Abstract

A fibration of $\mathbb{R}^n$ by oriented copies of $\mathbb{R}^p$ is called skew if no two fibers intersect nor contain parallel directions. Conditions on $p$ and $n$ for the existence of such a fibration were given by Ovsienko and Tabachnikov. A classification of smooth fibrations of $\mathbb{R}^3$ by skew oriented lines was given by Salvai, in analogue with the classification of oriented great circle fibrations of $S^3$ by Gluck and Warner. We show that Salvai’s classification has a topological variation which generalizes to characterize all continuous fibrations of $\mathbb{R}^n$ by skew oriented copies of $\mathbb{R}^p$. We show that the space of fibrations of $\mathbb{R}^3$ by skew oriented lines deformation retracts to the subspace of Hopf fibrations, and therefore has the homotopy type of a pair of disjoint copies of $S^2$. We discuss skew fibrations in the complex and quaternionic setting and give a necessary condition for the existence of a fibration of $\mathbb{C}^n$ ($\mathbb{H}^n$) by skew oriented copies of $\mathbb{C}^p$ ($\mathbb{H}^p$).

1 Introduction

A fibration of $\mathbb{R}^n$ by oriented copies of $\mathbb{R}^p$ is called skew if no two fibers intersect nor contain parallel directions. The study of such fibrations is motivated partly by the fact that they may arise as projections of fibrations of $S^n$ by great spheres $S^p$. In turn, spherical fibrations have been extensively studied due to their relationship with the Blaschke conjecture (see [18] for a recent and thorough summary on the current progress of this conjecture). Algebraic topology imposes severe restrictions on the possible dimensions of spherical fibrations. In particular, the Hopf fibrations

$$S^0 \to S^n \to \mathbb{R}P^n, \quad S^1 \to S^{2n+1} \to \mathbb{C}P^n, \quad S^3 \to S^{4n+3} \to \mathbb{H}P^n, \quad S^7 \to S^{15} \to S^8$$

provide examples of spherical fibrations, and the dimensions in the above list exhaust all possible dimensions for such fibrations.

Fibrations of $S^3$ by oriented great circles were completely classified by Gluck and Warner in [9]. An oriented great circle on $S^3$ corresponds to a unique oriented 2-plane in $\mathbb{R}^4$, hence represents a point in the oriented Grassmann manifold $\tilde{G}_2(4) \simeq S^2 \times S^2$ . Gluck and Warner show that a submanifold of $\tilde{G}_2(4)$ corresponds to a fibration of $S^3$ by oriented great circles if and only if it is the graph of a distance decreasing map from one copy of $S^2$ to the other. They proceed to show that the space of all great circle fibrations deformation retracts to the space of Hopf fibrations, which arise as the set of constant maps from one copy of $S^2$ to the other. Generalizations to higher dimensions have appeared; see for example [10], [17], and [24].

Given a fibration of $S^p$ by oriented great spheres $S^q$, the radial projection to any tangent hyperplane $\mathbb{R}^n$ induces a fibration of $\mathbb{R}^n$ by skew oriented copies of $\mathbb{R}^p$. For example, the lines in the image of the projection of the standard Hopf fibration $S^1 \to S^3 \to S^2$ form a family of nested hyperboloids of one sheet, together with a single vertical line through the origin, as shown in Figure 1.

We refer to a fibration of $\mathbb{R}^n$ by skew, oriented copies of $\mathbb{R}^p$ as a $(p, n)$-fibration, following Ovsienko and Tabachnikov in [15]. There, they show that a $(p, n)$-fibration exists if and only if $p \leq \rho(q) - 1$, where $q = n - p$ and $\rho(\cdot)$ represents the classical Hurwitz-Radon function,
defined as follows: if we decompose $q = 2b + 4c \cdot (2a + 1)$, with $0 \leq b < 4$, then $ho(q) := 2b + 8c$.

This function originally appeared in the independent works of Hurwitz and Radon in their studies of square identities ([11], [19]). It has since made prominent appearances in topology. In particular, an important and relevant result of Adams [1] is that the following two statements hold if and only if $p \leq \rho(q) - 1$:

- There exist $p$ linearly independent tangent vector fields on $S^{q-1}$.
- There exists a $(p + 1)$-dimensional vector space of $q \times q$ matrices with the property that every non-zero matrix in the space is nondegenerate.

A vector space satisfying the second property, dubbed Property $P$ by Adams (see also [2]), can be used to construct a $(p, n)$-fibration, giving sufficiency of the inequality. On the other hand, it can be shown that any $(p, n)$-fibration induces $p$ linearly independent tangent vector fields on $S^{q-1}$, which gives necessity of the inequality.

As far as we know, there has been only one previous result regarding the classification of flat fibrations: a 2009 paper [20] in which Salvai provides a classification of smooth, nondegenerate fibrations of $\mathbb{R}^3$ by oriented lines. The nondegeneracy condition may be seen as a local skewness, though it turns out that any nondegenerate smooth fibration is globally skew. Viewing the set of oriented lines in $\mathbb{R}^3$ as the affine oriented Grassmann $AG_1(3) \cong TS^2$, which may be equipped with a canonical pseudo-Riemannian metric of signature $(2, 2)$, Salvai shows that a surface $M \subset TS^2$ is the space of fibers of a nondegenerate smooth fibration of $\mathbb{R}^3$ by oriented lines if and only if $M$ is a closed definite connected submanifold. There is a certain duality with Gluck and Warner’s result that Salvai notes: $S^2 \times S^2$ admits a canonical pseudo-Riemannian metric, and a smooth surface $M \subset S^2 \times S^2$ is the space of fibers of a smooth fibration of $S^3$ by oriented great circles if and only if $M$ is a closed definite connected submanifold.

In an attempt to generalize the characterization to $(p, n)$-fibrations, one may first view the set of oriented $p$-planes in $\mathbb{R}^n$ as the affine oriented Grassmann $AG_p(n)$, but there is not necessarily a choice of metric that allows for a generalization of Salvai’s classification. However, Salvai offers a second characterization: a surface $M \subset TS^2$ is the space of fibers of a nondegenerate smooth fibration of $\mathbb{R}^3$ by oriented lines if and only if $M$ is the graph of a smooth vector field $v$ defined on an open convex subset $U \subset S^2$, such that $(\nabla v)_u$ has no real eigenvalues for all $u \in U$ and $|v(u_n)| \to \infty$ if $u_n \to u \in \partial U$ as $n \to \infty$. The first condition corresponds to nondegeneracy; the second ensures that the corresponding fibration covers all of $\mathbb{R}^3$.

In Theorem 1 we show that Salvai’s second formulation has a topological variation which generalizes to characterize all continuous $(p, n)$-fibrations. We proceed in Theorem 2 with a
new characterization of continuous (1,3)-fibrations as certain maps $B : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying properties analogous to those of Salvai’s vector fields. We exploit the explicit nature of this second classification in several ways. We see that the subset of (1,3)-fibrations which are induced by great circle fibrations (via central projection) sits naturally inside the set of (1,3)-fibrations as the maps $B$ which are additionally surjective. We use this to show that the space of (1,3)-fibrations deformation retracts to the subspace of (1,3)-fibrations induced by great circle fibrations, which in turn deformation retracts to the subspace of (1,3)-fibrations which occur as projections of Hopf fibrations (Theorem 3). This is a direct generalization of the result of Gluck and Warner.

Finally, we extend the notion of a flat fibration by studying fibrations of $\mathbb{F}^n$ by skew, oriented copies of $(1,2)$ are induced by great circle fibrations (via central projection) sits naturally inside the set of (1,3)-fibrations as the maps $B$ which are additionally surjective. We use this to show that the space of (1,3)-fibrations deformation retracts to the subspace of (1,3)-fibrations induced by great circle fibrations, which in turn deformation retracts to the subspace of (1,3)-fibrations which occur as projections of Hopf fibrations (Theorem 3). This is a direct generalization of the result of Gluck and Warner.

