Resistance values under transformations in regular triangular grids

Emily. J. Evans\textsuperscript{a}, Russell Jay Hendel\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Brigham Young University, Provo, Utah 84602, USA
\textsuperscript{b}Dept of Mathematics, Towson University, Towson Maryland 21252, USA

Abstract
In [9, 10] the authors investigated resistance distance in triangular grid graphs and observed several types of asymptotic behavior. This paper extends this work by studying the initial, non-asymptotic, behavior found when equivalent circuit transformations are performed, thus reducing the rows in the triangular grid graph one row at a time. The main conjecture characterizes when edge resistance values are less than, equal to, or greater than one after reducing an arbitrary number of times a triangular grid all of whose edge resistances are identically one. A special case of the conjecture is shown. The main theorem identifies patterns of repeating edge labels arising in diagonals of a triangular grid reduced \( s \) times provided the original grid had at least \( 4s \) rows of triangles. This paper also improves upon the notation and concepts introduced by the authors previously, and provides improved proof techniques.

Keywords: effective resistance, resistance distance, triangular grids, circuit simplification

1. Introduction
Resistance distance, also referred to as effective resistance, is a graph metric that has gained popularity in a wide variety of fields due to its ability to quantify the structural properties of a graph. Recently, there have been several papers that have considered resistance distance in structured graphs, including for example [3, 4, 5]. In addition to structured graphs, resistance distance has also been used in the field of chemistry to study chemical structures that have similar regularity patterns to those seen in graph theory [8, 9, 11]. Moreover, resistance distance is useful to those studying combinatorial matrix theory [2, 14], and spectral graph theory [1, 6, 7, 12].

In [3], a paper devoted to the family of straight linear 2–trees, the triangular grid graph was introduced, which heuristically consists of \( n \) rows of upright-oriented equilateral triangles with uniform one-edge resistances. The authors conjectured some results regarding the behavior of the maximal resistance distance in this graph as the number of rows increased. This conjecture was the result of empirical numerical calculations using the combinatorial Laplacian, rather than circuit analysis. Moreover [3, 9] introduced an algorithm for circuit reduction of the triangular grid graph, consisting of reducing the triangular graph one row at a time, but the authors made no attempt to determine the behavior of the resulting reduced graphs.

This computational approach for circuit reduction of the triangular grid was first studied in [10]. This approach provides additional numerical data strongly supporting several conjectures. For example, as \( n \) goes to infinity, the resistance of the edges of the single-triangle graph resulting from reducing the triangular graph with \( n \) rows \( n - 1 \) times unexpectedly approaches \( \frac{3}{2} \) in limit.

This paper complements the study of asymptotic behavior in [3] and [10] by studying non-asymptotic behavior arising in reduced grids. The main conjecture characterizes when resistance values are less than, equal to, or greater than one after reducing an arbitrary number of times a triangular grid all of whose resistance values were initially equal to one. A special case of the conjecture is proven. The main theorem identifies patterns of repeating edge labels arising in diagonals of a triangular grid reduced \( s \) times provided the original grid had at least \( 4s \) rows of triangles. This paper also improves upon the notation and concepts introduced by the authors previously, and provides improved techniques of proofs.

An outline of this paper is as follows. In Section 2 we review previous applications of circuit theory and previous notation and concepts which are extended as needed. In Section 3 we present a simply-formulated conjecture identifying edge resistances that are less than, equal to, or greater than one after reducing an arbitrary number of times an...
initial triangular grid with uniform labels of ones. This conjecture motivated the result, in Section 6 about repeating edge resistances found in the middle portions of reduced $n$-grids and in fact allows in Section 7 the proof of a special cases of the main conjecture. Section 6 is preceded by Section 4 which reviews and simplifies proofs methods from [10] and Section 5 which illustrates the techniques of Section 4. The paper concludes with Section 8 which presents a simple numerical pattern in these reduced $n$-grids, illustrative of the plethora of patterns that apparently abound in these triangular grids and their reductions, pointing to a fertile hunting ground for new patterns, techniques, and proof methods.

2. Definitions, Algorithms, Conventions and Notations

Much of the following comes from [10] and this attribution is not repeated at each item. However, where an idea is extended this is made explicit. More specifically, this section introduces

- Circuit transformations, and the $\Delta, Y$ functions,
- The definition of the $n$-Triangular-grid,
- Conventions about upright triangles,
- Notations for triangles and edges,
- The reduction algorithm,
- Notation for $Y$-legs and reduced triangles,
- $d$-rims and the upper left half,
- Subgrids, boundaries, and notation.

Circuit transformations.

Throughout the paper we use the following well-known circuit transformations

**Definition 2.1** (Series Transformation). Let $N_1, N_2,$ and $N_3$ be nodes in a graph where $N_2$ is adjacent to only $N_1$ and $N_3$. Moreover, let $R_A$ equal the resistance between $N_1$ and $N_2$ and $R_B$ equal the resistance between node $N_2$ and $N_3$. A series transformation transforms this graph by deleting $N_2$ and setting the resistance between $N_1$ and $N_3$ equal to $R_A + R_B$.

A $\Delta$–$Y$ transformation is a mathematical technique to convert resistors in a triangle ($\Delta$) formation to an equivalent system of three resistors in a “$Y$” format as illustrated in Figure 1. We formalize this transformation below.

**Definition 2.2** ($\Delta$–$Y$ transformation). Let $N_1, N_2, N_3$ be nodes and $R_A, R_B, R_C$ be given resistances as shown in Figure 7. The transformed circuit in the “$Y$” format as shown in Figure 4 has the following resistances:

\[
R_1 = \frac{R_B R_C}{R_A + R_B + R_C} \quad (1)
\]
\[
R_2 = \frac{R_A R_C}{R_A + R_B + R_C} \quad (2)
\]
\[
R_3 = \frac{R_A R_B}{R_A + R_B + R_C}. \quad (3)
\]

If $x, y, z$ are the resistances of sides of a triangle then we introduce the function $\Delta(x, y, z) = \frac{xy}{x+y+z}$. By convention, the first two arguments of $\Delta$ will be the labels of the two adjacent edges whose common vertex contains the $Y$-leg of the resulting computation.
Figure 1: $\Delta$ and $Y$ circuits with vertices labeled as in Definition 2.2.

