CONCERNING $L^p$ RESOLVENT ESTIMATES FOR SIMPLY CONNECTED MANIFOLDS OF CONSTANT CURVATURE

SHANLIN HUANG AND CHRISTOPHER D. SOGGE

Abstract. We prove families of uniform $(L^r, L^s)$ resolvent estimates for simply connected manifolds of constant curvature (negative or positive) that imply the earlier ones for Euclidean space of Kenig, Ruiz and the second author [6]. In the case of the sphere we take advantage of the fact that the half-wave group of the natural shifted Laplacian is periodic. In the case of hyperbolic space, the key ingredient is a natural variant of the Stein-Tomas restriction theorem.

1. Introduction and main results.

In a paper of Kenig, Ruiz and the second author [6], it was shown that for each $n \geq 3$ if $1 < r < s < \infty$ are Lebesgue exponents satisfying

\[ n\left(\frac{1}{r} - \frac{1}{s}\right) = 2 \quad \text{and} \quad \min\left(|\frac{1}{r} - \frac{1}{2}|, |\frac{1}{s} - \frac{1}{2}|\right) > \frac{1}{2n}, \]

then there is a uniform constant $C_{r,s} < \infty$ so that

\[ \|u\|_{L^s(\mathbb{R}^n)} \leq C_{r,s}\|\Delta_{\mathbb{R}^n} + \zeta\|_{L^r(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n), \]

where $\Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ denotes the standard Laplacian on Euclidean space. The first condition in (1.1) is dictated by scaling and the other part of (1.1) was also shown in [6] to be necessary for (1.2).

Euclidean space of course is the unique simply connected manifold of constant curvature equal to zero. The purpose of this paper is to prove sharp theorems for simply connected manifolds of constant curvature either $+\kappa$ or $-\kappa$, $\kappa > 0$, which naturally imply (1.2) when $\kappa \searrow 0$. In the case of positive curvature $+\kappa$, we recall that the manifold is the sphere endowed with the metric which in geodesic polar coordinates $r\theta$, $\theta \in S^{n-1}$ about any point is given by

\[ ds^2 = dr^2 + \left(\frac{\sin \sqrt{\kappa} r}{\sqrt{\kappa}} \right)^2 |d\theta|^2, \quad 0 < r < \frac{\pi}{\sqrt{\kappa}}, \]

while in the case of constant negative curvature $-\kappa$, $\kappa > 0$, it is $\mathbb{R}^n$ endowed with the metric which in any geodesic normal coordinate system takes the form

\[ ds^2 = dr^2 + \left(\frac{\sinh \sqrt{-\kappa} r}{\sqrt{-\kappa}} \right)^2 |d\theta|^2, \quad 0 < r < \infty, \]

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Theorem 1.2. Let \( r, s < \) imply (1.2) for a given pair of exponents 1 of the form (1.12) or (1.13) for all \( \zeta \). The case of (1.11) one uses the fact that \( r, s \) for such exponents \( \kappa \). When \( \kappa = 1 \), we in the case of the standard round unit sphere and write \( dV_{Sn} = dV_1 \) and \( \Delta_{Sn} = \Delta_1 \), while for \( \kappa = -1 \), we are in standard hyperbolic space and write \( dV_{-1} = dV_{\mathbb{H}^n} \) and \( \Delta_{-1} = \Delta_{\mathbb{H}^n} \).

We can now state our main results.

First, for the case of constant positive curvature we have the following

**Theorem 1.1.** Let \( R \subset \mathbb{C} \) be the region given by all \( \zeta \in \mathbb{C} \) satisfying

\[
\mathcal{R} = \{ \zeta \in \mathbb{C} : \Re \zeta \leq (\Im \zeta)^2 \}. \tag{1.9}
\]

Then for every \( n \geq 3 \) and \( 1 < r < s < \infty \) satisfying (1.1) there is a constant \( C_{r,s} \) so that

\[
\| u \|_{L^r(\mathbb{H}^n, dV_{\mathbb{H}^n})} \leq C_{r,s} \| ((\Delta_{\mathbb{H}^n} - \frac{n-1}{2})^2 + \zeta) u \|_{L^r(\mathbb{H}^n, dV_{\mathbb{H}^n})}, \quad u \in C^\infty(\mathbb{H}^n), \quad \zeta \in \mathcal{R}. \tag{1.10}
\]

For such exponents \( r, s \) and the same constant \( C_{r,s} \), we also have that for any \( \kappa > 0 \)

\[
\| u \|_{L^r(\mathbb{H}^n, dV_{\mathbb{H}^n})} \leq C_{r,s} \| ((\Delta - \kappa \frac{n-1}{2})^2 + \zeta) u \|_{L^r(\mathbb{H}^n, dV_{\mathbb{H}^n})}, \quad u \in C^\infty, \quad \zeta \in \mathcal{R}_\kappa = \{ \kappa \zeta : \zeta \in \mathcal{R} \} \text{ is the } \kappa\text{-dilate of } \mathcal{R}. \tag{1.11}
\]

For the case of constant negative curvature, we have the following

**Theorem 1.2.** Let \( n \geq 3 \). Then for every \( r, s \) as in (1.1) there is a constant \( C_{r,s} \), so that

\[
\| u \|_{L^r(\mathbb{H}^n, dV_{\mathbb{H}^n})} \leq C_{r,s} \| ((\Delta_{\mathbb{H}^n} + \frac{n-1}{2})^2 + \zeta) u \|_{L^r(\mathbb{H}^n, dV_{\mathbb{H}^n})}, \quad u \in C^\infty, \quad |\zeta| \geq 1. \tag{1.12}
\]

For such exponents \( r, s \) and the same constant \( C_{r,s} \), we also have that for each \( -\kappa, \kappa > 0 \)

\[
\| u \|_{L^r(\mathbb{H}^n, dV_{\mathbb{H}^n})} \leq C_{r,s} \| ((\Delta_{-\kappa} + \kappa \frac{n-1}{2})^2 + \zeta) u \|_{L^r(\mathbb{H}^n, dV_{\mathbb{H}^n})}, \quad u \in C^\infty, \quad |\zeta| \geq \kappa. \tag{1.13}
\]

In the case of \( n = 3 \), as we shall show at the end of §4, we can uniform obtain bounds of the form (1.12) or (1.13) for all \( \zeta \in \mathbb{C} \).

Note that, by sending \( \kappa \) to zero, it is straightforward to see that either (1.11) or (1.13) implies (1.2) for a given pair of exponents 1 < \( \frac{n-1}{2} \) and \( \zeta \) satisfying \( n(\frac{1}{r} - \frac{1}{s}) = 2 \). (In the case of (1.11) one uses the fact that \( \mathcal{R}_\kappa \) tends to \( \mathbb{C} \setminus \mathbb{R}_+ \) as \( \kappa \) goes to zero.) Therefore, since
it was shown in [6] that the second part of (1.1) is necessary for the Euclidean estimate, we conclude that our assumptions on the exponents in the above theorems dealing with the other constant curvature cases are also sharp.

It was shown by Bourgain, Shao, Yao and the second author [2] that when $s = \frac{2n}{n+2}$ and $r = \frac{2n}{n+2}$, the condition that $\zeta \in \mathcal{R}$ is sharp in the sense that the inequality cannot hold when $\mathcal{R}$ is replaced by any region of the form $\text{Re} \zeta \leq a((\text{Im} \zeta))(\text{Im} \zeta)^2$, where $a(\lambda) \rightarrow 0$ as $\lambda \rightarrow +\infty$. Since (1.10) for any such $r, s$ implies this special case involving these dual exponents, we conclude that the condition on $\mathcal{R}$ is sharp. See Figure 1 below to see the boundary of the region $\mathcal{R}$ and $\mathcal{R}_\kappa$ when $0 < \kappa \ll 1$. We shall show in §2 that the assumption (1.1), as in the Euclidean case, is sharp for $S^n$. As we shall see, (1.11) is a simple consequence of (1.10), and as $\kappa \searrow 0$, we can recover (1.2) from it. We have stated the results in Theorem 1.1 for the shifted Laplacian on $S^n$ since the spectrum of $\sqrt{-\Delta_{S^n}} + (\frac{n-1}{2})^2$ is $\{k + \frac{n-1}{2}\}_{k=0}^\infty$.

$$\begin{align*}
\text{Im} \zeta & \quad \mathcal{R} \\
\mathcal{R}_\kappa, 0 < \kappa \ll 1 & \quad \mathcal{R} \\
\text{Re} \zeta & \quad R\kappa, 0 < \kappa \ll 1
\end{align*}$$

**Figure 1.** Admissible regions for spheres of curvature $\kappa > 0$

For similar reasons, (1.12) is stated in terms of the shifted Laplacian, $\Delta_{H^n} + (\frac{n-1}{2})^2$, due to the fact that the spectrum of $\sqrt{-\Delta_{H^n}} - (\frac{n-1}{2})^2$ is $[0, \infty)$. Also, as we shall see later, the conditions on $r, s$ in Theorem 1.2 are necessary, and we can recover (1.2) from (1.13) by letting $-\kappa \nearrow 0$.

Let us now go over the main ideas in the proof of the theorems. As in [2], we shall use a formula which relates the resolvent to the solution of a Cauchy problem for the
wave equation involving the shifted Laplacians. Specifically, if \( P^2 \) is either the shifted Laplacian \(-\Delta_{S^n} + (\frac{n-1}{2})^2\) or \(-\Delta_{H^n} - (\frac{n-1}{2})^2\), then we have

\[
(P^2 + (\lambda + i\mu)^2)^{-1} = \frac{\text{sgn}\mu}{i(\lambda + i\mu)} \int_0^\infty e^{i(\text{sgn}\mu)\lambda t} e^{-|\mu| t} (\cos tP) \, dt.
\]

Here one can define \( \cos tP \) either via the spectral theorem, or use the fact that it gives the unique solution of the Cauchy problem

\[
(\partial_t^2 - P^2)u(t, x) = 0, \quad u(0, \cdot) = f(\cdot), \quad \partial_t u(0, \cdot) = 0.
\]

In [2] it was shown that favorable restriction type estimates can be used in conjunction with (1.14) to prove resolvent estimates.

