Further study on elliptic interpolation formulas for the elliptic Askey-Wilson polynomials and allied identities

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Abstract

In this paper, we introduce the so-called elliptic Askey-Wilson polynomials which are homogeneous polynomials in two special theta functions. With regard to the significance of polynomials of such kind, we establish some general elliptic interpolation formulas by the methods of matrix inversions and of polynomial representations. Furthermore, we find that the basis of elliptic interpolation space due to Schlosser can be uniquely characterized via the elliptic Askey-Wilson polynomials. As applications of these elliptic interpolation formulas, we establish some new elliptic function identities, including an extension of Weierstrass’ theta identity, a generalized elliptic Karlsson-Minton type identity, and an elliptic analogue of Gasper’s summation formula for very-well-poised $6_{2m}\phi_{5+2m}$ series.

\textbf{Keywords:} elliptic hypergeometric series; elliptic Askey-Wilson polynomial; theta function; triple product; elliptic interpolation; basis; matrix inversion; symmetric difference; polynomial representation; summation and transformation; Weierstrass’ theta identity.

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1. Introduction

Throughout this paper, we will adopt the standard notation and terminology for basic and elliptic hypergeometric series found in the book [12] by Gasper and Rahman. For

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instance, the $p$-shifted factorial with $|p| < 1$ is defined by

$$(x; p)_n := \prod_{n=0}^{\infty} (1 - xp^n)$$

and $$(x; p)_\infty := \frac{(x; p)_n}{(p^n x; p)_\infty}$$

for any integer $n$. Its multi-parameter form is compactly abbreviated to

$$(x_1, x_2, \ldots, x_m; p)_n := \prod_{k=1}^{m} (x_k; p)_n.$$ We also need the modified Jacobi theta function with argument $x \neq 0$ and norm $p$

$$\theta(x; p) := (x, p/x; p)_\infty.$$ (1.1)\n
The well known Jacobi triple product identity (cf. [12, (II.28)]) asserts that

$$\theta(x; p) = \frac{1}{(p; p)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n p^{n(n-1)/2} x^n.$$ (1.2)\n
By convention, the multi-parameter notation

$$\theta(x_1, x_2, \ldots, x_m; p) := \prod_{k=1}^{m} \theta(x_k; p).$$

We also adopt the notation

$$(x, q, p)_n := \prod_{k=0}^{n-1} \theta(xq^k; p)$$

for the theta $q, p$-shifted factorial (cf. [12, Eq. (11.2.5)]) together with

$$(x_1, x_2, \ldots, x_m; q, p)_n := \prod_{k=1}^{m} (x_k; q, p)_n.$$\n
An $r+1E_r$ theta hypergeometric series with base $q$, norm $p$ and argument $x$ is defined to be

$$r+1E_r \left[ \begin{array}{c} a_1, a_2, \ldots, a_r, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{array}; q, p; x \right] := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q, p)_n}{(q, b_1, b_2, \ldots, b_r; q, p)_n} x^n.$$ (1.3)\n
Further, if $a_1 q = a_2 b_1 = a_3 b_2 = \ldots = a_{r+1} b_r$ and

$$a_2 = qa_1^{1/2}, a_3 = -qa_1^{1/2}, a_4 = q(a_1/p)^{1/2}, a_5 = -q(a_1/p)^{1/2},$$

the $r+1E_r$ is called very-well-poised (VWP). In particular, when the norm $p = 0$ in (1.3), the $r+1E_r$ reduces to the usual basic hypergeometric VWP $r+1\phi_r$ series. Moreover, the $r+1V_r$ VWP elliptic hypergeometric series (cf. [12, Eq. (11.2.19)]) is defined to be

$$r+1V_r(a_1; a_6, \ldots, a_{r+1}; q, p; x) := \sum_{n=0}^{\infty} \frac{\theta(a_1 q^{2n}; p)}{\theta(a_1; p)} \frac{(a_1, a_6, \ldots, a_{r+1}; q, p)_n(qx)^n}{(q, a_1 q^2/a_6, \ldots, a_1 q/a_{r+1}; q, p)_n}. (1.4)$$
When $x = 1$, we denote such series in shorthand notation

$$r_{+1}V_r(a_1; a_6, \ldots, a_{r+1}; q, p).$$

The $r_{+1}V_r$ is elliptically balanced if and only if $(qa_6a_7 \cdots a_{r+1})^2 = (a_1q)^{r-5}$.

As it turns out, function expansions [29] and polynomial interpolations [10] are two rather old and fundamental subjects in Approximation Theory and Numerical Analysis. Both have also received great attention in the context of various (ordinary, basic, elliptic) hypergeometric series [1, 2, 6, 14, 18, 19, 27, 28] so far. In this regard, Ismail [14] established a $q$-Taylor theorem expanding of functions in terms of the Askey-Wilson monomial basis.

**Theorem 1.1** (Cf. [14, Theorem 1.3]). If $f(x)$ is a polynomial of degree $N$, then

$$f(x) = \sum_{k=0}^{N} f_k \phi_k(x; a), \quad (1.5)$$

where

$$f_k := \frac{(q - 1)^k}{(2a^k(q; q)_k)} q^{-k(k-1)/4} (D_q^{(k)} f)(x_k), \quad x_k := (aq^{k/2} + q^{-k/2}/a)/2.$$  

In the above, the Askey-Wilson monomials $\phi_k(x; a)$ are defined by

$$\phi_k(x; a) := (ae^{i\theta}, ae^{-i\theta}; q)_k, \quad x = \cos \theta$$

and the Askey-Wilson operator is defined to be

$$(D_q f)(x) := \frac{\tilde{f}(q^{1/2}e^{i\theta}) - \tilde{f}(q^{-1/2}e^{i\theta})}{\tilde{t}(q^{1/2}e^{i\theta}) - \tilde{t}(q^{-1/2}e^{i\theta})},$$

where $\tilde{t}(t) := (t + 1/t)/2, \tilde{f}(t) := f(t)$. As usual, for integer $k \geq 1, D_q^{(k)} = D_q(D_q^{(k-1)}), D_q^{(1)} = D_q$.

We should remark that it is this theorem by which Ismail [14] and Stanton [17] presented a unified approach to some basic results from basic hypergeometric series such as the $q$-Pfaff-Saalschutz $3\phi_2$ [12, (II.12)] and Jackson’s VWP $8\phi_7$ summation formulas [12, (II.22)], Sears’ $4\phi_3$ [12, (III.15)] and Watson’s $8\phi_7$ series transformations [12, (III.17)], while some new proofs were given in [18] by Ismail and Simeonov. Also, by the same theorem, Cooper [6] proved Watson’s VWP $6\phi_5$ summation formula [12, (II.20)]. In their paper [15] published in 2003, Ismail and Stanton extended the above polynomial $q$-Taylor theorem to one for entire functions of exponential growth. After that, by combining the above
$q$-Taylor theorem with the idea of polynomial interpolations, they [17] further put forward a Lagrange-type interpolation formula for polynomials. The meaning of interpolation is that any polynomial $f(x)$ of degree $N$ is completely determined by its evaluation at the $N+1$ special (interpolation) points, say $\xi(x_k), k = 0, 1, \ldots, N$.

**Theorem 1.2** (Cf. [17, Theorem 3.4]). With the same notation as above. For polynomial $f(x)$ of degree at most $N$ and with $x = \cos \theta$, we have the expansion

$$f(x) = \sum_{k=0}^{N} \frac{(q, a^2 q^2; q)_N}{(q a e^{i \theta}, q e^{-i \theta}; q)_N} \frac{1 - a^2 q^{2k}}{1 - a^2} \frac{(a^2, a e^{i \theta}, q^{-N}; q)_k}{(q, a e^{i \theta}, q e^{-i \theta}, a^2 q^{-N}; q)_k} q^{k(1+N)} f(a q^k).$$

(1.6)

One of keys to (1.6) is, as pointed out in [16, Section 12.2] by Ismail and Stanton, that the set $\{\phi_k(x; a)|0 \leq k \leq N\}$ forms a basis for the vector space of polynomials, named the Askey-Wilson basis. With regard to studying elliptic hypergeometric series, Schlosser [27] extended Ismail’s $q$-Taylor theorem to the elliptic analogues. One of his main results can be restated as follows.

