MODULARITY OF SOME NON–RIGID DOUBLE OCTIC CALABI–YAU THREEFOLDS

SLAWOMIR CYNK AND CHRISTIAN MEYER

INTRODUCTION

The modularity conjecture for Calabi–Yau manifolds predicts that every Calabi–Yau manifold should be modular in the sense that its $L$–series coincides with the $L$–series of some automorphic form(s). The case of rigid Calabi–Yau threefolds was (almost) solved by Dieulefait and Manoharmayum in [7, 6]. On the other hand in the non–rigid case it is even not clear which automorphic forms should appear.

Examples of non–rigid modular Calabi–Yau threefolds were constructed by Livné and Yui ([11]), Hulek and Verrill ([9, 10]) and Schütt ([17]). In these examples modularity means a decomposition of the associated Galois representation into two– and four–dimensional subrepresentations with $L$–series equal to $L(g_4, s)$, $L(g_2, s – 1)$ or $L(g_2 \otimes g_3, s)$, where $g_k$ is a weight $k$ cusp form. The summand with $L$–series equal to $L(g_2 \otimes g_3, s)$ is explained by a double cover of a product of a K3 surface and an elliptic curve (see [11]).

The $L$–series $L(g_2, s – 1)$ is the $L$–series of the product of the projective line $\mathbb{P}^1$ and an elliptic curve $E$ with $L(E, s) = L(g_2, s)$. A two–dimensional subrepresentation with such an $L$–series may be identified by a map $\mathbb{P}^1 \times E \rightarrow X$ which induces a non–zero map on the third cohomology (see [9]). Using an interpretation in terms of deformation theory we conjecture that a splitting of the Galois action into two–dimensional pieces can happen only for isolated elements of any family of Calabi–Yau threefolds.

In this paper we will study modularity of some non–rigid double octic Calabi–Yau threefolds, we will prove modularity of all examples listed in table [1] except $X_{154}$. We will use four methods for proving modularity, apart from the methods of Livné–Yui and Hulek–Verrill we will use two others based on giving a correspondence with a rigid Calabi–Yau
threefold or on an involution. We also observe that the splitting of the Galois action into two-dimensional pieces holds for those Calabi–Yau threefolds in the studied families having some additional geometric properties. The Calabi–Yau threefold $X_{154}$ is also the only one which we were not able to represent as a Kummer fibration associated to a fiber product of elliptic fibrations (cf. [15]).

1. Modular double octics with $h^{12} = 1$

Let $D$ be an arrangement of 8 planes in $\mathbb{P}^3$. If no six of the planes intersect in a point and no four in a line then the double covering of $\mathbb{P}^3$ branched along $D$ admits a resolution of singularities $X$ which is a smooth Calabi–Yau threefold (see [2]). The resolution of singularities is performed by blowing up singularities of the branch locus in the following order: fivefold points, fourfold points that do not lie on a triple line, triple lines, double lines. The Euler number of the resulting Calabi–Yau threefold can easily be expressed in numbers of different types of singularities. The Hodge number $h^{1,2}(X)$ (the dimension of the deformation space) can be computed as the dimension of the space of equisingular deformations of $D$ in $\mathbb{P}^3$; it can also be computed as the dimension of the equisingular ideal of $D$ (see [5]).

An extensive computer search in [12] produced 18 double octic Calabi–Yau threefolds with $h^{12} = 1$ (in 11 one-parameter families) for which

$$\text{tr}(\text{Frob}_p^* | H^3_{\text{ét}}(X)) = a_p + p \cdot b_p,$$

for all primes $5 \leq p \leq 97$, where $a_p$ (resp. $b_p$) are the coefficients of a weight four (resp. two) cusp form. This is a strong numerical evidence for modularity in the sense of splitting into two two-dimensional sub-representations. We list all these examples in Table 1. We include the no. of the arrangement (as in [12]), the equation, the expected modular form of level 4 and 2 (using W. Stein’s notation from [18]) and the Picard number $h^{11}$. Since the Calabi–Yau threefolds in the table coming from arrangements with the same no. are birational (see Lemma 3.1) we will use in this paper the notation $X_n$ for any Calabi–Yau threefold in the table constructed from arrangement no. $n$.

The Picard groups of all listed Calabi–Yau threefolds are generated by divisors defined over $\mathbb{Q}$, so Frobenius acts on $H^2_{\text{ét}}$ by multiplication with $p$. In fact, in all the examples except $X_{244}$, the skew-symmetric part of the Picard group is zero, whereas for $X_{244}$ it is generated by a divisor coming from the contact plane $x + y - z + t = 0$. 

| no. | equation: \( u^2 = xyzt \ldots \) | wt. 4 | wt. 2 | \( h^{11} \) |
|-----|---------------------------------|-------|-------|--------|
| 4   | \((x+y)(y+z)(x-y-z-t)(x+y-z-t)\) | 32k4A1 | 32A1  | 61     |
| 4   | \((x+y)(y+z)(x+2y+2z-t)(x+y+2z-t)\) | 32k4A1 | 32A1  | 61     |
| 4   | \(2(x+y)(y+z)(2x+y+z-2t)(2x+2y+z-2t)\) | 32k4A1 | 32A1  | 61     |
| 8   | \((x+y)(y+z)(-z+t)(3x-y-z+t)\) | 24k4A1 | 24A1  | 61     |
| 13  | \((x+y)(y+z)(x-z-t)(x-z-2t)\) | 32k4A1 | 32A1  | 61     |
| 13  | \((x+y)(y+z)(x-z-t)(x-z+t)\) | 32k4A1 | 32A1  | 61     |
| 21  | \((x+y)(y+z)(2x-y-t)(2x-z-2t)\) | 32k4A1 | 32A1  | 61     |
| 53  | \((x+y)(z+t)(x-y-z-t)(x+y-z-t)\) | 32k4B1 | 32A1  | 53     |
| 154 | \((x+y+z)(x+y+z-t)(-2x+y-3z+3t)(2x+3z-2t)\) | 8k4A1 | 72A1  | 41     |
| 244 | \((x+y+z+t)(x+y-z-t)(y-z+t)\) | 12k4A1 | 48A1  | 39     |
| 249 | \((x+y+z)(x+z+t)(2x+3y-z+2t)(y-z+2t)\) | 24k4A1 | 24A1  | 37     |
| 249 | \((x+y+z)(x+z+t)(2x-y+3z+2t)(-3y+3z+2t)\) | 24k4A1 | 24A1  | 37     |
| 267 | \((x+y-2z)(x-y-z+t)(2y-z+t)(x+y+z+t)\) | 96k4B1 | 96B1  | 37     |
| 267 | \((x+y+z)(x+2y-z+t)(-y+2z-2t)(2x+2y-z+2t)\) | 96k4B1 | 96B1  | 37     |
| 267 | \((2x+2y-z)(2x+y-2z+2t)(y+z-t)(x+y-2z+t)\) | 96k4B1 | 96B1  | 37     |
| 274 | \((x+y+z)(-x-z+t)(x+2y-z+t)(x+y-z+2t)\) | 96k4E1 | 96B1  | 37     |
| 275 | \((x+y+z)(2x-2z-t)(8y+4z+t)(2x+4y+t)\) | 96k4B1 | 96B1  | 37     |