Definition 2.3 (Y–$\Delta$ transformation). Let $N_1, N_2, N_3$ be nodes and $R_1, R_2$ and $R_3$ be given resistances as shown in Figure 1. The transformed circuit in the “$\Delta$" format as shown in Figure 1 has the following resistances:

\begin{align*}
R_A &= \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1} \\
R_B &= \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_2} \\
R_C &= \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_3}.
\end{align*}

(4)

(5)

(6)

If $x, y$ and $z$ are resistances of a $Y$ we define the function $Y(x, y, z) = \frac{xy + yz + zx}{x}$. By convention, the first argument of the $Y$ function will be resistance of the $Y$ leg perpendicular to the edge on the resulting $\Delta$ (triangle).

Definition of $n$-grid.

There are a variety of ways to define graphs including using an inductive approach, by general properties of graphs, and by the adjacency-matrix. We present a simple definition by vertices and edges.

Definition 2.4. An $n$-triangular-grid (usually abbreviated as an $n$-grid) is any graph that is (graph-) isomorphic to the graph, whose vertices are all integer pairs $(x, y) = (2r + s, s)$ in the Cartesian plane, with $r$ and $s$ integer parameters satisfying $0 \leq r \leq n, 0 \leq s \leq n - r$; and whose edges consist of any two vertices $(x, y)$ and $(x', y')$ with either i) $x' - x = 1, y' - y = 1$, ii) $x' - x = 2, y' - y = 0$, or iii) $x' - x = 1, y' - y = -1$.

Figure 2 illustrates the 3-grid.

Figure 2: A 3-grid with the upright oriented triangles labeled by row and diagonal.

Notation and Conventions 2.5. Upright–oriented triangles. Throughout the paper we will find it useful to use heuristics. We perceive the $n$-grid as a triangular grid consisting of $n$ rows of triangles. The $r$-th row of the $n$-grid has $2r - 1$ triangles of which $r$ are in the upright-oriented position and the remaining $r - 1$ triangles are in the downward
oriented position as illustrated in Figure 2. For purposes of describing the row-reduction step we are only concerned with the \( r \) upright-oriented equilateral triangles; consequently, throughout the paper when speaking about triangles, we will assume the reference is to upright-oriented triangles.

**Notation and Conventions 2.6. Triangle and edge notation.** The following terminology and notation will be used throughout the paper. We will refer to the left, right and base sides of the triangles. The left, right, and base sides of a triangle will also be called the 1-edge, 2-edge, and 3-edge. We let \( \langle r, d \rangle_n, 1 \leq d \leq r \), refer to the (upright oriented) triangle in row \( r \) and diagonal \( d \) with diagonals enumerated left to right, and rows enumerated top to bottom. If each edge in an \( n \)-grid is labeled with an electrical resistance (which is equivalent to the reciprocal of the edge weight), then, \( \langle r, d, e \rangle, e \in \{1, 2, 3\} \) refers to the resistance of edge \( e \) of triangle \( \langle r, d \rangle \).

**The Reduction Algorithm.**

We recall the algorithm presented in [9, 10] for performing reductions on the \( n \)-grid as defined by Definition 2.4 and using the Notation and Conventions 2.5 and 2.6. The algorithm will take an \( n \)-grid and reduce the number of rows by 1, resulting in an \( n - 1 \) grid. Since the algorithm uses circuit functions that produce equivalent electrical resistance this \( n - 1 \) grid is constructed so that the resistance between two corner vertices remains the same.

**Algorithm 2.7.**

Given an \( n \)-grid with labeled edges, the row-reduction of this labeled \( n \)-grid (to a labeled \( n - 1 \)-grid) refers to the sequential performance of the following steps:

- (Step A) Identify the edge-labels of relevant triangles in the original \( n \)-grid
- (Step B) Perform a \( \Delta - Y \) transformation, on each (upright-oriented) triangle.
- (Step C) Retain the 3 corner tails, that is, the edges with a degree 1 vertex (these are the dashed lines in Panel B) for possible later computation, however, discard them from the diagram for ease of understanding. This discarding does not affect the resistance labels of the edges of the reduced grid displayed in Panel E
- (Step D) Perform series transformations on all consecutive pairs of remaining boundary edges.
- (Step E) Perform \( Y - \Delta \) transformations on the remaining non-boundary edges.

The algorithm is illustrated in Figure 3

![Figure 3: Illustration of one row reduction, Algorithm 2.7, on a 3-grid all of whose edge-labels are 1. The panel labels correspond to the five steps of Algorithm 2.7](image)

**Remark 2.8.** The idea of retaining the corner tails for future computations, but omitting them in future reduction steps may seem unusual without appropriate context. In Section 3 of this paper as well as in [3, 9, 10] the problem of determining the resistance distance between the degree two vertices in the original graph was considered. For this problem it is essential that these tail values be retained to determine the resistance distance. In fact the determination of the resistance distance between these vertices led to the reduction algorithm, and the results detailed in this paper.
Definition 2.9. We call a single pass-through of Algorithm 2.7 (i.e., performing steps A-E one time) a reduction.

Notation and Conventions 2.10. Notation for Y-legs and reduced triangles. Throughout the paper we use the following additional notations and conventions.

- A $\Delta - Y$ transformation transforms a triangle into an upside down Y. We will abuse notation and let $Y_{12}, Y_8, Y_4$ refer to both the legs of an upside-down Y as well as their resistances. The meaning will always be clear from context and this should cause no confusion.