**Theorem 1.3** (Cf. [27, Theorem 4.2]). Let $W^N_e$ be the linear space spanned by the set

$$\left\{ \frac{g_k(x)}{(c x, c/x; q, p)_k} \right\}_{k=0}^{N},$$

(1.7)

where, for integer $k : 0 \leq k \leq N$,

$$g_k(x) = g_k(1/x),$$

$$g_k(px) = \frac{1}{p^k x^k} g_k(x).$$

(1.8)

Then, for any $f(x) \in W^N_e$, we have the expansion

$$f(x) = \sum_{k=0}^{N} f_k (ax, a/x; q, p)_k (c x, c/x; q, p)_k,$$

(1.9)

where

$$f_k := \frac{(-1)^k q^{-k(k-1)/4} \theta(q; p)^k}{(2a)^k (q, c/a, acq^{k-1}; q, p)} (D^{(k)}_{c,q,p} f)(aq^{k/2})$$

and the well-poised elliptic Askey-Wilson operator $D_{c,q,p}$ is defined by

$$(D_{c,q,p} f)(x) := 2q^{1/2} x \frac{\theta(c x q^{-1/2}, c x q^{1/2}, c q^{-1/2}/x, c q^{1/2}/x; p)}{\theta(q, x^2; p)} (f(q^{1/2} x) - f(q^{-1/2} x)).$$

Just like the Askey-Wilson basis which is crucial to Ismail’s expansion theorem, the set

$$\left\{ \frac{(ax, a/x; q, p)_k}{(c x, c/x; q, p)_k} \right\}_{k=0}^{N}$$
is proved to be a basis of $W^N_c$ in [27, Lemma 4.1] by Schlosser and plays an important role in Theorem 1.3. As further applications of his expansion theorem, Schlosser and Yoo [28] successfully extended Ismail and Stanton’s interpolation formula (1.6) to the following

**Theorem 1.4** (Cf. [28, Theorem 2.6]). If $f(x)$ is in $W^N_c$, then

$$\frac{(q, a^2 q, cx, c/x; q, p)_N}{(ac, c/a, aqx, aqx/x; q, p)_N} f(x) = \sum_{k=0}^{N} q^k \frac{\theta(a^2q^2k; p)}{\theta(a^2; p)} \frac{(a^2, aq/cx, ax, a/x, acq^N, q^{-N}; q, p)_k}{(q, ac, aqx, aqx/x, aq^{1-N}/c, a^2q^{N+1}; q, p)_k} f(aq^k).$$

In regard to applications of polynomial interpolations to $q$-series, one might not ignore a series of research works [3, 4, 9] by Chen, Fu and Lascoux. Indeed, as a discrete analogue of the aforementioned interpolation formulas, it is proved in [4] by Chen and Fu that

**Theorem 1.5** (Cf. [4, Theorem 1.1]). Let $N \geq 0$ be integer and $f(x)$ a theta function satisfying

$$f(x) = (px^2/c)^N f(c/(px)).$$

Then we have

$$f(x) = \sum_{k=0}^{N} C_k \prod_{i=1}^{k} \theta(x/b_i, c/(b_i x); p) \prod_{i=1}^{N-k} \theta(x/x_i, c/(x_i x); p),$$

where

$$C_k := \frac{f(b_1)}{\prod_{i=1}^{N-k+1} \theta(b_1/x_i, c/(x_i b_1); p)} \delta_{1(b)} \delta_{2(b)} \cdots \delta_{k(b)} \frac{\theta(b_{k+1}/x_{N-k+1}, c/(x_{N-k+1} b_{k+1}); p).}{\times \theta(b_{k+1}/x_{N-k+1}, c/(x_{N-k+1} b_{k+1}); p).}$$

Recall that the divided difference operator $\delta_{(a)}$ acting on the left of function $f$ in variables $\{a_n\}_{n \geq 1}$ is defined by

$$f(\ldots, a_i, a_{i+1}, \ldots) \delta_{(a)} = f(\ldots, a_i, a_{i+1}, \ldots) - f(\ldots, a_i, a_{i+1}, \ldots) \frac{\theta(a_{i+1}/a_i, c/(a_i a_{i+1}); p)}{\theta(a_{i+1}/a_i, c/(a_i a_{i+1}); p)}.$$

A full treatise on the operator $\delta_{(a)}$, acting on symmetric functions and applications to rational interpolation can be found in [21] due to Lascoux.

Recently, by means of the $(f, g)$-inversion formula, one of the authors set up in [30] that

**Theorem 1.6** (Cf. [30, Theorem 1.6]). Define

$$P(x) := \theta(-x^2; p^2)(-p^2; p)_\infty, \quad Q(x) := x \theta(-px^2; p^2)(-p; p)_\infty.$$
Then, for arbitrary polynomial $\sum_{k=0}^{N} \lambda_k x^k$ of degree at most $N$, we have

$$
\sum_{k=0}^{N} \lambda_k P(x)^k Q(x)^{N-k} = x^N \sum_{k=0}^{N} H_k(N) b_k \theta(x_k b_k, x_k/b_k; p) \\
\times \prod_{i=0}^{k-1} \theta(b_i x, b_i/x; p) \prod_{i=k+1}^{N} \theta(x_i x, x_i/x; p),
$$

where the coefficients

$$
H_k(N) := \sum_{j=0}^{N-k} \frac{\sum_{j=0}^{N} \lambda_j P(b_j)^j Q(b_j)^{N-j}}{\prod_{i=n}^{N} \theta(x_i b_k, x_i/b_k; p) \prod_{i=0, i \neq k}^{n} \theta(b_i b_k, b_i/b_k; p)}. 
$$

Especially noteworthy is that one of the most interesting results in [30] reveals a surprising fact: Weierstrass’ theta identity (cf. [12, Exercise 2.16(i)])

$$
\theta(xa, x/a, bc, b/c; p) - \theta(xc, x/c, ab, b/a; p) = \frac{b}{a} \theta(xb, x/b, ac, a/c; p) 
$$

is equivalent to the almost self-evident algebraic identity

$$(x-a)(b-c) + (x-b)(c-a) + (x-c)(a-b) = 0. 
$$

We refer the reader to Koornwinder’s paper [20] for the history and applications of Weierstrass’ theta identity to the theory of theta functions.

Before stating our main theorems, we first need to introduce a new kind of polynomials.

**Definition 1.7.** Let $P(x)$ and $Q(x)$ be given by (1.13). Any homogeneous polynomial in $P(x)$ and $Q(x)$ of degree $N$ in the form

$$
\sum_{k=0}^{N} \lambda_k P(x)^k Q(x)^{N-k}
$$

is called an elliptic Askey-Wilson polynomial of degree $N$. For brevity, we will use the notation $\mathcal{L}_N(P(x), Q(x))$ to denote the set of all homogeneous polynomials in $P(x)$ and $Q(x)$ of degree $N$.

**Remark 1.8.** The reason why we call (1.18) the elliptic Askey-Wilson polynomial is that

$$(ax, a/x; q, p)_k = \prod_{i=0}^{k-1} \theta(aq^i x, aq^i/x; p) = \frac{1}{x^k} \prod_{i=0}^{k-1} (P(aq^i)Q(x) - Q(aq^i)P(x)). 
$$

See Lemma 2.6 below for the second equality. The special case $p = 0$ leads us to the Askey-Wilson monomials $\phi_k(x; a) = (ax, a/x; q)_k$.

In the sense of Definition 1.7, it is easily seen that
Proposition 1.9. For any integers $m, n \geq 0$, if $f(x) \in \mathcal{L}_m(P(x), Q(x))$, $g(x) \in \mathcal{L}_n(P(x), Q(x))$, then $f(x)g(x) \in \mathcal{L}_{m+n}(P(x), Q(x))$.

The main purpose of this paper is, as further development of [30], to establish a few new interpolation formulas for any elliptic Askey-Wilson polynomials. At first, we can show a new interpolation formula which can be regarded as an explicit version of Theorem 1.5 of Chen and Fu.

Theorem 1.10. For any integer $N \geq 0$, let $\{x_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ be two sequences such that $x_k \neq b_i, b_i \neq b_j, 0 \leq i \neq j, k \leq N$. For any $f(x) \in \mathcal{L}_N(P(x), Q(x))$, we have

$$f(x) = x^N \sum_{k=0}^{N} b_k H_k(N) \prod_{i=0}^{k-1} \theta(b_i x, b_i / x; p) \prod_{i=1}^{N-k} \theta(x_i x, x_i / x; p),$$

(1.20)

where, for any $n \leq N$, the coefficient

$$H_n(N) = \sum_{k=0}^{n} \frac{f(b_k)}{b_k^{N+1}} \frac{\theta(x_{N-n+1} b_n, x_{N-n+1} / b_n; p)}{\theta(x_{N-n+1} b_k, x_{N-n+1} / b_k; p)} \prod_{i=1}^{N-n} \frac{1}{\theta(x_i b_k, x_i / b_k; p)} \prod_{i=0,i \neq k}^{n} \frac{1}{\theta(b_i b_k, b_i / b_k; p)}.$$  

(1.21)

Closely related with Theorem 1.6 and Weierstrass’ theta identity (1.16) is the following Lagrange-type interpolation formula.

