Recovery of Nonlinear Terms for Reaction Diffusion Equations from Boundary Measurements

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Communicated by D. Kinderlehrer

Abstract

In the present article we study the inverse problem of determining a general semilinear term for a class of nonlinear parabolic equations. We derive a new criterion for the unique and stable recovery of general semilinear terms for these type of equations from the knowledge of the parabolic Dirichlet-to-Neumann map associated with solutions of the equation having zero initial data.

1. Introduction

1.1. Statement

Let $\alpha \in (0, 1)$ and let $\Omega$ be a bounded and connected $C^{2+\alpha}$ domain of $\mathbb{R}^n$, $n \geq 2$. Fixing $T > 0$, we consider the initial boundary value problem

\[
\begin{align*}
\partial_t u - \Delta u + F(t, x, u) &= 0 \text{ in } (0, T) \times \Omega, \\
u \partial u &= g \text{ on } (0, T) \times \partial \Omega, \\
(0, 0) = 0 \text{ in } \Omega,
\end{align*}
\]

with $F \in C^1([0, T] \times \overline{\Omega} \times \mathbb{R})$ a well chosen semilinear term, and $g \in C([0, T]; C^{2+\alpha}(\partial \Omega)) \cap C^{1+\alpha/2}([0, T]; C(\partial \Omega))$ a Dirichlet boundary condition. Under suitable assumptions (see Sect. 1.4), there exists $T_1 \in (0, T]$ such that the problem (1.1) admits a unique solution $u \in C([0, T_1]; C^{2+\alpha}(\Omega)) \cap C^{1+\alpha/2}([0, T_1]; C(\overline{\Omega}))$. The goal of the present article is to prove the unique and stable recovery of the nonlinear term $F$ from partial knowledge of the parabolic Dirichlet-to-Neumann (DN in short) map $\nu \partial_u |_{(0, T_1) \times \partial \Omega}$ associated with (1.1), where $\nu(x)$ denotes the outward unit normal to $\partial \Omega$ computed at $x \in \partial \Omega$. 
1.2. Motivations

Reaction diffusion equations of the form (1.1) are often used to model several physical phenomenon with applications in several fields including chemistry, biology, geology, physics and ecology. For instance, problem (1.1) can be used for modelling the spreading of biological populations (e.g. [12]), the Rayleigh-Bénard convection (e.g. [31]) or models appearing in combustion theory (e.g. [34]). In this context, the inverse problem addressed in this paper can be seen as the determination of the underlying physical law of the system, associated with the nonlinear expression $F$ in (1.1), by applying a source and measuring the flux at the lateral boundary $(0, T) \times \partial \Omega$. Besides these physical motivations, there is a natural mathematical motivation for the study of such inverse problems which are highly nonlinear and ill-posed.

1.3. Known results

Inverse problems for nonlinear equations have received an important interest among the mathematical community these last decades. Among the different formulation of such problems, the recovery of nonlinear terms is one of the most challenging problems due to their ill-posedness and nonlinearity. For parabolic equations, one of the first results proving the unique recovery of general semilinear terms can be found in [15]. Here, the author proved the recovery of time independent semilinear terms of the form $F(x, u)$ depending on the space variable $x \in \Omega$ (i.e $F(x, u)$ with $x \in \Omega, u \in \mathbb{R}$) from knowledge of the parabolic DN map together with the final overdetermination, for solution of the equation having all possible initial conditions. The proof of [15] is based on the linearization of the inverse problem combined with results of recovery of time-dependent coefficients proved by the same author in [14]. In [5] the authors proved that the uniqueness result of [15] remains true while considering observation given by the parabolic DN map for solutions of (1.1) with all possible initial conditions. Moreover, the authors of [5] established a stability estimate associated with this problem. To our best knowledge, the only results available in the mathematical literature considering this problem with data restricted to the lateral boundary $(0, T) \times \partial \Omega$ (see e.g. [6,16,17]) have restricted their analysis to the determination of semilinear terms depending on the solution of the equation and space variable at the boundary of the domain (semilinear terms of the form $F(x, u)$, $x \in \partial \Omega, u \in \mathbb{R}$). We refer also to the works of [20,21] for similar problems stated with final time or lateral overdetermination. As far as we know, there is no result available in the mathematical literature proving the determination of general semilinear terms, depending on the space variable $x \in \Omega$, from lateral excitation and measurements.

Finally, we mention the works of [2,9,17,22] devoted to the recovery of quasilinear terms and the works of [7,8,11,18,19,23–26,29,30] devoted to similar problems for hyperbolic and elliptic equations.
1.4. Preliminary properties

Let us fix $T > 0$ and two non-decreasing maps $M \in C(\mathbb{R}_+; \mathbb{R}_+)$ and $T_1 \in C(\mathbb{R}_+; [0, T])$. From now on, we denote by $C^{\frac{q}{2}-\alpha}([0, T] \times X)$, with $X = \overline{\Omega}$ or $X = \partial \Omega$, the space of functions $f$ lying in $C([0, T] \times X)$ and satisfying that

$$\left| f(t, x) - f(s, y) \right| \frac{1}{\left| (x-y)^2 + |t-s| \right|^{\frac{q}{2}}} : (t, x), (s, y) \in [0, T] \times X, \ (t, x) \neq (s, y) < \infty.$$  

Then we define the space $C^{1+\frac{q}{2}, 2+\alpha}([0, T] \times X)$ as the set of functions $f$ lying in $C([0, T]; C^2(X)) \cap C^1([0, T]; C(X))$ such that

$$\partial_t f, \partial_x^\beta f \in C^{\frac{q}{2}-\alpha}([0, T] \times X), \ \beta \in \mathbb{N}^n, \ |\beta| = 2.$$ 

We consider on these spaces the usual norm and we refer to [4, pp. 4] for more details. Let us also introduce the following functional space

$$\mathcal{H}_0 := \{ g \in C^{1+\frac{q}{2}, 2+\alpha}([0, T] \times \partial \Omega) : g(0, \cdot) = \partial_t g(0, \cdot) = 0 \}.$$ 

We define the set $\mathcal{A}$ of functions $F \in C^1([0, T] \times \overline{\Omega} \times \mathbb{R})$ taking values in $\mathbb{R}$ such that the following condition is fulfilled:

(H) For all $r > 0$ and all $g \in \mathcal{H}_0$ satisfying that

$$\|g\|_{C^{1+\frac{q}{2}, 2+\alpha}([0, T] \times \partial \Omega)} \leq r,$$

the problem (1.1) admits a unique solution $u \in C^{1+\frac{q}{2}, 2+\alpha}([0, T_1(r)] \times \overline{\Omega})$ satisfying the estimate

$$\|u\|_{C^{1+\frac{q}{2}, 2+\alpha}([0, T_1(r)] \times \overline{\Omega})} \leq M(r). \quad (1.2)$$

According to [28, Theorem 6.1, pp. 452], [28, Theorem 2.2, pp. 429], [28, Theorem 4.1, pp. 443], [28, Lemma 3.1, pp. 535] and [28, Theorem 5.4, pp. 448], the condition (H) will be fulfilled if $F \in C^1([0, T] \times \overline{\Omega} \times \mathbb{R})$ satisfies the following conditions:

(P1) There exist a non-decreasing function $\mu \in C([0, +\infty); [0, +\infty))$ such that

$$|F(t, x, \lambda)| \leq \mu(|\lambda|), \quad t \in [0, T], \ x \in \overline{\Omega}, \ \lambda \in \mathbb{R}.$$ 

(P2) We have that

$$F(0, x, 0) = 0, \quad x \in \partial \Omega.$$ 

(P3) There exist two constants $b_1, b_2 \geq 0$ such that

$$F(t, x, \lambda) \lambda \geq -b_1 \lambda^2 - b_2, \quad t \in [0, T], \ x \in \overline{\Omega}, \ \lambda \in \mathbb{R}.$$

Assuming that conditions (P1)–(P3) are fulfilled, we obtain the existence of a global solution of problem (1.1) with $T_1(r) = T$, $r > 0$. However, it is well known (see e.g. [1]) that condition (H) is also fulfilled by some class of nonlinear terms $F$ satisfying (P1)–(P2) but not (P3). Solutions of such problems are local solutions that may blow-up at finite time (see e.g. [3, Proposition 5.4.1.]) for which the sign restriction (P3) is not required. In a similar way to [23], in the present article we consider this class of local solution of (1.1).
I.5. Main results