In the case of the sphere, let \( H_k : L^2(S^n) \to L^2(S^n) \) denotes projecting onto spherical harmonics of degree \( k = 0, 1, 2, \ldots \) (i.e., the projection operator onto the space of eigenfunctions satisfying \( P^2 e_k = (k + (\frac{n-1}{2}))^2 e_k \)). Then we shall verify the required estimate that

\[
\|H_k f\|_{L^r(S^n)} \leq C (1 + k) \|f\|_{L^s(S^n)}, \quad r, s, \text{ as in (1.1)}.
\]

In the case of hyperbolic space, we shall show that there is a constant \( C < \infty \) so that one has the uniform bounds

\[
\|\mathbb{I}_{[\lambda, \lambda + \varepsilon]}(P)\|_{L^2(\mathbb{H}^n) \to L^{\frac{2(n+1)}{n-1}}(\mathbb{H}^n)} \leq C \sqrt{\varepsilon} \lambda^{\frac{n-1}{2(n+1)}}, \quad \lambda, \varepsilon^{-1} \geq 1,
\]

if \( P = \sqrt{-\Delta_{\mathbb{H}^n} - (\frac{n-1}{2})^2} \).

Here \( \mathbb{I}_{[\lambda, \lambda + \varepsilon]} \) denotes the indicator function of the interval \([\lambda, \lambda + \varepsilon]\), and \( \mathbb{I}_{[\lambda, \lambda + \varepsilon]}(P) \) the function of \( P \) defined by this function and the spectral theorem. If \( P \) were the square root of minus the Euclidean Laplacian and the norms were taken over Euclidean space with Euclidean measure, then the analog of (1.16) would be equivalent to the Stein-Tomas restriction theorem for \( \mathbb{R}^n \) [20]. We shall prove (1.16) by adapting Stein’s proof of this result using his complex interpolation scheme. It seems that the requirement that \( \lambda \) be bounded away from zero is necessary for (1.16), unlike its Euclidean counterpart. Using (1.16), following an earlier argument of the second author, [11], we shall be able to adapt the proof of the Euclidean resolvent estimates (1.2) in [6] to obtain the hyperbolic variants (1.12) in Theorem 3.1.

The rest of the paper is organized as follows. In §2 we shall prove Theorem 1.1. Then §3 will be devoted to the proof of (1.16), and in §4 we shall show how they can be used to prove Theorem 1.2, the uniform \( L^r \to L^s \) resolvent estimates for hyperbolic space. Along the way, we shall use a few results that are well known to experts and essentially in the literature. For the sake of completeness, their proofs will be presented in an appendix.

2. Uniform resolvent estimates for \( S^n \).

In this section we shall give the proof of Theorem 1.1.

First, let us quickly go over some standard facts concerning Fourier analysis on \( S^n \). Further details can be found in many places, such as in Stein and Weiss [16] and [14].
We first recall that we have the orthogonal decomposition
\[ L^2(S^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k, \]
where \(\mathcal{H}_k\) are the spherical harmonics of degree \(k = 0, 1, 2, \ldots\). Thus \(\mathcal{H}_k\) is the space of restrictions to \(S^n\) of harmonic homogeneous polynomials of degree \(k\), and so
\[ (-\Delta_{S^n} + (\frac{n-1}{2})^2) e_k(x) = (k + (\frac{n-1}{2})^2) e_k(x), \quad \text{if } e_k \in \mathcal{H}_k, \]
and
\[ d_k = \dim \mathcal{H}_k = \binom{n+k}{n} - \binom{n+k-2}{n} = \frac{2kn-1}{(n-1)} + O(k^{n-2}). \]
If \(\{e_{k,j}\}_{j=1}^{d_k}\) is an orthonormal basis of \(\mathcal{H}_k\), then the projection operator \(H_k\) onto this space has kernel
\[ H_k(x, y) = \sum_{k=1}^{d_k} e_{k,j}(x) e_{k,j}(y). \]
Since \(e_{k,j}\) is the restriction to \(S^n\) of a homogeneous polynomial of degree \(k\), we have of course
\[ H_k(x, y^*) = (-1)^k H_k(x, y), \]
if
\[ y^* = -y, \quad y \in S^n \subset \mathbb{R}^{n+1} \]
is the antipodal map on \(S^n\).

We shall require asymptotics for the kernel \(H_k(x, y)\). The ones we require are essentially given by the classical Darboux formula for Jacobi polynomials (see [11], [17]), but we have not been able to find the exact results we need in the literature. Instead, in order to establish them, we shall use the fact that, since, by (2.1), the distinct eigenvalues (2.5) of
\[ P = \sqrt{-\Delta_{S^n} + (\frac{n-1}{2})^2} \]
is one, we have that
\[ H_k = \eta(\lambda_k - P) = \frac{1}{n} \int_{-\infty}^{\infty} \hat{\eta}(t)e^{it\lambda_k}e^{-itP} dt, \]
provided that \(\eta \in C_0^\infty(\mathbb{R})\) satisfies
\[ \eta(0) = 1, \quad \text{and } \eta(\tau) = 0, \quad |\tau| \geq 1/2. \]
Since \(P\) is positive, we have that \(\eta(\lambda_k + P) = 0\), by (2.8), if \(k = 1, 2, \ldots\). Thus, by (2.7) and Euler's formula, we have
\[ H_k(x, y) = \frac{1}{n} \int_{-\infty}^{\infty} \hat{\eta}(t)e^{it\lambda_k} (\cos tP) (x, y) dt, \]
where \((\cos tP)(x, y)\) denotes the kernel of \(\cos tP\). Note also that, by (2.5), we have that (2.10)
\[ \cos(t + 2\pi P) = (-1)^{n-1} \cos tP, \]
meaning that $\cos tP$ is $2\pi$-periodic when the dimension $n$ is odd and $4\pi$-periodic when $n$ is even.

As we shall see in the appendix, using the Hadamard parametrix, (2.3), (2.9) and (2.10) we can easily obtain the following

**Proposition 2.1.** Let $d_{S^n}(x, y)$ denote geodesic distance on $S^n$. We then have for $k = 0, 1, 2, \ldots$

\begin{equation}
|H_k(x, y)| \leq C(1 + k)^{n-1}.
\end{equation}

Moreover, for sufficiently large $k$, we can find functions $a_{\pm}(k; r)$ so that

\begin{equation}
H_k(x, y) = \lambda_k \sum_{\pm} a_{\pm}(k; d_{S^n}(x, y)) e^{\pm i \lambda_k d_{S^n}(x, y)}, \quad \text{if } \lambda_k^{-1} \leq d_{S^n}(x, y) \leq \frac{2\pi}{k},
\end{equation}

where for every $j = 0, 1, 2, \ldots$

\begin{equation}
|\partial^j a_{\pm}(k; r)| \leq C_j r^{-j}, \quad \text{if } \lambda_k^{-1} \leq d_{S^n}(x, y) \leq \frac{2\pi}{k}.
\end{equation}

We also have that, for the same functions $a_{\pm}(k; \cdot)$,

\begin{equation}
H_k(x, y) = (-1)^k \lambda_k \sum_{\pm} a_{\pm}(k; d_{S^n}(x, y^*)) e^{\pm i d_{S^n}(x, y^*)},
\end{equation}

\[ \text{if } \frac{\pi}{4} \leq d_{S^n}(x, y) \leq \pi - \lambda_k^{-1}. \]

In order to prove the desired bounds (1.15) for the harmonic projection operators and also to be able to prove uniform estimates for a localized version of the resolvent operators in (1.14), we require bounds for certain semi-classical Fourier integral operators (a.k.a. singular oscillatory integral operators). To state them for the generality we shall require throughout, let us assume that $g$ is a smooth Riemannian metric on $\mathbb{R}^n$ which is close to the Euclidean one. Assume further that the injectivity radius of $g$ is ten or more and let $d_g(x, y)$ be the Riemannian distance function (well defined near the diagonal) and $B_r(x) = \{ y \in \mathbb{R}^n : d_g(x, y) < r \}$ if, say, $0 < r < 10$. Then, as we shall see in the appendix, by using Stein’s oscillatory integral theorem [15] and Hörmander’s [7], it is not difficult to obtain the following

**Proposition 2.2.** Let $g$ be as above and assume that

\[ a(x, y) \in C^\infty(B_2(0) \times B_2(0) \backslash \{(x, x) : x \in B_2(0)\}) \]

satisfies

\begin{equation}
|a(x, y)| \leq C(d_g(x, y))^{2^{-n}}, \quad \text{if } d_g(x, y) \leq \lambda^{-1},
\end{equation}

and for every multi-index $\alpha$ we have

\begin{equation}
|\partial^\alpha x a(x, y)| \leq C_\alpha \lambda^{\frac{2^{-n}}{2}} (d_g(x, y))^{-\frac{n-1}{2} - |\alpha|}, \quad d_g(x, y) \geq \lambda^{-1},
\end{equation}

when $x, y \in B_2(0)$. We then have that if $r, s$ are as in (1.1)

\begin{equation}
\left\| \int e^{i\lambda d_g(x, y)} a(x, y) f(y) \, dy \right\|_{L^r(B_1(0))} \leq C_{r,s} \| f \|_{L^r(B_1(0))}, \quad f \in C_0^\infty(\mathbb{R}^n), \quad \text{supp } f \subset B_1(0).
\end{equation}
Additionally, if $g$ is sufficiently close to the Euclidean metric in the $C^\infty$ topology, the constant $C_{r,s}$ in (2.17) depends only on the constant $C$ in (2.15) and finitely many of the constants in (2.16).

Let us now see how we can use these two results to prove (1.15), which says that

\begin{equation}
\|H_k\|_{L^r(S^n) \to L^s(S^n)} \leq C_{r,s}(1 + k),
\end{equation}

if $r, s$ are as in (1.1). By compactness, it suffices to show that

\begin{equation}
\|H_k f\|_{L^r(S^n)} \leq C_{r,s}(1 + k)\|f\|_{L^s(S^n)}, \quad \text{if supp } f \subset B_1(x_0),
\end{equation}

with constant $C_{r,s}$ independent of the center $x_0 \in S^n$ of the unit-radius ball.

