INFINITE DIMENSIONAL ANALYSIS, REALIZATIONS OF INFINITE PRODUCTS, RUELLE OPERATORS AND WAVELET FILTERS

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Abstract. Using the notions and tools from realization in the sense of systems theory, we establish an explicit and new realization formula for families of infinite products of rational matrix-functions of a single complex variable. Our realizations of these resulting infinite products have the following four features: 1) Our infinite product realizations are functions defined in an infinite-dimensional complex domain. 2) Starting with a realization of a single rational matrix-function $M$, we show that a resulting infinite product realization obtained from $M$ takes the form of an (infinite-dimensional) Toeplitz operator with a symbol that is a reflection of the initial realization for $M$. 3) Starting with a subclass of rational matrix functions, including scalar-valued corresponding to low-pass wavelet filters, we obtain the corresponding infinite products that realize the Fourier transforms of generators of $L_2(\mathbb{R})$ wavelets. 4) We use both the realizations for $M$ and the corresponding infinite product to produce a matrix representation of the Ruelle-transfer operators used in wavelet theory. By “matrix representation” we refer to the slanted (and sparse) matrix which realizes the Ruelle-transfer operator under consideration. We also consider relations with Schur-Agler classes in an infinite number of variables.

1. Introduction

Among many applications of rational matrix valued functions are their use as filters in signal processing, and in the construction of classes of wavelets. In the latter case,

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the matrix function to be considered is made up of a prescribed system of scalar valued functions of a single complex variable. If \( N \) is a scaling number for the wavelet under consideration, then there are associated systems of \( N \) scalar valued functions represent each of the corresponding \( N \) frequency bands. Each such system produces a matrix valued function. This particular approach to wavelet filters was considered in [3] [8] [4] [3] [2]. The function corresponding to low-pass yields a father function for a wavelet when certain technical assumptions are imposed. Here we consider instead the matrix valued approach: it has the advantage that it allows one to treat the combination of individual bands in a single analysis. However the issues involving infinite products in the matrix valued case are more subtle, and we address them below. For example, to understand the infinite product formed from a rational matrix valued function of a single complex variable, one must introduce an infinite number of complex variables. We show that, under suitable assumptions, the infinite product-function in turn then also has a realization as a function of one variable. While our motivation derived initially from the study of wavelet filters, we note that there is a host of other applications of infinite products of rational matrix functions. Indeed the framework for our consideration of our infinite products goes beyond that of wavelet filters. We shall consider these more general settings in the last section of our paper. The latter non-wavelet applications are derived from the theory of systems. Indeed the theory of realization of systems is also a key tool in our analysis of infinite products.

In this setting, one refers to a state space model for an input-output formulation of a system under consideration. That is, given an input-output relationship for variables in a given system, a realization is a quadruple \( A, B, C, \) and \( D \) of time-varying, or time independent matrices. Here we limit our discussion to discrete time systems, so where the variables, input, output, and state, are time series, that is, functions on \( \mathbb{Z} \). A linear time-invariant system in this model will then be specified by a transfer matrix \( M(z) \), also called a transfer function. It is a rational matrix-valued function of a single complex variable. Moreover the complex variable \( z \) is dual to time, and so it represents frequency. A realization is then any system of four matrices \( A, B, C, \) and \( D \) of appropriate size, not necessarily square matrices such that

\[
M(z) = D + C(zI - A)^{-1}B
\]

holds.

In [3] we characterized wavelet filters as functions of the form

\[
M(z) = U(z^N)D(z)M_0
\]  

(1.1)

where we set (with \( \epsilon_N = e^{i\frac{2\pi}{N}} \))

\[
M_0 = \frac{1}{\sqrt{N}} \left( \epsilon_N^{-\ell j} \right)_{\ell,j=0,...,N-1}, \quad D(z) = \text{diag} (1, z^{-1}, \ldots, z^{-(N-1)}),
\]

and where \( U \) is a rational \( \mathbb{C}^{N \times N} \)-valued function which takes unitary values on the unit circle, with no poles outside the unit circle. We here take the normalization by
$U(1) = I_N$. To explain this, write $M$ as

$$M(z) = \begin{pmatrix}
m_0(z) & m_0(\epsilon_N z) & \cdots & m_0(\epsilon_{N-1}^N z) \\
m_1(z) & m_1(\epsilon_N z) & \cdots & \\
\vdots & \vdots & \ddots & \\
m_{N-1}(z) & m_{N-1}(\epsilon_N) & \cdots & m_{N-1}(\epsilon_{N-1}^N z)
\end{pmatrix} \quad (1.2)$$

and

$$\begin{pmatrix}
m_0(z) \\
m_1(z) \\
\vdots \\
m_{N-1}(z)
\end{pmatrix} = U(z^N) \begin{pmatrix}1 \\
\vdots \\
z^{-(N-1)}
\end{pmatrix} \quad (1.3)$$

The normalization $U(1) = I_N$ forces that $m_0(1) = 1$. This last condition is crucial to consider infinite products. Since multiplication on the left by a unitary constant does not change the property of being a wavelet filter, we find it convenient here to consider filters of the form

$$M(z) = M_0^{-1} U(z^N) D(z) M_0 \quad (1.4)$$

rather than (1.1). This forces the condition $M(1) = I_N$ and in particular $m_k(\epsilon_N) = 1$. This corresponds to $m_k$ being the filter that passes the $k$ band component of input signals. Set We set:

$$m(z) = m_0(z).$$

To follow the engineering literature we will assume that $M$ is analytic at infinity. The wavelet father function $\varphi(w)$ is given by its Fourier transform

$$\hat{\varphi}(w) = \prod_{k=1}^{\infty} m(e^{\frac{2\pi i w}{N^k}}) \quad (1.5)$$

For details, see e.g. [8] and [9].