We fix \( r > 0, \epsilon \in (0, 1), \delta_1 \in (0, T_1(r + \epsilon)) \) and \( \delta_2 > 0 \) such that there exists \( \chi \in C_0^\infty (\mathbb{R}_+ \times \partial \Omega; [0, +\infty)) \) satisfying \( \chi = \delta_2 \) on \([\delta_1, T_1(r + \epsilon)] \times \partial \Omega\) and

\[
\|\chi\|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \partial \Omega)} = 1.
\]

Then, for any \( \lambda \in [-r, r], h \in \mathcal{H}_0 \) satisfying that

\[
\|h\|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \partial \Omega)} \leq \epsilon,
\]

we consider the initial boundary value problem

\[
\begin{aligned}
\partial_t u - \Delta u + F(t, x, u) &= 0 \quad \text{in} \ (0, T_1(r + \epsilon)) \times \Omega, \\
\partial^2 u(t, x, u) &= \lambda \chi(t, x) + \tilde{h}(t, x) \quad (t, x) \in (0, T_1(r + \epsilon)) \times \partial \Omega, \\
\end{aligned}
\]

Assuming that \( F \in \mathcal{A} \), the assumption (H) implies that this problem admits a unique solution \( u_{\lambda, h} \in C^{1+\frac{\alpha}{2},2+\alpha}([0, T_1(r + \epsilon)] \times \Omega) \) satisfying the estimate

\[
\|u_{\lambda, h}\|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,T_1(r+\epsilon)] \times \Omega)} \leq M(r + \epsilon).
\]

We consider \( B_\epsilon := \{h \in \mathcal{H}_0 : h \text{ satisfies estimate (1.3)}\} \) and we define, for all \( \lambda \in [-r, r] \), the map

\[
\mathcal{N}_{\lambda, r, F} : B_\epsilon \ni h \mapsto \partial_t u_{\lambda, h}|_{(0,T_1(r+\epsilon)) \times \partial \Omega} \in L^2((0, T_1(r + \epsilon)) \times \partial \Omega).
\]

Our first main result can be stated as follows:

**Theorem 1.1.** For \( j = 1, 2 \), let \( F_j \in \mathcal{A} \cap C^1([0, T] \times \overline{\Omega}; C^4(\mathbb{R})) \). Assume that the conditions

\[
\begin{aligned}
F_1(t, x, 0) &= F_2(t, x, 0) = 0, \quad t \in (0, T), \quad x \in \Omega, \\
\partial^2_u F_1(t, x, u) &\leq 0, \quad \partial^2_u F_1(t, x, -u) \geq 0, \quad t \in (0, T), \quad x \in \Omega, \quad u \in [0, +\infty),
\end{aligned}
\]

are fulfilled. We fix also \( q \in W^{1, \infty}((0, T) \times \Omega) \) such that

\[
\max_{j=1,2} \partial_u F_j(t, x, 0) \leq q(t, x), \quad (t, x) \in (0, T) \times \Omega.
\]

Then, there exists a constant \( a_1 > 0 \), depending only on \( \Omega, T, \delta_1, q, n \) and \( \chi \), such that the condition

\[
\mathcal{N}_{\lambda, r, F_1} = \mathcal{N}_{\lambda, r, F_2}, \quad \lambda \in (-r, r)
\]

implies that

\[
F_1(t, x, s) = F_2(t, x, s), \quad (t, x) \in [\delta_1, T_1(r + \epsilon)] \times \Omega, \quad s \in [-ra_1, ra_1].
\]
We will also derive a stability result associated with the uniqueness result of Theorem 1.1. For this purpose, let us first recall that, according to [5, Proposition 6.1.], for \( F \in C^1(\overline{\Omega} \times [0, T]; C^3(\mathbb{R})) \cap A, r > 0 \) and \( \lambda \in [-r, r] \), the map \( \mathcal{N}_{\lambda,r,F} \) admits a Fréchet derivative at 0 denoted by \( \mathcal{N}_{\lambda,r,F}'(0) \). We consider the stable recovery of \( F \) from the data \( \mathcal{N}_{\lambda,r,F}'(0), \lambda \in [-r, r] \). For this purpose, we start by introducing the functional spaces

\[
\mathcal{K}_{0}^{T} := \{ H_{[0,T] \times \partial \Omega} : H \in L^{2}(0,T; H^{1}(\Omega)) \cap H^{1}(0,T; H^{-1}(\Omega)), H_{[0] \times \Omega} = 0 \},
\]

\[
\mathcal{K}_{T}^{T} := \{ H_{[0,T] \times \partial \Omega} : H \in L^{2}(0,T; H^{1}(\Omega)) \cap H^{1}(0,T; H^{-1}(\Omega)), H_{[T] \times \Omega} = 0 \},
\]

\[
S_{0} := \{ H \in L^{2}(0,T; H^{1}(\Omega)) \cap H^{1}(0,T; H^{-1}(\Omega)) : H_{[0] \times \Omega} = 0, \partial_{t} H - \Delta H = 0 \},
\]

\[
S_{T} := \{ H \in L^{2}(0,T; H^{1}(\Omega)) \cap H^{1}(0,T; H^{-1}(\Omega)) : H_{[T] \times \Omega} = 0, \partial_{t} H + \Delta H = 0 \}.
\]

According to [2, Proposition 2.1], for any \( h \in \mathcal{K}_{j}^{T}, j = 0, T, \) there exists a unique \( H \in S_{j} \) such that \( H_{[0,T] \times \partial \Omega} = h \). Therefore, we can associate with \( \mathcal{K}_{j}^{T}, j = 0, T \), the norm defined by

\[
\| H_{[0,T] \times \partial \Omega} \|_{\mathcal{K}_{j}^{T}}^{2} = \| H \|_{L^{2}(0,T; H^{1}(\Omega))}^{2} + \| H \|_{H^{1}(0,T; H^{-1}(\Omega))}^{2}, \quad H \in S_{j}.
\]

Then, combining [5, Proposition 6.1.] with [2, Proposition 2.3], we deduce that \( \mathcal{N}_{\lambda,r,F}''(0) = \mathcal{N}_{\lambda,r,F}''(0), \lambda \in [-r, r] \), can be uniquely extended to a bounded operator from \( \mathcal{K}_{0}^{T} \) to \( \mathcal{K}_{T}^{*} \), where \( \mathcal{K}_{T}^{*} \) denotes the dual space of \( \mathcal{K}_{T} \).

Using these properties, we can state our stability result as follows:

**Theorem 1.2.** For \( j = 1, 2 \), let \( F_{j} \in C^{1}(\{0, T\} \times \partial \Omega; C^{4}(\mathbb{R})) \cap A \) and let conditions (1.6)–(1.7) be fulfilled. Assume also that there exists a positive and non decreasing function \( \kappa \in C((0, +\infty)) \) such that

\[
\sum_{j=1}^{2} \sum_{k=0}^{2} \| \partial_{u}^{k} F_{j}(\cdot, u) \|_{W^{1,\infty}((0,T) \times \Omega)} \leq \kappa(|u|), \quad u \in \mathbb{R}. \tag{1.10}
\]

Then, there exists \( a_{2} > 0 \), depending on \( \kappa(0), T_{1}(r+\epsilon), \Omega, \delta_{1}, n, \chi \), such that, for all \( r > 0 \), we have that

\[
\sup_{(t,x) \in (\delta_{1}, T_{1}(r+\epsilon)) \times \Omega} \sup_{s \in (-r_{2}, r_{2})} | F_{1}(t, x, s) - F_{2}(t, x, s) |
\]

\[
\leq C_{r} \sup_{\lambda \in (-r, r)} \ln \left( 3 + \| \mathcal{N}_{\lambda,r,F}''(0) - \mathcal{N}_{\lambda,r,F}''(0) \|_{B(K_{0}^{T}, K_{T}^{*})}^{-1} \right)^{-\theta}, \tag{1.11}
\]

with \( C_{r} > 0 \) depending on \( \Omega, \kappa, r, M(r+\epsilon), T_{1}(r+\epsilon), T, \chi, n \) and \( \theta > 0 \) depending on \( n \).
1.6. Comments about our results

To the best of our knowledge, in Theorems 1.1 and 1.2 we obtain the first result of unique and stable recovery of general class of semilinear terms admitting variation inside the domain (semilinear terms of the form \( F(x, u), (x, u) \in \Omega \times \mathbb{R} \)) from excitation and measurements restricted to the lateral boundary \((0, T) \times \partial \Omega \) and associated with solutions having initial data fixed at zero. Indeed, it seems that all other similar results require at least measurements of solutions for an infinite number of different initial conditions (see [5,13,15]). Actually, Theorems 1.1 and 1.2 give a positive answer to the open problem stated in [13] (see [13, Problem 9.6, pp. 296]) for any time independent semilinear term \( F \) satisfying conditions (1.6)–(1.7) as well as condition (H) with a function \( T_1 \) such that

\[
\inf_{r > 0} T_1(r) > 0. \tag{1.12}
\]

