Finite-Dimensional Irreducible Modules of the Universal Askey–Wilson Algebra

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Received: 12 November 2014 / Accepted: 8 July 2015
Published online: 15 September 2015 – © Springer-Verlag Berlin Heidelberg 2015

Abstract: Since the introduction of Askey–Wilson algebras by Zhedanov in 1991, the classification of the finite-dimensional irreducible modules of Askey–Wilson algebras remains open. A universal analog $\triangle_q$ of the Askey–Wilson algebras was recently studied. In this paper, we consider a family of infinite-dimensional $\triangle_q$-modules. By the universal property of these $\triangle_q$-modules, we classify the finite-dimensional irreducible $\triangle_q$-modules when $q$ is not a root of unity.

1. Introduction

In his pioneering work of 1991, Zhedanov introduced the Askey–Wilson algebras [43], motivated by the Racah coefficients of $su_q(2)$ [13] and the hidden relations between the Askey–Wilson operator and the three-term recurrence relation of the Askey–Wilson polynomials [2]. These algebras are associative unital algebras over the complex number field involving a nonzero scalar $q$ and five parameters $\varrho, \varrho^*, \eta, \eta^*, \omega$. Given these data, the Askey–Wilson algebra $AW_q$ is defined by generators $K_0, K_1, K_2$ subject to the following relations:

$$q K_1 K_2 - q^{-1} K_2 K_1 = \omega K_1 + \varrho K_0 + \eta^*,$$

$$q K_2 K_0 - q^{-1} K_0 K_2 = \omega K_0 + \varrho^* K_1 + \eta,$$

$$q K_0 K_1 - q^{-1} K_1 K_0 = K_2.$$

Let us abbreviate $AW = AW_q$. For example, the quantum group $U'_q(\mathfrak{so}_3)$ [8,15–17,19,28–30,33] different from the Drinfeld–Jimbo type is the algebra $AW$ with $\varrho = 1, \varrho^* = 1, \eta = 0, \eta^* = 0, \omega = 0$ and the Bannai–Ito algebra [9–11,41] is the limit case $q \to -1$ of $AW$. Over two decades of research, the Askey–Wilson algebras have been found to

The research was supported by the National Center for Theoretical Sciences of Taiwan and the Council for Higher Education of Israel.
have applications to quantum integrable systems [4,5,12], the Drinfeld–Jimbo quantum group $U_q(s\ell_2)$ [6,14,39], the double affine Hecke algebras of rank one [20,24,25], the sixth Painlevé equation [27], discrete quantum mechanics [31,32] and so on. In this paper we study the representation theory of the Askey–Wilson algebras.

The first family of finite-dimensional AW-modules was constructed in [43, Section 2]. On these AW-modules, the elements $K_0$, $K_1$ act like Leonard pairs. Roughly speaking, the Leonard pair is a pair of diagonalizable linear transformations on a nonzero finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis of the other one [35, Definition 1.1]. According to their corresponding orthogonal polynomials, Leonard pairs were classified into the $q$-Racah, Racah and related types [37, Section 35]. The works [40,42] of Terwilliger and Vidunas gave a more comprehensive description of how Leonard pairs are related to AW-modules: Given a Leonard pair of $q$-Racah or other $q$-types, the underlying vector space supports an irreducible AW-module with appropriate parameters $q$, $q^*, \eta, \eta^*, \omega$ on which $K_0$, $K_1$ act as affine transformations of the Leonard pair. Conversely, assume that $V$ is a finite-dimensional irreducible AW-module on which each of $K_0$, $K_1$ is diagonalizable with all eigenspaces of dimension one. Then $K_0$, $K_1$ act on $V$ as a Leonard pair of $q$-type, provided that $q$ is not a root of unity.

The notion of Leonard pairs was extended to so-called Leonard triples by Curtin [7, Definition 2.1]. The finite-dimensional irreducible $U'_q(s\ell_3)$-modules for $q$ not a root of unity were classified by Havlíček and Pošta [16, Theorem 4]. Based on their results, it can be shown the action of $K_0$, $K_1$, $K_2$ on each of these irreducible $U'_q(s\ell_3)$-modules as a Leonard triple. The irreducible $U'_q(s\ell_3)$-modules at $q$ a root of unity were proved to be finite-dimensional and studied deeply in [16, Sections 5–7]. However the problem of classifying all finite-dimensional irreducible AW-modules with arbitrary parameters is still open. In a recent paper [38] of Terwilliger, the universal Askey–Wilson algebra $\Delta = \Delta_q$ with $q^4 \neq 1$ was introduced. The algebra $\Delta$ is generated by $A$, $B$, $C$ subject to the relations that assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

commutes with $A$, $B$, $C$. This algebra $\Delta$ is obtained from AW by the following two-step procedure. First, the algebra is renormalized by a mild change of generators. Second, the remaining parameters are interpreted as central elements in the algebra.

The purpose of this paper is to classify the finite-dimensional irreducible $\Delta$-modules for $q$ not a root of unity. We begin with an infinite-dimensional $\Delta$-module $M_\lambda(a,b,c)$ with four nonzero parameters $a$, $b$, $c$, $\lambda$, which is regarded as the Verma $\Delta$-module due to its significant universal property. Fix an integer $n \geq 0$. An $(n+1)$-dimensional $\Delta$-module $V_n(a,b,c)$ is explicitly constructed by taking a quotient of $M_\lambda(a,b,c)$ with $\lambda = q^n$. The irreducibility criterion for $V_n(a,b,c)$ can be simply characterized as

$$abc, ab^{-1}bc, abc^{-1} \notin \{q^{1-n}, q^{3-n}, \ldots, q^{n-1}\}.$$ 

Consider the set $T$ consisting of all such triples $(a,b,c)$. There is an action of the group $\{\pm 1\}^3$ on $T$ given by

$$(a,b,c)^{(-1,1,1)} = (a^{-1}, b, c),$$
$$(a,b,c)^{(1,-1,1)} = (a, b^{-1}, c),$$
$$(a,b,c)^{(1,1,-1)} = (a, b, c^{-1}).$$
for all \((a, b, c) \in T\). Let \(T/\{\pm 1\}^3\) denote the set of all \(\{\pm 1\}^3\)-orbits of \(T\). For \((a, b, c) \in T\) let \([a, b, c]\) denote the \(\{\pm 1\}^3\)-orbit of \(T\) that contains \((a, b, c)\). Define \(M\) to be the set of the isomorphism classes of irreducible \(\triangle\)-modules that have dimension \(n + 1\). By the universal property of Verma \(\triangle\)-modules we establish a bijection \(T/\{\pm 1\}^3 \rightarrow M\) given by
\[
[a, b, c] \mapsto \text{the isomorphism class of } V_n(a, b, c) \text{ for all } [a, b, c] \in T/\{\pm 1\}^3.
\]
This result gives a classification of the finite-dimensional irreducible \(\triangle\)-modules when \(q\) is not a root of unity.

Besides, we characterize on which \(\triangle\)-modules \(A, B, C\) give Leonard pairs or a Leonard triple and formulate the sufficient conditions for \(\triangle\)-modules to be unitary. Apply our classification to \(U'_q(\mathfrak{so}_3)\) and compare the result with [16, Theorem 4]. Determine how many \(U_q(\mathfrak{sl}_2)\)-modules on \(V_n(a, b, c)\) give the \(\triangle\)-module \(V_n(a, b, c)\) by pulling back via the homomorphism \(\triangle \rightarrow U_q(\mathfrak{sl}_2)\) given below [39, Proposition 1.1]. We close the paper with an illustration of how the Racah coefficients of \(U_q(\mathfrak{sl}_2)\) are related to the \(\triangle\)-modules.

2. Notation and Preliminaries

Before launching into the subject we lay some groundwork in preparation. Because our arguments are valid for any algebraically closed field \(\mathbb{F}\), we change the underlying field from the complex number field to \(\mathbb{F}\). Let \(\Lambda, Q, X, Y, Z\) denote five mutually commuting indeterminates over \(\mathbb{F}\). Let \(\mathbb{Z}\) denote the set of the nonnegative integers and \(\mathbb{N}^* = \mathbb{N}\setminus\{0\}\). Define
\[
\theta_i(\Lambda, Q; X) = \Lambda Q^{-2i}X^{-1} + \Lambda^{-1}Q^{2i}X \quad \text{for } i \in \mathbb{Z},
\]
\[
\phi_i(\Lambda, Q; X, Y, Z) = \Lambda Q X^{-1}Y^{-1}(Q^i - Q^{-i})(\Lambda^{-1}Q^{i-1} - \Lambda Q^{1-i})
\times (Q^{-i} - \Lambda^{-1}Q^{i-1}XYZ)(Q^{-i} - \Lambda^{-1}Q^{1-i}XYZ^{-1}) \text{ for } i \in \mathbb{Z},
\]
\[
\omega(\Lambda, Q; X, Y, Z) = (\Lambda Q + \Lambda^{-1}Q^{-1})(Z + Z^{-1}) + (X + X^{-1})(Y + Y^{-1}).
\]
Observe that
\[
\phi_i(\Lambda, Q; X, Y, Z) \in \mathbb{F}[\Lambda, Q; X, Y, Z + Z^{-1}] \quad \text{for } i \in \mathbb{Z}, \quad (2)
\]
\[
\omega(\Lambda, Q; X, Y, Z) \in \mathbb{F}[\Lambda Q + \Lambda^{-1}Q^{-1}, X + X^{-1}, Y + Y^{-1}, Z + Z^{-1}] \quad (3)
\]
and
\[
\theta_i(\Lambda^{-1}, Q^{-1}; X^{-1}) = \theta_i(\Lambda, Q; X) \quad \text{for } i \in \mathbb{Z}, \quad (4)
\]
\[
\phi_i(\Lambda^{-1}, Q^{-1}; X^{-1}, Y^{-1}, Z^{-1}) = \phi_i(\Lambda, Q; X, Y, Z) \quad \text{for } i \in \mathbb{Z}, \quad (5)
\]
\[
\omega(\Lambda, Q; X, Y, Z) = \omega(\Lambda, Q; X, Y, Z). \quad (6)
\]

Fix \(n \in \mathbb{N}\) throughout this paper. Let
\[
\{\theta_i(Q; X)\}_{i \in \mathbb{Z}}, \quad \{\phi_i(Q; X, Y, Z)\}_{i \in \mathbb{Z}}, \quad \omega(Q; X, Y, Z)
\]
denote the Laurent polynomials \(\{\theta_i(\Lambda, Q; X)\}_{i \in \mathbb{Z}}, \quad \{\phi_i(\Lambda, Q; X, Y, Z)\}_{i \in \mathbb{Z}}, \quad \omega(\Lambda, Q; X, Y, Z)\) with the substitution \(\Lambda = Q^n\), respectively. Observe that
\[
\theta_{n-i}(Q; X) = \theta_i(Q^{-1}; X) \quad \text{for } i \in \mathbb{Z}, \quad (7)
\]
\[
\phi_{n-i+1}(Q; X, Y, Z) = \phi_i(Q^{-1}; X, Y, Z) \quad \text{for } i \in \mathbb{Z}. \quad (8)
\]
Note that (2)–(8) will be used without further mention.
2.1. Three $\mathbb{N} \times \mathbb{N}$ matrices. A (possibly infinite) square matrix is said to be \textit{tridiagonal} if each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A square matrix is said to be \textit{lower} (resp. \textit{upper}) \textit{bidiagonal} if each nonzero entry lies on the diagonal or subdiagonal (resp. superdiagonal).

