THE MAPPING CLASS GROUP OF A SHIFT OFFINITE TYPE

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Dedicated to Roy Adler, in memory of his insight, humor and kindness

Abstract. Let \((X_A, \sigma_A)\) be a nontrivial irreducible shift of finite type (SFT), with \(M_A\) denoting its mapping class group: the group of flow equivalences of its mapping torus \(S X_A\), (i.e., self homeomorphisms of \(S X_A\) which respect the direction of the suspension flow) modulo the subgroup of flow equivalences of \(S X_A\) isotopic to the identity. We develop and apply machinery (flow codes, cohomology constraints) and provide context for the study of \(M_A\), and prove results including the following. \(M_A\) acts faithfully and \(n\)-transitively (for every \(n \in \mathbb{N}\)) by permutations on the set of circles of \(S X_A\). The center of \(M_A\) is trivial. The outer automorphism group of \(M_A\) is nontrivial. In many cases, \(\text{Aut}(\sigma_A)\) admits a nonspatial automorphism. For every SFT \((X_B, \sigma_B)\) flow equivalent to \((X_A, \sigma_A)\), \(M_A\) contains embedded copies of \(\text{Aut}(\sigma_B)/\langle \sigma_B \rangle\), induced by return maps to invariant cross sections; but, elements of \(M_A\) not arising from flow equivalences with invariant cross sections are abundant. \(M_A\) is countable and has solvable word problem. \(M_A\) is not residually finite. Conjugacy classes of many (possibly all) involutions in \(M_A\) can be classified by the \(G\)-flow equivalence classes of associated \(G\)-SFTs, for \(G = \mathbb{Z}/2\mathbb{Z}\). There are many open questions.

1. Introduction

Throughout this paper, \(T : X \to X\) denotes a homeomorphism of a compact zero dimensional metric space \(X\), and \(S(X, T)\) is the mapping torus of \(T\), which carries a natural suspension flow (detailed definitions are in Sec. 2). We usually write \(S X\) for

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1. Introduction

Throughout this paper, \(T : X \to X\) denotes a homeomorphism of a compact zero dimensional metric space \(X\), and \(S(X, T)\) is the mapping torus of \(T\), which carries a natural suspension flow (detailed definitions are in Sec. 2). We usually write \(S X\) for
$S(X,T)$. Let $\mathcal{F}(T)$ denote the group of self equivalences of the suspension flow on $\mathbb{S}X$, i.e., the homeomorphisms $\mathbb{S}X \to \mathbb{S}X$ which map orbits onto orbits, respecting the direction of the flow. Define the mapping class group of $T$, $\mathcal{M}(T)$, to be the group of isotopy classes of elements of $\mathcal{F}(T)$. By definition, for $h$ in $\mathcal{F}(T)$, the class $[h]$ is trivial in $\mathcal{M}(T)$ if there is a continuous map $\mathbb{S}X \times [0,1] \to \mathbb{S}X$, $(y,t) \mapsto h_t(y)$, with $h_0$ the identity, $h_1 = h$ and each $h_t$ in $\mathcal{F}(T)$. Because $X$ is zero dimensional, this condition forces each $h_t$ to map each flow orbit to itself. The automorphism group of $T$, $\text{Aut}(T)$, is the group of homeomorphisms $X \to X$ which commute with $T$.

For an irreducible matrix $A$ over $\mathbb{Z}_+$, let $\sigma_A : X_A \to X_A$ be the associated shift of finite type (SFT). We say an SFT is trivial if $X_A$ is a single finite orbit. Let $\mathcal{M}_A = \mathcal{M}(\sigma_A)$. In this paper we study $\mathcal{M}_A$, the mapping class group of an irreducible shift of finite type, introduced in [3]. (Several of the results, along with ingredients of some others, appeared in the Ph.D. thesis of S. Chuy surichay [13].)

Homeomorphisms $T,T'$ are flow equivalent if the suspension flows on their mapping tori are equivalent, i.e. there is a homeomorphism $h : \mathbb{S}X \to \mathbb{S}X'$ mapping orbits onto orbits, respecting the orientation of the flow. Here, $h$ induces an isomorphism $\mathcal{M}(T) \to \mathcal{M}(T')$. $\mathcal{M}(T)$ plays for flow equivalence the role that $\text{Aut}(T)$ plays for topological conjugacy. Flow equivalence is very naturally a part of unified algebraic framework for classifying SFTs (see e.g. [4]). A classification of SFTs up to flow equivalence is known; the classification, and some of the ideas involved, have been quite useful for the stable and unital classification of Cuntz-Krieger algebras (e.g. [23]) and more generally, graph $C^*$-algebras (e.g. [22]). The track record of utility for flow equivalence is another motivation for looking at $\mathcal{M}_A$.

We will see that for a nontrivial irreducible SFT $\sigma_A$, $\mathcal{M}_A$ contains naturally embedded copies of $\text{Aut}(\sigma_B)/\langle \sigma_B \rangle$, for every $\sigma_B$ flow equivalent to $\sigma_A$, where $\langle \sigma_B \rangle$ is the subgroup consisting of the powers of $\sigma_B$. Automorphism groups of SFTs are still poorly understood, despite longstanding interest (e.g. [30, 13, 35]): this relation to automorphism groups is another reason for our interest in $\mathcal{M}_A$, particularly given a resurgence of interest in automorphism groups of various symbolic systems (e.g. [19, 20, 21, 22, 31, 51, 60]).

We are also interested in $\mathcal{M}_A$ as a large (though countable) dynamically defined group. Some such groups arising from zero dimensional dynamics have turned out to be quite interesting as countable groups (e.g. [29, 32, 42].) And although the groups $\mathcal{M}_A$ are quite different from the mapping class groups of surfaces, it is not impossible that from the vast wealth of ideas and tools in the surface case (see [24]) some useful approach to $\mathcal{M}_A$ may be suggested.

We turn now to the organization of the paper. In Section 2 we give background. For a nontrivial irreducible SFT $\sigma_A$, the action of $\text{Aut}(\sigma_A)$ on finite invariant sets of periodic points has been a key tool for progress (e.g. in [35]). In Section 3 we show nothing like this is available to study $\mathcal{M}_A$: for every $n \in \mathbb{N}$, $\mathcal{M}_A$ acts $n$-transitively and faithfully on the set of circles in $\mathbb{S}X_A$. The other general tool which has proved useful for studying $\text{Aut}(\sigma_A)$ (especially with respect to its action on periodic points [35], via Wagoner’s Strong Shift Equivalence spaces [53]) is the dimension representation, $\rho_A$. The analogue of $\rho_A$ for $\mathcal{M}_A$ is the Bowen-Franks representation, $\beta_A$, which for a nontrivial irreducible SFT $\sigma_A$ maps $\mathcal{M}_A$ onto the group of group automorphisms of the Bowen-Franks group $\text{coker}(I - A)$ [3]. Among our questions: is the kernel of $\beta_A$ simple? finitely generated? sofic?
In Section 3, we also show the actions of $\mathcal{M}_A$ on circles of $S_X$ (by permutations) and on $\tilde{H}^1(S_X)$ are faithful, and prove an analogue of Ryan’s Theorem for $\text{Aut}(\sigma_A)$: the center of $\mathcal{M}_A$ is trivial.

In Section 4, we show $\mathcal{M}_A$ has a nontrivial outer automorphism group, and (extending work of [12]) for many mixing SFTs $\sigma_A$ construct a group isomorphism $\text{Aut}(\sigma_A) \to \text{Aut}(\sigma_A)$ which is not spatial: i.e., is not induced by a homeomorphism. We also show that spatial isomorphism of sufficiently rich subgroups is enough to imply flip conjugacy.

In Section 5, we describe how flow equivalences $S_X \to S_X$ with invariant cross sections are the flow equivalences induced by automorphisms of maps $S$ flow equivalent to $T$, and show that by this correspondence $\mathcal{M}_A$ contains embedded copies $\text{Aut}(\sigma_B)/\langle \sigma_B \rangle$ for any SFT $(X_B, \sigma_B)$ flow equivalent to $(X_A, \sigma_A)$. Appealing to a general extension result from [6], we also show that for any nontrivial irreducible SFT $(X_A, \sigma_A)$, there is an abundant supply of elements in $\mathcal{M}_A$ containing no flow equivalence with an invariant cross section. We also give a concrete example of such an element, not appealing to an extension theorem.

In Section 6, we show that $\mathcal{M}_A$ is not residually finite. In Section 7, we show $\mathcal{M}_A$ has solvable word problem. In Section 8, we give results on conjugacy classes of involutions in $\mathcal{M}_A$ by establishing a connection to the theory of $\mathbb{Z}_2$-SFTs. For example, if $\det(I - A)$ is odd, then only finitely many conjugacy classes in $\mathcal{M}_A$ can contain fixed point free involutions.

At points in the paper we make use of flow codes, a flow analogue of block codes, introduced in [7]. For Section 7, we also need to address composition of flow codes up to isotopy. The background and new work on flow codes is given in Appendix A.

In the course of the paper we make explicit several of the many open questions about $\mathcal{M}_A$.

2. Definitions and Background

There is more detailed background on the material below in [39] (for Sec. 2.1), [7] (for Secs. 2.2 and 2.3) and [3] (for Secs. 2.4 and 2.5).

2.1. Shifts of Finite Type. Let $A$ be an $n \times n$ nonnegative integral matrix. $A$ can be viewed as an adjacency matrix of a finite directed graph $G$ with $n$ ordered vertices and a finite edge set $E$ and $A_{ij}$ is the number of edges from vertex $i$ to vertex $j$. Let $X_A$ be the subspace of $E^\mathbb{Z}$ consisting of bi-infinite sequences $(x_i)$ such that for all $i \in \mathbb{Z}$, the terminal vertex of $x_i$ is the initial vertex of $x_{i+1}$. Then with the subspace topology from the product topology of $E^\mathbb{Z}$, $X_A$ is a compact metrizable space and the shift map $\sigma_A$ defined by the rule $(\sigma_A(x))_i = x_{i+1}$ is a homeomorphism from $X_A$ to $X_A$. $(X_A, \sigma_A)$ is a shift of finite type (SFT) defined by $A$. In general an SFT is any dynamical system topologically conjugate to some $(X_A, \sigma_A)$; in addition, $A$ can be chosen nondegenerate (no zero row or column). An SFT $(X_A, \sigma_A)$ is irreducible if it has a dense forward orbit; it is trivial if $X_A$ is a finite set. For $A$ nondegenerate, $(X_A, \sigma_A)$ is irreducible if and only if $A$ is an irreducible matrix. If $A$ is irreducible, then $(X_A, \sigma_A)$ is trivial if and only if $A$ is a cyclic permutation matrix.

2.2. Suspensions, Cross Sections, and Flow Equivalences. For a homeomorphism $T : X \to X$, we define its mapping torus $S(X,T) = S_X$ to be the quotient
space $(X \times \mathbb{R})/\sim$, where $(x, t) \sim (T^n(x), t - n)$ for $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. We write the image of $(x, t)$ in $S X$ as $[x, t]$. An element of $S X$ may be represented as $[x, t]$ for a unique $x$ in $X$ and $t$ in $[0, 1)$. For any $s \in \mathbb{R}$, the suspension flow $\alpha : S X \times \mathbb{R} \to S X$ is defined by $\alpha_s([x, t]) = [x, s + t]$. Two discrete dynamical systems $(X, T)$ and $(X', T')$ are flow equivalent if there is a homeomorphism $F : S X \to S X'$ mapping flow orbits onto flow orbits, respecting the direction of the flow. $F$ is called a flow equivalence. Any conjugacy of discrete dynamical systems induces a topological conjugacy of the corresponding suspension flows (and this is a flow equivalence), but in general flow equivalence is a much weaker equivalence relation.

A cross section $C$ of the suspension flow $\alpha$ on $S X$ is a closed set of $S X$ such that $\alpha : C \times \mathbb{R} \to S X$ is a local homeomorphism onto $S X$. It follows that every orbit hits $C$ in forward time and in backward time, the first return time defined by $f_c(x) = \inf\{s > 0 : \alpha_s(x) \in C\}$ is continuous and strictly positive on $C$, and the first return map $\rho_c : C \to C$ defined by $\rho_c(x) = \alpha_{f_c(x)}(x)$ is a homeomorphism. Discrete systems $(X, T)$ and $(X', T')$ are flow equivalent if and only if there is a flow $Y$ with two cross sections whose return maps are conjugate respectively to $T$ and $T'$.

We define the mapping class group of $T$, denoted by $\mathcal{M}(T)$, to be the group of flow equivalences $S X \to S X$ modulo the subgroup of flow equivalences which are isotopic to the identity in $\mathcal{F}(T)$. Two flow equivalences $F_0, F_1 : S X \to S X'$ are isotopic if $[F_1^{-1} F_0]$ is trivial in $\mathcal{M}(T)$. By definition, for $h$ in $\mathcal{F}(T)$, the class $[h]$ is trivial in $\mathcal{M}(T)$ if there is a continuous map $S X \times [0, 1] \to S X$, $(y, t) \mapsto h_t(y)$, with $h_0$ the identity, $h_1 = h$ and each $h_t$ in $\mathcal{F}(T)$. When $T$ is a shift of finite type, we may write $\mathcal{F}_A$ for $\mathcal{F}(T)$ and $\mathcal{M}_A$ for $\mathcal{M}(T)$.

Question 2.1. Does the epimorphism $\mathcal{F}(S X_A) \to \mathcal{M}_A$ given by $F \mapsto [F]$ split?

