Superconvergent HDG methods for Maxwell’s equations via the $M$-decomposition

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May 20, 2019

Abstract

The concept of the $M$-decomposition was introduced by Cockburn et al. in Math. Comp. vol. 86 (2017), pp. 1609-1641 to provide criteria to guarantee optimal convergence rates for the Hybridizable Discontinuous Galerkin (HDG) method for coercive elliptic problems. In that paper they systematically constructed superconvergent hybridizable discontinuous Galerkin (HDG) methods to approximate the solutions of elliptic PDEs on unstructured meshes. In this paper, we use the $M$-decomposition to construct HDG methods for the Maxwell’s equations on unstructured meshes in two dimension. In particular, we show the any choice of spaces having an $M$-decomposition, together with sufficiently rich auxiliary spaces, has an optimal error estimate and superconvergence even though the problem is not in general coercive. Unlike the elliptic case, we obtain a superconvergent rate for the curl of the solution, not the solution, and this is confirmed by our numerical experiments.

1 Introduction

A large number of computational techniques have been developed for solving Maxwell’s equations in both the frequency and time domains. In the frequency domain, and in the presence of inhomogeneous penetrable media, the finite element method is often used. It has an additional advantage compared to finite differences in that it can handle complex geometries.

Methods using $H(\text{curl}; \Omega)$-conforming edge elements have been widely studied, see for example [32, 37, 38, 40, 41, 46]. The implementation of the conforming method, particularly higher order elements, is complicated. Hence, non-conforming methods provide an interesting alternative for this kind of problem that may also be attractive for nonlinear problems. In particular, Discontinuous Galerkin (DG) methods have been used to approximate the solution of the Maxwell’s equations for a long time. The first DG method for solving Maxwell’s equations with high frequency was analyzed in [43]. A local discontinuous Galerkin (LDG) scheme was proposed for the time-harmonic Maxwell’s equations with low frequency was studied in [44] (see also [34] for this problem using mixed DG methods). These methods tend to have many more degrees of freedom than conforming methods so it is interesting to consider hybridizable methods.

This paper is concerned with developing a class of methods for Maxwell’s equations in 2D. Obviously Maxwell’s equations are usually studied in three dimensions, but if the domain and data functions are translation invariant in one direction, the full problem can be decoupled into a pair

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of problems posed in two dimensions. To see how this is possible, consider the usual time harmonic Maxwell system for the electric field $\mathbf{E}$ (a complex valued vector function):

$$\text{curl} \mu_r^{-1} \text{curl} \mathbf{E} - \kappa^2 \varepsilon_r \mathbf{E} = \mathbf{F}.$$ 

Here $\mu_r$ is the relative magnetic permeability, $\kappa > 0$ is the wave number, $\varepsilon_r$ is the relative electric permittivity which may be complex valued. In addition $\mathbf{F} = ik\varepsilon_0 \mathbf{j}$, where $\mathbf{j}$ is the given current density and $\varepsilon_0$ is the permittivity of vacuum.

If $\varepsilon_r$, $\mu_r$ and $\mathbf{F}$ are independent of $x_3$, and if we seek a solution $\mathbf{E}$ that is also independent of $x_3$ we obtain a simpler partial differential equation for $u = (E_1, E_2)^T$ given in (1.2a) below. In addition, a Helmholtz equation is obtained for $E_3$ but is not the subject of this paper (see [25,31]).

To define the problem for $u$ we need some notation. Because we are now working in two dimensions the curl operator can be defined in two ways depending on whether its argument is a scalar or a vector. We therefore introduce the following standard definitions where $\mathbf{v}$ is a smooth vector function and $p$ is a smooth scalar function:

$$\nabla \times \mathbf{v} = (-\partial_y, \partial_x)^T \mathbf{v}, \quad \nabla \times p = (\partial_y, -\partial_x)^T p,$$  \hspace{1cm} (1.1)

Similarly there are two definitions for the cross product again depending on the use of scalar or vector functions. If $\mathbf{n}$ is a unit vector (in practice the normal vector to a domain in $\mathbb{R}^2$), we define

$$\mathbf{n} \times \mathbf{v} = (-n_2, n_1)^T \mathbf{v}, \quad \mathbf{n} \times p = (n_2, -n_1)^T p,$$

We can now state the problem we shall study. Let $\Omega$ be a bounded simply-connected Lipschitz polygon in $\mathbb{R}^2$ with connected boundary $\partial \Omega$. Then a typical model problem for $u$ is to seek solutions of the following interior problem:

$$\nabla \times (\mu_r^{-1} \nabla \times u) - \kappa^2 \varepsilon_r u = f \quad \text{in } \Omega,$$  \hspace{1cm} (1.2a)

$$\mathbf{n} \times u = g \quad \text{on } \partial \Omega,$$  \hspace{1cm} (1.2b)

where the right hand side is $f = (F_1, F_2)^T$. To ensure the uniqueness of the solution to this problem (and hence existence via the Fredholm alternative), we assume that $\mu_r$ is real valued and positive. In addition, either $\Im(\varepsilon_r) > 0$, or $\Im(\varepsilon_r) = 0$ and $\kappa^2$ is not a Maxwell eigenvalue, where $\Im(\varepsilon_r)$ denotes the image part of $\varepsilon_r$. Note that using the vector form of the problem has been advocated for example in [3,5] and these papers motivate in part the current study.

In this paper we shall study hybridizable discontinuous Galerkin (HDG) methods applied to Maxwell’s equations (1.2). HDG methods for elliptic problems were first proposed in 2009 in [21] and an analysis using special projections was developed in [23]. HDG methods have several distinct advantages including: allowing static condensation and hence less global degree of freedoms, flexibility in meshing (inherited from DG methods), ease of design and implementation, and local conservation of physical quantities. As a result, HDG methods have been proposed for a large number of problems, see, e.g., [4,8,10,30,35,39,45].

An important property of HDG methods is the superconvergence of some quantities on unstructured meshes (after element by element post-processing). One way to guarantee the existence of an HDG projection and superconvergence is to ensure that the particular discretization spaces used in the HDG method satisfy an $M$-decomposition [20]. This reduces the problem of determining whether a choice of spaces will have good convergence properties to simply checking some inclusions and evaluating an index (see equation (3.4)). This method of analysis has been extended to other applications, see for example [15,19].
The HDG method has been applied to Maxwell’s equations in [42] but without an error analysis. Later on, an error analysis was provided in [11,12] for zero frequency and in [28,36] for impedance boundary conditions and high wave number. These papers did not use the $M$-decomposition and only considered simplicial elements.

The aim of this paper is to extend the concept of the $M$-decomposition to time-harmonic Maxwell’s equations with Dirichlet boundary condition in 2D. The main novelty of our paper is that we show that provided the HDG spaces satisfy the conditions for an $M$-decomposition, and certain auxiliary spaces contain constant piecewise polynomials, an optimal error estimate will hold as well as a super-convergence of the curl of the field (as was observed in [42]). Note that in our context superconvergence of the curl of the field is important because this implies that both the electric and magnetic fields can be approximated at the same rate. We then use the $M$-decomposition to exhibit finite element spaces with optimal convergence on triangles, parallelograms and squares. Our convergence theory is supported by numerical examples in each case.

The outline of the paper is as follows. In Section 2, we set some notation and give the HDG formulation of (1.2). In Section 3, we follow the seminal work [20] to introduce the concept of the $M$-decomposition for Maxwell’s equation. The error analysis is given in Section 4, we obtain optimal convergence rate for the electric field $u$ and superconvergence rate for $\nabla \times u$. The construction of example spaces and numerical experiments are provided to confirm our theoretical results in Section 5. We end with a conclusion.

2 The HDG method

We start by defining some notation. For any sufficiently smooth bounded domain $\Omega \subset \mathbb{R}^2$, let $H^m(\Omega)$ denote the usual $m$th-order Sobolev space of scalar functions on $\Omega$, and $\| \cdot \|_{m,\Omega}$, $| \cdot |_{m,\Omega}$ denote the corresponding norm and semi-norm. We use $(\cdot, \cdot)_{m,\Omega}$ to denote the complex inner product on $L^2(\Omega)$. Similarly, for the boundary $\partial \Omega$ of $\Omega$, we use $(\cdot, \cdot)_{\partial \Omega}$ to denote the $L^2$ inner product. Note that bold face fonts will be used for vector analogues of the Sobolev spaces along with vector-valued functions.

Recalling the definition of the curl operators in 2D in (1.1), for $\Omega \subset \mathbb{R}^2$ we next define

$$
\begin{align*}
H(\text{curl}; \Omega) &:= \{ u \in L^2(\Omega) : \nabla \times u \in L^2(\Omega) \}, & H_0(\text{curl}; \Omega) &:= \{ u \in H(\text{curl}; \Omega) : n \times u = 0 \text{ on } \partial \Omega \}, \\
H(\text{curl}; \Omega) &:= \{ u \in L^2(\Omega) : \nabla \times u \in L^2(\Omega) \}, & H_0(\text{curl}; \Omega) &:= \{ u \in H(\text{curl}; \Omega) : n \times u = 0 \text{ on } \partial \Omega \}, \\
H(\text{div}; \Omega) &:= \{ u \in L^2(\Omega) : \nabla \cdot u \in L^2(\Omega) \}, & H(\text{div}; \Omega) &:= \{ u \in H(\text{div}; \Omega) : \nabla \cdot u = 0 \}.
\end{align*}
$$

where $n$ is the unit outward normal vector on $\partial \Omega$.

Let $\mathcal{T}_h := \{ K \}$ denote a conforming mesh of $\Omega$, where $K$ is a Lipschitz polygonal element with finitely many edges. For each $K \in \mathcal{T}_h$, we let $h_K$ be the infimum of the diameters of circles containing $K$ and denote the mesh size $h := \max_{K \in \mathcal{T}_h} h_K$. We shall need more assumptions on the mesh to perform our analysis. These assumptions replace the usual “shape regularity” assumption but we delay a discussion of this point until Section 4.1. Let $\partial \mathcal{T}_h$ denote the set of edges $F \subset \partial K$ of the elements $K \in \mathcal{T}_h$ (i.e. edges of distinct triangles are counted separately) and let $\mathcal{E}_h$ denote the set of edges in the mesh $\mathcal{T}_h$. We denote by $h_F$ the length of the edge $F$. We abuse notation by using $\nabla \times$, $\nabla \cdot$ and $\nabla$ for broken curl, div and gradient operators with respect to mesh partition $\mathcal{T}_h$, respectively. To simplify the notation, we also define a function $h$ on $\mathcal{T}_h$, $\partial \mathcal{T}_h$ and $\mathcal{E}_h$ which dependanting on circumstances:

$$
h|_K = h_K, \quad \forall K \in \mathcal{T}_h, \quad h|_{\partial K} = h_K, \quad \forall K \in \mathcal{T}_h, \quad h|_F = h_F, \quad \forall F \in \mathcal{E}_h.$$

3
For \( u, v \in L^2(T_h) \) and \( \rho, \theta \in L^2(\partial T_h) \), we define the following inner product and norm
\[
(u, v)_{T_h} = \sum_{K \in T_h} (u, v)_K, \quad \|v\|^2_{T_h} = \sum_{K \in T_h} \|v\|^2_K, \quad (\rho, \theta)_{\partial T_h} = \sum_{K \in \partial T_h} (\rho, \theta)_{\partial K}, \quad \|\theta\|^2_{\partial T_h} = \sum_{K \in \partial T_h} \|\theta\|^2_{\partial K}.
\]

Given a choice of three finite dimensional polynomial spaces \( V(K) \subset H^1(K) \), \( W(K) \subset H(\text{curl}; K) \) and \( M(F) \subset L^2(F) \), where \( K \) is an arbitrary element in the mesh and \( F \) is an arbitrary edge, we define the global spaces by
\[
V_h := \{ v \in L^2(T_h) : w|_K \in V(K), K \in T_h \},
\]
\[
W_h := \{ w \in L^2(T_h) : w|_K \in W(K), K \in T_h \},
\]
\[
M_h := \{ \mu \in L^2(\mathcal{E}_h) : \mu|_F \in M(F), F \in \mathcal{E}_h \}.
\]

For later use, for any non-negative integer \( k \), let \( P_k(K) \) denote the standard space of polynomials in two variables have total degree less than or equal to \( k \).

Next, to give the HDG formulation of \([1.2]\), we need to rewrite it into a mixed form. Let \( q = \mu^{-1} \nabla \times u \) in \([1.2]\) to get the following mixed form
\[
\mu_r q - \nabla \times u = 0 \quad \text{in } \Omega, \tag{2.1a}
\]
\[
\nabla \times q - \kappa^2 \epsilon_r u = f \quad \text{in } \Omega, \tag{2.1b}
\]
\[
\nabla \times u = g \quad \text{on } \partial \Omega. \tag{2.1c}
\]

As usual for HDG, the upcoming method and analysis are based on the above mixed form.

For the convenience, we next give the following integration by parts formula for each curl operator in two-dimensions. The proof is followed by a standard density argument and hence we omit it here.

**Lemma 1.** Let \( K \) be an element in the mesh \( T_h \), and let \( u \in H(\text{curl}; K) \) and \( r \in H(\text{curl}; K) \). Then we have
\[
(\nabla \times u, r)_K = \langle \mathbf{n} \times u, r \rangle_{\partial K} + \langle u, \nabla \times r \rangle_K, \tag{2.2a}
\]
\[
(\nabla \times r, u)_K = \langle \mathbf{n} \times r, u \rangle_{\partial K} + \langle r, \nabla \times u \rangle_K, \tag{2.2b}
\]
where \( \mathbf{n} \) is the unit outward normal to \( K \).

We can now derive the HDG method for \([2.1]\) by multiplying each equation by the appropriate discrete test function, integrating element by element and use integration by parts (see \([2.2]\)) element by element in the usual way (c.f. \([21]\)). Summing the results over all elements, the HDG methods seeks an approximation to \( (q_h, u_h, \tilde{u}_h) \in V_h \times W_h \times M_h \), such that
\[
(\mu_r q_h, r_h)_{T_h} - (u_h, \nabla \times r_h)_{T_h} - \langle n \times \tilde{u}_h, r_h \rangle_{\partial T_h} = 0, \tag{2.3a}
\]
\[
(q_h, \nabla \times v_h)_{T_h} - \langle n \times \tilde{v}_h, v_h \rangle_{\partial T_h} - (\kappa^2 \epsilon_r u_h, v_h)_{T_h} = (f, v_h)_{T_h}, \tag{2.3b}
\]
\[
\langle n \times \tilde{q}_h, \tilde{v}_h \rangle_{\partial \Omega} = 0, \tag{2.3c}
\]
\[
\langle n \times \tilde{u}_h, n \times \tilde{v}_h \rangle_{\partial \Omega} = \langle g, n \times \tilde{v}_h \rangle_{\partial \Omega} \tag{2.3d}
\]
for all \( (r_h, v_h, \tilde{v}_h) \in V_h \times W_h \times M_h \), and the choice of \( n \times \tilde{q}_h \) follows the usual HDG pattern,
\[
n \times \tilde{q}_h = n \times q_h + \tau n \times (u_h - \tilde{u}_h) \times n, \tag{2.3e}
\]
where \( \tau \) is a penalization parameter taken to be positive and piecewise constant on the edges of the mesh (more details will be given later).
3 M-decompositions

In this section, we follow the seminal paper [20] to give the concept of the M-decomposition for Maxwell’s equation in two dimensions. To do this, we need an appropriate combined trace operator \( \text{tr} : V(K) \times W(K) \mapsto L^2(\partial K) \) defined as follows:

\[
\text{tr}(v, w) := (n \times v + n \times w \times n)|_{\partial K}.
\] (3.1)

**Definition 1.** We say that \( V(K) \times W(K) \) admits an M-decomposition when the following conditions are met:

\[
\begin{align*}
\n \times V(K) & \subset M(\partial K), & (3.2a) \\
\n \times W(K) \times \times n & \subset M(\partial K), & (3.2a)
\end{align*}
\]

and there exists a subspace \( \bar{V}(K) \times \bar{W}(K) \) of \( V(K) \times W(K) \) satisfying

\[
\begin{align*}
\n \times V(K) & \subset \bar{W}(K), & (3.2b) \\
\n \times W(K) & \subset \bar{V}(K), & (3.2b)
\end{align*}
\]

\[
\text{tr} : (\bar{V}^\perp(K) \times \bar{W}^\perp(K)) \mapsto M(\partial K)
\]

is an isomorphism,

\[
\begin{align*}
\n & \text{for any } \mu \in M(\partial K), \text{ if } n \times \mu = 0, \text{ it holds } \mu = 0. & (3.2d)
\end{align*}
\]

Here \( \bar{V}^\perp(K) \) and \( \bar{W}^\perp(K) \) are the \( L^2(K) \)-orthogonal complements of \( \bar{V}(K) \) in \( V(K) \), and \( \bar{W}(K) \) in \( W(K) \), respectively.

We notice that conditions (3.2a), (3.2b) and (3.2c) follow closely those in the M-decomposition in [20]. Condition (3.2d) is to ensure the uniqueness of \( \bar{u}_h \) determined by the HDG scheme [23].

