Spectral Properties of Neumann-Poincaré Operator and Anomalous Localized Resonance in Elasticity Beyond Quasi-Static Limit

Youjun Deng¹ · Hongjie Li² · Hongyu Liu³

Received: 22 August 2019 / Published online: 27 February 2020 © Springer Nature B.V. 2020

Abstract This paper is concerned with the polariton resonances and their application for cloaking due to anomalous localized resonance (CALR) for the elastic system within finite frequency regime beyond the quasi-static approximation. We first derive the complete spectral system of the Neumann-Poincaré operator associated with the elastic system in \( \mathbb{R}^3 \) within the finite frequency regime. Based on the obtained spectral results, we construct a broad class of elastic configurations that can induce polariton resonances beyond the quasi-static limit. As an application, the invisibility cloaking effect is achieved through constructing a class of core-shell-matrix metamaterial structures provided the source is located inside a critical radius. Moreover, if the source is located outside the critical radius, it is proved that there is no resonance.

Keywords Anomalous localized resonance · Negative elastic materials · Core-shell structure · Beyond quasistatic limit · Neumann-Poincaré operator · Spectral

Mathematics Subject Classification 35R30 · 35B30 · 35Q60 · 47G40

1 Introduction

Recently, there are considerable mathematical studies on various resonance phenomena associated to negative metamaterials in order to gain deep understandings about their distinc-
tive properties and investigate the potential applications. Metamaterials are a general terminology, usually referred to artificial materials with properties that do not exist in nature. Negative materials are a particular type of metamaterials that are engineered to have negative material properties, including negative permittivity and/or permeability in electromagnetism [40, 41] and negative elastic moduli in elasticity [19, 22, 29]. Under certain circumstances, the electromagnetic or elastic wave interacts with negative metamaterial structures to induce the so-called plasmon or polariton resonances, respectively. In the resonant state, due to the excitation of an appropriate source, the induced field exhibits highly oscillatory behaviors in a certain peculiar manner. Mathematically, the resonance phenomenon is connected to an infinite dimensional set of the so-called perfect plasmon/polariton waves, which are actually the kernel of a certain non-elliptic partial differential operator (PDO) arising from the underlying physical system. More specifically, the presence of the negative material parameters breaks the ellipticity of the aforementioned PDO and thus the PDO may have a nontrivial kernel space. On the other hand, through an integral reformulation via the potential-theoretic approach, the resonance can be connected to the spectral system of the so-called Neumann-Poincaré (N-P) operators, which are a certain type of boundary layer potential operators. Hence, in order to understand the plasmon/polariton resonances, one needs to achieve thorough understandings of the spectral properties of certain PDOS or integral operators in various scenarios that were unveiled before. Those connections make the mathematical study of plasmon resonances a fascinating topic. For related studies in the literature, we refer to [4, 5, 10, 18, 25, 35, 36] for the Helmholtz system, [1–3, 6, 12–14, 20, 27, 28, 30–33, 37–39] for the Maxwell system and [8, 9, 16, 17, 23, 24, 26] for the Lamé system.

One particularly interesting type of plasmon/polariton resonances is the anomalous localized resonance (ALR), which is also one of the focuses of the present study. There are several intriguing features of the ALR that make it a very appealing topic for investigation. First, the incorporation of negative materials is a critical ingredient for the delicate construction of material structures that can induce ALR. Second, the localized feature of ALR refers to the fact that the resonance is spatially localized; that is, the corresponding field only diverges in a certain region with a sharp boundary not defined by any discontinuities in the material parameters and outside that region, the field converges to a smooth one. Third, the ALR also heavily depends on the form as well as the location of the excitation source. In fact, the resonance region moves as the position of the source moves. For a fixed plasmon/polariton configuration, if the source is located inside a critical radius, then ALR occurs, whereas if it is located outside the critical radius, then resonance does not occur. Finally, the ALR has a practical feature that it can induce the invisibility cloaking effect. If ALR occurs, then both the underlying material structure and the source are invisible with respect to observations outside a certain region. This is referred to as cloaking due to anomalous localized resonance (CALR) in the literature. Moreover, if CALR occurs, small objects locating beside the material structure are also invisible to the external field observation. CALR was first observed and rigorously justified by Milton and Nicorovici in [31] and was further studied by Ammari et al. in [3] associated with electromagnetic waves. We also refer to the papers [3, 12, 18, 20, 25, 27, 31, 33, 35, 36, 38] and references therein for more related studies associated with acoustics and electromagnetism. The CALR was also investigated in elasticity [8, 9, 16, 23, 24, 26]. It is emphasized that due to the different physical nature, the studies on the polariton resonances (including the CALR) in elasticity possess some distinct features than those in the electromagnetism.

In this paper, we are mainly concerned with the polariton resonances and their application to CALR for the elastic system within the finite frequency regime that go beyond the quasi-static approximation. In what follows, we briefly summarize the major findings and
contributions of this work as well as discuss their connection to the existing results in the literature.

For all of the aforementioned existing studies concerning the polariton resonances for the elastic system [8, 9, 16, 23, 24, 26], the quasi-static approximation has played a critical role. By quasi-static approximation, we mean that the size of the material structure is much smaller compared to the underlying wavelength. In the mathematical term, it can be given as

$$\omega \cdot \text{diam}(\Omega) \ll 1,$$

where $\omega$ and diam($\Omega$), respectively, signify the wave frequency and the diameter of a material inclusion occupying the domain $\Omega$. In fact, [8, 9, 16, 23, 24] consider the static case by directly taking $\omega \equiv 0$ and [26] rigorously verifies the quasi-static approximation. The first major contribution of this work is the construction of a class material structures that can induce polariton resonance and in particular CALR in elasticity within the finite frequency regime beyond the quasi-static limit. The CALR construction takes the core-shell form with the negative elastic metamaterials located in the shell. As discussed earlier, the choice of the negative elastic material parameters is a critical ingredient for our study. Within the radial geometry, we derive a broad class of material choices that can induce the desired ALR effect. It is also shown that there exists a critical radius such that if the source is located inside this radius, then ALR occurs and otherwise ALR does not occur.

Second, our construction of the material structures is more general and feasible than those considered in the aforementioned literature. In [8, 9, 16, 17, 23, 24], the metamaterial parameters were constructed such that both the shear and bulk moduli are negative. In our study, the metamaterial parameters are constructed such that only one the shear modulus is required to be negative. It is noted that in [26], the resonant construction also only requires the negativity of any one of the two elastic moduli. However, the study in [26] is mainly concerned with the static case. Indeed, we show that the CALR construction in the current work includes the constructions in [16, 26] as special cases by taking the quasi-static limit.

Third, in order to establish the aforementioned results, we make essential use of spectral arguments. We derive the complete spectral system of the N-P operator associated to the elastic system with the finite-frequency regime. It is remarked that the corresponding derivation is highly nontrivial and the spectral results are of significant mathematical interest for their own sake.

Finally, two remarks regarding our main results are as follows. First, in order to achieve the resonance results in the current study that go beyond the quasi-static limit, in addition to the delicate and subtle construction of the material structure, we shall also need to impose a certain mild constraint on the form of the excitation sources. This is unobjectionable as it is also mentioned earlier that the ALR heavily depends on the form as well as the location of the source term. We shall discuss more about this point in Remark 5.2 in the following. We would also like to mention in passing several studies [12, 18, 25, 31, 36] on plasmon resonances in acoustics and electromagnetism that go beyond the quasi-static limit. Second, it is noted that we mainly work within the spherical geometry. Indeed, we shall require the exact spectral information of the N-P operator. Beyond the spherical geometry, it is rather unpractical to derive the required spectral results. In fact, even in the simplest electro-static case, only the radical geometry [3] and ellipse geometry [7] were considered. For more general geometries, one may resort to different constructions as well as different techniques; see [11, 12, 18, 20, 27] for related results in acoustics and electromagnetism. In what follows, we shall stick to the general geometry when discussing the polariton resonances in order
to emphasize that the study could hold in more general scenarios. We shall also explicitly specify the radial geometry if technically required.

The rest of the paper is organized as follows. Section 2 is devoted to the mathematical formulation of the polariton resonances in elasticity and some preliminary and auxiliary results for the Lamé system. In Sect. 3, the complete spectral system of the N-P operator is derived. Sections 4 and 5 are respectively devoted to the polariton resonance and CALR results.

2 Preliminaries and Auxiliary Results

2.1 Lamé System and Polariton Resonances

Let $C(x) := (C_{ijkl}(x))_{i,j,k,l=1}^3, x \in \mathbb{R}^3$ be a four-rank elastic material tensor defined by

$$C_{ijkl}(x) := \lambda(x) \delta_{ij} \delta_{kl} + \mu(x) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad x \in \mathbb{R}^3,$$

where $\delta$ is the Kronecker delta. In (2.1), $\lambda$ and $\mu$ are two scalar functions and referred to as the Lamé parameters. For a regular elastic material, the Lamé parameters satisfy the following two strong convexity conditions,

$$\text{i). } \mu > 0 \quad \text{and} \quad \text{ii). } 3\lambda + 2\mu > 0. \quad (2.2)$$

Physically, the first and second conditions in (2.2), respectively, signify that the shear modulus and the bulk modulus of a regular elastic material are both positive. As mentioned earlier, for an elastic metamaterial, either one of the two conditions (2.2) could be violated; that means, an elastic metamaterial can have negative shear and/or negative bulk moduli (cf. [19, 22]).

Let $D, \Omega \subset \mathbb{R}^3$ with $D \subset \Omega$ be two bounded domains with connected Lipschitz boundaries. Assume that the domain $\mathbb{R}^3 \setminus \overline{D}$ is occupied by a regular elastic material parameterized by the Lamé constants $(\lambda, \mu)$ satisfying the strong convexity conditions in (2.2). The shell $\Omega \setminus \overline{D}$ is occupied by a metamaterial whose Lamé parameters are given by $(\hat{\lambda}, \hat{\mu})$, where $(\hat{\lambda}, \hat{\mu}) \in C^2$ with $\Im \hat{\lambda} > 0, \Im \hat{\mu} > 0$, which shall be properly chosen in what follows. The imaginary parts of the shear and bulk moduli signify the loss of the elastic wave in the material. Finally, the inner core $D$ is occupied by a regular elastic material $(\bar{\lambda}, \bar{\mu})$ satisfying the strong convex conditions (2.2). Denote by $C_{\mathbb{R}^3 \setminus \overline{D}, \lambda, \mu}$ to specify the dependence of the elastic tensor on the domain $\mathbb{R}^3 \setminus \overline{D}$ and the Lamé parameters $(\lambda, \mu)$. The same notation also applies for the tensors $C_{\Omega \setminus \overline{D}, \hat{\lambda}, \hat{\mu}}$ and $C_{D, \bar{\lambda}, \bar{\mu}}$. Now we introduce the following elastic tensor

$$C_0 = C_{\mathbb{R}^3 \setminus \overline{D}, \lambda, \mu} + C_{\Omega \setminus \overline{D}, \hat{\lambda}, \hat{\mu}} + C_{D, \bar{\lambda}, \bar{\mu}}. \quad (2.3)$$

$C_0$ describes an elastic material configuration of a core-shell-matrix structure with the metamaterial located in the shell. Let $f \in H^{-1}(\mathbb{R}^3)^3$ signify an excitation elastic source that is compactly supported in $\mathbb{R}^3 \setminus \overline{D}$. The induced elastic displacement field $u = (u_i)_{i=1}^3 \in C^3$ corresponding to the configurations described above is governed by the following Lamé system

$$\nabla \cdot C_0 \nabla u(x) + \omega^2 u(x) = f \quad \text{in } \mathbb{R}^3, \quad u(x) \text{ satisfies the radiation condition}, \quad (2.4)$$
where $\omega \in \mathbb{R}_+$ is the angular frequency, and the operator $\nabla^s$ is the symmetric gradient given by

$$\nabla^s u := \frac{1}{2} (\nabla u + \nabla u^t),$$

(2.5)

with $\nabla u$ denoting the matrix $(\partial_j u_i)_{i,j=1}^3$ and the superscript $t$ signifying the matrix transpose. In (2.4), the radiation condition designates the following condition as $|x| \to + \infty$ (cf. [21]),

$$(\nabla \times \nabla \times u)(x) \times \frac{x}{|x|} - i k_s \nabla \times u(x) = O(|x|^{-2}),$$

(2.6)

where $i = \sqrt{-1}$ and $k_s = \omega / \sqrt{\mu}$, $k_p = \omega / \sqrt{\lambda + 2\mu}$,

(2.7)

Next we introduce the following functional for $w, v \in (H^1(\Omega \setminus D))^3$,

$$P_{\hat{\lambda}, \hat{\mu}}(w, v) = \int_{\Omega \setminus D} \nabla^s w : C_0 \nabla^s v(x) \, dx$$

$$= \int_{\Omega \setminus D} (\hat{\lambda} (\nabla \cdot w) (\nabla \cdot v)(x) + 2 \hat{\mu} \nabla^s w : \nabla^s v(x)) \, dx,$$

(2.8)

where $C_0$ and $\nabla^s$ are defined in (2.3) and (2.5), respectively. In (2.8) and also in what follows, $A : B = \sum_{i,j=1}^3 a_{ij} b_{ij}$ for two matrices $A = (a_{ij})_{i,j=1}^3$ and $B = (b_{ij})_{i,j=1}^3$. Henceforth, we define

$$E(u) = \Im P_{\hat{\lambda}, \hat{\mu}}(u, u),$$

(2.9)

which signifies the energy dissipation exists energy of the elastic system (2.4). We are now in a position to present the definition of CALR. We say that polariton resonance occurs if for any $M \in \mathbb{R}_+$,

$$E(u) \geq M,$$

(2.10)

where $u$ depends on the Lamé parameters $(\hat{\lambda}, \hat{\mu})$. In addition to (2.10), if the displacement field $u$ further satisfies the following boundedness condition,

$$|u| \leq C, \text{ when } |x| > \tilde{R},$$

(2.11)

for some $C, \tilde{R} \in \mathbb{R}_+$, which do not depend on $(\Im \hat{\lambda}, \Im \hat{\mu})$, then we say that CALR occurs. We refer to [3] and [31] for more relevant discussions on the implication of CALR to invisibility cloaking, which equally applies to the elastic case of the present study.

