Parameters, limits and higher derivative type II string corrections

Finn Gubay and Peter West
Department of Mathematics
King’s College, London WC2R 2LS, UK

String theory in $d$ dimensions has $n+1 = 11 - d$ parameters that may be thought of as being inherited from the geometry of an $n + 1$ torus which may be used to construct the theory using dimensional reduction from eleven dimensions. We give the precise relationship between these parameters and the expectation values of the scalar fields that parameterise the $E_{n+1}$ coset of the $d$ dimensional theory. This allows us to examine all possible limits of the automorphic forms which occur as the coefficient functions of the higher derivative corrections to the $d$ dimensional type II string effective action.
1. Introduction

The low-energy effective actions of the type IIA and IIB string theories are the IIA [1-3] and IIB [4-6] supergravity theories. Furthermore eleven-dimensional supergravity [7] is the low-energy effective action of one of the limits of M-theory. By virtue of the large amount of supersymmetry they possess the type IIA and type IIB supergravity theories contain all perturbative and non-perturbative string effects at low-energy. As a consequence their study has lead to many aspects of what we now know about string theory not least of which was the realisation that the underlying theory must contain branes on an equal footing to strings. Unfortunately, we know very little about branes, for example, we do not understand even their classical scattering and essentially nothing is known about their quantum properties.

The dimensional reduction of the IIA and IIB supergravity theories on an $n$ torus, or the eleven dimensional supergravity theory on an $n+1$ torus, lead to the unique maximal supergravity theory in $d = 10 - n$ dimensions which possesses a hidden $E_{n+1}$ duality symmetry [8-11]. The IIB supergravity theory possesses an $SL(2,R)$ symmetry [4]. The four-dimensional heterotic supergravity theory possesses an analogous $SL(2,R)$ symmetry and taking into account the fact that the brane charges are quantised [12,13] and rotated by this symmetry it was proposed [14,15] that the four dimensional heterotic string theory was invariant under an $SL(2,Z)$ symmetry which included a transformation that mixed perturbative to non-perturbative effects. This realisation was generalised to the $E_{n+1}$ symmetry of type II theories in $d = 10 - n$ dimensions in [16].

One might hope that this discrete $E_{n+1}$ group is a symmetry of string theory in $d$ dimensions in which case it must be a symmetry of the effective action in $d$ dimensions. Since the continuous $E_{n+1}$ group is a symmetry of the lowest energy theory, that is, the supergravity theory, it remains to show that it is a symmetry of the higher derivative string corrections. The terms in the effective action are polynomial in the Riemann tensor, the field strengths of the gauge fields and expressions containing the derivatives of the scalars. The coefficients of these terms are functions of the scalar fields and not their space-time derivatives. In the supergravity theory the terms are only bilinear in space-time derivatives and their coefficients are numerical factors. Indeed, the scalar fields in the supergravity theory are contained in a non-linear realisation of $E_{n+1}$ with the local subgroup being the maximal compact subgroup, denoted $I_c(E_{n+1})$; put another way the scalars belong to the coset $E_{n+1}/I_c(E_{n+1})$. As such the scalar fields in the supergravity theories appear through their derivatives which are contained in Cartan forms of $E_{n+1}$ and so the way the scalar fields can occur is highly constrained by the non-linear realisation of the continuous group. In the higher derivative terms in the effective action the derivatives of the scalars are also contained in the $E_{n+1}$ Cartan forms but the coefficients are functions of the naked scalars, although they must transform under the discrete $E_{n+1}$ symmetry which can be read off from the terms that they multiply.

The higher derivative corrections to string theory have been most studied in the context of the effective action of IIB string theory which should possess an $SL(2,Z)$ symmetry. For the terms with no more than twelve space-time derivatives and constructed from the Riemann curvature alone, it has been proposed that the coefficients are certain Eisenstein automorphic forms. Although these are not holomorphic functions of the two scalars they
are eigenvalues of the Laplacian acting on the coset space $SL(2,R)/SO(2)$. Such objects are highly constrained and this places very strong constraints on the theory [17-24]; indeed one finds restrictions of string scattering, such as non-renormalisation theorems for the perturbative corrections. Quite a number of these effects have been checked against known string results and have been found to be true; this provides strong evidence that the $SL(2,R)$ symmetry of the IIB supergravity theory [4], when discretised to $SL(2,Z)$, really is a symmetry of ten dimensional IIB string theory.

One of the simplest ways to see that the maximal supergravity theory in $d = 10 - n$ dimensions might have an $E_{n+1}$ symmetry is to compute the dependence on the diagonal components of the metric in the dimensional reduction on the $n$-torus of, for example, the ten dimensional IIB supergravity theory. These fields $\tilde{\phi}$ occur in the resulting action in the form $e^{i\sqrt{2}\tilde{\phi} \cdot \vec{w}}$ and one finds that the vectors $\vec{w}$ that occur are the roots of $E_{n+1}$ [25-27]. If one repeats such a calculation for the dimensional reduction of the higher derivative terms of the ten dimensional IIB effective action to $d$ dimensions then one does not find the roots of $E_{n+1}$. However, one does find that the corresponding vectors are certain weights of $E_{n+1}$ which can be accounted for if the coefficients in the effective action in $d$ dimensions are automorphic forms of $E_{n+1}$ constructed from the representations of $E_{n+1}$ that contain the weights that appear [27]. The dimensional reduction of the eleven dimensional M-theory effective action on an $n+1$ torus leads to a similar result [26]. The appearance of the weights of $E_{n+1}$ upon dimensional reduction of arbitrary higher derivative terms in the type IIA/B and M-theory effective actions to $d = 10 - n$ dimensions puts constraints on the automorphic forms that appear as the coefficients of the corresponding higher derivative terms in $d$ dimensions. In particular, the $E_{n+1}$ coefficient functions should contain the $E_{n+1}$ weights found when a given higher derivative term in type IIA/B string theory and M-theory are dimensionally reduced to $d$ dimensions, these weights suggest the coefficient functions contain automorphic forms constructed from particular representations of $E_{n+1}$ [28,29]. This provided evidence that the higher derivative string corrections do possess a discrete $E_{n+1}$ symmetry and that the coefficients are automorphic forms constructed from scalars that are contained in coset group elements of $E_{n+1}/I_{c}(E_{n+1})$ acting on the representations found.

Recently the higher derivative corrections of type II string theories in less than ten dimensions have been systematically studied and specific automorphic forms, constructed from particular representations of $E_{n+1}$, have been proposed for the coefficients of certain higher derivative terms which have fewer than twelve space-time derivatives [30-33]. This has lead to predictions for string scattering, especially for perturbative results, that have been checked against the known string theory results. This work provides further evidence for a discrete $E_{n+1}$ symmetry of string theory [30-33]. These papers have in particular generalised the previous results on non-renormalisation theorems [34-36]. Earlier results were found in eight dimensions [37,38] and more recently in [39], discussions can also be found in [40-42] and references therein.

Given the effective action for type IIA, or IIB, string theory in ten dimensions one can dimensionally reduce it on an $n$ torus to obtain terms in the effective action in $d = 10 - n$ dimensions. The scalars in the lower dimensional theory that arise from the diagonal components of the metric in the compactified directions parameterise, in a one to one
way, the volume of the \( n \)-torus and all the sub-tori that are used in the dimensional reduction. As explained above, these scalar fields together with the other scalar fields are encoded in the higher derivative effective action as a non-linear realisation of \( E_{n+1} \) with local subgroup \( I_<(E_{n+1}) \); the diagonal components of the metric, together with the ten dimensional dilaton, appear as the part of this group element constructed from the Cartan subalgebra of \( E_{n+1} \).

One can also carry out the dimensional reduction from the eleven dimensional effective action of M-theory to \( d = 10 - n \) dimensions on an \( n + 1 \) torus and then the \( n + 1 \) diagonal components of the metric parameterise the volumes of the \( n + 1 \) dimensional torus and all the possible sub-tori. In the dimensionally reduced theory these \( n + 1 \) scalar fields are encoded in a group element that belongs to the Cartan subalgebra of \( E_{n+1} \). As is well known, the volume of the sub-tori which sits in the eleventh dimension, essentially the radius of the circle in the eleventh direction, becomes the dilaton in the ten dimensional IIA string theory and it is related to the string coupling \( g_s \) in this theory.

