Special massive spin-2 on the de Sitter space

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Abstract. The theory of a massive spin-2 state on the de Sitter space—with the mass squared equal to one sixth of the curvature—is special for two reasons: (i) it exhibits an enhanced local symmetry; (ii) it emerges as a part of the model that gives rise to the self-accelerated Universe. The known problems of this theory are: either it cannot be coupled to a non-conformal conserved stress-tensor because of the enhanced symmetry, or it propagates a ghost-like state when the symmetry is constrained by the Lagrange multiplier method. Here we propose a solution to these problems in the linearized approximation.

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1. Introduction and summary

The main topic of the present paper is a theory of massive tensor perturbation on the de Sitter (dS) background, when there is a special relation between the mass of the perturbation, \( m^* \), and the curvature of the background, \( \bar{R} \):

\[
m^*_2 = \frac{\bar{R}}{6}.
\]  

(1)

Our interest in such a theory is twofold:

(I) It has been known since the works of Deser and Nepomechie [1], and Higuchi [2] (see also [3]), that the linearized theory of a massive graviton on the dS background with the mass–curvature relation (1) exhibits an additional local symmetry. The corresponding symmetry transformations are generalization of conformal transformations; the symmetry disappears for any other value of \( m^* \). As a consequence of this symmetry, the tensor mode (graviton) of mass (1) propagates on the dS background four on-shell helicity states [1,3]. However, such a theory cannot be coupled consistently to a conserved stress-tensor with a non-zero trace [2]. The question is whether this latter drawback can be improved.

(II) In the Dvali–Gabadadze–Porrati (DGP) model of modified gravity [4], there is a self-accelerated solution which could model the accelerated expansion of the Universe [5,6]. The acceleration there is due to ‘dark energy’ which is supplied by gravity itself; in particular, by a non-zero condensate of a massive graviton. Interestingly enough, the (lightest) graviton mass on the self-accelerated background is related to the curvature as in (1) just automatically [7]. This is because both the graviton mass and curvature there have the same origin. The model exhibits strongly coupled behavior at the nonlinear level [8,9] (see also [10,11]). In the linearized theory the problem of coupling to a conserved stress-tensor of non-zero trace, mentioned above, is absent. However, the corresponding linearized theory on the self-accelerated background has a ghost-like state [10], [12]–[15]. Because the theory is in the strongly coupled regime [9]–[11], the results of the linearized theory remain unwarranted in the full nonlinear theory [16] (see also [17]). Nevertheless, the presence of the ghost in the linearized theory is indicative [5,10] of potential negative mass solutions in the full nonlinear theory. Such negative mass solutions have been obtained in semi-exact [18] and exact nonperturbative settings in the DGP model [19] (see also, [20]). This suggest that the self-accelerated background should be unstable; however, it is not clear whether the instability is rapid or not: for instance, an explicit
calculation of the decay of the self-accelerated branch into the conventional branch shows that such a decay does not take place, at least in a quasi-classical approximation [21].

The present paper concerns a simpler issue. Here, we will consider the linearized theory and will try to decouple the ghost-like state from the physical sector. A consistent nonlinear theory of a massive graviton on the dS space with the mass–curvature relation (1), if it exists, should in the linearized approximation reduce to the Lagrangian given in the present paper.

We will show that it is possible to have a fully consistent linearized theory of massive spin-2 on the dS background with the mass–curvature relation (1). We show this by explicitly setting up the Lagrangian of such a theory and calculating the physical one-graviton exchange amplitude, from which the tachyon and ghost states decouple. As to its nonlinear completion, the full theory may or may not be strongly coupled, depending on the value of a certain coupling$^3$.

The organization of the paper is as follows. In section 2 we summarize the results on the enhanced symmetry of a massive graviton on the dS background with the mass–curvature relation (1). Then we discuss additional terms that appear in the low-energy Lagrangian of the DGP model on the self-accelerated background [13]; this section serves to clarify our computational tools, which in the case of tensor fields on curved background are subtle. In section 3 we discuss a further extension of the linearized Lagrangian which allows us to decouple the ghost and tachyons, and calculate the one-graviton exchange amplitude. We give brief comments on the issues associated with nonlinear interactions.

