Parquet Graph Resummation Method for Vortex Liquids

Joonhyun Yeo and M. A. Moore

Department of Physics, University of Manchester, Manchester, M13 9PL, United Kingdom.

(March 29, 2022)

We present in detail a nonperturbative method for vortex liquid systems. This method is based on the resummation of an infinite subset of Feynman diagrams, the so-called parquet graphs, contributing to the four-point vertex function of the Ginzburg-Landau model for a superconductor in a magnetic field. We derive a set of coupled integral equations, the parquet equations, governing the structure factor of the two-dimensional vortex liquid system with and without random impurities and the three-dimensional system in the absence of disorder. For the pure two-dimensional system, we simplify the parquet equations considerably and obtain one simple equation for the structure factor. In two dimensions, we solve the parquet equations numerically and find growing translational order characterized by a length scale \( R_c \) as the temperature is lowered. The temperature dependence of \( R_c \) is obtained in both pure and weakly disordered cases. The effect of disorder appears as a smooth decrease of \( R_c \) as the strength of disorder increases.

PACS: 74.20.De, 74.60.Ge

I. INTRODUCTION

Since the discovery of high-\( T_c \) superconductors, the nature of the mixed state in a type II superconductor has been a focus of theoretical and experimental investigation. In many high-\( T_c \) materials, thermal fluctuations are responsible for the melting of the vortex lattice phase predicted by the mean field theory into a vortex liquid phase. The presence of quenched random impurities in the vortex liquid or lattice phases also plays an important role, since it presents a possibility of a dissipation-free current flow due to pinning of flux lines. New theoretical phases such as the vortex glass phase and the Bose glass phase in the presence of extended defects have been proposed. As shown by Larkin and Ovchinnikov (LO)\(^5\), the quenched point disorder destroys, for spatial dimension \( d < 4 \), the long-range crystalline order of the vortex lattice. The system is described by some characteristic length scale \( R_c \) over which a short-range translational order exists.

In a previous paper\(^6\), we developed a nonperturbative scheme to calculate the structure factor of the two-dimensional (2D) vortex liquid in the absence of random impurities. The main ingredient of this nonperturbative method was the resummation of an infinite subset of Feynman diagrams contributing to the structure factor, the so-called parquet graphs.\(^7\) This is an analytic approach to 2D vortex liquid system which is sophisticated enough to predict growing crystalline order in the system as the temperature is lowered. The growth of the translational order was investigated in connection with the sharp peaks developing in the liquid structure factor. Within this scheme, we found no evidence for a finite temperature phase transition into the vortex lattice phase and the system remains as a liquid. The length scale, \( R_c \), characterizing this growing translational order seemed to diverge only in the zero temperature limit.

In this paper, we give detailed derivations of the parquet graph resummation technique which were omitted in the previous paper.\(^8\) In addition, we present a new simplified version of the parquet equations governing the structure factor, which were given in terms of a set of coupled nonlinear integral equations previously. In the present work, we were able to obtain one simple equation for the structure factor, which contains all the nonperturbative information for the 2D vortex liquid in the absence of disorder.

In this paper, we also apply the parquet graph resummation technique to the 2D vortex liquid in the presence of quenched disorder. We find that the sharp peaks which appeared in the structure factor of the pure system become broadened as the strength of the disorder increases. This is entirely consistent with one’s intuition that in the presence of disorder the length scale \( R_c \) describing the translational order becomes smaller as compared to the pure case. In the present work, since our pure system is always in the liquid phase, the effect of disorder appears as smooth deviations from the pure case. From the nonperturbative results, we find the temperature dependence of the length scale \( R_c \). It is, however, difficult to make any connection between our results and the LO type argument, since there exists no vortex lattice phase at any finite temperature within our nonperturbative scheme, while the LO argument always starts from the vortex lattice with a true long-range crystalline order.

We note that there exist recent theoretical studies\(^9\) based on the elastic theory of pinned lattices suggesting that the pinning by quenched disorder become less effective due to the periodicity of the lattice so that a quasi long-range order persist beyond the Larkin length scale \( R_c \). In the present nonperturbative analysis, the existence of such quasi long-range translational order has not been observed.

The point whether the 2D vortex liquid in the absence of disorder undergoes a finite temperature phase transition into a 2D vortex lattice is still controversial. Nu-
merical simulations seem to suggest a first-order phase transition. However, as shown in Ref. 3, these results depend crucially on boundary conditions. In a spherical geometry, the authors of Ref. 3 demonstrated the absence of a finite temperature phase transition. There also exists a recent experiment performed on a sample with very weak pinning, where no sign of a phase transition is detected. The present parquet approximation, which is an analytic theory on an infinite plane, seems to support the absence of a finite temperature phase transition between 2D vortex liquid and solid.

Unlike the 2D system, it is generally believed that a 3D vortex liquid undergoes a first-order phase transition into a vortex lattice. An experiment by Zeldov et al. on BSCCO has been accepted as a convincing evidence for the transition. However, there exists a recent claim that the results of Ref. 11 might be an artifact due to a particular sample geometry. Therefore, it would be interesting if one could develop a three-dimensional parquet graph resummation scheme and obtain nonperturbative information on the 3D vortex liquid system. We find that one can generalize the parquet equations to three dimensions without difficulty. Unfortunately, the equations become very complicated and we were not able to obtain a numerical solution to the 3D parquet equations.

In the next section, we introduce the structure factor of a two-dimensional disordered vortex liquid within the Ginzburg-Landau theory. In Secs. II and III, we present detailed derivations of the parquet equations which account for all the parquet graphs contributing to the structure factor for both pure and disordered cases. We also consider zero-dimensional models to discuss the validity of the parquet approximation in general. In the following section, we present the main results of our calculation and discuss the temperature dependence of the length scale $R_c$. In Sec. IV, we present the generalization of the parquet equations to three dimensions. Finally, we conclude with discussion on future directions.

II. THE STRUCTURE FACTOR

We begin our analysis with the Ginzburg-Landau free energy for a superconducting film in a perpendicular magnetic field $B = \nabla \times A$ in the presence of quenched random impurities;

$$F[\Psi] = \int d^2 r \left( \frac{1}{2m} \left( -i \hbar \nabla - e^* A \right) \Psi \right)^2 + \left( \alpha + \tau(r) \right) |\Psi|^2 + \frac{\beta}{2} |\Psi|^4, \tag{2.1}$$

where $\alpha$, $\beta$ and $m$ are phenomenological parameters. The random field $\tau(r)$ representing the quenched impurities satisfies the probability distribution, $\overline{\tau(r)\tau(r')} = 0$ and

$$\frac{\tau(r)\tau(r')}{\overline{(\tau(r)\tau(r'))}} = \lambda \delta^{(2)}(r-r'). \tag{2.2}$$

In this paper we neglect the fluctuations in the vector potential $A$ and restrict the order parameter $\Psi$ to the space spanned by the lowest Landau level (LLL) wavefunctions. In the symmetric gauge, where $A = \frac{B}{\pi} (-y, x)$, the LLL is fully described by an arbitrary analytic function of the variable $z = x + iy$ multiplied by an exponential factor; $\Psi(x,y) = \exp(-\mu^2 z^* z) \phi(z)$, where $\mu^2 \equiv e^* B/\hbar = 2\pi/Q$ and $Q$ is the area of the unit cell of the vortex lattice. In the LLL approximation, the GL free energy becomes

$$F[\phi] = \int dz^* dz \left( (\alpha_H + \tau(z, z^*)) \exp(-\mu^2 |z|^2) |\phi(z)|^2 + \frac{\beta}{2} \exp(-\mu^2 |z|^2) |\phi(z)|^4 \right), \tag{2.3}$$

where $\int dz^* dz$ denotes the integration over $x$-$y$ plane and $\alpha_H \equiv \alpha + he^* B/2m$ vanishes at the mean field transition temperature.

