Extinction and persistence of a stochastic SIRS model with nonlinear incidence rate and transfer from infectious to susceptible

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Abstract. In this paper, we propose a stochastic SIRS model with nonlinear incidence rate and transfer from infectious to susceptible. We first give the existence and uniqueness of the positive solution for this system. Moreover, the extinction of the disease and the persistence in the mean are established in the terms of a threshold value.

1. Introduction
Infectious diseases have always threatened the health of human beings and have brought enormous disasters to human beings. In order to protect human life and control the prevalence of infectious diseases, researchers have proposed various mathematical models. The first attempt is from Kermack and McKendrick [1], in which they studied the classical epidemic model known as SIR model to study epidemiology. Since then, various models have been used to describe various kinds of epidemics, and the dynamics of these systems have been investigated. These SIR models are proposed based on the hypothesis that the infected individuals can be recovered and immunized throughout one's life. However, it is not reasonable in the study of some communicable diseases, such as cholera, pertussis, influenza, and malaria, in which the recovery class may lose immunity after some time and back to the susceptible class. Therefore, Mena-Lorca and Hethcote [2] proposed the SIRS epidemic model. Moreover, in some cases, the infected individuals may recover after some treatments and go back directly to the susceptible class because of transient antibody. Li et al. [3] studied the following SIRS epidemic model with a nonlinear incidence rate and transfer from the infected class to the susceptible class

\[
\begin{align*}
\dot{S}(t) &= \Lambda - \mu S(t) - S(t)I(t) + \gamma_1 I(t) + \delta R(t), \\
\dot{I}(t) &= S(t)I(t) - (\mu + \gamma_1 + \gamma_2 + d)I(t), \\
\dot{R}(t) &= \gamma_2 I(t) - (\mu + \delta)R(t).
\end{align*}
\]

As we all know, real life is filled with randomness and unpredictability. For human disease related epidemics, the nature of epidemic growth and spread is random due to the unpredictability in person-to-person contacts. Therefore the variability and randomness of the environment is fed through the state of the epidemic. And in epidemic dynamics, stochastic models may be a more appropriate way of
modeling epidemics in many circumstances (see e.g., [4-9]). According to system (1), Song et al. [10] considered the following stochastic SIRS model with saturated incidence rate and transfer from infectious to susceptible

\[
\begin{align*}
S(t) &= \left( \Lambda - \mu S(t) - \frac{\beta S(t)I(t)}{1 + \alpha I(t)} + \gamma_1 I(t) + \delta R(t) \right)dt - \frac{\sigma S(t)I(t)}{1 + \alpha I(t)}dB(t), \\
I(t) &= \left( \frac{\beta S(t)I(t)}{1 + \alpha I(t)} - (\mu + \gamma_1 + \gamma_2 + d) I(t) \right)dt + \frac{\sigma S(t)I(t)}{1 + \alpha I(t)}dB(t), \\
R(t) &= \left( \gamma_2 I(t) - (\mu + \delta) R(t) \right)dt,
\end{align*}
\]

where \( \Lambda \) is the birth rate, \( \mu \) is the natural mortality rate, \( \beta \) is the average number of adequate contacts, \( \alpha \) denotes the half-saturation constant, \( \gamma_1 \) is the transfer rate from the infective individuals to the susceptible individuals, \( \gamma_2 \) is the recovery rate of the infective individuals, \( d \) is the mortality due to illness, \( \delta \) is the rate at which the recovered individuals lose of immunity and return to the susceptible individuals, \( B(t) \) is a standard Brownian motion with intensity \( \sigma^2 > 0 \).