If we choose $\alpha \in C^\infty(\mathbb{R}_+)$ satisfying

\begin{equation}
\alpha(r) = 1, \quad r \leq \delta, \quad \text{and } \alpha(r) = 0, \quad r \geq 2\delta,
\end{equation}

and let

\[ \tilde{H}_k(x, y) = \alpha(d_{S^n}(x, y))H_k(x, y), \]

then clearly, if $\delta > 0$ is small enough, by the above Propositions, the integral operator with this kernel $\tilde{H}_k$ satisfies

\begin{equation}
\|\tilde{H}_k f\|_{L^r(S^n)} \leq (1 + k)\|f\|_{L^s(S^n)}, \quad \text{supp } f \subset B_1(x_0).
\end{equation}

Similarly, since the map $x \to x^*$ mapping $S^n$ to itself preserves the volume element, we have that

\begin{equation}
\|\tilde{H}^*_k f\|_{L^r(S^n)} \leq C_{r,s}(1 + k)\|f\|_{L^s(S^n)}, \quad \text{supp } f \subset B_1(x_0).
\end{equation}

The remaining piece

\[ T_k = H_k - \tilde{H}_k - \tilde{H}^*_k, \]

by Proposition 2.1, is an oscillatory integral operator of the form

\[ T_k h(x) = (1 + k)^{\frac{1}{2}} \sum \int_{S^n} \alpha_\pm(k; x, y)e^{\pm i\lambda_k d_{S^n}(x, y)} h(y) dV_{S^n}(y), \]

where

\[ \alpha_\pm(k; x, y) = 0, \quad d_{S^n}(x, y) \notin [\delta, \pi - \delta], \]

and with bounds independent of $k$,

\[ |\nabla_{x,y}^\beta \alpha_\pm(k; x, y)| \leq C_\beta. \]

Therefore, the desired bounds for it are a consequence of the following special case of Stein’s oscillatory integral theorem [15].

**Lemma 2.3.** Let $(M, g)$ be an $n \geq 2$ dimensional Riemannian manifold. Assume that the injectivity radius of $M$ is larger than $R$ and that $M$ is either compact or of bounded geometry, and let $d_g(\cdot, \cdot)$ be the associated Riemannian distance function. Assume further that $\alpha \in C^\infty(M \times M)$ satisfies (in terms of covariant derivatives)

\begin{equation}
|\nabla_x^{\beta_1} \nabla_y^{\beta_2} \alpha(x, y)| \leq C_{\beta_1, \beta_2},
\end{equation}

and

\begin{equation}
\alpha(x, y) = 0, \quad \text{if } d_g(x, y) \notin [\delta, R - \delta],
\end{equation}

where $\delta$, $\delta$, and $\delta$ are as in (1.1), (1.5), and (2.16) respectively.
for some $\delta > 0$. Then if we set
\begin{equation}
I_\lambda f(x) = \int e^{i\lambda d_\theta(x,y)}\alpha(x,y) f(y) dV_g(y),
\end{equation}
we have
\begin{equation}
\|I_\lambda f\|_{L^q(M)} \leq C\lambda^{-\frac{n}{2}} \|f\|_{L^p(M)},
\end{equation}
where $C$ depends on finitely many of the constants in (2.23) and
\begin{equation}
1 \leq p \leq 2, \quad q = \frac{n+1}{n-1}p', \quad \frac{1}{p} + \frac{1}{p'} = 1.
\end{equation}

Since, by Gauss’ lemma the phase function $d_\theta(x,y)$ of the oscillatory integral in (2.26) satisfies the
$n \times n$ Carleson-Sjölin condition defined in [13, §2.2] on the support of $\alpha$ (by (2.24)), (2.26) follows from Corollary 2.2.3 in [13].

To see why this yields our claim that
\begin{equation}
\|T_k f\|_{L^r(S^n)} \leq C(1 + k) \|f\|_{L^r(S^n)},
\end{equation}
where $r, s$ are as in (1.1), we first note that, by Proposition 2.1,
\begin{equation}
I_k = (1 + k)^{-\frac{n+1}{n}} T_k
\end{equation}
is as in Lemma 2.3. Therefore, if we choose
\begin{equation}
p = \frac{2n}{n+1}, \quad q = \frac{n+1}{n-1}p' = \frac{2n(n+1)}{(n-1)^2},
\end{equation}
we have, by (2.26),
\begin{equation}
\|T_k f\|_{L^{\frac{2n(n+1)}{n-1}}(S^n)} \leq C(1 + k)^{\frac{n+1}{n}} (1 + k)^{-\frac{(n-1)^2}{2(n+1)n}} = C(1 + k)^{\frac{n+1}{n}} \|f\|_{L^{\frac{2n}{n-1}}(S^n)}.
\end{equation}

We also have the trivial bounds
\begin{equation}
\|T_k f\|_{L^\infty(S^n)} \leq C(1 + k)^{\frac{n+1}{n}} \|f\|_{L^{\frac{2n}{n-1}}(S^n)}.
\end{equation}
Since
\[
\begin{aligned}
&\frac{n+1}{n} = \frac{(n-1)^2}{2n(n-1)}, \\
&\frac{n+1}{n-1} = \frac{2n(n+1)}{(n-1)^2}, \\
&1 - \frac{(n-1)^2}{(n-1)^2} = \frac{4}{(n-1)^2}, \\
&1 = \frac{n-1}{n+1} \cdot \frac{(n+1)(n-3)}{(n-1)^2} + \frac{n-1}{2} \cdot \frac{4}{(n-1)^2},
\end{aligned}
\]
and
\[
1 = \frac{n-1}{n+1} \cdot \frac{(n+1)(n-3)}{(n-1)^2} + \frac{n-1}{2} \cdot \frac{4}{(n-1)^2},
\]
if we interpolate between (2.29) and (2.30), we conclude that
\begin{equation}
\|T_k\|_{L^{\frac{2n}{n-1}}(S^n)} \leq C(1 + k) \|f\|_{L^{\frac{2n}{n-1}}(S^n)}.
\end{equation}

Since, by the same argument, the adjoint of $T_k$ also enjoys these bounds, we conclude that we also have that
\begin{equation}
\|T_k f\|_{L^{\frac{2n}{n-1}}(S^n)} \leq C(1 + k) \|f\|_{L^{\frac{2n}{n-1}}(S^n)}.
\end{equation}

The estimate (2.29) corresponds to the point $A$ in Figure 2 below, and the last two estimates correspond to the points $\alpha$ and $\alpha'$, respectively. The points $(\frac{1}{r}, \frac{1}{2})$ on the open.
segment connecting these two points correspond to the exponents in (1.1), and so, by interpolation, (2.31) and (2.32) yield (2.28), which completes the proof of (2.18).

Let us now focus on the proof of Theorem 1.1. We require the following result whose proof we postpone until the appendix since it was essentially established in [4], [2] and [12]. Its proof is a routine stationary phase calculation.

**Proposition 2.4.** Fix an even function \( 0 \leq \rho \in C^\infty_0(\mathbb{R}) \) satisfying
\[
\rho(t) = 1, \ |t| \leq 1/2, \ \text{and} \ \rho(t) = 0, \ |t| \geq 1.
\]
Then if \( P = \sqrt{-\Delta_{S^n}} + \left(\frac{n-1}{2}\right)^2 \) and \( n \geq 3 \) and we set
\[
R_0^{\lambda, \mu} f = \frac{\text{sgn} \mu}{i(\lambda + i\mu)} \int_0^\infty \rho(t)e^{i(\text{sgn} \mu)\lambda t} e^{-|\mu|t} (\cos tP) \, dt,
\]
the kernel can be written as
\[
\sum_{\pm} a_{\pm}(\lambda; x, y)e^{\pm i\lambda d_{S^n}(x, y)} + O((d_{S^n}(x, y))^{2-n}),
\]
where \( a_{\pm} \) vanishes when \( d_{S^n}(x, y) \) is close to \( \pi \), and moreover, satisfies (2.15)-(2.16) with constants independent of \( \lambda \geq 1 \).

Using this result, Proposition 2.2, Lemma 2.3 and (2.19), we can prove the first part of Theorem 1.1.

Indeed, we first notice that by the first two of these results and the Hardy-Littlewood-Sobolev inequality we have that
\[
\|R_0^{\lambda, \mu} f\|_{L^r(S^n)} \leq C_{r,s} \|f\|_{L^s(S^n)},
\]
for $r,s$ as in (1.1). We also note that if we set
\[ m_{\lambda,\mu}(\tau) = \frac{\text{sgn } \mu}{i(\lambda + i\mu)} \int_0^\infty (1 - \rho(t)) e^{i(\text{sgn } \mu)\lambda \mu |t|} (\cos t \tau) \, dt, \]
then we clearly have for every $N = 1, 2, 3, \ldots$
\[ \lambda |m_{\lambda,\mu}(\tau)| \leq C_N (1 + |\lambda - \tau|)^{-N}, \quad \text{if } \tau \geq 0, \text{ and } \lambda, |\mu| \geq 1/2. \]
Therefore, by (1.14),
\[ R_{1,\lambda,\mu} f = (\Delta + (\frac{n-1}{2})^2)^{-1} f - R_{0,\lambda,\mu} f = \sum_{k=0}^\infty m_{\lambda,\mu}(\lambda_k) H_k, \]
must, by (2.19) satisfy for $\lambda, |\mu| \geq 1/2$,
\[ \|R_{1,\lambda,\mu} f\|_{L^r(S^n)} \leq \sum_{k=0}^\infty |m_{\lambda,\mu}| \|H_k f\|_{L^r(S^n)} \]
\[ \leq C \left( \sum_{k=0}^\infty \lambda^{-1} (1 + k)(1 + |\lambda - k|)^{-3} \right) \|f\|_{L^r(S^n)} \]
\[ \leq C' \|f\|_{L^r(S^n)}. \]

Based on this estimate we know that we have the uniform bounds in (1.10) provided that
\[ \zeta = (\lambda + i\mu)^2, \quad \lambda, |\mu| \geq 1/2. \]
Thus we have proven that we have the uniform bounds when $\zeta \in \mathcal{R}$ and $\text{Re } \zeta \geq 1$. Since the remaining cases follow from Sobolev estimates, the proof of (1.10) is complete.