Theorem 1.11. With the same assumption as Theorem 1.10. Then for any $f(x) \in \mathcal{L}_N(P(x), Q(x))$, we have

$$\frac{f(x)}{x^N} = \sum_{k=0}^{N} \frac{f(b_k)}{b_k^{N}} \prod_{i=0,i \neq k}^{n} \frac{\theta(b_i x, b_i / x; p)}{\theta(b_i b_k, b_i / b_k; p)}.$$  

(1.22)

Furthermore, using Theorem 1.10, we can establish

Theorem 1.12. For any $f(x) \in \mathcal{L}_N(P(x), Q(x))$, we have the expansion

$$\left( \frac{C}{x} \right)^N \frac{(q, C^2 q; q, p)_N}{(C x, C q / x; q, p)_N} f(x) = \sum_{k=0}^{N} q^k \frac{\theta(C^2 q^{2k}; p)}{\theta(C^2; p)} \frac{(C^2, C / x, C x, q^{-N}; q, p)_k}{(q, C x q, C q / x, C^2 q^{N+1}; q, p)_k} f(C q^k).$$  

(1.23)

Once taking Proposition 1.9 into account, we can show an even more general interpolation formula.

Theorem 1.13. Assume that $f(x) \in \mathcal{L}_{N_i}(P(x), Q(x))$. For $m$ integers $N_i \geq 0$, let $N = \sum_{i=0}^{m} N_i$. Then we have the expansion

$$\left( \frac{C}{x} \right)^N \frac{(q, C^2 q; q, p)_N}{(C x, C q / x; q, p)_N} \prod_{i=1}^{m} \left( \frac{(A_i x, A_i / x; q, p)_N}{(A_i C, A_i / C; q, p)_N} \right) f(x)$$

$$= \sum_{k=0}^{N} q^k \frac{\theta(C^2 q^{2k}; p)}{\theta(C^2; p)} \frac{(C^2, C / x, C x, q^{-N}; q, p)_k}{(q, C x q, C q / x, C^2 q^{N+1}; q, p)_k} f(C q^k) \prod_{i=1}^{m} \frac{(A_i C q^{N_i}, C q / A_i; q, p)_k}{(C q^{N_i+1} / A_i, A_i C; q, p)_k}.$$  

(1.24)
Notice that there is a subtle difference between Theorem 1.4 and the special case \( m = 1 \) of Theorem 1.13. It is this difference that inspires us to explore \( W^N_c \). Up to this point, we find a new characteristic of \( W^N_c \), which states that \( x^N g_N(x) \in L_N(P(x), Q(x)) \) being an elliptic Askey-Wilson polynomial of degree \( N \). For comparison purpose, we mention here that \( g_N(x) \) is referred to as \( BC_1 \) theta functions of degree one by Rains [24, Definition 1] and to as \( D_N \) theta functions by Rosengren and Schlosser [25, Definition 3.1], respectively. The reader may consult loc.cit. for their full exposition.

**Theorem 1.14.** Let \( P(x) \) and \( Q(x) \) be given by (1.13), and \( g_m(x) \) be such a function that

\[
g_m(x) \in W^N_c \quad (m \leq N).
\]

Then, there must exist a sequence of complex numbers \( \{\lambda_k\}_{k=0}^{N} \) being independent of \( x \), such that

\[
g_m(x) = \frac{1}{x^m} \left( \sum_{k=0}^{N} \lambda_k P(x)^k Q(x)^{N-k} \right) \prod_{k=0}^{N-m-1} \frac{1}{P(cq^{m+k})Q(x) - Q(cq^{m+k})P(x)}.
\]

(1.25)

In particular,

\[
g_N(x) = \frac{1}{x^N} \sum_{k=0}^{N} \lambda_k P(x)^k Q(x)^{N-k}.
\]

(1.26)

The rest of our paper is organized as follows. In succeeding section, some preliminary results about the \((f, g)\)-inversion formula and polynomial expansions are given in details. They are keys to Theorems 1.10 and 1.11. The full proofs of the main theorems are given in Section 3. Some elliptic function identities including an extension of Weierstrass’ theta identity and an elliptic analogue of Gasper’s summation formula for VWP \( \phi_{6+2m, 5+2m} \) series, as well as a generalized elliptic Karlsson-Minton type identity are presented in Section 4.

2. Preliminaries

One of our main ingredients for Theorems 1.10 and 1.11, instead of the Askey-Wilson operator \( D \) and the divided difference operator \( \delta_n \), is the technique of matrix inversions (in the sense of (2.3)). Among matrix inversions, a typical result is the following

**Lemma 2.1** (The \((f, g)\)-inversion formula. Cf. [22, Theorem 1.3]). Let \( A = (A_{n,k})_{n \geq k \geq 0} \) and \( B = (B_{n,k})_{n \geq k \geq 0} \) be a pair of infinite-dimensional lower-triangular matrices with entries
given by

\[ A_{n,k} = \frac{\prod_{i=0}^{n-1} f(x_i, b_k)}{\prod_{i=k+1}^{n} g(b_i, b_k)} \quad \text{and} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
To proceed further, we need two expansion formulas of polynomials. The first one is Lemma 3.2 of [30]. We record its proof below for completeness.

**Lemma 2.4.** Let \( \{x_n\}_{n \geq 0} \) and \( \{b_n\}_{n \geq 0} \) be two sequences such that \( x_k \neq b_i, b_i \neq b_j, 0 \leq i \neq j, k \leq N \). Then for any polynomial \( f(x) \) of degree at most \( N \), there holds

\[
f(x) = \sum_{k=0}^{N} \lambda_k \prod_{i=0}^{k-1} (b_i - x) \prod_{i=k+1}^{N} (x_i - x),
\]

where, for \( n \leq N \),

\[
\lambda_n = (x_n - b_n) \sum_{k=0}^{n} f(b_k) \prod_{i=n}^{k-1} \frac{1}{x_i - b_k} \prod_{i=0, i \neq k}^{n} \frac{1}{b_i - b_k}.
\]

**Proof.** It suffices to show that any polynomial \( f(x) \) in \( x \) of degree at most \( N \) can be expressed as a linear combination of polynomials

\[
\left\{ \prod_{i=0}^{k-1} (b_i - x) \prod_{i=k+1}^{N} (x_i - x) \right\}_{k=0}^{N}.
\]

To prove this, let us assume that

\[
f(x) = \sum_{k=0}^{N} \lambda_k \prod_{i=0}^{k-1} (b_i - x) \prod_{i=k+1}^{N} (x_i - x). \tag{2.11}
\]

It remains to determine the coefficients \( \lambda_k \). For this, we observe that (2.11) corresponds to the special case of Lemma 2.3, in which \( f(x, y) = g(x, y) = x - y \) and

\[
F(x) = \frac{f(x)}{\prod_{i=1}^{N} (x_i - x)}.
\]

Solving (2.11) for \( \lambda_k \) by Lemma 2.3, we obtain

\[
\lambda_n = (x_n - b_n) \sum_{k=0}^{n} F(b_k) \prod_{i=n}^{k-1} (x_i - b_k) \prod_{i=0, i \neq k}^{n} (b_i - b_k)
\]

\[
= (x_n - b_n) \sum_{k=0}^{n} \frac{f(b_k)}{\prod_{i=n}^{k} (x_i - b_k) \prod_{i=0, i \neq k}^{n} (b_i - b_k)}.
\]

The lemma is confirmed. \(\Box\)

The above conclusion suggests that the set

\[
\left\{ \prod_{i=0}^{k-1} (b_i - x) \prod_{i=k+1}^{N} (x_i - x) \middle| x_i \neq b_j, i \neq j, 0 \leq k \leq N \right\}
\]

is a basis of vector space of polynomials of degree at most \( N \). From this point forward, we proceed to find another basis for this vector space.
Lemma 2.5. Let $N \geq 0$ be integer and $(x_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two sequences such that $x_k \neq b_i, b_i \neq b_j, 0 \leq i \neq j, k \leq N$. Then for any polynomial $f(x)$ of degree at most $N$, there holds

$$f(x) = \sum_{k=0}^{N} \lambda_k \prod_{i=0}^{k-1} (b_i - x) \prod_{i=1}^{N-k} (x_i - x),$$

(2.12)

where, for $n \leq N$,

$$\lambda_n = \sum_{k=0}^{n} f(b_k) \frac{x_{N-n+1} - b_n}{x_{N-n+1} - b_k} \prod_{i=1}^{N-n} \frac{1}{x_i - b_k} \prod_{i=0, i \neq k}^{n} \frac{1}{b_i - b_k}.$$  

(2.13)

Proof. As previously, it suffices to show that any polynomial $f(x)$ in $x$ of degree at most $N$ can be expressed as a linear combination of polynomials

$$f(x) = \sum_{k=0}^{N} \lambda_k \prod_{i=0}^{k-1} (b_i - x) \prod_{i=1}^{N-k} (x_i - x).$$

(2.14)

All we need to do is to find the coefficients $\lambda_k$. Unlike the proof for Lemma 2.4, we proceed to find $\lambda_k$ via the use of recurrence relations. To that end, we take $x = b_n$ in (2.14) for $n = 0, 1, 2, \ldots, N$ in succession. Then we have a system of linear equations in $N + 1$ unknown $\lambda_k$’s as below:

$$f(b_n) = \sum_{k=0}^{n} \lambda_k \prod_{i=0}^{k-1} (b_i - b_n) \prod_{i=1}^{N-k} (x_i - b_n).$$

(2.15)