In this paper, by following the approach used in Mao et al. [11], we assume that the parameters involved in the model always fluctuate around some average value and the environmental noise is proportional to the variables \( S(t), I(t) \) and \( R(t) \). Therefore, we will study the following stochastic SIRS model with a specific nonlinear incidence rate and transfer from infectious to susceptible

\[
\begin{align*}
S(t) &= \left( \Lambda - \mu S(t) - \frac{\beta S(t)I(t)}{1 + \alpha S + \alpha_1 I + \alpha_2 SI} + \gamma_1 I(t) + \delta R(t) \right)dt + \sigma_1 S(t)dB_1(t), \\
I(t) &= \left( \frac{\beta S(t)I(t)}{1 + \alpha S + \alpha_1 I + \alpha_2 SI} - (\mu + \gamma_1 + \gamma_2 + d) I(t) \right)dt + \sigma_1 I(t)dB_2(t), \\
R(t) &= \left( \gamma_2 I(t) - (\mu + \delta) R(t) \right)dt + \sigma_3 R(t)dB_3(t),
\end{align*}
\]

where the incidence rate was introduced by Hattaf et al. [12], which includes various type of incidence rate existing in the literatures, especially the incidence rate in (2) when \( \alpha_1 = \alpha_2 = 0, B_i(t)(i = 1, 2, 3) \) are standard Brownian motions with intensity \( \sigma_i^2 > 0 \) which defined on complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We also let \( \mathbb{R}_+^d = \{ x \in \mathbb{R}^d : x_i > 0, 0 < i < d \} \) and \( a \wedge b = \min \{ a, b \} \). The purpose of this work is to investigate the asymptotic behaviors of system (3), and establish a threshold that determines the extinction and the persistence of the disease.

2. Existence and uniqueness of the positive solution

For biological rationality, in this section, we will derive the existence and uniqueness of the global positive solution.

**Theorem 2.1.** For any initial value \( X_0 = (S_0, I_0, R_0) \in \mathbb{R}_+^3 \), there is a unique positive solution \( X(t) = (S(t), I(t), R(t)) \) of system (3) on \( t \geq 0 \), and the solution will maintain in \( \mathbb{R}_+^3 \) with probability one.

**Proof.** Since the coefficients of system (3) satisfy the local Lipschitz condition, then for any initial value \( (S_0, I_0, R_0) \in \mathbb{R}_+^3 \), it admits a unique local solution \( (S(t), I(t), R(t)) \) on \( [0, \tau_e) \), where \( \tau_e \) is the explosion time [13]. To verify that this solution is global, we only need to show that \( \tau_e = \infty \) a.s. For this purpose, we define the following stopping time

\[
t = \inf \left\{ t > 0 : X(t) \notin \mathbb{R}_+^3 \right\}
\]
\[ \tau^* = \inf\{t \in [0, \tau_\varepsilon) : S(t) \leq 0, \text{ or } I(t) \leq 0 \text{ or } R(t) \leq 0\}, \]

where we set \( \inf \Phi = \infty \) (\( \Phi \) is empty set). Obviously, \( \tau^* \leq \tau_\varepsilon \), and it suffices to prove \( \tau^* = \infty \) a.s.

Next, we adopt the contradiction method to finish the proof. That is, we assume \( \tau^* < \infty \), then there exists a \( T > 0 \) such that \( \mathbb{P}\{\tau^* < T\} > 0 \).

Applying Itô’s formula on the function \( \ln S(t) + \ln I(t) + \ln R(t) \) for all \( t \in [0, \tau^*] \) and \( \omega \in \{\tau^* < T\} \), we have

\[
d(\ln S + \ln I + \ln R) = \left( \frac{\Lambda}{S} - \mu - \frac{\beta I}{1 + \alpha S + \alpha I + \alpha S I} + \frac{\gamma I}{S} + \frac{\delta R}{S} - \frac{\sigma_1^2}{2} \right) dt + \frac{\beta I}{1 + \alpha S + \alpha I + \alpha S I}(\mu + \gamma I + \gamma R + d) dt \]

\[
+ \left( \frac{\gamma I}{R} - (\mu + \delta) - \frac{\sigma_1^2}{2} \right) dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t) + \sigma_3 dB_3(t) \geq -\left( 3\mu + \gamma I + \gamma R + d + \delta + \frac{\beta}{\alpha S} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \right) t + \sigma_1 B_1(t) + \sigma_2 B_2(t) + \sigma_3 B_3(t). \tag{4} \]