The central quantity in this analysis is the structure factor of the two-dimensional vortex liquid. It is proportional to the Fourier transform of the density-density correlation function, $\chi(k) = \int d^2 R \ e^{ik \cdot R} \chi(r, r + R)$;

$$\chi(r, r') \equiv \langle |\Psi(r')|^2 |\Psi(r)|^2 \rangle - \langle |\Psi(r')|^2 \rangle \langle |\Psi(r)|^2 \rangle, \tag{2.4}$$

where the angular brackets denote the thermal averages. The structure factor $\Delta(k)$ is then defined by

$$Q \exp(-\frac{k^2}{2\mu^2}) \Delta(k) = \chi(k)/\langle |\Psi(r)|^2 \rangle^2. \tag{2.4}$$

A convenient way to deal with quenched averages is to introduce $n$ replicas of $Z$ and calculate the correlation functions with respect to

$$\overline{Z^n} = \prod_n \int d\phi_a d\phi_a^\dagger \exp \left[ - \int dz^* dz \right.$$

$$\left. \{ \alpha_H e^{-\mu^2 |z|^2/2} \sum_a |\phi_a(z)|^2 + \frac{\beta}{2} e^{-\mu^2 |z|^2} \sum_a |\phi_a(z)|^4 \right. - \frac{\lambda}{2} e^{-\mu^2 |z|^2} \sum_{a,b} |\phi_a(z)|^2 |\phi_b(z)|^2 \right]. \tag{2.5}$$

in the limit $n \to 0$. One can develop a standard perturbation theory for (2.3). The bare propagator arising from the perturbation expansion of (2.3) is given by (2.6)

$$G_0^{ab}(\zeta, z) \equiv \langle \langle \phi_a^\dagger(\zeta) \phi_b(z) \rangle \rangle_0 = \delta_{ab} \frac{1}{\alpha_H} \frac{\mu^2}{2\pi} \exp(\frac{\mu^2}{2} \zeta^2), \tag{2.6}$$

where the double bracket $\langle \langle \cdot, \cdot \rangle \rangle$ denotes the average with respect to $\overline{Z^n}$. It is important to note that, because of the simple quadratic term of the LLL free energy in (2.3), the renormalized propagator $G_\tau^{ab}$ is obtained by simply replacing $\alpha_H$ by the renormalized mass $\alpha_R$. This means
that the magnetic length $\mu^{-1}$ is the only length scale involved in the renormalized propagator. This is one of the simplifying features of the LLL approximation that makes the present nonperturbative calculation feasible.

As noticed in Ref. 13, the GL free energy obtained from (2.3) is not closed under renormalization. The renormalization effectively generates the quartic vertices of a general form;

$$
\tilde{g}(k) = \int d^2 z^* d z (|z|) \exp(\frac{i}{2}(k^* z + k z^*))
$$

and similarly $\tilde{w}(k)$, where $k, k^*$ are complex momenta, $k = k_1 + i k_2, k^* = k_1 - i k_2$ with $k = (k_1, k_2)$. Thus, $\tilde{g}_B(|z|) = \beta/2$ and $\tilde{w}_B(|k|) = \lambda/2$ are constants.

In order to calculate the structure factor, (2.4), we need to consider the renormalized four-point correlation function arising from Eq. (2.5):

$$
\langle \phi^*_a(z_1^*) \phi^*_b(z_2^*) \phi_c(z_3) \phi_d(z_4) \rangle_c = \frac{2}{2\pi} \int \frac{d^2 k}{(2\pi)} \frac{2\pi}{2} \delta_{ab} G_B(|k|) - \tilde{w}_B(|k|)
$$

where the last term denotes the contribution from all connected Feynman diagrams. To the lowest order of perturbation theory, the connected four-point correlation function can easily be calculated as

$$
\langle \phi^*_a(z_1^*) \phi^*_b(z_2^*) \phi_c(z_3) \phi_d(z_4) \rangle_c = -\delta_{ac} \delta_{bd} \frac{2}{\alpha_R} \rho_R^2 \frac{2}{2\pi} \exp(\frac{1}{2}(k^* z + k z^*))
$$

where the second term on the right hand side is the same as the first term with $c$ and $d$, and $z_3$ and $z_4$ interchanged. In (2.8), we absorbed the renormalization of the propagators using the renormalized mass $\alpha_R$. To the lowest order, one can easily evaluate the $k$-integrals in (2.8), but, in general, higher order corrections to the connected four-point function are represented in (2.8) by the departure of the quartic vertex functions from constants, $\tilde{g}_R(|k|) = \beta/2$ and $\tilde{w}_B(|k|) = \lambda/2$, to general $k$-dependent functions, $\tilde{g}_R(k)$ and $\tilde{w}_R(k)$. Therefore, the LLL approximation enables us to concentrate on the renormalized quartic vertex function $\tilde{g}_R(k)$ and $\tilde{w}_R(k)$, which depend only on one variable instead of three characterizing three independent channels in a usual field theory.

Now, one can put (2.8) for general $g_R(k)$ and $w_R(k)$ in a more symmetric form using scaled functions,

$$
f_R(k) = \frac{2}{\beta} \exp(-k^2/2\mu^2) \tilde{g}_R(k),
$$

and

$$
v_R(k) = \frac{2}{\lambda} \exp(-k^2/2\mu^2) \tilde{w}_R(k).
$$

The bare vertices are given by $f_B(k) = v_B(k) = \exp(-\frac{k^2}{2\mu^2})$. Interchanging $z_3$ and $z_4$ in the second term on the right hand side of (2.8) is equivalent to using, instead of $f_R(k)$ and $v_R(k)$, the transformed functions, $\tilde{f}_R(k)$ and $\tilde{w}_R(k)$, where $\tilde{f}(k)$ is defined for an arbitrary function $f(k)$ by

$$
\tilde{f}(k) = \frac{2\pi}{\mu^2} \int \frac{d^2 p}{(2\pi)^2} f(p) \exp(\frac{i}{\mu^2}(k_1 p_2 - k_2 p_1)),
$$

and

$$
f(k) = \frac{2\pi}{\mu^2} \int \frac{d^2 p}{(2\pi)^2} \tilde{f}(p) \exp(\frac{i}{\mu^2}(k_1 p_2 - k_2 p_1)).
$$

The general four-point function is then given by (see Appendix)

$$
\langle \phi^*_a(z_1) \phi^*_b(z_2) \phi_c(z_3) \phi_d(z_4) \rangle_c = -\frac{4}{\alpha_R} \rho_R^2 \frac{2}{2\pi} \exp(\frac{1}{2}(k^* z + k z^*))
$$

where

$$
\Gamma_{ab,cd}(k) = \frac{1}{2} \delta_{ac} \delta_{bd} \left[ \delta_{ab} f_R(k) - \theta v_R(k) \right]
$$

and $\theta = \lambda/\beta$ represents the strength of the disorder. Eq. (2.10) can be represented diagrammatically as in Fig. 14.

The structure factor $\Delta(k)$ is then obtained by joining two external legs of the four-point correlation functions, (2.9). From (2.9) and (2.10) and the definition (2.4) in the limit $n \to 0$, we obtain

$$
\Delta(k) = 1 - 2x \Gamma(k),
$$

where

$$
\Gamma(k) \equiv \frac{1}{2} \left[ f_R(k) + \tilde{f}_R(k) - \theta \left( v_R(k) + \tilde{v}_R(k) \right) \right],
$$

and $x \equiv \mu^2/2\pi \alpha_R^2$ is a dimensionless parameter which appears in the high-temperature perturbation expansion [13].
function also fixes the relationship between \( \alpha \) and \( v \), bare parameter \( \alpha \), renormalization of the propagator into the renormalized propagators. As mentioned earlier, we absorb any renormalization of the propagators. The thick dashed and dotted lines are for the self-energy \( \Sigma \). The solid lines represent the renormalized connected four-point correlation function. The label \( s \), · · · , \( z \), \( A \), respectively, and \( f_R \) and \( v_R \), as in Fig. 1, \( \Delta(\mathbf{k}) \) found in Ref. 5 and obtain a very simple equation for the structure factor \( \Delta(\mathbf{k}) \).

For both pure and disordered cases, we will present the analysis for the zero-dimensional analogues of (2.3), which can be integrated exactly. For the pure case, we find it convenient to introduce the parquet resummation scheme in a simple zero-dimensional model and then generalize to the two-dimensional problem. Furthermore, since there is no apparent expansion parameter involved in the parquet approximation, it would be instructive to apply the parquet resummation to the cases where exact solutions are known and to compare the approximate result with the exact solution.

### III. PARQUET GRAPH RESUMMATION: PURE SYSTEM \((\lambda = 0)\)

In order to calculate \( f_R(\mathbf{k}) \) and \( v_R(\mathbf{k}) \), one needs to evaluate the Feynman diagrams contributing to the four-point correlation function. In this and the next sections, we show in detail how to sum an infinite subset of such diagrams, the so-called parquet graphs.

We first consider the pure system where \( \lambda = 0 \), then generalize the result to the disordered case. The analysis of the pure case will mostly reproduce results given in Ref. 5. In the present work, we make a further simplification of the parquet equation for \( f_R(\mathbf{k}) \) found in Ref. 5 and obtain a very simple equation for the structure factor \( \Delta(\mathbf{k}) \).

