Real gas flow about a round cone

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Abstract. We propose a numerical method for a supersonic van der Waals gas flow about a round cone. This method enables one to calculate shock waves attached to the cone and flow parameters between the shock wave and the surface area of the cone. The advantages of the algorithm are demonstrated on some examples.

1. Introduction
We consider the supersonic van der Waals gas flow about a round cone. We assume that the upstream Mach number $M_\infty > 1$ is large enough and the shock wave is attached to the cone (see Figure 1).

The state equation of van der Waals gas has the form

$$p = \frac{RT}{V - V_0} - A \frac{V}{V^2},$$

where $p$ is the pressure, $V$ is the molar volume of the gas, $R$ is the universal gas constant, and $T$ is the temperature. The constants $V_0$ and $A$ provide the correct description of intermolecular forces. It is well known \cite{1,2} that the van der Waals gas can exist not only in the gaseous or liquid state but also in the two-phase state (Figure 2, see also \cite{3}).

We take

$$e = c_V^* T \left( 1 - \frac{A_1 T}{2} \right) - A \frac{V}{V},$$

as the equation for the internal energy. Here $e$ is the internal energy, $c_V^*$ is the heat capacity, $A_1$ is a constant. Usually, the equation of the internal energy is used with $A_1 = 0$, but the method proposed below enables one to use this equation in the more complicated form (1).

2. Equations and their reduction to a dimensionless form
The problem of a supersonic van der Waals gas flow about a round cone can be described by the system of PDE's of gas dynamics. But since the problem is axisymmetric this system is transformed into a system of ODE's. Following \cite{4}, we introduce some new constants for its reduction to a dimensionless form:

$$\alpha = \frac{A}{\hat{p}_\infty V_\infty^2}, \quad \beta = \frac{V_0}{V_\infty}, \quad 0 < \beta < 1,$$
\[ \gamma_{\infty} = 1 + \frac{R}{c_{V}^2 (1 - A \hat{T}_{\infty})}, \quad \gamma^* = 1 + \frac{R}{c_{V}^2}, \quad \gamma^* > \gamma_{\infty} > 0. \]

Then the system in a dimensionless form reads \((s = \theta - a, 0 \leq s \leq b - a = \delta, \text{see Figure 1}):\]

\[
\begin{align*}
\hat{u}'(s) & = \hat{v}(s), \\
\hat{v}'(s) + \hat{u}(s) & = \frac{\tilde{R}(s)}{M_\theta^2 - 1}, \\
\hat{V}(s) \hat{\rho}'(s) & = -\hat{v}(s) \frac{U_\infty^2}{\tilde{p}_{\infty} V_\infty M_\theta^2 - 1} \tilde{R}(s), \\
\hat{V}'(s) & = \frac{\hat{V}(s) \hat{v}(s) U_\infty^2}{c^2(s)} \frac{\tilde{R}(s)}{M_\theta^2 - 1}.
\end{align*}
\]

Here we use the following notations: \(\hat{u}(s), \hat{v}(s), \hat{\rho}(s), \tilde{\rho}(s)\) are the components of the dimensionless velocity, the dimensionless pressure and the dimensionless density respectively;

\[ \tilde{R}(s) = \hat{u}(s) + \hat{v}(s)\cot(s + a); \]

\[ M_\theta^2(s) = \frac{\tilde{M}_\infty^2 \hat{v}^2(s)}{C(s)}, \quad \tilde{M}_\infty^2 = M_\infty^2 \left( \frac{1 + \alpha}{1 - \beta} - 2\alpha \right); \]

\[ c^2(s) = \tilde{p}_\infty \hat{V}_\infty C(s), \quad C(s) = \frac{\tilde{\rho}(s) + \alpha \hat{\rho}^2(s)}{\tilde{\rho}(s)(1 - \beta \hat{\rho}(s))} \left( 1 + \frac{\gamma^* - 1}{1 + \alpha_1 \Lambda(s)} \right) - 2\alpha \tilde{\rho}(s); \]

\[ \tilde{\Lambda}(s) = (1 - \beta \tilde{\rho}(s))(\tilde{\rho}(s) + \alpha \hat{\rho}^2(s)) / \tilde{\rho}(s), \quad \tilde{\Lambda}(\delta) = \Lambda. \]

