Collapse of topological texture

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We study analytically the process of a topological texture collapse in the approximation of a scaling ansatz in the nonlinear sigma-model. In this approximation we show that in flat space-time topological texture eventually collapses while in the case of spatially flat expanding universe its fate depends on the rate of expansion. If the universe is inflationary, then there is a possibility that texture will expand eternally; in the case of exponential inflation the texture may also shrink or expand eternally to a finite limiting size, although this behavior is degenerate. In the case of power law noninflationary expansion topological texture eventually collapses. In a cold matter dominated universe we find that texture which is formed comoving with the universe expansion starts collapsing when its spatial size becomes comparable to the Hubble size, which result is in agreement with the previous considerations. In the nonlinear sigma-model approximation we consider also the final stage of the collapsing ellipsoidal topological texture. We show that during collapse of such a texture at least two of its principal dimensions shrink to zero in a similar way, so that their ratio remains finite. The third dimension may remain finite (collapse of \textit{cigar} type), or it may also shrink to zero similar to the other two dimensions (collapse of \textit{scaling} type), or shrink to zero similar to the product of the remaining two dimensions (collapse of \textit{pancake} type).

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I. INTRODUCTION

Global textures are scalar field configurations that arise in theories in which global symmetry $G$ is spontaneously broken to $H$ in such a way that the vacuum manifold $\mathcal{M} = G/H$ has nontrivial third homotopy group $\pi_3(\mathcal{M})$. Unlike in the case of lower dimensional defects—walls, strings, and monopoles—in a texture the scalar field can remain close to the vacuum manifold everywhere: textures do not possess cores. At the same time, texture configurations can be topologically nontrivial (in this case they are called topological textures). Such topological nontriviality guarantees that in the process of evolution texture eventually collapses [1,2]: there arises a shrinking spatial region in which its energy is concentrated. The details of such a phenomenon are discussed in the paper below.

Texture collapse events can set up the initial conditions for the structure formation in the early universe [3]. This is one of the reasons why thorough study of such a process is of importance. The evolution of texture configurations has been studied numerically in [4–6] and analytically in [7,8]. The main issue that was under investigation in these papers is the fate of texture depending on the so-called winding number $w$, which is defined as the fraction of the vacuum manifold spanned by the texture configuration. Typical result was that under each of the specific circumstances (flat space-time or expanding universe, spherical or nonspherical texture configuration, etc.) there exists a critical value $w_c < 1$ such that textures with $w > w_c$ collapse, while those with $w < w_c$ dissipate. In more recent paper [9] the authors investigated numerically the influence of the geometry and topology of the vacuum manifold on the dynamics of texture collapse. The dynamics of gravitational field during texture collapse were considered, for example, in [10,11].

In this paper we shall study analytically the case of a topological texture, for which $w$ is an integer, in a fixed background of flat space-time as well as in the expanding universe. Our goal will be to determine the conditions under which such a texture eventually collapses,
and to describe the possible regimes of collapse. Thus our investigation may be regarded as complementary to the numerical simulations of [4–6]. Throughout this paper we shall use a simple approximation. We shall remain in frames of the nonlinear sigma-model [described by the Lagrangian (3) below], and adopt a scaling ansatz that will reduce the complicated field evolution to the evolution of finite number of parameters. We hope that this approximation reflects the basic features of interest which we would like to reveal. First of all, in the preliminary Sec. II we describe generation of cosmic texture during the symmetry breaking. In Sec. III we describe topological texture and show that in flat space-time its collapse is inevitable. In Sec. IV we study texture collapse in a spatially flat expanding universe and investigate the conditions under which collapse takes place. In Sec. V we study the case of an ellipsoidal texture configuration and investigate possible spatial shape of the collapse region in the vicinity of the moment of collapse. In Sec. VI we summarize our results.

II. GENERATION OF COSMIC TEXTURE

Consider a theory of the scalar fields $\phi = \{\phi^A, A = 1, \ldots, N\}$ with the Lagrangian of type

$$L = \frac{1}{2} \sum_{A=1}^{N} (\partial \phi^A)^2 - V(\phi),$$

in which the potential $V(\phi)$ depends only on the value of $\phi^2 = \sum_A (\phi^A)^2$ and has a global minimum at $\phi^2 = \eta^2$. As an example one may consider

$$V(\phi) = \frac{g}{4} (\phi^2 - \eta^2)^2.$$  

The theory is invariant under the group $O(N)$ of transformations of the fields $\phi$. The manifold of vacuum values of $\phi$ in this case has topology of $(N-1)$-dimensional sphere. As is well known [2] (see also [12–14]), during thermal phase transition in which the symmetry becomes broken to $O(N-1)$ in the theory (1) there arise topological objects: if $N = 1$ these...
are domain walls, if \( N = 2 \)—(global) strings, if \( N = 3 \)—(global) monopoles, and, finally, if \( N = 4 \)—the so-called (global) textures. These latter objects, textures, will be the subject of our investigation. The above example Lagrangian \((\|)\) describes the situation of symmetry \( O(4) \) spontaneously breaking to \( O(3) \). However, the results of this and the following two sections will apply to a more general case of symmetry \( G \) breaking to \( H \) provided the third homotopy group of the vacuum manifold \( \mathcal{M} = G/H \) is nontrivial. In Sec. \( \Box \) we will use a spherically symmetric topological texture configuration, and the condition of existence of such configurations further restricts possible vacuum manifolds. For example, the theory with \( \mathcal{M} = S^3 \) will allow for spherically symmetric configurations, while the theory with \( \mathcal{M} = S^2 \) will not (see \([9]\) for details).