Let us observe that the results of Theorems 1.1 and 1.2 extend several results. First, Theorems 1.1 and 1.2 seem to be the first results of unique and stable recovery of semilinear terms admitting variation in \( t \in (0, T) \) and \( x \in \Omega \) independently of the solution. Indeed, it seems that all other results have been stated with semilinear terms of the form \( F(x, u), x \in \Omega, u \in \mathbb{R} \) (see [5,15]) or \( F(u), u \in \mathbb{R} \) (see [6,16,17]). For instance, Theorems 1.1 and 1.2 can be applied to the recovery of nonlinear terms associated with physical phenomenon admitting variation with respect to both time and space position. Secondly, in contrast to all other similar works that we know, including works for elliptic equations, the results of Theorems 1.1 and 1.2 can be applied to semilinear terms that are neither Lipschitz (or with derivative in \( u \) uniformly bounded in some suitable sense) with respect to \( u \in \mathbb{R} \) (as considered in [18,19]) nor lying on some specific classes of semilinear terms admitting an holomorphic extension with respect to \( u \in \mathbb{C} \) (as considered by [11,25,26,29,30]). For instance, fixing \( \epsilon_1 > 0, q_\pm, \gamma_\pm \in C^1([0, T] \times \overline{\Omega}) \) such that \( q_\pm \leq 0 \) and \( \gamma_\pm \geq 1 \), our result can be applied to any semilinear terms \( F \in C^1([0, T] \times \overline{\Omega}; C^4(\mathbb{R})) \) of the form

\[
F(t, x, \pm u) = q_\pm(t, x)(1 + |u|)\gamma_\pm(t, x), \quad (t, x, u) \in (0, T) \times \Omega \times [\epsilon_1, +\infty) \tag{1.13}
\]

for which condition (H) and (1.6)–(1.7) are fulfilled.

Let us remark that the proof of Theorems 1.1 and 1.2 are based on some asymptotic study of solutions of the nonlinear equation (1.1) with respect to a constant parameter \( \lambda \) applied on the lateral boundary \((0, T) \times \partial \Omega \). More precisely, following the arguments of [2,5] based on the first order linearization, which are recalled in Sect. 2, we obtain the identity (2.7) stated on the graph \( G \) of solutions of (2.4) with \( \lambda \in [-r, r] \), \( r > 0 \). In order to derive explicit determination of the nonlinear term under consideration from (2.7), in [2,5] the authors used the values of solutions at the initial time \( t = 0 \). In contrast to the analysis of [2,5], we obtain an identity associated with solutions of the equation vanishing at the initial time \( t = 0 \). We need therefore to consider a different approach for deriving the explicit uniqueness and the stability results of Theorems 1.1 and 1.2 from (2.7). For this purpose, we
develop a new method based on the study of the behavior of the solution \( v_{j,\lambda} \), \( j = 1, 2 \), of (2.4) with respect to the parameter \( \lambda \in [-r, r] \). This method combines several arguments including the second order linearization and properties of solutions of parabolic equations such as parabolic Harnack inequality, unique continuation and maximum principle. Combining these properties with a new criterion, stated in (1.7), that we impose to the semilinear terms under consideration we show Theorems 1.1 and 1.2 from the identity (2.7). This last criterion is fulfilled by nonlinear terms which are Lipschitz with respect to \( u \in \mathbb{R} \) but, in contrast to the results of [18,19], it is not limited to such class of nonlinear terms. In our analysis, we use the second linearization which is inspired by the approach of [33] as well as some recent development of inverse problems for nonlinear equations based on multiple linearization (see e.g. [11,22,25–27,29,30]). However, our application of the second linearization differs from the one of the previously mentioned articles.

The constants \( a_1 \) and \( a_2 \) appearing in the statement of Theorems 1.1 and 1.2 are explicitly given by formulas (2.19), (3.4). Namely, \( a_1 \) corresponds to the infimum of the solution \( w \) of the problem (2.18) on \((\delta_1, T) \times \Omega\) while \( a_2 \) denotes the infimum of the solution \( y \) of the problem (3.3) on \((\delta_1, T) \times \Omega\). Since here both \( \delta_1 \) and \( \chi \) depends on \( T_1(r+\epsilon) \), the constants \( a_1, a_2 \) depend also on \( T_1(r+\epsilon) \). This means that if \( T_1(r+\epsilon) \) is lower bounded by a positive constant independent of \( r > 0 \) the constants \( a_1, a_2 \) can be chosen independently of the parameter \( r \). Such a phenomenon can occur for instance if problem (1.1) admits a global solution. In that context, applying Theorem 1.1 one can deduce the global recovery of semilinear terms of the form \( F(x,u) \), \( x \in \Omega, u \in \mathbb{R} \), satisfying the conditions (1.6)–(1.7) from the data \( N_{\lambda,r,F}, r > 0, \lambda \in (-r,r) \). In the same way, from the result of Theorem 1.1 one can show the unique determination of semilinear terms of the form \( F(x,u) \), \( x \in \Omega, u \in \mathbb{R} \), for which there exists a sufficiently small parameter \( r_0 > 0 \) such that for all \( x \in \Omega \), the map \( u \mapsto F(x,u) \) is analytic in \((-\infty, r_0) \cup (r_0, +\infty)\)

### 1.7. Outline

The rest of this paper is organized as follows: In Sect. 2, we consider the proof of the uniqueness result stated in Theorem 1.1 while Sect. 3 is devoted to the proof of the stability result stated in Theorem 1.2.

### 2. The Uniqueness Result

This section is devoted to the proof of the uniqueness result stated in Theorem 1.1. For this purpose, we start with the linearization of the inverse problem.

#### 2.1. Linearization of the inverse problem

In this subsection we introduce a linearization procedure for the problem (1.4). Namely, we recall the first linearization for this inverse problem stated in [2,5,15].
For \( j = 1, 2 \), let \( F_j \) be given by Theorem 1.1 and satisfying (1.6)–(1.7). Fixing \( \lambda \in [-r, r] \), \( h \in B_{\epsilon} \) and \( s \in (-1, 1) \) we consider \( u_j = u_{j, \lambda, s, h} \) the solution of

\[
\begin{align*}
\partial_t u_j - \Delta u_j + F_j(t, x, u_j) &= 0 \quad \text{in } (0, T_1(r + \epsilon)) \times \Omega, \\
u_j(t, x) &= \lambda \chi(t, x) + s h(t, x) \quad (t, x) \in (0, T_1(r + \epsilon)) \times \partial\Omega, \\
u_j(0, x) &= 0 \\
\end{align*}
\]

(2.1) \( x \in \Omega \).

Let us also consider the linear problem

\[
\begin{align*}
\partial_t u^{(1)}_j - \Delta u^{(1)}_j + V_j,\lambda u^{(1)}_j &= 0 \quad \text{in } (0, T_1(r + \epsilon)) \times \Omega, \\
u^{(1)}_j &= h \quad \text{on } (0, T_1(r + \epsilon)) \times \partial\Omega, \\
u^{(1)}_j(0, x) &= 0 \\
\end{align*}
\]

(2.2) \( x \in \Omega \),

with

\[
V_{j,\lambda}(t, x) = \partial_h F_j(t, x, u_{j, \lambda, s, h}(t, x))|_{s=0}.
\]

Since \( V_{j,\lambda} \in C^1([0, T_1(r + \epsilon)] \times \overline{\Omega}) \), it is well known (see [28, Theorem 5.4, pp. 322]) that (2.2) admits a unique solution \( u^{(1)}_j \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T_1(r + \epsilon)] \times \overline{\Omega}) \). Then we introduce the parabolic DN map \( \Lambda_{V_{j,\lambda}} \) associated with (2.2) given by \( \Lambda_{V_{j,\lambda}} : h \mapsto \partial_h u^{(1)}_j|_{(0, T_1(r+\epsilon)) \times \partial\Omega} \). Following [5, Proposition 6.1], we can prove the following:

**Lemma 2.1.** For \( j = 1, 2 \), \( \lambda \in [-r, r] \), \( h \in B_{\epsilon} \), the map \( s \mapsto u_{j, \lambda, s, h} \) admits a Fréchet derivative at \( s = 0 \) in the sense of maps taking values in \( C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T_1(r + \epsilon)] \times \overline{\Omega}) \) and we have that

\[
\partial_s u_{j, \lambda, s, h}|_{s=0} = u^{(1)}_j.
\]

