THE BOUNDARY OF AMOEBAS

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Abstract. We define and characterize the extended boundary of an amoeba which is in contrast to the topological boundary sensitive to the degeneration of complement components of the amoeba. In the hypersurface case, complement components of amoebas contain crucial information about the defining polynomial. Our description of the extended boundary in particular strengthens Mikhalkin’s result on contours of amoebas in a sense, which allows to distinguish between the contour and the boundary. This gives rise to new structural results in amoeba theory and re-proves earlier statements by Forsberg, Passare, Tsikh as well as Mikhalkin, Rullgard as special cases.

Our characterization has some immediate applications. It allows to compute amoebas including their contour and their boundary in any dimension. In order to understand the amoeba of an ideal, we introduce the concept of amoeba bases. We show that our characterization of the boundary is essential for the computation of such amoeba bases and we illustrate the potential of this concept by constructing amoeba bases for linear systems of equations.

1. Introduction

Let $f \in \mathbb{C}[z^\pm] = \mathbb{C}[z_1^\pm, \ldots, z_n^\pm]$ with $n \geq 2$ be a Laurent polynomial defining a non-singular variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$. The amoeba $\mathcal{A}(f)$ of $f$, introduced by Gelfand, Kapranov and Zelevinsky in 1993 in [8], is the image of $\mathcal{V}(f)$ under the Log absolute map given by

$$\text{Log} \, | \cdot | : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|). \quad (1.1)$$

Amoebas are objects with an amazing amount of structural properties, which have been intensively studied during the last 20 years. In particular, amoebas reveal an intrinsic connection between classical algebraic geometry and tropical geometry [6, 14, 15]. Moreover, they appear naturally in various other fields of mathematics like complex analysis [7, 23], real algebraic curves [16] or statistical thermodynamics [22]. For a general overview about amoeba theory see [8] and e.g., [5, 17, 24, 27].

It has been observed by Gelfand, Kapranov and Zelevinsky that each complement component of an amoeba is a convex set [8, p. 195, Cor. 1.6.] and by Forsberg, Passare and Tsikh that these sets are open [7, Prop. 1.2]. This means that the amoeba itself is bounded – either up to points at infinity or, e.g., considered under a toric compactification [8, 16]. It turns out that except in the linear case [7] this boundary is very hard to describe – no explicit characterization is known so far.

However, Mikhalkin gave an explicit characterization of the contour $\mathcal{C}(f)$ of the amoeba $\mathcal{A}(f)$. For two smooth manifolds $\mathcal{M}_1 \subset \mathbb{R}^m, \mathcal{M}_2 \subset \mathbb{R}^n$ and a smooth map $g : \mathcal{M}_1 \to \mathcal{M}_2$ we call a point $v \in \mathcal{M}_1$ critical under $g$ if for an infinitesimal small neighborhood $U$ of $v$ holds $\dim g(U) < \min\{\dim \mathcal{M}_1, \dim \mathcal{M}_2\}$. The contour $\mathcal{C}(f)$ is defined as the Log $| \cdot |$ image of the set of critical points of $\mathcal{V}(f)$ under the Log $| \cdot |$ map. The contour is a closed real-analytic

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the Gauß map details) and hence \( C \) are given by the points with a real image under the \( T \) bijection with the normal field respectively with the tangent bundle \( \text{Log} \).

Mikhalkin’s results \([16, 18]\) state that the critical points of \( \text{Log} |_V(f) \) and thus also the contour, are given by the points with a real image under the \textit{logarithmic Gauß map}.

For a non-singular variety \( V(f) \subset (\mathbb{C}^*)^n \), interpreted as a complex, smooth \((n-1)\)-manifold, the \textit{Gauß map} is given by

\[
G : V(f) \to \mathbb{P}_{\mathbb{C}}^{n-1}, \quad z = (z_1, \ldots, z_n) \mapsto \left( \frac{\partial f}{\partial z_1}(z) : \ldots : \frac{\partial f}{\partial z_n}(z) \right).
\]

Geometrically, the Gauß map can be interpreted as follows: For every point \( v \in V(f) \) consider the tangent space \( T_v V(f) \) of \( V(f) \) at \( v \) in \( \mathbb{C}^n \). Then \( G(z) \) can be identified with the projectivization of the normalized normal vector of this \( \mathbb{C} \)-hyperplane. Hence, the image of the Gauß map is in bijection with the normal field respectively with the tangent bundle \( TV(f) \) of \( V(f) \). See Section \( 2.3 \) for details.

The \textit{logarithmic Gauß map} introduced by Kapranov \([12]\) is a composition of a branch of a holomorphic logarithm of each coordinate with the conventional Gauß map. It is given by

\[
\gamma : V(f) \to \mathbb{P}_{\mathbb{C}}^{n-1}, \quad z = (z_1, \ldots, z_n) \mapsto \left( z_1 \cdot \frac{\partial f}{\partial z_1}(z) : \ldots : z_n \cdot \frac{\partial f}{\partial z_n}(z) \right).
\]

For a given variety \( V(f) \) we define the set \( S(f) \) by

\[
S(f) = \{ z \in V(f) : \gamma(z) \in \mathbb{P}_{\mathbb{R}}^{n-1} \subset \mathbb{P}_{\mathbb{C}}^{n-1} \}.
\]

\textbf{Theorem 1.1} (Mikhalkin \([16, 18]\)). Let \( f \in \mathbb{C}[z^{\pm 1}] \) with \( V(f) \subset (\mathbb{C}^*)^n \). Then the set of critical points of the \( \text{Log} | \cdot | \) map equals \( S(f) \) and thus we have for the contour \( \mathcal{C}(f) = \text{Log} |S(f)| \).

Note that this implies if \( f \) is real then the real locus \( V_\mathbb{R}(f) \), i.e., the set of real points in \( V(f) \), is always contained in the contour \( \mathcal{C}(f) \).

Concerning the boundary of an amoeba, the theorem yields the following corollary, stating that a point \( w \in \mathbb{R}^n \) may only be a boundary point of an amoeba \( A(f) \), if there exists a point in the intersection of its fiber \( \mathbb{F}_w = \{ z \in (\mathbb{C}^*)^n : \text{Log} |z| = w \} \) and the variety \( V(f) \), which belongs to the set \( S(f) \).

\textbf{Corollary 1.2} (Mikhalkin). Let \( f \in \mathbb{C}[z^{\pm 1}] \) with \( V(f) \subset (\mathbb{C}^*)^n \) and let \( w \in \mathbb{R}^n \). Then

\[
w \in \partial A(f) \implies \mathbb{F}_w \cap V(f) \cap S(f) \neq \emptyset.
\]

In this article we give an explicit characterization of the \textit{extended boundary} of amoebas up to singular points of the contour by strengthening Corollary \([1, 2]\). Note that we call a point \( w \in A(f) \) an \textit{extended boundary point} if it can be transfered to the complement of the amoeba by an infinitesimal small perturbation of the coefficients of the defining polynomial. Thus, this set may exceed the topological boundary. See Section \( 2.1 \) for a motivation of this concept and the precise definition, and see also Example \( 2.1 \).

In detail, we show that a point \( w \in \mathbb{R}^n \) may only be a boundary point of an amoeba \( A(f) \), if \textit{every} point in the (non-empty) intersection of its fiber \( \mathbb{F}_w \) and the variety \( V(f) \) belongs to the set \( S(f) \). Furthermore, we show that this condition is also sufficient if \( w \) is a non-singular point of the contour.

\textbf{Theorem 1.3}. Let \( f \in \mathbb{C}[z^{\pm 1}] \) with \( V(f) \subset (\mathbb{C}^*)^n \) non-singular and let \( w \in \mathbb{R}^n \).

\[
\text{If } w \in \partial A(f) \text{ then there exists no } v \in \mathbb{F}_w \cap V(f) \text{ with } v \notin S(f).
\]
If \( w \) is in addition a non-singular point of the contour, then the implication in (1.3) is an equivalence.

The key idea to prove this theorem is to investigate the variety \( \mathcal{V}(f) \) as the intersection of the varieties of its real and imaginary part contained in \( \mathbb{R}^{2n} \). This realification technique was already used earlier, e.g., in [30].

In addition, we show that this theorem does not only provide a theoretical description of the boundary, but it may also be used to check efficiently whether a contour point of \( A(f) \) is a boundary point of \( A(f) \). We show that Theorem (1.3) yields another description of the boundary that is way less abstract and in particular much more useful from the computational point of view. We summarize this description of Theorem 4.7 as follows.

**Corollary 1.4.** Let \( f \in \mathbb{C}[z^{\pm 1}] \) with \( \mathcal{V}(f) \subset (\mathbb{C}^*)^n \) non-singular and \( w \) a non-singular point of the contour \( C(f) \). Then \( w \) is a boundary point if and only if every point \( v \in \mathcal{V}(f) \cap F_w \) has multiplicity greater than one for \( n = 2 \) respectively \( \mathcal{V}(f) \cap F_w \) is finite for \( n \geq 3 \). This is the case if and only if every real root of a particular ideal \( I \subset \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) determined by \( f \) and \( w \) has multiplicity greater than one for \( n = 2 \) respectively the real locus \( \mathcal{V}_\mathbb{R}(I) \) is finite for \( n \geq 3 \).

For \( n = 3 \) we give bounds on the number of connected components of \( \mathcal{V}(f) \cap F_w \) in dependence of the degree of \( f \). For \( n = 2 \) the set \( \mathcal{V}(f) \cap F_w \) generically is finite and we can even compute its cardinality exactly, which yields a decomposition of the amoeba. See Section 4, in particular Theorem 4.10.

Using Theorem 4.7 we can recover the description of amoebas of linear polynomials by Forsberg, Passare and Tsikh [7], see also Proposition 3.1. Also, Theorem 4.7 implies that for every Harnack curve [9] the contour and the boundary of the corresponding amoeba coincide, which has been proven earlier by Mikhalkin and Rullgård [19], see also Proposition 3.2.

The methods developed in this paper are essential for the comprehension and computation of final representatives of amoebas of ideals, so called *amoeba bases*, see Theorem 6.1. Similar to Gröbner bases for classical algebraic varieties, e.g., [4], and tropical bases for tropical varieties, e.g., [3, 10, 28], we call a set \( G = \{g_1, \ldots, g_s\} \) *amoeba basis* for a finitely generated ideal \( I \) if \( \langle g_1, \ldots, g_s \rangle = I \), \( A(I) = \bigcap_{j=1}^s A(g_j) \) and \( G \) is minimal with these properties, see Definition 6.1. Note that in contrast to the classical and the tropical case, amoeba bases are not well understood yet and even their existence is unclear in general. Here, we provide amoeba bases for a first, non-trivial class of ideals, namely for amoebas of ideals corresponding to full ranked systems of linear equations, see Theorem 6.2. The latter theorem has been obtained joint with Chris Manon and we are thankful for his approval to publish it in this paper.

The computation and approximation of amoebas was initialized by Theobald [29], where in particular a calculation of the contour in the plane is described using Mikhalkin’s Theorem 1.1. Later, approximation methods of amoebas based on (algebraic) certificates have been given by Purbhoo [26] using iterated resultants, by Theobald and the second author [30] via semidefinite programming (SDP) and sums of squares (SOS) and, recently, by Avendano, Kogan, Nisse and Rojas [1] via tropical geometry. We compare these approaches in Section 5. Here, we only point out that none of these methods can compute the boundary of an amoeba or decide membership of a point in an amoeba exactly (only non-membership can be certified) or give general degree bounds on certificates.
We provide a new algorithmical approach based on our theoretical results with the following key properties, see Section 5 for details.

