Completeness of first and second order ODE flows and of Euler-Lagrange equations

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Abstract

Two results on the completeness of maximal solutions to first and second order ordinary differential equations (or inclusions) over complete Riemannian manifolds, with possibly time-dependent metrics, are obtained. Applications to Lagrangian mechanics and gravitational waves are given.

1 Introduction

In this work we shall obtain some completeness result for maximal solutions to ordinary first and second order equations over a complete Riemannian manifold \((Q, a)\). We are concerned with the equations

\[
\dot{q} = \nu(t, q) \tag{1}
\]

\[
\frac{D}{dt} \dot{q} = f(t, q(t), \dot{q}(t)) \tag{2}
\]

where \(D\) is the affine connection of \((Q, a)\), \(\nu: \mathbb{R} \times Q \to TQ\) is a time-dependent vector field and \(f: \mathbb{R} \times TQ \to TQ\) is a time-dependent and velocity dependent vector field.

For \(Q = \mathbb{R}^n\) and Eq. (1) the problem is answered satisfactorily by Wintner’s theorem \([30,32]\) \[19, Theor. 5.1, Chap. 3\] on the completeness of first order flows on \(\mathbb{R}^n\), and by its refinements and variations \([5,7,10,14,28,31]\). This solution implies a straightforward answer to the analogous completeness problem for Eq. (2), indeed, this second order equation can be rewritten as a first order equation on \(TQ\), which for \(Q = \mathbb{R}^n\) is diffeomorphic to \(\mathbb{R}^{2n}\). In any case there have been other studies of Eq. (2) in \(\mathbb{R}^n\) and especially in \(\mathbb{R}\). For instance, Hartman and Wintner \([20,27]\) \[19, Theor. 5.2, Chap. 12\], taking advantage of some estimates by Bernstein and Nagumo, showed that if \(f\) has an almost quadratic asymptotic dependence on \(v\), and satisfies some other assumptions, then a forward complete solution exists which starts from any prescribed initial point \(q(0) = q_0\). However, in their theorem they could not impose an a priori value for \(\dot{q}(0)\). This limitation can be regarded as a consequence of the fact

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that the quadratic dependence on the velocity is just at the verge of spoiling completeness (Example 3.5).

If $Q$ is curved we still have that every coordinate chart $\{q^a\}$ on an open set $U \subset Q$ induces a chart $\{q^a, \dot{q}^\beta\}$ on $\pi^{-1}(U) \subset TQ$, where $\pi: TQ \to Q$ is the usual projection. Thus one can tackle the problem of completeness to maximal solutions to the second order differential equation (2) by studying the completeness of a flow on open sets of $\mathbb{R}^n$. Wintner’s theorem assures that under suitable assumptions the solution will be either complete or will reach the boundary of the coordinate patch (i.e. $q(t)$ will reach the boundary of $U$). In the latter case Wintner’s theorem can be applied once again, so that, inductively, either the solution is complete or it crosses infinite coordinate charts in finite time. The application of Wintner’s theorem to the general Riemannian case is not completely straightforward since one would like to formulate the hypothesis in a natural coordinate-free language, for instance expressing Wintner’s conditions on the asymptotic behavior of $f$ in terms of functions constructed from the Riemannian distance.

The problem of the completeness of maximal solutions to Eq. (2) in Riemannian spaces was considered by other authors who, apparently unaware of Wintner’s work, passed through similar ideas. The autonomous case for force fields $f$ of gradient type, was considered in [11, 16, 29], while the non autonomous case for more general fields, with an additional affine dependence on the velocity, has been recently considered in [6]. The results obtained in these studies do not seem to reach the generality that one would expect from the analogy with Wintner’s results in $\mathbb{R}^n$, particularly in connection with the possible superlinear growth of the fields $\nu$ and $f$, or of the dependence of $f$ on the velocity. In this work we shall remedy this situation, extending Wintner’s result to Riemannian manifolds and to second order ODE flows.

Actually, we shall also deal with an interesting related problem. Let $(Q, a_t)$ be a manifold endowed with a time-dependent Riemannian metric which is complete for every $t$. We are going to study the completeness of the maximal solutions to the equation

$$\frac{D(t)}{dt} \dot{q} = f(t, q(t), \dot{q}(t)), \tag{3}$$

where $D(t)$ is the affine connection of $(Q, a_t)$ and $f$ is as above. Apparently there is no difference between the two equations because the latter can be rewritten

$$\frac{D(0)}{dt} \dot{q} = f(t, q(t), \dot{q}(t)) + T(\dot{q}, \dot{q})$$

where $T = D(0) - D(t)$ is a tensor field of type $(1, 2)$ symmetric in the lower indices. However, we will be able to assure completeness in the case in which $f$ has a component which depends quadratically in the velocity only if the dependence is of the kind naturally embodied in Eq. (3). This fact singles Eq. (3) as particularly well shaped for our completeness study. Furthermore, and more importantly, the Euler-Lagrange equations of classical mechanics that are met in rheonomic mechanical systems are naturally written in this form where $a_t$ is the time dependent matrix appearing in the Lagrangian kinetic term [15,17] (see also section 3.1 below).
Our proof applies to the general case of Eq. (3) and is based on a result in Riemannian geometry which allows us to avoid the Nash embedding of $Q$ in $\mathbb{R}^n$ used in Gordon’s proof [16]. In Gordon’s work this embedding was used in order to construct a smooth proper function on the manifold. Actually, we do not need smoothness and so we can take the distance squared as proper function so as to give a straightforward geometrical meaning to our bounds.

2 Some distance inequalities

In this section we obtain some inequalities for the Riemannian distance. They will allow us to construct proper functions which are not necessarily smooth.

**Proposition 2.1.** Let $(Q, a_t)$ be a Riemannian manifold of class $C^3$, and suppose that for each $t$ the metric $a_t$ is $C^2$ with respect to the time and space coordinates. Let $\rho_t : Q \times Q \to [0, +\infty]$ be the Riemannian distance of $(Q, a_t)$, then $\rho_t(p, q)$ is continuous in $(t, p, q)$.

