On the action of the group of isometries on a locally compact metric space

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Abstract. In this short note we give an answer to the following question. Let \( X \) be a locally compact metric space with group of isometries \( G \). Let \( \{g_i\} \) be a net in \( G \) for which \( g_i x \) converges to \( y \), for some \( x, y \in X \). What can we say about the convergence of \( \{g_i\} \)? We show that there exist a subnet \( \{g_j\} \) of \( \{g_i\} \) and an isometry \( f : C_x \to X \) such that \( g_j \) converges to \( f \) pointwise on \( C_x \) and \( f(C_x) = C_y \), where \( C_x \) and \( C_y \) denote the pseudo-components of \( x \) and \( y \) respectively. Applying this we give short proofs of the van Dantzig-van der Waerden theorem (1928) and Gao-Kechris theorem (2003).

The main result and some applications

A few words about the notation we shall be using. In what follows, \( X \) will denote a locally compact metric space with group of isometries \( G \). If we endow \( G \) with the topology of pointwise convergence then \( G \) is a topological group \([2, \text{Ch. X, §3.5 Cor.}]\). On \( G \) there is also the topology of uniform convergence on compact subsets which is the same as the compact-open topology. In the case of a group of isometries these topologies coincide with the topology of pointwise convergence, and the natural action of \( G \) on \( X \) with \((g, x) \mapsto g(x), g \in G, x \in X\), is continuous \([2, \text{Ch. X, §2.4 Thm. 1 and §3.4 Cor. 1}]\). For \( F \subset G \), let \( K(F) := \{x \in X \mid \text{the set } Fx \text{ has compact closure in } X\} \). The sets \( K(F) \) are clopen \([6, \text{Lem. 3.1}]\).

Lemma 1. Let \( \Gamma = \{g_i\} \) be a net in \( G \) and \( x \in K(\Gamma) \) such that \( g_i x \) converges to \( y \) for some \( y \in X \). Then a subnet of \( \Gamma \) converges to an isometry \( f : K(\Gamma) \to X \) on \( K(\Gamma) \).

Proof. Let \( g_i|_{K(\Gamma)} \) denote the restriction of \( g_i \) on \( K(\Gamma) \). Arzela-Ascoli theorem implies that the set \( \{g_i|_{K(\Gamma)} : K(\Gamma) \to X\} \) has compact closure in the set of...
all continuous maps from $K(\Gamma')$ to $X$. Thus, there exist a subnet $\{g_j\}$ of $\{g_i\}$ and an isometry $f : K(\Gamma) \to X$ such that $g_j \to f$ on $K(\Gamma)$.

In [4] S. Gao and A. S. Kechris introduced the concept of pseudo-components. These are the equivalence classes $C_x$ of the following equivalence relation: $x \sim y$ if and only if $x$ and $y$, as also $y$ and $x$, can be connected by a finite sequence of intersecting open balls with compact closure. The pseudo-components are clopen [4, Prop. 5.3]. We call $X$ pseudo-connected if it has only one pseudo-component. An immediate consequence of the definitions is that $gC_x = C_{g x}$ for every $g \in G$. Another notion, that will be used in the proofs, is the radius of compactness $\rho(x)$ of $x \in X$ [4]. Let $B_r(x)$ denote the open ball centered at $x$ with radius $r > 0$. Then $\rho(x) := \sup\{r > 0 \mid B_r(x) \text{ has compact closure}\}$. If $\rho(x) = +\infty$ for some $x \in X$ then every ball has compact closure (i.e., $X$ has the Heine-Borel property), hence $\rho(x) = +\infty$ for every $x \in X$. If $\rho(x)$ is finite for some $x \in X$ then the radius of compactness is a Lipschitz map [4, Prop. 5.1]. Note that $\rho$ is $G$-invariant.

**Lemma 2.** Let $x, y \in X$ and $\{g_i\}_I$ be a net in $G$ with $g_i x \to y$. Then there is an index $i_0 \in I$ such that $C_x \subset K(F)$, where $F := \{g_i \mid i \geq i_0\}$.