From the definition of \( \tau^* \), we have that the solution of system (3) is positive on \([0, \tau^*]\) for almost all \( \omega \in \{\tau^* < T\} \) and \( S(t) \) \( I(t) \) \( R(t) = 0 \). Therefore,

\[
\lim_{t \to \tau^*} (\ln S(t) + \ln I(t) + \ln R(t)) = -\infty. \tag{5} \]

Integrating both sides of (4) from 0 to \( t(\leq \tau^*) \), then

\[
\ln S(t) + \ln I(t) + \ln R(t) - \ln S(0) - \ln I(0) - \ln R(0) \geq -\left( 3\mu + \gamma I + \gamma R + d + \delta + \frac{\beta}{\alpha S} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \right) t + \sigma_1 B_1(t) + \sigma_2 B_2(t) + \sigma_3 B_3(t). \tag{6} \]

Letting \( t \to \tau^* \) in (6) and using (5), we obtain

\[
-\infty \geq -\left( 3\mu + \gamma I + \gamma R + d + \delta + \frac{\beta}{\alpha S} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \right) \tau^* + \sigma_1 B_1(\tau^*) + \sigma_2 B_2(\tau^*) + \sigma_3 B_3(\tau^*) \geq -\infty, \]

which leads a contradiction. Thus, \( \tau^* = \tau_\varepsilon = +\infty \) a.s., and this completes the proof.

For the corresponding deterministic system of (3), it have been shown that the region

\[
\Omega = \left\{ (S, I, R) \in \mathbb{R}^3 : S + I + R \leq \frac{\Lambda}{\mu} \right\}
\]

is a positively invariant and attractive set [3]. Now, we will show that the region \( \Omega \) is almost surely positively invariant set of the stochastic system (3), which is crucial in the following estimations in the analysis of the dynamics for (3).

**Theorem 2.2.** \( \Omega \) is almost surely positively invariant set of system (3), i.e., if \( (S(0), I(0), R(0)) \in \Omega \), then

\[
\mathbb{P}\{S(t), I(t), R(t) \in \Omega \} = 1, \text{ for all } t \geq 0. \]

**Proof.** First, we define \( X(t) = (S(t), I(t), R(t)) \). Choose \( n_0 \) be a positive constant large enough such that each component of \( X(t) \) with initial value \( X(0) = (S(0), I(0), R(0)) \in \Omega \) lies in the interval

\[
\left( \frac{1}{n_0} \frac{\Lambda}{\mu} \right]. \]

For each integer \( n \geq n_0 \), we define the following stopping times
\[ \tau_n = \inf \left\{ t > 0 : X(t) \in \Omega, X(t) \notin \left( \frac{1}{n} \frac{\Lambda}{\mu} \right)^3 \right\}, \]
\[ \tau = \inf \left\{ t > 0 : X(t) \notin \Omega \right\}. \]

Therefore, we only need to prove that \( \mathbb{P}\{\tau < t\} = 0 \) for all \( t > 0 \). Obviously, \( \tau_n \leq \tau \), which infers that \( \mathbb{P}\{\tau < t\} \leq \mathbb{P}\{\tau_n < t\} \). Hence, it suffices to prove that \( \limsup_{n \to \infty} \mathbb{P}\{\tau_n < t\} = 0 \). For this purpose, we construct a \( C^2 \)-function \( F(X) : \mathbb{R}^3 \to \mathbb{R} \), defined by

\[ F(X) = \frac{1}{S} + \frac{1}{I} + \frac{1}{R}, \quad X = (S, I, R). \]

