Abstract

A model for the longitudinal structure function $F_L$ in the region of low $x$ and low $Q^2$ is discussed. It is constructed using the $k_T$ factorization theorem and a photon-gluon fusion mechanism suitably extrapolated to the region of low $Q^2$. A phenomenological model for higher twist is presented, which is based on the assumption that the contribution of quarks having limited transverse momenta is dominated by the soft pomeron exchange mechanism.
Introduction

The longitudinal structure function $F_L$ corresponds to the interaction of longitudinally polarized photons in the one–photon–exchange approximation of deep inelastic lepton-nucleon scattering. The experimental determination of $F_L$ is difficult since it usually requires measurement of the energy dependence of this process for fixed $x$ and $Q^2$. The structure function $F_L$ vanishes within the naive parton model [1] and also in the leading logarithmic approximation of QCD. In the next–to–leading order approximation the structure function $F_L$ acquires a leading twist contribution proportional to $\alpha_s(Q^2)$. The main process contributing to $F_L$ at low $x$ is photon–gluon fusion, $g\gamma \rightarrow q\bar{q}$. The longitudinal structure function $F_L$ is therefore mainly sensitive to the gluon distribution.

In practice it is the ratio $R(x, Q^2) = F_L(x, Q^2)/F_T(x, Q^2)$ which is determined experimentally (where $F_T(x, Q^2)$ is transverse structure function). Current measurements of $R(x, Q^2)$ (SLAC, NMC at CERN [2]) are very scarce, especially at low $x$. In the region of large $Q^2$ the structure function $F_L$ is fairly well described by perturbative QCD, while at lower values of $Q^2$ it is not well understood mainly because of the non–perturbative contributions. A better understanding of $F_L$ and $R$ in this region is important in order to implement the radiative corrections needed for the extraction of the $F_2(F_3)$ structure functions from deep inelastic scattering measurements. It is also crucial for the reliable extraction of the spin dependent structure function $g_1(x, Q^2)$ [3] from experimental data, since the uncertainties in $R$ are one of the main sources of systematic errors.

In this paper we present a model for the structure function $F_L$ which provides extrapolation into the region of low $Q^2$. We also propose a model for the higher twist term. It is based on the assumption that the contribution of quarks having limited transverse momenta is dominated by soft pomeron exchange. We introduce an effective coupling which can be determined from the background contribution to the structure function $F_T$ [4]. Calculations for $R(x, Q^2)$ in the region of low $Q^2$ are also presented. This paper is based on the results presented in [5].

$k_T$ factorization vs collinear factorization

The off–shell approximation is closely connected with the $k_T$ (or high energy) factorization theorem [4, 5] which can be presented in the following way,

$$F_L(x, Q^2) = \int_x^1 \frac{dx'}{x'} \int \frac{dk_T^2}{k_T^2} F_L^{g\gamma}(x', k_T^2, Q^2)f(x', k_T^2). \quad (1)$$

The dominant diagram for the $F_L$ is shown on Fig.1, where $x'$, $k_T$ are the longitudinal fraction of proton’s momentum and the transverse momentum of the gluon $g$ respectively. The function $F_L^{g\gamma}(x', k_T^2, Q^2)$ can be regarded as the longitudinal structure function of the off–shell gluon with...
virtually $k_T^2$, and $f(x, k_T^2)$ is the unintegrated gluon distribution which in principle should be obtained from the BFKL equation [3, 4]. In formula (1) there is double convolution over the variables $x'$ and $k_T$, and therefore it is called the off–shell approach. On the other hand the well known renormalization group analysis is basically connected with collinear factorization formula,

$$F(x, Q^2) = \sum_i \int_x^1 \frac{dx'}{x'} C_i(x/x', \alpha_s, Q^2/\mu^2)x'h_i(x', \mu^2).$$  

(2)

The $h_i(x', \mu^2)$ are the integrated parton distribution functions which satisfy conventional Altarelli-Parisi equations [10] and $C_i$ are the coefficient functions which can be calculated perturbatively. The summation is performed over different species of partons and $\mu^2$ is a factorization scale which usually is taken to be $Q^2$ in case of deep inelastic scattering.

