Abstract. In connection with cluster algebras, snake graphs and $q$-integers, Kyungyong Lee and Ralf Schiffler recently found a formula for computing the (normalized) Jones polynomials of rational links in terms of continued fraction expansion of rational numbers. Sophie Morier-Genoud and Valentin Ovsienko introduced $q$-deformed continued fractions, and showed that by using them each coefficient of the normalized Jones polynomial counted quiver representations of type $A_n$. In this paper we introduce $q$-deformed integers defined by pairs of coprime integers, which are motivated by the denominators and the numerators of their $q$-deformed continued fractions, and give an efficient algorithm for computing the (normalized) Jones polynomials of rational links. Various properties of $q$-integers defined by pairs of coprime integers are investigated and shown its applications.

For computation of the Jones polynomials for rational links, various formulas have already been known [2, 8, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21]. Among them Lee and Schiffler [14] study on the Jones polynomials of rational links from connection with cluster algebras and snake graphs, and give formulas for computation of them based on continued fractions. Morier-Genoud and Ovsienko [15] introduce a notion of $q$-deformed continued fractions and show that the normalized Jones polynomials whose constant term is 1 can be computed from their $q$-deformed rational numbers. Kogiso and the author [10, 11, 12] define the Kauffman bracket polynomials for Conway-Coxeter friezes of zigzag type, and show that the friezes correspond to the rational links up to isomorphism.

In the present paper we introduce a new algorithm for computation of the Jones polynomials of rational links and give some applications of them. Our approach is based on $q$-Farey sums by the negative continued fraction expansions of rational numbers, which are introduced by Morier-Genoud and Ovsienko [15], and the formula computing writhes of rational link diagrams given by Nagai and Terashima [17].

For a rational number $\alpha$ the normalized Jones polynomial $J_\alpha(q)$ of the corresponding rational link to $\alpha$ can be simply computed in comparison with the original Jones polynomial $V_\alpha(t)$ under the substitution of $t = -q^{-1}$. In fact, the normalized Jones polynomial can be inductively computed by $q$-Farey sums, which relate with hyperbolic geometry, cluster algebras, Euler-Ptolemy-Plücker relation, and so on. According to the original definition by Morier-Genoud...
and Ovsienko, for computing the $q$-Farey sum of $q$-rational numbers $[\frac{x}{a}]_q$ and $[\frac{y}{b}]_q$, the negative continued fraction expansion of the Farey sum $\frac{x}{a} + \frac{y}{b}$ is required. This gives rise to an interesting phenomenon as follows. For two rational numbers whose denominators are common, namely $r$, the denominators of their $q$-rational numbers does not coincide while they are $q$-deformations of the same integer $r$. To analyze this fact, we introduce $q$-deformations $(a, b)_q$ of a positive integer $k$ associated with pairs $(a, b)$ of coprime and positive integers such that $a + b = k$. In the present paper various properties and equations on $(a, b)_q$ are investigated. In particular, it is shown that the $q$-rational number of the Farey sum $\frac{x}{a} + \frac{y}{b}$ is given by $(x, y)_q(a, b)_q$. This means that the $q$-rational number $[\frac{x}{a} + \frac{y}{b}]_q$ can be obtained by computing from the numerators $x, y$ and the denominators $a, b$, separately. It is also given a formula of computing the normalized Jones polynomial $J_\alpha(q)$ from $(a, b)_q$. Though $(a, b)_q$ actually coincides with the numerator of $q$-rational number $[\frac{a+b}{b}]_q$, it would be expected that $(a, b)_q$ is useful and significant to study $q$-rational numbers and related areas, for instance, construction and analysis of $q$-deformations of Conway-Coxeter friezes.

When we wish to recover the original Jones polynomial from the normalized one, the most complicated factor is writhes. In [14] the formula for computing writhes of rational link diagrams by using even fraction expansions. However, the even fraction expansion of a rational number does not always exist though it can be transformed into one having even fraction expansion. Thanks to Nagai and Terashima’s formula [17], Kogiso and the author [12] gave a purely combinatorial formula of the writhe $wr(\alpha)$ based on the regular continued fraction expansion of $\alpha$. In the present paper we derive its negative version.

This paper is organized as follows. In Section 1 the definitions of rational links and their normalized Jones polynomials are recalled. A useful method of recovering the original Jones polynomials from the normalized ones is described. In Section 2 the above statement is proved. Moreover, based on negative continued fraction expansions, a combinatorial formula for computing writhes of rational link diagrams is given. In Section 3 we introduce $q$-deformations of integers derived from pairs of coprime integers. Various properties of them are investigated. In particular, we give a formula of computing $q$-Farey sums of $q$-rational numbers defined by Morier-Genoud and Ovsienko [15] from our $q$-integers derived from pairs of coprime integers. In the final section it is shown that the normalized Jones polynomials can be computed by $q$-integers derived from pairs of coprime integers. As an application of it, it is also shown that our $q$-integers actually are given by numerators of $q$-deformed rational numbers. So, the coefficients of the $q$-integers have a quiver theoretical meaning given in [15].

We use the following notations: For an $x \in \mathbb{R}$, we set $[x] := \min\{ n \in \mathbb{Z} \mid x \leq n \}$, $\lfloor x \rfloor := \max\{ n \in \mathbb{Z} \mid n \leq x \}$. We note that if $\alpha \not\in \mathbb{Z}$, then $\lfloor \alpha + 1 \rfloor = \lceil \alpha \rceil$. We refer to [3, 16] et al for basic theory of knots and links.
1. Rational links and their normalized Jones polynomials

Rational links are one of fundamental classes in knots and links. In some context they are called two-bridge links. They are defined by continued fraction expansions of rational numbers as follows.

A rational number $\alpha > 0$ can be represented by the following continued fraction:

$$[a_1, a_2, \ldots, a_n] := a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

where $a_1 \geq 0$ and $a_2, \ldots, a_n > 0$ are integers. This representation is unique if parity of $n$ is specified.

Let $\alpha \in \mathbb{Q}$ with $0 < \alpha < 1$, and write in the form $\alpha = [0, a_1, \ldots, a_n]$, where $n$ is odd. The link in the 3-sphere $S^3$ determined by the diagram $D(\alpha)$ below is called a rational link.

$$D(\alpha) = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
a_1 \\
a_2 \\
\vdots \\
\vdots \\
a_{n-1} \\
a_n \\
\end{array}
\end{array}$$

where

$$\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
-1 \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array} = \begin{cases}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{cases} \text{ if } n \geq 0,
\begin{cases}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{cases} \text{ if } n < 0.
\end{array}$$

If $\alpha > 1$, then $D(\alpha)$ is defined by $D(\alpha) := D(\alpha^{-1})$, and if $\alpha = 1$, then $D(\alpha) = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}$. According as the denominator of $\alpha$ is odd or even, $D(\alpha)$ is a knot or a two-component link, respectively. Orientations for $D(\alpha)$ are given as follows:

• If the denominator of $\alpha$ is odd, then

$$D(\alpha) = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}.$$

• If the denominator of $\alpha$ is even, then

$$D(\alpha) = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}.$$

In this case there is another orientation for $D(\alpha)$ which is given by
We denote the oriented link diagram by $D_{+-}(\alpha)$.

For an $\alpha \in \mathbb{Q}$ the oriented rational link determined by $D(\alpha)$ is denoted by $L(\alpha)$.

For a link diagram $D$, the Kauffman bracket polynomial $\langle D \rangle$ is defined. It is a regular isotopy invariant of $D$ and takes a value in the Laurent polynomial ring $\mathbb{Z}[A, A^{-1}]$. The Jones polynomial [7, 9] is an isotopy invariant for an oriented link $L$ in $S^3$, which is valued in $\mathbb{Z}[t^{\pm \frac{1}{2}}]$, and computed from the Kauffman bracket polynomial by the formula

$$V_L(t) = (-A^3)^{-\text{wr}(D)} \langle D \rangle \big|_{A=t^{-\frac{1}{2}}}.$$

where $\text{wr}(D)$ is the writhe of an oriented link diagram $D$ of $L$, which is the sum of signs $\pm 1$ running over all crossings when we correspond a sign to each crossing as in Figure 1:

![Figure 1. signs of crossings](image)

Example 1.1. Since $V_{\frac{1}{2}}(t) = -t^{\frac{3}{2}} + t^{\frac{1}{2}} - 3t^{\frac{1}{2}} + 2t^{-\frac{3}{2}} - 2t^{-\frac{1}{2}} + 2t^{-\frac{5}{2}} - t^{-\frac{9}{2}}$, $J_{\frac{1}{2}}(q) = 1 + q + 3q^2 + 2q^3 + 2q^4 + 2q^5 + q^6$.

By Lee and Schiffler [14], it is known that the Jones polynomial $V_{\alpha}(t)$ can be recovered from $J_{\alpha}(q)$. Their formula is given by even continued fractions. Unfortunately $\alpha$ often does not have an even continued fraction expansion though it can be always transformed into one having even continued fraction expansion [14, Proposition 2.3]. In this paper we give an alternative formula for it in terms of a negative continued fraction expansion of $\alpha$. Here, by a negative continued fraction we means an expression of a rational number as in the form

$$c_1 - \frac{1}{c_2 - \frac{1}{\ddots - \frac{1}{c_l}}}.$$

For a rational number $\alpha$, consider the following normalization $J_{\alpha}(q)$ of $V_{\alpha}(t) := V_{L(\alpha)}(t)$ [14]:

$$J_{\alpha}(q) := \pm t^{-h} V_{\alpha}(t) \big|_{t=-q^{-1}},$$

where $\pm t^h$ is the leading term of $V_{\alpha}(t)$. This indicates the normalization such that the constant term is 1 as a polynomial in $q$. 

