A MODEL-FREE FIRST-ORDER METHOD FOR LINEAR QUADRATIC REGULATOR WITH \( \tilde{O}(1/\varepsilon) \) SAMPLING COMPLEXITY

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Abstract. We consider the classic stochastic linear quadratic regulator (LQR) problem under an infinite horizon average stage cost. By leveraging recent policy gradient methods from reinforcement learning, we obtain a first-order method that finds a stable feedback law whose objective function gap to the optima is at most \( \varepsilon \) with high probability using \( \tilde{O}(1/\varepsilon) \) samples, where \( \tilde{O} \) hides polylogarithmic dependence on \( \varepsilon \). Our method is the first online (i.e., single trajectory) algorithm with this sampling complexity. The improved dependence on \( \varepsilon \) is achieved by showing the accuracy scales with the variance rather than the standard deviation of the gradient estimation error. Our developments that result in this improved sampling complexity fall in the category of actor-critic algorithms. The actor part involves a gradient descent-type method, while in the critic part, we utilize a conditional stochastic primal-dual method and show that the algorithm has an accelerated rate of convergence when paired with a shrinking multi-epoch scheme.

Key words. linear quadratic regulator, actor-critic, primal-dual methods, non-convex, Markovian noise

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1. Introduction. The linear quadratic regulator (LQR) problem is a classic problem in control theory studied extensively since the work of Kalman [16, 17]. Of interest is the infinite horizon linear-quadratic stochastic control problem. Techniques for solving this problem go back to the 1960s. It is well known that when this system is controllable, the optimal feedback law is a linear static state feedback of \( u_t^* = -K^* x_t \) for some gain matrix \( K^* \in \mathbb{R}^{m \times n} \), where \( x_t \in \mathbb{R}^n \) is the current state. The matrix \( K^* \) can be obtained by solving the algebraic Ricatti equation (ARE) \([1, 2]\). For computational aspects in regards to the solution of AREs, as well as connections to linear matrix inequalities, the classic references are \([35]\) and \([3]\), respectively. When the parameters of the system are unknown, one method is to obtain estimates using system identification techniques \([28]\) and the resulting uncertain system is then subject to robust control design \([50]\).

In recent years, methods from machine learning have been used for optimal control, in particular, for solving the LQR problem \([10, 30, 41, 43]\). For example, one can estimate the system parameters using regularized least-squares. The estimated system is then viewed as the nominal system and the solution to the LQR problem is obtained by solving the corresponding ARE. This is known as the certainty equivalence principle \([7, 32]\), which is a model-based method. In contrast, there are model-free methods, which aim to solve the problem without explicitly estimating the system parameters. Many model-free methods are inspired by problems in reinforcement learning (RL) \([42]\). RL tackles problems such as LQR using optimization methods that only require an oracle that can simulate the costs and evolution of the system. There are a plethora of model-free methods, and most can be categorized as either

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approximate dynamic programming [21, 43] or gradient-based (first- or zeroth-order) methods [24, 26, 40].

We focus on gradient-based methods. It is known LQR is non-convex (the set of stable feedback laws is non-convex [10]), so generally speaking at best we can only find locally optimal solutions. The ubiquitous gradient descent has been used in practice [45] while its convergence to a local optimal solution was proved in [47]. It was not until the breakthrough result of [10] that showed gradient descent converges to the global optimal solution by proving the objective function of LQR satisfies the so-called Polyak-Lojasiewicz-condition [29, 36], or PL-condition.

In this paper, we develop a first-order method to efficiently solve LQR based on an actor-critic method [18]. The outer loop, or the actor step, views the LQR problem as a nonlinear optimization problem with the PL-condition, or a gradient domination condition. To solve this, we present a method equivalent to natural gradient [10, 14]. In each actor step, one needs to solve the control Lyapunov equation, which is done by the critic. The critic solves this via a shrinking multi-epoch conditional stochastic primal-dual (CSPD) method.

Let us summarize our contributions. For the actor, we provide a novel analysis of the natural gradient method for LQR, where we bound the gradient estimation error from the critic step in a new way so that the error squared – rather than the error – needs be at most the accuracy $\varepsilon$. This reduces the accuracy needed from the critic and is one of the key ideas to improve sampling complexity. For the critic, our newly proposed shrinking multi-epoch CSPD solves min-max problems with Markovian noise (i.e., data generated from an ergodic Markov chain). By incorporating a primal predictive step to accelerate convergence, sampling procedure to mitigate temporal dependency, and multi-epoch scheme to reduce the overall error, our algorithm improves upon prior primal-dual methods and efficiently finds near optimal solutions. The resulting actor-critic method outputs a policy with function optimality gap at most $\varepsilon$ with probability $1 - \delta$ using $O(\varepsilon^{-1}(\ln(1/\varepsilon) + \ln(1/\delta))^7)$ samples (see Theorem 4.11 for the hidden constants). To the best of our knowledge, this result is new for the online setting, where data is generated from a single trajectory of an ergodic Markov chain. Most importantly, our result uses minimal assumptions.

To justify our claim of a novel sampling complexity, let us review prior works. A similar actor-critic method yields $O(\varepsilon^{-5})$ sampling complexity [48], which was later improved to $O(\varepsilon^{-3/2})$ [49] and then $O(\varepsilon^{-1}(\ln \varepsilon^{-1})^2)$ [51]. However, the third listed method is not for the online setting since it generates multiple independent trajectories from some fixed state. Therefore, our algorithm is the first to obtain $O(\varepsilon^{-1})$ sampling complexity in the online setting, which may be a more realistic setting for systems that cannot be reset to a fixed state. Additionally, our result makes weaker assumptions than prior works. The latter two works [49, 51] posit an almost sure bound on the mixing rate and norm of every policy generated by the algorithm, which may not be realistic since it is an assumption on the behavior of a stochastic algorithm. In contrast, we only assume the initial policy given to us is stable and $\varepsilon$ is not too large. We avoid these assumptions by obtaining high probability results and identifying a relationship between mixing rates, policy norm, and objective value.

Our discussion so far has focused on model-free methods. For model-based methods, a least-squares estimator can estimate the dynamics using $O(\varepsilon^{-2})$ samples [7]. This was later improved to $O(\varepsilon^{-3})$ by noticing the error scales as the square of the estimation error [32]. However, this method also requires generating multiple indepen-

\[^1\tilde{O}\) hides polylogarithmic dependence on $\varepsilon$.}
dent trajectories and is therefore not in the online setting. Thus, our proposed method shows the model-free case can be competitive with the model-based case in terms of the dependence on $\varepsilon$, even without generating independent trajectories. Moreover, one may prefer model-free since model-based methods can suffer from model bias, where insufficient samples and unaccounted-for uncertainties can lead to poorly fitted models and unstable policies [7, 8].

The paper is organized as follows. In section 2, we formulate the LQR optimization problem, explore its properties, and cover related works. In section 3, we present a gradient method to solve the LQR problem. This algorithm requires an accurate solution to the control Lyapunov equation, which is solved in section 4 by using a shrinking multi-epoch stochastic primal-dual method. We finish with some preliminary numerical experiments in section 5.

2. Preliminaries, problem formulation, and related works.

2.1. Notation. We refer to the norm $\| \cdot \|$ as the standard $\ell_2$-norm for vectors and the induced $\ell_2$-norm for matrices. The norm $\| \cdot \|_F$ is the Frobenius norm for a given matrix. The notation $\rho(\cdot)$ refers to the spectral radius of a given matrix. For symmetric matrices $A$ and $B$, we use the standard Loewner order of $A - B \succeq 0$ to mean that $A - B$ is semidefinite and $A - B \succ 0$ to mean that it is positive definite. For any symmetric matrix $X$, let $\text{vec}(X)$ be the vectorization of the upper triangular part of $X$ with the off-diagonals scaled by $\sqrt{2}$. Unless otherwise specified, we write $x = O(y)$ if there is an absolute constant $C$ (i.e., a scalar independent of $x$ and $y$) such that $x \leq C \cdot y$.

2.2. Linear systems and quadratic control. We consider a discrete-time, time-invariant linear dynamical system with disturbances,

$$x_{t+1} = Ax_t + Bu_t + w_t, \quad t \in \mathbb{Z}_+,$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the control input, and $w_t \in \mathbb{R}^n$ is the stochastic disturbance, as well as $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. It is assumed that the random variables $\{x_0, w_0, w_1, \ldots\}$ are jointly independent and Gaussian ($x_0 \sim \mathcal{N}(0, \Sigma_0)$ and $w_t \sim \mathcal{N}(0, \Psi)$) where the covariance matrices $\Sigma_0 \succ 0$, $\Psi \succ 0$ are known. The infinite horizon time-average cost to be minimized over causal (i.e., nonanticipative) policies is

$$J = \lim_{T \to \infty} T^{-1} \mathbb{E} \left[ \sum_{t=0}^{T-1} x_t^T Q x_t + u_t^T R u_t \right],$$

where $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ are symmetric positive definite matrices. We assume the matrices $(A, B)$ form a controllable pair. This is a classic problem in control theory whose solution can be reviewed in [5,22]. The average cost is minimized by a linear state feedback policy $u_t = -K^* x_t = -(R + B^T B)^{-1} B^T P A x_t$ for times $t \in \mathbb{Z}_+$, with $P \in \mathbb{R}^{n \times n}$ being the unique positive definite solution to the algebraic Riccati equation $P = Q + A^T PA - A^T P B (R + B^T B)^{-1} B^T P A$. The focus of this paper is on the model-free case, and we will restrict ourselves to randomized linear state feedback policies of the form

$$u_t = -K x_t + v_t, \quad v_t \sim \mathcal{N}(0, \sigma^2 I_m),$$

where $v_t$ is additive Gaussian noise with some noise level $\sigma \geq 0$ and $I_m$ is the identity matrix of dimension $m$. The additional feedback law disturbance, $v_t$, is often included to allow for exploration of the state and action space [7,21,48]. We also require that the
basic random variables \{x_0, u_0, v_0, w_1, v_1, \ldots \} are jointly independent. The resulting closed loop dynamics under (2.3) are

\begin{equation}
x_{t+1} = (A - BK)x_t + w_t + Bv_t.
\end{equation}

We denote the set of stable policies by

\begin{equation}
S = \{ K \in \mathbb{R}^{m \times n} \mid \rho(A - BK) < 1 \}.
\end{equation}

We will search in an iterative fashion for a feedback gain \( K \in S \) that yields an approximation to the optimum within a prespecified degree of fidelity. First, we rewrite the cost \( J \) as a function of \( K \) in closed-form\(^2\), which yields the following optimization problem for LQR:

\[
\min_{K} J(K) = \left\{ \begin{array}{ll}
\text{Tr}((Q + K^T R K) \Sigma_K + \sigma^2 R) & \text{if } K \in S \\
\infty & \text{otherwise}
\end{array} \right.,
\]

Denote \( K^* \in S \) as the optimal solution. Here, the matrix \( \Sigma_K \in \mathbb{R}^{n \times n} \) is the solution to the Lyapunov equation

\begin{equation}
\Sigma_K = (\Psi + \sigma^2 B B^T) + (A - BK) \Sigma_K (A - BK)^T,
\end{equation}

and it corresponds to the steady state covariance matrix of the state vector. The matrix \( P_K \in \mathbb{R}^{n \times n} \) is the unique positive definite solution to the Lyapunov equation \( P_K = Q + K^T R K + (A - BK)^T P_K (A - BK) \). Now we will establish some properties pertaining to the objective value \( J(K) \), mixing rate \( \rho(A - BK) \), and policy norm. We defer the proof to Appendix A.

**Lemma 2.1.** Let \( \rho \equiv \rho(A - BK) \). If \( K \in S \), then

\[
\| \Sigma_K \| \leq \text{Tr}(\Sigma_K) \leq \sigma_{\min}(Q)^{-1} J(K) \]
\[
\| P_K \| \leq \text{Tr}(P_K) \leq \sigma_{\min}(\Psi)^{-1} J(K) \]
\[
\| K \|^2_F \leq |\sigma_{\min}(\Psi)\sigma_{\min}(R)|^{-1} J(K) \]
\[
(1 - \rho^2)^{-1} \leq |\sigma_{\min}(\Psi)\sigma_{\min}(Q)|^{-1} J(K).
\]

The following lemma contains an expression for the policy gradient \( \nabla J(K) \).

