\[ \bigvee_i \mathbb{Y} x_i \text{ in } \mathbb{Q} \text{-Dist}(\mathbb{A}, \star_X) . \]
\( \mathbb{A} \) is conically cocomplete if it admits all conical colimits.

(2) The conical limit of a subset \( \{ x_i \} \subseteq \mathbb{A}_X \) is the infimum of the meet
\[ \bigwedge_i \mathbb{Y}^\dagger x_i \text{ in } \mathbb{Q} \text{-Dist}(\star_X, \mathbb{A}) . \]
\( \mathbb{A} \) is conically complete if it admits all conical limits.

**Proposition 3.5.9.** [Stu06] Let \( \mathbb{A} \) be a \( \mathbb{Q} \)-category, \( X \in \mathbb{Q}_0 \) and \( \{ x_i \} \subseteq \mathbb{A}_X \).

(1) If the conical colimit of \( \{ x_i \} \) exists, then it is also the join of \( \{ x_i \} \) in \( \mathbb{A}_X \), i.e.,
\[ \sup \bigvee_i \mathbb{Y} x_i = \bigvee_i x_i . \]
Conversely, if \( \mathbb{A} \) is cotensored, then the join of \( \{ x_i \} \) in \( \mathbb{A}_X \), when it exists, is also the conical colimit of \( \{ x_i \} \).

(2) If the conical limit of \( \{ x_i \} \) exists, then it is also the meet of \( \{ x_i \} \) in \( \mathbb{A}_X \), i.e.,
\[ \inf \bigwedge_i \mathbb{Y}^\dagger x_i = \bigwedge_i x_i . \]
Conversely, if \( \mathbb{A} \) is tensored, then the meet of \( \{ x_i \} \) in \( \mathbb{A}_X \), when it exists, is also the conical limit of \( \{ x_i \} \).
Proof. We prove (1) for example. If the conical colimit \( \sup_i Yx_i \) exists, then for all \( i \),
\[
\mathbb{A}(x, \sup_i Yx_i) = \mathbb{A}(\sup_i Yx_i, -) \searrow \mathbb{A}(x, -) \quad \text{(Remark 3.3.10(3))}
\]
\[
= \left( \mathbb{A} \uparrow \bigvee_i Yx_i \right) \searrow \mathbb{A}(x, -) \quad \text{(Example 3.4.5)}
\]
\[
\geq (\mathbb{A} \uparrow \mathbb{A}(-, x_i)) \searrow \mathbb{A}(x, -) \quad \text{(Proposition 3.1.5(3))}
\]
\[
= \mathbb{A}(x, -) \searrow \mathbb{A}(x, -) \quad \text{(Remark 3.3.10(6))}
\]
\[
= \mathbb{A}(x, x) \quad \text{(Remark 3.3.10(3))}
\]
\[
\geq 1_X.
\]
Thus \( x_i \leq \sup_i Yx_i \). Suppose that \( y \in \mathbb{A}_X \) and \( x_i \leq y \) for all \( i \), then
\[
\mathbb{A}\left(\sup_i Yx_i, y\right) = \mathbb{A}(-, y) \uparrow \bigvee_i Yx_i \quad \text{(Example 3.4.5)}
\]
\[
= \bigwedge_i (\mathbb{A}(-, y) \uparrow \mathbb{A}(-, x_i)) \quad \text{(Proposition 3.1.5(3))}
\]
\[
= \bigwedge_i \mathbb{A}(x_i, y) \quad \text{(Remark 3.3.10(2))}
\]
\[
\geq 1_X.
\]
Thus \( \sup_i Yx_i \leq y \). Therefore, \( \sup_i Yx_i = \bigvee_i Yx_i \).

Conversely, if \( \mathbb{A} \) is cotensored and the join \( \bigvee_i Yx_i \) in \( \mathbb{A}_X \) exists, then
\[
\mathbb{A}\left(\bigvee_i x_i, -\right) = \bigwedge_i \mathbb{A}(x_i, -) \quad \text{(Proposition 3.5.4)}
\]
\[
= \bigwedge_i (\mathbb{A} \uparrow \mathbb{A}(-, x_i)) \quad \text{(Remark 3.3.10)}
\]
\[
= \mathbb{A} \uparrow \bigvee_i Yx_i. \quad \text{(Proposition 3.1.5(3))}
\]
Thus \( \bigvee_i x_i = \sup_i Yx_i. \)

Corollary 3.5.10. Let \( \mathbb{A} \) be a \( \mathcal{Q} \)-category.
(1) If \( A \) is conically cocomplete, then \( A \) is order-complete. Conversely, if \( A \) is cotensored and order-complete, then \( A \) is conically cocomplete.

(2) If \( A \) is conically complete, then \( A \) is order-complete. Conversely, if \( A \) is tensored and order-complete, then \( A \) is conically complete.

**Theorem 3.5.11.** [Stu05a, Stu06, Stu13a] For a \( Q \)-category \( A \), the following conditions are equivalent:

(1) \( A \) is cocomplete.

(2) \( A \) is complete.

(3) Each \( \mu \in \mathcal{P}A \) has a supremum.

(4) Each \( \lambda \in \mathcal{P}^!A \) has an infimum.

(5) \( Y \) has a left inverse (up to isomorphism) \( \sup : \mathcal{P}A \rightarrow A \) in \( Q\text{-Cat} \).

(6) \( Y^! \) has a left inverse (up to isomorphism) \( \inf : \mathcal{P}^!A \rightarrow A \) in \( Q\text{-Cat} \).

(7) \( Y \) has a left adjoint \( \sup : \mathcal{P}A \rightarrow A \) in \( Q\text{-Cat} \).

(8) \( Y^! \) has a right adjoint \( \inf : \mathcal{P}^!A \rightarrow A \) in \( Q\text{-Cat} \).

(9) \( A \) is tensored and conically cocomplete.

(10) \( A \) is cotensored and conically complete.

(11) \( A \) is tensored, cotensored and order-complete.

In this case, for each \( \mu \in \mathcal{P}A \) and \( \lambda \in \mathcal{P}^!A \),

\[
\sup \mu = \bigvee_{a \in A_0} \mu(a) \otimes a, \quad \inf \lambda = \bigwedge_{a \in A_0} \lambda(a) \triangleright a,
\]

where \( \bigvee \) and \( \bigwedge \) denote respectively the join in \( A_{\mu} \) and the meet in \( A_{\lambda} \).
Proof. (1) \iff (3) and (2) \iff (4) follows immediately from Example 3.4.5 and Proposition 3.4.7.

(1) \implies (5) and (2) \implies (6): Proposition 3.4.8.

(5) \implies (7): Suppose that sup \circ Y \cong 1_A. Then for each \mu \in \mathcal{P} A,

\mu = \mathcal{P} A(Y - , \mu) \quad \text{(Yoneda lemma)}
\leq A(\sup \circ Y - , \sup \mu)
= A(-, \sup \mu) \quad \text{(Proposition 3.2.6)}
= Y \circ \sup \mu. \quad \text{(Formula (3.7))}

Thus 1_{\mathcal{P} A} \leq Y \circ \sup, and consequently sup \dashv Y : \mathcal{P} A \to A.

(6) \implies (8): Similar to (5) \implies (7).

(7) \implies (3): For all \mu \in \mathcal{P} A and x \in A_0,

A(\sup \mu, x) = \mathcal{P} A(\mu, Y x) \quad \text{(Proposition 3.2.11)}
= Y x \vee \mu \quad \text{(Equation (3.5))}
= A(-, x) \vee \mu \quad \text{(Formula (3.7))}
= (A \vee \mu)(x). \quad \text{(Remark 3.3.10(7))}

Thus each \mu \in \mathcal{P} A has a supremum \sup \mu.

(8) \implies (4): Similar to (7) \implies (3).

(5) \implies (6): Define inf = \sup \circ lb : \mathcal{P} \mathcal{P} A \to \mathcal{P} A \to A, then for all x \in A_0,

inf \circ Y \downarrow x = \sup \circ lb \circ Y \downarrow x
= \sup(A(x, -) \searrow A) \quad \text{(Example 3.4.3)}
= \sup(A(-, x)) \quad \text{(Remark 3.3.10(8))}
= \sup \circ Y x \quad \text{(Formula (3.7))}
\cong x.

(6) \implies (5): sup = \inf \circ ub : \mathcal{P} A \to \mathcal{P} \mathcal{P} A \to A is the required left inverse of Y in \mathcal{Q}\text{-}\mathcal{C}\text{at} by similar calculation to (5) \implies (6).

(3) \implies (9) and (4) \implies (10) are immediate consequences of Proposition 3.5.5 and Definition 3.5.8.
(9) $\implies$ (3): For each $\mu \in \mathcal{P}A$, by Proposition 3.5.9, we know that the conical colimit of $\{\mu(a) \otimes a \mid a \in A_0\} \subseteq A_{\mu}$ is the join of $\{\mu(a) \otimes a \mid a \in A_0\}$ in $A_{\mu}$, i.e.,

$$\sup_{a \in A_0} \bigvee A_{\mu}Y(\mu(a) \otimes a) = \bigvee_{a \in A_0} \mu(a) \otimes a.$$ 

It follows that

$$\bigvee_{a \in A_0} \mu(a) \otimes a - = \bigvee_{a \in A_0} A(\mu(a) \otimes a) \quad (\text{Example 3.4.3})$$

$$= \bigwedge_{a \in A_0} \bigvee A(-, \mu(a) \otimes a) \quad (\text{Proposition 3.1.5(3)})$$

$$= \bigwedge_{a \in A_0} A(\mu(a) \otimes a, -) \quad (\text{Remark 3.3.10(6)})$$

$$= \bigwedge_{a \in A_0} A(a, -) \bigvee \mu(a) \quad (\text{Definition 3.5.1})$$

$$= \bigvee A \mu. \quad (\text{Proposition 3.3.2(4)})$$

Thus $\sup_{a \in A_0} \bigvee A_{\mu}Y(\mu(a) \otimes a) = \bigvee_{a \in A_0} \mu(a) \otimes a = \sup \mu$.

(10) $\implies$ (4): For each $\lambda \in P^\dagger A$, by similar calculation to (9) $\implies$ (3) one obtains

$$\inf \bigwedge_{a \in A_0} A^\dagger(\lambda(a) \rightarrow a) = \bigwedge_{a \in A_0} \lambda(a) \rightarrow a = \inf \lambda.$$ 

Now we have (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) $\iff$ (5) $\iff$ (6) $\iff$ (7) $\iff$ (8) $\iff$ (9) $\iff$ (10). Finally we show that (3)+(4)+(9) $\implies$ (11) and (11) $\implies$ (9) to finish the proof.

(3)+(4)+(9) $\implies$ (11): Since (3) and (4) hold, by Proposition 3.5.5 we have that $A$ is tensored and cotensored. That $A$ is order-complete follows from (9) and Corollary 3.5.10.

(11) $\implies$ (9) is an immediate consequence of Corollary 3.5.10. 

Example 3.5.12. Let $A$ be a $Q$-category.

(1) $P\mathcal{A}$ is a complete $Q$-category in which

$$\sup_{\mu \in P\mathcal{A}} \Phi(\mu) \circ \mu = \Phi \circ (Y_A)_\xi$$
for all $\Phi \in \mathcal{P}(\mathcal{P}A)$ and

$$\inf \Psi = \bigwedge_{\mu \in \mathcal{P}A} \Psi(\mu) \vee \mu = \Psi \vee (Y_A)_{\sharp}$$

for all $\Psi \in \mathcal{P}(\mathcal{P}A)$, i.e., $\inf \Psi$ is the largest $\mathcal{Q}$-distributor $\mu : A \rightarrow \ast_{t\Psi}$ such that $\Psi \circ \mu \leq (Y_A)_{\sharp}$.

(2) $\mathcal{P}^1A$ is a complete $\mathcal{Q}$-category in which

$$\sup \Phi = \bigwedge_{\lambda \in \mathcal{P}^1A} \lambda \triangleright \Phi(\lambda) = (Y_A^1)_{\triangleright} \triangleright \Phi$$

for all $\Phi \in \mathcal{P}(\mathcal{P}^1A)$ and

$$\inf \Psi = \bigvee_{\lambda \in \mathcal{P}^1A} \lambda \circ \Psi(\lambda) = (Y_A^1)_{\triangleright} \circ \Psi$$

for all $\Psi \in \mathcal{P}(\mathcal{P}^1A)$. 
In particular, $\mathcal{P}X$ and $\mathcal{P}^1X$ are both complete $\mathcal{Q}$-categories for all $X \in \mathcal{Q}_0$.

**Proposition 3.5.13.** Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a type-preserving function between $\mathcal{Q}$-categories. If $\mathcal{A}$ and $\mathcal{B}$ are tensored, then $F$ is a $\mathcal{Q}$-functor if and only if

1. For all $x \in \mathcal{A}_0$ and $f \in \mathcal{P}(tx)$, $f \otimes B Fx \leq F(f \otimes A x)$;
2. $F : \mathcal{A}_0 \rightarrow \mathcal{B}_0$ is order-preserving.

Dually, if $\mathcal{A}$ and $\mathcal{B}$ are cotensored, then $F$ is a $\mathcal{Q}$-functor if and only if

1. For all $x \in \mathcal{A}_0$, $f \in \mathcal{P}^1(tx)$, $F(f \mapsto_A x) \leq f \mapsto_B Fx$;
2. $F : \mathcal{A}_0 \rightarrow \mathcal{B}_0$ is order-preserving.

**Proof.** We prove the case that $\mathcal{A}$ and $\mathcal{B}$ are tensored for example. Suppose that $F$ is a $\mathcal{Q}$-functor. Let $x \in \mathcal{A}_0$ and $f \in \mathcal{P}(tx)$, then

$$
\mathcal{B}(f \otimes_B Fx, F(f \otimes_A x)) = \mathcal{B}(F(x), F(f \otimes_A x)) \not\leq f \tag{Definition 3.5.1}
$$

$$
\geq \mathcal{A}(x, f \otimes_A x) \not\geq f \tag{Definition 3.2.3}
$$

$$
= \mathcal{A}(f \otimes_A x, f \otimes_A x) \tag{Definition 3.5.1}
$$

$$
\geq 1_{tx}. \tag{Definition 3.5.1}
$$

Thus $f \otimes_B Fx \leq F(f \otimes_A x)$.

Let $x, x' \in \mathcal{A}_0$. If $x \leq x'$, then $tx = tx' = X$ and

$$
1_X \leq \mathcal{A}(x, x') \leq \mathcal{B}(Fx, Fx').
$$

Thus $Fx \leq Fx'$. It follows that $F : \mathcal{A}_0 \rightarrow \mathcal{B}_0$ is order-preserving.

Conversely, for all $x, x' \in \mathcal{A}_0$, by Definition 3.5.1

$$
\mathcal{A}(\mathcal{A}(x, x') \otimes_A x, x') = \mathcal{A}(x, x') \not\leq \mathcal{A}(x, x') \geq 1_{tx'},
$$

thus $\mathcal{A}(x, x') \otimes_A x \leq x'$ in $\mathcal{A}_0$. Since $F : \mathcal{A}_0 \rightarrow \mathcal{B}_0$ is order-preserving, it follows that

$$
\mathcal{A}(x, x') \otimes_B Fx \leq F(\mathcal{A}(x, x') \otimes_A x) \leq Fx'.
$$
By Definition 3.5.1, this means that
\[ 1_{tx'} \leq B(A(x, x') \otimes_B Fx, Fx') = B(Fx, Fx') \uplus A(x, x'). \]
Thus \(A(x, x') \leq B(Fx, Fx').\) Hence \(F\) is a \(Q\)-functor. \(\Box\)

**Proposition 3.5.14.** [Stu06] Let \(F : A \rightarrow B\) be a type-preserving function between \(Q\)-categories. If \(A\) is tensored, then \(F\) is a left adjoint \(Q\)-functor in \(Q\text{-Cat}\) if and only if

1. \(F\) preserves tensors in the sense that \(F(f \otimes_A x) = f \otimes_B Fx\) for all \(x \in A_0\) and \(f \in P(tx)\);
2. \(F : A_0 \rightarrow B_0\) is a left adjoint in \(2\text{-Cat}\).

Dually, if \(A\) is cotensored, then \(F\) is a right adjoint \(Q\)-functor in \(Q\text{-Cat}\) if and only if

1. \(F\) preserves cotensors in the sense that \(F(f \rightarrow_A x) = f \rightarrow_B Fx\) for all \(x \in A_0\) and \(f \in P^\dagger(tx)\);
2. \(F : A_0 \rightarrow B_0\) is a right adjoint in \(2\text{-Cat}\).

**Proof.** We prove the case that \(A\) is tensored for example. Suppose that \(F : A \rightarrow B\) is a left adjoint \(Q\)-functor with a right adjoint \(G : B \rightarrow A\). Let \(x \in A_0\) and \(f \in P(tx)\), then

\[
B(F(f \otimes_A x), -) = A(f \otimes_A x, G-) \quad \text{(Proposition 3.2.11)}
\]

\[
= A(x, G-) \uplus f \quad \text{(Definition 3.5.1)}
\]

\[
= B(Fx, -) \uplus f. \quad \text{(Proposition 3.2.11)}
\]

Thus \(F(f \otimes_A x) = f \otimes_B Fx\).

Let \(x \in A_0\) and \(y \in B_0\), then by Proposition 3.2.11

\[
Fx \leq y \text{ in } B_0 \iff tx = ty = X \text{ and } 1_X \leq B(Fx, y)
\]

\[
\iff tx = ty = X \text{ and } 1_X \leq A(x, Gy)
\]

\[
\iff x \leq Gy \text{ in } A_0.
\]
Thus \( F : \mathcal{A}_0 \to \mathcal{B}_0 \) is a left adjoint of \( G : \mathcal{B}_0 \to \mathcal{A}_0 \) in 2-Cat.

Conversely, suppose that \( F : \mathcal{A}_0 \to \mathcal{B}_0 \) has a right adjoint \( G : \mathcal{B}_0 \to \mathcal{A}_0 \) in 2-Cat, then \( G \) is necessarily type-preserving because for all \( y \in \mathcal{B}_0 \),

\[
Gy \leq Gy \iff FGy \leq y
\]

implies \( ty = t(FGy) = t(Gy) \). In order to prove \( F \dashv G : \mathcal{A}_0 \to \mathcal{B} \), by Proposition 3.2.11 it suffices to show that \( \mathcal{B}(Fx, y) = \mathcal{A}(x, Gy) \) for all \( x \in \mathcal{A}_0 \) and \( y \in \mathcal{B}_0 \).

By Definition 3.5.1

\[
1_{ty} \leq \mathcal{A}(x, Gy) \not\leq \mathcal{A}(x, Gy) = \mathcal{A}(\mathcal{A}(x, Gy) \otimes_{\mathcal{A}} x, Gy).
\]

It follows that

\[
1_{ty} \leq \mathcal{B}(F(\mathcal{A}(x, Gy) \otimes_{\mathcal{A}} x), y) = \mathcal{B}(Fx, y) \not\leq \mathcal{A}(x, Gy),
\]

where the first inequality holds because \( F \dashv G : \mathcal{A}_0 \to \mathcal{B}_0 \) in 2-Cat, and the second equality follows from \( F(\mathcal{A}(x, Gy) \otimes_{\mathcal{A}} x) = \mathcal{A}(x, Gy) \otimes_{\mathcal{B}} Fx \). Thus \( \mathcal{A}(x, Gy) \leq \mathcal{B}(Fx, y) \).

For the reverse inequality, since \( F(\mathcal{B}(Fx, y) \otimes_{\mathcal{A}} x) = \mathcal{B}(Fx, y) \otimes_{\mathcal{B}} Fx \), we have

\[
1_{ty} \leq \mathcal{B}(Fx, y) \not\leq \mathcal{B}(Fx, y) = \mathcal{B}(F(\mathcal{B}(Fx, y) \otimes_{\mathcal{A}} x), y).
\]

It follows from \( F \dashv G : \mathcal{A}_0 \to \mathcal{B}_0 \) in 2-Cat that

\[
1_{ty} \leq \mathcal{A}(\mathcal{B}(Fx, y) \otimes_{\mathcal{A}} x, Gy) = \mathcal{A}(x, Gy) \not\leq \mathcal{B}(Fx, y).
\]

Thus \( \mathcal{B}(Fx, y) \leq \mathcal{A}(x, Gy) \), completing the proof.

The following corollary indicates that left adjoint \( \mathcal{Q} \)-functors between complete \( \mathcal{Q} \)-categories are exactly suprema-preserving \( \mathcal{Q} \)-functors, while right adjoint \( \mathcal{Q} \)-functors between complete \( \mathcal{Q} \)-categories are exactly infima-preserving \( \mathcal{Q} \)-functors.

**Corollary 3.5.15.** [Stu06] Let \( F : \mathcal{A} \to \mathcal{B} \) be a \( \mathcal{Q} \)-functor between \( \mathcal{Q} \)-categories, with \( \mathcal{A} \) complete.
(1) \( F : \mathcal{A} \to \mathcal{B} \) is a left adjoint in \( \mathcal{Q}\text{-Cat} \) if and only if \( F \) preserves suprema in the sense that \( F(\sup_{\mathcal{A}} \mu) = \sup_{\mathcal{B}} F^{-}(\mu) \) for all \( \mu \in \mathcal{P}_{\mathcal{A}} \).

\[
\begin{array}{ccc}
\mathcal{P}_{\mathcal{A}} & \xrightarrow{F^{-}} & \mathcal{P}_{\mathcal{B}} \\
\sup_{\mathcal{A}} & \downarrow & \sup_{\mathcal{B}} \\
\mathcal{A} & \xrightarrow{F} & \mathcal{B}
\end{array}
\]

(2) \( F : \mathcal{A} \to \mathcal{B} \) is a right adjoint in \( \mathcal{Q}\text{-Cat} \) if and only if \( F \) preserves infima in the sense that \( F(\inf_{\mathcal{A}} \mu) = \inf_{\mathcal{B}} F^{\leftarrow}(\mu) \) for all \( \mu \in \mathcal{P}^{\dagger}_{\mathcal{A}} \).

\[
\begin{array}{ccc}
\mathcal{P}^{\dagger}_{\mathcal{A}} & \xleftarrow{F^{\leftarrow}} & \mathcal{P}^{\dagger}_{\mathcal{B}} \\
\inf_{\mathcal{A}} & \downarrow & \inf_{\mathcal{B}} \\
\mathcal{A} & \xrightarrow{F} & \mathcal{B}
\end{array}
\]

Proof. We prove (1) for example. Suppose that \( F : \mathcal{A} \to \mathcal{B} \) has a right adjoint \( G : \mathcal{B} \to \mathcal{A} \) in \( \mathcal{Q}\text{-Cat} \). Then for all \( \mu \in \mathcal{P}_{\mathcal{A}} \),

\[
\mathbb{B}(F(\sup_{\mathcal{A}} \mu),-) = \mathbb{A}(\sup_{\mathcal{A}} \mu, G-) \quad (F \dashv G : \mathcal{A} \to \mathbb{B})
\]

\[
= \mathbb{A}(-, G-) \vee \mu \quad \text{(Example 3.3.5)}
\]

\[
= F_{\mathcal{A}} \vee \mu. \quad \text{(Proposition 3.3.4)}
\]

Thus \( F(\sup_{\mathcal{A}} \mu) = \colim_{\mu} F = \sup_{\mathcal{B}} F^{\rightarrow}(\mu) \).

Conversely, by Proposition 3.5.14 it suffices to show that \( F \) preserves tensors and \( F : \mathcal{A}_{0} \to \mathcal{B}_{0} \) is a left adjoint in \( 2\text{-Cat} \). For all \( x \in \mathcal{A}_{0} \) and \( f \in \mathcal{P}(tx) \),

\[
F(f \otimes_{\mathcal{A}} x) = F(\sup_{\mathcal{A}} (f \circ Y_{\mathcal{A}} x)) \quad \text{(Proposition 3.5.5)}
\]

\[
= \sup_{\mathcal{B}} F^{\rightarrow}(f \circ Y_{\mathcal{A}} x)
\]

\[
= \sup_{\mathcal{B}} (f \circ Y_{\mathcal{A}} x \circ F_{x}) \quad \text{(Definition 3.3.12)}
\]

\[
= \sup_{\mathcal{B}} (f \circ \mathbb{B}(-, F_{x}))
\]

\[
= \sup_{\mathcal{B}} (f \circ Y_{\mathcal{B}} \circ F_{x})
\]

\[
= f \otimes_{\mathcal{B}} F_{x}. \quad \text{(Proposition 3.5.5)}
\]
Thus $F$ preserves tensors.

Note that when $A$ is complete, $F : A_0 \rightarrow B_0$ is a left adjoint in $\textbf{2-Cat}$ if and only if $F : A_X \rightarrow B_X$ preserves underlying joins for each $X \in Q_0$. Let $X \in Q_0$ and $\{x_i\} \subseteq A_X$,

$$F\left(\bigvee i x_i\right) = F\left(\sup A \bigvee i Y_A x_i\right) \quad \text{(Proposition 3.5.9)}$$

$$= \sup B F^{-}\left(\bigvee i Y_A x_i\right)$$

$$= \sup B \left(\bigvee i Y_A x_i \circ F^{\sharp}\right) \quad \text{(Definition 3.3.12)}$$

$$= \sup B \left(\bigvee i (Y_A x_i \circ F^{\sharp})\right) \quad \text{(Proposition 3.3.2 and Formula (3.1))}$$

$$= \sup B \left(\bigvee i B(-, Fx_i)\right)$$

$$= \sup B \left(\bigvee i Y_B(Fx_i)\right)$$

$$= \bigvee i Fx_i. \quad \text{(Proposition 3.5.9)}$$

Thus $F : A_X \rightarrow B_X$ preserves underlying joins, completing the proof. \hfill \Box

### 3.6 Free cocompletion and completion

Let $A$ and $B$ be $Q$-categories. Each $Q$-functor $F : A \rightarrow \mathcal{P}^\dagger B$ corresponds to a $Q$-distributor $\Gamma F : A \leftrightarrow B$ given by

$$\Gamma F(x, y) = (Fx)(y) \quad (3.13)$$

for all $x \in A_0$ and $y \in B_0$, and each $Q$-functor $G : B \rightarrow \mathcal{P}A$ corresponds to a $Q$-distributor $\Gamma G : A \leftrightarrow B$ given by

$$\Gamma G(x, y) = (Gy)(x) \quad (3.14)$$

for all $x \in A_0$ and $y \in B_0$.

Conversely, each $Q$-distributor $\phi : A \leftrightarrow B$ corresponds to two $Q$-functors

$$\underline{\phi} : A \rightarrow \mathcal{P}^\dagger B \quad \text{and} \quad \bar{\phi} : B \rightarrow \mathcal{P}A$$
given by
\[ \phi x = \phi(x, -) \quad \text{and} \quad \phi y = \phi(-, y) \] (3.15)
for all \( x \in A_0 \) and \( y \in B_0 \).

**Proposition 3.6.1.** [Stu05a] Let \( A \) and \( B \) be \( Q \)-categories.

1. The correspondences \( \phi \mapsto \phi \) and \( F \mapsto \Gamma F \) establish an isomorphism of posets

\[ Q\text{-Dist}(A, B) \cong Q\text{-Cat}^{\alpha}(A, P^\dagger B), \]

where the symbol “\( \text{co} \)’’ means reversing order in the hom-sets.

2. The correspondences \( \phi \mapsto \bar{\phi} \) and \( F \mapsto \Gamma F \) establish an isomorphism of posets

\[ Q\text{-Dist}(A, B) \cong Q\text{-Cat}(B, P^\dagger A). \]

**Proof.** We prove (1) for example. It follows immediately from Equation (3.13) and (3.15) that

\[ Fx = \Gamma F^\gamma(x, -) = \Gamma \bar{\Gamma} x \]

and

\[ \phi(x, -) = \phi x = \Gamma \bar{\phi} (x, -) \]

for all \( x \in A_0 \). Thus the correspondences \( \phi \mapsto \phi \) and \( F \mapsto \Gamma F \) are mutual inverse. It remains to show that both of them are order-preserving. Indeed,

\[ \phi \leq \psi \text{ in } Q\text{-Dist}(A, B) \]

\[ \iff \forall x \in A_0, \phi x = \phi(x, -) \leq \psi(x, -) = \psi x \text{ in } Q\text{-Dist} \]

\[ \iff \forall x \in A_0, \phi x \geq \psi x \text{ in } (P^\dagger B)_0 \] (Remark 3.3.8)

\[ \iff \phi \leq \psi \text{ in } Q\text{-Cat}^{\alpha}(A, P^\dagger B) \]

and

\[ F \leq G \text{ in } Q\text{-Cat}^{\alpha}(A, P^\dagger B) \]

\[ \iff \forall x \in A_0, Fx \geq Gx \text{ in } (P^\dagger B)_0 \]

\[ \iff \forall x \in A_0, \Gamma F^\gamma(x, -) = Fx \leq Gx = \Gamma G^\gamma(x, -) \text{ in } Q\text{-Dist}(A, B) \]

\[ \iff \Gamma F^\gamma \leq \Gamma G^\gamma \text{ in } Q\text{-Dist}(A, B). \]
Given a $Q$-distributor $\phi : A \dashv B$, composing with $\phi$ yields two $Q$-functors

$$
\phi^\dagger : P^\dagger A \to P^\dagger B \quad \text{and} \quad \phi^* : PB \to PA
$$
defined by

$$
\phi^\dagger(\mu) = \phi \circ \mu \quad \text{and} \quad \phi^*(\lambda) = \lambda \circ \phi. \quad (3.16)
$$

Thus, the correspondence $\phi \mapsto \phi^\dagger$ induces a functor

$$
(\_\_)^\dagger : Q-\text{Dist} \to Q-\text{Cat},
$$

and the correspondence $\phi \mapsto \phi^*$ induces a contravariant functor

$$
(\_\_)^* : Q-\text{Dist} \to (Q-\text{Cat})^{\text{op}}.
$$

Recall that Proposition 3.3.3 gives rise to a functor

$$
(\_\_)^\natural : Q-\text{Cat} \to Q-\text{Dist}
$$

and a contravariant functor

$$
(\_\_)^\natural : (Q-\text{Cat})^{\text{op}} \to Q-\text{Dist}.
$$

**Proposition 3.6.2.** [Hey10]

1. $(\_\_)^\natural : Q-\text{Cat} \to Q-\text{Dist}$ is a left adjoint of $(\_\_)^\dagger : Q-\text{Dist} \to Q-\text{Cat}$.

2. $(\_\_)^\natural : (Q-\text{Cat})^{\text{op}} \to Q-\text{Dist}$ is a left adjoint of $(\_\_)^* : Q-\text{Dist} \to (Q-\text{Cat})^{\text{op}}$.

