Weighted Sobolev spaces of radially symmetric functions

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Abstract

We prove dilation invariant inequalities involving radial functions, polyharmonic operators and weights that are powers of the distance from the origin. Then we discuss the existence of extremals and in some cases we compute the best constants.

Keywords: Rellich inequality, Sobolev inequality, Caffarelli-Kohn-Nirenberg inequality, weighted biharmonic operator, dilation invariance.

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1 Introduction

The starting point of the present paper is the inequality

\[
\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p \, dx \geq c_{\alpha, k, p} \int_{\mathbb{R}^n} |x|^{\alpha - kp} |u|^p \, dx \quad \forall \ u \in C^k_{c,r}(\mathbb{R}^n \setminus \{0\}).
\]

Here \( n \geq 2 \) and \( k \geq 1 \) are integers, \( \alpha \in \mathbb{R}, \ p > 1, \ C^k_{c,r}(\mathbb{R}^n \setminus \{0\}) \) is the space of radially symmetric functions in \( C^k_c(\mathbb{R}^n \setminus \{0\}) \), and

\[
\nabla^k = \begin{cases} 
\Delta^m & \text{if } k = 2m \text{ is even,} \\
\nabla \Delta^n & \text{if } k = 2m + 1 \text{ is odd.}
\end{cases}
\]

Let us briefly describe our main results and our motivations.
First we find out the class of parameters $\alpha, k$ and $p$ such that (1.1) holds with a positive best constant $c_{\alpha,k,p}$. If $c_{\alpha,k,p} > 0$, then the space $D^{k,p}_r(\mathbb{R}^n; |x|^{\alpha}dx)$, defined as the completion of $C^{k,c}_{c, r}(\mathbb{R}^n \setminus \{0\})$ with respect to the norm
\[
\|u\|_{k,\alpha} = \left( \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p \, dx \right)^{1/p},
\]
(1.2)
is continuously embedded in $L^p(\mathbb{R}^n; |x|^{\alpha-kp}dx)$.

Notice that $D^{k,p}_r(\mathbb{R}^n; |x|^{\alpha}dx)$ is the natural ambient space in dealing with radially symmetric solutions to poliharmonic problems with weights. Having this application in mind, in the second part of the paper we study dilation invariant inequalities of the type
\[
\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p \, dx \geq S_{k,q,j}(\alpha) \left( \int_{\mathbb{R}^n} |x|^{-\beta_{k-j,q}} |\nabla^j u|^q \, dx \right)^{p/q}
\]
(1.3)
for $u \in D^{k,p}_r(\mathbb{R}^n; |x|^{\alpha}dx)$, and we show that the best constant $S_{k,q,j}(\alpha)$ is positive and achieved. Here $j = 0, \ldots, k - 1$ is an integer, $\nabla^0 u = u$, $q > p$ is given, and
\[
\beta_{k-j,q} = n - q \frac{n - (k - j)p + \alpha}{p}.
\]
(1.4)

We remark that (1.1) is closely related to the double weighted Hardy–Littlewood–Sobolev inequality (see e.g. Stein and Weiss [27] and Lieb [17]). The standard Hardy inequality in [15], [16] is recovered by choosing $k = 1$: it is well known that
\[
\int_{\mathbb{R}^n} |x|^\alpha |\nabla u|^p \, dx \geq \left| \frac{n + \alpha}{p} - 1 \right|^p \int_{\mathbb{R}^n} |x|^{\alpha-p} |u|^p \, dx \quad \forall u \in C^1_c(\mathbb{R}^n \setminus \{0\}),
\]
(1.5)
and that the constant in the right hand side can not be improved.

Second order dilation invariant inequalities have been largely studied since 1952, when Rellich showed in [24] (see also [25]) that
\[
\int_{\mathbb{R}^n} |\Delta u|^2 \, dx \geq \left( \frac{n(n-4)}{4} \right)^2 \int_{\mathbb{R}^n} |x|^{-4} |u|^2 \, dx \quad \forall u \in C^2_c(\mathbb{R}^n \setminus \{0\}).
\]

In the Hilbertian case $p = 2$, Ghoussoub and Moradifam proved in [14] that $c_{\alpha,2,2} = |(n - 4 + \alpha)(n - \alpha)/4|^2$ for any $\alpha \in \mathbb{R}$. We quote also [8], where a different approach is used.
Only partial results are available if \( p \neq 2 \) or \( k \geq 3 \), see for instance Mitidieri [19], Gazzola-Grunau-Mitidieri [12] (in a non-radial setting) and Adimurthi-Santra [2]. Related inequalities can be found in the above cited papers, in [1], [7], [9], [11], [20] and in the references therein.

In the present paper we compute \( c_{\alpha,k,p} \) for any \( \alpha,k \) and \( p \). We put

\[
H_{\alpha} = \frac{n + \alpha}{p} - 1, \quad \gamma_{\alpha,h} = \left( \frac{n + \alpha}{p} - h \right) \left( n - 2 + h - \frac{n + \alpha}{p} \right),
\]

(1.6)

where \( h = 2, \ldots, k \) is an integer. Notice that \( |H_{\alpha}|^p \) is the Hardy constant.

**Theorem 1.1** Let \( \alpha \in \mathbb{R}, p > 1 \) and \( k \geq 2 \). The best constant in (1.1) is given by

\[
c_{\alpha,k,p} = \begin{cases} 
\prod_{h=1}^{m} |\gamma_{\alpha,2h}|^p & \text{if } k = 2m \text{ is even}, \\
|H_{\alpha}|^p \prod_{h=1}^{m} |\gamma_{\alpha,2h+1}|^p & \text{if } k = 2m + 1 \text{ is odd}.
\end{cases}
\]

(1.7)

In case \( k = 2 \) Theorem 1.1 implies that

\[
\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx \geq \left( \frac{n + \alpha}{p} - 2 \right) \left( n - \frac{n + \alpha}{p} \right) \int_{\mathbb{R}^n} |x|^{\alpha - 2p} |u|^p dx
\]

(1.8)

for any \( u \in C^2_{c,r} (\mathbb{R}^n \setminus \{0\}) \). In fact (1.8) is a corollary of the next result (with \( m = 1 \)) and of the Hardy inequality. Here we agree that \( \Delta^0 u = u \).

**Theorem 1.2** Let \( \alpha \in \mathbb{R}, p > 1 \) and let \( m \geq 1 \) be a given integer. Then

\[
\int_{\mathbb{R}^n} |x|^\alpha |\Delta^m u|^p dx \geq n^{-\frac{n + \alpha}{p}} \int_{\mathbb{R}^n} |x|^{\alpha - p} |\nabla (\Delta^{m-1} u)|^p dx
\]

(1.9)

for any \( u \in C^2_{c,r} (\mathbb{R}^n \setminus \{0\}) \). The constant in the right hand side is sharp.

Notice that in the singular case \( \alpha = 2p - n \) the best constant in (1.8) vanishes, while

\[
\int_{\mathbb{R}^n} |x|^{2p-n} |\Delta u|^p dx \geq |n - 2|^p \int_{\mathbb{R}^n} |x|^{p-n} |\nabla u|^p dx
\]

for any \( u \in C^2_{c,r} (\mathbb{R}^n \setminus \{0\}) \).

Theorem 1.2 and (1.5) provide explicit best constants in inequalities of the type

\[
\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx \geq c_{\alpha,k,j,p} \int_{\mathbb{R}^n} |x|^{\alpha - (k-j)p} |\nabla^j u|^p dx \quad \forall \ u \in C^k_{c,r} (\mathbb{R}^n \setminus \{0\}),
\]

(1.10)
for any intermediate case \( j = 1, \ldots, k - 1 \), see Remark 2.3.

Weighted inequalities of order \( k \geq 2 \) for non radial functions are more involved. In [14] and [8], where \( p = 2 \) and \( k = 2 \) are assumed, it is proved that

\[
\int_{\mathbb{R}^n} |x|^{\alpha} |\Delta u|^2 \, dx \geq \min_{\gamma \in \mathbb{N} \cup \{0\}} \gamma_{\alpha,2} + i(n - 2 + i)^2 \int_{\mathbb{R}^n} |x|^{\alpha - 4} |u|^2 \, dx
\]

for any \( u \in C^2_c(\mathbb{R}^n \setminus \{0\}) \). Here \( \gamma_{\alpha,2} = \left( \frac{n-4+\alpha}{2} \right) \left( \frac{n-\alpha}{2} \right) \), accordingly with (1.6).

In particular, the best constant vanishes if \( -\gamma_{\alpha,2} \) is an eigenvalue of the Laplace-Beltrami operator on the sphere. The problem of finding the best constant for weighted Rellich type inequalities in a non radial setting and for general parameters \( \alpha, k, p \) is still open.

Next we direct our attention to semilinear inequalities. Assume \( c_{\alpha,k,p} > 0 \) and take an exponent \( q > p \). For any integer \( j \in \{0, \ldots, k - 1\} \) let \( S_{k,j}(\alpha) \) be the best constant in (1.3), that is,

\[
S_{k,j}(\alpha) = \inf_{u \in D^k_p(\mathbb{R}^n; |x|^\alpha \, dx)} \frac{\int_{\mathbb{R}^n} |x|^{\alpha} |\nabla^k u|^p \, dx}{(\int_{\mathbb{R}^n} |x|^{-\beta_{k-j,q}} |\nabla^j u|^q \, dx)^{p/q}}.
\] (1.11)

In Section 5.2 we prove the following existence result (see also Section 4.2 for shorter proofs in case \( k = 2, j \in \{0, 1\} \)).

**Theorem 1.3** If \( q > p > 1 \) and \( j \in \{0, 1, \ldots, k - 1\} \), then \( S_{k,j}(\alpha) \) is positive and achieved in \( D^k_p(\mathbb{R}^n; |x|^\alpha \, dx) \).

If \( n > kp, \alpha = 0, j = 0 \) and \( q = p^{k^*} := \frac{np}{n-kp} \), then \( \beta_{k,q} = 0 \) and \( S_{k,0}(0) \) coincides with the radial Sobolev constant \( S_{k,p}^{1*} \). Actually, (1.11) includes the \((k-1)^{th}\) best constants \( S_{k,p}^*, S_{k,p}^{w*}, \ldots, S_{k,p}^{(k-1)*} \), that are relative to the embeddings

\[
D^k_p(\mathbb{R}^n) \hookrightarrow D^j_{\frac{np}{n-(k-j)p}}(\mathbb{R}^n), \quad j = 1, \ldots, k - 1.
\]

We refer to Remark 5.9 for details on this subject.

If \( k = 1, \alpha > p-n \) and \( q > p \), then the infimum \( S_{1,q,0}(\alpha) \) is closely related to the celebrated Caffarelli-Kohn-Nirenberg inequalities in [5]. If in addition \( p = 2 \), then the minimizers of \( S_{1,q,0}(\alpha) \) are explicitly known since the paper [10] by Catrina and Wang (see also [17]). In the next theorem, that will be proved in Section 3.2, we extend the Catrina-Wang uniqueness result to the non Hilbertian case \( p \neq 2 \).
Theorem 1.4 If $\alpha \neq p - n$ and $q > p$, then problem
\[
- \text{div}(|x|^\alpha \nabla u)^{p-2} \nabla u = |x|^{-n+q\frac{\alpha+p-n}{p}} |u|^{q-2} u \quad \text{on } \mathbb{R}^n \tag{1.12}
\]
has a nontrivial radial solution $U$ in $\mathcal{D}^{1,p}_r(\mathbb{R}^n; |x|^\alpha dx)$ which is unique up to a change of sign and up to a transform of the type $u(x) \mapsto \rho^{\frac{n-1}{p-q}} u(\rho x)$, $\rho > 0$. Moreover, $U$ achieves the best constant $S_{1,q,0}(\alpha)$ and it is given by
\[
U(|x|) = \left( \frac{q}{p} \frac{|n-p+\alpha|}{(p-1)^{p-1}} \right)^{\frac{1}{q-p}} \left( 1 + |x|^{\frac{(n-p+\alpha)(q-p)}{p(p-1)}} \right)^{\frac{p}{p-q}}.
\]
Notice that no a-priori assumption on the sign of $u$ is needed in Theorem 1.4.

Now assume $k = 2$ and $\alpha \notin \{2p-n, np-n\}$. Then $c_{\alpha,2,p} > 0$ by Theorem 1.1, and for any $q > p$ the infima $S_{2,q,0}(\alpha)$, $S_{2,q,1}(\alpha)$ are both positive and achieved by Theorem 1.3. In Section 4.2 we prove our main result concerning the best constant $c_{\alpha,2,p}$.

