ON A UNIQUENESS PROPERTY OF CUSPIDAL UNIPOTENT REPRESENTATIONS

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Abstract. The formal degree of a unipotent discrete series character of a reductive group which is defined and unramified over a nonarchimedean local field, is a rational function of the cardinality $q$ of the residue field, whose poles and zeroes (apart from $q = 0$) are roots of unity. We prove that for absolutely almost simple groups of classical type, the formal degree of a cuspidal unipotent representation (in the sense of Lusztig [Lus1], [Lus2]) determines its Kazhdan-Lusztig-Langlands parameter, up to twisting by unramified characters. For such groups, a cuspidal unipotent formal degree, normalized appropriately, is the reciprocal of a polynomial whose zeroes are roots of unity of even order. We show that this property is very distinctive among unipotent formal degrees of general discrete series characters, and we show that if a unipotent formal degree has poles or zeroes of odd order, then the root of unity with the largest odd order that occurs in this way is a zero of the formal degree.

The main result of this article characterizes unramified Kazhdan-Lusztig-Langlands parameters which support a cuspidal local system in terms of formal degrees. The result implies the uniqueness of so-called cuspidal spectral transfer maps (as introduced in [Op1]) between unipotent affine Hecke algebras of classical type (up to twisting by unramified characters). In [Op2] the essential uniqueness of arbitrary unipotent spectral transfer maps was reduced to the cuspidal case.

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1. Introduction

Let $k$ be a non-archimedean local field with finite residue field $\mathbb{F}_q$ of characteristic $p > 0$. Fix a separable closure $k^s$ of $k$ and let $K \subset k^s$ be the maximal unramified extension of $k$. The Galois group $\text{Gal}(K/k)$ is isomorphic to $\hat{\mathbb{Z}}$, and we may choose as topological generator the geometric Frobenius element $\text{Frob}$, whose inverse induces the automorphism $x \mapsto x^q$. Let $I_k = \text{Gal}(k^s/K)$ be the inertia subgroup of $\text{Gal}(k^s/k)$ and let $W_k$ be the Weil group of $k$.

Let $G$ be a connected reductive group defined over $k$, and unramified. Then $\text{Gal}(K/k)$ naturally acts on $G := G(k) = G^{I_k}$. Let $F' \in \text{Aut}(G)$ be the automorphism given by the action of $\text{Frob}$. Recall that $G$ admits a maximal $K$-split torus $S \subset G$, which is defined over $k$ and maximally $k$-split, see [BT, 5.1.10]. Fix such a maximal torus $S$ of $G$, and denote by $S := S(k) \subset G = G(k)$ its group of $K$-points. In the Bruhat-Tits building of $G$ there exists an $F'$-stable apartment $A$, which, as an affine space, is isomorphic to $\mathbb{R} \otimes X_s(S)$ where $X_s(S)$ is the cocharacter lattice of $S$. Denote by $S_O = O_K^* \otimes X_s(S)$ the maximal bounded subgroup of $S$, we have $X_s(S) \simeq S/S_O$.

Let $G_1 = \langle S_O, G_{\text{der}} \rangle$ be the subgroup of $G$ generated by $S_O$ and by the derived group $G_{\text{der}}$ of $G$ (see [Op2, Corollary 2.2]). There exists an $F'$-stable alcove $C$ in $A$ (see e.g. [Tits]). A ($F'$-stable) standard parahoric subgroup of $G$ is a subgroup of the form $\text{Stab}_G(F_P) \cap G_1$, where $F_P \subset C$ is a ($F'$-stable) facet of $C$. A parahoric subgroup $P$ of $G$ is a subgroup conjugate to the standard parahoric subgroup. By [HR, Proposition 3] this definition is equivalent to the original definition in [BT]. We denote by $\mathcal{P}^{m.c.u}(G, F')$ the set of conjugacy classes of maximal $F'$-stable parahoric subgroups of $G$ which carry a cuspidal unipotent representation. Then $\mathcal{P}^{m.c.u}(G, F')$ is in natural bijection with the set of $G(k)$-conjugacy classes of maximal $G(k)$-parahoric subgroups carrying a cuspidal unipotent representation (by a version of Lang’s theorem, see e.g. [Lus1, Paragraph 1.3]). We will identify these two sets.

We refer to [Lus1, Paragraph 1.21] for the notion of unipotent representations of algebraic group over a non-archimedean local field (and its parahoric subgroups) (See also §12.1 of [Car]). We fix a $F'$-stable Iwahori subgroup $B \subset G$. A maximal cuspidal unipotent type $(P, \sigma)$ of $G$ consists of a maximal $F'$-stable parahoric subgroup $P$ of $G$, and a cuspidal unipotent representation $\sigma$ of $P$. In this situation, the compact induced representation $\text{c-Ind}_{P^{F'}}^G(k) \sigma$ is a finite direct sum of irreducible supercuspidal representations.
These irreducible summands are called **cuspidal unipotent representations of** $G(k)$, and these form the main subject of study of this paper.

Let $G$ be any connected reductive group over $k$, with maximal $k$-torus $S$ and Borel subgroup $B \supset S$. We will, admittedly unconventionally, denote by $\Sigma$ the based root datum of the identity component of the dual $L$-group $\check{L}$ (see below). Hence $\Sigma^\vee = (X_\ast, \Sigma_0^\vee, \Delta_0, X^\vee, \Sigma_0, \Delta)$ denotes the based root datum of $S \subset B \subset G$, where $X_\ast = X^\vee(S)$ is the character lattice of $S$.

Choose a pinning for the root subgroups of the simple roots for $G$. This induces a splitting of the canonical exact sequence $1 \to \text{Int}(G) \to \text{Aut}(G) \to \text{Aut}(\Sigma^\vee) \to 1$. The natural homomorphism $\gamma : \text{Gal}(k^s/k) \to \text{Aut}(G)$ induces in this way an action $\gamma_{qs} : \text{Gal}(k^s/k) \to \text{Aut}(G)$ which is an inner twist of $\gamma$, and such that $B$ is a $k$-subgroup with respect to $\gamma_{qs}$. In other words, we have defined a reductive group $G_{qs}$ over $k$ which is quasi split and which is an inner form of $G$. Let $G^\vee$ be a complex reductive group, with Borel $B$ and maximal torus $S^\vee \subset B^\vee$, such that the based root datum $\Sigma$ of $S^\vee \subset B^\vee \subset G^\vee$ is dual to $\Sigma^\vee$. Choose a pinning for the root subgroups of the simple roots for $G$ and $G^\vee$, inducing splittings of the canonical exact sequences $1 \to \text{Int}(G^\vee) \to \text{Aut}(G^\vee) \to \text{Aut}(\Sigma) \to 1$ as well. We can identify the groups $\text{Aut}(\Sigma)$ and $\text{Aut}(\Sigma^\vee)$ canonically, hence $\gamma_{qs}$ gives rise to an algebraic action $\gamma^\vee$ of $\text{Gal}(k^s/k)$ on $G^\vee$. Choose $\text{Gal}(k^s/k) \to \Gamma \to 1$, a finite quotient through which $\gamma^\vee$ factors. We set $LG := \Gamma \ltimes G^\vee$ (of course, the relevant notions in the theory do not depend on this choice of a finite form of $LG$).

Let $G$ be as above and unramified. In this situation we can take $\Gamma = \langle \theta \rangle$, the finite group generated by the automorphism $\theta$ of $G^\vee$ by which Frob acts. We recall that the isomorphism classes of $k$-rational structures of $G$ which are inner forms of $G$ are parameterized by the non-abelian cohomology set $H^1(k, G_{ad})$. As a consequence of the Steinberg vanishing theorem (see [DeRe, Theorem 2.2.1]), all inner $k$-forms of $G$ are $K$-split. If we denote by $F$ the automorphism of $G_{ad}$ (or $G$) corresponding to the quasi split action of Frob as constructed above, we have $H^1(k, G_{ad}) = H^1(F, G_{ad})$.

Vogan introduced the notion of pure inner forms, and formulated refined versions of the local Langlands conjectures in terms of these inner forms. Unless $G$ is of adjoint type, the set of equivalence classes of its pure inner forms is in general different from the set equivalence classes of its inner forms. The pure inner forms of $G$ correspond to cocycles $z \in Z^1(F, G)$. Note that such a cocycle is determined by the image $u = z(Frob)$ of the Frobenius element. The corresponding inner $k$-form of $G_{qs}$ is then defined by the functorial image $z_{ad} \in Z^1(F, G_{ad})$. The pure inner forms $G_{qs}$ of $G$ are defined by the twisted Frobenius action $F^u = F_u = \text{Ad}(u) \circ F$ on $G$, where we may choose (see [DeRe]) $u \in N_G(B)$. This yields an explicit realization $H^1(k, G) \simeq \Omega/(1 - \theta)\Omega$ of the pointed set $H^1(k, G)$ of equivalence classes of inner forms, where $\Omega = N_G(B)/B$. There exists a canonical isomorphism $\Omega = X^\ast/2\Sigma_0$ (cf. [DeRe]). The equivalence classes of inner forms are likewise parameterized by $H^1(k, G_{ad}) \simeq \Omega_{ad}/(1 - \theta)\Omega_{ad}$ with $\Omega_{ad} = N_{G_{ad}}(B_{ad})/B_{ad}$. 
The set of semisimple conjugacy classes of $L G := \langle \theta \rangle \ltimes G^\vee$ can be identified canonically with the orbit space $W_0 / T$ (cf. [Bo, Section 6]), where $W_0$ is the Weyl group of $G_{qs}$ with respect to the maximal $k$-split torus $(S^F)^0 \subset S$ (the relative Weyl group), and $T$ is the complex torus whose coordinate lattice is the lattice $Y_{qs} := X^*(S^F)^0$. Following [Spr, Sections 15.3.6-15.3.8] we can construct a root datum in this situation. The restriction map of characters of $S$ to $(S^F)^0$ induces an orthogonal projection (with respect to a suitable $W(S^F)^0$-invariant inner product) of the real vector space $V = \mathbb{R} \otimes X_*$ onto its $\theta$-invariant subspace $V^\theta$. The image $\Pi(X_*)$ of $X_*$ is $X^*(S)$ is $Y_{qs}$, and the projection $\Pi(\Sigma_0^\vee) \subset Y_{qs}$ is an integral, but possibly non-reduced root system. We define a reduced root system $R_0^\vee$, the set of unimultiplicable elements of $\Pi(\Sigma_0^\vee)$. Moreover $Y_{qs}$ is contained in $P(R_0^\vee)$, the weight lattice of $R_0^\vee$. The dual lattice is $X_{qs} = X^*(T) = X_*(S)^\theta = X^*(S)/\Theta$, and $T = S^\vee/(1-\Theta)(S^\vee)$. We introduce $R_0 \subset X_{qs}$ as the set of co-roots of $R_0^\vee$, and one can define a base $F_{qs}$ of the relative root system $R_0$ of $G_{qs}$ such that the projection of the cone spanned by $\Delta_0^\vee$ is the cone spanned by $F_{qs}^\vee$. Thus we have defined a based root datum $R_{qs}$ underlies the extended affine Iwahori-Matsumoto Hecke algebra $H^{IM}$ of $(G_{qs}, B_{qs})$. These remarks enables us to identify the set of semisimple conjugalcy classes of $L G$ with the spectrum $W_0 \setminus T$ of the center of $H^{IM}$.

From here onward we will assume that $G$ is a connected, absolutely almost simple unramified algebraic group over $k$. In this situation $\Omega^\theta = X^*/\Theta^\theta = X_{qs}/Q(R_0)$, and thus only depends on the relative root datum $R_{qs}$. Indeed, it is well known that the elements of $\Omega$ are in canonical bijection with the set $C \cap X^*$ by mapping $\omega \in \Omega$ to $\omega(0) \in C \cap X^*$. This maps $\Omega^\theta$ to $(C \cap X^*)^\theta$, hence every element of $\Omega^\theta$ can be represented by a $\Theta$-invariant element of $X^*$. If $X^*$ equals the weight lattice $X_0 = P(\Sigma_0)$ of $\Sigma_0$, it is an easy check to see that $X_{qs} = P(\Sigma_0)^\theta$ equals the weight lattice $P(R_m)$ of the root system $R_m$ introduced in [Op1, Definition 2.10] for the Hecke algebra $H^{IM}$. We denote this maximal group $P(R_m)/Q(R_0)$ of special diagram automorphisms of the affine Dynkin diagram of the affine extension of $(\Sigma_0^\vee)^{(1)}$ by $\Omega^\theta$. This case corresponds to the group $G_{ad}$ of adjoint type isogenous to $G$.

An important role is played by the group $G_{nr}$ of unramified complex characters of $G$, and the group $G(k)^{nr}_{\ast}$ of unramified complex characters of $G(k) := G^{F_u}$. These are finite abelian groups (since we have assumed that $G$ is almost simple). There is a natural identification of $G(k)^{nr}_{\ast}$ and $G_{qs}(k)^{nr}_{\ast}$. One also has a natural identification of $G^{nr}_{\ast}$ with $\Omega^*$. By the restriction map to $N_{G}(B)$ we have a canonical isomorphism of $G(k)^{nr}_{\ast}$ with $(\Omega^\vee)^\ast$, the Pontryagin dual of $\Omega^\theta$. This latter group acts on the set of unipotent characters of $G(k)$ by tensoring. This action is compatible with the action of this group on the Iwahori Hecke algebra $H^{IM} = H^{IM,a} \ltimes \Omega^\theta$ by algebra automorphisms. In terms of the relative root datum $R_{qs}$, we can write $G_{qs}(k)^{nr}_{\ast} \simeq (X_{qs}/Q_{qs})^* \subset W_0 \setminus T$. Observe that $(X_{qs}/Q_{qs})^* = (P_{qs}^\vee/Y_{qs})$ canonically.

The set $P^{m,c,u}(G_{sc}, u) := P^{m,c,u}(G_{sc}, F_u)$ (with $u \in N_{G_{ad}}(B_{ad})$ and $uB_{ad} := \omega \in \Omega_{ad}$) carries a natural action of $\Omega_{ad}^\theta$. On a combinatorial level, this can be understood by
observing that $\Omega^\theta_{ad}$ acts as an abelian group of $\theta$-invariant diagram automorphisms on the affine Dynkin diagram $(\Sigma^\vee)^{(1)}$ (namely, the group $\theta$-invariant special diagram automorphisms). This clearly acts on $\mathcal{P}^{m,c,u}(G_{sc}, u)$, since this set is in natural bijection with the set of maximal proper $\omega \circ \theta$-invariant subsets of $(\Sigma^\vee)^{(1)}$, which support a cuspidal unipotent representation. Observe that for $\mathbb{P}$ an $F_u$-invariant standard parahoric subgroup of $G_{sc}$, and $x \in \Omega_{ad}$, the standard parahoric $x(\mathbb{P})$ is $F_{u'}$ invariant, where $u$ and $u'$ cohomologous 1-cocycles. We have $u = u'$ if and only if $x \in \Omega^\theta_{ad}$.

On a more conceptual level, recall the exact sequence introduced by Kottwitz [Kot] (also see [Op2, Equation (2)])

\begin{equation}
1 \to G^F_{ad,1} \to G_{ad}(k) \to X^*(Z(G^\vee))^\theta \to 1
\end{equation}

Observe that we have a canonical isomorphism $X^*(Z(G^\vee))^\theta = \Omega_{ad}$, hence the above can be expressed in terms of the restricted root datum:

\begin{equation}
1 \to G^F_{ad,1} \to G_{ad}(k) \to \Omega^\theta_{ad} \to 1
\end{equation}

Since a semisimple group over $k$ of simply connected type is equal to its own derived group, it is easy to see that $G_{ad}(k)_{der}$ is the image of the simply connected covering map $\psi_{sc} : G_{sc}(k) \to G_{ad}(k)$. Hence by [Op2, Corollary 2.2] we have

\begin{equation}
G^F_{ad,1} = \langle \psi_{sc}(G_{sc}(k)), S_{ad}(O) \rangle
\end{equation}

Clearly all unramified characters of $G_{ad}(k)$ vanish on $G^F_{ad,1}$, and by (1.2) we see easily that

\begin{equation}
G^F_{ad,1} = \bigcap_{\chi \in G_{ad}(k)^{\mu}} \ker \chi \subset G_{ad}(k)
\end{equation}

As was remarked in [HR], the normal subgroup $G^F_{ad,1} \subset G_{ad}(k)$ is the subgroup generated by all parahoric subgroups of $G_{ad}(k)$, and in particular every parahoric subgroup of $G_{ad}(k)$ is contained in $G_{ad,1}(k)$. There is a bijective correspondence between the set of parahoric subgroups of $G_{ad}(k)$ and of $G_{sc}(k)$ by taking the $\psi_{sc}$-inverse image of a parahoric subgroup of $G_{ad}(k)$. The inverse map is defined by sending a parahoric $\mathbb{P}^F_u$ of $G_{sc}(k)$ to the normalizer $N^F_{G_{ad,1}}(\psi_{sc}(\mathbb{P}^F_u))$ of $\psi_{sc}(\mathbb{P}^F_u)$ in $G^F_{ad,1}$. (Indeed, this normalizer is a parahoric subgroup of $G_{ad}(k)$ by a straightforward application of the main result of [HR]). In particular, we may identify the sets $\mathcal{P}^\delta(G_{ad}, u)$ and $\mathcal{P}^\delta(G_{sc}, u)$ of $F_u$-stable parahoric subgroups of $G_{ad}$ and $G_{sc}$ respectively, since both sets are in natural bijection with the set of $F_{u}$-stable facets of the Bruhat-Tits building of $G_{ad}$ (cf. [HR]).

It is now clear how the action of $\Omega^\theta_{ad} = N_{G_{ad}(k)}(\mathbb{B}^F_u)/\mathbb{B}^F_u$ on $\mathcal{P}^{m,c,u}(G_{sc}, u)$ arises group theoretically from the conjugation action of $G_{ad}(k)$ on its normal subgroup $G_{ad,1}(k)$. Precisely, we have the natural action by conjugation of $G_{ad}(k)$ acting on the set $\mathcal{P}^\delta(G_{ad}, u)$. We can identify the latter set via the above natural bijection to $\mathcal{P}^\delta(G_{sc}, u)$, which comes equipped with the action by $G_{sc}(k)$. It follows easily from the construction of the bijection and (1.3) that the set $\mathcal{P}(G_{sc}, u)$ of $G_{sc}(k)$-conjugacy classes of $F_{u}$-stable parahorics of $G_{sc}$ can be naturally identified with the set of $G_{ad,1}$-orbits in $\mathcal{P}^\delta(G_{ad}, u)$. 

\begin{equation}
G^F_{ad,1} = \bigcap_{\chi \in G_{ad}(k)^{\mu}} \ker \chi \subset G_{ad}(k)
\end{equation}
Therefore the quotient $G_{\text{ad}}(k)/(G_{\text{ad}}^{F_{\text{sc}}}) = \Omega^\theta_{\text{ad}}$ acts on $P(G_{\text{sc}}, u)$, and the orbit space $P(G_{\text{ad}}, u) = \Omega^\theta_{\text{ad}} P(G_{\text{sc}}, u)$ is the set of $G_{\text{ad}}(k)$-conjugacy classes of $F_{\text{u}}$-stable parahorics of $G_{\text{ad}}$. Finally we remark that $P_{m,c,u}(G_{\text{sc}}, u)$ is a $\Omega^\theta_{\text{ad}}$-invariant subset of $P(G_{\text{sc}}, u)$. Since $\Omega^\theta_{\text{ad}}$ is an abelian group, the isotropy group $\Omega^\theta_{\text{ad}} P = P \cap P(G_{\text{sc}}, u)$ is the same for all parahorics in the same $\Omega^\theta_{\text{ad}}$-orbits. Hence we can also canonically attach this isotropy group to the orbit $\Omega^\theta_{\text{ad}} P \in P(G_{\text{ad}}, u)$ (and similarly for the $F_{\text{u}}$-stable parahoric subgroups of any intermediate group $G$ in the same isogeny class).

We remark that if $P_{\text{sc}} \in P^P(G_{\text{sc}}, u)$ and $P_{\text{ad}} \in P^P(G_{\text{ad}}, u)$ correspond to each other, then the sets of cuspidal unipotent representations of $P_{\text{sc}}$ and of $P_{\text{ad}}$ are in canonical bijection. Indeed, in view of the construction of the bijection, this statement reduces to the well known fact (cf. [Lus] or [Car, Section 12.1]) that the set of unipotent representations of a connected reductive group $G$ defined over the residue field $k$ of $k$ is independent of the type of $G$ within its isogeny class.

From here on $G$ is assumed to be (isogenous to) a classical group. For classical groups the set of irreducible cuspidal unipotent characters of $P$ consists of at most one element, so that the set of maximal unipotent types $\{(P, \sigma)\}$ is in natural bijection with the set $P_{m,c,u}(G, u)$. We denote by $U_{c,u}(G, u)$ (with $u \in H^1(k, \mathcal{G}_{\text{ad}})$) the set of irreducible cuspidal unipotent characters of $G(k) = G_{F_{\text{sc}}}$, and by $U_{c,u}(G)$ the disjoint union of the sets $U_{c,u}(G, u)$ where $u$ runs over a complete set of representatives of the equivalence classes of inner twists of $F$ (in other words, $[u] \in H^1(k, \mathcal{G}_{\text{ad}}) \cong \Omega_{\text{ad}}/(1-\theta)(\Omega_{\text{ad}})$). The set $U_{c,u}(G, u)$ is naturally partitioned in subsets $U_{c,u}^P(G, u, P, \sigma)$ consisting of the set of direct summands of the compact induction $c\text{-Ind}_{\mathcal{P}u}^{G(k)}(\sigma)$ of a cuspidal unipotent representation $\sigma$ of a maximal $F_{\text{u}}$-stable parahoric subgroup $\mathcal{P}$ of $G$, where $(P, \sigma)$ runs over the set of conjugacy classes of pairs of maximal $F_{\text{u}}$-stable parahoric subgroups $\mathcal{P}$ of $G(k)$, and $\sigma$ a cuspidal unipotent representation of $\mathcal{P}$ (or rather, its reducive quotient). In this cuspidal case we have an isomorphism $\mathcal{H}_u(P, \sigma) := \text{End}_{G(k)}(c\text{-Ind}_{\mathcal{P}u}^{G(k)}(\sigma)) \cong \mathbb{C}[\Omega^P, \theta]$ (cf. [Lus1]), where $\Omega_P, \theta = \Omega^\theta \cap \Omega^P_{\text{ad}}$. Thus we have

$$\begin{equation}
U_{c,u}(G) = \bigsqcup_{[u] \in H^1(k, \mathcal{G}_{\text{ad}})} \bigsqcup_{P \in P_{m,c,u}(G, u)} U_{c,u}^P(G, u, P, \sigma)
\end{equation}
$$

with

$$\begin{equation}
U_{c,u}^P(G, u, P, \sigma) \cong \text{Irr}(\Omega^P, \theta) = (\Omega^P, \theta)^*.
\end{equation}
$$

We remark that, similar to the discussion above, $P_{m,c,u}(G, u) \cong \Omega^\theta \setminus P(G_{\text{sc}}, u)$. Hence we see from the above decompositions on a combinatorial level that $U_{c,u}^P(G)$ carries an action of the abelian group $(\Omega^\theta)^* \times (\Omega^\theta/\Omega^\theta)$, whose set of orbits is naturally parameterized by the set

$$\begin{equation}
P_{m,c,u}(G_{\text{ad}}) := \bigsqcup_{[u] \in H^1(k, \mathcal{G}_{\text{ad}})} P_{m,c,u}(G_{\text{ad}}, u)
\end{equation}
$$

This can be given a representation theoretical basis, but we will not work out the details here.
Recall that an unramified local Langlands parameter for $G$ is a homomorphism
\[ \lambda : \langle \text{Frob} \rangle \times \text{SL}_2(\mathbb{C}) \rightarrow L^G \]
such that $\lambda(\text{Frob} \times \text{id}) = s \times \theta$ with $s \in G^\vee$ semisimple and such that $\lambda$ is algebraic on the $\text{SL}_2(\mathbb{C})$-factor. We call $\lambda$ discrete if $C_{G^\vee}(\lambda)$ is a finite group. We will denote the set of all unramified local Langlands parameters of $G$ by $\Lambda^{ulp}(G)$, and by $\Lambda^{dulp}(G)$ the set of all unramified discrete local Langlands parameters of $G$. Given $\lambda \in \Lambda^{dulp}(G)$ we denote by $[\lambda]$ its orbit in $\Lambda^{dulp}(G)$ for the action of $G^\vee$ by conjugation. We will write $\Lambda^{dulp}(\overline{G})$ for the set of orbits $[\lambda]$ in $\Lambda^{dulp}(G)$. The group of unramified complex characters $G(k)_{nr}^* \simeq (\Omega^\theta)^*$ of $G(k)$ acts naturally on $\Lambda^{dulp}(G)$ (cf. [Bo, Paragraph 8.5]), and it acts on the set of irreducible characters of $G(k)$ by taking tensor products. The local Langlands correspondence is expected the be equivariant for these actions [Bo, Paragraph 10.2]. We remark that a central isogeny $G_1 \rightarrow G_2$ of unramified semisimple groups defined over $k$ gives rise to a covering map $L^G_2 \rightarrow L^G_1$ which induces a surjective “functorial map” with finite fibres $\Lambda^{ulp}(G_2) \rightarrow \Lambda^{ulp}(G_1)$ such that the inverse image of $\Lambda^{dulp}(G_1)$ is $\Lambda^{dulp}(G_2)$ (the lifting of unramified local Langlands parameters is always possible because $W_k/I_k$ is infinite cyclic, and $\text{SL}_2$ is simply connected; and the second assertion is obvious). Since $(\Omega^\theta)^*$ is the central subgroup of the torus $T = \text{Hom}(X_{qs}, \mathbb{C}^\times) = \text{Hom}(X^* \theta, \mathbb{C}^\times)$, the orbits of $(\Omega^\theta)^* = X_{qs}/Q(R_0)$ on $\Lambda^{dulp}(G)$ are the fibres of the functorial map $\Lambda^{dulp}(G) \rightarrow \Lambda^{dulp}(G_{sc})$. We denote by $[\lambda_{\text{sc}}] = (\Omega^\theta)^* [\lambda]$ the image of $[\lambda]$ under this map, and by $[\lambda]_{\text{ad}} \in \Lambda^{dulp}(G_{ad})$ a lifting of $\lambda$ in $\Lambda^{dulp}(G_{ad})$. The isotropy subgroup of $\lambda_{\text{ad}}$ with respect to the action of the group $G_{ad}(k)^*_{nr} \simeq (\Omega^\theta)^*_{nr}$ only depends on $\lambda_{\text{sc}}$.

It was conjectured [III, Conjecture 1.4] (also see [GR, Conjecture 7.1]) that the local Langlands correspondence for $G$ has the following property regarding formal degrees. We normalize the Haar measures of $G^\circ(k) = G^\circ_{qs}$ and $G_{qs}(k) = G^\circ_{qs}$ as explained in [DeRe] (also see [Op2]). Let $\pi$ be an irreducible admissible discrete series character of $G^\circ(k)$, and let $\lambda$ denote the corresponding local Langlands parameter. Then $\lambda$ should be a discrete local Langlands parameter for $G$, and the formal degree $\text{fdeg}(\pi)$ of $\pi$ should be equal to a certain rational number independent of $q$ times the adjoint gamma factor $\gamma(\lambda)$ of the discrete Langlands parameter $\lambda$. In the present paper we are not interested in the rational factor. Ignoring this rational factor, we can express the above conjecture by saying that the $q$-rational factor $\text{fdeg}_q(\pi)$ of $\text{fdeg}(\pi)$ equals the $q$-rational factor $\gamma_q(\lambda)$ of $\gamma(\lambda)$. We remark that for a given maximal cuspidal unipotent pair $(\mathbb{P}, \sigma)$, all members of $\mathcal{U}^{c,u}(G, u, \mathbb{P}, \sigma) \simeq (\Omega^\theta, \theta)^*$ have the same formal degree.

Our main result is the following:

**Theorem 1.1.** Let $G$ be (isogenous to) an absolutely simple classical group, defined and unramified over a nonarchimedean local field $k$. Let $G_{qs}$ be a $k$-quasisplit inner form of $G$. Write $G = G(K)$ and denote by $F$ the action of Frobenius on $G$ corresponding to $G_{qs}$, and by $F_u := \text{Ad}(u) \circ F$ the action of Frobenius on $G$ corresponding to $G$, with $u \in Z^1(F, G_{ad})$. Let $(u, \mathbb{P}, \sigma)$ be a maximal $F_u$-stable cuspidal unipotent type of $G$. 


(a) There exists a unique $G(k)_{st}^1 = (\Omega^0)^* \lambda_{\text{sc}}$ such that for all $\pi \in \mathcal{U}^{c,u}(G, u, \mathbb{P}, \sigma)$, $\gamma_q(\lambda) = \text{fdeg}_q(\pi)$. We denote set of equivalence classes $[\lambda] \in \Lambda_{\text{dulp}}(G)$ such that there exists a cuspidal unipotent representation $\pi$ of $\mathbb{G}$ of rank $G$ for which $\gamma_q(\lambda) = \text{fdeg}_q(\pi)$ by $\Lambda_{\text{dulp}}(G)$. We refer to this set as the set of equivalence classes of discrete unramified Langlands parameters with cuspidal unipotent degree.

(b) The assignment $(u, \mathbb{P}, \sigma) \rightarrow [\lambda_{\text{sc}}]$ of (a) determines a canonical surjection from the set $\mathcal{P}_{\text{m,c,u}}^G(\mathbb{G})$ (cf. (1.7)) to $\Lambda_{\text{dulp}}(G_{\text{sc}})$.

(c) If $[\lambda_{\text{sc}}] \in \Lambda_{\text{dulp}}(G_{\text{sc}})$, then the set $\Lambda_{\text{dulp}}(G_{\text{sc}})$ corresponds to $(u, \mathbb{P}, \sigma)$ according to the surjection of (b), the isotropy group $(\Omega^0_{\text{ad}})^*|_{[\lambda_{\text{ad}}]}$ of $[\lambda_{\text{ad}}] \in \Lambda_{\text{dulp}}(G_{\text{ad}})$ contains $\gamma_q(\lambda_{\text{ad}}) \in (\Omega^0_{\text{ad}})^*$. Let us denote the index of this subgroup $d_{(u, \mathbb{P}, \sigma)}$. Then the number of cuspidal local systems supported by $[\lambda_{\text{ad}}]$ equals $\sum d_{(u, \mathbb{P}, \sigma)}$, where the sum runs over the fibre of the surjection of (b).

In particular, there exists an essentially unique (up to choosing an extension of each $\sigma$ from $\mathbb{P}$ to $N(G(k)(\mathbb{P}))$, which fixes the bijections $\mathcal{U}_{\text{c,u}}^G(G, u, \mathbb{P}, \sigma) \simeq \text{Irr}(\Omega^0_{\text{ad}})$, and up to a choice of $\lambda_{\text{ad}} \in \Lambda_{\text{dulp}}(G_{\text{ad}})$ above each $[\lambda_{\text{sc}}] \in \Lambda_{\text{dulp}}(G_{\text{sc}})$) the set of cuspidal local systems supported by the orbits in $\Lambda_{\text{dulp}}(G_{\text{ad}})$ which is compatible with the surjection (b).

Some remarks are in order.

Remark 1.2. The proof of Theorem 1.1 consists of a case by case analysis. For the proof of Theorem 1.1 it is clearly enough to consider $G$ of adjoint type.

Remark 1.3. The set of orbits for the action of $(\Omega^0)^* \times (\Omega^0_{\text{ad}}/\Omega^0)$ on $\mathcal{U}_{\text{c,u}}^G(G, u, \mathbb{P}, \sigma)$ as observed above, is in natural bijection with $\mathcal{P}_{\text{m,c,u}}^G(\mathbb{G}) \rightarrow \Lambda_{\text{dulp}}(G_{\text{sc}})$. As was mentioned above, one expects that the local Langlands correspondence maps each orbit in a $(\Omega^0)^*\text{-equivariant way}$ to the $(\Omega^0)^*\text{-set} \Lambda_{\text{dulp}}(G_{\text{sc}})$, which satisfies $\Lambda_{\text{dulp}}(G_{\text{sc}}) = \Lambda_{\text{dulp}}(G_{\text{sc}})$. The role of the action of $(\Omega^0_{\text{ad}}/\Omega^0)$ is less obvious.

Remark 1.4. If $G(k)$ is classical and not anisotropic, and $\pi$ is a cuspidal unipotent character, then it is easy to check that $\text{fdeg}_q(\pi)$ is the reciprocal of a product of even cyclotomic polynomials. This remark plays a key role in the case by case analysis we need to do, as it turns out that the condition on an unramified discrete local Langlands parameter $\lambda$ that $\gamma_q(\lambda)$ has only even cyclotomic polynomials both in its numerator and denominator, is already very restrictive.