The $\mathbb{N} \times \mathbb{N}$ matrices $L(\Lambda, Q; X), U(\Lambda, Q; X, Y, Z), T(\Lambda, Q; X, Y, Z)$ given below will be used to define the Verma $\Delta$-modules. The matrix $L(\Lambda, Q; X)$ is lower bidiagonal with

$$L(\Lambda, Q; X)_{ii} = \theta_i(\Lambda, Q; X) \quad \text{for} \quad i \in \mathbb{N},$$

$$L(\Lambda, Q; X)_{i,i-1} = 1 \quad \text{for} \quad i \in \mathbb{N}^*.$$  

The matrix $U(\Lambda, Q; X, Y, Z)$ is upper bidiagonal with

$$U(\Lambda, Q; X, Y, Z)_{ii} = \theta_i(\Lambda, Q; Y) \quad \text{for} \quad i \in \mathbb{N},$$

$$U(\Lambda, Q; X, Y, Z)_{i-1,i} = \phi_i(\Lambda, Q; X, Y, Z) \quad \text{for} \quad i \in \mathbb{N}^*.$$  

The matrix $T(\Lambda, Q; X, Y, Z)$ is tridiagonal with

$$T(\Lambda, Q; X, Y, Z)_{ii} = \frac{Q^{-1} \phi_{i+1}(\Lambda, Q; X, Y, Z) - Q \phi_i(\Lambda, Q; X, Y, Z)}{Q^2 - Q^{-2}} + \frac{\omega(\Lambda, Q; X, Y, Z) - \theta_i(\Lambda, Q; X) \theta_j(\Lambda, Q; Y)}{Q + Q^{-1}} \quad \text{for} \quad i \in \mathbb{N},$$

$$T(\Lambda, Q; X, Y, Z)_{i-1,i} = \frac{Q^{-1} \theta_i(\Lambda, Q; X) - Q \theta_{i-1}(\Lambda, Q; X)}{Q^2 - Q^{-2}} \phi_i(\Lambda, Q; X, Y, Z) \quad \text{for} \quad i \in \mathbb{N}^*,$$

$$T(\Lambda, Q; X, Y, Z)_{i,i-1} = \frac{Q^{-1} \theta_i(\Lambda, Q; Y) - Q \theta_{i-1}(\Lambda, Q; Y)}{Q^2 - Q^{-2}} \quad \text{for} \quad i \in \mathbb{N}^*.$$  

The $\mathbb{N} \times \mathbb{N}$ matrices $E(\Lambda, Q; X), S(\Lambda, Q; X, Y, Z)$ from two matrix equations will also be present in this paper. To solve the two equations we transform either of them into a recurrence relation. After straightforward observations and tedious verifications, the solutions are available below. Recall the notation

$$[i]_Q = \frac{Q^i - Q^{-i}}{Q - Q^{-1}} \quad \text{for} \quad i \in \mathbb{Z}$$

and the Gaussian binomial coefficients

$$\binom{j}{i}_Q = \prod_{h=1}^{i} \frac{[j - h + 1]_Q}{[h]_Q} \quad \text{for} \quad i \in \mathbb{N} \text{ and } j \in \mathbb{Z}.$$  

\textbf{Lemma 2.1.} (i) The upper triangular $\mathbb{N} \times \mathbb{N}$ matrix $E(\Lambda, Q; X)$ with $(i, j)$-entry

$$\binom{j}{i}_Q \prod_{h=1}^{j-i} (\Lambda^{-1} Q^{h+i-1} - \Lambda Q^{1-i-h})(Q^{1-h} X - Q^{h-1} X^{-1})$$

for all $i, j \in \mathbb{N}$ with $i \leq j$ is the unique matrix $E$ satisfying

$$L(\Lambda, Q; X) \cdot E = E \cdot L(\Lambda, Q; X^{-1})$$

with $E_{00} = 1$ and $E_{i0} = 0$ for all $i \in \mathbb{N}^*$.  

(ii) The upper triangular $\mathbb{N} \times \mathbb{N}$ matrix $S(\Lambda, Q; X, Y, Z)$ with $(i, j)$-entry

$$(-1)^j \Lambda^i Q^{-ij} Y^{-j} \left[ \prod_{Q=1}^{j-i} (\Lambda Q^{h-j} - \Lambda^{-1} Q^{j-h})(Q^{j-h} Y Z - \Lambda Q^{h-j-1} X^{-1}) \right]$$

for all $i, j \in \mathbb{N}$ with $i \leq j$ is the unique matrix $S$ satisfying

$$T(\Lambda, Q; X, Y, Z) \cdot S = S \cdot L(\Lambda, Q; Z)$$

with $S_{00} = 1$ and $S_{i0} = 0$ for all $i \in \mathbb{N}^*$.

While studying the finite-dimensional irreducible $\Delta$-modules, we will turn our attention to the submatrices

$$L(Q; X), \quad U(Q; X, Y, Z), \quad T(Q; X, Y, Z)$$

of $L(\Lambda, Q; X), U(\Lambda, Q; X, Y, Z), T(\Lambda, Q; X, Y, Z)$ indexed by the first $n + 1$ rows and $n + 1$ columns with the substitution $\Lambda = Q^n$, respectively. Also, we will see the $(n + 1) \times (n + 1)$ matrices $E(Q; X), S(Q; X, Y, Z)$ defined in the same way and the matrices $F(Q; X, Y, Z), P(Q; X, Y, Z)$ solved from the following matrix equations.

**Lemma 2.2.**

(i) The lower triangular $(n + 1) \times (n + 1)$ matrix $F(Q; X, Y, Z)$ with $(i, j)$-entry

$$\left[ \prod_{Q=1}^{j-i} (\theta_0(Q; Y) - \theta_{h-1}(Q^{-1}; Y)) \prod_{h=1}^{j} \frac{[n - i + h]_Q}{[n - h + 1]_Q} \phi_h(Q; X, Y, Z) \right]$$

for all $0 \leq j \leq i \leq n$ is the unique matrix $F$ satisfying

$$L(Q; X) \cdot F = F \cdot L(Q; X)$$

with $F_{i0} = \prod_{h=1}^{i} (\theta_0(Q; Y) - \theta_{h-1}(Q^{-1}; Y)) \prod_{h=1}^{n-i} \phi_h(Q; X^{-1}, Y, Z)$ for all $0 \leq i \leq n$.

(ii) The upper left triangular $(n + 1) \times (n + 1)$ matrix $P(Q; X, Y, Z)$ with $(n - i, j)$-entry

$$\left[ \prod_{Q=1}^{j-i} (\theta_0(Q; X) - \theta_{h-1}(Q^{-1}; X)) \prod_{h=1}^{j} \frac{[n - i + h]_Q}{[n - h + 1]_Q} \phi_h(Q; X, Y^{-1}, Z) \right]$$

for all $0 \leq j \leq i \leq n$ is the unique matrix $P$ satisfying

$$U(Q; X, Y, Z) \cdot P = P \cdot L(Q^{-1}; Y)$$

with $P_{n-i,0} = \prod_{h=1}^{i} (\theta_0(Q; X) - \theta_{h-1}(Q^{-1}; X))$ for all $0 \leq i \leq n$. 
2.2. The universal Askey–Wilson algebra. Define $\alpha$, $\beta$, $\gamma$ to be the central elements of $\Delta$ obtained from multiplying the elements in (1) by $q + q^{-1}$, respectively. In terms of $A$, $B$, $C$

\begin{align*}
C &= \frac{\gamma}{q + q^{-1}} - \frac{q AB - q^{-1} BA}{q^2 - q^{-2}}, \\
\alpha &= \frac{B^2 A - (q^2 + q^{-2}) BAB + AB^2 + (q^2 - q^{-2})^2 A + (q - q^{-1})^2 B \gamma}{(q - q^{-1})(q^2 - q^{-2})}, \\
\beta &= \frac{A^2 B - (q^2 + q^{-2}) ABA + BA^2 + (q^2 - q^{-2})^2 B + (q - q^{-1})^2 A \gamma}{(q - q^{-1})(q^2 - q^{-2})}.
\end{align*}

(9)

(10)

(11)

By (9) the elements $A$, $B$, $\gamma$ form a set of generators of $\Delta$. The Poincaré–Birkhoff–Witt theorem for $\Delta$ was proved in [38, Theorem 4.1]:

Lemma 2.3. The monomials $A^i C^j B^k \alpha^r \beta^s \gamma^t$ for all $i, j, k, r, s, t \in \mathbb{N}$ form a basis of $\Delta$.

Recall that the symmetry group $S_3$ of degree three has a presentation with generators $\sigma, \tau$ and relations $\sigma^2 = 1$, $\tau^2 = 1$, $(\sigma \tau)^3 = 1$. Define an action of $S_3$ on the set of all permutations $(R, S, T)$ of $A$, $B$, $C$ by

\[(R, S, T)\sigma = (T, S, R), \quad (R, S, T)\tau = (S, R, T)\.

Similarly define an action of $S_3$ on the set of all permutations $(r, s, t)$ of $\alpha$, $\beta$, $\gamma$ by

\[(r, s, t)\sigma = (t, s, r), \quad (r, s, t)\tau = (s, r, t).

Let

\[C^\vee = C + \frac{AB - BA}{q - q^{-1}}.\]

As a consequence of [38, Theorem 3.1] we have

Lemma 2.4. For each $g \in S_3$ there exists a unique automorphism $\tilde{g} : \Delta \rightarrow \Delta$ that sends

\[(A, B, C)^g \mapsto \begin{cases} (A, B, C) & \text{if the permutation } g \text{ is even,} \\ (A, B, C^\vee) & \text{if the permutation } g \text{ is odd}, \end{cases}

and sends $(\alpha, \beta, \gamma)^g \mapsto (\alpha, \beta, \gamma)$.

Given a $\Delta$-module $V$ and $g \in S_3$ the notation $V^g$ will stand for the $\Delta$-module obtained by pulling back $V$ via $\tilde{g}$.

3. The Verma $\Delta$-Module $M_\lambda(a, b, c)$

Let $a, b, c, \lambda$ denote any four nonzero scalars in $\mathbb{F}$. There is a $\Delta$-module $M_\lambda(a, b, c)$ that has a basis $\{m_i\}_{i \in \mathbb{N}}$ with respect to which the matrices representing $A$, $B$, $C$ are
respectively. To confirm the existence of this module, one can check that the above matrices satisfy the defining relations for $\triangle$. The central elements $\alpha, \beta, \gamma$ act on $M_{\lambda}(a, b, c)$ as scalar multiplications by
\[ \omega(\lambda, q; b, c, a), \omega(\lambda, q; c, a, b), \omega(\lambda, q; a, b, c) \]
respectively. The matrix representing $C^\lambda$ with respect to $\{m_i\}_{i \in \mathbb{N}}$ is equal to
\begin{equation}
T(\lambda^{-1}, q^{-1}; a^{-1}, b^{-1}, c^{-1}).
\end{equation}

For a semisimple Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic 0, by applying the universal property of Verma $\mathfrak{g}$-modules, every finite-dimensional irreducible $\mathfrak{g}$-module $V$ is shown as a quotient of the Verma $\mathfrak{g}$-module with the same highest weight as $V$. The $\triangle$-module $M_{\lambda}(a, b, c)$ plays a role as the Verma $\triangle$-module from the above viewpoint. The universal property of $M_{\lambda}(a, b, c)$ is presented in Sect. 3.1. Applying the universal property of $M_{\lambda}(a, b, c)$, in the proof of Theorem 4.7 we shall see that every finite-dimensional irreducible $\triangle$-module is a quotient of $M_{\lambda}(a, b, c)$ with appropriate $a, b, c, \lambda$ when $q$ is not a root of unity.

A connection between the Askey–Wilson polynomials and $M_{\lambda}(a, b, c)$ is displayed in Sect. 3.2. The four bases of $M_{\lambda}(a, b, c)$ mentioned in Sect. 3.3 are to prepare for Sect. 4.

3.1. The universal property of $M_{\lambda}(a, b, c)$. Let $I_{\lambda}(a, b, c)$ denote the left ideal of $\triangle$ generated by
\begin{equation}
B - \theta_0(\lambda, q; b),
\end{equation}
\begin{equation}
(\alpha - \omega(\lambda, q; b, c, a))[A - \theta_0(\lambda, q; a)] - \phi_1(\lambda, q; a, b, c),
\end{equation}
\begin{equation}
\beta - \omega(\lambda, q; c, a, b), \gamma - \omega(\lambda, q; a, b, c).
\end{equation}

**Lemma 3.1.** The $\mathbb{F}$-vector space $\triangle/I_{\lambda}(a, b, c)$ is spanned by $A^i + I_{\lambda}(a, b, c)$ for all $i \in \mathbb{N}$.