In Section 2 we formally state all of the results discussed above, and we provide a number of illustrative examples. Sections 3, 4, and 5 contain the proofs of these results, and Section 6 provides some additional discussion regarding $\mathbb{F}^2(p,n)$-fibrations.

2 Statement of Results

Given a $(p,n)$-fibration, each fiber $P$ in the oriented affine Grassmann manifold $\tilde{AG}_p(n)$ corresponds uniquely to a pair $(u,v)$ as follows: $v$ represents the closest point on $P$ to the origin of $\mathbb{R}^n$, and $u$ represents the point in the oriented (linear) Grassmann manifold $G_p(n)$ obtained by parallel translating $P$ from $v$ to the origin. Observe that $v$ necessarily lies on the copy of $\mathbb{F}^q$, $q = n - p$, which is both orthogonal to the plane $u$ and passing through the origin. Let $\xi_U$ be the (canonical) $\mathbb{F}^q$-bundle with base space $U \subset G_p(n)$ whose fiber over $u \in U$ is the orthogonal complement of $u$, and let $M \subset \tilde{AG}_p(n)$ be the set of planes from the fibration. Let $\pi : M \to G_p(n)$ be the projection onto the first coordinate: $\pi(u,v) = u$. By the skewness assumption, $\pi$ is injective. Therefore, if $U$ is the image of $\pi$, the set $M$ of skew fibers induces a section $v$ of the bundle $\xi_U$. The following result gives conditions for such a section to correspond to a $(p,n)$-fibration.

Theorem 1. A subset $M \subset \tilde{AG}_p(n)$ corresponds to a $(p,n)$-fibration if and only if $M$ is the graph of a section $v$ of the bundle $\xi_U$ for some $q$-dimensional, connected, contractible submanifold $U \subset G_p(n)$ with the following two properties:

- For all distinct $u_1, u_2 \in U$, $\dim(\text{Span}\{u_1, u_2, v(u_1) - v(u_2)\}) = 2p + 1$,
- If $\{u_n\} \subset U$ is a sequence approaching $u \in \partial U$, then $|v(u_n)| \to \infty$.

In the special case $(1,n)$, the fibers are oriented lines, and the association of a fiber $\ell$ with $(u,v)$ gives a correspondence of $\tilde{AG}_1(n)$ with $TS^{n-1}$. Explicitly, $u \in S^{n-1}$ is the unit oriented direction of $\ell$, $v$ is a point of $TuS^{n-1}$, the bundle $\xi_U$ over $U = \pi(M) \subset S^{n-1}$ is the tangent bundle $TU$, and $v$ is a tangent vector field. In this way we may view the $(1,3)$ case of Theorem 1 as a topological variation of Salvai’s result. As in his result, the first bullet point of Theorem 1 corresponds to skewness of the fibers, whereas the second ensures that the fibers cover the whole space.

Example 1. As depicted in Figure 1, the projection of the standard Hopf fibration of $S^3$ by oriented great circles gives a fibration of $\mathbb{R}^3$ by skew oriented lines. In this case, the subset $U \subset S^2$ is the open upper hemisphere and $v$ sends a point $\left(\sqrt{u_1^2 + u_2^2 + 1}^{-1} \cdot (u_1, u_2, 1)\right) \in U$ to the tangent vector $(-u_2, u_1, 0)$. This example may be manipulated to see the necessity of the second bullet point of Theorem 1. Restricting the domain of $v$ to a smaller spherical cap corresponds to removing some lines from the fibration, hence the resulting collection of lines are skew but do not cover $\mathbb{R}^3$. 

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Remark 1. Theorem may be reformulated with the observation that not only is each plane in a skew fibration unique, but that every unit vector contained in any plane occurs only once. Consider an orthonormal basis $u_1, \ldots, u_p$ of $u \in U \subset G_p(n)$. By contractibility of $U$, there is a choice of an orthonormal basis which varies continuously over all of $U$. At each $u$, the $p+1$ vectors $u_1, \ldots, u_p, v(u)$ are mutually orthogonal, hence the latter $p$ vectors lie in $T_u S^{n-1}$. In particular, the section $v$ of the bundle $\xi_U$ in Theorem may instead be viewed as some $q$-dimensional subset $\tilde{U} \subset S^{n-1}$ together with $p$ orthogonal sections of the tangent bundle $TU$. It is unclear to us whether there is any benefit to this reformulation.

Example 2. It is interesting to see how a $(1, n)$-fibration induces a nonzero tangent vector field on $S^{n-2}$ via the above characterization (cf. [13] and Example 4 below). Observe first that since $|v(u)|$ gives the distance from the fiber with direction $u$ to the origin of $\mathbb{R}^n$, there exists a unique $\tilde{u} \in U \subset S^{n-1}$ with $v(\tilde{u}) = 0$. Fix a small sphere $S^{n-2} \subset U$ centered at $\tilde{u}$, and for each $x \in S^{n-2}$, project $v(x)$ to $T_x S^{n-2}$. It follows from the first bullet point of Theorem that $v(x)$ is not in the plane spanned by $x$ and $\tilde{u}$, therefore it does not project to the 0 vector, so we have a nonzero tangent vector field on $S^{n-2}$.

We proceed by examining $(p, n)$-fibrations from a local perspective. Fix a fiber $\mathbb{R}^p$ and a point $z \in \mathbb{R}^p \subset \mathbb{R}^n$, and consider the copy of $\mathbb{R}^q$ through $z$ and orthogonal to $\mathbb{R}^p$. Every fiber close to $\mathbb{R}^p$ is the graph of an affine map $\mathbb{R}^p \to \mathbb{R}^q : t \mapsto B(y)t + y$, where $y$ is the coordinate in the transversal $\mathbb{R}^q$ and $B(y) : \mathbb{R}^p \to \mathbb{R}^q$ is a linear map defined for $y$ sufficiently close to $z$ (see Figure 2). Said differently, there is a continuous map $B$ defined in a neighborhood of $z$ in $\mathbb{R}^q$ and taking values in the set of $q \times p$ matrices, such that for a fixed $y$ in the neighborhood, the graph of the map $t \mapsto B(y)t + y$ is precisely the fiber through $y$. In particular, observe that $B(z) = 0$.

Let $A(y)$ be the $q \times (p + 1)$ matrix obtained by appending the column vector $y$ to $B(y)$.
Theorem 2. A subset $M \subset \tilde{A}G_1(3)$ corresponds to a $(1,3)$-fibration if and only if there exists a continuous map $B : \mathbb{R}^2 \to \mathbb{R}^2$ such that $B(0) = 0$, each point in $M$ is the graph of the function $t \mapsto B(y)t + y$ for some $y$, and the following two properties hold:

- For all distinct $y, z \in \mathbb{R}^2$, $\text{Ker}(A(y) - A(z)) = 0$ (here $A(y)$ is the $2 \times 2$ matrix with columns $B(y)$ and $y$),

- If $\{y_n\} \subset \mathbb{R}^2$ is a sequence with no accumulation points and $\ell_n$ represents the fiber through $y_n$, then $|\ell_n| \to \infty$.

Moreover, a $(1,3)$-fibration corresponds via central projection to a fibration of $S^3$ by oriented great circles if and only if $B$ is surjective.

As in Theorem 1, the first property corresponds to skewness, and a similar statement holds even for local fibrations of $\mathbb{F}^n$ by $\mathbb{F}^p$ (see Lemma 10). The second property ensures that the lines cover all of $\mathbb{R}^3$. The final statement may be reworded in the language of Theorem 1, a $(1,3)$-fibration corresponds to a great circle fibration if and only if the set of directions $U \subset S^2$ is an open hemisphere. This is the statement of Lemma 12.

