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About the limit state of deformable bodies

S Senashov¹ and I Savostyanova²

¹Professor, Reshetnev Siberian State University of Science and Technology 660037, Krasnoyarsk, Russia
²Docent, Reshetnev Siberian State University of Science and Technology 660037, Krasnoyarsk, Russia
E-mail: sen@sibsau.ru

Abstract. The theory of limit state deals with statically determinate condition of solids. In this case the system is closed due to the limit conditions, such properties of matter as viscosity, elasticity, etc. cannot influence the limit state. In other words, being at the limit state the nature of the relationship between stress and strain has no effect on the limit state. The article discusses systems of equations which correspond to the classical theory of plasticity. It is assumed that the components of the velocity vector depend on two spatial coordinates only. The constructed system can be used to describe the torsion of a parallelepiped around the three orthogonal axes. For the constructed system of equations group point symmetries, conservation laws were found. It is shown that the system admits 8-dimensional Lie algebra.

1. Introduction

Some of the problems of the ideal plasticity theory are fairly well studied. These are the so-called statically determinate problems. Among these are the problems of the twisting of prismatic bars and the problems of the plane deformation state. They belong to a wide class of problems – the problems of the limit state of deformable bodies. The theory of the limit state is one of the fundamental sections of deformable solids mechanics [1]. The theory of the limit state deals with statically determinate state of solid bodies. In this case the system is closed due to the limit conditions, and such characteristics of the material as viscosity, elasticity etc. cannot influence the limit state. In other words when reaching the limit state, the nature of relations between strains and deformations does not influence the limit state. Some of these systems are described in [1].

In the proposed paper the system of equations of plasticity that describes the limit state is presented. This system can be used to govern the plastic flow around the three orthogonal axes.

2. Setting of a problem

Let us assume that $x = x_1, y = x_2, z = x_3$ – orthogonal coordinates system, $u, v, w$ – strain-rate vector components, $e_{ij}$- strain-rate tensor components, $\sigma_{ij}$- stress tensor components. The stress tensor components satisfy the equations of balance

$$\partial_j \sigma_{ij} = 0$$ (1)

Over the repeated indices summation is assumed. The stress deviator and the strain-rate tensor are coaxial

$$\sigma_{ij} - \delta_{ij} p = \lambda e_{ij}$$ (2)

where $\sigma_{ij}$- Kronecker symbol, $\lambda$ - any nonnegative function, $3p = \sigma_{ii}$.

System of equations (1)-(2) is closed by the von Mises yield criterion

$$(\sigma_{11} - p)^2 + (\sigma_{22} - p)^2 + (\sigma_{33} - p)^2 + 2(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) = 2k_0^2$$ (3)

It is known [1] that in case that the prismatic bar is twisted around axis oz then the strain-rate field is written as

$$u = -yz, v = xz, w = w(x, y)$$ (4)

When enlarging relations (1) we claim that
Let us build up a system of equation corresponding to this strain-rate field. As a result we obtain the following system that will be studied in the paper

\[
\frac{\partial}{\partial t} \tau_{1}^1 + \frac{\partial}{\partial z} \tau_{2}^z = \frac{\partial}{\partial y} p, \quad \frac{\partial}{\partial x} \tau_{1}^1 + \frac{\partial}{\partial y} \tau_{1}^y = \frac{\partial}{\partial z} p
\]

\[
(\tau_1^1)^2 + (\tau_2^z)^2 + (\tau_3^z)^2 = k_s^2,
\]

where \(\sigma_{12} = \tau_1^1, \sigma_{13} = \tau_2^z, \sigma_{23} = \tau_3^z\).

System of equations (6) can in particular be used to govern the twisting of an orthographical parallelepiped in its plastic state around the three axes. See Fig. 1.

Figure 1. The torsion parallelepiped around the three axes

Let us assume that the parallelepiped is twisted around axes ox, oy, oz with equal and opposite couples of forces with moments \(M_1, M_2, M_3\). Wherein there are some limit moments \(M_1', M_2', M_3'\), when the parallelepiped goes into its plastic state and starts twisting. From system (6) we can see that such a problem is statically determinate and can be used to determine the value of the limit moments according to the formulas

\[
M_1' = \iint (y\tau_2^z - z\tau_1^1) dy dz, \\
M_2' = \iint (z\tau_1^1 - x\tau_2^z) dx dz, \\
M_3' = \iint (x\tau_2^z - y\tau_1^1) dy dz
\]

Apart from the moments (7) hydrostatic pressure has effect on the body as well

\[
P|_{\Sigma} = P_0,
\]

where \(\Sigma\) - parallelepiped’s lateral surface.

Let us study some of the attributes of the system (6).

