Can a Fingerprint be Modelled by a Differential Equation?

(When Galton Meets Poincaré)

by

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Handprints of children from a Moroccan kindergarten, celebrating author’s daughter first birthday.
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References (Let’s Twist Again, a parody song)
Abstract.
Some new directions to lay a rigorous mathematical foundation for the
phase-portrait-based modelling of fingerprints are discussed in the present work.
Couched in the language of dynamical systems, and preparing to a preliminary
modelling, a back-to-basics analogy between Poincaré’s categories of equilibria
of planar differential systems and the basic fingerprint singularities according
to Purkyné-Galton’s standards is first investigated. Then, the problem of the
global representation of a fingerprint’s flow-like pattern as a smooth deformation
of the phase portrait of a differential system is addressed. Unlike visualisation in
fluid dynamics, where similarity between integral curves of smooth vector fields
and flow streamline patterns is eye-catching, the case of an oriented texture like
a fingerprint’s stream of ridges proved to be a hard problem since, on the one
hand, not all fingerprint singularities and nearby orientational behaviour can
be modelled by canonical phase portraits on the plane, and on the other hand,
even if it were the case, this should lead to a perplexing geometrical problem
of connecting local phase portraits, a question which will be formulated within
Poincaré’s index theory and addressed via a normal form approach as a bivari-
te Hermite interpolation problem. To a certain extent, the material presented
herein is self-contained and provides a baseline for future work where, starting
from a normal form as a source image, a transport via large deformation flows
is envisaged to match the fingerprint as a target image.

1 Introduction

In 1892, Sir Francis Galton, the English Victorian scientist, published the first
book of his “trilogy” on fingerprints [1]. As said by the author, his attention had
been first drawn to ridges when preparing, some years ago, a Royal Institution
lecture on personal identification, which aimed at an account of the newly intro-
duced French anthropometric method of Alphonse Bertillon [2]. Realising both
how much had been done on the subject and how much there remained to do,
and being chiefly based on a thesis of the Czech physiologist, Jan Purkyné, at
the university of Breslau [3] - a very rare pamphlet on classification of papillary
ridges, he will become perhaps the first to place the fingerprint-based William
Herschel’s identification method [4] on a scientific footing and to lay securely
the foundation of a new branch of inquiry. Following a series of memoirs upon
the subject [5-8], a system for classifying fingerprint patterns into three broad
categories, which is very useful for rough preliminary purposes, is then mainly
used in Galton’s book and of which frequent reference will be made in this paper:
Arches, Loops and Whorls (ALW), in a sense to be specified.

In the same year, Henri Poincaré, the French universal scientist and the
father of the qualitative theory of differential equations, published the first vol-
ume of his “trilogy” on celestial mechanics [9], a masterpiece written during the
last decade of the nineteenth century, following the pioneering works of Cauchy,
Lagrange and Laplace, his own inaugural thesis [10] and a series of papers [11]
where different types of singularities have been named and studied: Nodes, Foci, Saddles and Centers.

The two scientists have probably never met in person but like Monsieur Jourdain who was speaking prose without knowing it, they were perhaps speaking the same language, as insinuated in Fig. fingerprinting when dressing phase portraits for Poincaré, and conversely, for Galton, solving differential equations when deciphering fingerprints!

The idea is then the following: to what extent can a fingerprint’s orientation image be visualised as (a smooth deformation of) the phase portrait (or more faithfully, in Poincaré’s language, a system of characteristics) of a planar dynamical system? In other terms, to what extent can a classification system of fingerprints be couched in the language of the qualitative theory of differential equations?

Contrary to what this introduction may be suggesting, the idea of using phase portraits in texture modelling is not new, as can be already seen through an interdisciplinary program initiated by the Semiconductor Research Corporation at the University of Michigan in the 80’s, whose goal was to develop a visual language for representing visual data in semiconductor wafer processing. The thesis by Rao [13], for instance, was part of a larger effort within the program to device a symbolic description of oriented texture patterns using the qualitative

\[1\] If one has to establish a far-fetched link between Galton and Poincaré, it would be perhaps Poincaré’s work of 1885 on the equilibrium figures of a fluid mass from which George Darwin, the son of Charles, who was Galton’s half-cousin, deduced what he believed to be a mechanism for the formation of the Moon!
theory of differential equations. Curiously, a large album of real texture images have been analysed, like an invigorating wood knots accord with notes of orange peel, enhanced by hints of brush strokes from Van Gogh paintings, but no fingerprint image seems to have been considered. Special mention must also be made of the thesis by Ford [14], at the University of Arizona, on 2-D fluid flow modelling and visualisation, whereby complex flows were split into simpler and easily described components, the latter being modelled by linear phase portraits and then combined to obtain a global model for the entire flow field. This idea has been finally applied to fingerprints by Li and Yau and recapitulated as a chapter in [15] (see also references therein) where, following Kass and Witkin’s scheme of squaring the gradient vectors in the computed oriented texture field [16], basic fingerprint singular points have been assimilated to either a Focus or a Saddle. However, amid the wealth of literature on fingerprints modelling in the last two decades, if we restrict ourselves solely to the Taylor expansion as a basis for phase portrait models, and while this clearly yields good results at a practical level, there are still some outstanding issues to be mathematically clarified. In fact, at a purely mathematical level, and without going into details for the moment, previous phase-portrait-based methods will be most certainly found to work near a singular point for a fingerprint orientation as long as the Hartman-Grobman theorem works for a nonlinear dynamical system near an equilibrium point. This can also be expressed in terms of Peixoto’s theorem within the scope of structural stability. Indeed, when dealing for example with a Saddle or a Focus, we know that a polynomial perturbation of the linear terms will not change the nature of these points (see the discussion in paragraph 5.2.1 on robustness of singular points against addition of nonlinear terms). However, for a Center, which is an unavoidable point in the Circular/Elliptical Whorl or the Cerclet in Loop modelling (see below), let alone a degenerate point (see the classification in paragraph 3.2), it is mathematically possible, but highly unlikely, that a center-like behaviour near a singular point will be preserved after addition of nonlinear terms, unless appropriate symmetries are shown by the global flow, a behaviour which is often observed in natural occurring flows, but certainly not on fingerprints. Besides, only nondegenerate singular points have been considered for the derived piecewise phase portrait models, then going to miss (nontrivial) degenerate points occurring in interesting bifurcations, next to the rich variety of phase portraits they offer (see Fig.7 below). Just in the nondegenerate case, a large part has been in fact devoted to the theory of centers by Poincaré [11, chap.XI], to show that the question has to be considered in its own right. At another level, the problem of interconnecting singular points should not be dealt with as if it were always solvable; obviously, this is true in fluid dynamics where connecting streamlines can be experimentally visualised and then topologically modelled. However, for a texture image like a fingerprint, and unless not more than a restricted and well-chosen number of relevant singular points are considered, such a connexion is generally impossible.

\[\text{For other variations, see for instance Wang et al. [17] for a Fourier-expansion-based approach and Ram et al. [18] for a Legendre-polynomial-based phase portrait model.}\]
as it comes out from a global index theory analysis of smooth vector fields on two-dimensional surfaces.

The present paper has no pretension to outperform previous works on phase-portrait-based modelling, but a targeted approach will be adopted to mobilising sophisticated results from dynamical systems, in line with Poincaré’s œuvre, to help overcome the limited capability of a basic linearisation approach and open up new possibilities for future works on fingerprints modelling. Nonhyperbolic and degenerate cases will be then carefully handled within a structure-preserving normal form approach, taking profitably the large variety of qualitative behaviours they host. As will be shown, if the difficulty can be partially overcome for a preliminary modelling within the ALW classification, in which no more than three singular points are taken into account, the general case involving many singularities with different natures, however, leads to a serious mathematical problem. Explicitly, even if all basic fingerprint patterns were modelled by appropriate phase portraits, the problem of associating a global phase portrait to a fingerprint would lead to an advanced bivariate Hermite interpolation problem for which an algebraic solution in the general case is hopeless. To obtain some elements of answer, the geometrical notion of connecting local phase portraits will be considered from an intuitive point of view, in the spirit of Poincaré’s question of how singular points of vector fields are distributed in the phase space, and how the study of a function defined in the vicinity of a singular point can be extended to the whole space. The geometrical notion of connecting local phase portraits, however, should be much easier to address than a direct attack of the interpolation problem. It will be then easily seen, within Poincaré’s index theory, that such a connexion do not always exist, and when it exists, obviously, it is not necessarily unique. But in general, there is no systematic approach to carry out a connexion, although a lot has been done by Poincaré on the sphere, after gnomonic projection, from which the subject can be shown to derive some strength and fruitfulness.

