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Recommended Citation

Ma, X., You, C., Adhikari, S., Matekole, E., Glasser, R., Lee, H., & Dowling, J. (2018). Sub-shot-noise-limited phase estimation via SU(1,1) interferometer with thermal states. *Optics Express, 26*(14), 18492-18504.  
[https://doi.org/10.1364/OE.26.018492](https://doi.org/10.1364/OE.26.018492)

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Sub-shot-noise-limited phase estimation via SU(1,1) interferometer with thermal states

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Abstract: We theoretically study the phase sensitivity of an SU(1,1) interferometer with a thermal state and a squeezed vacuum state as inputs and parity detection as the measurement. We find that the phase sensitivity can beat the shot-noise limit and approaches the Heisenberg limit, with increasing input photon number, in an ideal situation. We also consider the effect of various noises, including photon loss, dark counts, and thermal photon noise. Our results show that the phase sensitivity is below the shot-noise limit with photon loss and dark counts, but cannot beat the shot-noise limit in the presence of thermal noise.

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OCIS codes: (270.0270) Quantum optics; (190.4410) Nonlinear optics, parametric processes; (120.3180) Interferometry.

References and links
1. A. A. Michelson, and E. W. Morley, “On the Relative Motion of the Earth and of the Luminiferous Ether,” Sidereal Messenger 6, 306–310 (1887).
2. M. Xiao, L. A. Wu, and H. J. Kimble, “Precision measurement beyond the shot-noise limit,” Phys. Rev. Lett. 59(3), 278 (1987).
3. B. C. Barish, and R. Weiss, “LIGO and the detection of gravitational waves,” Phys. Today 52(10), 44 (1999).
4. H. Lee, P. Kok, and J. P. Dowling, “A quantum Rosetta stone for interferometry,” J. Mod. Opt. 49(14-15), 2325–2338 (2002).
5. A. N. Boto, P. Kok, D. S. Abrams, C. P. Williams, and J. P. Dowling, “Quantum interferometric optical lithography: exploiting entanglement to beat the diffraction limit,” Phys. Rev. Lett. 85(13), 2733 (2000).
6. B. M. Escher, R. L. de Matos Filho, and L. Davidovich, “General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology,” Nat. Phys. 7, 406–411 (2011).
7. V. Giovannetti, S. Lloyd, and L. Maccone, “Quantum-enhanced measurements: beating the standard quantum limit,” Science 306(5700), 1330–1336 (2004).
8. V. Giovannetti, S. Lloyd, and L. Maccone, “Advances in quantum metrology,” Nat. Photon. 5, 222–229 (2011).
9. R. Schnabel, N. Mavalvala, D. E. McClelland, and P. K. Lam, “Quantum metrology for gravitational wave astronomy,” Nat. Comm. 1, 121 (2010).
10. Z. Y. Ou, “Enhancement of the phase-measurement sensitivity beyond the standard quantum limit by a nonlinear interferometer,” Phys. Rev. A 85(2), 023815 (2012).
11. C. You, S. Adhikari, Y. Chi, M. LaBorde, C. Matyas, C. Zhang, Z. Su, T. Byrnes, C. Lu, J. P. Dowling, and J. P. Olson, “Multiparameter estimation with single photons - linearly-optically generated quantum entanglement beats the shot noise limit,” J. Opt. 19(12), 124002 (2017).
12. B. T. Gard, C. You, D. K. Mishra, R. Singh, H. Lee, T. R. Corbitt, and J. P. Dowling, “Nearly optimal measurement schemes in a noisy Mach-Zehnder interferometer with coherent and squeezed vacuum,” EPJ Quantum Technol. 4(1), 4 (2017).
13. J. Kong, F. Hudelist, Z. Y. Ou, and W. P. Zhang, “Cancellation of Internal Quantum Noise of an Amplifier by Quantum Correlation,” Phys. Rev. Lett. 111(3), 033608 (2013).
14. B. Chen, C. Qiu, S. Y. Chen, and J. X. Guo, and L. Q. Chen, Z. Y. Ou, and W. P. Zhang, “Atom-Light Hybrid Interferometer,” Phys. Rev. Lett. 115(4), 043602 (2015).