Remark 1.5. The set $\Lambda_{\text{dulp}}(G)$ of equivalence classes of discrete unramified local Langlands parameters of $G$ is in canonical bijection with the set $\text{Res}_L(\mathcal{R}_{qs}, \mathbf{m}_{qs})$ of Weyl group orbits of so-called residual points of the generic Iwahori-Matsumoto Hecke algebra $H_{LM}$ associated to the quasi split form of $G$, see [OS], [Op2, Section 2.3] (where $L = \mathbb{C}[v^\pm 1]$, with $v^2 = q$, and $\mathbf{m}_{qs}$ denotes the parameter function on $\mathcal{R}_{qs}$ associated to $H_{LM}$). By the main result of [Op2], if $[\lambda] \in \Lambda_{\text{dulp}}(G)$ corresponds to such an orbit of residual points.
\( W_0 \) with \( r \in T(L) = \text{Hom}(X_{qs}, L^\times) \), then
\[\gamma_q(\lambda) = \mu_q^{(r)}(r),\]
where \( \mu_q^{(r)}(r) \) denotes the \( q \)-rational factor of the “residue” of the \( \mu \)-function of \( H^{IM}_L \) at \( r \). In the language of [Op1], \( W_0 \) corresponds with a cuspidal spectral transfer morphism \( H^{u,(P,\sigma),e} \rightarrow H^{IM}_L \), where \( H^{u,(P,\sigma),e} \approx L \) is a direct summand of \( H^{u,(P,\sigma)} \approx L[\Omega^{P,\theta}] \) associated to an extension \( \tilde{\sigma} \) of \( \sigma \) to \( N_{G(k)}(P) \).

This enables us to reformulate Theorem 1.1 (a), (b) in the terminology of [Op1] as the existence and uniqueness of a spectral transfer morphism \( H^{u,(P,\sigma),e} \rightarrow H^{IM}(G_{sc}) \) for every cuspidal maximal parahoric subgroup \( P \) of any inner form \( G_u \) of \( G \).

Remark 1.6. The existence aspect of Theorem 1.1 (a), (b) is proved in [Op2] and [Fe1], [Fe2]. In this paper we are concerned with proving the uniqueness. We remark that analogous existence and uniqueness results are known for split exceptional groups by the work of Reeder [R1], [R2]. The remaining exceptional cases are treated in [Fe2].

Remark 1.7. The Langlands parameters in \( 1^\text{dual-\text{inf}}(G) \) have very special properties with respect to parabolic induction. Let \( H \subset G \) be a standard \( k \)-Levi subgroup of \( G \), and \( [\lambda] \in 1^\text{dual-\text{inf}}(H) \), and let \( \delta_{cu} \) be a cuspidal unipotent character of \( H \) corresponding to \( [\lambda] \) in the sense that its formal degree has the same \( q \)-rational factor as \( \gamma_H(\lambda_H) \). According to Theorem 1.1, there is an essentially unique matching residue point \( W_{H^\text{FL}} \) in the spectrum of the \( \Omega \)-Iwahori Hecke algebra of \( H_{qs} \). In view of [Op3], there exists an associated Iwahori-spherical discrete series character \( \delta \) of \( H_{qs} \) whose formal degree also has the same \( q \)-rational factor as \( \gamma_H(\lambda_H) \). By the uniqueness of \( [\lambda_H] \), the characters \( \delta_{cu} \) of \( H(k) \) and \( \delta \) of \( H_{qs}(k) \) should both belong to the \( L \)-packet of the parameter \( \lambda_H \). And indeed, it can be checked [Op2] that the spectral decomposition of the (generalized) principal series induced from the cuspidal representation \( \delta_{cu} \) of \( H(k) \) to \( G(k) \) on the one hand, and of the principal series induced from the discrete series \( \delta \) of \( H_{qs}(k) \) to \( G_{qs}(k) \) on the other hand, are closely related. The spectral decomposition of the former is described by the Plancherel decomposition of a unipotent affine Hecke algebra \( H^{u,(P,\sigma),e} \) (with \( P_H := P \cap H(k) \) a maximal parahoric subgroup of \( H(k) \), and \( P_H, \sigma P_H \) a cuspidal type as in Theorem 1.1), whereas the latter is described by taking a residue of \( \mu^{IM}(G_{qs}) \) along the residual coset \( r_H T^H \), where \( T^H \subset T \) is the image in \( T \) of the connected center of the dual \( \hat{H} \) of \( H \). The relation alluded to above between these two, implies essentially that there exists a unique spectral transfer morphism \( H^{u,(P,\sigma),e} \rightarrow H^{IM}(G_{qs}) \) in the sense of [Op1].

This point of view was worked out in [Op2], bringing the classification of unipotent discrete series characters of Lusztig [Lus1], [Lus2] within the framework of classical harmonic analysis. The fundamental fact on which this method relies is our present Theorem 1.1 (and its generalization to exceptional groups).

2. The \( \mu \)-function and the Plancherel measure of \( H \)

Our aim is to prove Theorem 1.1 for all unramified classical absolutely simple groups. The case of \( G = \text{PGL}_n \) is easy, and was treated in [Op2, 3.2.2]. In all other classical cases, up to spectral covering morphisms [Op2, Subsection 3.3.5] (which are irrelevant
here, since Theorem 1.1 only deals with the $q$-rational factors), the Iwahori-Matsumoto Hecke algebra of the quasi split form of $G$ is of the form $\mathcal{H}^{IM} = \mathcal{H}(\mathcal{R}, \mathfrak{m})$ with $\mathcal{R}$ a based root system of type $C_i^{(1)}$:

$$\mathcal{R} := (R_0, X, R_0^\vee, Y, F_0) = (B_l, \mathbb{Z}^l, C_l, \mathbb{Z}^l, F_0).$$

where $F_0$ is the basis $F_0 = \{e_1 - e_2, e_2 - e_3, \ldots, e_{l-1} - e_l, e_l\}$ of the root system of type $B_l$, with $(e_i)_{1 \leq i \leq n}$ the canonical basis of $\mathbb{Z}^l$. The main results in this paper deal with the above cases. In this article, we will more generally deal with Hecke algebras of general unipotent types $(\mathbb{P}, \sigma)$ of these classical simple groups. These are all, up to spectral covering morphisms, affine Hecke algebras of type $C_i^{(1)}$. Such Hecke algebras have three independent Hecke parameters, and we will discuss the various parameter values which arise in the above context.

Since $\alpha_i = 2e_i \in 2Y$, the elements $\pm 2\alpha_i$ are added to $R_0$ to form the non-reduced root system $R_{nr}$. The subset of inmultipliable elements of $R_{nr}$ is a root subsystem $R_1$,

$$R_1 = \{\alpha \in R_{nr} : 2\alpha \notin R_{nr}\} = \{\pm e_i \pm e_j : 1 \leq i < j \leq l\} \cup \{2e_i : 1 \leq i \leq l\},$$

which is of type $C_l$.

Let $L = \mathbb{C}[v^{\pm 1}]$ be the ring of complex Laurent polynomials in indeterminate $v$, and let $T$ denote the diagonalizable group scheme with character lattice $\mathbb{Z} \times X$, viewed as a scheme over $L$ via the homomorphism $L \to \mathbb{C}[Z \times X]$ given by $v \mapsto (1, 0)$. The fibre over $v \in \text{Spec}(L)$ is denoted by $T_v$, which is naturally isomorphic to the complex algebraic torus with character lattice $X$. We will denote by $t_i$ the character of $T$ associated to $(0, e_i)$, and by $v$ the character of $T$ associated with $(1, 0)$.

Let $W := W_0 \ltimes X$ be the extended affine Weyl group, and $R = C_l^{(1)}$ its set of affine roots. To a $W$-invariant integral valued parameter function $m_R$ (see [Op1, 2.1.5]) on $R$, we associate two half-integral $W_0$-invariant functions $\alpha \mapsto m_{\pm}(\alpha)$ on the root system $R_0 = B_l$ (cf. [Op1, 2.1.10]), subject to the rule that $m_{-}(\alpha) = 0$ unless $2\alpha \in R_{nr}$. The $W$-invariant function $m_R$ contains the same information as the pair of $W_0$-invariant functions $m_{\pm}$ on $R_0 = B_l$, and the based root datum as above together with the pair of functions $m_{\pm}$ determines an affine Hecke algebra associated to the root system of type $C_l$. This affine Hecke algebra is denoted by $\mathcal{H} = C_l(m_{-}, m_{+})[q^R]$ (see [Op2, p21]), where $b m_{-} := m_{-}(e_i)$, $b m_{+} := m_{+}(e_i)$ and $b := m_{+}(\pm e_i \pm e_j)$.

We firstly assign to each root $\alpha \in R_0$ a $c$-factor $c_{m, \alpha}$ defined by

$$c_{m, \alpha} = \frac{(1 + v^{-2}m_{-}(\alpha)\alpha^{-1})(1 - v^{-2}m_{+}(\alpha)\alpha^{-1})}{1 - \alpha^{-2}}.$$ 

Next, we form the $c$-function $c_m$ by multiplying all the $c$-factors $c_{m, \alpha}$ with $\alpha$ runs through all the positive roots in $R_0$:

$$c_m = \prod_{\alpha \in R_0^+} c_{m, \alpha}.$$
The $\mu$-function of $\mathcal{H} = C_l(m_-,m_+)[q^p]$ is a $W_0$-invariant rational function on $T$ (see [Op1, 2.2]) which is explicitly given by:

\begin{equation}
\mu_{R,m_{\pm}, d} = \mu_{m_-,m_+,d}(R) = v^{-2m_{\mathcal{W}}(w_0)} \frac{d}{\epsilon_{m_{\mathcal{W}}(m)}},
\end{equation}

where $m_{\mathcal{W}} : W \to \mathbb{Z}$ is given by

$$m_{\mathcal{W}}(w) = \sum_{\alpha \in R_+ \cap w^{-1}(R_-)} m_{R}(\alpha),$$

while $w_0$ is the longest element in the finite Weyl group $W_0$, and $d$ is a normalization factor which is closely related to the formal degree (see the discussion below).

Given an $L$-valued point $\vec{b} \in T(L)$ of $T$ we define, for $\epsilon = \pm 1$,

$$p_{\epsilon}(\vec{b}) = \{ \alpha \in R_0 \mid \epsilon \alpha(\vec{b}) = v^{-m_{\epsilon}(\alpha)} \}$$

and

$$z_{\epsilon}(\vec{b}) = \{ \alpha \in R_0 \mid \epsilon \alpha(\vec{b}) = 1 \}.$$

A residue point $\vec{r} \in T(L)$ is an $L$-valued point of $T$ such that:

$$\#p_{+}(\vec{r}) + \#p_{-}(\vec{r}) - \#z_{+}(\vec{r}) - \#z_{-}(\vec{r}) = 1.$$

We remark that the residue point is a dimension 0 residual coset. Indeed, every residual coset $L \subset T$ determines a root sub-system $R_L = \{ \alpha \subset R_0 : \alpha|_L \text{ is constant} \}$. This root sub-system is parabolic in the sense that $R_L$ is equal to $\mathbb{R}R_L \cap R_0$ (the intersection of the $\mathbb{R}$-span of $R_L$ with $R_0$). Let $T_L$ be the subtorus whose cocharacter lattice $Y_L$ equals to $Y \cap \mathbb{Q}R_L^\vee$ (Y in the root datum), and let $T_L$ be the subtorus whose cocharacter lattice $Y_L$ is $Y \cap (R_L)^\perp$. Then $L$ is a coset of $T_L$ inside $T$, and there is a residual point $\vec{r} \in T_L$ such that $L = \vec{r}T_L$. For more background on residual cosets we refer to [Op3].

It is useful and necessary to consider the regularization $\mu^L$ of the $\mu$-function along a residual coset $L$ (see [Op1, 2.3.3]). If $L = \{ \vec{r} \}$ is a residual point as defined above, then $\mu^L(\vec{r})$ is the rational function on $T$ defined by omitting the factors in the denominator and in the numerator of $\mu$ which are identically equal to zero when evaluated at $\vec{r} \in T(L)$. Hence $\mu^L(\vec{r})$ defines a nonzero rational function (denoted by $\mu^L(\vec{r})$) in $v$ when evaluating at $\vec{r}$ (compare to computing the residue of a rational function at a simple pole). Explicitly, we have:

\begin{equation}
\mu^L(\vec{r}) := \mu(\vec{r})(\vec{r})
= d \frac{v^{-2m_{\mathcal{W}}(w_0)}}{\prod_{\alpha \in R_0 \setminus \{ -1 \}} (1 + \alpha^{-1}(\vec{r})) \prod_{\alpha \in R_0 \setminus \{ 1 \}} (1 - \alpha^{-1}(\vec{r}))} \in \mathbb{K}_X.
\end{equation}

Here $\mathbb{K}$ is defined to be the fraction field of $L$. In our normalization of measure, $\mu^L(\vec{r})$ is regular at $v = 1$. For a general residual coset $L = \vec{r}LT_L$, we can decompose $\mu^L$ as follows. We can assume (if necessary, use an element $w \in W_0$ to translate $L$) that
$R_L = R_P$ is a standard parabolic root sub-system defined by $P \subset F_0$. Then we have the factorization

$$
\mu^{(L)} = \mu_{R_P}^{((\tau^1))}, \mu_{R_0 \setminus R_P} := \mu_{R_P} \prod_{a \in R_0, + \setminus R_P, +} (e_{m, \alpha} c_{m, \alpha}),
$$

where $w^a = w_0 w^{-1} \in W^P$ is the longest element (see [Op1, Proposition 2.20]). The smooth density $\mu^{(L)}(\tau^1 t, t) dL$ is known to be smooth on $L^{\text{temp}} := \mathcal{T}_0 T_0'$ and, up to $W_0$-invariant, locally constant rational factors, the sum of such densities over all residual cosets $L$ is equal to the push forward of the Plancherel measure on the the spectrum of the center $Z(\mathcal{H})$ of $\mathcal{H}$ (see [Op3]).

By interpreting the roots as characters of $T$ we can write the $\mu$-function as a rational function in $v$ and $t_i \in T$. For example, in our present situation where $R_0 = B_l$ and $X = \mathbb{Z}^l$, the root system can be expressed as

$$
\{ t_i^{\pm 1} t_j^{\pm 1} : 1 \leq i < j \leq l \} \cup \{ t_k^{\pm 1} : 1 \leq k \leq l \}.
$$

Therefore we can write the normalized $\mu$-function of $C_l(m_-, m_+)[q^b]$ as:

$$
\mu_{m, d}(B_l) = \frac{d}{v^{2b m w_{w_0}}(w_0)} \prod_{1 \leq i < j \leq l} v^{4b (1 - t_i t_j)(1 - t_i t_j)^2} (1 - v^{-2b t_i t_j}) (1 - v^{-2b t_i t_j}) \times \prod_{k=1}^{l} v^{2b(m_--m_+)(1 + t_k)} (1 + v^{2b m_--m_+} t_k) (1 - v^{-2b m_--m_+} t_k).
$$

To conclude this section we describe the normalization factors $d$ appearing in the $\mu$-function. Firstly, every generic affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, m)$ can be equipped with an $L$-valued trace defined by

$$
\tau^1(N_w) = \delta_{w, e}.
$$

Such defined $\tau^1$ gives a family of $\mathbb{C}$-valued traces $\mathbb{R}_{>1} \ni v \mapsto \tau_v^1$ on the $\mathbb{R}_{>1}$-family of algebras of $\mathcal{H}_v$, the specialization of $\mathcal{H}$ at $v$. There exists a conjugate linear anti-involution $\ast$ on $\mathcal{H}_v$ defined by $N_w^\ast = N_{w^{-1}}$. It is a fundamental fact that for all $v \in \mathbb{R}_{>1}$, the 3-tuple $(\mathcal{H}_v, \tau_v^1, \ast)$ defines a type I Hilbert algebra.

Let $M'$ be the subgroup of $K^\times$ generated by $\mathbb{Q}$, by $(v - v^{-1})$ and by the $q$-integers $[n]_q := \frac{v^n - v^{-n}}{v - v^{-1}}$. Note that for all $d \in M'$ we have $d(v^{-1}) = \pm d(v)$. Now let $M \subset M'$ be the subgroup such that

$$
M = \{ d \in M' \mid d(v) > 0 \text{ if } v > 1 \}.
$$

Observe that $M$ is a free abelian group. The following definition of normalized affine Hecke algebra is borrowed from [Op1, Definition 2.13].

**Definition 2.1.** Given a generic affine Hecke algebra $\mathcal{H}$ and an $L$-valued trace $\tau^1$, we define $\tau^d = d \tau^1$ with $d \in M$. The pair $(\mathcal{H}, \tau^d)$ is called a normalized affine Hecke algebra.

The formal degree of a discrete series representation of $\mathcal{H}$ depends on the normalization of the trace of $\mathcal{H}$. If $\mathcal{H} = \mathcal{H}^{\text{unip}}_{u, (F, \sigma)}$ is a unipotent affine Hecke algebra of rank $r$ associated
to a unipotent type \((P, \sigma)\) of an unramified semisimple group \(G^u\), then we normalize the trace \(\tau\) of \(H\) by
\[
\tau^d(1) = d = (\text{vol}(P^G))^{-1} \text{deg} \sigma
\]
Here the Haar measure of \(G^u(k)\) is normalized in such a way that \(\text{vol}(P^G)\) is equal to \(v^{-\dim(P^G)}\) times the cardinality of the group of \(\mathbb{F}_q\)-points of \(P^G\). Then \(d \in M\) (cf. [Op2]), \(d^0 := (v - v^{-1})^d\) is regular and nonzero at \(v = 1\), and by [Op1, Theorem 2.28] we have:

(i) \(\text{fdeg}(\pi) \in M\), for all discrete series \(\pi\) of \(H\).
(ii) \(\text{fdeg}(\pi)(v)\) is regular and nonzero at \(v = 1\), and \(\text{fdeg}(\pi)(v^{-1}) = \text{fdeg}(\pi)(v)\).
(iii) if \(r\) is a residual point, then up to a nonzero rational constant, \(\text{fdeg}(\pi)(v)\) is equal to \(\mu^{(T)}(v, \tau^d(v))\) for all \(v > 1\).

3. Residue points as images of standard STMs

The notion of spectral transfer morphism (abbreviated to STM) was discussed in [Op1, Section 3], thus we will not give the exact definition of STM verbatim. Instead we collect some useful facts on STM in this article, aiming at the relations of STM to the residual points and formal degrees.

Recall that the spectral transfer category \(\mathcal{C}\) has normalized affine Hecke algebras as objects and equivalence classes of spectral transfer maps as morphisms (see [Op1, Definition 3.9]). Choose an affine morphism
\[
\phi_T : T_1 \to T_2
\]
which represents the STM
\[
\phi : (H_1, \tau_1^d) \rightsquigarrow (H_2, \tau_2^d)
\]
and let \([\phi] = W_{2,0} \circ \phi_T\) be the associated class of morphisms. Then by the image \(\text{Im}([\phi])\) we mean the \(W_{2,0}\)-orbit of residual cosets \(W_{2,0} \phi_T(T_1)\) which lies in \(W_{2,0} \backslash T_2\). The rank and co-rank of \([\phi]\) are respectively defined to be \(\text{rk}([\phi]) = \dim(T_1) - 1(= \text{rk}(X))\) and \(\text{cork}([\phi]) = \dim(T_2) - \dim(T_1)\). Moreover \(\phi_T\) induces a morphism over \(L\)
\[
\phi_Z : W_{1,0} \backslash T_1 \to W_{2,0} \backslash T_2
\]
by sending \(W_{1,0}(t)\) to \(W_{2,0}(\phi_T(t))\) (cf. Corollary 3.5, loc. cit.).

A rank 0 affine Hecke algebra is simply \(L\). A normalized rank 0 Hecke algebra is equipped with a trace \(\tau^0\) on \(L\), given by the normalization factor \(d^0 = \tau^0(1) \in M\). The following result is straightforward from the definitions in [Op1]:

**Proposition 3.1.** [Op1, Proposition 3.24] Let \((H, \tau^d)\) be an arbitrary normalized affine Hecke algebra with \(H = H(R, m)\), and let \(\bar{\tau}\) be a generic residual point of \((R, m)\). Define
\[
d^0(v) = \mu^{(T)}(v, \tau(v)) \in K^\times
\]
where \(\mu\) is the \(\mu\)-function associated with \(H\). Then \(d^0 \in M\) (thus defines a normalization \(\tau^0\) of \(L\)), is regular and nonzero at \(v = 1\), and \(\phi_T(v) = \tau^d(v)\) defines a rank 0 STM \(\phi : (L, \tau^d) \rightsquigarrow (H, \tau^d)\). Conversely, all STMs \(\phi\) to \((H, \tau^d)\) of \(\text{rk}(\phi) = 0\) are of this form.
This sets up a canonical bijection between the set of rank 0 STM with target \((\mathcal{H}, \tau^d)\), and the set of \(W_0\)-orbits of residual points \(\overline{\mathcal{F}} \in T(L)\) for \(\mathcal{H}\).

In [Op2, Paragraphs 3.1.2, 3.1.3] it was pointed out that every spectral transfer morphism is induced from a rank 0 spectral transfer morphism to the semisimple quotient \(\mathcal{H}_L\) of a “standard Levi subalgebra” \(\mathcal{H}^L\) of \(\mathcal{H}\). In view of this fact, we shall place our emphasis on rank 0 spectral transfer morphisms. The converse is not true at all. An STM \(\phi : (L, \tau^0) \rightsquigarrow \mathcal{H}_L\) of rank 0 corresponds to an orbit of residual points \(W_L \overline{\mathcal{F}}_L \in T_L\). This induces the \(W_0\)-orbit of residual cosets of the form \(\overline{\mathcal{F}}_L \mathbb{T}^L\), and a Plancherel density as in (2.3). To say that this is the image of an STM of higher rank is essentially equivalent to saying that \(\mu^{(L)}\) as in (2.3) “is” the \(\mu\)-function of another affine Hecke algebra of rank \(\text{dim}(L)\). In general this cannot be the case, since \(\mu^{(L)}\) has too many poles for this to be possible (so apparently, if \(W_0L\) is the image of an STM, then many cancellations of poles and zeroes of \(\mu^{(L)}\) have to occur). We call \(\phi\) cuspidal if \(W_0L\) is the image of an STM. With this terminology we can summarize the above discussion by the statement that every STM is induced from a cuspidal STM of a standard sub-quotient algebra of the form \(\mathcal{H}_L\).

A truly remarkable fact is that if we restrict ourselves to the collection of unipotent normalized Hecke algebras related to a given unramified absolutely almost simple group \(G\) over \(k\) (so the normalized affine Hecke algebras associated to the unipotent types of the \(k\)-Levi subgroups of inner forms of \(G\)), then every rank 0 STM to an \(\mathcal{H}_L\) as above is cuspidal in the sense just explained. This statement follows from the main result of [Op2] and our Theorem 1.1.

As was pointed out above, we will concentrate on normalized affine Hecke algebras associated to the unitary, orthogonal and symplectic groups of adjoint types. In [Op2, Paragraph 3.2.4] an associated collection \(C_{\text{class}}\) of affine Hecke algebras was introduced, whose objects are normalized affine Hecke algebras of type \((G_l(m_-, m_+)[q^b], \tau_{m_-, m_+})\) with rank \(l \in \mathbb{Z}_{\geq 0}\). The pair of parameters \((m_-, m_+)\) lies in the parameter space \(V\), which is the collection of ordered pairs \((m_-, m_+)\) of elements \(m_\pm \in \mathbb{Z}/4\) such that \(2(m_- - m_+) \in \mathbb{Z}\). If both \(m_+ - m_- \in \mathbb{Z}\) then \(b = 1\), otherwise we put \(b = 2\).

Following [Op2, Paragraph 3.2.4] we now decompose \(V\) into six disjoint subsets \(V^\lambda\) with \(\lambda \in \{I, II, III, IV, V, VI\}\). If \(m_\pm \in \mathbb{Z} \pm \frac{1}{4}\) and \(m_\pm > 0\) we write

\[m_\pm = \kappa_\pm + \frac{2\epsilon_\pm - 1}{4}\]

with \(\epsilon_\pm \in \{0, 1\}\) and \(\kappa_\pm \in \mathbb{Z}_{\geq 0}\). Define \(\delta_\pm \in \{0, 1\}\) by \(\kappa_\pm = \delta_\pm + 2\mathbb{Z}\). In other words \(\delta_\pm\) indicates the parity of \(\kappa_\pm\). Now we can define the decomposition as follows:

- \((l; (m_-, m_+)) \in V^I\) iff \(m_\pm \in \mathbb{Z}/2\) and \(m_- - m_+ \notin \mathbb{Z}\).
- \((l; (m_-, m_+)) \in V^{II}\) iff \(m_\pm \in \mathbb{Z}/2\) and \(m_- - m_+ \in \mathbb{Z}\).
- \((l; (m_-, m_+)) \in V^{III}\) iff \(m_\pm \in \mathbb{Z}\) and \(m_- - m_+ \notin 2\mathbb{Z}\).
- \((l; (m_-, m_+)) \in V^{IV}\) iff \(m_\pm \in \mathbb{Z}\) and \(m_- - m_+ \in 2\mathbb{Z}\).
- \((l; (m_-, m_+)) \in V^{V}\) iff \(m_\pm \in \mathbb{Z} \pm \frac{1}{4}\) and \(\delta_- \neq \delta_+\).
- \((l; (m_-, m_+)) \in V^{VI}\) iff \(m_\pm \in \mathbb{Z} \pm \frac{1}{4}\) and \(\delta_- = \delta_+\).
These collections of unipotent affine Hecke algebras are associated to unipotent types \((P, \sigma)\) of the following isogeny classes of unramified classical groups (in the same order): Special unitary groups, special odd orthogonal groups, symplectic groups, special even orthogonal groups, symplectic groups and special even orthogonal groups. The last two of these cases are associated with unipotent types of the inner forms \(G^u\) where \(u\) corresponds (via Kottwitz’s theorem) to the action of the center \(LZ\) of \(LG\) in its spin representations. These cases play a special role, and the associated affine Hecke algebras are called extra-special.

We remark that both the type of parameters is independent of the rank \(l\), and the trace \(\tau_{m_-, m_+}\) depends on \(l\) in the simple uniform fashion

\[
\tau_{m_-, m_+}((v^b - v^{-b})^l) = d^0_{m_-, m_+}
\]

where \(d^0_{m_-, m_+}\) is independent of \(l\). The value of \(d^0_{m_-, m_+}\) is described in (3.2) below, in terms of degrees of cuspidal unipotent characters of finite groups of Lie type. The normalized affine Hecke algebra \((C_l(m_-, m_+)[q^b], \tau_{m_-, m_+})\) thus defined is often denoted by \((H^c_{m_-, m_+}, \tau_{m_-, m_+})\).

Lusztig has determined the formal degrees of cuspidal unipotent representations of reductive \(p\)-adic groups (based on the cuspidal unipotent representations of finite groups of Lie types). The cuspidal unipotent representations are rare, in particular, each of the classical groups of type \(A_l, 2A_l, B_l, C_l, D_l, 2D_l\) has at most one cuspidal unipotent representation (we do not consider type \(3D_4\) here). The conditions for these classical types having cuspidal unipotent representations are imposed onto their ranks \(l\). These conditions as well as the corresponding degrees of cuspidal unipotent representations can be found in \([\text{Car}, \S 13.7]\). They are rational functions in \(q\) with rational coefficients. For those various \(s\) which would provide cuspidal unipotent characters for each type as above, we write \(d^Y_s(q)\) for the formal degree of the cuspidal unipotent character for the adjoint group of type \(Y \in \{2A_l, B_l, C_l, D_l, 2D_l\}\) (where we normalize the volume of the group of \(\mathbb{F}_q\)-points to 1). We view \(d^Y_s(q)\) as a function in \(q\).

Table 1. Cuspidal unipotent representations for classical groups

| Types | rank \(l\) | No. of cuspidal unipotent characters |
|-------|------------|-----------------------------------|
| \(A_l\) | 0          | Only trivial representation       |
| \(2A_l\) | \(l = (s + 1)/2 - 1\) for some \(s\) | 1                                |
| \(B_l\) | \(l = s + 1\) for some \(s\) | 1                                |
| \(C_l\) | \(l = s + 1\) for some \(s\) | 1                                |
| \(D_l\) | \(l = s^2\) for some even integer \(s\) | 1                                |
| \(2D_l\) | \(l = s^2\) for some odd integer \(s\) | 1                                |

From Carter’s book \(op. cit\) we witness that the formal degrees of the cuspidal unipotent characters of \(B_l\) and \(C_l\) are equal and hence we write \(d^B_s\) for this degree. Also we write \(d^D_s\) for the degree of both type \(D\) and type \(2D\). We can prove the following properties of these formal degrees using elementary considerations.
Proposition 3.2. Suppose $s \in \mathbb{Z}_{\geq 0}$. We regard $d^S_a$ and $d^D_a$ functions in $s$.

(i) These formal degrees of cuspidal unipotent representations are elements of the free abelian group $\mathbf{M}$, and are linearly independent in $\mathbf{M}$.

(ii) Apart from a power of $q$, $d^S_a$ and $d^D_a$ are products of reciprocals of even cyclotomic polynomials $\Phi_{2m}$ (with $m \in \mathbb{Z}_+$).

As in [Op2, Equation (33)] we define:

$$d^0_{m-,m+} = \tau^0_{m-,m+}(1) := \begin{cases} 
\frac{d^{(2A)}_a(q)d^{(2A)}_b(q)}{d^{(2A)}_a(q)d^{(2A)}_b(q)} & \text{if } (m-,m+) \in \mathbb{V}^I \\
\frac{d^{(2A)}_a(q)d^{(2A)}_b(q)}{d^{(2A)}_a(q)d^{(2A)}_b(q)} & \text{if } (m-,m+) \in \mathbb{V}^II \\
\frac{d^{(2A)}_a(q)d^{(2A)}_b(q)}{d^{(2A)}_a(q)d^{(2A)}_b(q)} & \text{if } (m-,m+) \in \mathbb{V}^III \\
\frac{d^{(2A)}_a(q)d^{(2A)}_b(q)}{d^{(2A)}_a(q)d^{(2A)}_b(q)} & \text{if } (m-,m+) \in \mathbb{V}^IV \\
\frac{d^{(2A)}_a(q)d^{(2A)}_b(q)}{d^{(2A)}_a(q)d^{(2A)}_b(q)} & \text{if } (m-,m+) \in \mathbb{V}^V \\
\frac{d^{(2A)}_a(q)d^{(2A)}_b(q)}{d^{(2A)}_a(q)d^{(2A)}_b(q)} & \text{if } (m-,m+) \in \mathbb{V}^VI \\
\end{cases}$$

(3.2)

where $\{a,b : a,b \in \mathbb{Z}_{\geq 0}\}$ is determined by the following equalities of sets:

$$\begin{align*}
\{1/2 + a, 1/2 + b\} &= \{\lfloor m+ - m_{-}, |m+ + m_{-}| \rfloor \} & \text{if } (m-,m+) \in \mathbb{V}^I \\
\{2a, 1/2 + b\} &= \{\lfloor m+ - m_{-}, |m+ + m_{-}| \rfloor \} & \text{if } (m-,m+) \in \mathbb{V}^II \\
\{1/2 + a, 1/2 + b\} &= \{\lfloor m+ - m_{-}, |m+ + m_{-}| \rfloor \} & \text{if } (m-,m+) \in \mathbb{V}^III \\
\{2a, 2b\} &= \{\lfloor m+ - m_{-}, |m+ + m_{-}| \rfloor \} & \text{if } (m-,m+) \in \mathbb{V}^IV \\
\{1/2 + a, 1 + 2b\} &= \{\lfloor m+ - m_{-}, |m+ + m_{-}| \rfloor \} & \text{if } (m-,m+) \in \mathbb{V}^V \\
\{1/2 + a, 2b\} &= \{\lfloor m+ - m_{-}, |m+ + m_{-}| \rfloor \} & \text{if } (m-,m+) \in \mathbb{V}^VI
\end{align*}$$

(3.3)

Observe that these equalities determine $a$ and $b$ in case II, V and VI, and determines $a$ and $b$ up to order in the other cases. Thus the normalization (3.2) is always well defined.

We remark that the normalization for the last two cases V and VI (the ”extra-special cases”) can be conveniently summarized in the following uniform formula (cf. [Op2, Equation (36)]):

$$d^0_{m-,m+} = \prod_{i=1}^{\lfloor m_{-} - m_{+}\rfloor} \frac{q^{2|m_{-} - m_{+}| - 2i}}{(1 + q^{2|m_{-} - m_{+}| - 2i})^j} \prod_{j=1}^{\lfloor m_{+} + m_{+}| - 2j\rfloor} \frac{q^{2|m_{-} + m_{+}| - 2j}}{(1 + q^{2|m_{-} + m_{+}| - 2j})^j}.$$

(3.4)

Define

$$\begin{align*}
\eta_+ & : m+ \mapsto -m+, & m- \mapsto m- \\
\eta & : m+ \mapsto m-, & m- \mapsto m+
\end{align*}$$

Then the group $\text{Iso} = \langle \eta_+, \eta \rangle$ is isomorphic to the dihedral group of order 8. This group acts on $\mathcal{C}_{\text{class}}$ by spectral isomorphisms (cf. [Op1, Paragraph 3.3.3]). We define $\eta_- = \eta \circ \eta_+ \circ \eta$.