Thermal phase transitions take place in the theory of hot expanding universe. As a result of phase transition the scalar fields \( \phi \) acquire their values close to the vacuum manifold, in our example it is a three-sphere. Because during phase transition spatial regions that are sufficiently remote from each other do not have enough time to exchange signals the values of the scalar fields in such regions will be uncorrelated. Thus as a result of phase transition there arises a field configuration with the values of \( \phi(x) \) lying close to the vacuum manifold \([\phi^2(x) \text{ close to } \eta^2]\) but depending on the spatial point \( x \) and changing on the characteristic spatial scale \( \xi \simeq t_{ph} \), where \( t_{ph} \) is the characteristic time of the phase transition (the units are chosen in which the speed of light is equal to unity). At every spatial point \( x \) the scalar fields tend to oscillate around the closest vacuum value due to curvature of the potential \( V(\phi) \) in the “radial” direction in the space of \( \phi \). These oscillations fade away rather rapidly as a result of the universe expansion as well as due to excitation of the quanta of the fields \( \phi \) and/or of the other fields that interact with the fields \( \phi \). In what follows we shall not be interested in such oscillations, but turn to the space-time evolution of the vacuum values, which we denote as \( \phi_v(x,t) \), closest (in the vacuum manifold) to the values of \( \phi(x,t) \).

Since the fields \( \phi_v(x,t) \) acquire values in the vacuum manifold, a three-sphere in our
example, it is convenient to introduce arbitrary coordinates \( \varphi = \{ \varphi^a, \ a = 1, \ldots, 3 \} \) on this manifold and to describe the evolution of interest in terms of the fields \( \varphi(x, t) \). From Eq. (2) it then follows that the dynamics of the fields \( \varphi(x, t) \) is described by the Lagrangian of type

\[
L_\ast = \frac{1}{2} \sum_{a=1}^{3} G_{ab}(\varphi) (\partial_{\mu} \varphi^a) (\partial_{\nu} \varphi^b) g^{\mu\nu},
\]

where \( G_{ab}(\varphi) \) is the metric components of the vacuum manifold in the coordinates \( \varphi \), and \( g^{\mu\nu} \) is the inverse of the space-time metric tensor. The Lagrangian (3) is that of a nonlinear sigma-model. In what follows we shall neglect the back-reaction of the scalar fields on the space-time metric and regard the latter as fixed.

### III. TEXTURE COLLAPSE IN FLAT SPACE-TIME

To begin with, we consider the evolution of cosmic texture, i.e. of a configuration of the fields \( \varphi \), in flat space-time. Let the field configuration be of finite total energy and, moreover, let there exist a finite limit of \( \varphi \) along any ray that goes to infinity in the ordinary space. From finiteness of the energy it then follows

\[
\lim_{x \to \infty} \varphi(x) = \varphi_0 = \text{const}.
\]

We add infinitely remote point \( x_0 \) to the ordinary space and endow the space thus extended with the topology of a three-sphere. In this topology the map \( \varphi(x) \), defined also at the point \( x_0 \) as \( \varphi(x_0) = \varphi_0 \), will be continuous. Being a continuous map of a three-sphere (the extended ordinary space) into a three-sphere (the vacuum manifold) it can belong to a nontrivial homotopy class. If this happens the map \( \varphi(x) \) cannot be continuously deformed to become a constant map. This feature will have interesting consequences.

The energy of a field configuration is given by the expression

\[
E = \frac{1}{2} \int \sum_{ab} G_{ab}(\varphi) \left( \dot{\varphi}^a \dot{\varphi}^b + \sum_i \nabla_i \varphi^a \nabla_i \varphi^b \right) d^3x,
\]
in which the first part that depends on the time derivatives can be called kinetic energy, and the second part—gradient energy. In the process of evolution the total energy is conserved. Given a field configuration \( \varphi(x) \) in space the total energy will be minimized if the time derivative \( \dot{\varphi} \) is zero. Consider any of the homotopy classes of configurations \( \varphi(x) \). It is easy to see that configurations from this class can acquire arbitrarily small energy \([1]\). Indeed, let \( \varphi(x) \) be an arbitrary configuration from the homotopy class considered. Putting \( \dot{\varphi} \equiv 0 \) we minimize the energy for the configuration given. Then changing the configuration \textit{continuously} by the scaling transformation

\[
\varphi_{\lambda}(x) = \varphi(x/\lambda), \quad \lambda > 0, \tag{6}
\]

which depends on the parameter \( \lambda \), we can see from (5), that its gradient energy \( \Pi \) changes as

\[
\Pi_{\lambda} = \lambda \Pi, \tag{7}
\]

and acquires any positive value.