We shall show that this implies optimal convergence of the associated HDG scheme (under some extra conditions on \( \bar{V}(K) \) and \( \bar{W}(K) \)) and verify that several families of elements satisfy the M-decomposition. To give some idea of the form such elements can take, we refer to Table 1 for examples of M-decompositions for Maxwell’s equations in two-dimensions when

\[
M(\partial K) = \{ \mu : \mu|_F = n \times p_0 \text{ for some } p_0 \in P_0(F) \} \text{ and for each edge } F \text{ of } K. \]

(3.3)

| \( K \) | \( V(K) \) | \( W(K) \) | \( V(K) \) | \( W(K) \) | \( V^\perp(K) \) | \( W^\perp(K) \) |
|---|---|---|---|---|---|---|
| Triangle | \( P_0(K) \) | \( P_0(K) \) | \( P_0(K) \) | \( 0 \) | \( P_0(K) \) | \( 0 \) |
| Square | \( P_0(K) \) | \( P_0(K) \) \oplus \{ \{ x_y \} \} | \( P_0(K) \) | \( 0 \) | \( P_0(K) \) \oplus \{ \{ x_y \} \} | \( 0 \) |

To verify that a given space \( V(K) \times W(K) \) admits an M-decomposition, we need to construct the associated spaces \( \bar{V}(K) \) and \( \bar{W}(K) \) in Definition 1. However, this is difficult, hence we need a simple way to verify a given space \( V(K) \times W(K) \) admits an M-decomposition. Moreover, if the given space \( V(K) \times W(K) \) does not admit an M-decomposition, we need to understand how to find to build a new space from \( V(K) \times W(K) \) that admits an M-decomposition. Following [20], we define the M-index as follows:

\[
I_M(V(K) \times W(K)) := \dim M(\partial K) - \dim \{ n \times v|_{\partial K} : v \in V(K), \nabla \times v = 0 \} - \dim \{ n \times w \times n|_{\partial K} : w \in W(K), \nabla \times w = 0 \}.
\] (3.4)

Now, we state the main result in this section, the proof is found in Section 3.2.
Theorem 1. The spaces $V(K)$ and $W(K)$ admit an $M$-decomposition if and only if

\begin{align}
\mathbf{n} \times V(K) & \subset M(\partial K), \quad \mathbf{n} \times W(K) \times \mathbf{n} \subset M(\partial K), \quad (3.5a) \\
\nabla \times V(K) & \subset W(K), \quad \nabla \times W(K) \subset V(K), \quad (3.5b) \\
I_M(V(K) \times W(K)) & = 0, \quad (3.5c) \\
\text{For any } \mathbf{\mu} \in M(\partial K), \text{ if } \mathbf{n} \times \mathbf{\mu} = 0, \text{ it holds } \mathbf{\mu} = 0. \quad (3.5d)
\end{align}

Theorem 1 provides a simply way to check if any given choice of spaces $V(K) \times W(K)$ admits an $M$-decomposition by just verifying some inclusions and by calculating a single number, namely, $I_M(V(K) \times W(K))$. Of course the associated spaces $\tilde{V}(K)$ and $\tilde{W}(K)$ are essential to define an HDG projection for the a priori error analysis of the method and can be found once $V(K)$ and $W(K)$ are known.

Moreover, the conditions in Theorem 1 are “if and only if”, which means that if $I_M(V(K) \times W(K))$ is not zero, we need to add to $W(K)$ a space $\delta W$ of dimension $I_M(V(K) \times W(K))$ to obtain a new space admitting an $M$-decomposition.

3.1 Properties of the $M$-decomposition

We now prove a sequence of lemmas that culminate in the proof of Theorem 1.

Lemma 2. Let $V(K) \times W(K)$ admit an $M$-decomposition with associated spaces $\tilde{V}(K)$ and $\tilde{W}(K)$. Then we have the following orthogonality property:

$$M(\partial K) = \gamma \tilde{V}^\perp(K) \oplus \gamma \tilde{W}^\perp(K),$$

(3.6)

where $\gamma \tilde{V}^\perp(K) := \{\mathbf{n} \times v^\perp|_{\partial K} : v^\perp \in \tilde{V}^\perp(K)\}$ and $\gamma \tilde{W}^\perp(K) := \{\mathbf{n} \times w^\perp \times \mathbf{n} : w^\perp \in \tilde{W}^\perp(K)\}$.

Proof. By condition (3.2c) and the definition of the combined trace operator $\text{tr}$ in (3.1), we have

$$M(\partial K) = \gamma \tilde{V}^\perp(K) + \gamma \tilde{W}^\perp(K).$$

Hence, we only need to show that the sum is $L^2(\partial K)$-orthogonal.

For all $v^\perp \in \tilde{V}^\perp(K)$, $w^\perp \in \tilde{W}^\perp(K)$. On each edge $F$ of $\partial K$, it holds

$$w^\perp = \mathbf{n} \times w^\perp \times \mathbf{n} + (w^\perp \cdot \mathbf{n}) \mathbf{n}. \quad (3.7)$$

Using equations (3.7) and (2.2b), we get

$$\langle \mathbf{n} \times v^\perp, \mathbf{n} \times w^\perp \times \mathbf{n} \rangle_{\partial K} = \langle \mathbf{n} \times v^\perp, w^\perp - (w^\perp \cdot \mathbf{n}) \mathbf{n} \rangle_{\partial K}$$

$$= \langle \mathbf{n} \times v^\perp, w^\perp \rangle_{\partial K}$$

$$= (\nabla \cdot v^\perp, w^\perp)_K - (v^\perp, \nabla \times w^\perp)_K. \quad (3.8)$$

By equation (3.2b), we have $\nabla \times v^\perp \in \tilde{W}(K)$ and $\nabla \times w^\perp \in \tilde{V}(K)$, hence

$$\langle \mathbf{n} \times v^\perp, \mathbf{n} \times w^\perp \times \mathbf{n} \rangle_{\partial K} = 0. \quad (3.9)$$

This finishes the proof.

Lemma 3. [Uniqueness of $\tilde{V}(K)$] If $V(K) \times W(K)$ admits an $M$-decomposition with associated spaces $\tilde{V}(K)$ and $\tilde{W}(K)$, then the subspace $\tilde{V}(K)$ is unique. Moreover,

$$\tilde{V}(K) = \nabla \times W(K).$$

(3.10)
Proof. By (3.2b) we have $\nabla \times \mathbf{W}(K) \subset \tilde{V}(K)$, therefore, we only have to prove $[\nabla \times \mathbf{W}(K)]^\perp \cap \tilde{V}(K) = \{0\}$. We take $v \in [\nabla \times \mathbf{W}(K)]^\perp \cap \tilde{V}(K)$, then it satisfies

$$
(v, \nabla \times \mathbf{w})_K = 0 \text{ for all } \mathbf{w} \in \mathbf{W}(K).
$$

(3.11)

By the argument in (3.8) and (3.11) we have

$$
\langle \mathbf{n} \times v, \mathbf{n} \times \mathbf{w} \times \mathbf{n} \rangle_{\partial K} = 0 \text{ for all } \mathbf{w} \in \tilde{W}^\perp(K).
$$

(3.12)

For $\mathbf{n} \times v \in \mathbf{M}(\partial K)$, since $V(K) \times \mathbf{W}(K)$ admits an $M$-decomposition with associated spaces $\tilde{V}(K)$ and $\tilde{W}(K)$, then by equation (3.6), there exist $v^\perp \in \tilde{V}(K)$ and $\mathbf{w} \in \tilde{W}(K)$ such that

$$
\mathbf{n} \times v = -\mathbf{n} \times v^\perp + \mathbf{n} \times \mathbf{w} \times \mathbf{n},
$$

$$
\langle \mathbf{n} \times v^\perp, \mathbf{n} \times \mathbf{w} \times \mathbf{n} \rangle_{\partial K} = 0.
$$

By equation (3.12), we have

$$
\|\mathbf{n} \times (v \pm v^\perp)\|_{\partial K}^2 = \langle \mathbf{n} \times (v \pm v^\perp), \mathbf{n} \times \mathbf{w} \times \mathbf{n} \rangle_{\partial K} = 0.
$$

Thus, on $\partial K$, we have $\mathbf{n} \times (v \pm v^\perp) = 0$, therefore

$$
v \pm v^\perp = 0 \text{ on } \partial K.
$$

(3.13)

Next, for all $\mathbf{w} \in \mathbf{W}(K)$, by equations (2.2a), (3.11) and (3.13), we get

$$
(\nabla \times v, \mathbf{w})_K = -\langle v, \mathbf{n} \times \mathbf{w} \rangle_{\partial K} + (v, \nabla \times \mathbf{w})_K
$$

$$
= -\langle v, \mathbf{n} \times \mathbf{w} \rangle_{\partial K}
$$

$$
= \langle v^\perp, \mathbf{n} \times \mathbf{w} \rangle_{\partial K}
$$

$$
= -(\nabla \times v^\perp, \mathbf{w})_K + (v^\perp, \nabla \times \mathbf{w})_K.
$$

(3.14)

By equation (3.2b), we know $\nabla \times \mathbf{w} \in \tilde{V}(K)$. Moreover, $v^\perp \in \tilde{V}(K)$, hence we have

$$
(v^\perp, \nabla \times \mathbf{w})_K = 0.
$$

(3.15)

We combine equations (3.14) and (3.15) to get

$$
(\nabla \times (v \pm v^\perp), \mathbf{w})_K = 0 \text{ for all } \mathbf{w} \in \mathbf{W}(K).
$$

(3.16)

Finally, since $v \in \tilde{V}(K)$ and $v^\perp \in \tilde{V}(K)$, then by the equation (3.2b) we obtain $\nabla \times (v \pm v^\perp) \in \tilde{W}(K) \subset \mathbf{W}(K)$. We take $\mathbf{w} = \nabla \times (v \pm v^\perp)$ in the equation (3.16) to get

$$
\nabla \times (v \pm v^\perp) = 0.
$$

(3.17)

Combine equations (3.13) and (3.17) to get

$$
v \pm v^\perp = 0 \text{ on } K.
$$

Since $v \in \tilde{V}(K)$ and $v^\perp \in \tilde{V}(K)$, it follows that $v = -v^\perp = 0$. This proves that $[\nabla \times \mathbf{W}(K)]^\perp \cap \tilde{V}(K) = \{0\}$, i.e., the space $\tilde{V}(K)$ is unique and $\tilde{V}(K) \subset [\nabla \times \mathbf{W}(K)]^\perp = \nabla \times \mathbf{W}(K)$. Furthermore, by the equation (3.2b), we have

$$
\tilde{V}(K) = \nabla \times \mathbf{W}(K).
$$

$\square$
In **Lemma 3**, we have proved that if \( V(K) \times \mathbf{W}(K) \) admits an \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \tilde{\mathbf{W}}(K) \), then \( \tilde{V}(K) \) is unique and \( \tilde{V}(K) = \nabla \times \mathbf{W}(K) \). However, we do not have similar characterization of the space \( \tilde{\mathbf{W}}(K) \), i.e., the space \( \tilde{\mathbf{W}}(K) \) is not unique. Hence, it is important to provide a “canonical” \( M \)-decomposition.

First, we define the following space:

\[
\mathbf{W}_0(K) = \{ \mathbf{w} \in \mathbf{W}(K) : \nabla \times \mathbf{w} = 0 \text{ in } K, \mathbf{n} \times \mathbf{w} \times \mathbf{n} = 0 \text{ on } \partial K \}. \tag{3.18}
\]

**Lemma 4.** [The canonical \( M \)-decomposition] Let \( V(K) \times \mathbf{W}(K) \) admit an \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \tilde{\mathbf{W}}(K) \), then \( V(K) \times \mathbf{W}(K) \) admits an \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \tilde{\mathbf{W}}_c(K) \), in which

\[
\tilde{V}(K) = \nabla \times \mathbf{W}(K), \quad \tilde{\mathbf{W}}_c(K) = \nabla \times V(K) \oplus \mathbf{W}_0(K). \tag{3.19}
\]

In this case, we say that \( V(K) \times \mathbf{W}(K) \) admits the canonical \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \tilde{\mathbf{W}}_c(K) \).

The proof of **Lemma 4** follows from **Lemmas 3** and 5 to 7.

Next, we define the following space:

\[
\mathbf{X}(K) = \{ \tilde{\mathbf{w}}^\perp - \Pi_0 \tilde{\mathbf{w}}^\perp : \tilde{\mathbf{w}}^\perp \in \tilde{\mathbf{W}}_c(K) \},
\]

where \( \Pi_0 \) is the \( L^2 \)-projection onto the space \( \mathbf{W}_0(K) \).

**Lemma 5.** If \( V(K) \times \mathbf{W}(K) \) admits an \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \tilde{\mathbf{W}}(K) \), then we have \( \tilde{\mathbf{W}}_c(K) = \nabla \times V(K) \oplus \mathbf{W}_0(K) \subset \mathbf{X}^\perp(K) \).

**Proof.** For any \( \mathbf{w} \in \nabla \times V(K) \oplus \mathbf{W}_0(K) \) and \( \tilde{\mathbf{w}}^\perp \in \tilde{\mathbf{W}}_c(K) \), then there exist \( v \in V(K) \) and \( \mathbf{r} \in \mathbf{W}_0(K) \) such that \( \mathbf{w} = \nabla \times v + \mathbf{r} \). Hence, we have

\[
(\mathbf{w}, \tilde{\mathbf{w}}^\perp - \Pi_0 \tilde{\mathbf{w}}^\perp)_K = (\nabla \times v, \tilde{\mathbf{w}}^\perp - \Pi_0 \tilde{\mathbf{w}}^\perp)_K + (\mathbf{r}, \tilde{\mathbf{w}}^\perp - \Pi_0 \tilde{\mathbf{w}}^\perp)_K
= (\nabla \times v, \tilde{\mathbf{w}}^\perp)_K - (\nabla \times v, \Pi_0 \tilde{\mathbf{w}}^\perp)_K + (\mathbf{r}, \tilde{\mathbf{w}}^\perp - \Pi_0 \tilde{\mathbf{w}}^\perp)_K
= (\nabla \times v, \tilde{\mathbf{w}}^\perp)_K - (\nabla \times v, \Pi_0 \tilde{\mathbf{w}}^\perp)_K
= -(\nabla \times v, \Pi_0 \tilde{\mathbf{w}}^\perp)_K, \tag{3.20}
\]

where the last equality follows from definition \([3.2a]\).

Next, we estimate \( (\nabla \times v, \Pi_0 \tilde{\mathbf{w}}^\perp)_K \), since \( \Pi_0 \tilde{\mathbf{w}}^\perp \in \mathbf{W}_0(K) \), by \([2.2a]\) to obtain

\[
(\nabla \times v, \Pi_0 \tilde{\mathbf{w}}^\perp)_K = (v, \nabla \times \Pi_0 \tilde{\mathbf{w}}^\perp)_K - (v, \mathbf{n} \times \Pi_0 \tilde{\mathbf{w}}^\perp)_{\partial K} = 0. \tag{3.21}
\]

Combining \([3.20]\) and \([3.21]\) show that

\[
(\mathbf{w}, \tilde{\mathbf{w}}^\perp - \Pi_0 \tilde{\mathbf{w}}^\perp)_K = 0.
\]

This proves \( \mathbf{w} \in \mathbf{X}^\perp(K) \), i.e., \( \nabla \times V(K) \oplus \mathbf{W}_0(K) \subset \mathbf{X}^\perp(K) \).

**Lemma 6.** If \( V(K) \times \mathbf{W}(K) \) admits an \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \tilde{\mathbf{W}}(K) \), then \( V(K) \times \mathbf{W}(K) \) admits an \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \mathbf{X}^\perp(K) \).
Proof. By the Definition 1, we need to check conditions (3.2a)-(3.2d). It is obvious to see that (3.2a) and (3.2d) hold. Hence, we only need to check conditions (3.2b) and (3.2c).

First, by the Lemma 5, we have \( \nabla \times V(K) \oplus W_0(K) \subset X^\perp(K) \), which implies \( \nabla \times V(K) \subset X^\perp(K) \). Moreover, it is obvious that \( \nabla \times W(K) \subset \tilde{V}(K) \). This proves condition (3.2b).

Next, we check condition (3.2c). By the definition of \( X(K) \), it is obvious from the definition of \( W_0(K) \) in (3.18) that

\[
\gamma X(K) = \gamma \tilde{W}^\perp(K). \tag{3.22}
\]

Since \( \Pi_0 \) is the \( L^2 \)-projection onto space \( W_0(K) \) and we already proved that \( W_0(K) \subset X^\perp(K) \) in Lemma 5, we can use the definition of \( X(K) \) to get

\[
\dim X(K) \leq \dim \tilde{W}^\perp(K). \tag{3.23}
\]

Since \( V(K) \times W(K) \) admits an \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \tilde{W}(K) \), then by (3.2c) it holds that: \( \text{tr} \left( \left( \tilde{V}^\perp(K) \times \tilde{W}^\perp(K) \right) \right) \rightarrow M(\partial K) \) is an isomorphism. Then by (3.22) and (3.23) it follows that \( \text{tr} \left( \left( \tilde{V}^\perp(K) \times X(K) \right) \right) \rightarrow M(\partial K) \) is an isomorphism. Since \( X(K) \) and \( X^\perp(K) \) are complete, then \( X(K) = [X^\perp(K)]^\perp \). Therefore, \( V(K) \times W(K) \) admit an \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \tilde{X}(K) \).

\( \square \)

Lemma 7. If \( V(K) \times W(K) \) admits an \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \tilde{W}(K) \), then we have \( X^\perp(K) \subset \tilde{W}_c(K) = \nabla \times V(K) \oplus W_0(K) \).

The proof of Lemma 7 is similar to that of Lemma 3 and hence we omit it here.

