Luminosity distance–redshift relation for the LTB solution near the centre

Masayuki Tanimoto and Yasusada Nambu

Department of Physics, Graduate School of Science, Nagoya University, Chikusa, Nagoya 464-8602, Japan

E-mail: tanimoto@gravity.phys.nagoya-u.ac.jp and nambu@gravity.phys.nagoya-u.ac.jp

Received 2 March 2007, in final form 28 May 2007
Published 17 July 2007
Online at stacks.iop.org/CQG/24/3843

Abstract
Motivated by the inverse problem for the Lemaître–Tolman–Bondi dust solution in which problem the luminosity distance function $D_L(z)$ is taken as an input to select a specific model, we compute the function $D_L(z)$ of the LTB solution up to the third order of $z$. To perform the otherwise cumbersome computation, we introduce a new convenient form of the LTB solution, in which the solution is explicit and unified. With this form of the LTB solution we obtain the luminosity distance function with full generality. We, in particular, find that the function exactly coincides with that of a homogeneous and isotropic dust solution up to second order, if we demand that the solution be regular at the centre.

PACS numbers: 95.30.Sf, 95.36.+x, 98.80.Jk

1. Introduction
Modern observations show that the universe is presently in an accelerating phase and dominated by dark energy with negative pressure [1–5]. It is remarkable that there seems no obvious contradiction among the observations, but the true nature of dark energy is a great mystery. On the other hand, it is also true that this recognition is a result of the strict use of a Friedmann–Lemaître–Robertson–Walker (FLRW) homogeneous and isotropic model. If inhomogeneities are properly taken into account it might be possible to explain the observations without introducing dark energy. Recently, this possibility has renewed interest in inhomogeneous cosmological models, especially in the so-called Lemaître–Tolman–Bondi (LTB) solution [6–8], which represents a spherically symmetric dust-filled universe. Because of its simplicity this solution has been considered to be most useful to evaluate the effect of inhomogeneities in the observables like the luminosity distance–redshift relation.

Although this solution is simple, it has a great flexibility. Various models have been proposed that are consistent with the observations [9–15]. But, what is the best configuration
that can explain the observations of, e.g., luminosity distance–redshift relation for type Ia supernovae and still be consistent with other observations? To respond to this question one needs a systematic approach.

Our primary focus is the ‘inverse problem’ approach in which one takes a luminosity distance function $D_L(z)$ as an input to select a specific LTB model. Mustapha et al [16] argued that the free functions in the LTB solution can be chosen to match any given luminosity distance function $D_L(z)$ and source evolution function $m(z)$. Célérier [9] performed an expansion of $D_L(z)$ for the LTB solution of parabolic type to fit the $\Lambda$CDM model with $\Omega_m = 0.7$ and $\Omega_m = 0.3$, and argued that the model can explain the SNIa observation at least for $z \lesssim 1$. Vanderveld et al [17], however, showed that such a fitting with an accelerating model is only possible at the cost of occurrence of a weak singularity at the centre.

The main purpose of this paper is to find a Taylor expansion of $D_L(z)$ for the LTB solution in the most general form, not restricted to the parabolic type, with considerations for the condition to avoid the weak singularity. Since the differential equation for the inverse problem is, in general, singular at the origin $z = 0$ it is imperative to prepare a solution there using a different method. If we have the expansion of $D_L(z)$ for the general LTB solution, we can easily find this solution by comparing the given $D_L(z)$ and that of the LTB solution order by order.

To perform the computation we present a new way of representing the LTB solution. Although this is not our main purpose we would like to stress that this new representation would be very useful in various computations concerning the LTB solution. This solution is usually expressed with parametric functions, as is the FLRW dust solution. This parametric character of the solution often makes our analysis and considerations complicated and non-transparent. Moreover those functions change their forms depending on whether the solution is of spherical type, parabolic type or hyperbolic type. This split-into-cases character is also a drawback of the conventional expression of the solution. The new expression of the solution dissolves all these unwanted characters. It will play an essential role in achieving the complicated computations needed to compute $D_L(z)$ in the fully general setting.

The structure of this paper is as follows. In section 2, we introduce the new unified form of the LTB solution. In section 3, we perform the expansion of $D_L(z)$ using the unified expression. In section 4, we study how different the function $D_L(z)$ for the LTB solution is from that of the FLRW dust solution. Section 5 is devoted to conclusion. The appendix presents the result without the regularity condition at the centre.

2. The LTB solution in a unified new form

Let us first gather the conventional forms of the solution. The LTB metric is written in the form

$$ds^2 = -dt^2 + \frac{R'(t,r)^2}{1 + 2E(r)} dr^2 + R(t,r)^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

where primes denote derivatives with respect to the radial coordinate $r$ and $E(r)$ is a free function called the ‘energy function’. The conventional way of expressing the areal radius function $R(t,r)$ depends on the sign of $E(r)$ and is parametric. For $E(r) > 0$, we have

$$R(t,r) = \frac{M(r)}{2E(r)} (\cosh \eta - 1),$$

$$t - t_B(r) = \frac{M(r)}{(2E(r))^{1/2}} (\sinh \eta - \eta),$$

$$t - t_B(r) = \frac{M(r)}{(2E(r))^{1/2}} (\sinh \eta - \eta),$$

$$\frac{M(r)}{(2E(r))^{1/2}} (\sinh \eta - \eta),$$
where $M(r)$ and $t_B(r)$ are free functions called, respectively, the 'mass function' and the 'big bang function'. For $E(r) < 0$, we have

$$R(t, r) = \frac{M(r)}{-2E(r)}(1 - \cos \eta),$$  

(4)

$$t - t_B(r) = \frac{M(r)}{(-2E(r))^{3/2}}(\eta - \sin \eta).$$

(5)

Finally, for $E(r) = 0$, we have

$$R(t, r) = \left(\frac{9}{2}\right)^{1/3} M(r)^{1/3}(t - t_B(r))^{2/3}.$$  

(6)

(In this case the function $R(t, r)$ is explicit in terms of $t$ and $r$.)

