Variational inequalities for the fractional Laplacian

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Abstract

In this paper we study the obstacle problems for the fractional Laplacian of order $s \in (0, 1)$ in a bounded domain $\Omega \subset \mathbb{R}^n$, under mild assumptions on the data.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 1$. Given $s \in (0, 1)$, a measurable function $\psi$ and a distribution $f$ on $\Omega$, we consider the problem

\[
\begin{cases}
  u \geq \psi & \text{in } \Omega \\
  (-\Delta)^s u \geq f & \text{in } \Omega \\
  (-\Delta)^s u = f & \text{in } \{u > \psi\} \\
  u = 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

Our interest is motivated by the noticeable paper [19], where Louis E. Silvestre investigated (1.1) in case $\Omega = \mathbb{R}^n$, $f = 0$ and $\psi$ smooth. His results apply also to Dirichlet’s problems on balls, see [19, Section 1.3]. Besides remarkable results, in [19] the interested reader can find stimulating motivations for (1.1), arising from

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 mathematical finance. In addition, Signorini’s problem, also known as the lower dimensional obstacle problem for the classical Laplacian, can be recovered from (1.1) by taking $s = \frac{1}{2}$.

Among the papers dealing with (1.1) and related problems we cite also [1, 3, 4, 7, 15, 18] and references there-in, with no attempt to provide a complete reference list.

In the present paper we show that the free boundary problem (1.1) admits a solution under quite mild assumptions on the data, see Theorems 1.1 and 1.2 below. However, our starting interest included broader questions concerning the variational inequality

$$u \in K^s_{\psi}, \quad \langle (-\Delta)^s u - f, v - u \rangle \geq 0 \quad \forall v \in K^s_{\psi}, \quad (P(\psi, f))$$

where $f \in \tilde{H}^s(\Omega)'$ and

$$K^s_{\psi} = \left\{ v \in \tilde{H}^s(\Omega) \mid v \geq \psi \text{ a.e. on } \Omega \right\}.$$

Notation and main definitions are listed at the end of this introduction. We will always assume that the closed and convex set $K^s_{\psi}$ is not empty, also when not explicitly stated.

Problem $[P(\psi, f)]$ admits a unique solution $u$, that can be characterized as the unique minimizer for

$$\inf_{v \in K^s_{\psi}} \frac{1}{2} \langle (-\Delta)^s v, v \rangle - \langle f, v \rangle. \quad (1.2)$$

The variational inequality $[P(\psi, f)]$ and the free boundary problem (1.1) are naturally related. Any solution $u \in \tilde{H}^s(\Omega)$ to (1.1) coincides with the unique solution to $[P(\psi, f)]$ see Remark 3.5. Conversely, if $u$ solves $[P(\psi, f)]$ then $(-\Delta)^s u - f$ is a nonnegative distribution on $\Omega$, compare with Theorem 3.2. By analogy with the local case $s = 1$ one can guess that $(-\Delta)^s u = f$ outside the coincidence set $\{u = \psi\}$, at least when $u$ is regular enough. This is essentially the content of Section 3 in [19], where $f = 0$ and $\psi$ is a smooth, rapidly decreasing function on $\Omega = \mathbb{R}^n$, and of Theorems 1.1, 1.2 below.

To study the variational inequality $[P(\psi, f)]$ we took inspiration from the classical theory about the local case $s = 1$. In particular, we refer to the fundamental monograph [9] by Kinderlehrer and Stampacchia, and to the pioneering papers [2, 10, 11, 12, 13, 20, 21], among others.
Standard techniques do not apply directly in the fractional case, mostly because of the different behavior of the truncation operator $v \mapsto v^+$, $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$. Section 2 is entirely devoted to this subject; we collect there some lemmata that might have an independent interest.

We take advantage of the results in Section 2 to obtain equivalent and useful formulations for $\mathbb{P}(\psi, f)$ and to prove continuous dependence theorems upon the data $f$ and $\psi$, see Sections 3 and 4, respectively.

Some extra difficulties arise from having settled a nonlocal problem on a bounded domain, producing at least, but not only, the same (partially solved) technical difficulties as for the unconstrained problem $(-\Delta)^s u = f$, $u \in \tilde{H}^s(\Omega)$ (see for instance [6], [16], [17] and references there-in, for regularity issues).

Our main results proved in Section 5. They involve the unique solution $\omega_f$ to

$$(-\Delta)^s \omega_f = f \quad \text{in } \Omega, \quad \omega_f \in \tilde{H}^s(\Omega).$$

**Theorem 1.1** Assume that $\psi$ and $f \in \tilde{H}^s(\Omega)'$ satisfy the following conditions:

A1) $(\psi - \omega_f)^+ \in \tilde{H}^s(\Omega)$;

A2) $(-\Delta)^s(\psi - \omega_f)^+ - f$ is a locally finite signed measure on $\Omega$;

A3) $((-\Delta)^s(\psi - \omega_f)^+ - f)^+ \in L^p_{\text{loc}}(\Omega)$ for some $p \in [1, \infty]$.

Let $u \in \tilde{H}^s(\Omega)$ be the unique solution to $\mathbb{P}(\psi, f)$. Then the following facts hold.

i) $(-\Delta)^s u - f \in L^p_{\text{loc}}(\Omega)$;

ii) $0 \leq (-\Delta)^s u - f \leq ((-\Delta)^s(\psi - \omega_f)^+ - f)^+$ a.e. on $\Omega$;

iii) $(-\Delta)^s u = f$ a.e. on $\{u > \psi\}$.

In particular, $u$ solves the free boundary problem (1.1).

**Theorem 1.2** Assume that $\Omega$ is a bounded Lipschitz domain satisfying the exterior ball condition. Let $\psi \in C^0(\Omega)$ be a given obstacle, such that $K^\psi_{\psi}$ is not empty, $\psi \leq 0$ on $\partial \Omega$ and $f \in L^p(\Omega)$, for some exponent $p > n/2s$.