https://doi.org/10.1364/OE.26.018492
Journal © 2018 Received 25 Apr 2018; revised 6 Jun 2018; accepted 6 Jun 2018; published 3 Jul 2018
15. F. Hudelist, J. Kong, C. J. Liu, J. T. Jing, Z.Y. Ou and W. P. Zhang, “Quantum metrology with parametric amplifier-based photon correlation interferometers,” Nat. Comm. 5, 3049 (2014).
16. C. M. Caves, “Quantum-mechanical noise in an interferometer,” Phys. Rev. D 23(8), 1693–1708 (1981).
17. J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, “Optimal frequency measurements with maximally correlated states,” Phys. Rev. A 54(6), R4649–R4652 (1996).
18. J. P. Dowling, “Quantum optical metrology — the lowdown on high-N00N states,” Contemp. Phys. 49(2), 125-143 (2008).
19. B. Yurke, S. L. McCall, and J. R. Klauder, “SU (2) and SU (1, 1) interferometers,” Phys. Rev. A 33(6), 4033 (1986).
20. W. N. Plick, J. P. Dowling, and G. S. Agarwal, “Coherent-light-boosted, sub-shot noise, quantum interferometry,” New J. Phys. 12, 083014 (2010).
21. D. Li, C. H. Yuan, Z. Y. Ou, and W. P. Zhang, “The phase sensitivity of an SU (1, 1) interferometer with coherent and squeezed-vacuum light,” New J. Phys. 16, 073014 (2014).
22. D. Li, B. T. Gard, Y. Gao, C. H. Yuan, W. P. Zhang, H. Lee, and J. P. Dowling, “Phase sensitivity at the Heisenberg limit in an SU (1, 1) interferometer via parity detection,” Phys. Rev. A 94(6), 063840 (2016).
23. W. N. Plick, P. M. Anisimov, J. P. Dowling, H. Lee, and G. S. Agarwal, “Parity detection in quantum optical metrology without number-resolving detectors,” New J. Phys. 12, 113025 (2010).
24. S. S. Szegeti, R. J. Lewis-Swan, and S. A. Haine, “Pumped-up SU (1, 1) interferometry,” Phys. Rev. Lett. 118(15), 150401 (2017).
25. P. M. Anisimov, G. M. Raterman, A. Chiruvelli, W. N. Plick, S. D. Huver, H. Lee, and J. P. Dowling, “Quantum Metrology with Two-Mode Squeezed Vacuum: Parity Detection Beats the Heisenberg Limit,” Phys. Rev. Lett. 104(10), 103602 (2010).
26. H. Shibata, K. Shimizu, H. Takesue, and Y. Tokura, “Ultimate low system dark-count rate for superconducting nanowire single-photon detector,” Opt. Lett. 40(14), 3428-3431 (2015).
27. Z. X. Huang, K. R. Motes, P. M. Anisimov, J. P. Dowling, and D. W. Berry, “Adaptive phase estimation with two-mode squeezed vacuum and parity measurement,” Phys. Rev. A 95(5), 053837 (2017).
28. S. L. Braunstein, and P. V. Loock, “Quantum information with continuous variables,” Rev. Mod. Phys. 77(2), 513–577 (2005).
29. C. Weedbrook, S. Pirandola, and R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, “Gaussian quantum information,” Rev. Mod. Phys. 84(2), 621–669 (2012).

1. Introduction

Over the past decades, there has been much progress in both theoretical and experimental research on quantum metrology. This development has been due to the use of quantum resources in quantum metrology and has led us to surpass classical sensitivity (the shot-noise limit) [1–15]. One of the important tasks in quantum metrology is the estimation of optical phase. Mach-Zehnder interferometers (MZI), which consist of two mirrors, two beam splitters, and a phase shifter, are a conventional tool for phase estimation. Using only classical light, an MZI can achieve a phase sensitivity of $1/\sqrt{N}$, called the shot-noise limit (SNL), where $N$ is the mean number of photons that have passed through the MZI. Using quantum resources, however, the phase sensitivity can approach $1/N$, which is called the Heisenberg limit [16–18]. An MZI is also called an SU(2) interferometer, as Yurke et al. showed that the group SU(2) can naturally characterize an MZI [19]. Furthermore, in the same paper, the authors first proposed another type of interferometer, which is characterized by the group SU(1,1). They showed that this SU(1,1) interferometer can beat the SNL, and that it is similar to an SU(2) interferometer, except that the two beam splitters are replaced by two optical parametric amplifiers (OPAs).

Some new SU(1,1) interferometer schemes were proposed recently. Plick et al. [20] presented a theoretical scheme that uses a bright coherent source to boost the sensitivity of SU(1,1)
interferometer. Their scheme achieved scaling far below SNL for bright sources, and was recently implemented in the lab in 2012 by Ou [10]. To reach the same sensitivity and reduce the required intensities of the input states, Li et al. [21] considered a squeezed vacuum state to replace one of the two input coherent states. In addition, they used homodyne measurement as their detection, which is convenient for experimental detection of squeezing, and allows for near-unity detection efficiencies. Li et al. [22] also proposed another scheme that uses parity detection, which simply measures whether the output mode contains an even or odd number of photons, and can be implemented by using homodyne techniques [23]. More recently, Szigeti et al. [24] presented a modification of an SU(1,1) interferometer, where all the input particles participate in the phase measurement, and showed how this can be implemented in spinor Bose-Einstein condensates and hybrid atom-light systems. Besides the theoretical progresses, several experimental realizations were presented recently. A truncated optical SU(1,1) interferometer [25] without a second amplifier using seeded four-wave mixing in $^{85}$Rb vapor as nonlinear interaction and balanced homodyne detection as measurement, has demonstrated the possibility of a supersensitive phase measurement. At the same time, an unseeded SU(1,1) interferometer composed of two cascaded degenerate parametric amplifiers with direct detection at the output was investigated [26]. It achieves phase super sensitivity beating shot noise limit by 2.3 dB. Du et al. [27] studied a direct phase estimation of a FWM-based SU(1,1) interferometer which obtained 3 dB improvement in sensitivity for the SU(1,1) interferometer over the MZI.