Now, observe that in the $E > 0$ case, equation (3) shows that $\eta$ can be regarded as a function of $2E((t - t_B)/M)^{2/3}$. Therefore from equation (2) the function $R$ can be written in the form

$$R(t, r) = \frac{M}{2E} X \left(2E \left(\frac{t - t_B}{M}\right)^{2/3}\right)$$

(7)

using a certain function $X(x)$. To make the $E \to 0$ limit apparently regular, we then factor out $x$ from the function $X(x)$ and make

$$R(t, r) = \left(\frac{6}{M(r)}\right)^{1/3}(t - t_B(r))^{2/3}S\left(-2E(r) \left(\frac{t - t_B(r)}{6M(r)}\right)^{2/3}\right).$$

(8)

(The numerical factors inserted are our convention.)

The same observation is also applicable to both $E < 0$ and $E = 0$ cases, and as a result, we find that the form (8) may provide a desirable form of $R(t, r)$, which is explicit in terms of $t$ and $r$ and requires no separate considerations depending on the sign of $E(r)$. There is, however, still a remaining task, which is to confirm that the function $S(x)$ is smooth at $x = 0$. Otherwise, this form would be superficial.

To this, note that the function $S(x)$ can be expressed in parametric forms. For $x < 0$ (corresponding to $E > 0$), we have

$$S(x) = \frac{1}{6^{1/3}} \frac{\cosh \sqrt{-\zeta} - 1}{(\sinh \sqrt{-\zeta} - \sqrt{-\zeta})^{2/3}}, \quad x = \frac{-1}{6^{2/3}}(\sinh \sqrt{-\zeta} - \sqrt{-\zeta})^{2/3}.$$  

(9)

For $x > 0$ ($E < 0$), we have

$$S(x) = \frac{1}{6^{1/3}} \frac{1 - \cos \sqrt{\zeta}}{(\sqrt{\zeta} - \sin \sqrt{\zeta})^{2/3}}, \quad x = \frac{1}{6^{2/3}}(\sqrt{\zeta} - \sin \sqrt{\zeta})^{2/3}.$$  

(10)

For $x = 0$, we have

$$S(0) = \left(\frac{3}{4}\right)^{1/3}.$$  

(11)

The parameter $\zeta$ takes positive and negative values in accordance with the sign of $x$. (The explicit correspondence to $\eta$ is given by $\zeta = \eta^2$ for $E < 0$ and $\zeta = -\eta^2$ for $E > 0$.) We then find that $S$ and $x$ are expanded in powers of $\zeta$ in the same form for all signs of $x$, since by direct computation we can immediately confirm the following common expression:

$$S(x) = \frac{1}{6^{1/3}} \frac{1 - \frac{x}{6^{2/3}}(\zeta - \frac{\zeta^2}{3!} + \cdots)^{2/3}}{(\frac{1}{3!} - \frac{\zeta}{5!} + \cdots)^{2/3}}, \quad x = \frac{\zeta}{6^{2/3}} \left(\frac{1}{3!} - \frac{\zeta}{5!} + \cdots\right)^{2/3}.$$  

(12)
Proceeding with the series expansions,

\[
S(x) = \left(\frac{3}{4}\right)^{1/3} \left(1 - \frac{\zeta}{20} + \frac{\zeta^2}{1680} + O(\zeta^3)\right), \\
x = 6^{-4/3} \left(\zeta - \frac{\zeta^2}{30} + \frac{13\zeta^3}{25200} + O(\zeta^4)\right).
\]

Since both \(S\) and \(x\) are smooth (in fact, analytic) functions of \(\zeta\), in particular in the neighbourhood of \(\zeta = 0\), this shows that \(S(x)\) is smooth at \(x = 0\), as desired. (The smoothness of \(S(x)\) at \(x \neq 0\) is apparent.) Figure 1 shows a plot of the function \(S(x)\).

Expression (8) therefore does provide a desired form of the LTB solution, together with the conventional metric form (1). This expression will turn out to be extremely useful in performing most of the computations concerning the LTB solution, including the expansion of \(D_L(z)\) in the following section. In the rest of this section we briefly discuss some useful properties concerning the function \(S(x)\).

First, remember that \(R(t, r)\) satisfies the following generalized ‘Friedmann’ equation:

\[
\dot{R}^2 = \frac{2M}{R} + 2E, 
\]

where a dot denotes a derivative with respect to the proper time \(t\). Substituting equation (8) we immediately have the first-order ordinary differential equation (ODE) for \(S(x)\)

\[
\frac{4}{3} (S(x) + xS'(x))^2 + 3x - \frac{1}{S(x)} = 0, 
\]

where \(S' \equiv dS(x)/dx\). (We understand that primes attached to \(S\) always stand for derivatives with respect to its single argument, not to the radial coordinate \(r\)).

This ODE has a peculiar feature observed at \(x = 0\); at this point the term \(xS'(x)\) vanishes and therefore the equation degenerates into an algebraic equation that constrains \(S(0)\), provided that \(|S'(0)| < \infty\). As a result we obtain \(S(0) = (3/4)^{1/3}\) in accordance with equation (11). The finiteness condition of \(S'(x)\) at \(x = 0\) implies that the line \(y = S(x)\) in the \(x-y\) plane should intersect the \(x = 0\) axis transversely. The above feature therefore tells us that the function \(S(x)\) is the unique ‘transversal’ solution of the ODE (15). It may be useful to use this characterization to define \(S(x)\), instead of using the explicit parametric expressions (9)–(11).
Figure 2. Typical contours of $x(t, r)$ when $E(r)$ crosses zero. The decreasing bold curve corresponds to the big bang singularity ($t = t_B(r)$), while the vertical bold line corresponds to $E(r) = 0$ (this is not a singularity). Both bold lines correspond to $x = 0$. Contours are drawn for $t \geq t_B(r)$. (This example is intended to be helpful for a general discussion of the properties of $S(x)$, and not to imply a setting of the following section. The specific choice for this example is $E(r) = -r^2(1/2 - \theta_{\sigma}(r - 1)), M(r) = r^3$ and $t_B(r) = e^{-r^2}$. The function $\theta_{\sigma}(r)$ is a step function with the transition interval $[-\sigma, \sigma]$ with values $\theta_{\sigma}(r) = 1 (r \geq \sigma)$ and 0 ($r \leq -\sigma$).)