3. Characteristic surfaces of System (6)

System (6) contains the finite relation that relates values \(\tau_1^1, \tau_2^z, \tau_3^z\). Let us differentiate it with respect to \(x,y,z\) and as a result we obtain the system

\[
\frac{\partial}{\partial t} \tau_{1}^1 + \frac{\partial}{\partial z} \tau_{2}^z = \frac{\partial}{\partial y} p, \quad \frac{\partial}{\partial x} \tau_{1}^1 + \frac{\partial}{\partial y} \tau_{1}^y = \frac{\partial}{\partial z} p
\]

\[
(\tau_1^1)^2 + (\tau_2^z)^2 + (\tau_3^z)^2 = k_s^2,
\]

Let us write the characteristic surface equation of system of equations (8) as

\[
\psi = \psi(x,y,z) = 0
\]

The characteristic surfaces of system (8) are deduced from the determinant
\[
\begin{vmatrix}
\partial_x \psi & \partial_y \psi & \partial_z \psi & 0 \\
\partial_y \psi & \partial_z \psi & 0 & \partial_\tau \psi \\
\partial_z \psi & 0 & \partial_x \psi & \partial_\tau \psi \\
0 & \tau^1 & \tau^2 & \tau^3
\end{vmatrix} = 0.
\]

Note. It is not difficult to see that the last three equations of system (8) give the same rows in determinant (10) that is why they are omitted.

When expanding determinant (10) in the last row we obtain
\[
\tau^1 \partial_x \psi ((\partial_x \psi)^2 - (\partial_y \psi)^2 + (\partial_z \psi)^2) + \tau^2 \partial_y \psi ((\partial_x \psi)^2 - (\partial_y \psi)^2 - (\partial_z \psi)^2) + \tau^3 \partial_z \psi ((\partial_x \psi)^2 - (\partial_y \psi)^2 - (\partial_z \psi)^2) = 0.
\]

This equation can be written as
\[
\tau^1 n_3 (2n_3^2 - 1) + \tau^2 n_2 (2n_2^2 - 1) + \tau^3 n_1 (2n_1^2 - 1) = 0,
\]
where \(n_1 = \frac{\partial_x \psi}{\sqrt{\psi^2}}, n_2 = \frac{\partial_y \psi}{\sqrt{\psi^2}}, n_3 = \frac{\partial_z \psi}{\sqrt{\psi^2}}.\) One of the solutions of equation (11) that does not depend on values \(\tau^1, \tau^2, \tau^3\) is
\[
2n_i^2 = 1 \quad i = 1, 2, 3.
\]

It follows that the angle between the normal and the characteristic surface \(\psi(x,y,z) = 0\) and vector \(n\) is \(\mp \pi/4\). The set of the characteristic surface’s elements create opening of cone \(\pm \pi/4\) around the direction that is determined by the third root of equation (11) and depends on the stress condition.

4. Point symmetries of System of equations (6)

Because system (6) contains the finite relation then it is necessary to work with its conclusion, which is written as (9), where the following notation is introduced \(\partial_x \tau^1 = q_1, \partial_x \tau^2 = q_2, \partial_x \tau^3 = q_3, \partial_x \psi = q_1 \partial_x \psi, \partial_x \phi = q_2 \partial_x \phi, \partial_x \rho = q_3 \partial_x \rho, \partial_x \sigma = q_3 \partial_x \sigma.\)

\[
q_1 + q_2 = q_1, \quad q_1 + q_3 = q_2, \quad q_2 + q_3 = q_3,
\]
\[
\tau^1 q_1 + \tau^2 q_2 + \tau^3 q_3 = 0,
\]
\[
\tau^1 q_1 + \tau^2 q_2 + \tau^3 q_3 = 0,
\]
\[
\tau^1 q_2 + \tau^2 q_3 + \tau^3 q_3 = 0,
\]
\[
(\tau^1)^2 + (\tau^2)^2 + (\tau^3)^2 = k_2.
\]

Let us search the point symmetries regarding which the variety determined by system of equations (12) is invariant.