Table 1.


2. **Double quartic elliptic fibrations**

In this section we will shortly review some information about rational elliptic fibrations that can be realized as a resolution of a double covering of $\mathbb{P}^2$ branched along a sum of four lines. The structure of the elliptic fibration is determined by the choice of a point in $\mathbb{P}^2$. Some of these surfaces were described in [4]; we will omit here all the details explained in that paper.

The double covering is rational exactly when the lines do not intersect in one point. We can have the following combinations of singular fibers (the Picard number $\rho(S_w)$ of a generic fiber can be computed from the Zariski lemma):

| singular fibers | $\rho(S_w)$ |
|-----------------|-------------|
| $S_1$           | $D_4^*, D_4^*$ | 1 |
| $S_2$           | $I_2, I_2, D_6^*$ | 1 |
| $S_3$           | $I_2, I_2, I_4, I_4$ | 1 |
| $S_4$           | $I_2, I_2, I_2, D_4^*$ | 2 |
| $S_5$           | $I_2, I_2, I_2, I_2, I_4$ | 2 |
| $S_6$           | $I_2, I_2, I_2, I_2, I_2, I_2$ | 3 |

A double covering of $S_1$ branched along the two singular fibers is birational to a product of $\mathbb{P}^1$ and an elliptic curve $E$, and all smooth fibers are isomorphic to $E$. This elliptic fibration depends on the $j$–invariant of $E$.

The surfaces $S_2$ and $S_3$ are extremal, i.e. they have $\rho(S_w) = 1$. Consequently they are uniquely defined as fiber spaces. Moreover the parameters corresponding to the singular fibers of $S_3$ form a harmonic quadruple (i.e. their cross–ratio equals $-1$); they can be chosen as

$$
\frac{-1}{I_2} \quad \frac{0}{I_4} \quad \frac{1}{I_2} \quad \frac{\infty}{I_4}
$$

Denote by $S'_3$ the pullback of $S_3$ via the involution $t \mapsto \frac{t-1}{t+1}$ of $\mathbb{P}^1$, so $S'_3$ has the following singular fibers:

$$
\frac{-1}{I_4} \quad \frac{0}{I_2} \quad \frac{1}{I_4} \quad \frac{\infty}{I_2}
$$

Thus $S_3$ and $S'_3$ have singular fibers at the same points but of different types. There exists an isogeny $\Psi : S_3 \mapsto S'_3$ which is a degree 2 unbranched covering on a smooth fiber.

Fibration $S_4$ is not extremal, so we can chose arbitrary coordinates of singular fibers. The configuration of lines is not uniquely determined by the coordinates of singular fibers. In fact there are exactly two types: one with a triple point and one with a “vertical line”.


The Picard number of the generic fiber of Fibration $S_5$ equals two, so we cannot choose arbitrary coordinates of singular fibers. In fact there is an involution of $\mathbb{P}^1$ which preserves the fiber $I_4$ and exchanges two pairs of $I_2$’s. The configuration of lines is uniquely determined.

Fibration $S_6$ is the most complicated one. In this case the configuration of lines is not uniquely determined. There can be several choices coming from automorphisms of $\mathbb{P}^1$ preserving the singular fibers.

3. Kummer fibrations

All examples in table I except $X_{154}$ can be realized as a Kummer fibration associated to a fiber product of elliptic fibrations (cf. [15]). Contrary to Schoen we do not require that the involution on the fiber product lifts to a resolution, so the resulting Calabi–Yau threefold is not necessarily a blow–up of the Kummer fibration.

To see the fibration we reorder the planes such that the first four and the last four intersect in a point. Then after change of coordinates in $\mathbb{P}^3$ we may assume that these points of intersection are $(0, 0, 0, 1)$ and $(1, 0, 0, 0)$, or equivalently that the double octic is given in weighted projective space $\mathbb{P}(1, 1, 1, 1, 4)$ by the equation

$$w^2 = f_1(x, y, z) \cdot \ldots \cdot f_4(x, y, z)f_5(y, z, t) \cdot \ldots \cdot f_8(y, z, t).$$

Consequently the double octic is birational to the quotient of the fiber product of elliptic fibrations

$$u^2 = f_1(x, y, z) \cdot \ldots \cdot f_4(x, y, z)$$

and

$$v^2 = f_5(y, z, t) \cdot \ldots \cdot f_8(y, z, t)$$

by the involution

$$(x, y, z, t, u, v) \mapsto (x, y, z, t, -u, -v).$$

In the following table we list descriptions of Calabi–Yau threefolds from table I as Kummer fibrations. For each Kummer fibration we give coordinates and types of singular fibers. In some cases we were able to
find two different representations as a Kummer fibration.

