Non-Hermitian $\mathcal{PT}$ - symmetric relativistic quantum mechanics with a maximal mass in an external magnetic field

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Abstract

Starting with the modified Dirac equations for free massive particles with the $\gamma_5$-extension of the physical mass $m \rightarrow m_1 + \gamma_5 m_2$, we consider equations of relativistic quantum mechanics in the presence of an external electromagnetic field. The new approach is developing on the basis of existing methods for study the unbroken $\mathcal{PT}$ symmetry of Non-Hermitian Hamiltonians. The paper shows that this modified model contains the definition of the mass parameter, which may use as the determination of the magnitude scaling of energy $M$. Obviously that the transition to the standard approach is valid when small in comparison with $M$ energies and momenta. Formally, this limit is performed when $M \rightarrow \infty$, which simultaneously should correspond to the transition to a Hermitian limit: $m_2 \rightarrow 0$. Inequality $m \leq M$ may be considered and as the restriction of the mass spectrum of fermions considered in the model. Within of this approach, the effects of possible observability mass parameters: $m_1, m_2, M$ are investigated taking into account the interaction of the magnetic field with charged fermions together with the accounting of their anomalous magnetic moments.

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1 Introduction

As it is well-known the idea about existence of a maximal mass in a broad spectrum mass of elementary particles at the Planck mass was suggested by
Moisey Markov in 1965 [1]

\[ m \leq m_{\text{Planck}} \cong 10^{19}\text{GeV}. \]  

(1)

The particles with the limiting mass

\[ m = m_{\text{Planck}}, \]

named by the author "maximons" should play a special role in the world of elementary particles. However, Markov's original condition (1) was purely phenomenological and he used standard field theoretical techniques even for describing the maximon. In the end of the seventies of the last century a more radical approach was developed by V.G. Kadyshevsky [2]. His model contained a limiting mass \( \mathcal{M} \) as a new fundamental physical constant. Doing this condition of finiteness of the mass spectrum should be introduced by the relation:

\[ m \leq \mathcal{M}, \]  

(2)

where a new constant \( \mathcal{M} \) was named by the fundamental mass.

Really in the papers [2]-[13] the existence of mass \( \mathcal{M} \) has been understood as a new principle of Nature similar to the relativistic and quantum postulates, which was put into the ground of the new quantum field theory. At the same time the new constant \( \mathcal{M} \) is introduced in a purely geometric way, like the velocity of light is the maximal velocity in the special relativity.

Indeed, if one chooses a geometrical formulation of the quantum field theory (QFT), the adequate realization of the limiting mass hypothesis is reduced to the choice of the de Sitter geometry as the geometry of the 4-momentum space of the constant curvature with a radius equal to \( \mathcal{M} \) [2].

Besides the de Sitter space there is another space of constant curvature, breaking into a Minkowski space in small 4-momentum, which is called the space of anti-de Sitter. The detailed analysis of the different aspects of the construction of the modified quantum field theory with the maximal mass in the curved momentum anti-de Sitter space has allowed to obtain a number of interesting results. In particular, it has been shown that non-Hermitian fermionic Hamiltonians with the \( \gamma_5 \)-dependent mass term must arise in the modified field theory (see, for example, [8],[9]).

Now it is well-known fact, that the reality of the spectrum in models with a non-Hermitian Hamiltonian is a consequence of \( \mathcal{P}\mathcal{T} \)-invariance of the theory, i.e. a combination of spatial and temporary parity of the total
Hamiltonian: \([H, \mathcal{PT}] \psi = 0\). When the \(\mathcal{PT}\) symmetry is unbroken, the spectrum of the quantum theory is real. These results explain the growing interest in this problem. For the past a few years has been studied a lot of new non-Hermitian \(\mathcal{PT}\)-invariant systems [14] - [37]. It is important to note that the previous works which were devoted to studying pseudo-Hermitian quantum mechanics with \(\gamma_5\)-mass contribution [27] - [29] the restrictions of mass parameters were presented only as \(m_1^2 \geq m_2^2, m \leq m_1\).

This paper has the following structure. In section II the non-Hermitian approach to the construction of plane wave solutions is formulated for the case free massive particles. In the third section we study the basic characteristics of modified Dirac models with \(\gamma_5\)-massive contributions in the external magnetic field. Then, in the fourth section, we consider the modified Dirac-Schwinger-Pauli model in the magnetic field with non-Hermitian extension. This section also contains the discussion of the effects of the possible observability of the parameters: \(m_1, m_2\) and \(M\) taking into account the interaction with the magnetic field of charged fermions together with regard to their anomalous magnetic moments. The fifth section contains summary and conclusion.

2 Non-Hermitian extensions of plane waves

Let us now consider the solutions of modified Dirac equations for free massive particles following from procedures of the \(\gamma_5\)-extension of the mass \(m \rightarrow m_1 + \gamma_5 m_5\):

\[
\left( i \partial_\mu \gamma^\mu - m_1 - \gamma_5 m_2 \right) \tilde{\psi}(x, t) = 0.
\]

It is obvious that the Hamiltonian associated with this equation is non-Hermitian due to the appearance in it the \(\gamma_5\)-dependent mass components \((H \neq H^+)\).

First-order equations (3) can be transformed into equations of the second order by applying to (3) the operator:

\[
\Pi = i \partial_\mu \gamma^\mu + m_1 - \gamma_5 m_2.
\]

In a result the modified Dirac equation converts to the Klein-Gordon equation:

\[
\left( \partial^2 + m^2 \right) \psi(x, t) = 0
\]
where the physical mass of particle $m$ is expressed through the parameters $m_1$ and $m_2$

$$m^2 = m_1^2 - m_2^2. \quad (6)$$

It is easy to see from (6) that the mass $m$, appearing in the equation (5) is real, when the inequality

$$m_1^2 \geq m_2^2. \quad (7)$$

is accomplished.