According to Lemma 2.1, (1.9) implies that

\[
\Lambda_{V_{1,\lambda}} h = N'_{\lambda, r, F_1}(0) h = N'_{\lambda, r, F_2}(0) h = \Lambda_{V_{2,\lambda}} h, \quad \lambda \in [-r, r], \ h \in H_0.
\]

In a similar way to [2,5], combining this identity with [5, Theorem 1.1.] we deduce that

\[
\partial_u F_1(\cdot, u_{j, \lambda, s, h}|_{s=0}) = V_{1,\lambda} = V_{2,\lambda} = \partial_u F_2(\cdot, u_{j, \lambda, s, h}|_{s=0}), \quad \lambda \in [-r, r].
\]

(2.3)

For all \( \lambda \in [-r, r] \), let us consider \( v_{j, \lambda} \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T_1(r + \epsilon)] \times \overline{\Omega}) \) the solution of

\[
\begin{align*}
\partial_t v_{j,\lambda} - \Delta v_{j,\lambda} + F_j(t, x, v_{j,\lambda}) &= 0 \quad \text{in } (0, T_1(r + \epsilon)) \times \Omega, \\
v_{j,\lambda}(t, x) &= \lambda \chi(t, x) \quad (t, x) \in (0, T_1(r + \epsilon)) \times \partial\Omega, \\
v_{j,\lambda}(0, x) &= 0 \\
\end{align*}
\]

(2.4) \( x \in \Omega \).

Then, the identity (2.3) can be rewritten as

\[
\partial_u F_1(t, x, v_{1,\lambda}(t, x)) = \partial_u F_2(t, x, v_{2,\lambda}(t, x)), \quad (t, x, \lambda) \in (0, T_1(r + \epsilon)) \times \Omega \times [-r, r].
\]

(2.5)
Now let us consider \( v_{j,\lambda}^{(1)} \in C^{1+\frac{\beta}{2},\alpha}\) solving the problem
\[
\begin{align*}
\partial_t v_{j,\lambda}^{(1)} - \Delta v_{j,\lambda}^{(1)} + V_{j,\lambda} v_{j,\lambda}^{(1)} &= 0 \quad in \ (0, T_1(r + \epsilon)) \times \Omega, \\
v_{j,\lambda}^{(1)}(t, x) &= \chi(t, x) \quad (t, x) \in (0, T_1(r + \epsilon)) \times \partial\Omega, \\
v_{j,\lambda}^{(1)}(0, x) &= 0 \quad x \in \Omega.
\end{align*}
\]
(2.6)

Repeating the arguments used in Lemma 2.1, one can check that \( \partial_\lambda v_{j,\lambda} = v_{j,\lambda}^{(1)}, \) \( j = 1, 2, \lambda \in (-r, r). \) Moreover, the identity (2.3) implies that \( v_{1,\lambda}^{(1)} = v_{2,\lambda}^{(1)}, \) since they both solve the same initial boundary value problem which admits a unique solution. Combining this with the fact that condition (1.6) and the uniqueness of solution of (2.4) imply that \( v_{1,0} = v_{2,0} = 0, \) we deduce that
\[
v_{1,\lambda} = \int_0^\lambda v_{1,\tau}^{(1)} d\tau = \int_0^\lambda v_{2,\tau}^{(1)} d\tau = v_{2,\lambda}, \quad \lambda \in [-r, r].
\]
(2.7)

In view of this identity, the proof of Theorem 1.1 will be completed if we prove that there exists a constant \( a_1 > 0, \) depending only on \( \Omega, \delta_1, T, q, \) and \( \chi, \) such that \( \delta_1, (T_1(r + \epsilon)) \times \Omega \times [-a_1 r, a_1 r] \subset G. \) In order to prove this inclusion we will need to study the map \( (-r, r) \ni \lambda \mapsto v_{1,\lambda}(t, x), (t, x) \in \delta_1, (T_1(r + \epsilon)) \times \Omega. \) More precisely, we will develop a new method based on the second order linearization that we recall next. We introduce \( v_{1,\lambda}^{(2)} \) solving the following problem:
\[
\begin{align*}
\partial_t v_{1,\lambda}^{(2)} - \Delta v_{1,\lambda}^{(2)} + V_{j,\lambda} v_{1,\lambda}^{(2)} &= -\partial_\alpha^2 F_1(t, x, v_{1,\lambda}) \left(v_{1,\lambda}^{(1)}\right)^2 \quad \text{in} \ (0, T_1(r + \epsilon)) \times \Omega, \\
\left\{ \begin{array}{ll} 
\partial_\lambda v_{1,\lambda}^{(2)} = 0 & \text{on} \ (0, T_1(r + \epsilon)) \times \partial\Omega, \\
v_{1,\lambda}^{(2)}(0, x) &= 0 \end{array} \right. \\
x \in \Omega.
\end{align*}
\]
(2.8)

The second linearization of problem (2.4) can be stated as follows:

**Lemma 2.2.** The map \( \lambda \mapsto v_{1,\lambda}^{(1)} \) admits a Fréchet derivative in \((-r, r)\) and we have that
\[
\partial_\lambda v_{1,\lambda}^{(1)} = v_{1,\lambda}^{(2)}, \quad \lambda \in (-r, r),
\]
(2.9)
in the sense of functions taking values in \( C^{1+\frac{\beta}{2},\alpha}\) \( ([0, T_1(r + \epsilon)] \times \Omega). \)

**Proof.** We start by proving that the map \( (-r, r) \ni \lambda \mapsto v_{1,\lambda}^{(1)} \in C^{1+\frac{\beta}{2},\alpha}\) \( ([0, T_1(r + \epsilon)] \times \Omega) \) is continuous. For this purpose, we fix \( \lambda \in (-r, r), \) \( \delta \in (|\lambda| - r, r - |\lambda|) \setminus \{0\} \) and consider \( y = v_{1,\lambda+\delta}^{(1)} - v_{1,\lambda}^{(1)}\). It is clear that \( y \) solves
\[
\begin{align*}
\partial_t y - \Delta y + V_{j,\lambda} y &= G_{\delta,\lambda} \quad \text{in} \ (0, T_1(r + \epsilon)) \times \Omega, \\
y &= 0 \quad \text{on} \ (0, T_1(r + \epsilon)) \times \partial\Omega, \\
y(0, x) &= 0 \quad x \in \Omega.
\end{align*}
\]
(2.10)
with
\[ G_{\delta, \lambda} = (V_1, \lambda - V_{1, \lambda + \delta}) V_{1, \lambda + \delta}. \]

From now on we denote by \( \|\cdot\|_{2+\alpha, 1+\alpha/2} \) the norm of \( C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T_1(r+\epsilon)] \times \mathbb{R}). \) In view of (1.5), one can check that
\[ M_1(\lambda) = \sup_{\delta \in (|\lambda| - r, r - |\lambda|)} \left( \| V^{(1)}_{1, \lambda + \delta} \|_{2+\alpha, 1+\alpha/2} + \| V_{1, \lambda + \delta} \|_{2+\alpha, 1+\alpha/2} \right) < \infty. \] (2.10)

Moreover, the mean value theorem implies that
\[
\left\| V_{1, \lambda} - V_{1, \lambda + \delta} \right\|_{L^\infty((0, T_1(r+\epsilon)] \times \Omega)} \\
\leq \left( \sup_{(t, x) \in (0, T_1(r+\epsilon)] \times \Omega} \sup_{s \in [-M_1(\lambda), M_1(\lambda)]} \| \partial_n^2 F_1(t, x, s) \| \right) \left\| V_{1, \lambda + \delta} - V_{1, \lambda} \right\|_{L^\infty((0, T_1(r+\epsilon)] \times \Omega)} \\
\leq C(\lambda) \left\| V_{1, \lambda + \delta} - V_{1, \lambda} \right\|_{L^\infty((0, T_1(r+\epsilon)] \times \Omega)}.
\]

In the same way, we can prove that, for all \( k \in \mathbb{N} \) and all \( \beta \in \mathbb{N}^n \) such that \( |\beta| + k = 1 \), we have that
\[
\left\| \partial_x^k \partial_\lambda^{\beta} V_{1, \lambda} - \partial_x^k \partial_\lambda^{\beta} V_{1, \lambda + \delta} \right\|_{L^\infty((0, T_1(r+\epsilon)] \times \Omega)} \\
\leq 2 \left( \sup_{(t, x) \in (0, T_1(r+\epsilon)] \times \Omega} \sup_{s \in [-M_1(\lambda), M_1(\lambda)]} \| \partial_n^2 \partial_x^k \partial_\lambda^{\beta} F_1(t, x, s) \| + \| \partial_n^2 F_1(t, x, s) \| \right) \\
\left\| V_{1, \lambda + \delta} - V_{1, \lambda} \right\|_{L^\infty((0, T_1(r+\epsilon)] \times \Omega)} \\
+ M_1 \left( \sup_{(t, x) \in (0, T_1(r+\epsilon)] \times \Omega} \sup_{s \in [-M_1(\lambda), M_1(\lambda)]} \| \partial_n^3 F_1(t, x, s) \| \right) \\
\times \left\| V_{1, \lambda + \delta} - V_{1, \lambda} \right\|_{L^\infty((0, T_1(r+\epsilon)] \times \Omega)} \\
\leq C(\lambda) \left\| V_{1, \lambda + \delta} - V_{1, \lambda} \right\|_{L^\infty((0, T_1(r+\epsilon)] \times \Omega)}.
\]