Example 1.1. Since $V_{\alpha}(t) = -t^{\frac{3}{2}} + t^{\frac{1}{2}} - 3t^{\frac{1}{2}} + 2t^{-\frac{3}{2}} - 2t^{-\frac{1}{2}} + 2t^{-\frac{5}{2}} - t^{-\frac{9}{2}}$, $J_{\alpha}(q) = 1 + q + 3q^2 + 2q^3 + 2q^4 + 2q^5 + q^6$. 

By Lee and Schiffler [14], it is known that the Jones polynomial $V_{\alpha}(t)$ can be recovered from $J_{\alpha}(q)$. Their formula is given by even continued fractions. Unfortunately $\alpha$ often does not have an even continued fraction expansion though it can be always transformed into one having even continued fraction expansion [14, Proposition 2.3]. In this paper we give an alternative formula for it in terms of a negative continued fraction expansion of $\alpha$. Here, by a negative continued fraction we means an expression of a rational number as in the form

$$c_1 - \frac{1}{c_2 - \frac{1}{\ddots - \frac{1}{c_l}}}.$$
where $c_1, \ldots, c_l$ are integers which are greater than or equal to 2. We denote the fraction \([1.3]^{-}\) by \([c_1, \ldots, c_l]^{-}\), which is denoted by \([c_1, \ldots, c_l]\) in [15]. Any rational number $\alpha > 1$ can be uniquely represented by a negative continued fraction. Though the uniqueness does not hold, the fraction \([1.3]\) has a meaning even if $c_i = 1$ for some $i$.

**Theorem 1.2.** For a rational number $\alpha > 1$,
\[V_\alpha(t) = (-t)^{-\frac{1}{2} \text{wr}(\alpha) - \frac{3}{4} l + \frac{3}{4} l'} J_\alpha(-t^{-1}),\]
where $\text{wr}(\alpha) = \text{wr}(D(\alpha))$ and $l' = \sum_{j=1}^{l} (c_j - 2) + 1$ for $\alpha = [c_1, \ldots, c_l]$ with $c_j \geq 2$ for all $j = 1, \ldots, l$.

The theorem would be proved in the next section. Since the write \(\text{wr}(\alpha)\) can be purely computed in a combinatorial way by negative continued fraction expansion as shown in the next section, the Jones polynomial $V_\alpha(t)$ is recovered from the normalized Jones polynomial $J_\alpha(q)$ in a purely combinatorial way.

2. **Computation of the Jones polynomials for rational links based on negative continued fractions**

Irreducible fractions $\frac{x}{a}, \frac{y}{b}$ are said to be Farey neighbors if $ay - bx = 1$. In this paper the following conventions are assumed:

- $\infty = \frac{1}{0}$ is regarded as an irreducible fraction.
- For any irreducible fraction $\frac{x}{a}$, it is always $q \geq 0$.

If $\frac{x}{a}, \frac{y}{b}$ are Farey neighbors, then $\frac{x+y}{a+b} := \frac{\frac{x}{a} + \frac{y}{b}}{\frac{a}{a} + \frac{b}{b}}$, called the Farey sum, is also irreducible. On Farey neighbors the following lemmas hold:

**Lemma 2.1.**
1. Any non-negative rational number can be obtained from $\frac{0}{1}$ and $\frac{1}{0}$ applying $\#$ in finitely many times.
2. For any $\alpha \in \mathbb{Q}$, there are uniquely Farey neighbors $\frac{x}{a}, \frac{y}{b}$ such that $\alpha = \frac{x+y}{a+b}$. The pair $(\frac{x}{a}, \frac{y}{b})$ is called the parents of $\alpha$.

For a proof of the above lemma, see [1, Theorem 3.9] or [10, Lemma 3.5].

**Lemma 2.2.** If $\frac{x}{a}, \frac{y}{b} > 0$ are Farey neighbors, then
\[\left\lfloor \frac{x+y}{a+b} \right\rfloor = \begin{cases} \frac{x}{a} & \text{if } a > 1 \\ \frac{x}{a} + 1 & \text{if } a = 1 \end{cases} = \left\lfloor \frac{x}{a} + 1 \right\rfloor.
\]

**Proof.** If $a = 1$, then $y = bx + 1$. Since $\frac{x+y}{a+b} = x + \frac{r}{a+1}$, we have $\left\lfloor \frac{x+y}{a+b} \right\rfloor = x + 1 = \left\lfloor \frac{x}{a} \right\rfloor + 1$.

Next, consider the case $a > 1$. If $b = 1$, then $x = ay - 1$, and hence $\frac{x+y}{a+b} = y - \frac{1}{a+1}$. Thus, $\left\lfloor \frac{x+y}{a+b} \right\rfloor = y = \left\lfloor \frac{x}{a} \right\rfloor$. If $b > 1$, then $x, y$ are expressed as $x = ma + r, y = nb + s$ for some $r (0 < r < a), s (0 < s < b)$. 

If $m \geq n$, then
\[(*) \quad \frac{x + y}{a + b} = n + \frac{(m - n)a + (r + s)}{a + b}.
\]
Since $ay - bx = 1$, we have $ab(n - m) + sa - rb = (nb + s)a - (ma + r)b = 1$. Thus $(m - n)a = \frac{sa - rb - 1}{b}$, and it follows that the right-hand side of $(*)$ is less than $n + \frac{a}{b} < n + 1$. This implies that $\lceil \frac{x + y}{a + b} \rceil = n + 1 = \lceil \frac{a}{b} \rceil$. On the other hand, since $\frac{a}{b} < \frac{y}{b}$, we have $\lceil \frac{a}{b} \rceil \leq \lceil \frac{y}{b} \rceil$ while $\lceil \frac{a}{b} \rceil \geq \lceil \frac{y}{b} \rceil$ by $m \geq n$. Thus $m + 1 = \lceil \frac{x}{a} \rceil = \lceil \frac{y}{b} \rceil = n + 1$. This means that if $m \geq n$, then $m = n$, and $\lceil \frac{x + y}{a + b} \rceil = \lceil \frac{a}{b} \rceil = \lceil \frac{a}{b} \rceil$.

If $n > m$, then $\frac{x + y}{a + b} = m + \frac{(n - m)b + (r + s)}{a + b} = m + \frac{1 + r(a + b)}{a(a + b)} < m + 1$. Therefore, $\lceil \frac{x + y}{a + b} \rceil = m + 1 = \lceil \frac{a}{b} \rceil$.

From Lemma 2.1 we have a binary tree with extra vertices $\frac{0}{1}$ and two (dotted) edges, whose vertices are the non-negative rational numbers, depicted as in Figure 2. We call the tree the (extended) Stern-Brocot tree, which is essentially the same with the Farey tessellation appeared in hyperbolic geometry. We note that any non-negative rational number is appeared only once in the Stern-Brocot tree as a vertex.

Based on the Stern-Brocot tree, for each positive rational number $\alpha$ one can find a triangle $\text{YAT}(\alpha)$ which is shaped as $\text{YAT}(\alpha)$ and decomposed into small triangles according to the following rules:

(YAT1) All vertices are on two oblique edges.

(YAT2) For any small triangle such as $\beta_L \beta_R \beta$, $\beta = \beta_L \beta_R$.

The triangle $\text{YAT}(\alpha)$ is called the ancestor triangle of $\alpha$, and each small triangle in $\text{YAT}(\alpha)$ is called a fundamental triangle. This concept is introduced by Shuji Yamada [21] to study of
the Jones polynomials of two-bridge links. The same concept is also introduced by Hatcher and Ortel \[4\] from the more geometrical point of view.

Example 2.3. Let $\alpha = \frac{7}{3}$. Then $\frac{7}{3} = \frac{5 \# 2}{2 \# 3} = \frac{3 \# 2}{2 \# 3} = \frac{1 \# 2}{1 \# 3} = \frac{1}{1 \# 3} = \frac{0}{1 \# 3}$. Thus, $\text{YAT}(\frac{7}{3})$ consists of the blue parts in Figure 3.

Let $\alpha \in (0, 1) \cap \mathbb{Q}$. We then define a Laurent polynomial $\langle \Gamma_{\alpha} \rangle \in \mathbb{Z}[t, t^{-1}]$ as follows. First of all, for each fundamental triangle, we set $-t^{-1}$ and $-t$ on the left and right oblique edges, respectively. For an upward path $\gamma$ from $\alpha$ to $0$ or $1$ we compute the product $W(\gamma)$ of $-t^\pm 1$'s on all edges passed by $\gamma$. Finally, we set $\langle \Gamma_{\alpha} \rangle = \sum_{\gamma} W(\gamma)$ under replacing $t = A^{-4}$. By previous joint works with Kogiso, it is shown that $\langle \Gamma_{\alpha} \rangle$ coincides with the Jones polynomial $V_{\alpha}(t)$ up to multiplying $\pm t^k$ for some $k \in \mathbb{Z}$. More precisely we have the following:

**Proposition 2.4** ([12, Proposition 4.3]). For an $\alpha \in (0, 1) \cap \mathbb{Q}$,

\[(2.2) \quad V_{\alpha}(t) = (-A^{-3})^{\widetilde{\text{wt}}(\alpha)}|_{A^{-4}}\]

where $\widetilde{\text{wt}}(\alpha) = -\text{wr}(\alpha) - \text{wt}(\alpha)$, and $\text{wt}(\alpha) = -\sum_{j=1}^{l}(c_j - 2) + l - 1$ for $\alpha^{-1} = [c_1, \ldots, c_l]^{-}$ with $c_j \geq 2$ for all $j$.