It has been shown in [3, 11] that the leading twist part of the $k_T$ factorization formula can be recast into the collinear form (2) where all small $x'$ $(x/x')$ effects are resummed to all orders in $\alpha_s \ln(x'/x)$ ($\alpha_s \ln(x/x')$) in the splitting (and coefficient) functions. For instance the moments of the coefficient functions $C_L(\omega, \alpha_s)$

$$\bar{C}_L(\omega, Q^2, \alpha_s(Q^2)) \equiv \int_0^1 \frac{dz}{z} \omega^2 C_L(z, Q^2, \alpha_s(Q^2))$$  

(3)

are related to the double moment function $\bar{F}_L^{\gamma g}(\omega, \gamma)$ by

$$\bar{C}_L(\omega, Q^2, \alpha_s(Q^2)) = \gamma_{gg}(\alpha_s/\omega)\bar{F}_L^{\gamma g}(\omega, \gamma = \gamma_{gg}(\alpha_s/\omega))$$  

(4)

where $\bar{\alpha}_s = N_c\alpha_s/\pi$, and

$$\bar{F}_L^{\gamma g}(\omega, Q^2) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} d\gamma \bar{F}_L^{\gamma g}(\omega, \gamma) \left( \frac{Q^2}{k^2} \right)^\gamma$$  

$$\bar{F}_L^{\gamma g}(\omega, Q^2) = \int_0^1 \frac{dx}{x} x^\omega \bar{F}_L^{\gamma g}(x, Q^2).$$  

(5)

The quantity $\gamma_{gg}(\alpha_s/\omega)$ is the gluon anomalous dimension related to the appropriate splitting function by

$$\gamma_{gg}(\alpha_s/\omega) = \int_0^1 dzz^\omega P_{gg}(z, \alpha_s).$$  

(6)

Knowing the expansion of $\bar{F}_L^{\gamma g}$ in terms of the anomalous dimension and the expansion of $\gamma_{gg}$ in powers of $\alpha_s/\omega$, one can obtain the following formula for the coefficient function $C_L(z, \alpha_s)$ (in case of $F_L$),

$$C_L(z, \alpha_s) = \alpha_s[C_L^{(0)}(z) + \alpha_s C_L^{(1)}(z) + \alpha_s^2 \ln(\frac{1}{z}) C_L^{(2)}(z) + \alpha_s^3 \ln^2(\frac{1}{z}) C_L^{(3)}(z) + \ldots]$$  

(7)

where as $z \to 0$, $C_L^{(0)}(z) \to 0$ $C_L^{(i)}(z) \to \text{const} \neq 0$, $i \geq 1$.

If instead of the BFKL solution in (1), we take

$$f(x, k_T^2) = \left. \frac{\partial xg(x, Q^2)}{\partial \ln Q^2} \right|_{Q^2=k_T^2}$$  

(8)

2
where \( xg(x, Q^2) \) is a solution of the Altarelli-Parisi equations without small \( x \) corrections in \( P_{gg} \) but preserving the original phase space (no \( k_T \) ordering), then only the first three non-trivial corrections in small \( x \) are included in the expansion for \( C_L \). This approximation seems to be reliable, but not of course at very low values of \( x \) where the BFKL effects could play an important role. In our calculations we consider \( x > 10^{-4} \).

The photon - gluon fusion model for \( F_L \).

Here we show how after an expanding the integrand in the off–shell formula (1) around \( k_T^2 = 0 \) one can obtain the on–shell approximation [12] with additional power corrections. If we use the \( k_T \) factorization theorem to calculate \( F_L \), then we obtain the following form for \( F_L \) at low \( x \) and large \( Q^2 \):

\[
F_L(x, Q^2) = 2 \frac{Q^4}{\pi^2} \sum_q e_q^2 I_q(x, Q^2) 
\]  

(9)

where

\[
I_q(x, Q^2) = \int \frac{dk_T^2}{k_T^4} \int_0^1 d\beta \int d^2\kappa_T^2 \alpha_s(Q^2) \beta^2 (1 - \beta)^2 \frac{1}{2} \left( \frac{1}{D_{1q}} - \frac{1}{D_{2q}} \right)^2 f(x', k_T^2) 
\]  