Then the unique solution $u$ to $\mathbb{P}(\psi, f)$ is continuous on $\mathbb{R}^n$ and solves the free boundary problem (1.1).
Our results plainly cover the non-homogeneous Dirichlet’s free boundary problem
\[
\begin{cases}
  u \geq \psi & \text{in } \Omega \\
  (-\Delta)^s u \geq f & \text{in } \Omega \\
  (-\Delta)^s u = f & \text{in } \{u > \psi\} \\
  u = g & \text{in } \mathbb{R}^n \setminus \overline{\Omega},
\end{cases}
\]
under appropriate assumptions on the datum \(g\). Notice indeed that \(u\) solves the related variational inequality if and only if \(u - g\) solves \(\mathcal{P}(\psi - g, f + (-\Delta)^s g)\).

Free boundary problems for the operator \((-\Delta)^s u + u\) can be considered as well, with minor modifications in the statements and in the proofs.

**Notation** The definition of the fractional Laplacian \((-\Delta)^s\) involves the Fourier transform:
\[
\mathcal{F}[( - \Delta)^s u] = |\xi|^{2s} \mathcal{F}[u], \quad \mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i \xi \cdot x} u(x) \, dx.
\]
Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain. We adopt the standard notation
\[
H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid (-\Delta)^s u \in L^2(\mathbb{R}^n)\},
\]
\[
\tilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^n) \mid u \equiv 0 \text{ on } \mathbb{R}^n \setminus \overline{\Omega}\}.
\]
We endow \(H^s(\mathbb{R}^n)\) and \(\tilde{H}^s(\Omega)\) with their natural Hilbertian structures. We recall that the norm of \(u\) in \(\tilde{H}^s(\Omega)\) is given by the \(L^2(\mathbb{R}^n)\)-norm of \((-\Delta)^s u\).

We do not make any assumption on \(\Omega\). Thus \(\partial \Omega\) might be very irregular, even a fractal, and \(C^\infty(\Omega)\) might be not dense in \(\tilde{H}^s(\Omega)\). Notice that \(\tilde{H}^s(\Omega)\) coincides with \(\tilde{H}^s(\Omega')\), whenever \(\overline{\Omega} = \overline{\Omega'}\).

We denote by \((\cdot, \cdot)\) the duality product between \(\tilde{H}^s(\Omega)\) and its dual \(\tilde{H}^s(\Omega)'\). In particular, \((-\Delta)^s u \in \tilde{H}^s(\Omega)'\) for any \(u \in \tilde{H}^s(\Omega)\), and
\[
\langle (-\Delta)^s u, v \rangle = \int_{\mathbb{R}^n} (-\Delta)^s u \cdot (-\Delta)^s v \, dx = \int_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F}[u] \overline{\mathcal{F}[v]} \, d\xi.
\]

## 2 Truncations

For measurable functions \(v, w\) we put, as usual,
\[
\begin{align*}
v \vee w &= \max\{v, w\}, & v \wedge w &= \min\{v, w\}, & v^+ &= v \vee 0, & v^- &= -(v \wedge 0),
\end{align*}
\]
so that \(v = v^+ - v^-\). It is well known that \(v \vee w \in H^s(\mathbb{R}^n)\) and \(v \wedge w \in H^s(\mathbb{R}^n)\) if \(v, w \in H^s(\mathbb{R}^n)\).
Lemma 2.1 Let $v \in H^s(\mathbb{R}^n)$. Then

\[ \langle (-\Delta)^s v^+, v^- \rangle = \langle (-\Delta)^s v^-, v^+ \rangle \leq 0; \]

\[ \langle (-\Delta)^s v, v^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v^-|^2 \, dx \leq 0; \]

\[ \langle (-\Delta)^s v, v^+ \rangle - \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v^+|^2 \, dx \geq 0. \]

In addition, if $v \in H^s(\mathbb{R}^n)$ does not have constant sign, then all the above inequalities are strict.

Proof. In [14, Theorem 6], the Caffarelli-Silvestre extension argument [5] has been used to check that

\[ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v^+|^2 \, dx < \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v^-|^2 \, dx, \]

whenever $v$ changes sign. That is,

\[ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v^+ + v^-)|^2 \, dx < \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v^+ - v^-)|^2 \, dx. \]

The conclusion is immediate. \qed

Remark 2.2 One can use ii) in Lemma 2.1 to get the well known weak maximum principle, that is, if $u \in \tilde{H}^s(\Omega)$ and $(-\Delta)^s u \geq 0$ in $\Omega$ then $u \geq 0$ in $\Omega$.

Corollary 2.3 Let $v_h$ be a sequence in $H^s(\mathbb{R}^n)$ such that $v_h$ converges to a nonpositive function in $H^s(\mathbb{R}^n)$. Then $v_h^+ \to 0$ in $H^s(\mathbb{R}^n)$.

Proof. Statement iii) in Lemma 2.1 provides the estimate

\[ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v_h^+|^2 \, dx \leq \langle (-\Delta)^s v_h, v_h^+ \rangle, \quad (2.1) \]

that gives us the boundedness of the sequence $v_h^+$ in $H^s(\mathbb{R}^n)$. Since $v_h^+ \to 0$ in $L^2(\mathbb{R}^n)$, we have $v_h^+ \to 0$ weakly in $H^s(\mathbb{R}^n)$. Thus $\langle (-\Delta)^s v_h, v_h^+ \rangle \to 0$, as $(-\Delta)^s v_h$ converges in $H^s(\mathbb{R}^n)'$, and the conclusion follows from (2.1). \qed
Lemma 2.4 Let $v \in \tilde{H}^s(\Omega)$ and $m > 0$. Then $(v + m)^-, (v - m)^+, v \wedge m \in \tilde{H}^s(\Omega)$ and

\[ i) \quad \langle (-\Delta)^s v, (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^s (v + m)^-|^2 \, dx \leq 0; \]

\[ ii) \quad \langle (-\Delta)^s v, (v - m)^+ \rangle - \int_{\mathbb{R}^n} |(-\Delta)^s (v - m)^+|^2 \, dx \geq 0; \]

\[ iii) \quad \int_{\mathbb{R}^n} |(-\Delta)^s (v \wedge m)|^2 \, dx \leq \int_{\mathbb{R}^n} |(-\Delta)^s v|^2 \, dx - \int_{\mathbb{R}^n} |(-\Delta)^s (v - m)^+|^2 \, dx. \]

Proof. Clearly, $(v + m)^- \in L^2(\mathbb{R}^n)$ and $(v + m)^- \equiv 0$ outside $\Omega$. Fix a cutoff function $\eta \in C_0^\infty(\mathbb{R}^n)$, with $0 \leq \eta \leq 1$, and such that $\eta \equiv 1$ in a ball containing $\overline{\Omega}$. Then $(v + m)^- = (v + m\eta)^- \in \tilde{H}^s(\Omega)$, as trivially $m\eta \in H^s(\mathbb{R}^n)$.