**Proof.** We prove (1) for example. We show that the bijection of sets

$$
(\_\_) : Q-\text{Dist}(A, B) \cong Q-\text{Cat}(A, P^\dagger B)
$$
in Proposition 3.6.1 is natural in $A$ and $B$. This follows from the commutativity of the diagrams below for each $Q$-functor $F : A' \to A$ and $Q$-distributor $\psi : B \dashv B'$.

$$
\begin{array}{ccc}
Q-\text{Dist}(A, B) & \xrightarrow{(\_\_)} & Q-\text{Cat}(A, P^\dagger B) \\
\downarrow \text{Q-Dist}(F, B) & & \downarrow \text{Q-Cat}(F, P^\dagger B) \\
Q-\text{Dist}(A', B) & \xrightarrow{(\_\_)} & Q-\text{Cat}(A', P^\dagger B)
\end{array}
$$

(3.17)
Indeed, for each $\mathcal{Q}$-distributor $\phi : \mathcal{A} \hookrightarrow \mathcal{B}$ and $x' \in \mathcal{A}'_0$,

$$\phi \circ Fx' = \phi(Fx', -) \quad \text{(Equation (3.15))}$$

$$= (\phi \circ F_x)(x', -) \quad \text{(Proposition 3.3.11)}$$

$$= (\phi \circ F_x)x'. \quad \text{(Equation (3.15))}$$

Thus Diagram (3.17) commutes. For each $\mathcal{Q}$-distributor $\phi : \mathcal{A} \hookrightarrow \mathcal{B}$ and $x \in \mathcal{A}_0$,

$$(\psi^\dagger \circ \phi)x = \psi(\phi x) \quad \text{(Equation (3.16))}$$

$$= (\psi \circ \phi)(x, -) \quad \text{(Equation (3.15))}$$

$$= (\psi \circ \phi)x. \quad \text{(Equation (3.15))}$$

Thus Diagram (3.18) commutes. \(\square\)

Skeletal complete $\mathcal{Q}$-categories and left adjoint $\mathcal{Q}$-functors constitute a subcategory of $\mathcal{Q}$-$\mathcal{Cat}$ and we denote it by $\mathcal{Q}$-$\mathcal{CCat}$. Dually, we denote by $\mathcal{Q}$-$\mathcal{CCat}^\dagger$ the subcategory of $\mathcal{Q}$-$\mathcal{Cat}$ consisting of skeletal complete $\mathcal{Q}$-categories and right adjoint $\mathcal{Q}$-functors.

Recall that Proposition 3.3.16 gives rise to a functor $\mathcal{P} : \mathcal{Q}$-$\mathcal{Cat} \longrightarrow \mathcal{Q}$-$\mathcal{Cat}$ that sends a $\mathcal{Q}$-functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ to the left adjoint $\mathcal{Q}$-functor $F^\rightarrow : \mathcal{P}\mathcal{A} \longrightarrow \mathcal{P}\mathcal{B}$ between skeletal complete $\mathcal{Q}$-categories. Thus, $\mathcal{P}$ can be viewed as a functor

$$\mathcal{P} : \mathcal{Q}$-$\mathcal{Cat} \longrightarrow \mathcal{Q}$-$\mathcal{CCat}.$$

Similarly, the functor $\mathcal{P}^\dagger : \mathcal{Q}$-$\mathcal{Cat} \longrightarrow \mathcal{Q}$-$\mathcal{Cat}$ that sends a $\mathcal{Q}$-functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ to the right adjoint $\mathcal{Q}$-functor $F^\leftarrow : \mathcal{P}^\dagger\mathcal{A} \longrightarrow \mathcal{P}^\dagger\mathcal{B}$ between skeletal complete $\mathcal{Q}$-categories can be viewed as a functor

$$\mathcal{P}^\dagger : \mathcal{Q}$-$\mathcal{Cat} \longrightarrow \mathcal{Q}$-$\mathcal{CCat}^\dagger.$
Proposition 3.6.3. Let $\mathcal{P}$ and $\mathcal{P}^!$ be functors as defined above.

(1) $\mathcal{P}$ factor through $(-)^\natural$ via $(-)^*$, i.e., the diagram

\[
\begin{array}{ccc}
\mathcal{Q}\text{-Cat} & \xrightarrow{(-)^\natural} & (\mathcal{Q}\text{-Dist})^{\text{op}} \\
\downarrow{\mathcal{P}} & & \downarrow{(-)^*} \\
\mathcal{Q}\text{-CCat} & & \\
\end{array}
\]

commutes, where $(-)^*$ is viewed as a functor $(\mathcal{Q}\text{-Dist})^{\text{op}} \rightarrow \mathcal{Q}\text{-CCat}$.

(2) $\mathcal{P}^!$ factors through $(-)^!$ via $(-)^\natural$, i.e., the diagram

\[
\begin{array}{ccc}
\mathcal{Q}\text{-Cat} & \xrightarrow{(-)^\natural} & \mathcal{Q}\text{-Dist} \\
\downarrow{\mathcal{P}^!} & & \downarrow{(-)^!} \\
\mathcal{Q}\text{-CCat}^! & & \\
\end{array}
\]

commutes, where $(-)^!$ is viewed as a functor $\mathcal{Q}\text{-Dist} \rightarrow \mathcal{Q}\text{-CCat}^!$.

Proof. By Definition 3.3.12 it is easy to see that for each $\mathcal{Q}$-functor $F : \mathcal{A} \rightarrow \mathcal{B}$,

\[F^\rightarrow = (F^\natural)^* \quad \text{and} \quad F^\leftarrow = (F^\natural)^!\]

\[\square\]

Proposition 3.6.4. \textbf{Stu05a}

(1) $\mathcal{P} : \mathcal{Q}\text{-Cat} \rightarrow \mathcal{Q}\text{-CCat}$ is a left adjoint of the forgetful functor $\mathcal{Q}\text{-CCat} \rightarrow \mathcal{Q}\text{-Cat}$.

(2) $\mathcal{P}^! : \mathcal{Q}\text{-Cat} \rightarrow \mathcal{Q}\text{-CCat}^!$ is a left adjoint of the forgetful functor $\mathcal{Q}\text{-CCat}^! \rightarrow \mathcal{Q}\text{-Cat}$.

Proof. (1) By Proposition 3.3.17 $Y = \{Y_A\}$ is a natural transformation from the identity functor on $\mathcal{Q}\text{-Cat}$ to $\mathcal{P}$. We show that $Y$ is the unit of the desired adjunction.

Let $\mathcal{A}$ be a $\mathcal{Q}$-category, $\mathcal{B}$ a skeletal complete $\mathcal{Q}$-category and $F : \mathcal{A} \rightarrow \mathcal{B}$ a $\mathcal{Q}$-functor. We claim that there is a unique left adjoint $\mathcal{Q}$-functor $G : \mathcal{P}\mathcal{A} \rightarrow \mathcal{B}$
such that the following diagram commutes:

\[ \begin{array}{ccc} A & \xrightarrow{\mathcal{Y}_A} & \mathcal{P}A \\ \downarrow{F} & & \downarrow{G} \\ B \end{array} \] (3.19)

Define \( G = \text{sup}_B \circ F^\to : \mathcal{P}A \to \mathcal{P}B \to B \). By Proposition 3.3.14 and Theorem 3.5.11, \( G \) is the composition of the left adjoint \( \mathcal{Q} \)-functors \( F^\to \) and \( \text{sup}_B \), thus \( G \) is also a left adjoint \( \mathcal{Q} \)-functor. For all \( x \in A_0 \),

\[
G \circ \mathcal{Y}_A x = \text{sup}_B \circ F^\to \circ \mathcal{Y}_A x \quad \text{(Definition of } G) \\
= \text{sup}_B \circ \mathcal{Y}_B \circ Fx \quad \text{(Proposition 3.3.17)} \\
= Fx. \quad \text{(Proposition 3.4.8 and } B \text{ is skeletal)}
\]

Thus Diagram (3.19) commutes. Suppose that \( H : \mathcal{P}A \to B \) is another left adjoint \( \mathcal{Q} \)-functor making Diagram (3.19) commute. Then for each \( \mu \in \mathcal{P}A \),

\[
H(\mu) = H(\mu \circ A) \\
= H \left( \bigvee_{x \in A_0} \mu(x) \circ A(-, x) \right) \quad \text{ (Proposition 3.3.2)} \\
= H \left( \bigvee_{x \in A_0} \mu(x) \otimes \mathcal{P}_A \mathcal{Y}_A x \right) \quad \text{ (Example 3.5.6)} \\
= \bigvee_{x \in A_0} \mu(x) \otimes_B (H \circ \mathcal{Y}_A x) \quad \text{ (Proposition 3.5.14)} \\
= \bigvee_{x \in A_0} \mu(x) \otimes_B Fx. \quad \text{ (Diagram (3.19))}
\]
3.6. Free cocompletion and completion

Consequently,

\[ \mathbb{B}(H(\mu), -) = \mathbb{B} \left( \bigvee_{x \in A_0} \mu(x) \otimes_{\mathbb{B}} Fx, - \right) \]

\[ = \bigwedge_{x \in A_0} \mathbb{B}(\mu(x) \otimes_{\mathbb{B}} Fx, -) \quad \text{(Proposition 3.5.4)} \]

\[ = \bigwedge_{x \in A_0} \mathbb{B}(Fx, -) \vee \mu(x) \quad \text{(Definition 3.5.1)} \]

\[ = F \uparrow \mu \quad \text{(Proposition 3.3.2)} \]

\[ = \mathbb{B}(\text{colim}_\mu F, -) \quad \text{(Definition 3.4.1)} \]

\[ = \mathbb{B}(\text{sup}_{\mathbb{B}} \circ F^{-}(\mu), -) \quad \text{(Proposition 3.4.7)} \]

Since \( \mathbb{B} \) is skeletal, it follows that \( H = \text{sup}_{\mathbb{B}} \circ F^{-} = G \).

(2) Similar to (1), one can prove that \( Y^\dagger \) is the unit of the desired adjunction. Explicitly, for each \( \mathcal{Q} \)-category \( A \), skeletal complete \( \mathcal{Q} \)-category \( \mathbb{B} \) and \( \mathcal{Q} \)-functor \( F : A \rightarrow \mathbb{B} \), there is a unique right adjoint \( \mathcal{Q} \)-functor \( G = \text{inf}_{\mathbb{B}} \circ F^{-} : \mathcal{P}^\dagger A \rightarrow \mathcal{P}^\dagger \mathbb{B} \rightarrow \mathbb{B} \) making the following diagram commutes:

\[ \begin{array}{ccc}
A & \xrightarrow{Y^\dagger_A} & \mathcal{P}^\dagger A \\
\downarrow F & & \downarrow G \\
\mathbb{B} & \xrightarrow{G} & \mathbb{B}
\end{array} \]

Proposition 3.6.4 implies that \( \mathcal{P}A \) is the free cocompletion of a \( \mathcal{Q} \)-category \( A \), and \( \mathcal{P}^\dagger A \) is the free completion of \( A \). This means that

(1) each \( \mathcal{Q} \)-functor \( F : A \rightarrow \mathbb{B} \) into a cocomplete \( \mathcal{Q} \)-category factors uniquely (up to isomorphism) through the Yoneda embedding \( Y_A \) via a left adjoint \( \mathcal{Q} \)-functor \( \mathcal{P}A \rightarrow \mathbb{B} \);

(2) each \( \mathcal{Q} \)-functor \( F : A \rightarrow \mathbb{B} \) into a complete \( \mathcal{Q} \)-category factors uniquely (up to isomorphism) through the co-Yoneda embedding \( Y^\dagger_A \) via a right adjoint \( \mathcal{Q} \)-functor \( \mathcal{P}^\dagger A \rightarrow \mathbb{B} \).
3.7 Infomorphisms

In this section, we introduce the crucial notion in this dissertation, that of infomorphisms between $Q$-distributors. An infomorphism between $Q$-distributors is what a Chu transform between Chu spaces [Bar91, Pra95]. The terminology “infomorphism” is from computer science [BS97, Gan07].

**Definition 3.7.1.** Given $Q$-distributors $\phi : A \drawarrow B$ and $\psi : A' \drawarrow B'$, an infomorphism $(F, G) : \phi \Rightarrow \psi$ is a pair of $Q$-functors $F : A \rightarrow A'$ and $G : B \rightarrow B'$ such that $G^\natural \circ \phi = \psi \circ F^\natural$, or equivalently, $\phi(-, G-) = \psi(F-, -)$.

An adjunction $F \dashv G : A \leftrightarrow B$ in $Q$-$\text{Cat}$ is exactly an infomorphism from the identity $Q$-distributor on $A$ to the identity $Q$-distributor on $B$. Thus, infomorphisms are an extension of adjoint $Q$-functors.

$Q$-distributors and infomorphisms constitute a category $Q$-$\text{Info}$.

**Proposition 3.7.2.** Let $F : A \rightarrow B$ be a $Q$-functor, then

$$(F, F^\dashv) : ((Y_A)_\natural : A \hookrightarrow \mathcal{P}A) \rightarrow ((Y_B)_\natural : B \hookrightarrow \mathcal{P}B)$$

is an infomorphism.
3.7. Infomorphisms

Proof. For all \( x \in A_0 \) and \( \lambda \in P_B \),
\[
(Y_A)_!(x, F^\rightarrow(\lambda)) = \mathcal{P}_A(Y_A(x), F^\rightarrow(\lambda)) \\
= F^\rightarrow(\lambda)(x) \quad \text{(Yoneda lemma)} \\
= \lambda(Fx) \quad \text{(Proposition 3.3.15)} \\
= \mathcal{P}_B(Y_B(Fx), \lambda) \quad \text{(Yoneda lemma)} \\
= (Y_B)_!(Fx, \lambda).
\]
Hence the conclusion holds.

The above proposition gives rise to a fully faithful functor \( Y : Q\text{-Cat} \rightarrow Q\text{-Info} \) that sends each \( Q \)-category \( A \) to the graph \( (Y_A)_! \) of the Yoneda embedding.

**Proposition 3.7.3.** \( Y : Q\text{-Cat} \rightarrow Q\text{-Info} \) is a left adjoint of the forgetful functor \( U : Q\text{-Info} \rightarrow Q\text{-Cat} \) that sends an infomorphism
\[
(F, G) : (\phi : A \rightarrow B) \\ (\psi : A' \rightarrow B')
\]
to the \( Q \)-functor \( F : A \rightarrow A' \).

Proof. It is clear that \( U \circ Y = \text{id}_{Q\text{-Cat}} \), the identity functor on \( Q\text{-Cat} \). Thus \( \{1_A\} \) is a natural transformation from \( \text{id}_{Q\text{-Cat}} \) to \( U \circ Y \). It remains to show that for each \( Q \)-category \( A \), \( Q \)-distributor \( \psi : A' \rightarrow B' \) and \( Q \)-functor \( H : A \rightarrow A' \), there is a unique infomorphism
\[
(F, G) : Y(A) \\ (\psi : A' \rightarrow B')
\]
such that the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & U \circ Y(A) \\
\downarrow{H} & & \downarrow{U(F, G)} \\
A' & &
\end{array}
\]
is commutative. By definition, \( Y(A) \) is the graph \( (Y_A)_! : A \rightarrow \mathcal{P}_A \) and \( U(F, G) = F \). Thus, we only need to show that there is a unique \( Q \)-functor \( G : B' \rightarrow \mathcal{P}_A \) such that
\[
(H, G) : ((Y_A)_! : A \rightarrow \mathcal{P}_A) \\ (\psi : A' \rightarrow B')
\]
is an infomorphism.

Let $G = H^\leftarrow \circ \psi : \mathbb{B}' \to \mathcal{P} \mathbb{A}' \to \mathcal{P} \mathbb{A}$. Then
\[(H, G) : ((Y_\mathbb{A})_\sharp : \mathbb{A} \leftrightarrow \mathcal{P} \mathbb{A}) \to (\psi : \mathbb{A}' \leftrightarrow \mathbb{B}') \]
is an infomorphism since
\[(Y_\mathbb{A})_\sharp (x, G y') = (G y')(x) = H^\leftarrow \circ \psi(y')(x) = \psi(y')(H x) = \psi(H x, y') \]
for all $x \in \mathbb{A}_0$ and $y' \in \mathbb{B}_0'$. This proves the existence of $G$.

To see the uniqueness of $G$, suppose that $G' : \mathbb{B}' \to \mathcal{P} \mathbb{A}$ is another $\mathcal{Q}$-functor such that
\[(H, G') : ((Y_\mathbb{A})_\sharp : \mathbb{A} \leftrightarrow \mathcal{P} \mathbb{A}) \to (\psi : \mathbb{A}' \leftrightarrow \mathbb{B}') \]
is an infomorphism. Then for all $x \in \mathbb{A}_0$ and $y' \in \mathbb{B}_0'$,
\[(G' y')(x) = (Y_\mathbb{A})_\sharp (x, G' y') = \psi(H x, y') = \overline{\psi}(y')(H x) = H^\leftarrow \circ \overline{\psi}(y')(x) = (G y')(x), \]
hence $G' = G$. 

Similar to Proposition 3.7.2 one can check that sending a $\mathcal{Q}$-functor $F : \mathbb{A} \to \mathbb{B}$ to the infomorphism
\[(F^\leftarrow, F) : ((Y_\mathbb{B})^\dagger : \mathcal{P}^\dagger \mathbb{B} \leftrightarrow \mathbb{B}) \to ((Y_\mathbb{A})^\dagger : \mathcal{P}^\dagger \mathbb{A} \leftrightarrow \mathbb{A}) \]
induces a fully faithful functor $Y^\dagger : \mathcal{Q}\text{-}\text{Cat} \to (\mathcal{Q}\text{-}\text{Info})^{\text{op}}$.

**Proposition 3.7.4.** $Y^\dagger : \mathcal{Q}\text{-}\text{Cat} \to (\mathcal{Q}\text{-}\text{Info})^{\text{op}}$ is a left adjoint of the contravariant forgetful functor $(\mathcal{Q}\text{-}\text{Info})^{\text{op}} \to \mathcal{Q}\text{-}\text{Cat}$ that sends each infomorphism
\[(F, G) : (\phi : \mathbb{A} \leftrightarrow \mathbb{B}) \to (\psi : \mathbb{A}' \leftrightarrow \mathbb{B}') \]
to the $\mathcal{Q}$-functor $G : \mathbb{B}' \to \mathbb{B}$.

**Proof.** Similar to Proposition 3.7.3. 

\[\square\]
Chapter 4

\(\mathcal{Q}\)-closure spaces

The notions of \(\mathcal{Q}\)-closure operators and \(\mathcal{Q}\)-closure systems describe the structure of monads and their algebras in \(\mathcal{Q}\)-categories. A \(\mathcal{Q}\)-closure space is a \(\mathcal{Q}\)-category \(\mathbb{A}\) equipped with a monad on the \(\mathcal{Q}\)-category \(\mathcal{P}\mathbb{A}\) of contravariant presheaves, and has a similar structure to closure spaces in topology. We will discuss the relations between \(\mathcal{Q}\)-closure spaces and complete \(\mathcal{Q}\)-categories.

4.1 \(\mathcal{Q}\)-closure systems and \(\mathcal{Q}\)-closure operators

We first describe the monad and comonad structures on \(\mathcal{Q}\)-categories in the terminologies of \(\mathcal{Q}\)-closure operators and \(\mathcal{Q}\)-interior operators.

Definition 4.1.1. Let \(\mathbb{A}\) be a \(\mathcal{Q}\)-category.

1. An isomorphism-closed \(\mathcal{Q}\)-subcategory \(\mathbb{B}\) of \(\mathbb{A}\) is a \(\mathcal{Q}\)-closure system of \(\mathbb{A}\) if the inclusion \(\mathcal{Q}\)-functor \(I : \mathbb{B} \to \mathbb{A}\) is a right adjoint in \(\mathcal{Q}\text{-}\text{Cat}\).

2. An isomorphism-closed \(\mathcal{Q}\)-subcategory \(\mathbb{B}\) of \(\mathbb{A}\) is a \(\mathcal{Q}\)-interior system of \(\mathbb{A}\) if the inclusion \(\mathcal{Q}\)-functor \(I : \mathbb{B} \to \mathbb{A}\) is a left adjoint in \(\mathcal{Q}\text{-}\text{Cat}\).

Definition 4.1.2. Let \(\mathbb{A}\) be a \(\mathcal{Q}\)-category.

1. A \(\mathcal{Q}\)-functor \(F : \mathbb{A} \to \mathbb{A}\) is a \(\mathcal{Q}\)-closure operator on \(\mathbb{A}\) if \(1_\mathbb{A} \leq F\) and \(F^2 \cong F\).

2. A \(\mathcal{Q}\)-functor \(F : \mathbb{A} \to \mathbb{A}\) is a \(\mathcal{Q}\)-interior operator on \(\mathbb{A}\) if \(F \leq 1_\mathbb{A}\) and \(F^2 \cong F\).
Example 4.1.3. Let \( F \dashv G : \mathcal{A} \rightarrow \mathcal{B} \) be an adjunction in \( \mathcal{Q}\text{-Cat} \). Then \( G \circ F : \mathcal{A} \rightarrow \mathcal{A} \) is a \( \mathcal{Q} \)-closure operator and \( F \circ G : \mathcal{B} \rightarrow \mathcal{B} \) is a \( \mathcal{Q} \)-interior operator.

Proposition 4.1.4. Let \( \mathcal{A} \) be a \( \mathcal{Q} \)-category, \( \mathcal{B} \) an isomorphism-closed \( \mathcal{Q} \)-subcategory of \( \mathcal{A} \). The following conditions are equivalent:

1. \( \mathcal{B} \) is a \( \mathcal{Q} \)-closure system of \( \mathcal{A} \).

2. There is a \( \mathcal{Q} \)-closure operator \( F : \mathcal{A} \rightarrow \mathcal{A} \) such that \( \mathcal{B} = \{ x \in \mathcal{A}_0 : Fx \cong x \} \).

Proof. (1) \( \Rightarrow \) (2): If the inclusion \( \mathcal{Q} \)-functor \( I : \mathcal{B} \rightarrow \mathcal{A} \) has a left adjoint \( G : \mathcal{A} \rightarrow \mathcal{B} \), let \( F = I \circ G \), then \( F : \mathcal{A} \rightarrow \mathcal{A} \) is a \( \mathcal{Q} \)-closure operator. Since \( Fx = Gx \in \mathcal{B}_0 \) for all \( x \in \mathcal{A}_0 \) and \( \mathcal{B} \) is isomorphism-closed, it is clear that \( \{ x \in \mathcal{A}_0 : Fx \cong x \} \subseteq \mathcal{B}_0 \).

Conversely, for all \( x \in \mathcal{B}_0 \),

\[
\mathcal{B}(Fx, x) = \mathcal{B}(Gx, x) = \mathcal{A}(x, Ix) = \mathcal{A}(x, x) \geq 1_{tx},
\]

and \( \mathcal{B}(x, Fx) \geq 1_{tx} \) holds trivially, hence \( x \cong Fx \), as required.

(2) \( \Rightarrow \) (1): We show that the inclusion \( \mathcal{Q} \)-functor \( I : \mathcal{B} \rightarrow \mathcal{A} \) is a right adjoint. View \( F \) as a \( \mathcal{Q} \)-functor from \( \mathcal{A} \) to \( \mathcal{B} \), then \( 1_{\mathcal{A}} \leq I \circ F \). Since \( F^2 \cong F \), it follows that \( F \circ I \cong 1_\mathcal{B} \). Thus \( F \dashv I : \mathcal{A} \rightarrow \mathcal{B} \), as required. \( \square \)

Dually, we have the following proposition for \( \mathcal{Q} \)-interior systems and \( \mathcal{Q} \)-interior operators.

Proposition 4.1.5. Let \( \mathcal{A} \) be a \( \mathcal{Q} \)-category, \( \mathcal{B} \) an isomorphism-closed \( \mathcal{Q} \)-subcategory of \( \mathcal{A} \). The following conditions are equivalent:

1. \( \mathcal{B} \) is a \( \mathcal{Q} \)-interior system of \( \mathcal{A} \).

2. There is a \( \mathcal{Q} \)-interior operator \( F : \mathcal{A} \rightarrow \mathcal{A} \) such that \( \mathcal{B}_0 = \{ x \in \mathcal{A}_0 : Fx \cong x \} \).

Remark 4.1.6. For a \( \mathcal{Q} \)-category \( \mathcal{A} \), a \( \mathcal{Q} \)-closure operator \( F : \mathcal{A} \rightarrow \mathcal{A} \) is exactly a monad [ML98] on \( \mathcal{A} \). Proposition 4.1.4 states that a \( \mathcal{Q} \)-closure system of \( \mathcal{A} \) is exactly the category of algebras for a monad on \( \mathcal{A} \).
4.1. \(Q\)-closure systems and \(Q\)-closure operators

Dually, a \(Q\)-interior operator \(F : A \rightarrow A\) is exactly a comonad on \(A\). Proposition 4.1.5 states that a \(Q\)-interior system of \(A\) is exactly the category of coalgebras for a comonad on \(A\).

The terminologies “\(Q\)-closure operator” and “\(Q\)-interior operator” come from their similarity to closure operators and interior operators in topology.

**Proposition 4.1.7.** Each \(Q\)-closure system or \(Q\)-interior system of a complete \(Q\)-category is itself a complete \(Q\)-category.

**Proof.** Let \(B\) be a \(Q\)-closure system of a complete \(Q\)-category \(A\). By Proposition 4.1.4 there is a \(Q\)-closure operator \(F : A \rightarrow A\) such that \(B_0 = \{ x \in A_0 : Fx \approx x \}\). View \(F\) as a \(Q\)-functor from \(A\) to \(B\), then \(F \circ I \cong 1_B\), where \(I : B \rightarrow A\) is the inclusion \(Q\)-functor. Thus

\[
F(\sup_A I^{\rightarrow}(\mu)) = \sup_B F^{\rightarrow} \circ I^{\rightarrow}(\mu) \quad \text{(Corollary 3.5.15)}
\]

\[
= \sup_B (F \circ I)^{\rightarrow}(\mu) \quad \text{(Proposition 3.3.16)}
\]

\[
= \sup_B (1_B)^{\rightarrow}(\mu)
\]

\[
= \sup_B \mu
\]

for all \(\mu \in \mathcal{P}B\). Therefore, it follows from Proposition 3.5.11 that \(F(A)\) is a complete \(Q\)-category.

Similarly one can prove that each \(Q\)-interior system of a complete \(Q\)-category is itself a complete \(Q\)-category. \(\square\)

**Proposition 4.1.8.** Let \(A\) be a complete \(Q\)-category, \(B\) an isomorphism-closed \(Q\)-subcategory of \(A\), and \(I : B \rightarrow A\) the inclusion \(Q\)-functor.

1. \(B\) is a \(Q\)-closure system of \(A\) if and only if \(B\) is closed with respect to infima in \(A\) in the sense that \(\inf_A I^{\rightarrow}(\lambda) \in B_0\) for all \(\lambda \in \mathcal{P}B\).

2. \(B\) is a \(Q\)-interior system of \(A\) if and only if \(B\) is closed with respect to suprema in \(A\) in the sense that \(\sup_A I^{\rightarrow}(\mu) \in B_0\) for all \(\mu \in \mathcal{P}B\).

**Proof.** We prove (1) for example. Note that a \(Q\)-closure system is a complete \(Q\)-category by Proposition 4.1.7, thus the necessity follows immediately by applying Corollary 3.5.15 to the inclusion \(Q\)-functor \(I : B \rightarrow A\).
For the sufficiency, note that for all $\lambda \in P_1^\dagger B$ and $y \in B_0$,

$$B(y, \inf_A I \rightsquigarrow (\lambda)) = A(y, \inf_A I \rightsquigarrow (\lambda))$$

$$= I \rightsquigarrow (\lambda) \setminus A(y, -) \quad \text{(Example 3.4.5)}$$

$$= (I_\varepsilon \circ \lambda) \setminus A(y, -) \quad \text{(Definition 3.3.12)}$$

$$= \lambda \setminus (I^3 \circ A(y, -)) \quad \text{(Proposition 3.1.9(2))}$$

$$= \lambda \setminus I^t(y, -)$$

$$= \lambda \setminus A(y, I^-)$$

$$= \lambda \setminus B(y, -).$$

Thus $\inf_A I \rightsquigarrow (\lambda)$ is the infimum of $\lambda$ in $B$, i.e., $\inf_A I \rightsquigarrow (\lambda) = \inf_B \lambda$. Therefore, $B$ is a complete $\mathcal{Q}$-category and $I : B \rightarrow A$ preserves infima. Then it follows from Corollary 3.5.15 that $I$ is a right adjoint in $\mathcal{Q}\text{-Cat}$, and consequently $B$ is a $\mathcal{Q}$-closure system of $A$.

Proposition 4.1.9. Let $A$ be a complete $\mathcal{Q}$-category with tensor $\otimes$ and cotensor $\rightsquigarrow$, $B$ an isomorphism-closed $\mathcal{Q}$-subcategory of $A$. Then $B$ is a $\mathcal{Q}$-closure system of $A$ if and only if

1. for every subset $\{x_i\} \subseteq B_0$ of the same type $X$, the meet $\bigwedge_i x_i$ in $A_X$ belongs to $B_0$;

2. for each $x \in B_0$ and $f \in P_1^\dagger (tx)$, the cotensor $f \rightsquigarrow x$ in $A$ belongs to $B_0$.

Dually, $B$ is a $\mathcal{Q}$-interior system of $A$ if and only if

1. for every subset $\{x_i\} \subseteq B_0$ of the same type $X$, the join $\bigvee_i x_i$ in $A_X$ belongs to $B_0$;

2. for each $x \in B_0$ and $f \in P(tx)$, the tensor $f \otimes x$ in $A$ belongs to $B_0$.

Proof. We prove the case of $\mathcal{Q}$-closure system for example. Let $I : B \rightarrow A$ be the inclusion $\mathcal{Q}$-functor.
4.1. \(\mathcal{Q}\)-closure systems and \(\mathcal{Q}\)-closure operators

**Necessity.** By Proposition 4.1.7, \(\mathcal{B}\) is itself a complete \(\mathcal{Q}\)-category with cotensor \(\mapsto\) \(\mathcal{B}\). Since \(I\) is a right adjoint \(\mathcal{Q}\)-functor between complete \(\mathcal{Q}\)-categories, it follows from Proposition 3.5.14 that \(I\) preserves cotensors. Thus

\[
f \mapsto x = f \mapsto Ix = I(f \mapsto_B x) = f \mapsto_B x \in \mathbb{B}_0
\]

for each \(x \in \mathbb{B}_0\) and \(f \in \mathcal{P}^\dagger(tx)\).

That the meet \(\bigwedge_i x_i\) in \(\mathcal{A}_X\) belongs to \(\mathbb{B}_0\) for each subset \(\{x_i\} \subseteq \mathbb{B}_X\) can be obtained similarly.

**Sufficiency.** For each \(x \in \mathbb{B}_0\) and \(f \in \mathcal{P}^\dagger(tx)\), since the cotensor \(f \mapsto x\) in \(\mathcal{A}\) belongs to \(\mathbb{B}_0\), it follows that for each \(y \in \mathbb{B}_0\),

\[
\mathbb{B}(y, f \mapsto x) = \mathcal{A}(y, f \mapsto x) = f \leftarrow A(y, x) = f \leftarrow \mathbb{B}(y, x).
\]

This means that \(f \mapsto x\) is the cotensor of \(f\) and \(x\) in \(\mathbb{B}\), i.e., \(f \mapsto x = f \mapsto_B x\). Hence, \(\mathbb{B}\) is a cotensored \(\mathcal{Q}\)-category and it is clear that \(I\) preserves cotensors in \(\mathbb{B}\).

Similarly one can prove that if the meet \(\bigwedge_i x_i\) of a subset \(\{x_i\} \subseteq \mathbb{B}_X\) in \(\mathcal{A}_X\) belongs to \(\mathbb{B}_0\), then it is also the meet of \(\{x_i\}\) in \(\mathbb{B}_X\). Thus \(\mathbb{B}\) is order-complete and \(I\) preserves underlying meets in each \(\mathbb{B}_X\). This means that \(I : \mathbb{B}_0 \rightarrow \mathcal{A}_0\) is a right adjoint in \(\mathbf{2-Cat}\).

Therefore, \(I : \mathbb{B} \rightarrow \mathcal{A}\) is a right adjoint in \(\mathcal{Q}\text{-}\mathbf{Cat}\) by Proposition 3.5.14 and the conclusion thus follows.

An immediate consequence of Proposition 4.1.9 is that the infimum in a \(\mathcal{Q}\)-closure system \(\mathbb{B}\) of a complete \(\mathcal{Q}\)-category \(\mathcal{A}\) can be calculated as

\[
\inf_{\mathbb{B}} \lambda = \bigwedge_{b \in \mathbb{B}_0} (\lambda(b) \mapsto b) \tag{4.1}
\]

for \(\lambda \in \mathcal{P}^\dagger \mathbb{B}\), where the cotensors and meets are calculated in \(\mathcal{A}\).