In this article, we propose an interferometric scheme that performs phase estimation using a thermal state. All the coherent state based, or coherent and squeezed states based SU(1,1) interferometer schemes mentioned above were shown to have super-sensitivity in phase estimation. However, on the other hand, achieving super-sensitivity with thermal light comes as a surprise. Thermal states are a central concept in thermodynamics and statistical mechanics and have several important properties, such as they are equilibrium states, so typical dynamics will tend to evolve a state to be thermal, and it is the state that maximizes the entropy under constraints on physical quantities [28,29]. Lately, due to their astronomical, aerospace, defense, and other applications, the interests in the thermal states of light have been increasing ever more. Furthermore, the fact that thermal light achieves super sensitivity in an optical interferometer should intrigue researchers and open a new avenue towards sensitive detection of thermal light. Our paper on super-sensitive phase estimation with thermal light is an important step toward that goal and paves the way for practical quantum metrology with thermal sources in optical instruments, such as photometers, or at different wavelengths where the generation of quantum features, such as coherence, number states, squeezing or entanglement may be extremely challenging. There are three common detection schemes used in the experiments for phase estimation: direct, homodyne and parity detection. In our case, we use parity detection as the output measurement. As the means of the thermal state and the squeezed vacuum state are zero, homodyne detection is not applicable for our scheme. And after calculation, phase sensitivity via parity detection always beats SNL but it cannot approach the SNL using intensity measurement.

The structure of the present paper is as follows: In Sec. II. A, we introduce the model along with basic transformation rules of the SU(1,1) interferometer. The Heisenberg limit and shot-noise limit of the system is calculated in Sec. II. B. The parity detection method along with phase sensitivity obtained from this scheme is presented in Sec. II. C. Finally, comparison between phase sensitivity and Heisenberg limit is given in Sec. II. D. In Sec. III, we calculate the phase sensitivity of an imperfect, experimentally-realizable interferometer by including photon loss, dark counts and thermal noise. Lastly, we conclude with a brief summary in Sec. IV.
An SU(1,1) interferometer is similar to a MZI except the two beam splitters are replaced by two OPAs. $|\Psi_{th}\rangle$ and $|\Psi_{\xi}\rangle$ are thermal state and squeezed vacuum state respectively. $a_i$ and $b_i (i = 0, 1, 2)$ denote two light beams in the different processes. $\phi_1$ and $\phi_2$ describe the phase shift in the two paths.

2. SU(1,1) interferometer via parity detection

2.1. Model

An SU(1,1) interferometer is shown in Fig. 1, in which beam splitters of a traditional MZI are replaced by OPAs. Here, we consider a thermal state and a squeezed-vacuum state as inputs. Let $\hat{a}$ ($\hat{a}^{\dagger}$), $\hat{b}$ ($\hat{b}^{\dagger}$) be the annihilation (creation) operators of two modes respectively. After the first OPA, the upper path undergoes a phase shift $\phi_1$ and the lower path undergoes a phase shift $\phi_2$.

After these second OPA, we perform detection on the output to obtain the acquired phase difference between the two modes.

The evolution through an SU(1,1) interferometer is as follows. After the first OPA, the relation between modes $\hat{a}_1$, $\hat{b}_1$ and $\hat{a}_0$, $\hat{b}_0$ is described by

$$\begin{pmatrix} \hat{a}_1 \\ \hat{b}_1^\dagger \end{pmatrix} = \hat{T}_{\text{OPA1}} \begin{pmatrix} \hat{a}_0 \\ \hat{b}_0^\dagger \end{pmatrix}, \quad (1)$$

where $\hat{T}_{\text{OPA1}}$ is the transformation through first OPA.

After propagation through the SU(1,1) interferometer, the relation between input and output modes is described by

$$\begin{pmatrix} \hat{a}_2 \\ \hat{b}_2^\dagger \end{pmatrix} = \hat{T} \begin{pmatrix} \hat{a}_0 \\ \hat{b}_0^\dagger \end{pmatrix}. \quad (2)$$

$\hat{T}$ is the transformation through an SU(1,1) interferometer and represented by $\hat{T} = \hat{T}_{\text{OPA2}} \hat{T}_\phi \hat{T}_{\text{OPA1}}$, where $\hat{T}_\phi$ is the transformation through the phase shifters and $\hat{T}_{\text{OPA2}}$ is the transformation through second OPA.

Transformations through first OPA, phase shifter and second OPA are described by

$$\hat{T}_{\text{OPA1}} = \begin{pmatrix} u_1 & v_1 \\ v_1^* & u_1^* \end{pmatrix}, \quad (3)$$

$$\hat{T}_\phi = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}, \quad (4)$$

$$\hat{T}_{\text{OPA2}} = \begin{pmatrix} u_2 & v_2 \\ v_2^* & u_2^* \end{pmatrix}, \quad (5)$$

with $u_j = \cosh g_j$, $v_j = e^{i\theta_j} \sinh g_j$, where $\theta_j$ and $g_j$ are phase shift and parametrical strength in the OPAs ($j = 1, 2$).
2.2. Heisenberg limit and shot-noise limit

From Eq. (1), we can calculate the mean photon number $\bar{n}$ inside the SU(1,1) interferometer. Note that $\bar{n}$ is not the total input photon number because OPAs are nonlinear, exhibiting gain and there will be spontaneous photons emitted. Hence, the photon number inside SU(1,1) interferometer is amplified compared to input number. The mean photon number $\bar{n}$ inside SU(1,1) interferometer is given by

$$\bar{n} = \langle \Psi_{\text{in}} | (\hat{a}_{1}^\dagger \hat{a}_{1} + \hat{b}_{1}^\dagger \hat{b}_{1}) | \Psi_{\text{in}} \rangle,$$  \hspace{1cm} (6)

with $|\Psi_{\text{in}}\rangle = |\Psi_{\text{th}}\rangle \otimes |\Psi_{\xi}\rangle$, where $|\Psi_{\text{th}}\rangle$ is a thermal state and $|\Psi_{\xi}\rangle$ is a squeezed vacuum state.