- We can combine these clock positions with row diagonal vertices in subscripts, for example, $Y_{12,2,1}$ is the resistance of the $Y_{12}$ edge resulting from applying a $\Delta - Y$ transform to the triangle $\langle 2, 1 \rangle$.

- $T(k)$ (or $T_k$) is the k-grid all of whose edge resistances are 1.

- $T(k, k')$ or $T_{k,k'}$ is the k'-grid obtained from $T(k)$ by applying $k - k'$ reduction steps.

- $T(k)$ as well as $T(k, k')$ has physical isotropy (no real-world experiment can distinguish any direction); throughout this paper, physical isotropy is mathematically identified with vertical and rotational symmetry (by $\pi$ and $\frac{2\pi}{3}$) of the underlying grid regarded as a labeled graph.

- When performing a single reduction from $T_{k,k-1+i}$ to $T_{k,k-i}$ it is necessary to distinguish between triangles in the parent grid, $T_{k,k-1+i}$ and child grid $T_{k,k-i}$. In such a case we will use double subscript notation e.g. $\langle r, d \rangle_{k,k-i}$ is the triangle in $T_{k, k-i}$ in row r diagonal d.

$d$-rims, subgrids, and the upper left half.

Hendel [10, Equations (29),(30)] introduced the idea of perceiving the $n$-grid as a collection of concentric triangular annuli or triangular rims. He also introduced the idea of the upper left half of the $n$-grid. Prior to defining these concepts, we present a coordinate independent definition of the symmetries.

Definition 2.11 (Grid Symmetry). [10, Definition 9.1] When discussing $n$-grids we use either the terms isotropy or symmetry to describe a labeled $n$-grid possessing both vertical, rotational, and slide symmetry, where by vertical symmetry we mean that for $1 \leq r \leq n, 1 \leq d \leq r$,

$$\langle r, d, 1 \rangle = \langle r, r + 1 - d, 2 \rangle, \quad \langle r, d, 2 \rangle = \langle r, r + 1 - d, 1 \rangle, \quad \langle r, d, 3 \rangle = \langle r, r + 1 - d, 3 \rangle;$$

by rotational symmetry, we mean that for $1 \leq r \leq n, 1 \leq d \leq r$,

$$\langle r, d, 1 \rangle = \langle n + d - r, n + 1 - r, 2 \rangle, \quad \langle r, d, 2 \rangle = \langle n + d - r, n + 1 - r, 3 \rangle, \quad \langle r, d, 3 \rangle = \langle n + d - r, n + 1 - r, 1 \rangle,$$

and by slide symmetry we mean

$$\langle r, d, 1 \rangle = \langle n + d - r, d, 1 \rangle, \quad 1 \leq d \leq r \leq n.$$

It is straightforward to verify that an assumption of vertical and slide symmetry is equivalent to an assumption of vertical and rotational symmetry. Note also, that if the $n$-grid is represented by $n$-rows of equilateral triangles then the above definitions of vertical and rotational symmetry coincide with the coordinate definitions of vertical symmetry and clockwise rotation by $\frac{\pi}{3}$.

Definition 2.12 (The Upper Left Half). Given integers $n$ and $m$ with $0 \leq n \leq m - 1$, the upper left half of the $n$ grid, $T_{n,m}$ arising from $n - m$ reductions of $T_n$, is the set of triangles

$$S = \left\{ \langle r, d \rangle : 1 \leq d \leq \left\lfloor \frac{n + 2}{3} \right\rfloor, 2d - 1 \leq r \leq \left\lfloor \frac{n + d}{2} \right\rfloor \right\}$$

(7)

Example 2.13. If $n = 3$, (see panel A1 in Figure 3) the upper left half consists of the triangles $\langle r, d \rangle$, $d = 1, r = 1, 2$. If $n = 7$ (see Figure 3) the upper left half consists of the union of the triangle sets $\langle r, d \rangle$, $d = 1, 1 \leq r \leq 4$, $\langle r, d \rangle$, $d = 2, 3 \leq r \leq 4$, and $\langle 5, 3 \rangle$.
The importance of the upper left half is the following result which captures the implications of the symmetry of the \( n \)-grids [10, Corollary 9.6].

**Lemma 2.14.** With \( n, m \) defined as in Definition 2.12 the edge values of all edges in \( T_{n,m} \) are fixed once the edge values of the upper left half are fixed.

**Definition 2.15.** [10, Equation (29)] With \( n, m \) as in Definition 2.12 we define for \( s \geq 1 \) the \( s \)-subgrid of \( T_{n,m} \) to be the subgrid of \( T_{n,m} \) with corners \( \langle 2s-1, s \rangle, \langle n+1-s, s \rangle, \langle n+1-s, n+2-2s \rangle \).

The notation \( \partial T_{n,m}^{s} \) indicates the boundary of triangles of the \( s \)-subgrid \( T_{n,m}^{s} \). This boundary is also called the \( s \)-rim. The notation \( \partial^{2} T_{n,m}^{s} \) indicates the edge-boundary of the \( s \)-subgrid. When \( s \) is omitted we assume \( s = 1 \). The notation \( \text{Interior}(T_{n,m}^{s}) \) refers to all edges not on its edge-boundary.

![Figure 4: An illustration of the triangular rims. The edge boundaries of these rims are colored as follows: \( T^{1}_{4} \) is blue, \( T^{2}_{4} \) is red, \( T^{3}_{4} \) is green. The boundaries of each rim are solid, and the interior edges are dashed.](image)

**Example 2.16.** Consider the 7 grid shown in Figure 4. The 1-rim has corners \( \langle 1, 1 \rangle, \langle 7, 1 \rangle, \langle 7, 7 \rangle \) and is colored blue; the 2-rim has corners \( \langle 3, 2 \rangle, \langle 6, 2 \rangle, \langle 6, 5 \rangle \) and is colored red; the three rim is the singleton green triangle \( \langle 5, 3 \rangle \).