Dually, the supremum in a \(\mathcal{Q}\)-interior system \(\mathbb{B}\) of a complete \(\mathcal{Q}\)-category \(\mathcal{A}\) can be calculated as

\[
\sup_{\mathbb{B}} \mu = \bigvee_{b \in \mathbb{B}_0} (\mu(b) \otimes b) \tag{4.2}
\]

\(\mu \in \mathcal{P} \mathbb{B}\), where the tensors and joins are calculated in \(\mathcal{A}\).
4.2 \(\mathcal{Q}\)-closure spaces

In this section, we pay attention to monads on the \(\mathcal{Q}\)-category of contravariant presheaves, and introduce the notion of \(\mathcal{Q}\)-closure spaces.

**Definition 4.2.1.** A \(\mathcal{Q}\)-closure space is a pair \((A, C)\) that consists of a \(\mathcal{Q}\)-category \(A\) and a \(\mathcal{Q}\)-closure operator \(C : \mathcal{P}A \rightarrow \mathcal{P}A\). A continuous \(\mathcal{Q}\)-functor \(F : (A, C) \rightarrow (B, D)\) between \(\mathcal{Q}\)-closure spaces is a \(\mathcal{Q}\)-functor \(F : A \rightarrow B\) such that

\[F^{-} \circ C \leq D \circ F^{-}\]

The category of \(\mathcal{Q}\)-closure spaces and continuous \(\mathcal{Q}\)-functors is denoted by \(\mathcal{Q}\)-Cls.

**Remark 4.2.2.** If \(C\) and \(D\) are viewed as monads on \(\mathcal{P}A\) and \(\mathcal{P}B\) respectively, then a \(\mathcal{Q}\)-functor \(F : A \rightarrow B\) is continuous between \(\mathcal{Q}\)-closure spaces \((A, C)\) and \((B, D)\) if and only if \(F^{-} : \mathcal{P}A \rightarrow \mathcal{P}B\) is a lax map of monads from \(C\) to \(D\) in the sense of \([Lei04]\).

Note that for a \(\mathcal{Q}\)-closure space \((A, C)\), the \(\mathcal{Q}\)-closure operator \(C\) is idempotent since \(\mathcal{P}A\) is skeletal. Let \(C(\mathcal{P}A)\) denote the \(\mathcal{Q}\)-subcategory of \(\mathcal{P}A\) consisting of the fixed points of \(C\). Since \(\mathcal{P}A\) is a complete \(\mathcal{Q}\)-category, \(C(\mathcal{P}A)\) is also a complete \(\mathcal{Q}\)-category. A contravariant presheaf \(\mathcal{A} \rightarrow \mathcal{X}\) is said to be closed in the \(\mathcal{Q}\)-closure space \((A, C)\) if it belongs to \(C(\mathcal{P}A)\).

The following proposition states that continuous \(\mathcal{Q}\)-functors behave in a manner similar to the continuous maps between topological spaces: the inverse image of a closed contravariant presheaf is closed.

**Proposition 4.2.3.** A \(\mathcal{Q}\)-functor \(F : A \rightarrow B\) is continuous between \(\mathcal{Q}\)-closure spaces \((A, C)\) and \((B, D)\) if and only if \(F^{-}(\lambda) \in C(\mathcal{P}A)\) whenever \(\lambda \in D(\mathcal{P}B)\).

**Proof.** It suffices to show that \(F^{-} \circ C \leq D \circ F^{-}\) if and only if \(C \circ F^{-} \circ D \leq F^{-} \circ D\).

Suppose \(F^{-} \circ C \leq D \circ F^{-}\), then

\[F^{-} \circ C \circ F^{-} \circ D \leq D \circ F^{-} \circ F^{-} \circ D \leq D \circ D = D,\]
and consequently $C \circ F^\leftarrow \circ D \leq F^\leftarrow \circ D$.

Conversely, suppose $C \circ F^\leftarrow \circ D \leq F^\leftarrow \circ D$, then

$$C \leq C \circ F^\leftarrow \circ F^\rightarrow \leq C \circ F^\leftarrow \circ D \circ F^\rightarrow \leq F^\leftarrow \circ D \circ F^\rightarrow,$$

and consequently $F^\rightarrow \circ C \leq D \circ F^\rightarrow$.

Given a \(Q\)-category \(\mathbb{A}\), there are naturally two \(Q\)-closure spaces with \(\mathbb{A}\) being the underlying \(Q\)-category. One is the discrete \(Q\)-closure space \((\mathbb{A}, 1_{\mathbb{P}\mathbb{A}})\), in which every contravariant presheaf \(\mu \in \mathbb{P}\mathbb{A}\) is closed. The other one is the trivial \(Q\)-closure space \((\mathbb{A}, T_\mathbb{A})\) given by

$$T_\mathbb{A}(\mu)(x) = \top_{tx, t\mu}$$

for all \(\mu \in \mathbb{P}\mathbb{A}\) and \(x \in \mathbb{A}_0\), in which \(\mu \in \mathbb{P}\mathbb{A}\) is closed if and only if \(\mu\) is the largest \(Q\)-distributor in \(Q\)-Dist(\(\mathbb{A}, \ast_{t\mu}\))

It is easy to see that each \(Q\)-functor \(F : \mathbb{A} \rightarrow \mathbb{B}\) is continuous between the \(Q\)-closure spaces \((\mathbb{A}, 1_{\mathbb{P}\mathbb{A}})\) and \((\mathbb{B}, 1_{\mathbb{P}\mathbb{B}})\), and also continuous between the \(Q\)-closure spaces \((\mathbb{A}, T_\mathbb{A})\) and \((\mathbb{B}, T_\mathbb{B})\). Thus, we obtain a functor

\[
D : Q\text{-Cat} \rightarrow Q\text{-Cls}
\]

that sends a \(Q\)-category \(\mathbb{A}\) to the corresponding discrete \(Q\)-closure space \((\mathbb{A}, 1_{\mathbb{P}\mathbb{A}})\), and a functor

\[
T : Q\text{-Cat} \rightarrow Q\text{-Cls}
\]

that sends a \(Q\)-category \(\mathbb{A}\) to the corresponding trivial \(Q\)-closure space \((\mathbb{A}, T_\mathbb{A})\).

It is well known that the forgetful functor from the category of topological spaces to the category of (small) sets has a left adjoint creating the discrete topology on a set, and a right adjoint creating the trivial topology on a set. We have the following analogue conclusion for \(Q\)-closure spaces.

**Proposition 4.2.4.** The forgetful functor \(Q\text{-Cls} \rightarrow Q\text{-Cat}\) has a left adjoint \(D : Q\text{-Cat} \rightarrow Q\text{-Cls}\), and a right adjoint \(T : Q\text{-Cat} \rightarrow Q\text{-Cls}\).
Chapter 4. \(\mathcal{Q}\)-closure spaces

**Proof.** It is not difficult to verify that the correspondence \( F \mapsto F \) induces a bijection

\[
\mathcal{Q}\text{-Cat}(\mathcal{A}, \mathcal{B}) \cong \mathcal{Q}\text{-Cls}((\mathcal{A}, 1_{\mathcal{P}\mathcal{A}}), (\mathcal{B}, D))
\]

natural in each \(\mathcal{Q}\)-category \(\mathcal{A}\) and \(\mathcal{Q}\)-closure space \((\mathcal{B}, D)\). Also, the correspondence \( F \mapsto F \) induces a bijection

\[
\mathcal{Q}\text{-Cat}(\mathcal{A}, \mathcal{B}) \cong \mathcal{Q}\text{-Cls}((\mathcal{A}, C), (\mathcal{B}, T_{\mathcal{B}}))
\]

natural in each \(\mathcal{Q}\)-closure space \((\mathcal{A}, C)\) and \(\mathcal{Q}\)-category \(\mathcal{B}\). The conclusion thus follows. \(\Box\)

### 4.3 Relationship with complete \(\mathcal{Q}\)-categories

It follows from Proposition 4.2.3 that a continuous \(\mathcal{Q}\)-functor \( F : (\mathcal{A}, C) \longrightarrow (\mathcal{B}, D) \) between \(\mathcal{Q}\)-closure spaces induces a pair of \(\mathcal{Q}\)-functors

\[
F^\circ = D \circ F^\rightarrow : C(\mathcal{P}\mathcal{A}) \longrightarrow D(\mathcal{P}\mathcal{B}) \quad \text{and} \quad F^\circ = F^\leftarrow : D(\mathcal{P}\mathcal{B}) \longrightarrow C(\mathcal{P}\mathcal{A}).
\]

**Proposition 4.3.1.** If \( F : (\mathcal{A}, C) \longrightarrow (\mathcal{B}, D) \) is a continuous \(\mathcal{Q}\)-functor between \(\mathcal{Q}\)-closure spaces, then \( F^\circ \) and \( F^\circ \) are \(\mathcal{Q}\)-functors.

**Proof.** It is sufficient to check that

\[
\mathcal{P}\mathcal{B}(D \circ F^\rightarrow(\mu), \lambda) = \mathcal{P}\mathcal{B}(F^\rightarrow(\mu), \lambda)
\]

for all \(\mu \in C(\mathcal{P}\mathcal{A})\) and \(\lambda \in D(\mathcal{P}\mathcal{B})\) since it holds that \(\mathcal{P}\mathcal{A}(\mu, F^\leftarrow(\lambda)) = \mathcal{P}\mathcal{B}(F^\rightarrow(\mu), \lambda)\).

Indeed, since \(D\) is a \(\mathcal{Q}\)-closure operator,

\[
\mathcal{P}\mathcal{B}(F^\rightarrow(\mu), \lambda) \leq \mathcal{P}\mathcal{B}(D \circ F^\rightarrow(\mu), D(\lambda)) = \mathcal{P}\mathcal{B}(D \circ F^\rightarrow(\mu), \lambda) = \lambda \vee (D \circ F^\rightarrow(\mu)) \leq \lambda \vee F^\rightarrow(\mu) = \mathcal{P}\mathcal{B}(F^\rightarrow(\mu), \lambda),
\]

hence \(\mathcal{P}\mathcal{B}(D \circ F^\rightarrow(\mu), \lambda) = \mathcal{P}\mathcal{B}(F^\rightarrow(\mu), \lambda)\). \(\Box\)
The above proposition gives rise to a functor
\[ T : \text{Q-Cls} \rightarrow \text{Q-CCat} \]
that maps each continuous \( \text{Q} \)-functor
\[ F : (A, C) \rightarrow (B, D) \]
to the left adjoint \( \text{Q} \)-functor
\[ F^\circ : C(\mathcal{P}A) \rightarrow D(\mathcal{P}B) \]
between skeletal complete \( \text{Q} \)-categories.

For each complete \( \text{Q} \)-category \( A \), it follows from Theorem 3.5.11 and Example 4.1.3 that \( C_A = Y \circ \sup : \mathcal{P}A \rightarrow \mathcal{P}A \) is a \( \text{Q} \)-closure operator, hence \( (A, C_A) \) is a \( \text{Q} \)-closure space.

**Proposition 4.3.2.** If \( F : A \rightarrow B \) is a left adjoint \( \text{Q} \)-functor between complete \( \text{Q} \)-categories, then \( F : (A, C_A) \rightarrow (B, C_B) \) is a continuous \( \text{Q} \)-functor.

**Proof.** For all \( \mu \in \mathcal{P}A \),
\[
F^\rightarrow \circ C_A(\mu) = C_A(\mu) \circ F^\rightarrow
\leq B(F(\cdot, F(\sup_A \mu)) \circ F^\rightarrow
\leq B(\cdot, F(\sup_A \mu)) = F_B(\cdot, F(\sup_A \mu)) \circ F^\rightarrow
= B(\cdot, \sup_B F^\rightarrow(\mu)) \quad (F_1 \dashv F^\rightarrow : A \rightarrow B \text{ in } \text{Q-Dist})
\]
\[
= C_B \circ F^\rightarrow(\mu).
\]
Hence \( F : (A, C_A) \rightarrow (B, C_B) \) is continuous.

The above proposition gives a functor
\[ D : \text{Q-CCat} \rightarrow \text{Q-Cls}. \]
Proposition 4.3.3. The functor $T \circ D$ is naturally isomorphic to the identity functor on $\mathcal{Q}$-$\mathcal{CCat}$. In particular, each skeletal complete $\mathcal{Q}$-category $\mathcal{A}$ is isomorphic to $T \circ D(\mathcal{A})$.

Proof. For each $x \in \mathcal{A}_0$, since

$$Y_{\Lambda} x = Y_{\Lambda} \circ \sup_{\mathcal{A}} \circ Y_{\Lambda} x = C_{\Lambda} \circ Y_{\Lambda} x,$$

it follows that

$$T \circ D(\mathcal{A}) = C_{\Lambda}(\mathcal{P}_{\Lambda}) = \{ Y_{\Lambda} x \mid x \in \mathcal{A}_0 \}.$$ 

Thus we get that $Y_{\Lambda} x$ is closed in the $\mathcal{Q}$-closure space $(\mathcal{A}, C_{\Lambda})$. By Yoneda lemma, the correspondence $x \mapsto Y_{\Lambda} x$ induces a fully faithful $\mathcal{Q}$-functor $Y_{\Lambda} : \mathcal{A} \longrightarrow C_{\Lambda}(\mathcal{P}_{\Lambda})$. It is clear that $Y_{\Lambda}$ is surjective, hence an isomorphism of skeletal $\mathcal{Q}$-categories.

To see the naturality of $\{ Y_{\Lambda} \}$, for each left $\mathcal{Q}$-functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between skeletal complete $\mathcal{Q}$-categories, we prove the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{Y_{\Lambda}} & C_{\Lambda}(\mathcal{P}_{\Lambda}) \\
F \downarrow & & \downarrow T \circ D(F) = C_{\mathcal{B}} \circ F^{-} \\
\mathcal{B} & \xrightarrow{Y_{\mathcal{B}}} & C_{\mathcal{B}}(\mathcal{P}_{\mathcal{B}})
\end{array}$$

Indeed, by Proposition 3.3.17 and Equation (4.3), it follows immediately that

$$C_{\mathcal{B}} \circ F^{-} \circ Y_{\Lambda} = C_{\mathcal{B}} \circ Y_{\mathcal{B}} \circ F = Y_{\mathcal{B}} \circ F.$$

By Proposition 4.3.3, if we identify a skeletal complete $\mathcal{Q}$-category $\mathcal{A}$ with the $\mathcal{Q}$-subcategory $C_{\Lambda}(\mathcal{P}_{\Lambda})$ of $\mathcal{P}_{\Lambda}$, then the functor $T : \mathcal{Q}$-$\mathcal{Cls} \longrightarrow \mathcal{Q}$-$\mathcal{CCat}$ can be viewed as a left inverse of $D : \mathcal{Q}$-$\mathcal{CCat} \longrightarrow \mathcal{Q}$-$\mathcal{Cls}$.

Theorem 4.3.4. $T : \mathcal{Q}$-$\mathcal{Cls} \longrightarrow \mathcal{Q}$-$\mathcal{CCat}$ is a left inverse (up to isomorphism) and left adjoint of $D : \mathcal{Q}$-$\mathcal{CCat} \longrightarrow \mathcal{Q}$-$\mathcal{Cls}$. 

\end{proof}
Proof. It remains to show that $T$ is a left adjoint of $D$. Given a $Q$-closure space $(A, C)$, denote $C(PA)$ by $X$, then $D \circ T(A, C) = (X, C_X)$. Let $\eta_{(A, C)} = C \circ Y_A : A \longrightarrow X$. We show that $\eta = \{\eta_{(A, C)}\}$ is a natural transformation from the identity functor to $D \circ T$ and it is the unit of the desired adjunction.

**Step 1.** $\eta_{(A, C)} : (A, C) \longrightarrow (X, C_X)$ is a continuous $Q$-functor, i.e. $\eta_{(A, C)} \circ C \leq C_X \circ \eta_{(A, C)}$.

Firstly, we show that $C(\mu) = \sup X \circ \eta_{(A, C)}(\mu)$ for all $\mu \in PA$. Consider the diagram:

```
\[
P_{PA} \xrightarrow{Y_A^{-}} P(P_{PA}) \xrightarrow{\sup_{PA}} PA
\]
```

The commutativity of the left triangle follows from $\eta_{(A, C)} = C \circ Y_A$. Since $C : PA \longrightarrow X$ is a left adjoint in $Q$-$\text{Cat}$ (obtained in the proof of Proposition 4.1.4), it preserves suprema (Corollary 3.5.15), thus the right square commutes. The whole diagram is then commutative. For each $\mu \in PA$, we have that

\[
\mu = \mu \circ A = \mu \circ Y_A^{-} \circ (Y_A)\mu = Y_A^{-}(\mu) \circ (Y_A)\mu = \sup_{PA} \circ Y_A^{-}(\mu),
\]

where the second equality comes from the fact that the Yoneda embedding $Y_A$ is fully faithful and Proposition 3.3.5(1), while the last equality comes from Example 3.5.12. Consequently,

\[
C(\mu) = C \circ \sup_{PA} \circ Y_A^{-}(\mu) = \sup X \circ \eta_{(A, C)}(\mu)
\]

for all $\mu \in PA$. 

Secondly, we show that \( \eta_{(A,C)} \rightarrow (A,C)(\mu) \leq X(-,\mu) \) for each \( \mu \in X \). Indeed,

\[
\begin{align*}
\eta_{(A,C)} \rightarrow (A,C)(\mu) &= \mu \circ \eta_{(A,C)} \\
&= \mu \circ (C \circ Y_A)^2 \\
&= \mathcal{P} A Y_A(-,\mu) \circ Y_A^2 \circ C^2 \\
&= (Y_A^2 \dashv Y_A : A \rightarrow \mathcal{P} A \text{ in } \mathcal{Q}-\text{Dist}) \\
&\leq \mathcal{P} A (-,\mu) \circ C^2 \\
&\leq X(-,\mu) \circ C^2 \\
&= C^2(-,\mu) \\
&\leq X(-,\mu).
\end{align*}
\]

(\( C^2 \dashv C : \mathcal{P} A \rightarrow X \text{ in } \mathcal{Q}-\text{Dist} \))

Therefore, for all \( \mu \in \mathcal{P} A \),

\[
\eta_{(A,C)} \circ C(\mu) \leq Y_X \circ \sup X \circ \eta_{(A,C)}(\mu) = C_X \circ \eta_{(A,C)}(\mu),
\]

as desired.

**Step 2.** \( \eta = \{ \eta_{(A,C)} \} \) is a natural transformation. Let \( F : (A,C) \rightarrow (B,D) \) be a continuous \( \mathcal{Q} \)-functor, we must show that

\[
D \circ Y_B \circ F = \eta_{(B,D)} \circ F = D \circ T \circ F \circ \eta_{(A,C)} = D \circ F \rightarrow \circ C \circ Y_A.
\]

Firstly, since \( C \) is a \( \mathcal{Q} \)-closure operator, by Proposition 3.3.17

\[
Y_B \circ F = F \rightarrow \circ Y_A \leq F \rightarrow \circ C \circ Y_A,
\]

and consequently \( D \circ Y_B \circ F \leq D \circ F \rightarrow \circ C \circ Y_A \).

Secondly, the continuity of \( F \) leads to

\[
F \rightarrow \circ C \circ Y_A \leq D \circ F \rightarrow \circ Y_A = D \circ Y_B \circ F,
\]

hence \( D \circ F \rightarrow \circ C \circ Y_A \leq D \circ Y_B \circ F \).

**Step 3.** \( \eta_{(A,C)} : (A,C) \rightarrow (X,C_X) \) is universal in the sense that for any skeletal complete \( \mathcal{Q} \)-category \( B \) and continuous \( \mathcal{Q} \)-functor \( F : (A,C) \rightarrow (B,C_B) \),
4.3. Relationship with complete \( \mathcal{Q} \)-categories

there exists a unique left adjoint \( \mathcal{Q} \)-functor \( \mathcal{T} : \mathcal{X} \rightarrow \mathcal{B} \) that makes the following diagram commute:

\[
\begin{array}{ccc}
(A, C) & \xrightarrow{\eta_{(A,C)}} & (X, C_X) \\
\downarrow F & & \downarrow \mathcal{T} \\
(B, C_B) & & (X, C_X)
\end{array}
\] (4.5)

**Existence.** Let \( \mathcal{T} = \sup_B \circ F \rightarrow : \mathcal{X} \rightarrow \mathcal{B} \) be the following composition of \( \mathcal{Q} \)-functors

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\mathcal{P}A} & \mathcal{P}B \\
\mathcal{P}B & \xrightarrow{\sup_B} & \mathcal{B}
\end{array}
\]

First, \( \mathcal{T} : \mathcal{X} \rightarrow \mathcal{B} \) is a left adjoint in \( \mathcal{Q} \)-\( \text{Cat} \). Indeed, \( \mathcal{T} \) has a right adjoint \( G : \mathcal{B} \rightarrow \mathcal{X} \) given by \( G = F^\circ \circ \mathcal{Y}_B \). \( G \) is well-defined since \( \mathcal{Y}_B b \) is a closed in \( (\mathcal{B}, C_B) \) for each \( b \in \mathcal{B}_0 \). For all \( \mu \in \mathcal{X}_0 \) and \( y \in \mathcal{B}_0 \), it holds that

\[
\begin{align*}
\mathcal{B}(\mathcal{T}(\mu), y) &= \mathcal{B}(-, y) \triangleleft F^\circ(\mu) \\
&= \mathcal{B}(-, y) \triangleleft (\mu \circ F^\circ) \\
&= (\mathcal{B}(-, y) \circ F^\circ) \triangleleft \mu \quad \text{(Proposition 3.1.9(2))} \\
&= F^\circ(-, y) \triangleleft \mu \quad \text{(Remark 3.3.10(5))} \\
&= \mathcal{Y}_B(\mu, F^\circ \circ \mathcal{Y}_B y) \quad \text{(Definition of } F^\circ \text{ and } F^\circ) \\
&= \mathcal{X}(\mu, G y),
\end{align*}
\]

hence \( \mathcal{T} \) is a left adjoint of \( G \).

Second, \( F = \mathcal{T} \circ \eta_{(A,C)} \). Note that for all \( x \in \mathcal{A}_0 \),

\[
\begin{align*}
\mathcal{B}(Fx, -) &= F^\circ(x, -) \\
&= (\mathcal{B} \triangleleft F^\circ)(x, -) \quad \text{(Proposition 3.1.9(1))} \\
&= \mathcal{B} \triangleleft F^\circ(-, x) \\
&= \mathcal{B} \triangleleft (\mathcal{Y}_B \circ Fx) \\
&= \mathcal{B} \triangleleft (F^\circ \circ \mathcal{Y}_A x), \quad \text{(Proposition 3.3.17)}
\end{align*}
\]
thus $F = \sup_B \circ F \to \circ Y_A$. Consequently
\[
\overline{F} \circ \eta_{(A,C)} = \sup_B \circ F \to \circ C \circ Y_A
\leq \sup_B \circ C_B \circ F \to \circ Y_A \quad (F \text{ is continuous})
\]
\[
= \sup_B \circ Y_B \circ \sup_B \circ F \to \circ Y_A
= \sup_B \circ F \to \circ Y_A \quad (\sup_B \dashv Y_B : \mathcal{P}B \to B)
\]
\[
= F.
\]
Conversely, since $C$ is a $Q$-closure operator, it is clear that
\[
F = \sup_B \circ F \to \circ Y_A \leq \sup_B \circ F \to \circ C \circ Y_A = \overline{F} \circ \eta_{(A,C)},
\]
hence $F \cong \overline{F} \circ \eta_{(A,C)}$, and consequently $F = \overline{F} \circ \eta_{(A,C)}$ since $B$ is skeletal.

**Uniqueness.** Suppose $H : \mathcal{X} \to B$ is another left adjoint $Q$-functor that makes Diagram (4.5) commute. For each $\mu \in \mathcal{X}$, since $C : \mathcal{P}A \to \mathcal{X}$ is a left adjoint in $Q$-$\mathbf{Cat}$, we have
\[
\mu = C(\mu) = C(\mu \circ A) = C\left( \bigvee_{x \in A_0} \mu(x) \circ Y_Ax \right)
= \bigvee_{x \in A_0} \mu(x) \otimes_X C(Y_Ax),
\]
where the last equality follows from Example 3.5.6 and Proposition 3.5.14. It follows that
\[
H(\mu) = H\left( \bigvee_{x \in A_0} \mu(x) \otimes_X C(Y_Ax) \right)
= \bigvee_{x \in A_0} \mu(x) \otimes_B (H \circ \eta_{(A,C)})(x) \quad \text{(Proposition 3.5.14)}
= \bigvee_{x \in A_0} \mu(x) \otimes_B Fx.
\]
Consequently,
\[
\mathcal{B}(H(\mu), -) = \mathcal{B}\left( \bigvee_{x \in A_0} \mu(x) \otimes_B Fx, - \right)
= \bigwedge_{x \in A_0} \mathcal{B}(\mu(x) \otimes_B Fx, -) \quad \text{(Proposition 3.5.4)}
= \bigwedge_{x \in A_0} \mathcal{B}(Fx, -) \wedge \mu(x)
= F_{x} \wedge \mu.
Since $\mathcal{B}$ is skeletal, it follows that $H(\mu) = \colim_\mu F = \sup_\mathcal{B} \circ F \rightarrow (\mu)$. Therefore, $H = \sup_\mathcal{B} \circ F \rightarrow = \mathcal{T}$. \hfill \Box

The following corollary shows that the free cocompletion functor of $\mathcal{Q}$-categories factors through the category of $\mathcal{Q}$-closure spaces via a left adjoint functor $D : \mathcal{Q} \text{-Cat} \rightarrow \mathcal{Q} \text{-Cls}$ and a left adjoint functor $T : \mathcal{Q} \text{-Cls} \rightarrow \mathcal{Q} \text{-CCat}$.

**Corollary 4.3.5.** *Both the outer triangle and the inner triangle of the diagram*

![Diagram](image)

*commute, where $U$ denote the obvious forgetful functors.*

*Proof.* Follows immediately from the definitions of these functors. \hfill \Box
Chapter 5

Isbell Adjunctions in $\mathcal{Q}$-categories

The Isbell adjunction introduced in this chapter generalize the classical Isbell adjunction in category theory (see Proposition 2.2.2). Each $\mathcal{Q}$-distributor between $\mathcal{Q}$-categories gives rise to an Isbell adjunction between $\mathcal{Q}$-categories of contravariant presheaves and covariant presheaves, and hence to a monad. We prove that this process is functorial from the category of $\mathcal{Q}$-distributors and infomorphisms to the category of complete $\mathcal{Q}$-categories and left adjoint $\mathcal{Q}$-functors. Furthermore, the free cocompletion functor of $\mathcal{Q}$-categories factors through this functor.

5.1 Isbell Adjunctions

Given a $\mathcal{Q}$-distributor $\phi : \mathcal{A} \to \mathcal{B}$, define a pair of $\mathcal{Q}$-functors

$$\phi^\uparrow : \mathcal{P}\mathcal{A} \to \mathcal{P}\mathcal{B} \quad \text{and} \quad \phi^\downarrow : \mathcal{P}\mathcal{B} \to \mathcal{P}\mathcal{A}$$

by

$$\phi^\uparrow(\mu) = \phi \uparrow \mu \quad \text{and} \quad \phi^\downarrow(\lambda) = \lambda \downarrow \phi. \quad (5.1)$$

It should be warned that $\phi^\uparrow$ and $\phi^\downarrow$ are both contravariant with respect to local orders in $\mathcal{Q}\text{-}\text{Dist}$ by Remark 3.3.8 i.e.,

$$\forall \mu_1, \mu_2 \in \mathcal{P}\mathcal{A}, \mu_1 \leq \mu_2 \Rightarrow \phi^\uparrow(\mu_2) \leq \phi^\uparrow(\mu_1) \quad (5.2)$$

and

$$\forall \lambda_1, \lambda_2 \in \mathcal{P}\mathcal{B}, \lambda_1 \leq \lambda_2 \Rightarrow \phi^\downarrow(\lambda_2) \leq \phi^\downarrow(\lambda_1). \quad (5.3)$$
Proposition 5.1.1. $\phi \dashv \phi^\downarrow : \mathcal{P}A \to \mathcal{P}^\dagger B$ in $\mathcal{Q}\text{-Cat}$.

**Proof.** For all $\mu \in \mathcal{P}A$ and $\lambda \in \mathcal{P}^\dagger B$,

$$\mathcal{P}^\dagger B(\phi^\uparrow(\mu), \lambda) = \lambda \searrow \phi^\uparrow(\mu)$$

$$= \lambda \searrow (\phi \searrow \mu)$$

$$= (\lambda \searrow \phi) \searrow \mu$$

$$= \phi^\downarrow(\lambda) \searrow \mu$$

$$= \mathcal{P}A(\mu, \phi^\downarrow(\lambda)).$$

Hence the conclusion holds. \qed

We would like to stress that

$$\mu \leq \phi^\downarrow \circ \phi^\uparrow(\mu) \quad \text{and} \quad \lambda \leq \phi^\uparrow \circ \phi^\downarrow(\lambda) \quad (5.4)$$

for all $\mu \in \mathcal{P}A$ and $\lambda \in \mathcal{P}^\dagger B$ by Remark 3.3.8.

The Isbell adjunction presented in Proposition 3.4.4 is a special case of Proposition 5.1.1, i.e., letting $\mathcal{B} = \mathcal{A}$ and $\phi = \mathcal{A}$. So, the adjunction $\phi^\uparrow \dashv \phi^\downarrow : \mathcal{P}A \to \mathcal{P}^\dagger B$ is a generalization of the Isbell adjunction. As we shall see, all adjunctions between $\mathcal{P}A$ and $\mathcal{P}^\dagger B$ are of this form, and will be called Isbell adjunctions by abuse of language.

Recall that each $\mathcal{Q}$-functor $F : \mathcal{A} \to \mathcal{P}^\dagger B$ corresponds to a $\mathcal{Q}$-distributor $\Gamma F \neg : \mathcal{A} \leftrightarrow \mathcal{B}$ given by Equation (3.13), and each $\mathcal{Q}$-functor $G : \mathcal{B} \to \mathcal{P}A$ corresponds to a $\mathcal{Q}$-distributor $\Gamma G \neg : \mathcal{A} \leftrightarrow \mathcal{B}$ given by Equation (3.14).

Proposition 5.1.2. Let $\phi : \mathcal{A} \leftrightarrow \mathcal{B}$ be a $\mathcal{Q}$-distributor, then $\Gamma \phi^\uparrow \circ \mathcal{Y} \mathcal{A} \neg = \phi = \Gamma \phi^\downarrow \circ \mathcal{Y} \mathcal{B} \neg$. 


Proof. For all $x \in A_0$ and $y \in B_0$,

$$
\Gamma \phi \downarrow \circ Y \uparrow A(x, y) = (\phi \downarrow \circ Y \uparrow A)(y)
= (\phi \downarrow \circ (Y \uparrow A))(y)
= \phi(-, y) \downarrow \circ A(-, x)
= \phi(x, y)
= \mathbb{B}(y, -) \downarrow \phi(x, -)
= ((Y \uparrow B)y) \downarrow \phi(x)
= (\phi^+ \circ Y \uparrow B y)(x)
= \Gamma \phi^+ \circ Y \uparrow B \uparrow (x, y),
$$

showing that the conclusion holds.