Or equivalently, in terms of linear algebras,

$$AX = \beta,$$

(2.16)

where the $(n, k)$-entry of the coefficient matrix $A$

$$A_{n,k} = \prod_{i=0}^{k-1} (b_i - b_n) \prod_{i=1}^{N-k} (x_i - b_n),$$

$X$ and $\beta$ stand, respectively, for the $(N + 1)$-dimensional column vectors such that

$$X^T := (\lambda_0, \lambda_1, \ldots, \lambda_N), \quad \beta^T := (f(b_0), f(b_1), \ldots, f(b_N)).$$
where the superscript \( T \) denotes the transpose of vectors. As it is easily seen, the coefficient matrix \( A \) is lower-triangular with the \((n, n)\)-entry

\[
A_{n, n} = \prod_{i=0}^{n-1} (b_i - b_n) \prod_{i=1}^{N-n} (x_i - b_n) \neq 0.
\]

Hence the linear equations (2.16) has the unique solution. By solving (2.16) for \( \lambda_k \), we obtain

\[ X = A^{-1} \beta, \]

viz., for \( A^{-1} = (B_{n,k}) \),

\[
A_n = \sum_{k=0}^{n} B_{n,k} f(b_k). \tag{2.17}
\]

It remains to find any explicit expression for \( B_{n,k} \). To do this, we first set up certain recurrence relation for the entries of \((B_{n,k})\). What we obtained is that for any \( n - k \geq 1 \),

\[
B_{n,k} = -\sum_{i=k}^{n-1} B_{i,k} \prod_{j=i}^{n-1} \frac{x_{N-j} - b_n}{b_j - b_n}
\]

with the initial condition

\[
B_{k,k} = \prod_{i=0}^{k-1} \frac{1}{b_i - b_k} \prod_{i=1}^{N-k} \frac{1}{x_i - b_k}.
\]

As a matter of fact, (2.18) follows directly from the definition of inverse matrices (2.3) as below:

\[
(A_{n,k})(B_{n,k}) = E_{N+1},
\]

where \( E_{N+1} \) is the identity matrix of order \( N + 1 \). Written out in explicit terms, it means that for \( n - k \geq 1 \),

\[
A_{n,k} B_{k,k} + A_{n,k+1} B_{k+1,k} + \cdots + A_{n,n} B_{n,k} = 0.
\]

Based on this relation, it is not hard to find

\[
B_{n,k} = -\frac{A_{n,k}}{A_{n,n}} B_{k,k} - \frac{A_{n,k+1}}{A_{n,n}} B_{k+1,k} - \cdots - \frac{A_{n,n-1}}{A_{n,n}} B_{n-1,k}.
\]

Finally, by solving the recurrence relation (2.18) with the initial condition \( B_{k,k} \) by induction on \( n - k \geq 0 \), we have that

\[
B_{n,k} = \frac{x_{N-n+1} - b_n}{x_{N-n+1} - b_k} \prod_{i=1}^{N-n} \frac{1}{x_i - b_k} \prod_{i=0}^{n} \frac{1}{b_i - b_k}. \tag{2.19}
\]
Thus (2.13) follows after a direct substitution of (2.19) into (2.17). The lemma is proved. □

Another ingredient we will use is a basic fact about the symmetric-difference decomposition for the product of two theta functions. It is the foundation stone for our forthcoming discussions and has already been proved in [30] by the first author.

**Lemma 2.6** (Cf. [30, Lemma 3.1]). Let $P(x)$ and $Q(x)$ be given by (1.13). Then

\[ y\theta(xy, x/y; p) = P(x)Q(y) - P(y)Q(x). \] (2.20)

At the end of this section, we list some basic relations for the elliptic $q, p$-shifted factorials which will be used later.

**Lemma 2.7** (Cf. [12, p. 301]). The following properties hold for theta functions and elliptic $q, p$-shifted factorials.

(i) $\theta(x; p) = -x\theta(1/x; p)$, $\theta(px; p) = -1/x \theta(x; p)$.

(ii) The elliptic binomial coefficient

\[
\binom{N}{k}_{q, p} := \frac{(q; q, p)_N}{(q; q, p)_k(q; q, p)_{N-k}} = \frac{q^{kN}}{\tau_q(k)} \frac{(q^{-N}; q, p)_k}{(q; q, p)_k}. \] (2.21)

(iii)

\[
(ACq^{k}; q, p)_N = (AC; q, p)_N \frac{(ACq^N; q, p)_k}{(AC; q, p)_k}, \]
\[
(Aq^{-k}/C; q, p)_N = (A/C; q, p)_N q^{kN} \frac{(Cq/A; q, p)_k}{(Cq^{-N-1}/A; q, p)_k}. \] (2.22)

(iv)

\[
\theta(Cxq^k, Cq^k/x; p) = \frac{(Cx, C/x; q, p)_{k+1}}{(Cx, C/x; q, p)_k}. \] (2.23)

We will also need Frenkel and Turaev’s summation formula which is one of the most fundamental results for VWP-balanced elliptic hypergeometric series.

**Lemma 2.8** (Frenkel and Turaev’s summation formula. Cf. [8] or [12, Eq. (11.2.25)]). For any integer $n \geq 0$ and complex numbers $a, b, c, d, e$ with $a^2 q^{n+1} = bcde$, there holds

\[
_{10}V_9(a; b, c, d, e, q^{-n}; q, p) = \frac{(aq, aq/(bc), aq/(bd), aq/(cd); q, p)_n}{(aq/b, aq/c, aq/d, aq/(bcd); q, p)_n}. \] (2.24)

3. The proofs of the main theorems

3.1. The proofs of Theorems 1.10 and 1.11

As planned, we proceed to prove Theorem 1.10 first.
The proof of Theorem 1.10. We proceed as follows. At first, in light of Lemma 2.5, we only need to consider

\[ F(x) = \sum_{k=0}^{N} \Lambda_k x^k \]

and thereby have the expansion

\[ F(x) = \sum_{k=0}^{N} \Lambda_k(N) \prod_{i=0}^{k-1} (b_i - x) \prod_{i=1}^{N-k} (x_i - x), \tag{3.1} \]

where \( \Lambda_k(N) \) is uniquely given by (2.13). Next, with the same \( P(x) \) and \( Q(x) \) given by (1.13), we make the replacement or transformation of the parameters \( x_i, b_i \) and variable \( x \)

\[
\begin{align*}
  x_i &\to P(x_i)/Q(x_i) \\
  b_i &\to P(b_i)/Q(b_i) \\
  x &\to P(x)/Q(x),
\end{align*}
\]

for (3.1), thereby obtaining

\[
F(P(x)/Q(x)) = \sum_{k=0}^{N} \Lambda_k(N) \prod_{i=0}^{k-1} \frac{P(b_i)Q(x) - Q(b_i)P(x)}{Q(b_i)Q(x)} \prod_{i=1}^{N-k} \frac{P(x_i)Q(x) - Q(x_i)P(x)}{Q(x_i)Q(x)}
\]

\[
= \frac{1}{Q(x)^N} \sum_{k=0}^{N} \Lambda_k(N) \prod_{i=0}^{k-1} \frac{x\theta(x_i, x_i/x; p)}{Q(b_i)} \prod_{i=1}^{N-k} \frac{x\theta(x_i, x_i/x; p)}{Q(x_i)}. \tag{3.2}
\]

Note that the last equality results from Lemma 2.6. By considering

\[ f(x) := Q(x)^N F(P(x)/Q(x)) \in \mathcal{L}_N(P(x), Q(x)), \]

we obtain

\[ f(x) = x^N \sum_{k=0}^{N} \Lambda_k(N) \prod_{i=0}^{k-1} \frac{\theta(x_i, x_i/x; p)}{Q(b_i)} \prod_{i=1}^{N-k} \frac{\theta(x_i, x_i/x; p)}{Q(x_i)}. \tag{3.3} \]

Note that, under the same transformation (3.2), the coefficient \( \Lambda_n(N) \) given by (2.13) takes the form

\[
\Lambda_n(N) = \frac{b_n}{Q(b_n)} \sum_{k=0}^{n} \frac{Q(b_k)\theta(x_{N-n+1}b_n, x_{N-n+1}/b_n; p)}{b_k Q(b_n)\theta(x_{N-n+1}b_k, x_{N-n+1}/b_k; p)} F\left( \frac{P(b_k)}{Q(b_k)} \right) \prod_{i=1}^{N-n} \frac{Q(x_i)Q(b_k)}{b_k \theta(x_i b_k, x_i/b_k; p)} \prod_{i=0, i \neq k}^{n} \frac{Q(b_i)Q(b_k)}{b_k \theta(b_i b_k, b_i/b_k; p)}
\]

\[
= \frac{b_n}{Q(b_n)} \sum_{k=0}^{n} \frac{Q(b_k)\theta(x_{N-n+1}b_n, x_{N-n+1}/b_n; p)}{b_k^{N+1} \theta(x_{N-n+1}b_k, x_{N-n+1}/b_k; p)} f(b_k) \prod_{i=1}^{N-n} \frac{Q(x_i)}{\theta(x_i b_k, x_i/b_k; p)} \prod_{i=0, i \neq k}^{n} \frac{Q(b_i)}{\theta(b_i b_k, b_i/b_k; p)}.
\]
In the sequel, we redefine and then evaluate

\[
H_n(N) := \frac{Q(b_n)\Lambda_n(N)}{b_n \prod_{i=0}^{n} Q(b_i) \prod_{i=1}^{N-n} Q(x_i)}
\]

\[
= \sum_{k=0}^{n} \frac{1}{b_k^{N+1}} \frac{\theta(x_{N-n+1}b, x_{N-n+1}/b; p)}{\theta(x_Nb, x_N/b; p)} f(b_k) \prod_{i=1}^{N-n} \frac{1}{\theta(x_i; x_i/b; p)} \prod_{i=0, i \neq k}^{n} \frac{1}{\theta(b_i b_k, b_i/b_k; p)}.
\]

