An interaction index for multichoice games

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Abstract

Models in Multicriteria Decision Analysis (MCDA) can be analyzed by means of an importance index and an interaction index for every group of criteria. We consider first discrete models in MCDA, without further restriction, which amounts to considering multichoice games, that is, cooperative games with several levels of participation. We propose and axiomatize an interaction index for multichoice games. In a second part, we consider the continuous case, supposing that the continuous model is obtained from a discrete one by means of the Choquet integral.

Keywords: multicriteria decision analysis, interaction, multichoice game, Choquet integral

1 Introduction

An important issue in MultiCriteria Decision Analysis (MCDA) is to be able to analyse and explain a numerical model, obtained by elicitation of preferences of the decision maker. A classical way to do this is to assess the importance of each criterion (see a general approach to define an importance index in \(\text{Ridaoui et al.} (2017a)\)). This description of the model may appear to be sufficient in the case of simple models, which are additive in essence (e.g., additive utility models), as it is well known that they imply mutual preferential independence of criteria (\(\text{Keeney and Raiffa} (1976)\)). However, in case of more complex models, the preferential independence among criteria does not hold any more, and interaction appears among criteria, so that a description of the model by the sole importance indices is not sufficient any more. For example, for models where aggregation of preference is done through a Choquet integral w.r.t. a capacity, an interaction index is defined for any group of criteria (\(\text{Grabisch and Labreuche} (2010)\)), which is a generalization of the interaction index for pairs of criteria proposed by \(\text{Murofushi and Soneda} (1993)\). Roughly speaking, a positive interaction index induces a conjunctive behavior (like the minimum operator), while a negative interaction index induces a disjunctive behavior (maximum).
The aim of the paper is to propose an axiomatic foundation of an interaction index for a MCDA model with no special restriction (and in particular, mutual preferential independence is not supposed to hold). In a first step, the attributes are supposed to be defined on a finite universe. Then, such a model is equivalent to what is called a multichoice game in game theory (Hsiao and Raghavan, 1993), that is, a game on a set of players \( N \), where each player can play at a level of participation represented by an integer between 0 and \( k \). Up to our knowledge, there is no definition of an interaction index for multichoice games. Nevertheless, there exists a general form of interaction index for games on lattices (Grabisch and Labreuche, 2007), and multichoice games with \( k \) levels can be considered as games on the lattice \( (k+1)^N \). This interaction index is defined, however, for any element of the lattice \( x \in (k+1)^N \), i.e., any profile of participation of the players. This does not make sense for our purpose, since we are looking for an interaction index defined for groups of players/criteria. It is the contribution of this paper to provide such an index, and to give a characterization of it.

The paper is organized as follows. Section 2 introduces the necessary material and notation. Section 3 summarizes previous works on the interaction index (the case of classical games and the case of games on lattices). Our work on the importance index for multichoice games is summarized in Section 4 since some of the axioms are necessary for our approach. Section 5 gives the main result of the paper, which is the definition and characterization of an interaction index for multichoice games, and consequently for general discrete MCDA models. In Section 6, we address the continuous case, supposing that the model is obtained from a discrete one via the Choquet integral.

## 2 Preliminaries

Throughout the paper, the cardinality of sets will be denoted by corresponding lower case letters, i.e., \(|N| := n\), \(|S| := s\), etc. For notational convenience, we will omit braces for singletons, i.e., \( S \cup \{i\} \) is written \( S \cup i \), etc.

Let \( N = \{1, \ldots, n\} \) be a fixed and finite set which can be thought as the set of attributes or criteria (in MCDA), players (in cooperative game theory), etc., depending on the domain of application. In this paper, we will mainly focus on MCDA applications.

We suppose that each attribute \( i \in N \) takes values in a set \( L_i \), which is supposed to be finite and denoted by \( L_i = \{0, 1, \ldots, k_i\} \). The alternatives are represented as elements of the Cartesian product \( L := L_1 \times \ldots \times L_n \). An alternative is thus written as a vector \( x = (x_1, \ldots, x_n) \) where \( x_i \in L_i \) for all \( i \in N \). For each \( i \in N \), we denote by \( L_{-i} \) the set \( \times_{j \neq i} L_j \). For each \( y_{-i} \in L_{-i} \), and any \( \ell \in L_i \), \((y_{-i}, \ell_i)\) denotes the compound alternative \( x \) such that \( x_i = \ell \) and \( x_j = y_j, \forall j \neq i \). The vector \( 0_N = (0, \ldots, 0) \) is the null alternative of \( L \), and \( k_N = (k_1, \ldots, k_n) \) is the top element of \( L \). For each \( x \in L \), we denote by \( S(x) = \{i \in N \mid x_i > 0\} \) the support of \( x \), and by \( K(x) = \{i \in N \mid x_i = k_i\} \) the kernel of \( x \).

Let \( x, y \in L \) and \( T \subseteq N \setminus \{\emptyset\} \). \( x_T \) is the restriction of \( x \) to \( T \). We write \( x \leq y \) if \( x_i \leq y_i \) for every \( i \in N \), \( x_T < k_T \) if \( x_T \leq (k-1)_T \) and \( x_T > 0_T \) if \( x_T \geq 1_T \).

The preferences of a Decision Maker (DM) over the alternatives are supposed to be represented by a function \( v : L \rightarrow \mathbb{R} \). For the sake of generality, we do not make any
assumption on \( v \), except that
\[
v(0_N) = 0. \tag{1}
\]

For convenience, we assume from now on that all attributes have the same number of elements, i.e., \( k_i = k \) for every \( i \in N \) (\( k \in \mathbb{N} \)). Note that if this is not the case, we set \( k = \max_{i \in N} k_i \), and we extend \( v : L \to \mathbb{R} \) to \( v' : \{0, \ldots, k\}^N \to \mathbb{R} \) by
\[
v'(x) = v(y) \text{ where } y_i = \min(x_i, k_i) \quad \forall i \in N.
\]
This amounts to duplicating the last element \( k_i \) of \( L_i \) when \( k_i < k \). Under this assumption, we recover well-known concepts.

