Extension and trace results for doubling metric measure spaces and their hyperbolic fillings

Anders Björn
Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden
anders.bjorn@liu.se, ORCID: 0000-0002-9677-8321

Jana Björn
Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden
jana.bjorn@liu.se, ORCID: 0000-0002-1238-6751

Nageswari Shanmugalingam
Department of Mathematical Sciences, University of Cincinnati, P.O. Box 210025, Cincinnati, OH 45221-0025, U.S.A.; shanmun@uc.edu, ORCID: 0000-0002-2891-5064

Abstract In this paper we study connections between Besov spaces of functions on a compact metric space $Z$, equipped with a doubling measure, and the Newton–Sobolev space of functions on a uniform domain $X_{\varepsilon}$. This uniform domain is obtained as a uniformization of a (Gromov) hyperbolic filling of $Z$. To do so, we construct a family of hyperbolic fillings in the style of Bonk–Kleiner [9] and Bourdon–Pajot [13]. Then for each parameter $\beta > 0$ we construct a lift $\mu_{\beta}$ of the doubling measure $\nu$ on $Z$ to $X_{\varepsilon}$, and show that $\mu_{\beta}$ is doubling and supports a 1-Poincaré inequality. We then show that for each $\theta$ with $0 < \theta < 1$ and $p \geq 1$ there is a choice of $\beta = p(1-\theta)\log \alpha$ such that the Besov space $B_{\theta}^{p,p}(Z)$ is the trace space of the Newton–Sobolev space $N_{1,p}(X_{\varepsilon},\mu_{\beta})$ when $\varepsilon = \log \alpha$. Finally, we exploit the tools of potential theory on $X_{\varepsilon}$ to obtain fine properties of functions in $B_{\theta}^{p,p}(Z)$, such as their quasicontinuity and quasieverywhere existence of $L^{q}$-Lebesgue points with $q = s_{\nu}p/(s_{\nu} - p\theta)$, where $s_{\nu}$ is a doubling dimension associated with the measure $\nu$ on $Z$. Applying this to compact subsets of Euclidean spaces improves upon a result of Netrusov [42] in $\mathbb{R}^n$.

Key words and phrases: Gromov hyperbolic space, hyperbolic filling, Poincaré inequality, doubling measure, uniform space, uniformization, trace and extension, Besov space, Besov capacity, Sobolev capacity, quasicontinuity, Lebesgue point.

Mathematics Subject Classification (2020): Primary: 31E05, Secondary: 30L05, 53C23.

Contents

1 Introduction 2
2 Gromov hyperbolic spaces 6
3 Construction of hyperbolic fillings 9
4 The uniformized boundary $\partial_{\varepsilon}X$ 12
5 Uniformizing a hyperbolic filling with parameter $\varepsilon \leq \log \alpha$ 17
6 Roughly similar equivalence 19
7 Trees 22
8 Geodesics in the hyperbolic filling 24
9 Measures, function spaces and capacities 29
1. Introduction

Much of the current trend in first-order analysis (such as Sobolev type spaces and potential theory) on metric measure spaces assumes that the underlying space is at least locally compact, doubling and supports a Poincaré type inequality, see for example [2] and [27]. On doubling spaces that do not support any Poincaré inequality, there are other possible choices of function spaces that are, however, nonlocal. This means that the energy of functions in these spaces depends on their global behavior and can be nonzero on subsets where the functions vanish or are constant. Examples of such spaces include Besov, Triebel–Lizorkin and (fractional) Hajlasz–Sobolev spaces. As locality is a highly useful tool in potential theory and variational problems, it is desirable to seek an alternative approach to studying function spaces on nonsmooth metric spaces without Poincaré inequalities.

Using a construction termed hyperbolic filling, Bonk–Kleiner [9, Theorem 11.1] and Bourdon–Pajot [13] connected compact doubling metric spaces to Gromov hyperbolic spaces. Gromov hyperbolicity is a notion of negative curvature in the nonsmooth metric setting and, unlike Alexandrov curvature which covers all scales, it captures negative curvature at large scales by requiring that every point in a geodesic triangle is within a bounded distance from the other two sides. Gromov hyperbolicity has proven to be a highly useful tool in studying the conformal geometry of hyperbolic groups (Bridson–Haefliger [14] and Gromov [22]) and in understanding uniform domains (Bonk–Heinonen–Koskela [8], Bonk–Schramm [12] and Herron–Shanmugalingam–Xie [28]). We refer the interested reader to [14] and Buyalo–Schroeder [16] for more on synthetic notions of curvature in the metric setting and the hyperbolic filling technique, respectively.

In the current paper we contribute to the study of nonlocal analysis on compact doubling metric measure spaces by introducing measures to the above construction of hyperbolic fillings and by subsequently linking the nonlocal Besov spaces on compact doubling metric measure spaces to the Newtonian (Sobolev) spaces on uniformizations of their hyperbolic fillings, see Theorem 1.1.

One of the main results of the paper is that every Besov space $B_{p,p}^{\theta}(Z)$ with $0 < \theta < 1$ and $p \geq 1$ arises as a trace of a Newtonian space on a uniform domain, equipped with a doubling measure supporting a 1-Poincaré inequality. Specifically, we prove the following theorem.

**Theorem 1.1.** Let $Z$ be a compact metric space equipped with a doubling measure $\nu$, $X$ be a hyperbolic filling of $Z$ with parameters $\alpha, \tau > 1$, and $X_{\varepsilon}$ be the uniformization of $X$ with parameter $\varepsilon = \log \alpha$.

Then for each parameter $\beta > 0$ we can equip $X_{\varepsilon}$ with a measure $\mu_{\beta}$ induced by $\nu$ so that $\mu_{\beta}$ is doubling and supports a 1-Poincaré inequality both on $X_{\varepsilon}$ and its completion $\overline{X_{\varepsilon}}$. Moreover, for each $0 < \theta < 1$ and $1 \leq p < \infty$, the Besov space $B_{p,p}^{\theta}(Z)$ is the trace space of the Newtonian space $N_{1,p}(X_{\varepsilon}, \mu_{\beta})$ with $\beta = \varepsilon p (1 - \theta)$.

More precisely, $\text{Tr} \circ E$ is the identity map on $B_{p,p}^{\theta}(Z)$, where

$$\text{Tr} : N_{1,p}(X_{\varepsilon}, \mu_{\beta}) \to B_{p,p}^{\theta}(Z) \quad \text{and} \quad E : B_{p,p}^{\theta}(Z) \to N_{1,p}(X_{\varepsilon}, \mu_{\beta})$$

are bounded linear trace and extension operators, respectively.
In fact, we construct $\text{Tr}$ for each $\theta \leq 1 - \beta/\varepsilon p$ (see Theorem 11.3), and $E$ for each $\theta \geq 1 - \beta/\varepsilon p$ (see Theorem 12.1). Roughly speaking, $\mu_{\beta}$ is constructed so that its $\beta/\varepsilon$-codimensional Hausdorff measure is equivalent to the measure $\nu$ on $Z$. Thus Theorem 1.1 follows immediately from combining Theorems 11.3 and 12.1.

The smoothness exponent $\theta = 1 - \beta/\varepsilon p > 0$ exactly corresponds to, and generalizes, the case of $d$-sets in unweighted $\mathbb{R}^n$, considered by Jonsson–Wallin [29], [30]. We also show that for $\nu$-a.e. $z \in Z$ the trace $\text{Tr}(z)$ is achieved in three different ways, namely, as averaged pointwise limits (11.4) and (11.5), by Lebesgue point integral averages (11.17) and as a pointwise restriction from the Newtonian space $N^{1,p}(X_z)$.

Our study includes $p = 1$ and shows that the Besov space $B^\theta_{1,1}(Z)$ is the trace space of the Newtonian space $N^{1,1}(X_z, \mu_\beta)$ when $\beta = (1-\theta)e = (1-\theta)\log \alpha$. This is in contrast to the result of Gagliardo [19] that the trace space of the Sobolev space $W^{1,1}(\Omega)$ is $L^1(\partial \Omega)$ when $\Omega$ is a Lipschitz domain in $\mathbb{R}^n$. This trace operator is nonlinear, which is necessary according to a result due to Peetre [44]. See Malý [39, Section 7] and Malý–Shanmugalingam–Snipes [40, Theorem 1.2] for metric space analogs of this. The key difference between the setting of [40] and the current paper is that in [40] the measure on $\partial \Omega$ has codimension 1 relative to the measure on $\Omega$, while we have codimension $\beta/\varepsilon < p$ which precludes us from having $\beta/\varepsilon = 1$ when $p = 1$. See also [39, Section 7] for the importance of this difference.

To prove the above theorem, for each choice of $\alpha, \tau > 1$ and $0 < \varepsilon \leq \log \alpha$, we construct a hyperbolic filling $X$ of the metric space $Z$. Roughly speaking, is an infinite graph whose vertices correspond to maximal $\tau$-separated subsets of $Z$ for each positive integer $n$. The role of $\tau$ is to define nearness between two points in each of these sets as in (3.2). We then equip $X$ with the uniformized metric
\[ d_\varepsilon(x, y) = \inf_\gamma \int_\gamma e^{-\varepsilon d(\cdot, v_0)} \, ds, \]
where $d(\cdot, v_0)$ denotes the graph distance to the root $v_0$ of the hyperbolic filling and the infimum is taken over all curves in $X$ joining $x$ to $y$.

Along the way, we explore how the choice of parameters affects the structure of the hyperbolic fillings $X$ of $Z$ and their uniformizations $X_\varepsilon$:

- Hyperbolic fillings are Gromov hyperbolic for all $\alpha, \tau > 1$ (Theorem 3.4) but not when $\tau = 1$ (Example 8.8).
- The uniformization $X_\varepsilon$ is a uniform space for all $\varepsilon \leq \log \alpha$ (Theorem 5.1).
- The boundary of the uniformization $X_\varepsilon$, with $\varepsilon = \log \alpha$, is biLipschitz equivalent to $Z$ when $Z$ is compact (Proposition 4.4).

Subsequently, for a doubling measure $\nu$ on $Z$, we construct a lift of $\nu$ to a measure $\mu$ on $X$ which is uniformly locally doubling and supports a uniformly local 1-Poincaré inequality. We then show that for every $\beta > 0$, the corresponding weighted measure
\[ d\mu_{\beta}(x) = e^{-\beta d(x, x_0)} \, d\mu(x) \simeq \text{dist}_\varepsilon(x, \partial_\varepsilon X)^{\beta/\varepsilon} \, d\mu(x) \tag{1.1} \]
is globally doubling and supports a global 1-Poincaré inequality on $X_\varepsilon$ and its closure $\overline{X_\varepsilon}$, see Theorem 10.3. This gives us the flexibility to choose $\beta$ for each $0 < \theta < 1$ and $p \geq 1$ so that $\theta = 1 - \beta/p\varepsilon$ and thus see the nonlocal Besov space $B^{p}_{p,\beta}(Z)$ as the trace space of the Newtonian space $N^{1,p}(X_z, \mu_{\beta})$, with the advantage that the Newtonian energy is local, and that the theory for Newtonian spaces is more developed than the theory for Besov spaces on metric spaces.

Invoking the regularity properties of Newtonian spaces, we then easily obtain several regularity results for Besov functions on $Z$:
Corollary 1.2. Let $Z$ be a compact metric space equipped with a doubling measure $\nu$. Then for every $0 < \theta < 1$, Lipschitz functions are dense in $B^\theta_{p,p}(Z)$ and every function in $B^\theta_{p,p}(Z)$ has a representative that is quasicontinuous with respect to the Besov capacity.

If $p > s_\nu/\theta$, where $s_\nu$ is the growth exponent of $\nu$ from (10.13), then functions in $B^\theta_{p,p}(Z)$ have Hölder continuous representatives.

If $\nu$ is in addition reverse-doubling, then functions in $B^\theta_{p,p}(Z)$ belong to $L^q(Z)$ for $q = s_\nu p / (s_\nu - \theta)$ and have $L^q$-Lebesgue points outside a set of zero $B^\theta_{p,p}(Z)$-capacity.

Our results apply also to compact subsets of $\mathbb{R}^n$. On $\mathbb{R}^n$, the corresponding Sobolev-type higher integrability result is due to Peetre [43, Théorème 8.1], while Netrusov [42, Proposition 1.4] obtained the Lebesgue point result for $q < np/(n - \theta)$. Even though $\mathbb{R}^n$ is not compact, the above corollary allows us to improve upon Netrusov’s result in the Euclidean setting by including $q = np/(n - \theta)$, see Proposition 13.6. These results show that our exponent $q$ is optimal.

The above density and quasicontinuity results are known to hold when the Besov space is defined in terms of atomic decompositions or sequences of fractional Hajłasz gradients (see for example Han–Müller–Yang [23, Definition 5.29], Koskela–Yang–Zhou [34, Definitions 1.2 and 4.4] and Heikkinen–Koskela–Tuominen [24, Definition 2.5, Theorems 1.1 and 1.2]). Such spaces coincide with our Definition 9.7 when $Z$ is unbounded and “reverse-doubling”, by [34, Theorem 4.1]. In particular, the definitions are equivalent if $Z$ is uniformly perfect with $\nu$ doubling, for example when $Z = \mathbb{R}^n$. With our assumptions on $Z$, it is not clear whether those definitions agree with the nonlocal integral definition considered here. Note that the integral in Definition 9.7 coincides with the ones defining the classical fractional Sobolev spaces in Euclidean spaces and is naturally related to nonlocal minimization problems for the fractional $p$-Laplacian, as in Caffarelli–Silvestre [17] and Ferrari–Franchi [18].

Higher integrability and Hölder continuity of Besov functions for large $p$ appears also in Maly [39, Corollary 3.18], where it is obtained as a consequence of embeddings into Hajłasz–Sobolev spaces, provided by Lemma 6.1 in Gogatishvili–Koskela–Shanmugalingam [21]. Our approach based on hyperbolic fillings is different. Traces of Newtonian functions on uniform domains in metric spaces are also studied in [39, Theorem 1.1] by means of Lebesgue point averages and Poincaré inequalities. Our proof in the setting of hyperbolic fillings is more direct and rather elementary (albeit a bit lengthy) and is based only on the basic properties of upper gradients. In particular, the Poincaré inequality is not used. Moreover, we show that for any compact doubling space $Z$, the Besov space $B^\theta_{p,p}(Z)$ with $0 < \theta < 1$ is the trace of some Newtonian space, and that the trace can be obtained as a pointwise restriction from the Newtonian space $N^1,p(\overline{Z})$, see Theorem 1.1.

Let us compare our definition of Besov spaces with some other function spaces on boundaries of hyperbolic fillings. Assuming $Z$ to be uniformly perfect and Ahlfors $Q$-regular, the Besov space considered in Bourdon–Pajot [13] corresponds to our $B^\theta_{p,p}(Z)$ with $\theta = Q/p$ and is shown to be isomorphically equivalent to the first cohomology group $\ell_p H^1(X)$. For Ahlfors $Q$-regular spaces $Z$, the papers [10], [11] and [48] by Bonk, Saksman and Soto define certain function spaces on $Z$ by means of Poisson-type extensions as in our Theorem 12.1, and using the counting measure on the collection of all edges in the hyperbolic filling. In [10] they show that if $Z$ supports a $Q$-Poincaré inequality then their space $A^p(Z)$ coincides with the Hajłasz–Sobolev space $M^{1,Q}(Z)$. The function spaces considered in [11] are of Triebel–Lizorkin type, while the ones in [48] are identified with the Hajłasz–Besov spaces $N^s_{p,p}(Z)$, defined by atomic decompositions in Koskela–Yang–Zhou [34, Definition 1.2].

While these results are interesting, from our point of view it is somewhat unsat-
isfactory that the energy of functions considered in [10], [11] and [48] does not take into full account the measure \( \nu \) on \( Z \) and that \( \nu \) is not related to a measure on the hyperbolic filling.

Unlike in [48], our definition of Besov spaces is based on integrals directly on the metric space \( Z \), rather than on sequence spaces, see Definition 9.7. Moreover, the smoothness of the corresponding Poisson extension on the hyperbolic filling \( X \), is controlled by a measure on \( X \) that is compatible with the measure \( \nu \) on \( Z \).

The structure of this paper is as follows. The necessary background related to metric notions and Gromov hyperbolic spaces is given in Section 2, while notions related to Newtonian and Besov spaces are given in Section 9.

In Section 3 we describe the construction of the hyperbolic filling \( X \) of a general bounded metric space \( Z \), associated with the parameters \( \alpha, \tau > 1 \), and show that it indeed forms a Gromov hyperbolic space. Subsequently, in Section 4 we describe the uniformization \( X_\varepsilon \) of \( X \), with parameter \( \varepsilon > 0 \), in the style of Bonk–Heinonen–Koskela [8], adapted to the setting of hyperbolic fillings. In this section we also explore links between the boundary \( \partial_\varepsilon X \) of the uniformized space and the original metric space \( Z \). In particular, the results in Section 4 show why the bound \( \varepsilon \leq \log \alpha \) is natural.

The primary goal of Section 5 is to prove that the uniformization \( X_\varepsilon \) of \( X \) yields a uniform space when \( \varepsilon \leq \log \alpha \). The general results of [8] imply that \( X_\varepsilon \) is a uniform space for sufficiently small \( \varepsilon \), but our direct proof for hyperbolic fillings covers all \( \varepsilon \leq \log \alpha \), which is vital for our further results. Observe that general Gromov hyperbolic spaces do not always yield a uniform space when uniformized, see for example Lindquist–Shanmugalingam [35, Section 4].

Given that all \( \alpha, \tau > 1 \) generate a hyperbolic filling of \( Z \), it is worth exploring how the choice of these parameters affects the structure of the hyperbolic filling. The rough similarity between an arbitrary locally compact roughly starlike Gromov hyperbolic space \( X \) and the hyperbolic filling \( \hat{X} \) of its uniformized boundary \( \partial_\varepsilon X \) is for \( \varepsilon \) proved in Section 6, without any limitations on \( \alpha \) in terms of \( \varepsilon \). Trees and hyperbolic fillings of their uniformized boundaries, as well as some counterexamples, are considered in Section 7. In Section 8 we show that if \( \tau \geq (\alpha + 1)/(\alpha - 1) \), then we have good control over geodesics in the hyperbolic filling. As the discussion in these three sections is provided to explore the hyperbolic fillings and uniformization further, those who are interested only in the theory of Besov spaces may skip these three sections without confusion.

In the rest of the paper we consider a compact metric space \( Z \) equipped with a doubling measure \( \nu \), and its hyperbolic filling \( X \) as well as the uniformization \( X_\varepsilon \) for \( 0 < \varepsilon \leq \log \alpha \). Following the description of notions related to Newtonian and Besov spaces given in Section 9, we describe in Section 10 our method of lifting up the measure \( \nu \) on \( Z \) to a measure \( \mu \) on \( X \). In that section we also show that the uniformization \( \mu_\beta \) of the measure \( \mu \), given for \( \beta > 0 \) by (1.1), is globally doubling and globally supports a 1-Poincaré inequality on the uniformized space \( X_\varepsilon \) and its completion \( \overline{X}_\varepsilon \).

The trace and extension results from Theorem 1.1 are proved in their specific forms as Theorems 11.1, 11.3 and 12.1, respectively. Finally, the results stated in Corollary 1.2 are obtained in Section 13 by exploiting the perspective of the Besov spaces as traces of Newtonian spaces.

The third author communicated the results of this paper with Butler, who made use of some of the techniques of this paper together with the tools of the Buseman function to independently derive some of the results we obtain in Sections 3–5, with a focus on unbounded doubling metric spaces, see [15]. We do not address the issue of unbounded doubling metric spaces here, but the interested readers may consult [15]. However, his construction of the hyperbolic filling differs slightly from
ours in that he requires (3.4) instead of (3.3). The results in [15] also require that
\[ \tau \geq \min \left\{ 3, \frac{1}{1 - 1/\alpha} \right\} \]
(the parameter \( a \) in [15] corresponds to our \( 1/\alpha \)), but we do not require any such constraint except in Section 8.

Acknowledgement. Parts of this research project were conducted during 2017 and 2018 when N. S. was a guest professor at Linköping University, partially funded by the Knut and Alice Wallenberg Foundation, and during the parts of 2019 when A. B. and J. B. were Taft Scholars at the University of Cincinnati. The authors would like to thank these institutions for their kind support and hospitality. A. B. and J. B. were partially supported by the Swedish Research Council grants 2016-03424 resp. 621-2014-3974 and 2018-04106. N. S. was partially supported by the National Science Foundation (U.S.A.) grants DMS-1500440 and DMS-1800161.

2. Gromov hyperbolic spaces

In this section we will introduce Gromov hyperbolic spaces and uniform spaces and discuss relevant background results. In the later part of the paper we will need background results on upper gradients, Newtonian (Sobolev) spaces, Besov spaces, Poincaré inequalities etc. This background discussion will be provided in Section 9.

A curve is a continuous mapping from an interval. Unless stated otherwise, we will only consider curves which are defined on compact intervals. We denote the length of a curve \( \gamma \) by \( \ell(\gamma) \), and a curve is rectifiable if it has finite length. Rectifiable curves can be parametrized by arc length \( ds \).

A metric space \( X = (X, d) \) is geodesic if for each \( x, y \in X \) there is a curve \( \gamma \) with end points \( x \) and \( y \) and length \( \ell(\gamma) = d(x, y) \). \( X \) is a length space if
\[ d(x, y) = \inf_{\gamma} \ell(\gamma) \quad \text{for all} \ x, y \in X, \]
where the infimum is taken over all curves \( \gamma \) from \( x \) to \( y \).

A metric space is proper if all closed bounded sets are compact. We denote balls in \( X \) by \( B(x, r) = \{ y \in X : d(y, x) < r \} \) and the scaled concentric ball by \( \lambda B(x, r) = B(x, \lambda r) \). In metric spaces it can happen that balls with different centers and/or radii denote the same set. We will however adopt the convention that a ball comes with a predetermined center and radius. Sometimes (especially when dealing with several different spaces simultaneously) we will write \( B_X \) and \( d_X \) to indicate that these notions are taken with respect to the metric space \( X \). When we say that \( x \in \gamma \) we mean that \( x = \gamma(t) \) for some \( t \). If \( \gamma \) is noninjective, then this \( t \) may not be unique, but we are always implicitly referring to a specific such \( t \). If \( x_1, x_2 \in \gamma \), then \( \gamma_{x_1, x_2} \) denotes the subcurve between \( x_1 \) and \( x_2 \).

**Definition 2.1.** A complete unbounded geodesic metric space \( X \) is Gromov hyperbolic if there is a hyperbolicity constant \( \delta \geq 0 \) such that whenever \( [x, y], [y, z] \) and \( [z, x] \) are geodesics in \( X \), every point \( w \in [x, y] \) lies within a distance \( \delta \) of \( [y, z] \cup [z, x] \).

The ideal Gromov hyperbolic space is a metric tree, which is Gromov hyperbolic with \( \delta = 0 \). A metric tree is a tree where each edge is considered to be a geodesic of unit length.

**Definition 2.2.** An unbounded metric space \( X \) is roughly starlike if there are some \( x_0 \in X \) and \( M > 0 \) such that whenever \( x \in X \) there is a geodesic ray \( \gamma \) in \( X \), starting from \( x_0 \), such that \( \text{dist}(x, \gamma) \leq M \). A geodesic ray is a curve \( \gamma : [0, \infty) \to X \) with infinite length such that \( \gamma|_{[0,t]} \) is a geodesic for each \( t > 0 \).
If $X$ is a roughly starlike Gromov hyperbolic space, then the roughly starlike condition holds for every choice of $x_0$, although $M$ may change.

**Definition 2.3.** A nonempty open set $\Omega \subset X$ in a metric space $X$ is an $A$-uniform domain, with $A \geq 1$, if for every pair $x, y \in \Omega$ there is a rectifiable arc length parametrized curve $\gamma : [0, \ell(\gamma)] \to \Omega$ with $\gamma(0) = x$ and $\gamma(\ell(\gamma)) = y$ such that $\ell(\gamma) \leq \text{Ad}(x, y)$ and

$$d_\Omega(\gamma(t)) \geq \frac{1}{A} \min\{t, \ell(\gamma) - t\} \quad \text{for} \quad 0 \leq t \leq \ell(\gamma),$$

where

$$d_\Omega(z) = \text{dist}(z, X \setminus \Omega), \quad z \in \Omega.$$

The curve $\gamma$ is said to be an $A$-uniform curve. A noncomplete metric space $(\Omega, d)$ is $A$-uniform if it is an $A$-uniform domain in its completion.

The completion of a locally compact uniform space is always proper, by Proposition 2.20 in Bonk–Heinonen–Koskela [8]. Unlike the definition used in [8], we do not require uniform spaces to be locally compact.

**We assume for the rest of this section that $X$ is a roughly starlike Gromov $\delta$-hyperbolic space. We also fix a point $x_0 \in X$ and let $M$ be the constant in the roughly starlike condition with respect to $x_0$.**

The point $x_0$ will serve as a center for the uniformization $X_\varepsilon$ of $X$. Following Bonk–Heinonen–Koskela [8], we define, for a fixed $\varepsilon > 0$, the uniformized metric $d_\varepsilon$ on $X$ as

$$d_\varepsilon(x, y) = \inf_\gamma \int_0^{\ell(\gamma)} \rho_\varepsilon(s) \, ds,$$

where $\rho_\varepsilon(x) = e^{-\varepsilon d(x, x_0)}$ and the infimum is taken over all rectifiable curves $\gamma$ in $X$ joining $x$ to $y$. Note that if $\gamma$ is a compact curve in $X$, then $\rho_\varepsilon$ is bounded from above and away from 0 on $\gamma$, and in particular $\gamma$ is rectifiable with respect to $d_\varepsilon$ if and only if it is rectifiable with respect to $d$.

**Definition 2.4.** The set $X$, equipped with the metric $d_\varepsilon$, is denoted by $X_\varepsilon$ and called the uniformization of $X$, even when we do not know whether it is a uniform space. We let $\overline{X}_\varepsilon$ be the completion of $X_\varepsilon$, and let $\partial_\varepsilon X = \overline{X}_\varepsilon \setminus X_\varepsilon$ be the uniformized boundary of $X_\varepsilon$ (or $X$).

The uniformization $X_\varepsilon$ need not be a uniform space, as shown in Lindquist–Shanmugalingam [35, Section 4]. If $X$ is locally compact and $\varepsilon$ is sufficiently small, then $\partial_\varepsilon X$ as a set is independent of $\varepsilon$ and depends only on the Gromov hyperbolic structure of $X$, see e.g. [8, Section 3]. The notation adopted in [8] is $\partial G$. On the other hand, if $\varepsilon$ is large, then it is possible for $\partial_\varepsilon X$ to change, see for example Proposition 4.1 below.

When writing e.g. $B_\varepsilon$, $\text{diam}_\varepsilon$ and $\text{dist}_\varepsilon$, the subscript $\varepsilon$ indicates that these notions are taken with respect to $(\overline{X}_\varepsilon, d_\varepsilon)$. We also define

$$d_\varepsilon(x) = \text{dist}_\varepsilon(x, \partial_\varepsilon X).$$

The length of the curve $\gamma$ with respect to $d_\varepsilon$ is denoted by $\ell_\varepsilon(\gamma)$. The arc length $d\ell_\varepsilon$ with respect to $d_\varepsilon$ satisfies

$$d\ell_\varepsilon = \rho_\varepsilon \, ds.$$

It follows that $X_\varepsilon$ is a length space, and thus also $\overline{X}_\varepsilon$ is a length space. By a direct calculation (or [8, (4.3)]), $\text{diam}_\varepsilon \overline{X}_\varepsilon = \text{diam}_\varepsilon X_\varepsilon \leq 2/\varepsilon$.

The following important theorem is due to Bonk–Heinonen–Koskela [8]; see [5, Theorem 2.6] for this version. By the Hopf–Rinow theorem (see Gromov [22, p. 9] for a suitable version), $X$ is proper if and only if $X$ is locally compact.
Theorem 2.5. Assume that $X$ is locally compact. There is a constant $\varepsilon_0(\delta) > 0$, determined by $\delta$ alone, such that if $0 < \varepsilon \leq \varepsilon_0(\delta)$, then $X_\varepsilon$ is an $A$-uniform space for some $A$ depending only on $\delta$. Moreover, $X_\varepsilon$ is a compact geodesic space.

If $\delta = 0$, then $\varepsilon_0(0)$ can be chosen arbitrarily large.

There is also a converse, again due to Bonk–Heinonen–Koskela [8]. Namely, if $(Y, d)$ is a locally compact uniform space, then equipping $Y$ with the quasi-hyperbolic metric $k_Y$ gives a Gromov hyperbolic space, where

$$k_Y(x, y) := \inf_{\gamma} \int_0^1 \frac{ds(t)}{\text{dist}_Y(\gamma(t), \partial Y)}$$

for $x, y \in Y$, with the infimum taken over all rectifiable curves $\gamma$ in $Y$ with end points $x$ and $y$.

We recall, for further reference, the following key estimates from [8].