Given a $Q$-distributor $\phi : A \longrightarrow B$, it follows from Example 4.1.3 that $\phi^+ \circ \phi \downarrow : PA \longrightarrow PA$ is a $Q$-closure operator and $\phi \downarrow \circ \phi^+ : P^\uparrow B \longrightarrow P^\uparrow B$ is a $Q$-interior operator. For each $y \in B_0$, since

$$
\bar{\phi}y = \phi(-, y) = \phi^+ \circ Y \uparrow B y = \phi^+ \circ \phi \downarrow \circ \phi^+ \circ Y \uparrow B y,
$$

(5.5)

it follows that $\bar{\phi}y = \phi(-, y)$ is closed in the $Q$-closure space $(A, \phi^+ \circ \phi \downarrow)$. Dually, for all $x \in A_0$,

$$
\bar{\phi}x = \phi(x, -) = \phi \downarrow \circ Y \uparrow A x = \phi \downarrow \circ \phi^+ \circ \phi \downarrow \circ Y \uparrow A x
$$

(5.6)

is a fixed point of the $Q$-interior operator $\phi \downarrow \circ \phi^+$. These facts will be used in the proofs of Theorem 5.2.2 and Theorem 5.4.4.

**Theorem 5.1.3.** Let $A$ and $B$ be $Q$-categories. The correspondence $\phi \mapsto \phi \downarrow$ is an isomorphism of posets

$$
Q\text{-Dist}(A, B) \cong Q\text{-CCat}^{co}(PA, P^\uparrow B).
$$

Proof. Let $F : PA \longrightarrow P^\uparrow B$ be a left adjoint $Q$-functor. We show that the correspondence $F \mapsto \Gamma F \circ Y \uparrow A$ is an inverse of the correspondence $\phi \mapsto \phi \downarrow$, and thus they are both isomorphisms of posets between $Q\text{-Dist}(A, B)$ and $Q\text{-CCat}^{co}(PA, P^\uparrow B)$. 

Firstly, we show that both of the correspondences are order-preserving. Indeed,

\[ \phi \leq \psi \text{ in } Q\text{-Dist}(\mathbb{A}, \mathbb{B}) \]

\[ \iff \forall \mu \in \mathcal{P}\mathbb{A}, \phi_\downarrow(\mu) = \phi \not< \mu \leq \psi \not< \mu = \psi_\downarrow(\mu) \text{ in } Q\text{-Dist} \]

\[ \iff \forall \mu \in \mathcal{P}\mathbb{A}, \phi_\downarrow(\mu) \geq \psi_\downarrow(\mu) \text{ in } (\mathcal{P}^\dagger\mathbb{B})_0 \]

\[ \iff \phi_\downarrow \leq \psi_\downarrow \text{ in } Q\text{-CCat}^{\text{co}}(\mathcal{P}\mathbb{A}, \mathcal{P}^\dagger\mathbb{B}) \]

and

\[ F \leq G \text{ in } Q\text{-CCat}^{\text{co}}(\mathcal{P}\mathbb{A}, \mathcal{P}^\dagger\mathbb{B}) \]

\[ \iff \forall \mu \in \mathcal{P}\mathbb{A}, F(\mu) \geq G(\mu) \text{ in } (\mathcal{P}^\dagger\mathbb{B})_0 \]

\[ \iff \forall \mu \in \mathcal{P}\mathbb{A}, F(\mu) \leq G(\mu) \text{ in } Q\text{-Dist}(\mathbb{A}, \mathbb{B}) \]

\[ \Rightarrow \forall x \in \mathbb{A}_0, F \circ Y_{\mathbb{A}} x \leq G \circ Y_{\mathbb{A}} x \text{ in } Q\text{-Dist}(\mathbb{A}, \mathbb{B}) \]

\[ \iff \forall x \in \mathbb{A}_0, \Gamma F \circ Y_{\mathbb{A}} \odot (x, -) \leq \Gamma G \circ Y_{\mathbb{A}} \odot (x, -) \text{ in } Q\text{-Dist}(\mathbb{A}, \mathbb{B}) \]

\[ \iff \Gamma F \circ Y_{\mathbb{A}} \odot \leq \Gamma G \circ Y_{\mathbb{A}} \odot \text{ in } Q\text{-Dist}(\mathbb{A}, \mathbb{B}). \]

Secondly, \( F = (\Gamma F \circ Y_{\mathbb{A}} \odot)_\dagger \). For all \( \mu \in \mathcal{P}\mathbb{A} \), since \( F \) is a left adjoint in \( Q\text{-Cat} \), by Example 3.5.6 and Proposition 3.5.14 we have

\[
F(\mu) = F(\mu \circ \mathbb{A}) \\
= F\left( \bigvee_{x \in \mathbb{A}_0} \mu(x) \circ Y_{\mathbb{A}} x \right) \\
= \bigwedge_{x \in \mathbb{A}_0} (F \circ Y_{\mathbb{A}} x) \not< \mu(x) \\
= \Gamma F \circ Y_{\mathbb{A}} \not< \mu \\
= (\Gamma F \circ Y_{\mathbb{A}} \odot)_\dagger(\mu).
\]

Finally, \( \phi = \Gamma \phi_\downarrow \circ Y_{\mathbb{A}} \odot \). This is obtained in Proposition 5.1.2.

As a summary of Proposition 3.6.1 and Theorem 5.1.3 we have the following
5.2. Functoriality of the Isbell adjunction

In this section, we show that the construction of Isbell adjunctions is functorial from the category $\mathcal{Q}$-\textit{Info} to $\mathcal{Q}$-\textit{Cls}, and thus to $\mathcal{Q}$-$\text{CCat}$.

**Proposition 5.2.1.** Let $(F,G) : \phi \rightarrow \psi$ be an infomorphism between $\mathcal{Q}$-distributors $\phi : \mathbb{A} \rightarrow \mathbb{B}$ and $\psi : \mathbb{A}' \rightarrow \mathbb{B}'$. Then $F : (\mathbb{A}, \phi^\uparrow \circ \phi^\downarrow) \rightarrow (\mathbb{A}', \psi^\uparrow \circ \psi^\downarrow)$ is a continuous $\mathcal{Q}$-functor.

**Proof.** Consider the following diagram:

We must prove $F^\rightarrow \circ \phi^\downarrow \circ \phi^\uparrow \leq \psi^\downarrow \circ \psi^\uparrow \circ F^\rightarrow$. To this end, it suffices to check that

(a) the left square commutes if and only if $(F,G) : \phi \rightarrow \psi$ is an infomorphism; and

(b) $F^\rightarrow \circ \phi^\downarrow \leq \psi^\downarrow \circ G^\rightarrow$ if and only if $G^\downarrow \circ \phi \leq \psi \circ F^\rightarrow$. 

For (a), suppose $G^\downarrow \circ \phi \uparrow = \psi \circ F^\rightarrow$, then for all $x \in A_0$, 

\[
G^\uparrow \circ \phi (x, -) = G^\downarrow (\phi(x, -)) \tag{Definition 3.3.12}
\]

\[
= G^\downarrow (\phi \uparrow \circ Y_A x) \tag{Proposition 5.1.2}
\]

\[
= \psi \circ (F^\rightarrow \circ Y_A x) \tag{Definition 3.3.12}
\]

\[
= \psi \uparrow (F^\rightarrow \circ Y_A x) \tag{Equation (5.1)}
\]

\[
= (\psi \circ F_\downarrow) \uparrow A(-, x) \tag{Proposition 3.1.9(2)}
\]

\[
= \psi \circ F_\downarrow (x, -). \tag{Definition 3.3.12}
\]

Conversely, if $(F, G) : \phi \longrightarrow \psi$ is an infomorphism, then for all $\mu \in P.A$, 

\[
G^\downarrow \circ \phi \uparrow (\mu) = G^\uparrow (\phi \uparrow \mu) \tag{Equation (5.1)}
\]

\[
= (G^\uparrow \circ \phi) \uparrow \mu \tag{Proposition 3.1.9(3)}
\]

\[
= (\psi \circ F_\downarrow) \uparrow \mu \tag{Definition 3.7.1}
\]

\[
= \psi \uparrow (\mu \circ F^\uparrow) \tag{Proposition 3.1.9(2)}
\]

\[
= \psi \uparrow F^\rightarrow (\mu). \tag{Definition 3.3.12}
\]

For (b), suppose $F^\rightarrow \circ \phi \downarrow \leq \psi \downarrow \circ G^\uparrow$, then for all $y' \in B_0'$, 

\[
G^\uparrow (-, y') \circ \phi = G^\downarrow (y', -) \ \downarrow \phi \tag{Proposition 3.1.9(1)}
\]

\[
= \phi \downarrow (G^\downarrow (y', -)) \tag{Equation (5.1)}
\]

\[
\leq F^\rightarrow \circ F^\rightarrow \circ \phi \downarrow (G^\downarrow (y', -)) \tag{F^\rightarrow \dashv F^\rightarrow : P.A \rightarrow P.A'}
\]

\[
= F^\rightarrow \circ \psi \downarrow \circ G^\uparrow \circ G^\downarrow \circ Y_{B'} y'
\]

\[
= F^\rightarrow \circ \psi \downarrow \circ Y_{B'} y' \tag{Formula (3.10) and (5.3)}
\]

\[
= F^\rightarrow (\psi(-, y')) \tag{Proposition 5.1.2}
\]

\[
= \psi(-, y') \circ F_\downarrow. \tag{Definition 3.3.12}
\]
Conversely, if $G \circ \phi \leq \psi \circ F_z$, then for all $\lambda \in \mathcal{P}^\dagger \mathbb{B}$,

\[
F \circ \phi^\dagger(\lambda) = (\lambda \triangleleft \phi) \circ F^z
\]

\[
\leq ((G^z \circ \lambda) \triangleleft (G^z \circ \phi)) \circ F^z
\]

\[
\leq ((G^z \circ \lambda) \triangleleft (\psi \circ F_z)) \circ F^z
\]

\[
\leq (G^z \circ \lambda) \triangleleft (\psi \circ F_z \circ F^z)
\]

\[
\leq (G^z \circ \lambda) \triangleleft \psi
\]

\[
= \psi^\dagger \circ G^\tau(\lambda).
\]  

(Equation (5.11))

This completes the proof. \qed

By virtue of Proposition 5.2.1 we obtain a functor $U : \mathcal{Q} \text{-} \text{Info} \rightarrow \mathcal{Q} \text{-} \text{Cls}$ that sends an infomorphism

\[(F, G) : (\phi : A \rightarrow \mathbb{B}) \rightarrow (\psi : A' \rightarrow \mathbb{B}')\]

to a continuous $\mathcal{Q}$-functor

\[F : (A_0, \phi^\dagger \circ \phi^\dagger) \rightarrow (A'_0, \psi^\dagger \circ \psi^\dagger).\]

Given a $\mathcal{Q}$-closure space $(A, C)$, define a $\mathcal{Q}$-distributor $\zeta_C : A \rightarrow C(\mathcal{P}A)$ by

\[\zeta_C(x, \mu) = \mu(x)\]

for all $x \in A_0$ and $\mu \in C(\mathcal{P}A)$. It is clear that $\zeta_C$ is obtained by restricting the domain and the codomain of the $\mathcal{Q}$-distributor

\[\mathcal{P}^\dagger A \rightarrow \mathcal{P}A, \quad (\lambda, \mu) \mapsto \mu \circ \lambda.\]

(Equation (5.8))

Given a continuous $\mathcal{Q}$-functor $F : (A, C) \rightarrow (\mathbb{B}, D)$ between $\mathcal{Q}$-closure spaces, consider the $\mathcal{Q}$-functor $F^z : D(\mathcal{P}\mathbb{B}) \rightarrow C(\mathcal{P}A)$ that sends each closed contravariant presheaf $\lambda$ to $F^z(\lambda) = F^\tau(\lambda)$. Then similar to Proposition 3.7.2 one can check that

\[(F, F^z) : (\zeta_C : A \rightarrow C(\mathcal{P}A)) \rightarrow (\zeta_D : \mathbb{B} \rightarrow D(\mathcal{P}\mathbb{B}))\]

is an infomorphism. Thus, we obtain a functor $\mathcal{F} : \mathcal{Q} \text{-} \text{Cls} \rightarrow \mathcal{Q} \text{-} \text{Info}$. 

5.2. Functoriality of the Isbell adjunction
Theorem 5.2.2. \( \mathcal{F} : \mathcal{Q}\text{-Cls} \longrightarrow \mathcal{Q}\text{-Info} \) is a left adjoint and right inverse of \( \mathcal{U} : \mathcal{Q}\text{-Info} \longrightarrow \mathcal{Q}\text{-Cls} \).

Proof. Step 1. \( \mathcal{F} \) is a right inverse of \( \mathcal{U} \).

For each \( \mathcal{Q} \)-closure space \((A, C)\), by the definition of the functor \( \mathcal{F} \), \( \mathcal{F}(A, C) \) is the \( \mathcal{Q} \)-distributor \( \zeta_C : A \to C(\mathcal{P}A) \), where \( \zeta_C(x, \mu) = \mu(x) \) for all \( x \in A_0 \) and \( \mu \in C(\mathcal{P}A) \). In order to prove \( \mathcal{U} \circ \mathcal{F}(A, C) = (A, C) \), we show that \( C = \zeta^\downarrow \circ (\zeta_C)^\uparrow \).

For all \( \mu \in \mathcal{P}A \) and \( \lambda \in C(\mathcal{P}A) \), since \( C \) is a \( \mathcal{Q} \)-functor,

\[
\lambda \not\leq \mu = \mathcal{P}A(\mu, \lambda) \leq \mathcal{P}A(C(\mu), \lambda) = \lambda \not\leq C(\mu),
\]

and consequently \( C(\mu) \leq (\lambda \not\leq \mu) \not\leq \lambda \). Since \( C \) is a \( \mathcal{Q} \)-closure operator, we have

\[
(C(\mu) \not\leq \mu) \not\leq C(\mu) \leq 1_\mu \not\leq C(\mu) = C(\mu),
\]

hence

\[
C(\mu) = \bigwedge_{\lambda \in C(\mathcal{P}A)} (\lambda \not\leq \mu) \not\leq \lambda
= \bigwedge_{\lambda \in C(\mathcal{P}A)} (\zeta_C(-, \lambda) \not\leq \mu) \not\leq \zeta_C(-, \lambda)
= \bigwedge_{\lambda \in C(\mathcal{P}A)} (\zeta_C)^\uparrow(\mu)(\lambda) \not\leq \zeta_C(-, \lambda)
= \zeta^\downarrow \circ (\zeta_C)^\uparrow(\mu),
\]

as required.

Step 2. \( \mathcal{F} \) is a left adjoint of \( \mathcal{U} \).

For each \( \mathcal{Q} \)-closure space \((A, C)\), \( \text{id}_{(A,C)} : (A, C) \longrightarrow \mathcal{U} \circ \mathcal{F}(A, C) \) is clearly a continuous \( \mathcal{Q} \)-functor and \( \{\text{id}_{(A,C)}\} \) is a natural transformation from the identity functor on \( \mathcal{Q}\text{-Cls} \) to \( \mathcal{U} \circ \mathcal{F} \). Thus, it remains to show that for each \( \mathcal{Q} \)-distributor \( \psi : A' \longrightarrow B' \) and each continuous \( \mathcal{Q} \)-functor \( H : (A, C) \longrightarrow (A', \psi^+ \circ \psi^\uparrow) \), there is a unique infomorphism \( (F, G) : \mathcal{F}(A, C) \longrightarrow (\psi : A' \longrightarrow B') \).
such that the diagram

\[
\begin{array}{ccc}
(A, C) & \xrightarrow{\mathbf{id}_{A, C}} & U \circ F(A, C) \\
\downarrow H & & \downarrow U(F, G) \\
(A', \psi^+ \circ \psi^-) & \rightarrow & (A', \psi^+ \circ \psi^-)
\end{array}
\]

is commutative.

By definition, \( F(A, C) = \zeta_C : A \rightarrow C(\mathcal{P}A) \) and \( U(F, G) = F \), where \( \zeta_C(x, \mu) = \mu(x) \). Thus, we only need to show that there is a unique \( Q \)-functor \( G : \mathbb{B}' \rightarrow C(\mathcal{P}A) \) such that

\[
(H, G) : (\zeta_C : A \rightarrow C(\mathcal{P}A)) \rightarrow (\psi : A' \rightarrow \mathbb{B}')
\]

is an infomorphism.

Let \( G = H^a \circ \psi : \mathbb{B}' \rightarrow C(\mathcal{P}A) \). That \( G \) is well-defined follows from the fact that \( \overline{\psi}y' \in \psi^+ \circ \psi^- (\mathcal{P}A') \) for all \( y' \in \mathbb{B}'_0 \) by Equation (5.5) and that \( H : (A, C) \rightarrow (A', \psi^+ \circ \psi^-) \) is continuous. Now we check that

\[
(H, G) : (\zeta_C : A \rightarrow C(\mathcal{P}A)) \rightarrow (\psi : A' \rightarrow \mathbb{B}')
\]

is an infomorphism. This is easy since

\[
\zeta_C(x, Gy') = (Gy')(x) = H^a \circ \overline{\psi}(y')(x) = \overline{\psi}(y')(Hx) = \psi(Hx, y')
\]

for all \( x \in A_0 \) and \( y' \in \mathbb{B}'_0 \). This proves the existence of \( G \).

To see the uniqueness of \( G \), suppose that \( G' : \mathbb{B}' \rightarrow C(\mathcal{P}A) \) is another \( Q \)-functor such that

\[
(H, G') : (\zeta_C : A \rightarrow C(\mathcal{P}A)) \rightarrow (\psi : A' \rightarrow \mathbb{B}')
\]

is an infomorphism. Then for all \( x \in A_0 \) and \( y' \in \mathbb{B}'_0 \),

\[
(G'y')(x) = \zeta_C(x, G'y')
\]

\[
= \psi(Hx, y')
\]

\[
= \overline{\psi}(y')(Hx)
\]

\[
= H^a \circ \overline{\psi}(y')(x)
\]

\[
= (Gy')(x),
\]
hence $G' = G$.

\begin{hscode}
\source{104}
\begin{align*}
\text{Corollary 5.2.3.} & \text{ The category } Q\text{-Cls is a coreflective subcategory of } Q\text{-Info.}
\end{align*}
\end{hscode}

The composition of

\begin{align*}
\mathcal{U} : & Q\text{-Info} \rightarrow Q\text{-Cls} \\
\mathcal{T} : & Q\text{-Cls} \rightarrow Q\text{-CCat}
\end{align*}

and

\begin{align*}
\mathcal{M} = \mathcal{T} \circ \mathcal{U} : Q\text{-Info} \rightarrow Q\text{-CCat}
\end{align*}

(5.9)

\begin{hscode}
\source{104}
\begin{align*}
\text{that sends a } Q\text{-distributor } \phi : & A \to B \text{ to a complete } Q\text{-category } \phi_\perp \circ \phi_\uparrow (PA).
\end{align*}
\end{hscode}

Conversely, since $\mathcal{F}$ is a right inverse of $\mathcal{U}$ (Theorem 5.2.2) and $\mathcal{T}$ is a left inverse of $\mathcal{D}$ (up to isomorphism, Theorem 4.3.4), we have the following

\begin{hscode}
\source{104}
\begin{align*}
\text{Theorem 5.2.4.} & \text{ Every skeletal complete } Q\text{-category is isomorphic to } \mathcal{M}(\phi) \text{ for some } Q\text{-distributor } \phi.
\end{align*}
\end{hscode}

The following proposition shows that the free cocompletion functor of $Q$-categories factors through the functor $\mathcal{M}$.

\begin{hscode}
\source{104}
\begin{align*}
\text{Proposition 5.2.5.} & \text{ The diagram}
\end{align*}
\end{hscode}

\begin{hscode}
\source{104}
\begin{align*}
\text{commutes.}
\end{align*}
\end{hscode}
Proof. First, \(\mathcal{M}((\mathbb{Y}_\mathbb{A})_\sharp) = ((\mathbb{Y}_\mathbb{A})_\sharp)^\dagger \circ ((\mathbb{Y}_\mathbb{A})_\natural)^\natural = \mathcal{P}\mathbb{A}\) for each \(\mathbb{Q}\)-category \(\mathbb{A}\). To see this, it suffices to check that

\[
\mu = ((\mathbb{Y}_\mathbb{A})_\sharp)\circ ((\mathbb{Y}_\mathbb{A})_\natural)^\natural (\mu) = ((\mathbb{Y}_\mathbb{A})_\natural)^\natural \mu \leq (\mathbb{Y}_\mathbb{A})_\sharp \\
\text{for all } \mu \in \mathcal{P}\mathbb{A}.
\]

On one hand, by Yoneda lemma we have

\[
(\mathbb{Y}_\mathbb{A})_\sharp \mu = (\mathbb{Y}_\mathbb{A})_\sharp \leq (\mathbb{Y}_\mathbb{A})_\sharp (-, \mu) \geq \mathcal{P}\mathbb{A}(\mu, -),
\]

thus

\[
((\mathbb{Y}_\mathbb{A})_\natural)^\natural \leq \mathcal{P}\mathbb{A}(\mu, -) \leq (\mathbb{Y}_\mathbb{A})_\sharp = (\mathbb{Y}_\mathbb{A})_\sharp (-, \mu) = \mu.
\]

On the other hand, \(\mu \leq ((\mathbb{Y}_\mathbb{A})_\natural)^\natural \mu \leq (\mathbb{Y}_\mathbb{A})_\sharp\) holds trivially.

Second, it is trivial that for each \(\mathbb{Q}\)-functor \(F : \mathbb{A} \rightarrow \mathbb{B}\),

\[
\mathcal{M} \circ \mathbb{Y}(F) = F^\dagger = \mathcal{P}(F).
\]

Therefore, the conclusion holds.

5.3 \(\mathbb{Q}\)-state property systems

Corollary 5.2.3 says that the category \(\mathbb{Q}\)-Cls is a coreflective subcategory of \(\mathbb{Q}\)-Info. In this section we show that \(\mathbb{Q}\)-Cls is equivalent to a subcategory of \(\mathbb{Q}\)-Info. This equivalence is a generalization of that between closure spaces and state property systems in [ACVV99].

Definition 5.3.1. A \(\mathbb{Q}\)-state property system is a triple \((\mathbb{A}, \mathbb{B}, \phi)\), where \(\mathbb{A}\) is a \(\mathbb{Q}\)-category, \(\mathbb{B}\) is a skeletal complete \(\mathbb{Q}\)-category and \(\phi : \mathbb{A} \rightarrow \mathbb{B}\) is a \(\mathbb{Q}\)-distributor, such that

1. \(\phi(-, \inf_\mathbb{B} \lambda) = \lambda \backslash \phi \) for all \(\lambda \in \mathcal{P}\mathbb{B}\),

2. \(\mathbb{B}(y, y') = \phi(-, y') \backslash \phi(-, y) \) for all \(y, y' \in \mathbb{B}_0\).

\(\mathbb{Q}\)-state property systems and infomorphisms constitute a category \(\mathbb{Q}\text{-Sp}\), which is a subcategory of \(\mathbb{Q}\text{-Info}\).
Example 5.3.2. For each $Q$-closure space $(\mathbb{A}, C)$, $(\mathbb{A}, C(P\mathbb{A}), \zeta_C)$ is a $Q$-state property system. First, for all $\Psi \in P^1(C(P\mathbb{A}))$, it follows from Example 3.5.12 and Equation (4.1) that

$$
\zeta_C(-, \inf_{C(P\mathbb{A})} \Psi) = \inf_{C(P\mathbb{A})} \Psi
$$

$$
= \bigwedge_{\mu \in C(P\mathbb{A})} \Psi(\mu) \searrow \mu
$$

$$
= \bigwedge_{\mu \in C(P\mathbb{A})} \Psi(\mu) \searrow \zeta_C(-, \mu)
$$

$$
= \Psi \searrow \zeta_C.
$$

Second, it is trivial that

$$
C(P\mathbb{A})(\mu, \lambda) = \lambda \ngtr \mu = \zeta_C(-, \lambda) \ngtr \zeta_C(-, \mu)
$$

for all $\mu, \lambda \in C(P\mathbb{A})$.

Therefore, the codomain of the functor $F : Q\text{-Cls} \rightarrow Q\text{-Info}$ can be restricted to the subcategory $Q\text{-Sp}$.

Theorem 5.3.3. The functors $F : Q\text{-Cls} \rightarrow Q\text{-Sp}$ and $U : Q\text{-Sp} \rightarrow Q\text{-Cls}$ establish an equivalence of categories.

Proof. It is shown in Theorem 5.2.2 that $U \circ F = \text{id}_{Q\text{-Cls}}$, so, it suffices to prove that $F \circ U \cong \text{id}_{Q\text{-Sp}}$.

Given a $Q$-state property system $(\mathbb{A}, \mathbb{B}, \phi)$, we have by definition

$$
F \circ U(\mathbb{A}, \mathbb{B}, \phi) = (\mathbb{A}, \phi^+ \circ \phi_+(P\mathbb{A}), \zeta_{\phi^+ \circ \phi_+}).
$$

By virtue of Equation (5.7), the images of the $Q$-functor $\overline{\phi} : \mathbb{B} \rightarrow P\mathbb{A}$ are contained in $\phi^+ \circ \phi_+(P\mathbb{A})$, so, it can be viewed as a $Q$-functor $\overline{\phi} : \overline{\phi} : \mathbb{B} \rightarrow \phi^+ \circ \phi_+(P\mathbb{A})$. Since for any $x \in A_0$ and $y \in B_0$,

$$
\phi(x, y) = (\overline{\phi} y)(x) = \zeta_{\phi^+ \circ \phi_+}(x, \overline{\phi} y),
$$

it follows that $\eta_{\overline{\phi}} = (1_{A_0}, \overline{\phi})$ is an infomorphism from $\zeta_{\phi^+ \circ \phi_+} : A_0 \rightarrow \phi^+ \circ \phi_+(P\mathbb{A})$ to $\phi : A_0 \rightarrow \mathbb{B}$. Hence $\eta_{\overline{\phi}}$ is a morphism from $F \circ U(\mathbb{A}, \mathbb{B}, \phi)$ to $(\mathbb{A}, \mathbb{B}, \phi)$ in $Q\text{-Sp}$.
We claim that $\eta$ is a natural isomorphism from $\mathcal{F} \circ \mathcal{U}$ to the identity functor $\text{id}_{\mathcal{Q}\text{-Sp}}$.

Firstly, $\eta_{\phi}$ is an isomorphism. It suffices to show that\[
\phi : \mathcal{B} \to \phi \circ \phi_\tau(\mathcal{PA})
\]is an isomorphism between $\mathcal{Q}$-categories.

Since\[
\mathcal{B}(y, y') = \phi(-, y') \setminus \phi(-, y) = \mathcal{PA}(\phi y, \phi y')
\]for all $y, y' \in \mathcal{B}_0$, it follows that $\phi$ is fully faithful. For each $\mu \in \mathcal{PA}$, let $y = \inf_{\mathcal{B}} \phi_\tau(\mu)$, then\[
\phi y = \phi(-, y) = \phi(-, \inf_{\mathcal{B}} \phi_\tau(\mu)) = \phi_\tau(\mu) \setminus \phi = \phi^+ \circ \phi_\tau(\mu),
\]hence $\phi$ is surjective. Since $\mathcal{B}$ is skeletal, we deduce that $\phi : \mathcal{B} \to \phi^+ \circ \phi_\tau(\mathcal{PA})$ is an isomorphism.

Secondly, $\eta$ is natural. For this, we check the commutativity of the following diagram for any infomorphism $(F, G) : (\mathcal{A}, \mathcal{B}, \phi) \to (\mathcal{A}', \mathcal{B}', \psi)$ between $\mathcal{Q}$-state property systems:

\[
\begin{array}{ccc}
\mathcal{F} \circ \mathcal{U}(\mathcal{A}, \mathcal{B}, \phi) & \xrightarrow{(1_\mathcal{A}, \phi)} & (\mathcal{A}, \mathcal{B}, \phi) \\
(F, F^\circ) & & (F, G) \\
\mathcal{F} \circ \mathcal{U}(\mathcal{A}', \mathcal{B}', \psi) & \xleftarrow{(1_{\mathcal{A}'}, \psi)} & (\mathcal{A}', \mathcal{B}', \psi)
\end{array}
\]

In fact, the equality $F \circ 1_\mathcal{A} = 1_{\mathcal{A}'} \circ F$ is clear; and for all $x \in \mathcal{A}_0$ and $y' \in \mathcal{B}'_0$,
\[
\phi \circ G(y')(x) = \phi(x, G y') = \psi(F x, y') = \psi(y')(Fx) = F^\circ \circ \psi(y')(x),
\]thus the conclusion follows.

Together with Theorem 4.3.4 we have

**Corollary 5.3.4.** The composition
\[
\mathcal{T} \circ \mathcal{U} : \mathcal{Q}\text{-Sp} \to \mathcal{Q}\text{-CCat}
\]is a left adjoint of
\[
\mathcal{F} \circ \mathcal{D} : \mathcal{Q}\text{-CCat} \to \mathcal{Q}\text{-Sp}.
\]
5.4 Characterizations of \( M(\phi) \)

For each \( Q \)-distributor \( \phi : A \leftrightarrow B \), we have the following characterization of the complete \( Q \)-category \( M(\phi) \).

**Proposition 5.4.1.** Let \( X \) be a skeletal complete \( Q \)-category. The following conditions are equivalent:

1. \( X \) is isomorphic to \( M(\phi) \) for some \( Q \)-distributor \( \phi : A \leftrightarrow B \).
2. \( X \) is isomorphic to a \( Q \)-closure system of \( PA \) for some \( Q \)-category \( A \).
3. \( X \) is isomorphic to a quotient object of \( PA \) for some \( Q \)-category \( A \) in the category \( Q\text{-CCat} \), i.e., a subobject of \( PA \) in \( (Q\text{-CCat})^{\text{op}} \).

**Proof.**

(1) \( \Rightarrow \) (2): Trivial.

(2) \( \Rightarrow \) (1): It follows from Proposition 4.1.4 that \( X \) is isomorphic to \( C(PA) \) for some \( Q \)-closure space \( (A,C) \). Let \( \zeta_C = F(A,C) : A \leftrightarrow C(PA) \), since \( U \circ F = \text{id}_{Q\text{-Cls}} \) (by Theorem 5.2.2), we have

\[
M(\zeta_C) = T \circ U(\zeta_C) = T \circ U \circ F(A,C) = T(A,C) = C(PA).
\]

This implies \( X \cong M(\zeta_C) \).

(2) \( \iff \) (3): It is easily seen that \( B \) is a \( Q \)-closure system of \( PA \) if and only if the inclusion \( Q \)-functor \( I : B \rightarrow PA \) has a left adjoint \( F : PA \rightarrow B \), where \( F \) is epic in the category \( Q\text{-CCat} \), and equivalently a monic from \( B \) to \( PA \) in \( (Q\text{-CCat})^{\text{op}} \). \( \Box \)

In the rest this section, we present a characterization of \( M(\phi) \) for a \( Q \)-distributor \( \phi : A \leftrightarrow B \) through sup-dense and inf-dense \( Q \)-functors.

Given a \( Q \)-distributor \( \phi : A \leftrightarrow B \), let \( M_\phi(A,B) \) denote the set of pairs \( (\mu,\lambda) \in PA \times P^!B \) such that \( \lambda = \phi^!(\mu) \) and \( \mu = \phi^!(\lambda) \). \( M_\phi(A,B) \) becomes a \( Q \)-typed set if we assign \( t(\mu,\lambda) = t\mu = t\lambda \). For \( (\mu_1,\lambda_1), (\mu_2,\lambda_2) \in M_\phi(A,B) \), let

\[
M_\phi(A,B)((\mu_1,\lambda_1), (\mu_2,\lambda_2)) = PA(\mu_1,\mu_2) = P^!B(\lambda_1,\lambda_2),
\]

(5.10)
Then $\mathcal{M}_\phi(\mathbb{A}, \mathbb{B})$ becomes a $\mathcal{Q}$-category.