During evolution of a field configuration with arbitrary initial conditions \( \{\varphi(x), \dot{\varphi}(x)\} \) its gradient energy changes with time. From Eq. (7) it is clear that its values can in principle approach zero arbitrarily closely. When \( \Pi \) is close to zero then all over the space, with an exception of a bounded region, the field values approach a constant \( \varphi_0 \). However, if the initial configuration is topologically nontrivial then the map \( \varphi(x,t) \) at any moment of time must cover the whole three-sphere vacuum manifold. This means that in the spatial region in which the fields \( \phi_* \) change substantially (we are speaking now in terms of original fields \( \phi \)) their characteristic change is at least of order \( \eta \). If by \( R \) we denote the characteristic linear dimension of the spatial region of variation of the fields \( \phi_* \) then the value of the gradient energy

\[
\Pi \sim \eta^2 R \tag{8}
\]
decreases with the decrease of $R$. Thus energetically it is favourable for the value $R$ to decrease. At the same time the characteristic density $\rho_\Pi$ of the gradient energy in the region of field variation behaves as

$$\rho_\Pi \sim \left( \frac{\eta}{R} \right)^2,$$

and grows infinitely with the decrease of $R$. Such a situation, when the value of $R$ decreases in time and the gradient energy density increases in the region of field variation, is characterized as collapse of cosmic texture.

Now becomes clear the role which in our reasoning is played by the topological nontriviality of the initial field configuration. Were this configuration topologically trivial, then evolving continuously it could in principle approach a constant $\varphi_0$ uniformly in the space (the process which is sometimes called dissipation). Topological nontriviality guarantees that the map $\varphi(x,t)$ at every moment of time covers the whole three-sphere of the vacuum manifold. In such a case the gradient energy of the field configuration can decrease to zero only as a result of the collapse described above. If the texture is not topological, then it will collapse or it will dissipate without collapse depending to a large extent on the so-called winding number $w$ which is the fraction of the vacuum manifold covered by the texture. The corresponding analysis was performed in the papers [4–8].

In reality the energy density of a topological texture cannot increase unbounded. When the size of the collapse region becomes so small that the gradient energy density $\rho_\Pi$ is of the same order as the value $V(0)$ of the potential in its local maximum [we take this maximum to be at $\phi = 0$ as in the example (2)] the field dynamics becomes more complicated: the field values depart from the vacuum manifold rather far, in particular, they can acquire the values $\phi = 0$ and overcome the potential barrier that separates different homotopy classes.

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1 A topologically nontrivial field configuration is usually called *topological texture*. 

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The texture acquires the possibility to “unwind” itself. After this nothing prevents it from evolving uniformly to a constant $\phi_0$ (although before this happens several recollapses may occur, see [3]). From the above discussion it is clear that “unwinding” occurs when

$$\rho_\Pi \sim \left( \frac{\eta}{R} \right)^2 \sim V(0),$$

(10)

which for the potential (2) gives the estimate

$$R \sim \frac{1}{\eta \sqrt{g}}.$$  \hfill (11)

In order to describe qualitatively the dynamics of texture collapse in this work we shall adopt the scaling approximation. In the case of flat space-time we place the origin of the space-time coordinates at the centre of mass of the initial field configuration and choose the inertial coordinate system in which this centre is at rest. After that we consider the family of configurations (6) parametrized by $\lambda$. We shall take that the texture evolution proceeds within this family. Of course, this is not so in reality, but we hope this can serve as a reasonable approximation to the actual evolution. Note that an exact solution of the scaling type was found in the work [15] for the collapse of a spherically symmetric topological texture with infinite total energy.

For the family of configurations (6) the action as a functional of $\lambda(t)$ has the form

$$S[\lambda(t)] \equiv S[\varphi_{\lambda(t)}(x)] = \int \left( A\lambda^2 - B\lambda \right) dt,$$

(12)

where positive constants $A$ and $B$ are given by

$$A = \frac{1}{2} \int G_{ab}(\varphi)(x \cdot \nabla \varphi^a)(x \cdot \nabla \varphi^b) d^3 x,$$

(13)

$$B = \frac{1}{2} \int G_{ab}(\varphi)\nabla \varphi^a \cdot \nabla \varphi^b d^3 x.$$  \hfill (14)

Note that $\varphi = \varphi(x)$ is the initial field configuration (when $\lambda = 1$), and the dot denotes the scalar product in the three-dimensional euclidean metric. From the energy conservation law
\[ E = A\dot{\lambda}^2 + B\lambda = \text{const} \quad (15) \]

we get the equation

\[ \dot{\lambda} = \pm \sqrt{\frac{E - B\lambda}{A\lambda}}, \quad (16) \]

which can be integrated. If initially \( \dot{\lambda} < 0 \) then the texture starts shrinking immediately and eventually collapses. The case when initially \( \dot{\lambda} > 0 \) also leads to eventual collapse. Indeed, the value of \( \lambda \) cannot increase unbounded, and at the turning point \( \lambda = E/B \) we have \( \ddot{\lambda} = \frac{-B^2}{2AE} \neq 0 \) so that texture cannot eternally expand. Thus in the scaling approximation considered the texture eventually collapses under arbitrary initial conditions. The time of collapse from the turning point \( \lambda = E/B \) till \( \lambda = 0 \) is estimated as

\[ t_{\text{coll}} = \frac{E/B}{0} \int d\lambda \sqrt{\frac{A\lambda}{E - B\lambda}} = \frac{\pi E \sqrt{A}}{2B^{3/2}} \sim R_*, \quad (17) \]

where \( R_* \) is the spatial size of the texture at the turning point.