**Theorem 1.5.** Let \( f \in Q[i][z^{\pm 1}] \) (with \( i^2 = -1 \)) and \( v \in Q^n \). Then we can compute the contour and the boundary of \( A(f) \) as well as decide membership of \( \text{Log} |v| \in A(f) \).

In dimension two this approach can be implemented efficiently and the computation only uses Gröbner basis methods for zero-dimensional ideals.

This article is organized as follows. In Section 2 we introduce the notation and recall all relevant general facts about amoebas. In Section 3 we prove the Implication (1.3) of Theorem 1.3. In Section 4 we show that (1.3) is in fact an equivalence of statements if \( w \in R^n \) is a non-singular point of the contour. Furthermore, we prove Corollary 1.4 and discuss its consequences. In Section 5 we recall approximation methods for amoebas and develop an algorithm for computing the boundary and the contour. In Section 6 we introduce the concept of an amoeba basis, we give an overview of known facts and figure out, why a description of the boundary of amoebas is essential in order to be able to compute amoeba bases.

We remark that some of the results of this article, basically Section 3, are contained in the thesis of the second author [5].

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## 2. Preliminaries

### 2.1. Amoebas

In the following we always consider irreducible polynomials with a non-singular variety. The amoeba \( A(f) \subset R^n \) of an irreducible Laurent polynomial \( f \) is a full-dimensional topological space with open, convex complement components. Note that \( A(f) \) does not have to be full-dimensional if \( f \) is reducible, see, e.g., [27, p. 58, Figure 2]. Furthermore, every complement component corresponds uniquely to a lattice point \( \alpha \in \mathbb{Z}^n \cap \text{New}(f) \) via the order map [2]

\[
\text{ord} : \mathbb{R}^n \setminus A(f) \to \text{New}(f) \cap \mathbb{Z}^n, \quad \text{w} \mapsto \frac{1}{(2\pi i)^n} \int_{\log |z| = |\text{w}|} \frac{z_j \partial_j f(z)}{f(z)} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}, \quad 1 \leq j \leq n,
\]

which is invariant on all points of a particular complement component. We denote the complement component containing all points of order \( \alpha \in \mathbb{Z}^n \cap \text{New}(f) \) by \( E_{\alpha}(f) \), where we treat \( \alpha \in \mathbb{Z}^n \) equivalently as a vector and as a lattice point in a polytope with slight abuse of notation. Note that the order map can be interpreted as a multivariate analog of the classical argument
principle from complex analysis.

In order to characterize the boundary \( \partial A(f) \) of an amoeba \( A(f) \), we have to give an accurate definition of this object. For a finite support set \( A \subset \mathbb{Z}^n \) we define the parameter space \( \mathbb{C}^A \) as the set of all Laurent polynomials \( f = \sum_{\alpha \in A} b_{\alpha} z^\alpha \) with support set \( A \) and non-vanishing complex coefficients \( b_{\alpha} \in \mathbb{C}^* \). Note that we can and we will from now on always assume that

\[
A \subset \mathbb{N}^n,
\]

since we only consider polynomials with varieties \( \mathcal{V}(f) \subset (\mathbb{C}^*)^n \) and hence \( \mathcal{V}(f) \) and, in particular, \( A(f) \) are invariant under translations of support sets

\[
\tau_\beta : \mathbb{Z}^n \to \mathbb{Z}^n, \quad z^\alpha \mapsto z^{\alpha + \beta}
\]

with \( \beta \in \mathbb{Z}^n \). Furthermore, for every fixed \( A \subset \mathbb{N}^n \) we always assume that for the \( \alpha \in A \), which is minimal with respect to the lexicographic term ordering, the term \( b_{\alpha} \) equals 1. This can be done since varieties in \((\mathbb{C}^*)^n\) are invariant under scaling of the coefficients and hence we consider these polynomials as equivalent and only investigate equivalence classes (represented by \( \partial \)) in \( \mathbb{C}^A \). Note that these assumptions allow us to interpret \( \mathbb{C}^A \) as a \((\mathbb{C}^*)^{d-1}\) space, where \( d = \# A \), since we can identify every polynomial in \( \mathbb{C}^A \) with its coefficient vector.

We introduce a parameter metric \( d^A : \mathbb{C}^A \times \mathbb{C}^A \to \mathbb{R}_{\geq 0} \) in the following way. Let \( f = \sum_{\alpha \in A} b_{\alpha} \cdot z^\alpha \) and \( g = \sum_{\alpha \in A} c_{\alpha} \cdot z^\alpha \), then

\[
d^A(f,g) = \left( \sum_{\alpha \in A} |b_{\alpha} - c_{\alpha}|^2 \right)^{1/2}.
\]

For a given \( f \in \mathbb{C}^A \) we call a point \( w \in A(f) \) a boundary point of \( A(f) \) if there exists an \( \alpha \in \text{New}(f) \cap \mathbb{Z}^n \) such that for every \( \varepsilon > 0 \) there exists an \( g \in B^A_\varepsilon(f) \subset \mathbb{C}^A \) with \( w \in E_\varepsilon(g) \). Here, \( B^A_\varepsilon(f) \) denotes the open ball with radius \( \varepsilon \) with respect to the parameter metric on \( \mathbb{C}^A \) around the polynomial \( f \).

This definition of the boundary yields a set, which may exceed the topological boundary of the amoeba. In the following we motivate why this rather complicated definition is indeed natural. It guarantees that the boundary \( \partial A(f) \) of an amoeba \( A(f) \) is continuous under the change of coefficients – as the variety \( \mathcal{V}(f) \) as well as the amoeba \( A(f) \).

It is well known that the number of complement components is lower semicontinuous under arbitrary small perturbations of coefficients, see [7]. But, indeed, it happens that a new bounded complement component can appear somewhere inside an amoeba for an arbitrary small change of the coefficients in \( \mathbb{C}^A \), as in the following example.

**Example 2.1.** Let \( f = z_1^3 + z_2^3 + z_1z_2 + 1 \). Then \( A(f) \) is solid, i.e., it has only complement components corresponding to the three vertices of \( \text{New}(f) \) (via the order map). In particular, every sufficiently small neighborhood around the origin is contained in the interior of \( A(f) \). But, if we increase the coefficient of \( z_1z_2 \) by an arbitrary small \( \varepsilon > 0 \), then the origin is contained in a (bounded) complement component of \( f \) of order \((1,1)\). See Figure 1 and [5, 23, 31] for further details.

Hence, if we investigated the topological boundary of a family of amoebas given by polynomials \( f \) as in the Example 2.1 with coefficient \( c \) being changed continuously from e.g., 1.3 to 1, then we see that the boundary of the bounded complement component converges against the origin as a single, isolated point. But, obviously the origin is not part of the topological boundary of the amoeba of \( f = z_1^3 + z_2^3 + z_1z_2 + 1 \). Therefore, we use here this more complicated approach.
which characterizes such points as boundary points as well.

Recall that for every variety \( \mathcal{V}(f) \) one defines its real locus as \( \mathcal{V}_\mathbb{R}(f) = \mathcal{V}(f) \cap (\mathbb{R}^n)^n \). One essential tool to prove the main results of this article is to investigate the following realification of polynomials. For every polynomial \( f \in \mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n] \) we denote its real, respectively imaginary part as \( f^\text{re}, f^\text{im} \in \mathbb{R}[x,y] = \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n] \). So we have

\[
f(z) = f(x+iy) = f^\text{re}(x,y) + i \cdot f^\text{im}(x,y),
\]

see also e.g., \[30\]. With this notation \( \mathcal{V}(f) \subset (\mathbb{C}^n)^n \cong (\mathbb{R}^2 \setminus \{(0,0)\})^n \subset \mathbb{R}^{2n} \) obviously coincides with the intersection of the real loci \( \mathcal{V}_\mathbb{R}(f^\text{re}) \) and \( \mathcal{V}_\mathbb{R}(f^\text{im}) \) in \( (\mathbb{R}^2 \setminus \{(0,0)\})^n \) of the varieties \( \mathcal{V}(f^\text{re}), \mathcal{V}(f^\text{im}) \subset (\mathbb{C}^*)^{2n} \) of \( f^\text{re} \) and \( f^\text{im} \).

Note furthermore that if we assume that \( \mathcal{V}(f) \) is non-singular, then \( \mathcal{V}_\mathbb{R}(f^\text{re}) \) and \( \mathcal{V}_\mathbb{R}(f^\text{im}) \) are also non-singular after the embedding of \( \mathcal{V}(f) \) into \( \mathbb{R}^{2n} \).

**Example 2.2.** Let \( f = 2z_1^3 + (1 + 3i)z_1z_2 + 4iz_2 + 1 \). We set \( z_1 = x_1 + iy_1 \) and \( z_2 = x_2 + iy_2 \) and obtain

\[
f = 2(x_1^2 + y_1^2) + 4i x_1 y_1 + (1 + 3i)(x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)) + 4i(x_2 + iy_2) + 1
\]

\[
= 2(x_1^2 + y_1^2) + 4i x_1 y_1 + x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) + 3i(x_1 x_2 - y_1 y_2) - x_1 y_2
\]

\[
- x_2 y_1 + 4i x_2 - 4y_2 + 1.
\]

Hence, we have

\[
f^\text{re} = 2x_1^2 + x_1 x_2 - x_1 y_2 - x_2 y_1 + 2y_1^2 + y_1 y_2 - 4y_2 + 1 \quad \text{and}
\]

\[
f^\text{im} = 3x_1 x_2 + 4x_1 y_1 + x_1 y_2 + x_2 y_1 + 4x_2 - 3y_1 y_2.
\]

**2.2. Fibers.** Recall that a branch of the holomorphic logarithm is defined as

\[
\log_C : \mathbb{C}^* \to \mathbb{C}, \quad z \mapsto \log |z| + i \arg(z),
\]

where \( \arg(z) \) denotes the argument of the complex number \( z \). This means that the log absolute map \( \log \cdot \cdot \) is the real part of the complex logarithm. Since the multivariate case works componentwise like the univariate case, the holomorphic logarithm \( \log_C \) yields a fiber bundle structure for the log absolute map \( \log |\cdot\cdot| \) such that the following diagram commutes \([5\ 16\ 17]\):

\[
(C^n)^n \xrightarrow{\log_C} \mathbb{R}^n \times (S^1)^n.
\]
The fiber of each point \( w \in \mathbb{R}^n \) is a real \( n \)-torus, which we denote as \( F_w \), i.e.,

\[
F_w = \{ z \in (\mathbb{C}^*)^n : \text{Log} |z| = w \}.
\]

For us, the key fact about the fiber bundle structure is that the behavior of a point \( w \in \mathbb{R}^n \) is completely determined by the behavior of \( f \) on its fiber. This can be described in the following way \([5, 31]\). For \( f = \sum_{\alpha \in A} b_\alpha z^\alpha \) and \( v \in (\mathbb{C}^*)^n \) we define the fiber function

\[
f^{[v]} : (S^1)^n \rightarrow \mathbb{C}, \quad \phi \mapsto f(e^{\text{Log}|v| + i\phi}) = \sum_{\alpha \in A} b_\alpha \cdot |v|^{\alpha} \cdot e^{i(\alpha, \phi)},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product. Technically, \( f^{[v]} \) is \( f \) restricted to the fiber \( F_{|v|} \). More precisely, \( f^{[v]} \) is the function obtained by the pullback \( i_{[v]}^*(f) \) of \( f \) under the homeomorphism \( i_{[v]} : (S^1)^n \rightarrow F_{|v|} \subset (\mathbb{C}^*)^n \), which is induced by the fiber bundle structure. Hence, we have for the zero sets

\[
\mathcal{V}(f^{[v]}) = \mathcal{V}(i_{[v]}^*(f)) = \mathcal{V}(f) \cap F_{|v|},
\]

and in particular,

\[
\text{Log} |v| \in A(f) \Leftrightarrow \mathcal{V}(f^{[v]}) \neq \emptyset.
\]

We also need the Arg map given by

\[
\text{Arg} : (\mathbb{C}^*)^n \rightarrow (S^1)^n, \quad (z_1, \ldots, z_n) \mapsto (\text{arg}(z_1), \ldots, \text{arg}(z_n)),
\]

where \( \text{arg}(z) \) denotes the argument of a non-zero complex number \( z \). The Arg map can be seen as a natural counterpart of the \( \text{Log} |\cdot| \) map, since it is given by the componentwise projection on the imaginary part of the multivariate complex logarithm \( \text{Log}_\mathbb{C} \).