In the following sections, we will evaluate \( f_R(\mathbf{k}) \) and \( v_R(\mathbf{k}) \) nonperturbatively by summing the parquet graphs. As mentioned earlier, we absorb any renormalization of the propagator into the renormalized parameter \( \alpha_R \). But knowledge of the four-point vertex function also fixes the relationship between \( \alpha_R \) and the bare parameter \( \alpha_H \), thus completing the description of the system. This relation comes from the Dyson equation arising from (2.3) which is described diagrammatically in Fig. 2 for the self-energy \( \Sigma = G^{-1} - G_0^{-1} = 2\pi(\alpha_R - \alpha_H)/\mu^2 \):

\[
\alpha_T = \frac{1}{\sqrt{\pi}} \left[ 1 - x(2 - \theta) + x^2 \frac{2\pi}{\mu^2} \int \frac{d^2k}{(2\pi)^2} e^{-k^2/2\mu^2} \right. \\
\left. \times \left[ 2(1 - \theta)f_R(k) - \theta(2 - \theta)v_R(k) \right] \right], \tag{2.13}
\]

where \( \alpha_T = \alpha_H \sqrt{2\pi/\beta\mu^2} \) is the dimensionless temperature. Note that this is an exact relationship between the renormalized propagator and the quartic vertex functions. We will, however, use \( f_R(\mathbf{k}) \) and \( v_R(\mathbf{k}) \) obtained from the present nonperturbative approximation. The Hartree approximation used in the high-temperature perturbation expansion corresponds to neglecting terms that depend on the vertex functions, \( f_R \) and \( v_R \).

### A. \( d=0 \)

For \( \lambda = 0 \) and dimension \( d = 0 \), the partition function corresponding to (2.3) is a simple integral,

\[
Z(\alpha_H, \beta) = \int \frac{d\psi d\psi^*}{2\pi} \exp(-\alpha_H|\psi|^2 - \frac{\beta}{2}|\psi|^4) \\
= \sqrt{\frac{2\pi}{\beta}} \text{erfc}(\sigma) \exp(\sigma^2),
\]

where \( \sigma = \alpha_H/\sqrt{2\beta} \) and \( \text{erfc}(\sigma) \equiv 1 - \text{erf}(\sigma) \) is the complementary error function. In the parquet analysis, one calculates the renormalized four-point vertex graphs. 

FIG. 1. Diagrammatic representation of the renormalized connected four-point correlation function. The labels, 1, · · · , 4, denote \( z_1, z_2, z_3, z_4 \), respectively, and \( a, \cdots, d \) are replica indices.

FIG. 2. Diagrammatic representation of Dyson equation for the self-energy \( \Sigma \). The solid lines represent the renormalized propagators. The thick dashed and dotted lines are the renormalized vertex functions, \( f_R \) and \( v_R \), as in Fig. 1, while the thin dashed and dotted lines denote the bare vertex functions, \( f_B \) and \( v_B \), respectively.
\[
\Gamma = -\frac{\alpha_R^4}{4} \langle |\psi|^4 \rangle_c = \frac{\alpha_R^4}{2} \left[ \frac{\partial}{\partial \beta} \ln Z + \frac{1}{\alpha_R^2} \right],
\]

where \( \alpha_R^{-1} = \langle |\psi|^2 \rangle \). Although an exact solution can readily be found, one can construct the usual Feynman graph expansion for \( \Gamma \). To the lowest order, we have \( \Gamma = \Gamma_B = \beta/2 \). To the one-loop order, the diagrams can be represented as shown in Fig. 3. A convenient way to generate the next order diagrams is to replace each vertex in a one-loop diagram by the vertices obtained up to the one-loop order. In general, one can construct higher-order diagrams by replacing each vertex in the one-loop diagrams by the vertices obtained up to the current order of perturbation expansion. An example of such construction is shown in Fig. 4. The diagrams obtained in this way are called parquet graphs. Note that parquet graphs can be separated into two parts by cutting two propagator lines.

FIG. 3. (a) Three one-loop diagrams. The symmetry factors are 2, 4 and 4, respectively. The labels 1, \cdots, 4 are drawn for the general cases to be discussed later. For the zero-dimensional case, there is no distinction between the second and the third diagrams. But, it is important to separate this contribution into two equal parts. (b) An example of constructing successive parquet graphs.

Although parquet graphs cover an enormous number of diagrams, obviously not all diagrams can be constructed in this way. The non-parquet diagrams are generated in the above construction by the so-called totally irreducible vertices whose contribution here is denoted by \( R \). The totally irreducible vertex consists of the bare vertex and higher order \( O(\beta^4) \) vertices (see Fig. 4). There is no systematic ways of enumerating these higher order diagrams contributing to \( R \). The parquet approximation which we employ in this work corresponds to keeping only the bare vertex contribution in \( R \): \( R \simeq \Gamma_B = \beta/2 \) in the zero-dimensional case.

FIG. 4. The totally irreducible vertex \( R \).

\[
R = \begin{array}{c}
\times
\end{array} + \begin{array}{c}
\times
\end{array} + O(\beta^4)
\]

FIG. 5. The parquet decomposition of reducible vertex \( \pi_i \). Again, the labels 1, \cdots, 4 are drawn for the general cases. The same diagrammatic decomposition can be used in higher dimensional models for both pure and disordered cases.

If both bare vertices in a one-loop diagram are replaced by the full renormalized vertex in the above construction of graphs, then the diagrams become overcounted. There is a systematic way to eliminate this overcounting and to generate all the Feynman diagrams and the symmetry factors associated with each diagram, and that is to use the full vertex for the bare vertex on the right hand side of Fig. 3, but the so-called irreducible vertex for the one on the left hand side, which is defined as follows: Let \( \pi_i, i = 1, 2, 3 \) denote the diagrams constructed out of the three one-loop diagrams in Fig. 3 and \( \Lambda_i \) the corresponding irreducible vertices, we have from Fig. 5:

\[
\begin{align*}
\pi_1 &= -\frac{2}{\alpha_R} \Lambda_1 \Gamma, \\
\pi_2 &= -\frac{4}{\alpha_R} \Lambda_2 \Gamma, \\
\pi_3 &= -\frac{4}{\alpha_R} \Lambda_3 \Gamma,
\end{align*}
\]

where we absorb any renormalization on propagator lines into \( \alpha_R \). Now, for the irreducible vertex \( \Lambda_i \), we must include all the diagrams that do not belong to \( \pi_i \) to avoid
the overcounting, thus, \( \Lambda_i = \Gamma - \Pi_i \). Finally, the renormalized vertex \( \Gamma \) is given by the sum of all contributions;

\[
\Gamma = R + \sum_{i=1}^{3} \Pi_i. \tag{3.2}
\]

Therefore, \( \Lambda_i = R + \sum_{j \neq i} \Pi_j. \tag{3.3} \)

Note that (3.1)-(3.3) are exact relations for the renormalized vertex \( \Gamma \). But, as mentioned before, we use \( R \approx \beta/2 \).

One can easily simplify the above equations for \( R = \beta/2 \) as follows: From (3.1), one has

\[
\Pi_1 = \frac{2\Gamma^2}{\alpha_R^2 - 2\Gamma}, \quad \Pi_2 = \Pi_3 = \frac{4\Gamma^2}{\alpha_R^2 - 4\Gamma}.
\]

Therefore, from (3.2), we obtain

\[
\gamma = \rho - \frac{\gamma^2}{1 - \gamma} - \frac{2\gamma^2}{1 - 2\gamma}, \tag{3.4}
\]

where \( \gamma = 2\Gamma/\alpha_R^2 \) and \( \rho = \beta/\alpha_R^2 \). As a function of \( \rho \), there is only one solution \( \gamma(\rho) \) that satisfies the trivial condition: \( \Gamma = \gamma = 0 \) when \( \beta = \rho = 0 \).

From this relation using \( \gamma = \gamma(\rho) \) obtained from (3.4), one finds \( \rho \) as a function of \( \sigma \), and consequently we have the renormalized four-point vertex \( \gamma \) as a function of \( \sigma \). In Fig. 6, \( \gamma(\sigma) \) of the parquet approximation is compared with the exact solution. One can see there is excellent agreement between the two.