Question 2.1 asks whether $\mathcal{M}_A$ can be presented as a group of homeomorphisms.

2.3. The Parry-Sullivan Argument. A discrete cross section for a homeomorphism $T : X \to X$ is a closed subset $C$ of $X$ with a continuous function $r : C \to \mathbb{N}$ such that $r(x) = \min\{k \in \mathbb{N} : T^k(x) \in C\}$ and $X = \{T^k(x) : x \in C, k \in \mathbb{N}\}$. When $X$ is zero dimensional, the set $C$ must be clopen in $X$, by continuity of the return time function $r$.

The argument of Parry and Sullivan in [44] shows the following.

Theorem 2.2. (44; see [4] Theorem 4.1) Suppose $Y, Y'$ are one dimensional compact metric spaces with fixed point free flows $\gamma, \gamma'$ for which $C, C'$ are zero dimensional cross sections, with return maps $\rho_C, \rho_{C'}$. Suppose $h : Y \to Y'$ is a flow equivalence.

Then there are discrete cross sections $D, D'$ for $\rho_C, \rho_{C'}$, with $D \subset C$ and $D' \subset C'$, such that $h^{-1}(D') = F(D)$ for some $F$ isotopic to the identity, and $h$ is isotopic to a homeomorphism $Y \to Y'$ induced by a topological conjugacy $(D, \rho_D) \to (D', \rho_{D'})$.

Theorem 2.2 is implicit in the succinct paper [44]; see [4] for full details, generalization and related examples.

As a consequence of Theorem 2.2, we have the following fact.

Corollary 2.3. The mapping class group of a subshift $(X, \sigma)$ is countable.

Proof. Let $Y$ be the mapping torus of $X$. For any discrete cross section $D$ for $(X, \sigma)$, the system $(X, \rho_D)$ is expansive and therefore topologically conjugate to a
subshift. By Theorem 2.2 up to isotopy a flow equivalence $Y \to Y$ is determined by the choice of clopen sets $D, D'$ and a topological conjugacy $(D, \rho_D) \to (D', \rho_{D'})$ (which can be defined by a block code). There are only countably many clopen sets in $D$ and only countably many block codes. Therefore the mapping class group of $(X, \sigma)$ is countable. \hfill $\Box$

For a simple example in contrast to Corollary 2.3 note that $\mathcal{M}(T)$ is uncountable if $T$ is the identity map on a Cantor set.

2.4. **Positive Equivalence.** Let $A$ and $B$ be irreducible matrices. We embed $A$ and $B$ to the set of essentially irreducible infinite matrices over $\mathbb{Z}_+$, those which have only one irreducible component. Within the “positive K-Theory” approach to symbolic dynamics \cite{2}, there is the general “positive equivalence” method for constructing flow equivalences for SFTs (developed in \cite{1}, building on Franks’ work \cite{20}). (Flow codes, a flow equivalence analogue of block codes developed in \cite{7}, give a general presentation of flow equivalences up to isotopy for subshifts.)

A basic elementary matrix $E$ is a matrix in $\text{SL}(\mathbb{Z})$ which has off-diagonal entry $E_{ij} = 1$ where $i \neq j$ and 1 on the main diagonal and 0 elsewhere. We define four basic positive equivalences as follows: suppose $A_{ij} > 0$,

\[
(E, I) : I - A \to E(I - A), \quad (E^{-1}, I) : E(I - A) \to I - A
\]

\[
(A, E) : I - A \to (I - A)E, \quad (I, E^{-1}) : (I - A)E \to I - A.
\]

A positive equivalence is the composition of basic positive equivalences $(E_i, F_i)$, $(U, V) = (E_k \cdots E_1, F_1 \cdots F_k)$. We will only discuss the flow equivalence induced by the basic positive equivalence $(E, I) : I - A \to E(I - A)$. We can apply the same idea with the others. Define $A'$ from the equation $E(I - A) = I - A'$. Then $A$ and $A'$ agree except in row $i$, where we have

\[
A'_{ik} = A_{ik} + A_{jk} \quad \text{if} \quad j \neq k, \quad \text{and} \quad A'_{ij} = A_{ij} + A_{jj} - 1.
\]

Let $\mathcal{G}_A$ be a directed graph having $A$ as the adjacency matrix with edge set $\mathcal{E}_A$. We can describe a directed graph $\mathcal{G}_{A'}$ which has $A'$ as its adjacency matrix as follows.

Pick an edge $e$ which runs from a vertex $i$ to a vertex $j$ in $\mathcal{G}_A$ (exists because $A_{ij} > 0$ by assumption). The edge set $\mathcal{E}_{A'}$ will be obtained from $\mathcal{E}_A$ as follows:

a) remove $e$ from $\mathcal{E}_A$,

b) for each vertex $k$, for every edge $f$ in $\mathcal{E}_A$ from $j$ to $k$ add a new edge named $[ef]$ from $i$ to $k$.

Let $\mathcal{E}'_A$ be the set of new edges obtained from the above construction. Define a map $\gamma : \mathcal{E}_{A'} \to \mathcal{E}'_A$ by $\gamma(f) = f$ and $\gamma([ef]) = ef$. Then $\gamma$ induces a map $\hat{\gamma} : X_A \to X_{A'}$, defined by the rule

\[
\hat{\gamma} : \cdots x'_{-2}x'_{-1}x'_0x'_1 \cdots \mapsto \cdots \gamma(x'_{-2})\gamma(x'_{-1})\gamma(x'_0)\gamma(x'_1)\cdots
\]

Define $F_\gamma : \Sigma X_{A'} \to \Sigma X_A$ by setting, for $x$ in $X_{A'}$ and $0 \leq t < 1$,

\[
F_\gamma([x, t]) = \begin{cases} 
\hat{\gamma}(x), 2t, & \text{if } x \in X_{[ef]} \text{ for every edge of the form } [ef] \\
\hat{\gamma}(x), t, & \text{otherwise .} 
\end{cases}
\]

where for an edge $d$ in $\mathcal{E}_{A'}$, $X_d = \{x \in X_A : x_0 = d\}$.

Then $F_\gamma$ is a flow equivalence (in particular, surjective, even though $\hat{\gamma}$ is not).
2.5. The Bowen-Franks representation. The Bowen-Franks group of an $n \times n$ integral matrix $A$ is $\text{coker}(I-A) = \mathbb{Z}^n/(I-A)\mathbb{Z}^n$. For a shift of finite type $(X_A, \sigma_A)$, Parry and Sullivan [34] showed $\det(I-A)$ is an invariant of flow equivalence, Bowen and Franks [2] showed $\text{coker}(I-A)$ is an invariant of flow equivalence, and Franks [30] showed these invariants are complete for nontrivial irreducible shifts of finite type. There is a complete classification of general SFTs up to flow equivalence, due to Huang [8, 10], but the general invariant is much more complicated.

Let $(X_A, \sigma_A)$ be a nontrivial irreducible shift of finite type. Let $(U, V) : (I-A) \rightarrow U(I-A)V = I-A$ be a positive equivalence and let $F_{(U,V)}$ be an associated flow equivalence. (There can be many factorizations of $(U, V)$ into basic positive equivalences, and they can define isotopically distinct flow equivalences.) We define $F^*_{(U,V)} : \text{coker}(I-A) \rightarrow \text{coker}(I-A)$ by the rule $[u] \rightarrow [uV]$ (we use the $(U,V)$ action on row vectors to define $\text{coker}(I-A)$.) Then $F^*_{(U,V)}$ is an isomorphism. Let $\text{Aut}(\text{coker}(I-A))$ denote the group of group automorphisms of $\text{coker}(I-A)$. We define the map $\rho_A : M_A \rightarrow \text{Aut}(\text{coker}(I-A))$ by the rule $\rho_A : F_{(U,V)} \rightarrow F^*_{(U,V)}$ and call $\rho_A$ the Bowen-Franks representation (in [3], this is called the isotopy futures representation). It was proved in [3] that this rule gives a well defined group epimorphism. In contrast, it was proved in [35] that there can be automorphisms of the dimension module of $(X_A, \sigma_A)$ (as an ordered module) which are not induced by any element of $\text{Aut}(X_A)$.

3. Actions, Representations and Group Isomorphisms

The following result is fundamental for studying the mapping class group of an irreducible SFT.

**Theorem 3.1.** Suppose $(X_A, \sigma_A)$ is an irreducible SFT and $F \in \mathcal{F}_A$. Then the following are equivalent.

1. $F$ is isotopic to the identity.
2. $F(O) = O$ for all suspension flow orbits $O$ in $S^X_A$.
3. $F(C) = C$ for all circles
4. $F(C) = C$ for all but finitely many circles $C$ in $S^X_A$.

**Proof:** The implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) hold generally, i.e. with $(X,T)$ in place of $(X_A, \sigma_A)$, for $T$ a zero dimensional compact metric space $X$. In the case that $(X,T)$ is an irreducible SFT, the implication (2) $\implies$ (1) is [7, Theorem 6.2]. Given (3), it follows from [7, Theorem 6.1] that the flow equivalence $F$ up to isotopy is induced by an automorphism of the irreducible SFT. As recalled in the proof of [7, Theorem 6.2], an automorphism of an irreducible SFT which fixes all (or even all but finitely many) orbits must be a power of the shift [14, Theorem 2.5].

It remains to show (4) $\implies$ (3). Suppose $U$ is a word such that $\ldots UUU \ldots$ represents a periodic orbit of the irreducible SFT $\sigma_A$ such that for the corresponding circle $C(U)$ in $S^X_A$, $F(C(U)) \neq C(U)$. Then one can construct a word $W$ such that for all positive integers $n$, the words $WU^n$ represent distinct periodic orbits, with $F(C(WU^n)) \neq C(WU^n)$. So, if $F$ moves one circle outside itself, then $F$ moves infinitely many circles to different circles, and therefore (4) $\implies$ (3). \hfill $\Box$

**Remark 3.2.** The implication (2) $\implies$ (1) of Theorem [31] fails in general, even for some reducible SFTs and mixing sofic shifts (see [7, Example 6.1]).
Suppose $T : X \to X$ is a homeomorphism of a compact zero dimensional metric space. Then $T$ acts on $C(X, \mathbb{Z})$, the group of continuous functions from $X$ to the integers, by the rule $f \mapsto T \cdot f$. The following groups are isomorphic: the first Čech cohomology group $\check{H}^1(SX)$; the group $C(X, \mathbb{Z})/(I - T)C(X, \mathbb{Z})$; and the Bruschlinsky group $C(SX, S^1)/\sim$ of continuous maps from $SX$ to the circle modulo isotopy. (For some exposition, see [9].) The group $C(X, \mathbb{Z})/(I - T)C(X, \mathbb{Z})$ is of considerable interest for dynamics (see [11] [27] [35], their references and citations). A flow equivalence $F : SX \to SX$ induces a homeomorphism of each of these groups; for example, the automorphism of $C(SX, S^1)/\sim$ is defined by the obvious rule $[f] \mapsto [f \circ F]$.

**Corollary 3.3.** Suppose $(X_A, \sigma_A)$ is a nontrivial irreducible shift of finite type. Then the action (by permutations) of the mapping class group $M_A$ on the set of circles of $SX_A$ is faithful. The action of $M_A$ on the first Čech cohomology group of $SX_A$ is also faithful.

**Proof.** This follows from Theorem 3.1, since a homeomorphism moving a circle in $SX$ to another circle has nontrivial action on Čech cohomology. \qed

An important fact for analyzing the automorphism group of an irreducible SFT, and its actions, is that there are finite invariant sets (points of some period), whose union is dense. The next result (from [18]) shows in a strong way that we have nothing like that for the study of $M_A$.

**Theorem 3.4.** Suppose $(X_A, \sigma_A)$ is a nontrivial irreducible shift of finite type. Then $M_A$ acts $n$-transitively on the set of circles in $SX_A$ for all $n \in \mathbb{N}$.

**Proof.** Let $\{C_1, \ldots, C_n\}$ and $\{C'_1, \ldots, C'_n\}$ be sets of $n$ distinct circles. For each $i \in \{1, 2, \ldots, n\}$, let $x_i, x'_i$ be representatives of the circles $C_i, C'_i$ respectively. We take a $k$-block presentation of $(X_A, \sigma_A)$ where $k$ is large enough that any point of period $p$ comes from a path of length $p$ without repeated vertices except initial and terminal vertices and no two of these loops share a vertex. If one of these loops, say $L$, has length greater than 1, then we apply a basic positive equivalence which corresponds to cutting out an edge $e$ on the loop $L$ and replacing it with edges labeled $[e, f]$, for the edge $f$ following $e$. The new loop will have length $p - 1$ in the new graph. Continuing in the same fashion, we get a loop of length 1. Since no two of these loops share a vertex, we can apply the same idea to another loop without changing the former loop. Continuing in this way, we get a graph with loops $y_1, \ldots, y_n, y'_1, \ldots, y'_n$ of length 1, each of which comes from the loop containing $x_1, \ldots, x_n, x'_1, \ldots, x'_n$. If necessary we continue to apply basic positive equivalences until we get a graph $G_B$ with at least one point of least period $n$, for every positive integer $n$. Let $(X_B, \sigma_B)$ be the SFT induced by the graph $G_B$. $(X_B, \sigma_B)$ is flow equivalent to $(X_A, \sigma_A)$. Since $y_1, \ldots, y_n; y'_1, \ldots, y'_n$ are fixed points in $(X_B, \sigma_B)$ and $\sigma_B$ is mixing with points of all least periods, there is an inert automorphism $u \in \text{Aut}(\sigma_B)$ such that $u(y_i) = y'_i$ for all $i = 1, 2, \ldots, n$ [3]. Extend $u$ to a flow equivalence $\hat{u} : SX_B \to SX_B$ by $\hat{u}([x, t]) = [u(x), t]$. Let $G : SX_A \to SX_B$ be a flow equivalence arising from the construction. Then $F = G^{-1}\hat{u}G$ is the required flow equivalence, i.e., $F(C_i) = C'_i$ for all $i = 1, 2, \ldots, n$. \qed

In contrast to Theorem 3.4 note that if a flow equivalence $F$ maps a cross section $C$ onto a cross section $D$, then the return maps to these cross sections...
are topologically conjugate. The action of $\mathcal{F}_A$ on cross sections is very far from transitive.