It is easy to see that the \( \text{Log} |\cdot| \) and the Arg map including their fibrations naturally extend to the realified version of a given variety \( \mathcal{V}(f) \), which lives in \((\mathbb{R}^2 \setminus \{(0, 0)\})^n\). Here, we denote these maps as \( \text{Log}_\mathbb{R} |\cdot| \) and \( \text{Arg}_\mathbb{R} \). Later, if the context is clear, we just write \( \text{Log} |\cdot| \) and \( \text{Arg} \) with slight abuse of notation. Let \( \mathcal{R} : (\mathbb{C}^*)^n \rightarrow (\mathbb{R}^2 \setminus \{(0, 0)\})^n \) denote the realification homeomorphism. The reason for the natural extension is that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\text{univ. covering}} & (S^1)^n \\
\downarrow \text{Arg}_\mathbb{R} & & \downarrow \text{Arg} \\
(\mathbb{R}^2 \setminus \{(0, 0)\})^n & \xrightarrow{\mathcal{R}} & (\mathbb{C}^*)^n \\
\downarrow \text{Log}_\mathbb{R} |\cdot| & & \downarrow \text{Log} |\cdot| \\
\mathbb{R}^n & \xrightarrow{\text{Re}} & \mathbb{R}^n \times (S^1)^n.
\end{array}
\]

In the next section we make use of the following important fact. Let \( z \in (\mathbb{R}^2 \setminus \{(0, 0)\})^n \). In an infinitesimal open neighborhood \( \mathcal{B}_\varepsilon(z) \subset (\mathbb{R}^2 \setminus \{(0, 0)\})^n \) of \( z \) the maps \( \text{Log}_\mathbb{R} |\cdot| \) and \( \text{Arg}_\mathbb{R} \) behave like linear projections from \( \mathbb{R}^{2n} \) to \( \mathbb{R}^n \) with

\[
\begin{align*}
\mathcal{B}_\varepsilon(z) &= \mathcal{B}_\varepsilon(z) \setminus \text{Ker}(\text{Log}_\mathbb{R} |\cdot|) \oplus \mathcal{B}_\varepsilon(z) \setminus \text{Ker}(\text{Arg}_\mathbb{R} |\cdot|) \\
&= \text{Ker}(\text{Log}_\mathbb{R} |\cdot|) \oplus \text{Ker}(\text{Arg}_\mathbb{R} |\cdot|).
\end{align*}
\]

The reason is that \( \mathcal{R} \) is just an identification map and \( \text{Log}_\mathbb{C} \) as a holomorphic map is locally linear. The orthogonality comes from the fact that \( \text{Log}_\mathbb{R} |\cdot| \) and \( \text{Arg}_\mathbb{R} \) are induced by projecting on
The real and imaginary part of $\log(\mathbb{C}^*)$. Finally, since we only investigate an infinitesimal open neighborhood of $z$, it does not matter that $\log(\mathbb{R})$ is defined on $(\mathbb{R}^2 \setminus \{(0,0)\})^n$ instead of $\mathbb{R}^{2n}$.

2.3. The Gauss Map. In the introduction we stated that the Gauss map $G$, see [12], applied to a variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ can be interpreted as a mapping every smooth point $z \in \mathcal{V}(f)$ to the normal vector of its tangent space $T_z \mathcal{V}(f)$ up to a complex scalar.

Since $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$, and $(\mathbb{C}^*)^n$ is a Lie group, the tangent bundle $T \mathcal{V}(f)$ is trivial, i.e., $T \mathcal{V}(f) = \mathcal{V}(f) \times \mathbb{C}^{n-1}$. The reason is that there exists a natural isomorphism $T_z \mathcal{V}(f) \approx T_1 \mathcal{V}(f)$, with $1 = (1, \ldots, 1)$, which is induced by the group action on $(\mathbb{C}^*)^n$ given by the multiplication with $z^{-1}$. Thus, if $v$ is given it suffices to investigate the linear part of the affine space $T_z \mathcal{V}(f)$.

The tangent space $T_z \mathcal{V}(f)$ is given by all $p \in \mathbb{C}^n$ with $D f(p) = 0$, respectively $\sum_{j=1}^n \frac{\partial f}{\partial z_j}(v)p_j = 0$. Thus, we have

$$T_z \mathcal{V}(f) = \{ p \in \mathbb{C}^n : \langle G(\text{aff} \cdot p) = 0 \},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product and $G(\text{aff} \cdot p)$ denotes an affine embedding of $G(v)$. Since every $\mathbb{C}$-hyperplane $H$ is given as the kernel of a linear form $\langle t, \cdot \rangle$ where geometrically $t$ is the normalized normal vector of $H$, it makes indeed sense to identify $G(\text{aff} \cdot p)$ with the normal vector of $T_z \mathcal{V}(f)$ up to a complex scalar. We do this in the following. Analogously for the logarithmic Gauss map $\gamma$. For further details see e.g., [13, 17].

3. A Necessary Condition for Contour Points to be Boundary Points

The goal in this section is to prove Implication [13] of Theorem 1.3. This means that we want to show that a point $w \in \mathbb{R}^n$ can be a boundary point of an amoeba $\mathcal{A}(f)$ only if every point of the variety $\mathcal{V}(f)$ intersected with the fiber $\mathbb{F}_w$ is critical under the Log $|\cdot|$ map, respectively has real (projective) image under the logarithmic Gauss map. The key idea to obtain this is to study how the real and the imaginary part of $f, f^r$ and $f^i$, behave on a distinct fiber, in particular, how their varieties intersect.

Before we do this we want to briefly recall some of the known cases. We have claimed in the introduction that the boundary is well understood in the linear case. This is reflected by the following proposition.

Proposition 3.1 (Forsberg, Passare, Tsikh). If $f = 1 + \sum_{j=1}^n b_j z_j$ is linear, then $v \in E_e(f) \cap \partial \mathcal{A}(f)$ if and only if $|b_j v_j| = 1 + \sum_{k \in \{1, \ldots, n\} \setminus \{j\}} |b_k v_k|$, where $e_j \in \mathbb{Z}^n$ denotes the $j$-th standard vector. Analogously $v \in E_0(f) \cap \partial \mathcal{A}(f)$ if and only if $\sum_{j=1}^n |b_j v_j| = 1$.

The explicit description of the boundary of linear polynomials was given by Forsberg, Passare and Tsikh [7] following basically from a direct calculation.

In the case $n = 2$ it is known for an even way more general class that the boundary and the contour coincide due to a result by Mikhalkin and Rullgard [19]. Remember that a variety $\mathcal{V}(f)$ is called real if it is invariant under complex conjugation of the variables $z_1, \ldots, z_n$. For any lattice polytope $P \subset \mathbb{Z}^n$ we denote the volume of $P$ with respect to the standard volume form of $\mathbb{Z}^n$ (i.e., the standard simplex has exactly volume 1) as $\text{vol}(P)$, see [8], p. 182 et seq.). Furthermore, one denotes the area of an amoeba of a complex polynomial $f \in \mathbb{C}[z_1, z_2]$ with respect to the Lebesgue measure as $\text{area}(\mathcal{A}(f))$. $\mathcal{V}(f) \subset (\mathbb{C}^*)^2$ is called real up to scalar multiplication if there exist $a, b_1, b_2 \in \mathbb{C}^*$ such that $af(z_1/b_1, z_2/b_2)$ has real coefficients.

Proposition 3.2 (Mikhalkin, Rullgård). Let $f \in \mathbb{C}[z_1, z_2]$ with $\text{vol}(\text{New}(f)) > 0$. Then the following conditions are equivalent.
THE BOUNDARY OF AMOEBAS

(1) area(A(f)) = π² · vol(New(f))
(2) Log |V(f)| is at most 2 - 1 and V(f) is real up to scalar multiplication.
(3) V(f) is real up to scalar multiplication and its real locus V_R(f) is a (possibly singular) Harnack curve.

Furthermore, if any of these conditions is satisfied, then Log |V_R(f)| = ∂A(f).

Recall that we mentioned in the introduction that in general the boundary and the contour do not coincide. A couple of examples can be found in e.g., [16, 24, 29]. Later, we also see that for the amoeba introduced in Example 2.1 the boundary and the contour do not coincide for positive coefficients of the term z_1z_2, see Figure 3.

Now, we head over to the proof of the first part of Theorem 1.3. Let v ∈ (C*)^n with Log |v| = w. After the realification of f the fiber F_w is given by \{(x, y) ∈ R^{2n} : x_j^2 + y_j^2 = |v_j|^2 \text{ for } 1 ≤ j ≤ n\}, i.e., every fiber is a real variety in R^{2n} of codimension n. By construction f^{[w], \text{re}} = f^{[w]}_\text{re} analogously for the imaginary part. Thus, it does not matter if we investigate the real part f^{[w], \text{re}} of the fiber function f^{[w]} one by one of for if we first take the real part f^{\text{re}} of f and restrict it afterwards to the fiber F_w, since the bundle structure is preserved under the realification (see Section 2.2). Hence, we have, just like for f,

\[ V(f^{[w], \text{re}}) = V(f^{\text{re}}) ∩ F_w, \]

analogously for the imaginary part. The following lemma describes the structure of V(f^{[w], \text{re}}).

**Lemma 3.3.** Let f ∈ C[z] with V(f) ⊂ (C*)^n non-singular and Log |v| = w ∈ R^n with F_w ∩ V(f) ≠ ∅. Then V(f^{[w], \text{re}}) and V(f^{[w], \text{im}}) generically are real non-singular (n - 1)-manifolds.

Note that V(f^{[w], \text{re}}) and V(f^{[w], \text{im}}) are in general neither connected nor non-singular. Here, the term “generically” means that if V(f^{[w], \text{re}}) or V(f^{[w], \text{im}}) is singular or their intersection has dimension lower than n - 1, then the subset of all polynomials g in the parameter space C^4 with the same property, which are located in an infinitesimal neighborhood of f (with respect to the coefficient metric d_A), has codimension at least one.

**Proof.** We only show the real case. V(f^{\text{re}}) is a real, non-singular (2n - 1)-manifold in R^{2n}. We have seen before that the fiber F_w is given by n real, non-singular hypersurfaces in R^{2n} defined by x_j^2 + y_j^2 = |v_j|^2, 1 ≤ j ≤ n. Since F_w ∩ V(f^{\text{re}}) ≠ ∅ by assumption, it is generically the transverse, i.e., non-singular intersection of n + 1 real, non-singular (2n - 1)-manifolds, so it is a real, generically non-singular manifold of dimension (n + 1) · (2n - 1) - n · (2n) = n - 1. □

**Lemma 3.4.** Let f ∈ C[z] with V(f) ⊂ (C*)^n. Then a point v ∈ V(f) is critical under the Log |·| map if and only if it is critical under the Arg map.

This statement follows already (at least) implicitly from Mikhalkin’s argumentation on the logarithmic Gauß map [16] and was also observed by Nisse and Passare before [21]. For convenience, we give our own proof here. Note that we denote the tangent space of a smooth point v on a manifold M as T_vM and its orthogonal complement as in the given surrounding space of M as (T_vM)^c.