Besides these spectral isomorphisms, in [Op2, Paragraph 3.2.4] definitions were given of a set of basic for STMs between the Hecke algebras $\mathcal{C}_{\mathcal{I}}(m-,m_+)[q^n] \in \mathcal{C}^{\mathcal{V}}_{\text{class}}$ (with $\mathcal{X} \in \{\mathcal{I}, \mathcal{II}, \mathcal{III}, \mathcal{IV}\}$) and for the STMs of $\mathcal{C}^{\mathcal{V}}_{\text{class}}$ to $\mathcal{C}^{\mathcal{V}}_{\text{class}}$ and the “extra special cases” $\mathcal{C}^{\mathcal{V}}_{\text{class}}$ to $\mathcal{C}^{\mathcal{V}}_{\text{class}}$, and it was shown [Op2, Proposition 3.8] that these basic STMs generate all STMs between such types of normalized affine Hecke algebras.
Recall that the basic generating steps for the standard STMs are (reviewing some of the notations of \([\text{Op}2]\)) of the form:

\[
\begin{align*}
C_r(m_-, m_+)[q^2] &\rightsquigarrow C_{r+|m_+|-\frac{1}{2}}(m_- - \epsilon(m_-), m_+)[q^2] \quad \text{if } (m_-, m_+) \in \mathcal{V}_1, m_+ \notin \mathbb{Z} \\
C_r(m_-, m_+)[q^2] &\rightsquigarrow C_{r+2|m_+|-2}(m_-, m_+ - 2\epsilon(m_+))[q^2] \quad \text{if } (m_-, m_+) \in \mathcal{V}_1, m_+ \in \mathbb{Z} \\
C_r(m_-, m_+)[q] &\rightsquigarrow C_{r+|m_+|-\frac{1}{2}}(m_- - \epsilon(m_-), m_+)[q] \quad \text{if } (m_-, m_+) \in \mathcal{V}_I \\
C_r(m_-, m_+)[q] &\rightsquigarrow C_{r+2|m_+|-2}(m_-, m_+ - 2\epsilon(m_+))[q] \quad \text{if } (m_-, m_+) \in \mathcal{V}_{III} \\
C_r(m_-, m_+)[q^2] &\rightsquigarrow C_{2r+\frac{1}{2}a(a+1)+2b(b+1)}(\delta_-, \delta_+)[q] \quad \text{if } (m_-, m_+) \in \mathcal{V}_V \\
C_r(m_-, m_+)[q^2] &\rightsquigarrow C_{2r+\frac{1}{2}a(a+1)+2b^2-\delta_+ \delta_-}[q] \quad \text{if } (m_-, m_+) \in \mathcal{V}_VI
\end{align*}
\]

The underlying affine morphisms of algebraic tori for these generators were given in \([\text{Op}2, 3.2.5]\). The computations to verify that these maps indeed represent STMs were briefly indicated in \([\text{Op}2, 3.2.5]\). For more details we refer the reader to \([\text{Fe}1, \text{Fe}2]\) for more details.

Let us temporarily accept the fact that standard STMs exist. Then among the objects of \(\mathfrak{C}_{\mathcal{X}}^{class}\) with \(\mathcal{X} \in \{I, II, III, IV\}\), the minimal (or least) spectral isogeny classes of objects (in the sense of \([\text{Op}1, \text{Definition 3.25}]\)) are the following:

\[
\begin{align*}
[C_I(\frac{1}{2}, 0)[q^2]] \text{ and } [C_I(\frac{1}{2}, 1)[q^2]] & \quad \text{if } \mathcal{X} = I, \\
[C_I(\frac{1}{2}, \frac{1}{2})[q]] & \quad \text{if } \mathcal{X} = II, \\
[C_I(0, 1)[q]] & \quad \text{if } \mathcal{X} = III, \\
[C_I(0, 0)[q]] \text{ and } [C_I(1, 1)[q]] & \quad \text{if } \mathcal{X} = IV.
\end{align*}
\]

Using the standard generators indicated above, we can reach such minimal object by an STM from any of the Hecke algebras in \(\mathfrak{C}_{\mathcal{X}}^{class}\). Indeed, note first of all that using the action of the group of spectral isomorphism \(\text{Iso}\), we can interchange \(m_-\) and \(m_+\), and moreover map \(m_\pm\) to \(-m_\pm\). Hence we may assume that \(m_\pm\) are both nonnegative. In that situation the STMs listed here strictly decrease the value of \(m_- + m_+\). We can continue with this until we reach an object that is isomorphic to one of the four least objects listed above.

On the other hand, these minimal objects are closely related to the Iwahori-Matsumoto Hecke algebra of connected unramified quasi-split classical groups of special unitary, special odd orthogonal, symplectic and special even orthogonal type, via spectral covering morphisms (cf. \([\text{Op}1, 3.3.4, 3.3.5]\)).

We remark that the order in which we apply the basic translation STMs is not important since they commute, as can be easily checked by a direct computation using the explicit formulas of \([\text{Op}2, 3.2.5]\) (also see \([\text{Fe}2]\)). Also it follows from an easy direct verification that the commutation relations between these translation STMs, extra special STMs and elements of the group \(\text{Iso}\) of spectral isomorphisms, are simply via the action of \(\text{Iso}\) on the pair of parameters \((m_-, m_+)\) of the Hecke algebra. We will refer to the STMs of \(\mathfrak{C}_{\mathcal{X}}^{class}\) which are generated by the generators above and by the elements of \(\text{Iso}\): Standard STMs. The elementary arguments above show that a standard STM between two objects of \(\mathfrak{C}_{\mathcal{X}}^{class}\) as indicated above is essentially unique.
Thus for each object of $\mathfrak{C}_{\text{class}}$ there is an essentially unique (i.e. unique up to spectral automorphisms) standard STM to one of these least objects in $\mathfrak{C}_{\text{class}}^X$, with $X \in \{I, II, III, IV\}$. Note that the STMs from objects of type $X = V$ or $VI$ are in a different nature, and map in one step to a least object of type $\{III, IV\}$ respectively. These are called the extra special STMs. In Section 4 we describe the extra-special algorithms which allow us to deal with these extra-special STMs. Existence of these extra-special STMs will also be proved in detail in [Fe2].

**Remark 3.3.** The fact that all STMs between two objects of $\mathfrak{C}_{\text{class}}$ as indicated above are standard (and thus that the STMs above basic set together with the elements of Iso generates all STMs between these type of Hecke algebras) is a deeper fact that rests on our present Theorem 1.1 (cf. [Op2, Proposition 3.8] and its proof). We will not use this fact in the present paper, of course, we will only use the existence of standard STMs and their basic properties.

Note that in cases $X = II, IV$ these minimal objects admit a group $\text{Aut}_{es}$ of order 2 of spectral automorphisms, which arise from special affine diagram automorphisms (cf. [Lus1, 1.11]), and for the other cases there are no essentially strict spectral automorphisms.

Given an unramified, connected almost absolutely simple group $G_{qs} = G_{qs}(K)$, defined and quasi-split over $k$, of unitary, orthogonal or symplectic type, and of adjoint type in its isogeny class, let $R_{qs}$ be the relative based root datum which we have constructed in the introduction. Let $H^H(\mathfrak{C}_{qs}) = H(R_{qs}, m_{qs})$ be the Iwahori-Hecke algebra of $G_{qs}$.

**Proposition 3.4 ([Op2]).** There exists a spectral covering map $H^H(G_{qs}) \rightarrow H^X_{\min}$ to one of the minimal objects $H^X_{\min} = H(R, m^X_{\min})$ of $\mathfrak{C}_{\text{class}}^X$. This gives rise to a canonical bijection between the set of $\Omega^H_{ad} \ast$-orbits in $\text{Res}_L(R_{qs}, m_{qs})$ and the set of $\text{Aut}_{es}(H^X_{\min})$-orbits in $\text{Res}_L(R, m^X_{\min})$.

**Proof.** See the description in [Op2, 3.2.6]. The precise relation between the isogeny class of $G_{qs}$ and the type $X$, as well as the covering group of $H^H \rightarrow H^X_{\min}$ was described in [Op2, 3.2.6]. From that description we see easily that the elements of $\text{Aut}_{es}(H^X_{\min})$ always lift to elements of $\Omega^H_{ad} \ast$.

As a consequence, we can formulate Theorem 1.1(a), (b) in terms of $H^X_{\min}$ as follows. This is quite useful, because it reduces most of the work to considering the Hecke algebras of $\mathfrak{C}_{\text{class}}^X$.

**Proposition 3.5.** Let $H^H(G_{qs}) \rightarrow H^X_{\min}$ be as in Proposition 3.4. Let $G = G_{qs}^u$ be an inner form of $G_{qs}$. Let $(\mathcal{P}, \sigma)$ be a maximal $F_u$-stable cuspidal unipotent type of $G = G(K)$. Then there exists a unique $\text{Aut}_{es}(H^X_{\min})$-orbit $W_0 \mathcal{P} \in \text{Res}_L(R, m^X_{\min})$ such that $(\mu^X_{\min})_q((\tau_1)) = \text{fdeg}_q(\sigma)$.

This proposition will be proved in Section 6. Using the existence ([Op2], [Fe2]) of standard STMs to $H^X_{\min}$ we can still sharpen the formulation a little bit. Let $(\mathcal{P}, \sigma)$ be as above. Then, by definition of $\mathfrak{C}_{\text{class}}$, $H^V_0 := (L, d^0 = \text{fdeg}_q(\sigma))$ is a rank-0 object of
Corollary 3.6. Let $W_0\tau^* \in \text{Res}_L(\mathcal{R}, m^X_{\text{min}})$ such that $(\mu^X_{\text{min}}|q) = \text{fdeg}_q(\sigma)$ for some maximal $F_q$-stable unipotent type $(\mathcal{P}, \sigma)$ of some inner form $G = G_{\text{qs}}^u$ of $G_{\text{qs}}$. Then $W_0\tau^*$ represents a rank-0 standard STM $\phi : \mathcal{H}^0_0 \sim \mathcal{H}^X_{\text{min}}$, which is unique up to the action of $\text{Aut}_{\text{es}}(\mathcal{H}^X_{\text{min}})$, where $\mathcal{Y} = \mathcal{X}$ or $\mathcal{X} = \text{III}$ and $\mathcal{Y} = \text{V}$, or $\mathcal{X} = \text{IV}$ and $\mathcal{Y} = \text{VI}$.

There is another important simplification. Namely [Op2, Proposition 3.13] and the existence of the extra special STMs imply that all residue degrees of the form $(\mu^X_{\text{min}}|q) = \text{fdeg}_q(\sigma)$ for some unipotent cuspidal pair $(\mathcal{P}, \sigma)$.

4. DESCRIPTION OF THE EXTRA-SPECIAL ALGORITHMS

Let $P_{\text{odd, dist}}$ be the collection of all partitions (including the zero partition) with odd, distinct parts. Let

$$ R = \{(m, \rho) \mid m \in \mathbb{Z} \pm 1/4, m > 0 \text{ and } \rho \text{ a (possibly zero) partition}\}. $$

We will define two operations $E : P_{\text{odd, dist}} \to R$ and $D : R \to P_{\text{odd, dist}}$ and prove they are inverse to each other. We refer to $E$ as the extra-special algorithm.

We first describe $E : \lambda \mapsto (m, \rho)$. Recall the notion of an $m$-tableau (cf. [Slo], [HO]): For a real number $m$ and a partition $\lambda$, we define the $m$-tableau $T_m(\lambda)$ of $\lambda$ as the tableau with shape $\lambda$ with its box $b_{i,j}$ (where the coordinates $(i,j)$ have the same meaning as for matrix entries) filled with the nonnegative real number $|m - i + j|$. The algorithm $E$ produces a number $m \in (\mathbb{Z} \pm 1/4)_+$, and an $m$-tableau, whose shape we call $\rho$. The steps to produce $m$ and $T_m(\rho)$ from $\lambda \in P_{\text{odd, dist}}$ are as follows:

1. Write $\lambda$ as a non-negative integral sequence in decreasing order. Define $j = (\lambda - 1)/2$, where $(\lambda - 1)/2$ means subtracting 1 from all nonzero parts of $\lambda$, and then dividing each part by 2.

We stress that we do not regard $\lambda$ as a partition, but as a tuple of nonnegative integers, whose length is equal to the number of nonzero parts of $\lambda$.

2. Let $\kappa \geq 0$ be the excess number of parts of the dominant parity type (even or odd) of $j$. Put $\varepsilon = 1$ if the dominant parity type is odd or if $\kappa = 0$ (in which case we shall call the dominant parity type odd as well), otherwise put


\[ \varepsilon = 0. \]
\[ \varepsilon = 0. \text{ Let } m = \kappa + (2\varepsilon - 1)/4. \text{ This gives us the required number } m \in (\mathbb{Z} \pm 1/4)_+. \]

(3) Let \( j' = (\gamma_1, \ldots, \gamma_\kappa) \) be the sub-sequence in \( j \) of the \( \kappa \) smallest parts of dominant parity type.

(4) Removing \( j' \) from \( j \) and denote the remaining sub-sequence of \( j \) by \( j'' \). Thus \( j'' \) has an equal number parts of both parities. Arrange \( j'' \) in \( t \) pairs:

\[ j'' = (((\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)) \]

with \( \alpha_1 > \cdots > \alpha_t \) and \( \beta_1 > \cdots > \beta_t \), where for all \( i \), \( \alpha_i \) is of dominant parity type and \( \beta_i \) is of the other parity type.

(5) For every pair \((\alpha_i, \beta_i)\) we denote by \( T_m(H(\alpha_i, \beta_i)) \) the hook-shaped \( m \)-tableau whose hand (the box at the end of its arm) is filled with \((\alpha_i - 1/2)/2, \) and whose foot has filling \(|\beta_i - 1/2)/2|/2.\)

Note that we need to take the absolute value in the latter expression since it might happen that the smallest part of \( j \) is 0. If \( \varepsilon = 1 \) then this part 0 of \( j \) will appear as \( \beta_\ell = 0 \) in the smallest pair \((\alpha_\ell, \beta_\ell) \) of \( j' \). Also observe that if \( \varepsilon = 0 \) then \( \kappa > 0 \), and then this part 0 of \( j \) will appear as the smallest part \( \gamma_\kappa \) of \( j' \). In particular, we always have \((\alpha_i - 1/2)/2 > 0 \) for all \( i \).

Let \( T_m(H) \) be the \( m \)-tableau obtained by nesting the hook shaped tableaux \( T_m(H(\alpha_i, \beta_i)) \) in decreasing order. Observe that all hooks \( T_m(H(\alpha_i, \beta_i)) \) contain a box with filling \( m \) (namely the box at the corner) and a box with filling \( 1/4 \) (and if \( m = 1/4 \) then these two boxes coincide). We call such hooks \( m \)-hooks. Hence the leg of an \( m \)-hook has length at least \( \kappa \), since its corner box has filling \( m \) and the box with filling \( 1/4 \) is precisely \( \kappa \) boxes below that.

(6) We add horizontal strips \( S_i \) (which may be empty) to \( T_m(H) \) (for \( i = 1, \ldots, \kappa \)). If \( \gamma_i < 2(m - i + 1) + 1/2 \), then the strip \( S_i \) is empty. Otherwise \( S_i \) is the horizontal \((m - i + 1)\)-tableau whose rightmost extremity has filling \(|\gamma_i - 1/2)/2 (\text{again, we need the absolute value because it might happen that } \gamma_\kappa = 0, \text{ namely if } j \text{ contains 0 as a part, and if in addition } \varepsilon = 0). \text{ These horizontal strips can be added to } T_m(H) \text{ in a unique way such that the union is an } m \text{-tableau (so } S_1 \text{ is placed at the “armpit” of } T_m(H), \text{ and } S_{i+1} \text{ is placed just below } S_i \text{ for } i = 1, \ldots, \kappa - 1). \text{ Observe that for all } i \text{ the parity type of } \gamma_i \text{ is } \varepsilon. \text{ The smallest possible value of } \gamma_i \text{ equals } \gamma_i = 2(m - i + 1) - 3/2, \text{ corresponding to } S_i \text{ being empty. In particular, the smallest possible value of } j_\kappa \text{ is } j_\kappa = \varepsilon. \text{ If all strips } S_i \text{ are empty, we have:}

\[ j' = (\varepsilon + 2(\kappa - 1), \varepsilon + 2(\kappa - 2), \ldots, \varepsilon) \]

Denote the union of the strips \( S_i \) by \( T_m(S) \). Then \( T_m(S) \) is either an \( m \)-tableau or empty.
Finally, \( \rho \) is the partition whose Young diagram is the shape formed by the union of the \( m \)-hooks \( H \) and strips \( S_i \) in the way indicated above.

As mentioned above, we call the hook-shaped \( m \)-tableaux \( T_m(H(\alpha_i,\beta_i)) \) the \( m \)-hooks of \( T_m(\rho) \). Observe that the \( m \)-hooks of \( T_m(\rho) \) are precisely the hooks in \( T_m(\rho) \) which are \( m \)-tableaux, and contain \( 1/4 \). Equivalently, a hook of \( T_m(\rho) \) which is an \( m \)-tableau is an \( m \)-hook if and only if its leg length is at least \( \kappa \).

Example 4.1. Let \( \lambda = (21,19,13,9,5,3) \) be an odd distinct partition. We have already ordered the parts of \( \lambda \) in decreasing order. Then we have \( j = (10,9,6,4,2,1) \). We now have 4 even numbers and 2 odd numbers. Thus the dominant parity type is even (hence \( \varepsilon = 0 \)) and the excess number \( \kappa = 4 - 2 = 2 \). We get \( m = \kappa + (2\varepsilon - 1)/4 = 7/4 \).

Now \( j' = (4,2) \). Removing \( j' \) from \( j \) we obtain \( j'' = ((10,9),(6,1)) \). Therefore we will have 2 hook-shaped \( m \)-tableaux and 2 horizontal strips. The Young tableau is as follow:

\[
\begin{array}{cccccc}
7/4 & 11/4 & 15/4 & 19/4 \\
3/4 & 7/4 & 11/4 \\
1/4 & 3/4 & 7/4 \\
5/4 & 1/4 & 3/4 \\
9/4 \\
13/4 \\
17/4 \\
\end{array}
\]

The two horizontal strips are in colours.

We next describe the operation \( D \), namely how to recover the odd distinct partition \( \lambda \in P_{\text{odd, dist}} \) from \( (m,\rho) \in R \).

Recall that \( m \in (Z \pm 1/4)_+ \). Let \( \kappa \) be the closest integer to \( m \) and write \( m = \kappa + (2\varepsilon - 1)/4 \) with \( \varepsilon = 0 \) or \( 1 \). This uniquely determines a parity type \( \varepsilon \) and a nonnegative integer \( \kappa \). Define \( \delta \in \{0,1\} \) by \( \kappa = \delta + 2\mathbb{Z} \).

The \( m \)-tableau \( T_m(\rho) \) can be written as a disjoint union of nested \( T_m(\rho) \)-hooks which are themselves \( m \)-tableaux. If one of these hook shapes is an \( m \)-hook than all its predecessors are \( m \)-hooks too (since the condition for such hook to qualify as \( m \)-hook is that its leg length is at least \( \kappa \)). Hence \( T_m(\rho) \) has a unique decomposition as the union \( T_m(H) \cup T_m(S) \) of two \( m \)-tableaux (both possibly empty) such that \( H \) is the largest \( m \)-tableau contained in \( T_m(\rho) \) which is a union of \( m \)-hooks, and \( S \) is the complement of \( H \) in \( T_m(\rho) \). By the above we see that \( S \) is itself an \( m \)-tableau (or empty) without
We number the shapes of the nested $m$-hooks in $T_m(\rho)$ in decreasing order as $H_1, \ldots, H_t$. For every $i$, $T_m(H_i)$ defines unique pair of nonnegative integers $(\alpha_i, \beta_i)$ such that $T_m(H_i) = T_m(H(\alpha_i, \beta_i))$. Indeed, if the hand of $T_m(H_i)$ has filling $A_i \in (\mathbb{Z} \pm 1/4)_+$ and its foot has filling $B_i \in (\mathbb{Z} \pm 1/4)_+$, then $\alpha_i$ is the unique integer of parity type $\varepsilon$ nearest to $2A_i \in (\mathbb{Z} + 1/2)_+$ (which is easily seen to be $2A_i + 1/2$), and $\beta_i$ is the unique integer of parity type $1 - \varepsilon$ nearest to $2B_i \in (\mathbb{Z} + 1/2)_+$. In particular, every such pair $(\alpha_i, \beta_i)$ consists of nonnegative integers with opposite parity type.

Recall that $S$ itself is an $m$-tableaux (or empty) with at most $\kappa$ parts. Let $S_1, \ldots, S_\kappa$ denote the list of rows of $S$ (where some of the rows, or even all of them, may be empty). Let the rightmost box of $S_i$ be filled with $C_i \in (\mathbb{Z} \pm 1/4)_+$. If $S_i$ is empty, we define $C_i = |m - i| \in (\mathbb{Z} \pm 1/4)_+$ (this is the filling of the rightmost box of the $(t + i)$-th row of $T_m(\rho)$, provided that $t \neq 0$ (otherwise $S = T_m(\rho)$, in which case this row is empty by assumption)). Define $\gamma_i$ for $i = 1, \ldots, \kappa$, as the unique nonnegative integer of parity type $\varepsilon$ nearest to $2C_i \in (\mathbb{Z} \pm 1/2)_+$.

This determines a set of pairs of nonnegative integers (of opposite parity) $(\alpha_i, \beta_i)$, and a set of nonnegative integers $\gamma_j$ uniquely. Observe that these integers are mutually distinct.

Now we form the descending list $j$ of the numbers $(\alpha_i, \beta_i)$ (for $i = 1, \ldots, t$) and the $\gamma_j$ (for $j = 1, \ldots, \kappa$). Observe that the length of $j$ is at least $\kappa$ (namely, we have at least the numbers $\gamma_j$ for $i = 1, \ldots, \kappa$ in our list). Finally we define $\lambda := 2j + 1$. Observe that $\lambda$ has distinct, odd parts, and that $\delta$ (the parity of $\kappa$) is also the parity of the number of parts of $\lambda$.

We can also form the descending list $e$ of the numbers $(\alpha_i - 1/2)/2$ the $(\beta_i - 1/2)/2$ (for $i = 1, \ldots, t$) and the $(\gamma_j - 1/2)/2$ (for $j = 1, \ldots, \kappa$). Observe that the list $e$ may contain at most one negative number as an entry, namely $-1/4$. Then $j = 2e + 1/2$, and the list $|e|$ of absolute values of $e$ is the list consisting of the fillings of arms (the $A_i$) and feet (the $B_i$) of the $m$-hooks $H_i$ of $T_m(\rho)$ (for $i = 1, \ldots, t$), combined with the list of the $C_j$ (for $j = 1, \ldots, \kappa$).

**Theorem 4.2.** The operations $E : P_{\text{odd,dist}} \to R$ and $D : R \to P_{\text{odd,dist}}$ are inverse bijections.

We will not give the formal proof (which is not hard, see [Fe2]). Instead, we will give some illustrative examples.

**Example 4.3.** Let us give some examples of the operator $D$:

1. Let $m = 1/4$ and $\rho$ be zero. Then $\lambda$ is zero as well. On the other hand, if $m = 3/4$ and $\rho$ is zero, then we have no $m$-hooks, but since $\kappa = 1$ and $\varepsilon = 0$, we have one empty strip $S_1$, which yields $C_1 = 1/4$, and $\gamma_1 = 0$. Hence $e = (-1/4)$, and $\lambda = [1]$. 

Let \( m = 15/4 \) and \( \rho \) be the zero partition. Then \( m = 4 - 1/4 \) gives \( \kappa = 4 \) and \( \varepsilon = 0 \). We have no \( m \)-hooks, and \( \kappa = 4 \) strips which are all empty. Hence \( C_1 = m - 1 = 11/4 \), \( C_2 = m - 2 = 7/4 \), \( C_3 = m - 3 = 3/4 \) and \( C_4 = \lfloor m - 4 \rfloor = 1/4 \), and thus \( \gamma_1 = 6 \), \( \gamma_2 = 4 \), \( \gamma_1 = 2 \), \( \gamma_1 = 0 \). So \( e = (11/4, 7/4, 3/4, -1/4) \). We obtain the odd distinct partition \( \lambda = [1, 5, 9, 13] \) (let us agree that we may also denote a partition in increasing order, using square brackets as delimiters).

For a singleton \( m = 1/4 \) we have \( \kappa = 0 \), \( \varepsilon = 1 \). Hence we have one hook, and no strips (even no empty ones!). We find that \( e = (1/4, -1/4) \); thus we get \( \lambda = [1, 3] \).

Consider the following tableau:

\[
\begin{array}{c}
5/4 \\
1/4
\end{array}
\]

Here \( m = 5/4 = 1 + 1/4 \). Therefore \( \kappa = 1 \) and \( \varepsilon = 1 \). We have one \( m \)-hook, and one empty strip. Thus \((A_1, B_1) = (5/4, 1/4)\), and \(((\alpha_1 - 1/2)/2, (\beta_1 - 1/2)/2) = (5/4, -1/4)\). In addition the empty strip \( S_1 \) yields \( C_1 = \lfloor m - 1 \rfloor = 1/4 \), and thus \( \gamma_1 = 1 \). Thus we form the descending list \( e = (5/4, 1/4, -1/4) \) and recover the odd distinct partition \( \lambda = [1, 3, 7] \).

5. THE MULTIPLICITY OF ODD CYCLOTOMIC POLYNOMIALS

A remarkable property of cuspidal unipotent formal degrees of unramified unitary, orthogonal or symplectic groups is the fact that they have no odd cyclotomic polynomial factors in the numerator or the denominator (in the normalization of Haar measures we discussed before). This turns out to be a very selective property for unipotent discrete series characters of these classical groups. In the present section we will use a reduction argument and the tool of the extra-special STMs to limit the possible discrete Kazhdan-Lusztig-Langlands parameters of those unipotent discrete series characters whose formal degrees have this property, in the cases of unitary, symplectic and orthogonal groups (the types I to VI).

**Notation.** Let \( \mathbb{N} \) denote the set of strictly positive integers. For \( n \in \mathbb{N} \) we denote \( \Phi_n(q) = \prod_{d|n}(1-q^d)^{\mu(n/d)} \) the \( n \)-th cyclotomic polynomial, with \( \mu(\cdot) \) the Möbius function, defined by the rule below:

\[
\mu(a) = \begin{cases} 
1, & \text{if } a = 1, \\
(-1)^l, & \text{if } a \text{ is the product of } l \text{ different primes,} \\
0, & \text{if } a \text{ contains a square factor.}
\end{cases}
\]

All cyclotomic polynomials belong to the free abelian group \( \mathbb{M} \) and form a linearly independent set in \( \mathbb{M} \). By the above, the subgroup of \( \mathbb{M}_{ev} \) generated by the even cyclotomic polynomials \( \Phi_{2n} \) with \( n \in \mathbb{N} \) is equal to the subgroup generated by the
polynomials \((1 + q^k)\) with \(k \in \mathbb{N}\). Note also that \(\Phi_n(q^2) = \Phi_{2n}(q)\) if \(n\) is even, while \(\Phi_n(q^2) = \Phi_{2n}(q)\Phi_n(q)\) if \(n\) is odd. Therefore, for all \(n \in \mathbb{N}\), the congruence relation \(\Phi_n(q^2) \equiv \Phi_n(q) \pmod{M_{cv}}\) holds in \(M\).

5.1. Reduction to “real infinitesimal” central character. Recall that we have the notion of \(m\)-tableau (see Section 4) associated to a given pair \((m, \rho)\) where \(m \in \mathbb{Q}\) and \(\rho \vdash n\) (here \(n \in \mathbb{Z}_{\geq 0}\)). To \((m, \rho)\) we associate the \(W_0(B_n)\)-orbit of content vectors \(W_0(B_n)\xi(m, \rho) \in \mathbb{Q}^n\), the set of vectors \(\xi(m, \rho)\) in \(\mathbb{Q}^n\) such that the list of absolute values of coordinates of \(\xi(m, \rho)\) coincides with the list of box fillings of \(T_m(\rho)\) (counted with multiplicity) (here the root system \(B_n\) is realized as the set of linear functionals on \(\mathbb{Q}^n\) formed by \(\pm x_i\) (with \(i = 1, \ldots, n\), and \(x_1, \ldots, x_n\) the standard coordinates) together with the set \(\pm x_i \pm x_j\) (with \(i \neq j\)). We say that \(\rho\) is \(m\)-regular if the \(m\)-splitting \(S_m(\rho)\) of \(\Phi_n(\rho)\) (cf. [Slo, Definition 5.16]) is well defined. By [Slo, Lemma 5.17]), this is equivalent to the condition that a content vector \(\xi(m, \rho)\) is an \(m\)-linear residual point for the root system \(B_n\), in the sense of [HO] (also see [OS]). Yet another equivalent formulation of this ([Op3], [OS]) is the statement that \(W_0(B_n)(\xi(m, \rho))\) is the central character of a discrete series character of the graded affine Hecke algebra \(H_m(B_n)\) of the root system \(B_n\), with parameter \(1\) of the roots \(\pm x_i\) and \(m\) for the roots \(\pm x_i\).

Given a “base” \(q^0 \in \mathbb{R}_+\) (with \(q \in \mathbb{R}_+\), and \(b \in \mathbb{N}\)) and a pair \((m, \rho)\) as above, we form a set of vectors \(W_0(B_n)\mathcal{T}^{(b,m,\rho)}(q)\) with \(\mathcal{T}^{(b,m,\rho)}(q) = (q^{bc_1}, \ldots, q^{bc_n}) \in \mathbb{R}_+^n\). Here we denote by \(\mathcal{T}^{(b,m,\rho)}: = \exp(bk\xi(m, \rho))\) the generic residual point with coordinates \((q^{bc_1}, \ldots, q^{bc_n})\), where \(\xi(m, \rho) = (c_1, \ldots, c_n)\) is the \(m\)-linear residual point whose coordinates are the fillings of the boxes of \(T_m(\rho)\) (ordered arbitrarily), and \(k = \log(q)\). To lessen the burden of notation, we will often suppress the parameters \(m\) and \(b\) from the notation, and simply write \(\mathcal{T}^{\rho}\) instead of \(\mathcal{T}^{(b,m,\rho)}\) or even of \(\mathcal{T}^{(b,m,\rho)}(q)\) if there is little danger of confusion.

Recall from [OS] that the central character of a discrete series character of an affine Hecke algebra of type \(C_n(m_-, m_+)[q^0]\) can always be represented as the \(W_0(B_n)\)-orbit of a residual point \(\mathcal{T}^{\rho}(-\rho; -\rho) := (-\mathcal{T}^{(b,m_-; m_-)}(q), \mathcal{T}^{(b,m_+; m_+)}(q))\), where \(\rho_{\pm}\) denotes an \(m_+\)-regular partition of \(n_{\pm}\), with \(n_{-} + n_{+} = n\). We will always assume (without loss of generality) that \(m > 0\), that \(b \in \mathbb{N}\), \(m_{\pm} \in \mathbb{Q}_+\) are such that \(b(m_+ - m_-) \in \mathbb{Z}\) and \(b(m_+ - m_-) \in \mathbb{Z}\). This assumption is satisfied for unipotent affine Hecke algebras, and guarantees that all factors in the \(\mu\) function are of the form \((1 \pm q^n)\) with \(n \in \mathbb{Z}\).

We recall (see [HO], [Slo]) that for \(m \notin \{0, 1/2, 1, \ldots, n - 1\}\), all partitions \(\rho \vdash n\) are \(m\)-regular, implying that the set of \(W_0\)-orbits of the \(m\)-linear residual points in this situation is in canonical bijection to the set of all partitions \(\rho \vdash n\) of \(n\).

This is the case for example for type \(V\) and type \(VI\) classical unipotent affine Hecke algebras.

But for \(m \in (\mathbb{Z}/2)_{\geq 0}\) (both \(m_{\pm}\) are in this set, for all classical unipotent affine Hecke algebras of type \(I\) to \(IV\)) not all partitions \(\rho \vdash n\) are generic, in general. Slooten [Slo] has devised a general and uniform notion of a generalized ”distinguished unipotent class” in order to parameterize the set of \(W_0\)-orbits of the \(m\)-linear residual points in the cases
as well. The generalized distinguished unipotent classes for \( m \in \mathbb{Z}_{\geq 0} \) are partitions \( \lambda \) of \( 2n + m^2 \) with distinct odd parts, and with length at least \( m \). For \( m \in (\mathbb{Z} + 1/2)_{\geq 0} \) they are partitions \( \lambda \) of \( 2n + m^2 - 1/4 = 2n + |m|^2 \), with distinct even parts, of length at least \( m - 1/2 = |m| \). If \( \lambda \vdash 2n + |m|^2 \) is a distinguished unipotent \( m \)-class, then there exists a corresponding nonempty set of partitions \( \rho \vdash n \) whose sets of content vectors \( \xi(m, \rho) \) are equal and form a \( W_0(B_n) \)-orbit of \( m \)-linear residual points whose jump sequence ([HO], [Slo]) is equal to \((\lambda - 1)/2\). This characterizes Slooten’s parameterization alluded to above.

Given a distinguished unipotent \( m \)-class \( \lambda \vdash 2n + |m|^2 \) and a corresponding partition \( \rho \vdash n \) we will, by abuse of notation, often write \( \tau(\lambda, \rho) \) instead of \( \tau(m, \lambda, \rho) \).

Recall that \( M \subset K^\times \) is a free abelian group. Let \( M_0 \subset M \) be the subgroup generated by \( \mathbb{Q}^\times \) and the (linearly independent) set \( \{ \Phi_n \mid n \in \mathbb{N} \} \) of cyclotomic polynomials. When we speak of the multiplicity of a cyclotomic polynomial \( \Phi_n \) in an element \( f \in M_0 \) we simply mean the exponent \( \text{cycl}_f(n)/2 \) of \( \Phi_n \) as an irreducible factor of \( f \). Hence \( \text{cycl}_f \) is a \( \mathbb{Z} \)-valued function with finite support on \((\mathbb{Z}/2)_+\). (We apologize to the reader for this convention to divide the argument of cycl by 2, but this turns out to be convenient in the context of this paper.) If we say “\( f \) does not contain \( \Phi_n \) as a factor”, we mean that \( \text{cycl}(n/2) = 0 \).

