ROUGH TOPOLOGY ON APPROXIMATION SPACES

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Abstract: Though there are many studies on the topological properties of rough set approximations, only a few works have been done on the concept of rough topology. In this paper, a new definition of rough topology on an approximation space is proposed using the rough subsets of the extended approximation space. The basic concepts of a topological space are extended to the proposed rough topological space and the properties are investigated.

Keywords: lower approximation; upper approximation; rough set; topology; rough topology.

I. INTRODUCTION

The theory of rough sets was formulated by Z. Pawlak in 1982 [18]. The mathematical framework of this potential theory has been enriched by the contribution of many researchers in various dimensions. An interesting and natural research area is the study of the interconnections between rough set theory and topology theory. The topological structure of rough sets forms an important base for information analysis and knowledge processing [6,16,20,28].

There are mainly two streams of study connecting rough set theory and topology theory. The first one is regarding the topology induced by rough set approximations. In his seminal paper itself, Z. Pawlak [18] pointed out that the set of all equivalence classes in an approximation space formed a base for a topology on the set \( U \) and the set of all equivalence classes of the rough equality relation formed a base for a topology on the power set of \( U \). Both the topologies were found to be quasi discrete. Further studies on the topological properties of rough sets can be found in \([3,10,11,13,15,20,21,27,29]\). The second stream consists of the studies on the different types of approximation spaces induced by different topological spaces \([1,6,12,23,24,25]\). Some papers include results on both the streams \([2,14]\).

Regarding information systems, T. Herawan \([4,5]\) worked on the topology on an information system.

Though there are many studies on the relations between rough set approximations and topological spaces, only a few works are there on the concept of rough topology. Q. Wu et al. \([28]\) defined rough topology on a rough set by using a metric and then extended it to general topological space. M. L. Thivagar et al. \([26]\) introduced the concept of rough topology which consists of the null set, the universal set, the lower approximation, upper approximation and the boundary region of a subset of \( U \). B. P. Mathew and S. J. John \([17]\) introduced the concept of rough topology on a rough set as a pair of topologies of exact subsets of the lower and upper approximations of the rough set under consideration. A follow up work was done by M. Ravindran and A. J. Divya \([22]\) who studied the properties of compactness and connectedness and the separation axioms in rough topological spaces. But none of them regard rough topology as a rough subset of the power set.

In this paper, a new definition for rough topology on an approximation space is proposed using the rough subsets of the extended approximation space. The basic concepts of topology are extended to the proposed rough topological space and the conclusion is given in section 5.

II. PRELIMINARIES

In this section, some basic notions of rough set theory are recalled. Further details of rough set theory can be found in \([19]\). The basic concepts of topology theory are described in \([9]\).

A. Rough Set Theory

The theory of rough sets was introduced by Z. Pawlak \([18]\) in 1982. Consider the approximation space \((U, R)\), where \( U \) is a non-empty set of objects and \( R \) is an equivalence relation on \( U \). The equivalence classes of \( R \) are called elementary sets and sets which can be expressed as union of some equivalence classes are called composed sets.

The lower and upper approximations \([19]\) of \( A \subseteq U \), with respect to \( R \) are defined respectively as:

\[
\underline{R}(A) = \{ x \in U : [x]_R \subseteq A \} \quad (1)
\]

\[
\overline{R}(A) = \{ x \in U : [x]_R \cap A \neq \emptyset \} \quad (2)
\]

where, \([x]_R\) is the equivalence class of \( R \) containing \( x \). In other words, \( \underline{R}(A) \) is the union of equivalence classes that are contained in \( A \) and \( \overline{R}(A) \) is the union of equivalence classes that have non-empty intersection with \( A \). Thus, both lower and upper approximations are composed sets on \((U, R)\).