**Proof.** We choose a local branch of the holomorphic logarithm Log_C, and we identify (C*)^n with (R^2 \ {(0, 0)})^n. Furthermore, recall from (2.4) that Log |·| and Arg can in an ε-neighborhood
Let $v \in V(f)$. Consider the tangent space $T_v V(f)$ of $v$ in $V(f)$. Since $V(f)$ is a variety of complex codimension one, we have after realification dim $T_v V(f) = 2n - 2$.

First, we prove dim $\log |(T_v V(f))^c| = \dim \text{Arg}((T_v V(f))^c) = 1$. Namely, $(T_v V(f))^c$ is spanned by the (complex) normal vector $t$ of $T_v V(f)$ and hence we know that dim $\log |(T_v V(f))^c| + \dim \text{Arg}((T_v V(f))^c) = 2$. But we also have that dim $\log |(T_v V(f))^c| \leq 1$ as $\log |t|$ is given by $\text{Re}(\log C(t))$, which is locally a linear map. Thus, the claim follows.

Now, let $v \in V(f)$ be critical under the $\log |.|$ map. We investigate the situation locally such that $\log |.|$ can be treated as linear. $v$ critical implies that the Jacobian matrix of $\log |.|$ at $v$ does not have full rank, i.e., dim $\log |T_v V(f)| \leq n - 1$. Since furthermore dim $\log |T_v V(f)^c| = 1$ and $\log |.|$ is surjective it follows dim $\log |T_v V(f)| = n - 1$. Hence, for $\text{Ker}(\log |T_v V(f)|) \subset T_v V(f)$ holds $\text{dim Ker}(\log |T_v V(f)|) = n - 1$. So we can choose an orthogonal basis $B = (b_1, \ldots, b_{2n-2}) \subset \mathbb{R}^{2n}$ of $T_v V(f)$ with $b_1, \ldots, b_{n-1} \in \text{Ker}(\log |T_v V(f)|)$.

Due to (2.4), we have $\text{Ker}(\log |T_v V(f)|) \subset \text{Arg}(T_v V(f))$, i.e., in particular, $\text{Arg} |_{(b_1, \ldots, b_{n-1})}$ is an immersion. Moreover, as $\text{dim Log}|T_v V(f)| = n - 1$, also $\text{dim Log}|_{(b_1, \ldots, b_{2n-2})}$ is an immersion. Thus, with (2.4), this yields $b_1, \ldots, b_{2n-2}$ is contained in $\text{Ker}(\text{Arg}(T_v V(f)))$, i.e., $\text{dim Ker} (\text{Arg}(T_v V(f))) = n - 1$ and therefore $\text{dim Arg}(T_v V(f)) = n - 1$. Hence, $v$ is critical under the Arg map. Vice versa the argument works analogously. □

Lemma 3.5. Let $f \in \mathbb{C}[z]$ with $V(f) \subset (\mathbb{C}^*)^n$ non-singular and let $v \in V(f)$ be a non-critical point under the $\text{Arg}$ map with $\text{Arg}(v) = \phi \in [0, 2\pi)^n$. Then $T_{\phi} V(f^{[\phi],re}) \neq T_{\phi} V(f^{[\phi],im})$ and furthermore both $V(f^{[\phi],re})$ and $V(f^{[\phi],im})$ are non-singular at $\phi$.

Note that $T_{\phi} V(f^{[\phi],re})$ is the tangent space of the variety $V(f^{[\phi],re})$ of the real part of the fiber function $f^{[\phi]}$ at $\phi$ (analogously for the imaginary part). So, $T_{\phi} V(f^{[\phi],re})$ is a subset of real dimension $n - 1$ of $(S^1)^n$ (isomorphic to the fiber $\mathbb{F}_w$), and hence of codimension one if $V(f^{[\phi],re})$ is smooth at $\phi$.

Proof. Assume first $T_{\phi} V(f^{[\phi],re}) = T_{\phi} V(f^{[\phi],im})$ and $V(f^{[\phi],re}), V(f^{[\phi],im})$ non-singular at $\phi$. Since $T_{\phi} V(f^{[\phi]}) = T_{\phi} V(f^{[\phi],re}) \cap T_{\phi} V(f^{[\phi],im})$ and dim $V(f^{[\phi],re}) = \text{dim V}(f^{[\phi],im}) = n - 1$ by Lemma 3.3, we have dim $T_{\phi} V(f^{[\phi]}) = n - 1$. Since $V(f^{[\phi]}) \cong V(f) \cap \mathbb{F}_w$ for $w = \log |v|$ and hence, $T_{\phi} V(f^{[\phi]}) \cong T_{\phi} V(f) \cap \mathbb{F}_w$, there is an immersion of an $(n - 1)$-dimensional subspace of $T_v V(f)$ into $\text{Arg}(T_v V(f))$, which yields that $v$ is critical with the argumentation from Lemma 3.4.

Assume $V(f^{[\phi],re})$ is singular at $\phi$. Then $T_{\phi} V(f^{[\phi],re}) = [0, 2\pi)^n$ and hence, $T_{\phi} V(f^{[\phi]}) = T_{\phi} V(f^{[\phi],im})$. Since $V(f)$ is non-singular, $V(f^{[\phi],im})$ may not be singular at $\phi$ either. Hence, dim$(T_{\phi} V(f^{[\phi]}) = n - 1$. The rest works in the same way as above. □

With these lemmata we can prove our main result of this section, namely the first part of Theorem 1.3 given by the Implication 1.3. The idea of the proof is that if there exists a point $v \in V(f) \cap \mathbb{F}_w$ with $v \notin S(f)$, then the manifolds $V(f^{[\phi],re})$ and $V(f^{[\phi],im})$ intersect regularly in $\text{Arg}(v)$. But this means that they also intersect for every small perturbation of the coefficients of $f$. This is a contradiction to the assumption $w \in \partial A(f)$, which means that there is a small perturbation of the coefficients yielding $V(f) \cap \mathbb{F}_w = \emptyset$, i.e., $V(f^{[\phi],re}) \cap V(f^{[\phi],im}) = \emptyset$.

Proof. (Theorem 1.3 Implication 1.3) Since $V(f) \subset (\mathbb{C}^*)^n$ we can assume w.l.o.g. that $f \in \mathbb{C}[z]$. Let $w \in \partial A(f)$ and assume that there is $v \in V(f) \cap \mathbb{F}_w$ with $v \notin S(f)$. By Theorem 1.1
this means that $v$ is a non-critical point under the Log $|\cdot|$ map and hence, by Lemma 3.4, $v$ is a non-critical point under the Arg map, too. Thus, by Lemma 3.5 $T_{\phi}V(f|_{v}|_{\text{re}}) \neq T_{\phi}V(f|_{v}|_{\text{im}})$, and both $V(f|_{v}|_{\text{re}})$ and $V(f|_{v}|_{\text{im}})$ are regular at $\phi$, and thus also in a small neighborhood $U_{\phi} \subset (S^1)^n \equiv F_w$. Therefore, $V(f|_{v}|_{\text{re}})$ and $V(f|_{v}|_{\text{im}})$ intersect regularly in $U_{\phi}$. Hence, there exists a $\delta > 0$ such that in $U_{\phi}$ the intersection of every $\delta$-perturbation of $V(f|_{v}|_{\text{re}})$ and $V(f|_{v}|_{\text{im}})$ is not empty.

Since $w \in \partial A(f)$ we find some $g \in B_{\varepsilon}^A(f) \subset \mathbb{C}^A$ for every arbitrary small $\varepsilon > 0$ such that $w \notin A(g)$, i.e., $V(g) \cap F_w = \emptyset$, and thus in particular $V(g|_{v}|_{\text{re}}) \cap V(g|_{v}|_{\text{im}}) = \emptyset$. But $f|_{\text{re}}, f|_{\text{im}}$ are continuous under changing the coefficients of $f$. Hence, also the regular loci of $V(f|_{\text{re}}), V(f|_{\text{im}})$ are continuous under an infinitesimal changing the coefficients of $f$. Therefore, by definition of the fiber function, $V(g|_{v}|_{\text{re}})$ and $V(g|_{v}|_{\text{im}})$ are arbitrary small perturbations of $V(f|_{v}|_{\text{re}})$ and $V(f|_{v}|_{\text{im}})$ in a neighborhood of the regular point $\phi$. Thus, $V(f|_{v}|_{\text{re}})$ and $V(f|_{v}|_{\text{im}})$ may not intersect regularly in $\phi$. This is a contradiction. □

Figure 2. The behavior of $f = -2z_1^2 - 2z_1z_2^2 + 1, 5e^{i\pi}0.5z_1^{-1}z_2^{-1} + c$ on the fiber $F_{(0,0)}$ for $c \in \{-1.2, -2.7, -4.6, -4.9\}$. 
We finish this section with an example. For simplicity, we identify here a fiber \( F_{\log |v|} \) with the corresponding \((S^1)^n\) of the fiber function \( f^{|v|} : (S^1)^n \to (\mathbb{C}^*)^n\) with slight abuse of notation.

**Example 3.6.** Let \( f \) be a Laurent polynomial given by

\[
  f = -2z_1^2 - 2z_1z_2^2 + 1,5e^{ix-0.5}z_1^{-1}z_2^{-1} + c.
\]

Consider the fiber \( F_{(0,0)} \) of the point \( \log |(1,1)| \) for \( c = -1.2, -2.7, -4.6 \) and \(-4.9\) depicted in Figure 2. In all pictures the red curve corresponds to \( V(f^{(|(1,1)|,re}) \) and the green curve corresponds to \( V(f^{(|(1,1)|,im}) \) in \( F_{(0,0)} \). Hence, the points in the intersection of the red and the green curve are the points where the real and the imaginary part of \( f^{|(1,1)|} \) vanishes, i.e., these are the intersection points of the fiber \( F_{(0,0)} \) with the variety \( V(f) \).

The blue curve depicts the argument of points on the complex unit sphere, which are critical points under the logarithmic Gauß map, i.e., the critical points of \( \gamma \) on the fiber \( F_{(0,0)} \). Thus, by Corollary 1.2, \((0,0)\) is part of the contour if there is a point where the red, green and blue curve intersect and, by Theorem 1.3, \((0,0)\) may only be a boundary point if all intersection points of the red and the green curve also intersect the blue one. Note that in this example a change of the coefficient \( c \) along the real axis only changes the red curve.

Furthermore, observe that in the upper left with \( c = -1.2 \) of Figure 2 the red and green curve intersect regularly in several points and hence in this case \((0,0) \in A(f)\). On the upper right pictures with \( c = -2.7 \), there are two intersection points where all three curves intersect. But there are other points where (only) the red and the green curve intersect regularly. Thus, in this case \((0,0) \) is part of the contour but still \((0,0) \in A(f) \setminus \partial A(f)\). On the lower left picture with \( c = -4.6 \) the only two intersection points of the red and the green curve also intersect the blue one. Hence, in this case \((0,0)\) might be part of the boundary. On the lower right picture with \( c = -4.9 \), the red and the green curve do not intersect in any point anymore. Therefore, we have \((0,0) \notin A(f)\).

Note that the values \( c = -2.7 \) and \( c = -4.6 \) are not the exact values of \( c \) for \((0,0)\) to be in the contour, respectively in the boundary of \( A(f) \). They are approximations of the exact values in order to visualize the situation.