The projection

$$\pi_1 : \mathcal{M}_\phi(\mathbb{A}, \mathbb{B}) \rightarrow \mathcal{P}\mathbb{A}, \quad (\mu, \lambda) \mapsto \mu$$

is clearly a fully faithful $\mathcal{Q}$-functor. Since the image of $\pi_1$ is exactly the set of fixed points of the $\mathcal{Q}$-closure operator $\phi^\uparrow \circ \phi^\downarrow : \mathcal{P}\mathbb{A} \rightarrow \mathcal{P}\mathbb{A}$, we obtain that $\mathcal{M}_\phi(\mathbb{A}, \mathbb{B})$ is isomorphic to the complete $\mathcal{Q}$-category $\mathcal{M}(\phi) = \phi^\downarrow \circ \phi^\uparrow(\mathcal{P}\mathbb{A})$.

Similarly, the projection

$$\pi_2 : \mathcal{M}_\phi(\mathbb{A}, \mathbb{B}) \rightarrow \mathcal{P}^\uparrow\mathbb{B}, \quad (\mu, \lambda) \mapsto \lambda$$

is also a fully faithful $\mathcal{Q}$-functor and the image of $\pi_2$ is exactly the set of fixed points of the $\mathcal{Q}$-interior operator $\phi^\uparrow \circ \phi^\downarrow : \mathcal{P}^\uparrow\mathbb{B} \rightarrow \mathcal{P}^\uparrow\mathbb{B}$. Hence $\mathcal{M}_\phi(\mathbb{A}, \mathbb{B})$ is also isomorphic to the complete $\mathcal{Q}$-category $\phi^\uparrow \circ \phi^\downarrow(\mathcal{P}^\uparrow\mathbb{B})$, which is a $\mathcal{Q}$-interior system of the skeletal complete $\mathcal{Q}$-category $\mathcal{P}^\uparrow\mathbb{B}$.

Equation (5.10) shows that $\phi^\uparrow : \phi^\downarrow \circ \phi^\uparrow(\mathcal{P}\mathbb{A}) \rightarrow \mathcal{P}^\uparrow\mathbb{B}$ and $\phi^\downarrow : \phi^\uparrow \circ \phi^\downarrow(\mathcal{P}^\uparrow\mathbb{B}) \rightarrow \mathcal{P}\mathbb{A}$ are inverse to each other. Therefore, $\mathcal{M}(\phi)(= \phi^\downarrow \circ \phi^\uparrow(\mathcal{P}\mathbb{A}))$, $\phi^\uparrow \circ \phi^\downarrow(\mathcal{P}^\uparrow\mathbb{B})$ and $\mathcal{M}_\phi(\mathbb{A}, \mathbb{B})$ are isomorphic to each other.

**Definition 5.4.2.** Let $F : \mathbb{A} \rightarrow \mathbb{B}$ be a $\mathcal{Q}$-functor.

(1) $F$ is sup-dense if for any $y \in \mathbb{B}_0$, there is some $\mu \in \mathcal{P}\mathbb{A}$ such that $y = \sup_{\mathbb{B}} F^\rightarrow(\mu)$.

(2) $F$ is inf-dense if for any $y \in \mathbb{B}_0$, there is some $\lambda \in \mathcal{P}^\uparrow\mathbb{A}$ such that $y = \inf_{\mathbb{B}} F^\leftarrow(\lambda)$.

**Example 5.4.3.** For each $\mathcal{Q}$-category $\mathbb{A}$, the Yoneda embedding $Y : \mathbb{A} \rightarrow \mathcal{P}\mathbb{A}$ is sup-dense. Indeed, we have that

$$\mu = \sup_{\mathcal{P}\mathbb{A}} \circ Y^\rightarrow(\mu)$$
for all $\mu \in \mathcal{P}A$ (see Equation (4.4) in the proof of Theorem 4.3.4).

Dually, the co-Yoneda embedding $Y^\dagger : A \rightarrow \mathcal{P}^1A$ is inf-dense.

The following characterization of $\mathcal{M}_\phi(A, B)$ (hence $\mathcal{M}(\phi)$) extends Theorem 4.8 in [LZ09] to the general setting.

**Theorem 5.4.4.** Given a $\mathcal{Q}$-distributor $\phi : A \rightarrow B$, a skeletal complete $\mathcal{Q}$-category $X$ is isomorphic to $\mathcal{M}_\phi(A, B)$ if and only if there exist a sup-dense $\mathcal{Q}$-functor $F : A \rightarrow X$ and an inf-dense $\mathcal{Q}$-functor $G : B \rightarrow X$ such that $\phi = G^\circ \circ F^\circ = X(F^-, G^-)$.

**Proof. Necessity.** It suffices to prove the case $X = \mathcal{M}_\phi(A, B)$. Define $\mathcal{Q}$-functors $F : A \rightarrow X$ and $G : B \rightarrow X$ by

$$Fa = (\phi^\circ \circ \phi a, \phi a), \quad Gb = (\phi b, \phi^\circ \circ \phi b),$$

then $F$ and $G$ are well defined by Equations (5.5) and (5.6). It follows that

$$X(F^-, G^-) = \mathcal{P}A(\phi^\circ \circ \phi ^-, \phi^-)$$

$$= \mathcal{P}A(\phi^\circ \circ \phi ^-, \phi^\circ \circ Y^\dagger_B^-) \quad \text{(Equation (5.5))}$$

$$= \mathcal{P}^1B(\phi^\circ \circ \phi^\circ \circ \phi ^-, Y^\dagger_B^-) \quad \text{(Proposition 5.1.1)}$$

$$= \mathcal{P}^1B(\phi^-, Y^\dagger_B^-) \quad \text{(Equation (5.6))}$$

$$= (\phi^-)(-) \quad \text{(Yoneda lemma)}$$

$$= \phi.$$  

Now we show that $F : A \rightarrow X$ is sup-dense. For all $(\mu, \lambda), (\mu', \lambda') \in X_0$,

$$X((\mu, \lambda), (\mu', \lambda')) = \lambda' \searrow \lambda$$

$$= \lambda' \searrow \phi^\dagger(\mu)$$

$$= \lambda' \searrow (\phi \searrow \mu)$$

$$= (\lambda' \searrow \phi) \searrow \mu$$

$$= \mathcal{P}^1B(\phi^-, \lambda') \searrow \mu$$

$$= X(F^-, (\mu', \lambda')) \searrow \mu \quad \text{(Equation (5.10))}$$

$$= F^\circ_2(-, (\mu', \lambda')) \searrow \mu,$$
thus \((\mu, \lambda) = \colim \mu F = \sup X \circ F^{-}(\mu)\), as desired.

That \(G : B \rightarrow X\) is inf-dense can be proved similarly.

**Sufficiency.** We show that the type-preserving function

\[ H : X \rightarrow M_{\phi}(A, B), \quad Hx = (F_{3}(-, x), G^{2}(x, -)) \]

is an isomorphism of \(Q\)-categories.

**Step 1.** \(X = F_{3} \check{\uparrow} F_{3} = G^{3} \check{\leftarrow} G^{5}\).

For all \(x \in X_{0}\), since \(F : A \rightarrow X\) is sup-dense, there is some \(\mu \in \mathcal{P}A\) such that

\[ x = \sup X F \rightarrow (\mu) \]

thus \((\mu, \lambda) = \colim \mu F = \sup X \circ F^{-}(\mu)\), as desired.

That \(G : B \rightarrow X\) is inf-dense can be proved similarly.

\[ \begin{align*}
\mathsf{S} & = \mathsf{X} \check{\leftarrow} (F_{3} \uparrow (-, x)) \circ \mathsf{X}(x, x) \\
& = (F_{3} \check{\leftarrow} F_{3}(-, x)) \circ (F_{3}(-, x) \check{\leftarrow} \mu) \quad \text{(Equation (5.12))}\end{align*} \]

where the third equality follows from Proposition 3.1.9(2). Consequently

\[ \begin{align*}
\mathsf{X}(x, -) & \leq (F_{3} \check{\leftarrow} F_{3}(-, x)) \circ \mathsf{X}(x, x) \\
& = (F_{3} \check{\leftarrow} F_{3}(-, x)) \circ (F_{3}(-, x) \check{\leftarrow} \mu) \quad \text{(Equation (5.12))}\end{align*} \]

hence \(\mathsf{X}(x, -) = F_{3} \check{\leftarrow} F_{3}(-, x) = (F_{3} \check{\leftarrow} F_{3})(x, -)\).

Since \(G : B \rightarrow X\) is inf-dense, similar calculations lead to \(X = G^{3} \check{\leftarrow} G^{5}\).

**Step 2.** \(Hx \in M_{\phi}(A, B)\) for all \(x \in X_{0}\), thus \(H\) is well defined. Indeed,

\[ \begin{align*}
\phi_{\uparrow}(F_{3}(-, x)) &= \phi \check{\leftarrow} F_{3}(-, x) \\
& = (G^{2} \circ F_{3}) \check{\leftarrow} F_{3}(-, x) \quad \text{(\(\phi = G^{3} \circ F_{3}\))}
& = G^{2} \circ (F_{3} \check{\leftarrow} F_{3}(-, x)) \quad \text{(Proposition 3.1.9(3))}
& = G^{2} \circ \mathsf{X}(x, -) \quad \text{(Step 1)}
& = G^{2}(x, -).
\end{align*} \]

Similar calculation shows that \(\phi^{\downarrow}(G^{2}(x, -)) = F_{3}(-, x)\). Hence, \(Hx \in M_{\phi}(A, B)\).

**Step 3.** \(H\) is a fully faithful \(Q\)-functor. Indeed, for all \(x, x' \in X_{0}\), by Step 1,

\[ \begin{align*}
\mathsf{X}(x, x') &= F_{3}(-, x') \check{\leftarrow} F_{3}(-, x) = \mathcal{P}A(F_{3}(-, x), F_{3}(-, x')) \in M_{\phi}(A, B)(Hx, Hx').
\end{align*} \]
Step 4. \( H \) is surjective. For each pair \((\mu, \lambda) \in \mathcal{M}_\phi(A, B)\), we must show that there is some \( x \in \mathbb{X}_0 \) such that \( F_\natural(\_, x) = \mu \) and \( G_\natural(x, \_) = \lambda \). Indeed, let \( x = \sup_X F^{-1}(\mu) \), then

\[
G_\natural(x, \_) = G_\natural \circ \mathbb{X}(x, \_) \\
= G_\natural \circ (F_\natural \uplus \mu) \\
= (G_\natural \circ F_\natural) \uplus \mu \\
= \phi \uplus \mu \\
= \phi_\uplus(\mu) \\
= \lambda,
\]

and it follows that \( F_\natural(\_, x) = \phi_\uplus(G_\natural(x, \_)) = \phi_\uplus(\lambda) = \mu \).

\[\square\]

5.5 The MacNeille completion

Given a preordered set \( A \), the MacNeille completion \([\text{DP02, Mac37}]\) (or Dedekind-MacNeille completion) of \( A \) is the set \( \mathcal{M}(A) \) of pairs \((L, U)\) of subsets of \( A \), such that

\[
L = \text{lb } U \quad \text{and} \quad U = \text{ub } L,
\]

where \( \text{lb } U \) is the set of lower bounds of \( U \) and \( \text{ub } L \) is the set of upper bounds of \( L \). The preorder on \( \mathcal{M}(A) \) is given by

\[
(L_1, U_1) \leq (L_2, U_2) \iff L_1 \subseteq L_2 \iff U_2 \subseteq U_1.
\]

The notion of MacNeille completion has been extended to categories enriched over a commutative unital quantale \([\text{Bel04, LZ07, Wag94}]\) (see Chapter 7 for more discussion about unital quantales). In this section, we investigate the MacNeille completion of a \( \mathcal{Q} \)-category \( \mathbb{A} \) by considering the special \( \mathcal{Q} \)-distributor \( \mathbb{A} : \mathbb{A} \rightarrow \mathbb{A} \) and the induced Isbell adjunction

\[
\text{ub} \dashv \text{lb} : \mathcal{P}\mathbb{A} \rightarrow \mathcal{P}_\downarrow \mathbb{A}
\]

presented in Proposition \([3.4.4]\).
5.5. The MacNeille completion

Definition 5.5.1. A cut in a \( \mathcal{Q} \)-category \( \mathbb{A} \) is a pair \( (\mu, \lambda) \in \mathcal{P}\mathbb{A} \times \mathcal{P}^\dagger\mathbb{A} \) such that
\[
\lambda = \text{ub} \mu \quad \text{and} \quad \mu = \text{lb} \lambda.
\]

Definition 5.5.2. The MacNeille completion of a \( \mathcal{Q} \)-category \( \mathbb{A} \) is the \( \mathcal{Q} \)-category \( \text{M}(\mathbb{A}) \), in which

1. \( \text{M}(\mathbb{A})_0 \) is the set of cuts in \( \mathbb{A} \);
2. the type of a cut is \( t(\mu, \lambda) = t\mu = t\lambda \);
3. for all \( (\mu_1, \lambda_1), (\mu_2, \lambda_2) \in \text{M}(\mathbb{A})_0 \),
\[
\text{M}(\mathbb{A})((\mu_1, \lambda_1), (\mu_2, \lambda_2)) = \mathcal{P}\mathbb{A}(\mu_1, \mu_2) = \mathcal{P}^\dagger\mathbb{A}(\lambda_1, \lambda_2).
\]

It is obvious that the MacNeille completion \( \text{M}(\mathbb{A}) \) of a \( \mathcal{Q} \)-category \( \mathbb{A} \) is exactly the complete \( \mathcal{Q} \)-category \( \mathcal{M}_\mathbb{A}(\mathbb{A}, \mathbb{A}) \). In particular, the MacNeille completion \( \text{M}(\mathbb{A}) \) of a 2-category \( \mathbb{A} \) is just the MacNeille completion of preordered sets. Thus, the MacNeille completion of \( \mathcal{Q} \)-categories is a special case of the construction \( \mathcal{M}_\phi(\mathbb{A}, \mathbb{B}) \) for a \( \mathcal{Q} \)-distributor \( \phi : \mathbb{A} \rightarrow \mathbb{B} \).

Proposition 5.5.3. For each \( \mathcal{Q} \)-category \( \mathbb{A} \), the assignment
\[
x \mapsto (\mathcal{Y}x, \mathcal{Y}^\dagger x)
\]
gives rise to a fully faithful \( \mathcal{Q} \)-functor
\[
M : \mathbb{A} \rightarrow \text{M}(\mathbb{A}).
\]

Proof. It is easy to see that \( (\mathcal{Y}x, \mathcal{Y}^\dagger x) \) is a cut in \( \mathbb{A} \) for any \( x \in \mathbb{A}_0 \). Then the conclusion is an immediate consequence of the Yoneda lemma and Definition 5.5.2.

Proposition 5.5.4. For each \( \mathcal{Q} \)-category \( \mathbb{A} \), the \( \mathcal{Q} \)-functor \( M : \mathbb{A} \rightarrow \text{M}(\mathbb{A}) \) preserves all existing weighted colimits and weighted limits in \( \mathbb{A} \).
Proof. It suffices to show that $M$ preserves all existing suprema and infima in $\mathbb{A}$. Suppose that $\mu \in \mathcal{P}\mathbb{A}$ and $\text{sup}_{\mathbb{A}}\mu$ exists, we must show that

$$M(\text{sup}_{\mathbb{A}}\mu) = \text{sup}_{M(\mathbb{A})}M^{-}(\mu) = \text{colim}_{\mu}M.$$ 

Indeed, for all $(\mu', \lambda') \in M(\mathbb{A}),$

$$M(\mathbb{A})(M(\text{sup}_{\mathbb{A}}\mu), (\mu', \lambda')) = \lambda' \backslash \_\_ (\mathcal{Y}^{†} \circ \text{sup}_{\mathbb{A}}\mu)$$

$$= \lambda' \backslash \_\_ \mathbb{A}(\text{sup}_{\mathbb{A}}\mu, -)$$

$$= \lambda' \backslash \_\_ (\mathbb{A} \lor \mu)$$

$$= (\lambda' \backslash \_\_ \mathbb{A}) \lor \mu$$

$$= \mathcal{P}^{†}\mathbb{A}(\mathcal{Y}^{†} -, \lambda') \lor \mu$$

$$= M(\mathbb{A})(M-, (\mu', \lambda')) \lor \mu \quad (\text{Equation } (5.13))$$

$$= M_{\lor}(-, (\mu', \lambda')) \lor \mu.$$ 

Thus the conclusion follows. Similarly one can prove that

$$M(\text{inf}_{\mathbb{A}}\lambda) = \text{inf}_{M(\mathbb{A})}M^{-}(\lambda) = \text{lim}_{\lambda}M$$

whenever $\text{inf}_{\mathbb{A}}\lambda$ exists for some $\lambda \in \mathcal{P}^{†}\mathbb{A}$. \hfill \square

The MacNeille completion $M(\mathbb{A})$ is the “smallest” completion of a $\mathbb{Q}$-category $\mathbb{A}$ in the following sense.

**Proposition 5.5.5.** Let $F : \mathbb{A} \longrightarrow \mathbb{B}$ be a fully faithful $\mathbb{Q}$-functor, with $\mathbb{B}$ complete. Then $F$ factors through $M : \mathbb{A} \longrightarrow M(\mathbb{A})$ via a fully faithful $\mathbb{Q}$-functor $\mathcal{F} : M(\mathbb{A}) \longrightarrow \mathbb{B}$.

![Diagram](5.14)

In particular, if $\mathbb{A}$ is a skeletal complete $\mathbb{Q}$-category, then $M(\mathbb{A})$ is isomorphic to $\mathbb{A}$. Therefore, the process of the MacNeille completion is idempotent (up to isomorphism).
Proof. **Step 1.** For a fully faithful \( Q \)-functor \( \overline{F} : \mathcal{A} \to \mathcal{B} \), a fixed point \( \mu \) of \( \text{lb}_A \circ \text{ub}_A \) is also a fixed point of \( (\overline{F})^\uparrow \circ (\overline{F})^\downarrow \). On one hand,

\[
(A \not
preceq \mu) \circ ((F \not
preceq \mu) \searrow F) = ((F \not
preceq \mu) \searrow (F \not
preceq \mu) \searrow F) \quad \text{(Proposition 3.3.5)}
\]

\[
= F \circ (F \not
preceq \mu) \circ ((F \not
preceq \mu) \searrow F) \quad \text{(Proposition 3.1.9(3))}
\]

\[
\leq F \circ \overline{F} = A \quad \text{(Proposition 3.3.5)}
\]

and consequently

\[
(F \not
preceq \mu) \circ (F \not
preceq \mu) = (F \not
preceq \mu) \searrow F \leq (A \not
preceq \mu) \searrow A = \mu.
\]

On the other hand, \( \mu \leq (F \not
preceq \mu) \circ (F \not
preceq \mu) \) holds trivially. Thus \( \mu = (F \not
preceq \mu) \circ (F \not
preceq \mu) \).

**Step 2.** The existence of \( \overline{F} \). We identify \( M(A) \) with the \( Q \)-category \( \text{lb}_A \circ \text{ub}_A(\mathcal{P}A) \) of the fixed points of \( \text{lb}_A \circ \text{ub}_A \), then the functor \( M \) is exactly the Yoneda embedding \( Y_A \). Define

\[
\overline{F} = \sup_B \circ F^\to,
\]

then the commutativity of Diagram (5.14) follows immediately from Proposition 3.3.17 and 3.4.8. It remains to show that \( \overline{F} \) is fully faithful. Indeed, let \( \mu \) and \( \mu' \) be two fixed points of \( \text{lb}_A \circ \text{ub}_A \), then

\[
\mathbb{B}(\overline{F}(\mu), \overline{F}(\mu')) = \mathbb{B}(\sup_B \circ F^\to(\mu), \sup_B \circ F^\to(\mu')) = \mathbb{B}(\sup_B \circ F^\to(\mu'), -) \searrow \mathbb{B}(\sup_B \circ F^\to(\mu), -)
\]

\[
= (F \not
preceq \mu') \searrow (F \not
preceq \mu) \quad \text{(Proposition 3.4.7)}
\]

\[
= (F \not
preceq \mu') \searrow (F \not
preceq \mu') \not
preceq \mu \quad \text{(Step 1)}
\]

\[
= \mathcal{P}_A(\mu, \mu').
\]

**Step 3.** In particular, when \( \mathcal{A} \) is a skeletal complete \( Q \)-category, let \( F = 1_\mathcal{A} : \mathcal{A} \to \mathcal{A} \), then \( \overline{1_\mathcal{A}} = \sup_\mathcal{A} \). It follows Step 2 (or Proposition 3.4.8) that
\[ \text{sup}_A \circ Y_A = 1_A. \] Conversely, for all \( \mu \in \text{lb}_A \circ \text{ub}_A(P_A), \)

\[
Y_A \circ \text{sup}_A \mu = A(-, \text{sup}_A \mu) = A(\text{sup}_A \mu, -) \setminus A \\
= (A \sqcup \mu) \setminus A \\
= \text{lb}_A \circ \text{ub}_A \mu \\
= \mu.
\]

Thus \( Y_A \circ \text{sup}_A = 1_{M(A)}. \) This means that \( Y_A \) and \( \text{sup}_A \) are both isomorphisms between \( A \) and \( M(A). \) \( \square \)

Finally, we present a comparison of the free cocompletion \( P_A \) and the MacNeille completion \( M(A) \) (identified with \( \text{lb} \circ \text{ub}(P_A) \)) of a \( Q \)-category \( A. \)

**Proposition 5.5.6.** Let \( A \) be a \( Q \)-category.

1. The Yoneda embedding \( Y : A \to P_A \) is sup-dense.
2. If the codomain of \( Y \) is restricted to \( M(A) \), then \( Y : A \to M(A) \) is inf-dense.

**Proof.** (1) has been obtained in Example 5.4.3. For (2), note that \( Y : A \to M(A) \) is the composition of the inf-dense \( Q \)-functor given in Theorem 5.4.4 (Equation (5.11)) and the isomorphic projection from \( M_A(A, A) \) to \( \text{lb} \circ \text{ub}(P_A) \). The conclusion thus follows. \( \square \)

Recall that the \( Q \)-closure system \( M(A) = \text{lb} \circ \text{ub}(P_A) \) of \( P_A \) is closed with respect to infima in \( P_A \) (see Proposition 4.1.8). Therefore, in the case that \( A \) is skeletal, if we identify \( A \) with the \( Q \)-subcategory

\[ Y(A) = \{ Yx \mid x \in A_0 \} \]

of \( P_A \), then the above proposition implies that \( M(A) \) is the closure \[ AKSS \] \( [LZ07] \) of \( A \) under the formation of weighted limits in \( P_A \). As a comparison, \( P_A \) is the closure of \( A \) under the formation of weighted colimits in \( P_A \).
Chapter 6

Kan Adjunctions in $\mathcal{Q}$-categories

In this chapter, we first recall Kan extensions of $\mathcal{Q}$-functors presented in [Stu05a]. Then we introduce Kan adjunctions between $\mathcal{Q}$-categories of contravariant presheaves and covariant presheaves arise from $\mathcal{Q}$-distributors. We prove that this process is contravariant functorial from the category of $\mathcal{Q}$-distributors and infomorphisms to the category of complete $\mathcal{Q}$-categories and left adjoint $\mathcal{Q}$-functors. Also, the free cocompletion functor of $\mathcal{Q}$-categories factors through this functor.

6.1 Kan extensions of $\mathcal{Q}$-functors

Kan extensions of $\mathcal{V}$-functors introduced in Section 2.3 naturally give rise to the definitions of Kan extensions of $\mathcal{Q}$-functors.

Definition 6.1.1. Let $K : \mathcal{A} \to \mathcal{C}$ be a $\mathcal{Q}$-functor and $\mathcal{B}$ a $\mathcal{Q}$-category.

(1) The left Kan extension of a $\mathcal{Q}$-functor $F : \mathcal{A} \to \mathcal{B}$ along $K : \mathcal{A} \to \mathcal{C}$, if it exists, is a $\mathcal{Q}$-functor

$$\text{Lan}_K F : \mathcal{C} \to \mathcal{B}$$

satisfying

$$\text{Lan}_K F \leq S \iff F \leq S \circ K$$  \hspace{1cm} (6.1)

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for any other $\mathcal{Q}$-functor $S : \mathcal{C} \to \mathcal{B}$.

(2) The right Kan extension of a $\mathcal{Q}$-functor $F : \mathcal{A} \to \mathcal{B}$ along $K : \mathcal{A} \to \mathcal{C}$, if it exists, is a $\mathcal{Q}$-functor

$$\text{Ran}_K F : \mathcal{C} \to \mathcal{B}$$

satisfying

$$S \leq \text{Ran}_K F \iff S \circ K \leq F$$

for any other $\mathcal{Q}$-functor $S : \mathcal{C} \to \mathcal{B}$.

Given a $\mathcal{Q}$-functor $K : \mathcal{A} \to \mathcal{C}$ and another $\mathcal{Q}$-category $\mathcal{B}$, composing with $K$ yields an order-preserving function

$$K^+ : \mathcal{Q}\text{-Cat}(\mathcal{C}, \mathcal{B}) \to \mathcal{Q}\text{-Cat}(\mathcal{A}, \mathcal{B})$$

between the preordered sets of $\mathcal{Q}$-functors, which sends a $\mathcal{Q}$-functor $S : \mathcal{C} \to \mathcal{B}$ to $S \circ K : \mathcal{A} \to \mathcal{B}$.

If each $\mathcal{Q}$-functor $F : \mathcal{A} \to \mathcal{B}$ has a left Kan extension $\text{Lan}_K F : \mathcal{C} \to \mathcal{B}$ along $K$, then we obtain an adjunction in $2\text{-Cat}$

$$\text{Lan}_K \vdash K^+ : \mathcal{Q}\text{-Cat}(\mathcal{A}, \mathcal{B}) \to \mathcal{Q}\text{-Cat}(\mathcal{C}, \mathcal{B}).$$

(6.3)
Dually, if each \( Q \)-functor \( F : \mathcal{A} \to \mathcal{B} \) has a right Kan extension \( \text{Ran}_K F : \mathcal{C} \to \mathcal{B} \) along \( K \), then we obtain an adjunction in \( \text{2-Cat} \)

\[ K^* \dashv \text{Ran}_K : Q\text{-Cat}(\mathcal{C}, \mathcal{B}) \to Q\text{-Cat}(\mathcal{A}, \mathcal{B}). \tag{6.4} \]

Given a \( Q \)-functor \( F : \mathcal{A} \to \mathcal{B} \), if the codomain \( \mathcal{B} \) admits certain colimits (or equivalently, certain suprema), then the left Kan extension of \( F \) along some \( Q \)-functor \( K : \mathcal{A} \to \mathcal{C} \) can be constructed pointwise for each object \( c \in \mathcal{C}_0 \). Dually, if the codomain \( \mathcal{B} \) admits certain limits (or equivalently, certain infima), then the right Kan extension of \( F \) along \( K \) can be constructed pointwise.

**Proposition 6.1.2.** [Stu05a] Let \( F : \mathcal{A} \to \mathcal{B} \) and \( K : \mathcal{A} \to \mathcal{C} \) be \( Q \)-functors.

1. The left Kan extension of \( F \) along \( K \) can be computed by

\[ (\text{Lan}_K F)c = \text{colim}_{K^*(-,c)} F = \sup_{\mathcal{B}} F^\to(K^*(-, c)) \]

if the weighted colimit exists for each \( c \in \mathcal{C}_0 \).

2. The right Kan extension of \( F \) along \( K \) can be computed by

\[ (\text{Ran}_K F)c = \text{lim}_{K^*(-,c)} F = \inf_{\mathcal{B}} F^\to(K^*(c, -)) \]

if the weighted limit exists for each \( c \in \mathcal{C}_0 \).

**Proof.** We prove (1) for example. For each \( Q \)-functor \( S : \mathcal{C} \to \mathcal{B} \),

\[
\forall c \in \mathcal{C}_0, \text{colim}_{K^*(-,c)} F \leq Sc
\]

\[
\iff \forall c \in \mathcal{C}_0, 1_{tc} \leq B(\text{colim}_{K^*(-,c)} F, Sc)
\]

\[
\iff \forall c \in \mathcal{C}_0, 1_{tc} \leq F^\to(-, Sc) \vee K^* (-, c) \quad \text{(Definition 3.4.1)}
\]

\[
\iff \forall c \in \mathcal{C}_0, K^*(-, c) \leq F^\to(-, Sc) \quad \text{(Proposition 3.3.11)}
\]

\[
\iff \forall c \in \mathcal{C}_0, K^*(-, c) \leq (S \circ F^\to)(- , c)
\]

\[
\iff K^* \leq S \circ F^\to
\]

\[
\iff S^\circ K \leq F^\to \quad \text{(Formula 3.4)}
\]

Thus \( \text{colim}_{K^*(-,c)} F = (\text{Lan}_K F)c \).
By Proposition 6.1.2, we arrive at the following corollary that extends the equivalent characterizations of complete $\mathcal{Q}$-categories in Theorem 3.5.11.

**Corollary 6.1.3.** Let $\mathcal{B}$ be a $\mathcal{Q}$-category. The following conditions are equivalent:

1. $\mathcal{B}$ is complete.
2. The left Kan extension of each $\mathcal{Q}$-functor $F : \mathcal{A} \to \mathcal{B}$ along any $K : \mathcal{A} \to \mathcal{C}$ exists.
3. The right Kan extension of each $\mathcal{Q}$-functor $F : \mathcal{A} \to \mathcal{B}$ along any $K : \mathcal{A} \to \mathcal{C}$ exists.
4. The left Kan extension of each $\mathcal{Q}$-functor $F : \mathcal{A} \to \mathcal{B}$ along $\mathcal{Y}_A : \mathcal{A} \to \mathcal{P}A$ exists.
5. The right Kan extension of each $\mathcal{Q}$-functor $F : \mathcal{A} \to \mathcal{B}$ along $\mathcal{Y}_A^\dagger : \mathcal{A} \to \mathcal{P}^\dagger A$ exists.

In this case,

$$\operatorname{colim}_\mu F = (\operatorname{Lan}_{\mathcal{Y}_A} F)\mu \quad \text{and} \quad \operatorname{lim}_\lambda F = (\operatorname{Ran}_{\mathcal{Y}_A^\dagger} F)\lambda$$

for each $\mathcal{Q}$-functor $F : \mathcal{A} \to \mathcal{B}$ and $\mu \in \mathcal{P}A$, $\lambda \in \mathcal{P}^\dagger A$.

**Proof.** We prove (1) $\iff$ (2) $\iff$ (4) for example.

(1) $\implies$ (2): Proposition 6.1.2.

(2) $\implies$ (4): Trivial.

(4) $\implies$ (1): For each $\mathcal{Q}$-functor $F : \mathcal{A} \to \mathcal{B}$ and $\mu \in \mathcal{P}A$, 

$$\operatorname{colim}_\mu F = \operatorname{colim}_{(\mathcal{Y}_A)_\dagger(-,\mu)} F = (\operatorname{Lan}_{\mathcal{Y}_A} F)\mu,$$

where the first equality follows from the Yoneda lemma, and the second equality follows from Proposition 6.1.2. Thus $\mathcal{B}$ is complete.

If the $\mathcal{Q}$-functor $K : \mathcal{A} \to \mathcal{C}$ is fully faithful, then the pointwise Kan extension obtained in Proposition 6.1.2 is actually an extension, as indicated in the following conclusion.
Corollary 6.1.4. \cite{Stu05a} Let $F : \mathcal{A} \to \mathcal{B}$ and $K : \mathcal{A} \to \mathcal{C}$ be $\mathcal{Q}$-functors, with $K$ fully faithful.

(1) If the pointwise left Kan extension of $F$ along $K$ exists, then $(\text{Lan}_K F) \circ K \cong F$.

(2) If the pointwise right Kan extension of $F$ along $K$ exists, then $(\text{Ran}_K F) \circ K \cong F$.