Moreover, for fixed $p \in Q$, defined $R : \mathbb{R} \times Q \to [1, +\infty)$ and $E : \mathbb{R} \times TQ \to [1, +\infty)$ with (here $\|v\|_T^2 = a_t(v, v)$)

\[
R(t, q) = 1 + \rho_t(p, q), \quad (4)
\]

\[
E(t, q, v) = 1 + \rho_t(p, q)^2 + \|v\|_T^2, \quad (5)
\]

the functions $|t| + R(t, q)$ and $|t| + E(t, q, v)$, respectively over $\mathbb{R} \times Q$ and $\mathbb{R} \times TQ$, are proper.

**Proof.** Let us prove continuity at $(t, p, q)$. Let $\epsilon > 0$, using the $\sigma$-compactness of $Q$ we can find a metric $\hat{a}$ which is at every point larger than $a_t$, for every $t \in [t - \epsilon, t + \epsilon]$ (in the sense that the unit balls of $\hat{a}$ on the tangent space are contained in those of $a_t$) (the metric $\hat{a}$ need not be complete). The distance $\hat{\rho}$ is continuous and the topology induced by $\hat{\rho}$ coincides with the manifold topology. Using the triangle inequality for $t' \in [t - \epsilon, t + \epsilon]$ we obtain

\[
|\rho_t(p', q') - \rho_t(p, q)| \leq |\rho_t(p', q') - \rho_t(p', q)| + |\rho_t(p', q) - \rho_t(p, q)|
\]

\[
\leq |\rho_t(p', q') - \rho_t(p, q')| + |\rho_t(p, q') - \rho_t(p, q)|
\]

\[
+ |\rho_t(p, q) - \rho_t(p, q')| \leq \rho_t(p, p') + \rho_t(q, q') + |\rho_t(p, q) - \rho_t(p, q')|,
\]

which implies that we need only to prove that for fixed $p, q \in Q$, $\rho_t(p, q) \to \rho_t(p, q)$ if $t' \to t$.

Let $\sigma : [0, 1] \to Q, s \to \sigma(s)$, be a minimizing geodesic which connects $p$ to $q$ in $(Q, a_t)$. Let $v$ be any vector field over $\sigma([0, 1])$. The function $a_t(v)(v, v)$ is continuous in $(t', q')$ and hence uniformly continuous over the compact set $[t - \epsilon, t + \epsilon] \times \sigma([0, 1])$, a fact which implies the inequality for every $(t', q'), (t'', q'') \in [t - \epsilon, t + \epsilon] \times \sigma([0, 1])$

\[
|\rho_t(q')(v, v) - \rho_t(q')(v, v)| \leq \rho_t(q', q'') + |t'' - t'|.
\]
With \( q'' = q' \) the inequality states that for \( t' \to t, a_{t'}(v, v) \to a_t(v, v) \) uniformly over \( \sigma([0, 1]) \). Let \( v = d\sigma/ds \) and let \( l_t \) be the length functional for \( (Q, a_t) \), then \( l_t'(\sigma) = \rho_t(p, q) \), for \( t' \to t \), which implies that for every \( \delta > 0 \) we have for \( t' \) sufficiently close to \( t \), \( \rho_t(p, q) \leq l_t'(\sigma) \leq \rho_t(p, q) + \delta \). As \( \rho_t(p, q) \) stays bounded by \( \rho_t(p, q) + \delta \) in the limit, if by contradiction, \( \rho_t(p, q) \to \rho_t(p, q) \) does not hold, then we can find a sequence \( t_n \to t \) such that \( \rho_{t_n}(p, q) \neq \rho_t(p, q) \), \( L < \rho_t(p, q) \). As \( \delta \) is arbitrary, \( L \leq \rho_t(p, q) \), and hence \( L < \rho_t(p, q) \).

Let \( \exp \) be the exponential map for \( (Q, a_t) \) and let \( v_n \in T_{Q_p} \) be such that \( \exp_{p,v_n} = q \), thus \( \|v_n\|_{t_n} = \rho_{t_n}(p, q) \). Since \( a_{t_n}\|_p \to a_t\|_p \) the sequence \( v_n \) converges (pass to a subsequence if necessary) to some vector \( v \in T_{Q_p} \), and \( \|v_n\|_{t_n} \to \|v\|_t \), thus \( \|v\|_t = L \leq \rho_t(p, q) \). The geodesic equation is a first order differential equation over \( TQ \), thus standard results [19, Theor. 3.1] on the continuity of first order (on \( TQ \) in this case) differential equations with respect to initial conditions \( v \) in this case) and external parameters \( t \) in this case) imply that \( q = \exp_{p,v_n} \to \exp_{p,v} v \). Since \( \|v\|_t < \rho_t(p, q) \) we have that \( \exp_{p,v} v \neq q \). The contradiction proves that \( \rho_t(p, q) \to \rho_t(p, q) \).

Let us prove that \( F := \|t\| + E(t, q, v) \) is proper (the proof that \( |t| + R(t, q) \) is proper is contained in this one). Clearly \( F \) is continuous thus the inverse image of a compact set is closed. If there is a compact set \( K \) such \( F^{-1}(K) \) is not compact, then we can assume with no loss of generality \( K = [-B, B] \) for some \( B > 0 \), and we can find a sequence \( (t_n, q_n, v_n) \) which escapes every compact set of \( \mathbb{R} \times TQ \) and is such that \( |F(t_n, q_n, v_n)| \leq B \). However, due to the expression of \( F \), \( |t_n| \leq B \), thus we can assume with no loss of generality that \( t_n \to t \) for some \( t \in [-B, B] \). Moreover, \( \rho_{t_n}(p, q_n)^2 \leq B \) thus let \( w_n \in T_{Q_p} \) be a vector such that \( \exp_{p,v_n} w_n = q_n \); we have \( \|w_n\|_{t_n} = \rho_{t_n}(p, q_n) \leq B^{1/2} \). Since \( a_{t_n}\|_p \to a_t\|_p \) we can assume with no loss of generality (i.e. passing to a subsequence if necessary) that \( w_n \to w \in T_{Q_p} \). Using again [19, Theor. 3.1] we obtain \( q_n = \exp_{p} w_n \to \exp_{p} w =: \hat{q} \). As a consequence, the sequence \( q_n \) is contained in a compact set \( \hat{K} \supseteq \hat{q} \). We can find a metric \( \tilde{a} \) which is smaller than \( a_t \) in \( \hat{K} \) for \( t \in [-B, B] \) (in the sense clarified above). Thus the bound on \( F(t_n, q_n, v_n) \) implies \( \tilde{a}(v_n, v_n) \leq B \) which proves that the sequence \( (t_n, q_n, v_n) \) is actually contained in a compact set, a contradiction.