2.2 Layer Potential Technique

In this section, we present some preliminary knowledge on the layer potential technique as well as derive several auxiliary results for the Lamé system for our subsequent use. We first introduce the elastostatic operator $L_{\lambda, \mu}$ associated to the Lamé constants $(\lambda, \mu)$ as follows,

$$L_{\lambda, \mu} w := \mu \Delta w + (\lambda + \mu) \nabla \nabla \cdot w,$$

(2.12)
for $w \in \mathbb{C}^3$. The traction (the conormal derivative) of $w$ on $\partial \Omega$ is defined to be

$$\partial_\nu w = \lambda (\nabla \cdot w) v + 2\mu (\nabla^2 w) v,$$  \hspace{1cm} (2.13)

where $\nabla^s$ is defined in (2.5) and $v$ is the outward unit normal to the boundary $\partial \Omega$.

From [21], the fundamental solution $\Gamma^\omega = (\Gamma^\omega_{i,j})_{i,j=1}^3$ for the operator $L_{\lambda, \mu} + \omega^2$ in three dimensions is given by

$$\left(\Gamma^\omega_{i,j}\right)_{i,j=1}^3(x) = -\frac{\delta_{ij}}{4\pi \mu |x|} e^{ik_i|x|} + \frac{1}{4\pi \omega^2} \partial_i \partial_j \frac{e^{ik_i|x|} - e^{ik_j|x|}}{|x|},$$  \hspace{1cm} (2.14)

where $k_i$ and $k_j$ are defined in (2.7). Then the single-layer potential associated with the fundamental solution $\Gamma^\omega$ is defined as

$$S^\omega_{\delta \Omega}[\varphi](x) = \int_{\partial \Omega} \Gamma^\omega(x - y) \varphi(y) ds(y), \quad x \in \mathbb{R}^3,$$  \hspace{1cm} (2.15)

for $\varphi \in L^2(\partial \Omega)^3$. On the boundary $\partial \Omega$, the conormal derivative of the single-layer potential satisfies the following jump formula

$$\frac{\partial S^\omega_{\delta \Omega}[\varphi]}{\partial \nu}_\pm(x) = \left( \pm \frac{1}{2} I + \left( K^\omega_{\delta \Omega}\right)^* \right)[\varphi](x) \quad x \in \partial \Omega,$$  \hspace{1cm} (2.16)

where

$$\left( K^\omega_{\delta \Omega}\right)^*[\varphi](x) = \text{p.v.} \int_{\partial \Omega} \frac{\partial \Gamma^\omega}{\partial \nu(x)}(x - y) \varphi(y) ds(y),$$  \hspace{1cm} (2.17)

with p.v. standing for the Cauchy principal value and the subscript $\pm$ indicating the limits from outside and inside $\Omega$, respectively. The operator $(K^\omega_{\delta \Omega})^*$ is called to be the Neumann-Poincaré (N-P) operator. It is mentioned that the superscript $*$ of the N-P operator $(K^\omega_{\delta \Omega})^*$ signifies that it is an adjoint operator to the double-layer potential operator $K^\omega_{\eta \Omega}$ in the $L^2(\partial \Omega)$-topology, where $K^\omega_{\eta \Omega}$ is defined similarly to (2.17) with $\nu(x)$ replaced to be $\nu(y)$.

Let $\Phi(x)$ be the fundamental solution to the operator $\Delta + \omega^2$ in three dimensions given as follows

$$\Phi(x) = -\frac{e^{i\omega x}}{4\pi |x|}.$$  \hspace{1cm} (2.18)

For $\varphi \in L^2(\partial \Omega)$, we define

$$S^\omega_{\eta \Omega}[\varphi](x) = \int_{\partial \Omega} \Phi(x - y) \varphi(y) ds(y), \quad x \in \mathbb{R}^3.$$  \hspace{1cm} (2.19)

Next, to facilitate the exposition, we present some notations and useful formulas. Let $\mathbb{N}$ be the set of the positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Set $Y^m_n$ with $n \in \mathbb{N}_0$, $-n \leq m \leq n$ to be the spherical harmonic functions. Let $B_R$ denote a central ball of radius $R \in \mathbb{R}_+$ in $\mathbb{R}^3$ and $S_R := \partial B_R$. For simplicity, we set $S := S_1$. Furthermore, the operators $\nabla_S$, $\nabla^2_S$, and $\Delta_S$ designate the surface gradient, the surface divergence and the Laplace-Beltrami operator on the unit sphere $S$.

Let $j_n(t)$ and $h_n(t)$, $n \in \mathbb{N}_0$, denote the spherical Bessel and Hankel functions of the first kind of order $n$, respectively. The following asymptotic expansions shall be needed in what follows (cf. [15]),

$$j_n(t) = \frac{t^n}{(2n + 1)!} (1 + \tilde{j}(t)), \quad h_n(t) = \frac{(2n - 1)!}{i^{n+1}} (1 + \tilde{h}(t)),$$  \hspace{1cm} (2.20)
for \( n \gg 1 \), where
\[
\widetilde{j}(t) = O\left(\frac{1}{n}\right) \quad \text{and} \quad \widetilde{h}(t) = O\left(\frac{1}{n}\right);
\]
and for a fixed \( n \) with \( t \ll 1 \),
\[
j_n(t) = \frac{t^n}{(2n + 1)!!} (1 + O(t)), \quad h_n(t) = \frac{(2n - 1)!!}{i t^{n+1}} (1 + O(t)). \quad (2.21)
\]
Here and also in what follows, the double factorial is defined as
\[
(2n + 1)!! := 1 \cdot 3 \cdot (2n - 1) \cdot (2n + 1).
\]
The following three auxiliary lemmas shall be needed as well [34].

**Lemma 2.1** For a vector field \( w \in H^1(S)^3 \) and a scalar function \( v \in H^1(S) \), there hold the following relations
\[
\nabla_S \cdot (\nabla_S v \wedge \nu) = 0, \quad \triangle_S v = \nabla_S \cdot \nabla_S v,
\]
\[
\int_S \nabla_S v \cdot w \, ds = -\int_S v \nabla_S \cdot w \, ds,
\]
and
\[
\nabla_S \cdot (w v) = \nabla_S \cdot w v + w \cdot \nabla_S v.
\]

**Lemma 2.2** The spherical harmonic functions \( Y_n^m \) with \( n \in \mathbb{N}_0, -n \leq m \leq n \), are the eigenfunctions of the Laplace-Beltrami operator \( \triangle_S \) associated with the eigenvalue \(-n(n + 1)\), namely
\[
\triangle_S Y_n^m + n(n + 1)Y_n^m = 0.
\]

**Lemma 2.3** The family \((T_n^m, \mathcal{T}_n^m, \mathcal{N}_n^m)\), the vectorial spherical harmonics of order \( n \),
\[
\begin{align*}
T_n^m &= \nabla_S Y_{n+1}^m + (n + 1)Y_{n+1}^m \nu, \quad n \geq 0, \ n + 1 \geq m \geq -(n + 1), \\
\mathcal{T}_n^m &= \nabla_S Y_n^m \wedge \nu, \quad n \geq 1, \ n \geq m \geq -n, \\
\mathcal{N}_n^m &= -\nabla_S Y_{n-1}^m + nY_{n-1}^m \nu, \quad n \geq 1, \ n + 1 \geq m \geq -(n + 1),
\end{align*}
\]
forms an orthogonal basis of \((L^2(S))^3\).

From Lemma 2.3, one has that
\[
T_{n-1}^m = \nabla_S Y_n^m + nY_n^m \nu,
\]
which is a vectorial spherical harmonics of order \( n - 1 \). Thus \( T_{n-1}^m \) can be expressed by
\[
T_{n-1}^m = A_{n-1,m} Y_{n-1}^m, \quad (2.23)
\]
where
\[
Y_{n-1} = \begin{bmatrix} Y_{n-1}^{-(n-1)} \ldots, Y_{n-1}^{n-1} \end{bmatrix}^T.
\]
and $A_{n-1,m}$ is a $3 \times (2n - 1)$ matrix given by

$$A_{n-1,m} = [a_{n-1,m}^{-(n-1)}, \ldots, a_{n-1,m}^{n-1}].$$

Similarly, the vectorial spherical harmonics $N_{n+1}^m$ is of order $n + 1$. Hence, it can be expressed as

$$N_{n+1}^m = c_{n+1,m}^{n+1} Y_{n+1},$$  \hspace{1cm} (2.24)

where $C_{n+1,m}$ is a $3 \times (2n + 3)$ matrix given as follows

$$C_{n+1,m} = [c_{n+1,m}^{-(n+1)}, \ldots, c_{n+1,m}^{n+1}].$$

Next, we prove three important propositions.

**Proposition 2.1** The following identities hold

$$\int_S Y_{n-1}^q Y_n^m \nu ds = \frac{a_{n-1,m}^{q}}{2n+1}, \quad \int_S \nabla_S Y_{n-1}^q Y_n^m \nu ds = \frac{n+1}{2n+1} a_{n-1,m},$$

$$\int_S Y_{n+1}^q Y_n^m \nu ds = \frac{c_{n+1,m}^{q}}{2n+1}, \quad \int_S \nabla_S Y_{n+1}^q Y_n^m \nu ds = -\frac{n}{2n+1} c_{n+1,m},$$

and

$$\int_S Y_p^q Y_n^m \nu ds = 0, \quad \int_S \nabla_p^q Y_n^m \nu ds = 0, \quad \text{for } p \geq 0, \quad p \neq n - 1, n + 1,$$

where and also in what follows, the overline denotes the complex conjugate. Moreover, the coefficient vectors $a_{n,m}^q$ and $c_{n+1,q}^m$, defined in (2.23) and (2.24), satisfy the following identity

$$a_{n,m}^q = \frac{2n+3}{2n+1} c_{n+1,q}^m.$$  \hspace{1cm} (2.25)

**Proof** From Lemma 2.3 and the identities in (2.23) and (2.24), one has that

$$\nabla_S Y_n^m + n Y_n^m \nu = A_{n-1,m} Y_{n-1},$$

$$-\nabla_S Y_n^m + (n+1) Y_n^m \nu = C_{n+1,m} Y_{n+1}.$$  \hspace{1cm} (2.26)

Multiplying $Y_{n-1}^q$ on both sides of (2.26) and integrating on the unit sphere $S$ yield that

$$\int_S Y_{n-1}^q \nabla_S Y_n^m \nu ds + n \int_S Y_{n-1}^q Y_n^m \nu ds = a_{n-1,m}^q,$$  \hspace{1cm} (2.27)

and

$$-\int_S Y_{n-1}^q \nabla_S Y_n^m \nu ds + (n+1) \int_S Y_{n-1}^q Y_n^m \nu ds = 0.$$  \hspace{1cm} (2.28)

Solving the equations (2.27) and (2.28), one can obtain that

$$\int_S Y_{n-1}^q Y_n^m \nu ds = \frac{a_{n-1,m}^{q}}{2n+1}, \quad \int_S Y_{n-1}^q \nabla_S Y_n^m \nu ds = \frac{n+1}{2n+1} a_{n-1,m}.$$  \hspace{1cm} (2.29)
which are the first two identities in the proposition. By a similar argument, the other four integral identities can be proved.