**B. d=2**

We now consider the two-dimensional problem in the absence of disorder. The parquet equations can be constructed similarly to the \( d = 0 \) case. In fact, eqs. (3.1)-(3.3) take almost the same form as before. The only differences are the fact that the vertices now depend on the internal momentum \( k \) and the right hand sides of (3.1) should be obtained from a direct evaluation of diagrams in Fig. 4. In Appendix, we show explicitly how this calculation is done. The resulting equations corresponding to (3.1) are

\[
\Pi_1(k) = -x(\Lambda_1 \circ \Gamma)(k), \tag{3.6a}
\]
\[
\Pi_2(k) = -2x\Lambda_2(k)\Gamma(k), \tag{3.6b}
\]
\[
\Pi_3(k) = -2x(\Lambda_3 \ast \Gamma)(k), \tag{3.6c}
\]

where the operations \( \circ \) and \( \ast \) between two arbitrary functions \( f(k) \) and \( g(k) \) are defined by

\[
(f \circ g)(k) \equiv \frac{2\pi}{\mu^2} \int \frac{d^d p}{(2\pi)^2} f(k - p)g(p) \cos \left(\frac{k_1 p_2 - k_2 p_1}{\mu^2}\right),
\]
\[
(f \ast g)(k) \equiv \frac{2\pi}{\mu^2} \int \frac{d^d p}{(2\pi)^2} f(k - p)g(p).
\]

The remaining parquet equations are given by

\[
\Gamma(k) = R(k) + \sum_{i=1}^{3} \Pi_i(k), \tag{3.7}
\]
\[
\Lambda_i(k) = R(k) + \sum_{j \neq i} \Pi_j(k), \tag{3.8}
\]

where \( R(k) \) represents the totally irreducible part. In the parquet approximation, \( R(k) \) is equal to the bare vertex part: \( R(k) \simeq (1/2)(f_B + f_B) = \exp(-k^2/2\mu^2) \).

We can make a simplification on the parquet equations as in the zero-dimensional case. We first note the following identities:

\[
(f \circ g) = (f \circ g), \quad f \circ g = f \circ g = f \circ g,
\]

and \( f \ast g(k) = \hat{f}(k)\hat{g}(k) \). Inserting \( \Lambda_i(k) = \Gamma(k) - \Pi_i(k) \) into (3.6d) and (3.6c) and using \( \hat{\Gamma} = \Gamma \) from (2.11), we obtain

\[
\Pi_2(k) = \hat{\Pi}_3(k) = -\frac{2x\Gamma^2(k)}{1 - 2x\Gamma(k)}. \tag{3.9}
\]

Now, inserting this into (3.6a),

![Figure 6](https://example.com/figure6.png)

**FIG. 6.** The renormalized four-point vertex function \( \gamma \) for the pure zero-dimensional model as a function of temperature \( \sigma \). The solid line is the exact solution. The dashed line corresponds to the parquet approximation.

To complete the description of the system, one needs a relation between the bare (\( \alpha_H \) or \( \sigma \)) and renormalized (\( \alpha_R \) or \( \rho \)) mass. In the parquet approximation, we use an analogue of Dyson equation, (2.13), which, in this case, is

\[
\sigma = \frac{1}{2\rho} \left[ 1 - 2\rho(1 - \gamma) \right]. \tag{3.5}
\]
\[ \Pi_1(k) = -x ( (R + \Pi_1 + \Pi_3) \circ \Gamma)(k) \]
\[ = -x \left( (R + 2\Pi_2) \circ \Gamma \right)(k). \] (3.10)

Therefore, inserting (3.9) and (3.10) into (3.7), we obtain an equation for \( \Gamma(k) \). In terms of the structure factor \( \Delta(k) = 1 - 2x\Gamma(k) \), this equation becomes
\[ \frac{1 - \Delta(k)}{\Delta(k)} = xR(k) + x(R \circ \Delta)(k) \] (3.11)

This is our main equation mentioned in Sec. II that completely describes the structure factor of 2D vortex liquid in the absence of disorder. Note that we have kept the totally irreducible part \( R(k) \) in a general form. We note that, although this is an exact relation in a very simple form, it has little advantages for finding numerical solutions over the coupled parquet equations, (3.7) and (3.8) or another version to be described below. We believe, however, that this equation might open a possibility in the future for a nonperturbative analytic investigation on the 2D vortex liquid.

When we consider the disordered case in the next section, it will be necessary to decompose \( \Gamma(k) \) into \( f_R(k) \) and \( \tilde{f}_R(k) \) using (2.11) and consider the parquet equation in terms of \( f_R(k) \) arising from the faithful representation of Feynman diagrams as in Fig. II. Eq. (3.9) suggests the following decompositions:
\[ \Pi_1(k) = \frac{1}{2} \left( \Gamma_1(k) + \tilde{\Gamma}_1(k) \right), \]
\[ \Pi_2(k) = \tilde{\Pi}_3(k) = \frac{1}{2} \left( \Gamma_2(k) + \tilde{\Gamma}_3(k) \right), \]
\[ \Lambda_1(k) = \frac{1}{2} \left( I_1(k) + \tilde{I}_1(k) \right), \]
\[ \Lambda_2(k) = \tilde{\Lambda}_3(k) = \frac{1}{2} \left( I_2(k) + \tilde{I}_3(k) \right), \]
for some functions \( I_i \) and \( \Gamma_i \). Inserting these into (3.6), we have
\[ \Gamma_1(k) = -x (I_1 \circ f_R)(k) \] (3.12a)
\[ \Gamma_2(k) = -x \left( I_2(k) f_R(k) + I_2(k) \tilde{f}_R(k) \right) + \tilde{I}_3(k) f_R(k) \] (3.12b)
\[ \Gamma_3(k) = -x (I_3 \circ f_R)(k), \] (3.12c)

and
\[ f_R(k) = f_B(k) + \sum_{i=1}^{3} \Gamma_i(k), \] (3.12d)
\[ I_i(k) = f_B(k) + \sum_{j \neq i} \Gamma_j(k), \] (3.12e)

This version of the parquet equations has been previously given in Ref. [3].

IV. PARQUET GRAPH RESUMMATION: DISORDERED CASE

For the disordered case, one has to construct the parquet equation for the vertex function containing \( n \)-replica indices. Let us first consider the two-dimensional case directly. It is straightforward to generalize (3.11) to the present case. By putting the replica indices in the diagrams in Fig. III, we have
\[ \Pi_{ab,cd}^{(1)}(k) = -x \sum_{e,f} \left( \Lambda_{ab,ef}^{(1)} \circ \Gamma_{ef,cd} \right)(k), \] (4.1a)
\[ \Pi_{ab,cd}^{(2)}(k) = -2x \sum_{e,f} \left( \Lambda_{ae,cf}^{(2)} \right) \Gamma_{fb,cd}(k), \] (4.1b)
\[ \Pi_{ab,cd}^{(3)}(k) = -2x \sum_{e,f} \left( \Lambda_{ae,fd}^{(3)} \right) \Gamma_{fb,ce}(k). \] (4.1c)

The remaining equations follow from (3.7) and (3.8):
\[ \Gamma_{ab,cd}(k) = R_{ab,cd}(k) + \sum_{i=1}^{3} \Pi_{ab,cd}^{(i)}(k), \] (4.1d)
\[ \Lambda_{ab,cd}^{(i)}(k) = R_{ab,cd}(k) + \sum_{j \neq i} \Pi_{ab,cd}^{(j)}(k), \] (4.1e)

where
\[ R_{ab,cd}(k) \simeq \Gamma_{ab,cd}^{(B)}(k) = \delta_{a,b} \delta_{d,c} \delta_{\theta}(\theta - \theta \exp(-k^2/2\mu^2)) \]
in the parquet approximation.