For any integer $h \geq 1$ we set

\[ \eta_h(x) = \eta\left(\frac{x}{h}\right), \]

so that $\eta_h \to 1$ pointwise. A direct computation shows that

\[ (-\Delta)^s \eta_h(x) = h^{-2s}(-\Delta)^s \eta\left(\frac{x}{h}\right) \to 0 \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^n). \quad (2.2) \]

By $ii)$ in Lemma 2.1 we have that

\[
0 \geq \langle (-\Delta)^s (v + m\eta_h), (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^s (v + m)^-|^2 \, dx
\]

\[
= \langle (-\Delta)^s v, (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^s (v + m)^-|^2 \, dx + m \int_{\mathbb{R}^n} ((-\Delta)^s \eta)(v + m)^- \, dx
\]

\[
= \langle (-\Delta)^s v, (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^s (v + m)^-|^2 \, dx + o(1),
\]

by (2.2) and since $(v + m)^-$ has compact support in $\Omega$. Claim $i)$ is proved. To check $ii)$ notice that $(v - m)^+ = ((-v) + m)^-$ and then use $i)$ with $(-v)$ instead of $v$.

It remains to prove $iii)$. Notice that $v \wedge m = v - (v - m)^+$. Hence $v \wedge m \in \tilde{H}^s(\Omega)$.

Using $ii)$ we get

\[
\| (-\Delta)^s (v \wedge m) \|^2 = \| (-\Delta)^s v \|^2 - 2\langle (-\Delta)^s v, (v - m)^+ \rangle + \| (-\Delta)^s (v - m)^+ \|^2
\]

\[
\leq \| (-\Delta)^s v \|^2 - \| (-\Delta)^s (v - m)^+ \|^2.
\]

The proof is complete. \qed

6
3 Equivalent formulations

We start this section by introducing a crucial notion.

**Definition 3.1** A function $U \in \tilde{H}^s(\Omega)$ is a supersolution for $(-\Delta)^s v = f$ if

$$\langle (-\Delta)^s U - f, \varphi \rangle \geq 0 \quad \text{for any } \varphi \in \tilde{H}^s(\Omega), \varphi \geq 0.$$  

The above definition extends the usually adopted one in the local case $s = 1$, see [9, Definition 6.3]. A different definition of supersolution is used in [19] for $f = 0$. We refer to [19, Subsection 2.10], for a stimulating discussion on this subject.

**Theorem 3.2** Let $u \in K^s_\psi$. The following sentences are equivalent.

a) $u$ is the solution to problem $P(\psi, f)$.

b) $u$ is the smallest supersolution for $(-\Delta)^s v = f$ in the convex set $K^s_\psi$. That is, $U \geq u$ almost everywhere in $\Omega$, for any supersolution $U \in K^s_\psi$.

c) $u$ is a supersolution for $(-\Delta)^s v = f$ and

$$\langle (-\Delta)^s u - f, (v - u)^- \rangle = 0 \quad \text{for any } v \in K^s_\psi.$$  

d) $\langle (-\Delta)^s v - f, v - u \rangle \geq 0$ for any $v \in K^s_\psi$.

**Proof.** a) $\iff$ b). Assume that $u$ solves $P(\psi, f)$. Fix any nonnegative $\varphi \in \tilde{H}^s(\Omega)$. Testing $P(\psi, f)$ with $u + \varphi \in K^s_\psi$ one gets $\langle (-\Delta)^s u - f, \varphi \rangle \geq 0$, that proves that $u$ is a supersolution.

Next, take any supersolution $U \in K^s_\psi$. Then $u - (u - U)^+ = U \wedge u \in K^s_\psi$. Thus

$$\langle (-\Delta)^s u - f, -(u - U)^+ \rangle \geq 0.$$  

On the other hand, from $(-\Delta)^s U - f \geq 0$ we get

$$\langle (-\Delta)^s U - f, (u - U)^+ \rangle \geq 0.$$  

Adding the above inequalities we arrive at

$$0 \geq \langle (-\Delta)^s (u - U), (u - U)^+ \rangle \geq \int_{\mathbb{R}^n} |(-\Delta)^s (u - U)^+|^2 \, dx,$$  

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thanks to \( iii \) in Lemma 2.1. Thus \((u - U)^+ = 0\) almost everywhere in \( \Omega \), that is, \( u \leq U \) and proves that \( a) \) implies \( b) \).

Conversely, assume that \( u \) satisfies \( b) \) and let \( \tilde{u} \) be the solution to \( P(\psi, f) \). We already know that \( a) \Rightarrow b) \). Thus \( u \) and \( \tilde{u} \) must coincide, because both obey the condition of being the smallest supersolution to \((-\Delta)^su = f\) in \( K^s\psi \). Hence, \( a) \) holds.

\( a) \iff c) \). Let \( u \) be the solution to \( P(\psi, f) \). We already know that \( u \) is supersolution. Fix any function \( v \in K^s\psi \). Notice that

\[
(u + (v - u)^-) \geq u \geq \psi , \quad u - (v - u)^- = v \wedge u \geq \psi .
\]

Thus, testing \( P(\psi, f) \) with \( u \pm (v - u)^- \) we get \( \langle (-\Delta)^su, \pm(v - u)^- \rangle \geq 0 \), that is, \( c) \) holds.

Conversely, assume that \( u \) satisfies \( c) \). Let \( \tilde{u} \in K^s\psi \) be the solution to \( P(\psi, f) \). We already proved that \( \tilde{u} \) is the smallest supersolution in \( K^s\psi \). In particular, \( \tilde{u} \leq u \) and thus

\[
\langle (-\Delta)^su - f, u - \tilde{u} \rangle = 0
\]

by the assumption \( c) \) on \( u \). Since \( \tilde{u} \) solves \( P(\psi, f) \) we also get

\[
\langle (-\Delta)^s\tilde{u} - f, u - \tilde{u} \rangle \geq 0 .
\]

Subtracting, we infer \( \langle (-\Delta)^s(u - \tilde{u}), u - \tilde{u} \rangle \leq 0 \), that is, \( u = \tilde{u} \).