By Möbius inversion, \( \{ q^n - 1 \mid n \in \mathbb{N} \} \) is also a linearly independent set in \( M_0 \), which spans the same subgroup as \( \{ \Phi_n \mid n \in \mathbb{N} \} \). We define a \( \mathbb{Z} \)-valued function \( \text{mult}_f \) with finite support on \((\mathbb{Z}/2)_+\) such that \( \text{mult}_f(n/2) \) is the exponent of \( q^n - 1 \) if we express \( f \) as a product of a constant and powers of elements of \( \{ q^l - 1 \mid l \geq 1 \} \). If \( k \in (\mathbb{Z} + 1/2)_+ \), the multiplicity \( \text{cycl}(k) \) of the odd cyclotomic factor \( \Phi_{2k} \) in \( f \) can be expressed as follows in terms of the multiplicities \( \text{mult}(k') \) of the factors \( (1 - q^{2k'}) \) in \( f \):

\[
(5.1) \quad \text{cycl}(k) = \sum_{d \geq 1} \text{mult}(dk).
\]

We suppress the subscript \( f \) in \( \text{mult}_f \) and \( \text{cycl}_f \), if there is no danger of confusion.

Recall the normalization of the trace

\[
\tau_{m_- m_+}(1) = \frac{d_{m_- m_+}^0}{(v^d - v^{-d})^n} \nabla_{m_-, m_+} \quad \frac{d_{m_- m_+}^0}{(v^d - v^{-d})^n}
\]

of a unipotent affine Hecke algebra \((\mathcal{H}_{m_-, m_+}^m, \tau_{m_-, m_+})\) of type I to VI. It contains only one odd cyclotomic polynomial with nonzero multiplicity, namely \( \Phi_1 \) has multiplicity \(-n\). As a consequence we have the following obvious but important observation: Modulo even cyclotomic polynomial factors, rational constants and powers of \( q \), we have a factorization:

\[
(5.2) \quad f := \mu_{m_-, m_+} \cdot \tau_{(\rho_{-\rho_+})} \equiv \mu_{m_-, m_+} \cdot \tau_{(\rho_{-\rho_+})} \mu_{m_-, m_+} \cdot \tau_{(\rho_{0-\rho_+})} \cdot \tau_{(\rho_{0+\rho_+})} := f_{-\rho_+}
\]

Clearly, if the multiplicity of all odd cyclotomic polynomials in both factors on the right hand side is zero then the same thing is true on the left hand side. Remarkably, for classical unipotent affine Hecke algebras of type I to VI, the converse is also true:

**Proposition 5.1.** Let \( \tau_{(\rho_{-\rho_+})} \) be a residual point for a classical unipotent affine Hecke algebra \((\mathcal{H}_{m_-, m_+}^m, \tau_{m_-, m_+})\) of type I to VI. If the support \( \text{Supp}(\text{cycl}_{f_{-\rho_+}}) \) of \( \text{cycl}_{f_{-\rho_+}} \) is not
contained in $\mathbb{Z}$ for at least one of $f_-$ or $f_+$, let $p_\pm + 1/2 \in \text{Supp}(\text{cycl}_{f_\pm}) \cap (\mathbb{Z} + 1/2)$ denote the maximal element. In this case we have:

$$\text{cycl}_{f_\pm}(p_\pm + 1/2) > 0,$$

In particular the support of $\text{cycl}_f$ is not contained in $\mathbb{Z}$ in this situation either, and if $p + 1/2 \in \text{Supp}(\text{cycl}_f) \cap (\mathbb{Z} + 1/2)$ is the maximal element, then $\text{cycl}_f(p + 1/2) > 0$.

Consequently, if the left hand side of (5.2) has no odd cyclotomic polynomial factors, then the same is true for the two factors on the right hand side of (5.2).

In other words, the property of having no odd cyclotomic polynomial factors reduces to discrete series central characters $W_0(B_{n\pm}(\mathfrak{T}_{0,\rho_{\pm}}))$ with "real infinitesimal" central character. We also point out the obvious fact that:

$$\mu_{\frac{m_0}{\ell}, \ell_0}(\mathfrak{T}_{0,\rho_0}) = \mu_{\frac{m_0}{\ell}, \ell_0}(\mathfrak{T}_{0,\rho_0}),$$

modulo even cyclotomic factors, for any choice of $m_0' \in \mathbb{Z}/4$ such that the pair $(m_0', m_{\pm})$ also belongs to a type I to IV, whose base parameter $b'$ of $(m_0', m_{\pm})$ is the same as for $(m_\pm, m_{\pm})$ (thus $b' = b$). We often choose $m_\pm$ as small as possible. For example, for the types V and VI we choose $m_\pm = 1/4$, so that the expressions to analyze are both of the form $\mu_1,\mu_2(\mathfrak{T}_{0,\rho})$ with appropriate values for $m$ and $\rho$.

From here onward we may thus concentrate on the individual factors of the right hand side of (5.2), provided that we prove this positivity assertion on odd cyclotomic multiplicities for these factors, of course. We will omit the subscripts $\pm$ while we are focusing on these individual factors (actually, the indication $\pm$ will be used a lot below, but with entirely different meanings). We will also ignore powers of $q$ and rational constants since we are only interested in the $q$-rational factors (thus we can replace all factors of the form $(1 - q^a)$ by $(1 - q^{|a|})$).

### 5.2. The use of standard STMs.

We have already shown the existence of the standard STMs ([Op2], [Fe2]), hence we may use these as a tool for the task at hand.

In Section 3 we have recalled that from [Op1, Definition 3.10], given a (standard) STM

$$\Psi : \mathcal{H}_{(m_-, m_+)}^{\ell} \sim \mathcal{H}_{(\delta_-, \delta_+)}^{\ell}$$

of classical affine Hecke algebras of types I to VI, we have a corresponding morphism on the spectra of the centers of the corresponding Hecke algebras

$$\Psi_Z : \text{Spec} \mathcal{E}_{(m_-, m_+)}^{\ell} \rightarrow \text{Spec} \mathcal{H}_{(\delta_-, \delta_+)}^{\ell}$$

which essentially preserves the formal degree [Op1, Theorem 3.11]. Explicitly, given an orbit of residual points $W_{1,0}(\mathfrak{T}_{(\rho_-, \rho_+)}^{\ell})$ of $\mathcal{H}_{(m_-, m_+)}^{\ell}$ with $\mathfrak{T}_{(\rho_-, \rho_+)}^{\ell} = (-\mathfrak{T}_{(\delta_-; \delta_+)}^{\ell})$, there exists a unique orbit of residual points $W_{2,0}(\mathfrak{T}_{(\lambda_-, \lambda_+)}^{\ell})$ of $\mathcal{H}_{(\delta_-, \delta_+)}^{\ell}$ such that

$$\mathfrak{T}_{(\delta_-; \delta_+)}^{\ell} \equiv c. \mathfrak{T}_{(\delta_-; \delta_+)}^{\ell} \cdot \mathfrak{T}_{(\delta_-; \delta_+)}^{\ell}$$

where we may choose $\mathfrak{T}_{(\lambda_-, \lambda_+)}^{\ell}$ in the standard form: $\mathfrak{T}_{(\lambda_-, \lambda_+)}^{\ell} = (-\mathfrak{T}_{(\delta_-; \delta_+)}^{\ell})$, $\mathfrak{T}_{(\delta_-; \delta_+)}^{\ell}$).

The main property of STMs [Op1, Theorem 3.11] implies the equality of residues:

$$\mathfrak{H}_{\delta_-, \delta_+}^{\ell}(\mathfrak{T}_{(\lambda_-, \lambda_+)}^{\ell}) = c. \mu_{m_-, m_+}(\mathfrak{T}_{(\rho_-, \rho_+)}^{\ell})$$
for some constant \( c \in \mathbb{Q}^{\times} \). In this way we can analyze the \( q \)-rational parts of the formal degrees of \( \mathcal{H}_{\delta_{\pm},\delta_{\pm}} \) on either side. Since we have enough standard STMs to map all unipotent Hecke algebras to a minimal object of \( \mathcal{C}_{\text{class}} \), we may conclude that:

**Proposition 5.2.** In order to prove Proposition 5.1 it suffices to prove this Proposition for the minimal objects of (3.5).

For a standard translation STM \( \Psi \) [Op2, Section 3.2.4], the action of \( \Psi_Z \) on Weyl group orbits of residual points is very simple when we use Slooten’s parameterization of such orbits in terms of “unipotent partitions”. Namely, if \((\lambda_{-},\lambda_{+})\) is a pair of unipotent partitions of type I, II, III or IV with parameters \((m_{-},m_{+})\), and \( \Psi \) translates this parameter pair to a pair \((m'_{-},m'_{+})\), then \((\lambda_{-},\lambda_{+})\) is also a pair of unipotent partitions for the pair \((m'_{-},m'_{+})\), and

\[
\Psi_Z(W_0(-\tau^*(b,m_{-};\lambda_{-}),\tau^*(b,m_{+};\lambda_{+})) = W'_0(-\tau^*(b,m'_{-};\lambda_{-}),\tau^*(b,m'_{+};\lambda_{+}))
\]

Such standard translation STMs exist whenever \( m_{\pm} \in \mathbb{Z}/2 \), \( m'_{\pm} \) lies between \( m_{\pm} \) and 0, \( m_{\pm} - m'_{\pm} \in \mathbb{Z} \) if \( m_{\pm} \in \mathbb{Z} + 1/2 \), and \( m_{\pm} - m'_{\pm} \in 2\mathbb{Z} \) if \( m_{\pm} \in \mathbb{Z} \).

Apart from these translation STMs we have the spectral isomorphisms, which are generated by two isomorphisms which have the following effect on the orbits of residue points in their standard presentations:

\[
\eta_Z(W_0(-\tau^*(b,m_{-};\lambda_{-}),\tau^*(b,m_{+};\lambda_{+})) = W_0(-\tau^*(b,m_{+};\lambda_{-}),\tau^*(b,m_{-};\lambda_{+}))
\]

and

\[
\eta'_Z(W_0(-\tau^*(b,m_{-};\lambda_{-}),\tau^*(b,m_{+};\lambda_{+})) = W_0(-\tau^*(b,m_{-};\lambda_{+}),\tau^*(b,m_{+};\lambda_{-}))
\]

(where \( \lambda_{+}' \) is the conjugate of \( \lambda_{+} \)). Finally there are the extra special standard STMs, whose action on the residual points involves the bijections described by the extra special algorithms. Let \((m_{-},m_{+})\) be a pair of parameters of type V or VI. There exists a unique extra special standard STM \( \Psi^e : \mathcal{H}_{(m_{-},m_{+})} \hookrightarrow \mathcal{H}_{(\delta_{-},\delta_{+})} \), \((m_{-},\rho_{-}), (m_{+},\rho_{+})\) with \( \rho_{\pm} \) partitions of \( r_{\pm} \), where \( r = r_{-} + r_{+} \). Recall that the extra-special algorithm yields a bijection between the set \( R \) and the collection of odd distinct partitions \( P_{\text{odd, dist}} \). Let

\[
(m_{\pm},\rho_{\pm}) \leftrightarrow (\delta_{\pm},\lambda_{\pm})
\]

according to the extra special algorithm. Then

\[
\Psi_Z(W_{1,0}(-\tau^*(2,m_{-};\rho_{-}),\tau^*(2,m_{+};\rho_{+})) = W_{2,0}(-\tau^*(1,\delta_{-};\lambda_{-}),\tau^*(1,\delta_{+};\lambda_{+}))
\]

It is a very important fact (a consequence of the above, and the extra special bijections) that every pair \(((\delta_{-},\lambda_{-}), (\delta_{+},\lambda_{+}))\) arises in the image of a suitable extra special STM.

Consider types involving \( m \in \mathbb{Z}_{>0} \) (i.e. types I, III, IV), a minimal object has parameters \((\delta_{-},\delta_{+})\) in the set \( \{(0,1/2),(1/2,1),(0,1),(0,0),(1,1)\} \), hence at least one of \( \delta_{\pm} \) is in \( \{0,1\} \). Say \( \delta \in \{0,1\} \), and let \( \lambda \vdash 2n + \delta \) be a partition with odd, distinct parts. Let

\[
(m,\rho) \leftrightarrow (b,\lambda)
\]

correspond each other through the extra-special algorithm, where the left pair is expressed as a Young tableau \( T_m(\rho) \) and \( \lambda \) is an odd distinct partition whose number of parts is congruent to \( \delta \) modulo 2.
As mentioned above, all residue points of the form \( \overrightarrow{r}_{ b = 1, \delta; \lambda} \) are the image under an appropriate extra-special STM \( \Psi : \mathcal{H}^n_{(1/4, m)} \rightarrow \mathcal{H}^n_{(0, \delta)} \) of a residue point of the form \( \overrightarrow{r}_{(2, m, H)} \). Instead of analyzing formal degrees for the parameters \((0, \delta)\) (with \(b = 1\)) we can analyze the situation for the generic parameters \((1/4, m)\) (with \(b = 2\)), and this turns out to be an important simplification.

By the above reductions, in order to prove Proposition 5.1, it suffices to show that:

**Proposition 5.3.** For all pairs \((m, \rho)\) with \(\rho \vdash n\) a partition (possibly zero) and \(m \in (\mathbb{Z} \pm 1/4)_+\) such that \(\mu_{1/4, m}^{n, \{ \overrightarrow{r}_{(0, \rho)} \}}(\overrightarrow{r}_{0, \rho})\) has odd cyclotomic polynomial factors \(\Phi_{2p+1}\) with nonzero multiplicity, the multiplicity of the odd cyclotomic polynomial \(\Phi_{2p_{m+1}}\) with \(p_m \in \mathbb{N}\) maximal such that \(\Phi_{2p_{m+1}}\) has nonzero multiplicity in \(\mu_{1/4, m}^{n, \{ \overrightarrow{r}_{(0, \rho)} \}}(\overrightarrow{r}_{(0, \rho)})\) appears with positive multiplicity.

The same statement is true for all \(1/2\)-unipotent classes \(\lambda \vdash 2n\) (i.e. \(\lambda \vdash 2n\) has even, distinct parts), and factors of the form \(\Phi_{2p+1}\) of \(\mu_{1/2, 1/2}^{n, \{ \overrightarrow{r}_{(0, \lambda)} \}}(\overrightarrow{r}_{(0, \lambda)})\).

The proof of Proposition 5.3, hence as well as the classification of pairs \((m, \rho)\) (with \(m \in (\mathbb{Z} \pm 1/4)_+\) and \(\rho \vdash n \geq 0\) and \(1/2\)-unipotent classes \(\lambda\) for which no odd cyclotomic polynomials appear as a factor, will now be given in separate sections.

### 5.3. Counting odd cyclotomic polynomials for the case \(\delta = 1/2\)

Let \(\lambda \vdash 2n\) be a partition with even, distinct parts, and let \(\rho \vdash n\) be a partition of \(n\) such that the content vector \(\xi(1/2, \rho)\) of the \(1/2\)-tableau \(T_{1/2}(\rho)\) has jump sequence \((\lambda - 1)/2\).

Let \(h : (\mathbb{Z} + 1/2)_+ \rightarrow \mathbb{Z}_+\) denote the associated multiplicity function, that is, for all \(k \in (\mathbb{Z} + 1/2)_+\) the number of boxes in \(T_{1/2}(\rho)\) with filling \(k\) equals \(h(k)\). Let \(\overrightarrow{r} := (\emptyset, \overrightarrow{r}_{(b = 1, 1/2; \lambda)})\) be the corresponding positive residual point for \(\mathcal{H}_{(1/2, 1/2)}^n\). Recall from [HO, Section 4] that such function \(h\) defining a linear residual point for \(R_0 = B_n\) and \(m = 1/2\), is characterized by the following two conditions:

1. (A) Let \(p\) be the largest half integral number in the support of \(h\). Then \(h(p) = 1\).
2. (B) For all \(x \geq 1/2\) we have \(h(x) \in \{h(x + 1), h(x + 1) + 1\}\).

**Lemma 5.4.** The rational function \(\mu_{1/2, 1/2}^{n, \{ \overrightarrow{r}_{(0, \lambda)} \}}(\overrightarrow{r}_{(0, \lambda)})\) in \(q\) contains no odd cyclotomic polynomials if and only if \(h\) is given by

\[
h(x) = \begin{cases} 
p - x + 1 & \text{if } 0 \leq x \leq p \\
0 & \text{else} \end{cases}
\]

If \(h\) is not of the form (5.5) then the highest odd cyclotomic polynomial factor \(\Phi_{2j+1}\) of \(\mu_{1/2, 1/2}^{n, \{ \overrightarrow{r}_{(0, \lambda)} \}}\) appears in the numerator of \(\mu_{1/2, 1/2}^{n, \{ \overrightarrow{r}_{(0, \lambda)} \}}\).

**Proof.** Assume that \(\mu_{1/2, 1/2}^{n, \{ \overrightarrow{r}_{(0, \lambda)} \}}\) contains no odd cyclotomic polynomial factors. We ignore powers of \(q\) and rational constants, allowing us to rewrite all factors in the form \((1 - q^k)\) with \(k > 0\) (and we ignore factors of the form \((1 + q^k)\), since these only involve even cyclotomic factors). These factors are linearly independent over \(\mathbb{Z}\) in the abelian group \(K^\times\), and thus have a well defined multiplicity \(\text{mult}(k)\) in \(\mu_{1/2, 1/2}^{n, \{ \overrightarrow{r}_{(0, \lambda)} \}}\).
Assume that \( p = 2k \pm 1/2 \) and that \( 0 \leq i \leq k - 1 \). A simple book-keeping using (2.4) (with \( l \) replaced by \( n \)) determines \( \text{mult}(2p - 2i) \) in terms of \( h \) as follows (the inequality \( 2p - 2i \geq p + 3/2 \) insures that only the roots \((t_x t_y)^{\pm 1} \)) (with possibly \( x = y \)) contribute to this multiplicity):\[
\text{mult}(2p - 2i) = h(p - i)(h(p - i) - h(p - i - 1) + 1) + \sum_{x=1}^{i} h(p - i + x)(2h(p - i - x) - h(p - i - 1 - x) - h(p - i + 1 - x))
\]

Since \( 2p - 2i \geq p + 3/2 \), and since all factors of the form \((1 - q^k)^{r_2} \) of \( \mu_{n,((1/2)}) \) have order less or equal \( 2p + 1 \), it is clear that \( \text{mult}(2p - 2i) \) represents the multiplicity of the odd cyclotomic polynomial \( \Phi_{2p-2i} \) as an irreducible factor of \( \mu_{n,((1/2)}) \). By assumption this multiplicity must therefore be equal to 0.

For \( i = 0 \), the equation \( \text{mult}(2p) = 0 \) combined with (A) and (5.17) implies that \( h(p - 1) = 2 \).

Hence (5.5) holds for all \( x > p - 2 \). Now suppose by induction that (5.5) holds for all \( x > p - 2j \) for some integer \( 1 \leq j \leq k - 1 \). Using this induction hypothesis we see that all summands of \( \text{mult}(2p - 2j) \) for \( x \leq j - 2 \) vanish, and a simple computation shows that \( \text{mult}(2p - 2j) = 2j + 2 - h(p - 2j - 1) \). Hence we have \( h(p - 2j - 1) = 2j + 2 \), which implies, in view of (B) and \( h(p - 2j + 1) = 2j \), that \( h(p - 2j) = 2j + 1 \).

Hence we find that (5.5) holds for all \( x > p - 2j - 2 \) as well. By induction this proves that \( h \) satisfies (5.5) if \( p = 2k - 1/2 \) for some nonnegative integer \( k \), and it shows that (5.5) is satisfied for all \( x > 1/2 \) if \( p \) is of the form \( p = 2k + 1/2 \). And we see that if \( h \) does not satisfy (5.5) in this range of values for \( x \) then the numerator of \( \mu_{1/2,1/2}^{(1/2)} \) has an odd cyclotomic polynomial.

Finally, for \( p = 2k + 1/2 \) we need to rule out the possibility that \( h(1/2) = p - 1/2 \). So assume that \( h(1/2) = p - 1/2 \). We compute the multiplicity of the odd cyclotomic polynomial \( \Phi_{p+1/2} = \Phi_{2k+1} \), but now \( h(1/2) \) of the roots of the form \( t_x t_y^{-1} \) also contribute in the denominator. To compute this multiplicity, the easiest method is to compare \( \mu_{1/2,1/2}^{n,((1/2))} \) with the analogous product \( \mu_{1/2,1/2}^{n,((1/2))} \), where this time \( \overline{\tau}' \) has one extra coordinate equal to \( q^{1/2} \) compared to \( \tau \). We already know that the multiplicity of \( \Phi_{p+1/2} \) in \( \mu_{1/2,1/2}^{n,((1/2))} \) is 0 since \( h' \) (the multiplicity function of \( \overline{\tau}' \)) does satisfy (5.5). The difference with the multiplicity of \( \Phi_{p+1/2} \) in \( \mu_{1/2,1/2}^{n,((1/2))} \) consists of 2 extra factors in the numerator (coming from a factor of the form \((1 - q^{1/2} q^{2})^2 \)) and 3 more in the denominator (one coming from \((1 - qq q^{-1/2}) \) and two from \((1 - qq q^{-1/2} q^{1/2}) \)). Hence the multiplicity of \( \Phi_{p+1/2} \) in \( \mu_{1/2,1/2}^{n,((1/2))} \) is 1, which violates our assumption.
To sum up, we have shown in all cases that if $h$ does not satisfy (5.5) then highest odd cyclotomic polynomial $\Phi_{2j+1}$ (where we order the cyclotomic polynomials by the order of their associated roots) of $\mu^m_{1/2,1/2}$ will be a factor of the numerator. 

\[ \square \]

**Corollary 5.5.** For $\delta = 1/2$, there exists a positive residual point $\vec{r}$ such that $\mu^m_{1/2,1/2}$ contains no odd cyclotomic polynomials if, and only if, $n = r(r + 1)/2$ for some $r \geq 1$. In this case $\vec{r} = \vec{r}_{\{b=1,1/2,2\}}$, where $\lambda = [2, 2r - 2, 2r - 4, \ldots, 2] + 2n = r(r + 1)$. This residual point represents a cuspidal STM $\mathcal{H}^{j}_{1/2,m} \rightarrow \mathcal{H}^n_{1/2,1/2}$ where $m = r + 1/2$. Define $a, b \in \mathbb{N}$ by $\{2a, 2b + 1\} = \{r, r + 1\}$. If $r$ is even, then $a = b = r/2$, and if $r$ is odd, then $a = (r + 1)/2$ and $b = (r - 1)/2$. Then $\mu^m_{1/2,1/2} = d_{a,1/2}^j(q) d_{b,1/2}^j(q)$. Let $\pi$ be the corresponding cuspidal unipotent representation. Then $\pi$ is a representation of $SO_{2n+1}(k)$ if $n = r(r + 1)/2$ with $r \equiv 0, 3 \pmod{4}$, and $\pi$ is a representation of the nontrivial inner form of $SO_{2n+1}(k)$ if $r \equiv 1, 2 \pmod{4}$.

**Remark 5.6.** Observe the following relation between $h$ as in Lemma 5.4 and $\mu^m_{1/2,1/2}$:

$$\mu^m_{1/2,1/2} = \prod_{k \in \mathbb{Z}_+} (1 + q^k)^{-h(k+1)/2+h(k-1)/2)}$$

### 5.4. Counting odd cyclotomic polynomial factors in the case $m \in (\mathbb{Z} \pm 1/4)_+$. 

Recall that to a pair $(m, \rho)$ with a partition $\rho$ and parameter $m \in (\mathbb{Z} \pm 1/4)_+$ we attached the linear residual point $\xi$ whose coordinates are the fillings of the boxes of $T_m(\rho)$. Recall (cf. [OS, Sections 6.7.8, especially Theorem 8.7], or [Op2, Paragraph 3.3.2]) that the $W_0$-orbits of L-generic residual points of $\mathcal{H}^{j}_{1/4,m_+}$ are in natural bijection with partitions $\rho$ of $r$. A representative in the $W_0$-orbit of residual points is given by $\vec{r}_{\rho} := \exp(2k\xi)$ with $k = \log(q)$ (see Subsection 5.1).

Via the extra special algorithm we get the corresponding odd distinct partition $\lambda$. This gives rise to (cf. loc. cit., or [Slo]) to a residue point $\vec{r}_\lambda$ of $\mathcal{H}^{j}_{0,\delta}$. The coordinates of $\vec{r}_\lambda$ are given by $\exp(\xi)$, where $\xi$ is a linear residual point whose coordinates are given by $k$ times the fillings of the boxes of a $\delta$-tableau $T_\delta(\Lambda)$ where $\Lambda$ is a partition such that the tableau $T_\delta(\Lambda)$ has distinct $\delta$-extremities (in the sense of [Slo]) which are equal to $(\Lambda - 1)/2$. (Such $\Lambda$ always exist, and the Weyl group orbit of $\vec{r}_\Lambda$ does not depend on the choices of such $\Lambda$.)

### 5.5. Reduction to certain rectangular diagrams. 

Given a pair $(m, \rho)$ with $m \in (\mathbb{Z} \pm 1/4)_+$ and $\rho = r$ a partition, let us introduce some notations associated to $T_m(\rho)$.

We denote the entries of the upper-left, upper-right and lower-left cornered boxes of $T_m(\rho)$ by $m, p_+, p_-$ respectively. Note that $p_+ \geq m$ and $p_+ - m \in \mathbb{Z}$. Let $a_m \in [0, 1)$ be determined by $m - a_m \in \mathbb{Z}$. Then the $a_m$-diagonal inside the $m$-tableau indicates the change of congruence classes of the entries modulo $Z$. Note that all entries of $T_m(\rho)$ are in the same congruence class modulo $Z$ if and only if $p_+ - m \in \mathbb{Z}$. Denote the entry of the last box below $p_+$ by $r_+$, and the entry of the last box horizontal to the right of $p_-$ by $r_-$. Below is an example of a Young diagram with $p_+ = 15/4, m = 3/4 = a_m, p_- = 9/4$ and $r_+ = 7/4, r_- = 5/4$. 

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We will now study the odd cyclotomic polynomial factors in the \( q \)-rational part of residues \( \mu_{1/4,m}^{r,\{\tau_{0,\rho}\}}(\tau_{0,\rho}) \) of \( \mu_{1/4,m}^r \) at a positive (or “infinitesimally real”) residue point \( \tau_{0,\rho} \). Since we neglect even cyclotomic polynomials of the regularized value of \( \mu_{1/4,m}^r \) after substitution of \((t_1, \ldots, t_r)\) by \( \tau_{\rho} \) (whose coordinates all are odd powers of \( v \)) we can replace \( \mu_{1/4,m}^r \) by the following product (as usual, up to rational constants, powers of \( v \), and characters of \( T \)):

\[
\mu_{1/4,m}^r \equiv (1 - q)^{-r} \prod_{1 \leq i < j \leq n} \frac{(1 - t_it_j)^2(1 - t_i^{-1}t_j^{-1})^2}{(1 - q^2t_it_j)(1 - q^{-2}t_i^{-1}t_j^{-1})(1 - q^{-2}t_i^{-1}t_j^{-1})} \times \\
\prod_{z=1}^n \frac{(1 - t_z^2)^2}{(1 - q^{2m}t_z)(1 - q^{-2m}t_z)} \quad \text{mod } M_{\omega_v}
\]

(5.7)

Recall that the coordinates of \( \tau_{\rho} \) are of the form \( v^{4x} = q^{2x} \) where \( x \in \mathbb{Z} \pm 1/4 \). Since we are only interested in \( |x| \), and since the Weyl group allows us to change signs, we may and will choose \( \tau_{\rho} \) in such a way that all its coordinates are of the form \( q^{2x} \) with \( x > 0 \). We remark however that the congruence class modulo \( \mathbb{Z} \) of \( x \) and \(-x\) are distinct. This is a key fact in all that follows.

We choose Weyl group elements \( w_{\pm} \) such that \( w_{\pm}(\tau_{\rho}) \) has all its coordinates of the form \( v^{4x} \) with \( x \pmod{\mathbb{Z}} = \pm m \). Then the multiplicity \( h_{\pm}(x) \) of \( v^{4x} \) in \( w(\tau_{\rho}) \) is not dependent of the choice of \( w_{\pm} \). This defines two multiplicity functions \( h_{\pm} : \mathbb{Z} \pm m \to \mathbb{Z}_{\geq 0} \), which satisfy the obvious relation \( h_{-}(x) = h_{+}(-x) \) for all \( x \).

The coordinates of \( w_{\pm}(\tau_{\rho}) \) are the contents of the boxes of \( T_{\rho}(m, +) \), which is defined as the Young tableau of \( \rho \) with its boxes above or on the \( a_m \)-diagonal filled like those of \( T_{\rho}(m) \), but below the \( a_m \) diagonal we multiply the contents of the boxes of \( T_{\rho}(m) \) by \(-1\). Similarly, we define \( T_{\rho}(m, -) \) by multiplying the content of the boxes of \( T_{\rho}(m) \) above or on the \( a_m \)-diagonal by \(-1\), leaving the boxes below the \( a_m \) diagonal unchanged. Hence the multiplicity of the coordinate \( q^{2x} \) in \( w_{\pm}(\tau_{\rho}) \) is equal to the length of the \( x \)-diagonal in \( T_{\rho}(m, \pm) \).

**Remark 5.7.** Given \( m \in (\mathbb{Z} \pm 1/4)_+, \) for \( x \in \mathbb{Z} \pm 1/4 \) we adopt the notation \( x_{\pm} \) to denote the unique element \( x_{\pm} \in \{-x, x\} \) such that \( x_{\pm} \in \mathbb{Z} \pm m \). For example, \( m_{\pm} = \pm m \), and we always have \( p_+ = p_- \).

Now we will analyze the multiplicity functions \( \mult := \mult_f \) and \( \cyc(k) := \cyc_f(k) \) for \( f \) a residue of the form \( f = \mu_{1/4,m}^{r,\{\tau_{0,\rho}\}}(\tau_{0,\rho}) \), and \( k \geq (p_++p_-)/4 \). Let \( k \in \mathbb{Z}_{\geq 0} + 3/2 \) (we use 3/2 here, and not 1/2, because this turns out to be more convenient, and since we
already know that our normalization of Haar measures is such that the factor $\Phi_1 = q - 1$ has multiplicity 0 in the formal degree of any discrete series character. We will need to consider the functions $\text{mult}(dk)$ for $d = 1, 2$ or $d = 4$. As seen below, we distinguish between various contributions, coming from factors associated to different types of roots:

- $\text{mult}_+(k)$ and $\text{mult}_-(k)$: For the function $\text{mult}_\pm(k)$ we take into consideration the contribution to $(1 - q^{2k})$ from the $(1 - t_it_j)^2$-terms in the numerator, where $t_i, t_j$ are coordinates of $w_\pm(\overline{T}_\rho)$ (here $t_i = q^{2x_i}$, $t_j = q^{2y_j}$ and $x, y$ are both in the congruence class of $\pm m$ modulo $\mathbb{Z}$), the factors $(1 - q^2t_it_j)$ and $(1 - q^{-2}t_it_j)$ in the denominator for such $t_i, t_j$, as well as the factors $(1 - t_j^2)^2$. Recall that the multiplicity $h_\pm(x)$ of the coordinate $q^{2x}$ in $w_\pm(\overline{T}_\rho)$ is the length of the $x$-diagonal in $T_m(\rho, \pm)$.

In this way, any (unordered) pair of boxes of $T_\rho(m, \pm)$ with contents $x$ and $y$ such that $x + y \geq 3/2$ contributes $+2$ to $\text{mult}_+(x + y)$ and $-1$ to $\text{mult}_+(x + y + 1)$ and $\text{mult}_+(x + y - 1)$. Moreover, every single box of $T_m(\rho)$ contributes $+2$ to $\text{mult}_+(2x)$. Note that the maximal entry $p_\pm$ occurs only once in $T_m(\rho)$ (for $p_-$ we need to assume $p_- = p_-$ here). This implies in all cases easily that $\text{mult}_\pm(k) = 0$ for $k \geq 2p_\pm$.

- $\text{mult}_{+-}(2k)$: Consider factors of the form $(1 - q^{4k}) = (1 - q^{2k})(1 + q^{2k})$, where $4k$ equals twice an odd number. In $\text{mult}_{+-}(2k)$, we count such factors of $\mu_{1/4,m}(\overline{T}_\rho)$ arising from type D-roots, via factors of the form $(1 - t_it_j)^2$ (in the numerator) or $(1 - q^2t_it_j)$ or $(1 - q^{-2}t_it_j)$ (both in the denominator), with $t_i$ a coordinate of $w_+(\overline{T}_\rho)$, and $t_j$ a coordinate of $w_-(\overline{T}_\rho)$.

In such terms, the pair $\{i, j\}$ corresponds to a pair of boxes, one with entry $x^+$ of $T_m(\rho, +)$ and one with entry $y^-$ of $T_m(\rho, -)$, thus in different congruence classes modulo $\mathbb{Z}$. (If $T_m(\rho)$ contains entries below the $a_m$-diagonal, then $x^+$ is on or above the $a_m$-diagonal, and $y^-$ is below that diagonal in $T_m(\rho)$.) In the numerator terms we need that $2k = x^+ + y^-$ is an odd integer.

Then $t_i = q^{2x_i}$ is a coordinate of $w_+(\overline{T}_\rho)$, and $t_j = q^{2y_j}$ is a coordinate of $w_-(\overline{T}_\rho)$. The set of unordered pairs $\{i, j\}$ such that the corresponding coordinate pair $\{t_i, t_j\}$ of $\overline{T}_\rho$ satisfies $\{t_i, t_j\} = \{q^{2x_i}, q^{2y_j}\}$ (with $x^+ + y^-$ odd) equals $h_+(x^+)h_-(y^-)$. Each such pair contributes $+2$ to $\text{mult}_{+-}(x^+ + y^-)$. On the other hand, an unordered pair of such boxes with $x^+ + y^-$ even contributes $-1$ to $\text{mult}_{+-}(x^+ + y^- - 1)$ and to $\text{mult}_{+-}(x^+ + y^- + 1)$. Notice that $\text{mult}_{+-}(2k) = 0$ if $2k > p_+ + p_-$, for obvious reasons.