Combining this with Lemma 2.1, we deduce that
\[
\lim_{\delta \to 0} \left\| G_{\delta, \lambda} \right\|_{C^1([0, T_1(r+\epsilon)] \times \mathbb{R})} = 0. \] (2.11)

In addition, applying [28, Theorem 5.2, p. 320], we get that
\[
\left\| V^{(1)}_{1, \lambda + \delta} - V^{(1)}_{1, \lambda} \right\|_{2+\alpha, 1+\alpha/2} = \| y \|_{2+\alpha, 1+\alpha/2} \leq C \left( \left\| G_{\delta, \lambda} \right\|_{C^1([0, T_1(r+\epsilon)] \times \mathbb{R})} \right) \\
+ \left\| G_{\delta, \lambda} \right\|_{C^{\alpha/2}([0, T_1(r+\epsilon)] \times \mathbb{R})}.
\]

Combining this with (2.11), we deduce that \((-r, r) \ni \lambda \mapsto v^{(1)}_{1, \lambda} \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T_1(r+\epsilon)] \times \mathbb{R})\) is continuous. Now let us consider
\[
z = \frac{v^{(1)}_{1, \lambda + \delta} - v^{(1)}_{1, \lambda}}{\delta} - v^{(2)}_{1, \lambda}.
\]
It is clear that \( z \) solves the boundary value problem
\[
\begin{cases}
\partial_t z - \Delta z + V_{1,\lambda}(t,x)z = L_{\delta,\lambda} & \text{in } (0, T_1(r+\epsilon)) \times \Omega, \\
z = 0 & \text{on } (0, T_1(r+\epsilon)) \times \partial\Omega, \\
z(0, x) = 0 & \text{in } \Omega, 
\end{cases}
\tag{2.12}
\]
with
\[
L_{\delta,\tau} = \partial^2_u F_1(\cdot, v_{1,\lambda}) \left( v^{(1)}_{1,\lambda} \right)^2 + \frac{V_{1,\lambda} - V_{1,\lambda+\delta}}{\delta} v^{(1)}_{1,\lambda+\delta}.
\]
On the other hand, we have that
\[
L_{\delta,\tau} = \partial^2_u F_1(\cdot, v_{1,\lambda}) \left( v^{(1)}_{1,\lambda} \right)^2 - \left( \int_0^1 \partial^2_u F_1(\cdot, s v_{1,\lambda+\delta} + (1-s)v_{1,\lambda})ds \right) \times \left( \frac{v_{1,\lambda+\delta} - v_{1,\lambda}}{\delta} \right) v^{(1)}_{1,\lambda+\delta}.
\]
In addition, the continuity of the map \((-r, r) \ni \lambda \mapsto v^{(1)}_{1,\lambda} \in C([0, T_1(r+\epsilon)]; C^{2+\alpha}(\Omega)) \cap C^{1+\alpha/2}([0, T_1(r+\epsilon)]; \mathcal{C}(\Omega)) \) and Lemma 2.1 imply that
\[
\begin{align*}
\lim_{\delta \to 0} \left\| \frac{v_{1,\lambda+\delta} - v_{1,\lambda}}{\delta} - v^{(1)}_{1,\lambda} \right\|_{W^{1,\infty}((0, T_1(r+\epsilon)) \times \Omega)} &= 0, \\
\lim_{\delta \to 0} \left\| \frac{v_{1,\lambda+\delta} - v^{(1)}_{1,\lambda}}{\delta} \right\|_{W^{1,\infty}((0, T_1(r+\epsilon)) \times \Omega)} &= 0.
\end{align*}
\tag{2.13, 2.14}
\]
Applying the mean value theorem and Lemma 2.1, we deduce that
\[
\begin{align*}
&\left\| \int_0^1 \partial^2_u F_1(\cdot, s v_{1,\lambda+\delta} + (1-s)v_{1,\lambda})ds - \partial^2_u F_1(\cdot, v_{1,\lambda})ds \right\|_{W^{1,\infty}((0, T_1(r+\epsilon)) \times \Omega)} \\
&\leq \int_0^1 \left\| \partial^2_u F_1(\cdot, s v_{1,\lambda+\delta} + (1-s)v_{1,\lambda}) - \partial^2_u F_1(\cdot, v_{1,\lambda}) \right\|_{W^{1,\infty}((0, T_1(r+\epsilon)) \times \Omega)} ds \\
&\leq \left( \max_{|\beta|+|\lambda| \leq 1} \sup_{x \in \Omega} \sup_{\tau \in [-M_1(\lambda), M_1(\lambda)]} |\partial^3_u \partial^\beta \partial^\lambda_x F_1(t, x, \tau)| + |\partial^3_u F_1(t, x, \tau)| \right) \\
&\quad \times \left\| v_{1,\lambda+\delta} - v_{1,\lambda} \right\|_{W^{1,\infty}((0, T_1(r+\epsilon)) \times \Omega)} + M_1(\lambda) \left( \sup_{x \in \Omega} \sup_{\tau \in [-M_1(\lambda), M_1(\lambda)]} |\partial^4_u F_1(t, x, \tau)| \right) \left\| v_{1,\lambda+\delta} - v_{1,\lambda} \right\|_{L^\infty((0, T_1(r+\epsilon)) \times \Omega)}.
\end{align*}
\]
It follows that
\[
\lim_{\delta \to 0} \left\| \int_0^1 \partial^2_u F_1(\cdot, s v_{1,\lambda+\delta} + (1-s)v_{1,\lambda})ds - \partial^2_u F_1(\cdot, v_{1,\lambda}) \right\|_{W^{1,\infty}((0, T_1(r+\epsilon)) \times \Omega)} = 0.
\]
Combining this with (2.13)–(2.14), we deduce that
\[
\lim_{\delta \to 0} L_{\delta,\lambda} \left\|_{W^{1,\infty}((0, T_1(r+\epsilon)) \times \Omega)} = 0
\]
and applying again [28, Theorem 5.2, p.p. 320], we get that
\[
\left\| \frac{v_{1,\lambda+\delta}^{(1)} - v_{1,\lambda}^{(1)}}{\delta} - v_{1,\lambda}^{(2)} \right\|_{2^{+\alpha},1+\alpha/2} \leq C \left\| L_{\delta,\lambda} \right\|_{W^{1,\infty}((0,T_1(r+\epsilon)) \times \Omega)}.
\]

This last inequality clearly implies (2.9).

2.2. Completion of the proof of Theorem 1.1

Using the second linearization of our problem given by Lemma 2.2, we will complete the proof of Theorem 1.1 by showing that there exists a constant \( a_1 > 0 \), depending only on \( \Omega, T, q, \delta_1 \) and \( \chi \), such that \((\delta_1, T_1(r+\epsilon)) \times \Omega \times (-a_1 r, a_1 r) \subset G \). Here \( q \in W^{1,\infty}((0, T) \times \Omega) \) is given by (1.8). For this purpose, we fix \( x_0 \in \Omega \), \( t_0 \in (\delta_1, T_1(r+\epsilon)) \) and we will show that there exists a constant \( a_1 > 0 \), depending only on \( \Omega, T, q, \delta_1 \) and \( \chi \), such that the set \( G_{(t_0,x_0)} := \{ v_{1,\lambda}(t_0, x_0) : \lambda \in (-r, r) \} \) contains the set \((-a_1 r, a_1 r)\). Our approach is based on a suitable study of properties of the map \((-r, r) \ni \lambda \mapsto v_{1,\lambda}(t_0, x_0)\) that we derive from the second order linearization introduced in Lemma 2.2.

