A Distributed Algorithm for Polya-Gamma Data Augmentation

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Abstract. The Polya-Gamma data augmentation (PG-DA) algorithm is routinely used for Bayesian inference in logistic models. This algorithm has broad applications and outperforms other sampling algorithms in terms of numerical stability and ease of implementation. The Markov chain produced by the PG-DA algorithm is also known to be uniformly ergodic; however, the PG-DA algorithm is prohibitively slow in massive data settings because it requires passing through the whole data at every iteration. We develop a simple distributed extension of the PG-DA strategy using the divide-and-conquer technique that divides the data into sufficiently large number of subsets, performs PG-type data augmentation in parallel using a powered likelihood, and produces Monte Carlo draws of the parameter by combining Markov chain Monte Carlo (MCMC) draws of parameter obtained from each subset. The combined parameter draws play the role of MCMC draws from the PG-DA algorithm in posterior inference. Our main contributions are three fold. First, we develop the modified PG-DA algorithm with a powered likelihood in logistic models that is used on the subsets to obtain subset MCMC draws. Second, we modify the existing class of combination algorithms by introducing a scaling step. Finally, we demonstrate through diverse simulated and real data analyses that our distributed algorithm outperforms its competitors in terms of statistical accuracy and computational efficiency. We also provide theoretical support for our empirical observations.

Keywords: Data augmentation, Distributed computing, Divide-and-conquer, Location-scatter family, Polya-Gamma distribution.

1 Introduction

Logistic regression is a standard approach for classification. A variety of Bayesian and frequentist approaches exist to fit logistic regression models. Our focus is on Bayesian logistic regression using the PG-DA strategy (Polson et al., 2013). The PG-DA algorithm is easy to implement, numerically stable, and has theoretical guarantees (Choi and Hobert, 2013; Wang and Roy, 2018b,a); however, every iteration of the PG-DA algorithm generates a PG random variable specific to a training sample, which is extremely inefficient in applications involving massive data (Glynn et al., 2019). We develop a distributed Bayesian approach for posterior inference in logistic regression models by employing the PG-DA strategy in a divide-and-conquer setting. Our approach, called Distributed Polya-Gamma (DPG)-DA algorithm, retains the simplicity of the PG-DA strategy by employing it in parallel on many smaller subsets, which allows our algorithm to scale well in massive data settings.

Logistic regression is a popular choice in applications with categorical responses. A regression problem with a binary or multiple-class response can be solved using a single or a collection of logistic regression models. Such problems arise frequently in practice. In modern applications, however, there has been an explosion in the number of observations; for example, it is common to collect millions of categorical responses daily in the form of “likes” on Facebook, “retweets” on Twitter, and ratings on online movie databases and vendors. These applications motivate the development of automated methods for fitting logistic regression models that are easily implemented and leverage parallel computing. There are many optimization-based methods to accomplish the desired goal, but very few such methods exist under the Bayesian paradigm.
The inefficiency of posterior computations in massive data settings has motivated significant interest in developing general and scalable Bayesian sampling algorithms. The literature is still developing, but three main techniques are at the forefront. The first relies on analytic approximations of the posterior, such as expectation propagation (EP), variational Bayes (VB), and Laplace approximation (Rue et al., 2009; Gelman et al., 2014; Tan and Nott, 2014; Kucukelbir et al., 2015; Lee and Wand, 2016; Ranganath et al., 2016). The PG-DA strategy is known to outperform these methods in the ease of use and accuracy of uncertainty estimates (Polson et al., 2013). The second technique uses subsampling or efficient approximations of transition kernels to avoid computational bottlenecks in MCMC or sequential Monte Carlo (SMC) algorithms (Welling and Teh, 2011; Ahn et al., 2012; Korattikara et al., 2014; Lan et al., 2014; Shahbaba et al., 2014; Maclaurin and Adams, 2015; Bardenet et al., 2015; Johndrow et al., 2015; Alquier et al., 2016; Campbell and Broderick, 2018; Quiroz et al., 2018). In contrast, the PG-DA sampler requires no proposal tuning, thus motivating the design of scalable PG-DA-based extensions.

The third group of methods is based on the divide-and-conquer technique. This technique is not new, but its application for scalable Bayesian inference is recent. The methods in this group operate in three steps: randomly divide the full data into smaller subsets, run a modified form of an existing sampling algorithm in parallel on all the subsets, and combine the parameter draws from all the subsets. In the second step, the prior (Scott et al., 2016) or the likelihood (Minsker et al., 2014) is modified. Current methods ensure that the full data posterior and the probability distribution estimated in the combination step lead to the same inference in terms of parameter and uncertainty estimates (Neiswanger et al., 2014; Wang and Dunson, 2013; Wang et al., 2015; Minsker et al., 2017). The Wasserstein Posterior (WASP) is one such method that modifies the likelihood and combines the posterior distributions estimated on the subsets through their barycenter (Srivastava et al., 2015, 2018). A criticism of the WASP is that its combination algorithm requires solving a computationally expensive linear program. When the interest lies in a one-dimensional functional of the parameters, then the PIE algorithm circumvents this issue by exploiting the analytic form of the barycenter in terms of quantiles of the posterior distributions estimated on the subsets (Li et al., 2017; Savitsky and Srivastava, 2018; Guhaniyogi et al., 2017). Another alternative is the Double Parallel Monte Carlo (DPMC) algorithm that approximates the full data posterior using a mixture distribution obtained by a common centering of the subset MCMC draws (Xue and Liang, 2017). The DPG-DA algorithm allows for using either the WASP or the DPMC for combining the posterior draws obtained from the subsets using the PG-DA strategy.

Current Bayesian methods for scalable logistic regression rely on general MCMC approaches. On the contrary, DPG-DA algorithm relies on the WASP or DPMC for scalability and on the PG-DA strategy for avoiding analytic approximations, numerical integration, or Metropolis-Hastings algorithm in logistic regression (Polson et al., 2013). The DPG-DA algorithm deviates from the existing approaches in two main ways. The first is its successful coupling of the PG-DA strategy and a powered likelihood in terms of retaining PG-DA’s computational simplicity as well as its theoretical guarantees. Such a coupling fails with a modified prior. Second, DPG-DA’s combination algorithm is based on either extending DPMC algorithm using a scaling step or providing a computationally simpler approximation to the WASP, which reduces to the exact WASP under certain assumptions on the subset posterior distributions. Our experiments show that DPG-DA algorithm based on the WASP approximation outperforms the current state-of-the-art DPMC algorithm on simulated and a movie ratings dataset in terms of speed and accuracy.

2 Logistic Regression via Polya-Gamma Data Augmentation

We first review the PG-DA strategy for logistic regression. Let \( n \) be the sample size, \( \beta = (\beta_1, \ldots, \beta_p) \in \mathbb{R}^p \) be the regression coefficients, \( y_i \) be the number of successes, \( s_i \) be the number of trials, \( x_i = (x_{i1}, \ldots, x_{ip}) \in \mathbb{R}^p \) be the vector of predictors for sample \( i \) (\( i = 1, \ldots, n \)), \( y = (y_1, \ldots, y_n) \) be the response vector, \( X \in \mathbb{R}^{n \times p} \) be the design matrix with \( x_{i}^T \) as its \( i \)th row, and \( D = \{y, X\} \) be the full data. In logistic regression, \( y_i \), given \( \beta \) follows the binomial distribution \( \text{Binom}(s_i, 1/(1 + e^{-\psi_i})) \), where \( \psi_i = x_i^T \beta \) is the log odds of success, and \( \beta \) is assigned the \( \mathcal{N}(\mu_\beta, \Sigma_\beta) \) prior, where \( \mathcal{N} \) is the Gaussian distribution with mean \( \mu_\beta \) and covariance matrix \( \Sigma_\beta \). The PG sampler cycles between the imputation (I) and prediction (P) steps for a large number of iterations:
Distributed PG-DA

S.1 (I step) draw \( \omega_i \) given \( \beta \) and \( D \) from \( \text{PG}(s_i, |x_i^T \beta|) \) for \( i = 1, \ldots, n \), where \( \text{PG} \) is the Polya-Gamma distribution and \( \omega_i \) given \( \beta, x_i, s_i \), and \( y_i \) follows the PG distribution; and

S.2 (P step) draw \( \beta \) given \( \omega_1, \ldots, \omega_n \), and \( D \) from \( N(m_\omega, V_\omega) \), where \( V_\omega = (X^T \Omega X + \Sigma_\beta^{-1})^{-1}, m_\omega = V_\omega (X^T \kappa + \Sigma_\beta^{-1} \mu_\beta) \), \( \kappa = (y_1 - s_1/2, \ldots, y_n - s_n/2) \), and \( \Omega \) is the diagonal matrix of \( \omega_i \)s.

The Markov chain \( \Phi = \{\beta^{(t)}\}_{t=1}^\infty \) of the draws collected in step S.2, where \( t \) indexes the iterations, has \( \Pi(\cdot \mid D) \) as its invariant distribution (Choi and Hobert, 2013; Wang and Roy, 2018). The PG sampler is popular because it is easy to implement using a software for generating PG random variables.