The rest of the proof is to show the coefficient identity (2.25). Taking the complex conjugate on both sides of the equation (2.29) and replacing \( n \) with \( n + 1 \) yield that

\[
\int_{\mathcal{S}} Y_{n+1}^m Y_n^q \, ds = \frac{\overline{a}_{n,m}^q}{2n + 3}.
\]  

(2.30)

Comparing the equation (2.30) with the third integral identity of this proposition shows that

\[
a_{q n,m} = \frac{2n + 3}{2n + 1} c_{n+1,m},
\]

and this completes the proof. \( \square \)

**Proposition 2.2** The following identities hold

\[
\int_{\mathcal{S}} (\nabla_{\mathcal{S}} Y_{n-1}^q \cdot \nabla_{\mathcal{S}} Y_n^m) \, ds = \frac{(n + 1)(n - 1)}{2n + 1} a_{n-1,m}^q,
\]

\[
\int_{\mathcal{S}} (\nabla_{\mathcal{S}} Y_{n+1}^q \cdot \nabla_{\mathcal{S}} Y_n^m) \, ds = \frac{n(n + 2)}{2n + 1} c_{n+1,m}^q,
\]

and

\[
\int_{\mathcal{S}} (\nabla_{\mathcal{S}} Y_{p}^q \cdot \nabla_{\mathcal{S}} Y_n^m) \, ds = 0, \quad \text{for } p \geq 0, \ p \neq n - 1, n + 1,
\]

where the coefficient vectors \( a^q_{n,m} \) and \( c^q_{n,m} \) are defined in (2.23) and (2.24), respectively.

**Proof** From Lemmas 2.1 and 2.2, one has by direct calculations that

\[
\int_{\mathcal{S}} (\nabla_{\mathcal{S}} Y_{p}^q \cdot \nabla_{\mathcal{S}} Y_n^m) \, ds = \sum_{i=1}^{3} E_i \int_{\mathcal{S}} \nabla_{\mathcal{S}} Y_{p}^q \cdot \nabla_{\mathcal{S}} Y_n^m (v \cdot E_i) \, ds
\]

\[
= -\sum_{i=1}^{3} E_i \int_{\mathcal{S}} Y_{p}^q \nabla_{\mathcal{S}} \cdot (\nabla_{\mathcal{S}} Y_n^m (v \cdot E_i)) \, ds
\]

\[
= -\sum_{i=1}^{3} E_i \int_{\mathcal{S}} Y_{p}^q (\Delta_{\mathcal{S}} Y_n^m (v \cdot E_i) + \nabla_{\mathcal{S}} Y_n^m \cdot \nabla_{\mathcal{S}} (v \cdot E_i)) \, ds
\]

\[
= n(n + 1) \int_{\mathcal{S}} Y_{p}^q Y_n^m v - \int_{\mathcal{S}} \nabla_{\mathcal{S}} Y_n^m v \, ds,
\]

where and also in what follows, \( E_i, \ i = 1, 2, 3 \) are Euclidean unit vectors. With the help of Proposition 2.1, one can then obtain the integral identities of this proposition.

The proof is complete. \( \square \)

**Proposition 2.3** The following identities hold

\[
\int_{\mathcal{S}} \nabla_{\mathcal{S}} (\nabla_{\mathcal{S}} Y_{n-1}^q) \cdot \nabla_{\mathcal{S}} Y_n^m \, ds = \frac{-n(n + 1)(n - 1)}{2n + 1} a_{n-1,m}^q,
\]

\[
\int_{\mathcal{S}} \nabla_{\mathcal{S}} (\nabla_{\mathcal{S}} Y_{n+1}^q) \cdot \nabla_{\mathcal{S}} Y_n^m \, ds = \frac{n(n + 1)(n + 2)}{2n + 1} c_{n+1,m}^q,
\]
\[ \int_S \nabla_S (\nabla_S Y_p) \cdot \nabla_S Y_n^m \, ds = 0, \quad \text{for} \ p \geq 0, \ p \neq n - 1, n + 1, \]

where the coefficient vectors \( a_{n,m}^q \) and \( c_{n,m}^q \) are defined in (2.23) and (2.24), respectively.

**Proof** From Lemmas 2.1 and 2.2, one has that

\[
\int_S \nabla_S (\nabla_S Y_p) \cdot \nabla_S Y_n^m \, ds = \sum_{i=1}^3 e_i \int_S \nabla_S (\nabla_S Y_p \cdot e_i) \cdot \nabla_S Y_n^m \, ds
\]

\[
= n(n + 1) \int_S \nabla_S Y_p^d Y_n^m \, ds.
\]

Thus the integral identities in this proposition directly follow from Proposition 2.1.

The proof is complete.

## 3 Spectral Results of the Neumann-Poincaré Operator

In this section, we derive the complete spectral system of the N-P operator \( (K_0^\infty)^* \) in (2.17) for the elastic system within the finite-frequency regime. To that end, we first derive the spectral system of the single-layer potential and then utilize the jump formulation (2.16) to obtain the spectral system of the N-P operator.

From the expression of the fundamental solution \( \Gamma_0^\infty \) in (2.14), one can readily see that

\[
\Gamma_0^\infty = \Gamma_1^\infty + \Gamma_2^\infty,
\]

where

\[
\Gamma_1^\infty = -\frac{\delta_{ij}}{4\pi \mu |x|} e^{ik|x|} \quad \text{and} \quad \Gamma_2^\infty = \frac{1}{4\pi \omega^2} \partial_i \partial_j \frac{e^{ik|x|} - e^{ik|x|}}{|x|}.
\]

For the first part, one has \( \Gamma_1^\infty = \Phi(x) \delta_{ij} / \mu \), where \( \Phi(x) \) is the fundamental solution of the operator \( \Delta + \omega^2 \) defined in (2.18). Moreover, the spectral system of the operator \( S_k^S_{SR} \) defined in (2.19), associated with the kernel function \( \Phi(x) \), has been derived in [25]. For the convenience of readers, we include it in the following lemma.

**Lemma 3.1** The eigen-system of the single-layer potential operator \( S_k^S_{SR} \) defined in (2.19) is given as follows

\[
S_k^S_{SR}[Y_n^m](x) = -ik R^2 j_n(kR) h_n(kR) Y_n^m, \quad x \in S_R.
\]

Moreover, the following two identities hold

\[
S_k^S_{SR}[Y_n^m](x) = -ik R^2 j_n(k|x|) h_n(kR) Y_n^m \quad x \in B_R,
\]

and

\[
S_k^S_{SR}[Y_n^m](x) = -ik R^2 j_n(kR) h_n(k|x|) Y_n^m \quad x \in \mathbb{R}^3 \setminus B_R.
\]
Thus, we mainly focus on handling the second term $I^m_2$ given in (3.1). It is noted that the fundamental solution $\Phi(x - y)$ defined in (2.18) has the following expansion (cf. [15])

$$\Phi(x - y) = -ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n(k|x|)Y^m_n(\hat{x}) j_n(k|y|) \overline{Y^m_n(\hat{y})} \quad \text{for } |y| < |x|. $$

By direct calculations, there holds that

$$\nabla_y \Phi(x - y) = -ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n(k|x|)Y^m_n(\hat{x}) \nabla_y \left( j_n(k|y|) \overline{Y^m_n(\hat{y})} \right)$$

$$= -ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n(k|x|)Y^m_n(\hat{x}) \left( j''_n(k|y|)k^2 \overline{Y^m_n(\hat{y})} \hat{y} + j_n(k|y|) \nabla_y \overline{Y^m_n(\hat{y})} / |y| \right),$$

(3.3)

and

$$\frac{\partial}{\partial y} \nabla_y \Phi(x - y) = -ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n(k|x|)Y^m_n(\hat{x}) \left( j''_n(k|y|)k^2 \overline{Y^m_n(\hat{y})} \hat{y} + j_n(k|y|) \nabla_y \overline{Y^m_n(\hat{y})} / |y| \right)^2,$$

(3.4)

where

$$\frac{\partial}{\partial y} \nabla_y \Phi(x - y) = \nu_y \cdot \nabla^2 \Phi(x - y).$$

With the help of Propositions 2.1 and 2.2, one can derive the following important result.

**Proposition 3.1** There hold the following identities

$$\int_{\Sigma_R} \nabla^2_x \Phi(x - y) \cdot \nabla_y Y^m_n(\hat{y}) ds$$

$$= -ikh_{n-1}(k|x|) \left( j'_{n-1}(kR)kR \frac{n(n+1)}{2n+1} - j_{n-1}(kR) \frac{n(n+1)(n-1)}{2n+1} \right) N^m_{n-1}$$

$$- ikh_{n+1}(k|x|) \left( j'_{n+1}(kR)kR \frac{n(n+1)}{2n+1} + j_{n+1}(kR) \frac{n(n+1)(n+2)}{2n+1} \right) N^m_{n+1}. \quad (3.5)$$

and

$$\int_{\Sigma_R} \nabla^2_x \Phi(x - y) \cdot (Y^m_n(\hat{y}) \nu_y) ds$$

$$= -ikh_{n-1}(k|x|) \left( (j_{n-1}(kR) - j'_{n-1}(kR)kR) \frac{n-1}{2n+1} + j''_{n-1}(kR)kR^2 \frac{1}{2n+1} \right) T^m_{n-1}$$

$$- ikh_{n+1}(k|x|) \left( (j'_{n+1}(kR)kR - j_{n+1}(kR)) \frac{n+2}{2n+1} + j''_{n+1}(kR)kR^2 \frac{1}{2n+1} \right) N^m_{n+1}. \quad (3.6)$$
Proof Note that \( \nabla^2_x \Phi(x - y) = \nabla^2_y \Phi(x - y) \). Using integration by parts as well as Lemma 2.2, there holds

\[
\int_{\mathbb{S}^R} \nabla^2_x \Phi(x - y) \cdot \nabla Y^m_n(\hat{y}) \, ds = \int_{\mathbb{S}^R} \nabla^2_y \Phi(x - y) \cdot \nabla Y^m_n(\hat{y}) \, ds
\]

\[
= -\frac{1}{R} \int_{\mathbb{S}^R} \nabla_y \Phi(x - y) \Delta Y^m_n(\hat{y}) \, ds
\]

\[
= n(n + 1) \frac{1}{R} \int_{\mathbb{S}^R} \nabla_y \Phi(x - y) Y^m_n(\hat{y}) \, ds.
\]

Therefore, the integral identity (3.5) follows from Proposition 2.1 and the identity (3.3). For the other integral identity, one has by direct calculations that

\[
\int_{\mathbb{S}^R} \nabla^2_x \Phi(x - y) \cdot \left( Y^m_n(\hat{y}) v_y \right) \, ds
\]

\[
= \int_{\mathbb{S}^R} \left( \nabla_y \Phi(x - y) + \frac{\partial}{\partial y} \left( \nabla_y \Phi(x - y) \right) v_y \right) \cdot \left( Y^m_n(\hat{y}) v_y \right) \, ds
\]

\[
= \int_{\mathbb{S}^R} \frac{\partial}{\partial y} \left( \nabla_y \Phi(x - y) \right) Y^m_n(\hat{y}) \, ds.
\]

Finally, one can derive (3.6) from Proposition 2.1 and the identity (3.4).