According to Lie [2] procedure we search for the admissible operator of the point symmetry in the form of

\[
X = \xi^j \frac{\partial}{\partial x_j} + \eta^j \frac{\partial}{\partial \tau^j}, \quad j = 1, 2, 3; i = 1, 2, 3, 4
\]

We extend operator (13) to the first-order derivatives using the formulas

\[
\hat{X} = X + \epsilon_k^l \frac{\partial}{\partial q_k^l},
\]

where \(\epsilon_k^l = (\eta^l) - q_1^l D_k(\xi_1) + q_2^l D_k(\xi_2) + q_3^l D_k(\xi_3).\) Operator (14) influences the system of equations (12) and we go to the manifold given by this system. As a result we obtain quadratic expressions with respect to «internal» endogenous variables \(q_1^2, q_2^2, q_3^2.\) «External» exogenous variables \(q_k^i, q_k^i.\) are determined from the system (12) via endogenous variables. In the obtained quadratic expressions we set the coefficients at the first and second degrees of the endogenous variables to zero. This allows us to obtain an overdetermined system of linear differential equations with respect to coefficients \(\xi^j, \eta^j.\) When solving this system we obtain the following result

**Theorem.** System of equations (6) admits Lie algebra \(l_u\) generated by the operators

\[
X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = x_1 \frac{\partial}{\partial x_1}, X_3 = x_1 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} + \tau^1 \frac{\partial}{\partial \tau^1} - \tau^2 \frac{\partial}{\partial \tau^2},
\]
\[
X_3 = x_2 \frac{\partial}{\partial x_3} - x_2 \frac{\partial}{\partial x_2} + \tau^3 \frac{\partial}{\partial \tau^3} - \tau^2 \frac{\partial}{\partial \tau^2}.
\]
The presence of operators $X_i, i = 1, 2, 3, 4$ means that system (6) allows $x, y, z$ - direction translation and dilatation $x' = x + a_i, i = 1, 2, 3$, $x' = x$; exp$a_4$, translation for hydrostatic pressure $p' = p + a_g$, and also rotation around the three coordinate axes, where $a_i$ are group parameters.

5. Invariant solutions for System of equations (6)

1. Let us build an invariant solution with respect to the subalgebra generated by the operator $X_3 = \frac{\partial}{\partial z}$. Such a solution should be determined in the form of

$$\tau^1 = \tau^1(x, y), p = p(x, y).$$

Let us insert (16) into system (6) and we obtain

$$\tau^1 = \tau^1(x, y), \quad \frac{\partial \tau^1}{\partial x} = \frac{\partial p}{\partial x}, \quad \frac{\partial \tau^2}{\partial x} + \frac{\partial \tau^3}{\partial y} = \frac{\partial p}{\partial y},$$

$$\tau^1 + \tau^2 + \tau^3 = 0,$$  \hspace{1cm} (16)

From (16) we can easily obtain

$$\tau^1 = f(x + y) + g(x - y), \quad p = f(x + y) - g(x - y),$$

where $f(x + y), g(x - y)$ - some arbitrary differentiable functions.

Now functions $\tau^1, \tau^3$ will be determined from the system of equations

$$\frac{\partial \tau^1}{\partial x} + \frac{\partial \tau^3}{\partial y} = 0, \quad (\tau^2)^2 + (\tau^3)^2 = k_s^2.$$  \hspace{1cm} (18)

System of equations (18) regulates the rotation of the bar in conditions when yield limit depends on variables $x, y$. Such problems are described in [1].

2. Let us build an invariant solution regarding the subalgebra generated by operator $X_{12}$. In the cylindrical coordinate system $r \theta z$ this operator is written as $X_{12} = \frac{\partial}{\partial \theta}$. In this case System (6) will be written as

$$\frac{\partial \tau}{\partial r} + \frac{\partial \tau_r}{\partial r} = r \frac{\partial p}{\partial r}, \quad \frac{\partial \tau_r}{\partial r} + r \frac{\partial \tau_r}{\partial \theta} + r \frac{\partial \tau_r}{\partial z} = \frac{\partial p}{\partial \theta},$$

$$\tau_{r \theta} + \tau_{z \theta} + \tau_{z \theta} = k_s^2.$$  \hspace{1cm} (19)

In this case the invariant solution is determined from the following system

$$\frac{\partial \tau_r}{\partial r} = r \frac{\partial p}{\partial r}, \quad \frac{\partial \tau_r}{\partial \theta} + r \frac{\partial \tau_r}{\partial \theta} + 2 \frac{\partial \tau_r}{\partial z} = 0,$$

$$\tau_{r \theta} + \tau_{z \theta} + \tau_{z \theta} = k_s^2.$$  \hspace{1cm} (20)

Then $\tau_{r z}$ is determined from the linear differential equation

$$r \frac{\partial \tau_{r z}}{\partial r} + r^2 \frac{\partial \tau_{r z}}{\partial z} - \tau_{r z} + r^2 \frac{\partial \tau_{r z}}{\partial z} = 0.$$

The rest of the functions are determined from system (2).