Unlike the other scalar fields, the scalar fields in the \( d \) dimensional theory that are associated with the Cartan subalgebra in the theory in \( d \) dimensions appear in exponentials. The expectation values of these fields are parameters that define the theory in \( d \) dimensions. Indeed, the expectation value of one of the scalars \( \phi_d \) in the \( d \)-dimensional effective action is related to the string coupling in that dimension by \( g_d = e^{\phi_d} \). The perturbative corrections can be found by taking the limit \( g_d \rightarrow 0 \), or equivalently \( \phi_d \rightarrow -\infty \) in the effective action. To find these one must find the expression for the automorphic form in this limit and one can then check if one indeed finds a perturbation theory that is consistent with string theory; the simplest check being that it contains \( g_d^{-2+2n} \) for \( n = 0, 1, 2, \ldots \). This test was indeed carried out for the coefficient functions of the eight and twelve derivative terms constructed purely from the Riemann curvature with the proposed automorphic forms [30-33].

The scalar field which is related to the string coupling is just one of \( n + 1 \) scalar fields and we can also consider limits of their expectation values which correspond to various limits as certain sub-tori become large or small. Computing this limit for the automorphic form we can find the effect on the effective action in \( d \) dimensions. In particular if we let the radius of the circle in the \( d + 1 \)th direction become large then we are decompactifying the theory to \( d + 1 \) dimensions and the automorphic form of \( E_{n+1} \) becomes an automorphic form of \( E_n \) with coefficients that are powers of the exponential of the scalar involved in the limit. This was carried out in [31-33] for the eight, twelve and fourteen derivative terms constructed purely from the Riemann curvature in various dimensions. However, the precise connection to the parameters of the theory and the possible limits one can take in general have not been previously discussed.

In this paper we will consider the effective action in \( d \) dimensions and find the precise relationship between the scalar fields associated with the Cartan subalgebra of \( E_{n+1} \) and the parameters of the theory in ten dimensions, that is the string coupling and the tori sub-volumes. This will allow us to find a one to one correspondence between the nodes of the \( E_{n+1} \) Dynkin diagram and the parameters of the \( d \) dimensional string theory.

As mentioned earlier, upon dimensional reduction on a torus to \( d < 10 \) dimensions type IIA/B string theory and M-theory are equivalent and possess a hidden \( E_{n+1} \) symmetry.
The $d = 10 - n$ dimensional maximal supergravity theory may be viewed as arising from the dimensional reduction of type IIA/B string theory on an $n$ torus where the natural parameters to consider are the $d$ dimensional string coupling $g_d$ and the volume of the $n$ torus and the volume of its 1, 2, ..., $n - 1$ subtori. Alternatively, dimensional reduction of M-theory on an $m = n + 1$ torus leads to a $d = 10 - n$ dimensional supergravity theory with an $E_{n+1}$ symmetry where the natural parameters of the $d$ dimensional theory are the volume of the $m$ torus and the volume of its 1, 2, ..., $m - 1$ subtori.

For example, the volume $V_{m(M)}$ of the $m = n + 1$ torus, upon which M-theory is compactified, is associated with the exceptional node in the $E_m$ Dynkin diagram below. To be precise $V_{m(M)} = e^{3\phi_m}$, where $\phi_m$ is the field associated with Cartan subalgebra element $m$ in Chevalley basis, which in our notation corresponds to the exceptional node in the $E_m$ Dynkin Diagram in figure 1. The remaining nodes of the $E_m$ Dynkin diagram are associated with a specific combination of the $m$ parameters $V_{m(M)}$ and the $m - 1$ tori sub-volumes $V_j$, $j \leq m - 1$ as given below each node in figure 1.

![Figure 1](image1.png)

Figure 1. The $E_m$ Dynkin diagram labelled by the $d$ dimensional M-theory parameters

The powers of the parameters labelling node $i$ are such that they equal $e^{\phi_i}$, where $\phi_i$ is the field associated with Cartan subalgebra element $i$. The correspondence between the fields $\phi_i$ and the parameters is derived in section 2.

From the type IIB point of view, the nodes of the $E_{n+1}$ Dynkin diagram are associated with the $n + 1$ parameters that include the $d$ dimensional string coupling $g_d$, the volume $V_{n(B)}$ of the $n$ torus, upon which the type IIB theory is compactified and the $n - 1$ tori sub-volumes $V_j$, $j \leq n - 1$. The sub-volumes of the torus $V_j$, $j \leq n - 2$ are related to the Chevalley fields $\phi_i$ by $V_j = e^{\phi_j}$ and are identified with $n - 2$ nodes of the $E_{n+1}$ Dynkin diagram, as shown in figure 2. The specific combination of the $n + 1$ parameters associated with each node is given in figure 2.

![Figure 2](image2.png)

Figure 2. The $E_{n+1}$ Dynkin diagram labelled by the $d$ dimensional type IIB parameters
The nodes of the $E_{n+1}$ Dynkin diagram are similarly associated with the $n + 1$ parameters, of the type IIA theory dimensionally reduced on an $n$ torus. The $E_{n+1}$ Dynkin diagram labelled by these parameters is given in figure 3.

$$V_{n-1}^{\frac{1}{2}} e^{\frac{2}{8-n} g_d V_{n(A)} g_d}$$

Figure 3. The $E_{n+1}$ Dynkin diagram labelled by the $d$ dimensional type IIA parameters

These result allows us, in section 4, to discuss in general all possible limits of the automorphic form and the effective action in these limits.

To carry out these calculations we will use the $E_{11}$ formulation of the maximal supergravity theories. We will not rely on the conjecture that $E_{11}$ is a symmetry of the underlying theory of strings and branes [43] but results [43-48] that relate the non-linear realisations of $E_{11}$ at low levels to the maximal supergravity theories. One could also find these results by explicitly carrying out the dimensional reduction to find the $d$ dimensional theory, identifying the way the $E_{n+1}$ symmetry acts on the fields and then relating the parameters to the fields associated with the Cartan subalgebra of $E_{n+1}$. This is a lengthy process, however, using the $E_{11}$ approach it is simple to identify the $E_{n+1}$ symmetry and in particular the part of the group element that contains the Cartan subalgebra fields. The $E_{11}$ non-linear realisation also permits one to move between type IIA, type IIB and M-theory in a simple way.

2. Group interpretation of parameters in string theory and M-theory

Dimensional reduction of type IIA/B supergravity on an $n$ torus, or equivalently eleven dimensional supergravity on an $n + 1$ torus, leads to scalars belonging to a non-linear realisation of an $E_{n+1}$ symmetry. The coefficient functions of higher derivative terms in the effective action of type IIA/B string theory in $d=10-n$ dimensions transform as $E_{n+1}$ automorphic forms which are functions of these scalars. In this section we will use the $E_{11}$ formulation of type IIA/B and eleven dimensional supergravity theories to derive the relationship between these scalar fields and the physical parameters of these theories. In particular, we will find the relationship between the expectation values of the fields $\hat{\phi}_i$, $i = d + 1, ..., 11$, parameterising the Cartan subalgebra part of the $E_{n+1}$ group element and the physical parameters of type IIA/B and eleven dimensional supergravity when dimensionally reduced on an $n$ torus. This will lead us to a correspondence between the physical parameters and the nodes of the $E_{n+1}$ Dynkin diagram.

2.1 Review of parameters in string theory and M-theory

The low-energy effective actions of type IIA/B string theory in ten dimensions are the type IIA/B supergravity theories. As a consequence, the parameters of type IIA/B string
theory are related to those of corresponding supergravity theory in $d = 10$ dimensions. The type IIA/B supergravity theories have two parameters, the Newtonian coupling constant $\kappa_{10}$ and $e^{<\phi>}$, where $<\phi>$ is the expectation value of the type IIA/B dilaton $\phi$. Note that the type IIA dilaton and the Newtonian coupling are not equal to the type IIB dilaton and Newtonian coupling. Thus on each equation one should add the subscripts A or B to indicate what theory is being discussed. However, the generic discussion applies to each theory in the same way and in order to not over complicate the equations we will suppress the labels A and B, but they are to be understood to be present.

The $d$ dimensional Newtonian coupling constant $\kappa_d$ by definition appears in the action, multiplying the $d$ dimensional Einstein-Hilbert term in Einstein frame, in the form

$$\frac{1}{2\kappa_d^2} \int d^dx \sqrt{-g} R.$$  

(2.1.1)

The $d$ dimensional Planck length $l_d$ is defined by

$$l_d^{d-2} = 2\kappa_d^2,$$  

(2.1.2)

where the form of the relation is determined by dimensional analysis.