The center of the automorphism group of an irreducible shift of finite type is simply the powers of the shift [48]. The next result (from [18]) is the analogue for the mapping class group.

**Theorem 3.5.** Suppose $(X_A, \sigma_A)$ is a nontrivial irreducible shift of finite type. Then the center of $\mathcal{M}_A$ is trivial.

**Proof.** Let $C$ be a circle in $SX_A$ and $F$ be an element in the center of $\mathcal{M}_A$. Suppose that $F(C) \neq C$. Note that $F(C)$ is also a circle. Then there is a flow equivalence $G$ such that $G(C) = C$ and $G(F(C)) \neq F(C)$ by Theorem 3.4. Thus $FG(C) = F(C) \neq GF(C)$ which is a contradiction. Hence $F(C) = C$ for all circles $C$ in $SX_A$. Therefore, $F$ is isotopic to the identity by Theorem 3.1. $\square$

**Remark 3.6.** Suppose $\sigma_A$ and $\sigma_B$ are nontrivial irreducible SFTs. It is not known whether $\text{Aut}(\sigma_A)$ must embed as a subgroup of $\text{Aut}(\sigma_B)$. Kim and Roush proved the embedding does exist when $\sigma_A$ is a full shift [33]. With mapping class groups in place of automorphism groups, we do not have even the analogue of the Kim-Roush result. (Adapting the automorphism group argument of Kim and Roush to mapping class groups, using flow codes in place of block codes, is problematic.)

**Question 3.7.** Do all mapping class groups of nontrivial irreducible SFTs embed into each other?

Recall $\mathcal{M}_A^\rho$ denotes the kernel of the Bowen-Franks representation $\mathcal{M}_A \to \text{cok}(I - A)$. We are led to several questions about $\mathcal{M}_A^\rho$.

**Question 3.8.** Let $\sigma_A$ be a nontrivial irreducible SFT. Is $\mathcal{M}_A^\rho$ simple? perfect (i.e. equal to its commutator subgroup)? generated by involutions?

There has recently been a burst of results constraining the structure of an automorphism group of a subshift (usually assumed to be minimal) of low complexity (e.g. polynomial complexity, or even just zero entropy). (See [50, 19, 21, 22, 20] and their references.) Here degree $d$ polynomial complexity of a subshift means that the number of allowed words of length $n$ is bounded by a polynomial $p(n)$ of degree $d$. The classes of zero entropy shifts, degree $d$ polynomial complexity shifts and minimal shifts are each invariant under flow equivalence.

**Question 3.9.** Are there constraints on the structure of the mapping class group of a low complexity (minimal) shift, analogous to constraints on the automorphism group?

Some quite interesting full groups have been proved to be finitely generated or even finitely presented [32, 42].

**Question 3.10.** Let $\sigma_A$ be a nontrivial irreducible SFT. Is $\mathcal{M}_A^\rho$ finitely generated? Because $\rho_A$ is surjective, and the group of automorphisms of a finitely generated abelian group is itself finitely generated, we have that $\mathcal{M}_A$ is finitely generated if $\mathcal{M}_A^\rho$ is finitely generated. (In contrast, the group of automorphisms of the dimension module of $X_A$ is often but not always a finitely generated group [13].)
4. OUTER AND NONSPATIAL AUTOMORPHISMS

In this section we show that $\mathcal{M}_A$ has an outer automorphism. Extending work from [12], we give examples of $\text{Aut}(\sigma_A)$ with outer and nonspatial automorphisms, and derive consequences of spatiality of isomorphisms from sufficiently rich subgroups of $\text{Aut}(\sigma_A)$.

It is natural to suspect that nontrivial irreducible SFTs $\sigma_A, \sigma_B$ which are not flow equivalent cannot have isomorphic mapping class groups. (Although, given works of Riordam, Matsumoto and Matui (see [47, 41]), one could speculate that isomorphism of their Bowen-Franks groups alone might imply $\mathcal{M}_A \cong \mathcal{M}_B$.) Question 4.2 gives one standard approach to this possibility.

**Definition 4.1.** An isomorphism $\phi : G_1 \to G_2$ between groups of homeomorphisms is spatial if it is induced by some homeomorphism $H$ (i.e., $\phi(g) = H^{-1}gH$).

**Question 4.2.** Suppose $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are nontrivial irreducible shifts of finite type. Is every isomorphism $\psi : \mathcal{M}_A^0 \to \mathcal{M}_B^0$ spatial? Is every isomorphism $\psi : \mathcal{M}_A \to \mathcal{M}_B$ spatial?

**Remark 4.3.** A standard method for proving spatiality (and more) for a group $G$ of homeomorphisms of the Cantor set (e.g. within full groups [28, 29, 32]) appeals to $G$ having a sufficiently rich supply of maps which are the identity on large open sets. It’s problematic (perhaps impossible) to find some analogue of this approach for $\mathcal{M}_A$ or $\text{Aut}(\sigma_A)$. The only element of $\text{Aut}(\sigma_A)$ which is the identity on a nonempty open set is the identity; and if $F$ in $\mathcal{F}_A$ is the identity on an open neighborhood of a cross section, then $[F]$ is the identity in $\mathcal{M}_A$.

Recall, a system $(X,T)$ is indecomposable if $X$ is not the union of two disjoint nonempty $T$-invariant subsystems. Equivalently, $SX$ is connected.

**Definition 4.4.** For an indecomposable system $(X,T)$, the extended mapping class group of $T$, $\mathcal{M}^{\text{ext}}(T)$, is the group $\mathcal{H}(T)$ of all homeomorphisms $SX \to SX$, modulo the subgroup of those isotopic to the identity in $\mathcal{H}(T)$.

With $SX$ connected, an element of $\mathcal{H}(T)$ either respects orientation on all orbits or reverses orientation on all orbits. The mapping torus of $(X,T^{-1})$ can be identified with the mapping torus of $(X,T)$, but with its unit speed suspension flow moving in the opposite direction. With this identification, $\mathcal{M}(T) = \mathcal{M}(T^{-1})$. An orientation reversing homeomorphism $V$ of $SX$ is a flow equivalence from $T$ to $T^{-1}$. Such a $V$ always exists when $\sigma_A$ is a nontrivial irreducible SFT, because $(\sigma_A)^{-1}$ is conjugate to the transpose of $A$, and the complete invariants agree on $A$ and its transpose. Clearly $\mathcal{M}(T)$ is an index 2 normal subgroup of $\mathcal{M}^{\text{ext}}(T)$.

**Theorem 4.5.** Let $\sigma_A$ be a nontrivial irreducible SFT. The action of the extended mapping class group $\mathcal{M}^{\text{ext}}(\sigma_A)$ by permutations on the circles in $SX_A$ is faithful. Consequently, the center of $\mathcal{M}^{\text{ext}}(\sigma_A)$ is trivial and the outer automorphism group of $\mathcal{M}(\sigma_A)$ has cardinality at least two.

**Proof.** Suppose $F$ and $G$ are homeomorphisms of $SX_A$, with the same action by permutations on circles. If $FG^{-1}$ is orientation preserving, then $FG^{-1}$ is isotopic to the identity, by Corollary 3.3, so $[F] = [G]$ in $\mathcal{M}_A^{\text{ext}}$. Now suppose $F$ is orientation preserving and $G$ is orientation reversing. For definiteness, after passing to isotopic maps, we suppose they are given by flow codes. Let $W, V$ be distinct words such that
\( (WV^n W) \) is an \( X_A \)-word, for all \( n \). With \( O = WV^n \), consider a circle \( C \) which is the suspension of a periodic orbit for \( \sigma_A \) with defining block \( (O)V^N(OO)V^N(OOO)V^N \). For \( n \) sufficiently large, and then \( N \) sufficiently larger than \( n \), there will be large integers \( M, P \) and words \( \overline{O}, \overline{V}, \overline{O}, \overline{V} \) with \( \overline{V}^M \) much longer than \( OO \overline{O} \) and \( \overline{V}^P \) much longer than \( \overline{O} \overline{O} \overline{O} \), such that the circles \( FC \) and \( GC \) will be suspensions of \( \sigma_A \)-orbits with defining blocks of the following forms:

\[
\begin{align*}
\mathcal{B} & = (\overline{O})\overline{V}^M(\overline{O} \overline{O})\overline{V}^M(\overline{O} \overline{O} \overline{O})\overline{V}^M, & \text{for } FC, \\
\tilde{\mathcal{B}} & = (\tilde{\overline{O}})\tilde{\overline{V}}^P(\tilde{\overline{O}} \tilde{\overline{O}} \tilde{\overline{O}})\tilde{\overline{V}}^P, & \text{for } GC.
\end{align*}
\]

Now the blocks interrupting \( \overline{V} \)-periodicity in \( \mathcal{B} \) will have \( \ldots 123123123 \ldots \) as a periodic relative size pattern, while the blocks interrupting of \( \tilde{\overline{V}} \)-periodicity in \( \tilde{\mathcal{B}} \) will have \( \ldots 321321321 \ldots \) as a periodic relative size pattern. Thus \( FC \neq GC \). This finishes the proof of faithfulness. The proof of triviality of the center of \( \mathcal{M}_A^{\text{ext}} \) follows the proof of Theorem 3.5. Then, conjugation by an orientation reversing homeomorphism of \( SX_A \) defines an automorphism of \( M_A \) which is not an inner automorphism of \( M_A \).

We now turn to the automorphism group of \( \sigma_A \). The next definition formalizes a property used in [12], as recalled below.

**Definition 4.6.** An SFT \( \sigma_A \) is SIC if \( \text{Aut}(\sigma_A) \) is the internal direct sum of its center \( \langle \sigma_A \rangle \) and a complementary normal subgroup containing the inert automorphism subgroup \( \text{Aut}_0(\sigma_A) \).

We will show next that there are many examples of SIC SFTs. We say \( \lambda \) is rootless in \( R \) if \( \lambda = u^k \) with \( k \in \mathbb{N} \) and \( u \in R \) implies \( k = 1, \lambda = u \). For example, a positive integer is rootless in \( \mathbb{Q} \) if it is rootless in \( \mathbb{Z} \). A fundamental unit of a quadratic number ring \( R \) is rootless in \( R \). If \( \lambda \) is an algebraic number with infinite order, then it has a \( k \)th root in \( \mathbb{Q}(\lambda) \) for only finitely many \( k \).

**Proposition 4.7.** Suppose \( \sigma_A \) is a nontrivial irreducible SFT, and \( \lambda_A \), the Perron eigenvalue of \( A \), is rootless in \( \mathbb{Q}(\lambda_A) \). Then \( \sigma_A \) is SIC.

**Proof.** One part of the dimension representation \( \rho_A \) is the homomorphism \( \mu \) which sends an automorphism \( U \) to the positive number by which \( \rho_A(U) \) multiplies a Perron eigenvector of \( A \). The image group under multiplication, \( \mu(\text{Aut}(\sigma_A)) := H \), is finitely generated free abelian, with \( \mu(\sigma_A) = \lambda_A \), the Perron eigenvalue of \( A \). By the rootless assumption, \( H \) is the internal direct sum of \( \langle \lambda_A \rangle \) and some complementary group \( N \). The epimorphism \( \text{Aut}(\sigma_A) \to H/N \) splits (by \( [\lambda_A^n] \mapsto \sigma_A^n \)). Let \( K = \mu^{-1}(N) \). Because the complementary subgroup \( \langle \sigma_A \rangle \) is the center, the group \( \text{Aut}(\sigma_A) \) is the internal direct sum \( K \oplus \langle \sigma_A \rangle \).

**Proposition 4.8.** Suppose \( (X_A, \sigma_A) \) is a nontrivial SIC irreducible SFT, \( \text{Aut}(\sigma_A) \cong K \oplus \langle \sigma_A \rangle \cong K \oplus \mathbb{Z} \). Let \( \phi \) be the automorphism of \( \text{Aut}(\sigma_A) \) which in the latter notation is \( (k, n) \mapsto (k, -n) \). Then \( \phi \) is not spatial.

**Proof.** Suppose \( \phi \) is induced by a homeomorphism \( H \). It follows that \( H \) is a conjugacy from \( \sigma_A \) to its inverse, with \( HU = UH \) for every \( U \) in \( K \). First suppose \( \sigma_A \) is mixing. Then for any periodic point \( x \) of sufficiently large period, there is an inert automorphism \( U \) such that \( Ux = \sigma_A x \). (This follows e.g. from any of the three papers [8, 12, 43]; for a precise argument, see the proof of Proposition 4.11 below.)
Thus $H$ commutes with $\sigma_A$ on a dense set, and hence everywhere. This contradicts $H^{-1}\sigma_A H = \sigma_A^{-1}$.