6. Conservation laws of system of equations (6)

Let us determine the conservation laws of system of equations (6) in the form of

$$\frac{\partial A}{\partial x}(\tau^1, \tau^2, \tau^3, p) + \frac{\partial B}{\partial y}(\tau^1, \tau^2, \tau^3, p) + \frac{\partial C}{\partial z}(\tau^1, \tau^2, \tau^3, p) = 0.$$

This system should be valid for any solution of System of equations (6). From this the relations follow

$$X_{12}A - \tau^2 \frac{\partial p}{\partial x} + \tau^2 \frac{\partial C}{\partial x} = 0, X_{13}B - \tau^3 \frac{\partial p}{\partial x} + \tau^3 \frac{\partial C}{\partial x} = 0, X_{12}B - \tau^2 \frac{\partial p}{\partial x} = 0, X_{12}C - \tau^2 \frac{\partial p}{\partial x} = 0,$$

$$X_{13}C + \tau^3 \frac{\partial p}{\partial x} = 0, X_{13}A - \tau^3 \frac{\partial p}{\partial x} = 0.$$

Where $X_{12} = -\tau^2 \frac{\partial}{\partial r} - \tau^2 \frac{\partial}{\partial \theta}, X_{13} = -\tau^3 \frac{\partial}{\partial r} + \tau\frac{\partial}{\partial \theta}$.

Let us show that these equations are simultaneous. Let $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$ then one of the solutions of this system is an infinitely series of the conservation laws

$$\frac{\partial A(S)}{\partial x} + \frac{\partial B(S)}{\partial y} + \frac{\partial C(S)}{\partial z} = 0,$$

where $S = (\tau^1)^2 + (\tau^2)^2 + (\tau^3)^2$, $A(S), B(S), C(S)$ some arbitrary differentiable functions. Conservation laws can be used to solve a number of boundary value problems. For the plasticity equations we did this in the papers [3 – 5].

Note. Are there any other laws? It is not known but in the author’s opinion there are no other conservation laws.

7. About deformation fields of velocities of System (6).

System (6) allows us to find the stress condition. Let us assume that it is known. Then in order to find the three velocity vector components we have the three equations
\[ 2e_{12} = \partial_y u + \partial_x v, 2e_{13} = \partial_x u + \partial_y w, 2e_{23} = \partial_z u + \partial_y v, \]

where
\[ 2e_{12} = \partial_y u + \partial_x v, 2e_{13} = \partial_x u + \partial_y w, 2e_{23} = \partial_z u + \partial_y v. \]

Let us show that equations (21) can be solved in terms of the tensor component of the deformation velocities. As it is known, the deformation velocities tensor components apart from equations (21) must satisfy the compatibility equations as well. By virtue of relations (22) and (5) only six of them remain
\[
\begin{align*}
\partial^2_{xx} e_{12} &= 0, \partial^2_{yy} e_{13} = 0, \partial^2_{zz} e_{23} = 0, \\
\partial_x (\partial_x e_{23} - \partial_y e_{12} - \partial_y e_{13}) &= 0, \\
\partial_y (\partial_y e_{13} - \partial_x e_{12} - \partial_x e_{23}) &= 0, \\
\partial_z (\partial_z e_{12} - \partial_y e_{13} - \partial_x e_{23}) &= 0.
\end{align*}
\]

**Theorem.** The compatibility equations of the deformation velocities are satisfied identically.

In this case from (21) we have
\[
\begin{align*}
(\tau^1)^2 (e^1_{12} + e^1_{13} + e^1_{23}) &= k_z^2 e^1_{23}, \\
(\tau^2)^2 (e^2_{12} + e^2_{13} + e^2_{23}) &= k_z^2 e^2_{23}, \\
(\tau^3)^2 (e^3_{12} + e^3_{13} + e^3_{23}) &= k_z^2 e^3_{23}.
\end{align*}
\]

System of equations (24) is a system of homogeneous linear equations for variables \( e^1_{23}, e^2_{13}, e^3_{23} \). Its determinant is written as
\[
\begin{vmatrix}
(\tau^1)^2 - k_z^2 & (\tau^1)^2 & (\tau^1)^2 \\
(\tau^2)^2 & (\tau^2)^2 - k_z^2 & (\tau^2)^2 \\
(\tau^3)^2 & (\tau^3)^2 & (\tau^3)^2 - k_z^2
\end{vmatrix}
\]

This determinant is equal to zero because the sum of all the rows is equal to zero. This means that system (24) has only two independent equations for the three variables \( e^1_{23}, e^2_{13}, e^3_{23} \). For example value \( e^2_{23} \) can be selected randomly, that is why for the given stress condition found from system (6), the field of velocities is determined with a functional arbitrary rule.

**8. Conclusion**

The research performed has shown that the new system of equations describing the limiting state can be used to regulate the plastic flow around the three orthogonal axes. The system admits the eight-dimensional Lie algebra and an infinitely series of the conservation laws that allow building invariant solutions. The field of velocities for the given stress condition is determined with a functional arbitrary rule.

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