Lemma 2.6. ([8, Lemma 4.16]) Assume that $X$ is locally compact. Let $\varepsilon > 0$. If $x \in X$, then

$$e^{-\varepsilon d(x,x_0)} e^{-\varepsilon} \leq \text{dist}_\varepsilon(x, \partial \varepsilon X) =: d_\varepsilon(x) \leq C_0 e^{\varepsilon d(x,x_0)}$$

where $C_0 = 2e^{\varepsilon M} - 1$. In particular, $\varepsilon d_\varepsilon(x) \simeq \rho_\varepsilon(x)$, and $x \rightarrow \partial \varepsilon X$ with respect to $d_\varepsilon$ if and only if $d(x, x_0) \rightarrow \infty$.

Corollary 2.7. ([5, Corollary 2.9 and its proof]) Assume that $X$ is locally compact and that $0 < \varepsilon \leq \varepsilon_0(\delta)$, where $\varepsilon_0(\delta)$ is given by Theorem 2.5. Let $x, y \in X$. Then

$$\frac{d_\varepsilon(x,y)^2}{d_\varepsilon(x)d_\varepsilon(y)} \leq \exp(\varepsilon d(x, y)).$$

If $\varepsilon d(x, y) \geq 1$ then

$$\exp(\varepsilon d(x, y)) \simeq \frac{d_\varepsilon(x,y)^2}{d_\varepsilon(x)d_\varepsilon(y)},$$

where the comparison constants depend only on $\delta$, $M$ and $\varepsilon_0(\delta)$.

Here and later, we write $a \lesssim b$ if there is an implicit constant $C > 0$ such that $a \leq Cb$, and analogously $a \gtrsim b$ if $b \lesssim a$. We also use the notation $a \asymp b$ to mean $a \lesssim b \lesssim a$.

In the later part of the paper we will equip uniformizations of Gromov hyperbolic spaces and their boundaries with doubling measures. In the first part of the paper, the following metric doubling condition will instead play a role in a few places, but for most results no doubling assumption is needed.

A metric space $(Y, d)$ is doubling (or metrically doubling) if there is a constant $N_d \geq 1$ such that whenever $z \in Y$ and $r > 0$, the ball $B(z, r)$ can be covered by at most $N_d$ number of balls with radius $\frac{r}{2}$. Doubling is a uniform version of total boundedness. In particular, if $Y$ is complete and doubling, then $Y$ is also proper.

A Borel regular measure $\mu$ on $Y$ is doubling if there is a constant $C > 0$ such that

$$0 < \mu(2B) \leq C\mu(B) < \infty$$

for all balls $B \subset Y$.

If $Y$ carries a doubling measure, then $Y$ is necessarily doubling. The converse is not true in general. However, if $Y$ is a complete doubling measure space, then Luukkainen–Saksman [37] has shown that $Y$ carries a doubling measure. For more on doubling spaces and doubling measures, see Heinonen [25, Section 10.13].
3. Construction of hyperbolic fillings

The technique of hyperbolic fillings of doubling metric spaces was first considered in Buyalo–Schroeder [16, Chapter 6], and then used in Bourdon–Pajot [13] and Bonk–Saksman [10]. The constructions are different in these papers, see below.

We construct the hyperbolic filling as follows: We assume that a bounded metric space \( Z \) is given, and fix the parameters \( \alpha, \tau > 1 \) and a point \( z_0 \in Z \). By scaling we can assume that \( 0 \leq \text{diam} \ Z < 1 \). As mentioned in Section 2, we later want to equip \( Z \) with a doubling measure, but to begin with no such requirement is needed.

We set \( A_0 = \{ z_0 \} \) and note that \( Z = B_Z(z_0, 1) \). By a recursive construction using Zorn’s lemma or the Hausdorff maximality principle, for each positive integer \( n \) we can choose a maximal \( \alpha^{-n} \)-separated set \( A_n \subset Z \) such that \( A_n \subset A_m \) when \( m \geq n \geq 0 \). A set \( A \subset Z \) is \( \alpha^{-n} \)-separated if \( d_Z(z, z') \geq \alpha^{-n} \) whenever \( z, z' \in A \) are distinct. Then the balls \( B_Z(z, 1/\alpha^{-n}) \), \( z \in A_n \), are pairwise disjoint. Since \( A_n \) is maximal, the balls \( B_Z(z, \alpha^{-n}) \), \( z \in A_n \), cover \( Z \). Here and from now on, \( n \) and \( m \) will always be nonnegative integers.

Next, we define the “vertex set”

\[
V = \bigcup_{n=0}^{\infty} V_n, \quad \text{where} \ V_n = \{ (x, n) : x \in A_n \}. \tag{3.1}
\]

Note that a point \( x \in A_n \) belongs to \( A_k \) for all \( k \geq n \), and so shows up as the first coordinate in infinitely many points in \( V \). Given two different vertices \((x, n), (y, m) \in V\), we say that \((x, n)\) is a neighbor of \((y, m)\) (denoted \((x, n) \sim (y, m)\)) if and only if \([n - m] \leq 1\) and

\[
\tau B_Z(x, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-m}) \neq \emptyset, \quad \text{if} \quad m = n, \tag{3.2}
\]

\[
B_Z(x, \alpha^{-n}) \cap B_Z(y, \alpha^{-m}) \neq \emptyset, \quad \text{if} \quad m = n \pm 1. \tag{3.3}
\]

We let the hyperbolic filling \( X \) be the graph formed by the vertex set \( V \) together with the above neighbor relation (edges), and consider \( X \) to be a metric graph where the edges are unit intervals. As usual for graphs, we do not consider a vertex to be its own neighbor. The distance between two points in \( X \) is the length of the shortest curve between them. Since \( X \) is a metric graph, it is easy to see that these shortest curves exist, and thus \( X \) is a geodesic space.

If \((x, n) \sim (y, n + 1)\) we say that \((y, n + 1)\) is a child of \((x, n)\) while \((x, n)\) is a parent of \((y, n + 1)\). (We use this terminology also for rooted trees.) In general, each vertex has at least one child, and all vertices but for the root \( v_0 = (z_0, 0) \) have at least one parent. An edge \((x, n) \sim (y, m)\) is horizontal if \( m = n \) and vertical if \( m = n \pm 1 \).

We will show that the hyperbolic filling \( X \) is always a Gromov hyperbolic space, but first we compare our construction with those in Buyalo–Schroeder [16, Chapter 6], Bourdon–Pajot [13] and Bonk–Saksman [10]. In [13], Bourdon and Pajot use the same construction as we do with \( \alpha = \epsilon \) and \( \tau = 1 \). It is pointed out in [10] that the choice \( \tau = 1 \) causes problems in the proof of the hyperbolicity of the “hyperbolic filling”, more specifically in the proof of [13, Lemme 2.2]. Indeed, in Example 8.8 we construct a “hyperbolic filling” with \( \tau = 1 \) and \( \alpha = 2 \) which is not a Gromov hyperbolic space.

According to Bonk–Saksman [10], it is enough to enlarge the balls with a factor \( > 1 \), but they make the specific choices \( \alpha = \tau = 2 \). Their construction is however slightly different from ours: Instead of (3.3), they require that (with \( \tau = 2 \))

\[
\tau B_Z(x, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-m}) \neq \emptyset, \quad \text{even if} \quad m = n \pm 1. \tag{3.4}
\]

Thus, the hyperbolic fillings in [10] contain more vertical edges than those considered in this paper (with \( \alpha = \tau = 2 \)).
Buyalo and Schroeder, in [16, Chapter 6], use a similar construction with \( \alpha \geq 6 \) (i.e. \( \tau = 1/\alpha \leq \frac{1}{6} \) in their notation) and \( \tau = 2 \), but impose a different condition when \( m = n \pm 1 \), namely (with \( \tau = 2 \))
\[
\tau B_Z(y, \alpha^{-m}) \subset \tau B_Z(x, \alpha^{-n}), \quad \text{if } m = n + 1.
\]
Buyalo and Schroeder show that their hyperbolic filling is Gromov hyperbolic. Bonk and Saksman [10] refer to Bourdon–Pajot [13] for a proof, but mention that the proof in [13] is problematic for \( \tau = 1 \), as considered in [13]. Both [13] and [10] have stronger assumptions on \( Z \) than here.

When \( \tau \geq (\alpha + 1)/(\alpha - 1) \), we have more concrete information on geodesics in the hyperbolic filling, see Section 8. However, a wider choice of \( \tau > 1 \) yields a wider variety of hyperbolic fillings. For example, if \( Z \) is a Cantor set, obtained as the uniformized boundary of an infinite tree and equipped with the induced ultrametric, then its hyperbolic filling with the choice of \( \tau < \alpha \) gives back the original tree, whereas the choice of \( \tau \geq \alpha \) does not give a tree as the hyperbolic filling of \( Z \), see Section 7. On the other hand, the estimates regarding traces and extensions of Sobolev and Besov spaces in Sections 11–12 (related to the uniformization of hyperbolic fillings) are not affected by the precise values of \( \alpha \) and \( \tau \).

In the rest of the section, we assume that \( Z \) is a metric space with \( \text{diam} \, Z < 1 \) and that \( X \) is a hyperbolic filling, as constructed above, with parameters \( \alpha, \tau > 1 \).

We consider the projection maps \( \pi_1 : V \to Z \) and \( \pi_2 : V \to \{0, 1, 2, \ldots \} \) given by \( \pi_1((x, n)) = x \) and \( \pi_2((x, n)) = n \). We also set \( v_0 := (z_0, 0) \) and use the Gromov product
\[
(v|w)_n = \frac{1}{2}[d_X(v_0, v) + d_X(v_0, w) - d_X(v, w)], \quad v, w, v_0 \in V.
\]
It follows easily from the construction that the hyperbolic filling is connected. The following lemma is a more precise version of this.

**Lemma 3.1.** For all \( v \in V \) we have \( d_X(v, v_0) = \pi_2(v) \). In particular, \( V \) is connected in the graph sense, and \( X \) in the metric sense.

**Proof.** The first claim is clear if \( v = v_0 \). So suppose that \( v = (x, n) \) for some positive integer \( n \). By the construction of \( A_j \), there are \( x_j \in A_j \), such that \( x \in B_Z(x_j, \alpha^{-j}) \), \( j = 0, 1, \ldots, n \). In particular, \( x = x_n \),
\[
v_j := (x_j, j) \in V \quad \text{and} \quad x \in B_Z(x_j, \alpha^{-j}) \cap B_Z(x, \alpha^{-j})
\]
for \( j = 1, \ldots, n \). It follows that \( v_0 \sim v_1 \sim \ldots \sim v_n \), and thus \( d_X(v, v_0) \leq n \). As all other paths connecting these two points have length at least \( n \), we have the required conclusion of the first claim. The last part follows directly.

Using that every vertex has at least one child, the following consequence of the construction in the proof of Lemma 3.1 is immediate. We will use similar properties many times in this paper without further ado.

**Corollary 3.2.** (a) For every vertex \( v \) there is a geodesic ray starting at \( v_0 \) and containing \( v \).

(b) Every geodesic ray starting at \( v_0 \) consists solely of vertical edges.

(c) Any geodesic from any \( x \in X \) to the root \( v_0 \) contains at most a half of a horizontal edge.

(d) \( X \) is roughly startlike with \( M = \frac{1}{2} \).

Next, we provide a proof of the hyperbolicity for all parameters \( \alpha, \tau > 1 \). The ideas are similar to those in [13] and [16]. In particular the following lemma was inspired by [13, Lemme 2.2]. As mentioned above, when \( \tau = 1 \) it is possible for the “hyperbolic filling” to be nonhyperbolic, see Example 8.8 below.
Lemma 3.3. Let \( v = (z, n) \) and \( w = (y, m) \) be two vertices in \( X \). Then
\[
\alpha^{-(v|w)_{w_0}} \simeq d_Z(z, y) + \alpha^{-n} + \alpha^{-m},
\]
with comparison constants depending only on \( \alpha \) and \( \tau \).

Proof. Without loss of generality, we can assume that \( n \leq m \). If \( z = y \) then \( d_X(v, w) = m - n \) and therefore \( (v|w)_{w_0} = \frac{1}{2}(n + m - (m - n)) = n \), and so the statement holds in this case. Assume therefore that \( z \neq y \).

Let \( l \) be the smallest nonnegative integer such that \( \alpha^{-l} \leq \tau - 1 \), and \( k \) be the smallest nonnegative integer such that \( \alpha^{-k-1} < d_Z(z, y) \). For \( j = 0, 1, \ldots \), let \( z_j, y_j \in A_j \) be such that
\[
d_Z(z, z_j) < \alpha^{-j} \quad \text{and} \quad d_Z(y, y_j) < \alpha^{-j}.
\]
Clearly, we can choose \( z_j = z \) for \( j \geq n \) and \( y_j = y \) for \( j \geq m \). We shall distinguish two cases:

If \( k_0 := \min\{k-l, n\} \geq 0 \), then \( \alpha^{k_0-k} \leq \alpha^{-l} \leq \tau - 1 \), and the triangle inequality shows that
\[
d_Z(z, y_{k_0}) \leq d_Z(z, y) + d_Z(y, y_{k_0}) < \alpha^{-k} + \alpha^{-k_0} = (\alpha^{k_0-k} + 1)\alpha^{-k_0} \leq \tau \alpha^{-k_0}.
\]
Hence \( z \in \tau B_Z(z_{k_0}, \alpha^{-k_0}) \cap \tau B_Z(y_{k_0}, \alpha^{-k_0}) \), from which it follows that
\[
(z, n) \sim (z_{n-1}, n-1) \sim \ldots \sim (z_{k_0}, k_0) \sim (y_{k_0}, k_0) \sim \ldots \sim (y_{n-1}, m-1) \sim (y, m)
\]
where the middle edge may collapse into a single vertex. Thus,
\[
d_X(v, w) \leq n + m + 1 - 2k_0,
\]
and consequently, \( (v|w)_{w_0} = \frac{1}{2}(n + m - d_X(v, w)) \geq k_0 - \frac{1}{2} \). If \( k_0 < 0 \) then clearly \( (v|w)_{w_0} \geq 0 > k_0 \). In both cases we thus have that
\[
\alpha^{-(v|w)_{w_0}} \leq \alpha^{-(k_0+1)/2} \leq \alpha^{1/2}(\alpha^{-k} + \alpha^{-n}) \leq \alpha^{1/2}(d_Z(z, y) + \alpha^{-n} + \alpha^{-m}).
\]
Note that \( l \) only depends on \( \alpha \) and \( \tau \).

Conversely, let \( w_0 \sim w_1 \sim \ldots \sim w_N \) be a geodesic from \( v = w_0 \) to \( w = w_N \). Note that
\[
d_X(v, w) = N \geq m - n \quad \text{and} \quad (v|w)_{w_0} = \frac{1}{2}(n + m - N).
\]
Moreover, by the construction of the hyperbolic filling, \( \pi_Z(w_j) \geq n-j \) and \( \pi_Z(w_{N-j}) \geq m-j \) for all \( j, i = 0, 1, \ldots, N \). Therefore
\[
\alpha^{-\pi_Z(w_j)} \leq \alpha^{-j} \quad \text{and} \quad \alpha^{-\pi_Z(w_{N-j})} \leq \alpha^{i-m}.
\]
The triangle inequality then yields that for all \( k_1 = 0, 1, \ldots, N \),
\[
d_Z(z, y) + \alpha^{-n} + \alpha^{-m} \leq \sum_{j=1}^{N} d_Z(\pi_1(w_{j-1}), \pi_1(w_j)) + \alpha^{-n} + \alpha^{-m}
\leq \alpha^{-n} + \sum_{j=1}^{N} (\tau \alpha^{-\pi_Z(w_{j-1})} + \tau \alpha^{-\pi_Z(w_j)}) + \alpha^{-m}
\leq 2\tau \sum_{j=0}^{k_1-1} \alpha^{-j-n} + 2\tau \sum_{i=0}^{N-k_1} \alpha^{i-m}
\leq \frac{2\tau}{\alpha-1}(\alpha^{k_1-n} + \alpha^{N-k_1-m+1}).
\]
where the sum $\sum_{j=0}^{k_1-1}$ is empty when $k_1 = 0$. Choosing $k_1$ to be the smallest integer $\geq \frac{1}{2}(N + n - m)$ gives that

$$d_Z(z, y) + \alpha^{-n} + \alpha^{-m} < \frac{4\tau\alpha}{\alpha - 1} \alpha^{(N-n-m)/2} = \frac{4\tau\alpha}{\alpha - 1} \alpha^{-(v|w)_{v_0}}. \quad \square$$

**Theorem 3.4.** There is a constant $C \geq 0$, depending only on $\alpha$ and $\tau$, such that if $u$, $v$ and $w$ are three vertices in $V$, then

$$(v|w)_{v_0} \geq \min\{ (v|u)_{v_0}, (w|u)_{v_0} \} - C. \quad (3.5)$$

In particular, $X$ is Gromov hyperbolic.

**Proof.** It suffices to prove (3.5) since Gromov hyperbolicity is equivalent to it, see Bonk–Heinonen–Koskela [8] or Bridson–Haefliger [14, p. 411, Proposition 1.22].

Let $v = (z, n)$, $w = (y, m)$ and $u = (x, k)$. Then clearly,

$$d_Z(z, y) + \alpha^{-n} + \alpha^{-m} \leq (d_Z(z, x) + \alpha^{-n} + \alpha^{-k}) + (d_Z(x, y) + \alpha^{-k} + \alpha^{-m}),$$

and Lemma 3.3 implies that

$$\alpha^{-(v|w)_{v_0}} \lesssim \alpha^{-(v|u)_{v_0}} + \alpha^{-(u|w)_{v_0}} \leq 2\alpha^{-(v|v_0, (u,w)_{v_0})}.$$  

Taking logarithms concludes the proof. \quad \square

**4. The uniformized boundary $\partial_{\varepsilon}X$**

In this section, we assume that $Z$ is a metric space with $\text{diam} Z < 1$, and let $X$ be a hyperbolic filling of $Z$ with parameters $\alpha, \tau > 1$.

In this section we will look at how $Z$ relates to the uniformized boundary $\partial_{\varepsilon}X$ of $X$, see Section 2 for the definitions. We use the root $v_0 = (z_0, 0)$ as the uniformization center $x_0$. We will show that if $\varepsilon \leq \log \alpha$ and $Z$ is complete, then $\partial_{\varepsilon}X$ is snow-flake equivalent to $Z$, with exponent $\sigma = \varepsilon/\log \alpha$, see Proposition 4.4 for further details. In particular, $\partial_{\varepsilon}X$ and $Z$ are biLipschitz equivalent if $\varepsilon = \log \alpha$.

Before showing the equivalence of $\partial_{\varepsilon}X$ and $Z$, we take a look at the case $\varepsilon > \log \alpha$. Towards the end of the section we will also study how the degree of the vertices in $X$ depends on properties of $Z$. In this section we are not concerned with whether the uniformization $X_{\varepsilon}$ is a uniform domain or not. This question will be considered in Section 5.

Note that $\rho_{\varepsilon} \simeq \rho_{\varepsilon}(v)$ on every edge $[v, w] \subset X$ and thus, using also the basic facts from Corollary 3.2, for all $x \in X$,

$$d_{\varepsilon}(x) \simeq \int_{d_{\varepsilon}(x, v_0)}^{\infty} \rho_{\varepsilon} \, ds = \frac{1}{\varepsilon} \rho_{\varepsilon}(x), \quad (4.1)$$

with equality if $x$ is a vertex.

The following result shows that if $\varepsilon > \log \alpha$, then $\partial_{\varepsilon}X$ often becomes just one point.

**Proposition 4.1.** Assume that there is $L < \infty$ such that any two points in the $d_Z$-completion $\overline{Z}$ of $Z$ can be connected by a curve in $\overline{Z}$ of length at most $L$. If $\varepsilon > \log \alpha$, then $\partial_{\varepsilon}X$ consists of just one point.

This shows that when $\varepsilon > \log \alpha$ and $Z$ has at least two points, there is no natural connection between $Z$ and $\partial_{\varepsilon}X$. On the other hand, it is easy to see that if $Z$ consists of finitely many points, then $\partial_{\varepsilon}X$ is also finite and there is a natural biLipschitz map between these sets. (We leave the details regarding finite sets $Z$ to the interested reader.) In particular, some connectivity assumption is necessary in Proposition 4.1.
Proof. Let $F_n = \{x \in X : d_X(x, v_0) \leq n\}$ and let $x, x' \in X \setminus F_n$ be arbitrary. Then there are $d_X$-geodesics from $x$ and $x'$ to $v_0$ which contain vertices $v, v' \in V_n$, respectively. By the connectivity assumption on $Z$, there is a sequence $\{w_j\}_{j=0}^m$ of points in $Z$ such that $v = (w_0, n)$, $v' = (w_m, n)$, $m \leq 3L\alpha^n/(\tau - 1)$ and (if $m \geq 1$)

$$d_Z(w_j, w_{j-1}) \leq \frac{\varepsilon}{\tau - 1} \alpha^{-n}, \quad j = 1, \ldots, m.$$  

For each $j = 0, \ldots, m$ there are points $w'_j \in Z$ and $z_j \in A_n$ such that

$$d_Z(w'_j, w_j) \leq \frac{\varepsilon}{\tau - 1} \alpha^{-n} \quad \text{and} \quad d_Z(w'_j, z_j) < \alpha^{-n}.$$  

We can choose $z_0 = w'_0 = w_0$ and $z_m = w'_m = w_m$. As

$$d_Z(w'_j, z_{j-1}) \leq d_Z(w'_j, w_j) + d_Z(w_j, w_{j-1}) + d_Z(w_{j-1}, w'_{j-1}) + d_Z(w'_{j-1}, z_{j-1})$$  

$$< (\tau - 1) \alpha^{-n} + \alpha^{-n} = \tau \alpha^{-n}$$  

when $j \geq 1$, we see that $v = (z_0, n) \sim (z_1, n) \sim \ldots \sim (z_m, n) = v'$. It follows that

$$d_z(x, x') \leq d_z(x, v) + d_z(v, v') + d_z(v', x') < 2\int_n^\infty e^{-\varepsilon t} dt + m e^{-\varepsilon n}$$  

$$< \frac{2e^{-\varepsilon n}}{\varepsilon} + e^{-\varepsilon n} \frac{3L\alpha^n}{\tau - 1} = \frac{2e^{-\varepsilon n}}{\varepsilon} + \frac{L}{\tau - 1} e^{n(\log \alpha - \varepsilon)}.$$  

Taking supremum over all $x, x' \in X \setminus F_n$ shows that

$$\text{diam}_z \partial_z X \leq \text{diam}_z (X \setminus F_n) \to 0, \quad \text{as } n \to \infty,$$

since $\varepsilon > \log \alpha$. □

The easiest example of a space $Z$ applicable in Proposition 4.1 is $Z = [0, \frac{1}{2}]$. The following example shows that it is possible to apply this proposition to a noncompact complete space.

Example 4.2. Let $Z$ consist of countably many copies of $[0, \frac{1}{2}]$, all glued together at 0, and equipped with the inner length metric, so that $Z$ is geodesic. It is easy to see that $Z$ is noncompact and complete, and that $\text{diam} Z = \frac{1}{2}$. Thus Proposition 4.1 applies.

Before proceeding, we deduce the following lemma which will be used several times in this paper.

Lemma 4.3. Assume that $z, y \in Z$ are such that $d_Z(z, y) \leq \alpha^{-k}$ for some nonnegative integer $k$. For $j = 0, 1, \ldots$, let $z_j, y_j \in A_j$ be such that $d_Z(z, z_j) < \alpha^{-j}$ and $d_Z(y, y_j) < \alpha^{-j}$. Then

$$v_0 \sim (z_1, 1) \sim \ldots \sim (z_k, k) \sim (z_{k+1}, k+1) \sim \ldots \quad (4.2)$$

Let $l$ be the smallest nonnegative integer such that $\alpha^{-l} \leq \tau - 1$. Then, for any $m, n \geq h := \max\{k - l, 0\}$,

$$(z_n, n) \sim \ldots \sim (z_h, h) \sim (y_h, h) \sim \ldots \sim (y_m, m), \quad (4.3)$$

where the middle edge collapses into a single vertex if $z_h = y_h$. This path $\gamma$ has lengths $\ell_X(\gamma)$ and $\ell_z(\gamma)$ (with respect to $d_X$ resp. $d_z$) satisfying

$$d_X((z_n, n), (y_m, m)) \leq \ell_X(\gamma) \leq n + m + 1 - 2h \leq n + m + 1 + 2l - 2k, \quad (4.4)$$

$$d_z((z_n, n), (y_m, m)) \leq \ell_z(\gamma) \leq \frac{4}{\varepsilon} e^{-\varepsilon h} \leq \frac{4}{\varepsilon} e^{-\varepsilon(k-1)}.$$
Note that \( z_j \) and \( y_j \) exist by the construction of \( A_j \), and that \( y_0 = z_0 \) with \( z_0 \) being the unique element in \( A_0 \) as before.

**Proof.** Since \( z \in B_{\tau} (z_j, \alpha^j) \cap B_{\tau}(z_{\ell+1}, \alpha^{\ell+1}) \) for each \( j = 0, 1, \ldots \), we directly see that (4.2) holds. By the triangle inequality and the choice of \( i \) we see that

\[
 d_{\tau}(z, y_i) \leq d_{\tau}(z, y) + d_{\tau}(y, y_i) < \alpha^{-k} + \alpha^{-l-k} \leq \tau \alpha^{-k}.
\]

Moreover, \( z \in \tau B_{\tau}(z_h, \alpha^{-h}) \) and so \((z_h, h) \sim (y_h, h)\). Therefore (4.3) follows from (4.2) (and the corresponding path for \( y \)). The estimate (4.4) follows directly from (4.3).

To estimate \( \ell_{\varepsilon}(\gamma) \), recall that by Lemma 3.1, \( d_{\gamma}(v, w) = \pi_2(v) \) for all \( v \in V \) and that \( \rho_\varepsilon(x) = e^{-\varepsilon d_{\gamma}(x, w)} \) for all \( x \in X \). Hence

\[
 \ell_{\varepsilon}(\gamma) \leq \int_{h}^{n} e^{-e(k-1)} dt + \int_{h}^{m} e^{-e(t)} dt + 2 \int_{h}^{\frac{k+1}{2}} e^{-e(t)} dt \leq \frac{4}{\varepsilon} e^{-eh} \leq \frac{4}{\varepsilon} e^{-e(k-1)},
\]

where the last integral estimates the \( d_{\varepsilon} \)-length of the (possibly collapsed) horizontal edge \((z_h, h) \sim (y_h, h)\). \( \square \)

We next show that \( \partial_\varepsilon X \) is snowflake equivalent to the completion \( \overline{Z} \) of \( Z \). In particular, \( \partial_\varepsilon X \) is \( bi\)-Lipschitz equivalent to \( \overline{Z} \) when \( \varepsilon = \log \alpha \).

**Proposition 4.4.** Fix \( 0 < \varepsilon \leq \log \alpha \). Then for all vertices \( v, w \in X \),

\[
 d_{\tau}(\pi_1(v), \pi_1(w)) \sigma \leq C_{1} d_{\tau}(v, w), \quad \text{where } C_{1} = (2\tau \alpha)^{\sigma} \quad \text{and } \sigma = \frac{\varepsilon}{\log \alpha} \leq 1.
\]  

(4.5)

Moreover, \( \partial_\varepsilon X \) is snowflake-equivalent to the completion \( \overline{Z} \) of \( Z \), that is, there is a natural homeomorphism \( \Psi : \overline{Z} \to \partial_\varepsilon X \) such that for all \( z, y \in \overline{Z} \),

\[
 d_{\tau} (\Psi(z), \Psi(y)) \leq C_2 d_{\tau}(z, y)^{\sigma},
\]

(4.6)

where \( C_2 = 4\alpha^{(l+1)\sigma}/\varepsilon \) and \( l \) is the smallest nonnegative integer such that \( \alpha^{-l} \leq \tau^{-1} \).

**Proof.** Let \( w_0 \sim w_1 \sim \ldots \sim w_N \) be a path \( \gamma \) in \( X \) connecting \( w_0 = v \) to \( w_N = w \). We can assume without loss of generality that \( \pi_2(w_N) \leq \pi_2(w_0) \). Then by the construction of the hyperbolic filling,

\[
 d_{\tau}(\pi_1(w_0), \pi_1(w_N)) \leq \sum_{i=1}^{N} d_{\tau}(\pi_1(w_{i-1}), \pi_1(w_i)) \leq 2\tau \sum_{i=1}^{N} \alpha^{-\pi_2(w_i)}. \]

(4.7)

Moreover, for each \( i \),

\[
 \ell_{\varepsilon} ([w_{i-1}, w_i]) \geq \int_{\pi_2(w_i)}^{\pi_2(w_{i+1})} e^{-e(t)} dt \geq e^{-e(\pi_2(w_i)+1)} = e^{-e(\alpha^{-\pi_2(w_i)})^{\sigma}}.
\]

Summing over all \( i \) and using the elementary inequality \( (\sum_{i=1}^{N} a_i)^{\sigma} \leq \sum_{i=1}^{N} a_i^{\sigma} \) for \( \sigma \leq 1 \), together with (4.7), yields

\[
 \ell_{\varepsilon}(\gamma) = \sum_{i=1}^{N} \ell_{\varepsilon} ([w_{i-1}, w_i]) \geq e^{-\varepsilon} \sum_{i=1}^{N} (\alpha^{-\pi_2(w_i)})^{\sigma} \geq e^{-\varepsilon} \left( \frac{d_{\tau}(\pi_1(w_0), \pi(w_N))}{2\tau} \right)^{\sigma}.
\]
Taking the infimum over all such paths \( \gamma \) gives (4.5).