**Lemma 2.2.** For any \( K \in S \), the gradient of the cost function \( J(K) \) is \( \nabla J(K) = 2E_K \Sigma_K \) where the matrix \( E_K \), the natural gradient, is given by

\begin{equation}
E_K = (R + B^T P_K B)K - B^T P_K A.
\end{equation}

The proof of this lemma can be traced to [9] with minor adjustments to account for the fact that we are working in discrete time. The name *natural gradient* is from the reinforcement learning literature [10, 15]. Notice \( E_K \) does not depend on the control input noise \( \sigma \) used in (2.3).

If we can accurately estimate the gradient, we can characterize the change in objective value when we descend along the negative of this direction. The following result, also known as the performance difference lemma, quantifies this statement.

\(^2\)The case \( \sigma = 0 \) (i.e., \( u_t = -K x_t \)) is a classical result found in textbooks [5], while the case of \( \sigma > 0 \) can be derived by substituting in \( u_t = -K x_t + v_t \) into (2.2) and using (2.4) in place of (2.1) to deduce the steady state covariance matrix \( \Sigma_K \).
Lemma 2.3. For any \( K', K \in \mathcal{S} \), where \( E_{K'} \) is the exact natural gradient for \( K' \),

\[
J(K) - J(K') = 2\text{Tr}(\Sigma_K (K - K')^T E_{K'}) + \text{Tr}(\Sigma_K (K - K')^T (R + B^T P_K B) (K - K')).
\]

A proof can be found in [10, Lemma 6]. The above result is akin to the smoothness of a function, where the change in \( J(K') - J(K) \) can be upper bounded by a linear plus quadratic term. By fixing \( K' = K^* \), the above result derives the so-called PL-condition or gradient domination condition \([29, 36]\).

Lemma 2.4. For any \( K \in \mathcal{S} \),

\[
\frac{\sigma_{\min}(\Psi)}{\|R + B^T P_K B\|} \|E_K\|^2_F \leq J(K) - J(K^*) \leq \frac{\|\Sigma_K\|}{\sigma_{\min}(R)} \|E_K\|^2_F.
\]

A proof can be readily deduced from \([9]\). In view of Lemma 2.2, an accurate estimate of \( E_K \) requires an estimate of the solution \( P_K \) from the Lyapunov equation as well as the system parameters \( R, B, \) and \( A \). Alternatively, one can estimate \( E_K \) without such knowledge. First, it has a closed-form expression. Denote \( z = [x, u] \) as a state-action pair, and analogously let \( Q_K(z) = Q_K(x, u) \) and \( c(z) = c(x, u) \).

From \([48, Proposition 3.1]\), \( Q_K(z) = z^T \Theta(K)z - \sigma^2 \cdot \text{Tr}(P_K \Sigma_K) \), where \( \Theta(K) \in \mathbb{R}^{(n+m)\times(n+m)} \) is defined as

\[
\Theta(K) = \begin{bmatrix} A^T \\ B^T \end{bmatrix} P_K \begin{bmatrix} A & B \\ 0 & R \end{bmatrix}.
\]

The above decomposition makes evident the fact that \( \Theta(K) \succ 0 \). An important relation between the Q-function and natural gradient \( E_K \) is seen by

\[
E_K = \Theta(K)_{22} K - \Theta(K)_{21},
\]

where \( \Theta(K)_{ij} \) is the \((i, j)\)-th block of \( \Theta_K \) if we view it as a \( 2 \times 2 \) block matrix. Thus, by obtaining \( \Theta(K) \), we can construct \( E_K \). We will show one way to do so by exploiting the second useful property: the average reward Bellman equations (c.f. \([38, Theorem 8.2.6] \) and \([26, Equation 2.4] \)), which is a fixed-point equation:

\[
Q_K(z) = c(z) - J(K) + \mathbb{E}[Q_K(z')] \mid z, \forall z \in \mathbb{R}^{n+m},
\]

where \( z' \) is the next state-action pair with gain matrix \( K \) conditioned on the current state-action pair \( z \). Since this is an infinite-dimension linear system of equations, we reduce to a finite-dimension one. To do so, define

\[
\varphi(z) = \text{svect}(zz^T), \quad \theta(K) = \text{svect}(\Theta(K)), \quad \vartheta(K) = \begin{bmatrix} J(K) \\ \theta(K) \end{bmatrix} \in \mathbb{R}^{\binom{n+m}{2} + 1}.
\]

One can check \( Q_K(z) = \varphi(z)^T \theta(K) + (\text{constants independent of } z) \). Denote the stationary measure on the state and control space w.r.t. to the current gain matrix \( K \) as \( \Pi_K \). By multiplying (2.9) by \( \varphi(z) \), taking expectation w.r.t. \( z \sim \Pi_K \), and re-arranging terms, we arrive at the linear system of equations,

\[
H \vartheta(K) = b,
\]
where the matrix and vector are defined as
\[
(2.12) \quad H = \begin{bmatrix} 1 & 0 \\ \mathbb{E}_{\Pi_K}[\phi(z)] & \Xi_K \end{bmatrix}, \quad b = \begin{bmatrix} \mathbb{E}_{\Pi_K}[c(z)] \\ \mathbb{E}_{\Pi_K}[c(z)\phi(z)] \end{bmatrix},
\]
and \( \Xi_K = \mathbb{E}_{\Pi_K}[\phi(z)(\phi(z) - \phi(z'))^T] \). If \( \sigma \) is sufficiently large, then the solution to (2.11) is unique (since \( H \) is invertible from Lemma 4.5), and therefore it is also equal to the solution to (2.9). So by solving for \( \vartheta(K) \), we can extract \( \Theta(K) \) and construct the natural gradient \( E_K \) via (2.8).

2.3. Related Works. Two prominent paradigms of LQR being studied are the random initialization and noisy dynamics model [31]. The random initialization assumes only the initial state is random, and the remaining state transitions are deterministic. The noisy dynamics model, which we study in this paper, considers both a random initialization and perturbations in each state transition.

Within the noisy dynamics setting, an infinite-horizon average cost was studied by [48] using an actor-critic method. Around the same time, [21] used an approximate policy iteration from the RL literature. More recently, [49,51] utilize a stochastic gradient descent-type analysis to solve LQR. While they can get similar convergence rates, their results rely on a strong assumption that \( \rho(A - BK_t) \) is almost surely less than one for every \( t \), where \( K_t \) is the gain matrix at iteration \( t \). The aforementioned methods are model-free. As for model-based methods, [7] used a least-squares estimator for the transition dynamics and then use tools from system-level synthesis to obtain their controller. [32] later improved the sampling complexity by noticing the function gap scales as the square of the parameter error.

Similar problems include the infinite-horizon discounted objective value. The PhD thesis of [4] investigated the setting where there was no noise. In the noisy dynamic setting, [11] developed algorithms that find near-optimal solutions with high probability guarantees. The authors of [43] studied the sampling complexity of a least-squares temporal difference in this setting. A question of minimizing a discount factor of one on the random initialization setting was first shown to be solvable to global optimality by the breakthrough paper of [10]. Since their work, improvements in the sampling complexity have been proposed in the works of [31] and [33]. It should be mentioned these last two works explored the use of zeroth-order methods. This setting, while model-free, requires a generative model, where one can start running the linear dynamics from any initial state.

As mentioned in the introduction, our proposed shrinking multi-epoch conditional stochastic primal-dual algorithm can solve general min-max problems with Markovian noise. Similar primal-dual methods, under the guise of gradient temporal differencing [27, 44], have been proposed. But our method leverages several new tools to improve upon the prior art. First, we incorporate a primal predictive step to improve convergence [23]. Second, we exploit a sharpness condition [37] via the shrinking multi-epoch scheme to significantly reduce the deterministic error rates [12]. Third, we introduce a new sampling procedure borrowed from conditional temporal differencing to improve dependence on the mixing rates [19,34].

3. The actor: Policy optimization. Our goal is to solve the optimization problem
\[
\min_{K \in \mathcal{S}} J(K),
\]
where recall \( \mathcal{S} \) is the set of stable policies defined in (2.5). Since the optimal policy \( K^* \) lies in the interior of the open set \( \mathcal{S} \) [9], a necessary condition of optimality is
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\[ \nabla J(K^*) = 2E_K \cdot \Sigma_{K^*} = 0. \] A popular approach to minimize the gradient norm is by a gradient descent-type method. The natural policy gradient (NPG) from the reinforcement learning literature is a type of preconditioned gradient descent, where one moves in the negative direction of the natural gradient \( E_{K_t} \) rather than the gradient \( E_{K_t} \Sigma_{K_t} \) [15]; the advantages here are the need to only estimate \( E_{K_t} \) and possibly faster convergence. Each iteration of NPG first estimates the natural gradient \( E_{K_t} \) by solving the linear system of equations in (2.11). In the next section, we provide a method to complete this task. With \( E_{K_t} \), we get the next policy \( K_{t+1} \) (Line 4). We repeat this process for a total of \( T \) times and return the final policy \( K_T \).

Algorithm 3.1 Natural Policy Gradient Descent

1: Input: \( K_0, \eta \in \mathbb{R}_{++} \)
2: for \( t = 0, \ldots, T - 1 \) do
3: Solve \( H \left[ \begin{array}{c} J(K_t) \\ \theta(K_t) \end{array} \right] = b \) with Algorithm 4.2 and form \( E_{K_t} = \Theta(K_t)_{22} K_t - \Theta(K_t)_{21} \)
4: \( K_{t+1} = K_t - 2\eta E_{K_t} \)
5: end for
6: return \( K_T \)

Let us now move on to proving the convergence of Algorithm 3.1. To ease notation in our analysis, we define the following positive constants:

\[
C_1 := 2\|R\| + 2\sigma_{\text{min}}^{-1}(\Psi)\|B\|^2 J(K_0) \\
C_2 := \sigma_{\text{min}}(\Psi)\sigma_{\text{min}}(Q)\sigma_{\text{min}}(R) \\
C_3 := J(K_0)/\sigma_{\text{min}}(Q) \\
C_4 := \|A\| + \|B\| \sqrt{2J(K_0)/[\sigma_{\text{min}}(\Psi)\sigma_{\text{min}}(R)]}.
\]

We also define the natural gradient error,

\[
\delta_t := E_{K_t}^* - E_{K_t},
\]

where \( E_{K_t}^* \) is the true natural gradient and \( E_{K_t} \) is the stochastic estimate. We will now show a contraction property for the LQR problem.

**Proposition 3.1.** Suppose \( K_t \in \mathcal{S}, J(K_t) \leq 2J(K_0) \), Algorithm 3.1 uses constant step size \( \eta_t = 1/(2C_1) \), and let \( K_{t+1} \) be the computed (stochastic) policy. If \( K_{t+1} \in \mathcal{S} \), then

\[
J(K_{t+1}) - J(K^*) \leq \left( 1 - \frac{C_2}{2J(K^*)C_1} \right) [J(K_t) - J(K^*)] + \epsilon_t,
\]

where \( \frac{C_2}{2J(K^*)C_1} \in (0, \frac{1}{4}] \) and

\[
\epsilon_t := \frac{1}{C_1} \left[ \|\delta_t\|^2 F_{K_{t+1}} + \sigma_{\text{min}}(\Psi)\text{Tr}([E_{K_t}^*]^T \delta_t) \right].
\]
Proof. By the performance difference lemma (Lemma 2.3) and definition of $K_{i+1}$,

$$J(K_{i+1}) - J(K_i) = -4\eta \text{Tr}(\Sigma_{K_{i+1}}^T E_{K_i}^T \delta_i) - 4\eta \text{Tr}(\Sigma_{K_{i+1}} E_{K_i}^T (R + B^T P_{K_i} B) E_{K_i})$$

$$+ 4\eta^2 \text{Tr}(\Sigma_{K_{i+1}} E_{K_i}^T (R + B^T P_{K_i} B) E_{K_i})$$

$$\leq 2\eta \|E_{K_i} S^T\|_F^2 + 2\eta \|E_{K_i}^T \delta_i S^T\|_F^2 - (4\eta - 4\eta^2 \|R + B^T P_{K_i} B\|) \text{Tr}(\Sigma_{K_{i+1}} E_{K_i}^T E_{K_i})$$

$$\leq 2\eta \|E_{K_i} S^T\|_F^2 - (2\eta - 4\eta^2 R + B^T P_{K_i} B) \text{Tr}(\Sigma_{K_{i+1}} E_{K_i}^T E_{K_i})$$

$$\leq \frac{1}{C_1} \|\delta_i\|_F^2 \|\Sigma_{K_{i+1}}\| - \frac{1}{2C_1} \text{Tr}(\Sigma_{K_{i+1}} E_{K_i}^T E_{K_i})$$

$$\leq \frac{1}{C_1} \|\delta_i\|_F^2 \|\Sigma_{K_{i+1}}\| - \frac{\sigma_{\min}(\Psi)}{2C_1} \text{Tr}(E_{K_i}^T E_{K_i}),$$