The proof is complete. \( \square \)

Proposition 3.2 The following identity holds

\[
\int_{\mathbb{S}^R} \nabla^2_x \Phi(x - y) \cdot \left( \nabla Y^m_n(\hat{y}) \wedge v_y \right) \, ds = 0.
\]

Proof By using integration by parts, there holds

\[
\int_{\mathbb{S}^R} \nabla^2_x \Phi(x - y) \cdot \left( \nabla Y^m_n(\hat{y}) \wedge v_y \right) \, ds = \int_{\mathbb{S}^R} \nabla^2_y \Phi(x - y) \cdot \left( \nabla Y^m_n(\hat{y}) \wedge v_y \right) \, ds
\]

\[
= \frac{1}{R} \int_{\mathbb{S}^R} \nabla_y \left( \nabla_y \Phi(x - y) \right) \cdot \left( \nabla Y^m_n(\hat{y}) \wedge v_y \right) \, ds
\]

\[
= -\int_{\mathbb{S}^R} \nabla_y \Phi(x - y) \, \nabla_y \cdot \left( \nabla Y^m_n(\hat{y}) \wedge v_y \right) \, ds = 0,
\]

where the last identity follows from Lemma 2.1 and this completes the proof. \( \square \)

With the above preparations, we are in a position to derive the spectral system of the single-layer potential operator \( \mathbb{S}^2_{\mathbb{S}^R} \). To that end, we first show the following result about the single-layer potentials \( \mathbb{S}^2_{\mathbb{S}^R} [T^m_0], \mathbb{S}^2_{\mathbb{S}^R} [T^m_{n-1}] \) and \( \mathbb{S}^2_{\mathbb{S}^R} [N^m_{n+1}] \), which shall be critical to our subsequent analysis.
Theorem 3.1 The single-layer potentials associated with the density functions $T^m_n$, $T^m_{n-1}$, and $N^m_{n+1}$ are given as follows for $x \in \mathbb{R}^3 \setminus B_R$.

$$S_{S^R}^w[T^m_{n-1}](x) = -R^2i\left(\frac{(n+1)k_s j_{n-1,s} h_{n-1}(k_s|x|)}{\mu(2n+1)} + \frac{nk_p j_{n-1,p} h_{n-1}(k_p|x|)}{(\lambda + 2\mu)(2n+1)}\right)T^m_{n-1} - nR^2i\left(\frac{k_s j_{n-1,s} h_{n+1}(k_s|x|)}{\mu(2n+1)} - \frac{k_p j_{n+1,p} h_{n+1}(k_p|x|)}{(\lambda + 2\mu)(2n+1)}\right)N^m_{n+1},$$

$$S_{S^R}^w[N^m_{n+1}](x) = -(n+1)R^2i\left(\frac{k_s j_{n+1,s} h_{n-1}(k_s|x|)}{\mu(2n+1)} - \frac{k_p j_{n+1,p} h_{n-1}(k_p|x|)}{(\lambda + 2\mu)(2n+1)}\right)T^m_{n-1} - R^2i\left(\frac{nk_s j_{n+1,s} h_{n+1}(k_s|x|)}{\mu(2n+1)} + \frac{(n+1)k_p j_{n+1,p} h_{n+1}(k_p|x|)}{(\lambda + 2\mu)(2n+1)}\right)N^m_{n+1},$$

and

$$S_{S^R}^w[T^m_n](x) = -\frac{ik_s R^2 j_{n,s} h_n(k_s|x|)}{\mu}T^m_n,$$

where and also in what follows, we denote $j_n(k_s R)$, $j_n(k_p R)$, $h_n(k_s R)$ and $h_n(k_p R)$ by $j_{n,s}$, $j_{n,p}$, $h_{n,s}$ and $h_{n,p}$ for simplicity.

Proof The proof follows from the expression of the fundamental solution $\Gamma^w$ defined in (2.14), Lemma 3.1, and Propositions 3.1 and 3.2, along with straightforward (though tedious) calculations.

By a similar argument to Theorem 3.1, one can show

Proposition 3.3 For $x \in B_R$, the single-layer potentials $S_{S^R}^w[T^m_n]$, $S_{S^R}^w[T^m_{n-1}]$ and $S_{S^R}^w[N^m_{n+1}]$ are given as follows

$$S_{S^R}^w[T^m_{n-1}](x) = -R^2i\left(\frac{(n+1)k_s j_{n-1,s} h_{n-1}(k_s|x|)}{\mu(2n+1)} + \frac{nk_p j_{n-1,p} h_{n-1}(k_p|x|)}{(\lambda + 2\mu)(2n+1)}\right)T^m_{n-1} - nR^2i\left(\frac{k_s j_{n-1,s} h_{n+1}(k_s|x|)}{\mu(2n+1)} - \frac{k_p j_{n+1,p} h_{n+1}(k_p|x|)}{(\lambda + 2\mu)(2n+1)}\right)N^m_{n+1},$$

$$S_{S^R}^w[N^m_{n+1}](x) = -(n+1)R^2i\left(\frac{k_s j_{n+1,s} h_{n-1}(k_s|x|)}{\mu(2n+1)} - \frac{k_p j_{n+1,p} h_{n-1}(k_p|x|)}{(\lambda + 2\mu)(2n+1)}\right)T^m_{n-1} - R^2i\left(\frac{nk_s j_{n+1,s} h_{n+1}(k_s|x|)}{\mu(2n+1)} + \frac{(n+1)k_p j_{n+1,p} h_{n+1}(k_p|x|)}{(\lambda + 2\mu)(2n+1)}\right)N^m_{n+1},$$

and

$$S_{S^R}^w[T^m_n](x) = -\frac{ik_s R^2 j_{n,s} h_n(k_s|x|)}{\mu}T^m_n.$$

From Theorem 3.1 and the continuity of the single-layer potential operator $S_{S^R}^w$ from $x \in \mathbb{R}^3 \setminus B_R$ to $x \in S_R$, one can conclude that for $x \in S_R$

$$S_{S^R}^w[T^m_n](x) = b_n T^m_n, \quad S_{S^R}^w[T^m_{n-1}](x) = c_{1n} T^m_{n-1} + d_{1n} N^m_{n+1},$$

(3.8)

\(\text{ Springer}\)
and
\[ S_{S_R}^{\omega} \left[ N_{n+1}^m \right] (x) = c_{2n} x_{n-1}^m + d_{2n} N_{n+1}^m, \]  
(3.9)
where
\[ b_n = - \frac{ik_\nu R^2 j_{n,s} h_{n,s}}{\mu}, \]
\[ c_{1n} = - R^2 i \left( \frac{j_{n-1,s} h_{n-1,s} k_s (n + 1)}{\mu (2n + 1)} + \frac{j_{n-1,p} h_{n-1,p} k_p n}{(\lambda + 2\mu)(2n + 1)} \right), \]
\[ d_{1n} = - n R^2 i \left( \frac{j_{n+1,s} h_{n+1,s} k_s n}{\mu (2n + 1)} - \frac{j_{n+1,p} h_{n+1,p} k_p}{(\lambda + 2\mu)(2n + 1)} \right), \]
\[ c_{2n} = - (n + 1) R^2 i \left( \frac{j_{n+1,s} h_{n-1,s} k_s}{\mu (2n + 1)} - \frac{j_{n+1,p} h_{n-1,p} k_p (n + 1)}{(\lambda + 2\mu)(2n + 1)} \right), \]
\[ d_{2n} = - R^2 i \left( \frac{j_{n+1,s} h_{n+1,s} k_s n}{\mu (2n + 1)} + \frac{j_{n+1,p} h_{n+1,p} k_p (n + 1)}{(\lambda + 2\mu)(2n + 1)} \right). \]

The rest of the section is devoted to the derivation of the traction of the single-layer potential on the $S_R$, based on which, we can derive the spectral system of the N-P operator. First of all, we deduce the following two propositions.

**Proposition 3.4** The following identities hold for $n, p \in \mathbb{N}_0$:

\[ \nabla \cdot \left( h_n(k|x) \nabla_{S} Y_p^m \right) = - p (p + 1) h_n(k|x) Y_p^m / |x|, \]
\[ \nabla \cdot \left( h_n(k|x) Y_p^m v \right) = (kh'_n(k|x) + 2h_n(k|x) / |x|) Y_p^m, \]
and
\[ \nabla \cdot \left( h_n(k|x) \nabla_{S} Y_p^m \wedge v \right) = 0. \]

**Proof** By the vector calculus identity, one has that
\[ \nabla \cdot \left( h_n(k|x) \nabla_{S} Y_p^m \right) = \nabla h_n(k|x) \cdot \nabla_{S} Y_p^m + h_n(k|x) \nabla \cdot \nabla_{S} Y_p^m \]
\[ = h_n(k|x) \Delta_{S} Y_p^m / |x| = - p (p + 1) h_n(k|x) Y_p^m / |x|, \]
where the last two identities follow from Lemmas 2.1 and 2.2. Therefore, one can show the first identity of the proposition. The other two identities of the proposition can be shown in a similar manner. \( \square \)

**Proposition 3.5** The following identities hold for $n, p \in \mathbb{N}_0$:

\[ \nabla (h_n(k|x) \nabla_{S} Y_p^m) v = kh'_n(k|x) \nabla_{S} Y_p^m, \]
\[ \nabla (h_n(k|x) \nabla_{S} Y_p^m)^T v = -h_n(k|x) \nabla_{S} Y_p^m / |x|, \]
\[ \nabla (h_n(k|x) Y_p^m v) v = kh'_n(k|x) Y_p^m v, \]
\[ \nabla (h_n(k|x) Y_p^m v)^T v = kh'_n(k|x) Y_p^m v + h_n(k|x) / |x| \nabla_{S} Y_p^m. \]
\[
\n\nabla \left( h_n(k|x) \nabla_{\Sigma_p} Y^m \wedge \nu \right) \nu = kh_n'(k|x) \nabla_{\Sigma_p} Y^m \wedge \nu, \\
\n\nabla \left( h_n(k|x) \nabla_{\Sigma_p} Y^m \wedge \nu \right)^T \nu = -h_n(k|x)/|x| \nabla_{\Sigma_p} Y^m \wedge \nu.
\]

**Proof** In the following, we only give the proof of the first two identities and the other ones can be proved in a similar manner. First, one has

\[
\nabla \left( h_n(k|x) \nabla_{\Sigma_p} Y^m \wedge \nu \right) \nu = \left( \nabla h_n(\Sigma_p) \nabla_{\Sigma_p} Y^m \right)^T \nu = \left( \nabla h_n(\Sigma_p) \nabla_{\Sigma_p} Y^m \right)^T \nu = kh_n'(k|x) \nabla_{\Sigma_p} Y^m,
\]

where the last identity follows from the following fact

\[
\left( \nabla \nabla_{\Sigma_p} Y^m \right)^T \nu = \left( \frac{1}{|x|} \nabla \nabla_{\Sigma_p} Y^m \right)^T \nu = 0. \tag{3.10}
\]

Noting the symmetry of \( \nabla \nabla_{\Sigma_p} Y^m \) and rewriting (3.10) as

\[
\left( \nabla \nabla_{\Sigma_p} Y^m \right)^T \nu = \left( \nabla (|x| \nabla_{\Sigma_p} Y^m) \right)^T \nu = (|x| \nabla \nabla_{\Sigma_p} Y^m + \nabla_{\Sigma_p} Y^m \nu^T) \nu = 0,
\]

one can obtain that

\[
\nabla \nabla_{\Sigma_p} Y^m \nu = \left( \nabla \nabla_{\Sigma_p} Y^m \right)^T \nu = -\nabla_{\Sigma_p} Y^m / |x|. \tag{3.11}
\]

Similarly one has that

\[
\nabla \left( h_n(k|x) \nabla_{\Sigma_p} Y^m \wedge \nu \right)^T \nu = \left( \nabla h_n(\Sigma_p) \nabla_{\Sigma_p} Y^m \right)^T \nu = \left( \nabla h_n(\Sigma_p) \nabla_{\Sigma_p} Y^m \right)^T \nu = -h_n(k|x) \nabla_{\Sigma_p} Y^m / |x|,
\]

where the last identity follows from (3.11). Hence, we have shown the first two identities. The proof is complete. □

Next, we derive the tractions of the single-layer potentials \( S_{\Sigma_R}^{\omega} [T^m_n] \), \( S_{\Sigma_R}^{\omega} [T^m_{n-1}] \) and \( S_{\Sigma_R}^{\omega} [N^m_{n+1}] \) on \( \Sigma_R \).

**Proposition 3.6** The traction of the single-layer potentials \( S_{\Sigma_R}^{\omega} [T^m_n] \), \( S_{\Sigma_R}^{\omega} [T^m_{n-1}] \) and \( S_{\Sigma_R}^{\omega} [N^m_{n+1}] \) on \( \Sigma_R \) satisfy

\[
\partial_\nu S_{\Sigma_R}^{\omega} [T^m_n] (x) = b_n T^m_n, \tag{3.12}
\]

\[
\partial_\nu S_{\Sigma_R}^{\omega} [T^m_{n-1}] (x) = c_{n} T^m_{n-1} + \partial_{1n} N^m_{n+1}, \tag{3.13}
\]

\[
\partial_\nu S_{\Sigma_R}^{\omega} [N^m_{n+1}] (x) = c_{2n} T^m_{n-1} + \partial_{2n} N^m_{n+1}, \tag{3.14}
\]
where

\[ b_n = -i k_s R j_{n,s} (k_s R h_{n,s}' - h_{n,s}) , \]

\[ c_{1n} = -2(n - 1) \text{Ri} \left( \frac{j_{n-1}(k_s R) h_{n-1}(k_s R) k_s}{2n + 1} + \frac{j_{n-1}(k_p R) h_{n-1}(k_p R) k_p \mu n}{(\lambda + 2\mu)(2n + 1)} \right) \]

\[ + R^2 \left( \frac{j_{n-1}(k_s R) h_n(k_s R) k^2_n(n + 1) + j_{n-1}(k_p R) h_n(k_p R) k^2_p n}{2n + 1} \right), \]

\[ \vartheta_{1n} = 2n(n + 2) \text{Ri} \left( \frac{j_{n+1}(k_s R) h_{n+1}(k_s R) k_s}{2n + 1} - \frac{j_{n+1}(k_p R) h_{n+1}(k_p R) k_p \mu}{(\lambda + 2\mu)(2n + 1)} \right) \]

\[ + n R^2 \left( -\frac{j_{n-1}(k_s R) h_n(k_s R) k^2_s + j_{n-1}(k_p R) h_n(k_p R) k^2_p}{2n + 1} \right), \]

\[ c_{2n} = -2(n^2 - 1) \text{Ri} \left( \frac{j_{n+1}(k_s R) h_{n-1}(k_s R) k_s}{2n + 1} - \frac{j_{n+1}(k_p R) h_{n-1}(k_p R) k_p \mu}{(\lambda + 2\mu)(2n + 1)} \right) \]

\[ - (n + 1) R^2 \left( -\frac{j_{n-1}(k_s R) h_n(k_s R) k^2_s + j_{n-1}(k_p R) h_n(k_p R) k^2_p}{2n + 1} \right), \]

\[ \vartheta_{2n} = 2(n + 2) \text{Ri} \left( \frac{j_{n+1}(k_s R) h_{n+1}(k_s R) k_s n}{2n + 1} + \frac{j_{n+1}(k_p R) h_{n+1}(k_p R) k_p \mu(n + 1)}{(\lambda + 2\mu)(2n + 1)} \right) \]

\[ - R^2 \left( \frac{j_{n+1}(k_s R) h_n(k_s R) k^2_n + j_{n+1}(k_p R) h_n(k_p R) k^2_p(n + 1)}{2n + 1} \right). \]

Proof The proof follows from straightforward though tedious calculations along with the help of (2.13) and Propositions 3.4 and 3.5.