Now suppose $\sigma_A$ is irreducible with period $p > 1$. Then $\sigma_A$ induces a cyclic permutation of $p$ disjoint clopen sets $B, \sigma_A(B), \ldots, \sigma_A^{p-1}(B)$. After postcomposing $H$ with a power of $\sigma_A$, we may assume $H(B) = B$. The return map $\sigma_A^p|_B$ is a mixing SFT, and every inert automorphism of $\sigma_A^p|_B$ extends to an inert automorphism of $\sigma_A$. Thus $H|_B$ commutes with $\sigma_A^p|_B$. Because $\sigma_A^p|_B$ has infinite order, this contradicts $H^{-1}\sigma_A^p H = \sigma_A^{-p}$. \hfill $\square$

In [12 Proposition 4.2], the automorphism $\phi$ above was used to produce an example of a nonspatial automorphism of $\text{Aut}(\sigma_A)$, for a mixing SFT $\sigma_A$ such that $\text{Aut}(\sigma_A) \cong \text{Aut}_0(\sigma_A) \oplus \langle \sigma_A \rangle$ and $\sigma_A$ is not conjugate to its inverse. The proof in [12] was simply to note that spatiality of $\phi$ would require $\phi$ to be a (nonexistent) conjugacy from $\sigma_A$ to its inverse.

**Remark 4.9.** For a nontrivial SIC mixing SFT $\sigma_A$ which is topologically conjugate to its inverse (such as a rootless full shift), the outer automorphism group of $\text{Aut}(\sigma_A)$ has cardinality at least four. (There is the nonspatial involution, and another element of order two in $\text{Out}(\sigma_A)$ arising from conjugating by a topological conjugacy of $\sigma_A$ and its inverse, essentially by the argument proving Theorem 4.5.) The action on periodic points of conjugacies of $\sigma_A$ and $\sigma_A^{-1}$ is studied in [12, 37].

Although there can be nonspatial automorphisms of $\text{Aut}(\sigma_A)$, we do not know whether this is possible for various distinguished subgroups (such as the commutator). This motivates the following propositions.

**Proposition 4.10.** Suppose $\sigma_A$ is a nontrivial irreducible SFT, $(X,T)$ is a zero dimensional system and $H$ is a subgroup of $\text{Aut}(\sigma_A)$ satisfying the following:

1. $\{x \in X_A : \exists U \in H, Ux = \sigma_A x\}$ is dense in $X_A$.
2. The centralizer of $H$ in $\text{Aut}(\sigma_A)$ equals $\langle \sigma_A \rangle$.

Suppose $\phi : H \to \text{Aut}(T)$ is a spatial isomorphism to a subgroup of $\text{Aut}(T)$; i.e., $\phi : U \mapsto pU^{-1}p$, with $p : X_A \to X$ a homeomorphism. Then $p^{-1}Tp = \sigma_A$ or $\sigma_A^{-1}$, and $p$ induces a spatial automorphism $\text{Aut}(\sigma_A) \to \text{Aut}(T)$.

**Proof.** Let $\psi = p^{-1}Tp$. By (1), $\psi|_{\sigma_A} = \sigma_A \psi$ on a dense set, hence everywhere. By (2), $\psi \in (\sigma_A)$. Because $\psi$ and $\sigma_A$ have equal entropy, $\psi$ equals $\sigma_A$ or $\sigma_A^{-1}$. \hfill $\square$

**Proposition 4.11.** Suppose $\sigma_A$ is a nontrivial mixing SFT, and $H$ is a subgroup of $\text{Aut}(\sigma_A)$ containing the subgroup

$$K := \langle \{aba^{-1}b^{-1} : \{a,b\} \subset \text{Aut}_0(\sigma_A), a^2 = b^2 = 1d\} \rangle.$$

Then $H$ satisfies the conditions (1) and (2) of Proposition 4.10.

**Proof.** Let $\mathcal{P}_n$ be the set of $\sigma_A$ orbits of cardinality $n$. Pick $N$ such that $n \geq N$ implies $|\mathcal{P}_n| \geq 4$. Now suppose $n \geq N$. Given $x,y$ in distinct orbits in $\mathcal{P}_n$, we can choose an inert involution $U(x,y)$ which exchanges $x$ and $y$ and is the identity on points of period at most $n$ which are not in the orbits of $x$ and $y$. (This follows from [3] Lemma 2.3(a)], and the freedom to “vary the embedding" stated in its proof.) Suppose $x,y,z$ are in distinct orbits in $\mathcal{P}_n$. Let $a = U(x,y)$, $b = U(y,z)$, $k(x,y,z) = aba^{-1}b^{-1} \in K$. Then $k(x,y,z)$ cyclically permutes $x,y,z$ and is the identity map on points of period at most $n$ outside the orbits of $a,b$ and $c$. The map $k = k(\sigma_A(x),y,z)k(x,y,z)k(x,y,z)$ satisfies $k(x) = \sigma_A(x)$; this shows
$H$ satisfies (1). The maps $k(x,y,z)$ induce all 3-cycle permutations of $P_n$, and therefore $K$ induces all even permutations of $P_n$. Because $|P_n| \geq 4$, no nontrivial permutation of $P_n$ commutes with every even permutation. Thus an automorphism in the centralizer of $K$ maps $O$ to $O$, for all but finitely many of the finite orbits $O$, and thus must be a power of the shift. \hfill \Box

For mixing SFTs $\sigma_C$, let $G_C$ denote $\text{Aut}(\sigma_C)$ or $\text{Aut}_0(\sigma_C)$, and let $H_C$ denote some associated subgroup (such as the commutator, or the subgroup generated by involutions) such that (i) $H_C$ satisfies the containment assumption of Proposition 4.11 and (ii) any group isomorphism $G_A \to G_B$ must restrict to an isomorphism $H_A \to H_B$. Showing any isomorphism $H_A \to H_B$ must be spatial would show that the group isomorphism class of $H_A$ (and also the group isomorphism class of $G_A$) classifies $\sigma_A$ up to flip conjugacy.

5. Invariant cross sections and automorphisms

In this section we show how some elements of the mapping class group are induced by automorphisms of flow equivalent systems, and show for a nontrivial irreducible SFT $(X_A,\sigma_A)$ that these are (by far) not all of $M_A$. For $(X,T)$, let $\tilde{X}$ denote the cross section $\{[x,0] \in SX : x \in \hat{X}\}$.

**Definition 5.1.** If $u \in \text{Aut}(T)$, then $\tilde{u} : SX \to SX$ is the flow equivalence (actually a self-conjugacy of the suspension flow) defined by $\tilde{u} : [x,t] \mapsto [u(x),t]$, $0 \leq t < 1$.

**Definition 5.2.** Let $F : SX \to SX$ be a flow equivalence. A cross section $C$ of $SX$ is an invariant cross section for $F$ if $F(C) = C$.

For example, $\tilde{X}$ is an invariant cross section for $\tilde{u}$, for every $u$ in $\text{Aut}(T)$.

**Definition 5.3.** An equivalence $F : SX \to SX$ is induced by an automorphism $v$ of the return map $\rho_c$ to an invariant cross section $C$ if $F(y) = v(y)$ for all $y$ in $C$.

If flow equivalences $F,F'$ from $SX$ to $SX$ have the same invariant cross section $C$, and $F(y) = F'(y)$ for all $y$ in $C$, then $F$ and $F'$ are isotopic.

Now we can spell out a straightforward but useful correspondence.

**Theorem 5.4.** Let $T : X \to X$ be a homeomorphism of a compact zero dimensional metric space.

(1) Suppose $F : SX \to SX$ is a flow equivalence with an invariant cross section $C$. Then $F$ is a flow equivalence induced by an automorphism of the first return map $\rho_c$ under the suspension flow.

(2) Conversely, suppose $(X',T')$ is another system, and $\tilde{X}'$ denotes the cross section $\{[x,0] : x \in X'\}$ of $SX'$. Suppose $F : SX' \to SX$ is a flow equivalence. Then for every $u$ in $\text{Aut}(T')$, $F\tilde{u}F^{-1}$ is a flow equivalence $SX \to SX$; $F(\tilde{X}')$ is an invariant cross section for $F\tilde{u}F^{-1}$; and $(X',u)$ is topologically conjugate to the return map $F(\tilde{X}') \to F(\tilde{X}')$ under the suspension flow on $SX$. The map $u \mapsto F\tilde{u}F^{-1}$ induces a homomorphism $\phi_F : \text{Aut}(T') \to M(T)$.

**Proof.** For (1), let $u = F|_C$. Then $u : C \to C$ is a homeomorphism. Therefore $u \in \text{Aut}(\rho_c)$.

For (2), the homomorphism $\phi_F$ is a composition of group homomorphisms $\text{Aut}(T') \to \mathcal{F}(T') \to \mathcal{F}(T) \to M(T)$.
where $\mathcal{F}$ denotes the group of self flow equivalences. The second homomorphism is bijective and the third is surjective. \hfill \Box

**Theorem 5.5.** For a nontrivial irreducible SFT $(X_A, \sigma_A)$, let $\phi$ be the map $\text{Aut}(\sigma_A) \to \mathcal{M}_A$ defined by $u \mapsto \hat{u}$. Then $\text{Ker}(\phi) = \langle \sigma_A \rangle$, the cyclic group generated by $\sigma_A$.

**Proof.** Clearly $\text{Ker}(\phi) \supset \langle \sigma_A \rangle$. Now suppose $u \in \text{Ker}(\phi)$. By Theorem 3.1 for every circle $C$ in $SX_A$, $\hat{u}(C) = C$. It follows that the automorphism $u$ maps each finite $\sigma_A$ orbit to itself. Because $(X_A, \sigma_A)$ is an irreducible SFT, it follows from [11, Theorem 2.5], that $u$ is a power of the shift. \hfill \Box

**Theorem 5.6.** Suppose $(X_A, \sigma_A)$ is a nontrivial irreducible SFT. Then for every irreducible SFT $\sigma_B$ flow equivalent to $\sigma_A$, $\mathcal{M}_A$ contains a copy of $\text{Aut}(\sigma_B)/\langle \sigma_B \rangle$. Every flow equivalence $SX_A \to SX_A$ with an invariant cross section arises from an element of some such $\text{Aut}(\sigma_B)$, as described in Theorem 5.2.

**Proof.** This follows from Theorem 5.1, Theorem 5.5 and the fact that a homeomorphism flow equivalent to a nontrivial irreducible SFT must itself be a nontrivial irreducible SFT. \hfill \Box

**Example 5.7.** We do not know if there is any special algebraic relationship between the automorphism groups of flow equivalent nontrivial irreducible SFTs (versus arbitrary nontrivial irreducible SFTs). We show now that if $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are flow equivalent mixing SFTs, then it is not necessarily true that the groups $\text{Aut}(\sigma_A)/\langle \sigma_A \rangle$ and $\text{Aut}(\sigma_B)/\langle \sigma_B \rangle$ are isomorphic. Consider

$$
A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = [2].
$$

The matrices $B$ and $C$ define flow equivalent SFTs (if $D$ is $B$ or $C$, then $\text{coker}(I-D)$ is trivial and $\det(I-D) = -1$). The center of the automorphism group of an irreducible SFT is the powers of the shift [18]. But in $\text{Aut}(\sigma_B)$, the center has a square root (because $\sigma_{A^2}$ is conjugate to $(\sigma_A)^2$), while in $\text{Aut}(\sigma_C)$ and the center does not, because the 2-shift does not have a square root [38].

**Proposition 5.8.** Suppose $(X, T)$ contains a subsystem $(X', T')$ which is a nontrivial irreducible shift of finite type. Suppose $F \in \mathcal{F}(T)$ and $F$ maps $SX'$ (a subset of $SX$) into itself but not onto itself. Then no element of $[F]$ has an invariant cross section.

**Proof.** Any element of $[F]$ will also map $SX$ into itself but not onto itself. So it suffices to suppose there is an invariant cross section $C$ for $F$, and derive a contradiction. By Proposition 5.4, $F : SX \to SX$ is induced by an automorphism $u$ of the return map $\rho_c$ to $C$. The restriction $\rho'$ of $\rho_c$ to $C \cap SX'$ is an irreducible SFT, because it is flow equivalent to the irreducible SFT $(X', T')$, since $C \cap SX'$ is a cross section for the flow on $SX'$. Therefore the restriction of $u$ to $C \cap SX'$, being an injection into $C \cap SX'$ commuting with $\rho'$, must be a surjection. But this implies $F$ maps $SX'$ onto itself, which is a contradiction. \hfill \Box

The next result, generalizing a construction from [18], shows that flow equivalences satisfying the assumptions of Proposition 5.8 are abundant. We don’t understand much about them.
Theorem 5.9. Let \((\mathcal{X}_A, \sigma_A)\) be a nontrivial irreducible SFT. Let \((\mathcal{X}', \sigma')\) be a proper subsystem which is a nontrivial irreducible SFT. Then there is an infinite collection of flow equivalences \(F : \mathcal{S}_A \to \mathcal{S}_A\), representing distinct elements of \(\mathcal{M}_A\), such that \(F\) maps \(\mathcal{S}_X\) into itself but not onto itself (and therefore no element of \([F]\) has an invariant cross section).