We need a simple preliminary lemma (for a more general version see [11, Lemma 16.4]).

**Lemma 2.2.** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function whose right upper Dini derivative satisfies

\[
D^+ f(x) := \limsup_{h \to 0^+} \frac{f(x + h) - f(x)}{h} \leq g'(x),
\]

where \( g : [a, b] \to \mathbb{R} \) is a \( C^1 \) function, then \( f - g \) is non-increasing over the interval \( [a, b] \).

**Proof.** Let us define \( F := f - g \), so that \( D^+ F = D^+ f - g' \leq 0 \). Suppose by contradiction, that there are \( a', b' \in [a, b], a' < b' \), such that \( F(b') > F(a') \),
and let \( r = \frac{F'(b') - F'(a')}{b' - a'} > 0 \). The continuous function \( h = \frac{b}{2}(x - a') \) has a minimum at \( c \in [a', b'] \), and since \( h(b') = \frac{F(a') + F(b')}{2} > F(a') = h(a') \), we have \( c \neq b' \). Thus \( D^x F(c) = D^x h(c) + \frac{b}{2} \geq \frac{b}{2} > 0 \), a contradiction. \( \square \)

The proof of the next proposition would be considerably shortened under the assumption \( \partial_t a_t = 0 \). However, we shall need the following version in order to deal with a time dependent metric.

**Proposition 2.3.** Let \( (Q, a_t) \) and \( \rho_t \) be as in Prop. 2.1.

(i) Suppose that the time derivative of the metric \( a_t \) is bounded as follows:
there is a continuous function \( g : [1, +\infty) \rightarrow [1, +\infty) \) such that
\[
\pm (\partial_t a_t)(v, v) \leq 2g(1 + \rho_t(p, q)) a_t(v, v),
\tag{6}
\]
at every point \( (t, q, v) \in \mathbb{R} \times TQ \). Let \( q : I \rightarrow Q \) be a \( C^1 \) curve, then for \( \mathbf{I}, \mathbf{T} \in I, \mathbf{I} < \mathbf{T}, \) we have respectively
\[
\pm[\rho_T(p, q(\mathbf{T})) - \rho_I(p, q(\mathbf{I}))] \leq \int_{\mathbf{I}} \{||\dot{q}||_t + g(1 + \rho_t(p, q(t))) \rho_t(p, q(t))\} \, dt,
\tag{7}
\]
(ii) Suppose that the time derivative of the metric \( a_t \) is bounded as follows:
there is a continuous function \( g : [1, +\infty) \rightarrow [1, +\infty) \) such that
\[
\pm (\partial_t a_t)(v, v) \leq 2g(1 + \rho_t(p, q))^2 a_t(v, v),
\tag{8}
\]
at every point \( (t, q, v) \in \mathbb{R} \times TQ \). Let \( q : I \rightarrow Q \) be a \( C^1 \) curve, then for \( \mathbf{I}, \mathbf{T} \in I, \mathbf{I} < \mathbf{T}, \) we have respectively
\[
\pm[\rho_T(p, q(\mathbf{T}))^2 - \rho_I(p, q(\mathbf{I}))^2] \leq \int_{\mathbf{I}} \{\rho_t(p, q(t)||\dot{q}||_t + g(1 + \rho_t(p, q(t))^2) \rho_t(p, q(t))^2\} \, dt.
\tag{9}
\]

**Proof.** Let us prove the inequality Eqs. (6) and (8) with the left-hand side replaced respectively by \( \rho_t(p, q(\mathbf{T})) - \rho_t(p, q(\mathbf{I})) \) and \( \rho_t(p, q(\mathbf{T}))^2 - \rho_t(p, q(\mathbf{I}))^2 \). The other direction follows considering the curve \( q'(t) = q(-t), q' : [-\mathbf{T}, -\mathbf{I}] \rightarrow Q, \) and the Riemannian spaces \( (Q, a'_t), a'_t = a_{-t}, \) in such a way that \( \rho_t'(p, q'(t)) = \rho_{-t}(p, q(-t)). \)

For any positive integer \( n \), let \( \epsilon = (\mathbf{T} - \mathbf{I})/n \), and let \( t_k = \mathbf{I} + k\epsilon \). Let us abbreviate \( \rho_{t_k} \) with \( \rho_k \) and \( q(t_k) \) with \( q_k \). Then for \( k = 0, 1, \ldots, n - 1 \), we have