In the $(1,3)$ case, the first property implies that $\det(A(y) - A(z)) \neq 0$ for distinct $y$ and $z$. In particular, the map $K : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} : (y, z) \mapsto \det(A(y) - A(z))$ is zero precisely on the diagonal $\Delta \subset \mathbb{R}^2 \times \mathbb{R}^2$. Since $(\mathbb{R}^2 \times \mathbb{R}^2) - \Delta$ is path-connected, $K$ is either nonnegative or nonpositive. We refer to a fibration as positively-oriented or negatively-oriented according to the sign of $K$. Note that this formulation is reminiscent of Salvai’s definiteness property.

Example 3. Continuing Example 1, the map $H : \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to the (negatively-oriented) Hopf fibration sends $y = (y_1, y_2)$ to $iy = (-y_2, y_1)$. In particular, the fiber through the point $y$ is the graph of $t \mapsto (-y_2, y_1)t + (y_1, y_2)$. To be precise, this map $H$ may be defined on any oriented 2-plane through the origin of $\mathbb{R}^3$, so the set of negatively-oriented Hopf fibrations is actually a copy of $S^2$. Similarly, the map $y \mapsto -iy$ defines the positively-oriented Hopf fibrations.

Theorem 3. The space of $(1,3)$-fibrations deformation retracts, through the subspace of (projected) oriented great circle fibrations, to the subspace of (projected) oriented Hopf fibrations. Therefore each space has the homotopy type of a pair of disjoint copies of $S^2$.

Corollary 4. (Gluck-Warner 3) The space of oriented great circle fibrations deformation retracts to the subspace of oriented Hopf fibrations.

This result gives some additional motivation for studying the conjectured general case of Theorem 2. In particular, the explicit nature of such a characterization may allow for studying higher-dimensional spherical fibrations by studying their $(p,n)$-fibration counterparts.

Example 4. Continuing Example 2, we see how $p$ linearly independent vector fields on $S^{q-1}$ are induced by $B$, even if $B$ is only locally defined in a neighborhood $E$. Consider a small sphere $S^{q-1}$ in the transverse $\mathbb{R}^q$ and contained in $E$. By the (Lemma 10 version of the) first bullet point, to each $y \in S^{q-1}$ corresponds a $q \times (p + 1)$ matrix $A(y)$ with trivial kernel. Project the first $p$ column vectors onto $T_yS^{q-1}$. They are independent there, for otherwise some linear combination is equal to a multiple of $y$, but then $\text{Ker}(A(y))$ is nontrivial.

Let $\mathbb{F}$ be one of the classical division algebras over $\mathbb{R}$, that is, $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We will use the term $\mathbb{F}$-$(p,n)$-fibration to refer to a fibration of $\mathbb{F}^n$ by skew copies of $\mathbb{F}^p$. The local setup depicted in Figure 2 survives the extension to $\mathbb{C}$ and $\mathbb{H}$. In particular, let $|\mathbb{F}|$ be the dimension of $\mathbb{F}$ over $\mathbb{R}$ and consider the inner product $\langle \cdot, \cdot \rangle$ induced on $\mathbb{F}^q$ by the association of $\mathbb{F}^q$ with $\mathbb{R}^{|\mathbb{F}|}$. Fix a fiber $\mathbb{F}^p$ and let $\mathbb{F}^q$ refer to the hyperplane which is $\langle \cdot, \cdot \rangle$-orthogonal to the fiber at some point $z$. Then for each $y$ in a small neighborhood of $z$ in $\mathbb{F}^q$, there exists a $q \times p$ matrix $B(y)$ with entries in $\mathbb{F}$ such that the fiber through $y$ is the graph of the map $\mathbb{F}^p \to \mathbb{F}^q : t \mapsto B(y)t + y$. 

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Define the unit sphere $S^{[F]q-1}$ in $F^q$ as the set of vectors with unit $(\cdot, \cdot)$-length. For a point $y \in S^{[F]q-1}$, the $F$-tangent space at $y$ is the set of all $F$-lines that are $(\cdot, \cdot)$-orthogonal to $y$. Explicitly, the complex tangent space at $y \in S^{2q-1}$ is the copy of $R^{2q-2} = C^{q-1}$ containing vectors orthogonal to $y$ and $iy$, and the quaternionic tangent space at $y \in S^{4q-1}$ is the copy of $R^{4q-4} = H^{q-1}$ containing vectors orthogonal to $y, iy, jy, and ky$. In Example 4 above, we showed that an $R$-$(p,n)$-fibration induces $p$ linearly independent vector fields on $S^{q-1}$. A similar statement holds for $F$-$(p,n)$-fibrations.

**Theorem 5.** An $F$-$(p,n)$-fibration may exist only if there exist $p$ linearly independent (over $F$) sections of the $F$-tangent bundle on the unit sphere in $F^q$.

The problem of finding the maximum number of linearly independent sections of the $F$-tangent bundle on the unit sphere in $F^q$ has a fascinating history, thoroughly chronicled in Chapter 4 of [10]. Having already discussed the main results in the real case, we mention the relevant portions of the story in the complex and quaternionic cases. Around 1958, James showed (in [12], [13], [14]) that for every positive integer $p$, there exists an $F$-$(p,n)$-fibration of the unit sphere in $F^q$ if and only if $q$ is a multiple of a certain number $b_p$ (if $F = C$) or $c_p$ (if $F = H$), but he was not able to explicitly determine these numbers. Soon after, Atiyah and Todd showed in [1] that if the unit sphere in $C^q$ has $p$ linearly independent vector fields, then $q$ must be a multiple of a number $m_p$, which they determined explicitly. In 1965, Adams and Walker showed in [2] that this condition was sufficient by showing the equality $b_p = m_p$. Finally, in 1973, Sigrist and Suter solved the problem for $F = H$ in [21], where they explicitly determined the number $c_p$.

The corollaries below follow from the works of Adams and Walker (for the complex case) and Sigrist and Suter (for the quaternionic case).

**Corollary 6.** A $C$-$(p,n)$-fibration may exist only if for each integer $r$ with $0 \leq r \leq p$, the coefficient of $t^r$ in the power series expansion of

$$\left( \frac{t}{\ln(1 + t)} \right)^q$$

is an integer.

**Corollary 7.** An $H$-$(p,n)$-fibration may exist only if for each integer $r$ with $0 \leq r \leq p$, the coefficients of $t^r$ in the power series expansion of

$$\left( \frac{2}{\sqrt{t}} \cdot \sinh^{-1} \frac{\sqrt{t}}{2} \right)^q$$

is an integer for even $r$ and an even integer for odd $r$.

As far as we know, there is no construction of two or more linearly independent sections on the complex tangent bundle, nor is there a construction of a single section of the quaternionic tangent bundle. Accordingly, we know of no construction of a $C$-$(p,n)$-fibration for $p > 1$ nor a construction of any $H$-$(p,n)$-fibration. In particular, we do not know whether the converse of Theorem 5 holds. However, we are able to construct a $C$-$(1, 2k+1)$-fibration for $k \in \mathbb{N}$ (see Example 5).

We note that one failed attempt at constructing a $C$-$(p,n)$-fibration is based on an attempted generalization of Property P. Such a construction would require the existence of a vector space of complex matrices over $C$ such that every nonzero linear combination is nondegenerate. However, consider just two nonzero square matrices $A_1, A_2$. The quantity $\det(\lambda A_1 + A_2)$ is a polynomial in $\lambda$, so it must have some root in $C$, hence no such property exists. As a noteworthy aside, Adams, Lax, and Phillips studied vector spaces of complex and quaternionic matrices over $R$ for which every nonzero linear combination is nondegenerate [2].

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3 Topological properties of \((p, n)\)-fibrations

Each of the following three lemmas gives some information about the topology of a general \((p, n)\)-fibration. These lemmas are used primarily in the proof of Theorem 1.