Theorem 1.5 If $\alpha \notin \{2p-n, np-n\}$ and $q > p$, then the equation
\[
\Delta (|x|^\alpha \Delta U)^{p-2} \Delta U + \text{div} (|x|^{-\beta_1,q} |\nabla U|^{q-2} \nabla U) = 0 \tag{1.15}
\]
has a unique nontrivial solution $U \in \mathcal{D}^{1,p}_r(\mathbb{R}^n; |x|^\alpha dx)$. More precisely, $U$ achieves the best constant $S_{2,q,1}(\alpha)$ and it is given by
\[
U(x) = k \int_{|x|}^\infty t^{1-\frac{\alpha}{p-q}} \left( 1 + t^{\frac{(np-n-\alpha)(q-p)}{p(p-1)}} \right)^{\frac{p}{p-q}} dt \quad \text{if } \alpha > 2p-n
\]
\[
U(x) = k \int_0^{|x|} t^{1-\frac{\alpha}{p-q}} \left( 1 + t^{\frac{(np-n-\alpha)(q-p)}{p(p-1)}} \right)^{\frac{p}{p-q}} dt \quad \text{if } \alpha < 2p-n,
\]
where
\[
k = \left( \frac{q}{p} \frac{|np-n-\alpha|^p}{(p-1)^{p-1}} \right)^{\frac{1}{q-p}}.
\]
Thanks to Theorem 1.5 the sharp value of $S_{2, q, 1}(\alpha)$ is computable in terms of Gamma functions, as in [3], [10] and [28]. We refer to Remarks 4.11 and 4.12 for few comments about the "critical cases" $\alpha = 2p - n$, $\alpha = np - n$.

Let us point out a corollary of Theorem 1.5 that deals with the limiting embedding $D^{2,p}(\mathbb{R}^n) \hookrightarrow D^{1,p^*}(\mathbb{R}^n)$, where $p > 2n$ and $p^* = \frac{np}{n-p}$ is the (first order) critical exponent (see also Remark 4.14 in Section 4.2).

As usual, we denote by $\Delta_{p^*} = \text{div} (|\nabla \cdot |^{p^* - 2} \nabla \cdot )$ the $p^*$-Laplace operator.

**Corollary 1.6** Assume that $n > 2p$. Then the problem

$$
\Delta \left(|\Delta U|^{p-2} \Delta U\right) + \Delta_{p^*} U = 0 \quad (1.16)
$$

has a unique nontrivial solution $U \in D^{2,p}(\mathbb{R}^n)$. More precisely, $U$ achieves

$$
S_{2,p}^* := \inf_{u \in D^{2,p}(\mathbb{R}^n), u \neq 0} \frac{\int_{\mathbb{R}^n} |\Delta u|^p \, dx}{\left(\int_{\mathbb{R}^n} |\nabla u|^{p^*} \, dx\right)^{p/p^*}} \quad (1.17)
$$

and it is given by

$$
U(x) = n \frac{n-p}{p} \left(\frac{n(p-1)}{n-p}\right)^{\frac{n-p}{p}} \int_{|x|}^{\infty} s \left(1 + s^{p^*}\right)^{\frac{n-p}{p}} \, ds.
$$

There are a few ways to prove inequalities like (1.1) or (1.3). In [19] Mitidieri applied his powerful "simple approach" to compute $c_{\alpha, 2, p}$, among other best constants, when $\gamma_{\alpha, 2} \geq 0$. Pointwise estimates are frequently used to obtain integral inequalities, see for instance the papers [5] by Caffarelli, Kohn and Nirenberg and the more recent [1], [2]. In presence of symmetries, Calanchi-Ruf [6] and de Figueiredo-dos Santos-Miyagaki [11] obtained embedding results as corollaries of a radial lemma (in the spirit of [18] and [23]).

Here we use a different approach. We start in Section 2 by proving Theorems 1.2 and 1.1 via the Hardy inequality for functions of one real variable. No pointwise estimates are needed. To study (1.3) we argue in the opposite direction with respect to the above mentioned papers: first we prove certain embedding theorems, and then we infer the desired inequalities. We focus our attention on (1.3) even if we
can obtain also pointwise estimates, radial lemmas and informations on the regularity of radial functions, see Remark 5.7. Roughly speaking, to get more inequalities in case $c_{\alpha,k,p} > 0$ we define in Section 3 a $k$-th order Emden-Fowler transform $D_{k,p}^R(\mathbb{R}^n; |x|^\alpha dx) \to W^k_p(\mathbb{R})$ and we show that the induced norm on $W^k_p(\mathbb{R})$ is equivalent to the standard one. Then, classical results about the space $W^k_p(\mathbb{R})$ provide embedding theorems for $D_{k,p}^R(\mathbb{R}^n; |x|^\alpha dx)$, and the inequalities we are interested in readily follow.

The first step in this program consists in highlighting a suitable class of equivalent norms on the Sobolev spaces $W^k_p(\mathbb{R})$. We start with the lowest indexes $k = 1$ and $k = 2$ in Sections 3.1 and 4.1, respectively. The higher order case $k \geq 3$ will be briefly discussed in Section 5.1.

**Notation.** We denote by $c$ any nonnegative universal constant.

We set $\mathbb{R}_+ = (0, \infty)$. For any integer $n \geq 2$ we denote by $\omega_n$ the $(n-1)$ dimensional measure of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$.

If $\Omega \subseteq \mathbb{R}^n$ is a rotationally invariant domain and $k \geq 0$ is an integer, we denote by $C^k_c(\Omega)$ the space of radially symmetric functions $u \in C^k(\Omega)$. For $u \in C^1_c(\Omega)$ we let $u'$ be the radial derivative of $u$. Thus $|x|^q u'(x) = \nabla u(x) \cdot x$.

The exponent $q' = \frac{n}{q}$ is conjugate exponent to $q \in (1, \infty)$.

Let $\omega$ be a non-negative measurable function on a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$. The weighted Lebesgue space $L^q(\Omega; \omega(x) \, dx)$ is the space of measurable maps $u$ in $\Omega$ with finite norm $(\int_\Omega |u|^q \omega(x) \, dx)^{1/q}$. For $\omega \equiv 1$ we denote by $\|u\|_q$ the standard norm in $L^q(\Omega) = L^q(\Omega; \, dx)$.

The norm in the Sobolev space $W^{k,q}(\mathbb{R})$ is given by

$$\|g\|_{W^{k,q}} = \left( \int_\mathbb{R} |g^{(k)}| q \, ds + \int_\mathbb{R} |g|^q \, ds \right)^{1/q}.$$  

If $n > kp$ then the space $D^{k,p}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\| = \left( \int_{\mathbb{R}^n} |\nabla^k u|^p \, dx \right)^{1/p}.$$  

We put $D^{k,p}_r(\mathbb{R}^n) = \{ u \in D^{k,p}(\mathbb{R}^n) \mid u = u(|x|) \}$.

Let $p_{rb} = \frac{nq}{n-kp}$ be the $k$-th order critical exponent. The radial Sobolev constant

$$S_{k,p}^{rb} := \inf_{u \in D^{k,p}_r(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla^k u|^p \, dx}{\left( \int_{\mathbb{R}^n} |u|^{p_{rb}} \, dx \right)^{p/p_{rb}}}.$$
is positive and achieved (see also Theorem 1.3 and Remark 5.9 in Section 4.2 below).

Assume that \( k = 1 \) or \( p = 2 \). Then it is well known that \( S^*_{k,p} \) is the best constant in the embedding \( D^k,p(R^n) \rightarrow L^{p^*}(R^n) \), that is,

\[
S^*_{k,p} = \inf_{u \in D^k,p(R^n)} \frac{\int_{R^n} |\nabla^k u|^p dx}{\left( \int_{R^n} |u|^{p^*} dx \right)^{p/p^*}},
\]

see [3], [28] and for instance [13] for the poliharmonic case. The notation \( k^* \) means \( k^* \ldots k^* \ldots k^* \ldots \).

If \( k \in \{1, 2\} \) we write \( *, ** \) instead of \( 1^*, 2^* \), respectively.

## 2 Higher order Hardy-Rellich inequalities

Throughout this paper we will use the Hardy inequality for functions in \( C^1_c(R_+) \) several times. We recall that for any \( a \in \mathbb{R}, p > 1 \) and \( \omega \in C^1_c(R_+) \) the inequality

\[
\int_0^\infty r^a|\omega'|^p dr \geq \left| \frac{a + 1 - p}{p} \right|^p \int_0^\infty r^{a-p}|\omega|^p dr \tag{2.1}
\]

holds with a sharp and non achieved constant in the right hand side, see [15], [16]. We notice that (2.1) holds as well in the ”singular case” \( a = p - 1 \), with null best constant. For, use \( \omega_\varepsilon(r) = \omega(r^\varepsilon) \) as test function, where \( \omega \in C^1_c(R_+) \setminus \{0\} \) is fixed, to get

\[
\frac{\int_0^\infty r^{p-1}|\omega'_\varepsilon|^p dr}{\int_0^\infty r^{-1}|\omega'_\varepsilon|^p dr} = \varepsilon^p \frac{\int_0^\infty r^{p-1}|\omega'|^p dr}{\int_0^\infty r^{-1}|\omega|^p dr} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.
\]

We point out a simple but very useful corollary to the Hardy inequality.

**Lemma 2.1** Let \( \tau, \lambda \in \mathbb{R}, p > 1 \) and \( v \in C^2_c(R_+) \). Then the inequalities

\[
\int_0^\infty r^{\tau} |v'' + (\lambda - 1)r^{-1}v'|^p dr \geq \left| \frac{\tau + 1 - \lambda p}{p} \right|^p \int_0^\infty r^{\tau-p} |v'|^p dr \tag{2.2}
\]

\[
\int_0^\infty r^{\tau} |v'' + (\lambda - 1)r^{-1}v'|^p dr \geq \left| \frac{(\tau + 1 - \lambda p)(\tau + 1 - 2p)}{p^2} \right|^p \int_0^\infty r^{\tau-2p} |v|^p dr \tag{2.3}
\]

hold with sharp constants.
Proof. Inequalities (2.2), (2.3) are immediate consequences of (2.1), since for any \( v \in C^2_0(\mathbb{R}_+) \) we have that

\[
\int_0^\infty r^\tau |v'' + (\lambda - 1)r^{-1}v'|^p dr = \int_0^\infty r^{\tau-(\lambda-1)p} \left| r^{\lambda-1}v' \right|^p dr \\
\geq \frac{|\tau + 1 - \lambda p|}{p} \int_0^\infty r^{\tau-p} |v'|^p dr \\
\geq \frac{|\tau + 1 - \lambda p|}{p} \left| \int_0^\infty r^{\tau-2p} |v|^p dr \right|.
\]

To prove the sharpness of the constants in (2.2), (2.3) we fix a nontrivial function \( v \in C^2_0(\mathbb{R}_+) \) and we put

\[
v_\varepsilon(r) = \frac{1}{\varepsilon} r^{2+\frac{1+\tau}{p}} v(\varepsilon),
\]

where \( \varepsilon \rightarrow 0^+ \). To conclude, it suffices to compute

\[
\int_0^\infty r^\tau |v_\varepsilon'' + (\lambda - 1)r^{-1}v_\varepsilon'|^p dr = \frac{\int_0^\infty (\tau + 1 - \lambda p)(\tau + 1 - 2p) r^{-1}|v|^p dr + o(1)}{p^2} \\
\int_0^\infty r^{\tau-p} |v_\varepsilon'|^p dr = \frac{\tau + 1 - 2p}{p} \int_0^\infty r^{-1}|v|^p dr + o(1) \\
\int_0^\infty r^{\tau-2p} |v_\varepsilon|^p dr = \int_0^\infty r^{-1}|v|^p dr.
\]

\[\square\]

2.1 Proof of Theorem 1.2

Fix any \( u \in C^{2m}_{c,r}(\mathbb{R}^n \setminus \{0\}) \). Since

\[
\int_{\mathbb{R}^n} |x|^{\alpha} |\Delta u|^p dx = \omega_n \int_0^\infty r^{n-1+\alpha}|u'' + (n-1)r^{-1}u'|^p dr \\
\int_{\mathbb{R}^n} |x|^{\alpha-p} |\nabla u|^p dx = \omega_n \int_0^\infty r^{n-1+\alpha-p}|u'|^p dr,
\]

then from Lemma 2.1 it follows that

\[
\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx \geq \left| n - \frac{n + \alpha}{p} \right| \int_{\mathbb{R}^n} |x|^{\alpha-p} |\nabla u|^p dx
\]

and that the constant in the right hand side is the best possible. Thus (1.9) is proved when \( m = 1 \).