where $\Sigma_{K_{i+1}} = S^T S$ (since $\Sigma_{K_{i+1}}$ is symmetric positive semidefinite, or sym-psd). The first inequality used Cauchy-Schwarz and Young’s inequality to show $|\text{Tr}(M^T N)| \leq \|M\|_F \|N\|_F \leq \frac{\|M\|_F^2}{2} + \frac{\|N\|_F^2}{2}$ for matrices $M$ and $N$, and $|\text{Tr}(MN)| \leq \|M\| \|N\|$ when $N$ is sym-psd. The second inequality is by $\|\delta_i S^T\|_F^2 = \text{Tr}(\Sigma_{K_{i+1}} \delta_i^T \delta_i)$ and

$$\|R + B^T P_{K_i} B\| \leq \|R\| + \|B\| \|P_{K_i}\| \leq \|R\| + \sigma_{\min}^{-1}(\Psi) J(K_i) \|B\|^2 \leq C_1/2,$$

which is due to $J(K_i) \leq 2J(K_0)$ and Lemma 2.1. In the third inequality, we used $\text{Tr}(MN) \geq \sigma_{\min}(M) \text{Tr}(N)$ when $N$ is sym-psd and $\Sigma_{K_{i+1}} \geq \Psi > 0$ (c.f. (2.6)). Noting $\text{Tr}(\delta_i^T \delta_i) \geq 0$, we have $\text{Tr}(E_{K_i}^T E_{K_i}) \geq \text{Tr}([E_{K_i}^*]^T E_{K_i}^*) - 2\text{Tr}([E_{K_i}^*]^T \delta_i)$. Altogether,

$$J(K_{i+1}) - J(K_i) \leq - \frac{\sigma_{\min}(\Psi)}{2C_1} \text{Tr}([E_{K_i}^*]^T E_{K_i}^*) + \frac{1}{C_1} \|\delta_i\|_F^2 \|\Sigma_{K_{i+1}}\| + \sigma_{\min}(\Psi) \text{Tr}([E_{K_i}^*]^T \delta_i)$$

$$\leq - \frac{C_2}{2J(K^*)C_1} [J(K_i) - J(K^*)] + \epsilon_i.$$

where we used the PL-condition (Lemma 2.4) followed by the bound on $\|\Sigma_{K_i}\|$ (Lemma 2.1). Adding $J(K_i) - J(K^*)$ to both sides finishes the proof for (3.3).

The bound of $\frac{C_2}{2J(K^*)C_1} \epsilon_i \in (0, \frac{1}{4})$ follows by noticing all the constants are positive and bounded, and using $C_1$ and $C_2$ from (3.1), Lemma 2.1, and $K^* \in S$ to lower bound $2J(K^*)C_1 \geq 4J(K^*) \|R\| \geq 4\sigma_{\min}(R) \sigma_{\min}(\Psi) \sigma_{\min}(Q)/(1 - \rho(A - B K^*)) \geq 4C_2$.

The key idea of this result, highlighted by the inequality marked with ($\ast$) in the proof, is to bound the error term $\text{Tr}(\Sigma_{K_{i+1}} E_{K_i}^T \delta_i)$ by the squared norm $O(\|\delta_i\|_F^2)$ rather than $O(\|\delta_i\|_F)$ (c.f., the error scales with $O(\|\delta_i\|_F)$ in [48, Eq. (5.64)]). We then only need to bound $\|\delta_i\|_F^2 \leq \epsilon$, which is a weaker requirement than $\|\delta_i\|_F \leq \epsilon$ when $\epsilon$ is small.

The proposition above by itself is incomplete, though. First, it assumes $K_{i+1} \in S$ and second, the error $\epsilon_i$ depends on $\|\Sigma_{K_{i+1}}\|$. To resolve these issues, we introduce two results, adapted from [10] as Lemma 15 and 16.

**Lemma 3.2.** If $K_i \in S$, $E_{K_i}^*$ is the exact natural gradient for $K_i$, and the step size satisfies $\eta_i \leq \|R + B^T P_{K_i} B\|^{-1}$, then $K_{i+1} = K_i - 2\eta_i E_{K_i}^* \in S$.

**Lemma 3.3.** Suppose $K \in S$ and $\|K' - K\| \leq \frac{\sigma_{\min}(Q) \sigma_{\min}(\Sigma)}{4J(K)B^T(A - BK)^{(1)}}$ for some matrix $K'$. Then $K' \in S$ and $\|\Sigma_{K'} - \Sigma_K\| \leq 4\left(\frac{J(K)}{\sigma_{\min}(Q)}\right)^2 \|B\| \|A - B K\|^{(1)} \|K' - K\|.$
We will now apply the above results in the following proposition. Similar to $K_{t+1}$ from Line 4, denote the policy update with exact natural gradient as $K_{t+1}^* = K_t - 2\eta_t E_t^r$.

**Proposition 3.4.** Suppose the assumptions from Proposition 3.1 hold. Also suppose $\|K_{t+1} - K_{t+1}^*\|$ satisfies the bound in Lemma 3.3. Then

$$J(K_{t+1}) - J(K^*) \leq \left(1 - \frac{C_2}{4J(K^*)C_1}\right) [J(K_t) - J(K^*)] + \epsilon_t,$$

where

$$\epsilon_t = \frac{1}{C_1} \left[ \frac{\sigma_{\min}(\Psi)}{2C_2} \|\delta_t\|_F^2 + 2C_3 \|\delta_t\|_F^2 + \frac{16C_3^2 \|\Psi(1 + C_4)\| \|\delta_t\|_F^2}{\sigma_{\min}(\Psi) C_1} \right].$$

**Proof.** Using (3.5), we get $\eta_t = (2C_1)^{-1} \leq \frac{4\|R+B^TP_{K_t}B\|}{\sigma_{\min}(\Psi)}$. Then Lemma 3.2 tells us $K_{t+1}^* \in \mathcal{S}$. Now, using Lemma 2.1 to bound $\|K_{t+1}^*\|_F$,

$$\frac{\|B\| \|A - BK_{t+1}^*\| + 1}{\sigma_{\min}(\Sigma_{K_{t+1}^*})} \leq \frac{\|B\| \|A\| + \|B\| \|K_{t+1}^*\|_F + 1}{\sigma_{\min}(\Psi)} \leq \frac{\|B\| \|A\| + \|B\| \sqrt{J(K_{t+1}^*)} \sigma_{\min}(\Psi) + 1}{\sigma_{\min}(\Psi)} \leq \frac{\|B\| \|C_4\| + 1}{\sigma_{\min}(\Psi)}.$$

where we used $\Sigma_K \succeq \Psi \succ 0$ from (2.6) in the first line and $J(K_{t+1}^*) \leq J(K_t) \leq 2J(K_0)$ (from Proposition 3.1) and $C_4$ from (3.1) in the last line. In addition, Lemma 2.1 also derives $\|\Sigma_{K_{t+1}^*}\| \leq \frac{J(K_{t+1}^*)}{\sigma_{\min}(\Psi)} \leq \frac{2J(K_0)}{\sigma_{\min}(\Psi)} = 2C_3$. Then upon applying Lemma 3.3,

$$\left\|\Sigma_{K_{t+1}^*}\right\| \leq \left\|\Sigma_{K_{t+1}}\right\| \left\|\Sigma_{K_{t+1}} - \Sigma_{K_{t+1}^*}\right\| \leq 2C_3 + 4(2C_3)^2 \cdot \frac{\|B\| \|A\| + \|B\| \sqrt{J(K_{t+1}^*)} \sigma_{\min}(\Psi) + 1}{\sigma_{\min}(\Psi)} \|\delta_t\|,$$

with $\delta_t = E_t^r - E_t$. We have bounded the first term in $\epsilon_t$ from (3.4). We now bound the second term: $\sigma_{\min}(\Psi) \text{Tr}(E_{t+1}^r, \delta_t)/C_1$. By Young’s inequality,

$$\text{Tr}(E_{t+1}^r, \delta_t) \leq \frac{C_2}{4\|K^*\| \|R+B^TP_{K_t}B\|} \cdot \|E_{t+1}^r\|_F^2 + J(K^*) \|R+B^TP_{K_t}B\| \cdot \|\delta_t\|_F^2 \leq \frac{C_2}{4\sigma_{\min}(\Psi) J(K^*)} \cdot \left[ J(K_t) - J(K^*) \right] + \frac{J(K^*) C_1}{2C_2} \cdot \|\delta_t\|_F^2,$$

where in the last line we used the PL-condition (Lemma 2.4) and (3.5) to bound the two summands. We finish by applying both bounds above back into (3.3). \[\square\]

We can now prove convergence results for Algorithm 3.1. To simplify notation, define the condition number $\kappa = 8J(K^*)C_1/C_2 \geq 16$ (see Proposition 3.1).

**Theorem 3.5.** Let $l := \lceil \kappa \rceil$ and $\varepsilon \in (0, J(K_0))$. If $K_0 \in \mathcal{S}$, step size is $\eta_t = 1/(2C_1)$, and gradient error from (3.2) satisfies $\|\delta_t\|_F^2 \leq \min\{C_5, C_6, C_7\}$, where

$$C_5 := \left( \frac{\|B\| \sigma_{\min}(Q) \|R+B^TP_{K_t}B\|}{2(C_4 + 1)} \right)^2, \quad C_6 := \left( \frac{\sigma_{\min}(\Psi) \cdot C_2}{1920J(K_0) \|B\| C_3 (C_4 + 1)} \right)^{2/3},$$

$$C_7 := \frac{C_2}{60J(K_0)} \min\left( \frac{\sigma_{\min}(\Psi) J(K^*) C_1}{4C_3}, \frac{1}{C_6} \right),$$

(3.8)

then

$$\|K_{l+1} - K_{l+1}^*\|_F \leq \frac{C_2}{\sigma_{\min}(\Psi) J(K^*) C_1},$$

for all $l \geq 1$, and

$$\eta_t \leq \frac{\kappa}{4C_2} \sigma_{\min}(\Psi) J(K^*) C_1.$$
then the following holds for every $t = 0, \ldots, T$:
1. Stability: $K_t \in S$.
2. Monotonicity or convergence: either $J(K_t) \leq J(K_{t-1})$ or $J(K_t) - J(K^*) \leq \varepsilon$.
3. Linear rate: $J(K_t) - J(K^*) \leq 2^{-[t/l]} J(K_0)$ as long as $t \leq l \cdot \log_2(J(K_0)/\varepsilon)$.

Proof. We use mathematical induction, where the base case of $t = 0$ holds by our assumptions. Now consider some $t + 1 \leq T$. We first show $K_{t+1} \in S$. Recall $K_{t+1}$ from Line 4 and the error-free update, $K_{t+1}^* = K_t - 2\eta E_{K_t}$, where $\eta = 1/(2C_1)$. Then

$$
\|K_{t+1} - K_{t+1}^*\| = \frac{\|\delta_t\|_F}{C_1} \leq \frac{\|B\|\sigma_{\min}(Q)}{4\sigma^{-1}_\min(\Psi)} \frac{\|B\|2J(K_0)}{4J(K_t)} \cdot 2(C_4 + 1)
$$

where the first inequality is from recalling $C_1$ from (3.1) and using $\|\delta_t\|_F \leq C_5$, the second inequality is from $J(K_t) \leq \max\{J(K_0), J(K^*) + \varepsilon\} \leq 2J(K_0)$ (since $\varepsilon \leq J(K_0)$), and the third inequality can be shown similarly to (3.7). Therefore, the hypothesis for Lemma 3.3 is satisfied, and so $K_{t+1} \in S$.

Next, we prove monotonicity or convergence. Suppose $J(K_t) - J(K^*) > \varepsilon$. By the inductive hypothesis, $K_t \in S$ and we already showed $J(K_t) \leq 2J(K_0)$ and $K_{t+1} \in S$.

Invoking Proposition 3.4 and adding $J(K_t) - J(K_t)$,

$$
J(K_{t+1}) - J(K_t) \leq -\frac{C_2}{4J(K_t)^{C_1}} [J(K_t) - J(K^*)] + \epsilon_t
$$

where $\epsilon_t$ is defined in (3.6). By our bounds $\|\delta_t\|_F \leq \min\{C_6, C_7\}$, we ensure $\epsilon_t \leq \frac{C_7}{4J(K_t)^{C_1}}$. Plugging back into (3.9) and noticing $J(K_t) - J(K^*) > \varepsilon$, we get $J(K_{t+1}) \leq J(K_t)$.

