A derivative-free Milstein type approximation method for SPDEs covering the non-commutative noise case

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Abstract
We propose a derivative-free Milstein type scheme to approximate the mild solution of stochastic partial differential equations (SPDEs) that do not need to fulfill a commutativity condition for the noise term. The newly developed derivative-free Milstein type scheme differs significantly from schemes that are appropriate for the case of commutative noise. As a key result, the new derivative-free Milstein type scheme needs only two stages that are specifically tailored based on a technique that, compared to the original Milstein scheme, allows for a reduction of the computational complexity by one order of magnitude. Moreover, the proposed derivative-free Milstein scheme can flexibly be combined with some approximation method for the involved iterated stochastic integrals. As the main result, we prove the strong $L^2$-convergence of the introduced derivative-free Milstein type scheme, especially if it is combined with any suitable approximation algorithm for the necessary iterated stochastic integrals. We carry out a rigorous analysis of the error versus computational cost and derive the effective order of convergence for the derivative-free Milstein type scheme in the case that the truncated Fourier series algorithm for the approximation of the iterated stochastic integrals is applied. As a further novelty, we show that the use of approximations of iterated stochastic integrals based on truncated Fourier series together with the proposed derivative-free Milstein type scheme improves the effective order of convergence compared to that of the Euler scheme and the original Milstein scheme. This result is contrary to well known results in the finite dimensional SDE case where the use of merely truncated Fourier series does not improve the effective order of convergence in the $L^2$-sense compared to that of the Euler scheme.
Keywords  Stochastic partial differential equation · Milstein scheme · Numerical analysis · Non-commutative noise · Iterated stochastic integral · Stochastic evolution equation

1 Motivation

For the approximation of stochastic partial differential equations (SPDEs) with commutative noise, some higher order schemes such as the Milstein schemes in [1, 2, 9, 14, 15, 17], the derivative-free versions [18, 38], or the Wagner-Platen type scheme [3] were derived and implemented in the last years. Concerning equations that do not need to possess commutative noise, see [4, 16, 23, 29, 30, 34] for some examples with practical applications like, e.g., stochastic filtering, it was, however, an open question how to implement a higher order scheme due to the iterated stochastic integrals that are involved. Therefore, the numerical scheme of choice was so far some Euler scheme, for example, the exponential Euler or the linear implicit Euler, see [7, 13, 22, 25, 26]. Recently, the authors presented two algorithms to obtain an approximation of such stochastic integrals, see [21]. In [36], the Milstein scheme proposed by Jentzen and Röckner [9] has been analyzed for non-commutative equations in the case that it is combined with the algorithms proposed in [21]. However, as the main drawback the Milstein scheme requires the evaluation of the derivative of an operator in each time step. This is the reason that its computational complexity increases quadratically w.r.t. the dimension of the state space compared to the Euler scheme with linearly growing computational complexity. In the present paper, we propose a derivative-free numerical scheme to efficiently approximate the mild solution of SPDEs which do not need to have commutative noise, that is, the commutativity condition

\[(B'(v)(B(v)u))\tilde{u} = (B'(v)(B(v)\tilde{u}))u\]

for all \(v \in H_\beta, u, \tilde{u} \in U_0\) has not to be fulfilled. Our goal is to approximate the mild solution to SPDEs of type

\[dX_t = (AX_t + F(X_t))\, dt + B(X_t)\, dW_t, \quad t \in (0, T], \quad X_0 = \xi\]

with a scheme that obtains the same temporal order of convergence as the Milstein scheme, however, without the need to evaluate any derivative and with significantly reduced computational complexity which is of the same order of magnitude as for the Euler scheme, i.e., which depends only linearly on the dimension of the state space. For details on the notation, we refer to Sect. 2.1. In general, the Milstein scheme proposed in [9] applied to (2) using the notation \(Y_{m+1}^{\text{MIL}} = Y_{m+1}^{\text{MIL}:N,K,M}\) reads as \(Y_0^{\text{MIL}} = P_N\xi\) and

\[Y_{m+1}^{\text{MIL}} = P_N e^{Ah} \left( Y_m^{\text{MIL}} + h F(Y_m^{\text{MIL}}) + B(Y_m^{\text{MIL}}) \Delta W_{m+1}^{K,M} + \int_{s}^{t} B'(Y_m^{\text{MIL}}) \left( \int_{l}^{s} B(Y_m^{\text{MIL}}) \, dW_{r}^{K} \right) \, dW_{s}^{K} \right)\]
for some $K, M, N \in \mathbb{N}, h = \frac{T}{M}$, and $m \in \{0, \ldots, M - 1\}$. Numerical schemes that attain higher orders of convergence involve iterated stochastic integrals and it is not possible to rewrite these expressions such as

$$\int_{t}^{t+h} B'(X_t) \left( \int_{t}^{s} B(X_t) \, dW^K_r \right) \, dW^K_s$$

(4)

for $h > 0, t, t+h \in [0, T]$ and $K \in \mathbb{N}$ in terms of increments of the approximated $Q$-Wiener process $(W^K_t)_{t \in [0,T]}$ like in the commutative case, see [9]. Therefore, methods such as the derivative-free Milstein type scheme presented in [20], which was developed based on this assumption, are not applicable to approximate the mild solution of these equations. In [21], we introduced two methods to approximate iterated stochastic integrals

$$\int_{t}^{t+h} \Phi \left( \int_{t}^{s} dW_r \right) \, dW_s$$

(5)

with $t \geq 0, h > 0$ for some operators $\Psi \in L(H, L(U, H)_{U_0}), \Phi \in L(U, H)_{U_0}$, and a $Q$-Wiener process $(W_t)_{t \in [0,T]}$ of trace class. Therewith, it is possible to implement the Milstein scheme (3) from [9], we refer to [36] for details. However, the evaluation of the derivative in the Milstein scheme is costly. Precisely, the computational cost needed to evaluate this term is of order $O(N^2 K)$ in each time step, see [20, 36]. This computational effort can be reduced by one order of magnitude if the derivative is replaced by some customized approximation—see also the detailed discussion of this issue in [20].

In this work, we design a derivative-free numerical scheme to approximate the mild solution of Eq. (2) which can be combined with any method to simulate the iterated stochastic integrals involved in the scheme, see also [19]. As the main result, a two stages derivative-free Milstein type scheme is developed where the stages are constructed in such a way that, compared to the Milstein scheme (3) proposed in [9], the computational complexity is reduced by one order of magnitude. It is worth mentioning that the stages need to be chosen significantly different compared to the ones designed for the commutative noise case in [20]. The construction of the stages for the derivative-free Milstein type scheme in case of non-commutative noise is a nontrivial task as a naive choice of, e.g., finite differences does not lead to the achieved reduction of computational cost. The paper is organized as follows: First, we introduce the setting in which we work and state results on the convergence of the proposed scheme—both, with and without an approximation of the iterated integrals. The same theoretical order of convergence as for the Milstein scheme can be obtained. Moreover, we illustrate the advantages of such a higher order derivative-free scheme with a concrete example in Sect. 3. We combine the scheme with Algorithm 1 presented in [21], which is based on a truncated Fourier series expansion, and derive the effective order of convergence for this scheme—a concept that combines the theoretical order of convergence with the computational effort based on a cost model introduced in [20]. In terms of this effective order of convergence, the original Milstein scheme (3) is outperformed by the proposed derivative-free Milstein type scheme. Compared to the exponential Euler scheme, the proposed scheme obtains a higher effective order.
of convergence for a large set of parameter values when combined with Algorithm 1 from [21]. In Sect. 4, we analyze the $L^2$-error and the computational cost for the derivative-free Milstein type scheme numerically. The presented simulations confirm a higher effective order of convergence in contrast to the original Milstein scheme and at least the same or even higher effective order of convergence in contrast to the Euler scheme for the examples considered in Sect. 4. Finally, in Sects. 5 and 6, we give some concluding remarks and the proofs for the convergence results.

2 Approximation of solutions for SPDEs

In this section, we present a derivative-free Milstein type scheme for SPDE (2) which does not need to have commutative noise. Precisely, we introduce a scheme which can be coupled with an arbitrary method for the approximation of the involved iterated stochastic integrals. For example, when combined with the algorithms introduced in [21] for the simulation of twice-iterated integrals, the theoretical order of convergence of the original Milstein scheme can be maintained.

2.1 Framework

Throughout this work, we assume the framework presented in the following. Let $(H, \langle \cdot, \cdot \rangle_H)$ and $(U, \langle \cdot, \cdot \rangle_U)$ denote some separable real-valued Hilbert spaces and let $T \in (0, \infty)$ be some fixed time point. Further, let the operator $Q \in L(U)$ be non-negative, symmetric and have finite trace. Then, the subspace $U_0 \subset U$ is defined as $U_0 = Q^{1/2}U$. Moreover, we consider some complete probability space $(\Omega, \mathcal{F}, P)$ and a $U$-valued $Q$-Wiener process $(W_t)_{t \in [0, T]}$ with respect to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ which fulfills the usual conditions. In terms of the eigenvalues of $Q$, denoted as $\eta_j$, with corresponding eigenvectors $\tilde{e}_j$ for $j \in J$ with some countable index set $J$ forming an orthonormal basis $\{\tilde{e}_j : j \in J\}$ of $U$ (see [28]), we obtain the following series representation of the $Q$-Wiener process, see [28],

$$W_t = \sum_{j \in J, \eta_j \neq 0} \frac{\sqrt{\eta_j}}{\eta_j} \tilde{e}_j \beta^j_t, \quad t \in [0, T]. \quad (6)$$

In this representation, the stochastic processes $(\beta^j_t)_{t \in [0, T]}$ denote independent real-valued Brownian motions for all $j \in J$ with $\eta_j \neq 0$. Below, the following notation is used for different sets of linear operators. The space of linear and bounded operators mapping from $U$ to $H$ that are restricted to the subspace $U_0$ is called $(L(U, H)_{U_0}, \| \cdot \|_{L(U, H)})$ with $L(U, H)_{U_0} := \{ T : U_0 \to H \mid T \in L(U, H) \}$, by $L_{HS}(U, H)$, we denote the set of Hilbert-Schmidt operators mapping from $U$ to $H$ and, finally, we denote $L^{(2)}(U, H) = L(U, L(U, H))$ and $L^{(2)}_{HS}(U, H) = L_{HS}(U, L_{HS}(U, H))$.

For the existence and uniqueness of a mild solution of SPDE (2) and the validity of the proofs of convergence in Sect. 6, we assume the following conditions.
(A1) The linear operator \( A : \mathcal{D}(A) \subset H \to H \) is the generator of an analytic \( C_0 \)-semigroup \( S(t) = e^{tA} \) for all \( t \geq 0 \). We denote the eigenvalues of \(-A\) by \( \lambda_i \in (0, \infty) \) and the corresponding eigenvectors by \( e_i \) for \( i \in \mathcal{I} \) and some countable index set \( \mathcal{I} \), that is, \(-Ae_i = \lambda_i e_i \) for all \( i \in \mathcal{I} \). Furthermore, let \( \inf_{i \in \mathcal{I}} \lambda_i > 0 \) and let the eigenfunctions \( \{ e_i : i \in \mathcal{I} \} \) of \(-A\) form an orthonormal basis of \( H \), see [35], and

\[
Av = \sum_{i \in \mathcal{I}} -\lambda_i \langle v, e_i \rangle_H e_i
\]

for all \( v \in \mathcal{D}(A) \). We introduce the real Hilbert spaces \( H_r := \mathcal{D}((-A)^r) \) for \( r \in [0, \infty) \) with norm \( \| x \|_{H_r} = \| (-A)^r x \|_H \) for \( x \in H_r \).

(A2) Let \( \beta \in [0, 1) \) and assume that \( F : H_\beta \to H \) is twice continuously Fréchet differentiable with \( \sup_{v \in H_\beta} \| F'(v) \|_{L(H)} < \infty \) and \( \sup_{v \in H_\beta} \| F''(v) \|_{L^2(H,H_\beta)} < \infty \).

(A3) The operator \( B : H_\beta \to L(U, H)_{U_0} \) is assumed to be twice continuously Fréchet differentiable such that \( \sup_{v \in H_\beta} \| B'(v) \|_{L(H, L(U,H))} < \infty \) and \( \sup_{v \in H_\beta} \| B''(v) \|_{L^2(H=L(U,H))} < \infty \). Further, let \( B(H_\delta) \subset L(U, H_\delta) \) for some \( \delta \in (0, \frac{1}{2}) \) and assume that

\[
\| B(u) \|_{L(U,H_\delta)} \leq C (1 + \| u \|_{H_\delta}),
\| B'(v) B(v) - B'(w) P B(w) \|_{L^2(U_0,H)} \leq C \| v - w \|_H,
\| (-A)^{-\beta} B(v) Q^{-\alpha} \|_{L^2(U_0,H)} \leq C (1 + \| v \|_{H_r})
\]

for some constant \( C > 0 \), all \( u \in H_\delta \), \( v, w \in H_\gamma \), where \( \gamma \in [\max(\beta, \delta), \delta + \frac{1}{2}) \), \( \alpha \in (0, \infty) \), \( \beta \in (0, \frac{1}{2}) \), \( \beta \in [0, \delta + \frac{1}{2}) \), any orthogonal projection operator \( P : H \to \text{span} \{ e_i : i \in \tilde{\mathcal{I}} \} \subset H \) with finite index set \( \tilde{\mathcal{I}} \subset \mathcal{I} \) and the case that \( P \) is the identity.

(A4) The initial value \( \xi : \Omega \to H_\gamma \) is \( \mathcal{F}_0 \mathcal{B}(H_\gamma) \)-measurable and it holds \( E[\| \tilde{\xi} \|_{H_r}^4] < \infty \).

(A5) Assume that at least one of the following conditions is fulfilled:

(a) \( Q^\frac{1}{4} \) is a trace class operator,
(b) \( \| B''(v)(PB(u), PB(u)) \|_{L^2(U,L(U,H))} \leq C (1 + \| v \|_H + \| u \|_H) \) for all \( u, v \in H \), some \( C > 0 \) and any orthogonal projection operator \( P : H \to \text{span} \{ e_i : i \in \tilde{\mathcal{I}} \} \subset H \) with finite index set \( \tilde{\mathcal{I}} \subset \mathcal{I} \).

In this work, we do not make a difference between the operator \( B \) and its extension \( B : H \to L(U, H)_{U_0} \). The operator \( B \) is globally Lipschitz continuous as \( H_\beta \subset H \) is dense. With \( F \), we deal analogously. Note that assumptions (A1)–(A4) are the same as for the scheme for SPDEs with commutative noise introduced in [20] and similar to the conditions imposed in [9, 36] for the original Milstein scheme. However, the commutativity condition (1), which is essential in [9, 20], needs not to be fulfilled in our setting. On the other hand, assumption (A5) is required. Assumptions (A1)–(A4) assure the existence of a unique mild solution \( X : [0, T] \times \Omega \to H_\gamma \) for SPDE (2),
see [8, 9]. Moreover, it holds
\[
\sup_{t \in [0,T]} E \left[ \| X_t \|_{H^\gamma}^4 + \| B(X_t) \|_{L^{HS}(U_0,H_0)}^4 \right] < \infty
\]
and
\[
\sup_{s,t \in [0,T], s \neq t} \left( E \left[ \| X_t - X_s \|_{H^\gamma}^p \right] \right)^{\frac{1}{p}} |t - s|^{\min(\gamma - r, \frac{1}{2})} < \infty
\]
for every \( r \in [0, \gamma] \) and \( p \in [2, 4] \), see [8].

### 2.2 The derivative-free Milstein type scheme

In this section, we derive a numerical scheme to approximate the mild solution of SPDE (2). Since the Milstein scheme (3) is computationally expensive due to the derivative that has to be evaluated in each step, see also the discussion in [20, 36], the main goal is to develop a derivative-free Milstein type scheme with significantly reduced computational complexity and thus improved performance. In order to compare numerical methods in this work, we consider the so-called effective order of convergence first introduced in [33]. This number combines the theoretical order of convergence with the computational cost involved in the calculation of an approximation by a particular scheme. As in [20], the goal is to raise the effective order of convergence by means of a customized approximation for the term \( B' B \) that is free of derivatives and, in addition, computationally less expensive. Here, however, we cannot assume that the operator \( B' B \) fulfills a commutativity condition, that is, condition (1) is not required. Therefore, as the main challenge, a completely new and customized type of stages has to be developed for a derivative-free Milstein type scheme that does not assume the commutativity condition (1). It has to be pointed out that the design of these stages needs to be different compared to the stages of schemes for SPDEs with commutative noise, like the one proposed in [20], and is much harder to construct. In Theorem 2.1, we state that for the proposed derivative-free Milstein type scheme, the theoretical order of convergence is the same as for the Milstein scheme given in [9]. At the same time, compared to the Milstein scheme in [9], the computational effort is significantly reduced for the derivative-free Milstein type scheme. That means that the effective order of convergence is a priori larger for the proposed derivative-free method. Moreover, compared to the Euler schemes like in [7, 13, 22], the computational cost is of the same magnitude while the order of convergence w.r.t. step size \( h \) is at least the same or even significantly higher. Thus, the scheme that we derive in the following is more efficient in terms of the effective order of convergence than the Euler type schemes for many parameter sets determined by the SPDE under consideration if we combine it, for example, with the algorithms for the simulation of the iterated stochastic integrals introduced in [21], see Table 2. Precisely, compared to the Euler schemes, the increase in the computational cost that results from the approximation of the iterated stochastic integrals can be neglected and we get, in many cases, a significantly higher effective order of convergence due to the higher theoretical order of convergence in the time step that the derivative-free Milstein type scheme features.
At first, the infinite dimensional spaces have to be discretized. For the solution space $H$, we introduce the projection operator $P_N : H \rightarrow H_N$ that maps $H$ to the finite dimensional subspace $H_N := \text{span}\{e_i : i \in I_N\}$ for some fixed $N \in \mathbb{N}$ with some index set $I_N \subset I$ and $|I_N| = N$. We define this operator as

$$P_N x = \sum_{i \in I_N} \langle x, e_i \rangle_H e_i, \quad x \in H.$$  

Analogously, we define the projection operator $P_K : U \rightarrow U_K$ to approximate the $Q$-Wiener process for some fixed $K \in \mathbb{N}$ by

$$W^K_t := P_K W_t = \sum_{j \in \mathcal{J}_K} \sqrt{\eta_j} \beta^K_j, \quad t \in [0, T],$$

with $U_K := \text{span}\{\tilde{e}_j : j \in \mathcal{J}_K\}$ for some index set $\mathcal{J}_K \subset \mathcal{J}$, $|\mathcal{J}_K| = K$, and $\eta_j \neq 0$ for $j \in \mathcal{J}_K$. In order to discretize the time interval, we work with an equidistant time step for legibility of the representation. Let $h = \frac{T}{M}$ for some $M \in \mathbb{N}$ and define $t_m = m \cdot h$ for $m \in \{0, \ldots, M\}$. The increments of the approximated $Q$-Wiener process are then denoted by

$$\Delta W^K_{m,M} := W^K_{t_{m+1}} - W^K_{t_m} = \sum_{j \in \mathcal{J}_K} \sqrt{\eta_j} \Delta \beta^K_{m,j} \tilde{e}_j$$

where the increments of the real-valued Brownian motions are given by $\Delta \beta^K_{m,j} = \beta^K_{t_{m+1},j} - \beta^K_{t_m,j}$ for $m \in \{0, \ldots, M - 1\}$, $j \in \mathcal{J}_K$.

The key idea for the construction of the derivative-free Milstein type scheme is alike to that in the commutative case, see [20], which in turn is based on the work for the finite dimensional setting in [31–33]. The derivative-free Milstein type scheme yields a discrete time process which we denote by $(Y^{N,K,M}_m)_{m \in \{0,\ldots,M\}}$ such that $Y^{N,K,M}_m$ is $\mathcal{F}_m$-$\mathcal{B}(H)$-measurable for all $m \in \{0, \ldots, M\}$, $M \in \mathbb{N}$. We define the derivative-free Milstein type (DFM) scheme as $Y^{N,K,M}_0 = P_N \xi$ and

$$Y^{N,K,M}_{m+1} = P_N e^{Ah} \left( Y^{N,K,M}_m + h F(Y^{N,K,M}_m) + B(Y^{N,K,M}_m) \Delta W^K_{m,M} ight. \
+ \sum_{j \in \mathcal{J}_K} \left( B(Y^{N,K,M}_m) + \sum_{i \in \mathcal{J}_K} P_N B(Y^{N,K,M}_m) \tilde{e}_i I^{Q}_{(i,j),m} \right) \tilde{e}_j \
- \left. B(Y^{N,K,M}_m) \right) \tilde{e}_j \right)$$ (7)

for $m \in \{0, \ldots, M - 1\}$, $N$, $M$, $K \in \mathbb{N}$. For $i, j \in \mathcal{J}_K$ and $m \in \{0, \ldots, M - 1\}$, the term $I^{Q}_{(i,j),m} = I^{Q}_{(i,j),t_m,t_{m+1}}$ denotes the iterated stochastic Itô integral

$$I^{Q}_{(i,j),t_m,t_{m+1}} = \int_{t_m}^{t_{m+1}} \int_{t_m}^{s} \langle dW_r, \tilde{e}_i \rangle_U \langle dW_s, \tilde{e}_j \rangle_U.$$ (8)
The term containing the operator $B'B$ in the Milstein scheme (3) is approximated by customized stages in the derivative-free Milstein type scheme (7), similar to the stages used in Runge–Kutta schemes. These stages are developed in such a way that the overall computational cost for the DFM scheme is decreased by one order compared to the original Milstein scheme, see also the discussion in Sect. 3. However, the stage values have to be chosen differently compared to the commutative noise case. Unlike in [20], the main distinction is that we employ only two stages and that the iterated stochastic integrals are carefully placed within the stage. Note that the reduction of the cost for the newly developed derivative-free Milstein type scheme (7) results from a carefully tailored design of the derivative-free stages based on a technique first established in [31–33] for the finite-dimensional SDE setting. This kind of trick allows to make the number of necessary stages independent of the dimension of the driving Wiener process in the SDE setting which reduces the computational complexity considerably. However, applied to the SPDE setting, this technique even reduces the computational complexity of the scheme much more substantially such that a higher effective order of convergence is achieved. This improvement is analyzed in detail in Sect. 3.

At this point, we assume that the iterated stochastic integrals are given exactly in order to consider the error estimate independent of the approximation error for the iterated integrals. We consider the error resulting from the approximation of the mild solution of (2) without an approximation of the iterated stochastic integral since the method for the computation of the iterated stochastic integrals is interchangeable.

Then, in a second step, we conclude from Theorem 2.2 below that if an approximation of the iterated stochastic integral fulfills some specified conditions, this estimate remains valid.