The key idea in the PG-DA strategy lies in taking advantage of the conditional distribution of \( \beta \) given \( \omega_i \). Let \( p(y_i \mid \omega_i) \) be the conditional density of \( y_i \) given \( \omega_i \) and \( \beta \) follows \( \text{PG}(s_i, |x_i^T \beta|) \). Section 3.1 in Polson et al. (2013) shows that the conditional density of \( \beta \) given \( \omega_1, \ldots, \omega_n \) and \( y \) is

\[
p(\beta \mid \omega_1, \ldots, \omega_n, y) \propto \prod_{i=1}^n p(y_i \mid \omega_i, \beta)p(\beta) \propto \prod_{i=1}^n \exp \left\{ \kappa_i x_i^T \beta - \omega_i (x_i^T \beta)^2/2 \right\} p(\beta)
\]

\[
= \exp \left\{ -(z - X \beta)^T \Omega (z - X \beta)/2 \right\} p(\beta),
\]

where \( z = (\kappa_1/\omega_1, \ldots, \kappa_n/\omega_n) \), \( \Omega \) is defined in step S.2, and \( p(\beta \mid y, \omega_1, \ldots, \omega_n) \) yields a conditionally Gaussian likelihood for \( \beta \) with a working response \( z \), design matrix \( X \), and covariance matrix \( \Omega^{-1} \).

The computationally and conceptually simple PG sampler fails to scale to massive data settings. The main problem is that the algorithm passes through every sample in each iteration and generates \( \omega_i \)s, which is prohibitively slow in massive data applications. To bypass this problem, the DPG-DA algorithm developed next performs posterior computations in parallel using many smaller subsets created from the full data.

3 Distributed Polya-Gamma Data Augmentation Algorithm

3.1 First Step: Creation of Data Subsets

The first step of the DPG-DA algorithm creates \( k \) subsets using the full data. The default strategy is to obtain \( k \) subsets of the full data by random subsampling with replacement. Let \( D_j \) be the data on subset \( j \), \( m_j \) be the sample size of subset \( j \), \( \beta_j, y_{ji}, \omega_{ji}, s_{ji}, x_{ji} = (x_{ji1}, \ldots, x_{jip}) \) \( i = 1, \ldots, m_j \), \( y_j, X_j \) be the subset \( j \) equivalents of their full data counterparts \( \beta, y_i, \omega_i, s_i, x_i, y, X, D_j = \{ y_{ji}, X_j \} \), and \( D = \cup_{j=1}^k D_j \). The logistic regression model on subset \( j \) is \( y_{ji} \) given \( \beta_j \) follows \( \text{Binom}(s_{ji}, 1/(1 + e^{-x_{ji}^T \beta_j})) \), where \( \psi_{ji} = x_{ji}^T \beta_j \), and \( \beta_j \) is assigned the \( N(\mu_\beta, \Sigma_\beta) \) prior. The next step runs a modified PG-DA sampler on the \( k \) subsets.

3.2 Second Step: Modified Polya-Gamma Data Augmentation Using a Powered Likelihood

The second step of the DPG-DA algorithm is derived by modifying step S.2 of the original PG sampler. The I step for generating \( \omega_{j1}, \ldots, \omega_{jm_j} \) in the DPG-DA algorithm on subset \( j \) is identical to step S.1:

L.1 (I step on subset \( j \)) draw \( \omega_{ji} \) given \( \beta_j \) and \( D_j \) from \( \text{PG}(s_{ji}, x_{ji}^T \beta_j) \) for \( i = 1, \ldots, m_j \).

Since \( D_j \) has \((m_j/n)\)-fraction of the samples in \( D \), the asymptotic variance of the posterior distribution of \( \beta \), which conditions on \( D_j \), has an inflated variance by a factor of \( n/m_j \) relative to that of \( \Pi(\cdot \mid D) \). This problem is solved by raising the conditional likelihood of \( \beta \) on subset \( j \) to the power of \( n/m_j \), a strategy known as stochastic approximation (Minsker et al., 2014). Using (2.1), we have that

\[
\{p(\beta_j \mid y_j, \omega_{j1}, \ldots, \omega_{jm_j})\}^{n/m_j} \equiv \ell(\beta_j \mid y_j, \omega_{j1}, \ldots, \omega_{jm_j}) \propto e^{-(n/m_j)(z_j - X_j \beta_j)^T \Omega_j (z_j - X_j \beta_j)/2},
\]

where \( \ell(\beta_j \mid y_j, \omega_{j1}, \ldots, \omega_{jm_j}) \) is the stochastically-approximated conditionally Gaussian likelihood of \( \beta \), \( \int \ell(\beta_j \mid y_j, \omega_{j1}, \ldots, \omega_{jm_j})p(\beta_j)d\beta_j \) is finite, \( z_j = (z_{j1}, \ldots, z_{jm_j}) \), \( z_{ji} = \kappa_{ji}/\omega_{ji} \), \( \kappa_{ji} = y_{ji} - s_{ji}/2 \), and \( \Omega_j \) is
the diagonal matrix of \( w_{ji} \)s. The \( N(\mu_\beta, \Sigma_\beta) \) prior on \( \beta_j \) and the Bayes rule with the conditional likelihood in (3.1) gives the equivalent of step S.2 in the DPG-DA algorithm on subset \( j \):

\[ L.2 \text{ (P step on subset } j \text{)} \text{ draw } \beta_j \text{ given } \omega_{j1}, \ldots, \omega_{jm_j} \text{ and } D_j \text{ from } N(m_{\omega_j}, V_{\omega_j}), \text{ where } V_{\omega_j} = (\frac{1}{m_j} X_j^T \Omega_j X_j + \Sigma_\beta^{-1})^{-1}, m_{\omega_j} = V_{\omega_j}(\frac{1}{m_j} X_j^T \kappa_j + \Sigma_\beta^{-1} \mu_\beta) \text{ and } \kappa_j = (\kappa_{j1}, \ldots, \kappa_{jm_j}). \]

The theoretical results in Choi and Hobert (2013) imply that the \( k \) subset samplers in the DPG-DA algorithm are uniformly ergodic. Equation 2.2 in Choi and Hobert (2013) shows that the Markov transition density of the Markov chain \( \Phi_j = \{ \beta_j^{(t)} \}_{t=1}^\infty \) is

\[ K_j(\beta_j | \beta_j') = \int_{\mathbb{R}^p} \pi(\beta_j | y_j, \omega_j) \pi(\omega_j | y_j, \beta_j') d\omega_j, \quad \omega_j = (\omega_{j1}, \ldots, \omega_{jm_j}), \quad j = 1, \ldots, k, \tag{3.2} \]

where \( \pi(\beta_j | y_j, \omega_j) \) and \( \pi(\omega_j | y_j, \beta_j') \) are the full conditional densities in steps L.2 and L.1, respectively. This implies that the Markov chains \( \Phi_1, \ldots, \Phi_k \) generated by the subset samplers in DPG-DA algorithm are irreducible, aperiodic, and Harris recurrent. Additionally, the Markov chains are reversible because they are generated by two-step Gibbs samplers. The following proposition states that \( \Phi_1, \ldots, \Phi_k \) are uniformly ergodic.

**Proposition 3.1.** Let \( \beta_j, \beta_j' \in \mathbb{R}^p, K_j(\beta_j | \beta_j') \) be the Markov transition density defined in (3.2), \( \Sigma_{js} = \{ n/(2m_j) X_j^T X_j + \Sigma_\beta^{-1} \}^{-1}, s_j = (n/m_j) \Sigma_\beta^{-1/2} X_j^T \kappa_j + \Sigma_\beta^{-1/2} \mu_\beta \), \( \mu_{js} = \Sigma_{js}^{-1/2} s_j \), and \( \phi(\cdot; \mu_{js}, \Sigma_{js}) \) be the Gaussian kernel with mean \( \mu_{js} \) and covariance matrix \( \Sigma_{js} \) (\( j = 1, \ldots, k \)). Then, there exists \( 0 < \delta_j < 1 \) such that \( K_j(\beta_j | \beta_j') \) satisfies the minorization condition \( K_j(\beta_j | \beta_j') \geq \delta_j \phi(\beta_j; \mu_{js}, \Sigma_{js}) \) for every \( \beta_j \in \mathbb{R}^p \) and \( j = 1, \ldots, k \), where

\[ \delta_j = e^{-m_j/4 - m_j |\Sigma_{js}|^{1/2}/|\Sigma_\beta|^{1/2}} \exp\left( -\frac{1}{2} \mu_{js} \Sigma_{js}^{-1} \mu_{js} - \frac{1}{2} \delta_j^2 s_j \right). \tag{3.3} \]

If \( K_j^t \) is the \( t \)-step Markov transition density of the subset sampler on the \( j \)th subset in the DPG-DA algorithm, then the Markov chain \( \Phi_j \) is uniformly ergodic and the moment generating function for \( \beta_j \) exists; that is,

\[ \int_{\mathbb{R}^p} |K_j^t(\beta_j | \beta_j') - \pi(\beta_j | D_j)| d\beta_j \leq (1 - \delta_j)^t, \quad |K_j(\beta_j | \beta_j')| \leq \int_{\mathbb{R}^p} K_j^{t-1}(\beta_j | \beta_j'') K_j(\beta_j' | \beta_j') d\beta_j'', \tag{3.4} \]

where \( \pi(\cdot | D_j) \) is the posterior density of \( \beta \) given \( D_j \) obtained using \( \ell(\cdot | y_j, \omega_{j1}, \ldots, \omega_{jm_j}) \) in (3.1) as the conditional likelihood.