We assume that $M$ in (1.2), (1.4) is rational and for $m(z)$, its upper left entry we introduce a realization centered at infinity

$$m(z) = D + C(zI - A)^{-1} B$$

where we can assume that the realization is minimal, and that in particular $A$ has no spectrum on the unit circle since $M$ is analytic on the unit circle. Our use of the term realization, conforms to its common use in the theory of system from the study of dynamical systems and filters in engineering, and pioneered by Kalman and others; see [16, 15, 18].

The paper consists of eight sections besides the introduction, and its outline is as follows. In Section 2 we review some facts on rational functions and their realizations. Finite products of rational functions, each of a different variable, are considered in Section 3. Infinite products are considered in Section 4. As we will see in this section an important role is played by the Toeplitz operator with related symbol equal to

$$A + zB(I - zD)^{-1} C. \quad (1.6)$$

In Section 5 we compute the Markov parameter associated to $|m(z)|^2$ in terms of the given realization of $m$. In Section 6 we study the Ruelle operator and connections with
2. Rational function

As is well known, see e.g. [7], every \( C^{p \times q} \)-valued rational \( R(z) \) function analytic at infinity can be written as

\[
R(z) = D + C(zI - A)^{-1}B,
\]

for matrices \( A, B, C \) and \( D \) of appropriate sizes. Equation (2.1) is called a realization of \( R \). It is highly non unique.

In simple cases from linear algebra when \( M(z) \) is scalar valued, i.e., is a fraction of two polynomials, a minimal realization may be constructed from the companion matrix of the polynomial in the denominator.

The matrices in realizations for \( M(z) \) are typically of a larger size than that of the initial matrix-valued function \( M(z) \). To appreciate this fact, note that increase in matrix size obviously holds in all non-trivial cases when the initial transfer function \( M(z) \) is scalar valued, so when \( M(z) \) is a given rational complex function. Nonetheless, for general rational matrix functions, the matrix \( A \) in a realization must be a square matrix, but not necessarily the others, \( B, C, \) and \( D \). It is known that minimal realizations exist. Depending on the context there is a host of tools and algorithms available for finding realizations. The size of the matrix \( A \) in a minimal realization coincides with a certain winding number for \( M(z) \), and it is called the McMillan degree; see [7].

When \( d \) is minimal, the realization is then unique up to a similarity matrix, meaning that the only freedom in the choice of the realization is

\[
\begin{pmatrix}
A \\
C
\end{pmatrix} \to 
\begin{pmatrix}
T & 0 \\
0 & I_p
\end{pmatrix} 
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} 
\begin{pmatrix}
T^{-1} & 0 \\
0 & I_q
\end{pmatrix},
\]

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T & 0 \\
0 & I_p
\end{pmatrix} 
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} 
\begin{pmatrix}
T^{-1} & 0 \\
0 & I_q
\end{pmatrix},
\]

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where \( n = n_1 + n_2 \) and

\[
A = \begin{pmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{pmatrix}
\]  
\[
B = \begin{pmatrix} B_1 D_2 \\ B_2 \end{pmatrix}
\]  
\[
C = (C_1 \ D_1 C_2)
\]  
\[
D = D_1 D_2.
\]

An important tool in our argument is the counterpart of (2.3) when each function depends on a different variable. See Lemma 3.2.

3. Finite products

Factorization of rational matrix-valued functions of one variable is classical. In contrast, factorization theory of rational functions of several complex variables is not well developed. However, here we will consider rational functions of the form

\[
M(z) = \prod_{j=1}^{u} m_j(z_j)
\]

where \( m_1, \ldots, m_u \) are matrix-valued rational functions of appropriate sizes and analytic at infinity.

For future reference we mention the following result, whose proof is a direct verification, and will be omitted.

**Lemma 3.1.** Let \( H_1 \) and \( H_2 \) be two Hilbert spaces and let \( a : H_1 \to H_1, b : H_2 \to H_2, \) and \( c : H_2 \to H_1 \) be bounded linear operators, with \( a \) and \( b \) invertible. Then

\[
\begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & a^{-1} c b^{-1} \\ 0 & b^{-1} \end{pmatrix}.
\]

The following very simple lemma is the key to the formulas we develop:

**Lemma 3.2.**

\[
(D_1 + C_1(z_1 I_{n_1} - A_1)^{-1} B_1) \ (D_2 + C_2(z_2 I_{n_2} - A_2)^{-1} B_2) = D + C(\Lambda_2(z) - A)^{-1} B
\]

where \( n = n_1 + n_2 \) and

\[
\Lambda_2(z) = \begin{pmatrix} z_1 I_{n_1} & 0 \\ 0 & z_2 I_{n_2} \end{pmatrix}
\]  
\[
A = \begin{pmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{pmatrix}
\]  
\[
B = \begin{pmatrix} B_1 D_2 \\ B_2 \end{pmatrix}
\]  
\[
C = (C_1 \ D_1 C_2)
\]  
\[
D = D_1 D_2.
\]
Proof. We first note that by Lemma 3.1 we have:
\[
(A_2(z) - A)^{-1} = \begin{pmatrix}
(z_1I_{n_1} - A_1 & -B_1C_2 \\
0 & z_2I_{n_2} - A_2
\end{pmatrix}^{-1} = \begin{pmatrix}
(z_1I_{n_1} - A_1)^{-1} & (z_1I_{n_1} - A_1)^{-1}B_1C_2(z_2I_{n_2} - A_2)^{-1} \\
0 & (z_2I_{n_2} - z_2A_2)^{-1}
\end{pmatrix}.
\]
Therefore we have
\[
D + C(A_2(z) - A)^{-1}B = D_1D_2 + (C_1D_1C_2) \left( (z_1I_{n_1} - z_1A_1)^{-1} (z_1I_{n_1} - A_1)^{-1}B_1C_2(z_2I_{n_2} - A_2)^{-1} \right) (B_1D_2).\]
which is exactly the left side of (3.2). □

This formula can now be iterated to obtain a realization for a product (3.1). With
\[
r_j(z) = D_j + C_j(zI_{n_j} - A_j)^{-1}B_j, \quad j = 1, \ldots, u,
\]
and \(M(z) = r_1(z) \cdots r_u(z)\) we have
\[
M(z) = D + C(A(z) - A)^{-1}B,
\]
where
\[
A(z) = \text{diag} (z_1I_{n_1}, z_2I_{n_2}, \ldots, z_uI_{n_u}), \quad D = D_1D_2 \cdots D_u = M(\infty),
\]
where \(M(\infty) = \lim_{z \to \infty} M(z)\), and
\[
A = \begin{pmatrix}
A_1 & B_1C_2 & B_1D_2C_3 & \cdots & (B_1D_2 \cdots D_{u-1}C_u) \\
0 & A_2 & B_2C_3 & \cdots & (B_2D_3 \cdots D_{u-1}C_u) \\
0 & 0 & A_3 & \cdots & \text{ } \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & A_u
\end{pmatrix}
\]
(3.8)
\[
B = \begin{pmatrix}
(B_1D_2 \cdots D_u) \\
\vdots \\
B_{u-1}D_u \\
B_u
\end{pmatrix}
\]
\[
C = (C_1D_1C_2 \ D_1D_2C_3 \ \cdots \ (D_1 \cdots D_{u-1}C_u)).
\]
In other words, with \(A = (A_{ij})\) the natural block decomposition of \(A\), we have \(A_{jj} = A_j\) and the \((i, j)\) blocks of \(A\) with \(i < j\) is
\[
A_{ij} = B_i \left( \prod_{a=i+1}^{j-1} D_a \right) C_j,
\]
with the understanding that \(\prod_{a=i+1}^{j-1} = I\) when \(i + 1 > j - 1\). (The notation \(\cap\) means ordered product).

We note that the case where all the functions vanish at infinity leads to very simple formulas, which we gather in the following lemma.
Lemma 3.3. It holds that
\[ C_1(z_1I_{n_1} - A_1)^{-1}B_1C_2(z_2I_{n_2} - A_2)^{-1}B_2 \cdots C_u(z_uI_{n_u} - A_u)^{-1}B_u = C(Λ(z) - A)^{-1}B, \]
where
\[
A = \begin{pmatrix}
A_1 & B_1C_2 & 0 & \cdots & 0 \\
0 & A_2 & B_2C_3 & \cdots & 0 \\
& \vdots & \ddots & \ddots & \ddots \\
0 & & & \ddots & B_{u-1}C_{u} \\
0 & & & & A_u
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
0 \\
\vdots \\
0 \\
B_u
\end{pmatrix},
\]
\[
C = (C_1 & 0 & 0 & \cdots & 0).
\]

4. INFINITE PRODUCTS

While the framework of realizations is typically formulated for finite matrices, (as we point out below) a number of the results make sense for infinite matrices, hence for linear operators in Hilbert space. A case in point is the realizations we obtain now for our infinite products. We now wish to let \( u \to \infty \) in (3.8) when all the functions \( r_j \) coincide:
\[
r_1(z) = r_2(z) = \cdots = m(z),
\]
where \( m \) is a rational function analytic at infinity, with realization \( m(z) = D + C(zI - A)^{-1}B \). We assume that \( |m(z)| \leq 1 \) for all \( z \in \mathbb{T} \).