**Theorem 2.1 (Convergence of DFM scheme)** Assume that (A1)–(A4) and (A5) hold. Then, there exists a constant $C_{Q,T} \in (0, \infty)$, independent of $N$, $K$ and $M$, such that for $(Y_{m,N,K,M})_{0 \leq m \leq M}$, defined by the DFM scheme in (7), it holds

$$\max_{0 \leq m \leq M} \left( \mathbb{E} \left[ \left\| X_{tm} - Y_{m,N,K,M} \right\|_{H}^{2} \right] \right)^{1/2} \leq C_{Q,T} \left( \left( \inf_{i \in I \setminus I_{N}} \lambda_{i} \right)^{-\gamma} + \left( \sup_{j \in J \setminus J_{K}} \eta_{j} \right)^{\alpha} + M^{-q_{DFM}} \right)$$

(9)

for all $N$, $K$, $M \in \mathbb{N}$ and with $q_{DFM} = \min(2(\gamma - \beta), \gamma)$. The parameters are determined by assumptions (A1)–(A4).

**Proof** The proof of Theorem 2.1 is stated in Sect. 6. \qed

This is the same estimate (apart from the constant) as for the Milstein scheme (3) proposed in [9] or the derivative-free Milstein type scheme for SPDEs with commutative noise in [20]. The computational effort, however, increases compared to the schemes for SPDEs with commutative noise as the iterated stochastic integrals have to be simulated. We discuss this issue below.
2.3 Approximation of iterated integrals

In Sect. 2.2, we implicitly assumed that the iterated stochastic integrals can be computed exactly. However, up to now there exists no algorithm for the exact simulation of the iterated stochastic integrals in a setting with non-commutative noise. Therefore, the iterated integrals have to be approximated appropriately. We prove the following general result.

**Theorem 2.2** Let \( \bar{I}^Q_{(i,j),m}, i, j \in J_K, m \in \{0, \ldots, M - 1\} \), denote some approximations of the iterated stochastic integrals in (8) and let \( (\tilde{Y}_m)_{m \in \{0,\ldots,M\}} \) with \( \tilde{Y}_m = Y_m^{N,K,M} \) denote the discrete time process obtained by the DFM scheme (7) if the integrals \( I^Q_{(i,j),m} \) are replaced by the approximations \( \bar{I}^Q_{(i,j),m}, i, j \in J_K, m \in \{0, \ldots, M - 1\} \). Assume that conditions (A1)–(A5) are fulfilled and that

\[
\mathbb{E}\left[\left\|\int_{t_l}^{t_{l+1}} B'(\tilde{Y}_l) \left( \int_{t_l}^{t_s} P_N B(\tilde{Y}_l) \, dW^K_s \right) \, dW^K_l - \sum_{i,j \in J_K} \bar{I}^Q_{(i,j),l} B'(\tilde{Y}_l)(P_N B(\tilde{Y}_l)\hat{e}_i)\hat{e}_j \right\|_H^2\right]^{\frac{1}{2}} \leq E(M, K) \quad (10)
\]

for all \( l \in \{0, \ldots, M - 1\} \), \( K, M \in \mathbb{N} \) and some function \( E : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}_+ \). Further, in case of assumption (A5a) assume that

\[
\sum_{j \in J} \left( \mathbb{E}\left[\left( \sum_{i \in J} \left( \bar{I}^Q_{(i,j),t,t+h} \right)^2 \right)^2\right] \right)^{\frac{1}{4}} \leq C_Q h \quad (11)
\]

and in case of assumption (A5b) assume that

\[
\sum_{j \in J} \left( \mathbb{E}\left[\left( \sum_{i \in J} \left( \bar{I}^Q_{(i,j),t,t+h} \right)^2 \right)^q \right] \right)^{\frac{1}{2}} \leq C_Q h^q \quad (12)
\]

for \( q \in \{2, 3\} \), some \( C_Q > 0 \), all \( h > 0 \) and \( t \in [0, T-h] \). Then, there exists a constant \( C_{Q,T} \in (0, \infty) \), independent of \( N, K \) and \( M \), such that it holds

\[
\max_{0 \leq m \leq M} \left( \mathbb{E}\left[\left\|X_{t_m} - \tilde{Y}_m\right\|^2_H\right] \right)^{\frac{1}{2}} \leq C_{Q,T} \left( \left( \inf_{i \in J_N} \lambda_i \right)^{-\gamma} + \left( \sup_{j \in J_N \setminus J_K} \eta_j \right)^{\alpha} + M^{-q_{DFM}} + M^{\frac{1}{2}} E(M, K) \right)
\]

for all \( N, K, M \in \mathbb{N} \) and with \( q_{DFM} = \min(2(\gamma - \beta), \gamma) \).

**Proof** The proof of this theorem is stated in Sect. 6. □
Note that Theorem 2.2 applies to the Milstein scheme (3) as well, see also [36]. Now, we want to illustrate this statement with two exemplary choices—Algorithm 1 and Algorithm 2 as introduced in [21]. First, we consider Algorithm 1 which is based on a series representation of the iterated stochastic integral. This representation is truncated after $D$ summands for some $D \in \mathbb{N}$, see [12, 21], which yields the approximation. The numerical scheme (7) is called DFM-A1 if the iterated integrals are approximated by Algorithm 1—denoted as $\tilde{I}^{Q,(i,j),m}_{(i,j),m}$. For this method, there exists some constant $C_{Q,T} > 0$ such that (10) is fulfilled with

$$E(M, K) = E^{(D),(1)}(M, K) = C_{Q,T} \frac{1}{M \sqrt{D}}$$

for all $D, K, M \in \mathbb{N}$, see [21, Corollary 1]. If we approximate the integrals with Algorithm 2 instead, we denote the scheme (7) by DFM-A2 and the approximation $\tilde{I}^{Q,(i,j),m}_{(i,j),m}$ by $\tilde{I}^{Q,(D),(2)}_{(i,j),m}$. The series representation is not only truncated after $D$ summands, but the remainder is approximated by a multivariate normally distributed random vector, for details, we refer to [21]. Moreover, conditions (11) and (12) are fulfilled for Algorithm 1 and 2, which can be easily seen from the definition of the algorithms in [21].

In order to determine which of the two algorithms obtains a higher order of convergence, one has to analyze the computational costs that are involved, see also [21, 36] for a comparison. The goal is that the DFM scheme combined with Algorithm 1 or Algorithm 2 preserves the error estimate stated in Theorem 2.1. This requires a comparison. The goal is that the DFM scheme combined with Algorithm 1—denoted as $\tilde{I}^{Q,(i,j),m}_{(i,j),m}$. For this method, there exists some constant $C_{Q,T} > 0$ such that (10) is fulfilled with

$$E(M, K) = E^{(D),(1)}(M, K) = C_{Q,T} \frac{1}{M \sqrt{D}}$$

for all $D, K, M \in \mathbb{N}$ and some constant $C_{Q,T} > 0$, see [21, Corollary 2, Theorem 4]. This estimate shows that the error converges in $D$ with a higher order, compared to the estimate for Algorithm 1. Note that the error estimate also depends on the number $K$, which controls the accuracy of the approximation of the $Q$-Wiener process, and on the eigenvalues of the operator $Q$. For a proof of the error estimates (13) and (14), we refer to [21]. Moreover, conditions (11) and (12) are fulfilled for Algorithm 1 and 2, which can be easily seen from the definition of the algorithms in [21].

In order to determine which of the two algorithms obtains a higher order of convergence, one has to analyze the computational costs that are involved, see also [21, 36] for a comparison. The goal is that the DFM scheme combined with Algorithm 1 or Algorithm 2 preserves the error estimate stated in Theorem 2.1. This requires a choice of $D \geq D_1 = \lceil M^{2 \min(2(\gamma-\beta),\gamma)} \rceil$ for Algorithm 1, whereas for Algorithm 2 we need $D \geq D_2 = \min \{ K \sqrt{K-1}, (\min_{j \in J_K} \eta_j)^{-1} \} M^{\min(2(\gamma-\beta),\gamma)-\frac{1}{2}}$. Alternatively, one can choose $D \geq D_1 = \lceil M^{-1}(\sup_{j \in J_K \setminus J_K} \eta_j)^{-2\alpha} \rceil$ for the first algorithm and $D \geq D_2 = \lceil M^{-\frac{1}{2}} \min \{ K \sqrt{K-1}, (\min_{j \in J_K} \eta_j)^{-1} \} (\sup_{j \in J_K \setminus J_K} \eta_j)^{-\alpha} \rceil$ for the second algorithm. However, if all summands of the error estimate in Theorem 2.1 are optimally balanced, then $(\sup_{j \in J_K \setminus J_K} \eta_j)^{\alpha} = O(M^{-\min(2(\gamma-\beta),\gamma)})$ which results in the same orders of magnitude for the choice of $D_1$ and $D_2$, respectively. These considerations show that the computational effort for the two schemes DFM-A1 and DFM-A2 is determined by the parameters which in turn are specified by the equation. Therefore, the choice of the optimal scheme depends on the SPDE that has to be solved. From now on, we assume that $D \in \mathbb{N}$ is chosen such that the temporal order of convergence is not decreased, i.e., such that $D = D_1$ for Algorithm 1 or $D = D_2$ for Algorithm 2, respectively.
Remark 2.1 Note that Algorithm 1 and Algorithm 2 proposed in [21] merely represent examples and that Theorem 2.2 is valid if the derivative-free Milstein type scheme DFM is combined with any approximation for the iterated stochastic integrals such that conditions (10) together with (11) or (12) are fulfilled.

3 The effective order of convergence: a comparison

In the following, we compare the performance of the derivative-free Milstein type (DFM) scheme to the performance of the original Milstein (MIL) scheme (3), the exponential Euler (EXE) scheme and the linear implicit Euler (LIE) scheme. For example, one can combine the DFM scheme and the MIL scheme with Algorithm 1 or Algorithm 2 in order to approximate the solution of SPDEs that need not fulfill the commutativity condition (1). However, the analysis can be done similarly for any other approximation method for the iterated stochastic integrals as specified in Theorem 2.2. In the following, we restrict our analysis to Algorithm 1 as an example. The LIE scheme is considered in [13, 37] and the EXE scheme is introduced in [22] which are combined with a Galerkin approximation. The proof of the following theorem is detailed in [19] and the main idea can be found in [9].

Proposition 3.1 (Convergence of EXE scheme) Assume that (A1)–(A4) hold. Then, there exists a constant $C_{Q,T} \in (0, \infty)$, independent of $N$, $K$ and $M$, such that for the approximation process $(Y_{m}^{EXE})_{0 \leq m \leq M}$ with $Y_{m}^{EXE} = Y_{m}^{EXE:N,K,M}$, defined by the EXE scheme, it holds

$$
\max_{0 \leq m \leq M} \left( E \left[ \left\| X_{t_{m}} - Y_{m}^{EXE} \right\|_{H}^{2} \right] \right)^{1/2} \leq C_{Q,T} \left( \left( \inf_{i \in I \setminus I_{N}} \lambda_{i} \right)^{-\gamma} + \left( \sup_{j \in J \setminus J_{K}} \eta_{j} \right)^{\alpha} + M^{-q_{EXE}} \right)
$$

(15)

for all $N$, $K$, $M \in \mathbb{N}$ and with $q_{EXE} = \min(\frac{1}{2}, 2(\gamma - \beta), \gamma)$. The parameters are determined by assumptions (A1)–(A4).

Compared to the DFM scheme, the EXE scheme requires less restrictive assumptions as we do not need (A5) or conditions on the second derivative of $B$ and the estimate for $B'(v)PB(v) - B'(w)PB(w)$ can be omitted. For the LIE scheme, similar results as in Proposition 3.1 can be obtained analogously. Below, $q$ stands for the order of convergence w.r.t. the step size $h = \frac{T}{M}$ with $q_{DFM} = q_{MIL} = \min(2(\gamma - \beta), \gamma) \geq \min(\frac{1}{2}, 2(\gamma - \beta), \gamma) = q_{EXE}$. However, to compare the performance of the schemes we have to take into account their computational cost in combination with their error estimates as, e.g., iterated stochastic integrals have to be simulated for the higher order schemes DFM and MIL only.
3.1 The cost model

In order to compare the efficiency of different approximation algorithms, one is usually interested in the dependency of the errors on their computational cost. Therefore, we consider a theoretical cost model proposed in [20]. It is assumed that any standard arithmetic operation or evaluation of sine, cosine or exponential function etc. produces unit cost 1. Further, the simulation of any realization of an arithmetic operation or evaluation of sine, cosine or exponential function etc. produces \( \phi(v) \) of a functional valued random variable is assumed to produce at least cost one. However, the evaluation \( \phi(v) \) of a functional \( \phi \in V^* \) with \( V = H \) or \( V = U \) is assumed to be usually more costly with \( \text{cost}(\phi, v) = c \) for all \( v \in V \) and for some \( c \geq 1 \) where typically \( c \gg 1 \). Such functionals are needed for, e.g., the calculation of Fourier coefficients \( \phi_i(v) = \langle v, \hat{e}_i \rangle_V \) of \( v \in V \) for some ONB \( \{ \hat{e}_i \}_{i \in \mathbb{N}} \) of \( V \). Let \( L(H, E) = \{ T|H_N : T \in L(H, E) \} \) for some vector space \( E \) and let \( L_{HS}(U, H)_{K,N} = \{ P_N T|U_K : T \in L_{HS}(U, H) \} \). As a result, we obtain for any \( v, y \in H_N \) and \( u \in U_K \) the following computational costs due to \( |I_N| = N \) and \( |J_K| = K \) [20]:

(i) One evaluation of the mapping \( P_N \circ F : H \to H_N \) with

\[
P_N F(y) = \sum_{i \in I_N} \langle F(y), e_i \rangle_H e_i
\]

is determined by the functionals \( \langle F(y), e_i \rangle_H \) for \( i \in I_N \) which results in \( \text{cost}(P_N F) = \mathcal{O}(N) \).

(ii) Evaluating \( P_N \circ B(\cdot)|_{U_K} : H \to L_{HS}(U, H)_{K,N} \) with

\[
P_N B(y)u = \sum_{i \in I_N} \sum_{j \in J_K} \langle B(y)\tilde{e}_j, e_i \rangle_H \langle u, \tilde{e}_j \rangle_u e_i
\]

needs the evaluation of the functionals \( \langle B(y)\tilde{e}_j, e_i \rangle_H \) for \( i \in I_N \) and \( j \in J_K \) which results in \( \text{cost}(P_N \circ B(\cdot)|_{U_K}) = \mathcal{O}(NK) \).

(iii) For \( P_N \circ B'(\cdot)|_{H_{N,U_K}} : H \to L(U, L_{HS}(U, H)_{K,N})_N \) with

\[
P_N((B'(y)v)u) = \sum_{k,l \in I_N} \sum_{j \in J_K} \langle (B'(y)e_k)\tilde{e}_j, e_l \rangle_H \langle v, e_k \rangle_H \langle u, \tilde{e}_j \rangle_u e_l
\]

the functionals \( \langle (B'(y)e_k)\tilde{e}_j, e_l \rangle_H \) have to be evaluated for all \( k, l \in I_N \) and \( j \in J_K \) and it follows that \( \text{cost}(P_N \circ (B'(\cdot)(\cdot)|_{H_{N,U_K}}) = \mathcal{O}(N^2K) \).

Considering the computational cost for one time step of the Milstein scheme \( (3) \), one evaluation of \( P_N \circ F(\cdot) \), one of \( P_N \circ B(\cdot)|_{U_K} \), and one evaluation of \( P_N \circ B'(\cdot)|_{H_{N,U_K}} \) are needed. Then, the evaluated operators \( P_N \circ B(Y^\text{MIL}_m)|_{U_K} \in L(U, H)_{K,N} \), \( P_N \circ B'(Y^\text{MIL}_m)|_{H_{N,U_K}} \in L(U, L_{HS}(U, H)_{K,N})_N \), \( P_N \circ B'(Y^\text{MIL}_m)(v)|_{U_K} \in L_{HS}(U, H)_{K,N} \) and \( P_N \circ e^{Ah} \in L(U, H)_{K,N} \) have to be applied to the corresponding elements of the Hilbert spaces. Here, it has to be pointed out that calculating the Fourier coefficients of \( P_N B(Y^\text{MIL}_m)\tilde{e}_j \) for some basis element \( \tilde{e}_j \in U_K \) is for free because they are in the \( j \)-th column of the matrix representation \( P_N B(Y^\text{MIL}_m)|_{U_K} = \).
Theorem 2.1 and Proposition 3.1. However, it turns out that different schemes can
the needed computational cost instead of just comparing their error estimates w.r.t.
of different numerical schemes, one has to compare the accuracy of each scheme versus
computational cost, see also the discussion in [20]. In order to compare the performance
and the derivative-free Milstein type scheme both attain the same error estimate (9),
Table 1  Computational cost given by the number of evaluations of real-valued functionals and independent $N(0, 1)$-distributed random variables needed for each time step

| Scheme | # of evaluations of functionals | # of $N(0, 1)$ r. v. |
|--------|----------------------------------|----------------------|
| LIE    | $N$                              | $K N$                |
| EXE    | $N$                              | $K$                  |
| MIL-A1 | $N$                              | $K N^2$              |
| MIL-A2 | $N$                              | $K (1 + 2D_2) + \frac{1}{2} K (K - 1)$ |
| DFM-A1 | $N$                              | $K (1 + 2D_1)$       |
| DFM-A2 | $N$                              | $K (1 + 2D_2) + \frac{1}{2} K (K - 1)$ |

however, the computational effort for MIL is cost(MIL($N, K, M$)) = $O(N^2 K M)$ whereas for the DFM scheme it is only cost(DFM($N, K, M$)) = $O(N K M)$ if the random numbers are assumed not to be the dominating cost. Therefore, in this case, the DFM scheme performs a priori with a higher order of convergence compared to the MIL scheme if errors versus costs are considered. Compared with the LIE scheme and the EXE scheme, the DFM scheme belongs to the same class $O(N K M)$ of computational complexity, which is in some sense optimal for one-step approximations for SPDEs of type (2). Although the LIE scheme as well as the EXE scheme have worse error bounds given in (15) compared to the one for the DFM and the MIL scheme in (9), it is not clear which scheme should be preferred because the computational cost for simulating the iterated stochastic integrals for the DFM and the MIL scheme have to be taken into account as well. Therefore, we derive the effective order of convergence for each scheme under consideration. This concept is also detailed in [20].

3.2 Comparison of the effective orders of convergence

In order to compare the performance of different numerical schemes, we consider the so-called effective order of convergence which was proposed in [33] and also considered in [20]. In the following, we restrict our comparison to the schemes DFM-A1 and MIL-A1, both using Algorithm 1, as well as the EXE scheme in order to keep the analysis concise. Further, we assume that the simulation of a normally distributed real-valued random variable produces the same computational cost as the evaluation of a functional. For a detailed analysis and comparison of the effective order of convergence for the schemes MIL-A1, MIL-A2 and EXE we refer to [36]. Since the LIE scheme and the EXE scheme have the same order of convergence and similar computational cost, we restrict our analysis to the EXE scheme in the following because one can get exactly the same results for the LIE scheme. We want to point out that the focus of this article lies on the introduction and analysis of the derivative-free Milstein type scheme and a complete comparison taking into account further algorithms next to Algorithm 1 for the simulation of the iterated stochastic integrals would go beyond the scope of this article and may be object of future research.
For each scheme under consideration and its approximation process $\left( Y_{m}^{N,K,M} \right)_{m \in \{0, \ldots, M\}}$, we have to minimize the error term

$$\sup_{m \in \{0, \ldots, M\}} \left( \mathbb{E} \left[ \| X_{t_m} - Y_{m}^{N,K,M} \|_H^2 \right] \right)^{\frac{1}{2}}$$

over all $N$, $K$, $M \in \mathbb{N}$ under the constraint that the computational cost does not exceed some specified value $\bar{c} > 0$. Note that if $D$ is chosen as described in Sect. 2.3, then the computational cost of each scheme given in Table 1 depends on $N$, $K$ and $M$ only. In the following, we assume that $\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \eta_j = \mathcal{O}(K^{-\rho_Q})$ and $(\inf_{i \in \mathcal{I} \setminus \mathcal{I}_N} \lambda_i)^{-1} = \mathcal{O}(N^{-\rho_A})$ for some $\rho_A > 0$ and $\rho_Q > 1$. Then, we obtain the following expression for all $N$, $K$, $M \in \mathbb{N}$ and some $C > 0$, see also [20],

$$\text{err(SCHHEME)} = \sup_{m \in \{0, \ldots, M\}} \left( \mathbb{E} \left[ \| X_{t_m} - Y_{m}^{N,K,M} \|_H^2 \right] \right)^{\frac{1}{2}} \leq C \left( N^{-\gamma \rho_A} + K^{-\alpha \rho_Q} + M^{-q} \right). \tag{17}$$

Note that the parameter $q > 0$ is determined by the scheme that is considered. Given some computational cost $\bar{c} > 0$, the goal is to minimize the error under the constraint that the computational cost is bounded by $\bar{c}$. Solving this optimization problem yields the effective order of convergence, denoted by $\text{EOC(SCHHEME)}$, which is then given by an expression of the form

$$\text{err(SCHHEME)} = \mathcal{O} \left( \bar{c}^{-\text{EOC(SCHHEME)}} \right).$$

Next, we analyze the effective order of convergence for the DFM-A1 scheme and the MIL-A1 scheme, which make use of Algorithm 1 for the approximation of the iterated stochastic integrals, and the EXE scheme. To begin with, let $q := q_{\text{DFM}} = q_{\text{MIL}} = \min(2(\gamma - \beta), \gamma)$ for the scheme DFM-A1 and the scheme MIL-A1, see [36, Thm. 1] and [9, Thm. 1], and let $D = D_1 = \mathcal{O}(M^{2q-1})$ for Algorithm 1 unless otherwise stated.