When \( k = 1 \), \( v \) is a pseudo-Boolean function \( v : \{0,1\}^N \to \mathbb{R} \) vanishing at \( 0_N \). It can be put in the form of a function \( \mu : 2^N \to \mathbb{R} \), with \( v(\emptyset) = 0 \), which is a game in cooperative game theory. A capacity (Choquet, 1953) or fuzzy measure (Sugeno, 1974) is a monotone game, i.e., satisfying \( v(A) \leq v(B) \) whenever \( A \subseteq B \). For the general case (when \( k \geq 1 \)), \( v : L \to \mathbb{R} \) fulfilling 1 corresponds exactly to the concept of multichoice game (Hsiao and Raghavan, 1993), and the numbers \( 0, 1, \ldots, k \) in \( L_i \) are seen as the level of activity of the players. A \( k \)-ary capacity (Grabisch and Labreuche, 2003) is a multichoice game \( v \) satisfying the monotonicity condition: for each \( x, y \in L \) s.t. \( x \leq y \), \( v(x) \leq v(y) \) and the normalization condition: \( v(k, \ldots, k) = 1 \). Hence, a \( k \)-ary capacity represents a preference on \( L \) which is increasing with the value of the attributes. We denote by \( G(L) \) the set of multichoice games defined on \( L \).

The derivative of \( v \in G(L) \) at \( x \in L \) w.r.t. \( i \in N \) such that \( x_i < k \) is defined by
\[
\Delta_i v(x) = v(x + 1_i) - v(x).
\]
The derivative of \( v \in G(L) \) at \( x \in L \) w.r.t. \( T \subseteq N \setminus \{\emptyset\} \) such that \( \forall i \in T, x_i < k \) is defined recursively as follows,
\[
\Delta_T v(x) = \Delta_i(\Delta_{T \setminus i} v(x)).
\]
The general expression for the derivative of \( v \in G(L) \) is given by,
\[
\Delta_T v(x) = \sum_{A \subseteq T} (-1)^{|T| - |T|} v(x + 1_A), \forall T \subseteq N \setminus \{\emptyset\}, \forall i \in T, x_i < k.
\]

## 3 Values and interaction indices

### 3.1 The case of classical TU-games

In cooperative game theory, the notion of value or power index is one of the most important concepts. A value is a function \( \phi : G(2^N) \to \mathbb{R}^N \) which assigns a payoff vector to any game \( v \in G(2^N) \). In MCDA, values are interpreted as importance indices for criteria. The Shapley value (Shapley, 1953) of player \( i \in N \) is given by
\[
\phi_i(v) = \sum_{S \subseteq N\setminus i} \frac{(n - s - 1)!s!}{n!} (v(S \cup i) - v(S)), \forall v \in G(2^N).
\]
The concept of interaction index, which is an extension of that of value, was introduced axiomatically to measure the interaction phenomena among players in cooperative game theory or criteria in multicriteria decision analysis. For a game \( v \in \mathcal{G}(2^N) \), the interaction index \( I^v : 2^N \to \mathbb{R} \) that assigns to every coalition \( T \subseteq N \) its interaction degree.

\( \text{Grabisch} (1997) \) proposed a first axiomatization of the interaction index \( I^v \) for a pair of elements \( i, j \in N \) to estimate how well \( i \) and \( j \) interact. \( \text{Grabisch} \) and \( \text{Roubens} (1999) \) defined and extended the interaction index to coalitions containing more than two players. The interaction index \( I^v \) of a coalition \( S \subseteq N \) in a game \( v \in \mathcal{G}(2^N) \) is defined by

\[
I^v_S(S) = \sum_{T \subseteq N \setminus S} \frac{(n - t - s)!}{(n - s + 1)!} \sum_{K \subseteq S} (-1)^{s-k}v(K \cup T).
\]

Note that when \( S = \{i\} \), the interaction index coincides with the Shapley value.

A first axiomatization of the interaction index have been proposed by \( \text{Grabisch and Roubens} (1999) \), and it is axiomatized in a way similar to the Shapley value. The following axioms have been considered by \( \text{Grabisch and Roubens} : \)

- **Linearity axiom (L):** \( I^v(S) \) is linear on \( \mathcal{G}(2^N) \) for every \( S \subseteq N \).
- **Dummy axiom (D):** For any \( v \in \mathcal{G}(2^N) \), and any \( i \in N \) dummy for \( v \), \( I^v(S \cup i) = 0, \forall S \subseteq N \setminus i \).
- \( i \in N \) is said to be dummy for \( v \) if \( \forall S \subseteq N \setminus i, v(S \cup i) = v(s) + v(i) \).
- **Symmetry axiom (S):** For any \( v \in \mathcal{G}(2^N) \), any permutation \( \sigma \) on \( N \) and any \( S \subseteq N \setminus \emptyset \), \( I^v(S) = I^{v_{\sigma}}(S \sigma S) \).
- **Efficiency axiom (E):** For any \( v \in \mathcal{G}(2^N) \) and any \( i \in N \), \( \sum_{i \in N} I^v(i) = v(N) \).
- **Recursive axiom (R1):** For any \( v \in \mathcal{G}(2^N) \) and any \( S \subseteq N, s > 1 \),

\[
I^v(S) = I^{v^{-j}}(S \setminus j) - I^{v^{-j}}(S \setminus j), \forall j \in S,
\]

where, \( v^{-j} \) is the game \( v \) restricted to elements in \( N \setminus j \) defined by \( v^{-j}(S) = v(S), \forall S \subseteq N \setminus j \), and \( v^{-j}_{\setminus j} \) is the game on \( N \setminus j \) in the presence of \( j \) defined by \( v^{-j}_{\setminus j}(S) = v(S \cup j) - v(S), \forall S \subseteq N \setminus j \).

- **Recursive axiom (R2):** For any \( v \in \mathcal{G}(2^N) \) and any \( S \subseteq N, s > 1 \),

\[
I^v_{\text{Sh}}(S) = I^{v_{[S]}}([S]) - \sum_{K \subseteq N \setminus S, K \neq \emptyset, S} I^{v^{-K}}(S \setminus K),
\]

where, \( v_{[S]} \) is the game where all elements in \( S \) are considered as a single element denoted \( [S] \), it is defined by, for any \( K \subseteq N \setminus S \):

\[
v_{[S]}(K) = v(K),
\]

\[
v_{[S]}(K \cup [S]) = v(K \cup S).
\]
The axiom (R1) says that the interaction of the players in $S$ is equal to the interaction between the criteria in $S \setminus j$ in the presence of $j$ minus the interaction between the criteria of $S \setminus j$ in the absence of $j$. Axiom (R2) expresses interaction of $S$ in terms of all successive interactions of subsets. The authors have shown that (R1) and (R2) are equivalent under (L), (D) and (S) axioms.

