HOMOGENEOUS INTERPOLATION AND SOME CONTINUED FRACTIONS

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Abstract. We prove: if \( d/m < 2280/721 \), there is no curve of degree \( d \) passing through \( n = 10 \) general points with multiplicity \( m \) in \( \mathbb{P}^2 \). Similar results are given for other special values of \( n \). Our bounds can be naturally written as certain palindromic continued fractions.

1. Introduction

Denote by \( \mathcal{L}(d,n,m) \) the linear system of degree \( d \) curves in \( \mathbb{P}^2 \) passing through \( n \) general points \( P_1, \ldots, P_n \) with multiplicity at least \( m \). For \( n \geq 9 \), Nagata’s conjecture ([9]) predicts that \( \mathcal{L}(d,n,m) \) is empty if \( d < \sqrt{nm} \). The statement is clear if \( n = k^2 \) is a square, but remains widely open otherwise. A refined conjecture due to Harbourne-Hirschowitz further predicts that in fact \( \mathcal{L}(d,n,m) \) is *non-special*, i.e. of expected dimension \( \max\{-1, v\} \), where

\[
v = d(d+3)/2 - nm(m+1)/2
\]

is the virtual dimension of \( \mathcal{L}(d,n,m) \). We refer to ([2],[3],[4],[5]) for background on the problem and some recent results.

In this paper we prove:

**Main Theorem.** Let \( n \) be a non-square positive integer. Write \( n = k^2 + \alpha \) with \( k = \lfloor \sqrt{n} \rfloor \). Assume that either:

1) \( n = 8, 10, 12 \), or
2) \( k \geq 3 \), \( \alpha \) is even, \( \alpha \mid 2n \).

If the linear system \( \mathcal{L}(d,n,m) \) is nonempty, then \( d/m \geq c_n^{(2)} \), where

\[
c_n^{(2)} = k + \frac{1}{(2k/\alpha) + \frac{1}{\frac{1}{2k} + \frac{1}{(2k/\alpha)} + \frac{1}{k}}}
\]

Note that \( c_n^{(2)} \) is a palindromic continued fraction with rational coefficients. The value \( c_8^{(2)} = 48/17 \) is well-known to be sharp ([9]). The next few cases are \( c_{10}^{(2)} = 2280/721 \), \( c_{11}^{(2)} = 660/199 \), \( c_{12}^{(2)} = 336/97 \), \( c_{15}^{(2)} = 120/31 \) and \( c_{18}^{(2)} = 2448/677 \). Since \( \sqrt{10} - c_{10}^{(2)} \approx 3 \cdot 10^{-6} \), our bound for ten points is stronger than the bound \( 117/37 \) obtained by Eckl ([5]) and Ciliberto *et al.* ([2]).

In the case \( n = 10, 11, 12 \) we have a more refined result (Prop. 12.2). As a striking application, we are able show that the linear system \( \mathcal{L}(1499,10,474) \) is non-special (with \( v = -1 \)).

The proof of the Theorem consists of two degenerations. First, we specialize the \( n \) general points \( P_i \) in \( \mathbb{P}^2 \) to general points \( P_i \) on a fixed curve \( C \) of degree \( k \). The problem is naturally reduced to an interpolation problem on a ruled surface \( S = \mathbb{P}(\mathcal{E}) \) where \( \mathcal{E} \) is a semistable rank 2 vector bundle of degree \( \alpha \) on \( C \). Second, we specialize the \( n \) points in \( S \) to a curve \( \Gamma \) of self-intersection 0 (in general, this step requires a deformation of the underlying surface \( S \)). The methods in this paper

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extend our previous attempt in [10]. We were greatly influenced by the work of Ciliberto-Miranda in [4].

The paper is organized as follows. In Section 2 we introduce a Basic Lemma that will be useful throughout the paper. In Section 3 we give background on ruled surfaces and elementary transforms. In Section 4 we perform the first degeneration and obtain a certain weak bound on throughout the paper. In Section 3 we give background on ruled surfaces and elementary transforms. In Section 4 we perform the first degeneration and obtain a certain weak bound on throughout the paper.

Notation and Conventions. We work over \( \mathbb{C} \). Following EGA IV.4, for given a subscheme \( Y \subset X \) we denote by \( \mathcal{N}_{Y/X} \cong \mathcal{F}_Y/\mathcal{F}_Y^2 \) the conormal sheaf of \( Y \). For any coherent sheaf \( \mathcal{F} \) on \( X \), we denote \( \mathbb{P}(\mathcal{F}) = \text{Proj}(\oplus_{n \geq 0} \text{Sym}^n \mathcal{F}) \).

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2. Basic Lemma

The following elementary lemma is the key ingredient to several arguments in this paper.

**Lemma 2.1** (Basic Lemma). Let \( C \) be a nonsingular curve embedded in a nonsingular projective variety \( X \). Let \( D \) be an effective divisor on \( X \). Denote \( \mu = \text{mult}_{C/X}(D) \), the multiplicity of vanishing of \( D \) along \( C \). Then, there is a natural injective morphism of sheaves \( \mathcal{O}_C(-D) \rightarrow \text{Sym}^\mu \mathcal{N}_{C/X} \), where \( \mathcal{N}_{C/X} \) is the conormal bundle of \( C \) in \( X \).

**Proof.** Let \( \pi : X' \rightarrow X \) be the blowup of \( C \). Then \( \pi^*(D) = D' + \mu S \), where \( S = \mathbb{P}(\mathcal{N}_{C/X}) \) is the exceptional divisor of \( \pi \) and \( D' \) is the strict transform of \( D \). We have \( \mathcal{O}_S(S) = \mathcal{O}_S(-1) \), the tautological line bundle on \( S \). So, \( \pi_*(\mathcal{O}_S(S) = \mathcal{O}_S(-1)) = \pi_*(\mathcal{O}_C(D) = \mathcal{O}_C(-D) \) is naturally a subsheaf of \( \mathcal{O}_S(\mu) = \text{Sym}^\mu \mathcal{N}_{C/X} \) (also, it is a subsheaf iff \( D' \mid S \) has no vertical components). \( \square \)

We can use the lemma to give a lower bound on \( \text{mult}_{C/X}(D) \) based on the “local data” \( \mathcal{O}_C(-D) \). This takes a particularly simple form if \( \mathcal{N}_{C/X} \) is a semistable vector bundle. Recall that, by definition, a vector bundle \( \mathcal{E} \) on a curve \( C \) is semistable if and only if for any subsheaf \( \mathcal{F} \rightarrow \mathcal{E} \), we have \( \text{slope}(\mathcal{F}) \leq \text{slope}(\mathcal{E}) \). Here we denote \( \text{slope}(\mathcal{E}) = \deg(\mathcal{E})/\text{rank}(\mathcal{E}) \). For background on semistability, see e.g. [11].

**Corollary 2.2.** In the above setting, suppose that \( \mathcal{N}_{C/X} \) is semistable. Then

\[ \text{slope}(\mathcal{N}_{C/X}) \text{mult}_{C/X}(D) \geq -C \cdot D. \]

**Proof.** This follows from Basic Lemma together with the fact that, for any \( \mu \geq 0 \), \( \text{Sym}^\mu \mathcal{N}_{C/X} \) is semistable of slope \( \mu \cdot \text{slope}(\mathcal{N}_{C/X}) \) ([11], Thm. 10.2.1). \( \square \)

3. Ruled Surfaces and Semistability

In this section we give some background on ruled surfaces and elementary transforms. Our reference is ([11], Ch. V). Let \( C \) be a smooth curve of genus \( g \geq 0 \) and let \( S \) be a ruled surface over \( C \). Let \( C_0 \) be a minimal section of \( S \), i.e. a section of minimal self-intersection. The invariant \( e = C_0^2 \) is the degree of \( S \) (this differs by sign from Hartshorne’s notation). We have:

**Lemma 3.1.** Let \( \mathcal{E} \) be a rank 2 vector bundle on \( C \). Consider the ruled surface \( S = \mathbb{P}(\mathcal{E}) \). The following are equivalent:

(a) \( \mathcal{E} \) is semistable;
(b) \( e \geq 0 \);
(c) for any effective divisor \( D \) of \( S \), we have \( D^2 \geq 0 \).
Example 4.3. If also \([4\text{, Prop. 2.3}]\), there is a unique section, namely the union of 3 lines each passing through 2 points \([7\text{, Example V.5.7.1}]\). Hence \(D^2 \geq 0\).

Proof. (a) \(\Leftrightarrow\) (b) See \([7\text{, Exercise V.2.8}]\).

(b) \(\Rightarrow\) (c) Let \(C_0\) be a minimal section of \(S\). Suppose \(e = C_0^2 \geq 0\). Let \(D \sim \mu C_0 - b f\) be an effective divisor. Then, \(\frac{1}{2} e \mu - b \geq 0\) where \(b = \deg b\) (this follows from \([7\text{, Prop. V.2.20}]\) if \(e = 0\) and Prop. 21 if \(e > 0\)). Hence \(D^2 \geq 0\).

(c) \(\Rightarrow\) (b) is obvious. \(\square\)

If \(S = \mathbb{P}(\mathcal{E})\) with \(\mathcal{E}\) semistable, we will also say that the surface \(S\) is semistable. By the lemma above, \(S\) is semistable if and only if \(S\) is of degree \(e \geq 0\).

3.1. Elementary Transforms. Let \(S\) be a ruled surface over \(C\) and let \(P\) be a point on \(S\). We can create a new ruled surface \(S'\) by applying an elementary transform at \(P\) \([7\text{, Example V.5.7.1}]\).

We recall the construction. Denote by \(F\) the fiber of \(S\) through \(P\). Let \(\pi: \tilde{S} \to S\) be the blowup of \(P\) and let \(\tilde{F}\) be the strict transform of \(F\) in \(\tilde{S}\). Finally, let \(\pi': \tilde{S} \to S'\) be the contraction of the \((-1)\)-curve \(\tilde{F}\) in \(\tilde{S}\) (see Fig. 1).

Similarly, we can define elementary transforms for vector bundles. In the setting above, suppose that \(S = \mathbb{P}(\mathcal{E})\) where \(\mathcal{E}\) is a rank 2 vector bundle on \(C\). Consider the short exact sequence

\[
0 \to \mathcal{E}' \to \mathcal{E} \to \mathbb{C}(P) \to 0
\]

where the map on the right is just the evaluation map at \(P\). The kernel \(\mathcal{E}'\) is again a rank 2 vector bundle on \(C\). We can identify \(S' = \mathbb{P}(\mathcal{E}')\) with the surface constructed above.

The following lemma describes the behavior of semistability under elementary transforms.

Lemma 3.2. Let \(S = \mathbb{P}(\mathcal{E})\) be a semistable ruled surface. Let \(P\) be a general point on a fixed fiber \(F\) of \(S\) (the fiber \(F\) need not be general). Let \(S'\) be the ruled surface obtained from \(S\) by applying an elementary transform at \(P\). Then \(S'\) is semistable, unless \(S \cong C \times \mathbb{P}^1\) is the trivial ruled surface.

Proof. If \(S'\) is not semistable, there exists a section \(C'\) of \(S'\) with \((C')^2 < 0\). Denote by \(C\) be the strict transform of \(C'\) in \(S\). Since \(S\) is semistable, we have \(C^2 \geq 0\). It follows that \(C\) passes through \(P\) and \(C^2 = (C')^2 + 1 = 0\). Since \(P\) is a general point on a fiber \(F\), it follows that \(S \cong C \times \mathbb{P}^1\). \(\square\)

4. First Degeneration

We introduce the first degeneration. The method in this section extends author’s previous work in \([10]\). As an application we prove Theorem 4.1 below. We are unaware if the result has appeared previously in the literature in this form.