Finally, let us show linear rate of convergence. To simplify our notation, let us define the constant $Z = \frac{C_5}{5C_1}$. We already showed a bound on $\epsilon_t$ in the previous paragraph, which is equivalent to $\epsilon_t \leq \frac{Z}{5J(K_0)^{C_1}}$. Recalling that we consider iterations $t \leq l \cdot \log_2(J(K_0)/\varepsilon)$, this implies $\epsilon_t \leq 2^{-([t/l] - 2)}(Z/2)$. Thus, using our bound on $\epsilon_t$ and choice of $l$, the recursion (3.9) can be simplified into (c.f., [24, Lemma 11])

$$
J(K_{t+1}) - J(K^*) \leq 2^{-\left\lceil (t+1)/l \right\rceil} \left( J(K_0) - J(K^*) + \frac{5l \cdot Z}{16} \right) \leq 2^{-\left\lceil (t+1)/l \right\rceil} J(K_0),
$$

where the last inequality is because $\kappa \geq 1$, so $l/2 = \lfloor \kappa \rfloor/2 \leq \kappa$. This completes the proof of convergence and finishes our proof by induction.

A few remarks are in order. The assumption $\varepsilon \leq J(K_0)$ is mild, as otherwise the function gap is at most $\varepsilon$ with the initial policy. From the theorem, we need $O(J(K_0)^2 \ln(J(K_0)/\varepsilon))$ iterations to ensure $J(K_T) - J(K^*) \leq \varepsilon$, where big-O hides dependence on all constants except on $J(K_0)$ and $\varepsilon$. And an important consequence of the theorem is a uniform bound on the spectral radius of $A - BK_t$.

**Corollary 3.6.** If Theorem 3.5’s assumptions hold, then there exists a uniform bound on the spectral radius, $\max_{0 \leq t \leq T} \rho(A - BK_t) \leq \sqrt{1 - \frac{\sigma_{\min}(\Psi)\sigma_{\min}(Q)}{2J(K_0)}} < 1$.

The corollary can be proven with Lemma 2.1 and $J(K_t) \leq 2J(K_0)$ from Theorem 3.5. Next, we will define the critic method to make the gradient error sufficiently small.
4. The critic: Policy evaluation via stochastic primal-dual. Let \( K \in S \). Recall from (2.11) the natural gradient \( E_K \) can be derived by solving the linear system of equations induced by Bellman’s fixed-point equation,

\[
E_{\Pi_K}[\hat{H}]\vartheta = E_{\Pi_K}[\tilde{b}],
\]

where \( \hat{H} \) and \( \tilde{b} \) are the stochastic estimates of \( H = E_{\Pi_K}[\hat{H}] \) and \( b = E_{\Pi_K}[\tilde{b}] \) as defined in (2.12). We re-formulate the above problem as a residual minimization problem

\[
\min_{\vartheta \in X} \|E_{\Pi_K}[\hat{H}]\vartheta - E_{\Pi_K}[\tilde{b}]\|,
\]

which is equivalent to the min-max problem,

\[
\min_{\vartheta \in X} \{ f(\vartheta) := \max_{y \in Y} \{ \langle y, E_{\Pi_K}[\hat{H}]\vartheta - E_{\Pi_K}[\tilde{b}] \rangle \} \},
\]

where \( X = \mathbb{R}^{(n,m)+1} \) and \( Y = \{ y \in \mathbb{R}^N : \|y\| \leq 1 \} \) are the primal and dual spaces, respectively. The main advantage of the min-max formulation is one can efficiently get nearly unbiased estimates of the gradient, while one cannot as easily when using the \( l_2 \) norm from the residual minimization problem.

For the remainder of this section, we will establish some basic structural properties for \( \hat{H} \) and \( \tilde{b} \) followed by an efficient primal-dual algorithm which leverages these properties to solve (4.2). We finish by combining the primal-dual algorithm for policy evaluation with the previous natural policy gradient method to solve LQR.

4.1. Structural properties of Bellman’s fixed-point equation. Recall \( x_t \in \mathbb{R}^n \) is the state at time \( t \) of the linear dynamical system (2.1) while \( u_t = -K x_t + v_t \) is the feedback, where \( v_t \) is the policy disturbance introduced in (2.3). We let the stochastic process be \( \{ \xi_t \in \mathbb{R}^{2(n+m)} : \xi_t = [x_t, u_t, x_{t+1}, u_{t+1}] \}_t \), i.e., the current and next state-action pairs, and it is generated by some probability measure \( (\Omega, \mathcal{F}, P) \) on the state space \( S = \mathbb{R}^{2(n+m)} \). Define \( \mathcal{F}_t = \sigma(\xi_1, \xi_2, \ldots, \xi_t) \) as the \( \sigma \)-algebra generated by the first \( t \) random variables. Since the stochastic matrix/vector \( (\hat{H}, \tilde{b}) \) from (2.12) are functions of two consecutive state-action pairs \( (x_t, u_t) \) and \( (x_{t+1}, u_{t+1}) \), then we write the stochastic estimates as \( \hat{H}_t = \hat{H}(\xi_t) \) and \( \tilde{b}_t = \tilde{b}(\xi_t) \) to indicate their parameterization by the random variable at time \( t \).

Recall \( \Psi \) is the covariance matrix of the noise \( w_t \) and define the covariance matrices

\[
(3.3) \quad \Sigma_K^{(t)} = \begin{bmatrix} \Sigma_K^{(t)} & -K \Sigma_K^{(t)} K^T \\ -K \Sigma_K^{(t)} K^T + \sigma^2 I_m \end{bmatrix}, \quad \Sigma_K = \sum_{p=0}^{t-1} (A - BK)^p \Psi [(A - BK)^T]^p,
\]

where \( I_m \) is the identity matrix of size \( m \). By examining the policy (2.3) and dynamics (2.4), we can deduce the \( t \)-th state-action pair \( [x_t, u_t] \) is distributed according to a multivariate Gaussian with mean \( [x_0, -Kx_0] \) and covariance matrix \( \Sigma_K^{(t)} \).

Because \( [x_t, u_t] \) has unbounded support, then so do \( \hat{H}(\xi_t) \) and \( \tilde{b}(\xi_t) \). Thus, the stochastic estimates \( (\hat{H}(\xi_t), \tilde{b}(\xi_t)) \) can be arbitrarily large, in which case the stochastic estimation error is also arbitrarily large. To avoid such cases, we leverage the fact \( [x_t, u_t] \) is a multivariate Gaussian, which exhibits the following “light-tail” property.

This can be proven by the Hanson-Wright inequality (see [39]).

**Lemma 4.1.** Let \( \ell \sim \mathcal{N}(\mu, \Sigma) \). For any \( \delta \in (0, 1) \), we have

\[
P \left\{ \frac{1}{2} \| \ell \|^2 > \text{Tr}(\Sigma) + \frac{\sqrt{c_2} \| \Sigma \|}{\sqrt{c_1}} \log(2/\delta) + \frac{c_2 \| \Sigma \|}{c_1} \log(2/\delta) + \| \mu \|^2 \right\} \leq \delta,
\]

where \( c_1 \) and \( c_2 \) are some absolute positive constants.
Consider the event \([x_t, u_t] \text{ is bounded for some } \delta \in (0, 1/e)\):

\[
E_t(\delta) = \left\{ \|x_t, u_t\|^2 \leq 4 \left( \frac{c_2^2}{c_1} \|\tilde{\Sigma}^{(t)}_K\| + \frac{c_2^2}{c_1} \text{Tr}(\tilde{\Sigma}^{(t)}_K) + \|x_0, -Kx_0\|^2 \right) \ln \frac{1}{\delta} \right\}.
\]

By Lemma 4.1, we know \(\Pr(E_t(\delta)) \geq 1 - \delta\). Let \(\tau \geq 1\) be some parameter we call the \textit{mixing time}, which will be further elaborated in Lemma 4.4 and in subsection 4.2.

For the meantime, one can view this as a “buffer time” between state-action pairs to reduce their correlation. Let us define the intersection of the following events:

\[
E(\delta) = \bigcap_{1 \leq t \leq N} (E_t(\tau + 1 - 1) \cap E_t(\tau + 1))
\]

where \(N \geq 1\). While the dependence on \(N\) is omitted, it will be clear from context that \(N\) is the total number of observed samples \(\xi_t\). While the analysis of policy evaluation relies on \(E(\delta)\) taking place, which occurs with high probability (i.e., by union bound, \(\Pr(E(\delta)) \geq 1 - 2N\delta\)), there is still a nonzero probability \(E(\delta)\) will not occur. In such cases, we have the following solution, which is well-suited for the online (i.e., single trajectory) setting, where one cannot reset the system. Let \(\hat{t}\) be the first time \(E_{\hat{t}+1}(\delta)\) does not occur. Then we pretend to reset the system with a new initial state \(x_0 = x_{\hat{t}+1}\), i.e., use the same trajectory but re-index time so that \(\hat{t} + 1\) is the new starting time and adjust \(E_t(\delta)\) to account for this new initial state. Although the new initial state \(x_{\hat{t}+1}\) may be large, Lemma 4.1 says \(x_{\hat{t}+1}\) will almost surely be bounded (its norm will be influenced by the covariance matrix from (4.3)).

Equipped with these properties of the state-action pair, we will first show the stochastic estimates \(\tilde{\Sigma}_K\) can be controlled in the sense that the bias of the new starting time and adjust

\[
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\]

where \(N \geq 1\). While the dependence on \(N\) is omitted, it will be clear from context that \(N\) is the total number of observed samples \(\xi_t\). While the analysis of policy evaluation relies on \(E(\delta)\) taking place, which occurs with high probability (i.e., by union bound, \(\Pr(E(\delta)) \geq 1 - 2N\delta\)), there is still a nonzero probability \(E(\delta)\) will not occur. In such cases, we have the following solution, which is well-suited for the online (i.e., single trajectory) setting, where one cannot reset the system. Let \(\hat{t}\) be the first time \(E_{\hat{t}+1}(\delta)\) does not occur. Then we pretend to reset the system with a new initial state \(x_0 = x_{\hat{t}+1}\), i.e., use the same trajectory but re-index time so that \(\hat{t} + 1\) is the new starting time and adjust \(E_t(\delta)\) to account for this new initial state. Although the new initial state \(x_{\hat{t}+1}\) may be large, Lemma 4.1 says \(x_{\hat{t}+1}\) will almost surely be bounded (its norm will be influenced by the covariance matrix from (4.3)).

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\[
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then we have for any mixing time $\tau \geq 1$,
\[
\|H - E[\hat{H}_{t+\tau}|\mathcal{F}_{t-1}]\| \leq CM_H (\ln \frac{1}{\delta})^{3/2} \rho^\tau + O_H \sqrt{\delta}
\]
\[
\|b - E[b_{t+\tau}|\mathcal{F}_{t-1}]\| \leq CM_b (\ln \frac{1}{\delta})^{3/2} \rho^\tau + O_b \sqrt{\delta}.
\]

The proof is in Appendix B. This lemma illustrates that choosing a sufficiently large mixing time $\tau$ can help reduce the bias, where the bias is affected by the correlation between samples $\xi_t$. See subsection 4.2 for more details on the mixing time.

To measure progress towards minimizing $f$ from (4.2), we define the $Q$-gap function (where with some abuse of notation, $z \equiv (\tilde{\vartheta}, y)$ and $\tilde{z} \equiv (\tilde{\vartheta}, \tilde{y})$) is defined as
\[
Q(z, \tilde{z}) := (H\tilde{\vartheta} - b, y) - (H\vartheta - b, \tilde{y}),
\]
and the primal-dual gap function is
\[
g(z) := \max_{z \in X \times Y} Q(z, z).
\]
If there is a $\vartheta^*$ such that $H\vartheta^* = b$, then $g(z) \geq f(\tilde{\vartheta})$ for all $\tilde{y} \in Y$. In this case, minimizing (4.6) also minimizes $f$ from (4.2). In view of this definition, we show the primal-dual gap function satisfies a regularity condition related to sharpness, as first introduced by Polyak [37]. This condition also appears in some data science problems [6]. We will use it to help derive faster convergence rates. Recall $\sigma \geq 0$ is the added policy disturbance from (2.3), which can be arbitrarily set.

**Lemma 4.5.** Let $\rho \equiv \rho(A - BK)$. If $K \in \mathcal{S}$ and $\sigma^2 \geq \sigma_{\min}(\Psi)[1 + \|K\|^2]$, then
\[
\mu := (\sqrt{n + m + [\sigma_{\min}(\Psi)]^{-2}}/(1 - \rho^2))^{-1} > 0
\]
satisfies $\mu \|\vartheta - \vartheta^*\| \leq g(z)$ for all $z \equiv [\vartheta, y] \in X \times Y$.