(10)

and

\[
D_{1q} = \kappa_T^2 + \beta(1 - \beta)Q^2 + m_q^2 \\
D_{2q} = (\kappa_T - k_T)^2 + \beta(1 - \beta)Q^2 + m_q^2 
\]  

(11)

The transverse momentum \( \kappa_T' \) and the variable \( x' \) are defined as follows:

\[
\kappa_T' = \kappa_T - (1 - \beta)k_T \\
x' = x \left( 1 + \frac{\kappa_T^2 + m_q^2}{\beta(1 - \beta)Q^2 + k_T^2} \right) 
\]  

(12)

(13)

The integration over \( x' \) is implicit in (10). One can make it explicit by changing the variables from \( \kappa_T^2 \) to \( x' \) using the above formula (13). The integration limits in (10) are additionally constrained by the condition:

\[
x' < 1 
\]  

(14)

The condition \( x' > x \) demanded by the integration limit is automatically satisfied, see (13). In the on–shell approximation of the formulae (1,13) one expands integrands of the corresponding integrals around \( k_T = 0 \) and retains the leading term. The integrals \( I_q \) can then be expressed in terms of the conventional integrated gluon distribution \( g(y, Q^2) \):

\[
I_q(x, Q^2) = \pi \int_0^1 d\beta \int d\kappa_T^2 \alpha_s(Q^2) \beta^2 (1 - \beta)^2 \frac{\kappa_T^2}{D_q} yg(y, Q^2) 
\]  

(15)
where now

\[ y = x \left( 1 + \frac{\kappa_T^2 + m_q^2}{\beta(1 - \beta) Q^2} \right) \]  \hspace{1cm} (16)

and

\[ D_q = \kappa_T^2 + \beta(1 - \beta) Q^2 + m_q^2. \]  \hspace{1cm} (17)

In order to extrapolate formula (9) (with \( I_q \) given by (10) or (15)) to the low \( Q^2 \) region we have to freeze the evolution of \( g(y, Q^2) \) and the argument of \( \alpha_s(Q^2) \). To do this we substitute \( Q^2 + 4m_q^2 \) instead of \( Q^2 \) as the argument of \( \alpha_s \) and of \( yg(y, Q^2) \).

This substitution may be justified by analyticity arguments i.e. we want \( I_q \) to have threshold singularities for \( Q^2 < -4m_q^2 \). Moreover, for heavy quarks and for moderately large values of \( Q^2 \), it is the heavy mass squared and not \( Q^2 \) which should be taken as the scale of \( \alpha_s \) and of the parton distributions. It should be noted that after these modifications the structure function \( F_L \) can be continued down to \( Q^2 = 0 \), respecting the kinematical constraint \( F_L \to Q^4 \) as \( Q^2 \to 0 \). The formula (17) can be rearranged to give the Altarelli-Martinelli [13] integral over \( y \) but with the modified integration limits and additional power corrections. First we use \( y \) and \( \beta \) as the integration variables instead of \( \kappa_T^2 \) and \( \beta \). From (17) we find

\[ \kappa_T^2 = \beta(1 - \beta) Q^2 \left( \frac{y}{x} - 1 \right) - m_q^2, \]  \hspace{1cm} (18)

\[ D_q = \beta(1 - \beta) Q^2 \frac{y}{x}. \]  \hspace{1cm} (19)

From eq. (18) we can deduce various integration limits since

\[ \beta(1 - \beta) Q^2 \left( \frac{y}{x} - 1 \right) - m_q^2 > 0 \]  \hspace{1cm} (20)

and since \( 1 > \beta > 0 \),

\[ \frac{1}{4} > \beta(1 - \beta) > \frac{m_q^2}{Q^2 \left( \frac{y}{x} - 1 \right)}. \]  \hspace{1cm} (21)

From inequality (21) we obtain the lower limit for integration over \( y \)

\[ y > x \left( 1 + \frac{4m_q^2}{Q^2} \right) \]  \hspace{1cm} (22)

and of course,

\[ \beta(1 - \beta) > \frac{m_q^2}{Q^2 \left( \frac{y}{x} - 1 \right)}. \]  \hspace{1cm} (23)