To conclude the proof in case \( m > 1 \) it suffices to write down (2.4) with \( \Delta^{m-1} u \in C^2_{c,r}(\mathbb{R}^n \setminus \{0\}) \) instead of \( u \).

\[\square\]
2.2 Proof of Theorem 1.1

Let $\tilde{c}_{\alpha,k,p}$ be the best constant in (1.1), and let $c_{\alpha,k,p}$ be the constant defined in (1.7). We start by proving that $\tilde{c}_{\alpha,k,p} \geq c_{\alpha,k,p}$ in case $k = 2m$ is an even integer. We have to show that

$$\int_{\mathbb{R}^n} |x|^{\alpha} |\Delta^m u|^p \, dx \geq \left( \prod_{h=1}^{m} |\gamma_{\alpha,2h}|^p \right) \int_{\mathbb{R}^n} |x|^{\alpha-2mp} |u|^p \, dx \quad \forall u \in C_{c,r}^{2m}(\mathbb{R}^n \setminus \{0\}).$$

(2.5)

If $m = 1$ then (2.5) reduces to (1.8), that is an immediate consequence of (2.4) and of the Hardy inequality (1.5). Assume that (2.5) holds for some $m \geq 1$ and for any $\alpha \in \mathbb{R}$. Fix $u \in C_{c,r}^{2m+2}(\mathbb{R}^n \setminus \{0\})$ and use (1.8) to infer

$$\int_{\mathbb{R}^n} |x|^{\alpha-2mp} |\Delta u|^p \, dx \geq |\gamma_{\alpha,2m+2}|^p \int_{\mathbb{R}^n} |x|^{\alpha-2(m+1)p} |u|^p \, dx,$$

since $\gamma_{\alpha-2mp,2} = \gamma_{\alpha,2m+2}$. Thus we have that

$$\int_{\mathbb{R}^n} |x|^{\alpha} |\Delta^{m+1} u|^p \, dx = \int_{\mathbb{R}^n} |x|^{\alpha} |\Delta^m (\Delta u)|^p \, dx$$

$$\geq \left( \prod_{h=1}^{m} |\gamma_{\alpha,2h}|^p \right) \int_{\mathbb{R}^n} |x|^{\alpha-2mp} |\Delta u|^p \, dx$$

$$\geq \left( \prod_{h=1}^{m} |\gamma_{\alpha,2h}|^p \right) |\gamma_{\alpha,2m+2}|^p \int_{\mathbb{R}^n} |x|^{\alpha-2(m+1)p} |u|^p \, dx,$$

as desired. If $k = 2m + 1$ is odd we use the Hardy inequality and the first part of the proof to get

$$\int_{\mathbb{R}^n} |x|^{\alpha} |\nabla (\Delta^m u)|^p \, dx \geq |H_{\alpha}|^p \int_{\mathbb{R}^n} |x|^{\alpha-p} |\Delta^m u|^p \, dx$$

$$\geq |H_{\alpha}|^p \left( \prod_{h=1}^{m} |\gamma_{\alpha-p,2h}|^p \right) \int_{\mathbb{R}^n} |x|^{\alpha-p} |\Delta^m u|^p \, dx$$

for any $u \in C_{c,r}^{2m+1}(\mathbb{R}^n \setminus \{0\})$. Thus $\tilde{c}_{\alpha,k,p} \geq c_{\alpha,k,p}$ also in this case, since $\gamma_{\alpha-p,2h} = \gamma_{\alpha,2h+1}$ for any integer $h$.

To prove that $\tilde{c}_{\alpha,k,p} \leq c_{\alpha,k,p}$, fix a nontrivial function $u \in C_{c,r}^{k}(\mathbb{R}^n \setminus \{0\})$. For any $\varepsilon > 0$ define the radial function $u_{\varepsilon}(|x|) = |x|^{k-\frac{n+\alpha}{p}} u(|x|^{\varepsilon})$. Direct computations and
induction can be used to check that
\[
\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u_\varepsilon|^p \, dx = c_{\alpha,k,p} \varepsilon^{-1} \int_{\mathbb{R}^n} |x|^{-n} |u + \varepsilon \psi_\varepsilon|^p \, dx
\]
\[
\int_{\mathbb{R}^n} |x|^\alpha |\nabla^{k-1} u_\varepsilon|^p \, dx = \varepsilon^{-1} \int_{\mathbb{R}^n} |x|^{-n-2} |u|^p \, dx,
\]
where the radial function \( \psi_\varepsilon \in C^0_0(\mathbb{R}^n) \) is a linear combination of derivatives of \( u \), such that \( \sup_\varepsilon \| \psi_\varepsilon \|_\infty < \infty \). The conclusion readily follows. \( \Box \)

**Remark 2.2** An alternative proof of Theorem 1.1 is suggested in Remark 5.5.

**Remark 2.3** Similar computations allow us to find the best constant \( c_{\alpha,k,j,p} \) in (1.10). Theorem 1.2 and the Hardy inequality provide the values of \( c_{\alpha,k,j,p} \) for \( j = k - 1 \) and \( j = k - 2 \). For smaller indexes we put \( \delta_j = n - 1 - \frac{n+\alpha}{p} + k - j \). If \( k = 2m \) is even we have
\[
c_{\alpha,2m,j,p} = \begin{cases} 
\prod_{h=1}^{m-i} |\gamma_{2h}|^p & \text{if } j = 2i \\
|\delta_j|^p \prod_{h=1}^{m-i-1} |\gamma_{2h}|^p & \text{if } j = 2i + 1,
\end{cases}
\]
while if \( k = 2m + 1 \) it results that
\[
c_{\alpha,2m+1,j,p} = \begin{cases} 
|H_{\alpha}|^p \prod_{h=1}^{m-i} |\gamma_{2h+1}|^p & \text{if } j = 2i, \\
|H_\alpha\delta_j|^p \prod_{h=1}^{m-i-1} |\gamma_{2h+1}|^p & \text{if } j = 2i + 1.
\end{cases}
\]

### 3 The Emden-Fowler transform and first order inequalities

From now on we will assume that \( c_{\alpha,k,p} \) is positive, that is,
\[
\begin{cases}
\gamma_{2h} \neq 0 & \forall h = 1, \ldots, m & \text{if } k = 2m \text{ is even}, \\
\gamma_{2h+1} \neq 0 & \forall h = 1, \ldots, m \text{ and } H_\alpha \neq 0 & \text{if } k = 2m + 1 \text{ is odd}.
\end{cases}
\]

(3.1)
We study the properties of the Banach space $D^{k,p}_{r} (\mathbb{R}^n; |x|\alpha \, dx)$ by using a suitable transform $W^{k,p} (\mathbb{R}) \to D^{k,p}_{r} (\mathbb{R}^n; |x|\alpha \, dx)$ and by taking advantage of the results in the previous section. More precisely, for any $k \geq 0$ we define the (inverse) $k$-th order Emden-Fowler transform

$$T_k : C^k_c (\mathbb{R}) \to C^k_c, r (\mathbb{R}^n \setminus \{0\}) \quad (T_k g)(x) = |x|^{-H_{\alpha,k}} g(- \log |x|),$$

(3.2)

where we have set

$$H_{\alpha,k} = \frac{n + \alpha}{p} - k. \quad (3.3)$$

Notice that $H_{\alpha,1} = H_{\alpha}$, compare with (1.6). Then we show that $T_k$ extends to a bicontinuous isomorphism $W^{k,p} (\mathbb{R}) \to D^{k,p}_{r} (\mathbb{R}^n; |x|\alpha \, dx)$ for any $k \geq 1$. A crucial step in this program consists in finding out a large class of equivalent norms in $W^{k,p} (\mathbb{R})$.

We start with the lower order case $k = 1$. The cases $k = 2$ and $k \geq 3$ will be studied in the last two sections.

### 3.1 Equivalent norms on $W^{1,p} (\mathbb{R})$

We point out a simple lemma, based on the Hardy inequality for functions in $C^1_c (\mathbb{R}^+)$. 

**Lemma 3.1** Let $p > 1$ and $\lambda \in \mathbb{R}$. Then

$$M_p (\lambda) := \inf_{f \in W^{1,p} (\mathbb{R}), f \neq 0} \frac{\int_\mathbb{R} |f' - \lambda f|^p \, ds}{\int_\mathbb{R} |f|^p \, ds} = |\lambda|^p.$$

**Proof.** To any $f \in C^1_c (\mathbb{R})$ we associate the function $v(r) := r^\lambda f(- \log r)$. Then $v \in C^1_c (\mathbb{R}^+)$ and a direct computation shows that

$$\frac{\int_\mathbb{R} |f' - \lambda f|^p \, ds}{\int_\mathbb{R} |f|^p \, ds} = \frac{\int_0^\infty r^{(1-\lambda)p-1}|v'|^p \, dr}{\int_0^\infty r^{-\lambda p-1}|v|^p \, dr}.$$

The conclusion readily follows by using (2.1) and a density argument. \qed
Now we take an exponent \( q > p \) and we study the infimum
\[
M_{p,q}(\lambda) := \inf_{f \in W^{1,p}(\mathbb{R}), f \neq 0} \frac{\int_{\mathbb{R}} |f' - \lambda f|^p \, ds}{\left( \int_{\mathbb{R}} |f|^q \, ds \right)^{p/q}}. \tag{3.4}
\]

**Remark 3.2** A standard rescaling argument can be used to check that \( M_{p,q}(0) = 0 \).

The next proposition gives us the equivalent norms we need in case \( k = 1 \). Its proof is immediate, by Lemma 3.1 and by Sobolev embedding theorem.

**Proposition 3.3** Let \( p > 1 \) and \( \lambda \in \mathbb{R} \setminus \{0\} \). Then
\[
\|f\| := \left( \int_{\mathbb{R}} |f' - \lambda f|^p \, ds \right)^{1/p}
\]
is equivalent to the standard norm on \( W^{1,p}(\mathbb{R}) \). Thus, for any \( q > p \) the infimum \( M_{p,q}(\lambda) \) is positive.

**Remark 3.4** The minimization problem in (3.4) is non compact, due to translations in \( \mathbb{R} \). By nowadays standard arguments one can prove that for every bounded minimizing sequence \( f_h \), there exists a sequence \( s_h \) in \( \mathbb{R} \) such that \( f_h(\cdot - s_h) \) is relatively compact in \( W^{1,p}(\mathbb{R}) \). Hence, \( M_{p,q}(\lambda) \) is attained by some function \( f \neq 0 \) which solves
\[
- (|f'|^2 - \lambda |f|^p - (f' - \lambda f))^2 - \lambda |f'|^2 - \lambda |f|^p - (f' - \lambda f) = |f|^{q-2}f \quad \text{on } \mathbb{R} \tag{3.5}
\]
up to a Lagrange multiplier.

If \( p = 2 < q \), then extremals for
\[
M_{2,q}(\lambda) = \inf_{f \in H^1(\mathbb{R}), f \neq 0} \frac{\int_{\mathbb{R}} |f' - \lambda f|^2 \, ds}{\left( \int_{\mathbb{R}} |f|^q \, ds \right)^{2/q}} = \inf_{f \in H^1(\mathbb{R}), f \neq 0} \frac{\int_{\mathbb{R}} (|f'|^2 + \lambda^2 |f|^2) \, ds}{\left( \int_{\mathbb{R}} |f|^q \, ds \right)^{2/q}}
\]
give rise to nontrivial solutions of the Emden-Fowler (or Schrödinger) equation
\[
- f'' + \lambda^2 f = |f|^{q-2}f \quad \text{on } \mathbb{R}. \tag{3.6}
\]
It has been shown in [10] that, up to translations, equation (3.6) has a unique positive solution \( F \in H^1(\mathbb{R}) \), which is explicitly known. The interest of Catrina and Wang in the ODE (3.6) was motivated by its relevance with the Caffarelli-Kohn-Nirenberg inequalities in the Hilbertian case \( p = 2 \).

Now we state a uniqueness result for nontrivial solutions \( f \in W^{1,p}(\mathbb{R}) \) to (3.5). Notice that we do not require any sign assumption on \( f \). Thus some care is needed, as the exponents \( p, q \) might be smaller than 2.

We identify functions that coincide up to a translation and a change of sign.

**Theorem 3.5** Let \( q > p > 1 \) and \( \lambda \in \mathbb{R} \setminus \{0\} \). Then the ordinary differential equation (3.5) has a unique nontrivial solution \( F \in W^{1,p}(\mathbb{R}) \). More precisely, \( F \) achieves the best constant \( M_{p,q}(\lambda) \), and it is given by

\[
F(s) = \left( q \left( \frac{p}{p-1} \right)^{p-1} \left| \frac{\lambda}{2} \right|^{\frac{1}{q-p}} e^{\frac{\lambda(q-2)}{2(p-1)} s} \right)^{\frac{p}{p-2}}. \quad (3.7)
\]

**Proof.** Let \( f \in W^{1,p}(\mathbb{R}) \setminus \{0\} \) be a solution to (3.5), and put

\[
\varphi = |f' - \lambda f|^{p-2} (f' - \lambda f).
\]

The pair \( f, \varphi \) solves

\[
\begin{cases}
  f' - \lambda f = |\varphi|^{p'-2} \varphi \\
  -\varphi' - \lambda \varphi = |\varphi|^{q-2} \varphi
\end{cases} \quad (3.8)
\]

in the sense of distributions. Notice that \( p', q \) satisfy the standard anticoercivity assumption \((p' - 1)(q - 1) > 1\). Clearly, \( \varphi \in L^{p'}(\mathbb{R}) \) and \( \|\varphi\|_{p'} = \|f' - \lambda f\|_p \). Since

\[
-\varphi' = \lambda \varphi + |f|^{q-2} f \in L^{p'}(\mathbb{R})
\]

by Sobolev embeddings, then \( \varphi \in W^{1,p'}(\mathbb{R}) \). Thus \( \varphi \in C^1(\mathbb{R}) \), as \( f \) and \( \varphi \) are continuous function. But then also \( f \) is of class \( C^1 \), since \( f' = \lambda f + |\varphi|^{p'-2} \varphi \). Thus the pair \( f, \varphi \) is a classical homoclinic solution to (3.8).