Let \( z \in \mathbb{Z} \) and find a sequence \( z_j \in A_j \) such that \( d_Z(z_j, z) < \alpha^{-j} \). Then from Lemma 4.3 we see that \( v_0 \sim (z_1, 1) \sim \ldots \sim (z_j, j) \sim (z_{j+1}, j+1) \sim \ldots \). Moreover, when \( i > j \),
\[
\begin{align*}
  d_e((z_i, i), (z_j, j)) & \leq \int_j^{\infty} e^{-\epsilon t} dt = \frac{e^{-\epsilon j}}{\epsilon}.
\end{align*}
\]

It follows that \( \{(z_j, j)\} \) is a Cauchy sequence in \( X_e \). Set \( \Psi(z) = \lim_{j \to \infty}(z_j, j) \).

From the construction of \( X_e \) it is clear that \( \Psi(z) \not\in X_e \), and so this point belongs to \( \partial_e X \). If \( z^*_j \in A_j \) is such that \( d_Z(z^*_j, z) < \alpha^{-j} \), then \( z \in B_Z(z^*_j, \alpha^{-j}) \cap B_Z(z^*_j, \alpha^{-j-1}) \) and thus \( (z^*_j, j) \sim (z_j, j) \). Hence \( d_e((z^*_j, j), (z_j, j)) \lesssim e^{-\epsilon j} \) and so \( \lim_{j \to \infty}(z^*_j, j) = \lim_{j \to \infty}(z_j, j) \), which shows that \( \Psi(z) \) is well-defined and gives the map \( \Psi : \mathbb{Z} \to \partial_e X \).

When \( z, y \in \mathbb{Z} \) with \( z \neq y \) and \( k \) is a nonnegative integer such that \( \alpha^{-k-1} < d_Z(z, y) \leq \alpha^{-k} \), we pick \( z_j \) and \( y_j \) as in Lemma 4.3. Then applying Lemma 4.3, we obtain (upon noting that \( e^\epsilon = \alpha^\sigma \))
\[
\begin{align*}
  d_e(\Psi(z), \Psi(y)) & = \lim_{n \to \infty} d_e((z_n, n), (y_n, n)) \\
  & \leq \frac{4e\epsilon}{\epsilon} e^{-\epsilon k} < \frac{4\alpha^{(l+1)\sigma}}{\epsilon} d_Z(z, y)^l,
\end{align*}
\]
i.e. the last inequality in (4.6) holds.

Conversely, applying (4.5) to \( v = (y_n, n) \) and \( w = (z_n, n) \), and letting \( n \to \infty \), shows the first inequality in (4.6). In particular, \( \Psi \) is injective.

Finally, it only remains to show that \( \Psi \) is surjective. Suppose that \( \{x_j\}_{j=1}^\infty \) is a Cauchy sequence of points in \( X_e \) with limit \( x \in \partial_e X \). Then (4.1) shows that \( \lim_{j \to \infty} d_X(v_0, x_j) = \infty \). By the construction of \( X \), for each \( j \) we can find a vertex \( v_j \in X \) such that \( d_X(v_j, x_j) \leq \frac{1}{2^j} \). Then, with \( z_j := \pi_j(v_j) \in Z \), we know that \( \{v_j\}_{j=1}^\infty \) is a Cauchy sequence in \( X_e \), and so by (4.5) we also have that \( \{z_j\}_{j=1}^\infty \) is a Cauchy sequence in \( Z \). Hence, there is a point \( z_\infty \in Z \) such that \( \lim_{j \to \infty} d_Z(z_j, z_\infty) = 0 \). Lemma 4.3 again tells us that \( \lim_{j \to \infty} d_e(v_j, \Psi(z_\infty)) = 0 \), and so as \( \lim_{j \to \infty} d_e(v_j, x_j) = 0 \), we must have \( x = \Psi(z_\infty) \). Thus \( \Psi \) is surjective, completing the proof.

The construction of the hyperbolic filling as given above works for any bounded metric space, but the resulting hyperbolic filling can have vertices with infinite degree. The following proposition shows that if the metric space is doubling, then the degree is well-controlled. Recall that the degree of a vertex is the number of neighbors it has.

**Proposition 4.5.** The hyperbolic filling \( X \) has uniformly bounded degree if and only if \( Z \) is doubling.

**The uniformity and doubling constants depend only on \( \alpha, \tau \) and each other.**

**Proof.** Assume first that \( Z \) is doubling. Let \( (x, n) \in X \) and set \( A(x, n) \subset V \) to be the collection of all neighbors of \( (x, n) \). For \( (y, m) \in A(x, n) \) we know that \( |m - n| \leq 1 \) and \( \tau B_Z(x, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-m}) \) is nonempty. Hence \( d_Z(x, y) < \tau(\alpha + 1)\alpha^{-n} \) and so \( y \in \tau(\alpha + 1)B_Z(x, \alpha^{-n}) \). Since \( Z \) is assumed to be doubling, there is a positive integer \( N \) independent of \( n \), such that the ball \( \tau(\alpha + 1)B_Z(x, \alpha^{-n}) \) can be covered by balls \( B_1, \ldots, B_N \) of radius \( \frac{1}{8}\alpha^{-n-1} \), see Heinonen [25, p. 81].

Now, for each \( m \in \{n - 1, n, n + 1\} \), we have \( \frac{1}{2}\alpha^{-m} \geq \frac{1}{4}\alpha^{-n-1} \) and the balls \( B_Z(y, \frac{1}{2}\alpha^{-m}) \), \( y \in A_m \), are pairwise disjoint. It follows that each ball \( B_j \), \( j = 1, \ldots, N \), can contain at most one point from \( A_m \). Hence, there are at most \( N \) such \( y \in A_m \) satisfying \( (y, m) \in A(x, n) \). Since this is true for each \( m \in \{n, n \pm 1\} \), we conclude that the cardinality of \( A(x, n) \) is at most \( 3N \), that is, \( X \) is of uniformly bounded degree.
Conversely, assume that $X$ has a uniformly bounded degree. Let $\zeta \in Z$ and
$0 < r \leq 1$. Let $k$ be the smallest nonnegative integer such that $\min\{1, 3r\} \leq \alpha^{-k}$.
For this choice of $k$, let $n$ be the smallest integer such that $n \geq k$ and $\alpha^{-n} \leq \frac{3}{2}r$.
Note that $n \leq k + l'$ for some $l'$ depending only on $\alpha$.

Since the balls $B_Z(z, \alpha^{-n})$ with $z \in A_n$ cover $Z$, for every $\xi \in B_Z(\zeta, r)$ there is
$z \in A_n$ such that $\xi \in B_Z(z, \alpha^{-n}) \subset B_Z(z, \frac{1}{2}r)$.

Moreover,
$$d_Z(z, \zeta) \leq d_Z(z, \xi) + d_Z(\xi, \zeta) < \alpha^{-n} + r \leq \frac{3}{2}r.$$
It follows that the balls $B_Z(z, \alpha^{-n})$ with $z \in A := A_n \cap B_Z(\zeta, \frac{3}{2}r)$ cover $B(\zeta, r)$. To estimate the cardinality of $A$, note that any two points $z, z' \in A$ satisfy $d_Z(z, z') \leq \min\{1, 3r\} \leq \alpha^{-k}$. Lemma 4.3 and the above observation that $n \leq k + l'$ then imply
that (with $l$ as in Lemma 4.3),
$$d_X((z, n), (z', n)) \leq 2n + 1 + 2(l - k) \leq 1 + 2(l + l'),$$
which only depends on $\alpha$ and $r$. By assumption there is a uniform bound on the
degrees in $X$ and hence also on the number of vertices in balls with a fixed radius.
Thus there is a uniform bound on the cardinality of $A$, i.e. $Z$ is doubling.

\textbf{Proposition 4.6.} Let $\varepsilon > 0$ and let $X_\varepsilon$ be the uniformization of $X$, as defined in
Section 2. Then the following are equivalent:

(a) $Z$ is totally bounded;
(b) each vertex layer $V_n$ (as defined in (3.1)) is finite;
(c) every vertex in $X$ has finite degree;
(d) $X$, and equivalently $X_\varepsilon$, is locally compact;
(e) $\overline{X_\varepsilon}$ is compact.

If $\varepsilon \leq \log \alpha$, then the following condition is also equivalent to those above:
(f) $\partial X$ is compact.

Moreover, $\overline{X_\varepsilon}$ is geodesic whenever (a)–(e) hold.

Note that we do not require $X_\varepsilon$ to be uniform. Since (a)–(c) are independent of $\varepsilon$, so are (d) and (e). Furthermore, (f) is also independent of $\varepsilon$ provided that $\varepsilon \leq \log \alpha$. Example 4.2 shows that (f) is not equivalent to the other statements when $\varepsilon > \log \alpha$.

\textit{Proof.} (a) $\Leftrightarrow$ (b) It follows directly from the definition of total boundedness that
$Z$ is totally bounded if and only if all $A_n$ are finite sets, or equivalently all $V_n$ are finite.

(b) $\Rightarrow$ (c) Let $v = (x, n)$ be a vertex. Then all neighbors of $v$ belong to the finite
set $V_{n-1} \cup V_n \cup V_{n+1}$, i.e. $v$ has finite degree.

(c) $\Rightarrow$ (d) Let $m$ be the least index such that $V_m$ is infinite (which exists as
(b) fails). As $V_0 = \{v_0\}$, we must have $m \geq 1$. Each vertex in $V_m$ has at least
one parent in $V_{m-1}$. As $V_{m-1}$ is finite and $V_m$ is infinite, there must be a vertex in
$V_{m-1}$ which has infinitely many children, and hence has infinite degree.

(d) $\Leftrightarrow$ (e) This is easily seen to be true. Note that $X$ and $X_\varepsilon$ have the same
topology, and are thus simultaneously locally compact or not.

(e) $\Rightarrow$ (d) Let $F_0 = \{x \in X : d_X(x, v_0) \leq n\}$. Since each $V_j$ is finite, it follows
that $F_0$ is a union of finitely many compact intervals and so is compact.

Consider $x \in X \setminus F_0$ and let $\gamma$ be a geodesic from $x$ to $v_0$. As $d_X(x, v_0) > n$,
there is some point $v \in \gamma$ such that $d_X(v, v_0) = n$. Since $n$ is an integer, it follows
that $v \in V_n$. Hence
$$\text{dist}_\varepsilon(x, V_n) \leq \int_n^\infty e^{-\varepsilon t} \, dt = \frac{e^{-\varepsilon n}}{\varepsilon}.$$
This inequality also holds for \( x \in \partial_\varepsilon X \), since \( \partial_\varepsilon X \subset \overline{X \setminus F_\eta} \) (where the closure is with respect to the \( d_\varepsilon \) metric).

Let \( \eta > 0 \) and choose \( n \) so that \( e^{-\varepsilon n}/\varepsilon < \eta \). Then \( \overline{X_\varepsilon \setminus F_\eta} \subset \bigcup_{y \in V_\eta} B_\varepsilon(y, \eta) \), and as \( F_\eta \) is compact and \( V_\eta \) is finite we see that there is a finite \( \eta \)-net for \( \overline{X_\varepsilon} \). Since \( \eta > 0 \) was arbitrary, \( \overline{X_\varepsilon} \) is totally bounded and thus compact (as it is complete by definition).

\((e) \Rightarrow (b)\) As \( V_\eta \subset \overline{X_\varepsilon} \), it is also compact with respect to the metric \( d_\varepsilon \). The metrics \( d_X \) and \( d_\varepsilon \) are biLipschitz equivalent on \( V_\eta \) and thus \( V_\eta \) is compact also with respect to \( d_X \). Since distinct points in \( V_\eta \) are at least a distance 1 apart, it follows that \( V_\eta \) is a finite set.

Next, we assume that \( \varepsilon \leq \log \alpha \) and consider \((f)\).

\((e) \Rightarrow (f)\) This is trivial.

\((f) \Rightarrow (a)\) It follows from Proposition 4.4 that \( \overline{Z} \) is homeomorphic to \( \partial_\varepsilon X \), and is thus also compact. Hence \( \overline{Z} \) is totally bounded.

Finally, as \( \overline{X_\varepsilon} \) is a length space, it follows from Ascoli’s theorem that it is geodesic if it is compact. \(\square\)

5. Uniformizing a hyperbolic filling with parameter \( \varepsilon \leq \log \alpha \)

In this section, we assume that \( Z \) is a metric space with \( \text{diam} Z < 1 \), and let \( X \) be a hyperbolic filling of \( Z \) with parameters \( \alpha, \tau > 1 \).

The aim of this section is to show that the uniformization \( X_\varepsilon \) is a uniform domain when \( \varepsilon \leq \log \alpha \). (Recall Definition 2.3 of uniform spaces and uniform curves.) Since \( X \) is Gromov hyperbolic (by Theorem 3.4), it follows from Bonk–Heinonen–Koskela [8, Theorem 2.6] (see Theorem 2.5) that \( X_\varepsilon \) is a uniform space for sufficiently small \( \varepsilon > 0 \), when \( X \) is locally compact. In the later part of this paper we are interested in uniformizing (a locally compact) \( X \) with respect to \( \varepsilon = \log \alpha \), and so we cannot rely on Theorem 2.5 or [8]. Therefore we provide a direct proof here, which also avoids assuming local compactness. As we saw in Section 4, it is natural to assume that \( \varepsilon \leq \log \alpha \) in order for the boundary \( \partial_\varepsilon X \) to be homeomorphic to \( \overline{Z} \) (or its completion \( \overline{\overline{Z}} \) if \( \overline{Z} \) is not complete).

**Theorem 5.1.** For all \( 0 < \varepsilon \leq \log \alpha \), the uniformized space \( X_\varepsilon \) is uniform with the uniformity constant depending only on \( \alpha, \tau \) and \( \varepsilon \). Moreover, for all \( x', x'' \in X \),

\[
\begin{align*}
\varepsilon(x', x'') &\simeq e^{-\varepsilon(x')^{\nu_{\alpha}}_{\alpha}} \min\{d_X(x', x''), 1\},
\end{align*}
\]

with comparison constants depending only on \( \alpha, \tau \) and \( \varepsilon \).

**Proof.** First we show that curves given by Lemma 4.3, connecting vertices in \( X \), are quasiconvex curves (i.e. having length at most a constant multiple of the distance between their endpoints). These curves will be subsequently used to construct uniform curves connecting pairs of points that are “far apart”. Throughout the proof, we let \( l \) be the smallest nonnegative integer such that \( \alpha^{-l} \leq \tau - 1 \).

Let \( v = (z, n) \) and \( w = (y, m) \) be two distinct vertices in \( V \) with \( z \in A_n \subset Z \) and \( y \in A_m \subset Z \). We can assume that \( n \leq m \). Let \( k \) be the largest nonnegative integer such that \( k \leq n \) and \( d_Z(z, y) \leq \alpha^{-k} \). Consider a curve \( \gamma \) connecting \( v \) to \( w \), as in (4.3) of Lemma 4.3. Then by Lemma 4.3 we have

\[
\ell_\varepsilon(\gamma) \leq \frac{4}{\varepsilon} e^{-\varepsilon(k-1)} = \frac{4e^{(l+1)}}{\varepsilon} (\alpha^{-k-1})^\sigma, \quad \text{where} \quad \sigma = \frac{\varepsilon}{\log \alpha} \leq 1.
\]
We now show that $\ell_\varepsilon(\gamma)$ is comparable to $d_\varepsilon(v, w)$. If $k < n$ then the comparison follows from the choice of $k$ and from (4.5), which together imply that

$$
\ell_\varepsilon(\gamma) \lesssim (\alpha^{-k+1})^\sigma < d_\varepsilon(z, y)^\sigma \leq (2\tau\alpha)^\sigma d_\varepsilon(v, w).
$$

On the other hand, if $k = n$, then any injective curve $\gamma'$ connecting $v$ to $w$ starts with an edge $v \sim v'$, and thus

$$
\ell_\varepsilon(\gamma') \geq d_\varepsilon(v, v') \geq \int_{k}^{k+1} e^{-\varepsilon t} dt \geq e^{-\varepsilon(k+1)} = (\alpha^{-k+1})^\sigma \lesssim \ell_\varepsilon(\gamma).
$$

Taking infimum over all such curves $\gamma'$ shows that even when $k = n$ we have

$$
\ell_\varepsilon(\gamma) \leq d_\varepsilon(v, w). \quad (5.2)
$$

Now assume that $x', x'' \in X$ are two arbitrary distinct points and consider an injective curve $\tilde{\gamma}$ from $x'$ to $x''$ with $\ell_\varepsilon(\tilde{\gamma}) < 2d_\varepsilon(x', x'')$. If $\tilde{\gamma}$ contains at most one vertex, then $d_X(x', x'') < 2$, and by (4.1),

$$
\ell_\varepsilon(\tilde{\gamma}) \leq d_\varepsilon(x', x'') \leq e^{-\varepsilon d_X(x', x'')} d_X(x', x'') \leq d_\varepsilon(x) \quad \text{for all } x \in \tilde{\gamma}, \quad (5.3)
$$

and thus $\tilde{\gamma}$ is a uniform curve. By the triangle inequality,

$$
(x'|x'')_{v_0} = \frac{1}{2}\left[d_X(x', x_0) + d_X(x'', x_0) - d_X(x', x'')\right] \geq \frac{1}{2}\left[d_X(x', x_0) + d_X(x', x_0) - 2d_X(x', x'')\right] > d_X(x', x_0) - 2,
$$

and by the triangle inequality again, $(x'|x'')_{v_0} \leq d_X(x', x_0)$. Inserting this into (5.3) shows that (5.1) holds in this case.

If $\tilde{\gamma}$ contains at least two vertices, then let $v = (z, n)$ and $w = (y, m)$ be the first resp. last vertex in $\tilde{\gamma}$. Let $k$ be the largest nonnegative integer such that both $k \leq \min\{n, m\}$ and $d_Z(z, y) \leq \alpha^{-k}$. Let $\gamma$ be a curve as described at the beginning of this proof, connecting the vertices $v$ and $w$. The desired uniform curve $\tilde{\gamma}$ between $x'$ and $x''$ is then obtained by appending the segments $[x', v]$ and $[w, x'']$ to $\gamma$. By (5.2),

$$
\ell_\varepsilon(\tilde{\gamma}) \leq d_\varepsilon(x', v) + \ell_\varepsilon(\gamma) + d_\varepsilon(w, x'')
\leq d_\varepsilon(x', v) + d_\varepsilon(v, w) + d_\varepsilon(w, x'') = \ell_\varepsilon(\gamma) < 2d_\varepsilon(x', x'').
$$

We next show that $\tilde{\gamma}$ satisfies the twisted cone condition (2.1) in Definition 2.3. Note that $\tilde{\gamma}$ need not be injective and recall from Section 2 what $y \in \tilde{\gamma}$ and $\tilde{\gamma}_{y', w'}$ mean in such a case.

Let $v' = (z_h, h) \in \gamma$ and $w' = (y_h, h) \in \gamma$, where $h = \max\{k-l, 0\}$, $z_h$ and $y_h$ are as given for $v$ and $w$ by Lemma 4.3. Recall that $\gamma$ consists of two vertical segments, one from $v$ to $v'$ and the other from $w$ to $w'$, together with the (possibly collapsed) horizontal edge $[v', w']$. Let $x \in \tilde{\gamma}$ be arbitrary and consider the subcurve $\tilde{\gamma}_{x, v'}$ of $\tilde{\gamma}$ from $x$ to $x'$. We shall distinguish three basic situations and their symmetric equivalents. If $x \in \tilde{\gamma}_{x, v'}$, then

$$
\ell_\varepsilon(\tilde{\gamma}_{x, x'}) = d_\varepsilon(x, x') \leq e^{-\varepsilon(n-1)} \simeq d_\varepsilon(x).
$$

If $x \in \tilde{\gamma}_{y, v'}$, then $\tilde{\gamma}_{v, x}$ is a vertical segment, and hence, using (4.1) and that $\pi_2(v) = n \geq d_X(x, v_0)$,

$$
\ell_\varepsilon(\tilde{\gamma}_{x, x'}) = d_\varepsilon(x', v) + \int_{d_X(x, v_0)}^n e^{-\varepsilon t} dt \leq e^{-\varepsilon(n-1)} + \int_{d_X(x, v_0)}^\infty e^{-\varepsilon t} dt \lesssim d_\varepsilon(x).
$$
If \( x \in \tilde{\gamma}_{w',x''} \), then \( \pi_2(v') = h \leq n \), and thus
\[
\ell_\varepsilon(\tilde{\gamma}_{x,x''}) \leq d_\varepsilon(x', v) + \int_0^1 e^{-\varepsilon t} dt + 2 \int_0^{1/2} e^{-\varepsilon t} dt \\
\leq e^{-\varepsilon(n-1)} + 3 \int_0^\infty e^{-\varepsilon t} dt \lesssim e^{-\varepsilon} \simeq d_\varepsilon(x).
\]
The case when \( x \in \tilde{\gamma}_{w,x''} \) is treated similarly. Thus, \( \tilde{\gamma} \) is a uniform curve.

To prove (5.1) also in this case, note that by Lemma 4.3, and using that \( k \) is the largest nonnegative integer \( \leq \min\{n, m\} \) such that \( d_Z(z, y) \leq \alpha^{-k} \),
\[
d_\varepsilon(x', x'') \leq d_\varepsilon(x', v) + d_\varepsilon(v, w) + d_\varepsilon(w, x'') \leq e^{-\varepsilon n} + e^{-\varepsilon k} + e^{-\varepsilon m} \\
\lesssim d_\varepsilon(z, y) + \alpha^{-\varepsilon n} + \alpha^{-\varepsilon m}.
\]
Conversely, (4.5) shows that
\[
2d_\varepsilon(x', x'') \geq \ell_\varepsilon(\tilde{\gamma}) \geq d_\varepsilon(v, w) \gtrsim d_\varepsilon(z, y)^\sigma.
\]
Since also \( d_\varepsilon(v, w) \gtrsim e^{-\varepsilon n} + e^{-\varepsilon m} = \alpha^{-\varepsilon n} + \alpha^{-\varepsilon m} \), we get by Lemma 3.3 that
\[
d_\varepsilon(x', x'') \simeq d_\varepsilon(z, y)^\sigma + \alpha^{-\varepsilon n} + \alpha^{-\varepsilon m} \simeq \alpha^{-\varepsilon(n+w)} \simeq e^{-\varepsilon(x'|x'')}^{\varepsilon n}.
\]

6. Roughly similar equivalence

In this section, we want to show that every locally compact roughly starlike Gromov hyperbolic space is roughly similar to any hyperbolic filling of its uniformized boundary when the uniformization index \( \varepsilon \) is small enough to guarantee that the uniformized space is a uniform space.

Let \( X \) be a locally compact roughly starlike Gromov hyperbolic space. For \( 0 < \varepsilon \leq \varepsilon_0(\delta) \), where \( \varepsilon_0(\delta) \) is given by Theorem 2.5, we uniformize \( X \), with uniformization point \( x_0 \), to obtain \( X_\varepsilon \) equipped with the metric \( d_\varepsilon \) and having boundary \( \partial_\varepsilon X \). It will be convenient to use the scaled metric
\[
\tilde{d}_\varepsilon = \frac{\varepsilon}{\varepsilon}d_\varepsilon
\]
on \( X_\varepsilon \), and correspondingly \( \tilde{\text{dist}}_\varepsilon \) and \( \tilde{\text{diam}}_\varepsilon \). A consequence is that \( \tilde{d}_\varepsilon(x_0) = 1/e \) and
\[
\frac{1}{e} \leq \tilde{\text{diam}}_\varepsilon \partial_\varepsilon X \leq \frac{2}{e} < 1
\]
and thus \( \tilde{\text{diam}}_\varepsilon \partial_\varepsilon X \leq 2/e < 1 \). Note that if \( X \) is the half-line, then \( \partial_\varepsilon X \) consists of just one point and thus has diameter 0.

**Definition 6.1.** Let \( (W, d_W) \) and \( (Y, d_Y) \) be metric spaces. A (not necessarily continuous) map \( \Phi : W \to Y \) is a rough similarity if there are \( C \geq 0 \) and \( L \geq 1 \) such that every point in \( Y \) is at most a distance \( C \) from \( \Phi(W) \), and for all \( x, x' \in W \),
\[
Ld_W(x, x') - C \leq d_Y(\Phi(x), \Phi(x')) \leq Ld_W(x, x') + C.
\]

We refer interested readers to Bonk–Schramm [12, Section 2] for more on rough similarity between Gromov hyperbolic spaces.

**Theorem 6.2.** Assume that \( X \) is a locally compact roughly starlike Gromov hyperbolic space and that \( 0 < \varepsilon \leq \varepsilon_0(\delta) \), where \( \varepsilon_0(\delta) \) is given by Theorem 2.5. Let \( Z = \partial_\varepsilon X \) be the uniformized boundary of \( X \) equipped with the metric \( \tilde{d}_\varepsilon \). Let \( \tilde{X} \) be
any hyperbolic filling of $Z$, constructed with parameters $\alpha, \tau > 1$ and the maximal $\alpha^{-n}$-separated sets $A_n \subset Z$.

Consider the mapping $\Phi : X \to \hat{X}$, defined for $x \in X$ with

$$\alpha^{-n-1} < \hat{d}_\varepsilon(x) \leq \alpha^{-n}, \quad n = 0, 1, \ldots,$$

by $\Phi(x) = (z, n)$, where $z$ is chosen to be a nearest point in $A_n$ to $x$, i.e. $\hat{d}_\varepsilon(x, z) = \text{dist}_\varepsilon(x, A_n)$.

Then $\Phi$ is a rough similarity with $L = (\log \alpha)/\varepsilon = 1/\sigma$ and $C$ depending only on $\alpha, \tau, \varepsilon_0(\delta), \delta$ and $M$.

By combining Bonk–Schramm [12, Theorem 8.2] with Proposition 4.4 we know that $X$ and $\hat{X}$ are roughly similar if $\varepsilon > 0$ is small enough to guarantee that $\partial X_\varepsilon$ is snowflake equivalent to the visual boundary of $X$ as in Bonk–Heinonen–Koskela [8]. However, as this rough similarity is obtained from [12] via comparison with the cone $\text{Con}(\partial Y_\varepsilon)$, here we give a more direct construction of the rough similarity between $X$ and $\hat{X}$. In so doing, we also demonstrate how the parameters $\alpha, \tau$ and $\varepsilon$ affect the rough similarity constants.

We chose to point out the dependence on $\varepsilon_0(\delta)$ separately in Theorem 6.2 even though $\varepsilon_0(\delta)$ is supposed to be determined solely by $\delta$, because should a better upper bound for $\varepsilon$ in Theorem 2.5 and Corollary 2.7 be known in the future, the result in this theorem will also be valid for the enlarged range of $\varepsilon$. Note that $Z = \partial X$ is compact by Theorem 2.5, and thus so is $A_n$, which shows that the nearest points above exist. Proposition 7.3 shows that $Z$ can be non doubling.

To prove this theorem, we need the following lemma. Let $N_0$ be the smallest integer $\geq 1/\log \alpha$. For each $n \geq N_0$ we set

$$S_n = \{x \in X_\varepsilon : \hat{d}_\varepsilon(x) = \alpha^{-n}\}. \quad (6.3)$$

By the choice of $N_0$ we know that $S_n \neq \emptyset$.

**Lemma 6.3.** Suppose that the assumptions in Theorem 6.2 hold. Fixing $n \geq N_0$, let $z \in \partial X$ and let $x$ be a nearest point in $S_n$ to $z$. Then

$$\alpha^{-n} = \hat{d}_\varepsilon(x) \leq \hat{d}_\varepsilon(z, x) \leq \alpha^{-n} e.$$

As noted above, $\overline{X}_\varepsilon$ is compact. Therefore $S_n$ is also compact, and hence the nearest points referred to above do exist.