We are in a position to present the spectral system of the N-P operator \((K_{S,R}^\omega)^*\).

**Theorem 3.2** The spectral system of the N-P operator \((K_{S,R}^\omega)^*\) is given as follows

\[ (K_{S,R}^\omega)^*[\mathcal{T}_n^m] = \lambda_{1,n} \mathcal{T}_n^m, \]  
\[ (K_{S,R}^\omega)^*[\mathcal{U}_n^m] = \lambda_{2,n} \mathcal{U}_n^m, \]  
\[ (K_{S,R}^\omega)^*[\mathcal{V}_n^m] = \lambda_{3,n} \mathcal{V}_n^m, \]  

where

\[ \lambda_{1,n} = b_n - 1/2, \]

and if \(\vartheta_{1n} \neq 0,\)

\[ \lambda_{2,n} = \frac{\vartheta_{1n} + \vartheta_{2n} - 1 + \sqrt{(\vartheta_{2n} - \vartheta_{1n})^2 + 4 \vartheta_{1n} \vartheta_{2n}}}{2} \]

\[ \lambda_{3,n} = \frac{\vartheta_{1n} + \vartheta_{2n} - 1 - \sqrt{(\vartheta_{2n} - \vartheta_{1n})^2 + 4 \vartheta_{1n} \vartheta_{2n}}}{2} \]

\[ \mathcal{U}_n^m = (\vartheta_{1n} - \vartheta_{2n} + \sqrt{(\vartheta_{2n} - \vartheta_{1n})^2 + 4 \vartheta_{1n} \vartheta_{2n}}) \mathcal{T}_{n-1}^m + 2 \vartheta_{1n} \mathcal{V}_{n+1}^m, \]

\[ \mathcal{V}_n^m = (\vartheta_{1n} - \vartheta_{2n} - \sqrt{(\vartheta_{2n} - \vartheta_{1n})^2 + 4 \vartheta_{1n} \vartheta_{2n}}) \mathcal{T}_{n-1}^m + 2 \vartheta_{1n} \mathcal{V}_{n+1}^m, \]
if $\partial_{1n} = 0$,
\[
\lambda_{2,n} = \epsilon_{1n} - 1/2, \quad \lambda_{3,n} = \partial_{2n} - 1/2,
\]
\[
\mathcal{U}^m_n = \mathcal{I}^m_{n-1}, \quad \mathcal{V}^m_n = \epsilon_{2n}\mathcal{I}^m_{n-1} + (\partial_{2n} - \epsilon_{1n})\mathcal{N}^m_{n+1},
\]
with $\mathcal{I}^m_n$, $\mathcal{I}^m_{n-1}$, and $\mathcal{N}^m_{n+1}$ given in Lemma 2.3, and the parameters $\mathcal{b}_n$, $\epsilon_{1n}$, $\partial_{1n}$, $\epsilon_{2n}$ and $\partial_{2n}$ defined in (3.15).

**Proof** From the jump formula (2.16) and the identity (3.12), one can directly have that
\[
(K^o_{S_R})^* \left[ \mathcal{I}^m_n \right] = \frac{\partial}{\partial y} S^o_{S_R} \left[ \mathcal{I}^m_n \right] - \frac{1}{2} \mathcal{I}^m_n = (\mathcal{b}_n - 1/2) \mathcal{I}^m_n.
\]
Hence, the first identity (3.16) is proved. For the other two ones, namely (3.17) and (3.18), by noting (3.13) and (3.14), one sees that the eigenfunctions should be the linear combinations of $\mathcal{I}^m_{n-1}$, and $\mathcal{N}^m_{n+1}$. Hence, we can assume that the eigenfunctions have the following form $a\mathcal{I}^m_{n-1} + \mathcal{N}^m_{n+1}$, namely,
\[
(K^o_{S_R})^* \left[ a\mathcal{I}^m_{n-1} + \mathcal{N}^m_{n+1} \right] = \lambda \left( a\mathcal{I}^m_{n-1} + \mathcal{N}^m_{n+1} \right).
\]
Again from the jump formula (2.16) and the identities (3.13) and (3.14), one has that
\[
(K^o_{S_R})^* \left[ \mathcal{I}^m_{n-1} \right] = (\epsilon_{1n} - 1/2) \mathcal{I}^m_{n-1} + \partial_{1n} \mathcal{N}^m_{n+1},
\]
\[
(K^o_{S_R})^* \left[ \mathcal{N}^m_{n+1} \right] = \epsilon_{2n} \mathcal{I}^m_{n-1} + (\partial_{2n} - 1/2) \mathcal{N}^m_{n+1}.
\]
Substituting the last two equations into (3.19) and comparing the coefficient on both sides yield that
\[
a^2 \partial_{1n} + a(\partial_{2n} - \epsilon_{1n}) - \epsilon_{2n} = 0.
\]
If $\partial_{1n} \neq 0$, solving the equation (3.21) gives that
\[
a = \frac{\epsilon_{1n} - \partial_{2n} \pm \sqrt{(\partial_{2n} - \epsilon_{1n})^2 + 4\partial_{1n} \epsilon_{2n}}}{2\partial_{1n}}.
\]
Therefore, the two identities (3.17) and (3.18), follow from substituting (3.22) into (3.19). If $\partial_{1n} = 0$, from the equation (3.20), one can directly have that
\[
(K^o_{S_R})^* \left[ \mathcal{I}^m_{n-1} \right] = (\epsilon_{1n} - 1/2) \mathcal{I}^m_{n-1},
\]
which signifies that $\mathcal{I}^m_{n-1}$ is one of the eigenfunctions of the N-P operator $K^o_{S_R}$ corresponding to the eigenvalue $\epsilon_{1n} - 1/2$. For the other eigenfunction containing $\mathcal{N}^m_{n+1}$, solving the equation (3.21) yields that
\[
a = \frac{\epsilon_{2n}}{\partial_{2n} - \epsilon_{1n}}.
\]
Substituting the last equation into (3.19) yields that
\[
(K^o_{S_R})^* \left[ \mathcal{N}^m_{n+1} \right] = (\partial_{2n} - 1/2) \mathcal{N}^m_{n+1},
\]
where
\[
\mathcal{N}^m_n = \epsilon_{2n} \mathcal{I}^m_{n-1} + (\partial_{2n} - \epsilon_{1n}) \mathcal{N}^m_{n+1}.
\]
The proof is complete. □

Remark 3.1 By taking \( \omega \to +0 \) in the spectral results in Theorem 3.2 and applying the asymptotic properties of the spherical Bessel and Hankel functions, \( j_\nu(t) \) and \( h_\nu(t) \), for \( t \ll 1 \) in (2.21), one can obtain after straightforward though tedious calculations the spectral system of the N-P operator \((K^0_{\partial \Omega})^*\) in the static case, which coincides with that established in [16].

4 Polariton Resonance Beyond the Quasi-Static Approximation

In this section, using the spectral results established in the previous section, we construct a broad class of elastic structures of the form \( C_0 \) in (2.3) with no core, namely \( D = \emptyset \) that can induce polariton resonances. Suppose that a source term \( f \in H^{-1}(\mathbb{R}^3) \) is compactly supported outside \( \Omega \), then the elastic system (2.4) can be simplified as the following transmission problem

\[
\begin{cases}
L_{\lambda,\mu} u(x) + \omega^2 u(x) = 0, & x \in \Omega \\
L_{\lambda,\mu} u(x) + \omega^2 u(x) = f, & x \in \mathbb{R}^3 \setminus \Omega \\
u(x)|_\partial = u(x)|_+, & x \in \partial \Omega \\
\partial \nu u(x)|_- = \partial \nu u(x)|_+, & x \in \partial \Omega,
\end{cases}
\tag{4.1}
\]

where \( \partial \nu \) is given in (2.13), \( L_{\lambda,\mu} \) is defined in (2.12) and \( u \) satisfies the radiation condition (2.6). In (4.1) and also in what follows, \( L_{\lambda,\mu} \) and \( \partial \nu \) denote the Lamé operator and the traction operator associated with the Lamé parameters \( \lambda \) and \( \mu \), and the same notations hold for the single-layer potential operator \( S^0_{\partial \Omega} \) and the N-P operator \((K^0_{\partial \Omega})^*\).

Using the single-layer potential defined in (2.15), the solution to the system (4.1) can be written as

\[
\begin{cases}
\hat{S}^0_{\partial \Omega} \psi_1(x), & x \in \Omega, \\
S^0_{\partial \Omega} \psi_2(x) + F, & x \in \mathbb{R}^3 \setminus \Omega,
\end{cases}
\tag{4.2}
\]

where

\[
F(x) := \int_{\mathbb{R}^3} \Gamma \omega(x - y)f(y)dy, \quad x \in \mathbb{R}^3,
\tag{4.3}
\]

is called the Newtonian potential of the source \( f \) and \( \psi_1, \psi_2 \in L^2(\partial \Omega)^3 \). One can readily verify that the solution defined in (4.2) satisfy the first two conditions in (4.1). For the third and forth condition in (4.1) across \( \partial \Omega \), namely the transmission condition, one can obtain that

\[
\begin{cases}
\hat{S}^0_{\partial \Omega} \psi_1 - S^0_{\partial \Omega} \psi_2 = F, & x \in \partial \Omega. \\
\partial \nu \hat{S}^0_{\partial \Omega} \psi_1|_- - \partial \nu S^0_{\partial \Omega} \psi_2|_+ = \partial \nu F,
\end{cases}
\tag{4.4}
\]

With the help of the jump formula (2.16), the equation (4.4) can be rewritten as

\[
A^\omega \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} F \\ \partial \nu F \end{bmatrix},
\tag{4.5}
\]

where

\[
A^\omega = \begin{bmatrix}
\hat{S}^0_{\partial \Omega} & -S^0_{\partial \Omega} \\
-1/2I + (\hat{K}^0_{\partial \Omega})^* & -1/2I - (K^0_{\partial \Omega})^*
\end{bmatrix}.
\tag{4.6}
\]
In what follows and throughout the rest of this section, we always assume that the domain \( \Omega \) is a ball \( B_R \), namely \( \Omega = B_R \) and \( \partial \Omega = S_R \). Since the source term \( f \) is supported outside \( B_R \), there exists \( \epsilon > 0 \) such that when \( x \in B_{R+\epsilon} \), the Newtonian potential \( F \) defined in (4.3) satisfies

\[
L_{\lambda, \mu} \mathbf{F} + \omega^2 \mathbf{F} = 0.
\]

Thus \( F \) can be written as

\[
F = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( f_{1,n,m} j_n(k_s|x|)T^m_n + f_{2,n,m} S^m_{S_R} \left[ T^m_{n-1} \right] + f_{3,n,m} S^m_{S_R} \left[ N^m_{n+1} \right] \right),
\]

for \( x \in B_{R+\epsilon} \), which follows from Lemma 2.3 and Proposition 3.3.

Our main result in this section is stated in the following theorem. It characterizes the polariton resonance for the configuration without a core.

**Theorem 4.1** Consider the configuration \( C_0 \) with \( D = \emptyset \) defined in (2.3). Suppose that the source term \( f \in H^{-1}(\mathbb{R}^3) \) is compactly supported outside the domain \( \Omega \), whose Newtonian potential \( \mathbf{F} \) is defined in (4.7) with \( f_{1,n_0,m} \neq 0 \) for some \( n_0 \in \mathbb{N} \). For any \( M \in \mathbb{R}_+ \), if the Lamé parameter \( \hat{\mu} \) inside the domain \( \Omega \) is chosen such that

\[
\frac{\Im(\hat{\mu})}{|\tilde{\psi}_{1,n_0,m}|^2} > M,
\]

where \( \tilde{\psi}_{1,n_0,m} \) is defined in (4.14), then the polariton resonance occurs.