For all \( t > 0 \) and \( u \in [0, t \wedge \tau_n] \), utilizing Itô’s formula on \( F \), we have

\[ dF(S(u), I(u), R(u)) \]
\[ = \left( -\frac{\Lambda}{S^2(u)} + \frac{\mu}{S(u)} + \frac{\beta I(u)}{S(u)(1 + \alpha S + \alpha_I + \alpha_S SI)} - \frac{\gamma I(u)}{S^2(u)} + \frac{\delta R(u)}{S^2(u)} + \frac{\sigma^2}{S(u)} \right) du \]
\[ + \left( \frac{\mu + \gamma I(u)}{I(u)} - \frac{\beta S(u)}{I(u)(1 + \alpha S + \alpha_I + \alpha_S SI)} + \frac{\sigma^2}{I(u)} \right) du \]
\[ + \left( \frac{\mu + \gamma}{R(u)} - \frac{\gamma I(u)}{R^2(u)} + \frac{\sigma^2}{R(u)} \right) du - \frac{\sigma_1}{S(u)} dB_1(u) - \frac{\sigma_2}{I(u)} dB_2(u) - \frac{\sigma_3}{R(u)} dB_3(u). \]

Due to \( u \leq \tau_n \), we have that \( I(u) \leq \frac{\Lambda}{\mu} \), which implies that

\[ dF(S(u), I(u), R(u)) \leq \rho F(X(u)) du - \frac{\sigma_1}{S(u)} dB_1(u) - \frac{\sigma_2}{I(u)} dB_2(u) - \frac{\sigma_3}{R(u)} dB_3(u), \]  \( \text{where} \)

\[ \rho := \max \left\{ \mu + \frac{\beta \Lambda}{\mu} + \sigma_1^2, \mu + \gamma + \gamma_2 + d + \sigma_2^2, \mu + \delta + \sigma_3^2 \right\}. \]

Solving the differential inequality and taking expectations on both sides of (7), we have

\[ EF(X(u)) \leq F(X(0)) + \rho \int_0^u EF(X(s)) ds. \]

Applying Gronwall inequality, we obtain for \( \forall u \in [0, t \wedge \tau_n] \),

\[ EF(X(u)) \leq F(X(0))e^{\rho u}. \]

Hence,

\[ EF(X(t \wedge \tau_n)) \leq F(X(0))e^{\rho (t \wedge \tau_n)} \leq F(X(0))e^{\rho t}, \quad \forall t > 0. \]

Due to the definition of \( \tau_n \), it infers that at least one of \( S(\tau_n), I(\tau_n), R(\tau_n) \) is less than or equal to \( \frac{1}{n} \), which implies that

\[ EF(X(t \wedge \tau_n)) \geq E \left[ F(X(\tau_n))X_{\{\tau_n \leq t\}} \right] \geq n \mathbb{P}(\tau_n < t). \]
From (8) and (9), we have that for all \( t > 0 \),
\[
\mathbb{P}(\tau_n < t) \leq \frac{F(X(0))e^{\alpha t}}{n}.
\]

Taking superior limit on the above inequality, we have \( \limsup_{n \to \infty} \mathbb{P}(\tau_n < t) = 0 \), and this completes the proof.

3. Extinction of the disease

In this section, we will establish the sufficient conditions for the extinction of the disease, which is very important in the prevention and control of diseases. It is easy to see that the basic reproduction number of the corresponding deterministic system of (3) is
\[
R_0^D = \frac{\beta \Lambda}{(\mu + \alpha \Lambda)(\mu + \gamma_1 + \gamma_2 + d)},
\]
which characterizes the dynamical behaviors of the deterministic system and determines the extinction and persistence of the disease. Similarly, we define the following value of the stochastic model (3) as follows
\[
R_0^S = R_0^D - \frac{\sigma_2^2}{2(\mu + \gamma_1 + \gamma_2 + d)}.
\]

Obviously, \( R_0^S \leq R_0^D \), and \( R_0^S = R_0^D \) when \( \sigma_2 = 0 \).

For simplicity, we denote \( \langle x(t) \rangle = \frac{1}{t} \int_0^t x(u)du \).