A. Mustafazadeh identified the necessary and sufficient requirements of reality of eigenvalues for pseudo-Hermitian and $\mathcal{PT}$-symmetric Hamiltonians and formalized the use these Hamilton operators in his papers \[17,18\] and \[30-36\]. According to the recommendations of this works we can define Hermitian operator $\eta$, which transform non-Hermitian Hamiltonian by means of invertible transformation to the Hermitian-conjugated one. It is easy to see that with Hermitian operator

$$\eta = e^{\gamma_5 \theta} \quad (8)$$

we can obtain

$$\eta H \eta^{-1} = H^+, \quad (9)$$

where

$$H = \alpha p + \beta(m_1 + \gamma_5 m_2) \quad (10)$$

and

$$H^+ = \alpha p + \beta(m_1 - \gamma_5 m_2). \quad (11)$$

Here matrices $\alpha_i = \gamma_0 \cdot \gamma_i$, $\beta = \gamma_0$, and $\theta = \text{arctanh}(m_2/m_1)$

In addition, multiplying the Hamilton operator $H$ from left to $e^{\theta \gamma_5}/2$ and taking into account that matrices $\gamma_5$ commute with matrices $\alpha_i$ and anti-commute with $\beta$, we can obtain

$$e^{\gamma_5 \theta/2} H = H_0 e^{\gamma_5 \theta/2}, \quad (12)$$

where $H_0 = \alpha p + \beta m$ is a ordinary Hermitian Hamiltonian of a free particle.

The mathematical sense of the action of the operator (8) it turns out, if we notice that according to the properties of $\gamma_5$ matrices, all the even degree of $\gamma_5$ are equal to 1, and all odd degree are equal to $\gamma_5$. Given that $\cosh(x)$ decomposes on even and $\sinh(x)$ odd degrees of $x$, the expressions (9)-(12)
can be obtained by representing non-unitary exponential operator $\eta$ in the form
\[ \eta = e^{\gamma_5 \vartheta} = \cosh \vartheta + \gamma_5 \sinh \vartheta, \] (13)
where
\[ \cosh \vartheta = m_1/m; \quad \sinh \vartheta = m_2/m. \] (14)

The region of the unbroken $\mathcal{PT}$-symmetry of (10) can be found in the form (7). However, it is not apparent that the area with undisturbed $\mathcal{PT}$-symmetry defined by such a way does not include the regions, corresponding to the some unusual particles, description of which radically distinguish from traditional one.

A feature of the model with $\gamma_5$-mass contribution is that it may contain any additional restrictions for mass parameters besides (7). Indeed while that for the physical mass $m$ one may be constructed by infinite number combinations of $m_1$ and $m_2$, satisfying to (6), however besides it need to provide and the rules of conformity of this parameters in the Hermitian limit.

In particular, the simple a linear scheme may be easy constructed if one takes the obvious restriction for the mass spectrum of the fermions in the form
\[ m \leq M_1, \] (15)
where $M_1$ - the fixed value of mass parameter $m_1 (M_1 = m_1)$. Using this approach we can in principle describe the whole spectrum of fermions when $m \leq M_1$ by means of defining appropriate values $m_2$. According to (6) one can obtain the expression
\[ m = M_1 \sqrt{1 - m_2^2/M_1^2}. \] (16)

With the help (16) we can see, that when the mass $m_2$ is increased, the values of the physical mass tends to zero. The equality of parameters $m_2 = m_1$ corresponds to the case of massless fermions. But it should be noted that in this model the Hermitian limit $m_2 \to 0$ may be reached only in the case of the particles with maximal mass $m = M_1$. At the same time, the Hermitian limit is absent for all other mass values.

Therefore the procedure of limitations of the physical mass spectrum by the inequality $m \leq M_1$ has the essential drawback since in this frame is not possible to describe all ordinary fermions, respecting the Principle of Conformity, except, $m = M_1$. 

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These considerations make search in the frame of mass restriction of \( m \leq m_1 \) the existence of more complicated non-linear dependence of limiting mass value
\[
m \leq M(m_1, m_2),
\]
which meets the requirements of the Principle of Conformity:

i) The Dirac limit must exist for all ordinary fermions for which the condition \((17)\) is satisfied.

ii) In Hermitian limit the parameter \( m_1 \) must coincide with the physical mass \( m \).

The fulfillment of these conditions lets to find the most appropriate scheme restriction of the mass fermions, for which exist the consistency with the ordinary Dirac theory.

Possible explicit expression for \( M(m_1, m_2) \), may be obtained from the simple mathematical theorem about the arithmetical average of two non-negative real numbers \( a \) and \( b \) which is always greater than or equal to the geometrical mean of the same numbers. Really, let \( a = m^2 \) and \( b = m_2^2 \) then using
\[
\frac{m^2 + m_2^2}{2} \geq \sqrt{m^2 \cdot m_2^2}
\]
and substitution \((6)\), we can get the inequality
\[
m \leq m_1^2/2m_2 = M(m_1, m_2).
\]
Values \( M \) now is defined by two parameters \( m_1, m_2 \) and in the limit \( m_2 \to 0 \), value of the maximal mass \( M \) tends to infinity. It is very important that in this limit the restriction of mass value of particles completely disappear. In such a way a natural transition of the modified model to the ordinary Dirac theory is demonstrated for any values of the physical mass.

Using \((6)\) and expression \((18)\) we can also obtain the system of two equations
\[
\begin{align*}
  m &= m_1^2 - m_2^2 \\
  M &= m_1^2/2m_2
\end{align*}
\]
(19)
The solution of this system relative to the parameters \( m_1 \) and \( m_2 \) may be represented in the form
\[
m_1^\pm = \sqrt{2}M \sqrt{1 \mp \sqrt{1 - m^2/M^2}},
\]
(20)
\[ m_2^\pm = M \left( 1 \mp \sqrt{1 - m^2/M^2} \right). \] (21)

It is easy to verify that the obtained values of the mass parameter satisfy the conditions (6) and (7) regardless of which sign is chosen. Besides it should be emphasized that, formulas (20), (21) in the case of the upper sign are agreed with conditions \( m_2 \to 0 \) and \( m_1 \to m \) when \( M \to \infty \), i.e. there are a so-called Hermitian and Dirac limits, which determined in the conditions i) and ii). However, if one choose a lower sign (i.e. for the \( m_1^+ \) and \( m_2^+ \)) the such limits are absent. Thus we can see that the nonlinear scheme of mass restrictions (see 18) additionally contains the solutions to satisfy the requirements of the ”exotic” particles. However this ”exotic” solutions should be considered only as an indication of the principal possibility of the existence of such particles. In this case, as follows from (20), (21) for each ordinary particles may be exist some ”exotic” partners, possessing the same mass, but a number of unique properties.