**Proof.** Since $X$ is locally compact there exists an index $i_0$ such that the set $F(x)$ has compact closure, where $F := \{g_i \mid i \geq i_0\}$. We claim that for every $z \in C_x$ the set $F(z)$ also has compact closure, hence $C_x \subset K(F)$. The strategy is to start with an open ball $B_r(x)$ with radius $r < \rho(x)$ and prove that $F(z)$ has compact closure for every $z \in B_r(x)$. Then our claim follows from the definition of $C_x$. To prove the claim take a sequence $\{g_n z\} \subset F$. Since the closure of $F(x)$ is compact we may assume, upon passing to a subsequence, that $g_n x \to w$ for some $w$ in the closure of $F(x)$. Assume that $\rho(x)$ is finite and take a positive number $\varepsilon$ such that $r + \varepsilon < \rho(x)$. Then for $n$ big enough

$$d(g_n z, w) \leq d(g_n z, g_n x) + d(g_n x, w) = d(z, x) + d(g_n x, w) < r + \varepsilon < \rho(x).$$

Recall that the radius of convergence is a continuous map, and since $g_n x \to w$ then $\rho(x) = \rho(w)$. So, the sequence $\{g_n z\}$ is contained eventually in a ball of $w$ with compact closure, hence it has a convergence subsequence. The same also holds in the case where $\rho(x) = +\infty$.

**Theorem 3.** Let $X$ be a locally compact metric space with group of isometries $G$ and let $\{g_i\}$ be a net in $G$ for which $g_i x$ converges to $y$, for some $x, y \in X$. Then there exist a subnet $\{g_j\}$ of $\{g_i\}$ and an isometry $f : C_x \to X$ such that $g_j$ converges to $f$ pointwise on $C_x$ and $f(C_x) = C_{f(x)}$

**Proof.** By Lemma 2 there is an index $i_0 \in I$ such that $C_x \subset K(F)$, where $F := \{g_i \mid i \geq i_0\}$. Hence, by Lemma 1, there exists a subnet $\{g_j\}$ of $\{g_i\}$ which converges to an isometry $f : K(F) \to X$ on $K(F)$. Therefore, $g_j \to f$ on $C_x$. Let us show that $f(C_x) = C_{f(x)}$. Since $d(x, g_j^{-1} f(x)) = d(g_j x, f(x)) \to 0$ it follows that $g_j^{-1} f(x) \to x$. Hence, by repeating the previous procedure, there exist a subnet $\{g_k\}$ of $\{g_j\}$ and an isometry $h : C_{f(x)} \to X$ such that $g_k^{-1} \to h$.
pointwise on $C_f(x)$ and $h(f(x)) = x$. Note that $g_kx \in C_f(x)$ eventually for every $k$, since $g_kx \to f(x)$ and $C_f(x)$ is clopen. Therefore, $g_kC_x = Cg_kx = C_f(x)$. Take a point $z \in C_x$. Then, $g_kz \to f(z)$ and since $C_f(x)$ is clopen then $f(z) \in C_f(x)$, so $f(C_x) \subset C_f(x)$. By repeating the same arguments as before, it follows that $hC_f(x) \subset C_x$. Take now a point $w \in C_f(x)$. Then $h(w) \in C_x$, hence $g_k^{-1}(w) \in C_x$ eventually for every $k$. So, $w = g_kg_k^{-1}(w) \to f(h(w)) \in f(C_x)$ from which follows that $C_f(x) \subset f(C_x)$.