2.1. Grid Equality

**Definition 2.17.** Grid and triangle equality. If \( S \) and \( S' \) are any collection of triangles and/or edges in \( T_{n} \) for some \( n \) then we say \( S \) equals \( S' \), \( S = S' \), to indicate that the edge labels of all corresponding edges (assuming the correspondence is clear) are equal.

3. Motivation

There is a two-fold motivation for this paper. First, prior work in [3], observed asymptotic behavior in the sequence of the maximal resistance distances in a sequence of \( n \)-grids. In particular, let \( T_{n} \) be the \( n \)-grid with \( n \) rows and \( m = n^{2} \) triangles. Moreover, let \( a \) and \( b \) be any pair of distinct vertices with degree 2, and let \( r_{n}(a, b) \) be the resistance distance between \( a \) and \( b \) in \( T_{n} \). Then

\[
\lim_{n \to \infty} \exp(r_{n+1}(a, b)) - \exp(r_{n}(a, b)) = C > 0.
\]

Subsequent to this result, Hendel in [10], discovered a pattern in the tails of the transformed \( n \)-grid. He observed that as \( n \) goes to infinity, the resistance of the edges of the single-triangle graph resulting from reducing the triangular graph with \( n \) rows \( n-1 \) times unexpectedly approaches \( \frac{2}{e} \) in limit.

It is straightforward to derive the formula \( r_{n} = 2 \times \sum_{i=1}^{n} t_{1}(n, i) \). By decomposing the total resistance into summands, we can use information about these summands. As illustrated and conjectured in [10], \( t_{1}(n, i) = \frac{1}{2} t_{1}(n, 1) \).
provided \( n \) is large enough and \( i \) is relatively constant. Using these considerations, we were able to refine [3, Conjecture 7.8] to \( r_n = \sum_{i=1}^{n-1} \frac{1}{2} i \).

Second, in an attempt to understand these results, we began to look at the “initial behavior” of the edge resistances as an \( n \)-grid undergoes these transformations. We noticed that with each transformation, the location of edges with resistance equal to one decreased in a predictable pattern. Moreover we noticed that the non-one edges also maintained a specific pattern dependent on their location in a rim. We formalize these observations in the following conjecture.

**Conjecture 3.1** (Vanishing Ones Conjecture). *For integer \( n \geq 1 \), we have the following:

(a) \( T_{1,n}^{n,n} = 1 \).

(b) For \( 1 \leq s \leq \left\lfloor \frac{n+1}{4} \right\rfloor \), \( \text{Interior}(T_{n,n-s}^{n,s}) \) is equal to one.

(c) With \( s \) as in (b), for an edge in the complement of the interior of an \( s \)-subgrid: its edge value is strictly less than one if it lies on the edge boundary of some \( s' \)-grid, \( 1 \leq s' < s \); its edge value is strictly greater than one otherwise.

(d) For any \( s \), \( \left\lfloor \frac{n+1}{4} \right\rfloor < s \leq n - 1 \) there are no edges with label 1 in \( T_{n,n-s} \) (the ones “vanish”).

### 4. Proof Methods.

All results in this paper are proven applying Algorithm 2.7 which in turn involves computing \( \Delta \) and \( Y \) transformations. We will follow the technique used by Hendel [10] who approached each proof using the five steps (panels) A-E presented in Figure 3. In other words, each proof will consist of (Step A) identification of the triangles and edge values in some \( n \)-grid used in the computation, (Steps B, C) application of the \( \Delta - Y \) transforms and ignoring any tails, (Step D) performing any relevant series computations (if border edges are involved), and (Step E) performing \( Y - \Delta \) transformations resulting in the edge value of a triangle in the reduced \( (n-1) \)-grid. The proofs are typically summarized in a Figure sequentially showing Steps A-E.

Unlike [10] where a typical proof applied to all edges of a triangle, proofs in this paper will very often focus on specific edges. It turns out there are only 4 cases that have to be considered to develop a comprehensive suite of lemmas that can cover all proofs. These four cases are presented in this section, Each lemma is given a mnemonical name to facilitate reference later in the paper.

**Lemma 4.1.** [Base Edge] Given integers \( n \geq 3 \), \( r \leq n - 2 \), \( 1 \leq d \) to compute \( \langle r, d, 3 \rangle_{n,n-1} \) it suffices to use the triangles, edge values, and functions presented in Figure 5.

**Proof.** Algorithm 2.7. To clarify the proof method we note that Panel A of Figure 5 identifies the 9 edges of the parent grid needed to compute the target base edge of the child grid. This identification is in fact what is required by Step A of 2.7. Then Steps B,C requires discarding tails and performing \( \Delta - Y \) transforms as in fact shown in Panels B,C. Finally, panels D,E illustrate the required computations of Steps D,E, the computation of \( Y - \Delta \) transformations. The function derived is all that is needed to perform the computations in future sections.

**Lemma 4.2.** [Boundary Edge] Given integers \( n \geq 2 \), \( r \leq n - 1 \), to compute \( \langle r, 1, 1 \rangle_{n,n-1} \) it suffices to use the triangles, edge values, and functions presented in Figure 6.

**Proof.** Algorithm 2.7.

**Lemma 4.3.** [Left Edge] Given integers \( n \geq 2 \), \( r \leq n - 1 \), \( d \geq 2 \) to compute \( \langle r, d, 1 \rangle_{n,n-1} \) it suffices to use the triangles, edge values, and functions presented in Figure 7.

**Proof.** Algorithm 2.7.

**Lemma 4.4.** [Right Edge] Given integers \( n \geq 2 \), \( r \leq n - 1 \), \( d \leq r - 1 \) to compute \( \langle r, d, 2 \rangle_{n,n-1} \) it suffices to use the triangles, edge values, and functions presented in Figure 8.

**Proof.** Algorithm 2.7.
5. Illustrations and Easy Consequences of the Proof Methods

This section begins by presenting two straightforward consequences of the proof methods of Section 4. We first present a corollary illustrating how the figures associated with the lemmas in Section 4 naturally provide explicit computations to verify any result.