First, we consider the scheme DFM-A1. The computational cost for the calculation of one trajectory with $M$ time steps amounts to $\bar{c} = \mathcal{O}(MKN) + \mathcal{O}(KM^{2q})$, see Table 1 and the discussion in the last section. Solving the above mentioned optimization problem results in $M = \mathcal{O} \left( \bar{c}^{-\alpha \rho_Q \gamma \rho_A} \right)$, $K = \mathcal{O}(\bar{c}^{-\gamma \rho_A})$ and $N = \mathcal{O}(\bar{c}^{-\alpha \rho_Q})$ for some $z > 0$, such that all summands in (17) are balanced. Then, one determines $z$ from $\bar{c} = \mathcal{O}(MKN)$ or $\bar{c} = \mathcal{O}(KM^{2q})$, depending on which of the two terms dominates the total computational cost. If $2q - 1 \leq 0$ or if $M^{2q-1} = \mathcal{O}(N)$, then $\bar{c} = \mathcal{O}(MKN)$ and we calculate that $z = (\alpha \rho_Q + \gamma \rho_A)q + \alpha \rho_Q \gamma \rho_A$. This is the case if $\alpha \rho_Q \gamma \rho_A (2q - 1) \leq \alpha \rho_Q$ is fulfilled. On the other hand, if $2q - 1 > 0$ and if $N = \mathcal{O}(M^{2q-1})$, then $\bar{c} = \mathcal{O}(KM^{2q})$ are the dominating cost and we get $z = (1 + 2 \alpha \rho_Q) \gamma \rho_A q$, which is the case if $\alpha \rho_Q \gamma \rho_A (2q - 1) \geq \alpha \rho_Q$.
Now, two cases have to be distinguished: If $\gamma\rho_A(2q - 1) \leq q$ is fulfilled, then $\bar{c} = \mathcal{O}(MKN)$. We solve the optimization problem and obtain

$$ M = \mathcal{O} \left( \bar{c} \frac{\gamma\rho_A\rho Q}{(\alpha\rho Q + \gamma\rho_A\eta + \alpha\rho Q\gamma\rho A)} \right), \quad N = \mathcal{O} \left( \bar{c} \frac{\alpha\rho Q\rho A}{(\alpha\rho Q + \gamma\rho_A\eta + \alpha\rho Q\gamma\rho A)} \right), \quad K = \mathcal{O} \left( \bar{c} \frac{\gamma\rho A}{(\alpha\rho Q + \gamma\rho_A\eta + \alpha\rho Q\gamma\rho A)} \right). $$

(18)

Further, the effective order of convergence is given by

$$ \text{err}(DFM-A1) = \mathcal{O} \left( \bar{c} - \frac{\gamma\rho_A\rho Q}{(\alpha\rho Q + \gamma\rho_A\eta + \alpha\rho Q\gamma\rho A)} \right), $$

(19)

which is the same result as for the derivative-free Milstein type scheme in the case of SPDEs with commutative noise, see the computations in [20]. On the other hand, if $\gamma\rho_A(2q - 1) \geq q$ holds, then $\bar{c} = \mathcal{O}(KM^2q)$ and optimization yields

$$ M = \mathcal{O} \left( \bar{c} \frac{\alpha\rho Q}{(2\alpha\rho Q + 1)\gamma\rho A} \right), \quad N = \mathcal{O} \left( \bar{c} \frac{\alpha\rho Q}{(2\alpha\rho Q + 1)\gamma\rho A} \right), \quad K = \mathcal{O} \left( \bar{c} \frac{1}{2\alpha\rho Q + 1} \right). $$

(20)

In this case, we obtain the effective order of convergence from

$$ \text{err}(DFM-A1) = \mathcal{O} \left( \bar{c} - \frac{\alpha\rho Q}{2\alpha\rho Q + 1} \right). $$

(21)

Next, we consider the Milstein scheme MIL-A1. Here, the computational effort for the computation of one trajectory is $\bar{c} = \mathcal{O}(MKN^2) + \mathcal{O}(KM^2q)$, compare Table 1. Again, two cases have to be considered: If $\gamma\rho_A(2q - 1) \leq 2q$, then $\bar{c} = \mathcal{O}(MKN^2)$ and solving the optimization problem yields

$$ M = \mathcal{O} \left( \bar{c} \frac{\gamma\rho_A\rho Q}{(\alpha\rho Q + \gamma\rho_A\eta + \alpha\rho Q\gamma\rho A)} \right), \quad N = \mathcal{O} \left( \bar{c} \frac{\alpha\rho Q\rho A}{(\alpha\rho Q + \gamma\rho_A\eta + \alpha\rho Q\gamma\rho A)} \right), \quad K = \mathcal{O} \left( \bar{c} \frac{\gamma\rho A}{(\alpha\rho Q + \gamma\rho_A\eta + \alpha\rho Q\gamma\rho A)} \right). $$

(22)

As a result of this, we obtain the effective order of convergence from

$$ \text{err}(MIL-A1) = \mathcal{O} \left( \bar{c} - \frac{\gamma\rho_A\rho Q\rho A}{(\alpha\rho Q + \gamma\rho_A\eta + \alpha\rho Q\gamma\rho A)} \right), $$

(23)

which is also the same effective order of convergence as for the Milstein scheme if it is applied to some SPDE with commutative noise, see also [20]. However, in the case of $\gamma\rho_A(2q - 1) \geq 2q$ the computational effort for the MIL-A1 scheme is $\bar{c} = \mathcal{O}(KM^2q)$ and we obtain the same choice for $M$, $N$ and $K$ as given in (20) and also the same effective order of convergence as given by (21), see also [36].
Finally, we consider the EXE scheme where the optimal choice for $M$, $N$ and $K$ is given by (18), however, with $q = q_{\text{EXE}} = \min(\frac{1}{2}, 2(\gamma - \beta), \gamma)$ and the effective order of convergence for the EXE scheme was computed in [20] and is given by

$$\text{err}(\text{EXE}) = \mathcal{O}\left(c^{-\frac{\gamma \rho A \alpha \rho Q}{2\alpha \rho Q + \gamma \rho A \alpha \rho Q} q_{\text{EXE}}^{1/2 - \gamma \rho A \alpha \rho Q} \right).$$

(24)

Here, we note that the same holds for the LIE scheme.

In order to determine the scheme which is most efficient for the approximation of the solution for an SPDE of type (2) that needs not to fulfill a commutativity condition for the noise, we have to compare the effective orders of convergence according to the distinct parameter settings for the schemes DFM-A1, MIL-A1 and EXE.

If $\gamma \rho A (2q - 1) \leq q$ and $q \leq \frac{1}{2}$, then, it follows that $q = q_{\text{EXE}}$. Thus, the EXE scheme and the DFM-A1 scheme have the same effective order of convergence given in (19) and (24), whereas the MIL-A1 scheme obviously has a lower effective order of convergence given in (23).

If $\gamma \rho A (2q - 1) \leq q$ and $q > \frac{1}{2}$, then, it follows that $q > q_{\text{EXE}} = \frac{1}{2}$. Here, the DFM-A1 scheme has obviously a higher effective order of convergence compared to the one of the EXE scheme. Further, comparing the effective order of convergence of the EXE scheme and the MIL-A1 scheme results in

$$\frac{q_{\text{MIL}} \gamma \rho A \alpha \rho Q}{(2 \alpha \rho Q + \gamma \rho A) q_{\text{MIL}} + \gamma \rho A \alpha \rho Q} \leq \frac{1}{2} \gamma \rho A \alpha \rho Q}{(\alpha \rho Q + \gamma \rho A) \frac{1}{2} + \gamma \rho A \alpha \rho Q}.$$

Here, it follows that the EXE scheme has a higher order of convergence than the MIL-A1 scheme and thus the DFM-A1 scheme attains the highest effective order of convergence in this case.

If $q \leq \gamma \rho A (2q - 1) \leq 2q$, then, it holds for the effective orders of convergence of the EXE, the MIL-A1 and the DFM-A1 scheme that

$$\frac{1}{2} \gamma \rho A \alpha \rho Q}{(\alpha \rho Q + \gamma \rho A) \frac{1}{2} + \gamma \rho A \alpha \rho Q} \leq \frac{q_{\text{MIL}} \gamma \rho A \alpha \rho Q}{(2 \alpha \rho Q + \gamma \rho A) q_{\text{MIL}} + \gamma \rho A \alpha \rho Q} \leq \frac{\alpha \rho Q}{2 \alpha \rho Q + 1}.$$

Thus, the DFM-A1 scheme is the one with the highest effective order of convergence in the present case.

If $2q \leq \gamma \rho A (2q - 1)$, it holds that $q > q_{\text{EXE}} = \frac{1}{2}$. In this case, the DFM-A1 and the MIL-A1 scheme attain the same effective order of convergence given in (21). As a result of this, a comparison of the effective order of the EXE scheme with the one of the MIL-A1 scheme and DFM-A1 scheme results in

$$\frac{1}{2} \gamma \rho A \alpha \rho Q}{(\alpha \rho Q + \gamma \rho A) \frac{1}{2} + \gamma \rho A \alpha \rho Q} \leq \frac{\alpha \rho Q}{2 \alpha \rho Q + 1}.$$

In this case, the DFM-A1 scheme and the MIL-A1 scheme have the same effective order of convergence which is higher than the one of the EXE scheme.
Table 2 For a given parameter set, the conditions in this table have to be checked in order to determine the optimal scheme among the schemes DFM-A1, MIL-A1 and EXE

| Conditions               | Optimal scheme | Optimal $M, N, K$ | EOC                  |
|--------------------------|----------------|-------------------|----------------------|
| $q \leq \frac{1}{2}$    | DFM-A1 = EXE   | (18)              | $\gamma q A + \alpha q A$ |
| $\gamma A (2q - 1) \leq q \wedge q > \frac{1}{2}$ | DFM-A1         | (18)              | $\gamma q A + \alpha q A$ |
| $q \leq \gamma A (2q - 1) \leq 2q$ | DFM-A1         | (20)              | $\gamma q A + \alpha q A$ |
| $2q \leq \gamma A (2q - 1)$ | DFM-A1 = MIL-A1 | (20)              | $\gamma q A + \alpha q A$ |

Here, $q = q_{DFM} = q_{MIL}$

The same holds true if the EXE scheme is replaced by the LIE scheme. We summarize the results of our comparison in Table 2 which shows that the DFM-A1 scheme always attains the highest possible effective order of convergence. However, in the case of $q_{DFM} = q_{EXE} \leq \frac{1}{2}$, although the EXE scheme and the DFM-A1 scheme have the same effective order of convergence, one may prefer the EXE scheme because it requires less computational effort compared to the DFM-A1 scheme, see Table 1. On the other hand, in the case of $2q \leq \gamma A (2q - 1)$, both the DFM-A1 and the MIL-A1 scheme have the same optimal effective order of convergence, which is higher than that of the EXE scheme. Here, one may prefer the DFM-A1 scheme because it needs less computational effort compared to the MIL-A1 scheme, see Table 1, and because it is derivative-free whereas one has to calculate the derivative of the operator $B$ for the MIL-A1 scheme.

Finally, it has to be pointed out that the maximal effective order of convergence that can be attained is always bounded by $1/2$ independent of the given parameters whenever Algorithm 1 is applied to simulate the iterated stochastic integrals.

For completeness, we want to note that assumption (A5) as well as parts of (A3) do not have to be fulfilled for the exponential Euler scheme. This means that there are parameter sets that are valid for the EXE scheme but not for the DFM scheme and in these situations the exponential Euler scheme would be the method of choice. Moreover, it is not clear if the obtained upper error bounds are sharp and thus if the effective order of convergence may be further improved.

3.3 The case of a finite-dimensional $Q$-Wiener process

If the $Q$-Wiener process $W$ is finite-dimensional, i.e., if $\{\{ j \in J : \eta_j \neq 0 \} \} < \infty$, the error estimate only depends on $M$ and $N$ provided we choose $K = \{\{ j \in J : \eta_j \neq 0 \} \}$. Then, we obtain new solutions for $M$ and $N$ solving the optimization problem that minimizes the error under the constraint of a prescribed computational cost budget $\bar{c}$. Therefore, we compare once more the DFM-A1 scheme, the MIL-A1 scheme and the EXE scheme. Now, the computational cost required to approximate one trajectory of the solution of SPDE (2) by the DFM-A1 scheme becomes $\bar{c} = \mathcal{O}(MN) + \mathcal{O}(M^{2q})$, for the MIL-A1 scheme we get $\bar{c} = \mathcal{O}(MN^2) + \mathcal{O}(M^{2q})$ and for the EXE scheme it is $\bar{c} = \mathcal{O}(MN)$.
If $\gamma \rho_A(2q - 1) \leq q$, the computational cost for the DFM-A1 scheme is $ar{c} = O(MN)$ and solving the optimization problem yields

$$M = O\left(\bar{c}^{\frac{\gamma \rho_A}{\gamma \rho_A + q}}\right), \quad N = O\left(\bar{c}^{\frac{q}{\gamma \rho_A + q}}\right).$$  \tag{25}$$

Then, the effective order of convergence is given by

$$\text{err}(\text{DFM-A1}) = O\left(\bar{c}^{1 - \frac{\gamma \rho_A q}{\gamma \rho_A + q}}\right).$$  \tag{26}$$

If $\gamma \rho_A(2q - 1) \geq q$, then $q > \frac{1}{2}$ and $ar{c} = O(M^{2q})$ for the DFM-A1 scheme. Here, optimization results in

$$M = O\left(\bar{c}^{\frac{\gamma \rho_A}{2\gamma \rho_A + q}}\right), \quad N = O\left(\bar{c}^{\frac{q}{2\gamma \rho_A + q}}\right),$$  \tag{27}$$

and the effective order of convergence can be calculated as

$$\text{err}(\text{DFM-A1}) = O\left(\bar{c}^{1 - \frac{1}{2}}\right).$$  \tag{28}$$

Considering the MIL-A1 scheme, again two cases have to be distinguished: If $\gamma \rho_A(2q - 1) \leq 2q$, the MIL-A1 has computational cost $ar{c} = O(MN^2)$ and optimization yields

$$M = O\left(\bar{c}^{\frac{\gamma \rho_A}{\gamma \rho_A + 2q}}\right), \quad N = O\left(\bar{c}^{\frac{q}{\gamma \rho_A + 2q}}\right),$$  \tag{29}$$

and the effective order of convergence is given by

$$\text{err}(\text{MIL-A1}) = O\left(\bar{c}^{1 - \frac{\gamma \rho_A q}{\gamma \rho_A + 2q}}\right).$$  \tag{30}$$

If $2q \leq \gamma \rho_A(2q - 1)$, it follows that $q > \frac{1}{2}$ and that the MIL-A1 scheme attains the same computational cost $ar{c} = O(M^{2q})$ as the DFM-A1 scheme in the second case. Thus, we also get (27) for $M$ and $N$, and also the same effective order of convergence as given by (28).

Clearly, for the EXE scheme it holds $q_{\text{EXE}} \leq \frac{1}{2}$ and thus we get the same results as for the DFM-A1 scheme given in (25) for $M$ and $N$ as well as by (26) for the effective order of convergence with $q = q_{\text{EXE}}$.

Finally, comparing the effective orders of convergence for the schemes under consideration, we easily derive the results presented in Table 3. Here, again the DFM-A1 scheme performs better or at least as good as one of the other schemes. Clearly, in the case of $q_{\text{DFM}} = q_{\text{MIL}} = q_{\text{EXE}} \leq \frac{1}{2}$ one may prefer the EXE scheme or the LIE scheme although they have the same effective order of convergence as the DFM-A1 scheme because they are easier to implement. However, in the case of $2q \leq \gamma \rho_A(2q - 1)$ where the DFM-A1 scheme and the MIL-A1 scheme attain the same effective order
Table 3 In case of a finite-dimensional $Q$-Wiener process and $K = |\{ j \in J : \eta_j \neq 0\}| < \infty$, the conditions in this table have to be checked in order to determine the optimal scheme among the schemes DFM-A1, MIL-A1 and EXE:

| Conditions | Optimal scheme | Optimal $M, N$ | EOC |
|------------|----------------|---------------|-----|
| $q \leq \frac{1}{2}$ | DFM-A1 = EXE | (25) | $\frac{\gamma \rho_A}{\gamma \rho_A + q}$ |
| $\gamma \rho_A (2q - 1) \leq q \wedge q > \frac{1}{2}$ | DFM-A1 | (25) | $\frac{\gamma \rho_A}{\gamma \rho_A + q}$ |
| $q \leq \gamma \rho_A (2q - 1) \leq 2q$ | DFM-A1 | (27) | $\frac{1}{2}$ |
| $2q \leq \gamma \rho_A (2q - 1)$ | DFM-A1 = MIL-A1 | (27) | $\frac{1}{2}$ |

Here, $q = q_{DFM} = q_{MIL}$.

of convergence one may prefer the DFM-A1 scheme because it needs less computational effort and because no derivative of the operator $B$ is needed by the DFM-A1 scheme. Again, the effective order of convergence is always bounded by $1/2$ as for the infinite-dimensional noise case.

3.4 An illustrative example for the performance of the DFM scheme

In order to illustrate the improvement of the effective order of convergence for the DFM-A1 scheme compared to the MIL-A1 and the EXE scheme, we consider as an example the important case where $A = \Delta$ is the Laplace operator. This covers the stochastic heat equation and many reaction-diffusion type equations. Thus, it holds $\rho_A = 2$ and we assume for simplicity that $\beta = 0$ and that $\delta \in (0, \frac{1}{2})$ is maximal. Then, for any $F$ and $B$ fulfilling the corresponding assumptions in Sect. 2.1, we get $q_{DFM} = q_{MIL} = \gamma$ and $q_{EXE} = \min(\gamma, \frac{1}{2})$. Therefore, we only need to analyze the effective order of convergence subject to the values of $\gamma \in (0, 1)$ and $\alpha \rho_Q > 0$. In Fig. 1, we plot the effective order of convergence (EOC) versus the parameter $\gamma$ for the schemes DFM-A1, EXE and MIL-A1 if they are applied to such an SPDE in case of an infinite-dimensional $Q$-Wiener process for $\alpha \rho_Q = 1$ on the left and in case of a finite-dimensional $Q$-Wiener process on the right.

![Fig. 1](image.png)

**Fig. 1** Effective order of convergence (EOC) vs $\gamma$ for the schemes DFM-A1, EXE and MIL-A1 for an SPDE driven by an infinite-dimensional (left) and a finite-dimensional (right) $Q$-Wiener process.
In case of an infinite-dimensional $Q$-Wiener process, the EOC for the DFM-A1 scheme is given by (19) if $\gamma < \frac{3}{4}$ and by (21) if $\gamma \geq \frac{3}{4}$. Therefore, the maximal possible EOC for the DFM-A1 scheme is $p_{\text{DFM}} = \frac{\alpha \rho Q}{2\alpha \rho Q + 1}$ which is attained for any $\gamma \geq \frac{3}{4}$. For the EXE scheme, the EOC is determined by (24) and the upper bound for the EOC is given by $p_{\text{EXE}} = \frac{2\alpha \rho Q}{\alpha \rho Q + 2 + 4\alpha \rho Q}$ which is approached by the EXE scheme as $\gamma \to 1$. Moreover, the EOC for the MIL-A1 scheme is given by (23) for all $\gamma < 1$. Thus, an upper bound for the EOC of the MIL-A1 scheme is given by $p_{\text{MIL-A1}} = p_{\text{DFM-A1}} = \frac{\alpha \rho Q}{2\alpha \rho Q + 1}$ which is approached as $\gamma \to 1$. For $\alpha \rho Q = 1$, these results are presented in the left diagram of Fig. 1. If the SPDE is driven by a finite-dimensional $Q$-Wiener process, then following the discussion in Sect. 3.3, the EOC for the DFM-A1 scheme is given by (26) if $\gamma < \frac{3}{4}$ and the EOC is $\frac{1}{2}$ if $\gamma \geq \frac{3}{4}$. For the EXE scheme, the EOC coincides with the one given in (26) if $\gamma < \frac{1}{2}$ and then changes to the value $\frac{\gamma}{2\gamma + \frac{1}{2}}$ if $\gamma \geq \frac{1}{2}$. An upper bound for the EOC of the EXE scheme is given by $\frac{2}{5}$, which is approached as $\gamma \to 1$. Finally, the EOC for the MIL-A1 scheme is determined by (30) for any $\gamma < 1$ and an upper bound for the EOC is given by $\frac{1}{2}$ which is again approached as $\gamma \to 1$. Note that the EOC of all schemes under consideration does not depend on the term $\alpha \rho Q$ in case of a finite-dimensional $Q$-Wiener process and the results are presented in the right diagram of Fig. 1. The left diagram displays the case $\alpha \rho Q = 1$, however, the qualitative characteristics remain the same if different values of $\alpha \rho Q > 0$ are considered. One can even see that the left diagram continuously approaches the right diagram and for the upper bounds it holds that $p_{\text{DFM}} = p_{\text{MIL-A1}} \to \frac{1}{2}$ whereas $p_{\text{EXE}} \to \frac{2}{5}$ as the value of $\alpha \rho Q$ increases.

Comparing the EOC of the different schemes, we can see that for $\gamma \in (0, \frac{1}{2}]$ the schemes EXE and DFM-A1 attain exactly the same EOC that is always higher than the EOC of the MIL-A1 scheme. For $\gamma \in \left(\frac{1}{2}, \frac{3}{4}\right]$, the regime for the EOC of the EXE schemes changes due to $q_{\text{EXE}} = \min\left(\frac{1}{2}, \gamma\right)$ in this setting. Here, the EOC of the EXE scheme is still higher than the EOC of the MIL-A1 scheme. Moreover, the DFM-A1 scheme has the highest EOC and thus outperforms the EXE as well as the MIL-A1 scheme. In case of $\gamma \in \left(\frac{3}{4}, 1\right]$, the regime for the EOC of the DFM-A1 scheme changes as the cost for the computation of the iterated stochastic integrals are now dominating the overall cost. Further, now the MIL-A1 schemes has a higher EOC than the EXE scheme. However, the DFM-A1 scheme still has the highest EOC compared to the other two schemes under consideration. Note that the DFM-A1 scheme clearly outperforms the MIL-A1 scheme for all $\gamma \in (0, 1)$, which is due to the reduction of the computational cost for the DFM scheme compared to the MIL scheme irrespective of the computational cost for the iterated stochastic integrals. This reduction of the cost for the DFM scheme results from a carefully tailored design of the derivative-free stages for the DFM scheme. Summarizing, the newly proposed DFM scheme combined with Algorithm 1 for the computation of the iterated stochastic integrals attains for any $\gamma \in (0, 1)$ always the highest possible EOC compared to the EXE scheme and the MIL-A1 scheme. Moreover, the maximal possible EOC for the EXE scheme is $\frac{2}{5}$ whereas the maximal possible EOC for the DFM-A1 scheme is $\frac{1}{2}$.\[Springer]
4 Numerical analysis

In this section, we compare the DFM-A1 scheme to the MIL-A1 and the EXE schemes to demonstrate the theoretical results presented above, summarized in Tables 2 and 3, by numerical computations for some examples. In the following, we approximate the mild solution of SPDE (2), that is,

\[ X_t = e^{At} \xi + \int_0^t e^{A(t-s)} F(X_s) \, ds + \int_0^t e^{A(t-s)} B(X_s) \, dW_s, \quad t \in (0, T). \]

For the numerical analysis, we consider the following setting. We fix \( H = U = L^2((0, 1), \mathbb{R}) \), set \( T = 1 \), and \( \mathcal{I} = \mathcal{J} = \mathbb{N} \). Let \( A \) be the Laplace operator with Dirichlet boundary conditions. To be precise, \( A = \frac{\Delta}{100} \) with eigenvalues \( \lambda_i = \frac{\pi^2 i^2}{100} \) of \( -A \) and eigenvectors \( e_i = \sqrt{2} \sin(i \pi x) \) for \( i \in \mathbb{N}, x \in (0, 1) \) and on the boundary, we have \( X_t(0) = X_t(1) = 0 \) for all \( t \in (0, T) \). The covariance operator \( Q \) is defined by the eigenvalues \( \eta_j = j^{-p_0} \) for some \( p_0 > 1 \) which is given separately for each example below and \( \tilde{e}_j = \sqrt{2} \sin(j \pi x) \) for \( j \in \mathbb{N}, x \in (0, 1) \). For the operator \( B \), we present the general setting introduced for the numerical analysis in [20]. Define the functionals \( \mu_{ij} : H_\beta \to \mathbb{R}, \phi_{ij}^k : H_\beta \to \mathbb{R} \) for \( i, k \in \mathcal{I}, j \in \mathcal{J} \) such that \( \phi_{ij}^k \) is the Fréchet derivative of \( \mu_{ij} \) in direction \( e_k \) and let

\[ B(y) u = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \mu_{ij}(y) ( u, \tilde{e}_j ) U e_i, \]

as well as

\[ (B' (y))(B(y) v)) u = \sum_{i, k \in \mathcal{I}} \sum_{j, r \in \mathcal{J}} \phi_{ij}^k (y) \mu_{kr} (y) ( v, \tilde{e}_r ) U ( u, \tilde{e}_j ) U e_i \]

for \( y \in H_\beta \) and \( u, v \in U_0 \), see [20, Sec. 5.3] for details.