**Proof.** By Lemma 2.3 the cosets $A^i C^j B^k a^r \beta^s \gamma^t + I_{\lambda}(a, b, c)$ for all $i, j, k, r, s, t \in \mathbb{N}$ span $\triangle/I_{\lambda}(a, b, c)$. By (13) and (15), $A^i C^j B^k a^r \beta^s \gamma^t + I_{\lambda}(a, b, c) \in \mathbb{F} A^i C^j + I_{\lambda}(a, b, c)$ for all $i, j, k, r, s, t \in \mathbb{N}$. Therefore it suffices to show that for any $i, j \in \mathbb{N}$ the coset $A^i C^j + I_{\lambda}(a, b, c)$ is contained in the $\mathbb{F}$-vector space spanned by $A^h + I_{\lambda}(a, b, c)$ for all $h \in \mathbb{N}$.

To see this we proceed by induction on $j$. There is nothing to prove for $j = 0$. Assume that $j \geq 1$. By (9), $A^i C^j$ is a linear combination of $A^i C^j AB, A^i C^j BA$. By (13), $A^i C^j AB + I_{\lambda}(a, b, c) \in \mathbb{F} A^i C^j + I_{\lambda}(a, b, c)$. By (15), $A^i C^j - \gamma + I_{\lambda}(a, b, c) \in \mathbb{F} A^i C^j + I_{\lambda}(a, b, c)$. By (13) and (14), $A^i C^j AB + I_{\lambda}(a, b, c)$ is a linear combination of $A^i C^j AB + I_{\lambda}(a, b, c)$ and $A^i C^j BA + I_{\lambda}(a, b, c)$. Combining the above comments $A^i C^j + I_{\lambda}(a, b, c)$ is a linear combination of $A^i C^j - \gamma + I_{\lambda}(a, b, c)$ and $A^i C^j - \gamma + I_{\lambda}(a, b, c)$. The result now follows by induction hypothesis. $\square$

**Proposition 3.2.** There exists a unique $\triangle$-module isomorphism $\triangle/I_{\lambda}(a, b, c) \rightarrow M_{\lambda}(a, b, c)$ that sends $1 + I_{\lambda}(a, b, c)$ to $m_0$.  

Here we identify the Askey–Wilson polynomials with the following properties. The three-term recurrence relation for the Askey–Wilson polynomials, provided that \( q \) is not a root of unity and
\[ q^{-2i} \neq \lambda^{-2}, \quad \lambda^{-2}q^{-2}b^2, \quad \lambda^{-1}qabc, \quad \lambda^{-1}qabc^{-1} \quad \text{for all } i \in \mathbb{N}, \]
where \( q \) is the basic hypergeometric series. Recall from [2, 23] that \( \{p_0(X)\}_{i \in \mathbb{N}} \) are the Askey–Wilson polynomials with the following properties. The three-term recurrence relation for \( \{p_0(X)\}_{i \in \mathbb{N}} \) is
\[ Xp_i(X) = a_i p_{i+1}(X) + c_i p_i(X) + b_i p_{i-1}(X) \quad \text{for } i \in \mathbb{N}, \]
where \( p_{-1}(X) \) is interpreted to be 0 and
\[
\begin{align*}
a_i &= \frac{(\lambda^{-1}q^i b - \lambda q^{-i}b^{-1}) \phi_{i+1}(\lambda, q; a, b, c)}{(q^{i+1} - q^{-1-i})(\lambda^{-1}q^{2i} b - \lambda q^{-2i}b^{-1})(\lambda^{-1}q^{2i+1} b - \lambda q^{-2i}b^{-1})}, \\
b_i &= \frac{(q^i b - q^{-i}b^{-1}) \phi_i(\lambda, q; a^{-1}, b, c)}{(\lambda^{-1}q^{-1-i} - \lambda q^{1-i})(\lambda^{-1}q^{2i-1} b - \lambda q^{-2i-1}b^{-1})(\lambda^{-1}q^{2i} b - \lambda q^{-2i}b^{-1})}, \\
c_i &= \theta_0(\lambda, q; a) - a_i - b_i
\end{align*}
\]
for \( i \in \mathbb{N} \). The Askey–Wilson operator \( \mathcal{D} \) is the linear transformation \( \mathbb{F}(Y) \to \mathbb{F}(Y) \) defined by
\[
\mathcal{D} f(Y) = A(Y) f(q^2 Y) - (A(Y) + A(Y^{-1}) - \lambda b^{-1} - \lambda^{-1} b) f(Y) + A(Y^{-1}) f(q^{-2} Y)
\]
for \( f(Y) \in \mathbb{F}(Y) \), where
\[
A(Y) = \frac{\lambda(1 - \lambda^{-1} a Y)(1 - \lambda^{-1} a^{-1} Y)(1 - q b c Y)(1 - q b c^{-1} Y)}{b (1 - Y^2)(1 - q^2 Y^2)}.
\]
For each \( i \in \mathbb{N} \) the polynomial \( p_i(X) \) is an eigenfunction of \( \mathcal{D} \) with respect to the eigenvalue \( \theta_i(\lambda, q; b) \). By [43, Introduction] the polynomial ring \( \mathbb{F}[X] \) supports a \( \Delta \)-module on which
\[
\begin{align*}
\mathbb{F}[X] & \to \mathbb{F}[X] \\
A : & p(X) \mapsto X p(X), \\
B : & p(X) \mapsto \mathcal{D} p(X)
\end{align*}
\]
for all \( p(X) \in \mathbb{F}[X] \) and \( \gamma \) acts as scalar multiplication by \( \omega(\lambda, q; a, b, c) \). By the universal property of \( M_\lambda(a, b, c) \) it is routine to show that there is a \( \Delta \)-module isomorphism
\[
M_\lambda(a, b, c) \to \mathbb{F}[X]
\]
\[
m_i \mapsto \prod_{h=1}^i (X - \theta_{h-1}(\lambda, q; a)) \quad \text{for all } i \in \mathbb{N}.
\]

3.3. Four bases of \( M_\lambda(a, b, c) \). For convenience the basis \( \{m_i\}_{i \in \mathbb{N}} \) of \( M_\lambda(a, b, c) \) will be said to be canonical. In this section we make use of the universal property of the Verma \( \Delta \)-modules to obtain four bases of \( M_\lambda(a, b, c) \) on which the action of \( A, B, C \) is similar to that on the canonical basis. To present the symmetry of these bases, we define an action of the Klein group \( V_4 = \{\pm 1\} \times \{1, \sigma\} \) on the set consisting of all 5-tuples \( (\lambda, q^\epsilon; a, b, c) \in \mathbb{F}^5 \) with \( a, b, c, \lambda \neq 0 \) and \( \epsilon \in \{\pm 1\} \) by
\[
(\lambda, q^\epsilon; a, b, c)^{-1,1} = (\lambda, q^{-\epsilon}; a^{-1}, b, c^{-1}),
\]
\[
(\lambda, q^\epsilon; a, b, c)^{-1,1} = (\lambda^{-1}, q^{-\epsilon}; c^{-1}, b, a^{-1}).
\]

**Proposition 3.3.** For each \( (\epsilon, g) \in V_4 \) there exists a unique basis \( \{m_i^{(\epsilon, g)}\}_{i \in \mathbb{N}} \) of \( M_\lambda(a, b, c) \) with \( m_0^{(\epsilon, g)} = m_0 \) and with respect to which the matrices representing \( (A, B, C)^g \) are
\[
L(\Lambda, Q; X), \quad U(\Lambda, Q; X, Y, Z), \quad T(\Lambda, Q; X, Y, Z)
\]
with \( (\Lambda, Q; X, Y, Z) = (\lambda, q; a, b, c)^{\epsilon, g} \), respectively.

**Proof.** (Uniqueness): From the matrix \( L(\lambda, q; a, b, c) \) representing \( A \) with respect to the basis \( \{m_i^{(1,1)}\}_{i \in \mathbb{N}} \) of \( M_\lambda(a, b, c) \), we see that
\[
m_i^{(1,1)} = \prod_{h=1}^i (A - \theta_{h-1}(\lambda, q; a)) m_0^{(1,1)} \quad \text{for all } i \in \mathbb{N}^*.
\]
Since $m_0^{(1,1)}$ is fixed these vectors $m_i^{(1,1)}$ for all $i \in \mathbb{N}^*$ are uniquely determined. Therefore the uniqueness of $\{m_i^{(1,1)}\}_{i \in \mathbb{N}}$ follows. By similar arguments the uniqueness of $\{m_i^{(\varepsilon,\sigma)}\}_{i \in \mathbb{N}}$ follows for each $(\varepsilon, \sigma) \in V_4$.

(Existence): The canonical basis of $M_\lambda(a, b, c)$ is exactly $\{m_i^{(1,1)}\}_{i \in \mathbb{N}}$. Since $V_4$ is generated by $(-1, 1)$ and $(1, \sigma)$, it is enough to show the existence of $\{m_i^{(-1,1)}\}_{i \in \mathbb{N}}$ and $\{m_i^{(1,\sigma)}\}_{i \in \mathbb{N}}$. We first prove the existence of $\{m_i^{(-1,1)}\}_{i \in \mathbb{N}}$. It is routine to verify that

$$ (B - \theta_1(\lambda, q; b))(A - \theta_0(\lambda, q; a^{-1}))m_0 = \phi_1(\lambda, q; a^{-1}, b, c^{-1})m_0. $$

The central elements $\alpha, \beta, \gamma$ act on $M_\lambda(a, b, c)$ as scalar multiplications by

$$ \omega(\lambda, q; b, c^{-1}, a^{-1}), \quad \omega(\lambda, q; c^{-1}, a^{-1}, b), \quad \omega(\lambda, q; a^{-1}, b, c^{-1}) $$

respectively. Let $\{u_i\}_{i \in \mathbb{N}}$ denote the canonical basis of $M_\lambda(a^{-1}, b, c^{-1})$. By the universal property of $M_\lambda(a^{-1}, b, c^{-1})$ there exists a $\Delta$-module isomorphism $M_\lambda(a^{-1}, b, c^{-1}) \to M_\lambda(a, b, c)$ that sends $u_i$ to

$$ \prod_{h=1}^i (A - \theta_{h-1}(\lambda, q; a^{-1}))m_0 \quad \text{for all } i \in \mathbb{N}. \quad (17) $$

The matrices representing $A, B, C$ with respect to the basis $\{u_i\}_{i \in \mathbb{N}}$ of $M_\lambda(a^{-1}, b, c^{-1})$ are $L(\lambda, q; a^{-1}), U(\lambda, q; a^{-1}, b, c^{-1}), T(\lambda, q; a^{-1}, b, c^{-1})$ respectively. Thus $(17)$ is the basis $\{m_i^{(-1,1)}\}_{i \in \mathbb{N}}$ of $M_\lambda(a, b, c)$.