**Corollary 5.1.** Let $n \geq 2$. Then $\bar{v}^2 T_{n,n-1} = \frac{2}{3}$, and $\text{Interior}(T_{n,n-1}) = 1$. In words, the edge boundary of $T_{n,n-1}$ is identically $\frac{2}{3}$ while all other edges are 1.
Proof. To acclimate the reader to the proof methods of Section 4, we present two proofs.

First, we prove this result by appealing directly to the reduction algorithm and calculations. By Algorithm 2.7, using a $\Delta - Y$ transformation, we start the proof by converting the $n$ rows of $T_n$ to $n$ rows of $Y_n$.

Since $\Delta(1, 1, 1) = \frac{1}{3}$, all $Y$-edges have weight $\frac{1}{3}$. By Algorithm 2.7, there are now two cases to consider. Along the boundary, the edges of $T_{n,n-1}$ arise from a series transformation and hence have weight $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. All remaining edges in $T_{n,n-1}$ arise from $Y - \Delta$ transformations and hence have weight $Y(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$.

The second proof reflects the fact that the algorithmic descriptions have already been provided in the figures accompanying the proofs of Section 4. First consider the left-boundary edges of $T_{n,n-1}$. By the Boundary Edge Lemma, $\Delta(1, 1, 1)$ in $T_{n,n-1} = \Delta(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) + \Delta(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{2}{3}$. For the remaining edges of $T_{n,n-1}$ by the Left, Right, and Base edge Lemmas and accompanying Figures, the computation of the edge values arises from a $Y$ function applied to three arguments each of which is a $\Delta$ function whose three arguments are identically one. The proof is therefore completed by noting that $\Delta(1, 1, 1) = \frac{1}{3}$ and $Y(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$. \qed

Remark 5.2. In the sequel, we will suffice with proofs based on the appropriate lemmas and accompanying figures of Section 4.

The following corollary is based on [10, Section 10].

Corollary 5.3 (Six triangles). Given $r, d, n$ with $n \geq 4, 1 \leq r \leq n - 2, 1 \leq d \leq r - 1$, with $r, d$ lying in the upper-left-half (Definition 2.12), to calculate $\langle r, d, e \rangle_{n,n-1}$ it suffices to know the edge values, where defined, of $\langle r', d' \rangle_n$ for
(r', d') ∈ \{(r, d - 1), (r, d), (r, d + 1), (r + 1, d), (r + 1, d + 1), (r + 2, d + 2)\}

Proof. By the lemmas and figures of Section 4.

Remark 5.4. To clarify the phrase “where defined”, consider a case where \(d = 1\). In this case, \(d - 1 = 0\) and some of the six triangles are not defined, but the corollary remains true.

Remark 5.5. To clarify the need for the requirement “in the upper left half” consider the base edges of the triangles in the bottom row of the child triangle. The parent grid has only one more row than the child grid. But the Base Edge Lemma requires triangles two rows below the row of the target triangle at the bottom of the child grid. Thus the Base Edge Lemma does not suffice for computation here. However, the lemmas of Section 2 do suffice for computations in the upper left half; then the edge values of the base edges of the triangles in the bottom row of the entire grid are determined by the upper left half (Lemma 2.14).

Remark 5.6. As indicated earlier, Conjecture 3.1 motivated the other results in this paper. To tie Corollary 5.3 to the Main Conjecture notice that this corollary indicates that at most four triangles in the parent grid determine a triangle in the child grid. This gives insight into how fast, defined as how many reductions are needed, before the resistance value on a given edge is not equal to one. More specifically, for any \(s\), the resistances on edges in the center of the \(T_{n,n-s}\) will remain equal to one until there is a change from all ones in the surrounding triangles in the parent grid. The corollary, however, implies that on each reduction the only triangles in the child grid that will change are those
whose parents have edge resistance not equal to one. Thus per reduction only one “rim” of the triangle can change at a time.

**Corollary 5.7 (All-One).** If the edge values of the at most six triangles in the parent grid determining the edge value of a triangle in the child grid are identically 1, then the edges of the triangle in the child grid are identically one.

**Proof.** This follows immediately by the lemmas of Section 4 and the computational facts that \( \Delta(1, 1, 1) = \frac{1}{4} \) and \( Y(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1 \).

In the sequel of the paper, we explore several results naturally motivated by Corollary 5.1. We begin with the very beautiful Uniform Center Theorem. To motivate this elegant result we need first a definition and remark.

**Definition 5.8.** The pre-type of a triangle \( \langle r, d \rangle \) refers to the ordered triple of edge values listed in clockwise order starting from the left-most edge. Two triangles, \( T_1 = \langle r, d \rangle, T_2 = \langle r', d' \rangle \) are said to be of identical type (or of the same type or of uniform type) if either i) their pretypes are equal (as ordered triplets) or ii) the labeled triangle \( T_2 \) is the result of application of vertical and rotational symmetries applied to the labeled triangle \( T_1 \).

**Remark 5.9.** To clarify the effect of symmetry consider the two triangles \( \langle 2, 1 \rangle_{n,n-1}, \langle 2, 2 \rangle_{n,n-1} \), whose pre-types are \( \frac{2}{3}, 1, 1 \) and \( 1, \frac{2}{3}, 1 \) respectively. Although these two ordered triplets are distinct, the triangle \( \langle 2, 2 \rangle \) whose associated pre-type is \( 1, \frac{2}{3}, 1 \) arises from a vertical symmetry applied to triangle \( \langle 2, 1 \rangle \) whose associated pre-type is \( \frac{2}{3}, 1, 1 \); the vertical symmetry is consistent with the labels. Therefore, Definition 5.8 justifies our saying that \( \langle 2, 1 \rangle_{n,n-1} \) and \( \langle 2, 2 \rangle_{n,n-1} \) are of the same type.