The following theorem was shown by Grabisch and Roubens (1999).

**Theorem 1.** Under axioms (L), (D), (D), (E), and ((R1) or (R2)), for all $v \in G(2^N)$,

$$I^v(S) = \sum_{T \subseteq N \setminus S} \frac{(n-t-s)!!}{(n-s+1)!!} \sum_{K \subseteq S} (-1)^{s-k}v(K \cup T), \forall S \subseteq N.$$

### 3.2 The case of games on lattices

Grabisch and Labreuche (2007) generalized the notion of interaction defined for criteria modelled by capacities, by considering functions defined on lattices. The interaction (Grabisch and Labreuche, 2007) is based on the notion of derivative of a function defined on a lattice. For this, they introduce the following definitions:

Let $i = (0_{-j}, i_j)$ with $i_j \in L_j, j \in N$. Let $x, y \in L$ with $y = \vee_{k=1}^n i_k$ and $v \in G(L)$. The derivative of $v$ w.r.t. $i$ at point $x \in L$ is given by:

$$\Delta_i v(x) = v(x \vee i) - v(x),$$

and the derivative of $v$ w.r.t. $y$ at $x$ is given by:

$$\Delta_y v(x) = \Delta_{i_1}(\Delta_{i_2}(\ldots \Delta_{i_n}v(x)\ldots)).$$

The following definition has been proposed by Grabisch and Labreuche (Grabisch and Labreuche, 2007):

**Definition 1.** Let $J \subseteq N$, and $x = \vee_{j \in J} i_j$, with $i_j = (0_{-j}, \ell_j), \ell_j \in L_j \setminus \{0\}$.

$$I^v(x) = \sum_{y \in A(x)} \alpha^j_{h(y)} \Delta_x v(y),$$

where, $A(x) = \{y \in L | y_j = k \text{ or } 0 \text{ if } j \notin J, y_j = x_j - 1 \text{ else } \}$, $h(y)$ is the number of components of $y$ to $k$ and $\alpha^j_{h(y)} = \frac{(n-j-h(y))!(n-y_j+1)!}{(n-j+1)!}.$

### 4 Characterization of the importance index for multichoice games

In this section, we present the importance index (value) for multichoice games defined by Ridaoui et al. (2017b) together with its axiomatization. Let $\phi$ be a value defined for any $v \in G(L)$.

**Linearity axiom (L)**: $\phi$ is linear on $G(L)$, i.e., $\forall v, w \in G(L), \forall \alpha \in \mathbb{R},$

$$\phi_i(v + \alpha w) = \phi_i(v) + \alpha \phi_i(w), \forall i \in N.$$
An attribute \( i \in N \) is said to be **null** for \( v \in \mathcal{G}(L) \) if
\[
v(x + 1_i) = v(x), \forall x \in L, x_i < k.
\]

**Null axiom (N):** If an attribute \( i \) is null for \( v \in \mathcal{G}(L) \), then
\[
\phi_i(v) = 0.
\]

Let \( \sigma \) be a permutation on \( N \). For all \( x \in L \), we denote \( \sigma(x)_{\sigma(i)} = x_i \). For all \( v \in \mathcal{G}(L) \), the game \( \sigma \circ v \) is defined by \( \sigma \circ v(\sigma(x)) = v(x) \).

**Symmetry axiom (S):** For any permutation \( \sigma \) of \( N \),
\[
\phi_{\sigma(i)}(\sigma \circ v) = \phi_i(v), \forall i \in N.
\]

**Invariance axiom (I):** Let us consider two games \( v, w \in \mathcal{G}(L) \) such that, for some \( i \in N \),
\[
v(x + 1_i) - v(x) = w(x) - w(x - 1_i), \forall x \in L, x_i \notin \{0, k\}
\]
\[
v(x_{-i}, 1_i) - v(x_{-i}, 0_i) = w(x_{-i}, k_i) - w(x_{-i}, k_i - 1), \forall x_{-i} \in L_{-i},
\]
then \( \phi_i(v) = \phi_i(w) \).

**Efficiency axiom (E):** For all \( v \in \mathcal{G}(L) \),
\[
\sum_{i \in N} \phi_i(v) = \sum_{x \in L, x_i < k} (v(x + 1_N) - v(x)).
\]

Ridaoui et al. (2017a) have shown the following result.

**Theorem 2.** Let \( \phi \) be a value defined for any \( v \in \mathcal{G}(L) \).

(i) If \( \phi \) fulfills (L) and (N) then there exists a family of real constants \( \{b^i_x, x \in L\} \) such that
\[
\phi_i(v) = \sum_{x \in L, x_i < k} b^i_x (v(x + 1_i) - v(x)), \forall i \in N. \tag{2}
\]

(ii) If \( \phi \) fulfills (L), (N) and (I) then
\[
\phi_i(v) = \sum_{x_{-i} \in L_{-i}} b^i_{x_{-i}} (v(x_{-i}, k) - v(x_{-i}, 0)), \forall i \in N. \tag{3}
\]

(iii) If \( \phi \) fulfills (L), (N), (I) and (S) then
\[
\phi_i(v) = \sum_{x_{-i} \in L_{-i}} b_{n(x_{-i})} (v(x_{-i}, k) - v(x_{-i}, 0)), \forall i \in N, \tag{4}
\]
where \( n(x_{-i}) = (n_0, n_1, \ldots, n_k) \) with \( n_j \) the number of components of \( x_{-i} \) being equal to \( j \).