**Remark 2.5.** Note that the above replacement $t = A^{-4}$ is inverse of that in the papers [11,12].

Let $\alpha \in (0, 1) \cap \mathbb{Q}$, and define two paths $\gamma_L$ and $\gamma_R$ as in Figure 4 that is, $\gamma_L$ is the path from $\alpha$ to $\frac{1}{2}$ along the left oblique edge, and $\gamma_R$ is the path from $\alpha$ to $\frac{1}{2}$ along the right oblique edge. We note that $\frac{1}{2}$ is always on the right oblique side in YAT($\alpha$).

By counting the number of edges on the left and right oblique sides, respectively, $W(\gamma_L)$ and $W(\gamma_R)$ are given as in the following lemma.

**Lemma 2.6.** For an $\alpha \in (0, 1) \cap \mathbb{Q}$ with expression $\alpha^{-1} = [c_1, \ldots, c_l]^{-}$

\[(2.3) \quad W(\gamma_L) = (-t^{-1})^l, \quad W(\gamma_R) = (-t)^l,\]
where \( l' := \sum_{j=1}^{l} (c_j - 2) + 1 \). They are the lowest and highest term of \( \langle \Gamma_\alpha \rangle \), respectively.

By using the lemma Theorem 1.2 can be proved as follows.

**Proof of Theorem 1.2.** By Proposition 2.4 \( V_\alpha(t) = V_{\alpha^{-1}}(t) = (-t^{\frac{3}{4}}) \tilde{w}_t(\alpha^{-1}) \langle \Gamma_{\alpha^{-1}} \rangle \). So, by Lemma 2.6 the leading term of \( V_\alpha(t) \) is given by \( (-t^{\frac{3}{4}}) \tilde{w}_t(\alpha^{-1}) W(\gamma_R) = (-t^{\frac{3}{4}}) \tilde{w}_t(\alpha^{-1}) w_\gamma + l' \). This implies the equation (1.4). \( \square \)

Let us explain that some relationship between negative continued fractions and ancestor triangles. For an \( \alpha \in (0, 1) \cap \mathbb{Q} \), we express its inverse as \( \alpha^{-1} = [c_1, \ldots, c_l]^{-} \). Then the vertices in the both oblique sides of \( \text{YAT}(\alpha) \) are as in Figure 5.

![Figure 5. Vertices of YAT(\alpha) and negative continued fractions](image)

For each \( j \) the number of the fundamental triangles in the region enclosed by \( E_{j-1}, E_j \), the both oblique sides is \( c_1 \) for \( j = 1 \), and is \( (c_j - 1) \) for the others \( j \). We note that if \( c_l = 2 \), then end points of \( E_{l-1} \) and \( E_l \) coincide since \([c_1, \ldots, c_l - 1]^{-} = [c_1, \ldots, c_{l-1} - 1]^{-}\).

**Example 2.7.** If \( \alpha = \frac{7}{11} \), then \( \alpha^{-1} = [4]^{-} = [2, 3, 2, 2]^{-} \) and \( \text{YAT}(\frac{7}{11}) \) is given by Figure 6.

By a direct application of Nagai and Terashima’s results \([17]\) one can obtain a recursive formula \([12, \text{Theorem 5.2}]\) to compute the writhe \( w_r(\alpha) \) from the regular continued fraction expansion of \( \alpha \). By using the conversion between regular and negative continued fraction expansions given in \([5, 6, 15]\), we have another recursive formula to compute \( w_r(\alpha) \). To explain the
The rational numbers are classified into three types such as $\frac{1}{1}$, $\frac{1}{0}$, $\frac{0}{1}$-types. A rational number $\frac{x}{a}$ is called $\frac{1}{1}$-type if $x \equiv 1, a \equiv 1 \pmod{2}$, and $\frac{1}{0}$-type and $\frac{0}{1}$-type are similarly defined.

Let $\alpha$ be a rational number in $(0, 1)$. All of types appear in the vertices in each fundamental triangle of YAT($\alpha$). A Seifert path of $\alpha$ is a downward path in YAT($\alpha$), which is started from $\frac{1}{0}$ to $\alpha$ satisfying the following condition: The end points of any edge in the path consist of $\frac{1}{1}$- and $\frac{1}{0}$-types, or consist of $\frac{1}{0}$- and $\frac{0}{1}$-types. If the denominator of $\alpha$ is odd, then a Seifert path is unique. We denote it by $\gamma_\alpha$. If the denominator of $\alpha$ is even, namely $\alpha$ is of type $\frac{1}{0}$, then there are exactly two Seifert paths. In this case we denote by $\gamma_\alpha$ the Seifert path whose vertices consist of $\frac{1}{0}$- and $\frac{0}{1}$-types, and denote by $\gamma'_\alpha$ the remaining Seifert path.

Following [17] let us explain how to define a sign $t_\alpha(\Delta)$ for each fundamental triangle $\Delta$ in YAT($\alpha$). We consider the successive sequence of the fundamental triangles of YAT($\alpha$) whose initial term is the fundamental triangle with the edge joining the vertices $\frac{1}{0}$ and $\frac{1}{1}$. If $\Delta$ is the initial triangle, then we set

$$t_\alpha(\Delta) := \begin{cases} 1 & \text{if } \alpha \text{ is of } \frac{1}{1}\text{-type}, \\ -1 & \text{otherwise}. \end{cases}$$

Assume that for the previous triangle $\Delta_-$ of $\Delta$, the sign $t_\alpha(\Delta_-)$ is defined. Then $t_\alpha(\Delta)$ is defined by

$$t_\alpha(\Delta) = \begin{cases} t_\alpha(\Delta_-) & \text{if } \gamma_\alpha \text{ does not pass through between } \Delta_- \text{ and } \Delta, \\ -t_\alpha(\Delta_-) & \text{otherwise}. \end{cases}$$

**Theorem 2.8** (Nagai and Terashima [17 Theorem 4.4]). For an $\alpha \in (0, 1) \cap \mathbb{Q}$ the writhe of $D(\alpha)$ is given by

$$-\text{wr}(\alpha) = \sum_{\text{the fundamental triangles } \Delta \text{ in } YAT(\alpha)} t_\alpha(\Delta). \tag{2.4}$$
If $\alpha \in (0,1) \cap \mathbb{Q}$ is $1_0$-type, then one can also define another sign $t'_\alpha(\Delta)$ for each fundamental triangle $\Delta$ in $\mathrm{YAT}(\alpha)$ by using $\gamma'_\alpha$ as follows. For the initial triangle $\Delta$ we set $t'_\alpha(\Delta) := -1$.
Assume that for the previous triangle $\Delta_-$ of $\Delta$, the sign $t'_\alpha(\Delta_-)$ is defined. Then $t'_\alpha(\Delta)$ is defined by

$$t'_\alpha(\Delta) = \begin{cases} t'_\alpha(\Delta_-) & \text{if } \gamma'_\alpha \text{ does not pass through between } \Delta_- \text{ and } \Delta, \\ -t'_\alpha(\Delta_-) & \text{otherwise.} \end{cases}$$

Then by the same way of proof of Theorem 2.8 it is shown that the equation

$$-\mathrm{wr}_{+}(\alpha) = \sum_{\text{the fundamental triangles } \Delta \text{ in } \mathrm{YAT}(\alpha)} t'_\alpha(\Delta)$$

holds, where $\mathrm{wr}_{+}(\alpha) = \mathrm{wr}(D_{+}(\alpha))$. By rewriting the formulas (2.4) and (2.5) we have the following proposition.

**Proposition 2.9.** Let $\alpha \in \mathbb{Q} \cap (0,1)$ with expression $\alpha^{-1} = [c_1, \ldots, c_l]^-$. Let $E_0$ be the edge between $0$ and $1$, and for each $j = 1, \ldots, l$ let $E_j, F_j$ be the edges between $\frac{1}{[c_1, \ldots, c_j]}$ and $\frac{1}{[c_1, \ldots, c_j^{-1}]}$, and between $\frac{1}{[c_1, \ldots, c_j^{-1}]^{-1}}$ and $\frac{1}{[c_1, \ldots, c_j^{-1}]}$, respectively. Denoted by $z_j$ is the number of times that the Seifert path $\gamma_\alpha$ crosses $\mathrm{YAT}(\alpha)$ while it starts from $\frac{1}{0}$ and comes first to one of the edge points of $E_j$, where if $\gamma_\alpha$ passes through $E_0$, then we count it.