We end this section by the following remark. It is impossible that as a result of a phase transition in the infinite space there arises a field configuration of finite energy. Actually it is only the average energy density that will be finite. Nevertheless, in the process of evolution the field configuration tends to become constant in as large spatial regions as possible. Thus spatial islands may arise in which fields change substantially, surrounded by the sea of practically constant fields. It is such island regions to which our analysis will then be applicable.

**IV. TEXTURE COLLAPSE IN A SPATIALLY FLAT EXPANDING UNIVERSE**

As it was shown in the previous section in the scaling approximation, in flat space-time any topological texture eventually collapses. In an expanding universe texture collapse is not inevitable. In this section we investigate the conditions under which it takes place.
Consider evolution of a cosmic texture in a spatially flat expanding universe. As we did in the previous section, we shall use the scaling approximation, that is, we restrict texture configurations to the family (3) with \( \varphi(x) \) being the initial configuration. Moreover, in this section we shall regard the motion of the texture centre of mass with respect to the cosmological background to be negligible. Then the action as a functional of \( \lambda(t) \) will be

\[
S[\lambda(t)] = \int \left( Aa^3\lambda \dot{\lambda}^2 - B a \lambda \right) dt ,
\]

(18)

where \( a(t) \) is the universe scale factor, and the constants \( A \) and \( B \) are given respectively by (13) and (14).

Physical size of the texture is proportional to the product \( a\lambda \). We shall proceed therefore to a new variable

\[
\mu = a\lambda .
\]

(19)

The Lagrangian in terms of this variable acquires the form

\[
L = A\mu (\dot{\mu} - H\mu)^2 - B\mu ,
\]

(20)

and the equations of motion are

\[
2\mu \ddot{\mu} + \dot{\mu}^2 - \left( 2\dot{H} + 3H^2 \right) \mu^2 + b = 0 ,
\]

(21)

where \( H \equiv \dot{a}/a \) is the Hubble parameter, and \( b = B/A \). One can get rid of the first time derivative in Eq. (21) by making another change of variable

\[
\nu = \mu^{3/2} .
\]

(22)

For the value \( \nu(t) \) we then get the equation

\[
\frac{4}{3} \ddot{\nu} = f(t) \nu - b\nu^{-1/3} ,
\]

(23)

where
\[ f(t) \equiv 2\dot{H} + 3H^2. \] (24)

Consider one important class of the universe expansion laws

\[ a(t) \propto t^\alpha, \quad \alpha > 0, \] (25)

which is realized in the standard cosmological model. Thus \( \alpha = 1/2 \) if the universe is radiation dominated, and \( \alpha = 2/3 \) if it is dominated by nonrelativistic matter ("dust"). For this class we have

\[ f(t) = \frac{\alpha(3\alpha - 2)}{t^2} = \frac{\beta}{t^2}, \] (26)

in which we made notation \( \beta = \alpha(3\alpha - 2) \). If \( \alpha \leq 1 \) then the texture always eventually collapses, that is, \( \nu(t) \) acquires the value of zero in a finite time. Indeed, for \( \alpha \leq 2/3 \) we have \( f(t) \leq 0 \), and from Eq. (23) it follows immediately that the texture eventually collapses. Now let \( 2/3 < \alpha < 1 \), and let us assume that there exists solution \( \nu(t) \) which never becomes zero in the future. This solution should be bounded from above by the one which is obtained by dropping the second term from the right-hand side of Eq. (23) and imposing the same initial conditions. Thus at \( t \to \infty \) we should have \( \nu(t)t^{-3/2} \to 0 \). But for such a behavior of \( \nu(t) \) at sufficiently large values of \( t \) the second term in the right-hand side of (23) should become dominating, so that the texture expansion must be followed by its contraction and collapse, what contradicts the assumption made. Somewhat more lengthy analysis (see Appendix A) shows that texture collapse is inevitable also for \( \alpha = 1 \) (Milne’s cosmology).

If \( \alpha > 1 \) then also \( \beta > 1 \); in this case there exists solution

\[ \nu(t) = \left( \frac{\beta}{\beta - 1} \right)^{3/4} t^{3/2}, \] (27)

which corresponds to eternally expanding texture.

Consider now two important cases in which Eq. (21) can be easily integrated.
A. The case of \( a(t) \propto t^{2/3} \)

In the case of \( a(t) \propto t^{2/3} \) we have \( f = 2\dot{H} + 3H^2 \equiv 0 \). The equation (21) for \( \mu(t) \) in this case is easily integrated and the result can be put in a parametric form

\[
\mu = c\sqrt{b}(1 - \cos \tau),
\]

\[
t = c(\tau_0 + \tau - \sin \tau),
\]

where \( c \) and \( \tau_0 \) are the integration constants and \( \tau \) the parameter. The solution describes a cycloid. The beginning of texture collapse, i.e. the turning point, corresponds to \( \tau = \pi \). The physical values at this moment will be labelled by an asterisk. At this moment the ratio of the texture size \( R_\ast = r\mu_\ast \) to the Hubble radius \( H_*^{-1} = 3t_\ast/2 \) is

\[
R_\ast H_* = r\mu_* H_* = \frac{4r\sqrt{b}}{3(\pi + \tau_0)},
\]

where \( r \) is the coordinate size of the configuration \( \phi(x) \).