Theorem 4.1.
(i) Assume that
\[
\lim_{k \to \infty} \|D^k\|^{1/k} < 1. \tag{4.1}
\]
Then, the operators
\[
\mathcal{A} = \begin{pmatrix}
A & BC & BDC & BD^2C & \cdots \\
0 & A & BC & BDC & \cdots \\
0 & 0 & A & BC & \cdots \\
& \vdots & \ddots & \ddots & \ddots \\
& & & \ddots & BD_u \\
& & & & B
\end{pmatrix}, \quad \ell_2(\mathbb{N}) \otimes \mathbb{C}^m \to \ell_2(\mathbb{N}) \otimes \mathbb{C}^m \tag{4.2}
\]
\[
\mathcal{B} = \begin{pmatrix}
BD^2 \\
BD \\
B
\end{pmatrix}, \quad \mathbb{C}^m \to \ell_2(\mathbb{N}) \otimes \mathbb{C}^m, \tag{4.3}
\]
\[
\mathcal{C} = \begin{pmatrix}
C & DC & D^2C & \cdots \\
\end{pmatrix}, \quad \ell_2(\mathbb{N}) \otimes \mathbb{C}^m \to \mathbb{C}^m, \tag{4.4}
\]
are bounded.
(ii) \( \mathcal{A} \) in (4.2) is the block-Toeplitz operator with symbol
\[
A + zB(I - zD)^{-1}C.
\]
Proof. We have
\[ A + zB(I - zD)^{-1}C = A + zBC + z^2BDC + z^3BD^2C + \cdots, \]
and hence the function \( \phi(z) = A + zB(I - zD)^{-1}C \) is the symbol of the block Toeplitz operator (4.2). We note that, in view of (4.1)
\[ \|A\| = \|A + zB(I - zD)^{-1}C\| < \infty, \] (4.5)
and so the block Toeplitz operator \( A \) is bounded. We use the fact that a block Toeplitz operator with symbol \( \phi(z) \) has norm \( \|\phi\|_\infty \); see [12]. □

Remark 4.2. The same result holds \textit{mutatis mutandis} when the alternative realization \( m(z) = D + zC(I - zA)^{-1}B \) is chosen.

Remark 4.3. We note that
\[ B = O(D, B) \quad \text{and} \quad C = C(D, C) \]
in (4.3) and (4.4) are the observability and controlability operators (see [18]).

Theorem 4.4. Assume that \( m \) has no singularity at the point \( z = 1 \) and that \( m(1) = 1 \), and let \( (z_k)_{k \in \mathbb{N}_0} \) be a sequence of complex numbers which are not poles of \( m \) and such that
\[ \sum_{k=0}^{\infty} |1 - z_k| < \infty. \] (4.6)

Then it holds that
\[ \prod_{k=1}^{\infty} m(z_k) = C(\Lambda(z) - A)^{-1}B \] (4.7)
where \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) are defined by (4.2)-(4.4), and where
\[ \Lambda(z) = \text{diag}(z_1I_n, z_2I_n, \ldots). \]

Proof. Since the realization of \( m \) is assumed minimal, 1 is not in the spectrum of \( A \) and we have \( m(1) = D + C(I_n - A)^{-1}B \). Thus
\[ m(z) - m(1) = D + C(zI_n - A)^{-1}B - D - C(I_n - A)^{-1}B \]
\[ = C(zI_n - A)^{-1}B - C(I_n - A)^{-1}B \]
\[ = (1 - z)C(zI - A)^{-1}(I - A)^{-1}B. \]

Furthermore, (4.6) implies in particular that \( \lim_{k \to \infty} z_k = 1 \). Let
\[ K_0 = \max_{z \in V} \left\{ \|C(zI_n - A)^{-1}(I_n - A)^{-1}B\| \right\}, \]
where \( V \) is a closed neighborhood of 1 in which \( m \) has no pole, and let
\[ K = \max_{\text{where the } z_u \not\in V} \left\{ K_0, \|C(z_uI_n - A)^{-1}(I_n - A)^{-1}B\| \right\}. \]

Then we have
\[ |m(z_k) - 1| \leq K \cdot |1 - z_k| \]
and hence the result. □
Corollary 4.5. Assume that \( m(1) = 1 \), and let \((\theta_k)\) be a sequence of numbers on the real line that
\[
\sum_{k=0}^{\infty} |\theta_k| < \infty.
\]
Then the infinite product
\[
\prod_{k=0}^{\infty} m(e^{i\theta_k}), \quad t \in \mathbb{R},
\]
converges for all real \( t \).

Proof. Since \( |e^{i\theta} - 1| \leq |\theta| \) for \( \theta \) real we have
\[
|m(e^{i\theta_k}) - 1| \leq K_1 \cdot |\theta_k|
\]
where now we can take
\[
K_1 = \max_{\theta \in [0, 2\pi]} \|C(e^{i\theta}I_n - A)^{-1}(I_n - A)^{-1}B\|,
\]
and hence the result. \( \square \)

Corollary 4.6. In the notation and hypothesis of the previous proposition, the product
\[
\prod_{k=0}^{\infty} m(e^{i2\pi w N_k})
\]
converges for every \( w \in \mathbb{R} \).

5. Markov parameters

Let \( m \) be analytic in the exterior of the closed unit disk, with minimal realization \( m(z) = D + C(zI - A)^{-1}B \). In particular \( \sigma(A) \subset \mathbb{D} \). Let \( h_0 + \sum_{k=1}^{\infty} \frac{h_k}{z^k} \) be the Laurent expansion at infinity of \( m \). The coefficients \( h_1, \ldots \) are called the Markov parameters of \( m \). They are given by \( h_0 = D \) and
\[
h_k = CA^{k-1}B, \quad k = 1, 2, \ldots
\]
(5.1)

We extend the sequence \( h_k \) by
\[
h_u = 0, \quad u < 0.
\]
We assume that the spectral radius of \( A \) is strictly less than 1 and set
\[
\Gamma = \sum_{u=0}^{\infty} A^u C^* C A^u.
\]
(5.2)

Note that \( \Gamma \) is called the observability Gramian, and is the unique solution of the matrix equation (usually called a Stein equation)
\[
\Gamma - A^* \Gamma A = C^* C.
\]
We set
\[
Y = (D^* C + B^* \Gamma A).
\]
(5.3)

In view of the next result we recall that a rational function \( r \) with no poles on the unit circle belongs to the Wiener algebra of the disk, that is can be written as
\[
r(z) = \sum_{n \in \mathbb{Z}} z^n r_n.
\]
where $\sum_{n \in \mathbb{Z}} |r_n| < \infty$. See for instance [13, Corollary 3.2].