Let $t_\alpha([E_{j-1}, E_j])$ be the recursively defined number as follows:

$$t_\alpha([E_0, E_1]) := \begin{cases} -c_1 & \text{if } z_1 = 0, \\ c_1 & \text{if } z_1 = 1, \\ c_1 - 2 & \text{if } z_1 = 2, \end{cases}$$

and for $j \geq 2$

$$t_\alpha([E_{j-1}, E_j]) := \begin{cases} (-1)^{z_j-1}(c_j - 1) & \text{if } \gamma_\alpha \text{ does not pass through } F_j, \\ (-1)^{z_j}(c_j - 3) & \text{otherwise.} \end{cases}$$

![Figure 7. vertices and edges of $\mathrm{YAT}(\alpha)$](image-url)
Then we have

\[ \text{wr}(\alpha) = \sum_{j=1}^{l} t_{\alpha}([E_{j-1}, E_{j}]). \]  

Furthermore, if \( \alpha \) is \( \frac{1}{0} \)-type, then \( \text{wr}_{++}(\alpha) \) can be computed as follows. Denoted by \( z'_{j} \) is the number of times that the Seifert path \( \gamma'_{\alpha} \) crosses \( \text{YAT}(\alpha) \) while it starts from \( \frac{1}{0} \) and comes first to one of the edge points of \( E \). Let \( t'_{\alpha}([E_{j-1}, E_{j}]) \) be the recursively defined number as follows:

\[ t'_{\alpha}([E_{j-1}, E_{j}]) := (-1)^{z'_{j}-1}(c_{j} - 1). \]

Then we have

\[ \text{wr}_{++}(\alpha) = \sum_{j=1}^{l} t'_{\alpha}([E_{j-1}, E_{j}]). \]

**Proof.** To prove (2.6), for each \( j = 1, \ldots, l \), let \( \gamma^{(j)}_{\alpha} \) be the subpath of \( \gamma_{\alpha} \) from the edge point of \( E_{j-1} \) first reached to the edge point of \( E_{j} \) first reached. We also denote by \([E_{j-1}, E_{j}]\) the quadrilateral enclosed by \( E_{j-1}, E_{j} \) and the both oblique sides. By Theorem 2.8 it is sufficient to show that \(-t_{\alpha}([E_{j-1}, E_{j}])\) coincides with the sum of signs of the fundamental triangles in \([E_{j-1}, E_{j}]\). For \( j > 1 \) it can be verified by dividing into the following four cases:

1. \( z_{j} = z_{j-1} \),
2. \( z_{j} = z_{j-1} + 1 \) and \( \gamma^{(j)}_{\alpha} \) passes through \( E_{j-1} \),
3. \( z_{j} = z_{j-1} + 1 \) and \( \gamma^{(j)}_{\alpha} \) passes through \( F_{j} \),
4. \( z_{j} = z_{j-1} + 2 \).

In the case of Part (1), \( \gamma^{(j)}_{\alpha} \) does not pass through \( E_{j-1} \) and \( F_{j} \). So, the sum of signs of the fundamental triangles in \([E_{j-1}, E_{j}]\) is \((-1)^{z_{j}}(c_{j} - 1)\), and therefore multiplying it by \(-1\) we have \((-1)^{z_{j}-1}(c_{j} - 1) = (-1)^{z_{j}-1}(c_{j} - 1) = t_{\alpha}([E_{j-1}, E_{j}])\). In all other cases, by the same argument, we see that the desired equation holds.

For \( j = 1 \) the desired equation can be also verified by dividing into the three cases where \( z_{j} = z_{j-1} \), \( z_{j} = z_{j-1} + 1 \) and \( z_{j} = z_{j-1} + 2 \).

The equation (2.7) can be obtained by a quite similar argument. \( \square \)

**Remark 2.10.** By \([12, \text{Lemma 4.1}(3)]\), we also have

\[ \text{wr}_{+-}(\alpha) = -\text{wr}(1 - \alpha). \]

By Proposition 2.8 we have:

**Theorem 2.11.** Let \( \alpha \in \mathbb{Q} \cap (0,1) \), and express its inverse as \( \alpha^{-1} = [c_{1}, \ldots, c_{l}] \). Define \( E_{j}, F_{j} \) and \( z_{j} \) as in Proposition 2.9 and set \( \alpha_{j} := \frac{1}{|c_{1}, \ldots, c_{j}|} \), \( \beta_{j} := \frac{1}{|c_{1}, \ldots, c_{j-1}|} \), \( z_{j}(\alpha_{j}) := z_{j} \) for \( j = 1, \ldots, l \). Then the writhe \( \text{wr}(\alpha) \) can be recursively computed as follows:
(1) If \( l = 1 \), that is, \( \alpha = \frac{1}{c_1} \), then \( \text{wr}(\alpha) = (-1)^{c_1}c_1 \). In addition, if \( c_1 \) is even, then 
\( \text{wr}_+(\alpha) = -c_1 \).

(2) If \( l = 2 \), that is, \( \frac{1}{\alpha} = [c_1, c_2]^- \), then \( \text{wr}(\alpha) = (-1)^{(c_1-1)c_2}c_1 + (-1)^{c_1}c_2 - 1 \). In addition, 
if both \( c_1, c_2 \) are odd, then \( \text{wr}_+(\alpha) = -c_1 + c_2 - 1 \).

(3) Let \( l \geq 3 \) and set
\[
\varepsilon(c_l) := \begin{cases} 
+ & \text{if } c_l \text{ is even,} \\
- & \text{if } c_l \text{ is odd,}
\end{cases}
\]
and \( \text{wr}_+(\alpha) := \text{wr}(\alpha) \). Then

(i) if \( \alpha_{l-1} \) is \( \frac{1}{0} \)-type and \( \alpha \) is \( \frac{1}{1} \)-type, then
\[
\text{wr}(\alpha) = \text{wr}_+(\alpha_{l-1}) + (-1)^{z_{l-1}(\alpha_{l-1})} + \frac{1+(-1)^{\gamma_l}}{2} + 1(c_l - 1),
\]

(ii) if \( \alpha_{l-1} \) is \( \frac{1}{1} \)-type and \( \alpha \) is \( \frac{1}{0} \)-type, then
\[
\text{wr}(\alpha) = \text{wr}_+(\alpha_{l-1}) + (-1)^{z_{l-2}(\alpha_{l-2})} + \frac{1+(-1)^{\gamma_l}}{2} + 1(-c_l + c_l + 2),
\]

(iii) if \( \alpha_{l-1} \) is \( \frac{1}{0} \)-type and \( \alpha \) is \( \frac{0}{0} \)-type, then
\[
\text{wr}(\alpha) = \text{wr}(\alpha_{l-1}) + (-1)^{z_{l-1}(\alpha_{l-1})} + \frac{1+(-1)^{\gamma_l}}{2} (c_l - 1),
\]

(iv) \( \alpha_{l-1} \) is \( \frac{1}{0} \)-type and \( \alpha \) is \( \frac{0}{0} \)-type, then
\[
\text{wr}(\alpha) = \text{wr}(\alpha_{l-2}) + (-1)^{z_{l-2}(\alpha_{l-2})} + \frac{1+(-1)^{\gamma_l}}{2} (-c_l + c_l + 2),
\]

(v) if \( \alpha_{l-1} \) is \( \frac{0}{1} \)-type and \( \alpha \) is \( \frac{0}{0} \)-type, then
\[
\text{wr}(\alpha) = \text{wr}(\alpha_{l-1}) + (-1)^{z_{l-2}(\alpha_{l-2})} + \frac{1+(-1)^{\gamma_l}}{2} (c_l - 1),
\]
\[
\text{wr}_+(\alpha) = \text{wr}_+(\alpha_{l-2})(\alpha_{l-2}) + (-1)^{z_{l-2}(\alpha_{l-2})} + \frac{1+(-1)^{\gamma_l}}{2} + 1\frac{(-1)^{\gamma_l}}{2} (c_l - 1 + c_l - 2),
\]

(vi) if \( \alpha_{l-1} \) is \( \frac{1}{0} \)-type and \( \alpha \) is \( \frac{0}{1} \)-type, then
\[
\text{wr}(\alpha) = \text{wr}(\alpha_{l-2}) + (-1)^{z_{l-2}(\alpha_{l-2})} + \frac{1+(-1)^{\gamma_l}}{2} (-c_l + c_l + 2),
\]
\[
\text{wr}_+(\alpha) = \text{wr}(\alpha_{l-1}) + (-1)^{z_{l-2}(\alpha_{l-2})} + \frac{1+(-1)^{\gamma_l}}{2} + 1\frac{(-1)^{\gamma_l}}{2} + 1(c_l - 1).
\]

**Proof.** It can be easily verified in the case where \( n = 1, 2 \). So, let \( n \geq 3 \). Here, we only give the proof of (ii) since other cases are shown by a quite similar argument. In the case of (ii), the types of \( \beta_{l-1} \) and \( \alpha_{l-2} \) are \( \frac{1}{0} \)-type or \( \frac{1}{1} \)-type, and they are different. Thus, \( \gamma_{\alpha} \) is expressed as
\[
\gamma_{\alpha} = \begin{cases} 
\gamma_{\alpha_{l-2}} * (\alpha_{l-2} \rightarrow \beta_{l-1} \rightarrow \cdots \rightarrow \alpha) & \text{if } \beta_{l-1} \text{ is } \frac{1}{0} \text{-type,} \\
\gamma'_{\alpha_{l-2}} * (\alpha_{l-2} \rightarrow \beta_{l-1} \rightarrow \cdots \rightarrow \alpha) & \text{if } \beta_{l-1} \text{ is } \frac{1}{1} \text{-type.}
\end{cases}
\]
We see that

\[
\begin{align*}
z_l(\alpha) = z_{l-1}(\alpha) &= \begin{cases} 
z_{l-2}(\alpha) + 1 & \text{if } \beta_{l-1} \text{ is } \frac{1}{0}\text{-type and } \beta_{l-2} \text{ is } \frac{1}{0}\text{-type,} \\
z_{l-2}(\alpha) + 2 & \text{if } \beta_{l-1} \text{ is } \frac{1}{0}\text{-type and } \beta_{l-2} \text{ is } \frac{1}{0}\text{-type,} \\
z_{l-2}(\alpha) + 1 & \text{if } \beta_{l-1} \text{ is } \frac{1}{1}\text{-type and } \beta_{l-2} \text{ is } \frac{1}{0}\text{-type,} \\
z_{l-2}(\alpha) + 2 & \text{if } \beta_{l-1} \text{ is } \frac{1}{1}\text{-type and } \beta_{l-2} \text{ is } \frac{1}{1}\text{-type,}
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
z_{l-2}(\alpha) &= \begin{cases} 
z_{l-2}(\alpha_{l-2}) & \text{if } \beta_{l-1} \text{ is } \frac{1}{1}\text{-type,} \\
z_{l-2}(\alpha_{l-2}) - 1 & \text{if } \beta_{l-1} \text{ is } \frac{1}{1}\text{-type.}
\end{cases}
\end{align*}
\]