### 6. The Uniform Center Theorem

Using the concept of type we can generalize Corollary 5.1 by first observing the follows patterns confirmed in the associated figures.

- Figure 10 presents \( T_{n,n-1} \). In this figure distinct letters in the interiors of the triangles correspond to distinct types. It is easy to see that D corresponds to the pre-type 1, 1, 1; A corresponds to the pre-type \( \frac{2}{3}, \frac{2}{3}, 1 \), and B corresponds to the pre-type \( 2 \), \( \frac{2}{3}, 1 \). Notice that the types on the diagonal \( d = 1 \) are in the “center” of the one diagonal (that is on non-corners). In fact, for \( n \geq 4 \), in \( T_{n,n-1} \), the triangles \( \langle r, 1 \rangle, 2 \leq r \leq n - 2 \) all have the same type.

- For \( n \geq 12 \), in \( T_{n,n-3} \) the triangles \( \langle r, 3 \rangle, 6 \leq r \leq n - 6 \) all have the same type, as shown in the triangles labeled G in Figure 9.

- This generalizes for a fixed \( s \) as: For \( n \geq 4s, 1 \leq d \leq s \), the triangles \( \langle r, d \rangle \), where \( s + d \leq r \leq n - 2s \) all have the same type in \( T_{n,n-s} \).

This example motivates the concept of the uniform center.

**Definition 6.1.** Fix \( s \geq 1 \). If \( n \geq 4s, 1 \leq d \leq s \) the uniform center of \( T_{n,n-s} \) refers to the closure under vertical symmetry of all triangles \( \langle r, d \rangle_{n,n-s} \), where

\[
s + d \leq r \leq n - 2s. \tag{8}
\]

The Uniform Center Theorem simply captures the heuristic that the types of the triangles on each diagonal \( d, 1 \leq d \leq s \) are uniform in a central region of \( T_{n,n-s} \). Unexpectedly, the theorem can be proven without computations.

**Theorem 6.2 (Uniform Centers).** For any \( s \geq 1 \), and for \( n \geq 4s \),

(a) The triangles on each diagonal \( d, 1 \leq d \leq s \) of the uniform center, that is satisfying (8), are of the same type.

(b) For each defined \( s \)-subgrid \( T_{n,n-s} \), Definition 2.13 the interior resistance labels are uniformly 1, and the boundary edge resistance labels are all equal.

(c) For triangles in the uniform center, that is satisfying (8), the right and base resistance labels are identical.
Figure 9: The triangle types of the first 7 rows of $T_{n,n-3}$ after 3 reductions of $T_n$. Here, $n$ is very large, say $n \geq 23$.

**Remark 6.3.** Part (c) trivially follows from the observation that $T_{n,n-s}^{s+1} \subseteq \text{Interior}(T_{n,n-s}^s)$. It remains to prove parts (a) and (b).

**Notation and Conventions 6.4.** Prior to presenting the proof we introduce a notational convenience. Consider, for example Figure 8 which asserts

$$R = Y(\Delta(g,e,f), \Delta(h,i,j), \Delta(b,c,a)) = F(a,b,c,e,f,g,h,i,j),$$

where $F$ is the function implicitly defined by the equality for the value of $R$. $F$ is a function of 9 variables and hence notationally cumbersome. It is notationally convenient to introduce the notation

$$R = F(T_1, T_2, T_3),$$

with $T_1 = \langle r, d \rangle$, $T_2 = \langle r, d+1 \rangle$, and $T_3 = \langle r+1, d+1 \rangle$, that is, triangles are listed in clockwise order starting from the leftmost (smallest diagonal). This equation is interpreted by replacing each triangle by its sides listed in clockwise order starting from the left-most side.

The point of this notational convenience is that if two edge values are functions of the same triangles (including order) then the edge values must be equal.

**Proof.** We begin with the proof of (a), by induction. Throughout the proof we refer to Figure 9 to illustrate the proof steps. We note that in this figure, $s = 3$ and that by (8) the uniform center for diagonals 1, 2, and 3 begin in rows $s + 1 = 4$, $s + 2 = 5$, and $s + 2 = 6$ as illustrated by the triangles labeled $E$, $G$, and $I$ respectively.

For an induction assumption we assume that for some $s \geq 1$, the following is true:

- $\text{Interior}(T_{n,n-s}^s) = 1$.
- For each diagonal, $1 \leq d \leq s$ the triangles $\langle r, d \rangle$, with $r$ satisfying (8) have identical type.

By Corollary 5.1, this induction assumption is true for the base case, $s = 1$. For the induction step, we must prove the itemized bullets true with $s$ replaced by $s + 1$.

We start the proof by considering the specific diagonal, $d = 2$. By (8)

- The uniform center for diagonal 2 begins on row $s + 2$, (which corresponds to the $G$’s on the second diagonal beginning in row 5 in the figure);
- The uniform center for diagonal 3 begins on row $s + 3$, (which correspond to the $I$’s on the third diagonal beginning in row 6 in the figure);
• The uniform center for diagonal 1 begins on row \(s + 1\), (which correspond to the \(E\)s on the first diagonal beginning in row 4 in the figure);

Using these values in the parent grid \(T_{n,n-\delta}\) we next compute right, base, and left edge values in the child grid \(T_{n,n-(s+1)}\)

• By the Right Edge Lemma, the right edge value of \(\langle r,2 \rangle_{n,n-(s+1)}, r = 5,6, \ldots\) is \(F(\langle r,2 \rangle, \langle r,3 \rangle, \langle r+1,3 \rangle)\) (corresponding to \(F(G,H,I)\) in the figure) and hence these right edge values are identical.

• By the Base Edge Lemma, the base edge value of \(\langle r,2 \rangle_{n,n-(s+1)}, r = 6,7, \ldots\) is \(F(\langle r+1,2 \rangle, \langle r+1,3 \rangle, \langle r+2,3 \rangle)\) (corresponding to \(F(G,I,I)\) in the figure) and hence these base edge values are identical.