Theorem 4.1. Let \(n\) be a non-square positive integer. Write \(n = k^2 + \alpha\) with \(k = \lfloor \sqrt{n} \rfloor\). Assume that either: (i) \(\alpha\) is even, or (ii) \(k \geq 3\). If the linear system \(\mathcal{L}(d,n,m)\) is nonempty, then \(d/m \geq e_n^{(1)}\), where

\[
e_n^{(1)} = k + \frac{1}{(2k/\alpha) + 1}.
\]

In particular, the theorem applies for \(n = 3, 6, 8\) and any \(n \geq 9\).

Example 4.2. For \(n = 3\), we have \(e_3^{(2)} = \frac{3}{2}\), which is sharp. The linear system \(\mathcal{L}(3,3,2)\) has a unique section, namely the union of 3 lines each passing through 2 points \([7\text{, remark on p.772}]\); see also \([4\text{, Prop. 2.3}]\).

Example 4.3. If \(n = 6\), we have \(e_6^{(2)} = \frac{12}{5}\), which is also sharp. The linear system \(\mathcal{L}(12,6,5)\) has a unique section, namely the union of six conics each passing through 5 of the 6 points \((\text{Loc. cit.})\).
Proof of the Theorem. Assume \( \mathcal{L}(d, n, m) \) is nonempty. The idea is to specialize the \( n \) general points in \( \mathbb{P}^2 \) to a smooth curve \( C \) of degree \( k \) and then apply Basic Lemma to estimate the multiplicity of vanishing of a general curve in \( \mathcal{L}(d, n, m) \) along \( C \).

Step 1. Let \( \Delta \) be the open unit disk over \( \mathbb{C} \) and let \( X = \mathbb{P}^2 \times \Delta \). We view \( X \) as a relative plane over \( \Delta \). For any \( t \in \Delta \), denote the fiber \( X_t = X \times \{t\} \). Fix a smooth curve \( C \subset X_0 \) of degree \( k \). We have the following split exact sequence

\[
0 \to \mathcal{N}_{X_0/X|C} \to \mathcal{N}_{C/X} \to \mathcal{N}_{C/X_0} \to 0
\]

where \( \mathcal{N}_{X_0/X|C} \cong \mathcal{O}_C \) and \( \mathcal{N}_{C/X_0} \cong \mathcal{O}_C(-kH) \). Consider the ruled surface \( S' = \mathbb{P}(\mathcal{N}_{C/X}) \).

Let \( C' \) be the section of \( S' \) corresponding to the short exact sequence above. Note that \( C' \sim \mathcal{O}_{S'}(1) \) and \( \mathcal{O}_{S'}(C') \otimes \mathcal{O}_{C'} \cong \mathcal{O}_C(-kH) \).

Step 2. Choose any set of \( n \) distinct points \( P_i \) on \( C \) (here we do not require the points \( P_i \) to be general). Next, we construct a set of \( n \) relative points \( \mathcal{P}_i \to \Delta \) in \( X \) specializing to \( P_i \) in a general way. Denote by \( P'_i \) the images of \( \mathcal{P}_i \) in \( S' \). Thus, each \( P'_i \) is a general point on the fiber above \( P_i \).

Let \( \tilde{X} \to X \) be the blowup of the relative points \( \mathcal{P}_i \) and let \( E_i \) denote the corresponding exceptional divisors. Denote by \( \tilde{C} \) the strict transform of \( C \) in \( \tilde{X} \). We have the following short exact sequence:

\[
0 \to \mathcal{N}_{\tilde{X}_0/\tilde{X}|\tilde{C}} \to \mathcal{N}_{\tilde{C}/\tilde{X}} \to \mathcal{N}_{\tilde{C}/\tilde{X}_0} \to 0
\]

We have \( \mathcal{N}_{\tilde{X}_0/\tilde{X}|\tilde{C}} \cong \mathcal{O}_C \) and \( \mathcal{N}_{\tilde{C}/\tilde{X}_0} \cong A \), where \( A = \mathcal{O}_C(\sum P_i - kH) \) is a line bundle of degree \( \alpha = n - k^2 \) on \( C \). The short exact sequence corresponds to a certain element \( \xi \in \text{Ext}^1(A, \mathcal{O}_C) \).

Next, consider the ruled surface \( S = \mathbb{P}(\mathcal{N}_{\tilde{C}/\tilde{X}}) \).

We identify \( C \) with the section of \( S \) corresponding to the above exact sequence. Note that \( C \sim \mathcal{O}_S(1) \) and \( \mathcal{O}_S(C) \otimes \mathcal{O}_C \cong A \).

**Figure 1.** Elementary transforms

The conormal bundles \( \mathcal{N}_{C/X} \) and \( \mathcal{N}_{\tilde{C}/\tilde{X}} \) are related by elementary transforms at the points \( P_i \):

\[
0 \to \mathcal{N}_{C/X} \to \mathcal{N}_{\tilde{C}/\tilde{X}} \to \oplus_{i=1}^n \mathcal{O}(P_i) \to 0
\]
Proof. The idea is to consider the construction in Step 2 in reversed order. Start with any extension
\[ 0 \to \mathcal{O}_C \to \mathcal{E} \to A \to 0 \]
and let \( S = \mathbb{P}(\mathcal{E}) \). As before, we identify \( C \) with the section of \( S \) corresponding to the above exact sequence. Next, we construct the vector bundle \( \mathcal{E}' \) by applying elementary transforms at the points \( P_i \) on \( C \):
\[ 0 \to \mathcal{E}' \to \mathcal{E} \to \oplus_{i=1}^n C(P_i) \to 0 \]
Let \( S' = \mathbb{P}(\mathcal{E}') \) and let \( P'_1, \ldots, P'_n \) be as on Fig. 1. It follows that \( \mathcal{E}' \) is realized as an extension
\[ 0 \to \mathcal{O}_C \to \mathcal{E}' \to \mathcal{O}_C(-kH) \to 0 \]
Now, the key observation is that
\[ \text{Ext}^1(\mathcal{O}_C(-kH), \mathcal{O}_C) \cong H^1(C, \mathcal{O}_C(kH)) = 0 \]
so the above extension is trivial. This allows us to identify \( \mathcal{E}' \cong \mathcal{O}_C \oplus \mathcal{O}_C(-kH) \) with \( \mathcal{N}_{C/X} \), and so \( S' \) with \( \mathbb{P}(\mathcal{N}_{C/X}) \). Finally, we choose the relative points \( \mathcal{P}_i \) to pass through \( P'_i \) in \( S' \). This identifies \( \mathcal{E}' \) with \( \mathcal{N}_{C/X} \), and so \( S \) with \( \mathbb{P}(\mathcal{N}_{C/X}) \).

Corollary 4.5. If the specialization of \( \mathcal{P}_i \) to \( P_i \) is general enough, the conormal bundle \( \mathcal{N}_{C/X} \) is semistable of slope \( \alpha/2 \).

This follows from Lemma 4.4 and the following general fact:

Lemma 4.6. Let \( C \) be a curve of genus \( g \geq 0 \). Let \( A \) be a line bundle on \( C \) of degree \( \alpha \geq 0 \). Assume that either: (i) \( \alpha \) is even, or (ii) \( g \geq 1 \). Then, a general element \( \xi \in \text{Ext}^1(A, \mathcal{O}_C) \) corresponds to a semistable rank 2 vector bundle \( \mathcal{E} \) on \( C \).

Proof. The set of elements \( \xi \in \text{Ext}^1(A, \mathcal{O}_C) \) that correspond to semistable vector bundles \( \mathcal{E} \) is open (this follows from [3, Thm. 2.8]). So, it suffices to show that the set is nonempty. Now, if \( g = 0 \) and \( \alpha = 2\alpha_0 \) is even, then \( \mathcal{O}_{P_0}(\alpha_0) \oplus \mathcal{O}_{P_0}(\alpha_0) \) is semistable. If \( g \geq 1 \), one can prove the statement by induction on \( \alpha \) by using Lemma 3.2. We leave this as an exercise.

Step 4. We complete the proof of the theorem. Since \( \mathcal{L}(d, n, m) \) is nonempty by assumption, there is a flat family of curves \( \overline{\mathcal{C}} \to \Delta \), where \( \overline{\mathcal{C}} \) is a nontrivial section of \( |\mathcal{O}_{\overline{X}}(dH - \sum mE_i)| \). We are interested in estimating \( \mu = \text{mult}_{\overline{C}/X}(\mathcal{E}) \), which of course is the same as \( \mu = \text{mult}_{\overline{C}/X}(\mathcal{E}) \) where \( \mathcal{E} \) is the image of \( \overline{\mathcal{C}} \) in \( X \). Obviously,
\[ \mu \leq \frac{d}{k}. \]  
(3)
By Lemma 4.5 and Cor. 2.2 we have:
\[ \text{slope}(\mathcal{N}_{C/X}) \cdot \mu \geq \frac{\overline{\mathcal{C}} \cdot \overline{\mathcal{E}}}{nm-kd}. \]  
(4)
Combining (3) and (4), we get:
\[ \frac{\alpha d}{2k} \geq nm - kd. \]
One can easily check that this is equivalent to the inequality in the theorem. See also Lemma B.1(a) in the Appendix.
5. Reduction of Interpolation Problems to Ruled Surfaces

We formalize some results from the previous section. Our result here is Theorem 5.5 which will be used through the rest of the paper.

**Notation 5.1.** A marked surface \((S; P_1, \ldots, P_n)\) is simply a surface \(S\) together with \(n\) distinct points \(P_1, \ldots, P_n\) on \(S\).

**Notation 5.2.** Let \(S = (\mathbb{P}(\mathfrak{C}); P_1, \ldots, P_n)\) be a marked ruled surface over \(C\). For any integers \((\mu, b, \hat{m})\) and a line bundle \(\mathfrak{E}\) of degree \(b\) on \(C\), we denote the line bundle

\[
\mathcal{L}_S(\mu, b, \hat{m}) = \mathcal{O}_S(\mu - bf - \sum \hat{m}e_i)
\]

on the blowup \(\pi: \mathcal{S} \rightarrow S\) at the points \(P_1, \ldots, P_n\), with exceptional divisors \(e_1, \ldots, e_n\). Here we denote \(\mathcal{O}_{\mathcal{S}}(\mu) = \pi^* \mathcal{O}_S(\mu)\).

**Lemma 5.3.** We have

\[
\chi(\mathcal{L}_S(\mu, b, \hat{m})) = (\mu + 1)(\binom{\hat{m}}{2} - b + 1) - n(\hat{m} + 1)
\]

where \(\alpha = \deg(\mathfrak{C})\) and \(g\) is the genus of \(C\).