\( a) \iff d) \). Clearly \( a) \) implies \( d) \) because

\[
\langle (-\Delta)^sv - f, v - u \rangle = \langle (-\Delta)^su - f, v - u \rangle + \langle (-\Delta)^s(v - u), v - u \rangle \geq \langle (-\Delta)^su - f, v - u \rangle .
\]

Now assume that \( u \) satisfies \( d) \) and fix any \( v \in K^s\psi \). From \( \frac{u + v}{2} \in K^s\psi \) and \( d) \) we obtain

\[
0 \leq 2\langle (-\Delta)^s\left(\frac{v + u}{2}\right) - f, \frac{v + u}{2} - u \rangle = \frac{1}{2}\langle (-\Delta)^s(v + u), v - u \rangle - \langle f, v - u \rangle
\]

\[
= \left(\frac{1}{2}\langle (-\Delta)^sv, v \rangle - \langle f, v \rangle\right) - \left(\frac{1}{2}\langle (-\Delta)^su, u \rangle - \langle f, u \rangle\right) .
\]

Thus \( u \) solves the minimization problem [1.2], that is, \( u \) solves \( P(\psi, f) \) \( \square \)
Adding and taking $i$-th derivative, we can use $v_i$-th derivative of $P$. On the other hand, we can test $u_i$ to be the solution to $P(\psi, f_i)$, $i = 1, 2$. If $f_1 \geq f_2$ in the sense of distributions, then $u_1 \geq u_2$ a.e. in $\Omega$.

**Proof.** The function $u_1$ is a supersolution for $(-\Delta)^s u = f_2$ and $u_1 \in K^s_\psi$. Hence $u_1 \geq u_2$, by statement b) in Theorem 3.2. \hfill $\Box$

**Remark 3.5** Let $u \in \bar{H}^s(\Omega)$ be a solution to (1.1). Then $(-\Delta)^s u - f$ can be identified with a nonnegative Radon measure on $\Omega$ having support in $\{u = \psi\}$. If $v \in K^s_\psi$, then $(v - u)^- \text{ vanishes on } \{u = \psi\}$. Thus $\langle (-\Delta)^s u - f, (v - u)^- \rangle = 0$, hence $u$ solves $P(\psi, f)$ by Theorem 3.3.

4 Continuous dependence results

**Theorem 4.1** Let $\psi_1, \psi_2$ be given obstacles, $f \in \bar{H}^s(\Omega)'$ and let $u_i$ be the solution to $P(\psi, f)$, $i = 1, 2$. If $\psi_1 - \psi_2 \in L^\infty(\Omega)$, then $u_1 - u_2$ is bounded, and

\begin{align*}
i)\, \| (u_1 - u_2)^+ \|_\infty &\leq \| (\psi_1 - \psi_2)^+ \|_\infty, \quad \text{ii) } \| (u_1 - u_2)^- \|_\infty \leq \| (\psi_1 - \psi_2)^- \|_\infty. \end{align*}

**Proof.** Put $m := \| (\psi_1 - \psi_2)^+ \|_\infty$. Since $(u_2 - u_1 + m)^- \in \bar{H}^s(\Omega)$ by Lemma 2.4, then

$v_1 := u_1 - (u_2 - u_1 + m)^- = (u_2 + m) \land u_1 \in K^s_{\psi_1}.$

Hence we can use $v_1$ as test function in $P(\psi_1, f)$ to get

$\langle (-\Delta)^s u_1 - f, -(u_2 - u_1 + m)^- \rangle \geq 0.$

On the other hand, we can test $P(\psi_2, f)$ with $u_2 + (u_2 - u_1 + m)^- \in K^s_{\psi_2}$. Hence

$\langle (-\Delta)^s u_2 - f, (u_2 - u_1 + m)^- \rangle \geq 0.$

Adding and taking i) of Lemma 2.4 into account, we arrive at

$-\int_{\mathbb{R}^n} |(-\Delta)^s (u_2 - u_1 + m)^-|^2 \, dx \geq \langle (-\Delta)^s (u_2 - u_1), (u_2 - u_1 + m)^- \rangle \geq 0.$

Hence, $(u_2 - u_1 + m)^- = 0$. We have proved that $(u_1 - u_2)^+ \leq m$ a.e. in $\Omega$, hence i) holds. Inequality ii) can be proved in the same way. \hfill $\Box$
Corollary 4.2 Let $\psi \in L^\infty(\Omega)$ and $f \in L^p(\Omega)$, with $p \in (1, \infty)$, $p > n/2s$. Let $u \in \tilde{H}^s(\Omega)$ be the unique solution to $[\mathcal{P}(\psi, f)]$. Then $u \in L^\infty(\Omega)$ and

$$\psi \vee \omega_f \leq u \leq \|\psi^+\|_\infty + c\|f^+\|_p \quad \text{a.e. in } \Omega,$$

where $\omega_f$ solves (1.3) and $c$ depends only on $n, s, p$ and $\Omega$. In particular, if $f = 0$ then

$$\psi^+ \leq u \leq \|\psi^+\|_\infty.$$

Proof. First of all, notice that $f \in \tilde{H}^s(\Omega)'$ by Sobolev embedding theorem. Since $u$ is supersolution of (1.3), the first inequality in (4.1) follows by the maximum principle in Remark 2.2.

Denote by $\omega_{f^+}$ the unique solution to (1.3) with $f$ replaced by $f^+$. If $n > 2s$ we use convolution to define

$$U = c_1 |x|^{2s-n} * (f^+ \cdot \chi).$$

For proper choice of the constant $c_1$, $U$ solves $(-\Delta)^s U = f^+ \cdot \chi$ in $\mathbb{R}^n$. Convolution estimates give $U \leq c\|f^+\|_p$ on $\mathbb{R}^n$. By the maximum principle, $\omega_{f^+} \leq U$ on $\Omega$, hence $\omega_{f^+} \leq c\|f^+\|_p$. For $n = 1 \leq 2s$ this inequality also holds, see, e.g., [16, Remark 1.5].