**Theorem 5.1.** Let $(c_n)_{n \in \mathbb{Z}}$ be defined by

$$|m(z)|^2 = \sum_{n \in \mathbb{Z}} c_n z^n, \quad z \in \mathbb{T}.$$  \hfill (5.4)

Then,

$$c_n = \sum_{j \in \mathbb{Z}} h_j^* h_{j+n}, \quad n \in \mathbb{Z}.$$  \hfill (5.5)

**Proof.** For $n = 0$ we have

$$c_0 = D^* D + \sum_{k=1}^{\infty} B^* A^*(k-1) C^* C A^{k-1} B$$

$$= D^* D + B^* \left( \sum_{k=1}^{\infty} A^*(k-1) C^* C A^{k-1} \right) B$$

$$= D^* D + B^* \Gamma B,$$

where $\Gamma$ is the Gramian matrix from (5.2).

We now assume $n < 0$. Then,

$$c_n = h_0^* h_n + \sum_{k=1}^{\infty} h_k^* h_{k+n}$$

$$= D^* C A^{n-1} B + \sum_{k=1}^{\infty} B^* A^{k-1} C^* C A^{k+n-1} B$$

$$= D^* C A^{n-1} B + B^* \Gamma A^n B.$$

Finally, for $n < 0$, we have:

$$c_n = \sum_{k=-n}^{\infty} h_k^* h_{k+n}$$

$$= \sum_{u=0}^{\infty} h_u^* h_{u-n}$$

$$= B^* A^{*(n-1)} C^* D + \sum_{u=1}^{\infty} B^* A^{*(u-n-1)} C^* C A^{u-1} B$$

$$= B^* A^{*(n-1)} C^* D + B^* A^{*(-n)} \Gamma B.$$

$\square$
Corollary 5.2. Let $d$ be the size of $A$ (that is, $A \in \mathbb{C}^{d \times d}$). Let $Y$ be defined by \eqref{5.3} and

$$C_L(A, B) = (B \ AB \ A^2B \ \cdots \ A^{d-1}B)$$

Then

$$c_n = Y A^{n-1} B, \quad n = 0, 1 \ldots$$ (5.6)

and

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_d \end{pmatrix} = Y \cdot C_L(A, B).$$ (5.7)

Theorem 5.3. There exists complex numbers $a_0, \ldots, a_{d-1}$ such that

$$a_0 c_1 + a_1 c_2 + \cdots + a_{d-1} c_d + c_{d+1} = 0,$$ (5.8)

and more generally, for any $p \geq 1,$

$$a_0 c_p + a_1 c_{p+1} + \cdots + a_{d-1} c_{d+p-1} + c_{d+p} = 0.$$ (5.9)

Proof. By the Cayley-Hamilton theorem there exists numbers $a_0, a_1, \ldots, a_{d-1}$ such that

$$a_0 + a_1 A + a_2 A^2 + \cdots + a_{d-1} A^{d-1} + A^d = 0.$$ It then follows from (5.6) that we have (5.8). □

Formulas (5.5) take a simpler form in a number of cases, which we mention as remarks:

Remark 5.4. If $A$ is nilpotent (and then $m$ is a polynomial in $1/z$) we have

$$c_{d+1} = c_{d+2} = \cdots = 0.$$

Remark 5.5. Assume $D = 0.$ Then (5.5) become:

$$c_n = \begin{cases} B^* A^{(-n)} \Gamma B, & n \leq 0, \\ B^* \Gamma A^n B, & n > 0. \end{cases}$$ (5.10)

Remark 5.6. We recall that the observability Gramian $\Gamma$ is invertible if and only if the pair $(C, A)$ is observable, meaning that

$$\cap_{u=1}^{d-1} A^u = \{0\}$$

Then one assume $\Gamma = I_d$ by taking $T = \Gamma^{1/2}$ as similarity matrix in \eqref{2.2}. When furthermore $D = 0$ we then have

$$c_n = \begin{cases} B^* A^{(-n)} B, & n \leq 0, \\ B^* A^n B, & n > 0. \end{cases}$$ (5.11)

Proposition 5.7. Assume $m$ rational. Then the coefficients $c_n$ satisfy the estimates of the form

$$|c_k| \leq C e^{-\alpha |k|}, \quad k \in \mathbb{Z}.$$ (5.12)

for every $\alpha > \rho(A).$
6. RUELLE OPERATOR

The Ruelle operator (or transfer operator) is defined by
\[(Rf)(z) = \sum_{w \in T} \sum_{w^N = z} \sum_{m} (m(w))^2 f(w).\] (6.1)

See [8, p. 156]. In terms of the coefficients (5.4) in Theorem 5.1 it is the operator between appropriate subspaces of \(\ell_2(\mathbb{Z})\) and with matrix representation
\[r_{\ell,j} = c_{N\ell-j},\]
that is
\[(Rf)_{\ell} = \sum_{k \in \mathbb{Z}} c_{N\ell-k} f_k.\] (6.2)

Example 6.1. When \(N = 2, D = 0\) and \(\Gamma = I_d\) the matrix representation of the Ruelle operator is
\[
\begin{pmatrix}
\cdots & B^* B & B^* AB & B^* A^2 B & B^* A^3 B & B^* A^4 B & \cdots \\
\cdots & B^* A^2 B & B^* AB & B^* A^3 B & B^* A^4 B & B^* A^5 B & \cdots \\
\cdots & B^* A^3 B & B^* A^2 B & B^* AB & B^* A^4 B & B^* A^5 B & \cdots \\
\cdots & B^* A^4 B & B^* A^3 B & B^* A^2 B & B^* AB & B^* A^4 B & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
where the box denotes the \((0, 0)\) element.