A few words about properness. A continuous action of a topological group $H$ on a topological space $Y$ is called proper (or Bourbaki proper) if the map $H \times Y \to Y \times Y$ with $(g, x) \mapsto (g \cdot x, x)$ for $g \in H$ and $x \in Y$, is proper, i.e., it is continuous, closed and the inverse image of a singleton is a compact set [1, Ch. III, §4.1 Def. 1]. In terms of nets, a continuous action is proper if and only if whenever we have two nets $\{g_i\}$ in $H$ and $\{x_i\}$ in $Y$, for which both $\{x_i\}$ and $\{g_ix_i\}$ converge, then $\{g_i\}$ has a convergent subnet. For isometric actions, it is easy to see that a continuous action is proper if and only if whenever we have a net $\{g_i\}$ in $H$ for which $\{g_ix\}$ converges for some $x \in Y$, then $\{g_i\}$ has a convergent subnet. If $H$ is locally compact and $Y$ is Hausdorff, then $H$ acts properly on $Y$ if and only if for every $x, y \in Y$ there exist neighborhoods $U$ and $V$ of $x$ and $y$, respectively, such that the set $\{g \in H \mid gU \cap V \neq \emptyset\}$ has compact closure in $H$ [1, Ch. III, §4.4 Prop. 7]. Observe that if $H$ acts properly on a locally compact space $Y$ then $H$ is also locally compact.

A direct implication of Theorem 3 is the van Dantzig-van der Waerden Theorem [3]. The advantage of our proof, comparing to the proofs given in the original work of van Dantzig-van der Waerden or in [5, Thm. 4.7, pp. 46–49], is that it is considerably shorter.

**Corollary 4.** (van Dantzig-van der Waerden theorem 1928) Let $X$ be a connected locally compact metric space with group of isometries $G$. Then $G$ acts properly on $X$ and is locally compact.

Another application of Theorem 3 is that we can rederive the results of Gao and Kechris in [4, Thm. 5.4 and Cor. 6.2].

**Corollary 5.** (Gao-Kechris theorem 2003) Let $X$ be a locally compact metric space with finitely many pseudo-components. Then the group of isometries $G$ of $X$ is locally compact. If $X$ is pseudo-connected, then $G$ acts properly on $X$.

**Proof.** Let $C_1, C_2, \ldots, C_n$ denote the pseudo-components of $X$ and take points $x_1 \in C_1, x_2 \in C_2, \ldots, x_n \in C_n$ and open balls $B_r(x_m) \subset C_m, m = 1, 2, \ldots , n, r > 0$ such that all $B_r(x_m)$ have compact closures. We will show that the set $V := \bigcap_{m=1}^{n} \{g \in G \mid gx_m \in B_r(x_m)\}$ is an open neighborhood of the identity in $G$ with compact closure. Indeed, take a net $\{g_i\}$ in $V$. Since each $B_r(x_m)$ has compact closure there exist a subnet $\{g_{j_l}\}$ of $\{g_i\}$ and points $y_1 \in C_1, y_2 \in C_2, \ldots, y_n \in C_n$ such that $g_jx_m \to y_m$ for every $m = 1, 2, \ldots , n$. Theorem 3 implies that there exist a subnet $\{g_l\}$ of $\{g_{j_l}\}$ and isometries $f_m : C_m \to X$ such that $g_l \to f_m$ on $C_m$ and $f_m(C_m) = C_m$ for all $m$. The last implies that $\{g_l\}$ converges to an isometry on $X$, hence $V$ has compact closure.
If $X$ is pseudo-connected the proof of the statement follows directly from Theorem 3.

\[\square\]

Remark 6. Note that in Corollary 5 we do not require that $X$ is separable as in [4, Thm 5.4 and Cor. 6.2]. This is not a real improvement since if $X$ has countably many pseudo-components then it is separable. Indeed, we define a relation on $X$ by $xSy$ if and only if there exist separable balls $B_r(x)$ and $B_l(y)$ with $y \in B_r(x)$ and $x \in B_l(y)$. Let $U(x)$ be the equivalence class of $x$ in the transitive closure of the relation $S$. Then, each $U(x)$ is a separable clopen subset of $X$ [5, Lem. 3 in App. 2]. By construction $C_x \subset U(x)$, therefore $X$ is separable.

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