**Proof.** We prove (1) for example. For each $a \in \mathcal{A}_0$,

$$
\mathbb{B}((\text{Lan}_K F) \circ Ka, -) = \mathbb{B}((\text{colim}_{K_a(-)}^{-} F), -) \quad \text{(Proposition 6.1.2)}
$$

$$
= \mathbb{B}((\text{colim}_C K_{-}(-) F), -)
$$

$$
= \mathbb{B}((\text{colim}_{\mathcal{A}_(-)}^{-} a F), -) \quad \text{ (K is fully faithful)}
$$

$$
= F_a \vee \mathcal{A}(\cdot, a) \quad \text{(Definition 3.4.1)}
$$

$$
= F_a(a, -)
$$

$$
= \mathbb{B}(Fa, -).
$$

Thus $(\text{Lan}_K F) \circ K \cong F$.

**6.2 Kan adjunctions**

Given a $\mathcal{Q}$-distributor $\phi : \mathcal{A} \leftrightarrow \mathcal{B}$, recall that composing with $\phi$ yields two $\mathcal{Q}$-functors

$$
\phi^* : \mathcal{P}_\mathcal{B} \to \mathcal{P}_\mathcal{A} \quad \text{and} \quad \phi^! : \mathcal{P}^!_\mathcal{A} \to \mathcal{P}^!_\mathcal{B}
$$

defined by Equation (3.16), i.e.,

$$
\phi^*(\lambda) = \lambda \circ \phi \quad \text{and} \quad \phi^!(\mu) = \phi \circ \mu.
$$

Define another two $\mathcal{Q}$-functors

$$
\phi_* : \mathcal{P}_\mathcal{A} \to \mathcal{P}_\mathcal{B} \quad \text{and} \quad \phi^! : \mathcal{P}^!_\mathcal{B} \to \mathcal{P}^!_\mathcal{A}
$$

by

$$
\phi_*(\mu) = \mu \vee \phi \quad \text{and} \quad \phi^!(\lambda) = \phi \wedge \lambda. \quad (6.5)
$$
We remind the readers that \( \phi^\dagger \) and \( \phi_\dagger \) are both covariant with respect to local orders in \( \mathcal{Q} \)-\text{Dist}. Indeed,

\[
\mu_1 \leq \mu_2 \text{ in } \mathcal{Q} \text{-Dist} \iff \mu_2 \leq \mu_1 \text{ in } (\mathcal{P}^\dagger A)_0 \quad \text{(Remark 3.3.8)}
\]

\[
\Rightarrow \phi^\dagger(\mu_2) \leq \phi^\dagger(\mu_1) \text{ in } (\mathcal{P}^\dagger B)_0
\]

\[
\iff \phi^\dagger(\mu_1) \leq \phi^\dagger(\mu_2) \text{ in } \mathcal{Q} \text{-Dist}.
\]

Similarly one can deduce that

\[
\lambda_1 \leq \lambda_2 \text{ in } \mathcal{Q} \text{-Dist} \implies \phi_\dagger(\lambda_1) \leq \phi_\dagger(\lambda_2) \text{ in } \mathcal{Q} \text{-Dist}.
\]

**Proposition 6.2.1.** \( \phi^* \dashv \phi_* : \mathcal{P} B \to \mathcal{P} A \) and \( \phi_\dagger \dashv \phi^\dagger : \mathcal{P}^\dagger B \to \mathcal{P}^\dagger A \) in \( \mathcal{Q} \text{-Cat} \).

**Proof.** For all \( \lambda \in \mathcal{P} B \) and \( \mu \in \mathcal{P} A \),

\[
\mathcal{P} A(\phi^*(\lambda), \mu) = \mu \not\succ \phi^*(\lambda)
\]

\[
= \mu \not\succ (\lambda \circ \phi)
\]

\[
= (\mu \not\succ \phi) \not\succ \lambda
\]

\[
= \phi_*(\mu) \not\succ \lambda
\]

\[
= \mathcal{P} B(\lambda, \phi_*(\mu)).
\]

Thus \( \phi^* \dashv \phi_* : \mathcal{P} B \to \mathcal{P} A \). Similarly, for all \( \lambda \in \mathcal{P}^\dagger B \) and \( \mu \in \mathcal{P}^\dagger A \),

\[
\mathcal{P}^\dagger A(\phi_\dagger(\lambda), \mu) = \mu \not\prec \phi_\dagger(\lambda)
\]

\[
= \mu \not\prec (\phi \not\prec \lambda)
\]

\[
= (\phi \not\prec \mu) \not\prec \lambda
\]

\[
= \phi^\dagger(\mu) \not\prec \lambda
\]

\[
= \mathcal{P}^\dagger B(\lambda, \phi^\dagger(\mu)).
\]

Thus \( \phi_\dagger \dashv \phi^\dagger : \mathcal{P}^\dagger B \to \mathcal{P}^\dagger A \).

We would like to stress that

\[
\lambda \leq \phi_* \circ \phi^*(\lambda) \quad \text{and} \quad \phi^* \circ \phi_*(\mu) \leq \mu \quad (6.6)
\]
for all $\lambda \in P_B$ and $\mu \in P_A$; whereas
\[ \nu \leq \phi_! \circ \phi^!(\nu) \quad \text{and} \quad \phi^! \circ \phi_!(\gamma) \leq \gamma \] (6.7)
for all $\nu \in P^!_A$ and $\gamma \in P^!_B$ by Remark 3.3.8.

**Proposition 6.2.2.** Let $\phi : A \rightarrow B$ be a $Q$-distributor, then $\phi^* \circ Y_B^\neg = \phi = \phi^! \circ Y_A^\neg$.

**Proof.** For all $x \in A_0$ and $y \in B_0$,
\[
\phi^* \circ Y_B^\neg(x, y) = (\phi^* \circ Y_B y)(x) \\
= (Y_B y \circ \phi)(x) \\
= B(-, y) \circ \phi(x, -) \\
= \phi(x, y) \\
= \phi(-, y) \circ A(x, -) \\
= (\phi \circ Y_A^!)(y) \\
= (\phi^! \circ Y_A^!)(y) \\
= \phi^! \circ Y_A^\neg(x, y),
\]
showing that the conclusion holds. \( \square \)

Given a $Q$-distributor $\phi : A \rightarrow B$, for each $y \in B_0$, since
\[ \overline{\phi y} = \phi(-, y) = \phi^* \circ Y_B y = \phi^* \circ \phi_+ \circ \phi^* \circ Y_B y, \] (6.8)
it follows that $\overline{\phi y} = \phi(-, y)$ is a fixed point of the $Q$-interior operator $\phi^* \circ \phi_+ : P_B \longrightarrow P_B$. Similarly, for all $x \in A_0$,
\[ \underline{\phi x} = \phi(x, -) = \phi^! \circ Y_A^! x = \phi^! \circ \phi_+ \circ \phi^! \circ Y_A^! x \] (6.9)
is a fixed point of the $Q$-closure operator $\phi^! \circ \phi_+ : P^!_B \longrightarrow P^!_B$.

If $\phi : A \rightarrow B$ is itself a left adjoint $Q$-distributor, then $\phi^*$ and $\phi^!$ are not only left adjoint $Q$-functors, but also right adjoint $Q$-functors as asserted in the following proposition. It is also an analogue of Proposition 3.3.4 for $Q$-distributors.
Proposition 6.2.3. Let $\phi : A \rightarrow B$ and $\psi : B \rightarrow A$ be a pair of $\mathcal{Q}$-distributors. The following conditions are equivalent:

(1) $\phi \dashv \psi : A \rightarrowtail B$ in $\mathcal{Q}$-Dist.

(2) $\phi^* = \psi_*$. 

(3) $\psi^* \dashv \phi^* : \mathcal{P}A \rightarrowtail \mathcal{P}B$ in $\mathcal{Q}$-Cat.

(4) $\psi_* \dashv \phi_* : \mathcal{P}B \rightarrowtail \mathcal{P}A$ in $\mathcal{Q}$-Cat.

(5) $\phi^\dagger = \psi^\dagger$. 

(6) $\psi^\dagger \dashv \phi^\dagger : \mathcal{P}^\dagger A \rightarrowtail \mathcal{P}^\dagger B$ in $\mathcal{Q}$-Cat.

(7) $\psi^\dagger \dashv \phi^\dagger : \mathcal{P}^\dagger B \rightarrowtail \mathcal{P}^\dagger A$ in $\mathcal{Q}$-Cat.

Proof. (1) $\implies$ (2): By Proposition 3.1.9(1), for all $\lambda \in \mathcal{P}B$,

$$\phi^*(\lambda) = \lambda \circ \phi = \lambda \vee \psi = \psi_*(\lambda).$$

(2) $\implies$ (1): We must show that $A \leq \psi \circ \phi$ and $\phi \circ \psi \leq B$. Indeed, for all $x \in A_0$ and $y \in B_0$,

$$\psi(-, x) \circ \phi = \phi^*(\psi(-, x))$$

$$= \psi_* \circ \psi^* \circ YAx = \psi_\lambda \lambda \circ \phi \circ \psi = \psi_*(\lambda).$$

(Proposition 6.2.2)

$$\geq 1_{\mathcal{P}A} \circ YAx = \mathcal{A}(-, x)$$

(Equation (6.6))

and

$$\phi(-, y) \circ \psi = \psi^*(\phi(-, y))$$

$$= \psi^* \circ \phi^* \circ YBy = \psi^*(\phi(-, y))$$

(Proposition 6.2.2)

$$= \psi^* \circ \psi_* \circ YBy$$

$$\leq 1_{\mathcal{P}B} \circ YBy = \mathcal{B}(-, y).$$

(Equation (6.6))
(1) $\iff$ (3) $\iff$ (4): Follows immediately from (1) $\iff$ (2) and Corollary 3.1.10.

(1) $\iff$ (5): Similar to (1) $\iff$ (2).

(1) $\iff$ (6) $\iff$ (7): Follows immediately from (1) $\iff$ (5) and Corollary 3.1.10.

Therefore, if a $Q$-distributor $\phi$ has a right adjoint $\psi$ in $Q$-Dist, then $\phi^*$ has both a right adjoint $\phi_*$ and a left adjoint $\psi^*$ in $Q$-Cat.

In particular, given a $Q$-functor $F : A \to B$, since the cograph $F^\natural : B \to A$ of $F$ is the right adjoint of the graph $F_\flat : A \to B$ of $F$, it follows that $(F^\natural)_* = (F_\flat)^*$ is the right adjoint of $(F^\natural)^*$, and $(F_\flat)^{\dagger} = (F^\natural)^{\dagger}$ is the left adjoint of $(F_\flat)^\dagger$.

Since $F^\leftarrow : P^B \to P^A$ and $F^\leftarrow : P^{\dagger}B \to P^{\dagger}A$ are respectively the counterparts of the functor $- \circ F$ and $F \circ -$ for $Q$-categories, we arrive at the following conclusion which asserts that the adjunctions $\phi^* \dashv \phi_*$ and $\phi^\dagger \dashv \phi^{\dagger}$ generalize Kan extensions (Corollary 2.3.3) in category theory.

**Theorem 6.2.4.** For each $Q$-functor $F : A \to B$, it holds that

$$F^\leftarrow = (F^\natural)^* \dashv (F^\natural)_* = F^\leftarrow = (F_\flat)^* \dashv (F_\flat)_*$$

and

$$(F^\natural)^{\dagger} \dashv (F^\natural)^\dagger = F^\leftarrow = (F_\flat)^{\dagger} \dashv (F_\flat)^\dagger.$$

**Proof.** It remains to prove $(F^\natural)_* = F^\leftarrow$ and $(F^\natural)^{\dagger} = F^\leftarrow$. Indeed, for each $\lambda \in P^B$ and $\gamma \in P^{\dagger}B$, by Definition 3.3.12

$$F^\leftarrow(\lambda) = \lambda \circ F_\flat = (F^\natural)^*(\lambda) \quad \text{and} \quad F^\leftarrow(\gamma) = F^\natural \circ \gamma = (F^\natural)^{\dagger}(\gamma).$$

So, adjunctions of the forms $\phi^* \dashv \phi_* : P^B \to P^A$ and $\phi^{\dagger} \dashv \phi^{\dagger} : P^{\dagger}B \to P^{\dagger}A$ will be called Kan adjunctions by abuse of language.

**Remark 6.2.5.** In Section 6.1 we have seen that if each $Q$-functor $F : A \to B$ has left and right Kan extensions along $K : A \to C$, then the order-preserving
function “composing with $K$”

$$K^* : Q\text{-Cat}(C, B) \longrightarrow Q\text{-Cat}(A, B)$$

is both a left and right adjoint in $2\text{-Cat}$ (not in $Q\text{-Cat}$, since $Q\text{-Cat}(A, B)$ is not a $Q$-category in general).

However, if we consider the graph $F_\natural$ of $F$ instead, then the $Q$-functor “composing with $F_\natural$”

$$(F_\natural)^* : P^B \longrightarrow P^A$$

is both a left and right adjoint in $Q\text{-Cat}$, as presented in Theorem 6.2.4.

**Corollary 6.2.6.** [PZ14] Let $F : A \longrightarrow B$ and $G : B \longrightarrow A$ be a pair of $Q$-functors. The following conditions are equivalent:

1. $F \dashv G : A \Rightarrow B$.
2. $F \Rightarrow G : P^A \Rightarrow P^B$.
3. $G^\Rightarrow : F^\Rightarrow : P^B \Rightarrow P^A$.
4. $F^\Rightarrow : G^\Rightarrow : P^{\dag A} \Rightarrow P^{\dag B}$.
5. $G^\Rightarrow : F^\Rightarrow : P^{\dag B} \Rightarrow P^{\dag A}$.

**Proof.** This is an immediate consequence of Proposition 3.3.4, Proposition 6.2.3 and Theorem 6.2.4. We prove $(1) \iff (2)$ for example.

$$F \Rightarrow G : A \Rightarrow B$$

$$\iff G^\Rightarrow \Rightarrow F^\Rightarrow : A \Rightarrow B$$

(Proposition 3.3.4(4))

$$\iff (F^\Rightarrow)^* \Rightarrow (G^\Rightarrow)^* : P^A \Rightarrow P^B$$

(Proposition 6.2.3(3))

$$\iff F^\Rightarrow \Rightarrow G^\Rightarrow : P^A \Rightarrow P^B.$$ 

(Theorem 6.2.4)

The following theorem states that all adjunctions between $P^B$ and $P^A$ are of the form $\phi^* \Rightarrow \phi_*$, and all adjunctions between $P^{\dag B}$ and $P^{\dag A}$ are of the form $\phi_\dag \Rightarrow \phi^\dag$. 


Theorem 6.2.7. Let $A$ and $B$ be $Q$-categories.

1. The correspondence $\phi \mapsto \phi^*$ is an isomorphism of posets

$$Q\text{-Dist}(A, B) \cong Q\text{-CCat}(P_B, P_A).$$

2. The correspondence $\phi \mapsto \phi^\dagger$ is an isomorphism of posets

$$Q\text{-Dist}(A, B) \cong (Q\text{-CCat}^\dagger)^\circ(P^\dagger_A, P^\dagger_B).$$

Proof. (1) Let $F : P_B \longrightarrow P_A$ be a left adjoint $Q$-functor. We show that the correspondence $F \mapsto \Phi \Phi : F \mapsto \phi^*$ is an inverse of the correspondence $\phi \mapsto \phi^*$, and thus they are both isomorphisms of posets between $Q\text{-Dist}(A, B)$ and $Q\text{-CCat}(P_B, P_A)$.

Firstly, we show that both of the correspondences are order-preserving. Indeed,

$$\phi \leq \psi \text{ in } Q\text{-Dist}(A, B)$$

$$\iff \forall \lambda \in P_B, \phi^*(\lambda) = \lambda \circ \phi \leq \lambda \circ \psi = \psi^*(\lambda) \text{ in } Q\text{-Dist}$$

$$\iff \forall \lambda \in P_B, \phi^*(\lambda) \leq \psi^*(\lambda) \text{ in } (P_A)_0$$

$$\iff \phi^* \leq \psi^* \text{ in } Q\text{-CCat}(P_B, P_A)$$

and

$$F \leq G \text{ in } Q\text{-CCat}(P_B, P_A)$$

$$\iff \forall \lambda \in P_B, F(\lambda) \leq G(\lambda) \text{ in } (P_A)_0$$

$$\iff \forall \lambda \in P_B, F(\lambda) \leq G(\lambda) \text{ in } Q\text{-Dist}(A, B)$$

$$\iff \forall y \in B_0, F \circ Y_B y \leq G \circ Y_B y \text{ in } Q\text{-Dist}(A, B)$$

$$\iff \forall y \in B_0, (F \circ Y_B) \dag(-, y) \leq (G \circ Y_B) \dag(-, y) \text{ in } Q\text{-Dist}(A, B)$$

$$\iff (F \circ Y_B) \dag \leq (G \circ Y_B) \dag \text{ in } Q\text{-Dist}(A, B).$$

Secondly, $F = (\Gamma F \circ Y_B) \dag$. For all $\lambda \in P_B$, since $F$ is a left adjoint in $Q\text{-Cat}$,
by Example 3.5.6 and Proposition 3.5.14 we have

$$F(\lambda) = F(\lambda \circ B)$$

$$= F\left( \bigvee_{y \in B_0} \lambda(y) \circ Y_B y \right)$$

$$= \bigvee_{y \in B_0} \lambda(y) \circ (F \circ Y_B y)$$

$$= \lambda \circ \gamma F \circ Y_B^{-1}$$

$$= (\gamma F \circ Y_B^{-1})^*(\lambda).$$

Finally, $\phi = \gamma \phi^* \circ Y_B^{-1}$. This is obtained in Proposition 6.2.2.

(2) Let $F : \mathcal{P}^\dagger A \to \mathcal{P}^\dagger B$ be a right adjoint $Q$-functor. Similar to (1), one can show that the correspondence $F \mapsto \gamma F \circ Y_B^{-1}$ is an inverse of the correspondence $\phi \mapsto \phi^\dagger$, and thus they are both isomorphisms of posets between $Q\text{-Dist}(A, B)$ and $(Q\text{-CCat}^\dagger)^{\text{co}}(\mathcal{P}^\dagger A, \mathcal{P}^\dagger B)$.

Theorem 6.2.7 adds two more isomorphisms of posets to (5.7):

- $Q\text{-Dist}(A, B) \cong Q\text{-Cat}^{\text{co}}(A, \mathcal{P}^\dagger B)$
- $Q\text{-Cat}(\mathcal{P}^\dagger A, \mathcal{P}^\dagger B)$
- $Q\text{-CCat}^{\text{co}}(\mathcal{P}^\dagger A, \mathcal{P}^\dagger B)$
- $Q\text{-CCat}(\mathcal{P}^\dagger B, \mathcal{P}^\dagger A)$
- $(Q\text{-CCat}^\dagger)^{\text{co}}(\mathcal{P}^\dagger A, \mathcal{P}^\dagger B)$.

The following corollary reveals the connection between Kan adjunctions and pointwise Kan extensions.

**Corollary 6.2.8.** Let $F : A \to B$ be a $Q$-functor.

1. $(F^\natural)^* = \text{Lan}_{\mathcal{V}_A} (Y_B \circ F)$, $(F^\natural)_\dagger = \text{Ran}_{\mathcal{V}_A} (Y_B^\dagger \circ F)$.

2. If the pointwise left Kan extension of $F$ along $K : A \to \mathcal{C}$ exists, then

$$\text{Lan}_K F(c) = B \nabla (F^\natural)^*(K^\natural(c, -)).$$
If the pointwise right Kan extension of $F$ along $K : A \to C$ exists, then

$$\text{Ran}_K F(c) = (F_\sharp)(K^\natural(c, -)) \downarrow B.$$  

**Proof.** (1) Since $\mathcal{P}B$ is complete, the pointwise left Kan extension of $Y_B \circ F : A \to \mathcal{P}B$ along $Y_A : A \to \mathcal{P}A$ exists. For all $\mu \in \mathcal{P}A$,

$$(F^\natural)^*(\mu) = \mu \circ F^\natural$$

$$(F^\natural)^*(\mu) = \mu \circ (Y_B \circ F)^\natural \circ (Y_B)_\sharp$$

$$(F^\natural)^*(\mu) = \sup_{\mathcal{P}B} (Y_B \circ F)^\to(\mu)$$

$$(F^\natural)^*(\mu) = \sup_{\mathcal{P}B} (Y_B \circ F)^\to((Y_A)_\sharp(-, \mu))$$

$$(F^\natural)^*(\mu) = \text{Lan}_{Y_A} (Y_B \circ F)(\mu).$$

Thus $(F^\natural)^* = \text{Lan}_{Y_A} (Y_B \circ F)$. That $(F_\sharp)^* = \text{Ran}_{Y_A} (Y_B^\natural \circ F)$ can be proved similarly.

(2) and (3) are immediate consequences of Proposition 6.1.2. \[ \Box \]

**Remark 6.2.9.** Consider the two-element Boolean algebra $2$ as a quantaloid. Then every set can be regarded as a discrete $2$-category. Given sets $A$ and $B$, a $2$-distributor $F : A \leftrightarrow B$ is essentially a relation from $A$ to $B$, or a set-valued map $A \to 2B$. If we write $F^\op$ for the dual relation of $F$, then both $F_*$ and $(F^\op)^*$ are maps from $2B$ to $2A$. Explicitly, for each $V \subseteq B$,

$$F_*(V) = \{x \in A \mid F(x) \subseteq V\} \quad \text{and} \quad (F^\op)^*(V) = \{x \in A \mid F(x) \cap V \neq \emptyset\}.$$  

If both $A$ and $B$ are topological spaces, then the upper and lower semi-continuity of $F$ (as a set-valued map) \cite{Ber63} can be phrased as follows: $F$ is upper (resp. lower) semi-continuous if $F_*(V)$ (resp. $(F^\op)^*(V)$) is open in $A$ whenever $V$ is open in $B$. In particular, if $F$ is the graph of some map $f : A \to B$, then $(F^\op)^*(V) = F_*(V) = f^{-1}(V)$ for all $V \subseteq B$, hence $f$ is continuous iff $F$ is lower semi-continuous iff $F$ is upper semi-continuous \cite{Ber63}.

The following corollary shows that for a fully faithful $Q$-functor $F : A \to B$, both $(F^\sharp)^*$ and $(F_\sharp)_*$ can be regarded as extensions of $F$ \cite{Law73}. 

---

(3) If the pointwise right Kan extension of $F$ along $K : A \to C$ exists, then
Corollary 6.2.10. If $F : A \rightarrow B$ is a fully faithful $Q$-functor, then for all $\mu \in PA$, it holds that $(F^\natural)^*(\mu) \circ F_\natural = \mu$ and $(F^\natural)_*(\mu) \circ F_\natural = \mu$.

Proof. The first equality is a reformulation of Proposition \ref{prop:3.3.5}(1). For the second equality,

$$(F^\natural)_*(\mu) \circ F_\natural = (\mu \not\leq F_\natural) \circ F_\natural = (F^\natural \circ F_\natural) (Proposition \ref{prop:3.1.9}(4)) = (\mu \not\leq A) (Proposition \ref{prop:3.3.5}(1)) = \mu.$$  

This completes the proof. \hfill $\Box$

### 6.3 Functoriality of the Kan adjunction

In this section, we show that the construction of Kan adjunctions $\phi^* \dashv \phi_*$ is contravariant functorial from the category $Q$-Info to $Q$-Cls, and thus to $Q$-CCat.

Since $\phi_* \circ \phi^* : PB \rightarrow PB$ is a $Q$-closure operator for each $Q$-distributor $\phi : A \xrightarrow{\to} B$, it follows that $(B, \phi_* \circ \phi^*)$ is a $Q$-closure space.

**Proposition 6.3.1.** Let $(F,G) : (\phi : A \xrightarrow{\to} B) \rightarrow (\psi : A' \xrightarrow{\to} B')$ be an infomorphism. Then $G : (B', \psi_* \circ \psi^*) \rightarrow (B, \phi_* \circ \phi^*)$ is a continuous $Q$-functor.

Proof. Consider the following diagram:

We must prove $G^\rightarrow \circ \psi_* \circ \psi^* \leq \phi_* \circ \phi^* \circ G^\rightarrow$. To this end, it suffices to check that

(a) the left square commutes if and only if $(F,G) : \phi \rightarrow \psi$ is an infomorphism; and
(b) if $G^\circ \phi \leq \psi \circ F$, then $G \circ \psi_* \leq \phi_* \circ F$.

For (a), suppose $F \circ \psi^* = \phi^* \circ G$, then for all $y' \in \mathbb{B}_0$,

\[ G^\circ (\cdot, y') \circ \phi = \mathbb{B}' (\cdot, y') \circ G^\circ \circ \phi \]

\[ = \phi^* (Y_{\mathbb{B}} y' \circ G^\circ) \quad \text{(Equation 3.16)} \]

\[ = \phi^* (G \circ \circ Y_{\mathbb{B}} y') \quad \text{(Definition 3.3.12)} \]

\[ = F \circ (\psi^* \circ Y_{\mathbb{B}} y') \quad \text{(Definition 3.3.12)} \]

Conversely, if $(F, G) : \phi \longrightarrow \psi$ is an infomorphism, then for all $\lambda' \in \mathcal{P}_{\mathbb{B}'}$,

\[ F \circ \psi^* (\lambda') = \lambda' \circ \psi \circ F \quad \text{(Equation 3.16)} \]

\[ = \lambda' \circ G^\circ \circ \phi \quad \text{(Definition 3.7.1)} \]

\[ = \phi^* \circ G \circ (\lambda'). \quad \text{(Equation 3.16)} \]

For (b), suppose $G^\circ \circ \phi \leq \psi \circ F$, then for all $\mu' \in \mathcal{P}_{\mathbb{A}'}$,

\[ G \circ \psi_* (\mu') = (\mu^\circ \psi) \circ G^\circ \quad \text{(Equation 6.5)} \]

\[ \leq (\mu^\circ \psi) \circ ((\psi \circ F) \circ \phi) \]

\[ \leq ((\mu^\circ \psi) \circ \psi \circ F) \circ \phi \]

\[ \leq (\mu^\circ F) \circ \phi \]

\[ = \phi_* \circ F \circ (\mu'). \quad \text{(Equation 6.5)} \]

This completes the proof.

By virtue of Proposition 6.3.1 we obtain a functor $\mathcal{V} : (\mathcal{Q-Info})^{\text{op}} \longrightarrow \mathcal{Q-Cls}$ that sends an infomorphism

\[ (F, G) : (\phi : \mathbb{A} \rightarrow \mathbb{B}) \longrightarrow (\psi : \mathbb{A}' \rightarrow \mathbb{B}') \]
to a continuous $Q$-functor

$$G : (\mathbb{B}', \psi_* \circ \psi^*) \longrightarrow (\mathbb{B}, \phi_* \circ \phi^*).$$

The composition of

$$\mathcal{V} : (Q\text{-Info})^{\text{op}} \longrightarrow Q\text{-Cls}$$

and

$$\mathcal{T} : Q\text{-Cls} \longrightarrow Q\text{-CCat}$$

gives a functor

$$\mathcal{K} = \mathcal{T} \circ \mathcal{V} : (Q\text{-Info})^{\text{op}} \longrightarrow Q\text{-CCat} \quad (6.10)$$

that sends each $Q$-distributor $\phi : A \leftarrow \mathbb{B}$ to the complete $Q$-category $\mathcal{K}(\phi) = \phi_* \circ \phi^* (\mathcal{P}\mathbb{B})$.

The following conclusion asserts that the free cocompletion functor of $Q$-categories factors through $\mathcal{K}$. 

**Proposition 6.3.2.** If $F : A \longrightarrow \mathbb{B}$ is a fully faithful $Q$-functor, then $\mathcal{K}(F^\natural) = \mathcal{P}A$.

In particular, the diagram

$$Q\text{-Cat} \overset{\mathcal{V}^!}{\longrightarrow} (Q\text{-Info})^{\text{op}} \overset{\mathcal{K}}{\longrightarrow} Q\text{-CCat}$$

commutes.

**Proof.** In order to see that $\mathcal{K}(F^\natural) = (F^\natural)_* \circ (F^\natural)^* (\mathcal{P}A) = \mathcal{P}A$, it suffices to check
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that \((F^\natural)_* \circ (F^\natural)^*(\mu) = \mu\) for all \( \mu \in \mathcal{P}A \). Indeed,

\[
(F^\natural)_* \circ (F^\natural)^*(\mu) = (F^\natural)_* \circ (F^\natural)^*(\mu) = (F^\natural \circ F^\natural)^*(\mu) = A^*(\mu) = \mu.
\]

Furthermore, it is easy to verify that \( K \circ Y^\dagger(G) = G^\to = \mathcal{P}(G) \) for each \( Q \)-functor \( G : A \to B \). Thus, the conclusion follows.

\[ \square \]

6.4 When \( Q \) is a Girard quantaloid

Theorem 5.2.4 shows that every skeletal complete \( Q \)-category is of the form \( \mathcal{M}(\phi) \).

It is natural to ask whether every skeletal complete \( Q \)-category can be written of the form \( \mathcal{K}(\phi) \) for some \( Q \)-distributor \( \phi \). A little surprisingly, this is not true in general. This fact was pointed out in \([LZ09]\) in the case that \( Q \) is a unital commutative quantale. However, the answer is positive when \( Q \) is a Girard quantaloid.

Let \( Q \) be a Girard quantaloid with a cyclic dualizing family

\[ \mathcal{D} = \{d_X : X \to X \mid X \in Q_0\}. \]

For all \( f \in Q(X,Y) \), let

\[ \neg f = d_X \vee f = f \wedge d_Y : Y \to X. \]

Then \( \neg \neg f = f \) since \( \mathcal{D} \) is a dualizing family. For each \( Q \)-category \( A \), set

\[ (\neg A)(y, x) = \neg A(x, y) \]

for all \( x, y \in A_0 \). It is easy to verify that \( \neg A : A \to A \) is a \( Q \)-distributor.

**Proposition 6.4.1.** \([Ros96]\) If \( Q \) is a Girard quantaloid, then \( Q\text{-Dist} \) is a Girard quantaloid.
Proof. If $\mathcal{D} = \{d_X : X \twoheadrightarrow X \mid X \in \mathcal{Q}_0\}$ is a cyclic dualizing family of $\mathcal{Q}$, we show that

$$\mathcal{D}' = \{\neg A : A \rightarrowtail A \mid A \text{ is a } \mathcal{Q}\text{-category}\}$$

is a cyclic dualizing family of $\mathcal{Q}\text{-Dist}$. Indeed, for each $\mathcal{Q}\text{-distributor } \phi : A \rightarrowtail B$,

$$\neg A \upharpoonright \phi = \bigwedge_{x \in A_0} (A(-,x) \downarrow d_{tx}) \upharpoonright \phi(x,-)$$

$$= \bigwedge_{x \in A_0} A(-,x) \downarrow (d_{tx} \upharpoonright \phi(x,-))$$

$$= A \downarrow \neg \phi$$

$$= \neg \phi \upharpoonright B$$

$$= \bigwedge_{y \in B_0} (\phi(-,y) \downarrow d_{ty}) \upharpoonright B(y,-)$$

$$= \bigwedge_{y \in B_0} \phi(-,y) \downarrow (d_{ty} \upharpoonright B(y,-))$$

$$= \phi \downarrow \neg B,$$

and it follows that

$$ \neg A \upharpoonright \phi \upharpoonright \neg A = (A \downarrow \neg \phi) \downarrow \neg A $$

$$ = \neg \phi \downarrow \neg A $$

$$ = \neg \neg \phi \upharpoonright A $$

$$ = \phi,$$

as desired. \hfill \Box$

Therefore, by assigning $\neg \phi = \neg A \upharpoonright \phi = \phi \downarrow \neg B$ for each $\mathcal{Q}\text{-distributor } \phi : A \rightarrowtail B$, we obtain a functor $\neg : \mathcal{Q}\text{-Info} \rightarrow (\mathcal{Q}\text{-Info})^{op}$ that sends an infomorphism

$$(F,G) : (\phi : A \rightarrowtail B) \rightarrow (\psi : A' \rightarrowtail B')$$

to

$$(G,F) : (\neg \psi : B' \rightarrowtail A') \rightarrow (\neg \phi : B \rightarrowtail A).$$

It is clear that $\neg \circ \neg = \text{id}_{\mathcal{Q}\text{-Info}}$. We leave it to the reader to check that $(-\phi)(y,x) = -\phi(x,y)$ for any $\mathcal{Q}\text{-distributor } \phi : A \rightarrowtail B$ and $x \in A_0, y \in B_0$. 