**Theorem 3.1.** Let \((S(t), I(t), R(t))\) be the solution of system (3) with initial value \((S(0), I(0), R(0)) \in \Omega\). If \( R_0^S < 1 \), then
\[
\limsup_{t \to \infty} \frac{\ln I(t)}{t} \leq (\mu + \gamma_1 + \gamma_2 + d)(R_0^S - 1) < 0 \quad \text{a.s.}
\]
That is, the disease dies out exponentially with probability one. Moreover,
\[
\lim_{t \to \infty} \langle S(t) \rangle = \frac{\Lambda}{\mu}, \quad \lim_{t \to \infty} R(t) = 0 \quad \text{a.s.}
\] (10)

**Proof.** Firstly, by using Itô’s formula on \( I(t) \), we have
\[
d \ln I(t) = \left[ \frac{\beta S(t)}{1 + \alpha_1 S(t) + \alpha_2 I(t) + \alpha_3 SI(t)} - \left( \mu + \gamma_1 + \gamma_2 + d + \frac{\sigma_2^2}{2} \right) \right] dt + \sigma_2 dB_2(t). \] (11)

Noting that \( \frac{\beta S}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI} \) has the following decomposition:
\[
\frac{\beta S}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI} = \frac{\beta \Lambda}{\mu + \alpha_1 \Lambda} - \frac{\beta \mu}{(1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI)(\mu + \alpha_1 \Lambda)} \left( \frac{\Lambda - S}{\mu} \right) - \frac{\beta \alpha_1 \Lambda}{(1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI)(\mu + \alpha_1 \Lambda)} I
\]
\[
- \frac{\beta \alpha_2 \Lambda}{(1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI)(\mu + \alpha_1 \Lambda)} SI
\] (12)
then, we have
\[
\frac{\beta S}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI} \leq \frac{\beta \Lambda}{\mu + \alpha_1 \Lambda}.
\]
Substituting the above inequality into (11), we have
\[ d \ln I(t) \leq \left[ \frac{\beta \Lambda}{\mu + \alpha_i \Lambda} - \left( \mu + \gamma_1 + \gamma_2 + d + \frac{\sigma_2^2}{2} \right) \right] dt + \sigma_2 dB_2(t). \]

Integrating on both sides of above equality from 0 to \( t \) and dividing by \( t \), we obtain that
\[ \frac{\ln I(t)}{t} \leq \frac{\beta \Lambda}{\mu + \alpha_i \Lambda} - \left( \mu + \gamma_1 + \gamma_2 + d + \frac{\sigma_2^2}{2} \right) + \frac{\sigma_2 B_2(t)}{t} + \frac{\ln I(0)}{t}, \tag{13} \]

Since \( B_2(t) \) is a continuous square-integrable martingale and its quadratic variation \( \{B_2, B_2\}_t = t \) for all \( t \geq 0 \), combined with the strong law of large numbers, we have
\[ \lim_{t \to \infty} \frac{B_2(t)}{t} = 0 \text{ a.s.} \]

Hence, if \( R_0^S < 1 \), from (13) we have
\[ \limsup_{t \to \infty} \frac{\ln I(t)}{t} \leq (\mu + \gamma_1 + \gamma_2 + d)(R_0^S - 1) < 0 \text{ a.s.}, \]

which implies that \( \lim I(t) = 0 \) a.s. By system (3), it is easily concluded that \( \lim R(t) = 0 \) a.s.

Finally, we show the asymptotic behavior of \( S(t) \). Applying Itô’s formula on \( S(t) + I(t) + R(t) \), we obtain
\[ d(S(t) + I(t) + R(t)) = [\Lambda - \mu(S(t) + I(t) + R(t)) - dI(t)]dt + \sigma_1 S(t)dB_1(t) + \sigma_2 I(t)dB_2(t) + \sigma_3 R(t)dB_3(t). \]

Consequently, we have
\[ S(t) + I(t) + R(t) \leq S(0) + I(0) + R(0) \]
\[ = \frac{\Lambda}{\mu} - \frac{\mu + d}{\mu} \left( I(t) - I(0) - R(t) \right) \]
\[ + \frac{\sigma_1}{t} \int_0^t S(u)dB_1(u) + \frac{\sigma_2}{t} \int_0^t I(u)dB_2(u) + \frac{\sigma_3}{t} \int_0^t R(u)dB_3(u). \]

