CONVERGENCE ANALYSIS OF INEXACT TWO-GRID METHODS: A THEORETICAL FRAMEWORK

XUEFENG XU† AND CHEN-SONG ZHANG‡

Abstract. Multigrid is one of the most efficient methods for solving large-scale linear systems that arise from discretized partial differential equations. As a foundation for multigrid analysis, two-grid theory plays an important role in motivating and analyzing multigrid algorithms. For symmetric positive definite problems, the convergence theory of two-grid methods with exact solution of the Galerkin coarse-grid system is mature, and the convergence factor of exact two-grid methods can be characterized by an identity. Compared with the exact case, the convergence theory of inexact two-grid methods (i.e., the coarse-grid system is solved approximately) is of more practical significance, while it is still less developed in the literature (one reason is that the error propagation matrix of inexact coarse-grid correction is not a projection). In this paper, we develop a theoretical framework for the convergence analysis of inexact two-grid methods. More specifically, we present two-sided bounds for the energy norm of the error propagation matrix of inexact two-grid methods, from which one can readily obtain the identity for exact two-grid convergence. As an application, we establish a unified convergence theory for multigrid methods, which allows the coarsest-grid system to be solved approximately.

Key words. Multigrid, inexact two-grid methods, convergence factor, eigenvalue analysis

AMS subject classifications. 65F08, 65F10, 65N55, 15A18

1. Introduction. Multigrid is a powerful solver, with linear or near-linear computational complexity, for a large class of linear systems arising from discretized partial differential equations; see, e.g., [11, 36, 37]. The idea of multigrid originated with Fedorenko in the 1960s [15, 16], while it received scant attention until the work of Brandt in the 1970s [5, 6]. Some fundamental elements for the convergence analysis of multigrid methods are attributed to Hackbusch [18, 19]. Other representative works on the early development of multigrid methods can be found in [20, 46, 11, 36] and the references therein. Since the early 1980s, multigrid has been extensively studied and applied in scientific and engineering computing; see, e.g., [36, 37].

The foundation of multigrid methods is a two-grid scheme, which combines two complementary processes: smoothing (or local relaxation) and coarse-grid correction. The smoothing process is typically a simple iterative method, such as the (weighted) Jacobi and Gauss-Seidel iterations. In general, these classical methods are efficient at eliminating high-frequency (i.e., oscillatory) error, while low-frequency (i.e., smooth) error components cannot be eliminated effectively [11, 36]. To remedy this defect, a coarse-grid correction strategy is used in the two-grid scheme: the low-frequency error can be further reduced by solving a coarse-grid system. The coarse-grid correction process involves two intergrid operators that transfer information between fine- and coarse-grids: one is a restriction matrix that restricts the residual formed on a fine-grid to a coarser-grid; the other is a prolongation (or interpolation) matrix \( P \in \mathbb{R}^{n \times n_c} \) (n...
and \( n_c \) are the numbers of fine and coarse variables, respectively) with full column rank that extends the correction computed on the coarse-grid to the fine-grid. Typically, the restriction matrix is taken to be \( P^T \), the transpose of \( P \), as considered in this paper. The so-called Galerkin coarse-grid matrix is defined as \( A_c := P^T AP \in \mathbb{R}^{n_c \times n_c} \), which gives a coarse representation of the fine-grid matrix \( A \in \mathbb{R}^{n \times n} \) (\( A \) is assumed to be symmetric positive definite (SPD)).

Regarding two-grid analysis, most previous works (see, e.g., [14, 47, 30, 42, 10]) focus on exact two-grid methods, with some exceptions like [29, 37]. An identity has been established to characterize the convergence factor of exact two-grid methods [41, 14]. In practice, however, it is often too costly to solve the Galerkin coarse-grid system exactly, especially when its size is large. Instead, one may solve the coarse-grid system approximately as long as the convergence speed is satisfactory. A recursive call (e.g., the V- and W-cycles) of two-grid procedure yields a multigrid method, which can be regarded as an inexact two-grid scheme. It is well known that two-grid convergence is sufficient to assess the W-cycle multigrid convergence; see, e.g., [20, 36]. Based on the idea of hierarchical basis [2] and the minimization property of Schur complements (see, e.g., [1, Theorem 3.8]), Notay [29] derived an upper bound for the convergence factor of inexact two-grid methods. With this estimate, Notay [29, Theorem 3.1] showed that, if the convergence factor of exact two-grid method at any level is uniformly bounded by \( \sigma < 1/2 \), then the convergence factor of the corresponding W-cycle multigrid method is uniformly bounded by \( \sigma/(1 - \sigma) \).

Besides theoretical considerations, two-grid theory can be used to guide the design of multigrid algorithms. The implementation of a multigrid scheme on large-scale parallel machines is still a challenging topic, especially in the era of exascale computing. For instance, stencil sizes (the number of nonzero entries in a row) of the Galerkin coarse-grid matrices tend to increase further down in the multilevel hierarchy of algebraic multigrid methods [9, 7, 33], which will increase the cost of communication. As problem size increases and the number of levels grows, the overall efficiency of parallel algebraic multigrid methods may decrease dramatically [12]. Motivated by the inexact two-grid theory in [29], Falgout and Schroder [12] proposed a non-Galerkin coarsening strategy to improve the parallel efficiency of algebraic multigrid algorithms. Some other sparse approximations to \( A_c \) can be found, e.g., in [8, 35, 34].

Algebraic multigrid constructs the coarsening process in a purely algebraic manner (that is, the explicit knowledge of geometric properties is not required), which has been widely applied in scientific and engineering problems associated with complex domains, unstructured grids, jump coefficients, etc; see, e.g., [37, 42]. As stated in [29], it is possible to prove optimal convergence properties of multigrid methods via some smoothing and approximation properties or via the theory of subspace correction methods; see, e.g., [3, 4, 20, 28, 23, 40, 46, 32]. However, convergence bounds derived by these approaches do not, in general, give satisfactory predictions of actual convergence speed [36, page 96]. Moreover, for algebraic multigrid methods, it may be difficult to check some required assumptions [29]. In fact, two-grid analysis is still a main strategy for assessing and analyzing algebraic multigrid methods [22, 30].

In this paper, we develop a theoretical framework for the convergence analysis of inexact two-grid methods, in which the Galerkin coarse-grid matrix \( A_c \) is replaced by a general SPD matrix \( B_c \in \mathbb{R}^{n_c \times n_c} \). More precisely, we present lower and upper bounds for the energy norm of the error propagation matrix of inexact two-grid methods, from which one can readily get the identity for exact two-grid convergence. The new upper bounds are sharper than the existing one in [29] (see Remark 3.4). As an application of the framework, we establish a unified convergence theory for multigrid methods, in
which the coarsest-grid system is not required to be solved exactly.

The rest of this paper is organized as follows. In section 2, we review some fundamental properties of two-grid methods and an elegant identity for the convergence factor of exact two-grid methods. In section 3, we present a theoretical framework for the convergence analysis of inexact two-grid methods. In section 4, we establish a unified convergence theory for multigrid methods based on the proposed framework. In section 5, we give some concluding remarks.

2. Preliminaries. In this section, we review some useful properties of two-grid methods, which play a fundamental role in the convergence analysis of inexact two-grid methods. For convenience, we list some notation used in the subsequent discussions.

- $I_n$ denotes the $n \times n$ identity matrix (or $I$ when its size is clear from context).
- $\lambda_{\min} (\cdot)$, $\lambda_{\min}^n (\cdot)$, and $\lambda_{\max} (\cdot)$ denote the smallest eigenvalue, the smallest positive eigenvalue, and the largest eigenvalue of a matrix, respectively.
- $\lambda (\cdot)$ denotes the spectrum of a matrix.
- $\rho (\cdot)$ denotes the spectral radius of a matrix.
- $(\cdot, \cdot)$ denotes the standard Euclidean inner product of two vectors.
- $\| \cdot \|_2$ denotes the spectral norm of a matrix.
- $\| \cdot \|_A$ denotes the energy norm induced by an SPD matrix $A \in \mathbb{R}^{n \times n}$; for any $v \in \mathbb{R}^n$, $\| v \|_A = \langle Av, v \rangle^\frac{1}{2}$; for any $B \in \mathbb{R}^{n \times n}$, $\| B \|_A = \max_{v \in \mathbb{R}^n, \langle v, v \rangle = 1} \frac{\| B v \|_A}{\| v \|_A}$.
- $\kappa_A (\cdot)$ denotes the condition number, with respect to $\| \cdot \|_A$, of a matrix.

2.1. Two-grid methods. Consider solving the linear system

$$ Au = f, \quad (2.1) $$

where $A \in \mathbb{R}^{n \times n}$ is SPD, $u \in \mathbb{R}^n$, and $f \in \mathbb{R}^n$. Given an initial guess $u^{(0)} \in \mathbb{R}^n$ and a nonsingular smoother $M \in \mathbb{R}^{n \times n}$, we perform the following iteration:

$$ u^{(k+1)} = u^{(k)} + M^{-1} (f - A u^{(k)}) \quad k = 0, 1, \ldots \quad (2.2) $$

From (2.2), we have

$$ u - u^{(k+1)} = (I - M^{-1} A) (u - u^{(k)}), $$

which leads to

$$ \| u - u^{(k)} \|_A \leq \| I - M^{-1} A \|_A^k \| u - u^{(0)} \|_A. $$

For any initial guess $u^{(0)}$, if $\| I - M^{-1} A \|_A < 1$, then

$$ \lim_{k \to +\infty} \| u - u^{(k)} \|_A = 0. $$

Since

$$ \| (I - M^{-1} A) v \|_A^2 = \| v \|_A^2 - \langle (M + M^T - A) M^{-1} A v, M^{-1} A v \rangle \quad \forall v \in \mathbb{R}^n, $$

a sufficient and necessary condition for the iteration (2.2) to be $A$-convergent, that is, $\| I - M^{-1} A \|_A < 1$, is that $M + M^T - A$ is SPD.

For an $A$-convergent smoother $M$, we define two symmetrized variants:

$$ \overline{M} := M (M + M^T - A)^{-1} M^T, \quad (2.3a) $$

and

$$ \widetilde{M} := M^T (M + M^T - A)^{-1} M. \quad (2.3b) $$
It is easy to check that
\[ I - \overline{M}^{-1}A = (I - M^{-T}A)(I - M^{-1}A), \]
\[ I - \tilde{M}^{-1}A = (I - M^{-1}A)(I - M^{-T}A), \]
from which one can easily deduce that both $\overline{M} - A$ and $\tilde{M} - A$ are symmetric positive semidefinite (SPSD).