Let’s consider the ”normalized” parameter of the modified model with the maximal mass \( M \):

\[ x = \frac{m_1}{M} = 2\frac{m_2}{m_1} \] (22)

and using (16) we can obtain

\[ \frac{m_2}{M} = \frac{x^2}{2}, \] (23)

\[ \frac{m}{M} = x\sqrt{1 - x^2/4}. \] (24)

At Fig.1 we can see dependence of the the normalized parameters \( m/m_1, M/m_1 \) and \( m/M \) on the parameter \( x = m_1/M = 2m_2/m_1 \). In particular, maximum value of the particle mass \( m/M = 1 \) is achieved at the value of \( x = \sqrt{2} \). In this point parameter \( m_2/M \) is also equal to unit. Further increasing of \( x \), leads to decreasing of \( m/M \) and at the point \( x = 2 \) this value

\footnote{As the exotic particles do not agree in the limit \( m_2 \to 0 \) with the ordinary Dirac expressions then one can assume that in this case we deal with a description of some new particles, properties of which have not yet been studied. This fact for the first time has been fixed by V.G.Kadyshevsky in his early works in the geometric approach to the development of the theory with a ”fundamental mass” \[2\]. Besides in \[8,9\] it was noted that the most intriguing prediction of the new approach is the possible existence of exotic fermions with no analogues in the SM, which may be candidates for dark matter constituents.}
Figure 1: Dependence of $m/M, M/m_1,$ and $m/m_1$ from the parameter $x = m_1/M = 2m_2/m_1$

is equal to zero. Thus, the region of $x > \sqrt{2}$ ($m_2/M > 1$) corresponds to the description of the "exotic particles", for which there is not transition to Hermitian limit.

In the frame of the inequality (7) we can see three specific sectors of unbroken $\mathcal{PT}$-symmetry of the Hamiltonian (10) in the plane $\nu_1 = m_1/M, \nu_2 = m_2/M$. Thus the plane $\nu_1, \nu_2$ may be divided by the three groups of the inequalities:

\begin{align*}
I. & \quad \nu_1/\sqrt{2} \leq \nu_2 \leq \nu_1, \\
II. & \quad -\nu_1/\sqrt{2} < \nu_2 < \nu_1/\sqrt{2}, \\
III. & \quad -\nu_1 \leq \nu_2 \leq -\nu_1/\sqrt{2}.
\end{align*} \quad (25)

Only the area $II.$ corresponds to the description of ordinary particles, while the $I.$ and $III.$ agree with the description of some as yet unknown particles. This conclusion is not trivial, because in contrast to the geometric approach, where the emergence of new unusual properties of particles associated with the presence in the theory a new degree of freedom (sign of the
fifth component of the momentum $\varepsilon = p_5/|p_5|$ [21]), in the case of a simple extension of the free Dirac equation due to the additional $\gamma_5$-mass term, the satisfactory explanation of this fact is not there yet.

Then we can establish the limits of change of parameters. At preset values of $m$ and $M$ as it follows from the (20), (21) the limits of variation of parameters $m_1$ and $m_2$ are the following:

$$m \leq m_1 \leq 2M \quad ; \quad -2M \leq m_2 \leq 2M.$$  \hfill (26)

In the areas of change of these parameters has a point in which we have

$$m_1 = \sqrt{2}M \quad ; \quad m_2 = M.$$  \hfill (27)

In this point the physical mass $m$ reaches its maximum value $m = M$ and corresponds to mass of the maximon.

If we used the standard representation of $\gamma$-matrixes the non-Hermitian Hamiltonian $H$ can be written in the following matrix form

$$H = \begin{pmatrix}
    m_1 & 0 & p_3 - m_2 & p_1 - ip_2 \\
    0 & m_1 & p_1 + ip_2 & -m_2 - p_3 \\
    m_2 + p_3 & p_1 - ip_2 & -m_1 & 0 \\
    p_1 + ip_2 & m_2 - p_3 & 0 & -m_1
\end{pmatrix},$$

where $p_i$ are components of momentum.

It is clear that

$$H \psi = E \psi.$$  

The condition $\det (H - E) = (-E^2 + m_1^2 - m_2^2 + p_\perp^2 + p_3^2)^2 = 0$ results in the eigenvalues of $E$ which are represented in the form:

$$E = \pm \sqrt{m_1^2 - m_2^2 + p_\perp^2 + p_3^2},$$  \hfill (28)

where $p_\perp = \sqrt{p_1^2 + p_2^2}$ and $m_1^2 - m_2^2 = m^2$, that coincide with the eigenvalues of energy of Hermitian operator $H_0$.

Let us now consider the state of a free particle with certain values of the momentum and energy, which is described by a plane wave and can be written as

$$\tilde{\psi} = \frac{1}{\sqrt{2E}} \tilde{u} e^{-ipx}.$$  \hfill (29)