The number of fundamental triangles in the triangle whose vertices are \(\alpha_{l-1}, \beta_{l-1}, \alpha\) is \(c_l - 1\). So, \(\beta_{l-1}\) is \(\frac{1}{1}\)-type if and only if \(c_l\) is odd. Therefore, \(z_{l-2}(\alpha)\) can be written as

\[
z_{l-2}(\alpha) = z_{l-2}(\alpha_{l-2}) - \frac{1 - (-1)^{c_l}}{2}.
\]

Similarly, considering the number of fundamental triangles in the region enclosed by \(E_{l-2}, E_{l-1}\) and the both oblique sides, we see that \(z_{l-1}(\alpha)\) can be written as

\[
z_{l-1}(\alpha) = z_{l-2}(\alpha) + 1 + \frac{1 + (-1)^{c_l-1}}{2}.
\]

Substituting the above two equations into

\[
\text{wr}(\alpha) = \text{wr} \times (c_l)(\alpha_{l-2}) + t([E_{l-2}, E_{l-1}]) + t([E_{l-1}, E_l])
\]

\[
= \text{wr} \times (c_l)(\alpha_{l-2}) + (-1)^{z_{l-1}(\alpha)}(c_{l-1} - 3) + (-1)^{z(\alpha)-1}(c_l - 1),
\]

we have the equation in (ii).

\[\square\]

Remark 2.12. The type of the rational number \(\alpha = [c_1, \ldots, c_l]^{-}\) with \(c_j \geq 2\) for all \(j\) can be inductively determined as follows: Let \(n(\alpha)\) and \(d(\alpha)\) be the parities of the numerator and the denominator of \(\alpha\), respectively. We set \(\alpha_j = [c_1, \ldots, c_j]^{-}\) for \(j = 1, \ldots, l\). Then we have

\begin{enumerate}
\item \(n(\alpha_1) = \frac{1 - (-1)^{c_1}}{2}, \quad d(\alpha_1) = 1,\)
\item \(n(\alpha_2) = \frac{1 + (-1)^{c_1}c_2}{2}, \quad d(\alpha_2) = \frac{1 - (-1)^{c_2}}{2},\)
\item \(\text{for } j \geq 3 \quad n(\alpha_j) = n(\alpha_{j-1}) \frac{1 - (-1)^{c_j}}{2} + n(\alpha_{j-2}),\)
\item \(d(\alpha_j) = d(\alpha_{j-1}) \frac{1 - (-1)^{c_j}}{2} + d(\alpha_{j-2}).\)
\end{enumerate}

Example 2.13. (1) Let us consider the case \(\alpha = \frac{13}{21}\). Since \(\frac{21}{13} = [2, 3, 3, 2]^{-}\) and \(\alpha_3 = \frac{1}{[2,3,3]}^{-} = \frac{8}{13}, \quad \alpha_2 = \frac{1}{[2,3]}^{-} = \frac{2}{5}, \quad \alpha_1 = \frac{1}{[2]}^{-} = \frac{1}{2},\) we will apply the formula of Theorem 2.11(ii). Then we have \(\text{wr}(\frac{13}{21}) = \text{wr}(\frac{2}{5}) + (-1)^{z_2(\alpha_2)}\). Here, \(z_2(\alpha_2) = 1, \quad \text{wr}(\frac{2}{5}) = 0\). Thus \(\text{wr}(\frac{13}{21}) = -1\). Since \(l = 4, \quad l' = 3\) and

\[
J_{\frac{13}{21}}(q) = 1 + 3q + 3q^2 + 4q^3 + 4q^4 + 3q^5 + 2q^2 + q^7,
\]
by Theorem 1.2 we have
\[ V_{\frac{13}{16}}(t) = -t^3 + 3t^2 - 3t + 4 - 4t^{-1} + 3t^{-2} - 2t^{-3} + t^{-4}. \]

(2) Let us consider the case \( \alpha = \frac{9}{16} \). Since \( \frac{16}{9} = [2, 5, 2] \) and \( \alpha_2 = \frac{1}{2} \), we will apply the formula of Theorem 2.11(3)(vi). Then we have
\[ \text{wr}(\frac{9}{16}) = \text{wr}(\frac{1}{2}) - (-1)^{z_1(\alpha_1)} = 3. \]
Since \( l = 3 \), \( l' = 4 \) and
\[ J_{\frac{9}{16}}(q) = 1 + 2q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + q^6 + q^7, \]
we have
\[ V_{\frac{9}{16}}(t) = (-A^{-3})^{-2}(t^4 - 2t^3 + 2t^2 - 3t + 3 - t^{-1} + t^{-2} - t^{-3}) \]
\[ = t^5 - 2t^4 + 2t^3 - 3t^2 + 3t^{-1} - 3t^{-2} - 3t^{-3} - t^{-4} - t^{-5}. \]

3. q-deformed integers derived from pairs of coprime integers

In this section we introduce q-deformed integers derived from pairs of coprime integers. This concept is motivated by results on q-deformed rational numbers due to Morier-Genoud and Ovsienko.

For a positive integer \( a \) we set
\[ [a]_q := 1 + q + q^2 + \cdots + q^{a-1} \in \mathbb{Z}[q], \]
and \([0]_q := 0\). In the quotient field \( Q(\mathbb{Z}[q]) \) the equation \([a]_q = \frac{1-q^a}{1-q}\) holds.

Morier-Genoud and Ovsienko [15] introduced q-deformations of rational numbers by using continued fractions. Let \( c_1, \ldots, c_l \) be finite sequence of integers which are greater than or equal to 2. Then \([c_1, \ldots, c_l]_q \in Q(\mathbb{Z}[q])\) is defined by
\[ [c_1]_q - \frac{q^{c_1-1}}{[c_2]_q - \frac{q^{c_2-1}}{\cdots - \frac{q^{c_l-2-1}}{[c_l]_q - \frac{q^{c_{l-1}-1}}{[c_{l-1}]_q - \frac{q^{c_{l-2}-1}}{\cdots - \frac{q^{c_{l-2}}}{[c_{l-2}]_q - \frac{q^{c_{l-3}}}{\cdots - [c_2]_q - \frac{q^{c_1-1}}{[c_1]_q}}}}}}}}}. \]

(3.2)

For an \( \alpha \in (1, \infty) \cap \mathbb{Q} \), by expanding as \( \alpha = [c_1, \ldots, c_l]^{-} \), we set
\[ [\alpha]_q := [c_1, \ldots, c_l]^{-}_q, \]
and call it the q-rational number of \( \alpha \). By reducing (3.2) without multiplying elements of \( \mathbb{Z}[q] \), \([\alpha]_q \) is uniquely represented by the form
\[ [\alpha]_q = \frac{N_q(\alpha)}{D_q(\alpha)} \]
for some \( N_q(\alpha), D_q(\alpha) \in \mathbb{Z}[q] \), where
(i) $N_q(\alpha), D_q(\alpha)$ are coprime in $\mathbb{Z}[q]$, and
(ii) if $\alpha = \frac{r}{s}$ with $(r, s) = 1$, $r, s \geq 1$, then $N_1(\alpha) = r$, $D_1(\alpha) = s$.

For convenience we set $N_q(\frac{1}{1}) = 1$, $D_q(\frac{1}{1}) = 0$.

**Theorem 3.1 (Morier-Genoud and Ovsienko [15]).** Let $\alpha, \beta \geq 1$ be Farey neighbors, and express the Farey sum as $\alpha \sharp \beta = [c_1, \ldots, c_l]^-$. Then

$$N_q(\alpha \sharp \beta) = N_q(\alpha) + q^{c_l-1}N_q(\beta),$$

$$D_q(\alpha \sharp \beta) = D_q(\alpha) + q^{c_l-1}D_q(\beta).$$

So, we define $[\alpha]_q \oplus [\beta]_q$ by

$$[\alpha]_q \oplus [\beta]_q := \frac{N_q(\alpha) + q^{c_l-1}N_q(\beta)}{D_q(\alpha) + q^{c_l-1}D_q(\beta)}.$$  \hspace{1cm} (3.5)

Then

$$[\alpha]_q \oplus [\beta]_q = [\alpha \sharp \beta]_q.$$  \hspace{1cm} (3.6)

**Remark 3.2.** The integer $c_l$ in the above theorem is given by

$$c_l - 1 = \left\lceil \frac{N(\alpha)}{N(\beta)} \right\rceil,$$  \hspace{1cm} (3.7)

where $N(\alpha), N(\beta)$ are the numerators of $\alpha, \beta$, respectively. In fact, if we write $\alpha = \frac{p}{q}$, $\beta = \frac{r}{s}$, then $r(\alpha \sharp \beta) = \frac{p + rs}{q} = \frac{p}{q} + 1$, where $r$ is the operator defined in [12, Lemma 1.3]. On the other hand, it also given by $r(\alpha \sharp \beta) = [c_1, \ldots, c_l]^- = c_l - \frac{1}{[c_{l-1}, \ldots, c_1]}$. Thus, $\frac{p}{q} + 1 = c_l - \frac{1}{[c_{l-1}, \ldots, c_1]}$. By comparing the integer parts of both sides of this equation, (3.7) is obtained.

