Non-spreading matter-wave packets in a ring

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Received 10 May 2019, revised 5 June 2019
Accepted for publication 20 June 2019
Published 13 August 2019

Abstract

Non-spreading wave packets and matter-wave packets in ring traps have both attracted great research interest for quite a long time due to their remarkable physical properties and intriguing applications. Here, we prove that there is only one set of non-spreading matter-wave packets in a free ring, and this set of wave packets has been found analytically. These non-spreading matter-wave packets can be realized in a toroidal trapped Bose–Einstein condensate system with the help of Feshbach resonance to eliminate contact interaction between atoms. Since experimentally residual interaction noise will always exist, its effect on the stability of these non-spreading wave packets is also examined. Qualitatively, under weak residual interaction noise, these non-spreading wave packets can preserve their shape for quite a long time, while a stronger interaction noise will induce shape breathing of the wave packets. The shape-keeping abilities of these wave packets are further studied quantitatively. We found that this set of wave packets has the same shape-keeping ability against interaction noise, and the shape-keeping ability is linearly related to the strength of the interaction noise.

Keywords: non-spreading matter-wave, ring trap, Bose–Einstein condensate

(Some figures may appear in colour only in the online journal)

1. Introduction

Matter-waves with substantial intensity are crucial in applications such as atom lithography [1–3], ultra-sensitive magnetometry [4, 5], and matter-wave interferometry [6–8]. When a matter-wave packet propagates in free space, it will spread due to dispersion, and its intensity will decrease [9, 10]. Thus, non-spreading matter-wave packets are attracting great research interest. It is well known that self-focusing nonlinearity can be introduced to overcome the dispersion spreading, and form bright matter-wave solitons [11–14] which are non-spreading wave packets. Soliton-based matter-wave interferometers have been elaborated in many theoretical works [15–19], and also experimentally implemented with an increase of interference fringe visibility having been observed [20, 21]. However, nonlinearity may also do harm to the performance of interferometers, for example, it could induce a phase diffusion, which will reduce the coherent time [22, 23]. And recently it has also been reported that the best sensitivity of a matter-wave interferometer is reached in the linear regime [24, 25]. (We note that the relation between nonlinearity and the performance of an interferometer is quite subtle. Besides the above-mentioned effects, nonlinearity also gives rise to non-classical correlations and squeezed states [26, 27]. Taking advantage of these properties, a standard quantum limit surpassed interferometers can be realized [28, 29].) So, achieving linear non-spreading matter-wave packets could be an interesting research subject.

In 1979, Berry and Balazs showed that the free particle linear Schrödinger equation has a nontrivial Airy function solution which holds its shape during an accelerating propagation [30]. After this pioneering work, such linear non-spreading waves were extensively studied (for a review see reference [31]), and first experimentally demonstrated in an optical system in 2007 [32]. Soon after, linear non-spreading matter-waves were also realized in electron beams [33]. Atomic Bose–Einstein condensate (BEC), because of its macroscopic quantum properties and highly controllable feature, is an ideal system for exploring matter-wave optical phenomena. Recently, the generation of similar non-spreading BEC wave packets by amplitude or phase imprinting techniques [34] and time dependent harmonic traps [35] have also been proposed.
Due to potential applications in realizing a matter-wave Sagnac interferometer [36–40], atomic analogy of SQUID circuits [41–43], persistent current [44–47] and quantum time crystal [48–50], and non-spreading matter-wave packets in a ring also merit considerable research interest. Now we are inquisitive about whether non-spreading wave packets can exist in a free ring. In this paper, first we analytically found a set of non-spreading wave packets in a free ring, and at the same time we also proved that they are the only set. Then the realization of these non-spreading wave packets in an atomic BEC system with the help of the Feshbach resonance technique is discussed. Lastly, although in principle inter-atom interaction can be totally eliminated by the Feshbach resonance technique, practically it is impossible to operate the Feshbach magnetic field with infinite precision, and there will always be some residual interactions [22, 23, 53], therefore the stability of these non-spreading wave packets against residual interaction noise is also studied.