The set \( \overline{R}(A) \) is called the positive region, \( U - \overline{R}(A) \) is called the negative region and the set \( \overline{R}(A) - \underline{R}(A) \) is called the boundary region. The properties of the rough set approximations are as follows:

1. \( \underline{R}(\emptyset) = \overline{R}(\emptyset) = \emptyset \)
2. \( \underline{R}(x) = \underline{R}(\{x\}) = x \)
3. \( \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \)
4) \( \overline{\overline{R(A)}} = \overline{R(A)} \)
5) \( \overline{R(A)} = \overline{R(A)} \)
6) \( R(A \cap B) = \overline{R(A)} \cap \overline{R(B)} \)
7) \( \overline{R(A \cap B)} = \overline{R(A)} \cap \overline{R(B)} \)
8) \( \overline{R(A \cup B)} = \overline{R(A)} \cup \overline{R(B)} \)
9) \( \overline{R(A \cup B)} = \overline{R(A)} \cup \overline{R(B)} \)
10) \( \overline{R(A)} = \overline{(R(A))^c} \)
11) \( \overline{R(A)} = \overline{(R(A))^c} \)
12) \( A \subseteq B \Rightarrow \overline{R(A)} \subseteq \overline{R(B)} \)
13) \( A \subseteq B \Rightarrow \overline{R(A)} \subseteq \overline{R(B)} \)
14) \( R([x]_R) = \{x\} \) for all \( x \in X \)

B. Operations on Rough Sets

The term ‘rough set’ has been used in two different viewpoints by Z. Pawlak. According to his original proposal, the equivalence classes of the rough equality relation were termed as rough sets [18]. Later, in [19], a subset \( A \subseteq U \) with \( \overline{R(A)} \neq \overline{R(A)} \) was called rough set. Rough sets may also be described using rough membership functions [19]. There is yet another approach to rough set theory proposed by T. B. Iwinski [7]. Throughout this paper, the word rough set refers to a pair \( < X_L, X_U > \), where \( X_L \) and \( X_U \) are subsets of \( U \) such that \( X_L = \overline{R(A)} \) and \( X_U = \overline{R(A)} \), for some \( X \subseteq U \) [8]. For convenience, a rough set may be denoted by \( < R(A), \overline{R(A)} > \), where \( A \subseteq U \).

The rough inclusion, rough union, rough intersection and rough complement operations of rough sets are defined in [8] as:

\[ \begin{align*}
\left< R(A), \overline{R(A)} \right> & \subseteq \left< R(B), \overline{R(B)} \right> \\
\iff & \left< R(A) \subseteq R(B), \overline{R(A)} \subseteq \overline{R(B)} \right> \\
\left< R(A), \overline{R(A)} \right> & > \left< R(B), \overline{R(B)} \right> \\
\iff & \left< R(A) \cup R(B), \overline{R(A)} \cup \overline{R(B)} \right> \\
\left< R(A), \overline{R(A)} \right> & > \left< R(B), \overline{R(B)} \right> \\
\iff & \left< R(A) \cap R(B), \overline{R(A)} \cap \overline{R(B)} \right> \\
\left< R(A), \overline{R(A)} \right> & > \left< R(B), \overline{R(B)} \right> \\
\iff & \left< R(A), \overline{R(A)} \right> = \left< (R(A))^c, (R(A))^c \right> \\
\end{align*} \]

respectively.

B. Topology and Rough Sets

Let \( (U, R) \) be an approximation space. Then, the lower approximation operator satisfies the properties of an interior operator and hence it induces a topology \( \tau_R \) on \( U \) [1]. A subset \( A \in \tau_R \) if and only if \( R(A) = A \). Hence, \( \tau_R \) consists of all composed sets on \( U \). By properties (10) and (11), a set \( A \) will be closed if \( \overline{R(A)} = A \). Hence all open subsets are closed and \( \tau_R \) is the quasi discrete topology. The family of all equivalence classes form a basis for \( \tau_R \) [1].