Since \( \lambda \mapsto v_{1,\lambda}(t_0, x_0) \in C([-r, r]) \), the proof will be completed if we prove that
\[
\pm v_{1,\pm r}(t_0, x_0) \geq \pm a_1 r. \tag{2.15}
\]

We start by proving (2.15) for \( \pm \) replaced by +. We fix \( \lambda \in [-r, r] \) and we consider \( M_{\lambda} = \left\| V_{1,\lambda} \right\|_{L^{\infty}((0, T_1(r+\epsilon)) \times \Omega)} \) and \( \tilde{v}_{1,\lambda}^{(1)} = e^{-M_\lambda t} v_{1,\lambda}^{(1)} \). One can check that \( \tilde{v}_{1,\lambda}^{(1)} \) satisfies
\[
\begin{cases}
\partial_t \tilde{v}_{1,\lambda}^{(1)} - \Delta \tilde{v}_{1,\lambda}^{(1)} + (V_{1,\lambda} + M_{\lambda}) \tilde{v}_{1,\lambda}^{(1)} = 0 \text{ in } (0, T_1(r+\epsilon)) \times \Omega, \\
\tilde{v}_{1,\lambda}^{(1)}(t,x) = e^{-M_\lambda t} \chi(t,x) \quad (t, x) \in (0, T_1(r+\epsilon)) \times \partial \Omega, \\
\tilde{v}_{1,\lambda}^{(1)}(0,x) = 0 \quad x \in \Omega.
\end{cases}
\]

Using the fact that \( V_{1,\lambda} + M_{\lambda} \geq 0, \chi \geq 0 \) and applying the weak maximum principle for parabolic equations (e.g. [10, Theorem 9, pp. 369]) to \( \tilde{v}_{1,\lambda}^{(1)} \) we deduce that
\[
v_{1,\lambda}^{(1)}(t,x) = e^{M_\lambda t} \tilde{v}_{1,\lambda}^{(1)}(t,x) \geq 0, \quad \lambda \in [-r, r], \, x \in \Omega, \, t \in (0, T_1(r+\epsilon)).
\]

Combining this with Lemma 2.1 and the fact that, thanks to (1.6), \( v_{1,\lambda}|_{\lambda=0} = 0 \), we find
\[
v_{1,\lambda}(t,x) \geq 0, \quad \lambda \in [0,r], \, x \in \Omega, \, t \in (0, T_1(r+\epsilon)).
\]

Thus, in light of (1.7), we have that
\[
\partial_y^2 F_1(t,x,v_{1,\lambda}(t,x)) \leq 0, \quad \lambda \in [0,r], \, x \in \Omega, \, t \in (0, T_1(r+\epsilon)).
\]

Then, in view of (2.8), the weak maximum principle for parabolic equations (e.g. [10, Theorem 9, pp. 369]) combined with the above argumentation imply that
\[
v_{1,\lambda}^{(2)}(t,x) \geq 0, \quad \lambda \in [0,r], \, x \in \Omega, \, t \in (0, T_1(r+\epsilon)).
\]
Combining this with Lemma 2.2, we deduce that
\[ v_{1,\lambda}^{(1)}(t_0, x_0) \geq v_{1,0}^{(1)}(t_0, x_0), \quad \lambda \in [0, r] \]
and we find that
\[ v_{1,\lambda}(t_0, x_0) = \int_0^\lambda v_{1,\tau}^{(1)}(t_0, x_0)\,d\tau \geq v_{1,0}^{(1)}(t_0, x_0)\lambda, \quad \lambda \in [0, r]. \tag{2.16} \]

In view of (2.16), the proof will be completed if we show that there exists a constant \( a_1 > 0 \), depending only on \( \Omega, T, q, \delta_1 \) and \( \chi \), such that
\[ v_{1,0}^{(1)}(t_0, x_0) \geq a_1. \tag{2.17} \]

For this purpose, let us fix \( w \in H^1(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H^1(\Omega)) \) solving
\[
\begin{cases}
\partial_t w - \Delta w + qw = 0 \text{ in } (0, T) \times \Omega, \\
w(t, x) = \chi(t, x) \quad (t, x) \in (0, T) \times \partial \Omega, \\
w(0, x) = 0 \quad x \in \Omega.
\end{cases}
\tag{2.18}
\]

Since \( q \in W^{1,\infty}((0, T) \times \Omega), \) [28, Theorem 5.4, pp. 322] implies that \( w \in C^{1+\frac{\alpha}{2},2+\alpha}([0, T] \times \Omega) \). We fix \( w_1 = v_{1,0}^{(1)} - w \) and we notice that \( w_1 \) solves
\[
\begin{cases}
\partial_t w_1 - \Delta w_1 + V_{1,0}v_1 = (q - V_{1,0})w \text{ in } (0, T_1(r+\epsilon)) \times \Omega, \\
w_1(t, x) = 0 \quad (t, x) \in (0, T_1(r+\epsilon)) \times \partial \Omega, \\
w_1(0, x) = 0 \quad x \in \Omega.
\end{cases}
\]

On the other hand, we have that
\[ q(t, x) - V_{1,0}(t, x) = q(t, x) - \partial_a F_1(t, x, v_{1,0}(t, x)), \quad (t, x) \in (0, T_1(r+\epsilon)) \times \Omega \]
and from (1.6) we deduce that \( v_{1,0} = 0 \). Thus, condition (1.8) implies that
\[ q(t, x) - V_{1,0}(t, x) = q(t, x) - \partial_a F_1(t, x, 0) \geq 0, \quad (t, x) \in (0, T_1(r+\epsilon)) \times \Omega. \]

Moreover, since \( \chi \geq 0 \), applying the weak maximum principle we deduce that \( w \geq 0 \). It follows that \( (q - V_{1,0})w \geq 0 \) and the weak maximum principle implies that \( v_{1,0}^{(1)} - w = w_1 \geq 0 \). Therefore, we have \( v_{1,0}^{(1)}(t_0, x_0) \geq w(t_0, x_0) \) and the estimate (2.17) holds true with
\[ a_1 := \inf_{(t, x) \in (\delta_1, T) \times \Omega} w(t, x). \tag{2.19} \]

Therefore, in order to complete the proof of the theorem we only need to show that \( a_1 > 0 \). For this purpose, let us assume the contrary. Since \( w \in \dot{C}([\delta_1, T] \times \Omega) \), there exists \( (t_1, x_1) \in [\delta_1, T] \times \Omega \) such that \( a_1 = w(t_1, x_1) \). Since \( w \geq 0 \), we have \( a_1 = w(t_1, x_1) = 0 \). Using the fact that
\[ w(t, x) = \chi(t, x) = \delta_2 > 0, \quad (t, x) \in [\delta_1, T] \times \partial \Omega, \]
we deduce that $x_1 \in \Omega$. Thus, we can fix $\delta_3 > 0$ such that, for $B(x_1, \delta_3) := \{x \in \mathbb{R}^n : |x - x_1| < \delta_3\}$, we have $\overline{B(x_1, \delta_3)} \subset \Omega$. Using the fact that $w \geq 0$ and applying the parabolic Harnack inequality (e.g. [10, Theorem 10 pp. 370]), we deduce that, for all $t_2 \in (\delta_3^2, \delta_1)$, we get that

$$\sup_{x \in B(x_1, \delta_3)} w(t_2, x) \leq C \inf_{x \in B(x_1, \delta_3)} w(t_1, x) = 0,$$

where $C$ depends on $t_1, t_2, x_1, \delta_3, \Omega$ and $q$. Therefore, we have

$$w(t, x) = 0, \quad x \in B(x_1, \delta_3), \quad t \in \left(\frac{\delta_1}{2}, \delta_1\right). \quad (2.20)$$

Using the fact that $\Omega$ is connected and applying results of unique continuation for parabolic equations (e.g. [32, Theorem 1.1]), we deduce from (2.20) that

$$w(t, x) = 0, \quad x \in \Omega, \quad t \in \left(\frac{\delta_1}{2}, \delta_1\right).$$

It follows that

$$w(t, x) = 0, \quad x \in \partial \Omega, \quad t \in \left(\frac{\delta_1}{2}, \delta_1\right),$$

and, using the fact that $w \in C([0, T] \times \overline{\Omega})$, we get from (2.18) that

$$\chi(\delta_1, x) = \lim_{t \to \delta_1^{-}} \sup_{t \in \overline{\Omega}} w(t, x) = 0, \quad x \in \partial \Omega.$$

This contradicts the fact that $\chi(\delta_1, x) = \delta_2 > 0, x \in \partial \Omega$. Thus, we have $a_1 > 0$ and the proof of (2.15) for $\pm$ replaced by $+$ is completed.