Proof. From the complete invariants for flow equivalence of nontrivial irreducible SFTs, and Krieger’s Embedding Theorem, one can find a sequence \(X_1, X_2, \ldots\) of distinct (even disjoint) nontrivial irreducible SFTs which are proper subsystems of \(\mathcal{X}'\) and are flow equivalent to \(\mathcal{X}'\). By the Extension Theorem in [6], a flow equivalence \(F_n : \mathcal{S}_X' \to \mathcal{S}_X \subset \mathcal{S}_A\) extends to a flow equivalence \(F_n : \mathcal{S}_A \to \mathcal{S}_A\). The classes \([F_n]\) are distinct, because the images \(F_n(\mathcal{S}_X')\) are distinct. \(\square\)

Next we exhibit an example, not relying on an appeal to an extension theorem, of a flow equivalence \(F\) such that no element of \([F]\) has an invariant cross section.

Example 5.10. Let \(\sigma : \mathcal{X} \to \mathcal{X}\) be the full shift on three symbols \(\{0, 1, 2\}\). If \(W = W_1W_2\ldots\) is any sequence on these symbols and \(W_1 \neq 2\), then \(W\) has a unique prefix in the set \(\mathcal{W} = \{00, 01, 02, 1\}\); likewise, \(W\) has a unique prefix in the set \(\mathcal{W}' = \{10, 11, 12, 0\}\). Let \(W \to W'\) be the bijection given by \(00 \mapsto 0, 01 \mapsto 10, 02 \mapsto 12, 1 \mapsto 11\). We claim there is a flow equivalence \(F : \mathcal{S}_X \to \mathcal{S}_X\) corresponding to the change \(2W \to 2W'\) wherever \(W \in \mathcal{W}\) and \(2W'\) occurs in a point of \(\mathcal{X}\). Let \(\mathcal{X}' \subset \mathcal{X}\) be the full 2-shift on symbols \(\{1, 2\}\); let \(\mathcal{X}''\) be the points of \(\mathcal{X}'\) in which the word 212 does not occur. Then \(F\) maps \(\mathcal{S}_X'\) onto \(\mathcal{S}_X''\), a proper subset of \(\mathcal{S}_X'\), so no element of \([F]\) has an invariant cross section.

To be precise, we will construct \(F\) as a flow code, as described in the appendix. First, we define a discrete cross section \(\mathcal{C}\) of \(\mathcal{X}\) as the disjoint union of two “state sets” \(V_0\) and \(V_1\), with \(V_0 = \{x \in \mathcal{X} : x_{-1} = 2\}\), \(V_1 = \{x \in \mathcal{X} : x_{-2}x_{-1} \in \{21, 00, 01, 10, 11\}\}\). If \(x \in \mathcal{C}\), and \(k\) is the least positive integer such that \(\sigma^k(x) \in \mathcal{C}\), then \(x_0 \ldots x_{k-1}\) is a \(C\)-return word \(W\), of length \(k\) (here \(k\) is 0 or 1). Whether \(\sigma^k(x)\) is in \(V_0\) or \(V_1\) is determined by the state set containing \(x\) and the return word \(W\). Thus the return words can be used to label edges of a directed graph with states \(V_0, V_1\). The adjacency matrix \(\tilde{A}\) of this word-labeled graph (whose entries are formal sums of labeling words), and the adjacency matrix \(A\) of the underlying graph, are as follows:

\[
\tilde{A} = \begin{pmatrix}
2 + 02 \\
+ 01 + 1 \\
2 + 0 + 1 + 1 \\
0 + 1 \\
+ 0 + 1 + 1 \\
0 + 1
\end{pmatrix}, \quad
A = \begin{pmatrix}
2 3 \\
1 2
\end{pmatrix}.
\]

Similarly, we define another discrete cross section, \(\mathcal{C}'\), as the disjoint union of state sets \(V_0' = \{x \in \mathcal{X} : x_{-1} = 2\}\), \(V_1' = \{x \in \mathcal{X} : x_{-2}x_{-1} \in \{20, 00, 01, 10, 11\}\}\). As happened with \(\mathcal{C}\), the \(\mathcal{C}'\) return words label edges of a graph with states \(V_0'\) and \(V_1'\), with labeled and unlabeled adjacency matrices

\[
\tilde{A}' = \begin{pmatrix}
2 + 12 \\
+ 0 + 0 + 1 + 1 \\
2 + 0 + 1 + 1 \\
0 + 1 \\
+ 0 + 1 + 1 \\
0 + 1
\end{pmatrix}, \quad
A' = A = \begin{pmatrix}
2 3 \\
1 2
\end{pmatrix}.
\]

Now we may define a homeomorphism \(\phi : \mathcal{C} \to \mathcal{C}'\), taking \(V_0\) to \(V_0'\) and \(V_1\) to \(V_1'\), by a \(C, C'\) word block code \(W_0' \mapsto W_0'\) described by an input-output automaton which simply changes word labels:

\[
\begin{pmatrix}
2 \to 2, 02 \to 12 \\
2 \to 2, 00 \to 0, 01 \to 10, 1 \to 11
\end{pmatrix}.
\]
This $\phi$ is a conjugacy of the return maps to $C$ and $C'$ (each of which is conjugate to the SFT $\sigma_A$). The induced map $S\phi : SX \to SX$ is the flow equivalence $F$ we require.

**Question 5.11.** Is the mapping class group of a nontrivial irreducible SFT generated by elements which have an invariant cross section?

**Proposition 5.12.** Let $(X_A, \sigma_A)$ be a nontrivial irreducible SFT, with $F \in \mathcal{F}_A$. If there is a circle $C$ such that $\{F^n(C) : n \in \mathbb{N}\}$ is an infinite collection of circles then no element of $[F]$ has an invariant cross section.

**Proof.** If $F$ has an invariant cross section $C$, then $F$ is determined up to isotopy by an automorphism $U$ of the return map $\rho_C$. As $\rho_C$ is another irreducible SFT, every periodic point of $\rho_C$ lies in a finite $U$-invariant set, so every circle in $X_A$ lies in a finite $F$-invariant set of circles. $\square$

We do not know if the converse to Proposition 5.12 is true.

**Example 5.13.** In Example 5.10, the forward $F$ orbit of the circle through the periodic orbit $(21)^\infty$ is the union of infinitely many circles (those through the periodic orbits of $(21^n)^\infty$, $n \geq 1$).

### 6. Residual finiteness

**Definition 6.1.** Let $G$ be a group. $G$ is residually finite if for every pair of distinct elements $g, h$ in $G$, there is a homomorphism $\phi$ from $G$ to a finite group such that $\phi(g) \neq \phi(h)$.

The automorphism group of a subshift need not be residually finite. There is a minimal subshift whose automorphism group contains a copy of $\mathbb{Q}$, and therefore is not residually finite. At another extreme, we thank V. Salo for pointing out to us residual finiteness often fails to hold for reducible systems, as in work in progress of Salo and Schraudner, and examples such as the following, related to examples in [49]. Let $S_\infty$ denote the increasing union of the groups $S_n$, the permutations of $\{1, 2, \ldots, n\}$, identified with the permutations $\pi$ of $\mathbb{N}$ such that $\pi(k) = k$ if $k > n$. Then $S_\infty$ contains $A_\infty$, the increasing union of the alternating groups $A_n$. Because $A_\infty$ is an infinite simple group, it is not residually finite. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. One easily checks that Aut($\sigma_A$) contains a copy of $S_\infty$, and thus is not residually finite.

In contrast, the automorphism group of an irreducible shift of finite type (or any subshift with dense periodic points) is residually finite [13].

**Theorem 6.2.** Let $X_A$ be a nontrivial irreducible SFT. Then $\mathcal{M}_A$ is not residually finite.

**Proof.** For a proof, it suffices to define a monomorphism $S_\infty \to \mathcal{M}_A$. After passing from $X_A$ to a topologically conjugate shift, we may assume that there is a symbol $\alpha$ such that there are infinitely many distinct words $V_1, V_2, \ldots$ such that for all $k$, $\alpha V_k \alpha$ is an allowed word and $\alpha$ does not occur in $V_k$. Informally, an element $\pi$ of $S_\infty$ will act simply by replacing words $\alpha V_k \alpha$ with $\alpha V_{\pi(k)} \alpha$.

To make this precise we use flow codes (described in Appendix A). For $n$ in $\mathbb{N}$, define $\ell(n) = |V_n| + 1$, and $K_n = \{x \in X_A : x_0 \ldots x_{\ell(n)} = \alpha V_n \alpha\}$. Given $N$, define
a discrete cross section

\[ C_N = X_A \setminus \left( \bigcup_{n=1}^{N} \bigcup_{j=1}^{f(n)-1} \sigma_A^j K_n \right) . \]

Let \( W_N \) be the set of return words to \( C_N \). Given \( \pi \in S_N \), define a word block code

\[ \Phi_\pi : W_N \to W_N \]

\[ aV_j \mapsto aV_{\pi(j)} , \quad 1 \leq j \leq N , \]

\[ W \mapsto W , \quad \text{if } W \text{ is a symbol} . \]

\( \Phi_\pi \) defines a continuous map \( \phi_\pi : C_N \to C_N \). The rule \( \pi \mapsto \phi_\pi \) defines a monomorphism from \( S_N \) into the group of homeomorphisms \( C_N \to C_N \), and therefore \( \pi \mapsto S\phi_\pi \) defines a group monomorphism \( S_N \to F_A \). It is then easy to see (from distinct actions on periodic orbits) that \( \pi \mapsto [\phi_\pi] \) is a group monomorphism \( S_N \to M_A \). Finally, the definition of \( \phi_\pi \) does not change with increasing \( N \), so we have an embedding \( S_\infty \to F_A \) producing the embedding \( S_\infty \to M_A \). \( \square \)

The sofic groups introduced by Gromov are an important simultaneous generalization of amenable and residually finite groups. (See e.g. \cite{17, 45, 55} for definitions and a start on the large literature around sofic groups) So far, no countable group has been proven to be nonsofic. The mapping class group of a nontrivial irreducible SFT \( \sigma_A \) is not residually finite, and it is not amenable (as \( M_A \) contains a copy of \( \text{Aut}(\sigma_A) / \langle \sigma_A \rangle \), which contains free groups \cite{13}).

**Question 6.3.** Is \( M^o_A \) a sofic group?

**Remark 6.4.** With a somewhat more complicated proof appealing to canonical covers, we expect that the basic idea of Theorem 6.2 can be used to show that the mapping class group of a positive entropy sofic shift is not residually finite. Likewise, we expect a subshift which is a positive entropy synchronized system \cite{1} will have a mapping class group which is not residually finite.

## 7. Solvable word problem

The purpose of this section is to prove Theorem 7.10 which shows that the mapping class group of an irreducible SFT has solvable word problem. We begin with definitions and context.

The *alphabet* \( \mathcal{A}(T) \) of a subshift \( (X,T) \) is its symbol set. For \( j \leq k \), \( W(X,j,k) \) denotes \( \{ x_j \ldots x_k : x \in X \} \), the words of length \( k - j + 1 \) occurring in points of \( X \). The language of a subshift \( (X,T) \) is \( \bigcup_{n \geq 0} W(X,0,n) \).

**Definition 7.1.** A subshift \( (X,T) \) has a decidable language if there is an algorithm which given any finite word \( W \) on \( \mathcal{A}(T) \) decides whether \( W \) is in the language of \( X \).

**Definition 7.2.** A group \( G \) has solvable word problem if for every finite subset \( E \) of \( G \) there is an algorithm which given any product \( g = g_m \ldots g_1 \) of elements of \( E \) decides whether \( g \) is the identity.

An old observation of Kitchens \cite{13} notes that the automorphism group of a shift of finite type has a solvable word problem. We thank Mike Hochman for communicating to us the following sharper result.

**Proposition 7.3.** Suppose \( (X,T) \) is a subshift with decidable language (for example, any shift of finite type). Then \( \text{Aut}(T) \) has solvable word problem.
Proof. Given $E = \{\phi_1, \ldots, \phi_m\} \subset \text{Aut}(T)$, there are $N \in \mathbb{N}$ and functions $\Phi_i : W(-N,N)(T) \to A(T)$, $1 \leq i \leq m$, such that $\Phi_i$ defines $\phi_i$ as a block code, i.e. for all $x$ and $n$, $(\phi_i x)_n = \Phi_i(x_{n-N} \ldots x_{n+N})$. Suppose $k \in \mathbb{N}$ and $\phi = \phi_j \circ \cdots \circ \phi_j$. Then for all $x$ in $X$, $(\phi x)_i = \Phi(x_{i-kN} \ldots x_{i+kN})$, where $\Phi$ is a rule mechanically computed from the rules $\Phi_k, \ldots, \Phi_1$ [59]. However, the domain of $\Phi$ might properly contain the set $W(-kN,kN)$ (even when the set $W(-N,N)$ used to define the $\Phi_i$ is known). The map $\phi$ is the identity if and only if $\Phi(x_{-kN} \ldots x_{kN}) = x_0$ for all words $x_{-kN} \ldots x_{kN}$ in $W(-kN,kN)$; because $(X,T)$ has decidable language, this set is known and can be checked. \hfill\Box

**Definition 7.4.** A locally constant function $p$ on $X$ is given by an explicit rule if for some $N$ there is given a function $P$ from some superset of $W(X,-N,N)$ to $\mathbb{Z}$ such that for all $x$ in $X$, $p(x) = P(x_{-N} \ldots x_N)$ (or if $p$ is given by data from which such a $P$ could be algorithmically produced).

**Definition 7.5.** A subshift $(X,T)$ has solvable $\mathbb{Z}$-cyclic triviality problem if there is an algorithm which decides for any explicitly given continuous (i.e. locally constant) function $p : X \to \mathbb{Z}$ whether there is a continuous function $q : X \to \mathbb{Z}$ such that $p = (q \circ T) - q$ (i.e., $p$ is a coboundary in $C(X,\mathbb{Z})$, with transfer function $q$).