This paper is organized as follows. In section 2, formulas describing non-spreading wave packets in a ring are derived, and the only set of non-spreading matter-wave packets is found analytically. In section 3, the experimental realization of the non-spreading wave packets in BEC systems is discussed briefly. In section 4, the stability of the wave packets against residual interaction noise is studied numerically. Lastly, the work is summarized in section 5.

2. Non-spreading matter-wave packets in a ring

To generally discuss the problem of non-spreading wave packets in a ring, we begin with the following dimensionless Schrödinger equation:

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \psi(\theta, t) = -\frac{1}{2} \frac{\partial^2}{\partial \theta^2} \psi(\theta, t),
\]

where \( \theta \) is the azimuthal angle which takes a value in the range of \(-\pi\) to \(\pi\). Here, we are interested in the case of a free ring, therefore in the equation only the kinetic energy term exists, while the external potential term is not included. A continuous wave function and its first order derivative require

\[
\psi(\theta_0, t) = \psi(\theta_0 + 2n\pi, t),
\]

where \( \theta_0 \) is an arbitrary azimuthal angle, \( n = 0, \pm 1, \pm 2, \ldots \) is an integer number, and \( \psi'(\theta_0, t) = \frac{\partial \psi(\theta, t)}{\partial \theta} \bigg|_{\theta = \theta_0} \) is the first order derivative of \( \psi(\theta, t) \) at point \( \theta = \theta_0 \).

To find a shape preserving wave packet, we rewrite the wave function \( \psi(\theta, t) \) as follows \([54, 55]\):

\[
\psi(\theta, t) = A(\theta, t) e^{iS(\theta, t)},
\]

where \( A \) and \( S \) are two real number functions, representing the probability density and phase distribution, respectively. Under such a form, if function \( A(\theta, t) \) can be expressed as

\[
A(\theta, t) = A(\phi)
\]

the wave packet will preserve its shape during the propagation. Here, \( \phi \) can be regarded as the trajectory function of the wave packet, with \( f(t) \) being a real number function.

Inserting the Madelung transformed wave function (4) into a Schrödinger equation (1), and splitting the real and imaginary parts of the equation, we get the following two equations:

\[
-\frac{\partial A}{\partial \phi} \frac{df}{dt} = -\frac{1}{2} \left[ 2 \frac{\partial A}{\partial \phi} \frac{\partial S}{\partial \phi} + A \frac{\partial^2 S}{\partial \phi^2} \right],
\]

and

\[
-\frac{\partial S}{\partial \theta} = -\frac{1}{2} \left[ \frac{\partial^2 A}{\partial \theta^2} - A \left( \frac{\partial S}{\partial \theta} \right)^2 \right].
\]

Equation (6) can be rearranged into the following form:

\[
\frac{\partial}{\partial \theta} \left( \frac{df}{dt} - \frac{\partial S(\theta, t)}{\partial \theta} A^2(\phi) \right) = 0,
\]

which means that \( \frac{df}{dt} - \frac{\partial S(\theta, t)}{\partial \theta} A^2(\phi) \) is independent of the azimuthal angle \( \theta \). So it can written as

\[
\left[ \frac{df(t)}{dt} - \frac{\partial S(\theta, t)}{\partial \theta} \right] A^2(\phi) = c(t),
\]

with \( c(t) \) being an arbitrary function depending on time variable \( t \) only.

First, it can obviously be seen that equation (8) admits a trivial solution

\[
A(\phi) = \text{cons}, \quad S(\theta, t) \propto \theta.
\]

Equation (7) and the wave function continuous condition, equations (2) and (3), are also considered, and the corresponding wave function reads as

\[
\psi(\theta, t) = Ce^{i\theta + it/\hbar^2},
\]

which is a trivial solution with a uniformly distributed density in the ring at all times.