We next prove the existence of $\{m_i^{(1,\sigma)}\}_{i \in \mathbb{N}}$. A direct calculation yields that

$$ (B - \theta_1(\lambda, q; b))(C^\vee - \theta_0(\lambda, q; c))m_0 = \phi_1(\lambda, q; c, b, a)m_0. $$

The central elements $\alpha, \beta, \gamma$ act on $M_\lambda(a, b, c)$ as scalar multiplications by

$$ \omega(\lambda, q; c, b, a), \quad \omega(\lambda, q; a, c, b), \quad \omega(\lambda, q; b, a, c) $$

respectively. By Lemma 2.4 the automorphism $\tilde{\sigma}$ of $\Delta$ sends

$$ (A, B, C, \alpha, \beta, \gamma) \mapsto (C^\vee, B, A, \gamma, \beta, \alpha). $$

Let $\{v_i\}_{i \in \mathbb{N}}$ denote the canonical basis of $M_\lambda(c, b, a)$. By the universal property of $M_\lambda(c, b, a)$ there exists a unique $\Delta$-module homomorphism $M_\lambda(c, b, a) \to M_\lambda(a, b, c)$ that sends $v_0$ to $m_0$. Pulling back via $\tilde{\sigma}$ we obtain a $\Delta$-module homomorphism $i : M_\lambda(c, b, a)^\sigma \to M_\lambda(a, b, c)$ that sends $v_i$ to

$$ \prod_{h=1}^i (C - \theta_{h-1}(\lambda, q; c))m_0 \quad \text{for all } i \in \mathbb{N}. \quad (18) $$

The tridiagonal matrix $T(\lambda, q; a, b, c)$ representing $C$ with respect to the basis $\{m_i\}_{i \in \mathbb{N}}$ of $M_\lambda(a, b, c)$ has nonzero subdiagonal entries

$$ T(\lambda, q; a, b, c)_{i, i-1} = -\lambda q^{1-2i}b^{-1} \quad \text{for all } i \in \mathbb{N}^*. $$

Therefore $(18)$ is a basis of $M_\lambda(a, b, c)$ and $i$ is an isomorphism. The matrices representing $C, B$ with respect to the basis $\{v_i\}_{i \in \mathbb{N}}$ of $M_\lambda(c, b, a)^\sigma$ are equal to $L(\lambda^{-1}, q^{-1}; c^{-1}), U(\lambda^{-1}, q^{-1}; c^{-1}, b^{-1}, a^{-1})$ respectively. By $(12)$ the matrix representing $A$ with respect to the basis $\{v_i\}_{i \in \mathbb{N}}$ of $M_\lambda(c, b, a)^\sigma$ is $T(\lambda^{-1}, q^{-1}; c^{-1}, b^{-1}, a^{-1})$. Therefore $(18)$ is the basis $\{m_i^{(1,\sigma)}\}_{i \in \mathbb{N}}$ of $M_\lambda(a, b, c)$.
By Lemma 2.1 we have

**Lemma 3.4.** For each \((ε, g) \in V_4\) the transition matrices from \(\{m_i^{(ε,g)}\}_{i \in \mathbb{N}}\) to \(\{m_i^{(-ε,g)}\}_{i \in \mathbb{N}}\) and \(\{m_i^{(ε,g)\alpha}\}_{i \in \mathbb{N}}\) are \(E(κ, Q; X)\) and \(S(κ, Q; X, Y, Z)\) with \((κ, Q; X, Y, Z) = (λ, q; a, b, c)^{(ε,g)}\), respectively.

### 4. Finite-Dimensional Irreducible Δ-Modules

We are going to classify the finite-dimensional irreducible \(Δ\)-modules when \(q\) is not a root of unity. To do this, from now on we set \(λ = q^n\) and start with a quotient \(Δ\)-module of \(M_λ(a, b, c)\).

#### 4.1. The quotient Δ-module \(V_n(a, b, c)\) of \(M_λ(a, b, c)\)

Define \(N_λ(a, b, c)\) to be the \(F\)-subspace of \(M_λ(a, b, c)\) spanned by the \(m_i\) for all \(i \geq n + 1\). It is equivalent to say \(N_λ(a, b, c) = K_α(A)M_λ(a, b, c)\), where

\[
K_Y(X) = \prod_{i=0}^{n} (X - \theta_i(q; Y)).
\]

By construction \(N_λ(a, b, c)\) is \(A\)-invariant. By the setting \(λ = q^n\) the \((n, n + 1)\)-entry \(φ_{n+1}(q; a, b, c)\) of \(U(λ, q; a, b, c)\) is zero. It follows that \(N_λ(a, b, c)\) is \(B\)-invariant. Since \(Δ\) is generated by \(A, B, γ\) the \(F\)-space \(N_λ(a, b, c)\) is a \(Δ\)-module and hence

\[
V_n(a, b, c) = M_λ(a, b, c)/N_λ(a, b, c)
\]

is a \(Δ\)-module of dimension \(n + 1\). A criterion for an irreducible \(Δ\)-module to be isomorphic to \(V_n(a, b, c)\) immediately arises from the definition of \(V_n(a, b, c)\).

**Lemma 4.1.** Let \(V\) denote an \((n + 1)\)-dimensional irreducible \(Δ\)-module. Assume that

(i) there is a nontrivial \(Δ\)-module homomorphism \(ι : M_λ(a, b, c) \to V\);

(ii) \(K_α(X)\) is the characteristic polynomial of \(A\) on \(V\).

Then there is a \(Δ\)-module isomorphism \(V_n(a, b, c) \to V\) given by

\[
m + N_λ(a, b, c) \mapsto ι(m) \quad \text{for all} \quad m \in M_λ(a, b, c).
\]

**Proof.** Since \(V\) is irreducible \(ι\) is surjective. By the Cayley-Hamilton theorem \(K_α(A)\) vanishes on \(V\) and thus \(N_λ(a, b, c)\) is contained in the kernel of \(ι\). This lemma follows. \(\square\)

For each \((ε, g) \in V_4\) let

\[
v_i^{(ε,g)} = m_i^{(ε,g)} + N_λ(a, b, c) \quad (0 \leq i \leq n).
\]

Clearly \(\{v_i^{(1,1)}\}_{i=0}^n\) is a basis of \(V_n(a, b, c)\). The matrix \(E(λ, q; a)\) is upper triangular with nonzero diagonal entries and so is \(S(λ, q; a, b, c)\). Therefore, by Lemma 3.4, \(\{v_i^{(ε,g)}\}_{i=0}^n\) is a basis of \(V_n(a, b, c)\) for each \((ε, g) \in V_4\).

**Lemma 4.2.** The matrices representing \(A, B, C\) with respect to the basis \(\{v_i^{(ε,g)}\}_{i=0}^n\) of \(V_n(a, b, c)\) for each \((ε, g) \in V_4\) are as follows:

\[
\text{(Details omitted for brevity.)}
\]
4.2. The irreducibility conditions for $V_n$

Proof. If $\Lambda = Q^n$ then all entries on the $(n + 1)$th column of $E(\Lambda, Q; X)$ are zero except for the diagonal entry and so is $S(\Lambda, Q; X, Y, Z)$. Therefore, by Lemma 3.4, $m^{(e,g)}_{n+1} \in N_\lambda(a, b, c)$ for each $(e, g) \in V_4$. The result now follows by Proposition 3.3. □

Lemma 4.3. On the $\Delta$-module $V_n(a, b, c)$

(i) the characteristic polynomials of $A, B, C$ are $K_a(X), K_b(X), K_c(X)$ respectively;
(ii) the traces of $A, B, C$ are $[n + 1]_q(a + a^{-1}), [n + 1]_q(b + b^{-1}), [n + 1]_q(c + c^{-1})$ respectively.

Proof. (i) By Lemma 4.2 the matrices representing $A, B$ with respect to $\{v_i^{(1,1)}\}_{i=0}^n$ are lower and upper bidiagonal with $\{\theta_i(q; a)\}_{i=0}^n$, $\{\theta_i(q; b)\}_{i=0}^n$ on the diagonal entries, respectively. Therefore $K_a(X), K_b(X)$ are the characteristic polynomials of $A, B$ on $V_n(a, b, c)$, respectively. By Lemma 4.2 the matrix representing $C$ with respect to $\{v_i^{(1,\sigma)}\}_{i=0}^n$ is lower bidiagonal with $\{\theta_i(q; c)\}_{i=0}^n$ on the diagonal entries. Therefore $K_c(X)$ is the characteristic polynomial of $C$ on $V_n(a, b, c)$.

(ii) By (i) we have to show that

$$\sum_{i=0}^n \theta_i(Q; X) = [n + 1]_q(X + X^{-1}).$$

To get the equality, apply the summation formula of the geometric series. □

4.2. The irreducibility conditions for $V_n(a, b, c)$. This section is devoted to proving the following necessary and sufficient conditions for $V_n(a, b, c)$ to be irreducible.

Theorem 4.4. The $\Delta$-module $V_n(a, b, c)$ is irreducible if and only if the following conditions hold:

(i) $q^{2i} \neq 1$ for $1 \leq i \leq n$.
(ii) $abc, a^{-1}bc, ab^{-1}c, abc^{-1} \notin \{q^{1-n}, q^{3-n}, \ldots, q^{n-1}\}$.

Proof. (Necessity): By Lemma 4.2 the matrices representing $A, B$ with respect to $\{v_i^{(1,1)}\}_{i=0}^n$ are $L(q; a), U(q; a, b, c)$ respectively. Thus, if there exists $1 \leq i \leq n$ such that the $(i - 1, i)$-entry $\phi_i(q; a, b, c)$ of $U(q; a, b, c)$ is zero, then $\sum_{h=i}^n F v_h^{(1,1)}$ is $A$- and $B$-invariant and therefore is a $\Delta$-module, contrary to the irreducibility of $V_n(a, b, c)$. Therefore

$$\phi_i(q; a, b, c) \neq 0 \quad (1 \leq i \leq n). \quad (19)$$
The matrices representing $A, B$ with respect to \{v_i^{(−1,1)}\}_{i=0}^n$ are $L(q; a^{-1}), U(q; a^{-1}, b, c^{-1})$ respectively. Thus a similar argument leads to

$$\phi_i(q; a^{-1}, b, c^{-1}) \neq 0 \quad (1 \leq i \leq n).$$

(20)

Conditions (i), (ii) are equivalent to (19), (20).

(Sufficiency): Let

$$R = \prod_{h=1}^{n}(B - \theta_h(q; b)), \quad S_i = \prod_{h=1}^{n-i}(A - \theta_{h-1}(q^{-1}; a)) \quad (0 \leq i \leq n).$$

By Lemma 4.2 we have $Rv \in \mathbb{F}v_0^{(1,1)}$ for all $v \in V_n(a, b, c)$. Let $F$ denote the $(n + 1) \times (n + 1)$ matrix such that

$$RS_iv_j^{(1,1)} = F_{ij}v_0^{(1,1)} \quad (0 \leq i, j \leq n).$$

A routine calculation shows that $F$ satisfies the conditions given in Lemma 2.2(i) with $(Q; X, Y, Z) = (q; a, b, c)$. Therefore $F = F(q; a, b, c)$ is lower triangular. By (i), (ii)

$$F_{ii} = \prod_{h=1}^{i}[n - i + h]_q \phi_h(q; a, b, c) \prod_{h=1}^{n-i}\phi_h(q; a^{-1}, b, c) \quad (0 \leq i \leq n)$$

are nonzero and thus $F$ is invertible.

Now let $V$ denote a nonzero $\Delta$-submodule of $V_n(a, b, c)$. We show that $V = V_n(a, b, c)$. Pick a nonzero vector $v \in V$ and consider

$$RS_iv = a_i v_0^{(1,1)} \quad (0 \leq i \leq n),$$

(21)

where $a_i \in \mathbb{F}$. Since $V$ is a $\Delta$-module $a_i v_0^{(1,1)} \in V$ for each $0 \leq i \leq n$. On the other hand, write $v = \sum_{i=0}^{n}b_i v_i^{(1,1)}$ where $b_i \in \mathbb{F}$ and express the Eq. (21) as the matrix equation

$$F(b_0, b_1, \ldots, b_n)' = (a_0, a_1, \ldots, a_n)'$$

where “$\tau$” means the transpose. Since at least one of $b_i$ for all $0 \leq i \leq n$ is nonzero, the invertibility of $F$ implies that at least one of $a_i$ for all $0 \leq i \leq n$ is nonzero. Therefore $v_0^{(1,1)} \in V$ which generates $V_n(a, b, c)$ as a $\Delta$-module, as claimed. \qed

4.3. 24 Bases of irreducible $V_n(a, b, c)$. When the $\Delta$-module $V_n(a, b, c)$ is irreducible, there are 24 bases of $V_n(a, b, c)$, described in Proposition 4.5, with respect to which the matrices representing $A, B, C$ have similar forms to those in Lemma 4.2. Some of these bases will be used in the proofs of Lemma 4.8, Theorems 5.2 and 5.6.