It may be worth commenting that we should not think of the ODE (15) as a usual ‘evolution equation’, for the following two reasons. First, as discussed previously, this ODE has no freedom to choose an initial value. Note that the solution of the generalized Friedmann equation (14) does possess freedom to choose an initial datum, which is the function $t_B(r)$. This function is however already incorporated in the form (8). The resulting equation therefore cannot contain further freedom to choose a solution. Second, the gradient of the variable $x(t, r)$ for the ODE is in general not timelike. In other words, the contours of $x(t, r)$ are not always spacelike. This is most noticeable when $E(r)$ crosses zero at some spatial points. Figure 2 shows an example where $E(r)$ crosses zero at one point ($r = 1$).

Note that the form (8) already contains the parabolic solution as one factor, with the remaining factor being $S(x)$. We can think of $|x|$ as a parameter that measures the deviation from the parabolic evolution. It is obvious that $x = 0$ corresponds to both big bang time $t = t_B(r)$ and comoving points $E(r) = 0$. In the above example, these two regions are shown as the bold lines forming an inverted ‘T’-shape (figure 2). Let us consider a neighbourhood of this ‘T’-shaped region such that for a small positive $\epsilon$, $|x| < \epsilon$. Then we can say that the time evolution of the points contained in this spacetime region is close to that of the parabolic solution. Actually, for small $t - t_B(r)$ this is a well-known fact. (Both elliptic and hyperbolic solutions asymptotically approach the parabolic-type evolution $(t - t_B(r))^{2/3}$ near the big bang time.) It is also apparent that if $|E(r)|$ is small enough the evolution of those points is close to that of the parabolic solution (for a finite time interval). The variable $x$ is therefore a deviation parameter, rather than an evolution parameter.

Now, let us return to the discussion of useful properties of $S(x)$. Remember that equation (14) is an integral of the following second-order partial differential equation:

$$2R\ddot{R} + \dot{R}^2 - 2E = 0. \quad (16)$$
From this we have the second order ODE for $S(x)$

$$x(2S(x)S''(x) + S^2(x)) + 5S(x)S'(x) + \frac{9}{4} = 0.$$ (17)

For $x \neq 0$, this equation is useful in case we want to eliminate higher derivatives of $S(x)$ than first order.

When $x = 0$, equation (17) itself does not determine $S''(0)$, due to the multiplied factor $x$. However, we can regard this equation as an equation that determines the values of arbitrarily higher derivatives of $S(x)$ at $x = 0$, in terms of $S(0)$. One of the useful applications of this property is to obtain a series expansion of $S(x)$ about $x = 0$. Since $S(0)$ is given in equation (11) (or determined from equation (15)), we have

$$S(x) = \left(\frac{3}{4}\right)^{1/3} \left(1 - \frac{3}{5} \left(\frac{3}{4}\right)^{1/3} x - \frac{27}{350} \left(\frac{9}{2}\right)^{1/3} x^2 + O(x^3)\right).$$ (18)

As discussed previously, this expansion can provide a ‘parabolic approximation’ of the solution.

We may be interested in the significance of the general solutions of the ODE (17). To avoid confusions let us top tilde, like $\tilde{S}(x)$, to denote a solution of the ODE which does not necessarily coincide with $S(x)$. First, the above observation immediately tells us that a solution of the ODE which intersects the $x = 0$ axis transversely is specified only by one parameter $s \equiv \tilde{S}'(0)$, since $\tilde{S}'(x)$ is not free at $x = 0$. We can easily confirm that all those transversal solutions are generated from $S(x)$ by rescaling,

$$\tilde{S}_\alpha(x) = \alpha S\left(\frac{x}{\alpha^2}\right),$$ (19)

where $\alpha = s/S(0) = (4/3)^{1/3}x$. This rescaled function is also a solution of the first-order ODE (15) with the modification $1/S \to 1/\alpha^2 S$. These rescaling of the function and modification of the equation just correspond to the rescaling $M(r) \to \alpha^2 M(r)$, and therefore although the rescaled function $\tilde{S}_\alpha(x)$ does generate a dust solution, it is equivalent to that by the original $S(x)$. Thus, we may not be interested in the general transversal solution $\tilde{S}_\alpha(x)$.

Non-transversal solutions of the ODE (17) in general have two parameters. An interesting one-parameter special solution is

$$\tilde{V}_\beta(x) = \sqrt{-x} - \frac{\beta}{x},$$ (20)

where $\beta$ is an arbitrary constant parameter. $\tilde{V}_\beta(x)$ approaches $S(x)$ as $x \to -\infty$, $\tilde{V}_\beta(x) \to S(x)$ (as $x \to -\infty$).

In particular, $\tilde{V}_0(x) = \sqrt{-x}$ approaches $S(x)$ from below (figure 1), and is often useful for various estimates of $S(x)$ for large $-x$. We remark that the function $\tilde{V}_\beta(x)$ satisfies the first-order ODE (15) if the $1/S(x)$ term in the equation is neglected. This in effect corresponds to taking the limit $M(r) \to 0$, and therefore the metric generated by this function (through equation (8) with $S(x)$ replaced by $\tilde{V}_\beta(x)$) represents a vacuum solution. In fact, by a direct computation of the Riemann tensor we find that it all represents the Minkowski solution.

Finally, it is worth mentioning that, slightly modifying $\tilde{V}_0(x)$, the function

$$S_C(x) \equiv \sqrt{x_C - x},$$ (22)
which is no longer a solution of any of the ODEs, provides a good approximation\(^1\) of \(S(x)\) for all domains of \(x \leq x_C\). The function \(S_C(x)\) asymptotically approaches \(S(x)\) from above as \(x \to -\infty\) (figure 1). An elementary estimate actually confirms the inequality

\[
S(x) \leq S_C(x),
\]

where the equality holds for \(x = x_C\). Also, another elementary estimate establishes

\[
xS'(x) < \frac{S(x)}{2}
\]

for the first derivative.