Globally, the paper is structured as follows. In the first part, a brief survey of Galton’s book is first given, followed by a (partial, yet sufficient) classification of singular points on the plane. A gallery of local phase portraits is then presented from which a collection of typical singularities will be hand-picked to match - whenever possible - the basic fingerprint patterns to be modelled. Then, an attempt at a preliminary modelling of fingerprints is made according to Purkyně-Galton’s standards, where some hand-drawn phase portraits are given as approximations. As will be seen, the main obstruction caused by a delta-like pattern will be overcome by integrating in the phase plane a cusp-like singular point from the well-known Bogdanov-Takens bifurcation. In the second part, some directions for building global phase portraits from local ones are discussed within Poincaré’s index theory and, when possible, a structure-preserving normal form - in a sense to be specified - will be computed as a bivariate Hermite interpolation problem and attributed to some subclasses within the ALW system. As said in the abstract, such a normal form will be the starting point for
future work where, to obtain a final signature, a transport via large deformation flows is envisaged to carry away the normal form as a source image in order to match the fingerprint as a target image.

Part I
Preliminary phase-portrait-based modelling of fingerprints

2 A brief survey of Galton’s book

In [1], and not without a wicked sense of humour, Galton begins with these words: “The palms of the hands and the soles of the feet are covered with two totally distinct classes of marks. The most conspicuous are the creases or folds of the skin which interest the followers of palmistry, but which are no more significant to others than the creases in old clothes; they show the lines of most frequent flexure, and nothing more. The least conspicuous marks, but the most numerous by far, are the so-called papillary ridges; they form the subject of the present book. If they had been only twice as large as they are, they would have attracted general attention and been commented on from the earliest times. Had Dean Swift known and thought of them, when writing about the Brobdingnags,
whom he constructs on a scale twelve times as great as our own, he would certainly have made Gulliver express horror at the ribbed fingers of the giants who handled him. The ridges on their palms would have been as broad as the thongs of our coach-whips."

After treating of the previous use of fingerprints, from superstition of personal contact to the modern regular official employment of Herschel, various methods of making good prints and enlarging them are described as they are adopted at the author’s own anthropometric laboratory. Then these preliminary topics having been disposed of, the author begins with a discussion of the various patterns formed by the lineations, emphasizing the independent ones that appear upon the bulbs of the fingers, and where plates of the principal varieties of patterns are given as a visual support. A useful classification system for rough preliminary study is then presented into Arches, for which we have no pattern strictly speaking; Loops, where we have a system of ridges that bends back upon itself and in which no one ridge turns through a complete circle; Whorls, for which at least one ridge turns through a complete circle (Fig.3). Of course, chapters dealing with evidential values, methods of indexing, personal identification or heredity are beyond the interest of the present paper.

An important passage of the book should be however highlighted: a translation in part from the Latin of the Commentatio of Purkyně, made by the author himself, which a copy has been procured from the United States to the Library of the Royal College of Surgeons. The following nomenclature was then established according to the nine principal varieties of curvature observed by Purkyně, and presented here in the same order as they appear in Plate 12 of Galton’s book (see Fig.5): 1) Transverse flexures; 2) Central Longitudinal Stria; 3) Oblique Stria; 4) Oblique Sinus; 5) Almond; 6) Spiral; 7) Ellipse, or Elliptical Whorl; 8) Circle, or Circular Whorl; 9) Double Whorl. Following Purkyně, all these forms have been concisely described by Galton, within Arches for 1-3, Loops for 4-5 and Whorls for 6-9. We prefer to refer to diagrams for explanation at this stage, while more detail will be given for each variety when proceeding to modelling.
Figure 4: Purkyně’s Commentatio (U.S. National Library of Medicine).

Figure 5: The standard patterns of Purkyně, as captured from Galton’s book.
To start the study in section 4, and for purely aesthetic reasons, I'll choose from the above configurations the Spiral and the Elliptical/Circular Whorl as basic models to be identified to the phase portrait of an autonomous planar system of ordinary differential equations. For such a purpose, I'll try to establish an analogy between some categories of singular points of a planar vector field and the basic singular points composing the fingerprint's orientation image being analysed. Reference is made to Galton's book (and mainly to the translation of Purkyně's thesis therein) when it comes to using terminology from fingerprints, except for the term Delta which is borrowed from the well-known Henry Classification System \[19\] and will be preferred to Purkyně-Galton's Triangle. As for terminology from dynamical systems, reference will be mainly made to Poincaré, whose the œuvre is the basis of the planar classification of singular points given below.

3 Poincaré’s imprint

In July 2012, celebrating 100 years after Poincaré, Alain Chenciner, from the Institut de Mécanique Céleste de l’Observatoire de Paris, delivered in the cemetery of Montparnasse a moving and eloquent speech, with a riveting and poetic survey of all Poincaré's œuvre. Besides the man and his heritage, one of the interesting things to which attention could be drawn in Chenciner’s speech, is especially an “opposition” of Poincaré’s spirit and nature to Charles Hermite’s, the French realistic and anti-geometer mathematician. In fact, within the present paper, the way in which is addressed the problem of assigning (a smooth deformation of) a differential equation’s phase portrait to a fingerprint, i.e. whether via the intuitive notion of connecting local phase portraits or, rigorously, as an algebraic multivariate interpolation problem, could be seen as a certain expression of the opposition between these two natures, and will be sometimes perceptible as work progresses.

3.1 Singular points of the first species

Following [9-11], let consider curves defined by an equation of the form

\[
\frac{dx}{X} = \frac{dy}{Y} \tag{1}
\]

where \(X\) and \(Y\) are analytic functions in \(x\) and \(y\). Such curves are called characteristics by Poincaré. As we are not concerned with the study of infinite branches for the moment, we will not consider the gnomonic projection on the sphere as Poincaré usually does, restricting ourselves to (a subdomain of) the phase plane \((x, y)\). Following Cauchy [20], Briot and Bouquet [21], and himself

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3Expanding on Galton’s classification system, this is another interesting book which, by order of the government of (British) India, and for bureaucratic settings, was published in 1900 by Edward Henry as being a former member of the civil service at the presidency of Fort William in Bengal.
Figure 6: Classification of singular points of planar systems (E. Izhikevich, Scholarpedia (2007)).

Poincaré gave a complete description of the characteristics near an isolated nondegenerate singular point, that is for which $X = Y = 0$ and the Jacobian matrix $J$ (of first-order partial derivatives of $X$ and $Y$ at the singular point) has no zero eigenvalues. Depending on the distribution of these eigenvalues in the complex plane, a classification into four categories is given: Nodes, Foci, Saddles, for the hyperbolic case (no eigenvalues with zero real part), and Centers, for the nonhyperbolic case. Poincaré calls them singular points of the first species. In Fig[4] a diagram of bifurcation is given according to the trace $\tau$ and the determinant $\Delta$ of the Jacobian matrix, the half-axis $\tau = 0, \Delta > 0$ and the axis $\Delta = 0$ corresponding to nonhyperbolic singularities that arise at Andronov-Hopf and Saddle-Node Bifurcation, respectively.

### 3.2 Singular points of the second species

The case when the curves $X = Y = 0$ intersect in many combined points leads to singular points of the second species, which are indeed nonhyperbolic and can be considered as the limit of a system of singular points of the first species, combined together. They are sometimes referred to as multiple singular points since they can be made to split into a number of hyperbolic critical points under suitable perturbation of $X$ and/or $Y$.

As said by Poincaré [11] in p. 393, such points are of too numerous and too diverse particularities to be studied in details. In Perko [22], following Poincaré [11], Bendixson [23] and more recently Andronov et al. [24], a collection of interesting results on nonhyperbolic singular points of planar analytic systems
is recalled and a gallery of phase portraits is plotted for different singular points with sometimes unusual behaviour of nearby trajectories (Fig. 7).

In the case when $J$ has at least one zero eigenvalue (the degenerate case), but $J \neq 0$, it is shown that there are at most $2(m+1)$ directions $\theta$ along which a solution curve of (1) may approach the singular point (supposed put at the origin, without loss of generality), provided the function

$$f(\theta) = \cos \theta X_m(\cos \theta, \sin \theta) - \sin \theta Y_m(\cos \theta, \sin \theta)$$

is not identically zero, $X_m$ and $Y_m$ being the $m$th-degree terms with which begin the Taylor series of $X$ and $Y$, respectively. These directions are given by solutions of the equation $f(\theta) = 0$ and then the notion of sector become fundamental for the classification. In fact, a sufficiently small neighborhood of the origin will be divided by these curves into a finite number of open regions (sectors), each of them being either of a hyperbolic, a parabolic or an elliptic type (Fig. 8). This is to be understood in a topological sense, no regard being paid to the direction of the flow.

The trajectories which lie on the boundary of a hyperbolic sector are then called separatrices, and it is shown that, besides the usual types of singular points for planar analytic systems, i.e. Nodes, Foci, Saddles and Centers, the only other types of singular points that can occur for (1) (when $J \neq 0$) are: Saddle-Nodes, for which we have two hyperbolic sectors and one parabolic

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Figure 7: Some degenerate singular points, as collected from Perko’s book.\textsuperscript{4}

\textsuperscript{4}May I ever be excused for “stealing” these phase portraits from [22], without permission from Prof. Lawrence Perko.
Figure 8: *From left to right, a hyperbolic, a parabolic and an elliptic sector [23].*

(Fig.7a): Cusps, occurring in Bogdanov-Takens bifurcation and for which we have two (and only two) hyperbolic sectors (Fig.7b); Singular points with an elliptic domain, for which we have one elliptic sector, one hyperbolic sector and two parabolic sectors (Fig.7c).