If a subshift $(X,T)$ has solvable word problem, then for an explicitly given $p$ in $C(X,\mathbb{Z})$ known to be a coboundary there is a procedure which will produce an explicitly defined $q$ such that $p = (q \circ T) - q$ (enumerate the possible $q$ and test them).

For a positive integer $j$, a subshift $(X,T)$ with language $\mathcal{L}$ is a $j$-step shift of finite type if for all words $U,V,W$ in $\mathcal{L}$, if $V$ has length $j$ and $UV \in \mathcal{L}$ and $VW \in \mathcal{L}$, then $UVW \in \mathcal{L}$.

**Remark 7.6.** As is well known, for an irreducible $j$-step shift of finite type $(X,T)$, and $p$ defined by $P,N$ as in Definition [7.4], the following are equivalent.

1. There is a continuous $q : X \to \mathbb{R}$ such that $p = (q \circ T) - q$.
2. There is a continuous $q : X \to \mathbb{Z}$ such that $p = (q \circ T) - q$, and for all $x$, $q(x)$ depends only on the word $x_{-N} \ldots x_N$.
3. If $x \in X$ and $T^n(x) = x$ and $k \leq M := \max\{j + 1, 2N + 1\}$, then $\sum_{n=0}^{k-1} p(T^n x) = 0$.

(Here, (1) $\implies$ (2) because $T$ has a dense orbit. After passage to the $M$-block presentation (see e.g. [39] Prop. 1.5.12)), $p$ presents as an edge labeling on an edge SFT, and the implication (3) $\implies$ (1) reduces to an old graph argument (recalled in [37] Lemma 6.1)), which also gives a decent algorithm for producing the transfer function $q$ of (2).)

Clearly, an irreducible SFT has solvable $\mathbb{Z}$-cyclic triviality problem.

To prove Theorem [7.10] we emulate the proof of Proposition [7.3] using flow codes in place of block codes. There are two difficulties. First, we need for flow codes a computational analogue of composition of block codes. This is addressed in Appendix [A]. Second, we need an algorithm to determine triviality of $[F]$ in $M(T)$ when $F$ is given by a flow code. We address the latter issue now.

A subshift $(X,T)$ is infinite if the set $X$ contains infinitely many points. A subshift is transitive if it has a dense orbit.
Lemma 7.7. Suppose \((X,T)\) is a subshift, \(C\) is an explicitly given discrete cross section for \((X,T)\) and \(\phi:C \rightarrow D\) is a flow code defined by an explicitly given word code \((\Phi,C)\).

Then the following are equivalent.
1. \([\mathcal{S}\phi]\) is trivial in \(\mathcal{M}(T)\).
2. There is a continuous function \(b:C \rightarrow \mathbb{Z}\) such that for all \(x \in C\), the following hold:
   - \((a)\) The word \(W'_0(x)\) equals the word \(x_{b(x)} \cdots x_{b(x)+|W'_0|−1}\).
   - \((b)\) \(b(x) + |W'_0(x)| = |W_0(x)| + b(T|W_0(x)|)(x)\).

Proof. Let \(\alpha_t\) denote the time \(t\) map of the suspension flow on \(\mathcal{S}X\). Let \(\Phi:W_{-N} \cdots W_N \rightarrow W'\) be the explicitly given word code for \(\phi\), mapping \((2N+1)\)-blocks of \(C\)-return words to a return word for \(D\). For \(x \in C\) with return block \(W_{-N}(x)\), there is a concrete description of return times of \(x\) to \(C\) and \(S\phi(\phi(x))\) to \(D\):

\[
\tau_C(x) = |W_0(x)| \quad \text{and} \quad \tau_D(S\phi(\phi(x))) = |W'_0(x)|.
\]

The condition \((2)\)(b) states that the functions \(x \mapsto \tau_C(x)\) and \(x \mapsto \tau_D(S\phi(\phi(x)))\) are cohomologous in \(C(C,\mathbb{Z})\), with respect to the return map \(\rho_C:x \mapsto T|W_0(x)|(x)\).

For a flow equivalence \(F:\mathcal{S}X \rightarrow \mathcal{S}X\) which maps each orbit to itself, and maps a cross section \(C\) onto a cross section \(D\), the following conditions are equivalent (see e.g. [7, Theorem 3.1]):
1. \(F\) is trivial in \((T)\).
2. There is a continuous function \(\beta:\mathcal{S}X \rightarrow \mathbb{R}\) such that \(F:y \mapsto \alpha_{\beta(y)}(y)\), for all \(y \in S\mathcal{X}\).

In the case \(F = S\phi\), given the second condition, \(\beta\) must assume integer values on \(C\). Define \(b:C \rightarrow \mathbb{Z}\) by \(b:x \mapsto \beta([x,0])\). Then \(b\) satisfies the conditions \((a),(b)\) of the Lemma. The (cocycle) condition \((b)\) is a consistency condition: each side of \((b)\) gives a computation of the number \(t\) (strictly greater than \(b(x)\)) such that \(S\phi\) maps the orbit interval \(\{\alpha_s([x,0]) : 0 \leq s \leq |W_0(x)|\}\) onto the orbit interval \(\{\alpha_{s'}([x,0]) : b(x) \leq s' \leq t\}\).

Conversely, suppose \(b:C \rightarrow \mathbb{Z}\) is a continuous function satisfying \((a)\) and \((b)\). By induction, using the given word block code, we see that for all \(x\) in \(C\) and all nonnegative integers \(k\), for \(s = \sum_{i=0}^k |W'_i(x)|\) we have \((\phi x)_0 \cdots (\phi x)_k = x_{b(x)} \cdots x_{b(x)+|W'_0|−1}\). Because the return map to \(C\) is a homeomorphism, we then have for \(r = \sum_{i=0}^{-1} |W'_i(x)|\) that \((\phi x)_{r−1} \cdots (\phi x)_0 = x_{b(x)−r} \cdots x_{b(x)−1}\). Consequently, for all \(x \in C\), \(S\phi:[x,0] \rightarrow [T^{b(x)},0]\), so \(S\phi\) maps each flow orbit to itself. Finally, from \(b\) we can define the continuous function \(\beta\) of condition \((2)\), as follows. For \(x \in C\), set \(r(x) = |W'_0(x)|/|W_0(x)|\) and

\[
\beta : [x,t) \mapsto [x,b(x)+tr(x)], \quad x \in C, 0 \leq t < |W_0(x)|.
\]

This rule defines \(\beta\) on the entire mapping torus. The piecewise linearity of \(\beta\) on the flow segments between returns to the cross section agrees with the flow code definition.

\[\square\]

Lemma 7.9. Suppose \((X,T)\) is a transitive subshift (for example, any irreducible SFT) with decidable language and solvable \(\mathbb{Z}\)-cocycle triviality problem. Suppose \(C\) is an explicitly given discrete cross section for \((X,T)\) and \(\phi:C \rightarrow \overline{C}\) is a flow code defined by an explicitly given word code \((\Phi,C)\).
Then there is a procedure which decides whether $S\phi$ is a flow equivalence $SX \to SX$ such that $|S\phi|$ is trivial in $\mathcal{M}(T)$.

**Proof.** We will decide whether there is a function $b \in C(X, \mathbb{Z})$ satisfying the conditions (a),(b) of Lemma 7.7. We are explicitly given the locally constant return time functions $\tau_C(x) = |W_0(x)|$ and $\tau_D(\phi x) = |W_0(\phi x)|$. Because there is a dense $T$ orbit, a solution $b$ to (b) is unique up to an additive constant. Thus, either every solution to (b) also satisfies (a), or no solution to (b) also satisfies (a).

By the $\mathbb{Z}$-cocyle triviality and solvable word problem assumptions, there is an algorithm which produces $b \in C(X, \mathbb{Z})$ such that $|W_0'(x)| - |W_0(x)| = b(T^{|W_0(x)|}(x)) - b(x)$, if such a $b$ exists. So, suppose we have $b \in C(X, \mathbb{Z})$ satisfying condition (b) of Lemma 7.7. From the explicitly given $C$ and word code $\phi$, we can compute a positive integer $M$ such that for all $x$, the word $x_{-M} \ldots x_M$ determines $W_{-N}(x)$ (and thus $W_0'(x)$), and also $M > \max |b| + \max |W_0'|$. Now whether (a) holds can be detected by testing each word $x_{-M} \ldots x_M$ occurring in $X$. By the assumed decidability of the language, there is an algorithm to list these words. \hfill $\Box$

**Theorem 7.10.** Suppose $(X, T)$ is a transitive subshift with decidable language and solvable $\mathbb{Z}$-cocyle triviality problem (for example, any irreducible SFT). Then $\mathcal{M}(T)$ has solvable word problem.

**Proof.** Given $[F_1], \ldots, [F_k]$ in $\mathcal{M}(T)$, for $1 \leq i \leq k$ let $\Phi_i$ be an explicitly given word code defining a homeomorphism of discrete cross sections, $\phi_i : C_i \to D_i$, with $S\phi_i$ isotopic to $F_i$. Suppose $[F] = [F_{i_1}] [F_{i_2}] \cdots [F_{i_l}]$ in $\mathcal{M}(T)$.

By Proposition A.3 there is an algorithm which computes a rule $\Phi$, defining a homeomorphism $\phi : C \to D$ of explicitly given cross sections of $(X, T)$, such that $|S\phi| = [F]$. By Lemma 7.9 there is then a procedure which decides whether $|S\phi|$ is trivial in $\mathcal{M}(T)$. \hfill $\Box$

### 8. Conjugacy classes of involutions

Throughout this section, $A$ is a matrix defining a nontrivial irreducible SFT. We will prove and exploit Theorem 8.1 which shows how conjugacy classes of many involutions in $\mathcal{M}_A$ are classified as $G$-flow equivalence classes of mixing $G$-SFTs, for $G = \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

We prepare for the statement of Theorem 8.1 with some definitions and background. In this paper, by a $G$-SFT we mean a shift of finite type together with a continuous (not necessarily free) action of a finite group $G$ by homeomorphisms which commute with the shift. A $G$-SFT is mixing (irreducible) if it is mixing (irreducible) as an SFT. A continuous $G$ action on an SFT $X_A$ lifts to a continuous $G$ action on its mapping torus $SX_A$. Two $G$-SFTs are $G$-flow equivalent if there is an orientation preserving homeomorphism between their mapping tori which intertwines the induced $G$ actions.

Recall, if $C$ is a cross section for a flow equivalence $F : SX_A \to SX_A$, and $\rho_C : C \to C$ is the return map to $C$ under the flow, then $\rho_C$ is flow equivalent to $\sigma_A$ and in particular is a nontrivial irreducible SFT. If $C$ is also invariant under an involution $V$ in $\mathcal{F}_A$, then the pair $T = (\rho_C, V|_C)$ is a $\mathbb{Z}_2$-SFT; we say this $\mathbb{Z}_2$-SFT is associated to $V$, and to $SX_A$.

**Theorem 8.1.** For $i = 1, 2$: suppose $V_i$ in $\mathcal{F}_A$ is an involution. Then $V_i$ has an invariant cross section. Let $C_i$ be any invariant cross section for $V_i$, with $T_i$ the associated $\mathbb{Z}_2$-SFT. Then the following are equivalent.
(1) $[V_1]$ and $[V_2]$ are conjugate elements in $\mathcal{M}_A$.

(2) The $\mathbb{Z}_2$-SFTs $T_1$ and $T_2$ are $\mathbb{Z}_2$-flow equivalent.

Proof. The involutions $V_1, V_2$ have invariant cross sections by Lemma 8.4. By Lemma 8.4, there is an involution $V$ in $\mathcal{F}_A$ which equals $V_2$ on $C_2$ (and therefore defines the same associated $\mathbb{Z}_2$-SFT), such that there is a flow equivalence $J$ such that $J^{-1}V_1J = V$. This shows the two $\mathbb{Z}_2$-SFTs are $\mathbb{Z}_2$-flow equivalent. 

If $V$ is an involution in $\mathcal{F}_A$, then the fixed point set of its restriction to an invariant cross section $C$ will, as a subsystem of $(C, \rho_C)$, be an SFT. Theorem 8.1 shows that the flow equivalence class of this SFT is an invariant of the conjugacy class.

Lemma 8.4, there is an involution $V$ defines the same associated $\mathcal{M}_A$-SFT, such that there is a flow equivalence $J$ such that $\mathcal{M}_A$-SFT, even though there can be other elements $W$ in $[V]$ (but not other involutions) with fixed point set containing a submapping torus whose intersection with $C$ properly contains $C \cap \text{Fix}(V)$ and represents a different flow equivalence class.

Question 8.2. Suppose $[F]$ is an involution in $\mathcal{M}_A$. Is there an involution $V$ such that $[F] = [V]$?

If the answer to Question 8.2 is yes, then Theorem 8.1 applies to all order two elements of the mapping class group; if the answer is no, then the quotient map $\mathcal{F}_A \to \mathcal{M}_A$ does not split.

Below, for visual simplicity, where a point $x$ in $X_A$ denotes a point in $\mathcal{S}X_A$, it denotes $[x, 0]$. We similarly abuse notation for sets.

Lemma 8.3. Suppose $V \in \mathcal{F}_A$ is an involution. Then $V$ has an invariant cross section.