Lemma 8. Let \( V(K) \times W(K) \) admit the canonical \( M \)-decomposition with associated spaces \( \tilde{V}(K) \) and \( \tilde{W}_c(K) \), then there holds

\[
\dim \gamma \tilde{V}^\perp(K) = \dim \tilde{V}^\perp(K), \quad \dim \gamma \tilde{W}_c^\perp(K) = \dim \tilde{W}_c^\perp(K). \tag{3.24}
\]

Proof. Let \( v^\perp \in \tilde{V}^\perp(K) \), if \( v^\perp = 0 \) on \( K \), then obviously we have \( n \times v^\perp = 0 \) on \( \partial K \); if \( n \times v^\perp = 0 \) on \( \partial K \), then it holds \( v^\perp = 0 \) on \( \partial K \). For any \( w \in W(K) \), by condition (3.2b) we know \( \nabla \times w \in \tilde{V}^\perp(K) \), which implies \( (v^\perp, \nabla \times w)_{\partial K} = 0 \). Hence, by parts leads to

\[
0 = \langle n \times v^\perp, w \rangle_{\partial K} = (\nabla \times v^\perp, w)_K - (v^\perp, \nabla \times w)_K = (\nabla \times v^\perp, w)_K. \tag{3.25}
\]

We take \( w = \nabla \times v^\perp \) in (3.25) to get \( \nabla \times v^\perp = 0 \) on \( K \), combining this with \( v^\perp = 0 \) on \( \partial K \) imply \( v^\perp = 0 \) on \( K \).

Let \( w^\perp \in \tilde{W}_c^\perp(K) \), if \( w^\perp = 0 \) on \( K \), then we have \( n \times w^\perp \times n = 0 \) on \( \partial K \); if \( n \times w^\perp \times n = 0 \), for any \( v \in V(K) \), we have

\[
0 = \langle n \times w^\perp \times n, v \times n \rangle_{\partial K} = (w^\perp - (w^\perp \cdot n)n, v \times n)_{\partial K} = \langle w^\perp \cdot v \rangle_{\partial K} - (w^\perp, \nabla \times v)_K \tag{3.26}
\]

We take \( v = \nabla \times w^\perp \in V(K) \) in (3.26) to get \( \nabla \times w^\perp = 0 \) on \( K \), then combining this with \( n \times w^\perp \times n = 0 \) on \( \partial K \) to get \( w^\perp \in W_0(K) \). By (3.19), we have \( w^\perp \in \tilde{W}_c(K) \), remembering that \( w^\perp \in \tilde{W}_c^\perp(K) \), then \( w^\perp = 0 \).

\( \square \)
3.2 Proof of Theorem 1

Proof. The proof of only if: Let $V(K) \times W(K)$ admits an $M$-decomposition, then by the Definition 1 we know (3.5a), (3.5b) and (3.5d) hold. Moreover, Lemma 4 implies that $V(K) \times W(K)$ admits the canonical $M$-decomposition with the associated spaces $V(K)$ and $\tilde{W}_c(K)$. By (3.2c) and (3.24) we have

\[ \dim V(K) + \dim W(K) = \dim \tilde{V}(K) + \dim \tilde{W}_c(K) + \dim M(\partial K). \]  

(3.27)

This gives

\[ \dim M(\partial K) = \dim V(K) + \dim W(K) - \dim \tilde{V}(K) - \dim \tilde{W}_c(K). \]

Then we use (3.10) and (3.19) to get

\[ \dim M(\partial K) = \dim V(K) + \dim W(K) - \dim \nabla \times W(K) - \dim [\nabla \times V(K) \oplus W_0(K)] \\
= [\dim V(K) - \dim \nabla \times V(K)] + [\dim W(K) - \dim \nabla \times W(K) - \dim W_0(K)] \\
= \dim \{ n \times v \mid_{\partial K} : v \in V(K), \nabla \times v = 0 \} \\
+ \dim \{ n \times w \times n \mid_{\partial K} : w \in W(K), \nabla \times w = 0 \}, \]

this proves (3.5c).

The proof of if: If (3.5a), (3.5b), (3.5c) and (3.5d) hold, then we only need to prove (3.2b) and (3.2c). If we define $\tilde{V}(K)$ and $\tilde{W}(K)$ as

\[ \tilde{V}(K) = \nabla \times W(K), \]
\[ \tilde{W}(K) = \nabla \times V(K) \oplus W_0(K), \]

then (3.2b) holds.

Next, we prove that (3.2c) hold. Use $I_M(V(K) \times W(K)) = 0$ to get

\[ 0 = I_M(V(K) \times W(K)) \\
= \dim M(\partial K) - \dim \{ n \times v \mid_{\partial K} : v \in V(K), \nabla \times v = 0 \} \\
- \dim \{ n \times w \times n \mid_{\partial K} : w \in W(K), \nabla \times w = 0 \} \\
= \dim M(\partial K) - [\dim V(K) - \dim \nabla \times V(K)] - [\dim W(K) - \dim \nabla \times W(K) - \dim W_0(K)] \\
= \dim M(\partial K) - [\dim V(K) + \dim W(K) - \dim \nabla \times W(K) - \dim \nabla \times V(K) \oplus W_0(K)] \\
= \dim M(\partial K) + \dim \tilde{V}(K) + \dim \tilde{W}(K) - \dim V(K) - \dim W(K), \]

which combines with (3.24) to get (3.2c). This finishes our proof.

4 Error Analysis

In this section, we present our main result, an error analysis for the HDG approximation to Maxwell’s equations given by (2.3). To simplify the derivation we shall assume that $\mu_r$ and $\epsilon_r$ are constants. First we discuss the extra conditions on the spaces $V(K)$ and $W(K)$ needed for this analysis. These conditions arise because at this point each element $K \in T_h$ is a general polygon, yet we need certain properties for functions in these spaces (that hold for standard elements including triangles, parallelograms and squares that are considered later in this paper). For triangles these conditions follow if the mesh is assumed to be regular, and the spaces $V(K)$ and $W(K)$ are sufficiently rich. After this discussion, we consider an adjoint problem needed for the analysis and finally present the error analysis.
4.1 Additional assumptions on the approximation spaces

Throughout this section we assume that the following conditions on the local spaces $V(K)$ and $W(K)$ hold:

1. Most importantly, we assume that the space $V(K) \times W(K)$ admits an $M$-decomposition.

2. The spaces $V(K)$ and $W(K)$ must satisfy 

$$\mathcal{P}_0(K) \in \tilde{V}(K) \text{ and } [\mathcal{P}_0(K)]^2 \in \tilde{W}(K),$$

for all elements $K$. In addition, we assume that if $\Pi_0$ (respectively $\Pi_0^\ell$) denotes the $L^2(K)$ (respectively $L^2(K)$) orthogonal projection onto $V(K)$ (respectively $W(K)$) then the following estimates hold:

$$\|w - \Pi_0 w\|_K \leq Ch^s_K \|w\|_{H^s(K)}$$

and

$$\|p - \Pi_0 p\|_K \leq Ch^s_K \|p\|_{H^s(K)}$$

for any sufficiently smooth $w$ or $p$ and $0 \leq s \leq 1$.

3. Let $T^s_h$ be a refined mesh of $T_h$ consisting of simplices obtained by subdividing each element $K \in T_h$ using triangles. We assume that the number of triangles used in each element is bounded independent of $h$ (i.e. there is a fixed maximum number of triangles covering each $K$ independent of $h$). Next we define $W^s_h = \{u \in L^2(\Omega) : u|_K \in [\mathcal{P}_\ell(K)]^2, \forall K \in T^s_h\}$ and $\ell \geq 1$ is some integer such that $W_h \subset W^s_h$. We assume that $T^s_h$ is shape-regular. This assumption implies that standard scaling estimates can be used for $V(K)$ and $W(K)$, because scaling can be used triangle by triangle on the $T^s_h$. In addition, standard finite element spaces constructed on this mesh have the usual approximation properties.

Note that this notion of shape regularity for the general mesh is what is the analogue of that used to define shape regularity for a quadrilateral mesh in [29].

4.2 The dual problem

Consider the following dual problem: find $(\Psi, \Phi) \in H(\text{curl}; \Omega) \times [H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)]$ such that

$$\mu_r \Psi - \nabla \times \Phi = 0 \quad \text{in } \Omega,$$

$$\nabla \times \Psi - \kappa^2 \epsilon_r \Phi = \Theta \quad \text{in } \Omega,$$

$$n \times \Phi = 0 \quad \text{on } \Gamma,$$

where $\Theta \in H(\text{div}; \Omega)$ and $\nabla \cdot \Theta = 0$ and $\epsilon_r^\star$ is the complex conjugate of $\epsilon_r$. Under our assumptions on $\mu_r$, $\epsilon_r$, and $\kappa$, this problem has a unique solution. The regularity of the solution of (4.2) is given in [Theorem 3].

We recall the following result, where $L_0^2(\Omega)$ denotes the space of functions in $L^2(\Omega)$ with average value zero.

**Lemma 9** (c.f [29, Corollary 2.4]). Let $\Omega$ be a bounded connected Lipschitz domain in $\mathbb{R}^2$, then for any $f \in L_0^2(\Omega)$, there exists a $\mathbf{v} \in H^1_0(\Omega)$ such that

$$\nabla \cdot \mathbf{v} = f, \quad \|\mathbf{v}\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$
Lemma 10. Let \( \Omega \) be a bounded connected Lipschitz domain in \( \mathbb{R}^2 \), then for any \( f \in L^2(\Omega) \), there exists a \( v \in H^1(\Omega) \) such that
\[
\nabla \cdot v = f, \quad \|v\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.
\]

Proof. Let \( \bar{f} = \frac{1}{|\Omega|} \int_{\Omega} f \) be the mean value of \( f \), then \( f - \bar{f} \in L^2(\Omega) \). By Lemma 9, there exists a \( w = (w_1, w_2)^T \in H^1_0(\Omega) \), such that \( \nabla \cdot w = f - \bar{f} \), and \( \|w\|_{H^1(\Omega)} \leq C\|f - \bar{f}\|_{L^2(\Omega)} \).

Let \( (x_0, y_0) \) be a point in the domain \( \Omega \). Define \( v = (v_1, v_2)^T \) with \( v_1 = w_1 + (x-x_0)\bar{f}, \quad v_2 = w_2 \), then \( v \in H^1(\Omega) \) and
\[
\nabla \cdot v = \nabla \cdot w + \nabla \cdot ((x-x_0)\bar{f}, 0)^T = f,
\]
\[
\|v\|_{H^1(\Omega)} \leq \|w\|_{H^1(\Omega)} + \|(x-x_0)\bar{f}\|_{H^1(\Omega)}
\]
\[
\leq C\|f - \bar{f}\|_{L^2(\Omega)} + \|(x-x_0)\bar{f}\|_{L^2(\Omega)} + \|\bar{f}\|_{L^2(\Omega)}
\]
\[
\leq C\|f\|_{L^2(\Omega)}.
\]

The previous result can be used to prove the existence of a vector potential as follows

Lemma 11. Let \( f \in L^2(\Omega) \), then there exists a function \( w \in H^1(\Omega) \), such that
\[
\nabla \times w = f, \quad \|w\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.
\]

Proof. By Lemma 10, there exists a \( v = (v_1, v_2)^T \in H^1(\Omega) \), such that \( \nabla \cdot v = f \) and \( \|v\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)} \). We take \( w = (w_1, w_2)^T \) with \( w_1 = -v_2 \) and \( w_2 = v_1 \), hence
\[
\nabla \times w = -\partial_y w_1 + \partial_x w_2 = \partial_y v_2 + \partial_x v_1 = \nabla \cdot v = f,
\]
\[
\|w\|_{H^1(\Omega)} = \|v\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.
\]

The proof of the next theorem follows that of [1, Proposition 3.7] and [38, Theorem 3.50], which deal with the 3D case.

Theorem 2. Let \( \Omega \) be a simply connected Lipschitz domain in \( \mathbb{R}^2 \), then the space \( H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) is imbedded in the space \( H^s(\Omega) \) with some \( s \in (\frac{1}{2}, 1] \), and the following estimate holds
\[
\|u\|_{H^s(\Omega)} \leq C \left( \|\nabla \times u\|_{L^2(\Omega)} + \|\nabla \cdot u\|_{L^2(\Omega)} \right), \quad (4.3)
\]
for all \( u \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \).

Proof. Let \( \mathcal{O} \) be a smooth open set with a connected boundary (a circle for instance), which contains \( \bar{\Omega} \). Let \( u \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \), we extend \( u \) to \( \mathcal{O} \) by zero, so \( u \in H(\text{curl}; \mathcal{O}) \), therefore \( \nabla \times u \in L^2(\mathcal{O}) \). From Lemma 11, there exists a function \( w \in H^1(\mathcal{O}) \) such that \( \nabla \times w = \nabla \times u \) in \( \mathcal{O} \) and \( \|w\|_{H^1(\mathcal{O})} \leq C\|\nabla \times u\|_{L^2(\mathcal{O})} \). Since \( \nabla \times (u - w) = 0 \) in \( \mathcal{O} \), then there is a function \( \chi \in H^1(\mathcal{O}) \) such that \( u - w = \nabla \chi \) in \( \mathcal{O} \). Since \( u = 0 \) in \( \mathcal{O} \setminus \Omega \), we have \( -w = \nabla \chi \) in \( \mathcal{O} \setminus \Omega \), therefore, \( \chi \in H^2(\mathcal{O} \setminus \Omega) \). Then \( \chi \in H^1(\Omega) \setminus \mathbb{R} \) satisfies
\[
\Delta \chi = \nabla \cdot u - \nabla \cdot w \quad \text{in} \quad \Omega,
\]
where $\nabla \chi|_{\partial \Omega}$ takes the exterior value of $\nabla \chi = -w$. So $\chi|_{\partial \Omega} \in H^{1+s}(\partial \Omega)$ with some $s \in (1/2, 1]$. By [26, Corollary 18.15], we have $\chi \in H^{1+s}(\Omega)$ and

$$
\|\chi\|_{H^{1+s}(\Omega)} \leq C \left( \|\nabla \cdot u - \nabla \cdot w\|_{H^{1+s}(\Omega)} + \|\nabla \chi\|_{H^{-\frac{1}{2}+s}(\partial \Omega)} \right).
$$

Therefore, using the fact that $\nabla \chi = -w$ on $\mathcal{O} \setminus \Omega$ it holds

$$
\|u\|_{H^{s}(\Omega)} = \|w + \nabla \chi\|_{H^{s}(\Omega)} \\
\leq \|w\|_{H^{1}(\Omega)} + C \left( \|\nabla \cdot u - \nabla \cdot w\|_{H^{1+s}(\Omega)} + \|\nabla \chi\|_{H^{-\frac{1}{2}+s}(\partial \Omega)} \right) \\
\leq C \left( \|w\|_{H^{1}(\Omega)} + \|\nabla \cdot u\|_{L^2(\Omega)} \right) \\
\leq C \left( \|w\|_{H^{1}(\Omega)} + \|\nabla \cdot u\|_{L^2(\Omega)} \right) \\
\leq C \left( \|\nabla \times u\|_{L^2(\Omega)} + \|\nabla \cdot u\|_{L^2(\Omega)} \right).
$$

Thus we finish our proof. \qed

Now we can state a complete regularity result for the adjoint problem:

**Theorem 3.** Let $\mu_r$ be a smooth function, then we have the following regularity for the solution of problem (4.2)

$$
\|\Psi\|_{H^1(\Omega)} + \|\Phi\|_{H^s(\Omega)} \leq C \|\Theta\|_{L^2(\Omega)},
$$

for some $s \in (1/2, 1]$ depending on $\Omega$.

**Proof.** To simplify the notation, we define

$$
a^+(u, v) = (\mu_r^{-1} \nabla \times u, \nabla \times v)_\Omega + (u, v)_\Omega.
$$

Let $\Phi \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega)$ be the solution of

$$
a^+(\Phi, v) = (\Theta, v)_\Omega, \quad \forall v \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega).
$$

Let $K : L^2(\Omega) \to H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega)$ be such that for any $w \in L^2(\Omega)$ the function $Kw$ satisfies

$$
a^+(Kw, v) = -(\kappa^2 \tau_r - 1)(w, v)_\Omega, \quad \forall v \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega).
$$

Obviously, $\Phi$ and $K$ are well-defined and

$$
a^+((I + K)\Phi, v) = a^+(\Phi, v).
$$

This gives

$$
(I + K)\Phi = \Phi.
$$

Form (4.5) and (4.6), we get

$$
\|\Phi\|_{L^2(\Omega)} + \|\nabla \times \Phi\|_{L^2(\Omega)} \leq C \|\Theta\|_{L^2(\Omega)},
$$

$$
\|K\Phi\|_{L^2(\Omega)} + \|\nabla \times K\Phi\|_{L^2(\Omega)} \leq C \|\Phi\|_{L^2(\Omega)}.
$$
From Theorem 2, we know that \( H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) is compactly imbedded in the space \( L^2(\Omega) \). Using [4.6], we see that \( K \) is a self-adjoint and compact operator on \( L^2(\Omega) \). Hence, since our assumptions on \( \epsilon_r \) and \( \kappa^2 \) guarantee at most one solution, by the Fredholm Alternative, (4.7) has a unique solution. Therefore,

\[
\| \Phi \|_{L^2(\Omega)} = \| (I + K)^{-1} \Phi \|_{L^2(\Omega)} \leq C \| \Theta \|_{L^2(\Omega)},
\]

\[
\| \nabla \times \Phi \|_{L^2(\Omega)} \leq \| \nabla \times K \Phi \|_{L^2(\Omega)} + \| \nabla \times \Phi \|_{L^2(\Omega)} \leq C \left( \| \Theta \|_{L^2(\Omega)} + \| \Phi \|_{L^2(\Omega)} \right) \leq C \| \Theta \|_{L^2(\Omega)}.
\]

By the Equation (4.2), we have

\[
\| \nabla \times (\mu_r^{-1} \nabla \times \Phi) \|_{L^2(\Omega)} \leq C \| \Theta \|_{L^2(\Omega)} + C \| \Phi \|_{L^2(\Omega)} \leq C \| \Theta \|_{L^2(\Omega)}.
\]

Since \( \mu \) is smooth, then we have \( \nabla \times \Phi \in H^1(\Omega) \) and

\[
\| \nabla \times \Phi \|_{H^1(\Omega)} \leq C \| \Theta \|_{L^2(\Omega)}.
\]

Since Theorem 2 ensures \( H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) is imbeded in the space \( H^s(\Omega) \) with \( (\frac{1}{2}, 1) \), then we have

\[
\| \Phi \|_{H^s(\Omega)} \leq C \left( \| \Phi \|_{L^2(\Omega)} + \| \nabla \times \Phi \|_{L^2(\Omega)} \right) \leq C \| \Theta \|_{L^2(\Omega)}.
\]

This finishes our proof. \( \square \)

Throughout this section, we use \( C \) to denote a positive constant independent of mesh size, which may take on different values at each occurrence. Let \( P_V, P_{\tilde{V}}, P_W \) and \( P_{\tilde{W}} \) are the \( L^2 \)- projections on the spaces \( V_h, \tilde{V}_h, W_h \) and \( \tilde{W}_h \), respectively.