**Lemma 8.** If \(M \subset \tilde{A}G_p(n)\) corresponds to a \((p, n)\)-fibration, then \(M\) is closed in \(\tilde{A}G_p(n)\). Moreover, if \((u_n, v_n) \in M\) is a sequence with no accumulation point in \(M\), then \(|v_n| \to \infty\).

**Proof.** Let \(P_n = (u_n, v_n) \in M\) be a sequence converging to a point \(P = (u, v) \in \tilde{A}G_p(n)\). Let \(V : \mathbb{R}^n \to \tilde{G}_p(n)\) be the map sending a point in \(\mathbb{R}^n\) to the unique oriented plane from the fibration through that point. Viewing \(v_n\) as a point in \(\mathbb{R}^n\) (specifically, the point on \(P_n\) nearest to the origin), we have \(u_n = V(v_n)\). This sequence approaches \(V(v)\) by continuity of \(V\). Therefore, \(u_n\) approaches both \(u\) and \(V(v)\), so \(u = V(v)\). Geometrically, this means that the plane through \(v\) is precisely \(u\), so \((u, v)\) is contained in \(M\), thus \(M\) is closed.

Consider again a sequence \((u_n, v_n)\) in \(M\). We prove the contrapositive of the latter statement. If the sequence of distances \(|v_n|\) does not approach \(\infty\), then there is a bounded subsequence, and hence a convergent further subsequence \(|v_{n_j}|\). Thus the subsequence \((u_{n_j}, v_{n_j})\) of the original sequence is contained in a compact subset of \(A\tilde{G}_p(n)\), and so it has an accumulation point, which is contained in \(M\) by closure.

**Lemma 9.** The set \(U\) is a \(q\)-dimensional, connected, contractible submanifold of \(\tilde{G}_p(n)\).

**Proof.** Choose a plane \(P\) from \(M\), let \(Q\) be the \(q\)-dimensional hyperplane orthogonal to \(P\) and passing through the origin, and let \(z\) be the intersection point of \(P\) and \(Q\). There is an open neighborhood \(E \subset \tilde{Q}\) of \(z\) such that all the planes from the fibration intersect \(E\) transversely. The map \(V : E \to \tilde{G}_p(n)\) which takes a point \(y\) to the oriented plane through \(y\) is a homeomorphism onto its image, a \(q\)-dimensional subset of \(\tilde{G}_p(n)\). Letting \(P\) range over \(\tilde{M}\), this gives a collection of charts which cover \(U\).

Consider the map \(V\) defined on \(\mathbb{R}^n\). By definition, \(U = V(\mathbb{R}^n)\), so \(U\) is connected. Moreover, the preimage of every \(u \in U\) is a copy of \(\mathbb{R}^p\). That is, we have the structure of a fiber bundle

\[\mathbb{R}^p \to \mathbb{R}^n \xrightarrow{V} U,\]

which in turn induces a long exact sequence of homotopy groups:

\[\cdots \to \pi_{n+1}(\mathbb{R}^p) \to \pi_{n+1}(\mathbb{R}^n) \xrightarrow{V_*} \pi_{n+1}(U) \to \pi_n(\mathbb{R}^p) \to \cdots \to \pi_0(\mathbb{R}^p) \to \pi_0(\mathbb{R}^n) \to 0.\]

It follows that \(V\) induces isomorphisms \(\pi_k(\mathbb{R}^n) \approx \pi_k(U)\) for all \(k > 0\), so by Whitehead’s Theorem, \(V\) is a homotopy equivalence and \(U\) is contractible.

The following lemma was originally shown for \((p, n)\)-fibrations in [15], but we take the opportunity to observe that the result holds for \(\mathbb{F}-\text{(p,n)}\)-fibrations. We assume the local setup depicted in Figure 2 and discussed for \(\mathbb{F}-\text{(p,n)}\)-fibrations prior to the statement of Theorem 5. Recall that \(A(y)\) refers to the \(q \times (p + 1)\) matrix obtained by appending the column vector \(y\) to \(B(y)\).

**Lemma 10.** The fibers through \(y\) and \(z\) are skew if and only if \(\text{Ker}(A(y) - A(z)) = 0\).

**Proof.** The fibers through \(y\) and \(z\) intersect if and only if \(B(y)\eta + y = B(z)\eta + z\) has a solution, or equivalently, if and only if

\[\begin{pmatrix} A(y) - A(z) \\ 0 \end{pmatrix} \begin{pmatrix} \eta \\ 1 \end{pmatrix} = 0.\]

The fibers through \(y\) and \(z\) contain parallel directions if and only if the equation \(B(y)\eta = B(z)\eta\) has a nonzero solution, or equivalently, if and only if

\[\begin{pmatrix} A(y) - A(z) \\ 0 \end{pmatrix} \begin{pmatrix} \eta \\ 0 \end{pmatrix} = 0.\]

\[\square\]
4 Geometry and topology of (1, 3)-fibrations

Here we study (1, 3)-fibrations in great detail. The following four lemmas provide the geometric insights necessary for the proofs of Theorems 2 and 3.

We begin with the following convexity result, which was shown in the smooth (1, 3) case by Salvai in [20]. We offer a separate proof based on a different geometric idea. A generalization of either proof to even the (1, n) case would be of great interest, since it is the only missing link in generalizing the classification in Theorem 2 to (1, n)-fibrations, a result which may allow for studying great circle fibrations of $S^n$ by studying their (1, n)-fibration counterparts.

**Lemma 11.** In the case of a (1, 3)-fibration, $U$ is a convex subset of $S^2$.

*Proof.* Let $P_0$ be a 2-plane through the origin in $\mathbb{R}^3$. Observe that the intersection of $P_0$ with $U$ is the set of directions in the fibration parallel to $P_0$; therefore, the line in the fibration corresponding to one of these points must be contained in $P_0$ or one of its parallel translates $P_t$. Also note that, by the assumption of skewness, each plane can contain at most one line.

Given a plane $P$, if $P$ contains a line $t_P$ from the fibration, define $S_P$ as the set of points contained in $t_P$; otherwise, let $S_P$ be empty. Then for all $t$, we have a homeomorphism

$$P_t \setminus S_P \approx U \setminus (U \cap P_0)$$

by the map sending a point in $P_t \setminus S_P$ to the direction of the (necessarily transverse) line through that point. This gives a homeomorphism from $P_0 \setminus S_{P_0}$ to $P_t \setminus S_{P_t}$. This implies that all $S_{P_t}$ are lines, or all are empty, since otherwise we would have a homeomorphism between a connected set and a disconnected set.

In particular, this establishes the following fact: for any 2-plane $P_0$ through the origin of $\mathbb{R}^3$, either each translate $P_t$ is transverse to all fibers, or each $P_t$ contains exactly one line from the fibration. In the first case, $U \cap P_0$ is empty; in the second, we have a homeomorphism from $\mathbb{R}$ to $U \cap P_0$ sending $t$ to the direction of the line contained in $P_t$, so that $U \cap P_0$ is an open, connected segment of a great circle. Since $U$ cannot contain antipodal points, convexity follows: for any two points in $U$, the shorter segment of the great circle connecting those points must also lie in $U$. \hfill \Box

Any fibration of $S^3$ by oriented great circles induces a (1, 3)-fibration by central projection. In [20], Salvai shows that in this case, the set of directions is an open hemisphere $U$. The idea is that in any equatorial copy of $S^2 \subset S^3$, there is exactly one circle from the fibration contained in $S^2$. This splits the remainder of $S^2$ into two connected components: an open hemisphere $U$ and its antipode $-U$. The intersection of any great circle $C$ from the fibration transverse to $S^2$ intersects $S^2$ in two points: $u \in U$ and $-u \in -U$. The central projection of $C$ to a copy of $\mathbb{R}^3$ parallel to $S^2$ yields a line whose unit direction (up to orientation) is precisely $u$. Projecting the entire fibration therefore yields a flat fibration whose set of directions is an open hemisphere.