Now let us consider (2.15) for $\pm$ replaced by $-$. Repeating the above argumentation and applying (1.7), we deduce that

$$\frac{\partial}{\partial t} F_1(t, x, v_{1, \lambda}(t, x)) \geq 0, \quad \lambda \in (-r, 0), \quad x \in \Omega, \quad t \in (0, T_1(r + \epsilon)).$$

Applying the weak maximum principle for parabolic equations, we get

$$v_{1, \lambda}^{(2)}(t, x) \leq 0, \quad \lambda \in (-r, 0), \quad x \in \Omega, \quad t \in (0, T_1(r + \epsilon))$$

and Lemma 2.2 implies that

$$v_{1, \lambda}^{(1)}(t_0, x_0) \geq v_{1, 0}^{(1)}(t_0, x_0), \quad \lambda \in (-r, 0).$$

Thus, we find that

$$v_{1, \lambda}(t_0, x_0) = \int_{0}^{\lambda} v_{1, \tau}^{(1)}(t_0, x_0) d\tau \leq v_{1, 0}^{(1)}(t_0, x_0) \lambda, \quad \lambda \in [-r, 0], \quad (2.21)$$

which, combined with (2.17), imply (2.15). Therefore, for all $x_0 \in \Omega, t_0 \in (\delta_1, T_1(r + \epsilon))$ we have $[-a_1 r_1, a_1 r] \subset G(t_0, x_0)$ and we deduce that $(\delta_1, T_1(r + \epsilon)) \times \Omega \times [-a_1 r_1, a_1 r] \subset G$. This completes the proof of Theorem 1.1.
3. Stable recovery of semilinear term

This section is devoted to the proof of the stability result stated in Theorem 1.2 associated with the uniqueness result stated in Theorem 1.1. For this purpose, we assume that the conditions of Theorem 1.1 are fulfilled and we would like to prove (1.11). In a similar way to Theorem 1.1, we prove Theorem 1.2 by mean of a new method based on properties of the map \((−r, r) \ni \lambda \mapsto v_{1, \lambda}(t, x), (t, x) \in (\delta_1, T_1(r + \epsilon)) \times \Omega \) that we obtain below by applying the second order linearization and several properties of solutions of parabolic equations. We start by considering the following intermediate result:

**Lemma 3.1.** Let the conditions of Theorem 1.1 and (1.10) be fulfilled, fix \( r > 0 \) and consider \( v_{1, \lambda}, \lambda \in [−r, r] \) the solution of (2.4). Then, there exists \( a_2 > 0 \), depending only on \( \Omega, T, \delta_1, \chi \) and \( \kappa(0) \), such that, for any \( (t, x) \in (\delta_1, T_1(r + \epsilon)) \times \Omega \) and any \( s \in [−a_2r, a_2r] \), there exists \( \lambda_{t, x, s} \in [−r, r] \) with \( v_{1, \lambda_{t, x, s}}(t, x) = s \)

**Proof.** Let us fix \( (t, x) \in (\delta_1, T_1(r + \epsilon)) \times \Omega \) and recall that the map \( \lambda \mapsto v_{1, \lambda}(t, x) \in C([−r, r]) \). Thus, the proof of the lemma will be completed if we show that

\[ \pm v_{1, \pm r}(t, x) \geq a_2r. \]  

(3.1)

Let us first consider the above estimate with \( \pm \) replaced by \(+\). In view of (2.16), the proof of (3.1) will be completed if we show that there exists a constant \( a_2 > 0 \), depending only on \( \Omega, T, \kappa(0), \delta_1 \) and \( \chi \), such that

\[ v_{1, 0}^{(1)}(t_0, x_0) \geq a_2. \]  

(3.2)

For this purpose, let us fix \( y \in C([0, T]; C^{2+\alpha}(\overline{\Omega})) \cap C^{1+\alpha/2}([0, T]; C(\overline{\Omega})) \), solving that

\[
\begin{aligned}
\partial_t y - \Delta y + \kappa(0)y &= 0 \quad \text{in } (0, T) \times \Omega, \\
y(t, x) &= \chi(t, x) \quad \text{for } (t, x) \in (0, T) \times \partial\Omega, \\
y(0, x) &= 0 \quad \text{for } x \in \Omega.
\end{aligned}
\]  

(3.3)

We set \( z = v_{1, 0}^{(1)} - y \) and we notice that \( z \) solves

\[
\begin{aligned}
\partial_t z - \Delta z + \kappa(0)z &= (\kappa(0) - V_{1, 0})y \quad \text{in } (0, T_1(r + \epsilon)) \times \Omega, \\
z(t, x) &= 0 \quad \text{for } (t, x) \in (0, T_1(r + \epsilon)) \times \partial\Omega, \\
z(0, x) &= 0 \quad \text{for } x \in \Omega.
\end{aligned}
\]

On the other hand, we have that

\[ \kappa(0) - V_{1, 0}(t, x) = \kappa(0) - \partial_u F_1(t, x, v_{1, 0}(t, x)), \quad (t, x) \in (0, T_1(r + \epsilon)) \times \Omega \]

and from (1.6) we find \( v_{1, 0} = 0 \). Thus, condition (1.10) implies that

\[ \kappa(0) - V_{1, 0}(t, x) \geq 0, \quad (t, x) \in (0, T_1(r + \epsilon)) \times \Omega. \]

Moreover, since \( \chi \geq 0 \), applying the weak maximum principle we deduce that \( y \geq 0 \). It follows that \( (\kappa(0) - V_{1, 0})y \geq 0 \) and the weak maximum principle implies...
that \( v_{1,0}^{(1)} - y = z \geq 0 \). Therefore, we have \( v_{1,0}^{(1)}(t, x) \geq y(t, x) \) and the estimate (3.2) holds true with

\[
a_2 := \inf_{(t, x) \in (\delta_1, T) \times \Omega} y(t, x).\tag{3.4}
\]

Here it is clear that \( a_2 \) depends only on \( \Omega, T, \delta_1, \chi \) and \( \kappa(0) \) and the proof of the lemma will be completed if we show that \( a_2 > 0 \). This last property can be deduced by using arguments similar to the ones used in the proof of Theorem 1.1 for showing that \( a_1 > 0 \). This proves (3.1) with \( \pm \) replaced by \( + \). In the same way, we can complete the proof of (3.1) with \( \pm \) replaced by \( - \).

\[\square\]

Applying Lemma 3.1, we are now in position to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let \( r > 0 \), fix \( s \in [-a_2r, a_2r] \) and consider \( (t, x) \in (0, T_1(r + \epsilon)) \times \Omega \). Consider also \( \lambda_{t, x, s} \) given by Lemma 3.1. Then, we have that

\[
|F_1(t, x, s) - F_2(t, x, s)|
= |F_1(x, v_{1,\lambda_{t, x, s}}(t, x)) - F_2(t, x, v_{1,\lambda_{t, x, s}}(t, x))|
\leq |F_1(t, x, v_{1,\lambda_{t, x, s}}(t, x)) - F_2(x, v_{1,\lambda_{t, x, s}}(t, x))| + |F_2(t, x, v_{1,\lambda_{t, x, s}}(t, x)) - F_2(t, x, v_{2,\lambda_{t, x, s}}(t, x))|.
\tag{3.5}
\]

We fix

\[
I := |F_1(t, x, v_{1,\lambda_{t, x, s}}(t, x)) - F_2(x, v_{2,\lambda_{t, x, s}}(t, x))|,

II := |F_2(t, x, v_{1,\lambda_{t, x, s}}(t, x)) - F_2(t, x, v_{2,\lambda_{t, x, s}}(t, x))|.
\]

Using estimate (3.5), we will complete the proof of the theorem by proving that the expressions \( I \) and \( II \) can be estimated by the right hand of (1.11). We start with \( I \).