Now we state the main result in this section. The proof is found in Section 4.4.

**Theorem 4.** Suppose the spaces \((V_h, W_h, M_h)\) have an \( M \)- decomposition and the assumptions in Section 4.1 are satisfied. Let \((q, u) \in H(\text{curl}; \Omega) \times H(\text{curl}; \Omega)\) and \((q_h, u_h) \in V_h \times W_h \times M_h\) be the solution of (2.1) and (2.3), respectively. Then there exists an \( h_0 > 0 \) such that for all \( h \leq h_0 \), we have the error estimate

\[
\| q - q_h \|_{T_h} \leq C \left( \| q - P_V q \|_{T_h} + \| u - P_W u \|_{T_h} \right),
\]

\[
\| u - u_h \|_{T_h} \leq C \left( \| q - P_V q \|_{T_h} + \| u - P_W u \|_{T_h} \right).
\]

Furthermore, the post processed solution \( u_h^* \in W^*(T_h) \) defined later in (4.55) satisfies the estimate

\[
\| \nabla \times (u - u_h^*) \|_{T_h} \leq C \left( \| q - P_V q \|_{T_h} + \| u - P_W u \|_{T_h} + \inf_{w_h^* \in W^*(T_h)} \| \nabla \times (u - w_h^*) \|_{T_h} \right).
\]

### 4.3 The HDG Projection

An appropriate HDG projection plays a key role in the derivation of optimal error estimates and superconvergence (see for example [6, 7, 9, 13, 14, 22, 24]). In the case of Maxwell’s equations, we define the following HDG projection: find \((\Pi_V q, \Pi_W u) \in V(K) \times W(K)\) such that

\[
(\Pi_V q, v_h)_K = (q, v_h)_K \quad \forall v_h \in V(K), \quad (4.9a)
\]

\[
(\Pi_W u, w_h)_K = (u, w_h)_K \quad \forall w_h \in W(K), \quad (4.9b)
\]

\[
(\Pi_V q - \tau n \times \Pi_W u, n \times \mu_h)_F = (q - \tau n \times u, n \times \mu_h)_F \quad \forall \mu_h \in M(F) \text{ and for all edges } F \subset \partial K. \quad (4.9c)
\]
The following theorem proves that the above definition uniquely specifies the projections and provides optimal error estimates for this projection.

**Theorem 5.** System (4.9) defines a unique projection \((\Pi_V q, \Pi_W u)\). Moreover, we have the following error estimate:

\[
\|\Pi_W u - u\|_K \leq C \left( \|u - P_W u\|_K + h_K \|\nabla \times q - P_W \nabla \times q\|_K + h_K^{1/2} \|n \times (u - P_W u)\|_{\partial K} \right),
\]

(4.10a)

\[
\|\Pi_V q - q\|_K \leq C \left( h_K^{1/2} \|P_V q\|_{\partial K} + h_K^{1/2} \|n \times (\Pi_W u - u)\|_{\partial K} + \|q - P_V q\|_K \right).
\]

(4.10b)

We only give a proof for (4.10a) in the following three lemmas, since (4.10b) is very similar.

**Lemma 12** (Existence and Uniqueness). System (4.9) defines a unique projection \((\Pi_V q, \Pi_W u)\).

**Proof.** By Definition 1 we have

\[
\dim V(K) + \dim W(K) = \dim \tilde{V}(K) + \dim \tilde{W}(K) + \dim M(\partial K).
\]

This means that system (4.9) is square, hence we only need to prove uniqueness. We set the right hand sides of (4.9) to zero, i.e., \(q = 0\) and \(u = 0\). By (4.9a) and (4.9b), we have

\[
\Pi_V q \in \tilde{V}^\perp(K) \quad \text{and} \quad \Pi_W u \in \tilde{W}^\perp(K).
\]

(4.11)

Since \(n \times W(K) \times n \subset M(\partial K)\), then we can take \(\mu_h = n \times \Pi_W u \times n\) in (4.9c) to get

\[
\langle \tau n \times \Pi_W u, n \times (n \times \Pi_W u \times n) \rangle_{\partial K} = \langle \Pi_V q, n \times (n \times \Pi_W u \times n) \rangle_{\partial K}
\]

\[
= \langle \Pi_V q, n \times \Pi_W u \rangle_{\partial K}
\]

\[
= \langle \Pi_V q, \nabla \times \Pi_W u \rangle_K - (\nabla \times \Pi_V q, \Pi_W u)_K
\]

\[
= 0.
\]

Since \(\tau\) is piecewise constant and positive, then

\[
n \times \Pi_W u \times n = 0 \quad \text{on} \quad \partial K.
\]

(4.12)

Moreover, \(n \times \Pi_W u = 0\) on \(\partial K\). Since \(n \times V(K) \subset M(\partial K)\), then we can take \(\mu_h = n \times \Pi_V q\) in (4.9c) to get

\[
n \times \Pi_V q = 0 \quad \text{on} \quad \partial K.
\]

(4.13)

We combine (4.11), (4.12), (4.13) and (3.2c) to conclude that \(\Pi_W u = 0\) and \(\Pi_V q = 0\). This proves the system (4.9) defines a unique projection \((\Pi_V q, \Pi_W u)\). \(\square\)

To estimate \(\Pi_W u - u\), we decouple the projection \(\Pi_W\) from \(\Pi_V\) in (4.9) as follows.

**Lemma 13.** The projection \(\Pi_W u\) satisfies

\[
(\Pi_W u, v_h)_K = (u, v_h)_K, \quad \forall v_h \in \tilde{W}(K),
\]

\[
\langle \tau n \times \Pi_W u \times n, w_h \rangle_{\partial K} = \langle \nabla \times q, w_h \rangle_K + \langle \tau n \times u \times n, w_h \rangle_{\partial K}, \quad \forall w_h \in \tilde{W}^\perp(K).
\]

(4.14a)

(4.14b)
Lemma 14. For any \( s \) holds when this assumption follows from our assumption on the auxiliary mesh \( T \).

Therefore, (4.9b), (4.16) and (4.18) give the system (4.14).

Next, we use the integration by parts identity (2.2b) to get

\[
\langle \tau n \times \Pi_W u, n \times w_h \rangle_{\partial K} = \langle \nabla \times (q - \Pi_V q), w_h \rangle_K - \langle q - \Pi_V q, \nabla \times w_h \rangle_K
\]

Then, for all \( w_h \in \tilde{W}^r(K) \), by (3.2b) and (4.9a), we have

\[
\langle \nabla \times \Pi_V q, w_h \rangle_K = 0,
\]

\[
(q - \Pi_V q, \nabla \times w_h)_{K} = 0.
\]

Next, we can give the proof of (4.10a).

Proof of (4.10a). By the definition of \( P_W \) and \( \tilde{P}_W \), we can rewrite equation (4.14) as follows:

\[
\langle \Pi_W u - P_W u, v_h \rangle_K = 0, \quad \forall v_h \in \tilde{W}(K),
\]

\[
\langle \tau n \times (\Pi_W u - P_W u), n \times w_h \rangle_{\partial K} = \langle \nabla \times (q - \Pi_V q), w_h \rangle_K - \langle q - \Pi_V q, \nabla \times w_h \rangle_K.
\]

By the same arguments as in the proof of Lemma 12, we can prove that \( \Pi_W u - P_W u \in W(K) \) is uniquely determined by the right hand side of (4.19). Using a standard scaling estimate (this can be used because of the assumption on \( T_h^* \) in Section 4.1) we have

\[
\| \Pi_W u - P_W u \|_K \leq C h_K \tau^{-1}(\nabla \times q - \Pi_W \nabla \times q) \|_K + h_K^{1/2} \| n \times (u - P_W u) \|_{\partial K}.
\]

Thus, the triangle inequality gives the desired result.

Next, we extend the error estimates (4.10) to fractional order Sobolev spaces. To do this we use a local inverse inequality. For any function \( w_h \in W(K) \) or \( p_h \in V(K) \) the following inverse estimate holds:

\[
\| w_h \|_{H^s(K)} \leq C h_K^{-s} \| w_h \|_K, \quad \text{and} \quad \| p_h \|_{H^s(K)} \leq C h_K^{-s} \| p_h \|_K
\]

with \( 0 \leq s \leq 1 \). The constant \( C \) is independent of the function, element and mesh size. Note that this assumption follows from our assumption on the auxiliary mesh \( T_h^* \) when \( s = 1 \) and trivially holds when \( s = 0 \). Hence by interpolation it holds for general \( 0 \leq s \leq 1 \).

Lemma 14. For any \( s \in [0, 1] \), we have

\[
\Pi W u - u \|_{H^s(K)} \leq C h_K^{-s} \left( \| u - P_W u \|_K + h_K \| \nabla \times q - P_W \nabla \times q \|_K + h_K^{1/2} \| n \times (u - P_W u) \|_{\partial K} \right)
\]

\[
+ \| P_W u - u \|_{H^s(K)},
\]

\[
\Pi V q - q \|_{H^s(K)} \leq C \left( h_K^{1/2-s} \| q - P_V q \|_{\partial K} + h_K^s \| q - P_V q \|_K + \| q - P_V q \|_{H^s(K)} + \| u - P_W u \|_{H^s(K)} \right)
\]

\[
+ C \left( h_K^{-s} \| u - P_W u \|_K + h_K^{1/2-s} \| \nabla \times q - P_W \nabla \times q \|_K + h_K^{1/2-s} \| n \times (u - P_W u) \|_{\partial K} \right).
\]
Proof. Using the fact that $P_W$ is the $L^2$ orthogonal projection on $W(K)$ and applying the local inverse inequality discussed before the statement of the lemma, we get

\[
\|\Pi_W u - u\|_{H^s(K)} = \|\Pi_W u - P_W u + P_W u - u\|_{H^s(K)} \\
\leq \|\Pi_W u - P_W u\|_{H^s(K)} + \|P_W u - u\|_{H^s(K)} \\
\leq Ch_K^{-s}\|\Pi_W u - P_W u\|_K + \|P_W u - u\|_{H^s(K)} \\
\leq Ch_K^{-s}\|\Pi_W u - u\|_K + Ch_K^{-s}\|P_W u - u\|_K + \|P_W u - u\|_{H^s(K)}.
\]

Combining the estimate (4.10a) and the above inequality we have

\[
\|\Pi_W u - u\|_{H^s(K)} \leq Ch_K^{-s} \left(\|u - P_W u\|_K + h_K\|\nabla \times q - P_W \nabla \times q\|_K + h_K^{1/2}\|n \times (u - P_W u)\|_{\partial K}\right) \\
+ \|P_W u - u\|_{H^s(K)}.
\]

This proves (4.20a).

Next, we prove (4.20b). By the same arguments we have

\[
\|\Pi_V q - q\|_{H^s(K)} \leq Ch_K^{-s}\|\Pi_V q - q\|_K + Ch_K^{-s}\|P_V q - q\|_K + \|P_V q - q\|_{H^s(K)}.
\]

By Lemma 7.2 in [27] to get

\[
\|\Pi_W u - u\|_{\partial K} \leq C \left(h_K^{-1/2}\|\Pi_W u - u\|_K + h_K^{-1/2}\|\Pi_W u - u\|_{H^s(K)}\right). \tag{4.21}
\]

Using estimates (4.10b), (4.10a), (4.21) and (4.20a), we can obtain (4.20b). \hfill \square

Since $P_0(K) \in \tilde{V}(K)$ and $[P_0(K)]^2 \in \tilde{W}(K)$ with appropriate projection error bounds (see Section 4.1), by Theorem 4 and Lemma 14, we have the following corollary.

**Corollary 1.** Let $(\Psi, \Phi) \in H(\text{curl}; \Omega) \times [H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)]$ be the solution of (4.2) and assume that the regularity result (4.4) holds, then for $s \in (1/2, 1]$, we have

\[
\|\Pi_W \Phi - \Phi\|_{\tau_h} + \|\Pi_V \Psi - \Psi\|_{\tau_h} \leq Ch^s\|\Theta\|_{\tau_h}, \tag{4.22a}
\]

\[
\|\Pi_W \Phi - \Phi\|_{H^s(\tau_h)} + \|\Pi_V \Psi - \Psi\|_{H^s(\tau_h)} \leq C\|\Theta\|_{\tau_h}. \tag{4.22b}
\]

We can now prove our main result: Theorem 4.

### 4.4 Proof of Theorem 4

First, we define the following HDG operator $\mathcal{B} : [V_h \times W_h \times M_h]^2 \rightarrow \mathbb{C}$

\[
\mathcal{B}(q_h, u_h, \tilde{u}_h; r_h, v_h, \tilde{v}_h) = (\mu q_h, r_h)_{\tau_h} - (u_h, \nabla \times r_h)_{\tau_h} - (n \times \tilde{u}_h, r_h)_{\partial \tau_h} \\
+ (\nabla \times q_h, v_h)_{\tau_h} + (q_h, n \times \tilde{v}_h)_{\partial \tau_h} \\
+ (\tau n \times (u_h - \tilde{u}_h), n \times (v_h - \tilde{v}_h))_{\partial \tau_h}. \tag{4.23}
\]

By the definition of $\mathcal{B}$ in (4.23), we can rewrite the HDG formulation of the system (2.3) in a compact form, as follows:

**Lemma 15.** The HDG method seeks $(q_h, u_h, \tilde{u}_h) \in V_h \times W_h \times M_h^0$ such that

\[
\mathcal{B}(q_h, u_h, \tilde{u}_h; r_h, v_h, \tilde{v}_h) - (\kappa^2 r_h, u_h, v_h)_{\tau_h} = (f, v_h)_{\tau_h}, \tag{4.24}
\]
for all \((r_h, v_h, \hat{v}_h) \in V_h \times W_h \times M_0^h\), in which \(M_0^h\) and \(M_0^0\) are defined as
\[
M_0^h = \{ \mu \in M_h : n \times \mu |_{\partial\Omega} = (P_M(n \times g)) \times n \}, \\
M_0^0 = \{ \mu \in M_h : n \times \mu |_{\partial\Omega} = 0 \},
\]
where \(P_M\) denotes the \(L^2\)-projection from \(L^2(F)\) onto space \(M(F)\). Thus if \(u \in L^2(F)\) then \(P_M u \in M(F)\) satisfies
\[
\langle P_M u, v_h \rangle_F = \langle u, v_h \rangle_F \quad \forall v_h \in M(F).
\]
Next, we give some properties of the operator \(B\) below, the proof of the following lemma is very simple and we omit it here.

**Lemma 16.** For any \((q_h, u_h, \tilde{u}_h, r_h, v_h, \hat{v}_h) \in [V_h \times W_h \times M_h]^2\), we have
\[
B(q_h, u_h, \tilde{u}_h; r_h, -v_h, -\hat{v}_h) = B(r_h, v_h, \hat{v}_h; q_h, -u_h, -\tilde{u}_h).
\]

**Lemma 17.** If for all \(r_h \in V_h\), \((q_h, u_h, \tilde{u}_h) \in V_h \times W_h \times M_h\) satisfies
\[
B(q_h, u_h, \tilde{u}_h; r_h, 0, 0) = (G, r_h)_T_h,
\]
where \(G \in L^2(\Omega)\), then we have
\[
||\nabla \times u_h||_{T_h} \leq C \left( ||q_h||_{T_h} + ||h^{-1/2}n \times (u_h - \tilde{u}_h)||_{\partial T_h} + ||G||_{T_h} \right).
\]

**Proof.** By the definition of \(B\) in (4.23), we have
\[
(\mu q_h, r_h)_T_h - \langle u_h, \nabla \times r_h \rangle_{T_h} - \langle n \times \tilde{u}_h, r_h \rangle_{\partial T_h} = (G, r_h)_T_h.
\]
We take \(r_h = \nabla \times u_h\) in (4.27) and integrate by parts to get
\[
(\mu q_h, \nabla \times u_h)_T_h - \langle \nabla \times u_h, \nabla \times u_h \rangle_{T_h} - \langle n \times (u_h - \tilde{u}_h), \nabla \times u_h \rangle_{\partial T_h} = (G, \nabla \times u_h)_T_h.
\]
After apply the Cauchy-Schwartz inequality and the local inverse inequality can get our desired result. \(\square\)

Now, we give the proof of Theorem 4 splitting it into three steps.