• By the Left Edge Lemma, the left edge value of \(\langle r,2 \rangle_{n,n-(s+1)}, r = 5,6, \ldots\) is \(F(\langle r,1 \rangle, \langle r,2 \rangle, \langle r+1,2 \rangle)\) (corresponding to \(F(E,G,G)\) in the figure) and hence these left edge values are identical.

We have just proven that the pre-type – left, right, and base edge values – of \(\langle r,2 \rangle_{n,n-(s+1)}, r = 6,7, \ldots\) is uniform for \(r \geq s + 1 + 2\) (corresponding to row 6 in the figure) consistent with [8]; hence we have proven part (a) for the case \(d = 2\) The proofs for \(2 \leq d \leq s - 1\) are highly similar and hence omitted. The proofs for \(d = 1\) and \(d = s\) are also similar requiring adjustments for example for the left boundary edge. This complete the proof of part (a).

![Figure 10: The triangle types of the first 7 rows of \(T_{n,n-1}\) after 1 reduction of \(T_{n,n} \geq 3\).](image)

We next must prove part (b) of Theorem 6.2 which we will do by induction on \(s\).

Assume by induction that for some \(s \geq 1\) that

(i) \(\text{Interior}(T^s_{n,n-s}) = 1\),
(ii) \(c^2T^s_{n,n-s}\) (corresponding to the triangles marked \(D\) in Figure 10) has a uniform resistance label.

The base case, \(s = 1\), has been established in Corollary 5.1 and is illustrated in Figure 10. We must prove the above two bullets for the case \(s + 1\). We first prove statement (i).

Recall, that the top corner triangle of the \((s + 1)\)-rim, \(T^s_{n,n-s}\) occurs at \(r = 2s + 1, d = s + 1\) (corresponding to the \(D\) triangle in row 3 diagonal 2 in Figure 10).

We proceed to calculate (non-boundary) left, (non-boundary) right, and base edge values in the child grid \(T^{s+1}_{n,n-(s+1)}\)

• By the Base Edge Lemma, a base edge of \(\langle r,d \rangle\) in the interior of \(T^{s+1}_{n,n-(s+1)}\) is \(F(\langle r+1,d \rangle, \langle r+1,d+1 \rangle, \langle r+2,d+1 \rangle)\) and since these triangle arguments are identically 1 (corresponding to the \(D\) triangles in the figure), the base edge is identically one by the All-One Corollary.

• By the Right Edge Lemma, a right edge of \(\langle r,d \rangle, d < r\) in the interior of \(T^{s+1}_{n,n-(s+1)}\) is \(F(\langle r,d \rangle, \langle r,d+1 \rangle, \langle r+1,d+1 \rangle)\) and since these triangle arguments are identically 1 (corresponding to the \(D\) triangles in the figure), the base edge is identically one by the All-One Corollary.

13
• By the Left Edge Lemma, a left edge of \( \langle r, d \rangle, d \neq 1 \) in the interior of \( T_{n,n-(s+1)}^{s+1} \) is \( F(\langle r, d \rangle, \langle r+1, d+1 \rangle) \), and since these triangle arguments are identically 1 (corresponding to the \( D \) triangles in the figure), the left edge is identically 1 by the All-One Corollary.

This shows that the \( \text{Interior}(T_{n,n-(s+1)}^{s+1}) = 1 \).

We next prove statement (ii).

We have to compute the left resistance labels on the boundary of the \( s + 1 \) subgrid. It suffices to compute the left resistance labels on the triangles on diagonal \( d = s + 1 \) since the right and base edge values then follow by symmetry considerations (Lemma 2.1). By the left-edge lemma the left edges in the child grid are computed using triangles on the diagonals \( d = s, d = s + 1 \) in the parent grid. However, by statement (i) just shown the triangles on diagonal \( d = s + 1 \) are identically 1. Thus, by Equation (6) the uniform center for diagonal \( d = s \) begins on row \( s + d = s + s = 2s \) and hence the triangles \( \langle r, d \rangle, r \geq 2s + 1 \) are of the same type. It then immediately follows by the left edge lemma that all left edge resistance labels are \( F(T, 1, 1) \) where \( T \) is a triangle on diagonal \( d = s \) in the uniform center and 1 indicates a triangle uniformly labeled 1. This completes the proof of part (b).

We have left to prove part (c) of the uniform center theorem. Say the target triangle (whose right and edge resistance labels we wish to prove equal) lies on row \( r \) and diagonal \( d > 1 \). The proof for \( d = 1 \) is similar and hence omitted.

By hypothesis we work in the uniform center and hence the triangles on diagonal \( d \) and \( d + 1 \) are uniformly labeled. By the right edge lemma the right edge is \( F(\langle r, d \rangle, \langle r, d+1 \rangle, \langle r+1, d+1 \rangle) \), say \( Y(\Delta(h, i, j), \Delta(b, c, a), \Delta(g, e, f)) = Y(\Delta(h, i, j), \Delta(b, c, a), \Delta(h, i, j)) \), where we have exploited the uniformity along diagonal \( d + 1 \).

Similarly by the base edge lemma the base resistance label is \( F(\langle r + 1, d \rangle, \langle r + 1, d + 1 \rangle, \langle r + 2, d + 1 \rangle) \), say \( Y(\Delta(b, c, a), \Delta(h, i, j), \Delta(h, i, j)) \), where we have exploited the uniformity along diagonals \( d + 1 \).

Thus the proof of equality of the right-edge label and base-edge label is reduced to showing the equality of two functions of nine variables. This equality follows from Definition 2.3 which shows that permuting the second and third arguments of \( Y \) does not alter the value.

This completes the proof of part (c) and the proof of the uniform center theorem is complete.