Recall that

\[
I = \int_{-r}^{r} \partial_\tau F_1(t, x, v_{1,\tau}(t, x))v_{1,\tau}^{(1)}(t, x) - \partial_\tau F_2(t, x, v_{2,\tau}(t, x))v_{2,\tau}^{(1)}(t, x) \, d\tau
\leq \int_{-\lambda_{t, x, s}}^{\lambda_{t, x, s}} |V_{1,\tau}(t, x)v_{1,\tau}^{(1)}(t, x) - V_{2,\tau}(t, x)v_{2,\tau}^{(1)}(t, x)| \, d\tau
\leq \int_{-r}^{r} |V_{1,\tau}(t, x) - V_{2,\tau}(t, x)| \left|v_{1,\tau}^{(1)}(t, x)\right| \, d\tau
+ \int_{-r}^{r} \left|v_{2,\tau}^{(1)}(t, x)\right| \left|v_{2,\tau}^{(1)}(t, x) - v_{1,\tau}^{(1)}(t, x)\right| \, d\tau.
\tag{3.6}
\]

In view of condition (H), applying (1.10), we obtain

\[
\left\|V_{j, \tau}\right\|_{W^{1,\infty}((0, T_1(r + \epsilon)) \times \Omega)}
\leq \sup_{\lambda \in [-M(r + \epsilon), M(r + \epsilon)]} \left\|F_j(\cdot, \lambda)\right\|_{W^{1,\infty}((0, T_1(r + \epsilon)) \times \Omega)}
+ M(r + \epsilon) \sup_{\lambda \in [-M(r + \epsilon), M(r + \epsilon)]} \left\|\partial_\tau F_j(\cdot, \lambda)\right\|_{W^{1,\infty}((0, T_1(r + \epsilon)) \times \Omega)}
\leq \kappa(M(r + \epsilon))(M(r + \epsilon) + 1), \quad \tau \in [-r, r].
\tag{3.7}
\]
Combining (3.7) with (3.6), we obtain
\begin{equation}
I \leq 2rM(r + \epsilon) \sup_{\tau \in [-r, r]} \| V_{1, \tau} - V_{2, \tau} \|_{L^\infty((0, T_1(r+\epsilon)) \times \Omega)} + 2r(M(r + \epsilon) + 1)\kappa(M(r + \epsilon)) \sup_{\tau \in [-r, r]} \| v_{1, \tau}^{(1)} - v_{2, \tau}^{(1)} \|_{L^\infty((0, T_1(r+\epsilon)) \times \Omega)}.
\end{equation}

On the other hand, using (3.7) and considering the stability estimate of \cite[Theorem 1.1]{} with the geometric optics solutions of \cite[Proposition 4.3, 4.4]{2}, we get that
\begin{equation}
\lambda \leq \kappa_0^0 \mathcal{N}_{r, r, F_1}^{1}(0) - \mathcal{N}_{r, r, F_2}^{1}(0) \leq C_r \ln \left( 3 + \sqrt{\lambda_{V_{1, \lambda}} - \lambda_{V_{2, \lambda}}} \right) \mathcal{B}(\kappa_1^{r+\epsilon} ; \kappa_1^{r+\epsilon})^{-\theta_1},
\end{equation}

where \( C_r > 0 \) depends on \( r, \Omega, \kappa, M(r + \epsilon), T_1(r + \epsilon), \chi, n \) and \( \theta_1 > 0 \) depends on \( n \). Recalling that \( \mathcal{N}_{r, r, F}^{1}(0) = \lambda_{V_{j, \lambda}}^{1} \) and applying a classical interpolation result (see e.g. \cite[Lemma AppendixB.1]{5}) as well as (3.7), for all \( \lambda \in [-r, r] \), we obtain
\begin{align}
\| V_{1, \lambda} - V_{2, \lambda} \|_{L^\infty((0, T_1(r+\epsilon)) \times \Omega)} & \leq C \| V_{1, \lambda} - V_{2, \lambda} \|_{C^1(\{0, T_1(r+\epsilon)\} \times \Omega)} \| V_{1, \tau} - V_{2, \tau} \|_{L^\infty((0, T_1(r+\epsilon)) \times \Omega)} \\leq C_r \ln \left( 3 + \sqrt{\mathcal{N}_{r, r, F_1}^{1}(0) - \mathcal{N}_{r, r, F_2}^{1}(0)} \right) \mathcal{B}(\kappa_1^{r+\epsilon} ; \kappa_1^{r+\epsilon})^{-\theta},
\end{align}

with \( C_r > 0 \) depending on \( r, \Omega, T, \kappa, M(r + \epsilon), T_1(r + \epsilon), \chi, n \) and with \( \theta > 0 \) depending on \( n \). It follows that
\begin{align}
\sup_{\lambda \in (-r, r)} \| V_{1, \lambda} - V_{2, \lambda} \|_{L^\infty((0, T_1(r+\epsilon)) \times \Omega)} & \leq C_r \sup_{\lambda \in (-r, r)} \ln \left( 3 + \sqrt{\mathcal{N}_{r, r, F_1}^{1}(0) - \mathcal{N}_{r, r, F_2}^{1}(0)} \right) \mathcal{B}(\kappa_1^{r+\epsilon} ; \kappa_1^{r+\epsilon})^{-\theta}.
\end{align}

According to the weak maximum principle, we find that
\begin{equation}
\| v_{1, \lambda}^{(1)} - v_{2, \lambda}^{(1)} \|_{L^\infty((0, T_1(r+\epsilon)) \times \Omega)} \leq 1, \quad \lambda \in [-r, r].
\end{equation}

Fixing \( w = v_{1, \lambda}^{(1)} - v_{2, \lambda}^{(1)} \), we deduce that \( w \) solves
\begin{align}
\partial_t w - \Delta w + V_{1, \lambda} w = (V_{2, \lambda} - V_{1, \lambda})v_{1, \lambda}^{(1)} \quad & \text{in } (0, T_1(r + \epsilon)) \times \Omega, \\
w = 0 & \quad \text{on } \partial(0, T_1(r + \epsilon)) \times \Omega, \\
w(x, 0) = 0 & \quad x \in \Omega.
\end{align}

Therefore, in view of \cite[Theorem 9.1, pp. 341]{28}, fixing \( p > n + 1 \) and applying (3.7), (3.10) as well as the Sobolev embedding theorem, we deduce that
\begin{align}
\| v_{1, \lambda}^{(1)} - v_{2, \lambda}^{(1)} \|_{L^\infty((0, T_1(r+\epsilon)) \times \Omega)} & \leq C \| w \|_{W^{1, p}(0, T_1(r+\epsilon)) \times \Omega} \\leq C_r \| V_{1, \lambda} - V_{2, \lambda} \|_{L^\infty((0, T_1(r+\epsilon)) \times \Omega)}, \quad \lambda \in [-r, r].
\end{align}
Then, (3.9) implies that

$$\sup_{\lambda \in (-r, r)} \left\| v_1^{(1)}(1) - v_2^{(1)}(2) \right\|_{L^\infty((0,T_1(t + \epsilon)) \times \Omega)} \leq C_r \sup_{\lambda \in (-r, r)} \ln \left( 3 + \left\| \mathcal{N}_{\lambda, r, F_1}^\prime(0) - \mathcal{N}_{\lambda, r, F_2}^\prime(0) \right\|_{\mathcal{B}(K_0^{T_1(t + \epsilon)}; K_1^{T_1(t + \epsilon)})} \right)^{-\theta}$$  (3.11)

Combining this with (3.8) and (3.9), we get

$$I \leq C_r \sup_{\lambda \in (-r, r)} \ln \left( 3 + \left\| \mathcal{N}_{\lambda, r, F_1}^\prime(0) - \mathcal{N}_{\lambda, r, F_2}^\prime(0) \right\|_{\mathcal{B}(K_0^{T_1(t + \epsilon)}; K_1^{T_1(t + \epsilon)})} \right)^{-\theta}$$  (3.12)

Now let us consider $II$. Applying the mean value theorem and (1.10), we get that

$$II \leq \sup_{\lambda \in [-M(r + \epsilon), M(r + \epsilon)]} \left\| \partial_u F_2(\cdot, \lambda) \right\|_{L^\infty((0,T_1(t + \epsilon)) \times \Omega)} \left| v_1^{(1)}(1, t, x) - v_2^{(1)}(1, t, x) \right| \leq \kappa (M(r + \epsilon)) 2r$$  

$$\sup_{\lambda \in (-r, r)} \left\| v_1^{(1)}(1, t, x) - v_2^{(1)}(1, t, x) \right\|_{L^\infty((0,T_1(t + \epsilon)) \times \Omega)} \leq \kappa (M(r + \epsilon)) 2r$$  

Then, applying (3.11), we deduce that

$$II \leq C_r \sup_{\lambda \in (-r, r)} \ln \left( 3 + \left\| \mathcal{N}_{\lambda, r, F_1}^\prime(0) - \mathcal{N}_{\lambda, r, F_2}^\prime(0) \right\|_{\mathcal{B}(K_0^{T_1(t + \epsilon)}; K_1^{T_1(t + \epsilon)})} \right)^{-\theta}$$

Combining this estimate with (3.12) and (3.5) we deduce (1.11). This completes the proof of Theorem 1.2. \qed

**Acknowledgements.** The work of Y.K is partially supported by the French National Research Agency ANR (project MultiOnde) grant ANR-17-CE40-0029. The research of G.U. is partially supported by NSF, a Walker Professorship at UW and a Si-Yuan Professorship at IAS, HKUST. Part of the work was supported by the NSF grant DMS-1440140 while G.U. were in residence at MSRI in Berkeley, California, during Fall 2019 semester.

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(Received January 8, 2021 / Accepted December 9, 2022)
Published online January 9, 2023
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