*Proof.* Denote by \(C_1\) the class \(c_1(\mathcal{O}_S(1))\). It follows from ([7], Lemma V.2.10), that

\[
K_S = -2C_1 + (2g - 2 + \alpha)f.
\]

By the Riemann-Roch formula,

\[
\chi(\mathcal{O}_S(\mu C_1 - bf)) = \frac{1}{2}(\mu C_1 - bf) \cdot (\mu C_1 - bf - K_S) + 1 + p_a(S).
\]

We have \(p_a(S) = -g\) ([7], Cor. V.2.5). The lemma now follows from \(C_1^2 = \alpha, C_1 \cdot f = 1\) and \(f^2 = 0\). \(\square\)

**Notation 5.4.** Let \(C\) be a smooth curve, \(A\) a line bundle on \(C\) and let \(\xi \in \text{Ext}^1(A, \mathcal{O}_C)\) corresponding to an extension

\[
0 \rightarrow \mathcal{O}_C \rightarrow \mathfrak{C} \rightarrow A \rightarrow 0.
\]

We denote by

\[
S(C, A, \xi)
\]

the ruled surface \(S = \mathbb{P}(\mathfrak{C})\) and we identify \(C\) with the section determined by the short exact sequence. Note that \(C \sim \mathcal{O}_S(1)\) and \(\mathcal{O}_S(1) \otimes \mathcal{O}_C \cong A\).

The following theorem allows to reduce interpolation problems on \(\mathbb{P}^2\) to certain interpolation problems on ruled surfaces.

**Theorem 5.5.** Let \(n\) be a non-square positive integer. Write \(n = k^2 + \alpha\) with \(k \geq 1\) and \(\alpha \geq 0\). Consider the linear system \(\mathcal{L}(d, n, m)\) for some positive integers \(d\) and \(m\). Fix a smooth curve \(C\) of degree \(k\) in \(\mathbb{P}^2\). Let \(S = S(C, A, \xi; \{P_i\})\) be a marked ruled surface where:

- \(A\) is any line bundle of degree \(\alpha\) on \(C\);
- \(\xi \in \text{Ext}^1(A, \mathcal{O}_C)\) is any element;
- \(P_1, \ldots, P_n\) are distinct points on \(C \subset S\) such that \(\sum P_i \sim A + kH\).

Then, for any \(\mu\), we have

\[
h^0(\mathcal{L}_{\mathbb{P}^2}(d, n, m)) \leq h^0(\mathcal{L}_S(\mu, b, \hat{m})) + h^0(\mathcal{O}_{\mathbb{P}^2}(\epsilon H)) - h^0(\mathcal{O}_C(\epsilon H))
\]

where:

- \(b = \mathcal{O}_C(\sum mP_i - dH)\) of degree \(b = nm - kd\);
- \(\hat{m} = \mu - m\);
- \(\epsilon = d - k\mu\).

The following lemma justifies our definition of \(\mathcal{L}_S(\mu, b, \hat{m})\):
Lemma 5.6. In the setting of the theorem, we have $\mathcal{L}_S(\mu, b, \tilde{m}) \otimes \mathcal{O}_{\tilde{C}} \cong \mathcal{O}_C(\epsilon H)$, where $\tilde{C}$ is the strict transform of $C$ in $\tilde{S}$.

Proof. This is an easy computation. We have:

$\mathcal{O}_S(\mu C - bf) \otimes \mathcal{O}_C \cong \mathcal{O}_C(\mu A - b) \cong \mathcal{O}_C(\sum \tilde{m}P_i + \epsilon H)$.

The lemma follows. $\square$

Proof of Theorem. We will use the same construction as in the proof of Theorem 4.1.

Step 1. Consider the threefold $X = \mathbb{P}^2 \times \Delta$. We identify $C$ with a curve on $X_0$. Given $\xi \in \text{Ext}^1(A, \mathcal{O}_C)$, we specialize the $n$ relative points $\mathcal{P}_i$ to $P_i \in C$ as in Lemma 4.4. As before, let $\tilde{X} \to X$ be the blowup of the $\mathcal{P}_i$, and let $\tilde{C}$ be the strict transform of $C$ in $X$. We have the short exact sequence

$$0 \to \mathcal{N}_{X_0/\tilde{X}}|_C \to \mathcal{N}_{\tilde{C}/\tilde{X}} \to \mathcal{N}_{\tilde{C}/X_0} \to 0$$

where $\mathcal{N}_{X_0/\tilde{X}}|_C \cong \mathcal{O}_C$ and $\mathcal{N}_{\tilde{C}/X_0} \cong A$. By construction, the above extension corresponds to $\xi$. In particular, $S \cong \mathbb{P}(\mathcal{N}_{\tilde{C}/\tilde{X}})$, where $S = S(C, A, \xi)$ is the ruled surface we started with.

Step 2. Consider the threefold $Y$ obtained from $X = \mathbb{P}^2 \times \Delta$ by first blowing up $C$ (with exceptional divisor $S' = \mathbb{P}(\mathcal{N}_{C/X})$), followed by blowing up the strict transforms of the relative points $\mathcal{P}_1, \ldots, \mathcal{P}_n$ (with exceptional divisors $E_1, \ldots, E_n$). We view $Y \to \Delta$ as a flat family with general fiber $Y_0 \cong \tilde{X}_i$. The special fiber $Y_0$ is the union of two surfaces $\tilde{S} \cup X_0$ meeting transversely along $\tilde{C} \cong C$. $\daleth$

This construction is related to the construction in Step 1 as follows. Let $\tilde{X}'$ be the threefold obtained from $\tilde{X}$ by blowing up $\tilde{C}$ (with exceptional divisor $S = \mathbb{P}(\mathcal{N}_{\tilde{C}/\tilde{X}})$). Then, $Y$ can be obtained from $\tilde{X}'$ by applying $(-1)$-transfers to the exceptional curves $e_1, \ldots, e_n$ as on Fig. 2 (see also [3], Section 4.1). The induced map $\pi: \tilde{S} \to S$ coincides with the corresponding map on Fig. 1.

![Figure 2. (-1) transfers between Y and $\tilde{X}'$](image)

Step 3. For a given $\mu$, consider the following line bundle on $Y$:

$$\mathcal{L}_Y \cong \mathcal{O}_Y(dH - \sum mE_i - \mu \tilde{S}).$$

We view $\mathcal{L}_Y$ as a flat family of line bundles with general fiber $\mathcal{L}_{Y_i} \cong \mathcal{O}_{\tilde{X}_i}(dH - \sum mE_i)$. The special fiber $\mathcal{L}_{Y_0}$ is described by the following short exact sequence:

$$0 \to \mathcal{L}_{Y_0} \to \mathcal{L}_\tilde{S} \oplus \mathcal{L}_{X_0} \to \mathcal{L}_C \to 0 \quad (*)$$

where

$$\mathcal{L}_\tilde{S} \cong \mathcal{L}_Y \otimes \mathcal{O}_\tilde{S}; \quad \mathcal{L}_{X_0} \cong \mathcal{O}_{\mathbb{P}^2}(\epsilon H); \quad \mathcal{L}_C \cong \mathcal{O}_C(\epsilon H).$$

Lemma 5.7. We have:

$$\mathcal{L}_\tilde{S} \cong \mathcal{L}_S(\mu, b, \tilde{m}).$$

1We denote by $\tilde{C}$, resp. $C$, the same curve in $Y$ viewed as a divisor in $\tilde{S}$, resp. $X_0$. 

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**Homogeneous Interpolation and Some Continued Fractions**

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Homogeneous Interpolation and Some Continued Fractions

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Homogeneous Interpolation and Some Continued Fractions

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Proof. The line bundle \( \mathcal{L}_S \) has the following properties:
\[
\mathcal{L}_S \otimes \mathcal{O}_C \cong \mathcal{O}_C(eH); \quad \mathcal{L}_S \cdot f = \mu; \quad \mathcal{L}_S \cdot e_i = \hat{m}.
\]
From Lemma 5.6, \( \mathcal{L}_S(\mu, b, \hat{m}) \) has the same properties. It follows that \( \mathcal{L}_S \cong \mathcal{L}_S(\mu, b, \hat{m}) \). □

To complete the proof of the theorem, take cohomology in (*):
\[
0 \to H^0(\mathcal{L}_{Y_0}) \to H^0(\mathcal{L}_S) \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(eH)) \to H^0(\mathcal{O}_C(eH)) \to H^1(\mathcal{L}_{Y_0}).
\]
The restriction \( H^0(\mathcal{O}_{\mathbb{P}^2}(eH)) \to H^0(\mathcal{O}_C(eH)) \) is surjective. Hence, the coboundary map \( \delta = 0 \). The theorem now follows from the semicontinuity principle applied to \( h^0(\mathcal{L}_{Y_1}) \). □

**Corollary 5.8.** Assume the above setting.

(a) For any \( \mu \), we have:
\[
\chi(\mathcal{L}_{\mathbb{P}^2}(d, n, m)) = \chi(\mathcal{L}_S(\mu, b, \hat{m})) + \chi(\mathcal{O}_{\mathbb{P}^2}(eH)) - \chi(\mathcal{O}_C(eH)).
\]

(b) For any \( \mu \leq d/k \) (hence \( e \geq 0 \)), we have:
\[
\chi(\mathcal{L}_{\mathbb{P}^2}(d, n, m)) = \chi(\mathcal{L}_S(\mu, b, \hat{m})) + h^1(\mathcal{O}_C(eH)).
\]

**Proof.** (a) This follows from the short exact sequence (*) and the fact that \( \chi(\mathcal{L}_{Y_1}) \) is a constant function of \( t \).

(b) This follows from (a). □

**Corollary 5.9.** Suppose the linear system \( \mathcal{L}(d, n, m) \) is nonempty. Then \( |\mathcal{L}_S(\mu, b, \hat{m})| \) is nonempty with \( \mu = [d/k] \).

**Proof.** Consider the long exact cohomology sequence associated to (*). If \( \mu = [d/k] \), the restriction \( H^0(\mathcal{O}_{\mathbb{P}^2}(eH)) \to H^0(\mathcal{O}_C(eH)) \) is an isomorphism. The claim follows. □

### 6. Families of Ruled Surfaces

We can use the degeneration technique from the previous section to reduce an interpolation problem on \( \mathbb{P}^2 \) to an interpolation problem on a certain ruled surface \( S \). We would like to perform further degenerations to study the later problem. Our first goal is to define an object \( \mathcal{S}(Z, \mathcal{A}, \xi) \to \Delta \) which is a relative analogue of \( S(C, A, \xi) \). We conclude with a technical result (Prop. 6.4) which will be used in Sections 10 and 13. As usual, \( \Delta \) denotes the open unit disk over \( \mathbb{C} \).

**Notation 6.1.** Let \( \mathcal{A} \) be a torsion-free sheaf of rank 1 on \( C \times \Delta \). Assume that \( \mathcal{A} = \mathcal{A}' \otimes \mathcal{I}_W \) where \( \mathcal{A}' \) is invertible and \( \mathcal{I}_W \) is the ideal sheaf of a l.c.i. zero-dimensional subscheme \( W \subset C \times \Delta \) (possibly \( W = 0 \)). Let \( \xi \in \text{Ext}^1(\mathcal{A}, \mathcal{O}_{C \times \Delta}) \) be an element corresponding to an extension
\[
0 \to \mathcal{O}_{C \times \Delta} \to \mathcal{E} \to \mathcal{A} \to 0
\]
such that \( \mathcal{E} \) is locally free. We denote by
\[
\mathcal{S}(Z, \mathcal{A}, \xi) \to \Delta
\]
the relative ruled surface \( \mathcal{S} = \mathbb{P}(\mathcal{E}) \to \Delta \) together with the subscheme \( Z = \mathbb{P}(\mathcal{A}) \) defined by the above exact sequence. In particular, \( Z \) is a divisor of \( \mathcal{S} \) with \( Z \sim \mathcal{O}_\mathcal{E}(1) \).