Furthermore, if \( n_0 \gg 1 \) is large enough such that the spherical Bessel and Hankel functions, \( j_n(t) \) and \( h_n(t) \), enjoy the asymptotic expression shown in (2.20), then one can choose the Lamé parameter \( \hat{\mu} \) inside the domain \( \Omega \) as follows

\[
\hat{\mu} = -\mu + i \frac{1}{M} + p_{1,n_0},
\]

where \( p_{1,n_0} \) should satisfy

\[
p_{1,n_0} + q_{1,n_0} = \mathcal{O}\left( \frac{1}{M} \right),
\]

with \( q_{1,n_0} \) defined via (4.18), to ensure the occurrence of the polariton resonance.

**Proof** Following Propositions 3.3 and 3.6, one can conclude that the displacement and traction of the term \( j_n(k_s|x|)T^m_n \) on the boundary \( B_R \) are orthogonal to both the corresponding components of the other two terms, namely \( S^m_{S_R} \left[ T^m_{n-1} \right] \) and \( S^m_{S_R} \left[ N^m_{n+1} \right] \). Therefore, in order to show the polariton resonance, it suffices to consider the source only containing the terms \( j_n(k_s|x|)T^m_n \), namely

\[
\mathbf{F} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( f_{1,n,m} j_n(k_s|x|)T^m_n \right).
\]
Thanks to the orthogonality of the functions $T^m_n$, $T^m_n$ and $N^m_n$, the density functions in (4.2) have the following expressions

$$
\psi_1 = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \psi_{1,n,m} T^m_n,
$$

(4.12)

$$
\psi_2 = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \psi_{2,n,m} T^m_n.
$$

From the jump formula (2.16), and Propositions 3.3 and 3.6, the equation (4.5) can be written as

$$
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
  \psi_{1,n,m} \\
  \psi_{2,n,m}
\end{bmatrix} =
\begin{bmatrix}
  f_{1,n,m} j_n(\hat{k}_s R) \\
  g_{1,n,m}
\end{bmatrix},
$$

(4.13)

where

$$
a_{11} = -i \hat{k}_s R^2 j_n(\hat{k}_s R) h_n(\hat{k}_s R) / \mu,
$$

$$
a_{12} = i \hat{k}_s R^2 j_n(\hat{k}_s R) h_n(\hat{k}_s R) / \mu,
$$

$$
a_{21} = -i \hat{k}_s R^2 h_n(\hat{k}_s R) (\hat{k}_s R j_n'(\hat{k}_s R) - j_n(\hat{k}_s R)),
$$

$$
a_{22} = -1 + i \hat{k}_s R^2 h_n(\hat{k}_s R) (\hat{k}_s R j_n'(\hat{k}_s R) - j_n(\hat{k}_s R)),
$$

and

$$
g_{1,n,m} = f_{1,n,m} \mu (k_s R j_n'(k_s R) - j_n(k_s R)) / R,
$$

with $\hat{k}_s = \omega / \sqrt{\mu}$. With the help of the Wronskian identity

$$
j_n(t) h_n'(t) - j_n'(t) h_n(t) = \frac{i}{t^2}, \quad \text{for} \quad t > 0,
$$

solving the equation (4.13) yields that

$$
\psi_{1,n,m} = \frac{f_{1,n,m} j_n(\hat{k}_s R)}{\psi_{1,n,m}},
$$

(4.14)

where

$$
\tilde{\psi}_{1,n,m} = \left( (\mu - \hat{\mu}) j_n(\hat{k}_s R) + \hat{k}_s m R j_n'(\hat{k}_s R) h_n(k_s R)
\right.

\left. - k_s m R j_n(\hat{k}_s R) h_n'(k_s R) \right) k_s \hat{k}_s R^3 j_n(k_s R) h_n(\hat{k}_s R) R.
$$

Next we calculate the dissipation energy $E(u)$. From the definition of the functional $P_{\lambda,\mu}(u,u)$ given in (2.8) and the following identity

$$
\nabla \cdot u = \nabla \cdot \hat{S}_{\omega \hat{\mu}}[T^m_n] = 0,
$$

there holds that

$$
E(u) = \Im P_{\lambda,\hat{\mu}}(u,u) = \Im \left( \hat{\mu} P_{\lambda,\hat{\mu},1}(u,u) \right)
$$

$$
= \Im(\hat{\mu}) \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} (|\psi_{1,n,m}|^2 P_{\lambda,\hat{\mu},1}(\hat{S}_{\omega \hat{\mu}}[T^m_n], \hat{S}_{\omega \hat{\mu}}[T^m_n])).
$$

(4.15)
Thus if there exists \( n_0 \) such that for any \( M \in \mathbb{R}_+ \)
\[
\Im(\hat{\mu})|\psi_{1,n_0,m}|^2 > M,
\]
(4.16)
then resonance occurs. From the expression of \( \psi_{1,n,m} \) in (4.14), the condition (4.16) is equivalent to the following one
\[
\frac{\Im(\hat{\mu})}{|\tilde{\psi}_{1,n_0,m}|^2} > M,
\]
(4.17)
since \( f_{1,n_0,m} \neq 0 \).

Next we perform some asymptotic analysis for the left-hand side of the condition (4.17) for the large \( n_0 \gg 1 \). From the asymptotic expression of the spherical Bessel and Hankel functions, \( j_n(t) \) and \( h_n(t) \) in (2.20), one can obtain that
\[
\tilde{\psi}_{1,n_0,m} = C(\hat{\mu} + \mu + q_{1,n_0})
\]
(4.18)
where \( C \) is a constant and
\[
q_{1,n_0} = O\left(\frac{1}{n_0}\right).
\]
Thus if the parameter \( \hat{\mu} \) inside the domain \( \Omega \) is chosen as stated in the theorem that
\[
\hat{\mu} = -\mu + i/M + p_{1,n_0},
\]
(4.19)
where
\[
p_{1,n_0} + q_{1,n_0} = O\left(1/M\right),
\]
(4.20)
with \( q_{1,n_0} \) defined in (4.18), then the left-hand side of the condition (4.17) can be simplified as
\[
\frac{\Im(\hat{\mu})}{|\tilde{\psi}_{1,n_0,m}|^2} \geq M.
\]
(4.21)
Thus the polariton resonance occurs and the proof is complete. □

Remark 4.1 In Theorem 4.1, we only impose a constraint on the Lamé parameter \( \hat{\mu} \) (by (4.8) and (4.9), respectively) and there is no restriction on the Lamé parameter \( \hat{\lambda} \). This indicates that only the first strong convexity condition in (2.2) might need to be violated. That is, for the elastic metamaterial in \( \Omega \), only the real part of its shear modulus might be required to be negative (according to (4.8) or (4.9)). According to the two generic examples in Remarks 4.2 and 4.3 in what follows, it is indeed that the shear modulus needs to be negative in order to have the polariton resonance.

Remark 4.2 We present a numerical example to illustrate that the condition (4.8) can indeed be achieved generically. The parameters are chosen as follows
\[
n_0 = 5, \quad \omega = 5, \quad R = 1, \quad \mu = 1, \quad \Re(\hat{\mu}) = -1.87988,
\]
(4.22)
which is the case beyond the quasi-static approximation from the values of \( \omega \) and \( R \). The absolute value of the LHS quantity in (4.8) in terms of the parameter \( \Im(\hat{\mu}) \) is plotted in Fig. 1, which evidently demonstrates that the condition (4.8) is fulfilled. Finally, we would like to point out that according to (4.22), the real part of the shear modulus of the material occupying \( \Omega \) is negative.
Fig. 1 The absolute value of the LHS quantity in (4.8) in terms of the parameter $\Im(\hat{\mu})$. Horizontal axis: value of $\Im(\hat{\mu})$; Vertical axis: value of $|\tilde{\psi}_{1,n_0,m}|$ where $\tilde{\psi}_{1,n_0,m}$ is defined in (4.14).

Fig. 2 The absolute value of the LHS quantity in (4.10) in terms of the parameter $p_{1,n_0}$. Horizontal axis: value of $p_{1,n_0}$; Vertical axis: value of $|p_{1,n_0} + q_{1,n_0}|$, where $q_{1,n_0}$ is defined via (4.18).

Remark 4.3 The condition (4.10) is easy to fulfil. Since the parameter $q_{1,n_0}$ defined in (4.18) is of $O(1/n_0)$, therefore one could choose $p_{1,n_0} = O(1/n_0)$ to fulfil the condition (4.10). Moreover, we conduct a numerical experiment to demonstrate that the condition (4.10) can be fulfilled generically. The parameters are chosen as follows

$$n_0 = 100, \quad \omega = 5, \quad R = 1, \quad \mu = 1, \quad M = 10^{10} \quad \text{and} \quad \hat{\mu} = -\mu + i/M + p_{1,n_0}. \quad (4.23)$$

One can easily verify that this is the case beyond the quasi-static approximation. The absolute value of the LHS quantity in (4.10) in terms of the parameter $p_{1,n_0}$ is plotted in Fig. 2, which apparently demonstrates that the condition (4.10) is satisfied with $p_{1,n_0} \approx -0.02779005 = O(1/n_0)$. Clearly, according to (4.23), the real part of the shear modulus of the material in $\Omega$ is negative.

5 CALR Beyond the Quasi-Static Approximation

In this section, we consider the cloaking effect induced by anomalous localized resonance. Henceforth, we let $D = B_{r_i}$ and $\Omega = B_{r_e}$ with $0 < r_i < r_e < +\infty$. To save the notations, we first define the following two functions

$$\hat{j}_n(t) = tj_n'(t) - j_n(t),$$
$$\hat{h}_n(t) = th_n'(t) - h_n(t), \quad (5.1)$$

where $j_n'(t)$ and $h_n'(t)$ are the derivatives of the functions $j_n(t)$ and $h_n(t)$, respectively. Set

$$\hat{k}_s = \omega/\sqrt{\hat{\mu}}, \quad \text{and} \quad \tilde{k}_s = \omega/\sqrt{\tilde{\mu}}.$$
and we also introduce the following notations,

\begin{align}
 j_{n0i} &= j_n(k_i r_i), & j_{n1i} &= j_n(\hat{k}_i r_i), & j_{n2i} &= j_n(\hat{k}_i r_i), \\
 j_{n0e} &= j_n(k_e r_e), & j_{n1e} &= j_n(\hat{k}_e r_e), & j_{n2e} &= j_n(\hat{k}_e r_e),
\end{align}

(5.2)

and, the same notations hold for the spherical Hankel function \( h_n(t) \), the derivative of the Bessel and Hankel functions, \( j'_n(t) \) and \( h'_n(t) \), the functions \( \tilde{j}_n(t) \) and \( \tilde{h}_n(t) \) defined in (2.20), and the functions \( \tilde{j}_n(t) \) as well as \( \tilde{h}_n(t) \) defined in (5.1). Moreover, we let \( L_{\lambda, \mu} \), \( \partial_\nu \), \( \hat{S}_{\lambda D} \) and \( (\hat{K}_{\lambda D}^o)^* \), respectively, denote the Lamé operator, the associated conormal derivative, the single-layer potential operator and the N-P operator associated with the Lamé parameters \((\lambda, \mu)\).