The uniform ergodicity of the Markov chains \( \Phi_1, \ldots, \Phi_k \) guarantees the existence of central limit theorems and consistent estimators of the associated asymptotic variances; see Section 3 in Choi and Hobert (2013) for details. An important consequence of this result is that if \( g : \mathbb{R}^p \to \mathbb{R} \) is a square integrable with respect to the \( j \)th subset posterior density, \( \pi(\cdot | D_j) \), and \( g_j = \mathbb{E}(g(\beta_j) | D_j) \) is the posterior mean of \( g(\cdot) \) with respect to \( \pi(\cdot | D_j) \), then there is a \( \tau_j^2 > 0 \) such that

\[ \sqrt{T} \left( \hat{g}_j - g_j \right) \xrightarrow{d} N(0, \tau_j^2), \quad \hat{g}_j = T^{-1} \sum_{t=1}^T g(\beta_j^{(t)}), \quad j = 1, \ldots, k. \tag{3.5} \]

We remark that the stationary distributions of the Markov chains \( \Phi_1, \ldots, \Phi_k \) differ from that of the original PG-DA sampler studied in Choi and Hobert (2013). This precludes naively using these chains for posterior inference on \( \beta \). Our next section addresses this issue by introducing a step that combines the \( k \) Markov chains such that the combined chain is close to the target distribution of the PG-DA sampler; see Theorems 4.1 and 4.2 for the precise result.
The posterior draws of $\beta_1, \ldots, \beta_k$ are obtained by repeating steps L.1 and L.2 in parallel on the $k$ subsets. Let $T$ be the total number of post burn-in iterations on every subset, $\langle \omega_j^{(t)}, \beta_j^{(t)} \rangle$ be the posterior draw of $(\omega_j, \beta_j)$ at the $t$th iteration, and $\Pi(\cdot | D_j)$ be the stochastically-approximated posterior distribution of $\beta$ given $D_j$ with density $\pi(\beta_j | D_j)$ in (3.4). The distribution $\Pi(\cdot | D_j)$ is called the $j$th subset posterior distribution, $\beta_j^{(1)}, \ldots, \beta_j^{(T)}$ are called the $j$th subset posterior draws of $\beta$, and Proposition 3.1 implies that $\beta_j^{(t)}$ is approximately distributed as $\Pi(\cdot | D_j)$ for every $j$ and $t$. Proposition 3.1 also demonstrates the theoretical and practical advantages of modifying the likelihood over the prior in divide-and-conquer posterior computations. If the prior on $\omega_j$ is modified, then the full conditional density in step L.1 does not follow a PG distribution. This destroys the simplicity of posterior computations in the PG sampler and prevents a direct use of the results in Choi and Hobert (2013).

The first two steps of subset samplers in the DPG-DA algorithm do overcome the hurdles in using the original PG-DA sampler in massive data settings while retaining its advantages. First, if we assume that each subset has sample size $m$, then subset posterior computations scale as $O(m)$, which is smaller than the complexity of original PG-DA sampler by a factor of $k$. Second, step L.1 is identical to step S.1 and step L.2 is a slight modification of step S.2, so the subset samplers retain the simplicity of the PG sampler. In particular, replacing $n$, $X$, and $\kappa$ in step L.2 of the PG sampler by $m_j$, $(n/m_j)^{1/2}X_j$, and $(n/m_j)^{1/2}\kappa_j$ yields step S.2 of a subset sampler in the DPG-DA algorithm.

Any existing method can be used to combine the subset posterior draws, but we focus on extending DPMC and developing an approximation to the WASP. We seek an approximation, as the linear program in terms of empirical measures supported on subset posterior draws and efficient algorithms replaces $\Pi(\cdot | D)$ at the $t$th iteration, and $\Pi(\cdot | D_j)$ in (3.4). The distribution $\Pi(\cdot | D_j)$ be the stochastically-approximated posterior distribution of $\beta$, and Proposition 3.1 implies that $\beta_j^{(t)}$ is approximately distributed as $\Pi(\cdot | D_j)$ for every $j$ and $t$. Proposition 3.1 also demonstrates the theoretical and practical advantages of modifying the likelihood over the prior in divide-and-conquer posterior computations. If the prior on $\omega_j$ is modified, then the full conditional density in step L.1 does not follow a PG distribution. This destroys the simplicity of posterior computations in the PG sampler and prevents a direct use of the results in Choi and Hobert (2013).

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Any existing method can be used to combine the subset posterior draws, but we focus on extending DPMC and developing an approximation to the WASP. We seek an approximation, as the linear program for estimating the WASP of $\beta$ has a computational complexity of $O(T^3)$, which is prohibitively slow in practice. We develop an algorithm that combines DA-based draws from the subset posteriors to yield an approximation to the WASP, and show that the empirical measures supported on the draws from the true WASP and that resulting from our algorithm are asymptotically equivalent. In our simulated and real data analyses, this algorithm outperforms its competitors in accuracy and computational efficiency. On the other hand, the current DPMC algorithm is easy to implement but fails to account for the settings where the covariance matrices of the subset posterior distributions significantly differ, which often happens when the fraction of ones and zeros are imbalanced. Exploiting the flexibility of DPMC, we extend the original algorithm by introducing a scaling step.

### 3.3 Third Step: Combination of Subset MCMC Draws

We require two concepts from the theory of optimal transport for approximating the WASP. The first is that of the Wasserstein barycenter. Let $\| \cdot \|_2$ be the Euclidean metric, $P(\mathbb{R}^p)$ be the set of all probability measures on $\mathbb{R}^p$, the Wasserstein space of order 2 be $P_2(\mathbb{R}^p) = \{ \nu \in P(\mathbb{R}^p) : \int \| \beta \|_2^2 \nu(d\beta) < \infty \}$, and the Wasserstein distance of order 2 between $\nu_1, \nu_2 \in P_2(\mathbb{R}^p)$ be $W_2(\nu_1, \nu_2) = \left\{ \inf_{\pi \in \mathcal{L}(\nu_1, \nu_2)} \int_{\mathbb{R}^p \times \mathbb{R}^p} \| x - y \|_2^2 \pi(dx, dy) \right\}^{1/2}$, where $\mathcal{L}(\nu_1, \nu_2)$ is the set of all probability measures on $\mathbb{R}^p \times \mathbb{R}^p$ with marginals $\nu_1$ and $\nu_2$. Assume that $\nu_1, \ldots, \nu_k \in P_2(\mathbb{R}^p)$, then their Wasserstein barycenter with weights $(w_1, \ldots, w_k)$ is

$$\nu = \arg\min_{\nu \in P_2(\mathbb{R}^p)} \sum_{j=1}^{k} \frac{w_j}{2} W_2^2(\nu, \nu_j), \quad \sum_{j=1}^{k} w_j = 1, \ w_1, \ldots, w_k > 0,$$  \hfill (3.6)

where $\nu$ exists uniquely (Agueh and Carlier, 2011). In scalable Bayesian applications, $\Pi(\cdot | D_1), \ldots, \Pi(\cdot | D_k)$ play the role of $\nu_1, \ldots, \nu_k$, respectively. Their Wasserstein barycenter is the WASP, denoted as $\bar{\Pi}(\cdot | D)$, and replaces $\Pi(\cdot | D)$ for inference on $\beta$ (Srivastava et al., 2015). The optimization problem in (3.6) is posed as a linear program in terms of empirical measures supported on subset posterior draws and efficient algorithms exist to obtain an empirical approximation of $\bar{\Pi}(\cdot | D)$ (Li et al., 2017; Staib et al., 2017; Srivastava et al., 2018). We fix $w_j$ at $1/k$. Proposition 3.1 and (3.4) imply that the moment generating function of $\Pi(\cdot | D_j)$ exists for every $j$, so $\Pi(\cdot | D_j) \in P_2(\mathbb{R}^p)$ ($j = 1, \ldots, k$), which in turn implies that $\bar{\Pi}(\cdot | D) \in P_2(\mathbb{R}^p)$.