The parameters of type IIA/B supergravity in $d = 10 - n$ dimensions may be found by dimensional reduction of these theories on an $n$ torus. To be general we consider a theory in $D$ dimensions and reduce it on an $n$ torus to $d = D - n$ using the ansatz

$$ds^2 = e^{2\alpha\tilde{\rho}} ds^2 + e^{2\beta(\tilde{\rho} + <\rho>)} G_{ij} \left( dx^i + A^i_\mu dx^\mu \right) \left( dx^j + A^j_\mu dx^\mu \right),$$  

(2.1.3)

where, $\tilde{\rho} = \rho - <\rho>$, $det(G) = 1$ and the constants $\alpha$ and $\beta$ are given by

$$\alpha = \sqrt{\frac{n}{2(d-2)(D-2)}}, \quad \beta = -\frac{(d-2)\alpha}{n}.$$  

(2.1.4)

This ansatz is designed so that the dimensionally reduced theory in $d$ dimensions has an Einstein term that involves just the graviton, but no scalar fields such as $\tilde{\rho}$. Equation (2.1.3) is a generalisation of the ansatz used in our previous papers [26-30] that carefully takes into account the expectation values of the fields which will turn out to appear in the definitions of the physical parameters of the various theories. Although many of the results in this section are well known, we are not aware of a very careful discussion and as this is central to our paper we have tried to present this material here in a diligent way.

Compactifying a $D$ dimensional theory to $d = D - n$ dimensions, via this ansatz, gives the $d$ dimensional theory in Einstein frame. The coordinate reparameterisation invariant length of a torus cycle in the $i$ direction is

$$\int_0^{l_D} \hat{e}_i^i dx^i = 2\pi r_i,$$  

(2.1.5)

where we have parameterised the circle by the $D$ dimensional Planck length $l_D$ and $r_i$ is the physical radius in the $i$ direction. Note that the vielbein of the internal metric $\hat{e}_i^i$ is
taken to be independent of the compactified coordinates, therefore the integration is trivial and one finds

\[ <\hat{e}_i^i> = \frac{2\pi r_i}{l_{D}}. \tag{2.1.6} \]

This allows one to express the volume of an \( n \) torus which is given by \( \sqrt{<\text{det}(\hat{G})>} \), where the components of \( \hat{G} \) are \( \hat{G}_{ij} = e^{2\beta(\hat{\rho}+<\rho>)}G_{ij} \), as

\[
\sqrt{<\text{det}(\hat{G})>} = <\hat{e}_D^D><\hat{e}_{D-1}^{D-1}>...<\hat{e}_{d+1}^{d+1>} = (2\pi)^n \frac{r_{D}r_{D-1}...r_{d+1}}{(l_{D})^n}, \tag{2.1.7}
\]

where our chosen vielbein frame has lower triangular components for \( \hat{e} \) set to zero. Since \( \text{det}(G) = 1 \), the expression in equation (2.1.7) is given by

\[
\sqrt{<\text{det}(\hat{G})>} = e^{n\beta <\rho>}. \tag{2.1.8}
\]

We note that we have taken the vielbein in Einstein frame. It will turn out to be simpler to take a rescaled definition of the volume \( V_n \) of the \( n \) torus by defining

\[
V_n = (2\pi)^n \frac{r_{D}r_{D-1}...r_{d+1}}{(l_{d})^n} = (2\pi)^n \left( \frac{l_{D}}{l_{d}} \right)^n \frac{r_{D}r_{D-1}...r_{d+1}}{(l_{D})^n}, \tag{2.1.9}
\]

where the factor of \((2\pi)^n\) is chosen to simplify later expressions.

We will also be interested in subtori of the \( n \) torus. The dimensionless volume of the \( j \) subtorus, where \( j < n \), is defined by

\[
V_j = (2\pi)^j \frac{r_{d+1}r_{d+2}...r_{d+j}}{(l_{d})^j}. \tag{2.1.10}
\]

The dimensional reduction of \( D \) dimensional supergravity on a circle of radius \( r_{D} \) gives

\[
\frac{1}{2\kappa_{D}^2} \int d^D x \sqrt{-g} \hat{R} + ... = \frac{1}{2\kappa_{D}^2} \int \hat{e}_D^D dx^D \int d^{D-1} x \sqrt{-g} R + ...
\]

\[
= 2\pi r_{D} \frac{1}{2\kappa_{D}^2} \int d^{D-1} x \sqrt{-g} R + ... \tag{2.1.11}
\]

where ... denotes all terms in the action beyond the Einstein-Hilbert term and fields possessing a \( \hat{\ } \) are \( D \) dimensional fields while fields without are \( D - 1 \) dimensional fields. Comparing the coupling of the \( D - 1 \) dimensional theory on the the right hand side of equation (2.1.11) with the expected coupling in the action of the \( D - 1 \) dimensional theory given by

\[
\frac{1}{2\kappa_{D-1}^2} \int d^{D-1} x \sqrt{-g} R + ...
\]

one finds that the relationship between the \( D \) dimensional Newtonian coupling constant and the \( D - 1 \) dimensional Newtonian coupling constant is

\[
(k_D)^2 = 2\pi r_{D} (k_{D-1})^2. \tag{2.1.13}
\]
One may rewrite equation (2.1.13) in terms of the Planck length rather than the Newtonian coupling constant, this leads to

\[ (l_D)^{D-2} = 2\pi r_D (l_{D-1})^{D-3}. \quad (2.1.14) \]

We may write the ratio of the \( D \) dimensional Planck length \( l_D \) to the \( d \) dimensional Planck length \( l_d \) by iterating equation (2.1.14) \( n \) times and dividing by \( (l_D)^n \), this gives

\[ \left( \frac{l_D}{l_d} \right)^{d-2} = (2\pi)^n \frac{r_D r_{D-1} \cdots r_{d+1}}{(l_D)^n}. \quad (2.1.15) \]

Substituting equation (2.1.15) back into (2.1.9) and using equations (2.1.7) and (2.1.8) to express the result in terms of \( \langle \rho \rangle \) one finds that the volume \( V_n \) of the \( n \) torus as a function of the field \( \rho \) is given by

\[ V_n = e^{\left( \frac{D-2}{D-1} \right) n \beta \langle \rho \rangle}. \quad (2.1.16) \]

A generic feature of supergravity theories compactified on an \( n \) torus to \( d = D - n \) dimensions is the scaling of the \( d \) dimensional gauge fields by factors of the dimensionless volume \( V_n \) of the \( n \) torus. For example, the ten dimensional Einstein-Hilbert term dimensionally reduced on an \( n \) torus with the ansatz (2.1.3) gives the \( d = 10 - n \) dimensional action

\[
\begin{align*}
&\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} R \\
= &\frac{1}{(2\pi)^7 l_{10}^8} l_1^n e^{n\beta \langle \rho \rangle} \int d^d x \sqrt{-g} \left( R - \frac{1}{4} e^{2(\beta + \langle \rho \rangle) - \alpha \tilde{\rho}} G_{ij} F_{\mu \nu}^{i} F^{j \mu \nu} - S_{mi}^{j} S_{j}^{\mu i} \\
- &\gamma^2 \partial_{\mu} \rho \partial^{\mu} \rho - 2(n\beta + (d-1)\alpha) \nabla^2 \rho \right) \\
= &\frac{1}{(2\pi)^7 l_{10}^8} V_{n}^{\frac{d-n}{8}} \int d^d x \sqrt{-g} \left( R - \frac{1}{4} V_{n}^{\frac{d}{2}} e^{2(\beta - \alpha) \tilde{\rho}} G_{ij} F_{\mu \nu}^{i} F^{j \mu \nu} - S_{mi}^{j} S_{j}^{\mu i} - \gamma^2 \partial_{\mu} \rho \partial^{\mu} \rho \\
- &2(n\beta + (d-1)\alpha) \nabla^2 \rho \right). \quad (2.1.17)
\end{align*}
\]

So we see that in this case each field strength \( F_{\mu \nu}^{i} \) constructed from the gauge field \( A_{\nu}^{i} \) appears with a factor of \( V_{n}^{\frac{d-n}{8}} \). In general one finds that each gauge field \( A_{\nu}^{i...k} \) carrying \( k \) contracted downstairs compact indices is scaled by a factor of \( V_{n}^{-\frac{k}{8}} \). Similarly, each gauge field carrying \( k \) contracted upstairs compact indices is scaled by a factor of \( V_{n}^{\frac{k}{8}} \).