Then,
\[ \{S(t)\} = \frac{\Lambda}{\mu} - \frac{\mu + d}{\mu} \{I(t)\} - \{R(t)\} - \phi(t), \tag{14} \]

where
\[ \phi(t) = \frac{S(t) + I(t) + R(t) - S(0) + I(0) + R(0)}{\mu t} \frac{\sigma_1}{\mu t} \int_0^t S(u)dB_1(u) - \frac{\sigma_2}{\mu t} \int_0^t I(u)dB_2(u) - \frac{\sigma_3}{\mu t} \int_0^t R(u)dB_3(u). \]

Since \( S(t), I(t), R(t) \in \Omega \), we have
\[ \lim_{t \to \infty} \phi(t) = 0 \text{ a.s.} \tag{15} \]

From (14), (15) and the acquired asymptotic behavior of \( I(t), R(t) \), we have
\[ \lim_{t \to \infty} \{S(t)\} = \frac{\Lambda}{\mu} \text{ a.s.} \]

The proof is complete.
4. Persistence
In this section, we will focus on the conditions for the persistence of the disease. First, we recall the
definition of persistence in the mean.

**Definition 4.1.** System (3) is said to be persistence in the mean, if
\[
\lim \inf_{t \to \infty} \langle I(t) \rangle > 0 \text{ a.s.}
\]

Further, we have the following lemma (see Lemma 5.1. in [14]).

**Lemma 4.2.** Let \( f \in C([0, \infty) \times \Omega, (0, \infty)) \) and \( F \in C([0, \infty) \times \Omega, \mathbb{R}) \) such that
\[
\lim_{t \to \infty} \frac{F(t)}{t} = 0 \text{ a.s.}
\]

If there exist positive constants \( \lambda_0, \lambda \) such that for all \( t \geq 0, \)
\[
\ln f(t) \geq \lambda t - \lambda_0 \int_0^t f(u) \, du + F(t) \text{ a.s.}
\]

Then
\[
\lim \inf_{t \to \infty} \langle f(t) \rangle \geq \frac{\lambda}{\lambda_0} \text{ a.s.}
\]

**Theorem 4.3.** If \( R^S_0 > 1 \), then the infected individual \( I(t) \) is persistence in the mean, that is,
\[
\lim \inf_{t \to \infty} \langle I(t) \rangle \geq \frac{(\mu + \gamma_1 + \gamma_2 + d)(R^S_0 - 1)(\mu + \alpha_3 \Lambda)}{\beta \mu + \mu + \Lambda \left( \frac{\alpha_2 + \alpha_3 \Lambda}{\mu} \right)} > 0.
\]

**Proof.** From Theorem 2.2, we have that \( S(t) \leq \frac{\Lambda}{\mu} \text{ a.s.} \) Combined this estimation with (12), we have
\[
\frac{\beta S}{1 + \alpha_3 S + \alpha_2 I + \alpha_3 S I} \geq \frac{\beta \Lambda}{\mu + \alpha_3 \Lambda} \left( 1 - \frac{1}{1 + \alpha_2 S + \alpha_3 I} \right)
\]

\[
+ \frac{\beta \mu S}{(1 + \alpha_2 S + \alpha_3 I)(\mu + \alpha_3 \Lambda)} - \frac{\beta \Lambda}{\mu + \alpha_3 \Lambda} \left( \frac{\alpha_2 + \alpha_3 \Lambda}{\mu} \right) I.
\]

Hence, we have
\[
d \ln I(t) \geq \left[ \frac{\beta \Lambda}{\mu + \alpha_3 \Lambda} \left( \mu + \gamma_1 + \gamma_2 + d + \frac{\sigma_2^2}{2} \right) \right] dt - \frac{\beta \Lambda}{\mu + \alpha_3 \Lambda} \left( \frac{\alpha_2 + \alpha_3 \Lambda}{\mu} \right) I dt + \sigma_2 dB_2(t).
\]