2. Enhanced symmetry versus a ghost

Consider an expansion about the conventional de Sitter background of curvature $R = 12H^2$. The perturbations of the metric are defined as follows:

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}, \quad (2)$$

where $\gamma_{\mu\nu}$ denotes the background dS metric. The Lagrangian that we are interested in takes the form [2] (we set $8\pi G_N = 1$, with $G_N$ being the Newton constant)

$$\mathcal{L}_s \equiv \mathcal{L}^{(2)} - \frac{1}{16}m_*^2 \left( h_{\mu\nu}^2 - \gamma_{\mu\nu}^2 \right), \quad (3)$$

where $m_*$ is the graviton mass and $\mathcal{L}^{(2)}$ denotes the Einstein–Hilbert Lagrangian expanded up to the quadratic order about the dS background:

$$\mathcal{L}^{(2)} \equiv \frac{1}{2}(\nabla_\mu h^{\mu\nu})^2 + \frac{1}{4}h_{\mu\nu} \Box h^{\mu\nu} - \frac{1}{2}h \Box h + \frac{1}{2}h^{\mu\nu} \nabla_\mu h \nabla_\nu h - \frac{1}{2}H^2 \left( h_{\mu\nu}^2 + \frac{1}{2}h^2 \right). \quad (4)$$

All the covariant derivatives above are taken w.r.t. the background dS metric, and the covariant d’Alambertian is denoted by $\Box \equiv \gamma^{\mu\nu} \nabla_\mu \nabla_\nu$.

Because of the mass term, the Lagrangian (3) is not invariant w.r.t. linearized reparameterizations, $h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu$, with $\zeta_\mu$ being a transformation parameter. However, the symmetry can be restored by using the St"uckelberg method: redefine the field as $h_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} - \nabla_\mu V_\nu - \nabla_\nu V_\mu$; the resulting Lagrangian written in terms of $\bar{h}$ and $V$ will be invariant under the simultaneous transformations $\bar{h}_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} + \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu$.

$^3$ The perturbative stability of massive gravity on de Sitter has also been considered, albeit within the context of bi-gravity theories, in [22].
and $V_\mu \rightarrow V_\mu + \zeta_\mu$. Hence, we can regard the Lagrangian (3) as a gage fixed version of the reparameterization invariant theory in which the gage condition $V_\mu = 0$ had been enforced.

The Lagrangian (3) exhibits an additional local symmetry [1] when

$$m_s^2 = 2H^2. \tag{5}$$

The corresponding symmetry transformation is

$$h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x) + (\nabla_\mu \nabla_\nu + H^2 \gamma_{\mu\nu}) \rho(x). \tag{6}$$

As a consequence of this symmetry, there are four on-shell degrees of freedom propagated by such a graviton; moreover, no ghost or tachyon states appear in the spectrum [1]–[3]. For $m_s < 2H^2$, on the other hand, a ghost appears and the theory is inconsistent [2], while for $m_s > 2H^2$ the graviton propagates five helicity states and the theory is unitary. The case with the mass–curvature relation (5) (or with (1)) is the main subject of this paper.

So far we have been discussing pure gravity. How about its coupling to other fields? The coupling can be introduced by adding to (3) the conventional term $h_{\mu\nu}T^{\mu\nu}$, where the stress-tensor $T^{\mu\nu}$ is covariantly conserved, $\nabla_\mu T^{\mu\nu} = 0$. This does not affect our arguments about the linearized reparameterizations. However, the coupling breaks the invariance of the theory w.r.t. (6), unless $T = 0$. One could introduce a coupling a la Stückelberg ($h_{\mu\nu} - (\nabla_\mu \nabla_\nu + H^2 \gamma_{\mu\nu}) \sigma)T^{\mu\nu}$, where $\sigma$ is a field that also transforms as $\delta \sigma(x) = \rho(x)$, when the metric is transformed according to (6). Such a coupling would not violate the symmetry of the theory. However, the equation of motion of the $\sigma$ field would require that $T^{\mu}_\mu = 0$.