The second concept is the location-scatter family of probability measures. It is defined as follows:

**Definition 3.1** (Location-scatter family; Álvarez-Esteban et al. (2016)). Let $X_0$ be a random vector with probability law $G_0 \in P_2(\mathbb{R}^p)$ such that $E(X_0) = 0$ and $\text{var}(X_0) = I$, where $I$ is a $p \times p$ identity matrix, $\mathcal{L}(W)$
be the probability distribution of a random variable \( W \), and \( \mathcal{M}_+^{p \times p} \) be the set of \( p \times p \) positive definite matrices. The family \( \mathcal{F}(G_0) = \{ \mathcal{L}(\Sigma^{1/2}X_0 + \mu) : \Sigma \in \mathcal{M}_+^{p \times p}, \mu \in \mathbb{R}^p \} \) of probability laws induced by positive definite affine transformations from \( G_0 \) is called a location-scatter family, where \( \Sigma^{1/2} \) is the symmetric square-root of \( \Sigma \).

The family \( \mathcal{F}(G_0) \) contains distributions parameterized in terms of their mean \( \mu \) and covariance matrix \( \Sigma \); elliptical families are canonical examples. Theorem 4.2 in Álvarez-Esteban et al. (2016) implies that if \( \nu_1, \ldots, \nu_k \in \mathcal{F}(G_0) \) for some \( G_0 \) and \( \alpha_j \) and \( B_j \) are the mean vector and covariance matrix of \( \nu_j \) \( (j = 1, \ldots, k) \), then their Wasserstein barycenter, denoted as \( \mathfrak{v} \), also belongs to \( \mathcal{F}(G_0) \) under general assumptions and its mean vector \( \mathfrak{v} = \frac{1}{k} \sum_{j=1}^k \alpha_j \) and the covariance matrix \( \mathcal{B} \) is the fixed point of sequence \( \{S_t\}_{t=0}^\infty \) defined as

\[
S_{t+1} = S_t^{-1/2} \left\{ \sum_{j=1}^k \left( 1/k \right) \left( S_t^{1/2} B_j S_t^{1/2} \right)^{1/2} \right\} S_t^{-1/2}, \quad t = 0, 1, 2, \ldots, \quad S_0 = I_p. \tag{3.7}
\]

The third step of DPG-DA algorithm defines the mean vector and covariance matrix of the combined posterior distribution based on the results for the location-scatter family. Let \( \mu_j, \Sigma_j \) and \( \Sigma = \Sigma_{\infty} \), where \( \{\Sigma_t\}_{t=0}^\infty \) is defined based on a numerically stable version of (3.7) as

\[
\Sigma_{t+1} = \Sigma_t^{-1/2} \left\{ \frac{1}{k} \sum_{j=1}^k \left( \Sigma_t \Sigma_j \right)^{1/2} \right\} \left( \frac{1}{k} \sum_{j=1}^k \left( \Sigma_t \Sigma_j \right)^{1/2} \right)^T \Sigma_t^{-1/2}, \quad t = 0, 1, 2, \ldots, \quad \Sigma_0 = I_p. \tag{3.8}
\]

In practice, \( \mu_j \)s and \( \Sigma_j \)s are unknown, so the DPG-DA algorithm replaces them by their Monte Carlo estimates based on the subset posterior draws of \( \beta \) as

\[
\hat{\mu}_j = \frac{1}{T} \sum_{t=1}^T \beta_j^{(t)}, \quad \hat{\Sigma}_j = \frac{1}{T} \sum_{t=1}^T \left( \beta_j^{(t)} - \hat{\mu}_j \right) (\beta_j^{(t)} - \hat{\mu}_j)^T, \quad \hat{\mu} = \frac{1}{k} \sum_{j=1}^k \hat{\mu}_j, \quad \hat{\Sigma} = \hat{\Sigma}_{\infty}, \tag{3.9}
\]

where \( \{\Sigma_t\}_{t=0}^\infty \) is obtained by replacing \( \Sigma_j \) with \( \hat{\Sigma}_j \) in (3.8). While there is no guarantee that \( \Pi(\cdot \mid D_j) \), \( \Pi(\cdot \mid D_k) \) is members of a location-scatter family, the DPG-DA algorithm assumes this to be true and defines the mean vector and covariance matrix of the combined posterior as \( \hat{\mathfrak{v}} \) and \( \hat{\Sigma} \). This suggests a simple algorithm for transforming the subset posterior draws into draws from the combined posterior: (i) center and scale \( j \)th subset posterior draws as \( q_j^{(t)} = \hat{\Sigma}_j^{-1/2} (\beta_j^{(t)} - \hat{\mu}_j) \) and (ii) rescale and recenter \( q_j^{(t)} \)'s to obtain draws following the combined posterior distribution as \( \hat{\mathfrak{v}} + \hat{\Sigma}^{1/2} q_j^{(t)} \) for \( t = 1, \ldots, T \) and \( j = 1, \ldots, k \). This heuristic is summarized in Algorithm 1 and justified theoretically in the next section; therefore, the DPG-DA algorithm provides an approximation to the WASP that reduces to the true WASP if the subset posterior distributions belong to a common location-scatter family.

The DPG-DA based on the approximate WASP is easily extended to the DPMC algorithm, which approximates the full data posterior using a mixture of subset posterior distributions. Let \( \pi(\cdot \mid D_j) \) be the density of the \( j \)th subset posterior distribution. Then, DPMC uses \( \sum_{j=1}^k (1/k) \pi(\beta - \mathfrak{v} + \mu_j \mid D_j) \) for posterior inference on \( \beta \). Similar to the approximate WASP method, DPMC also replaces \( \mathfrak{v} \) and \( \mu_j \) with their Monte Carlo estimates in (3.9) and uses \( \beta_j^{(t)} - \hat{\mu}_j + \hat{\mathfrak{v}} \) \( (j = 1, \ldots, k; t = 1, \ldots, T) \) for posterior inference on \( \beta \).

Unlike the approximate WASP method, there is no scaling step because the DPMC assumes that sample size on every subset is large enough to justify the asymptotic normality of the subset posterior distributions. This assumptions often fails in practice. To bypass this problem, we first note that if \( \pi_j(\beta) = \pi(\beta - \mathfrak{v} + \mu_j \mid D_j) \) and \( \beta \sim \sum_{j=1}^k (1/k) \pi_j(\beta) \), then the mean vector and covariance matrix of \( \beta \) are

\[
\hat{\mu} = \frac{1}{k} \sum_{j=1}^k \int \beta \pi_j(\beta) d\beta = \mathfrak{v}, \quad \hat{\Sigma} = \mathbb{E}(\beta \beta^T) - \hat{\mu} \hat{\mu}^T = \frac{1}{k} \sum_{j=1}^k \Sigma_j + \frac{1}{k} \sum_{j=1}^k \mu_j \mu_j^T - \hat{\mathfrak{v}} \hat{\mathfrak{v}}^T. \tag{3.11}
\]
Algorithm 1 DPG-DA algorithm based on a WASP approximation and a scaling-based DPMC extension.

1. INPUT
   (a) Subset posterior draws for \( \beta, \beta_j^{(t)} \) \( (j = 1, \ldots, k); t = 1, \ldots, T \), and a known function of \( \beta, f(\beta) \).
   (b) Mean vectors and covariance matrices of the subset posterior distributions, the approximate WASP posterior distribution, and the scaled DPMC posterior distribution of \( \beta, \tilde{\mu}_j, \tilde{\mu}, \tilde{\Sigma}_j, \tilde{\Sigma}, \tilde{\Sigma} \) \( (j = 1, \ldots, k) \).

2. DO
   (a) Center and scale the subset posterior draws for \( j = 1, \ldots, k \) and \( t = 1, \ldots, T \) to define \( \tilde{q}_j^{(t)} = \tilde{\Sigma}_j^{-1/2}(\beta_j^{(t)} - \tilde{\mu}_j) \).
   (b) For \( j = 1, \ldots, k \) and \( t = 1, \ldots, T \), define the \( t' \)th combined draw using the \( t \)th subset posterior draw of \( \beta \) as

\[
\bar{\beta}^{(t')} = \bar{\mu} + \tilde{\Sigma}^{1/2} \tilde{q}_j^{(t)} \quad \text{and} \quad \tilde{\beta}^{(t')} = \tilde{\mu} + \tilde{\Sigma}^{1/2} \tilde{q}_j^{(t)} , \quad t' = (j - 1)T + t, \tag{3.10}
\]

where \( \bar{\beta}^{(t')} \) and \( \tilde{\beta}^{(t')} \) are the approximate WASP and scaling-based DPMC draws for \( \beta \), respectively.
   (c) For \( t' = 1, \ldots, kT \), define the \( t' \)th approximate WASP and scaling-based DPMC draws for \( f(\beta) \) as \( f(\bar{\beta}^{(t')}) \) and \( f(\tilde{\beta}^{(t')}) \), respectively.