Now let $u_1$ be the unique solution of $\mathcal{P}(\psi, f^+)$. Then $u_1 \geq u$ by Corollary 3.4. Finally, we can consider $\omega_{f^+}$ as the solution of the problem $\mathcal{P}(\omega_{f^+}, f^+)$. Theorem 4.1 gives

$$u \leq (u_1 - \omega_{f^+})^+ + \omega_{f^+} \leq \|\psi - \omega_{f^+})^+\|_\infty + \omega_{f^+},$$

and the last inequality in (4.1) follows. □

Roughly speaking, Theorem 4.1 concerns the continuity of $L^\infty \ni \psi \mapsto u \in L^\infty$. The next result gives the continuity of the arrow $L^\infty \ni \psi \mapsto u \in \tilde{H}^s(\Omega)$.

Theorem 4.3 Let $\psi_h \in L^\infty(\Omega)$ be a sequence of obstacles and let $f \in \tilde{H}^s(\Omega)'$ be given. Assume that there exists $v_0 \in \tilde{H}^s(\Omega)$, such that $v_0 \geq \psi_h$ for any $h$.

Denote by $u_h$ the solution to the obstacle problem $\mathcal{P}(\psi_h, f)$. If $\psi_h \to \psi$ in $L^\infty(\Omega)$, then $u_h \to u$ in $\tilde{H}^s(\Omega)$, where $u$ is the solution to the limiting problem $\mathcal{P}(\psi, f)$.

Proof. Let $u$ be the solution to $[\mathcal{P}(\psi, f)]$. We already know from Theorem 4.1 that $\|u - u_h\|_\infty \leq \|\psi - \psi_h\|_\infty$. Hence, in particular, $u_h \to u$ a.e. in $\Omega$. Now, test $\mathcal{P}(\psi_h, f)$
with $v_0$ to obtain that
\[
\langle (-\Delta)^s u_h, u_h \rangle \leq \langle (-\Delta)^s u_h - f, v_0 \rangle + \langle f, u_h \rangle.
\]

Hence, the sequence $u_h$ is bounded in $\tilde{H}^s(\Omega)$. Therefore, $u_h \to u$ weakly in $\tilde{H}^s(\Omega)$. To prove that $u_h \to u$ in the $\tilde{H}^s(\Omega)$ norm we only need to show that
\[
\limsup_{h \to \infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2 \leq \| (-\Delta)^{\frac{s}{2}} u \|_2.
\]

For any $\varepsilon > 0$ we introduce the function
\[
v_\varepsilon = u + (v_0 - u) \wedge \varepsilon.
\]

Since $\psi_h \to \psi$ in $L^\infty(\Omega)$, we have $v_\varepsilon \geq \psi_h$ for $h$ large enough. Using $v_\varepsilon$ as test function in $\mathcal{P}(\psi_h, f)$ we get
\[
\langle (-\Delta)^s u_h - f, u + (v_0 - u) \wedge \varepsilon - u_h \rangle \geq 0,
\]
and hence
\[
\| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 = \langle (-\Delta)^s u_h, u_h \rangle \leq \langle (-\Delta)^s u_h - f, u + (v_0 - u) \wedge \varepsilon \rangle + \langle f, u_h \rangle.
\]

Letting $h \to \infty$ we infer
\[
\limsup_{h \to \infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 \leq \langle (-\Delta)^s u - f, u + (v_0 - u) \wedge \varepsilon \rangle + \langle f, u \rangle
\]
\[
= \| (-\Delta)^{\frac{s}{2}} u \|_2^2 + \langle (-\Delta)^s u - f, (v_0 - u) \wedge \varepsilon \rangle. \tag{4.2}
\]

Now we let $\varepsilon \to 0$. Clearly $(v_0 - u) \wedge \varepsilon \to -(v_0 - u)^-$ in $L^2(\Omega)$. In addition, the functions $(v_0 - u) \wedge \varepsilon$ are uniformly bounded in $\tilde{H}^s(\Omega)$ by $iii)$ in Lemma 2.4. Thus $(v_0 - u) \wedge \varepsilon \to -(v_0 - u)^-$ weakly in $\tilde{H}^s(\Omega)$. Thus, from (4.2) we get
\[
\limsup_{h \to \infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 \leq \| (-\Delta)^{\frac{s}{2}} u \|_2^2 - \langle (-\Delta)^s u - f, (v_0 - u)^- \rangle = \| (-\Delta)^{\frac{s}{2}} u \|_2^2
\]
since $u$ solves $\mathcal{P}(\psi, f)$ and therefore it satisfies condition $c)$ in Theorem 3.2. Thus $u_h \to u$ in $\tilde{H}^s(\Omega)$. \qed
Next we deal with the continuity of the arrow $H^s \ni \psi \mapsto u \in \tilde{H}^s$.

**Theorem 4.4** Let $\psi_h \in H^s(\mathbb{R}^n)$ be a sequence of obstacles such that $\psi_h^+ \in \tilde{H}^s(\Omega)$, and let $f_h$ be a sequence in $\tilde{H}^s(\Omega)'$. Assume that

$$
\psi_h \to \psi \quad \text{in } H^s(\mathbb{R}^n), \quad f_h \to f \quad \text{in } H^s(\Omega)'.
$$

Denote by $u_h$ the solution to the obstacle problem $P(\psi_h, f_h)$. Then $u_h \to u$ in $\tilde{H}^s(\Omega)$, where $u$ is the solution to the limiting obstacle problem $P(\psi, f)$.

**Proof.** We can assume that $f_h, f = 0$. If not, replace the obstacles $\psi_h$ and $\psi$ with $\psi_h - \omega f_h$ and $\psi - \omega f$, respectively, see (1.3).