The same problem arises if one looks at the Einstein equations following from (3): the Bianchi identities then enforce the condition $(\nabla^{\mu}h_{\mu\nu} - \nabla_\nu h) = 0$, and, on this condition, the trace of the Einstein equation reads

$$(m_s^2 - 2H^2)h = -2T. \tag{7}$$

Therefore, for the special value of the mass (5), the theory does not admit coupling to a non-conformal source, just like the Maxwell theory does not admit coupling to a non-conserved source.

How can this be changed? Our approach below is motivated by the low-energy effective Lagrangian of the DGP model on the self-accelerated solution which was derived in [13]. We consider the Lagrangian of the following form (see also [16]):

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_s(h_{\mu\nu}) - \phi \mathcal{O}^{\mu\nu} h_{\mu\nu} + h_{\mu\nu} T^{\mu\nu}, \tag{8}$$

where we have introduced a Lagrange multiplier field $\phi$, and used the notation

$$\mathcal{O}_{\mu\nu} \equiv \nabla_\mu \nabla_\nu - \gamma_{\mu\nu} \Box - 3H^2 \gamma_{\mu\nu}. \tag{9}$$

Irrespective of its origin, the form of $\mathcal{O}_{\mu\nu}$ can be motivated by the requirement that the Bianchi identities be free of the field $\phi$ (see below).

Variation w.r.t. $\phi$ gives

$$\mathcal{O}^{\mu\nu} h_{\mu\nu} = 0. \tag{10}$$
The Einstein equation becomes

$$G^\text{dS}_{\mu\nu} - \frac{m_2^2}{2} (h_{\mu\nu} - \gamma_{\mu\nu} h) - \hat{\mathcal{O}}_{\mu\nu} \phi = -T_{\mu\nu},$$

(11)

where the Einstein tensor on the dS space is defined in the usual way:

$$G^\text{dS}_{\mu\nu} = \frac{1}{2} \left( \Box h_{\mu\nu} - \nabla_\mu \nabla_\alpha h^\alpha_{\nu} - \nabla_\nu \nabla_\alpha h^\alpha_{\mu} + \nabla_\mu \nabla_\nu h \right) - \frac{1}{2} \gamma_{\mu\nu} \left( \Box h - \nabla_\alpha \nabla_\beta h^{\alpha\beta} \right) - H^2 \left( h_{\mu\nu} + \frac{1}{2} \gamma_{\mu\nu} h \right).$$

(12)

Note that $\nabla_\mu G^\text{dS}_{\mu\nu} = 0$ and $\nabla_\mu \hat{\mathcal{O}}_{\mu\nu} \text{(scalar)} = 0$; then, the Bianchi identities mentioned above follow from (11)$^4$, and $h = 0$ from (10). Therefore, $h_{\mu\nu}$ is transverse and traceless. Thus, out of the ten components of $h_{\mu\nu}$ only five are unrestricted. These correspond to five degrees of freedom of a massive spin-2 state. However, in addition there is a sixth degree of freedom propagated by the $\phi$ field. As we will see below, this degree of freedom is ghost-like, and moreover, it does couple to the trace of the stress-tensor, making the theory inconsistent.

To calculate the field $h_{\mu\nu}$ produced by the source $T_{\mu\nu}$, it is useful to introduce a decomposition of the tensors in their transverse traceless (TT), pure trace (PT) and symmetric trace-free (ST) parts:

$$T_{\mu\nu} = T^{TT}_{\mu\nu} + \frac{1}{4} \gamma_{\mu\nu} T + \frac{1}{3} P_{\mu\nu} \frac{1}{Q} T,$$

(13)

and we use the notations

$$P_{\mu\nu} \equiv \nabla_\mu \nabla_\nu - \frac{1}{4} \gamma_{\mu\nu} \Box, \quad Q \equiv -\Box - 4H^2, \quad S \equiv -\Box + 4H^2.$$  

(14)

Noticing that

$$P_{\mu\nu} (\Box + 8H^2) \varphi = \Box P_{\mu\nu} \varphi,$$

(15)

for any scalar $\varphi$, we get that $P_{\mu\nu} f(Q) \varphi = f(S) P_{\mu\nu} \varphi$, for any smooth function $f$.