**Corollary 6.5.** Conjecture 3(b) is true.

7. **Proof of the Vanishing One Conjecture in a Special Case**

While we have shown the first two parts of Conjecture 3, a proof of the remaining two parts remains elusive. It is, however, possible to show the proof of parts (c) and (d) for certain edges, specifically those boundary edges in the 1-rim of the reduced triangle. In this section, we use the notation \( L_s = \langle s, 1, 1 \rangle_{n,n-s} \), to refer to the left boundary edge of the triangle at row \( s \) diagonal 1, of \( T_{n,n-s} \) which by the Uniform Center Theorem is the unique edge label of the constant center of diagonal 1 in the \( s \)-th reduction of the original \( n \) grid. \( B_s \) is defined similarly.

**Lemma 7.1.** The sequence \( L_s, s = 1, 2, 3, \ldots \) is monotone decreasing.

**Proof.** Using Definition 6.A the Boundary Edge Lemma, and its accompanying Figure 6, summarized in Figure 11, we have

\[
L_{s+1} = F(A, A) = \frac{2L_sB_s}{L_s + 2B_s}.
\]

However, clearing denominators and taking like terms, this last equation is equivalent to the assertion that

\[
L_{s+1}L_s = 2B_s(L_s - L_{s+1}).
\]

The proof is completed by noting that i) all resistances are positive and consequently, ii) the left side of the last equation is positive implying that iii) the parenthetical expression is positive which iv) is equivalent to a statement of monotonicity.

The following corollary proves parts (c) and (d) of Conjecture 3 for these boundary edges.

**Corollary 7.2.** For all \( s, L_s < 1 \).
Proof. The corollary follows from the lemma and the fact that $L_1 = \frac{3}{4}$ (Corollary 5.1).

One might ask if the other edges in the 1-rim can be determined. Since the boundary edges of the one rim in $T_{n,n-(s+1)}$ are only functions of those triangles in the 1-rim of $T_{n,n-s}$, if the complete sequence $(L_s)$ in the constant center was known, all of the edge resistances would be known in the constant center of the 1-rim would be known.

![Diagram](image)

**Figure 11**: Two left boundary triangles referred to in Lemma 7.1. These two triangles appear in $T_{n,n-s}$ and are in row $r$ and $r+1$ where $s < r < n - 2s$ (i.e., these two triangles are in the constant center of diagonal 1 after $s$ reductions).

**Corollary 7.3.** Let $B_s$ denote the edge resistance on the base and right edge of those triangles in the 1-rim after $s$ reductions. Then,

$$B_{s+1} = 2 \times \left( \frac{1}{L_{s+1}} - \frac{1}{L_s} \right).$$

8. Conclusion

The following section presents one further conjecture as well as directions for future investigations.

**Conjecture 8.1.** With (i) $\gcd$ standing for greatest common divisor, (ii) $L_i^{(i)}$, $i \in \{1, 2\}$ defined as in Section 7 and (iii) $\text{num}$ standing for the numerator of a maximally reduced fraction, we have

$$\gcd(\text{num}_{L_s^{(1)}}, \text{num}_{L_{s+1}^{(2)}}) > 1, \quad s \geq 1.$$

**Remark 8.2.** The conjecture was tested on several dozen initial values for which it is true. The conjecture has implications of edge values in consecutive reductions of an initial grid.

This conjecture is a simple example of the plethora of patterns and conjectures that loom in the $n$-grids and their reduction. In fact these reduction patterns of the $n$-grids lie in a 5-dimensional space. The five dimensions are as follows.

- $n$ is the number of rows in the initial $n$-grid
- $k$ is the number of row reductions that produce an $n - k$-grid under study
- The $n - k$ grid is further dimensionalized by
  - rows, $r$
  - diagonals $d$
  - edges $e$.

The five dimensions allow a richness of perspectives. This paper showed that many patterns abound in these $n$-grids. It is hoped that [10] and this paper will inspire other researchers to delve more deeply into the fascinating patterns that seem to lurk in these $n$-grids.
References

[1] B. Bapat. Resistance distance in graphs. Math. Student, 68(1-4):87–98, 1999.

[2] R. B. Bapat and Somit Gupta. Resistance distance in wheels and fans. Indian. J. Pure App. Math., 41(1):1–13, Feb 2010.

[3] Wayne Barrett, Emily. J. Evans, and Amanda E. Francis. Resistance distance in straight linear 2-trees. Discrete Appl. Math., 258:13–34, 2019.

[4] Wayne Barrett, Emily J. Evans, and Amanda E. Francis. Resistance distance and spanning 2-forest matrices of linear 2-trees. Linear Algebra Appl., 606:41–67, 2020.

[5] Wayne Barrett, Emily J. Evans, Amanda E. Francis, Mark Kempton, and John Sinkovic. Spanning 2-forests and resistance distance in 2-connected graphs. Discrete Applied Mathematics, 284:341–352, 2020.

[6] Bela Bollobas. Modern graph theory, volume 184 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.

[7] Haiyan Chen and Fuji Zhang. Resistance distance and the normalized Laplacian spectrum. Discrete Appl. Math., 155(5):654–661, 2007.

[8] Peter G. Doyle and J. Laurie Snell. Random walks and electric networks, volume 22 of Carus Mathematical Monographs. Mathematical Association of America, Washington, DC, 1984.

[9] E.J. Evans and A.E. Francis. Algorithmic techniques for finding resistance distances on structured graphs. Discrete Applied Mathematics, 320:387–407, 2022.

[10] Russell Jay Hendel. Limiting behavior of resistances in triangular graphs, 2021.

[11] D. J. Klein and M. Randic. Resistance distance. J. Math. Chem., 12(1):81–95, Dec 1993.

[12] Daniel A. Spielman and Nikhil Srivastava. Graph sparsification by effective resistances. SIAM J. Comput., 40(6):1913–1926, 2011.

[13] William Stevenson. Elements of Power System Analysis. McGraw Hill, New York, 3 edition, 1975.

[14] Yujun Yang and Douglas J. Klein. A recursion formula for resistance distances and its applications. Discrete Appl. Math., 161(16-17):2702–2715, November 2013.