**Proof.** Find a sequence $z_k \in X$ such that $z_k \to z$ with respect to $\hat{d}_\varepsilon$. Since $X$ is roughly starlike, there is a sequence of arc length parametrized $d_X$-geodesic rays $\gamma_k : [0, \infty) \to X$ starting from $x_0$, and a sequence of points $w_k \in \gamma_k$, such that $\text{dist}_X(z_k, w_k) \leq M$. For all $y \in [z_k, w_k]$ (where $[z_k, w_k]$ is any $d_X$-geodesic from $z_k$ to $w_k$),

$$d_X(y, x_0) \geq d_X(z_k, x_0) - M \quad \text{and hence} \quad \rho_\varepsilon(y) \leq \varepsilon^M \rho_\varepsilon(z_k).$$

It follows that

$$\hat{d}_\varepsilon(z_k, w_k) \leq \varepsilon e \int_{[z_k, w_k]} \rho_\varepsilon ds \leq \frac{M \varepsilon}{e} \varepsilon^M \rho_\varepsilon(z_k) \to 0, \quad \text{as} \ k \to \infty,$$

showing that $w_k \to z$. The ray $\gamma_k$ intersects $S_n$ and for sufficiently large $k$ there exists $y_k = \gamma_k(t_k) \in \gamma_k \cap S_n$ such that $w_k \in \gamma_k([t_k, \infty))$. As $y_k \in S_n$, using (2.2) we see that

$$\alpha^{-n} = \hat{d}_\varepsilon(y_k) = \frac{\varepsilon}{e} \hat{d}_\varepsilon(y_k) \geq \frac{\varepsilon^{-\varepsilon d_X(y_k, x_0)}}{e^2} = \frac{e^{-\varepsilon t_k}}{e^2}. $$
Consequently,
\[ \hat{d}_x(w_k, y_k) \leq \frac{\varepsilon}{e} \int_{t_k}^{\infty} e^{-at} \, dt = \frac{e^{-at_k}}{e} \leq \alpha^{-n} e. \]
This finally implies, since \( w_k \to z, \) that
\[ \text{dist}_x(z, S_n) \leq \hat{d}_x(z, y_k) \leq \hat{d}_x(z, w_k) + \hat{d}_x(w_k, y_k) \leq \hat{d}_x(z, w_k) + \alpha^{-n} e \to \alpha^{-n} e, \]
as \( k \to \infty, \)
which proves the last inequality in the statement of the lemma. The remaining (in)equalities are clear from the definitions.

\( \square \)

**Proof of Theorem 6.2.** Let \( x, x' \in X. \) Let \( n \) and \( m \) be the largest integers such that \( \hat{d}_x(x) \leq \alpha^{-n} \) and \( \hat{d}_x(x') \leq \alpha^{-m}, \) respectively. Note that \( n, m \geq 0 \) by (6.1). Let \( v = \Phi(x) = (z, n) \) and \( w = \Phi(x') = (y, m), \) with \( z \in A_n \) and \( y \in A_m. \) Note that
\[ \hat{d}_x(x, z) < 2\alpha^{-n} \quad \text{and} \quad \hat{d}_x(x', y) < 2\alpha^{-m}. \tag{6.4} \]
The triangle inequality and Lemma 3.3 imply that
\[ \hat{d}_x(x, x') \leq \hat{d}_x(z, y) + 2(\alpha^{-n} + \alpha^{-m}) \approx \alpha^{-(v,w)\circ_0}. \]
It then follows from Corollary 2.7 that either \( \varepsilon \hat{d}_X(x, x') < 1 \) or
\[ \exp(\varepsilon \hat{d}_X(x, x')) \simeq \frac{\hat{d}_x(x, x')^2}{\hat{d}_x(x) \hat{d}_x(x')} \leq \frac{\alpha^{-2(v,w)\circ_0}}{\hat{d}_x(x) \hat{d}_x(x')} \leq \frac{\alpha^{-(n+m-\hat{d}_x(x, w))}}{\alpha^{-n} \alpha^{-m-1}} \approx \alpha^{d_X(v, w)}. \]
Taking logarithms proves the first inequality in (6.2) of Definition 6.1.

For the second inequality we distinguish two cases. Assume, without loss of generality, that \( n \leq m. \) If \( \hat{d}_x(x, x') > (1/\alpha) \hat{d}_x(x), \) then \( \alpha^{-n} < \alpha \hat{d}_x(x) \lesssim \hat{d}_x(x, x') \) and hence, by Lemma 3.3 and (6.4),
\[ \alpha^{-(v,w)\circ_0} \approx \hat{d}_x(z, y) + \alpha^{-n} + \alpha^{-m} < \hat{d}_x(x, x') + 3(\alpha^{-n} + \alpha^{-m}) \lesssim \hat{d}_x(x, x'). \]
This together with Corollary 2.7 implies that
\[ \alpha^{d_X(v, w)} = \frac{\alpha^{-2(v,w)\circ_0}}{\alpha^{-n} \alpha^{-m}} \lesssim \frac{\hat{d}_x(x, x')^2}{\hat{d}_x(x) \hat{d}_x(x')} \lesssim \exp(\varepsilon \hat{d}_X(x, x')). \]
Taking logarithms proves the second inequality in (6.2) in this case.

If \( \hat{d}_x(x, x') \leq (1 - 1/\alpha) \hat{d}_x(x), \) then
\[ \alpha^{-m} \geq \hat{d}_x(x') \geq \hat{d}_x(x) - \hat{d}_x(x, x') \geq \frac{\hat{d}_x(x)}{\alpha} \geq \alpha^{-n-2}, \]
and hence \( n + 1 \geq m \geq n. \) Let \( l \) and \( t \) be the smallest nonnegative integers such that \( \alpha^{-l} \leq \tau - 1 \) and \( 5 \leq \alpha^t. \) Then, by (6.4),
\[ \hat{d}_x(z, y) \leq \hat{d}_x(z, x) + \hat{d}_x(x, x') + \hat{d}_x(x', y) < 2\alpha^{-n} + \hat{d}_x(x) + 2\alpha^{-m} \leq 5\alpha^{-n} \leq \alpha^{l-n}. \]
If \( n \geq t, \) then by Lemma 4.3 (with \( k = n - t \geq 0), \)
\[ \hat{d}_X(v, w) \leq n + m + 1 + 2l - 2(n - t) \leq 2(l + t + 1). \]
If on the other hand \( n < t, \) then
\[ \hat{d}_X(v, w) \leq n + m \leq 2n + 1 < 2l + 1 \leq 2(l + t + 1). \]
Thus, the second inequality in (6.2) holds also in the case \( \hat{d}_X(x, x') \leq (1 - 1/\alpha)\hat{d}_X(x) \) by choosing \( C \geq 2(l + t + 1) \).

To verify that some \( C' \)-neighborhood of \( \Phi(X) \) contains \( \hat{X} \), note that every point in \( \hat{X} \) is within a distance \( \frac{1}{2} \) of the set \( V \) of vertices in \( \hat{X} \). So it suffices to show that if \( (y, n) \in V \), then there is some \( x \in X \) such that \( \hat{d}_X((y, n), \Phi(x)) \leq C'' \).

As before, let \( l \) and \( t \) be the smallest nonnegative integers such that \( \alpha^{-l} \leq \tau - 1 \) and \( 5 \leq \alpha^t \). Note that \( t \geq 1 \). Recall the definition of \( S_\alpha \) from (6.3). Let \( x \) be a nearest point in \( S_{n+t+l} \) to \( y \). Then \( \hat{d}_X(y, x) \leq \alpha^{-n-l-t}e \) by Lemma 6.3. By the construction of \( \Phi \), the point \( x \) has a nearest point \( z_{n+l+t} \) in \( A_{n+l+t} \) such that \( \hat{d}_X(x, z_{n+l+t}) < 2\alpha^{-n-l-t} \). For \( j = n, \ldots, n + l + t - 1 \), find \( z_j \in A_j \) such that \( \hat{d}_X(z_j) < \alpha^{-j} \). Hence, by the choice of \( l \) and \( t \),

\[
\hat{d}_X(y, z_n) \leq \hat{d}_X(y, x) + \hat{d}_X(x, z) + \hat{d}_X(z, z_n) < \alpha^{-n-l-t}e + 2\alpha^{-n-l-t} + \alpha^{-n} < \tau\alpha^{-n}.
\]

Since also \( y \in \tau B_Z(y, \alpha^{-n}) \) and \( z \in B_Z(z_j, \alpha^{-j}) \), \( j = n, \ldots, n + l + t \), we see that

\[
(y, n) \sim (z_n, n) \sim (z_{n+1}, n + 1) \sim \cdots (z_{n+l+t}, n + l + t) = \Phi(x),
\]

where the first edge collapses into a single vertex if \( y = z_n \). This implies that \( d_X((y, n), \Phi(x)) \leq l + 1 \).

\[\Box\]

7. Trees

In this section, we will obtain a sharper version of Theorem 6.2 in the case when \( X \) is a tree, namely we get an isometry rather than a mere rough similarity, provided that the parameters are chosen appropriately. Note that since \( X \) is a rooted tree, \( \delta = 0 \) and we can choose \( \epsilon(0) \) arbitrarily, by Theorem 2.5, so in this case there is no upper bound on \( \epsilon \) in Theorem 6.2. Recall that an isometry is a 1-bilipschitz map, i.e. a rough similarity with \( L = 1 \) and \( C = 0 \).

**Theorem 7.1.** Let \( X \) be a metric tree, rooted at \( x_0 \), such that every vertex \( x \in X \) has at least one child. Consider the uniformized boundary \( \partial_x X \) of \( X \), with parameter \( \epsilon > 0 \), and let \( 1 < \tau < \alpha = e^\epsilon \) be fixed. Let \( Z = \partial_x X \) be equipped with the metric

\[
d_Z(\zeta, \xi) := \frac{\epsilon\tau}{2\alpha} d_x(\zeta, \xi).
\]

Then \( X \) is isometric to any hyperbolic filling of \( Z \), constructed with the parameters \( \alpha \) and \( \tau \).

Note that \( \text{diam } Z \leq \tau/\alpha < 1 \). In Remark 7.2 below we show that if \( \tau \geq \alpha \), then the hyperbolic filling is never a tree. Thus the range \( 1 < \tau < \alpha \) for \( \tau \) in Theorem 7.1 is optimal.

**Proof.** Let \( \hat{X} \) be a hyperbolic filling of \( Z \), constructed from a maximal \( \alpha^{-n} \)-separated set \( A_n \subset Z \), with parameters \( \tau \) and \( \alpha \). It suffices to show that the sets of vertices in \( X \) and \( \hat{X} \), respectively, are isometric, since the extension to the edges is straightforward.

To start with, note that if \( \zeta, \xi \in Z \) have a common ancestor \( x \in X \) at distance \( n \geq 0 \) from the root \( x_0 \), then

\[
d_Z(\zeta, \xi) \leq \frac{\epsilon\tau}{\alpha} \int_n^\infty e^{-ct} dt = \tau\alpha^{-n-1} < \alpha^{-n}, \quad (7.1)
\]

with equality if and only if \( \zeta \) and \( \xi \) do not have a common ancestor at distance \( n + 1 \) from \( x_0 \).
It follows that for every $n \geq 0$, the metric space $Z$ can be written as a finite union of open balls of radius $\alpha^{-n}$, namely those consisting exactly of all descendants in $Z$ of some vertex $x \in X$ with $d_X(x, x_0) = n$. Moreover, every two points in such a ball satisfy (7.1), and these balls are disjoint and can also be written as balls centered at any of the points in it, with radius $\tau \alpha^{-n}$. Indeed, if $d_Z(\zeta, \eta) < \tau \alpha^{-n}$, then we know by (7.1) and the comment after it that $\zeta$ and $\eta$ have a common ancestor at distance $n$ from the root $x_0$, and so $d_Z(\zeta, \eta) \leq \tau \alpha^{-n}) < \alpha^{-n}$. Thus, $A_n$ contains exactly one point in each of these balls and this correspondence defines a bijection $F$ between the vertices in $X$ at level $n$ and the set $A_n \subset Z$. More precisely, $F(x)$ is the unique descendant of $x$ belonging to $A_n$. Define the mapping $\hat{F}$ from vertices in $X$ to vertices in $\hat{X}$ as

$$\hat{F}(x) = (F(x), d_X(x, x_0)).$$

To show that $\hat{F}$ is an isometry between the two sets of vertices, assume that $x \sim y$ in $X$. Without loss of generality, we may assume that $x$ is a parent of $y$ and that $d_X(x, x_0) = n$. Then both $F(x)$ and $F(y)$ have $x$ as a common ancestor and hence by (7.1), $d_Z(F(x), F(y)) < \alpha^{-n}$, which yields

$$F(y) \in B_Z(F(x), \alpha^{-n}) \cap B_Z(F(y), \alpha^{-n-1}).$$

Therefore $\hat{F}(x) = (F(x), n) \sim (F(y), n+1) = \hat{F}(y)$.

Conversely, assume that $\hat{F}(x) \sim \hat{F}(y)$. Then $|n - m| \leq 1$, where $n = d_X(x, x_0)$ and $m = d_X(y, x_0)$. By the construction of $\hat{X}$, there exists $\zeta \in Z$ such that

$$d_Z(\zeta, F(x)) < \tau \alpha^{-n} \quad \text{and} \quad d_Z(\zeta, F(y)) < \tau \alpha^{-m}.$$

As pointed out after (7.1), the first inequality implies that $\zeta$ and $F(x)$ have a common ancestor at distance $n$ from the root $x_0$ and this ancestor must be $x$ since there is only one ray in $X$ from $x_0$ to $F(x)$. Similarly, $\zeta$ and $F(y)$ have $y$ as a common ancestor at distance $m$ from the root $x_0$.

Because there is only one ray in $X$ from $x_0$ to $\zeta$, this implies that $x = y$ when $m = n$ and contradicts the assumption $\hat{F}(x) \sim \hat{F}(y)$. Consequently, there are no horizontal edges in $\hat{X}$. If $m \neq n$, then we can assume that $m = n + 1$ and conclude that $x$ is the parent of $y$ in the above ray, and so $x \sim y$. Thus, $\hat{F} : X \to \hat{X}$ is an isometry.

**Remark 7.2.** If $Z$ is not a singleton in Theorem 7.1 (that is, $X$ is not the half-line), then for $\tau \geq \alpha$ and a scaling of $d_Z$ so that diam $Z < 1$, the hyperbolic filling $\hat{X}$ of $Z$ will always contain horizontal edges and is thus not a tree.

More precisely, let $d_Z(\zeta, \xi) := \frac{1}{2} K \text{diam}_Z(\zeta, \xi)$, where $0 < \kappa < 1$, so that diam $Z \leq \kappa < 1$. Let $l \geq 0$ be the smallest integer such that $\kappa < \alpha^{-l}$. Then (7.1) becomes $d_Z(\zeta, \xi) \leq \kappa \alpha^{-n} < \alpha^{-n-l}$, and so $Z$ can be written as a finite disjoint union of open balls of radius $\alpha^{-n-l}$, each consisting exactly of all descendants in $Z$ of some vertex $x \in X$ with $d_X(x, x_0) = n$. Thus, $A_{n+l}$ contains exactly one point in each of these balls.

However, if $\zeta, \xi \in A_{n+l}$ are descendants of two distinct vertices $x, y \in X$ at distance $n$ from the root, having the same parent, then because $\tau \geq \alpha$, we have

$$d_Z(\zeta, \xi) \leq \kappa \alpha^{1-n} < \alpha^{1-n-l} \leq \tau \alpha^{n-l}.$$ 

We therefore see that $\zeta \in \tau B_Z(\zeta, \alpha^{-n-l}) \cap \tau B_Z(\xi, \alpha^{-n-l})$ and thus the vertices $(\zeta, n+l)$ and $(\xi, n+l)$ in $\hat{X}$ will be neighbors connected by a horizontal edge.

We also give the following characterizations.
Proposition 7.3. Let $X$ be a rooted tree such that every vertex $x \in X$ has at least one child, and let $\varepsilon > 0$. Then the following are true:

(a) The uniformized boundary $\partial_\varepsilon X$ is compact if and only if every vertex has a finite number of children.

(b) The uniformized boundary $\partial_\varepsilon X$ is doubling if and only if there is a uniform bound on the number of children for each vertex.

Proof. Let $1 < \tau < \alpha = e^\varepsilon$ and let $\hat{X}$ be any hyperbolic filling of $\partial_\varepsilon X$, with parameters $\alpha$ and $\tau$. By Theorem 7.1, $\hat{X}$ is isometric to $X$. Part (a) now follows from Proposition 4.6, while part (b) follows from Proposition 4.5.

8. Geodesics in the hyperbolic filling

In this section, except for Examples 8.7 and 8.8, we fix an arbitrary parameter $\alpha > 1$ and assume that

\[
\tau \geq \frac{\alpha + 1}{\alpha - 1}. \tag{8.1}
\]

We also assume that $Z$ is a metric space with $\text{diam} \ Z < 1$, and let $X$ be a hyperbolic filling of $Z$ with the parameters $\alpha$ and $\tau$.

In this section we study how the geodesics in a hyperbolic filling look like under the above restriction relating $\tau$ and $\alpha$. We do not use these precise properties of geodesics in the rest of the paper, and so in the other sections we do not require this limit on $\tau$. However, in other applications the structure of the geodesics is quite useful to know. As we gain control of the geodesics in a straightforward manner under the above constraint on $\tau$, we have included this study here as well for the convenience of the reader and for possible future applications. In Example 8.7 we show that most of the properties obtained in this section can fail when $\tau$ is close to 1. We end the section with Example 8.8 showing that when $\tau = 1$ it is possible for the “hyperbolic filling” to be nonhyperbolic.

Uniformizations will not play any role in this section. We will only study geodesics between vertices in $X$ and all the geodesics are with respect to the $d_X$-metric.

Lemma 8.1. Assume that (8.1) holds. If $(x, n) \sim (z, n+1) \sim (y, n+1)$ is a segment in a path with $x \neq y$, then $(x, n) \sim (y, n)$ and the path is not a geodesic.

Proof. By the hypothesis of this lemma, we have $B_Z(x, \alpha^{-n}) \cap B_Z(z, \alpha^{-(n+1)}) \neq \emptyset$ and so, using (8.1),

\[
d_Z(x, z) < \frac{\alpha + 1}{\alpha} \alpha^{-n} < \tau \alpha^{-n} \quad \text{and similarly,} \quad d_Z(y, z) < \tau \alpha^{-n}.
\]

It follows that $z \in \tau B_Z(x, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-n})$ and consequently $(x, n) \sim (y, n)$. So the length of the path can be reduced by one by replacing the segment $(x, n) \sim (z, n+1) \sim (y, n)$ with the edge $(x, n) \sim (y, n)$. Hence it cannot be a geodesic.

Lemma 8.2. Assume that (8.1) holds. If $(x_1, n) \sim (x_2, n+1) \sim (y_2, n+1) \sim (y_1, n)$ is a segment in a path with $x_1 \neq y_1$, then $(x_1, n) \sim (y_1, n)$ and the path is not a geodesic.

Proof. Since $(x_2, n+1) \sim (y_2, n+1)$, there exists

\[
b \in \tau B_Z(x_2, \alpha^{-(n+1)}) \cap \tau B_Z(y_2, \alpha^{-(n+1)}).
\]
Similarly, as \((x_1, n) \sim (x_2, n + 1)\), we have \(B_Z(x_1, \alpha^{-n}) \cap B_Z(x_2, \alpha^{-(n+1)}) \neq \emptyset\), and so
\[
d_Z(x_1, x_2) < \frac{\alpha + 1}{\alpha} \alpha^{-n}.
\]
Therefore,
\[
d_Z(x_1, b) \leq d_Z(x_1, x_2) + d_Z(x_2, b) < \frac{\alpha + 1}{\alpha} \alpha^{-n} + \tau \alpha^{-(n+1)} \leq \tau \alpha^{-n}.
\]
Similarly, \(d_Z(y_1, b) < \tau \alpha^{-n}\). Hence \(b \in \tau B_Z(x_1, \alpha^{-n}) \cap \tau B_Z(y_1, \alpha^{-n})\), which shows that \((x_1, n) \sim (y_1, n)\).

Finally, replacing the segment \((x_1, n) \sim (x_2, n + 1) \sim (y_2, n + 1) \sim (y_1, n)\) with the edge \((x_1, n) \sim (y_1, n)\) reduces the length of the path by 2, and thus the original path is not a geodesic. \(\square\)

Next, we show that there are no geodesics going first down (i.e. away from the root) and then back up. The first part of this lemma will also be used when proving the structure Lemma 8.6.

**Lemma 8.3.** Assume that (8.1) holds. If \((x, n) \sim (y, n+1) \sim (z, n+1)\) is a segment in a geodesic, then there is some \(y' \in A_n\) such that \((x, n) \sim (y', n) \sim (z, n+1)\), and replacing the segment \((x, n) \sim (y, n+1) \sim (z, n+1)\) with \((x, n) \sim (y', n) \sim (z, n+1)\) also gives a geodesic.

Consequently, if \((x_0, n_0) \sim (x_1, n_1) \sim \ldots \sim (x_m, n_m)\) is a geodesic, then there are no indices \(0 \leq i < j < k \leq m\) with \(n_j = \max\{n_i, n_k\}\).

**Proof.** By the choice of \(A_n\), there is some \(y' \in A_n\) such that \(d_Z(z, y') < \alpha^{-n}\). We immediately have that \((y', n) \sim (z, n+1)\). Since we started with a geodesic, we see that \(x \neq y'\). Lemma 8.2 therefore implies that \((x, n) \sim (y', n)\). Since \((x, n) \sim (y, n+1) \sim (z, n+1)\) is a geodesic segment, it follows that \((x, n) \sim (y', n) \sim (z, n+1)\) is also a geodesic segment, which proves the first claim.

Next, assume that there would exist a geodesic violating the second part. Because of Lemma 8.1, after restricting to a subpath we may assume that it is of the form
\[
(x_0, n) \sim (x_1, n+1) \sim \ldots \sim (x_{m-1}, n+1) \sim (x_m, n).
\]
Applying the first part of the lemma iteratively shows that there are \(y_1, \ldots, y_{m-2}\) such that
\[
(x_0, n) \sim (y_1, n) \sim \ldots \sim (y_{m-2}, n) \sim (x_{m-1}, n+1) \sim (x_m, n).
\]
As it has the same length as (8.2), it is also a geodesic, but this contradicts Lemma 8.1. \(\square\)

**Lemma 8.4.** Assume that (8.1) holds. Let \(v = (x, n) \in V\). Then the following are true:
(a) If
\[
v_0 \sim v_1 \sim \ldots \sim v_n = v\quad \text{and} \quad v_0 \sim w_1 \sim \ldots \sim w_n = v
\]
are two geodesics, then for \(j = 1, \ldots, n-1\) we have \(v_j \sim w_j\).
(b) If \(\bar{v} = (y, n-1) \sim v\), then there is a geodesic \(v_0 \sim v_1 \sim \ldots \sim v_n = v\) such that \(v_{n-1} = \bar{v}\).

In Example 8.7 below we show that (a) can fail drastically if \(\tau\) is close to 1. Recall that here we assume that \(\tau\) satisfies (8.1), and that \(v_0\) is the root of \(X\). Part (b) holds for any hyperbolic filling, also without the requirement (8.1).
Proof. To verify (a), note that by Lemma 3.1, \( v_j = (x_j, j) \) and \( w_j = (y_j, j) \) for some \( x_j, y_j \in A_j \), \( j = 0, \ldots, n \). Then
\[
 w_{n-1} = (y_{n-1}, n-1) \sim v = (x, n) \sim (x_{n-1}, n-1) = v_{n-1}
\]
and it follows from Lemma 8.1 that \( v_{n-1} \sim w_{n-1} \). Now an inductive application of Lemma 8.2 gives the desired conclusion.

The second claim follows from the fact that the concatenation of the edge \( v \sim \hat{v} \) to any of the geodesics connecting \( \hat{v} \) to the root vertex \( v_0 \) gives a geodesic. \( \square \)

**Lemma 8.5.** Assume that (8.1) holds. Let \( n_0 \) be the smallest positive integer such that
\[
 n_0 \alpha^{1-n_0} \leq \frac{1}{\alpha + 1}.
\]
Then there is no horizontal geodesic of length \( \geq 2n_0 \), i.e., if \( m \geq 2n_0 \) and
\[
 (y_0, n) \sim (y_1, n) \sim \ldots \sim (y_m, n),
\]
then (8.3) is not a geodesic.

If we drop the assumption (8.1) then the proof below shows that the same conclusion holds provided that \( n_0 \) is the smallest positive integer such that \( 2n_0 \alpha^{1-n_0} \leq 1 - 1/\alpha \).

Proof. We may assume that \( m = 2n_0 \). For \( 0 \leq j \leq n \), there is some \( x_j \in A_j \) so that \( y_0 \in B_Z(x_j, \alpha^{-j}) \). Necessarily, \( x_0 = z_0 \) and \( x_n = y_0 \). Then for each \( j = 0, \ldots, n-1 \) we have that \( y_0 \in B_Z(x_j, \alpha^{-j}) \cap B_Z(x_{j+1}, \alpha^{-j-1}) \), and so
\[
 (y_0, n) \sim (x_{n-1}, n-1) \sim (x_{n-2}, n-2) \sim \ldots \sim (x_1, 1) \sim v_0.
\]
Similarly, we can find \( z_j \in A_j \) so that \( y_m \in B_Z(z_j, \alpha^{-j}) \), \( 1 \leq j < n \), and
\[
 (y_m, n) \sim (z_{n-1}, n-1) \sim (z_{n-2}, n-2) \sim \ldots \sim (z_1, 1) \sim v_0.
\]
If \( n \leq n_0 - 1 \), then combining (8.4) and (8.5) gives us a path from \( (y_0, n) \) to \( (y_m, n) \) (through the root \( v_0 \)) of length at most \( 2n \leq 2(n_0 - 1) < m \), and thus (8.3) is not a geodesic.

Now assume that \( n \geq n_0 \). Since
\[
 d_Z(y_j, y_{j+1}) < 2\tau\alpha^{-n} \quad \text{for} \quad 0 \leq j \leq n_0 - 1,
\]
we get that \( d_Z(y_0, y_{n_0}) < 2n_0\tau\alpha^{-n} \). Let \( k = n + 1 - n_0 \). Then \( 1 \leq k \leq n \) and
\[
 d_Z(x_k, y_{n_0}) \leq d_Z(x_k, y_0) + d_Z(y_0, y_{n_0}) < \alpha^{-k} + 2n_0\tau\alpha^{-n} = \alpha^{-k}(1 + 2n_0\tau\alpha^{1-n_0}) \leq \tau\alpha^{-k},
\]
and in particular \( y_{n_0} \in \tau B_Z(x_k, \alpha^{-k}) \). Similarly, \( y_{n_0} \in \tau B_Z(z_k, \alpha^{-k}) \), and thus
\[
 (x_k, k) \sim (z_k, k).
\]
It follows that
\[
 (y_0, n) \sim (x_{n-1}, n-1) \sim \ldots \sim (x_k, k) \sim (z_k, k) \sim \ldots \sim (y_{2n_0}, n)
\]
is a path of length \( 2(n - k) + 1 = 2n_0 - 1 < 2n_0 \) showing that (8.3) is not a geodesic. \( \square \)

More general geodesics can be more complicated. However, we have the following lemma, which can be used to obtain potentially simpler geodesics.
Lemma 8.6. Assume that (8.1) holds. If $v = (x, k_0)$ and $w = (y, k_m)$ are two distinct vertices with $d_X(v, w) = m$, then there is a geodesic

$$v = (x_0, k_0) \sim (x_1, k_1) \sim \ldots \sim (x_{m-1}, k_{m-1}) \sim (x_m, k_m) = w$$

consisting of at most two vertical and one horizontal segments. More precisely, there exist integers $0 \leq j_0 \leq j_1 \leq m$, with $j_1 - j_0 \leq 2n_0 - 1$ where $n_0$ is as in Lemma 8.5, such that

$$k_{j+1} = k_j - 1 \text{ for } 0 \leq j < j_0,$$

$$k_{j+1} = k_j + 1 \text{ for } j_1 < j \leq m,$$

$$k_j = k_{j_0} = k_{j_1} \text{ for } j_0 \leq j \leq j_1.$$

This geodesic minimizes $\sum_j k_j$ over all the geodesics between $v$ and $w$, and has a similar shape to the path identified in the latter part of Lemma 4.3.

Proof. As $X$ is connected, there is a geodesic between $v$ and $w$. By the second part of Lemma 8.3 this geodesic does not contain any subpath going first down and then up. We can therefore modify this geodesic iteratively using the first part of Lemma 8.3 to obtain a geodesic of the type described above. That $j_1 - j_0 \leq 2n_0 - 1$ follows from Lemma 8.5.

We end this section with two examples. In the first one we show that, in contrast to the fact obtained in Lemma 8.4(a), the distance between different geodesics connecting a pair of points can be large when $\tau$ is small in comparison with $\alpha$.

The first example is tailored so that it can be used iteratively in the second example, producing a nonhyperbolic “hyperbolic filling” when $\tau = 1$. This illustrates the dependence of the Gromov constant $\delta$ on $\alpha$ and $\tau$.