3. RETURN
   \( \bar{\beta}^{(1)}, \ldots, \bar{\beta}^{(kT)}, f(\bar{\beta}^{(1)}), \ldots, f(\bar{\beta}^{(kT)}), \tilde{\beta}^{(1)}, \ldots, \tilde{\beta}^{(kT)}, f(\tilde{\beta}^{(1)}), \ldots, f(\tilde{\beta}^{(kT)}) \) as the approximate WASP draws and \( \tilde{\beta}^{(1)}, \ldots, \tilde{\beta}^{(kT)} \), \( f(\tilde{\beta}^{(1)}), \ldots, f(\tilde{\beta}^{(kT)}) \) as the scaling-based DPMC draws.

We also replace \( \mu_j \) and \( \Sigma_j \) in (3.11) by their Monte Carlo estimates to obtain \( \hat{\mu} = \tilde{\mu} \) and \( \hat{\Sigma} \). The DPG-DA algorithm based on DPMC introduces two scaling steps: (i) identical to the first step in approximate WASP and (ii) rescale and recenter \( q_j^{(t)} \)s to obtain draws following the combined posterior distribution as \( \hat{\mu} + \hat{\Sigma}^{1/2} q_j^{(t)} \) for \( t = 1, \ldots, T \) and \( j = 1, \ldots, k \). The theoretical guarantee for DPG-DA algorithm based on DPMC follows from Theorem 1 and Corollary 1 in Xue and Liang (2017) and its steps are summarized in Algorithm 1.

4 Theoretical Properties

While the PG-DA part of the DPG-DA algorithm is exclusively for logistic regression, Algorithm 1 with a suitably replaced subset sampling scheme makes it applicable to a broad class of likelihoods. While the computational efficiency of the PG-DA is paramount in practice, only its geometric ergodicity is relevant in the theoretical analysis of Algorithm 1. For these reasons, our theoretical analysis is stated in the setting of a broad class of likelihoods, with the geometric ergodicity of the subset samplers as a requirement. Henceforth, Algorithm 1 is to be understood in the above stated broader setting with a suitable subset sampling scheme. Furthermore, since the theoretical validity of Algorithm 1 with the DPMC extension follows from arguments similar to that in Xue and Liang (2017), we focus here solely on the combination scheme based on the WASP approximation.

In the following, \( P_{\beta}^n \) is the joint distribution of the training data based a likelihood satisfying the assumptions stated in Appendix A and \( \beta_0 \) is the true value of \( \beta \). Let \( \Pi \) be the full-data posterior, \( \overline{\Pi} \) be the combined posterior in Algorithm 1 based on the WASP approximation, and \( \mu, \overline{\mu} \) and \( \Sigma, \overline{\Sigma} \) be the means and covariance matrices of \( \Pi, \overline{\Pi} \). While \( \Pi \) and \( \overline{\Pi} \) are both analytically intractable, it is more efficient to obtain \( \beta \) draws from \( \overline{\Pi} \) using Algorithm 1 than from a general sampling scheme for \( \Pi \). One source of error in using draws from Algorithm 1 for posterior inference on \( \beta \) is statistical in nature, which arises from the use of \( \overline{\Pi} \) instead of \( \Pi \). We quantify this error by \( W_2^2(\Pi, \overline{\Pi}) \), which is independent of the subset sampling scheme. Algorithm 1 is motivated by the fact that if \( \Pi \) and \( \overline{\Pi} \) belong to the same location-scatter family, then

\[
W_2^2(\Pi, \overline{\Pi}) = \| \mu - \overline{\mu} \|_2^2 + \text{tr}(\Sigma + \overline{\Sigma} - 2(\Sigma^{1/2} \Sigma \Sigma^{1/2})^{1/2}); \tag{4.1}
\]
see Theorem 2.3 in Álvarez-Esteban et al. (2016). For the convenience of theoretical analysis, we make this an assumption on \( \Pi \) and the subset posterior distributions. If we show that the two terms on the right are \( o(n^{-1}) \) in \( P_{\beta_0} \)-probability, then \( W_2(\Pi, \hat{\Pi}) \) is \( o(n^{-1/2}) \) in \( P_{\beta_0} \)-probability, implying that the statistical error decays to 0 at the parametric optimal \( n^{-1/2} \) rate. The next theorem shows that this is indeed true under certain regularity assumptions.

**Theorem 4.1.** If Assumptions 1–4 in the appendix hold and \( k = O(1) \), then as \( n, m \to \infty \)

\[
W_2(\Pi, \hat{\Pi}) = o(n^{-1/2}) \text{ in } P_{\beta_0} \text{-probability.}
\]

In typical settings \( \Pi \) is not analytically tractable and one resorts to working with the empirical distribution constructed from the MCMC draws. Algorithm 1 provides an alternate way of arriving at an empirical distribution, which serves as an approximation to the latter. We posit that the distribution between these two empirical distributions is a relevant measure of Monte Carlo error. Theorem 4.2 below uses the rate of convergence of the Monte Carlo error to give a guidance on the choice of \( T \). Note that these two empirical measures are random quantities, and hence in Theorem 4.2 we derive the asymptotic order for a certain coupled versions of these measures. We describe this coupling below.

Based on Definition 3.1, let \( G_0 \) be the distribution with mean 0 and covariance matrix \( I_\beta \) that defines the location-scatter family of subset posterior distributions, and \( \beta_j' (i = 1, \ldots, kT) \) denote a \( kT \) independent draws from \( G_0 \). If we assume that the subset posterior distributions belongs to the location-scatter family defined by \( G_0 \), then define \( \beta_j(i) = \mu_j + \Sigma_j^{1/2} \beta_j(j-1)_{T+i} \) for \( t = 1, \ldots, T \) as a random sample of size \( T \) from the \( j \)th subset posterior \( (j = 1, \ldots, k) \), where \( \beta_j \)'s are unobserved and \( \beta_j(i) \)'s are the \( j \)th subset MCMC draws. Denote the empirical means and the covariance matrices computed using \( \beta_j(i) \) as \( \hat{\mu}_j \) and \( \hat{\Sigma}_j \), respectively, and the uniform empirical measure supported on atoms

\[
\hat{\pi} + \sum_j^{1/2} \hat{\Sigma}_j^{-1/2} (\hat{\mu}_j - \hat{\mu}_j) + \sum_j^{1/2} \hat{\Sigma}_j^{-1/2} \hat{\Sigma}_j^{1/2} \beta_j(j-1)_{T+i}, \quad i = 1, \ldots, T,
\]

as \( \hat{\Pi}_j \), where \( \hat{\pi} \) and \( \hat{\Sigma} \) are defined in (3.9). The Monte Carlo based approximation of \( \Pi \) using \( kT \) MCMC draws resulting from Algorithm 1 is denoted as \( \hat{\Pi} \), and equals the uniform mixture of \( \hat{\Pi}_j, \ j = 1, \ldots, k \). Similarly, we define \( \hat{\Pi} \) to be the uniform discrete distribution on the observations

\[
\pi + \sum_j^{1/2} \beta_j', \quad i = 1, \ldots, kT,
\]

where \( \pi = \frac{1}{k} \sum_{j=1}^k \mu_j \) and \( \Sigma \) is defined in (3.8). Let \( Q \) be the probability measure corresponding to a \( n \) sample from \( P_{\beta_0} \) and \( \beta_j(i) (i = 1, \ldots, kT) \). Then, the following theorem defines the Monte Carlo error as \( W_2(\hat{\Pi}, \hat{\Pi}) \) and quantifies its rate of decay as \( n, T \) tend to infinity.

**Theorem 4.2.** Let \( \hat{\Pi} \) and let \( \hat{\Pi} \) be as defined above. Under Assumptions 1–5, \( n = o(\sqrt{T}) \) and \( n \to \infty \),

\[
W_2^2(\hat{\Pi}, \hat{\Pi}) = o_Q(n^{-1}).
\]

## 5 Experiments

### 5.1 Setup

We evaluate the performance of Algorithm 1 using the full data posterior distribution as the benchmark. The original PG-DA strategy generates MCMC draws from the full data posterior distribution. There is no divide-and-conquer method focusing on PG-DA, so we use the first two steps of DPG-DA algorithm in Sections 3.1 and 3.2 for generating subset posterior MCMC draws. The BayesLogit R package developed by Polson et al. (2013) for generating PG random variables is not maintained actively, so we have used the
R code from the package’s website\textsuperscript{1} in all our samplers. Every sampling algorithm in our simulated and real data analyses runs for 10,000, and we discard the first 5000 draws as burn-ins and thin the chain by collecting every fifth draw. The convergence of the chains is assessed using trace plots. The simulated and real data analyses are performed on a cluster and the original PG sampler is assigned four times more memory resources than the subset posterior samplers. We collect all subset posterior draws and combine them using the third step of DPG-DA algorithm based on the approximate WASP and extended DPMC algorithms following Algorithm 1.