We choose \( \mu_{ij}(y) = \frac{(y, e_j)_H}{i^p + j^p} \) for all \( i \in \mathcal{I}, j \in \mathcal{J}, y \in H \) and some \( p > 1 \) that differs in the examples presented below, which leads to \( \phi_{ij}^k(y) = \begin{cases} 0, & k \neq j \\ \frac{1}{i^p + j^p}, & k = j \end{cases} \) for all \( i, k \in \mathcal{I}, j \in \mathcal{J}, y \in H \). This is the setting considered in [36]. Assumption (A1) is obviously fulfilled and assumptions (A2) and (A4) can be easily verified to be fulfilled in the following examples. Next, we elaborate on assumption (A3). By the definition of the \( L(U, H_\delta) \)-norm and the operator \( B \), we obtain

\[ \| B(y) \|_{L(U, H_\delta)} = \sup_{\| u \|_U = 1} \left\| \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \lambda_i^\delta \mu_{ij}(y) ( u, \tilde{e}_j ) U e_i \right\|_H. \]
In the next steps, we employ the Parseval equality and the triangle inequality

\[ \| B(y) \|_{L(U, H_b)} = \sup_{\| u \|_U = 1} \left( \sum_{i \in I} \sum_{j \in J} \lambda_i^\delta |\mu_{ij}(y)(u, \tilde{e}_j)_U| \right)^{\frac{1}{2}} \leq \sup_{\| u \|_U = 1} \left( \sum_{i \in I} \left( \sum_{j \in J} |\lambda_i^\delta| \cdot |\mu_{ij}(y)| \cdot |(u, \tilde{e}_j)_U| \right)^2 \right)^{\frac{1}{2}}. \]

It holds by Parseval’s equality that

\[ \| y \|_{H_b}^2 = \| (A)^\delta y \|_H^2 = \sum_{i \in I} |\lambda_i^\delta (y, e_i)_H|^2 \]

and therewith

\[ |(y, e_j)_H|^2 = \lambda_j^{-2\delta} |\lambda_j^\delta (y, e_j)_H|^2 \leq \lambda_j^{-2\delta} \| y \|_{H_b}^2 \] (31)

for all \( j \in J \). As \( |(u, \tilde{e}_j)_U|^2 \leq 1 \) by Parseval, we obtain

\[ \| B(y) \|_{L(U, H_b)} \leq \left( \sum_{i \in I} \lambda_i^\delta \left( \sum_{j \in J} |\mu_{ij}(y)| \right) \right)^{\frac{1}{2}} \leq \left( \sum_{i \in I} \frac{\pi^{4\delta} i^{4\delta}}{100^{2\delta}} \left( \sum_{j \in J} |(y, e_j)_H| \right)^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i \in I} \frac{\pi^{4\delta} i^{4\delta}}{100^{2\delta}} \left( \sum_{j \in J} \lambda_j^{-\delta} \| y \|_{H_b} \right)^2 \right)^{\frac{1}{2}}. \]

Then, for some \( \varepsilon \in (0, 2\delta) \), some \( C_1 = C_1(\varepsilon, \delta) > 0 \) and with \( r = \frac{4}{3+\varepsilon+2\delta} > 1 \), \( q = \frac{4}{1+\varepsilon-2\delta} > 1 \) such that \( \frac{1}{r} + \frac{1}{q} = 1 \), Young’s inequality gives the estimate

\[ \| B(y) \|_{L(U, H_b)} \leq C_1 \left( \sum_{i \in I} i^{4\delta - \frac{3+\varepsilon+2\delta}{2}} \right)^{\frac{1}{2}} \| y \|_{H_b} . \]

If for \( \delta \in (0, \frac{1}{2}) \) it holds that \( p > \frac{2+8\delta}{3+2\delta} \), then it follows that \( \| B(y) \|_{L(U, H_b)} \leq C (1 + \| y \|_{H_b}) \) for all \( y \in H_b \).

Next, we compute the term

\[ \| (-A)^{-\gamma} B(y) Q^{-\alpha} \|_{LHS(U_0, H)} = \left( \sum_{j \in J} \| (-A)^{-\gamma} B(y) Q^{-\alpha} + \frac{1}{2} \tilde{e}_j \|_H^2 \right)^{\frac{1}{2}}. \]
for all $y \in H_y$. We rewrite the expression above to obtain

$$
\|(-A)^{-\theta} B(y) Q^{-\alpha}\|_{L_{HS}(U_0, H)} = \left( \sum_{i \in I} \sum_{j \in J} \left| \langle (-A)^{-\theta} B(y) Q^{-\alpha+\frac{1}{2}} \tilde{e}_j, e_i \rangle \right| H \right)^{\frac{1}{2}}
$$

$$
= \left( \sum_{i \in I} \sum_{j \in J} \left| \lambda_i^{-\theta} \langle B(y) Q^{-\alpha+\frac{1}{2}} \tilde{e}_j, e_i \rangle \right| H \right)^{\frac{1}{2}}
$$

$$
= \left( \sum_{i \in I} \sum_{j \in J} \lambda_i^{-2\theta} \left| \langle B(y) \eta_j^{-\alpha+\frac{1}{2}} \tilde{e}_j, e_i \rangle \right| H \right)^{\frac{1}{2}}.
$$

Here, we employed the definition of the operators $A$ and $Q$. In the next step, we insert the definition of the operator $B$

$$
\|(-A)^{-\theta} B(y) Q^{-\alpha}\|_{L_{HS}(U_0, H)} = \left( \sum_{i \in I} \sum_{j \in J} \lambda_i^{-2\theta} \eta_j^{-2\alpha+1} |\mu_{ij}(y)|^2 \right)^{\frac{1}{2}}
$$

$$
= \left( \sum_{i \in I} \sum_{j \in J} \pi_{-4\theta} i^{-4\theta} j^{-2\alpha-1} \rho_Q \left| \langle y, e_j \rangle \right| H \left| \langle y, e_j \rangle \right| H \right)^{\frac{1}{2}}
$$

By Parseval’s equality and calculations as in (31), we obtain for some $C_2 > 0$ that

$$
\|(-A)^{-\theta} B(y) Q^{-\alpha}\|_{L_{HS}(U_0, H)} \leq C_2 \left( \sum_{i \in I} \sum_{j \in J} i^{-4\theta} j^{-2\alpha-1} \rho_Q \left| \langle y, e_j \rangle \right| H \left| \langle y, e_j \rangle \right| H \right)^{\frac{1}{2}}.
$$

Then, for all $\varepsilon \in (4\theta - 1, 4\theta - 1 + 2p)$ with $r = \frac{2p}{1+\varepsilon-4\theta} > 1$, $q = \frac{2p}{2p-1-\varepsilon+4\theta} > 1$, Young’s inequality yields that

$$
\|(-A)^{-\theta} B(y) Q^{-\alpha}\|_{L_{HS}(U_0, H)} \leq C_2 \left( \sum_{i \in I} \sum_{j \in J} i^{-4\theta} j^{-2\alpha-1} \rho_Q \left| \langle y, e_j \rangle \right| H \right)^{\frac{1}{2}}
$$

$$
= \left( \sum_{i \in I} \sum_{j \in J} i^{-4\theta} j^{-2\alpha-1} \rho_Q \left| \langle y, e_j \rangle \right| H \right)^{\frac{1}{2}}
$$

with $C_3 = C_3(\varepsilon, \theta, p) > 0$. Therefore, $\|(-A)^{-\theta} B(y) Q^{-\alpha}\|_{L_{HS}(U_0, H)} \leq C(1 + \|y\|_{H_y})$ holds for all $y \in H_y$ and some $C > 0$ if $\alpha < \frac{7+\rho_Q+4\gamma}{2\rho_Q} + \frac{2(\min(0,4\theta-1)-\varepsilon)}{\rho_Q}$ for some arbitrary small $\hat{\varepsilon} > 0$, $p > \max \left( \frac{1-4\theta}{2}, 1 \right)$ and if $\varepsilon \in (\max(0,4\theta-1), 4\theta - 1 + 2p)$. In the following examples, $p > \max \left( \frac{2+8\gamma}{3+2\alpha}, 1 \right)$ and $\rho_Q$ are specified and we
select \( \gamma \) and \( \alpha \) to be maximal. We do not state any other condition given in (A3) as these do not pose a restriction on the parameters but note that these are fulfilled as well. Finally, we examine the commutativity condition (1). On the one hand, it holds that

\[
\sum_{k \in I} \phi^k_{im}(y) \mu_{kn}(y) = \frac{1}{i^p + m^4} \langle y, e_n \rangle_H
\]

but on the other hand, it holds that

\[
\sum_{k \in I} \phi^k_{in}(y) \mu_{km}(y) = \frac{1}{i^p + n^4} \langle y, e_m \rangle_H
\]

for all \( y \in H \) and all \( i \in I, m, n \in J \). Obviously, these two expressions differ for some choice of \( m, n \in J \). Thus, the considered example does not fulfill the commutativity condition (1).

In the following examples, we have \( \rho_A = 2 \) and we choose the parameter \( p \) such that assumption (A3) is always fulfilled and such that different cases in Table 2 are addressed. All simulations are computed with an Intel Xeon E3-1245 v5 CPU at 3.50 GHz and 32 GB of memory using Matlab version R2021b. Further, the iterated stochastic integrals are approximated by Algorithm 1 from [21] and we make use of the implementation that can be found in the toolbox [10]. First, for each example the orders of convergence w.r.t. the time discretization \( q_{DFM}, q_{MIL} \) and \( q_{EXE} \) that are given by Theorems 2.1, 2.2 and Proposition 3.1 are analyzed. Note that from the convergence result [36, Thm. 1], we observe that \( q_{MIL} = q_{DFM} \), see also [9, Thm. 1]. Then, the much more relevant effective order of convergence is analyzed where the error versus computational effort is considered. Here, the computational cost \( \tilde{c} \) is computed as \( \tilde{c}(DFM-A1) = MN + 2MNK + MK(1 + 2M^{2q-1}) \), \( \tilde{c}(MIL-A1) = MN + MNK + MN^2K + MK(1 + 2M^{2q-1}) \), and \( \tilde{c}(EXE) = MN + MNK + MK \), see also Table 1. The effective order of convergence is a good indicator for the performance of numerical schemes in practice. The numerical results for the effective order of convergence are confirmed by a comparison of the corresponding averaged measured CPU times that are based on a huge number of simulations. However, one has to keep in mind that in general CPU time measurements may critically depend on, e.g., the concrete implementation of a scheme which may lead to significantly differing results. Therefore, considering a theoretical cost model and the corresponding effective order of convergence is an attempt to have a more objective performance indicator. That is why we concentrate our analysis on the effective order of convergence.

### 4.1 Example 1

In the first example, we set the parameters to \( p = \frac{4}{3}, \rho_Q = 3 \) and we choose \( F(y) = 1 - y, y \in H \). This allows for \( \beta \in [0, 1) \) and we choose \( \beta = 0 \). Moreover, we set the initial value \( \xi(x) = X_0(x) = 0 \) for all \( x \in (0, 1) \). From condition \( p > \max\left(\frac{2 + 8\delta}{3 + 2\delta}, 1\right) \), it follows that \( \delta \in (0, \frac{1}{8}) \). Therefore, we set \( \delta = \frac{3}{8} - \varepsilon \delta \) for some arbitrarily small
\( \varepsilon_\delta > 0 \). From these parameter values, we compute \( \gamma \in \left[ \frac{3}{8} - \varepsilon_\delta, \frac{7}{8} - \varepsilon_\delta \right] \) and we thus choose \( \gamma = \frac{7}{8} - \varepsilon_\delta - \varepsilon_\gamma \) for some arbitrarily small \( \varepsilon_\gamma > 0 \). As a result of this, it follows that \( q = q_{DFM} = q_{MIL} = \frac{7}{8} - \hat{\varepsilon} \) with \( \hat{\varepsilon} = \varepsilon_\delta + \varepsilon_\gamma > 0 \) arbitrarily small. Finally, since we can choose \( \vartheta \in (0, \frac{1}{4}) \) arbitrarily for this example, we take \( \vartheta = \frac{1}{4} \) for simplicity. Then, from the condition \( \alpha < \frac{7 + \rho_Q + 4\gamma}{2\rho_Q} \), we directly get that \( \alpha \in (0, \frac{27}{12} - \frac{2}{3}\hat{\varepsilon}) \) and we choose \( \alpha = \frac{27}{12} - \varepsilon_\alpha \) for some arbitrarily small \( \varepsilon_\alpha > \frac{2}{3}\hat{\varepsilon} > 0 \). Thus, assumption (A3) holds, as discussed above. Furthermore, condition (A5a) is fulfilled as \( \rho_Q > 2 \).

With these parameters, we can identify the scheme that is superior. For this example, it holds that \( q < 2\gamma \rho_A (2q - 1) \leq 2q \) for sufficiently small \( \hat{\varepsilon} > 0 \). Thus, the DFM-A1 scheme is optimal, i.e., it is the scheme with the highest effective order of convergence according to Table 2. In order to compare the DFM-A1 scheme to the other schemes under consideration, we calculate the effective orders of convergence for each of the schemes. We expect that the scheme DFM-A1 obtains the highest effective order of convergence in this setting with

\[
\text{error}(\text{DFM-A1}) = O \left( c^{-\frac{27-12\varepsilon_\alpha}{58-24\varepsilon_\alpha}} \right)
\]

given by (21), i.e., EOC(DFM-A1) \( \approx \frac{27}{58} \). Moreover, we fix some arbitrary \( N \in \mathbb{N} \) and compute the relation \( M = N^2 \) and \( K = \left\lceil \frac{N^{2/3} - 3\varepsilon_\alpha}{\varepsilon_\alpha} \right\rceil \approx \left\lceil N^{\frac{7}{12}} \right\rceil \) as given in (20) for the implementation of the DFM-A1 scheme.

Considering the scheme MIL-A1, the effective order of convergence for this scheme is given by (23) with

\[
\text{error}(\text{MIL-A1}) = O \left( c^{-\frac{189}{460} \frac{7-2\hat{\varepsilon}}{125} \varepsilon_\alpha + 3\varepsilon_\alpha} \right),
\]

i.e., EOC(MIL-A1) \( \approx \frac{189}{460} \). For this example, the relations between \( N, K \) and \( M \) for the MIL-A1 scheme given in (22) are exactly the same as for the DFM-A1 scheme.

On the other hand, for the EXE scheme we obtain from (18) for some arbitrarily fixed \( N \in \mathbb{N} \) the relation \( M = \left\lceil N^{\frac{7}{12} - 4\hat{\varepsilon}} \right\rceil \approx \left\lceil N^{\frac{7}{12}} \right\rceil \) and \( K = \left\lceil N^{\frac{7}{12} - 3\varepsilon_\alpha} \right\rceil \approx \left\lceil N^{\frac{7}{12}} \right\rceil \) as an optimal choice. The effective order of convergence for the EXE scheme is given as

\[
\text{error}(\text{EXE}) = O \left( c^{-\frac{189}{514} \frac{7-2\hat{\varepsilon}}{125} \varepsilon_\alpha - 2\hat{\varepsilon} + 3\varepsilon_\alpha} \right)
\]
as stated in (24), i.e., it holds EOC(EXE) \( \approx \frac{189}{514} \).

For the numerical evaluation, we compare the approximations of the schemes DFM-A1, MIL-A1 and EXE to an approximation computed with the linear implicit Euler scheme with \( N = 2^6 \), \( K = \left\lceil 2^{\frac{14}{3}} \right\rceil \) and \( M = \left\lceil 2^{\frac{35}{2}} \right\rceil \) that serves as the reference solution. We simulate 500 paths with each scheme and for each \( N \in \{2, 4, 8, 16, 32\} \) to compare the \( L^2 \)-error at time \( T = 1 \) versus number of time steps as well as versus...
computational cost, see Fig. 2. Here, log–log scales are used such that the absolute value of the slope of the graph indicates the temporal order of convergence with respect to step size (left figure) and the effective order of convergence with respect to computational effort (right figure).

Considering the parameters used in this example, the temporal order of convergence is \( q_{\text{EXE}} = \frac{1}{2} \) for the EXE scheme and \( q_{\text{DFM}} = q_{\text{MIL}} = \frac{7}{8} - \hat{\varepsilon} \) for the MIL-A1 and DFM-A1 schemes for some arbitrarily small \( \hat{\varepsilon} > 0 \). Thus, the MIL-A1 and DFM-A1 schemes possess a significantly higher order of convergence than the EXE scheme which is confirmed by the left diagram in Fig. 2 and the corresponding results given in Table 4.

Further, it holds that \( \text{EOC}(\text{EXE}) < \text{EOC}(\text{MIL-A1}) < \text{EOC}(\text{DFM-A1}) \) and thus the DFM-A1 scheme has a higher effective order of convergence than the other schemes. This is confirmed by the numerical simulation results presented in the right diagram in Fig. 2 and in Table 4 where \( \bar{c} \) denotes the computational cost measured by counting the number of functional evaluations and normally distributed random numbers needed by each scheme. Additionally, we also specify the corresponding average measured CPU times in Table 5, where the DFM-A1 scheme takes the lowest CPU times as accuracy increases. These results substantiate the cost model and the results for the effective order of convergence in Fig. 2. As a result of this, for this example the DFM-A1 scheme performs better than the MIL-A1 and the EXE scheme.

4.2 Example 2

Here, we choose a smaller value \( p = \frac{44}{41} \) and the same covariance operator \( Q \) as in Example 1 with \( \rho_Q = 3 \). Thus, condition (A5a) is fulfilled. As in Example 1, we consider the function \( F(y) = 1 - y \), \( y \in H \) and choose \( \beta = 0 \). Again, the initial value is chosen as \( \xi(x) = X_0(x) = 0 \) for all \( x \in (0, 1) \). Further, we calculate the condition \( \delta \in (0, \frac{5}{24}) \) and choose \( \delta = \frac{5}{24} - \varepsilon_\delta \) for some arbitrarily small \( \varepsilon_\delta > 0 \). Then, we get \( \gamma \in [\frac{5}{24}, \frac{17}{24} - \varepsilon_\delta] \) and we set \( \gamma = \frac{17}{24} - \varepsilon_\delta - \varepsilon_\gamma \) for some arbitrarily small \( \varepsilon_\gamma > 0 \). This implies \( q = q_{\text{DFM-A1}} = q_{\text{MIL-A1}} = \frac{17}{24} - \hat{\varepsilon} \) with \( \hat{\varepsilon} = \varepsilon_\delta + \varepsilon_\gamma > 0 \) arbitrarily small. Moreover, one can choose \( \vartheta \in (0, \frac{1}{2}) \) arbitrarily and here we choose \( \vartheta = \frac{1}{4} \) for

\[ \text{log-log scales} \]

Fig. 2 \( L^2 \)-error at \( T = 1 \) against number of time steps (left) and against computational cost (right) for Example 1 computed from 500 paths for \( N \in \{2, 4, 8, 16, 32\} \) in log–log scale, respectively.
Table 4 Computational cost $\bar{c}$, $L^2$-error and corresponding standard deviation for Example 1 obtained from 500 paths

| $N$ | $M$ | $K$     | DFM-A1 scheme | MIL-A1 scheme |
|-----|-----|---------|---------------|---------------|
|     |     |         | $\bar{c}$ | Error | Std | $\bar{c}$ | Error | Std |
| 2   | 4   | $[2^{27}]$ | 94          | $3.65 \cdot 10^{-2}$ | $2.77 \cdot 10^{-3}$ | 110   | $3.65 \cdot 10^{-2}$ | $2.76 \cdot 10^{-3}$ |
| 4   | $2^4$ | $[2^{27}]$ | 864         | $2.95 \cdot 10^{-2}$ | $9.97 \cdot 10^{-4}$ | 1248  | $2.95 \cdot 10^{-2}$ | $1.02 \cdot 10^{-3}$ |
| 8   | $2^6$ | $[2^{27}]$ | 8481        | $1.81 \cdot 10^{-2}$ | $2.47 \cdot 10^{-4}$ | 15,649 | $1.81 \cdot 10^{-2}$ | $2.57 \cdot 10^{-4}$ |
| 16  | $2^8$ | $[2^{27}]$ | 127,744     | $6.81 \cdot 10^{-3}$ | $8.66 \cdot 10^{-5}$ | 312,064 | $6.81 \cdot 10^{-3}$ | $8.17 \cdot 10^{-5}$ |
| 32  | $2^{10}$ | $[2^{35}]$ | 1,344,631   | $1.86 \cdot 10^{-3}$ | $7.04 \cdot 10^{-5}$ | 4,392,055 | $1.86 \cdot 10^{-3}$ | $6.95 \cdot 10^{-5}$ |

| $N$ | $M$ | $K$     | EXE scheme | $\bar{c}$ | Error | Std |
|-----|-----|---------|------------|-----------|--------|-----|
| 2   | $[2^2]$ | $[2^{27}]$ | 96         |           | $2.71 \cdot 10^{-2}$ | $2.16 \cdot 10^{-3}$ |
| 4   | $2^7$ | $[2^{27}]$ | 1792       |           | $3.07 \cdot 10^{-2}$ | $1.29 \cdot 10^{-3}$ |
| 8   | $[2^{21}]$ | $[2^{21}]$ | 37,674     |           | $1.82 \cdot 10^{-2}$ | $5.43 \cdot 10^{-4}$ |
| 16  | $2^{14}$ | $[2^{28}]$ | 1,097,728  |           | $6.77 \cdot 10^{-3}$ | $1.17 \cdot 10^{-4}$ |
| 32  | $[2^{35}]$ | $[2^{35}]$ | 24,282,684 |           | $1.85 \cdot 10^{-3}$ | $7.27 \cdot 10^{-5}$ |
Table 5: Average measured CPU times for Example 1 that correspond to the simulation results in Table 4 obtained from 500 computed paths, respectively

|   | DFM-A1 scheme | MIL-A1 scheme | EXE scheme |
|---|---------------|---------------|------------|
| 2 | $3.77 \cdot 10^{-4}$ s | $3.37 \cdot 10^{-4}$ s | $2.93 \cdot 10^{-4}$ s |
| 4 | $9.15 \cdot 10^{-4}$ s | $7.90 \cdot 10^{-4}$ s | $2.36 \cdot 10^{-3}$ s |
| 8 | $4.08 \cdot 10^{-3}$ s | $4.12 \cdot 10^{-3}$ s | $3.28 \cdot 10^{-2}$ s |
| 16 | $2.92 \cdot 10^{-2}$ s | $3.12 \cdot 10^{-2}$ s | $7.36 \cdot 10^{-1}$ s |
| 32 | $1.73 \cdot 10^{-1}$ s | $2.63 \cdot 10^{-1}$ s | $14.29$ s |

simplicity. Then, we calculate that $\alpha \in (0, \frac{77}{36} - \frac{2}{3} \hat{\varepsilon})$ and therefore set $\alpha = \frac{77}{36} - \varepsilon_\alpha$ with $\varepsilon_\alpha > \frac{2}{3} \hat{\varepsilon} > 0$ arbitrarily small. Thus, assumption (A3) is fulfilled.