Usually, the iteration (2.2) can only eliminate high-frequency error effectively. To further reduce the remaining low-frequency modes, a coarse-grid correction strategy is used in two-grid scheme. Let $P \in \mathbb{R}^{n \times n_c}$ be a prolongation (or interpolation) matrix of rank $n_c$, where $n_c (< n)$ is the number of coarse variables. The Galerkin coarse-grid matrix takes the form $A_c = P^TAP \in \mathbb{R}^{n_c \times n_c}$. Let $u^{(f)} \in \mathbb{R}^n$ be an approximation to $u = A^{-1}f$, e.g., $u^{(f)}$ is generated from (2.2). The (exact) coarse-grid correction can be described as follows:

\[ u^{(f+1)} = u^{(f)} + PA_c^{-1}P^T(f - Au^{(f)}). \]

Let
\[ \Pi_A = PA_c^{-1}P^T A. \]

Then
\[ u - u^{(f+1)} = (I - \Pi_A)(u - u^{(f)}). \]

Note that $I - \Pi_A$ is an $A$-orthogonal projection along (or parallel to) range($P$) onto null($P^T A$). Thus,
\[ (I - \Pi_A)e = 0 \quad \forall e \in \text{range}(P), \]
which suggests that $I - \Pi_A$ can remove the error components contained in the coarse space range($P$). That is, an efficient coarse-grid correction will be achieved if range($P$) can cover most of the low-frequency error.

With the iterations (2.2) and (2.5), a symmetric two-grid scheme for solving (2.1) can be described by Algorithm 2.1. If the SPD coarse-grid matrix $B_c$ in Algorithm 2.1 is taken to be $A_c$, then the algorithm is called an exact two-grid method; otherwise, it is called an inexact two-grid method.

**Algorithm 2.1 Two-grid method**

1. Presmoothing: $u^{(1)} \leftarrow u^{(0)} + M^{-1}(f - Au^{(0)})$ \{ $M + M^T - A \in \mathbb{R}^{n \times n}$ is SPD \}
2. Restriction: $r_c \leftarrow P^T(f - Au^{(1)})$ \{ $P \in \mathbb{R}^{n \times n_c}$ has full column rank \}
3. Coarse-grid correction: $e_c \leftarrow B_c^{-1}r_c$ \{ $B_c \in \mathbb{R}^{n_c \times n_c}$ is SPD \}
4. Prolongation: $u^{(2)} \leftarrow u^{(1)} + Pe_c$
5. Postsmoothing: $u_{\text{ITG}} \leftarrow u^{(2)} + M^{-T}(f - Au^{(2)})$

From Algorithm 2.1, we have
\[ u - u_{\text{ITG}} = E_{\text{ITG}}(u - u^{(0)}), \]
where
\[ E_{\text{ITG}} = (I - M^{-T}A)(I - PB_c^{-1}P^T A)(I - M^{-1}A). \]
It is referred to as the iteration matrix (or error propagation matrix) of Algorithm 2.1, which can be expressed as

\[ E_{\text{ITG}} = I - B_{\text{ITG}}^{-1} A \]  

with

\[ B_{\text{ITG}}^{-1} = \overline{M}^{-1} + (I - M^{-T} A)P\tilde{B}_{c}^{-1}P^{T}(I - A M^{-1}). \]

Since \( \overline{M} \) and \( B_{c} \) are SPD, we deduce from (2.9) that \( B_{\text{ITG}} \) is an SPD matrix, which is called the inexact two-grid preconditioner. In view of (2.8), we have

\[ \| E_{\text{ITG}} \|_{A} = \rho \left( E_{\text{ITG}} \right) = \max \{ \lambda_{\text{max}} \left( B_{\text{ITG}}^{-1} A \right) - 1, 1 - \lambda_{\text{min}} \left( B_{\text{ITG}}^{-1} A \right) \}, \]

which is referred to as the convergence factor of Algorithm 2.1.

2.2. Convergence of exact two-grid methods. The convergence properties of Algorithm 2.1 with \( B_{c} = A_{c} \) have been well studied by the multigrid community. For its algebraic analysis, we refer to [37, 22, 30] and the references therein.

Denote the iteration matrix of Algorithm 2.1 with \( B_{c} = A_{c} \) by \( E_{\text{TG}} \). Then

\[ E_{\text{TG}} = (I - M^{-T} A)(I - \Pi A)(I - M^{-1} A), \]

where \( \Pi A \) is given by (2.6). Similarly, \( E_{\text{TG}} \) can be expressed as

\[ E_{\text{TG}} = I - B_{\text{TG}}^{-1} A \]

with

\[ B_{\text{TG}}^{-1} = \overline{M}^{-1} + (I - M^{-T} A)P A_{c}^{-1}P^{T}(I - A M^{-1}). \]

The SPD matrix \( B_{\text{TG}} \) is called the exact two-grid preconditioner.

The following theorem provides an identity for the convergence factor \( \| E_{\text{TG}} \|_{A} \) [14, Theorem 4.3], which is a two-level version of the XZ-identity [41, 47].

**Theorem 2.1.** Let \( \widetilde{M} \) be defined by (2.3b), and let

\[ \Pi_{\widetilde{M}} = P(P^{T} \overline{M} P)^{-1} P^{T} \overline{M}. \]

Then, the convergence factor of Algorithm 2.1 with \( B_{c} = A_{c} \) can be characterized as

\[ \| E_{\text{TG}} \|_{A} = 1 - \frac{1}{K_{\text{TG}}}, \]

where

\[ K_{\text{TG}} = \max_{v \in \mathbb{R}^{n} \backslash \{0\}} \frac{\|(I - \Pi_{\overline{M}}) v\|_{\overline{M}}^{2}}{\|v\|_{A}^{2}}. \]

**Remark 2.2.** The matrix \( \Pi_{\overline{M}} \) given by (2.14) is an \( \overline{M} \)-orthogonal projection onto \( \text{range}(P) \). That is, \( \Pi_{\overline{M}}^{2} = \Pi_{\overline{M}} \), \( \text{range}(\Pi_{\overline{M}}) = \text{range}(P) \), and \( \Pi_{\overline{M}} \) is self-adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_{\overline{M}} := \langle \overline{M}, \cdot, \cdot \rangle \).
Remark 2.3. The expression (2.11) implies that $A^\frac{1}{2}E_TA^{-\frac{1}{2}}$ is an SPSD matrix with smallest eigenvalue 0. Since
\[
A^\frac{1}{2}E_TA^{-\frac{1}{2}} = I - A^\frac{1}{2}B_{TG}^{-1}A^\frac{1}{2},
\]
we get that $B_{TG} - A$ is also SPSD and $\lambda_{\text{max}}(B_{TG}^{-1}A) = 1$. Due to
\[
1 - \frac{1}{K_{TG}} = \|E_{TG}\|_A = \rho(E_{TG}) = \lambda_{\text{max}}(E_{TG}) = 1 - \lambda_{\text{min}}(B_{TG}^{-1}A),
\]
it follows that
\[
\lambda_{\text{min}}(B_{TG}^{-1}A) = \frac{1}{K_{TG}}.
\]
As a result, we have
\[
K_{TG} = \frac{\lambda_{\text{max}}(B_{TG}^{-1}A)}{\lambda_{\text{min}}(B_{TG}^{-1}A)}.
\]
This shows that $K_{TG}$ is the corresponding condition number when Algorithm 2.1 with $B_c = A_c$ is treated as a preconditioning method.

As is well known, the aim of two-grid methods is to balance the interplay between smoother and coarse space (or interpolation). For a fixed smoother $M$ (e.g., the Jacobi or Gauss–Seidel type), an optimal interpolation can be obtained by minimizing $K_{TG}$. In practice, however, it is often too costly to compute the optimal interpolation, because it requires the explicit knowledge of eigenvectors corresponding to small eigenvalues of the generalized eigenvalue problem $Ax = \lambda Mx$; see [42, 10] for details. To find a cheap alternative to the optimal interpolation, one may minimize a suitable upper bound for $K_{TG}$.

Let $R$ be an $n_c \times n$ matrix with the property $RP = I_{n_c}$, and let $Q = PR$. Clearly, $Q \in \mathbb{R}^{n \times n}$ is a projection onto range($P$). In light of (2.16), we have
\[
K_{TG} = \max_{v \in \mathbb{R}^n \setminus \{0\}} \min_{v_c \in \mathbb{R}^{n_c}} \frac{\|v - PV_c\|_M^2}{\|v\|_A^2} \leq \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\|(I - Q)v\|_M^2}{\|v\|_A^2} =: K,
\]
which, together with (2.15), yields
\[
\|E_{TG}\|_A \leq 1 - \frac{1}{K}.
\]
By minimizing $K$ over all interpolations, one can obtain an ideal interpolation [13, 44], which gives a strategy for designing an interpolation with sparse or simple structure; see, e.g., [25, 26, 44, 24]. In particular, if $R = (P^TMP)^{-1}P^TM$, then $K = K_{TG}$; see [44] for a quantitative relation between $K$ and $K_{TG}$. Hence, the ideal interpolation can be viewed as a generalization of the optimal one.

3. Convergence of inexact two-grid methods. In this section, we develop a general framework for the convergence analysis of Algorithm 2.1. More specifically, lower and upper bounds for the convergence factor of Algorithm 2.1 are established.

According to (2.10), the main task of estimating $\|E_{TG}\|_A$ is to bound the extreme eigenvalues of $B_{TG}^{-1}A$. It was proved by Notay [29, Theorem 2.2] that
\[
\begin{align*}
\lambda_{\text{max}}(B_{TG}^{-1}A) &\leq \max \left\{1, \lambda_{\text{max}}(B_c^{-1}A_c)\right\} \lambda_{\text{max}}(B_{TG}^{-1}A), \\
\lambda_{\text{min}}(B_{TG}^{-1}A) &\geq \min \left\{1, \lambda_{\text{min}}(B_c^{-1}A_c)\right\} \lambda_{\text{min}}(B_{TG}^{-1}A),
\end{align*}
\]
Algorithm 2.1 gives nothing but a trivial upper bound 1, from which one cannot explicitly determine whether the limiting algorithm is convergent. This suggests that the estimate (3.2) is not sharp in some situations.

In what follows, we establish a new convergence theory for Algorithm 2.1 based on some technical eigenvalue identities and the well-known Weyl’s theorem.

We first give several important eigenvalue identities, which will be frequently used in the subsequent analysis.