Let \( X =< X_L, X_U > \) be a rough subset of the approximation space \( (U, R) \). Let \( \tau_L \) and \( \tau_U \) be any two topologies which contain only exact subsets of \( X_L \) and \( X_U \) respectively. Then the pair \( < \tau_L, \tau_U > \) said to be a rough topology on \( U \) [17].

III. ROUGH TOPOLOGY ON APPROXIMATION SPACES

Let \( (U, R) \) be an approximation space, where \( U \) is a non-empty set of objects and \( R \) is an equivalence relation on \( U \). Consider the rough equality relation \( R \) on \( \mathcal{P}(U) \), given by \( (A, B) \in R \iff R(A) = R(B) \) and \( \overline{R(A)} = \overline{R(B)} \) (7)

Then, \( R \) is an equivalence relation on \( \mathcal{P}(U) \) and the pair \( (\mathcal{P}(U), R) \) is called the extended approximation space corresponding to \( (U, R) \) [1]. Hence, we can extend the definition of rough set approximations to any subfamily \( \mathcal{F} \subseteq \mathcal{P}(U) \).

Definition 3.1:

Let \( (U, R) \) be an approximation space and \( R \) be the rough equality relation. Then, the \( R \)-lower and \( R \)-upper approximations of a subfamily \( \mathcal{F} \subseteq \mathcal{P}(U) \) are given by

\[ \begin{align*}
\mathcal{R}(\mathcal{F}) &= \{ A \in \mathcal{P}(U) : A \subseteq \mathcal{F} \} \\
\overline{\mathcal{R}(\mathcal{F})} &= \{ A \in \mathcal{P}(U) : A \cap \mathcal{F} \neq \emptyset \}
\end{align*} \]

Lemma 3.1:

In the extended approximation space \( (\mathcal{P}(U), R) \), \( \forall x \in U \), \([x]_R = \{x\} \).

Proof:

Since \( R \) is reflexive, \([x]_R \subseteq \{x\} \). Using property (14), \([x]_R = \overline{R([x]_R)} \). Using eqn (7), if \( A \in \{x\} \), then \( R(A) = \overline{R([x]_R)} = \overline{R([x]_R)} = \overline{R(A)} \). Thus \( A = \overline{R(A)} \). Therefore, \([x]_R = \{x\} \).

Theorem 3.1:

The \( R \)-lower and \( R \)-upper approximations of the quotient set \( U/R \) are \( U/R \) itself.

Proof:

We have, \( \mathcal{R}(U/R) \subseteq U/R. \) If \( A \in U/R, \) then \( A = \overline{R(A)} \) for some \( x \in U \). So, \([x]_R \subseteq \{x\} \subseteq \{x\} \subseteq U/R \), using lemma 3.1. Hence, \( A \in \mathcal{R}(U/R) \). Thus, \( \mathcal{R}(U/R) = U/R \). Similarly, \( \overline{\mathcal{R}(U/R)} = U/R \).

Lemma 3.2:

If \( A \) is a composed set in \( (U, R) \), then in the extended approximation space \( (\mathcal{P}(U), R) \), \([A]_R = \{A\} \).

Proof:

Obviously, \( A \in \{A\} \). Also, by the definition of \( R \), if \( B \in \{A\} \), then \( R(A) = \overline{R(B)} \) and \( \overline{R(A)} = \overline{R(B)} \). Since \( A \) is a composed set, \( R(A) = A = \overline{R(A)} \). It follows that, \( B = A \). Therefore, \([A]_R = \{A\} \).

Theorem 3.2:

If \( \mathcal{F} \subseteq \mathcal{P}(U) \) is a family of composed subsets of \( U \), then \( \mathcal{R}(\mathcal{F}) \subseteq \mathcal{F} \).

Proof:

By property (3), \( \mathcal{R}(\mathcal{F}) \subseteq \mathcal{F} \). If \( A \in \mathcal{F} \), then \( A \) is a composed set and by Lemma 3.2, \([A]_R \subseteq \mathcal{F} \). Hence, \( A \in \mathcal{R}(\mathcal{F}) \) and so, \( \mathcal{F} \subseteq \mathcal{R}(\mathcal{F}) \). Thus, \( \mathcal{R}(\mathcal{F}) = \mathcal{F} \). Similarly, \( \overline{\mathcal{R}(\mathcal{F})} = \mathcal{F} \).