More precisely, if \( J \) has exactly one zero eigenvalue, the underlying singular point is either a Node, a Saddle, or a Saddle-Node; and if \( J \) has two zero eigenvalues (remember, \( J \neq 0 \)), the singular point is either a Focus, a Center, a Node, a Saddle, a Saddle-Node, a Cusp, or a singular point with an elliptic domain. Note that we have only taken into account analytic vector fields, so according to a theorem by Dulac (1923) on the finitude of the number of limit cycles (i.e. isolated closed trajectories, a question intimately linked to the famous and still unsolved Hilbert’s sixteenth problem), the case of the Center-Focus is to be excluded: this corresponds to a singular point surrounded by an infinite number of accumulating limit cycles. As will be seen, however, this is not a big loss since patterns of a spiral or a circular form can be merely represented by a Focus or a Center.

The case \( J = 0 \) is a bit exceptional, and the behaviour near the singular point can be more complex (Fig.7(d-g)). All we recall here is a (consequence of a) beautiful result due to Bendixson [23], within index theory (cf. subsection 5.1.1), and which can be summarised as follows:

\[
e \equiv h \pmod{2}
\]

\( e \) and \( h \) being, respectively, the number of elliptic sectors and the number of hyperbolic ones. In other terms, \( e \) and \( h \) have the same parity. Note that all degenerate singular points (a-g) in Fig.7 agree well with this formula. One can of course endeavour to enlarge our collection by including other exotic points in the degenerate case; this would not be however of much use since, as we shall see in the next section, the formula above is quite sufficient to test whether an imaginary singular point (suggested by a pattern) can match a real singular point or not.
4 Preliminary modelling

In this section, we will consider Purkyně-Galton’s basic fingerprint patterns as models to be mathematically represented by the phase portrait of an autonomous planar system of ordinary differential equations, which in an equivalent form can be derived from (1) and written as

\[ \dot{x} = f(x) \]  

(2)

According to Newton’s notation, the dot above \( x \) means derivative with respect to the independent variable (time, in general) and \( f \) is a function which the class of regularity is at least \( C^1 \) on \( \mathbb{R}^2 \). We know that, up to an appropriate rescaling of time, system (2) defines a dynamical system on the plane, that is for which solutions are uniquely defined for all times. Practically, only a restriction of it will be however considered, namely a restriction to a compact, simply-connected subset \( U \) of the plane which will be identified to the cylindrical projection upon that plane of the inner surface of the last phalanx of the finger in question (cf. chapter III of Galton’s book on methods of printing). For preliminary modelling, by mathematical representation we roughly mean existence of a sufficiently smooth, not necessarily computable, one-to-one transformation, by means of which trajectories of (2) on the “restricted phase space” \( U \) could be mapped onto a simplified version of the fingerprint’s streams of ridges, i.e. the emerging fingerprint’s general orientation feature and corresponding “trajectories” obtained, for example, via a gradient-based or, in a more sophisticated way, a filter-based method. In fact, a large literature is available on the subject of computer-based methods for fingerprints features extraction, but for our sense of vision, whose accuracy in pattern recognition cannot in principle be matched by any computer-based method, Galton already pointed out that when contemplating a fingerprint, the (unaided) eye is guided merely by the general appearance, while actually the object under study can be much more complex. Moreover, what may still bias the study is that a complex pattern is capable of suggesting various readings, as the figuring on a wall-paper may suggest a variety of forms and faces to those who have such fancies. A simplified version, whatever it may be, remains in fact a purely subjective notion. This, however, does not prevent it from being of a certain utility for primary classification purposes, and as we shall see, even in simplified form, and no regard being paid to the fact whether the wanted transformation is preserving or not orientation along trajectories, it will make it certain in most cases that finality will never be perfectly reached, as the following subsection shows.

\(^5\)The term smooth deformation of phase portraits will be used in the remaining part of the paper. Continuity of the transformation and its inverse is the minimum required. However, as we shall see, to preserve the nature (but not necessarily the position) of each singular point, the degree of regularity has to be increased when a classification up to homeomorphism (topological equivalence) cannot distinguish between two singular points, as for example a Node and a Focus.

\(^6\)We don’t give a survey as it would load these pages too heavily to present such technical methods here.
4.1 Searching for the Delta

To model the last six categories of fingerprints according to Purkyně’s standards, and besides core points which will be discussed later, a quite obvious (or at least what it seems to be) singular point is the Delta, as called in Henry’s classification and appears in Fig.9 such a point being a basic singular point in impressions of the Loop and Whorl types. It may be formed either by the bifurcation of a single ridge, or by the abrupt divergence of two ridges that hitherto had run side by side. And as pointed out by Galton, following Purkyně, in the Spiral, the Ellipse, the Circle and the Double Whorl, “triangles” may be seen at the points where the divergence begins between the transverse and the arched lines, and at both sides. In the language of dynamical systems, a Delta can be roughly seen as a singular point with three hyperbolic sectors. So, the question is the following: is there any singular point with three hyperbolic sectors?

Of course, in the first species class of singularities, there is no such point. Note that a Saddle is a point with (a deleted neighborhood consisting of) four hyperbolic sectors. As for singular points of the second species, at least those from Fig.7, none looks like a Delta. In fact, as said before, without searching to draw up an exhaustive list, such a point cannot exist as it follows immediately from Bendixson formula above:

**Proposition 4.1** If a singular point of system (2) has three (and in general, an odd number of) sectors, they can not all be of the hyperbolic type.

To take a closer look, the curious reader can suppose given a singular point with three hyperbolic sectors and a defined direction of the flow for one of the three sectors. He will then necessarily find himself faced with incompatibly
oriented trajectories, that is a situation where opposing tangent vectors appear in an infinitely small region of the phase space. Mathematically, denoting by $S_i$, $i = 1, 2, 3$, the three separatrices, there exist $i \in \{1, 2, 3\}$, $y \in S_i$ and $\varepsilon > 0$ such that, for any neighborhood $U$ of $y$, there exist $\delta > 0$ and $z \in U$ satisfying

$$\|x(t, y) - x(t, z)\| > \varepsilon \text{ for all } t \in [-\delta, \delta].$$

$x(t, y)$ being the solution of the initial value problem

$$\begin{cases}
\dot{x} = f(x) \\
x(0) = y
\end{cases}$$

and $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^2$. This contradicts the regularity of the vector field $f$ and then the fundamental theorem on dependence on initial conditions (and parameters, eventually) which, being based on Gronwall’s lemma, states that this dependence is - roughly speaking - as continuous as the function $f$. The general result for an arbitrary odd number $n > 3$ of hyperbolic sectors holds by the same reasoning.

The fact that there is no three-hyperbolic-sector point in smooth dynamical systems is at the same time frustrating and fascinating. Frustrating because the modelling process, still at an embryonic stage of realisation, seems to be blocked. Fascinating when you realise that life has a lot more imagination in printing than a differential equation can do!

### 4.2 The Cusp, a faulty point but the best available

As it has been mathematically shown, no singular point can match the Delta. To overcome the problem, a well-known clever trick consists in doubling the fingerprint’s orientation field, thus transforming a Delta into a Saddle and vice versa, after reconstruction. The same idea works for a special type of cores, namely what is called “Single” by Galton, transforming it into a Focus, which is undoubtedly one of the best points to be securely distributed by two Saddles, i.e. without changing its nature. However, a Focus is not always to be expected from an orientation doubling if the rich variety of cores is taken into account (see Fig.10), thus showing the limitation of this technique. Trying to see if the task can otherwise be achieved, and as a three-sector singular point, on could have thought for instance of the Saddle-Node as an approximation; however, as can be easily seen from the structure of the Spiral or the Elliptical/Circular Whorl, the parabolic sector of the Saddle-Node is not appropriate to approach.

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7 At the beginning of chapter IV on ridges and their use, Galton talks about minute ridges that bear a superficial resemblance to those made on sand by wind or flowing water. Indeed, it is a deceptive resemblance as it is known for example in fluid dynamics, and unless there is an obstacle, there can be no delta-like motion in a natural occurring flow.