The type IIA supergravity is the low energy effective theory for IIA string theory. As mentioned earlier, the parameters of the type IIA supergravity theory in ten dimensions are the Newtonian coupling constant \( \kappa_{10(A)} \) and \( e^{\langle \phi_A \rangle} \). The type IIA string theory in ten dimensions also has two parameters, the string length \( l_{s(A)} \) and the string coupling constant \( g_{s(A)} \), where the string length may be defined in terms of \( \alpha'_A \) as \( l_{s(A)} = \sqrt{\alpha'_A} \). Similar statements hold for the IIB theory and for simplicity we have omitted the A or B labels on the parameters. The parameters of interest to us in taking various limits of
type IIA and IIB string theory and M-theory compactified to $d = 10 - n$ dimensions are the $d$ dimensional Planck length $l_d$, the string coupling $g_d$ in $d = 10 - n$ dimensions, the physical radii $r_i$, $i = d + 1, \ldots, D$, of the $n$ torus and the volume of the torus upon which type IIA/B string theory are compactified on, denoted $V_{n(A)}$ and $V_{n(B)}$ respectively, along with the volume $V_{m(M)}$ of the torus upon which M-theory is compactified upon.

The relationship between the type IIA and IIB supergravity coupling $\kappa_{10}$ and the string length $l_s$ is found by comparing the supergravity action and the effective action derived from type IIA and IIB string theory, this yields [49, 50]

$$g_s = e^{<\phi>}, \quad \text{and} \quad 2\kappa_{10}^2 = (2\pi)^7 l_s^8 g_s^2.$$  \hspace{1cm} (2.1.18)

Each of the symbols in this equation should carry the label A or B denoting the string theory to which it belongs, but here, and in much of what follows below, we have suppressed these for simplicity. The ten dimensional Planck length $l_{10}$ in type IIA or type IIB string theory is related to the ten dimensional Newtonian coupling constant through equation (2.1.2) so one has

$$l_{10}^8 = (2\pi)^7 l_s^8 g_s^2.$$  \hspace{1cm} (2.1.19)

Our aim is to derive an expression for the effective coupling $g_d$ of type IIA/B string theory compactified on a torus to $d = 10 - n$ dimensions. The string coupling $g_d$ is the $d$ dimensional analogue of the ten dimensional type IIA/B string coupling $g_s$. The ten dimensional type IIA/B supergravity action in Einstein frame takes the schematic form

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10} x \sqrt{-gR} + \ldots$$  \hspace{1cm} (2.1.20)

Rewriting the action in terms of the ten dimensional string parameters $l_s$ and $g_s$, using equation (2.1.19), the low-energy effective action of type IIA/B string theory in ten dimensions takes the form

$$S = \frac{1}{(2\pi)^7 l_s^8 g_s^2} \int d^{10} x \sqrt{-gR} + \ldots$$  \hspace{1cm} (2.1.21)

Dimensionally reducing the low-energy effective action of equation (2.1.21) on an $n$ torus via the ansatz (2.1.3), using equations (2.1.7) and (2.1.8), one finds

$$S = \frac{1}{(2\pi)^7 l_s^8 e^{2<\phi>} (l_{10})^n e^{n\beta<\rho>} \int d^d x \sqrt{-gR} + \ldots$$

$$= \frac{1}{(2\pi)^7} \frac{l_s^n e^{\frac{2}{l_{10}}}<\phi> e^{n\beta<\rho>}}{l_s^8 e^{2<\phi>}} \int d^d x \sqrt{-gR} + \ldots$$

$$= \frac{1}{(2\pi)^7 l_s^{8-n} \left[e^{\frac{2}{l_{10}}}<\phi>-\left(\frac{n\beta<\rho>}{2}\right)^2\right]} \int d^d x \sqrt{-gR} + \ldots$$  \hspace{1cm} (2.1.22)

We now define $g_d$ as the $d = 10 - n$ dimensional analogue of $g_s$ in equation (2.1.21). The compactified $d = 10 - n$ action takes the form

$$S = \frac{1}{(2\pi)^7 l_s^{8-n} g_d^2} \int d^d x \sqrt{-gR} + \ldots$$  \hspace{1cm} (2.1.23)
Comparing the expected $d = 10 - n$ dimensional action in equation (2.1.20) with the dimensionally reduced action in equation (2.1.21) one observes that the $d$ dimensional effective coupling $g_d$ is given by

$$g_d = e^{\frac{8-n}{8} \langle \phi \rangle - \frac{n}{2} \beta \langle \rho \rangle} V_n^{-\left(\frac{8-n}{16}\right) g_s \left(\frac{8-n}{8}\right)}$$,

(2.1.24)

where $g_s$ is the ten dimensional type IIA/B string coupling and $V_n$ is the volume of the $n$ torus of equation (2.1.16). Defining the shifted $d$ dimensional dilaton $\phi_d$ by $\phi_d = \left(\frac{8-n}{8}\right) \phi - \frac{n}{2} \beta \rho$, one may write $g_d = e^{\langle \phi_d \rangle}$. Note that in $d$ dimensions, the components of the Einstein frame metric $g_{\mu \nu}$ are related to the components of the string frame metric $g_{(S)\mu \nu}$ by $g_{\mu \nu} = e^{-\frac{4}{d-2} \tilde{\phi}_d} g_{(S)\mu \nu}$, where $\tilde{\phi}_d = \phi_d - \langle \phi_d \rangle$, that is it does not involve the expectation value of $\rho$.

Our expressions for the volume $V_n$, coupling constant $g_d$ and the ten dimensional Planck length $l_{10}$ then allows us to write

$$l_d^{d-2} = g_d^2 l_{10}^{d-2}.$$  

(2.1.25)

In principle one has different type IIA and type IIB string couplings in $d$ dimensions, $g_{s(A)}$ and $g_{s(B)}$. However, there is a unique maximal supergravity theory in $d < 10$ dimensions and the type IIA and type IIB string theories in $d < 10$ dimensions are related by T-duality. Moreover, T-duality provides a map between the ten dimensional type IIA dilaton $\phi$ and the moduli of the torus compactifying the type IIA theory with their type IIB counterparts, as a result one may show that the string coupling $g_d$ takes the same form for both the type IIA and type IIB theories. Similarly the $d$ dimensional Planck length, or equivalently the $d$ dimensional Newtonian coupling, takes the same form for both the compactified type IIA and type IIB theories. We will revisit the correspondence between the physical fields of type IIA and type IIB supergravity in $d < 10$ dimensions in section three. The same conclusion applies to the Newtonian coupling constant $\kappa_d$ and so the Planck length in $d$ dimensions $l_d$.

Eleven dimensional supergravity is conjectured to be the low-energy effective action of M-theory. The IIA supergravity theory was derived [43,51] from the eleven dimensional supergravity theory using dimensional reduction on a circle using a special case of the torus ansatz (2.1.3) and is given by

$$ds^2 = e^{-\frac{4}{d-2} \tilde{\phi}} g_{(S)\mu \nu} dx^\mu dx^\nu + e^{\frac{2}{(d-2)}} (dx^{11} + A_\mu dx^\mu)^2,$$

(2.1.26)

where the subscript $S$ indicates that the ansatz is written in terms of the string frame ten dimensional metric and $\tilde{\phi} = \phi - < \phi >$. From the ansatz, we may identify $e^{\frac{2}{d-2} (\tilde{\phi} + < \phi >)}$ as the vielbein on the circle $e_{11}^{\mu}$ and so using equations (2.1.6) and (2.1.18) one finds

$$r_{11} = g_{s(A)}^{\frac{2}{d-2}}.$$

(2.1.27)

It then follows that upon compactification of the eleventh dimension, we have

$$2\kappa_{11}^2 = 2\pi r_{11} \kappa_{10} = 2\pi r_{11} l_{10} g_{s(A)}^2,$$

(2.1.28)
where $\kappa_{11}$ is the $d = 11$ supergravity coupling constant. Using our expression for the eleven dimensional Planck length $l_{11}$ in terms of the IIA string coupling $g_{s(A)}$ and the radius of the eleventh dimension $r_{11}$, we find, from the above expression,

$$r_{11} = g_{s} l_{s}. \quad (2.1.29)$$

We may use equation (2.1.15) to write the $d = 11 - m$ dimensional Planck length $l_{d}$ as a function of the volume $V_{m(M)}$ of the torus used to dimensionally reduce M-theory and the eleven dimensional Planck length $l_{11}$, this yields

$$l_{d} = l_{11} V_{m(M)}^{-\frac{1}{d-1}}. \quad (2.1.30)$$

Reinstating the labels denoting type IIA and type IIB quantities we have, in summary, that the Planck length in ten and eleven dimensions are related to the string length and IIA/IIB coupling by

$$l_{11} = g_{s(A)} l_{s}, \quad (2.1.31)$$

$$l_{10(A)} = g_{s(A)}^{\frac{1}{2}} l_{s}, \quad (2.1.32)$$

$$l_{10(B)} = g_{s(B)}^{\frac{1}{2}} l_{s}, \quad (2.1.33)$$

$$r_{11} = g_{s(A)} l_{s}, \quad (2.1.34)$$

By dimensional analysis, an arbitrary higher derivative term in the $d$ dimensional Einstein frame effective action of type IIA/B string theory and M-theory compactified on a torus to $d = 10 - n$ dimensions contains a factor of $l_{d}^{p-d}$, where $p$ is the number of derivatives in the term. To examine the various limits in the parameters of type IIA/B string and M-theory compactified to $d = 10 - n$ dimensions we will rewrite the $d$ dimensional Planck length $l_{d}$ in terms of these parameters.