The proof is given in Appendix B.

We will now utilize these properties to derive a primal-dual method to solve the policy evaluation problem.

**4.2. A primal-dual algorithm for policy evaluation under Markovian noise.** We use a primal-dual method, Algorithm 4.1, to find a near optimal primal-dual solution $z \equiv [\vartheta, y] \in X \times Y$ for the min-max problem (4.2). In each iteration, the algorithm separately collects $\tau$ samples to estimate gradients for the primal and dual steps. We refer to $\tau \geq 1$, which can be arbitrarily set, as the mixing time, since its effect on convergence is related to the mixing time of ergodic Markov chains. We call this method the conditional stochastic primal-dual method since $\tau$ must satisfy a condition (Proposition 4.10) for the final solution to attain a desired accuracy [19]. With some abuse of notation, we write the $t$-th stochastic estimates for the dual step as $\hat{H}_{t,Y}$ and $b_{t,Y}$ and for primal step, $\hat{H}_{t,X}$ and $b_{t,X}$ (see Lines 4 and 6). The reason for collecting $\tau$ samples to estimate one gradient is to reduce the bias (e.g., correlation across time) between the primal and dual steps. After completing $k$ primal and dual steps, we return a weighted average of all past $k$ points.

We provide the general convergence result.

**Proposition 4.6.** Let $\{\gamma_t, \eta_t, \lambda_t, \zeta_t\}_{t=1}^k$ be a set of non-negative reals satisfying $\gamma_{t-1}\eta_{t-1} \leq \gamma_t\eta_t$, $\gamma_{t-1}\lambda_{t-1} \leq \gamma_t\lambda_t$, and $\gamma_t\zeta_t = \gamma_{t-1}$, and let there exist some $p, q \in (0, 1)$ satisfying $\|H\|^2/(p\lambda_t) - q\eta_t \leq 0$ for all $t$. Then
\[
\sum_{t=1}^k \gamma_t Q(z_t, z) \leq \gamma_k \eta_k D_X^2 + \gamma_k \lambda_k D_Y^2 + \sum_{t=1}^k \lambda_t(\vartheta, y),
\]
Algorithm 4.1 Conditional stochastic primal-dual (CSPD)

1: Input: $X, Y \subseteq \mathbb{R}^N$; $\vartheta_{-1} = \vartheta_0 \in X, y_0 \in Y$; $k, \tau \in \mathbb{Z}_{++}$; $\{\eta_t, \lambda_t, \zeta_t\}_{t=1}^k$
2: for $t = 1, \ldots, k$ do
3: $g_t = \vartheta_{t-1} + \zeta_t(\vartheta_{t-1} - \vartheta_{t-2})$
4: Collect $\tau$ samples, use $\tau$-th sample to set $(\tilde{H}_{t,Y}, \tilde{b}_{t,Y}) = (\tilde{H}_{(2t-1),Y}, \tilde{b}_{(2t-1),Y})$
5: $y_t = \operatorname{argmin}_{y \in Y} \{(-\tilde{H}_{t,Y} g_t + \tilde{b}_{t,Y}) + \frac{\lambda_t}{2} \|y - y_{t-1}\|^2\}$
6: Collect $\tau$ more samples, use $\tau$-th sample to set $\tilde{H}_{t,X} = \tilde{H}_{2t}$
7: $\vartheta_t = \operatorname{argmin}_{\vartheta \in X} \{(y_t, \tilde{H}_{t,X} \vartheta) + \frac{\mu_t}{2} \|\vartheta - \vartheta_{t-1}\|^2\}$
8: end for
9: $(\tilde{\vartheta}_k, \tilde{y}_k) = \frac{2}{k(k+1)} \sum_{t=1}^k t \cdot (\vartheta_t, y_t)$
10: return $\tilde{\vartheta}_k$

where $z_t = [\vartheta_t, y_t]$, $D_X^2 := \max_{x,x' \in X} \|x - x'\|^2$ (and similarly for $D_Y^2$) and

\[
\Lambda_t(\vartheta, y) := -\frac{\gamma_t \lambda_t}{2} (1 - \rho) \|y_t - y_{t-1}\|^2 - \frac{(1 - \rho) \gamma_t \eta_t}{2} \|\vartheta_t - \vartheta_{t-1}\|^2 \\
+ \gamma_t ((\tilde{H}_{t,Y} - H) y_t - (\tilde{b}_{t,Y} - b), y_t - y) - \gamma_t ((\tilde{H}_{t,X} - H)^T y_t, \vartheta_t - \vartheta) \\
\]

A proof can be found within [23, Theorem 3.8]. Next, we bound the stochastic error on a convex set $U$ (where $U$ can either be $X$ or $Y$). Due to space constraints, we only show the main steps of the proof and provide references to auxiliary parts.

**Proposition 4.7.** Let $\{\gamma_t, c_t\}$ be non-negative scalars satisfying $\gamma_{t-1} c_{t-1} \leq \gamma_t c_t$. Let $\tilde{G}_{t\tau}$ be an $\mathcal{F}_{t\tau}$-measurable stochastic vector such that

\[
\|\tilde{G}_{t\tau} - \mathbb{E}[\tilde{G}_{t\tau}|\mathcal{F}_{(t-1)\tau}]\| \leq \varsigma \text{ w.p. } 1 - \delta,
\]

\[
\|G - \mathbb{E}[\tilde{G}_{t\tau}|\mathcal{F}_{(t-1)\tau}]\|_\ast \leq C M \rho^\tau + O\sqrt{\delta},
\]

where $C, M, \varsigma, \tau \in \mathbb{R}_{++}$, $\rho, \delta \in [0,1]$, and $G \in \mathbb{R}^n$ are constants. Then for any vectors $\{u_t, v_t \in U\}$ where $u_t$ and $v_t$ are $\mathcal{F}_{t\tau}$-measurable, with probability at least $1 - (k+1)\delta$,

\[
\sum_{t=1}^k \gamma_t (G - \tilde{G}_{t\tau}, u_t - u) - \frac{c_t \gamma_t}{2} \|u_t - u_{t-1}\|^2 \leq \gamma_k c_k D_U^2 \\
+ \sum_{t=1}^k \gamma_t \left((CM \rho^\tau + O\sqrt{\delta})D_U + \frac{4((CM \rho^\tau)^2 + O^2 \delta + \varsigma^2)}{c_t}\right) + \varsigma D_U \sqrt{8 \ln(1/\delta)} \sum_{t=1}^k \gamma_t^2 ,
\]

where $D_U^2 = \max_{u,v \in U} \|u - v\|^2$.

**Proof.** With $u_t^\ast := u_0$ and $\Delta_t := G - \tilde{G}_{t\tau}$, let $u_t^\ast := \operatorname{argmin}_{u \in U} \{(\Delta_t, u) + \frac{C_t}{2} \|u - u_{t-1}\|^2\}$. Since $G$ is a constant and $\tilde{G}_{t\tau}$ is assumed to be $\mathcal{F}_{t\tau}$-measurable, a simple proof by induction ensures $u_t^\ast$ is $\mathcal{F}_{t\tau}$-measurable. Now, we decompose

\[
\sum_{t=1}^k \gamma_t (\Delta_t, u_t - u) - \frac{C_t \gamma_t}{2} \|u_t - u_{t-1}\|^2 \\
= \sum_{t=1}^k \gamma_t (\Delta_t, u_t - u_{t-1}) - \frac{C_t \gamma_t}{2} \|u_t - u_{t-1}\|^2 + \gamma_t (\Delta_t, u_{t-1} - u_{t-1}) + \gamma_t (\Delta_t, u_{t-1}^\ast - u).
\]
We have the following bound, which holds with probability $1 - k\delta$ by union bound:

$$
\sum_{t=1}^{k} A_t \leq \sum_{t=1}^{k} \frac{\gamma_t}{2c_t} \| G - \tilde{G}_{t\tau} \|^2 \\
\leq \sum_{t=1}^{k} \frac{2\gamma_t}{c_t} \left( \| G - \mathbb{E}[\tilde{G}_{t\tau} | \mathcal{F}_{(t-1)\tau}] \|^2 + \| \mathbb{E}[\tilde{G}_{t\tau} | \mathcal{F}_{(t-1)\tau}] - \tilde{G}_{t\tau} \|^2 \right) \\
\leq \sum_{t=1}^{k} \frac{2\gamma_t}{c_t} (CM\rho^\tau)^2 + O^2\delta + \zeta^2,
$$

which is due to Young’s inequality, $(a + b)^2 \leq 2a^2 + 2b^2$, and then (4.7) and (4.8).

Next, we can bound with probability $1 - \delta$,

$$
\sum_{t=1}^{k} B_t = \sum_{t=1}^{k} (B_t - \mathbb{E}[B_t | \mathcal{F}_{(t-1)\tau}]) + \mathbb{E}[B_t | \mathcal{F}_{(t-1)\tau}] \\
\leq \zeta D_U \left[ 8 \ln(1/\delta) \sum_{t=1}^{k} \gamma_t \right]^{-1/2} + |CM\rho^\tau| O\sqrt{\delta} D_U \sum_{t=1}^{k} \gamma_t,
$$

where we used the Azuma-Hoeffding inequality to bound the martingale-difference sequence $B_t - \mathbb{E}[B_t | \mathcal{F}_{(t-1)\tau}]$ (and used the fact $\Delta_t$ is $\mathcal{F}_{t\tau}$-measurable while $u_{t-1} - u_{t-1}^\tau$ are $\mathcal{F}_{(t-1)\tau}$-measurable as well as (4.7) and definition of $D_U$; see [44, Lemma 7] for full details) and (4.8) to bound $\mathbb{E}[B_t | \mathcal{F}_{(t-1)\tau}]$. Finally, using $\gamma_{t-1}c_{t-1} \leq \gamma_tC_t$, we get

$$
\sum_{t=1}^{k} C_t \leq \gamma_k c_k D_U \sum_{t=1}^{k} \frac{\gamma_t}{c_t} \| \Delta_t \|^2 \leq \gamma_k c_k D_U \sum_{t=1}^{k} \frac{\gamma_t}{c_t} ((CM\rho^\tau)^2 + O^2\delta + \zeta^2),
$$

where the first and second inequality can be shown similarly to [23, Lemma 4.10] and the bound on $A_t$, respectively. Combining everything finishes the proof.

We are now ready to complete the proof of convergence. Recall the gap function $g(\bar{z}) \equiv \max_{z \in \mathcal{X} \times Y} Q(\bar{z}, z)$ and the weighted average solution $\bar{z}_k \equiv [\bar{x}_k, \bar{y}_k]$.

**Theorem 4.8.** Suppose $K \in \mathcal{S}$ and $\mathcal{F}(\delta)$ take place for some $\delta \in (0, 1/e)$. Under the same assumptions as Proposition 4.6, then with probability $1 - 2(k + 1)\delta$,

$$
\left( \sum_{t=1}^{k} \gamma_t \right) g(\bar{z}_k) \leq 2\gamma_k (\eta_k D_X^2 + \lambda_k D_Y^2) + 2(D_X M_X + D_Y M_Y) \sqrt{8 \ln \frac{1}{\delta} \sum_{t=1}^{k} \gamma_t^2} \\
+ \sum_{t=1}^{k} \gamma_t \left[ C(D_X M_X + D_Y M_Y)\rho^\tau + (O_X D_X + O_Y D_Y)\sqrt{\delta} + \frac{16M_X^2}{\eta_k(1-q)} + \frac{16M_Y^2}{\lambda_t(1-p)} \\
+ 4C^2 \left( \frac{M_X^2}{\eta_k(1-q)} + \frac{M_Y^2}{\lambda_t(1-p)} \right)\rho^{2\tau} + 4 \left( \frac{O_X^2}{\eta_k(1-q)} + \frac{O_Y^2}{\lambda_t(1-p)} \right) \delta \right],
$$

where $\zeta := \max_{t} \zeta_t$, $D_U^2 := \max_{u, u' \in U} \| u - u' \|^2$ for some space $U$, and

$$
\Omega_Y := \max \{ \| y \| : y \in Y \}, \quad \Omega_X := \| \theta^* \| + (1 + \zeta) \sqrt{2D_X}, \\
M_X := M_H \Omega_Y \left( \ln \frac{1}{\delta} \right)^2, \quad M_Y := (M_H \Omega_X + M_b) \left( \ln \frac{1}{\delta} \right)^2, \\
O_X := O_H \Omega_Y, \quad O_Y := O_H \Omega_X + O_b.
$$