8 The title is inspired by the paragraph “Fraternity, a faulty word but the best available”, from Galton’s chapter XI on heredity.
the center of the Whorl. In fact, in this quest for the Delta, it becomes more and more certain that the goal will never be reached by the path hitherto pursued, namely, to seek at all costs to identify the Delta (and any pattern in general) to the “whole” of a singular point. I then had the idea of considering only a “portion” of it, and I almost immediately realised that, for example, from a well-chosen cut of the Cusp, it emerges something close to a Delta, as appears in Fig. 12. Then, as a preliminary modelling of the corresponding fingerprint, one can place two Cusps “face to face” on the boundary of a finger-shaped domain and put in the middle a Center for the Elliptical/Circular Whorl, and a Focus for the Spiral. The result is the following:

4.3 Preliminary modelling of the Circular/Elliptical Whorl, the Spiral, the Arch and the Loop

The Circular/Elliptical Whorl and the Spiral

In Fig. 12 (in which, as well as in all sketched phase portraits, and to better imitate a fingerprint, solution curves are deliberately represented by irregular discontinuous lines), we have two (partial) local phase portraits corresponding to a Cusp, and a local phase portrait corresponding to a Center. These three local phase portraits have to be connected to obtain a global phase portrait, whence the notion of connexion mentioned in the introduction. Intuitively, this fact can be achieved only if the curves on either side of the Center meet symmetrically one-to-one to form closed orbits, the separatrices acting as heteroclinic orbits joining the Cusps and forming what is called a separatrix cycle. We obtain in fact a similar configuration as for the stable equilibrium of the undamped pendulum, where the Center is now served by Cusps instead of Saddles. As for the Spiral in Fig. 13, a possible configuration is simply that of Fig. 1, but with Cusps instead of Saddles, the origin being a global attractor for all trajectories, except those corresponding to the stable manifolds of the Cusps. Another modelling phase portrait which, up to a smooth deformation, and if we exclude phase portraits with (bifurcating) limit cycles, seems to be the only other possibility for a configuration where a Focus is trapped between two symmetric Cusps, is that for which the separatrix cycle is approached by the unbounded curves and the inner spiral as a limit set (the set of cluster points of the forward/backward orbit), no regard being paid to the direction of the “twist” or to the number of “turns”. Theoretically, the state of the limit set is reached after an infinite number of forwards or backwards turns. As the case may be, we will have a stable, unstable or semistable separatrix cycle. It may be asked why the well-known phase portraits corresponding to an undamped pendulum or to a simple case (zero drive strength) of the damped pendulum have not been directly con-

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9 According to Definition 4.1 (cf. paragraph 4.4), I am currently trying, however, to identify a family of Saddle-Node points arising in the codimension-two Bogdanov-Takens bifurcation (Fig. 11) $\mu_1 = 0$ with some parted cores (Fig. 10, 44–46).

10 Such a cut can be justified only if one agrees that the hidden area can be easily guessed and topologically reconstructed for all types of fingerprints.
Figure 10: Cores to Loops, as captured from Plate 8 of Galton’s book. At top left, the Single.
Figure 11: As captured from Perko’s book: bifurcation set and corresponding phase portraits for the system $\dot{x} = \mu_1 + \mu_2 \dot{x} + x^2 + \ddot{x}$, where TB, H, SN and HL being for Takens-Bogdanov, Hopf, Saddle-Node and Homoclinic-Loop bifurcations, respectively.
Figure 12: Preliminary modelling of the Circular/Elliptical Whorl.
Figure 13: Preliminary modelling of the Spiral.
sidered as preliminary models for the Elliptical/Circular Whorl or the Spiral, respectively. The reason in fact why I did not adopt such a configuration is that, unlike a Cusp, Saddle' separatrices do not meet tangentially as required by the enveloping ridges of a pattern, a behaviour that can be easily observed from a close-up of the Delta in Fig.9. Besides, considering a Saddle instead of a Cusp supposes neglecting a whole hyperbolic sector, whereas the two sectors clearly appear in the case of a Cusp, only small portions of (eventually closed) curves have been ignored. Letting Galton comment on these figures, he would probably say: What seemed before to be a vague and bewildering maze of lineations over which the glance wandered distractedly, seeking in vain for a point on which to fix itself, now suddenly assumes the shape of a sharply-defined phase portrait. 

The question that remains however is the following: Given in general fragments of phase portraits, how to rigorously carry out (all possible cases of) a connexion? And once it has been accomplished, how to validate it, that is to construct algebraically a dynamical system whose phase portrait is a smooth deformation of the connexion? This is the question to which we try to provide some elements of answer in section 5. In the following, we resume the preliminary modelling for the simple cases in Arches and Loops, namely the Transverse flexures, the Central Longitudinal Stria and the Oblique Sinus, where the Cusp will be found to give considerable help. The remaining cases of the Oblique Stria, the Almond, the Composite Spiral (in a sense to be specified) and the Double Whorl will be discussed in the next subsection.

The Arch

For fingerprints of the Arch type, especially those with the transverse flexures for example, where ridges are arranged transversely in beautiful order, and as a preliminary modelling, a straightforward topological similitude can be made with straight curves running transversely from one side of the phalanx to the other. In terms of the well-known rectification theorem from dynamical systems, the topologically equivalent phase portrait can be seen as a magnification of the flow near a nonsingular point \(x\), after rectification, that is after having applied a change of coordinates for a region around \(x\) where the vector field \(f\) becomes a series of parallel vectors of the same magnitude. In other words, a Plain Arch can be merely seen as a smooth deformation of a line. However, at another level, a pattern of Arch type can be a bit more complex, as for example it is the case for the Central Longitudinal Stria, where the configuration is nearly the same as in the previous case, but with a small difference: a perpendicular stria is enclosed within the transverse furrows, as if it were a nucleus. In Galton’s language, this case, extremely rare on the thumb, can be included within the Tent Arch for which an approximate, but apparently fairly correct, phase portrait can be expressed by a Cusp (Fig.14).

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11In fact, in [1, p.69], Galton was speaking about the outlines of patterns and how they can be accurately drawn. I just took the liberty to replace “figure” with “phase portrait”!

12I do not remember ever to have seen it there, says Galton in [1, p.75].
Figure 14: Tented Arch on male finger (Wikimedia Commons) and associated phase portrait.

The Loop

For the Oblique Sinus, where an oblique line recurves towards the side from which it started, accompanied by several others, all recurved in the same way, and as a preliminary modelling within the principal head of Loops, a simplified version can be seen as a smooth deformation of a phase portrait where a Cusp cohabits with an infinite singular point\(^\text{13}\) with (at least) an elliptic sector. To be placed along the vertex of the oblique sinus of the furrows distribution, the specific need of an infinite point comes from the fact that the unbounded Cusp’ separatrices have to be flexed in order to cover the recurve lines as enveloping ridges. And for the elliptic sector, as the name suggests, it seems that no solution curve can simulate the shape of a loop better than that of an elliptic sector. The general feature of a Loop’s phase portrait is sketched in Fig.15.

The main complaint one might have concerning this model, however, is the lack of the core point. For a more accurate phase portrait, in the case of a Single core, another adjustable slightly sloping Cusp can be integrated within the picture. Otherwise, getting even more accurate, but requiring a larger phase space, an interesting idea to be elaborated in future work is proposed in subsection 4.4.3, where the same problem is encountered in the Double Whorl modelling.

\(^{13}\)With all due respect to Hermite who, unlike Poincaré, opposes a vocabulary which he finds too colourful, like points at infinity in projective geometry!
Figure 15: Preliminary modelling of the Loop.
4.4 Preliminary modelling of the Oblique Stria, the Almond and the Composite Spiral

The Oblique Stria

The pattern is defined as Transverse flexures where a solitary line runs from one or other of the two sides of the finger, passing obliquely between the transverse curves, and ending near the middle. In the language of dynamical systems, this can be achieved by considering (a smooth deformation of) the phase portrait associated to the Tented Arch, but with a deleted separatrix for the Cusp. Doing so, and for reasons that will become clear later, we introduce a modified definition of what will be meant from now on by a phase portrait, whose a smooth deformation is to be associated to the ridge flow of a fingerprint:

Definition 4.1 Given a compact, simply-connected domain $U \subset \mathbb{R}^2$, and a nonempty finite subset $V$ of $U$, a phase portrait on $U$ with respect to $V$ is defined to be the set

$$P = \{ \varphi_{\pm t}(x), \ t \geq 0, \ x \in V \} \cap U$$

where $\varphi_t$ is the (one-parameter) flow generated by a vector field $f \in C^1(\mathbb{R}^2)$, and the symbol $\pm$ means that, for an initial condition $x$ and positive times $t$, one has the choice to consider the solution curve either of the system $\dot{x} = f(x)$ or $\dot{x} = -f(x)$.

Remember, $U$ is practically identified to the cylindrical projection upon a plane of the inner surface of the last phalanx of the finger at issue; it has been roughly considered so far as a restricted phase space in the preliminary modelling of fingerprints. As for the underlying vector field, it should be stressed that $f$ is defined on the whole plane, the modelling phase portrait being created by the flow generated by the restriction of $f$ to $U$, according to Definition 4.1. In this regards, it does not matter how $f$ behaves outside $U$ as long as it works inside, i.e. it gives the desired phase portrait on $U$.