To complete the proof of the theorem (modulo the proofs of the Propositions), we just need to show that (1.11) follows from (1.10) and a simple scaling argument. To wht, we claim that if we have for a certain $r,s$ as in (1.1) and $\zeta \in \mathbb{C}$
\begin{equation}
(2.36) \quad \|u\|_{L^r(S^n, dV_{S^n})} \leq C_{r,s} \left( \|\Delta_{S^n} - (\frac{n-1}{2})^2 + \zeta\|_{L^r(S^n, dV_{S^n})} \right), \quad u \in C^\infty(S^n),
\end{equation}
then for the same constant $C_{r,s}$, we must have for a given $\kappa > 0$
\begin{equation}
(2.37) \quad \|u\|_{L^r(S^n, dV_{S^n})} \leq C_{r,s} \left( \|\Delta_{S^n} - \kappa (\frac{n-1}{2})^2 + \kappa \zeta\|_{L^r(S^n, dV_{S^n})} \right), \quad u \in C^\infty.
\end{equation}

We recall that $\Delta_{S^n} = \Delta_1$ and that $dV_{S^n} = dV_1$. To use this we note that, for, say, $u_1 \in C^\infty_0((0, \pi) \times S^{n-1})$, if we set
\[ u_\kappa(\cdot, \theta) = u_1(\sqrt{\kappa} \cdot, \theta) \in C^\infty_0((0, \pi/\sqrt{\kappa}) \times S^{n-1}), \]
then for $0 < r < \pi$, we have
\[ (\partial_r^2 + (n-1) \cot r \partial_r + (\csc r)^2 \Delta_{S^{n-1}}) u_1(r, \theta) \]
\[ = \kappa^{-1} (\partial_r^2 u_\kappa)(r/\sqrt{\kappa}, \theta) + (\sqrt{\kappa})^{-1} \cot(r)(\partial_r u_\kappa)(r/\sqrt{\kappa}, \theta) + (\csc r)^2 \Delta_{S^{n-1}} u_\kappa(r/\sqrt{\kappa}, \theta) \]
\[ = \kappa^{-1} \left[ (\partial_r^2 u_\kappa)(t, \theta) + \sqrt{\kappa} \cot(\sqrt{\kappa} t)(\partial_r u_\kappa)(t, \theta) + (\sqrt{\kappa} \csc(\sqrt{\kappa} t))^2 \Delta_{S^{n-1}} u_\kappa(t, \theta) \right], \]
with $t = r/\sqrt{\kappa}, 0 < t < \pi/\sqrt{\kappa}$. Thus, if $u \in C^\infty(S^n)$ and $u_1(r, \theta)$ and $u_\kappa(r, \theta)$ are its polar coordinates representation about a point in $S^n$ with respect to the metric of constant curvature 1 and $\kappa > 0$, respectively, we have
\[ (\Delta_1 + z) u_1(\cdot, \theta) = \kappa^{-1} (\Delta_\kappa + \kappa z) u_\kappa(\cdot/\sqrt{\kappa}, \theta), \quad z \in \mathbb{C}. \]
Therefore, by (1.5),
\[
\int_{S^n} |(\Delta_1 - (n-1)^2 + \zeta)u|^{r} \, dV_1
\]
\[
= \int_{S^{n-1}} \int_{0}^{\pi} |(\Delta_1 - (n-1)^2 + \zeta)u_1(t, \theta)|^{r} (\sin t)^{n-1} \, dt \, d\theta
\]
\[
= \sqrt{\kappa} \int_{S^{n-1}} \int_{0}^{\pi/\sqrt{\kappa}} |\kappa^{-1}(\Delta_\kappa - \kappa^{(n-1)^2} + \kappa\zeta)u_\kappa(t, \theta)|^{r} (\sin(\sqrt{\kappa}t))^{n-1} \, dt \, d\theta
\]
\[
= (\sqrt{\kappa})^n \int_{S^{n-1}} \int_{0}^{\pi/\sqrt{\kappa}} |\kappa^{-1}(\Delta_\kappa - \kappa^{(n-1)^2} + \kappa\zeta)u_\kappa(t, \theta)|^{r} \left(\sin(\sqrt{\kappa}t)\right)^{n-1} \, dt \, d\theta
\]
\[
= \kappa^{\frac{n}{2}} \kappa^{-r} \int_{S^n} |(\Delta_\kappa - \kappa^{(n-1)^2} + \kappa\zeta)u|^{r} \, dV_\kappa.
\]
Similarly,
\[
\int_{S^n} |u|^{s} \, dV_1 = \kappa^{\frac{s}{2}} \int_{S^n} |u|^{s} \, dV_\kappa.
\]
Therefore, if we assume that (2.36) is valid, we have
\[
\|u\|_{L^r(dV_\kappa)} = \kappa^{-\frac{r}{2}} \|u\|_{L^r(dV_1)}
\]
\[
\leq C_{r,s} \kappa^{-\frac{s}{2}} \|((\Delta_1 + \zeta)u)\|_{L^{r}(dV_1)}
\]
\[
= C_{r,s} \kappa^{-\frac{s}{2}} \kappa^{-\frac{r}{2}} \kappa^{-1} \|(\Delta_\kappa - \kappa^{(n-1)^2} + \kappa\zeta)u\|_{L^{r}(dV_\kappa)},
\]
which yields (2.37) as claimed by the first part of our assumption in (1.1).

3. Stein-Tomas estimates for $\mathbb{H}^n$.

If $P = \sqrt{-\Delta_{\mathbb{H}^n} - (\frac{n-1}{2})^2}$, then the main result in this section is the following analogue of the Stein-Tomas restriction theorem for Euclidean space.

**Theorem 3.1.** There is a uniform constant $C$ so that for all $T$, $\lambda \geq 1$ we have

(3.1) \[
T^{\frac{n}{2}} \| \mathbb{1}_{[\lambda, \lambda+T^{-1}]}(P) \|_{L^2(\mathbb{H}^n) \rightarrow L^{\frac{2(n+1)}{n+2}}(\mathbb{H}^n)} \leq C \lambda^{\frac{n-1}{2(n+1)}},
\]
and

(3.2) \[
T \| \mathbb{1}_{[\lambda, \lambda+T^{-1}]}(P) \|_{L^{\frac{2(n+1)}{n+2}}(\mathbb{H}^n) \rightarrow L^{\frac{2(n+1)}{n+2}}(\mathbb{H}^n)} \leq C \lambda^{\frac{n-1}{2}}.
\]

By duality, (3.1) is equivalent to

(3.3) \[
T^{\frac{n}{2}} \| \mathbb{1}_{[\lambda, \lambda+T^{-1}]}(P) \|_{L^{\frac{n+1}{2(n+1)}}(\mathbb{H}^n) \rightarrow L^2(\mathbb{H}^n)} \leq C \lambda^{\frac{n-1}{2}}, \quad \lambda, T \geq 1,
\]
and it is clear that this estimate along with (3.1) yields (3.2). Conversely, by a standard $TT^*$ argument, (3.2) implies (3.1).

Before proving the Theorem, let us make a couple more observations about these estimates. First, by the spectral theorem and (3.3), we have that there is a uniform constant $C$ so that

(3.4) \[
T^{\frac{n}{2}} \| m(T(P - \lambda)) \|_{L^{\frac{2(n+1)}{n+2}}(\mathbb{H}^n) \rightarrow L^2(\mathbb{H}^n)} \leq C \| m \|_{L^\infty} \lambda^{\frac{n-1}{2(n+1)}}, \quad \lambda, T \geq 1,
\]
if $m \in C(\mathbb{R})$ and $\text{supp } m \subset [0, 1]$. 
From this and (3.1) we immediately obtain the following result which will be useful in
the sequel:

**Corollary 3.2.** There is a constant $C$ which is independent of $\lambda$, $T \geq 1$ so that if $m$ is
as in (3.4) we have

$$T\|m(T(P - \lambda))\|_{L^{2(n+1)/(n+1)}(\mathbb{H}^n)} \leq C\|m\|_{L^\infty} \lambda^{n-1}.$$  

To prove the theorem, let us first prove the special case corresponding to $T = 1$, which
we can obtain by local techniques:

**Lemma 3.3.** There is a uniform constant $C$ so that

$$\|\rho(\lambda, \lambda + 1)(P)\|_{L^2(\mathbb{H}^n) \to L^{2(n+1)/(n+1)}(\mathbb{H}^n)} \leq C(1 + \lambda) \chi(n) \lambda^{n-1}, \quad \lambda \geq 0.$$  

If $\lambda$ is bounded by a fixed constant, the estimate (3.6) is a simple consequence of the
Sobolev estimates for $\Delta_n$ that can be found, for instance in §3 of [1], which say that if
$1 < p < q < \infty$ then for powers of the unshifted Laplacian, $-\Delta_n = P^2 + (\frac{n-1}{2})^2$, we
have

$$\left\|\left(P^2 + (\frac{n-1}{2})^2\right)^{-n} \right\|_{L^p(\mathbb{H}^n) \to L^q(\mathbb{H}^n)} < \infty.$$  

Therefore, in proving (3.6) we shall assume that $\lambda$ is large. By duality and the spectral
theorem, this then would follow from showing that

$$\|\rho(\lambda - P)\|_{L^2(\mathbb{H}^n) \to L^{2(n+1)/(n+1)}(\mathbb{H}^n)} \leq C\lambda \chi(n) \lambda^{n-1}, \quad \lambda \gg 1,$$

assuming that

$$\rho \in S(\mathbb{R}), \quad \rho(0) = 1, \quad \text{and supp } \hat{\rho} \subset [1/2, 1].$$

We then write

$$\rho(\lambda - P) = \frac{1}{\pi} \int \hat{\rho}(t)e^{i\lambda t} \cos tP dt + \rho(\lambda + P).$$

Since by duality, the spectral theorem and the Sobolev estimates (3.7), we have

$$\|\rho(\lambda + P)\|_{L^2(\mathbb{H}^n) \to L^{2(n+1)/(n+1)}(\mathbb{H}^n)} = O(\lambda^{-N}), \quad \forall N, \lambda \geq 1,$$

it suffices to show that

$$\left\|\int \hat{\rho}(t)e^{i\lambda t} \cos tP f dt\right\|_{L^2(\mathbb{H}^n) \to L^{2(n+1)/(n+1)}(\mathbb{H}^n)} = O(\lambda^{n-1}) \chi(n), \quad \lambda \geq 1.$$  