Let $u_h$ solve $P(\psi_h, 0)$ and let $u$ be the solution to the limiting problem $P(\psi, 0)$. Recall that $u$ is the unique minimizer for

$$
\inf_{v \in K^s_\psi} \langle (-\Delta)^s v, v \rangle. \quad (4.3)
$$

Since $u \vee \psi_h = u + (\psi_h - u)^+$ and $\psi_h - u \to \psi - u \leq 0$, then

$$
u \vee \psi_h \to u \quad \text{in } \tilde{H}^s(\Omega) \quad (4.4)
$$

by Corollary 2.3. Moreover, $u \vee \psi_h \in K^s_{\psi_h}$ and thus from $P(\psi_h, 0)$ we infer

$$
\langle (-\Delta)^s u_h, u \vee \psi_h \rangle \leq \langle (-\Delta)^s u_h, u \vee \psi_h \rangle. \quad (4.5)
$$

Inequality (4.5) guarantees the boundedness of the sequence $u_h$ in $\tilde{H}^s(\Omega)$. Hence we can assume that $u_h \to \tilde{u}$ weakly in $\tilde{H}^s(\Omega)$. Since $\psi_h \to \psi$ and $u_h \to \tilde{u}$ a.e. in $\Omega$, clearly $\tilde{u} \in K^s_\psi$.

Next, by weak lower semicontinuity, (4.5) and (4.4) we get

$$
\langle (-\Delta)^s \tilde{u}, \tilde{u} \rangle \leq \liminf_{h \to \infty} \langle (-\Delta)^s u_h, u \vee \psi_h \rangle \leq \limsup_{h \to \infty} \langle (-\Delta)^s u_h, u \vee \psi_h \rangle \leq \langle (-\Delta)^s \tilde{u}, u \rangle. \quad (4.6)
$$

Thus

$$
\| (-\Delta)^s \tilde{u} \|_2^2 \leq \| (-\Delta)^s \tilde{u} \|_2^2 - \| (-\Delta)^s \tilde{u} \|_2^2. \quad (4.7)
$$

Hence, $\tilde{u} = u$, as the minimization problem (4.3) admits a unique solution, and (4.6) implies $\| (-\Delta)^s u_h \|_2 \to \| (-\Delta)^s u \|_2$. Hence $u_h \to u$ strongly in $\tilde{H}^s(\Omega)$. \qed
5 Proof of the main results

We start with a preliminary theorem of independent interest, that gives distributional bounds on \((-\Delta)^s u - f\) under mild assumptions on the data.

**Theorem 5.1** Let \(\psi\) and \(f \in \tilde{H}^s(\Omega)'\) satisfying assumptions A1) and A2) in Theorem 1.1. Let \(u \in \tilde{H}^s(\Omega)\) be the unique solution to \(P(\psi, f)\). Then

\[
0 \leq (-\Delta)^s u - f \leq ((-\Delta)^s (\psi - \omega_f)^+ - f)^+ \quad \text{in the distributional sense on } \Omega.
\]

**Proof.** The main tool was inspired by the penalty method by Lewy-Stampacchia [10] and already used for instance in [18] under smoothness assumptions on the data and on the solution.

In order to simplify notations we start the proof with some remarks. First, we can assume that \(f = 0\), as we did in the proof of Theorem 4.4. Thus \((-\Delta)^s u \geq 0\) and \(u \geq \psi\), that imply \(u \geq \psi^+\), use the maximum principle in Remark 2.2. Clearly \(u\) is the smallest supersolution to \((-\Delta)^s v = 0\) in \(K_{\psi^+}\), and hence it solves the obstacle problem \(P(\psi^+, 0)\). In conclusion, it suffices to prove Theorem 5.1 in case \(f = 0\) and \(\psi \geq 0\) in \(\mathbb{R}^n\). Our aim is to show that

\[
0 \leq (-\Delta)^s u \leq ((-\Delta)^s (\psi)^+)^+ \quad \text{in the distributional sense on } \Omega, \quad (5.1)
\]

for \(\psi \in \tilde{H}^s(\Omega), \psi \geq 0\), such that \((-\Delta)^s \psi\) is a measure on \(\Omega\).

The proof of (5.1) will be achieved in few steps.

**Step 1** Assume \((-\Delta)^s \psi \in L^p(\Omega)\) for any large exponent \(p > 1\). Then (5.1) holds.

We take \(p \geq \frac{2n}{n+2s}\), that is needed only if \(n > 2s\). Then \(\tilde{H}^s(\Omega) \hookrightarrow L^p(\Omega)\) and \(L^p(\Omega) \subset \tilde{H}^s(\Omega)'\) by Sobolev embeddings. In particular \(((\Delta)^s \psi)^+ \in \tilde{H}^s(\Omega)'\).

Take a function \(\theta_\varepsilon \in C^\infty(\mathbb{R})\) such that \(0 \leq \theta_\varepsilon \leq 1\), and

\[
\theta_\varepsilon(t) = 1 \quad \text{for } t \leq 0, \quad \theta_\varepsilon(t) = 0 \quad \text{for } t \geq \varepsilon.
\]

By standard variational methods we have that there exists a unique \(u_\varepsilon \in \tilde{H}^s(\Omega)\) that weakly solves

\[
(-\Delta)^s u_\varepsilon = \theta_\varepsilon (u_\varepsilon - \psi) ((-\Delta)^s \psi)^+ \quad \text{in } \Omega.
\]
We claim that
\[ u \leq u_\varepsilon \leq u + \varepsilon \text{ a.e. in } \Omega. \]

By \textit{iii)} in Lemma 2.1 we can estimate
\[
\left\| (-\Delta)^{\frac{s}{2}} (\psi - u_\varepsilon)^+ \right\|^2 \leq \langle (-\Delta)^{s} (\psi - u_\varepsilon), (\psi - u_\varepsilon)^+ \rangle \\
\leq \int_{\Omega} \left((-\Delta)^{s} \psi \right)^+ (1 - \theta_\varepsilon (u_\varepsilon - \psi))(\psi - u_\varepsilon)^+ \, dx = 0.
\]