3. Expansion of \(D_L(z)\)

Motivated by the inverse problem, we in this section perform the expansion of the function \(D_L(z)\), the luminosity distance as a function of the redshift, for the LTB solution.

The luminosity distance for the LTB solution is simply given by [19, 20]

\[
D_L = (1 + z)^2 R,
\]

where the areal radius function \(R = R(t(z), r(z))\) must be evaluated along the light ray emitted from the light source and caught by the central observer at \((t, r) = (t_0, 0)\). With this formula we can expand \(D_L(z)\) if we expand \(R\).

Before proceeding the expansion, we note that the LTB solution has the gauge freedom of choosing the radial coordinate \(r\). In fact, it is easy to see that the coordinate transformation \(r \to \tilde{r} = \tilde{r}(r)\) retains the characteristic form of the solution if we redefine the free functions \(E(r), tB(r)\), and \(M(r)\) suitably. The significance of this freedom is that it enables us to fix the function \(M(r)\) for many cases.

One of the popular gauge choices is (e.g., [9, 14, 17]) to take \(M(r) = M_0 r^3\) with \(M_0\) being a positive constant. We call this choice the FLRW gauge, since the standard form of the FLRW dust solution is realized with this choice with the other functions being \(E(r) = -(k/2)r^2\) and \(tB(r) = 0\). \((k\) is the curvature constant.\)

Another excellent choice is [16, 21] to choose the radial coordinate \(r\), so that the light rays coming into the observer are simply expressed by

\[
t = t_0 - r.
\]

This is equivalent to the condition that \(dr/dt = -1\) must hold along those light rays, and therefore from the metric (1) it is equivalent to imposing

\[
\frac{R'(t_0 - r, r)}{\sqrt{1 + 2E(r)}} = 1,
\]

where \(R'(t_0 - r, r) \equiv (\partial R(t, r)/\partial r)|_{t = t_0, r = r}\). This gauge choice determines the function \(M(r)\) only simultaneously with \(E(r)\) and \(tB(r)\). We call this choice of \(r\) or the resulting choice of \(M(r)\) the light cone gauge.

We in the following adopt the light cone gauge, which, together with formula (8), makes our procedure of expansion systematic and straightforward. In fact, with condition (26) we can firstly expand \(R\) in powers of \(r\) in a straightforward manner, putting \(R = R(t_0 - r, r)\) in

\(^1\) It is interesting to note that in the special case of the FLRW dust solution, the approximated solution with \(S_C(x)\) corresponds to the ‘renormalized solution’ [18] of a solution obtained in the long wavelength approximation of Einstein’s equation.
formula (8). To convert the result into expansion in terms of $z$, we need to find $r(z)$ in powers of $z$. This is possible from formula [9]

$$\frac{dr}{dz} = \frac{\sqrt{1 + 2E(r)}}{1 + z}R'(t_0 - r, r).$$

(28)

This would complete our expansion, but we still need to consider an extra condition, which is the regularity at $r = 0$. As pointed out in [17, 22], if the first derivative with respect to $r$ of the matter density $\rho(t, r)$ does not vanish at $r = 0$, the spacetime has a weak singularity there. We wish to eliminate this undesirable feature; therefore we impose

$$\rho'(t, 0) = 0.$$  

(29)

The expansion with this condition imposed will give the final form of our result.

We are now in a position to proceed the expansion, following the procedure outlined above. We wish to expand $D_L(z)$ up to the third order, which leads us to expand the free functions as follows:

$$E(r) = \frac{e_2}{2}r^2 + \frac{e_3}{3!}r^3 + \frac{e_4}{4!}r^4 + O(r^5),$$

$$t_B(r) = b_1r + \frac{b_2}{2} + O(r^3),$$

$$M(r) = \frac{m_3}{3!}r^3 + \frac{m_4}{4!}r^4 + \frac{m_5}{5!}r^5 + O(r^6),$$

(30)

where the differential coefficients $e_i \equiv E^{(i)}(0)$, $b_i \equiv t^{(i)}_B(0)$ and $m_i \equiv M^{(i)}(0)$ are constants, with $m_3 > 0$. Our task is then to express the expansion of $D_L(z)$ in terms of these coefficients.

It turns out to be convenient to, prior to the general computations, finish the computation for the first order of $D_L(z)$ to determine the Hubble constant $H_0$ in comparison with the definition $D_L(z) = z/H_0 + O(z^2)$. This is not very difficult, and we find

$$H_0 = \frac{2}{3m_3} \left(1 + x_0 \frac{S'(x_0)}{S(x_0)}\right), \quad x_0 = -e_2 \left(\frac{t_0}{m_3}\right)^{2/3}. $$

(31)

(As remarked, the prime attached to $S(x)$ stands for the derivative with respect to $x$, not with respect to $r$; $S'(x_0) \equiv (dS(x)/dx)|_{x=x_0}$.)

The gauge condition (27) is equivalent to the vanishing of the function

$$G(r) \equiv \sqrt{1 + 2E(r) - R'(t_0 - r, r)}.$$  

(32)

We expand this function up to second order and demand that the coefficients be equal to zero. First, from the 0th coefficient we have

$$S(x_0) = t_0^{-2/3}m_3^{-1/3},$$

(33)

and together with equation (31), we also have

$$S'(x_0) = \frac{2 - 3H_0t_0^2}{2e_2t_0^{4/3}m_3^{1/3}}.$$  

(34)

In the following we will use equations (33) and (34) to eliminate $S(x_0)$ and $S'(x_0)$ from various equations. We remark that as we will see, equation (34) is valid not only for the case $e_2 \neq 0$ but also for the $e_2 = 0$ case, if we understand that an appropriate limit is taken. Note also that equations (33) and (34) are not independent, since neither are $S(x)$ and $S'(x)$. Substituting the equations into equation (15) we obtain the relation

$$\frac{e_2}{H_0^2} = 1 - \frac{m_3}{3H_0^2}.$$  

(35)
Since \( m_3 > 0 \), this also means
\[
\frac{e_2}{H_0^2} < 1. 
\tag{36}
\]