Here we show the converse statement.

**Lemma 12.** A (1, 3)-fibration corresponds via central projection to a great circle fibration if and only if $U \subset S^2$ is an open hemisphere.

*Proof.* By the discussion above, we must only show the backwards implication. Assume that the fibered $\mathbb{R}^3$ is positioned as the tangent hyperplane to the north pole $(0, 0, 0, 1) \in S^3 \subset \mathbb{R}^4$. We also assume, by rotating and translating, if necessary, that $\mathbb{R}^3$ is oriented so that the set of directions $U$ is comprised of vectors of the form $\frac{1}{\sqrt{u_1^2 + u_2^2 + 1}}(u_1, u_2, 1, 0)$, with the direction $(0, 0, 1, 0)$ occurring at the origin of the fibered $\mathbb{R}^3$, or equivalently, at $(0, 0, 0, 1) \in \mathbb{R}^4$.

The inverse central projection bijects to the open upper hemisphere of $S^3$ (points with final coordinate positive); we may complete each great semicircle to a great circle. This will cover the open lower hemisphere, as well as the pairs of antipodal points in the equatorial
copy of $S^2$ which correspond to directions from the fibration. In total, this leaves a single great circle $C$ in $S^3$ uncovered, specifically the circle spanned by $(1,0,0,0)$ and $(0,1,0,0)$.

We must show that adding $C$ yields a continuous fibration of $S^3$ by oriented great circles. In particular, to show continuity we must show the following:

**Claim 1.** If $x_n \in S^3 \setminus C$ is a sequence of points converging to some point $x \in C$, then the sequence of fibers $C_n$ through $x_n$ converge to $C$.

Each $C_n$ is the image, under the inverse central projection, of a line $(u_n, v_n)$ from the flat fibration. Said differently, $C_n$ is spanned by the orthonormal vectors $u_n$ and $\frac{v_n}{|v_n|}$. Indeed, $u_n$ is the direction of the line and therefore lies on $C_n$, and the point $v_n$ on the line projects to the point $\frac{v_n}{|v_n|}$. We study Claim 1 by first showing the following.

**Claim 2.** If for any convergent subsequence of $u_n$ and $\frac{v_n}{|v_n|}$, the limits $u$ and $v$ are in $C$, then $C_n \to C$.

We prove the contrapositive. The space of oriented great circles on $S^3$ is equivalent to the compact space $G_2(4)$, so if $C_n \not\to C$, then some subsequence $C_{n_k}$ converges to a great circle $D \neq C$. By compactness of $S^3$, the subsequences $u_{n_k}$ and $\frac{v_{n_k}}{|v_{n_k}|}$ have further convergent subsequences. Since $C_{n_k} \to D$, the limits of these further subsequences provide an orthonormal basis of $D$, and therefore cannot both lie on $C$.

The now-proven Claim 2 reduces Claim 1 to the following.

**Claim 3.** If $x_n \in S^3 \setminus C$ is a sequence of points converging to some point $x \in C$, then the limit $(u, v)$ of any convergent subsequence of $(u_n, \frac{v_n}{|v_n|})$ is an orthonormal basis of $C$.

To avoid stacking indices, we will use the notation $(u_k, \frac{v_k}{|v_k|})$ to refer to a convergent subsequence of $(u_n, \frac{v_n}{|v_n|})$. Specifically, to prove Claim 3, we now assume that $x_k \to x \in C$ and that $(u_k, \frac{v_k}{|v_k|}) \to (u, v)$, and we show that $u$ and $v$ must be points of $C$.

Projecting the parametrized flat fiber

$$v_k + tu_k = \begin{pmatrix} v_{k1} \\ v_{k2} \\ v_{k3} \\ 1 \end{pmatrix} + t \begin{pmatrix} u_{k1} \\ u_{k2} \\ u_{k3} \\ 0 \end{pmatrix}$$

to $S^3$ yields a parametrized great circle

$$\cos \theta \cdot u_k + \frac{v_k}{|v_k|} = \cos \theta \begin{pmatrix} u_{k1} \\ u_{k2} \\ u_{k3} \\ 0 \end{pmatrix} + \sin \theta \begin{pmatrix} v_{k1} \\ v_{k2} \\ v_{k3} \\ 1 \end{pmatrix}.$$ 

Since $x_k \in C_k$, there exists $\theta_k$ such that $x_k = \cos \theta_k u_k + \sin \theta_k \frac{v_k}{|v_k|}$, so the statement $x_k \to x \in C$ can be rewritten as

$$\cos \theta_k \begin{pmatrix} u_{k1} \\ u_{k2} \\ u_{k3} \\ 0 \end{pmatrix} + \sin \theta_k \frac{v_{k1}}{|v_k|} \begin{pmatrix} v_{k1} \\ v_{k2} \\ v_{k3} \\ 1 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix}. \quad (1)$$

**Proof** that $u \in C$: Assume for contradiction that $\lim u_k = u \notin C$. Then $u \in U$, so by continuity of the map $v : U \to TU$, we have $|v(u_k)| \to |v(u)|$, hence $v(u_k)$ is bounded. Together with the fourth coordinate in equation (1), this yields $\sin \theta_k \to 0$. But the sequence $\frac{v_{k3}}{|v_k|}$ is also bounded, so in the third coordinate of (1) we have

$$\sin \theta_k \frac{v_{k3}}{|v_k|} \to 0,$$

and hence $\cos \theta_k \cdot u_{k3} \to 0$. 


Therefore \( u_k = \lim u_{k_j} = 0 \), so \( u \in C \) and we have arrived at the contradiction.

**Proof that \( v \in C \):** Since \( u_k = 0 \), the sequence \( u_k \) has no accumulation point in \( U \), therefore by Lemma 8, \( |u_k| \to \infty \), and so \( \frac{u_k}{|v_k|} = v \) is a point of the equator \( S^2 \). In particular, if \( D \) is the circle spanned by \( u \) and \( v \), then \( D \) lies entirely on \( S^2 \). Therefore \( D \) is not the image of a line from the flat fibration, since none of these image circles lie completely on the equator.

Since \( v \in S^2 \), either \( v \in \pm U \) or \( v \in C \). Assume for contradiction that \( v \in \pm U \). Through each point of \( \pm U \), there is already an image circle from the fibration; call \( E \) the image circle passing through \( v \). Then applying continuity of the fibration to \( \frac{u_k}{|v_k|} \to v \) yields \( C_k \to E \), but this is impossible since \( C_k \to D \) and \( D \) is not an image circle. Thus we have arrived at the contradiction, and we conclude that \( v \in C \).

This completes the proof of Claim 3, which shows that the induced great circle fibration is continuous. It remains to show that there is a choice of orientation for the great circle \( C \) which orients the fibration. Orientation of the original flat fibration is passed to the great circles, so we can choose to order the basis of \( C \) by \( u_n, v_n \) so that, near \( C = \text{Span} \{(1,0,0,0),(0,1,0,0)\} \), the quantity

\[
\det \begin{pmatrix}
  u_{n_1} & u_{n_2} & 1 & 0 \\
  v_{n_1} & v_{n_2} & 1 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 1 & 1 \\
\end{pmatrix} = \det \begin{pmatrix}
  u_{n_1} & u_{n_2} \\
  v_{n_1} & v_{n_2} \\
\end{pmatrix}
\]

is always positive or always negative. We may choose an ordering of the basis \( (1,0,0,0), (0,1,0,0) \) accordingly. \( \square \)

Let \( B \) be the set of maps \( \mathbb{R}^2 \to \mathbb{R}^2 \) with the properties of Theorem 2. In the discussion following the statement of Theorem 2 we showed that \( B \in \mathcal{B} \) is either negatively- or positively-oriented, based on the sign of the corresponding map \( (y,z) \mapsto \det(A(y) - A(z)) \), where here \( A(y) \) represents the \( 2 \times 2 \) matrix with columns \( B(y) \) and \( y \). We label these subspaces \( \mathcal{B}_- \) and \( \mathcal{B}_+ \), and we note that these spaces are equivalent by the map \( B \mapsto -B \) and that there is no path from \( \mathcal{B}_- \) to \( \mathcal{B}_+ \).