The $d$ dimensional Planck length $l_{d}$ is related to the $d + 1$ dimensional quantities $l_{d+1}$ and the radius $r_{d+1}$ on which a $d + 1$ dimensional theory is compactified on, relevant to the decompactification of a single dimension limit, by

$$l_{d} = r_{d+1}^{-\frac{1}{d-2}} (l_{d+1})^{\frac{d-1}{d-2}}, \quad (2.1.35)$$

where we have made use of equation (2.1.14). The $d$ dimensional Planck length $l_{d}$ is related to the volume of the $n$ torus $V_{n(A)}$ upon which $D = 10$ type IIA string theory is compactified by

$$l_{d} = l_{10(A)} V_{n(A)}^{-\frac{1}{d}}, \quad (2.1.36)$$

this may be derived by iterating equation (2.1.14) $n$ times. Similarly, the $d$ dimensional Planck length $l_{d}$ is related to the volume of the $n$ torus $V_{n(B)}$ upon which $D = 10$ type IIB string theory is compactified by

$$l_{d} = l_{10(B)} V_{n(B)}^{-\frac{1}{d}}, \quad (2.1.37)$$
While the $d$ dimensional Planck length $l_d$ is related to the volume of the $m$ torus $V_{m(M)}$ upon which M-theory is compactified by

$$l_d = l_{11} V_{m(M)}^{\frac{1}{11}},$$

(2.1.38)

again, this may be derived by iterating equation (2.1.14) $m$ times. The $d$ dimensional Planck length $l_d$ is related to the volume of a $j$ dimensional subtorus $V_j$ by

$$l_d = l_{d+j} V_j^{\frac{1}{d+j}}.$$  

(2.1.39)

Finally, the transition from $d$ dimensional Einstein frame to $d$ dimensional string frame, relevant to the $d$ dimensional perturbative limit, is given by

$$g_{\mu\nu} = e^{-\frac{1}{d-2} \phi} g_{(S)\mu\nu},$$

(2.1.40)

where $g_{\mu\nu}$ are the components of the $d$ dimensional Einstein frame metric and $g_{(S)\mu\nu}$ are the components of the $d$ dimensional string frame metric.

### 2.2. Parameters and fields in M-theory

Upon dimensional reduction on an $n$ torus to $d = 10 - n$ dimensions the type IIA and Type IIB supergravity theories lead to the same theory which possess an $E_{n+1}$ symmetry. The scalars that appear belong to a non-linear realisation of this $E_{n+1}$ symmetry. As such the scalar fields appear as parameters of an $E_{n+1}$ group element, strictly a coset element. Indeed, the diagonal components of the metric and the dilaton, in the case of the IIA and IIB theories, parameterise the part of this group element that is in the Cartan subgroup of $E_{n+1}$. In the previous section we described the connection between the expectation values of certain scalar fields in type IIA/B supergravity compactified on an $n$ torus, or equivalently eleven dimensional supergravity on an $m = n + 1$ torus, and the parameters of the compactified $d$ dimensional maximal supergravity theory, or equivalently string theory in $d$ dimensions. By certain scalar fields we mean the scalar fields just mentioned, namely, the diagonal components of the metric and the dilaton, in the case of the IIA and IIB theories. In this section we will derive the connection between these scalar fields and the nodes of the $E_{n+1}$ Dynkin diagram. As a result, we will find that the parameters in type IIA/B supergravity compactified on an $n$ torus, or equivalently eleven dimensional supergravity on an $n + 1$ torus, are associated with specific nodes of the $E_{n+1}$ Dynkin diagram.

The eleven dimensional, IIA and IIB supergravity theories, as well as the maximal type II supergravity theories in lower dimensions, can be formulated as non-linear realisations of the Kac-Moody algebra $E_{11}$ [43-48]. The different theories arise by taking the different decompositions of $E_{11}$ into the subalgebras that arise from deleting the different nodes in the $E_{11}$ Dynkin diagram. The fields of these theories appear as the parameters of an $E_{11}$ group element, or more precisely a coset element, and it turns out that the fields are in one to one correspondence with the generators of the Borel subalgebra of $E_{11}$. As such $E_{11}$ encodes the fields of each of these
theories and, as there is only one $E_{11}$ algebra, it provides us with a way of relating the fields in the different supergravity theories to each other [52]. As we now explain this connection is particularly simple for the subcase of interest to us in this paper.

The $E_{11}$ algebra is formulated in terms of generators that are the multiple commutators of the so called Chevalley generators. There are three Chevalley generators associated with every node of the $E_{11}$ Dynkin diagram and they obey conditions that are encoded in the Cartan matrix of $E_{11}$. In this paper, we will not need a detailed knowledge of this construction, as we will be interested in the $E_{11}$ group element restricted to the Cartan subgroup. As such it will suffice to know that the Chevalley generators contain the Cartan subalgebra of $E_{11}$ and we will denote these generators by $H_{\hat{a}}, \hat{a} = 1, 2, \ldots, 11$; the generator $H_{\hat{a}}$ being associated with node $a$ in the $E_{11}$ Dynkin diagram. Thus the non-linear realisation of $E_{11}$ restricted to the Cartan subgroup is of the form $e^{\sum_{\hat{a}} \phi_{\hat{a}} H_{\hat{a}}}$ where $\phi_{\hat{a}}$ will be called the Chevalley fields. It is important to realise that the generators $H_{\hat{a}}$ are essentially uniquely specified by the above construction and so as a consequence are the fields $\phi_{\hat{a}}$. We will denote the Cartan subalgebra part of the same group element in Weyl basis as $e^{\vec{\phi} \cdot \vec{H}}$, where $\vec{H}$ are the Cartan subalgebra generators in Weyl basis and $\vec{\phi}$ are the corresponding fields.

The maximal supergravity theories in $d$ dimensions are described in terms of the fields that have been used to describe the propagating degree of freedom since long ago, such as the graviton and gauge fields, these include the fields of interest to us, that is the diagonal components of the metric and the dilaton, and we will refer to these as the physical fields. However, these are not the same as the fields that arise in the $E_{11}$ non-linear realisation when we formulate the algebra using the Chevalley generators and in particular the Chevalley fields $\phi_{\hat{a}}$ that arise when we write the Cartan subalgebra in terms of the generators $H_{\hat{a}}$. However since the $E_{11}$ non-linear realisation leads to the supergravity theories there is a one to one relationship between the Chevalley fields $\phi_{\hat{a}}$ and the physical fields of interest to us. We note that in the different supergravity theories one finds different sets of physical fields but only one set of Chevalley fields. In this section we will find the relationship between the physical fields consisting of the diagonal components of the metric and dilaton, where present, in the eleven dimensional, type IIA and type IIB theories in ten dimensions and their reduction on tori and the Chevalley fields. As such we will be able to connect the physical parameters of the various theories with the nodes of the Dynkin diagram.

One could also derive the results of this section in a more conventional way by dimensionally reducing the eleven and ten dimensional theories to $d$ dimensions on tori, identifying the fields that belong to the $E_{n+1}$ coset and finding the relations between the fields in the resulting $d$ dimensional theories. However, this requires the use of dualisations to find all the scalars and is generally rather complicated. The $E_{11}$ approach has the advantage that it is rather technically simple and that the presence of the $E_{n+1}$ symmetry is very transparent. We also do not rely on the $E_{11}$ conjecture although there is now good evidence for this. See [47,53] for a short review.