It is convenient to make another change of the integration variables:

\[ \beta = \frac{1}{2} + \lambda. \]  \hspace{1cm} (24)

The inequality (21) gives the following limits for the variable \( \lambda \)

\[ -\sqrt{\frac{1}{4} - \frac{m_q^2}{Q^2 \left( \frac{y}{x} - 1 \right)}} < \lambda < \sqrt{\frac{1}{4} - \frac{m_q^2}{Q^2 \left( \frac{y}{x} - 1 \right)}}. \]  \hspace{1cm} (25)
Finally we get the following representation for $F_L$:

$$F_L = 2 \sum_q e_q^2 (J_q^{(1)} - 2 \frac{m_q^2}{Q^2} J_q^{(2)})$$  \hspace{1cm} (26)$$

where

$$J_q^{(1)} = \frac{\alpha_s}{\pi} \int_{x_q}^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 \left( 1 - \frac{x}{y} \right) \sqrt{1 - \frac{4m_q^2x}{Q^2(y-x)}} y g(y, Q^2)$$  \hspace{1cm} (27)$$

$$J_q^{(2)} = \frac{\alpha_s}{\pi} \int_{x_q}^1 \frac{dy}{y} \left( \frac{x}{y} \right)^3 \ln \left( \frac{1 + \sqrt{1 - \frac{4m_q^2x}{Q^2(y-x)}}}{1 - \sqrt{1 - \frac{4m_q^2x}{Q^2(y-x)}}} \right) y g(y, Q^2)$$  \hspace{1cm} (28)$$

and

$$\bar{x}_q \equiv x \left( 1 + \frac{4m_q^2}{Q^2} \right).$$  \hspace{1cm} (29)$$

It should be noted that expressions (26,27,28) reduce to the following formula

$$F_L(x, Q^2) = \frac{\alpha_s(Q^2)}{\pi} \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 \left( 1 - \frac{x}{y} \right) y g(y, Q^2)$$  \hspace{1cm} (30)$$

given by [13] in the $Q^2 \to \infty$ limit. The expression $\frac{m_q^2}{Q^2} J_q^{(2)}$ represents the power correction enhanced by the logarithmic factor $\ln Q^2/m_q^2$. The modification of the lower limit becomes obvious, if we consider the energy–momentum conservation constraint for the subprocess $\gamma g \to q \bar{q}$.

**A phenomenological model for the higher–twist contribution to $F_L$**

The kinematical constraint restricts the behaviour of $F_L$ in the region of small values of $Q^2$. Analysing the longitudinal cross section for deep inelastic scattering one concludes that in order to remove the possible singularities of this observable at $Q^2 \to 0$, (i.e. $\sigma_L \sim F_L/Q^2$), $F_L$ should behave as $Q^4$. This implies that $\sigma_L \sim Q^2$, so it vanishes in the real photoproduction limit as expected. Phenomenological analysis of the quantity $R$ in the region of moderately large $Q^2$ ($Q^2 > 1 GeV^2$) implies that the higher twist effects (i.e. terms that behave like $1/Q^2$ as $Q^2 \to \infty$) should play an important role [13].

In this paragraph we introduce a phenomenological model for the higher twist term which satisfies the constraint $F_L \sim Q^4$ as $Q^2 \to 0$. The idea is to treat the contributions from low and high quark momenta in a different way. We divide the integration over $\kappa_T^2$ (in (14) for the off–shell model (or in (27),(28) for the on–shell model ) into two parts – the region $0 < \kappa_T^2 < \kappa_{WT}^2$,...
and $\kappa_T^2 > \kappa_{0T}^2$ – where $\kappa_{0T}^2$ is an arbitrary cut–off, chosen to be of order $1GeV^2$. The region of high momenta is treated in the usual way (using off–shell [9,10] or on–shell [26,27,28] formula) whereas for low momenta we assume the existence of an effective coupling constant:

$$\alpha_s(Q^2)zg(z,Q^2) \to A.$$  

(31)