**Lemma 13.** The set \( \mathcal{B} \) consists of two path-connected components: \( \mathcal{B}_- \) and \( \mathcal{B}_+ \).

**Proof.** We show that \( \mathcal{B}_- \) is path-connected by homotoping any element to the negatively-oriented Hopf fibration \( H : (y_1, y_2) \mapsto (-y_2, y_1) \), as introduced in Example 3. Define the straight-line homotopy \( B_s(y) = s \cdot H(y) + (1 - s) \cdot B(y) \). We must check that for each \( s \), \( B_s \in \mathcal{B}_- \). We have \( B_s(0) = 0 \), and using \( A_s(y) \) for the \( 2 \times 2 \) matrix with columns \( B_s(y) \) and \( y \), we have

\[
\det(A_s(y) - A_s(z)) = s \cdot \det(A_1(y) - A_1(z)) + (1 - s) \cdot \det(A_0(y) - A_0(z)) \leq 0,
\]

with equality if and only if \( y = z \).

If \( y_n \) is a sequence with no accumulation point and \( \ell_n \) is the fiber through \( y_n \) corresponding to \( B \), we have \( |\ell_n| \to \infty \). We must show that this property is maintained for each \( B_s \). Given \( y \in \mathbb{R}^2 \), let \( \ell_s \) represent the fiber through \( y \) induced by \( B_s \). Since the property holds for \( B = B_0 \), it is enough to show that for all \( y \) and all \( s \), \( |\ell_s| \geq |\ell_0| \). Geometrically, the idea is that the Hopf map \( H \) achieves, for each point \( y \in \mathbb{R}^2 \), the maximum possible (shortest) distance from the fiber to the origin, namely \(|y|\). Therefore, taking some average of any other map \( B \) with the Hopf map \( H \) does not decrease the distance.

Explicitly, a parametrization for \( \ell_s \) is given by

\[
\ell_s = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -y_2 \\ y_1 \\ 1 \end{pmatrix} + (1 - s) \begin{pmatrix} B_1 \\ B_2 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R},
\]
where \( B_1 \) and \( B_2 \) are the components of the map \( B \) and we have suppressed the argument \( y \) of \( B_i \).

This parametrization is of the form

\[
\begin{pmatrix}
  y_1 \\
  y_2 \\
  0
\end{pmatrix} + t
\begin{pmatrix}
  z_1 \\
  z_2 \\
  1
\end{pmatrix},
\]

which is closest to the origin when

\[
t = -\frac{y_1 z_1 + y_2 z_2}{1 + z_1^2 + z_2^2},
\]

with squared minimum distance to the origin given by

\[
y_1^2 + y_2^2 - \frac{(y_1 z_1 + y_2 z_2)^2}{1 + z_1^2 + z_2^2}.
\]

Replacing appropriately for \( z_1 \) and \( z_2 \) gives the squared minimum distance from \( \ell_s \) to the origin:

\[
|\ell_s|^2 = y_1^2 + y_2^2 - \frac{(1-s)^2(y_1 B_1 + y_2 B_2)^2}{s^2(y_1^2 + y_2^2) + (1-s)^2(B_1^2 + B_2^2) + 2s(1-s)(y_1 B_1 - y_2 B_2) + 1}.
\]

When \( s = 0 \), this reduces to

\[
|\ell_0|^2 = y_1^2 + y_2^2 - \frac{(y_1 B_1 + y_2 B_2)^2}{B_1^2 + B_2^2 + 1}.
\]

To check that \( |\ell_s|^2 \geq |\ell_0|^2 \), we compare the fractional terms. Multiplying the top and bottom of the latter fraction by \((1-s)^2\) and comparing denominators yields the desired inequality, since \( y_1 B_1 - y_2 B_2 = \det(A(y)) < 0 \).

Based on Lemma 12, the following lemma says that for all \( s > 0 \), \( B_s \) corresponds to a great circle fibration.

**Lemma 14.** For all \( B \in \mathcal{B}_- \) and all \( s \in (0,1] \), the map \( B_s = sH + (1-s)B \) is surjective.

**Proof.** Component-wise, we have \( B_{s_1} = -sy_2 + (1-s)B_1 \) and \( B_{s_2} = sy_1 + (1-s)B_2 \). For \( y \neq 0 \) we compute

\[
|B_s|^2 = s^2(y_1^2 + y_2^2) + (1-s)^2(B_1^2 + B_2^2) - 2s(1-s)\det(A(y))
\]

Therefore \( |B_s|^2 \geq s^2(y_1^2 + y_2^2) \), since \( \det(A(y)) < 0 \). The map \( B_s \) is continuous and injective, where injectivity follows from the fact that no two fibers are parallel. Therefore, by Invariance of Domain, it is a homeomorphism onto its image. In particular, the image of the circle \( S^1(r) \) is a simple closed curve \( C \) lying outside the circle \( S^1(rs^2) \), and the image of the corresponding disk \( D^2(r) \) is the region enclosed by \( C \). Therefore any point \( w \) of the codomain \( \mathbb{R}^2 \) is in the image of the disk \( D^2(|w|/s^2) \).

## 5 Characterization of \((p, n)\)-fibrations

Here we prove the main theorems, with the help of Lemmas 8 and 14.

**Proof of Theorem 1.** “\( \Rightarrow \)”: By Lemma 9, the set \( U \) of oriented \( p \)-planes appearing in the fibration \( M \) is a \( q \)-dimensional submanifold of \( \tilde{G}_p(n) \). Each \( p \)-plane from the fibration is achieved exactly once, so we can write each \((u, v) \in M \) as \((u, v(u))\) where \( v \) is a section of the bundle \( \xi_U \). Since any pair of distinct \( p \)-planes \( u_1 \) and \( u_2 \) contain no parallel directions,
their span is 2p-dimensional, and so it only remains to show that \( v(u_1) - v(u_2) \) is not in the span of \( u_1 \) and \( u_2 \). If it were, we could write \( v(u_1) + C_1 x_1 = v(u_2) + C_2 x_2 \), where \( x_1 \) and \( x_2 \) are vectors in the planes \( u_1 \) and \( u_2 \), respectively. But this means that the planes intersect.

The second bullet point follows from Lemma 8, since a sequence approaching a boundary point has no accumulation point in \( M \).

“\( \Leftarrow \)”: It follows from the first bullet point that the collection of planes is pairwise skew, so we only must show that the planes cover all of \( \mathbb{R}^n \). Let \( D: U \to \mathbb{R} \) be the map \( u \mapsto |v(u)| \), and let \( m \) be the infimum of the image of \( D \). There is a sequence \( u_n \) with \( |v(u_n)| \) approaching \( m \), and this sequence must converge to some point \( u \in U \) by the second bullet point. Thus \( D \) achieves its minimum \( m \) at the point \( u \). Assume that \( m \) is not 0, so that \( u \) does not contain the origin of \( \mathbb{R}^n \).