Corollary 3.1:

\( \overline{R(\tau_R)} = \tau_R = \overline{R(\tau_R)} \), where \( \tau_R \) is the topology on \( U \) induced by \( R \).

Proof:

Being the topology on \( U \) induced by \( R, \tau_R \) consists of all composed subsets of \( U \). Hence by theorem 3.2, \( \overline{R(\tau_R)} = \tau_R = \overline{R(\tau_R)} \).

Definition 3.2:

Let \( \mathcal{T} \) be a subfamily of \( \mathcal{P}(U) \). The pair \( < \mathcal{T}(\mathcal{T}), \overline{\mathcal{T}(\mathcal{T})} > \) is called an \( R \)-rough topology on \( U \), if both \( \mathcal{T}(\mathcal{T}) \) and \( \overline{\mathcal{T}(\mathcal{T})} \) are topologies on \( U \). The set \( U \) together with the \( R \)-rough topology is called an \( R \)-rough topological space.
**Definition 3.3:**
Let \( T \) be a subfamily of \( \mathcal{P}(U) \). The pair \(< \mathcal{R}(T), \overline{\mathcal{R}(T)} > \) is called a lower \( \mathcal{R} \) -rough topology on \( U \), if \( \mathcal{R}(T) \) is a topology on \( U \).

**Definition 3.4:**
Let \( T \) be a subfamily of \( \mathcal{P}(U) \). The pair \(< \mathcal{R}(T), \overline{\mathcal{R}(T)} > \) is called an upper \( \mathcal{R} \) -rough topology on \( U \), if \( \overline{\mathcal{R}(T)} \) is a topology on \( U \).

**Definition 3.5:**
Let \(< \mathcal{R}(T), \overline{\mathcal{R}(T)} > \) be an \( \mathcal{R} \) -rough topology on \( U \). The rough set \(< \mathcal{R}(A), \overline{\mathcal{R}(A)} > \) is called lower \( \mathcal{R} \) -rough open if \( \mathcal{R}(A) \) is an open set in \( \mathcal{R}(T) \), upper \( \mathcal{R} \) -rough open if \( \overline{\mathcal{R}(A)} \) is an open set in \( \overline{\mathcal{R}(T)} \) and \( \mathcal{R} \) -rough open if it is both lower \( \mathcal{R} \) -rough open and upper \( \mathcal{R} \) -rough open.

**Example 3.1:**
Let \( U = \{a, b, c, d\} \) and consider the equivalence relation given by \( U/R = \{(a, b), (b, c), (c, d)\} \). The induced equivalence relation on \( \mathcal{P}(U) \) by \( \mathcal{P}(U)/R = \{\emptyset, (U, (a, b), (b, c), (c, d))\} \) is a topology on \( U \).

**Example 3.2:**
Let \( T = \{\emptyset, U, \{a, c, d\}, \{c, d\}\} \) in the previous example. Clearly, both \( \mathcal{R}(T_1) = \{\emptyset, U, \{a, c, d\}, \{b, c, d\}, \{c, d\}\} \) and \( \mathcal{R}(T_2) = \{\emptyset, U, \{a, c, d\}, \{b, c, d\}\} \) are topologies on \( U \). Therefore, \( \mathcal{R}(T_1), \mathcal{R}(T_2) \) are \( \mathcal{R} \) -rough topologies on \( U \). Hence, \( \mathcal{R}(T_1), \mathcal{R}(T_2) \) are \( \mathcal{R} \) -rough topologies on \( U \).

**Proposition 3.1:**
If \( \mathcal{R}(A), \overline{\mathcal{R}(A)} > \) is a lower \( \mathcal{R} \) -rough open set, then \( \mathcal{R}(A), \overline{\mathcal{R}(A)} \) is an \( \mathcal{R} \) -rough open set.