| X_4 | 0 1 2 3 ∞ | −1 0 1 ∞ |
|-----|------------|----------|
| I_2 I_2 I_2 I_2 I_4 | I_4 I_2 I_4 I_2 |
| I_0 I_2 I_2 I_0 D_6 * | D_6 * I_2 I_0 I_2 |
| 0 1 4 ∞ | |
| D_4 * I_2 I_2 I_2 | I_2 I_2 I_0 D_6 * |

| X_8 | |
|-----|---|
| I_2 I_2 I_0 D_6 * | |

| X_13 | 0 1 ∞ | −1 0 1 ∞ |
|-----|--------|----------|
| D_4 * I_2 I_4 I_0 | I_2 I_4 I_2 I_4 |
| I_2 I_2 D_6 * | I_0 D_4 * I_0 D_4 * |
| −1 0 1 ∞ | −1 0 1 ∞ |
| I_2 I_2 D_4 * I_2 I_2 | I_2 I_2 D_4 * I_2 I_2 |

| X_21 | |
|-----|---|
| I_2 I_2 I_0 D_6 * | |

| X_244 | −1 0 1 2 ∞ | −1 0 1 3 ∞ |
|-------|------------|------------|
| I_0 I_2 I_4 I_2 I_4 | I_2 I_2 I_4 I_0 I_2 |
| I_4 I_2 I_4 I_0 I_2 | |
| −1 0 1 3 ∞ |

| X_249 | |
|-----|---|
| I_0 I_2 I_2 I_4 I_2 I_2 | |
| I_2 I_4 I_0 I_2 I_0 I_4 |

| X_267 | |
|-----|---|
| I_2 I_2 I_2 I_2 I_2 I_2 | |

| X_274 | |
|-----|---|
| I_2 I_4 I_2 I_0 I_2 | |
| I_4 I_2 I_2 I_0 I_2 |

| X_275 | |
|-----|---|
| I_2 I_2 I_2 I_2 I_2 | |
| I_2 I_2 I_2 I_2 I_2 |

**Lemma 3.1.** The Calabi–Yau threefolds in table 1 defined by arrangements of the same type are birational. The Calabi–Yau threefolds X_21
and $X_{53}$ are birational; and the Calabi–Yau threefolds $X_{267}$ and $X_{275}$ are birational. There exists a correspondence between the Calabi Yau–threefolds $X_8$ and $X_{249}$.

**Proof.** From the explicit description of the fiber products in local co-ordinates it easily follows that the Calabi–Yau threefolds defined by arrangements of the same type with different parameters are in fact projectively equivalent.

Arrangement no. 21 is projectively equivalent to

$$x(x - z)(x + z)(x + y)(t + z)(t - z)(t + y) = 0.$$ 

Substituting the birational involution of $\mathbb{P}^3$ given by

$$(x, y, z, t) \mapsto (yz, xz, xy, tx)$$

we obtain

$$(xyz^2)x(x - z)(x + z)(x + y)z(t + y)(t - y)(t + z) = 0,$$

and since arrangement no. 53 is projectively equivalent to

$$x(x - z)(x + z)(x + y)z(t + y)(t - y)(t + z) = 0,$$

we conclude that the resulting Calabi–Yau threefolds are birational.

To prove that $X_{267}$ and $X_{275}$ are birational, observe that the corresponding arrangements are projectively equivalent to

**Arr. no. 267:**

$$x(x - z)(2x - 2z + y)(2x - z - y) \times t(t + z - y)(2y - z - 2t)(2z - y + 2t) = 0$$

**Arr. no. 275:**

$$x(x - z)(2x - 2z + y)(2x - z - y) \times t(2t - y)(2t - z)(3t - y - z) = 0.$$ 

Simple computations show that the cross ratios of the quadruples

$$0, \quad y - 1, \quad y - \frac{1}{2}, \quad \frac{1}{2}y - z$$

$$0, \quad \frac{1}{2}y, \quad \frac{1}{2}z, \quad \frac{1}{3}y + \frac{1}{3}z$$

are equal so there is a birational transformation in $y, z, t$ that maps one of them to the other.

To see the correspondence between the Calabi–Yau threefolds $X_8$ and $X_{249}$, first pull back arrangement no. 8 by the map $t \mapsto (\frac{t + 1}{t - 1})^2$, obtaining

$$\begin{array}{ccccccc}
-1 & 0 & \frac{1}{3} & 1 & 3 & \infty \\
I_0 & I_2 & I_2 & I_4 & I_2 & I_2 \\
I_4 & I_2 & I_0 & I_4 & I_0 & I_2
\end{array}$$
Now it is enough to compose this map with the isogeny of the elliptic fibration with fibers $I_4, I_4, I_2, I_2$ that exchanges $I_2$ fibers with $I_4$ fibers (see [4]).

**Remark 3.2.** Arrangements no. 267 and 275 are not projectively equivalent, they come from different twisted self–fiber products of the same elliptic fibration. The self–fiber product (without twist) of this elliptic fibration gives a non–birational Calabi–Yau threefold with $h^{12} = 2$ (see example [1]).

4. Ruled surface over elliptic curves

In this section we will use elliptic ruled surfaces to prove modularity of four Calabi–Yau threefolds from table [1].

**Proposition 4.1.** The Calabi–Yau threefolds $X_4$, $X_8$, $X_{244}$ and $X_{249}$ are modular, with modular forms as listed in table [7].

Consider a Calabi–Yau threefold $X$ such that an $L$–series of the form $L(g_2, s−1)$ (where $g_2$ is a weight two modular form corresponding to an elliptic curve $E$) appears in the Galois representation. Then by the Tate Conjecture we can expect that there is a correspondence between $X$ and the product $E \times \mathbb{P}^1$ which induces the isomorphism of representations.

Hulek and Verrill proved in [9] that when a smooth ruled surface over an elliptic curve $S \rightarrow E$ is contained in a Calabi–Yau threefold $X$ then the map on third cohomology $H^3(X) \rightarrow H^3(S)$ is surjective. The map can be represented by a direct sum of $H^1(T_X) \rightarrow H^1(N_{S|X})$ and its complex conjugate. The map $H^1(T_X) \rightarrow H^1(N_{S|X})$ associates to a deformation of $X$ the obstruction to lift it to a deformation of $E$ (inside $X$). Therefore if this map is non-zero then $E$ deforms inside $X$ only over a codimension one submanifold of the Kuranishi space of $X$.