Lemma 6.4.2. Suppose $Q$ is a Girard quantaloid. Then for any $Q$-distributor $\phi : A \rightarrow B$, it holds that $\phi^* = \neg \circ (\neg \phi)_\uparrow$ and $\phi_* = (\neg \phi)^\downarrow \circ \neg$.

Proof. For all $\lambda \in P B$ and $\mu \in P A$, we have

$$\phi^*(\lambda) = \lambda \circ \phi$$
$$= \lambda \circ (\neg \phi \searrow \neg A) \quad \text{(Proposition 6.4.1)}$$
$$= (\neg \phi \not\!
earrow \lambda) \searrow \neg A \quad \text{(Proposition 3.1.15(3))}$$
$$= \neg \circ (\neg \phi)_\uparrow(\lambda)$$

and

$$\phi_*(\mu) = \mu \not\!
earrow \phi$$
$$= \mu \not\!
earrow (\neg \phi \searrow \neg A) \quad \text{(Proposition 6.4.1)}$$
$$= (\neg A \not\!
earrow \mu) \searrow \neg \phi \quad \text{(Proposition 3.1.15(5))}$$
$$= \neg \mu \searrow \neg \phi \quad \text{(Proposition 6.4.1)}$$
$$= (\neg \phi)^\downarrow \circ \neg \mu.$$  

The conclusion thus follows. \qed

Proposition 6.4.3. Suppose $Q$ is a Girard quantaloid. Then $V = U \circ \neg$ and it has a left adjoint right inverse given by

$$G = \neg \circ F : Q \cdot \text{Cls} \rightarrow (Q \cdot \text{Info})^{\text{op}}.$$  

Therefore, every skeletal complete $Q$-category is isomorphic to $\mathcal{K}(\phi)$ for some $Q$-distributor $\phi$.

Proof. This is an immediate consequence of Theorem 5.2.2 and Lemma 6.4.2. \qed

6.5 Towards a characterization of $\mathcal{K}(\phi)$

In the case that $Q$ is a Girard quantaloid, the results in Section 6.4 asserts that

$$\mathcal{K}(\phi) = M(\neg \phi)$$
for each \(Q\)-distributor \(\phi : A \to B\). Thus, the characterizations of \(K(\phi)\) can be obtained directly from that of \(M(\phi)\) presented in Section 5.4.

For a general quantaloid \(Q\), characterizing \(K(\phi)\) is more complicated. In this section, we provide a characterization of \(K(\phi)\) parallel to Proposition 5.4.1. However, we fail to find a counterpart of Theorem 5.4.4 for \(K(\phi)\).

Given a \(Q\)-distributor \(\phi : A \to B\), let \(K(\phi)(A, B)\) denote the set of pairs \((\mu, \lambda) \in \mathcal{P}A \times \mathcal{P}B\) such that \(\lambda = \phi_\ast(\mu)\) and \(\mu = \phi^* (\lambda)\). \(K(\phi)(A, B)\) becomes a \(Q\)-typed set if we assign \(t(\mu, \lambda) = t(\mu) = t(\lambda)\). For \((\mu_1, \lambda_1), (\mu_2, \lambda_2) \in K(\phi)(A, B)\), let

\[
K(\phi)(A, B)((\mu_1, \lambda_1), (\mu_2, \lambda_2)) = \mathcal{P}A(\mu_1, \mu_2) = \mathcal{P}B(\lambda_1, \lambda_2), \tag{6.11}
\]

Then \(K(\phi)(A, B)\) becomes a \(Q\)-category.

The projection

\[
\pi_2 : K(\phi)(A, B) \longrightarrow \mathcal{P}B, \quad (\mu, \lambda) \mapsto \lambda
\]

is clearly a fully faithful \(Q\)-functor. Since the image of \(\pi_2\) is exactly the set of fixed points of the \(Q\)-closure operator \(\phi_\ast \circ \phi^* : \mathcal{P}B \to \mathcal{P}B\), we obtain that \(K(\phi)(A, B)\) is isomorphic to the complete \(Q\)-category \(M(\phi) = \phi_\ast \circ \phi^*(\mathcal{P}B)\).

Similarly, the projection

\[
\pi_1 : K(\phi)(A, B) \longrightarrow \mathcal{P}A, \quad (\mu, \lambda) \mapsto \mu
\]

is also a fully faithful \(Q\)-functor and the image of \(\pi_1\) is exactly the set of fixed points of the \(Q\)-interior operator \(\phi^* \circ \phi_\ast : \mathcal{P}A \to \mathcal{P}A\). Hence \(K(\phi)(A, B)\) is also isomorphic to the complete \(Q\)-category \(\phi^* \circ \phi_\ast(\mathcal{P}A)\), which is a \(Q\)-interior system of the skeletal complete \(Q\)-category \(\mathcal{P}A\).

Equation (6.11) shows that

\[
\phi^* : \phi_\ast \circ \phi^*(\mathcal{P}B) \longrightarrow \phi^* \circ \phi_\ast(\mathcal{P}A)
\]

and

\[
\phi_\ast : \phi^* \circ \phi_\ast(\mathcal{P}A) \longrightarrow \phi^* \circ \phi_\ast(\mathcal{P}B)
\]

are inverse to each other. Therefore, \(K(\phi)(= \phi_\ast \circ \phi^*(\mathcal{P}B))\), \(\phi^* \circ \phi_\ast(\mathcal{P}A)\) and \(K(\phi)(A, B)\) are isomorphic to each other.
We present the following characterization of the complete $\mathcal{Q}$-category $K(\phi)$, and we remind the readers to compare it with Proposition 5.4.1.

**Proposition 6.5.1.** Let $X$ be a skeletal complete $\mathcal{Q}$-category. The following conditions are equivalent:

1. $X$ is isomorphic to $K(\phi)$ for some $\mathcal{Q}$-distributor $\phi : \mathbb{A} \leftrightarrow \mathbb{B}$.

2. $X$ is isomorphic to a $\mathcal{Q}$-interior system of $\mathcal{P}\mathbb{A}$ for some $\mathcal{Q}$-category $\mathbb{A}$.

3. $X$ is isomorphic to a subobject of $\mathcal{P}\mathbb{A}$ for some $\mathcal{Q}$-category $\mathbb{A}$ in the category $\mathcal{Q}$-$\text{CCat}$.

**Proof.** (1) $\implies$ (2) and (2) $\iff$ (3): Trivial.

(2) $\implies$ (1): Let $F : \mathcal{P}\mathbb{A} \rightarrow \mathcal{P}\mathbb{A}$ be a $\mathcal{Q}$-interior operator and $\mathbb{B} = F(\mathcal{P}\mathbb{A})$. Define a $\mathcal{Q}$-distributor $\zeta_F : \mathbb{A} \leftrightarrow \mathbb{B}$ by

$$\zeta_F(x, \mu) = \mu(x),$$

which is also obtained by restricting the domain and the codomain of the $\mathcal{Q}$-distributor [5.8]. We prove $F = (\zeta_F)^* \circ (\zeta_F)_*$, and consequently $K(\zeta_F)$ is isomorphic to $F(\mathcal{P}\mathbb{A})$.

For all $\mu \in \mathcal{P}\mathbb{A}$ and $x \in \mathbb{A}_0$, let $\lambda \in \mathbb{B}_0$, then

$$\mathcal{P}\mathbb{A}(\lambda, \mu) \leq \mathcal{P}\mathbb{A}(F(\lambda), F(\mu)) = \mathcal{P}\mathbb{A}(\lambda, F(\mu)) \leq F(\mu)(x) \not\leq \lambda(x),$$

and consequently

$$F(\mu)(x) \geq \mathcal{P}\mathbb{A}(\lambda, \mu) \circ \lambda(x) = \left( \bigwedge_{a \in \mathbb{A}_0} \mu(a) \not\leq \lambda(a) \right) \circ \lambda(x).$$

If $f : tx \rightarrow t\mu$ satisfies

$$f \geq \left( \bigwedge_{a \in \mathbb{A}_0} \mu(a) \not\leq F(\mu)(a) \right) \circ \lambda(x)$$

for all $\lambda \in \mathbb{B}_0$, then

$$f \geq \left( \bigwedge_{a \in \mathbb{A}_0} \mu(a) \not\leq F(\mu)(a) \right) \circ F(\mu)(x) \geq 1_{t\mu} \circ F(\mu)(x) = F(\mu)(x),$$
hence

\[ F(\mu)(x) = \bigvee_{\lambda \in B_0} \left( \bigwedge_{a \in A_0} \mu(a) \lor \lambda(a) \right) \circ \lambda(x) \]

\[ = \bigvee_{\lambda \in B_0} \left( \bigwedge_{a \in A_0} \mu(a) \lor \zeta_F(a,\lambda) \right) \circ \zeta_F(x,\lambda) \]

\[ = (\zeta_F)^* \circ (\zeta_F)_*(\mu)(x), \]

as required.

We wish to give a direct characterization of \( K_\phi(A,B) \) like Theorem 5.4.4 for \( M_\phi(A,B) \). However, although there exists a sup-dense \( Q \)-functor \( G : B \rightarrow K_\phi(A,B) \), the following Proposition 6.5.3 states that there is not an inf-dense \( Q \)-functor \( F : A \rightarrow K_\phi(A,B) \) in general.

**Proposition 6.5.2.** There exists a sup-dense \( Q \)-functor \( G : B \rightarrow K_\phi(A,B) \).

**Proof.** Let \( X = K_\phi(A,B) \). Define a \( Q \)-functor \( G : B \rightarrow X \) by

\[ Gb = (\overline{\phi}b, \phi_* \circ \overline{\phi}b), \]

then \( G \) is well defined by Equation (6.8). We show that \( G \) is sup-dense. For all \((\mu, \lambda), (\mu', \lambda') \in X_0\),

\[ X((\mu, \lambda), (\mu', \lambda')) = \mu' \not\sim \mu \]

\[ = \mu' \not\sim \phi^*(\lambda) \]

\[ = \mu' \not\sim (\lambda \circ \phi) \]

\[ = (\mu' \not\sim \phi) \not\sim \lambda \]

\[ = \mathcal{P}A(\overline{\phi^*}, \mu') \not\sim \lambda \]

\[ = X(G^-, (\mu', \lambda')) \not\sim \lambda \quad \text{(Equation 6.11)} \]

\[ = G_\omega(-, (\mu', \lambda')) \not\sim \lambda, \]

thus \((\mu, \lambda) = \text{colim}_\lambda G = \sup_X G^{-}(\lambda), \) as desired.

**Proposition 6.5.3.** Let \( Q \) be a quantaloid with the identity arrow \( 1_X : X \rightarrow X \) being the top element of \( Q(X,X) \) for all \( X \in Q_0 \) and \( \mathcal{D} = \{ \bot_{X,X} \mid X \in Q_0 \} \) a cyclic family. Then the following conditions are equivalent:
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(1) \( \mathcal{D} = \{ \bot_{X,X} \mid X \in \mathcal{Q}_0 \} \) is a dualizing family, hence \( \mathcal{Q} \) is a Girard quantaloid.

(2) There is an inf-dense \( \mathcal{Q} \)-functor \( F : A \to \mathcal{K}_\phi(A, B) \) for each \( \mathcal{Q} \)-distributor \( \phi : A \to \mathcal{B} \).

Proof. (1) \( \Rightarrow \) (2): For each \( \mathcal{Q} \)-distributor \( \phi : A \to B \), by Proposition \[6.4.3\] we have \( \mathcal{K}_\phi(A, B) = \mathcal{M}_\phi(A, B) \), thus the conclusion follows from Theorem \[5.4.4\].

(2) \( \Rightarrow \) (1): For each \( X \in \mathcal{Q}_0 \), consider the \( \mathcal{Q} \)-distributor \( \phi : *_X \to *_X \) given by \( \phi(*, *) = 1_X \). It is easy to see that \( \mathcal{K}(\phi) = \mathcal{P}X \), thus there is an inf-dense \( \mathcal{Q} \)-functor \( F : *_X \to \mathcal{P}X \). This means that for all \( f \in \mathcal{P}X \), there is some \( g \in \mathcal{P}(*_X) = \mathcal{P}X \) such that \( f = \inf_{\mathcal{P}X} F^{-}(g) \), i.e.,

\[
\mathcal{P}X(-, f) = F^{-}(g) \setminus \mathcal{P}X
= (\mathcal{P}X(F*, -) \circ g) \setminus \mathcal{P}X
= g \setminus \mathcal{P}X(-, F*)
= \mathcal{P}X(-, g \setminus F*),
\]

and consequently \( f = g \setminus F* \). Hence \( (-) \setminus F* : \mathcal{P}X \to \mathcal{P}X \) is a surjective \( \mathcal{Q} \)-functor. Since \( (-) \setminus F* : (\mathcal{P}X)_X \to (\mathcal{P}X)_X \) is order-preserving, we obtain that \( \bot_{X,X} = 1_X \setminus \bot_{X,X} = F* \). This means that the \( \mathcal{Q} \)-functor \( (-) \setminus \bot_{X,X} : \mathcal{P}X \to \mathcal{P}X \) is surjective, hence for all \( f \in \mathcal{Q}(X, Y) \), there is some \( g \in \mathcal{Q}(Y, X) \) such that \( f = g \setminus \bot_{X,X} \). Note that for all \( f \in \mathcal{P}X \) and \( g \in \mathcal{P}X \),

\[
\mathcal{P}X(\bot_{X,X} \lor f, g) = g \setminus (\bot_{X,X} \lor f) = (g \setminus \bot_{X,X}) \lor f = \mathcal{P}X(f, g \setminus \bot_{X,X}),
\]

thus \( \bot_{X,X} \lor (-) \setminus \bot_{X,X} : \mathcal{P}X \to \mathcal{P}X \), and therefore

\[
(\bot_{X,X} \lor f) \setminus \bot_{X,X} = (\bot_{X,X} \lor (g \setminus \bot_{X,X})) \setminus \bot_{X,X} = g \setminus \bot_{X,X} = f,
\]

as desired. \( \square \)
Chapter 7

Applications in fuzzy set theory

Isbell adjunctions and Kan adjunctions in $\mathcal{Q}$-categories correspond closely to the theories of formal concept analysis and rough set theory in computer science. Our results obtained in Chapter 5 and 6 can be applied to two special quantaloids: the one-object quantaloid (i.e., a unital quantale), and the quantaloid generated by a divisible unital quantale. Through this way we develop the theories of formal concept analysis and rough set theory on fuzzy relations between fuzzy sets.

7.1 Preordered sets valued in a unital quantale

Recall that a unital quantale is a one-object quantaloid. Explicitly, a unital quantale is a complete lattice $\mathcal{Q}$ equipped with a binary operation $\& : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{Q}$ such that

(i) $(\mathcal{Q}, \&)$ is a monoid with a unit $I$;

(ii) $a \& (\lor b_i) = \lor (a \& b_i)$ and $(\lor b_i) \& a = \lor (b_i \& a)$ for all $a, b_i \in \mathcal{Q}$.

In a unital quantale $\mathcal{Q}$, the left implication $/$ and the right implication $\backslash$ are two binary operations on $\mathcal{Q}$ determined by the adjoint property

$$c \leq a \backslash b \iff a \& c \leq b \iff a \leq b / c$$

for all $a, b, c \in \mathcal{Q}$.  

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A unital quantale $Q$ is *commutative* if $(Q, \&)$ is a commutative monoid. A commutative unital quantale $Q$ is called a *complete residuated lattice* if the unit $I$ is the largest element of $Q$.

**Example 7.1.1.** We list some basic examples of unital quantales.

1. Each frame is a commutative unital quantale.
2. Each complete BL-algebra is a commutative unital quantale. In particular, each continuous t-norm on the unit interval $[0, 1]$ is a commutative unital quantale.
3. $([0, \infty]^\text{op}, +)$ is a commutative unital quantale in which $b/a = a \backslash b = \max\{0, b-a\}$.

Consider a unital quantale $Q$ as a one-object quantaloid, then each (crisp) set $A$ can be viewed as a discrete $Q$-category (see Example 3.2.4(2)), in which

$$A(x, y) = \begin{cases} 1, & x = y; \\ 0, & x \neq y. \end{cases}$$

**Definition 7.1.2.** A $Q$-relation (or *fuzzy relation*) $\phi : A \leftrightarrow B$ between (crisp) sets $A$ and $B$ is a $Q$-distributor between discrete $Q$-categories $A$ and $B$.

A $Q$-relation $\phi : A \leftrightarrow B$ is exactly a function

$$\phi : A \times B \rightarrow Q.$$ 

Each $Q$-relation $\phi : A \leftrightarrow B$ has a dual $Q$-relation $\phi^\text{op} : B \leftrightarrow A$ given by $\phi^\text{op}(y, x) = \phi(x, y)$ for all $x \in A$ and $y \in B$.

**Definition 7.1.3.** Let $\phi : A \leftrightarrow A$ be a $Q$-relation on a (crisp) set $A$.

1. $\phi$ is *reflexive* if $A \leq \phi$.
2. $\phi$ is *transitive* if $\phi \& \phi \leq \phi$.
3. $\phi$ is *symmetric* if $\phi = \phi^\text{op}$.
7.1. Preordered sets valued in a unital quantale

(4) \( \phi \) is separated if \( \phi(x, x) \leq \phi(x, y) \) and \( \phi(y, y) \leq \phi(y, x) \) implies \( x = y \).

**Definition 7.1.4.** A \( Q \)-preorder (or fuzzy preorder) on a (crisp) set \( A \) is a reflexive and transitive \( Q \)-relation \( \mathbb{A} : A \rightarrow A \). The pair \((A, \mathbb{A})\) is called a \( Q \)-preordered set.

Explicitly, a \( Q \)-preorder on a (crisp) set \( A \) is a function \( \mathbb{A} : A \times A \rightarrow Q \) such that

1. \( I \leq \mathbb{A}(x, x) \) for each \( x \in A \) (reflexivity);
2. \( \mathbb{A}(y, z) \& \mathbb{A}(x, y) \leq \mathbb{A}(x, z) \) for all \( x, y, z \in A \) (transitivity).

A \( Q \)-preordered set \((A, \mathbb{A})\) is separated if the \( Q \)-relation \( \mathbb{A} \) is separated.

**Proposition 7.1.5.** A \( Q \)-preordered set \((A, \mathbb{A})\) is separated if and only if it satisfies

3. \( I \leq \mathbb{A}(x, y) \) and \( I \leq \mathbb{A}(y, x) \) implies \( x = y \) (anti-symmetry).

**Proof.** For the non-trivial direction, suppose that \( I \leq \mathbb{A}(x, y) \) and \( I \leq \mathbb{A}(y, x) \), we have

\[
\mathbb{A}(x, x) \leq \mathbb{A}(x, y) \setminus \mathbb{A}(x, y) \leq I \setminus \mathbb{A}(x, y) = \mathbb{A}(x, y).
\]

and

\[
\mathbb{A}(y, y) \leq \mathbb{A}(y, x) \setminus \mathbb{A}(y, x) \leq I \setminus \mathbb{A}(y, x) = \mathbb{A}(y, x).
\]

Thus \( x = y \). \( \square \)

In a \( Q \)-preordered set \((A, \mathbb{A})\), the value \( \mathbb{A}(x, y) \) can be interpreted as the degree to which \( x \) is less than or equal to \( y \). Compared with Definition 3.2.2, it is easy to see that \( Q \)-preordered sets are exactly categories enriched over the unital quantale \( Q \), and separated \( Q \)-preordered sets are skeletal \( Q \)-categories. We abbreviate the pair \((A, \mathbb{A})\) to \( \mathbb{A} \) and write \( \mathbb{A}_0 = A \) if there is no confusion.

**Definition 7.1.6.** An order-preserving map \( F : \mathbb{A} \rightarrow \mathbb{B} \) between \( Q \)-preordered sets is a \( Q \)-functor between \( Q \)-categories \( \mathbb{A} \) and \( \mathbb{B} \).
Given a $Q$-preordered set $\mathbb{A}$, a lower set of $\mathbb{A}$ is a contravariant presheaf $\mu \in \mathcal{P}\mathbb{A}$, and an upper set of $\mathbb{A}$ is a covariant presheaf $\lambda \in \mathcal{P}^\dagger \mathbb{A}$. The $Q$-preordered sets $\mathcal{P}\mathbb{A}$ and $\mathcal{P}^\dagger \mathbb{A}$ are called respectively the $Q$-powerset and the dual $Q$-powerset of $\mathbb{A}$.

In particular, consider a (crisp) set $A$ as a discrete $Q$-category, the $Q$-category $\mathcal{P}A$ of contravariant presheaves on $A$ is the $Q$-powerset of $A$. Explicitly, the objects of $\mathcal{P}A$ are the maps $A \rightarrow Q$, i.e., $(\mathcal{P}A)_0 = Q^A$, and the $Q$-preorder on $\mathcal{P}A$ is given by

$$\mathcal{P}A(\mu, \lambda) = \bigwedge_{x \in A} \lambda(x)/\mu(x).$$

Dually, the $Q$-category $\mathcal{P}^\dagger A$ of covariant presheaves on $A$ is the dual $Q$-powerset of $A$. The objects of $\mathcal{P}^\dagger A$ are also the maps $A \rightarrow Q$, but the $Q$-preorder on $\mathcal{P}^\dagger A$ is given by

$$\mathcal{P}^\dagger A(\mu, \lambda) = \bigwedge_{x \in A} \lambda(x)\setminus\mu(x).$$

A separated $Q$-preordered set $\mathbb{A}$ is a complete $Q$-lattice if it is a skeletal complete $Q$-category. For each $Q$-preordered set $\mathbb{A}$, $\mathcal{P}\mathbb{A}$ and $\mathcal{P}^\dagger \mathbb{A}$ are both complete $Q$-lattices.

### 7.2 Preordered fuzzy sets valued in a divisible unital quantale

In order to derive the theory of preordered fuzzy sets (not only preordered crisp sets), we add one more requirement to the unital quantale $Q$, i.e., being divisible. This assumption makes sense because all the unital quantales given in Example 7.1.1 are divisible, which cover most of the important truth tables in fuzzy set theory.

A unital quantale $Q$ is divisible if it satisfies one of the equivalent conditions in the following Proposition 7.2.1.

**Proposition 7.2.1.** [PZ12, TLZ12] For a unital quantale $Q$, the following conditions are equivalent:
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(1) \( \forall a, b \in Q, a \leq b \) implies \( x \land b = a = b \land y \) for some \( x, y \in Q \).

(2) \( \forall a, b \in Q, a \leq b \) implies \( a = b \& (b \setminus a) = (a/b) \& b \).

(3) \( \forall a, b, c \in Q, a, b \leq c \) implies \( a \& (c \setminus b) = (a/c) \& b \).

(4) \( \forall a, b \in Q, (b/a) \& a = a \land b = a \& (a \setminus b) \).

In this case, the unit \( I \) must be the top element \( 1 \) in \( Q \).

Proposition 7.2.2. \([HK11, PZ12]\) Each divisible unital quantale \( Q \) gives rise to a quantaloid \( Q \) that consists of the following data:

(1) objects: elements \( X, Y, Z, \ldots \) in \( Q \);

(2) morphisms: \( Q(X,Y) = \{ \alpha \in Q : \alpha \leq X \land Y \} \);

(3) composition: \( \beta \circ \alpha = \beta \& (Y \setminus \alpha) = (\beta/Y) \& \alpha \) for all \( \alpha \in Q(X,Y) \) and \( \beta \in Q(Y,Z) \);

(4) implication: for all \( \alpha \in Q(X,Y), \beta \in Q(Y,Z) \) and \( \gamma \in Q(X,Z) \),

\[
\gamma \mathbin{\Rightarrow} \alpha = Y \land Z \land (\gamma/ (Y \setminus \alpha)) \quad \text{and} \quad \beta \mathbin{\Rightarrow} \gamma = X \land Y \land ((\beta/Y) \setminus \gamma);
\]

(5) the unit \( 1_X \) of \( Q(X,X) \) is \( X \);

(6) the partial order on \( Q(X,Y) \) is inherited from \( Q \).

In Example 3.1.13(3) we have seen that if \( Q \) is a Boolean algebra, then the quantaloid \( Q \) induced by Proposition 7.2.2 is a Girard quantaloid. However, we point out that \( Q \) might not be a Girard quantaloid if \( Q \) is a general Girard quantale. For example, the quantaloid induced by a Lukasiewicz \( t \)-norm in this way is not a Girard quantaloid.

In this section, \( Q \) is always assumed to be a divisible unital quantale and \( Q \) the associated quantaloid given by Proposition 7.2.2 if not otherwise specified.

Definition 7.2.3. A \( Q \)-subset (or fuzzy set) is a \( Q \)-typed set.
Explicitly, a $Q$-subset is a pair $(A_0, A)$, where $A_0$ is the underlying (crisp) set and $A : A_0 \rightarrow Q$ is the type map. For each $x \in A_0$, the type $Ax$ is interpreted as the membership degree of $x$ in $A_0$. We abbreviate the pair $(A_0, A)$ to $A$ if no confusion would arise. $Q$-subsets and type-preserving functions constitute the slice category $\text{Set} \downarrow Q$.

An element $x$ in a $Q$-subset $A$ is global if $Ax = 1$. A $Q$-subset is global if every element in it is global. A global $Q$-subset is exactly a crisp set.

By Example 3.2.4(2), each $Q$-subset $A$ can be viewed as a discrete $Q$-category, in which

$$A(x, y) = \begin{cases} Ax, & x = y, \\ 0, & x \neq y. \end{cases}$$

**Definition 7.2.4.** A $Q$-relation (or fuzzy relation) $\phi : A \leftrightarrow B$ between $Q$-subsets is a $Q$-distributor between discrete $Q$-categories $A$ and $B$.

Explicitly, a $Q$-relation $\phi : A \leftrightarrow B$ is a map $\phi : A \times B \rightarrow Q$ such that

$$\phi(x, y) \leq Ax \land By$$

for all $x \in A_0$ and $y \in B_0$. Each $Q$-relation $\phi : A \leftrightarrow B$ has a dual $Q$-relation $\phi^{\text{op}} : B \leftrightarrow A$ given by $\phi^{\text{op}}(y, x) = \phi(x, y)$ for all $x \in A_0$ and $y \in B_0$.

Given $Q$-relations $\phi : A \leftrightarrow B$ and $\psi : B \leftrightarrow C$, the composition of $\psi$ and $\phi$

$$\psi \circ \phi : A \leftrightarrow C$$

is defined by the composition of $Q$-distributors, i.e.,

$$\psi \circ \phi(x, z) = \bigvee_{y \in B_0} \psi(y, z) \land (By \backslash \phi(x, y)) = \bigvee_{y \in B_0} (\psi(y, z) / By) \land \phi(x, y).$$

This formula can be regarded as a many-valued reformulation of the statement that $x, z$ are related if there exists some $y$ in $B$ such that $x, y$ are related and $y, z$ are related.

**Definition 7.2.5.** Let $\phi : A \leftrightarrow A$ be a $Q$-relation on a $Q$-subset $A$.

(1) $\phi$ is reflexive if $A \leq \phi$. 
(2) \( \phi \) is \textit{transitive} if \( \phi \circ \phi \leq \phi \).

(3) \( \phi \) is \textit{symmetric} if \( \phi = \phi^{op} \).

(4) \( \phi \) is \textit{separated} if \( \phi(x,x) \leq \phi(x,y) \) and \( \phi(y,y) \leq \phi(y,x) \) implies \( x = y \).

\textbf{Definition 7.2.6.} A \textit{\( \mathbb{Q} \)-preorder} (or \textit{fuzzy preorder}) on a \( \mathbb{Q} \)-subset \( A \) is a reflexive and transitive \( \mathbb{Q} \)-relation \( A : A \rightarrow \mathbb{Q} \).

Explicitly, a \( \mathbb{Q} \)-preorder on a \( \mathbb{Q} \)-subset \( A \) is a map \( A : A_0 \times A_0 \rightarrow \mathbb{Q} \) such that for all \( x,y,z \in A_0 \),

(i) \( A(x,y) \leq A(x,x) \land A(y,y) \),

(ii) \( A(x,x) \leq A(x,x) \),

(iii) \( A(y,z) \land (A(y,y) \setminus A(x,y)) = (A(y,z)/A(y,y)) \land A(x,y) \leq A(x,z) \).

It can be inferred from (i) and (ii) that \( A(x,x) = A(x) \) for all \( x \in A_0 \). Therefore, a \( \mathbb{Q} \)-preordered \( \mathbb{Q} \)-subset can be described as a pair \( (A,A) \), where \( A \) is a (crisp) set and \( A : A \times A \rightarrow \mathbb{Q} \) is a map, such that for all \( x,y,z \in A \),

(1) \( A(x,y) \leq A(x,x) \land A(y,y) \),

(2) \( A(y,z) \land (A(y,y) \setminus A(x,y)) = (A(y,z)/A(y,y)) \land A(x,y) \leq A(x,z) \).

A \( \mathbb{Q} \)-preordered \( \mathbb{Q} \)-subset \( (A,A) \) is \textit{separated} if the \( \mathbb{Q} \)-relation \( A \) is separated. It is easy to see that \( (A,A) \) is separated if and only if \( A(x,x) = A(y,y) = A(x,y) = A(y,x) \) implies \( x = y \).

Compared with \textbf{Definition 3.2.2}, we obtain that \( \mathbb{Q} \)-preordered \( \mathbb{Q} \)-subsets are exactly categories enriched over the quantaloid \( \mathbb{Q} \), and separated \( \mathbb{Q} \)-preordered \( \mathbb{Q} \)-subsets are skeletal \( \mathbb{Q} \)-categories. We abbreviate the pair \( (A,A) \) to \( A \) and write \( A_0 = A \) if there is no confusion. For each \( \mathbb{Q} \)-preordered \( \mathbb{Q} \)-subset \( A \), we denote by \( |A| \) the underlying \( \mathbb{Q} \)-subset of \( A \), with \( |A|_0 = A_0 \) and \( |A|x = A(x,x) \) for all \( x \in A_0 \).

A \( \mathbb{Q} \)-preordered \( \mathbb{Q} \)-subset \( A \) is \textit{global} if the underlying \( \mathbb{Q} \)-subset \( |A| \) is global, i.e., \( A(x,x) = 1 \) for all \( x \in A_0 \). A global \( \mathbb{Q} \)-preordered \( \mathbb{Q} \)-subset is exactly a
Q-preordered set. Thus, our discussion in Section \[ \text{Section 7.1} \] is a special case of this section when \( Q \) is a divisible unital quantale.

**Definition 7.2.7.** An order-preserving map \( F : \mathbb{A} \to \mathbb{B} \) between Q-preordered \( Q \)-subsets is a \( Q \)-functor between \( Q \)-categories \( \mathbb{A} \) and \( \mathbb{B} \).

Given a \( Q \)-preordered \( Q \)-subset \( \mathbb{A} \), a **potential lower set** of \( \mathbb{A} \) is a contravariant presheaf \( \mu \in \mathcal{P}\mathbb{A} \), and a **potential upper set** of \( \mathbb{A} \) is a covariant presheaf \( \lambda \in \mathcal{P}^{\dagger}\mathbb{A} \). The \( Q \)-preordered \( Q \)-subsets \( \mathcal{P}\mathbb{A} \) and \( \mathcal{P}^{\dagger}\mathbb{A} \) are called respectively the \( Q \)-powerset and the dual \( Q \)-powerset of \( \mathbb{A} \). We point out that the underlying \( Q \)-subset \( |\mathcal{P}\mathbb{A}| \) of \( \mathcal{P}\mathbb{A} \) is given by the following data:

1. \( |\mathcal{P}\mathbb{A}|_0 \) is the set of potential lower sets \( \mu \in \mathcal{P}\mathbb{A} \).
2. The type map \( |\mathcal{P}\mathbb{A}| : |\mathcal{P}\mathbb{A}|_0 \to Q \) sends each \( \mu \in \mathcal{P}\mathbb{A} \) to \( t\mu \).