## 4. Characterization of the Boundary

As an initial key step in this section we finish the proof of Theorem 1.3 by showing that the implication formulated in (1.3) in fact is an equivalence at least for all non-singular points of the contour of an amoeba. Afterwards, we point out that the approach used to prove Theorem 1.3 allows an alternative description of amoebas, their contour and their boundary, by using the real locus of a variety of a particular ideal instead of critical points of the \( \log | \cdot | \) map and the logarithmic Gauß map. We formulate these results in Theorem 4.7. In particular, this alternative approach yields a natural decomposition of the amoeba basis space \( \mathbb{R}^n \) given by the 0-th Betti number of the real locus of the variety of the ideal mentioned above. In Theorem 4.8 we show that the intersections of the closures of the cells of the decomposition form exactly the contour of the corresponding amoeba respectively its boundary if one of the cells have 0-th Betti number 0.

In order to be able to complete the proof of Theorem 1.3 we need to show respectively recall some important facts. First, we give a statement about the critical locus.
Lemma 4.1. Let \( f \in \mathbb{C}[z] \) with \( \mathcal{V}(f) \subset (\mathbb{C}^*)^n \). Then the critical locus \( S(f) \subset \mathcal{V}(f) \) is a real algebraic \((n-1)\)-variety in \( \mathbb{R}^{2n} \).

Proof. By Mikhalkin’s Theorem [13], the critical locus \( S(f) \) of \( f \) is given by all points in \( \mathcal{V}(f) \), which have projective real image under the logarithmic Gauss map. Hence, a point \( v \in \mathbb{R}^{2n} \) is contained in \( S(f) \) if and only if \( f^e(v) = f^m(v) = 0 \) and for all \( j, k \in \{1, \ldots, n\} \) with \( j \neq k \) it holds that

\[
\left( ||v_j \frac{\partial f}{\partial z_j}(v)||^2 \cdot v_k \frac{\partial f}{\partial z_k}(v) + ||v_k \frac{\partial f}{\partial z_k}(v)||^2 \cdot v_j \frac{\partial f}{\partial z_j}(v) \right) \cdot \left( ||v_j \frac{\partial f}{\partial z_j}(v)||^2 \cdot v_k \frac{\partial f}{\partial z_k}(v) - ||v_k \frac{\partial f}{\partial z_k}(v)||^2 \cdot v_j \frac{\partial f}{\partial z_j}(v) \right) = 0,
\]

which means that the \( j\)-th and the \( k\)-th entry of the image of \( v \) under the logarithmic Gauss map only differ by a real scalar. Since all equations are given by polynomials in \( \mathbb{R}[x, y] \), the critical locus is a real algebraic variety. For every smooth point \( v \) of \( S(f) \) the tangent space \( T_v S(f) \) of the critical locus at the point \( v \) is given by the subset of \( T_v \mathcal{V}(f) \) of \( \mathcal{V}(f) \), which keeps the argument of the logarithmic Gauss image invariant. Thus, the real dimension is \( n-1 \). \( \square \)

The next lemma is a statement about the logarithmic Gauss map by Mikhalkin [16], see also e.g., [24].

Lemma 4.2. Let \( A \subset \mathbb{Z}^n \) finite \( f \in \mathbb{C}^n \) with \( \mathcal{V}(f) \subset (\mathbb{C}^*)^n \) and \( t \in \mathbb{P}^{n-1} \). Let \( \gamma \) denote the logarithmic Gauss map. Then \( \gamma \) has the degree \( \text{vol}(\text{New}(f)) \). I.e., if the pair \( (f, t) \) is generic, then the set

\[
N_t = \{ z \in (\mathbb{C}^*)^n : z \in \mathcal{V}(f) \cap \gamma^{-1}(t) \},
\]

has cardinality \( \text{vol}(\text{New}(f)) \) and hence is in particular finite.

For convenience, we recall the proof here.

Proof. Let w.l.o.g. \( t_1, \ldots, t_s = 0 \) and \( t_{s+1}, \ldots, t_n \neq 0 \). Then \( v \in N_t \) if and only if \( v \) satisfies the following system of equations

\[
f(v) = 0, \quad z_1 \frac{\partial f}{\partial z_1}(v) = 0, \quad \ldots, \quad z_s \frac{\partial f}{\partial z_s}(v) = 0,
\]

\[
\frac{z_{s+1}}{t_{s+1}} \frac{\partial f}{\partial z_{s+1}}(v) = \ldots = \frac{z_n}{t_n} \frac{\partial f}{\partial z_n}(v).
\]

This is a system of \( n \) equations in \( n \) variables and since furthermore all involved polynomials have the same Newton polytope it follows by the Kouchnirenko Theorem, see, e.g., [8] p. 201], which states that generically the number of solutions is \( \text{vol}(\text{New}(f)) \). \( \square \)

As a third ingredient we recall the structure of the contour itself, see, e.g., [24].

Lemma 4.3. Let \( f \in \mathbb{C}[z] \) with \( \mathcal{V}(f) \subset (\mathbb{C}^*)^n \). Then the contour \( \mathcal{C}(f) \) is a closed real-analytic hypersurface in \( \mathcal{A}(f) \subset \mathbb{R}^n \).

Note that this particular lemma justifies to talk about tangent spaces, regular and singular points with respect to the contour. As a last step, before we can finish the proof of Theorem [1.3] we need to show how the critical locus, the contour, the logarithmic Gauss map and the Gauss map interact. Partially or implicitly the following proposition is stated in several papers, e.g., [16] [17] and [24] p. 277, but to the best of our knowledge it was nowhere stated in this form before.
Proposition 4.4. Let \( f \in \mathbb{C}[z] \) with \( V(f) \subset (\mathbb{C}^\ast)^n \). For the regular locus of \( S(f) \) and of the contour \( C(f) \) the following diagram commutes.

\[
\begin{array}{ccc}
S(f) & \xrightarrow{\text{Log} \ |} & C(f), \\
& \searrow & \downarrow \gamma \\
& & \mathbb{P}^{n-1}_\mathbb{R} \\
& \nearrow & \downarrow G
\end{array}
\]

where \( G \) denotes the Gauß map and \( \gamma \) denotes the logarithmic Gauß map. In particular, for \( w \in C(f) \) regular, all points in \( V(f) \cap \mathbb{F}_w \cap S(f) \) have the same image under the logarithmic Gauß map and therefore \( V(f) \cap \mathbb{F}_w \cap S(f) \) is generically finite.

Proof. Let \( v \in S(f) \subset V(f) \) non-singular and \( w = \text{Log} \ | v| \) with \( w \) non-singular in \( C(f) \). Since \( C(f) \) is a real smooth hypersurface and \( w \) is regular in \( C(f) \) the tangent space \( T_wC(f) \) is real \((n-1)\)-dimensional.

Let \( (b_1, \ldots, b_{n-1}) \subset (\mathbb{C}^\ast)^n \) be a basis of the tangent space \( T_vV(f) \) and \( t \) its normal vector. On the one hand we have shown in the proof of Lemma 4.4 that \( | \text{Log} \ | T_vV(f) \rangle \) is exactly spanned up by \( \text{Log} \ | b_1|, \ldots, \text{Log} \ | b_{n-1} \rangle \in \mathbb{R}^n \) and hence always a real \((n-1)\)-dimensional hyperplane with normal vector \( \text{Log} \ | t \rangle \). On the other hand \( v \) is in \( S(f) \). Thus, \( \gamma(v) = c \cdot \text{Log} \ | t \rangle \in \mathbb{P}^{n-1}_\mathbb{C} \) with \( c \in \mathbb{C} \). But since we furthermore showed in Lemma 4.1 that \( S(f) \) is a real \((n-1)\)-manifold and for every point in a small neighborhood of \( v \) in \( S(f) \) the logarithmic Gauß map needs to remain real, it follows that \( T_vS(f) \) is also spanned by \( c'|b_1|, \ldots, c'|b_{n-1}| \) for some \( c' \in \mathbb{C}^\ast \). Hence,

\[ T_wC(f) = \text{Log} \ | T_vV(f) \rangle \]

Thus, the Diagram (4.1) commutes since the normal vector of \( T_wC(f) \) is \( \text{Log} \ | t \rangle \) and therefore \( G(w) = \gamma(v) \). This also shows that all points of \( S(f) \), which are located in \( \mathbb{F}_w \) have the same image under the logarithmic Gauß map. The finiteness of \( V(f) \cap \mathbb{F}_w \cap S(f) \) follows from Mikhalkin’s Lemma 4.2.

Corollary 4.5. Let \( f \in \mathbb{C}[z] \) with \( V(f) \subset (\mathbb{C}^\ast)^n \) and \( \{w_1, \ldots, w_k\} \subset C(f) \subset \mathbb{R}^n \) the set of points in the contour of \( f \), which are mapped to a generic point \( \lambda \in \mathbb{F}_n^\mathbb{R} \) under the Gauß map. Then we have

\[ \sum_{j=1}^k \#(S(f) \cap \mathbb{F}_{w_j}) = \text{vol}(\text{New}(f)). \]

Proof. Follows immediately from Lemma 4.2 and Proposition 4.4.

Now, we can finish the proof of Theorem 1.3

Proof. (Equivalence statement of Theorem 1.3) Let \( f \in \mathbb{C}[z] \) with \( V(f) \subset (\mathbb{C}^\ast)^n \) and \( w \in C(f) \) such that the contour \( C(f) \) is not singular at \( w \). Assume furthermore that for every \( z \in V(f) \cap \mathbb{F}_w \) it holds \( z \in S(f) \), i.e., the right hand side of (1.3) in Theorem 1.3 is satisfied. By Proposition 4.4 the complex normal vectors of the tangent spaces of all \( z \in V(f) \cap \mathbb{F}_w \) coincide. Now, we modify the coefficients of \( f \) in a small \( \varepsilon \)-neighborhood \( B^A_\varepsilon(f) \) with respect to the parameter metric \( d^A \) such that the variety \( V(f) \) is locally translated in the negative direction of the unique complex normal vector of all points in \( V(f) \cap \mathbb{F}_w \). This is possible since in an infinitesimal neighbourhood around each point in \( \mathbb{F}_w \cap V(f) \) the variety behaves due to its smoothness like a linear polynomial in the coefficients \( f \). After this translation all critical points are not contained in the translated
variety \( \mathcal{V}(\tilde{f}) \) anymore, where \( \tilde{f} \) denotes the translated polynomial. Let \( \text{Log} |v| = w \). Since we have only modified the coefficients by an arbitrary small \( \varepsilon > 0 \) with respect to the coefficient metric \( d_A \) and as \( f^{[v], re} \) and \( f^{[v], im} \) are continuous under the change of coefficients, no new intersection points of \( \mathcal{V}(f^{[v], re}) \) and \( \mathcal{V}(f^{[v], im}) \) may appear in the \((S^1)^n\) isomorphic to the fiber \( F_w \). Thus, \( \mathcal{V}(\tilde{f}) \cap F_w = \emptyset \).

In Theorem 4.3 we required that \( w \in \mathbb{R}^n \) is a non-singular point of the contour. The reason is that we cannot guarantee to find a proper direction in the parameter space for changing the coefficient in order to transform a particular boundary point into a complement point as we have done it in the latter part of the proof. We depict this fact in the following example.

**Example 4.6.** Let \( f = z_1^3 + z_3^3 + z_1z_2 + 1 \). Although \( \mathcal{V}(f) \) is non-singular the contour \( \mathcal{C}(f) \) has a singular point in the origin given by three intersecting branches. See Figure 3 in Section 5. Via considering a small neighborhood of the origin we can conclude that by continuity reasons in the fiber over the origin there will exist three different normal directions of tangent directions of the parameter space such that all intersections in the fiber vanish. Theoretically it might happen that every small perturbation of coefficients, which lets one intersection points of \( \mathcal{V}(f^{[0], re}) \) and \( \mathcal{V}(f^{[0], im}) \) vanish turns another one into a regular intersection. Indeed, in this particular case the origin is a boundary point and a suitable shifting direction of the parameter space is given by increasing the \( z_1z_2 \) term. See Figure 3 again.