Proof. Suppose $X_A \cap V(X_A)$ is nonempty (if it is empty, then $X_A \cup V(X_A)$ is an invariant cross section for $V$). Fix $\epsilon > 0$ small enough that the image under $V$ of any orbit interval of length $2\epsilon$ has length less than 1. For a clopen subset $C$ of $X_A$ containing $X_A \cap V(X_A)$, with $V(C) \subset X_A \times (-\epsilon, \epsilon)$, define $C'$ to be the clopen-in-$X_A$ set of points $x'$ such that for some $t$ in $(-\epsilon, \epsilon)$ and some $x$ in $C$, $V(x) = [x', t]$. Fix $C$ small enough that we also have $V(C') \subset X_A \times (-\epsilon, \epsilon)$, and set $D = C \cup C'$. Now there is a continuous involution $h : D \to D$ with $h(C) = C'$, and a continuous function $\gamma : D \times (-\epsilon, \epsilon) \to \mathbb{R}$, such that for all $[x, t]$ in $D \times (-\epsilon, \epsilon)$,

$$V : [x, t] \to [h(x), t + \gamma([x, t])] .$$

For every $x$ in $D$, $V$ maps the interval $\{[x, t] : -\epsilon < t < \epsilon\}$ by an orientation preserving homeomorphism to some orbit interval of length less than 1. In particular, if $h(x) = x$, then $\gamma(x) = 0$ (otherwise, $V$ would map the orbit segment between $x$ and $Vx$ onto itself reversing endpoints, and thus reversing orientation). Define

$$K = \{x \in D : \gamma(x) \geq 0\} \cup \{[h(x), \gamma(h(x))] : x \in D, \gamma(x) \geq 0\} ,$$

$$L = X_A \setminus D ,$$

$$E = K \cup L \cup VL .$$

We will show $E$ is an invariant cross section for $V$. Invariance is clear, since for $x$ in $D$, we have $V(x) = [h(x), \gamma(h(x))]$.

Suppose $x \in D$. Let $K(x) = K \cap \{x\} \times (-\epsilon, \epsilon)$; then $K = \bigcup_{x \in D} K(x)$. Let $y = h(x)$. We have $K(x) \subset \{x, [x, \gamma(y)]\}$. Either both $\gamma(y)$ and $\gamma(x)$ are zero, or
they are nonzero with opposite sign. Thus

\[ K(x) = \{ x, [x, \gamma(y)] \} = \{ x \} , \quad \text{if } \gamma(x) = 0 , \]
\[ = \{ x \} , \quad \text{if } \gamma(x) > 0 , \]
\[ = \{ [x, \gamma(y)] \} , \quad \text{if } \gamma(x) < 0 . \]

For \( x \) in \( D \), define \( \kappa(x) = \max\{ \gamma(x), \gamma(h(x)) \} \). It follows that \( K = \{ [x, \kappa(x)] : x \in D \} \), the graph of a continuous function on \( D \). The sets \( K, L, VL \) are disjoint. It is now straightforward to verify that \( E \) is closed, \( E \) intersects every flow orbit and the return time function on \( E \) is continuous. Thus \( E \) is a cross section.

\[ \square \]

Below, by the normalized suspension flow over a cross section \( C \), we mean the suspension flow after a time change such that points move at unit speed and points in \( C \) have return time 1. This can be achieved by a flow equivalence from the mapping torus of the return map \( \rho_C \).

**Lemma 8.4.** Suppose \( U \) and \( W \) are involutions \( SX_A \to SX_A \), with \( [U] \) and \( [W] \) conjugate in \( \mathcal{M}_A \). Then \( [W] \) contains an involution \( V \) such that the following hold.

1. There is an invariant cross section \( C \) for \( W \) and for \( V \) such that \( V = W \) on \( C \).
2. \( V \) is an isomorphism of a normalized suspension flow over an invariant cross section.
3. There is a flow equivalence \( J : SX_A \to SX_A \) such that \( J^{-1}UJ = V \).

**Proof.** By Lemma 8.3, \( W \) has an invariant cross section; without loss of generality, we assume it is \( X_A \). Given \( x \) in \( X \), let \( \gamma_x : [0, 1] \to [0, 1] \) be the homeomorphism such that \( W : [x, t] \mapsto [W(x), \gamma_x(t)] \), \( 0 \leq t \leq 1 \). Then define \( B : SX_A \to SX_A \) to be \( [x, t] \mapsto [x, \gamma_x^{-1}(t)] \), \( 0 \leq t \leq 1 \), and set \( V = UB \). \( V \) is the required map.

By assumption there is \( F \) such that \( [F]^{-1}[U][F] = [V] \), so \( F^{-1}UF = VH \), where there is a continuous function \( \beta : SX_A \to \mathbb{R} \) such that \( H \) is a flow equivalence defined by \( H(x) = \alpha_{\beta(x)}(x) \) (for this see e.g. [7, Theorem 3.1]). It suffices to find a flow equivalence \( G \) such that \( G(VH)G^{-1} = V \). Define functions \( b = \max\{ \beta, 0 \} \) and \( c = \min\{ \beta, 0 \} \). Then define \( H_+ \) and \( H_- \) on \( SX_A \) by \( H_+(x) = \alpha_{b(x)}(x) \) and \( H_-(x) = \alpha_{c(x)}(x) \).

The set in an orbit on which \( \beta \) is nonzero is a disjoint union of intervals; on each, \( \beta \) has constant sign, and on each, \( H \) is a surjective self-homeomorphism respecting the flow orientation. Now by continuity of the functions \( b \) and \( c \), \( H_+ \) and \( H_- \) are flow equivalences of \( SX_A \), isotopic to the identity. Clearly \( H = H_+H_- = H_-H_+ \).

We have \( V = VH \), because \( V = V(VHVH) = V^2(HVH) \). Next, we show that \( H_+VH_+ = V \). For all \( x \),

\[ VH(x) = V(\alpha_{b(x)}(x)) = \alpha_{\beta(x)}(Vx) \equiv z . \]

Then \( H(VH(x)) = \alpha_{\beta(x)}(z) \), and

\[ x = VH(VH(x)) = V(\alpha_{\beta(x)}(z)) = \alpha_{\beta(z)}(Vz) = \alpha_{\beta(z)}V\alpha_{\beta(x)}(Vx) \]
\[ = \alpha_{\beta(z)}\alpha_{\beta(x)}(V^2x) = \alpha_{\beta(z)+\beta(x)}(x) . \]

Thus \( \beta(z) + \beta(x) = 0 \) on the dense set of aperiodic points, hence everywhere. Because the sign of \( \beta(z) \) is the same as the sign of \( \beta \) on \( H(z) = HVH(x) = V(x) \), it follows that \( \beta \) is nonzero at \( x \) if and only if \( \beta \) is nonzero with opposite sign at
Theorem 8.6. Suppose in $M$ is an odd integer. Then $\text{Sm}(G)$ has a normal form (slightly unconventional, following [15], to address sign and achieve $B$ equivalent) to a unique Smith normal form, which we denote $\text{Sm}(G)$. This section that tools for $G$ for general free $G$-SFTs are of some use for learning about conjugacy classes $G$-flow equivalence. Still, we will see with the remainder of the section that tools for $G$-SFTs are of some use for learning about conjugacy classes $G$-flow equivalence.

Finally, let $G = H_+$. Then

$$G(VH)G^{-1} = H_-(VH+H_-)(H_-)^{-1} = H_-VH_+ = V.$$ 

We give more information now on the $G$-SFTs.

A free $G$-SFT is a $G$-SFT for which the $G$-action is free. By a construction of Parry explained in [15] (also see [14, Appendix A]), free $G$-SFTs can be presented by square matrices with entries in $\mathbb{Z}_+G$, the set of elements $\sum_n n_g g$ in the integral group ring $\mathbb{Z}G$ with every $n_g$ a nonnegative integer. Let $\text{El}(n, \mathbb{Z}G)$ be the group of $n \times n$ elementary matrices over the integral group ring $\mathbb{Z}G$.

**Theorem 8.5.** [15] Suppose $A, B$ are square matrices over $\mathbb{Z}_+G$, presenting nontrivial irreducible free $G$-SFTs. Then the following are equivalent.

1. The $G$-SFTs are $G$-flow equivalent.
2. $I - A$ and $I - B$ are stably elementary equivalent over $\mathbb{Z}_+G$, in the following sense: there exist integers $j, k, n$ and identity matrices $I_j, I_k$ with sizes $j, k$ such that there are matrices $U, V$ in $\text{El}(n, \mathbb{Z}_+G)$ such that $U((I - A) \oplus I_j)V = (I - B) \oplus I_k$.

The classification statement Theorem 8.5 is the main special case of [15, Theorem 6.4], given one additional simplifying remark. When a $\mathbb{Z}_+G$ matrix $A$ defines a nontrivial irreducible $G$-SFT $T$, the matrix $A$ must be essentially irreducible (so, [15, Theorem 6.4] applies) and its “weights group” in the statement of [15, Theorem 6.4] must be all of $G$. (Otherwise, as an SFT $T$ could not have a dense orbit; see [5, Prop.D.7] for detail.)

There is also a complete (more complicated) classification of $G$-flow equivalence for general free $G$-SFTs, in [5]. In the nonfree case, significant invariants are known, but the classification problem is open. Still, we will see with the remainder of the section that tools for $G$-SFTs are of some use for learning about conjugacy classes in $\mathcal{M}_A$. Define

$$\mathcal{C}_A = \{ [V] \in \mathcal{M}_A : V^2 = Id \text{ and } V \text{ is associated to a free } \mathbb{Z}_2 - \text{SFT} \}.$$ 

It follows from Proposition 8.4 that

$$\mathcal{C}_A = \{ [V] \in \mathcal{M}_A : V \text{ is a fixed point free involution} \}.$$ 

We will say an $n \times n$ matrix $D$ over $\mathbb{Z}$ is a Smith normal form if $D$ is a diagonal matrix $\text{diag}(d_1, d_2, \ldots, d_n)$ satisfying the following conditions: $d_{i+1}$ divides $d_i$ whenever $1 \leq i < n$ and $d_{i+1} \neq 0$; $d_{i+1} = 0$ implies $d_i = 0$; and $d_i \geq 0$ if $i > 1$. It is well known that any $n \times n$ matrix $B$ over $\mathbb{Z}$ is $\text{SL}(n, \mathbb{Z})$ equivalent (hence $\text{El}(n, \mathbb{Z})$ equivalent) to a unique Smith normal form, which we denote $\text{Sm}(B)$. (Our “Smith normal form” is slightly unconventional, following [15], to address sign and achieve $\text{Sm}(B \oplus I_k) = \text{Sm}(B) \oplus I_k$.) Note, $\det(B) = \det(\text{Sm}(B))$.

**Theorem 8.6.** Suppose $A$ is a nontrivial irreducible $SFT$ and $\det(I - A)$ is an odd integer. Then $\mathcal{C}_A$ is the union of finitely many conjugacy classes in $\mathcal{M}_A$. 

$V(x)$. So, if $\beta(x) > 0$, then $H_-VH_+(x) = HVH(x) = V(x)$. If $\beta(y) < 0$, then $y = Vx$ with $\beta(Vx) > 0$, so

$$H_-VH_+(y) = H_-VH_+(Vx) = V(Vx) = x = V(y).$$ 

Finally, let $G = H_-$. Then

$$G(VH)G^{-1} = H_-(VH+H_-)(H_-)^{-1} = H_-VH_+ = V.$$ 

$\square$
Proof. Let \( C \) be a matrix over \( \mathbb{Z} \) presenting a free \( \mathbb{Z}_2 \)-SFT which is \( \mathbb{Z}_2 \)-flow equivalent to a \( \mathbb{Z}_2 \)-SFT associated to a free involution in \( F_A \). Let \( C = eX + pY \), with \( X \) and \( Y \) over \( \mathbb{Z} \) and \( G = \{ e, g \} \). The matrix \( F = \left( \begin{smallmatrix} e & f \\ 0 & e \end{smallmatrix} \right) \) defines an SFT flow equivalent to \( \sigma_A \), so \( \det(I - A) = \det(I - F) \), and therefore

\[
\det(I - A) = \det(I - (X + Y)) \det(I - (X - Y)).
\]

In our special situation, with \( G = \mathbb{Z}_2 \) and \( \det(I - F) \) is odd, by [15] Theorem 8.1 the stable \( \text{El}(ZG) \) equivalence class of \( C \) (and thus the \( G \)-flow equivalence class of the \( G \)-SFT it presents) is determined by the pair \((\text{Sm}(I - (X + Y)), \text{Sm}(I - (X - Y)))\). Because \( \det(I - A) \neq 0 \), it follows from (8.7) that only finitely many pairs are possible.

Let us say a \( \mathbb{Z}_2 \)-SFT is \text{inert} if the involution defining the \( \mathbb{Z}_2 \) action is an inert automorphism of the underlying SFT. This is the class of greatest interest to us. (For example, these involutions induce involutions on the mapping torus which are in the kernel of the Bowen-Franks representation.)

Theorem 8.8. Suppose \( \sigma_A \) is a nontrivial mixing SFT and there is an inert \( \mathbb{Z}_2 \)-SFT associated to a free involution \( V \) in \( F_A \). Then \( \det(I - A) \) is an odd integer.

Proof. Suppose \((\sigma_B, U)\) is an SFT with free inert involution \( U \) associated to \( V \). Ulf Fiebig proved that a finite order automorphism \( U \) of \( \sigma_A \) is inert if and only if the homeomorphism induced by the shift on the quotient space of \( U \)-orbits has the same zeta function as \( \sigma_A \) [24] Theorem B]. This condition forces \( \det(I - tB) = 1 \) mod 2, by the argument of [31] Lemma 2.2], so \( \det(I - B) \) is odd. Because \( \sigma_A \) and \( \sigma_B \) are flow equivalent, \( \det(I - B) = \det(I - A) \).