Now, according to Definition 4.1, any initial condition can be chosen to generate an Oblique Stria phase portrait, except a point located on the Cusp’ separatrix to be ignored. Especially, denoting by $W^s$ (resp. $W^u$) the stable (resp. unstable) manifold of the Cusp and by $\partial U$ the boundary of $U$, if we let $x_0$ be a nonsingular point so that

$$W^s \cap \partial U = \{ x_0 \} \quad \text{(resp. } W^u \cap \partial U = \{ x_0 \} \),$$

the trajectory

$$T^+ = \{ \varphi_t(x_0), \ t \geq 0 \} \quad \text{(resp. } T^- = \{ \varphi_t(x_0), \ t \leq 0 \}$$

will stand for a smooth deformation of the oblique stria (the ridge, not the pattern), depending on which side of the finger the solitary line is running from.
The Almond

What is called Almond by Purkyně and described as an Oblique Sinus enclosing an almond-shaped figure, blunt above, pointed below, and formed of concentric furrows, is in fact a compound pattern called Circlet in Loop by Galton, or, as the case may be, Spiral in Loop. As pointed out by Galton, Whorls enclosed within Loops are by far the commonest pattern among the compound category. So, trying to encrust a whorled pattern into a Loop, remember the interesting cases we have met in the codimension-two Bogdanov-Takens bifurcation (Fig.11, $\mu_1 < 0$), where a Focus is connected to a Saddle, directly by a heteroclinic orbit (Fig.16), via a homoclinic loop or a limit cycle, a configuration which can be used for the Spiral in Loop. The case of the Circlet in Loop can be approached in the same manner where a Center is connected to a Saddle via a homoclinic loop (Fig.17). To complete the picture, a Cusp has to be added, where the lower separatrix has to be connected to an infinite Node in order to delimit the pattern, and according to Definition 4.1, a Saddle’ separatrix has to be deleted, when some trajectories (within the same phase portrait) have to be drawn backwards.
Figure 17: Preliminary modelling of the Circlet in Loop.
The Composite Spiral

Remember, the term “Spiral” has been used so far in the usual geometric sense. However, as pointed out by Galton, if the term conveys a well-defined general idea, there are four concrete forms of it which admit of being verbally distinguished: the (simple) Spiral, the Twist, the Plait and the Deep Spiral. In addition to Circles and Ellipses, they appear as Cores to Whorls in Plate 8 - Fig.15 of Galton’s book (see Fig.18).

In Purkyně’s Commentatio, only the Spiral and the Twist are considered as spirals, the former being classified as simple and the latter as composite. The Plait is called Double Whorl therein and no reference seems to be made to the Deep Spiral. Trying to identify each of the “complex” spirals with a singular point, I realised that they were ranked in ascending order of difficulty, as if Galton had already tried to identify them with a phase portrait! In fact, I could not do anything for the Deep Spiral, however, for the Twist, and to a lesser extent the Plait, two local phase portraits have been found to be of some help, as will be explained in the following.

The Twist  The spiriform of the pattern is described by Purkyně as being made up of several lines proceeding from the same centre, or of lines branching at intervals and twisted upon themselves. The best I could do for such a pattern is to draw the phase portrait of an Improper Node, a singular point known for being a transitional case between a Node and a Focus. In fact, algebraically, for linear differential planar systems (Fig.6), the Improper Node is located on the parabola \( \tau^2 - 4\Delta = 0 \), for which a double root is exhibited by the characteristic polynomial, passing continuously from the case of two distinct roots with the same sign to complex conjugate ones. And as said before, this is the reason why a homeomorphism cannot distinguish between a Node and a Focus. Geometrically, through the Maple-drawn phase portrait (Fig.19) and the hand-drawn preliminary model (Fig.20), the judgement is left to the reader on whether an Improper Node can be a good candidate for a Twist or not. Finally, to obtain the whole picture, we don’t forget to place (and connect) the Improper Node between two Cusps, as said by Purkyně when describing his Composite Spiral: At either side, where the spiral is contiguous to the place at which the straight and curved lines begin to diverge, in order to enclose it, two triangles are formed, just like the single one that is formed at the side of the Oblique Sinus.
Figure 19: Computer-drawn phase portrait of an Improper Node.
Figure 20: Preliminary modelling of the Twist.
The Double Whorl, or the need to increase dimension  Also a Plait (and sometimes an Overlap) for Galton, it is described as a curious effect where two systems of ridges that roll together, end bluntly, the end of the one system running right into a hollow curve of the other, and there stopping short; it seems, at the first glance, to run beneath it, as if it were a plait. This mode of ending forms a singular contrast to the Spiral and the Twist, where the ridges twist themselves into a point. And for Purkyně, one portion of the transverse lines runs forward with a bend and recurses upon itself with a half turn, and is embraced by another portion which proceeds from the other side in the same way. This produces a doubly twisted figure which is rarely met with except on the thumb, fore, and ring fingers.

After numberless observations, I failed completely in trying to translate this in terms of solution curves of a planar differential system; but I have been vaguely conscious of remembering that a similar figure can be obtained on a Poincaré section when considering a two-degree-of-freedom Hamiltonian, i.e. a four dimensional conservative system. Indeed, such a type of behaviour can be encountered in classical mechanics when dealing, for example, with the Lagrange problem of equal masses, a special case of the well-known restricted planar three-body problem when Coriolis’ force is neglected. Without going into details, the Double Whorl, roughly expressed in Fig.21, is taken from an international postgraduate course given by A. Deprit in August 1960 at the Université Libre de Bruxelles. The plait-like behaviour is undergone by a special family of solution curves on the invariant \((x, y)\) plane for the system with the Lagrangian function

\[
L = \frac{1}{2}(\dot{x} + \dot{y}) + \frac{1}{4}(r_{13}^2 + r_{23}^2) + \frac{1}{2}(1/r_{13} + 1/r_{23})
\]

\(r_{13}\) being the distance of the moving particle to the origin and \(r_{23}\) the distance between the two masses within the Keplerian motion. The mode of ending perfectly described by Galton corresponds in fact to the position of the equal masses, and the two systems of ridges that roll together are portrayed by a family of six asymptotic orbits, beginning with an ejection orbit from one of the two masses, passing through orbits that have missed the double-shock, thus continuing to revolve around that same mass, ending finally with an ejection orbit from the other mass. At another energy level, solution curves behaviour could be also of some help for the preliminary modelling of the Loop, where the missing core in Fig.15 is now portrayed by a collision orbit with one of the two masses, then reaching back to join asymptotically one of the five well-known Lagrange points (Fig.22). However, to complete the picture, and if all is confined in an ellipse with the equal masses positions as focal points, I don’t know if there exist off-ellipse initial conditions leading to a (partial) delta-like motion, let alone how to proceed to a connexion. Carrying out a connexion from given data in the general case is the main subject of the following section, and as can be suggested for

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14 As a young researcher, I have met the late Prof. André Deprit for the first time in the summer of 1998 in Prague. The private communications that I had the honor to tie with him, mainly on normal form theory and celestial mechanics, have left their imprint until today.
future work, besides index theory on a two-dimensional surface and multivariate interpolation theory, bifurcation theory is another framework within which the problem could be formulated and dealt with. More precisely, the rather fascinating question of studying all possible connections of local phase portraits could be seen as a bifurcation subproblem, i.e. a global bifurcation problem with the constraint that all involved singular points are preset to predetermined geographical positions and behavioural natures.\footnote{Obviously, the term bifurcation, at least in its dynamical sense, appears nowhere in Galton’s book, but when the author talks about sets of concentric circles or ellipses, pointing out in p. 77 that \textit{they are rarely so in a strict sense throughout the pattern, usually breaking away into a more or less spiriform arrangement}, we dare to wonder if he was already anticipating the well-known Andronov-Hopf bifurcation! Better yet, the \textit{transitional cases} concisely described by Galton, as those between a Tented Arch and a Loop, could be expressed in a dynamical context as a bifurcation problem.}

Figure 21: \textit{A plait-like behaviour of trajectories on a Poincaré section from a special case of the restricted planar three-body-problem.}
Figure 22: A loop-like behaviour of trajectories on a Poincaré section from a special case of the restricted planar three-body-problem. Compare this pattern to the one on César Baldaccini’s Thumb from the Theme of the Hand exhibition!
Part II
Connecting singular points within structure-preserving normal forms

5 Some directions in connecting phase portraits

Recapitulating, the way hitherto pursued in modelling consisted in locating basic fingerprint patterns, identifying them to (portions of) local phase portraits, then, guided by intuition making a connexion to obtain a global phase portrait. Although this results in interesting phase portraits, it must be recognised that the whole process is a craftsman’s task, and no systematic approach has been proposed. Besides, there is no guarantee for the correspondence via a smooth deformation between the object thus constructed and the phase portrait of an explicitly written (in terms of elementary functions) dynamical system. As for the problem of existence of such a phase portrait in the general case, i.e. for any number (and nature!) of singular points, nothing is known so far, especially in case we fail in the connexion process. In the next paragraph, we deal with the problem within Poincaré’s index theory to show that a connexion is generally impossible, but when it exists, obviously, a connexion is not necessarily unique, as it will be shown through a simple illustrating example. In this regards, some results from index theory will be briefly recalled to deal with the question of existence in the general case, and as it will be seen, the explicit construction of the underlying dynamical system, i.e. the ultimate goal of associating a (smooth deformation of a) differential system’s phase portrait to a fingerprint, according to Definition 4.1, leads to a dreadful and generally insoluble problem.