Proof. See (5.11). \(\square\)

Let
\[\mathcal{E}_r = \left\{ (f_n)_{n \in \mathbb{Z}} : \sum_{n \in \mathbb{Z}} e^{r|n|} |f_n| < \infty \right\}\]
and
\[\mathcal{E}_r^{(2)} = \left\{ (f_n)_{n \in \mathbb{Z}} : \|f\|_{r,1}^2 = \sum_{n \in \mathbb{Z}} e^{r|n|} |f_n|^2 < \infty \right\}\]
See [19], [8, p. 158] for these last spaces. We note that an element of \(\mathcal{E}_r\) satisfies an estimate of the form
\[|f_n| \leq Ke^{-r|n|}, \quad n \in \mathbb{Z},\] (6.3)
for some \(K > 0\). More precisely, one has the following lemma, whose elementary proof will be omitted.

Lemma 6.2. Let
\[\mathcal{F}_r = \left\{ (f_n)_{n \in \mathbb{Z}} : \exists C > 0, \text{ such that } |f_n| \leq Ce^{-r|n|}, \forall n \in \mathbb{Z} \right\}\]
Then
\[\mathcal{E}_r^{(2)} \subset \mathcal{F}_r \subset \mathcal{E}_{r-\epsilon}, \quad \forall \epsilon > 0.\]

Theorem 6.3. Assume that the coefficients \(c_n\) satisfy the estimate (5.12). Then for every choice of \(\beta\) and \(\beta'\) such that
\[\alpha < \beta \quad \text{and} \quad \beta' < N\alpha,\] (6.4)
the Ruelle operator is continuous from \(\mathcal{E}_\beta\) into \(\mathcal{E}_{\beta'}\) and from \(\mathcal{E}_\beta^{(2)}\) into \(\mathcal{E}_{\beta'}^{(2)}\).
Proof. Let \( f = (f_n)_{n \in \mathbb{Z}} \) be an element of \( E_\beta \), with \( \beta \) as in (6.3). The \( \ell \) component of the vector \( Rf \) is given by (6.2),

\[
(Rf)_\ell = \sum_{k \in \mathbb{Z}} c_{N\ell-k} f_k
\]

and, using (5.12) and (6.3) can be bounded as:

\[
|(Rf)_\ell| \leq \sum_{k \in \mathbb{Z}} |c_{N\ell-k}| \cdot |f_k| \\
\leq \sum_{k \in \mathbb{Z}} Ce^{-\alpha|N\ell-k|} \cdot e^{-\beta|k|} \\
\leq \sum_{k \in \mathbb{Z}} CKe^{-\alpha|N\ell-k|+|k|} \cdot e^{(\alpha-\beta)|k|} \\
\leq \sum_{k \in \mathbb{Z}} CKe^{-\alpha|N\ell|} \sum_{k \in \mathbb{Z}} e^{(\alpha-\beta)|k|}. 
\]

Hence for any \( \beta' < N\alpha \),

\[
\sum_{z} |(Rf)_\ell| \cdot e^{\beta'|\ell|} < \infty,
\]

and so \( Rf \in E_{\beta'} \).

Let now \( \epsilon > 0 \) and \( f, g \in E_\beta \) and such that \( \|f - g\|_{\beta, 1} < \epsilon \). Then the above argument shows that

\[
\|Rf - Rg\|_{\beta', 1} < K_1 \epsilon,
\]

where the constant

\[
K_1 = K\left(\sum_{\ell \in \mathbb{Z}} e^{-\alpha|N\ell|}\right)\left(\sum_{k \in \mathbb{Z}} e^{(\alpha-\beta)|k|}\right)
\]

is independent of \( f \) and \( g \), and the continuity of \( R \) follows.

The case of the spaces \( E^{(2)}_\beta \) and \( E^{(2)}_{\beta'} \) is proved in the same way.

For a result related to the following theorem in the non rational case, see [8, p. 158].

The first item in the next theorem is taken from [8, p.156-159], [11].

**Theorem 6.4.**

1. The Ruelle operator has finite trace, and its trace is given by the formula

\[
\text{Tr} \ R = \sum_{k \in \mathbb{Z}} c_{(N-1)k} = \sum_{\substack{w \in T \\ w^N = 1}} |m(w)|^2.
\]

2. In the rational case, we have

\[
\text{Tr} \ R = DD^* + B^* \Gamma B + Y(I - A)^{-1} B^* + B(I - A^*)^{-1} Y
\]

Proof. The second item is a direct consequence of formulas [5.5].
7. Wavelets and rational filters

In this section we show that starting from a rational wavelet filter, the infinite product \((1.5)\) is indeed in \(L_2(\mathbb{R}, dx)\). To that purpose it is enough to prove that

\[ R1 \leq 1 \]

for the corresponding Ruelle operator, where, by definition of \(R\),

\[
(R1)(z) = \sum_{w \in \mathbb{T}} |m(w)|^2.
\]

Let \(M\) be a rational wavelet filter, that is a function of the form \((1.1)\). Recall that its first column is given by \((1.3)\).