Assume that the source \( f \in H^{-1}(\mathbb{R}^3) \) is compactly supported outside \( \Omega \), then the elastic system (2.4) can be expressed as the following equation system

\[
\begin{align}
 L_{\lambda, \mu} u(x) + \omega^2 u(x) &= 0, & \text{in } D, \\
 L_{\lambda, \mu} u(x) + \omega^2 u(x) &= 0, & \text{in } \Omega \setminus \overline{D}, \\
 L_{\lambda, \mu} u(x) + \omega^2 u(x) &= f, & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\
 u_- = u_+ & \quad \partial_\nu u_- = \partial_\nu u_+ & \text{on } \partial D, \\
 u_- = u_+ & \quad \partial_\nu u_- = \partial_\nu u_+ & \text{on } \partial \Omega.
\end{align}
\]

(5.3)

With the help of the potential theory, the solution to the equation system (5.3) can be represented by

\[
 u(x) = \begin{cases} 
 \hat{S}_{\lambda D}^o [\varphi_1](x), & x \in D, \\
 \hat{S}_{\lambda D}^o [\varphi_2](x) + \hat{S}_{\lambda D}^o [\varphi_3](x), & x \in \Omega \setminus \overline{D}, \\
 \hat{S}_{\lambda D}^o [\varphi_4](x) + F(x), & x \in \mathbb{R}^3 \setminus \overline{\Omega},
\end{cases}
\]

(5.4)

where \( \varphi_1, \varphi_2 \in L^2(\partial D)^3 \), \( \varphi_3, \varphi_4 \in L^2(\partial \Omega)^3 \) and \( F \) is the Newtonian potential of the source \( f \) defined in (4.3). One can easily see that the solution given (5.4) satisfies the first two condition in (5.3) and the last two conditions on the boundary yield that

\[
\begin{align}
 &\hat{S}_{\lambda D}^o [\varphi_1] = \hat{S}_{\lambda D}^o [\varphi_2] + \hat{S}_{\lambda D}^o [\varphi_3], & \text{on } \partial D, \\
 &\partial_\nu \hat{S}_{\lambda D}^o [\varphi_1]_- = \partial_\nu (\hat{S}_{\lambda D}^o [\varphi_2] + \hat{S}_{\lambda D}^o [\varphi_3])_+, & \text{on } \partial D, \\
 &\hat{S}_{\lambda D}^o [\varphi_2] + \hat{S}_{\lambda D}^o [\varphi_3] = \hat{S}_{\lambda D}^o [\varphi_4] + F, & \text{on } \partial \Omega, \\
 &\partial_\nu (\hat{S}_{\lambda D}^o [\varphi_2] + \hat{S}_{\lambda D}^o [\varphi_3])_- = \partial_\nu (\hat{S}_{\lambda D}^o [\varphi_4] + F)_+, & \text{on } \partial \Omega.
\end{align}
\]

(5.5)

With the help of the jump formula in (2.16), the equation system (5.5) further yields the following integral system,

\[
\begin{bmatrix}
 -\frac{1}{2} + (\hat{K}_{\lambda D}^o)^* & -\hat{S}_{\lambda D}^o & -\hat{S}_{\lambda D}^o & 0 \\
 0 & 0 & \hat{S}_{\lambda D}^o & 0 \\
 0 & \partial_\nu \hat{S}_{\lambda D}^o & -\frac{1}{2} + (\hat{K}_{\lambda D}^o)^* & -\frac{1}{2} - (\hat{K}_{\lambda D}^o)^* \\
 & & & \\
 \end{bmatrix}
\begin{bmatrix}
 \varphi_1 \\
 \varphi_2 \\
 \varphi_3 \\
 \varphi_4 \\
 \end{bmatrix}
=
\begin{bmatrix}
 0 \\
 0 \\
 F \\
 \partial_\nu F \\
\end{bmatrix},
\]

(5.6)

where \( \partial_\nu \) and \( \partial_\nu \) signify the conormal derivatives on the boundaries of \( D \) and \( \Omega \), respectively.
In the following, we assume that the Newtonian potential \( F \) of the source \( \mathbf{f} \) has the following expression
\[
F = \sum_{n=N}^{\infty} \sum_{m=-n}^{n} (f_{1,n,m} j_n(k_s |x|) T_n^m) \quad \text{for} \ x \in \Omega,
\]
where \( N \) is large enough such the spherical Bessel and Hankel functions, \( j_n(t) \) and \( h_n(t) \), fulfill the asymptotic expansions shown in (2.20). From the Theorem 3.1 and the orthogonality of the functions \( T_n^m, T_{n-1}^m \) and \( N_{n+1}^m \), one can deduce that the density functions \( \varphi_i, \ i = 1, 2, 3, 4 \) can be written as follows
\[
\varphi_1 = \sum_{n=N}^{\infty} \sum_{m=-n}^{n} \varphi_{1,n,m} T_n^m, \quad \varphi_2 = \sum_{n=N}^{\infty} \sum_{m=-n}^{n} \varphi_{2,n,m} T_n^m, \quad \varphi_3 = \sum_{n=N}^{\infty} \sum_{m=-n}^{n} \varphi_{3,n,m} T_n^m, \quad \varphi_4 = \sum_{n=N}^{\infty} \sum_{m=-n}^{n} \varphi_{4,n,m} T_n^m.
\]
(5.8)

With the help of the equation (3.8) as well as the Theorem 3.2 and by substituting the expressions in (5.7) into the equation system (5.6), the integral system can be reduced the following equation system
\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} & 0 \\
    a_{21} & a_{22} & a_{23} & 0 \\
    0 & a_{32} & a_{33} & a_{34} \\
    0 & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
    \varphi_{1,n,m} \\
    \varphi_{2,n,m} \\
    \varphi_{3,n,m} \\
    \varphi_{4,n,m}
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0 \\
    f_{1,n,m} j_{n0e} \\
    g_{1,n,m}
\end{bmatrix},
\]
(5.9)
where
\[
a_{11} = -i \hat{k} r^2 j_{n11} h_{n1i}, \quad a_{12} = -i \hat{k} r^2 j_{n12} h_{n2i}, \quad a_{13} = -i \hat{k} r^2 j_{n11} h_{n2e},
\]
\[
a_{21} = -i \hat{k} r_1 j_{n11} h_{n11}, \quad a_{22} = -i \hat{k} r_1 j_{n12} h_{n21}, \quad a_{23} = -i \hat{k} r_1 j_{n21} h_{n2e},
\]
\[
a_{32} = -i \hat{k} r_2 j_{n21} h_{n2e}, \quad a_{33} = -i \hat{k} r_2 j_{n22} h_{n2e}, \quad a_{34} = i \hat{k} r_2 j_{n0e} h_{n0e},
\]
\[
a_{42} = -i \hat{k} r_1 j_{n21} h_{n2e}, \quad a_{43} = -i \hat{k} r_2 j_{n22} h_{n2e}, \quad a_{44} = i \hat{k} r_1 j_{n0e} h_{n0e},
\]
and
\[
g_{1,n,m} = f_{1,n,m} \mu (j_{n0e}^2 - j_{n0e})/r_e.
\]
Solving the equation system (5.9) gives that
\[
\varphi_{1,n,m} = \frac{\tilde{\varphi}_{1,n,m}}{d_{n,m}}, \quad \varphi_{2,n,m} = \frac{\tilde{\varphi}_{2,n,m}}{d_{n,m}}, \quad \varphi_{3,n,m} = \frac{\tilde{\varphi}_{3,n,m}}{d_{n,m}}, \quad \varphi_{4,n,m} = \frac{\tilde{\varphi}_{4,n,m}}{d_{n,m}},
\]
(5.10)
where
\[
\tilde{\varphi}_{1,n,m} = -i k^2 r^2 f_{1,n,m} h_{n23} j_{n0e} j_{n21} (\hat{h}_{n0e} - j_{n0e} h_{n0e}) (\hat{h}_{n21} j_{n21} - j_{n21} h_{n21}).
\]
To simplify the exposition, we introduce the following two notations

\[
\eta_{n2e} = n - 1 + n\tilde{j}_{n2e} - \tilde{j}_{n2e},
\]

and

\[
\gamma_{n2e} = n + 2 + (n + 1)\tilde{h}_{n2e} + \tilde{h}_{n2e},
\]

where \(\tilde{j}_{n2e}, \tilde{j}_{n2e}, \tilde{h}_{n2e}\) and \(\tilde{h}_{n2e}\) are defined in (5.2). The same notations also hold for \(\eta_{n1i}, \eta_{n1i}, \gamma_{n0e}\) and \(\gamma_{n2i}\). We also define the following function

\[
q_{2,n}(\tilde{\mu}, \hat{\mu}, r_i, r_e) = (\hat{\mu} + \tilde{\mu})(\mu + \tilde{\mu})n^2 r_e^2 + (\tilde{\mu}r_i - \hat{\mu}r_e)(\mu r_i - \tilde{\mu}r_e)n^2 \rho^{2n}
\]

\[
- \tilde{\mu}r_e(1 + \tilde{h}_{n0e})\tilde{\mu}r_{\eta_{11}}(\gamma_{n2e}\rho^{2n}(1 + \tilde{\gamma}_{n2i}) + (1 + \tilde{\gamma}_{n2i})\eta_{n2e})
\]

\[
- \hat{\mu}(1 + \tilde{\gamma}_{n1i})(r_i\gamma_{n2e}\eta_{n2i}\rho^{2n} - r_e\gamma_{n2e}\eta_{n2e})
\]

\[
- \tilde{\mu}\rho_{n0e}(\tilde{\mu}r_{\eta_{11}}(r_e(1 + \tilde{\gamma}_{n2i})(1 + \tilde{\gamma}_{n2e}) - r_i\rho^{2n}(1 + \tilde{\gamma}_{n2e})(1 + \tilde{\gamma}_{n2i})
\]

\[
\times \hat{\mu}(1 + \tilde{\gamma}_{n1i})(r_i^2\rho^{2n}(1 + \tilde{\gamma}_{n2e})\eta_{n2i} + r_e^2(1 + \tilde{\gamma}_{n2e})\gamma_{n2i}),
\]

where and also in what follows, \(\rho := r_i/r_e\).

With the above preparation, we are in a position to show the CALR result, which is concluded in the following theorem.

**Theorem 5.1** Consider the configuration \((C_0, f)\) where \(C_0\) is given in (2.3). Suppose that the Newtonian potential \(F\) of the source term \(f\) has the expression shown in (5.7) with \(f_{1,n_0,m} \neq 0\) for some \(n_0 \in \mathbb{N}\). For any \(M \in \mathbb{R}_+\), if the parameters in \(C_0\) are chosen as follows

\[
\tilde{\mu} = \mu, \quad \text{and} \quad \hat{\mu} = -\mu + i\rho^{n_0} + p_{2,n_0},
\]

such that

\[
p_{2,n_0}^2 + q_{2,n_0} = O(\rho^{2n_0})
\]
and
\[ n_0 \left( 1 + \tau_1 \frac{k^2 r_3^2}{r_1} \right)^{n_0} > M, \]  
(5.16)

where \( q_{2,n_0} \) is defined in (5.13) and \( \tau_1 \in \mathbb{R}_+ \) is given in (5.23), then the phenomenon of the CALR could occur if the source supported inside the critical radius \( r_* = \sqrt{r_3^3/r_1} \). Moreover, if the source is supported outside \( B_{r_*} \), then there is no resonant result.

**Proof** We first show the polariton resonance, namely the condition (2.10). For notational convenience of the proof, we set
\[ \tilde{f}_{1,n,m} := \frac{f_{1,n,m}(2n + 1)!}{1}, \quad n \geq N. \]

When \( N \) is large enough such that the spherical Bessel and Hankel functions, \( j_n(t) \) and \( h_n^{(1)}(t) \), enjoy the asymptotic expression shown in (2.20), direct calculations show that the coefficients satisfy the following estimates
\[
|\tilde{\varphi}_{2,n,m}| \approx \frac{f_{1,n,m}(\hat{k}_s r_i)^n}{(2n + 1)!}, \quad |\tilde{\varphi}_{3,n,m}| \approx \frac{f_{1,n,m}r_{0}(\hat{k}_s r_e)^n}{(2n + 1)!},
\]  
(5.17)
\[
|\tilde{\varphi}_{4,n,m}| \leq \frac{f_{1,n,m}(kr_e)^n}{(2n + 1)!}.
\]  
(5.18)

Moreover, the condition (5.15) yields that when \( n = n_0 \),
\[ |d_{n_0,m}| \approx \rho^{2n_0}, \]
and when \( n \neq n_0 \),
\[ |d_{n,m}| \geq \rho^{2n_0} + \rho^{2n}. \]
(5.20)

Thus from (5.4), the displacement field \( u \) to the system (5.3) in the shell \( \Omega \setminus \overline{D} \) can be represented as
\[
u = \tilde{S}_{ad}[\varphi_2](x) + \tilde{S}_{ad}[\varphi_3](x)
= \sum_{n=N}^{\infty} \sum_{n=-n}^{n} - \frac{i\hat{k}_s}{\mu} (\varphi_{2,n,m}r_i^2 j_{n2}^2 h_n(\hat{k}_s|x|) + \varphi_{3,n,m}r_e^2 h_{2e}^2 j_n(\hat{k}_s|x|)) T_n^m,
\]  
(5.21)

where \( \varphi_{2,n,m} \) and \( \varphi_{3,n,m} \) are defined in (5.10).

Next we give the estimate of the dissipation energy \( E(u) \). From the definition of the dissipation energy \( E(u) \) in (2.9) and with the help of Green’s formula, one can have the following estimate
\[
E(u) = \Im \frac{\lambda_{\tilde{f}_{1,n,m}}(u, u)}{\nu} = \Im \left( \int_{\partial \Omega} \partial \varphi \bar{u} \, ds - \int_{\partial D} \partial \varphi \bar{u} \, ds \right) \geq \tilde{f}_{1,n_0,m} \left( \frac{k^2 r_3^2}{r_1} \right)^{n_0}
\]  
(5.22)

If the source \( f \) is supported inside the critical radius \( r_* = \sqrt{r_3^3/r_1} \), by (5.7) and the asymptotic property of \( j_n(t) \) in (2.20), one can verify that there exists \( \tau_1 \in \mathbb{R}_+ \) such that
\[
\limsup_{n \to \infty} (\tilde{f}_{1,n,m})^{1/n} = \sqrt{\frac{r_1}{k^2 r_3^3} + \tau_1}.
\]  
(5.23)
Combining (5.22) as well as (5.23) and together with the help of condition (5.16), one can obtain that

\[ E(u) \geq n_0 \left( \frac{r_i}{k_2 r_e^3} + \tau_1 \right)^{n_0} \left( \frac{k_2 r_e^3}{r_i} \right)^{n_0} > M, \]

which exactly shows that the polariton resonance occurs, namely the condition (2.10) is fulfilled.