### 4.4.1 Step 1: Error equations and energy arguments

**Lemma 18.** Let \((q, u) \in H(\text{curl}; \Omega) \times [H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)]\) be the weak solution of (2.1), then for all \((r_h, v_h, \hat{v}_h) \in V_h \times W_h \times M_0^h\), we have
\[
B(\Pi V q, \Pi W u, P_M u; r_h, v_h, \hat{v}_h) = (\mu(\Pi V q - q), r_h)_T_h + (\nabla \times q, v_h)_T_h.
\]

**Proof.** By the definition of \(B\) in (4.23) and use (3.2a) we have
\[
B(\Pi V q, \Pi W u, P_M u; r_h, v_h, \hat{v}_h)
= (\mu \Pi V q, r_h)_T_h - (\Pi W u, \nabla \times r_h)_T_h - \langle n \times P_M u, r_h \rangle_{\partial T_h}
+ \langle \nabla \times \Pi V q, v_h \rangle_{T_h} + \langle \Pi V q, n \times (n \times \tilde{v}_h) \rangle_{\partial T_h}
+ \langle (\nabla \times (\Pi W u - P_M u), n \times (v_h - \tilde{v}_h)) \rangle_{\partial T_h}
= (\mu \Pi V q, r_h)_T_h - \langle u, \nabla \times r_h \rangle_{T_h} - \langle n \times u, r_h \rangle_{\partial T_h}
+ \langle (\Pi V q, \nabla \times v_h)_T_h + (\Pi V q, n \times (\tilde{v}_h - v_h)) \rangle_{\partial T_h}
+ \langle (\tau n \times (\Pi W u - u), n \times (v_h - \tilde{v}_h)) \rangle_{\partial T_h}
= (\mu \Pi V q - \nabla \times u, r_h)_T_h
+ \langle (q, \nabla \times v_h)_T_h + (\Pi V q, n \times (\tilde{v}_h - v_h)) \rangle_{\partial T_h}
+ \langle (\tau n \times (\Pi W u - u), n \times (v_h - \tilde{v}_h)) \rangle_{\partial T_h}.
\]
Since \( \hat{v}_h \) is single valued on interior edges and equal to zero on boundary faces, then \( (q, n \times \hat{v}_h)_{\partial T_h} = 0 \). Moreover, by (2.2a) we have
\[
(q, \nabla \times v_h)_{\partial T_h} + \langle \Pi_V q, n \times (\hat{v}_h - v_h) \rangle_{\partial T_h} = 0.
\]
Next, we take \((4.24)\) from (4.28) to get the following error equations.

This implies
\[
\mathcal{B}(\Pi_V q, \Pi_W u, P_M u; r_h, v_h, \hat{v}_h) = (\mu \Pi_V q - \nabla \times u, r_h)_{\partial T_h} + (\nabla \times q, v_h)_{\partial T_h} = (\mu \Pi_V q - q, r_h)_{\partial T_h} + (\nabla \times q, v_h)_{\partial T_h},
\]
and completes the proof.

To simplify notation, we define
\[
\varepsilon^q_h = \Pi_V q - q_h, \quad \varepsilon^u_h = \Pi_W u - u_h, \quad \varepsilon^\mu_h = P_M u - \hat{u}_h.
\]
We subtract (4.24) from (4.28) to get the following error equations.

**Lemma 19.** Using the notation (4.29), for any \((r_h, v_h, \hat{v}_h) \in V_h \times W_h \times M^0_h\), we have
\[
\mathcal{B}(\varepsilon^q_h, \varepsilon^u_h, \varepsilon^\mu_h, r_h, v_h, \hat{v}_h) = (\mu \Pi_V q - q, r_h)_{\partial T_h} + (\kappa^2 \varepsilon^\mu_h, v_h)_{\partial T_h}.
\]

**Proof.** By the definition of \(\mathcal{B}\) in (4.23) and Lemma 18, we get
\[
\mathcal{B}(\varepsilon^q_h, \varepsilon^u_h, \varepsilon^\mu_h, r_h, v_h, \hat{v}_h) = (\mu \Pi_V q - q, r_h)_{\partial T_h} + (\kappa^2 \varepsilon^\mu_h, v_h)_{\partial T_h}.
\]
where we used (2.1b) in the last inequality.

**Lemma 20.** Using definition (4.29), we have the error estimate
\[
\| \sqrt{\mu} \varepsilon^q_h \|_{T_h} + \| \sqrt{\tau} n \times (\varepsilon^u_h - \varepsilon^\mu_h) \|_{\partial T_h} \leq C (\| q - \Pi_V q \|_{T_h} + \| u - \Pi_W u \|_{T_h} + \| \varepsilon^\mu_h \|_{T_h}).
\]

**Proof.** First, we take \((r_h, v_h, \hat{v}_h) = (\varepsilon^q_h, 0, 0)\) in (4.30) to get
\[
\mathcal{B}(\varepsilon^q_h, \varepsilon^u_h, \varepsilon^\mu_h, \varepsilon^q_h, 0, 0) = (\mu \Pi_V q - q, \varepsilon^q_h)_{\partial T_h}.
\]
Next, we take \((r_h, v_h, \hat{v}_h) = (0, \varepsilon^u_h, \varepsilon^\mu_h)\) in (4.30) to get
\[
\mathcal{B}(\varepsilon^q_h, \varepsilon^u_h, \varepsilon^\mu_h, 0, \varepsilon^u_h, \varepsilon^\mu_h) = (\mu \Pi_V q - q, \varepsilon^u_h, \varepsilon^\mu_h)_{\partial T_h}.
\]
By the equations (4.32) and (4.33), we get
\[
\mathcal{B}(\varepsilon_h, \varepsilon_h, \theta, \varphi, 0, 0) + \mathcal{B}(\varepsilon_h, \tilde{\varepsilon}_h, \varepsilon_h, \theta, \varphi) - (\kappa^2 \varepsilon_r \varepsilon_h, \varepsilon_h)_{\tau_h} = (\mu(\Pi \nabla q - q), \varepsilon_h)_{\tau_h} + (\kappa^2 \varepsilon_r (u - \Pi W u), \varepsilon_h)_{\tau_h}.
\] (4.34)

On the other hand, by the definition of \(\mathcal{B}\) in (4.23), to get
\[
\mathcal{B}(\varepsilon_h, \varepsilon_h, \theta, \varphi, 0, 0) + \mathcal{B}(\varepsilon_h, \tilde{\varepsilon}_h, \varepsilon_h, \theta, \varphi) = \|\sqrt{\mu} \varepsilon_h\|_{\tau_h}^2 + \|\sqrt{\nabla} \times (\varepsilon_h - \tilde{\varepsilon}_h)\|_{\tau_h}^2.
\] (4.35)

Hence, by the equation (4.34) and (4.35), we have
\[
\|\sqrt{\mu} \varepsilon_h\|_{\tau_h}^2 + \|\sqrt{\nabla} \times (\varepsilon_h - \tilde{\varepsilon}_h)\|_{\tau_h}^2 = (\mu(\Pi \nabla q - q), \varepsilon_h)_{\tau_h} + (\kappa^2 \varepsilon_r (u - \Pi W u), \varepsilon_h)_{\tau_h}
\]
\[
\leq C\|\Pi \nabla q - q\|_{\tau_h}\sqrt{\|\varepsilon_h\|_{\tau_h}} + C\|u - \Pi W u\|_{\tau_h}\|\varepsilon_h\|_{\tau_h}.
\]

Use of Young’s inequality gives our desired result.

\[\square\]

4.4.2 Step 2: Duality argument

Similarly to Lemma 18 we have:

**Lemma 21.** Let \((\Psi, \Phi) \in H(\text{curl}; \Omega) \times [H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)]\) be the solution of (4.2), then for all \((r_h, v_h, \widehat{v}_h) \in V_h \times W_h \times M_h^0\), we have
\[
\mathcal{B}(\Pi \nabla \Psi, \Pi W \Phi, P_M \Phi; r_h, v_h, \widehat{v}_h) = (\mu(\Pi \nabla \Psi - \Psi), r_h)_{\tau_h} + (\nabla \times \Psi, v_h)_{\tau_h}.
\]

The next lemma gives a partial error estimate:

**Lemma 22.** Assume \(\Theta \in H(\text{div}; \Omega), \nabla \cdot \Theta = 0\) and that the regularity estimate (4.4) holds, then we have
\[
(\varepsilon_h^u, \Theta)_{\tau_h} \leq Ch^s (\|q - \Pi V q\|_{\tau_h} + \|u - \Pi W u\|_{\tau_h}) \|\Theta\|_{\tau_h} + Ch^s \varepsilon_h^u \|\Theta\|_{\tau_h}.
\]

**Proof.** First, we take \((r_h, v_h, \widehat{v}_h) = (-\Pi \nabla \Psi, \Pi W \Phi, P_M \Phi)\) in (4.30) we get
\[
\mathcal{B}(\varepsilon_h^u, \varepsilon_h, \theta, \varphi, -\Pi \nabla \Psi, \Pi W \Phi, P_M \Phi) - (\kappa^2 \varepsilon_r \varepsilon_h^u, \Pi W \Phi)_{\tau_h}
\]
\[
= - (\mu(\Pi \nabla q - q), \Pi \nabla \Psi)_{\tau_h} + (\kappa^2 \varepsilon_r (u - \Pi W u), \Pi W \Phi)_{\tau_h}.
\] (4.36)

By Lemma 16 and Lemma 21 and using (4.2b) we have
\[
\mathcal{B}(\varepsilon_h^u, \varepsilon_h, \theta, \varphi, -\Pi \nabla \Psi, \Pi W \Phi, P_M \Phi)
\]
\[
= \mathcal{B}(\Pi \nabla \Psi, \Pi W \Phi, P_M \Phi; -\varepsilon_h^u, \varepsilon_h, \theta, \varphi)
\]
\[
= - (\varepsilon_h^u, \mu(\Pi \nabla \Psi - \Psi))_{\tau_h} + (\varepsilon_h, \nabla \times \Psi)_{\tau_h}
\]
\[
= - (\varepsilon_h^u, \mu(\Pi \nabla \Psi - \Psi))_{\tau_h} + (\varepsilon_h^u, \Theta + \kappa^2 \varepsilon_r \Phi)_{\tau_h}.
\] (4.37)

Comparing (4.36) with (4.37) to get
\[
(\varepsilon_h^u, \Theta)_{\tau_h} = (\mu \varepsilon_h^u, \Pi \nabla \Psi - \Psi)_{\tau_h} - (\mu(\Pi \nabla q - q), \Pi \nabla \Psi)_{\tau_h}
\]
\[
+ (\kappa^2 \varepsilon_r (u - \Pi W u), \Pi W \Phi)_{\tau_h} - (\kappa^2 \varepsilon_r (\Phi - \Pi W \Phi), \varepsilon_h^u)_{\tau_h}
\]
\[
= T_1 + T_2 + T_3 + T_4.
\]
Next, we estimate \( \{ T_i \}_{i=1}^4 \) one by one. For the terms \( T_1 \) and \( T_4 \), by (4.22a) and estimate for \( \varepsilon_h^u \) in Lemma 20, we have

\[
|T_1| \leq Ch^s\|\Theta\|_{\Gamma_h}(q - \Pi V q)\|_{\tau_h} + \|u - \Pi W u\|_{\tau_h} + \|\varepsilon_h^u\|_{\tau_h},
\]
\[
|T_4| \leq Ch^s\|\Theta\|_{\Gamma_h}\|\varepsilon_h^u\|_{\tau_h}.
\]

For the remain two terms \( T_2 \) and \( T_3 \), since \( P_0(K) \in \tilde{V}(K) \) and \([P_0(K)]^2 \in \tilde{W}(K)\) with appropriate estimates for the projection (see Section 4.1), then by (4.22b) we have

\[
T_2 = |(\Pi V q - q, \mu\Pi V \Psi - \Pi_0(\mu\Pi V \Psi))|_{\tau_h} \leq Ch^s\|q - \Pi V q\|_{\tau_h}\|\Pi V \Psi\|_{H^s(\Omega)}
\]
\[
- Ch^s\|q - \Pi V q\|_{\tau_h}\|\Theta\|_{\tau_h},
\]
\[
T_3 = |(u - \Pi W u, k^2\partial_n \Pi W \Phi - \Pi_0 k^2\partial_n \Pi W \Phi)|_{\tau_h} \leq Ch^s\|u - \Pi W u\|_{\tau_h}\|\Pi W \Phi\|_{H^s(\Omega)}
\]
\[
- Ch^s\|u - \Pi W u\|_{\tau_h}\|\Theta\|_{\tau_h}.
\]

By the above estimates of \( \{ T_i \}_{i=1}^4 \) we get

\[
(\varepsilon_h^u, \Theta)|_{\tau_h} \leq Ch^s(\|q - \Pi V q\|_{\tau_h} + \|u - \Pi W u\|_{\tau_h})\|\Theta\|_{\tau_h} + Ch^s\|\varepsilon_h^u\|_{\tau_h}\|\Theta\|_{\tau_h}. \tag{4.38}
\]

This completes the proof. \( \square \)

We cannot set \( \Theta = \varepsilon_h^u \) to get an estimate of \( \varepsilon_h^u \) since \( \varepsilon_h^u \notin H(\text{div}; \Omega) \), hence we need to modify the analysis.

Recall the shape-regular submesh \( T_h^* \) defined in Section 4.1. We define \( W_h^* = \{ u \in L^2(\Omega) : u|_K \in [P_{\ell}(K)]^2, \forall K \in T_h^* \} \) and \( \ell \geq 1 \) is some integer such that \( W_h \subset W_h^* \).

Next, we recall the \( H(\text{curl}; \Omega) \) conforming element in 2D. For any \( v \in H(\text{curl}; K) \), with \( K \) being a simplex, find \( \Pi_{K,\ell}^{\text{curl}} v \in P_{\ell}(K) \) such that

\[
\langle n \times \Pi_{K,\ell}^{\text{curl}} v, p_{\ell} \rangle_E = \langle n \times v, p_{\ell} \rangle_E, \quad \forall p_{\ell} \in P_{\ell}(E), \tag{4.39a}
\]
\[
\langle \Pi_{K,\ell}^{\text{curl}} v, \nabla \times p_{\ell-1} \rangle_K = \langle v, \nabla \times p_{\ell-1} \rangle_K, \quad \forall p_{\ell-1} \in P_{\ell-1}(K), \tag{4.39b}
\]

and, when \( \ell \geq 2 \)

\[
\langle \Pi_{K,\ell}^{\text{curl}} v, \nabla (b_K p_{\ell-2}) \rangle_K = \langle v, \nabla (b_K p_{\ell-2}) \rangle_K, \quad \forall p_{\ell-2} \in P_{\ell-2}(K) \tag{4.39c}
\]

for all edges \( E \) of \( K \), where \( b_K \) is the bubble function of \( K \) of order three.

Following a standard procedure in [2] Lemma 3.2, Theorem 3.1, we have the following theorem:

**Theorem 6.** Equation (4.39) defines a unique \( \Pi_{K,\ell}^{\text{curl}} v \in P_{\ell}(K) \), and the following estimate holds:

\[
\|\Pi_{K,\ell}^{\text{curl}} v - v\|_{0,K} \leq Ch_{K}^{m}\|v\|_{m,K}, \tag{4.40}
\]

with \( v \in H^m(\Omega) \), and \( m \in \{ \frac{1}{2}, \ell + 1 \} \). We define \( \Pi_{K,\ell}^{\text{curl}} = \Pi_{h,\ell}^{\text{curl}} |_K \) for all \( K \in T_h^* \), then \( \Pi_{h,\ell}^{\text{curl}} v \in H(\text{curl}; \Omega) \). In addition, \( n \times v|_{\partial\Omega} = 0 \) implies that \( n \times \Pi_{h,\ell}^{\text{curl}} v|_{\partial\Omega} = 0 \).

Furthermore, the previously defined interpolation operator commutes with curl.

**Lemma 23.** Let \( \Pi_{K,\ell-1} \) be the \( L^2 \) projection onto space \( P_{\ell-1}(K) \), then we have the commutativity property

\[
\nabla \times \Pi_{K,\ell}^{\text{curl}} v = \Pi_{K,\ell-1} \nabla \times v. \tag{4.41}
\]
Proof. For any $p_{\ell-1} \in \mathcal{P}_{\ell-1}(K)$, we have
\[
(\nabla \times \Pi_{K,\ell}^\mathrm{curl} \mathbf{v}, p_{\ell})_K = (\Pi_{K,\ell}^\mathrm{curl} \mathbf{v}, \nabla \times p_{\ell})_K + \langle n \times \Pi_{K,\ell}^\mathrm{curl} \mathbf{v}, p_{\ell} \rangle_{\partial K}
\]
\[
= (\mathbf{v}, \nabla \times p_{\ell-1})_K + \langle n \times \mathbf{v}, p_{\ell-1} \rangle_{\partial K}
\]
\[
= (\nabla \times \mathbf{v}, p_{\ell-1})_K.
\]
\[

\]

Following the same techniques in [33] Proposition 4.5] of 3D case, we have the following result for 2D.

Lemma 24 (c.f [33] Proposition 4.5]). For any $\mathbf{v}_h \in W_h$, there exists $\Pi_{h}^\mathrm{curl,c} \mathbf{v}_h \in W_h^* \cap H_0(\text{curl}; \Omega)$ such that
\[
\| \mathbf{v}_h - \Pi_{h}^\mathrm{curl,c} \mathbf{v}_h \|_{\mathcal{T}_h} \leq C \| h^{1/2} [ n \times \mathbf{v}_h ] \|_{\mathcal{E}_h},
\]
\[
\| \nabla \times (\mathbf{v}_h - \Pi_{h}^\mathrm{curl,c} \mathbf{v}_h) \|_{\mathcal{T}_h} \leq C \| h^{-1/2} [ n \times \mathbf{v}_h ] \|_{\mathcal{E}_h},
\]

where $W^*_h = \{ \mathbf{u} \in L^2(\Omega) : \mathbf{u}|_K \in [\mathcal{P}_\ell(K)]^2, \forall K \in \mathcal{T}_h^* \}$ and $\ell$ is some integer such that $W_h \subset W^*_h$.