Consider in \( \mathbb{R}^n \) the copy of \( \mathbb{R}^3 \) through the origin and orthogonal to \( u \). Let \( E \) be a neighborhood of \( u \) contained in \( U \), small enough so that for each \( p \)-plane \( u' \in E \), \( u' \) intersects \( \mathbb{R}^3 \) transversely. Define the map \( f: E \to \mathbb{R}^3 \) to take each \( u' \) to its intersection point with \( \mathbb{R}^3 \). The map \( f \) is continuous and injective, so by associating \( E \) with an open subset of \( \mathbb{R}^3 \), we can apply invariance of domain to see that \( f(E) \) is an open subset of \( \mathbb{R}^3 \). But in such a subset, there is a closer point to the origin than \( f(u) \), a contradiction. Thus \( m \) must be 0, and so \( u \) passes through the origin of \( \mathbb{R}^n \). This argument holds for any point in \( \mathbb{R}^n \), and so there is some plane passing through each point of \( \mathbb{R}^n \), completing the proof. \( \square \)

**Proof of Theorem 2** “\( \Rightarrow \)”: Since \( U \) is a convex set (Lemma 11), there is a plane \( P \) in \( \mathbb{R}^3 \) which is transverse to all lines from the fibration. We may choose \( P \) so that there is a fiber from the fibration orthogonal to \( P \). Situating \( P \) so that this fiber passes through the origin of \( P \) induces a globally-defined \( B: P = \mathbb{R}^2 \to \mathbb{R}^2 \), with \( B(0) = 0 \), and with the first bullet point of Theorem 2 following from Lemma 10.

Let \( y_n \) be a sequence in \( P \) and let \( \ell_n \) be the fiber containing \( y_n \). If \( |\ell_n| \to \infty \), Lemma 8 gives a convergent subsequence \( \ell_{n_k} \) to some fiber \( \ell \). Let \( y \) be the intersection of \( P \) with \( \ell \), which exists because \( P \) is transverse to all the fibers. Then in each neighborhood of \( y \) in \( P \), there is a fiber from the subsequence \( \ell_{n_k} \), so \( y_{n_k} \) is in the neighborhood. Hence \( y \) is an accumulation point of \( y_n \).

“\( \Leftarrow \)”: The existence of a matrix \( B \) with the property in the first bullet gives a collection of skew lines in \( \mathbb{R}^3 \). We must show that the lines cover the whole space. Let \( P_0 \) represent the copy of \( \mathbb{R}^2 \) on which \( B \) is defined, and choose a plane \( P \) parallel to \( P_0 \). The map \( g: P_0 \to P \) which sends \( y \) to the intersection of \( P \) with the fiber through \( y \) is a continuous injective map, hence a homeomorphism onto its image. In particular, the image is open; we must also show that it is closed. Let \( z_n \) be a sequence in \( g(P_0) \) converging to a point \( z \). Let \( y_n \) be the preimage of \( z_n \) and \( \ell_n \) the corresponding fiber. Since \( z_n \) lies on \( \ell_n \) and \( z_n \to z \), \( |\ell_n| \) is a bounded sequence, so by the second hypothesis, the sequence \( y_n \) has a subsequence \( y_{n_k} \) converging to \( y \). By continuity of \( g \), \( z_{n_k} = g(y_{n_k}) \to g(y) \). But \( z_{n_k} \) also approaches \( z \), so \( g(y) = z \). Hence \( g(P_0) \) is a closed set, so \( g(P_0) = P \).

The final statement of the theorem follows from Lemma 12. \( \square \)

The idea of the proof of Theorem 3 is as follows. Given a \((1, 3)\)-fibration, we may always find an oriented 2-plane transverse to all the fibers and define \( B \) accordingly. A map \( B \) constructed in this way is not necessarily unique. Unless the set of directions is an open hemisphere, there are many choices for the transverse plane \( P \). Additionally, shifting \( P \) in the normal direction yields a different \( B \). We showed in Lemma 13 that any map \( B \) with the properties of Theorem 2 is homotopic to the similarly-oriented Hopf fibration defined on the same plane. Recall that the set of Hopf fibrations is a disjoint union of two copies of \( S^2 \). To exhibit the deformation retract, we only must show that the Hopf fibration at the end of the homotopy can be chosen to depend continuously on the choice of the \((1, 3)\)-fibration.

**Proof of Theorem 3** Fix a \((1, 3)\)-fibration \( M \). The image \( U \) of the map \((u, v) \mapsto u \) is a convex subset of \( S^2 \). We may define the map \( c \) to take \( M \) to the circumsphere of the set \( U \); that is, the center of the smallest ball containing \( U \). This map varies continuously with \( M \).
homotope to the similarly-oriented Hopf fibration on \( \tilde{v} \) with trivial kernel. Consider each of the first 2 columns as a vector in \( \mathbb{F}^{\mathbb{F}^q} \) and project to the \( \mathbb{F} \)-tangent space at \( y \). If the projections are \( \mathbb{F} \)-linearly dependent, then some \( \mathbb{F} \)-linear combination is equal to some \( \mathbb{F} \)-multiple of \( y \), contradicting that \( A(y) \) has trivial kernel.

6 Complex and Quaternionic skew fibrations

Proof of Theorem 5. We only must extend the idea of Example 4. Consider the fiber \( \mathbb{F}^p \) through the origin of \( \mathbb{F}^n \) and consider a small copy \( S \) of the sphere in the orthogonal \( \mathbb{F}^q \). By (the \( \mathbb{F} \)-generalization of) Lemma 10, we have, for each \( y \in S \), a \( q \times (p + 1) \) matrix \( A(y) \) with trivial kernel. Consider each of the first \( p \) columns as a vector in \( \mathbb{F}^{\mathbb{F}^q} \) and project to the \( \mathbb{F} \)-tangent space at \( y \). If the projections are \( \mathbb{F} \)-linearly dependent, then some \( \mathbb{F} \)-linear combination is equal to some \( \mathbb{F} \)-multiple of \( y \), contradicting that \( A(y) \) has trivial kernel.

We observe that \( p \) linearly independent complex vector fields on \( S^{2q-1} \) induce \( 2p + 1 \) linearly independent real vector fields on \( S^{2r-1} \): the \( 2p \) that come directly from the \( p \) complex vector fields, along with the “Hopf” vector field obtained by multiplying the normal vector by \( i \). Arguing similarly in the quaternionic case, we have the following results.

- A \( \mathbb{C}-(p, n) \)-fibration may exist only if an \( \mathbb{R}-(2p + 1, 2n + 1) \)-fibration exists.
- An \( \mathbb{H}-(p, n) \)-fibration may exist only if a \( \mathbb{C}-(2p + 1, 2n + 1) \)-fibration exists.
- An \( \mathbb{H}-(p, n) \)-fibration may exist only if an \( \mathbb{R}-(2p + 3, 2n + 3) \)-fibration exists.

Though interesting to note, these obstructions are less restrictive than those in Corollaries 6 and 7 which we now examine carefully. Starting with the complex case, we have the power series:

\[
\frac{t}{\ln(1 + t)} = 1 + \frac{1}{2}t - \frac{1}{12}t^2 + \frac{1}{24}t^3 - \frac{19}{720}t^4 + \frac{9}{160}t^5 + \ldots,
\]

\[
\left( \frac{t}{\ln(1 + t)} \right)^2 = 1 + t + \frac{1}{12}t^2 - \frac{1}{240}t^4 + \frac{1}{240}t^5 + \ldots.
\]

For the coefficient of \( t \) in the \( q \)th power of the first series to be integral, it is necessary that \( q \) is even. Hence it suffices to consider powers of the latter series, the \( r \)th power of which is

\[
1 + rt + \frac{6r^2 - 5r^2}{12}t^2 + \frac{2r^3 - 5r^2 + 3r^3}{12}t^3 + \frac{60r^4 - 300r^3 + 485r^2 - 251r}{1440}t^4
\]

\[+ \frac{12r^5 - 100r^4 + 305r^3 - 401r^2 + 190r}{1440}t^5 + \ldots.\]

To obtain possible dimensions \( n \) for a \( \mathbb{C}-(p, n) \)-fibration, we seek values of \( q = 2r \) for which the coefficients of the first \( p \) powers of \( t \) are integral. For example, a fibration by complex lines may exist for any \( r \), or equivalently any even \( q \). Specifically, such fibrations take the form \((1, 2k + 1)\). The coefficient of \( t^2 \) is integral if and only if \( r \) is divisible by 12 (one may check 3, and 2, then 4), equivalently \( q \) by 24. This condition is also sufficient for the coefficient of \( t^3 \) to be integral.