**Lemma 3.1.** The extreme eigenvalues of \((I - \tilde{M}^{-1}A)(I - \Pi_A)\) and \((I - \tilde{M}^{-1}A)\Pi_A\) have the following properties:

\begin{align}
(3.3a) & \quad \lambda_{\text{min}}((I - \tilde{M}^{-1}A)(I - \Pi_A)) = 0, \\
(3.3b) & \quad \lambda_{\text{max}}((I - \tilde{M}^{-1}A)(I - \Pi_A)) = 1 - \frac{1}{K_{TG}}, \\
(3.3c) & \quad \lambda_{\text{min}}((I - \tilde{M}^{-1}A)\Pi_A) = 0, \\
(3.3d) & \quad \lambda_{\text{max}}((I - \tilde{M}^{-1}A)\Pi_A) = 1 - \lambda_{\text{min}}(\tilde{M}^{-1}A\Pi_A).
\end{align}

**Proof.** Since \(\Pi_A^2 = \Pi_A\) and \(\text{rank}(\Pi_A) = n_c\), there exists a nonsingular matrix \(X \in \mathbb{R}^{n \times n}\) such that

\[X^{-1}\Pi_AX = \begin{pmatrix} I_{n_c} & 0 \\ 0 & 0 \end{pmatrix}.\]

Let

\[X^{-1}\tilde{M}^{-1}AX = \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{pmatrix},\]

where \(\tilde{X}_{ij} \in \mathbb{R}^{m_i \times m_j}\) with \(m_1 = n_c\) and \(m_2 = n - n_c\). Then

\begin{align}
(3.4a) & \quad X^{-1}(\tilde{M}^{-1}A(I - \Pi_A) + \Pi_A)X = \begin{pmatrix} I_{n_c} & \tilde{X}_{12} \\ 0 & \tilde{X}_{22} \end{pmatrix}, \\
(3.4b) & \quad X^{-1}(I - \tilde{M}^{-1}A)(I - \Pi_A)X = \begin{pmatrix} 0 & -\tilde{X}_{12} \\ 0 & I - \tilde{X}_{22} \end{pmatrix}, \\
(3.4c) & \quad X^{-1}(I - \tilde{M}^{-1}A)\Pi_AX = \begin{pmatrix} I - \tilde{X}_{11} & 0 \\ -\tilde{X}_{21} & 0 \end{pmatrix}, \\
(3.4d) & \quad X^{-1}\tilde{M}^{-1}A\Pi_AX = \begin{pmatrix} \tilde{X}_{11} & 0 \\ \tilde{X}_{21} & 0 \end{pmatrix}.
\end{align}
Using (2.4a) and (2.13), we obtain
\[
B_{TG}^{-1}A = \overline{M}^{-1}A + (I - M^{-T}A)\Pi_A(I - M^{-1}A) \\
= I - (I - M^{-T}A)(I - \Pi_A)(I - M^{-1}A),
\]
which, together with (2.4b), yields
\[
\lambda(B_{TG}^{-1}A) = \lambda(I - (I - M^{-1}A)(I - M^{-T}A)(I - \Pi_A)) \\
= \lambda(I - (I - \tilde{M}^{-1}A)(I - \Pi_A)) \\
= \lambda(\tilde{M}^{-1}A(I - \Pi_A) + \Pi_A).
\]
According to Remark 2.3 and (3.4a), we deduce that
\[
\lambda(\tilde{x}_{22}) \subset \left[\frac{1}{K_{TG}}, 1 \right] \quad \text{and} \quad \lambda_{\min}(\tilde{x}_{22}) = \frac{1}{K_{TG}}.
\]
Then, by (3.4b), we have
\[
\lambda_{\min}(I - \tilde{M}^{-1}A)(I - \Pi_A) = \min\{0, 1 - \lambda_{\max}(\tilde{x}_{22})\} = 0,
\]
\[
\lambda_{\max}(I - \tilde{M}^{-1}A)(I - \Pi_A) = \max\{0, 1 - \lambda_{\min}(\tilde{x}_{22})\} = 1 - \frac{1}{K_{TG}}.
\]
Since $\tilde{M} - A$ is PSD and $\Pi_A = PA_{e}^{-1}P^{T}A$, it follows that
\[
\lambda(\tilde{M}^{-1}A\Pi_A) \subset [0, 1],
\]
which, combined with (3.4d), yields $\lambda(\tilde{x}_{11}) \subset [0, 1]$. In light of (3.4c), we have
\[
\lambda_{\min}(I - \tilde{M}^{-1}A)(I - \Pi_A) = \min\{0, 1 - \lambda_{\max}(\tilde{x}_{11})\} = 0,
\]
\[
\lambda_{\max}(I - \tilde{M}^{-1}A)(I - \Pi_A) = \max\{0, 1 - \lambda_{\min}(\tilde{x}_{11})\} = 1 - \lambda_{\min}(\tilde{x}_{11}).
\]
Note that both $A^{-1} - \tilde{M}^{-1}$ and
\[
(A^{-1} - \tilde{M}^{-1})^{+}A(A^{-1} - \tilde{M}^{-1})^{+} - (A^{-1} - \tilde{M}^{-1})^{+}A\Pi_A(A^{-1} - \tilde{M}^{-1})^{+}
\]
are PSD. We then have
\[
\lambda_{\max}(I - \tilde{M}^{-1}A)(I - \Pi_A) = \lambda_{\max}((A^{-1} - \tilde{M}^{-1})^{+}A\Pi_A(A^{-1} - \tilde{M}^{-1})^{+}) \\
\leq \lambda_{\max}((A^{-1} - \tilde{M}^{-1})^{+}A(A^{-1} - \tilde{M}^{-1})^{+}) \\
= 1 - \lambda_{\min}(\tilde{M}^{-1}A) < 1.
\]
The above inequality, together with (3.5), leads to $\lambda_{\min}(\tilde{x}_{11}) > 0$. Thus,
\[
\lambda_{\max}(I - \tilde{M}^{-1}A)(I - \Pi_A) = 1 - \lambda_{\min}(\tilde{x}_{11}) = 1 - \lambda_{\min}(\tilde{M}^{-1}A\Pi_A).
\]
This completes the proof. \(\square\)

Let $S_{1} \in \mathbb{R}^{n \times n}$ and $S_{2} \in \mathbb{R}^{n \times n}$ be symmetric matrices. Denote the spectra of $S_{1}$, $S_{2}$, and $S_{1} + S_{2}$ by $\{\lambda_{i}(S_{1})\}_{i=1}^{n}$, $\{\lambda_{i}(S_{2})\}_{i=1}^{n}$, and $\{\lambda_{i}(S_{1} + S_{2})\}_{i=1}^{n}$, respectively. For each $k = 1, \ldots, n$, the Weyl’s theorem (see, e.g., [21, Theorem 4.3.1]) states that
\[
\lambda_{k-j+1}(S_{1}) + \lambda_{j}(S_{2}) \leq \lambda_{k}(S_{1} + S_{2}) \leq \lambda_{k+\ell}(S_{1}) + \lambda_{n-\ell}(S_{2})
\]
(3.6)
for all $j = 1, \ldots, k$ and $\ell = 0, \ldots, n-k$, where $\lambda_i(\cdot)$ denotes the $i$-th smallest eigenvalue of a matrix. In particular, one has

\begin{align}
(3.7a) \quad & \lambda_{\min}(S_1 + S_2) \geq \lambda_{\min}(S_1) + \lambda_{\min}(S_2), \\
(3.7b) \quad & \lambda_{\min}(S_1 + S_2) \leq \min\{\lambda_{\min}(S_1) + \lambda_{\max}(S_2), \lambda_{\max}(S_1) + \lambda_{\min}(S_2)\}, \\
(3.7c) \quad & \lambda_{\max}(S_1 + S_2) \geq \max\{\lambda_{\max}(S_1) + \lambda_{\min}(S_2), \lambda_{\min}(S_1) + \lambda_{\max}(S_2)\}, \\
(3.7d) \quad & \lambda_{\max}(S_1 + S_2) \leq \lambda_{\max}(S_1) + \lambda_{\max}(S_2).
\end{align}

It is worth noting that the Weyl’s theorem can also be applied to the nonsymmetric matrix $(I - \widetilde{M}^{-1}A)(I - r\Pi_A)$ with parameter $r$. Indeed, $(I - \widetilde{M}^{-1}A)(I - r\Pi_A)$ has the same spectrum as the symmetric matrix $(A^{-1} - \widetilde{M}^{-1})^{1/2}A(I - r\Pi_A)(A^{-1} - \widetilde{M}^{-1})^{1/2}$. One can first apply the Weyl’s theorem to the symmetric one, and then transform the result into a form related to $(I - \widetilde{M}^{-1}A)(I - \Pi_A)$, or $(I - \widetilde{M}^{-1}A)(I - \Pi_A)$. For example, if $r \geq 0$, we get from (3.7d) that

\[
\lambda_{\max}((I - \widetilde{M}^{-1}A)(I - r\Pi_A)) = \lambda_{\max}((A^{-1} - \widetilde{M}^{-1})^{1/2}A(I - r\Pi_A)(A^{-1} - \widetilde{M}^{-1})^{1/2}) \\
\leq \lambda_{\max}((A^{-1} - \widetilde{M}^{-1})^{1/2}A(A^{-1} - \widetilde{M}^{-1})^{1/2}) \\
- r\lambda_{\min}((A^{-1} - \widetilde{M}^{-1})^{1/2}A(I^{-1} - \widetilde{M}^{-1})^{1/2}) \\
= \lambda_{\max}(I - \widetilde{M}^{-1}A) - r\lambda_{\min}((I - \widetilde{M}^{-1}A)\Pi_A).
\]

For brevity, such a trick will be implicitly used in the subsequent discussions. We are now in a position to present a new convergence theory for Algorithm 2.1.

Theorem 3.2. Let

\[ r_1 = \lambda_{\min}(B_{\varepsilon}^{-1}A_{\varepsilon}) \quad \text{and} \quad r_2 = \lambda_{\max}(B_{\varepsilon}^{-1}A_{\varepsilon}). \]

Under the assumptions of Algorithm 2.1, the convergence factor $\|E_{\text{ITG}}\|_A$ satisfies the following estimates.

(i) If $r_2 \leq 1$, then

\[ L_1 \leq \|E_{\text{ITG}}\|_A \leq \mathcal{U}_1, \]

where

\[ L_1 = 1 - \min\left\{ \frac{1}{K_{\text{ITG}}}, \frac{1}{\lambda_{\min}(\widetilde{M}^{-1}A) + r_2(1 - \lambda_{\min}(\widetilde{M}^{-1}A\Pi_A))} \right\}, \]

\[ \mathcal{U}_1 = 1 - \frac{r_1}{K_{\text{ITG}}} - (1 - r_1)\lambda_{\min}(\widetilde{M}^{-1}A). \]

(ii) If $r_1 \leq 1 < r_2$, then

\[ L_2 \leq \|E_{\text{ITG}}\|_A \leq \max\{\mathcal{U}_1, \mathcal{U}_2\}, \]

where

\[ L_2 = 1 - \min\left\{ \lambda_{\max}(\widetilde{M}^{-1}A), \frac{r_2}{K_{\text{ITG}}} - (r_2 - 1)\lambda_{\min}(\widetilde{M}^{-1}A) \right\}, \]

\[ \mathcal{U}_2 = (r_2 - 1)(1 - \lambda_{\min}^+(\widetilde{M}^{-1}A\Pi_A)). \]

(iii) If $1 < r_1$, then

\[ \max\{L_2, L_3\} \leq \|E_{\text{ITG}}\|_A \leq \mathcal{U}_3, \]

where

\[ L_3 = 1 - \min\left\{ \frac{1}{\lambda_{\max}(\widetilde{M}^{-1}A) + r_1(1 - \lambda_{\min}(\widetilde{M}^{-1}A\Pi_A))} \right\}. \]
where

\[ Z_3 = r_1 - 1 - \min \left\{ r_1 \lambda_{\min}^+ (\widetilde{M}^{-1} A \Pi_A) - \lambda_{\min} (\widetilde{M}^{-1} A), (r_1 - 1) \lambda_{\max} (\widetilde{M}^{-1} A) \right\}, \]

\[ \mathcal{U}_3 = \max \left\{ 1 - \frac{1}{K_{ITG}}, (r_2 - 1) (1 - \lambda_{\min}^+ (\widetilde{M}^{-1} A \Pi_A)) \right\}. \]