- $\text{mult}_{+-}(4k)$: Similar as for $\text{mult}_{+-}(2k)$, but now counting the multiplicity of the factors of the form $(1 - q^{8k}) = (1 + q^{4k})(1 + q^{2k})(1 - q^{2k})$, where $8k$ is four times an odd number. For factors in the numerator we are thus searching $x, y$ such that $4k = x^+ + y^- \equiv 2 \pmod{4}$. In the denominator terms we should solve $4k = x^+ + y^- \pm 1 \equiv 2 \pmod{4}$. (Since our overall assumption will be that $4k \geq (p_+ + p_-)$, $\text{mult}_{+-}(4k)$ is possibly nonzero only for the smallest values of $k$ in our range, depending on the congruence of $p_+ + p_-$ modulo 4.)
The terms \(-h_\pm(k \mp m)\), coming from the denominator in the second line of (5.7), as well as the possible contributions from this denominator to \((1 - q^{2k})\) via a contribution to \((1 - q^{4k})\) and \((1 - q^{8k})\), in other words the terms \(-h_\pm(2k \pm m)\) and \(-h_\pm(4k \pm m)\).

In all we have for \(k \in (\mathbb{Z} + 1/2)_+\):

\[
\text{mult}(k) := (\text{mult}_+(k) - h_+(k - m)) + (\text{mult}_-(k) - h_-(k + m))
\]

while for \(n \in \mathbb{N}\) we have:

\[
\text{mult}(n) := (\text{mult}_+(n) - h_-(n - m)) - h_+(n + m)
\]

We shall now compute each of these functions in terms of \(h_+\) and \(h_-\). Note that both \(2p_+ - k\) and \(2p_- - k\) are integers since \(p_+, p_- \in \mathbb{Z} \pm 1/4\) and \(k \in (\mathbb{Z}_+ + 1/2)_+\).

First look at the factors

\[
\frac{(1 - t_it_j)^2}{(1 - q^{2t_it_j})(1 - q^{-2t_it_j})}.
\]

To find out the multiplicity \(\text{mult}_+(k)\) of the factor \((1 - q^{2k})\), we separate two cases, based on the parity of \(2p_+ - k\).

Case (i): \(2p_+ - k = 2i\) is even. Then

\[
\text{mult}_+(k)
= 2 \sum_{x=1}^{i} h_+(p_+ - i + x)h_+(p_+ - i - x) + 2 \times \frac{1}{2} h_+(p_+ - i)[h_+(p_+ - i) - 1] + 2h_+(p_+ - i) - \sum_{x=0}^{i} h_+(p_+ - i + x)h_+(p_+ - i - x - 1).
\]

Because of \(t_it_j = q^{2(c(i)+c(j))}\) with \(i < j\), we need to count twice the multiplicities of the entries in the Young diagram such that their sum is equal to \(k\) (corresponding to \((1-t_it_j)^2\), an subtract once those with sum \(k+1\) (corresponding to \(1-q^{2t_it_j}\)) and similar for \(k-1\) (corresponding to \(1-q^{-2t_it_j}\)). For the first case we add the terms \(2h_+(x)h_+(y)\) for \((x, y)\) a solution to \(x + y = k\) with \(p_+ \geq x \geq y\) and \(x, y \equiv p_+ \mod \mathbb{Z}\). The solutions to this equation yield the first 2 terms in the formula (the second term corresponds to \(x = y = k/2 = p-i\). The third term correspond to the factor \((1-t_2^2)^2\) in the \(\mu\)-function. The terms with minus sign correspond to the factor \((1-q^{2t_it_j})(1-q^{-2t_it_j})\).

We introduce the \(\Delta\)-operator which is defined by

\[
\Delta h(x) = 2h(x) - h(x + 1) - h(x - 1),
\]

and the “jump function” for all \(x\):

\[
J_\pm(x) = h_\pm(x) - h_\pm(x + 1) \in \{-1, 0, 1\}.
\]
With these notations we can rewrite the formula of \( \text{mult}_+(k) \) into a neat way (if \( k = 2p_+ - 2i \)):

\[
\text{mult}_+(k) = \sum_{x=1}^{i} h_+(p_+ - i + x) \Delta h_+(p_+ - i - x) + h_+(p_+ - i)[1 - J_+(p_+ - i - 1)].
\]

Case (ii): \( 2p_+ - k = 2i - 1 \) is odd. In this case for the numerator we lose the terms corresponding to \((k/2, k/2)\) because of parity, and also \((1 - t_z^2)^2\) does not contribute because of parity. But for the denominator we have the contributions corresponding to \((x, y)\) with \( x = y = (k + 1)/2 \) and \( x = y = (k - 1)/2 \). So the formula becomes:

\[
\text{mult}_+(k)
= 2 \sum_{x=1}^{i} h_+(p_+ - i + x)h_+(p_+ - i - x + 1)
- \sum_{x=1}^{i} h_+(p_+ - i + x)h_+(p_+ - i - x) - \frac{1}{2} h_+(p_+ - i)[h_+(p_+ - i) - 1]
- \sum_{x=1}^{i-1} h_+(p_+ - i + x + 1)h_+(p_+ - i - x + 1) - \frac{1}{2} h_+(p_+ - i + 1)[h_+(p_+ - i + 1) - 1]
\]

Using \( \Delta \) and \( J \) we can rewrite this formula as (for \( k = 2p_+ - 2i + 1 \))

\[
\text{mult}_+(k) = \sum_{x=1}^{i} h_+(p_+ - i + x) \Delta h_+(p_+ - i - x + 1) + \frac{1}{2} [h_+(p_+ - i + 1) + h_+(p_+ - i)][1 - J_+(p_+ - i)].
\]

We can formally write

\[
(5.13) \quad \text{mult}_+(k) = \sum_{x+y=k, p_+ \geq x > y \atop x \equiv y \equiv m(\mathbb{Z})} h_+(x) \Delta h_+(y) + R_+(k) =: M_+(k) + R_+(k),
\]

where the “remainder” \( R_+(k) \) denotes the term which is not under the summation symbol of the above formulae. Observe that \( R_+(k) \geq 0. \)

By virtue of symmetry we obtain the formula of \( \text{mult}_-(k) \) from the formula of \( \text{mult}_+(k) \) by replacing \( + \) by \( - \). But it is important to observe here that the contributions of \((1 - t_z^2)^2\) of the form \((1 - q^{2k})\) with \( k \equiv 2m \) (mod \( 2\mathbb{Z} \)) contribute to \( \text{mult}_+(k) \), while those with \( k \equiv -2m \) (mod \( 2\mathbb{Z} \)) contribute to \( \text{mult}_-(k) \). Together this uses all such contributions coming from this factor of the \( \mu \)-function exactly once. Hence we may write

\[
(5.14) \quad \text{mult}_-(k) = \sum_{x+y=k, p_- \geq x > y \atop x \equiv y \equiv -m} h_-(x) \Delta h_-(y) + R_-(k),
\]

with \( R_-(k) \geq 0. \)

**Remark.** Observe that \( J_-(x) + J_+(-x - 1) = 0 \) for all \( x \in \mathbb{Z} \pm 1/4. \)
Next, we consider the term
\[
\frac{(1-t_it_j)^2}{(1-q^2t_it_j)(1-q^{-2}t_it_j)}
\]
where we take \(t_i\) and \(t_j\) from opposite congruence classes modulo \(Z\). In this case \(c(i) + c(j) \in Z\). The contribution to an odd cyclotomic factor \((1-q^{2k})\) (with \(2k \in 2\mathbb{Z}+1\)) from terms of this kind comes from their contribution to the factor of the form \((1-q^{2(c(i)+c(j))})\) with \(c(i) + c(j) = 2k\) or \(4k\) (and not \(k\), because \(c(i) + c(j) \in Z\) now). This readily yields:

\[(5.15) \quad \text{mult}_{+,-}(2k) = \sum_{a+b=2k, a \equiv m(Z), b \equiv -m(Z)} h_+(a) \Delta h_-(b) = \sum_{a+b=2k, a \equiv m(Z), b \equiv -m(Z)} \Delta h_+(a) h_-(b) + \text{mult}(2k)\]

and a similar expression for \(\text{mult}_{+,-}(2k)\). The second equality holds by symmetry.

Finally, in our range \(k \geq (p_+ + p_-)/4\), we need subtract the multiplicity of the odd cyclotomic polynomials occurring in the factors \((1-q^{2k})\), \((1-q^{4k})\) and \((1-q^{8k})\) appearing in the denominator of the factors

\[
\frac{1}{(1-q^{2mt_z})(1-q^{-2mt_z})}\]

This yields eight contributions to \(\text{cycl}(k)\): \(-h_+(k \equiv m), -h_+(2k \equiv m)\) and \(-h_+(4k \equiv m)\).

**Lemma 5.8.** Assume that \((m, \rho)\) (with \(\rho\) a partition of the rank \(r\)) is such that for the odd cyclotomic factors of \(f = \mu_{1/4,m}^{(r)}(\tau_{0,\rho})\) we have \(\text{cycl}(k) = 0\) for all \(k \in (\mathbb{Z}+1/2)_+\) such that \(k > p_+.\) If \(r = 1\) then \(m = 1/4\). Otherwise \(T_m(\rho)\) contains boxes below the \(a_m\)-diagonal (so \(p_- = p_-\) if \(r > 1\)).

**Proof.** If \(r = 1\) then \(q \equiv (q-1)^{-1}(q^{4m} - 1)\) modulo even cyclotomic factors. Hence unless \(4m = 1\) we have \(\text{cycl}(2m) = 1\) and \(k = 2m > p_+ = m\), contradicting the assumption.

Suppose that \(r > 1\), and that \(T_m(\rho)\) has no entries below the \(a_m\)-diagonal. Put \(k = p_+ + r_+ \in (\mathbb{Z}+1/2)_+\). Now \((\text{mult}_{+,-}(k) - h_+(k-m)) \geq 1\) by (5.13) (use that \(\Delta(h_+(r_+)) \geq 1\) and that \(\Delta(h_+(r_+)) \geq 1\) and that for all \(y\) such that \(r_+ < y \leq p_+ + m\), we have \(\Delta(h_+)(y) = 0\). Since clearly \(2k > p_+ + p_-\) we have \(\text{mult}_{+,-}(dk) = 0\) for all \(d > 0\). Also \(2k \equiv m > p_+ > p_-,\) so \(h_+/(2k \equiv m) = 0\), and \(k + m > p_\) so \(h_-/(k + m) = 0\). Hence (5.1), (5.8), (5.9) imply that \(\text{cycl}(k) = \text{mult}(k) > 0\), contradicting the assumption. \(\square\)

**Corollary 5.9.** Assume that \((m, \rho)\) is as above, and \(r > 1\). Then \(p_+ + p_- \geq 1\). If \(k \in (\mathbb{Z}+1/2)_+\) satisfies \(\text{mult}(2k) \neq 0\) then \(k \leq (p_+ + p_- + 1)/2\) if \(p_+ + p_- \in \mathbb{Z}\) is even, and \(k \leq (p_+ + p_-)/2\) otherwise.

**Proof.** If \(r > 1\) then \(p_- = p_-\), i.e. \(p_-\) is in a different class modulo \(Z\) than \(p_+.\) It is immediate that \(p_+ + p_- \geq 1\). Looking at (5.9), (5.15) we see easily that for \(k\) above the indicated bounds, \(\text{mult}(2k) = 0\) (use that \(p_+ \geq m\), and that \(p_- = p_-\)). \(\square\)

**Lemma 5.10.** Assume that \((m, \rho)\) is as above, and \(r > 1\). Then \(p_+ > p_-\).
Lemma 5.11. Assume that \((m, \rho)\) is such that \(\text{cycl}(k) = 0\) for all \(k \in (\mathbb{Z} + 1/2)_+\) with \(k \geq (p_+ + p_-)/4\). Then all parts of \(\rho\) are equal.

Proof. In other words, we need to prove that \(T_m(\rho)\) is a rectangular tableau. Clearly we may assume without loss of generality that \(r > 1\). Assume first that \(r_+\) is above the \(a_m\)-diagonal. Then \(k_1^+ := p_+ + r_+ > p_+\) (the notation \(k_1^+\) is not to be confused with the notation \(x^+\) for \(x \in \mathbb{Z} \pm 1/4\) as in Remark 5.7), and \((\text{mult}_+(k^+) - h_+(k^+ - m)) > 0\). As we saw before, it follows from (5.9) that the largest argument \(k'/m\) for which \(\text{mult}(k') \neq 0\) is either \(k_1^+\) or \(k_1^- := p_- + r_-\) if \(k_1^- > k_1^+\). In any case, \(k_1 := \max\{k_1^+, k_1^-\} \geq k_1^+ = p_+ + r_+ > (p_+ + p_-)/2\) (the inequality follows from Lemma 5.10), and \(\text{mult}(k_1) > 0\).

Our assumption \(\text{cycl}(k_1) = 0\) and (5.1) now force that \(\text{mult}(2k_1) = 0\). Corollary 5.9 then implies that \(k_1 \leq (p_+ + p_- + 1)/2\) (if \(p_+ + p_-\) is even) or \(k_1 \leq (p_+ + p_-)/2\) (if \(p_+ + p_-\) is odd). In the second case we reach a contradiction with the above, and we conclude that \(p_+ + p_-\) is even, and that \(k_1 = k_1^+ = (p_+ + p_- + 1)/2\). Then (5.9) and (5.15) imply that \(\text{mult}(2k_1) = (\text{mult}(2k_1) - h_-(2k_1 - m)) \leq h_-(2k_1 - p_+ - 1) = 1\), so that \(k_1^+ = k_1^-\) is not allowed (since that would imply that \(\text{cycl}(k_1) = 2\), so that \(\text{cycl}(k_1) \geq 1\), a contradiction). Thus \(k_1^- < k_1^+\). Now notice that \(2k_1^+ = 2p_+ + 2r_+ = p_+ + p_- + 1\), implying that \(p_+ + 2r_+ = p_- + 1\), so that \(0 < 2r_+ = 1 + p_- - p_+ \leq 1\). It follows that \(r_+ = 1/4\), and \(p_+ = p_- + 1/2\). Suppose now that \(r_-\) is still below the \(a_m\)-diagonal, then \(r_- \geq 3/4\) and \(k_1^- = p_- + r_- \geq p_+ + 1/4 = k_1^+\), contradicting our earlier conclusion that \(k_1^- < k_1^+\). Thus \(r_-\) must be above the \(a_m\)-diagonal, and in fact we must have \(r_- = r_+ = 1/4\). We finally conclude that \(T_m(\rho)\) is a square diagram with \(m = 1/4\), and \(p_+ = n + 1/4\), \(p_- = n - 1/4\) and \(r_+ = r_- = 1/4\). This finishes the case where \(r_+\) is above the diagonal.

Next assume that \(r_+\) is below the \(a_m\)-diagonal. Then \(r_-\) is below the \(a_m\)-diagonal as well. This implies in particular that either \(r_- = r_+\) (which is what we want to show) or otherwise \(r_- - r_+ \geq 2\).

We have seen that the largest argument \(k_1\) for which \(\text{mult}(k_1) > 0\) is \(k_1 = \max\{k_1^+, k_1^-\}\), where this time (because of the congruence classes of \(r_-\) and \(r_+\) modulo \(\mathbb{Z}\)) \(k_1^+ = p_+ - r_+\) and \(k_1^- = p_- + r_-\). Since \(\text{cycl}(k_1) = 0\) we must have that \(\text{mult}(2k_1) = 0\), which implies by Corollary 5.9 that \(k_1 \leq (p_+ + p_- + 1)/2\) (if \(p_+ + p_-\) is even) and \(k_1 \leq (p_+ + p_-)/2\) (if \(p_+ + p_-\) is odd) as before. We note that \(\text{mult}(p_+ + p_- + 1) = 1\) in the first case \((p_+ + p_-\) even), while \(\text{mult}(p_+ + p_-) = 2\) in the second case. Assume that \(p_+ + p_-\) is even. Then
\( p_+ + p_- + (r_+ - r_-) = k_1^+ + k_1^- \leq 2k_1 \leq p_+ + p_- + 1 \). Thus \( r_+ = r_- \), as we intended to show. Next assume that \( p_+ + p_- \) is odd. Then \( p_+ + p_- + (r_+ - r_-) = k_1^+ + k_1^- \leq 2k_1 \leq p_+ + p_- \). Again it follows that \( r_+ = r_- \), and we are done. \( \square \)

**Corollary 5.12.** In the notations of the proof of Lemma 5.11, put \( r_- = r_+ := r \). We have \( m \geq r \), and \( r \equiv m \pmod{Z} \) if and only if \( m = r = 1/4 \).

**Proof.** It was shown in the proof of Lemma 5.11 that \( r \equiv m \pmod{Z} \) implies that \( m = r = 1/4 \). If \( r \not\equiv m \pmod{Z} \) then Lemma 5.8 and Lemma 5.11 imply that \( p_- \equiv r \pmod{Z} \), and by Lemma 5.11 it follows that \( p_+ - m = p_- - r \), or \( m - r = p_+ - p_- \). The assertion now follows from Lemma 5.10. \( \square \)

**Theorem 5.13.** Assume that \((m, \rho)\) is such that \( \text{cycl}(k) = 0 \) for all \( k \in (Z + 1/2)_+ \) with \( k \geq (p_+ + p_-)/4 \). Then \((m, \rho)\) is one of the following possibilities:

(a) \( m \) is arbitrary and \( \rho \) is zero.

(b) \( m = 1/4 \) and \( \rho \) is a square diagram, so that \( r_- = r_+ = 1/4 \).

(c) \( m = 3/4 \), and \( \rho \) is a rectangular diagram such that \( r_- = r_+ = 1/4 \). In this case we can write \( p_+ = n + 3/4, p_- = n + 1/4 \) for some \( n \in Z_{\geq 0} \).

(d) \( m = 5/4 \), and \( \rho \) is a rectangular diagram such that \( r_- = r_+ = 3/4 \). In this case we can write \( p_+ = 2n + 5/4, p_- = 2n + 3/4 \) for some \( n \in Z_{\geq 0} \).

(e) \( m = 7/4 \), and \( \rho \) is a rectangular diagram such that \( r_- = r_+ = 1/4 \). In this case we can write \( p_+ = 2n + 7/4, p_- = 2n + 1/4 \) for some \( n \in Z_{\geq 0} \).

If \((m, \rho)\) does not belong to one of these cases, the largest \( k_{(m, \rho)} \in Z + 1/2 \) for which \( \text{cycl}(k_{(m, \rho)}) \neq 0 \) satisfies \( k \geq (p_+ + p_-)/4 \) and \( \text{cycl}(k_{(m, \rho)}) > 0 \).

**Proof.** We have seen in the proof of Lemma 5.11 that if \( \rho \) is not zero and \( r_+ \) is above the \( a_m \)-diagonal, then we are in case (b).

Hence we now assume that \( \rho \) is not zero and \( r := r_+ = r_- \) is below the \( a_m \)-diagonal. We are left with the task of proving that we are in one of the cases (c) to (e).

Since we are assuming \( k \geq (p_+ + p_-)/4 \), we see that \( \text{cycl}(k) = \text{mult}(k) + \text{mult}(2k) + \text{mult}(3k) + \text{mult}(4k) \). For the same reason, we see that \( \text{cycl}(3k) = \text{mult}(3k) = 0 \) (the latter equality holds by assumption). Hence

\[
\text{cycl}(k) = \text{mult}(k) + \text{mult}(2k) + \text{mult}(4k)
\]

We will see that \( \text{mult}(k) \) for \( k \geq 1/2 \) is given by the remarkably simple explicit formula \( (5.17) \), and that \( \text{mult}(k) \geq 0 \). In the proof below we analyze the implications of the requirement that \( \text{mult}(2k) + \text{mult}(4k) \) cancels \( \text{mult}(k) \) for \( k \geq (p_+ + p_-)/4 \).

Following the steps and notations of the proof of Lemma 5.11 we put \( k_1 = \text{max}\{k_1^-, k_1^+\} \), with \( k_1^+ = p_+ \mp r \). Then \( k_1 \) is the largest argument for which \( \text{mult}(k_1) \neq 0 \). Using \( (5.13), \) \( (5.14), \) and \( (5.8) \) the rectangular shape of \( T_m(\rho) \) implies that for all \( k \geq 1/2 \):

\[
\text{mult}(k) = h_+(k + r) + h_-(k - r) \geq 0.
\]

The simplicity of this formula is somewhat deceptive. One uses that \( \Delta(h_\pm)(y) = 1 \) if \( y = \pm m \) or \( y = \mp r \), \( \Delta(h_\pm)(y) = -1 \) if \( y = \pm (p_+ + 1) \) or \( y = \mp (p_- + 1) \), and \( \Delta(h_\pm)(y) = 0 \) otherwise. For example, to see that \( \text{mult}_+(k) - h_+(k - m) = h_+(k + r) \) for all \( k \geq 1/2 \), one checks that for \( k > 2m \) we have

\[
M_+(k) := h_+(k - m) + h_+(k + r) - h_+(k + p_- + 1) = h_+(k - m) + h_+(k + r)
\]
and $R_+(k) = 0$. For $m - r < k \leq 2m$ we have
\[ M_+(k) := h_+(k + r) - h_+(k + p_ - + 1) = h_+(k + r), \]
and $R_+(k) = h_+(m) = h_+(k - m)$, while for $1/2 \leq k \leq m - r$ we have $M_+(k) = h_+(k + r) - h_+(k + p_ - + 1)$ and $R_+(k) = h_+(m)$, where we observe that $h_+(k - m) = h_+(m) - h_+(k + p_ - + 1)$. Similar observations apply to show that
\[ \text{mult}_-(k) - h_-(k + m) = h_-(k - r). \]

The function $h_+(k + r)$ is zero for $k > k_1^+ = p_+ - r$, then equals the linear function $1 + k_1^+ - k$ for $m - r \leq k \leq k_1^+ + 1$, and then is constant for $1/2 \leq k \leq m - r$.

The function $h_-(k - r)$ is zero for $k > k_1^- = p_+ + r$, then equals the linear function $1 + k_1^- - k$ for $2r \leq k \leq k_1^- + 1$, and then is constant for $1/2 \leq k \leq 2r$.

Let us now look at $\text{mult}(2k)$. By (5.9) and (5.15) we have in the rectangular case:
\[ \text{mult}(2k) = \text{mult}_+,-(2k) - h_-(2k - m) - h_+(2k + m) = -h_-(2k - p_+ - 1) \]
for all $k \geq (p_+ + p_-)/4$. Here we used that $h_+(2k + m) = 0$ if $k \geq (p_+ + p_-)/4$, because $(p_+ + p_-)/2 + m = p_+ + (m + r)/2 > p_+$. To describe the function $\text{mult}(2k)$ in this range, we distinguish the following cases for later reference.

(a) $p_+ + p_-$ and $p_+ + r$ are both even. Then the function $\text{mult}(2k)$ is zero for $k > (p_+ + p_- + 1)/2$, it equals the positive linear function $2k - 2 - p_+ - p_-$ for $k^2_2 := (p_+ + r + 1)/2 \leq k \leq (p_+ + p_- + 1)/2 := k_2$, and is constant for $(p_+ + p_-)/4 \leq k \leq (p_+ + r + 1)/2 := k^b_2$ (so $k^b_2 = k^2_2$ here).

(b) $p_+ + p_-$ even and $p_+ + r$ odd. Then the function $\text{mult}(2k)$ is zero for $k > (p_+ + p_- + 1)/2$, it equals the positive linear function $2k - 2 - p_+ - p_-$ for $k^2_2 := (p_+ + r + 2)/2 \leq k \leq (p_+ + p_- + 1)/2 := k_2$, and is constant for $(p_+ + p_-)/4 \leq k \leq (p_+ + r + 1)/2 := k^b_2$ (so $k^b_2 = k^2_2$ here).

(c) $p_+ + p_-$ odd and $p_+ + r$ even. Then the function $\text{mult}(2k)$ is zero for $k > (p_+ + p_-)/2$, it equals the positive linear function $2k - 2 - p_+ - p_-$ for $k^2_2 := (p_+ + r + 1)/2 \leq k \leq (p_+ + p_-)/2 := k_2$, and is constant for $(p_+ + p_-)/4 \leq k \leq (p_+ + r + 1)/2 := k^b_2$ (so $k^b_2 = k^2_2$ here).

(d) $p_+ + p_-$ and $p_+ + r$ are both odd. Then the function $\text{mult}(2k)$ is zero for $k > (p_+ + p_-)/2$, it equals the positive linear function $2k - 2 - p_+ - p_-$ for $k^2_2 := (p_+ + r + 2)/2 \leq k \leq (p_+ + p_-)/2 := k_2$, and is constant for $(p_+ + p_-)/4 \leq k \leq (p_+ + r + 1)/2 := k^b_2$ (so $k^b_2 = k^2_2$ here).

We will also need to consider the largest value $k_4 \in \mathbb{Z} + 1/2$ such that $\text{mult}(4k_4) \neq 0$, assuming that this $k_4$ satisfies $k_4 \geq (p_+ + p_-)/4$. By (5.15) we have:
\[ \text{mult}(4k) = (\text{mult}_+,-(4k) - h_-(4k - m)) = -h_-(4k - p_+ - 1) \]
Hence the summand $\text{mult}(4k)$ is possibly nonzero only for the smallest values of $k$ in the range $k \geq (p_+ + p_-)/4$, since $h_-(4k - p_+ - 1) \neq 0$ implies that $4k \leq p_+ + p_- + 1$. Considering that $4k_4 = 2 \pmod{4}$ we make this more precise: Let $k_4$ be the largest $k \in (\mathbb{Z} + 1/2)_+$ for which $\text{mult}(4k_4) \neq 0$.

(i) If $p_+ + p_- = 1 \pmod{4}$ then $k_4 = (p_+ + p_- + 1)/4$, and $\text{mult}(4k_4) = -1$;
(ii) If $p_+ + p_- = 2 \pmod{4}$ then $k_4 = (p_+ + p_-)/4$, and $\text{mult}(4k_4) = -h_-(p_- - 1)$;
(iii) If $p_+ + p_- = 0 \pmod{4}$ then $k_4 = (p_+ + p_- - 2)/4$, and so is not in our range;
(iv) If \( p_+ + p_- = 3 \pmod{4} \) then \( k_4 = (p_+ + p_- - 1)/4 \), (and so is not in our range):

We see that always \( k_2 > k_4 \), and \( k_4 \) is either not in our range or it is the smallest argument in our range.

A function defined on \( \mathbb{Z} + 1/2 \) is the restriction of a unique continuous piecewise linear function on \( \mathbb{R} \) whose intervals of linearity have end points in \( \mathbb{Z} + 1/2 \). For \( \text{mult}(2k) \) we distinguished the following “events” \( k_2, k_2^a, k_2^b \in (\mathbb{Z} + 1/2)_+ \): It is zero for \( k > k_2 \), it is negative and linear on \([k_2^a, k_2] \), and it is constant again on \([k_4, k_2^b]\), with \( 0 \leq k_2^a - k_2^b \leq 1 \).

We define \( k_2^b \in \mathbb{Z} + 1/2 \) to be the largest argument such that \( \text{mult}(2k_2^b) \) is strictly larger than the value of the linear function \( l_2(k) := 2k - 2 - p_+ - p_- \) at \( k_2^b \) describing \( \text{mult}(2k) \) on the interval \([k_2^b, k_2] \). Then \( k_2^b = k_2^a - 1 \) in cases (a) and (c), while \( k_2^b = k_2 \) in cases (b) and (d). We define the \( \text{mult}(2k) \)-deficit \( d_2 = \text{mult}(2k_2^b) - l_2(k_2^b) > 0 \) at \( k_2^b \) to be the difference between \( \text{mult}(2k_2^b) \) and this linear expression evaluated at \( k_2^b \). Hence the deficit equals 2 for the cases (a) and (c), and 1 for the cases (b) and (d).

It is immediate that the condition \( \text{cycl}(k) = 0 \) for all \( k \geq (p_+ + p_-)/4 \) implies that \( k_1 := \max\{k_1^+, k_1^-\} = k_2 \). Notice that we also have \( k_1^+ + k_1^- = 2k_2 - 1 \) in the cases (a) and (b), and \( k_1^+ + k_1^- = 2k_2 \) in the cases (c) and (d). Hence in the cases (a) and (b) we have \( |k_1^+ - k_1^-| = 1 \), while for the cases (c) and (d) we have \( k_1^+ = k_1^- \). We see that in these cases, the linear expression representing \( \text{mult}(k) \) just below \( k_1 \), and the linear expression of \( \text{mult}(2k) \) just below \( k_2 \) indeed cancel each other. We also see that if \( k_1 \neq k_2 \), then \( k_1 > k_2 \), and thus \( \text{cycl}(k_1) = \text{mult}(k_1) > 0 \).

We denote by \( k_1^\ell \) the largest argument for which there is a positive deficit \( d_1 = \ell_1(k_1^\ell) - \text{mult}(k_1^\ell) > 0 \) between \( \text{mult} \) and the linear expression \( \ell_1 \) valid for \( \text{mult}(k) \) with \( k_1^\ell < k \leq k_1 + 1 \). From (5.17) and the text below that formula, we see that \( k_1^\ell = \max(m - r, 2r) - 1 \), and that \( d_1 = 2 \) if \( m - r = 2r \), while \( d_1 = 1 \) else. We now have more possibilities to distinguish.

(a') As in case (a), where we impose in addition that \( k_1^+ = k_1^- + 1 = k_1 = k_2 \). This implies that \( k_1^\ell = p_\pm \mp r = (p_+ + p_- \pm 1)/2, \) thus \( p_+ - p_- = 1 + 2r = m - r \) (recall that, from the rectangular shape of \( T_m(p) \), we have \( p_+ - m = p_- - r \)), so that \( m = 3r + 1 \). Hence \( k_1^\ell = \min(m - r, 2r) = 2r \), with deficit \( d_1 = 1 \). Moreover, 
\[
k_2^\ell = (p_+ + r - 1)/2 = (p_+ + p_- + 4r - 1)/4 \quad \text{with deficit} \quad d_2 = 2.
\]
Notice that 
\[
k_2^\ell - k_1^\ell = (p_+ - 3r - 1)/2 = (p_+ - m) \geq 0.
\]

(a'') As in case (a), where we impose in addition that \( k_1^+ = k_1^- + 1 = k_1 = k_2 \). This implies that \( k_1^\ell = p_\pm \mp r = (p_+ + p_- \mp 1)/2, \) thus \( p_+ - p_- = -1 + 2r = m - r \), so that \( m = 3r - 1 \). Hence \( k_1^\ell = \min(m - r, 2r) = 2r - 1 \), with deficit \( d_1 = 1 \). Moreover, 
\[
k_2^\ell = (p_+ - r - 1)/2 = (p_+ + p_- + 4r - 3)/4 \quad \text{with deficit} \quad d_2 = 2.
\]
Observe that 
\[
k_2^\ell - k_1^\ell = (p_+ - 3r + 1)/2 = (p_+ - m) \geq 0.
\]

(b') As in case (b), where we impose in addition that \( k_1^+ = k_1^- + 1 = k_1 = k_2 \). This implies that \( k_1^\ell = p_\pm \mp r = (p_+ + p_- \pm 1)/2, \) thus \( p_+ - p_- = 1 + 2r = m - r \), so that \( m = 3r + 1 \). In this case \( k_1^\ell = \min(m - r, 2r) = 2r \), with deficit \( d_1 = 1 \).
Moreover, \( k_2^c = (p_+ + r)/2 = (p_+ + p_- + 4r + 1)/4 \) with deficit \( d_2 = 1 \). Hence \( k_2^c - k_1^c = (p_+ - 3r)/2 = (p_+ - m + 1) > 0 \).

(b") As in case (b), where we impose in addition that \( k_1^- = k_1^+ + 1 = k_1 = k_2 \). This implies that \( k_1^\pm = p_\pm + r = (p_+ + p_- + 1)/2 \), thus \( p_+ - p_- = 1 + 2r = m - r \), so that \( m = 3r - 1 \). In this case \( k_1^\pm = \min(m - r, 2r) = 2r - 1 \) with deficit \( d_1 = 1 \). Moreover, \( k_2^c = (p_+ + r)/2 = (p_+ + p_- + 4r - 1)/4 \) with deficit \( d_2 = 1 \). Hence \( k_2^c - k_1^c = (p_+ - 3r + 1)/2 = (p_+ - m + 2) > 0 \).

(c') As in case (c), where we impose in addition that \( k_1^- = k_1^+ = k_1 = k_2 \). This implies that \( k_1^\pm = p_\pm + r = (p_+ + p_-)/2 \), thus \( p_+ - p_- = 2r = m - r \), so that \( m = 3r \). In this case \( k_1^\pm = m - r - 1 = 2r - 1 \) with deficit \( d_1 = 2 \). Moreover, \( k_2^c = (p_+ + r - 1)/2 = (p_+ + p_- + 4r - 2)/4 \) with deficit \( d_2 = 2 \). Hence \( k_2^c - k_1^c = (p_+ - 3r + 1)/2 = (p_+ - m + 1) > 0 \).