Then we consider the case when the source is supported outside the critical radius \( r_\ast \).

Thus there exists \( \tau_2 > 0 \) such that

\[ \limsup_{n \to \infty} (\tilde{f}_{1,n,m})^{1/n} \leq \frac{1}{kr_\ast + \tau_2}, \]

and the dissipation energy \( E(u) \) can be estimated as follows

\[ E(u) \leq \sum_{n \geq N} \frac{\tilde{f}_{1,n,m}^2 (kr_e)^{2n} \rho^{n_0}}{\rho^{2n_0} + \rho^{2n}} \leq \sum_{n \geq N} \tilde{f}_{1,n,m}^2 \left( \frac{k_2 r_e^3}{r_i} \right)^{n_0} \leq C, \]

which means that resonance does not occur.

Next we prove the boundedness of the solution \( u \) when \(|x| > r_e^3/r_i^2\). From (5.4), (5.8) and (5.10), the displacement field \( u \) in \( \mathbb{R}^3 \setminus \Omega \) can be represented as

\[ u = \sum_{n=N}^{\infty} \sum_{m=-n}^{n} \frac{-ik_m}{\mu} (\varphi_{i,n,m} r_e^2 j_{n} j_{n} (k_\ast |x|)) \mathcal{T}^m_n + F(x), \]  

(5.24)

Moreover, from (5.18), (5.19) and (5.20), one can obtain that

\[ |u| \leq \sum_{n=N}^{\infty} \sum_{m=-n}^{n} |\tilde{f}_{1,n,m}| (kr_e)^{n_0} \left( \frac{r_e^3}{r_i^2} \right)^{n} \frac{1}{r^n} + |F| \leq C, \]  

(5.25)

when \(|x| > r_e^3/r_i^2\).

This completes the proof. \( \square \)

**Remark 5.1** Similar to Remark 4.1, in Theorem 5.1, we only impose a constraint on the Lamé parameter \( \hat{\mu} \) (of the form in (5.14)) and there is no restriction on the Lamé parameter \( \hat{\lambda} \). According to (5.14), for the elastic metamaterial in the shell \( \Omega \setminus \hat{D} \), only the real part of its shear modulus is required to be negative. This is also confirmed by the subsequent numerical example in Remark 5.3.

**Remark 5.2** In Theorem 5.1, we impose a constraint on the excitation source \( f \), namely its Newtonian potential \( F \) should have the expression in (5.7). This is mainly due to a technical reason. In fact, we require that \( N \) in (5.7) should be large enough in order to apply the asymptotic properties of the spherical Bessel and Hankel functions, \( j_n(t) \) and \( h_n(t) \), to prove the resonance condition (2.10) and the boundedness condition (2.11). However, noting that the ALR is a spectral phenomenon at the accumulating point of the eigenvalues of the N-P operator, we believe that the constraint could be relaxed, and CALR should hold for more general sources. A rigorous proof of such a conjecture is fraught with difficulties and we leave it for future consideration.
Remark 5.3 We present a numerical example to show that the condition (5.15) can be fulfilled. The parameters are chosen as follows

\[ n_0 = 50, \quad \omega = 5, \quad r_i = 0.8, \quad r_e = 1, \quad \bar{\mu} = \mu = 1 \quad \text{and} \quad \rho^{2n_0} = (r_i/r_e)^{2n_0} \approx 2 \times 10^{-10}. \]

From the values of \( \omega \) and \( r_e \), one can readily verify that this is the case beyond the quasi-static approximation. The absolute value of the LHS quantity in (5.15) in terms of the parameter \( p_{2,n_0} \) is plotted in Fig. 3, which clearly indicates that the condition (5.15) can be fulfilled for a certain \( p_{2,n_0} \). Finally, it can be directly seen that the real part of the shear modulus of the elastic material in the shell \( \Omega \setminus D \) (cf. (5.14)) is negative in the resonance case.

Acknowledgements The authors would like to express their gratitudes to the anonymous referees and the Associate Editor who handled our manuscript, as well as the Editor-in-Chief Professor Fosdick for many insightful and constructive comments and suggestions, which have significantly improved the results and presentation of the paper. The work of Y. Deng was supported by NSF grant of China No. 11971487, PSCF of Hunan No. 18YBQ077 and RFEB of Hunan No. 18B337. The work of Hongyu Liu was support by a startup fund of City University of Hong Kong, and Hong Kong RGC General Research Funds, 12302017, 12301218 and 12302919.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. Ammari, H., Ciraolo, G., Kang, H., Lee, H., Milton, G.W.: Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance II. Contemporary Math. 615, 1–14 (2014)
2. Ammari, H., Ciraolo, G., Kang, H., Lee, H., Milton, G.W.: Anomalous localized resonance using a folded geometry in three dimensions. Proc. R. Soc. A 469, 20130048 (2013)
3. Ammari, H., Ciraolo, G., Kang, H., Lee, H., Milton, G.W.: Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance. Arch. Ration. Mech. Anal. 208, 667–692 (2013)
4. Ammari, H., Deng, Y., Millien, P.: Surface plasmon resonance of nanoparticles and applications in imaging. Arch. Ration. Mech. Anal. 220, 109–153 (2016)
5. Ammari, H., Millien, P., Ruiz, M., Zhang, H.: Mathematical analysis of plasmonic nanoparticles: the scalar case. Arch. Ration. Mech. Anal. 224, 597–658 (2017)
6. Ammari, H., Ruiz, M., Yu, S., Zhang, H.: Mathematical analysis of plasmonic resonances for nanoparticles: the full Maxwell equations. J. Differential Equations 261, 3615–3669 (2016)
7. Ando, K., Kang, H.: Analysis of plasmon resonance on smooth domains using spectral properties of the Neumann-Poincaré operator. J. Math. Anal. Appl. 435, 162–178 (2016)
8. Ando, K., Ji, Y., Kang, H., Kim, K., Yu, S.: Spectral properties of the Neumann-Poincaré operator and cloaking by anomalous localized resonance for the elastostatic system. European J. Appl. Math. 29, 189–225 (2018)
9. Ando, K., Kang, H., Kim, K., Yu, S.: Cloaking by anomalous localized resonance for linear elasticity on a coated structure. arXiv:1612.08384
10. Ando, K., Kang, H., Liu, H.: Plasmon resonance with finite frequencies: a validation of the quasi-static approximation for diametrically small inclusions. SIAM J. Appl. Math. 76, 731–749 (2016)
11. Bruno, O., Lintner, S.: Superlens-cloaking of small dielectric bodies in the quasistatic regime. Journal of Applied Physics 102(12), 124502 (2007)
12. Blåsten, E., Li, H., Liu, H., Wang, Y.: Localization and geometrization in plasmon resonances and geometric structures of Neumann-Poincaré eigenfunctions. ESAIM: Math. Model. Numer. Anal. https://doi.org/10.1051/m2an/2019091
13. Bouchitté, G., Schweizer, B.: Cloaking of small objects by anomalous localized resonance. Quart. J. Mech. Appl. Math. 63, 438–463 (2010)
14. Bruno, O.P., Lintner, S.: Superlens-cloaking of small dielectric bodies in the quasistatic regime. J. Appl. Phys. 102, 124502 (2007)
15. Colton, D., Kress, R.: Inverse Acoustic and Electromagnetic Scattering Theory, 2nd edn. Springer, Berlin (1998)
16. Deng, Y., Li, H., Liu, H.: On spectral properties of Neumann-Poincaré operator and plasmonic cloaking in 3D elastostatics. J. Spectr. Theory 9(3), 767–789 (2019)
17. Deng, Y., Li, H., Liu, H.: Analysis of surface polariton resonance for nanoparticles in elastic system. SIAM J. Math. Anal. (2020), in press, arXiv:1804.05480
18. Kettunen, H., Lassas, M., Ola, P.: On absence and existence of the anomalous localized resonance without the quasi-static approximation. SIAM J. Appl. Math. 78, 609–628 (2018)
19. Kochmann, D.M., Milton, G.W.: Rigorous bounds on the effective moduli of composites and inhomogeneous bodies with negative-stiffness phases. J. Mech. Phys. Solids 71, 46–63 (2014)
20. Kohn, R.V., Lu, J., Schweizer, B., Weinstein, M.I.: A variational perspective on cloaking by anomalous localized resonance. Comm. Math. Phys. 328, 1–27 (2014)
21. Kupradze, V.D.: Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity. North-Holland, Amsterdam (1979)
22. Lakes, R.S., Lee, T., Bersie, A., Wang, Y.: Extreme damping in composite materials with negative-stiffness inclusions. Nature 410, 565–567 (2001)
23. Li, H., Liu, H.: On anomalous localized resonance for the elastostatic system. SIAM J. Math. Anal. 48, 3322–3344 (2016)
24. Li, H., Liu, H.: On three-dimensional plasmon resonance in elastostatics. Annali di Matematica Pura ed Applicata 196, 1113–1135 (2017)
25. Li, H., Liu, H.: On anomalous localized resonance and plasmonic cloaking beyond the quasi-static limit. Proc. R. Soc. A 474, 20180165 (2018)
26. Li, H., Li, J., Liu, H.: On novel elastic structures inducing polariton resonances with finite frequencies and cloaking due to anomalous localized resonance. Journal de Mathématiques Pures et Appliquées 120, 195–219 (2018)
27. Li, H., Li, J., Liu, H.: On quasi-static cloaking due to anomalous localized resonance in $\mathbb{R}^3$. SIAM J. Appl. Math. 75(3), 1245–1260 (2015)
28. Li, H., Li, S., Liu, H., Wang, X.: Analysis of electromagnetic scattering from plasmonic inclusions beyond the quasi-static approximation and applications. ESAIM: Math. Model. Numer. Anal. 53(4), 1351–1371 (2019)
29. Li, H., Liu, H., Zou, J.: Minnaert resonances for bubbles in soft elastic materials. arXiv:1911.03718
30. McPhedran, R.C., Nicorovici, N.-A.P., Botten, L.C., Milton, G.W.: Cloaking by plasmonic resonance among systems of particles: cooperation or combat? C.R. Phys. 10, 391–399 (2009)
31. Milton, G.W., Nicorovici, N.-A.P.: On the cloaking effects associated with anomalous localized resonance. Proc. R. Soc. A 462, 3027–3059 (2006)
32. Milton, G.W., Nicorovici, N.-A.P., McPhedran, R.C., Cherednichenko, K., Jacob, Z.: Solutions in folded geometries, and associated cloaking due to anomalous resonance. New. J. Phys. 10, 115021 (2008)
33. Milton, G.W., Nicorovici, N.-A.P., McPhedran, R.C., Podolsky, V.A.: Proof of superlensing in the quasistatic regime, and limitations of superlenses in this regime due to anomalous localized resonance. Proc. R. Soc. A 461, 3999–4034 (2005)
34. Nédélec, J.C.: Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems. Springer, New York (2001)
35. Nguyen, H.: Cloaking an arbitrary object via anomalous localized resonance: the cloak is independent of the object. SIAM J. Math. Anal. 49, 3208–3232 (2017)
36. Nguyen, H.: Cloaking via anomalous localized resonance for doubly complementary media in the finite frequency regime. arXiv:1511.08053
37. Nicorovici, N.-A.P., McPhedran, R.C., Enoch, S., Tayeb, G.: Finite wavelength cloaking by plasmonic resonance. New J. Phys. 10, 115020 (2008)
38. Nicorovici, N.-A.P., McPhedran, R.C., Milton, G.W.: Optical and dielectric properties of partially resonant composites. Phys. Rev. B 49, 8479–8482 (1994)
39. Nicorovici, N.-A.P., Milton, G.W., McPhedran, R.C., Botten, L.C.: Quasistatic cloaking of two-dimensional polarizable discrete systems by anomalous resonance. Optics Express 15, 6314–6323 (2007)
40. Smith, D.R., Pendry, J.B., Wiltshire, M.C.K.: Metamaterials and negative refractive index. Science 305, 788–792 (2004)
41. Veselago, V.G.: The electrodynamics of substances with simultaneously negative values of $\epsilon$ and $\mu$. Sov. Phys. Usp. 10, 509–514 (1968)