Now, if we have ruled surfaces $E_1, \ldots, E_r$, with $r = h^{21}(X)$, such that the map

\[(1) \quad H^3(X) \rightarrow \bigoplus_i H^3(E_i)\]

is surjective then the obstructions are independent and the surfaces do not deform simultaneously over any subvariety of the Kuranishi space of $X$ of positive dimension. It is an explanation why in a family there were always only finitely many examples were one was able to prove modularity in that way.

If we have several ruled surface over elliptic curves, it is usually difficult to determine whether the map (1) is surjective. In case we know the Kuranishi space of $X$ we can try to invert the above argument. For
each elliptic fibration we consider the hypersurface $V_i$ of the Kuranishi space over which $E_i$ deforms, knowing that the kernel of (1) is the tangent to the intersection of the $V_i$’s plus its complex conjugate (see example at the end of this section).

To use this method in our examples we need to find elliptic fibrations inside the double octics. If a plane $S$ in $\mathbb{P}^3$ contains two double lines and the other four arrangement planes intersect at a point in $S$, then the pullback of $S$ to the double covering is an elliptic fibration. On the Kummer fibration these planes are recognized as corresponding to the product of fibers $I_0$ and $I_4$.

We were able to find such a plane only for two arrangements:

**Arrangement no. 4:** the plane $S$ has equation $x - z = 0$ resp. $y + 2z - t = 0$ resp. $2x + y - 2t = 0$ (for the three arrangements in the table).

**Arrangement no. 244:** the plane $S$ has equation $x + y + z - t = 0$.

To prove modularity of $X_8$ and $X_{249}$ we will study an auxiliary Calabi–Yau threefold $X_{269}$ with $h^{12} = 2$. Modularity of this Calabi–Yau threefold follows from existence of some elliptic ruled surfaces and their behavior under deformations.

**Example 1.** Consider the double octic Calabi–Yau threefold $X_{269}$ defined by the following arrangement of eight planes (arrangement no. 269 in [12]):

$$xyz(x + y + z)(x + 2y - z + t)(y + z - t)(x + y - 2z + t) = 0$$

It has $h^{2,1}(X_{269}) = 2$. Substituting $y = y - z, z = z + t$ we can represent this Calabi–Yau threefold as the following Kummer fibration:

$$
\begin{array}{cccccc}
-1 & 0 & \frac{1}{3} & 1 & 3 & \infty \\
I_2 & I_2 & I_2 & I_2 & I_2 & I_2 \\
I_2 & I_2 & I_2 & I_2 & I_2 & I_2 \\
\end{array}
$$

On the other hand substituting $x = x + 2z - 4y, z = x - 2y$ we can also obtain the following Kummer fibration:

$$
\begin{array}{cccccc}
-1 & 0 & \frac{1}{3} & 1 & 3 & \infty \\
I_0 & I_2 & I_4 & I_2 & I_2 & I_2 \\
I_4 & I_2 & I_0 & I_4 & I_0 & I_2 \\
\end{array}
$$

Hence using the isogeny between $S_3$ and $S_2^5$ from section [2] we can find correspondences between this Calabi–Yau threefold and the Calabi–Yau threefolds $X_8$ and $X_{249}$.

Observe that the planes $z = x + 2y$ and $y = 2z - t$ contain two double lines and a fourfold point, so they give two ruled surfaces $E_1, E_2$ over an elliptic curve with conductor 24.
The Kuranishi space of the Calabi–Yau threefold $X_{269}$ may be parametrized by the equation

$$xyzt(x + y + z)(Bx + Cy - Az + At) \times \times (y + z - t)(Bx + By + (-A + B - C)z + At) = 0.$$ 

By [10] both elliptic fibrations give non-zero maps

$$H^3(X) \longrightarrow H^3(E_i)$$

so they deform over curves in $\mathbb{P}^2$. One easily checks that they deform over the lines given by

$$A + B - C = 0, \quad C = 2B,$$

which intersect only at the point $(1, 1, 2)$ corresponding to the equation we started with. Consequently the obstructions are independent and the map

$$H^3(X) \longrightarrow H^3(E_1) \oplus H^3(E_2)$$

is surjective, giving a splitting of the representation on $H^3$ into two-dimensional pieces. Counting points over $\mathbb{F}_p$ for $p \leq 97$ one checks that $X$ is modular and that the coefficients of the $L$-series are given by $b_p + 2pc_p$, where $b_p$ resp. $c_p$ are the coefficients of the unique cusp form of level 24 and weight 4 resp. 2.

There is a degree two correspondence between the above Calabi–Yau threefold and $X_{249}$, hence also $X_8$. These correspondences prove the modularity of $X_8$ and $X_{249}$.

5. CORRESPONDENCES WITH RIGID DOUBLE OCTICS

In this section we will use correspondences between rigid and non–rigid Calabi–Yau threefolds to prove modularity of the latter.

**Proposition 5.1.** The Calabi–Yau threefolds $X_4$, $X_{21}$, $X_{53}$ and $X_{244}$ are modular, with modular forms as listed in table 1.

In [3] we checked the modularity and computed modular forms of some rigid double octic Calabi–Yau threefolds. Now we will use correspondences between some rigid and non–rigid Calabi–Yau threefolds to show the modularity of the latter.

We first recall the considered rigid examples. As before we will use the equations and numbers of arrangements from [12] (in brackets we give the numbers from [3]).