Similar to Sect. 3.3, in order to describe the symmetry among the 24 bases, we define an action of $\{±1\}^2 \rtimes S_3$ on the set consisting of all 4-tuples $(q^\varepsilon; a, b, c) \in \mathbb{F}^4$ with $a, b, c \neq 0$ and $\varepsilon \in \{±1\}$ by

$$(q^\varepsilon; a, b, c)^{(−1,1,1)} = (q^\varepsilon; a^{-1}, b, c^{-1}), \quad (q^\varepsilon; a, b, c)^{(1,1,1)} = (q^{-\varepsilon}; c^{-1}, b^{-1}, a^{-1}),$$

$$\quad (q^\varepsilon; a, b, c)^{(−1,1,−1)} = (q^\varepsilon; a, b^{-1}, c^{-1}), \quad (q^\varepsilon; a, b, c)^{(1,1,−1)} = (q^{-\varepsilon}; b^{-1}, a^{-1}, c^{-1}),$$

where $\{±1\}^2 \rtimes S_3$ is the semidirect product of $S_3$ by $\{±1\}^2$ with respect to the group homomorphism $S_3 \to \text{Aut}(\{±1\}^2)$ defined by

$$(−1, 1)^\sigma = (−1, 1), \quad (1, −1)^\sigma = (−1, 1), \quad (−1, 1)^\tau = (1, −1), \quad (1, −1)^\tau = (−1, 1).$$
Proposition 4.5. Assume that the $\Delta$-module $V_n(a, b, c)$ is irreducible. For each $(\varepsilon_0, \varepsilon_1, g) \in \{\pm\}^2 \rtimes S_3$, up to scalar multiplication, there exists a unique basis $\{v_{i}^{(\varepsilon_0, \varepsilon_1, g)}\}_{i=0}^{n}$ of $V_n(a, b, c)$ with respect to which the matrices representing $(A, B, C)$ are

$$L(Q; X), \quad U(Q; X, Y, Z), \quad T(Q; X, Y, Z)$$

with $(Q; X, Y, Z) = (q; a, b, c)^{(\varepsilon_0, \varepsilon_1, g)}$, respectively.

Proof. (Uniqueness): From the matrix $L(q; a)$ representing $A$ with respect to $\{v_{i}^{(1,1,1)}\}_{i=0}^{n}$, we see that

$$v_{i}^{(1,1,1)} = \prod_{h=1}^{i} (A - \theta_{h-1}(q; a))v_{0}^{(1,1,1)} \quad (1 \leq i \leq n).$$

The first column of $U(q; a, b, c)$ implies that $v_{0}^{(1,1,1)}$ is an eigenvector of $B$ on $V_n(a, b, c)$ with respect to $\theta_0(q; b)$. The uniqueness of $\{v_{i}^{(1,1,1)}\}_{i=0}^{n}$ is now immediate from this lemma:

Lemma 4.6. Each eigenspace of $A, C$ on $V_n(a, b, c)$ is of dimension one. Moreover, if $V_n(a, b, c)$ is irreducible then each eigenspace of $B$ on $V_n(a, b, c)$ is of dimension one.

Proof. By Lemma 4.2 the matrix representing $A$ (resp. $C$) with respect to the basis $\{v_{i}^{(1,1)}\}_{i=0}^{n}$ (resp. $\{v_{i}^{(1,\sigma)}\}_{i=0}^{n}$) of $V_n(a, b, c)$ is lower bidiagonal with nonzero subdiagonal entries. Therefore each eigenspace of $A, C$ on $V_n(a, b, c)$ is of dimension one. Suppose that $V_n(a, b, c)$ is irreducible. The matrix representing $B$ with respect to the basis $\{v_{i}^{(1,1)}\}_{i=0}^{n}$ of $V_n(a, b, c)$ is the upper bidiagonal matrix $U(q; a, b, c)$. By Theorem 4.4 the superdiagonal entries of $U(q; a, b, c)$ are nonzero. Therefore each eigenspace of $B$ on $V_n(a, b, c)$ is of dimension one. \qed

By similar arguments the uniqueness of $\{v_{i}^{g}\}_{i=0}^{n}$ follows for each $g \in \{\pm\}^2 \rtimes S_3$.

(Existence): By Lemma 4.2, $\{v_{i}^{(\varepsilon, g)}\}_{i=0}^{n}$ is the basis $\{v_{i}^{(1,1,1)}\}_{i=0}^{n}$ of $V_n(a, b, c)$ for each $(\varepsilon, g) \in V_4$. Since $\{\pm\}^2 \rtimes S_3$ is generated by $(-1, 1, 1), (1, -1, 1), (1, 1, \sigma), (1, 1, \tau)$, it remains to show the existence of $\{v_{i}^{(1,1,1)}\}_{i=0}^{n}$ and $\{v_{i}^{(1,1,\tau)}\}_{i=0}^{n}$. We first show the existence of $\{v_{i}^{(1,1,1)}\}_{i=0}^{n}$. It is routine to verify that

$$u = \sum_{i=0}^{n} \prod_{h=1}^{n-i} \phi_{h}(q; a^{-1}, b^{-1}, c) \prod_{h=1}^{i} (\theta_{0}(q; b^{-1}) - \theta_{h-1}(q^{-1}; b^{-1}))v_{i}^{(1,1)} \quad (22)$$

satisfies $Bu = \theta_{0}(q; b^{-1})u$ and $(B - \theta_{1}(q; b^{-1}))(A - \theta_{0}(q; a))u = \phi_{1}(q; a, b^{-1}, c^{-1})u$. The central elements $\alpha, \beta, \gamma$ act on $V_n(a, b, c)$ as scalar multiplications by

$$\omega(q; b^{-1}, c^{-1}, a), \quad \omega(q; c^{-1}, a, b^{-1}), \quad \omega(q; a, b^{-1}, c^{-1})$$

respectively. Let $\{u_{i}\}_{i \in \mathbb{N}}$ denote the canonical basis of $M_{\Delta}(a, b^{-1}, c^{-1})$. Combining the above comments, the universal property of $M_{\Delta}(a, b^{-1}, c^{-1})$ implies that there is a $\Delta$-module homomorphism $M_{\Delta}(a, b^{-1}, c^{-1}) \rightarrow V_n(a, b, c)$ that sends $u_0 \mapsto u$. Recall from Lemma 4.3(i) that $K_{\alpha}(X)$ is the characteristic polynomial of $A$ on $V_n(a, b, c).$ By
Lemma 4.1 there is a $\Delta$-module isomorphism $V_n(a, b^{-1}, c^{-1}) \rightarrow V_n(a, b, c)$ that sends $u_i + N_\lambda(a, b^{-1}, c^{-1})$ to

$$\prod_{h=1}^{i} (A - \theta_{h-1}(q; a))u \quad (0 \leq i \leq n). \quad (23)$$

By Lemma 4.2, with respect to the basis $\{u_i + N_\lambda(a, b^{-1}, c^{-1})\}_{i=0}^{n}$ of $V_n(a, b^{-1}, c^{-1})$ the matrices representing $A, B, C$ are $L(q; a), U(q; a, b^{-1}, c^{-1}), T(q; a, b^{-1}, c^{-1})$ respectively. Therefore (23) is the basis $\{v_i^{(1,1,1)}\}_{i=0}^{n}$ of $V_n(a, b, c)$.

We next prove the existence of $\{v_i^{(1,1,1)}\}_{i=0}^{n}$. A direct calculation yields that

$$v = \sum_{i=0}^{n} \prod_{h=1}^{i} (\theta_0(q; a) - \theta_{h-1}(q^{-1}; a))v_{n-i}^{(1,1)} \quad (24)$$

satisfies $Av = \theta_0(q; a)v$ and $(A - \theta_1(q; a))(B - \theta_0(q; b))v = \phi_1(q; b, a, c)v$. The central elements $\alpha, \beta, \gamma$ act on $V_n(a, b, c)$ as scalar multiplications by

$$\omega(q; c, b, a), \quad \omega(q; a, c, b), \quad \omega(q; b, a, c)$$

respectively. Recall from Lemma 2.4 that the automorphism $\tilde{\tau}$ of $\Delta$ maps

$$(A, B, C, \alpha, \beta, \gamma) \mapsto (B, A, C^\vee, \beta, \alpha, \gamma).$$

Let $\{v_i\}_{i \in \mathbb{N}}$ denote the canonical basis of $M_\lambda(b, a, c)$. Combining the above comments, the universal property of $M_\lambda(b, a, c)$ implies that there exists a $\Delta$-module homomorphism $M_\lambda(b, a, c) \rightarrow V_n(a, b, c)\tau$ that maps $v_0$ to $v$. By Lemma 4.3(i), $K_b(X)$ is the characteristic polynomial of $A$ on $V_n(a, b, c)\tau$. By Lemma 4.1 there is a $\Delta$-module isomorphism $V_n(b, a, c) \rightarrow V_n(a, b, c)\tau$ that maps $v_0 + N_\lambda(b, a, c)$ to $v$. Pulling back via $\tilde{\tau}$ we obtain a $\Delta$-module isomorphism $V_n(b, a, c)\tau \rightarrow V_n(a, b, c)$ that sends $v_i + N_\lambda(b, a, c)$ to

$$\prod_{h=1}^{i} (B - \theta_{h-1}(q; b))v \quad (0 \leq i \leq n). \quad (25)$$

By Lemma 4.2 the matrices representing $B, A$ with respect to the basis $\{v_i + N_\lambda(b, a, c)\}_{i=0}^{n}$ of $V_n(b, a, c)\tau$ are $L(q^{-1}; b^{-1}), U(q^{-1}; b^{-1}, a^{-1}, c^{-1})$ respectively. By (12) the matrix representing $C$ with respect to $\{v_i + N_\lambda(b, a, c)\}_{i=0}^{n}$ is $T(q^{-1}; b^{-1}, a^{-1}, c^{-1})$. Therefore (25) is the basis $\{v_i^{(1,1,1)}\}_{i=0}^{n}$ of $V_n(a, b, c)$. \hfill \square

We end this section with the remark that for each $g \in [\pm 1]^2 \rtimes \mathfrak{S}_3$, up to scalar multiplication the transition matrices from $\{v_i^g\}_{i=0}^{n}$ to $\{v_i^{g(-1,1,1)}\}_{i=0}^{n}$, $\{v_i^{g(1,-1,1)}\}_{i=0}^{n}$, $\{v_i^{g(1,1,1)}\}_{i=0}^{n}$, $\{v_i^{g(1,1,1)}\}_{i=0}^{n}$ are

$$E(Q; X), \quad F(Q; X, Y^{-1}, Z), \quad S(Q; X, Y, Z), \quad P(Q; X, Y, Z)E(Q^{-1}; Y)$$

with $(Q; X, Y, Z) = (q; a, b, c)$, respectively. To see this, without loss we assume that $g = (1, 1, 1)$. By Lemma 3.4 the transition matrices from $\{v_i^{(1,1,1)}\}_{i=0}^{n}$ to $\{v_i^{(-1,1,1)}\}_{i=0}^{n}$ and $\{v_i^{(1,1,1)}\}_{i=0}^{n}$ are as claimed. Let $F$ denote the transition matrix from $\{v_i^{(1,1,1)}\}_{i=0}^{n}$
of all triples of irreducible △-modules. Assume that q is not a root of unity. Let Vn be the set of all triples (a, b, c) of nonzero scalars in \( \mathbb{F} \) that satisfy Theorem 4.4(ii):

\[
abc, \quad a^{-1}bc, \quad ab^{-1}c, \quad abc^{-1} \notin \{q^{1-n}, q^{3-n}, \ldots, q^{n-1}\}.
\]

Define an action of the group \( \{\pm 1\}^3 \) on \( T \) by

\[
(a, b, c)^{(-1,1,1)} = (a^{-1}, b, c),
\]

\[
(a, b, c)^{(1,-1,1)} = (a, b^{-1}, c),
\]

\[
(a, b, c)^{(1,1,-1)} = (a, b, c^{-1})
\]

for all \( (a, b, c) \in T \). Let \( T/\{\pm 1\}^3 \) denote the set of the \( \{\pm 1\}^3 \)-orbits of \( T \). For \( (a, b, c) \in T \) let \( [a, b, c] \) denote the \( \{\pm 1\}^3 \)-orbit of \( T \) that contains \( (a, b, c) \). Then there is a bijection \( T/\{\pm 1\}^3 \to M \) given by

\[
[a, b, c] \mapsto \text{the isomorphism class of } V_n(a, b, c) \text{ for } [a, b, c] \in T/\{\pm 1\}^3.
\]

**Proof.** We begin with a lemma, the “if” part of which implies the existence of the map \( T/\{\pm 1\}^3 \to M \) and the “only if” part implies that the map is injective.