5.1 Index theory revisited

5.1.1 Towards Poincaré index theorem

To give a complete account of the theory would require a chapter, so I have opted for a sketch. Following Poincaré, we recall the definition of the index

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16By intuition, we allude to all these rules from dynamical systems to be respected when drawing up phase portraits, beginning with the fundamental theorem of existence and unicity, and arriving at the most elaborate theories, like Poincaré’s index, consequents, contacts, centers and limit cycles theories [9-11]. For example, it would be inconceivable to draw a cycle without any singularity inside, to delimit a Center by a limit cycle, or to try to connect a Node and a Focus without resorting to other singular points.

17A link to smooth deformations could be made from Galton’s work, as for example when he says in p. 75: *Perhaps the best general rule in selecting standard outlines, is to limit them to such as cannot be turned into any other by viewing them in an altered aspect, as upside down or from the back, or by magnifying or deforming them, whether it be through stretching, shrinking, or puckering any part of them.*

18As the reader will notice, the purpose for which the theory is recalled, i.e. the problem of connecting local phase portraits, has nothing whatever to do with the way Poincaré’s index is used for singular points extraction from fingerprints coarse orientation fields.
$I_f(x_0)$ of an isolated singular point of a $C^1$ vector field $f$, defined on an open subset $U$ of $\mathbb{R}^2$, as being the index $I_f(C)$ of any Jordan curve $C \subset U$, containing $x_0$ and no other singular point of $f$ on its interior, which is given by

$$I_f(x_0) = I_f(C) = \frac{\Delta \theta}{2\pi}$$

$\Delta \theta$ being the total change in the angle $\theta$ that the vector $f = (f_1, f_2)^T$ makes with respect to the $x$-axis, i.e. the change in

$$\theta(x, y) = \arctan \frac{f_1(x, y)}{f_2(x, y)}$$

as the point $(x, y)$ traverses $C$ exactly once in the positive direction. Explicitly, this can be computed as

$$I_f(x_0) = \frac{1}{2\pi} \oint_C \frac{f_1 \, df_2 - f_2 \, df_1}{f_1^2 + f_2^2}$$

In index language, with the same notation, the previously mentioned Bendixson result (on the number of elliptic and hyperbolic sectors of a singular point) can be stated for analytic $f$ as follows:

**Theorem 5.1 (Bendixson)**

$$I_f(x_0) = 1 + \frac{e - h}{2}$$

It follows that, for instance, in the nondegenerate case, $I_f(x_0)$ is $-1$ or $+1$ according to whether the singular point is or is not a (topological) Saddle. Note also that the index of a Saddle-Node is zero.

With respect of the vector field $f$, the index theory is extended to a two-dimensional surface $S$ (i.e. a compact, two-dimensional, differentiable manifold of class $C^2$, or nappe as would say Poincaré in French), on which $f$ is supposed to have a finite number of singular points $x_1, ..., x_m$. $I_f(S)$ is then defined as the sum of the indices at each of the singular points:

$$I_f(S) = \sum_{i=1}^{n} I_f(x_i)$$

$I_f(x_i)$ being defined relatively to the restriction of $f$ to some chart, into the detail of which it is unnecessary here to enter.

As shown by Poincaré [11] in chapter XVIII, following a work in two parts of Kronecker [25], it is one of the most interesting facts of the index theory that $I_f(S)$ is independent of the vector field $f$ and only depends on the topology of the surface:
Theorem 5.2 (Poincaré Index Theorem)

\[ I_f(S) = \chi(S) \]

where \( \chi(S) = T + v - l \) is the Euler characteristic associated to a decomposition of \( S \) into a number \( T \) of curvilinear triangles, with a number \( v \) of vertices and a number \( l \) of edges.

It can also be shown that

\[ \chi(S) = 2(1 - p) \]

where \( p \) is the genus of \( S \) (i.e. the maximum number of nonintersecting closed curves than can be drawn on \( S \) without dividing it into two separate regions), thus leading to the topological invariance of \( \chi(S) \) and then of \( I_f(S) \). As examples we have \( I(S^2) = 2 \) for the sphere, \( I(T^2) = 0 \) for the two-dimensional torus, \( I(P) = 1 \) for the projective plane or \( I(K) = 0 \) for the Klein bottle.

An immediate consequence of the Poincaré Index Theorem is the following:

Corollary 5.1 (Poincaré) Suppose that \( f \) is an analytic vector field on an analytic, two-dimensional surface \( S \) of genus \( p \) and that \( f \) has only hyperbolic singular points, i.e. isolated Saddles, Nodes and Foci, on \( S \). Then

\[ n + f - s = 2(1 - p) \]

where \( n, f \) and \( s \) are the number of Nodes, Foci and Saddles on \( S \) respectively.

5.1.2 Link to the connexion problem

Now, as an answer to the existence problem of a connexion, at least in the hyperbolic case, it can be easily seen that for a given surface and a set of Nodes, Foci and Saddles, which the respective numbers do not satisfy condition of Corollary 5, it is impossible to carry out a connexion. For the nonhyperbolic, eventually degenerate, case, any collection of singular points whose the sum of indices does not match the Euler characteristic of the underlying surface can not be connected. In other terms, a connexion of index-theory-incompatible singular points with respect to a surface is simply a faulty phase portrait on that surface. As for uniqueness, we simply consider a portion of a phase portrait made up of local phase portraits from a Saddle, a Node and a Focus, but in three qualitatively different ways (Fig.23).

In (a), the Focus is joined by heteroclinic orbits from the Node and the Saddle, the laters being themselves joined by two heteroclinic orbits corresponding to the stable manifolds of the Saddle; in (b), the Focus is enclosed within an homoclinic orbit corresponding to a separatrix cycle; and in (c), the Focus is
surrounded by an unstable limit cycle. The three phase portraits, obviously, are not topologically equivalent.

As for the general existence problem, it seems mathematically possible, but highly unlikely, that a (large) number of pre-assigned singularities, randomly distributed on a surface, yet index-theory-compatible, can be connected. In other terms, we have a necessary condition from Poincaré Index Theorem which seems to be sufficient (to build a global phase portrait) only up to a redistribution of the singular points on the surface. The second part of the problem is, once a connection is proved possible from the given data, find an explicitly written differential system whose phase portrait is a smooth deformation of one of the possible connections. Such a reverse problem from index theory will be explicitly stated in the next subsection as an interpolation problem. Consequently, whenever possible, fingerprints discussed in section 4 will be assigned a simple planar differential system as a primary model within the ALW classification.

5.2 A bivariate interpolation problem

5.2.1 Robustness of singular points against addition of nonlinear terms

Let there be an open subset $U$ of the plane in which a finite number $n$ of pre-assigned singular points $x_i$ have been successfully connected. The problem of finding an explicit dynamical system

$$\dot{x} = f(x), \ x \in \mathbb{R}^2$$

with the same singularities on $U$ and whose phase portrait is a smooth deformation of the connection in the vicinity of each singular point (but not necessarily

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19 These phase portraits are taken from an anonymous pdf document; I apologise to the unknown author(s) for using these drawings to illustrate the non-uniqueness of a connexion.

20 By pre-assigned singularities, we mean data singular points which are intrinsically considered as such, i.e. as independent geometrical objects, or as local phase portraits, like those in Fig[4], no regard being paid for the moment to the explicit form of vector fields exhibiting such singularities.
the case elsewhere), amounts to find a closed-form sufficiently smooth (say, analytic) vector valued function $f$ whose zeros in $U$ are exactly the $x_i$’s and Taylor polynomial at $x_i$, given with a sufficiently high degree, defines a (local) differential system whose trajectories exhibit the same behaviour as already pre-assigned for $x_i$, for all $i = 1, ..., n$. By sufficiently high degree, we mean the smallest order at which $f$ should be expanded so that the nature of the singular point would be completely decided, and no additional high-order terms can destroy this nature. In fact, as well-known within structural stability, under the addition of smooth nonlinear terms to a linear system with a (global) Center, the singularity may become a Focus or even a Center-Focus (in the nonanalytic case). Besides, a homeomorphism does not distinguish between a Node and a Focus, where a Saddle remains a (topological) Saddle under continuously differentiable perturbations, as follows from the Hartman-Grobman Theorem (1959). However, it follows from a result shown in Coddington and Levinson [20] (resp. a theorem by Hartman (1960)) that a Focus (resp. a Node) remains diffeomorphically unchanged under the addition of (resp. twice) continuously differentiable nonlinear terms. This, indeed, is the reason why the approach by Li and Yau [15] works, cores and Deltas being respectively modelled by Foci and Saddles. As for the Center, a large part has been devoted to the subject by Poincaré [11], chap. XI, where necessary and sufficient (yet difficult to implement) conditions were derived to ensure the preservation of the center-like nature. Following Poincaré, the simplest sufficient condition for a nonlinear planar system to preserve the center nature is a symmetry with respect to one or both of the axes, and this is probably the reason why the approach by Ford [14] works for typical flow configurations in fluid dynamics. Therefore, recapitulating, if we restrict ourselves to the nondegenerate case, given a distribution of Centers $C_i$, Foci $F_i$, Nodes $N_i$ and Saddles $S_i$ in $U$, it is clear that if there exists a symmetric and twice continuously differentiable vector field $f$, vanishing in $U$ only at these points, and for which $C_i$ (resp. $F_i$, $N_i$, $S_i$) is a Center (resp. Focus, Node, Saddle) of the linearised vector field (i.e. defined by Taylor polynomial of degree 1), then the phase portrait of the flow generated by $f$ stands for a connexion of the pre-assigned data of singular points. A simple example in the nondegenerate case is given below.