Checking the conditions in Table 2, we are in the case of $q > \frac{1}{2}$ and $\gamma \rho_A (2q - 1) \leq q$ for sufficiently small $\hat{\varepsilon} > 0$. In this case, the optimal effective order of convergence is obtained by the DFM-A1 scheme according to Table 2. For the DFM-A1 scheme, we get from (19) that

$$\text{error(DFM-A1)} = O\left(\frac{-\frac{1309}{288} \frac{17}{8} \varepsilon_\alpha - \frac{17}{18}}{\hat{\varepsilon}^{1309/288} - 9 \varepsilon_\alpha - 2 \hat{\varepsilon}}\right),$$

i.e., $\text{EOC(DFM-A1)} \approx \frac{1309}{2976}$. The optimal choice of $M$ and $K$ given some $N \in \mathbb{N}$ is then determined in (18), which results in $M = N^2$ and $K = \lceil N^{\frac{17}{12} - 3 \hat{\varepsilon}} \rceil \approx \lceil N^{\frac{17}{12}} \rceil$.

Considering the MIL-A1 scheme, we obtain from (23) the effective order of convergence

$$\text{error(MIL-A1)} = O\left(\frac{-\frac{1309}{288} \frac{17}{8} \varepsilon_\alpha - \frac{17}{18}}{\hat{\varepsilon}^{1309/288} - 12 \varepsilon_\alpha - 2 \hat{\varepsilon}}\right),$$

i.e., it holds that $\text{EOC(MIL-A1)} \approx \frac{1309}{5900}$. Given some $N \in \mathbb{N}$, the optimal choice for $M$ and $K$ is given in (22) and yields the same results as for the DFM-A1 scheme in this example.

For the Euler scheme, it holds $q_{\text{EXE}} = \frac{1}{2}$ which in turn yields with (18) that $M = \lceil N^{\frac{17}{12} + 4 \hat{\varepsilon}} \rceil \approx \lceil N^{\frac{17}{12}} \rceil$ and $K = \lceil N^{\frac{17}{12} - 3 \hat{\varepsilon}} \rceil \approx \lceil N^{\frac{17}{12}} \rceil$. For the effective order of convergence, we obtain

$$\text{error(EXE)} = O\left(\frac{-\frac{1309}{288} \frac{17}{8} \varepsilon_\alpha - \frac{17}{18}}{\hat{\varepsilon}^{1309/288} - 6 \varepsilon_\alpha - 3 \hat{\varepsilon}}\right),$$

i.e., it holds $\text{EOC(EXE)} \approx \frac{1309}{3746}$.

Now, the performance of the schemes DFM-A1, MIL-A1 and EXE is analyzed for this example by numerical simulations. Therefore, a reference solution is computed by the linear implicit Euler scheme. Precisely, we choose $N = 2^6$, $K = \lceil 2^{102} \rceil$ and
In the left log–log plot of Fig. 3, the $L^2$-errors versus the number of time steps are compared. Here, we can see that the corresponding theoretical temporal order of convergence $q_{DFM} = q_{MIL} = \frac{17}{24} - \hat{\varepsilon}$ for the DFM-A1 and the MIL-A1 scheme as well as $q_{EXE} = \frac{1}{2}$ for the EXE scheme are validated by the numerical simulations. The respective numerical results are also listed in Table 6.

In the right diagram of Fig. 3, we analyze the effective orders of convergence for the schemes under consideration. For this example, we expect $EOC(MIL-A1) < EOC(EXE) < EOC(DFM-A1)$ due to the cost model. Again, the DFM-A1 scheme performs best with the highest effective order of convergence and the original Milstein scheme MIL-A1 has the lowest effective order of convergence, which is even less than that of the EXE scheme, which is also confirmed by the numerical results in the right diagram. The simulation results are also given in Table 6. Supplementary, the average measured CPU times corresponding to the simulation results in Table 6 are given in Table 7 where the DFM-A1 has the lowest computing times. These measurements underpin the results for the effective order of convergence in Fig. 3. Thus, for this example the DFM-A1 scheme outperforms the MIL-A1 and the EXE scheme.

4.3 Example 3

The following example has been considered for the first time in [36] for an analysis of the original Milstein scheme. Here, we choose the parameters $p = 4$, $\rho_Q = 3$ and we consider $F(y) = 1 - y$ for $y \in H$. Then, as in the previous examples, we can choose $\beta \in [0, 1)$ arbitrarily and therefore set $\beta = 0$. The initial condition is given by $\xi(x) = X_0(x) = 0$ for all $x \in (0, 1)$. Then, the condition $p > \max\left\{\frac{2 + \delta}{3 + \frac{8}{3}, 1}\right\}$ is fulfilled for any $\delta \in (0, \frac{1}{2})$ and we choose a maximal $\delta = \frac{1}{2} - \epsilon_\delta$ for some arbitrarily small $\epsilon_\delta > 0$. We have $\gamma \in \left[\frac{1}{2} - \epsilon_\delta, 1 - \epsilon_\delta\right)$, i.e., we can choose $\gamma = 1 - \epsilon_\delta - \epsilon_\gamma$ for some arbitrarily small $\epsilon_\gamma > 0$. Thus, for the temporal order of convergence we get $q = q_{DFM} = q_{MIL} = 1 - \hat{\epsilon}$ for some arbitrarily small $\hat{\epsilon} = \epsilon_\delta + \epsilon_\gamma > 0$. Note that this
is the maximal possible temporal order for the DFM and the MIL scheme. In order to determine \( \alpha \) to be maximal, we choose \( \vartheta = \frac{1}{4} \) for simplicity from \( \vartheta \in (0, \frac{1}{2}) \) such that \( \alpha < \frac{r + \rho A + 2q}{2\rho Q} \) needs to be fulfilled. Thus, we can choose \( \alpha = \frac{7}{3} - \varepsilon_\alpha \) for some arbitrarily small \( \varepsilon_\alpha > \frac{2}{3} \hat{\epsilon} > 0 \). As a result of this, assumption (A3) is fulfilled as well as condition (A5a).

Analyzing the effective order of convergence, we see that \( \gamma \rho A (2q - 1) = 2q \) as \( \hat{\epsilon} \to 0 \). Therefore, both schemes DFM-A1 and MIL-A1 have the same optimal effective order of convergence for this example, see also Table 2. From (21) it follows that

\[
\text{error}(\text{DFM-A1}) = \text{error}(\text{MIL-A1}) = O \left( \bar{c}^{-\frac{7-3\varepsilon_\alpha}{15-6\varepsilon_\alpha}} \right)
\]

and we get \( \text{EOC}(\text{DFM-A1}) = \text{EOC}(\text{MIL-A1}) \approx \frac{7}{15} \). Then, for \( N \in \mathbb{N} \) we compute the relation \( M = N^2 \) and \( K = \left\lceil N^{\frac{2-2\hat{\epsilon}}{3\varepsilon_\alpha}} \right\rceil \approx N^{\frac{7}{15}} \) due to (20) for the implementation of the DFM-A1 and the MIL-A1 scheme.

Further, we obtain for the EXE scheme that \( q_{\text{EXE}} = \frac{1}{2} \) and the effective order of convergence can be calculated from (24) as

\[
\text{error}(\text{EXE}) = O \left( \bar{c}^{-\frac{7-3\varepsilon_\alpha - 2\hat{\epsilon} + 3\varepsilon_\alpha}{15 - 6\varepsilon_\alpha}} \right)
\]

and thus we have \( \text{EOC}(\text{EXE}) \approx \frac{14}{37} \). Then, given some \( N \in \mathbb{N} \) and following (18), we get \( M = N^{4-4\hat{\epsilon}} \approx N^4 \) and \( K = \left\lceil N^{\frac{2-2\varepsilon_\alpha}{3\varepsilon_\alpha}} \right\rceil \approx N^{\frac{7}{15}} \).

Now, we study the performance of the schemes under consideration by using a reference solution computed by the linear implicit Euler scheme with \( N = 2^6, K = \left\lceil 2^{\frac{12}{7}} \right\rceil \) and \( M = 2^{24} \). Then, for each \( N \in \{2, 4, 8, 16, 32\} \) the \( L^2 \)-error for each scheme at time \( T = 1 \) is considered based on 500 computed paths.

For this example, the maximal possible theoretical temporal order of convergence \( q_{\text{DFM}} = q_{\text{MIL}} = 1 - \hat{\epsilon} \) for the DFM-A1 and the MIL-A1 scheme and \( q_{\text{EXE}} = \frac{1}{2} \) for the EXE scheme is achieved. This order is validated by the numerical simulation results presented in the left log-log plot of Fig. 4 and in Table 8.
| $N$ | $M$ | $K$ | DFM-A1 scheme | MIL-A1 scheme |
|-----|-----|-----|---------------|---------------|
|     |     |     |  $\bar{c}$  |  $c$         |  $\bar{c}$  |  $c$         |
|     |     |     | Error | Std     | Error | Std     |
| 2   | $2^2$ | $[\sqrt{2^2}]$ | 77 | $4.50 \cdot 10^{-2}$ | $6.06 \cdot 10^{-3}$ | 93 | $4.50 \cdot 10^{-2}$ | $6.10 \cdot 10^{-3}$ |
| 4   | $2^4$ | $[\sqrt{2^4}]$ | 556 | $3.63 \cdot 10^{-2}$ | $3.07 \cdot 10^{-3}$ | 940 | $3.63 \cdot 10^{-2}$ | $3.05 \cdot 10^{-3}$ |
| 8   | $2^6$ | $[\sqrt{2^6}]$ | 4137 | $2.20 \cdot 10^{-2}$ | $1.49 \cdot 10^{-3}$ | 11,305 | $2.20 \cdot 10^{-2}$ | $1.49 \cdot 10^{-3}$ |
| 16  | $2^8$ | $[\sqrt{2^8}]$ | 31,314 | $8.89 \cdot 10^{-3}$ | $5.71 \cdot 10^{-4}$ | 154,194 | $8.89 \cdot 10^{-3}$ | $5.70 \cdot 10^{-4}$ |
| 32  | $2^{10}$ | $[\sqrt{2^{10}}]$ | 342,791 | $3.15 \cdot 10^{-3}$ | $2.74 \cdot 10^{-4}$ | 3,390,215 | $3.15 \cdot 10^{-3}$ | $2.76 \cdot 10^{-4}$ |

| $N$ | $M$ | $K$ | EXE scheme |   |
|-----|-----|-----|           |   |
|     |     |     |  $\bar{c}$  |  $c$         |   |
|     |     |     | Error | Std     |   |
| 2   | $2^{\sqrt{2}}$ | $[\sqrt{2\cdot2}]$ | 64 | $3.60 \cdot 10^{-2}$ | $5.70 \cdot 10^{-3}$ |   |
| 4   | $2^{2\cdot2}$ | $[\sqrt{2\cdot2\cdot2}]$ | 714 | $3.68 \cdot 10^{-2}$ | $2.52 \cdot 10^{-3}$ |   |
| 8   | $2^{4\cdot2}$ | $[\sqrt{2\cdot2\cdot2\cdot2}]$ | 9438 | $2.26 \cdot 10^{-2}$ | $1.19 \cdot 10^{-3}$ |   |
| 16  | $2^{8\cdot2}$ | $[\sqrt{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2}]$ | 129,050 | $9.24 \cdot 10^{-3}$ | $6.12 \cdot 10^{-4}$ |   |
| 32  | $2^{16\cdot2}$ | $[\sqrt{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2}]$ | 2,409,221 | $3.08 \cdot 10^{-3}$ | $3.10 \cdot 10^{-4}$ |   |
For the effective order of convergence, we have $\text{EOC}(\text{EXE}) < \text{EOC}(\text{MIL-A1}) = \text{EOC}(\text{DFM-A1})$ due to the cost model. These theoretical results are confirmed by the right diagram of Fig. 4 where the negative slope of each graph reveals the effective order of convergence of the corresponding scheme. The presented simulation results are also given in Table 8. As for the previous examples, we also report on the average measured CPU times that correspond to the simulations performed by each scheme, see Table 9. Again, it can be seen that the DFM-A1 schemes performs best and despite the approximation of iterated stochastic integrals the DFM-A1 and MIL-A1 scheme perform significantly better than the EXE scheme. Again, the measured CPU times support the results for the effective order of convergence presented in Fig. 4.

### 4.4 Example 4

Compared to the other examples, we choose a different nonlinearity $F$ for this example in order to obtain restrictions for the parameter $\beta$. Therefore, we consider the mapping $F : H_\beta \to H$ given by

$$F(v) = \sum_{i \in \mathcal{I}} f_i(v) e_i$$

for $v \in H_\beta$ with some $f_i : H_\beta \to \mathbb{R}$ for $i \in \mathcal{I}$. In this example, we choose $f_i(v) = i^{-s} \sin(i^r \langle v, e_i \rangle_H)$ for $v \in H_\beta$, $s > \frac{1}{2}$, $r \leq \min(s, 2\beta + \frac{1}{2})$ and $i \in \mathcal{I}$. Then, we get

$$\|F(v)\|_H^2 = \sum_{i \in \mathcal{I}} |f_i(v)|^2 = \sum_{i \in \mathcal{I}} \frac{|\sin(i^r \langle v, e_i \rangle_H)|^2}{i^{2s}} < \infty.$$ 

Further, $F$ is twice continuously Fréchet differentiable and it holds

$$\sup_{v \in H_\beta} \|F'(v)\|_{L(H)}^2 = \sup_{v \in H_\beta} \sup_{\|u\|_H = 1} \sum_{i \in \mathcal{I}} \left| \frac{\partial f_i}{\partial v_k} (v) \langle u, e_k \rangle_H \right|^2 = \sup_{v \in H_\beta} \sup_{\|u\|_H = 1} \sum_{i \in \mathcal{I}} i^{2(r-s)} |\cos(i^r \langle v, e_i \rangle_H)|^2 \langle u, e_i \rangle_H^2$$
| $N$ | $M$ | $K$   | DFM-A1 scheme | MIL-A1 scheme |
|-----|-----|-------|--------------|--------------|
|     |     |       | $\bar{c}$  | $\bar{c}$    | $\bar{c}$  | $\bar{c}$    |
|     |     |       | Error | Std | Error | Std |
| 2   | $2^2$ | $[2^7]$ | 112   | $3.08 \cdot 10^{-2}$ | $1.86 \cdot 10^{-3}$ | 128 | $3.08 \cdot 10^{-2}$ | $4.30 \cdot 10^{-3}$ |
| 4   | $2^4$ | $[2^7]$ | 1376  | $2.52 \cdot 10^{-2}$ | $2.19 \cdot 10^{-3}$ | 1760 | $2.52 \cdot 10^{-2}$ | $5.70 \cdot 10^{-3}$ |
| 8   | $2^6$ | $[2^7]$ | 19,072 | $1.67 \cdot 10^{-2}$ | $4.20 \cdot 10^{-4}$ | 26,240 | $1.67 \cdot 10^{-2}$ | $9.24 \cdot 10^{-5}$ |
| 16  | $2^8$ | $[2^7]$ | 422,656 | $6.27 \cdot 10^{-3}$ | $9.45 \cdot 10^{-5}$ | 606,976 | $6.27 \cdot 10^{-3}$ | $2.65 \cdot 10^{-5}$ |
| 32  | $2^{10}$ | $[2^{10}]$ | 6,523,904 | $1.59 \cdot 10^{-3}$ | $2.03 \cdot 10^{-5}$ | 9,571,328 | $1.59 \cdot 10^{-3}$ | $5.12 \cdot 10^{-6}$ |

| $N$ | $M$ | $K$   | EXE scheme | Error | Std |
|-----|-----|-------|------------|-------|-----|
|     |     |       | $\bar{c}$ |       |     |
| 2   | $2^4$ | $[2^7]$ | 128 | $2.13 \cdot 10^{-2}$ | $4.53 \cdot 10^{-3}$ |
| 4   | $2^8$ | $[2^7]$ | 3584 | $2.65 \cdot 10^{-2}$ | $6.06 \cdot 10^{-4}$ |
| 8   | $2^{12}$ | $[2^7]$ | 106,496 | $1.66 \cdot 10^{-2}$ | $1.30 \cdot 10^{-4}$ |
| 16  | $2^{16}$ | $[2^7]$ | 4,390,912 | $6.11 \cdot 10^{-3}$ | $2.26 \cdot 10^{-5}$ |
| 32  | $2^{20}$ | $[2^7]$ | 137,363,456 | $1.52 \cdot 10^{-3}$ | $3.42 \cdot 10^{-6}$ |
respectively

Table 9 Average measured CPU times for Example 3 that correspond to the simulation results in Table 8 obtained from 500 computed paths, respectively

| N  | DFM-A1 scheme     | MIL-A1 scheme     | EXE scheme      |
|----|-------------------|-------------------|-----------------|
| 2  | 2.86 · 10^{-4} s  | 3.36 · 10^{-4} s  | 3.35 · 10^{-4} s|
| 4  | 8.08 · 10^{-4} s  | 7.14 · 10^{-4} s  | 4.31 · 10^{-3} s|
| 8  | 3.80 · 10^{-3} s  | 3.94 · 10^{-3} s  | 8.63 · 10^{-2} s|
| 16 | 3.26 · 10^{-2} s  | 3.38 · 10^{-2} s  | 2.80 s          |
| 32 | 2.47 · 10^{-1} s  | 3.34 · 10^{-1} s  | 78.91 s         |

\[
\leq \sup_{u \in H} \sum_{i \in \mathcal{I}} |\langle u, e_i \rangle_H|^2 = 1,
\]

because \( r \leq s \). Next, considering the second Fréchet derivative, we get

\[
\sup_{v \in H_{\beta}} \| F''(v) \|_{L^2(H_{\beta}, H)}^2 \sup_{v \in H_{\beta}} \sup_{u, w \in H_{\beta}} \frac{\partial^2 f_i}{\partial v_k \partial v_l} (v) \langle u, e_k \rangle_H \langle w, e_l \rangle_H
\]

\[
= \sup_{v \in H_{\beta}} \sup_{u, w \in H_{\beta}} \frac{\partial^2 f_i}{\partial v_k \partial v_l} (v) \langle u, e_k \rangle_H \langle w, e_l \rangle_H \sup_{u, w \in H_{\beta}} \frac{\partial^2 f_i}{\partial v_k \partial v_l} (v) \langle u, e_k \rangle_H \langle w, e_l \rangle_H
\]

\[
= \sup_{u, w \in H_{\beta}} \sup_{u, w \in H_{\beta}} \sum_{i \in \mathcal{I}} i^{4r-2s} \sin(i^r \langle v, e_i \rangle_H) |\langle u, e_i \rangle_H|^2 |\langle w, e_i \rangle_H|^2
\]

\[
\leq \sup_{u, w \in H_{\beta}} \sup_{u, w \in H_{\beta}} \sum_{i \in \mathcal{I}} i^{4r-2s-2\beta} |\langle u, e_i \rangle_H|^2
\]

\[
\leq \frac{100^2}{\pi^4} \sup_{u \in H_{\beta}} \sum_{i \in \mathcal{I}} i^{4r-2s-4\beta} |\langle u, e_i \rangle_H|^2
\]

\[
\leq \frac{100^4}{\pi^8} \sup_{u \in H_{\beta}} \| u \|^2_{H_{\beta}} = \frac{100^4}{\pi^8} < \infty,
\]

since \( \| z \|^2_{H_{\beta}} = \frac{\pi^4}{100^4} \sum_{i \in \mathcal{I}} i^{4\beta} |\langle z, e_i \rangle_H|^2 \) for any \( z \in H_{\beta} \) and because \( r \leq \min(s, 2\beta + 3) \). Thus, assumption (A2) is fulfilled.

Again, we choose \( \rho_Q = 3 \). Moreover, we select \( r = s = \frac{7}{2} \) in the definition of \( F \) and \( p = 4 \) in the definition of the operator \( B \). As the initial value, we choose \( X_0 = \xi \in H \) with \( \langle \xi(x), e_i \rangle_H = i^{-2} \) for \( x \in (0, 1) \) and \( i \in \mathcal{I} \). First, we calculate \( \beta \in \left[ \frac{7}{8}, 1 \right) \) from the condition \( r \leq \min(s, 2\beta + 3) \). Therefore, we choose \( \beta = \frac{7}{8} \) minimal possible. Analogously to Example 1, we derive \( \delta, \vartheta \in (0, 1) \) and choose \( \delta = \frac{1}{2} - \epsilon_\delta \) and \( \vartheta = \frac{1}{2} - \epsilon_\vartheta \) for arbitrarily small \( \epsilon_\delta, \epsilon_\vartheta > 0 \). Then, we choose \( \gamma \in \left[ \frac{7}{8}, 1 - \epsilon_\gamma \right) \) maximal,
i.e., we choose \( \gamma = 1 - \varepsilon_\delta - \varepsilon_\gamma \) for arbitrarily small \( \varepsilon_\gamma > 0 \). Let \( \hat{\varepsilon} = \varepsilon_\delta + \varepsilon_\gamma > 0 \) be arbitrarily small. It follows that \( q = q_{\text{DFM-A1}} = q_{\text{MIL-A1}} = \frac{1}{4} - 2\hat{\varepsilon} \). Finally, we calculate that \( \alpha \in (0, \frac{7}{3} - \frac{2}{3}\hat{\varepsilon}) \) and we set \( \alpha = \frac{7}{3} - \varepsilon_\alpha \) for some arbitrarily small \( \varepsilon_\alpha > \frac{2}{7}\hat{\varepsilon} > 0 \).

Since we have \( q \leq \frac{1}{2} \), the optimal schemes are the EXE scheme and the DFM-A1 scheme, both attaining the same effective order of convergence for this example, see Table 2. Taking into account all parameters, we get from (19) and (24) that

\[
\text{error(EXE/DFM-A1)} = \mathcal{O} \left( \frac{\frac{2-2\varepsilon}{4}}{\frac{12}{7} - \frac{2\hat{\varepsilon}}{4} - \frac{12\varepsilon_\alpha}{7} + \frac{12\varepsilon_\alpha\hat{\varepsilon}}{4} + \frac{4\varepsilon_\alpha^2}{7}} \right),
\]

i.e., for the effective order of convergence it holds that \( \text{EOC(EXE)} = \text{EOC(DFM-A1)} \approx 14 \). For some arbitrarily fixed \( N \in \mathbb{N} \), we obtain for the EXE scheme as well as for the DFM-A1 scheme from (18) that \( M = \left\lceil \frac{N^2 - 2\hat{\varepsilon}}{4} \right\rceil \approx N^8 \) and \( K = \left\lceil \frac{N^2 - 8\varepsilon_\alpha}{7} \right\rceil \approx \lceil N^2 \rceil \) as the optimal choice. In this case, the computation of the double integrals is not expensive as it holds that \( D \geq D_1 = M^{-\frac{1}{2} - \varepsilon} \) for some \( \varepsilon > 0 \) such that \( D = 1 \) can be fixed or it can even be neglected. On the other hand, for the MIL-A1 scheme, we compute from (23) that

\[
\text{error(MIL-A1)} = \mathcal{O} \left( \frac{\frac{7}{18} - \frac{32\varepsilon_\alpha^2}{15} - \frac{18\varepsilon_\alpha\hat{\varepsilon}}{5} + \frac{18\varepsilon_\alpha\hat{\varepsilon}^2}{5} + \frac{4\varepsilon_\alpha^2}{5}}{\frac{12}{7} - \frac{2\hat{\varepsilon}}{4} - \frac{12\varepsilon_\alpha}{7} + \frac{12\varepsilon_\alpha\hat{\varepsilon}}{4} + \frac{4\varepsilon_\alpha^2}{7}} \right),
\]

which gives us the effective order of convergence \( \text{EOC(MIL-A1)} \approx \frac{7}{36} \). Moreover, the optimal choice for \( M \) and \( K \) given some \( N \in \mathbb{N} \) can be calculated from (22) to be exactly the same as for the EXE scheme and the DFM-A1 scheme. For this example, we have \( \text{EOC(MIL-A1)} < \text{EOC(EXE)} = \text{EOC(DFM-A1)} \). Further, the computational effort involved in computing a convergence plot is very high due to the relation \( M \approx N^8 \). Therefore, we do not present a convergence plot for this setting.