As one can see from (2.10), in order to take \( n \rightarrow 0 \) limit, one needs to decompose the above equations as in the previous section and to get equations analogous to (3.12). First, we note that
\[ \tilde{\Lambda}_{ab,cd}^{(3)}(k) = \Lambda_{ab,cd}^{(2)}(k), \quad \tilde{\Pi}_{ab,cd}^{(3)}(k) = \Pi_{ab,cd}^{(2)}(k). \]

Therefore, we can write, for some functions \( \Gamma_i, \Xi_i, I_i \) and \( J_i \),
\[ \Pi_{ab,cd}^{(1)}(k) = \frac{1}{2} \delta_{ac} \delta_{bd} \left[ \delta_{ab} \Gamma_1(k) - \theta \Xi_1(k) \right] + \frac{1}{2} \delta_{ad} \delta_{bc} \left[ \delta_{ab} \Gamma_1(k) - \theta \Xi_1(k) \right], \]
\[ \Pi_{ab,cd}^{(2)}(k) = \tilde{\Pi}_{ab,cd}^{(3)}(k) = \frac{1}{2} \delta_{ac} \delta_{bd} \left[ \delta_{ab} \Gamma_2(k) - \theta \Xi_2(k) \right] + \frac{1}{2} \delta_{ad} \delta_{bc} \left[ \delta_{ab} \Gamma_3(k) - \theta \Xi_3(k) \right], \]
and
\[ \Lambda_{ab,cd}^{(1)}(k) = \frac{1}{2} \delta_{ac} \delta_{bd} \left[ \delta_{ab} I_1(k) - \theta J_1(k) \right] + \frac{1}{2} \delta_{ad} \delta_{bc} \left[ \delta_{ab} I_1(k) - \theta J_1(k) \right], \]
\[ \Lambda_{ab,cd}^{(2)}(k) = \tilde{\Lambda}_{ab,cd}^{(3)}(k) = \frac{1}{2} \delta_{ac} \delta_{bd} \left[ \delta_{ab} I_2(k) - \theta J_2(k) \right] + \frac{1}{2} \delta_{ad} \delta_{bc} \left[ \delta_{ab} I_3(k) - \theta J_3(k) \right]. \]
Inserting these expressions and (2.10) into (4.1) and eliminating one term that contains an explicit factor of \( n \), we get the following set of parquet equations in the presence of disorder:

\[
\Gamma_1(k) = -x(I_1 \circ f_R - \theta I_1 \circ v_R - \theta J_1 \circ f_R)(k),
\]

\[
\Gamma_2(k) = -x \left( I_2(k)f_R(k) + J_2(k)f_R(k) - \theta \bar{v}_R(k) \right),
\]

\[
\Gamma_3(k) = -x(I_3 \ast f_R - \theta I_3 \ast v_R - \theta J_3 \ast f_R)(k),
\]

and

\[
X_1(k) = x\theta(J_1 \circ v_R)(k),
\]

\[
X_2(k) = -x \left( J_2(k)v_R(k) + J_2(k)f_R(k) \right),
\]

\[
X_3(k) = x\theta(J_3 \ast v_R)(k)
\]

with

\[
f_R(k) = f_B(k) + \sum_{i=1}^3 \Gamma_i(k), \tag{4.2a}
\]

\[
v_R(k) = v_B(k) + \sum_{i=1}^3 X_i(k), \tag{4.2b}
\]

and

\[
I_i(k) = f_B(k) + \sum_{j \neq i} \Gamma_j(k), \tag{4.3a}
\]

\[
J_i(k) = v_B(k) + \sum_{j \neq i} X_j(k). \tag{4.3b}
\]

Before solving these complicated equations, we consider first the zero-dimensional toy model where the parquet equations reduce to a set of algebraic equations as in (3.4).

**A. \( d=0 \)**

For \( d = 0 \), the partition function is again a simple integral,

\[
Z(\alpha_H, \tau, \beta) = \int \frac{d\psi d\psi^*}{2\pi} \exp \left( -\langle \alpha_H + \tau \rangle |\psi|^2 - \frac{\beta}{2} |\psi|^4 \right)
\]

\[
= \sqrt{\frac{\pi}{2\beta}} \text{erfc} \left( \frac{\alpha_H + \tau}{\sqrt{2\beta}} \right) \exp \left( \frac{(\alpha_H + \tau)^2}{2\beta} \right).
\]

Averaging over \( \tau \) with respect to (2.2), we have

\[
-\ln Z(\sigma, \beta, \theta) = \frac{1}{2} \ln \beta - \sigma^2 - \frac{\theta}{2}
\]

\[
- \frac{1}{\sqrt{\theta \pi}} \int_{-\infty}^{\infty} d\sigma e^{-(\sigma - \sigma')^2/\theta} \ln \text{erfc}(\sigma'),
\]

where \( \sigma = \alpha_H / \sqrt{2\beta} \), and \( \theta = \lambda / \beta \) as before.

The quantities corresponding to \( f_R(k) \) and \( v_R(k) \) in the zero-dimensional case are defined by

\[
f_R \equiv -\frac{\alpha_H^4}{2\beta} \left( \langle |\psi|^4 \rangle - 2\langle |\psi|^2 \rangle \right)
\]

\[
= -\frac{\alpha_H^4}{\beta} \left[ \frac{\partial}{\partial \beta} + \frac{\partial^2}{\partial \alpha_H^2} \right] \ln Z,
\]

\[
v_R \equiv \frac{\alpha_H^2}{\beta} \left( \langle |\psi|^2 \rangle^2 - \langle |\psi|^4 \rangle \right)
\]

\[
= -\frac{\alpha_H^2}{\beta} - \frac{\alpha_H^4}{\beta} \left[ 2 \frac{\partial}{\partial \beta} + \frac{\partial^2}{\partial \alpha_H^2} \right] \ln Z,
\]

where \( \alpha_H^{-1} \equiv \langle |\psi|^2 \rangle = -\partial \ln Z / \partial \alpha_H \). These quantities can easily be evaluated as functions of \( \sigma \) and \( \theta \) using (17).

On the other hand, the parquet equations for \( f_R \) and \( v_R \) take exactly the same form as (4.2), (4.3), if we make the following simplifications: the binary operations \( \circ \) and \( \ast \) become multiplications between two factors; the transformation \( \tilde{f} \) has no effect, \( \tilde{f} = f \); \( x \) is replaced by \( \rho = \beta / \nu_R \); and finally \( f_B = v_B = 1 \). It is then straightforward to eliminate \( I_i \) (or \( \Gamma_i \)) and \( J_i \) (or \( X_i \)) from the parquet equations in favor of \( f_R' \equiv \rho f_R \) and \( v_R' \equiv \rho v_R \). The parquet equations then reduce to the following coupled algebraic equations.

\[
\frac{2f_R'}{(1 - f_R' + v_R')(1 + v_R')} \tag{4.5}
\]

\[
+ \frac{f_R'}{(1 - f_R' + v_R')(1 - 2f_R' + v_R')} - 2f_R' = \rho,
\]

\[
\frac{2v_R'}{1 + v_R'} + \frac{v_R'}{(1 - 2f_R' + v_R')^2} - 2v_R' = \rho. \tag{4.6}
\]

For given \( \theta \) and \( \rho \), one can find the corresponding solutions \( f_R' \) and \( v_R' \) to these equations. We can then use the Dyson equation (2.13):

\[
\sigma = \frac{1}{\sqrt{2\rho}} \left[ 1 - \rho(2 - \theta) + \rho[2(f_R' - v_R') - \theta(2f_R' - v_R')] \right].
\]

To find a relation between \( \rho \) and \( \sigma \): \( \rho = \rho(\sigma) \) for given \( \theta \). Using this, we obtain \( f_R'(\sigma) \) and \( v_R'(\sigma) \) for given \( \theta \). These are shown in Fig. 7 together with the exact solutions. For \( \theta \lesssim 1 \), the parquet results show excellent agreement with the exact solutions. As \( \theta \) becomes larger, however, the discrepancies between two solutions grow. This analysis for the zero-dimensional model suggests that the parquet approximation is in general very good when the strength of disorder is moderate. But, the diagrams omitted in the parquet approximation might produce quantitative errors in the strong disorder regime. We note, however, that physical quantities in this approximation remain smooth as functions of \( \theta \) unlike other approximation methods on this system.
The minimum temperature where the disorder prevents ordering on long length scales. In Fig. 11, we show the height of the first peak as a function of temperature for \( \theta \geq 0.1, 0.2 \) and 0.3, respectively.

In Figs. 3 and 10, the structure factor is plotted for various values of \( \alpha_T \) and \( \theta \). The structure factor develops a collection of peaks around the reciprocal lattice vectors (RLV) of the triangular lattice. As the temperature is lowered for fixed \( \theta \), one can clearly see from Fig. 3 that the first peak grows with decreasing width. This can be interpreted as a growing short-range translational order in the disordered vortex liquid. The length scale \( R_c \) over which this order exists can be obtained from the inverse width of the first peak, or equivalently from the peak height. For fixed temperatures, we find that the length scale decreases as the strength of disorder increases. (See Fig. 11.) The result is consistent with an intuitive picture where the disorder prevents ordering on long length scales. In Fig. 11, we show the height of the first peak as a function of temperature for \( \theta = 0.1, 0.2 \) and 0.3. We find that the length scale \( R_c \) grows as \( |\alpha_T| \) with decreasing proportionality constant for increasing \( \theta \). We recall that, in the absence of disorder, the length scale characterizing a growing crystalline order also grows as \( |\alpha_T| \). But, from Fig. 11, we find that the rate of this growth gets smaller arbitrarily large values of \( \theta \). Again, one needs to start from the actual solution for \( \theta \) close to the value for which the structure factor is to be calculated. In the present analysis, we were able to obtain \( \Delta(K) \) for three values of \( \theta; \theta = 0.1, 0.2 \) and 0.3.