5.2.2 Poisedness of the problem

Let begin by considering the following actual situation: given a configuration in the plane $(x, y)$ where a Center is placed at the origin and two Saddles at the opposite points $\pm(\pi, 0)$, find a connexion in the square $U = \{(x, y) \in \mathbb{R}^2 / |x| \text{ and } |y| \leq 4\}$. Mathematically, this amounts to find a symmetric system,

$$
\begin{cases}
\dot{x} = f(x, y) \\
\dot{y} = g(x, y)
\end{cases}
$$

38
that is a system which is invariant under the transformation \((t, y) \rightarrow (-t, -x)\) and/or \((t, y) \rightarrow (-t, -y)\), where \(f\) and \(g\) are \(C^2\)-functions such that
\[
f(0, 0) = f(\pm \pi, 0) = g(0, 0) = g(\pm \pi, 0) = 0 \quad (3)
\]
and there is no other points of \(U\) in which \(f\) and \(g\) vanish simultaneously. Moreover, (cf. diagram of bifurcation, Fig.6),
\[
det(J_{\pm \pi}) < 0, \quad \text{tr}(J_0) = 0, \quad \det(J_0) > 0, \quad (4)
\]
\(J_{\pm \pi}\) and \(J_0\) being the Jacobian matrices at \(\pm (\pi, 0)\) and the origin, respectively.

As can be seen, the problem can be solved for \(f(x, y) = y\) and \(g(x, y) = -\sin x\). It can also be shown in fact that the phase portrait of the flow generated by such a vector field is - up to a smooth deformation - the unique connexion for a Center and two Saddles. However, as the reader has certainly noticed, we purposely considered a problem of which we already know the solution, namely the phase portrait of the undamped pendulum. But in a more general context, condition (3) can be seen as a bivariate Lagrange interpolation problem which is poised in the Haar space \(C^1(U)\), i.e. (allowing some flexibility in the definition) it can be solved in \(C^1(U)\) for any given data of isolated points in \(U\), but not in the subspace of \(C^2(U)\) of symmetric vector fields. As for conditions (3-4), they can be seen as an advanced bivariate Hermite interpolation problem which is not poised in \(C^1(U)\), as follows from index theory. It should be stressed that, for both problems, however, we are not necessarily dealing with polynomial interpolation on the one hand, and on the other, no values have been assigned to the partial derivatives of \(f\) and \(g\) at the singular points, only a special distribution in the trace-determinant plane of the Jacobian matrices is required.

The situation is dramatically different in the degenerate case, where high-order derivatives are required to decide on the nature of the singular point. And it is even more difficult to formulate such a problem in the general case where a mixture of degenerate and nondegenerate points are given as connected pre-assigned data. As will be shown, however, the problem can be solved in some special situations where a restrictive number of singularities are considered, as for the configurations “one Center/Focus/Improper Node - two Cusps”, but no simple form has yet been found for the Loop (due to the interpolation point at infinity) and the Double Whorl (due to the narrowness of the phase plane to harbour such a behaviour).

### 5.3 Introducing normal forms for fingerprints
#### 5.3.1 A normal form for nilpotent planar systems

A starting point to solve the interpolation problem for a primary modelling of fingerprints is to find the “simplest” class of parameter-dependent planar vector fields which cover the widest possible set of singular points. For the
approximating phase portraits given in Section 4, such a simple class is mainly supposed to cover the Focus, the Center, the (Improper) Node and the Cusp. An interesting result in this regards was shown by Andronov et al. [24] and is reported here from Perko [22]:

Let assume that the origin is an isolated singular point of the planar system

$$\begin{align*}
\dot{x} &= P(x, y) \\
\dot{y} &= Q(x, y)
\end{align*}$$

(5)

where $P$ and $Q$ are analytic in some neighborhood of the origin. To cover the Cusp, we consider the case when the Jacobian matrix $J$ at the origin has two zero eigenvalues, but $J \neq 0$. Following Andronov et al., system (5) can be put in the simple form, called normal form:

$$\begin{align*}
\dot{x} &= y \\
\dot{y} &= a_kx^k(1 + h(x)) + b_nx^ny(1 + g(x)) + y^2R(x, y)
\end{align*}$$

(6)

where $h(x)$, $g(x)$ and $R(x, y)$ are analytic in a neighborhood of the origin, $h(0) = g(0) = 0$, $k \geq 2$, $a_k \neq 0$ and $n \geq 1$. Then we have the following:

**Theorem 5.3** [24] Let $k = 2m + 1$ with $m \geq 1$ in (6) and let $\lambda = b_n^2 + 4(m + 1)a_k$. Then if $a_k > 0$, the origin is a (topological) Saddle. If $a_k < 0$, the origin is (1) a Focus or a Center if $b_n = 0$ and also if $b_n \neq 0$ and $n > m$ or if $n = m$ and $\lambda < 0$, (2) a Node if $b_n \neq 0$, $n$ is an even number and $n < m$ and also if $b_n \neq 0$, $n$ is an even number, $n = m$ and $\lambda \leq 0$ and (3) a critical point with an elliptic domain if $b_n \neq 0$, $n$ is an odd number and $n < m$ and also if $b_n \neq 0$, $n$ is an odd number, $n = m$ and $\lambda \geq 0$.

Let $k = 2m$ with $m \geq 1$ in (6). Then the origin is (1) a Cusp if $b_n = 0$ and also if $b_n \neq 0$ and $n \geq m$ and (2) a Saddle-Node if $b_n \neq 0$ and $n < m$.

### 5.3.2 Some normal forms within the ALW classification

As an interesting case, to compute a normal form for the Elliptical/Circular Whorl, that is a simple planar system whose phase portrait is a smooth deformation of the connexion given in Fig. 12, we put a Center at the origin of the $(x, y)$ plane and two face-to-face Cusps at the opposite points $\pm(1, 0)$; then, we compute a vector field of the form (6), for which the origin and the opposite points are the only singularities, and whose the respective Taylor polynomials at these points define a local differential system of the form

$$\begin{align*}
\dot{x} &= y \\
\dot{y} &= \alpha x + o(x^2)
\end{align*}$$

\footnote{In fact, at a purely formal level, simpler normal forms can be reached for nilpotent systems, i.e. systems with a nilpotent linear part (see for instance Stolovitch [27] for a Carleman-linearisation-based approach). As for the explicit computation of a normal form, it would be horrible to try to conduct by hand high-order expansions. In this regards, a Lie-series-based Maple package for symbolic computation of Poincaré-Dulac normal forms in the general case is available at Elsevier’s CPC Library [28].}
Figure 24: Normal form of the Elliptical/Circular Whorl.

for the Center, with $\alpha < 0$, and

\[
\begin{cases}
\dot{x} &= y \\
\dot{y} &= \pm \beta (x \pm 1)^2 + o((x \pm 1)^3)
\end{cases}
\]

for the Cusps, according to Theorem 6, with $k = 2$, $b_n = 0$, $h(x) = 0$ and $\beta > 0$.

To preserve the center nature of the origin, we only considered the case where $\dot{y}$ is an univariate function $f(x)$, thus obtaining symmetric systems with respect to the x-axis. Letting $f$ vanish in 0 with $f'(0) = -1$ and in $\pm 1$ with $f'(\pm 1) = 0$, $f''(1) < 0$ and $f''(-1) > 0$, leads to an univariate Hermite interpolation problem, for which a solution is given by

$$f(x) = -x(x^2 - 1)^2$$

We obtain the normal form

\[
\begin{cases}
\dot{x} &= y \\
\dot{y} &= -x(x^2 - 1)^2
\end{cases}
\]

whose a Maple-drawn phase portrait is given in Fig.24.

The same reasoning holds for the Spiral where a Focus is to be placed at the origin instead of a Center. The phase portraits to be connected are associated to the flows generated by systems of the following form:

\[
\begin{cases}
\dot{x} &= y \\
y &= \alpha x + \beta y + o(x^6)
\end{cases}
\]
for the Focus, with $4\alpha + \beta^2 < 0$, and

$$\begin{cases}
\dot{x} &= y \\
\dot{y} &= \pm \gamma (x \pm 1)^2 + o((x \pm 1)^3)
\end{cases}$$

for the Cusps, according to Theorem 6, with $k = 2$, $b_n = 0$, $h(x) = 0$ and $\gamma > 0$.

$$\begin{cases}
\dot{x} &= y \\
\dot{y} &= (\gamma + \delta y)(x - 1)^2 + o((x - 1)^3)
\end{cases}$$

Fixing $y$ and solving the corresponding univariate Hermite interpolation problem, with flexible parameters, leads to the normal form

$$\begin{cases}
\dot{x} &= y \\
\dot{y} &= (y - x/2)(x^2 - 1)^2
\end{cases}$$

whose (reversed) phase portrait looks the same as the damped pendulum’s in Fig.1, but with Cusps instead of Saddles.