\[1\]An heuristic way of obtaining Eq. (9) consists in working out \( \partial_t [\rho_t^2(p, q(t))] \), expressing the squared distance as an ‘energy’ action integral over a minimal geodesic. The reader has to use the Cauchy-Schwarz inequality for the scalar product that there appears and then integrate. This method is not rigorous since even for \( \partial_t a_t = 0 \), \( \rho_t(p, q(t)) \) is not always differentiable when \( q(t) \) passes through a cut point to \( p \). This problem cannot be easily amended since the cut points are not necessarily isolated.
by the triangle inequality
\[
\rho_{k+1}(p, q_k+1) - \rho_k(p, q_k) \leq \rho_{k+1}(p, q_{k+1}) - \rho_{k+1}(p, q_k) + \rho_{k+1}(p, q_k) - \rho_k(p, q_k) \\
\leq A_k + B_k,
\]
\[
\rho_{k+1}(p, q_k+1)^2 - \rho_k(p, q_k)^2 = [\rho_{k+1}(p, q_{k+1}) - \rho_k(p, q_k)] [\rho_{k+1}(p, q_{k+1}) + \rho_k(p, q_k)] \\
\leq \{ [\rho_{k+1}(p, q_{k+1}) - \rho_{k+1}(p, q_k)] + [\rho_{k+1}(p, q_k) - \rho_k(p, q_k)] \} \\
[\rho_{k+1}(p, q_{k+1}) + \rho_k(p, q_k)] \\
\leq 2[A_k + B_k]C_k,
\]
where
\[
A_k = \rho_{k+1}(q_k, q_k) + 1, \\
B_k = \rho_{k+1}(p, q_k) - \rho_k(p, q_k), \\
C_k = [\rho_{k+1}(p, q_{k+1}) + \rho_k(p, q_k)]/2.
\]
Summing over \(k\) and taking the limit for \(n \to +\infty\)
\[
\rho_{k+1}(p, q_k+1) - \rho_k(p, q_k) \leq \lim_{n \to +\infty} \sum_{k=0}^{n} A_k + \lim_{n \to +\infty} \sum_{k=0}^{n} B_k,
\]
\[
\rho_t(p, q(t))^2 - \rho_{\tilde{t}}(p, q(\tilde{t}))^2 \leq 2 \lim_{n \to +\infty} \sum_{k=0}^{n} A_k C_k + 2 \lim_{n \to +\infty} \sum_{k=0}^{n} B_k C_k.
\]
By the continuity of \(\rho_t(p, q(t))\), there is a point \(\tilde{t}_k \in [t_k, t_{k+1}]\) such that
\[
\rho_{\tilde{t}_k}(p, q(\tilde{t}_k)) = C_k.
\]
Moreover, there is some \(\tilde{t}_k \in [t_k, t_{k+1}]\) such that
\[
A_k = \rho_{k+1}(q_k, q_k+1) \leq \int_{t_k}^{t_{k+1}} ||\dot{q}||_{t_{k+1}} \ dt = ||\dot{q}(\tilde{t}_k)||_{t_{k+1}} \epsilon,
\]
thus
\[
2A_k C_k \leq 2\rho_{\tilde{t}_k}(p, q(\tilde{t}_k)) ||\dot{q}(\tilde{t}_k)||_{t_{k+1}} \epsilon,
\]
and
\[
\lim_{n \to +\infty} \sum_{k=0}^{n-1} A_k \leq \lim_{n \to +\infty} \sum_{k=0}^{n-1} ||\dot{q}(\tilde{t}_k)||_{t_{k+1}} \frac{1}{n},
\]
\[
2 \lim_{n \to +\infty} \sum_{k=0}^{n-1} A_k C_k \leq \lim_{n \to +\infty} \sum_{k=0}^{n-1} 2\rho_{\tilde{t}_k}(p, q(\tilde{t}_k)) ||\dot{q}(\tilde{t}_k)||_{t_{k+1}} \frac{1}{n}.
\]
We obtain the first term in the integral argument in the right-hand side of Eqs. (7)-9. Indeed, the right-hand side of the above equation is the Riemann integral of a continuous function 22 (observe that \(\sqrt{a'}(\tilde{q}(t), \tilde{q}(t))\) regarded as a function of \((t', t)\) is continuous and hence uniformly continuous over the
compact set $[t_k, t_{k+1}]$, thus in the previous expression $(t_{k+1}, \hat{t}_k)$ can be replaced by $(\hat{t}_k, t_k)$ with a total error which can be made arbitrarily small.

As for the remaining term, let $l(t)$ be the length functional of $(Q, a_t)$. For any fixed $C^1$ curve $\eta(s)$ the function $l_t[\eta]$ is $C^1$ in $t$ because (here $d\eta/ds$ is denoted $\eta'$)

$$\partial_t l_t[\eta] = \int_\eta \partial_t \sqrt{a_t(\eta', \eta')} \, ds = \int_\eta \frac{1}{2 \sqrt{a_t(\eta', \eta')}} (\partial_t a_t)(\eta', \eta') \, ds.$$ 

We wish to bound $D_t^+ \rho_t(p, r)$ at any time $t$ where $r$ is arbitrary and does not depend on $t$.

Let $\gamma_t$ be a minimizing geodesic of $(Q, a_t)$ which connects $p$ to $r$. We have

$$\limsup_{\epsilon \to 0^+} \frac{1}{\epsilon} [\rho_{t+\epsilon}(p, r) - \rho_t(p, r)] \leq \limsup_{\epsilon \to 0^+} \frac{1}{\epsilon} \{l_{t+\epsilon}[\gamma] - l_t[\gamma]\} = \partial_t l_t[\gamma],$$

where we have used the fact that at time $t + \epsilon$, for $\epsilon > 0$, $\gamma_t(s)$ is not necessarily minimizing. Thus, using Eq. (6) or (8), we obtain (in cases (i) and (ii) respectively)

$$D_t^+ \rho_t(p, r) \leq \partial_t l_t[\gamma] = \int_{\gamma_t} \frac{1}{2 \sqrt{a_t(\gamma'_t, \gamma_t)}} (\partial_t a_t)(\gamma'_t, \gamma_t) \, ds \leq \int_{\gamma_t} g(1 + \rho_t(p, \gamma_t(s))) \sqrt{a_t(\gamma'_t, \gamma_t)} \, ds.$$ 

$$D_t^+ \rho_t(p, r) \leq \partial_t l_t[\gamma] = \int_{\gamma_t} \frac{1}{2 \sqrt{a_t(\gamma'_t, \gamma_t)}} (\partial_t a_t)(\gamma'_t, \gamma_t) \, ds \leq \int_{\gamma_t} g(1 + \rho_t(p, \gamma_t(s))^2) \sqrt{a_t(\gamma'_t, \gamma_t)} \, ds.$$ 

But as at time $t$ the geodesic $\gamma_t$ is minimizing and starts from $p$, the function $\rho_t(p, \gamma_t(s))$ grows with $s$, and hence (in cases (i) and (ii) respectively)

$$D_t^+ \rho_t(p, r) \leq g(1 + \rho_t(p, r)) \int_{\gamma_t} \sqrt{a_t(\gamma'_t, \gamma_t)} \, ds = g(1 + \rho_t(p, r)) \rho_t(p, r),$$

$$D_t^+ \rho_t(p, r) \leq g(1 + \rho_t(p, r)^2) \int_{\gamma_t} \sqrt{a_t(\gamma'_t, \gamma_t)} \, ds = g(1 + \rho_t(p, r)^2) \rho_t(p, r).$$