To take the \( e_2 \to 0 \) limit in various equations that will appear in the rest of our computation, it is useful to have an expression of \( H_{00} \) in powers of \( e_2/H_0^2 \). First, from equations (31) and (18) we have
\[
H_{00} = \frac{2}{3} \left( 1 - \frac{3}{2} \right) x_0 - \frac{117}{350} \left( \frac{2}{5} \right) x_0^2 + O(x_0^3). 
\tag{37}
\]

On the other hand, from equation (35) (and the definition of \( x_0 \) (31)) we have
\[
x_0 = - \frac{e_2}{H_0^2} \left( \frac{H_{00}/3}{1 - e_2/H_0^2} \right)^{2/3} \tag{38}
\]

Substituting this into equation (37), we obtain an equation for \( H_{00} \). We can solve this and find
\[
H_{00} = \frac{2}{3} \left( 1 + \frac{e_2}{5 H_0^2} + \frac{3}{35} \left( \frac{e_2}{H_0^2} \right)^2 + O\left( \left( \frac{e_2}{H_0^2} \right)^3 \right) \right). 
\tag{39}
\]

This is the expression we wanted. In particular, this implies as \( e_2 \to 0 \),
\[
\frac{2 - 3H_{00}}{e_2/H_0^2} \to - \frac{2}{5},
\tag{40}
\]

\[
\left( \frac{2 - 3H_{00}}{e_2/H_0^2} + \frac{2}{5} \right) \frac{1}{e_2/H_0^2} \to - \frac{6}{35}. 
\tag{41}
\]

With the first of these we can confirm that equation (34) provides the right value for the limit \( e_2 \to 0 \), and therefore equation (34) is, as mentioned, valid also for \( e_2 = 0 \).

Returning to the gauge condition (32), from its first-order coefficient we have
\[
m_4 = \frac{4m_3}{t_0} - \frac{4e_2 m_3^{1/3} S'(x_0)}{t_0^{4/3} S(x_0)} = 6H_0 m_3. 
\tag{42}
\]

Similarly, from the second order we have
\[
m_5 = \frac{m_3}{2} \left( 15 H_0^2 + \frac{5 e_2}{e_2} + \frac{5}{3(1 - H_{00})} \left( 13 H_0^2 + 12 H_0 b_2 + 2 e_2 - \frac{e_4}{e_2} \right) \right). 
\tag{43}
\]

(The second derivative \( S''(x_0) \) appears in the coefficient, which we eliminate using equation (17).) In particular, for \( e_2 = 0 \) we have
\[
m_5 = m_3 \left( 10 H_0(4 H_0 + 3 b_2) - \frac{e_4}{H_0^2} \right). 
\tag{44}
\]

In the following we will eliminate \( m_4 \) and \( m_5 \) with equations (42) and (43).

To find the regularity condition at \( r = 0 \) we expand
\[
4\pi \rho(t, r) = \frac{M'(r)}{R^2(t, r) R'(t, r)}
\]
in powers of \( r \) up to first order with \( t \) fixed,
\[
4\pi \rho(t, r) = \frac{1}{S^3(x(t))} \left[ \frac{1}{2t^2} + \frac{4b_1}{3t^3} - \frac{4b_1 e_2}{3m_3^{2/3} t^{4/3}} \frac{S'(x(t))}{S(x(t))} + \frac{(2e_3 m_3 - e_2 m_4)}{3m_3^{2/3} t^{4/3}} \frac{S'(x(t))}{S(x(t))} \right] r + O(r^2),
\tag{46}
\]
where
\[ x(t) \equiv -\frac{e_2 t^{2/3}}{m_3^{2/3}}. \]  
(47)

Condition (29) then implies
\[ b_1 = 0, \quad e_3 = \frac{e_2 m_4}{2m_3} = 3H_0 e_2. \]  
(48)

In the last equality we have used equation (42). (If we had employed the FLRW gauge, which implies \( m_4 = 0 \), we would have had \( e_3 = 0 \) and \( b_1 = 0 \), which is consistent with the condition derived in [17].)

From equation (46) we can immediately find the energy density at the central observer. Using equation (33) we have
\[ 8\pi \rho(t_0, 0) = m_3. \]  
(49)

Therefore, it may be natural to write
\[ \frac{m_3}{3H_0^2} = \Omega_{M,0}, \quad \frac{e_2}{H_0^2} = \Omega_{\xi,0}, \]  
(50)

where \( \Omega_{M,0} \) and \( \Omega_{\xi,0} \) are density parameters for the central observer. Then we can interpret equation (35) as the usual relation for the dust FLRW model,
\[ \Omega_{M,0} + \Omega_{\xi,0} = 1. \]  
(51)

The final task before presenting our main result is to obtain \( r(z) \) up to third order. Note that equation (28) is integrable and we find
\[ 1 + z = \exp \int_0^r \frac{\dot{R}(t_0 - r', r')}{\sqrt{1 + 2E(r')}} \, dr'. \]  
(52)

From this equation, we have
\[ r(z) = \frac{z}{H_0} - \frac{1}{4H_0} \left( 5 - \frac{e_2}{H_0^2} \right) z^2 + \frac{1}{48H_0} \left[ 109 - \frac{35e_2^2 + 3e_4}{H_0^2 e_2} + \frac{4e_2^2 + e_4}{H_0^2} \right] \frac{z^3}{3} + O(z^4). \]  
(53)

In particular, for \( e_2 = 0 \) we have
\[ r(z) = \frac{z}{H_0} - \frac{5}{4H_0} z^2 + \frac{1}{6H_0} \left( \frac{35}{4} - \frac{9b_2}{2H_0} - \frac{e_4}{10H_0^2} \right) \frac{z^3}{3} + O(z^4). \]  
(54)

The regularity conditions (48) have been imposed in these equations, as well as the gauge conditions (42) and (43).

We are now ready to compute the expansion of \( R \) in the light cone gauge in powers of \( r \) with the regularity at \( r = 0 \), and convert it to powers of \( z \) using formulae (53) or (54). As mentioned, the expansion of \( R = R(t_0 - r, r) \) is straightforward if we write \( R \) using the function \( S(\lambda) \) as in equation (8). Moreover, the use of equations (33) and (34), together with the ODE (17), enables us to eliminate the function \( S(\lambda) \) and its derivatives to simplify the equations.