(iv) If \( \phi \) fulfills (L), (N), (I), (S) and (E) then
\[
\phi_i(v) = \sum_{x_{-i} \in L_{-i}} \frac{(n - \sigma(x_{-i}) - 1)! \kappa(x_{-i})!}{(n + \kappa(x_{-i}) - \sigma(x_{-i}))!} (v(x_{-i}, k) - v(x_{-i}, 0)), \forall i \in N \tag{5}
\]
5 Axiomatization of the interaction index

In this section we intend to define axiomatically the interaction index of multichoice games. The approach presented here is based on a recursion formula, starting from the importance index (value) defined in Section 4, as in (Grabisch and Roubens, 1999). An interaction index of the $k$-ary multichoice game $v \in \mathcal{G}(L)$ is a function $I^v : 2^N \to \mathbb{R}$.

The first axiom (L) is trivially generalized for the interaction index.

**Linearity axiom (L)**: $I^v$ is linear on $\mathcal{G}(L)$, i.e., $\forall v, w \in \mathcal{G}(L), \forall \alpha \in \mathbb{R},$

$$I^{v+\alpha w} = I^v + \alpha I^w.$$  

**Proposition 1.** Under (L), for every $T \subseteq N \setminus \{\emptyset\}$, there exists real constants $a^T_x$, for all $x \in L$, such that for every $v \in \mathcal{G}(L)$

$$I^v(T) = \sum_{x \in L} a^T_x v(x).$$ (6)

**Proof.** It is easy to check that the above formula satisfies the linearity axiom. Conversely, we consider $I^v$ satisfying (L). We have $\forall v \in \mathcal{G}(L), v = \sum_{x \in L} v(x) \delta_x$. Then by (L),

$$I^v(T) = \sum_{x \in L} v(x) I^v(T), \forall T \subseteq N \setminus \{\emptyset\}$$

Setting $a^T_x = I^v(T), \forall x \in L, \forall T \subseteq N$, we obtain the wished result.  

**Remark 1.** Let $i \in N$ be a null criterion for $v \in \mathcal{G}(L)$. We have,

$$\forall T \subseteq N, x \in L, x + 1_T \leq k_T, \Delta^i_T v(x) = 0,$$

$$\forall T \subseteq N, x \in L, x + 1_j \leq k, \Delta^j_T v(x) = 0.$$  

**Null axiom (N):** If a criterion $i$ is null for $v \in \mathcal{G}(L)$, then for all $T \subseteq N$ such that $T \ni i$, $I^v(T) = 0$.

**Proposition 2.** Under axioms (L) and (N), for every $T \subseteq N \setminus \{\emptyset\}$, there exist real constants $b^T_x$, for all $x \in L$, with $x + 1_T \leq k_T$, such that for every $v \in \mathcal{G}(L)$

$$I^v(T) = \sum_{x \in L, x_T < k_T} b^T_x \Delta_T v(x).$$ (7)

To prove this result, the following lemmas are useful.

**Lemma 1.** Let $A \subseteq N$.

$$a_{(x_A, x_{-A})} = \sum_{C \subseteq A} (-1)^{a-c} \sum_{\ell_C = x}^{k_A} a_{(\ell_C, x, \ell_A, x)} a_{(\ell, x_A, x_{-A})}, \forall x_A \in L_A \setminus \{k_A\}.$$
Proof. Let $A \subseteq N$. We proceed by recurrence on $|A|$. The relation is obviously true for $|A| = 0$. Let us suppose that the relation is true for any set of at most $|A| - 1$ elements, and try to show it is also true for any set of $|A|$ elements. We have, for all $x_A \in L_A \setminus \{k\}^A$,

$$a(x_A, x_{-A}) = a(x_{A \setminus i}, x_i, x_{-A})$$

$$= \sum_{C \subseteq A \setminus i} (-1)^{a-c-1} \sum_{\ell_C = x_C} k_{A \setminus i} a(\ell_C, \ell_{A \setminus C,i}, x_i, x_{-A})$$

$$= \sum_{C \subseteq A \setminus i} (-1)^{a-c-1} \sum_{\ell_C = x_C} k_{A \setminus i} a(\ell_C, \ell_{A \setminus C,i}, x_i, x_{-A}) - \sum_{\ell_C = x_C} k_{A \setminus i} a(\ell_C, \ell_{A \setminus C,i}, x_i, x_{-A})$$

$$= \sum_{C \subseteq A \setminus i} \left( (-1)^{a-c-1} \sum_{\ell_C = x_C} k_{A \setminus i} a(\ell_C, \ell_{A \setminus C,i}, x_i, x_{-A}) + (-1)^{a-c} \sum_{\ell_C = x_C} k_{A \setminus i} a(\ell_C, \ell_{A \setminus C,i}, x_i, x_{-A}) \right)$$

$$= \sum_{C \subseteq A \setminus i} (-1)^{a-c} \sum_{\ell_C = x_C} k_{A \setminus i} a(\ell_C, \ell_{A \setminus C,i}, x_i, x_{-A})$$

Lemma 2.

$$\sum_{x \in L \setminus k_N} b_x \sum_{A \subseteq N} (-1)^{n-a} v(x + 1_A) = \sum_{A \subseteq N} \sum_{B \subseteq N \setminus A} (-1)^{a+b-c} \theta(x_A - 1_C, 0_B, (k-1)N \setminus A \cup B) v(x_A, 0_B, k_N \setminus A \cup B)$$

Proof. We shall proceed by induction on $n$. For simplicity, we denote $N \setminus i$ by $S$, $(x_A, 0_B, k_N \setminus A \cup B)$ by $x_{A,B}^S$, and $(x_A - 1_C, 0_B, (k-1)N \setminus A \cup B)$ by $x_{A,C,B}^S$, with $C \subseteq A$. The relation is obviously true for $n = 1$. Let us suppose that the relation is true for any set of at most $n - 1$ elements, and try to show it is also true for any set of $n$ elements. We
have
\[
\sum_{x \in L \atop x < k_N} b_x \sum_{A \subseteq N} (-1)^{n-a}v(x + 1_A) \\
= \sum_{x \in L \atop x < k_N} b_x \sum_{A \subseteq N \setminus i} (-1)^{n-a} \left( v(x + 1_A) - v(x + 1_{A \cup i}) \right) \\
= \sum_{x_i < k_i \atop x_{i-1} \in L_{i-1}} \sum_{x_i < k_i} b_{x_{i-1},x_i} \sum_{A \subseteq S} (-1)^{a-b} \left( v(x_{A,B} + 1_A) - v(x_{A,B} + 1_{A \cup i}) \right)
\]
\[
= \sum_{A \subseteq S} \sum_{B \subseteq S \setminus A \atop C \subseteq A} \left( (-1)^{a+b-c} b_{x_{A,C,B},x_i} \left( v(x_{A,B} + 1_A) - v(x_{A,B} + 1_{A \cup i}) \right) \right)
\]
\[
= \sum_{A \subseteq S} \sum_{B \subseteq S \setminus A \atop C \subseteq A} \left( (-1)^{a+b-c} b_{x_{A,B},k_i} \left( v(x_{A,B} + 1_A) - v(x_{A,B} + 1_{A \cup i}) \right) \right)
\]
\[
= \sum_{A \subseteq S} \sum_{B \subseteq S \setminus A \atop C \subseteq A} \left( (-1)^{a+b-c} b_{x_{A,B},k_i} \left( v(x_{A,B} + 1_A) - v(x_{A,B} + 1_{A \cup i}) \right) \right)
\]
which is the desired result.