**Lemma 4.8.** Assume that \( q^{2^i} \neq 1 \) for \( 1 \leq i \leq n \). For \( (a, b, c) \) and \( (r, s, t) \) in \( T \), \( V_n(a, b, c) \) is isomorphic to \( V_n(r, s, t) \) if and only if \( [a, b, c] = [r, s, t] \).

**Proof.** (Necessity): Any two isomorphic finite-dimensional △-modules have the same trace map. By Lemma 4.3(ii) this part follows.

(Sufficiency): It suffices to show that \( V_n(a, b, c) \) is isomorphic to \( V_n(a^{-1}, b, c), V_n(a, b^{-1}, c), V_n(a, b, c^{-1}) \). Observe that \( U(Q; X, Y, Z) \) and \( T(Q; X, Y, Z) \) are invariant when \( Z \) is replaced by \( Z^{-1} \). Therefore, by Lemma 4.2 with \( (\varepsilon, g) = (1, 1) \) and \( (\varepsilon, g) = (-1, 1) \) we see that \( V_n(a, b, c) \) is isomorphic to \( V_n(a, b, c^{-1}) \) and \( V_n(a^{-1}, b, c) \), respectively. By Proposition 4.5 with \( (\varepsilon_0, \varepsilon_1, g) = (1, -1, 1) \) the \( \Delta \)-module \( V_n(a, b, c) \) is isomorphic to \( V_n(a, b^{-1}, c) \). \( \square \)

To see that the map is surjective, we assume that \( V \) is an \( (n + 1) \)-dimensional irreducible △-module and show that there exists \( (a, b, c) \in T \) such that \( V \) is isomorphic to \( V_n(a, b, c) \). For any \( \theta \in \mathbb{F} \) and any \( S \in \Delta \) define \( V_S(\theta) = \{v \in V \mid Sv = \theta v\} \). The nonzero scalar \( b \in \mathbb{F} \) is chosen to satisfy

\[
V_B(\theta_{-1}(q; b)) = 0, \quad V_B(\theta_0(q; b)) \neq 0.
\]

(26)

To see the existence of \( b \) one may apply this lemma:
Lemma 4.9. For i, j in \( \mathbb{Z} \), \( \theta_i(Q; X) = \theta_j(Q; X) \) if and only if \( Q^{2i} = Q^{2j} \) or \( X^2 = Q^{2(n-i-j)} \).

Proof. Factor \( \theta_i(Q; X) - \theta_j(Q; X) = (Q^{i-j} - Q^{j-i})(Q^{i+j-n}X - Q^{n-i-j}X^{-1}) \). \( \square \)

Pick an eigenvalue \( \theta \) of \( B \) on \( V \). Since \( \mathbb{F} \) is algebraically closed there exists a nonzero scalar \( s \in \mathbb{F} \) such that \( \theta = \theta_0(q; s) \). Under the assumption that \( q \) is not a root of unity, Lemma 4.9 implies that either \( \{\theta_i(q; s)\}_{i \in \mathbb{Z}} \) are pairwise distinct or \( \theta_i(q; s) = \theta_j(q; s) \) if and only if \( i + j \) is equal to a specific integer. In either case there are only finitely many \( i \in \mathbb{Z} \) with \( V_B(\theta_i(q; s)) \neq 0 \). Therefore there exists \( i \in \mathbb{Z} \) with \( V_B(\theta_{i-1}(q; s)) = 0 \) and \( V_B(\theta_i(q; s)) \neq 0 \); in other words the scalar \( b = \gamma q^{2i} \) satisfies (26). Similarly the nonzero scalar \( a \in \mathbb{F} \) is chosen to satisfy

\[
V_A(\theta_{i-1}(q; a)) = 0, \quad V_A(\theta_0(q; a)) \neq 0.
\]

Note that every central element in \( \triangle \) acts on \( V \) as a scalar. Since \( \mathbb{F} \) is algebraically closed and \( q^{2n+2} \neq -1 \), there exists a nonzero scalar \( c \in \mathbb{F} \) such that \( \gamma \) acts on \( V \) as \( \omega(q; a, b, c) \).

We invoke Lemma 4.1 to show that \( V_n(a, b, c) \) is isomorphic to \( V \). Taking the commutator with \( B \) on either side of (10), we obtain that

\[
B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = -(q^2 - q^{-2})^2(BA - AB).
\] (27)

Pick any vector \( v \in V_B(\theta_0(q; b)) \). Applying \( v \) to either side of (27), we find that \( (B - \theta_{i-1}(q; b))(B - \theta_0(q; b))(B - \theta_1(q; b)) \) vanishes at \( Av \). By (26) it reduces to \( (B - \theta_0(q; b))(B - \theta_1(q; b))Av = 0 \). This shows that \( V_B(\theta_0(q; b)) \) is \( (B - \theta_1(q; b))A \)-invariant. Let \( v_0 \) denote an eigenvector of \( (B - \theta_1(q; b))A \) in \( V_B(\theta_0(q; b)) \). Recurrently define

\[
v_i = (A - \theta_{i-1}(q; a))v_{i-1} \quad \text{for all } i \in \mathbb{N}^*.
\] (28)

We now proceed by induction to show that

\[
(B - \theta_1(q; b))v_i \in \sum_{h=0}^{i-1} \mathbb{F}v_h \quad \text{for } i \in \mathbb{N}.
\] (29)

By the choice of \( v_0 \), (29) holds for \( i = 0, 1 \). Suppose \( i \geq 2 \). Observe that

\[
\theta_i(Q; X) = (Q^2 + Q^{-2})\theta_{i-1}(Q; X) - \theta_{i-2}(Q; X).
\] (30)

To get (29), we apply \( v_{i-2} \) to either side of (11) and simplify the resulting equation by using (28), (30) and induction hypothesis. To see that \( \{v_i\}_{i=0}^n \) is a basis of \( V \), we suppose on the contrary that there exists \( 0 \leq i \leq n - 1 \) such that \( v_{i+1} \in \sum_{h=0}^{i} \mathbb{F}v_h \). Then \( \sum_{h=0}^{i} \mathbb{F}v_h \) is \( A \)- and \( B \)-invariant by (28) and (29), respectively. This leads to a contradiction to the irreducibility of \( V \).

Similarly, choose \( w_0 \) to be an eigenvector of \( (A - \theta_1(q; a))B \) in \( V_A(\theta_0(q; a)) \) and recurrently define

\[
w_i = (B - \theta_{i-1}(q; b))w_{i-1} \quad \text{for all } i \in \mathbb{N}^*.
\]

Then \( \{w_i\}_{i=0}^n \) is a basis of \( V \) with respect to which the matrix representing \( A \) is upper triangular with \( \theta_i(q; a) \) on the \((i, i)\)-entry for each \( 0 \leq i \leq n \). Therefore \( K_a(X) \) is the
characteristic polynomial of $A$ on $V$. By the Cayley-Hamilton theorem $K_n(A)$ vanishes on $V$. In particular $K_n(A)v_0 = 0$ and hence

$$Av_n = \theta_n(q; a)v_n. \quad (31)$$

For each $1 \leq i \leq n$ let $\phi_i$ denote the $(i - 1, i)$-entry of the matrix representing $B$ with respect to the basis $\{v_i\}_{i=0}^n$ of $V$. Apply $v_{i-1}$ ($1 \leq i \leq n$) to either side of (11) and simplify the resulting equation by using (28), (31). Comparing the coefficient of $v_i$ on either side we then obtain that $\phi_{i+1} = (q^2 + q^{-2})\phi_i + \phi_{i-1}$ is equal to

$$(q^2 + q^{-2})(\theta_i(q; a)\theta_i(q; b) + \theta_{i-1}(q; a)\theta_{i-1}(q; b))$$

for each $1 \leq i \leq n$, where $\phi_0$ and $\phi_{n+1}$ are interpreted as 0. A direct calculation yields that $\phi_i = \phi_i(q; a, b, c)$ for all $1 \leq i \leq n$ satisfy the above recurrence relation. Since $q^{4n+4} \neq 1$ there are no other scalars $\phi_i$ ($1 \leq i \leq n$) satisfying the recurrence relation. In particular we have

$$(B - \theta_1(q; b))(A - \theta_0(q; a))v_0 = \phi_1(q; a, b, c)v_0. \quad (32)$$

Applying $v_0$ to either side of (10) and using (32) to simplify the resulting equation, we see that $\alpha v_0 = \omega(q; b, c, a)v_0$. A similar argument shows that $\beta v_0 = \omega(q; c, a, b)v_0$. Therefore the central elements $\alpha, \beta$ act on $V$ as the scalars $\omega(q; b, c, a), \omega(q; c, a, b)$ respectively.

Combining the above comments, the universal property of $M_2(a, b, c)$ implies that there exists a unique $\Delta$-module homomorphism $M_2(a, b, c) \to V$ that maps $m_0$ to $v_0$. By Lemma 4.1 this induces a $\Delta$-module isomorphism $V_n(a, b, c) \to V$. By Theorem 4.4 the triple $(a, b, c) \in T$, as claimed.

As a consequence of Lemma 4.3(ii) and Theorem 4.7 we see that

**Corollary 4.10.** Assume that $q$ is not a root of unity. Let $V$ denote an $(n+1)$-dimensional irreducible $\Delta$-module. Let $\text{tr} A, \text{tr} B, \text{tr} C$ denote the traces of $A, B, C$ on $V$ respectively. Then $V$ is isomorphic to $V_n(a, b, c)$ if and only if $a, b, c$ are the roots of the quadratic polynomials

$$[n + 1]_q X^2 - \text{tr} AX + [n + 1]_q,$$

$$[n + 1]_q X^2 - \text{tr} BX + [n + 1]_q,$$

$$[n + 1]_q X^2 - \text{tr} CX + [n + 1]_q,$$

respectively.

**5. Applications**

On the finite-dimensional AW-module constructed by Zhedanov in [43, Section 2], the elements $K_0, K_1$ act like a Leonard pair but in a less mathematically rigorous treatment. Motivated by the $P$- and $Q$-polynomial association schemes [3, Section 3.5], Terwilliger independently introduced Leonard pairs [35, Definition 1.1]. The classification of Leonard pairs [35, Theorem 1.9] can be considered to be a linear algebraic version of Leonard theorem [26], which gave a characterization of the $q$-Racah and related polynomials in the Askey scheme [23, Chapter 3]. This led to the name of Leonard
pairs. Afterward the notion of Leonard pairs was naturally extended to Leonard triples by Curtin in [7, Definition 1.2].