**Proof.** By (2.7) and (2.8), we have

\[ B_{ITG}^{-1} A = I - (I - M^{-T} A)(I - P B_{c}^{-1} P^T A)(I - M^{-1} A). \]

Then

\[ \lambda(B_{ITG}^{-1} A) = \lambda(I - (I - M^{-1} A)(I - M^{-T} A)(I - P B_{c}^{-1} P^T A)) \]

\[ = \lambda(I - (I - \widetilde{M}^{-1} A)(I - P B_{c}^{-1} P^T A)), \]

which yields

\[ \lambda_{\max}(B_{ITG}^{-1} A) = 1 - \lambda_{\min}((I - \widetilde{M}^{-1} A)(I - P B_{c}^{-1} P^T A)), \]

\[ \lambda_{\min}(B_{ITG}^{-1} A) = 1 - \lambda_{\max}((I - \widetilde{M}^{-1} A)(I - P B_{c}^{-1} P^T A)). \]

Note that \((I - \widetilde{M}^{-1} A)(I - P B_{c}^{-1} P^T A)\) has the same eigenvalues as the symmetric matrix \((A^{-1} - \widetilde{M}^{-1})^T A(I - P B_{c}^{-1} P^T A)(A^{-1} - \widetilde{M}^{-1})^T\). In view of (3.8), we have

\[ \lambda_{\min}(A^{-1} - \widetilde{M}^{-1})^T A(I - P B_{c}^{-1} P^T A)(A^{-1} - \widetilde{M}^{-1})^T \leq 1. \]

(3.12a) \[ -s_1 \leq \lambda_{\max}(B_{ITG}^{-1} A) - 1 \leq -s_2, \]

(3.12b) \[ t_2 \leq 1 - \lambda_{\min}(B_{ITG}^{-1} A) \leq t_1, \]

where

\[ s_k = \lambda_{\min}((I - \widetilde{M}^{-1} A)(I - r_k \Pi_A)), \]

\[ t_k = \lambda_{\max}((I - \widetilde{M}^{-1} A)(I - r_k \Pi_A)). \]

According to (2.10), (3.12a), and (3.12b), we deduce that

\[ \max \{-s_1, t_2\} \leq \| E_{ITG} \|_A \leq \max \{-s_2, t_1\}. \]

Next, we are devoted to establishing the upper bounds for \(s_1\) and \(t_1\), as well as the lower bounds for \(s_2\) and \(t_2\). The remainder of this proof is divided into three parts corresponding to the cases \(r_2 \leq 1, r_1 \leq 1 < r_2, \) and \(1 < r_1. \)

**Case 1:** \(r_2 \leq 1.\) By (3.7b), we have that

\[ s_1 = \lambda_{\min}((I - \widetilde{M}^{-1} A)(I - r_1 \Pi_A)) \]

\[ \leq \lambda_{\min}((I - \widetilde{M}^{-1} A)(I - \Pi_A)) + (1 - r_1) \lambda_{\max}((I - \widetilde{M}^{-1} A) \Pi_A) \]

\[ = 1 - r_1 - (1 - r_1) \lambda_{\min}^{+}(\widetilde{M}^{-1} A \Pi_A) \]

and

\[ s_1 = \lambda_{\min}((I - \widetilde{M}^{-1} A)(I - r_1 \Pi_A)) \]

\[ \leq \lambda_{\min}(I - \widetilde{M}^{-1} A) - r_1 \lambda_{\min}((I - \widetilde{M}^{-1} A) \Pi_A) \]

\[ = 1 - \lambda_{\max}(\widetilde{M}^{-1} A), \]
where we have used the facts (3.3a), (3.3c), and (3.3d). Then

\[ s_1 \leq 1 - \max \left\{ r_1 + (1 - r_1)\lambda_{\min}(\widetilde{M}^{-1}AP_A), \lambda_{\max}(\widetilde{M}^{-1}A) \right\}. \]

By (3.7a), we have

\[ s_2 = \lambda_{\min}((I - \widetilde{M}^{-1}A)((1 - r_2)I + r_2(I - P_A))) \]
\[ \geq (1 - r_2)\lambda_{\min}(I - \widetilde{M}^{-1}A) + r_2\lambda_{\min}((I - \widetilde{M}^{-1}A)(I - P_A)), \]

which, together with (3.3a), yields

\[ s_2 \geq (1 - r_2)(1 - \lambda_{\max}(\widetilde{M}^{-1}A)). \]

Using (3.7d), we obtain

\[ t_1 = \lambda_{\max}((I - \widetilde{M}^{-1}A)((1 - r_1)I + r_1(I - P_A))) \]
\[ \leq (1 - r_1)\lambda_{\max}(I - \widetilde{M}^{-1}A) + r_1\lambda_{\max}((I - \widetilde{M}^{-1}A)(I - P_A)). \]

The above inequality, combined with (3.3b), yields

\[ t_1 \leq 1 - \frac{r_1}{K_TG} - (1 - r_1)\lambda_{\min}(\widetilde{M}^{-1}A). \]

By (3.3b)–(3.3d) and (3.7c), we have that

\[ t_2 = \lambda_{\max}((I - \widetilde{M}^{-1}A)(I - P_A + (1 - r_2)P_A)) \]
\[ \geq \lambda_{\max}((I - \widetilde{M}^{-1}A)(I - P_A)) + (1 - r_2)\lambda_{\min}((I - \widetilde{M}^{-1}A)P_A) \]
\[ = 1 - \frac{1}{K_TG} \]

and

\[ t_2 = \lambda_{\max}((I - \widetilde{M}^{-1}A)(I - r_2P_A)) \]
\[ \geq \lambda_{\max}(I - \widetilde{M}^{-1}A) - r_2\lambda_{\max}((I - \widetilde{M}^{-1}A)P_A) \]
\[ = 1 - \lambda_{\min}(\widetilde{M}^{-1}A) - r_2(1 - \lambda_{\min}(\widetilde{M}^{-1}A)P_A). \]

We then have

\[ t_2 \geq 1 - \min \left\{ \frac{1}{K_TG}, \lambda_{\min}(\widetilde{M}^{-1}A) + r_2(1 - \lambda_{\min}(\widetilde{M}^{-1}A)P_A) \right\}. \]

Combining (3.13)–(3.17), we can arrive at the estimate (3.9) immediately.

**Case 2:** \( r_1 \leq 1 < r_2 \). Note that the inequalities (3.14) and (3.16) still hold due to \( r_1 \leq 1 \). We next focus on the lower bounds for \( s_2 \) and \( t_2 \). By (3.7a), we have

\[ s_2 = \lambda_{\min}((I - \widetilde{M}^{-1}A)(I - P_A + (1 - r_2)P_A)) \]
\[ \geq \lambda_{\min}((I - \widetilde{M}^{-1}A)(I - P_A)) + (1 - r_2)\lambda_{\max}((I - \widetilde{M}^{-1}A)P_A), \]

which, together with (3.3a) and (3.3d), yields

\[ s_2 \geq (1 - r_2)(1 - \lambda_{\min}(\widetilde{M}^{-1}A)P_A)). \]
In light of (3.3b), (3.3c), and (3.7c), we have that
\[ t_2 = \lambda_{\text{max}}((I - \tilde{M}^{-1}A)(I - r_2\Pi_A)) \]
\[ \geq \lambda_{\text{min}}(I - \tilde{M}^{-1}A) - r_2\lambda_{\text{min}}((I - \tilde{M}^{-1}A)\Pi_A) \]
\[ = 1 - \lambda_{\text{max}}(\tilde{M}^{-1}A) \]
and
\[ t_2 = \lambda_{\text{max}}((I - \tilde{M}^{-1}A)((1 - r_2)I + r_2(I - \Pi_A))) \]
\[ \geq (1 - r_2)\lambda_{\text{max}}(I - \tilde{M}^{-1}A) + r_2\lambda_{\text{max}}((I - \tilde{M}^{-1}A)(I - \Pi_A)) \]
\[ = 1 - \frac{r_2}{K_{TG}} + (r_2 - 1)\lambda_{\text{min}}(\tilde{M}^{-1}A). \]
Hence,
\[ t_2 \geq 1 - \min \left\{ \lambda_{\text{max}}(\tilde{M}^{-1}A), \frac{r_2}{K_{TG}} - (r_2 - 1)\lambda_{\text{min}}(\tilde{M}^{-1}A) \right\}. \tag{3.19} \]

The estimate (3.10) then follows by combining (3.13), (3.14), (3.16), (3.18), and (3.19).

**Case 3:** \( 1 < r_1 \). In this case, the estimates (3.18) and (3.19) still hold. We then consider the upper bounds for \( s_1 \) and \( t_1 \). Using (3.3a), (3.3d), and (3.7b), we get that
\[ s_1 = \lambda_{\text{min}}((I - \tilde{M}^{-1}A)(I - r_1\Pi_A)) \]
\[ \leq \lambda_{\text{max}}(I - \tilde{M}^{-1}A) - r_1\lambda_{\text{max}}((I - \tilde{M}^{-1}A)\Pi_A) \]
\[ = 1 - \lambda_{\text{min}}(\tilde{M}^{-1}A) - r_1(1 - \lambda_{\text{min}}(\tilde{M}^{-1}A)\Pi_A) \]
and
\[ s_1 = \lambda_{\text{min}}((I - \tilde{M}^{-1}A)((1 - r_1)I + r_1(I - \Pi_A))) \]
\[ \leq (1 - r_1)\lambda_{\text{min}}(I - \tilde{M}^{-1}A) + r_1\lambda_{\text{min}}((I - \tilde{M}^{-1}A)(I - \Pi_A)) \]
\[ = 1 - r_1 + (r_1 - 1)\lambda_{\text{max}}(\tilde{M}^{-1}A). \]
Hence,
\[ s_1 \leq 1 - r_1 + \min \left\{ r_1\lambda_{\text{max}}(\tilde{M}^{-1}A)\Pi_A - \lambda_{\text{min}}(\tilde{M}^{-1}A), (r_1 - 1)\lambda_{\text{max}}(\tilde{M}^{-1}A) \right\}. \tag{3.20} \]

By (3.7d), we have
\[ t_1 = \lambda_{\text{max}}((I - \tilde{M}^{-1}A)(I - \Pi_A + (1 - r_1)\Pi_A)) \]
\[ \leq \lambda_{\text{max}}((I - \tilde{M}^{-1}A)(I - \Pi_A)) + (1 - r_1)\lambda_{\text{min}}((I - \tilde{M}^{-1}A)\Pi_A), \]
which, combined with (3.3b) and (3.3c), gives
\[ t_1 \leq 1 - \frac{1}{K_{TG}}. \tag{3.21} \]
In light of (3.13) and (3.18)–(3.21), we conclude that the estimate (3.11) is valid. \( \square \)
Remark 3.3. In particular, if $B_c = A_c$, then the lower and upper bounds in (3.9) become

$$L_1 = 1 - \min \left\{ \frac{1}{K_{TG}}, 1 + \lambda_{\min}(\tilde{M}^{-1}A) - \lambda_{\min}^+(\tilde{M}^{-1}A) \right\},$$

$$U_1 = 1 - \frac{1}{K_{TG}}.$$ 

By (3.3b), (3.3d), and (3.7d), we have

$$2 - \frac{1}{K_{TG}} - \lambda_{\min}^+(\tilde{M}^{-1}A) \geq \lambda_{\max}(I - \tilde{M}^{-1}A)(I - A) + (I - \tilde{M}^{-1}A)A$$

which yields

$$1 + \lambda_{\min}(\tilde{M}^{-1}A) - \lambda_{\min}^+(\tilde{M}^{-1}A) \geq \frac{1}{K_{TG}}.$$ 

Then

$$L_1 = 1 - \frac{1}{K_{TG}}.$$ 

Hence, the estimate (3.9) will reduce to the identity (2.15) when $B_c = A_c$.