By Lemma 2.2, we have for $\hat{t} \leq \hat{t}$ (in cases (i) and (ii) respectively)

$$\rho_t(p, r) - \hat{\rho}_t(p, r) \leq \int_{\hat{t}}^t g(1 + \rho_t(p, r)) \rho_t(p, r) \, dt,$$

$$\rho_t(p, r) - \hat{\rho}_t(p, r) \leq \int_{\hat{t}}^t g(1 + \rho_t(p, r)^2) \rho_t(p, r) \, dt.$$
Choosing \( \hat{t} = t_k, \check{t} = t_{k+1}, r = q_k \) we obtain (in cases (i) and (ii) respectively)

\[
B_k \leq \int_{t_k}^{t_{k+1}} g(1 + \rho_t(p, q_k))\rho_t(p, q_k) dt \leq g(1 + \rho_t(p, q_k))\rho_t(p, q_k) \frac{1}{n},
\]

\[
B_k \leq \int_{t_k}^{t_{k+1}} g(1 + \rho_t(p, q_k)^2)\rho_t(p, q_k) dt \leq g(1 + \rho_t(p, q_k)^2)\rho_t(p, q_k) \frac{1}{n},
\]

for some (different) \( t' \in [t_k, t_{k+1}] \). Finally, (in cases (i) and (ii) respectively)

\[
\sum_{k=0}^{n-1} B_k \leq \sum_{k=0}^{n-1} \frac{g(1 + \rho_t(p, q_k))\rho_t(p, q_k)}{n},
\]

\[
2 \sum_{k=0}^{n-1} B_k C_k \leq 2 \sum_{k=0}^{n-1} \rho_{t_k}(p, q(t_k))g(1 + \rho_t(p, q_k)^2)\rho_t(p, q_k) \frac{1}{n}.
\]

Using the continuity and hence uniform continuity of \( g(1 + \rho_t(p, q_k))\rho_t(p, q_k) \) as a function of \( t \) on the compact set \([\underline{t}, \overline{t}]\), or of \( \rho_t(p, q(t))g(1 + \rho_t(p, q_k)^2)\rho_t(p, q_k) \) as a function of \((t, t')\) on the compact set \([\underline{t}, \overline{t}]^2\) we conclude that in the limit \( n \to \infty \) the right-hand side of the previous inequalities converge respectively to the Riemann integrals \( \int_{\underline{t}}^{\overline{t}} [g(1 + \rho_t(p, q(t)))\rho_t(p, q(t))] dt \) and

\[
2 \int_{\underline{t}}^{\overline{t}} [g(1 + \rho_t(p, q(t))^2)\rho_t(p, q(t))^2] dt.
\]

\[\square\]

### 2.1 Generalization of Wintner’s theorem to Riemannian manifolds

In this section we generalize Wintner’s theorem to Riemannian spaces with possibly time-dependent metrics.

Let us recall LaSalle-Bihari \[4,24\] generalization of Gronwall’s inequality. We give here a kind of two-sided generalization.

**Theorem 2.4.** Let \( x : I \to [0, +\infty) \) be a continuous function which satisfies the inequality:

\[
\pm [x(t') - x(t)] \leq \beta \int_{t}^{t'} \Psi(s) \omega(x(s)) ds, \quad t, t' \in I, t < t',
\]

where \( \beta > 0, \Psi : \mathbb{R} \to [0, +\infty), \) and \( \omega : [0, +\infty) \to (0, +\infty) \) is continuous and non-decreasing. Then we have the respective estimates

\[
\pm [\Phi(x(t')) - \Phi(x(t))] \leq \beta \int_{t}^{t'} \Psi(s) ds, \quad t, t' \in I, t < t',
\]
where $\Phi : \mathbb{R} \to \mathbb{R}$ is given by

$$
\Phi(u) := \int_{u_0}^u \frac{ds}{\omega(s)}, \quad u \in \mathbb{R}.
$$

Proof. The minus version can be obtained from the usual plus version by considering the function $\tilde{x}(t) = x(-t)$, and applying the plus version to $\tilde{x}$.

Remark 2.5. With $g$ or $g_r$, we denote a non-decreasing $C^0$ function $g_r : [1, +\infty) \to [1, +\infty)$ with the property that the increasing function $G_r : [1, +\infty) \to [0, +\infty)$ diverges for $y \to +\infty$. The typical choice will be $g = \text{cnst.} \geq 1$, but there are choices that strengthen the next theorems, e.g. $g = \ln(\eta + x)$ or $g = \ln(\eta + \ln(\eta + x))$ and so on, where $\eta = e - 1$.

On first reading one can just consider the simple case $g_r = \text{cnst.} \geq 1$, $\partial_t a_t = 0$. We are ready to generalize Wintner’s theorem to Riemannian manifolds.

Theorem 2.6. Let $(Q, a_t)$ be a 1-parameter family of complete Riemannian manifolds as in Prop. 2.1, and let $f : \mathbb{R} \times Q \to TQ$ be a $C^0$ field. Let $p \in Q$ and let $R : \mathbb{R} \times Q \to [1, +\infty)$ be given by

$$
R(t, q) = 1 + \rho_t(p, q).
$$

Suppose that for every compact interval $[-r, r] \subset \mathbb{R}$ there is a function $g_r$ as in remark 2.5 such that for every $(t, q) \in [-r, r] \times Q$

$$
\pm(\partial_t a_t)(v, v) \leq 2g_r(R(t, q)) a_t(v, v),
$$

$$
\|\nu(t, q)\| \leq g_r(R(t, q)) R(t, q),
$$

then the maximal solutions to the first order equation

$$
\dot{q} = \nu(t, q(t))
$$

are complete in the forward (resp. backward) direction.

Remark 2.7. Since we do not need the uniqueness of the solution, we just ask $\nu$ to be continuous rather than Lipschitz [18]. The existence of some maximal solutions is assured by [19] Theor. 3.1. The theorem can be easily generalized to differential inclusions, i.e. to the case in which $\nu$ is a lower semi-continuous set valued mapping and $\nu(t, q)$ is a convex set of $TQ$, for each $(t, q)$. In this case the existence of some maximal solutions is assured by [2] Theor. 1, Chap. 2.