Example 8.7. Let $n \geq 3$, $\alpha = 2$, $1 \leq \tau < \frac{5}{4}$ and $\frac{1}{4}(\tau - 1) \leq \rho < 2^{-n-1}$ with $\rho > 0$ as well when $\tau = 1$. Set

$$z_0 = 0, \quad z_1 = \frac{1}{2}, \quad z_\pm = \frac{1}{2} \pm \rho \quad \text{and} \quad Z = [z_0, z_-] \cup [z_+, z_1] =: Z_- \cup Z_+.$$

Next we choose, $A_0, A_1, \ldots$, as follows:

$$A_0 = \{z_0\}, \quad A_1 = A_2 = \{z_0, z_1\},$$

$$A'_j = \{2^{-j}k : k = 0, 1, \ldots, 2^{j-1}\} \cap Z, \quad j = 3, 4, \ldots,$$

$$A_j = A'_j \setminus \{z_+\} \cup \{z_\pm\}, \quad j = 3, \ldots, n - 1,$$

$$A_j = \left(A'_j \setminus \{z_+\}\right) \cup \{z_\pm\}, \quad j = n, n + 1, \ldots,$$

where $z'_j$ is the point in $A'_j$ closest to $z_\pm$. We then construct a “hyperbolic filling” based on this. In the first three levels we have

$$(z_0, 0) \sim (z_0, 1) \sim (z_1, 1) \sim (z_0, 0),$$

$$(z_0, 2) \sim (z_0, 1) \sim (z_1, 2) \quad \text{and} \quad (z_0, 2) \sim (z_1, 1) \sim (z_1, 2).$$

Next, for $j = 2, \ldots, n - 1$, the distance between $A_j \cap Z_-$ and $z_+$, as well as between $A_j \cap Z_+$ and $z_-$, is

$$|z'_j - z_-| = |z'_j - z_+| = 2^{-j} + \rho \geq 2^{-j}.$$

Hence, there are no horizontal edges between the vertices $(z, j)$ and $(y, i)$ with

$$z \in A_j \cap Z_-, \quad y \in A_i \cap Z_+ \quad \text{for } 2 \leq i, j \leq n - 1. \quad (8.6)$$
On the other hand, since \( z_+ \in B_Z(z_-, 2^{-n}) \cap B_Z(z_+^{n-1}, 2^{1-n}) \), we see that
\[
(z_-, n) \sim (z_+^{n-1}, n - 1) \quad \text{and similarly,} \quad (z_+, n) \sim (z_+^{n-1}, n - 1).
\] (8.7)
Hence, there are (at least) two upward-directed geodesics \( \gamma_\pm \) between \((z_-, n)\) and \((z_0, 1)\), with \( \gamma_- \) passing only through vertices with the first coordinate in \( Z_- \), while the vertices in \( \gamma_+ \) have the first coordinate in \( Z_+ \), except for the starting and ending vertices. It follows that the midpoints in \( \gamma_+ \) and \( \gamma_- \) have distance \( \frac{1}{2}(n - 1) \) to \( \gamma_+ \) and \( \gamma_- \), respectively, and so the Gromov constant \( \delta \geq \frac{1}{2}(n - 1) \). (If \( n \) is even the midpoints of \( \gamma_\pm \) are not vertices.)

With \( n = 3 \), this also shows that (at least) for \( \tau < \frac{3}{4} \), it can happen that
\[
(z_0, 2) \sim (z_-, 3) \sim (z_1, 2), \quad \text{while} \quad (z_0, 2) \not\sim (z_1, 2),
\]
i.e. both conclusions in Lemma 8.1, the last conclusion in Lemma 8.3, as well as Lemma 8.4 (a) all fail in this case. Similarly, since
\[
(z_0, 2) \sim (z_+, 3) \sim (z_1, 2)
\]
and there is no \( y \) such that \((z_0, 2) \sim (y, 2) \sim (z_1, 3)\), the first conclusions in both Lemma 8.2 and 8.3 fail.

When \( n = 4 \), the geodesics \((\frac{1}{3}, 3) \sim (z_+, 4) \sim (\frac{2}{3}, 3)\) are the only geodesics between \((\frac{1}{3}, 3)\) and \((\frac{2}{3}, 3)\), and thus Lemma 8.6 fails. Moreover, the geodesic
\[
(z_0, 3) \sim (\frac{1}{3}, 4) \sim (z_-, 4) \sim (\frac{2}{3}, 3)
\]
vio-lates both conclusions of Lemma 8.2 and the last conclusion of Lemma 8.3.

**Example 8.8.** Let \( \alpha = 2 \) and \( \tau = 1 \). Let \( \{n_j\}_{j=0}^{\infty} \) be an increasing sequence of positive integers \( n_j \geq 3 \) and let \( N_k = \sum_{j=0}^{k} n_j \). Choose \( \rho_j < 2^{-n_j-1} \) and repeat the construction in Example 8.7 with \( n = n_j \) and \( \rho = \rho_j \), and call the resulting space \( Z_j \), \( j = 0, 1, \ldots \).

Now, replace the interval \([0, 2^{-N_0-1}] \subset Z_0\) by a \( 2^{-N_0}\)-scaled copy of \( Z_1 \) to form the new space
\[
Z_1' = Z_0 \setminus (z'_-, z'_+), \quad \text{where} \quad z'_\pm := 2^{-N_0}(\frac{1}{4} \pm \rho_1).
\]
The first two points \( z_0 \) and \( 2^{-N_0-1} \) in \( A_{N_0+1} \subset Z_1' \) are still next to each other and the corresponding vertices form the horizontal edge
\[
(z_0, N_0 + 1) \sim (2^{-N_0-1}, N_0 + 1)
\]
similarly to \((z_1, 1) \sim (z_1, 1)\) in \( Z_0 \).

On the other hand, in the following levels \( j = N_0 + 2, \ldots, N_1 - 1 \), similarly to (8.6), there are no horizontal edges between the left-most interval \([0, z'_-]\) and the rest of \( Z_j' \). At the same time, similarly to (8.7), the points \( z'_\pm \in A_{N_j} \) have upward-directed edges both to the interval \([0, z'_-]\) and the second interval in \( Z_j' \). Consequently, the vertex \((z'_-, N_1)\) has two upward-directed geodesics \( \gamma'_\pm \) to \((z_0, N_0 + 1)\), such that the midpoints of \( \gamma'_- \) and \( \gamma'_+ \) have distance \( \frac{1}{2}(n_1 - 1) \) to \( \gamma'_- \) and \( \gamma'_+ \), respectively.

Next, the interval \([0, 2^{-N_1-1}] \subset Z_1' \) can be replaced by a \( 2^{-N_1}\)-scaled copy of \( Z_2 \), i.e. we get the new space \( Z_2' = Z_1' \setminus (2^{-N_1}(\frac{1}{4} - \rho_2), 2^{-N_1}(\frac{1}{4} + \rho_2)) \). Continuing in this way, we obtain a compact doubling space
\[
Z' = \bigcap_{j=1}^{\infty} Z_j'.
\]
Moreover, if \( z \in Z' \) and \( 0 < r < \frac{1}{4} \), then \( m(B_Z(z, r) \cap Z') \simeq m(B_Z(z, r)) \), where \( m \) denotes the Lebesgue measure.

Since \( \lim_{j \to \infty} n_j = \infty \), we can for each \( k \) find two vertices having two upward-directed geodesics \( \tilde{\gamma}'_\pm \) between them such that the midpoint of \( \tilde{\gamma}'_+ \) has distance \( \geq k \) to \( \tilde{\gamma}'_- \), i.e. the hyperbolic filling of \( Z' \) does not satisfy the Gromov \( \delta \)-condition, and is thus not Gromov hyperbolic.
9. Measures, function spaces and capacities

In this section, we assume that $1 \leq p < \infty$ and that $(Y, d)$ is a metric space equipped with a complete Borel measure $\nu$ such that $0 < \nu(B) < \infty$ for all balls $B \subseteq Y$. We call $Y = (Y, d, \nu)$ a metric measure space.

In the rest of the paper we are interested in studying the metric space $Z$, considered in the previous sections, together with a doubling measure on $Z$ and Besov spaces on $Z$ with respect to this measure. In particular, as mentioned in the introduction, we wish to associate Besov functions on $Z$ with upper gradient-based Sobolev functions on the uniformization $X$, of the hyperbolic filling $X$ of $Z$. In this section we will explain the notions related to measures and function spaces.

We follow Heinonen–Koskela [26] in introducing upper gradients as follows (they are referred to as very weak gradients in [26]). For proofs of the facts on upper gradients and Newtonian functions discussed in this section, we refer the reader to Björn–Björn [2] and Heinonen–Koskela–Shanmugalingam–Tyson [27].

**Definition 9.1.** A Borel function $g : Y \to [0, \infty]$ is an upper gradient of a function $u : Y \to [-\infty, \infty]$ if for each nonconstant compact rectifiable curve $\gamma$ in $Y$, we have

$$|u(x) - u(y)| \leq \int_{\gamma} g ds.$$  \hspace{1cm} (9.1)

Here $x$ and $y$ denote the two endpoints of $\gamma$. The above inequality should be interpreted as also requiring that $\int_{\gamma} g ds = \infty$ if at least one of $u(x)$ and $u(y)$ is not finite. If $g$ is a nonnegative measurable function on $Y$ and if (9.1) holds for $p$-almost every curve (see below), then $g$ is a $p$-weak upper gradient of $u$.

We say that a property holds for $p$-almost every curve if the family $\Gamma$ of all nonconstant compact rectifiable curves for which the property fails has zero $p$-modulus, i.e. there is a Borel function $0 \leq \rho \in L^p(Y)$ such that $\int_{\gamma} \rho ds = \infty$ for every curve $\gamma \in \Gamma$. The $p$-weak upper gradients were introduced in Koskela–MacManus [33]. It was also shown therein that if $g \in L^p(Y)$ is a $p$-weak upper gradient of $u$, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of $u$ such that $\|g_j - g\|_{L^p(Y)} \to 0$.

If $u$ has an upper gradient in $L^p(Y)$, then it has a minimal $p$-weak upper gradient $g_u \in L^p(Y)$ in the sense that $g_u \leq g$ a.e. for every $p$-weak upper gradient $g \in L^p(Y)$ of $u$, see Shanmugalingam [46]. The minimal $p$-weak upper gradient is well defined up to a set of measure zero.

Following Shanmugalingam [45], we define a version of Sobolev spaces on $Y$.

**Definition 9.2.** A function $u : Y \to [-\infty, \infty]$ is in the Newtonian space $\tilde{N}^{1,p}(Y)$ if $\int_Y |u|^p \, d\nu < \infty$ and $u$ has a $p$-weak upper gradient $g \in L^p(Y)$. This space is a vector space and a lattice, equipped with the seminorm $\|u\|_{\tilde{N}^{1,p}(Y)}$ given by

$$\|u\|_{\tilde{N}^{1,p}(Y)} := \left( \int_Y |u|^p \, d\nu + \inf_g \int_Y g^p \, d\nu \right)^{1/p},$$

where the infimum is taken over all upper gradients $g$ of $u$, or equivalently all $p$-weak upper gradients $g$ of $u$ (see the comments above).

The Newtonian space $N^{1,p}(Y) = \tilde{N}^{1,p}(Y)/\sim$, where $\sim$ is the equivalence relation on $\tilde{N}^{1,p}(Y)$ given by $u \sim v$ if and only if $\|u - v\|_{\tilde{N}^{1,p}(Y)} = 0$. To specify the measure with respect to which the Newtonian space is taken, we will also write $\tilde{N}^{1,p}(Y, \nu)$ and $N^{1,p}(Y, \nu)$. 

Definition 9.3. The $C_p^Y$-capacity of a set $E\subset Y$ is defined as
\[ C_p^Y(E) = \inf_u \|u\|_{N^{1,p}(Y)}^p, \]
where the infimum is taken over all $u \in \tilde{N}^{1,p}(Y)$ satisfying $u \geq 1$ on $E$.

Note that since functions in $\tilde{N}^{1,p}(Y)$ are defined pointwise everywhere, the requirement $u \geq 1$ on $E$ in the definition of $C_p^Y(E)$ makes sense for an arbitrary set $E \subset Y$.

A property is said to hold quasieverywhere (q.e. or $C_p^Y$-q.e.) if the set of all points at which the property fails has $C_p^Y$-capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in \tilde{N}^{1,p}(Y)$, then $v \sim u$ if and only if $v = u$ q.e. Moreover, if $u, v \in \tilde{N}^{1,p}(Y)$ and $u = v$ a.e., then $u = v$ q.e. That means that the equivalence classes in $N^{1,p}(Y)$ are precisely made up of functions which are equal q.e., and not a.e. as in the usual Sobolev spaces. By an abuse of notation, just as for $L^p$-spaces, we will often not distinguish between a function in $\tilde{N}^{1,p}(Y)$ and the corresponding equivalence class in $N^{1,p}(Y)$.

**Definition 9.4.** We say that $\nu$ supports a $p$-Poincaré inequality if there exist $C, \lambda > 0$ such that for each ball $B = B(x, r)$ and for all integrable functions $u$ and upper gradients $g$ of $u$ on $\lambda B$,
\[ \int_B |u - u_B| \, dv \leq Cr \left( \int_B g^p \, dv \right)^{1/p}, \]
where $u_B := \frac{1}{|B|} \int_B u \, dv = \nu(B)^{-1} \int_B u \, dv$.

See Björn–Björn [2] and Heinonen–Koskela–Shanmugalingam–Tyson [27] for equivalent formulations of the $C_p^Y$ capacity and the $p$-Poincaré inequality.

**Remark 9.5.** We will primarily be interested in Newtonian spaces on the uniformization $X_\varepsilon$ of a hyperbolic filling of $Z$, and on its closure $\overline{X}_\varepsilon$, in both cases equipped with the measure $\mu_\beta, \beta > 0$, defined by (10.7) below. In particular, each edge in $X$ is measured by a multiple of the Lebesgue measure. It is then quite easy to see that the only family of nonconstant compact rectifiable curves in $X_\varepsilon$ which has zero $p$-modulus (with respect to $\mu_\beta$) is the empty family. Functions in Newtonian spaces are absolutely continuous on $p$-almost every line, see Shanmugalingam [45]. Thus all functions $u \in \tilde{N}^{1,p}(X_\varepsilon, \mu_\beta)$ are continuous on $X_\varepsilon$ and absolutely continuous on each edge. Moreover, $g_u = |du|/ds_\varepsilon$ a.e. on each edge, where $ds_\varepsilon$ denotes the arc length with respect to $d_\varepsilon$. In particular, each equivalence class in $N^{1,p}(X_\varepsilon, \mu_\beta)$ contains just one function, and that function is continuous. Moreover, points in $X_\varepsilon$ have positive capacity.

For functions on $\overline{X}_\varepsilon$, the situation is not quite as simple, but the following result will be useful. A function $u$ on $Y$ is $C_p^Y$-quasicontinuous if for each $\eta > 0$ there is an open set $G \subset Y$ with $C_p^Y(G) < \eta$ such that $u|_{Y \setminus G}$ is continuous. Note that any $E \subset \overline{X}_\varepsilon$ with $C_p^Y(E) = 0$ must satisfy $E \subset \partial_\varepsilon X$.

**Theorem 9.6.** (Björn–Björn–Shanmugalingam [4]) Assume that $Y$ is complete and that $\nu$ is doubling and supports a $p$-Poincaré inequality. Then every $u \in \tilde{N}^{1,p}(Y)$ is $C_p^Y$-quasicontinuous. Moreover, $C_p^Y$ is an outer capacity, i.e.
\[ C_p^Y(E) = \inf_{G \supset E} \sup_{G \text{ open}} C_p^Y(G). \]
We will use these facts together with our trace and extension results to show that Besov functions on $Z$ have $\text{Cap}_{B^p_{p,p}(Z)}$-quasicontinuous representatives (which is defined just as $C^p_{\nu}$-quasicontinuous), see Proposition 13.3. Similarly, we will obtain density of Lipschitz functions and existence of Hölder continuous representatives in Besov spaces using our trace and extension results, and corresponding theorems for Newtonian functions.

We now give the definition of Besov spaces on metric measure spaces.

Definition 9.7. Let $\theta > 0$. We say that $u \in L^p(Y)$ is in the Besov space $B^\theta_{p,p}(Y)$ if
\[
\|u\|_{B^\theta_{p,p}} := \int_Y \int_Y \frac{|u(\zeta) - u(\xi)|^p}{d(\zeta, \xi)^{p\theta}} \frac{\,d\nu(\xi)}{\nu(B(\zeta,d(\zeta,\xi)))} < \infty.
\]

Remark 9.8. Note that $B^\theta_{p,p}(Y)$ is a Banach space with the norm given by
\[
\|u\|_{B^\theta_{p,p}(Y)} = \|u\|_{B^\theta_{p,p}} + \|u\|_{L^p(Y)}.
\]

Indeed, it is clear that this function space is a normed vector space. To see that it is complete, we argue as follows. Let $\{u_k\}_{k=1}^\infty$ be a Cauchy sequence in $B^\theta_{p,p}(Y)$. Then it is a Cauchy sequence in $L^p(Y)$, and hence it is convergent to some function $u \in L^p(Y)$. By passing to a subsequence if necessary, we also ensure that $u_k \to u$ $\nu$-a.e. in $Y$. Setting the measure $\nu_0$ on $Y \times Y$ by
\[
\nu_0(E) = \int_E \frac{d(v \times \nu)(\xi, \zeta)}{d(\zeta, \xi)^{p\theta} \nu(B(\zeta,d(\zeta,\xi)))},
\]
and defining $v_k : Y \times Y \to \mathbb{R}$ as $v_k(\xi, \zeta) = u_k(\xi) - u_k(\zeta)$, we note that
\[
\|u_k\|_{B^\theta_{p,p}} = \|v_k\|_{L^p(Y \times Y, \nu_0)}.
\]
Thus, $\{v_k\}_{k=1}^\infty$ is also a Cauchy sequence in the complete space $L^p(Y \times Y, \nu_0)$, and so converges therein to a function $v : Y \times Y \to \mathbb{R}$. Again, by passing to yet another subsequence if necessary, we may also assume that $v_k \to v$ $\nu_0$-a.e. in $Y \times Y$. Setting $w : Y \times Y \to \mathbb{R}$ by $w(\xi, \zeta) = u(\xi) - u(\zeta)$, we see that necessarily $v = w$ $\nu_0$-a.e. in $Y \times Y$. Therefore $v_k \to w$ in $L^p(Y \times Y, \nu_0)$, that is, $u_k \to u$ in $B^\theta_{p,p}(Y)$.

We recall the following lemma. For a proof see Gogatishvili–Koskela–Shanmugalingam [21, Theorem 5.2 and (5.1)] (where the factor of 2 should be replaced with $\alpha > 1$).

Lemma 9.9. Assume that $\nu$ is doubling and $\theta > 0$. If $u \in B^\theta_{p,p}(Y)$, then
\[
\|u\|_{B^\theta_{p,p}} \approx \sum_{n=0}^{\infty} \int_Y \int_{B(\zeta,\alpha^{-n})} \frac{|u(\zeta) - u(\eta)|^p}{\alpha^{-n\theta p}} \,d\nu(\zeta) \,d\nu(\eta).
\]

Definition 9.10. We set the Besov capacity of $E \subset Y$ to be the number
\[
\text{Cap}_{B^\theta_{p,p}(Y)}(E) := \inf_u (\|u\|_{B^\theta_{p,p}} + \|u\|_{L^p(Y)}),
\]
where the infimum is taken over all $u \in B^\theta_{p,p}(Y)$ satisfying $u \geq 1$ a.e. on a neighborhood of $E$.

10. Lifting doubling measures from $Z$ to its hyperbolic filling $X$

From now on, we let $X$ be a hyperbolic filling, constructed with parameters $\alpha, \tau > 1$, of a compact metric space $Z$ with $0 < \text{diam} Z < 1$ and equipped with a doubling
measure ν. In this section, we also let X be the uniformization of X with parameter 0 < ε ≤ log α.

We now focus on lifting up ν from Z as follows: Recall that the vertices in X are denoted v = (z, n), where z belongs to the maximal α−n-separated set A₀ ⊂ Z. Note that n is the graph distance from the root v₀ := (z₀, 0) to (z, n). For (z, n) ∈ V we set
\[ \hat{\mu}(\{(z, n)\}) = \nu(B_Z(z, \alpha^{-n})). \]  
(10.1)
The measure μ on X is then given by “smearing out” \( \hat{\mu} \) to X: for a Borel set \( A \subset X \),
\[ \mu(A) = \sum_{v \in V} \sum_{w \sim v} (\hat{\mu}(\{v\}) + \hat{\mu}(\{w\})) \mathcal{L}(A \cap [v, w]), \]  
(10.2)
where \([v, w]\) denotes the unit interval that connects the two vertices \( v \) and \( w \), and \( \mathcal{L} \) denotes the Lebesgue measure.

Strictly speaking, it is \( \hat{\mu}(v) \deg(v) \) that is smeared out, but because X has uniformly bounded degree (by Proposition 4.5) this is comparable to \( \hat{\mu}(v) \).

Note that the vertex set \( V_\alpha \) of points in X that are at level n from the root is composed of a maximal \( \alpha^{-n} \)-separated set of points from Z. So by the work of Gill and Lopez [20], [36], we know that \( V_\alpha \), equipped with the neighborhood relationship inherited from V and with the measure \( \hat{\mu}|_{V_\alpha} \), is doubling and that a subsequence converges in the pointed measured Gromov–Hausdorff sense to a measure \( \hat{\mu}_\infty \) on Z such that \( \hat{\mu}_\infty \sim \nu \).

**Lemma 10.1.** Let \((z, n), (y, m) \in V\) with \((y, m) \sim (z, n)\). Then
\[ \frac{1}{C_d^N} \hat{\mu}(\{(z, n)\}) \leq \hat{\mu}(\{(y, m)\}) \leq C_d^N \hat{\mu}(\{(z, n)\}), \]
where \( N \) is the smallest integer such that \( 2^N \geq \alpha(1 + \tau) + \tau \) and \( C_d \) is the doubling constant associated with \( \nu \).

**Proof.** Since the two points are neighbors, we have that \(|n - m| \leq 1\) and thus \( \alpha^{-n} \leq \alpha^{1-m} \). Since \( \tau B_Z(z, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-m}) \neq \emptyset \) by the construction of the hyperbolic filling, every \( \zeta \in B_Z(z, \alpha^{-n}) \) satisfies
\[ d_Z(\zeta, y) \leq d_Z(\zeta, z) + d_Z(z, y) \leq \alpha^{-n} + \tau(\alpha^{-n} + \alpha^{-m}) \leq 2^N \alpha^{-m}, \]
and so \( B_Z(z, \alpha^{-n}) \subset 2^N B_Z(y, \alpha^{-m}) \). The doubling property of \( \nu \) then implies that
\[ \hat{\mu}(\{(z, n)\}) = \nu(B_Z(z, \alpha^{-n})) \leq C_d^N \nu(B_Z(y, \alpha^{-m})) = C_d^N \hat{\mu}(\{(y, m)\}). \]
Reversing the roles of \( z \) and \( y \) in the above argument gives the desired double inequality. \( \square \)

**Theorem 10.2.** Let \( Y \) be a metric graph equipped with the length metric \( d \) such that all edges have length 1 and assume that \( Y \) has uniformly bounded degree, i.e., every vertex has at most \( K \) neighbors. Let \( \mu \) be a discrete measure defined on the vertices of \( Y \) such that \( \hat{\mu}(v) \simeq \hat{\mu}(w) \) whenever \( v \sim w \), with comparison constants independent of \( v \) and \( w \). Consider the smeared out measure \( \mu \) on \( Y \) given by (10.2).

Then for each \( R_0 > 0 \) there is a constant \( C_0 \geq 1 \) such that for all balls \( B = B(x, r) \) with \( r \leq R_0 \), and every integrable function \( u \) and upper gradient \( g \) of \( u \) on \( B \),
\[ \mu(2B) \leq C_0 \mu(B) \quad \text{and} \quad \int_B |u - u_B| \, d\mu \leq C_0 r \int_B g \, d\mu. \]  
(10.3)
The constant \( C_0 \) depends only on \( R_0 \), \( K \) and the comparison constants in \( \hat{\mu}(v) \simeq \hat{\mu}(w) \).
\textbf{Proof.} Since $Y$ is a length space, it follows from Lemma 3.5 and Theorem 5.3 in Björn–Björn–Shanmugalingam \cite{5} that it suffices to consider only the case $R_0 = \frac{1}{2}$.

Let $v$ be a nearest vertex to the center $x$ of $B$, i.e. $d(x, v) = \text{dist}(x, V)$. As $r \leq \frac{1}{2}$, the ball $2B$ contains at most one vertex, namely $v$. Hence,

$$B \subset 2B \subset \bigcup_{w \sim v} [v, w] \quad \text{and} \quad d\mu = (\hat{\mu}(\{v\}) + \hat{\mu}(\{w\})) ds \simeq \hat{\mu}(\{v\}) ds \text{ on each } [v, w],$$

by Lemma 10.1. Thus

$$\mu(2B) \lesssim Kr\hat{\mu}(\{v\}) \lesssim \mu(B).$$

To prove the 1-Poincaré inequality in (10.3), observe that if $v \notin B$, then $B$ is an interval, and so the 1-Poincaré inequality for $B$ follows from the 1-Poincaré inequality on $\mathbb{R}$ and the fact that $d\mu = C_\beta d\mathcal{L}$ on $B$. On the other hand, if $v \in B$, then

$$B = \bigcup_{w \sim v} I_w, \quad \text{where } I_w = B \cap [v, w].$$

We therefore obtain from (10.4) and the definition of upper gradients that for each $w \sim v$,

$$\int_{I_w} |u - u(v)| d\mu \leq \int_{I_w} \int_{I_w} g(s) ds d\mu = \mu(I_w) \int_{I_w} g ds \lesssim r\hat{\mu}(\{v\}) \int_{I_w} g ds \simeq r \int_{I_w} g d\mu.$$ 

Summing over all $w \sim v$ yields

$$\int_B |u - u(v)| d\mu = \sum_{w \sim v} \int_{I_w} |u - u(v)| d\mu \lesssim \sum_{w \sim v} r \int_{I_w} g d\mu = r \int_B g d\mu.$$ 

A standard argument based on the triangle inequality allows us to replace $u(v)$ on the left-hand side by $u_B$ at the cost of an extra factor 2 on the right-hand side. \hfill \Box

To obtain a doubling measure on $X_\varepsilon$ with respect to the uniformized metric $d_\varepsilon$, we can equip $X_\varepsilon$ with the uniformized measure

$$d\hat{\rho}_\beta(x) = \rho_\beta(x) d\mu(x), \quad \text{where } \rho_\beta(x) = e^{-\beta d_{X_\varepsilon}(x, z_0)} \simeq d_\varepsilon(x)^{3/\varepsilon}.$$ (10.5)

Theorems 4.9 and 6.2 in Björn–Björn–Shanmugalingam \cite{5} (which hold for general Gromov hyperbolic spaces) then guarantee that for sufficiently large $\beta$, the obtained measure $\hat{\rho}_\beta$ is doubling and supports a 1-Poincaré inequality on $X_\varepsilon$ as well as on $X_\varepsilon$. More precisely, this holds for $\beta > \beta_0$, where $\beta_0$ is determined by the limitations given in [5, Theorems 4.9 and 6.2] based on the doubling constant from Theorem 10.2.

In considering the specific case of hyperbolic fillings, we are able to show that the requirement $\beta > \beta_0$ can be omitted. Just as we showed in Theorem 5.1 that the full range $0 < \varepsilon \leq \log \alpha$ is possible, we can allow for the full range $\beta > 0$ in our setting. This will be important in Sections 11–13 for our trace and extension results, and their applications.

Since we consider measures (on hyperbolic fillings) that are constructed from doubling measures on $Z$, it is possible to start directly with the weighted discrete measure

$$\rho_\beta(v)\hat{\mu}(\{v\}) = e^{-\beta n} \nu(B_Z(z, \alpha^{-n})) \quad \text{for } v = (z, n) \in V,$$ (10.6)

defined on the vertices in $X$, and smear it out as in (10.2):

$$\mu_\beta(A) = \sum_{v \in V} \sum_{w \sim v} (\rho_\beta(v)\hat{\mu}(\{v\}) + \rho_\beta(w)\hat{\mu}(\{w\})) \mathcal{L}(A \cap [v, w]).$$ (10.7)

Since $\rho_\beta(v) \simeq \rho_\beta(w)$ whenever $v \sim w$, the measures $\mu_\beta$ and $\hat{\mu}_\beta$ are clearly comparable, and the following result also holds for $\hat{\mu}_\beta$. 

Theorem 10.3. For every $\beta > 0$, $\mu_\beta$ is doubling and supports a 1-Poincaré inequality on $X_\varepsilon$ as well as on $\overline{X}_\varepsilon$.

Furthermore, for all $\zeta \in \partial X$ and $0 < r \leq 2 \text{diam}_\varepsilon X_\varepsilon \leq 4/\varepsilon$,

$$\mu_\beta(B_\varepsilon(\zeta, r)) \simeq (\varepsilon r)^{\beta/\varepsilon} \nu(B_\varepsilon(\zeta, (\varepsilon r)^{1/\sigma})), \quad \text{where } \sigma = \frac{\varepsilon}{\log \alpha} \leq 1 \quad (10.8)$$

and the comparison constants depend only on $\varepsilon$, $\beta$, $\alpha$, $\tau$ and the doubling constant associated with $\nu$.

In particular, $\nu$ is comparable to the $\beta/\varepsilon$-codimensional measure on $\partial_\varepsilon X$ generated by $\mu_\beta$.

Lemma 10.4. Let $0 < \varepsilon \leq \log \alpha$. Then

$$B_X\left(x, \frac{C_1 r}{\varepsilon d_\varepsilon(x)}\right) \subset B_\varepsilon(x, r) \subset B_X\left(x, \frac{C_2 r}{\varepsilon d_\varepsilon(x)}\right), \quad \text{if } x \in X \text{ and } 0 < r \leq \frac{1}{4} d_\varepsilon(x),$$

where $C_1, C_2 > 0$ are independent of $\varepsilon$.