If \( A(\phi) \) is not a constant, by adapting its background value, the angle \( \phi_0 \) always exists making \( A(\phi_0) = 0 \). And note that the right-hand side of equation (8) is only a function of \( t \), this is, for all \( \phi \) the following equation needs to be satisfied:

\[
\left[ \frac{df(t)}{dt} - \frac{\partial S(\theta, t)}{\partial \theta} \right] A^2(\phi) = c(t) = 0.
\]

From this equation, we get the following relationship between \( f(t) \) and \( S(\theta, t) \):

\[
\frac{df(t)}{dt} = \frac{\partial S(\theta, t)}{\partial \theta} = \sum_{i=0}^{\infty} c_i t^i.
\]

Here, since \( \frac{df(t)}{dt} \) is a function only depending on time variable \( t \), we expand it into a Taylor series with \( c_i \) being the \( i \)th order coefficient. In the following, depending on the highest order of this series, we discuss the problem in three distinct cases:
(i) If series (10) only has the constant term, i.e.,
\[
\frac{df(t)}{dt} = \frac{\partial S(\theta, t)}{\partial \theta} = c_0,
\]
then it is straightforward to get
\[
f(t) = c_0 t,
\]
\[
S(t) = c_0 \theta + g(t),
\]
with \(g(t)\) being an arbitrary function. Substituting them into equation (7), we have
\[
-A(\phi) \frac{\partial S(\theta, t)}{\partial t} = -\frac{1}{2} \left[ \frac{\partial^2 A(\phi)}{\partial \phi^2} - A(\phi) c_0^2 \right].
\]
Because the right-hand side of this equation does not explicitly depend on time variable \(t\), the equality requires \(\frac{\partial S(\theta, t)}{\partial t} = \frac{\partial g(t)}{\partial t}\) is also explicitly time independent. That is, function \(g(t)\) must take the following linear form:
\[
g(t) = G_1 t + G_0,
\]
where \(G_0\) and \(G_1\) are two free constant parameters. Lastly, equation (7) becomes
\[
\frac{\partial^2 A(\phi)}{\partial \phi^2} + (c_0^2 - 2G_1)A(\phi) = 0,
\]
from which the shape of a non-spreading wave packet can be determined
\[
A(\phi) = C \cos(m\phi).
\]
Here, \(m = \sqrt{l_0^2 - 2G_1} \) is an integer number which is restricted by the boundary condition, and \(C\) is a normalization constant. At the same time, the boundary conditions also require that \(c_0\) be an integer number \(c_0 = l\). Lastly, putting the above expressions of \(f, g, A,\) and \(S\) together, a full non-spreading solution can be constructed as
\[
\psi(\theta, t) = C \cos(m\theta - lt) \exp \left[ il\theta + \frac{1}{2}(m^2 + l^2)t \right].
\]
The physical meanings of \(m\) and \(l\) are obvious. \(2m\) is the number of nodes of the wave function, and \(1/l\) describes the moving speed of the wave packet along the ring. We also note that the previous trivial solution (9) is just the \(m = 0\) specification of solution (11).

(ii) If series (10) has terms up to the first order, i.e.,
\[
\frac{df(t)}{dt} = \frac{\partial S(\theta, t)}{\partial \theta} = c_0 + c_1 t,
\]
function \(f(t)\) and \(S(\theta, t)\) are integrated to be
\[
f(t) = c_0 t + \frac{1}{2} c_1 t^2,
\]
\[
S(t) = c_0 \theta + c_1 \theta t + g(t).
\]
Then equation (7) becomes
\[
\frac{1}{2} \frac{\partial^2 A(\phi)}{\partial \phi^2} - c_1 \phi A(\phi) - \frac{1}{2} c_0^2 A(\phi) = A(\phi)[g'(t) + 2c_0c_1 t + c_1^2 t^2].
\]
As in case (i), the left-hand side of this equation also does not explicitly depend on variable \(t\). And in this case, the equality between the left- and right-hand sides can be met by the following function of \(g(t)\):
\[
g(t) = -\frac{1}{3} c_1^2 t^3 + c_0 c_1 t^2 + G_1 t + G_0.
\]
Lastly, the equation governing function \(A(\phi)\) reads as
\[
\frac{1}{2} \frac{\partial^2 A(\phi)}{\partial \phi^2} - c_1 \phi A(\phi) - \left( \frac{1}{2} c_0^2 + G_1 \right) A(\phi) = 0,
\]
which promises an Airy function solution. But an Airy function cannot fulfill the periodical boundary conditions (2) and (3), therefore a non-spreading wave packet cannot exist in this situation.