Remark 3.4. With the notation in (3.8), the estimate (3.2) reads

$$(3.22) \quad \|E_{ITG}\|_A \leq \begin{cases} 1 - \frac{r_1}{K_{TG}} & \text{if } r_2 \leq 1, \\ \max \left\{ 1 - \frac{r_1}{K_{TG}}, r_2 - 1 \right\} & \text{if } r_1 \leq 1 < r_2, \\ \max \left\{ 1 - \frac{1}{K_{TG}}, r_2 - 1 \right\} & \text{if } 1 < r_1. \end{cases}$$

It is easy to see that the upper bounds in (3.9)–(3.11) are smaller than that in (3.22). On the other hand, if $B_c = \alpha I_{n_c}$ with $\alpha > 0$, then

$$\lim_{\alpha \to +\infty} r_1 = \lim_{\alpha \to +\infty} \frac{\lambda_{\min}(A_c)}{\alpha} = 0 \quad \text{and} \quad \lim_{\alpha \to +\infty} r_2 = \lim_{\alpha \to +\infty} \frac{\lambda_{\max}(A_c)}{\alpha} = 0.$$ 

One can readily check that both $L_1$ and $U_1$ tend to $1 - \lambda_{\min}(\tilde{M}^{-1}A)$ as $\alpha \to +\infty$, which is exactly the convergence factor of the limiting algorithm. That is, the estimate (3.9) has fixed the defect of (3.2) indicated at the outset of this section. Besides improved upper bounds, Theorem 3.2 provides new lower bounds for $\|E_{ITG}\|_A$, which give necessary conditions for a fast convergence speed.

As mentioned earlier, the Galerkin coarse-grid matrix may affect the parallel efficiency of algebraic multigrid algorithms. To improve the parallel performance, Falgout and Schroder [12] proposed a non-Galerkin coarsening strategy, which is motivated by the following result. Define

$$(3.23) \quad \theta := \|I - A_c^{-1}B_c\|_2,$$

where $B_c \in \mathbb{R}^{n_c \times n_c}$ is a general SPD approximation to $A_c$. If $\theta < 1$, then

$$(3.24) \quad \kappa_A(B^{-1}_{ITG}A) \leq \frac{1 + \theta}{1 - \theta} \kappa_A(B^{-1}_{TG}A) = \frac{1 + \theta}{1 - \theta} K_{TG}.$$
and
\[
\|E_{ITG}\|_A \leq \max \left\{ \frac{\theta}{1-\theta}, 1 - \frac{1}{(1+\theta)K_{TG}} \right\}.
\]

The definition (3.23) implies that
\[
\theta \geq \rho(I - A_{c}^{-1}B_{c}) = \max \{ \lambda_{\max}(A_{c}^{-1}B_{c}) - 1, 1 - \lambda_{\min}(A_{c}^{-1}B_{c}) \},
\]
and hence
\[
1 - \theta \leq \lambda_{\min}(A_{c}^{-1}B_{c}) \leq \lambda_{\max}(A_{c}^{-1}B_{c}) \leq 1 + \theta.
\]

With the notation in (3.8), we have
\[
\frac{1}{1+\theta} = r_1 \leq r_2 \leq \frac{1}{1-\theta},
\]
which contains the following three cases:

\[ C_1 : \frac{1}{1+\theta} \leq r_1 \leq r_2 \leq 1; \]
\[ C_2 : \frac{1}{1+\theta} \leq r_1 \leq 1 < r_2 \leq \frac{1}{1-\theta}; \]
\[ C_3 : 1 < r_1 \leq r_2 \leq \frac{1}{1-\theta}. \]

From (3.12a) and (3.12b), we have
\[
\lambda(B_{ITG}^{-1}A) \subset [1 - t_1, 1 - s_2] \subset (0, +\infty),
\]
where we have used the facts (3.16) and (3.21). Then
\[
\kappa_A(B_{ITG}^{-1}A) = \frac{\lambda_{\max}(B_{ITG}^{-1}A)}{\lambda_{\min}(B_{ITG}^{-1}A)} \leq \frac{1 - s_2}{1 - t_1}.
\]

According to (3.15), (3.16), (3.18), and (3.21), we deduce that
\[
\kappa_A(B_{ITG}^{-1}A) \leq \begin{cases} 
1 - \frac{1}{1+\theta}K_{TG}\lambda_{\min}(M^{-1}A) & \text{if } C_1 \text{ holds}, \\
\frac{1-\theta}{1+\theta}\lambda_{\min}(M^{-1}A) + \frac{1}{1+\theta}K_{TG} & \text{if } C_2 \text{ holds}, \\
\frac{1-\theta}{1+\theta}\lambda_{\max}(M^{-1}A) & \text{if } C_3 \text{ holds}.
\end{cases}
\]

Furthermore, using (3.9)–(3.11), we obtain that
\[
\|E_{ITG}\|_A \leq \begin{cases} 
1 - \frac{1}{1+\theta}K_{TG}\lambda_{\min}(M^{-1}A) & \text{if } C_1 \text{ holds}, \\
\max \left\{ 1 - \frac{1+\theta}{(1+\theta)K_{TG}}\lambda_{\min}(\tilde{M}^{-1}A), \frac{\theta-\theta\lambda_{\min}(\tilde{M}^{-1}A)}{1-\theta} \right\} & \text{if } C_2 \text{ holds}, \\
\max \left\{ 1 - \frac{\theta}{K_{TG}}, \frac{\theta\lambda_{\min}(\tilde{M}^{-1}A)}{1-\theta} \right\} & \text{if } C_3 \text{ holds}.
\end{cases}
\]

It is easy to see that the estimates (3.26) and (3.27) are sharper than (3.24) and (3.25), respectively.
4. An application of inexact two-grid theory. In practice, it is often too costly to solve the Galerkin coarse-grid system exactly, especially when its size is large. Instead, without essential loss of convergence speed, one may replace the coarse-grid matrix by a suitable approximation. A natural way to obtain such an approximation is to apply Algorithm 2.1 recursively in the coarse-grid correction steps. To describe the resulting (multigrid) algorithm conveniently, we give some notation and assumptions.

- The algorithm involves \( L + 1 \) levels with indices \( 0, \ldots, L \), where \( 0 \) and \( L \) correspond to the coarsest- and finest-levels, respectively.
- \( n_k \) denotes the number of unknowns at level \( k \) (\( n = n_L > n_{L-1} > \cdots > n_0 \)).
- For each \( k = 1, \ldots, L \), \( P_k \in \mathbb{R}^{n_k \times n_{k-1}} \) denotes a prolongation matrix from level \( k - 1 \) to level \( k \), and \( \text{rank}(P_k) = n_{k-1} \).
- Let \( A_L = A \). For each \( k = 0, \ldots, L - 1 \), \( A_k := P_{k+1}^T A_{k+1} P_{k+1} \) denotes the Galerkin coarse-grid matrix at level \( k \).
- Let \( \hat{A}_0 \in \mathbb{R}^{n_0 \times n_0} \) be an SPD approximation to \( A_0 \), and let \( \hat{A}_0 - A_0 \) be SPD.
- For each \( k = 1, \ldots, L \), \( M_k \in \mathbb{R}^{n_k \times n_k} \) denotes a nonsingular smoother at level \( k \) with \( M_k + M_k^T - A_k \) being SPD (or, equivalently, \( \| I - M_k^{-1} A_k \|_{A_k} < 1 \)).
- \( \gamma \) denotes the cycle index involved in the coarse-grid correction steps.

Given an initial guess \( u_k^{(0)} \in \mathbb{R}^{n_k} \), the standard multigrid scheme for solving the linear system \( A_k u_k = f_k \) (with \( f_k \in \mathbb{R}^{n_k} \)) can be described by Algorithm 4.1. The symbol MG\( \gamma \) in Algorithm 4.1 means that the multigrid scheme will be carried out \( \gamma \) iterations. In particular, \( \gamma = 1 \) corresponds to the V-cycle and \( \gamma = 2 \) to the W-cycle.

**Algorithm 4.1 Multigrid method at level \( k \):** \( u_{\text{IMG}} \leftarrow \text{MG}(k, A_k, f_k, u_k^{(0)}) \)

1. Presmoothing: \( u_k^{(1)} \leftarrow u_k^{(0)} + M_k^{-1}(f_k - A_k u_k^{(0)}) \)
2. Restriction: \( r_{k-1} \leftarrow P_k^T (f_k - A_k u_k^{(1)}) \)
3. Coarse-grid correction: \( e_{k-1} \leftarrow \begin{cases} \hat{A}_0^{-1} r_0 & \text{if } k = 1, \\ \text{MG}(k-1, A_{k-1}, r_{k-1}, 0) & \text{if } k > 1. \end{cases} \)
4. Prolongation: \( u_k^{(2)} \leftarrow u_k^{(1)} + P_k e_{k-1} \)
5. Postsmoothing: \( u_{\text{IMG}} \leftarrow u_k^{(2)} + M_k^{-T} (f_k - A_k u_k^{(2)}) \)

The iteration matrix of Algorithm 4.1 is

\[
E_{\text{IMG}}^{(k)} = (I - M_k^{-T} A_k) \left[ I - P_k \left( I - (E_{\text{IMG}}^{(k-1)})^\gamma A_{k-1}^{-1} P_k^T A_k \right) \right] (I - M_k^{-1} A_k),
\]

which satisfies

\[
u_k - u_{\text{IMG}} = E_{\text{IMG}}^{(k)} (u_k - u_k^{(0)}).
\]

In particular,

\[
E_{\text{IMG}}^{(1)} = (I - M_1^{-T} A_1) \left( I - P_1 \hat{A}_0^{-1} P_1^T A_1 \right) (I - M_1^{-1} A_1).
\]