As a next step we want to give a statement equivalent to the characterization of the boundary of Theorem 4.3 which is in particular more convenient in order to tackle the contour and the boundary from a computational point of view. This theorem was already partially stated as Corollary 1.4 in the introduction.

**Theorem 4.7.** Let \( n \geq 2 \), \( f \in \mathbb{C}[v] \) with \( \mathcal{V}(f) \subset (\mathbb{C}^*)^n \). Let \( \text{Log} |v| = w \in \mathbb{R}^n \) a non-singular point of the contour such that \( \gamma^{-1}(G(w)) \) is finite. Let \( I = \{f^{re}, f^{im}, x_v^2 + y_v^2 = |v_1|^2, \ldots, x_n^2 + y_n^2 = |v_n|^2\} \), where \( f^{re}, f^{im} \) are given by the realification \( z_j = x_j + i y_j \) of variables. Then \( \mathcal{V}(I) \) has dimension at most \( n - 2 \) and

1. \( w \notin \mathcal{A}(f) \) if and only if \( \mathcal{V}_R(I) = \emptyset \),
2. \( w \in \mathcal{C}(f) \) if and only if
   a. \( \mathcal{V}_R(I) \) contains roots of multiplicity greater than one for \( n = 2 \),
   b. \( \mathcal{V}_R(I) \) contains an isolated point for \( n \geq 3 \),
3. \( w \in \partial \mathcal{A}(f) \) if and only if
   a. every root in \( \mathcal{V}_R(I) \) has multiplicity greater than one for \( n = 2 \),
   b. \( \mathcal{V}_R(I) \) is finite for \( n \geq 3 \),
4. \( w \in \mathcal{A}(f) \) else.

Note that the assumptions on \( w \) are very weak. Both the singular points and the points with \( \gamma^{-1}(G(w)) \) non-finite form a subset of the contour with codimension at least one by Lemma 4.2, Lemma 4.3 and Proposition 4.4. Therefore, we can denote \( w \) as a generic point of the contour.

**Proof.** \( \mathcal{V}(I) \) is given by \( n + 2 \) independent polynomial equations in \( 2n \) variables. Hence, \( \mathcal{V}(I) \) has expected dimension \( n - 2 \). Claim (1) follows since \( \mathcal{V}(f) \cap F_w \cong \mathcal{V}(f^{[v], re}) \cap \mathcal{V}(f^{[v], im}) = \mathcal{V}(f^{[v], re}) \cap \mathcal{V}(f^{[v], im}) \) is isomorphic to the real locus of the variety of the polynomials \( x_v^2 + y_v^2 = |v_1|^2, \ldots, x_n^2 + y_n^2 = |v_n|^2 \). Hence, we have in total \( \mathcal{V}_R(I) \cong \mathcal{V}(f) \cap F_w \) by construction. By Lemma 5.3 \( \mathcal{V}(f^{[v], re}) \) and \( \mathcal{V}(f^{[v], im}) \) are real \( n - 1 \) hypersurfaces in a real \( n \)-dimensional
variety. The point $w$ is contained in the contour if and only if $V_\mathbb{R}(I) \cong V(f) \cap \mathbb{F}_w$ contains a critical point $\phi$ of the $\log | \cdot |$ map (to keep the notation simple, we do not distinguish between $\phi$ in complex space and after realification; see Section 2). By Lemma 3.4 and Lemma 3.5 this is the case if and only if $T_\phi(V(|v|^{\text{re}})) = T_\phi(V(|v|^{\text{im}}))$, i.e., the manifolds $V(|v|^{\text{re}})$ and $V(|v|^{\text{im}})$ intersect non-transversally in $\phi$. For $n = 2$ this is the case if and only if $(w, \phi)$ is a multiple root. For $n \geq 3$ this is the case if and only if all points of $V_\mathbb{R}(I)$ in a neighborhood of $\phi$ are critical. And since we assumed $\gamma^{-1}(G(w))$ is finite, $\phi$ has to be an isolated point in $V_\mathbb{R}(I)$. Hence, Claim (2) follows. Statement (3) is a direct consequence of Theorem 1.3, which yields that $w$ is a boundary point if and only if all points in $V(f) \cap \mathbb{F}_w \cong V_\mathbb{R}(I)$ are critical, i.e., if and only if they are all multiple roots respectively isolated points. Statement (4) follows from (1)–(3). \hfill \square

We give a couple of remarks on Theorem 4.7. First observe that the ideal $I$ is of dimension zero if and only if $n = 2$, which is extremely relevant for the computation of amoebas, their boundary and their contour. See Section 3 for further details. Second notice that one consequence of Theorem 4.7 is that if the $\log | \cdot |$ map restricted to the variety of a curve in $\mathbb{C}[z_1, z_2]$ is 2 to 1, then the contour coincides with the boundary of the amoeba since in this case $\#V_\mathbb{R}(I) = 2$ and hence either all roots are multiple roots or none of them. This fact partially re-proves Proposition 3.2 by Mikhalkin and Rullgård. Third we remark that the characterization of the boundary of linear polynomials by Forsberg, Passare and Tsikh as in Proposition 3.1 is a consequence of Theorem 4.7. We give a proof in the following.

Proof. (Proposition 3.1) Let $f = 1 + \sum_{j=1}^{n} b_j z_j$. After a parameter transformation we can assume that every $b_j$ is real. Since vol(New($f$)) = 1 the logarithmic Gauss map is 1 to 1 by Lemma 4.2 and hence, by Theorem 1.3 and Corollary 4.5, if $w \in \partial A(f)$, then $\#(\mathbb{F}_w \cap V(f)) = 1$. Thus, for $\log |v| = w$ the fiber function $f^{\text{re}}$ vanishes at a single point.

It is easy to see by the linearity of $f$ that the image of the fiber function $f^{\text{re}}$ is a closed annulus around the constant term 1, where the inner circle may have radius zero. Thus, the only uniquely attained values are the two intersection points of the outer ring of the annulus with the real line. These extremal points are obviously attained if the condition $|b_k v_k| = 1 + \sum_{j \in \{1, \ldots, n\} \setminus \{k\}} |b_j v_j|$ is satisfied for some $k$ at a point $v \in (\mathbb{C}^*)^n$. \hfill \square

In the rest of this section we discuss further consequences of Theorem 4.7. First, we have a closer look at the topology of the variety $V(f)$ restricted to a certain fiber $\mathbb{F}_{\log |v|}$, which is isomorphic to $V(f^{\text{re}})$. By Lemma 3.3 we know that the realification of $V(f^{\text{re}})$ is a real $(n - 2)$-dimensional algebraic set. But, as we mentioned before, this set is in general not connected. Thus, for every $f \in \mathbb{C}[z]$ there exists a natural, non-trivial map

\begin{equation}
B : \mathbb{R}^n \to \mathbb{Z}, \quad w \mapsto b_0(V(f) \cap \mathbb{F}_w),
\end{equation}

where $b_0(V(f) \cap \mathbb{F}_w)$ denotes the 0-th Betti number of $V(f) \cap \mathbb{F}_w$. I.e., $B$ maps every point $w$ of the amoeba ground space to the number of connected components of the variety $V(f)$ restricted to the fiber of $w$ with respect to the $\log | \cdot |$ map. In particular the zero set of $B$ is exactly the complement of the amoeba. Note that for $n = 2$ we have $b_0(V(f) \cap \mathbb{F}_w) = #(\mathbb{F}_w \cap V(f))$ counted without multiplicity, since by Theorem 4.7 $\mathbb{F}_w \cap V(f)$ is isomorphic to a finite arrangement of points in $(S^1)^2$ in this case.

Obviously, $B$ yields a partition of $\mathbb{R}^n$, respectively $A(f)$, which we call $b_0$-decomposition or Betti decomposition. The following theorem shows that the Betti decomposition is an alternative way to describe the contour.
Theorem 4.8. Let \( f \in \mathbb{C}[z] \) and \( w \in \mathbb{R}^n \). If \( w \) is not an isolated singular point of \( C(f) \), then \( w \in C(f) \) if and only if there exists no \( \varepsilon > 0 \) such that the map \( B \) is invariant in the \( \varepsilon \)-neighborhood \( B_\varepsilon(w) \) of \( w \).

Geometrically, the latter theorem means that the contour of a polynomial \( f \) is the union of the boundary of (the closure of) the cells given by the Betti decomposition.

Proof. Assume that there exists no \( \varepsilon > 0 \) such that \( B \) is invariant in the \( \varepsilon \)-neighborhood \( B_\varepsilon(w) \) of \( w \). By definition, this means that the number of connected components of the real locus of \( V(f^{\text{re}} \cap V(f^{\text{im}}) \cap \mathbb{F}_w) \) is not constant for all \( w' \in B_\varepsilon(w) \). This means that there exists a point \( w' \in B_\varepsilon(w) \) such that for \( \text{Log } |V'| = w' \) the varieties \( V(f^{\text{re}}|V') \) and \( V(f^{\text{im}}|V') \) do not intersect transversally on some connected component on \( \mathbb{F}_{w'} \), which means that \( w' \in C(f) \) by Theorem 4.7. Since this is true for every \( \varepsilon > 0 \), it follows that \( w' = w \).

Assume the other way around that there exists an \( \varepsilon \)-neighborhood of \( w \) where \( B \) is invariant and \( w \) is a contour point, which is not an isolated singularity. First, assume \( w \) is a regular point of the contour. It follows from Lemma 4.4 and the corresponding proof that the connected component of the real locus of \( V(f^{\text{re}} \cap V(f^{\text{im}}) \cap \mathbb{F}_w) \) is invariant in the corresponding normal direction. This is a contradiction to the assumption that \( B \) is invariant.

If \( w \) is a singular point of the contour but not isolated, then every \( \varepsilon \)-neighborhood contains regular points of the contour. Hence, by the same argumentation as above, the invariance of \( B \) is contradicted.

Corollary 4.9. Let \( f \in \mathbb{C}[z] \). Then \( C(f) = \partial A(f) \) up to singular points if and only if \( B \) is constant on \( A(f) \setminus \partial A(f) \), i.e., the \( b_0 \)-partition has exactly one component on \( A(f) \setminus \partial A(f) \).

Now, we show that we can bound the map \( B \) from above for \( n = 2, 3 \). The Harnack inequality, e.g., [9, [11, Cor. 5.4], tells us that a real algebraic curve of genus \( g \) can have at most \( g + 1 \) real connected components. By a result due to Castelnuovo, e.g., [20, 25], the genus of a real algebraic curve of degree \( d \) in \( \mathbb{P}^n \) is bounded from above by Castelnuovo's bound \( C(d, n) \). \( C(d, n) \) is defined as follows: for \( d \geq n \geq 2 \) there exist unique \( m, p \in \mathbb{N} \) such that \( d - 1 = m(n - 1) + p \) with \( 0 \leq p \leq n - 1 \). Then \( C(d, n) = m \frac{m - 1}{d} + 2p \). For \( n = 2 \) this bound specializes to the well-known result \( C(d, 2) = \frac{(d - 1)(d - 2)}{2} \), which can also be obtained by the adjunction formula for \( g \).

Theorem 4.10. Let \( f \in \mathbb{C}[z] \) with \( \deg(f) = d \), \( \deg(f^{\text{re}}) = d^{\text{re}} \) and \( \deg(f^{\text{im}}) = d^{\text{im}} \). Let \( B \) denote the map yielding the associated \( b_0 \)-partition of \( \mathbb{R}^n \) as in (4.2). Let \( w \in \mathbb{R}^n \) such that \( V(f) \) and \( \mathbb{F}_w \) do not have a common irreducible component.