The examples presented above confirm the theoretical analysis that we conducted in Sect. 3. The numerical experiments show that the derivative-free Milstein type scheme for equations with non-commutative noise defined in (7), in combination with Algorithm 1, has always at least the same and in many cases an even higher effective order of convergence compared to the exponential Euler scheme and the original Milstein scheme.

5 Conclusion

We proposed the derivative-free Milstein type scheme DFM for the approximation of the mild solution of SPDEs that need not fulfill a commutativity condition for the noise and we proved an upper bound for the \( L^2 \)-error. As the main novelty, the introduced DFM scheme is derivative-free and has computational cost \( \mathcal{O}(NKM) \) which is of the
same magnitude as for the Euler schemes EXE and LIE. This is a significant reduction of the computational complexity compared to the original Milstein scheme MIL that is not derivative-free and which has computational cost \( O(N^2 K M) \). In addition, the convergence of the DFM method is proved if it is combined with any suitable simulation method for the iterated stochastic integrals. As an example, the effective order of convergence of the DFM scheme combined with Algorithm 1 in [21] for the simulation of the iterated stochastic integrals is analyzed in detail. For Algorithm 1, the effective order of convergence of the DFM scheme is at least that for the Euler schemes or the Milstein scheme MIL and turns out to be even significantly higher for many parameter settings depending on the specific SPDE to be approximated. Thus, in many cases the proposed DFM scheme outperforms the Euler schemes as well as the original Milstein scheme.

The maximal possible effective order of convergence that can be attained by the DFM scheme combined with Algorithm 1 is bounded by 1/2, which is in accordance with the upper bound for the order of strong convergence in case of finite-dimensional SDEs if Algorithm 1 is applied, see also [5]. However, in contrast to the finite-dimensional SDEs setting, for SPDEs the Euler schemes often attain some effective order of convergence less than 1/2. This gap in the order of convergence for the Euler schemes is the reason why the use of higher order approximation methods can be reasonable and which is in strong contrast to the finite-dimensional SDE setting. To the best of the authors knowledge, this is the first attempt to give a rigorous analysis of the error versus computational cost for higher order approximation methods applied to SPDEs without any commutativity condition where the computational cost for the approximation of iterated stochastic integrals is incorporated within the framework of a cost model. It remains an open question whether the application of higher order numerical methods that incorporate further iterated stochastic integrals from the stochastic Taylor expansion may close the gap for the order of convergence to the upper bound of 1/2 if, e.g., naive approximations like Algorithm 1 are applied for the approximation of these iterated stochastic integrals. As a result of this, higher order approximation methods may be of strong interest, especially in the case of SPDEs. On the other hand, it may be possible to overcome the upper bound of 1/2 for the order of convergence if some more sophisticated algorithm for the simulation of the iterated stochastic integrals is combined with the DFM scheme, see e.g., Algorithm 2 in [21] and the recently proposed algorithm in [24].

6 Proofs

Here, we give the proof of the convergence result for the derivative-free Milstein scheme (7) as stated in Theorem 2.1. Moreover, we prove the estimate given in Theorem 2.2 which incorporates the approximation of the stochastic double integrals additionally. In the following, we always denote \( Y_m = Y_m^{N,K,M} \) for simplicity and let \( I_{(i,j),I} = (\eta_i \eta_j)^{-1} I_{(i,j),I}^Q \). Attention should be paid to the fact that for ease of notation the constants in our proofs may differ from line to line even though their denomination is not changed. We need the following estimate on the moments of the
approximation process \((Y_m)_{m \in \{0, \ldots, M\}}\) for the proof of Theorem 2.1. Note that, without loss of generality, we present the proofs with an equidistant time step \(h = h_m\) for all \(m \in \{0, \ldots, M - 1\}\).

**Lemma 6.1** Assume that (A1)–(A4) and (A5) hold. Then, it holds that

\[
\sup_{m \in \{0, \ldots, M\}} \left( \mathbb{E}\left[ \|Y_m\|^p_{H^\delta} \right] \right)^{\frac{1}{p}} \leq C_{p, Q, T, \delta} \left( 1 + \left( \mathbb{E}\left[ \|X_0\|^p_{H^\delta} \right] \right)^{\frac{1}{p}} \right)
\]

for all \(p \in [2, \infty)\) in case of (A5a) and for \(p = 2\) in case of (A5b) for some arbitrary \(N, K, M \in \mathbb{N}\) and some constant \(C_{p, Q, T, \delta} > 0\) independent of \(N, K, M\).

**Proof of Lemma 6.1** We conduct the proof of this lemma iteratively. Fix some \(N, K, M \in \mathbb{N}\) and let \(p \in [2, \infty)\). The statement obviously holds for \(m = 0\). Then, for some \(m \in \{1, \ldots, M\}\), we assume that the statement is true for all \(Y_l\) with \(l \in \{0, \ldots, m - 1\}\).

By the triangle inequality, we get

\[
\left( \mathbb{E}\left[ \|Y_m\|^p_{H^\delta} \right] \right)^{\frac{2}{p}} \leq \left( C \left( \mathbb{E}\left[ \|X_0\|^p_{H^\delta} \right] \right)^{\frac{1}{p}} \right)^{\frac{2}{p}} + \sum_{l=0}^{m-1} \left( \mathbb{E}\left[ \int_{t_l}^{t_{l+1}} e^{A(t_m-t_l)} F(Y_l) \, ds \right]^p_{H^\delta} \right)^{\frac{1}{p}}
\]

\[
\quad + \left( \mathbb{E}\left[ \int_{t_0}^{t_m} \sum_{l=0}^{m-1} e^{A(t_m-t_l)} B(Y_l) \mathbbm{1}_{[t_l, t_{l+1})}(s) \, dW^K_s \right]^p_{H^\delta} \right)^{\frac{1}{p}}
\]

\[
\quad + \left( \mathbb{E}\left[ \sum_{l=0}^{m-1} e^{A(t_m-t_l)} \sum_{j \in J_K} B \left( Y_l + \sum_{i \in J_K} P_N B(Y_l) \tilde{e}_i I_{Q(i,j),l} \right) \tilde{e}_j \right]^p_{H^\delta} \right)^{\frac{1}{p}}
\]

\[
\quad - B(Y_l) \tilde{e}_j \right]^p_{H^\delta} \right)^{\frac{1}{p}} \right)^2.
\]

Case 1: Assume that assumption (A5a) is fulfilled. We estimate the individual terms by a Burkholder–Davis–Gundy type inequality [6, Theorem 4.37], and a Taylor expansion of the difference approximation. Precisely, we use

\[
B \left( Y_l + \sum_{i \in J_K} P_N B(Y_l) \tilde{e}_i I_{Q(i,j),l} \right) = B(Y_l)
\]

\[
+ \int_0^1 B' \left( \xi(Y_l, j, u) \right) \left( \sum_{i \in J_K} P_N B(Y_l) \tilde{e}_i I_{Q(i,j),l} \right) du \quad (32)
\]
for some $\xi(Y_l, j, u) = Y_l + u \sum_{i \in J_K} P_N B(Y_l) \tilde{e}_i I_Q^{(i,j),l} \in H_\beta$, $l \in \{0, \ldots, m - 1\}$, $j \in J_K$, $u \in [0, 1]$. Therewith, we get

\[
\left( \mathbb{E} \left[ \| Y_m \|_{H_\delta}^p \right] \right)^{\frac{2}{p}} \leq C_p \left( \mathbb{E} \left[ \| X_0 \|_{H_\delta}^p \right] \right)^{\frac{2}{p}} + \left( \sum_{l=0}^{m-1} \left( \mathbb{E} \left[ \left( \int_{t_l}^{t_{l+1}} \| (-A)^{\delta} e^{A(t_m - t_l)} F(Y_l) \|_H \, ds \right)^p \right] \right)^{\frac{1}{p}} \right)^2 + \int_{t_0}^{t_m} \mathbb{E} \left[ \left( \sum_{l=0}^{m-1} e^{A(t_m - t_l)} B(Y_l) \mathbb{I}_{[t_l, t_{l+1})}(s) \right)^p \right]_{L_H S(U_0, H_\delta)} \, ds

+ \left( \mathbb{E} \left[ \sum_{l=0}^{m-1} (-A)^{\delta} e^{A(t_m - t_l)} \sum_{j \in J_K} \int_0^1 B' \left( \xi(Y_l, j, u) \right) \times \left( \sum_{i \in J_K} P_N B(Y_l) \tilde{e}_i I_Q^{(i,j),l} \right) \tilde{e}_j \, du \right]^p \right)^{\frac{2}{p}}.
\]

The estimates on the analytic semigroup, see Lemma 6.3 and 6.13 in [27, Ch.2], and assumptions (A2), (A3), yield

\[
\left( \mathbb{E} \left[ \| Y_m \|_{H_\delta}^p \right] \right)^{\frac{2}{p}} \leq C_p \left( \mathbb{E} \left[ \| X_0 \|_{H_\delta}^p \right] \right)^{\frac{2}{p}} + C_p \delta M \sum_{l=0}^{m-1} \left( h^p (t_m - t_l)^{-\delta} \right)^{\frac{2}{p}} \left( \mathbb{E} \left[ \| F(Y_l) \|_{H_\delta}^p \right] \right)^{\frac{2}{p}} + C_p \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} \mathbb{E} \left[ \left( \sum_{k=0}^{m-1} e^{A(t_m - t_k)} B(Y_k) \mathbb{I}_{[t_k, t_{k+1})}(s) \right)^p \right]_{L_H S(U_0, H_\delta)} \, ds

+ C_p \delta M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \left( \mathbb{E} \left[ \left( \sum_{j \in J_K} \int_0^1 B' \left( \xi(Y_l, j, u) \right) \times \left( \sum_{i \in J_K} P_N B(Y_l) \tilde{e}_i I_Q^{(i,j),l} \right) \tilde{e}_j \, du \right)^p \right] \right)^{\frac{2}{p}} \leq C_p \left( \mathbb{E} \left[ \| X_0 \|_{H_\delta}^p \right] \right)^{\frac{2}{p}} + C_p, T, \delta h \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \left( 1 + \mathbb{E} \left[ \| Y_l \|_{H_\delta}^p \right] \right)^{\frac{2}{p}}.
\]
\[ + C_p \sum_{l=0}^{m-1} \left( E\left[ \| B(Y_l) \|_{L^{H_S}(U_0, H_S)}^p \right] \right)^{\frac{2}{p}} \int_{t_l}^{t_{l+1}} \|(A)^{-\delta} \|_{L(H)}^2 \|(A)^{\delta} e^{A(t_m - t_l)} \|_{L(H)}^2 \, ds \\
+ C_{p, \delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \times \left( \sum_{j \in J_K} \left( E\left[ \left( \int_0^1 \| B'(\xi(Y_l, j, s)) \|_{L(H,L(U,H))} \, ds \right)^p \right] \right)^{\frac{1}{p}} \right)^2 \times \left( \sum_{j \in J_K} \left( E\left[ \| B(Y_l) \|_{L^{H_S}(U_0, H_S)}^p \right] \right)^{\frac{1}{p}} \right)^2 \times \left( E\left[ \| B(Y_l) \|_{L(H)}^p \right] \right)^{\frac{1}{p}} \left( E\left[ \sum_{i \in J_K} \left( I_{(i,j)}, l \right)^{2 \frac{p}{2}} \right] \right)^{\frac{1}{p}} \right)^2 \].

This expression can further be simplified by the distributional properties of \( I_{(i,j)} \), \( i, j \in J_K \), see [11]. Therewith, we obtain

\[ \left( E\left[ \| Y_m \|_{H_S}^p \right] \right)^{\frac{2}{p}} \leq C_p \left( E\left[ \| X_0 \|_{H_S}^p \right] \right)^{\frac{2}{p}} + C_{p, T, \delta} h^{-2\delta} \sum_{l=0}^{m-1} (m - l)^{-2\delta} \left( 1 + \left( E\left[ \| Y_l \|_{H_S}^p \right] \right)^{\frac{2}{p}} \right) \]

\[ + C_{p, Q, \delta} h \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \left( 1 + \left( E\left[ \| Y_l \|_{H_S}^p \right] \right)^{\frac{2}{p}} \right) \]

\[ + C_{p, \delta} M h^{-2\delta} \sum_{l=0}^{m-1} (m - l)^{-2\delta} \times \left( \left( 1 + E\left[ \| Y_l \|_{H_S}^p \right] \right)^{\frac{1}{p}} \sum_{i, j \in J_K} \left( E\left[ |I_{(i,j)}, l| \sqrt{\eta_i} \sqrt{\eta_j} |H_S|^p \right] \right)^{\frac{1}{p}} \right)^2 \]

\[ \leq C_p \left( E\left[ \| X_0 \|_{H_S}^p \right] \right)^{\frac{2}{p}} + C_{p, Q, T, \delta} h^{-2\delta} \sum_{l=0}^{m-1} (m - l)^{-2\delta} \left( 1 + \left( E\left[ \| Y_l \|_{H_S}^p \right] \right)^{\frac{2}{p}} \right) \]

\[ + C_{p, \delta} M h^{-2\delta} \sum_{l=0}^{m-1} (m - l)^{-2\delta} \left( \left( 1 + E\left[ \| Y_l \|_{H_S}^p \right] \right)^{\frac{1}{p}} \sum_{i, j \in J_K} \sqrt{\eta_i} \sqrt{\eta_j} h \right)^2 .\]
Case 2: Assume $p = 2$ and that assumption (A5b) is fulfilled. Again, we estimate the individual terms by a Burkholder-Davis-Gundy type inequality [6, Theorem 4.37], but using a first order Taylor expansion of the difference approximation. Thus, we use

\[
B\left(Y_l + \sum_{i \in \mathcal{J}_K} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}^Q\right)
= B(Y_l) + B'(Y_l)\left(\sum_{i \in \mathcal{J}_K} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}^Q\right)
+ \int_0^1 \int_0^u B''(\xi(Y_l, j, r))\left(\sum_{i \in \mathcal{J}_K} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}^Q\right) \text{d}r \text{d}u
\]

for some $\xi(Y_l, j, r) = Y_l + r \sum_{i \in \mathcal{J}_K} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}^Q \in H_\beta$, $l \in \{0, \ldots, m - 1\}$, $j \in \mathcal{J}_K$, $r \in [0, 1]$.

With estimates on the analytic semigroup, see Lemma 6.3 and 6.13 in [27, Ch.2], we get that

\[
\left(E\left[\|Y_m\|_{H_2}^p\right]\right)^{\frac{2}{p}} \leq C_p \left(\left(E\left[\|X_0\|_{H_2}^p\right]\right)^{\frac{2}{p}} + \left(\sum_{l=0}^{m-1} \left(E\left[\int_{t_l}^{t_{l+1}} (-A)^{\delta} e^{A(t_m-t_l)} F(Y_l) \|\|_H \text{d}s\right]^p\right]\right)^{\frac{1}{p}}\right)^2
\]

\[
+ \int_{t_0}^{t_m} \left( E\left[\|\sum_{l=0}^{m-1} e^{A(t_m-t_l)} B(Y_l) 1_{[t_l,t_{l+1})}(s)\|_{L^p_{HS}(U_0,H_\beta)}\right]\right)^{\frac{2}{p}} \text{d}s
\]

\[
+ \left(E\left[\|\sum_{l=0}^{m-1} (-A)^{\delta} e^{A(t_m-t_l)} \sum_{j \in \mathcal{J}_K} \left(\sum_{i \in \mathcal{J}_K} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}^Q\right) \tilde{e}_j\right]\right)^{\frac{2}{p}}
\]

\[
\times \left(\sum_{i \in \mathcal{J}_K} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}^Q, \sum_{i \in \mathcal{J}_K} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}^Q \tilde{e}_j \text{d}r \text{d}u\right)^{\frac{2}{p}}
\]

\[
\leq C_p \left(E\left[\|X_0\|_{H_2}^p\right]\right)^{\frac{2}{p}} + C_{p,\delta} M \sum_{l=0}^{m-1} \left(h_p^{(t_m-t_l)^{-\delta}}\right)^{\frac{2}{p}} \left(E\left[\|F(Y_l)\|_{H_2}^p\right]\right)^{\frac{2}{p}}
\]

\[
+ C_p \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} \left(E\left[\|\sum_{k=0}^{m-1} e^{A(t_m-t_k)} B(Y_k) 1_{[t_k,t_{k+1})}(s)\|_{L^p_{HS}(U_0,H_\beta)}\right]\right)^{\frac{2}{p}} \text{d}s
\]

\[
+ C_{p,\delta} M \sum_{l=0}^{m-1} (t_m-t_l)^{-2\delta} \left(E\left[\|\sum_{i,j \in \mathcal{J}_K} I_{(i,j),l}^Q B'(Y_l)(P_N B(Y_l) \tilde{e}_i) \tilde{e}_j\|_H^p\right]\right)^{\frac{2}{p}}
\]
\[ + C_{\rho, \delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \left( E \left[ \sum_{j \in J_K} \int_0^1 \int_0^u B''(\xi(Y_l, j, r)) \right] \right)^{2/p}. \]

Making use of \( p = 2 \), assumptions (A2), (A3) and (A5b) yield

\[
E\left[ \| Y_m \|_{H_2}^2 \right] 
\leq C E\left[ \| X_0 \|_{H_2}^2 \right] + C_{T, \delta} \sum_{l=0}^{m-1} h(t_m - t_l)^{-2\delta} \left( 1 + E\left[ \| Y_l \|_{H_2}^2 \right] \right)
\]

\[
+ C \sum_{l=0}^{m-1} E\left[ \| B(Y_l) \|_{L(H(S(U_0, H_2)))} \right] \int_{t_l}^{t_{l+1}} \| (-A)^{-\delta} \|_{L(H)}^2 \| (-A)^{\delta} e^{A(t_m - t_l)} \|_{L(H)}^2 ds
\]

\[
+ C_{\delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \sum_{i_1, i_2, j_1, j_2 \in J_K} E\left[ I^{Q}_{(i_1,j_1),l} I^{Q}_{(i_2,j_2),l} \right]
\]

\[
\times \left[ \left( \sum_{j \in J_K} \left( \int_0^1 \int_0^u \| B''(\xi(Y_l, j, r)) (P_N B(Y_l), P_N B(Y_l)) \|_{L^2(U, L(U, H))} \right) \right]^2 \right]^{1/2}
\]

\[
\leq C E\left[ \| X_0 \|_{H_2}^2 \right] + C_{T, \delta} \sum_{l=0}^{m-1} h(t_m - t_l)^{-2\delta} \left( 1 + E\left[ \| Y_l \|_{H_2}^2 \right] \right)
\]

\[
+ C \sum_{l=0}^{m-1} E\left[ \| B(Y_l) \|_{L(H(S(U_0, H_2)))} \right] \| (-A)^{-\delta} \|_{L(H)}^2 \| (-A)^{\delta} e^{A(t_m - t_l)} \|_{L(H)}^2 h
\]

\[
+ C_{\delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \sum_{i_1, i_2, j_1, j_2 \in J_K} E\left[ I^{Q}_{(i_1,j_1),l} I^{Q}_{(i_2,j_2),l} \right]
\]

\[
\times E\left[ \left( B'(Y_l)(P_N B(Y_l)\tilde{e}_{i_1})\tilde{e}_{j_1}, B'(Y_l)(P_N B(Y_l)\tilde{e}_{i_2})\tilde{e}_{j_2} \right)_H \right]
\]

\[
+ C_{\delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \left( \sum_{j \in J_K} \left( \int_0^1 \int_0^u (1 + \| \xi(Y_l, j, r) \|_H + \| Y_l \|_H) \right) \right)
\]
\[
\times \sum_{i \in J_K} \left( I_{(i,j),i}^Q \right)^2 dr du \right)^2 \right) ^{\frac{1}{2}}^2.
\]