Finally, the total photon number in SU(1,1) interferometer is given by

$$\bar{n} = (n_{\text{OPA}} + 1)(n_{\text{th}} + n_{s}) + n_{\text{OPA}},$$  \hspace{1cm} (7)

where, $n_{\text{th}}$ is the input photon number of the thermal state, $n_{s} = \sinh^2 r$ is the photon number of the squeezed vacuum, $n_{\text{OPA}} = 2\sinh^2 g$ is the emitted photon number from the first OPA and $g = g_{1} = g_{2}$ is the parametrical strength of the two OPAs. We can see that there are two contributions in increasing the mean photon number inside SU(1,1) interferometer. The first contribution comes from the amplification process of the input photon number and the second contribution appears due to the amplification of the input vacuum state, a process called the spontaneous process. Using the total photon number inside our SU(1,1) scheme, with thermal and squeezed vacuum state inputs, we calculate the SNL and HL to be

$$\Delta \phi_{\text{SNL}} = \frac{1}{\sqrt{\bar{n}}} = \frac{1}{\sqrt{(n_{\text{OPA}} + 1)(n_{\text{th}} + n_{s}) + n_{\text{OPA}}}},$$  \hspace{1cm} (8)

$$\Delta \phi_{\text{HL}} = \frac{1}{\bar{n}} = \frac{1}{(n_{\text{OPA}} + 1)(n_{\text{th}} + n_{s}) + n_{\text{OPA}}}. \hspace{1cm} (9)$$

2.3. Phase sensitivity via parity detection

We consider parity measurement as our output detection. Parity detection is a single mode measurement and parity operator detection on output mode $b$ is defined as

$$\hat{\Pi}_{b} \equiv (-1)^{\hat{b}_{1}^\dagger \hat{b}_{2}}.$$  \hspace{1cm} (10)

According to [23], parity measurement satisfies $\langle \hat{\Pi}_{b} \rangle = \pi W(0,0)$, that is, the expectation value of parity measurement is given by the value of Wigner function at the origin of the phase space. This property is very useful for calculating the parity signal. The minimum detectable phase i.e., the phase sensitivity, is given by

$$\Delta \phi = \Delta \hat{\Pi}_{b} \Big|_{\Theta(\hat{\Pi}_{b})/\partial \phi}.$$  \hspace{1cm} (11)

where, $\Delta \hat{\Pi}_{b} = \sqrt{\langle \hat{\Pi}_{b}^2 \rangle - \langle \hat{\Pi}_{b} \rangle^2} = \sqrt{1 - \langle \hat{\Pi}_{b} \rangle^2}$ with the property that $\langle \hat{\Pi}_{b} \rangle = 1$. The phase sensitivity via parity detection for a SU(1,1) interferometer with thermal and squeezed vacuum states is calculated to be

$$\Delta \phi = \sqrt{\frac{2}{n_{\text{OPA}}(n_{\text{OPA}} + 2)(1 + 2n_{s})(1 + 2n_{\text{th}})}},$$  \hspace{1cm} (12)

with the assumption, $\phi = 0$. For an arbitrary value of $\phi$, the phase fluctuation is shown in eqs. (B2) and (B3) (See Appendix B).

Then we compare the phase sensitivity via parity detection with HL and SNL. First, we
Parity Detection
HL
SNL
0.0 0.5 1.0 1.5 2.0 2.5 3.0
0.001
0.010
0.100
1
10
100
g
Δϕ

Fig. 2. Phase sensitivity with parity detection as a function of \( g \) with constraint \( r = 0 \), \( n_{th} = 0 \) on the input.

Parity Detection
HL
SNL
0 5 10 15 20 25 30
0.001
0.005
0.010
0.050
0.100
0.500
\( n_{th} \)
Δϕ

Fig. 3. Phase sensitivity with parity detection as a function of \( n_{th} \) with \( r = 0 \), \( g = 2 \). The phase sensitivity at \( n_{th} = 0 \) is equal to the sensitivity at \( g = 2 \) in Fig. 2 and becomes greater with increasing \( n_{th} \).

Parity Detection
HL
0 5 10 15 20 25 30
5 \times 10^{-4}
\( n_{th} \)
Δϕ

Fig. 4. Phase sensitivity with parity detection as a function of \( n_{th} \) with constraints \( r = 2 \), \( g = 2 \). Comparing the values at \( n_{th} = 20 \) in Fig. 3, both the phase sensitivity and HL are lower due to replacing vacuum state with squeezed vacuum state. Moreover, the phase sensitivity is much higher.

consider two vacuum states as our inputs. The phase sensitivity with parity detection is found to be \( \Delta \phi_v = 1 / \sqrt{n_{OPA}(n_{OPA} + 2)} \), while the corresponding Heisenberg limit is \( \Delta \phi_{HL} = 1 / n_{OPA} \), and the shot noise limit is \( \Delta \phi_{SNL} = 1 / \sqrt{n_{OPA}} \). We plot the phase sensitivity \( \Delta \phi \) as a function of \( g \) in Fig. 2 along with \( \Delta \phi_{SNL} \) and \( \Delta \phi_{HL} \). In this case, phase sensitivity with parity detection always beats SNL and HL, which was proved by Anisimov et al. [30]. Moreover, when \( g \leq 0.6 \), the SNL is below the HL because the total inside photon number, \( n_{OPA} < 1 \).