Let us introduce a convenient method to compute $[\alpha \sharp \beta]_q$ by Euclidean algorithm. For a pair $(a, b)$ of positive and coprime integers we define $(a, b)_q$ by

$$(a, b)_q := \begin{cases} (a - r, r)_q + q(a, b - a)_q & \text{if } a < b, \\ (a - b, b)_q + q^{\frac{r}{s} - 1}(r, b - r)_q & \text{if } a > b, \end{cases}$$  \hspace{1cm} (3.8)

where $r$ is the remainder when $b$ is divided by $a$ in case where $a < b$, and when $a$ is divided by $b$ in case where $a > b$, and also $(1, n)_q = (n, 1)_q = [1 + n]_q$ for any non-negative integer $n$.

**Theorem 3.3.** If $\alpha = \frac{a}{b}$, $\beta = \frac{b}{n} \geq 1$ are Farey neighbors, then

$$D_q(\alpha \sharp \beta) = (a, b)_q, \quad N_q(\alpha \sharp \beta) = (x, y)_q.$$  \hspace{1cm} (3.9)

Thus

$$[\alpha \sharp \beta]_q = \frac{(x, y)_q}{(a, b)_q}. \hspace{1cm} (3.10)$$
By the formula
\[ x = \frac{a}{1}, \quad \beta = \frac{a+1}{1}. \]
Since \( \alpha, \beta = [a+1, 2] \), we have
\[ \alpha, \beta = \frac{a+1}{2}q = \frac{(a+1)+q}{2}q = \frac{(a,a+1)}{q}. \]

**Proof.** The theorem can be verified by induction on the number of the operation \( \frac{2}{3} \).

I. If \( a + b = 1 \), then \( \alpha = \frac{a}{1}, \quad \beta = \frac{1}{1} \), and \( (3.10) \) can be easily verified. If \( a + b = 2 \), then \( \alpha = \frac{a}{1}, \quad \beta = \frac{a+1}{1} \). Since \( \alpha, \beta = [a+1, 2] \), we have
\[ \alpha, \beta = \frac{a+1}{2}q = \frac{(a+1)+q}{2}q = \frac{(a,a+1)}{q}. \]

II. Let \( \alpha, \beta \geq 1 \) be Farey neighbors. Under the assumption \( a + b \geq 3 \), it is enough to show that if \( \alpha, \beta \) satisfy \( (3.9) \), then \( \alpha, \beta \) does. Under this assumption we note that \( a < b \) if and only if \( x < y \).

Let us consider the case where \( a < b \), and therefore \( x < y \). Let \( r \) be the remainder when \( b \) is divided by \( a \), and \( s \) be the remainder when \( y \) is divided by \( x \). Then, the pair \( \frac{a}{1}, \frac{b-a}{s} \) and the pair \( \frac{x}{a}, \frac{x}{b-a} \) are Farey neighbors, respectively, and \( \frac{b-a}{s} = \frac{x}{a}, \frac{a}{b-a} = \frac{x}{b} \). By induction hypothesis,
\[ D_q(\alpha) = (a-r, r), \quad N_q(\alpha) = (x-s, s), \quad D_q(\beta) = (a-b-a), \quad N_q(\beta) = (x, y-x). \]

Since \( \left\lfloor \frac{x}{y} \right\rfloor = 1 \), by Theorem 3.1 and (3.8) we have
\[ D_q(\alpha, \beta) = (a-r, r), \quad q(a, a-x, s) = (a, b-a), \quad N_q(\alpha, \beta) = (x, y-x). \]

Next, let us consider the case where \( a > b \), and therefore \( x > y \). Let \( r \) be the remainder when \( a \) is divided by \( b \), and \( s \) be the remainder when \( x \) is divided by \( y \). Then, the pair \( \frac{a}{b}, \frac{r-s}{s} \) and the pair \( \frac{x}{a}, \frac{y-s}{s} \) are Farey neighbors, respectively, and \( \frac{r-s}{s} = \frac{x}{b}, \frac{x-a}{b} = \frac{x-s}{y} \). By the same argument above, we see that \( D_q(\alpha, \beta) = (a, b), \quad N_q(\alpha, \beta) = (x, y) \). Moreover, since \( x \) can be written as \( x = my + s \) for some \( 0 < s < y \) and \( 1 \leq a - mb \leq b \), it follows that \( \left\lfloor \frac{x}{y} \right\rfloor = m+1 = \left\lfloor \frac{a}{b} \right\rfloor \).

This implies that \( (3.9) \) holds for \( \alpha, \beta \).

We use the following convention: For a positive integer \( n \)
\[ (n)_q = [n]_q = 1 + q + \cdots + q^{n-1}, \quad (0)_q = 0. \]

**Example 3.4.** By the formula \( (3.10) \), \( \left[ \frac{17}{5} \right]_q = \frac{(107)_q}{(3,2)_q} = \frac{(3,7)_q}{(3,2)_q} \). Since \( (3,4)_q = (3) + q(3,4)_q \), \( (3,7)_q = (3) + q(3,4)_q \), we have
\[ \left[ \frac{17}{5} \right]_q = \frac{(1+q+q^2)(3)_q + (1+q)q^2(4)_q}{(3)_q + q^2(2)_q}. \]

In the sequel, we will investigate properties of \( (a, b)_q \).

**Lemma 3.5.** Let \( (a, x) \) be a pair of coprime integers with \( 1 \leq a \leq x \), and write in the form \( x = ca + r \) (\( 0 \leq r < a \)). Then
\[ (a, x)_q = (a-r, r)_q = q^r(a,r)_q, \]
\[ \deg(x-a, a)_q = \deg(a, x)_q - 1, \]
\[ \deg(a-r, r)_q = \deg(a, x)_q - \left\lfloor \frac{x}{a} \right\rfloor, \]
\[ \deg(a, x)_q = \deg(x, a)_q, \]
\[ (x, a)_q = q^\deg(x, a)_q(a, x)_q - 1. \]
\textbf{Proof.} The equation (3.12) can be directly derived from the definition of \((a, x)_q\).

We will show the equations (3.13) and (3.14). To do this, let \(k \geq 2\) be an integer, and by induction on \(k\) we show that

\[(*) \quad \deg(x, a)_q = \deg(a, x)_q = \deg(x - a, a)_q + 1\]

for all pairs \((a, x)\) of coprime integers satisfying \(1 \leq a \leq x\) and \(a + x = k\).

When \(k = 2\), we have \(a = x = 1\), and when \(k = 3\), we have \(a = 1, x = 2\). In the both cases thus the equation (\(*\)) holds.

Assume that \(k > 2\) and the equation (\(*\)) holds for all pairs \((a', x')\) of coprime integers satisfying \(1 \leq a' \leq x', a' + x' < k\). Let \((a, x)\) be a pair of coprime integers satisfying \(1 \leq a \leq x\), \(a + x = k\).

We express \(x\) as the form \(x = ca + r (0 \leq r < a)\). Since \(\deg(a, r)_q = \deg(r, a)_q = \deg(a - r, r)_q + 1\) by induction hypothesis, we have \(\deg((a - r, r)_q(c)_q) = c + d - 2\), \(\deg((a - a, a)_q) = c + d\) by setting \(d := \deg(a, r)_q\). Thus \(\deg(a, x)_q = c + d = \deg(a - r, r)_q + 1 + c\), and in particular, \(\deg(a, x)_q = \deg(a, x - a)_q + 1\) by combining it with \(\deg(a, x)_q > \deg(a - r, r)_q\) and \((a, x)_q = (a - r, r)_q + q(a, x - a)_q\). Since \((x, a)_q = (x - a, a)_q + q^{\lceil \frac{x}{a} \rceil}(r, a - r)_q\), we have

\[
\begin{align*}
\deg(x - a, a)_q &= \deg(a, x - a)_q = \deg(a, x)_q - 1 \\
&= \deg(a - r, r)_q + c = \deg(r, a - r)_q + c.
\end{align*}
\]

If \(a > 1\), then \(\lceil \frac{x}{a} \rceil = c + 1\). It follows that

\[
\deg(x - a, a)_q < \deg(r, a - r)_q + c + 1 = \deg(r, a - r)_q + \left\lceil \frac{x}{a} \right\rceil.
\]

Therefore, we obtain \(\deg(x, a)_q = \lceil \frac{x}{a} \rceil + \deg(r, a - r)_q = c + d = \deg(a, x)_q\).

If \(a = 1\), then \((x, a)_q = (x + 1)_q = (a, x)_q\), and hence \(\deg(x, a)_q = \deg(a, x)_q\).

This completes the induction argument, and therefore the equation (\(*\)) holds for all pairs \((a, x)\) of coprime integers with \(1 \leq a \leq x\).

The equation (3.14) can be obtained from \((x, a)_q = (x - a, a)_q + q^{\lceil \frac{x}{a} \rceil}(r, a - r)_q\) and \(\deg(a, x)_q = \deg(x, a)_q > \deg(x - a, a)_q\).

Finally, we will derive the equation (3.16). To do this, let \(k \geq 2\) be an integer, and we show that by induction on \(k\) the two equations (3.16) and

\[\begin{align*}
(3.17) \quad (a, x)_q &= q^{\deg(a, x)_q} (x, a)_q^{-1}
\end{align*}\]

hold for all pairs \((a, x)\) of coprime integers satisfying \(1 \leq a \leq x\), \(x + a \leq k\) at the same time.