Proof. Let the initial condition be $q(t_0) = q_0$, and suppose by contradiction that $q(t)$ is a maximal solution whose interval of definition $I$ is bounded on the
right, i.e. \( I \subset (-\infty, B] \) (resp. on the left \( I \subset [-B, +\infty) \)) for some \( B > 0 \). Let us observe that

\[
\| \dot{q} \|_t = \| \nu \|_t \leq g_B(R(t, q(t))) R(t, q(t)),
\]

thus plugged in Eq. (7) with \( g = g_B \), we obtain for every \( t_1, t_2 \in I, t_1 \leq t_2 \),

\[
\pm [R(t_2, q(t_2)) - R(t_1, q(t_1))] \leq 2 \int_{t_1}^{t_2} g_B(R(s, q(s))) R(s, q(s)) \, ds. \tag{16}
\]

Thus, by LaSalle-Bihari generalization of Gronwall’s inequality (Theor. 2.4) we have for \( t \in I, t > t_0 \) (resp. \( t < t_0 \))

\[
R(t, q(t)) \leq G_B^{-1}(G_B(R(t_0, q(t_0))) + \beta |t - t_0|) \\
\leq G_B^{-1}(G_B(R(t_0, q(t_0))) + \beta 2B) < +\infty.
\]

As \( R \) is proper and \( |t| \) is bounded by \( B \) in the forward (resp. backward) direction, \( q(t) \) cannot escape every compact set and hence it must be complete in that direction \cite[Theor. 3.1, Chap. II]{19}, a contradiction.

3 The completeness of maximal solutions to second order ODEs

The following notation will simplify the statement of the theorem. In this section the functions \( g_r \) and \( G_r \) are defined as in the previous section. Let \( F_t : TQ \to TQ \) be a time dependent 2-form field. By \( F_t^\# : TQ \to TQ \) we denote the corresponding endomorphism of the tangent space defined by \( F_t^\#(v) = a^{-1}(F_t(\cdot), v) \). On first reading one can just consider the simple case \( g_r = \text{cnst.} \geq 1, F_t = 0, \partial_t a_t = 0 \). We are ready to state the theorem.

**Theorem 3.1.** Let \((Q, a_t)\) be a 1-parameter family of complete Riemannian manifolds as in Prop. 2.1, and let \( f : \mathbb{R} \times TQ \to TQ \) be a \( C^0 \) field. Let \( p \in Q \) and let \( E : \mathbb{R} \times TQ \to [1, +\infty) \) be given as above by

\[
E(t, q, v) = 1 + \|v\|_t^2 + \rho_t(p, q)^2.
\]

Suppose that for every compact interval \([-r, r] \subset \mathbb{R} \) there are a constant \( K(r) > 0 \), a function \( g_r \) as in remark 2.5 and a continuous 2-form field \( F_t \), such that for every \((t, q, v) \in [-r, r] \times TQ \)

\[
\pm |(\partial_t a_t)(v, v)| \leq 2 g_r \rho_t(p, q)^2 a_t(v, v), \tag{17}
\]

\[
\| f(t, q, v) - F_t^\#(v) \|_t \leq g_r E(t, q, v) \frac{E(t, q, v)}{K + \|v\|_t} \tag{18}
\]

Then the maximal solutions to the second order equation

\[
\frac{D^{(i)}}{dt} \dot{q} = f(t, q(t), \dot{q}(t)),
\]

are complete in the forward (resp. backward) direction.
Remark 3.2. The inequality (18) mentions \(F^q_t\) in order to clarify that the inclusion of force field components of ‘electromagnetic’ or ‘Coriolis’ type can only enlarge the domain of the maximal solutions (see also [6]). Actually, the left-hand side of Eq. (18) could be replaced, just in the study of forward completeness, by \(\|f(t,q,v) - F^q_t(v) + h(t,q,v)\|\), where \(h : \mathbb{R} \times TQ \to [0, +\infty)\). This can be easily understood from inspection of Eq. (20) below. This fact proves that the introduction of force components which represent friction forces proportional to the velocity can only enlarge the domain of the maximal solutions and hence can only make it easier to attain forward completeness, see also [24]. However, these maximal solutions could be incomplete in the backward direction.

Remark 3.3. Clearly, the assumptions of the theorem are satisfied if \(\partial_3 a_t = 0\) and for each \(s\) there are positive constants \(K_0(r), K_1(r), K_2(r), K_3(r), K_4(r)\), such that for \((t,q,v) \in [-r,r] \times TQ\)

\[
\|f(t,q,v)\| \leq K_0 + K_1 \rho(p,q) + K_2\|v\| + K_3 \frac{\rho(p,q)^2}{K_4 + \|v\|}
\]

It suffices to choose \(g_r\) to be a sufficiently large constant and \(K_4 = K\). Thus, in this case the second order ODE is complete. Of course the asymptotic behavior of \(f\) could be faster than linear for instance of the form \(\sim \rho \ln(q + \rho)\) it suffices to consider a non-trivial \(g_r\). Observe that we did not impose any type of dependence of \(f\) on \(v\), e.g. linear, as in previous works, thus \(f(v)\) could be quite general. Also we obtain a new type of sufficient asymptotic bound, expressed by the last term of the previous expression, which has been noticed here for the first time.

Remark 3.4. As in the previous section, since we do not need the uniqueness of the solution, we ask \(f\) to be just continuous rather than Lipschitz. The existence of some maximal solutions is assured by [19 Theor. 3.1]. Theorem 3.1 can be easily generalized to differential inclusions, i.e. to the case in which \(f\) is a lower semi-continuous set valued mapping and \(f(t,q,v)\) is a convex set of \(TQ\) for each \((t,q,v)\). In this case the existence of some maximal solutions is assured by [2 Theor. 1, Chap. 2].