The constant $A$ is not a free parameter. We can estimate it from $F_2$ assuming that the non-perturbative contribution to $F_2$ in the scaling region also comes from the region of low values of $\kappa_T^2$ and is controlled by the same parameter. The contribution to $F_L$ from the region $0 < \kappa_T^2 < \kappa_{0T}^2$ is calculated using on–shell formula, making the substitution [31] and integrating over $\kappa_T^2$,

$$\Delta F_L(Q^2) = \frac{2A}{\pi} Q^4 \sum_q e_q^2 \int_0^1 d\beta \beta^2 (1-\beta)^2 \left\{ \frac{1}{6 \beta (1-\beta)Q^2 + m_q^2} \right. \left. \frac{1}{1 - \frac{\beta (1-\beta)Q^2 + 3m_0^2}{\beta (1-\beta)Q^2 + m_q^2 + \kappa_{0T}^2}} \right\}.$$  

(32)

This formula vanishes like $1/Q^2$ for large values of $Q^2$, therefore it has the form of a higher twist term. It also obeys the kinematical constraint for the structure function $F_L$, i.e. vanishes like $Q^4$ for $Q^2 \to 0$.

To evaluate the constant $A$ we consider the corresponding formula for $F_T$,

$$F_T(x,Q^2) = 2 \sum_q e_q^2 Q^2 \int_0^1 d\beta \int d^2 \kappa_T' \alpha_s(\kappa_T^2 + m_0^2) \times$$

$$\left\{ \left[ \beta^2 + (1-\beta)^2 \right] \left[ \frac{\kappa_T^2}{D_{1q}^2} - \frac{\kappa_T \cdot (\kappa_T - \kappa_T')}{D_{1q} D_{2q}} \right] \right\} f(x',k_T'^2)$$  

(33)

where $\kappa_T'$, $D_{1q}$ and $D_{2q}$ are defined by eqs. [11, 12]. If we rewrite the integrands to be symmetric in variables $\kappa_T$ and $\kappa_T - \kappa_T'$ we obtain,

$$\left[ \frac{\kappa_T^2}{D_{1q}^2} - \frac{\kappa_T \cdot (\kappa_T - \kappa_T')}{D_{1q} D_{2q}} \right] = \frac{1}{2} \left[ \frac{\kappa_T}{D_{1q}} - \frac{\kappa_T - \kappa_T'}{D_{2q}} \right]^2$$

$$m_q^2 \left[ \frac{1}{D_{1q}^2} - \frac{1}{D_{1q} D_{2q}} \right] = \frac{m_q^2}{2} \left[ \frac{1}{D_{1q}^2} - \frac{1}{D_{2q}^2} \right]^2$$  

(34)

If we now expand the right hand side of the eq.[34] in powers of $k_T^2/Q^2$ (on–shell approximation) we obtain the following formula,

$$\left[ \frac{\kappa_T}{D_{1q}} - \frac{\kappa_T - \kappa_T'}{D_{2q}} \right]^2 \simeq \left[ \frac{k_T}{D_q} - \frac{2k_T' (\kappa_T' \cdot \kappa_T')}{D_q} \right]^2 =$$

$$= k_T^2 \left( \frac{1}{D_q^2} - \frac{4 \cos^2 \phi k_T'^2}{D_q^3} + \frac{4 \cos^2 \phi k_T'^2}{D_q^4} \right)$$  

(35)
where we have dropped terms containing odd powers of $\cos \phi$ which give vanishing contribution after performing integration over $d\phi$. Then $F_T$ takes the form

$$F_T(x, Q^2) = 2 A \sum_q e_q^2 \frac{Q^2}{4\pi} \int_0^1 d\beta \int_0^{\kappa_0^2} d\kappa_T^2 \times \left\{ \frac{1}{2} [\beta^2 + (1 - \beta)^2] \left( \frac{1}{D_q^2} - \frac{2 \kappa_T^2}{D_q^4} + \frac{2 \kappa_T^4}{D_q^4} \right) + \frac{m_q^2 \kappa_T^2}{D_q^4} \right\}. \quad (36)$$

Performing the integration over $\kappa_T^2$, substituting (31) and taking the limit $Q^2 \to \infty$, we get the following form for the formula for the structure function $F_L$,

$$A = \frac{2 \pi F_T^{bg}}{\sum_q e_q^2 \left\{ -\frac{2}{3} \ln \frac{m_q^2}{m_q^2 + \kappa_0^2} + \frac{1}{3} \frac{\kappa_0^2}{m_q^2 + \kappa_0^2} \right\}}. \quad (37)$$

The value for $F_T^{bg}$ is taken from the phenomenological model for $F_T$ [4] where $F_T$ is assumed to have the form,

$$F_2 \simeq F_T = F_T^{bg} + F_T^{BFL} \quad (38)$$

where $F_T^{bg}$ is the background originating from non-perturbative region and its value is fixed by the data to be around 0.4. Assuming $\kappa_0^2 = 1 GeV^2$ one obtains $A \simeq 1.96$.