Next, let us introduce the Lichnerowicz operator $\Delta_L$, which acts differently on the TT, PT and ST parts of the stress-tensor:

$$(\Delta_L - 4H^2) T^{TT}_{\mu\nu} = S T^{TT}_{\mu\nu},$$

$$(\Delta_L - 4H^2) \gamma_{\mu\nu} \varphi = \gamma_{\mu\nu} Q \varphi,$$

$$(\Delta_L - 4H^2) P_{\mu\nu} \varphi = P_{\mu\nu} Q \varphi,$$

(16)

where $\varphi$ is an arbitrary scalar.

Using the above relations, one can calculate the field. The result reads

$$\frac{1}{2} h_{\mu\nu} = \frac{1}{12} \left( \frac{T^{(1/3)}_{\mu\nu}}{\Box + 4H^2} - \frac{T_{\mu\nu}}{12} \Box + 4H^2 \right) + \frac{1}{3} \left( \nabla_\mu \nabla_\nu - \frac{1}{4} \gamma_{\mu\nu} \Box \right) \frac{T}{(\Box + 4H^2)^2},$$

(17)

where we introduced the notation

$$T^{(1/3)}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{3} \gamma_{\mu\nu} T,$$

(18)

and all the operators act from left to right.

$^4$ Naively, there is no kinetic term for $\phi$, but it can be generated by transformation (6) with $\rho = \alpha \phi$. The additional term in the Lagrangian (8) is

$$\alpha 3H \phi (\Box + 4H^2) \phi.$$
It is also convenient to rewrite the expression (17) in the following form:

$$\frac{1}{2} h_{\mu\nu} = \frac{1}{\Delta L - 4H^2} T^{(1/3)}_{\mu\nu} - \frac{1}{3} \left( \nabla_\mu \nabla_\nu + \gamma_{\mu\nu}H^2 \right) \frac{T}{(\Box + 4H^2)^2}. \quad (19)$$

We can decipher the needed information about the spectrum from expression (19). The first term on the rhs of (19) represents a contribution of the helicity-2 and helicity-0 components of the massive graviton.

In the second term of the rhs of (19), there is a structure with covariant derivatives $\nabla_\mu \nabla_\nu$ that does not contribute to the gage invariant physical amplitude, $A \equiv \int d^4x \sqrt{\gamma} T_{\mu\nu} h_{\mu\nu}$, because of the conservation of the stress-tensor $T_{\mu\nu}$. Thus, in the linearized theory these terms carry no information (they will be important, though, in the nonlinear theory—see section 3). However, there is a very last term in (19) which is proportional to $\gamma_{\mu\nu}$ and contains a double pole $(\Box + 4H^2)^2$. The latter can be decomposed into a sum of single poles giving rise to a tachyon and a ghost-like state!

The goal of the next section will be to decouple these states entirely from the physical sector of the linearized theory.

### 3. Decoupling the ghost

To solve the problems of the previous section we will consider the Lagrangian of the tensor field $h_{\mu\nu}$ and scalar $\phi$ with the mixing and quadratic kinetic terms:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_s(h_{\mu\nu}) - \phi \hat{O}^{\mu\nu} h_{\mu\nu} + \phi \hat{K} \phi + h_{\mu\nu} T_{\mu\nu} + q \phi T. \quad (20)$$

Here, $\hat{K}$ is some operator which can contain up to two derivatives as well as terms proportional to $H^2$ (which will be specified below) and $q$ is a constant determining the strength of the scalar coupling to the trace of the stress-tensor.