We will assume that \( W \) is supported on \( C \times \{0\} \). In applications, \( W \) will be reduced; however, everything we say in this section holds in the more general setting.

**Lemma 6.2.** In the above setting, the projection \( p : Z \to C \times \Delta \) is just the blowup of \( W \subset C \times \Delta \). Denote by \( F \) the exceptional divisor of \( p \). Then \( p^* \mathcal{A} \cong p^* \mathcal{A}' \otimes \mathcal{O}_Z(-F) \cong \mathcal{O}_\mathcal{E}(Z) \otimes \mathcal{O}_Z \).

**Proof.** Since \( \mathcal{A}' \) is invertible, \( Z = \mathbb{P}(\mathcal{A}') = \mathbb{P}(\mathcal{I}_W) \) which is exactly the definition of a blowup. The rest is clear. □
Consider the following question: given \( \mathcal{A} \) as above, which elements \( \xi \in \text{Ext}^1(\mathcal{A}, \mathcal{O}_{C \times \Delta}) \) correspond to a locally free extension \( \mathcal{E} \)? Clearly, if \( W = \emptyset \), then \( \mathcal{A} \) is locally free and so any \( \xi \) will do. In the general case, we have:

**Proposition 6.3.** Let \( \mathcal{A} = \mathcal{A}' \otimes \mathcal{F}_W \) be as above. Then, there is a natural exact sequence

\[
0 \to H^1(C \times \Delta, (\mathcal{A}')^{-1}) \to \text{Ext}^1(\mathcal{A}, \mathcal{O}_{C \times \Delta}) \to H^0(\mathcal{E}\text{xt}^1(\mathcal{A}, \mathcal{O}_{C \times \Delta})) \to 0
\]

Moreover, we have:

(a) \( \mathcal{E}\text{xt}^1(\mathcal{A}, \mathcal{O}_{C \times \Delta}) \cong \mathcal{E}\text{xt}^2(\mathcal{O}_W, \mathcal{O}_{C \times \Delta}) \cong \mathcal{O}_W \).

(b) The extension \( \mathcal{E} \) corresponding to \( \xi \in \text{Ext}^1(\mathcal{A}, \mathcal{O}_{C \times \Delta}) \) is locally free if and only if \( \delta(\xi) \) generates the sheaf \( \mathcal{E}\text{xt}^1(\mathcal{A}, \mathcal{O}_{C \times \Delta}) \).

**Proof.** See [6], Chapter 2, p. 36–37. The exact sequence follows from the local-to-global spectral sequence

\[
E_2^{ij} = H^i(\mathcal{E}\text{xt}^j(\mathcal{A}, \mathcal{O}_{C \times \Delta})) \Rightarrow \text{Ext}^{i+j}(\mathcal{A}, \mathcal{O}_{C \times \Delta}).
\]

Note that \( E_2^{0,0} = H^0(\mathcal{A}')^{-1} = H^0(\mathcal{A}'^{-1}) = H^0(\Delta, R^i\pi_*((\mathcal{A}')^{-1})) \) where \( \pi : C \times \Delta \to \Delta \) is the projection. In particular, \( E_2^{0,0} = 0 \) for \( i \geq 2 \).

(a) Ibid., Lemma 7 (note that the isomorphisms are not canonical).

(b) Ibid., Theorem 8.

Consider a relative ruled surface \( \mathcal{S}(Z, \mathcal{A}, \xi) \to \Delta \). For any \( t \in \Delta \), \( \mathcal{S}_t = \mathcal{P}(\mathcal{E}_t) \) is a ruled surface over \( C \times \{t\} \) where \( \mathcal{E}_t \) arises as an extension

\[
0 \to \mathcal{O}_{C \times \{t\}} \to \mathcal{E}_t \to \mathcal{A}_t \to 0
\]

We have \( Z_t \sim \mathcal{E}_{\mathcal{S}_t}(1) \). For a general \( t \), the subscheme \( Z_t \) is a section of \( \mathcal{S}_t \). On the special fiber, we have \( Z_0 = C_0 \cup F_0 \), where \( C_0 \) is a section of \( \mathcal{S}_0 \) and \( F_0 = F \times \{0\} \) is the vertical component of \( Z_0 \).

If \( F_0 \neq \emptyset \), we will say that \( Z_0 \) is a degenerate section of \( \mathcal{S}_0 \).

Next, we will show that any degenerate section can be smoothed, in the following sense.

**Proposition 6.4.** Let \( \mathcal{A} = \mathcal{A}' \otimes \mathcal{F}_W \) be as above, with \( W \) supported on \( C \times \{0\} \). Let

\[
0 \to \mathcal{O}_{C \times \{0\}} \to \mathcal{E}_0 \to \mathcal{A}_0 \to 0
\]

be any extension, with \( \mathcal{E}_0 \) locally free. Then, the exact sequence can be extended to

\[
0 \to \mathcal{O}_{C \times \Delta} \to \mathcal{E} \to \mathcal{A} \to 0
\]

with \( \mathcal{E} \) locally free on \( C \times \Delta \).

**Proof.** Consider the commutative diagram with exact rows:

\[
\begin{array}{c}
0 \to H^1(C \times \Delta, (\mathcal{A}')^{-1}) \to \text{Ext}^1(\mathcal{A}, \mathcal{O}_{C \times \Delta}) \to H^0(\mathcal{E}\text{xt}^1(\mathcal{A}, \mathcal{O}_{C \times \Delta})) \to 0 \\
0 \to H^1(C \times \{0\}, (\mathcal{A}_0)^{-1}) \to \text{Ext}^1(\mathcal{A}_0, \mathcal{O}_{C \times \{0\}}) \to H^0(\mathcal{E}\text{xt}^1(\mathcal{A}_0, \mathcal{O}_{C \times \{0\}})) \to 0
\end{array}
\]

Note that \( \mathcal{A}_0 \cong \mathcal{A}_0'(-W_0) \oplus \mathcal{O}_{W_0} \) where \( W_0 = W \times \{0\} \). Therefore, \( (\mathcal{A}_0)^{-1} \cong (\mathcal{A}_0'(-W_0))^{-1} \). It follows that the bottom row of the diagram splits. Now, the map on the left factors through

\[
\begin{array}{ccc}
H^1(C \times \Delta, (\mathcal{A}')^{-1}) & \to & H^1(C \times \{0\}, (\mathcal{A}_0)^{-1}) \\
H^0(\Delta, R^i\pi_*((\mathcal{A}')^{-1})) & \to & (R^i\pi_*((\mathcal{A}')^{-1}))_{\{0\}}
\end{array}
\]

which is surjective. The map on the right is just the restriction

\[
H^0(\mathcal{E}_W) \to H^0(\mathcal{O}_{W_0})
\]

which is also surjective. By the Short Five Lemma, the map in the middle is surjective as well. Hence, any given extension \( \xi_0 \in \text{Ext}^1(\mathcal{A}_0, \mathcal{O}_{C \times \{0\}}) \) can be lifted to \( \xi \in \text{Ext}^1(\mathcal{A}, \mathcal{O}_{C \times \Delta}) \). The resulting \( \mathcal{E} \) is locally free by Prop. 6.3(b).
7. Main Result – Overview

The following theorem was announced in the introduction. The proof will occupy the rest of the paper. We will consider a certain refinement in Section 12.

Main Theorem. Let \( n \) be a non-square positive integer. Write \( n = k^2 + \alpha \) with \( k = \lfloor \sqrt{n} \rfloor \). Assume that either:

i) \( n = 8, 10, 12 \), or

ii) \( k \geq 3 \), \( \alpha \) is even, \( \alpha \mid 2n \).

If the linear system \( \mathcal{L}(d, n, m) \) is nonempty, then \( d/m \geq c_n^{(2)} \).

The proof of the theorem consists of the four steps outlined below.

7.1. Setup. We assume \( \mathcal{L}(d, n, m) \) is nonempty. Fix a smooth curve \( C \) of degree \( k \) in \( \mathbb{P}^2 \). By Cor. 5.9, the linear system \( |\mathcal{L}_S(\mu, B, \tilde{m})| \) is nonempty, with \( \mu = \lfloor d/k \rfloor \), for any marked ruled surface \( S(C, A, \xi; \{P_i\}) \) as in Theorem 5.5.

7.2. Degeneration. We will construct a relative marked ruled surface \( \mathfrak{S}(Z, \mathcal{A}, \xi; \{\mathcal{P}_i\}) \) over \( \Delta \) such that the general fiber \( \mathfrak{S}_t \) satisfies the assumptions of Theorem 5.5, i.e.:

- \( \mathcal{A} \) is a line bundle on \( C \times \Delta \) of relative degree \( \alpha = \deg \mathcal{A}_t \).
- \( \mathcal{P}_1, \ldots, \mathcal{P}_n \) lie on \( Z \). We denote by \( \tilde{\mathcal{P}}_i \) the projection of \( \mathcal{P}_i \) under \( Z \to C \times \Delta \).
- For \( t \in \Delta \) general, we have \( \sum \mathcal{P}_{i,t} \sim \mathcal{A}_t + kH \) on \( Z_t \cong C \).

We now describe the special fiber \( \mathfrak{S}_0 \). We assume that \( \mathfrak{S}_0 \) is semistable. Next, we assume that there is a (possibly disconnected) curve \( \Gamma \) on \( \mathfrak{S}_0 \) with the following two properties:

- \( \Gamma \) meets \( Z_0 \) transversely at \( n \) distinct points points \( P_i = \mathcal{P}_{i,0} \);
- \( \Gamma^2 = 0 \).

This determines uniquely the numerical class of \( \Gamma \):

\[
\Gamma \equiv \frac{2n}{\alpha} (Z_0 - \frac{\alpha}{2} f)
\]

where \( Z_0 \sim \mathcal{O}_{\mathfrak{S}_0}(1) \). In particular, a necessary condition for the existence of \( \Gamma \) is that \( \alpha \mid 2n \).

Let \( \Gamma = \sum_{i=1}^n \Gamma_i \) where each \( \Gamma_i \) is a smooth irreducible curve, with \( \Gamma_i \cdot \Gamma_j = 0 \) for \( i \neq j \). Since \( \Gamma^2 = 0 \) and \( \mathfrak{S} \) is semistable, \( \Gamma \) lies on the boundary of the effective cone of \( \mathfrak{S}_0 \) (this follows from Lemma 3.2). Therefore, \( \Gamma_i \equiv \lambda_i \Gamma \) for some \( \lambda_i \in \mathbb{Q} \) with \( \sum \lambda_i = 1 \) (in fact, in applications we will always have \( \lambda_1 = \cdots = \lambda_n = 1/s \)).

7.3. Semistability. Denote by \( \pi : \tilde{\mathfrak{S}} \to \mathfrak{S} \) the blowup of \( \mathcal{P}_1, \ldots, \mathcal{P}_n \) and let \( E_1, \ldots, E_n \) be the corresponding exceptional divisors. Denote by \( \tilde{\Gamma} \) the strict transform of \( \Gamma \). We make the following hypothesis:

- for each \( i \), the conormal bundle \( \mathcal{N}_{\Gamma_i/\tilde{\mathfrak{S}}} \) is semistable of slope \( \frac{1}{2} \lambda_i n \).