The result of the expansion of \( R \) is in such a simple form that it does not depend on the higher differential coefficients \( e_4 \) and \( b_2 \),
\[ R(t_0 - r, r) = r - \frac{H_0}{2} r^2 - \frac{H_0^2}{3} \left( 1 - \frac{e_2}{H_0^2} \right) r^3 + O(r^4). \]  
(55)
Substituting equation (53) and taking into account equation (25), we have the final form of regular $D_L(z)$, which is

$$D_L(z) = \frac{z}{H_0} + \frac{1}{4H_0^2} \left( 1 + \frac{e_2}{H_0^2} \right) z^2 - \frac{1}{12H_0^4} \left[ 1 - \frac{e_2}{H_0^2} \left( 13 + \frac{2e_2}{H_0^2} + \frac{12b_2}{H_0} \right) \right]$$

$$- 33 + \frac{e_2}{H_0^2} \left( 7 \right) - \frac{4e_2}{H_0^2} + \frac{1}{1 - H_0t_0} \left( H_0t_0 + \frac{2 - 3H_0t_0}{e_2/H_0^2} \right) \frac{e_4}{H_0^2} ] z^3 + O(z^4).$$

(56)

(This looks singular when $1 - H_0t_0 = 0$, but the use of equations (31) and (24) immediately shows $1 - H_0t_0 > 0$.) In particular, for $e_2 = 0$:

$$D_L(z) = \frac{z}{H_0} + \frac{z^2}{4H_0} \left( 1 + \frac{6b_2}{H_0} + \frac{2e_4}{15H_0^2} \right) z^3 + O(z^4).$$

(57)

A comparison with equation (56) immediately gives

$$H_0 = \frac{1}{I_1}$$

(59)

$$\frac{e_2}{H_0^2} = 2 \frac{I_2}{I_1} - 1$$

(60)

and

$$\frac{1}{1 - H_0t_0} \left( H_0t_0 + \frac{2 - 3H_0t_0}{e_2/H_0^2} \right) \frac{e_4}{H_0^2} + \frac{1 - e_2/H_0^2}{H_0} \frac{12b_2}{H_0}$$

$$= -8 \frac{I_3}{I_1} + 33 - \frac{e_2}{H_0^2} \left( 7 \right) - \frac{4e_2}{H_0^2} - \frac{1}{1 - H_0t_0} \left( 13 + \frac{2e_2}{H_0^2} \right).$$

(61)

Equation (59) determines $H_0$, which constrains the parameters $e_2, m_3$ and $t_0$ through equation (31). Equation (60) determines $e_2$, which in turn determines $m_3$ through equation (35). Then, the present time $t_0$ for the central observer is implicitly determined from condition (33) with $e_2$ and $m_3$. This way we can determine the parameters $e_2, m_3$ and $t_0$ for given $I_1$ and $I_2$. We can see that the right-hand side of equation (61) is now a function of $I_1, I_2$ and $I_3$. This equation represents a constraint for $e_4$ and $b_2$, only one of which is determined from the other through this equation. This is a result of the fact that the LTB solution is highly degenerate in that it can represent inequivalent multiple models that give rise to the same $D_L(z)$. If we determine one of $e_4$ and $b_2$ according to a separate consideration, we can determine the other. All the other parameters are determined from equations (42), (43) and (48). We remark that this procedure is doable only for $I_2/I_1 < 1$ due to the inequality (36). (See the following section for its significance.)
4. Comparison with the FLRW limit

To understand the significance of the expansion (56) itself, it is profitable to consider how much different it is from the FLRW limit. As mentioned, one of the choices that realize the FLRW dust solution in the LTB solution is to take

\[ E(r) = -\frac{k}{2}r^2 = \frac{E''(0)}{2}r^2, \quad M(r) = M_0 r^3 = \frac{M''(0)}{3!}r^3, \quad t_B(r) = 0, \]

\[ (t_B(r) \text{ can be any constant for, which we take as zero.)} \]

However, this coordinate choice is different from the one we have been employing. A general gauge-independent condition to give the FLRW solution is obtained by eliminating the explicit \( r \)-dependences from the above expressions,

\[ E(r) = \frac{E''(0)}{2} \left( \frac{6}{M''(0)} M(r) \right)^{2/3}, \quad t_B(r) = 0. \]

(The constant factors have been determined so as to be valid for any \( M(r) = O(r^3) \).) The functions that satisfy both these equations and gauge condition (27) provide the functions that realize the FLRW solution in the light cone gauge. Substituting expansions (30) into the above equations we have

\[ e_3 = \frac{e_2 m_4}{2 m_3}, \quad e_4 = \frac{e_2}{60m_3^2} (-5m_4^2 + 24m_3 m_5). \]

We must solve four equations, the above two with equations (42) and (43), for the four variables \( e_3, e_4, m_4 \) and \( m_5 \). We have

\[ e_3 = 3H_0 e_2, \quad e_4 = \left( 13H_0^2 + 2e_2 \right) e_2, \]

\[ m_4 = 6H_0 m_3, \quad m_5 = 5(8H_0^2 + e_2^2)m_3, \]

which, together with \( b_i = 0 \), give the FLRW limit for the differential coefficients. Among the above four equations, only the equation for \( e_4 \) is essential; the equation for \( m_4 \) is the same as the general gauge condition (42), the one for \( m_5 \) is equivalent to the general gauge condition (43) with the equation for \( e_4 \) imposed, and the one for \( e_3 \) is the same as the regularity condition (48). (The regularity at the centre is therefore automatically guaranteed for the FLRW limit as it should be.)