It is easy to see that the wave amplitude $\tilde{u}$ is determined by bispinor, normalization of which now needs an additional explanation.
Really using (3) and taking into account properties of matrices $\gamma_0, \vec{\gamma}, \gamma_5$, we can write also complex-conjugate equation from equation (3)

$$\left(-p_0 \tilde{\gamma}_0 - \vec{p} \tilde{\gamma} - m_1 - \gamma_5 m_2\right) \tilde{\psi}^* = 0,$$

where $\tilde{\gamma}_\mu$ are transpose matrix. Rearranging function $\tilde{\psi}^*$ and introducing new bispinor $\tilde{\psi} = \tilde{\psi}^* \gamma_0$, we can obtain

$$\tilde{\psi} (\gamma p + m_1 - \gamma_5 m_2) = 0. \tag{31}$$

The operator $p$ is assumed here acts on the function, standing on the left of it. Using (8) we can write equation (3), (31) in the following form

$$\tilde{\psi} (\gamma p + m_1 - \gamma_5 m_2) = 0 \tag{32}$$

$$\tilde{\psi} (\gamma p + m_\eta^{-1}) = 0 \tag{33}$$

Multiplying (32) on the left of the $\tilde{\psi} e^{-\theta \gamma_5}$ and the equation (33) on the right of the $e^{\theta \gamma_5} \tilde{\psi}$ and summing up the resulting expressions, one can obtain

$$\tilde{\psi} e^{-\theta \gamma_5/2} \gamma_\mu e^{\theta \gamma_5/2} (p \tilde{\psi}) + (p \tilde{\psi}) e^{-\theta \gamma_5/2} \gamma_\mu e^{\theta \gamma_5/2} = p_\mu \left(\tilde{\psi} e^{-\theta \gamma_5/2} \gamma_\mu e^{\theta \gamma_5/2} \tilde{\psi}\right) = 0 \tag{34}$$

Here brackets indicate which of the function are subjected to the action of the operator $p$. The obtained equation has the form of the continuity equation

$$\partial_\mu j_\mu = 0, \tag{35}$$

where

$$j_\mu = \tilde{\psi} e^{-\theta \gamma_5/2} \gamma_\mu e^{\theta \gamma_5} \tilde{\psi} = \left(\tilde{\psi}^* e^{\theta \gamma_5} \tilde{\psi}, \tilde{\psi}^* \gamma_0 e^{\theta \gamma_5} \tilde{\psi}\right) \tag{36}$$

Thus here the value of $j_\mu$ is a 4-vector of current density of particles in the model with $\gamma_5$-mass extension. It is very important that its temporal component

$$j_0 = \tilde{\psi}^* e^{\theta \gamma_5} \tilde{\psi} \tag{37}$$

does not change in time (see (35) and positively defined. It is easy to see from the following procedure. Let us construct the norm of any state for considered model for arbitrary vector, taking into account the weight operator $\eta$ (13):

$$\tilde{\psi} = \left(\begin{array}{c} x + iy \\ u + iv \\ z + iw \\ t + ip \end{array}\right).$$

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Using (14), (37), in a result we have

$$\tilde{\psi}^\ast \eta = \left( \frac{m_1 + m_2}{m}(x - iy), \frac{m_1 + m_2}{m}(u - iv), \frac{m_1 - m_2}{m}(z - iw), \frac{m_1 - m_2}{m}(t - ip) \right).$$

Then

$$\langle \tilde{\psi}^\ast \eta | \tilde{\psi} \rangle = \frac{m_1 + m_2}{m}(x^2 + y^2) + \frac{m_1 + m_2}{m}(u^2 + v^2), \frac{m_1 - m_2}{m}(z^2 + w^2), \frac{m_1 - m_2}{m}(t^2 + p^2)$$

is explicitly non negative, because $m_1 \geq m_2$ in the area of unbroken $PT$-symmetry (7).

With the help of (3),(7) and properties commutation of $\gamma$-matrix one can obtain that components of new bispinor amplitudes which must satisfy the following system of algebraic equations:

$$\left( \gamma_p - m e^{\gamma_5} \right) \tilde{u} = 0; \quad \tilde{\psi}$$

$$\bar{u} \left( \gamma_p - m e^{-\gamma_5} \right) = 0,$$

where $\bar{u} = \tilde{u}^\ast \gamma_0$.

According to (12),(8) one can write bispinor amplitudes in the form

$$\tilde{\psi} = \sqrt{2m} \begin{pmatrix} A_1 w \\ A_2 w \end{pmatrix}; \quad \tilde{\psi}$$

$$\bar{u} = \sqrt{2m} \begin{pmatrix} A_1 w^\star, & -A_2 w^\ast \end{pmatrix},$$

where the notations are used:

$$A_1 = \cosh \frac{\vartheta}{2} \cosh \frac{\beta}{2} + \sinh \frac{\vartheta}{2} \sinh \frac{\beta}{2} (\mathbf{n}\sigma);$$

$$A_2 = \sinh \frac{\vartheta}{2} \cosh \frac{\beta}{2} + \cosh \frac{\vartheta}{2} \sinh \frac{\beta}{2} (\mathbf{n}\sigma).$$

In addition, we have relations (13),(14) and $\cosh \beta = E/m, \sinh \beta = p/m$. And also $w$ - two-component spinor, satisfying the normalization condition

$$w^\ast w = 1.$$
Besides need to note that $\sigma$ are $2 \times 2$-Pauli matrices and $\mathbf{n} = \mathbf{p}/p$ - a unit vector in the direction of the momentum.

The explicit form of these spinors can be found using the condition that spiral states correspond to the plane wave in which spinors $w$ is an eigenfunction of the operator $(\sigma \mathbf{n})$

$$\sigma \mathbf{n} w^\zeta = \zeta w^\zeta.$$ 

Therefore we get

$$w^1 = \begin{pmatrix} e^{-i\varphi/2} \cos \theta/2 \\ e^{i\varphi/2} \sin \theta/2 \end{pmatrix}, \quad w^{-1} = \begin{pmatrix} -e^{-i\varphi/2} \sin \theta/2 \\ e^{i\varphi/2} \cos \theta/2 \end{pmatrix},$$

where $\theta$ and $\varphi$ - polar and azimuthal angles that determine the direction $\mathbf{n}$ concerning to the axes $x_1, x_2, x_3$.