I. When \(a = x = 1\), it can be immediately verified that the desired equations hold.

II. Let \(k \geq 2\) be an integer, and assume that (3.16) and (3.17) hold for all coprime pairs \((a, x)\) of integers satisfying \(1 \leq a \leq x\), \(x + a \leq k\). Let \((a, x)\) be a pair of coprime integers satisfying \(1 \leq a \leq x\), \(a + x = k + 1\), and \(r\) be the remainder of \(x\) divided by \(a\). Since \((a - r) + r = a < a + x\), \(a + (x - a) = x < a + x\), by induction hypothesis we have \((a, x)_q = (a - r, r)_q + q(a, x - a)_q = q^{\deg(a - r, r)_q}(r, a - r)_q^{-1} + q^{1 + \deg(x - a, a)_q}(x - a, a)_q^{-1}\). By (3.13) and (3.14)
the right-hand side is rewritten as \( q^{\deg(a,x)}(q^{-\lceil \frac{x}{a} \rceil}(r,a-r)q^{-1} + (x-a,a)q^{-1}) = q^{\deg(a,x)}(x,a)q^{-1} \). Replacing \( q \) in the equation with \( q^{-1} \) we have \((a,x) q^{-1} = q^{-\deg(a,x)}(x,a)q\). This is equivalent to \((3.16)\). This completes the induction argument.

\[ \blacksquare \]

**Lemma 3.6.** Let \((a, x), (b, y)\) be pairs of coprime integers such that \( 1 \leq a \leq x, 1 \leq b \leq y \) and \( \frac{a}{x}, \frac{y}{b} \) are Farey neighbors. Then

\begin{align*}
(3.18) & \quad \deg(a + b, x + y) = \deg(b, y) + \left\lfloor \frac{x}{y} \right\rfloor, \\
(3.19) & \quad \deg(a + b, x + y) = \deg(a, x) + \left\lfloor \frac{b}{a} + 1 \right\rfloor.
\end{align*}

**Proof.** This lemma can be proved by induction on \( k \geq 3 \) when we set \( x + y = k \).

I. Let \( k = 3 \). Then \((x, y) = (1, 2)\) and \((a, b) = (1, 1)\). Thus \((3.18), (3.19)\) become \( \deg(2, 3)_q = \deg(1, 2)_q + 1 \), \( \deg(2, 3)_q = \deg(1, 1)_q + 2 \), respectively. It can be easily see that these equations hold.

II. Let \( k \geq 3 \). Assume that \((3.18), (3.19)\) hold for all positive integers \( a', b', x', y' \) such that \( x' + y' < k \), \( a'y' - b'x' = 1 \), \( 1 \leq a' \leq x' \), \( 1 \leq b' \leq y' \). Let us consider positive integers \( a, b, x, y \) which satisfy \( x + y = k \) and \( ay - bx = 1 \), \( 1 \leq a \leq x \), \( 1 \leq b \leq y \), and let \( r \) be the remainder of \( x + y \) divided by \( a + b \). Since \( a + b < x + y \), we have

\[ (a + b, x + y)_q = (a + b - r, r)_q + q(a + b, (x + y) - (a + b))_q. \]

By \((3.14)\) we have

\[ \text{(*) } \deg(a + b, x + y)_q = 1 + \deg(a + b, (x + y) - (a + b))_q. \]

(1) In the case where \( a < x - a \) and \( b < y - b \), by induction hypothesis

\[ \deg(a + b, (x + y) - (a + b))_q = \deg(a + b, (x - a) + (y - b))_q \]

\[ = \deg(b, y - b)_q + \left\lfloor \frac{x - a}{y - b} \right\rfloor \]

\[ = \deg(a, x - a)_q + \left\lfloor \frac{b}{a} + 1 \right\rfloor. \]

Applying \((3.13)\) and Lemma 2.2 we have \((3.18), (3.19)\).

(2) The case where \( a < x - a \) and \( b \geq y - b \) does not occur since \( 2b - y = 2b - \frac{xb + 1}{a} = \frac{2a - x - b - 1}{a} \leq 0 \), which is a contradiction. Similarly, the case where \( a \geq x - a \) and \( b < y - b \) does not occur.

(3) In the case where \( a \geq x - a \) and \( b \geq y - b \), we see that \( a > x - a \) and \( b > y - b \).
Let \((3.19)\) by easy computation. □

Using \((3.13)\) and \((3.20)\) we have \(\deg(a + b, (x − a) + (y − b))_q = \deg((y − b) + (x − a), b + a)_q\)
\[
= \deg(x − a, a)_q + \left\lfloor \frac{b}{a} \right\rfloor
= \deg(y − b, b)_q + \left\lfloor \frac{x − a}{y − b} + 1 \right\rfloor.
\]

Using \((3.13)\) and \((3.20)\) we have \(\deg(a + b, x + y)_q = \deg(a, x)_q + \left\lfloor \frac{b}{a} \right\rfloor\).

We see that \(a > 1\) and \(\frac{b}{a} \notin \mathbb{Z}\). So, \([\frac{b}{a}] = [\frac{b}{a} + 1]\), and therefore, \(\deg(a + b, x + y)_q = \deg(a, x)_q + \left\lfloor \frac{b}{a} + 1 \right\rfloor\). Similarly, by \((3.13)\), Lemma \((2.2)\) and \((3.20)\), we have
\[
\deg(a + b, x + y)_q = 1 + \deg(y − b, b)_q + \left\lfloor \frac{x − a}{y − b} + 1 \right\rfloor
= \deg(b, y)_q + \left\lfloor \frac{x − a}{y − b} + 1 \right\rfloor.
\]

- If \(y − b > 1\), then \(x − a, y − b\) are coprime, and hence \(\frac{x − a}{y − b} \notin \mathbb{Z}\). It follows that \([\frac{x − a}{y − b} + 1] = [\frac{x − a}{y}]\) by Lemma \((2.2)\) and hence \(\deg(a + b, x + y)_q = \deg(b, y)_q + \left\lfloor \frac{x}{y} \right\rfloor\).
- If \(y − b = 1\), then \([\frac{x − a}{y − b} + 1] = [\frac{x − a}{y}] + 1 = [\frac{x}{y}]\) by Lemma \((2.2)\) and hence \(\deg(a + b, x + y)_q = \deg(b, y)_q + \left\lfloor \frac{x}{y} \right\rfloor\).

(ii) If \(y − b = 0\), then \(y = b = 1\). However, this is a contradiction since the Farey neighbors \(\frac{x}{a}, \frac{y}{b} = 1\) should be greater than or equal to 1.

(iii) If \(x − a = 0\), then \(x = a = 1\) and \(y = b = 1\). In this case we can directly prove \((3.18)\), \((3.19)\) by direct computation.

\[\frac{b}{a} \notin \mathbb{Z}\] 

Theorem 3.7. Let \((a, x), (b, y)\) be pairs of coprime integers such that \(1 ≤ a ≤ x, 1 ≤ b ≤ y\) and \(\frac{x}{a}, \frac{y}{b}\) are Farey neighbors. Then
\begin{align*}
(a + b, x + y)_q &= (a, x)_q + q\left\lfloor \frac{x}{y} \right\rfloor (b, y)_q, \\
(y + x, b + a)_q &= (y, b)_q + q\left\lfloor \frac{y}{x} \right\rfloor (x, a)_q.
\end{align*}

Proof. First, we will show that the equation \((3.20)\) holds. If \(a + b = 2\), that is, \(a = b = 1\), then \(y = x + 1\). In this case, \((3.20)\) is written as \((2, 2x + 1)_q = (1, x)_q + q(1, x + 1)_q\). This equation can be obtained by direct computation.

Next we consider the case \(a + b ≥ 3\). We note that \(a ≤ x, b ≤ y\) by assumption of the theorem and \(a + b ≥ 3\).

1. Let us consider the case \(x < y\). We prove the equation
\[
\frac{(a + b, x + y)_q}{(a, x)_q} = 1 + q\left\lfloor \frac{x}{y} \right\rfloor \frac{(b, y)_q}{(a, x)_q}.
\]
which is equivalent to (3.20). Since \( \lceil \frac{x}{y} \rceil = 1 \) and \( b < y \), we have

\[
(\text{R.H.S. of (3.22)}) = 1 + q \left\lfloor \frac{b}{a} \right\rfloor = qD_q \left( \frac{b}{a} \right) + qD_q \left( \frac{b}{x} \right).
\]

On the other hand, since \( \lceil \frac{a+b}{x+y} \rceil = 1 \) from \( a \leq x, b < y \), it follows that

\[
(\text{L.H.S. of (3.22)}) = qD_q \left( \frac{a}{b} \right) + qN_q \left( \frac{a}{b} \right) + qN_q \left( \frac{x}{y} \right).
\]

(2) Let us consider the case \( x > y \). The equation (3.20) is rewritten as

\[
(3.23) \quad \left( \frac{a+b}{x+y}, \frac{x}{y} \right)_q = \left( \frac{a}{x}, \frac{b}{y} \right)_q + q \left\lfloor \frac{x}{y} \right\rfloor.
\]

To prove the equation, applying (3.16) and the result of (1), we have

\[
\frac{(a+b, x+y)_q}{(b, y)_q} = \frac{(a, x)_q}{(b, y)_q} + q \left\lfloor \frac{x}{y} \right\rfloor,
\]

and hence

\[
(\text{L.H.S. of (3.23)}) = \frac{(a+b, x+y)_q}{(b, y)_q} = \frac{(a, x)_q}{(b, y)_q} + q \left\lfloor \frac{x}{y} \right\rfloor.
\]

By Lemma 3.6 and (3.16) the above value coincides with the right-hand side of (3.23).