Remark 8.9. If \( \det(I - A) \) is an odd negative squarefree integer, then \( \sigma_A \) is flow equivalent to a full shift with a free inert involution, and there is a free inert \( \mathbb{Z}_2 \)-SFT associated to an involution of \( S \mathbf{X}_A \). We expect it is possible to prove such involutions exist whenever \( \det(I - A) \) is odd, by direct construction or by appealing to the following difficult result of Kim and Roush.

Theorem 8.10. [31] Theorem 7.2 and Lemma 2.2] Let \( \sigma_A \) be a mixing shift of finite type and let \( p \) be a prime. Then the following are equivalent.

1. There is an SFT \( \sigma_B \) shift equivalent to \( \sigma_A \), and an order \( p \) automorphism \( U \) of \( \sigma_B \), and a factor map \( \pi : X_A \to X_B \) for which the fiber over every point is a cardinality \( p \) orbit of \( U \).

2. For all positive integers \( n \),

\[
o_n \geq \frac{p - 1}{p} o_{n/p} + \frac{p - 1}{p^2} o_{n/p^2} + \frac{p - 1}{p^3} o_{n/p^3} + \cdots
\]

where \( o_n \) denotes the number of \( \sigma_A \) orbits of cardinality \( k \).

The condition (2) above implies \( \det(I - tA) = 1 \) mod \( p \). Condition (2) holds for all \( n \) if it holds up to a computable bound. (See [31] Sections 1-2 for more explanation.) The automorphism \( U \) in (1) must be inert (by [24] Theorem B]), so there will be an inert \( \mathbb{Z}_p \)-SFT associated to a free \( \mathbb{Z}_p \) action on \( S \mathbf{X}_B \). The shifts \( \sigma_A \) and \( \sigma_B \) in (1) are flow equivalent, so there will also be an inert \( \mathbb{Z}_p \)-SFT associated to a free \( \mathbb{Z}_p \) action on \( S \mathbf{X}_A \).
While the classification of non-free $\mathbb{Z}_2$ SFTs is open, we can consider the invariant which is the flow equivalence class of the fixed point shift. The following result of Long [40, Theorem 1.1] is an instrument for creating examples.

**Theorem 8.11.** (Long) Let $f$ be an inert automorphism of a mixing shift of finite type $X$ with $\text{Fix}_f(X) \subset Y$ where $Y \neq X$ and $Y$ is a $f$-invariant subshift of finite type in $X$. Suppose $n \geq 2$ and $n$ is the smallest positive integer such that $f^n = \text{Id}$. If the restriction of $f$ to $Y$ is inert, then there exists an inert automorphism of $X$, $U$, such that $Y$ is the fixed point shift of $U$, where $U^n = \text{Id}$ and $n$ is the minimal positive integer $k$ such that $U^k = \text{Id}$.

For example, let $f$ be the inert involution of the full shift on symbols 0, 1, 2 which exchanges the symbols 0 and 1. For a positive integer $n$, let $T_n$ be the subshift with language $(\{0,1\}^2)^n$ (words of length $n$ on $\{0,1\}$ alternate with the symbol 2). Then $T_n$ is invariant under $f$, and one can check the restriction of $f$ to $T_n$ is inert.

By Long’s theorem, $T_n$ is the fixed point shift of some inert involution of the 3-shift. $T_n$ is an irreducible SFT with Bowen-Franks group $\mathbb{Z}/(2^n-1)\mathbb{Z}$. So, infinitely many flow equivalence classes occur as the fixed shift of an inert $\mathbb{Z}_2 - \text{SFT}$ associated to an involution of $S X_A$, and those involutions must represent distinct elements of $\mathcal{M}_A$.

One can more generally produce infinitely many distinct flow equivalence classes of inert involutions of $\mathbb{Z}_2$-SFTs associated to $S X_A$, whenever there is a free $\mathbb{Z}_2$-SFT associated to $S X_A$, by combining some of Long’s results ([40, Theorem 1.1, Theorem 1.2, Lemma B.2]) and some construction work (e.g., for $k \in \mathbb{N}$ embed into $X_A 2k$ disjoint copies of an SFT admitting an inert involution, say using [8]).

**Appendix A. Flow codes**

Flow codes were developed in [7] as a flow map analogue of block codes. In [7], flow codes were considered for not necessarily invertible flow maps. In this appendix, for simplicity we only consider flow equivalences, and “flow code” means “flow code” for a flow equivalence.

First we recall some definitions from [7]. Let $C$ be a discrete cross section for a subshift $X$. Given $C$, the **return time bisequence** of a point $x$ in $C$ is the bisequence $(r_n)_{n \in \mathbb{Z}}$ (with $r_n = r_n(x)$) such that

1. $\sigma^j(x) \in C$ if and only if $j = r_n$ for some $n$,
2. $r_n < r_{n+1}$ for all $n$, and
3. $r_0 = 0$.

A return word is a word equal to $x[0, r_1(x))$ for some $x \in C$. Given $x \in C$ and $n \in \mathbb{Z}$, $W_n = W_n(x)$ denotes the return word $x[r_n, r_{n+1})$. In the context of a given $C$, when we write $x = \ldots W_{-1}W_0W_1 \ldots$ below, we mean $x \in C$ and $W_n = W_n(x)$. Given $x \in C$ and $i \leq j$, the tuple $(W_n(x))_{n=i}^j$ is the $[i,j]$ return block of $x$, also denoted $W_{ij}(x)$, and $\mathcal{W}(i, j, C) = \{W_{ij}(x) : x \in C\}$. To know this return block is to know the word $W = W_i \cdots W_j$ together with its factorization as a concatenation of return words.

**Definition A.1.** Suppose $C$ is a discrete cross sections of a subshift $(X, T)$. A **C word block code** is a function $\Phi : \mathcal{W}(-N, N, C) \to \mathcal{W}_0$, where $\mathcal{W}_0$ is a set of words and $N$ is a nonnegative integer. A **word block code** is a C word block code for some $C$. The function $\phi$ from $C$ into a subshift given by $\Phi$ is defined to map
$x = (W_n)_{n \in \mathbb{Z}}$ to the concatenation $x' = (W'_n)_{n \in \mathbb{Z}}$, with $W'_n = \Phi(W_{n-M}, \ldots, W_{n+M})$ and $x'[0, \infty) = W'_0W'_1\ldots$.

For $D$ a discrete cross section of a subshift $(X', T')$, a $C, D$ flow code is a $C$ word block code $\Phi$ defined as above, with the following additional properties:

1. $W'_0$ is the set of $D$ return words.
2. The induced map $\phi$ is a homeomorphism $\phi : C \to D$ which is a topological conjugacy of the return maps of $C$ and $D$ (with respect to $T$ and $T'$).

In this case we refer to $(\Phi, C, D)$ as a flow code defining $\phi$. This code induces a flow equivalence $S\phi : SX \to SX'$ by the following rule, in which $r(x) = |W'_0(x)|/|W_0(x)|$:

$S\phi : [x, t] \mapsto [\phi(x), t\tau(x)]$, if $x \in C$ and $0 \leq t < |W_0(x)|$.

By [7] Theorem 5.1, for every flow equivalence of subshifts, $F : SX \to SX'$, there is a flow code $(\Phi, C, D)$ inducing $S\phi : SX \to SX'$ such that $S\phi$ is isotopic to $F$.

Now suppose flow codes $(\Phi_1, C_1, D_1)$ and $(\Phi_2, C_2, D_2)$ induce $\phi_1 : C_1 \to D_1$ and $\phi_2 : C_2 \to D_2$, with $S\phi_1 : SX_1 \to SX_2$ and $S\phi_2 : SX_2 \to SX_3$. If $D_1 = C_2$, then we can compose the word block codes as easily as we compose block codes to obtain a flow code $(\Phi, C, D)$ defining $\phi : C_1 \to D_2$ such that $S\phi = (S\phi_2) \circ (S\phi_1)$. But if $C_1 \neq D_1$, we need another ingredient to produce a flow code as a function of the given data such that $[S\phi] = [(S\phi_2) \circ (S\phi_1)]$.

Given a discrete cross section $C$ for $(X, T)$ and $x \in X$, define $\tau(x, C) = \min\{i \geq 0 : T^i(x) \in C\}$. The next proposition adapts the Parry-Sullivan argument ([44]; see Theorem 2.2) to discrete cross sections. As the argument is very similar to arguments given in [7], we will leave a proof as an exercise (perhaps after reviewing [7]).

Proposition A.2. Suppose $C$ and $D$ are discrete cross sections for a subshift $(X, T)$. Define

$K = K(C, D) = (C \cap D) \bigcup \{x \in C \setminus D : \tau(x, D) < \tau(x, C)\} \subset C$

$L = L(C, D) = \{\sigma^{\tau(x, D)}(x) : x \in K\} \subset D$

$\delta : K \to L$, $\delta : x \mapsto T^{\tau(x, D)}$.

Then

1. $K$ and $L$ are discrete cross sections for $(X, T)$.
2. $\delta$ is a well defined homeomorphism.
3. $\delta$ is a topological conjugacy of the return maps to $K$ and $L$ under $T$, i.e. $\delta\rho_K = \rho_L\delta$.
4. $\delta$ is given by a word block code $\Delta : W(K, 0, 1) \to W(L)$.
5. $\delta^{-1}$ is given by a word block code $\Psi : W(L, -1, 0) \to W(K)$.
6. $[S\delta]$ is the identity element in $\mathcal{M}(T)$.

Suppose the subshift $(X, T)$ has decidable language. Let there be given $N \in \mathbb{N}$ and word sets $V_C, V_D$ such that $C = \{x \in X : x_{-N} \ldots x_N \in V_C\}$ and $D = \{x \in X : x_{-N} \ldots x_N \in V_D\}$. Then there is an algorithm to determine $M$ in $\mathbb{N}$ and the following:

(a) $V_K \subset W(X)$ such that $K = \{x \in X : x_{-N} \ldots x_{M+N} \in V_K\}$.
(b) $V_L \subset W(X)$ such that $L = \{x \in X : x_{-(M+N)} \ldots x_{M+N} \in V_L\}$.
(c) The word codes $\Delta$ and $\Psi$. 

In Part (6) above, the decidability of the language lets us find an upper bound to the return time to $K$.

We say a discrete cross section $C$ for a subshift $(X,T)$ is explicitly given if there is given $N$ in $\mathbb{N}$ and a subset $V_C$ of the language of $X$ such that $C = \{ x \in X : x[-N,N] \in V \}$ (or if $C$ is given by data from which such a set $V$ could be algorithmically produced). Similarly, a flow code $(\Phi,C,D)$ is explicitly given if $C$ is explicitly given and for some $M$, $\Phi$ is given as a function from a subset of $W(C,-M,M)$ (or by algorithmically equivalent information).

**Proposition A.3.** For $i = 1,\ldots,k+1$, let $(X_i,T_i)$ be a subshift with decidable language. Suppose for $1 \leq i \leq k$ that $S\phi_i : S\xi_i \to S\xi_{i+1}$ is a flow equivalence defined from a homeomorphism $\phi_i : C_i \to D_i$ defined by an explicitly given flow code $(\Phi_i,C_i,D_i)$. Then there is an algorithm which produces an explicitly given flow code $(\Phi,E,\overline{E})$, with $E \subset C_1$ and $\overline{E} \subset D_k$, inducing $\phi : E_1 \to E_k$ such that $(S\phi_k) \circ (S\phi_1)$ and $S\phi$ are isotopic.

Note, we are not claiming to produce a $\phi$ such that $(S\phi_k) \circ (S\phi_1) = S\phi$.

**Proof of Proposition A.3** By induction, it suffices to prove the proposition assuming $k = 2$.

For the explicitly given discrete cross sections $D_1$ and $C_2$ of $(X_2,T_2)$, we have explicitly from Lemma A.2, $K = K(D_1,C_2)$, $L = L(D_1,C_2)$ and a flow code $(\Delta, K, L)$ for $\delta$. Define $E = \phi_2^{-1}(K)$ and $\overline{E} = \phi_2(L)$. Set $\psi_1 = \phi_1|E$, $\psi_2 = \phi_2|L$ and $\phi = \psi_2 \delta \psi_1$. Now $S\phi_1$ and $S\psi_1$ are isotopic, for $i = 1,2$, and $S\delta$ is isotopic to the identity, by Lemma A.2 Therefore $(S\phi_2) \circ (S\phi_1)$ is isotopic to $S\phi$.

From the explicitly given word codes for $\phi_1$ and $\phi_2$, we can compute explicitly a flow code $(\Psi_1,E,K)$ for $\psi_1$ and a flow code $(\Psi_2,L,\overline{E})$ for $\psi_2$. Now the discrete cross sections align, and we can compose the word codes $(\Psi_2,L,\overline{E})$, $(\Delta, K, L)$, $(\Psi_1,E,K)$ to obtain a block word code rule $(\Phi,E,\overline{E})$ for $\phi : E \to \overline{E}$, with $\Phi$ defined for some $M$ on a set $W$ containing $\{ W_M^x(x) : x \in E \}$. (Moreover, by solvability of the word problem for $(X_1,T_1)$, we may then choose to shrink $W$ so that the containment becomes equality.)

\[ \square \]

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