7.4. Invariants. Assuming the construction above can be realized, we complete the proof of the theorem. Consider the line bundle

\[
\mathcal{L}_{\tilde{\mathfrak{S}}}(\mu, \mathcal{B}, \tilde{m}) = \mathcal{O}_{\tilde{\mathfrak{S}}}(\mu - \mathcal{B} f - \sum \tilde{m} E_i)
\]

on the blowup \( \pi : \tilde{\mathfrak{S}} \to \mathfrak{S} \), where

- \( \mu = \lfloor d/k \rfloor \);
- \( \mathcal{B} \sim \mathcal{O}_{C \times \Delta}(\sum m \tilde{\mathcal{P}}_i - dH) \) of relative degree \( b = \deg \mathcal{B}_t = nm - kd \).
- \( \tilde{m} = \mu - m \).

By construction, there is a flat family of curves \( \tilde{\mathcal{C}} \to \Delta \) in \( \tilde{\mathfrak{S}} \), where \( \tilde{\mathcal{C}} \) is a section of \( |\mathcal{L}_{\tilde{\mathfrak{S}}}(\mu, \mathcal{B}, \tilde{m})| \). Denote by \( \mathcal{C} \) the projection of \( \tilde{\mathcal{C}} \) in \( \mathfrak{S} \). The following is a key computation:
Lemma 7.1. Define \( \gamma = -\frac{1}{n} \bar{\Gamma} \cdot \bar{\mathcal{C}} \). Then
\[
\gamma = \frac{n + k^2}{\alpha} m - \frac{2k}{\alpha} d.
\]

**Proof.** Using the fact that \( \Gamma \cdot Z_0 = n, \Gamma \cdot f = \frac{2n}{\alpha} \) and \( \mathcal{C}_0 \sim \mu Z_0 - b f \), we find:
\[
\gamma = -\frac{1}{n} \bar{\Gamma} \cdot \bar{\mathcal{C}} \\
= -\frac{1}{n} (\Gamma \cdot \mathcal{C} - n \hat{m}) \\
= -(\mu - \frac{2}{\alpha} b) + \hat{m} \\
= -m + \frac{2}{\alpha} b.
\]

Finally, substitute \( b = n m - k d \) and \( n = k^2 + \alpha \).

To complete the proof of the theorem, we will estimate \( \mu_i = \text{mult}_{\Gamma_i/\mathcal{S}}(\mathcal{C}) \) in two ways. First, there is an obvious upper bound which comes from the numerical class of \( \Gamma \):
\[
\sum \lambda_i \mu_i \leq \frac{\mu}{2n/\alpha}.
\]
Since \( \mu \leq d/k \), this becomes:
\[
\sum \lambda_i \mu_i \leq \frac{d}{2kn/\alpha}. \tag{\#}
\]
By Basic Lemma and the semistability hypothesis, we have:
\[
\text{slope}(\mathcal{E}/\mathcal{S}, \lambda_i \mu_i) \geq \frac{\bar{\Gamma} \cdot \bar{\mathcal{C}}}{\lambda_i \gamma}
\]
i.e.
\[
\frac{1}{2} \lambda_i \mu_i \geq \gamma. \tag{♭}
\]
From (\#) and (♭), and using \( \sum \lambda_i = 1 \), we get:
\[
\frac{1}{2} \frac{d}{2kn/\alpha} \geq \gamma = \frac{n + k^2}{\alpha} m - \frac{2k}{\alpha} d.
\]
It turns out that this is equivalent to the inequality in the theorem. We will check this explicitly for specific values of \( n \). For the general case, see Lemma B.1(b) in the Appendix.

8. Eight Points

We verify Main Theorem in the case \( n = 8 \). The value \( c_8^{(2)} = \frac{48}{17} \) is well-known to be sharp (see example below). We include this case for illustration purposes.

8.1. Setup. Since \( k = 2 \), we take \( C \) to be a smooth conic in \( \mathbf{P}^2 \). The line bundle \( A \cong \mathcal{O}_{\mathbf{P}^1}(4) \) on \( C \) is of degree \( \alpha = 4 \). Consider an extension
\[
0 \to \mathcal{O}_C \to \mathcal{E} \to A \to 0
\]
corresponding to a general \( \xi \in \text{Ext}^1(A, \mathcal{O}_C) \). It follows that
\[
\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(2).
\]
Hence, \( S = \mathbf{P}(\mathcal{E}) \cong C \times \mathbf{P}^1 \cong \mathbf{P}^1 \times \mathbf{P}^1 \). We identify \( C \) with the section of \( S \) corresponding to the short exact sequence. It follows that \( C \sim C_0 + 2f \) where \( C_0 \) is a horizontal section of \( S \).
8.2. Degeneration. Let $S = S \times \Delta$ and $Z = C \times \Delta \subset S$. We identify $S$ with the special fiber of $\mathcal{S} \to \Delta$. We take $\Gamma = \Gamma_1 + \cdots + \Gamma_4$ on $S$, where each $\Gamma_i \sim C_0$ is a general horizontal section of $S$. Let $\Gamma_i \cap C = \{P_{2i-1}, P_{2i}\}$. Next we specialize the eight relative points $\mathcal{P} \subset Z$ to $P_i \in C$ in a general way.

8.3. Semistability. Let $\mathcal{S}$ be the blowup of $S$ along the relative points $\Gamma_i$. We have to show that, for each $i$, the conormal bundle $N_{\mathcal{S}^i/\mathcal{S}}$ is semistable (of slope 1). Denote by $P_i'$ the image of $P_i$ in the corresponding $P(\mathcal{M}_{\mathcal{S}^i/\mathcal{S}}) \cong \Gamma \times \mathbf{P}^1$. Then, $\mathcal{P}(\mathcal{M}_{\mathcal{S}^i/\mathcal{S}})$ is obtained from $\mathcal{P}(\mathcal{M}_{\mathcal{S}^i/\mathcal{S}})$ by performing elementary transforms at the points $P_{2i-1}^i, P_{2i}^i$. If the specialization is general enough, the points $P_{2i-1}^i, P_{2i}^i$ do not belong to the same horizontal section of $\mathcal{P}(\mathcal{M}_{\mathcal{S}^i/\mathcal{S}})$. It follows that $N_{\mathcal{S}^i/\mathcal{S}} \cong \mathcal{O}_{\mathcal{P}^i_2}(1) \oplus \mathcal{O}_{\mathcal{P}^i_2}(1)$, which is semistable of slope 1.

8.4. Invariants. Denote $\mu_i = \text{mult}_{\mathcal{S}^i/\mathcal{S}}(\mathcal{E})$. By symmetry, $\mu_1 = \cdots = \mu_4$. Since $\Gamma_i \sim C_0$, we have the following upper bound:

$$\mu_1 \leq \frac{\mu}{4} \leq \frac{d}{4 \cdot 2}. \quad (\sharp)$$

The lower bound from Basic Lemma is:

$$\frac{1}{2} \mu_1 \geq \gamma = 3m - d. \quad (\flat)$$

Combining ($\sharp$) and ($\flat$), we get:

$$\frac{1}{2} \cdot \frac{d}{8} \geq 3m - d \iff 17d \geq 48m,$$

q.e.d.

The bound is sharp:

**Example 8.1.** The linear system $\mathcal{L}(8, 8, 17)$ has a unique section, namely the union of 8 curves of degree 6 each passing through 7 points with multiplicity 2 and 1 point with multiplicity 3 ([9], remark on p. 772; [4], Prop. 2.3).

9. Ten Points

Here we prove Main Theorem for $n = 10$ points.

9.1. Setup. Since $k = 3$, we take $C$ to be a smooth cubic in $\mathbf{P}^2$. Fix a point $W$ on $C$ such that $9W \sim 3H$.

For example, we can take $W$ to be a Weierstrass point of $C$ (however, later in Section 12 we will require that $3W \sim H$). Let $A = \mathcal{O}_C(W)$ which is of degree $\alpha = 1$. Since $h^1(A^\vee) = 1$, there is a unique nontrivial extension

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \rightarrow A \rightarrow 0$$

The surface $S = \mathbf{P}(\mathcal{E})$ is an indecomposable elliptic ruled surface of degree 1 (see Appendix A for background). The short exact sequence determines a minimal section of $S$ which we identify with the curve $C$.
9.2. **Degeneration.** Let $S = S \times \Delta$ and $Z = C \times \Delta \subset S$. We identify $S$ with the special fiber of $\mathcal{S} \to \Delta$. Next, we take $\Gamma = \Gamma_1 + \cdots + \Gamma_5$, where each $\Gamma_i$ is a general section of the pencil $| - 2K_S|$. By Prop. A.1, each $\Gamma_i \cong 4C - 2f$ is a smooth elliptic curve. Denote $\Gamma_i \cap C = \{P_{2i-1}, P_{2i}\}$. Note that $P_{2i-1} + P_{2i} \sim 2A$. Since $3A \sim H$, we have

$$\sum P_i \sim 10A \sim A + 3H.$$ 

Finally, we specialize the ten relative points $\mathcal{P}_1, \ldots, \mathcal{P}_{10}$ in $Z$ to $P_1, \ldots, P_{10}$ in a general way such that

$$\sum \mathcal{P}_{i,t} \sim A + 3H$$

for any $t \in \Delta$.

9.3. **Semistability.** We claim that, for each $i$, $N_{\tilde{\mathcal{G}}}/\tilde{\mathcal{S}}$ is semistable (of slope 1). Denote by $P_2'_{i}$ the image of $P_i$ in $\mathbb{P}(\mathcal{N}_{\Gamma_i}/\mathcal{S}) \cong \mathbb{P}(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma) \cong \Gamma \times \mathbb{P}^1$. Now, $\mathbb{P}(\mathcal{N}_{\Gamma_i}/\mathcal{S})$ is obtained from $\mathbb{P}(\mathcal{N}_{\Gamma_i}/\mathcal{S})$ by performing elementary transforms at the points $P_2'_{i-1}, P_2'_{i}$. If the specialization is general enough, the points $P_2'_{i-1}, P_2'_{i}$ do not belong to the same horizontal section of $\mathbb{P}(\mathcal{N}_{\Gamma_i}/\mathcal{S})$. It follows that $N_{\tilde{\mathcal{G}}}/\tilde{\mathcal{S}} \cong \mathcal{O}_{\Gamma_i}(P_2'_{i-1}) \oplus \mathcal{O}_{\Gamma_i}(P_2'_{i})$, which is semistable of slope 1.

![Figure 3. Specialization to $\Gamma$ ($n = 10$)](image)

9.4. **Invariants.** Denote $\mu_i = \text{mult}_{\Gamma_i/\mathcal{S}}(\mathcal{E})$. By symmetry, $\mu_1 = \cdots = \mu_5$. Since $\Gamma_i \cong 4C - 2f$, we have the following upper bound:

$$\mu_1 \leq \frac{\mu}{5 \cdot 4} \leq \frac{d}{5 \cdot 4 \cdot 3}.$$ (9)

The lower bound from Basic Lemma is:

$$\frac{1}{2} \mu_1 \geq \gamma = 19m - 6d.$$ (9)

From (9) and (9), we get:

$$\frac{1}{2} \cdot \frac{d}{60} \geq 19m - 6d \iff 721d \geq 2280m,$$

q.e.d.

10. **Eleven Points**

We prove Main Theorem for $n = 11$ points. This is the first time when we study an interpolation problem $\mathcal{Z}_S(\mu, b, \hat{m})$ by deforming the underlying surface $S$ itself.