It is easy to verify by the direct multiplication that really

$$\bar{u} \tilde{u} = 2m.$$ 

This result however in advance obvious, because there are the connection between bispinor amplitudes of modified equations $\bar{u}, \tilde{u}$ and corresponding solutions of the ordinary Dirac equations:

$$\bar{\tilde{u}} = e^{-\gamma_5 \vartheta/2} u$$

$$\bar{u} = \bar{u} e^{\gamma_5 \vartheta/2}.$$ 

Taking into account that the Dirac bispinor amplitudes as usually [38] are normalized by invariant condition $\bar{u} u = 2m$. Hence we have

$$\bar{u} \tilde{u} = \bar{u} u = 2m. \quad (43)$$

By using (43) we can also obtain

$$\bar{u} e^{-\gamma_5 \vartheta} \gamma_\mu \tilde{u} = 2p_\mu.$$ 

Taking into account (29) one easily finds

$$\bar{\tilde{\psi}} \gamma_\mu e^{\gamma_5 \vartheta} \tilde{\psi} = \{1, p/E\},$$

whence it follows that the operator $\eta$ in full compliance with Mostafazadeh’s result (see, for example, [18],[17]) induces the inner product

$$\bar{\tilde{\psi}} \eta \tilde{\psi} = 1$$

for $\tilde{\psi} \neq 0$. 

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3 Dirac modified models with $\gamma_5$-massive contributions in the external homogenies magnetic field

As it is known wave Dirac equations provide a basis for relativistic quantum mechanics and quantum electrodynamics of spinor particles in external electromagnetic fields. Solutions of relativistic wave equation are referred to as one-particle wave functions which allow the development of the approach known as the Furry picture. This method incorporates study the interactions with the external field exactly, regardless of the field intensity \[38\]. Beside there is no regular methods of describing such an arbitrariness explicitly. The physically most important exact solutions of the Dirac equations are: an electron in a Coulomb field, in a uniform magnetic field and in the field of a plane wave. In this connection, is of interest from investigating a modified non-Hermitian Dirac models which describe an alternative formulation of relativistic quantum mechanics where the Furry picture may be realized too.

Consider a uniform magnetic field $H = (0, 0, H)$ directed along the $x_3$ axis ($H > 0$). The electromagnetic potentials are chosen in the gage \[38\]

\[
A_0 = 0, \quad A_3 = 0, \quad A_1 = 0, \quad A_2 = Hx_1.
\]

We can write the modified Dirac equations in the form

\[
(\gamma_\mu P^\mu - meb_5)\Psi = 0,
\]

were $P^\mu = i\partial^\mu - eA^\mu$ ; $e = -|e|$ and $\gamma$- matrixes still chosen in the standard representation. In the field under consideration, the operators $P_0$, $P_2$ and $P_3$ are mutually commuting integrals of motion $[D, P_0] = 0, [D, P_2] = 0, [D, P_3] = 0$, where $D = (\gamma_\mu P^\mu - meb_5)$.

Let present the function $\Psi$ in the form

\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} e^{-iEt}
\]

and use Hamilton’s form of Dirac equations

\[
H\psi = E\psi
\]
where

\[ H = (\alpha \mathbf{p}) + \beta m_1 + \beta \gamma_5 m_2. \]

It is useful to introduce the change of variables \[38\]

\[ \psi(x_1, x_2, x_3) = e^{ip_2 x_2 + ip_3 x_3} \Phi(x_1) \]

we can obtain the following system of equations:

\[ (E \mp m_1) \Phi_{1,3} + iR_2 \Phi_{4,2} - (p_3 \mp m_2) \Phi_{3,1} = 0; \]  (47)

where \( R_2 = \left[ \frac{\partial}{\partial x_1} + (p_2 + eH) \right] \);

\[ (E \mp m_1) \Phi_{2,4} + iR_1 \Phi_{3,1} + (p_3 \pm m_2) \Phi_{4,2} = 0; \]  (48)

where \( R_1 = \left[ \frac{\partial}{\partial x_1} - (p_2 + eH) \right] \), and top mark relates to the components of the wave function with the first indexes, and the lower - to the components with the second indexes.

Next convenient to go to the dimensionless variable

\[ \rho = \sqrt{\gamma} x_1 + p_2 / \sqrt{\gamma}, \]  (49)

where \( \gamma = |e| H \), and equations (47), (48) take the form

\[ (E \mp m_1) \Phi_{1,3} + i\sqrt{\gamma} \left( \frac{d}{d\rho} + \rho \right) \Phi_{4,2} - (p_3 \mp m_2) \Phi_{3,1} = 0; \]  (50)

\[ (E \mp m_1) \Phi_{2,4} + i\sqrt{\gamma} \left( \frac{d}{d\rho} - \rho \right) \Phi_{3,1} + (p_3 \pm m_2) \Phi_{4,2} = 0. \]  (51)

General solution of this system can be represented in the form of the Hermite functions

\[ u_n(\rho) = \left( \frac{\gamma^{1/2}}{2^n n! \pi^{1/2}} \right) e^{-\rho^2 / 2} H_n(\rho), \]

where \( H_n(x) \) is standardizing the Hermite polynomials:

\[ H_n(x) = (-1)^n e^{x^2 / 2} \frac{d^n}{dx^n} e^{-x^2 / 2}. \]

In should be noted that Hermite function are satisfied to the recurrence relation:

\[ \left( \frac{d}{d\rho} + \rho \right) u_n = (2n)^{1/2} u_{n-1}; \]  (52)
\[
\left( \frac{d}{d\rho} - \rho \right) u_{n-1} = -(2n)^{1/2} u_n. \quad (53)
\]

It is easy to see from (52), (53) that
\[
\left( \frac{d}{d\rho} + \rho \right) \left( \frac{d}{d\rho} - \rho \right) u_n = -2nu_n
\]
and hence (see, for example [38])
\[
R_1 R_2 = -2\gamma n, \quad (54)
\]
where \( n = 0, 1, 2, \ldots \).