**Theorem 3.3:**
Let \( \mathcal{R}(T), \overline{\mathcal{R}(T)} \) be an \( \mathcal{R} \) -rough topology on \( U \). Then, the family of all \( \mathcal{R} \) -rough open sets on \( U \) is a topology on \( \mathcal{P}(U) \).
If \( < R(A), \overline{R}(A) > \) is an \( R \)-rough open set with respect to \( T_D \), then, using theorem 3.5, \( < R(A), \overline{R}(A) > \) is \( R \)-rough open with respect to \( T_D \). Therefore, \( \mathcal{T}_{T_D} \subseteq \mathcal{T}_{\overline{R}} \). Conversely, if \( < R(B), \overline{R}(B) > \in \mathcal{T}_{\overline{R}} \), then, \( R(B) \in R(t_D) \) and \( \overline{R}(B) \in \overline{R}(t_D) \), because \( \overline{R}(t_D) = T_D = \overline{R}(t_D) \). So, \( < R(B), \overline{R}(B) > \in \mathcal{T}_{T_D} \). Therefore, \( \mathcal{T}_{T_D} \subseteq \mathcal{T}_{\overline{R}} \). Thus, \( \mathcal{T}_{T_D} \subseteq \mathcal{T}_{\overline{R}} \). Hence, \( < R(t_D), \overline{R}(t_D) > \) is equivalent to \( < \overline{R}(t_D), \overline{R}(t_D) > \). Therefore, \( < R(t_D), \overline{R}(t_D) > \) is a discrete \( R \)-rough topology on \( U \).

**Theorem 3.8:**

The pair \( < R(t_1), \overline{R}(t_1) > \) is an \( R \)-rough topology on \( U \), where \( t_1 \) is the indiscrete topology on \( U \).

**Proof:**

The indiscrete topology consists of only \( \emptyset \) and \( U \), which are composed sets on \( U \). Hence, \( \overline{R}(t_1) = t_1 = \overline{R}(t_1) \). Therefore, \( \overline{R}(t_1) \) and \( \overline{R}(t_1) \) are topologies on \( U \). Thus, \( < \overline{R}(t_1), \overline{R}(t_1) > \) is an \( R \)-rough topology on \( U \).

**Definition 3.9:**

Any \( R \)-rough topology on \( U \) which is equivalent to \( < \overline{R}(t_1), \overline{R}(t_1) > \) is called an indiscernible \( R \)-rough topology.

**Remark 3.1:**

The rough sets \( \emptyset, \emptyset \) and \( U, U \) are \( R \)-rough open sets in the indiscrete \( R \)-rough topology on \( U \). However, if \( \emptyset, U \) and \( U, \emptyset \) are rough sets, then they are also \( R \)-rough open sets in the indiscrete \( R \)-rough topology on \( U \).

IV. ROUGH CLOSED SETS, ROUGH CLOSURE AND ROUGH INTERIOR

In this section, some basic concepts of a topological space such as rough closed set, rough interior and rough closure are extended to the proposed rough topological space and the properties are investigated.

**Definition 4.1:**

Let \( < \overline{R}(T), \overline{R}(T) > \) be an \( R \)-rough topology on \( U \). Then the rough set \( < R(A), \overline{R}(A) > \) is respectively called lower \( R \)-rough closed, upper \( R \)-rough closed and \( R \)-rough closed if \( < R(A), \overline{R}(A) > \) is lower \( R \)-rough open, upper \( R \)-rough open or \( R \)-rough open.

**Theorem 4.1:**

A rough set \( R(A), \overline{R}(A) > \) is \( R \)-rough closed iff its rough complement is \( R \)-rough open.