For the rest assume that $q$ is not a root of unity. To study the finite-dimensional irreducible $\Delta$-modules, it is now enough to consider the $\Delta$-module $V_n(a, b, c)$ for $(a, b, c) \in T$ by Theorem 4.7. In Sect. 5.1 we give the equivalent conditions for $A, B, C$ on $V_n(a, b, c)$ as Leonard pairs or a Leonard triple. In Sect. 5.2 we discuss the sufficient conditions for $V_n(a, b, c)$ to be unitary under the constraint of $A, B, C$ as a Leonard triple. In Sect. 5.3 we apply Theorem 4.7 to classify the finite-dimensional irreducible $U_q(sl_2)$-modules and compare the result with [16, Theorem 4]. In [1] Al-najjar described how to obtain the Leonard pairs of $q$-Racah and other $q$-types from $U_q(sl_2)$-modules. Improving the result Terwilliger gave an $F$-algebra homomorphism $\Delta \to U_q(sl_2)$ below [39, Proposition 1.1]. The purpose of Sect. 5.4 is to determine how many $U_q(sl_2)$-modules on $V_n(a, b, c)$ give the $\Delta$-module $V_n(a, b, c)$ by pulling back via the homomorphism. The work [13] of Granovski˘ı and Zhedanov showed how the Racah coefficients of $su_q(2)$ are relevant to the AW-modules. The idea was recently extended to $sl_{-1}(2)$ and $osp_q(1|2)$ in [9, 11] respectively. Inspired by the performances, the end of Sect. 5.5 is to display an $F$-algebra homomorphism $\Delta \to U_q(sl_2) \otimes U_q(sl_2) \otimes U_q(sl_2)$ and explain its connection to the Racah coefficients of $U_q(sl_2)$.

5.1. Leonard pairs and Leonard triples. A tridiagonal matrix is said to be irreducible if each entry on the subdiagonal and superdiagonal is nonzero. Let $V$ denote a nonzero finite-dimensional vector space. A pair (resp. triple) of linear transformations on $V$ is called a Leonard pair (resp. triple) whenever for each of the two (resp. three) transformations, there exists a basis of $V$ with respect to which the matrix representing that transformation is diagonal and the matrices representing the other transformations are irreducible tridiagonal.

Lemma 5.1. For $(a, b, c) \in T$ the following are equivalent:

(i) $A$ (resp. $B$) (resp. $C$) is diagonalizable on $V_n(a, b, c)$.
(ii) $\theta_i(q; a)$ (resp. $\theta_i(q; b)$) (resp. $\theta_i(q; c)$) for all $0 \leq i \leq n$ are pairwise distinct.
(iii) $a^2$ (resp. $b^2$) (resp. $c^2$) is not among $q^{2n-2}, q^{2n-4}, \ldots, q^{2-2n}$.

Proof. (i) $\iff$ (ii) follows from Lemma 4.6. (ii) $\iff$ (iii) follows from Lemma 4.9. \Box

Theorem 5.2. For $(a, b, c) \in T$ the following are equivalent:

(i) $A, B$ (resp. $A, C$) (resp. $B, C$) act on $V_n(a, b, c)$ as a Leonard pair.
(ii) $A, B$ (resp. $A, C$) (resp. $B, C$) are diagonalizable on $V_n(a, b, c)$.
(iii) Neither of $a^2, b^2$ (resp. $a^2, c^2$) (resp. $b^2, c^2$) is among $q^{2n-2}, q^{2n-4}, \ldots, q^{2-2n}$.

Proof. (ii) $\iff$ (iii) follows from Lemma 5.1. By Proposition 4.5 the matrices representing $B, C$ with respect to the basis $\{v^{(1, 1, \sigma\tau)}_i\}_{i=0}^n$ of $V_n(a, b, c)$ are $L(q; b), U(q; b, c, a)$ respectively. By [18, Lemma 7.3] and [37, Theorem 7.2], (i) $\iff$ (iii) holds for the pair $B, C$. By similar arguments (i) $\iff$ (iii) holds for the pairs $A, B$ and $A, C$. \Box

As a consequence of [18, Theorem 14.5] and Theorem 5.2 we have

Theorem 5.3. For $(a, b, c) \in T$ the following are equivalent:

(i) $A, B, C$ act on $V_n(a, b, c)$ as a Leonard triple.
(ii) Any two of $A, B, C$ act on $V_n(a, b, c)$ as a Leonard pair.
(iii) $A, B, C$ are diagonalizable on $V_n(a, b, c)$.
(iv) None of $a^2, b^2, c^2$ is among $q^{2n-2}, q^{2n-4}, \ldots, q^{2-2n}$. 


5.2. The unitary structure on $V_n(a,b,c)$. In this section we assume that there is an involution $*: F \to F$. Recall that given an $F$-vector space $V$ a map $(,) : V \times V \to F$ is called a $*$-form if

$$(u + v, w) = (u, w) + (v, w), \quad (u, v + w) = (u, v) + (u, w),$$

$$(\mu u, v) = \mu (u, v), \quad (u, \mu v) = \mu^*(u, v)$$

for all $u, v, w \in V$ and $\mu \in F$. A $*$-form $(,) : V \times V \to F$ is said to be degenerate if there exists a nonzero vector $u \in V$ such that $(u, v) = 0$ for all $v \in V$ or $(v, u) = 0$ for all $v \in V$. For example the Hermitian form is a nondegenerate $*$-form over the complex number field with $*$ as complex conjugation. Given an $F$-algebra $A$, a $*$-involution $\dagger$ on $A$ means a map $\dagger : A \to A$ satisfying

$$(R + S)\dagger = R\dagger + S\dagger, \quad (RS)\dagger = S\dagger R\dagger, \quad 1\dagger = 1,$$

$$(\mu R)\dagger = \mu^* R\dagger, \quad (R^\dagger)\dagger = R$$

for all $R, S \in A$ and all $\mu \in F$. Given an $F$-algebra $A$ endowed with a $*$-involution $\dagger$, an $A$-module $V$ is said to be $\dagger$-unitary or unitary [34, Definition 2.3.1] if there exists a nondegenerate $*$-form $(,)$ with

$$(Su, v) = (u, S^\dagger v) \quad \text{for all } u, v \in V \text{ and } S \in A.$$

Under some additional hypotheses on $q, a, b, c$ we have a unitary structure on the irreducible $\triangle$-module $V_n(a,b,c)$ with two of $A, B, C$ as a Leonard pair. Without loss of generality say the action of $A, B$ on $V_n(a,b,c)$ as a Leonard pair. Under the assumption $q^* = q^{-1}$, Lemma 2.4 with $g = \tau$ implies that there is a unique $*$-involution $\dagger : \triangle \to \triangle$ that maps

$$(A, B, C, \alpha, \beta, \gamma) \mapsto (B, A, C^\vee, \beta, \alpha, \gamma).$$

For each $0 \leq i \leq n$, let $u_i$ and $v_i$ denote the eigenvectors of $A, B$ on $V_n(a,b,c)$ with respect to $\theta_i(q; a)$ and $\theta_i(q; b)$ respectively. By Theorem 5.2, $\{u_i\}_{i=0}^n$ and $\{v_i\}_{i=0}^n$ are two bases of $V_n(a,b,c)$. If $a^* = b^{-1}$ and $c^* \in \{c, c^{-1}\}$ then it is routine to check that $V_n(a,b,c)$ is $\dagger$-unitary with respect to the nondegenerate $*$-form $(,) \text{ defined by}$

$$(u_i, v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

By virtue of Theorem 4.7 we can conclude that the irreducible $\triangle$-module $V_n(a,b,c)$ with $A, B, C$ as a Leonard triple admits a unitary structure, provided that $q^* = q^{-1}$ and one of the following holds:

- $a^* \in \{b, b^{-1}\}$ and $c^* \in \{c, c^{-1}\}$.
- $b^* \in \{c, c^{-1}\}$ and $a^* \in \{a, a^{-1}\}$.
- $c^* \in \{a, a^{-1}\}$ and $b^* \in \{b, b^{-1}\}$.
5.3. A feedback to $U_q'(\mathfrak{so}_3)$-modules. Recall from Introduction that $U_q'(\mathfrak{so}_3)$ is a long-studied example of $\mathbb{AW}$ with the defining relations
\[
q K_1 K_2 - q^{-1} K_2 K_1 = K_0, \\
q K_2 K_0 - q^{-1} K_0 K_2 = K_1, \\
q K_0 K_1 - q^{-1} K_1 K_0 = K_2.
\]

The quantum group $U_q'(\mathfrak{so}_3)$ is not the Drinfeld–Jimbo type but plays the crucial roles in the nuclear spectroscopy [15], the quantum Laplace operator [19,30], the $(2+1)$-dimensional quantum gravity [28,29] and so on. In this section we restrict Theorem 4.7 of the finite-dimensional irreducible

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Theorem 4] of the finite-dimensional irreducible $U_q'(\mathfrak{so}_3)$-modules.

By the universal property of $\triangle$ there is a unique $\mathbb{F}$-algebra homomorphism $\triangle \rightarrow U_q'(\mathfrak{so}_3)$ that sends
\[
(A, B, C, \alpha, \beta, \gamma) \mapsto \left( (q^{-2} - q^2) K_0, (q^{-2} - q^2) K_1, (q^{-2} - q^2) K_2, 0, 0, 0 \right).
\]
Assume that $V$ is an $(n+1)$-dimensional irreducible $U_q'(\mathfrak{so}_3)$-modules. The pullback of $V$ via the above homomorphism gives an irreducible $\triangle$-module structure on $V$. By Theorem 4.7 there exists a unique $\{a, b, c\} \in \mathbb{T}/(\pm 1)^3$ such that $V$ is isomorphic to $V_n(a, b, c)$ as a $\triangle$-module. Solving for $a, b, c$ with the vanishment of $\alpha, \beta$, $\gamma$ on $V_n(a, b, c)$ it yields that two possible cases:

- $a^2 = b^2 = c^2 = -1$.
- $a + a^{-1} = -\varepsilon_0(q^{n+1} + q^{-n-1}), b + b^{-1} = -\varepsilon_1(q^{n+1} + q^{-n-1}), c + c^{-1} = -\varepsilon_2(q^{n+1} + q^{-n-1})$ where $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ with $\varepsilon_i = \varepsilon_{i-1} \varepsilon_{i+1}$ for all $i \in \mathbb{Z}/3\mathbb{Z}$.

In terminology of [16, Theorem 4], Corollary 4.10 implies that the two cases correspond to the classical and nonclassical irreducible $U_q'(\mathfrak{so}_3)$-modules respectively. By Theorem 5.3 the action of $K_0, K_1, K_2$ on each finite-dimensional irreducible $U_q'(\mathfrak{so}_3)$-module forms a Leonard triple. By the result of Sect. 5.2 the finite-dimensional irreducible $U_q'(\mathfrak{so}_3)$-modules are unitary when $q^* = q^{-1}$.

5.4. A connection to $U_q(\mathfrak{sl}_2)$-modules. The quantum group $U_q(\mathfrak{sl}_2)$ is an associative unital $\mathbb{F}$-algebra generated by $e, f, k^{\pm 1}$ subject to the relations
\[
kk^{-1} = k^{-1} k = 1, \\
ke = q^2 ek, \quad kf = q^{-2} fk, \\
enf - fne = \frac{k - k^{-1}}{q - q^{-1}}.
\]

We review a classification of the finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules from [22, Theorem 2.6]. For each $\varepsilon \in \{\pm 1\}$ there exists an $(n+1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module $V_n, \varepsilon$ that has a basis with respect to which the matrices representing $e, f, k$ are superdiagonal, subdiagonal, diagonal respectively with
\[
e_{i-1, i} = \varepsilon[n - i + 1]_q \quad (1 \leq i \leq n),
\]
\[
f_{i, i-1} = [i]_q \quad (1 \leq i \leq n),
\]
\[
k_{ii} = \varepsilon q^{n-2i} \quad (0 \leq i \leq n).
\]
Every \((n+1)\)-dimensional irreducible \(U_q(\mathfrak{sl}_2)\)-module is shown to be isomorphic to \(V_{n,1}\) or \(V_{n,-1}\). If \(\text{char } \mathbb{F} = 2\) the \(U_q(\mathfrak{sl}_2)\)-modules \(V_{n,1}\) and \(V_{n,-1}\) are isomorphic. In what follows, the parameter \(\varepsilon\) will be called the type of \(V_{n,\varepsilon}\) and the above basis of \(V_{n,\varepsilon}\) will be said to be canonical. Observe that the Casimir element
\[
\Lambda = ef + \frac{q^{-1}k + qk^{-1}}{(q - q^{-1})^2}
\]
acts on \(V_{n,\varepsilon}\) as the scalar
\[
\varepsilon \frac{q^{n+1} + q^{-n-1}}{(q - q^{-1})^2}.
\]
It follows that

**Lemma 5.4.** Let \(V, W\) denote two finite-dimensional irreducible \(U_q(\mathfrak{sl}_2)\)-modules. If \(\Lambda\) acts on \(V\) and \(W\) as the same scalar then \(V\) and \(W\) are isomorphic.