By (4.1), we have

\[
A_k^\frac{1}{\gamma} E_{\text{IMG}}^{(k)} A_k^{-\frac{1}{\gamma}} = N_k^T \left[ I - A_k^\frac{1}{\gamma} P_k A_k^{-\frac{1}{\gamma}} \left( I - (A_{k-1}^{-\frac{1}{\gamma}} E_{\text{IMG}}^{(k-1)} A_{k-1}^{-\frac{1}{\gamma}})^\gamma A_{k-1}^{-\frac{1}{\gamma}} P_k^T A_k^{\frac{1}{\gamma}} \right) \right] N_k
\]
with

\[
N_k = I - A_k^\frac{1}{\gamma} M_k^{-1} A_k^{\frac{1}{\gamma}}.
\]
By induction, one can show that $A_k^\frac{1}{2} E_{\text{IMG}}^{(k)} A_k^{-\frac{1}{2}}$ is symmetric and 
\[ \lambda(E_{\text{IMG}}^{(k)}) = \lambda \left( A_k^\frac{1}{2} E_{\text{IMG}}^{(k)} A_k^{-\frac{1}{2}} \right) \subset [0, 1) \quad \forall k = 1, \ldots, L, \]
which lead to 
\[ \|E_{\text{IMG}}^{(k)}\|_{A_k} < 1. \]

As a result, $E_{\text{IMG}}^{(k)}$ can be expressed as
\[ E_{\text{IMG}}^{(k)} = I - B_k^{-1} A_k, \]
where $B_k \in \mathbb{R}^{n_k \times n_k}$ is SPD and $B_k - A_k$ is SPD. Combining (4.1) and (4.2), we can obtain the recursive relation
\[ B_k^{-1} = \underbar{M}_k^{-1} + (I - M_k^{-T} A_k) P_k \left( \sum_{j=0}^{\gamma-1} (I - B_{k-1}^{-1} A_{k-1})^j \right) B_{k-1}^{-1} P_k^T (I - A_k M_k^{-1}), \]
where
\[ \underbar{M}_k := M_k \left( M_k + M_k^T - A_k \right)^{-1} M_k^T. \]
Interchanging the roles of $M_k$ and $M_k^T$ in (4.3) yields another symmetrized smoother:
\[ \underbar{M}_k := M_k^T \left( M_k + M_k^T - A_k \right)^{-1} M_k. \]
It is easy to verify that both $\underbar{M}_k - A_k$ and $\underbar{M}_k - A_k$ are SPD.

Comparing (4.1) with (2.7), we can see that Algorithm 4.1 is essentially an inexact two-grid method with $M = M_k$, $A = A_k$, $P = P_k$, and
\[ B_c = A_{k-1} \left( I - (E_{\text{IMG}}^{(k-1)})^\gamma \right)^{-1}. \]

Remark 4.1. From (4.5), we have
\[ B_c = A_{k-1}^\frac{1}{2} \left( A_{k-1}^{-\frac{1}{2}} - (E_{\text{IMG}}^{(k-1)})^\gamma A_{k-1}^{-\frac{1}{2}} \right)^{-1} A_{k-1}^\frac{1}{2} \left( A_{k-1}^{-\frac{1}{2}} - (E_{\text{IMG}}^{(k-1)})^\gamma A_{k-1}^{-\frac{1}{2}} \right)^{-1} \]
\[ = A_{k-1}^\frac{1}{2} \left( A_{k-1}^{-\frac{1}{2}} - (E_{\text{IMG}}^{(k-1)})^\gamma A_{k-1}^{-\frac{1}{2}} \right)^{-1} A_{k-1}^\frac{1}{2} \left( A_{k-1}^{-\frac{1}{2}} - (E_{\text{IMG}}^{(k-1)})^\gamma A_{k-1}^{-\frac{1}{2}} \right)^{-1} A_{k-1}^\frac{1}{2} \left( A_{k-1}^{-\frac{1}{2}} - (E_{\text{IMG}}^{(k-1)})^\gamma A_{k-1}^{-\frac{1}{2}} \right)^{-1} A_{k-1}^\frac{1}{2}. \]
Due to the fact that $A_{k-1}^\frac{1}{2} E_{\text{IMG}}^{(k-1)} A_{k-1}^{-\frac{1}{2}}$ is symmetric and $\lambda(E_{\text{IMG}}^{(k-1)}) \subset [0, 1)$, $B_c$ given by (4.5) is SPD.

Define
\[ \sigma_{\text{YG}}^{(k)} := \|E_{\text{YG}}^{(k)}\|_{A_k} \quad \text{and} \quad \sigma_{\text{IMG}}^{(k)} := \|E_{\text{IMG}}^{(k)}\|_{A_k}, \]
which are the convergence factors of the (exact) two-grid method and (inexact) multigrid method at level $k$, respectively. In view of (3.8) and (4.5), we have
\[ r_1 = \lambda_{\min} \left( I - (E_{\text{IMG}}^{(k-1)})^\gamma \right) = 1 - (\lambda_{\max} (E_{\text{IMG}}^{(k-1)}))^\gamma = 1 - (\sigma_{\text{IMG}}^{(k-1)})^\gamma; \]
\[ r_2 = \lambda_{\max} \left( I - (E_{\text{IMG}}^{(k-1)})^\gamma \right) = 1 - (\lambda_{\min} (E_{\text{IMG}}^{(k-1)}))^\gamma \leq 1. \]
Using (3.9), we obtain

\[
\sigma^{(k)}_{\text{IMG}} \leq 1 - \frac{1}{K^{(k)}_{\text{TG}}} + (\sigma^{(k-1)}_{\text{IMG}})^\gamma \left( \frac{1}{K^{(k)}_{\text{TG}}} - \lambda_{\min}(\overline{M}^{-1}_k A_k) \right),
\]

where

\[
K^{(k)}_{\text{TG}} = \max_{v_k \in \mathbb{R}^{n_k}} \frac{\| (I - \Pi_{\overline{M}_k}) v_k \|_2^2}{\| v_k \|_A^2} \quad \text{with} \quad \Pi_{\overline{M}_k} = P_k (P^T_k \overline{M}_k P_k)^{-1} P^T_k \overline{M}_k.
\]

It follows that

\[
\sigma^{(k)}_{\text{IMG}} \leq \sigma^{(k)}_{\text{TG}} + (\sigma^{(k-1)}_{\text{IMG}})^\gamma \left( 1 - \sigma^{(k)}_{\text{TG}} - \lambda_{\min}(\overline{M}^{-1}_k A_k) \right),
\]

where we have used the fact \(\sigma^{(k)}_{\text{TG}} = 1 - \frac{1}{K^{(k)}_{\text{TG}}}\).

Remark 4.2. The lower bound in (3.9) yields

\[
\sigma^{(k)}_{\text{IMG}} \geq \sigma^{(k)}_{\text{TG}}.
\]

Thus, a well converged multigrid method entails that the corresponding (exact) two-grid method has a fast convergence speed.

Define

\[
\sigma_L := \max_{1 \leq k \leq L} \sigma^{(k)}_{\text{TG}},
\]

\[
\varepsilon_L := \min_{1 \leq k \leq L} \lambda_{\min}(\overline{M}^{-1}_k A_k).
\]

In view of (4.6) and (4.10), we have

\[
K^{(k)}_{\text{TG}} = \lambda_{\max}(A^{-1}_k \overline{M}_k (I - \Pi_{\overline{M}_k})) \leq \lambda_{\max}(A^{-1}_k \overline{M}_k) = \frac{1}{\lambda_{\min}(\overline{M}^{-1}_k A_k)} \leq \frac{1}{\varepsilon_L}.
\]

Then

\[
\sigma^{(k)}_{\text{TG}} = 1 - \frac{1}{K^{(k)}_{\text{TG}}} \leq 1 - \varepsilon_L \quad \forall k = 1, \ldots, L,
\]

and hence

\[
0 \leq \sigma_L \leq 1 - \varepsilon_L.
\]

We remark that the extreme cases \(\sigma_L = 0\) and \(\sigma_L = 1 - \varepsilon_L\) seldom occur in practice.

In what follows, we only consider the nontrivial case

\[
0 < \sigma_L < 1 - \varepsilon_L.
\]

To analyze the convergence of Algorithm 4.1, we first prove a technical lemma.

Lemma 4.3. Let \(\sigma_L\) and \(\varepsilon_L\) be defined by (4.9) and (4.10), respectively. Then

\[
(1 - \sigma_L - \varepsilon_L) x^\gamma - x + \sigma_L = 0 \quad (0 < x < 1)
\]

has a unique root \(x^\gamma\) in \([\sigma_L, \sigma_L^{1+\varepsilon_L}]\), and \(\{x^\gamma\}_{\gamma=1}^{+\infty}\) is a strictly decreasing sequence with limit \(\sigma_L\).
Proof. When \( \gamma = 1 \), one can readily see that \( \frac{\sigma_L}{\sigma_L + \varepsilon_L} \) is the root of (4.12). Let
\[
F_\gamma(x) = (1 - \sigma_L - \varepsilon_L)x^\gamma - x + \sigma_L \quad (\gamma \geq 2).
\]
Then
\[
\frac{dF_\gamma(x)}{dx} = \gamma(1 - \sigma_L - \varepsilon_L)x^{\gamma-1} - 1.
\]

- If \( \gamma(1 - \sigma_L - \varepsilon_L) \leq 1 \), then \( \frac{dF_\gamma(x)}{dx} < 0 \) in \((0, 1)\), that is, \( F_\gamma(x) \) is a strictly decreasing function in \((0, 1)\). Due to

\[
F_\gamma(\sigma_L) = (1 - \sigma_L - \varepsilon_L)\sigma_L \sigma_L > 0,
\]

it follows that \( F_\gamma(x) = 0 \) has a unique root \( x_\gamma \) in \((\sigma_L, \frac{\sigma_L}{\sigma_L + \varepsilon_L})\).

- If \( \gamma(1 - \sigma_L - \varepsilon_L) > 1 \), then

\[
\begin{cases}
\frac{dF_\gamma(x)}{dx} < 0 & \text{if } 0 < x < (\gamma(1 - \sigma_L - \varepsilon_L))^{\frac{1}{\gamma-1}}, \\
\frac{dF_\gamma(x)}{dx} > 0 & \text{if } (\gamma(1 - \sigma_L - \varepsilon_L))^{\frac{1}{\gamma-1}} < x < 1.
\end{cases}
\]

The existence and uniqueness of \( x_\gamma \in \left(\sigma_L, \frac{\sigma_L}{\sigma_L + \varepsilon_L}\right) \) follow immediately from the facts \( F_\gamma(\sigma_L) > 0, F_\gamma\left(\frac{\sigma_L}{\sigma_L + \varepsilon_L}\right) < 0 \), and \( F_\gamma(1) < 0 \).

Since \( x_\gamma < 1 \), it holds that
\[
F_{\gamma+1}(x_\gamma) = (1 - \sigma_L - \varepsilon_L)x_\gamma^{\gamma+1} - x_\gamma + \sigma_L < F_\gamma(x_\gamma) = 0,
\]
which, together with \( F_{\gamma+1}(\sigma_L) > 0 \), yields
\[
\sigma_L < x_{\gamma+1} < x_\gamma.
\]
In addition, we deduce from \( F_\gamma(x_\gamma) = 0 \) that
\[
x_\gamma = (1 - \sigma_L - \varepsilon_L)x_\gamma^\gamma + \sigma_L,
\]
which leads to
\[
\lim_{{\gamma \to +\infty}} x_\gamma = \sigma_L.
\]
This completes the proof. \( \square \)

Using (3.9), (4.7), and Lemma 4.3, we can obtain the following convergence estimate.