Proof. This was proved when $0 < \varepsilon \leq \varepsilon_0(\delta)$ for general Gromov hyperbolic spaces (where $\delta$ is the Gromov hyperbolicity constant) in Theorem 2.1 in Björn–Björn–Shanmugalingam [5], with Remark 2.11 in [5] showing that $C_1, C_2 > 0$ are independent of $\varepsilon$. That proof only relies on the following facts which hold for hyperbolic fillings for all $0 < \varepsilon \leq \log \alpha$:

- That $X$ is geodesic, which follows from the definition.
- That $X$ is locally compact and $\overline{X}_\varepsilon$ is geodesic, which follows from Proposition 4.6, as $Z$ is equipped with a doubling measure.
- Lemma 2.6 (i.e. [8, Lemma 4.16]) which holds for arbitrary $\varepsilon > 0$.

Proof of Theorem 10.3. We first concentrate on the doubling property. We need to consider three types of balls, namely, subWhitney balls, balls centered in (or near) $\partial_\varepsilon X$ and intermediate balls. Recall that the graph $X$ has uniformly bounded degree, by Proposition 4.5.

1. For subWhitney balls, that is, balls $B_\varepsilon(x, r)$ with $r \leq \frac{1}{4} d_\varepsilon(x)$, we note that $2C_2 r/\varepsilon d_\varepsilon(x) \leq C_2/2\varepsilon$. Hence, Lemmas 10.1, 10.4 and Theorem 10.2 imply that

$$\mu_\beta(B_\varepsilon(x, 2r)) \leq \mu_\beta\left(B_X\left(x, \frac{2C_2 r}{\varepsilon d_\varepsilon(x)}\right)\right) \simeq \mu_\beta(x) \mu_\beta\left(B_X\left(x, \frac{2C_2 r}{\varepsilon d_\varepsilon(x)}\right)\right) \simeq \mu_\beta\left(B_X\left(x, \frac{C_1 r}{\varepsilon d_\varepsilon(x)}\right)\right) \leq \mu_\beta(B_\varepsilon(x, r)).$$

2. If $x \in \overline{X}_\varepsilon$ and $r \geq 2d_\varepsilon(x)$, then for some $\zeta \in \partial_\varepsilon X$,

$$B_\varepsilon(x, 2r) \subset B_\varepsilon(\zeta, \frac{2}{\varepsilon} r) \quad \text{and} \quad B_\varepsilon(\zeta, \frac{1}{\varepsilon} r) \subset B_\varepsilon(x, r).$$

It therefore suffices to estimate $\mu_\beta(B_\varepsilon(\zeta, r))$ for all $\zeta \in \partial_\varepsilon X$ and $0 < r \leq \text{diam}_\varepsilon X_\varepsilon$. From the construction of $\mu_\beta$, and using also the uniformly bounded degree of $X$, it is clear that

$$\mu_\beta(B_\varepsilon(\zeta, r)) \simeq \sum_{v \in V \cap B_\varepsilon(\zeta, r)} e^{-\beta \pi_\varepsilon(v)} \hat{\mu}(v).$$

Let $v = (z, n) \in V \cap B_\varepsilon(\zeta, r)$. Then $e^{-\varepsilon n} = \varepsilon d_\varepsilon(v) < \varepsilon r$ by (4.1), and hence $n \geq N$, where $N$ is the smallest nonnegative integer such that

$$N \geq \frac{1}{\varepsilon} \log \frac{1}{\varepsilon r} \quad (10.9)$$
Let \( z_j \in A_j \subset Z \) be such that \( d_Z(z_j, \zeta) < \alpha^{-j}, j = 0, 1, \ldots \). Proposition 4.4 shows that
\[
d_Z(z, z_j)^\sigma \leq (2\tau\alpha)^\sigma d_\varepsilon(v, (z_j, j)) \leq (2\tau\alpha)^\sigma (d_\varepsilon(v, \zeta) + d_\varepsilon(\zeta, (z_j, j))).
\]
Since the path \( (z_0, 0) \sim \ldots \sim (z_j, j) \sim \ldots \) is a geodesic ray in \( X \) ending at \( \zeta \) (see Lemma 3.1 and Proposition 4.4), we have
\[
d_\varepsilon(\zeta, (z_j, j)) = e^{-\varepsilon j}/\varepsilon. \tag{10.10}
\]
Letting \( j \to \infty \) then shows that
\[
d_Z(z, \zeta) \leq d_Z(z, z_j) + d_Z(z_j, \zeta) \leq 2\tau\alpha \left( r + \frac{e^{-\varepsilon j}}{\varepsilon} \right)^{1/\sigma} + \alpha^{-j} \to 2\tau\alpha r^{1/\sigma}.
\]
We therefore obtain that
\[
\mu_\beta(B_\varepsilon(\zeta, r)) \lesssim \sum_{\varepsilon \in \mathbb{A}_n \cap B_\varepsilon(\zeta, \varepsilon r^{1/\sigma})} e^{-\beta n} \sum_{n \geq N} \nu(B_\varepsilon(z, \alpha^{-n})).
\]
Since \( \alpha^{-n} = (e^{-\varepsilon n})^{1/\sigma} < (\varepsilon r)^{1/\sigma} \), the bounded overlap of the balls \( B_\varepsilon(z, \alpha^{-n}) \) in \( A_n \subset Z \) and the doubling property of \( \nu \) now imply that
\[
\mu_\beta(B_\varepsilon(\zeta, r)) \lesssim \sum_{n \geq N} e^{-\beta n} \nu(B_\varepsilon(\zeta, (\varepsilon r)^{1/\sigma})) \approx (\varepsilon r)^{\beta/\sigma} \nu(B_\varepsilon(\zeta, (\varepsilon r)^{1/\sigma})). \tag{10.11}
\]
To verify the reverse comparison, observe that by (10.9) and (10.10) we have \( d_\varepsilon(\zeta, (z_N, N)) = e^{-\varepsilon N}/\varepsilon < r \). It follows that the edge \( (z_N, N) \sim (z_{N+1}, N + 1) \) is contained in \( B_\varepsilon(\zeta, r) \). Consequently, using the doubling property of \( \nu \) and the fact that \( d_Z(\zeta, z_N) \leq \alpha^{-N} = (e^{-\varepsilon N})^{1/\sigma} \approx (\varepsilon r)^{1/\sigma} \), we conclude from (10.7) that
\[
\mu_\beta(B_\varepsilon(\zeta, r)) \lesssim e^{-\beta N} \mu(\{(z_N, N)\})
\approx (\varepsilon r)^{\beta/\sigma} \nu(B_\varepsilon(z, \alpha^{-N})) \approx (\varepsilon r)^{\beta/\sigma} \nu(B_\varepsilon(\zeta, (\varepsilon r)^{1/\sigma})),
\]
which, together with (10.11), proves (10.8).

3. If \( x \in X_\varepsilon \) and \( \frac{1}{4}d_\varepsilon(x) \leq r \leq 2d_\varepsilon(x) \), then clearly
\[
B_\varepsilon(x, \frac{1}{4}d_\varepsilon(x)) \subset B_\varepsilon(x, r) \subset B_\varepsilon(x, 2r) \subset B_\varepsilon(x, 4d_\varepsilon(x)).
\]
From Proposition 4.6 we know that \( X_\varepsilon \) is compact, and thus there is \( \zeta \in \partial_X \) such that \( d_\varepsilon(\zeta, x) = d_\varepsilon(x) \). Let \( \gamma \) be a \( d_\varepsilon \)-geodesic from \( x \) to \( \zeta \) and let \( v = (z, k) \) be the vertex in \( \gamma \) nearest to \( x \). As in case 1, using (10.7), Lemma 10.1, and the uniform boundedness of the degrees, we see that
\[
\mu_\beta(B_\varepsilon(x, \frac{1}{4}d_\varepsilon(x))) \simeq \rho_\beta(x) \mu(\{v\}) = \rho_\beta(x) \nu(B_\varepsilon(x, \alpha^{-k})). \tag{10.12}
\]
On the other hand, by (10.8) (proved when considering case 2 above),
\[
\mu_\beta(B_\varepsilon(x, 4d_\varepsilon(x))) \leq \mu_\beta(B_\varepsilon(\zeta, 5d_\varepsilon(x))) \lesssim (\varepsilon d_\varepsilon(x))^{\beta/\varepsilon} \nu(B_\varepsilon(\zeta, (5\varepsilon d_\varepsilon(x))^{1/\sigma})).
\]
Note that the right-hand side of the above is comparable to the right-hand side of (10.12) since \( d_\varepsilon(z, \zeta) \leq d_\varepsilon(z, v) + d_\varepsilon(v, \zeta) \leq 2d_\varepsilon(x) \), \( \nu \) is doubling, and
\[
(\varepsilon d_\varepsilon(x))^{\beta/\varepsilon} \simeq \rho(x)^{\beta/\varepsilon} = \rho_\varepsilon(x) \quad \text{and} \quad \alpha^{-k} = e^{-\varepsilon k/\sigma} \simeq \rho_\varepsilon(x)^{1/\sigma} \approx (\varepsilon d_\varepsilon(x))^{1/\sigma}.
\]
The doubling property of \( \mu_\beta \) now follows directly in all three cases from the above estimates.
To show that \( \mu_\beta \) supports a 1-Poincaré inequality on \( \overline{X}_\varepsilon \), we proceed as in the proof of Lemma 6.1 in Björn–Björn–Shanmugalingam [5]. This is possible for all \( \beta > 0 \), not only \( \beta > \beta_0 \) as in [5], since we already know that \( \mu_\beta \) is doubling on \( X_\varepsilon \). Together with Theorem 10.2, it shows that there exists \( c_0 > 0 \) such that (10.3) holds for all subWhitney balls \( B = B_\varepsilon(x,r) \) with \( x \in X \), the measure \( \mu_\beta \), and \( 0 < r \leq c_0d_\varepsilon(x) \). Since \( X_\varepsilon \) is a uniform length space and \( \mu_\beta \) is doubling on \( X_\varepsilon \), we conclude from [5, Proposition 6.3] that \( \mu_\beta \) supports a 1-Poincaré inequality on \( X_\varepsilon \) as well as on \( \overline{X}_\varepsilon \). □

**Corollary 10.5.** With the assumptions as in Theorem 10.3, we have for all \( x \in \overline{X}_\varepsilon \) and \( 0 < r \leq 2 \text{diam}_\varepsilon X_\varepsilon \),

\[
\mu_\beta(B_\varepsilon(x,r)) \sim \begin{cases} 
(r\varepsilon)^{\beta/\varepsilon} \nu(B_\varepsilon(\zeta,(r\varepsilon)^{1/\varepsilon})), & \text{if } r \geq d_\varepsilon(x) \text{ and } \zeta \in Z \text{ is a nearest point to } x, \\
{r}(\varepsilon d_\varepsilon(x))^{\beta/\varepsilon - 1} \hat{\mu}(\{v\}), & \text{if } r \leq d_\varepsilon(x) \text{ and } v \in X \text{ is a nearest vertex to } x.
\end{cases}
\]

In both cases, the nearness is with respect to the metric \( d_\varepsilon \).

Recall from (4.1) that \( \alpha^{-n} = (e^{-n})^{1/\varepsilon} = (\varepsilon d_\varepsilon(v))^{1/\varepsilon} \) if \( v = (z, n) \in V \). Moreover, if \( \zeta \in Z \) and \( v = (z, n) \in V \) are nearest points to \( x \) in \( Z \) and \( V \), respectively, then by Proposition 4.4,

\[
d_\varepsilon(\zeta, z)^n \simeq d_\varepsilon(\zeta, z) \lesssim d_\varepsilon(x) \simeq d_\varepsilon(v).
\]

It therefore follows from (10.1) and the doubling property of \( \nu \) that

\[
\hat{\mu}(\{v\}) = \nu(B_\varepsilon(z,(\varepsilon d_\varepsilon(v))^{1/\varepsilon})) \simeq \nu(B_\varepsilon(\zeta,(\varepsilon d_\varepsilon(x))^{1/\varepsilon})),
\]

which further simplifies the formula in Corollary 10.5. Also note that since \( \nu \) is doubling, \( B_\varepsilon(\zeta,(\varepsilon d_\varepsilon(x))^{1/\varepsilon}) \) can be replaced by any ball \( B_\varepsilon(\xi,(\varepsilon d_\varepsilon(x))^{1/\varepsilon}) \) with \( \xi \in Z \) such that \( d_\varepsilon(\xi, x) \lesssim d_\varepsilon(x) \), and that \( \nu(B_\varepsilon(\zeta,(\varepsilon d_\varepsilon(x))^{1/\varepsilon})) \simeq \nu(B_\varepsilon(\zeta,(\varepsilon d_\varepsilon(x))) \).

**Proof of Corollary 10.5.** The first case follows directly from (10.8) together with the inclusions

\[
B_\varepsilon(\zeta, r) \subset B_\varepsilon(x, 2r) \subset B_\varepsilon(\zeta, 3r)
\]

and the doubling property of \( \mu_\beta \) and \( \nu \).

In the second case, Lemma 10.4 implies that

\[
B_\varepsilon(x, \frac{r}{2}) \subset B_X \left(x, \frac{C_2r}{2\varepsilon d_\varepsilon(x)} \right) \subset B_X \left(x, \frac{C_2}{2\varepsilon} \right)
\]

Recall from Proposition 4.5 that the graph \( X \) has uniformly bounded degree. Therefore we have by (10.7) and Lemma 10.1 that

\[
\mu_\beta(B_\varepsilon(x, \frac{r}{2})) \simeq \frac{r}{\varepsilon d_\varepsilon(x)} \left( \rho_\beta(v)\hat{\mu}(\{v\}) + \sum_{w \in V \cap B_\varepsilon(x, r/2)} \rho_\beta(w)\hat{\mu}(\{w\}) \right)
\]

\[
\simeq \frac{r}{\varepsilon d_\varepsilon(x)} \rho_\beta(v)\hat{\mu}(\{v\}).
\]

Since \( \rho_\beta(v) \simeq (\varepsilon d_\varepsilon(x))^{\beta/\varepsilon} \), the doubling property of \( \mu_\beta \) concludes the proof. □

**Lemma 10.6.** Assume that the measure \( \nu \) on \( Z \) satisfies for all \( \zeta \in Z \) and \( 0 < r' \leq r \leq 2 \text{diam} \ Z \),

\[
\frac{\nu(B_\varepsilon(\zeta, r'))}{\nu(B_\varepsilon(\zeta, r))} \gtrsim \left( \frac{r'}{r} \right)^{s_\nu}.
\]

(10.13)
Let $\beta > 0$ and $\varepsilon = \log \alpha$. Then the measure $\mu_\beta$, defined by (10.6) and (10.7), satisfies for all $x \in X_\varepsilon$ and $0 < r' \leq r \leq 2 \operatorname{diam} X_\varepsilon$,

$$
\frac{\mu_\beta(B_z(x, r'))}{\mu_\beta(B_z(x, r))} \geq \left(\frac{r'}{r}\right)^{s_\beta}, \quad \text{where } s_\beta = \max\{1, \beta/\varepsilon + s_\nu\}.
$$

(10.14)

It is well known that every doubling measure $\nu$ satisfies (10.13) for some $s_\nu > 0$, see for example [25, (4.16)].

**Proof.** Note that $s = 1$. We shall distinguish three cases:

1. If $r \leq d_z(x)$ then the second case in Corollary 10.5 applies both to $r$ and $r'$ and hence

$$
\frac{\mu_\beta(B_z(x, r'))}{\mu_\beta(B_z(x, r))} \simeq \frac{r'}{r}.
$$

2. If $r' \geq d_z(x)$ then the first case in Corollary 10.5 applies both to $r$ and $r'$ and hence

$$
\frac{\mu_\beta(B_z(x, r'))}{\mu_\beta(B_z(x, r))} \simeq \frac{(\varepsilon r')^{\beta/\varepsilon} \mu(B_z(\zeta, \varepsilon r'))}{(\varepsilon r)^{\beta/\varepsilon} \mu(B_z(\zeta, \varepsilon r))} \geq \left(\frac{r'}{r}\right)^{\beta/\varepsilon + s_\nu}.
$$

3. If $r' \leq d_z(x) \leq r$ then by the already proved cases 1 and 2,

$$
\frac{\mu_\beta(B_z(x, r'))}{\mu_\beta(B_z(x, r))} = \frac{\mu_\beta(B_z(x, r'))}{\mu_\beta(B_z(x, d_z(x)))} \frac{\mu_\beta(B_z(x, d_z(x)))}{\mu_\beta(B_z(x, r))} \geq \frac{r'}{d_z(x)} \left(\frac{d_z(x)}{r}\right)^{\beta/\varepsilon + s_\nu}.
$$

If $\beta/\varepsilon + s_\nu \geq 1$ then

$$
\frac{r'}{d_z(x)} \left(\frac{d_z(x)}{r}\right)^{\beta/\varepsilon + s_\nu} \geq \left(\frac{r'}{d_z(x)}\right)^{\beta/\varepsilon + s_\nu} \left(\frac{d_z(x)}{r}\right)^{\beta/\varepsilon + s_\nu} = \left(\frac{r'}{r}\right)^{\beta/\varepsilon + s_\nu},
$$

and if $\beta/\varepsilon + s_\nu \leq 1$, then

$$
\frac{r'}{d_z(x)} \left(\frac{d_z(x)}{r}\right)^{\beta/\varepsilon + s_\nu} \geq \frac{r'}{d_z(x)} \frac{d_z(x)}{r} = \frac{r'}{r}.
$$

From the above three cases we conclude that (10.14) holds. \square

11. **Traces to $Z$ from the hyperbolic filling $X$**

Recall the standing assumptions from Section 10. Here and in the rest of the paper, we also let $1 \leq p < \infty$ and consider the uniformized space $X_\varepsilon$ equipped with the measure $\mu_\beta$ where $\varepsilon = \log \alpha$ and $\beta > 0$.

Theorem 5.1 shows that $X_\varepsilon$ is a uniform space. From Proposition 4.4 with $\varepsilon = \log \alpha$ we know that $\partial_\varepsilon X$ is biLipschitz equivalent to $Z$. Hence we can replace $\partial_\varepsilon X$ by $Z$ as well, since the Besov spaces are biLipschitz invariant. Of course, the measure on $Z$ is pushed forward to $\partial_\varepsilon X$ via the biLipschitz identification $\Psi : Z \to \partial_\varepsilon X$. In the following, we shall therefore not distinguish between $(\partial_\varepsilon X, d_z)$ and $(Z, d_Z)$.

We equip the uniformized space $X_\varepsilon$ with the doubling measure $\mu_\beta$, obtained in (10.7). Equivalently, the uniformized measure $\hat{\mu}_\beta$ from (10.5), based on the smeared out measure $\mu$ from (10.2), can be used.

For the vertices in $X$, consider the projections $\pi_1((z, n)) = z$ and $\pi_2((z, n)) = n$. Whenever a nonvertex $x \in X$ belongs to the edge $[v, w] \subset X$, let

$$
\pi_2(x) := \min\{\pi_2(v), \pi_2(w)\}.
$$

(11.1)
Theorem 11.1. Let \( u \in \tilde{N}^{1,p}(X,\mu_\beta) \) and \( 0 < \theta \leq 1 - \beta/\varepsilon p \). Then \( u \) has a trace \( \tilde{u} \in B^{0,\theta}_{p,p}(Z) \) given by (11.4) and (11.5) below, with the (semi)norm estimates
\[
\|\tilde{u}\|_{\theta,p} \lesssim \|g_u\|_{L^p(X,\mu_\beta)} \quad (11.2)
\]
and
\[
\|\tilde{u}\|_{L^p(Z)} \lesssim |u(v_0)| + \|g_u\|_{L^p(X,\mu_\beta)} \lesssim \|u\|_{N^{1,p}(X,\mu_\beta)}. \quad (11.3)
\]
If \( u \in \text{Lip}(X) \) then \( \tilde{u} = \tilde{u}|_Z \), where \( \tilde{u} \) is the unique Lipschitz extension of \( u \) to \( \overline{X} \).

Note that the equivalence classes in \( N^{1,p}(X,\mu_\beta) \) consist of one function each, see Remark 9.5.

Proof. Let \( u \in \tilde{N}^{1,p}(X,\mu_\beta) \) with an upper gradient \( g \in L^p(X,\mu_\beta) \). For \( \zeta \in Z \) and \( n = 0, 1, \ldots \), let \( A_n(\zeta) = A_n \cap B_\epsilon(\zeta,\alpha^{-n}) \). We define
\[
\tilde{u}_n(\zeta) = \frac{1}{\#A_n(\zeta)} \sum_{z \in A_n(\zeta)} u((z,n)), \quad (11.4)
\]
where \( \#A_n(\zeta) \) is the cardinality of \( A_n(\zeta) \). Note that the construction of \( A_n \), together with the doubling property of \( Z \), shows that \( 1 \leq \#A_n(\zeta) \leq K \) for some \( K \) independent of \( n \) and \( \zeta \).

For each fixed \( z \in A_n \), the function \( \chi_{B_\epsilon(z,\alpha^{-n})} \) is lower semicontinuous and thus \( \nu \)-measurable. Hence also the linear combinations
\[
\sum_{z \in A_n(\zeta)} u((z,n)) = \sum_{z \in A_n} u((z,n))\chi_{B_\epsilon(z,\alpha^{-n})}(\zeta) \quad \text{and} \quad \#A_n(\zeta) = \sum_{z \in A_n} \chi_{B_\epsilon(z,\alpha^{-n})}(\zeta)
\]
are \( \nu \)-measurable, and hence so is \( u_n \). We shall show that for \( \nu \)-a.e. \( \zeta \in Z \), the limit
\[
\tilde{u}(\zeta) = \lim_{n \to \infty} u_n(\zeta) \quad (11.5)
\]
exists and defines the trace \( \tilde{u} : Z \to \mathbb{R} \). To this end, note that \( (z,j) \sim (y,j+1) \) whenever \( z \in A_j(\zeta) \) and \( y \in A_{j+1}(\zeta) \), \( j = 0,1,\ldots \), since \( \zeta \in B(z,\alpha^{-j}) \cap B(y,\alpha^{-j-1}) \). Also,
\[
\frac{1}{\#A_j(\zeta)\#A_{j+1}(\zeta)} \leq 1.
\]
We then have for each \( j \),
\[
|u_j(\zeta) - u_{j+1}(\zeta)| \leq \sum_{z \in A_j(\zeta)} \sum_{y \in A_{j+1}(\zeta)} |u((z,j)) - u((y,j+1))| \\
\leq \sum_{z \in A_j(\zeta)} \sum_{y \in A_{j+1}(\zeta)} \int_{[(z,j),(y,j+1)]} g \, ds_z. \quad (11.6)
\]
On the edge \( E = [(z,j),(y,j+1)] \), we have by (10.7) that
\[
ds_z \simeq e^{-\epsilon j} \, d\mathcal{L} = \frac{\alpha^{-j} \, d\mu_\beta}{\mu_\beta(E)}. \quad (11.7)
\]
If \( p > 1 \), then (11.7) and Hölder’s inequality applied to (11.6) give
\[
|u_j(\zeta) - u_{j+1}(\zeta)| \leq \alpha^{-j} \sum_{z \in A_j(\zeta)} \sum_{y \in A_{j+1}(\zeta)} \int_{[(z,j),(y,j+1)]} g \, d\mu_\beta \\
\leq \alpha^{-j} \sum_{z \in A_j(\zeta)} \sum_{E \in \mathcal{E}(z,j)} \left( \int_E g^p \, d\mu_\beta \right)^{1/p}, \quad (11.8)
\]
where $\mathcal{E}(z,j)$ consists of all downward-directed edges emanating from the vertex $(z,j)$. Choose $0 < \kappa < \theta_p$ and insert $a^{-j\kappa/p} \alpha^{-j/p}$ into (11.8). Summing over $j$, together with another use of Hölder’s inequality, this time on the sum, shows that for all $m > n \geq 0$,

\[
|u_n(\zeta) - u_m(\zeta)| \leq \sum_{j=n}^{m-1} |u_j(\zeta) - u_{j+1}(\zeta)|
\]

(11.9)

\[
\leq \sum_{j=n}^{\infty} a^{-j\kappa/p} \alpha^{-j(1-\kappa/p)} \sum_{z \in A_j(\zeta)} \sum_{E \in \mathcal{E}(z,j)} \left( \int_E g^p \, d\mu_\beta \right)^{1/p}
\]

\[
\lesssim a^{-nk/p} \left( \sum_{j=n}^{\infty} a^{-j(p-\kappa)} \sum_{z \in A_j(\zeta)} \sum_{E \in \mathcal{E}(z,j)} \int_E g^p \, d\mu_\beta \right)^{1/p},
\]

where we have also used the fact that

\[
\left( \sum_{j=n}^{\infty} a^{-j(p-\kappa)}/p \right)^{1-1/p} \lesssim a^{-nk/p},
\]

together with $\# A_j(\zeta) \leq K$ and $\# \mathcal{E}(z,j) \leq K$. For $p = 1$ the estimate is simpler and Hölder’s inequality is not needed, and the above estimate holds as well. We shall now see that (11.9) tends to zero as $m \to \infty$ for $\nu$-a.e. $\zeta \in Z$. Thus, the sequence $\{u_n(\zeta)\}_{n=0}^\infty$ is a Cauchy sequence, and has a limit as $n \to \infty$, for $\nu$-a.e. $\zeta \in Z$.

To this end, note that for $E \in \mathcal{E}(z,j)$,

\[
\int_E g^p \, d\mu_\beta \simeq \frac{\alpha^{\beta/\varepsilon}}{\nu(B_Z(z, \alpha^{-j}))} \int_{X_V} g(x)^p \chi_E(x) \, d\mu_\beta(x),
\]

(11.10)

where $X_V$ denotes the union of all vertical edges in $X$. Also, $z \in A_j(\zeta)$ if and only if $z \in A_j$ and $\zeta \in B_Z(z, \alpha^{-j})$. Integrating over all $\zeta \in Z$ we then obtain from (11.9) by means of Tonelli’s theorem that

\[
\int_Z |u_m(\zeta) - u_n(\zeta)|^p \, d\nu(\zeta)
\]

\[
\lesssim a^{-nk} \int_Z \sum_{j=n}^{\infty} \sum_{z \in A_j} a^{-j(p-\beta/\varepsilon-\kappa)} \nu(B_Z(z, \alpha^{-j})) \chi_{B_Z(z, \alpha^{-j})}(\zeta)
\]

\[
\times \sum_{E \in \mathcal{E}(z,j)} \int_{X_V} g(x)^p \chi_E(x) \, d\mu_\beta(x) \, d\nu(\zeta)
\]

\[
= a^{-nk} \int_{X_V} g(x)^p \sum_{j=n}^{\infty} \sum_{z \in A_j} a^{-j(p-\beta/\varepsilon-\kappa)}
\]

\[
\times \sum_{E \in \mathcal{E}(z,j)} \int_{Z} \frac{\chi_{B_Z(z, \alpha^{-j})}(\zeta)}{\nu(B_Z(z, \alpha^{-j}))} \, d\nu(\zeta) \chi_E(x) \, d\mu_\beta(x).
\]

(11.11)

The integral over $Z$ is clearly equal to 1. Moreover, for a.e. $x \in X$,

\[
\chi_E(x) \neq 0 \quad \text{only if} \quad x \in E \in \mathcal{E}(z,j) \text{ with } j = \pi_2(x),
\]

and so for a.e. $x \in X$ we have

\[
\sum_{z \in A_j} \sum_{E \in \mathcal{E}(z,j)} \int_{Z} \frac{\chi_{B_Z(z, \alpha^{-j})}(\zeta)}{\nu(B_Z(z, \alpha^{-j}))} \, d\nu(\zeta) \chi_E(x) = \sum_{z \in A_j} \sum_{E \in \mathcal{E}(z,j)} \chi_E(x)
\]

\[
= \chi_{\{y \in X_V: \pi_2(y) = j\}}(x).
\]

(11.12)
We therefore conclude that
\[
\int_Z |u_m(\xi) - u_n(\xi)|^p \, d\nu(\xi) \lesssim \alpha^{-\kappa}\int_{[y \in X : \tau_z(y) \geq n]} g(x)^p \alpha^{-\pi_2(x)(p-\beta/\varepsilon - \kappa)} \, d\mu_\beta(x).
\]
Since \(p - \beta/\varepsilon - \kappa \geq \theta p - \kappa > 0\), we obtain that
\[
\int_Z |u_m(\xi) - u_n(\xi)|^p \, d\nu(\xi) \lesssim \alpha^{-n(p-\beta/\varepsilon)} \int_{X_\nu} g^p \, d\mu_\beta \to 0, \quad \text{as } m > n \to \infty.
\]
Hence, the sequence \(\{u_n\}_{n=0}^\infty\) is a Cauchy sequence both in \(L^p(\nu)\) and \(\nu\text{-a.e. in } Z\) (recall that we have \(m > n\) in the above computations). The limit (11.5) therefore exists for \(\nu\text{-a.e. } \xi \in Z\), and \(\tilde{u} \in L^p(\nu)\).