Constants $M_H, M_b, C, O_H, O_b$ are from Lemma 4.2 and Lemma 4.4.
\textbf{Proof.} We first decompose the error \( \sum_{t=1}^{k} \Lambda_t(\vartheta, y) \) defined in Proposition 4.6 as
\[
\sum_{t=1}^{k} \Lambda_t(\vartheta, y) = \sum_{t=1}^{k} \gamma_t \langle (\tilde{H}_{t,Y} g_t - \tilde{b}_{t,Y}) - (H g_t - b), y_t - y \rangle - \gamma_t \lambda_t (1 - p) \frac{\|y_t - y_{t-1}\|^2}{2}
\]
\[
- \gamma_t \langle \tilde{H}_{t,Y}^T y_t - \tilde{H}^T y_t, \vartheta_t - \vartheta \rangle - \frac{(1 - q)\gamma_t \eta_t}{2} \|\vartheta_t - \vartheta_{t-1}\|^2.
\]

We wish to use Proposition 4.7 to bound the first two summands (the last two summands can be similarly shown), which requires verification of (4.7) and (4.8). First, because \( \tilde{H}_{t,Y} \) and \( \tilde{b}_{t,Y} \) are generated with the \((2t-1)\tau\)-th sample in Algorithm 4.1 and \( g_t = \vartheta_{t-1} + \zeta_t (\vartheta_{t-1} - \vartheta_{t-2}) \), then \( \tilde{G}_{t,Y} \) is \( F_{(2t-1)\tau} \)-measurable. Denoting \( F' = F_{(2t-2)\tau} \), then with probability \( 1 - \delta \),
\[
\|\tilde{G}_{t,Y} - \mathbb{E} [\tilde{G}_{t,Y} | F']\|_* \leq \|\tilde{H}_{t,Y}\|_* + \|\mathbb{E} [\tilde{H}_{t,Y} | F']\|_* (\|\vartheta^*\| + \|\vartheta_{t-1} - \vartheta^*\| + \zeta_t \|\vartheta_{t-1} - \vartheta_{t-2}\|)
\]
\[
+ \|\tilde{b}_{t,Y}\|_* + \mathbb{E} [\|\tilde{b}_{t,Y} | F']\|_* + \mathbb{E} [\|\tilde{H}_{t,Y} | F']\|_* (\|\vartheta^*\| + (1 + \zeta_t)D_X) + \|\tilde{b}_{t,Y}\|_* + \mathbb{E} [\|\tilde{b}_{t,Y} | F']\|
\]
\[
\leq 2M_H (\ln \frac{1}{\delta})^2 (\|\vartheta^*\| + (1 + \zeta_t)\sqrt{2}D_X) + 2M_b (\ln \frac{1}{\delta})^2 \leq 2M_Y,
\]
where the second inequality is by definition of \( D_X^2 \) and Jensen’s inequality, and the last line used Lemma 4.2 (recall we assumed \( K \in S \) and \( E(\delta) \) occur). This verifies (4.7) with \( \zeta = 2M_Y \) up to a shift in the index \( \tau t \). One can similarly show \( \|G_Y - \mathbb{E} [\tilde{G}_{t,Y} | F_{(2t-2)(\tau+1)}]\|_* \leq C M_Y \rho^\tau + O_Y \sqrt{\delta} \) using Lemma 4.4, verifying (4.8).

Fixing \( U = Y, u_t = y_t, u = y, \) and \( c_t = \lambda_t \) so that \( \gamma_{t-1} c_{t-1} \leq \gamma_t c_t \) (from the assumption in Proposition 4.6), we can use Proposition 4.7 to bound the first two summation in the definition of \( \Lambda_t \). After applying a similar bound to the latter two summations in \( \sum_{t=1}^{k} \Lambda_t \), we plug the resulting bound back in Proposition 4.6, which bounds the primal-dual gap. The proof is complete by invoking convexity of \( Q \) and dividing through by the sum over \( \tau_t \). \hfill \Box

We can now specify an explicit step size to obtain the following convergence rate. The proof is similar to [23, Corollary 4.3], so we skip it.

\textbf{Corollary 4.9.} Let \( \eta_t = \frac{3\sqrt{2}}{2\sqrt{2}D_X d} t^{3/2} \), \( \lambda_t = \frac{3\sqrt{2}}{2\sqrt{2}D_Y t^{3/2}} \), and \( c_t = \frac{t-1}{t} \), where \( M_I \) and \( O_I \) for \( I \in \{X, Y\} \) are defined in Theorem 4.8. If \( K \in S \) and \( E(\delta) \) take place for some \( \delta \in (0, 1/e) \), then with probability \( 1 - 2(k + 1)\delta \),
\[
g(\tilde{z}_k) \leq \frac{12\|H\|D_X D_Y}{k + 1} + \frac{2(48 + 3\sqrt{2} + \frac{16\sqrt{2}}{\sqrt{3}}) (D_X M_X + D_Y M_Y)}{\sqrt{k}}
\]
\[
+ \frac{24}{\sqrt{k}} \left[ C^2(D_X M_X + D_Y M_Y)\rho^{2\tau} + \left( \frac{D_X O_X^2}{2M_X} + \frac{D_Y O_Y^2}{2M_Y} \right) \delta \right]
\]
\[
+ C(D_X M_X + D_Y M_Y)\rho^\tau (D_X O_X + D_Y O_Y)\sqrt{\delta}.
\]

In view of Lemma 4.5 and assuming the mixing time \( \tau \) and added noise \( \sigma \) are sufficiently large, the above result tells us \( \tilde{z}_k \) satisfies \( \|\tilde{z}_k - z^*\|^2 \leq O(\frac{\|H\|^2}{\mu_k^2} + \frac{\sigma}{\mu_k}) \)
with high probability, where \( z^* = [\vartheta^*; y^*] \) is the optimal primal-dual solution to (4.2), “var” bounds stochastic estimation errors, and \( k \) is the number of iterations. It turns out one can further modify the algorithm to decrease the overall error faster.

To do so, we introduce a shrinking multi-epoch algorithm in Algorithm 4.2. It repeatedly calls Algorithm 4.1 and uses the solution to the previous call to warm-start the next call. We name each call an epoch. Each epoch \( s \) will use an updated feasible set for the primal variable \( X_s \) that is shrunk from the previous epoch. The feasible set for the dual variable \( Y \) remains unchanged. A key feature is that the optimal solution to (4.2) is still feasible every epoch, i.e., \( \vartheta^* \in X_s \). Since \( X_s \) gets smaller, we also update the diameter \( D_{X_s} \). The choice of step sizes still adheres to those prescribed in Corollary 4.9 up to the difference in the diameter \( D_{X_s} \), while the number of iterations \( k_s \) increases each epoch.

**Algorithm 4.2 Shrinking multi-epoch CSPD**

Input: \( X \subseteq \mathbb{R}^n; p_0 \in X; D_0 \in \mathbb{R}_{++}; k, \tau, S \in \mathbb{Z}_{++} \)

for \( s = 1, \ldots, S \) do

\[
\begin{align*}
D_s^2 &:= 2^{-(s-1)}D_0^2 \\
X_s &:= \{ \vartheta \in X : ||p_{s-1} - \vartheta||^2 \leq D_s^2 \} \\
p_s &\leftarrow \text{Algorithm 4.1 with } X_s, p_{s-1}, k_s, \tau_s, \delta_s, \{\eta_t, \lambda_t, \zeta_t\}_t \text{ specified by Corollary 4.9 and Proposition 4.10} \\
end for \\
return \ p_S
\]

We now establish the main convergence result, similar to [23, Lemma 4.5].

**Proposition 4.10.** Consider the same assumptions and parameters as Corollary 4.9. Suppose \( ||p_0 - \vartheta^*||^2 \leq D_0^2 \) and \( \sigma \) satisfies the bound from Lemma 4.5. Set

\[
\begin{align*}
k_s &= \left[ 400 \max \left\{ \frac{||H||D_Y}{\mu}, \frac{4000 + 256\ln(1/\delta)}{\mu^2} \left( M_X^2 + \frac{D_Y^2 M_Y^2}{D_0^2} \cdot 2^s \right) \right\} \right] \\
\tau_s &\equiv \tau = \left[ \frac{\frac{1}{2} \ln(\frac{2N_{S+1}C}{\mu^2}) (D_X M_X + D_Y M_Y), \ln(\frac{2N_{S+1}C}{\mu^2}) (D_X M_X + D_Y M_Y)}{\ln(1/\rho)} \right] \\
\delta_s &\equiv \delta \leq \min \left\{ \sqrt{\frac{\epsilon \mu^2}{18}} \left( \frac{D_X O_X^2}{2M_X} + \frac{D_Y O_Y^2}{2M_Y} \right), \frac{\epsilon}{18\mu} (D_X O_X + D_Y O_Y)^2 \right\}
\end{align*}
\]

Choosing the number of epochs as \( S = \ceil{\log_2 \frac{D_0^2}{\epsilon^2}} \), then \( ||p_S - \vartheta^*||^2 \leq \epsilon \) with probability at least \( 1 - 2(2N_S + S)\delta \), and the total iterations is at most

\[
N_S := O(1) \left( \frac{||H||D_Y}{\mu} \ln \left( \frac{D_0}{\epsilon} \right) + \frac{1 + \ln(1/\delta)}{\mu^2} \left( M_X^2 \ln \left( \frac{D_0}{\epsilon} \right) + \frac{D_Y^2 M_Y^2}{\epsilon} \right) \right),
\]

while the sample complexity is \( 2N_S \tau \). Here, \( O(1) \) is some absolute constant.

**Proof.** We will show by mathematical induction that for all \( s = 0, \ldots, S \), \( ||p_s - \vartheta^*|| \leq 2^{-s}D_0^2 \) with probability \( 1 - \sum_{i=1}^{s} 2(k_i + 1)\delta \). The base case of \( s = 0 \) is true by the assumption on \( p_0 \).

Now suppose the claim is true for \( s - 1 \). By the inductive hypothesis, \( ||p_{s-1} - \vartheta^*||^2 \leq 2^{-s+1}D_0^2 \) with probability \( 1 - \sum_{i=1}^{s-1} 2(k_i + 1)\delta \). Therefore, the optimal solution
is still in the feasible region, i.e., $\vartheta^* \in X_s$. Now, conditioned on the success of the previous epochs, Corollary 4.9 combined with our choice in parameters $k_s$, $\tau_s$, and $\delta_s$ guarantees the primal-dual gap function respects with probability $1 - 2(k_s + 1)\delta$, $g([p_s; y_s]) \leq \mu \sqrt{2^{-s}D_0^2}$, where $[p_s; y_s]$ is primal-dual solution from the $s$-th call to Algorithm 4.2. Squaring both sides and using Lemma 4.5 (recall we assumed $K \in S$ from Corollary 4.9), we conclude $\mu^2\|p_s - \vartheta^*\|^2 \leq g([p_s; y_s])^2 \leq \mu^2 \cdot 2^{-s}D_0^2$. By union bound, this holds unconditionally with probability $1 - \sum_{i=1}^s 2(k_s + 1)\delta$, and this completes the proof by induction.

Finally, the total number of iterations can be derived by summing $k_s$ across all epochs $s$.

The above result improves upon Corollary 4.9 by reducing the error term corresponding to $\|H\|$ (i.e., deterministic error) without worsening the stochastic estimation errors. Next, we apply this generic primal-dual method to the setting of estimating the natural gradient $E_k$ within the natural policy gradient method.

### 4.3. Combining natural policy gradient with the primal-dual method to solve LQR

We can now prove $O(1/\varepsilon)$ samples suffice to solve LQR. First, define the radius $R_s = (\|Q\|_F + \|R\|_F) + (\|A\|_F^2 + \|B\|_F^2) \cdot \frac{J(K_0)}{\sigma_{\min}(\Psi)}$.

**Theorem 4.11.** Suppose $K_0 \in S$ and $\varepsilon \in (0, J(K_0)]$, and set $\sigma^2 = \sigma_{\min}(\Psi) + 2\sigma_{\min}(R)^{-1}J(K_0)$ in (2.3). Run Algorithm 3.1 with the step size from Theorem 3.5 and $T = O(J(K_0)^2 \ln(1/\varepsilon))$. To compute the natural gradient via Algorithm 4.2, use the parameters from Proposition 4.10, $D_0 = 2R_s$, $\epsilon = O(\varepsilon/J(K_0)^4)$, $\delta \in (0, 1/\varepsilon)$, and choose any initial solution $p_0$ satisfying $\|p_0\| \leq R_s$. Then Algorithm 3.1 outputs $K_T$ s.t. $J(K_T) - J(K^*) \leq \varepsilon$ with probability $1 - 8\tilde{N}\delta$, and the total number of samples is

$$\tilde{N} := O\left(\frac{(J(K_0)^{23} \cdot m^4(n + m) \cdot \ln(1/\delta) + \ln(1/\varepsilon))^7}{\varepsilon}\right).$$

Big-O hides dependence on absolute constants; $A$, $B$, and $\Psi$ from (2.1); $Q$ and $R$ from (2.2); and logarithmic dependence on $J(K_0)$, $n$, and $m$.