I could not find, however, a normal form for the configuration in which the separatrix cycle is approached by the inner spiral as a limit set. In fact, building a separatrix cycle falls within the scope of the global theory of dynamical systems and cannot be dealt with as an interpolation problem. I proceeded by trial and error within Theorem 6, considering the Focus as a degenerate singular point ($k = 5$, $b_n \neq 0$, $n = 5 > m = 2$, $h(x) = g(x) = 0$ and $\alpha < 0$) of the system

$$\begin{cases}
\dot{x} &= y \\
\dot{y} &= \alpha(1 + y)x^5 + o(x^6)
\end{cases}$$

This led to the normal form

$$\begin{cases}
\dot{x} &= y \\
\dot{y} &= -x^3(x^2 - 1)^2(1 + y(1 + x)^3)
\end{cases}$$

but with no separatrix cycle. Global bifurcations, which are more difficult to understand than local ones, can be considered within the well-known Melnikov theory for perturbed planar analytic systems, where parameters values can be found to characterise bifurcations experienced at homoclinic or heteroclinic loops. In a more general context, however, we know that Coppel’s problem of determining all possible phase portraits for just a quadratic planar system and classifying them by means of algebraic inequalities on the coefficients is insoluble. In other terms, if finding geometrically all possible connexions - up to a smooth deformation - is already a difficult task, associating algebraically a normal form to each connexion is of another kind of difficulty.
For the Twist, where an Improper Node is distributed by two Cusps, and with the same reasoning as above, the following normal form has been found:

\[
\begin{align*}
  \dot{x} &= y \\
  \dot{y} &= (2y - x)(x - 1)^2
\end{align*}
\]

and for Plain or Tented Arches, the straight or tented curves can be merely seen as a smooth deformation of the phase portrait of the flow generated by a trivial system with zero \( y \) and constant \( x \), or, as the case may be, a simple system corresponding to a topsy-turvy Cusp, as shown in Fig.14. Finally, the lack of normal forms for Loops and for some special cases of Whorls should be noted, a question to be sought not only from a traditional planar interpolation viewpoint, but also involving infinite interpolation points or, eventually, considering the qualitative behaviour on invariant planes of high dimensional dynamical systems.

5.4 Patched phase portraits, matched fingerprints: Some perspective

Now that local phase portraits have been patched together within a structure-preserving normal form connexion, that is a connexion preserving the nature of singular points, but not necessarily their position, as for example in the case of the Elliptical/Circular Whorl, the problem is to find the simplest deformation allowing transportation from the normalised phase portrait as a source image to the underlying fingerprint’s stream of ridges as a target image (Fig.25). This idea, from the academic discipline of computational anatomy, will be developed in future work. Specifically, instead of roughly considering smooth deformations, we will be using the terminology of large deformation diffeomorphic metric mapping (LDDMM), as follows from earlier works by Christensen et al. and Trouvé on pattern matching in image analysis. An intrinsic Cartesian coordinate system will be then associated to each category of fingerprints, where, in the case of a whorled pattern for example, Deltas are maintained fixed along the \( x \)-axis and the core, say a Center, is to be displaced from the origin to its real position along the \( y \)-axis via a LDDMM. Besides Deltas and cores, other reference points (and, more generally, portions of curves) can be considered within the landmark matching problem, mainly those corresponding to the maximal curvature of the enveloping ridges of the pattern, i.e. mathematically speaking, the common separatrices of the Cusps. Many LDDMM algorithmic codes are available in the literature and should prove valuable for such a purpose, but whichever software suite is being used, the lesser the input data the better is modelling, as diffeomorphic mapping parameters have to be added to those of the normal form to optimally encode the fingerprint.

The increment in dimension, however, doesn’t necessarily mean increment in behavioural complexity. For example, no strange attractor is required in fingerprints modelling, and until proved otherwise, no individual has been identified with the Lorenz butterfly on the thumb!
Even more to be welcomed, though difficult to express in terms of elementary functions, then to implement, is to consider nonautonomous differential systems, normal form source image and fingerprint’s orientation target image being seen as screenshots of a moving nonlinear phase portrait. To encode the targeted orientation field, only model’s coefficients and exact time of simulation are required. It is then no violent misuse of metaphor to compare, for example, a moving Focus of a nonautonomous dynamical system to a moving cyclone in a meteorological forecast map. Interesting, but easier said than done!

6 Conclusion: Elementary, my dear Galton

Let us go back now to the main question stated at header level: can a fingerprint be modelled by a differential equation? Obviously, if that is meant to be a model which faithfully reproduces the fingerprint, right down to the finest detail about the papillary ridges (Fig.26), the answer is certainly a no. In fact, according to Definition 4.1, such a modelling should ideally be starting from a finite set of initial conditions, i.e. a set of reference points in the pattern, as for example those where ridges stop abruptly, then, by some fundamental law reproducing exactly the stream of ridges of the fingerprint under study. We already know, however, that if solving differential equations in terms of elementary functions is generally impossible, conversely, trying to assign an explicitly written differential equation to an observed system of curves is most often illusory. Put another way, the family album of our elementary functions is too small to allow closed-form expressions to be derived for any curve encountered in nature. As an approximation, following the work by Ford on visualisation in fluid dynamics, the least squares method can be used to adjust the coefficients of a polynomial model vector fields to minimise the difference between its integral curves and the observed flow streamline. Although practically interesting, at a purely mathematical level, some hidden problems from dynamical systems theory cannot however come to light if the modelling is entrusted to a mere least-squares-based approach. For example, as a first step for a primary classification, the choice of a diffeomorphic conjugation of the fingerprint directional field as a symmetric normalised vector field was guided by an intention to keep under control and mitigate the risk (of changing in nature) faced by nonhyperbolic singular points when proceeding to a connexion whose linearisation escapes the control of the Hartman-Grobman theorem. Of course, a symmetry condition is too restrictive for a vector field and is poorly adapted to the nonsymmetric character trait of a fingerprint orientation field. However, on the one hand, and as said by the author in, necessary and sufficient Poincaré’s conditions to ensure the preservation of the center-like nature of a

---

23 This amounts to writing down one’s unique genetic code as a differential equation!
24 Like smooth functions, which in terms of categories have been shown by Banach to be negligible for the class of continuous functions, i.e. almost all continuous functions are everywhere nondifferentiable, curves that can be expressed in terms of elementary functions, allowing special ones and any combination of them, should prove exotic among a collection of curves taken for example from a child’s drawing!
Figure 25: Landmark matching problem for the Elliptical/Circular Whorl.
Figure 26: Characteristic peculiarities in ridges, about 8 times the natural size, as captured from Plate 3 of Galton’s book.

singular point are too cumbersome so that it would be difficult to proceed to modelling within an asymmetric framework. On the other hand, it should not be overlooked that, basically, the symmetric normal form is just an example of a starting point in the modelling process, a stepping-stone which seems in every respect preferable, but any structure-preserving connexion could be safely used as a source image to be diffeomorphically carried towards the targeted fingerprint orientation image.

Another framework within which the present work could have been conducted would be a reading where curves under study are considered as contour lines of a Poincaré’s topographic system \( \text{25} \) double-points of compound cycles being assigned to terrain’s passes and singular points not belonging to a cycle corresponding to bottoms and summits. This however should lead to the same order of difficulty as discussed above, and in fact, building an algebraic relief surface whose projected contour lines form a carbon copy of the fingerprint pattern would be as complicated as writing down a differential system whose solution curves align with the fingerprint stream of ridges.

Finally, beyond biometry, forensic science, or the question of whether the qualitative theory of differential equations should be taught to budding Sherlock Holmes, the main beneficiary of the present study is perhaps mathematics itself.

\( \text{25} \) As already pointed out by Galton in [1, p.77], a curious optical effect is connected with the circular forms, which becomes almost annoying when many specimens are examined in succession. They seem to be cones standing bodily out from the paper. This singular appearance becomes still more marked when they are viewed with only one eye; no stereoscopic guidance then correcting the illusion of their being contour lines.
In fact, as the reader may have guessed, fingerprints (and any oriented texture in general) were merely a pretext for raising interesting questions in dynamical systems theory, as for example, the rather formidable connexion problem for a set of randomly distributed and natured singular points in space.

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When I look at the Twist, I almost feel like singing Chubby Checker’s “Let’s Twist Again” from the ’60s, with some modifications in the lyrics..

Come on everybody!
Print your hands!
(Inky) All you looking good!
I’m gonna (rock’n) roll your thumb
It won’t take long!
We’re gonna print the Twist
And it goes like this:

Come on let’s twist again,
Like it is on the last phalanx!
Yeaah, let’s twist again,
Like we did for the last Whori!

(...)

50