1. If \( n = 2 \) then \( B(w) \leq 4d^{\text{re}}d^{\text{im}} \leq 4d^2 \) for all \( w \in \mathbb{R}^2 \).
2. If \( n = 3 \) then \( B(w) \leq C(4d^2, 6) + 1 \) for all \( w \in \mathbb{R}^3 \).

Proof. Intersection results like Bézout's Theorem for projective varieties also hold in our affine situation, since we work with compact affine varieties (due to the fiber torus \( \mathbb{F}_w \)).

By definition of \( B \) and by Theorem 4.7, \( B(w) \) equals the number of connected components of the real locus of the variety of the ideal \( I = \langle f^{\text{re}}, f^{\text{im}}, x_i^2 + y_i^2 - |v_i|^2, \ldots, x_i^2 + y_i^2 - |v_n|^2 \rangle \), where \( \text{Log } |v| = w \). Since \( V(f) \) and \( \mathbb{F}_w \) do not have a common irreducible component by assumption the defining polynomials are independent. Thus, for \( n = 2 \) the intersection \( V(f) \cap \mathbb{F}_w \) is zero-dimensional and, by application of Bézout's Theorem, \( B(w) \) has degree \( D \leq 4d^{\text{re}}d^{\text{im}} \leq 4d^2 \). In
the case \( n = 3 \) the variety \( \mathcal{V}(f) \cap \mathbb{F}_w \) is a curve, hence we can use Castelnuovo’s bound for the real locus of the variety \( \mathcal{V}(I) \).

5. Computation of the Boundary

The computation of amoebas, its contour and its boundary was initialized by Theobald in \cite{29} and tackled by different authors during the last decade as well. In this section, we first give an overview about the state of the art of the computation of amoebas. Afterwards we exploit our own results, particularly Theorem 4.7 to provide a new algorithmic approach allowing to decide membership of points in amoebas exactly and yielding a distinction between boundary points and contour points. In dimension two this approach allows an efficient computation of the boundary and the contour of the amoeba and even the Betti decomposition of amoebas. In particular, to the best of our knowledge no other known algorithm can decide membership exactly and can compute the boundary.

We start with a comparison of the existing approximation methods for amoebas. As mentioned before, the first method was given by Theobald \cite{29} in dimension two and can be generalized to higher dimensions. The key idea is to use Mikhalkin’s Theorem 1.1 to compute the contour of an amoeba and thus obtain an approximation. Practically, for \( f \in \mathbb{C}[z] \) this can be done by computing the roots of all ideals

\[
I_s = \left\langle f, z_n \frac{\partial f}{\partial z_n} - s_1 \cdot z_1 \cdot \frac{\partial f}{\partial z_1}, \ldots, z_n \cdot \frac{\partial f}{\partial z_n} - s_{n-1} \cdot z_{n-1} \cdot \frac{\partial f}{\partial z_{n-1}} \right\rangle
\]

with \( s = (s_1, \ldots, s_{n-1}) \in \mathbb{R}^{n-1} \). Theorem 1.1 guarantees that all points in \( \mathcal{V}(I) \) are critical and all critical points are of the Form (5.1). Since finally \( \mathcal{V}(I) \) is zero-dimensional one can obtain an approximation of the contour and hence of the amoeba by computing \( \mathcal{V}(I) \) for suitable many different \( s \in \mathbb{R}^{n-1} \), see Lemma 4.2.

Later approaches rely on solving the following membership problem.

**Problem 5.1.** Let \( f \in \mathbb{C}[z] \) and \( w \in \mathbb{R}^n \). Decide, whether \( w \in \mathcal{A}(f) \).

Although already described in \cite{29}, this approach was first used by Purbhoo \cite{26} to provide an approximation method for amoebas based on a lopsidedness certificate. This certificate checks whether – evaluated at one specific point – the absolute value of one monomial of \( f \) is larger than the sum of the absolute values of all other monomials. It is easy to see that if this is the case, then the point is contained in the complement of the amoeba. In general this condition is not necessary (beside of the case of linear polynomials – see Proposition 3.1). But Purbhoo shows that it is possible to investigate iterated resultants of the original polynomial in order to approximate the amoeba. The iteration process keeps the amoeba invariant, but guarantees that every point becomes lopsided if the iteration level is sufficiently high \cite{26}.

Theobald and the second author have shown in \cite{30} that the membership problem can be transformed into a feasibility problem of a particular semidefinite optimization problem via realification and using the Real Nullstellensatz. This allows an approximation of amoebas via SDP-methods, which turn out to be of at most the same complexity as Purbhoo’s approach.

Recently, Avendano, Kogan, Nisse and Rojas \cite{1} provided limit bounds on the (Hausdorff-) distance between the amoeba of a polynomial \( f = \sum_{j=1}^d b_j z^{\alpha(j)} \) and the tropical hypersurface of the tropical polynomial archtrop(\( f \)) = \( \bigoplus_{j=1}^d \log |b_j| \odot z^{\alpha(j)} \). Recall that the tropical polynomial archtrop(\( f \)) is defined over the tropical semi-ring \( (\mathbb{R} \cup \{-\infty\}, \oplus, \odot) = (\mathbb{R} \cup \{-\infty\}, \max, +) \), i.e.,
archtop\((f) = \max_{j=1}^d \log |b_j| + \langle z, \alpha(j) \rangle\), where \(\langle \cdot, \cdot \rangle\) denotes the standard inner product. Recall furthermore that the tropical variety of a tropical polynomial is defined as the subset of \(\mathbb{R}^n\) where the tropical polynomial has non-smooth image, i.e., as subset of \(\mathbb{R}^n\), where the maximum is attained at least twice. Since the tropical hypersurface archtop\((f)\) is easy to compute, this result yields a new rough but quick way to approximate the amoeba via finding certificates for non-containment in the amoeba for certain points. For an introduction to tropical geometry see e.g., [14].

Now, we head over to our approach. The main improvement is that we are able to describe the boundary of amoebas and that we are able to answer the membership question exactly. The other approaches only yield certificates for non-membership of points in the amoeba, but cannot certify membership. The way to do this is to use Theorem 4.7, which guarantees for \(n = 2\) that every fiber \(\mathbb{F}_w\) only contains finitely many points of \(\mathcal{V}(f)\) and, furthermore, it allows to distinguish between regular points of the amoeba, contour points and boundary points.

**Corollary 5.2.** Let \(f \in \mathbb{C}[z]\) such that the real and imaginary parts of the coefficients of \(f\) are rational, and \(w \in \mathbb{R}^n\) such that \(\exp(w) \in \mathbb{Q}^n\). Then we can decide with symbolical methods, whether \(w \in \mathcal{A}(f)\), \(w \in \partial \mathcal{A}(f)\) or \(w \in \mathcal{C}(f)\).

Note that this corollary means in particular that we can solve the membership problem exactly.

*Proof.* Let \(v = (v_1, \ldots, v_n) \in \mathbb{Q}^n\) such that \(\log |v| = w \in \mathbb{R}^n\). We also set \(z_j = x_j + i \cdot y_j\) for \(j \in \{1, \ldots, n\}\).

Suppose \(n = 2\). We compute the (zero-dimensional) variety of \(I = \langle f^{\text{re}}, f^{\text{im}}, x_2^2 + y_2^2 - |v_1|^2, x_2^2 + y_2^2 - |v_2|^2 \rangle\) symbolically via computing a Gröbner basis. Using Theorem 4.7 we can read out from the real locus where \(w\) is located.

If \(n > 2\) we also consider the ideal \(I = \langle f^{\text{re}}, f^{\text{im}}, x_2^2 + y_2^2 - |v_1|^2, \ldots, x_n^2 + y_n^2 - |v_n|^2 \rangle\). The generic dimension of \(I\) is \(n - 2 > 0\), but by Theorem 4.7 it suffices to determine the dimension of the real locus of \(\mathcal{V}(I)\). This can be decided by quantifier elimination methods, see e.g., [2, Algorithm 14.10].

With Corollary 5.2 we can now immediately obtain an approximation of the contour and the boundary of an amoeba \(\mathcal{A}(f)\) of a given polynomial \(f\) in dimension two. First, we compute the contour \(\mathcal{C}(f)\) - e.g., either using Theobald’s method solving the ideal \(I\) from 5.1. Alternatively, we can e.g., compute all contour points along a one-dimensional affine subspace of \(\mathbb{R}^2\) given by fixing the absolute value of \(z_1\) (respectively \(z_2\)), i.e., by computing the variety of the ideal \(I = \langle f^{\text{re}}, f^{\text{im}}, x_2 + y_2^2 - |v_1|^2, x_2 + y_2^2 - |v_2|^2 \rangle\) in \(\mathbb{R}[x_1, x_2, y_1, y_2]\), where the third polynomial guarantees that we only investigate critical points and the fourth polynomial ensures that the absolute value of \(z_1\) equals \(\lambda_1 \in \mathbb{R}\). In the following example we present the result of a prototype implementation of our algorithm.

For \(n > 2\) an implementation to approximate the boundary would be also possible using Theobald’s method and Corollary 5.2, but would result in very long runtimes.

**Example 5.3.** We approximate the amoebas of \(f = z_1^2 z_2 + z_1 z_2^2 + c z_1 z_2 + 1\) with \(c = 1.5\) and \(c = 1\) via computing their contour and their boundary. Here, we use Theobald’s method (see 5.1) to compute the contour. Afterwards we distinguish between contour and boundary with the approach described in Corollary 5.2. The result is depicted in Figure 3.
Corollary 5.2 allows us furthermore to compute an arbitrary exact approximation of the Betti decomposition of the amoeba of an arbitrary polynomial in dimension two. Recall that in dimension two the map $B$, see (4.2), maps $w \in \mathbb{R}^n$ to the number of roots in $\mathcal{V}(f) \cap F_w$, i.e., to the number of real roots in a fiber ideal as given in Theorem 4.7. Hence, we can immediately compute all values of $B$ via Corollary 4.9, which yields — in addition to information provided by the image of $B$ itself — another method to approximate the amoeba, the contour and the boundary.

Example 5.4. We compute the Betti decomposition of the polynomials $f = z_1^3 + z_2^3 + cz_1 z_2 + 1$ with $c = 1$ and $c = 1.5$. In order to do so, we compute the variety of every fiber ideal of points contained in the set $\{20 \cdot (w_1, w_2) \in \mathbb{Z}^2 : -2 \leq w_1, w_2 \leq 2\}$. The plots are depicted in Figure 4. Note that for $c = -4$ the polynomial $f$ defines a Harnack curve and hence $\text{Log} | \cdot |$ is 2 to 1 by Proposition 3.2 or by Theorem 4.7. Thus, by Corollary 4.9 we have $\partial \mathcal{A}(f) = \mathcal{C}(f)$ in this case. Therefore, the left picture in Figure 4 also provides an example for this Corollary 4.9.

6. Impact on Amoeba Bases

In contrast to amoebas of Laurent polynomials, which have been extensively studied and are well understood in many aspects, amoebas of ideals, i.e., the $\text{Log} | \cdot |$ image of varieties of ideals, have merely been investigated and are almost completely not understood. To the best of our knowledge, the most remarkable result is by Purbhoo, see [26], stating that for every polynomial ideal $I$ holds

$$\mathcal{A}(I) = \bigcap_{f \in I} \mathcal{A}(f).$$

A genuine question arising from this result, which was already mentioned by Purbhoo himself, is if the intersection on the right hand side can be restricted to a finite subset of the polynomials in the ideal — an amoeba basis.