Since the kernel of this operator vanishes when the distance between $x$ and $y$ is larger
than one, to prove this estimate it suffices to verify that

$$\left\|\int \hat{\rho}(t)e^{i\lambda t} \cos tP f dt\right\|_{L^{2(n+1)/(n+1)}(B)} \leq C\lambda \chi(n) \|f\|_{L^2(\mathbb{H}^n)}, \quad \lambda \geq 1,$$

whenever $B$ is a geodesic ball in $\mathbb{H}^n$ of radius one. If we choose geodesic normal coordinates
about the center, this follows from the proof of (5.1.3') in [13], which completes
the proof of (3.6).
Proof of Theorem 3.1. Since, as we noted before (3.1) and (3.2) are equivalent, it suffices to prove the former. Repeating the first part of the proof of Lemma 3.3, it suffices to show that if $\rho$ is as in (3.8) then

$$T^{\frac{1}{2}}\|\rho(T(\lambda-P))\|_{L^2(\mathbb{H}^n) \to L^{\frac{2(n+1)}{n-1}}(\mathbb{H}^n)} \leq C\lambda^{\frac{n-1}{2}}, \quad T, \lambda \geq 1.$$ 

If $\chi = |\rho|^2 \in S(\mathbb{R})$, this follows from showing that

$$T\|\chi(\lambda-P)\|_{L^{2(n+1)}(\mathbb{H}^n) \to L^{2(n+1)}(\mathbb{H}^n)} \leq C\lambda^{\frac{n-1}{2}}, \quad \lambda, T \geq 1,$$

and since, by Sobolev estimates $\chi(T(\lambda + P))$ has $L^{\frac{2(n+1)}{n-1}}(\mathbb{H}^n) \to L^{\frac{2(n+1)}{n-1}}(\mathbb{H}^n)$ operator norm which is $O((\lambda T)^{-N})$ for every $N$, we would be done if we could show that for $\lambda, T \geq 1$ we have the uniform bounds

$$\left\| \int \hat{\chi}(t/T)e^{i\lambda t} \cos tP dt \right\|_{L^{2(n+1)}(\mathbb{H}^n) \to L^{2(n+1)}(\mathbb{H}^n)} \leq C\lambda^{\frac{n-1}{2}}, \quad \lambda, T \geq 1.$$

If $\beta \in C_0^\infty(\mathbb{R})$ satisfies $\beta(s) = 1$, $|s| \leq 1$, then it follows from the Lemma 3.3, duality and orthogonality that

$$\left\| \int \int (1 - \beta(t))\hat{\chi}(t/T)e^{i\lambda t} \cos tP dt \right\|_{L^{2(n+1)}(\mathbb{H}^n) \to L^{2(n+1)}(\mathbb{H}^n)} \leq C\lambda^{\frac{n-1}{2}}, \quad \lambda, T \geq 1.$$

Therefore, we would be done if we could show that

$$\left\| \int_0^\infty (1 - \beta(t))\hat{\chi}(t/T)e^{i\lambda t} \cos tP dt \right\|_{L^{2(n+1)}(\mathbb{H}^n) \to L^{2(n+1)}(\mathbb{H}^n)} \leq C\lambda^{\frac{n-1}{2}}, \quad \lambda, T \geq 1,$$

since the same argument will give this bound if the integral is taken over $(-\infty, 0]$.

To prove this, as in Stein’s argument for the Euclidean case, we shall use analytic interpolation. Define the analytic family of operators

$$S^{z}_{\lambda,T} = e^{z^2} \int_0^\infty t^z \ (1 - \beta(t))\hat{\chi}(t/T)e^{i\lambda t} \cos tP dt.$$ 

Then $S^0_{\lambda,T}$ is the operator in (3.9), and so, by Stein’s analytic interpolation theorem, we would have this estimate if we could show that

$$\|S^{z}_{\lambda,T}\|_{L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)} \leq C, \quad \Re z = -1, \quad \lambda, T \geq 1,$$

as well as

$$\|S^{z}_{\lambda,T}\|_{L^1(\mathbb{H}^n) \to L^\infty(\mathbb{H}^n)} \leq C\lambda^{\frac{n-1}{2}}, \quad \Re z = \frac{n-1}{2}, \quad \lambda, T \geq 1.$$

The estimate (3.10) follows from the spectral theorem and the fact that the Fourier transforms of $e^{z^2} t^z / \Gamma(z+1)$ are continuous functions whose $L^\infty$ norms which are bounded independent of $z$ if $\Re z = -1$.

To prove (3.11), let us first assume that $n$ is odd. Then the kernel of $S^{\frac{n-1}{2}+i\sigma}_{\lambda,T}, \sigma \in \mathbb{R}$, is a constant $c_n$ times

$$e^{((n-1)/2+is)z^2} \int_0^\infty \left| \frac{1}{\sinh \frac{t}{2}} \int e^{i\lambda t} \hat{\chi}(t/T) e^{i\lambda t} \right|_{t=d(x,y)}.$$ 

(12)
where \(d(x, y)\) denotes the distance between \(x\) and \(y\) coming from the hyperbolic metric. Since \((1 - \beta(t))\) vanishes near the origin, and \(t/\sinh t < 1, t > 0\), it is easy to see that this expression is bounded by a fixed multiple of \(\lambda^{-2}\) when \(\lambda \geq 1\) and \(\sigma \in \mathbb{R}\), which means that we have (3.11) when \(n\) is even.

To finish the proof, we have to establish (3.11), when \(n\) is even. In this case we can use the fact that the kernel is given by the formula

\[
(3.13) \quad c_n e^{((n-1)/2+\sigma)^2} \int_t^\infty \frac{\sinh s}{\sqrt{\cosh s - \cosh t}} \left(\frac{1}{\sinh s} d_s\right)^{\frac{n}{2}} \left[(1 - \beta(s))s^{\frac{n-1}{2} + i\sigma} \hat{\chi}(s/T)e^{i\lambda s}\right] ds,
\]

where, as before, \(t = d(x, y)\).

To prove this we note that since \(\beta\) equals one near the origin and \(\lambda, T \geq 1\), if we use Leibniz’s rule, we can write the integral as

\[
(3.14) \quad \lambda^{\frac{n}{2}} \int_t^\infty \frac{a(T, \sigma, \lambda; s)}{\sqrt{1 + e^{-2s} - e^{t-s} - e^{-t-s}}} e^{i\lambda s} ds,
\]

where for constants which are independent of \(\sigma \in \mathbb{R}\) and \(T, \lambda \geq 1\) we have

\[
(3.15) \quad |a(T, \sigma, \lambda; s)| + |\partial_s a(T, \sigma, \lambda; s)| \leq C(1 + s)^{-2}, \text{ if } 1 - \beta(s) \neq 0, s > 0.
\]

Note also that we have that for \(1 - \beta(s) \neq 0\) we have

\[
(3.16) \quad 1 + e^{-2s} - e^{t-s} - e^{-t-s} \geq c(s - t), \quad \text{if } s \in [t, t + 1], t \geq 0,
\]

for some fixed \(c > 0\). Using this estimate and the bound for the first term in the left of (3.15), we deduce that

\[
\lambda^{\frac{n}{2}} \int_t^{t+\lambda^{-1}} \frac{a(T, \sigma; s)(1 - \beta(s))}{\sqrt{1 + e^{-2s} - e^{t-s} - e^{-t-s}}} e^{i\lambda s} ds \leq C\lambda^{\frac{n+1}{2}},
\]

as posited in (3.11), independent of \(\sigma, T\) and \(\lambda\) as above. To handle the remaining piece, we note that we also have that there also must be a fixed \(c > 0\) so that if \(1 - \beta(s) \neq 0\) and \(t > 0\) then

\[
1 + e^{-2s} - e^{t-s} - e^{-t-s} \geq c, \quad \text{if } s \geq t + 1.
\]

Using this bound as well as (3.15)-(3.16), we conclude that the remaining piece of (3.14) must be bounded independent of \(\sigma, T, \lambda\) as above since after integrating by parts it is dominated by

\[
\lambda^{\frac{n}{2} - 1} \left| \frac{a(T, \sigma; s)(1 - \beta(s))}{\sqrt{1 + e^{-2s} - e^{t-s} - e^{-t-s}}} \right|_{s=t+\lambda^{-1}}^{t+\lambda^{-1}} + \lambda^{\frac{n}{2} - 1} \int_{t+\lambda^{-1}}^\infty \left| \frac{\partial}{\partial s} \left( \frac{a(T, \sigma; s)(1 - \beta(s))}{\sqrt{1 + e^{-2s} - e^{t-s} - e^{-t-s}}} \right) \right| ds = O(\lambda^{\frac{n+1}{2}}).
\]

\[\square\]
4. **Uniform \((L^r, L^s)\) resolvent bounds for \(\mathbb{H}^n\).**

The goal of this section is to prove the uniform resolvent bounds

\[
\| (\Delta_{\mathbb{H}^n} + (\frac{n-1}{2})^2 + z^2)^{-1} \|_{L^r(\mathbb{H}^n) \to L^s(\mathbb{H}^n)} \leq C, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \ |z| \geq 1,
\]

assuming that, as in \([6]\),

\[
n\left(\frac{1}{r} - \frac{1}{s}\right) = 2, \quad \text{and} \quad \min\left(\left|\frac{1}{r} - \frac{1}{s}\right|, \left|\frac{1}{s} - \frac{1}{2}\right|\right) > \frac{1}{2n}.
\]

This is the first part of Theorem 1.2. By the same argument that (1.10) implies (1.11), one sees that (4.1) implies the other part of the theorem, (1.13).

Clearly, by letting \(\kappa \to 0\), one sees that this inequality implies the earlier Euclidean estimates (1.2) in \([6]\), and based on this, one sees that (4.2) is the sharp range of exponents for all of these estimates. Pictorially, we have (4.2) if \(\left(\frac{1}{r}, \frac{1}{s}\right)\) is on the open line segment connecting \(\alpha\) and \(\alpha'\) in Figure 2.

Note also, that if we write \(z = \lambda + i\mu\), then it suffices to verify that we have (4.1) if

\[
z = \lambda + i\mu, \quad \text{with} \quad \lambda \geq 1, \ \mu \neq 0,
\]

since the remaining cases of (4.1) follow from Sobolev estimates.