Definition 2. Let $Q^*_h = H^1_0(\Omega) \cap \mathcal{P}_{\ell+1}(\mathcal{T}^*_h)$ be a finite element space with respect to the mesh $\mathcal{T}^*_h$ (therefore, $\nabla Q^*_h \subset H_0(\text{curl}; \Omega) \cap W^*_h$) with $\sigma_h \in H^1_0(\Omega) \cap Q^*_h$ satisfy
\[
(\nabla \sigma_h, \nabla q_h)_{\mathcal{T}_h} = (\Pi_{h}^\mathrm{curl,c}(\mathbf{u}_h - \Pi_{W} \mathbf{u}), \nabla q_h)_{\mathcal{T}_h}
\]
for all $q_h \in H^1_0(\Omega) \cap Q^*_h$. Then we define
\[
\Pi_{W}^m(\mathbf{u}, \mathbf{u}_h) = \Pi_{W} \mathbf{u} + \nabla \sigma_h.
\]

It is easy to check the following lemma using Definition 2 and Lemma 24, hence we omit the proof.

Lemma 25. We have
\[
\nabla \times \Pi_{W}^m(\mathbf{u}, \mathbf{u}_h) = \nabla \times \Pi_{W} \mathbf{u}, \quad [n \times \Pi_{W}^m(\mathbf{u}, \mathbf{u}_h)] = [n \times \Pi_{W} \mathbf{u}],
\]
\[
\nabla \times \Pi_{h}^\mathrm{curl,c}(\mathbf{u}_h - \Pi_{W} \mathbf{u}), \nabla q_h)_{\mathcal{T}_h} = 0, \quad \forall q_h \in H^1_0(\Omega) \cap Q^*_h.
\]

In addition, we have the following estimates:

Lemma 26. We have the following estimates:
\[
\| \nabla \times (\Pi_{h}^\mathrm{curl,c}(\Pi_{W}^m(\mathbf{u}, \mathbf{u}_h)) - \mathbf{u}_h) \|_{\mathcal{T}_h}
\]
\[
\leq C \left( \| h^{-1/2}(q - \Pi_V q) \|_{\mathcal{T}_h} + \| h^{-1/2}(u - \Pi_{W} \mathbf{u}) \|_{\mathcal{T}_h} + \| h^{-1/2} \mathbf{e}_h \|_{\mathcal{T}_h} \right),
\]
\[
\| (\Pi_{W}^m(\mathbf{u}, \mathbf{u}_h) - \mathbf{u}_h) - \Pi_{h}^\mathrm{curl,c}(\Pi_{W}^m(\mathbf{u}, \mathbf{u}_h) - \mathbf{u}_h) \|_{\mathcal{T}_h}
\]
\[
\leq C \left( \| h^{1/2}(q - \Pi_V q) \|_{\mathcal{T}_h} + \| h^{1/2}(u - \Pi_{W} \mathbf{u}) \|_{\mathcal{T}_h} + \| h^{1/2} \mathbf{e}_h \|_{\mathcal{T}_h} \right).
\]
Proof. We use definition [Definition 2] and [Lemma 24]. By the definition of $\Pi_h^m$ in (4.44) and the approximation property of $\Pi_h^{\text{curl},c}$ in [Lemma 24] to get
\[
\|\nabla \times (\Pi_h^{\text{curl},c}(\Pi_h^m(u, u_h) - u_h))\|_{\mathcal{T}_h} \\
\leq \|\nabla \times (\Pi_h^{\text{curl},c}(\Pi_h^m(u, u_h) - u_h)) - \nabla \times (\Pi_h^m(u, u_h) - u_h)\|_{\mathcal{T}_h} \\
+ \|\nabla \times (\Pi_h^m(u, u_h) - u_h)\|_{\mathcal{T}_h} \\
= \|\nabla \times (\Pi_h^{\text{curl},c}(\Pi_h^m u - u_h)) - \nabla \times (\Pi_h^m u - u_h)\|_{\mathcal{T}_h} \\
+ \|\nabla \times (\Pi_h^m u - u_h)\|_{\mathcal{T}_h} \quad \text{by (4.45b)} \\
\leq C\|h^{-1/2}[n \times (\Pi_h^m u - u_h)]\|_{\mathcal{E}_h} + \|\nabla \times (\Pi_h^m u - u_h)\|_{\mathcal{T}_h} \quad \text{by (4.42b)} \\
= C\|h^{-1/2}[n \times \varepsilon_h]\|_{\mathcal{E}_h} + \|\nabla \times \varepsilon_h\|_{\mathcal{T}_h}.
\]
Since $\varepsilon_h^n$ is single valued on the interior faces and zero on the boundary, then we have
\[
\|\nabla \times (\Pi_h^{\text{curl},c}(\Pi_h^m(u, u_h) - u_h))\|_{\mathcal{T}_h} \leq C\|h^{-1/2}[n \times (\varepsilon_h^n - \varepsilon_h^n)]\|_{\mathcal{E}_h} + \|\nabla \times \varepsilon_h^n\|_{\mathcal{T}_h}.
\]
Hence, (4.26) and (4.31) give the proof of (4.46a).

Next, by the approximation of $\Pi_h^{\text{curl},c}$ in [Lemma 24] and (4.45a) to get
\[
\|\Pi_h^m(u, u_h) - \Pi_h^{\text{curl},c}(\Pi_h^m(u, u_h) - u_h)\|_{\mathcal{T}_h} \\
\leq C\|h^{1/2}[n \times (\Pi_h^m(u, u_h) - u_h)]\|_{\mathcal{E}_h} \\
= C\|h^{1/2}[n \times (\varepsilon_h^n - \varepsilon_h^n)]\|_{\mathcal{E}_h}.
\]
Finally, (4.31) gives the proof of (4.46b). \(\square\)

Next, we prove the following lemma which is similar in [32] Lemma 4.5.

Lemma 27. Let $\Theta \in H_0(\text{curl}; \Omega)$ satisfy
\[
\nabla \times \Theta = \nabla \times w_h, \quad \nabla \cdot \Theta = 0 \quad \text{in} \quad \Omega, \quad (4.47)
\]
where $w_h \in H_0(\text{curl}; \Omega) \cap W^*_\ast$ and $(w_h, \nabla q_h)_{\Omega} = 0$ for all $q_h \in Q^*_h$. Then we have
\[
\|w_h - \Theta\|_{L^2(\Omega)} \leq C h^s\|\nabla \times \Theta\|_{L^2(\Omega)}, \quad (4.48)
\]
where $s \in (\frac{1}{2}, 1]$ is defined in [Theorem 2]. The following stability result also holds:
\[
\|\Theta\|_{L^2(\Omega)} \leq C\|w_h\|_{L^2(\Omega)}. \quad (4.49)
\]
Proof. We define $\Pi_{\ell-1}\vert_K := \Pi_{K,\ell-1}$, then the following holds
\[
\nabla \times (w_h - \Pi_h^{\text{curl}}(\Theta)) = \nabla \times w_h - \Pi_{\ell-1}\nabla \times \Theta = \nabla \times w_h - \Pi_{\ell-1}\nabla \times w_h = 0. \quad (4.50)
\]
Thus there is a $q_h \in Q^*_h = \mathcal{P}_{\ell+1}(T^*_h) \cap H^1_0(\Omega)$ such that $w_h - \Pi_h^{\text{curl}}(\Theta) = \nabla q_h$. By a direct calculation, one can obtain
\[
\|w_h - \Theta\|^2_{L^2(\Omega)} = (w_h - \Theta, w_h - \Theta)_\Omega \\
= (w_h - \Theta, w_h - \Pi_h^{\text{curl}}(\Theta) + \Pi_h^{\text{curl}}(\Theta - \Theta))_\Omega \\
= (w_h - \Theta, \nabla q_h + \Pi_h^{\text{curl}}(\Theta - \Theta))_\Omega \\
= (w_h - \Theta, \Pi_h^{\text{curl}}(\Theta - \Theta))_\Omega,
\]
Finally, (4.31) gives the proof of (4.46b). \(\square\)
where we have used that \( w_h \) is discrete divergence free, and \( \Theta \) is divergence free. Now using Theorems 2 and \( \Theta \) we get
\[
\| w_h - \Theta \|_{L^2(\Omega)}^2 \leq Ch^s \| w_h - \Theta \|_{L^2(\Omega)} \| \Theta \|_{H^s(\Omega)}
\]
\[
\leq Ch^s \| w_h - \Theta \|_{L^2(\Omega)} \| \nabla \times \Theta \|_{L^2(\Omega)},
\]
where \( s \in \left( \frac{1}{2}, 1 \right] \) is specified in Theorem 2.

By the Helmholtz decomposition in two dimensions, there is a \( \phi \in H^1_0(\Omega) \) and \( \psi \in H^1(\Omega) \) such that
\[
\Theta = \nabla \phi + \nabla \times \psi, \quad \| \phi \|_{H^1(\Omega)} \leq \| \Theta \|_{L^2(\Omega)}, \quad \| \psi \|_{H^1(\Omega)} \leq \| \Theta \|_{L^2(\Omega)}.
\]

Then we use the integration by parts and (4.47) to get
\[
\| \Theta \|_{L^2(\Omega)}^2 = (\Theta, \Theta)_\Omega
\]
\[
= (\nabla \phi + \nabla \times \psi, \Theta)_\Omega
\]
\[
= -(\phi, \nabla \cdot \Theta)_\Omega + (\psi, \nabla \times \Theta)_\Omega
\]
\[
= (\psi, \nabla \times w_h)_\Omega
\]
\[
= (\nabla \times \psi, w_h)_\Omega
\]
\[
\leq \| \Theta \|_{L^2(\Omega)} \| w_h \|_{L^2(\Omega)}.
\]

Thus we obtain our result. \( \square \)

**Lemma 28.** Let \((q, u) \in H(\text{curl}; \Omega) \times H(\text{curl}; \Omega) \) and \((q_h, u_h) \in V_h \times W_h \) be the solution of (2.1) and (2.3), respectively. Then there exists an \( h_0 > 0 \) such that for all \( h \leq h_0 \), we have the error estimate
\[
\| q - q_h \|_{\tau_h} \leq C \left( \| q - \Pi_V q \|_{\tau_h} + \| u - \Pi_W u \|_{\tau_h} \right),
\]
\[
\| u - u_h \|_{\tau_h} \leq C \left( h^{s-1/2} \| q - \Pi_V q \|_{\tau_h} + C \| u - \Pi_W u \|_{\tau_h} \right).
\]

**Proof.** First, let \( \Theta \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) be the solution of
\[
\nabla \times \Theta = \nabla \times (\Pi^\text{curl,c}_h (\Pi^m_W(u, u_h) - u_h)), \quad \nabla \cdot \Theta = 0 \quad \text{in} \ \Omega.
\]

By Lemma 27 and (4.46a), one has
\[
\| \Theta - (\Pi^\text{curl,c}_h (\Pi^m_W(u, u_h) - u_h)) \|_{\tau_h}
\]
\[
\leq C \| h \nabla \times (\Pi^\text{curl,c}_h (\Pi^m_W(u, u_h) - u_h)) \|_{\tau_h}
\]
\[
\leq Ch^{s-1/2} \left( \| q - \Pi_V q \|_{\tau_h} + \| u - \Pi_W u \|_{\tau_h} + \| \varepsilon_u \|_{\tau_h} \right) .
\]

Therefore, by the triangle inequality, (4.49) and Definition 2 we have
\[
\| \Theta \|_{\tau_h} \leq \| (\Pi^\text{curl,c}_h (\Pi^m_W(u, u_h) - u_h)) \|_{\tau_h} \leq 2 \| \Pi^\text{curl,c}_h (\Pi^m_W u - u_h) \|_{\tau_h} \leq C \| \varepsilon_u \|_{\tau_h} .
\]
Next, we rewrite \( \| \varepsilon_h^u \|_{T_h}^2 \) as follows:
\[
\| \varepsilon_h^u \|_{T_h}^2 = (\varepsilon_h^u, \Pi_h^{\text{curl}}(\Pi_m^W(u, u_h) - u_h) - \Theta)_{T_h} + (\varepsilon_h^u, \Theta)_{T_h} \\
+ (\varepsilon_h^u, (\Pi_m^W(u, u_h) - u_h) - \Pi_h^{\text{curl}}(\Pi_m^W(u, u_h) - u_h))_{T_h} \\
+ (\varepsilon_h^u, \Pi_W^W u - \Pi_W^W u_h)_{T_h} \\
= (\varepsilon_h^u, \Pi_h^{\text{curl}}(\Pi_m^W(u, u_h) - u_h) - \Theta)_{T_h} + (\varepsilon_h^u, \Theta)_{T_h} \\
+ (\varepsilon_h^u, (\Pi_m^W(u, u_h) - u_h) - \Pi_h^{\text{curl}}(\Pi_m^W(u, u_h) - u_h))_{T_h} \\
+ (\varepsilon_h^u, \Pi_h^{\text{ curls}}(\Pi_m^W(u, u_h) - u_h))_{T_h} \\
= S_1 + S_2 + S_3 + S_4.
\]

The first three terms \( S_1, S_2 \) and \( S_3 \) have been estimated in Lemma 22, Lemma 22, and Lemma 46, respectively. We next estimate the last term \( S_4 \) by taking \((r_h, v_h, \Theta) = (0, \nabla \sigma_h, \nabla \sigma_h - (n \cdot \nabla \sigma_h)n)\) in (4.24) to get
\[
(k^2 \varepsilon, u_h, \nabla \sigma_h)_{T_h} = - (f, \nabla \sigma_h)_{T_h}.
\]

Moreover, we have \( -(f, \nabla \sigma_h)_{T_h} = (k^2 \varepsilon, u, \nabla \sigma_h)_{T_h} \), since \( \varepsilon \) is a constant therefore
\[
(u_h - u, \nabla \sigma_h)_{T_h} = 0.
\]

This implies
\[
|S_4| = |(u_h - u, \nabla \sigma_h)_{T_h} + (u - \Pi_W^W u, \nabla \sigma_h)_{T_h}| = |(u - \Pi_W^W u, \nabla \sigma_h)_{T_h}| \leq C \|u - \Pi_W^W u\|_{T_h} \|\varepsilon_h^u\|_{T_h}.
\]

By the above estimations of \( S_4 \), there exists an \( h_0 > 0 \) such that for all \( h \leq h_0 \), we have
\[
\|\varepsilon_h^u\|_{T_h} \leq C \left( h^{s-1/2} \|q - \Pi_V q\|_{T_h} + \|u - \Pi_W^W u\|_{T_h} \right).
\]

By the above estimate and Lemma 20 we get
\[
\|\varepsilon_h^q\|_{T_h} \leq C \left( \|q - \Pi_V q\|_{T_h} + \|u - \Pi_W^W u\|_{T_h} \right).
\]

Combining the above estimates with the triangle inequality gives the desired result. 

\[\square\]

### 4.4.3 Step 3: Post-processing

Let \( W^*(K) \) be a finite element space, we first define the following space:
\[
V^*(K) = \{v : \nabla v \in W^*(K)\}.
\]

The post-processing method reads: we seek \( u_h^* \in W^*(K) \) such that
\[
(\nabla \times u_h^*, \nabla \times w)_K = (q_h, \nabla \times w)_K, \quad \text{for all } w \in W^*(K), \tag{4.55a}
\]
\[
(u_h^*, \nabla v)_K = (u_h, \nabla v)_K, \quad \text{for all } v \in V^*(K). \tag{4.55b}
\]

Now, we state the main result in this section.

**Lemma 29.** Let \((q, u)\) be the solution of (2.1). Then the system (4.55) is well-defined and there exists an \( h_0 > 0 \) such that for all \( h \leq h_0 \), we have the error estimate
\[
\|\nabla \times (u - u_h^*)\|_{T_h} \leq C \left( \|q_h - q\|_{T_h} + \inf_{w_h \in W^*(T_h)} \|\nabla \times (u - w_h)\|_{T_h} \right).
\]
Lemma 30. The post-processing problem (4.55) is equivalent to the following system: find \((u_h^*, \eta_h, \gamma_h) \in W^*(K) \times V^*(K) \times P_0(K)\), such that

\[
\begin{align}
(\nabla \times u_h^*, \nabla \times w)_K + (\nabla \eta_h, w)_K &= (q_h, \nabla \times w)_K & \text{for all } w \in W^*(K), \\
(u_h^*, \nabla v)_K + (\gamma_h, v)_K &= (u_h, \nabla v)_K & \text{for all } v \in V^*(K), \\
(\eta_h, s)_K &= 0 & \text{for all } s \in P_0(K).
\end{align}
\]

Proof. To prove this, we only need to prove (4.57) is well-defined and \(\eta_h = \gamma_h = 0\). It is obvious to see that the system (4.57) is a square system, hence we only need to prove the uniqueness. We take \(w = \nabla \eta_h, v = \gamma_h\) and \(s = 1\) in (4.57) to get \(\nabla \eta_h = 0, \gamma_h = 0\) and \((\eta_h, 1) = 0\). Hence \(\eta_h = \gamma_h = 0\).