It is convenient to perform the additional transformation

\[
\begin{align*}
W(s) & = \sin(s + a)\hat{u}(s) + \cos(s + a)\hat{v}(s), \\
U(s) & = \cos(s + a)\hat{u}(s) - \sin(s + a)\hat{v}(s) - 1.
\end{align*}
\]
In these terms system (2)–(5) becomes

\[ W'(s) = \frac{\text{ctg}(s + a)}{M_\theta^2 - 1} W(s), \]
\[ U'(s) + \text{tg}(s + a)W'(s) = 0, \]
\[ \hat{p}'(s) = -\hat{p}(s)M_\theta^2 \left[ \frac{W^2(s) + U^2(s)}{2} + U(s) \right], \]
\[ \hat{\rho}'(s) = \frac{\hat{\rho}(s)}{C(s)}. \]

This system is supplemented with the condition of impermeability

\[ \cos(a)W(0) - \sin(a)U(0) = \sin(a) \]

on the surface area of cone and the four conditions

\[ W(\delta) = \cos(b) \sin(b)k, \]
\[ \hat{p}(\delta) = 1 + M_\infty^2 \sin^2(b)k, \]
\[ \hat{\rho}(\delta) = \frac{1}{1 - k}, \]
\[ \sin(b)W(\delta) + \cos(b)U(\delta) = 0 \]

on the shock front which follow from the Rankine–Hugoniot conditions. Here \( k \) is the root of the function \( Q(k, b), 0 < k < 1 - \beta, 0 < \beta < 1: \)

\[ Q(k, b) = \Lambda \left( 1 + \frac{\alpha_1}{2} \Lambda \right) - \Lambda_\infty \left( 1 + \frac{\alpha_1}{2} \Lambda_\infty \right) - (\gamma^* - 1)k \left( \frac{\alpha}{1 - k} + \frac{\hat{\rho}(\delta) + 1}{2} \right) = 0. \]

3. Numeric method

Let \( M_\infty, a, \alpha, \beta, \gamma_\infty \) and \( \gamma^* \) be given constants. Because of the nonlinearity of the problem the proposed algorithm has an iterative structure.

3.1. Iterations

If the parameters \( k \) and \( b \) are fixed, then the boundary value problem (6)–(13) has a usual form and can be solved in any convenient way. For this propose we use a special form of equations (6)–(7). Assuming that the function \( M_\theta \) is known (in practice, we get \( M_\theta \) from the previous iteration), the boundary value problem (6)–(7), (10)–(11) can be solved explicitly:

\[ W(s) = \frac{\sin^2(a)\hat{\mu}}{\sin(s + a)\bar{I}(s)}, \]
\[ U(s) = \frac{\hat{\mu}}{\cos(a)} + \sin^2(a)\hat{\mu}R(s) - \text{tg}(s + a)W(s), \]

where

\[ \bar{I}(s) = \exp \left\{ \int_0^s \frac{\text{ctg}(\tau + a)M_\theta^2(\tau)}{1 - M_\theta^2} d\tau \right\}, \]
\[ R(s) = \int_0^s \frac{d\tau}{\sin(\tau + a) \cos(\tau + a)\bar{I}(\tau)}, \]
\[ \hat{\mu} = -\frac{\cos^2(a)\sin^2(a)R(\delta)}{1 + \cos(a)\sin^2(a)R(\delta)}, \]
\[ \tilde{\mu} = -\frac{\cos(a)}{1 + \cos(a)\sin^2(a)R(\delta)}. \]
Note that in these terms the boundary condition (14) is reduced to the equation

\[ k = \frac{\sin^2(a) \hat{\mu}}{\cos(b) \sin^2(b) \bar{I}(\delta)}. \]  

(16)

We then use the calculated functions \( U(s) \) and \( W(s) \) by solving numerically the Cauchy problem (8)–(9), (12)–(13).

For calculating the values \( k \) and \( b \) for the next iteration we find the point of the intersection of the graphs of the function \( \bar{k}(b) \) defined by (16) and the implicit function \( \tilde{k}(b) \) defined by (15).

### 3.2. First approximation

To built the first approximation for the solution we use the same fact connected with the intersection of curves (15) and (16), but in this case we put \( M_\theta(s) \equiv 0 \). It follows that

\[ \bar{I}(s) \equiv 1, \quad R(s) = \frac{1}{2} \ln \frac{1 - \cos^2(a)}{1 - \cos^2(a + s)} + \frac{1}{\cos(a + s)} - \frac{1}{\cos(a)}, \]

\[ \hat{\mu}(s) = -\frac{\cos^2(a) \sin^2(a) R(a + s)}{1 + \cos(a) \sin^2(a) R(a + s)}, \quad \tilde{\mu}(s) = -\frac{\cos(a)}{1 + \cos(a) \sin^2(a) (a + s)}. \]

The typical examples of such functions (15) and (16) are given in Figure 3. A lot of numerical experiments show that this way of choosing the initial values of \( k \) and \( b \) gives the fast convergence of the algorithm.