The system (3.8) is conservative, with with Hamiltonian energy

\[
H(f, \varphi) = \lambda f \varphi + \frac{1}{q} |f|^q + \frac{1}{p'} |\varphi|^{p'}.
\]

In particular, (3.8) is equivalent to

\[
\begin{cases}
  f' = \partial_2 H(f, \varphi) \\
  \varphi' = -\partial_1 H(f, \varphi)
\end{cases} \quad (3.9)
\]
From $f \in W^{1,p}(\mathbb{R}), \varphi \in W^{1,p'}(\mathbb{R})$ one infers that $f, \varphi$ vanish at infinity, and therefore

$$\lambda f \varphi + \frac{1}{q} |f|^q + \frac{1}{p'} |\varphi|^{p'} = 0.$$ (3.10)

Notice that $\lambda f \varphi < 0$ on the set $\{f \neq 0\} = \{\varphi \neq 0\}$. We can assume that $f$ achieves its positive maximum at some point $s_0$. Using $f'(s_0) = 0$, (3.8) and (3.10) one can uniquely compute the values of $f(s_0) > 0$ and $\lambda \varphi(s_0) < 0$. Since for any initial datum $f_0 > 0, \varphi_0 \neq 0$ the Cauchy problem for (3.9) has a unique local solution, to conclude the proof we only have to show that the pair $F, \Phi$ solves (3.8), where

$$\Phi = |F' - \lambda F|^{p-2}(F' - \lambda F).$$

In order to avoid long computations one can argue as follows. Put

$$k = \left( q \left( \frac{p}{p-1} \right)^{p-1} \left| \frac{\lambda}{2} \right|^p \right)^{\frac{1}{q-p}}, \quad c_1 = \frac{\lambda(p-2)}{2(p-1)}, \quad c_2 = \frac{\lambda(q-p)}{2(p-1)},$$

so that $F(s) = ke^{c_1s} (\cosh c_2s)^{p-q}$, and compute

$$\Phi = - \left( k \frac{p}{2(p-1)} \right)^{p-1} |\lambda|^{p-2} \lambda e^{(p-1)(c_1+c_2)s} \cosh c_2s \left( \frac{p-1}{p-q} \right).$$

Now it is easy to check that the pair $F, \Phi$ satisfies the conservation law (3.10), that is sufficient to conclude that $F, \Phi$ solves (3.8), as desired. □

### 3.2 The space $D^{1,p}_r(\mathbb{R}^n; |x|\alpha dx)$

If $\alpha \neq p - n$, then the Banach space $D^{1,p}_r(\mathbb{R}^n; |x|\alpha dx)$, endowed with the norm

$$\|u\|_{1,\alpha}^p = \int_{\mathbb{R}^n} |x|^{\alpha} |\nabla u|^p \, dx,$$

is continuously embedded into $L^p(\mathbb{R}^n; |x|^{\alpha-p} dx)$ by the Hardy inequality.

For $g \in C^1_c(\mathbb{R})$ we put $$(T_1 g)(x) = |x|^{-H_\alpha} g(- \log |x|),$$ where $H_\alpha = \frac{\alpha + \alpha}{p} - 1$. Then clearly $T_1 : C^1_c(\mathbb{R}) \to C^1_{c,r}(\mathbb{R}^n \setminus \{0\})$ is a linear, invertible transform.

**Lemma 3.6** Assume $\alpha \neq p - n$.

i) The transform $T_1$ can be extended in a unique way to a bicontinuous isomorphism $W^{1,p}(\mathbb{R}) \to D^{1,p}_r(\mathbb{R}^n; |x|\alpha dx)$.  

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ii) If \( \alpha > p - n \) then \( C^1_c(\mathbb{R}^n) \subset D_1^{1,p}(\mathbb{R}^n; |x|^\alpha dx) \). In particular, if \( p < n \) then \( D_1^{1,p}(\mathbb{R}^n; |x|^0 dx) = D_1^{1,p}(\mathbb{R}^n) \).

Proof. Notice that
\[
\int_{\mathbb{R}^n} |x|^\alpha |\nabla (T_1 g)|^p \, dx = \omega_n \int_{\mathbb{R}} |g' + H_\alpha g|^p \, ds
\]
for any \( g \in C^1_c(\mathbb{R}) \). Therefore i) follows from Proposition 3.3 as \( H_\alpha \neq 0 \).

To prove ii), take any \( u \in C^1_c(\mathbb{R}^n) \), and let \( g = T_1^{-1} u \). Then \( g \equiv 0 \) for \( s << 0 \) and \( g(s), g'(s) = O(e^{-H_\alpha s}) \) for \( s \to \infty \). Since \( H_\alpha > 0 \) then \( g \) and \( g' \) decay exponentially at infinity, and therefore \( g \in W^{1,p}(\mathbb{R}) \). Thus \( u \in D_1^{1,p}(\mathbb{R}^n; |x|^\alpha) \) by i), as desired.

Proof of Theorem 1.4. Using the definitions and the results in Section 3.1, it is easy to compute
\[
S_{1,q,0}(\alpha) = \inf_{u \in D_1^{1,p}(\mathbb{R}^n; |x|^\alpha dx), \ u \neq 0} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\nabla u|^p \, dx}{\left( \int_{\mathbb{R}^n} |x|^{-\beta_{1,q}} |u|^q \, dx \right)^{p/q}} = \omega_n^{2-p} M_{p,q}(-H_\alpha) ,
\]
where \( \beta_{1,q} \) is defined in (1.4). Moreover, \( u = T_1 g \) solves (1.12) if and only if \( g \) is a weak solution to (3.5), where \( \lambda = -H_\alpha \). The conclusion follows by Theorem 3.5.

Remark 3.7 The uniqueness result in Theorem 1.4 has been already stated and used in [4], without proof.

4 Second order inequalities

The main results in this section concerns the infima \( S_{2,q,0}(\alpha) \), \( S_{2,q,1}(\alpha) \) and their extremals. We follow the same scheme as in the previous section, that is, we first prove few results about the space \( W^{2,p}(\mathbb{R}) \); we will turn our attention to \( D_1^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \) in subsection 4.2.
4.1 Equivalent norms on $W^{2,p}(\mathbb{R})$

We start with a preliminary result.

**Lemma 4.1** Let $p > 1$ and $A, \gamma \in \mathbb{R}$ with $A^2 + \gamma \geq 0$. Then

$$I_p(A, \gamma) := \inf_{\substack{g \in W^{2,p}(\mathbb{R}) \\ g \neq 0}} \frac{\int_{\mathbb{R}} |g'' - 2Ag' - \gamma g|^p \, ds}{\int_{\mathbb{R}} |g|^p \, ds} = |\gamma|^p$$

and $I_p(A, \gamma)$ is not achieved.

**Proof.** Put $\lambda := 2 + 2\sqrt{A^2 + \gamma}$, $b := 2 + \sqrt{A^2 + \gamma} - A$, so that $(b - 2)(\lambda - b) = \gamma$.

To any $g \in C^2_0(\mathbb{R})$ we associate the function $v(r) := r^2 - b g(-\log r)$. By direct computation and using Lemma 2.1 we easily infer

$$I_p(A, \gamma) = \inf_{\substack{v \in C^2_0(\mathbb{R}_+) \\ v \neq 0}} \frac{\int_{0}^{\infty} r^{p-1} |v'' + (\lambda - 1)r^{-1}v'|^p \, dr}{\int_{0}^{\infty} r^{p-2p-1}|v|^p \, dr} = |(b - \lambda)(b - 2)|^p = |\gamma|^p,$$

and the first claim is proved. By contradiction, assume that there exists $g \in W^{2,p}(\mathbb{R})$ such that $\int_{\mathbb{R}} |g|^p \, ds = 1$ and $\int_{\mathbb{R}} |g'' - 2Ag' - \gamma g|^p \, ds = |\gamma|^p$. Then $\gamma \neq 0$, since $g \equiv 0$ is the only function in $W^{2,p}(\mathbb{R})$ that solves $-g'' + 2Ag' = 0$ on $\mathbb{R}$. Thus we also have that $b \neq 2$ and $b \neq \lambda$. Now we put $v(r) = r^{2-b} g(-\log r)$ as before, and we compute

$$\int_{0}^{\infty} r^{(b-\lambda+1)p-1} |(r^{\lambda-1}v')'|^p \, dr = \int_{\mathbb{R}} |g'' - 2Ag' - \gamma g|^p \, ds = |\gamma|^p.$$

Then we estimate via the Hardy inequality (2.1)

$$\int_{0}^{\infty} r^{(b-\lambda)p-1} |(r^{\lambda-1}v')'|^p \, dr \geq |b - 2|^p \int_{0}^{\infty} r^{(b-2)p-1} |v|^p \, dr = |b - 2|^p \int_{\mathbb{R}} |g|^p \, ds = |b - 2|^p.$$

This is impossible, as $\lambda - b = \frac{\gamma}{b-2}$ and since the best constant in the Hardy inequality

$$\int_{0}^{\infty} r^{(b-\lambda+1)p-1} |\omega'|^p \, dr \geq |\lambda - b|^p \int_{0}^{\infty} r^{(b-\lambda)p-1} |\omega|^p \, dr$$

is not achieved. The lemma is completely proved. $\square$
Now we take an exponent $q > p$ and we study the infimum

$$I_{p,q}(A, \gamma) := \inf_{g \in \mathcal{W}^2,p(\mathbb{R}) \setminus \{0\}} \int_{\mathbb{R}} \left( g'' - 2Ag' - \gamma g \right)^p ds \left( \int_{\mathbb{R}} |g|^q ds \right)^{p/q}.$$

**Remark 4.2** If $\gamma = 0$ then $I_{p,q}(A,0) = 0$ for any $A \in \mathbb{R}$. For, take $g \in C^2_c(\mathbb{R}) \setminus \{0\}$ and test $I_{p,q}(A,0)$ with $g_\varepsilon(s) = g(\varepsilon s)$, where $\varepsilon \to 0^+$, to get

$$I_{p,q}(A,0) \leq \int_{\mathbb{R}} \left( |g''_\varepsilon - 2Ag'_\varepsilon|^p ds \right)^{p/q} \left( \int_{\mathbb{R}} |g_{\varepsilon}|^q ds \right)^{p/q} \varepsilon^p = o(1).$$

**Proposition 4.3** Let $p > 1$ and $A, \gamma \in \mathbb{R}$ with $A^2 + \gamma \geq 0$. If $\gamma \neq 0$ then

$$\|g\|_{A,\gamma} := \left( \int_{\mathbb{R}} |g'' - 2Ag' - \gamma g|^p ds \right)^{1/p}$$

is an equivalent norm on $W^{2,p}(\mathbb{R})$. Moreover, for any $q > p$ the infimum $I_{p,q}(A,\gamma)$ is positive and achieved in $W^{2,p}(\mathbb{R})$.

**Proof.** Fix a small $\varepsilon > 0$ such that $|2A|\varepsilon \leq 1/2$. We recall that there exists a constant $C_\varepsilon > 0$ such that $\|g'\|_p \leq \varepsilon \|g''\|_p + C_\varepsilon \|g\|_p$ for any $g \in W^{2,p}(\mathbb{R})$. Using also Lemma 4.1 we find that

$$\|g\|_{W^{2,p}} \leq 2 \left( 1 + |2A|C_\varepsilon + \frac{|\gamma| + 1}{|\gamma|} \right) \|g'' - 2Ag' - \gamma g\|_p$$

$$\|g'' - 2Ag' - \gamma g\|_p \leq (2 + |2A|C_\varepsilon + |\gamma|) \|g\|_{W^{2,p}}.$$

Thus the norm $\|\cdot\|_{A,\gamma}$ is equivalent to the standard one.

Since $W^{2,p}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ by Sobolev embedding, then $I_{p,q}(A,\gamma) > 0$ by the first part of the proof. By nowadays standard arguments, it is easy to prove that every bounded minimizing sequence for $I_{p,q}(A,\gamma)$ is relatively compact in $W^{2,p}(\mathbb{R})$ up to translations in $\mathbb{R}$. In particular, $I_{p,q}(A,\gamma)$ is attained in $W^{2,p}(\mathbb{R})$. \qed
Now we focus our attention on the inclusions $W^{2,p}(\mathbb{R}) \hookrightarrow W^{1,q}(\mathbb{R})$, where $q \geq p$.

We start with the "linear" case $q = p$.

**Lemma 4.4** Let $p > 1$, $A, \gamma, H \in \mathbb{R}$ with $A^2 + \gamma \geq 0$. Let

$$J_p(A, \gamma, H) := \inf_{g \in W^{2,p}(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} |g'' - 2Ag' - \gamma g|^p \, ds}{\int_{\mathbb{R}} |g'|^p \, ds}.$$ 

i) If $\gamma \neq 0$ then $J_p(A, \gamma, H) > 0$.

ii) $J_p(A, 0, 0) = |2A|^p$.

iii) If $H \neq 0$ and $H^2 + 2AH - \gamma = 0$, then $J_p(A, \gamma, H) = \frac{\gamma^p}{H^p}$.