We use two metrics for comparing the performance of sampling algorithms. Let \( \Pi \) be the MCMC-based approximation of the full data posterior using the PG sampler and \( \tilde{\Pi} \) be the approximation of \( \Pi \) obtained using DPMC or DPG-DA algorithms. Motivated from (4.1), the first metric quantifies the approximation error in using \( \tilde{\Pi} \) instead of \( \Pi \) for inference on using DPMC or DPG-DA algorithms. Informally, effective sample size is the number of independent samples obtained from a sampling algorithm and is computed using R package coda (Plummer et al., 2006). Following Johndrow et al. (2016), we define the MCMC computational complexity of PG-DA and DPMC or DPG-DA algorithms as

\[
\text{Computational Gain} = \log_2\left\{ \frac{T_{e/\tilde{t}}}{(T_e/t)} \right\},
\]

(5.2)

A positive value of computational gain indicates that the divide-and-conquer extension of PG-DA strategy produces more independent samples of \( \beta \) per unit time.

### 5.2 Simulated Data Analysis: The Balanced Case

Motivated by Johndrow et al. (2016), our simulated data analysis has two cases for binomial regression. First, we present the case where there is no imbalance in the fractions of ones and zeros in the observed responses. We set \( s_i \) to be 15 for every sample, simulate the entries of \( X \) as independently from \( N(0,1) \), and the entries of \( \beta \) as \(-2, 2\) alternately. We vary \( n \) as \( 10^4, 10^5 \), vary \( p \) as 10, 20, and simulate \( y_i \) as Binom\(\{s_i, 1/(1+e^{-\psi_i})\}\), where \( \psi_i = x_i^T \beta \) for every combination of \( n \) and \( p \). This simulation setup is replicated 10 times. The posterior draws of \( \beta \) are obtained in every replication using the PG-DA strategy following the steps outlined in Section 5.1.

We obtain subset posterior draws using two partitioning schemes. We set \( k \) as 50, 100, and 150 in both schemes. First, we randomly partition the samples into \( k \) disjoint subsets for every \( n \) and \( p \). Second, we partition the samples into \( k \) overlapping subsets by selecting samples with replacement. For every value of \( k \), we set \( m = 500 \) and \( m = 1000 \), respectively. The values of \( k \) are small relative to \( m \) when \( m = 1000 \) or \( n = 10^5 \), which satisfies conditions in Theorem 4.1. The subset posterior sampler iterates between steps L.1–L.2 in parallel on the \( k \) subsets. The post burn-in \( \beta \) draws are collected from the \( k \) subsets and aggregated using DPMC and the third step of DPG-DA algorithm.

The approximation error and computational gain of DPMC and DPG-DA (DPMC) are more sensitive to the subset sample size than DPG-DA (WASP) (Figure 3). In overlapping partitions, \( m \) is large compared to \( p \) in all simulation settings, which justifies the theoretical assumptions of DPMC and DPG-DA (WASP). This results in similar accuracy and computational gains for DPMC and DPG-DA (WASP), whereas DPG-DA (DPMC) has slightly lower accuracy but comparable computational gain (Figures 1 and 2). In contrast, as \( k \) increases \( m \) decreases in disjoint partitions. When \( n = 10^4 \), \( k = 150 \) and \( p = 20 \), every subset size has about

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\textsuperscript{1}https://github.com/jwindle/BayesLogit/tree/master/Code/R/PG.R
70 samples. This violates the assumptions of DPMC and DPG-DA, resulting in inaccurate approximation of the full data posterior distribution (Figure 3); however, as \( m \) decreases, the accuracy and computational gain of DPMC and DPG-DA (DPMC) decreases very rapidly than that of DPG-DA (WASP). We conclude from this simulation that if \( m \) is very large compared to \( p \), then DPMC, DPG-DA (DPMC) and DPG-DA (WASP) have similar performance but DPG-DA (WASP) is a better choice when this assumption is not justified.

5.3 Simulated Data Analysis: The Unbalanced Case

Our second simulation replicates the setup of unbalanced binomial regression in Johndrow et al. (2016). We set \( s_i \) to be \( 10^3 \) for every sample, \( p = 20 \), and \( n = 10^3 \). The entries in the first column of \( X \) are set to be 1 and the remaining entries of \( X \) are simulated from Unif(\(-1, 1\)). The first entry of \( \beta \) is set to \( \alpha \) and the remaining entries of \( \beta \) are simulated independently from \( N(0, 1) \). We set \( \alpha = -5 \) and \( \alpha = -10 \), where the former and latter values result in medium and extreme imbalance in the fraction of ones and zeroes in the observed responses. The \( y_i \)'s are simulated as \( \text{Binom}\{s_i, 1/(1+e^{-\psi_i})\} \), where \( \psi_i = x_i^T \beta \) for every combination of \( n \) and \( p \). This simulation setup is replicated 10 times for the two values of \( \alpha \). The full data posterior draws using the PG-DA strategy and the draws from DPMC and DPG-DA are obtained following the three steps outlined in Section 5.2.
DPG-DA (WASP) outperforms DPMC and DPG-DA (DPMC) in binomial regression with medium and extreme imbalance (Figures 4 and 5). Similar to our previous simulation results, we observe that the accuracy of DPMC, DPD DA (DPMC) and DPG-DA (WASP) increases as $m$ increases in the medium and extreme imbalance cases (4a, 4c, 5a, 5c); however, DPG-DA (WASP) is more accurate than DPMC and DPG-DA (DPMC) in all the simulation settings. On the hand, DPG-DA (WASP) and DPG-DA (DPMC) offer significantly higher computational gains than DPMC across all simulation settings irrespective of the choice of $m$ (4b, 4d, 5b, 5d). We conclude from this simulation that if $m$ is chosen to significantly larger than $k$ and $p$, then DPG-DA (WASP) offers an attractive alternative for divide-and-conquer Bayesian inference in binomial regression where the PG strategy cannot be used due to imbalance.

5.4 MovieLens Data Analysis

We use MovieLens ratings data with 1 million ratings to illustrate the application of DPG-DA algorithm. This data contains the ratings for about 65 thousand movies from about 72 thousand users of the online movie ratings database named MovieLens. Every record in the database contains information about the user, movie name, rating of the movie by the user ranging from 0.5 to 5 in the increments of 0.5, time of the rating, and genres of the movie rated, where a movie can belong to one or more of the 17 predefined genres. The response is defined as 1 if the rating equals 5 and 0 otherwise. If a movie belongs to $g$ genres, then each of these $g$ genre values is assigned the value $1/g$ and a genre predictor is defined as the genre value minus the value of animation genre.

We have also added predictors capturing popularity of movie and the mood of the user following Perry (2017). The popularity of a movie is defined as logit ${n_\text{lik} + 0.5 \over n_\text{rat} + 1.0}$, where logit$(x) = \log\{x/(1-x)\}$ and $n_\text{lik}$ and $n_\text{rat}$ are the number of users who rated the movie above 3 and who rated the movie in 30 or fewer most recent observations for the movie. The user’s mood equals 1 if the previous movie rating assigned by the user is above 3. Finally, we added a column of ones to the design matrix $X$ and randomly divided the full data into 10 parts for performing 10 replications of the real data analysis with $p = 20$ and $n \approx 10^6$.

Following previous two sections, we obtain the posterior draws of $\beta$ using the PG-DA strategy, DPMC, and DPG-DA. The PG-DA strategy is used for obtaining MCMC draws of $\beta$ from the full data posterior distribution. We set $k = 500$ so that $m$ is close to 2000 when the full data is randomly partitioned into disjoint partitions and set $m = 2000$ for overlapping partitions. We obtain subset posterior draws of $\beta$ in parallel using the second step of DPG-DA algorithm for both partitioning schemes and combine the post burn-in $\beta$ draws from the $k$ subsets in both partitioning schemes using DPMC and the third step of DPG-DA.

Agreeing with the simulation results, DPG-DA (WASP) outperforms DPMC and DPG-DA (DPMC) in computational gain and accuracy in both partitioning schemes (Figure 6). A notable departure from our
Figure 4: Approximation errors (a), (c) and computational gains (b), (d) of DPMC and DPG-DA in unbalanced binomial regression using overlapping partitioning scheme for medium imbalance ($\alpha = -5$) and extreme imbalance ($\alpha = -10$) cases.

Simulation results is that DPG-DA (WASP) has slightly higher computational gains over DPG-DA (DPMC). DPMC’s and DPG-DA’s accuracy and computational complexity are independent of the partitioning scheme, but the accuracy and computational gain of DPG-DA (WASP) are significantly higher than that of DPMC. Both the results agree with our simulation results and we conclude that DPG-DA (WASP) leads to more accurate and efficient posterior inference on $\beta$ than DPMC and DPG-DA (DPMC) irrespective of the partitioning schemes.

6 Discussion

We have presented the DPG-DA (WASP) and DPG-DA (DPMC) algorithms as a divide-and-conquer extensions of the Polya-Gamma data augmentation strategy so that it can be used in massive data applications. Additionally, our simulations in Section 5.3 show that DPG-DA (WASP) algorithm can be used for binomial regression where the number of 1s and 0s is imbalanced and the original Polya-Gamma data augmentation strategy fails to produce accurate posterior inference on $\beta$. If $m$ is large relative to $k$ and $p$, then DPG-DA (WASP) algorithm has high accuracy and computational efficiency.