(iii) If the series have terms equal to or higher than second order, i.e.,
\[
\frac{df(t)}{dt} = \frac{\partial S(\theta, t)}{\partial \theta} = c_0 + c_1 t + c_2 t^2 + \ldots,
\]
we have
\[
f(t) = c_0 t + \frac{1}{2} c_1 t^2 + \frac{1}{3} c_2 t^3 + \ldots,
\]
\[
S(t) = \theta (c_0 + c_1 t + c_2 t^2 + \ldots) + g(t).
\]
Similar as in cases (i) and (ii), we rewrite equation (7) as follows:
\[
\frac{1}{2} \frac{\partial^2 A}{\partial \phi^2} - c_1 \phi A - \frac{1}{2} c_0^2 A = A \left[ \frac{(c_0 c_1 + 2c_2 \phi)t}{2} \right.
\]
\[
+ \left. (c_1^2 + 2c_0 c_2) t^2 + \ldots + g'(t) \right].
\]
The left-hand side of this equation is still \(t\) independent. But because of the existence of the new term \(2c_2 \phi t\) (and some other higher order ones), no function \(g(t)\) exists which can make the right-hand side also be time independent. Thus, this equation has no solution. No non-spreading wave packet solutions exist in such a case.

Now, in a brief summary, combining all of the three above cases, we can conclude that the only set of non-spreading
wave packets in a ring is given by equation (11). This set of non-spreading wave packets travels along the ring with a constant velocity. Moreover, owing to the quantum feature, these non-spreading wave packets can only propagate in some quantized fractional velocities. And unlike in free space, self-accelerating non-spreading wave packets with an Airy function-like form cannot exist in a ring configuration.

3. Realization in a toroidal trapped BEC system

The non-spreading matter-wave packets found in section 2 can be experimentally realized in a quasi-one-dimensional toroidal trapped dilute atomic BEC system, see figure 1. In a BEC system, because of the collision between atoms, besides the kinetic energy and external trap potential, there will also exist a nonlinear term, i.e., the system is described by the following nonlinear Schrödinger equation (Gross–Pitaevskii equation):

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(r, \theta, z) \psi + g |\psi|^2 \psi, \]  

where \( \hbar \) is the reduced Planck constant, \( m \) is the mass of atom, \( g = 4\pi \hbar^2 a_s/m \) is the interaction strength with \( a_s \) being the s-wave scattering length, and

\[ V(r, \theta, z) = \frac{1}{2} m \omega_z^2 z^2 + \frac{1}{2} m \omega_r^2 r^2 + V_0 \exp \left[ -\frac{2r^2}{w_0^2} \right], \]

is the toroidal trapping potential which is experimentally realized by a magnetic trap and a plug laser beam with Gaussian shape [46] (there are also some other schemes to realize a toroidal trap, such as the ones in [56–59]) having its minimum at \( z = 0 \) and \( r = R \), with

\[ R^2 = \frac{w_0^2}{2} \ln \left( \frac{4V_0}{m \omega_r^2 w_0^2} \right). \]

Expanding the trapping potential around \( r = R \) into a Taylor series, and neglecting the third and higher orders of \( r - R \), the potential approximately reads as

\[ V(r, \theta, z) = \frac{1}{2} m \omega_z^2 z^2 + \frac{1}{2} m \omega_r^2 (r - R)^2, \]

with \( \omega_r = 2\omega R/w_0 \) being the effective radial trapping frequency.