In the rest of this paper we are really working with the expectation values of the fields, but in order to not clutter the equations we will not explicitly show the the expectation value, but it is to be understood to be present.
The $E_{11}$ Kac Moody algebra is encoded in the Dynkin diagram given in figure 4.

\[ \begin{array}{cccccccccc}
& & & & & & & & & & 11 \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
1 & \cdots & 6 & 7 & 8 & 9 & 10 \\
\end{array} \]

Figure 4. The $E_{11}$ Dynkin diagram with eleven dimensional supergravity labeling

The eleven dimensional supergravity theory emerges from the $E_{11}$ non-linear realisation if we decompose the $E_{11}$ algebra in terms of the algebra that results from deleting the exceptional node labelled eleven, namely the algebra $GL(11)$. This subalgebra has the generators $K_{\hat{a} \hat{b}}$, $\hat{a}, \hat{b} = 1, \ldots, 11$ and it includes all the Cartan subalgebra generators of $E_{11}$. In particular the relation to the Chevalley generators $H_{\hat{a}}$, $\hat{a} = 1, \ldots, 11$ being

\[ H_{\hat{a}} = K_{\hat{a} \hat{a}} - K_{\hat{a} + 1 \hat{a} + 1}, \quad \hat{a} = 1, \ldots, 10, \]

\[ H_{11} = -\frac{1}{3} (K_{11} + \ldots + K_{88}) + \frac{2}{3} (K_{99} + K_{1010} + K_{1111}). \] (2.2.1)

The first ten generators being the Cartan subalgebra generators of $SL(11)$.

The contribution of the $GL(11)$ subgroup to the $E_{11}$ group element in the non-linear realisation is of the form

\[ g = e^{h_{\hat{a} \hat{b}} K_{\hat{a} \hat{b}}}, \] (2.2.2)

where we have added the space-time translation generators $P_{\hat{a}}$. The non-linear realisation of $GL(11)$ and the space-time translations is known to give rise to eleven dimensional gravity and as a result, when dealing with the eleven dimensional theory, the line in the above Dynkin diagram that is from nodes one to ten inclusive is known as the gravity line. In the non-linear realisation of $GL(11)$ together with the translations one finds that the Cartan form contains the object

\[ g^{-1} dx^{\hat{a}} P_{\hat{a}} g = dx^{\hat{a}} \hat{e}_{\hat{a}} (\det e_{\hat{a}}^{\hat{a}})^{-\frac{1}{2}} P_{\hat{a}} \] (2.2.3)

and it turns out that $e_{\hat{a}}^{\hat{a}} = (e^h)^{\hat{a}}_{\hat{b}}$ is the eleven-dimensional vielbein. Thus the fields $h_{\hat{a}}^{\hat{a}}$ are related to the gravity fields and can be thought of as physical fields. In equation (2.2.3) we have used the relation $[K_{\hat{a} \hat{b}}, P_{\hat{c}}] = -\delta_{\hat{a}}^{\hat{b}} K_{\hat{a} \hat{c}} + \frac{1}{2} \delta_{\hat{a}}^{\hat{b}} K^{\hat{c} \hat{c}}$ [54]. This commutator between the lowest level $E_{11}$ generators, that is the $K_{\hat{a} \hat{b}}$, and the lowest level generators of the fundamental representation associated with node one, that is $P_{\hat{a}}$, follows from the fact that this is a highest weight representation.

We are interested in the diagonal components of the metric, or equivalently the fields $h_{\hat{a}}^{\hat{a}}$. As already mentioned the generators $K_{\hat{a} \hat{a}}$ span the Cartan subalgebra of $E_{11}$ and the group element of $E_{11}$ restricted to the Cartan subalgebra can also be written in the form $e^{\sum_{\hat{a}} h_{\hat{a}}^{\hat{a}} K_{\hat{a} \hat{a}}}$. However, the generators $H_{\hat{a}}$ also span the Cartan subalgebra and so we
can, as discussed above, also write the group element of $E_{11}$ restricted to the Cartan subalgebra in the form $e^{\phi_a H_a}$. These are just two different ways of parameterising the group element and we may equate them to find

$$e^{\sum_a \phi_a H_a} = e^{\sum_a h_a^a K^a_a}. \quad (2.2.4)$$

Comparing coefficients of $K^a_a$ using equations (2.2.1) we find the following relations between the physical fields and the Chevalley fields

$$\phi_i = h_1^1 + h_2^2 + ... + h_i^i - \frac{i}{2} \sum_{j=1}^{11} h_j^j, \quad \text{for} \quad 1 \leq i \leq 8,$$

$$\phi_9 = h_1^1 + h_2^2 + ... + h_9^9 - 3 \sum_{j=1}^{11} h_j^j, \quad \phi_{10} = h_1^1 + h_2^2 + ... + h_{10}^{10} - 2 \sum_{j=1}^{11} h_j^j,$$

$$\phi_{11} = \frac{3}{2} \sum_{j=1}^{11} h_j^j. \quad (2.2.5)$$

It is possible to find the correspondence between the physical fields and Chevalley fields beyond those associated with the Cartan subalgebra. In the next section we carry out the identification of the physical and Chevalley fields, when restricted to the Cartan subalgebra, in the dimensionally reduced theory.

**2.2.1. Dimensionally reduced M-theory**

The maximal supergravity theory in $d$ dimensional theory can be found by dimensional reduction of eleven dimensional supergravity on an $m = n + 1$ torus. This theory also appears in the non-linear realisation of $E_{11}$ if we decompose $E_{11}$ into the subalgebra that arises when we delete node $d$ of the Dynkin diagram, as shown in figure 5; that is the subalgebra $GL(d) \otimes E_{n+1}$.

![Dynkin Diagram](image)

Figure 5. The $E_{11}$ Dynkin diagram appropriate to maximal supergravity in $d < 10$ dimensions

The $GL(d)$ algebra is responsible for $d$ dimensional gravity and restricted to the Cartan subalgebra of $E_{11}$ these are given by $K^a_a$, $a = 1, ..., d$. The embedding of the $E_m$ algebra in $E_{11}$ is fixed by requiring that it commutes with the $GL(d)$ subalgebra of the gravity line and the space-time translations $P_a$ in the $d$ dimensions. Looking at the Dynkin diagram
of figure 5 we see that the $E_m$ algebra contains an $GL(m)$ algebra corresponding to nodes $d + 1$ to 10 and the generators of this subalgebra are given by

$$
\dot{K}^i_j \equiv K^i_j - \frac{1}{9 - m} \delta^i_j \sum_{a=1}^{d} K^a, \quad i, j = d + 1, \ldots, 11.
$$

(2.2.6)

One may verify that these generators obey the condition $[P_a, \dot{K}^i_j] = 0$, for all $a = 1, \ldots, d$ and $i, j = d+1, \ldots, 11$. In deriving this equation we have again used the equation $[K^a\hat{a}, P_c] = -\delta^b_a K^\hat{a}_b + \frac{1}{2} \delta^b_a K^\hat{c}_c$ [54]. The Chevalley generators of $E_m$ belonging to its Cartan subalgebra element are given by

$$
T_i = \dot{K}^{i+1}_{i+1} - K^{i+2}_{i+2} \quad i = d + 1, \ldots, 10,
$$

$$
T_{11} = -\frac{1}{3} \left( \dot{K}^{d+1}_{d+1} + \ldots + \dot{K}^8_{8} \right) + \frac{2}{3} \left( \dot{K}^{9}_{9} + \dot{K}^{10}_{10} + \dot{K}^{11}_{11} \right).
$$

(2.2.7)

Substituting the expressions of equation (2.2.6) into equation (2.2.7) one finds that

$$
H_i = T_i
$$

(2.2.8)

for $i = d + 1, \ldots, 11$, that is the Chevalley generators $T_i$ in $E_m$ are equal to the Chevalley generators of $E_{11}$ associated with nodes $d + 1$ to 10.

The Cartan subalgebra of $E_{11}$ consists of the generators $K^a_a$ and the generators $T_i$, $i = d + 1, \ldots, 11$ and so we can write the group element of $E_{11}$ restricted to the Cartan subalgebra in the form

$$
k_M = e \sum_{a=1}^{d} h^a_a K^a_a + e_3 e \sum_{a=1}^{d} K^a_a \sum_{a=d+1}^{11} \dot{\varphi}_a T_a.
$$

(2.2.9)

where $\dot{\varphi}_a$ are $E_m$ Chevalley fields and $e_3$ is a constant.