In Fig. 8, we first present the renormalized propagator \( \sqrt{x} \sim \alpha_R^{-1} \) as a function of temperature \( \alpha_T \) for three values of \( \theta \). Compared to the pure (\( \theta = 0 \)) case, \( \alpha_T(x) \) as a function of \( x \) shows very little deviations. In general, one finds that the same value of parameter \( x \) represents a slightly higher temperature as \( \theta \) increases.

FIG. 7. The renormalized four-point vertex functions, \( f_{R}^{\prime} \) and \( v_{R}^{\prime} \) for the disordered zero-dimensional model as functions of temperature \( \sigma \). The filled triangles, circles and squares are obtained from the parquet approximation for \( \sigma = 0, 1.0 \) and 5.0, respectively. The solid lines are the corresponding exact solutions.

FIG. 8. The renormalized propagator \( (\mu^2/2\pi\alpha_R \sim \sqrt{x}) \) as a function of temperature \( \alpha_T \) in \( d = 2 \). The filled squares are obtained from the parquet approximation in the pure case. The open triangles, filled triangles and open squares correspond to the parquet approximation for \( \theta = 0.1, 0.2 \) and 0.3, respectively.

V. RESULTS AND ANALYSIS

For the case of the two-dimensional system, one has to solve a set of coupled integral equations for \( f_{R}(k) \) and \( v_{R}(k) \), (4.2) and (4.3), containing two parameters \( x \) and \( \theta \). We consider a rotationally symmetric case where all vertex functions depend only on \( K = |k|/\mu \).

We apply a similar numerical technique to the one used in Ref. 5. In practice, we find it convenient to work with \( h_{R}(K) \equiv f_{R}(K) - \theta v_{R}(K) \) and \( v_{R}(K) \). For given set of irreducible parts, \{I_{i}(K)\} and \{J_{i}(K)\}, eqs. (4.2) and (4.3) are coupled linear integral equations for \( h_{R} \) and \( v_{R} \). We first solve these equations for \( h_{R} \) for fixed \( v_{R} \) by numerically inverting a matrix and then solve for \( v_{R} \) using the solution \( h_{R} \). We then update the irreducible parts using (4.3). The solution to the parquet equations are obtained by iterating this procedure.

A fast convergence can be obtained if we choose the initial functions, \{I_{i}\}, \{J_{i}\}, and \( v_{R} \) close to the actual solutions. At high enough temperatures, it is sufficient to start from \( I_{1} = J_{1} = v_{R} = \exp(-K^2/2) \). As one goes into the low temperature regime, it is necessary to use the solution at a temperature close to the desired temperature as initial functions. We face the same numerical difficulty as in Ref. 5 as the temperature is lowered, namely one has to use a finer mesh in \( K \) space as well as a larger cutoff in order to get a low temperature solution. In this case, we have a coupled set of equations, which requires an additional computing time. The minimum temperature we used was \( \alpha_T \approx -8.5 \) where the cutoff was at \( K = 15 \) and number of mesh points was 600. We also find it difficult to solve the parquet equations directly for arbitrarily large values of \( \theta \). Again, one needs to start from the actual solution for \( \theta \) close to the value for which the structure factor is to be calculated. In the present analysis, we were able to obtain \( \Delta(K) \) for three values of \( \theta; \theta = 0.1, 0.2 \) and 0.3.
as the disorder gets stronger. In the next subsection, we show that one can derive this behavior analytically from the parquet equations.

![Graph](image)

FIG. 9. The structure factor for fixed $\theta = 0.2$ and for various temperatures (dashed line: $\alpha_T = -3.78$, dotted line: $\alpha_T = -5.34$, dot-dashed line: $\alpha_T = -6.51$ and solid line: $\alpha_T = -8.36$). The arrows indicate the positions of RLV of the triangular lattice. Since the second and the third RLV are closely spaced, our solution could not resolve these peaks.

A. Parquet Equations for $\theta \ll 1$

For very weak disorder, $\theta \ll 1$, one can extract from the parquet equations the small-$\theta$ behavior of the structure factor $\Delta(k)$ around the pure ($\theta = 0$) solution. One can in fact derive the temperature dependence of the length scale $R_c$ for small $\theta$ from the parquet equations. In order to do that, we need to find the parquet equation analogous to [4.4], [5.2] and [5.3] for $\Gamma(k) = (1/2)[h_R(k) + \hat{h}_R(k)]$. Out of the functions used in [4.4] and [5.2], we define

$$\Lambda_1(k) = \frac{1}{2} \left[ I_1(k) + \tilde{I}_1(k) - \theta \left( J_1(k) + \tilde{J}_1(k) \right) \right],$$

$$\Lambda_2(k) = \frac{1}{2} \left[ I_2(k) + \tilde{I}_2(k) - \theta \left( J_2(k) + \tilde{J}_2(k) \right) \right],$$

$$\Lambda_3(k) = \Lambda_2(k).$$

and

$$\Pi_1(k) = \frac{1}{2} \left[ \Gamma_1(k) + \tilde{\Gamma}_1(k) - \theta \left( \Xi_1(k) + \tilde{\Xi}_1(k) \right) \right],$$

$$\Pi_2(k) = \frac{1}{2} \left[ \Gamma_2(k) + \tilde{\Gamma}_2(k) - \theta \left( \Xi_2(k) + \tilde{\Xi}_2(k) \right) \right],$$

$$\Pi_3(k) = \Pi_2(k).$$

Using these functions and the original parquet equations, [L2] and [L3], one can find a set of equations for $\Gamma(k)$ as follows:

$$\Gamma(k) = \Gamma_B(k) + \sum_{i=1}^{3} \Pi_i(k), \quad (5.1a)$$

$$\Lambda_i(k) = \Gamma(k) - \Pi_i(k), \quad (5.1b)$$

$$\Pi_1(k) = -x(\Lambda \circ \Gamma)(k), \quad (5.1c)$$

$$\Pi_2(k) = -2\pi \left[ \Lambda_2(k) \Gamma(k) - \frac{\theta^2}{4} J_2(k) v_R(k) \right], \quad (5.1d)$$

$$\Pi_3(k) = \hat{\Pi}_2(k), \quad (5.1e)$$

where

$$\Gamma_B(k) = (1 - \theta) \exp(-k^2/2\mu^2). \quad (5.2)$$

and we have used the above-mentioned identities involving the binary operations, $\circ$ and $\ast$. Note that these equations are almost the same as those given in the pure case, [L6], [L7] and [L8]. One extra term in (5.1d) comes from the subtraction of the diagram containing an explicit factor of $n$. There is of course the second equation for $v_R$, which couples nontrivially to these equations.

![Graph](image)

FIG. 10. The structure factor for fixed $x = 30$, (or slightly varying $\alpha_T; -6.67 < \alpha_T < -6.45$) and for various values of $\theta$: $\theta = 0$ (thick line), 0.1, 0.2 and 0.3. The peaks are getting smaller as the strength of disorder ($\theta$) increases.