We have established (1.21). As a last step, by substituting \( H_n(N) \) for \( \Lambda_n(N) \) in (3.3), we readily find that

\[
f(x) = x^N \sum_{k=0}^{N} H_k(N) \left( b_k \prod_{i=0}^{k-1} Q(b_i) \prod_{i=1}^{N-k} Q(x_i) \right) \prod_{i=0}^{k-1} \frac{\theta(b_i, x_i/b; p)}{Q(b_i)} \prod_{i=1}^{N-k} \theta(x_i; x_i/b; p).
\]

Hence, we have proven the theorem. \( \square \)

**Remark 3.1.** As far as we are aware, the transformation (3.2) is of value since it may serve as a bridge connecting theta or elliptic function identities such as (1.16) and polynomial identities like (1.17). We refer the reader to [30] for more applications of (3.2).

**Remark 3.2.** Analyzing the proofs of Theorems 1.6 and 1.10, we think that any base of the ring of polynomials can be used to elliptic interpolation, provided that the expansion coefficients can be found easily via the technique of matrix inversions while the transformation (3.2) is available.

In regard to Theorem 1.6, we remark that in [30], the author established Theorem 1.6 with the help of Lemma 2.4 and Transformation (3.2) but under a prior condition that two sequences \( x_i \neq b_j \) for all \( 1 \leq i, j \leq N \). This restriction arises from (2.10) of Lemma 2.4. As a matter of fact, if \( x_i = b_j \) for all \( 1 \leq i = j \leq N \), then we have a interpolation formula, viz., Theorem 1.11. The proof is given as follows.

**The proof of Theorem 1.11.** It suffices to note that in (2.10) of Lemma 2.4, when \( x_n = b_n \), the limitation

\[
\Lambda_n = \lim_{x_n \to b_n} \left\{ \sum_{k=0}^{n} f(b_k) \prod_{i=0}^{n} \frac{1}{x_i - b_k} \prod_{i=0, i \neq k}^{n} \frac{1}{b_i - b_k} \right\} = f(b_n) \prod_{i=0, i \neq n}^{n} \frac{1}{b_i - b_n}.
\]

As such, we rediscover the Lagrange interpolation formula

\[
f(x) = \sum_{k=0}^{N} f(b_k) \prod_{i=0, i \neq k}^{N} \frac{b_i - x}{b_i - b_k}.
\]
As been done for Theorem 1.10, we apply Transformation (3.2) to both sides of (3.4), obtaining

\[ f(P(x)/Q(x)) = \sum_{k=0}^{N} \frac{f(P(b_k)/Q(b_k))}{b_k^N} \prod_{i=0,i \neq k}^{N} \frac{P(b_i)Q(x) - Q(b_i)P(x)}{Q(b_i)Q(x)} \prod_{i=0,i \neq k}^{N} \frac{Q(b_i)Q(b_k)}{P(b_i)Q(b_k) - Q(b_i)P(b_k)}. \]

By abbreviating \( Q(x)^N f(P(x)/Q(x)) \in \mathcal{L}_N(P(x), Q(x)) \) with \( f(x) \), we obtain (1.22) at once.

When the base \( p = 0 \) in (1.22), we achieve at once an interesting interpolation formula for rational functions at points \( b_0, b_1, \ldots, b_N \).

**Example 3.3.** For nonnegative integers \( m \leq N \), there holds

\[
\left( \frac{1 + x^2}{x} \right)^m = \sum_{k=0}^{N} \left( \frac{1 + b_k^2}{b_k} \right)^m \prod_{i=0,i \neq k}^{N} \frac{1 - b_i x/(1 - b_i / b_k)}{1 - b_i x/(1 - b_i / b_k)}. \]

### 3.2. Two proofs for Theorem 1.12

Now we are in a good position to show Theorem 1.12 in full details. As we will see below, it is implied by both Theorem 1.6 and Theorem 1.10.

**The first proof of Theorem 1.12.** It is a direct application of Theorem 1.10. To make this clear, we first exchange the order of summations in (1.20) to get

\[
f(x) = x^N \sum_{i=0}^{N} \frac{f(b_i)}{b_i^{N+1}} \prod_{i=0}^{N} \frac{\theta(b_i x, b_i / x; p)}{\theta(b_i b_i, b_i / b_i; p)} S_{N,i},
\]

where

\[
S_{N,i} := \sum_{k=i}^{N} b_k \prod_{j=i+1}^{N-k} \frac{\theta(x_{N-k+1} b_k, x_{N-k+1} / b_k; p)}{\theta(x_{N-k+1} b_i, x_{N-k+1} / b_i; p)} \prod_{j=i+1}^{N-k} \frac{\theta(x_{j} x_{i} / x_{j}; p)}{\theta(x_{j} b_i, x_{i} / b_i; p)} \prod_{j=i+1}^{k} \frac{\theta(b_j b_i, b_j / b_i; p)}{\theta(b_j b_i, b_j / b_i; p)}.
\]

Next we choose

\[
x_i = Aq^{-i}, b_i = Cq^i.
\]

In this case, it is easy to check that

\[
\prod_{i=1}^{N-k} \frac{\theta(x_i x_i / x_i; p)}{\theta(x_i b_i, x_i / b_i; p)} = \frac{(xAq^{-N}, Aq^{-N} / x; q, p)_{N-k}}{(ACq^{i-k-N}, Aq^{-i-k-N} / C; q, p)_{N-k}}.
\]

\[
\prod_{i=1}^{k} \frac{\theta(b_{i-1} x, b_{i-1} / x; p)}{\theta(b_i b_i, b_i / b_i; p)} = \frac{(xCq^i, Cq^i / x; q, p)_{k-i}}{(q, C^2q^{2i+1}; q, p)_{k-i}}.
\]
Therefore, upon changing the index $k$ to $K$ by the relation $K = k - t$, we are able to compute

$$
S_{N,t} = Cq^t \sum_{k=0}^{N-t} q^k \theta(ACq^{2k+2N-1}, Aq^{-N-1}/C; p) \\
\times (xAq^{K+1-N}, Aq^{K+1-N}/C; q, p)_{N-t-K} \times (xCq^t, Cq^t/x; q, p)_K.
$$

Since

$$
\frac{(xAq^{K+1-N}, Aq^{K+1-N}/C; q, p)_{N-t-K}}{(ACq^{K+2N-1}, Aq^{K-N}/C; q, p)_{N-t-K}} = \frac{(xAq^{2N-1}, Aq^{-N-1}/x, q, p)_{N-t}}{(ACq^{2N-1}, Aq^{-N-1}/C; q, p)_{N-t}} \times \frac{(x, q, p)_{N-t}}{(xAq^{2N-1}, Aq^{-N-1}/x, q, p)_{N-t}}.
$$

it is easily found that

$$
S_{N,t} = Cq^t \frac{(xAq^{2N-1}, Aq^{-N-1}/x, q, p)_{N-t}}{(ACq^{2N-1}, Aq^{-N-1}/C; q, p)_{N-t}} \times \frac{\sum_{k=0}^{N-t} q^k \theta(ACq^{2k+2N-1}, Aq^{-N-1}/C; p)}{\theta(ACq^{2N-1}, Aq^{-N-1}/C; p)} \times \frac{(xCq^t, Cq^t/x; q, p)_K}{(q, C^2q^{2t+1}; q, p)_K}.
$$

Using the relation

$$
\frac{\theta(ACq^{2N-1}, Aq^{-N-1}/C; p)}{\theta(ACq^{K+2N-1}, Aq^{K-N}/C; p)} = \frac{(ACq^{2N-1}, Aq^{-N-1}/C; q, p)_{N-t}}{(ACq^{K+2N-1}, Aq^{K-N}/C; q, p)_{N-t}}
$$

and then recasting the last sum in standard notation of elliptic hypergeometric series, we arrive at

$$
S_{N,t} = Cq^t \frac{(xCq^{*+t}, Cq^{*+1}/x; q, p)_{N-t}}{(q, C^2q^{2t+1}; q, p)_{N-t}} \times 10V_0(ACq^{2N-1}; Aq^{-N-1}/C, Cq^t x, Cq^t/x, ACq^t, q^{-(N-t)}, q, p).
$$