Substituting next in (50), (51), we have
\[
\Phi = \begin{pmatrix}
C_1 u_{n-1}(\rho) \\
C_2 u_n(\rho) \\
C_3 u_{n-1}(\rho) \\
C_4 u_n(\rho)
\end{pmatrix},
\]
and one can find that coefficients \( C_i \) \( (i = 1, 2, 3, 4) \) is determined by algebraic equations
\[
(E \mp m_1)C_{1,3} - (2\gamma n)^{1/2}C_{4,2} - (p_3 \mp m_2)C_{3,1} = 0;
\]
\[
(E \mp m_1)C_{2,4} - (2\gamma n)^{1/2}C_{3,1} + (p_3 \pm m_2)C_{4,2} = 0.
\]
The equality to zero of the determinant of this system leads to a spectrum of energy of the non-Hermitian Hamiltonian in the form
\[
E = \sqrt{m_1^2 - m_2^2 + 2\gamma n + p_3^2}, \quad (55)
\]
that with take into account \( m^2 = m_1^2 - m_2^2 \) coincide with the eigenvalues of energy of Hermitian Hamiltonian in the magnetic field [38].

The coefficients \( C_i \) may be determined if in as operator of polarization to choose the component of the tensor polarization in the direction of the magnetic field
\[
\mu_3 = m_1 \sigma_3 + \rho_2 [\vec{\sigma} \vec{P}] \quad (56)
\]
where matrices
\[
\sigma_3 = \begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix}; \quad \rho_2 = \begin{pmatrix}
0 & -iI \\
iI & 0
\end{pmatrix}.
\]
It is easy to see, that bispinor \( C \) can be written as

\[
\begin{pmatrix}
  C_1 \\
  C_2 \\
  C_3 \\
  C_4
\end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix}
  \cosh(\vartheta/2)\Phi_1 + \sinh(\vartheta/2)\Phi_3 \\
  \cosh(\vartheta/2)\Phi_2 + \sinh(\vartheta/2)\Phi_4 \\
  \sinh(\vartheta/2)\Phi_1 + \cosh(\vartheta/2)\Phi_3 \\
  \sinh(\vartheta/2)\Phi_2 + \cosh(\vartheta/2)\Phi_4,
\end{pmatrix}
\]

(57)

where

\[
\begin{align*}
\Phi_1 &= \sqrt{1 + \zeta m/p_\perp} \sin(\pi/4 + \lambda/2) \\
\Phi_2 &= \zeta \sqrt{1 - \zeta m/p_\perp} \sin(\pi/4 - \lambda/2) \\
\Phi_3 &= \zeta \sqrt{1 + \zeta m/p_\perp} \sin(\pi/4 - \lambda/2) \\
\Phi_4 &= \sqrt{1 - \zeta m/p_\perp} \sin(\pi/4 + \lambda/2).
\end{align*}
\]

Here \( \mu_3 \psi = \zeta k \psi \), \( k = \sqrt{p_\perp^2 + m^2} \) and \( \zeta = \pm 1 \) that is corresponding to the orientation of the fermion spin: along (+1) or opposite (−1) to the magnetic field, and parameter \( \lambda \) obey to the condition \( \cos \lambda = p_3/E \).

4 Non-Hermitian modified Dirac-Schwinger-Pauli model in the magnetic field

In this section, we will touch upon a question of describing the motion of Dirac particles, if their own magnetic moment is different from the Bohr magneton. As it was shown by Schwinger [39], that the Dirac equation of particles in the external electromagnetic field \( A_{\text{ext}} \) taking into account the radiative corrections may be represented in the form

\[
(\mathcal{P}\gamma - m)\Psi(x) - \int \mathcal{M}(x, y | A_{\text{ext}})\Psi(y)dy = 0,
\]

(58)

where \( \mathcal{M}(x, y | A_{\text{ext}}) \) is the mass operator of fermion in external field. From equation (58) by means of expansion of the mass operator in series according to \( eA_{\text{ext}} \) with precision not over then linear field terms one can obtain the modified equation (see, for example, [38]). This equation preserves the relativistic covariance and consistent with the phenomenological equation of Pauli obtained in his early papers.

Now let us consider the model of massive fermions with \( \gamma_5 \)-extension of mass \( m \rightarrow m_1 + \gamma_5 m_2 \) taking into account the interaction of their charges and anomalous magnetic moment (AMM) with the electromagnetic field \( F_{\mu\nu} \):
\[
\left( P_\mu \gamma^\mu - m_1 - \gamma_5 m_2 - \frac{\Delta \mu}{2} \sigma^{\mu\nu} F_{\mu\nu} \right) \Psi(x) = 0, \tag{59}
\]

where \( \Delta \mu = (\mu - \mu_0)/\mu_0 \) - AMM of fermion, \( \mu_0 = |e|/2m \) - the Bohr magneton, \( \sigma^{\mu\nu} = i/2(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \). Thus phenomenological constant which was introduced by Pauli \( \Delta \mu \), is part of the equation and gets the interpretation with the point of view quantum field theory (QFT).

The Hamiltonian form of (59) in the homogeneous magnetic field is the following
\[
i \frac{\partial}{\partial t} \Psi(r, t) = H_{\Delta \mu} \Psi(r, t), \tag{60}
\]

where
\[
H_{\Delta \mu} = \alpha P + \beta(m_1 + \gamma_5 m_2) + \Delta \mu \beta(\bar{\sigma}H). \tag{61}
\]

Given the quantum electrodynamics contribution in AMM with accuracy up to \( e^2 \) order we have \( \Delta \mu = \frac{\alpha}{2e} \mu_0 \), where \( \alpha = e^2 = 1/137 \) - the fine-structure constant and we still believe that the potential of an external field satisfies to the free Maxwell equations.