(d') As in case (d), where we impose in addition that \( k_1^- = k_1^+ = k_1 = k_2 \). This implies that \( k_1^\pm = p_\pm + r = (p_+ + p_-)/2 \), thus \( p_+ - p_- = 2r = m - r \), so that \( m = 3r \). In this case \( k_1^\pm = m - r - 1 = 2r - 1 \) with deficit \( d_1 = 2 \). Moreover, \( k_2^c = (p_+ + r)/2 = (p_+ + p_- + 4r)/4 \) with deficit \( d_2 = 1 \). Hence \( k_2^c - k_1^c = (p_+ - 3r + 2)/2 = (p_+ - m + 2) > 0 \).

This shows that in all cases, \( k_2^c \geq k_1^c \), and if \( k_1^c = k_2^c \) then \( d_1 < d_2 \). This implies that always \( \text{mult}(k_2^c) + \text{mult}(2k_2^c) > 0 \) if \( k_2^c \geq (p_+ + p_-)/4 \) (in other words, if \( k_2^c \) is in our range \( k \geq (p_+ + p_-)/4 \)). Therefore, \( \text{cycl}(k) = 0 \) for all \( k \geq (p_+ + p_-)/4 \) could only happen if one of the following possibilities hold: Either \( k_2^c < (p_+ + p_-)/4 \), or else \( (p_+ + p_-)/4 \leq k_2^c \leq k_1 \), and \( \text{cycl}(k) = (\text{mult}(k_2^c) + \text{mult}(2k_2^c)) + \text{mult}(4k_2^c) = 0 \). We have already mentioned above that \( k_4 \geq (p_+ + p_-)/4 \) implies that \( k_4 \) is the smallest element of \( Z + 1/2 \) that satisfies this inequality. In other words, the second possibility mentioned is equivalent to \( (p_+ + p_-)/4 \leq k_2^c \leq k_1 \).

If \( k_2^c < (p_+ + p_-)/4 \) we must be in case (a") or (c') with \( r = 1/4 \). In the first case that would imply \( m < 0 \), which is absurd. In the case (c') we get a solution, with \( r = 1/4 \), \( m = 3/4 \), \( p_+ = (2n - 1) + 3/4 \), \( p_- = (2n - 1) + 1/4 \). This is the odd rank case of Theorem 5.13(c).

Next we analyze the case \( (p_+ + p_-)/4 \leq k_2^c = k_4 \). One reads off from the above list of cases that this implies that we are either in the case (i) and (d'), with \( r = 1/4 \), \( m = 3/4 \), \( p_+ = 2n + 3/4 \), and \( p_- = 2n + 1/4 \) (here \( d_2 = 1 = \text{mult}(k_4) \)), which is the even rank case of Theorem 5.13(c), or in case (ii) and (a'), with \( r = 1/4 \) and \( m = 7/4 \), which gives Theorem 5.13(e), or in case (ii) and (a'), with \( r = 3/4 \) and \( m = 5/4 \), which is case Theorem 5.13(d). In the latter two cases we have \( d_2 = 2 \), and \( \text{mult}(k_4) = 2 \) if \( k_1^c < k_2^c \), while \( \text{mult}(k_4) = 1 = d_1 \) if \( k_1^c = k_2^c \).

Observe that in the course of the proof, we also checked that if a pair \((m, \rho)\) does not belong to one of the cases listed in Theorem 5.13(a)-(e), then there exist \( k \in Z + 1/2 \) such that \( k \geq (p_+ + p_-)/4 \) and \( \text{cycl}(k) \neq 0 \), the largest \( k_{(m, \rho)} \) of which satisfies \( \text{cycl}(k_{(m, \rho)}) > 0 \).

This concludes the proof of all assertions. \( \square \)

5.6. **Counting odd cyclotomic polynomial factors for the cases** \( \delta \in \{0, 1\} \). As we have already explained in Subsection 5.2, the statements about odd cyclotomic factors
of $\mu_{0,\delta}^{(\{\bar{\tau},\lambda\})}$ follow from the results of the previous paragraph by application of the extra-special STMs. We now translate the results to the cases $\delta \in \{0,1\}$ using the extra-special bijection.

**Corollary 5.14.** Suppose that $\delta \in \{0,1\}$ and let $\bar{\tau}$ be a positive residual point such that $\mu_{0,\delta}^{n,\{(\bar{\tau})\}}$ has no odd cyclotomic factors. Then $\bar{\tau}$ is of the form $\bar{\tau} = \bar{\tau}_\lambda$ with $\lambda \vdash \delta + 2n \alpha$ partition with odd, distinct parts. The pair $(\delta, \lambda)$ must belong one of the following cases:

(a) Choose $m \in \mathbb{Z} + 1/4)$ and define $\kappa \in \mathbb{Z}_{\geq 0}$ and $\epsilon \in \{0,1\}$ by writing $m = \kappa + (2\epsilon - 1)/4$. Define $\lambda = [1 + 2\epsilon, 5 + 2\epsilon, \ldots, 4(\kappa - 1) + 1 + 2\epsilon]$, and $\delta \in \{0,1\}$ by $\kappa \equiv \delta \pmod{2}$. Define $n$ by $2n + \delta = 2\kappa^2 + (2\epsilon - 1)\kappa$. Then $\bar{\tau}_\lambda$ represents a cuspidal (extra-special) unipotent STM $\mathcal{H}_{1/4,m}^0 \simeq \mathcal{H}_0^n$. In particular, modulo powers of $q$ and rational constants, we have $\mu_{0,\delta}^{n,\{(\bar{\tau})\}} = d_{n/4,m}^0$ (with $d_{n/4,m}^0$ as in (3.4)), and for all these cases $\mu_{0,\delta}^{n,\{(\bar{\tau})\}}$ indeed has no odd cyclotomic factors.

(b) For $r \in \mathbb{Z}_{\geq 0}$, put $\lambda = [1,3,\ldots, 4r + 1, 4r + 3] \vdash 2n$ with $n = 2(r + 1)^2$, and put $\delta = 0$. Then $\bar{\tau} = \bar{\tau}_\lambda$ represents a cuspidal unipotent STM $\mathcal{H}_{0,m}^0 \simeq \mathcal{H}_0^n$ with $m = 2(r + 1)$, and $\mu_{0,0}^{n,\{(\bar{\tau})\}} = (d_{a}^{D}(q))^2$ with $a = (r + 1)$. In particular, for all these cases $\mu_{0,0}^{n,\{(\bar{\tau})\}}$ indeed has no odd cyclotomic factors. (Note: We consider the empty diagram with $\delta = 0$ as belonging to case (a).)

(c) For $r \in \mathbb{Z}_{\geq 0}$, put $\lambda = [1,3,\ldots, 4r + 3, 4r + 5] \vdash 2n + 1$ with $n = 2(r + 1)(r + 2)$, and put $\delta = 1$. Then $\bar{\tau} = \bar{\tau}_\lambda$ represents a cuspidal unipotent STM $\mathcal{H}_{0,m}^0 \simeq \mathcal{H}_0^n$ with $m = 2r + 3$, and $\mu_{0,1}^{n,\{(\bar{\tau})\}} = (d_{b}^{B}(q))^2$ with $b = (r + 1)$. In particular, for all these cases $\mu_{0,1}^{n,\{(\bar{\tau})\}}$ indeed has no odd cyclotomic factors.

(d) Let $r \in \mathbb{Z}_{\geq 0}$, put $n = 8(r + 1)^2 - 1$, $\lambda = [3,5,7,\ldots, 8r + 5,8r + 7] \vdash 2n + 1$ and $\delta = 1$. In this case, modulo powers of $v$ and rational constant, we have

$$
\mu_{0,1}^{n,\{(\bar{\tau})\}} = \frac{(1 - q^{4(r+1)})(1 - q^{4(r+1)})}{(1 - q)} \mu_{0,0}^{n,1,\{(\bar{\tau})\}}
$$

where $\bar{\tau}' = \bar{\tau}_\lambda$, with $\lambda' = [1,3,5,\ldots, 8r + 7]$ (then $\bar{\tau}' = (1,\bar{\tau})$, and $a = 2(r+1)$. In particular, $\mu_{0,1}^{n,\{(\bar{\tau})\}}$ has no odd cyclotomic polynomials as factors if and only if $r = 2s - 1$ for some $s \geq 0$.

(e) Let $r \in \mathbb{Z}_{\geq 0}$, put $n = 8(r + 1)^2 + 1$, $\delta = 0$, and $\lambda = [1,3,5,\ldots, 8r + 5,8r + 9] \vdash 2n$. Then $\bar{\tau}_\lambda = (\bar{\tau}_\lambda', q^{4(r+1)})$, where $\lambda' = [1,3,5,\ldots, 8r + 7]$. Since $\bar{\tau}_\lambda$ represents a standard translation morphism, we see that $\bar{\tau}$ is the image under the standard translation STM $\mathcal{H}_{0,m}^0 \simeq \mathcal{H}_0^n$ of the residual point $t = (q^{4(r+1)})$, where $m = 4(r + 1)$. Consequently, modulo powers of $v$ and rational constants we have:

$$
\mu_{0,0}^{n,\{(\bar{\tau})\}} = \frac{(1 - q^{4(r+1)})(1 - q^{4(r+1)})(1 - q)}{(1 + q^{4(r+1)})^2} \mu_{0,0}^{n,\{(\bar{\tau})\}}
$$

with $a = 2(r + 1)$. In particular, $\mu_{0,0}^{n,\{(\bar{\tau})\}}$ has no odd cyclotomic polynomials as factors if and only if $r = 2s - 1$ for some $s \geq 0$. 

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Together with the results of Theorem 5.13 the above results imply that in all cases, if a residue of the form $\mu_{0,\delta}^n(\{\mathcal{P}\})$ or of the form $\mu_{1/4,m}^r(\{\mathcal{P}\})$ contains odd cyclotomic polynomials with nonzero multiplicity, then the highest odd cyclotomic factor which has nonzero multiplicity in the residue in fact has positive multiplicity.

Proof. The last assertion follows from the expressions given in parts (a)-(e). Parts (a)-(c) follow directly from the results of Theorem 5.13 by application of the extra-special bijection and STMs. The expression for residues in terms of unipotent degrees follows from the relation between the parameters $\{a,b\}$ and $\{m_-,m_+\}$, the normalization of unipotent affine Hecke algebras, and the defining properties of STMs, as explained in Section 3 (also see [Op2]).

The results mentioned in (e) follow from an application of a standard translation STM (as explained there).

The result of (d) is obtained by a direct computation, using that the residue point $\mathcal{P}'$ represents a standard STM, and is constructed from $\mathcal{P}$ simply by adding one coordinate with the value 1. Using this, it is an easy direct computation to check the formula given in (d). \hfill \Box

Remark 5.15. This finishes the proof of Proposition 5.3, and thus of Proposition 5.1.

6. Proof of the main theorem

6.1. Computing residues. In the present section we will prove the main Theorem 1.1 using the techniques we have developed so far. In the course of the analysis we will need to be able to compute residues of the form

\begin{equation}
(\mu_{\min}^X(\{\mathcal{P}\})) := (\mu_{\delta_-,\delta_+}^q(\{\mathcal{P}_{(\lambda_-,\lambda_+)}\}))
\end{equation}

where $X \in \{I, II, III, IV\}$, with $\delta_{\pm} \in \{0, 1/2, 1\}$ and $b \in \{1, 2\}$ determined such that $\mathcal{H}_{\min}^X = C_\mathcal{P}(\delta_-, \delta_+)[g^b]$ (see (3.5)), and where $\mathcal{P} = (\mathcal{P}_{(\lambda_-,\lambda_+)} = (-\mathcal{P}_{(b,\delta_-:\lambda_-)}, \mathcal{P}_{(b,\delta_+:\lambda_+)}))$ is a $C_\mathcal{P}$-residual point for these parameters, i.e. $\lambda_{\pm} \vdash 2n_{\pm} + [\delta_{\pm}]$ is a partition with distinct parts of parity type $2\delta_{\pm} - 1$, such that $n_- + n_+ = n$.

Suppose that $(\mathcal{P}, \sigma)$ is a unipotent cuspidal pair of an inner form of a connected, unramified, almost absolutely simple group $G_{qs} = G_{qs}(K)$, defined and quasi-split over $k$, of adjoint type in its isogeny class, for which there exists a spectral covering map $\mathcal{H}_{\min}^X(G_{qs}) \hookrightarrow \mathcal{H}_{\min}^X$.

Recall (Proposition 3.5, Corollary 3.6) that it is our task to show that the set of orbits $W_0 \mathcal{P}$ of residual points of $\mathcal{H}_{\min}^X$ satisfying the equation

\begin{equation}
(\mu_{\min}^X(\{\mathcal{P}\})) = \text{fdeg}_q(\sigma)
\end{equation}

consists of a single $\text{Aut}_{\text{es}}(\mathcal{H}_{\min}^X)$-orbit (which is then necessarily an orbit of images of standard STMs, since we already know the existence of enough standard STMs).

Recall from Section 5 that if $W_0 \mathcal{P}$ is a solution of (6.2) with $\mathcal{P}$ of the form $\mathcal{P} = (-\mathcal{P}_{(b,\delta_-:\lambda_-)}, \mathcal{P}_{(b,\delta_+:\lambda_+)}))$, then $\lambda_{\pm}$ have to belong to the set of unipotent partitions described in Lemma 5.4 (for $\delta_{\pm} = 1/2$) or in Corollary 5.14 (for $\delta_{\pm} \in \{0, 1\}$). Hence it
suffices to show that the assignment $W_0 \varphi \rightarrow (\mu_{\text{min}}^X)_{\{\varphi\}}(\mathcal{F})$ is injective on this set of orbits of residual points, modulo the action of Aut$(\mathcal{H}_{\text{min}}^X)$.

Define $\Phi^c \subset \Phi$ the subgroup generated by the rational constants and expressions of the form $v^k + v^{-k}$ for $k \in \mathbb{Z}_+$. For $f \in \Phi^c$, there exists a unique function $m^e_f : \mathbb{N} \rightarrow \mathbb{Z}$ such that modulo the subgroup of $K^*$ generated by powers of $v$ and rational constants, we have

$$f \equiv \prod_{n \in \mathbb{Z}_+} (1 + q^n)^{m^e_f(n)} \quad (6.3)$$

Hence it makes sense, for parameters $(b; \delta-, \delta_+)$ such that $\mathcal{H}_{\text{min}}^X = C_n(\delta-, \delta_+)[q^b]$ and for distinguished unipotent partitions $\lambda\pm$ for $\delta_\pm$ as listed in Lemma 5.4 (for $\delta_\pm = 1/2$) or in Corollary 5.14 (for $\delta_\pm \in \{0,1\}$), to define:

**Definition 6.1.** We put

$$\text{mult}_{(b; \delta-, \delta_+)}(\lambda_-; \lambda_+) = m^e_{(\mu_{\text{min}}^X, \delta_0 = 1)((\mathcal{F}))}$$

with $(\delta-, \delta_+)$ and $b$ as above, and $\mathcal{F}' = (-\mathcal{F}(b; \delta-, \lambda_-), \mathcal{F}(b; \delta_+, \lambda_+))$ as above. Here $\mu_{\text{min}}^{X, \delta_0 = 1}$ denotes the $\mu$-function of $C_n(\delta-, \delta_+)[q^b]$ normalized by $(v^b - v^{-b})^{-n}$ (the actual normalization factor is $(v^b - v^{-b})^{-n}(v + v^{-1})^{-\delta_1, \delta_2, \delta_3}$, which has an additional factor $d_{\text{2D}}^2 = (v + v^{-1})^{-1}$ if $\delta_\pm = 1$, due to the anisotropic factor of the maximal torus of $2D_n$).

6.2. **Remarks on residues at residual points with positive coordinates.** For the residue points $\mathcal{F}' = \mathcal{F}(b; 1, \delta, \lambda)$ with positive coordinates as listed in the above Corollary 5.14 we computed the residues $\mu^{n,\{\varphi\}}_{0, \delta}$. We could alternatively compute the residues $\mu^{n,\{\varphi\}}_{1, \delta}$, and for later reference we will express the relation between these two residues now, and use the opportunity to introduce some useful notations.

In general, let $\delta \in \{0,1/2,1\}$ and let $\lambda \vdash 2n + [\delta]$ be a distinguished unipotent partition of type $\delta$, i.e. $\lambda$ has distinct parts of parity type $1 - 2\delta$. Let $\mathcal{F}' = \mathcal{F}(b; \delta, \lambda)$ be the corresponding residue point with positive coordinates.

**Definition 6.2.** For $x \in \mathbb{Q}$, let $h_{b; \lambda}(x) \in \mathbb{Z}_{\geq 0}$ denote the number of coordinates of $\mathcal{F}' = \mathcal{F}(b; \delta, \lambda)$ equal to $q^x$ or to $q^{-x}$. Note that this is independent of the choice of $\mathcal{F}'$ in its $W_0(B_n)$-orbit. In particular $h_{b; \lambda}(bx) = h_{1; \lambda}(x)$, and $h_{b; \lambda}$ is an even function supported on $b(\mathbb{Z} + \delta)$. Observe that $h_{b; \lambda}(0)$ equals twice the number of coordinates of $\mathcal{F}'$ which are equal to 1. If $b = 1$ then we usually drop it from the notation and simply write $h_\lambda$.

For the residue points listed in Corollary 5.14 we restrict ourselves to $b = 1$ and $\delta \in \{0,1\}$. Then it is obvious from the definition of $h_\lambda$ and from (2.4) that, modulo powers of $q$ and rational constant factors, we have:

$$\mu^{n,\{\varphi\}}_{1, \delta}(\mathcal{F}) = \prod_{x \in \mathbb{Z}_+} (1 + q^x)^{\Delta_1(h_\lambda)(x)\mu^{n,\{\varphi\}}_{0, \delta}(\mathcal{F})} \quad (6.4)$$

where $\Delta_1(h_\lambda)(x) = 2h_\lambda(x) - h_\lambda(x - y) - h_\lambda(x + y)$. (If $y = 1$ we will often suppress it from the notation, and simply write $\Delta$).
Equivalently, using the notations we introduced above, we can express this result as:

$$(6.5) \quad \text{mult}^{(0,\lambda)}_{(\delta, \delta)} = \text{mult}^{(0,\lambda)}_{(0,\delta)} + \Delta_{\delta} (\tilde{h}_\lambda)(x).$$

A function defined on $\mathbb{Q}$ with finite support can be viewed canonically as an element in the groups algebra of $\mathbb{Q}$ (viewed as a discrete group). As such we consider the convolution product $m_1 * m_2(z) = \sum_{x+y=z} m_1(x)m_2(y)$, which gives this set of functions the structure of a commutative, unital associative algebra. We use the Dirac delta $\delta_x$ to denote the basis element corresponding to $x \in \mathbb{Q}$, so that $\delta_0$ is the identity of convolution. If $T_xm$ is the (left) translation of $m$, i.e., $(T_x m)(y) = m(y - x)$, then $T_xm = \delta_x * m$. We have the obvious rules $\Delta(m_1 * m_2) = \Delta(m_1) * m_2 = m_1 * \Delta(m_2)$, also $\Delta(m) = \Delta(\delta_0) * m$.

In order to compute the residues $\mu^{p,((\tau),1)}_{\lambda,\delta}$ for the collection of residue points $\tau$ of Corollary 5.14, it suffices to list the functions $\tilde{h}_\lambda$ and $\Delta(\tilde{h}_\lambda)$ for these cases. In addition we will compute $\text{mult}^{(0,\lambda)}_{(0,\delta)}$ in terms of $\tilde{h}_\lambda$.

**Proposition 6.3.** Using the notations of Corollary 5.14, we have

(a) Let $m \in (\mathbb{Z} \pm 1/4)_+$, written as usual as $m = \kappa + (2\epsilon - 1)/4$. Let $\delta \in \{0, 1\}$ be the parity of $\kappa$. Put $p = 2(\kappa - 1) + \epsilon$, the largest jump in the jump sequence of $\lambda$. Thus $\lambda = \lambda_p$ and $\delta$ are both determined by $p$, and for $k \in \mathbb{Z}$ we have:

$$\tilde{h}_\lambda(k) = \begin{cases} 0 & \text{if } k = 0, \\ \max\{0, \lfloor (p + 2 - |k|)/2 \rfloor \} & \text{else}. \end{cases}$$

We have $\text{mult}^{(0,\lambda)}_{(0,\delta)} = -\tilde{h}_\lambda$, and

$$\Delta(\tilde{h}_\lambda) + \Delta_{\delta}(\delta_0) = -S_{p+1} + \delta_0,$$

where

$$S_{p+1}(k) = \begin{cases} (-1)^{p+1-k} & \text{if } |k| \leq p + 1, \\ 0 & \text{else}. \end{cases}$$

(b), (c) The cases (b) ($\delta = 0$) and (c) ($\delta = 1$) can be treated uniformly. Let $p = 2r + 1 + \delta$, thus $\lambda = \lambda_p$ and $\delta$ are both determined by $p$, and for $k \in \mathbb{Z}$ we have:

$$\tilde{h}_\lambda(k) = \begin{cases} 2\lfloor (p + 1 - \delta)/2 \rfloor = p + 1 - \delta & \text{if } k = 0, \\ \max\{0, p + 1 - |k|\} & \text{else}. \end{cases}$$

We have $\text{mult}^{(0,\lambda)}_{(0,\delta)} = -2\tilde{h}_\lambda$, and

$$\Delta(\tilde{h}_\lambda) + \Delta_{\delta}(\delta_0) = \Delta_{(p+1)}(\delta_0) = -\delta_{(p+1)} - \delta_{-(p+1)} + 2\delta_0.$$  

(d) In this case $\delta = 1$ and $p = 2a + 1$, with $a = 2(r + 1)$ a power of 2. Hence $\lambda = \lambda_p$ is determined by $p$, and for $k \in \mathbb{Z}$ we have:

$$\tilde{h}_\lambda(k) = \begin{cases} p - 1 & \text{if } k = 0, \\ \max\{0, p + 1 - |k|\} & \text{else}. \end{cases}$$

We have $\text{mult}^{(0,\lambda)}_{(0,\delta)} = -2\tilde{h}_\lambda + \delta_1 + \delta_2 + \delta_4 + \cdots + \delta_a$, and

$$\Delta(\tilde{h}_\lambda) + \Delta_{\delta}(\delta_0) = -\delta_{(p+1)} - \delta_{-(p+1)} + \delta_{-1} + \delta_{+1}.$$
(e) In this case $\delta = 0$ and $p = 2a$, with $a = 2(r + 1)$ a power of 2. Hence $\lambda = \lambda_p$ is determined by $p$, and for $k \in \mathbb{Z}$ we have:

$$\tilde{h}_\lambda(k) = \begin{cases} p & \text{if } k = 0, \\ \max\{0, p - |k|\} & \text{if } 0 < |k| < p, \\ 1 & \text{if } |k| = p, \\ 0 & \text{else.} \end{cases}$$

We have $\text{mult}(0, \lambda) = -2\tilde{h}_\lambda + \delta_1 + \delta_2 + \delta_4 + \cdots + \delta_p$, and

$$\Delta(\tilde{h}_\lambda) = (-\delta(p+1) + \delta_p - \delta(p-1)) + (-\delta(-(p-1) + \delta_p - \delta(-(p+1))) + 2\delta_0).$$

Proof. We use the rules (cf. [OS, Proposition 6.6]) describing a linear residual point $\xi$ for $B_n$ with parameter $\delta \in \{0, 1\}$ and $k = \log(q)$, in terms of a distinguished $\delta$-partition $\lambda$. The number $m_0$ of coordinates equal to 0 is equal to $[(m_1 + 1 - \delta)/2]$, where $m_1$ is the number of coordinates equal to $\pm k$. In terms of $\tilde{h}_\lambda$, this means that $\tilde{h}_\lambda(0) = (\tilde{h}_\lambda(1)+1-\delta)/2$. For positive integers $n$, $\tilde{h}_\lambda(n)$ equals the number of jumps in the jump sequence $j = (\lambda - 2)/2$ which are larger than or equal to $n$. Since we extended the function $\tilde{h}_\lambda$ as an even function, this shows that $\tilde{h}_\lambda$ is determined by $(\delta, \lambda)$ and can be read of simply from the information provided in Corollary 5.14. \qed

6.3. Residues at arbitrary residual points. We continue to assume that $\tilde{r} = (-\tilde{r}'_{(b, \delta_-, \lambda_+)}$, $\tilde{r}'_{(b, \delta_+, \lambda_-)})$ is a residual point for $\mathcal{H}^\lambda_{\min}$ such that the residue $\langle \mu^\lambda_{\min}, q \rangle$ has no odd cyclotomic factors.

By the combined results of Lemma 5.4 (for $\delta_\pm = 1/2$), Corollary 5.14 (for $\delta_\pm \in \{0, 1\}$) and Proposition 6.3, the degree $\langle \mu^\lambda_{\min}, q \rangle$, or equivalently the multiplicity function $\text{mult}(\lambda_-, \lambda_+)$, is determined if one of $\lambda_-$ or $\lambda_+$ is the zero partition. In general we need to also take into account the ”mixed” factors of this residue, which arise from the type D-roots which are a product of a coordinate (or its reciprocal) of $-\tilde{r}'_{(b, \delta_-, \lambda_-)}$ and a coordinate (or its reciprocal) of $\tilde{r}'_{(b, \delta_+, \lambda_+)}$. Since each nonzero factor of (2.4) obviously either comes from such ”mixed” root or not, it easy follows that:

**Proposition 6.4.** Suppose that $b \in \{1, 2\}$. For all $k \in (\mathbb{Z}/b)_+$ we have:

$$\text{mult}(\lambda_-, \lambda_+)(bk) = \Delta(\tilde{h}_\lambda)(k) + \text{mult}(0, \lambda_+)(bk) + \text{mult}(0, \lambda_-)(bk)$$

It is also easy to see that we always have:

$$\text{mult}(0, \lambda_\pm)(bk) = (\Delta_{\delta_\pm}(\delta_0) * \tilde{h}_\lambda)(k) + \text{mult}(0, \lambda_\pm)(bk)$$

Combined with the results of Remark 5.6, Corollary 5.14 and Proposition 6.3 this formula enables us to express $\text{mult}(0, \lambda_\pm)(bk)$ in terms of $\tilde{h}_\lambda$ and $\tilde{h}_\lambda$.

We now have enough tools to prove case by case that the solutions of (6.2) indeed constitute an $\text{Aut}_c(\mathcal{H}^\lambda_{\min})$-orbit of images of standard STM.

6.4. Proof of the main Theorem for odd orthogonal groups. We consider $G_{qs} = \text{SO}_{2n+1}$. This is type II, and this is by far the simplest case.
6.4.1. Proof of Theorem 1.1(a), (b). The right hand side of (6.2) is of the form \( fde_{q}(\sigma) = d_{a}^{B}(q)d_{b}^{B}(q) \) (see (3.2)), and a glance at the tables in [Car, 13.7] readily shows that the multiplicity functions \( m_{a,b}^{e}(k) \) for \( d_{a,b} := d_{a}^{B}(q)d_{b}^{B}(q) \) is of the form (where \( k \in \mathbb{Z}_{+} \)):

\[
(6.6) \quad m_{a,b}^{e}(k) = -(\max\{0, 2a - k\} + \max\{0, 2b + 1 - k\})
\]

On the other hand, we have \( \delta_{\pm} = 1/2 \) in this case, and \( b = 1 \). Lemma 5.4 implies that \( \lambda_{\pm} = [2, 4, \ldots, 2r_{\pm}] \), and that the corresponding multiplicity functions \( \tilde{h}_{\pm} \) are supported on \( \mathbb{Z} + 1/2 \) and are given by \( h_{\pm}(x) = \max\{0, p_{\pm} + 1 - |x|\} \) where \( p_{\pm} = r_{\pm} - 1/2 = m_{\pm} - 1 \).

We warn the reader that from here onwards we are using the notation \( \lambda_{\pm}, m_{\pm}, p_{\pm} \) etc. in the sense of Subsections 5.1 and 5.2, but not as in the Subsection 5.4. Applying Proposition 6.3, Lemma 5.4, and Remark 5.6 we have (with \( \tilde{h}_{\pm} := \tilde{h}_{\lambda_{\pm}} \) and \( k \in \mathbb{Z}_{+} \)):

\[
\mu_{\lambda_{\pm}, \lambda_{\pm}}^{m}(k) = \Delta(\tilde{h}_{\pm} + \tilde{h}_{\pm})(k) = \tilde{h}_{\pm}(k) - \tilde{h}_{\pm}(k - 1/2) - \tilde{h}_{\pm}(k + 1/2) - \tilde{h}_{\pm}(k - 1/2) - \tilde{h}_{\pm}(k + 1/2)
\]

\[
= \tilde{h}_{\pm}(k) - \tilde{h}_{\pm}(k - 1/2) - \tilde{h}_{\pm}(k + 1/2) - \tilde{h}_{\pm}(k - 1/2) - \tilde{h}_{\pm}(k + 1/2) = -(\max\{0, p_{\pm} + 2 - k\} + \max\{0, p_{\pm} - 2 - k\})
\]

so that necessarily \( \{2a, 2b + 1\} = \{p_{\pm} + 2, [p_{\pm} - 2]\} = \{m_{\pm} + 1, |m_{\pm} - 1|\} \), as was to be proved (compare with (3.3)), and using the standard STM \( \phi_{(d, +), T}^{(m_{\pm} + 1, m_{\pm})} : T_{d} \rightarrow T_{d+m_{\pm}+1/2} \) of [Op2, Subsection 3.2.5]).

It is instructive to carefully analyze the last step in the above computation. If \( k \) is sufficiently large then \( m_{a,b}^{e}(k) \) and \( \mu_{\lambda_{\pm}, \lambda_{\pm}}^{m}(k) \) are zero. Decreasing \( k \) to 1 we meet three interesting arguments of \( \mu_{\lambda_{\pm}, \lambda_{\pm}}^{m}(k) \), namely \( k_{1} = p_{\pm} + p_{\pm} + 1 \) (this is the first argument where \( \mu_{\lambda_{\pm}, \lambda_{\pm}}^{m}(k) \) is nonzero), \( k_{2} = p_{\pm} + 1/2 \) (this is the first argument after \( T_{p_{\pm}+1} \) reached its maximum) and \( k_{3} = |p_{\pm} - p_{\pm}| - 1 \) (this is the argument where either \( T_{p_{\pm}+1} \) starts to increase (if \( p_{\pm} < p_{\pm} \)) or where \( T_{p_{\pm}+1} \) stops to decrease (if \( p_{\pm} > p_{\pm} \)). Now compare to \( m_{a,b}^{e}(k) \). We see that \( k_{1} = \max\{2a - 2b\} \), \( k_{2} = \min\{2a - 1, 2b\} \). At \( k_{3} \), we see that \( -\tilde{h}_{\pm}(k) - \tilde{h}_{\pm}(k + 1/2) \) exactly reaps the linear behavior of \( T_{p_{\pm}+1} \) at \( k_{3} \). Finally, for \( k \leq k_{3} \) we obtain (in either case) an additional linear contribution, implying that \( k_{3} = \min\{2a - 1, 2b\} \).

Observe that \( \tilde{P}_{\lambda_{\lambda_{\lambda}}} \) represents an STM \( \phi \) which is a composition of the standard translation STMs described in [Op2, Subsection 3.2.5] (also see [Fe1], [Fe2]):

\[
(6.7) \quad \phi = \phi_{(d, +), T}^{(m_{\pm} - 1/2, m_{\pm} + 1/2)} \circ \phi_{(d, +), T}^{(m_{\pm} + 1/2, m_{\pm} - 1/2)} \circ \cdots \circ \phi_{(d, +), T}^{(m_{\pm} - 1/2)} \circ \phi_{d, +, T}^{(0)}
\]

6.4.2. Proof of Theorem 1.1(c). We start by making some general observations on the isotropy groups of orbits of discrete unipotent local Langlands parameters \( [\lambda_{ad}] \) under the action of \( \langle \Omega_{d}^{\theta} \rangle^{*} \). These remarks will be a basis of our analysis for all cases below.

First of all, via [Bo, Section 6] (see also the Introduction of the present paper) it follows that the set of such \( [\lambda_{ad}] \) are in canonical bijection with the set of central characters of \( \mathcal{H}^{L,M}(G_{qs}) \) which support discrete series characters of \( \mathcal{H}^{L,M}(G_{qs}) \) (where \( G_{qs} \) is quasi-split,
and of adjoint type). Since this is important for understanding the isotropy groups, let us discuss this in some detail.

The spectral diagram (cf. [Op1], [Op2]) of $H^{IM}(G_{qs})$ is easily seen to be the untwisted version of the Kac diagram $D(\theta G^\vee)$ (see [R1]) of the based root datum $\Sigma^\vee$ with respect to the diagram automorphism $\theta$ of $F_0$, whose vertices $v$ are labelled with parameters $m_v, b \log(q)$ such that the group of spectral diagram automorphisms and the group of Kac-diagram automorphisms can be identified (the multiples of $m_v$ reflect the twisting information). The set of central characters of $H^{IM}(G_{qs})$ supporting discrete series representations was described in [OS, Section 8], which we will now recall (also see the Introduction). The lattice $Y_{qs}$ is the projection of $X_\ast = X^\ast(S)$ onto the vector space $V^\theta$ of $\theta$-invariants (notations as in the Introduction), and this is the coroot lattice $Q_{qs}$ when $G_{qs}$ is of adjoint type. Hence, the subgroup $\Gamma = Y_{qs}/Q_{qs}^\vee \subseteq \Gamma_{spec}$ (see [OS, Section 8]) of the full group $\Gamma_{spec}$ of special diagram automorphisms $\Gamma_{spec} := P_{qs}/Q_{qs}^\vee$ of $D(\theta G^\vee)$, is trivial: $\Gamma = \Gamma_{ad} = 1$ in our present situation.