We then consider thermal and vacuum state as inputs. As depicted in Fig. 3, under the condition \( (r = 0 \text{ and } g = 2) \), phase sensitivity via parity detection always beats SNL but does not approach HL. In this case, as the total inside photon number is always larger than 1, SNL is always greater than HL. Note that the values of phase sensitivity and HL at \( n_{th} = 0 \) are equal to the corresponding values at \( g = 2 \) in Fig. 2. The phase sensitivity increases by increasing the mean photon number of the thermal state.
Lastly, we inject thermal and squeezed vacuum states as inputs. Comparing Eq. (9) with Eq. (12), the necessary optimal condition for approaching HL is found to be

\[ n_{\text{th}} = \sinh^2 g - n_s \left\frac{2}{\cosh^2 (2g)} \right. \]  

The above expression guarantees that the phase sensitivity for input thermal state \( n_{\text{th}} \), input squeezed vacuum state \( n_s \) and the OPA process \( g \) approaches HL. One can prove that regardless the values of \( n_s \) and \( g \), the optimal mean photon number \( n_{\text{th}} \) of the thermal state is no more than 1. Fig. 4 shows phase sensitivity as a function of \( n_{\text{th}} \) under the condition \( r = 2, g = 2 \). The phase sensitivity with parity detection is always below the SNL but cannot beat the HL. Comparing Fig. 4 with Fig. 3, we see that the phase sensitivity becomes larger and closer to HL by replacing the input vacuum state with squeezed vacuum state. Additionally, we find that the phase sensitivity approaches the HL with increasing \( n_{\text{th}} \).

If we replace the squeezed vacuum state in the second mode by a thermal state or a coherent state, will the phase sensitivity be smaller? The answer is no. The phase sensitivity even can not beat the shot noise limit in these two cases. In Fig. 5 and Fig. 6, We compare the phase sensitivity with SNL and HL as a function of the mean photon in the second mode. From the Figs, we can see the phase sensitivity is higher than SNL and becomes larger with the increasing of the mean photon in the second mode.

If we use other output detection schemes, such as homodyne or intense detection, will the phase sensitivity be better? Homodyne detection measures the mean value of any arbitrary super position of the state in phase space. As the means of thermal state and squeezed vacuum state are zero, it will not have any phase information after the transformation through the interferometer and thus we cannot use homodyne detection as our measurement in this particular situation. We compare intensity measurement and parity detection for our scheme. The result shows that parity detection is better than intensity measurement for an SU(1,1) interferometer with thermal seeding. As depicted in the Fig. 7, under the condition \( r=5, g=2 \), phase sensitivity via parity detection always beats SNL but it cannot approach SNL using intensity measurement.

3. Phase sensitivity in a lossy SU(1,1) interferometer

For any realistic model, one must consider the effect of various noises, including photon loss, dark counts, and thermal photon noise. In this section, we will discuss the model of a lossy SU(1,1) interferometer.
3.1. Photon loss

First we consider the model of photon loss to the environment inside the interferometer. This can be modeled by placing a fictitious beam splitter, of different transmissivity, inside the interferometer with vacuum state and a mode from the interferometer as inputs and tracing over one of the output modes, to mimic loss of photons. Fig. 8 shows the sensitivity of SU(1,1) interferometer with photon loss $L = 1\%$ and photon loss $L = 10\%$. Obviously, one can see that if the mean photon number is larger than about 15, the sensitivity can beat SNL in the case of low photon loss. And from the Fig, we can infer that with enough photons, the phase sensitivity will beat SNL in the case of high photon loss. That means photon loss can be compensated by injecting more input photons.
Fig. 9. Phase sensitivity with parity detection as a function of $n_{th}$ under the condition $r = 2$, $g = 2$ and dark counts $d = 0.01$.

Fig. 10. Phase sensitivity with parity detection as a function of $n_{th}$ under the condition $r = 2$, $g = 2$ and thermal photon noise is $10^{-20}$ or 1.

3.2. Dark counts

Another common realistic factor is dark counts that exist in the photon-number-resolving detector. For a detector with 5000 APD image elements, dark counts of each APD less than 100 c/s and the width of sampling gate as 20 ns, the rate of dark counts can be deduced to be $d = 10^{-2}$ Hz. This low rate of dark counts was realized and reported by [31]. The probability of $n$ dark counts follows the Poisson distribution $P(n) = e^{-d}d^n/n!$, where $d$ is the rate of dark counts. Thus the output signal with dark counts can be rewritten as [32]

$$\langle \hat{\Pi}_b \rangle_{DC} = e^{-2d} \langle \hat{\Pi}_b \rangle.$$  \hspace{1cm} (14)

Fig. 9 shows the sensitivity of SU(1,1) interferometer with the rate of dark counts $d = 0.01$. One can see that dark counts do not have much effect on the sensitivity which is still below SNL in the case of thermal photons $n_{th} \geq 2$.