Now, by appealing to Frenkel and Turaev’s summation formula (2.25), we readily find that

$$
S_{N,t} = Cq^t \frac{(xCq^{*+t}, Cq^{*+1}/x; q, p)_{N-t}}{(q, C^2q^{2t+1}; q, p)_{N-t}} \times \frac{(ACq^{2N-1}, Cxq^{*+1}, Cq^{*+1}/x, Aq^{-N}/C; q, p)_{N-t}}{(C^2q^{2t+1}, Aq^{-N}/x, Axq^{-N}, q, p)_{N-t}}.
$$

A substitution of this expression simplifies the preceding expansion (3.5) to

$$
f(x) = x^N \sum_{t=0}^{N} \frac{f(Cq^t)}{(Cq^t)^N (C^2q^t, q^t; q, p)_{N-t} (q, C^2q^{2t+1}; q, p)_{N-t}} \times \frac{\theta(Cq^{2t}; p) f(Cq^t)}{(C^2q^t, q, p)_{N-t} \theta(Cq^t, Cq^t/x; p)}. \tag{3.6}
$$
Again, referring to (2.24), it is clear that
\[
\frac{1}{\theta(C x^q, C q^t / x; p)} = \frac{1}{\theta(C, C q^t; p)} (C x^q, q^t p)_t/(C x^q, C q^t / q^t p)_t.
\]
Therefore, we reduce (3.6) to
\[
f(x) = \left(\frac{x}{C}\right)^N (C x^q, C x / q^t p)_N \sum_{i=0}^{N} \frac{\theta(C^2 q^t, p)}{\theta(C^2 q^t, p)} (C^2, C x, q^t / C x, q^t p)_t \frac{\tau_i(q^t)}{(q^t, q^t p)_t} f(C q^t, p) q^t.
\]
This is precisely what we want. The theorem is proved.

The next derivation for Theorem 1.12 is based on Theorem 1.6.

*The second proof of Theorem 1.12.* It only needs to specialize Theorem 1.6 to the case
\[
x_i = B q^{i-1}, b_i = C q^i.
\]
Consequently, the expansion (1.14) reduces to
\[
\frac{f(x)}{(B, B/x; q, p)_N} = x^N \sum_{k=0}^{N} H_k(N) (C x, C / x, q, p)_k (B, B/x; q, p)_k,
\]
where the coefficient $H_k(N)$ is given by (1.15), viz.,
\[
H_k(N) = C q^k \theta(B C q^{2k-1}, B/(C q); p)
\]
\[
\times \sum_{i=0}^{k} \frac{1}{(C q^t)^{N+1}} \frac{f(C q^t) \theta(C^2 q^t, p)}{(B C q^{2k-1}, B q^{k-1-t} / C q, q, p)_q q^{k+1} (C^2 q^t; q, p)_q q^{k-1}} \prod_{i=0,i \neq i}^{k} \frac{1}{\theta(q^{t-1}; p)}.
\]
Note that
\[
\prod_{i=0,i \neq i}^{k} \frac{1}{\theta(q^{t-1}; p)} = \frac{\tau_i(t) q^t}{(q, q, p)_t (q, q, p)_{k-1}}.
\]
Therefore,
\[
H_k(N) = \frac{q^k}{C^N} \theta(B C q^{2k-1}, B/(C q); p)
\]
\[
\times \sum_{i=0}^{k} \frac{f(C q^t) \theta(C^2 q^t, p)}{(B C q^{2k-1}, B q^{k-1-t} / C q, q, p)_q q^{k+1} (C^2 q^t; q, p)_q q^{k-1}} \tau_i(t) q^{Nt}.
\]
Substituting the above and exchanging the order of summations gives rise to
\[
\frac{f(x)}{(B, B/x; q, p)_N} = \left(\frac{x}{C}\right)^N \theta(B/(C q); p) \sum_{i=0}^{N} \tau_i(t) q^{Nt} \frac{\theta(C^2 q^t, p)}{(q, q, p)_t} f(C q^t) T_{N,t},
\]
where, we write $T_{N,t}$ for the corresponding inner sum, i.e.,
\[
T_{N,t} := \sum_{k=0}^{N} \frac{\theta(B C q^{2k-1}; p)}{(B C q^{2k-1}, B q^{k-1-t} / C q, q, p)_q q^{k+1}} \frac{C x, C / x, q, p)_k q^k}{(B, B/x; q, p)_k (q, q, p)_{k-1}}.
\]
Further, by changing the index $k$ of summation to $K$ by the relation $K = k - t$, we arrive at

\[
T_{N,t} = \frac{\theta(BCq^{2r-1}; p)}{(BCq^{2r-1}, Bq^{-1}/C; q, p)_{N-t+1}(C^2 q^t; q, p), (Bx, B/x; q, p)_t} q^j
\times \sum_{k=0}^{N-1} \frac{\theta(BCq^{2r+2-1}; p)}{(BCq^{2r-1}, Bq^{-1}/C; q, p)_{K}} \frac{\theta(q, C^2 q^{2r+1}; q, p)}{(q, C^2 q^t; q, p)_{K}} \frac{(Bx, B/x; q, p)_{K}}{(Bxq^t, Bq^{t}/x; q, p)_{K}} q^K
\]

\[
= \frac{\theta(BCq^{2r-1}; p)}{(BCq^{2r-1}, Bq^{-1}/C; q, p)_{N-t+1}(C^2 q^t; q, p), (Bx, B/x; q, p)_t} q^j
\times 10 V_0(BCq^{2r-1}; Bq^{-1}/C, Cxq^t, Cq^t/x; BCq^{2N+1}, q^{-(N-1)}; q, p).
\]

Now, by Frenkel and Turaev’s summation formula (2.25), we can evaluate $T_{N,t}$ in closed form:

\[
T_{N,t} = \frac{\theta(BCq^{2r-1}; p)}{(BCq^{2r-1}, Bq^{-1}/C; q, p)_{N-t+1}(C^2 q^t; q, p), (Bx, B/x; q, p)_t} q^j
\times \frac{\theta(q, C^2 q^{2r+1}; q, p)}{(q, C^2 q^t; q, p)_{N-t}} \theta(Cxq^t, Cq^t/x; (C^2 q^{2N+1}; q, p)_t)
\]

In the sequel, on substituting this computational result into (3.7), we have the expansion

\[
f(x) = \frac{x^N}{(Bx, B/x; q, p)_N} \left(\frac{Cx, C/x; q, p)_N}{(C^2 q; q, p)_{N+1}(Bx, B/x; q, p)_N}\right)
\times \sum_{t=0}^{N} \frac{\tau_q(t)q^{j-Nt}}{(q, q, p)_t(q, q, p)_{N-t}} \frac{\theta(C^2 q^{2t}; p)(C^2 q^t; q, p)_t}{\theta(Cxq^t, Cq^t/x; (C^2 q^{2N+1}; q, p)_t)} f(Cq^t).
\]

At this stage, we utilize the properties (2.21) and (2.24) to simplify the last expansion, obtaining

\[
f(x) = \frac{x^N}{(C, C/x; q, p)_N} \left(\frac{C, Cq/x; q, p)_N}{(C^2 q; q, p)_N}\right)
\times \sum_{t=0}^{N} \frac{\theta(C^2 q^{2t}; p)}{\theta(C^2; p)} \frac{(C^2, C/x, Cx, q^{N}; q, p)_t}{(q, Cxq^t, Cq^t/x, C^2 q^{2N+1}; q, p)_t} f(Cq^t) q^t.
\]

It gives the complete proof of the theorem. \[\square\]

**Remark 3.4.** It is worth pointing out that (1.23) is independent of $A$ and $B$, although we derive (1.23) by setting either $x_i = Aq^{-1}$ or $x_i = Bq^{-1}$. We believe that other choices for $x_i$ and $b_i$ also deserve further study, in order to find interpolation formulas in closed form.
3.3. The proof of Theorem 1.13 and a characterization of $W_c^N$

Indeed, a combination of Theorem 1.12 with the basic relation (1.19) or Lemma 2.6 leads us to a proof of Theorem 1.13.

The proof of Theorem 1.13. Given such $f(x) \in \mathcal{L}_{N_0}(P(x), Q(x))$, we only need to consider the theta function
\[ F(x) = f(x) \prod_{i=1}^{m} x^{N_i}(A_i x, A_i / x; q, p)_{N_i}. \] (3.8)

In this case, by Proposition 1.9, we see that $F$ is the theta function. By (2.22) and (2.23), it is easy to check that
\[ F = q^{-k(N-N_0)} \prod_{i=1}^{m} (A_i C, A_i / C; q, p)_{N_i} \prod_{i=1}^{m} (A_i C q^N, C q / A_i; q, p)_k \frac{(C q^{-N+1}/A_i, A_i C; q, p)_k}{(C q^{-N+1}/A_i, A_i C; q, p)_k}. \]

In conclusion, we obtain
\[ f(x) = \frac{\left(\frac{x}{C}\right)^{N_0} (C x, C q / x; q, p)_N}{(q, C^2 q; q, p)_N} \prod_{i=1}^{m} (A_i C, A_i / C; q, p)_{N_i} \prod_{i=1}^{m} (A_i C q^N, C q / A_i; q, p)_k \frac{(C q^{-N+1}/A_i, A_i C; q, p)_k}{(C q^{-N+1}/A_i, A_i C; q, p)_k}. \]

This completes the proof of the theorem.