**Proof.** Our goal is to apply Theorem 3.5. To do so, we need to verify the natural gradient estimation error is $\|E_{K_t} - E_{K_t}^*\|^2 \leq \epsilon' := \min\{C_5, C_6, C_7\} = O(\varepsilon/J(K_0)^3)$, where the constants are defined in Theorem 3.5. If this holds, then by our choice in $T$, Theorem 3.5 guarantees the final iterate $K_T$ satisfies $J(K_T) - J(K^*) \leq \varepsilon$.

To show the desired accuracy, recall from (2.8), (2.10), and (2.11) the relation between the desired natural gradient $E_{K_t}$ and an approximate solution $\vartheta(K_t)$ for (2.11) returned by Algorithm 4.2 (denote $\vartheta^* \equiv \vartheta(K_t)^*$ as the optimal solution):

$$\|E_{K_t} - E_{K_t}^*\|^2 = \|\Theta(K_t)_{22}K_t - \Theta(K_t)_{21} - \Theta(K_t)^*_{22}K_t - \Theta(K_t)^*_{21}\|_F^2$$

$$\leq 2\|\Theta(K_t)_{22} - \Theta(K_t)^*_{22}\|_F^2\|K_t\|_F^2 + 2\|\Theta(K_t)_{21} - \Theta(K_t)^*_{21}\|_F^2$$

$$\leq 2\|K_t\|_F^2 + 1)\|\vartheta(K_t) - \vartheta(K_t)^*\|^2$$

$$\leq 2\|\vartheta(K_t) - \vartheta(K_t)^*\|^2.$$

Hence, we require accuracy $\|\vartheta(K_t) - \vartheta(K_t)^*\|^2 \leq \epsilon := \epsilon'/(2(\|K_t\|_F^2 + 1))$.

For us to use Proposition 4.10 to show $\|\vartheta(K_t) - \vartheta(K_t)^*\|^2 \leq \epsilon$, we must verify its assumptions. Suppose event $E(\delta)$ from (4.5) occurs (we will bound the probability of failure at the end of the proof). We know $K_t \in S$ at iteration $t$ by Theorem 3.5. And
by choice of $\sigma$, we can write the assumption for Lemma 4.5 because

$$\sigma^2 = \sigma_{\min}(\Psi)(1 + 2[\sigma_{\min}(\Psi)]^{-1}J(K_0)) \geq \sigma_{\min}(\Psi)(1 + \|K_i\|^2_F),$$

where the first inequality used $J(K_i) \leq 2J(K_0)$ from Theorem 3.5 and the second inequality applied Lemma 2.1. Finally, the choice in $D_0$ and assumption $\|p_0\| \leq R_*$ guarantees $\|p_0 - \theta^*\|^2 \leq D_0^2$ ($\|\theta^*\|^2 \leq R_*^2$ can be shown by [48, Eqn 5.4] and Lemma 2.1). So, we have verified all assumptions required by Proposition 4.10.

Having verified the assumptions for Proposition 4.10, let us now bound $N_S$, $\tau$, and $\delta$ to achieve accuracy $\epsilon$. In view of $N_S$ from Proposition 4.10, then combining the constants derived in Theorem 4.8, Lemma 4.2, the assumption $\|\theta_0\| \leq R_*$, Lemma 4.3, Lemma 4.4, Lemma 4.5, choice of $\sigma^2$, and $D_{\mathcal{X}_*} \leq D_0 = O(J(K_0))$ (since we shrink the feasible region), then $N_S = O\left(\frac{\|K_i\|^2_F - J(K_0)^2}{\epsilon^2}m^2(n + m)(\ln(1/\delta))^{5/6}\right)$. To bound $\tau$ from Proposition 4.10, we use Corollary 3.6, $\frac{1}{\log(1/u)} \leq \frac{1}{1-u}$ for $u \in (0,1)$, and Lemma 4.4 to show $\tau = O\left(\frac{\ln(n/\epsilon)}{\log(1/\rho(A-BK_1))}\right) \leq O\left(\frac{\ln(n/\epsilon)}{1-\rho(A-BK_1)}\right)$. Finally, combining Corollary 3.6, Theorem 4.8, and Proposition 4.10, we need $\delta \leq O(\epsilon)$, and so we choose $\delta$ so that $\delta \leq O(\epsilon)$ (this will not affect the complexity in terms of big-O notation, as shown below).

Altogether, the total number of samples in each call to Algorithm 4.2 is

$$N_S \cdot \tau \leq O\left(\frac{\|K_i\|^8_F \cdot J(K_0)^2 \cdot m^4(n + m)(\ln(1/\delta) + \ln(n/\epsilon))^{6/\epsilon}}{(1 - \rho(A - BK_1))^3}\right),$$

where the second line used Corollary 3.6 and $(1 - \sqrt{1-u})^{-1} \leq 2u^{-1}$ for $u \in (0,1)$, $\epsilon = O(\epsilon/J(K_0)^2)$, $\epsilon \leq O(\epsilon/J(K_0)^2)$, Lemma 2.1, and $J(K_i) \leq 2J(K_0)$ from Theorem 3.5 (and assumption $\epsilon \leq J(K_0)$). Now, since Algorithm 4.2 is called during each of the $T = O(J(K_0)^2\ln(1/\epsilon))$ iterations in Algorithm 3.1, then the total number of samples is $\hat{N} = N_S \cdot \tau \cdot T$, and the probability of success is $1 - T[2(N_S + S)\delta + 2(N_S + S)\delta] \geq 1 - 8\hat{N}\delta$, where the first term $2(N_S + S)\delta$ is the failure from Proposition 4.10 and the second term is from event $\mathcal{E}(\delta)$ (from (4.5) not occurring. This completes the proof for estimating every natural gradient $E_{K_i}$ up to accuracy $\epsilon^r$, and from Theorem 3.5 it guarantees $J(K_T) - J(K^*) \leq \epsilon$.

Some comments are in order. Assuming $\epsilon \leq J(K_0)$ is mild, otherwise the initial $K_0$ suffices since $J(K_0) - J(K^*) < \epsilon$. So the only non-trivial assumption we make is $K_0$ being stable. In contrast, prior works either arbitrarily postulate a uniform bound for $\hat{K} = \max_{0 \leq t \leq T-1} \|K_t\|$ and $\rho_T = \max_{0 \leq t \leq T-1} \rho(A - BK_t)$ [49, 51] or have their sampling complexity depend on $\hat{K}$ and $\rho_T$ [48]. We do neither. Instead, our algorithm ensures $\hat{K}^2$ and $\rho_T$ are bounded by $O(J(K_0))$. This is why our dependence on $J(K_0)$ is so large. We believe this dependence may be overconservative, as in our experiments we observed a less acute dependence on $J(K_0)$.

5. Numerical Experiments. We now run numerical experiments with our actor-critic algorithm. The code is at https://github.com/jucaleb4/online-lqr. First, let us describe the implementation details.

5.1. Implementation details. We implemented three algorithms: multi-epoch natural policy gradient (NPG), which is the combination of NPG Algorithm 3.1 (actor) with the multi-epoch scheme Algorithm 4.2 (critic); single-epoch NPG which uses the
single-epoch critic Algorithm 4.1; and two-time scale actor-critic (AC) method [49], which achieves the sampling complexity of $O(\varepsilon^{-3/2})$ for the online (single trajectory) setting. We fine tuned all methods using a simple grid search similarly to [25], which includes the parameters (whenever applicable to the method) for the step sizes, mixing times $\tau$, initial diameter $D_0$, and number of epochs $S$. See our code for the exact parameter values. We also applied a mini-batch to the gradient estimates for both the single- and multi-epoch scheme (but not the two-time scale actor-critic method) to reduce variance and produce higher accuracy results. More specifically, our mini-batch scheme collects $m_{\text{batch}} = 10$ (found by manually tuning) estimates of the gradient, each of which involves $\tau$ samples, sequentially along the same trajectory and then averages the $m_{\text{batch}}$ estimates to form a single estimate. Thus, the number of samples for a single gradient increases by a factor of $m_{\text{batch}}$.

5.2. Simple synthetic problem. Consider the synthetic problem from [7]:

$$A = \begin{bmatrix} 1.01 & 0.01 & 0 \\ 0.01 & 1.01 & 0.01 \\ 0 & 0.01 & 1.01 \end{bmatrix}, \quad B = R = \Psi = I_3, \quad Q = 10^{-3}I_3,$$

where $I_3$ is the identity matrix of dimension $3 \times 3$. We also set the added noise as $\sigma = 1$. We call this problem the simple environment. The initial controller is set as $I_3$. We also implemented a large simple environment, which enlarges the Toeplitz matrix and identity matrices from the simple environment from dimension $n = 3$ to $n = 100$, to test a larger problem. All constants remain the same.

The results from both simple environments are shown in the two left hand side plots of Figure 1. While the two-time scale actor-critic (AC) method has fast initial convergence, a majority of the seeds (19/32) yield unstable policies after 1200 samples of the simple environment. In contrast, both the single- and multi-epoch NPG decrease the objective further and more gradually, and moreover, all the seeds output a stable policy. The robust performance of both NPG methods highlights the importance of removing the assumption that the policy is almost surely stable at every iteration (see the discussion after Theorem 4.11). Moreover, the multi-epoch NPG has equal or better performance than the single-epoch. This further supports the theoretical advantage of the former method over the latter (Proposition 4.10). Finally, both NPG methods exhibit fast initial convergence followed by slower and stagnated convergence for the simple and large simple environments, respectively. This slower performance is from the policy evaluation (i.e., critic) error, which propagates to the actor and stalls convergence.

5.3. Longitudinal control of a wide-body aircraft. We consider a state-feedback control problem of the longitudinal dynamics of a Boeing 747 aircraft. The continuous time nonlinear dynamics are linearized around trim conditions and subsequently discretized as in [20]. The resulting matrices in the state recursion are

$$A = \begin{bmatrix} 1 & -1.13 & -0.65 & -8.07 & 1.59 \\ 0 & 0.77 & 0.32 & -0.98 & -2.97 \\ 0 & 0.12 & 0.02 & -0.00 & -0.36 \\ 0 & 0.01 & 0.01 & -0.03 & -0.04 \\ 0 & 0.14 & -0.09 & 0.29 & 0.76 \end{bmatrix}, \quad B = \begin{bmatrix} 89.20 & -50.17 & 1.13 & -19.35 \\ 5.22 & 6.36 & 0.23 & -0.32 \\ -9.47 & 5.93 & -0.12 & 0.99 \\ -0.32 & 0.32 & -0.01 & -0.01 \\ -4.53 & 3.21 & -0.14 & 0.09 \end{bmatrix}.$$

Here, the $n$-dimensional ($n = 5$) state is $x_t = [\alpha_t, v_t, v^2_t, \theta_t, \phi_t]$, where the variables correspond, respectively, to the deviation in altitude (positive is down), deviation in
velocity along the aircraft’s axis (forward is positive), deviation in velocity orthogonal to the aircraft’s axis (positive is down), deviation in angle between the aircraft’s axis and the X-axis, and angular velocity of the aircraft (pitch rate). We set the cost matrices $Q$ and $R$ as identity matrices, controller noise as $\sigma = 1$, and covariance matrix as a symmetric positive definite matrix that attempts to capture dependencies in the noise (e.g., a positive correlation between the deviation in altitude $\alpha_t$ and orthogonal velocity $v^\perp_t$ from wind pushing the plane down). Due to space constraints, we omit the matrix (it can be found in the code). Finally, a matrix of all ones scaled by $1/200$ is used as the initial stabilizing controller.

The results are shown in the right hand side of Figure 1. Similar to the simple synthetic problems, the single- and multi-epoch NPG can decrease the function gap at a near linear rate before slowing down. However, this problem requires about five times more samples than the simple synthetic problems to achieve a similar relative decrease in the function gap. This can be possibly explained by the large initial value $J(K_0) = 52499$, which affects the sampling complexity (Theorem 4.11). Overall, the multi-epoch NPG outperforms the single-epoch NPG, while the two-time scale actor-critic can only make a small improvement before diverging.