3.3. Thermal noise

In addition to photon loss and dark counts, we also model the effect with thermal noise from the environment. In this case, we need to consider the environment modes. Similar to the case of photon loss, this process can also be modeled by placing a fictitious beam splitter inside the interferometer. The difference is that the vacuum input of beam splitter is replaced by thermal state in this case. Hence the two input modes turn into four input modes and the four-by-four transformation matrices are replaced by eight-by-eight matrices. The loss can be modeled by adding two fictitious beam splitters on the two arms of the interferometer and the detailed calculation can be found in Appendix C. We plot Fig. 10 with beam splitter transmissivity $T = 99\%$ in the cases of thermal photon noise $N = 10^{-20}$ at room temperature and thermal photon noise $N = 1$ which can be obtained in microwave frequency. The sensitivity in this case is similar to that of the photon loss and dark counts cases. From the Fig., we can see the sensitivity can not approach HL but can beat SNL with enough input thermal photons.
4. Conclusion

We have studied the phase sensitivity of an SU(1,1) interferometer. If two vacuum states are considered at the input, the phase sensitivity can approach HL due to the total photon number inside the interferometer being relatively small. However, if one input state is a thermal state, the phase sensitivity worsens and at best beats the SNL but does not reach the HL. If we replace the vacuum state by a squeezed vacuum state, different strategies will give higher phase sensitivity. These strategies include, increasing photon number of the thermal state or increasing the parametric strength $g$ in the OPA or increasing the squeezed strength $r$ which all achieve higher phase sensitivity and approaches HL. We also calculate the phase sensitivity of thermal states in both inputs or a thermal state in one input and a coherent state in the other input. In both cases, the phase sensitivities even can not beat SNL. Compared with these two cases, a thermal state in one input and a squeezed vacuum state in the other input can get a better phase sensitivity.

A. Parity detection signal

For analysis of our interferometer, we use Wigner functions as our mathematical model. First, we focus on evolution of mean values and covariance matrix of quadrature operators in SU(1,1) interferometer. The quadrature operators can be defined as

\[ \hat{x}_{aj} = \hat{a}_j + \hat{a}_j^\dagger, \quad \hat{p}_{aj} = -i(\hat{a}_j - \hat{a}_j^\dagger), \quad (A1) \]

\[ \hat{x}_{bj} = \hat{b}_j + \hat{b}_j^\dagger, \quad \hat{p}_{bj} = -i(\hat{b}_j - \hat{b}_j^\dagger), \quad (A2) \]

and the quadrature column vector is defined as

\[ X_j = (\hat{X}_{j,1}, \hat{X}_{j,2}, \hat{X}_{j,3}, \hat{X}_{j,4})^T = (\hat{x}_{aj}, \hat{p}_{aj}, \hat{x}_{bj}, \hat{p}_{bj})^T. \quad (A3) \]

The mean and covariances of quadrature operators are given by

\[ \bar{X}_j = (\langle \hat{X}_{j,1} \rangle, \langle \hat{X}_{j,2} \rangle, \langle \hat{X}_{j,3} \rangle, \langle \hat{X}_{j,4} \rangle)^T, \quad (A4) \]

\[ \Gamma_{kl} = \frac{1}{2} \text{Tr}[(\bar{X}_{j,k}\bar{X}_{j,l} + \bar{X}_{j,l}\bar{X}_{j,k})\rho], \quad (A5) \]

where, $\bar{X}_{j,k} = \hat{X}_{j,k} - \langle \hat{X}_{j,k} \rangle$, $\bar{X}_{j,l} = \hat{X}_{j,l} - \langle \hat{X}_{j,l} \rangle$ and $\rho$ is a density matrix of the input state.

Transformations through the first OPA, phase shifter and the second OPA in phase space are described by

\[ S_{\text{OPA1}} = \begin{pmatrix} \cosh g & 0 & \sinh g & 0 \\ 0 & \cosh g & 0 & -\sinh g \\ \sinh g & 0 & \cosh g & 0 \\ 0 & -\sinh g & 0 & \cosh g \end{pmatrix}, \quad (A6) \]

\[ S_{\phi} = \begin{pmatrix} \cos(\frac{\phi}{2}) & -\sin(\frac{\phi}{2}) & 0 & 0 \\ \sin(\frac{\phi}{2}) & \cos(\frac{\phi}{2}) & 0 & 0 \\ 0 & 0 & \cos(\frac{\phi}{2}) & -\sin(\frac{\phi}{2}) \\ 0 & 0 & \sin(\frac{\phi}{2}) & \cos(\frac{\phi}{2}) \end{pmatrix}, \quad (A7) \]
\[
S_{\text{OPA2}} = \begin{pmatrix}
cosh g & 0 & -\sinh g & 0 \\
0 & \cosh g & 0 & \sinh g \\
-\sinh g & 0 & \cosh g & 0 \\
0 & \sinh g & 0 & \cosh g \\
\end{pmatrix},
\]
(A8)

with, \(g_1 = g_2 = g\), \(\phi_1 = \phi_2 = \phi/2\), \(\theta_1 = 0\) and \(\theta_2 = \pi\). Hence, the propagation of \(\vec{X}_0\) and \(\Gamma_0\) through SU(1,1) interferometer is determined by \(S = S_{\text{OPA2}}S_\phi S_{\text{OPA1}}\). And the evolution of mean values and covariance matrix of quadrature operators in SU(1,1) interferometer is given by

\[
\vec{X}_2 = S\vec{X}_0, \quad \Gamma_2 = S\Gamma_0S^T.
\]

(A9) (A10)

We obtain the mean value and covariance matrix of the outputs from the above calculation. Then, with the mean value and the covariance matrix of the lower output, we can get the measurement signal. According to [34], the parity detection is

\[
\langle \hat{\Pi}_b \rangle = \exp(-\vec{X}_2^T \cdot \Gamma_2^{-1} \cdot \vec{X}_2) \sqrt{\det \Gamma_2}.
\]

(A11)

where, \(\vec{X}_{22} = (\hat{X}_{2,3}, \hat{X}_{2,4})^T\) and \(\Gamma_{22} = \begin{pmatrix}
\Gamma_{33} & \Gamma_{34} \\
\Gamma_{43} & \Gamma_{44}
\end{pmatrix}\).