**Proof.** It is easy to check that the formula satisfies the axioms. Conversely, we consider \(I^v\) satisfying \((L)\) and \((N)\). Let \(v \in G(L)\), and \(T \in 2^N \setminus \{\emptyset\}\). By Proposition \([1]\), there exists \(u^T_x \in \mathbb{R}\), for all \(x \in L\), such that,
\[
I^v(T) = \sum_{x \in L} a^T_x v(x).
\]
Then,
\[
I^v(T) = \sum_{x_{i-1} \in L_{i-1}} \sum_{x_i \in L_i} a^T_{(x_{i-1},x_i)} v(x_{i-1}, x_i).
\]
Assume now that $i$ is null criterion for $v$. We have $v(x_{-i}, x_i) = v(x_{-i}, 0_i)$. Hence,

$$I^v(T) = \sum_{x_{-i} \in L_{-i}} \sum_{x_i \in L_i} a^T_{(x_{-i}, x_i)} v(x_{-i}, 0_i).$$

By (N), we have, for all $i \in T$ null, and for all $x_{-i} \in L_{-i}$,

$$\forall x_{-T} \in L_{-T}, \text{let } A \subseteq T, B \subseteq T \setminus A, \text{ and set}$$

$$b^T_{(x_A, 0_{B}, (k-1)_T \setminus A \cup B, x_{-T})} = (-1)^b \sum_{\ell_A = (x+1)_A}^{k_A} a^T_{(\ell, x_{A \cup C}, 0_{B \setminus T \cup A \cup B, x_{-T}})}, \forall x_A \in L_A \setminus \{0, k\}.$$

Then, we have, $\forall x_{-T} \in L_{-T}, \forall A \subseteq T, \forall B \subseteq T \setminus A, \forall x_A \in L_A \setminus \{0, k\}$,

$$a^T_{(x_A, 0_{B}, (k-1)_T \setminus A \cup B, x_{-T})} = \sum_{C \subseteq A} \sum_{\ell_C = x_C}^{k_A} a^T_{(\ell_C, x_{A \cup C}, 0_{B \setminus T \cup A \cup B, x_{-T}})}, \forall x_A \in L_A \setminus \{0, k\}.$$

Therefore, it suffices to replace the values of $a^T_x$ in the formula (4), and then the result is established.

$$I^v(T) = \sum_{x_{-T} \in L_{-T}} \sum_{x_T \in L_T} a^T_{(x_T, x_{-T})} v(x_T, x_{-T})$$

$$= \sum_{x_{-T} \in L_{-T}} \sum_{A \subseteq T} \sum_{0_A < x_A < k_A} \sum_{B \subseteq T \setminus A} a^T_{(x_A, 0_{B}, k_{(k-1)_T \setminus A \cup B, x_{-T}})} v(x_A, 0_B, k_{T \setminus A \cup B, x_{-T}})$$

$$= \sum_{x_{-T} \in L_{-T}} \sum_{A \subseteq T} \sum_{0_A < x_A < k_A} \sum_{B \subseteq T \setminus A} (-1)^b \sum_{\ell_A = (x+1)_A}^{k_A} a^T_{(\ell, x_{A \cup C}, 0_{B \setminus T \cup A \cup B, x_{-T}})} v(x_A, 0_B, k_{T \setminus A \cup B, x_{-T}})$$

$$= \sum_{x_{-T} \in L_{-T}} \sum_{A \subseteq T} \sum_{0_A < x_A < k_A} \sum_{B \subseteq T \setminus A} (-1)^{t-a} b^T_{x} \sum_{A \subseteq T} (-1)^{t-a} v(x + 1_A).$$

$\blacksquare$
Proposition 3. Under axioms (L), (N) and (I), \( \forall v \in G(L) \), \( \forall i \in N \),

\[
I^v(T) = \sum_{x_{-T} \in L_{-T}} b^T_{x_{-T}} \sum_{S \subseteq T} (-1)^{l-s} v(0_S, k_{T \setminus S}, x_{-T}).
\]

**Proof.** It is easy to check that the above formula satisfies the axioms. Conversely, we consider \( I^v \) satisfying (L), (N) and (I). Let \( v, w \in G(L) \), and \( T \subseteq N \). By Proposition 2 and the axiom (I), we have, for any \( i \in T \)

\[
I^v(T) = \sum_{x \in L_i \atop x_{T} < k_T} b^T_x \Delta_T v(x)
\]

\[
= \sum_{x \in L_i \atop x_{T} < k_T} \left( b^T((0, x_{-i}) \Delta_{T \setminus i} v(0_i, x_{-i}) + \sum_{x_i \in L_i \atop x_i \not\in (0, k)} b^T x_{T \setminus i} \Delta_i v(x) \right)
\]

\[
= \sum_{x \in L_i \atop x_{T} < k_T} \left( b^T((0, x_{-i}) \Delta_{T \setminus i} (k_{i-1}, x_{-i}) + \sum_{x_i \in L_i \atop x_i \not\in (0, k)} b^T x_{T \setminus i} \Delta_i w(x_{i-1}) \right)
\]

\[
= \sum_{x \in L_i \atop x_{T} < k_T} \left( b^T((0, x_{-i}) \Delta_{T \setminus i} (k_{i-1}, x_{-i}) + \sum_{x_{i+1, x_{-i}} \in L_{i+1, x_{-i}} \atop x_i \not\in (0, k)} b^T x_{T \setminus i} \Delta_i w(x) \right),
\]