It seems interesting that we are able to characterize, by making use of Schlosser and Yoo’s relevant results and our argument, the set $W_c^N$ in terms of $P(x)$ and $Q(x)$.

The proof of Theorem 1.14. We begin with proving (1.26) first. For this, recall that for any integer $N \geq 0$, by Schlosser’s conclusion [27, Lemma 4.1], we have the expansion
\[ \frac{g_N(x)}{(c x, c/x; q, p)_N} = \sum_{k=0}^{N} f_k \frac{(a x, a / x; q, p)_k}{(c x, c/x; q, p)_k}. \] (3.9)
It can be reformulated as
\[ g_N(x) = \sum_{k=0}^{N} f_k(ax, a/x; q, p) (cxq^k, cq^k/x; q, p)_{N-k}. \]  
(3.10)

Observe that
\[ (ax, a/x; q, p)_k = \prod_{i=0}^{k-1} \theta(aq^i x, aq^i/x; p) = \frac{1}{x^k} \prod_{i=0}^{k-1} (P(aq^i)Q(x) - Q(aq^i)P(x)), \]
\[ (cxq^k, cq^k/x; q, p)_{N-k} = \prod_{i=0}^{N-k-1} \theta(cq^{k+i} x, cq^{k+i}/x; p) \]
\[ = \frac{1}{x^{N-k}} \prod_{i=0}^{N-k-1} (P(cq^{k+i})Q(x) - Q(cq^{k+i})P(x)). \]

Referring to Definition 1.7 and Proposition 1.9, it is easily seen that
\[ x^N(ax, a/x; q, p)_{k}(cxq^k, cq^k/x; q, p)_{N-k} \]
is an elliptic Askey-Wilson polynomial of degree \( N \). Assume further that
\[ x^N(ax, a/x; q, p)_{k}(cxq^k, cq^k/x; q, p)_{N-k} = \sum_{i=0}^{N} \mu_{N,k;i} P(x)^i Q(x)^{N-i}, \]  
(3.11)
where the coefficients \( \mu_{N,k;i} \) are independent of \( x \). Finally we substitute (3.11) into (3.10), arriving at
\[ g_N(x) = \frac{1}{x^N} \sum_{k=0}^{N} f_k \sum_{i=0}^{N} \mu_{N,k;i} P(x)^i Q(x)^{N-i} \]
\[ = \frac{1}{x^N} \sum_{i=0}^{N} \left( \sum_{k=0}^{N} f_k \mu_{N,k;i} \right) P(x)^i Q(x)^{N-i}. \]

By choosing \( \lambda_i = \sum_{k=0}^{N} f_k \mu_{N,k;i} \), we get (1.26) of our theorem. For the case \( m \leq N \), we only need to replace \( g_N(x) \) of (1.26) with
\[ g_m(x) \frac{(cx, c/x; q, p)_N}{(cx, c/x; q, p)_m} = g_m(x)(cq^m x, cq^m/x; q, p)_{N-m} \]
and apply Lemma 2.6 to the factor \((cq^m x, cq^m/x; q, p)_{N-m}\). Then (1.25) follows. This completes the proof of the theorem.

\[ \square \]

**Remark 3.5.** By (1.26) and the definitions of \( P(x) \) and \( Q(x) \), it is easily verified that \( g_N(x) \) satisfies (1.8), viz.,
\[ \left\{ \begin{array}{l} g_N(x) = g_N(1/x), \\
g_N(px) = \frac{1}{p^{N-2N}} g_N(x). \end{array} \right. \]

Likewise, we readily find that the elliptic Askey-Wilson polynomial \( f(x) \in \mathcal{L}_N(P(x), Q(x)) \) possesses the properties
\[ f(1/x) = f(px) = \frac{1}{x^{2N}} f(x). \]  
(3.12)
4. Applications

In this section, we will pursue some specific cases of our interpolation formulas, viz., Theorems 1.11, 1.12, and 1.13. We begin by establishing a generalization of Weierstrass’ theta identity. It is an application of Theorem 1.11 with Theorem 1.6 utilized.

**Corollary 4.1** (Generalized Weierstrass theta identity). Let \( \{x_n\}_{n \geq 0} \) and \( \{b_n\}_{n \geq 0} \) be two sequences such that \( b_i \neq b_j, 0 \leq i \neq j \leq N \). Then, for integers \( N \geq k \geq 0 \), there holds

\[
\sum_{n=k}^{N} b^n \frac{\theta(x_n b_n, x_n / b_n; p)}{\theta(x_n b_k, x_n / b_k; p)} \prod_{i=n+1}^{N} \frac{\theta(x_i x, x_i / x; p)}{\theta(x_i b_k, x_i / b_k; p)} \times \frac{\prod_{i=0,i \neq k}^{n-1} \theta(b_i x, b_i / x; p)}{\prod_{i=0,i \neq k}^{n} \theta(b_i b_k, b_i / b_k; p)} = b_k \prod_{i=0,i \neq k}^{N} \frac{\theta(b_i x, b_i / x; p)}{\theta(b_i b_k, b_i / b_k; p)}.
\]

**Proof.** It suffices to consider two expansions for any \( f(x) \in L_N(P(x), Q(x)) \). At first, according to Theorem 1.6, we have the expansion

\[
f(x) = \sum_{n=0}^{N} H_n(N) b_n \theta(x_n b_n, x_n / b_n; p) \prod_{i=0}^{n-1} \theta(b_i x, b_i / x; p) \prod_{i=n+1}^{N} \theta(x_i x, x_i / x; p),
\]

where the coefficients

\[
H_n(N) := \sum_{k=0}^{n} \frac{f(b_k)}{b_k^{n+1}} \prod_{i=n}^{N} \frac{1}{\theta(x_i b_k, x_i / b_k; p)} \prod_{i=0,i \neq k}^{n} \frac{1}{\theta(b_i b_k, b_i / b_k; p)}.
\]

By exchanging the order of summations, we have

\[
f(x) = \sum_{k=0}^{N} f(b_k) \frac{b_k^{N}}{b_k^{N+1}} \Omega_{k,N},
\]

where

\[
\Omega_{k,N} := \frac{1}{b_k} \sum_{n=k}^{N} b_n \theta(x_n b_n, x_n / b_n; p) \prod_{i=0}^{n-1} \theta(b_i x, b_i / x; p) \prod_{i=n+1}^{N} \theta(x_i x, x_i / x; p) \prod_{i=0,i \neq k}^{n} \theta(b_i b_k, b_i / b_k; p).
\]

On the other hand, from Theorem 1.11 it follows that

\[
f(x) = \sum_{k=0}^{N} f(b_k) \frac{b_k^{N}}{b_k^{N+1}} \prod_{i=0,i \neq k}^{N} \frac{\theta(b_i x, b_i / x; p)}{\theta(b_i b_k, b_i / b_k; p)}.
\]

By comparing (4.2) and (4.3) and taking the arbitrariness of \( f(x) \) into account, we have

\[
\Omega_{k,N} = \prod_{i=0,i \neq k}^{N} \frac{\theta(b_i x, b_i / x; p)}{\theta(b_i b_k, b_i / b_k; p)}.
\]

Written out in full, (4.4) is identified with (4.1). The proof is finished. \( \square \)
We should make some remarks on the implications of Corollary 4.1.

**Remark 4.2.** It is worth mentioning that Corollary 4.1 offers a general version of Weierstrass’ theta identity (1.16). To make this clear, we only need to specialize (4.1) to the case $N = 2$ and $k = 1$, obtaining

$$b_1 \frac{\theta(x_2 x, x_2/x; p)}{\theta(x_2 b_1, x_2/x; p)} \frac{\theta(b_0 x, b_0/x; p)}{\theta(b_0 b_1, b_0/b_1; p)} + b_2 \frac{\theta(x_2 b_2, x_2/x; p)}{\theta(x_2 b_1, x_2/b_1; p)} \frac{\theta(b_0 x, b_0/x; p)}{\theta(b_0 b_1, b_0/b_1; p)} \frac{\theta(b_1 b_1, b_1/x; p)}{\theta(b_2 b_1, b_2/b_1; p)} = b_1 \frac{\theta(b_0 x, b_0/x; p)}{\theta(b_0 b_1, b_0/b_1; p)} \frac{\theta(b_2 x, b_2/x; p)}{\theta(b_2 b_1, b_2/b_1; p)}.$$

After some routine simplification, we have

$$\theta(b_2 x, b_2/x, x_2 b_1, x_2/b_1; p) - \theta(x_2 x, x_2, b_2 b_1, b_2/b_1; p) = \frac{b_2}{b_1} \theta(x_2 b_2, x_2/b_2, b_1 x, b_1/x; p).$$

It is in agreement with (1.16) after relabeling the parameters.