We expect that the advantages of the DPG-DA algorithm carry over to models with random effects. This is so because it is known that the Polya-Gamma sampler can be used in any logistic model, especially binomial mixed effects models and spatial models (Polson et al., 2013); therefore, it is interesting to investigate the analogues of Theorems 4.1 and 4.2 in such models.
**Distributed PG-DA**

Figure 5: Approximation errors (a), (c) and computational gains (b), (d) of DPMC and DPG-DA in unbalanced binomial regression using disjoint partitioning scheme for medium imbalance ($\alpha = -5$) and extreme imbalance ($\alpha = -10$) cases.

Figure 6: Approximation errors (a) and computational gains (b) of DPMC and DPG-DA in MovieLens data analysis for overlapping and disjoint partitioning schemes with 500 subsets.

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Appendix A: Assumptions for the Theoretical Analysis

For the theoretical analysis of Algorithm 1 based on the approximate WASP, we employ the following assumptions:

1. The $y_1, \ldots, y_n$ are independent and identically distributed as $P_{\beta_0}$, where $\beta_0$ is the true value of $\beta$.

2. The subset posterior and full data posterior distributions belong to a location scatter family with $P_{\beta_0}$-probability 1.
3. The regularity assumptions of Laplace approximation hold. Let $B_3(\beta)$ denote an open ball of radius $\delta$ centered at $\beta$. Let the log likelihood of $\beta$ given $y_1, \ldots, y_n$ be $\ell_n(\beta)$, $\hat{\beta}_n$ be the maximum likelihood estimate of $\beta$, and $D^2\ell_n(\beta)$ be its Hessian. Further, suppose that there exists positive numbers $\epsilon$, $M$, and $\eta$ and an integer $n_0$ such that for all $n \geq n_0$: (a) for every $\beta \in B_1(\hat{\beta}_n)$ and all $1 \leq j_1, \ldots, j_d \leq p$ with $1 \leq d \leq 6$, $|\partial_{j_1,\ldots,j_d}\ell_n(\beta)| < M$; (b) $\det\{D^2\ell_n(\beta)\} > \eta$; and (c) for every $\delta$ satisfying $0 < \delta < \epsilon$, $B_3(\hat{\beta}_n) \subseteq \mathbb{R}^p$ and
\[
\limsup_{n \to \infty} \sup_{\beta \in \mathbb{R}^p \setminus B_3(\hat{\beta}_n)} \{\ell_n(\hat{\beta}_n) - \ell_n(\beta)\} < 0
\]

4. The number of subsets $k$ satisfies $k = O(1)$, and the subsets are disjoint and equal in size such that $km = n$, where we have assumed that $m = m_1 = \cdots = m_k$.

5. The number of MCMC iterations $T$ satisfies $n = o(T^{1/2})$ and the MCMC estimates of $\mu_j$ and $\Sigma_j$ satisfy $\sqrt{T}(\hat{\mu}_j - \mu_j) = O_Q(T^{-1/2})$ and $\sqrt{T}(\hat{\Sigma}_j - \Sigma_j) = O_Q(T^{-1/2})$ ($j = 1, \ldots, k$).

Assumption 2 is required for obtaining an analytic expression for the $W_2$-distance between the full data posterior distribution and the approximate WASP. Assumption 3 is based on those required for the validity of the Laplace approximation for the full data and subset posterior distributions; see Kass et al. (1990). Our results generalize to cases where the subset sizes differ, but requiring a common subset sample size in Assumption 4 simplifies the analysis. Assumption 5 is satisfied when the subset sampling scheme is geometrically ergodic; for example, Proposition 3.1 guarantees this in the case of modified PG-DA strategy.

**Appendix B: Proof of Proposition 3.1**

Replace $n, X$, and $\kappa$ by $m_j, (n/m_j)^{1/2}X_j$, and $(n/m_j)^{1/2}\kappa_j$ in Propositions 3.1 and 3.2 of Choi and Hobert (2013) to obtain (3.3) and the last equation in (3.4), respectively. Use Equations 3.2 and 3.1 in Choi and Hobert (2013) and (3.3) to obtain (3.4). The proposition is proved.

**Appendix C: Proof of Theorem 4.1**

Assumption 2 implies that $\Pi(\cdot \mid D)$ and $\Pi(\cdot \mid D)$ belong to the same location-scatter family, and using Theorem 2.3 in Álvarez-Esteban et al. (2016) we have that
\[
W_2^2\{\Pi(\cdot \mid D), \Pi(\cdot \mid D)\} = \|\mu - \mu\|_2^2 + \text{tr}\{\Sigma + \Sigma - 2(\Sigma^{1/2}\Sigma\Sigma^{1/2})^{1/2}\}. \tag{C.1}
\]
If $k = O(1)$, then Lemma E.1 and Lemma E.4 together imply that the expression on the right is $o(n^{-1})$ in $P_{\beta_0}$-probability. The theorem is proved.

**Appendix D: Proof of Theorem 4.2**

We begin by observing that by the definition of the Wasserstein distance we have,
\[
W_2^2(\hat{\Pi}, \hat{\Pi}) \leq \frac{1}{k} \sum_{j=1}^{k} \frac{1}{T} \sum_{i=1}^{T} \|a_j + B_j\beta_{(j-1)T+i}\|^2 \tag{D.1}
\]
\[
\leq \frac{2}{k} \sum_{j=1}^{k} \|a_j\|^2 + \frac{2}{kT} \sum_{j=1}^{k} \sum_{i=1}^{T} \|B_j\beta_{(j-1)T+i}\|^2 \tag{D.2}
\]
\[
\leq \frac{2}{k} \sum_{j=1}^{k} \|a_j\|^2 + 2 \max_{1 \leq j \leq k} \|B_j\|^2 \frac{1}{kT} \sum_{i=1}^{kT} \|\beta_i\|^2 \tag{D.3}
\]
where \(\|B\|\) for a matrix \(B\) denotes the operator norm,

\[
a_j := \hat{\mu} - \mu + \frac{\hat{\Sigma}}{n} \hat{\Sigma}_j^{-1/2} (\mu_j - \hat{\mu}_j), \quad \text{and} \quad B_j := \frac{\hat{\Sigma}}{n} \hat{\Sigma}_j^{-1/2} \hat{\Sigma}_j^{1/2} - \frac{\hat{\Sigma}}{n} \hat{\Sigma}^{1/2}.
\]

Note that by the law of large numbers, in view of the above, it suffices to show that

\[
a_j = o_Q(n^{-1/2}), \quad \text{and} \quad \|B_j\| = o_Q(n^{-1/2}). \tag{D.4}
\]

We begin by establishing the former statement, and towards this note that since \(\|\hat{\mu} - \mu\| = O_Q(T^{-1/2})\), we have

\[
\|\hat{\mu} - \mu\| \leq \frac{1}{k} \sum_{j=1}^k \|\hat{\mu}_j - \mu_j\| = O_Q(T^{-1/2}).
\]

Moreover, we note that

\[
\|\hat{\Sigma}^{1/2} \hat{\Sigma}_j^{-1/2}\| \leq \|n \hat{\Sigma}^{1/2} \| (n \hat{\Sigma}_j^{-1})^{1/2} \leq \left(\frac{1}{k} \sum_{j=1}^k \|n \hat{\Sigma}_j\|\right) \|\|n \hat{\Sigma}_j\| (n \hat{\Sigma}_j^{-1})^{1/2} \| \leq O_Q(1). \tag{Theorem 9 of Bhatia et al. (2018)}
\]

Combining the previous two observations, we have

\[
\|a_j\|^2 \leq 2\|\hat{\mu} - \mu\|^2 + 2\|\hat{\Sigma}^{1/2} \hat{\Sigma}_j^{-1/2} (\mu_j - \hat{\mu}_j)\|^2 \leq O_Q(T^{-1}) + 2\|\hat{\Sigma}^{1/2} \hat{\Sigma}_j^{-1/2}\|^2 \|\mu_j - \hat{\mu}_j\|^2 \leq O_Q(T^{-1}) + O_Q(1)O_Q(T^{-1}) = O_Q(T^{-1}) = o_Q(n^{-1/2}).
\]