Let us estimate the value of \( b \). Using (13) and (14) we obtain

\[
A \sim \eta^2r^3, \quad B \sim \eta^2r,
\]

and

\[
b = \frac{B}{A} \sim r^{-2}.
\]

Thus for the value (30) we have

\[
R_\ast H_* \sim \frac{1}{\pi + \tau_0}.
\]

Let texture form at the moment of time \( t_0 \) with its characteristic size \( R_0 \) at this moment much larger than the Hubble size,

\[
R_0 H_0 \gg 1.
\]
At the moment of its generation let texture expand with the universe, that is, $\dot{\lambda}(t_0) = 0$, and, therefore,

$$\dot{\mu}(t_0) = H_0 \mu_0, \quad (35)$$

where the subscript “0” labels various quantities at the moment $t_0$. Then it is easy to see that the integration constant $\tau_0$ in (29) must be much smaller than unity, hence

$$R_* H_* \sim \frac{1}{\pi}. \quad (36)$$

This result implies that the texture starts collapsing when its linear dimension compares with the Hubble size, which is in agreement with the reasonable expectations (see [3]).

**B. The case** $a(t) \propto \exp(H t)$, $H \equiv \text{const}$

According to cosmological inflation scenarios [12] in the very early universe there might have been a period of (almost) exponential expansion during which

$$a(t) \propto \exp(H t), \quad H \approx \text{const}. \quad (37)$$

Textures also might form during this period as a result of spontaneous symmetry breaking. We shall investigate their possible subsequent evolution in the scaling approximation.

In the case of $H \equiv \text{const}$ the equation (21) can be integrated with the result

$$\dot{\mu} = \pm \sqrt{H^2 \mu^2 - b + C/\mu}, \quad (38)$$

where $C$ is the integration constant, and $b = B/A$ as before. Depending on the value of $C$ the cubic three-nom

$$H^2 \mu^3 - b \mu + C \quad (39)$$

can:
(i) have no positive roots; this case is realized if \( C > C_0 \);

(ii) have only one positive root; this case is realized if \( C = C_0 \) or \( C \leq 0 \);

(iii) have two positive roots: this case is realized if \( 0 < C < C_0 \).

The constant \( C_0 \) above is equal to

\[
C_0 = \frac{2}{H} \left( \frac{b}{3} \right)^{3/2}.
\]  

(40)

In the case (i) the texture expands unbounded or monotonically collapses depending on the sign in (38). In the case (iii) the cubic three-nom (33) has two positive roots \( \mu_1 \) and \( \mu_2 > \mu_1 \). In this case the region \( \mu_1 < \mu < \mu_2 \) of the values of \( \mu \) is forbidden. If initially \( 0 < \mu < \mu_1 \), then for the initial time derivative \( \dot{\mu} < 0 \) the texture collapses monotonically. If, however, initially \( \dot{\mu} > 0 \), then the texture first expands to the value of \( \mu = \mu_1 \), after which the derivative \( \dot{\mu} \) changes its sign and the texture collapses. The expansion to the point of \( \mu = \mu_1 \) proceeds in a finite time since in the vicinity of this point we have \( \dot{\mu} \propto \sqrt{\mu_1 - \mu} \).

The case \( \mu > \mu_2 \) is analysed in a similar way. In this case the texture eventually expands unbounded independently of the sign of the initial time derivative \( \dot{\mu} \). The case (ii) for \( C \leq 0 \) is totally analogous to the case (iii) in the region \( \mu > \mu_2 \). In a particular case \( C = 0 \) the solution is expressed through elementary functions

\[
\mu(t) = \frac{\sqrt{b}}{H} \text{ch} \left[ H(t - t_0) \right].
\]  

(41)

Consider the remaining case of \( C = C_0 \). The positive root of the three-nom (33) in this case is twofold degenerate, \( \mu_1 = \mu_2 = \frac{\sqrt{b/3}}{H} \). Let initially we have \( \mu < \mu_1 \). Then, if \( \dot{\mu} < 0 \) the texture will collapse monotonically. If \( \dot{\mu} > 0 \) it will expand to the size \( \mu = \mu_1 \). Since the root \( \mu_1 \) is twofold degenerate, in its vicinity we will have \( \dot{\mu} \propto (\mu_1 - \mu) \) and the value \( \mu = \mu_1 \) is attained in infinite time. Thus for \( \mu < \mu_1 \) and \( \dot{\mu} > 0 \) the texture expands eternally approaching the value of \( \mu = \mu_1 \). In the case \( \mu > \mu_1 \) the texture will expand unbounded if

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\( \dot{\mu} > 0 \), and contract eternally approaching the value of \( \mu = \mu_1 \) if \( \dot{\mu} < 0 \). Note that such a behavior is also highly unlikely since in order for it to be realized the values of \( \mu \) and of its time derivative \( \dot{\mu} \) are to be fine tuned from the beginning.