Examining the ansatz of equation (2.1.3) we see that the gravity fields in $d$ dimensions are scaled by powers of $e^\rho$ and so to arrive at the correct gravity fields in $d$ dimensions using the non-linear realisation we must incorporate this shift into the way the group element is written. As such we consider rewriting the group element of equation (2.2.2) in the form

$$
g = e \sum_{a=1}^{d} h^a_a K^a_a + e_1 e \sum_{a=1}^{d} K^a_a \sum_{i=d+1}^{11} h^i_i K^i_i + e_2 e \sum_{i=d+1}^{11} K^i_i,
$$

(2.2.10)

where $e_1$ and $e_2$ are constants. Proceeding in an analogous way to the steps leading to equation (2.2.3) we find that for the new parameterisation of equation (2.2.10) the vielbein is given by

$$
(dx^\mu \dot{e}_\mu^a P_a + dx^i \dot{e}_i^j P_j)(\det \dot{e}_\mu^a)^{-\frac{1}{2}} = g^{-1} (dx^\mu P_a + dx^i P_i) g
$$

$$
= \sum_{a=1}^{d} dx^\mu e_\mu^a (\det e_\mu^a)^{-\frac{1}{2}} (\det e_i^j)^{-\frac{1}{2}} e^{\rho e_1 e^{-\frac{\mu}{2}(\mu e_1 + me_2)} P_a
$$

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\[ + \sum_{i=d+1}^{11} dx^j e_j^i (\det e_\mu^a)^{-\frac{1}{2}} (\det e_\mu^j)^{-\frac{1}{2}} e^{\rho e_2} e^{-\frac{2}{\phi(e_2+e_1)}} P_j, \quad (2.2.11) \]

where we have defined

\[ e_i^j = e^{h_i^j} \delta_i^j, \quad i = d + 1, \ldots, 11 \quad \text{and} \quad e_\mu^a = e^{h_\mu^a} \delta_\mu^a, \quad (2.2.12) \]

consistent with the fact that we have a diagonal metric. The condition that the internal metric satisfies \( \det(G) = \det e_i^j = 1 \) simplifies the above equation and corresponds to the constraint \( \sum_{i=d+1}^{11} h_i^j = 0 \). Taking the resulting metric to be of the form

\[ \hat{e}_\mu^a = e^{\alpha \rho} e_\mu^a \quad \text{and} \quad \hat{e}_j^i = e^{\beta \rho} e_j^i \quad (2.2.13) \]

and substituting into the first line of equation (2.2.11) we find the last line of the same equation but with \( \alpha \) and \( \beta \) replaced by \( e_1 \) and \( e_2 \).

We can now consider the same maneuver but with the group element of equation (2.2.9) and we find that \( k_M^{-1} (dx^a P_a + dx^j P_j) k_M \) leads to a \( P_a \) term with a factor \( e^{-\frac{2}{\phi(e_2+e_1)}}. \) Thus to find the same result as the group element of equation (2.2.10) as given in equation (2.2.11) we must choose \( (d-2)e_3 = (d-2)e_1 + me_2. \)

Let us first consider the ansatz of equation (2.1.3) for the part of \( \rho \) that has zero expectation value, that is \( \bar{\rho} \). As the above discussion makes clear, we will find the ansatz of equation (2.1.3) provided we take \( e_1 = \alpha(D = 11), \) \( e_2 = \beta(D = 11) \). We note that if \( (d-2)e_1 + me_2 = 0 \) then \( \hat{e}_\mu^a (\det \hat{e}_\mu^a)^{-\frac{1}{2}} \) contains no \( \rho \), which also implies that the product of two inverse vielbeins times the determinant of the vielbein contains no \( \rho \). However, this is just the condition that the dimensional reduction leads to an Einstein term with no \( \rho \) and so is in Einstein frame. We recognise this as the same condition as that of equation (2.1.4). From the viewpoint of the group element of equation (2.2.9) we will recover the ansatz of equation (2.1.3) for the part \( \bar{\rho} \) if we set \( e_3 = 0. \)

Let us now apply the above discussion to the part of the ansatz of equation (2.1.3) that involves the expectation value of \( \rho \), that is \( < \rho > \); this is the case of interest to us in this paper. In this case \( e_2 = \beta \) has the same value as above, as it multiplies \( \bar{\rho} + < \rho > \) in the compactified part, but \( e_1 = \alpha = 0. \) We note that in this case we do find that the Einstein term contains \( < \rho > \). From the viewpoint of the group element of equation (2.2.9), we now must take \( (d-2)e_3 = me_2. \) We note that the value of the coefficient \( e_3 \) only affects the uncompactified part.

We now proceed to compare the \( E_{11} \) group element, restricted to the Cartan subalgebra, when written in terms of the physical fields, that is in equation (2.2.10), and the Chevalley fields, that is \( e_\mu^a \). For \( d < 9 \) dimensions and only considering the part of the group element that is associated with the \( n \) torus to the group element in equation (2.2.10) we find that

\[ e(h_{d+1}^{d+1} + e_2 \rho) K_{d+1}^{d+1} \cdots e(h_9 + e_2 \rho) K_9 \cdots e(h_{10}^{10} + e_2 \rho) K_{10}^{11} \cdots e(h_{11}^{11} + e_2 \rho) K_{11}^{11} = e^{\phi_1} T_i = \]

\[ = e^{\phi_{d+1}} (K_{d+1}^{d+1} - K_{d+2}^{d+2}) \cdots e^{\phi_{10}} (K_{10}^{10} - K_{11}^{11}) e^{\phi_{11}} \left( \frac{1}{2} \sum_{a=d+1}^{11} K_{a}^{a} - \sum_{a=d+1}^{8} K_{a}^{a} \right). \quad (2.2.14) \]
where we have used equations (2.2.7). Equating the coefficients of the generators gives the relations between the physical fields in the dimensionally reduced theory and the $E_{n+1}$ Chevalley fields

$$h^{d+1}_{d+1} + e_2 \rho = \dot{\phi}_{d+1} - \frac{1}{3} \dot{\phi}_{11}, \quad h^{d+2}_{d+2} + e_2 \rho = -\dot{\phi}_{d+1} + \dot{\phi}_{d+2} - \frac{1}{3} \dot{\phi}_{11}, \ldots$$

$$h^8 + e_2 \rho = -\dot{\phi}_7 + \dot{\phi}_8 - \frac{1}{3} \dot{\phi}_{11}, \quad h^9 + e_2 \rho = -\dot{\phi}_8 + \dot{\phi}_9 + \frac{2}{3} \dot{\phi}_{11},$$

$$h^{10}_{10} + e_2 \rho = -\dot{\phi}_9 + \dot{\phi}_{10} + \frac{2}{3} \dot{\phi}_{11}, \quad h^{11}_{11} + e_2 \rho = -\dot{\phi}_{10} + \frac{2}{3} \dot{\phi}_{11}. \quad (2.2.15)$$

Solving these equations for the $E_m$ fields in terms of the scalars $h^{d+1}_{d+1}, \ldots, h^{11}_{11}$ and $\rho$ found upon dimensional reduction we find

$$\dot{\phi}_i = h^{d+1}_{d+1} + h^{d+2}_{d+2} + \ldots + h^8 + (m - 11 + i) \frac{9}{9 - m} e_2 \rho, \quad d + 1 \leq i < 8.$$

$$\dot{\phi}_9 = h^{d+1}_{d+1} + h^{d+2}_{d+2} + \ldots + h^9 + \frac{6 (m - 3)}{9 - m} e_2 \rho,$$

$$\dot{\phi}_{10} = h^{d+1}_{d+1} + h^{d+2}_{d+2} + \ldots + h^{10} + \frac{3 (m - 3)}{9 - m} e_2 \rho, \quad \dot{\phi}_{11} = \frac{3}{9 - m} e_2 \rho. \quad (2.2.16)$$

The reader may also like to find the analogous equations for the uncompactified part. For example, the coefficient of $K^1_1$ implies the equation $h^1_1 = h^1_1 + e_3 \rho - \frac{1}{3} \dot{\phi}_{11}$. Using the relation $(d - 2)e_3 = me_2$ and the value of $\dot{\phi}_{11}$ given in equation (2.2.16) we find that this equation is automatically satisfied.