It should be noted that now the operator projection of the fermion spin at the direction of its movement \(- \bar{\sigma}P \) is not commute with the Hamiltonian (61) and hence it is not the integral of motion. The operator, which is commuting with this Hamiltonian and it is an integral of motion remains \( \mu_3 \) (see (56)). Subjecting the wave function \( \psi \) to requirement to be eigenfunction of the operator polarization (56) and Hamilton operator (61) we can obtain:
\[
\mu_3 \psi = \zeta k \psi, \quad \mu_3 = \begin{pmatrix}
m_1 & 0 & 0 & P_1 - iP_2 \\
0 & -m_1 & -P_1 - iP_2 & 0 \\
0 & -P_1 + iP_2 & m_1 & 0 \\
P_1 + iP_2 & 0 & 0 & -m_1
\end{pmatrix}, \tag{62}
\]

where \( \zeta = \pm 1 \) are characterized the projection of fermion spin at the direction of the magnetic field.

\[
H_{\Delta \mu} \psi = E \psi,
\]

\[
H_{\Delta \mu} = \begin{pmatrix}
m_1 + H\Delta \mu & 0 & P_3 - m_2 & P_1 - iP_2 \\
m_1 - H\Delta \mu & P_1 + iP_2 & m_1 - H\Delta \mu & 0 \\
0 & m_1 + H\Delta \mu & P_1 - iP_2 & -m_2 - P_3 \\
P_1 + iP_2 & m_2 - P_3 & -m_1 - H\Delta \mu & 0
\end{pmatrix}. \tag{63}
\]
Performing calculations in many ways reminiscent of similar calculations carried out in the previous section for the fermion energy may be find the exact solution:

\[ E(\zeta, p_3, 2\gamma n, H) = \sqrt{p_3^2 - m_2^2 + \left[ \sqrt{m_1^2 + 2\gamma n + \zeta \Delta \mu H} \right]^2} \]  \hspace{1cm} (64)

and eigenvalues of the operator polarization we can write in the form

\[ k = \sqrt{m_1^2 + 2\gamma n}. \]  \hspace{1cm} (65)

The expression analogous to (64), in the frame of ordinary Dirac-Schwinger-Pauli approach was previously obtained in exact form in the paper [40]. Direct comparison of modified formula (64) in the Hermitian limit \( m_2 \to 0 \) (and \( m_1 \to m \)) with the analogical result [40] shows their complete coincidence. It should also emphasize that the expression (64) contains dependence on mass parameters \( m_1 \) and \( m_2 \), which are not combined into a mass of particles \( m \).
Figure 3: The dependence of parameters $^+m_1/m, ^+m_2/m$ on the $x = m/M$.

Thus, in contrast to the cases described in the previous sections, here accounting of interaction AMM of fermions with the magnetic field allow to raise the question about the possibility of experimental studies of the influence of Non-Hermitian extensions of the fermion mass. In particular if to suggest that $m_2 = 0$ and hence $m_1 = m$, we return as it was noted early to the Hermitian limit. But taking into account the expressions (20) and (21) we obtain that the energetic spectrum dependence (64) is expressed through the fermion mass $m$ and the value of the maximal mass $M$. Taking into account that the AMM removes the degeneracy on spin we can obtain the energy of the ground fermion state $n = 0, p_3 = 0, \zeta = -1$ in the form

$$E(-1, 0, 0, H, x) = m \sqrt{-\left(\frac{1 \mp \sqrt{1 - x^2}}{x}\right)^2 + \left(\frac{\sqrt{2} \sqrt{1 \mp \sqrt{1 - x^2}}}{x} - \frac{\Delta \mu H}{m}\right)^2},$$

where $x = m/M$ and the upper sign corresponds to the ordinary particles and
the lower sign defines their "exotic" partners (see the footnote after formula (22)).

Through decomposition of functions $-m_1$ and $-m_2$ we can obtain

$$-\frac{m_1}{m} = \begin{cases} 1 + \frac{x^2}{8} + \frac{7x^4}{128}, & x \ll 1 \\ \frac{\sqrt{2}}{x} & x \rightarrow 1 \end{cases}$$

$$-\frac{m_2}{m} = \begin{cases} \frac{x}{2} + \frac{x^3}{8} + \frac{x^5}{16}, & x \ll 1 \\ \frac{1}{x} & x \rightarrow 1 \end{cases}$$

Similarly, for $+m_1$ and $+m_2$ we have

$$+\frac{m_1}{m} = \begin{cases} \frac{2}{x} - \frac{x}{4} - \frac{5x^3}{64}, & x \ll 1 \\ \frac{\sqrt{2}}{x} & x \rightarrow 1 \end{cases}$$

$$+\frac{m_2}{m} = \begin{cases} \frac{2}{x} - \frac{x}{2} - \frac{x^3}{8}, & x \ll 1 \\ \frac{1}{x} & x \rightarrow 1 \end{cases}$$

(67)

Let us now turn to a more detailed consideration of the fermion energy in the ground state in the external field. As follows from (67) and (68) function (64) not trivial depends on the parameters $x = m/M$ and $H$. For reasons outlined above, the effect of magnetic field on the lowest-energy state of the fermion with the small mass $x \ll 1$ (for example, the mass of ordinary electron $m$) and also considering the smallness of the constants of interaction AMM with magnetic field $\Delta \mu = \alpha/2\pi\mu_0$ we can write

$$E(-1, 0, 0, H, x) = m \left[ 1 - \frac{\alpha}{4\pi} \frac{H}{H_c} (1 + x^2/8 + x^4/128) \right],$$

(69)

where $H_c = m^2/|e| = 4.41 \cdot 10^{13}$ G-the critical quantum electrodynamic field.

On the other hand, for the case of the "exotic" particles (with a mass of electron $m$ and $\Delta \mu = \alpha/2\pi \cdot \mu_0$) in this limit $x \ll 1$ one can obtain a result which significantly different from previous one (see (68) and Fig.3)

$$E(-1, 0, 0, H, x) = m \left( 1 - \frac{\alpha}{2\pi} \frac{H}{xH_c} \right).$$

(70)

From (70) should be that the field corrections in this case is increased substantially. Note that under fixed parameters intensity of the field and the mass of the fermion from (70) one can judge about of a maximal mass value of $M$. 

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We can see from (67), (68) that the changes of the parameters $\pm m_1$ and $\pm m_2$ occur such a way that in the point $x = 1$ ($m = M$) functions corresponding to the ordinary and exotic particles are crossed. At Fig. 2 and Fig. 3 dependencies $m_1^\mp /m$ and $m_2^\pm /m$ on the parameter $x = m/M$ are represented and one may also clearly see the justice of this fact.