V. COLLAPSE OF AN ELLIPSOIDAL TEXTURE

In this section we consider collapse of an ellipsoidal texture configuration [see Eq. (12) below] in a theory which allows for spherically symmetric topological texture configurations (see [9] for details). For instance, this can be the theory with Lagrangian (1) in which the symmetry \( O(4) \) is spontaneously broken to \( O(3) \). In the vicinity of the collapse point the cosmological curvature of the space-time geometry can be disregarded, and we shall take the space-time to be flat. As to the scaling ansatz we set

\[
\varphi_{\{\lambda\}}(x_1, x_2, x_3) = \chi \left( \frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2}, \frac{x_3}{\lambda_3} \right),
\]

where \( x_i \) are the spatial Cartesian coordinates, \( \lambda_i \) are parameters analogous to the \( \lambda \) in Eq. (3), and \( \chi(x) \) is a spherically symmetric topological configuration, such that \( \lim_{x \to \infty} \chi(x) = \varphi_0 \). The parameters \( \lambda_i \) determine the dimensions of the ellipsoidal configuration (12). By adopting the approximation according to which the texture evolution proceeds within the three-parameter family (12) we obtain after straightforward calculation the action as a functional of \( \{\lambda_i(t)\} \):

\[
S[\{\lambda_i(t)\}] = \int \left( \sum_{ij} A_{ij} h_i h_j - B \sum_i \lambda_i^{-2} \right) \prod_k \lambda_k \, dt,
\]

where

\[
h_i = \frac{\dot{\lambda}_i}{\lambda_i},
\]

and, in view of the spherical symmetry imposed,

\[
A_{ij} = (3A_1 + 2A_2) \delta_{ij} + (A_2 - A_1),
\]
\[ A_1 = \frac{1}{60} \int G_{ab}(\chi) \sum_{ij} \nabla_i \chi^a \nabla_j \chi^b \left( r^2 \delta_{ij} - x_i x_j \right) d^3 x > 0 , \quad (46) \]

\[ A_2 = \frac{1}{30} \int G_{ab}(\chi) \sum_{ij} \nabla_i \chi^a x_i x_j d^3 x > 0 , \quad (47) \]

\[ B = \frac{1}{6} \int G_{ab}(\chi) \sum_i \nabla_i \chi^a \nabla_i \chi^b d^3 x > 0 . \quad (48) \]

Above we made a notation \( r^2 = \sum_i x_i^2 \). In (46)–(48) \( \chi = \chi(x) \)—the spherically symmetric configuration. Note that the constant \( B \) introduced in (48) should not be confused with the analogous constant in Sections III and IV.

From the action (43) we obtain the equations of motion

\[ 10 \dot{h}_k + 10 h_k \sum_i h_i - s \sum_i h_i^2 - p \left( \sum_i h_i \right)^2 - 10 b \lambda_k^{-2} + b(4 + q) \sum_i \lambda_i^{-2} = 0 , \quad (49) \]

where

\[ s = 3q + 2 , \quad p = 1 - q , \quad q = \frac{A_1}{A_2} , \quad b = \frac{B}{sA_2} , \quad (50) \]

and also the expression for the total energy

\[ E = A_2 \prod_j \lambda_j \left[ s \sum_i h_i^2 + p \left( \sum_i h_i \right)^2 + sb \sum_i \lambda_i^{-2} \right] . \quad (51) \]

Let \( t_0 \) denote the time moment of collapse, when one of the \( \lambda_i \) turns to zero, and let this be \( \lambda_1 \). From the expression (51) for the energy, and from the energy conservation law, it then follows that at least one of the remaining two parameters, say \( \lambda_2 \), must also turn to zero at this moment of time. If \( \lambda_3 \) remains finite then from the energy conservation law it also follows that

\[ \lim_{t \to t_0} \frac{\dot{\lambda}_1}{\dot{\lambda}_2} = \zeta \neq 0 , \quad (52) \]

and that the values of \( \dot{\lambda}_1 \) and \( \dot{\lambda}_2 \) must be bounded in the vicinity of \( t_0 \). It can be shown (see Appendix B) that in the case considered in fact \( \zeta = 1 \) in Eq. (52).
Now consider the case when at \( t = t_0 \) all three parameters \( \lambda_i \) vanish. Let us look for the solutions which in the vicinity of \( t = t_0 \) have asymptotic behavior

\[
\lambda_i \sim \Lambda_i (t_0 - t)^{\alpha_i}, \quad i = 1, 2, 3, \tag{53}
\]

with positive constants \( \Lambda_i \) and \( \alpha_i \). Then in the vicinity of \( t = t_0 \)

\[
h_i \sim -\frac{\alpha_i}{t_0 - t}, \quad \dot{h}_i \sim -\frac{\alpha_i}{(t_0 - t)^2}. \tag{54}
\]

Taking the sum of Eq. (49) over the index \( k \), multiplying the result by \( (t_0 - t)^2 \), then taking the limit of \( t \to t_0 \) and using (53), (54) we obtain

\[
-10 \sum_i \alpha_i + (10 - 3p) \left( \sum_i \alpha_i \right)^2 - 3s \sum_i \alpha_i^2 + sb \lim_{t \to t_0} \left[ (t_0 - t)^2 \sum_i \Lambda_i^{-2}(t_0 - t)^{-2\alpha_i} \right] = 0. \tag{55}
\]