Note that the structure factor $\Delta(k)$ is just given by $\Delta = 1 - 2\pi \Gamma$, and we can simplify the above equations as done in the pure case to get one equation for the structure factor. From (5.1d) and $J_2$ equation in [L3b], we have

$$J_2(k) = v_R(k) \frac{1 + 2\pi \Lambda_2(k)}{\Delta(k) - 2\theta xv_R(k)}.$$
Inserting this into (5.14), we obtain
\[
\Lambda_2(k) = \frac{1}{2x} \left[ -1 + \frac{\Delta(k) - 2\theta xv_R(k)}{\Delta(k) - \theta xv_R(k)} \right].
\]

Now using (5.14) and the identities mentioned above, we find
\[
\frac{\Delta(k) - 2\theta xv_R(k)}{\Delta(k) - \theta xv_R(k)} = 1 + x\Gamma_B(k) + x(\Gamma_B \circ \Delta)(k)
- \left( \left[ \Delta - 2 + \frac{\Delta - 2\theta xv_R}{\Delta - \theta xv_R} \right] \circ \Delta \right)(k).
\]

When \( \theta = 0 \), this reduces to (3.11). So far we have not made any assumptions on the magnitude of \( \theta \). For small \( \theta \), one may consider the lowest order perturbation in the structure factor as compared to the pure solution. Note that, up to \( O(\theta) \), the equation takes the same form as in the \( \theta = 0 \) case, except that \( \Gamma_B(k) \) in (5.2) contains an additional factor of \( (1 - \theta) \). This means that for \( \theta < 1 \), the solution for given \( \theta \) and \( x \) is the same as the pure solution, denoted by \( \Delta_0 \), at \( x(1 - \theta) \):
\[
\Delta(k; x, \theta) \simeq \Delta_0(k; x(1 - \theta)), \quad \theta \ll 1.
\]

![Diagram](image.png)

FIG. 11. The height \( \Delta_1 \) of the first peak as a function of temperature. The filled circles, triangles, filled squares and open squares correspond to the solutions at \( \theta = 0, 0.1, 0.2 \) and 0.3, respectively. The solid lines are linear fits.

For given \( x \) and \( \theta \), \( \alpha_T \) is determined via (2.13). If we denote by \( \alpha^0_T(x) \) the relation between \( \alpha_T \) and \( x \) for \( \theta = 0 \), then from (2.13) and (5.4) we deduce that to \( O(\theta) \)
\[
\alpha_T(x) \simeq \alpha^0_T(x(1 - \theta)) - \frac{\theta}{2} \frac{1}{(2\sqrt{x} + \alpha^0_T(x))}.
\]

We note that, for \( \theta = 0 \), \( \alpha^0_T(x) = (1 - x\beta_A(x))/\sqrt{x} \sim -\sqrt{x}\beta_A \) as \( \alpha_T \rightarrow -\infty \), where the generalized Abrikosov ratio changes from 2 at high temperatures to some low temperature limit, \( \beta_A \). In the pure case, we found that the length scale grows as \( \sim |\alpha^0_T(x)| \sim \sqrt{\beta_A(x)} \). Using (5.4) and (5.3), one can derive that
\[
R_\circ \sim \beta_A \sqrt{(1 - \theta) x} \sim (1 - \theta \left( \frac{1}{2} + \frac{1}{\beta_A} \right)) |\alpha_T| \quad (5.6)
\]
as \( \alpha_T \rightarrow -\infty \). As mentioned before, this behavior is already captured in our numerical solution (see Fig. 11).

VI. GENERALIZATION TO THREE DIMENSIONS

In this section, we apply the parquet resummation method to a three-dimensional vortex liquid system. We shall demonstrate that, while the generalization of the parquet equations to three dimensions is straightforward, it is virtually impossible to solve the equations using the same numerical technique as in the 2D case. We shall then discuss a possible analytic approach to the problem.

We consider only the pure case for simplicity. The order parameter \( \Psi(x, y, r_\perp) \) in 3D depends on the coordinate \( r_\perp \) perpendicular to the \((x, y)\) plane. In the lowest Landau level approximation, the order parameter takes the form
\[
\Psi(x, y, r_\perp) = \phi(z, r_\perp) \exp(-\frac{\mu^2}{4}|z|^2).
\]
The GL free energy analogous to (2.3) is given by
\[
F[\phi] = \int dz_\perp d\phi^* dz \left( (\partial_\perp \phi^2 + \alpha_H |\phi|^2) e^{-\mu^2|z|^2/2} + \frac{\beta}{2} \exp(-\mu^2|z|^2) |\phi(z, r_\perp)|^4 \right),
\]
where \( \partial_\perp = \partial/\partial r_\perp \). The renormalized propagator can be written as
\[
G(\zeta^*, z; q) = \frac{\mu^2}{2\pi} G(q) \exp(\frac{\mu^2}{2} \zeta^*), \quad (6.1)
\]
where \( q \) is the Fourier momentum corresponding to \( r_\perp \). For the bare propagator, \( G_0(q) = (q^2 + \alpha_H)^{-1} \). But, the renormalization drives \( G(q) \) into a general function. This is in contrast to the two-dimensional case where the propagator is completely described by one parameter \( \alpha_H \).

Similarly, the renormalized four-point vertex carries four momenta, \( q_1, q_2, q_3, \) and \( q_4 \), in addition to \( k \) describing the correlation in \((x, y)\) plane. Among the four momenta only three will be independent due to the momentum conservation \((q_1 + q_2 = q_3 + q_4) \). Using, for example, new variables, \( s \equiv q_1 + q_2, t \equiv q_1 - q_3 \) and \( u \equiv q_1 - q_4 \), one can describe the renormalized four-point vertex function in terms of \( \Gamma(s, t; u; k) \). The parquet equations for \( \Gamma \) follow from Fig. 5. By evaluating the diagrams in the Fig. 5 using the momentum conservation on each vertex, one finds that
the above parquet equations as in the 2D case is the fact that the propagator is given as an unknown function \( G(q) \) in this case. In the 2D problem, we were able to determine the unknown constant \( \alpha_R \) (or \( \alpha \)) self-consistently using Dyson equation. In this case, we have a functional self-consistent equation for the self-energy \( \Sigma(q) = G^{-1}(q) - G_0^{-1}(q) \) as follows:

\[
\Sigma(q) = 2\eta \int \frac{dq'}{2\pi} \frac{G(q')}{G(q - q')} - 2\eta^2 \frac{2\pi}{\mu^2} \int \frac{d^2k}{(2\pi)^2} \frac{dq' dq''}{(2\pi)^2} e^{-k^2/2\mu^2} G(q + q'-q'') \Gamma(q + q', q - q'', q'' - q; k)
\]

Even if one uses this self-consistent relation, the parquet equations are coupled integral equations for the function \( \Gamma \) of four independent variables \((s, t, u \text{ and } k)\). Storing all the data for \( \Gamma \) and for all the irreducible parts, \( \Lambda_i \) in the numerical calculation will be a formidable task. Furthermore, the integrations in the equations become multiple sums and therefore the previous numerical method of inverting a matrix become more cumbersome. One may have to resort to a direct iteration, which converges slower than the numerical inversion of matrices.

These numerical difficulties, however, should not discourage one from attempting to get some nonperturbative information from the parquet equations. In 2D, the parquet equations seem to be a minimal set of equations that predicts the growing translational order in \((x, y)\) plane, whose length scale is characterized by \( R_c \). Thus, we expect that the above 3D parquet equations contain among other things nonperturbative information on various length scales characterizing the low temperature regime of the system. In addition to \( R_c \), the 3D system may have growing length scales for the correlation in the \( r_\perp \) direction. We denote by \( \xi_{L} \) the length scale arising from the propagator \( G \) and by \( L_c \) the one associated with the 3D structure factor \( S(k, q) \equiv (2\pi/\mu^2) \exp(k^2/2\mu^2) \tilde{\chi}(k, q) \) (see \([2, 4]\)), where

\[
S(k, q) = \int \frac{dq'}{2\pi} G(q') G(q + q') - 2\eta \int \frac{dq' dq''}{(2\pi)^2} G(q') G(q'') \times G(q + q') G(q + q'') \Gamma(q + q', q - q'', q'' - q; k)
\]

If one uses in the parquet equations some kind of low-temperature asymptotic forms for \( G(q) \) and \( S(k, q) \) which are characterized by the length scales, \( \xi_{L} \), \( \xi_{R} \) and \( R_c \), one might be able to extract nonperturbative relations for these length scales and the temperature. We have tried a simple ansatz where \( G \) and \( S \) are represented as delta-function-like sharp peaks around \( q = 0 \) and \( k = G \) (RLV). The length scales are set to be equal to the inverse width of the corresponding peaks. But, we found that these forms are too simplified to produce sufficient amount of information on the temperature dependences of each length scale. We believe that more sophisticated asymptotic forms are needed for the analysis of the 3D parquet equations. This point will be discussed again in the next section.