The elements
\[
x = k^{-1} - q^{-1}(q - q^{-1})ek^{-1}, \quad y^{\pm 1} = k^{\pm 1}, \quad z = k^{-1} + (q - q^{-1})f
\]
form a set of generators of \(U_q(\mathfrak{sl}_2)\), which are called the equitable generators of \(U_q(\mathfrak{sl}_2)\) [21, Definition 2.2]. The matrices representing \(x, y, z\) with respect to the canonical basis of \(V_{n,\varepsilon}\) are upper bidiagonal, diagonal, lower bidiagonal respectively with
\[
x_{ii} = \varepsilon q^{2i-n} \quad (0 \leq i \leq n), \quad x_{i-1,i} = q^{2i-n-1}(q^{i-n-1} - q^{n-i+1}) \quad (1 \leq i \leq n),
\]
\[
y_{ii} = \varepsilon q^{n-2i} \quad (0 \leq i \leq n),
\]
\[
z_{ii} = \varepsilon q^{2i-n} \quad (0 \leq i \leq n), \quad z_{i,i-1} = q^i - q^{-i} \quad (1 \leq i \leq n).
\]
Each of \(x, y, z\) is diagonalizable on \(V_{n,\varepsilon}\) with pairwise distinct eigenvalues \(\varepsilon q^{n-2i}\) for all \(0 \leq i \leq n\). By [21, Theorem 7.5] or applying Lemma 5.4 we see that

**Lemma 5.5.** For each finite-dimensional irreducible \(U_q(\mathfrak{sl}_2)\)-module \(V\), there exists an invertible linear transformation \(L\) on \(V\) such that
\[
L^{-1}xL = y, \quad L^{-1}yL = z, \quad L^{-1}zL = x.
\]

We are ready to prove the main result of this section.

**Theorem 5.6.** For each \((a, b, c) \in T\) there are exactly
\[
h = \begin{cases} 
2 & \text{if char } \mathbb{F} \neq 2 \text{ and } a^2 = b^2 = c^2 = -1, \\ 
1 & \text{otherwise} 
\end{cases}
\]
distinct \(U_q(\mathfrak{sl}_2)\)-modules on \(V_n(a, b, c)\) satisfying
\[
A = ax + a^{-1}y + bc^{-1} \frac{xy - yx}{q - q^{-1}}, \quad (33)
\]
\[
B = by + b^{-1}z + ca^{-1} \frac{yz - zy}{q - q^{-1}}, \quad (34)
\]
\[
C = cz + c^{-1}x + ab^{-1} \frac{zx - xz}{q - q^{-1}}. \quad (35)
\]
Moreover these \(U_q(\mathfrak{sl}_2)\)-modules are irreducible, one of which is of type 1 and if \(h = 2\) then the other one is of type \(-1\).
Proof. The irreducibility of $V_n(a, b, c)$ forces that the $U_q(\mathfrak{sl}_2)$-modules on $V_n(a, b, c)$ with (33)–(35) are irreducible. We first show that there exists a unique irreducible $U_q(\mathfrak{sl}_2)$-module on $V_n(a, b, c)$ of type 1 satisfying (33)–(35).

(Existence): Let

$$c_i = \prod_{h=1}^{i} (q^h - q^{-h})(b^{-1} - q^{n-2h+1}a^{-1}c) \quad (0 \leq i \leq n).$$

Since $q$ is not a root of unity and $(a, b, c) \in \mathbb{T}$ the scalars $c_i$ are nonzero for all $0 \leq i \leq n$. Define an $U_q(\mathfrak{sl}_2)$-module $V$ on $V_n(a, b, c)$ such that the action of $x, y, z$ on $\{c_i^{-1}v_i^{(1,-1,\tau)}\}_{i=0}^{n}$ is the same as that on the canonical basis of $V_{n,1}$. A direct calculation yields that $V$ satisfies (33)–(35). This shows the existence.

(Uniqueness): Suppose that $V$ is any irreducible $U_q(\mathfrak{sl}_2)$-module on $V_n(a, b, c)$ of type 1 with (33)–(35). To see the uniqueness, it is enough to show that the $q^{n-2i}$-eigenspaces of $x, y, z$ on $V$ for $0 \leq i \leq n$ are only determined by $V_n(a, b, c)$. Let $\{v_i\}_{i=0}^{n}$ denote the canonical basis of $V$. Then $Fv_i (0 \leq i \leq n)$ is the $q^{n-2i}$-eigenspace of $y$ on $V$. The action of $A, B, C$ on $\{c_i v_i^n\}_{i=0}^{n}$ is the same as that on $\{v_i^{(1,-1,\tau)}\}_{i=0}^{n}$. By Proposition 4.5, $Fv_i = Fv_i^{(1,-1,\tau)}$ for each $0 \leq i \leq n$. Let $L$ denote the linear transformation on $V$ from Lemma 5.5. Then $FLv_i$ and $FL^{-1}v_i (0 \leq i \leq n)$ are the $q^{n-2i}$-eigenspaces of $x, z$ on $V$ respectively. By symmetry $FLv_i = Fv_i^{(-1,1,\sigma)}$ and $FL^{-1}v_i = Fv_i^{(-1,,-1,\sigma \tau \sigma)}$ for each $0 \leq i \leq n$. The uniqueness follows.

Now assume that $\text{char} \mathbb{F} \neq 2$. On any $U_q(\mathfrak{sl}_2)$-modules on $V_n(a, b, c)$ of type 1 with (33)–(35), the traces of $A, B, C$ are equal to $-\lfloor n+1 \rfloor q(a+a^{-1})$, $-\lfloor n+1 \rfloor q(b+b^{-1})$, $-\lfloor n+1 \rfloor q(c+c^{-1})$ respectively. By Lemma 4.3(ii) these $U_q(\mathfrak{sl}_2)$-modules exist only if $a^2 = b^2 = c^2 = -1$. Conversely, if $a^2 = b^2 = c^2 = -1$ then a similar argument shows that there exists a unique $U_q(\mathfrak{sl}_2)$-module of type 1 on $V_n(a, b, c)$ with (33)–(35). □

5.5. An application to the Racah coefficients of $U_q(\mathfrak{sl}_2)$. Recall that $U_q(\mathfrak{sl}_2)$ has a coalgebra structure with comultiplication $\Delta : U_q(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ given by

$$\Delta(e) = e \otimes 1 + k \otimes e,$$

$$\Delta(f) = f \otimes k^{-1} + 1 \otimes f,$$

$$\Delta(k) = k \otimes k.$$

Denote by $\Delta_1 = \Delta$ and recurrently define

$$\Delta_{n+1} = (\Delta \otimes 1 \otimes 1 \otimes \cdots \otimes 1) \circ \Delta_n \quad (1 \text{ appearing } n \text{ times})$$

for all $n \in \mathbb{N}^*$. Therefore any $U_q(\mathfrak{sl}_2)^{\otimes n}$-module ($n \geq 2$) can be treated as a $U_q(\mathfrak{sl}_2)$-module by pulling back via $\Delta_{n-1}$.

Roughly speaking, the Racah coefficients of $U_q(\mathfrak{sl}_2)$ are used to describe the change of two natural bases of the $U_q(\mathfrak{sl}_2)$-module $U \otimes V \otimes W$ for any finite-dimensional $U_q(\mathfrak{sl}_2)$-modules $U, V, W$. By [22, Theorem 2.9] the finite-dimensional $U_q(\mathfrak{sl}_2)$-modules are completely reducible provided that $\text{char} \mathbb{F} \neq 2$. For any $n \in \mathbb{N}$ and $\varepsilon \in \{\pm 1\}$ the irreducible $U_q(\mathfrak{sl}_2)$-module $V_{n,\varepsilon}$ is isomorphic to $V_{0,0} \otimes V_{n,1}$ and $V_{n,1} \otimes V_{0,\varepsilon}$. Thus it is enough to work on the $U_q(\mathfrak{sl}_2)$-modules $V_{n,1}$ in general. Henceforth we denote by
$V_n = V_{n,1}$ for simplicity. The Clebsch–Gordan formula [22, Section 5A.8] decomposes $V_m \otimes V_n \ (m, n \in \mathbb{N})$ into

$$
\bigoplus_{i=0}^{\min\{m,n\}} V_{m+n-2i}.
$$

As an application of the Clebsch–Gordan formula we obtain two bases of the $U_q(\mathfrak{sl}_2)$-module

$$V_m \otimes V_n \otimes V_p \ (m, n, p \in \mathbb{N}).$$

The first one comprises the canonical bases of the irreducible components of $V_{m+n-2i} \otimes V_p$ for all $0 \leq i \leq \min\{m,n\}$. The second one is obtained by the same procedure beginning with the decomposition of $V_n \otimes V_p$. Denote by $\{u_i\}_{i=0}^N, \{v_i\}_{i=0}^N$ the two bases of $V_m \otimes V_n \otimes V_p$ respectively, where $N = (m+1)(n+1)(p+1) - 1$. The Racah coefficients of $U_q(\mathfrak{sl}_2)$ are defined to be the entries of the transition matrix from $\{u_i\}_{i=0}^N$ to $\{v_i\}_{i=0}^N$. The so-called Racah problem for $U_q(\mathfrak{sl}_2)$ is to find an explicit expression for Racah coefficients of $U_q(\mathfrak{sl}_2)$.

Inspired by the works [9,11,13] on the Racah coefficients of $\mathfrak{sl}_{-1}(2), \mathfrak{osp}_q(1|2)$ and $\mathfrak{su}_q(2)$, one may predict that

**Theorem 5.7.** There exists a unique $\mathbb{F}$-algebra homomorphism $\triangle \rightarrow U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ that sends

$$
\begin{align*}
\frac{A}{(q - q^{-1})^2} & \mapsto \Delta(\Lambda) \otimes 1, \\
\frac{B}{(q - q^{-1})^2} & \mapsto 1 \otimes \Delta(\Lambda), \\
\frac{\gamma}{(q - q^{-1})^4} & \mapsto \Lambda \otimes 1 \otimes \Lambda + (1 \otimes \Lambda \otimes 1) \cdot \Delta_2(\Lambda).
\end{align*}
$$

Pulling back via the homomorphism shown in Theorem 5.7 the $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$-module $V_m \otimes V_n \otimes V_p$ admits a $\triangle$-module structure. By Lemma 5.4 the bases $\{u_i\}_{i=0}^N, \{v_i\}_{i=0}^N$ are the eigenbases of $A, B$ on $V_m \otimes V_n \otimes V_p$ respectively. Applying the relations (10), (11) it follows that $A, B$ act on $\{u_i\}_{i=0}^N, \{v_i\}_{i=0}^N$ in tridiagonal fashions respectively. Moreover the $\triangle$-module $V_m \otimes V_n \otimes V_p$ is completely reducible and $A, B$ act on each irreducible component as a Leonard pair. All of the details will be covered in a future paper.

As a result, the Racah problem for $U_q(\mathfrak{sl}_2)$ can be expanded to determine the transition matrices between the two eigenbases of an arbitrary Leonard pair. A solution in terms of hypergeometric series and their $q$-analogues can be found in [36].

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Communicated by Y. Kawahigashi