In our scheme, the inputs are thermal and squeezed vacuum state and the initial mean value and covariance matrix of quadrature operators are

\[
\vec{X}_0 = 0, \quad \Gamma_0 = \sigma_{\text{th}} \oplus \sigma_{\text{sqz}},
\]

(A12)

as both of the input states have zero mean and their covariance matrices are

\[
\sigma_{\text{th}} = (2n_{\text{th}} + 1) \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad \sigma_{\text{sqz}} = \begin{pmatrix}
e^{2r} & 0 \\
0 & e^{-2r}
\end{pmatrix}.
\]

(A13) (A14)

According to Eq. (A9) and Eq. (A10), the final mean value and covariance matrix of quadrature operators become

\[
\vec{X}_2 = 0, \quad \Gamma_2 = \begin{pmatrix}
\gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\
\gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \\
\gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} \\
\gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44}
\end{pmatrix},
\]

(A15) (A16)

where,

\[
\gamma_{11} = e^{-2r} \sin^2(\frac{\phi}{2}) \sinh^2(2g) + \frac{1}{4} \left[ 3 + \cosh(4g) - 2 \cos \phi \sinh^2(2g) \right] (1 + 2n_{\text{th}}),
\]

(A17)

\[
\gamma_{13} = \gamma_{31} = \sin^2(\frac{\phi}{2}) \sinh(4g) (-\cosh r + \sinh r) (\cosh r + e^r n_{\text{th}}),
\]

(A18)

\[
\gamma_{14} = \gamma_{41} = e^{-2r} \cosh g \sin \phi \sinh g \left[ 1 + e^{2r} (1 + 2n_{\text{th}}) \right],
\]

(A19)
After plugging the values into Eq. (A11), we get the parity detection signal, which is given as

\[ \gamma_{22} = e^{2r} \sin^2 \left( \frac{\phi}{2} \right) \sinh^2 (2g) + \frac{1}{4} \left[ 3 + \cosh (4g) - 2 \cos \phi \sinh^2 (2g) \right] (1 + 2n_{th}), \]  
(A20)

\[ \gamma_{23} = \gamma_{32} = \cos g \sin \phi \ sinh g \left( 1 + e^{2r} + 2n_{th} \right), \]  
(A21)

\[ \gamma_{24} = \gamma_{42} = \frac{1}{2} \sin^2 \left( \frac{\phi}{2} \right) \sinh (4g) \left( 1 + e^{2r} + 2n_{th} \right), \]  
(A22)

\[ \gamma_{33} = \frac{1}{2} e^{2r} (1 + \cos \phi) + e^{-2r} \cosh^2 (2g) \sin^2 \left( \frac{\phi}{2} \right) + \sin^2 \left( \frac{\phi}{2} \right) \sinh^2 (2g) (1 + 2n_{th}), \]  
(A23)

\[ \gamma_{34} = \gamma_{43} = \cosh (2g) \sin \phi \ \sinh (2r), \]  
(A24)

\[ \gamma_{44} = \frac{1}{2} e^{-2r} (1 + \cos \phi) + e^{2r} \cosh^2 (2g) \sin^2 \left( \frac{\phi}{2} \right) + \sin^2 \left( \frac{\phi}{2} \right) \sinh^2 (2g) (1 + 2n_{th}). \]  
(A25)

With Eq. (A15) and Eq. (A16), we know the mean value and the covariance matrix of the lower output \( b \) which is given as

\[ \mathbf{X}_{22} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{\Gamma}_{22} = \begin{pmatrix} \gamma_{33} & \gamma_{34} \\ \gamma_{43} & \gamma_{44} \end{pmatrix}. \]  
(A26)

After plugging the values into Eq. (A11), we get the parity detection signal, which is given as

\[ \langle \Pi_b \rangle = \frac{8}{\sqrt{T}}, \]  
(A27)

where,

\[ T = e^{-2r} \{ -7 + 50e^{2r} - 7e^{4r} + (1 + e^{2r})^2 \{ 4 \cosh (4g) + 3 \cosh (8g) + 8 \cos (2\phi) \sinh^4 (2g) \\
- 8 \cos \phi \sinh^2 (4g) \} + 32e^{-2r} \sin^2 \left( \frac{\phi}{2} \right) \sinh^2 (2g) n_{th} \{(1 + e^{4r})[3 + \cosh (4g) \\
- 2 \cos \phi \sinh^2 (2g)] + 8e^{2r} \sin^2 \left( \frac{\phi}{2} \right) \sinh^2 (2g) (1 + n_{th}) \} \}. \]  
(A28)

If \( \phi = 0 \), the outputs will be same as the inputs and the signal will reduce to \( \langle \Pi_b \rangle = 1 \) which coincides with the theory that for one-mode squeezed vacuum, the parity signal is 1.