**Proof:**

We have, \( < R(A), \overline{R}(A) > \) is \( R \)-rough closed
\[ \iff < \overline{R}(A), \overline{R}(A) > \) is \( R \)-rough open
\[ \iff < R(A), (R(A)) > \) is \( R \)-rough open
\[ \iff < R(A), (R(A)) > \) is \( R \)-rough open

**Corollary 4.1:**

The rough sets \( \emptyset, \emptyset \) and \( U, U \) are \( R \)-rough closed in any \( R \)-rough topology on \( U \).

**Proof:**

As \( < \emptyset, \emptyset > \) is \( R \)-rough open sets with respect to any \( R \)-rough topology on \( U \), \( < \emptyset, \emptyset > \) and \( < U, U > \) are \( R \)-rough closed in any \( R \)-rough topology on \( U \).

**Theorem 4.2:**

Let \( < \overline{R}(T), \overline{R}(T) > \) be an \( R \)-rough topology on \( U \) and \( \mathcal{F}_T \) denote the family of all \( R \)-rough closed subsets of \( U \). Then,

1. \( \emptyset, \emptyset \in \mathcal{F}_T \)
2. \( \mathcal{F}_T \) is closed under finite rough union.
3. \( \mathcal{F}_T \) is closed under arbitrary rough intersection.

**Proof:**

From theorem 4.1, \( < R(A), \overline{R}(A) > \) is \( R \)-rough closed iff \( < (R(A), \overline{R}(A)) > \) is \( R \)-rough open.

**Definition 4.2:**

Let \( < \overline{R}(T), \overline{R}(T) > \) be an \( R \)-rough topology on \( U \). Then the \( R \)-rough interior denoted by \( \mathcal{R}int(< R(A), \overline{R}(A) >) \) of a rough set \( < R(A), \overline{R}(A) > \) is defined as the union of all \( R \)-rough open sets contained in \( < R(A), \overline{R}(A) > \).

**Definition 4.3:**

Let \( < \overline{R}(T), \overline{R}(T) > \) be an \( R \)-rough topology on \( U \). Then the \( R \)-rough closure denoted by \( \mathcal{R}cl(< R(A), \overline{R}(A) >) \) of a rough set \( < R(A), \overline{R}(A) > \) is defined as the intersection of all \( R \)-rough closed sets containing \( < R(A), \overline{R}(A) > \).

**Remark 4.1:**

From the definition, \( \mathcal{R}int(< R(A), \overline{R}(A) >) \) is the largest \( R \)-rough open set contained in \( < R(A), \overline{R}(A) > \) and \( \mathcal{R}cl(< R(A), \overline{R}(A) >) \) is the smallest \( R \)-rough closed set containing \( < R(A), \overline{R}(A) > \).

**Theorem 4.3:**

Let \( < \overline{R}(T), \overline{R}(T) > \) be an \( R \)-rough topology on \( U \) and \( < R(A), \overline{R}(A) > \) be a rough subset of \( U \). Then,

1. \( < R(A), \overline{R}(A) > \in \mathcal{F}_T \)
2. \( \mathcal{R}int(< R(A), \overline{R}(A) >) = < (R(A), \overline{R}(A)) > \)
3. \( \mathcal{R}int(< R(A), \overline{R}(A) >) \)
4. \( \mathcal{R}int(< R(A), \overline{R}(A) >) \cap < \overline{R}(B), \overline{R}(B) > = < \overline{R}(A), \overline{R}(A) > \cap < \overline{R}(B), \overline{R}(B) > \)

**Proof:**

1. \( < R(A), \overline{R}(A) > \in \mathcal{F}_T \)
2. \( \mathcal{R}int(< R(A), \overline{R}(A) >) = < (R(A), \overline{R}(A)) > \)
3. \( \mathcal{R}int(< R(A), \overline{R}(A) >) \cap < \overline{R}(B), \overline{R}(B) > = < \overline{R}(A), \overline{R}(A) > \cap < \overline{R}(B), \overline{R}(B) > \)
2. Both $\emptyset, \emptyset$ and $U, U$ are $R$-rough open sets. Hence, from (1) we get, $Rint \emptyset, \emptyset = = \emptyset, \emptyset$ and $Rint U, U = = U, U$.