Now we establish the second statement of (D.4). Towards this end we note that,

\[
\|B_j\| \leq n^{-1/2} \left(\|I_{\beta_0}^{-1/2} - (n \hat{\Sigma})^{1/2}\| + \|I_{\beta_0}^{-1/2} \hat{\Sigma}_j^{-1/2} \hat{\Sigma}_j^{1/2} - I_{\beta_0}^{-1/2}\| + \|\|n \hat{\Sigma}^{1/2} \hat{\Sigma}_j^{-1/2}\| (n \hat{\Sigma}_j^{-1})^{1/2} - I_{\beta_0}^{-1/2} \hat{\Sigma}_j^{-1/2} \hat{\Sigma}_j^{1/2}\|\right) \leq n^{-1/2} \left(\|I_{\beta_0}^{-1/2} - (n \hat{\Sigma})^{1/2}\| + \|I_{\beta_0}^{-1/2}\| \|\hat{\Sigma}_j^{-1/2} \hat{\Sigma}_j^{1/2} - I\| + \|\|n \hat{\Sigma}^{1/2} \hat{\Sigma}_j^{-1/2}\| (n \hat{\Sigma}_j^{-1})^{1/2} - I_{\beta_0}^{-1/2} \hat{\Sigma}_j^{-1/2} \hat{\Sigma}_j^{1/2}\|\right), \tag{D.5}
\]

where we have used \(I_{\beta_0}\) to denote the Fisher information matrix. Note that it suffices to show that the term within parenthesis in (D.5) is \(o_Q(1)\). For the first term we observe using Lemma E.2, Lemma E.3, and (E.6) that,

\[
\|\hat{\Sigma}_j^{-1/2} \hat{\Sigma}_j^{1/2} - I\| \leq \sqrt{\|\hat{\Sigma}_j^{-1}\| \|\hat{\Sigma}_j\|^{-1} - \hat{\Sigma}_j^{1/2}/2} \leq \sqrt{\|\hat{\Sigma}_j\|^{-1} \|\hat{\Sigma}_j\|^{-1}} \leq O_Q(1). \tag{D.6}
\]

For the second term within parenthesis in (D.5), we note that our Assumption 5 implies that

\[
\|\hat{\Sigma}_j^{-1/2} \hat{\Sigma}_j^{1/2} - I\| \leq \sqrt{\|\hat{\Sigma}_j^{-1}\| \|\hat{\Sigma}_j\|^{-1} - \hat{\Sigma}_j^{1/2}/2} \leq \sqrt{\|\hat{\Sigma}_j\|^{-1} \|\hat{\Sigma}_j\|^{-1}} \leq O_Q(1),
\]

where \(\|B\|_F\) is the Frobenius norm of the matrix \(B\). For the last term within parenthesis in (D.5), an argument mimicking that in (D.6) and using Assumption 5 confirms that it is of order \(o_Q(1)\).
Appendix E: Technical Lemmas

The following lemma states that asymptotic order of the first term on the right hand side in (C.1).

**Lemma E.1.** Let $\mu$ and $\bar{\mu}$ be the means of $\Pi(\cdot | D)$ and $\bar{\Pi}(\cdot | D)$. If Assumptions 1–4 hold and $k = O(1)$, then as $n,m \to \infty$

$$
\| \mu - \bar{\mu} \|^2_2 = o(n^{-1}) \text{ in } \mathcal{P}_n^{\beta_0}-\text{probability}.
$$

**Proof.** The proof follows from the proof Theorem 1 in Xue and Liang (2017) because our assumptions include all the regularity assumptions required for Theorem 1 in Xue and Liang (2017) to hold. \qed

In the following, we define $d(\cdot, \cdot)$ as

$$
d(A, B) := \sqrt{\text{tr}(A + B - 2(A^{1/2}BA^{1/2})^{1/2})},
$$

where $A$ and $B$ are two $p \times p$ positive semidefinite matrices. In Bhatia et al. (2018) (see page 3 therein) it is shown that $d(\cdot, \cdot)$ defines a metric on the space of positive semidefinite matrices. By the Wasserstein mean of $K$ positive semidefinite matrices $A_k$, $k = 1, \ldots, K$, we mean the variance-covariance matrix of the Wasserstein barycenter of $N(0, A_k)$, $k = 1, \ldots, K$.

**Lemma E.2.** Let $A_k$, $k = 1, \ldots, K$ be a sequence of $p \times p$ positive definite matrices, and let their Wasserstein mean be denoted by $\bar{A}$. Then for another positive definite matrix $A_0$ we have,

$$
d(\bar{A}, A_0) \leq 2 \sqrt{\frac{p}{K} \sum_{k=1}^K \| A_k - A_0 \|} \leq 2 \sqrt{\frac{p}{K} \sum_{k=1}^K \| A_k - A_0 \|_F}. \tag{E.1}
$$

**Proof.** By the definition of $\bar{A}$, or see (57) in Bhatia et al. (2018), we have

$$
\bar{A} := \arg\min_X \sum_{k=1}^K d^2(X, A_k).
$$

This implies that

$$
\frac{1}{K} \sum_{k=1}^K d^2(\bar{A}, A_k) \leq \frac{1}{K} \sum_{k=1}^K d^2(A_0, A_k).
$$

Now we have by use of the triangle inequality and the AM-GM inequality that

$$
d^2(\bar{A}, A_0) \leq 2 \left[ \frac{1}{K} \sum_{k=1}^K d^2(\bar{A}, A_k) + \frac{1}{K} \sum_{k=1}^K d^2(A_0, A_k) \right] \tag{E.2}
$$

$$
\leq \frac{4}{K} \sum_{k=1}^K d^2(A_0, A_k) \leq \frac{4}{K} \sum_{k=1}^K \| A_0^{1/2} - A_k^{1/2} \|_F^2, \tag{E.3}
$$

$$
\leq \frac{4p}{K} \sum_{k=1}^K \| A_0^{1/2} - A_k^{1/2} \|_F^2, \tag{E.4}
$$

where the last inequality follows from Theorem 1 of Bhatia et al. (2018). Now since $\sqrt{\cdot}$ is operator monotone, using Theorem X.1.1 of Bhatia (2013) with the above inequality yields the first inequality of (E.1). The final inequality of (E.1) follows by the fact that the Frobenius norm upper bounds the operator norm. \qed

**Lemma E.3.** For two $p \times p$ positive semi-definite matrices $A$ and $B$, we have

$$
\| A - B \|_F \leq d(A, B) \left( \sqrt{\text{tr}[A]} + \sqrt{\text{tr}[B]} \right)
$$
Proof. Let us define for any positive semi-definite $p \times p$ matrix $C$,

$$\mathcal{F}(C) := \{M_{p \times p} : C = MM^T\}.$$ 

Let $M, N$ be members of $\mathcal{F}(A)$ and $\mathcal{F}(B)$, respectively. Then we have,

$$\|A - B\|_F = \|MM^T - NN^T\|_F \leq \|M\|_F\|M^T - N^T\|_F + \|N^T\|_F\|M - N\|_F \leq \|M - N\|_F (\sqrt{\text{tr}(A)} + \sqrt{\text{tr}(B)})$$

Using the above with Theorem 1 of Bhatia et al. (2018) yields,

$$\|A - B\|_F \leq \left(\sqrt{\text{tr}(A)} + \sqrt{\text{tr}(B)}\right) \min_{M \in \mathcal{F}(A); N \in \mathcal{F}(B)} \|M - N\|_F = d(A, B) \left(\sqrt{\text{tr}(A)} + \sqrt{\text{tr}(B)}\right).$$

\square

The following lemma states that asymptotic order of the second term on the right hand side in (C.1).

**Lemma E.4.** Let $\Sigma$ and $\overline{\Sigma}$ be the covariance matrices of $\Pi(\cdot \mid D)$ and $\overline{\Pi}(\cdot \mid D)$. If Assumptions 1–4 hold, then as $n, m \to \infty$

$$d^2(\Sigma, \overline{\Sigma}) = o_p(n^{-1}).$$

**Proof.** Let $\Sigma_j = \text{var}(\beta_j \mid D_j)$. Assumption 3, the existence of moment generating function, and Theorem 4 in Kass et al. (1990) imply via the Laplace approximation of the posterior and the subset posteriors that

$$\Sigma_j = \frac{I_j^{-1}}{n} + O_{p_{\beta_0}}(n^{-2}), \quad \text{and} \quad \Sigma = \frac{I_n^{-1}}{n} + O_{p_{\beta_0}}(n^{-2}),$$

(E.5)

where $I_j$ and $I_n$ are the Fisher information matrices evaluated at the maximum likelihood estimators computed using subset $j$ and full data, respectively. Since the maximum likelihood estimates are consistent estimates of $\beta_0$ and matrix inversion is a continuous operator on the subspace of invertible matrices, we have

$$I_j^{-1} = I_{\beta_0}^{-1} + O_{p_{\beta_0}}(1), \quad \text{and} \quad I_n^{-1} = I_{\beta_0}^{-1} + O_{p_{\beta_0}}(1).$$

Combining the above observations we have

$$n\Sigma_j - I_{\beta_0}^{-1} = o_{p_{\beta_0}}(1), \quad n\Sigma - I_{\beta_0}^{-1} = o_{p_{\beta_0}}(1), \quad \text{and} \quad n\Sigma_j - n\Sigma = o_{p_{\beta_0}}(1).$$

(E.6)

Now using Lemma E.2 we have

$$nd^2(n\Sigma, n\overline{\Sigma}) = d^2(n\Sigma, n\overline{\Sigma}) \leq \frac{4p}{K} \sum_{k=1}^{K} \|n\Sigma_j - n\Sigma\|_F = o_{p_{\beta_0}}(n^{-1}).$$

\square