Dually, the underlying \( Q \)-subset \( |\mathcal{P}^{\dagger}\mathbb{A}| \) of \( \mathcal{P}^{\dagger}\mathbb{A} \) is given by the following data:

1. \( |\mathcal{P}^{\dagger}\mathbb{A}|_0 \) is the set of potential upper sets \( \lambda \in \mathcal{P}^{\dagger}\mathbb{A} \).
2. The type map \( |\mathcal{P}^{\dagger}\mathbb{A}| : |\mathcal{P}^{\dagger}\mathbb{A}|_0 \to Q \) sends each \( \lambda \in \mathcal{P}^{\dagger}\mathbb{A} \) to \( t\lambda \).

In particular, Let \( \mathbb{A} \) be a \( Q \)-subset. Consider \( \mathbb{A} \) as a discrete \( Q \)-category, then the \( Q \)-category \( \mathcal{P}\mathbb{A} \) is called the \( Q \)-powerset of \( \mathbb{A} \). In elementary words, \( (\mathcal{P}\mathbb{A})_0 \) consists of pairs \( (\mu, a) \in Q^{A_0} \times A_0 \) satisfying

\[
\forall x \in A_0, \mu(x) \leq Ax \land a,
\]

and

\[
\mathcal{P}\mathbb{A}((\mu, a), (\lambda, b)) = a \land b \land \bigwedge_{x \in A_0} \lambda(x)/(a \land \mu(x))
\]

for all \( (\mu, a), (\lambda, b) \in (\mathcal{P}\mathbb{A})_0 \). Dually, the dual \( Q \)-powerset \( \mathcal{P}^{\dagger}\mathbb{A} \) of \( \mathbb{A} \) has the same underlying \( Q \)-subset \( |\mathcal{P}^{\dagger}\mathbb{A}| = |\mathcal{P}\mathbb{A}| \), but

\[
\mathcal{P}^{\dagger}\mathbb{A}((\mu, a), (\lambda, b)) = a \land b \land \bigwedge_{x \in A_0} (\lambda(x)/b \land \mu(x))
\]

for all \( (\mu, a), (\lambda, b) \in (\mathcal{P}^{\dagger}\mathbb{A})_0 \).

A \( Q \)-preordered \( Q \)-subset \( \mathbb{A} \) is **complete** if it is a complete \( Q \)-category. For each \( Q \)-preordered \( Q \)-subset \( \mathbb{A} \), \( \mathcal{P}\mathbb{A} \) and \( \mathcal{P}^{\dagger}\mathbb{A} \) are both separated and complete.
7.3 Formal concept analysis on fuzzy sets

Formal concept analysis [CR04, DP02, GSW05, GW99] is a useful tool for qualitative data analysis. A formal context is a triple \((A, B, R)\), where \(A, B\) are sets and \(R \subseteq A \times B\) is a relation between \(A\) and \(B\). Elements in \(A\) are interpreted as objects and those in \(B\) as attributes. Each relation \(R \subseteq A \times B\) induces a pair of operators \(R^\uparrow : 2^A \to 2^B\) and \(R^\downarrow : 2^B \to 2^A\) as follows:

\[
R^\uparrow(U) = \{ y \in B : \forall x \in U, (x, y) \in R \},
\]

\[
R^\downarrow(V) = \{ x \in A : \forall y \in V, (x, y) \in R \}.
\]

This pair of operators is a contravariant Galois connection in the sense that

\[
U \subseteq R^\downarrow(V) \iff V \subseteq R^\uparrow(U).
\]

This Galois connection plays a fundamental role in formal concept analysis.

A formal concept of a formal context \((A, B, R)\) is a pair \((U, V) \in 2^A \times 2^B\), where \(U\) is the extent and \(V\) the intent, such that \(U = R^\downarrow(V)\) and \(V = R^\uparrow(U)\). The fundamental theorem of formal concept analysis states that the formal concepts of a formal context form a complete lattice (called the formal concept lattice) and every complete lattice is the formal concept lattice of some formal context.

From the viewpoint of category theory, a formal context \((A, B, R)\) is just a 2-distributor between discrete 2-categories \(A\) and \(B\), while the contravariant Galois connection \((R^\uparrow, R^\downarrow)\) is exactly an Isbell adjunction between \(2^A\) and \((2^B)^{op}\). That is to say, formal concept analysis is essentially a theory on distributors between categories.

As demonstrated in Section 7.1 and 7.2, fuzzy relations between (crisp) sets can be viewed as distributors between quantale-enriched categories, while fuzzy relations between fuzzy sets and can be viewed as distributors between quantaloid-enriched categories. Thus, the theory of formal concept analysis can be generalized to a theory on fuzzy relations between crisp sets [Bel01, Bel02, Bel04, GP04, LZ09, Pop04, SZ13], and it is possible to extend the theory further to the realm of fuzzy relations between fuzzy sets.
From now on we assume that $Q$ is a divisible unital quantale. A "fuzzy context" is a triple $(A, B, \phi)$, where $A$ and $B$ are $Q$-subsets, and $\phi : A \rightarrow B$ is a $Q$-relation. Fuzzy contexts and infomorphisms constitute a category $Q\text{-Ctx}$. It is easy to see that $Q\text{-Ctx}$ is a subcategory of $Q\text{-Info}$ containing only $Q$-distributors between discrete $Q$-categories. It is noteworthy to point out that $(Q\text{-Ctx})^{op} = Q\text{-Ctx}$.

We point out that if $A$ and $B$ are global, i.e., $A$ and $B$ are crisp sets, then the fuzzy context $(A, B, \phi)$ is a $Q$-context (also called fuzzy context in some literatures) in the sense of [Bel04, LZ09, SZ13]. Therefore, formal concept analysis and rough set theory (introduced in the next section) on $Q$-contexts are special cases of our discussion here.

Each fuzzy context $(A, B, \phi)$ gives rise to an Isbell adjunction

$$\phi^* \dashv \phi_1 : \mathcal{P}A \rightarrow \mathcal{P}^1B.$$ 

Let $\mathcal{M}_\phi(A, B)$ denote the $Q$-preordered $Q$-subset given by

1. $|\mathcal{M}_\phi(A, B)|_0$ consists of pairs $(\mu, \lambda) \in (\mathcal{P}A)_0 \times (\mathcal{P}^1B)_0$ such that
   $$\lambda = \phi^*(\mu) \quad \text{and} \quad \mu = \phi_1(\lambda);$$
2. for each $(\mu, \lambda) \in |\mathcal{M}_\phi(A, B)|_0$,
   $$|\mathcal{M}_\phi(A, B)|((\mu, \lambda)) = t\mu = t\lambda;$$
3. for each $(\mu, \lambda), (\mu', \lambda') \in |\mathcal{M}_\phi(A, B)|_0$,
   $$\mathcal{M}_\phi(A, B)((\mu, \lambda), (\mu', \lambda')) = \mathcal{P}A(\mu, \mu') = \mathcal{P}^1B(\lambda, \lambda').$$

Then $\mathcal{M}_\phi(A, B)$ is called the formal concept lattice of the fuzzy context $(A, B, \phi)$, and the $Q$-subset $|\mathcal{M}_\phi(A, B)|$ is called the fuzzy set of formal concepts of $(A, B, \phi)$. Elements $(\mu, \lambda)$ in $|\mathcal{M}_\phi(A, B)|_0$ are called "potential formal concepts" of $(A, B, \phi)$, and $|\mathcal{M}_\phi(A, B)|((\mu, \lambda)$ is interpreted as the degree that $(\mu, \lambda)$ is a formal concept.

The following conclusions are direct consequences of the results in Chapter 5. In particular, we extend the fundamental theorem of formal concept analysis to the fuzzy setting.
Theorem 7.3.1. For any fuzzy context \((A, B, \phi)\), \(\mathcal{M}_\phi(A, B)\) is a separated and complete \(Q\)-preordered \(Q\)-subset. Conversely, each separated and complete \(Q\)-preordered \(Q\)-subset \(A\) is isomorphic to the formal concept lattice \(\mathcal{M}_\phi(A, B)\) of some fuzzy context \((A, B, \phi)\).

The category \(Q\)-CCat consists of the separated and complete \(Q\)-preordered \(Q\)-subsets as objects, and the left adjoint order-preserving maps as morphisms.

Proposition 7.3.2. Let \(X\) be a separated and complete \(Q\)-preordered \(Q\)-subset. The following conditions are equivalent:

1. \(X\) is isomorphic to the formal concept lattice \(\mathcal{M}_\phi(A, B)\) for some fuzzy context \((A, B, \phi)\).

2. \(X\) is isomorphic to a \(Q\)-closure system of \(PA\) for some \(Q\)-subset \(A\).

3. \(X\) is isomorphic to a quotient object of \(PA\) for some \(Q\)-subset \(A\) in the category \(Q\)-CCat.

Proof. Note that all the equivalences follow from Proposition 5.4.1 except (2) \(\implies\) (1). To this end, suppose that \(X\) is isomorphic to \(C(PA)\) for some \(Q\)-subset \(A\) and \(Q\)-closure operator \(C : PA \rightarrow PA\). Then similar to Step 1 of Theorem 5.2.2 one can deduce that \((A, B, \zeta_C)\) given by \(B_0 = C(PA)_0\), \(B(\mu) = PA(\mu, \mu)\) and \(\zeta_C(x, \mu) = \mu(x)\) is the desired fuzzy context.

Theorem 7.3.3. A separated and complete \(Q\)-preordered \(Q\)-subset \(X\) is isomorphic to the formal concept lattice \(\mathcal{M}_\phi(A, B)\) of some fuzzy context \((A, B, \phi)\) if and only if there exists a sup-dense order-preserving map \(F : A \rightarrow X\) and an inf-dense order-preserving map \(G : B \rightarrow X\) such that \(\phi(x, y) = X(Fx, Gy)\) for all \(x \in A_0\), \(y \in B_0\).

The functoriality of the formal concept lattice is established as follows.

Proposition 7.3.4. \(\mathcal{M} : Q\text{-Ctx} \rightarrow Q\text{-CCat}\) is a functor that sends each fuzzy context \((A, B, \phi)\) to the formal concept lattice \(\mathcal{M}_\phi(A, B)\). This functor is obtained
by restricting to domain of $\mathcal{M} : \text{Q-Info} \longrightarrow \text{Q-CCat}$ defined by Equation (5.9) and identifying $\mathcal{M}_\phi(A, B)$ with $\mathcal{M}(\phi)$.

\[ \text{Q-Info} \xrightarrow{\mathcal{M}} \text{Q-CCat} \]

\[ \text{Q-Ctx} \]

### 7.4 Rough set theory on fuzzy sets

Rough set theory [Paw82, Pol02, Yao04] is another important tool for qualitative data analysis. Given a formal context $(A, B, R)$, the relation $R \subseteq A \times B$ induces another pair of operators $R_\exists : 2^A \longrightarrow 2^B$ and $R^\forall : 2^B \longrightarrow 2^A$ as follows:

\[
R_\exists(U) = \{ y \in B : \exists x \in U, (x, y) \in R \},
\]

\[
R^\forall(V) = \{ x \in A : \forall y \in B, (x, y) \in R \text{ implies } y \in V \}.
\]

This pair of operators is a covariant Galois connection in the sense that

\[
R_\exists(U) \subseteq V \iff U \subseteq R^\forall(V).
\]

This Galois connection plays a fundamental role in rough set theory.

A property oriented concept of a formal context $(A, B, R)$ is a pair $(U, V) \in 2^A \times 2^B$, where $U$ is the object and $V$ the property, such that $U = R^\forall(V)$ and $V = R_\exists(U)$. From the viewpoint of category theory, the covariant Galois connection $(R_\exists, R^\forall)$ is exactly a Kan adjunction between $2^A$ and $2^B$. That is to say, rough set theory is also a theory on distributors between categories, thus can be generalized to a theory on fuzzy relations between crisp sets [DG02, LZ09, Yao04], and further to that on fuzzy relations between fuzzy sets.

Suppose that $Q$ is a divisible unital quantale. Each fuzzy context $(A, B, \phi)$ gives rise to a Kan adjunction

\[
\phi^* \dashv \phi_* : \mathcal{P}B \rightarrow \mathcal{P}A.
\]

Let $\mathcal{K}_\phi(A, B)$ denote the $Q$-preordered $Q$-subset given by
(1) \( |K_\phi(A, B)|_0 \) consists of pairs \((\mu, \lambda) \in (\mathcal{P}A)_0 \times (\mathcal{P}^1B)_0 \) such that 
\[
\lambda = \phi_*(\mu) \quad \text{and} \quad \mu = \phi^*(\lambda);
\]
(2) for each \((\mu, \lambda) \in |K_\phi(A, B)|_0 \), 
\[
|K_\phi(A, B)|_0(\mu, \lambda) = t\mu = t\lambda;
\]
(3) for each \((\mu, \lambda), (\mu', \lambda') \in |K_\phi(A, B)|_0 \), 
\[
K_\phi(A, B)((\mu, \lambda), (\mu', \lambda')) = \mathcal{P}A(\mu, \mu') = \mathcal{P}B(\lambda, \lambda').
\]

Then \( K_\phi(A, B) \) is called the property oriented concept lattice of the fuzzy context \((A, B, \phi)\), and the \( Q \)-subset \( |K_\phi(A, B)| \) is called the fuzzy set of property oriented concepts of \((A, B, \phi)\). Elements \((\mu, \lambda) \) in \( |K_\phi(A, B)|_0 \) are called potential property oriented concepts of \((A, B, \phi)\), and \( |K_\phi(A, B)|_0(\mu, \lambda) \) is interpreted as the degree that \((\mu, \lambda) \) is a property oriented concept.

The following conclusions are direct consequences of the results in Chapter 6.

**Proposition 7.4.1.** For any fuzzy context \((A, B, \phi)\), \( K_\phi(A, B) \) is a separated and complete \( Q \)-preordered \( Q \)-subset.

**Proposition 7.4.2.** If \( Q \) is a Boolean algebra, then each separated and complete \( Q \)-preordered \( Q \)-subset \( \mathcal{A} \) is isomorphic to the property oriented concept lattice \( K_\phi(A, B) \) of some fuzzy context \((A, B, \phi)\).

**Proof.** Follows immediately from Example 3.1.13(3) and Proposition 6.4.3.

**Proposition 7.4.3.** Let \( \mathcal{X} \) be a separated and complete \( Q \)-preordered \( Q \)-subset. The following conditions are equivalent:

(1) \( \mathcal{X} \) is isomorphic to the property oriented concept lattice \( K_\phi(A, B) \) for some fuzzy context \((A, B, \phi)\).

(2) \( \mathcal{X} \) is isomorphic to a \( Q \)-interior system of \( \mathcal{P}A \) for some \( Q \)-subset \( A \).
(3) \( X \) is isomorphic to a subobject of \( \mathcal{P}A \) for some \( Q \)-subset \( A \) in the category \( Q\text{-CCat} \).

Proof. For (2) \( \implies \) (1), suppose that \( X \) is isomorphic to \( F(\mathcal{P}A) \) for some \( Q \)-subset \( A \) and \( Q \)-interior operator \( F : \mathcal{P}A \longrightarrow \mathcal{P}A \), then similar to Proposition 6.5.1 one can deduce that \( (A, B, \zeta_F) \) given by \( B_0 = F(\mathcal{P}A)_0 \), \( B(\mu) = \mathcal{P}A(\mu, \mu) \) and \( \zeta_F(x, \mu) = \mu(x) \) is the desired fuzzy context. \( \square \)

We present below the functoriality of the property oriented concept lattice.

Proposition 7.4.4. \( \mathcal{K} : Q\text{-Ctx} \longrightarrow Q\text{-CCat} \) is a functor that sends each fuzzy context \( (A, B, \phi) \) to the property oriented concept lattice \( \mathcal{K}_\phi(A, B) \). This functor is obtained by restricting to domain of \( \mathcal{K} : (Q\text{-Info})^\text{op} \longrightarrow Q\text{-CCat} \) defined by Equation (6.10) and identifying \( \mathcal{K}_\phi(A, B) \) with \( \mathcal{K}(\phi) \).

\[
\begin{array}{ccc}
(Q\text{-Info})^\text{op} & \xrightarrow{\mathcal{K}} & Q\text{-CCat} \\
\mathcal{K} & & \downarrow \mathcal{K} \\
Q\text{-Ctx} = (Q\text{-Ctx})^\text{op} & & \\
\end{array}
\]
Bibliography

[ACVV99] D. Aerts, E. Colebunders, A. Van Der Voorde, and B. Van Steirteghem. State property systems and closure spaces: A study of categorical equivalence. *International Journal of Theoretical Physics*, 38:359–385, 1999.

[AHS90] J. Adámek, H. Herrlich, and G. E. Strecker. *Abstract and concrete categories: the joy of cats*. Wiley, New York, 1990.

[AK88] M. H. Albert and G. M. Kelly. The closure of a class of colimits. *Journal of Pure and Applied Algebra*, 51(1-2):1–17, 1988.

[Awo10] S. Awodey. *Category Theory*, volume 52 of *Oxford Logic Guides*. Oxford University Press, Oxford, second edition, 2010.

[Bar91] M. Barr. ∗-autonomous categories and linear logic. *Mathematical Structures in Computer Science*, 1:159–178, 1991.

[BC82] R. Betti and A. Carboni. Cauchy-completion and the associated sheaf. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 23(3):243–256, 1982.

[BCSW83] R. Betti, A. Carboni, R. Street, and R. F. C. Walters. Variation through enrichment. *Journal of Pure and Applied Algebra*, 29(2):109–127, 1983.

[BD98] J. C. Baez and J. Dolan. Categorification. In E. Getzler and M. Kapranov, editors, *Higher Category Theory*, volume 230 of *Con-
 temporary mathematics, pages 1–36. American Mathematical Society, Providence, Rhode Island, 1998.

[Bêl01] R. Bělohlávek. Lattices of fixed points of fuzzy Galois connections. Mathematical Logic Quarterly, 47(1):111–116, 2001.

[Bêl02] R. Bělohlávek. Fuzzy Relational Systems: Foundations and Principles, volume 20 of IFSR International Series on Systems Science and Engineering. Kluwer Academic Publishers, Dordrecht, 2002.

[Bêl04] R. Bělohlávek. Concept lattices and order in fuzzy logic. Annals of Pure and Applied Logic, 128(1-3):277–298, 2004.

[Bén67] J. Bénabou. Introduction to bicategories. In Reports of the Midwest Category Seminar, volume 47 of Lecture Notes in Mathematics, pages 1–77. Springer Berlin Heidelberg, 1967.

[Ber63] C. Berge. Topological Spaces: Including a Treatment of Multi-valued Functions, Vector Spaces, and Convexity. Oliver & Boyd, Edinburgh and London, 1963.

[Ber11] T. Bertrand. Lectures on dg-categories. In Topics in Algebraic and Topological K-Theory, Lecture Notes in Mathematics, pages 243–302. Springer Berlin Heidelberg, 2011.

[Bir48] G. Birkhoff. Lattice Theory, volume 25 of American Mathematical Society Colloquium Publications. American Mathematical Society, New York, revised edition, 1948.

[Bor94a] F. Borceux. Handbook of Categorical Algebra: Volume 1, Basic Category Theory, volume 50 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994.

[Bor94b] F. Borceux. Handbook of Categorical Algebra: Volume 2, Categories and Structures, volume 51 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994.
[Bor94c] F. Borceux. *Handbook of Categorical Algebra: Volume 3, Sheaf Theory*, volume 52 of *Encyclopedia of Mathematics and its Application*. Cambridge University Press, Cambridge, 1994.

[BS97] J. Barwise and J. Seligman. *Information Flow: The Logic of Distributed Systems*, volume 44 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 1997.

[BW95] M. Barr and C. Wells. *Category Theory for Computing Science*. Prentice-Hall International Series in Computer Science. Prentice Hall, 1995.

[CKWW08] A. Carboni, G. M. Kelly, R. F. C. Walters, and R. J. Wood. Cartesian bicategories II. *Theory and Applications of Categories*, 19(6):93–124, 2008.

[CR04] C. Carpineto and G. Romano. *Concept Data Analysis: Theory and Applications*. Wiley, New Jersey, 2004.

[CW87] A. Carboni and R. F. C. Walters. Cartesian bicategories I. *Journal of Pure and Applied Algebra*, 49:11–32, 1987.

[DG02] I. D"untsch and G. Gediga. Modal-style operators in qualitative data analysis. In V. Kumar, S. Tsureno, X. Wu, P. S. Yu, and N. Zhong, editors, *Proceedings of the 2002 IEEE International Conference on Data Mining*, pages 155–162, Los Alamitos, 2002. IEEE Computer Society.

[DL07] B. J. Day and S. Lack. Limits of small functors. *Journal of Pure and Applied Algebra*, 210(3):651–663, 2007.

[DP02] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, Cambridge, second edition, 2002.

[Dub70] E. J. Dubuc. *Kan extensions in enriched category theory*, volume 145 of *Lecture notes in mathematics*. Springer-Verlag, 1970.
[FS90] P. J. Freyd and A. Scedrov. Categories, Allegories, volume 39 of North-Holland Mathematical Library. Elsevier Science, North-Holland, 1990.

[Gan07] B. Ganter. Relational Galois connections. In S. O. Kuznetsov and S. Schmidt, editors, Formal Concept Analysis, volume 4390 of Lecture Notes in Computer Science, pages 1–17. Springer Berlin Heidelberg, 2007.

[GHK+03] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. Continuous Lattices and Domains, volume 93 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2003.

[GMdZ10] J. G. García, I. Mardones-Pérez, M. A. de Prada Vicente, and D. Zhang. Fuzzy Galois connections categorically. Mathematical Logic Quarterly, 56(2):131–147, 2010.

[Gog67] J. A. Goguen. L-fuzzy sets. Journal of Mathematical Analysis and Applications, 18(1):145–174, 1967.

[Gol84] R. Goldblatt. Topoi: The Categorial Analysis of Logic, volume 98 of Studies in Logic and the Foundations of Mathematics. Elsevier Science, North-Holland, revised edition, 1984.

[GP97] R. Gordon and A. J. Power. Enrichment through variation. Journal of Pure and Applied Algebra, 120(2):167–185, 1997.

[GP04] G. Georgescu and A. Popescu. Non-dual fuzzy connections. Archive for Mathematical Logic, 43:1009–1039, 2004.

[Gra74] J. W. Gray. Formal category theory: adjointness for 2-categories, volume 391 of Lecture Notes in Mathematics. Springer-Verlag, 1974.
[GSW05] B. Ganter, G. Stumme, and R. Wille, editors. *Formal Concept Analysis: Foundations and Applications*, volume 3626 of Lecture Notes in Computer Science. Springer-Verlag, Berlin Heidelberg, 2005.

[GW99] B. Ganter and R. Wille. *Formal Concept Analysis: Mathematical Foundations*. Springer, Berlin, 1999.

[Hey10] H. Heymans. *Sheaves on quantales as generalized metric spaces*. PhD thesis, Universiteit Antwerpen, Belgium, 2010.

[HK11] U. Höhle and T. Kubiak. A non-commutative and non-idempotent theory of quantale sets. *Fuzzy Sets and Systems*, 166:1–43, 2011.

[HS97] P. J. Hilton and U. Stammbach. *A Course in Homological Algebra*, volume 4 of Graduate Texts in Mathematics. Springer Verlag, 1997.

[HS11] H. Heymans and I. Stubbe. Symmetry and cauchy completion of quantaloid-enriched categories. *Theory and Applications of Categories*, 25(11):276–294, 2011.

[Joh02a] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium, Volume 1*, volume 43 of *Oxford Logic Guides*. Clarendon Press, 2002.

[Joh02b] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium, Volume 2*, volume 44 of *Oxford Logic Guides*. Clarendon Press, 2002.

[JT84] A. Joyal and M. Tierney. An extension of the Galois theory of Grothendieck. *Memoirs of the American Mathematical Society*, 51(309), 1984.

[Kan58] D. M. Kan. Adjoint functors. *Transactions of the American Mathematical Society*, 87(2):294–329, 1958.
[Kel82] G. M. Kelly. Basic concepts of enriched category theory, volume 64 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1982.

[KS05] G. M. Kelly and V. Schmitt. Notes on enriched categories with colimits of some class. Theory and Applications of Categories, 14(17):399–423, 2005.

[KW10] T. Kenney and R. J. Wood. Tensor products of sup-lattices and generalized sup-arrows. Theory and Applications of Categories, 24(11):266–287, 2010.

[Lac10] S. Lack. A 2-categories companion. In J. C. Baez and J. P. May, editors, Towards Higher Categories, volume 152 of The IMA Volumes in Mathematics and its Applications, pages 105–191. Springer New York, 2010.

[Law73] F. W. Lawvere. Metric spaces, generalized logic and closed categories. Rendiconti del Seminario Matematico e Fisico di Milano, 43:135–166, 1973.

[Law86] F. W. Lawvere. Taking categories seriously. Revista Colombiana de Matemáticas, XX, pages 147–178, 1986.

[Lei98] T. Leinster. Basic bicategories. arXiv:math/9810017, 1998.

[Lei02] T. Leinster. Generalized enrichment of categories. Journal of Pure and Applied Algebra, 168:391–406, 2002. Category Theory 1999: selected papers, conference held in Coimbra in honour of the 90th birthday of Saunders Mac Lane.

[Lei04] T. Leinster. Higher Operads, Higher Categories, volume 298 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2004.
[LR03] F. W. Lawvere and R. Rosebrugh. Sets for Mathematics. Cambridge University Press, Cambridge, 2003.

[LZ06] H. Lai and D. Zhang. Fuzzy preorder and fuzzy topology. Fuzzy Sets and Systems, 157(14):1865–1885, 2006.

[LZ07] H. Lai and D. Zhang. Complete and directed complete Ω-categories. Theoretical Computer Science, 388:1–25, 2007.

[LZ09] H. Lai and D. Zhang. Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory. International Journal of Approximate Reasoning, 50(5):695–707, 2009.

[Mac37] H. M. MacNeille. Partially ordered sets. Transactions of the American Mathematical Society, 42(3):416–460, 1937.

[ML98] S. Mac Lane. Categories for the Working Mathematician, volume 5 of Graduate Texts in Mathematics. Springer, New York, second edition, 1998.

[Paw82] Z. Pawlak. Rough sets. International Journal of Computer & Information Sciences, 11(5):341–356, 1982.

[Pit88] A. M. Pitts. Applications of sup-lattice enriched category theory to sheaf theory. Proceedings of the London Mathematical Society, s3-57(3):433–480, 1988.

[Pol02] L. Polkowski. Rough Sets: Mathematical Foundations, volume 15 of Advances in Intelligent and Soft Computing. Springer-Verlag Berlin Heidelberg, 2002.

[Pop04] A. Popescu. A general approach to fuzzy concepts. Mathematical Logic Quarterly, 50(3):265–280, 2004.

[Pra95] V. Pratt. Chu spaces and their interpretation as concurrent objects. In J. Leeuwen, editor, Computer Science Today, volume 1000 of
Lecture Notes in Computer Science, pages 392–405. Springer Berlin Heidelberg, 1995.

[PZ12] Q. Pu and D. Zhang. Preordered sets valued in a GL-monoid. Fuzzy Sets and Systems, 187(1):1–32, 2012.

[PZ14] Q. Pu and D. Zhang. Categories enriched over a quantaloid: algebras. preprint, 2014.

[Rie14] E. Riehl. Categorical Homotopy Theory. New Mathematical Monographs. Cambridge University Press, Cambridge, 2014.

[Ros90] K. I. Rosenthal. Quantales and their applications, volume 234 of Pitman research notes in mathematics series. Longman, Harlow, 1990.

[Ros92] K. I. Rosenthal. Girard quantaloids. Mathematical Structures in Computer Science, 2:93–108, 1992.

[Ros96] K. I. Rosenthal. The Theory of Quantaloids, volume 348 of Pitman Research Notes in Mathematics Series. Longman, Harlow, 1996.

[Stu03] I. Stubbe. Categorical structures enriched in a quantaloid: categories and semicategories. PhD thesis, Université Catholique de Louvain, Belgium, 2003.

[Stu05a] I. Stubbe. Categorical structures enriched in a quantaloid: categories, distributors and functors. Theory and Applications of Categories, 14(1):1–45, 2005.

[Stu05b] I. Stubbe. Categorical structures enriched in a quantaloid: Orders and ideals over a base quantaloid. Applied Categorical Structures, 13:235–255, 2005.
[Stu06] I. Stubbe. Categorical structures enriched in a quantaloid: tensored and cotensored categories. *Theory and Applications of Categories*, 16(14):283–306, 2006.

[Stu13a] I. Stubbe. The double power monad is the composite power monad. *preprint*, 2013.

[Stu13b] I. Stubbe. An introduction to quantaloid-enriched categories. *Fuzzy Sets and Systems*, 2013. http://dx.doi.org/10.1016/j.fss.2013.08.009.

[SZ13] L. Shen and D. Zhang. The concept lattice functors. *International Journal of Approximate Reasoning*, 54(1):166–183, 2013.

[TLZ12] Y. Tao, H. Lai, and D. Zhang. Quantale-valued preorders: Globalization and cocompleteness. *Fuzzy Sets and Systems*, 2012. http://dx.doi.org/10.1016/j.fss.2012.09.013.

[Wag94] K. R. Wagner. *Solving Recursive Domain Equations with Enriched Categories*. PhD thesis, Carnegie Mellon University, Pittsburgh, 1994.

[Wal81] R. F. C. Walters. Sheaves and Cauchy-complete categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 22(3):283–286, 1981.

[Yao04] Y. Yao. Concept lattices in rough set theory. In *Proceedings of 2004 Annual Meeting of the North American Fuzzy Information Processing Society (NAFIPS 2004)*, volume 2, pages 796–801, 2004.

[Zad65] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[Zad71] L. A. Zadeh. Similarity relations and fuzzy orderings. *Information Sciences*, 3(2):177–200, 1971.
Acknowledgement

First of all, I would like to express my deepest gratitude to my supervisor, Professor Dexue Zhang. During the six years under his supervision, I received careful guidance and constant encouragement both in life and study from Professor Zhang. His illuminating suggestions have significantly improved the quality of this dissertation.

Second, I am sincerely grateful to Professor Hui Kou, who not only taught me knowledge, but also instructed and encouraged me with valuable discussions. I am also grateful to Professor Shuguo Zhang, who inspired me with his persistence in a research field. Meanwhile, I would like to thank all the teachers who taught me lessons when I was studying in Sichuan University.

Moreover, I wish to thank Professor Liangang Peng and Professor Bing Hu for their guidance and support for my career. I am grateful to the seniors Hongliang Lai, Yingming Chai, Lingqiang Li, Piwei Chen, Yayan Yuan, Jiachao Wu, Chengling Fang, Hui Wang and Wei Zhang for their helpful discussions and communications with me both in life and study. I would like to thank my friends and schoolmates Qiang Pu, Haoran Zhao, Liping Xiong, Zhongwei Yang and Chengyong Du for sharing the graduate study. I would like to thank the juniors Yuanye Tao, Wei Li, Jialiang He, Zhiqiang Shen and Junsheng Qiao who are important members of our group. Furthermore, my thanks go out to faculty members, staff members and all the graduate students in the School of Mathematics, for their friendship and help in different ways.

Finally, I would like to express my gratitude for the support from my wife Xiaojuan Zhao, my parents and every member of my family.