Integrating both sides of the above inequality from 0 to \( t \) and by (14), we have
\[
\ln I(t) \geq \left[ \frac{\beta \Lambda}{\mu + \alpha_3 \Lambda} \left( \mu + \gamma_1 + \gamma_2 + d + \frac{\sigma_2^2}{2} \right) \right] t - \frac{\beta}{\mu + \alpha_3 \Lambda} \left( \mu + d + \Lambda \left( \frac{\alpha_2 + \alpha_3 \Lambda}{\mu} \right) \right) \langle I(t) \rangle t + \Theta(t),
\]

where
\[
\Theta(t) = - \frac{\beta \mu}{\mu + \alpha_3 \Lambda} \left( \langle R(t) + \phi(t) \rangle t + \sigma_2 B_2(t) + \ln I(0) \right).
\]

It follows (10), (15) and the theorem of large numbers for martingales that
\[
\lim_{t \to \infty} \frac{\Theta(t)}{t} = 0 \text{ a.s.}
\]

Using Lemma 4.2, we obtain that
\[ \liminf_{t \to \infty} \langle I(t) \rangle \geq \left( \mu + \gamma_1 + \gamma_2 + d \right) (R_0^2 - 1) \left( \mu + \alpha_i \Lambda \right) > 0. \]

This completes the proof.

5. Numerical simulations and conclusions

In this section, we give some numerical simulations in order to illustrate the theoretical results. The numerical simulation is based on the Euler-Maruyama (EM) method [15]. We choose the coefficients in system (3) as \( \Lambda = 0.3, \mu = 0.1, \gamma_1 = 0.4, \gamma_2 = 0.1, d = 0.1, \alpha_1 = 0.1, \alpha_2 = 0.02, \alpha_3 = 0.03, \delta = 0.1, \sigma_1 = \sigma_3 = 0.01 \), then (3) becomes as follows:

\[
\begin{align*}
    dS(t) &= \left( 0.3 - 0.1S(t) - \frac{\beta S(t)I(t)}{1 + 0.1S + 0.02I + 0.03SI} + 0.4I(t) + 0.1R(t) \right) dt + 0.01S(t)dB_1(t), \\
    dI(t) &= \left( \frac{\beta S(t)I(t)}{1 + 0.1S + 0.02I + 0.03SI} - 0.7I(t) \right) dt + \sigma_2 I(t)dB_2(t), \\
    dR(t) &= (0.1I(t) - 0.2R(t))dt + 0.01R(t)dB_3(t).
\end{align*}
\]

At first, we set \( \beta = 0.2, \sigma_2 = 0.1 \). By calculation, we have \( R_0^S = 0.6522 < 1 \). From Theorem 3.1, we obtain that the disease dies out. Figure 1 illustrates this result.

Next, we choose \( \beta = 0.5, \sigma_2 = 0.1 \). In this case, \( R_0^S = 1.6412 > 1 \). Hence, it follows from Theorem 4.3 that system (3) is persistence in the mean, which means that the disease persists in the population (see Figure 2).

![Figure 1](image1.png)

Figure 1. Dynamics of system (3) when \( R_0^S = 0.6522 < 1 \).
At last, when we set $\beta = 0.4$, the basic reproduction number for the corresponding deterministic system of (3) is 1.3187. Then the deterministic system is persistence, while the red line in Figure 3 demonstrates the result. When we increase $\sigma_2$ to 0.7, $R_0^S = 0.9687 < 1$, then the disease dies out. The numerical simulation in Figure 3 illustrates this result.

In this paper, we consider the asymptotic properties of a stochastic SIRS model with nonlinear incidence rate and transfer from infectious to susceptible. We establish sufficient conditions for extinction and persistence in the mean of the epidemic. The threshold between persistence in the mean and extinction of system (3) is obtained. Compared with the corresponding deterministic model, the threshold affected by the white noise is smaller than the basic reproduction number of the deterministic system, which infers that the white noise can suppress the break of the disease.

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