5 Construction and numerical experiments

In this section, we shall present some concrete examples of spaces \(V(K), W(K), M(\partial K)\) and associated spaces \(\tilde{V}(K)\) and \(\tilde{W}(K)\) that satisfy the definition of the \(M\)-decomposition; see Definition 1. In addition the spaces \(\tilde{V}(K)\) and \(\tilde{W}(K)\) need to satisfy (4.1). The approach to constructing the upcoming spaces follows [15]. It is straightforward to check that the examples in Table 3 have an \(M\)-decomposition. In this section we consider higher order families of elements.

Based on the construction below, conditions (3.2a), (3.2b) and (3.2d) are easy to check, while condition (3.2c) is not always obvious. Fortunately, the following equivalent of condition (3.2c) is easy to check in our construction.

Lemma 31. Assume the conditions (3.2a), (3.2b) and (3.2d) hold, then (3.2c) is equivalent to

\[
\begin{align}
\gamma &\text{ is injective on the space } \tilde{V}^\perp(K), \\
\gamma &\text{ is injective on the space } \tilde{W}^\perp(K), \\
\gamma \tilde{V}^\perp(K) &\oplus \gamma \tilde{W}^\perp(K) = M(\partial K),
\end{align}
\]

where \(\gamma \tilde{V}^\perp(K) := \{ n \times v^\perp |_{\partial K} : v^\perp \in \tilde{V}^\perp(K)\}\) and \(\gamma \tilde{W}^\perp(K) := \{ n \times w^\perp \times n |_{\partial K} : w^\perp \in \tilde{W}^\perp(K)\}\).
Proof. By (3.9), it is easy to see that
\[
\text{tr} : \left( \tilde{V}^\perp(K) \times \tilde{W}^\perp(K) \right) = \gamma \tilde{V}^\perp(K) \oplus \gamma \tilde{W}^\perp(K).
\] (5.2)

Assuming (5.1a)-(5.1c) hold, we need to prove that (3.2c) holds. By (5.1c) and (5.2) it holds
\[
\text{tr} : \left( \tilde{V}^\perp(K) \times \tilde{W}^\perp(K) \right) = M(\partial K).
\] (5.3)

This proves the mapping \text{tr} is surjective. Next, we prove \text{tr} is injective. Let \( q \in M(\partial K) \), then there exist \( v_1, v_2 \in \tilde{V}^\perp(K) \) and \( w_1, w_2 \in \tilde{W}^\perp(K) \) such that
\[
q = n \times v_1 + n \times w_1 \times n,
q = n \times v_2 + n \times w_2 \times n.
\]

Therefore,
\[
n \times (v_1 - v_2) + n \times (w_1 - w_2) \times n = 0.
\]

Using (3.9) we get \( n \times (v_1 - v_2) = 0 \) and \( n \times (w_1 - w_2) \times n = 0 \). By (5.1b) and (5.1c), it holds \( v_1 = v_2 \) and \( w_1 = w_2 \). This proves \text{tr} is injective on the space \( \tilde{V}^\perp(K) \times \tilde{W}^\perp(K) \) and hence we finish the proof of condition (3.2c).

On the other hand, assume that condition (3.2c) holds so we need to prove (5.1a)-(5.1c) hold. By (3.6), the condition (5.1c) holds. Next, we prove (5.1a) and (5.1b). Let \( v_1, v_2 \in \tilde{V}^\perp(K) \) and \( w_1, w_2 \in \tilde{W}^\perp(K) \) such that
\[
n \times v_1 = n \times v_2,
n \times w_1 = n \times w_2 \times n.
\]

Then it holds
\[
n \times (v_1 - v_2) + n \times (w_1 - w_2) \times n = 0
\] (5.4)

Since \text{tr} is an injective on space \( \tilde{V}^\perp(K) \times \tilde{W}^\perp(K) \), then it holds \( v_1 = v_2 \) and \( w_1 = w_2 \). This proves that (5.1a) and (5.1b) hold and hence completes proof.

In this section, we shall show some numerical experiments for each choice of element. In all numerical experiments, we take \( \mu = 1 \), and \( \kappa^2 \epsilon_r = 10 \). The exact solution is
\[
u_1 = \sin(2\pi x) \sin(2\pi y), \quad u_2 = \sin(\pi x) \sin(\pi y), \quad q = \pi \cos(\pi x) \sin(\pi y) - 2\pi \sin(2\pi x) \cos(2\pi y).
\]

Boundary data is chosen so that the above functions satisfy (1.2)

The post-processing spaces in all experiments are taken as
\[
V^*(K) = \mathcal{P}_{k+2}(K), \quad W^*(K) = \mathcal{P}_{k+1}(K).
\]

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5.1 Triangle Mesh

We assume that the mesh $\mathcal{T}_h$ consists of shape regular triangles and choose $\mathcal{T}_h^* = \mathcal{T}_h$ (see Section 4.1). We might hope that standard $P_k$ polynomial spaces could work, and this is indeed the case as shown in the next lemma.

Lemma 32. For any integer $k \geq 1$, let

\[ V(K) = \mathcal{P}_k(K), \quad W(K) = \mathcal{P}_k(K), \]
\[ M(\partial K) = \{ \mu : \mu|_F = n \times p_k, \text{ for some } p_k \in \mathcal{P}_k(F) \text{ and for each edge } F \subset \partial K \}, \]

then we have

\[ I_M(V(K) \times W(K)) = 0. \]

Proof. It is easy to see that

\[ \dim \{ n \times v|_{\partial K} : v \in V(K), \nabla \times v = 0 \} = 1, \quad \dim M(\partial K) = 3(k+1). \]

Next, we give more details to compute

\[ \dim \{ n \times w \times n|_{\partial K} : w \in W(K), \nabla \times w = 0 \} \]
\[ = \dim \{ n \times (\nabla p_{k+1}) \times n : p_{k+1} \in \mathcal{P}_{k+1}(K) \} \]
\[ = \dim \{ \nabla p_{k+1} : p_{k+1} \in \mathcal{P}_{k+1}(K) \} \]
\[ - \dim \{ p_{k+1} \in \mathcal{P}_{k+1}(K) : n \times (\nabla p_{k+1}) \times n = 0 \} \]
\[ = \dim \{ \nabla p_{k+1} : p_{k+1} \in \mathcal{P}_{k+1}(K) \} \]
\[ - \dim \{ \lambda_1 \lambda_2 \lambda_3 p_{k-2} : p_{k-2} \in \mathcal{P}_{k-2}(K) \} \]
\[ = \binom{k+3}{2} - 1 - \binom{k}{2}. \]

By the definitions of $I_M$ in (3.4) to get

\[ I_M(V(K) \times W(K)) = 0. \]

By Theorem 1 and Lemma 32, we know the finite element spaces $\mathcal{P}_k(K) \times \mathcal{P}_k(K)$ admits an $M$-decomposition, i.e., there exist two spaces $\tilde{V}(K)$ and $\tilde{W}(K)$ satisfy the Definition 1. In the following Lemma 33, we give a concrete construction of the spaces $\tilde{V}(K)$ and $\tilde{W}(K)$.

Lemma 33. For any integer $k \geq 1$, let

\[ V(K) = \mathcal{P}_k(K), \quad W(K) = \mathcal{P}_k(K), \quad M(\partial K) = \{ \mu : \mu|_F = n \times \mathcal{P}_k(F) \text{ for each face } F \subset \partial K \}, \]
\[ \tilde{V}(K) = \mathcal{P}_{k-1}(K), \quad \tilde{W}(K) = \mathcal{P}_{k-1}(K). \]

Then $V(K)$ and $W(K)$ admit an $M$-decomposition with respect the space $\tilde{V}(K)$ and $\tilde{W}(K)$.

Proof. We notice that (3.2a), (3.2b) and (3.2d) obviously hold. We only have to prove (3.2c). By Lemma 31, we need to check the following three conditions:
(1) \( \gamma \) is injective on the space \( \tilde{V}^\perp(K) \): Let \( v^\perp \in \tilde{V}^\perp(K) \), if \( n \times v^\perp |_{\partial K} = 0 \), we have \( v^\perp |_{\partial K} = 0 \). If \( k \leq 2 \), then \( v^\perp |_{K} = 0 \). If \( k \geq 3 \), we can set \( v^\perp = p_{k-3} \lambda_1 \lambda_2 \lambda_3 \), where \( p_{k-3} \in \mathcal{P}_{k-3}(K) \). Then \( (v^\perp, p_{k-3})_K = (p_{k-3} \lambda_1 \lambda_2 \lambda_3, p_{k-3})_K = 0 \), this gives \( p_{k-3} = 0 \), hence \( v^\perp = 0 \). This proves \( \gamma \) is an injective, also \( \gamma \) is an injective; therefore, \( \gamma \) is an isomorphism, which implies

\[
\text{dim } \gamma \tilde{V}^\perp(K) = \text{dim } \tilde{V}^\perp(K).
\]

(2) \( \gamma \) is injective on the space \( \tilde{W}^\perp(K) \): Let \( w^\perp \in \tilde{W}^\perp(K) \) satisfy \( n \times w^\perp \times n = 0 \), this implies \( n \times w^\perp = 0 \). For all \( v \in V(K) \), we have \( \nabla \times v \in \nabla \times V(K) \subset \tilde{W}(K) \), then

\[
0 = (w^\perp, \nabla \times v)_K = (\nabla \times w^\perp, v)_K + \langle w^\perp, n \times v \rangle_{\partial K} = (\nabla \times w^\perp, v)_K.
\]

Since \( \nabla \times W(K) \subset V(K) \), we take \( v = \nabla \times w^\perp \) in the above equation to get \( \nabla \times w^\perp = 0 \), then there is a \( p_{k-2} \in \mathcal{P}_{k-2}(K) \) such that

\[
w^\perp = \nabla (\lambda_1 \lambda_2 \lambda_3 p_{k-2}).
\]

For any \( f_{k-1} \in \mathcal{P}_{k-1}(K) = \tilde{W}(K) \), we have

\[
0 = (\nabla (p_{k-2} \lambda_1 \lambda_2 \lambda_3), f_{k-1})_K = (p_{k-2} \lambda_1 \lambda_2 \lambda_3, \nabla \cdot f_{k-1})_K.
\]

Since \( \nabla \cdot \mathcal{P}_{k-1}(K) = \mathcal{P}_{k-2}(K) \), we can take \( f_{k-1} \) such that \( \nabla \cdot f_{k-1} = p_{k-2} \), then we have \( p_{k-2} = 0 \), and therefore, \( w^\perp = 0 \), hence \( \gamma \) is an injective; thus \( \gamma \) is an isomorphism, which implies

\[
\text{dim } \gamma \tilde{W}^\perp(K) = \text{dim } \tilde{W}^\perp(K).
\]

(3) \( \gamma \tilde{V}^\perp(K) \oplus \gamma \tilde{W}^\perp(K) = M(\partial K) \): on the one hand \( \gamma \tilde{V}^\perp(K) \oplus \gamma \tilde{W}^\perp(K) \subset M(\partial K) \); on the other hand

\[
\text{dim } M(\partial K) - \text{dim } \gamma \tilde{V}^\perp(K) - \text{dim } \gamma \tilde{W}^\perp(K)
= \text{dim } M(\partial K) - \text{dim } \tilde{V}^\perp(K) - \text{dim } \tilde{W}^\perp(K)
= \text{dim } M(\partial K) - (\text{dim } V(K) - \text{dim } \tilde{V}(K)) - (\text{dim } W(K) - \text{dim } \tilde{W}(K))
= 0.
\]

Hence \( \gamma \tilde{V}^\perp(K) \oplus \gamma \tilde{W}^\perp(K) = M(\partial K) \) holds.

Next, we give the canonical construction on a triangle element.

**Lemma 34** (Canonical Construction). For any integer \( k \geq 0 \), let

\[
V(K) = \mathcal{P}_k(K), \quad W(K) = \mathcal{P}_k(K),
\]

\[
M(\partial K) = \{ \mu: \mu|_F = n \times p_k \text{ for some } p_k \in \mathcal{P}_k(F) \text{ and for each edge } F \subset \partial K \},
\]

and

\[
\tilde{V}(K) = \mathcal{P}_{k-1}(K), \quad \tilde{W}(K) = \nabla \times V(K) \oplus W_0(K),
\]

where

\[
W_0(K) = \{ w_0 \in W(K): \nabla \times w_0 = 0, n \times w_0 \times n = 0 \}.
\]

Then \( V(K) \) and \( W(K) \) admit the canonical \( M \)-decomposition with respect the spaces \( \tilde{V}(K) \) and \( \tilde{W}(K) \).
Proof. Similar with Lemma 33 we only need to check the three conditions in Lemma 31. Moreover, the condition (5.1a) is the same with Lemma 33. Hence we check the conditions (5.1b) and (5.1c) in the following.

(1) \( \gamma \) is injective on the space \( \widetilde{W}^\perp(K) \): Let \( w^\perp \in \widetilde{W}^\perp(K) \) satisfy \( n \times w^\perp \times n = 0 \), by the proof in (2) in Lemma 33 we get \( \nabla \times w^\perp = 0 \). Therefore, \( w^\perp \in W_0(K) \subset \widetilde{W}(K) \). This implies \( w^\perp \in \widetilde{W}^\perp(K) \cap \widetilde{W}(K) = \{0\} \), hence \( \gamma \) is an injective; thus \( \gamma \) is an isomorphism, which implies

\[
\dim \gamma \widetilde{W}^\perp(K) = \dim \widetilde{W}^\perp(K).
\]

(2) \( \gamma \widetilde{V}^\perp(K) \oplus \gamma \widetilde{W}^\perp(K) = M(\partial K) \): Since \( \gamma \widetilde{V}^\perp(K) \oplus \gamma \widetilde{W}^\perp(K) \subset M(\partial K) \), then we only need to prove

\[
\dim M(\partial K) = \dim \gamma \widetilde{V}^\perp(K) + \dim \gamma \widetilde{W}^\perp(K)
= \dim V(K) - \dim \widetilde{V}(K) + \dim W(K) - \dim \widetilde{W}(K)
= \dim V(K) - \dim \widetilde{V}(K) + \dim W(K) - \dim (\nabla \times V(K)) - \dim W_0(K),
\]

i.e., we need to prove \( \dim (W_0(K)) = \binom{k}{2} \).

For any \( w_0 \in W_0(K) \), since \( \nabla \times w_0 = 0 \), there exists \( q_{k+1} \in P_{k+1}(K) \) such that \( w_0 = \nabla q_{k+1} \). Therefore,

\[
0 = n \times w_0|_{\partial K} \times n = n \times \nabla q_{k+1}|_{\partial K} \times n.
\]

Therefore, there is a constant \( C_0 \) and \( p_{k-2} \in P_{k-2}(K) \) such that \( q_{k+1} = p_{k-2} \lambda_1 \lambda_2 \lambda_3 + C_0 \), i.e., \( \dim W_0(K) = \binom{k}{2} \).

In Table 2, we show numerical results on the unit square with a uniform triangular mesh. We obtain an optimal convergence rate for the solution \( u \) and superconvergence rate for \( \nabla \times u \).

| \( k \) | \( \frac{\sqrt{2}}{n} \) | \( \|u - u_b\|_{\mathcal{T}_h} \) | \( \|\nabla \times (u - u_b)\|_{\mathcal{T}_h} \) | \( \|q - q_b\|_{\mathcal{T}_h} \) | \( \|u - u_b^*\|_{\mathcal{T}_h} \) | \( \|\nabla \times (u - u_b^*)\|_{\mathcal{T}_h} \) |
|---|---|---|---|---|---|---|
| 1 | \( 2^3 \) | 2.14e-1 | 2.96 | 8.07e+0 | 1.71 | 1.71e-1 | 2.73 |
| 1 | \( 2^4 \) | 4.43e-2 | 2.27 | 3.53e+0 | 1.19 | 3.58e-2 | 2.26 |
| 1 | \( 2^5 \) | 1.03e-2 | 2.10 | 1.67e+0 | 1.08 | 8.46e-3 | 2.08 |
| 1 | \( 2^6 \) | 2.51e-3 | 2.04 | 8.10e-1 | 1.04 | 2.09e-3 | 2.02 |
| 1 | \( 2^7 \) | 6.18e-4 | 2.02 | 4.00e-1 | 1.02 | 5.20e-4 | 2.00 |
| 2 | \( 2^3 \) | 2.20e-2 | 3.31 | 1.47e+0 | 2.25 | 1.46e-2 | 3.17 |
| 2 | \( 2^4 \) | 2.51e-3 | 3.13 | 3.40e-1 | 2.12 | 1.81e-3 | 3.02 |
| 2 | \( 2^5 \) | 3.00e-4 | 3.06 | 8.15e-2 | 2.06 | 2.26e-4 | 3.00 |
| 2 | \( 2^6 \) | 3.67e-5 | 3.03 | 2.00e-2 | 2.03 | 2.82e-5 | 3.00 |
| 2 | \( 2^7 \) | 4.54e-6 | 3.02 | 4.94e-3 | 2.02 | 3.52e-6 | 3.00 |
5.2 Parallelogram Mesh

The mesh $\mathcal{T}_h$ is assumed to consist of parallelograms. For this mesh we construct $\mathcal{T}_h^*$ by subdividing each parallelogram into two subtriangles. The triangular mesh is assumed to be shape regular so satisfying the requirements from Section 4.1.