and,

\[
I^w(T) = \sum_{x \in L_i \atop x_{T} < k_T} \left( b^T((0, x_{-i}) \Delta_{T \setminus i} (k_{i-1}, x_{-i}) + \sum_{x_i \in L_i \atop x_i \not\in (0, k)} b^T x_{T \setminus i} \Delta_i w(x) \right),
\]

then, \( b^T_{x_i, x_{-i}} = b^T_{x_{i+1}, x_{-i}}, \forall x_{-i} \in L_{-i}, \forall x_i \in L_i \setminus \{k, k-1\} \) and any \( i \in T \). Hence, \( b^T_{x_{T}, x_{-T}} = b^T_{x_{T}, x_{-T}} \), for all \( x_{T} \in L_{-T} \) and for all \( x_{T} \in L_T \) such that \( x_{T} < k_T \). We conclude that the coefficient \( b^T_{x_{T}, x_{-T}} \) does not depend on \( x_{T} \).

We set thus \( b^T_{x_{T}, x_{-T}} := b^T_{x_{T}, x_{-T}} \). Hence, for any \( v \in G(L) \), and for any \( T \subseteq N \), we have,

\[
I^v(T) = \sum_{x \in L} b^T_x \Delta_T v(x)
\]

\[
= \sum_{x_{T} \in L_{-T}} b^T x_{T} \sum_{x_{T} \in L_T \atop x_{T} < k_T} \Delta_T v(x)
\]

\[
= \sum_{x_{T} \in L_{-T}} b^T x_{T} \sum_{S \subseteq T} (-1)^{l-s} v(0_T \setminus S, k_S, x_{-T})
\]

\[\square\]
We introduce the Symmetry axiom.

**Symmetry axiom (S):** For all \( v \in \mathcal{G}(L) \), for all permutation \( \sigma \) on \( N \),
\[
I^{\sigma v}(\sigma(T)) = I^v(T), \forall T \subseteq N.
\]

**Proposition 4.** Under axioms (L), (N), (S), \( \forall v \in \mathcal{G}(L), \forall T \subseteq N, \)
\[
I^v(T) = \sum_{x \in L, x_T < k_T} b^T_{x_T; n_0, \ldots, n_k} \Delta_T v(x),
\]
where \( b^T_{x_T; n_0, \ldots, n_k} \in \mathbb{R} \), and \( n_j = \{ \ell \in N \setminus T, x_\ell = j \} \)

**Proof.** Let \( v \in \mathcal{G}(L) \) and let \( \sigma \) be a permutation on \( N \). For every \( x \in L \), we put \( y = \sigma^{-1}(x) \). From Proposition \( \mathbb{2} \) we have \( \forall T \subseteq N \)
\[
I^v(T) = \sum_{y \in L, y_T < k_T} b^T_y \Delta_T v(y),
\]
and,
\[
I^{\sigma v}(\sigma(T)) = \sum_{x \in L, x_{\sigma(T)} < k_{\sigma(T)}} b^T_x \Delta_{\sigma(T)} \sigma \circ v(x)
\]
\[
= \sum_{y \in L, y_T < k_T} b^T_{\sigma(y)} \Delta_T v(y).
\]
Then, from the symmetry axiom, we have for all \( y \in L \) such that \( y_T < k_T \): \( b^T_{\sigma(y)} = b^T_y \).

For every \( y \in L \) such that \( y_T < k \), we can write, \[
b^T_{(y_T; y - T)} = b^T_y = b^T_{\sigma(y)} = b^T_{\sigma(y)} = b^T_{\sigma(y)} = b^T_{\sigma(y)} = b^T_{\sigma(y)}
\]
Assuming that \( \sigma(T) = T \), then,
\[
b^T_{(y_T; y - T)} = b^T_{(y_T; y - T)}
\]
For a fixed \( T \), \( b^T_{(y_T; y - T)} \) depends only on \( n(y - T) \), with \( n(y - T) = \{ n_0(y - T), n_1(y - T), \ldots, n_k(y - T) \} \), and \( n_j(y - T) = |\{ \ell \in N \setminus T | y_\ell = j \}| \).
\[
p^T_{(y_T; y - T)} = p^T_{(y_T; n(y - T))}
\]
Suppose now that \( \sigma(T) = S \) (with \( S \neq T \)), and \( \sigma(\ell) = \ell, \forall \ell \in N \setminus S \cup T \), then,
\[
b^T_{(y_T; y - T)} = b^T_{(y_T; n(y - T))}
\]
we can conclude that the value \( b^T_{y_T; n(y - T)} \) does not depend on the exponent \( T \). We denote by \( b_{y_T; n(y - T)} \) this value. \( \square \)
Proposition 5. Under axioms (L), (N), (I) and (S), for any \( v \in \mathcal{G}(L) \), \( \forall T \subseteq N \),

\[
I^v(T) = \sum_{x_T \in L_T} b_{n(x_T)} \sum_{S \subseteq T} (-1)^{|T| - s} v(0_T \setminus S, k_S, x_T),
\]

where \( b_{n(x_T)} \in \mathbb{R} \), \( n(x_T) = (n_0, n_1, \ldots, n_k) \) and \( n_j = |\{ \ell \in N \setminus T, x_\ell = j \}| \)

**Efficiency axiom (E):** For all \( v \in \mathcal{G}(L) \),

\[
\sum_{i \in N} I^v(i) = \sum_{x \in L} (v(x + 1_N) - v(x)).
\]

We introduce now the **Recursivity axiom** which is the exact counterpart of the one for classical games in (Grabisch and Roubens, 1999). For this, we introduce the following definitions:

Let \( v \) be a multichoice game in \( \mathcal{G}(L) \) and \( S \subseteq N \). We introduce the restricted multichoice game \( v^{-S} \) of \( v \), which is defined on \( N \setminus S \) as follows

\[
v^{-S}(x_{-S}) = v(x_{-S}, 0_S), \forall x_{-S} \in L_{-S}.
\]