Next, we seek multiples of 12 that make \( 60r^4 - 300r^3 + 485r^2 - 251r \) divisible by 1440 = \( 2^5 \cdot 3^2 \cdot 5 \). Reduction mod 5 yields \( r \equiv 0 \mod 5 \). Reduction mod 8 gives \( 4r^4 + 4r^3 + 5r^2 + 5r \); testing 0 and 4 yields \( r \equiv 0 \mod 8 \). Similar arguments show divisibility by 9, 16, and 32, so \( r \) must be divisible by 1440. Observe that this condition is also sufficient for the coefficient of \( t^5 \) to be integral.
In the quaternionic case we have the power series:

\[
\left(\frac{2}{\sqrt{t}} \cdot \sinh^{-1} \frac{\sqrt{t}}{2}\right)^2 = 1 - \frac{1}{12} t + \frac{1}{90} t^2 - \frac{1}{560} t^3 + \frac{1}{3150} t^4 - \frac{1}{16632} t^5 + \cdots ,
\]

\[
\left(\frac{2}{\sqrt{t}} \cdot \sinh^{-1} \frac{\sqrt{t}}{2}\right)^{2r} = 1 - \frac{1}{12} r t + \frac{11r + 5r^2}{1440} t^2 + \frac{-382r - 231r^2 - 35r^3}{362880} t^3 + \frac{14982r + 10181r^2 + 2310r^3 + 175r^4}{87091200} t^4 + \cdots .
\]

The strategy outlined above for testing divisibility is algorithmic and hence amenable to the use of software. In the table below, we use the notation of James (see the discussion following the statement of Theorem 5), although our index differs from his by 1. The interpretation of the table is that a \( C(p,n) \)-fibration (resp. \( H(p,n) \)-fibration) may exist only if \( n \) is of the form \( b_p k + p \) (resp. \( c_p k + p \) for \( k \in \mathbb{N} \)). We use \( a_p \) as the corresponding number for an \( R(p,n) \)-fibration (cf. the table in [15]).

| \( p \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| \( a_p \) | 2 | 4 | 4 | 8 | 8 | 8 | 8 | 16 |
| \( b_p \) | 24 | 24 | 2880 | 2880 | 362880 | 362880 | 362880 | 29030400 |
| \( c_p \) | 24 | 1440 | 362880 | 14515200 | 958003200 | 1.57 \times 10^{13} | 6.28 \times 10^{13} | 2.56 \times 10^{17} |

| \( p \) | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|
| \( a_p \) | 32 | 64 | 64 | 128 | 128 | 128 |
| \( b_p \) | 29030400 | 958003200 | 958003200 | 3.14 \times 10^{13} | 3.14 \times 10^{13} | 6.28 \times 10^{13} |
| \( c_p \) | 9.20 \times 10^{20} | 1.01 \times 10^{24} | 9.31 \times 10^{25} | 6.10 \times 10^{29} | 1.22 \times 10^{30} | 2.12 \times 10^{33} |

Several comments are in order. The bold numbers correspond to fibrations that are known to exist. The numbers multiplied by powers of 10 are estimates. Restrictions on possible dimensions of complex and quaternionic fibrations are much more severe than in the real case; however, the complex and quaternionic are related. As Sigrist and Suter note in [21], we always have either \( c_p = b_{2p+1} \) or \( c_p = \frac{1}{2} b_{2p+1} \), with the latter equality always true for even \( p \).

The intersection of a local \((p,n)\)-fibration with a hyperplane transverse to all fibers yields a local \((p-1,n-1)\)-fibration. This leads Ovsienko and Tabachnikov to define a dominant \((p,n)\)-fibration as one for which a \((p+1,n+1)\)-fibration does not exist. For example, an \((8,24)\)-fibration is dominant since there is no \((9,25)\)-fibration, yet there is a chain of fibrations dimensions from \((7,23)\) to \((1,17)\). This idea extends to local \( F(p,n) \)-fibrations.

The example below gives an explicit construction of a \( C(1,2k+1) \)-fibration.

**Example 5.** We begin by constructing a \( C(1,3) \)-fibration. Define \( B : \mathbb{C}^2 \to \mathbb{C}^2 : (y_1, y_2) \mapsto (\overline{y_2}, -\overline{y_1}) \). In particular, the fiber through \( y \in \mathbb{C}^2 \) is the graph of the map \( \mathbb{C}^1 \to \mathbb{C}^2 : t \mapsto B(y)t + y \). We must show that for every \((t, \eta) \in \mathbb{C}^1 \times \mathbb{C}^2 = \mathbb{C}^4\), there is a unique \( y \in \mathbb{C}^2 \) such that the fiber through \( y \) passes through \((t, \eta)\). In particular, that there exists \( y \) with

\[
\begin{pmatrix}
\overline{y_2} & y_1 \\
-\overline{y_1} & y_2
\end{pmatrix}
\begin{pmatrix}
t \\
1
\end{pmatrix}
= \begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix}.
\]

Equating \( C \) with \( \mathbb{R}^2 \) allows us to rewrite (2) as

\[
\begin{pmatrix}
1 & 0 & \text{Re } t & \text{Im } t \\
0 & 1 & \text{Im } t & -\text{Re } t \\
-\text{Re } t & -\text{Im } t & 1 & 0 \\
-\text{Im } t & \text{Re } t & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\text{Re } y_1 \\
\text{Im } y_1 \\
\text{Re } y_2 \\
\text{Im } y_2
\end{pmatrix}
= \begin{pmatrix}
\text{Re } \eta_1 \\
\text{Im } \eta_1 \\
\text{Re } \eta_2 \\
\text{Im } \eta_2
\end{pmatrix}.
\]
The determinant of the matrix above is $(1 + |t|^2)^2$, so it is an isomorphism for every $t$. Hence a choice of $t$ and $\eta$ guarantee the existence of a unique $y$.

This construction extends to the $C-(1, n+1)$ case for even $n$. Consider the map

$$B : (y_1, y_2, y_3, y_4, \ldots, y_{n-1}, y_n) \mapsto (y_2, -y_1, y_4, -y_3, \ldots, y_n, -y_{n-1}).$$

Generalizing yields a $2n \times 2n$ matrix with the above $4 \times 4$ block appearing down the diagonal, and zeros elsewhere. This matrix has determinant $(1 + |t|^2)^n$, so indeed this map yields a $C-(1, n+1)$-fibration.

There is an obvious question of whether the converse to Theorem 5 is true. In particular, is the existence of $p$ linearly independent sections of the $\mathbb{F}$-tangent bundle on the unit sphere in $\mathbb{F}^q$ a sufficient condition for the existence of an $\mathbb{F}-(p, n)$-fibration?

Aside from constructing an $\mathbb{F}-(p, n)$-fibration directly from linearly independent sections, one possible method of attack is as follows. By generalizing Theorem 1 one could give a bundle-theoretic condition for the existence of an $\mathbb{F}-(p, n)$-fibration. It may be possible that methods similar to those of Adams-Walker and Sigrist-Suter could then be used to study the existence problem in the complex and quaternionic cases.

We conclude by briefly calling attention to a related notion: totally skew embeddings of manifolds into Euclidean space. A submanifold is called totally skew if the tangent spaces at any pair of distinct points are skew (see [5], [8], [22], [23], and most recently, [6] and [7]; the latter deals with complex skew embeddings.)

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