**Arrangement no. 3** (old no. 6) is given by the equation

$$xyzt(x + y)(y + z)(z + t)(t + x) = 0.$$
The corresponding fiber product of elliptic fibrations has singular fibers

\[
\begin{array}{cccc}
I_4 & I_4 & I_2 & I_2 \\
D_6^* & I_2 & I_2 & I_0
\end{array}
\]

**Arrangement no. 19** (old no. 23) is given by the equation

\[
xyz(x+y)(y+z)(x-z-t)(x+y+z-t) = 0.
\]

The corresponding fiber product of elliptic fibrations has singular fibers

\[
\begin{array}{cccc}
I_2 & I_2 & I_4 & I_4 \\
I_0 & D_6^* & I_2 & I_2
\end{array}
\]

**Arrangement no. 239** (old no. 86\textsuperscript{a}) is given by the equation

\[
xyz(x+y+z)(x+y+t)(x+z+t)(y+z+t) = 0.
\]

The corresponding fiber product of elliptic fibrations has singular fibers

\[
\begin{array}{cccc}
I_2 & I_2 & I_4 & I_4 \\
I_0 & I_4 & I_2 & I_4
\end{array}
\]

**Lemma 5.2.** There are correspondences between the Calabi–Yau threefolds given by the following arrangements:

1. No. 4 and no. 19,
2. No. 21 and no. 3,
3. No. 53 and no. 3,
4. No. 244 and no. 239.

**Proof.** All the correspondences are in fact defined on the level of the fiber products of elliptic fibrations. They are given by applying the isogeny of the elliptic fibration with fibers \(I_2, I_2, I_4, I_4\) that exchanges the fibers \(I_2\) and \(I_4\). \(\square\)

Assume that we have a generically finite correspondence between two Calabi–Yau threefolds \(X\) and \(Y\). Then this correspondence induces an isomorphism between \(H^{3,0}(X)\) and \(H^{3,0}(Y)\) coming from a pullback of the canonical form. If \(Y\) is rigid then taking this isomorphism plus its complex conjugate we obtain a splitting of the Galois representation on \(H^3(X)\) into a two-dimensional representation isomorphic to \(H^3(Y)\) and its complement. Using the correspondences from the above lemma and counting points in \(\mathbb{F}_p\) for \(p \leq 97\) we obtain proposition 5.1.

### 6. Kummer construction

In this section we will use the Kummer construction studied by Livné and Yui (\cite{LivneYui}).

**Proposition 6.1.** The Calabi–Yau threefold \(X_{13}\) is modular, with modular forms as listed in table 7.
We will consider a two-dimensional family of double octic Calabi–Yau threefolds which are the quotient by an involution of a product of a K3 surface studied in [1] and an elliptic curve. Take the elliptic curve
\[ E_\mu = \{ (x, t, u) \in \mathbb{P}(1, 1, 2) : u^2 = (x - t)(x^2 - \mu t^2)t \} \]
and the K3 surface
\[ S_\lambda = \{ (y, z, t, v) \in \mathbb{P}(1, 1, 1, 3) : v^2 = yzt(y + t)(z + t)(y + \lambda z) \} . \]

On the product \( Y_{\lambda, \mu} := E_\mu \times S_\lambda \) we have a natural involution
\[(x, t, u), (y, z, t, v) \mapsto (x, t, -u), (y, z, t, -v) . \]

The quotient \( X_{\lambda, \mu} \) of \( Y_{\lambda, \mu} \) by this involution has a Calabi–Yau nonsingular model. To show this observe that \( Y_{\lambda, \mu} \) is birational to the double covering of \( \mathbb{P}^3 \) branched along the octic \( D_{\lambda, \mu} \) given by the equation
\[(x - t)(x^2 - \mu t^2)yz(y + t)(z + t)(y + \lambda z) = 0 . \]

The birational map can be given by in appropriate affine coordinates \((t = 1)\) by
\[(x, 1, u), (y, z, 1, v) \mapsto (x, y, z, uv) . \]

The octic itself is defined over \( \mathbb{Q} \). Over \( \mathbb{Q}[\sqrt{\mu}] \) it splits into a sum of eight planes (for general \( \mu \), two of them are not defined over \( \mathbb{Q} \)). Using [2] we conclude that \( X_{\lambda, \mu} \) has a nonsingular model \( \tilde{X}_{\lambda, \mu} \) which is a Calabi–Yau threefold.

For general values of \( \lambda \) and \( \mu \), the arrangement \( D_{\lambda, \mu} \) is arrangement no. 52 in [12], so \( \tilde{X}_{\lambda, \mu} \) has the invariants \( h^{11}(\tilde{X}_{\lambda, \mu}) = 56 \) and \( h^{12}(\tilde{X}_{\lambda, \mu}) = 2 \).

For \( \lambda \neq 0, -1 \), the rank of the symmetric part of the Picard group of the K3 surface \( S_\lambda \) is 19; denote by \( H^2_{\text{skew}}(S) \) the three-dimensional skew-symmetric part. Thus there is a Shioda–Inose structure on \( S_\lambda \), namely there exists an involution on \( S_\lambda \) such that the quotient of \( S_\lambda \) by that involution is a Kummer surface.

In [11] it is proved that the surface \( S_\lambda \), with \( \lambda \in \mathbb{Q} \setminus \{0, -1\} \), is modular exactly when \( \lambda \in \{1, 8, 1/8, -4, -1/4, -64, -1/64\} \), and the modular form for \( S_\lambda \) is computed. We have the following diagram of rational maps
\[ Y_{\lambda, \mu} \longrightarrow \tilde{X}_{\lambda, \mu} \]
\[ X_{\lambda, \mu} \]

The rational map \( Y_{\lambda, \mu} \longrightarrow \tilde{X}_{\lambda, \mu} \) can be resolved by blowing up at points and lines so it induces a well defined map in cohomologies \( H^3(X_{\lambda, \mu}) \longrightarrow \)
$H^3(Y_{\lambda,\mu})$. The image of the map is invariant under the involution on $Y_{\lambda,\mu}$, so in fact we obtain a map $H^3(X_{\lambda,\mu}) \to H^1(E_{\mu}) \otimes H^2_{skew}(S_{\lambda})$. From the description of deformations of double coverings of smooth algebraic varieties \(^5\) it follows that this map is surjective, moreover both vector spaces have dimension 6, so it is an isomorphism. We obtain

**Proposition 6.2.** $H^3(X_{\lambda,\mu}) \cong H^1(E_{\mu}) \otimes H^2_{skew}(S_{\lambda})$.

**Corollary 6.3.** The Calabi–Yau threefold $X_{\lambda,\mu}$ is modular for $\lambda \in \{1, 8, 1/8, -4, -1/4, -64, -1/64\}$ and $\mu \in \mathbb{Q} \setminus \{0, 1\}$.