Proof. Let the initial condition be \(q(t_0) = q_0\), \(\dot{q}(t_0) = v_0\) and suppose by contradiction that \(q(t)\) is a maximal solution whose interval of definition \(I\) is bounded from above, i.e. \(I \subset (-\infty, B]\) for some \(B > 0\) (resp. from below, i.e. \(I \subset [-B, +\infty)\) for some \(B > 0\)). Let us consider the curve for \(t \in \hat{I} = [-B,B] \cap I\), we can take \(B\) sufficiently large so that \(t_0 \in [-B,B]\). We are going to prove that if \(t_1,t_2 \in \hat{I}\), \(t_1 < t_2\),

\[
\pm[E(t_2,q(t_2),\dot{q}(t_2)) - E(t_1,q(t_1),\dot{q}(t_1))] \leq \beta \int_{t_1}^{t_2} g_B(E(s,q(s),\dot{q}(s))) E(s,q(s),\dot{q}(s)) ds. \tag{19}
\]

for a constant \(\beta > 0\). Thus, defined \(E_0 = E(t_0,q(t_0),\dot{q}(t_0))\), by LaSalle-Bihari generalization of Gronwall’s inequality (Theor. 24) we have for \(t \in \hat{I}, t > t_0\)
As a consequence, as $F(t, q(t), \dot{q}(t)) = |t| + E$ and $F$ is proper by Prop. 2.1 (q(t), \dot{q}(t))$ cannot escape every compact set in the forward (resp. backward) direction and hence this solution must be complete in that direction [19] Theor. 3.1, Chap. II, a contradiction. In order to prove the inequality (19), let us observe that

\[
\pm \frac{d}{dt} a(t, q, \dot{q}) = \pm a(t, q, D^{(i)} \dot{q}) = \pm (\partial_{ta}(q, \dot{q})) = \pm a(t, q, f) \pm (\partial_{ta}(q, \dot{q})) \\
\leq \pm 2a(t, q, f - F^1_t(q)) + 2g_B(1 + \rho_t(p, q(t))^2) \|\dot{q}\|^2_t \\
\leq 2 \|s\|_s \|f - F^1_t(q)\|_s + 2g_B(1 + \rho_t(p, q(t))^2) \|\dot{q}\|^2_t,
\]

which once integrated gives for $t_1, t_2 \in I, t_1 < t_2$,

\[
\pm[a_{t_1}(q, \dot{q})(t_2) - a_{t_1}(q, \dot{q})(t_1)] \leq 2 \int_{t_1}^{t_2} \|s\|_s \|f - F^1_t(q)\|_s + g_B(1 + \rho_s(p, q(s))^2) \|\dot{q}\|^2_s \, ds.
\]

Summing it to Eq. (19) where we make the choice $g = g_B, \tilde{t} = t_2, \tilde{t} = t_1$, we obtain

\[
\pm[E(t_2, q(t_2), \dot{q}(t_2)) - E(t_1, q(t_1), \dot{q}(t_1))] \leq 2 \int_{t_1}^{t_2} \Lambda(s) g_B(E)E \, ds,
\]

where we shortened the notation introducing the function

\[
\Lambda(s) = \frac{x(s)\|\dot{q}\|_s + g_B(1 + x(s)^2) x(s)^2 + \|f - F^1_t(q)\|_s \|\dot{q}\|_s + g_B(1 + x(s)^2) \|\dot{q}\|^2_s}{g_B(E(q(s), q(s))) E(s, q(s), \dot{q}(s))},
\]

where $x(s) := \rho_t(p, q(s))$. Let us also define

\[
\Omega(x, y) = \frac{xy + x^2 g_B(1 + x^2) + (1 + x^2 + y^2) x^2}{(1 + x^2 + y^2)g_B(1 + x^2 + y^2)},
\]

and

\[
\beta = \sup_{x, y > 0} 2\Omega(x, y) \leq 6 < +\infty.
\]

The bound on $f$ implies $\Lambda(s) \leq \Omega(\rho_t(p, q(s)), \|\dot{q}\|_s)$ thus

\[
\pm[E(t_2, q(t_2), \dot{q}(t_2)) - E(t_1, q(t_1), \dot{q}(t_1))] \leq 2 \int_{t_1}^{t_2} \Omega(\rho(s, p, q(s)), \|\dot{q}\|_s) g_B(E)E \, ds
\]

\[
\leq \beta \int_{t_1}^{t_2} g_B(E(s, q(s), \dot{q}(s))) E(s, q(s), \dot{q}(s)) \, ds.
\]
Example 3.5. It is useful to study the differential equation on \( \mathbb{R} \) given by

\[ \ddot{q} = \beta \frac{q^2}{\alpha + q}, \]

\( \alpha, \beta > 0 \), also because of the analogy between this type of force and the type of limiting behavior of form \( K_3 \frac{q^2}{K_4 + |q|} \) allowed by theorem [3.1]. It is easy to check that for \( \beta = 1 \) it admits complete solutions of the form \( q(t) = c_2 \exp(c_1 t) - \alpha \), and hence with any prescribed initial position \((q_0 \neq -\alpha)\) and velocity, while for \( \beta = 2 \), the solutions is \( q(t) = c_1 t - c_2 - \alpha \) and hence it admits a future complete solution (i.e. \( c_2 < 0 \)) only if \( q(0) \) and \( \dot{q}(0) \) have opposite signs (consistently with [20]). This example shows that if the force is quadratic in the velocity, we need further assumptions in order to establish completeness, even if the proportionality constant expressing such dependence is inversely proportional to the distance.