In general we have a natural short exact sequence

$$1 \to \Gamma \to \Gamma_{spec} \to (\Omega^\theta)^\ast \to 1,$$

and in the present situation this implies that $(\Omega^\theta_{ad})^\ast = \Gamma_{spec}$ is the full group of diagram automorphisms of the spectral diagram of $H^{IM}(G_{qs})$ (or equivalently, of the Kac-diagram).

In [OS, Section 8] it was explained that the set of central characters of $H^{IM}(G_{qs})$ which support a discrete series representation, is in bijection with the set of pairs $(\Gamma s(e), W_{s(e)}\xi)$ of a $\Gamma$-orbit of vertices $e$ of the spectral diagram (i.e. a node $e$ of $D(\theta G^\vee)$ in our present situation) and an orbit of linear residual points $W_{s(e)}\xi$ of the graded affine Hecke algebra $H_e$ associated to $\Gamma e$. The latter orbits $W_{s(e)}\xi$ are in canonical bijection with the set of distinguished nilpotent orbits $N \subseteq g^{0,s(e)}$ of the Lie algebra of fixed points $g^{0,s(e)}$ corresponding to the torsion element $s(e) \in T^\theta$ associated with the node $e$ (cf. [OS]).

This setup shows that this set of central characters is in natural bijection with the set of discrete unipotent Langlands parameters $\lambda_{ad}$. Indeed, $\lambda_{ad}$ is completely determined by a semisimple conjugacy class in $L^G$ of the image $\theta.s = \lambda_{ad}(Frob)$ of the Frobenius, and a distinguished nilpotent orbit $\xi_s$ in the Lie algebra of the centralizer $C_{L^G}(\theta.s)$. We know that we can choose the semisimple element $s$ in the fundamental dual alcove $C^\vee$ (see [R1, Theorem 3.7]). We need in addition that the Lie algebra $g^{0,s}$ admits distinguished nilpotent orbits, i.e. is semisimple. This forces $s = s(e)$ to be a vertex of $C^\vee$. Hence in terms of diagrams, the set of conjugacy classes of discrete unipotent local Langlands parameters $\lambda_{ad}(Frob)$ is therefore also in natural bijection with the set of pairs $(e, N)$, where $e$ is a vertex of the Kac diagram (see [R1, Section 3.3, 3.4]) and $N$ a distinguished nilpotent orbit of $g^{0,s(e)}$.

An element $\gamma \in (\Omega^\theta_{ad})^\ast = \Gamma_{spec}$ fixes $[\lambda_{ad}]$ if and only if $\gamma$ fixes $s(e)$, and in addition the action of $\gamma$ on $g^{0,s(e)}$ should fix $N$ (or equivalently $W_{s(e)}\xi$). An important remark is that linear residual orbits $W_{s(e)}\xi$ of a graded affine Hecke algebra $H$ associated to an irreducible based root system $(R, F)$ will be invariant for all diagram automorphisms of $(R, F)$ (see [Op3, Theorem 7.14(i)]). Moreover, if the orbit of residual points $W_0 \tau$
which corresponds to \((e, W_{s(e)}\xi)\) as above, represents a standard STM then it is obvious (from the definitions of the standard STMs in Section 3) that if \(R_e(e)\) contains isomorphic irreducible components then the components of \(W_{s(e)}\xi\) attached to these components correspond to each other under a (hence any) isomorphism between these components. In short, the isotropy group \(\Gamma_{s,\text{spec},W}\ol{\tau}\) is simply equal to the isotropy group \(\Gamma_{\text{spec},e} = (\Omega_{\text{ad}})^{\ast} (\tau)\).

On the arithmetic side we see similar phenomena (see the Introduction), since a cuspidal unipotent pair \((P, \sigma)\) is known to be inert for twisting by automorphisms of \(\mathbb{F}\) defined over \(\mathbb{F}_q\). This implies that the isotropy subgroup of \((P, \sigma)\) in \(\Omega_{\text{ad}}^\theta\) is simply the stabilizer subgroup \(\Omega_{\text{ad}}^{\text{st}}(\tau)\) of \(\mathbb{F}\).

Let us now return to the case at hand, the case of odd orthogonal groups. We established that the equation (6.2) with \(\text{fdeg}_q(\sigma) = d_2^D(q)\delta_2^B(q)\) forces \(\tilde{\tau} = (-\tilde{\tau}_-^\circ, \tilde{\tau}_+^\circ) = (\tau_1^\circ, \tau_2^\circ), \tau_\pm^\circ\) are the residual points as described in Corollary 5.5, with parameters \(m_{\pm}\) which are related to \(a, b\) by the equation of sets \(\{2a, 2b + 1\} = \{m_+ + m_+, |m_+ - m_-|\}.\) In this case \(\mathcal{H}_{\mathbb{Z}}(G_{qs}) \sim \mathcal{H}_{\mathbb{Z}}^{\mathbb{Q}}\) in fact an isomorphism (see [Op2, Subsection 3.2.6]), and we may associate a unique unramified discrete Langlands parameter \([\lambda_{ad}]\) to the pair \((\lambda_-^\circ, \lambda_+^\circ)\) via the correspondence discussed in Remark 1.5 and above.

Recall that the spectral map diagram of the STM \(\mathcal{H}_{\mathbb{Z}}^0(m, m_+) \sim \mathcal{H}_{\mathbb{Z}}(G_{qs})\) defines a maximal proper subdiagram of the spectral diagram of the latter Hecke algebra, which has two (possibly empty) components of type \(C_{n_{\pm}}\) with \(n_{\pm} = (m_\pm + 1/2)(m_\pm - 1/2)\).

On the other hand, the parahoric subgroup \(\mathbb{P}_{a,b}\) associated to a pair \(a, b\) is defined by a maximal proper subdiagram of its arithmetic diagram, which consists of two (possibly empty) components of type \(D_{n/2}\) and of type \(B_{n/2+b}\).

One observes that the isotropy group of \([\lambda_{ad}]\) in \((\Omega_{ad})^\ast\) is trivial except when \(m_+ = m_-\). This happens if and only if \(a = 0\), and this happens if and only if \(\Omega_{ad}^\theta\) is trivial, as desired.

6.5. **Proof of the main Theorem for unitary groups.** In this subsection we consider the case \(G_{qs} = PU_{2n}\) or \(G_{qs} = PU_{2n+1}\), which are the cases in type I. In the first case (the odd rank case) we have a spectral covering map \(\mathcal{H}^I(G_{qs}) = B_n(2, 1)[q] \sim C_n(\frac{1}{2}, 0)[q^2]\) (a double cover), and in the second case (the even rank case) we have a (spectral) isomorphism \(\mathcal{H}^I(G_{qs}) \sim C_n(\frac{1}{2}, 1)[q^2]\).

6.5.1. **Proof of Theorem 1.1(a), (b).** This is type I. The right hand side of (6.2) is of the form \(\text{fdeg}_q(\sigma) = d_2^A(q)\delta_2^A(q) := d_{s,t}\) (see (3.2)), and a glance at the tables in [Car, 13.7] readily shows that the multiplicity function \(m_{s,t}^\circ\) for \(d_{s,t}\) is supported on the odd integers, and is of the form \((k \in (\mathbb{Z} + 1/2)_+)\):

\[ m_{s,t}^\circ(2k) = -\max\{0, s + 1/2 - k\} + \max\{0, t + 1/2 - k\} \tag{6.8} \]

On the other hand, we have \(\delta_- = 1/2\) and \(\delta_+ \in \{0, 1\}\) in this case, and \(b = 2\), so Lemma 5.4 implies that \(\lambda_- = [2, 4, \ldots, 2r_-]\) and \(\lambda_+\) is as described in Proposition 6.3. The corresponding multiplicity function \(h_-\) is supported on \(\mathbb{Z} + 1/2\), and \(h_+\) is supported on \(\mathbb{Z}\). We know we can write \(h_-(x) = \max\{0, p_+ + 1 - |x|\} \) where \(p_+ = m_- - 1\) and
\( m_+ = r_- + 1/2 \), but at this point there are still various choices for \( h_+ = h_{\lambda_+} \) possible. Applying Proposition 6.3, Lemma 5.4, and Remark 5.6 we have (with \( \tilde{h}_+ := h_{\lambda_+} \) and \( k \in (\mathbb{Z}/2)_+ \)):

\[
\text{mult}_{\frac{\lambda_+ - \lambda_+}{2; (1/2, \delta)}(2k)} = \Delta(\tilde{h}_- + \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) + \text{mult}_{\frac{\lambda_+ - \lambda_+}{2; (1/2, \delta)}(2k)}(0, \lambda_+) + \text{mult}_{\frac{\lambda_+ - \lambda_+}{2; (1/2, \delta)}(2k)}(0, \delta_+) \quad (3.3)
\]

Observe that \( m_+^c(s, t)2k \) is nonzero only if \( k \in \mathbb{Z} + 1/2 \). This implies that for all \( k \in \mathbb{Z} \) we must have \( 2\tilde{h}_+(k) = \text{mult}_{\frac{\lambda_+ - \lambda_+}{2; (1/2, \delta)}(2k)}(0, \lambda_+) \), since all other terms in the expression are obviously equal to zero for \( k \in \mathbb{Z} \). Checking the cases in Proposition 6.3 we see that this happens if and only if \( \lambda_+ \) is of the form described in cases (b) or (c). Hence we have \( \lambda_+ = [1, 3, \ldots, 1 + 2p_+] \) with \( p_+ = m_+ - 1 \) and \( m_+ = 2r_+ + 2 + \delta_+ \), and

\[
\text{mult}_{\frac{\lambda_+ - \lambda_+}{2; (1/2, \delta)}(2k)}(k) = \text{mult}_{\frac{\lambda_+ - \lambda_+}{2; (1/2, \delta)}(2k)}(0, \lambda_+) = \Delta(\tilde{h}_- + \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) + \text{mult}_{\frac{\lambda_+ - \lambda_+}{2; (1/2, \delta)}(2k)}(0, \delta_+)
\]

From here on the argument is completely analogous to the odd orthogonal case Subsection 6.4. We see that necessarily \( \{s - 1/2, t - 1/2\} = \{p_+ + p_- + 1, |p_+ - p_-| - 1\} \) or \( \{s + 1/2, t + 1/2\} = \{m_+ + m_+, |m_+ - m_-|\} \), as was to be proved (compare with (3.3), and with Subsection 5.2).

6.5.2. Proof of Theorem 1.1(c). In this case \( \mathcal{H}_{\Omega}(G_{qs}) \rightarrow \mathcal{H}_{\lambda}^{\text{cusp}} \) is in fact an isomorphism if rank is even and it is a double cover if the rank is odd (see [Op2, Subsection 3.2.6]).

The group \( \Gamma_{\text{spec}} = (\Omega_{ad}^\theta)^* \) is trivial in the first case, and equal to \( C_2 \) in the second.

As discussed above, we may associate a unique unramified discrete local Langlands parameter \( [\lambda_{ad}] \) to the pair \( (\lambda_-, \lambda_+) \), via the correspondence discussed in Remark 1.5 and above in Subsection 6.4. Recall that we have \( \{s + 1/2, t + 1/2\} = \{m_+ + m_- - 1, m_+ - m_-\} \) with \( m_- \in (\mathbb{Z}+1/2) \) and \( m_+ \in \mathbb{Z}_+ \), where \( \mathbb{P}_{s, t} \) corresponds to a maximal proper \( \theta \)-stable subdiagram of the affine Dynkin diagram of \( G_{qs} \) with two (possibly empty) components, of type \( 2A_2(s/2)^{(s-1)} \) and \( 2A_2(t/2)^{(t-1)} \), while the spectral map diagram of the STM \( \mathcal{H}_{\Omega}^{\text{cusp}} \) is induced by \( \mathcal{H}_{\Omega}(G_{qs}) \) (the complement of a vertex \( e \) of the spectral diagram), which has two (possibly empty) components, one of type \( D_{m_+^2/2} \) (odd rank case) or \( C_{(m_+ - 1)(m_+ + 1)/2} \) (even rank case), and one of type \( B_{(m_- - 1/2)(m_+ - 1/2)/2} \).

Regarding isotropy subgroups of \( \Omega_{\mathbb{P}_{s, t}}^{\text{cusp}} \) and \( \Gamma_{\text{spec}}^{\text{cusp}} \), we see that in the even rank case there is nothing to prove, while in the odd rank case the former group is trivial unless \( s = t \), in which case it equals \( \Omega^\theta = C_2 \). At the other side, \( \Gamma_{\text{spec}}^{\text{cusp}} \) is \( C_2 \), unless \( m_+ = 0 \), which happens if and only if \( s = t \), as was to be proven.

6.6. Proof of the main Theorem for symplectic and even orthogonal groups.

In this subsection we treat the cases with \( G_{qs} \) of type \( \text{PCSp}_{2n} \), \( \mathbb{P}(\text{CO}_{2n}^0) \) or \( \mathbb{P}((\text{CO}_{2n+2})^0) \), which are of type III (first case) or of type IV (the last two cases). We have spectral covering maps \( \mathcal{H}_{\Omega}(G_{qs}) \approx C_n(\delta_-, \delta_+) \), where \( (\delta_-, \delta_+) = (0, 0), (0, 1), (1, 0) \) or \( (1, 1) \) respectively, represented by a covering map of algebraic tori of degree 2 in the first three
cases (cf. [Op1, 3.3.5]) (of course, the cases two and three are isomorphic, but it is convenient to allow both possibilities at this point), and which is an isomorphism in the last case.

Before we embark on the proof, let us recall the smallest unipotent partitions \( \lambda \) which belong of the cases (a)-(e) of Corollary 5.14 and Proposition 6.3. We always denote by \( p \) the largest element of the support of the function \( \hat{h}_\lambda \), and we formally extend this to cases where \( \lambda \) is the zero partition (compatible with our formulas for residues). If \( \lambda = 0 \) then we put \( p = -1 \) if \( \delta = 0 \) and \( p = 0 \) if \( \delta = 1 \). These can be viewed as a case (a), or as case (b) (if \( \delta = 0 \)) or as case (c) (if \( \delta = 1 \)). If \( p = 1 \) then \( \lambda = [3] \), and this is a case (a) cuspidal partition, or \( \lambda = [1, 3] \), which is case (b). If \( p = 2 \) then \( \lambda = [1, 5] \), which is a case (a), or \( \lambda = [1, 3, 5] \), which is case (c). The smallest case (d) is \([3, 5, 7]\) (with \( p = 3 \)), and the smallest case (e) is \([1, 3, 5, 9]\) (with \( p = 4 \)).

6.6.1. **Proof of Theorem 1.1(a), (b).** The right hand side of (6.2) is of the form

\[
\text{fdeg}_q(\sigma) = d_a^B(q)d_b^B(q) \quad \text{or} \quad \text{fdeg}_q(\sigma) = d_a^B(q^2)d_b^A(q)
\]

in the first case, or of the form

\[
\text{fdeg}_q(\sigma) = d_a^D(q)d_b^D(q) \quad \text{or} \quad \text{fdeg}_q(\sigma) = d_a^D(q)d_b^A(q)
\]

in the last two cases (see (3.2)). A glance at the tables in [Car, 13.7] readily shows that the multiplicity function \( m_{e,\text{ord},\text{III}}^{e,\text{ord},\text{III}} \) for \( d_{a,b}^{\text{ord},\text{III}} := d_a^B(q)d_b^B(q) \) is of the form (where \( k \in \mathbb{Z}_+ \)):

\[
(6.9) \quad m_{a,b}^{e,\text{ord},\text{III}}(k) = -(\max\{0, 2a + 1 - k\} + \max\{0, 2b + 1 - k\})
\]

while \( m_{a,b}^{e,\text{extra},\text{III}} \) for \( d_{a,b}^{\text{extra},\text{III}} := d_a^B(q^2)d_b^A(q) \) is of the form

\[
(6.10) \quad m_{a,b}^{e,\text{extra},\text{III}}(k) = -(\max_{\text{int}}\{0, (4a + 2 - k)/2\} + \max_{\text{int}}\{0, (2b + 1 - k)/2\})
\]

where \( \max_{\text{int}} \) is defined as the largest integer of a finite set of rational numbers, or 0 if the set contains no integer.

Similarly, for the parameter type IV, the multiplicity function \( m_{a,b}^{e,\text{ord},\text{IV}} \) for \( d_{a,b}^{\text{ord},\text{IV}} := d_a^D(q)d_b^D(q) \) is of the form (where \( k \in \mathbb{Z}_+ \)):

\[
(6.11) \quad m_{a,b}^{e,\text{ord},\text{IV}}(k) = -(\max\{0, 2a - k\} + \max\{0, 2b - k\})
\]

while \( m_{a,b}^{e,\text{extra},\text{IV}} \) for \( d_{a,b}^{\text{extra},\text{IV}} := d_a^D(q^2)d_b^A(q) \) is of the form

\[
(6.12) \quad m_{a,b}^{e,\text{extra},\text{IV}}(k) = -(\max_{\text{int}}\{0, (4a - k)/2\} + \max_{\text{int}}\{0, (2b + 1 - k)/2\})
\]

If \( \lambda = (-\lambda_{-\delta}, \lambda_{+\delta}) \) is a solution of (6.2) then \( \lambda_{\pm} \) must both belong to the cases listed in Corollary 5.14, with \( \delta_{\pm} \) determined by \( G_{qs} \). Using Proposition 6.3 and Proposition 6.4 we will compute the multiplicity function \( \text{mult}^{(\lambda_{-\delta}, \lambda_{+\delta})}(k) \) where \( \lambda_{\pm} \) belongs to one of these cases (a)-(e) as listed in Corollary 5.14, to study whether such combinations could provide solutions of equation (6.2).

Let us first assume that \( \lambda_{\pm} \) both belong to case (b) (if \( \delta_{\pm} = 0 \)) or to case (c) (if \( \delta_{\pm} = 1 \). Note that it is strictly speaking not necessary to treat these cases, since we
already know that they all represent standard STMs. Yet it is useful and instructive to do the calculations.

Recall that if \( (\delta_-, \delta_+) = (1,1) \) then the normalization factor \( d_{1,1}^0 \) of the \( \mu \)-function of \( \mathcal{H}^{1M}(G_{qs}) \) is \( d_{1,1}^0 = (1 + q)^{-1} \) (due to the fact that in this case, a maximal \( K \) split torus of \( G_{qs} \) which is defined over \( k \) and maximal \( k \)-split has a \( k \)-anisotropic factor of dimension 1 which splits over an unramified quadratic extension of \( k \); this gives rise to the even cyclotomic factor \( (1 + q) \) in the volume of Iwahori subgroup). In all other cases \( d_{1,1}^0 = 1 \). In order to take this extra factor into account we need to subtract \( \delta_- \delta_+ \delta_1(k) \) from the multiplicity function \( \text{mult}^{(\lambda_-, \lambda_+)}_{(\delta_-, \delta_+)}(k) \) (see definition 6.1). Thus we have, using Proposition 6.3, and analogous to the odd orthogonal cases, for all \( k \in \mathbb{Z}_+ \):

\[
\text{mult}^{(\lambda_-, \lambda_+)}_{(\delta_-, \delta_+)}(k) - \delta_- \delta_+ \delta_1(k)
\]

\[
= \Delta(\hat{h}_+ \ast \hat{h}_+)(k) - \hat{h}_-(k) + \Delta_{\delta_+}(\hat{h}_+)(k) - \hat{h}_+(k) + \Delta_{\delta_-}(\hat{h}_+)(k) - \delta_- \delta_+ \delta_1(k)
\]

\[
= \Delta(\hat{h}_- \ast \hat{h}_+)(k) - \hat{h}_-(k) + \Delta_{\delta_-}(\hat{h}_-)(k) - \hat{h}_+(k) + (\Delta_{\delta_-}(\delta_0) \ast \hat{h}_+)(k) - \delta_- \delta_+ \delta_1(k)
\]

\[
= ((\Delta(\hat{h}_-)) + \Delta_{\delta_-}(\delta_0) \ast \hat{h}_+)(k) - 2\hat{h}_+(k) - 2\hat{h}_-(k) + \Delta_{\delta_-}(\hat{h}_-)(k) - \delta_- \delta_+ \delta_1(k)
\]

\[
= - T_{-(p_++1)} \hat{h}_+(k) - T_{+(p_++1)} \hat{h}_+(k) - 2\hat{h}_-(k) - \delta_+ \delta_+(p_++1)(k) + \delta_- \delta_+ \delta_1(k) - \delta_- \delta_+ \delta_1(k)
\]

\[
= -(\max\{0, p_- + p_+ + 2 - k\} + \max\{0, p_+ - p_- - k\})
\]

so that this could be a solution of (6.2), but clearly only if the right hand side is of the form \( \mu_{a,b}^{e, \text{ord}, \text{IV}}(k) \) (if the parities of \( p_- \) and \( p_+ \) are equal). In the first case we see that \( \hat{t} \) is a solution of (6.2) if and only if \( \{2a+1, 2b+1\} = \{p_+ + p_- + 2, |p_+ - p_-|\} = \{m_+ + m_-, |m_+ - m_-|\} \), and in the second case if and only if \( \{2a, 2b\} = \{p_+ + p_- + 2, |p_+ - p_-|\} = \{m_+ + m_-, |m_+ - m_-|\} \). These solutions correspond exactly to the ordinary standard translation STMs (compare with (3.3), and with Subsection 5.2).

The second case we will consider is that of the cuspidal extra special STMs, that is, when both \( \lambda_{\pm} \) belong to case (a). (Again, strictly speaking this is not necessary, since all cases are known to represent cuspidal extra special STMs.) We compute, using Proposition 6.3 and Proposition 6.4, for \( k \in \mathbb{Z}_+ \):

\[
\text{mult}^{(\lambda_-, \lambda_+)}_{(\delta_- \delta_+)}(k) - \delta_- \delta_+ \delta_1(k)
\]

\[
= \Delta(\hat{h}_- \ast \hat{h}_+)(k) - \hat{h}_-(k) + \Delta_{\delta_+}(\hat{h}_-)(k) - \hat{h}_+(k) + \Delta_{\delta_-}(\hat{h}_+)(k) - \delta_- \delta_+ \delta_1(k)
\]

\[
= \Delta(\hat{h}_- \ast \hat{h}_+)(k) - \hat{h}_-(k) + \Delta_{\delta_-}(\hat{h}_-)(k) - \hat{h}_+(k) + (\Delta_{\delta_-}(\delta_0) \ast \hat{h}_+)(k) - \delta_- \delta_+ \delta_1(k)
\]

\[
= - (S_{p_-+1} \ast \hat{h}_+)(k) - \hat{h}_-(k) + \delta_+ (-S_{p_-+1} - \Delta_{\delta_-}(\delta_0) + \delta_0)(k) - \delta_- \delta_+ \delta_1(k)
\]

\[
= - (S_{p_-+1} \ast \hat{h}_+)(k) - \hat{h}_-(k) + \delta_+ (-S_{p_-+1} + \delta_1)(k) - \delta_- \delta_+ \delta_1(k)
\]

\[
= - (S_{p_-+1} \ast \hat{h}_+)(k) - \hat{h}_-(k) - \delta_+ S_{p_-+1}(k)
\]

Now observe that there exists a unique symmetric function \( \hat{H}_+ \) supported on \( \mathbb{Z}_+ + 1/2 \) and a remainder \( R_{m_+} \) such that we have \( \hat{h}_+ = (\delta_{-1/2} + \delta_{+1/2}) \ast \hat{H}_+ + R_{m_+} \delta_0 \). One checks easily
that in fact $R_{m_0} = (-1)^s + 2([(p_+ - 1)/4] + \epsilon + \delta_\pm)$ and $\tilde{H}_+(k) = \max_{\text{int}}\{0, (2p_+ - 2|k| + 3)/4\}$. We may continue the computation as follows:

\[
\text{mult}(\lambda_-^{\delta_-\delta_+^1})(k) - \delta_-\delta_+^1(k) \\
= - (S_{p_-+1} \ast \tilde{h}_+)(k) - \tilde{h}_-(k) - \delta_+ S_{p_-+1}(k) \\
= - S_{p_-+1} (\delta_-1/2 + \delta_+1/2) \ast \tilde{H}_+(k) - R_{m_0} S_{p_-+1}(k) - \tilde{h}_-(k) - \delta_+ S_{p_-+1}(k) \\
= - T_{p_-+3/2} \tilde{H}_+(k) - T_{-(p_-+3/2)} \tilde{H}_+(k) - (R_{m_0} + \delta_+) S_{p_-+1}(k) - \tilde{h}_-(k) \\
= - T_{p_-+3/2} \tilde{H}_+(k) - T_{-(p_-+3/2)} \tilde{H}_+(k) - (-1)^s + 2([(p_+ + 2)/4] + \delta_+) S_{p_-+1}(k) - \tilde{h}_-(k) \\
= \max_{\text{int}}\{0, (p_- + p_+ + 3 - k)/2\} + \max_{\text{int}}\{0, (|p_- - p_+| - k)/2\}
\]

where the last line follows from carefully checking what happens at $k = p_- + 2, p_- + 1, p_-, \ldots$ and at $|p_- + p_+| = 1, |p_- + p_+| = 2, \ldots$ at the various congruence classes of $p_+$ modulo 4. Now use that in case (a), we have $p_\pm = 2(\kappa_\pm - 1) + \epsilon = 2m_\pm - 3/2$. Hence we see that $\vec{r}$ is a solution of (6.2) if and only if the right hand side is of type $d_{a,b}^{\text{extra,III}}$, in which case we need that $\{m_- + m_+, |m_- - m_+|\} = \{2a + 1, b + 1/2\}$, or of type $d_{a,b}^{\text{extra,IV}}$, in which case we need $\{m_- + m_+, |m_- - m_+|\} = \{2a, b + 1/2\}$. These are indeed the residual points representing the cuspidal extra special STMs.

Let us now try to find solutions of (6.2) with $\lambda_-$ of type (b) or (c), and $\lambda_+$ of type (a). We allow any pair $(\delta_-, \delta_+) \in \{0, 1\}^2$ here. Similar to the previous computations, we have:

\[
\text{mult}(\lambda_-^{\delta_-\delta_+^1})(k) - \delta_-\delta_+^1(k) \\
= \Delta(\tilde{h}_- \ast \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta\delta_+^1(\tilde{h}_+)(k) - \tilde{h}_+(k) + \Delta\delta_-^1(\tilde{h}_+)(k) - \delta_-\delta_+^1(k) \\
= ((\Delta(\tilde{h}_-) + \Delta\delta_-^1(\tilde{h}_+)) \ast \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta\delta_+^1(\tilde{h}_+)(k) - \tilde{h}_+(k) - \delta_-\delta_+^1(k) \\
= - T_{-(p_-+1)} \tilde{h}_+(k) - T_{p_-+1} \tilde{h}_+(k) + \tilde{h}_+(k) - 2\tilde{h}_-(k) + \Delta\delta_+^1(\tilde{h}_+)(k) - \delta_-\delta_+^1(k) \\
= - T_{p_-+1} \tilde{h}_+(k) + \tilde{h}_+(k) - 2\tilde{h}_+(k) - T_{-(p_-+1)} \tilde{h}_+(k) - \delta_-\delta_+^1(k)
\]

If both $p_- > p_+$, then the above expression has value $-1$ at $k = p_- + p_+ + 1$ and value $-1$ at $k = p_- + p_+$, and from the form of the available cuspidal formal degrees we easily conclude that if $\vec{r}$ solves (6.2) in this case, then the right hand side has to be of the form $d_{a,b}^{\text{extra,III}}$, or $d_{a,b}^{\text{extra,IV}}$ where $a$ and $b$ are such that one of the corresponding extraspecial parameters $(m_-^\pm, m_+^\pm)$ equals 1/4. Indeed, if $\lambda_-^\pm, \lambda_+^\pm$ are the unipotent partitions of type (a) such that the image of the corresponding extra special STM $\vec{r}(\lambda_-^\pm, \lambda_+^\pm)$ also solves of (6.2) (with the same right hand side), then the values of the two highest multiplicities we just discussed imply that $p_-^\pm + p_+^\pm + 1$, and $|p_-^\pm - p_+^\pm| - 2$ differ by one (see the formula for the multiplicities of the residue we derived in the case where $\lambda_-$ and $\lambda_+$ are both type (a)), from which we conclude that one of $p_-^\pm$ equals $-1$, or equivalently $m_-^\pm = 1/4$. Let us say that $p_-^\pm = -1$, and write $p_-^\pm = p_+^\pm$. Since $p_- + p_+ + 1 = p_-^\pm + p_+^\pm + 1$ we have $p_-^\pm = p_- + p_+ + 1$. On the other hand, looking at above formula, it is clear that for all $k > p_- + 1$, the term $\tilde{h}_+(k) - 2\tilde{h}_-(k) - \delta_+ \delta_{p_-+1}(k) - T_{-(p_-+1)} \tilde{h}_+(k)$ to vanish, while at $k = p_- + 1$ this expression needs to be 0, $-1$, or $-2$ (depending on the congruence
class of $p_+$ modulo 4), and at $k = p_-$ it always needs to be $-1$ (we invite the reader to draw the graph of $T_{p_+ - 1}$ for $k = p_-, p_+ - 1, p_+ - 1, p_+ + 2$ and compare this with the graph of $-h_{\lambda}$ to see this). It follows that $p_- + 1 = p_+$ or $p_- = p_+$.

In the first case we have $p_{\text{max}}^c = p_- + p_+ + 1 = 2p_+$. Now we compute the rank $n$ in terms of $(p_-, p_+)$ and of $(p_-, p_+^c)$. The first method gives $n = n_- + n_+ + 1$

$$n_- = \begin{cases} 
\frac{m_-^2}{2} = (p_- + 1)^2/2, & \text{type (b)} \\
(m_- - 1)(m_- + 1)/2 = p_- (p_- + 2)/2, & \text{type (c)}
\end{cases}$$

Hence $n_- \leq p_+^2/2$. For $n_+$ we have:

$$n_+ = [\kappa + (\kappa - 1) + 2\kappa - 1)/2] = (p_+ + 2 - \epsilon)(p_+ + 1 + \epsilon)/4]$$

$$= (p_+ + 1)(p_+ + 2)/4 \leq (p_+ + 1)(p_+ + 2)/4.$$ 

Thus $n \leq p_+^2/2 + (p_+ + 1)(p_+ + 2)/4 = 3p_+(p_+ + 1)/4 + 1/2$. The second method gives $n = n_- + n_+ = n_- = [(2p_+ + 1)(2p_+ + 2)/4] \geq p_+(p_+ + 3/2)$. We conclude that $p_+(p_+ + 3/2) \leq 3p_+(p_+ + 1)/4 + 1/2$, which implies $p_+ \leq 1$, hence $p_- \leq 0$, a contradiction.

In the second case we have $p_{\text{max}}^c = 2p_+ + 1$. We now find that

$$n \leq (p_+ + 1)^2/2 + (p_+ + 1)(p_+ + 2)/4 = (p_+ + 1)(3p_+ + 4)/4$$

$$= 3p_+(p_+ + 1)/4 + 1,$$ 

and $n = n_+ = [(2p_+ + 2)(2p_+ + 3)/4] \geq p_+(p_+ + 5/2) + 1$. We thus see that $p_+(p_+ + 5/2) \leq 3p_+(p_+ + 1)/4$, which implies $p_+ \leq -6$, also a contradiction.

In other words, have shown that (6.2) has no solutions if $\lambda_+$ is of type (a) and $\lambda_-$ is of type (b) or (c).

Let us now consider the cases where $\lambda_+$ is of type (a) and $\lambda_-$ is of type (d). We use the notation $\Sigma_{\delta}^d = \delta_1 + \delta_2 + \delta_4 + \cdots + \delta_{a_-}$, where $a_- = 2^{s+1} - 1$. Also recall that $\delta_- = 1$, and $p_- = 2^{s+2} - 1$ with $s \geq 0$ in case (d), and we will use the notation $r_- = 2^s - 1$.

Similar to the previous case (with $\lambda_-$ is of type (c), and $\lambda_+$ of type (a)) we have for all $k > 0$:

$$\text{mult}_{\delta_\lambda}(k) - \delta_- \delta_\lambda (k)$$

$$= \Delta(h_+ - \hat{h}_+)(k) - 2\hat{h}_+(k) + \Sigma_{\delta}^d(k) + \Delta_{\delta_\lambda}(h_-)(k) - \hat{h}_+(k) + \Delta(h_+)(k) - \delta_\lambda(k)$$

$$= (\Delta(h_+ - \hat{h}_+)(k) - 2\hat{h}_+(k) + \Sigma_{\delta}^d(k) + \Delta_{\delta_\lambda}(h_-)(k) - \hat{h}_+(k) - \delta_\lambda(k)$$

$$= - T_{-(p_- + 1)} \hat{h}_+(k) - T_{p_- + 1} \tilde{h}_+(k) + \hat{h}_+(k) + \Delta(h_+)(k)$$

$$+ \Sigma_{\delta}^d(k) - 2\hat{h}_+(k) + \Delta_{\delta_\lambda}(h_-)(k) - \delta_\lambda(k)$$

$$= - T_{p_- + 1} \hat{h}_+(k) + \hat{h}_+(k) + S_{p_- + 1} + \Sigma_{\delta}^d(k) - 2\hat{h}_+(k) - T_{-(p_- + 1)} \hat{h}_+(k)$$

$$- \delta_\lambda(k)$$

Like in the previous case, we conclude first of all that the right hand side of (6.2) needs to be of the form $\delta_{a,b}^{\text{extra,IV}}$, or $\delta_{a,b}^{\text{extra,IV}}$ where $a$ and $b$ are such that one of the corresponding extraspecial parameters $(m_{\chi, k}^c, m_{\chi, k}^c)$ equals 1/4. Let us use the same notations as in the previous case. We again have $\{p_{\text{max}}, p_+^c\} = \{-1, p_{\text{max}}^c\}$ for some $p_{\text{max}}^c \in \mathbb{Z}_+$. We see that
\( p'_{\text{max}} = p_- + p_+ + 1 \). As before we need that \( T(k) := \tilde{h}_+(k) + S_{p_-+1} + \Sigma_-(k) - 2\tilde{h}_-(k) - T_{-(p_+ + 1)}\tilde{h}_+(k) - \delta_+\delta_{p_-+1}(k) + (\delta_+ - 1)\delta_1(k) = 0 \) for all \( k > p_- + 1 \), from which we conclude that \( p_+ < p_- + 1 \) (because of the presence of the function \( S_{p_-+1} \)). As before, we need that \( T(p_-) = -1 \), implying that \( p_+ = p_- - 1 \) this time.