## B. Phase Estimation

According to Eq. (11), Eq. (A11) and Eq. (A26), one can calculate the phase sensitivity via parity detection with thermal and squeezed vacuum as input states in SU(1,1) interferometer using

\[ \Delta \phi = \left. \frac{\Delta \hat{\Pi}_b}{\partial \langle \Pi_b \rangle / \partial \phi} \right|_{\phi}, \]  
(B1)

where,

\[ \Delta \hat{\Pi}_b = \{ 1 - 64 / (e^{-2r} \{ -7 + 50e^{2r} - 7e^{4r} + (1 + e^{2r})^2 \{ 4 \cosh (4g) + 3 \cosh (8g) + 8 \cos (2\phi) \sinh^4 (2g) \\
- 8 \cos \phi \sinh^2 (4g) \} + 32e^{-2r} \sin^2 \left( \frac{\phi}{2} \right) \sinh^2 (2g) n_{th} \{(1 + e^{4r})[3 + \cosh (4g) - 2 \cos \phi \sinh^2 (2g)] \\
+ 8e^{2r} \sin^2 \left( \frac{\phi}{2} \right) \sinh^2 (2g) (1 + n_{th}) \}) \} \}^{1/2}. \]  
(B2)
\[ \frac{\partial \langle \hat{\Omega}_b \rangle}{\partial \phi} = -128 \sinh^2(2g) [-2 \sin(2\phi) \sinh^2(2g)] (\cosh^2 r + n_{th} (1 + \cosh(2r) + n_{th}) ) + \sin(\phi) (4 \cosh^2(2g) \cosh^2 r + 4 n_{th} [\cosh^2(2g) \cosh(2r) + \sinh^2(2g) (1 + n_{th}) ] ) \} / \{ e^{-2r} ( -7 + (1 + e^{2r}) [4 \cosh(4g) + 3 \cosh(8g) + 8 \cos(2\phi) \sinh^2(2g) - 8 \cos(\phi) \sinh^2(4g)] - 7 e^{4r} + 50 e^{2r} + 32 \sin^2(\frac{\phi}{2}) \sinh^2(2g)(1 + e^{4r}) (3 + \cosh(4g) - 2 \cos(\phi) \sinh^2(2g)) + 8 e^{2r} \sin^2(\frac{\phi}{2}) \sinh^2(2g)(1 + n_{th}) ] \}^{3/2}. \] 

One can verify that the phase sensitivity with \( \phi = 0 \) is given as in Eq. (12).

\section*{C. PHASE ESTIMATION WITH THERMAL NOISE}

The matrices of inputs, OPAs, phase shifter and the virtual BS in thermal noise coupling situation are

\[ \Gamma_0^* = \left( \begin{array}{cc} \Gamma_0 & 0_4 \\ 0_4 & \sigma_{th} \oplus \sigma_{th} \end{array} \right)_{8 \times 8}, \]  

\[ S_{\text{OPA}_i}^* = \left( \begin{array}{cc} S_{\text{OPA}_i} & 0_4 \\ 0_4 & I_4 \end{array} \right)_{8 \times 8}, \]  

\[ S_\phi^* = \left( \begin{array}{cc} S_\phi & 0_4 \\ 0_4 & I_4 \end{array} \right)_{8 \times 8}, \]  

\[ S_{\text{VBS}}^* = \left( \begin{array}{cc} \sqrt{T} I_4 & \sqrt{1 - T} I_4 \\ \sqrt{1 - T} I_4 & -\sqrt{T} I_4 \end{array} \right)_{8 \times 8}, \]

where 0_4 is four-by-four zero matrix and I_4 is four-by-four identity matrix. Hence, the whole the propagation of \( \bar{X}_0^* \) and \( \Gamma_0^* \) through SU(1,1) interferometer is determined by \( S^* = S_{\text{OPA}_2}^* S_{\text{VBS}}^* S_{\phi}^* S_{\text{OPA}_1}^* \). And the transformation relation are

\[ \bar{X}_2^* = S \bar{X}_0^*, \]  

\[ \Gamma_2^* = S \Gamma_0^* (S^*)^T, \]

where \( \bar{X}_0^* \) is a zero vector since both thermal state and squeezed vacuum state have a zero mean vector.

Similarly with the ideal case, the parity detection is given by

\[ \langle \hat{\Omega}_b \rangle^* = \frac{\exp(-\bar{X}_{22}^* \Gamma_{22}^{-1} \cdot \bar{X}_{22}^*)}{\sqrt{|\Gamma_{22}^*|}}, \]

where \( \bar{X}_{22}^* = (0,0)^T \) and \( \Gamma_{22}^* = \left( \begin{array}{cc} \Gamma_{22}^{33} & \Gamma_{22}^{34} \\ \Gamma_{22}^{43} & \Gamma_{22}^{44} \end{array} \right) \).

\section*{Funding}

National Natural Science Foundation of China (NSFC) (61575180, 61701464, 11475160, 61640009); Natural Science Foundation of Shandong Province (ZR2014AQ026, ZR2014AM023); Economic Development Assistantship from Louisiana State University System Board of Regents; Air Force Office of Scientific Research (AFOSR); Army Research Office (ARO); Defense Advanced Research Project Agency (DARPA); National Science Foundation (NSF).