10.1. **Setup.** As before, $C$ is a smooth cubic in $\mathbb{P}^2$. Fix a point $W$ on $C$ such that $9W \sim 3H$. Let $A$ be any line bundle of degree $\alpha = 2$ on $C$. It is easy to see that for a general $\xi \in \text{Ext}^1(A, \mathcal{O}_C)$, the ruled surface $S(C, A, \xi)$ is decomposable of degree 0. Denote by $C_{(i)}, i = 0, 1$, the two minimal sections of $S$. It follows that $C \equiv C_{(i)} + f$. 


10.2. Degeneration. We will construct a relative marked ruled surface $\mathcal{S}(Z, \mathcal{A},\xi;\{\mathcal{P}_{ij}\})$ such that:

- The special fiber $\mathcal{S}_0$ is simply $C \times \mathbb{P}^1$.
- The special section $Z_0 = C_0 \cup F$; here $C_0$ is a horizontal section of $\mathcal{S}_0$ and $F$ is the fiber of $\mathcal{S}_0$ above $W$.
- The relative points $\{\mathcal{P}_i\}$ on $Z$ are such that $\sum \mathcal{P}_{i,t} \sim \mathcal{A}_t + kH$ on $Z_t \cong C$, for general $t$.
- Each limit point $\mathcal{P}_i = \mathcal{P}_{i,0}$ is a general point on $F$.

The construction is done as follows. First, we choose relative points $\mathcal{P}_i$ in $C \times \Delta$ specializing to $W \times \{0\}$ in a general way. Let $\mathcal{A}'_0 = \mathcal{O}_{C \times \Delta}(\sum \mathcal{P}_i - 3H)$; $\mathcal{A}_0 = \mathcal{A}'_0 \otimes \mathcal{I}_{W \times \{0\}}$.

Since $9W \sim 3H$, it follows that $\mathcal{A}'_0 \cong \mathcal{O}_C(11W - 3H) \cong \mathcal{O}_C(2W)$; $\mathcal{A}_0 \cong \mathcal{O}_C(W) \oplus \mathcal{O}_W(W)$.

Consider the following short exact sequence on $C \times \{0\}$:

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(W) \oplus \mathcal{O}_C(W) \rightarrow \mathcal{O}_C(W) \oplus \mathcal{O}_W(W) \rightarrow 0$$

By Prop. 6.4, the sequence can be extended to

$$0 \rightarrow \mathcal{O}_{C \times \Delta} \rightarrow \mathcal{E} \rightarrow \mathcal{A} \rightarrow 0$$

with $\mathcal{E}$ locally free. Let $\mathcal{E} = \mathcal{P}(\mathcal{E})$ and $Z = \mathcal{P}(\mathcal{A})$. Hence $\mathcal{E}_0 = \mathcal{P}(\mathcal{O}_C(W) \oplus \mathcal{O}_C(W)) = C \times \mathbb{P}^1$. Now, $Z \rightarrow C \times \Delta$ is just the blowup of $W \times \{0\}$ with exceptional divisor $F$. Finally, we take $\mathcal{P}_i$ to be the strict transform of $\mathcal{P}_i$ in $Z$.

![Figure 4. The blowup $Z \rightarrow C \times \Delta$](image)

10.3. Semistability. We take $\Gamma = \Gamma_1 + \cdots + \Gamma_{11}$ where $\Gamma_i$ is the horizontal section of $\mathcal{E}_0 = C \times \mathbb{P}^1$ through $P_i$ (see Fig. 4). Consider the blowup $\mathcal{E} \rightarrow \mathcal{E}$ at the relative points $\mathcal{P}_i$. We claim that for each $i = 1, \ldots, 11$, $\mathcal{N}_{\Gamma_i/\mathcal{E}}$ is indecomposable of degree 1 (hence semistable of slope $1/2$). First, we will show that $\mathcal{N}_{\Gamma_i/\mathcal{E}}$ is indecomposable of degree 0.

We will need some deformation theory. Let $D = \mathbb{C}[t]/t^2$ be the ring of dual numbers. Let $\mathcal{E}' = \mathcal{E} \times_D D$ viewed as an infinitesimal deformation of $\mathcal{E}_0 \cong C \times \mathbb{P}^1$ over $D$. We will say that a section $T$ of $\mathcal{E}_0$ is (infinitesimally) unobstructed if and only if $T$ can be extended to a subscheme $T'$ of $\mathcal{E}'$ flat over $D$. 

Lemma 10.1. Assume the above setting.

(a) $T$ is unobstructed if and only if the following short exact sequence splits:

$$0 \to N_{\mathcal{E}/\mathcal{E}}|_T \to \mathcal{N}_T/\mathcal{E} \to \mathcal{N}_T/\mathcal{E}_0 \to 0$$

(b) Suppose there are 3 disjoint horizontal sections $T_1, T_2, T_3$ of $\mathcal{S}_0 \cong C \times \mathbb{P}^1$ that are unobstructed. Then, $\mathcal{E}'$ is an infinitesimally trivial deformation, i.e. $\mathcal{E}' \cong \mathcal{S}_0 \times D$. In particular, any section of $\mathcal{S}_0$ is unobstructed.

(c) $C_0$ is obstructed.

Proof. (a) This is clear.

(b) We have $\mathcal{E}' = \mathbb{P}(\mathcal{E}')$ where $\mathcal{E}' = \mathcal{E} \otimes D$. For any $i = 1, 2, 3$, the embedding $f_i : T'_i \to \mathcal{S}'$ induces a surjective morphism $\mathcal{E}' \to \mathcal{L}'$ where $\mathcal{L}' = f'_i \mathcal{E}'(1)$ is a line bundle on $C' = C \times D$.

By Nakayama’s lemma, for any $i \neq j$, the induced map $\mathcal{E}' \to \mathcal{L}'_i \oplus \mathcal{L}'_j$ is an isomorphism. Hence $\mathcal{L}'_i \cong \mathcal{L}'_2 \cong \mathcal{L}'_3 \cong \text{coker}(\mathcal{E}' \to \mathcal{L}'_1 \oplus \mathcal{L}'_2 \oplus \mathcal{L}'_3)$, i.e. $\mathcal{E}' \cong \mathcal{S}_0 \times D$.

(c) Consider the short exact sequence

$$0 \to N_{\mathcal{E}/\mathcal{E}}|_{C_0} \to N_{\mathcal{E}_0/\mathcal{E}} \to N_{\mathcal{E}_0/\mathcal{E}_0} \to 0$$

where $N_{\mathcal{E}/\mathcal{E}}|_{C_0} \cong \mathcal{O}_C(-W)$ and $N_{\mathcal{E}_0/\mathcal{E}_0} \cong \mathcal{O}_C(W)$. Since $h^0(C, \mathcal{O}_C(W)) = 1$, it is clear that $N_{\mathcal{E}_0/\mathcal{E}} \not\cong \mathcal{O}_C \oplus \mathcal{O}_C$. Now consider the short exact sequence from part (a):

$$0 \to N_{\mathcal{E}_0/\mathcal{E}}|_{C_0} \to N_{\mathcal{E}_0/\mathcal{E}} \to N_{\mathcal{E}_0/\mathcal{E}_0} \to 0$$

where $N_{\mathcal{E}_0/\mathcal{E}}|_{C_0} \cong \mathcal{O}_C$ and $N_{\mathcal{E}_0/\mathcal{E}_0} \cong \mathcal{O}_C$. It follows that $N_{\mathcal{E}_0/\mathcal{E}_0}$ is indecomposable of degree 0. Hence, $C_0$ is obstructed.

Part (c) of the lemma implies that $\mathcal{E}'$ is not an infinitesimally trivial deformation. By part (b) and by symmetry, $\Gamma_i$ is obstructed for any $i$. It follows that the conormal bundle $N_{\mathcal{E}_i/\mathcal{S}}$ is indecomposable of degree 0.

Denote by $P'_i$ the image of $\mathcal{P}_i$ in $\mathbb{P}(N_{\mathcal{E}_i/\mathcal{S}})$. Clearly, $P'_i$ does not belong to the unique minimal section of $\mathbb{P}(N_{\mathcal{E}_i/\mathcal{S}})$ (because $\mathcal{P}_i$ meets $\mathcal{S}_0$ transversely). Finally, $\mathbb{P}(N_{\mathcal{E}_i/\mathcal{S}})$ is obtained from $\mathbb{P}(N_{\mathcal{E}_i/\mathcal{S}})$ by performing an elementary transform at $P'_i$. It follows that $N_{\mathcal{E}_i/\mathcal{S}}$ is indecomposable of degree 1.

10.4. Invariants. Denote $\mu_i = \text{mult}_{\mathcal{E}_i/\mathcal{S}}(\mathcal{E})$. By symmetry, $\mu_1 = \cdots = \mu_{11}$. Since $\Gamma_i \cong C_0$, we have the following upper bound:

$$\mu_1 \leq \frac{\mu}{11} \leq \frac{d}{11 + 3}. \tag{2}$$

The lower bound from Basic Lemma is:

$$\frac{1}{2} \mu_1 \geq \gamma = 10m - 3d. \tag{3}$$

Combining (2) and (3), we get:

$$\frac{1}{2} \cdot \frac{d}{33} \geq 10m - 3d \iff 199d \geq 660m.$$

This completes the proof for eleven points.

Remark 10.2. It might be also profitable to study the behavior of $\mathcal{E}$ along the fiber $F$. We have $N_{\mathcal{E}/\mathcal{S}} \cong \mathcal{O}_{\mathbb{P}^1}(10) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, which follows from the split exact sequence

$$0 \to N_{\mathcal{E}/\mathcal{S}}|_F \to \mathcal{N}_{\mathcal{F}/\mathcal{E}} \to \mathcal{N}_{\mathcal{F}/\mathcal{Z}} \to 0$$

with $N_{\mathcal{E}/\mathcal{S}}|_F \cong \mathcal{O}_{\mathbb{P}^1}(10)$ and $\mathcal{N}_{\mathcal{F}/\mathcal{Z}} \cong \mathcal{N}_{\mathcal{F}/\mathcal{Z}} \cong \mathcal{O}_{\mathbb{P}^1}(1)$. The fact that $N_{\mathcal{F}/\mathcal{E}}$ is unstable causes certain multiplicity and tangency conditions on the limit curve $\mathcal{Z}_0$ at the point $W = C_0 \cap F$. A more careful analysis of the situation is beyond the scope of this paper.
11. Twelve Points

In this section we prove Main Theorem for \( n = 12 \). This case is similar to \( n = 10 \).

11.1. Setup. As before, \( C \) is a smooth cubic in \( \mathbb{P}^2 \). Fix a point \( W \) on \( C \) such that \( 9W \sim 3H \). Take \( A = \mathcal{O}_C(3W) \) which is of degree \( \alpha = 3 \). It is easy to see that for a general \( \xi \in \text{Ext}^1(A, \mathcal{O}_C) \), \( S(C, A, \xi) \) is an indecomposable elliptic ruled surface of degree 1. It follows that \( C \equiv C_0 + f \) where \( C_0 \) is a minimal section of \( S \).

11.2. Degeneration. Let \( \mathcal{S} = S \times \Delta \) and \( Z = C \times \Delta \subset \mathcal{S} \). We identify \( S \) with the special fiber of \( \mathcal{S} \). Take \( \Gamma = \Gamma_1 + \Gamma_2 \) where each \( \Gamma_i \) is a general section of the pencil \( |-2K_S| \). In particular, \( \Gamma_i \cdot C = (4C_0 - 2f) \cdot (C_0 + f) = 6 \). Let \( C \cap \Gamma_1 = \{ P_1, \ldots, P_6 \} \) and \( C \cap \Gamma_2 = \{ P_7, \ldots, P_{12} \} \). It follows that

\[
\sum P_i \sim 4A \sim A + 3H.
\]

Next, we specialize \( \mathcal{P}_i \) in \( Z \) to \( P_i \) in a general way such that

\[
\sum \mathcal{P}_{i,t} \sim A + 3H
\]

for any \( t \in \Delta \).