The equation (3.21) can be obtained from (3.20) by using Lemma 3.5 as follows:

\[
(y + x, b + a)_q = q^{\deg(x+y,a+b)_q - \deg(y,b)_q} \left( (a, x)_q + q \left\lfloor \frac{b}{y} \right\rfloor (b, y)_q \right).
\]

Corollary 3.8. Let \( (a, x), (b, y) \) be pairs of coprime integers such that \( \frac{x}{y}, \frac{b}{y} \) are Farey neighbors. We define \( (a, x)_q \oplus (b, y)_q \) by

\[
(3.24) \quad (a, x)_q \oplus (b, y)_q := \begin{cases} (a, x)_q + q \left\lfloor \frac{x}{y} \right\rfloor (b, y)_q & \text{if } 1 \leq a \leq x, \\ (a, x)_q + q \left\lfloor \frac{b}{y} + 1 \right\rfloor (b, y)_q & \text{if } 1 \leq y \leq b. \end{cases}
\]

Then \( (a, x)_q \oplus (b, y)_q = (a + b, x + y)_q \).

Proof. The result is a direct consequence of Theorem 3.7. □
4. The normalized Jones polynomials based on $q$-Farey sums and its applications

By Proposition A.1 and $[\alpha + 1]_q = \frac{D_{\alpha}(\alpha) + qN_{\alpha}(\alpha)}{D_{\alpha}(\alpha)}$ we see that the normalized Jones polynomial $J_\alpha(q)$ can be computed by the following formula:

**Lemma 4.1.** For a rational number $\alpha > 1$,

\[
J_\alpha(q) = qN_\alpha(\alpha) + (1 - q)D_\alpha(\alpha).
\]

Furthermore, we have:

**Lemma 4.2.** The normalized Jones polynomials $J_\alpha(q)$ can be inductively computed from the initial data $J_1(q) = 1$, $J_\infty(q) = q$ and the formula

\[
J_{\alpha \# \beta}(q) = J_\alpha(q) + q \left\lfloor \frac{N(\alpha)}{N(\beta)} \right\rfloor J_\beta(q)
\]

for all Farey neighbours $\alpha, \beta > 1$, where $N(\alpha), N(\beta)$ are the numerators of $\alpha, \beta$, respectively.

**Proof.** The equation (4.2) is obtained from combining Theorem 3.1, Lemma 4.1 and the equation (3.7).

The normalized Jones polynomial $J_\alpha(q)$ can be computed from $(a, b)_q$ as follows.

**Theorem 4.3.** Let $(a, x)$ be a pair of coprime integers with $1 \leq a < x$. Then

\[
J_{\frac{a}{x}}(q) = q^2(a, x - a)_q - (q - 1)(a, x)_q,
\]

\[
(a, x)_q = J_{\frac{a}{x}}(q) + q(a - r, r)_q,
\]

where $r$ is the remainder when $x$ is divided by $a$.

**Proof.** Let us prove the equation (4.3). It is proved by induction on the number of times of the operation $\#$.

First of all, let us consider the case $a = 1$. Dividing into the two cases where $b = 1$ or not, we can easily see that the equation (4.3) holds.

Next, let $\alpha = \frac{b}{a}$, $\beta = \frac{d}{c} \geq 1$ be Farey neighbors, and assume that they satisfy (4.3). Then we show that for $\alpha \# \beta = \frac{b+d}{a+c}$ the equation (4.1) holds by dividing into the four cases as follows.

1. In the case where $a = b$, we have $a = b = 1$ and $c = 1$, $d = 2$. So, $\alpha \# \beta = \frac{3}{2}$ and hence the right-hand side of (4.3) is $q^2(2, 3 - 2)_q - (q - 1)(2, 3)_q = 1 + q + q^3$. On the other hand, by $\frac{3}{2} = \frac{1 + \frac{2}{1}}{1}$ we have $J_{\frac{3}{2}}(q) = J_1(q) + q \left\lfloor \frac{1}{2} \right\rfloor J_2(q) = 1 + q + q^3$. Thus, in this case (4.3) holds.

2. In the case where $a < b$ and $d - c > 1$, by Lemmas 2.2, 4.2, induction hypothesis and (3.10) we have

\[
J_{\alpha \# \beta}(q) = q^2(a, b - a)_q - (q - 1)(a, b)_q + q \left\lfloor \frac{b}{a} \right\rfloor (q^2(c, d - c)_q - (q - 1)(c, d)_q)
\]

\[
= q^2((a, b - a)_q + q \left\lfloor \frac{b}{a} \right\rfloor (c, d - c)_q)
\]

\[= (q - 1)((a, b)_q + q \left\lfloor \frac{b}{a} \right\rfloor (c, d)_q) \quad \left(\frac{b}{d} - \frac{b - a}{d - c}, \quad d - c > 1\right)\]
\[ q^2(a + c, (b - a) + (d - c))_q - (q - 1)(a + c, b + d)_q. \]

It follows that (4.3) holds for \( \alpha \# \beta \).

(3) In the case where \( a < b \) and \( d - c = 1 \), as the same argument of (2), one can verify the equation (4.3) for \( \alpha \# \beta \).

(4) In the case where \( a < b \) and \( d = c \), we have \( c = d = 1 \). This leads to \( a = b + 1 \) by \( ad - bc = 1 \), which is a contradiction for \( a \leq b \). Thus, this case does not occur.

This completes the induction argument.

The equation (4.4) can be derived from (4.3) as follows. For the case \( a = b = 1 \), the desired equation can be easily verified. So, we consider the case of \( a < b \). By definition of \( (a, b)_q \), we have 
\[ (a, b)_q = (a - r, r)_q + q(a, b - a)_q. \]
This implies that 
\[ J_b(a)_q + q(a - r, r)_q = J_a(q) + q(a, b)_q - q(a, b - a)_q = (a, b)_q. \]

As an application of Theorem 4.3 we have:

**Theorem 4.4.** For a pair \( (a, x) \) of coprime integers with \( 1 \leq a \leq x \),
\begin{align*}
(4.5) & \quad N_q\left(\frac{x}{a}\right) = (a, x - a)_q, \\
(4.6) & \quad D_q\left(\frac{x}{a}\right) = (a - r, r)_q
\end{align*}
where \( r \) is the remainder when \( x \) is divided by \( a \).

**Proof.** The equation (4.5) is proved by induction of the number of times of \( \# \) as follows. In the case of \( a = 1 \) the equation (4.3) holds since \( N_q\left(\frac{n}{1}\right) = [n]_q = (n)_q = (1, n - 1)_q \). Let \( \frac{x}{a}, \frac{y}{b} \) be Farey neighbors consisting of rational numbers greater than 1, and assume that they satisfy \( N_q\left(\frac{x}{a}\right) = (a, x - a)_q, \quad N_q\left(\frac{y}{b}\right) = (b, y - b)_q \). Then
\begin{align*}
(a + b, x + y - (a + b))_q &= (a, x - a)_q + q\left[\frac{x}{a}\right] (b, y - b)_q \quad \text{(Theorem 3.7)} \\
&= N_q\left(\frac{x}{a}\right) + q\left[\frac{x}{a}\right] N_q\left(\frac{y}{b}\right) \quad \text{(induction hypothesis)} \\
&= N_q\left(\frac{x + y}{a + b}\right) \quad \text{(Theorem 3.1 and 3.3)} \\
&= N_q\left(\frac{x + y}{a + b}\right),
\end{align*}
and hence the equation (4.5) holds for the Farey sum \( \frac{x + y}{a + b} \). This completes the induction argument.

The equation (4.6) can be reduced from (4.5) as follows. By Lemma 4.1 and (4.4) the following equation holds:
\[ (a, p)_q = qN_q\left(\frac{p}{a}\right) + (1 - q)D_q\left(\frac{p}{a}\right) + q(a - r, r)_q. \]
By (4.5) this implies that
\[(1 - q)D_q\left(\frac{p}{a}\right) = (a, p)_q - q(a - r, r)_q - q(a, p - a)_q = (1 - q)(a - r, r)_q,\]
and hence we have (4.6). \qed

If positive integers \(a, x\) are coprime, then so are \(a, a + x\). By applying Theorem 4.4 we have:

**Corollary 4.5.** For a pair \((a, x)\) of coprime integers with \(1 \leq a \leq x\),
\[(a, x)_q = N_q\left(\frac{a + x}{a}\right).\]

**Remark 4.6.** By Corollary 4.5 and Morier-Genoud and Ovsienko’s results [15] we see that coefficients of \((a, x)_q\) have a quiver theoretical meaning as follows. Let \(G\) be an oriented graph, and \(V(G)\) denote the set of vertices of \(G\). A subset \(C \subset V(G)\) is an \(\ell\)-vertex closure if \(\sharp C = \ell\) and there are no edges from vertices in \(C\) to vertices in \(V(G) - C\). A vertex in a 1-vertex closure is called a sink. For a pair \((a, x)\) of coprime integers with \(1 \leq a \leq x\), we write in the form \(\frac{x}{a} = [a_1, a_2, \ldots, a_{2m-1}, a_{2m}]\). Consider the quiver
\[
G\left(\frac{x}{a}\right) : \begin{array}{cccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\xrightarrow{a_1} & \xrightarrow{a_2} & \xrightarrow{a_{2m-1}} & \xrightarrow{a_{2m-1}} \\
\end{array}
\]
By [15] then \((a, x)_q\) can be expressed as
\[(a, x)_q = 1 + \rho_1q + \rho_2q^2 + \cdots + \rho_{n-1}q^{n-1} + q^n,
\]
where \(n = a_1 + a_2 + \cdots + a_{2m}\), and
\[
\rho_i = \sharp\{\text{the } i\text{-vertex closures of }G\left(\frac{x}{a}\right)\}
\]
for each \(1 \leq i \leq n - 1\).

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