For the seven values of $\lambda$ the $L$–series of $S_\lambda$ corresponds to a cusp form for $S_3(\Gamma_1(8))$, $S_3(\Gamma_1(16))$, $S_3(\Gamma_1(12))$, $S_3(\Gamma_1(7))$ (for $\lambda$ and $1/\lambda$ the $L$–series differ only by a twist). They are the only $\eta$–product weight 3 modular forms. The modular form of the surface $S_\lambda$ corresponds to the symmetric power of the modular form associated to the elliptic curve $E_{\lambda,\mu}^3$ (see \([1]\)). For the seven special values of $\lambda$ the elliptic curve $E_{\lambda,\mu}^3$ has complex multiplication. Denoting by $a_p$ resp. $b_p$ the Fourier coefficients of the level 2 (resp. level 3) modular forms we get

$$b_p = \begin{cases} a_p^2 - 2p, & \frac{-(\lambda+1)}{p} = 1 \\ 0, & \frac{-(\lambda+1)}{p} = -1. \end{cases}$$

The Fourier coefficient of the $L$–series of $S_\lambda$ equals $\left(\frac{-(\lambda+1)}{p}\right)(b_p + p)$.

The third symmetric power of a weight 2 form yields also a weight 4 modular form with Fourier coefficients

$$c_p = a_p^3 - 3pa_p,$$

so we obtain

$$a_pb_p = c_p + pa_p.$$

Consequently we get much better modularity properties for the threefolds $X_\lambda := X_{\lambda,\mu}^3$.

**Proposition 6.4.** The $L$–series of the Calabi–Yau threefold $X_\lambda$ has Fourier coefficients equal to

$$c_p + 2pa_p.$$

In the table we collect the data for the four Calabi–Yau threefolds the $L$–series of which do not only differ by a twist:
**6.1. Singular K3.** From the above considerations we excluded the case of $\lambda = -1$. There are two reasons for this. First, in this case all divisors on the K3 surface are symmetric and consequently $h^{12}(\tilde{X}_{-1,\mu}) = 1$ (this is arr. no 13). Second, $\frac{1}{\lambda + 1}$ makes no sense. We can however take in that case also the curve $E_{1/g}$, as the modular forms appearing in $S_{-1}$ and $S_8$ are the same. Hence for the Calabi–Yau threefold $\tilde{X}_{-1,1/g}$ the modular form has coefficients $c_p + p\alpha_p$, where $c_p$ resp. $\alpha_p$ are coefficients of a weight 4 resp. 2 level 32 newform.

In the above considerations we can replace the elliptic curve $E_{\frac{1}{11}}$ by another elliptic curve with the same modular form, or replace both $E_\mu$ and $S_\lambda$ by some twist.

Now fix $\lambda \in \{1, 8, \frac{1}{2}, -4, -\frac{1}{4}, -64, -\frac{1}{64}\}$. Using [1] we can compute the characteristic polynomial of Frobenius on $H^3$ for the Calabi–Yau threefold $\tilde{X}_{\lambda,\mu}$ for any rational $\mu \neq 0, -1$. Denoting by $\alpha_p, \beta_p$ resp. $\bar{\alpha}_p, \bar{\beta}_p$ the eigenvalues of Frobenius on $H^1(E_{\lambda,\mu})$ resp. $H^1(E_{\mu,\mu})$ we find that the characteristic polynomial of Frobenius acting on $H^3(\tilde{X}_{\lambda,\mu})$ is (up to sign)

$$(T - p\beta_p)(T - p\bar{\beta}_p)(T - \alpha_p^2\beta_p)(T - \alpha_p^2\bar{\beta}_p)(T - \bar{\alpha}_p^2\beta_p)(T - \bar{\alpha}_p^2\bar{\beta}_p).$$

This polynomial splits over $\mathbb{Z}$ into the characteristic polynomial of the Frobenius action on $H^2(\mathbb{P}^1 \times E_{\mu,\mu})$ and the degree 4 polynomial $(T - \alpha_p^2\beta_p)(T - \alpha_p^2\bar{\beta}_p)(T - \bar{\alpha}_p^2\beta_p)(T - \bar{\alpha}_p^2\bar{\beta}_p)$. In the construction, this splitting comes from the cartesian product of $E_\mu$ and a transcendental cycle on the K3 surface $S_\mu$: it should have a better geometric interpretation via the Shioda–Inose structure.

If the elliptic curves $E_\lambda$ and $E_\mu$ are non–isogenous, the degree 4 polynomial does not divide by the characteristic polynomial of $\mathbb{P}^1 \times E$, for any elliptic curve $E$. To see this, denote the eigenvalues of Frobenius on $H^1(E)$ by $\gamma_p, \bar{\gamma}_p$ and assume that $p\gamma_p = \bar{\beta}_p\alpha_p^2$. Multiplying by $\beta_p$