5 Summary and Conclusions.

In the researches, presented in the previous sections, we have shown that the Dirac Hamiltonian of a particle with $\gamma_5$-mass term has the unbroken $\mathcal{PT}$-symmetry in the area $m_1^2 \geq m_2^2$ which however has a number of sub-regions (see (25)). In this regard, more informative is the transition from the variables $m_1, m_2$ to variables $m, M$ according to (20), (21). In particular, one is obtained the restriction of the particle mass in this model: $m \leq M$. This conditions describe the new boundaries definition of the unbroken region $\mathcal{PT}$-symmetry of the modified Hamiltonians.

In addition, we have shown that the introduction of the postulate about the limitations of the mass spectrum, lying in the ground of the geometric approach to the development of the modified QFT (see, for example [8], [9]) leads to the appearance of non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians in the fermion sector of the model with the Maximal Mass. But conversely: using of non-Hermitian $\mathcal{PT}$-symmetric quantum theory with $\gamma_5$ mass term may be considered as conditions of the appearance of the limitation of the mass particle in a fermion sector of the model.

In particular, this applies to the modified Dirac equation in which produced the substitution $m \rightarrow m_1 + \gamma_5 m_2$. Into force of the ambiguity of the definition of parameters $m_1, m_2$ the inequality $m_1 \geq m_2 \geq 0$ describes a particle of two types. In the first case, it is about ordinary particles, when mass parameters are limited by the terms

$$0 \leq m_2 \leq m_1 / \sqrt{2}. \quad (71)$$

In the second area we are dealing with so-called exotic partners of ordinary particles, for which is still accomplished (7), but one can write

$$m_1 / \sqrt{2} \leq m_2 \leq m_1. \quad (72)$$

Intriguing difference between particles of the second type from traditional fermions is that they are described by the other modified Dirac equations.
So, if in the first case, the equations of motion have a limit transition when \( m_2 \to 0 \) that leads to the standard Dirac equation, however in the inequality \( (72) \) the such a limit is not there.

Thus, it is shown that the main progress, is obtained by us in the algebraic way of the construction of the fermion model with \( \gamma_5 \)-mass term is consists of describing the new energy scale, which is defined by the parameter \( M = m_1^2 / 2m_2 \). This value on the scale of the masses and serves as a point of transition from the ordinary particles to exotic. Furthermore, the possibility of describing of the exotic particles are turned out essentially the same as in the model with a maximal mass, which was investigated by V.G.Kadyshevsky with colleagues on the basis of geometrical approach.

We have presented a number of examples of non-Hermitian models with \( \gamma_5 \)-extension mass in relativistic quantum mechanics including in presence of external electromagnetic field for which the Hamiltonian \( H \) has a real spectrum. Although the energy spectra of the fermions in some cases were makes them indistinguishable from the spectrum of corresponding Hermitian Hamiltonian \( H_0 \) we found example, in which the energy of fermions is clearly dependent on non-Hermitian characteristics. We are talking about the consideration of the interaction of the anomalous magnetic moment of fermions with a magnetic field. In this case we obtained the exact solution for the energy of fermions (see \( (64) \)).

It should be noted that the formula \( (64) \) is a valid not only for charged fermions, but and for the neutral particles possessing AMM. In this case you must simply replace the square of quantized transverse momentum of a charged particle in a magnetic field on the ordinary value \( 2\gamma n \to p_1^2 + p_2^2 \). Thus, for the case of ultra cold polarized exotic electron neutrino we can write

\[
E(-1, 0, 0, H, x) = m \left( 1 - \frac{\mu_{\nu e}}{\mu_0} \frac{MH}{m_{\nu e} H e} \right). \tag{73}
\]

It is well known \[42], [43] that in the minimally extended Standard Model the one-loop radiative correction generates neutrino magnetic moment which is proportional to the neutrino mass

\[
\mu_{\nu} = \frac{3}{8\sqrt{2} \pi^2} |e| G_F m_{\nu} = \left( 3 \cdot 10^{-19} \right) \mu_0 \left( \frac{m_{\nu}}{1 eV} \right), \tag{74}
\]

where \( G_F \)-Fermi coupling constant and \( \mu_0 \) is Bohr magneton. However, so far, the most stringent laboratory constraints on the electron neutrino magnetic
moment come from elastic neutrino-electron scattering experiments: $\mu_{\nu_e} < (1.5 \cdot 10^{-10})\mu_0$.

Besides the discussion of problem of measuring the mass of neutrinos (either active or sterile) show that for an active neutrino model we have $\sum m_\nu = 0.320\text{eV}$, whereas for a sterile neutrino $\sum m_\nu = 0.06\text{eV}$ [45]. One can also estimate the change of the border of region of unbroken $\mathcal{PT}$-symmetry due to the shift of the lowest-energy state in the magnetic field, using formula (73)

$$\frac{\mu_{\nu_e} \cdot MH}{\mu_0 m_{\nu_e} H_c} \leq 1.$$ 

Indeed let us take the following parameters of neutrino: the neutrino mass equal to $m_\nu = 1\text{eV}$ and magnetic moment equal to (74). If we assume that the values of mass and magnetic moment of exotic neutrino identical to parameters of ordinary neutrinos, and also assuming that the magnetic fields is equal to $H_c$ we can obtain the estimate of the border area undisturbed $\mathcal{PT}$ symmetry in the form

$$M \frac{m_{\nu}}{m_{\nu}} = \mu_0.$$  (75)

Omitting the numerical coefficients from (75),(74) we can find the magnitude of the limit value of the parameter $M$:

$$M = \frac{1}{2G_F \cdot m} = 10^5 \cdot \frac{m_n^2}{2m} \approx 10^8 \text{GeV},$$  (76)

where $m_n$ - the nucleon mass and $m$ - mass of an electron.

The obtained formulas (73),(75) convincingly shows, that with their help it is possible, in principle, experimentally evaluate the possibility of the existence of the Maximal Mass at low energies. However it should be noted that a considerable increase of investigated field corrections is connected with regard to the possible contributions from the so-called exotic particles, but justice applied calculation methods in these conditions as a matter of fact require a discussion.

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