Hence, \( u_\varepsilon \geq \psi \). Since \((-\Delta)^{s} u_\varepsilon \geq 0\), then \( u_\varepsilon \geq u \) by \textit{b)} in Theorem 3.2. Next, we use \textit{iii)} in Lemma 2.4 and \((-\Delta)^{s} u \geq 0\) to estimate
\[
\left\| (-\Delta)^{\frac{s}{2}} (u_\varepsilon - u - \varepsilon)^+ \right\|^2 \leq \langle (-\Delta)^{s} (u_\varepsilon - u), (u_\varepsilon - u - \varepsilon)^+ \rangle \\
\leq \int_{\Omega} \left((-\Delta)^{s} \psi \right)^+ \theta_\varepsilon (u_\varepsilon - \psi) (u_\varepsilon - u - \varepsilon)^+ \, dx = 0.
\]

Thus \( u_\varepsilon \leq u + \varepsilon \), and the claim is proved. In particular, we have that \( \|u_\varepsilon - u\|_\infty \to 0 \) as \( \varepsilon \to 0 \). Therefore, for any nonnegative test function \( \varphi \in C_0^\infty (\Omega) \) we have that
\[
\langle (-\Delta)^s u, \varphi \rangle = \int_{\Omega} u (-\Delta)^s \varphi \, dx = \int_{\Omega} u_\varepsilon (-\Delta)^s \varphi \, dx + o(1)
\]
\[
= \langle (-\Delta)^s u_\varepsilon, \varphi \rangle + o(1) \leq \langle (-\Delta)^s \psi, \varphi \rangle + o(1),
\]
that readily gives \((-\Delta)^s u \leq (-\Delta)^s \psi^+\) in the distributional sense in \( \Omega \).

\textbf{Step 2} Approximation argument.

Fix a small \( \varepsilon > 0 \) and put \( \Omega_\varepsilon := \{ x \in \Omega \mid \text{dist}(x, \Omega) < \varepsilon \} \). The convex set
\[ K_\varepsilon = \{ v \in \tilde{H}^s(\Omega_\varepsilon) \mid v \geq \psi \text{ a.e. on } \mathbb{R}^n \} \]
contains \( K_\psi^s \), hence it is not empty. We denote by \( u_\varepsilon \) the unique solution to the variational inequality
\[ u_\varepsilon \in K_\varepsilon, \quad \langle (-\Delta)^s u_\varepsilon, v - u_\varepsilon \rangle \geq 0 \quad \forall v \in K_\varepsilon, \quad (\mathcal{P}_\varepsilon) \]
so that \( u_\varepsilon \in \tilde{H}^s(\Omega_\varepsilon) \) and is nonnegative. Next we prove that
\[ 0 \leq (-\Delta)^s u_\varepsilon \leq (-\Delta)^s \psi^+ \text{ in the distributional sense on } \Omega. \quad (5.2) \]
For, we approximate $\psi$ in a standard way, via convolution. Let $(\rho_h)_h$ be a sequence of mollifiers such that $\text{supp}(\rho_h) \subset B_{\frac{1}{h}}$ and put $\psi_h = \psi \ast \rho_h$. Notice that for $h$ large enough, $\psi_h = 0$ outside $\Omega_\varepsilon$. Therefore

$$\psi_h \in \tilde{H}^s(\Omega_\varepsilon), \quad \psi_h \to \psi \quad \text{in} \quad H^s(\mathbb{R}^n). \quad (5.3)$$

The convex set $K_{\varepsilon,h} := \{ v \in \tilde{H}^s(\Omega_\varepsilon) \mid u \geq \psi_h \}$ is not empty, as it contains $\psi_h$. The variational inequality

$$u_h \in K_{\varepsilon,h}, \quad \langle (-\Delta)^s u_h, v - u_h \rangle \geq 0 \quad \forall v \in K_{\varepsilon,h}, \quad (P_{\varepsilon,h})$$

has a unique solution $u_h \in \tilde{H}^s(\Omega_\varepsilon)$. Theorem 4.4 readily gives that $u_h \to u_\varepsilon$ in $\tilde{H}^s(\mathbb{R}^n)$. Since $(-\Delta)^s \psi_h \in L^p(\mathbb{R}^n)$ for any $p \geq 1$, then Step 1 applies. In particular

$$0 \leq (-\Delta)^s u_h \leq ((-\Delta)^s \psi_h)^+ \quad \text{in the distributional sense on } \Omega. \quad (5.4)$$

Next, $((-\Delta)^s \psi)^+ \ast \rho_h$ is a nonnegative smooth function, and

$$((-\Delta)^s \psi)^+ \ast \rho_h \geq ((-\Delta)^s \psi) \ast \rho_h = (-\Delta)^s \psi_h.$$

Thus $((-\Delta)^s \psi)^+ \ast \rho_h \geq ((-\Delta)^s \psi_h)^+$, and (5.3) implies

$$0 \leq (-\Delta)^s u_h \leq ((-\Delta)^s \psi_h)^+ \ast \rho_h \quad \text{in the distributional sense on } \Omega.$$

Claim (5.2) follows, since $((-\Delta)^s \psi)^+ \ast \rho_h \to ((-\Delta)^s \psi)^+$ in the sense of measures, and $(-\Delta)^s u_h \to (-\Delta)^s u_\varepsilon$ in the sense of distributions.

**Step 3 Conclusion of the proof.**

The last step in the proof consists in passing to the limit along a sequence $\varepsilon \to 0$. First, we notice that $u \in \tilde{H}^s(\Omega_\varepsilon)$ and in particular $u \in K_{\varepsilon}$. Therefore, using the variational characterization of the unique solution $u_\varepsilon$ to $(P_\varepsilon)$ we find

$$\frac{1}{2} \langle ((-\Delta)^s u_\varepsilon, u_\varepsilon \rangle \leq \frac{1}{2} \langle ((-\Delta)^s u, u \rangle. \quad (5.5)$$