For sufficiently strong confinement in the axial and radial directions, the wave function can be assumed to have a variables separated form \( \psi = \phi_0(r)\phi_0(z)\psi(\theta, t) \) with

\[ \phi_0 = [m \omega_r/(\pi \hbar)]^{1/4} \exp \left[ -\frac{m \omega_r^2 z^2}{2\hbar} \right], \]

\[ \rho_0 = [m \omega_r/(\pi \hbar^2)]^{1/4} \exp \left[ -\frac{m \omega_r (r - R)^2}{2\hbar} \right] \]

being the ground state wave function of the axial and radial traps. Inserting it into equation (12), the system can be reduced to one dimension. The azimuthal wave function \( \psi(\theta, t) \) obeys the equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2mR^2} \partial^2 \psi + g |\psi|^2 \psi. \]

Here, \( g = m \sqrt{\omega_z \omega_r} g/(2\pi R) \) is the quasi-one-dimensional effective contact interaction strength. Taking the transformation \( t \rightarrow mR^2t/\hbar \), equation (16) can be rescaled to the following dimensionless form:

\[ \frac{\partial \psi}{\partial t} = -\frac{1}{2} \partial^2 \psi + g_s |\psi|^2 \psi, \]

with \( g_s = gR^2\hbar/2m = 2\pi R a_s \sqrt{\omega_z \omega_r}/h \). Considering, for instance, the BEC of \( ^{87}\text{Rb} \) atoms, with atomic number \( N = 10^3 \), s-wave scattering length \( a_s = 33a_0 \) (where \( a_0 \) is the Bohr radius), trapped in a toroidal trap with parameters \( R = 10 \mu \text{m}, \omega_r = \omega_r = 1000 \text{ Hz} \), the interaction strength is \( g_s \approx 22 \). Thus, the interaction will usually play a significant role. Fortunately, it can be eliminated by the Feshbach resonance technique. According to Feshbach resonance theory [51, 52], the s-wave scattering length \( a_s \) can be tuned by applying a magnetic field

\[ a_s(B) = a_s(\infty) \frac{B - B_0}{B - B_r}, \]

where \( a_s(\infty) \) is the asymptotic s-wave scattering length in the case of far from resonance, \( B \) is the magnetic induction of the applied magnetic field, \( B_r \) is the resonant magnetic induction, and \( B_0 \) is the value of the magnetic induction for a vanishing s-wave scattering. That is, the inter-atom interaction of BEC can be totally eliminated when a magnetic field whose induction equals \( B_0 \) is applied. And we get the same equation as (1) in section 2. So, we propose the non-spreading wave packets can be realized in a toroidal trapped BEC system.
4. Stability against residual interaction noise

According to Feshbach resonance theory [51, 52], in principle the strength of the inter-atom interaction can be tuned to 0 by the magnetic field \( B_0 \). However, practically the noise of the magnetic field around \( B_0 \) is unavoidable, thus a small residual interaction noise will always be left

\[
g_\xi = g_\xi(t) \quad (19)
\]

Here, \( g_\xi \) is the strength of the residual interaction noise, and \( \xi(t) \) is a random function. White noise is assumed, and \( \xi(t) \) is uniformly distributed in the range of \([-1, 1]\). According to the experiment in reference [22], the interaction can be reduced by a factor of \( 10^3 \) with a 100 mG magnetic field. So the typical strength of the residual interaction noise is \( g_\xi \approx 0.022 \) (recall that without Feshbach resonance the interaction strength was estimated to be \( g_\xi = 22 \) in the previous section). If the magnetic field is operated more precisely (in experiment [23], the magnetic field can be controlled to about 1 mG), an even smaller value of \( g_\xi \) can be obtained. Thus, here we typically choose \( g_\xi \) on the order of \( 10^{-2} \), and to study the limit behaviors, \( g_\xi = 0.002 \) and \( g_\xi = 0.5 \) are also examined.