The Einstein equation reads as follows:

$$G_{\text{dS}}^{\mu\nu} - \frac{m_s^2}{2} (h_{\mu\nu} - \gamma_{\mu\nu} h) - \hat{O}_{\mu\nu} \phi = -T_{\mu\nu}, \quad (21)$$

while variation w.r.t. $\phi$ gives

$$\hat{O}^{\mu\nu} h_{\mu\nu} - 2 \hat{K} \phi = qT. \quad (22)$$

The Bianchi identities give, as before, $\nabla_\mu h_{\mu\nu} = \nabla_\nu h$, reducing the number of independent components of $h_{\mu\nu}$ from ten down to six. Using the Bianchi identities in the equation of motion of $\phi$ (22), we find an equation that is linear in $h$, has no derivatives of $h$, and therefore, determines $h$ in terms of $\phi$. Finally, the trace of the Einstein equation determines $\phi$. The corresponding expressions for $\phi$ and $h$ read

$$\phi = \frac{1}{3Q} T, \quad h = -\frac{1}{3H^2} \left( q + \frac{2 \hat{K}}{3Q} \right) T. \quad (23)$$

Hence, there are six independent degrees of freedom, five in $h_{\mu\nu}$ and one in $\phi$. However, as we will see below, for a special choice of the coefficient in (20) the sixth degree of freedom decouples from the physical sector in the linearized approximation. Alternatively, one could have excluded $\phi$ in favor of $h$ using (22) and the Bianchi identities; then, there would remain six unconstrained degrees of freedom in $h_{\mu\nu}$. 


The physical metric that couples to the source, and determines the physical amplitude, is

\[ h_{\mu\nu}^{\text{phys}} = h_{\mu\nu} + \gamma_{\mu\nu} q \phi = h_{\mu\nu} - \gamma_{\mu\nu} q \frac{T}{3(\Box + 4H^2)}. \]  

(24)

After some algebra, which uses the properties of the Lichnerowicz operator (16), one can find the following expression for the physical field:

\[ \frac{1}{2} h_{\mu\nu}^{\text{phys}} = \frac{1}{\Delta_L - 4H^2} T^{(1/3)}_{\mu\nu} + \gamma_{\mu\nu} (q - a) \frac{T}{3Q} + \frac{\nabla_\mu \nabla_\nu}{3H^2} \left( \frac{q}{2} + a \right) \frac{T}{Q}, \]  

where we have used an explicit form for \( \hat{\mathcal{K}} \), namely

\[ \hat{\mathcal{K}} = 3H^2 + 3aQ. \]  

(26)

The first term in (26) is necessary to cancel the double pole!

In (25) there is still freedom in choosing \( a \). This should be used to remove the tachyonic pole in the second term in (25). Hence, we can set \( a = q \). Then the expression for the physical metric reads

\[ \frac{1}{2} h_{\mu\nu}^{\text{phys}} = \frac{1}{\Delta_L - 4H^2} T^{(1/3)}_{\mu\nu} - q \frac{\nabla_\mu \nabla_\nu}{2H^2} \frac{T}{\Box + 4H^2}. \]  

(27)

We achieved our goal—there are no ghosts or tachyons in the part of (27) that couples to a conserved stress-tensor; the last term in (27) does not contribute to physical amplitudes in the linearized approximation. That term would, however, contribute to Feynman diagrams with nonlinear self-interactions of gravitons as long as \( q \neq 0 \). It would give rise to strong coupling behavior in a full nonlinear theory. Then, it should be possible to reconcile the predictions of such a theory with the observations [8, 9] (for a brief review, see [23]). However, for \( q \neq 0 \) there might be a danger that the sixth degree of freedom would contribute to the physical sector in the full nonlinear theory. This, however, may be possible to avoid if \( \phi \) corresponds to a Nambu–Goldstone boson of a nonlinearly realized symmetry, similar to the brane bending mode in brane induced gravity.

Alternatively, for \( q = 0 \) one would not expect to have strongly coupled behavior in the nonlinear theory. This would be similar to ‘softly massive’ gravity [24] (see also [25]). Then, one would have to overcome the vDVZ discontinuity [26]. Some new ideas are needed for this.

It is also possible that there exist nonlinear theories that give rise to self-accelerated backgrounds, which do not necessarily respect the relation (1). Some of these might be possible to construct by pursuing further the proposal of [27].

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