11.3. Semistability. We have to show that if the specialization of the points \( \mathcal{P}_i \) is general enough, the conormal bundle \( \mathcal{N}_{\Gamma_i/\mathcal{S}} \) is semistable (of slope 3). Since semistability is an open property ([8], Thm. 2.8), it suffices to describe a particular specialization for which \( \mathcal{N}_{\Gamma_i/\mathcal{S}} \) is semistable. This is not hard. In fact, we claim that we can specialize the points in such a way that

\[
\mathcal{N}_{\Gamma_1/\mathcal{S}} \cong \mathcal{O}_{\Gamma_1}(P_1 + P_2 + P_3) \oplus \mathcal{O}_{\Gamma_1}(P_4 + P_5 + P_6)
\]

and similarly for \( \Gamma_2 \). This can be achieved by moving the triples of points \( \{ \mathcal{P}_{3i-2}, \mathcal{P}_{3i-1}, \mathcal{P}_{3i} \} \) “in parallel” while being assigned to the same section of the pencil \( |-2K_S| \).

![Specialization to \( \Gamma \) (\( n = 12 \))](image)

11.4. Invariants. Denote \( \mu_i = \text{mult}_{\Gamma_i/\mathcal{S}}(\mathcal{C}) \). By symmetry, \( \mu_1 = \mu_2 \). Since \( \Gamma_i \equiv 4C_0 - 2f \), we have the following upper bound:

\[
\mu_1 \leq \frac{\mu}{2 \cdot 4} \leq \frac{d}{2 \cdot 4 \cdot 3} \quad (\sharp)
\]

The lower bound from Basic Lemma is:

\[
\frac{1}{2} \mu_1 \geq \gamma = 7m - 2d \quad (\natural)
\]

Combining (\#) and (\natural), we get:

\[
\frac{1}{2} \cdot \frac{d}{24} \geq 7m - 2d \iff 97d \geq 336m.
\]

This completes the proof for twelve points.
12. A Refinement

Here we prove a certain refinement of the Main Theorem in the case of $n = 10, 11$ and $12$ points. We work in the setting of the previous sections. The idea is to show that, under some additional assumptions, the inequality $(b)$ can be replaced by a stronger inequality $(bb)$. First, we have:

**Lemma 12.1.** Let $n = 10, 11$ or $12$. Consider the degeneration described above for the particular value of $n$. Let $b := B_0 = \mathcal{O}_C(\sum mP_i - dH)$.

(a) We have $b - bW \sim d(3W - H)$. In particular, $3b \sim 3bW$.

(b) Assume there is an equality in $(b)$, i.e. $\frac{1}{2}\mu_1 = \gamma$. Then $b \sim bW$.

**Proof.** (a) We have $\sum P_i \sim nW$. Therefore

$$b - bW \sim (nmW - dH) - (nm - 3d)W \sim d(3W - H).$$

Since $9W \sim 3H$, it follows that $3b \sim 3bW$.

(b) Assume $\frac{1}{2}\mu_1 = \gamma$. By Basic Lemma, there is an injective morphism:

$$\mathcal{O}_{\tilde{F}_1}(-\tilde{\mathcal{E}}) \hookrightarrow \text{Sym}^{2\gamma} \mathcal{N}_{\tilde{F}_1/\tilde{\mathcal{E}}}. $$

Now the idea is to show that the vector bundle on the right hand side decomposes as a direct sum of line bundles of the same degree as $\mathcal{O}_{\tilde{F}_1}(-\tilde{\mathcal{E}})$. It will follow that $\mathcal{O}_{\tilde{F}_1}(-\tilde{\mathcal{E}})$ is isomorphic to one of the summands. Below we consider each case for $n$ separately.

**Case** $n=10$. Here $\mathcal{O}_{\tilde{F}_1}(-\tilde{\mathcal{E}})$ is of degree $2\gamma$. More precisely, since $\mathcal{O}_{\tilde{F}_1}(C) \cong \mathcal{O}_{\tilde{F}_1}(P_1 + P_2)$, we have:

$$\mathcal{O}_{\tilde{F}_1}(-\tilde{\mathcal{E}}) \cong \mathcal{O}_{\tilde{F}_1}(-m(P_1 + P_2) + bf).$$

Recall that $\mathcal{N}_{\tilde{F}_1/\tilde{\mathcal{E}}} \cong \mathcal{O}_{\tilde{F}_1}(P_1) \oplus \mathcal{O}_{\tilde{F}_1}(P_2)$. Therefore:

$$\text{Sym}^{2\gamma} \mathcal{N}_{\tilde{F}_1/\tilde{\mathcal{E}}} \cong \bigoplus_{i+j=2\gamma} \mathcal{O}_{\tilde{F}_1}(iP_1 + jP_2).$$

It follows that $\mathcal{O}_{\tilde{F}_1}(-\tilde{\mathcal{E}})$ must be isomorphic to one of the summands. Hence $bf \sim iP_1 + jP_2$ on $\Gamma_1$ for some $i, j$ with $i + j = 4b$. Since $\Gamma_1 \to C$ is an isogeny of degree $4$, it follows that $4b \sim iP_1 + jP_2$ on $C$. By symmetry, $4b \sim jP_1 + iP_2$. Therefore, $8b \sim 4b(P_1 + P_2) \sim 8bW$ on $C$. Since $3b \sim 3bW$ and $\gcd(8, 3) = 1$, it follows that $b \sim bW$.

**Case** $n=11$. Here $\mathcal{O}_{\tilde{F}_1}(-\tilde{\mathcal{E}})$ is of degree $\gamma$. More precisely, since $\mathcal{O}_{\tilde{F}_1}(Z_0) \cong \mathcal{O}_{\tilde{F}_1}(P_1)$, we find:

$$\mathcal{O}_{\tilde{F}_1}(-\tilde{\mathcal{E}}) \cong \mathcal{O}_{\tilde{F}_1}(-mP_1 + bf) \cong \mathcal{O}_C(-mW + b).$$

Recall that $\mathcal{N}_{\tilde{F}_1/\tilde{\mathcal{E}}}$ is indecomposable of degree $1$ (determinant $W$). From the results in Appendix $[X] \text{Sym}^{2\nu} \mathcal{N}_{\tilde{F}_1/\tilde{\mathcal{E}}}$ is a direct sum of line bundles of the form $\mathcal{O}_C(\nu W + L_i)$ where $L_i^{\otimes 2} \cong \mathcal{O}_C$. We conclude that $\mathcal{O}_{\tilde{F}_1}(-\tilde{\mathcal{E}})$ is isomorphic to one of the summands. It follows that $2b \sim 2bW$. Since $3b \sim 3bW$ and $\gcd(2, 3) = 1$, we conclude that $b \sim bW$.

**Case** $n=12$. This is similar to the case of ten points. We leave the details to the reader.

The following result is a refinement of the Main Theorem. Note that it only applies when $3 \nmid d$.

**Proposition 12.2.** Let $n = 10, 11$ or $12$. If $\mathcal{L}(d, n, m)$ is nonempty and $3 \nmid d$, then $\kappa_n \geq 0$ with

$$\begin{align*}
\kappa_{10} &= 721d - 2280m - 60 \\
\kappa_{11} &= 199d - 660m - 33 \\
\kappa_{12} &= 97d - 336m - 24.
\end{align*}$$
Proof. Fix $W$ so that $9W \sim 3H$ but $3W \not\sim H$. Since $3 \nmid d$, part (a) of the lemma implies that $b \approx bW$. From part (b), we get:

$$\frac{1}{2} \mu_1 \geq \gamma + \frac{1}{2}. \quad (\ast)$$

Finally, $(\ast)$ together with $(\ast \ast)$ imply the desired inequality. \hfill \square

**Corollary 12.3.** The following linear systems $\mathcal{L}(d, n, m)$ with $v = -1$ are empty, hence non-special:

| $d$ | $n$ | $m$ | $\chi_{P^2}$ | $\mu$ | $\epsilon$ | $b$ | $\hat{m}$ | $\chi_S$ | $\gamma$ | $\kappa_n$ |
|-----|-----|-----|---------------|------|---------|-----|-------|------|------|--------|
| 1499| 10  | 474 | 0             | 499  | 2       | 243 | 25    | 0    | 12   | -1     |
| 778 | 10  | 246 | 0             | 259  | 1       | 126 | 13    | 0    | 6    | -2     |
| 428 | 11  | 129 | 0             | 142  | 2       | 135 | 13    | 0    | 6    | -1     |
| 229 | 11  | 69  | 0             | 76   | 1       | 72  | 7     | 0    | 3    | -2     |
| 215 | 12  | 62  | 0             | 71   | 2       | 99  | 9     | 0    | 4    | -1     |
| 118 | 12  | 34  | 0             | 39   | 1       | 54  | 5     | 0    | 2    | -2     |

**Remark 12.4.** The assumption $3 \nmid d$ in the proposition cannot be dropped. For example, consider the nonempty linear system $\mathcal{L}(57, 10, 18)$ (with $v = 0$, $\kappa_{10} = -3$ and $3 \mid d$). Unfortunately, it is not clear to us how to extend the proposition in the case $3 \mid d$. Our discussion will not be complete without mentioning the following interesting open problems: $\mathcal{L}(2220, 10, 702)$, $\mathcal{L}(627, 11, 189)$ and $\mathcal{L}(312, 12, 90)$ (with $v = 0$, $\kappa_n = 0$ and $3 \mid d$).

13. **The Remaining Case**

Here we prove the Main Theorem in the case when $k \geq 3$, $\alpha$ is even, $\alpha \mid 2n$. This generalizes the case of eleven points (in fact, the proof can be also applied in the case of eight points).

13.1. **Setup.** Let $C$ be a smooth plane curve of degree $k$. We will make the following assumption: there is a divisor $W = W_1 + \cdots + W_2$ on $C$, where $W_i$’s are distinct points, such that

$$\frac{2k^2}{\alpha} W \sim kH.$$  

Here is one way to construct such a curve. Fix a line $\ell \subset P^2$ and let $W_1, \ldots, W_2$ be distinct points on $\ell$. Now, take $C$ to be any smooth curve of degree $k$ which is tangent to $\ell$ to order $\frac{2k}{\alpha}$ at each of the points $W_i$. It follows that $\frac{2k}{\alpha} W \sim H$, which satisfies the assumption.

13.2. **First Degeneration.** It will be convenient to re-index the $n$ relative points as $\{\mathcal{P}_{ij}\}$ where $i = 1, \ldots, \frac{2n}{\alpha}$ and $j = 1, \ldots, \frac{\alpha}{2}$. We will construct a relative marked ruled surface $\mathcal{G}(Z, \mathcal{A}, \xi; \{\mathcal{P}_{ij}\})$ such that:

- The special fiber $\mathcal{G}_0$ is simply $C \times P^1$.
- The special section $Z_0 = C_0 \cup F_1 \cup \cdots \cup F_2$; here $C_0$ is a horizontal section of $\mathcal{G}_0$ and $F_j$ is the fiber of $\mathcal{G}_0$ above $W_j$, for each $j$.
- The relative points $\{\mathcal{P}_{ij}\}$ on $Z$ are such that $\sum \mathcal{P}_{ij,t} \sim \mathcal{A}_t + kH$ on $Z_t \cong C$, for general $t$.
- Each limit point $P_{ij} = \mathcal{P}_{ij,0}$ is a general point on the fiber $F_j$.