**Theorem 4.4.** Let \( \sigma_L \) and \( \varepsilon_L \) be defined by (4.9) and (4.10), respectively. Let \( x_\gamma \) be the (unique) root of (4.12) contained in \((\sigma_L, \frac{\sigma_L}{\sigma_L + \varepsilon_L})\). If

\[
\lambda(\tilde{A}_0^{-1}A_0) \subset \left[ \frac{1 - \varepsilon_L - x_\gamma}{1 - \varepsilon_L - \sigma_L}, 1 \right],
\]

then
\[
\sigma^{(k)}_{\text{MG}} \leq x_\gamma \quad \forall k = 1, \ldots, L.
\]
Theorem 4.4

Corollary 4.5

and for the cases that follows by induction.

Theorem (4.14)

From (4.7), we have

$$
\sigma_{\text{IMG}}^{(k)} \leq \left(1 - \sigma_{\text{IMG}}^{(k-1)}\right)\sigma_{\text{TG}}^{(k)} + \sigma_{\text{IMG}}^{(k-1)}\left(1 - \lambda_{\min}(M_k^{-1}A_k)\right)
$$

$$
\leq \left(1 - \sigma_{\text{IMG}}^{(k-1)}\right)\max_{1 \leq k \leq L} \sigma_{\text{TG}}^{(k)} + \sigma_{\text{IMG}}^{(k-1)}\left(1 - \min_{1 \leq k \leq L} \lambda_{\min}(M_k^{-1}A_k)\right)
$$

$$
= \left(1 - \sigma_{\text{IMG}}^{(k-1)}\right)\sigma_{\max} + (1 - \sigma_{\text{IMG}}^{(k-1)})\gamma
$$

$$
= \sigma_{\text{IMG}} + (1 - \sigma_{\text{IMG}}^{(k-1)})\gamma.
$$

If $\sigma_{\text{IMG}}^{(k-1)} \leq x_\gamma$, then

$$
\sigma_{\text{IMG}}^{(k)} \leq (1 - \sigma_{\text{IMG}}^{(k-1)})x_\gamma + \sigma_{\text{IMG}} = F_\gamma(x_\gamma) + x_\gamma = x_\gamma.
$$

The estimate (4.14) then follows by induction.

The following corollary particularizes Theorem 4.4 for the cases $\gamma = 1$ and $\gamma = 2$.

**Corollary 4.5.** Under the assumptions of Theorem 4.4, it holds that

$$
\sigma_{\text{IMG}}^{(k)} \leq \begin{cases} 
\frac{\sigma_L}{\sigma_L + \varepsilon_L} & \text{if } \gamma = 1, \\
\frac{2\sigma_L}{1 + \sqrt{(1 - 2\sigma_L)^2 + 4\sigma_L\varepsilon_L}} & \text{if } \gamma = 2,
\end{cases}
$$

\forall k = 1, \ldots, L.

**Remark 4.6.** For the V-cycle multigrid methods, if $\sigma_L \leq C\varepsilon_L$, then

$$
\sigma_{\text{IMG}}^{(k)} \leq \frac{\sigma_L}{\varepsilon_L + 1} \leq \frac{C}{C + 1}.
$$

For the W-cycle multigrid methods, if $\sigma_L \leq \sigma < 1$ for a level-independent quantity $\sigma$, we deduce from Corollary 4.5 that

$$
\sigma_{\text{IMG}}^{(k)} \leq \frac{2\sigma_L}{1 + \sqrt{(1 - 2\sigma_L)^2 + 4\sigma_L\varepsilon_L}} \leq \frac{\sigma_L}{1 - \sigma_L} \leq \frac{\sigma}{1 - \sigma},
$$

that is, our result improves the existing one in [29, Theorem 3.1].

The next theorem gives an upper bound for $\sigma_{\text{IMG}}^{(k)}$ that depends on the level index $k$, which sharpens the bound in (4.14).
Theorem 4.7. Let $\sigma_k$ and $\varepsilon_k$ be defined by (4.9) and (4.10), respectively. Let $x_\gamma$ be the (unique) root of (4.12) contained in $[\sigma_L, \frac{\sigma_L}{\sigma_L+\varepsilon_L}]$. If $\sigma_{IMG}^{(1)} < x_\gamma$, then

$$
\sigma_{IMG}^{(k)} \leq x_\gamma - (x_\gamma - \sigma_{IMG}^{(1)}) \left(1 - \sigma_L - \varepsilon_L\right) x_\gamma^{\gamma-1} \sum_{j=0}^{\gamma-1} \left(\frac{\delta_j}{x_\gamma}\right)^j, \tag{4.15}
$$

where

$$
\delta_L := \min_{1 \leq k \leq L} \sigma_{TG}^{(k)}. \tag{4.16}
$$

Proof. Similarly to the proof of Theorem 4.4, one can prove that $\sigma_{IMG}^{(k)} < x_\gamma$ for all $k = 1, \ldots, L$. Due to

$$
\sigma_{IMG}^{(k)} \leq \sigma_L + (1 - \sigma_L - \varepsilon_L)(\sigma_{IMG}^{(k-1)})^\gamma \quad \text{and} \quad x_\gamma = \sigma_L + (1 - \sigma_L - \varepsilon_L)x_\gamma^\gamma,
$$

it follows that

$$
x_\gamma - \sigma_{IMG}^{(k)} \geq (1 - \sigma_L - \varepsilon_L)\left(x_\gamma^\gamma - \left(\sigma_{IMG}^{(k-1)}\right)^\gamma\right),
$$

which yields

$$
\frac{x_\gamma - \sigma_{IMG}^{(k)}}{x_\gamma - \sigma_{IMG}^{(k-1)}} \geq (1 - \sigma_L - \varepsilon_L)x_\gamma^{\gamma-1} \sum_{j=0}^{\gamma-1} \left(\frac{\sigma_{IMG}^{(k-1)}}{x_\gamma}\right)^j.
$$

By (4.8) and (4.16), we have

$$
\sigma_{IMG}^{(k-1)} \geq \sigma_{TG}^{(k-1)} \geq \delta_L.
$$

Then

$$
\frac{x_\gamma - \sigma_{IMG}^{(k)}}{x_\gamma - \sigma_{IMG}^{(k-1)}} \geq (1 - \sigma_L - \varepsilon_L)x_\gamma^{\gamma-1} \sum_{j=0}^{\gamma-1} \left(\frac{\delta_j}{x_\gamma}\right)^j.
$$

Hence,

$$
\frac{x_\gamma - \sigma_{IMG}^{(k)}}{x_\gamma - \sigma_{IMG}^{(1)}} = \prod_{i=2}^{k} \frac{x_\gamma - \sigma_{IMG}^{(i-1)}}{x_\gamma - \sigma_{IMG}^{(i)}} \geq \left((1 - \sigma_L - \varepsilon_L)x_\gamma^{\gamma-1} \sum_{j=0}^{\gamma-1} \left(\frac{\delta_j}{x_\gamma}\right)^j\right)^{k-1},
$$

which leads to the estimate (4.15). \qed

Remark 4.8. Observe that

$$
(1 - \sigma_L - \varepsilon_L)x_\gamma^{\gamma-1} \sum_{j=0}^{\gamma-1} \left(\frac{\delta_j}{x_\gamma}\right)^j = \frac{(1 - \sigma_L - \varepsilon_L)x_\gamma^\gamma - (1 - \sigma_L - \varepsilon_L)\delta_L^\gamma}{x_\gamma - \delta_L} = \frac{x_\gamma - \sigma_L - (1 - \sigma_L - \varepsilon_L)\delta_L^\gamma}{x_\gamma - \delta_L} \in (0, 1).
$$

Thus, the upper bound in (4.15) is a strictly increasing function with respect to $k$.

The condition $\sigma_{IMG}^{(1)} < x_\gamma$ in Theorem 4.7 will be satisfied if $\tilde{A}_0$ is simply chosen as $A_0$, in which case the convergence factor is denoted by $\sigma_{MG}^{(k)}$. This yields the following corollary.
CONVERGENCE ANALYSIS OF INEXACT TWO-GRID METHODS

**Corollary 4.9.** Let $\sigma_L$, $\varepsilon_L$, and $\delta_L$ be defined by (4.9), (4.10), and (4.16), respectively. Let $x_\gamma$ be the (unique) root of (4.12) contained in $[\sigma_L, \frac{\sigma_L}{\sigma_L + \varepsilon_L}]$. If $\hat{A}_0 = A_0$, then

$$\sigma^{(k)}_{MG} \leq x_\gamma - (x_\gamma - \sigma_L) \left(1 - \sigma_L - \varepsilon_L\right)x_\gamma^{-1} \sum_{j=0}^{\gamma-1} \left(\frac{\delta_L}{x_\gamma}\right)^j \left(1 \right)^{k-1}.$$  

In particular, one has

$$\sigma^{(k)}_{MG} \leq \begin{cases} x_1 \left(1 - \sigma_L - \varepsilon_L\right)^k & \text{if } \gamma = 1, \\ x_2 - (x_2 - \sigma_L) \left(1 - \sigma_L - \varepsilon_L\right)(x_2 + \delta_L) & \text{if } \gamma = 2, \end{cases}$$

where

$$x_1 = \frac{\sigma_L}{\sigma_L + \varepsilon_L} \text{ and } x_2 = \frac{2\sigma_L}{1 + \sqrt{(1 - 2\sigma_L)^2 + 4\sigma_L \varepsilon_L}}.$$

**Proof.** If $\hat{A}_0 = A_0$, then

$$\sigma^{(1)}_{MG} = \sigma^{(1)}_{TG} \leq \sigma_L < x_\gamma.$$

The estimate (4.17) then follows immediately from (4.15).

**Remark 4.10.** For some simple smoothers, $\varepsilon_L$ may be very small. A solution is to use more powerful smoothers, like the SIF (structured incomplete factorization) and eSIF preconditioners in [39, 38]. For the SIF-type smoothers, $\varepsilon_L$ is a controllable quantity, which will not be tiny if a reasonable truncation tolerance is used. Note that our theory is valid as long as $\sigma_L < 1 - \varepsilon_L$. If $\varepsilon_L$ is very small, then multigrid methods can carry over the convergence properties of two-grid methods under a very weak constraint on two-grid convergence speed. In the extreme case when $\varepsilon_L$ is zero, we deduce from (4.18) that

- the V-cycle multigrid satisfies

$$\sigma^{(k)}_{MG} \leq 1 - (1 - \sigma_L)^k;$$

- the W-cycle multigrid satisfies

$$\sigma^{(k)}_{MG} \leq \begin{cases} \frac{\sigma_L}{1 - \sigma_L} - \frac{\sigma_L^2}{1 - \sigma_L}\left(\sigma_L + (1 - \sigma_L)\delta_L\right)^{k-1} & \text{if } \sigma_L < \frac{1}{2}, \\ 1 - (1 - \sigma_L)^k(1 + \delta_L)^{k-1} & \text{if } \frac{1}{2} \leq \sigma_L < 1. \end{cases}$$

We remark that the estimates (4.19) and (4.20) are applicable for $\sigma_L < 1$.