Now we fix $\varepsilon_0 > 0$. Thanks to (5.5), we get that the sequence $u_\varepsilon$ is bounded in $\tilde{H}^s(\Omega_{\varepsilon_0})$, and therefore we can assume that $u_\varepsilon \to \tilde{u}$ weakly in $\tilde{H}^s(\Omega_{\varepsilon_0})$. From (5.5) we readily get

$$\frac{1}{2} \langle ((-\Delta)^s \tilde{u}, \tilde{u} \rangle \leq \frac{1}{2} \langle ((-\Delta)^s u, u \rangle. \quad (5.6)$$
On the other hand, $u_\varepsilon \to \tilde{u}$ almost everywhere. Hence $\tilde{u} \in \tilde{H}^s(\Omega)$ and $\tilde{u} \geq \psi$ on $\Omega$, that is, $\tilde{u} \in K^s_\psi$. Using the characterization of $u$ as the unique solution to the minimization problem (4.3), from (5.6), (5.5) we get that $\tilde{u} = u$ and $u_\varepsilon \to u$ in $\tilde{H}^s(\Omega_\varepsilon)$. In particular, $\langle (-\Delta)^s u_\varepsilon, \varphi \rangle \to \langle (-\Delta)^s u, \varphi \rangle$ for any $\varphi \in C^\infty_0(\Omega)$. Now, from (5.2) we know that $\langle (-\Delta)^s \psi^+, (-\Delta)^s u_\varepsilon \rangle$ is a nonnegative distribution on $\Omega$. Thus $\langle (-\Delta)^s \psi^+, (-\Delta)^s u \rangle$ is nonnegative as well, and (5.1) is proved. □

**Proof of Theorem 1.1**

Statements $i)$ and $ii)$ hold by Theorem 5.1. It remains to prove the last claim.

It is not restrictive to assume $f \equiv 0$. Hence $u$ solves $\mathcal{P}(\psi, 0)$, $(-\Delta)^s u \geq 0$ by Theorem 3.2 and $u$ is nonnegative in $\Omega$, see Remark 2.2. Actually $u$ is lower semicontinuous and positive by the strong maximum principle, see for instance [8, Theorem 2.5]. Thus $u \geq \psi^+$ and $\{u > \psi\} = \{u > \psi^+\}$.

Next we use c) in Theorem 3.2 with $v = \psi^+ \in \tilde{H}^s(\Omega)$, to get

$$\langle (-\Delta)^s u, u - \psi^+ \rangle = 0.$$ (5.7)

Let $\Omega'$ be any domain compactly contained in $\Omega$. We claim that

$$\int_{\Omega'} (-\Delta)^s u \cdot (u - \psi^+) \, dx = 0.$$ (5.7)

Since $(-\Delta)^s u \cdot (u - \psi^+)$ is a measurable nonnegative function then the integral in (5.7) is nonnegative. To prove the opposite inequality we put $g_m = (u - \psi^+) \wedge m$, $m \geq 1$. Let $\varphi$ be any nonnegative cut off function, with $\varphi \in C^\infty_0(\Omega)$ and $\varphi \equiv 1$ on $\Omega'$. Since $(-\Delta)^s u \geq 0$, $(-\Delta)^s u \in L^1_{loc}(\Omega)$, $u - \psi^+ \geq \varphi g_m$ and $\varphi g_m \in L^\infty(\Omega)$ has compact support in $\Omega$, we have that

$$0 = \langle (-\Delta)^s u, u - \psi^+ \rangle \geq \langle (-\Delta)^s u, \varphi g_m \rangle = \int_{\Omega} (-\Delta)^s u \cdot (\varphi g_m) \, dx \geq \int_{\Omega'} (-\Delta)^s u \cdot g_m \, dx.$$

Next, use the monotone convergence theorem to get

$$0 \geq \lim_{m \to \infty} \int_{\Omega'} (-\Delta)^s u \cdot g_m \, dx = \int_{\Omega'} (-\Delta)^s u \cdot (u - \psi^+) \, dx,$$

that concludes the proof of (5.7).

Now, since $\Omega'$ was arbitrarily chosen and $(-\Delta)^s u \cdot (u - \psi^+) \geq 0$, equality (5.7) implies that $(-\Delta)^s u \cdot (u - \psi^+) = 0$ a.e. in $\Omega$, and $iii)$ is proved. □
Remark 5.2 \textit{Theorem 1.1 holds with the same proof also in the local case $s = 1$. Notice that no regularity assumptions on $\Omega$ are needed, and the cases $p = 1, p = \infty$ are included as well.}

Remark 5.3 \textit{To obtain better regularity results for $u$, one can apply the regularity theory for}
\[ (-\Delta)^s u = g \in L^p(\Omega) \quad \text{in} \quad \Omega, \quad u \in \tilde{H}^s(\Omega). \]
\textit{In particular, if $p > \frac{n}{2s}$ and $\Omega$ is Lipschitz and satisfies the exterior ball condition, then $u$ is Hölder continuous in $\Omega$. See for example [16, Proposition 1.4] and [17, Proposition 1.1].}

\textbf{Proof of Theorem 1.2}
As usual, we can assume $f = 0$. Fix a small $\varepsilon > 0$, and let $\psi^\varepsilon_h$ be a mollification of $\psi - \varepsilon$. Then $\psi^\varepsilon_h$ is smooth on $\overline{\Omega}$, $\psi^\varepsilon_h < 0$ on $\partial \Omega$ and $\psi^\varepsilon_h \to \psi - \varepsilon$ uniformly on $\overline{\Omega}$, as $h \to \infty$.

By Theorem 1.1, the solution $u_h^\varepsilon \in \tilde{H}^s(\Omega)$ to $\mathcal{P}(\psi^\varepsilon_h, 0)$ satisfies $(-\Delta)^s u_h^\varepsilon \in L^p(\Omega)$ and therefore $u_h^\varepsilon$ is Hölder continuous, see Remark 5.3. Moreover, the estimates in Theorem 4.1 imply that $u_h^\varepsilon \to u^\varepsilon$ uniformly on $\Omega$, where $u^\varepsilon$ solves $\mathcal{P}(\psi - \varepsilon, 0)$. In particular, $u^\varepsilon \in C^0(\Omega)$. Finally, use again Theorem 4.1 to get that $u^\varepsilon \to u$ uniformly, where $u$ solves $\mathcal{P}(\psi, 0)$. In particular, $u$ is continuous on $\mathbb{R}^n$.

To check the last statement notice that the set $\{u > \psi\} \subseteq \Omega$ is open; for any test function $\varphi \in C^\infty(\{u > \psi\})$ we have that $u \pm t\varphi \in K^s_{\psi}$ and therefore $t((-\Delta)^s u, \pm \varphi) \geq 0$ for $|t|$ small enough. The conclusion is immediate. \hfill $\Box$

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