### Numerical results

In this section we present the numerical analysis for the structure function $F_L$. We shall also determine the ratio $R$ and compare it with SLAC parametrisation [13]. We include $u, d, s, c$ quark contributions, with quark masses $0.35, 0.35, 0.5, 1.5 GeV$ respectively. We have used both GRV [12] and MRS(A) [17] parton distributions. In GRV the LO gluons and coupling constant were used. The LO coupling constant has the following form,

$$\frac{\alpha_s(Q^2)}{2\pi} = \frac{2}{\beta_0 \ln \frac{Q^2}{\Lambda^2}} \quad (39)$$

where $\beta_0 = 11 - 2N_f/3$, $N_f$ is the number of flavours, and the value of the constant $\Lambda$ depends on number of active flavours

$$\Lambda_0^{(3,4)} = 0.232, 0.200 \ GeV. \quad (40)$$

We include the charm quark production threshold i.e. if $Q^2 > 4m_c^2$ then $N_f = 4$, otherwise $N_f = 3$.

In order to extrapolate the structure function $F_L$ to low $Q^2$ we use the method of freezing the argument of the coupling and of the gluon distributions as explained before. For the off–shell formula we have used the prescription proposed in [18]. In order to regularise the integration over $k_T^2$ we have introduced the cut-off $k_0^2$. The contribution of integration from 0 to $k_0^2$
is calculated using the approximate formula (26,27,28) with the substitution \( yg(y, k^2_{0T}) \). The magnitude of the structure function \( F_L \) turns out to depend rather weakly on the chosen cut-off \( k^2_{0T} \). The dominant contribution to \( F_L \) is given by the photon-gluon fusion subprocess \( \gamma g \to q\bar{q} \) (so the gluon distribution is the most important at low \( x \)) but the contributions from quarks have also been included with the modification \( x \to x(1 + 4m^2_i/Q^2) \) for the lower limit of the integration over \( y \).

\[
\Delta F_L(x, Q^2) = \sum_i c_i^2 \frac{\alpha_s(Q^2 + 4m^2_i)}{\pi} \frac{4}{3} \int_{\bar{x}_q}^{1} \frac{dy}{y} \left[ q_i(y, Q^2 + 4m^2_i) + \bar{q}_i(y, Q^2 + 4m^2_i) \right] (41)
\]

where \( \bar{x}_q \) is given by (29). In Fig.2, we compare our results for massive and massless quarks. As can be observed the quark masses play an important role in the region of low \( Q^2 \). But even at moderate values of \( Q^2 \) the difference is visible. This is because the on–shell model (26,27,28) with masses contains the power correction term multiplied by the logarithmic factor, \( \sim m^2/Q^2 \ln(Q^2/m^2) \). Note also that the difference between the on–shell and off–shell calculations is very small.

In Fig.3 we present \( F_L \) as a function of \( x \), with and without higher twist corrections. We have chosen the cut-off parameter \( \kappa^2_{0T} \) to be 1.0 GeV\(^2\), but there is no significant dependence on this quantity (within the range of 0.8 < \( \kappa^2_{0T} < 2.0 \) GeV\(^2\)). Higher twist gives higher values for \( F_L \), especially in the region where \( x \) is not very small (\( x > 0.001 \)). On the other hand it gives lower values for small \( x \) than the model without higher twist. This is due to the fact that the contribution coming from low values of the quark momenta is now described by the soft pomeron exchange. The difference between these calculations decreases as \( Q^2 \) increases.