To prove (4.1), as in \([2]\), we shall use the formula

\[
(\Delta_{\mathbb{H}^n} + (\frac{n-1}{2})^2 + (\lambda + i\mu)^2)^{-1} = \frac{\text{sgn} \mu}{i(\lambda + i\mu)} \int_0^\infty e^{i(\text{sgn} \mu) \lambda t} e^{-|\mu|t} (\cos t P) dt,
\]

where now \(P\) is the square root of the shifted Laplacian on \(\mathbb{H}^n\), i.e., \(P = \sqrt{-\Delta_{\mathbb{H}^n} - (\frac{n-1}{2})^2}\). Thus

\[
u(t, \cdot) = \cos t P
\]
solves the Cauchy problem for the shifted Laplacian

\[
(\partial_t^2 - \Delta_{\mathbb{H}^n} - (\frac{n-1}{2})^2)u(t, x) = 0, \quad u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = 0.
\]

To prove this, choose a Littlewood-Paley bump function \(\beta \in C_0^\infty((1/2, 2))\) satisfying

\[
\sum_{j \in \mathbb{Z}} \beta(2^{-j} t) = 1, \ t > 0.
\]

It then follows that

\[
\beta_0(t) = 1 - \sum_{j=0}^\infty \beta(2^{-j} t)
\]
equals one for \(t > 0\) near the origin and vanishes when \(t \geq 2\). Thus, the kernel of

\[
S_0 = \frac{\text{sgn} \mu}{i(\lambda + i\mu)} \int_0^\infty \beta_0(t) e^{i(\text{sgn} \mu) \lambda t} e^{-|\mu|t} (\cos t P) dt,
\]

vanishes when \(d_{d_{\mathbb{H}^n}}(x, y) \geq 2\). As we shall see, its kernel is similar to the corresponding local operator \(R^{\alpha, \mu}_{\mathbb{H}^n}\) that we encountered in our bounds for \(S^n\). Specifically, in the appendix we shall prove the following
Proposition 4.1. Let $S_0^\lambda(x, y) = S_0(x, y)$ denote the kernel of the operator in (4.5). Then

$$S_0(x, y) = \sum_{\pm} a_\pm(\lambda; x, y) e^{\pm i\lambda d(x, y)} + O((d_{\mathbb{H}^n}(x, y))^{2-n}),$$

where

$$a_\pm(\lambda; x, y) = 0 \quad \text{if} \quad d_{\mathbb{H}^n}(x, y) \geq 2,$$

and, moreover, with constants independent of $\lambda \geq 1$

$$|a_\pm(\lambda; x, y)| \leq C(\lambda d_{\mathbb{H}^n}(x, y))^{2-n}, \quad \text{if} \quad d_{\mathbb{H}^n}(x, y) \geq \lambda^{-1},$$

and

$$|\nabla x^\alpha \nabla y^\beta a_\pm(\lambda; x, y)| \leq C_{\alpha, \beta, \lambda} \lambda^{-\frac{n-3}{2}}(\lambda d_{\mathbb{H}^n}(x, y))^{-\frac{n-1}{2}+\alpha_1+\alpha_2}, \quad \text{if} \quad d_{\mathbb{H}^n}(x, y) \geq \lambda^{-1}.$$

Due to this Proposition, it is clear that we can use the proof of (2.35) to show that if $r, s$ are as in (1.1) then there is a constant $C_{r, s}$ so that for all $\lambda \geq 1$

$$\|S_0 f\|_{L^r(\mathbb{H}^n)} \leq C_{r, s}\|f\|_{L^r(\mathbb{H}^n)}.$$ 

This implies that the non-local part of the resolvent is actually bounded for exponents $(\frac{1}{r}, \frac{1}{s})$ on the closed segment joining $\alpha$ and $\alpha'$ in the Figure 2, i.e., we have

$$\|\beta\Delta_{\mathbb{H}^n} + \alpha_1^2 + (\lambda + i\mu)^2\|_{L^r(\mathbb{H}^n)} \leq C, \quad \lambda, \mu \in \mathbb{R}, \quad \lambda \geq 1,$$

and $n(\frac{1}{r} - \frac{1}{s}) = 2, \quad \frac{2n}{n+3} \leq r \leq \frac{2n}{n+1}$.

To prove (4.11), we shall use an interpolation argument. The three ingredients we require are that there is a uniform constant $C$ so that for $k = 1, 2, \ldots$

$$\|S_k\|_{L^2(\mathbb{H}^n)} \leq C 2^{-k} \lambda^{-\frac{n+3}{2(n+1)}},$$

as well as for all $k, N \in \mathbb{N}$, there is a constant $C_N$, depending only on $N$ so that

$$\|S_k\|_{L^1(\mathbb{H}^n)} \leq C_N 2^{-kN} \lambda^{\frac{n-3}{2}},$$

and

$$\|S_k\|_{L^\infty(\mathbb{H}^n)} \leq C_N 2^{-kN} \lambda^{\frac{n-3}{2}}.$$

Since $\lambda^{\frac{n+3}{2(n+1)}} = \lambda^{-1 + \frac{n-1}{2(n+1)}}$, (4.13) follows from the formula for $S_k$ and (3.4).
In odd dimensions, both (4.14) and (4.15) follow immediately from the fact that the kernel of $S_k$ is given by

$$c_n \frac{\text{sgn} \mu}{i(\lambda + i\mu)} \left( \frac{1}{\sinh t} \right) \frac{2^{-1}}{\sinh t} \left[ \beta(2^{-k}t) e^{i(\text{sgn} \mu)\lambda t} e^{-|\mu|t} \right], \quad t = d_{\mathbb{H}^n}(x, y).$$

In even dimensions, one establishes these two bounds using the fact that the kernel is given by

$$c_n \frac{\text{sgn} \mu}{i(\lambda + i\mu)} \int_{t}^{\infty} \frac{\sinh s}{\sqrt{\cosh s - \cosh t}} \left( \frac{1}{\sinh s} \frac{2^{-1}}{ds} \right) \left[ \beta(2^{-k}s) e^{i(\text{sgn} \mu)\lambda s} e^{-|\mu|s} \right] ds, \quad t = d_{\mathbb{H}^n}(x, y),$$

by the same argument that established (3.11) from (3.13).

Since $\frac{(n-1)^2}{2n(n+1)} = \frac{n-1}{2(n+1)} \cdot \frac{n-1}{n}$, and

$$- \frac{2}{n+1} = \frac{n-3}{2} \cdot \frac{1}{n} - \frac{n+3}{2(n+1)} \cdot \frac{n-1}{n},$$

if we interpolate between (4.13) and (4.14), we conclude that

$$\|S_k\|_{L^2_{\mathbb{H}^n} \to L^2_{\mathbb{H}^n}} = O_N(2^{-kN} \lambda^{-\frac{2}{2n}}).$$

This is a bound which corresponds to the point $A$ in the figure.

To get the bound (4.11) corresponding to one the endpoints

$$\|S_k\|_{L^2_{\mathbb{H}^n} \to L^2_{\mathbb{H}^n}} \leq C2^{-k}$$

which corresponds to the point $\alpha = (\frac{n+1}{2n}, \frac{n-3}{2n})$ in the figure we need to interpolate between (4.16) and (4.15). We first note that

$$\frac{n-3}{2n} = \frac{(n-1)^2}{2n(n+1)} \cdot \theta,$$

with

$$\theta = \frac{(n+1)(n-3)}{(n-1)^2}.$$

Since for this $\theta$, a calculation shows that

$$0 = \frac{n-3}{2} \cdot (1-\theta) - \frac{2}{n+1} \cdot \theta,$$

we conclude that (4.17) does indeed follow from (4.15) and (4.16) via interpolation.

From this we obtain all of (4.11), since by duality we have from (4.17) that

$$\|S_k\|_{L^{2n}_{\mathbb{H}^n} \to L^{2n}_{\mathbb{H}^n}} \leq C2^{-k},$$

which corresponds to the point $\alpha'$ in the figure and yields (4.11) for the remaining exponents if we interpolate with (4.17).
4.1. Improved estimates for $\mathbb{H}^3$. Let us now give the simple argument showing that in three dimensions we can obtain the following improvement over Theorem 1.2:

\[ \|u\|_{L^s(\mathbb{H}^3, dV_{\mathbb{H}^3})} \leq C_{r,s} \left\| \left( \Delta - \kappa + \sqrt{\kappa} + \zeta \right) u \right\|_{L^r(\mathbb{H}^3, dV_{\mathbb{H}^3})}, \quad u \in C_0^\infty, \quad \zeta \in \mathbb{C}. \]

By a straightforward variant of the scaling argument at the end of §2, these bounds which are uniform both in $\zeta$ and in the curvature $-\kappa, \kappa > 0$, would just follow from the special case where the curvature is $-1$, i.e.,

\[ \|u\|_{L^s(\mathbb{H}^3, dV_{\mathbb{H}^3})} \leq C_{r,s} \left\| \left( \Delta_{\mathbb{H}^3} + 1 + \frac{\zeta}{2} \right) u \right\|_{L^r(\mathbb{H}^3, dV_{\mathbb{H}^3})}, \quad u \in C_0^\infty, \quad \zeta \in \mathbb{C}. \]

In proving this, we may assume that $\zeta = (\lambda + i\mu)^2$ with $\lambda \in \mathbb{R}, \mu > 0$, and then we just use the fact that the kernel of $\left( \Delta_{\mathbb{H}^3} + 1 + z^2 \right)^{-1}$ equals

\[ \frac{1}{4\pi \sinh(d_{\mathbb{H}^3}(x,y))} e^{i\lambda_k t} e^{-itP} \]

(see [19, p. 105]). Because of this, we obtain (4.20) via the Hardy-Littlewood-Sobolev inequality and Young’s inequality.

5. Appendix: Proof of the Propositions.

We shall conclude matters by proving Propositions 2.1, 2.2, 2.4 and 4.1. As we noted before, each is essentially in the literature (e.g., [2], [4], [6], [9], [11], [12], and [13]).

5.1. Proof of Proposition 2.1. Let us start with Proposition 2.1, which concerns asymptotics for the kernel of projection onto spherical harmonics of degree $k$, which involve multiples of the zonal functions on $S^n$. As we noted, before, the asymptotics that we require are essentially in the classical Darboux formula (see [17]).