6. Conclusion. When applied to LQR, our proposed actor-critic method outputs a policy $\hat{K}$ such that $J(\hat{K}) - J(K^*) \leq \varepsilon$ with high probability and $O(\varepsilon^{-1})$ sampling complexity. The actor uses natural policy gradient [15], and we provide a novel analysis on the effect of the gradient estimation error. The critic is our novel conditional stochastic primal-dual method. Unlike previous model-free methods, our method only samples the system along a single trajectory (i.e., the online setting), and it does not reset the system nor assume some arbitrary behavior from the algorithm. In other words, we match the best rate from prior works (up to polylogarithmic factors) for a harder problem and with less restrictive assumptions.

While our work studies the average reward setting, our work should be amenable to the infinite-horizon discounted setting as well, since they both share some properties such as the PL-condition and the natural gradient being partially defined by a solution to a linear system of equations.
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Appendix A. Proofs from section 2.

Proof of Lemma 2.1. The first three inequalities follow from the definitions of
for some positive integer $\Sigma_K$, $P_K$, and $J(K)$, respectively, and basic trace inequalities (c.f. [9, Lemma 3.8]).

To show the last inequality, we first define the identity operator $I$ and linear symmetric operator $T(X) := (A - BK)X(A - BK)^T$. Since $\rho(A - BK) < 1$, then the Neumann series $(I - T)^{-1}(X) = \sum_{t=0}^\infty (A - BK)^t X [(A - BK)^T]^t$ is well-defined [10, Lemma 18]. Denote $T^t$ as the composition of $T$ with itself $t$ times. Now, by definition of $\Sigma_K$ in (2.6), we have $(I - T)(\Sigma_K) = \Psi + \sigma^2 B B^T$, and so

$$\sigma_{\min}(Q)^{-1} J(K) \geq \text{Tr}[\Sigma_K] = \text{Tr}[(I - T)^{-1}(\Psi + \sigma^2 B B^T)]$$

$$\geq \sigma_{\min}(\Psi) \text{Tr}[(I - T)^{-1}(I)]$$

$$= \sigma_{\min}(\Psi) \sum_{t \geq 0} \text{Tr}[T^t(I)]$$

$$\geq \sigma_{\min}(\Psi) \sum_{t \geq 0} \rho^t(A - BK)^2t$$

$$= \sigma_{\min}(\Psi)/(1 - \rho(A - BK)^2),$$

where the third line is by definition of $(I - T)^{-1}$ and exchanging summations, while the fourth line is from the inequality $\text{Tr}[(A - BK)^t ((A - BK)^T)^t] \geq \|(A - BK)^t\|^2 \geq \rho^t(A - BK)^2t$.

**Appendix B. Proofs from section 4.**

**Proof of Lemma 4.2.** Recall from (2.10) that $\phi_t \equiv \phi(x_t, u_t) = \text{svec}(z_t z_t^T)$ and $z_t = [x_t, u_t]$. Using the definition of the stochastic estimate $\hat{H}_t$ for $H$ from (2.12), $\|\hat{H}_t\| \leq 2 + 2 \|z_t\|^4 + \|z_t\|^2\|z_{t+1}\|^2 \leq 2 + 4 \max\{\|z_t\|^4, \|z_{t+1}\|^4\}$. By definition of $\mathcal{E}_t(\delta)$ in (4.4) and the covariance matrix $\Sigma^{(t)}$ from (4.3), then for any $t$,

$$\|z_t\|^2 \leq 4\left(\frac{c_3^2}{c_1} \|\Sigma^{(t)}_K\| + \frac{c_3^2}{c_1} \text{Tr}(\Sigma^{(t)}_K) + \|z_t\|^2\right) \ln \frac{1}{\delta}$$

(B.1)

$$\leq 4(1 + \|K\|^2)\left(\frac{c_3^2}{c_1} \|\Sigma^{(t)}_K\| + \sigma^2 + \frac{c_3^2}{c_1} \text{Tr}(\Sigma^{(t)}_K) + \sigma^2 \cdot m + \|z_t\|^2\right) \ln \frac{1}{\delta}$$

$$\leq 4(1 + \|K\|^2)\left(\frac{\sqrt{c_3}}{c_1} \cdot \frac{\text{J}(K)}{\sigma_{\min}(Q)} + \sigma^2 \cdot (m + 1) + \|z_t\|^2\right) \ln \frac{1}{\delta},$$

where the last line used $\|\Sigma^{(t)}_K\| \leq \text{Tr}(\Sigma^{(t)}_K) \leq \text{Tr}(\Sigma_K) \leq J(K)/\sigma_{\min}(Q)$ thanks in part to Lemma 2.1. Combining the last two inequalities gets us the upper bound on $\|\hat{H}_t\|$. One can similarly bound $\|\tilde{b}_t\|$ by recalling $c(x_t, u_t) = x_t^T Q x_t + u_t^T R u_t$.

In order to prove Lemma 4.4, we will first need the geometrically fast mixing property of a linear dynamical system (e.g., evolution of the state-action pair $[x_t, u_t]$ induced by (2.3) and (2.4)).

**PROPOSITION B.1 (Proposition 3.1 [43]).** Take the linear dynamical system $X_{t+1} = F X_t + w_t$, where $\omega_t \sim \mathcal{N}(0, \Lambda)$ and $\Lambda > 0$. Suppose $\|F^k\| \leq \Gamma \rho^k$ for all $k \geq 0$, where $\Gamma > 0$ and $\rho \in (0, 1)$. Let $\mathbb{P}_{X_k} (\cdot \mid X_0 = x)$ be the conditional distribution of $X_k$ given $X_0 = x$. Then for all $k \geq 0$ and any distribution $\nu_0$ over $x \in \mathbb{R}^n$,

$$\mathbb{E}_{x \sim \nu_0} [\|\mathbb{P}_{X_k} (\cdot \mid X_0 = x) - \nu_\infty\|_{tv}] \leq \frac{\Gamma}{2} \sqrt{\mathbb{E}_{\nu_0} [\|x\|^2]} + \frac{n}{\Gamma - \rho^2} \rho^k,$$

where $\|\cdot\|_{tv}$ is the total-variation distance.

We need to bound higher moments of a mean-zero Gaussian random vector.

**LEMMA B.2.** Let $X \sim \mathcal{N}(0, \Sigma)$ and $I = \{i_j \in [n]\}_{j=1}^{2p}$ be a set of arbitrary indices for some positive integer $p$. Then $\mathbb{E}[\prod_{j=1}^{2p} X_{i_j}] \leq (3 \cdot 5 \cdot \ldots \cdot (2p - 1))\|\Sigma\|^p$. 

Proof. Since \(|I| = 2p\) is even, Isserlis’ Theorem \([13, 46]\) derives us the closed-form expression \(E[\prod_{i=1}^{2p} X_{i,j}] = \sum_{\sigma \in \Pi(I)} \prod_{(i,j) \in \sigma} \text{Cov}(X_i, X_j) \leq |\Pi(I)| \cdot \|\Sigma\|^p\), where \(\Pi(I)\) denotes the set of all partitions of \(I\) into (disjoint) pairs. Here, it is known the cardinality of \(\Pi(I)\) is \(|\Pi(I)| = (2p)!/(2^p p!) = 1 \cdot 3 \cdot \ldots \cdot (2p - 1)\).

Proof of Lemma 4.4. Let \(z_t = [x_t, u_t]\) be the state-action pair and \(dP_{t', t}\) be the probability density function for \(z_t\) conditioned on \(z_t\). Recall \(H = E_{z \sim \Pi_K} \tilde{H}(\xi)\), and denote \(\tilde{H}_1\) as the stochastic estimate of \(\tilde{H}\) without the constant value “1” in the leftmost column of the first row. Writing \(\mathcal{E}_{t+\tau} = \mathcal{E}_{t+\tau}(\delta)\), then

\[
\|H - E[\tilde{H}_{t+\tau}|\mathcal{F}_{t-1}, \mathcal{E}_{t+\tau}]\|_* \\
= \bigg\| \int_{\xi \in \mathcal{E}_{t+\tau}} \tilde{H}_1(\xi) [d\Pi_K(\xi) - dP_{t+\tau}|(\xi)] + \int_{\xi \notin \mathcal{E}_{t+\tau}} \tilde{H}_1(\xi) d\Pi_K(\xi) \bigg\|_* \\
\leq \int_{\xi \in \mathcal{E}_{t+\tau}} \|\tilde{H}_1(\xi)\|_* [d\Pi_K(\xi) - dP_{t+\tau}|(\xi)] + \int_{\xi \notin \mathcal{E}_{t+\tau}} \|\tilde{H}_1(\xi)\|_* d\Pi_K(\xi) \\
\leq (\frac{M_H}{2} \sqrt{|z_t|^2 + \frac{n + m}{1 - \rho^2}}) \ln \frac{1}{\delta} \rho^r + \sqrt{E_{\Pi_K}[\|\tilde{H}_1\|^2] \sqrt{1 - P(\mathcal{E}_{t+\tau})}} \\
\leq (\frac{M_H}{2} M_{1/4}^H + \frac{n + m}{1 - \rho^2}) \left( \ln \frac{1}{\delta} \right)^{3/2} \rho^r + \sqrt{E_{\Pi_K}[\|\tilde{H}_1\|^2] \sqrt{\delta}},
\]

where the bound on the first term in \((i)\) is by Proposition B.1 (where \(\rho \equiv \rho(A - BK) < 1\) by assumption \(K \in \mathcal{S}\)) and Lemma 4.2 and the second term by Cauchy-Schwarz. Inequality \((ii)\) is by \(|z_t|^2 \leq \sqrt{M_H} \ln(1/\delta)\), where we used \((B.1)\) and \(M_H\) from Lemma 4.2, as well as Lemma 4.1 and \((4.4)\) to bound the probability term. By definition of \(\tilde{H}(\xi)\) in \((2.12)\) and denoting \(E = E_{\Pi_K}\),

\[
E[\|\tilde{H}_1\|^2] \leq E[\|\tilde{H}\|_F^2] \\
\leq 10(n + m)^2 \max_{i,j,a} \left\{ E([z_t]'[z_t])^2, E([z_t]'[z_t]'[z_t]'[z_t])^2, E([z_t]'[z_t]'[z_t+1]'[z_t+1])^2 \right\} \\
\leq 40(n + m)^2 \max_i \{ E([z_t])^4, E([z_t])^8, E([z_t+1])^8 \} \\
\leq (65(n + m) \max_i \{ ||\tilde{\Sigma}_K||^2, ||\tilde{\Sigma}_K||^4 \})^2 \equiv O_H^2,
\]

where the last line is by Lemma B.2. Combining the last two results yields the bound on the bias for \(\tilde{H}\). By noting the similarity in definition between \(\tilde{H}\) and \(\tilde{b}\) in \((2.12)\), one can similarly bound \(|b - E[b_{t+\tau}|\mathcal{F}_{t-1}]|_*\).

Proof of Lemma 4.5. Recall the matrix \(H = E_{\Pi_K} \tilde{H}\) from \((2.12)\). Our goal is to show the smallest singular value of \(H\), denoted by \(\sigma_{\text{min}}(H)\), is greater than zero, or upper bound \(|H^{-1}| = (\sigma_{\text{min}}(H))^{-1}\). Assuming this is true, because \(|H|\) is bounded (by \(K \in \mathcal{S}\) and Lemma 4.3), then \(H\) is invertible. When \(H\) is invertible, then there exists a \(\theta^*\) such that \(H \theta^* = b\), which in view of the discussion prior to the statement of Lemma 4.5, implies \(g(\theta, y) \geq f(\theta) = \|H \theta - b\| \geq \sigma_{\text{min}}(H)\|\theta - \theta^*\|\). Therefore, when \(\sigma_{\text{min}}(H)\) is positive, the sharpness constant \(\mu\) can be set to \(\sigma_{\text{min}}(H)\).

Towards that, using the condition \(\sigma^2 \geq \sigma_{\text{min}}[1 + \|K\|^2]\) and the policy \(K \in \mathcal{S}\) within the proof of \([48, \text{Lemma B.2}]\), one can show

\[
|H^{-1}|^2 \leq \frac{1}{(1 - \rho^2)^2} \left( \frac{1}{\sigma_{\text{min}}(\Psi)^2} + \frac{n + m}{\sigma_{\text{min}}(\Psi)^2} + (1 + \rho^2) \right) \left( \frac{n + m}{\sigma_{\text{min}}(\Psi)^2} + (1 + \rho^2) \right)^2,\]

where we used the fact \(\rho \equiv \rho(A - BK) \in (0, 1)\) since \(K \in \mathcal{S}\).