The construction generalizes the case of eleven points. Namely, we first choose relative points $\mathcal{P}_{ij}$ in $C \times \Delta$ specializing to $W_j \times \{0\}$ in a general way. Next, let $\mathcal{A}' = \mathcal{O}_C \times \Delta(\sum \mathcal{P}_{ij} - kH)$ and $\mathcal{A} = \mathcal{A}' \otimes I_{W \times \{0\}}$. It follows that

$$\mathcal{A}_0' \cong \mathcal{O}_C(\frac{2\alpha}{\alpha} W - kH) \cong \mathcal{O}_C(2W);$$

$$\mathcal{A}_0 = \mathcal{O}_C(W) \oplus \mathcal{O}_W(W).$$

Consider the short exact sequence on $C \times \{0\}$:

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(W) \oplus \mathcal{O}_C(W) \rightarrow \mathcal{O}_C(W) \oplus \mathcal{O}_W(W) \rightarrow 0$$
(Note that \( h^0(C, \mathcal{O}_C(W)) = 1 \), so the sequence is unique). By Prop. 6.4 the sequence can be extended to
\[
0 \to \mathcal{O}_{C \times \Delta} \to \mathcal{E} \to \mathcal{A} \to 0
\]
over \( C \times \Delta \), where \( \mathcal{E} \) is locally free. Finally, we take \( \mathcal{S} = P(\mathcal{E}) \) and \( Z = P(\mathcal{A}) \). It follows that \( \mathcal{S}_0 \cong P(\mathcal{O}_C \oplus \mathcal{O}_C) \cong C \times P^1 \). We take \( \mathcal{P}_{ij} \) to be the strict transform of the \( \mathcal{P}_{ij} \) on the blowup \( Z \to C \times \Delta \) at \( W \times \{0\} \).

Next, we will distinguish between two cases: \( \alpha = 2 \) and \( \alpha \geq 4 \).

13.3. **Semistability** (\( \alpha = 2 \)). Let \( \Gamma_i \) be the horizontal section of \( \mathcal{S}_0 \) through \( P_{i,1} \). Just as in the case of eleven points, we can show that the conormal bundle \( \mathcal{N}_{\Gamma_i/S} \) is semistable of slope \( 1/2 \).

13.4. **Second Degeneration** (\( \alpha \geq 4 \)). In this case, we perform another degeneration on the trivial ruled surface \( S = C \times P^1 \). Fix \( \frac{2n}{\alpha} \) general horizontal sections \( \Gamma_1, \ldots, \Gamma_{2n/\alpha} \) of \( S \). Let \( P_{ij} = \Gamma_i \cap F_j \) for \( i = 1, \ldots, \frac{2n}{\alpha} \) and \( j = 1, \ldots, \frac{\alpha}{2} \). Next, we specialize the \( n \) relative points \( \mathcal{P}_{ij} \) to \( P_{ij} \) by “sliding” them along the corresponding fibers \( F_j \), in a general way.

13.5. **Semistability** (\( \alpha \geq 4 \)). Denote by \( \widetilde{S} \times \Delta \) the blowup of \( S \times \Delta \) at the relative points \( \mathcal{P}_{ij} \) constructed in the previous step. Now, \( P(\mathcal{N}_{\Gamma_i/S \times \Delta}) \) is obtained from \( P(\mathcal{N}_{\Gamma_i/S \times \Delta}) = P(\mathcal{O}_C \oplus \mathcal{O}_C) \) by applying \( \alpha/2 \) elementary transforms at general points on the fixed fibers through \( W_1, \ldots, W_{\frac{\alpha}{2}} \). Since \( \alpha/2 \geq 2 \), it follows that the resulting vector bundle is semistable of slope \( \alpha/4 \) (the proof is similar to that of Lemma 3.2).

13.6. **Invariants.** The computation of invariants was carried out in Section 7. This completes the proof of the Main Theorem.

**APPENDIX A. THE INDECOMPOSABLE ELLIPTIC RULED SURFACE OF DEGREE 1**

Below we summarize some facts about the indecomposable elliptic ruled surface of degree 1. Our references are [11] and (7, Chapter V.2).

Let \( C \) be an elliptic curve. Let \( \mathcal{E} \) be an indecomposable rank 2 vector bundle of degree 1 on \( C \). Then, \( \mathcal{E} \) arises as the unique nontrivial extension
\[
0 \to \mathcal{O}_C \to \mathcal{E} \to A \to 0
\]
where \( A = \text{det}(\mathcal{E}) \).

Let us compute the symmetric powers of \( \mathcal{E} \). By (11, Lemma 22 on p.439), we have:
\[
\mathcal{E}^* \otimes \mathcal{E} \cong \mathcal{O}_C \oplus L_1 \oplus L_2 \oplus L_3
\]
where the \( L_i \) are the nontrivial line bundles with \( L_i^{\otimes 2} \cong \mathcal{O}_C \). Also, by (11, Cor. to Thm. 7 on p.434), we have \( \mathcal{E} \otimes L_i \cong \mathcal{E} \) and \( \mathcal{E}^* \cong \mathcal{E} \otimes A^{-1} \). Finally, we have the Clebsch-Gordan formula (11, p.438) for a rank 2 vector bundle:
\[
\text{Sym}^m \mathcal{E} \otimes \mathcal{E} \cong \text{Sym}^{m+1} \mathcal{E} \oplus (A \otimes \text{Sym}^{m-1} \mathcal{E}).
\]

Using the above, we find:
\[
\begin{align*}
\text{Sym}^2 \mathcal{E} & \cong A \otimes (L_1 \oplus L_2 \oplus L_3) \\
\text{Sym}^3 \mathcal{E} & \cong A \otimes (\mathcal{E} \otimes \mathcal{E}) \\
\text{Sym}^4 \mathcal{E} & \cong A \otimes (\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E}) \\
& \quad \vdots
\end{align*}
\]

In general, if \( m = 2k \) is even, \( \text{Sym}^m \mathcal{E} \) decomposes as a sum of line bundles that are isomorphic to \( A^\otimes k \) or \( A^\otimes k \otimes L_i \). If \( m = 2k + 1 \) is odd, \( \text{Sym}^m \mathcal{E} \) decomposes as a sum of \( k + 1 \) copies of \( A^\otimes k \otimes \mathcal{E} \).
Next, consider the ruled surface $S = \mathbf{P}(E)$ together with the projection $\pi : S \to C$. We identify $C$ with the unique section of $|\mathcal{O}_S(1)|$. The anticanonical class of $S$ is

$$-K_S \sim 2C - \pi^*(A).$$

Note that

$$K_S^2 = 0.$$

We have:

**Proposition A.1.** Let $E$ be the indecomposable vector bundle of rank 2 and degree 1, $\det(E) = A$. Consider the ruled surface $S = \mathbf{P}(E)$.

(a) For any $x \in \text{Pic}^0(C)$, there is a unique curve $C_x \sim C + \pi^*(x)$ on $S$.

(b) There are precisely 3 curves $\Gamma_i$, $i = 1, 2, 3$, on $S$ that are numerically equivalent to $-K_S$.

(c) The linear system $|{-2K_S}|$ sweeps a base-point free pencil on $S$. There are 3 nonreduced sections, namely $2\Gamma_i$, $i = 1, 2, 3$. Any other section is a smooth elliptic curve $\Gamma$ isomorphic to $C$ (the natural projection $\Gamma \to C$ being the usual multiplication-by-2 map).

**Proof.** a) This follows from the fact that $E \otimes \pi^*(x)$ is the unique indecomposable rank 2 vector bundle with determinant $A + 2x$.

b) This follows from $\text{Sym}^2 E \cong A \oplus (L_1 \oplus L_2 \oplus L_3)$.

c) We have $\text{Sym}^2 E \cong A \oplus (\mathcal{O}_C^2 \oplus L_1 \oplus L_2 \oplus L_3)$. Therefore $h^0(-2K_S) = h^0(\text{Sym}^2 E \otimes A^2) = 2$, i.e. $|{-2K_S}|$ sweeps a pencil on $S$. Next, let $\Gamma$ any section of $|{-2K_S}|$ other than $2\Gamma_i$, $i = 1, 2, 3$. From part b), $\Gamma$ is irreducible. Since $\Gamma$ is of arithmetic genus 1, it follows that $\Gamma$ is a smooth elliptic curve. One can show that $\Gamma$ is isomorphic to $C$ as follows. First, one shows that every irreducible section of $|{-2K_S}|$ is isomorphic to the fixed section $\Gamma$. Next, one checks that, for any $i = 1, 2, 3$, $\Gamma$ admits a 2:1 cover to $\Gamma_i$. It follows that $\Gamma \to \Gamma_i$ is the isogeny dual to $\Gamma_i \to C$. \hfill $\square$

**Appendix B. Some Continued Fractions**

Let $n = k^2 + \alpha$ where $k > 0$ and $\alpha > 0$. Consider the matrix

$$M_1 = \begin{bmatrix} k & n \\ 1 & k \end{bmatrix}.$$

For any positive integer $i$, define

$$M_i = \begin{bmatrix} p_i & q_i \\ r_i & p_i \end{bmatrix} = \alpha^{-i/2}(M_1)^i.$$

Note that $\det(M_{2i-1}) = -\alpha$ and $\det(M_{2i}) = 1$.

For example,

$$p_1 = k; \quad q_1 = n; \quad r_1 = 1;$$

and

$$p_2 = \frac{n + k^2}{\alpha}; \quad q_2 = \frac{2nk}{\alpha}; \quad r_2 = \frac{2k}{\alpha}.$$

The ratios $p_i/r_i$ and $q_i/p_i$ have natural expansions as continued fractions approximating $\sqrt{\alpha}$ (the later are palindromic). In particular,

$$\frac{q_2}{p_2} = \epsilon_n^{(1)} \quad \text{and} \quad \frac{q_4}{p_4} = \epsilon_n^{(2)}$$

are precisely the constants in Theorem [4.1] and the Main Theorem.
Lemma B.1. Let $d$ and $m$ be any real numbers.

(a) The following are equivalent:

$$\frac{\alpha d}{2p_1} \geq q_1m - p_1d \iff p_2d - q_2m \geq 0.$$ 

(b) The following are equivalent:

$$\frac{1}{2} \frac{d}{q_2} \geq p_2m - r_2d \iff p_4d - q_4m \geq 0.$$ 

Proof. (a) Multiply both sides by $2p_1$ and substitute $\alpha = -p_1^2 + q_1r_1$:

$$(\alpha + 2p_1^2)d \geq 2p_1q_1m \implies \left(\frac{p_1^2 + q_1r_1}{\alpha p_2}\right) d \geq \frac{2p_1q_1 m}{\alpha q_2}.$$ 

(b) Multiply both sides by $2q_2$ and substitute $1 = p_2^2 - q_2r_2$:

$$(1 + 2q_2r_2)d \geq 2p_2q_2m \implies \left(\frac{p_2^2 + q_2r_2}{p_4}\right) d \geq \frac{2p_2q_2 m}{q_4}.$$ 

\[\square\]

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