The restriction of \( v \) to \( i \in N \) in the presence of \( i \) denoted by \( v_i^{-i} \) is the multichoice game on \( L_{-i} \) defined by

\[
v_i^{-i}(x_{-i}) = v(x_{-i}, k_i) - v(0_{-i}, k_i), \forall x_{-i} \in L_{-i}.
\]

**Recursivity axiom (R):** For any \( v \in \mathcal{G}(L) \),

\[
I^v(T) = I^{v^{-i}}(T \setminus i) - I^{v^{-i}}(T \setminus i), \forall T \subseteq N \setminus \{\emptyset\}, \forall i \in T.
\]

Lemma 3. Under axioms (L), (N), (I) (S) and (R), for any \( v \in \mathcal{G}(L) \), \( \forall T \subseteq N \setminus \{\emptyset\} \),

\[
I^v(T) = \sum_{A \subseteq T} (-1)^{|T| - |A|} I^{v_{[A]}^{(-T)\cup[A]}}([A]),
\]

with \( v_{[A]}^{(-T)\cup[A]} \) is the reduced multichoice game of \( v \) to \( T \) with respect to \( A \) defined on the set \( \{0, \ldots, k\}^{(N \setminus T)\cup[A]} \) as follows:

\[
v_{[A]}^{(-T)\cup[A]}(x_{-T}, \ell_{[A]}) = v(x_{-T}, \ell_{[A]}, 0_{T \setminus A}), \ell \in \{0, \ldots, k\}.
\]

**Proof.** We suppose that the axioms (L), (N), (I), (S) and (R) are satisfied. We proceed by induction on \( |T| \). The formula is true for \( |T| = 1 \). Let us assume it is true up to \( |T| = t - 1 \), and try to prove it for \( t \) elements. By induction assumption we have, for any \( v \in \mathcal{G}(L) \), and \( i \in T \),

\[
I^{v^{-i}}(T \setminus i) = \sum_{A \subseteq T \setminus i} (-1)^{|T| - |A| - 1} I^{v_{[A]}^{(-T)\cup[A]}}([A]),
\]
Let $A \subseteq T \setminus i$ such that $A \neq \emptyset$. From Proposition 5

$$I_{[A],i}^{\ell(T),[A]}([A]) = \sum_{x_T \subseteq L_T} b_n(x_T) \left( v^{\ell(T),[A]}(x_T, k_A, 0_{T \setminus A}) - v^{\ell(T),[A]}(x_T, 0_T) \right)$$

with $v^{\ell(T),[A]}(x_T, k_A, 0_T) = v(x_T, \ell_A, k_i, 0_{T \setminus A}) - v(0_{T \setminus A}, k_i, \ell \in \{0, \ldots, k\}$.

By (R), we have

\[
I^v(T) = I_i^{v^{-1}}(T \setminus i) - I_i^{v^{-1}}(T \setminus i)
\]

\[
= \sum_{A \subseteq T \setminus i, A \neq \emptyset} (-1)^{t-a-1} I_{[A],i}^{\ell(T),[A]}([A]) - \sum_{A \subseteq T \setminus i, A \neq \emptyset} (-1)^{t-a-1} I_{[A],i}^{\ell(T),[A]}([A])
\]

\[
= \sum_{A \subseteq T \setminus i, A \neq \emptyset} (-1)^{t-a-1} \left( I_{[A \cup i]}^{\ell(T),[A \cup i]}([A \cup i]) - I_{[A]}^{\ell(T),[A]}([A]) \right) - \sum_{A \subseteq T \setminus i, A \neq \emptyset} (-1)^{t-a-1} I_{[A],i}^{\ell(T),[A]}([A])
\]

\[
= \sum_{A \subseteq T \setminus i, A \neq \emptyset} (-1)^{t-a-1} \left( I_{[A \cup i]}^{\ell(T),[A \cup i]}([A \cup i]) - I_{[A]}^{\ell(T),[A]}([A]) \right) - I_{[A],i}^{\ell(T),[A]}([A]) \sum_{A \subseteq T \setminus i, A \neq \emptyset} (-1)^{t-a-1}
\]

\[
= \sum_{A \subseteq T \setminus i, A \neq \emptyset} (-1)^{t-a-1} \left( I_{[A \cup i]}^{\ell(T),[A \cup i]}([A \cup i]) - I_{[A]}^{\ell(T),[A]}([A]) \right) + (-1)^{t-a-1} I_{[A],i}^{\ell(T),[A]}([i])
\]

\[
= \sum_{A \subseteq T \setminus i, A \neq \emptyset} (-1)^{t-a} I_{[A]}^{\ell(T),[A]}([A])
\]

\[
\square
\]

**Theorem 3.** Under axioms (L), (N), (I), (S), (E) and (R), $\forall v \in G(L), \forall T \subseteq N \setminus \emptyset$,

\[
I^v(T) = I^v_s(T) := \sum_{x_T \subseteq L_T} \frac{(n - s(x_T) - t)!k(x_T)!}{(n - s(x_T) + k(x_T) - t + 1)!} \sum_{A \subseteq T} (-1)^{t-a} I_{[A \cup i]}^{\ell(T),[A \cup i]}([A \cup i])
\]

\[
\sum_{A \subseteq T} (-1)^{t-a} I_{[A],i}^{\ell(T),[A]}([A])
\]

\[
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\]
where

\[ \mu \]

**Proposition 6.** For every \( v \in G(L) \), and \( T \in N \setminus \{\emptyset\} \). By axioms (L), (N), (I), (S) and (E), we have

\[
I^v_{[A]}([A]) = \sum_{x_T \in L_T} b_n(x_T)(v^{(-T)\cup[A]}(x_T, k[A]) - v^{(-T)\cup[A]}(x_T, 0[A])),
\]

with \( b_n(x_T) = \frac{(n - t - s(x_T))!k(x_T)!}{(n - t + 1 + k(x_T) - s(x_T))!} \).