3. $Rint(<R(A), R(A)>)$ is an $R$-rough open set. So, $Rint(Rint(<R(A), R(A)>) ) = Rint(<R(A), R(A)>)$

4. $Rint(<R(A), R(A)>) \cap Rint(<R(B), R(B)>) = Rint(<R(A), R(A)>) \cap Rint(<R(B), R(B)>)$

Then, $<R(X), \overline{R(B)}>$ is an $R$-rough closed set containing $<R(A), \overline{R(A)}>$ and $<R(B), \overline{R(B)}>$.

Thus, $RC((<R(A), \overline{R(A)}>) \cup Rint(<R(B), \overline{R(B)}>)) \supseteq RC((<R(A), \overline{R(A)}>) \cup Rint(<R(B), \overline{R(B)}>)$. Again, $RC((<R(A), \overline{R(A)}>) \cup Rint(<R(B), \overline{R(B)}>))$ is an $R$-rough closed set containing $<R(A), \overline{R(A)}>$ and $<R(B), \overline{R(B)}>$. Therefore, $RC((<R(A), \overline{R(A)}>) \cup Rint(<R(B), \overline{R(B)}>)) = RC((<R(A), \overline{R(A)}>) \cup Rint(<R(B), \overline{R(B)}>))$.

Definition 4.3:
Let $<R(T), \overline{R(T)}>$ be an $R$-rough topology on $U$. Then, the rough set $<R(A), \overline{R(A)}>$ is said to be $R$-dense, if $RC((<R(A), \overline{R(A)}>) = <U, U>$.

Theorem 4.5:
Let $<R(T), \overline{R(T)}>$ be an $R$-rough topology on $U$. Then, a rough set $<R(A), \overline{R(A)}>$ is $R$-dense iff every non-empty $R$-rough open set has non-empty rough intersection with $<R(A), \overline{R(A)}>$.

Proof:
First assume that $<R(A), \overline{R(A)}>$ is $R$-dense. Then, $RC((<R(A), \overline{R(A)}>) = <U, U>$. Consider a non-empty $R$-rough open set $<R(B), \overline{R(B)}>$.

Thus, $<R(B), \overline{R(B)}>$ is an $R$-rough closed set containing $<R(A), \overline{R(A)}>$. Hence, $RC((<R(B), \overline{R(B)}>) \subseteq <R(B), \overline{R(B)}>$.

Therefore, $<U, U> \subseteq <R(B), \overline{R(B)}>$. So, $<U, U> = <R(B), \overline{R(B)}>$. Thus, $<R(B), \overline{R(B)}>= <\emptyset, \emptyset>$. This is a contradiction. Therefore, $<R(A), \overline{R(A)}> \cap <R(B), \overline{R(B)}> = <\emptyset, \emptyset>$.

Conversely, let every non-empty $R$-rough open set has non-empty rough intersection with $<R(A), \overline{R(A)}>$. Since, $RC((<R(A), \overline{R(A)}>)$ is an $R$-closed set containing $<R(A), \overline{R(A)}>$, $(RC((<R(A), \overline{R(A)}>))$ is an $R$-open set which does not intersect $<R(A), \overline{R(A)}>$. This is possible only when $(RC((<R(A), \overline{R(A)}>)) = \emptyset$. Then, $RC((<R(A), \overline{R(A)}>) = <U, U>$. Therefore, $<R(A), \overline{R(A)}>$ is $R$-dense.

V. CONCLUSION
Though there are many studies on the topological properties of rough set approximations, only a few works have been done on the concept of rough topology. In this paper, an attempt has been made to define rough topology on an approximation space as a rough subset of the extended approximation space. The basic concepts of a topological...
space such as rough open set, rough closed set, rough interior and rough closure were extended to the proposed rough topological space and the properties were investigated.

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