Due to \( E[I_{(i_1,j_1),i_1}^Q I_{(j_2,j_2),i_2}^Q] = \frac{1}{2} \eta_{i_1} \eta_{j_2} h_i^2 \) if \( i_1 = i_2 \) and \( j_1 = j_2 \) and \( E[I_{(i_1,j_1),i_1}^Q I_{(j_2,j_2),i_2}^Q] = 0 \) otherwise, we get

\[
E[\|Y_m\|_{H_0}^2] \\
\leq CE[\|X_0\|_{H_0}^2] + C_{T,\delta} \sum_{l=0}^{m-1} h(t_m - t_l)^{-2\delta} (1 + E[\|Y_l\|_{H_0}^2]) \\
+ C \sum_{l=0}^{m-1} E[\|B(Y_l)\|_{L(U,H_0)}^2] \|(-A)^{-\delta}\|_{L(H)}^2 \|(-A)^{\delta} e^{A(t_m - t_l)}\|_{L(H)}^2 h \\
+ C_{\delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \sum_{i,j \in J_K} \eta_i \eta_j h^2 E[\|A'(Y_l)(P_N B(Y_l) \tilde{e}_i \tilde{e}_j)\|_{H}^2] \\
+ C_{\delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \\
\times \left( \sum_{j \in J_K} \left( E\left[ (\int_0^1 \int_0^\mu 1 + 2\|Y_l\|_H + r \left( P_N B(Y_l) \sum_{i \in J_K} \tilde{e}_i I_{(i,j),i}^Q, \tilde{e}_j I_{(i,j),j}^Q \right) \right] \right)^{\frac{1}{2}} \right)^2 \\
\leq CE[\|X_0\|_{H_0}^2] + C_{T,\delta} \sum_{l=0}^{m-1} h(t_m - t_l)^{-2\delta} (1 + E[\|Y_l\|_{H_0}^2]) \\
+ C \sum_{l=0}^{m-1} E[\|\text{tr} Q B(Y_l)\|_{L(U,H_0)}^2] \|(-A)^{-\delta}\|_{L(H)}^2 \|(-A)^{\delta} e^{A(t_m - t_l)}\|_{L(H)}^2 h \\
+ C_{Q,\delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} h^2 E[\|B'(Y_l) P_N B(Y_l)\|_{L(U,L(U,H))}^2] \\
+ C_{\delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \left( \sum_{j \in J_K} \left( E\left[ 1 + \|Y_l\|_{H_0} + \|(-A)^{-\delta}\|_{L(H)} \|B(Y_l)\|_{L(U,H_0)} \right] \right)^{\frac{1}{2}} \right)^2 \\
\times \left( \sum_{i \in J_K} \left( I_{(i,j),i}^Q \right)^2 \right)^{\frac{1}{2}}^2 \left( \sum_{i \in J_K} \left( I_{(i,j),i}^Q \right)^2 \right)^{\frac{1}{2}}^2 \\
\leq CE[\|X_0\|_{H_0}^2] + C_{T,\delta} \sum_{l=0}^{m-1} h(t_m - t_l)^{-2\delta} (1 + E[\|Y_l\|_{H_0}^2])
\]
\[ + C_{Q,h} \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \left( 1 + E\left[ \|Y_l\|_{H_b}^2 \right] \right) \]
\[ + C_{Q,T,h} \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} E\left[ \|B'(Y_l)\|_{L(H,L(U,H))}^2 \|(-A)^{-\delta} \|_{L(U,H)}^2 \|B(Y_l)\|_{L(U,H)}^2 \right] \]
\[ + C_{\delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \left( \sum_{j \in \mathcal{J}_K} \left( E\left[ \left( \sum_{i \in \mathcal{J}_K} (I_{(i,j),l})^2 \right)^2 \right] \right) \right) \]
\[ + \left( \sum_{i \in \mathcal{J}_K} (I_{(i,j),l})^2 \right)^3 \left( 1 + E\left[ \|Y_l\|_{H_b}^2 \right] \right)^2 \]
\[ \leq C E\left[ \|X_0\|_{H_b}^2 \right] + C_{Q,T,h} h^{1-2\delta} \sum_{l=0}^{m-1} (m - l)^{-2\delta} \left( 1 + E\left[ \|Y_l\|_{H_b}^2 \right] \right) \]
\[ + C_{\delta} M \sum_{l=0}^{m-1} (t_m - t_l)^{-2\delta} \left( \sum_{j \in \mathcal{J}_K} \left( E\left[ \left( \sum_{i \in \mathcal{J}_K} (I_{(i,j),l})^2 \right)^2 \right] \right) \right) \]
\[ + \left( \sum_{i \in \mathcal{J}_K} (I_{(i,j),l})^2 \right)^3 \left( 1 + E\left[ \|Y_l\|_{H_b}^2 \right] \right)^2 \]
\[ \leq C E\left[ \|X_0\|_{H_b}^2 \right] + C_{Q,T,h} h^{1-2\delta} \sum_{l=0}^{m-1} (m - l)^{-2\delta} \left( 1 + E\left[ \|Y_l\|_{H_b}^2 \right] \right) \]
\[ + C_{Q,h} h^{-2\delta} \sum_{l=0}^{m-1} (m - l)^{-2\delta} \]
\[ \times \left( \text{tr} \left( (\text{tr} \ Q)^2 h^4 + (\text{tr} \ Q)^3 h^6 \right) \right)^{\frac{1}{2}} \left( 1 + E\left[ \|Y_l\|_{H_b}^2 \right] \right) \]
\[ \leq C E\left[ \|X_0\|_{H_b}^2 \right] + C_{Q,T,h} h^{1-2\delta} \sum_{l=0}^{m-1} (m - l)^{-2\delta} \left( 1 + E\left[ \|Y_l\|_{H_b}^2 \right] \right) \]
\[ + C_{Q,T,h} h^{1-2\delta} \sum_{l=0}^{m-1} (m - l)^{-2\delta} (h^2 + h^4) \left( 1 + E\left[ \|Y_l\|_{H_b}^2 \right] \right). \]

Now, we continue with the final estimates in case 1 and case 2 simultaneously having in mind that case 2 is restricted to \( p = 2 \).

Interpreting the terms \( \sum_{l=0}^{m-1} (m - l)^{-2\delta} \) as lower Darboux sums, we estimate these expressions as in the proof of the scheme for SPDEs with commutative noise in [20], see also [7], for \( \delta \in (0, \frac{1}{2}) \) and all \( m \in \{1, \ldots, M\}, M \in \mathbb{N} \)

\[ \sum_{l=0}^{m-1} (m - l)^{-2\delta} = \sum_{l=1}^{m} l^{-2\delta} \leq 1 + \int_1^M r^{-2\delta} \, dr \leq \frac{M^{1-2\delta}}{1-2\delta}. \]
This yields

\[
\left( \mathbb{E}[\| Y_m \|^p_{H^\delta}] \right)^{\frac{2}{p}} \leq C_p \left( \mathbb{E}[\| X_0 \|^p_{H^\delta}] \right)^{\frac{2}{p}} + C_{p, Q, T, \delta} + h^{1-2\delta} C_{p, Q, T, \delta} \sum_{l=0}^{m-1} (m-l)^{-2\delta} \left( \mathbb{E}[\| Y_l \|^p_{H^\delta}] \right)^{\frac{2}{p}}
\]

in a first step. Further, the discrete Gronwall Lemma implies the boundedness of the moments

\[
\left( \mathbb{E}[\| Y_m \|^p_{H^\delta}] \right)^{\frac{2}{p}} \leq \left( C_p \left( \mathbb{E}[\| X_0 \|^p_{H^\delta}] \right)^{\frac{2}{p}} + C_{p, Q, T, \delta} \sum_{l=0}^{m-1} (m-l)^{-2\delta} h^{1-2\delta} \right) \leq C_{p, Q, T, \delta} \left( 1 + \left( \mathbb{E}[\| X_0 \|^p_{H^\delta}] \right)^{\frac{2}{p}} \right)
\]

for all \( m \in \{1, \ldots, M\}, M \in \mathbb{N}, \) for \( p \in [2, \infty) \) in case of (A5a) and for \( p = 2 \) in case of (A5b).

We address the proof of Theorem 2.1 now and show that the scheme converges with the specified order. This estimate does not yet involve any approximation of the stochastic iterated integrals.

**Proof of Theorem 2.1** First, we express the mild solution of (2) as

\[
X_{tm} = e^{Atm} X_0 + \sum_{l=0}^{m-1} \int_{tl}^{tl+1} e^{A(t_{l+1}-s)} F(X_s) \, ds + \sum_{l=0}^{m-1} \int_{tl}^{tl+1} e^{A(t_{l+1}-s)} B(X_s) \, dW_s
\]

for all \( m \in \{0, \ldots, M\}, M \in \mathbb{N} \) to align the components with the corresponding terms in the approximation below. We define the following auxiliary processes for \( m \in \{0, \ldots, M\}, M, N, K \in \mathbb{N} \)

\[
\hat{Y}_{m}^{\text{MIL}} = P_N \left( e^{Atm} X_0 + \sum_{l=0}^{m-1} \int_{tl}^{tl+1} e^{A(t_{l+1}-s)} F(Y_l) \, ds + \sum_{l=0}^{m-1} \int_{tl}^{tl+1} e^{A(t_{l+1}-s)} B(Y_l) \, dW_s \right)
\]

\[
\tilde{Y}_m = P_N \left( e^{Atm} X_0 + \sum_{l=0}^{m-1} \int_{tl}^{tl+1} e^{A(t_{l+1}-s)} F(Y_l) \, ds + \sum_{l=0}^{m-1} \int_{tl}^{tl+1} e^{A(t_{l+1}-s)} B(Y_l) \, dW_s \right)
\]

\[
\hat{Y}_{m}^{\text{MIL}} = P_N \left( e^{Atm} X_0 + \sum_{l=0}^{m-1} \int_{tl}^{tl+1} e^{A(t_{l+1}-s)} F(Y_l) \, ds \right)
\]

\[
+ \sum_{l=0}^{m-1} \int_{tl}^{tl+1} e^{A(t_{l+1}-s)} B(Y_l) \, dW_s
\]

\[
\tilde{Y}_m = P_N \left( e^{Atm} X_0 + \sum_{l=0}^{m-1} \int_{tl}^{tl+1} e^{A(t_{l+1}-s)} F(Y_l) \, ds \right)
\]

\[
+ \sum_{l=0}^{m-1} \int_{tl}^{tl+1} e^{A(t_{l+1}-s)} B(Y_l) \, dW_s
\]
\[ + \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} e^{A(t_{m-l})} B' (Y_l) \left( P_N \int_{t_l}^{s} B(Y_l) \, dW^K_l \right) \, dW^K_s \].

The discrete process \((Y_m)_{m \in \{0, \ldots, M\}}\) denotes the approximation obtained by the DFM scheme in (7). The auxiliary processes are introduced in order to split the approximation error such that we can employ some known prior estimates. We analyze the following terms separately

\[
\left( E\left[ \| X_{t_m} - Y_m \|_{H}^2 \right] \right)^{\frac{1}{2}} \leq \left( E\left[ \| X_{t_m} - \hat{Y}_{MIL}_m \|_{H}^2 \right] \right)^{\frac{1}{2}} + \left( E\left[ \| \hat{Y}_{MIL}_m - Y_m \|_{H}^2 \right] \right)^{\frac{1}{2}}
\]

\[
\leq \left( E\left[ \| X_{t_m} - \hat{Y}_{MIL}_m \|_{H}^2 \right] \right)^{\frac{1}{2}} + \left( E\left[ \| \hat{Y}_{MIL}_m - \tilde{Y}_{MIL}_m \|_{H}^2 \right] \right)^{\frac{1}{2}} + \left( E\left[ \| \tilde{Y}_{MIL}_m - Y_m \|_{H}^2 \right] \right)^{\frac{1}{2}} \tag{34}
\]

for all \(m \in \{0, \ldots, M\}, M \in \mathbb{N}\). The first term is similar to the error that results from the approximation of (2) with the Milstein scheme by Jentzen and Röckner presented in [9]. A slight difference arises as we introduce the projection operator \(P_N\) in the definition of \(\hat{Y}_{MIL}\), see the computations in [18, 20]. The main reasoning, however, is the same. In the error analysis in [9], the commutativity condition is not needed—it is only employed to facilitate implementation—whereas all conditions required in the proof in [9] are fulfilled due to assumptions (A1)–(A4). Therefore, the estimate

\[
\sup_{m \in \{0, \ldots, M\}} \left( E\left[ \| X_{t_m} - \hat{Y}_{MIL}_m \|_{H}^2 \right] \right)^{\frac{1}{2}} \leq C_{Q,T} \left( \left( \inf_{i \in I \setminus I_N} \lambda_i \right)^{-\gamma} + \left( \sup_{j \in J \setminus J_K} \eta_j \right)^{\alpha} + M^{-\min(2(\gamma-\beta),\gamma)} \right) \tag{35}
\]

for arbitrary \(N, M, K \in \mathbb{N}\) is valid. For details, we refer to [9].

The error estimate of the second term in (34), \(E[\| \hat{Y}_{MIL}_m - \tilde{Y}_{MIL}_m \|_{H}^2], m \in \{0, \ldots, M\}, M \in \mathbb{N}\), can be obtained by the same means as in the proof of convergence of the Milstein scheme in [9] and mainly relies on the Lipschitz properties of the operators. We transfer this reasoning from [9, Section 6.3], which yields

\[
E\left[ \| \hat{Y}_{MIL}_m - \tilde{Y}_{MIL}_m \|_{H}^2 \right] \leq C_T h \sum_{l=0}^{m-1} E\left[ \| \hat{Y}_{MIL}^l - Y_l \|_{H}^2 \right] \tag{36}
\]

for all \(m \in \{0, \ldots, M\}, M, N, K \in \mathbb{N}\).

Next, we analyze the third term in (34) which represents the error that results from the approximation of the derivative. We can show that the theoretical order of convergence that the Milstein scheme obtains is not reduced by this approximation.

We rewrite the expression in (34) as

\[ E[\| \hat{Y}_{MIL} - Y_m \|_{H}^2] \]
Then, we obtain with Lemma 6.1 in the case that condition (A5a) is valid that

\[ E \left[ \left( \sum_{l=0}^{m-1} \sum_{i,j \in J_K} \sqrt{\eta_j} \sqrt{\eta_i} e^{A(l-m-t)} B(Y_l)(P_N B(Y_l) \tilde{e}_i I_{(i,j),l}) \right) \right] ^2 \]

\[ - P_N \left( \sum_{l=0}^{m-1} e^{A(l-m-t)} \sum_{j \in J_K} B(Y_l + \sum_{i \in J_K} \sqrt{\eta_j} \sqrt{\eta_i} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}) \tilde{e}_j \right) \]

\[ - B(Y_l) \tilde{e}_j \right) \right] ^2 \].

We employ a Taylor approximation of first order for the second term, see (33), such that the first order derivatives cancel. Moreover, the triangle inequality and assumption (A3) imply

\[ E \left[ \left\| \sum_{l=0}^{m-1} e^{A(l-m-t)} \right\|_H \right] \]

\[ \leq E \left[ \left( \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \int_0^u \int_0^v B''(Y_l + r \sum_{i \in J_K} \sqrt{\eta_j} \sqrt{\eta_i} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}) \right) \tilde{e}_j \, dr \, du \right)^2 \right] \]

\[ \leq C_T E \left[ \left( \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \int_0^u \int_0^v B''(Y_l + r \sum_{i \in J_K} \sqrt{\eta_j} \sqrt{\eta_i} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}) \right) \tilde{e}_j \, dr \, du \right)^2 \right] \]

\[ \leq C_T E \left[ \left( \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \int_0^u \int_0^v B'(Y_l) \tilde{e}_j \, dr \, du \right)^2 \right] \]

Case 1: Assume that assumption (A5a) is fulfilled, i.e., Lemma 6.1 is valid for any \( p \geq 2 \). Thus, it follows from (37) that

\[ E \left[ \left\| \sum_{l=0}^{m-1} e^{A(l-m-t)} \right\|_H \right] \]

\[ \leq E \left[ \left( \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \int_0^u \int_0^v B''(Y_l + r \sum_{i \in J_K} \sqrt{\eta_j} \sqrt{\eta_i} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}) \right) \tilde{e}_j \, dr \, du \right)^2 \right] \]

\[ \leq C_T E \left[ \left( \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \int_0^u \int_0^v B'(Y_l) \tilde{e}_j \, dr \, du \right)^2 \right] \]

Then, we obtain with Lemma 6.1 in the case that condition (A5a) is valid that

\[ E \left[ \left\| \sum_{l=0}^{m-1} e^{A(l-m-t)} \right\|_H \right] \]
\[ \leq C_T E \left[ \left( \sum_{l=0}^{m-1} \sum_{j \in J_K} \| B(Y_l) \|_{L(U,H)}^2 \sum_{i \in J_K} \sqrt{\eta_i} \sqrt{\eta_i} \tilde{e}_i I_{(i,j),l} \right)^2 \right] \]

\[ \leq C_T E \left[ \left( \sum_{l=0}^{m-1} \sum_{j \in J_K} \| B(Y_l) \|_{L(U,H_0)}^2 \sum_{i \in J_K} \eta_i \eta_i I_{(i,j),l} \right) \right] \]

\[ = C_T \left( \sum_{l=0}^{m-1} \sum_{j \in J_K} \sum_{i \in J_K} (E[\| B(Y_l) \|_{L(U,H_0)}^4])^{\frac{1}{2}} \left( E\left[ (\eta_i \eta_i I_{(i,j),l})^2 \right] \right)^{\frac{1}{2}} \right)^2 \]

\[ \leq C_{Q,T,\delta} \left( \sum_{l=0}^{m-1} \sum_{j \in J_K} \sum_{i \in J_K} \eta_i \eta_i (E[I_{(i,j),l}^4])^{\frac{1}{2}} \right)^2 . \]

Finally, we get

\[ E[\| \tilde{Y}_m^\text{MIL} - Y_m \|_H^2] \leq C_{Q,T,\delta} \left( \sum_{l=0}^{m-1} (\tr Q)^2 h^2 \right)^2 \leq C_{Q,T,\delta} h^2 (\tr Q)^4 \leq C_{Q,T,\delta} h^2 \]

by the distributional properties of \( I_{(i,j),l}, l \in \{0, \ldots, m - 1\}, i, j \in J_K \) for all \( m \in \{1, \ldots, M\}, M, K \in \mathbb{N} \), see [11].

Case 2: If assumption (A5b) is fulfilled, then Lemma 6.1 is valid for \( p = 2 \). Therefore, we need a customized proof to proceed and we get for \( p = 2 \) from (33) and (37) that

\[ E[\| \tilde{Y}_m^\text{MIL} - Y_m \|_H^2] \]

\[ \leq E\left[ \left( \sum_{l=0}^{m-1} e^{A(lu-\delta)} \sum_{j \in J_K} \int_0^u \int_0^u B''(Y_l + r \sum_{i \in J_K} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}) \right) \right] \]

\[ \times \left( \sum_{i \in J_K} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}^Q, \sum_{i \in J_K} P_N B(Y_l) \tilde{e}_i I_{(i,j),l}^Q, \tilde{e}_j \right) \]
\[
\begin{align*}
\leq C_T M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u \| B''(\xi(Y_t, j, r)) (P_N B(Y_t), P_N B(Y_t)) \|_{L(2)(U,L(U,H))} \right. \right. \right. \right.
\end{align*}
\]

Due to \( E \left[ \left( \sum_{i \in \mathcal{J}_K} \| \hat{e}_i I_Q^{(i,j)} \|_{L(U)} \right)^2 \right] \right)^2 \)

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| \xi(Y_t, j, r) \|_H + \| Y_t \|_H) \right. \right. \right. \right.
\end{align*}
\]

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| \xi(Y_t, j, r) \|_H + \| Y_t \|_H) \right. \right. \right. \right.
\end{align*}
\]

Due to \( E[Y_m^2 - Y_m^2] = \frac{1}{2} \eta_1 \eta_2 h_t^2 \) if \( i_1 = i_2 \) and \( j_1 = j_2 \) and \( E[Y_m^2] = 0 \) otherwise, we get

\[
E[Y_m^2 - Y_m^2] \leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| Y_t \|_H + \| (-A)^{-\delta} \|_{L(H)} \| B(Y_t) \|_{L(U,H_0)} \right. \right. \right. \right.
\end{align*}
\]

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| Y_t \|_H + \| (-A)^{-\delta} \|_{L(H)} \| B(Y_t) \|_{L(U,H_0)} \right. \right. \right. \right.
\end{align*}
\]

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| Y_t \|_H + \| (-A)^{-\delta} \|_{L(H)} \| B(Y_t) \|_{L(U,H_0)} \right. \right. \right. \right.
\end{align*}
\]

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| Y_t \|_H + \| (-A)^{-\delta} \|_{L(H)} \| B(Y_t) \|_{L(U,H_0)} \right. \right. \right. \right.
\end{align*}
\]

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| Y_t \|_H + \| (-A)^{-\delta} \|_{L(H)} \| B(Y_t) \|_{L(U,H_0)} \right. \right. \right. \right.
\end{align*}
\]

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| Y_t \|_H + \| (-A)^{-\delta} \|_{L(H)} \| B(Y_t) \|_{L(U,H_0)} \right. \right. \right. \right.
\end{align*}
\]

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| Y_t \|_H + \| (-A)^{-\delta} \|_{L(H)} \| B(Y_t) \|_{L(U,H_0)} \right. \right. \right. \right.
\end{align*}
\]

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| Y_t \|_H + \| (-A)^{-\delta} \|_{L(H)} \| B(Y_t) \|_{L(U,H_0)} \right. \right. \right. \right.
\end{align*}
\]

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| Y_t \|_H + \| (-A)^{-\delta} \|_{L(H)} \| B(Y_t) \|_{L(U,H_0)} \right. \right. \right. \right.
\end{align*}
\]

\[
\leq C_T, M \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{J}_K} \left( E \left[ \left( \int_0^1 \int_0^u (1 + \| Y_t \|_H + \| (-A)^{-\delta} \|_{L(H)} \| B(Y_t) \|_{L(U,H_0)} \right. \right. \right. \right.
\end{align*}
\]
\[ \leq C_{T,\delta} (\text{tr } Q)^{\frac{4}{3}} h \sum_{l=0}^{m-1} (h^2 + \text{tr } Q h^4) \left( 1 + \mathbb{E} \left[ \| X_0 \|_{H_{\delta}}^2 \right] \right) \]
\[ \leq C_{Q,T,\delta} h^2. \]  

(39)

Now, we proceed for both cases similarly. A combination of estimates (35), (36) and (38) or (39) for case 1 and case 2, respectively, with (34), and Gronwall’s Lemma imply

\[ \mathbb{E} \left[ \| \hat{Y}_{m}^{\text{MIL}} - Y_{m} \|_{H}^2 \right] \leq C_{T} h \sum_{l=0}^{m-1} \mathbb{E} \left[ \| \hat{Y}_{l}^{\text{MIL}} - Y_{l} \|_{H}^2 \right] + C_{Q,T,\delta} h^2 \leq C_{Q,T,\delta} h^2. \]

This results in

\[
\left( \mathbb{E} \left[ \| X_{t_m} - Y_{m} \|_{H}^2 \right] \right)^{\frac{1}{2}} \leq C_{Q,T,\delta} \left( \left( \inf_{i \in I \setminus I_{N}} \lambda_i \right)^{-\gamma} + \left( \sup_{j \in J \setminus J_{K}} \eta_j \right)^{\alpha} + M^{-\min(2(\beta - \gamma),\gamma)} \right)
\]

for the overall error.

Remark 6.1 Under the assumption that

\[
\sum_{j \in J_{K}} \left( \mathbb{E} \left[ \left( \sum_{i \in J_{K}} (\tilde{I}_{(i,j),t,t+h})^2 \right)^{p} \right] \right)^{\frac{1}{p}} \leq C_{Q} h
\]

for \( p \in [2, \infty) \) in case of assumption (A5a) or

\[
\sum_{j \in J_{K}} \left( \mathbb{E} \left[ \left( \sum_{i \in J_{K}} (\tilde{I}_{(i,j),t,t+h})^2 \right)^{q} \right] \right)^{\frac{1}{q}} \leq C_{Q} h^q
\]

for \( q = 2, 3 \) in case of assumption (A5b) for any \( h > 0 \) and \( t \in [0, T-h] \), a statement similar to Lemma 6.1 also holds for the process \( (\tilde{Y}_l)_{l \in \{0, \ldots, M\}} \) which includes the approximation of the stochastic double integral, i.e., it holds

\[
\sup_{m \in \{0, \ldots, M\}} \left( \mathbb{E} \left[ \| \tilde{Y}_{m} \|_{H_{\delta}}^p \right] \right)^{\frac{1}{p}} \leq C_{Q,T,\delta} \left( 1 + \left( \mathbb{E} \left[ \| X_0 \|_{H_{\delta}}^p \right] \right)^{\frac{1}{p}} \right),
\]

however with the restriction \( p = 2 \) in case of (A5b).
Proof of Theorem 2.2  From the proof of Theorem 2.1 we get an estimate for \( \left( E[\| X_m - Y_m \|_H^2] \right)^{\frac{1}{2}} \). It remains to prove the expression for the error caused by the approximation of the iterated stochastic integrals, that is,

\[
\left( E[\| Y_m - \bar{Y}_m \|_H^2] \right)^{\frac{1}{2}} \leq \left( E[\| Y_m - Y_{m,\bar{Y}} \|_H^2] \right)^{\frac{1}{2}} + \left( E[\| Y_{m,\bar{Y}} - \bar{Y}_m \|_H^2] \right)^{\frac{1}{2}} (40)
\]

where

\[
Y_{m,\bar{Y}} = P_N\left( e^{A_{tm} X_0} + \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} e^{A(t_{m-l})} F(\bar{Y}_l) \, ds \right.
\]

\[
+ \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} e^{A(t_{m-l})} B(\bar{Y}_l) \, dW_s^K
\]

\[
+ \sum_{l=0}^{m-1} \sum_{j \in J} e^{A(t_{m-l})} \left( B(\bar{Y}_l) + \sum_{i \in J} P_N B(\bar{Y}_l) \bar{e}_i I_{(i,j),l} \right) \bar{e}_j - B(\bar{Y}_l) \bar{e}_j \biggr). \]

For the terms inside the two integrals, we employ Taylor approximations of first order of the difference operators as in (33) where \( \xi(Y_l, j, r) = Y_l + r \sum_{i \in J_k} B(Y_l) \bar{e}_i I_{(i,j),l} \) for all \( j \in J_k, l \in \{0, \ldots, m - 1\} \) and \( r \in [0, 1] \); below \( \tilde{\xi}(\bar{Y}_l, j, r) \) is defined analogously.