**Proof.** If $\gamma \neq 0$ then $||g||_{A, \gamma}$ is an equivalent norm on $W^{2,p}(\mathbb{R})$ by Proposition 4.3. Hence $J_p(A, \gamma, H) > 0$, since $W^{2,p}(\mathbb{R}) \hookrightarrow W^{1,p}(\mathbb{R})$.

To check ii) one can reproduce the trick in the proof of Lemma 4.1 or can argue as follows. First notice that $J_p(A, 0, 0) \geq |2A|^p$ by Lemma 3.1. To prove the opposite inequality use a rescaling argument, as in Remark 4.2. For the convenience of the reader we repeat here the proof. Take any $g \in C_c^2(\mathbb{R}) \setminus \{0\}$ and test $J_p(A, 0, 0)$ with $s \mapsto g(\varepsilon s)$, where $\varepsilon \to 0^+$. The conclusion is readily achieved, as

$$J_p(A, 0, 0) \leq \frac{\int_{\mathbb{R}} |\varepsilon g'' - 2Ag'|^p \, ds}{\int_{\mathbb{R}} |g'|^p \, ds} = |2A|^p + o(1).$$

It remains to check iii). Notice that

$$J_p(A, \gamma, H) = \inf_{g \in W^{2,p}(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \left| (g' + Hg)' - \frac{\gamma}{H} (g' + Hg) \right|^p \, ds}{\int_{\mathbb{R}} |g'|^p \, ds} \geq \frac{\gamma^p}{H^p}$$

by Lemma 3.1. Then use rescaling as before to prove the opposite inequality. \[\square\]
In the remaining part of this section we direct our attention to "semilinear" inequalities. For any \( q > p \), \( \lambda \in \mathbb{R} \), the infimum \( M_{p,q}(\lambda) \) has been defined in (3.4).

**Proposition 4.5** Let \( q > p > 1 \), \( A, \gamma, H \in \mathbb{R} \) with \( A^2 + \gamma \geq 0 \), and put

\[
J_{p,q}(A, \gamma, H) := \inf_{g \in W^{2,p} (\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} |g'' - 2Ag' - \gamma g|^p \, ds}{\left( \int_{\mathbb{R}} |g' + Hg|^q \, ds \right)^{p/q}}.
\]

i) If \( \gamma \neq 0 \) then \( J_{p,q}(A, \gamma, H) \) is positive and it is achieved in \( W^{2,p}(\mathbb{R}) \).

ii) \( J_{p,q}(A,0,-2A) = 0 \) for any \( A \in \mathbb{R} \).

iii) \( J_{p,q}(A,0,0) = M_{p,q}(2A) \) and it is not achieved.

**Proof.** If \( \gamma \neq 0 \) then \( \|g\|_{A,\gamma} \) is an equivalent norm on \( W^{2,p}(\mathbb{R}) \) by Proposition 4.3. Thus i) readily follows, as \( W^{2,p}(\mathbb{R}) \hookrightarrow W^{1,q}(\mathbb{R}) \). To prove that \( J_{p,q}(A, \gamma, H) \) is attained use standard arguments in translation-invariant problems, as for Proposition 4.3. Equality \( J_{p,q}(A,0,-2A) = 0 \) can be proved via rescaling, since

\[
J_{p,q}(A,0,-2A) = \inf_{g \in W^{2,p} (\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} |(g' - 2Ag')'|^p \, ds}{\left( \int_{\mathbb{R}} |g' - 2Ag'|^q \, ds \right)^{p/q}}.
\]

To check iii) we first notice that

\[
J_{p,q}(A,0,0) \geq \inf_{f \in W^{1,p} (\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} |f' - 2Af|^p \, ds}{\left( \int_{\mathbb{R}} |f|^q \, ds \right)^{p/q}} = M_{p,q}(2A).
\]

Next, for any function \( f \in C^1_c(\mathbb{R}) \), \( f \neq 0 \) we put \( g(s) = \int_{-\infty}^{s} f(t) \, dt \). Then \( g \) is bounded, \( g(s) \equiv 0 \) for \( s < < 0 \) and \( g(s) \) is a constant for \( s >> 0 \), so that in general \( g \notin L^p(\mathbb{R}) \). Take a function \( \eta \in C^2(\mathbb{R}) \), such that \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) on \((-\infty,1)\) and \( \eta \equiv 0 \) on \((2,\infty)\). We test \( J_{p,q}(A,0,0) \) with the function \( g_h(s) = \eta(h^{-1}s)g(s) \), where \( h \geq 1 \) is an integer. Notice that \( g_h \in W^{2,p}(\mathbb{R}) \) since it is smooth and it has
compact support. It is not difficult to show that \( g_h' \to g' = f \) in \( L^p(\mathbb{R}) \) and in \( L^q(\mathbb{R}) \), \( g_h'' \to g'' = f' \) in \( L^p(\mathbb{R}) \) as \( h \to \infty \). Thus

\[
J_{p,q}(A, 0, 0) \leq \int_{\mathbb{R}} \left| g_h'' - 2Ag_h' \right|^p ds = \int_{\mathbb{R}} \left| f' - 2Af \right|^p ds + o(1).
\]

Thus \( J_{p,q}(A, 0, 0) = M_{p,q}(2A) \), as \( f \) was arbitrarily chosen. It remains to check that \( J_{p,q}(A, 0, 0) \) is not attained. Assume that \( g \in W^{2,p}_{\text{loc}}(\mathbb{R}) \) is a non constant function such that \( g' \in L^p(\mathbb{R}) \cap L^q(\mathbb{R}), \int_{\mathbb{R}} |g'|^q ds = 1, g'' \in L^p(\mathbb{R}) \) and

\[
\int_{\mathbb{R}} \left| g'' - 2Ag' \right|^p ds = J_{p,q}(A, 0, 0) = M_{p,q}(2A).
\]

Then \( g' \in W^{1,p}(\mathbb{R}) \) achieves \( M_{p,q}(2A) \), and hence \( g' \) has constant sign (use a standard convexity argument or Theorem 3.5). In particular \( g \) is monotone, that implies that \( g \not\in L^p(\mathbb{R}) \). Thus \( g \) does not achieve \( J_{p,q}(A, 0, 0) \). \( \square \)

Remark 4.6 If \( A \neq 0 \) then \( J_{p,q}(A, 0, 0) = M_{p,q}(2A) > 0 \) by Proposition 3.3. In Theorem 3.5 we proved that the infimum \( M_{p,q}(2A) \) is achieved by a unique and positive function \( F \in W^{1,p}(\mathbb{R}) \). Therefore, any primitive \( g \) of \( F \) satisfies \( g', g'' \in L^p(\mathbb{R}) \) and \( (4.1) \). However, \( g \not\in W^{2,p}(\mathbb{R}) \) since \( g \) is increasing on \( \mathbb{R} \).

In order to simplify notations we introduce the differential operators

\[
B_+ \eta = \eta' + H\eta, \quad B_- \eta = \eta' - H\eta, \\
L_+ \eta = -\eta'' + 2A\eta' + \gamma\eta, \quad L_- \eta = -\eta'' - 2A\eta' + \gamma\eta.
\]

Remark 4.7 Assume \( \gamma \neq 0 \). Then any minimizer \( g \) for \( J_{p,q}(A, \gamma, H) \) is, up to a Lagrange multiplier, a weak solution to the fourth order differential equation

\[
L_- (|L_+ g|^{p-2}L_+ g) + B_- (|B_+ g|^{q-2}B_+ g) = 0 \quad \text{on} \ \mathbb{R}.
\]

In the next result we identify functions that coincide up to a change of sign and composition with translations in \( \mathbb{R} \).
Theorem 4.8 Let $p > 1$, $q > p$, $A, H, \gamma \in \mathbb{R}$ with $\gamma, H \neq 0$, $A^2 + \gamma \geq 0$ and

$$H^2 + 2AH - \gamma = 0. \tag{4.3}$$

Then (4.2) has a unique nontrivial solution $G \in W^{2,p}(\mathbb{R})$. More precisely, $G$ achieves the best constant $J_{p,q}(A, \gamma, H)$, $J_{p,q}(A, \gamma, H) = M_{p,q}(\gamma H)$, and

$$G(s) = k e^{-Hs} \int_{-\infty}^{\infty} e^{Ht} F(t) \, dt \quad \text{if } H > 0$$

$$G(s) = k e^{-Hs} \int_{-\infty}^{\infty} e^{Ht} F(t) \, dt \quad \text{if } H < 0,$$

where

$$k = \left( q \left( \frac{p}{p-1} - \frac{\gamma |H|}{H} \right) \right)^{\frac{1}{q-p}}.$$

**Proof.** First of all one has to prove that $G$ is a $W^{2,p}(\mathbb{R})$-solution to (4.2). We indicate here a way to minimize computations. We notice that

$$G(s) =
\begin{cases} 
  e^{-Hs} \int_{e^{-s}}^{\infty} e^{Ht} F(t) \, dt & \text{if } H > 0 \\
  e^{-Hs} \int_{0}^{e^{-s}} e^{Ht} F(t) \, dt & \text{if } H < 0,
\end{cases}$$

where $F \in W^{1,p}(\mathbb{R})$ is the function defined in (3.7) with $\lambda = \gamma H$. Thus by Theorem 3.5 we know that $F = G' + HG$ achieves the infimum $M_{p,q}(\gamma H)$ and solves

$$- \left( |f'| - \frac{\gamma}{H} f |f|^{p-2} (f' - \frac{\gamma}{H} f) \right)' - \frac{\gamma}{H} |f'|^{p-2} (f' - \frac{\gamma}{H} f) = |f|q-2 f. \tag{4.4}$$

Since $G$ decays exponentially at $\pm \infty$, then clearly $G \in L^p(\mathbb{R})$. Hence $G \in W^{2,p}(\mathbb{R})$, as $G' = F - HG \in L^p(\mathbb{R})$. Now we use (4.3) to get

$$F' - \frac{\gamma}{H} F = G'' - 2AG' - \gamma G = -\mathcal{L}_+ G.$$

Hence, we have showed that $G$ solves

$$\left( |\mathcal{L}_+ G|^{p-2}(\mathcal{L}_+ G) \right)' + \frac{\gamma}{H} |\mathcal{L}_+ G|^{p-2}(\mathcal{L}_+ G) = |G' + HG|q-2 (G' + HG)$$

$$= |\mathcal{L}_+ G|q-2 \mathcal{B}_+ G. \tag{4.5}$$

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Finally, we apply the operator $-\mathcal{B}_-$ to both sides of (4.5) and we use again (4.3) to get that $G$ is a solution to (4.2).

Now, assume that $g \in W^{2,p}(\mathbb{R}) \setminus \{0\}$ solves (4.2), and put

$$f := g' + Hg = \mathcal{B}_+g \in W^{1,p}(\mathbb{R}).$$

We have to show that $f$ solves (4.4). By (4.3) we have that

$$\varphi := \left| f' - \frac{\gamma}{H} f \right|^{p-2} \left( f' - \frac{\gamma}{H} f \right) = -|\mathcal{L}+g|^{p-2} \mathcal{L}+g \in L^{p'}(\mathbb{R})$$

with pointwise a.e. equalities. Since $g$ solves (4.2) then $\varphi$ is a distributional solution to

$$\mathcal{L}_- \varphi = \mathcal{B}_- \left( |f|^{q-2}f \right),$$

(4.6) that is,

$$\int_{\mathbb{R}} \varphi (\mathcal{L}_+ \eta) \, ds = -\int_{\mathbb{R}} |f|^{q-2} f (\mathcal{B}_+ \eta) \, ds \quad \text{for any } \eta \in W^{2,p}(\mathbb{R})$$

(use a density argument). Now take any $w \in C_c^\infty(\mathbb{R})$ and put

$$\eta(s) = \begin{cases} 
  e^{-Hs} \int_{-\infty}^s e^{Ht} w(t) \, dt & \text{if } H > 0 \\
  -e^{-Hs} \int_{s}^{+\infty} e^{Ht} w(t) \, dt & \text{if } H < 0.
\end{cases}$$

Notice that $\eta \in W^{2,p}(\mathbb{R})$ since $H \neq 0$ and $\eta' + H\eta = w$. Moreover, it holds that

$$\int_{\mathbb{R}} |\mathcal{B}_+ \eta|^p \, ds \leq c ||\eta||_{W^{1,p}}^p \leq c \int_{\mathbb{R}} |w|^p \, ds,$$

by (4.3), and $|f|^{q-1} \in L^{p'}(\mathbb{R})$ as $g \in W^{2,p}(\mathbb{R}) \hookrightarrow W^{1,\tau}(\mathbb{R})$ for any $\tau \geq p$. Thus

$$\left| \int_{\mathbb{R}} \varphi w' \, ds \right| \leq \left| \int_{\mathbb{R}} \varphi (-\mathcal{L}_+ \eta + \frac{\gamma}{H} \eta w) \right| \, ds \leq \left| \int_{\mathbb{R}} |f|^{q-2} f (\mathcal{B}_+ \eta) \right| + c \left( \int_{\mathbb{R}} |w|^p \, ds \right)^{1/p} \leq c \left( \int_{\mathbb{R}} |w|^p \, ds \right)^{1/p}.$$ 

Hence $\varphi \in W^{1,p'}(\mathbb{R})$ and $\varphi$ is a weak solution to (4.6). On the other hand, in the dual $W^{-1,p}(\mathbb{R})$ we can compute

$$\mathcal{L}_- \varphi = -\varphi'' - 2A\varphi' + \gamma \varphi$$

$$= -\left( \varphi' + \frac{\gamma}{H} \varphi \right)' + H \left( \varphi' + \frac{\gamma}{H} \varphi \right) = -\mathcal{B}_- \left( \varphi' + \frac{\gamma}{H} \varphi \right),$$

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thanks to (4.3). Thus we have shown that

\[-B_-(\varphi' + \frac{\gamma}{H} \varphi) = B_- (|f|^{q-2} f)\,.