By Lemma (3), we have

\[
I^v(T) = \sum_{A \subseteq T \setminus \emptyset} (-1)^{t-a} I^v_{[A]}([A])
\]

\[
= \sum_{A \subseteq T \setminus \emptyset} (-1)^{t-a} \sum_{x_T \in L_T} b_n(x_T)\left(v(x_T, k[A], 0_{T \setminus A}) - v(x_T, 0_T)\right)
\]

\[
= \sum_{x_T \in L_T} b_n(x_T) \sum_{A \subseteq T \setminus \emptyset} (-1)^{t-a} \left(v(x_T, k[A], 0_{T \setminus A}) - v(x_T, 0_T)\right)
\]

\[
= \sum_{x_T \in L_T} b_n(x_T) \sum_{A \subseteq T \setminus \emptyset} (-1)^{t-a} v(k[A], 0_{T \setminus A}, x_T).
\]

\[
\blacksquare
\]

### 6 Interaction indices for the Choquet integral

We propose in this section an interpretation of the interaction in continuous spaces, that is, after extending \( v \) to the continuous domain \([0, k]^N\). The most usual extension of \( v \) on \([0, k]^N\) is the Choquet integral with respect to \( k \)-ary capacities (Grabisch and Labreuche, 2003).

Let \( z \in [0, k]^N \), and \( q \in L \) such that \( q = \lfloor z \rfloor \) (the floor integer part of \( z \)). The Choquet integral w.r.t. a \( k \)-ary capacity \( v \) at point \( z \) is defined by

\[
C_v(z) = v(q) + C_{\mu_q}(z - q),
\]

where \( \mu_q \) is a capacity given by

\[
\mu_q(A) = v((q + 1)A, q_A) - v(q), \forall A \subseteq N.
\]

**Proposition 6.** For every \( v \in G(L) \),

\[
I^v_s(T) = \sum_{x \in \{0, \ldots, k-1\}^N} I^v_{sh}(T), \forall T \subseteq N \setminus \{\emptyset\}.
\]

To prove this result, the following combinatorial result is useful.

**Lemma 4.**

\[
\sum_{S \subseteq [A,B]} \frac{(n - s - 1)!s!}{n!} = \frac{(n - b - 1)!a!}{(n - b + a)!}, \forall A, B \subseteq N, A \subseteq B
\]

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Proof. Let \( A, B \subseteq N \), such that \( A \subseteq B \),

\[
\sum_{S \subseteq [A, B]} \frac{(n - s - 1)!s!}{n!} = \sum_{S \subseteq [\emptyset, B \setminus A]} \frac{(n - s - a - 1)!(s + a)!}{n!}
\]

\[
= \sum_{s=0}^{b-a} \binom{b - a}{s} \frac{(n - s - a - 1)!(s + a)!}{n!}
\]

\[
= \sum_{s=0}^{b-a} \binom{b - a}{s} \int_0^1 x^{n - s - a - 1}(1 - x)^{s + a} dx
\]

\[
= \int_0^1 x^{n - b - 1}(1 - x)^a \sum_{s=0}^{b-a} \binom{b - a}{s} x^{b - a - s}(1 - x)^s dx
\]

\[
= \int_0^1 x^{n - b - 1}(1 - x)^a dx
\]

\[
= \frac{(n - b - 1)!a!}{(n - b + a)!}
\]

\[
\square
\]

We now prove Proposition 6.

Proof. Let \( T \subseteq N \setminus \{\emptyset\} \).

\[
\sum_{x \in \{0, \ldots, k - 1\}^N} I^\mu_x(T) = \sum_{x \in \{0, \ldots, k - 1\}^N} \sum_{S \subseteq N \setminus T} \frac{(n - s - t)!s!}{(n - t + 1)!} \Delta_T \mu_x(S)
\]

\[
= \sum_{x \in \{0, \ldots, k - 1\}^N} \sum_{S \subseteq N \setminus T} \frac{(n - s - t)!s!}{(n - t + 1)!} \Delta_T v(x + 1 s)
\]

\[
= \sum_{z \in L} \Delta_T v(z) \sum_{S \subseteq N \setminus T, \forall j \in S, z_j > 0, \forall j \in N \setminus S, z_j < k} \frac{(n - s - t)!s!}{(n - t + 1)!}
\]

\[
= \sum_{z \in L} \Delta_T v(z) \sum_{S \subseteq S(z-\tau) \setminus S(z-\tau), S \supseteq K(z-\tau)} \frac{(n - s - t)!s!}{(n - t + 1)!}
\]

\[
= \sum_{z \in L} \Delta_T v(z) \sum_{S \subseteq S(z-\tau) \setminus S(z-\tau), S \supseteq K(z-\tau)} \frac{(n - s - t)!s!}{(n - t + 1)!}
\]

\[
= \sum_{z \in L} \frac{(n - s(z-\tau) - t)!k(z-\tau)!}{(n - s(z-\tau) + k(z-\tau) - t + 1)!} \Delta_T v(z).
\]

\[
\square
\]

The interaction index on continuous domain \([0, k]^N\) takes the form of the total over the domain \(\{0, \ldots, k - 1\}^N\) of the classical interaction index, it means that for each
elementary cell in the grid L, the interaction index corresponds to the usual interaction index.

**Theorem 4.** Let $v$ a $k$-ary capacity.

$$I^v_s(T) = \int_{[0,k]^n} \frac{\partial |T| C_v}{\partial z_T} (z) \, dz, \forall T \subseteq N.$$ 

**Proof.** Let $v$ a $k$-ary capacity. For every $T \subseteq N$. $\forall x \in \{0, \ldots, k-1\}^N$, we have,

$$I^v_s(T) = \sum_{x \in \{0, \ldots, k-1\}^N} I^v_{Sh}(T)$$

$$= \sum_{x \in \{0, \ldots, k-1\}^N} \int_{[0,1]^n} \frac{\partial |\mu_{\alpha}(z)|}{\partial z_T} (z) \, dz$$

$$= \sum_{x \in \{0, \ldots, k-1\}^N} \int_{[x,x+1]} \frac{\partial |\mu_{\alpha}(z)|}{\partial z_T} (z-x) \, dz$$

$$= \sum_{x \in \{0, \ldots, k-1\}^N} \int_{[x,x+1]} \frac{\partial C_v}{\partial z_T} (z) \, dz$$

$$= \int_{[0,k]^n} \frac{\partial C_v}{\partial z_T} (z) \, dz. \quad \square$$

The interaction index on continuous domain appears as the mean of relative amplitude of the range of $C_v$ w.r.t. $T$, when the remaining variables take uniformly random values. The partial derivative is the local interaction of $C_v$ at point $z$.

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