It is then necessary to study the stability of those non-spreading matter-wave packets found in section 2 against such noises. Here, this is done by numerically solving equation (17) with an operator splitting method. The nonlinear part is directly integrated (since this is a stochastic integral, the Ito stochastic integral formula [60] is used) in x-space, while the second order differential term is handled in momentum space by means of fast Fourier transformation.

In figure 2, we plot the evolution of some non-spreading wave packets (with \( m, \ l = 1, 1; 2, 3; \) and \( 5, 5 \)) under the influence of residual interaction noise with both a very weak strength \( g_\xi = 0.002 \) and a little stronger one of \( g_\xi = 0.05 \) (here, we mean stronger compared to \( g_\xi = 0.002 \). But, it is still in fact a weak interaction. The ’kinetic’ energy of a non-

\[
\begin{align*}
\langle \theta - lt \rangle (m & = 1), \ (a1) \\
\langle \theta - lt \rangle (m & = 2), \ (a2) \\
\langle \theta - lt \rangle (m & = 5), \ (a3)
\end{align*}
\]

\[
\begin{align*}
\langle \theta - lt \rangle (l = 1), \ (b1) \\
\langle \theta - lt \rangle (l = 3), \ (b2) \\
\langle \theta - lt \rangle (l = 5), \ (c1)
\end{align*}
\]

wave packet is

\[
E_k = -\frac{1}{2} \int_0^{2\pi} \psi^* \frac{\partial^2}{\partial \psi^2} \psi^2 \, d\theta = l^2 + m^2, \quad \text{while the interaction energy is}
E_i = \frac{g_\xi}{2} \int_0^{2\pi} |\psi|^2 \, d\theta = \frac{3g_\xi}{8\pi}. \quad \text{Thus, for} \ g_\xi = 0.05, \ E_i \ll E_k \text{still holds.) In the figure, to conveniently compare the shapes of a wave packet at different times during the evolution, the vertical coordinate is set to} \ \phi = \theta - lt/m, \ \text{i.e., the wave packet is shifted to its initial location. From this figure, we can see that for very weak noise, the non-spreading wave packet can keep its shape for quite a long time, while a stronger noise will induce a periodical breathing of the wave packet shape after a long time. For both cases, the periodical travel of the wave packet in the ring is not affected by the interaction noise.}

The numerically found shape breathing periods for different wave packets are listed in table 1. From the table, it is natural to conclude that breathing period is totally determined by the wave packet parameters \( m \), and has nothing to do with \( l \) and \( g_\xi \), and its value is approximately \( T \approx \pi/(2m^2) \). This oscillating period can easily be understood by considering the formation mechanism of a breathing mode. For a wave packet with quantum number \( m \), the interaction noise will induce a transition to state with quantum number \( 3m \), and thus excites a breathing mode. Taking \( m = 1 \) for an example, in figure 3, we show the formation of such a breathing mode by adding wave packet \( \psi_1 = \cos(\theta) \) and a small perturbation \( \Delta \psi_1 = \delta \cos(3\theta) \exp[i\phi] \). When \( \psi_1 \) and \( \Delta \psi_1 \) are in the phase \( \phi = 0 \), the excitation will have a suppressing effect on the width of the wave packet, while \( \psi_1 \) and \( \Delta \psi_1 \) have a phase difference of \( \pi (\phi = \pi) \), the excitation will spread the wave packet. Thus, for a wave packet with quantum number \( m \), the breathing period will be

\[
T = \frac{2\pi}{(3m - m)^2} = \frac{\pi}{2m^2}. \quad (20)
\]

And if there exists a considerable large interaction noise, higher order oscillating modes will be excited as shown in
their phases are opposite, the wave packet is broadened. and \( \delta \psi \) have the same phase, the wave packet is suppressed; while their phases are opposite, the wave packet is suppressed; while their phases are opposite, the wave packet is broadened.