To prove that $H_k(x, y)$ satisfies (2.11)–(2.14), we first note that the first bound is just a special case of sup-norm estimates for spectral clusters. See, e.g. [14, (3.2.5)–(3.2.6)].

To obtain the off-diagonal assertions in (2.12)–(2.14), we note that we can write, with

\[ P = \sqrt{-\Delta_{S^n} + (\frac{n-1}{2})^2}, \quad \lambda_k = k + \frac{n-1}{2} \]

\[ H_k(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\rho}(t) e^{i\lambda_k t} e^{-itP} dt \]

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \hat{\rho}(t) e^{i\lambda_k t} (\cos tP)(x, y) dt + \rho(\lambda_k + P)(x, y), \]

if $\rho \in C_0^\infty(\mathbb{R})$ satisfies

\[ \rho(\tau) = 1, |\tau| \leq 1/2, \quad \rho(\tau) = 0, |\tau| \geq 3/4. \]

Since the last term and all of its derivatives or $O_N((1 + \lambda_k)^{-N})$, for every $N$, it suffices to show that

\[ \tilde{H}_k(x, y) = \int_{-\infty}^{\infty} \hat{\rho}(t) e^{i\lambda_k t} (\cos tP)(x, y) dt \]

is as in (2.12)–(2.14).
If $n$ is odd then $\cos tP$ is $2\pi$-periodic. Since $\hat{\rho} \in \mathcal{S}(\mathbb{R})$, it then follows that if we set
\[
\psi_{\text{odd}}(t) = \sum_{j \in \mathbb{Z}} \hat{\rho}(t - 2\pi j),
\]
then $\psi_{\text{odd}}$ is smooth and $2\pi$-periodic, and, therefore,
\[
\tag{5.3}
\tilde{H}_k(x, y) = \int_{-\pi}^{\pi} \psi_{\text{odd}}(t) e^{i\lambda_k t} (\cos tP)(x, y) \, dt, \quad n \text{ odd}.
\]
Similarly, since $\cos tP$ is $4\pi$-periodic when $n$ is even, if we set
\[
\psi_{\text{even}}(t) = \sum_{j \in \mathbb{Z}} \hat{\rho}(t - 4\pi j),
\]
then $\psi_{\text{even}}$ is smooth and $4\pi$ periodic and we have
\[
\tag{5.4}
\tilde{H}_k(x, y) = \int_{-2\pi}^{2\pi} \psi_{\text{even}}(t) e^{i\lambda_k t} (\cos tP)(x, y) \, dt, \quad n \text{ even}.
\]
To proceed, let us first assume that $n$ is odd. We then fix $\eta \in C_0^\infty(\mathbb{R})$ satisfying
\[
\eta(t) = 1, \ |t| \leq 1/2, \quad \text{and} \quad \eta(t) = 0, \ |t| \geq 1.
\]
We then can write
\[
\tilde{H}_k(x, y) = \int_{-\pi}^{\pi} \eta(\pi - |t|) \psi_{\text{odd}}(t) e^{i\lambda_k t} (\cos tP)(x, y) \, dt
\]
\[
+ \int_{-\pi}^{\pi} (1 - \eta(\pi - |t|)) \psi_{\text{odd}}(t) e^{i\lambda_k t} (\cos tP)(x, y) \, dt.
\]
The proof of [13, Lemma 5.1.3] shows that the first term is as in (2.12)–(2.13), and, since for odd $n$ $(\cos tP)(x, y) = 0$ if $d_{S^n}(x, y) \neq |t|, \ 0 < |t| < \pi$ (see [19]) the second term vanishes when $d_{S^n}(x, y) \leq \pi - 1$, which means that for odd dimensions we have all but (2.14) in Proposition 2.1. Since (2.14) just follows from what we have done and the fact that $H_k(x, y^*) = (-1)^k H_k(x, y)$, the proof of Proposition 2.1 for odd $n$ is complete.

The proof for even $n$ is similar. In this case one splits
\[
\tilde{H}_k(x, y) = \int_{-2\pi}^{2\pi} \eta(\pi - t) \psi_{\text{even}}(t) e^{i\lambda_k t} (\cos tP)(x, y) \, dt
\]
\[
+ \int_{-2\pi}^{2\pi} \eta(\pi + t) \psi_{\text{even}}(t) e^{i\lambda_k t} (\cos tP)(x, y) \, dt
\]
\[
+ \int_{-2\pi}^{2\pi} (1 - \eta(\pi - t) - \eta(\pi + t)) \psi_{\text{even}}(t) e^{i\lambda_k t} (\cos tP)(x, y) \, dt,
\]
and uses the fact that
\[
(\cos tP)(x, y) = - (\cos(t + 2\pi)P)(x, y)
\]
and
\[
(\cos tP)(x, y) \text{ is smooth if } d_{S^n}(x, y) \neq |t| \text{ mod } \pi.
\]
By the latter fact, the first two terms are smooth with derivatives bounded independent of $\lambda_k$ if $d_{S^n}(x, y) \leq 3\pi/4$. Using these facts one can also use the proof of [13, Lemma 5.1.3] to see that, under this assumption, the last term in the decomposition is as in
This implies the first part of Proposition 2.1 for even $n$, and since, as before, (2.14) trivially follows from this, the proof is complete. □

Since the proof of Proposition 2.4 is similar to the above, let us now turn to it.

5.2. Proof of Proposition 2.4. Recall that for $\lambda \geq 1$ the Euclidean resolvent kernels

\begin{equation}
\frac{\text{sgn } \mu}{i(\lambda + i\mu)} \int_0^\infty e^{i(\text{sgn } \mu)\lambda t} e^{-|\mu|t} \left( \cos t \sqrt{-\Delta_{\mathbb{R}^n}} \right) (x) \, dt
= \frac{(2\pi)^{-n}}{i(\lambda + i\mu)} \int_0^\infty \int_{\mathbb{R}^n} e^{i(\text{sgn } \mu)\lambda t} e^{-|\mu|t} \cos (t|\xi|) \, e^{ix \cdot \xi} \, d\xi \, dt
= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{ix \cdot \xi}}{-|\xi|^2 + (\lambda + i\mu)^2} \, d\xi = K_{\lambda,\mu}(|x|),
\end{equation}

can be written as

\begin{equation}
K_{\lambda,\mu}(|x|) = \sum_{\pm} a_{\pm}(\lambda; |x|) e^{\pm i\lambda |x|} + O(|x|^{2-n}),
\end{equation}

where

\begin{equation}
|\partial^j r a_{\pm}(\lambda; r)| \leq C_j \lambda^{\frac{n-3}{2}} r^{-\frac{n+1}{2}-j}, \quad r \geq \lambda^{-1},
\end{equation}

and

\begin{equation}
K_{\lambda,\mu}(|x|) \leq C|x|^{2-n}, \quad |x| \leq \lambda^{-1},
\end{equation}

where the constants in (5.7)–(5.8) are independent of $\mu \in \mathbb{R}$ and $\lambda \geq 1$. This follows easily from stationary phase, and it also follows from writing the kernel in terms of Bessel potentials. See e.g., [6, p. 338–339].

To prove that when $P = \sqrt{-\Delta_{\mathbb{R}^n} + \left(\frac{n-1}{2}\right)^2}$

\begin{equation}
\frac{\text{sgn } \mu}{i(\lambda + i\mu)} \int_0^\infty \rho(t) e^{i(\text{sgn } \mu)\lambda t - |\mu|t} \left( \cos tP \right)(x, y) \, dt
\end{equation}

has similar behavior if $0 \leq \rho \in C_0^\infty(\mathbb{R})$ satisfies $\rho(t) = 1$, $|t| \leq 1/2$ and $\rho(t) = 0$, $|t| \geq 1$, we shall use the Hadamard parametrix (see [8], [14]) which says that we can write for say, $|t| \leq 1$,

\begin{equation}
\left( \cos tP \right)(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{id_{\mathbb{R}^n}(x,y)1} \alpha(t, x, y; |\xi|) \cos t|\xi| \, d\xi,
\end{equation}

modulo a smooth function, where

$$1 = (1, 0, \ldots, 0),$$

and $\alpha \in S^0$ (zero order symbol) satisfies

\begin{equation}
|\partial^j_{\xi, x, y, \theta} \alpha(t, x, y; r)| \leq C_{\beta, j} (1 + r)^{-j}.
\end{equation}

In fact, modulo a symbol of order $-2$, $\alpha$ just equals a smooth function of $x$ and $y$ which is independent of $|\xi|$. Plugging this into (5.5), it is easy to see that the top order part of the parametrix contributes to a term satisfying the analog of (5.6)-(5.8) with $|x|$ being replaced by $d_{\mathbb{R}^n}(x, y)$. The smooth error term in the parametrix clearly contributes to a term which is $O(1)$ with bounds independent of $\lambda \geq 1$. 
Let us now give the argument that the full parametrix also gives rise to a term satisfying the analog of (5.6)-(5.8). In other words, if \( \alpha \) is as in (5.9), we shall show that when \( \lambda \geq 1 \)

\[
\frac{1}{\lambda + i\mu} \int_0^\infty \int_{\mathbb{R}^n} e^{idS_n(x,y)} \xi \rho(t)e^{i(sgn \mu)\lambda t - |\mu|t} \alpha(t, x, y; |\xi|) \cos t|\xi| \, d\xi dt,
\]

is as in (5.6)-(5.8).

We first choose \( \beta \in C_0^\infty((1/4, 2]) \) satisfying

\[
\beta(r) = 1, \quad 1/2 \leq r \leq 3/2.
\]

Then, by a simple integration by parts argument, the difference between (5.10) and

\[
\frac{1}{\lambda + i\mu} \int_0^\infty \int_{\mathbb{R}^n} e^{idS_n(x,y)} \xi \rho(t)e^{i(sgn \mu)\lambda t - |\mu|t} \beta(|\xi|/\lambda) \alpha(t, x, y; |\xi|) \cos t|\xi| \, d\xi dt,
\]

is \( O((dS_n(x,y))^{-n}) \), independent of \( \lambda \geq 1 \). Also, it is straightforward to see that (5.11) is \( O(\lambda^{-n-2}) \), and so we only need to check that this term is of the desired form when \( dS_n(x,y) \geq \lambda^{-1} \).