The case $k = 0$ of (4.1) yields another theta identity being equivalent to (1.16).

**Example 4.3.** For any integer $N \geq 0$, there holds

$$\sum_{n=0}^{N} b_n \frac{\theta(b_0 x, b_0/x, x_n b_n, x_n/b_n; p)}{\theta(x_n b_0, x_n/b_0, b_n x, b_n/x; p)} \prod_{i=n+1}^{N} \frac{\theta(b_i b_0, b_i/b_0, x_i x_i/x; p)}{\theta(x_i b_0, x_i/b_0, b_i x, b_i/x; p)} = b_0. \quad (4.5)$$

In regard to applications of Theorem 1.12, two easy cases arise naturally.

**Example 4.4.** For any integer $N \geq 0$, we have

$$\left( \frac{C}{x} \right)^N \frac{(q, C^2 q, q, p)_N}{(C x, C q/x, q, p)_N} \theta^N(-x^2; p^2) = \sum_{k=0}^{N} \frac{(C^2 q^2k; p)}{(C^2; p)} \frac{(C^2, C x, C x/q^{-N}; q, p)_k}{(q, C x, q, C^2 q^{N+1}; q, p)_k} \theta^N(-C^2 q^{2k}; p^2) q^k, \quad (4.6)$$

$$\left( \frac{C x, C q/x, q, p)_N}{(q, C^2 q/x, q, p)_N} \theta^N(-p x^2; p^2) \right) = \sum_{k=0}^{N} \left[ \frac{N}{k} \right]_{q, p} \frac{(C^2 q^2k; p)}{(C^2; p)} \frac{(C^2, C x, C x/q; q, p)_k}{(C x, C q/x, C^2 q^{N+1}; q, p)_k} \theta^N(-p C^2 q^{2k}; p^2) \tau_q(k) q^k. \quad (4.7)$$

**Proof.** It suffices to specialize (1.23) to the cases $f(x) = P(x)^N = \theta^N(-x^2; p^2)(-p; p)_{\infty}^N$ and $f(x) = Q(x)^N = (x\theta(-x^2; p^2)(-p; p)_{\infty})^N$. Then we have (4.6) and (4.7) correspondingly.

As a very good illustration of Theorem 1.12, we prefer to reconsider Example 3.5 of [30]. It gives a special result of Frenkel and Turaev’s well-known summation formula (2.25) given by Lemma 2.8.
Example 4.5. For any integer \( N \geq 0 \), there holds
\[
10V_9(C^2; C/x, Cx, Cq/A, ACq^N, q^{-N}; q, p) = \frac{(q, C^2q, Ax, A/x; q, p)_N}{(AC, A/C, Cxq, Cq/x; q, p)_N}.
\] (4.8)

Proof. It only needs to apply Theorem 1.12 to \( f(x) = x^N(Ax, A/x; q, p)_N \), since \( f(x) \in \mathcal{L}_N(P(x), Q(x)) \) as indicated in Remark 1.8 (also see Lemma 2.6). By making use of the relations (2.22) and (2.23), we deduce from (1.23) that
\[
(A, A/x; q, p)_N = \frac{(Cqx, Cq/x; q, p)_N}{(q, Cq^2; q, p)_N} \times \sum_{k=0}^{N} \frac{\theta(C^2q^{2k}; p)}{\theta(C^2; p)} \frac{(C^2, C/x, Cx, q^{-N}; q, p)_k}{(q, Cqx, Cq/x, C^2q^{N+1}; q, p)_k} (ACq^k, Aq^{-k}/C; q, p)_N q^{k+N}
\]
\[
= \frac{(Cqx, Cq/x, AC, A/C; q, p)_N}{(q, Cq^2; q, p)_N} \times \sum_{k=0}^{N} \frac{\theta(C^2q^{2k}; p)}{\theta(C^2; p)} \frac{(C^2, C/x, Cx, q^{-N}; q, p)_k}{(q, Cqx, Cq/x, C^2q^{N+1}; q, p)_k} (ACq^N, Cq/A; q, p)_k q^{k},
\]
which coincides with (4.8).
\[
\]

With the help of Theorem 1.12, we can extend the elliptic Karlsson-Minton type identity [28, Corollary 2.8] given by Schlosser and Yoo to the following

Theorem 4.6 (Generalized elliptic Karlsson-Minton type identity). For any \( f(x) \in \mathcal{L}_N(P(x), Q(x)) \), we have the expansion
\[
\left( \frac{C^m}{x} \right)^{N_0} \frac{(q, C^2q, q, p)_{m+N_0}}{(Cqx, Cq/x, q, p)_{m+N_0}} f(x) \prod_{i=1}^{m} \theta(A_i x, A_i x; p)
\]
\[
= \sum_{k=0}^{m+N_0} q^{k(m+1)} \frac{\theta(C^2 q^{2k}; p)}{\theta(C^2; p)} \frac{(C^2, C/x, Cx, q^{-(m+N_0)}; q, p)_k}{(q, Cqx, Cq/x, C^2q^{m+N_0+1}; q, p)_k} f(Cq^{k}) \prod_{i=1}^{m} \theta(A_i C q^k, A_i q^{-k}/C; p).
\] (4.9)

Proof. To establish (4.9), we only need to apply Theorem 1.12 to the elliptic Askey-Wilson polynomial
\[
F(x) = f(x) \prod_{i=1}^{m} x \theta(A_i x, A_i x; p).
\]

Note that \( F(x) \) is of degree \( m + N_0 \). The expansion corresponding to (1.23) becomes
\[
\left( \frac{C^m}{x} \right)^{N_0} \frac{(q, C^2q, q, p)_{m+N_0}}{(Cqx, Cq/x, q, p)_{m+N_0}} f(x) \prod_{i=1}^{m} x \theta(A_i x, A_i x; p)
\]
\[
= \sum_{k=0}^{m+N_0} \frac{\theta(C^2 q^{2k}; p)}{\theta(C^2; p)} \frac{(C^2, C/x, Cx, q^{-(m+N_0)}; q, p)_k}{(q, Cqx, Cq/x, C^2q^{m+N_0+1}; q, p)_k} q^{k} f(Cq^{k})(Cq^{k})^m \prod_{i=1}^{m} \theta(A_i C q^k, A_i q^{-k}/C; p).
\]

A bit simplification leads us to (4.9).
\[
\]
Corollary 4.7 (The elliptic Karlsson-Minton type identity. Cf. [28, Corollary 2.8]). For any integer \( m \geq 0 \), there holds

\[
\frac{(q, C^2 q; q, p)_m}{(C x q, C q/x; q, p)_m} \prod_{i=1}^{m} \theta(A_i x, A_i/x; p)
= \sum_{k=0}^{m} q^{k(m+1)} \frac{\theta(C^2 q^{2k}; p)}{\theta(C^2; p)} \frac{(C^2, C/x, C x, q^{-m}; q, p)_k}{(q, C x q, C q/x, C^2 q^{m+1}; q, p)_k} \prod_{i=1}^{m} \theta(A_i C q^k, A_i q^{-k}/C; p).
\]

Next is a direct consequence of Theorem 4.6 with the choice \( f(x) = Q(x)^{N_0} \).

Corollary 4.8. For any integers \( m, N_0 \geq 0 \), there holds

\[
\frac{(q, C^2 q; q, p)_{m+N_0}}{(C x q, C q/x; q, p)_{m+N_0}} \prod_{i=1}^{m} \theta(A_i x, A_i/x; p)
= \sum_{k=0}^{m+N_0} q^{k(m+N_0+1)} \frac{\theta(C^2 q^{2k}; p)}{\theta(C^2; p)} \left( \frac{\theta(-p C^2 q^{2k}; p^2)}{\theta(-p x^2; p^2)} \right)^{N_0} \prod_{i=1}^{m} \theta(A_i C q^k, A_i q^{-k}/C; p).
\]

We conclude our paper with the following elliptic analogue of Gasper’s summation formula for VWP \( 6+2m \phi 5+2m \) series. The reader may consult [11] or [12, Exercise 2.33(i)] for details and Rosengren and Warnaar’s survey [26, Eq. (1.3.7)] for its multivariate version.

Corollary 4.9. For any integers \( m \geq 1, N_i \geq 0, N = \sum_{i=1}^{m} N_i \), there holds

\[
8+2m V_{2+2m}(C^2, C/x, C x, q^{-N}, A_1 C q^{N_1}, \ldots, A_m C q^{N_m}, C q/A_1, \ldots, C q/A_m; q, p)
= \frac{(q, C^2 q; q, p)_N}{(C x q, C q/x; q, p)_N} \prod_{i=1}^{m} \frac{(A_i x, A_i/x; q, p)_{N_i}}{(A_i C, A_i/C; q, p)_{N_i}}.
\]

Proof. It is a direct consequence of Theorem 1.13 by taking \( f(x) = 1 \) (i.e., \( N_0 = 0 \)) in (1.24).

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