Before we define amoeba bases, we briefly recall the definition of Gröbner bases and tropical bases in order to demonstrate the analogy in the definition of amoeba bases in (6.1) afterwards.
In algebraic geometry and computer algebra Gröbner bases play a fundamental role, since they are the generic tool to solve (non-linear) systems of polynomial equations. For a finitely generated ideal $I = \langle f_1, \ldots, f_r \rangle \subseteq \mathbb{C}[z]$ with variety $V(I)$ a Gröbner bases $G \subset I$ (with respect to a given monomial ordering $\prec$) is a finite system of polynomials $g_1, \ldots, g_s$ such that the ideal $\text{lt}_\prec(I)$ of leading terms of polynomials in $I$ is generated by the leading terms of the elements of the Gröbner basis, i.e.,

$$\text{lt}_\prec(I) = \langle \text{lt}_\prec(g_1), \ldots, \text{lt}_\prec(g_s) \rangle.$$  

Gröbner bases exist for every finitely generated polynomial ideal and monomial ordering and can be computed efficiently, e.g., via Buchberger’s algorithm. For an introduction to the topic see e.g., [4].

Similarly, in tropical geometry there exist tropical bases. Let $I \subset \mathbb{K}[z]$ be an ideal generated by finitely many polynomials in a polynomial ring $\mathbb{K}[z]$ over a real valuated field $\mathbb{K}$, e.g., the field of Puiseux series. One defines for every $f = \sum_{\alpha \in A} b_\alpha z^\alpha \in \mathbb{K}[z]$ with $A \subset \mathbb{Z}^n$ the corresponding tropical polynomial $\text{trop}(f)$ as

$$\text{trop}(f) = \bigoplus_{\alpha \in A} -\text{val}(b_\alpha) \oplus z^\alpha,$$

where $\text{val}$ denotes the natural valuation map from the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$ to $\mathbb{R} \cup \{\infty\}$. As before, the tropical variety $\mathcal{T}(\text{trop}(f))$ is defined as the subset of $\mathbb{R}^n$ where $\text{trop}(f)$ attains its maximum at least twice and the tropical variety $\mathcal{T}(I)$ of the ideal $I$ is given by $\mathcal{T}(I) = \bigcap_{f \in I} \mathcal{T}(\text{trop}(f))$. With this notation one calls $G = (g_1, \ldots, g_r) \subset \mathbb{K}[z]$ a tropical basis of $I$ if $\langle g_1, \ldots, g_r \rangle = I$ and

$$\mathcal{T}(I) = \bigcap_{j=1}^r \mathcal{T}(\text{trop}(g_j)).$$

For more details on tropical bases see e.g., [3] [10] [14] [28].
As an analog to Gröbner bases from algebraic geometry and tropical bases from tropical geometry, we define an amoeba basis in the following way. Let \( I \subset \mathbb{C}[z] \) be a finitely generated ideal. Then we call \((g_1, \ldots, g_s) \subset I\) an amoeba basis if

\[
\begin{align*}
(1) & \quad \mathcal{A}(I) = \bigcap_{j=1}^{s} \mathcal{A}(g_j), \\
(2) & \quad \mathcal{A}(I) \subset \bigcap_{j \in \{1, \ldots, s\} \setminus \{i\}} \mathcal{A}(g_j) \text{ for every } 1 \leq i \leq s, \\
(3) & \quad \langle g_1, \ldots, g_s \rangle = I.
\end{align*}
\]

Although the question of the existence and (possible) form of amoeba bases has a, say, folkloric character in the mathematical (amoeba) community, to the best of our knowledge no formal definition of an amoeba basis was given elsewhere before. Therefore, we want to point out here, while axiom (2) only requires minimality of an amoeba basis and hence is no proper restriction, it is not clear whether it makes sense to require axiom (3) in general. We do this here basically since we want to create an object, which is truly analogous to Gröbner bases and tropical bases.

In general, it is highly unclear for which ideals amoeba bases always exist. We remark that Purbhoo claims that they do not exist in general [26, p. 25], but he does not give a formal proof. Furthermore, if amoeba bases exist for certain ideals, then it is completely unclear, how many elements they have, how they can be computed, or if they are unique in any sense. In this section we show that if an amoeba basis exists for some given ideal, then the amoeba of the ideal intersects the boundary of the amoeba of every basis element, see Theorem 6.1. Thus, an understanding of the boundary of amoebas indeed is essential in order to find amoeba bases.

In Theorem 6.1 we have assumed that amoeba bases exist. But, as already mentioned in the beginning of the section, the question about the existence of amoeba bases is completely open. Obviously, amoeba bases always exist for principal ideals and hence in particular for ideals of univariate polynomials, but this is trivial since for every such ideal \( I = \langle f \rangle \), we have \( \mathcal{A}(I) = \mathcal{A}(f) \). We show in the following that in general there also exist amoeba bases for non-trivial cases by proving their existence and computability for full ranked systems of linear equations. The following theorem is joint work with Chris Manon. The initial proof strategy was obtained by him and the second author.

**Theorem 6.1.** Let \( I \subset \mathbb{C}[z] \) be a finitely generated ideal. Assume there exists an amoeba basis \((g_1, \ldots, g_s)\) for \( I \). Then \( \partial \mathcal{A}(g_i) \cap \mathcal{A}(I) \neq \emptyset \) for every \( 1 \leq i \leq s \).

**Proof.** Since \((g_1, \ldots, g_s)\) is an amoeba basis of \( I \), we have \( \mathcal{A}(I) = \bigcap_{j=1}^{s} \mathcal{A}(g_j) \) and therefore \( \mathcal{A}(I) \subset \mathcal{A}(g_i) \) for every \( i \). Assume \( \partial \mathcal{A}(g_i) \cap \mathcal{A}(I) = \emptyset \) for some \( i \). Then \( \mathcal{A}(I) \subset (\mathcal{A}(g_i) \setminus \partial \mathcal{A}(g_i)) \) and thus \( \mathcal{A}(I) = \bigcap_{j \in \{1, \ldots, s\} \setminus \{i\}} \mathcal{A}(g_j) \), which is a contradiction to the minimality assumption of amoeba bases.

As a first result of this section, we show that the boundary of an amoeba plays a key role for the comprehension of amoeba bases.

**Theorem 6.2.** Let \( I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{C}[z] \), where the \( f_j \) are linear polynomials with generic coefficients (i.e., the corresponding system of linear equations has full rank). Then one can compute a linear amoeba basis of length \( n + 1 \).
Proof. If the \( f_j \) have generic coefficients, then \( \mathcal{V}(I) \) contains a unique point \( v \in (\mathbb{C}^*)^n \) and hence \( \mathcal{A}(I) = \{w\} \) with \( \text{Log } |v| = w \in \mathbb{R}^n \). For \( 0 \leq j \leq n \) we define linear polynomials

\[
\begin{align*}
g_0 &= 1 + \frac{1}{||v||_1} \sum_{k=1}^{n} -e^{-i \arg(v_k)} z_k, \\
g_j &= 1 + \sum_{k \in \{1, \ldots, n\} \setminus \{j\}} e^{-i \arg(v_k)} z_k - \frac{1 + ||v||_1 - |v_j|}{v_j} \cdot z_j \quad \text{for } 1 \leq j \leq n,
\end{align*}
\]

where \( ||v||_1 = \sum_{k=1}^{n} |v_k| \). First, notice that every \( g_j(v) = 0 \) by construction and since all \( g_j \) are linear this implies \( \mathcal{V}(\langle g_0, \ldots, g_n \rangle) = v \). Since furthermore all \( f_j \) and \( g_j \) are linear, \( I \) and \( \langle g_0, \ldots, g_n \rangle \) equal their radical ideals and thus \( \langle g_0, \ldots, g_n \rangle = I \).

Furthermore, with this choice of coefficients for every \( g_j(v) \) the norm of the term in \( z_j \) (respectively the constant term for \( j = 0 \)) equals the sum of the norms of all other terms of \( g_j(v) \). More specific, we have

\[
\frac{1}{||v||_1} \sum_{k=1}^{n} | -e^{-i \arg(v_k)} \cdot v_k | = \frac{1}{||v||_1} \sum_{k=1}^{n} |v_k | = 1 \quad \text{for } g_0
\]

and

\[
1 + \sum_{k \in \{1, \ldots, n\} \setminus \{j\}} | -e^{-i \arg(v_k)} \cdot v_k | = 1 + ||v||_1 - |v_j| = \left| 1 + \frac{||v||_1 - |v_j|}{v_j} \cdot v_j \right| \quad \text{for } g_j.
\]

By Proposition 3.1 this implies \( w \in E_e(g_j) \cap \partial \mathcal{A}(g_j) \) for every \( 0 \leq j \leq n \) (recall at this point that \( E_e(g_j) \) denotes the complement component with order given by the \( j \)-th standard vector \( e_j \), where \( e_0 = 0 \), of the amoeba of \( g_j \)). Thus, we have \( w \in \bigcap_{k=0}^{n} \mathcal{A}(g_k) \) and hence \( \langle g_0, \ldots, g_n \rangle \) indeed forms an amoeba basis, if we can show that \( \bigcap_{k=0}^{n} \mathcal{A}(g_k) \) contains no other points. Assume, there exists another \( u \in (\mathbb{C}^*)^n \) with \( \text{Log } |u| \neq w \) and \( \text{Log } |u| \in \bigcap_{k=0}^{n} \mathcal{A}(g_k) \subset \mathbb{R}^n \). Then either \( ||u||_1 < ||v||_1 \) or there exists an entry \( u_j \) with \( |u_j| > |v_j| \). But this means either for \( g_0(u) \) that \( 1 > \sum_{k=1}^{n} |u_k| \) or for some \( g_j(u) \) that \( |u_j| > 1 + \sum_{k \in \{1, \ldots, n\} \setminus \{j\}} |u_k| \). Again, by Proposition 3.1 this implies that there exists some \( j \in \{0, \ldots, n\} \) with \( \text{Log } |u| \in E_e(g_j) \), which is a contradiction to the assumption that \( \text{Log } |u| \in \bigcap_{k=0}^{n} \mathcal{A}(g_k) \). Since \( v \) is computable with the Gauß algorithm, the computability of the amoeba basis follows. \( \square \)

Example 6.3. Let \( I = \langle f_1, \ldots, f_n \rangle \) be a zero-dimensional ideal such that all \( f_j \) are linear and the coefficient matrix of the defining set is stochastic, i.e., the entries of the coefficient vector of each \( f_j \) are positive real numbers summing up to 1. Hence, since the constant term is always 1, in particular \( \mathcal{V}(I) = \{-1\} \). Then \( (1 + \frac{1}{n} \sum_{k=1}^{n} z_k, 1 - \sum_{k=2}^{n} z_k + n z_1, \ldots, 1 - \sum_{k=1}^{n-1} z_k + n z_n) \) is an amoeba basis for \( I \). See Figure 6 for the case \( n = 2 \).

We close the section with an outlook on amoeba bases. For more general ideals a couple of problems arise. The first, and probably, main problem is that for general (non-linear) polynomials the boundary of an amoeba is not given by the lopsidedness equation. Thus, although it is now due to the results of this paper possible to describe the boundary and to compute it in dimension two, it remains unclear so far how to create polynomials such that their amoeba have certain points located on their boundary. A second problem is that for arbitrary ideals we need to place (at least) multiple points on the boundary. It is unclear how to guarantee this in general, in particular since the position of one point relative to the others is not unique if the ideal has at least \( n + 2 \) zeros, where \( n \) is the number of variables of the polynomial ring where
Figure 5. The amoeba basis \((1 + 0.5z_1 + 0.5z_2, 1 + 2z_1 - z_2, 1 - z_1 + 2z_2)\) for the zero-dimensional ideal given by linear polynomials with stochastic \(2 \times 2\) coefficient matrix. The figure shows that the amoebas of the basis elements only intersect in the origin, which is exactly \(\log |(-1, -1)| = A(I)\).

the ideal is defined in. So, even for ideals with dimension zero it remains unclear so far if an amoeba basis always exists, and, if it exists, how many polynomials have to be contained in it or which degree respectively support they need to have.

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