This yields

\[
E[\| Y_m - Y_{m,\bar{Y}} \|_H^2]
\]

\[
= E\left[ \left\| P_N \left( \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} e^{A(t_{m-l})} (F(Y_l) - F(\bar{Y}_l)) \right) ds \right. \right.
\]

\[
+ \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} e^{A(t_{m-l})} (B(Y_l) - B(\bar{Y}_l)) \, dW_s^K
\]

\[
+ \sum_{l=0}^{m-1} \sum_{j \in J_k} e^{A(t_{m-l})} \left( B'(Y_l) + \sum_{i \in J} P_N B(Y_l) \bar{e}_i I_{(i,j),l} \right) \bar{e}_j
\]

\[
- B'(\bar{Y}_l) \left( \sum_{i \in J_k} P_N B(\bar{Y}_l) \bar{e}_i I_{(i,j),l} \right) \bar{e}_j + \sum_{l=0}^{m-1} \sum_{j \in J_k} e^{A(t_{m-l})}
\]

\[
\times \left( \int_0^1 \int_0^u B''(\xi(Y_l, j, r)) \right.
\]

\[
\times \left( \sum_{i \in J_k} P_N B(Y_l) \bar{e}_i I_{(i,j),l} \right) \sum_{i \in J_k} P_N B(Y_l) \bar{e}_i I_{(i,j),l} \bar{e}_j \, dr \, du
\]

\[
- \left. \int_0^1 \int_0^u B''(\bar{\xi}(\bar{Y}_l, j, r)) \right) \]
\[
\begin{align*}
&\times \left( \sum_{i \in J_K} P_N B(\tilde{Y}_i) \tilde{e}_i I^0_{(i,j),l}, \sum_{i \in J_K} P_N B(\tilde{Y}_i) \tilde{e}_i I^0_{(i,j),l} \right) \tilde{e}_j \, dr \, du \right) \|_{H}^2
\end{align*}
\]

\[
\leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} \mathbb{E} \left[ \| \dot{Y}_l - \ddot{Y}_l \|_{H}^2 \right] + \mathbb{E} \left[ \sum_{l=0}^{m-1} \sum_{j \in J_K} e^{A(t_m-t_l)} \right.
\]

\[
\times \left( \sum_{i \in J_K} P_N B(Y_i) \tilde{e}_i I^0_{(i,j),l}, \sum_{i \in J_K} P_N B(Y_i) \tilde{e}_i I^0_{(i,j),l} \right) \tilde{e}_j \, dr \, du \right) \|_{H}^2
\]

\[
\leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} \mathbb{E} \left[ \| \dot{Y}_l - \ddot{Y}_l \|_{H}^2 \right] + CM \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \mathbb{E} \left[ \| B''(\xi(Y_l, j)) \|_{L^2(H,H(U,H))]^2} \sum_{i \in J_K} B(Y_i) \tilde{e}_i I^0_{(i,j),l} \right] \right)^{1/2} \right)^2
\]

\[
\times \left( \sum_{i \in J_K} P_N B(\tilde{Y}_i) \tilde{e}_i I^0_{(i,j),l}, \sum_{i \in J_K} P_N B(\tilde{Y}_i) \tilde{e}_i I^0_{(i,j),l} \right) \tilde{e}_j \, dr \, du \right) \|_{H}^2
\]

\[
(41)
\]

where in the second step the computations are the same as in [9, Section 6.3], see also (36). This estimate mainly employs the Lipschitz continuity of the involved operators.

Case 1: Assume that assumption (A5a) is fulfilled, i.e., Lemma 6.1 is valid for any \( p \geq 2 \). Then, by the triangle inequality, the norm properties as well as assumption (A3), (41) results in

\[
\mathbb{E} \left[ \| Y_m - Y_m, \tilde{Y}_l \|_{H}^2 \right]
\]

\[
\leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} \mathbb{E} \left[ \| \dot{Y}_l - \ddot{Y}_l \|_{H}^2 \right]
\]

\[
+ CM \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \mathbb{E} \left[ \| B''(\xi(Y_l, j)) \|_{L^2(H,H(U,H))]^2} \sum_{i \in J_K} B(Y_i) \tilde{e}_i I^0_{(i,j),l} \right] \right)^{1/2} \right)^2
\]

\[
+ CM \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \mathbb{E} \left[ \| B''(\tilde{\xi}(Y_l, j)) \|_{L^2(H,H(U,H))]^2} \sum_{i \in J_K} B(\tilde{Y}_i) \tilde{e}_i I^0_{(i,j),l} \right] \right)^{1/2} \right)^2
\]
Lemma 6.1 and Remark 6.1 imply

\[ I_{Q} \]\n
\[ \text{Theorem 2.1} \]

By applying the triangle inequality, we get analogously to case 2 in the proof of Lemma 6.1 is valid for \( m \in \{1, \ldots, M\} \) and assumption (A3). Furthermore, (A5a), (A5b), Lemma 6.1 and Remark 6.1 imply

\[ E\left[ \left\| Y_{m} - Y_{m, \tilde{y}} \right\|_{H}^{2} \right] \]

\[ \leq C_{Q, T, \beta, \gamma} h \sum_{l=0}^{m-1} E\left[ \left\| Y_{l} - \tilde{Y}_{l} \right\|_{H}^{2} \right] \]

\[ + C_{Q} M \sum_{l=0}^{m-1} \left( \sum_{j \in J_{K}} \left( E\left[ \left( \sum_{i \in J_{K}} I_{Q}^{i} \right)^{2} \left\| B(Y_{l}) \right\|_{L(U, H)}^{4} \right] \right) \right)^{2} \]

\[ \leq C_{Q, T, \beta, \gamma} h \sum_{l=0}^{m-1} E\left[ \left\| Y_{l} - \tilde{Y}_{l} \right\|_{H}^{2} \right] \]

\[ + C_{Q} M \sum_{l=0}^{m-1} \left( \sum_{j \in J_{K}} \left( E\left[ \left( \sum_{i \in J_{K}} I_{Q}^{i} \right)^{2} \left\| B(Y_{l}) \right\|_{L(U, H)}^{4} \right] \right) \right)^{2} \]

\[ \leq C_{Q, T, \beta, \gamma} h \sum_{l=0}^{m-1} E\left[ \left\| Y_{l} - \tilde{Y}_{l} \right\|_{H}^{2} \right] + C_{Q, T, \beta, \gamma} \sum_{l=0}^{m-1} h^{3} \]

\[ \leq C_{Q, T, \beta, \gamma} h \sum_{l=0}^{m-1} E\left[ \left\| Y_{l} - \tilde{Y}_{l} \right\|_{H}^{2} \right] + C_{Q, T, \beta, \gamma} h^{2}. \]

Case 2: If assumption (A5b) is fulfilled, then Lemma 6.1 is valid for \( p = 2 \).

By applying the triangle inequality, we get analogously to case 2 in the proof of Theorem 2.1 that

\[ E\left[ \left\| Y_{m} - Y_{m, \tilde{y}} \right\|_{H}^{2} \right] \]

\[ \leq C_{Q, T, \beta, \gamma} h \sum_{l=0}^{m-1} E\left[ \left\| Y_{l} - \tilde{Y}_{l} \right\|_{H}^{2} \right] + C_{M} \sum_{l=0}^{m-1} \left( \sum_{j \in J_{K}} E\left[ \int_{0}^{1} \int_{0}^{u} \right] \right). \]
\[ B''(\xi(Y_l, j, r)) \left( \sum_{i \in J_K} P_N B(Y_l)\tilde{c}_i I_Q^{(i, j), l} + \sum_{i \in J_K} P_N B(Y_l)\tilde{c}_i I_Q^{(i, j), l} \tilde{e}_j \, dr \, du \right) \] 

\[ \leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} E\left[ \| Y_l - \bar{Y}_l \|_H^2 \right] + CM \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \left( E\left[ \left( \int_0^1 \int_0^u B''(\xi(\bar{Y}_l, j, r)) \left( P_N B(Y_l) - \bar{Y}_l \right) \| \tilde{e}_j \|_U \, dr \, du \right)^2 \right] \right)^{1/2} \right)^2 

\leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} E\left[ \| Y_l - \bar{Y}_l \|_H^2 \right] + CM \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \left( E\left[ \left( \int_0^1 \int_0^u \| B''(\xi(\bar{Y}_l, j, r)) \left( P_N B(Y_l) - \bar{Y}_l \right) \| \tilde{e}_j \|_U \, dr \, du \right)^2 \right] \right)^{1/2} \right)^2 

Making use of the distributional characteristics of \( I_Q^{(i, j), l} \), we get 

\[ E\left[ \| Y_m - Y_m,\bar{Y} \|_H^2 \right] \leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} E\left[ \| Y_l - \bar{Y}_l \|_H^2 \right] + CM \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \left( E\left[ \left( \int_0^1 \int_0^u \left( 1 + \| \xi(Y_l, j, r) \|_H + \| \tilde{Y}_l \|_H \right) \sum_{i \in J_K} \left( I_Q^{(i, j), l} \| \tilde{e}_i \|_U \right) \right)^2 \right] \right)^{1/2} \right)^2. \]
\[ \times \sum_{i \in J_K} \left( I_{(i,j),l}^Q \right)^2 dr du \right)^2 \right) \right)^{\frac{1}{2}} \]

\[ + \sum_{j \in J_K} \left( \mathbb{E} \left[ \left( \sum_{i \in J_K} \left( I_{(i,j),l}^Q \right)^2 \right)^2 \right] \right) \left( 1 + \mathbb{E} \left[ \| Y_l \|_{H}^2 \right] \right) \]

\[ \times \sum_{i \in J_K} \left( I_{(i,j),l}^Q \right)^2 dr du \right)^2 \right) \right)^{\frac{1}{2}} \right)^2 \]

\[ \leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} \mathbb{E} \left[ \| Y_l - \tilde{Y}_l \|_{H}^2 \right] + C_{Q,T,\delta} M \]

\[ \times \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \mathbb{E} \left[ \left( \sum_{i \in J_K} \left( I_{(i,j),l}^Q \right)^2 \right)^2 \right] \right)

\[ + \mathbb{E} \left[ \left( \sum_{i \in J_K} \left( I_{(i,j),l}^Q \right)^2 \right)^3 \right] \right) \left( 1 + \mathbb{E} \left[ \| Y_l \|_{H}^2 \right] \right) \]

\[ \times \left( 1 + \mathbb{E} \left[ \| \tilde{Y}_l \|_{H}^2 \right] \right) \right) \right)^{\frac{1}{2}} \right)^2 \]

\[ \leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} \mathbb{E} \left[ \| Y_l - \tilde{Y}_l \|_{H}^2 \right] \]

\[ + C_{Q,T,\delta} M \sum_{l=0}^{m-1} \left( \text{tr} Q \left( (\text{tr} Q)^2 h^4 + (\text{tr} Q)^3 h^6 \right) \right)^{\frac{1}{2}} \]

\[ \left( 1 + \mathbb{E} \left[ \| Y_l \|_{H_2}^2 \right] + \mathbb{E} \left[ \| \tilde{Y}_l \|_{H_2}^2 \right] \right) \]

\[ \leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} \mathbb{E} \left[ \| Y_l - \tilde{Y}_l \|_{H}^2 \right] \]

\[ + C_{Q,T,\delta} \left( \text{tr} Q \right)^4 h \sum_{l=0}^{m-1} \left( h^2 + \text{tr} Q h^4 \right) \left( 1 + \mathbb{E} \left[ \| X_0 \|_{H_2}^2 \right] \right) \]

\[ \leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} \mathbb{E} \left[ \| Y_l - \tilde{Y}_l \|_{H}^2 \right] + C_{Q,T,\delta} h^2. \quad (43) \]

Summarizing, we have \( \mathbb{E} \left[ \| Y_m - Y_m,\tilde{Y} \|_{H}^2 \right] \leq C_{Q,T,\beta,\gamma} h \sum_{l=0}^{m-1} \mathbb{E} \left[ \| Y_l - \tilde{Y}_l \|_{H}^2 \right] + C_{Q,T,\delta} h^2 \) for both cases.
Finally, we analyze the second term in (40). We basically employ the same techniques as for the previous term. At first, we replace the difference operator by a first order Taylor expansion.

\[
\begin{align*}
E[\|Y_{m,\tilde{F}} - \tilde{Y}_m\|^2_H] &= E\left[\left\| P_N \sum_{t=0}^{m-1} \sum_{j \in J_K} e^{A(t_{m-t})_j}\left( (B(\tilde{Y}_t + \sum_{i \in J_K} P_N B(\tilde{Y}_t) \tilde{e}_i I^Q_{(i,j),t}) \tilde{e}_j - B(\tilde{Y}_t) \tilde{e}_j ) \ight) \right\|^2_H\right] \\
&= E\left[\left\| P_N \sum_{t=0}^{m-1} \sum_{j \in J_K} e^{A(t_{m-t})_j}(B'(\tilde{Y}_t) \left( \sum_{i \in J_K} P_N B(\tilde{Y}_t) \tilde{e}_i I^Q_{(i,j),t} \right) \tilde{e}_j \\
- \left( B'(\tilde{Y}_t) \left( \sum_{i \in J_K} P_N B(\tilde{Y}_t) \tilde{e}_i I^Q_{(i,j),t} \right) \tilde{e}_j \\
- \int_0^1 \int_0^u B''(\tilde{\xi}(\tilde{Y}_t, j, r)) \left( \sum_{i \in J_K} P_N B(\tilde{Y}_t) \tilde{e}_i I^Q_{(i,j),t} \right) \tilde{e}_j \, dr \, du \\
- \int_0^1 \int_0^u B''(\tilde{\xi}(\tilde{Y}_t, j, r)) \\
\times \left( \sum_{i \in J_K} P_N B(\tilde{Y}_t) \tilde{e}_i I^Q_{(i,j),t} \right) \tilde{e}_j \, dr \, du \right) \right\|^2_H\right].
\end{align*}
\]

As above, we obtain for the terms involving the second derivative

\[
\begin{align*}
E[\|Y_{m,\tilde{F}} - \tilde{Y}_m\|^2_H] &\leq CE\left[\left\| \sum_{t=0}^{m-1} e^{A(t_{m-t})_j}\left( \int_{t_{l+1}}^{t_l+1} B'(\tilde{Y}_t) \left( \int_{t_l}^s P_N B(\tilde{Y}_t) \, dW^K_r \right) \, dW^K_s \\
- \sum_{i, j \in J_K} \tilde{I}^Q_{(i,j),t} B'(\tilde{Y}_t)(P_N B(\tilde{Y}_t) \tilde{e}_j) \right) \right\|^2_H\right] \\
&\quad + CE\left[\left\| \sum_{t=0}^{m-1} e^{A(t_{m-t})_j} \sum_{j \in J_K} \\
\times \left( \int_0^1 \int_0^u B''(\tilde{\xi}(\tilde{Y}_t, j, r)) \\
\times \left( \sum_{i \in J_K} P_N B(\tilde{Y}_t) \tilde{e}_i I^Q_{(i,j),t} \right) \tilde{e}_j \, dr \, du \\
- \int_0^1 \int_0^u B''(\tilde{\xi}(\tilde{Y}_t, j, r)) \right) \right\|^2_H\right].
\end{align*}
\]
\[
\times \left( \sum_{i \in \mathcal{K}} P_N B(\bar{Y}_l) \tilde{e}_i \tilde{I}^Q_{(i,j),l}, \sum_{i \in \mathcal{K}} P_N B(\bar{Y}_l) \tilde{e}_i \tilde{I}^Q_{(i,j),l} \right) \tilde{e}_j \, dr \, du \right\|_H^2 \right] \\
\leq C \sum_{l=0}^{m-1} \mathbb{E} \left[ \left\| \int_{t_l}^{t_{l+1}} B'(\bar{Y}_l) \left( \int_{t_l}^s P_N B(\bar{Y}_l) \, dW^K_r \right) \, dW^K_s \\
- \sum_{i,j \in \mathcal{K}} \tilde{I}^Q_{(i,j),l} B'(\bar{Y}_l)(P_N B(\bar{Y}_l) \tilde{e}_i) \tilde{e}_j \right\|_H^2 \right] \\
+ C \left( \sum_{l=0}^{m-1} \mathbb{E} \left[ \left\| e^{A(t_m-t_l)} \sum_{j \in \mathcal{K}} \int_0^1 \int_0^u B''(\tilde{\xi}(\bar{Y}_l, j, r)) \right. \right. \\
\times \left( \sum_{i \in \mathcal{K}} P_N B(\bar{Y}_l) \tilde{e}_i I^Q_{(i,j),l}, \sum_{i \in \mathcal{K}} P_N B(\bar{Y}_l) \tilde{e}_i I^Q_{(i,j),l} \right) \tilde{e}_j \, dr \, du \right\|_H^2 \right) \right]^{\frac{1}{2}} \\
+ \sum_{l=0}^{m-1} \mathbb{E} \left[ \left\| e^{A(t_m-t_l)} \sum_{j \in \mathcal{K}} \int_0^1 \int_0^u B''(\tilde{\xi}(\bar{Y}_l, j, r)) \right. \right. \\
\times \left( \sum_{i \in \mathcal{K}} P_N B(\bar{Y}_l) \tilde{e}_i \tilde{I}^Q_{(i,j),l}, \sum_{i \in \mathcal{K}} P_N B(\bar{Y}_l) \tilde{e}_i \tilde{I}^Q_{(i,j),l} \right) \tilde{e}_j \, dr \, du \right\|_H^2 \right]^{\frac{1}{2}} \right) ^2. \tag{44}
\]

The first term is the error that results from the approximation of the iterated stochastic integral. Depending on the choice of the scheme, this error estimate may differ. Assumption (10) states that

\[
\left( \mathbb{E} \left[ \left\| \int_{t_l}^{t_{l+1}} B'(\bar{Y}_l) \left( \int_{t_l}^s P_N B(\bar{Y}_l) \, dW^K_r \right) \, dW^K_s \\
- \sum_{i,j \in \mathcal{K}} \tilde{I}^Q_{(i,j),l} B'(\bar{Y}_l)(P_N B(\bar{Y}_l) \tilde{e}_i) \tilde{e}_j \right\|_H^2 \right] \right) ^{\frac{1}{2}} \leq \mathcal{E}(M, K)
\]

for all \( l \in \{0, \ldots, m - 1\} \), \( m \in \{1, \ldots, M\} \), \( h > 0 \) and \( M, K \in \mathbb{N} \).

Case 1: Assume that assumption (A5a) is fulfilled, i.e., Lemma 6.1 is valid for any \( p \geq 2 \). Analogously to the calculations in (42), we get for (44)

\[
\mathbb{E} \left[ \left\| Y_{m,\bar{Y}} - \bar{Y}_m \right\|_H^2 \right] \leq C \sum_{l=0}^{m-1} \mathcal{E}(M, K) ^2 \\
+ CM \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{K}} \left( \mathbb{E} \left[ \left( \sum_{i \in \mathcal{K}} (I^Q_{(i,j),l})^2 \right) \left\| B(\bar{Y}_l) \right\|_{L(U,H)}^4 \right] \right)^{\frac{1}{2}} \right)^2 \\
+ CM \sum_{l=0}^{m-1} \left( \sum_{j \in \mathcal{K}} \left( \mathbb{E} \left[ \left( \sum_{i \in \mathcal{K}} (\tilde{I}^Q_{(i,j),l})^2 \right) \left\| B(\bar{Y}_l) \right\|_{L(U,H)}^4 \right] \right)^{\frac{1}{2}} \right)^2.
\]
By assumption (11) and the properties of $I_{(i,j),l}^Q$ as well as Remark 6.1, we obtain

$$E[\|Y_{m,\bar{Y}} - \bar{Y}_{m}\|_H^2] \leq C \sum_{l=0}^{m-1} E(M, K)^2$$

$$+ CM \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \left( E\left( \left( \sum_{i \in J_K} (I_{(i,j),l}^Q)^2 \right)^2 \right) E\left( \|B(\bar{Y}_l)\|^4_{L(U, H)} \right) \right)^{1/2} \right)^2$$

$$+ CM \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \left( E\left( \left( \sum_{i \in J_K} (\bar{I}_{(i,j),l})^2 \right)^2 \right) E\left( \|B(\bar{Y}_l)\|^4_{L(U, H)} \right) \right)^{1/2} \right)^2$$

$$\leq C \sum_{l=0}^{m-1} E(M, K)^2 + C Q M \sum_{l=0}^{m-1} \left( h^2 \left( E[\|B(\bar{Y}_l)\|^4_{L(U, H)}] \right)^{1/2} \right)^2$$

$$\leq C M E(M, K)^2 + C_{Q,T,\delta} h^2,$$

which completes the proof for case 1.

Case 2: If assumption (A5b) is fulfilled, then Lemma 6.1 is valid for $p = 2$. Analogously to the computations in (43), we get with assumption (12) that

$$E[\|Y_{m,\bar{Y}} - \bar{Y}_{m}\|_H^2] \leq C \sum_{l=0}^{m-1} E(M, K)^2$$

$$+ C M \sum_{l=0}^{m-1} \left( \sum_{j \in J_K} \left( E\left( \left( \sum_{i \in J_K} (I_{(i,j),l}^Q)^2 \right)^2 \right) E\left( \|B(\bar{Y}_l)\|^4_{L(U, H)} \right) \right)^{1/2} \right)^2$$

$$+ \sum_{j \in J_K} \left( E\left( \left( \sum_{i \in J_K} (\bar{I}_{(i,j),l})^2 \right)^2 \right) \right)^{1/2} \left( 1 + E[\|Y_{\bar{Y}}\|_H^2] \right)$$

$$\leq C \sum_{l=0}^{m-1} E(M, K)^2 + C_T (\text{tr } Q)^4 h \sum_{l=0}^{m-1} \left( h^2 + \text{tr } Q h^4 \right) \left( 1 + E[\|X_0\|_{H_5}^2] \right)$$

$$\leq C M E(M, K)^2 + C_{Q,T,\delta} h^2.$$

This proves the statement for case 2. \qed

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