For the parallelogram mesh we have the following which shows that $\mathcal{P}_k$ is not sufficient on such elements:

**Lemma 35.** For any integer $k \geq 1$, let

$$V(K) = \mathcal{P}_k(K), \quad W(K) = \mathcal{P}_k(K),$$

$$M(\partial K) = \{ \mu : \mu|_F = n \times p_k, \text{ for some } p_k \in \mathcal{P}_k(F) \text{ and for each edge } F \subset \partial K \},$$

then we have

$$I_M(V(K) \times W(K)) = 2.$$

**Proof.** By Lemma 32 we have

$$\dim \{ n \times v|_{\partial K} : v \in V(K), \nabla \times v = 0 \} = 1, \quad \dim M(\partial K) = 4k + 4.$$ 

For $k \geq 1$, we have

$$\dim \{ n \times w \times n|_{\partial K} : w \in W(K), \nabla \times w = 0 \}$$

$$= \dim \{ n \times (\nabla p_{k+1}) \times n : p_{k+1} \in \mathcal{P}_{k+1}(K) \}$$

$$= \dim \{ \nabla p_{k+1} : p_{k+1} \in \mathcal{P}_{k+1}(K) \} - \dim \{ p_{k+1} \in \mathcal{P}_{k+1}(K) : n \times (\nabla p_{k+1}) \times n = 0 \}$$

$$= \dim \{ \nabla p_{k+1} : p_{k+1} \in \mathcal{P}_{k+1}(K) \} - \dim \{ p_{k-3} \Pi_{i=1}^4 (a_i \lambda_i + b_i \lambda_i + c_i) : p_{k-3} \in \mathcal{P}_{k-3}(K) \}$$

$$= \left( k + 3 \right) \frac{3}{2} - 1 - \left( k - 1 \right) \frac{1}{2}$$

$$= 4k + 1.$$ 

This implies our result. \(\square\)

By enriching the space we can arrive at an $M$-decomposable set of spaces (note that since $I_M = 2$ in Lemma 35 we add just two functions to the spaces in that lemma):

**Lemma 36 (Enriched Construction I).** For any integer $k \geq 0$, let

$$V(K) = \mathcal{P}_k(K), \quad W(K) = \mathcal{P}_k(K) + \nabla \{ x^{k+1} y, x y^{k+1} \},$$

$$M(\partial K) = \{ \mu : \mu|_F = n \times p_k, \text{ for some } p_k \in \mathcal{P}_k(F) \text{ and for each edge } F \subset \partial K \},$$

$$\tilde{V}(K) = \mathcal{P}_{k-1}(K), \quad \tilde{W}(K) = \nabla \times V(K) \oplus W_0(K).$$

Then $V(K)$ and $W(K)$ admit an $M$-decomposition with respect the spaces $\tilde{V}(K)$ and $\tilde{W}(K)$.

**Proof.** As discussed in Lemma 33 we only need to prove the condition of (5.1c). Since $\gamma \tilde{V}^\bot(K) \oplus \gamma \tilde{W}^\bot(K) \subset n \times \mathcal{P}_k(\partial K)$, then we only need to prove

$$\dim M(\partial K) = \dim V(K) - \dim \tilde{V}(K) + \dim W(K) - \dim (\nabla \times V(K)) - \dim (W_0(K)),$$
i.e., we need to prove \( \dim(W_0(K)) = \binom{k-1}{2} \).

Since \( \nabla \times \mathbf{w}_0 = 0 \), there exists \( q_{k+1} \in \mathcal{P}_{k+1}(K) \), and constants \( a, b \), such that \( \mathbf{w}_0 = \nabla q_{k+1} + a \nabla x^{k+1}y + b \nabla xy^{k+1} \). Therefore, by a direct calculation we can get

\[
0 = \mathbf{n} \times \mathbf{w}_0|_{\partial K} \times \mathbf{n} = \mathbf{n} \times (\nabla q_{k+1} + a \nabla x^{k+1}y + b \nabla xy^{k+1})|_{\partial K} \times \mathbf{n} \\
= \nabla_{\partial K} q_{k+1} + a \nabla_{\partial K} x^{k+1}y + b \nabla_{\partial K} xy^{k+1}.
\]

Then \( a = -b \) when \( k = 0 \), \( a = b = 0 \) when \( k \geq 1 \), and \( q_{k+1} \) is a constant on \( \partial K \). Therefore, there is a constant \( C_0 \) and \( p_{k-3} \in \mathcal{P}_{k-3}(K) \), such that \( q_{k+1} = p_{k-3} \Pi_i^{k+3}(a_ix + b_i y + c_i) + C_0 \). Then it holds

\[
\mathbf{w}_0 = \nabla (p_{k-3} \Pi_i^{k+3}(a_ix + b_i y + c_i)).
\]

This implies

\[
\dim(W_0(K)) = \binom{k-1}{2}.
\]

Now, we give another construction:

**Lemma 37** (Enriched Construction II). For any integer \( k \geq 0 \), let

\[
V(K) = \mathcal{P}_k(K), \quad W(K) = \mathcal{P}_k(K) + \nabla \{ x^{k+1}y, xy^{k+1} \} + \binom{y}{z} \overline{\mathcal{P}}_k(K),
\]

\[
M(\partial K) = \{ \mathbf{u} \mid \mathbf{u}|_F = \mathbf{n} \times p_k \text{ for some } p_k \in \mathcal{P}_k(F) \text{ and for each edge } F \subset \partial K \},
\]

\[
\overline{W}(K) = \nabla \times V(K) \oplus W_0(K).
\]

Then \( V(K) \) and \( W(K) \) admit an \( M \)-decomposition with respect the spaces \( \overline{V}(K) \) and \( \overline{W}(K) \).

In Tables 3 and 4 we show numerical results on a parallelogram with a uniform parallelogram mesh. We obtain the optimal convergence rate for the solution \( \mathbf{u} \) and superconvergence rate for \( \nabla \times \mathbf{u} \). In terms of accuracy, and order of convergence, the enriched space in Lemma 37 does not offer any advantages over the space in Lemma 36 as is to be expected since the space in Lemma 36 was already sufficiently enriched to have an \( M \)-decomposition.

### Table 3: Results for parallelogram mesh and enriched case I on \( \Omega = \{(x, y) : 0 \leq x - \sqrt{3}y \leq 1, 0 \leq y \leq 1/2\}\)

| \( k \) | \( \sqrt{2} \) | \( n \) | \( \| \mathbf{u} - \mathbf{u}_h \|_{\mathcal{T}_h} \) | \( \| \nabla \times (\mathbf{u} - \mathbf{u}_h) \|_{\mathcal{T}_h} \) | \( \| q - q_h \|_{\mathcal{T}_h} \) | \( \| \mathbf{u} - \mathbf{u}_h^* \|_{\mathcal{T}_h} \) | \( \| \nabla \times (\mathbf{u} - \mathbf{u}_h^*) \|_{\mathcal{T}_h} \) |
|---|---|---|---|---|---|---|---|
| 1 | \( 2^3 \) | 1.30e+1 | 2.98 | -2.58 | 1.15e+2 | -3.63 | 4.35e+0 | 1.29e+1 | 4.35e+0 |
| | \( 2^4 \) | 1.57e-1 | 6.37 | 5.11e+0 | 4.49 | 6.49e-2 | 6.07 | 1.54e-1 | 6.39 | 6.49e-2 | 6.07 |
| | \( 2^5 \) | 1.48e-2 | 3.41 | 2.11e+0 | 1.28 | 1.08e-2 | 2.59 | 1.32e-2 | 3.55 | 1.08e-2 | 2.59 |
| | \( 2^6 \) | 2.52e-3 | 2.55 | 1.02e+0 | 1.04 | 2.37e-3 | 2.19 | 1.93e-3 | 2.77 | 2.37e-3 | 2.19 |
| | \( 2^7 \) | 5.23e-4 | 2.27 | 5.07e-1 | 1.01 | 5.71e-4 | 2.05 | 3.34e-4 | 2.53 | 5.71e-4 | 2.05 |
| 2 | \( 2^3 \) | 7.19e-1 | 2.86 | 9.18e+0 | 2.74 | 1.40e-1 | 4.45 | 7.15e-1 | 2.84 | 1.40e-1 |
| | \( 2^4 \) | 1.46e-2 | 5.62 | 8.16e-1 | 3.49 | 2.68e-3 | 5.71 | 1.43e-2 | 5.65 | 2.68e-3 | 5.71 |
| | \( 2^5 \) | 1.35e-3 | 3.44 | 1.66e-1 | 2.29 | 3.23e-4 | 3.05 | 1.31e-3 | 3.45 | 3.23e-4 | 3.05 |
| | \( 2^6 \) | 1.49e-4 | 3.18 | 3.85e-2 | 2.11 | 4.03e-5 | 3.00 | 1.44e-4 | 3.18 | 4.03e-5 | 3.00 |
| | \( 2^7 \) | 1.76e-5 | 3.08 | 9.29e-3 | 2.05 | 5.04e-6 | 3.00 | 1.70e-5 | 3.09 | 5.04e-6 | 3.00 |
Table 4: Results for parallelogram mesh and enriched case II on $\Omega = \{(x, y) : 0 \leq x - \sqrt{3}y \leq 1, 0 \leq y \leq 1/2\}$

| $n$ | $\sqrt{\frac{h}{n}}$ | $\|u - u_h\|_{T_h}$ | Rate | $\|\nabla \times (u - u_h)\|_{T_h}$ | Rate | $\|q - q_h\|_{T_h}$ | Rate | $\|u - u_h^*\|_{T_h}$ | Rate | $\|\nabla \times (u - u_h^*)\|_{T_h}$ | Rate |
|-----|-----------------|-----------------|------|-----------------|------|-----------------|------|-----------------|------|-----------------|------|
| 1   | $2^4$           | 1.20e+1         | -0.64| 1.75e+2         | -0.54| 3.61e+0         | 1.66 | 1.19e+1         | -0.68| 3.61e+0         |       |
|     | $2^2$           | 1.60e-1         | 6.23 | 5.18e+0         | 5.08 | 6.52e-2         | 5.79 | 1.56e-1         | 6.24 | 6.52e-2         | 5.79 |
|     | $2^2$           | 1.47e-2         | 3.44 | 2.11e-0         | 1.30 | 1.08e-2         | 2.59 | 1.31e-2         | 3.58 | 1.08e-2         | 2.59 |
|     | $2^2$           | 2.51e-3         | 2.55 | 1.02e-0         | 1.05 | 2.37e-3         | 2.19 | 1.91e-3         | 2.78 | 2.37e-3         | 2.19 |
|     | $2^2$           | 5.22e-4         | 2.27 | 5.05e-1         | 1.01 | 5.71e-4         | 2.05 | 3.32e-4         | 2.53 | 5.71e-4         | 2.05 |

5.3 Rectangle Mesh

The mesh $T_h$ is assumed to consist of squares. For this mesh we construct $T_h^*$ by subdividing each square into two subtriangles. The triangular mesh is shape regular so satisfying the requirements from Section 4.1 (for a general rectangular mesh, the triangular mesh must be shape regular).

In this section, we assume that all elements $K$ are squares with edges parallel to the coordinate axes. We denote by $Q_k$ the standard space of polynomials in two variables with maximum degree $k$ in each variable. Unlike in the parallelogram case, we consider the use of $Q_k$ based elements as these are often used for square elements. Our first lemma shows that simple $Q_k$ elements alone do not suffice.

**Lemma 38.** For any integer $k \geq 1$, let

$$V(K) = Q_k(K), \quad W(K) = Q_k(K),$$

$$M(\partial K) = \{ \mu : \mu|_F = n \times p_k, \text{ for some } p_k \in P_k(F) \text{ and for each edge } F \subset \partial K \}.$$

We have

$$I_M(V(K) \times W(K)) = 2.$$

**Proof.** It is easy to see that

$$\dim \{ n \times v|_{\partial K} : v \in V(K), \nabla \times v = 0 \} = 1, \quad \dim M(\partial K) = 4k + 4.$$

Moreover, we have

$$\dim \{ n \times w \times n|_{\partial K} : w \in W(K), \nabla \times w = 0 \}$$

$$= \dim \{ n \times (\nabla p_{k+1}) \times n : p_{k+1} = x^\alpha y^\beta, \alpha \leq k, \beta \leq k; \alpha = k + 1, \beta = 0; \alpha = 0, \beta = k + 1 \}$$

$$= \dim \{ \nabla p_{k+1} : p_{k+1} = x^\alpha y^\beta, \alpha \leq k, \beta \leq k \} - \dim \{ p_{k+1} = x^\alpha y^\beta, \alpha \leq k - 2, \beta \leq k - 2 \}$$

$$= (k + 1)^2 + 2 - 1 - (k - 1)^2$$

$$= 4k + 1.$$

This implies that $I_M(V(K) \times W(K)) = 2$ and completes our proof. \hfill $\Box$

33
The previous result shows that we must add two basis functions to the space. A possible choice is given by the following lemma:

**Lemma 39 (Enriched Construction I).** For any integer $k \geq 1$, let

\[
V(K) = \mathcal{Q}_k(K), \quad W(K) = \mathcal{Q}_k(K) + \nabla \text{span}\{x^{k+1}, y^{k+1}\},
\]

\[
M(\partial K) = \{ \mu \mid \mu|_F = n \times p_k \text{ for some } p_k \in \mathcal{P}_k(F) \text{ and for each edge } F \subset \partial K \},
\]

\[
\tilde{V}(K) = \mathcal{Q}_{k-1}(K), \quad \tilde{W}(K) = \nabla \times V(K) \oplus W_0(K).
\]

Then $V(K)$ and $W(K)$ admit an $M$-decomposition with respect to the spaces $\tilde{V}(K)$ and $\tilde{W}(K)$.

We omit the proofs of this and the following lemma, since they are similar to the proofs in the previous section. An alternative choice of enriched space is given next.

**Lemma 40 (Enriched Construction II).** For any integer $k \geq 0$, let

\[
V(K) = \mathcal{Q}_k(K), \quad W(K) = \mathcal{Q}_k(K) + \nabla \text{span}\{x^{k+1}, y^{k+1}\} + \text{span}\left\{\left\{x^{k+1}\right\}\right\},
\]

\[
M(\partial K) = \{ \mu \mid \mu|_F = n \times p_k \text{ for each edge } F \subset \partial K \},
\]

\[
\tilde{V}(K) = \mathcal{Q}_k(K), \quad \tilde{W}(K) = \nabla \times V(K) \oplus W_0(K).
\]

Then $V(K)$ and $W(K)$ admit an $M$-decomposition with respect to the spaces $\tilde{V}(K)$ and $\tilde{W}(K)$.

In Table 5 we show the numerical results on unit square with rectangle mesh and we obtain optimal convergence rate for the solution $u$ and superconvergence rate for $\nabla \times u$ using Enrichment Construction I elements. Numerical results for Enrichment Construction II elements show that exactly the same error is observed so we do not reproduce the results here.

| $k$ | $\frac{\sqrt{2}}{n}$ | $\|u - u_h\|_{T_h}$ | $\|\nabla \times (u - u_h)\|_{T_h}$ | $\|q - q_h\|_{T_h}$ | $\|u - u^*_h\|_{T_h}$ | $\|\nabla \times (u - u^*_h)\|_{T_h}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
|     | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 1   | 2$^4$  | 1.78e-1 | 2.60 | 6.93e+0 | 1.46 | 1.41e-1 | 2.65 | 4.09e-2 | 3.82 | 2.17e-1 |
|     | 2$^4$  | 3.90e-2 | 2.19 | 3.05e+0 | 1.19 | 2.88e-2 | 2.29 | 5.38e-3 | 2.93 | 5.11e-2 | 2.09 |
|     | 2$^5$  | 9.19e-3 | 2.08 | 1.44e+0 | 1.09 | 6.83e-3 | 2.08 | 9.12e-4 | 2.56 | 1.26e-2 | 2.02 |
|     | 2$^6$  | 2.23e-3 | 2.04 | 6.98e-1 | 1.04 | 1.68e-3 | 2.02 | 1.96e-4 | 2.22 | 3.14e-3 | 2.00 |
|     | 2$^7$  | 5.51e-4 | 2.02 | 3.44e-1 | 1.02 | 4.20e-4 | 2.00 | 4.70e-5 | 2.06 | 7.85e-4 | 2.00 |
| 2   | 2$^4$  | 1.86e-2 | 2.87 | 8.44e-1 | 2.32 | 8.50e-3 | 3.09 | 1.37e-2 | 2.08 | 2.44e-2 |
|     | 2$^4$  | 1.10e-3 | 4.08 | 1.37e-1 | 2.62 | 8.76e-4 | 3.28 | 4.76e-4 | 4.84 | 3.09e-3 | 2.98 |
|     | 2$^5$  | 1.34e-4 | 3.05 | 3.31e-2 | 2.05 | 1.09e-4 | 3.00 | 5.64e-5 | 3.08 | 3.88e-4 | 2.99 |
|     | 2$^6$  | 1.65e-5 | 3.02 | 8.13e-3 | 2.02 | 1.36e-5 | 3.00 | 6.90e-6 | 3.03 | 4.85e-5 | 3.00 |
|     | 2$^7$  | 2.04e-6 | 3.01 | 2.02e-3 | 2.01 | 1.70e-6 | 3.00 | 8.53e-7 | 3.02 | 6.07e-6 | 3.00 |

**6 Conclusion**

In this paper we have shown that the $M$-decomposition, together with sufficiently rich auxiliary spaces, is sufficient to guarantee optimal order convergence for the vector 2D problem arising from Maxwell’s equations. This can be used to evaluate and construct HDG schemes on two commonly occurring elements (triangles and squares).
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An interesting extension which we have not yet completed would be to devise spaces on general quadrilateral elements that have an $M$-decomposition. More interesting is to devise a similar theory for the full Maxwell’s equations in three dimensions. Not only is this more complicated, but it is also essentially different compared to 2D. This will be explored in our future work.

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