These limit conditions motivate us to put

\[ e_4 = \left( 13H_0^2 + 2e_2 \right) e_2 + H_0^4 e_4, \]

where \( e_4 \) is a dimensionless parameter which becomes zero in the FLRW limit. Substituting this into equation (56), we have

\[ D_L(z) = D_L^{(\text{hom})}(z) - \frac{1}{48H_0(1 - H_00)} \left[ H_00 + \frac{2 - 3H_00}{e_2/H_0^2} \right] e_4 \]

\[ + \left( 1 - \frac{e_2}{H_0^2} \right) \frac{12b_2}{H_0^2} z^3 + O(z^4), \]

where the \( D_L(z) \) for the FLRW limit is

\[ D_L^{(\text{hom})}(z) = \frac{z}{H_0} + \frac{1}{4H_0} \left( 1 + \frac{e_2}{H_0^2} \right) z^2 - \frac{1}{8H_0} \left( 1 - \frac{e_2}{H_0^2} \right) z^3 + O(z^4). \]

A striking feature of our result is that it shows that the luminosity distance function \( D_L(z) \) for an LTB solution which is regular at the centre exactly coincides with that of an FLRW dust solution, up to second order.
In particular, if we, as in [23], define the deceleration parameter \( q_0 \) in comparison with the \( D_L(z) \) for a general FLRW model:

\[
D_L(z) = \frac{1}{H_0} \left( z + \frac{1 - q_0}{2} z^2 + O(z^3) \right),
\]

we have \( q_0 > 0 \), as in the FLRW dust solution. In fact, comparing with equation (56) we have

\[
q_0 = \frac{1}{2} \left( 1 - \frac{\epsilon_2}{H_0^2} \right) > 0,
\]

because of equation (36). This reconfirms the claim in [17, 24, 25].

5. Conclusions

We have first given a new way of expressing the LTB solution, which is explicit (i.e., not parametric) and requires no separate considerations depending on the sign of the energy function \( E(r) \). This has been done using a ‘special’ function \( S(x) \), which can be defined as the unique ‘transversal’ solution of the first-order ODE (15). Using this monotonic function, we can write the areal radius function \( R(t, r) \) for the LTB metric in the concise form (8).

To simplify expressions involving higher derivatives of \( S(x) \) it is most useful to use the second-order ODE (17).

Taking advantage of this concise expression, we have computed the luminosity distance function \( D_L(z) \) for the LTB solution up to the third order of \( z \). We have found that if we impose the regularity condition at the centre, the function degenerates into an FLRW dust case up to second order, and differences only appear from the third order.

The second-order coincidence with the FLRW dust solution tells us that we cannot choose a set of LTB functions \( \{ E(r), H_b(r), M(r) \} \) so that the resulting \( D_L(z) \) fits an FLRW model which shows an accelerating expansion like the \( \Lambda \)CDM model, as long as the regularity condition is imposed. This is however of course not to say that we cannot find an LTB model that explains the observations, since for this it is not necessary to have an exact fitting of \( D_L(z) \) near the centre with an accelerating FLRW model.

Perhaps, the simplest way to guarantee the regularity and still have a good fit of \( D_L(z) \) with the observations is to choose the model so that it exactly coincides with an FLRW dust solution (of perhaps negative curvature) for \( z \) smaller than a certain small (but finite) value \( z_1 \), and use the full flexibility of the LTB solution for \( z > z_1 \) to fit the observations. This approach however has the drawback that the \( D_L(z) \) chosen this way inevitably becomes a non-analytic function, since if it were analytic it would be that of the FLRW solution. To be sure, it is possible to approximate such a non-analytic function with an appropriate analytic function. For example, one might use \( \tanh x \) to approximate a step function, but \( \tanh x \) never becomes constant for large \( x \), as opposed to the step function. Like this, an analytic \( D_L(z) \) chosen to approximate a \( D_L(z) \) that is endowed with the above property deviates from that of FLRW near the centre.

To summarize, if one wants \( D_L(z) \) to be analytic (or of any arbitrary form), the inverse problem at the centre is nontrivial. Our result (56) gives, as we have seen, a solution to this problem. A further study based on the present work is under progress.

Acknowledgment

We thank Satoshi Gonda for stimulating discussions.
Appendix. $D_L(z)$ without regularity condition

In this appendix, we present the luminosity distance function for the LTB solution that is not necessarily regular at the centre. To distinguish from the regular one, we write $D_L^{(all)}(z)$ to denote this general function. Let $D_L(z)$ be the regular part of $D_L^{(all)}(z)$ as in equation (56) or (67), and $d_L(z)$ be the difference from the regular part, i.e.,

$$D_L^{(all)}(z) = D_L(z) + d_L(z).$$

(A.1)

The difference part $d_L(z)$ should vanish when the conditions (48) are satisfied. Motivated by the condition for $\varepsilon_3$, we rewrite $\varepsilon_3$ using the dimensionless parameter $\varepsilon_3$ defined by

$$\varepsilon_3 = 3H_0e_2 + H_0^2\varepsilon_3.$$  

(A.2)

The parameter $\varepsilon_3$ vanishes when $\varepsilon_3$ satisfies the regularity condition.

Then, we have

$$d_L(z) = \frac{-A_2}{12H_0(1 - H_0t_0)}z^2 + \frac{A_3}{288H_0(1 - H_0t_0)^3}z^3 + O(z^4),$$  

(A.3)

where

$$A_2 \equiv \left( \frac{H_0t_0 + 2 - 3H_0t_0}{e_2/H_0} \right) \varepsilon_3 + 6 \left( 1 - \frac{e_2}{H_0} \right) b_1$$  

(A.4)

and

$$A_3 \equiv 36(4 - 5H_0t_0) \left( 1 - \frac{e_2}{H_0} \right)^2 b_1^2 + 12 \left[ 2H_0t_0(8 - 7H_0t_0) - 4 + \frac{e_2}{H_0} \left( \frac{2 - 3H_0t_0}{e_2/H_0} \right)^2 - H_0t_0(4 - 5H_0t_0) \right] b_1 \varepsilon_3$$

$$+ \left[ H_0^2t_0^2(4 - 5H_0t_0) - \frac{2}{5} \left( \frac{2 - 3H_0t_0}{e_2/H_0} \right) (5 - H_0t_0(21 - 20H_0t_0)) + \left[ \frac{2 - 3H_0t_0}{e_2/H_0} + \frac{2}{5} \frac{10 - H_0t_0(22 - 13H_0t_0)}{e_2/H_0} \right] \varepsilon_3^2 \right]$$

$$+ 4(1 - H_0t_0) \left\{ 6 \left( 1 - \frac{e_2}{H_0} \right) \left( 9 - 11H_0t_0 - 2(3 - 4H_0t_0) e_2/H_0 \right) b_1 + \left[ \frac{2 - 3H_0t_0}{e_2/H_0} (15 - 17H_0t_0) + 33H_0t_0(1 - H_0t_0) - 4 - 2H_0t_0(3 - 4H_0t_0) \frac{e_2}{H_0} \right] \varepsilon_3 \right\}. $$