The operator $B_- : W^{1,p}(\mathbb{R}) \to W^{-1,p}(\mathbb{R})$ is invertible, and therefore it holds that $-\varphi' - \frac{\gamma}{H} \varphi = |f|^{q-2} f$, that is, $f$ solves (4.4). By Theorem 3.5 we can assume that $f$ coincides with the function $F = G' + HG$. Thus $g = G$, as $g, G \in W^{1,p}(\mathbb{R})$. The theorem is completely proved. \hfill \Box

4.2 The space $\mathcal{D}^{2,p}_r(\mathbb{R}^n; |x|^\alpha dx)$

In order to simplify notation we put

$$H_2 = H_{\alpha,2} = \frac{n + \alpha}{p} - 2, \quad \gamma_2 = \gamma_{\alpha,2} = \left(\frac{n + \alpha}{p} - 2\right) \left(n - \frac{n + \alpha}{p}\right),$$

compare with (3.3) and (1.3). We need also the constant

$$A_2 = \frac{n - 2}{2} - H_2.$$

Notice that

$$A_2^2 + \gamma_2 = \left(\frac{n - 2}{2}\right)^2 \geq 0, \quad H_2^2 + 2A_2H_2 - \gamma_2 = 0. \quad (4.7)$$

Assume that $\gamma_2 \neq 0$, that is, $\alpha \notin \{2p - n, np - n\}$. Then $\mathcal{D}^{2,p}_r(\mathbb{R}^n; |x|^\alpha dx)$ is a Banach space with norm

$$||u||_{2,p}^p = \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p \, dx.$$

Moreover, $\mathcal{D}^{2,p}_r(\mathbb{R}^n; |x|^\alpha dx)$ is continuously embedded into $L^p(\mathbb{R}^n; |x|^{\alpha-2p} dx)$ and into $\mathcal{D}^{1,p}_r(\mathbb{R}^n; |x|^{\alpha-p} dx)$ by (1.8), (2.4).

For $g \in C_c^2(\mathbb{R})$ we put $(T_2 g)(x) = |x|^{-H_2} g(-\log |x|)$. Then $T_2$ is a linear, invertible transform $C_c^2(\mathbb{R}) \to C_c^{2,p}(\mathbb{R}^n \setminus \{0\})$.

Now we prove the second-order version of Lemma 3.6.

**Lemma 4.9** Assume $\alpha \notin \{2p - n, np - n\}$. 

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i) The transform $T_2$ can be extended in a unique way to a bicontinuous isomorphism $W^{2,p}(\mathbb{R}) \to D^{2,p}(\mathbb{R}; |x|^\alpha dx)$.

ii) If $\alpha > 2p - n$ then $C^2_c(\mathbb{R}^n) \subset D^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$. In particular, if $n > 2p$ then $D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) = D^{2,p}(\mathbb{R})$.

**Proof.** By direct computation one can check that

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta (T_2 g)|^p dx = \omega_n \int_{\mathbb{R}} |g'' - 2A_2 g' - \gamma_2 g|^p ds$$

for any $g \in C^2_c(\mathbb{R})$. Therefore Proposition 4.3 immediately implies i). To prove ii) fix $u \in C^2_c(\mathbb{R}^n)$, put $g = T_2^{-1} u$ and then argue as in Lemma 3.6.

Now we fix an exponent $q > p$ and we use Lemma 4.9 together with the results in Section 4.1 to study the best constants $S_{2,q,0}(\alpha)$ and $S_{2,q,1}(\alpha)$. Notice that

$$\int_{\mathbb{R}^n} |x|^{-\beta_{2,q}} |T_2 g|^q dx = \omega_n \int_{\mathbb{R}} |g|^q ds$$

$$\int_{\mathbb{R}^n} |x|^{-\beta_{1,q}} |\nabla (T_2 g)|^q dx = \omega_n \int_{\mathbb{R}} |g' + H_2 g|^q ds$$

for any $g \in W^{2,p}(\mathbb{R})$ where, accordingly with (1.4),

$$\beta_{2,q} = n - q \frac{n - 2p + \alpha}{p}, \quad \beta_{1,q} = n - q \frac{n - p + \alpha}{p}.$$

First we use Lemma 4.9 and the above computations to observe that the minimization problems

$$S_{2,q,0}(\alpha) = \inf_{\substack{u \in D^{2,p}(\mathbb{R}; |x|^\alpha dx) \backslash \{0\}}} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx}{\left( \int_{\mathbb{R}^n} |x|^{-\beta_{2,q}} |u|^q dx \right)^{p/q}}$$

$$I_{p,q}(A_2, \gamma_2) = \inf_{\substack{g \in W^{2,p}(\mathbb{R}) \backslash \{0\}}} \frac{\int_{\mathbb{R}} |g'' - 2A_2 g' - \gamma_2 g|^p ds}{\left( \int_{\mathbb{R}} |g|^q ds \right)^{p/q}}$$

are equivalent. Thus Proposition 4.3 immediately implies the next existence result, which is indeed included in the more general Theorem 1.3.
**Theorem 4.10** If $\alpha \notin \{2p - n, np - n\}$ and $q > p$, then $S_{2,q,0}(\alpha)$ is positive and attained in $\mathcal{D}^{2,p}({\mathbb{R}}^n;|x|^\alpha dx)$.

We notice that any minimizer for $S_{2,q,0}(\alpha)$ satisfies the Euler-Lagrange equation

$$\Delta \left(|x|^\alpha |\nabla u|^{p-2} \nabla u\right) = |x|^{-\beta_{2,q}} |u|^{q-2} u,$$

which is equivalent to the well known and largely studied Hénon-Lane-Emden system

$$\begin{cases} -\Delta u = |x|^a |v|^{p' - 2} v \\ -\Delta v = |x|^b |u|^{q-2} u, \end{cases}$$

where $a = -\frac{\alpha}{p-1}$ and $b = -\beta_{2,q}$. In particular Theorem 4.10 provides the existence of solutions to the above system, if $a, b \neq -n$ and $p', q$ lie on the "critical hyperbola"

$$\frac{a + n}{p'} + \frac{b + n}{q} = n - 2.$$ 

We quote [22] for details and additional results.

We are in position to prove one of the main results in the introduction.

**Proof of Theorem 1.5.** For any $f \in W^{2,p}({\mathbb{R}})$ put

$$\mathcal{L} f = -f'' + 2A_2 f + \gamma_2 f, \quad \mathcal{B} f = f' + H_2 f.$$ 

Direct computations lead to

$$-\Delta \left(\mathcal{T}_2 f\right) = \mathcal{T}_0 \left(\mathcal{L} f\right), \quad \left(\mathcal{T}_2 f\right)' = -\mathcal{T}_1 \left(\mathcal{B} f\right),$$

see (3.2) for the definition of $\mathcal{T}_h$. In particular, for any pair $U = \mathcal{T}_2 G, \varphi = \mathcal{T}_2 f$ we have that

$$\int_{\mathbb{R}^n} |x|^\alpha |\nabla U|^{p-2} \nabla U \Delta \varphi \, dx = \omega_n \int_{\mathbb{R}} |\mathcal{L} G|^{p-2} (\mathcal{L} G)(\mathcal{L} f) \, ds$$

$$\int_{\mathbb{R}^n} |x|^{-\beta_{1,q}} |\nabla U|^{p-2} \nabla U \cdot \nabla \varphi \, dx = \omega_n \int_{\mathbb{R}} |\mathcal{B} G|^{q-2} (\mathcal{B} G)(\mathcal{B} f) \, ds.$$ 

To conclude the proof use Theorem 4.8 with $A = A_2$ and

$$\frac{\gamma}{H} = \frac{\gamma_2}{H_2} = \frac{np - n + \alpha}{p}.$$ 

$\square$
Remark 4.11 Here we take $\alpha = np - n$ and we define

$$S_{2,q,0}(np - n) := \inf_{\substack{u \in C^2_c(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}}} \int_{\mathbb{R}^n} |x|^{np-n}|\Delta u|^p dx \
\left( \int_{\mathbb{R}^n} |x|^{-n+q(n-2)}|u|^q dx \right)^{p/q}$$

$$S_{2,q,1}(np - n) := \inf_{\substack{u \in C^2_c(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}}} \int_{\mathbb{R}^n} |x|^{np-n}|\Delta u|^p dx \
\left( \int_{\mathbb{R}^n} |x|^{-n+q(n-1)}|\nabla u|^q dx \right)^{p/q}.$$ 

Thanks to the Emden-Fowler transform $(T_g)(x) = |x|^{2-n}g(-\log |x|)$, and using Remark 4.2 and ii) in Proposition 4.5, it is easy to check that

$$S_{2,q,0}(np - n) = \omega_n^{2-p} I_{p,q} \left( -\frac{n-2}{2}, 0 \right) = 0$$

$$S_{2,q,1}(np - n) = \omega_n^{2-p} J_{p,q} \left( -\frac{n-2}{2}, 0, n-2 \right) = 0.$$ 

Remark 4.12 Here we take $n \geq 3$ and $\alpha = 2p - n$ (notice that for $n = 2$ the two "degenerate cases" $\alpha = np - n$ and $\alpha = 2p - n$ coincide). Now we observe that $H_2 = 0$ and we use $(T_g)(x) = g(-\log |x|)$ to get

$$S_{2,q,0}(2p - n) := \inf_{\substack{u \in C^2_c(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}}} \int_{\mathbb{R}^n} |x|^{2p-n}|\Delta u|^p dx \
\left( \int_{\mathbb{R}^n} |x|^{-n}|u|^q dx \right)^{p/q} = \omega_n^{2-p} I_{p,q} \left( -\frac{n-2}{2}, 0 \right) = 0$$

by Remark 4.2 Next we define

$$S_{2,q,1}(2p - n) := \inf_{\substack{u \in C^2_c(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}}} \int_{\mathbb{R}^n} |x|^{2p-n}|\Delta u|^p dx \
\left( \int_{\mathbb{R}^n} |x|^{-n+q(n-1)}|\nabla u|^q dx \right)^{p/q}.$$ 

We have that

$$S_{2,q,1}(2p - n) = \omega_n^{2-p} J_{p,q} \left( -\frac{n-2}{2}, 0, 0 \right) = M_{p,q}(n-2)$$

by iii) in Proposition 4.5. Thus the value of $S_{2,q,1}(2p - n)$ is explicitly known, thanks to Theorem 3.5. In particular, $S_{2,q,1}(2p - n) > 0$, but no function in $L^p(\mathbb{R}^n; |x|^{-n}dx)$

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achieves $S_{2,q,1}(2p - n)$, use again Proposition 4.5. See also Remark 4.14 below for the case $\alpha = 2p - n = 0$, $q = p^*$. 