In Fig.4 we present the results for the ratio \( R(x, Q^2) = F_L(x, Q^2)/(F_2(x, Q^2) - F_L(x, Q^2)) \) together with SLAC [15] parametrisation. For both parton sets (GRV and MRS(A)) the structure function \( F_2 \) has been calculated from the VMD model [13]. We also observe the same effect as for the longitudinal structure function: the model with higher twist predicts bigger values for the ratio \( R \) at high \( x \).

For intermediate values of \( Q^2 \) this model coincides with SLAC parametrisation, whereas for lower values of \( Q^2 \) it vanishes quickly as \( Q^2 \to 0 \). It should be noted that the difference between the parametrisations in the low \( Q^2 \) region are not very large, they both decrease very fast with decreasing \( Q^2 \). This suggests that the ratio \( R \) and the structure function \( F_L \) are not very sensitive to the differences in parton parametrisations in this region. Because the SLAC model is not applicable in the low \( Q^2 \) region, using it as an estimate of the ratio \( R \) may cause significant errors.

**Summary**

We have analysed the proton structure function \( F_L \) in the low \( Q^2 \) and low \( x \) region. We have studied the off–shell formula which originates from the high energy factorization theorem and
we have proved that in the limit when $k_T^2 \to 0$ it reduces to the on–shell approximation. The numerical differences between these two approaches are not very significant. We have also studied the structure function $F_L$ in the very low $Q^2$ region, where we have proposed the method of continuing the formulae for the structure functions down to $Q^2 = 0$. The non-zero values of quark masses have strong consequences in the structure function behaviour. Both the off–shell and on–shell formulae contain power corrections connected with quark masses (i.e. terms like $m^2/Q^2 \ln(Q^2/m^2)$). We have also proposed a model for the higher twist term which is based on the assumption that the contribution of quarks having limited transverse momenta is dominated by soft pomeron exchange. This results in change of the $x$ dependence of the structure function $F_L$. We have also performed calculations for $R$. The model for the ratio $R$ enables us to estimate this quantity in the low $Q^2$ region. It is particularly important because the uncertainty in the ratio $R$ is the main source of systematic errors when extracting other structure functions from experimental data.

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Figure captions

Fig.1 The diagramatic representation of the $k_T$ factorization formula for $F_L$.

Fig.2 $F_L$ plotted as a function of $x$ for different values of $Q^2$. We use GRV partons \cite{16}. Solid line: off–shell approximation with $k_{0T}^2 = 0.23 GeV^2$, dashed line: off–shell approximation with $k_{0T}^2 = 1.0 GeV^2$, dotted line: on–shell approximation, dashed-dotted line on–shell approximation with massless quarks $u,d,s$.

Fig.3 $F_L$ as a function of $x$ for different values of $Q^2$ calculated in the on–shell approximation: comparison of higher twist effect, partons distributions are obtained using the GRV parametrisation. Solid and dotted lines correspond to the results without and with the higher twist term respectively.

Fig.4 The ratio $R$ plotted as a function of $Q^2$ for different values of $x$ with the higher twist effect included and compared with the SLAC parametrisation (thick solid lines). Solid and dashed
lines correspond to the off–shell and on–shell approximation with the GRV parametrisation of the parton distributions [16]. The dotted and dashed-dotted lines correspond to off–shell and on–shell approximation calculated using MRS(A) partons [17].
Fig. 1
Fig. 2: $F_L(x, Q^2)$

- $Q^2 = 2 \text{ GeV}^2$
- $Q^2 = 4.5 \text{ GeV}^2$
- $Q^2 = 8.5 \text{ GeV}^2$
- $Q^2 = 12 \text{ GeV}^2$
- $Q^2 = 15 \text{ GeV}^2$
- $Q^2 = 20 \text{ GeV}^2$

$x$-axis: $10^{-4}$ to $10^2$
Fig. 3  \( F_L(x, Q^2) \)

- \( Q^2 = 2 \text{ GeV}^2 \)
- \( Q^2 = 4.5 \text{ GeV}^2 \)
- \( Q^2 = 8.5 \text{ GeV}^2 \)
- \( Q^2 = 12 \text{ GeV}^2 \)
- \( Q^2 = 15 \text{ GeV}^2 \)
- \( Q^2 = 20 \text{ GeV}^2 \)
Fig. 4: $R(x,Q^2)$