To finish the proof of Proposition 2.4 and show that (5.11) has the desired form, we recall that the Fourier transform of surface measure on the \((n-1)\)-sphere can be written

\[
\int_{S^{n-1}} e^{i\omega \cdot x} dS(\omega) = |x|^{-\frac{n-1}{2}} \sum_{\pm} a_\pm(|x|) e^{i\pm|x|},
\]

where

\[
|\partial_r^j a_\pm(r)| \leq Cr^{-j}, \quad r \geq 1.
\]

Writing \( \xi = r\omega \) in polar coordinates, we find that for \( dS_n(x,y) \geq \lambda^{-1} \) we have that (5.11) can be rewritten as the sum over \( \pm \) of

\[
\frac{\lambda^{n+1}}{\lambda + i\mu} (dS_n(x,y))^{-\frac{n-1}{2}} \int_0^\infty \int_0^\infty \beta(r) \rho(t)e^{i(sgn \mu)\lambda t - |\mu|t} e^{\pm i\lambda r dS_n(x,y)} \cos(\lambda tr) \, r \frac{n-1}{2} \, drdt
\]

\[
= \frac{\lambda^{n+1}}{\lambda + i\mu} (dS_n(x,y))^{-\frac{n-1}{2}} \times b_\pm(\lambda; dS_n(x,y)),
\]

where

\[
b_\pm(\lambda; \tau) = \int_0^\infty \int_0^\infty \beta(r) \rho(t)e^{i(sgn \mu)\lambda t - |\mu|t} e^{\pm i\lambda (r-\tau)} \cos(\lambda tr) \, r \frac{n-1}{2} \, drdt.
\]

Since clearly

\[
b = O(\lambda^{-1}),
\]

and, moreover, a simple integration by parts argument shows that

\[
|\partial_r^j b(\lambda; \tau)| \leq C_j \lambda^{-1} \tau^{-j}, \quad \tau \geq \lambda^{-1},
\]

we conclude that (5.11) has the desired form, which completes the proof. \( \square \)
5.3. Proof of Proposition 4.1. Since $\mathbb{H}^n$ has bounded geometry it is clear that the proof of Proposition 2.4 just given can be used to show that the kernel of the local resolvent operator in (4.5) satisfies (4.6)-(4.9). Note that (4.7) just follows from Huygens principal.

Alternately, in odd dimensions one could just use the fact that

\[ S_0(x,y) = c_n \frac{\text{sgn } \mu}{i(\lambda + i\mu)} \int_t^\infty \frac{1}{\sinh s - \cosh t} \left( \beta_0(s) e^{i(\text{sgn } \mu)\lambda s - |\mu|s} \right) ds, \quad t = d_{\mathbb{H}^n}(x,y), \]

while in even dimensions one can reach the conclusions of Proposition 4.1 by using the fact that

\[ S_0(x,y) = c_n \frac{\text{sgn } \mu}{i(\lambda + i\mu)} \int_t^\infty \frac{1}{\sinh s} \left( \beta_0(s) e^{i(\text{sgn } \mu)\lambda s - |\mu|s} \right) ds, \quad t = d_{\mathbb{H}^n}(x,y). \]

\[ \square \]

5.4. Proof of Proposition 2.2. We need to show that

\[ \|T_\lambda f\|_{L^r(B_1(0))} \leq C_{r,s} \|f\|_{L^s(B_1(0))}, \quad \text{supp } f \subset B_1(0), \]

assuming that

\[ T_\lambda f(x) = \int e^{i\lambda d_\mu(x,y)} a(x,y) f(y) dy, \]

where $a(x,y)$ vanishes if $d_\mu(x,y) > 4$, and, moreover,

\[ |\partial^\alpha_{x,y} a(x,y)| \leq C a \lambda^{-n/2} (d_\mu(x,y))^{-n/2-|\alpha|}, \quad d_\mu(x,y) \geq \lambda^{-1}, \]

assuming that

\[ |a(x,y)| \leq C (d_\mu(x,y))^{2-n}, \quad d_\mu(x,y) \leq \lambda^{-1}, \]

and $r, s$ are as in (1.1). We are also assuming that $g$ is a smooth metric on $\mathbb{R}^n$ with injectivity radius 10 or more, and $B_r(x)$ denotes the geodesic ball of radius $r$ centered at $x$ if, say, $0 < r < 10$.

To prove these bounds, as before, choose $\beta \in C^\infty_0((1/2, 2))$ satisfying $\sum_{j \in \mathbb{Z}} \beta(2^{-j}t) = 1, t > 0$. We then set

\[ (T^k_\lambda f)(x) = \int e^{i\lambda d_\mu(x,y)} \beta(\lambda 2^{-k} d_\mu(x,y)) a(x,y) f(y) dy, \quad k = 1, 2, 3, \ldots. \]

Then if

\[ T_\lambda^0 = T_\lambda - \sum_{k=1}^\infty T^k_\lambda, \]

it follows from (5.12)-(5.13), the Hardy-Littlewood-Sobolev inequality and the fact that $n(\frac{1}{r} - \frac{1}{s}) = 2$ that

\[ \|T_\lambda^0 f\|_{L^r} \leq C_{r,s} \|f\|_{L^s}. \]

Therefore, we would be done if we could show that whenever $r, s$ are as in (1.1) there is a

\[ \sigma_{r,s} > 0 \]
After a simple change of scale argument, we see that this is equivalent to showing that
\[ \text{supp } f \subset B_{\lambda^{-1}2^k}(0). \]

After a simple change of scale argument, we see that this is equivalent to showing that
\[ \|S_k f\|_{L^p} \leq C r_{\ast} 2^{-k\sigma_{\ast}} \|f\|_{L^r}, \]
if
\[ S_k f(x) = 2^{\frac{n-3}{2}} \int e^{ik\phi_k(x,y)} b_k(x,y) f(y) \, dy, \]
where
\[ \phi_k(x,y) = \varepsilon^{-1}d_k(\varepsilon x, \varepsilon y), \quad \varepsilon = \lambda^{-1}2^k, \]
and
\[ b_k(x,y) = 2^{\frac{n-3}{2}} \lambda^{-(n-2)} \beta(\phi_k(x,y)) a(\lambda^{-1}2^k x, \lambda^{-1}2^k y). \]

Note that if \( g_{\varepsilon} \) is the “stretched” metric \( g_{ij}(\varepsilon x) \) then \( \phi_k(x,y) \) is just the Riemannian distance function for this metric with \( \varepsilon \) as above, i.e.,
\[ \phi_k(x,y) = d_{g_{\varepsilon}}(x,y), \quad \varepsilon = \lambda^{-1}2^k. \]
The amplitude \( b_k \) then vanishes if \( d_{g_{\varepsilon}}(x,y) \notin [1/2, 2] \), and (5.15) is equivalent to \( \text{supp } f \subset B_{1}(0) \) for this metric if \( \varepsilon = \lambda^{-1}2^k \). Also, by (5.12), the amplitudes satisfy
\[ |\partial_{x,y}^\alpha b_k(x,y)| \leq C_{\alpha}, \]
for every multi-index \( \alpha \). Thus, by the special case of Stein’s oscillatory integral theorem [15], (2.26), we have that
\[ \|S_k f\|_{L^p} \leq C 2^{\frac{n-3}{2}k - \frac{n}{2}k} \|f\|_{L^p}, \quad \text{if } 1 \leq p \leq 2, \quad q = \frac{n+1}{n-p}, \]
Since the stretched metrics \( g_{\varepsilon} \) tend to the constant coefficient metric \( g_{ij}(0) \) as \( \varepsilon \searrow 0 \), it is clear that we can choose the constant \( C \) to be independent of \( k \). In particular, if, as before
\[ p = \frac{2n}{n+1}, \quad q = \frac{2n(n+1)}{(n-1)^2}, \]
which corresponds to the point \( A \) in Figure 2, we have
\[ \|S_k f\|_{L_{\frac{2n}{n+1}}} \leq C 2^{\frac{n-3}{2}k - \frac{(n-1)^2}{2(n+1)}} \|f\|_{L_{\frac{2n}{n+1}}} \leq C 2^{\frac{2n}{n+1}} \|f\|_{L_{\frac{2n}{n+1}}}. \]

We also, of course, have the trivial bounds
\[ \|S_k f\|_{L^\infty} \leq C 2^{\frac{n-3}{2}k} \|f\|_{L_{\frac{2n}{n+1}}}. \]

Since
\[ n - 3 = \frac{(n-1)^2}{2n(n+1)} - \frac{(n+1)(n-3)}{(n-1)^2}, \]
\[ 1 - \frac{(n+1)(n-3)}{(n-1)^2} = \frac{4}{(n-1)^2}, \]
and
\[ 0 = -\frac{2}{n+1} \cdot \frac{(n+1)(n-3)}{(n-1)^2} + \frac{4}{(n-1)^2} \cdot \frac{n-3}{2}, \]
if we interpolate between these two estimates, we conclude that
\begin{equation}
\|S_k \lambda f\|_{L^{\frac{2n}{n-k}}} \leq C \|f\|_{L^{\frac{2n}{n+2}}},
\end{equation}
for a uniform constant $C$, which is independent of $k = 1, 2, \ldots$. The pair of exponents in this inequality corresponds to the point $\alpha$ in Figure 2, which is one of the endpoints for the range in (1.1).

If we use Hörmander’s oscillatory integral theorem [7] (see [13, Theorem 2.1.1]) then we can also obtain the following estimate for the dual exponents satisfying (1.1),
\begin{equation}
\|S_k \lambda f\|_{L^{\frac{2n}{n-k}}} \leq C 2^{\frac{n-k}{2}k} 2^{-(n-1)\frac{n-2}{2n}} \|f\|_{L^{\frac{2n}{n+2}}} = C 2^{-\frac{n}{n+2}} \|f\|_{L^{\frac{2n}{n+2}}}.
\end{equation}
If we interpolate between (5.17) and (5.18) we conclude that we have (5.14) for some $(5.17)$

If we use Hörmander’s oscillatory integral theorem [7] (see [13, Theorem 2.1.1]) then we can also obtain the following estimate for the dual exponents satisfying (1.1),

If we interpolate between these two estimates, we conclude that we have (5.14) for some $(5.18)$

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Department of Mathematics, Huazhong University of Science and Technology, Wuhan, China

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218