**Proposition 7.1.** It holds that

\[ R1 = 1 \quad (7.1) \]

**Proof.** From \((1.3)\) we have

\[
m_0(z) = \sum_{j=0}^{N-1} U_{0j}(z^N) z^j,
\]

and so for \(z \in \mathbb{T}\),

\[
\frac{1}{N} \sum_{w \in \mathbb{C}} \sum_{w^N = z} |m_0(z)|^2 = \frac{1}{N} \sum_{w \in \mathbb{C}} \sum_{w^N = z} |\sum_{j=0}^{N-1} U_{0j}(w^N) w^j|^2 = \frac{1}{N} \sum_{w \in \mathbb{C}} |\sum_{j,k=0}^{N-1} U_{0j}(z) U_{0k}(z^*) w^{j-k} (\text{note that } w^N = z)|
\]

\[
= \sum_{j,k=1}^{N} U_{0j}(z) U_{0k}(z^*) \delta_{jk} \quad (\text{since } \frac{1}{N} \sum_{w \in \mathbb{C}} \sum_{w^N = z} w^{j-k} = \delta_{jk})
\]

\[
= \sum_{j} |U_{0j}(z)|^2 = 1 \quad (\text{since } U \text{ takes unitary values on } \mathbb{T})
\]

\[ \square \]

**Theorem 7.2.** Let \(m\) be the upper left entry of a wavelet filter. Then the infinite product \((1.5)\) converges to an \(L_2(\mathbb{R})\) function

**Proof.** We proceed in a number of steps.

**STEP 1:** The infinite product \((1.5)\) converges pointwise

This follows from Corollary \((1.5)\) with \(\theta_k = \frac{2\pi w}{N^k}\) since \(M(1) = I_N\) and hence \(m_0(1) = 1\).

We now set

\[
f_k(w) = 1_{[-\frac{2\pi k}{N^k}, \frac{2\pi k}{N^k}]}(w) \prod_{\ell=1}^{k} m(e^{i\frac{2\pi \ell}{N^k}}), \quad k = 1, 2, \ldots \quad (7.2)
\]
The key to the proof is to establish the identity
\[
\int_{-N^k}^{N^k} |f_k(w)|^2 dw = \int_{-N^{k-1}}^{N^{k-1}} (R1(w)|f_{k-1}(w)|^2 dw
\]  \hspace{1cm} (7.3)

STEP 2: (7.3) holds:
Indeed,
\[
\int_{-N^k}^{N^k} |f_k(w)|^2 dw = \int_{-N^k}^{N^k} \left( \prod_{t=1}^{k} |m \left( e^{i \frac{w \cdot N^t}{2}} \right) |^2 \right) dw
\]
\[
= \int_{-N^{k-1}}^{N^{k-1}} \left( \prod_{t=1}^{k-1} |m \left( e^{i \frac{w \cdot N^t}{2}} \right) |^2 \right) \left( \sum_{p=0}^{N} |m \left( e^{i \frac{w + pN^{k-1}}{N^{k-1}}} \right) |^2 \right) dw
\]
\[
= \int_{-N^{k-1}}^{N^{k-1}} \left( \prod_{t=1}^{k-1} |m \left( e^{i \frac{w \cdot N^t}{2}} \right) |^2 \right) \left( \sum_{p=0}^{N-1} |m \left( e^{i \frac{w + pN^{k-1}}{N^{k}}} \right) |^2 \right) dw
\]
This is R1
\[
= \int_{-N^{k-1}}^{N^{k-1}} \left( \prod_{t=1}^{k-1} |m \left( e^{i \frac{w \cdot N^t}{2}} \right) |^2 \right) dw \quad \text{(since R1 = 1)}
\]
\[
= \int_{-N^{k-1}}^{N^{k-1}} |f_{k-1}(w)|^2 dw
\]
\[
= \int_{-\frac{1}{2}}^{\frac{1}{2}} |m(e^{it})|^2 dt
\]
\[
= 1.
\]

STEP 3: The pointwise limit function \( f(w) = \prod_{t=1}^{\infty} m \left( e^{i \frac{w \cdot N^t}{2}} \right) \) belongs to \( L_2(\mathbb{R}) \).
Indeed, by Fatou’s lemma,
\[
\int_{\mathbb{R}} |f(w)|^2 dw = \int_{\mathbb{R}} \lim_{k \to \infty} |f_k(w)|^2 dw
\]
\[
= \int_{\mathbb{R}} \liminf_{k \to \infty} |f_k(w)|^2 dw
\]
\[
\leq \liminf_{k \to \infty} \int_{\mathbb{R}} |f_k(w)|^2 dw < \infty
\]
in view of the previous step. \( \square \)

Remark 7.3. The preceding arguments hold still in the case \( R1 \leq 1 \). This covers important cases of rational filters for which the function \( U \) in (1.1) is only contractive as opposed to unitary.