In order that the energy (51) remains nonzero and finite in the limit of \( t \to t_0 \), and also that the limit in (55) is finite, it is necessary that

\[
\sum_i \alpha_i = 2; \quad \text{and} \quad \alpha_i \leq 1 \quad \text{for all} \quad i. \tag{56}
\]

If \( \alpha_i < 1 \) for all \( i \) then from (53) we find the unique solution

\[
\alpha_i = \frac{2}{3} \quad \text{for all} \quad i. \tag{57}
\]

It is easy to check that the asymptotic expressions (53) with these values of \( \alpha_i \) satisfy the system (49) to the leading approximation. We thus recover the scaling collapse which we studied in Sec. III. In this case

\[
\lim_{t \to t_0} (\lambda_1 : \lambda_2 : \lambda_3) = \Lambda_1 : \Lambda_2 : \Lambda_3. \tag{58}
\]

If, for example, \( \alpha_3 = 1 \) then using the system of equations (49) we obtain the solution

\[
\alpha_1 = \alpha_2 = \frac{1}{2}, \quad \Lambda_3 = \sqrt{2b}. \tag{59}
\]
In this case

\[ \lim_{t \to t_0} \frac{\lambda_1}{\lambda_2} = \frac{\Lambda_1}{\Lambda_2}, \quad \lim_{t \to t_0} \frac{\lambda_1 \lambda_2}{\lambda_3} = \frac{\Lambda_1 \Lambda_2}{\Lambda_3}, \]  \hspace{1cm} (60)

Thus an ellipsoidal texture always collapses similar in two directions. In the third direction its dimension can behave in three possible ways: (i) remain finite, then Eq. (52) is valid with \( \zeta = 1 \), (ii) shrink similar to the other two dimensions, so that Eq. (58) is valid, (iii) shrink faster, so that Eq. (60) is satisfied.

Note that our analysis was performed in frames of the nonlinear sigma-model with the Lagrangian (3). As it was emphasized in Sec. III this approximation breaks down as soon as one of the texture spatial dimensions becomes as small as

\[ R \sim \frac{1}{\eta \sqrt{g}}, \]  \hspace{1cm} (61)

where, we remember, \( g \) is the coupling constant in (2). At this moment texture unwinding is expected to occur.

It is worth noting that the solution in the form (53) becomes exact if one neglects the last two terms in Eq. (49). One may raise the issue about the role of these two terms. It is easy to see that in all the cases considered above their influence will be in the direction of reducing the texture spherical asymmetry. This is indeed observed in numerical simulations [4,5]. However, simulations in [4,5] are not limited to the case of nonlinear sigma-model (3), they are based on the original Lagrangian (1), which means that field variation in radial direction (in the space of fields) may also be of significance in producing the effect discussed in [4,5].

VI. SUMMARY

In this paper we studied analytically the process of topological texture collapse using the approximation of scaling ansatz in the nonlinear sigma-model. Our investigation may
be regarded as complementary to the numerical simulations of [4–6,9]. We have seen that in flat space-time topological texture eventually collapses while in the case of spatially flat expanding universe its fate depends on the rate of expansion. If the universe is inflationary, $a(t) \propto t^\alpha$ with $\alpha > 1$, or $a(t) \propto \exp(\text{H}t)$, then there is a possibility that texture will expand eternally. In the case of exponential inflation there is also a possibility of eternal shrinking or expansion to a finite limiting size, however, this solution is degenerate. In the case of power law universe expansion, $a(t) \propto t^\alpha$, with $0 < \alpha \leq 1$ topological texture eventually collapses. In the case of cold matter dominated universe, $a(t) \propto t^{2/3}$, we have seen that texture which is formed comoving with the universe expansion starts collapsing when its spatial size becomes comparable to the Hubble size, which is in agreement with the previous considerations. We considered also the final stage of collapsing ellipsoidal topological texture. Remaining in the approximation of the nonlinear sigma-model with the Lagrangian (3) we have seen that during collapse of such a texture at least two of its principal dimensions shrink to zero in a similar way, so that their ratio remains finite. The third dimension may remain finite [collapse of cigar type], or it may also shrink to zero similar to the other two dimensions [collapse of scaling type, Eq. (58)], or shrink to zero similar to the product of the remaining two dimensions [collapse of pancake type, Eq. (60)].

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In this Appendix we shall investigate Eq. (23), in the case of Milne’s cosmology:

\[ a(t) \propto t, \]  

for which the function \( f(t) \) is given by the expression (20) with \( \beta = 1 \). We shall show that in this case any solution describes eventual collapse, that is, \( \nu(t) \) attains the value of zero.

In Eq. (23) with \( \beta = 1 \) we make substitution

\[ t = \sqrt{\frac{\tau}{b}}, \quad \nu = \sigma \tau^{-1/4}. \]  

As a result we get the equation for \( \sigma(\tau) \):

\[ \ddot{\sigma} = -\frac{3}{16} \left( \frac{\sigma \tau^2}{4} \right)^{-1/3}, \]  

in which the dot denotes the derivative with respect to \( \tau \).