This also shows, by letting \(n = 0\) and \(u_m \to \tilde{u}\), that
\[
\left(\int_Z |\tilde{u} - u(v_0)|^p \, d\nu\right)^{1/p} \lesssim \int_{X_\nu} g^p \, d\mu_\beta,
\]
where \(v_0 = (z_0, 0)\). Thus the first inequality
\[
\|\tilde{u}\|_{L^p(\nu)} \lesssim \|u(v_0)\| + \|g\|_{L^p(X, \mu_\beta)}
\]
holds in (11.3). For the second inequality, recall the notion of capacity from Definition 9.3. Since \(u(v_0)|^p C_{\beta}^X(\{v_0\}) \leq \|u\|^p_{X^{\nu, p}(X, \mu_\beta)}\) by the definition of \(C_{\beta}^X(\{v_0\})\), and \(C_{\beta}^X(\{v_0\}) > 0\), by Remark 9.5, we conclude that the second inequality in (11.3) holds as well.

To estimate \(\|\tilde{u}\|_{\theta, p}\), we let \(m \to \infty\) in (11.9) to obtain for \(\nu\text{-a.e. } \xi \in Z\) and any \(n \geq 0\),
\[
|\tilde{u}(\xi) - u_n(\xi)|^p \lesssim \alpha^{-n\kappa} \sum_{j=n}^{\infty} \alpha^{-j(p-\kappa)} \sum_{z \in A_j(\xi)} \sum_{E \in E(z, \xi)} \int_E g^p \, d\mu_\beta. \tag{11.13}
\]
A similar estimate holds for \(\nu\text{-a.e. } \xi \in Z\). As in Lemma 4.3, we let \(l \geq 0\) be the smallest integer such that \(\alpha^{-l} \leq \tau - 1\). Also let
\[
Z_n(\xi) = \{\xi \in Z : \alpha^{-n-l-1} < d_Z(\xi, \xi) \leq \alpha^{-n-l}\}, \quad n = 1, 2, \ldots,
\]
and \(Z_0(\xi) = Z \setminus B_Z(\xi, \alpha^{-l-1})\). Note that \(\xi \in Z_n(\xi)\) if and only if \(\xi \in Z_n(\xi)\), in which case also \(\xi \in \tau B_Z(z, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-n})\) and thus \((z, n) \sim (y, n)\) for all \(z \in A_n(\xi)\) and \(y \in A_n(\xi)\) with \(y \neq z\). Hence for all \(\xi \in Z_n(\xi)\), \(n = 1, \ldots\),
\[
|u_n(\xi) - u_n(\xi)|^p \lesssim \sum_{z \in A_n(\xi)} \sum_{y' \neq z} |u((z, n)) - u((y, n))|^p, \tag{11.14}
\]
while \(u_0(\xi) = u(v_0) = u_0(\xi)\) for all \(\xi, \xi' \in Z\). Hölder’s inequality and (11.7) with \(E = [(z, n), (y, n)]\) give
\[
|u((z, n)) - u((y, n))|^p \leq \left(\int_E g \, ds\right)^p \lesssim \alpha^{-np} \int_E g^p \, d\mu_\beta. \tag{11.15}
\]
Next, note that
\[
|\tilde{u}(\xi) - \tilde{u}(\xi)|^p \lesssim |\tilde{u}(\xi) - u_n(\xi)|^p + |u_n(\xi) - u_n(\xi)|^p + |u_n(\xi) - \tilde{u}(\xi)|^p, \tag{11.16}
\]
and that each of the three terms can be estimated with the aid of (11.13)–(11.15). We shall insert (11.16) into the Besov norm
\[
\|	ilde{u}\|_{\theta, p}^p = \int_Z \int_{Z \setminus \{\xi\}} \frac{|\tilde{u}(\xi) - \tilde{u}(\xi)|^p}{d_Z(\xi, \xi)^{\theta p}} \, d\nu(\xi) \, dv(\xi) d\nu(\xi).
\]
and obtain three terms corresponding to the three terms on the right-hand side of (11.16). We next use the comparisons $d_Z(\xi, \zeta) \nu \simeq \alpha^{-nP}$ and

$$\nu(B_Z(\zeta, d_Z(\xi, \zeta))) \simeq \nu(B_Z(\xi, d_Z(\xi, \zeta))) \simeq \nu(B_Z(\zeta, \alpha^{-n})) \simeq \nu(B_Z(\xi, \alpha^{-n}))$$

whenever $\xi \in Z_n(\zeta)$ (or equivalently, $\zeta \in Z_n(\xi)$). We then get $\|\tilde{u}\|_{\nu,p}^p \lesssim I_0 + I_{II_0}$, where

$$I_0 := \int_Z \sum_{n=0}^{\infty} \int_{Z_n(\zeta)} \frac{\|\tilde{u}(\zeta) - u_n(\zeta)\|_p}{\alpha^{-nP}} \nu(B_Z(\zeta, \alpha^{-n})) \nu(\zeta) d\nu(\zeta),$$

$$I_{II_0} := \int_Z \sum_{n=0}^{\infty} \int_{Z_n(\zeta)} \frac{\|\tilde{u}(\zeta) - u_n(\zeta)\|_p}{\alpha^{-nP}} \nu(\zeta) d\nu(\zeta),$$

Observe that $I_{II_0}$ is the same as $I_0$ once the roles of $\zeta$ and $\xi$ are switched, and so it suffices to find estimates for $I_0$ and $I_{II_0}$. Using (11.13)–(11.15), we find that

$$I_0 \lesssim \int_Z \sum_{n=0}^{\infty} \frac{\alpha^{-n(\kappa - \theta p)}}{\nu(B_Z(\zeta, \alpha^{-n}))} \int_{Z_n(\zeta)} \sum_{j=n}^{\infty} \frac{\alpha^{-j(p - \kappa)}}{\nu(\zeta)} \times \sum_{z \in A_j(\zeta)} \sum_{E \in \mathcal{E}(z, j)} g(x)^p d\mu_\beta(x) \nu(\zeta) d\nu(\zeta) =: I,$$

$$I_{II_0} \lesssim \int_Z \sum_{n=0}^{\infty} \frac{\alpha^{-n(p - \theta p)}}{\nu(B_Z(\zeta, \alpha^{-n}))} \times \int_{Z_n(\zeta)} \sum_{z \in A_n(\zeta)} \sum_{y \in A_n(\zeta)} \sum_{y \neq z} g(x)^p d\mu_\beta(x) \nu(\zeta) d\nu(\zeta) =: II.$$

To estimate $I$, we use (11.10) and that $z \in A_j(\zeta)$ if and only if $z \in A_j$ and $\zeta \in B_Z(z, \alpha^{-j})$. Now an argument using Tonelli’s theorem as in the verification of (11.11) yields that

$$I \lesssim \sum_{n=0}^{\infty} \frac{\alpha^{-n(\kappa - \theta p)}}{\nu(B_Z(\zeta, \alpha^{-n}))} \sum_{j=n}^{\infty} \frac{\alpha^{-j(p - \beta z - \kappa)}}{\nu(\zeta)} \times \int_Z \nu(\zeta) d\nu(\zeta) \int_{Z_n(\zeta)} \frac{\nu(\zeta)}{\nu(B_Z(\zeta, \alpha^{-n}))} \nu(\zeta) \int_{X_v} g(x)^p \chi_E(x) d\mu_\beta(x).$$

Since $Z_n(\zeta) \subset B_Z(\zeta, \alpha^{-n})$, the integral over $Z_n(\zeta)$ followed by the integral over $Z$ is clearly at most 1. Another use of Tonelli’s theorem therefore implies that

$$I \lesssim \int_{X_v} g(x)^p \sum_{n=0}^{\infty} \frac{\alpha^{-n(\kappa - \theta p)}}{\nu(B_Z(\zeta, \alpha^{-n}))} \sum_{j=n}^{\infty} \frac{\alpha^{-j(p - \beta z - \kappa)}}{\nu(\zeta)} \sum_{z \in A_j} \chi_E(x) d\mu_\beta(x).$$

The last three sums are simplified using the last identity in (11.12) and we obtain

$$I \lesssim \int_{X_v} g(x)^p \alpha^{-\tau_2(x)(p - \beta z - \kappa)} \sum_{n=0}^{\infty} \alpha^{-n(\kappa - \theta p)} d\mu_\beta(x) \approx \int_{X_v} g(x)^p \alpha^{\tau_2(x)(p - \beta z - \theta p)} d\mu_\beta(x),$$
because of the choices $\kappa < \beta p$ and $\alpha > 1$. Since $p - \beta/\varepsilon - \theta p \geq 0$, this yields

$$I \lesssim \int_{X_H} g^p \, d\mu_\beta.$$  

To estimate $II$, we proceed similarly. As in (11.10), we have that when $(z, n) \sim (y, n)$,

$$\int_{\{z, n\} \times \{y, n\}} g^p \, d\mu_\beta \lesssim \frac{\alpha^{n\beta/\varepsilon}}{\nu(B_Z(z, \alpha^{-n}))} \int_{X_H} g(x)^p \chi_{\{z, n\}, \{y, n\}}(x) \, d\mu_\beta(x),$$

where $X_H$ denotes the union of all horizontal edges in $X$.

Moreover, $z \in A_n(\zeta)$ and $y \in A_n(\xi)$ if and only if $z, y \in A_n$, $\zeta \in B_Z(z, \alpha^{-n})$ and $\xi \in B_Z(y, \alpha^{-n})$. Tonelli’s theorem then yields that

$$II \simeq \sum_{n=1}^{\infty} \alpha^{-n(p - \beta/\varepsilon - \theta p)} \sum_{z, y \in A_n \atop (z, n) \sim (y, n)} \int_Z \chi_{B_Z(z, \alpha^{-n})}(\zeta) \int_{Z_A(\zeta)} \chi_{B_Z(y, \alpha^{-n})}(\xi) \, d\nu(\xi) \, d\nu(\zeta) \int_{X_H} g(x)^p \chi_{\{z, n\}, \{y, n\}}(x) \, d\mu_\beta(x).$$

Since $Z_A(\zeta) \subset B_Z(\zeta, \alpha^{-n})$, the integral over $Z_A(\zeta)$ followed by the integral over $Z$ is at most 1, and another use of Tonelli’s theorem shows that

$$II \lesssim \int_{X_H} g(x)^p \sum_{n=1}^{\infty} \alpha^{-n(p - \beta/\varepsilon - \theta p)} \sum_{z, y \in A_n \atop (z, n) \sim (y, n)} \chi_{\{z, n\}, \{y, n\}}(x) \, d\mu_\beta(x).$$

Moreover, $\chi_{\{z, n\}, \{y, n\}}(x) \neq 0$ if only if $x \in ([z, n], (y, n])$, in which case also $n = \pi_2(x)$. We therefore conclude that

$$II \lesssim \int_{X_H} g(x)^p \alpha^{-\pi_2(x)}(p - \beta/\varepsilon - \theta p) \, d\mu_\beta(x) \leq \int_{X_H} g^p \, d\mu_\beta,$$

because $p - \beta/\varepsilon - \theta p \geq 0$. Combining the estimates for $I$ and $II$ gives the desired bound (11.2).

The fact that $\bar{u} = \bar{u}|_Z$ when $u$ is Lipschitz continuous on $X_\varepsilon$ follows from the definition of $\bar{u}$ and the fact that $u$ has a unique Lipschitz extension to $\overline{X_\varepsilon}$.  

Recall the notion of capacity from Definition 9.3. The following proposition shows that the boundary measure $\nu$ on $Z = \partial_\varepsilon X$ is absolutely continuous with respect to the $C^\infty_p$-capacity. Note that points in $X$ have positive $C^\infty_p$-capacity, but that it is possible to have nonempty subsets of $\partial_\varepsilon X = \overline{Z}$ with zero capacity.

**Proposition 11.2.** Let $E \subset \partial_\varepsilon X$. If $p > \beta/\varepsilon$ and $C^\infty_p(E) = 0$, then $\nu(E) = 0$.

**Proof.** Since $C^\infty_p$ is an outer capacity by Theorem 9.6, there are open sets $G_j \supset E$ such that $C^\infty_p(G_j) < 1/j$. Then

$$E' := \bigcap_{j=1}^{\infty} G_j \supset E$$

is a Borel set with zero capacity. Let $K \subset E'$ be compact. Because $\mu_\beta$ is doubling and supports a $p$-Poincaré inequality on $\overline{X_\varepsilon}$, it follows from Kallunki-Shanmugalingam [31, Theorem 1.1] (or [2, Theorem 6.7 (xii)]) that there are $u_k \in \text{Lip}(\overline{X_\varepsilon})$
such that $u_k = 1$ on $K$ and $\|u_k\|_{N^{1,p}(X_e,\mu_\beta)} < 1/k$, $k = 1, 2, \ldots$. By the last part of Theorem 11.1 with $\theta = 1 - \beta/\varepsilon p > 0$,

$$\nu(K) \leq \lim_{k \to \infty} \|u_k\|_{L^p(Z)} \leq \lim_{k \to \infty} \|u_k\|_{N^{1,p}(X_e,\mu_\beta)} = 0,$$

i.e., $\nu(K) = 0$. Since $E'$ is a Borel set and $\nu$ is a Borel regular measure, we conclude that

$$\nu(E) \leq \nu(E') = \sup_{K \subset E \text{ compact}} \nu(K) = 0. \quad \square$$

The following result is a refinement of Theorem 11.1. In the case of regular trees, it provides a more precise trace result than Proposition 6.1 in Björn–Björn–Gill–Shanmugalingam [3]. Recall that by Theorem 7.1, every rooted tree $X$ can be seen as a hyperbolic filling of its uniformized boundary $\partial_e X$.

**Theorem 11.3.** Let $u \in \hat{N}^{1,p}(X_e,\mu_\beta)$ and $0 < \theta \leq 1 - \beta/\varepsilon p$. Then $u$ has an extension $\hat{u} \in \hat{N}^{1,p}(X_e,\mu_\beta)$. Furthermore, the restriction $\hat{u} := \hat{u}_Z$ agrees with the trace of $u$ defined earlier $\nu$-a.e. in $Z$, and belongs to $B^{d}_{p,p}(Z)$ with the (semi)norm estimates

$$\|\hat{u}\|_{\theta,p} \lesssim \|u\|_{L^p(Z)}$$

and

$$\|\hat{u}\|_{L^p(Z)} \lesssim |u(x_0)| + \|g_u\|_{L^p(X_e,\mu_\beta)} \lesssim \|u\|_{N^{1,p}(X_e,\mu_\beta)}.$$  

Moreover, for $C^{\infty}_p\nu$-a.e. and thus $\nu$-a.e. $\xi \in Z$ we have that

$$\lim_{r \to 0^+} \int_{X_e \cap B_r(\xi,\varepsilon)} |u - \hat{u}(\xi)|^p \, d\mu_\beta = 0. \quad (11.17)$$

Note that the extension $\hat{u}$ is not unique, but it is unique up to sets of capacity zero and thus $\nu$-a.e. (by Proposition 11.2), since if $\hat{u}_1$ and $\hat{u}_2$ are two extensions, then they are equal $\mu_\beta$-a.e., and thus $C^{\infty}_p\nu$-q.e. We may therefore take the restriction of any such extension. The key observation that makes the above statement true is that the representatives in Newtonian spaces are equal q.e., not just a.e. as for standard Sobolev spaces.

The last claim of the above theorem tells us that the trace of a function in $N^{1,p}(X_e,\mu_\beta)$, as constructed in Theorem 11.1, agrees with other notions of traces in the current literature, see e.g. Malý [39].

**Proof.** By Theorems 5.1 and 10.3, $X_e$ is a uniform domain in $X_e$ and $\mu_\beta$ is doubling and supports a $p$-Poincaré inequality on $X_e$. Thus by Proposition 5.9 in Björn–Shanmugalingam [7], $X_e$ is an extension domain, and thus $u$ has an extension to $X_e$, denoted $\hat{u} \in \hat{N}^{1,p}(X_e,\mu_\beta)$.

By Shanmugalingam [45, Theorem 4.1 and Corollary 3.9], there is a sequence $u_j \in \text{Lip}(X_e)$ such that $\|u_j - \hat{u}\|_{N^{1,p}(X_e,\mu_\beta)} \to 0$ and $u_j(x) \to \hat{u}(x)$ for $C^{\infty}_p\nu$-q.e. $x \in X_e$, as $j \to \infty$. Let $\hat{u} := \hat{u}_Z$ and $\hat{u}_j := u_j|_Z$. By Proposition 11.2, $\hat{u}_j(\xi) \to \hat{u}(\xi)$ for $\nu$-a.e. $\xi \in Z$. Moreover, by Theorem 11.1, $\{\hat{u}_j\}_{j=1}^\infty$ is a Cauchy sequence in the norm $\|\cdot\|_{B^{d}_{p,p}(Z)}$. By Remark 9.8, $B^{d}_{p,p}(Z)$ is complete with respect to $\|\cdot\|_{B^{d}_{p,p}(Z)}$, and hence we see that $\|\hat{u}_j - \hat{u}\|_{B^{d}_{p,p}(Z)} \to 0$ as $j \to \infty$, with the (semi)norm estimates from Theorem 11.1 preserved for $\hat{u}$.

By [2, Theorem 5.62] or [27, Theorem 9.2.8] (for $p > 1$) and Kinnunen–Korte–Shanmugalingam–Tuominen [32, Theorem 4.1 and Remark 4.7] (for $p = 1$, see below) we know that $C^{\infty}_p\nu$-q.e. point in $X_e$ is an $L^p(\mu_\beta)$-Lebesgue point of $\hat{u}$; hence (11.17) holds because $\mu_\beta(\partial_e X) = 0$.

In [32, p. 404] it is assumed that $\mu(X) = \infty$, which is used in their proof of the boxing inequality. In Mäkeläinen [38], the boxing inequality is proved also when $\mu(X) < \infty$, and thus the Lebesgue point result in [32] holds also here where $\mu_\beta(X_e) < \infty$. \quad \square
12. Extension from $Z$ to its hyperbolic filling $X$

Recall the standing assumptions from Sections 10 and 11.

Theorem 11.3 related the Newtonian space $N^{1,p}(X, \mu_\beta)$ to a certain range of Besov spaces of functions on $Z \equiv \partial_\varepsilon X$. The principal goal of this section is to find a counterpart of this theorem in the opposite direction. This is the purpose of the theorem below.

**Theorem 12.1.** For $f \in B^p_{\theta,p}(Z)$, consider the extension

$$Ef((z, n)) := \int_{B^*_\varepsilon(z, \alpha^{-n})} f \, d\nu, \quad \text{if } (z, n) \in V \subset X,$$

extended piecewise linearly (with respect to $d_\varepsilon$) to each edge in $X_\varepsilon$, and then to the boundary $\partial_\varepsilon X$ by letting

$$Ef(\zeta) = \limsup_{r \to 0^+} \int_{B_\varepsilon(\zeta, r)} Ef \, d\mu_\beta, \quad \zeta \in \partial_\varepsilon X, \quad (12.1)$$

so that $Ef : X_\varepsilon \to [-\infty, \infty]$.

If $\theta \geq 1 - \beta/p$, then $Ef \in N^{1,p}(\overline{X_\varepsilon}, \mu_\beta)$ with

$$\int_{\overline{X_\varepsilon}} |Ef|^p \, d\mu_\beta \lesssim \|f\|_{\theta,p}^p \quad \text{and} \quad \int_{\overline{X_\varepsilon}} |Ef|^p \, d\mu_\beta \lesssim \int_Z |f|^p \, d\nu. \quad (12.2)$$

Moreover, if $\zeta \in Z$ is an $L^q(\nu)$-Lebesgue point of $f$ for some $q \geq 1$ then $\zeta$ is an $L^q(\mu_\beta)$-Lebesgue point of $Ef$, and

$$Ef(z) = f(z).$$

Let $\zeta$ be an $L^1(\nu)$-Lebesgue point of $f$. Then for any choice of $z_n \in A_n$ with $d_\varepsilon(z_n, \zeta) \leq \alpha^{-n}$ for $n = 1, 2, \ldots$, we have,

$$\lim_{n \to \infty} Ef((z_n, n)) = f(\zeta).$$

Note that $E$ is a linear operator.

**Proof.** If $v = (z, n) \sim (y, m) = w$, then $|m - n| \leq 1$ and so, by the choice of $\varepsilon = \log \alpha$,

$$d_\varepsilon(v, w) \sim e^{-\varepsilon n} = \alpha^{-n}.$$

The function given for $x \in [v, w]$ by

$$g_{[v, w]}(x) := \frac{|Ef(v) - Ef(w)|}{d_\varepsilon(v, w)}$$

is an upper gradient of $Ef$ on $[v, w]$ with respect to the uniformized metric $d_\varepsilon$. Note that $g_{[v, w]}$ is a constant function. Because of $|m - n| \leq 1$ and $(z, n) \sim (y, m)$, we have for all $\eta \in B_Z(z, \alpha^{-n})$ that

$$B_Z(y, \alpha^{-m}) \subset 4\varepsilon B_Z(\eta, \alpha^{1-n}).$$

Hence,

$$g_{[v, w]} \simeq \alpha^n \left( \int_{B_Z(z, \alpha^{-n})} f(\zeta) \, d\nu(\zeta) - \int_{B_Z(y, \alpha^{-m})} f(\eta) \, d\nu(\eta) \right)$$

$$\lesssim \alpha^n \int_{B_Z(z, \alpha^{-n})} \int_{B_Z(y, \alpha^{-m})} |f(\zeta) - f(\eta)| \, d\nu(\eta) \, d\nu(\zeta)$$

$$\lesssim \alpha^n \int_{B_Z(z, 4\varepsilon \alpha^{1-n})} \int_{B_Z(\eta, 4\varepsilon \alpha^{1-n})} |f(\zeta) - f(\eta)| \, d\nu(\eta) \, d\nu(\zeta).$$
Now by Hölder’s inequality, we see that
\[ g^p_{[v,w]} \lesssim \alpha^{n(p(1-\theta))} \int_{B_Z(z,4\tau\alpha^{1-n})} \int_{B_Z(\eta,4\tau\alpha^{1-n})} \frac{|f(\zeta) - f(\eta)|^p}{\alpha^{-np\theta}} d\nu(\eta) d\nu(\zeta). \]
Therefore, letting \( g : X \to \mathbb{R} \) be given by \( g = g_{[v,w]} \) on each edge \([v,w]\), and noting from \((10.6)-(10.7)\) that
\[ d\mu_\beta \simeq \alpha^{-\beta n/\varepsilon} B(Z(z,\alpha^{-n})) d\mathcal{L} \quad \text{on } [v,w] \text{ with } n = \pi_2(v), \]
we obtain
\[
\int_{[v,w]} g^p \, d\mu_\beta \lesssim \alpha^{n(p(1-\theta) - \beta/\varepsilon)} \int_{B_Z(z,4\tau\alpha^{1-n})} \int_{B_Z(\eta,4\tau\alpha^{1-n})} \frac{|f(\zeta) - f(\eta)|^p}{\alpha^{-np\theta}} d\nu(\eta) d\nu(\zeta),
\]
where \( z = \pi_1(v) \in A_n \subset Z \). For each nonnegative integer \( n \) set \( X(n) := \{ x \in X : n \leq \pi_2(x) < n + 1 \} \),
where \( \pi_2(x) \) is as in (11.1). By Proposition 4.5, each vertex in \( X \) has degree at most \( K \) and thus we get integrating over \( X(n) \) that
\[
\int_{X(n)} g^p \, d\mu_\beta \lesssim \sum_{z \in A_n} \sum_{V \ni w \sim (z,n)} \int_{(z,n),w} g^p \, d\mu_\beta \lesssim \alpha^{n(p(1-\theta) - \beta/\varepsilon)} \int_{B_Z(z,4\tau\alpha^{1-n})} \int_{B_Z(\eta,4\tau\alpha^{1-n})} \frac{|f(\zeta) - f(\eta)|^p}{\alpha^{-np\theta}} d\nu(\eta) d\nu(\zeta)
\]
\[
\lesssim \alpha^{n(p(1-\theta) - \beta/\varepsilon)} \int_{Z} \int_{B_Z(\eta,4\tau\alpha^{1-n})} \frac{|f(\zeta) - f(\eta)|^p}{\alpha^{-np\theta}} d\nu(\eta) d\nu(\zeta).
\]
In the last line we used the fact that the balls \( B_Z(z,4\tau\alpha^{1-n}) \), \( z \in A_n \), have a bounded overlap in \( Z \) because of the doubling property of \( Z \). As \( X = \bigcup_{n=0}^\infty X(n) \) with \( X(n) \cap X(m) = \emptyset \) if \( m \neq n \), it follows that
\[
\int_{X(n)} g^p \, d\mu_\beta \lesssim \sum_{n=0}^\infty \alpha^{n(p(1-\theta) - \beta/\varepsilon)} \int_{Z} \int_{B_Z(\eta,4\tau\alpha^{1-n})} \frac{|f(\zeta) - f(\eta)|^p}{\alpha^{-np\theta}} d\nu(\eta) d\nu(\zeta).
\]
At \( \theta \geq 1 - \beta/p\varepsilon \), it then follows from Lemma 9.9 that
\[
\int_{X(n)} g^p \, d\mu_\beta \lesssim \|f\|_{p,\beta}^p < \infty.
\]
As \( \mu_\beta \) supports a 1-Poincaré inequality on \( X_\varepsilon \), by Theorem 10.3, and \( X_\varepsilon \) is bounded, it follows that \( Ef \in N^{1,p}(X_\varepsilon,\mu_\beta) \).

As in the proof of Theorem 11.3, we have an extension \( u \in N^{1,p}(\overline{X}_\varepsilon,\mu_\beta) \) of \( Ef \), and \( C_p \)-q.e. point in \( \overline{X}_\varepsilon \) is a Lebesgue point of \( u \). As \( \mu_\beta(\partial_2 X) = 0 \), we see that \( u(x) = Ef(x) \) for \( C_p \)-q.e. \( x \in \partial_2 X \), where \( Ef|_{\partial_2 X} \) is given by (12.1). Hence \( Ef \) is also in \( N^{1,p}(\overline{X}_\varepsilon,\mu_\beta) \). Since \( X_\varepsilon \) is open in \( \overline{X}_\varepsilon \), we see that the minimal \( p \)-weak upper gradients of \( Ef \) with respect to \( \overline{X}_\varepsilon \) and \( X_\varepsilon \) coincide almost everywhere in \( X_\varepsilon \), and thus the first inequality in (12.2) follows from (12.3).

To control the \( L^p \)-norm of \( Ef \) as stated in the theorem, note that for \( v = (z,n) \sim w = (y,m) \),
\[
\int_{[v,w]} |Ef|^p \, d\mu_\beta \leq \mu_\beta([v,w]) |Ef(v)|^p + |Ef(w)|^p
\]
\[
\leq \mu_\beta([v,w]) \left( \int_{B_Z(z,\alpha^{-n})} |f|^p \, d\nu + \int_{B_Z(y,\alpha^{-m})} |f|^p \, d\nu \right)
\]
\[
\lesssim \mu_\beta([v,w]) \int_{B_Z(z,4\tau\alpha^{1-n})} |f|^p \, d\nu.
\]
Therefore, for each nonnegative integer $n$, with $X(n)$ as above, we have that
\[
\int_{X(n)} |Ef|^p \, d\mu_\beta \lesssim \alpha^{-n\beta/\varepsilon} \sum_{z \in A_n} \nu(B_Z(z, \alpha^{-n})) \int_{B_Z(z, 4\tau \alpha^{1-\varepsilon})} |f|^p \, d\nu
\]
\[
\lesssim \alpha^{-n\beta/\varepsilon} \int_Z |f|^p \, d\nu.
\]
It follows that
\[
\int_{X_n} |Ef|^p \, d\mu_\beta \lesssim \|f\|_{L^p(Z)}^p \sum_{n=0}^\infty \alpha^{-n\beta/\varepsilon} \lesssim \|f\|_{L^p(Z)}^p
\]
as desired.