Remark 4.13 Assume $\alpha > 2p - n$, $\alpha \neq n(p - 1)$. Then the weight $|x|^{\alpha - 2p}$ is locally integrable and the space $C^2_{c,r}(\mathbb{R}^n)$ is dense in $\mathcal{D}^{2,p}_r(\mathbb{R}^n; |x|^\alpha dx)$ by ii) in Lemma 4.9. In particular, inequalities (1.3) and (2.4) hold in $C^2_{c,r}(\mathbb{R}^n)$. We also have that

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx \geq S_{2,q,0}(\alpha) \left( \int_{\mathbb{R}^n} |x|^{-n+q \frac{n-2p+\alpha}{p}} |u|^q dx \right)^{p/q} \forall u \in C^2_{c,r}(\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx \geq S_{2,q,1}(\alpha) \left( \int_{\mathbb{R}^n} |x|^{-n+q \frac{n-p+\alpha}{p}} \nabla u|^q dx \right)^{p/q} \forall u \in C^2_{c,r}(\mathbb{R}^n).$$

Remark 4.14 Take $\alpha = 0$. If $n = 2p$ then $p^* = n$, the constant $S_{2,n,1}(0)$ in Remark 4.12 is positive and any radially symmetric function $\tilde{U}$ satisfying

$$\tilde{U}'(r) = kr(1 + r^n)^{-1}, \quad k = 2p^* (2(p - 1))^{1/p} = n(n - 2)/n$$

is indeed a solution to the Euler-Lagrange equation (1.16). For, it is convenient to notice that $\Delta \left( |\Delta \tilde{U}|^{p-2} \Delta \tilde{U} \right) + \Delta \tilde{U} = \nabla (\Phi |x|^{-1} x)$, where

$$\Phi := \left( |\Delta \tilde{U}|^{p-2} \Delta \tilde{U} \right) + |\tilde{U}'|^{p^*-2} \tilde{U}'.$$

We compute $\Delta \tilde{U} = nk(1 + r^n)^{-2} = n^2(n - 2)^2/n(1 + r^n)^{-2}$, so that

$$\left( |\Delta \tilde{U}|^{p-2} \Delta \tilde{U} \right)' = -n^{n-1}(n - 2)^{\frac{n(n-1)}{n}} r^{n-1}(1 + r^n)^{-1-n} = -|\tilde{U}'|^{p^*-2} \tilde{U}'.
$$

Thus $\Phi \equiv 0$ and therefore $\tilde{U}$ solves (1.16). We also point out that

$$\int_{\mathbb{R}^n} |\Delta \tilde{U}|^p dx < \infty, \quad \int_{\mathbb{R}^n} |x|^{-p} |\nabla \tilde{U}|^p dx < \infty, \quad \int_{\mathbb{R}^n} |x|^{-2p} |\tilde{U}|^p dx = \infty,$$

as $\tilde{U}$ is radially increasing and the weight $|x|^{-2p} = |x|^{-n}$ is not integrable at 0, nor at infinity. For instance, for any constant $c \in \mathbb{R}$ the function $\tilde{U}(x) = 2\sqrt{2} \arctan |x|^2 + c$ solves

$$\Delta^2 \tilde{U} + \Delta \tilde{U} = 0 \quad \text{on } \mathbb{R}^4.$$ 

5 Higher order inequalities

In this last section we study the case $k > 2$ and we prove Theorem 1.3. We start by investigating the standard space $W^{k,p}(\mathbb{R})$. 

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5.1 Equivalent norms on $W^{k,p}(\mathbb{R})$

Let $m \geq 1$ be an integer and let

$$\vec{A} = (A_1, \ldots, A_m) \in \mathbb{R}^m, \quad \vec{\gamma} = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m$$

be given $m$-vectors. We define the $m + 1$ differential operators

$$L_h g = g'' - 2A_h g' - \gamma_h g, \quad \mathbb{L}_{\vec{A}, \vec{\gamma}} = L_1 \circ \cdots \circ L_m,$$

so that $\mathbb{L}_{\vec{A}, \vec{\gamma}}$ has order $2m$.

We distinguish the ”even case” $k = 2m$ from the ”odd” one, when $k = 2m + 1$.

**Proposition 5.1** Assume that $A_h^2 + \gamma_h \geq 0$ for any $h = 1, \ldots, m$. Then

$$I_p(\vec{A}, \vec{\gamma}) := \inf_{g \in W^{2m,p}(\mathbb{R})} \frac{\int_{\mathbb{R}} |\mathbb{L}_{\vec{A}, \vec{\gamma}} g|^p ds}{\int_{\mathbb{R}} |g|^p ds} = \prod_{h=1}^m |\gamma_h|^p.$$ 

Moreover, $\|g\|_{\vec{A}, \vec{\gamma}} := \|\mathbb{L}_{\vec{A}, \vec{\gamma}} g\|_p$ is an equivalent norm on $W^{2m,p}(\mathbb{R})$ provided that $\gamma_h \neq 0$ for any $h = 1, \ldots, m$.

**Proof.** To check that $I_p(\vec{A}, \vec{\gamma}) \leq \prod_{h=1}^m |\gamma_h|^p$ we use a rescaling argument, as in Lemma 4.4. Take any $g \in C^k_\infty(\mathbb{R}) \setminus \{0\}$ and test $I_p(\vec{A}, \vec{\gamma})$ with $g_\varepsilon(s) := g(\varepsilon s)$, where $\varepsilon \to 0^+$. By direct computation one gets that

$$\int_{\mathbb{R}} |\mathbb{L}_{\vec{A}, \vec{\gamma}} g_\varepsilon|^p ds = \varepsilon^{-1} \left( \prod_{h=1}^m |\gamma_h|^p \right) \int_{\mathbb{R}} |g|^p ds + o(\varepsilon^{-1}), \quad \int_{\mathbb{R}} |g_\varepsilon|^p ds = \varepsilon^{-1} \int_{\mathbb{R}} |g|^p ds.$$

The conclusion readily follows. The opposite inequality can be proved by induction, starting with the case $k = 2$ that has already been discussed in Section 4.1. □

The same arguments lead to the next result. We omit the details of the proof.

**Proposition 5.2** Assume that $A_h^2 + \gamma_h \geq 0$ for any $h = 1, \ldots, m$ and let $\lambda \in \mathbb{R}$. Then

$$M_p(\vec{A}, \vec{\gamma}; \lambda) := \inf_{g \in W^{2m+1,p}(\mathbb{R})} \frac{\int_{\mathbb{R}} |\mathbb{L}_{\vec{A}, \vec{\gamma}} g'|^p ds}{\int_{\mathbb{R}} |g|^p ds} = |\lambda|^p \prod_{h=1}^m |\gamma_h|^p.$$ 

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Moreover, \( \|g\|_{A,\gamma,\lambda} := \|L_{A,\gamma} g' - \lambda L_{A,\gamma} g\|_p \) is an equivalent norm on \( W^{2m+1,p}(\mathbb{R}) \) provided that \( \lambda \neq 0 \) and \( \gamma_h \neq 0 \) for any \( h = 1, ..., m \).

**Remark 5.3** One can get more inequalities by taking advantage of the embeddings \( W^{k,p}(\mathbb{R}) \hookrightarrow W^{j,q}(\mathbb{R}) \) for \( h = 1, ..., k-1 \) and \( q \geq p \).

### 5.2 The space \( D^{k,p}(\mathbb{R}^n; |x|^\alpha dx) \)

To simplify notation we put

\[
H_h = H_{\alpha,h} = \frac{n + \alpha}{p} - h, \quad \gamma_h = \gamma_{\alpha,h} = H_h (n - 2 - H_h)
\]

for any integer \( h \geq 1 \), compare with (3.3) and (1.6). We introduce also the constants

\[
A_h = \frac{n - 2}{2} - H_h.
\]

Notice that

\[
A_h^2 + \gamma_h = \left( \frac{n - 2}{2} \right)^2 \geq 0, \quad H_h^2 + 2A_h H_h - \gamma_h = 0.
\]

In this section we will always assume that (3.1) is satisfied. In particular, we have that \( D^{k,p}(\mathbb{R}^n; |x|^\alpha dx) \) is a well defined Banach space with norm

\[
\|u\|_{k,\alpha}^p = \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx,
\]

and it is continuously embedded into \( L^p(\mathbb{R}^n; |x|^{\alpha-kp} dx) \) by Theorem 1.1.

Accordingly with (3.2), we put \( (T_k g)(x) = |x|^{-H_k} g(-\log |x|) for \( g \in C^k_c(\mathbb{R}) \).

**Lemma 5.4** The transform \( T_k \) can be extended in a unique way to a bicontinuous isomorphism \( W^{k,p}(\mathbb{R}) \to D^{k,p}(\mathbb{R}^n; |x|^\alpha dx) \).

**Proof.** Recall that \( (T_h g)(x) = |x|^{-n+\alpha+h} g(-\log |x|) \) for any \( h \geq 0 \) and notice that

\[
\Delta(T_h g) = T_h g'' - 2A_h g' - \gamma_h g
\]

for any \( h \geq 2 \) (use an induction argument).

We distinguish the case \( k = 2m \) from the case of odd order operators.
Even poliharmonic operators. Assume that $k = 2m$ is an even integer and that $\gamma_{2h} \neq 0$ for any $h = 1, \ldots, m$. We adopt the notation in Section 5.1 with

$$\vec{A} = (A_2, \ldots, A_{2m}), \quad \vec{\gamma} = (\gamma_2, \ldots, \gamma_{2m}),$$

$$L_h g = g'' - 2A_{2h} g' - \gamma_{2h} g, \quad \mathbb{L}_{\vec{A}, \vec{\gamma}} = L_1 \circ \cdots \circ L_m.$$ Using (5.1) it is easy to prove by induction that

$$\Delta^m (T_{2m} g) = T_0 \left( \mathbb{L}_{\vec{A}, \vec{\gamma}} g \right)$$

for any $g \in C^k_c(\mathbb{R})$. Therefore, for $u = T_{2m} g$ the following equality holds

$$\|u\|_{2m, \alpha}^p = \int_{\mathbb{R}^n} |x|^\alpha |\Delta^m u|^p dx = \omega_n \int_{\mathbb{R}} \left| \mathbb{L}_{\vec{A}, \vec{\gamma}} g \right|^p ds.$$ The conclusion in the even case follows by Proposition 5.1.

Odd poliharmonic operators. When $k = 2m+1$ is odd we assume that $\gamma_{2h+1} \neq 0$ for any $h = 1, \ldots, m$ and that the Hardy constant is positive. Now we put

$$\vec{A} = (A_3, \ldots, A_{2m+1}), \quad \vec{\gamma} = (\gamma_3, \ldots, \gamma_{2m+1}),$$

$$L_h g = g'' - 2A_{2h+1} g' - \gamma_{2h+1} g, \quad \mathbb{L}_{\vec{A}, \vec{\gamma}} = L_1 \circ \cdots \circ L_m.$$ Using (5.1) one can prove by induction that

$$\Delta^m (T_{2m+1} g) = T_1 \left( \mathbb{L}_{\vec{A}, \vec{\gamma}} g \right)$$

for any $g \in C^k_c(\mathbb{R})$. Therefore, for $u = T_{2m+1} g$ it is easy to compute

$$\|u\|_{2m+1, \alpha}^p = \int_{\mathbb{R}^n} |x|^\alpha |\nabla (\Delta^m u)|^p dx = \omega_n \int_{\mathbb{R}} \left| \mathbb{L}_{\vec{A}, \vec{\gamma}} g' + H_\alpha \mathbb{L}_{\vec{A}, \vec{\gamma}} g \right|^p ds.$$ The conclusion follows by Proposition 5.2.

Remark 5.5 The computations in the proof of Lemma 5.4 together with Propositions 5.1, 5.2 provide an alternative proof of Theorem 1.1.

Remark 5.6 Assume $\alpha > kp - n$. One can adapt the arguments already used in the lowest order case $k = 1$ to show that $C^k_{c, r}(\mathbb{R}^n) \subset D^{k, p}_r(\mathbb{R}^n; |x|^{\alpha} dx)$. 

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Remark 5.7 Thanks to Lemma 5.4 we can identify the spaces $D^{k,p}_{r}(\mathbb{R}^n; |x|^\alpha dx)$ and $W^{k,p}(\mathbb{R})$ through $\mathcal{T}_h$. In particular, every function $u \in D^{k,p}_{r}(\mathbb{R}^n; |x|^\alpha dx)$ has continuous derivatives up to the order $k - 1$, and partial derivatives of order $k$ (in the classical sense) exist for almost every $|x| > 0$.

Now we focus our attention on Sobolev type embeddings. For $j \in \{0, ..., k\}$ and $q > p$ define $\beta_{k-j,q}$ accordingly with (1.4).

Remark 5.8 Fix an index $j = 1, ..., k - 1$, and notice that
\[
\tilde{H}_j := \frac{n - \beta_{k-j,q}}{q} - h = \frac{n + \alpha}{p} - (k - j + h) = H_{k-j+h},
\]
so that $\tilde{H}_j = H_k$. Moreover, $\tilde{\gamma}_h := \tilde{H}_h(n - 2 + \tilde{H}_h)$ satisfies (3.1) with $\alpha, k, p$ replaced by $\beta_{k-j,q}, j$ and $q$, respectively. Thus $\mathcal{T}_h$ can be used to identify $W^{j,q}(\mathbb{R})$ with the Banach space $D^{j,q}_{r}(\mathbb{R}^n; |x|^{-\beta_{k-j,q}} dx)$ by Lemma 5.4.

We are in position to prove Theorem 1.3.

Proof of Theorem 1.3 By Lemma 5.4 Sobolev embedding theorem and Remark 5.8 we have the following chain of continuous arrows:
\[
D^{k,p}_{r}(\mathbb{R}^n; |x|^\alpha dx) \overset{\mathcal{T}_h^{-1}}{\longrightarrow} W^{k,p}(\mathbb{R}) \hookrightarrow W^{j,q}(\mathbb{R}) \overset{\mathcal{T}_h}{\longrightarrow} D^{j,q}_{r}(\mathbb{R}^n; |x|^{-\beta_{k-j,q}} dx).
\]
Thus $D^{k,p}_{r}(\mathbb{R}^n; |x|^\alpha dx) \hookrightarrow D^{j,q}_{r}(\mathbb{R}^n; |x|^{-\beta_{k-j,q}} dx)$ with a continuous embedding, and hence $S_{k,j,q}(\alpha)$ is positive.

To prove that $S_{k,j,q}(\alpha)$ is achieved one can study an equivalent minimization problem for functions in $W^{k,p}(\mathbb{R})$, as we did in Section 4.2 for the case $k = 2$. Here we adopt a direct approach. The strategy is essentially the same of [21], and it was originally inspired to the author by the famous paper [26] by Sacks and Uhlenbeck.