Hence in this situation we have \( p_+^2 = p_- + p_+ + 1 = 2p_- \). We have \( n = n_- + n_+ \) with \( n_- = 8(r_- + 1)^2 = (p_- + 1)^2/2 \), and \( n_+ = \lceil \kappa_+(2(\kappa_+ - 1) + 2\epsilon - 1)/2 \rceil = [(p_+ + 2 - \epsilon)(p_+ + 1 + \epsilon)/4] = [p_- (p_- + 1)/4] \), hence \( n_+ \leq p_- (p_- + 1)/4 \). Taken together we have \( n \leq p_- (p_- + 1)^2/2 + p_- (p_- + 1)/4 = (p_- + 1)(3p_- + 2)/4 \).

On the other hand, we have \( n = n_+^\epsilon \geq p_- (p_- + 3/2) \). Hence we conclude that \( p_- (p_- + 3/2) \leq (p_- + 1)(3p_- + 2)/4 \), which implies that \( p_-^2/4 + p_- / 4 \leq 1/2 \), hence \( p_- \leq 1 \). But then \( p_- \leq 0 \), a contradiction.

Hence we have shown that (6.2) has no solutions for \( \lambda_+ \) of type (a) and \( \lambda_- \) of type (d).

Next let us consider the cases with \( \lambda_+ \) of type (a) and \( \lambda_- \) of type (e). We now have \( r_- = 2^s - 1 \) for some \( s \geq 0 \), and \( a_- = 2(r_- + 1), p_- = 2a_- = 2^s + 2 \geq 4 \). Moreover, \( \delta_- = 0 \) now. We write \( \Sigma_-^e = \delta_1 + \delta_2 + \delta_4 + \cdots + \delta_{p_-} \). Similar to the previous cases (with \( \lambda_- \) is of type (b), and \( \lambda_+ \) of type (a)) we have for all \( k > 0 \):

\[
\begin{align*}
\mult_{(\lambda_-^e, \lambda_+)}(\lambda_-^e, \lambda_+) \cdot (\delta_-^e, \delta_+) & = \Delta(\tilde{h}_- \ast \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Sigma_-^e(k) + \Delta_{\delta_+}^e(\tilde{h}_-)(k) - \tilde{h}_+(k) \\
& = \Delta(\tilde{h}_- \ast \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Sigma_-^e(k) + \Delta_{\delta_+}^e(\tilde{h}_-)(k) - \tilde{h}_+(k) \\
& = -T_{-(p_+ + 1)}\tilde{h}_+(k) + T_p \tilde{h}_+(k) - T_{p_- - 1}\tilde{h}_+(k) \\
& = -T_{-(p_+ + 1)}\tilde{h}_+(k) + T_{-p_- \tilde{h}_+}(k) - T_{-(p_- - 1)}\tilde{h}_+(k) \\
& = \tilde{h}_+(k) + \Sigma_-^e(k) + \Delta_{\delta_+}^e(\tilde{h}_-)(k) - 2\tilde{h}_-(k)
\end{align*}
\]

If we assume that \( p_+ \geq 3 \) then the maximum of the support of the sum \( S_6 \) of the first 6 terms (the translations of \( \tilde{h}_+ \), but not including the non translated term \( \tilde{h}_+ \)) is \( k = p_- + p_+ + 1 \), and the maximum of the support of the remaining terms \( T_R := \Sigma_-^e(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - 2\tilde{h}_-(k) \) (excluding \( \tilde{h}_+ \)) is equal to \( p_- + 1 \) (if \( \delta_+ = 1 \)) or \( p_- \) (if \( \delta_+ = 0 \)). Since \( p_+ = 3 \) corresponds to \( \delta_+ = 0 \), we see that \( T_R = 0 \) on \( p_- + p_+ + 1, p_- + p_+ + p_- - 1, p_- + p_+ - 2 \). The values of \( S_6 \) there are easily seen to be \(-1, 0, -2, -1 \) respectively. Since \( \tilde{h}_+ \) is also 0 at these arguments (since \( p_- + p_+ - 2 > p_+ \)) these values are the multiplicities of the factors \((1 + q^k)\) in \( \mu_{(\lambda_-^e, \lambda_+)} \) for these values of \( k \). If \( \mathcal{F} \) is a solution of (6.2) we conclude that the right hand side \( d_{\text{rhs}} \) of (6.2) is of the form \( d_{a,b}^\text{extra,III} \), or \( d_{a,b}^\text{extra,IV} \) where \( a \) and \( b \) are such that one of the corresponding extraspecial parameters \((m_-^e, m_+^e)\) equals \( 3/4 \). Hence we now know \( d_{\text{rhs}} \) exactly in terms of \( p_- \) and \( p_+ \).

Now consider the behavior of \( S_6 - d_{\text{rhs}} \) at the arguments \( k = p_- + 1, \), \( p_- - 1, \) \( p_- - 2 \).

Looking at the three positively translated graphs of \( \tilde{h}_+ \) for all possible congruence classes of \( p_+ \) modulo 4, we see that these values are \( \delta_+, 1 - \delta_+, 1 + \delta_+, 2 \) respectively, and the value of \( T_R \) at these arguments is \(-\delta_+, \delta_+-1, -\delta_+ - 1, \Sigma_-^e(p_- - 2) - 2 \). The sum of these gives the values \( 0, 0, -1, \Sigma_-^e(p_- - 2) - 2, -2 \), and if we add \( \tilde{h}_+ \) to this we should get 0 everywhere. We conclude that \( p_+ = p_- - 1 \) and \( p_- = 4 \), otherwise this is not
possible. However, then $p_+ = 3$ which corresponds to $\delta_+ = 0$, so that $(\delta_-,\delta_+) = (0,0)$. But $d_{e} = d_{e}^{\text{extra}, \text{III}}$, with corresponding values $m_{e}^\text{e} = 3/4$ and $m_{e}^\text{e} = 9/4$, which yields $\delta_{+}^e = 1$ and $\delta_{-}^e = 0$. One checks that $d_{e}$ is an extra special cuspidal unipotent degree for the symplectic group $\text{PCSp}_{36}$, while the “solution” $\tilde{\tau}$ of (6.2) is a residual point for $\text{P}(\text{CO}_{28})$, so this is not really a solution.

For $p_+ \leq 2$, the value $\tilde{\delta}_+(k)$ is zero at $k$ equal to $p_- + p_+ + 1, \ldots, p_- - 1$. If $p_+ = 2$ we find that $S_0$ has the values $-1, 0, -2, 0, -2$ there, while $\delta_+ = 0$ and $T_R$ has the values $0, 0, 0, -2, -1, -2$, which yields the total $-1, 0, -2, -1, -4$, which are not the multiplicities of a cuspidal unipotent formal degree, and for $p_+ = 1$ (with $\delta_+ = 1$) we get for these values of $S_0$ the sequence $-1, 1, -2, 1$ while for $T_R$ we get $0, -1, 0, -3$ which yields the total $-1, 0, -2, -2$, which is also not a sequence of multiplicities of a cuspidal unipotent formal degree. We may finally conclude that there are no solutions to (6.2) of the form $\tilde{\tau}_{\lambda_-,\lambda_+}$ with $\lambda_-$ of type (e) and $\lambda_+$ of type (a).

Next let us consider the cases with $\lambda_-$ of type (b), (c) and $\lambda_+$ of type (d). This implies that $\delta_+ = 1$, and $p_+ \geq 3$. We compute in a similar way to the case where both $\lambda_\pm$ are of type (b) or (c):

$$\text{mult}_{(\delta_- \delta_+)}(k) - \delta_- \delta_+ \delta_1(k)$$

$$= \Delta(\tilde{\delta}_- * \tilde{\delta}_+)(k) - 2\tilde{\delta}_-(k) + \Delta_\delta_\delta_+(\tilde{\delta}_-)(k) - 2\tilde{\delta}_+(k) + \Sigma_\delta_\delta_+(k) + \Delta_\delta_\delta_- (\tilde{\delta}_+)(k) - \delta_- \delta_+ \delta_1(k)$$

$$= ((\Delta(\tilde{\delta}_-) + \Delta_\delta_\delta_\delta_-(\delta_0)) * \tilde{\delta}_+)(k) - 2\tilde{\delta}_+(k) + \Sigma_\delta_\delta_+(k) - 2\tilde{\delta}_-(k) + \Delta(\tilde{\delta}_-)(k) - \delta_- \delta_1(k)$$

$$= - T_{(p_- + 1, p_- + 1)}(k) - T_{+ (p_- + 1)}(k) + \Sigma_\delta_\delta_+(k) - 2\tilde{\delta}_-(k) - \delta_{(p_- + 1)}(k)$$

$$= - (\max\{0, p_- + p_+ + 2 - k\} + \max\{0, |p_- - p_+| - k\}) + R(k)$$

where we see that $R(k) = 0$ for $k > p_- + 1$, and $R(p_- + 1) > 0$.

This does not represent a cuspidal unipotent formal degree, and therefore there are no solutions of (6.2) with $\lambda_-$ of type (b) or (c) and $\lambda_+$ of type (d).

Next let us consider the cases with $\lambda_-$ of type (b), (c) and $\lambda_+$ of type (e). This implies that $\delta_+ = 0$, and $p_+ \geq 4$. We again compute in a similar way to the case where both $\lambda_\pm$ are of type (b) or (c):

$$\text{mult}_{(\delta_- \delta_+)}(k) - \delta_- \delta_+ \delta_1(k)$$

$$= \Delta(\tilde{\delta}_- * \tilde{\delta}_+)(k) - 2\tilde{\delta}_-(k) + \Delta_\delta_\delta_+(\tilde{\delta}_-)(k) - 2\tilde{\delta}_+(k) + \Sigma_\delta_\delta_+(k) + \Delta_\delta_\delta_- (\tilde{\delta}_+)(k)$$

$$= ((\Delta(\tilde{\delta}_-) + \Delta_\delta_\delta_\delta_-(\delta_0)) * \tilde{\delta}_+)(k) - 2\tilde{\delta}_+(k) + \Sigma_\delta_\delta_+(k) - 2\tilde{\delta}_-(k)$$

$$= - T_{(p_- + 1)}(k) - T_{+ (p_- + 1)}(k) + \Sigma_\delta_\delta_+(k) - 2\tilde{\delta}_-(k)$$

We see that the values at $k$ equal to $p_- + p_+ + 1, p_- + p_+, p_- + p_+ - 1, p_- + p_+ - 2$ are $-1, -1, -2, -3$. This does not represent a cuspidal unipotent formal degree, and therefore there are no solutions of (6.2) with $\lambda_-$ of type (b) or (c) and $\lambda_+$ of type (e).
Next let us consider the cases with $\lambda_\pm$ both of type (d). This implies that $\delta_\pm = 1$, and $p_\pm \geq 3$. We may assume that $p_+ \leq p_-$. We again perform similar computations:

$$\text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) - \delta_1(k)$$

$$= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) - 2\tilde{h}_+(k) + \Sigma^d_+(k) + \Sigma^d_-(k)$$

$$= -T_{-(p_+-1)}\tilde{h}_+(k) - T_{p_-}\tilde{h}_+(k) - T_{p_-1}\tilde{h}_+(k)$$

$$- T_{-(p_-1)}\tilde{h}_+(k) - T_{p_-}\tilde{h}_+(k)$$

$$- 2\tilde{h}_-(k) + \Sigma^e_+(k) + \Sigma^e_-(k)$$

This can not represent a solution of (6.2) since, looking at the values of this sum at the arguments $k = p_+ + p_+ + 2, p_- + p_+ + 1, p_- + p_+ - 1, p_- + p_+ - 2$, we get 0, -1, -1, -2, -3, which does not correspond to a cuspidal formal degree. Hence there is no solution $\tilde{T}_{\lambda_-, \lambda_+}$ of (6.2) with $\lambda_+\pm$ of type (d).

Finally, let us consider the cases with $\lambda_-$ of type (d), and $\lambda_+$ of type (e). This implies that $\delta_- = 1$, and $p_\pm \geq 3$. We again perform similar computations:

$$\text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k)$$

$$= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Sigma^d_+(k) + \Sigma^d_-(k)$$

$$= -T_{-(p_+-1)}\tilde{h}_+(k) - T_{+(p_-1)}\tilde{h}_+(k) - 2\tilde{h}_-(k) - \Delta(\tilde{h}_+)(k)$$

$$- 2\tilde{h}_-(k) + \Sigma^e_+(k) + \Sigma^e_-(k)$$

If $p_- > 3$, this can not represent a solution of (6.2) since in this case, looking at the values of this sum at the arguments $k = p_+ + p_+ + 2, p_- + p_+ + 1, p_- + p_+ - 1, p_- + p_+ - 2$, we get 0, -1, -1, -2, -3, which does not correspond to a cuspidal formal degree. If $p_- = 3$ the values of the sum on $k = p_+ + 5, p_+ + 4, p_+ + 3, p_+ + 2, p_+ + 1$, $p_+$ are as follows 0, -1, -1, -2, -2, -4, which also does not correspond to a cuspidal unipotent formal degree. Hence there is no solution $\tilde{T}_{\lambda_-, \lambda_+}$ of (6.2) with $\lambda_-$ of type (d) and $\lambda_+$ of
We have finally checked all possibilities, which completes the proof of Theorem 1.1(a),(b) for symplectic and even orthogonal groups. Together with the results of Subsection 6.4 and Subsection 6.5 we have now finally completed the proof of Theorem 1.1(a),(b).

6.6.2. Proof of Theorem 1.1(c). Recall that we are considering $G_{qs}$ of type PCSp$_{2n}$, $P(CO_{2n}^0)$ or $P((CO_{2n+2})^0)$, which are of type III (first case) or of type IV (the last two cases). We have spectral covering maps $H^1(G_{qs}) \rightarrow C_n(\delta_-,\delta_+)[q]$ with $(\delta_-,\delta_+) = (0,1)$, (0,0), (1,1), which are given by degree two coverings in the first two cases, and which is an isomorphism in the last case.

The group $\Omega = \Omega_{ad}$ is equal to $C_2 = \{1, \rho\}$ for $G = PCSp_{2n}$. For $G = P(CO_{2n}^0)$ we use the notation $\Omega = \{1, \rho, \eta, \eta\rho\}$, which is isomorphic to $C_2 \times C_2$ if $n$ is even, and to $C_4$ of $n$ is odd. We also fix $\theta$, a nontrivial finite type $D_n$ subdiagram automorphism of order two, and take $\eta$ and $\rho$ such that: $\eta$ is $\theta$-invariant, and $[\rho, \theta] = [\rho\eta, \theta] = \eta$. Then $\rho^2 = 1$ if $n$ is even, and $\rho^2 = \eta$ is $n$ is odd. Thus for $G = P((CO_{2n+2})^0)$ we have $\Omega^\theta = \{1, \eta\}$ and $\Omega/(1-\theta)\Omega = \{1, \eta\}$. If $\Omega = \{1, \rho, \eta, \eta\rho\}$ then we denote the elements of the dual group $\Omega^*$ by $\Omega^* = \{1, \epsilon, \gamma, \epsilon\gamma\}$, where $\epsilon(\rho) = \epsilon^2$, $\epsilon(\eta) = -1$, and $\gamma(\rho) = -1$ and $\gamma(\eta) = 1$.

The relevant groups of spectral diagram automorphisms $\Gamma_{spec}(\Omega_{ad})$ is denoted by: $C_2 = \{1, \gamma\}$ in the first case, $\{1, \epsilon, \gamma, \epsilon\gamma\}$ in the second case (where for $n$ even we have $\epsilon^2 = 1$, and in the case $n$ odd we have $\epsilon^2 = \gamma$), and $C_2 = \{\gamma\}$ in last case.

As discussed above, we may associate a unique unramified discrete local Langlands parameter $[\lambda_{ad}]$ to a pair $(\lambda_-, \lambda_+)$, via the correspondence discussed in Remark 1.5 and above in Subsection 6.4.

Given a type $\chi \in \{\text{III, IV, V, IV}\}$ and a pair of nonnegative integers $\{a, b\}$ (subject to requirements we will make explicitly below) we can associate a conjugacy class of maximal parahoric subgroups $P_{a,b}^{\chi}$ of a well defined inner form $G^u$ of $G_{qs}$ (with $u \in H^1(k, G) \simeq \Omega/(1-\theta)\Omega$) supporting a cuspidal unipotent character. The conjugacy classes of such maximal parahoric subgroups $P_{a,b}^{\chi}$ of $G_{qs}$ are represented by the diagram of an apartment of the building of $G = G(K)$, with a maximal proper subdiagram of boxed vertices which forms the complement of a $\theta,u$-orbit, where the boxed subdiagram (with Frobenius acting via $\theta,u$) is the root diagram of a finite group of Lie type supporting a cuspidal unipotent character.

First assume that $G = PCSp_{2n}$ (so $\theta = 1$, and $n \geq 2$). In this case, for $u = 1$, we are considering $P_{a,b}^{\chi}$ whose boxed subdiagram is a maximal proper subdiagram of the affine Dynkin diagram of $G_{qs}$ with two (possibly empty) components, of type $C_{a^2+1}$ and $C_{b^2+b}$. We define $\{m_-, m_+\}$ by $\{1 + 2a, 1 + 2b\} = \{m_+ + m_-\}$ and obtain via Theorem 1.1(b) the corresponding residue point $\tau := \tau(\lambda_-, \lambda_+)$, where $\lambda_-$ is of type Corollary 5.14(b) and $\lambda_+$ of type Corollary 5.14(c). We see that $(\Gamma_{spec})_{W_0} \tau = C_2$ unless $m_- = 0$, in which case $(\Omega^\theta)_{W_0} \tau = 1$. On the other hand, we have $(\Omega^\theta)_{P_{a,b}} = 1$ unless $a = b$, in which case $\Omega^\theta_{P_{a,b}} = C_2$. This proves the desired result in this case.

Next, assume that we take unipotent cuspidal pairs of the inner form defined $u = \rho$. Then we are considering $P_{a,b}^{\chi}$ of type $C_{b^2+b}[q^2]$ (by which we mean the restriction of
scalars group defined over \( \mathbb{F}_q \), whose group of \( \mathbb{F}_q \)-points is the group \( \text{Sp}_{2(2a+b)}(\mathbb{F}_q) \): this appears as two isomorphic components of the boxed subdiagram which are interchanged by \( \rho \) and a component of type \( 2A_{(a/2)(a+1)-1}(q^2) \). This determines \( m_\pm \) by the rule 
\[
\{1/2 + a, 1 + 2b\} = \{m_+ + m_- | m_+ - m_-\}.
\]
We obtain a residual point \( \vec{r} = \vec{r}(\lambda_-, \lambda_+) \) where \( \lambda_\pm \) are both of type Corollary 5.14(a), of length \( l_\pm := \kappa_\pm \). We have \( \Omega^{\theta, P_{a,b}} = \Omega^\theta = C_2 = \{1, \rho\} \) in all cases, hence we have two such cuspidal characters. On the other hand, 
\[
(\Omega^\theta)_{W_0}^\ast = C_2 \text{ unless } m_- = 1/4 \text{ (which holds if and only if } \lambda_- \text{ is the zero partition, in this situation).}
\]
Observe that if \( \mathbb{P}^V_{a,b} \) supports a cuspidal character then clearly \( n > 2 \), hence \( |\lambda_-| + |\lambda_+| \geq 7 \), and \( L + l_+ \geq 3 \) (the smallest solution is \( n = 3 \), with \( \lambda_- = [1, 5] \) and \( \lambda_+ = [1] \)). If \( \lambda_- \neq 0 \), the centralizer group \( A_{[\lambda_{ad}]} \) is a group of type \( 2^{(l_- + l_+-3)+2} \), according to [Op2, Proposition 13.4]. Here we use the notation \( 2^{(2a)+b} \) to indicate an extra special \( 2 \)-group of dimension \( 2^{a+b} \), with \( 2^{2a+b-1} \) representations of dimension 1, and \( 2^{b-1} \) irreducible representations of dimension \( 2^a \). Hence if \( \lambda_- \neq 0 \), there are two representations of \( A_{[\lambda_{ad}]} \) of dimension \( 2^{(l_-+l_+-3)/2} \) which restrict to \( \rho \) times the identity on \( kZ \simeq (\Omega/(1-\theta)\Omega)^\ast \), and these two “extra special” cuspidal local systems correspond to the two cuspidal characters. In the case where \( \lambda_- = 0 \), then \( (\Omega^\theta)_{W_0}^\ast = 1 \).

Now \( A_{[\lambda_{ad}]} \) is a group of type \( 2^{(l_+-1)+1} \), which has one representation of dimension \( 2^{(l_-+l_+-1)/2} \), which corresponds to a cuspidal local system. Now we have two Langlands parameters \( [\lambda_{ad}] \) and \( [\lambda'_{ad}] \) over \( [\lambda_{sc}] \) (a single \( (\Omega^\theta)^\ast \)-orbit), each of which accommodates one of the two cuspidal characters. This completes the proof for \( G = \text{PSp}_{2a} \).

Now consider \( G = \text{P}(\text{CO}_{2n}) \), with \( n \geq 4 \). First we consider \( u = 1 \). Then we are considering \( \mathbb{P}^V_{a,b} \) whose boxed subdiagram is a maximal proper subdiagram of the affine Dynkin diagram of \( G_{qs} \) with two (possibly empty) components, of type \( D_{a2} \) and \( D_{b2} \), with \( a, b \in 2\mathbb{Z}_{\geq 0} \). Hence \( \Omega^{\theta, P_{a,b}} = C_2 \), except when \( a = 0 \) or \( b = 0 \) in which case \( \Omega^{\theta, P_{a,b}} = 1 \), or when \( a = b \), in which case \( \Omega^{\theta, P_{a,b}} = C_2 \times C_2 \) (indeed, this can occur only if \( n \) is even). We define \( \{m_-, m_+\} \) by \( \{2a, 2b\} = \{m_+ + m_-, |m_+ - m_-|\} \) and obtain via Theorem 1.1(b) the corresponding residue point \( \vec{r} := \vec{r}(\lambda_-, \lambda_+) \), where \( \lambda_\pm \) is of type Corollary 5.14(b). We see that \( (\Omega^\theta)_{W_0}^\ast \Omega^\ast = C_2 \) unless \( m_- = 0 \), in which case \( (\Omega^\theta)_{W_0}^\ast \Omega^\ast = \Omega^\ast \). Hence in all these cases we have the equality \( (\Omega^\theta)_{W_0}^\ast = (\Omega^\theta)^\ast \). Also, in each such case \( A_{[\lambda_{ad}]} \) has one cuspidal local system (corresponding to a pair of Lusztig-Shoji symbols each with one empty row). This proves the result for \( u = 1 \).

Now consider the inner form with \( u = \eta \). We are considering \( \mathbb{P}^V_{a,b} \) whose boxed subdiagram is a maximal proper subdiagram of the affine Dynkin diagram of \( G_{qs} \) with two (possibly empty) components, of type \( D_{a2} \) and \( D_{b2} \), with \( a, b \in 2\mathbb{Z}_{\geq 0} + 1 \). Hence \( \Omega^{\theta, P_{a,b}} = C_2 \), except when \( a = b \), in which case \( \Omega^{\theta, P_{a,b}} = C_2 \times C_2 \) (indeed, this can occur only if \( n \) is even). We define \( \{m_-, m_+\} \) by \( \{2a, 2b\} = \{m_+ + m_-, |m_+ - m_-|\} \) and obtain via Theorem 1.1(b) the corresponding residue point \( \vec{r} := \vec{r}(\lambda_-, \lambda_+) \), where \( \lambda_\pm \) is of type Corollary 5.14(b). We see that \( (\Omega^\theta)_{W_0}^\ast = C_2 \) unless \( m_- = 0 \), in which case \( (\Omega^\theta)_{W_0}^\ast = \Omega^\ast \). Hence in all these cases we also have the equality \( (\Omega^\theta)_{W_0}^\ast = (\Omega^\theta)^\ast \). Again, in each such case \( A_{[\lambda_{ad}]} \) has one cuspidal local system (corresponding to a pair
of Lusztig-Shoji symbols in the sense of Slooten [Slo] each with one empty row). This proves the result for \( u = 1 \).

Now consider the inner forms with \( u = \rho \) or \( u = \eta \rho \) (the extra special cases). We treat \( u = \rho \) (the case \( u = \rho \eta \) gives similarly complete results). Then we are considering \( \mathbb{P}^\rho_{a,b} \), with one (possibly empty) component of type \( D_q[q^2] \) (by which we mean the restriction of scalars group defined over \( F_q \), whose group of \( \mathbb{P}^q \)-points is the group \( SO_{2q}(F_q) \) (if \( n \) is even) or \( SO_{2q}^+(F_q) \) (if \( n \) is odd); this appears as two isomorphic components of the boxed subdiagram which are interchanged by \( \rho \) and a (possibly empty) component of type \( 2A_{(a/2)(a+1)} - 1(q^2) \). Then \( \Omega^\rho_{P,a,b} \) equals \( \Omega \), unless \( n \) is even and \( b = 0 \), in which case \( \Omega^\rho_{P,a,b} = \{1, \rho\} = C_2 \).

We define \( \{m_-, m_+\} \) by \( \{1/2 + a, 2b\} = \{m_+ + m_-, |m_+ - m_-|\} \). Observe that for \( n \) even, we have \( b \) even and \( a \equiv 0, 3 \) (mod 4), while for \( n \) odd, we have \( b \) odd and \( a \equiv 1, 2 \) (mod 4). This implies that \( \kappa_\pm \equiv 2Z \). We obtain via Theorem 1.1(b) the corresponding residue point \( \vec{r} := \vec{r}(\lambda_-, \lambda_+) \), where \( \lambda_\pm \) is of type Corollary 5.14(a). Recall that the length \( l_\pm \) of the partition \( \lambda_\pm \) equals \( \kappa_\pm \), hence the lengths \( l_\pm \) are both \( \text{even} \). We see that \( \Omega^\rho_{P,a,b} \Omega \) equals \( C_2 \) except when \( m_- = 1/4 \) (which is equivalent to \( \lambda_- = 0 \)), in which case \( \Omega^\rho_{P,a,b} = \Omega \).

If \( \lambda_- = 0 \) then \( \Omega^\rho_{P,a,b} = \Omega \). We have 4 orbits of Langlands parameters \( [\lambda_{ad}] \) above \( \lambda_{sc} \), which form one \( \Omega^- \)-orbit. In this case, \( A_{[\lambda_{ad}]} \) is of type \( 2(l_+ - 2)^+2 \), which carries 1 cuspidal local systems which restrict to \( \rho \) times the identity on \( LZ \simeq \Omega^* \) (and also one which restricts to \( \rho \eta \) times the identity). Together these correspond to the 4 cuspidal characters of this type. Indeed, the number of cuspidal characters supported by \( \mathbb{P}^\rho_{a,b} \) equals \( 4 \) (one orbit under \( \Omega^\rho \simeq \Omega^* \)).

If \( \lambda_- \neq 0 \) and \( m_- \neq m_+ \), then \( \Omega^\rho_{P,a,b} = \Omega \) and we again have 4 cuspidal characters with the same formal degree of this inner form. By Theorem 1.1(b) these correspond to \( W_0 \vec{r}(\lambda_-, \lambda_+) \) and \( W_0 \vec{r}(\lambda_+, \lambda_-) \), where \( \lambda_- \neq \lambda_+ \). Now [Op2, Proposition 3.14] implies that \( A_{[\lambda_{ad}]} \) is of type \( 2(l_- + l_+ - 4)^+3 \), and both these orbits of Langlands parameters carry 2 cuspidal local systems which restrict to \( \rho \) times the identity on \( LZ \).

Finally, if \( m_- = m_+ \), then \( \eta = 0 \), and \( \Omega^\rho_{P,a,b} = \Omega \), and \( \mathbb{P}^\rho_{a,b} \) carries 2 cuspidal characters in this case. On the other hand, now \( \Omega^\rho_{P,a,b} = \Omega \), and we have one orbit of Langlands parameter \( [\lambda_{ad}] \) with \( A_{[\lambda_{ad}]} \) of type \( 2(2l_+ - 4)^+3 \). This has 2 irreducible representations which restrict to \( \rho \) times the identity on \( LZ \). This proves the result for \( G = P(CO_{2n}^\rho) \).

Finally consider \( G = P(CO^*)_{2n+2} \), with \( n \geq 3 \). We have \( \Omega^\rho = \{1, \eta\} \), \( \Omega^\rho = \{1, \tau\} \), and \( \Omega/(1 - \theta) \Omega = \Omega \{1, \bar{\rho}\} \). First we consider \( u = 1 \). Then we are considering \( \mathbb{P}^\rho_{a,b} \) whose boxed subdiagram is a maximal proper subdiagram of the affine Dynkin diagram of \( G_{qs} \) with two (possibly empty) components, of type \( D_{a/2} \) and \( D_{b/2} \), with \( a \) even and with \( b \) odd, such that \( a^2 + b^2 = n + 1 \). Hence \( \Omega^\rho_{P,a,b} \) equals \( C_2 \), except when \( a = 0 \) in which case \( \Omega^\rho_{P,a,b} = 1 \). We define \( \{m_-, m_+\} \) by \( \{2a, 2b\} = \{m_+ + m_-, |m_+ - m_-|\} \) and obtain via Theorem 1.1(b) the corresponding residue point \( \vec{r} := \vec{r}(\lambda_-, \lambda_+) \), where \( \lambda_{\pm} \) is of type Corollary 5.14(b). We see that \( \Omega^\rho_{P,a,b} = 1 \) unless \( m_+ = m_+ \), in which case
$$(\Omega^\theta)^*_{W_0,\tau} = C_2.$$ In all these cases we have the equality $$(\Omega^\theta)^*_{W_0,\tau} = (\Omega^\theta / \Omega^{\theta,a,b})^*.$$ Also, in each such case $A_{[\lambda_{ad}]}$ has one cuspidal local system (corresponding to a pair of Lusztig-Shoji symbols each with one empty row). This proves the result for $u = 1$.

Now consider the inner form with $u = 7$ (the extra special cases). Then we are considering $P_{a,b}$, maximal proper with one (possibly empty) component of type $D_{2r}[q^2]$ (by which we mean the restriction of scalars group defined over $\mathbb{F}_q$, whose group of $\mathbb{F}_q$-points is the group $SO_{2r}(\mathbb{F}_q)$ (if $n$ is even) or $SO_{2r}^+(\mathbb{F}_q)$ (if $n$ is odd): this appears as two isomorphic components of the boxed subdiagram which are interchanged by $\rho$) and a (possibly empty) component of type $2A_{(a/2)(a+1)-1}(q^2)$. Then $\Omega^\theta,a,b$ equals $\Omega$, unless $n$ is even and $b = 0$, in which case $\Omega^\theta,a,b = \{1, \rho\} = C_2$.

We define $\{m_-, m_+\}$ by $\{1/2 + a, 2b\} = \{m_+ + m_-, |m_+ - m_-|\}$, and this time we see that in all cases $\kappa_\pm$ are both odd. Via Theorem 1.1(b) we obtain the corresponding residue point $\overline{\tau} := \overline{\tau}(\lambda_-, \lambda_+)$, where $\lambda_\pm$ is of type Corollary 5.14(a), both of odd length $l_\pm$. We see that $(\Omega^\theta)^*_{W_0,\tau} = 1$, except when $m_- = m_+$, in which case $(\Omega^\theta)^*_{W_0,\tau} = (\Omega^\theta)^*$. If $m_- \neq m_+$, then $\Omega^\theta \overline{\tau}^\pm_{a,b} = \Omega$ and we again have two cuspidal characters with the same formal degree of this inner form. By Theorem 1.1(b) these correspond to $W_0^1 \overline{\tau}(\lambda_-, \lambda_+)$ and $W_0^1 \overline{\tau}(\lambda_+, \lambda_-)$, where $\lambda_- \neq \lambda_+$. Now $A_{[\lambda_{ad}]}$ is of type $2^{(l_- + l_+ - 2) + 1}$, hence both these orbits of Langlands parameters carry 1 cuspidal local system which restricts to $\overline{\tau}$ times the identity on $LZ$.

Finally, if $m_- = m_+$ then $n$ is even, and $\Omega^\theta \overline{\tau}^\pm_{a,b} = 1$, hence $P_{a,b}$ carries one cuspidal characters in this case. On the other hand, now $(\Omega^\theta)^*_{W_0,\tau} = (\Omega^\theta)^*$, and we have one orbit of Langlands parameter $[\lambda_{ad}]$ with $A_{[\lambda_{ad}]}$ of type $2^{(2l_+ - 2) + 1}$. This has one irreducible representation which restrict to $\overline{\tau}$ times the identity on $LZ$. This proves the result for $G = P(CO_{2b})$.

We thus completed the proof of Theorem 1.1, finally, completely.\]

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