VII. DISCUSSION AND SUMMARY

In this paper, we applied the parquet resummation method to the 2D vortex liquid with and without quenched impurities and to the 3D vortex liquid in the absence of disorder. In the 2D system, we were able to solve the parquet equations numerically and find the length scale \( R_c \). The temperature dependence of \( R_c \) was also obtained. In the pure 2D vortex liquid, we found in the previous paper, the asymptotic forms, \( \Delta(k) \sim 2\pi\mu^2 \sum_{q \neq 0} \delta^{(2)}(k - G) \), or equivalently \( f_R(k) \sim -1/(1 - \pi\mu^2 \sum_{q \neq 0} \delta^{(2)}(k - G)) \) solves the parquet equations if the inverse width of the delta-function peaks, or the length scale \( R_c \) behaves like \( \sqrt{\tau} \sim |\alpha_T| \) as \( \alpha_T \to -\infty \). In the disordered case, one can obtain similar low-temperature asymptotic forms for \( f_R(k) \) and \( v_R(k) \) in terms of delta functions around \( k \sim G \). One can then obtain expressions for \( R_c \) similar to \([5,6]\). These simple asymptotic forms, however, have some limitations. First of all, the amplitude of each delta-function peak cannot be determined from the parquet equations. It is also not clear whether one can use these representations in the
strong disorder regime. In the previous section, we argued that an ansatz like $S(k, q) \sim \delta(q) \sum_{G \neq 0} \delta^{(2)}(k - G)$ was not enough to produce any useful nonperturbative information for the three dimensional solution. We believe that finding appropriate low temperature asymptotic forms for the vertex functions is a first step towards a more fruitful use of the parquet equations. To this end, a simple equation like (3.11) might be useful.

The effect of random impurities on the mixed state of a type-II superconductor is usually described by the so-called Larkin-Ovchinnikov argument, which states that for a dimension $d < 4$ the long-range crystalline order of the mixed state is destroyed by a weak disorder. For thin films ($d = 2$), one can estimate the Larkin length scale $R_c$ over which a short-range order persists as $\lambda R_c \approx |\alpha_T|/\lambda$. Although the $|\alpha_T|$-dependence of $R_c$ is the same as the present result, we note that there is a basic difference between our result and the LO type argument. Since the LO argument starts from a perfect crystalline state where $\lambda = 0$, $R_c = \infty$, a very small amount of disorder makes an abrupt change as can be seen from $1/\sqrt{\lambda}$ dependence of $R_c$. Within the parquet approximation, there is no 2D crystalline phase and the pure system has a finite $R_c$ at any finite temperature. As we have seen in the previous sections, the effect of small disorder is represented as a smooth decrease of this length scale.

In this paper, we have concentrated on a rotationally symmetric liquid state where the vertex functions depend only on the magnitude of $k$. But, for example, in the presence of magnetoelastic interactions between vortices, one must consider general $k$-dependent vertex functions in the parquet equations. We recall that the input parameters of the parquet equations are the temperature and the bare vertex functions, $f_B$ and $\nu_B$. It would be interesting to study a situation where the bare vertex functions possess a nontrivial $k$-dependence.

One could also generalize the present parquet resummation technique to study the dynamics of type-II superconductors. We expect that the parquet equations for the time-dependent Ginzburg Landau theory can be obtained without much difficulty. But, the vertex functions will depend on the frequency as well as the usual momenta. Thus, one faces similar numerical difficulty to the 3D case. In addition, one has to be careful in taking the LLL limit.\[\]

**ACKNOWLEDGMENTS**

We would like to thank T. Blum, A. J. Bray, M. J. W. Dodgson and T. J. Newman for useful discussions.

**APPENDIX: EVALUATION OF DIAGRAMS**

Here we give some examples of evaluating the Feynman diagrams involved in the present work. Similar analysis can be found in the Appendix of Ref. [2]. We consider only the pure case ($\lambda = 0$) for simplicity. The results can easily be generalized to the disordered case carrying the replica indices. Feynman diagrams are constructed by connecting the propagator lines represented by (2.6) with the quartic vertices, which can be obtained from (2.7) as (see Fig. 12)

$$-\frac{\beta}{2} \int \frac{dk}{(2\pi)^2} \Gamma(k)e^{[k^2 + \mu^2]/2} \exp \left[ -\frac{i}{2} (k^2 (\zeta - \zeta') + (\zeta^* - \zeta'^*) k) \right].$$

By joining four propagator lines (starting from $z_1$ and $z_2$ and ending at $z_3$ and $z_4$) with this vertex, we have

$$-4(\beta^2/2\pi)^4 \int \frac{dk}{(2\pi)^2} \Gamma(k)e^{[k^2 + \mu^2]/2} \int \prod_{i=1}^2 d\zeta_i d\zeta_i^* e^{-\mu^2([\zeta_i]^2 + [\zeta_i^*]^2)/2} \exp \left[ \frac{\mu^2}{2} (z_1\zeta_1 + z_2\zeta_2 + \zeta_1^* z_3 + \zeta_2^* z_4) - \frac{i}{2} (k\zeta_1 - \zeta_2 + (\zeta_1^* - \zeta_2^*) k) \right].$$

After performing the Gaussian integrals using

$$\int \prod_{i=1}^m d\zeta_i d\zeta_i^* \exp[-\mu^2(\zeta_i M_{ij} \zeta_j - a_i^* \zeta_i^* - \zeta_i \zeta_j^*)] = (\frac{\pi}{\mu^2})^m (\det M)^{-1} \exp[\mu^2(a_i^* M_{ij}^{-1} b_j)],$$

one obtains (2.9).

Let us evaluate the one-loop diagrams and derive (3.6). We consider only the first diagram of Fig. 3 and thus derive (3.6a). The remaining two diagrams can be evaluated in the same manner. The first one-loop diagram gives

$$8(\beta^2/2\pi)^4 \int \frac{dk_1}{(2\pi)^2} \frac{d\zeta_j d\zeta_j^*}{(2\pi)^2} \Lambda_i(k_1) \Gamma(k_2) e^{(k_1^2 + k_2^2)/2} \int \prod_{j=1}^4 d\zeta_j d\zeta_j^* \exp \left( -\frac{\mu^4}{2} \sum_{j=1}^4 |\zeta_j|^2 \right)$$
\[\times \exp \left[ \frac{\mu^2}{2} \left( z_1^* \zeta_1 + z_2^* \zeta_2 + \zeta_3^* z_3 + \zeta_4^* z_4 + \zeta_1^* \zeta_3 + \zeta_2^* \zeta_4 \right) \right] \]

\[\times \exp \left[ -i \left( k_1^* (\zeta_1 - \zeta_2) + (\zeta_1^* - \zeta_2^*) k_1 + k_2^* (\zeta_3 - \zeta_4) + (\zeta_3^* - \zeta_4^*) k_2 \right) \right].\]

Integrating over \(\zeta\) and \(\zeta^*\) using (A1) and changing the variables \(k_1 + k_2 \to k\), and \(k_1 \to p\), we obtain

\[8 \left( \frac{\beta \mu^2}{2 \pi} \right)^2 \frac{1}{\alpha_R^2} e^{\mu^2 (z_1^* z_3 + z_2^* z_4)/2} \int \frac{dkdk^* dpdp^*}{(2\pi)^4} \Lambda_1(p) \Gamma(k-p) e^{-|k|^2/2\mu^2} \]

\[\exp \left[ -i \left( k^* (z_3 - z_4) + (z_1^* - z_2^*) k \right) + \frac{1}{2\mu^2} (k^* p - p^* k) \right].\]

Comparing this with the general expression (2.9), we find that the one-loop contribution from this diagram to the vertex function is

\[-\frac{\beta \mu^2}{2\pi \alpha_R^2} \int dp dp^* \Lambda_1(p) \Gamma(k-p) \exp \left( \frac{1}{2\mu^2} (k^* p - p^* k) \right),\]

which is equal to the right hand side of (3.6a).

---

FIG. 12. Diagrammatic representation of the propagator and the four-point vertex.
In this paper, we assume that all the vertex functions have a reflection symmetry; e.g. $f_R(-k) = f_R(k)$ so that the exponential in the definition of $\hat{f}_R$ can be replaced by a cosine. This is certainly true for the liquid state where the system has a rotational symmetry and also true for a possible crystalline state having, for example, a triangular lattice structure. In this work, we consider only the liquid state, but the equations presented here are valid for a general case with a reflection symmetry.

The value of the parameter $x$ is twice the one used in Ref. 16 and Ref. 5.

The numerical difficulty is mainly due to the presence of $\Delta$ in the denominators of the equation. If one tries to solve the equation iteratively, the procedure becomes too sensitive to the region where $\Delta \approx 0$.

A. J. Bray, T. McCarthy, M. A. Moore, J. D. Reger and A. P. Young, Phys. Rev. B 36, 2212 (1987).

W. H. Press et al., *Numerical Recipes*, p.782, Cambridge University Press, Cambridge (1992).

T. J. Newman and M. A. Moore, preprint.