**Figure 3.** The functions \( \bar{k}(b) \) (solid) and \( \tilde{k}(b) \) (dashed) by \( M_\theta \equiv 0, M_\infty = 5, \gamma_\infty = 1.8, \gamma^* = 1.9, \alpha = 4, \beta = 1/3, a = 10^\circ \).

### 3.3. Alternative version of the algorithm

If we fix the angle \( b \) instead of \( M_\infty \), then we can use the same idea of constructing the algorithm for finding the Mach number associated with the shock wave with the angle \( b \).

### 4. Numerical examples

Let us put \( M_\infty = 5, \gamma_\infty = 1.8, \gamma^* = 1.9, \alpha = 4, \beta = 1/3, a = 10^\circ \). We perform the algorithm described in sections 3.1, 3.2 with the discretization step 0.01\(^\circ\). As the result we get the two
angles for the shock wave \( b_1 = 19.29^\circ \) and \( b_2 = 88.34^\circ \). For the given precision the algorithm converges in 2–3 iterations.

Using the algorithm modification from section 3.3 we find pairs \((b, M_\infty)\) for several angles of the cone (see Figure 4). The minimal upstream Mach number for which the shock wave is attached to the cone can be defined by calculating of the minimums of the function \( M_\infty(b) \) (see Figure 5).

![Figure 4](image)

**Figure 4.** Dependence of Mach number on the shock wave angle for the cone angles \( a = 3^\circ, 5^\circ, \ldots, 21^\circ \).

![Figure 5](image)

**Figure 5.** Dependence of the minimal upstream Mach number for which the shock wave is attached to the vertex of the cone on the cone angle.

### 4.1. Atmosphere

Since Nitrogen occupies the largest part of the Earth atmosphere (78%), for modeling the conic flow in the atmosphere we use the following data for this gas:

\[
A = 1.348 \times 10^{-6} \text{atm.cm}^6/\text{moll}^2, \quad V_0 = 3.86 \times 10^{-5} \text{m}^3/\text{moll}, \quad c_V = 4.45 \times 10^2 \text{J kg}^{-1} \text{K}^{-1}.
\]

The parameter \( A_1 \) is not defined in the literature, and we thus put \( A_1 = 0 \). The influence of a nonzero value \( A_1 \) is demonstrated below as well.

The pressure and the density of the gas on the height 0 m are

\[
\hat{p}_\infty = 1.01325 \times 10^5 \text{ Pa}, \quad \hat{\rho}_\infty = 1.225 \text{ kg/m}^3.
\]

We consider the case \( a = 3^\circ, M_\infty = 2 \). The discretization step is 0.01°. As the result of modeling we get two shock waves with the angles \( b_1 = 30.0134^\circ \) (weak shock wave) and \( b_2 = 89.9240^\circ \) (strong shock wave). The solutions are showed in Figure 6.

Other simulations show the following. The numerical solution corresponding to the height of 10000 m (\( \hat{p}_\infty = 2.64999 \times 10^4 \text{ Pa}, \hat{\rho}_\infty = 0.41351 \text{ kg/m}^3 \)) is slightly different from the described above. The increase of the Mach number decreases the angle of the weak shock wave (to \( b_1 = 7.6019 \) by \( M_\infty = 8 \)) and increases the jump of the pressure and the density corresponding to the strong shock wave. A positive nonzero value \( A_1 \) decreases the angle of the strong shock wave (to \( b_2 = 75.9763^\circ \) by \( A_1 = 4 \cdot 10^{-6} \)) and significantly changes the values of the pressure and the density (see Figure 7).
Figure 6. The values of the pressure $\hat{p}$ (solid line) and the density $\hat{\rho}$ (dashed line) are pictured at the top, and the velocity components $\hat{\nu}$ (solid line), $\hat{u}$ (dashed line) are pictured at the bottom. The solution corresponding to $b_1$ is pictured at the left, and the solution corresponding to $b_2$ at the right.

Figure 7. The values of the pressure $\hat{p}$ (solid line), and the density $\hat{\rho}$ (dashed line) corresponding to $b_1 = 30.3514^\circ$ are pictured at the left, and ones corresponding to $b_2 = 75.9763^\circ$ at the right. $A_1 = 4 \cdot 10^{-6}$.

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