| $\lambda$ | $\lambda = 1$ | $\lambda = 8$ | $\lambda = -4$ | $\lambda = -64$ |
|----------|----------------|----------------|----------------|----------------|
| wt 2 form | $256k^2D$ | $32k^2A$ | $144k^2B$ | $49k^2A$ |
| wt 3 form | $8k^3A[1, 1]$ | $16k^3A[1, 0]$ | $12k^3A[0, 1]$ | $7k^3A[3]$ |
| wt 4 form | $256k^4H$ | $32k^4A$ | $144k^4A$ | $49k^4D$ |
| $b_p = a_p^2 - 2p$ | $p \equiv 1, 3(8)$ | $p \equiv 3(4)$ | $p \equiv 1(3)$ | $p \equiv 1, 2, 4(7)$ |
| $b_p = 0$ | $p \equiv 5, 7(8)$ | $p \equiv 1(4)$ | $p \equiv 2(3)$ | $p \equiv 3, 5, 6(7)$ |
| $\eta$–products (wt 3) | $\eta^2(z)\eta^2(2z)$ | $\eta^6(4z)$ | $\eta^3(2z)\eta^3(6z)$ | $\eta^3(z)\eta^3(7z)$ |
| $\eta$–products (wt 2) | $\eta^2(4z)\eta^2(8z)$ | $\eta^2(8z)\eta^2(4z)$ | $\eta^{12}(12z)\eta^4(24z)\eta^4(6z)$ | $-$ |
and dividing by $p = |\beta|^2$ we get $\beta p \gamma = \alpha_p^2$. Since $E_\lambda$ has complex multiplication, looking at the sets of primes $p$ for which the coefficients $\alpha_p$, $\beta_p$, and $\gamma_p$ equal $\pm ip^{1/2}$ we easily see that the other two elliptic curves have complex multiplication by the same quadratic field and so up to a twist the three weight two forms coincide. In particular $E_\lambda$ and $E_\mu$ are isogenous.

7. Involution

In this section we will use an involution on a Calabi–Yau threefold to split the cohomology group $H^3$. Note that van Geemen and Nygaard ([8]) were the first to use an automorphism of a Calabi–Yau manifold to split the Galois representation and prove modularity.

**Proposition 7.1.** Calabi–Yau threefolds $X_{53}$, $X_{244}$, $X_{267}$, $X_{274}$ and $X_{275}$ are modular, with modular forms as listed in table 7.

On some of the Calabi–Yau threefolds considered in this paper we can find an involution. On the middle cohomology the involution may have only eigenvalues $\pm 1$. If both 1 and $-1$ are eigenvalues then the map gives us a splitting of $H^3$. Since the splitting is compatible with the Frobenius morphism it is in fact a splitting of the Galois representation into two–dimensional subrepresentations.

We can use the Lefschetz formula to compute the trace of Frobenius composed with the involution. This trace is equal to the trace of Frobenius on the $+1$–eigenspace minus the trace of Frobenius on the $-1$–eigenspace. Together with the trace of Frobenius on $H^3$ this gives the traces on the two subspaces.

Assume that we have a $\mathbb{Q}$–linear involution on $\mathbb{P}^3$ which preserves the arrangement of eight planes. This map induces an involution $\Phi : X \rightarrow X$ on the Calabi–Yau threefold $X$ defined by this arrangement. We will compute the trace

$$d_p = \text{tr}((\text{Frob}_p \circ \Phi)^*|H^3(\bar{X}_p, \mathbb{Q}_l))$$

of Frobenius composed with $\Phi$. Since this map acts by multiplication with $\pm p$ on $H^2$ and with $\pm p^2$ on $H^4$ the Lefschetz fixed–point formula relates $d_p$ to the number $N_p$ of fixed points of $\text{Frob}_p \circ \Phi$.

**Lemma 7.2.** If $\Phi$ is a linear involution on $\mathbb{P}^N(\mathbb{F}_p)$ defined over $\mathbb{F}_p$ then the fixed points of $\text{Frob}_p \circ \Phi$ are $\mathbb{F}_{p^2}$–rational.

**Proof.** The Frobenius morphism $\text{Frob}_p$ commutes with any linear involution defined over $\mathbb{F}_p$, so any fixed point of $\text{Frob}_p \circ \Phi$ is also a fixed point of $\text{Frob}^p$.
Using the Lemma we reduce the counting of fixed points over the infinite field $\mathbb{F}_p$ to counting of points over the finite field $\mathbb{F}_{p^2}$, which can easily be done using a computer.

From the representation as a Kummer fibration we can easily recognize some linear involutions preserving the arrangement:

**Arr. no. 53:** $(x, y, z, t) \mapsto (y, x, -t, -z)$

**Arr. no. 244:** $(x, y, z, t) \mapsto (y, x, -t, -z)$

**Arr. no. 267:** $(x, y, z, t) \mapsto (t, -z, -y, x)$

**Arr. no. 274:** $(x, y, z, t) \mapsto (z, -t, x, -y)$

Simple computations show that the above involutions are not equal to identity on the deformation space $H^1(T_X) \cong H^{12}(X)$, hence they split the Galois representations. In fact it is easy to observe that $H^{12}(X) \oplus H^{21}(X)$ must be $(-1)$–eigenspaces. Counting fixed points on the singular double octic yields for all primes $5 \leq p \leq 97$:

\[
\begin{align*}
X_{53} : & \quad 1 + p^3 - a_p + pb_p + p^2 + p \\
X_{244} : & \quad \begin{cases} 
1 + p^3 - a_p + pb_p + 2p^2 - p, & p \equiv 1 \mod 4 \\
1 + p^3 - a_p + pb_p + 3p, & p \equiv 3 \mod 4
\end{cases} \\
X_{267} : & \quad 1 + p^3 - a_p + pb_p + p^2 - p \\
X_{274} : & \quad \begin{cases} 
1 + p^3 - a_p + pb_p + p^2 - p, & p \equiv 1 \mod 4 \\
1 + p^3 - a_p + pb_p + p^2 + 3p, & p \equiv 3 \mod 4
\end{cases}
\end{align*}
\]

Analyzing the action of Frobenius on the generators of the Picard group and the space of curves $H^4$ gives the traces of Frobenius of the two–dimensional Galois subrepresentations. Applying the Faltings–Serre-Livn´e method finishes the proof.

M. Sch¨ utt suggested to us that counting points in $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ we can compute the characteristic polynomial, which factors into two degree two polynomials. Since we know that the representation splits we get the traces of both actions. It is however not straightforward that the numbers $a_p$ resp. $pb_p$ will correspond to the $+1$–eigenspace resp. the $-1$–eigenspace.