Table 1. Shape breathing period of some non-spreading wave packets subjected to interaction noise. The values of the breathing period are collected from numerical simulations, which are in perfect agreement with equation (20).

| Wave packet parameters | Noise strength \( g_\xi \) | Breathing period \( T \) |
|------------------------|--------------------------|--------------------------|
| \( m \) \( l \)       |                          |                          |
| 1 \( 1 \)              | 0.01 \ 1.5674            | \( \approx \pi/2 \)      |
| 2 \( 2 \)              | 0.02 \ 1.5678            | \( \approx \pi/8 \)      |
| 3 \( 3 \)              | 0.05 \ 0.3930            | \( \approx \pi/18 \)     |
| 4 \( 2 \)              | 0.05 \ 0.3932            |                          |
| 5 \( 5 \)              | 0.05 \ 0.0628            | \( \approx \pi/50 \)     |

Figure 3. Schematic of the formation of a breathing mode excitation on the non-spreading wave packet with \( m=1 \). The superposition of the main wave function \( \psi = \cos(\theta) \) and small excitation wave function \( \Delta \psi = \delta \cos(\theta) \exp(i\phi) \) forms a breathing mode. When \( \psi \) and \( \delta \psi \) have the same phase, the wave packet is suppressed; while their phases are opposite, the wave packet is broadened.

Figure 4. High order oscillating modes excited by interaction noise with a considerably large strength \( g_\xi \). The top panel is a heat map plot of \( |\psi(\phi, t)|^2 \) with \( \phi = \theta - lt/m \). The bottom panel is a plot of \( |\psi(0, t)|^2 \). The wave packet parameters are \( m=1, l=1 \).

Figure 5. Evolution of the shape differences of a non-spreading wave packet subjected to residual interaction noise with different strengths. The mean values of the shape difference \( D_s(t) \) for 500 individual simulations are plotted for interaction noises with strengths \( g_\xi = 0.01, 0.02, 0.03, 0.04, \) and 0.05 (represented by different colors as labeled in the figure). The black lines are the corresponding square root function \( D_s(t) = D_\xi \sqrt{t} \) fits of the data, and the fitting parameters are \( D_\xi = 4.23, 8.45, 13.4, 16.9, 21.2 \times 10^{-4} \), respectively. The wave packet parameters are \( m=1, l=1 \) for all lines.

To quantitatively measure the shape stability of the non-spreading wave packets, we introduce the following quantity:

\[
D_s(t) = \frac{\int |\psi(\theta - \theta_{ci}, t)|^2 - |\psi(\theta, 0)|^2| d\theta}{\int |\psi(\theta, 0)|^2 d\theta}, \tag{21}
\]

to describe the shape difference of the wave packet between time \( t \) and 0 (initial). Here, wave packet at time \( t \) is shifted by \( \theta_{ci} \) (azimuthal angle the wave packet have passed since \( t=0 \)) to line up with the initial wave packet. If \( D_s = 0 \), the profiles of the wave packet at time \( t \) and 0 are absolutely the same. And a larger value of \( D_s \) indicates a bigger shape change. It is reasonable to expect that the amount of shape difference will positively relate to the strength of the interaction noise, see figure 5, where the shape differences of the non-spreading wave packet with \( m=1, l=1 \) are plotted for different strengths of residual interaction noises. From the figure, the numerical results also suggest a square root formula of the shape difference

\[
D_s(t) = D_\xi \sqrt{t}. \tag{22}
\]