Assume that there exists solution \( \nu(\tau) \) without collapse. Then from the equations (23) and (A2) it follows that

\[ \lim_{\tau \to \infty} \sigma(\tau) = \infty. \]  

According to Eq. (A3) the derivative \( \dot{\sigma}(\tau) \) monotonically decreases, and according to Eq. (A4) it is bounded by zero from below. Hence, there exists a finite limit

\[ \lim_{\tau \to \infty} \dot{\sigma}(\tau) = v \geq 0. \]  

If we assume that \( v > 0 \), then asymptotically \( \sigma \sim v \tau \), and integration of Eq. (A3) would yield \( \dot{\sigma} \sim \text{const} \cdot \log \tau \) at large \( \tau \) which is incompatible with Eq. (A4). Therefore in the equality (A5) there must be \( v = 0 \).

Using the de l’Hospital rule we calculate the limit
Thus
\[
\lim_{\tau \to \infty} \frac{\sigma(\tau)}{\tau} = \lim_{\tau \to \infty} \dot{\sigma}(\tau) = 0 .
\]  (A6)

\[
\sigma(\tau) = \tau y(\tau) , \quad \lim_{\tau \to \infty} y(\tau) = 0 ,
\]  (A7)

whence, with Eq. (A6) taken into account, it follows
\[
\lim_{\tau \to \infty} \tau \dot{y}(\tau) = 0 .
\]  (A8)

From Eqs. (A3), (A7) we obtain the equation for \( y(\tau) \):
\[
\tau^2 \ddot{y} + 2\tau \dot{y} = -\frac{3}{16y^{4/3}} ,
\]  (A9)

from which with (A7), (A8) taken into account it follows
\[
\lim_{\tau \to \infty} \tau^2 \ddot{y} = -\infty .
\]  (A10)

Applying once again the de l’Hospital rule and using (A8), (A10), we get
\[
0 = \lim_{\tau \to \infty} \tau \dot{y}(\tau) = \lim_{\tau \to \infty} \frac{\dot{y}(\tau)}{(1/\tau)} = \lim_{\tau \to \infty} \frac{\ddot{y}(\tau)}{(1/\tau)} = -\lim_{\tau \to \infty} \tau^2 \ddot{y}(\tau) = \infty .
\]  (A11)

Therefore, the initial assumption of the absence of collapse has lead to contradiction. Hence it is proven, within the scaling approximation, that in the universe that expands as \( a(t) \propto t \) any texture eventually collapses.

**APPENDIX B: THE VALUE OF \( \zeta \)**

Here we show that the value of \( \zeta \) in Eq. (52) is equal to one. It is reasonable to assume that the functions \( \lambda_i(t) , i = 1, 2, 3 \), together with their first and second derivatives behave monotonically in the vicinity of the collapse moment of time \( t_0 \). This guarantees the existence of all the limits below. As it was mentioned after Eq. (52), the values of \( \dot{\lambda}_i , i = 1, 2 \), are bounded in the vicinity of the collapse moment \( t_0 \), hence...
\[ \lim_{t \to t_0} \dot{\lambda}_i = \omega_i, \quad i = 1, 2. \]  

(B1)

By the de l’Hospital rule we have
\[ \lim_{t \to t_0} \frac{\lambda_1}{\lambda_2} = \lim_{t \to t_0} \frac{\dot{\lambda}_1}{\dot{\lambda}_2} = \zeta. \]  

(B2)

Now applying the de l’Hospital rule in the third equality just below we get
\[ \omega_i \omega_j = \lim_{t \to t_0} \left( \dot{\lambda}_i \dot{\lambda}_j \right) = \lim_{t \to t_0} \left( \frac{\lambda_i \dot{\lambda}_j}{t - t_0} \right) = \lim_{t \to t_0} \left( \lambda_i \dot{\lambda}_j + \lambda_i \ddot{\lambda}_j \right) = \omega_i \omega_j + \lim_{t \to t_0} \left( \lambda_i \ddot{\lambda}_j \right), \]  

(B3)

hence
\[ \lim_{t \to t_0} \left( \lambda_i \ddot{\lambda}_j \right) = 0. \]  

(B4)

The existence of the last limit is guaranteed by the assumed monotonic behavior of the functions’ second derivatives. Using (B1) and (B4) we obtain
\[ \lim_{t \to t_0} \lambda_i h_j = \omega_i, \quad \lim_{t \to t_0} \lambda_i^2 h_j = -\omega_i^2, \]  

(B5)

where, we remember, \( h_k \equiv \dot{\lambda}_k / \lambda_k \), and the indices \( i, j \) independently of each other take on the values 1, 2. From the law of conservation of energy (51) it also follows that the values of \( \lambda_i h_3, i = 1, 2 \), have finite limits as \( t \to t_0 \).

Finally, we take the equations of motion (49) with \( k = 1 \) and \( k = 2 \), multiply each of them by \( \lambda_1^2 \) and take the limit of \( t \to t_0 \). Using (B2), (B3) and finiteness of the product \( \lambda_1 h_3 \) we obtain \( \zeta = 1 \).
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