8. The Schur-Agler class of the infinite polydisk
Formula (4.7) suggests connections with infinite dimensional analysis (see for instance [14, 17] for background on the latter). We discuss some of these links here. When one
consider functions analytic at the origin rather than at infinity, realizations of the form (4.7) are replaced by realizations of the form

$$ \mathcal{C}(I - \Lambda(z)A)^{-1}\Lambda(z)B. $$

When a finite number of variables is involved, such realizations appear in the study of Schur-Agler functions. See [1, 6]. In the case of an infinite number of variables, formulas (4.7) and (8.1) suggest connections with infinite dimensional analysis. In the paper [5, §7] the following classes of functions were studied:

**Definition 8.1.** A $\mathbb{C}^{n \times m}$-valued function $s$ defined in the infinite polydisk $\mathbb{D}^N$ is said to be in the Schur-Agler class if there exist $\mathbb{C}^{n \times n}$-valued functions $k_1(z, w), k_2(z, w), \ldots$ positive definite in $\mathbb{D}^N$ and such that

$$ I_n - s(z)s(w)^* = \sum_{j=1}^{\infty} (1 - z_j \overline{w_j})k_j(z, w), \quad z, w \in \mathbb{D}^N. $$

In [5] and in [6, §7] it is proved that $s$ is in the Schur-Agler class if and only if it can be written as

$$ s(z) = D + C(I - \Lambda(z)A)^{-1}\Lambda(z)B, $$

where the operator matrix

$$ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : (\oplus \mathcal{H}(k_j)) \oplus \mathbb{C}^m \rightarrow (\oplus \mathcal{H}(k_j)) \oplus \mathbb{C}^n $$

is coisometric (and where $\mathcal{H}(k_n)$ denotes the reproducing kernel Hilbert space with reproducing kernel $k_j$).

**Theorem 8.2.** Consider in the space $((\oplus \mathcal{H}(k_j)) \oplus \mathbb{C}^m) \times ((\oplus \mathcal{H}(k_j)) \oplus \mathbb{C}^n)$ the linear relation $\mathcal{R}$ spanned by the pairs

$$ \begin{pmatrix} (\overline{w_1}k_1(\cdot, w)c) \\ (\overline{w_2}k_2(\cdot, w)c) \\ \vdots \\ d \end{pmatrix} \rightarrow \begin{pmatrix} (k_1(\cdot, w) - k_1(\cdot, 0)c) \\ (k_2(\cdot, w) - k_2(\cdot, 0)c) \\ \vdots \\ k_1(\cdot, 0) \\ k_2(\cdot, 0) \end{pmatrix}, $$

where $c \in \mathbb{C}^m$, $d \in \mathbb{C}^n$ and $(w_1, w_2, \ldots) \in \mathbb{D}^N$. Then $\mathcal{R}$ extends to an isometric relation from the closed linear span $\mathcal{N} \subset ((\oplus \mathcal{H}(k_j)) \oplus \mathbb{C}^n)$

of the functions

$$ \begin{pmatrix} \overline{w_1}k_1(\cdot, w)c \\ \overline{w_2}k_2(\cdot, w)c \\ \vdots \\ d \end{pmatrix} $$

into $((\oplus \mathcal{H}(k_j)) \oplus \mathbb{C}^n)$. This relation is the graph of an isometric operator $V$ from $\mathcal{N}$ into $((\oplus \mathcal{H}(k_j)) \oplus \mathbb{C}^n)$. Let

$$ U = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} $$
be a unitary extension of $V$ in some Hilbert space $K$. From (8.3) we have, with
\[
\hat{A}(\Lambda(w)^* k(\cdot, w)c) = k(\cdot, w) - k(\cdot, 0)c
\]
\[
\hat{B} = k(\cdot, 0)d
\]
\[
\hat{C}(\Lambda(w)^* k(\cdot, w)c) = (s(w)^* - s(0)^*)c
\]
\[
\hat{D} = D^* c.
\]

**Proof.** For the convenience of the reader we recall the main idea in the proof.

It follows that $k(\cdot, w)c = (I - \hat{A}(\Lambda(w))^(-1)(k(\cdot, 0)c)$ and so
\[
s(w)^*c = s(0)^*c + (s(w)^* - s(0)^*)c
\]
\[
= D^*c + \hat{C}(\Lambda(w)^*k(\cdot, w)c)
\]
\[
= D^*c + \hat{C}(\Lambda(w)^*(I - \hat{A}(\Lambda(w))^(-1)(k(\cdot, 0)c)
\]
\[
= D^*c + \hat{C}(\Lambda(w)^*(I - \hat{A}(\Lambda(w))^(-1)\hat{B}c.
\]

Taking adjoint we get
\[
s(w) = s(0) + \hat{B}^*(I - \Lambda(w)\hat{A}^*)^{-1}\Lambda(w)\hat{C}^*
\]
and thus (8.2) holds. \qed

9. Remarks

We noted in Section 6 that realization of the filter $m(z)$ in one complex variable lead to spectral data for $L_2(\mathbb{R})$-wavelets. On the other hand, the realization arguments presented here with $z$ replaced by $(z_1, z_2, \cdots, z_d)$ do carry over to the multivariate case. These computations and realizations of a filter $m(z_1, \ldots, z_d)$ in $d$ complex variables lead to spectral and insights into $L_2(\mathbb{R}^d)$-wavelets. See for instance [8, 10] for the latter. While it is known that multivariable realizations exist (for $d > 1$) which are analogous to the realizations we used in Sections 4-6, the implications for $L_2(\mathbb{R}^d)$-wavelets analysis will be postponed to a future paper.

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