3.1 Application to Lagrangian mechanics and gravitational waves

Let \( Q \) be a \( d \)-dimensional manifold (the space) endowed with the (possibly time dependent) positive definite metric \( a_t \), 1-form field \( b_t \) and potential function \( V(t, q) \) (all \( C^r, r \geq 2 \)). On the classical spacetime \( E = T \times Q \), \( T \) connected interval of the real line, let \( t \) be the time coordinate and let \( e_0 = (t_0, q_0) \) and \( e_1 = (t_1, q_1) \) be events, the latter in the future of the former i.e. \( t_1 > t_0 \). Consider the action functional of classical mechanics

\[ S_{e_0, e_1}[q] = \int_{t_0}^{t_1} L(t, q(t), \dot{q}(t)) dt, \quad (21) \]

where

\[ L(t, q, v) = \frac{1}{2} a_t(v, v) + b_t(v) - V(t, q), \quad (22) \]

on the space \( C^1_{e_0, e_1} \) of \( C^1 \) curves \( q : [t_0, t_1] \rightarrow Q \) with fixed endpoints \( q(t_0) = q_0, \ q(t_1) = q_1 \). Let \( F_i := db_i \), where \( d \) is the exterior differentiation on \( Q \) (thus \( d \) does not differentiate with respect to \( t \)), that is, the 2-form whose components in local coordinates are \( F_{i, ij} = \partial_i b_{t, j} - \partial_j b_{t, i} \). The \( C^1 \) stationary points are smoother than the Lagrangian (namely \( C^{r+1} \), see [23] Theor. 1.2.4). By the Hamilton’s principle, they solve the Euler-Lagrange equation (e.g. [12, 17, Eq. (2-39)])

\[ a_t(\cdot, D^{(t)}(\cdot, \dot{q})) = F_i(\cdot, \dot{q}) - (\partial_t a_t)(\cdot, \dot{q}) - (\partial_i b_t + \partial_q V), \quad (23) \]

where, as in previous sections, we denoted with \( D^{(t)} \) the affine connection of \( a_t \) at the given time.

Historically this has proved to be one of the most important variational problems because the mechanical systems of particles subject to (possibly time dependent) holonomic constraints move according to Hamilton’s principle with a Lagrangian given by [22] (see [15]).
Theorem 3.1 allows us to establish the completeness of the maximal solutions to the above Euler-Lagrange equations.

**Corollary 3.6.** Let \( Q \) be a d-dimensional manifold (the space) endowed with the (possibly time dependent) positive definite metric \( a_t \), 1-form field \( b_t \) and potential function \( V(t,q) \) (all \( C^r \), \( r \geq 2 \)). Let us suppose that \( (Q,a_t) \) are complete for each \( t \) and let \( \rho_t(p,q) \) be the corresponding Riemannian distance. Let us fix \( p \in Q \) and let us suppose that for every time interval \([-r,r]\) we can find a continuous non-decreasing function \( g_r : [1, +\infty) \to [1, +\infty) \) with the property that

\[
G_r(y) := \int_1^y \frac{1}{xg_r(x)} \, dx,
\]

diverges for \( y \to +\infty \) (the typical choice will be \( g_r = \text{cnst.} \geq 1 \)) and such that for every \((t,q,v) \in [-r,r] \times TQ\)

\[
\pm [\partial_1 a_t(v,v)] \leq 2g_r(1 + \rho_t(p,q)^2) a_t(v,v), \quad (24)
\]

\[
\|\partial_1 b_t(t,q)\|_t, \quad \|\partial_2 V(t,q)\|_t \leq g_r(1 + \rho_t(p,q)^2)[1 + \rho_t(p,q)], \quad (25)
\]

then the E.-L. flow is complete in the forward (resp. backward) direction.

**Corollary 3.7.** Let \( Q \) be a d-dimensional manifold (the space) endowed with the (possibly time dependent) positive definite metric \( a_t \), 1-form field \( b_t \) and potential function \( V(t,q) \) (all \( C^r \), \( r \geq 2 \)). If \( Q \) is compact then the E.-L. flow of (23) is complete.

**Proof.** By the Hopf-Rinow theorem each Riemann space \((Q,a_t)\) is complete. At any point \((t,q) \in [-r,r] \times Q\) we can choose a sufficiently large constant \( g_r \) so as to satisfy the inequalities of Cor. 3.6 for arbitrary \( v \) at that point. Thus by compactness of \([-r,r] \times Q\) we can find a sufficiently large constant \( g_r \) such that the assumptions of that corollary hold.

**Remark 3.8.** This Lagrangian problem has the following application to gravitational waves. Let \( M := T \times Q \times \mathbb{R} \), \( T = \mathbb{R} \), with \( Q \) as above, and let an element of \( M \) be denoted by \((t,q,y)\). Let \( M \) be endowed with the Lorentzian metric

\[
g = a_t - dt \otimes (dy - b_t) - (dy - b_t) \otimes dt - 2Vdt^2, \quad (26)
\]

where the fields \( a_t, b_t \) and \( V \) are as above and the time orientation is given by the global timelike vector \( W = -[V - \frac{1}{2}]\partial_y + \partial_t \), \( g(W,W) = -1 \). The future directed lightlike vector \( n = \partial_y \) can be shown to be covariantly constant. In fact, these spacetimes can be characterized as those spacetimes which admit a covariantly constant null vector which generates a \((\mathbb{R}, +)\)-principal fiber bundle. Spacetimes of this form are called *generalized gravitational waves* [25] (more restrictive families are considered in [3] and [13], the difference being essentially that passing between *general* and *natural* mechanical systems [17], see next paragraph).

Eisenhart [12] realized that the spacelike geodesics of \((M,g)\) project to \( E := T \times Q \) into solutions of the above Euler-Lagrange equation and that any such
solution can be regarded as such projection. The author showed that the same can be said with \textit{spacelike} replaced by \textit{lightlike}, a fact particularly useful for its connection with causality theory \cite{21,25,26}. Thus there is a very fruitful one-to-one correspondence between generalized gravitational waves and rheonomic mechanical systems, which allows one to import methods and ideas from one field to the other \cite{26}.

For instance, under the assumption that \((Q, a_t)\) are complete, the completeness of the E.-L. flow is equivalent to the geodesic completeness of \((M, g)\), thus Corollary \cite{3.6} establishes conditions for the geodesic completeness of \((M, g)\). However, we shall leave the details of this result to a different work \cite{26}.

4 Conclusions

We have generalized Wintner’s theorem on first order ODE flows over \(\mathbb{R}^n\) to Riemannian manifolds with possibly time-dependent metrics. We have also given a second order version which is particularly well shaped for application to Lagrangian mechanics. The proofs are based on some inequalities for the Riemannian distance which allowed us to build non-smooth proper functions over the manifold. The lack of smoothness was handled using the LaSalle-Bihari generalization of Gronwall’s inequality. Our results can be easily generalized to second order inclusions so as to deal with other interesting aspects of classical mechanics, such as static friction. An application to the theory of exact gravitational waves and lightlike dimensional reduction was also commented.

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