We introduce the energies $e_{k,p}, e_{j,q}$ and $E : D^{k,p}_{r}(\mathbb{R}^n; |x|^\alpha dx) \to \mathbb{R}$ by putting
\[
e_{k,p}(u) = \frac{1}{p} \int_{\mathbb{R}^n} |x|^\alpha |\nabla u|^p dx, \quad e_{j,q}(u) = \frac{1}{q} \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u|^q dx, \quad E(u) = e_{k,p}(u) - e_{j,q}(u),
\]
where \( \beta = \beta_{k-j,q} \). The functional \( E \) is of class \( C^1 \) and minimizers for \( \tilde{S} := S_{k,q,j}(\alpha) \) give rise to critical points of \( E \). Using Ekeland’s variational principle we can select a minimizing sequence \( u_h \) such that

\[
E'(u_h) \cdot v \to 0 \quad \text{uniformly for } v \text{ in bounded subsets of } D^k_p(\mathbb{R}^n; |x|^\alpha dx).
\] (5.2)

In particular we have \( o(\|u_h\|_{k,\alpha}) = E'(u_h) \cdot u_h = \|u_h\|^p_{k,\alpha} + O(\|u_h\|^q_{k,\alpha}) \). Thus \( u_h \) is bounded and we may suppose, without loss of generality, that

\[
\int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u_h|^q \, dx = \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u_h|^p \, dx + o(1) = \frac{2}{S^q} + o(1).
\] (5.3)

Since the ratio in (1.11) is invariant with respect to dilations, we can assume that

\[
\int_{\{x<2\}} |x|^{-\beta} |\nabla^j u_h|^q \, dx = \left( \frac{1}{2} \right)^q \frac{2}{S^q}.
\] (5.4)

We have to prove that, up to a subsequence, \( u_h \) converges weakly to some nontrivial limit \( u \). Then, a standard convexity argument shows that \( u \) achieves \( \tilde{S} \). Assume by contradiction that \( u_h \to 0 \) weakly in \( D^k_p(\mathbb{R}^n; |x|^\alpha dx) \). By Lemma \((\ref{lem:convexity})\) we have that

\[
|\nabla^j u_h| \to 0 \text{ in } L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\})
\]

and therefore

\[
\int_{\{x<1\}} |x|^{-\beta} |\nabla^j u_h|^q \, dx = \left( \frac{1}{2} \right)^q \frac{2}{S^q} + o(1)
\] (5.5)

by (5.3). In essence, the idea is to fix a function \( \varphi \in C^k_{c,r}(\mathbb{R}^n) \) such that \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 1 \) on the unit ball, and to use \( E'(u_h) \cdot (\varphi^p u_h) = o(1) \) to reach a contradiction with (5.4). However this can not be done if \( p < k \), as \( \varphi^p \) is not of class \( C^k(\mathbb{R}^n) \). Thus we define

\[
\Phi_\varepsilon(x) = (\varepsilon^2 + \varphi(x)^2)^{p/2} - \varepsilon^p,
\]

where \( \varepsilon \in (0,1) \) is fixed. Notice that \( \Phi_\varepsilon \in C^k_{c,r}(\mathbb{R}^n) \) and that \( \Phi_\varepsilon \) is a constant in a neighborhood of 0. Thanks to Lemma \((\ref{lem:epsilon})\) we can compute \( E'(u_h) \cdot (\Phi_\varepsilon u_h) \). Since the family \( \Phi_\varepsilon u_h \) is uniformly bounded in \( D^k_p(\mathbb{R}^n; |x|^\alpha dx) \) as \( h \to \infty \), then (5.2) gives

\[
e_{k,p}(u) \cdot (\Phi_\varepsilon u_h) = e_{j,q}(u) \cdot (\Phi_\varepsilon u_h) + o(1).
\] (5.5)

We apply (A.2) in Lemma \((\ref{lem:critical})\) to get

\[
e'_{k,p}(u) \cdot (\Phi_\varepsilon u_h) = \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u_h|^{p-2} \nabla^k u_h \cdot \nabla^k (\Phi_\varepsilon u_h) \, dx
\]

\[
= \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u_h|^p \Phi_\varepsilon \, dx + o(1).
\]
From $\Phi \geq \varphi^p - \varepsilon^p$, we infer
\[
e'_{k,p}(u) \cdot (\Phi \varepsilon u_h) \geq \int_{\mathbb{R}^n} |x|^\alpha |\varphi \nabla^k u_h|^p \, dx - c\varepsilon^p + o(1)
\]
\[
= \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k (\varphi u_h)|^p \, dx - c\varepsilon^p + o(1)
\]
by (A.1), where $c = \sup_h \|u_h\|_{k,\alpha}$. Since $\varphi u_h \in D_{k,\alpha}^p(\mathbb{R}^n, |x|^\alpha \, dx)$ by Lemma A.1, from the definition of $\tilde{S}$ we get
\[
e'_{k,p}(u) \cdot (\Phi \varepsilon u_h) \geq \tilde{S} \left( \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j (\varphi u_h)|^q \, dx \right)^{p/q} - c\varepsilon^p + o(1). \tag{5.6}
\]
To estimate the right hand side of (5.5) we use (A.2) with $\alpha,k,p$ replaced by $\beta,j,q$, respectively, to get
\[
e'_{j,q}(u) \cdot (\Phi \varepsilon u_h) = \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u_h|^{q-2} \nabla^j u_h \cdot \nabla^j (\Phi \varepsilon u_h) \, dx
\]
\[
= \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u_h|^{q-p} |\nabla^j u_h|^p \Phi \varepsilon \, dx + o(1).
\]
Thus
\[
e'_{j,q}(u) \cdot (\Phi \varepsilon u_h) \leq \left( \int_{\{ |x| < 2 \}} |x|^{-\beta} |\nabla^j u_h|^q \, dx \right)^{2+\beta} \left( \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u_h|^q (\Phi \varepsilon)^{\frac{q}{p}} \, dx \right)^{\frac{p}{q}} + o(1)
\]
by Hölder inequality. From (5.3), Lemma A.5 and (A.1) we obtain
\[
e'_{j,q}(u) \cdot (\Phi \varepsilon u_h) \leq \frac{1}{2} \tilde{S} \left( \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u_h|^q (\varphi^q + c\varepsilon) \, dx \right)^{\frac{q}{p}} + o(1)
\]
\[
= \frac{1}{2} \tilde{S} \left( \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j (\varphi u_h)|^q \, dx + c\varepsilon \right)^{p/q} + o(1)
\]
\[
= \frac{1}{2} \tilde{S} \left( \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j (\varphi u_h)|^q \, dx \right)^{p/q} + c\varepsilon^{p/q} + o(1). \tag{5.7}
\]
Comparing (5.7), (5.5) and (5.6) we conclude that
\[
c\epsilon^{p/q} + \frac{1}{2} \tilde{S} \left( \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j (\varphi u_h)|^q \, dx \right)^{p/q}
\]
\[
\geq e'_{j,q}(u) \cdot (\Phi \varepsilon u_h) + o(1) = e'_{k,p}(u) \cdot (\Phi \varepsilon u_h) + o(1)
\]
\[
\geq \tilde{S} \left( \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j (\varphi u_h)|^q \, dx \right)^{p/q} - c\varepsilon^p + o(1).
\]
Since $\varepsilon^p = o(\varepsilon^{p/q})$, we infer

$$
\int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j (\varphi u_h)|^q \, dx \leq c \varepsilon + o(1),
$$

that together with (5.4) implies

$$
\left( \frac{1}{2} \tilde{S} \right)^{\frac{q-p}{q-p}} + o(1) = \int_{\{|x|<1\}} |x|^{-\beta} |\nabla^j u_h|^q \, dx \leq \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j (\varphi u_h)|^q \, dx \leq c \varepsilon + o(1),
$$
as $\varphi \equiv 1$ on the unit ball. The desired contradiction is achieved by choosing $\varepsilon > 0$ small enough. The proof is complete. \qed

Remark 5.9 Theorem 1.3 includes $k$ Sobolev constants without weights. Assume $n > kp$, take an integer $j \in \{0, ..., k-1\}$ and $\alpha = 0$. In addition, assume that $q$ equals the $(k-j)$th order critical exponent

$$
p^{(k-j)*} := \frac{np}{n-(k-j)p}.
$$

Then $\beta_{k-j,q} = 0$. Taking Remark 5.6 into account, by Theorem 1.3 we have that the radial Sobolev constant

$$
S_{k,p}^{(k-j)*} := \inf_{\substack{u \in D^{k,p}_r(\mathbb{R}^n) \setminus \{0\}}} \frac{\int_{\mathbb{R}^n} |\nabla^k u|^p \, dx}{\left( \int_{\mathbb{R}^n} |\nabla^j u|^{\frac{pn}{n-(k-j)p}} \, dx \right)^{\frac{n-(k-j)p}{n}}}
$$
is positive and achieved in $D^{k,p}_r(\mathbb{R}^n)$.

Appendix

Here we prove some compactness and technical results that have been used in the proof of Theorem 1.3. We always assume that (3.1) is satisfied.

Lemma A.1 Let $u \in D^{k,p}_r(\mathbb{R}^n; |x|^\alpha \, dx)$ and $\Phi \in C^k_c(\mathbb{R}^n)$ such that $\Phi$ is constant in a neighborhood of $0$. Then $\Phi u \in D^{k,p}_r(\mathbb{R}^n; |x|^\alpha \, dx)$ and

$$
\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k (\Phi u)|^p \, dx \leq c_\Phi \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p \, dx,
$$

where the constant $c_\Phi$ does not depend on $u$. 35
Proof. Let $g = T_k^{-1}u \in W^{k,p}(\mathbb{R})$ and put $\tilde{\Phi}(s) := \Phi(e^{-s}) \in C^k(\mathbb{R})$. Notice that $\tilde{\Phi}(s) \equiv 0$ for $s \ll 0$, $\tilde{\Phi}(s)$ is a constant for $s \gg 0$. Thus $\tilde{\Phi}g \in W^{k,p}(\mathbb{R})$ and hence $\Phi_u = T_k(\tilde{\Phi}g) \in \mathcal{D}^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$. Finally, from Lemma 5.4 we infer that

$$
\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k(\Phi_u)|^p dx \leq c\|\tilde{\Phi}g\|_{W^{k,p}}^p \leq c\Phi \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx,
$$

and the Lemma is proved. \hfill \Box

Next we need few results on weak convergence in $\mathcal{D}^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$. Notice that $\mathcal{D}^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ is reflexive because it is topologically equivalent to $W^{k,p}(\mathbb{R})$ (or because its norm is uniformly convex).

**Lemma A.2** Let $\Omega$ be a domain such that $\overline{\Omega} \subset \mathbb{R}^n \setminus \{0\}$. Then $\mathcal{D}^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ is compactly embedded into $W^{k-1,\tau}(\Omega)$ for any $\tau \geq p$.

**Proof.** Use the Emden-Fowler transform and Rellich theorem for $W^{k,p}(I)$, where $I$ is an appropriate bounded interval. \hfill \Box

**Remark A.3** Lemma A.2 is closely related to Theorem II.1 in [15]. Actually, the same argument shows that $\mathcal{D}^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ is compactly embedded into $C^{k-1}(\overline{\Omega})$.

**Lemma A.4** Let $\Phi \in C^k_c(\mathbb{R}^n)$ be a given function, such that $\Phi$ is constant in a neighborhood of 0. If $u_h \to 0$ in $\mathcal{D}^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ then

$$
\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k(\Phi_u_h) - \Phi \nabla^k u_h|^p dx = o(1) \quad (A.1)
$$

$$
\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u_h|^{2p} - 2\nabla^k u_h \cdot \nabla^k(\Phi_u_h) dx = \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u_h|^p \Phi dx + o(1). \quad (A.2)
$$

**Proof.** Note that $\nabla^k(\Phi u_h) = \Phi \nabla^k u_h + \Psi D_{k-1} u_h$, where $\Psi \in C^0_c(\mathbb{R}^n \setminus \{0\})$ and $D_{k-1}$ is a differential operator of order $k-1$. Thus (A.1) holds by Lemma A.2. To prove (A.2) use (A.1) and Hölder inequality. \hfill \Box

We conclude this appendix with an elementary lemma.

**Lemma A.5** Let $\varepsilon, \varphi \in [0,1]$, $1 < p < q$. Then there exists a constant $c \geq 0$ such that

$$
\left[ (\varepsilon^2 + \varphi^2)^{p/2} - \varepsilon^p \right]^{q/p} \leq \varphi^q + c\varepsilon.
$$

**Proof.** Fix $\varphi$ and put $\Phi(\varepsilon) = \left[ (\varepsilon^2 + \varphi^2)^{p/2} - \varepsilon^p \right]^{q/p}$. If $p \leq 2$ then $\Phi$ is non increasing and hence the conclusion in the lemma holds with $c = 0$. If $2 < p < q$ it suffices to notice that $\Phi$ is differentiable at $\varepsilon = 0$. \hfill \Box
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