Thus, \( D_\xi \) can be interpreted as a parameter to describe the shape-keeping ability of the non-spreading wave packets. A larger value of \( D_\xi \) indicates that the wave packet will deviate from its initial shape more quickly, while a smaller value of \( D_\xi \) means the wave packet will keep its initial shape for a longer time. The numerical results show a linear dependence of \( D_\xi \) on the interaction noise strength \( g_\xi \), see figure 6. Further numerical results show that these conclusions also hold for non-spreading wave packets with other values of \( m \) and \( l \).
We also examined the shape differences for different non-spreading wave packets during their evolution. In figure 7, we plot the evolution of $D_s$ for wave packets with parameters $m, l = 1, 1; 2, 3; 3, 2; 5, 5$. As all the lines almost overlap with each other and at the same time the square root function $D(t) = D_\xi \sqrt{t}$ fits of the data, and the fitting parameters are $D_\xi = 2.09, 2.16, 2.18,$ and $2.14 \times 10^{-3}$, respectively. The interaction noise strength is $g_\xi = 0.05$ for all lines.

![Figure 6. Shape-keeping ability of non-spreading wave packets against residual interaction noise strength. The ‘+’ are the data points of $(g_\xi, D_s)$ obtained from the numerical results. The solid line is a linear fit of the data points. The wave packet parameters are $m = 1$ and $l = 1$.](image1)

![Figure 7. Evolution of shape differences for different non-spreading states. Mean values of shape differences for 500 individual simulations are plotted for different states $m, l = 1, 1; 2, 3; 3, 2; 5, 5$ (represented by different colors as labeled in the figure). The black lines are the corresponding square root function $D(t) = D_\xi \sqrt{t}$ fits of the data, and the fitting parameters are $D_\xi = 2.09, 2.16, 2.18,$ and $2.14 \times 10^{-3}$, respectively. The interaction noise strength is $g_\xi = 0.05$ for all lines.](image2)

We also examined the shape differences for different non-spreading wave packets during their evolution. In figure 7, we plot the evolution of $D_s$ for wave packets with parameters $m, l = 1, 1; 2, 3; 3, 2; 5, 5$. As all the lines almost overlap with each other and at the same time the square root function fitting parameters $D_\xi = 2.09, 2.16, 2.18$, $2.14 \times 10^{-3}$ are very close to each other, we conclude that all the non-spreading wave packets are equally stable against residual interaction noise.

Lastly, we point out that the square root increasing formula (22) only holds for a small value of $D_s$. From the definition, the value of $D_s$ will never be larger than 1. Therefore, when $D_s$ becomes large, it will attain saturation. In figure 8, an example is shown for the parameters $g_\xi = 0.5, m = 1, l = 1$. From the figure, one see that when $t < 400$ formula (22) fits the data well. However, when $t > 1400$ the increase of $D_s$ becomes saturated.

![Figure 8. Saturation of shape differences. Mean values of the shape differences $D_s$ for 500 individual simulations are plotted for wave packets $m = 1, l = 1$. The strength of the residual interaction noise is $g_\xi = 0.5$. The solid line represents the numerical results, and the black dashed line represents the square root fit.](image3)

5. Conclusion

In conclusion, we examined non-spreading wave packets in a ring. We found that only one set of non-spreading wave packets exists in a ring, and it was found analytically. The realization of these packets in ultra-cold atom systems (with the assistance of the Feshbach resonance technique to eliminate contact interaction between atoms) is discussed. The stability of these wave packets against residual interaction noise is examined numerically. It is found that these non-spreading wave packets are stable against weak interaction noise, and a strong interaction noise will induce periodical shape breathing of the wave packets. All these wave packets have the shape-keeping ability. And the shape-keeping ability is linearly related to the strength of the interaction noise.

While we focus on the non-spreading wave packet and its main features in the present work, it will also be interesting to further build a matter-wave interferometer using such non-spreading wave packets. Due to the linearity feature, the splitting and recombination processes in operating an interferometer are not expected to destroy the non-spreading property. A detailed splitting, recombination scheme, and the interference pattern may be discussed in future work. Other interesting extensions of the present work are to study the persistent current and quantum time crystal-related phenomena.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11847059 and 11874127).

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