**Remark 7.3.** The described involutions act on singular double octics. Since the resolution of singularities of a double octic is not unique (it depends on the order in which we blow up lines in a triple point) it may happen that an involution maps to a birational Calabi–Yau threefold. Since two smooth models differ by a sequence of flops, we can compose the involution with these flops or we can consider a threefold that dominates both smooth models. The action on $H^3$ is well defined.
If we know that we can choose such a resolution of singularities of the double covering to which the involution lifts, then the quotient will be (after resolution) a rigid Calabi–Yau threefold.

**Example 2.** Consider the arrangement of planes (arr. no. 287 in [12]) given by

\[
xyzt(x + y + z - 3t)(x + y - 3z + t)(x - 3y + z + t)(-3x + y + z + t) = 0.
\]

The corresponding Calabi–Yau threefold \(X_{287}\) has Hodge numbers \(h_{11}(X_{287}) = 37, h_{12}(X_{287}) = 3\). Counting points in \(\mathbb{F}_p\) shows that, for \(5 \leq p \leq 97\), the trace of Frobenius on the middle cohomology equals \(a_p + 3b_p\), where \(a_p\) resp. \(b_p\) are the coefficients of the weight 4 level 6 resp. weight 2 level 24 cusp form. The arrangement has many linear symmetries. We can use the induced involutions on \(X\) to decompose the Galois representation.

We can also use the elliptic fibrations on \(X\) described in [12] and apply the deformation argument from example 1 to prove modularity of \(X_{287}\).

In fact the full permutation group \(S_4\) acts on this Calabi–Yau threefold. If we consider the action of permutations of order 3, then the eigenvalues will be defined in \(\mathbb{F}_p\) only for some \(p\), so the decomposition of Frobenius action will depend on \(p\).

**Acknowledgements.** The work on this paper was done during the first named author’s stays at the Institutes of Mathematics of the Johannes Gutenberg-Universität Mainz and the Universität Hannover. He would like to thank both institutions for their hospitality. The authors also would like to thank Prof. Duco van Straten, Prof. Klaus Hulek and Matthias Schütt for their help.

**References**

[1] S. Ahlgren, K. Ono, D. Penniston, *Zeta functions of an infinite family of K3 surfaces*. Amer. J. Math. 124 (2002), no. 2, 353–368.

[2] S. Cynk, *Double coverings of octic arrangements with isolated singularities*, Adv. Theor. Math. Phys. 3 (1999), 217–225.

[3] S. Cynk, Ch. Meyer, *Geometry and Arithmetic of Certain Double Octic Calabi–Yau Manifolds*, Canadian Math. Bull. 48 (2005), no. 2, 180–194.

[4] S. Cynk, Ch. Meyer, *Modular Calabi–Yau threefolds of level eight*, preprint [math.AG/0504070]

[5] S. Cynk, D. van Straten, *Infinitesimal deformations of smooth algebraic varieties*, to appear in Math. Nachrichten, preprint [math.AG/0303329]

[6] L. Dieulefait, *From potential modularity to modularity for integral Galois representations and rigid Calabi-Yau threefolds*, preprint [math.NT/0409102]
[7] L. Dieulefait, J. Manoharmayum, *Modularity of rigid Calabi-Yau threefolds over* $\mathbb{Q}$ *in*: N. Yui, J. D. Lewis (eds.): Calabi-Yau Varieties and Mirror Symmetry (Toronto 2001), Fields Inst. Comm. 38, AMS (2003), 159–166.

[8] B. van Geemen, N. Nygaard, *On the geometry and arithmetic of some Siegel modular threefolds*, J. Number Theory 53 (1995), no. 1, 45–87.

[9] K. Hulek, H. Verrill, *On modularity of rigid and nonrigid Calabi-Yau varieties associated to the root lattice* $A_4$, to appear in Nagoya Math. Journal, preprint [math.AG/0304169](http://arxiv.org/abs/math.AG/0304169).

[10] K. Hulek, H. Verrill, *On the modularity of Calabi-Yau threefolds containing elliptic ruled surfaces*, preprint [math.AG/0502158](http://arxiv.org/abs/math.AG/0502158).

[11] R. Livné, N. Yui, *The modularity of certain non-rigid Calabi-Yau threefolds*, preprint [math.AG/0304497](http://arxiv.org/abs/math.AG/0304497).

[12] Ch. Meyer, *A Dictionary of Modular Threefolds*, Thesis, Mainz (2005).

[13] R. Miranda, U. Persson, *On extremal rational elliptic surfaces*, Math. Z. 193 (1986), no. 4, 537–558.

[14] M.-H. Saito, N. Yui, *The modularity conjecture for rigid Calabi-Yau threefolds over* $\mathbb{Q}$, J. of Math. Kyoto Univ. 41 (2001), no. 2, 403–419.

[15] C. Schoen, *On Fiber Products of Rational Elliptic Surfaces with Section*, Math. Z. 197 (1988), 177–199.

[16] M. Schütt, *New examples of modular rigid Calabi-Yau threefolds*, Collectanea Mathematica 55 (2004), no. 2, 219–228.

[17] M. Schütt, *On the modularity of three Calabi-Yau threefolds* with *bad reduction at* $11$, to appear in Canadian Math. Bull., preprint [math.AG/0405450](http://arxiv.org/abs/math.AG/0405450).

[18] W. A. Stein, *Modular forms database*, [http://modular.fas.harvard.edu/](http://modular.fas.harvard.edu/)

[19] N. Yui, *Update on the modularity of Calabi-Yau varieties*, with an appendix by H. Verrill. Fields in: N. Yui, J. D. Lewis (eds.): Calabi-Yau Varieties and Mirror Symmetry (Toronto 2001), Fields Inst. Comm. 38, AMS (2003), 307–362.

**Instytut Matematyki, Uniwersytetu Jagiellońskiego, ul. Reymonta 4, 30–059 Kraków, Poland**

**Current address**: Institut für Mathematik, Universität Hannover, Welfengarten 1, D–30060 Hannover, Germany

**E-mail address**: s.cynk@im.uj.edu.pl

**Fachbereich Mathematik und Informatik, Johannes Gutenberg-Universität, Staudingerweg 9, D–55099 Mainz, Germany**

**E-mail address**: cm@mathematik.uni-mainz.de