To compare the performances of (4.19), (4.20), and the existing estimate in [29, Theorem 3.1], we give a numerical example: the 2D Poisson’s equation with homogeneous Dirichlet boundary condition on a unit square (using the P1-finite element on a quasi-uniform grid with one million degrees of freedom). The resulting linear system is solved by the classical algebraic multigrid method [9, 33] in a standard setting: the classical coarsening and the direct interpolation are exploited. In the experiments, we set the number of pre- and post-smoothing steps to be 1, the finest-level index $L$ to be 5, and the strong threshold to be $\frac{1}{4}$ (no truncation is applied). The coarsest-grid systems are solved by a sparse direct solver. The asymptotic convergence factor of a multigrid method is computed when the energy norm of error is less than $10^{-12}$.

Table 1 displays that our estimates improve the existing one in [29]. Moreover, the W-cycle multigrid method may carry over two-grid convergence even when $\sigma_L > \frac{1}{2}$.
Table 1

| Smoother     | \( \delta_L \) | \( \sigma_L \) | Cycle | Conv. factor | Existing | New   |
|--------------|----------------|----------------|-------|--------------|----------|-------|
| Gauss–Seidel | 0.232          | 0.462          | V     | 0.876        | N/A      | 0.955 |
|              |                 |                | W     | 0.556        | 0.859    | 0.812 |
| \( \omega \)-Jacobi (\( \omega = 0.5 \)) | 0.292          | 0.625          | V     | 0.905        | N/A      | 0.993 |
|              |                 |                | W     | 0.639        | Fail     | 0.979 |

5. Conclusions. In this paper, we present a theoretical framework for the convergence analysis of inexact two-grid methods (for SPD problems), which improves and extends the existing theory for two-grid methods. A natural question is how to construct the coarse-grid matrix \( B_c \) or approximate the Galerkin coarse-grid matrix \( A_c \), which serves as a motivation for designing new multigrid-based algorithms. As an application of the framework, we establish a unified convergence theory for standard multigrid methods, which allows the coarsest-grid system to be solved approximately. Furthermore, the framework can be used to analyze hybrid multilevel methods, like the VW- and WV-cycle multigrid methods in [45]. In the future, we expect to analyze the convergence of inexact two-grid methods for nonsymmetric problems, which is an interesting topic that deserves in-depth study; see, e.g., [26, 24, 27, 17, 31].

Acknowledgment. The authors would like to thank the anonymous referees for their valuable comments and suggestions, which greatly improved the original version of this paper.

REFERENCES

[1] O. Axelsson, *Iterative Solution Methods*, Cambridge University Press, Cambridge, 1994.
[2] R. E. Bank, T. F. Dupont, and H. Yserentant, *The hierarchical basis multigrid method*, Numer. Math., 52 (1988), pp. 427–458.
[3] D. Braess, *The convergence rate of a multigrid method with Gauss–Seidel relaxation for the Poisson equation*, in Multigrid Methods (W. Hackbusch and U. Trottenberg, eds), Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 960 (1982), pp. 368–386.
[4] D. Braess and W. Hackbusch, *A new convergence proof for the multigrid method including the V-cycle*, SIAM J. Numer. Anal., 20 (1983), pp. 967–975.
[5] A. E. Brandt, *Multi-level adaptive technique (MLAT) for fast numerical solution to boundary value problems*, Proceedings of the Third International Conference on Numerical Methods in Fluid Mechanics, Lecture Notes in Physics (H. Cabannes and R. Temam, eds), Springer, Berlin, Heidelberg, 18 (1973), pp. 82–89.
[6] A. E. Brandt, *Multi-level adaptive solutions to boundary-value problems*, Math. Comp., 31 (1977), pp. 333–390.
[7] A. E. Brandt, *Algebraic multigrid theory: The symmetric case*, Appl. Math. Comput., 19 (1986), pp. 23–56.
[8] A. E. Brandt, *General highly accurate algebraic coarsening*, Electron. Trans. Numer. Anal., 10 (2000), pp. 1–20.
[9] A. E. Brandt, S. F. McCormick, and J. W. Ruge, *Algebraic multigrid (AMG) for sparse matrix equations*, in Sparsity and Its Applications (Loughborough, 1983), Cambridge University Press, Cambridge, (1985), pp. 257–284.
[10] J. Brannick, F. Cao, K. Kahl, R. D. Falgout, and X. Hu, *Optimal interpolation and compatible relaxation in classical algebraic multigrid*, SIAM J. Sci. Comput., 40 (2018), pp. A1473–A1493.
[11] W. L. Briggs, V. E. Henson, and S. F. McCormick, *A Multigrid Tutorial*, SIAM, Philadelphia, PA, 2nd ed., 2000.
[12] R. D. Falgout and J. B. Schroder, *Non-Galerkin coarse grids for algebraic multigrid*, SIAM
CONVERGENCE ANALYSIS OF INEXACT TWO-GRID METHODS

J. Sci. Comput., 36 (2014), pp. C309–C334.

[13] R. D. FALGOUT and P. S. VASILEVSKI, On generalizing the algebraic multigrid framework, SIAM J. Numer. Anal., 42 (2004), pp. 1669–1693.

[14] R. D. FALGOUT, P. S. VASILEVSKI, and L. T. ZIKATANOV, On two-grid convergence estimates, Numer. Linear Algebra Appl., 12 (2005), pp. 471–494.

[15] R. P. FEDORENKO, A relaxation method for solving elliptic difference equations, USSR Comp. Math. Math. Phys., 1 (1962), pp. 1092–1096.

[16] R. P. FEDORENKO, The speed of convergence of one iterative process, USSR Comp. Math. Math. Phys., 4 (1964), pp. 227–235.

[17] L. GARCÍA RAMOS, R. KEHl, and R. NABBEN, Projections, deflation, and multigrid for nonsymmetric matrices, SIAM J. Matrix Anal. Appl., 41 (2020), pp. 83–105.

[18] W. HACKBUSCH, Convergence of multi-grid iterations applied to difference equations, Math. Comp., 34 (1980), pp. 425–440.

[19] W. HACKBUSCH, On the convergence of multi-grid iterations, Beiträge Numer. Math., 9 (1981), pp. 213–239.

[20] W. HACKBUSCH, Multi-Grid Methods and Applications, Springer-Verlag, Berlin, Heidelberg, 1985.

[21] R. A. HORN and C. R. JOHNSON, Matrix Analysis, Cambridge University Press, Cambridge, 2nd ed., 2013.

[22] S. P. MACLACHLAN and L. N. OLSON, Theoretical bounds for algebraic multigrid performance: review and analysis, Numer. Linear Algebra Appl., 21 (2014), pp. 194–220.

[23] J. MANDEL, S. F. MCCORMICK, and J. W. RUGE, An algebraic theory for multigrid methods for variational problems, SIAM J. Numer. Anal., 25 (1988), pp. 91–110.

[24] T. A. MANTTEUFFEL, S. MÜNZENMAIER, J. W. RUGE, and B. S. SOUTHWORTH, Nonsymmetric correction-based algebraic multigrid, SIAM J. Sci. Comput., 41 (2019), pp. S242–S268.

[25] T. A. MANTTEUFFEL, L. N. OLSON, J. B. SCHRODER, and B. S. SOUTHWORTH, A root-node–based algebraic multigrid method, SIAM J. Sci. Comput., 39 (2017), pp. S723–S756.

[26] T. A. MANTTEUFFEL, J. W. RUGE, and B. S. SOUTHWORTH, Nonsymmetric algebraic multigrid based on local approximate ideal restriction (lAIR), SIAM J. Sci. Comput., 40 (2018), pp. A4105–A4130.

[27] T. A. MANTTEUFFEL and B. S. SOUTHWORTH, Convergence in norm of nonsymmetric algebraic multigrid, SIAM J. Sci. Comput., 41 (2019), pp. S269–S296.

[28] S. F. MCCORMICK, Multigrid methods for variational problems: General theory for the V-cycle, SIAM J. Numer. Anal., 22 (1985), pp. 634–643.

[29] Y. NOTAY, Convergence analysis of perturbed two-grid and multigrid methods, SIAM J. Numer. Anal., 45 (2007), pp. 1035–1044.

[30] Y. NOTAY, Algebraic theory of two-grid methods, Numer. Math. Theor. Meth. Appl., 8 (2015), pp. 168–198.

[31] Y. NOTAY, Analysis of two-grid methods: The nonnormal case, Math. Comp., 89 (2020), pp. 807–827.

[32] P. OSMALD, Multilevel Finite Element Approximation: Theory and Applications, Teubner Skripten zur Numerik, Vieweg+Teubner Verlag, Wiesbaden, 1994.

[33] J. W. RUGE and K. STÜBEN, Algebraic multigrid, in Multigrid Methods, Frontiers Appl. Math., SIAM, Philadelphia, 3 (1987), pp. 73–130.

[34] H. D. STERCK, R. D. FALGOUT, J. W. NOLTING, and U. M. YANG, Distance-two interpolation for parallel algebraic multigrid, Numer. Linear Algebra Appl., 15 (2008), pp. 115–139.

[35] H. D. STERCK, U. M. YANG, and J. J. HEYS, Reducing complexity in parallel algebraic multigrid preconditioners, SIAM J. Matrix Anal. Appl., 27 (2006), pp. 1019–1039.

[36] U. TROTENBERG, C. W. OOSTERLEE, and A. SCHÜLLER, Multigrid, Academic Press, New York, 2001.

[37] P. S. VASILEVSKI, Multilevel Block Factorization Preconditioners: Matrix-based Analysis and Algorithms for Solving Finite Element Equations, Springer-Verlag, New York, 2008.

[38] J. XIA, Robust and effective sIF preconditioning for general SPD matrices, arXiv:2007.03729, (2020).

[39] J. XIA and Z. XIN, Effective and robust preconditioning of general SPD matrices via structured incomplete factorization, SIAM J. Matrix Anal. Appl., 38 (2017), pp. 1298–1322.

[40] J. XU, Iterative methods by space decomposition and subspace correction, SIAM Rev., 34 (1992), pp. 581–613.

[41] J. XU and L. T. ZIKATANOV, The method of alternating projections and the method of subspace corrections in Hilbert space, J. Amer. Math. Soc., 15 (2002), pp. 573–597.

[42] J. XU and L. T. ZIKATANOV, Algebraic multigrid methods, Acta Numer., 26 (2017), pp. 591–721.
[43] X. Xu, Algebraic theory of multigrid methods, Ph.D. Thesis (in Chinese), University of Chinese Academy of Sciences, (2019).

[44] X. Xu and C.-S. Zhang, On the ideal interpolation operator in algebraic multigrid methods, SIAM J. Numer. Anal., 56 (2018), pp. 1693–1710.

[45] X. Xu and C.-S. Zhang, Convergence analysis of multigrid methods with alternating cycles, submitted, (2021).

[46] H. Yserentant, Old and new convergence proofs for multigrid methods, Acta Numer., 2 (1993), pp. 285–326.

[47] L. T. Zikatanov, Two-sided bounds on the convergence rate of two-level methods, Numer. Linear Algebra Appl., 15 (2008), pp. 439–454.