(A.5)

To take the limit $e_2 \to 0$ of this expression, one may need both equations (40) and (41). As a result, for $e_2 = 0$ we have

$$d_L(z) = -\frac{B_2}{H_0} z^2 + \frac{B_3}{H_0} z^3 + O(z^4).$$

(A.6)

where

$$B_2 \equiv \frac{\varepsilon_3}{15} + \frac{3b_1}{2}.$$  

(A.7)
and
\[ B_3 \equiv \frac{7e_3}{30} + \frac{13e_3^2}{1575} + \frac{e_3 b_1}{2} + \frac{5b_1}{4} + \frac{9b_1^2}{4}. \]  
(A.8)

The deceleration parameter \( q_0 \) for the central observer is therefore
\[ q_0 = \frac{1}{2} \left( 1 - \frac{e_3}{H_0^2} \right) + \frac{1}{6(1 - H_0^2)} \left[ \left( \frac{H_0^2}{e_3} + 2 \right) \frac{H_0^2}{e_3} \right] \left( 1 - \frac{e_3}{H_0^2} \right) b_1. \]  
(A.9)

In particular, for the \( e_3 = 0 \) case,
\[ q_0 = \frac{1}{2} + \frac{2}{15} e_3 + 3b_1, \]  
(A.10)

which is apparently positive unless at least one of \( e_3 \) or \( b_1 \) is nonzero. This is also the case for any \( e_2 \) including \( e_2 \neq 0 \), as remarked in section 4. Nonzero \( e_3 \) or \( b_1 \) however leads to a weak singularity at \( z = 0 \).

We comment that our \( D_L^{\text{fall}}(z) \) is a generalization of the result of Célérier [9] which corresponds to the case \( e_3 = 0 \). To confirm the consistency, however, there are two remarks to make. (i) [9] employs the FLRW gauge, which is different from ours. To compare, an appropriate transformation is needed among the differential coefficients of the three LTB functions. (ii) Equation (45) of [9] contains two wrong numerical factors; 7 and 10 should read, respectively, 6 and 9. We have confirmed that our result is consistent with [9] after taking these two points into account.

References

[1] Riess A G et al (Supernova Search Team Collaboration) 1998 Astron. J. 116 1009 (Preprint astro-ph/9805201)
[2] Perlmutter S et al (Supernova Cosmology Project Collaboration) 1999 Astrophys. J. 517 565 (Preprint astro-ph/9812133)
[3] Riess A G et al (Supernova Search Team Collaboration) 2004 Astrophys. J. 607 665 (Preprint astro-ph/0402512)
[4] Spergel D N et al (WMAP Collaboration) 2003 Astrophys. J. Suppl. 148 175 (Preprint astro-ph/0302209)
[5] Tegmark M et al (SDSS Collaboration) 2004 Phys. Rev. D 69 103501 (Preprint astro-ph/0310725)
[6] Lemaitre G 1933 Ann. Soc. Sci. Bruxelles A 53 51
[7] Tolman R C 1934 Proc. Natl Acad. Sci. 20 169
[8] Bondi H 1947 Mon. Not. R. Astron. Soc. 107 410
[9] Célérier M-N 2000 Astron. Astrophys. 353 63 (Preprint astro-ph/9907206)
[10] Iguchi H, Nakamura T and Nakao K I 2002 Prog. Theor. Phys. 108 809 (Preprint astro-ph/0112419)
[11] Añez H, Amaruiguini M and Gron O 2006 Phys. Rev. D 73 083519 (Preprint astro-ph/0512006)
[12] Bolejko K 2005 Preprint astro-ph/0512103
[13] Garfinkle D 2006 Class. Quantum Grav. 23 4811 (Preprint gr-qc/0605088)
[14] Biswas T, Mansouri R and Notari A 2006 Preprint astro-ph/0606703
[15] Enqvist K and Mattsson T 2007 J. Cosmol. Astropart. Phys. JCAP02(2007)019 (Preprint astro-ph/0609120)
[16] Mustapha N, Hellaby C and Ellis G F R 1997 Mon. Not. Roy. Astron. Soc. 292 817 (Preprint gr-qc/9808079)
[17] Vanderveld R A, Flanagan É É and Wasserman I 2006 Phys. Rev. D 74 023506 (Preprint astro-ph/0602476)
[18] Nambu Y and Yamaguchi Y Y 1999 Phys. Rev. D 60 104011 (Preprint gr-qc/9904053)
[19] Ellis G F R 1971 General Relativity and Cosmology (Proc. Int. School of Physics ‘Enrico Fermi’, Course XLVII)
[20] ed R K Sachs (New York: Academic) p 104
[21] Partovi M H and Mashhoon B 1984 Astrophys. J. 276 4
[22] Mustapha N, Bassett B A, Hellaby C and Ellis G F R 1998 Class. Quantum Grav. 15 2363 (Preprint gr-qc/9708043)
[23] Mustapha N and Hellaby C 2001 Gen. Rel. Grav. 33 455 (Preprint astro-ph/0006083)
[24] Barausse E, Matarrese S and Riotto A 2005 Phys. Rev. D 71 063537 (Preprint astro-ph/0501152)
[25] Flanagan É É 2005 Phys. Rev. D 71 103521 (Preprint hep-th/0503202)
[26] Hirata C M and Seljak U 2005 Phys. Rev. D 72 083501 (Preprint astro-ph/0503582)