Assume that $q \geq 1$ and that $\zeta \in Z$ is an $L^q(\nu)$-Lebesgue point of $f$. Let $N \geq 0$ be a fixed but arbitrary integer and consider all $x \in X$ such that $d_x(x, \zeta) < r := \alpha^{-N}/\varepsilon$. If $x$ belongs to an edge $[v, w]$, then at least one of the vertices also belongs to $B_x(\zeta, r)$, say $w = (y, m)$, and hence
\[
\alpha^{-m} = e^{-\varepsilon m} = \varepsilon d_x(w) \leq \varepsilon d_x(w, \zeta) < \varepsilon r = \alpha^{-N},
\]
from which it follows that $m \geq N + 1$ and thus $n \geq N$, where $v = (z, n)$. In particular, $d_x(z, v) = d_x(v) = e^{-\varepsilon n}/\varepsilon \leq r$ and $d_x(v, x) \leq e^{-\varepsilon N} = \varepsilon r$. Proposition 4.4 then yields
\[
d_Z(z, \zeta) \leq 2\tau \alpha d_x(z, \zeta) \leq 2\tau \alpha (d_x(z, v) + d_x(v, x) + d_x(x, \zeta)) < 2\tau \alpha (2 + \varepsilon)r,
\]
and similarly $d_Z(y, \zeta) < 2\tau \alpha (2 + \varepsilon)r$. Since $Ef(x)$ is a convex combination of $Ef(v)$ and $Ef(w)$, we have
\[
\int_{[v, w]} |Ef - f(\zeta)|^q \, d\mu_\beta \leq (|Ef(v) - f(\zeta)|^q + |Ef(w) - f(\zeta)|^q)\mu_\beta([v, w]),
\]
where by the definition of $Ef$ and Hölder’s inequality,
\[
|Ef(v) - f(\zeta)|^q = \left| \int_{B_Z(z, \alpha^{-n})} f \, d\nu - f(\zeta) \right|^q \leq \int_{B_Z(z, \alpha^{-n})} |f - f(\zeta)|^q \, d\nu.
\]
Noting that
\[
\mu_\beta([v, w]) \simeq \rho_\beta(v)\hat{\mu}(v) = e^{-\beta n}\nu(B_Z(z, \alpha^{-n}))
\]
and that every vertex belongs to at most a bounded number of edges (by Proposition 4.5), we conclude that
\[
\int_{B_x(\zeta, r)} |Ef - f(\zeta)|^q \, d\mu_\beta \lesssim \sum_{n \geq N} e^{-\beta n} \sum_{z \in A_n \cap B_Z(\zeta, 2\tau \alpha (2 + \varepsilon)r)} \int_{B_Z(z, \alpha^{-n})} |f - f(\zeta)|^q \, d\nu.
\]
Since for each $n \geq N$ we have that $\alpha^{-n} \leq \alpha^{-N} = \varepsilon r$ and the balls $B_Z(z, \alpha^{-n})$, $z \in A_n$, have bounded overlap in $Z$, we obtain
\[
\int_{B_x(\zeta, r)} |Ef - f(\zeta)|^q \, d\mu_\beta \lesssim \sum_{n \geq N} e^{-\beta n} \int_{B_Z(\zeta, 4\tau \alpha (1 + \varepsilon)r)} |f - f(\zeta)|^q \, d\nu
\]
\[
\simeq e^{-\beta N} \int_{B_Z(\zeta, 4\tau \alpha (1 + \varepsilon)r)} |f - f(\zeta)|^q \, d\nu,
\]
where $e^{-\beta N} = (\alpha^{-N})^{\beta/\varepsilon} = (\varepsilon r)^{\beta/\varepsilon}$. Dividing by $\mu_\beta(B_x(\zeta, r)) \simeq (\varepsilon r)^{\beta/\varepsilon}\nu(B_Z(\zeta, \varepsilon r))$ (because of Corollary 10.5) and letting $N \to \infty$ shows that $\zeta$ is an $L^q(\mu_\beta)$-Lebesgue
point of $Ef$. That $Ef(\zeta) = f(\zeta)$ now follows directly from the definition of $Ef(\zeta)$ in (12.1) and by considering the case $q = p$ in the above discussion (recall that $f$ is necessarily in $L^p(\nu)$ and so $\nu$-a.e. point in $Z$ is an $L^p(\nu)$-Lebesgue point of $f$).

Moreover, with $z_n$ as in the final claim of the theorem for $n = 1, 2, \ldots$, the doubling property of $\nu$ and the fact that $B_Z(z_n, \alpha^{-n}) \subset B_Z(\zeta, 2\alpha^{-n})$ yield

$$|Ef((z_n, n)) - f(\zeta)| = \int_{B_Z(z_n, \alpha^{-n})} |f(y) - f(\zeta)| \, d\nu(y) \lesssim \int_{B_Z(\zeta, 2\alpha^{-n})} |f(y) - f(\zeta)| \, d\nu(y) \to 0, \quad \text{as } n \to \infty. \quad \square$$

13. Properties of Besov functions on $Z$

Recall the standing assumptions from Sections 10 and 11. Since $\epsilon = \log \alpha$, from Proposition 4.4 we know that $\partial \mathcal{X}$ can be identified with $Z$ in a biLipschitz fashion. In this section we also fix $0 < \theta < 1$ and let $\beta = \epsilon p(1 - \theta)$.

In this section we will only consider the Besov spaces on $\mathcal{X}$ as in Theorem 1.1. This makes it possible to derive various regularity properties for $B^\theta_{p,p}(Z)$ from the theory of Newtonian spaces.

**Proposition 13.1.** If $0 < \theta < 1$ then Lipschitz functions are dense in $B^\theta_{p,p}(Z)$.

**Proof.** Equip the uniformized hyperbolic filling $X_\epsilon$ with the measure $\mu_\beta$, where $\beta = \epsilon p(1 - \theta)$. Theorems 11.3 and 12.1 tell us that $B^\theta_{p,p}(Z)$ is the trace space of $N^{1,p}(X_\epsilon, \mu_\beta)$, with comparable norms.

Since $\mu_\beta$ is doubling and supports a 1-Poincaré inequality on $X_\epsilon$, it follows from Shanmugalingam [45, Theorem 4.1 and Corollary 3.9] that Lipschitz functions are dense in $N^{1,p}(X_\epsilon, \mu_\beta)$. Their restrictions to $Z$ are then dense in $B^\theta_{p,p}(Z)$. \square

**Proposition 13.2.** Let $E \subset Z$. Then $\text{Cap}_{B^\theta_{p,p}(Z)}(E) \simeq C^\theta (E)$.

**Proof.** Let $u \in B^\theta_{p,p}(Z)$ be admissible in the definition of $\text{Cap}_{B^\theta_{p,p}(Z)}(E)$, i.e. $u \geq 1$ $\nu$-a.e. in an open neighborhood $G \subset Z$ of $E$. By truncation and redefinition on a set of $\nu$-measure zero, we may assume that $u \equiv 1$ in $G$. Let $Eu \in N^{1,p}(X_\epsilon, \mu_\beta)$ be its extension as guaranteed by Theorem 12.1. As all points in $G$ are Lebesgue points for $u$, we see that $Eu \equiv u \equiv 1$ in $G$. Hence $Eu$ is admissible in computing $C^\theta (E)$, and so by Theorem 12.1,

$$C^\theta (E) \leq \|Eu\|_{N^{1,p}(X_\epsilon, \mu_\beta)}^p \lesssim \|u\|_{B^\theta_{p,p}(Z)}^p.$$  

Taking infimum over all $u$ admissible in the definition of $\text{Cap}_{B^\theta_{p,p}(Z)}(E)$ proves one inequality in the statement of the lemma.

Conversely, since $C^\theta$ is an outer capacity by Theorems 9.6 and 10.3, for each $\eta > 0$ we can find an open set $U \subset X_\epsilon$ with $E \subset U$ and a function $u \in N^{1,p}(X_\epsilon, \mu_\beta)$ such that $u \geq 1$ on $U$ and

$$\|u\|_{N^{1,p}(X_\epsilon, \mu_\beta)} \leq C^\theta (E) + \eta.$$  

By Theorem 11.3, the function $f = u|_Z \in B^\theta_{p,p}(Z)$ with

$$\|f\|_{L^p(Z)} + \|f\|_{\theta,p,Z} \leq \|u\|_{N^{1,p}(X_\epsilon, \mu_\beta)} \lesssim C^\theta (E) + \eta.$$  

As $f \geq 1$ on the relatively open set $G := U \cap Z$, letting $\eta \to 0$ concludes the proof. \square
We have now shown that for subsets of $Z$ the two capacities are comparable. Next we turn our attention to the matter of continuity properties of Besov functions. The following result shows that functions in $B^\theta_{p,p}(Z)$ have representatives that are quasicontinuous with respect to the Besov capacity, i.e. such that for each $\eta > 0$ there is an open set $G \subset Z$ with $\text{Cap}_{\theta\cdot p,p}(Z)(G) < \eta$ such that $f|_{Z\setminus G}$ is continuous.

**Proposition 13.3.** Let $f_0 \in B^\theta_{p,p}(Z)$. Then there is a $\text{Cap}_{\theta\cdot p,p}(Z)$-quasicontinuous function $f \in B^\theta_{p,p}(Z)$ such that $f = f_0$ $\nu$-a.e. in $Z$.

*Proof.* Given such a function $f_0$, let $E f_0 \in \mathcal{N}^{1,p}(\overline{\mathbb{X}},\mu_\beta)$ be its extension given by Theorem 12.1. Then $f := E f_0|_Z = f_0$ $\nu$-a.e. By Theorems 9.6 and 10.3, $E f_0$ is $C^\theta_{p,p}$-quasicontinuous, i.e. for each $\eta > 0$ there is an open set $U \subset \overline{\mathbb{X}}$ with $C^\theta_{p,p}(U) < \eta$ such that $f|_{Z\cap U}$ is continuous. By choosing $G = U \cap Z$, we get that $f|_{Z\setminus G}$ is continuous. Moreover, by Proposition 13.2, we see that

$$\text{Cap}_{\theta\cdot p,p}(Z)(G) \approx C^\theta_{p,p}(G) \leq C^\theta_{p,p}(U) < \eta,$$

which completes the proof. \qed

The following result shows that Besov functions have Lebesgue points q.e., provided that the measure on $Z$ satisfies a reverse-doubling property.

**Proposition 13.4.** Assume that $\nu$ satisfies (10.13) with $s_\nu > 0$ and that there is some $\eta > 0$ such that

$$\frac{\nu(B_Z(\zeta,r'))}{\nu(B_Z(\zeta,r))} \lesssim \left(\frac{r'}{r}\right)^{s_\nu} \quad (13.1)$$

for all $\zeta \in Z$ and all $0 < r' \leq r \leq 2\text{diam }Z$. Let $1 \leq q \leq s_\nu/p/(s_\nu - p\theta)$ and $\tilde{u} \in B^\theta_{p,p}(Z)$. Then there is a function $u \in L^q(Z)$ such that $u = \tilde{u}$ $\nu$-a.e. and

$$\lim_{r \to 0^+} \int_{B_z(\zeta,r)\cap Z} |u - u(\zeta)|^q \, d\nu = 0 \quad \text{for } \text{Cap}_{\theta\cdot p,p}(Z)\text{-q.e. } \zeta \in Z.$$

Since $\nu$ is doubling, condition (13.1) is equivalent to $Z$ being uniformly perfect, see Martín–Ortiz [41, Lemma 7]. See [47] for a weaker Lebesgue point result when $Z$ is not necessarily uniformly perfect. Embeddings of Besov spaces into $L^q$ spaces were also obtained in Malý [39, Corollary 3.18(i)] via embeddings into Hajlasz–Sobolev spaces. Proposition 13.4 will follow from our trace and extension results and the following two-weighted Poincaré type inequality, which is a special case of Björn–Kahsa [6, Theorem 3.1].

**Proposition 13.5.** Let $\nu$ and $\mu_\beta$ be doubling measures on $Z = \partial X$ and $\overline{\mathbb{X}}$, respectively. Assume moreover that $\mu_\beta$ supports a $p$-Poincaré inequality on $\overline{\mathbb{X}}$ with dilation $\lambda$ and that $\nu$ satisfies the reverse-doubling condition (13.1). Let $1 \leq p < q < \infty$ and $u \in \mathcal{N}^{1,p}(\overline{\mathbb{X}},\mu_\beta)$ be such that $\nu$-a.e. $z \in Z$ is a $\mu_\beta$-Lebesgue point of $u$. Then for all balls $B = B_z(\zeta,r)$ with $\zeta \in Z$, \n
$$\left(\int_{B\cap Z} |u - u_{B,\mu_\beta}|^q \, d\nu\right)^{1/q} \lesssim \Theta_q(r) \left(\int_{2\lambda B} g^p_{\nu}\, d\mu_\beta\right)^{1/p},$$

where \n
$$\Theta_q(r) := \sup_{0<r\leq r} \sup_{\zeta \in B\cap Z} \frac{\nu(B_z(\zeta,\rho))^{1/q}}{\mu_\beta(B_z(\zeta,\rho))^{1/p}}.$$
Proof of Proposition 13.4. By Hölder’s inequality, it suffices to consider the case of \( q > p \). Using Theorem 12.1, we can find a function \( u \in N^{1,p}(\mathbb{X}, \mu_\beta) \) such that \( u = \tilde{u} \) \( \nu \)-a.e. on \( Z \). By [27, Lemma 9.2.4] and Theorem 11.3, we know that for \( \mathcal{C}_p^n \)-q.e. \( \zeta \in Z \),

\[
\lim_{r \to 0^+} r^p \int_{B_r(\zeta,r)} g^p \, d\mu_\beta = 0 \quad \text{and} \quad \lim_{r \to 0^+} \int_{B_r(\zeta,r)} u \, d\mu_\beta = u(\zeta). \tag{13.2}
\]

In particular, Proposition 11.2 shows that \( \nu \)-a.e. \( z \in Z \) is a \( \mu_\beta \)-Lebesgue point of \( u \).

Proposition 13.2 shows that \((13.2)\) holds for \( \mathcal{C}_p^n(Z)\)-q.e. \( \zeta \in Z \). For such \( \zeta \), we have by the Minkowski inequality and Proposition 13.5 that

\[
\left( \int_{B_r(\zeta, r) \cap Z} |u - u(\zeta)|^q \, d\nu \right)^{1/q} \lesssim \int_{B_r(\zeta, r)} u \, d\mu_\beta - u(\zeta) + \Theta_q(r) \frac{\mu_\beta(B_r(\zeta, 2\lambda r))^{1/p}}{\nu(B_r(\zeta, r))^{1/q}} \left( \int_{B_r(\zeta, 2\lambda r)} g^p \, d\mu_\beta \right)^{1/p}. \tag{13.3}
\]

In view of \((13.2)\) and the definition of \( \Theta_q(r) \), it suffices to show that for all \( 0 < \rho \leq r \leq 2 \text{diam}_x X_e \) and \( z \in B_2(\zeta, r) \cap Z \),

\[
\frac{\rho \nu(B_2(z, \rho))^{1/q}}{\mu_\beta(B_2(z, \rho))^{1/p}} \simeq r \tag{13.4}
\]

with a comparison constant independent of \( z, \rho \) and \( r \). Theorem 10.3 and the doubling property show that

\[
\frac{\rho \nu(B_2(z, \rho))^{1/q}}{\mu_\beta(B_2(z, \rho))^{1/p}} \simeq \rho^{1 - \beta/\epsilon p} \nu(B_2(z, \rho))^{1/q - 1/p - 1/q - 1/p - 1/q - 1/q - 1/q - 1/q}. \]

Since \( 1/q - 1/p < 0 \) and \( \nu \) satisfies \((10.13)\), the required estimate \((13.4)\) holds because

\[
1 - \frac{\beta}{\epsilon p} + s \nu \left( \frac{1}{q} - \frac{1}{p} \right) \geq 0.
\]

The conclusion \( u \in L^q(Z) \) follows by applying \((13.3)\) to \( r = 2 \text{diam}_x X_e \). \( \square \)

Even though we have so far only considered compact \( Z \), we can now apply Proposition 13.4 to obtain the following improvement of Netrusov’s result [42, Proposition 1.4] in \( \mathbb{R}^n \), which was obtained for \( q < np/(n - p\theta) \). The lift from the compact to the unbounded case is somewhat subtle since the Besov norm is nonlocal.

Proposition 13.6. Assume that \( n \geq 1 \) and that \( p\theta < n \). Let \( q = np/(n - p\theta) \), \( \tilde{u} \in B_{p,p}^\theta(\mathbb{R}^n) \), \( u(\zeta) := \limsup_{r \to 0^+} \tilde{u}_{B_r(\zeta, r)} \), and

\[
E = \left\{ \zeta : \limsup_{r \to 0^+} \int_{B_r(\zeta, r)} |u(x) - u(\zeta)|^q \, dx > 0 \right\}
\]

be the set of non-\( L^q \)-Lebesgue points for \( u \). Then \( \text{Cap}_{B_{p,p}^\theta(\mathbb{R}^n)}(E) = 0 \).

Proof. For \( r > 0 \), let \( Z_r = \overline{B(0,r)} \). Observe that for \( x \in Z_r \) and \( 0 < \rho \leq 2r \) we have \( m(B(x, \rho) \cap Z_r) \simeq m(B(x, \rho)) \simeq \rho^n \). It follows that \( B_{p,p}^\theta(\mathbb{R}^n) \subset B_{p,p}^\theta(Z_r) \)
for all \( r > 0 \). Since \( \text{Cap}_{B^p_{p,p}(R^n)} \) is comparable to a countably subadditive Besov capacity on \( R^n \), see Adams–Hedberg [1, Propositions 2.3.6 and 4.4.3], it suffices to show that \( \text{Cap}_{B^p_{p,p}(R^n)}(E \cap B(0,r)) = 0 \) for all \( r > 1 \). Fix \( \varepsilon > 0 \) and let \( R \geq 3r \). As \( u \in B^p_{p,p}(Z_R) \), it follows from Proposition 13.4 that \( \text{Cap}_{B^p_{p,p}(Z_R)}(E \cap B(0,r)) = 0 \). Proposition 13.2, together with [2, Lemma 6.15 and Theorem 4.21] and Theorem 11.3, now implies that there is a function \( v : R^n \to [0,1] \) with \( \text{supp} v \subset B(0,2r) \) such that \( v \geq 1 \) in an open neighborhood of \( E \cap B(0,r) \) and \( \|v\|_{B^p_{p,p}(Z_R)}^p < \varepsilon \).

Now

\[
\|v\|_{L^p(R^n)}^p + \|v\|_{L^p(R^n)}^p \lesssim \left\| \begin{array}{l}
v \\nu_{B^p_{p,p}(Z_R)}^p \\int_{\mathbb{R}^n} \int_{B(0,2r)} |v(z)| d\xi d\eta \\
\mathbb{R}^n \setminus Z_R \end{array} \right\| \lesssim \varepsilon + \frac{v(\zeta)}{(2r)^n},
\]

as \( R \to \infty \). Hence \( \text{Cap}_{B^p_{p,p}(R^n)}(E \cap B(0,r)) \lesssim \varepsilon \), and letting \( \varepsilon \to 0 \) concludes the proof.

The following result extends Proposition 6.6 in Björn–Björn–Gill–Shanmugalingam [3] to general compact doubling metric measure spaces \( Z \), and essentially recovers Corollary 3.18 (iii) in Malý [39].

**Proposition 13.7.** Assume that the measure \( \nu \) on \( Z \) satisfies (10.13) for all \( \zeta \in Z \) and \( 0 < r < d \) diam \( Z \), with exponent \( s_\nu \). Let \( s_\beta = \max \{ 1, p(1 - \theta) + s_\nu \} \). If \( p > s_\beta \) then every \( f \in B^\theta_{p,p}(Z) \) has a \( \nu \)-a.e. representative which is \( (1 - s_\beta/p) \)-Hölder continuous on \( Z \).

**Proof.** By Theorem 12.1, there is a function \( u \in N^{1,p}(X, \mu_\beta) \) such that \( u|_Z = f \) \( \nu \)-a.e. By Lemma 10.6 and by \( \beta = \varepsilon p(1 - \theta) \), \( \mu_\beta \) satisfies the dimension condition (10.14).

Since \( p > s_\beta \), functions in \( N^{1,p}(X, \mu_\beta) \) are \( (1 - s_\beta/p) \)-Hölder continuous with respect to \( d_\varepsilon \), by [2, Corollary 5.49] or [27, Theorem 9.2.14]. (There is a missing local compactness assumption in [27, Theorem 9.2.14].) It follows that the trace \( u|_Z \) is \( (1 - s_\beta/p) \)-Hölder continuous with respect to \( d_\varepsilon \).

Elementary calculations show that Proposition 13.7 applies in the following two cases with \( 0 < \theta < 1 \) and \( p > s_\nu / \theta \):

- If \( p \geq (1 - s_\nu)/(1 - \theta) \), then \( s_\beta = p(1 - \theta) + s_\nu \geq 1 \) and \( 1 - s_\beta/p = \theta - s_\nu/p \).
  Hence every \( f \in B^\theta_{p,p}(Z) \) has a \( \nu \)-a.e. representative which is \( (\theta - s_\nu/p) \)-Hölder continuous.

- If \( 1 < p < (1 - s_\nu)/(1 - \theta) \) (which necessarily implies that \( s_\nu < \theta \)), then \( p(1 - \theta) + s_\nu < 1 = s_\beta \) and every \( f \in B^\theta_{p,p}(Z) \) has a \( \nu \)-a.e. representative which is \( (1 - 1/p) \)-Hölder continuous.

If \( \theta \geq 1 \) then \( B^\theta_{p,p}(Z) \subset B^\theta_{p,p}(Z) \) for every \( 0 < \theta' < 1 \) and the above two cases with \( \theta \) replaced by \( \theta' \) imply (upon letting \( \theta' \to 1 \)):

- If \( p > s_\nu \geq 1 \), then every \( f \in B^\theta_{p,p}(Z) \) has a \( \nu \)-a.e. representative which is \( \eta \)-Hölder continuous for any \( 0 < \eta < 1 - s_\nu/p \).

- If \( 0 < s_\nu < 1 < p \), then every \( f \in B^\theta_{p,p}(Z) \) has a \( \nu \)-a.e. representative which is \( (1 - 1/p) \)-Hölder continuous.
References

1. Adams, D. R. and Hedberg, L. I., Function Spaces and Potential Theory, Springer, Berlin–Heidelberg, 1996. 50
2. Björn, A. and Björn, J., Nonlinear Potential Theory on Metric Spaces, EMS Tracts in Mathematics 17, European Math. Soc., Zürich, 2011. 2, 29, 30, 42, 43, 50
3. Björn, A., Björn, J., Gill, J. and Shanmugalingam, N., Geometric analysis on Cantor sets and trees, J. Reine Angew. Math. 725 (2017), 63–114. 43, 50
4. Björn, A., Björn, J. and Shanmugalingam, N., Quasicontinuity of Newton–Sobolev functions and density of Lipschitz functions on metric spaces, Houston J. Math. 34 (2008), 1197–1211. 30
5. Björn, A., Björn, J. and Shanmugalingam, N., Bounded geometry and $p$-harmonic functions under uniformization and hyperbolization, to appear in J. Geom. Anal. arXiv:1908.04644 7, 8, 33, 34, 36
6. Björn, J. and Kalamajska, A., Poincaré inequalities, dimensions and compact embeddings from Sobolev type spaces into weighted $L^q$ spaces on metric spaces, In preparation. 48
7. Björn, J. and Shanmugalingam, N., Poincaré inequalities, uniform domains and extension properties for Newton–Sobolev functions in metric spaces, J. Math. Anal. Appl. 332 (2007), 190–208. 43
8. Bonk, M., Heinonen, J. and Koskela, P., Uniformizing Gromov hyperbolic spaces, Astérisque 270 (2001). 2, 5, 7, 8, 12, 17, 20, 34
9. Bonk, M. and Kleiner, B., Quasisymmetric parametrizations of two-dimensional metric spheres, Invent. Math. 150 (2002), 127–183. 1, 2
10. Bonk, M. and Saksman, E., Sobolev spaces and hyperbolic fillings, J. Reine Angew. Math 737 (2018), 161–187. 4, 5, 9, 10
11. Bonk, M., Saksman, E. and Soto, T., Triebel–Lizorkin spaces on metric spaces via hyperbolic fillings, Indiana Univ. Math. J. 67 (2018), 1625–1663. 4, 5
12. Bonk, M. and Schramm, O., Embeddings of Gromov hyperbolic spaces, Geom. Funct. Anal. 10 (2000), 266–306. 2, 19, 20
13. Bourdon, M. and Pajot, H., Cohomologie $l_p$ et espaces de Besov, J. Reine Angew. Math. 558 (2003), 85–108. 1, 2, 4, 9, 10
14. Bridson, M. and Haefliger, A., Metric Spaces of Non-positive Curvature, Grundlehren der mathematischen Wissenschaften 319, Springer, Berlin, 1999. 2, 12
15. Butler, C., Uniformizing Gromov hyperbolic spaces with Busemann functions, Preprint, 2020. arXiv:2007.11143 5, 6
16. Buyalo, S. and Schroeder, V., Elements of Asymptotic Geometry, EMS Monographs in Mathematics 3, European Math. Soc., Zürich, 2007. 2, 9, 10
17. Caffarelli, L. and Silvestre, L., An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007), 1245–1260. 4
18. Ferrari, F. and Franchi, B., Harnack inequality for fractional sub-Laplacians in Carnot groups, Math. Z. 279 (2015), 435–458. 4
19. Gagliardo, E., Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in $n$ variabili, Rend. Semin. Mat. Univ. Padova 27 (1957), 284–305. 3
20. Gill, J. T. and Lopez, M., Discrete approximations of metric measure spaces of controlled geometry, J. Math. Anal. Appl. 431 (2015), 73–98. 32
21. Gogatishvili, A., Koskela, P. and Shanmugalingam, N., Interpolation properties of Besov spaces defined on metric spaces, *Math. Nachr.* **283** (2010), 215–231.

22. Gromov, M., *Metric Structures for Riemannian and Non-Riemannian Spaces*, Progress in Math. **152**, Birkhäuser, Boston, MA, 1999.

23. Han, Y., Müller, D. and Yang, D., A theory of Besov and Triebel–Lizorkin spaces on metric measure spaces modeled on Carnot–Carathéodory spaces, *Abstract and Applied Analysis* **2008**, Art. ID 0102739.

24. Heikkinen, T., Koskela, P. and Tuominen, H., Approximation and quasicontinuity of Besov and Triebel–Lizorkin functions, *Trans. Amer. Math. Soc.* **369** (2017), 3547–3573.

25. Heinonen, J., *Lectures on Analysis on Metric Spaces*, Springer, New York, 2001.

26. Heinonen, J. and Koskela, P., Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** (1998), 1–61.

27. Heinonen, J., Koskela, P., Shanmugalingam, N. and Tyson, J. T., *Sobolev Spaces on Metric Measure Spaces*, New Mathematical Monographs **27**, Cambridge Univ. Press, Cambridge, 2015.

28. Herron, D., Shanmugalingam, N. and Xie, X., Uniformity from Gromov hyperbolicity, *Illinois J. Math.* **52** (2008), 1065–1109.

29. Jonsson, A. and Wallin, H., The trace to subsets of $\mathbb{R}^n$ of Besov spaces in the general case, *Anal. Math.* **6** (1980), 223–254.

30. Jonsson, A. and Wallin, H., *Function Spaces on Subsets of $\mathbb{R}^n$*, Math. Rep. **2:1**, Harwood, London, 1984.

31. Kallunki [Rogovin], S. and Shanmugalingam, N., Modulus and continuous capacity, *Ann. Acad. Sci. Fenn. Math.* **26** (2001), 455–464.

32. Kinnunen, J., Koskela, P., Shanmugalingam, N. and Tuominen, H., Lebesgue points and capacities via the boxing inequality in metric spaces, *Indiana Univ. Math. J.* **57** (2008), 401–430.

33. Koskela, P. and MacManus, P., Quasiconformal mappings and Sobolev spaces, *Studia Math.* **131** (1998), 1–17.

34. Koskela, P., Yang, D. and Zhou, Y., Pointwise characterizations of Besov and Triebel–Lizorkin spaces and quasiconformal mappings, *Adv. Math.* **226** (2011), 3579–3621.

35. Lindquist, J. and Shanmugalingam, N., Rough isometry between Gromov hyperbolic spaces and uniformization, to appear in *Ann. Acad. Sci. Fenn. Math.*

36. López, M., *Discrete Approximations of Metric Measure Spaces with Controlled Geometry*, Ph.D. Thesis, University of Cincinnati, Cincinnati, OH, 2015.

37. Luukkainen, J. and Saksman, E., Every complete doubling metric space carries a doubling measure, *Proc. Amer. Math. Soc.* **126** (1998), 531–534.

38. Mäkäläinen, T., Adams inequality on metric measure spaces, *Rev. Mat. Iberoam.* **25** (2009), 533–558.

39. Malý, L., Trace and extension theorems for Sobolev-type functions in metric spaces, *Preprint*, 2017. *arXiv*:1704.06344, 3, 4, 43, 48, 50.

40. Malý, L., Shanmugalingam, N. and Snipes, M., Trace and extension theorems for functions of bounded variation, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **18** (2018), 313–341.

41. Martín, J. and Ortiz, W. A., A Sobolev type embedding for Besov spaces defined on doubling metric spaces, *J. Math. Anal. Appl.* **479** (2019), 2302–2337.

42. Netrusov, Yu. V., Sets of singularities of functions in spaces of Besov and Lizorkin–Triebel type, *Tr. Mat. Inst. Steklova* **187** (1989).
162–177 (in Russian). English translation in Proc. Steklov Inst. Math. 187 (1990), 185–203. 1, 4, 49
43. Peetre, J., Espaces d’interpolation et théorème de Soboleff, Ann. Inst. Fourier (Grenoble) 16:1 (1966), 279–317. 4
44. Peetre, J., A counterexample connected with Gagliardo’s trace theorem, Comment. Math. Special Issue 2 (1979), 277–282. 3
45. Shanmugalingam, N., Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, Rev. Mat. Iberoam. 16 (2000), 243–279. 29, 30, 43, 47
46. Shanmugalingam, N., Harmonic functions on metric spaces, Illinois J. Math. 45 (2001), 1021–1050. 29
47. Shanmugalingam, N., Shibahara, H. and Snipes, M., Neumann boundary value problems and a construction of fractional $p$-Laplacian on compact doubling metric measure spaces, In preparation. 48
48. Soto, T., Besov spaces via hyperbolic fillings, Preprint, 2016. arXiv:1606.08082 4, 5