2.2.2 M-theory Parameters

Determining the fields $\dot{\phi}_i$ in terms of the scalars $h^{d+1}_{d+1}, \ldots, h^{11}_{11}$ and $\rho$, in equations (2.2.16) allows one to express the M-theory parameters given in section 2.1, namely the volume of the $m$ torus $V_{m(M)}$ and the ratio of the radius $r_{d+1}$ of a compact dimension to the $d$ dimensional Planck length, in terms of the $E_m$ fields $\dot{\phi}_i$. For the volume of the M-theory torus $V_{m(M)}$ one finds from equations (2.1.16) and (2.2.16),

$$V_{m(M)} = (2\pi)^m \frac{a_1 a_2 \ldots a_{d+1}}{l_d m} = e^{\frac{a}{g_s^2}} = e^{\frac{a}{g_s^2}}. \quad (2.2.17)$$

We therefore find that the volume $V_{m(M)}$ is closely related to the expectation value (not explicitly shown) of the field $\dot{\phi}_{11}$ which is itself associated with the node eleven of the Dynkin diagram in figure 5.

Equations (2.1.7) and (2.1.29) may be used to show that the ratio of the radius of the circle in the $d + 1$ direction $r_{d+1}$ to the $d$ dimensional Planck length $l_d$ is

$$\frac{r_{d+1}}{l_d} = \frac{l_{11}}{l_d} \frac{r_{d+1}}{l_{11}} = e^{\frac{a}{g_s^2}} = e^{\frac{a}{g_s^2}}. \quad (2.2.18)$$
Thus $\frac{r_{d+1}}{l_d}$ is closely related to the expectation value of the field $\hat{\varphi}_{d+1}$ which is itself associated with the node $d+1$ of the Dynkin diagram in figure 5.

Furthermore, nodes $d+1$ to 8 are associated with subtori of the $m$ torus. The dimensionless volume of the $j$ torus $V_j$ contained within the $m$ torus $V_{m(M)}$ is defined in equation (2.1.10) and may be written

$$V_j = (2\pi)^j \left( \frac{l_{11}}{l_d} \right)^j \frac{r_d + 1 + r_d + 2 + \ldots + r_d + j}{(l_{11})^j} = e^j \left( \frac{a}{g_{m-m}} \right)^{\beta \rho + h^{d+1}} d \ldots d + h^{d+2} d \ldots d + \ldots + h^{d+1} d \ldots d = e^{\hat{\varphi}_{d+j}},$$

where we have made use of equations (2.1.15), (2.1.16), (2.1.6) and equations (2.2.16) to express the $d$ dimensional M-theory physical fields in terms of the $E_{n+1}$ Chevalley field.

Similarly, the volume of the $m-2$ subtorus is given by

$$V_{m-2} = (2\pi)^{m-2} \left( \frac{l_{11}}{l_d} \right)^{m-2} \frac{r_d + 1 + r_d + 2 + \ldots + r_9}{(l_{11})^{m-2}} = e^{(m-2) \left( \frac{a}{g_{m-m}} \right)^{\beta \rho + h^{d+1}} d \ldots d + h^{d+2} d \ldots d + \ldots + h^9}$$

$$= e^{\hat{\varphi}_9 + \hat{\varphi}_{11}},$$

and the volume of the $m-1$ subtorus is

$$V_{m-1} = (2\pi)^{m-1} \left( \frac{l_{11}}{l_d} \right)^{m-1} \frac{r_d + 1 + r_d + 2 + \ldots + r_{10}}{(l_{11})^{m-1}} = e^{(m-1) \left( \frac{a}{g_{m-m}} \right)^{\beta \rho + h^{d+1}} d \ldots d + h^{d+2} d \ldots d + \ldots + h^{10}}$$

$$= e^{\hat{\varphi}_{10} + 2\hat{\varphi}_{11}}.$$

It follows that each node of the $E_m$ Dynkin diagram is associated with a specific combination of the $m$ parameters $V_i$, $i = 1, \ldots, m$, as given below each node in figure 6.

$$V_{m(M)}^j$$

$$\bullet$$

$$\frac{r_{d+1}}{l_d}$$

$$V_{m-4} \quad V_{m-3} \quad V_{m-2} V_{m(M)}^{-\frac{1}{2}} \quad V_{m-1} V_{m(M)}^{-\frac{2}{3}}$$

Figure 6. The $E_m$ Dynkin diagram labelled by the $d$ dimensional M-theory parameters

### 2.3 Parameters and fields in type IIA supergravity

Let us now consider the ten dimensional IIA supergravity theory which can be obtained from the supergravity theory in eleven dimensions by dimensional reduction on a circle. In this process, the diagonal components of the eleven dimensional metric result in the diagonal components of the ten dimensional metric and a scalar $\phi$, which is the dilaton of the IIA theory. In terms of the $E_{11}$ non-linear realisation we obtain the IIA theory by decomposing $E_{11}$ into the algebra that results from deleting nodes ten and eleven of the...
Dynkin diagram below (see figure 7), that is the subalgebra $GL(10) \otimes GL(1)$ algebra. The $GL(10)$ algebra leads to ten dimensional gravity and the $GL(1)$ factor leads to the IIA dilaton.

\[
\begin{array}{ccccccc}
11 & 10 & \bullet & \bullet & \mid & \mid & \\
1 & 6 & 7 & 8 & 9 & - & \ldots & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet
\end{array}
\]

Figure 7. The $E_{11}$ Dynkin diagram appropriate to type IIA supergravity

The gravity line is now the horizontal line of the Dynkin diagram of figure 7.

Let us denote the generators of $GL(10)$ by $K^a_{b}$, $a, b = 1, \ldots, 10$ and let $\tilde{R}$ be the $GL(1)$ generator. These contain the generators of the Cartan subalgebra of $E_{11}$. The group element in the Cartan subalgebra of $E_{11}$ can therefore be written in the form

\[
g = e^{\sum a \tilde{K}^a_a K^a_{a+1}} e^{\tilde{\sigma} \tilde{R}}. \tag{2.3.1}
\]

The tilde distinguishes the fields and generators from those used in eleven dimensions. However, in terms of the Chevalley generators in the Cartan subalgebra of $E_{11}$, the group element has the form $g = e^{\sum a \phi_a H_a}$. This is the same form as in eleven dimensions as the $E_{11}$ algebra has essentially a unique set of generators $H_a$.

The derivation of the IIA supergravity theory as a non-linear realisation leads to the following relation between the Cartan sub-algebra generators $H_a$ of the $E_{11}$ algebra and those in the $GL(10) \otimes GL(1)$ algebra [45]

\[
H_a = \tilde{K}_a^a - \tilde{K}^a_{a+1}, \quad a = 1, \ldots, 9,
\]

\[
H_{10} = -\frac{1}{8} \left( \tilde{K}^1_1 + \ldots + \tilde{K}^9_9 \right) + \frac{7}{8} \tilde{K}^{10}_{10} - \frac{3}{2} \tilde{R},
\]

\[
H_{11} = -\frac{1}{4} \left( \tilde{K}^1_1 + \ldots + \tilde{K}^8_8 \right) + \frac{3}{4} \left( \tilde{K}^9_9 + \tilde{K}^{10}_{10} \right) + \tilde{R}. \tag{2.3.2}
\]

Equating the group element $g$ in the Cartan subalgebra written in terms of the two different set of generators we find that

\[
g = e^{\sum_{a=1}^{10} \tilde{K}^a_a K^a_{a+1}} e^{\sigma \tilde{R}} = e^{\sum_{a=1}^{11} \phi_a H_a} = e^{\phi_1 (\tilde{K}^1_1 - \tilde{K}^2_2)} \ldots e^{\phi_9 (\tilde{K}^9_9 - \tilde{K}^{10}_{10})}
\times e^{\phi_{10} (\tilde{K}^1_1 + \ldots + \tilde{K}^9_9) + \frac{7}{8} \tilde{K}^{10}_{10} - \frac{3}{4} \tilde{R}} e^{\phi_{11} (\tilde{K}^1_1 + \ldots + \tilde{K}^8_8) + \frac{3}{4} (\tilde{K}^9_9 + \tilde{K}^{10}_{10}) + \tilde{R}}. \tag{2.3.3}
\]

using equations (2.3.2). Comparing the coefficients of the generators $\tilde{R}$ and $\tilde{K}^a_a$ we find the physical fields are related to the the $E_{11}$ Chevalley fields $\phi_a$ by

\[
\tilde{\sigma} = -\frac{3}{2} \phi_{10} + \phi_{11}, \quad \tilde{h}_1^1 = \phi_1 - \frac{1}{8} \phi_{10} - \frac{1}{4} \phi_{11},
\]

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\[
\tilde{h}_i = -\phi_{i-1} + \phi_i - \frac{1}{8} \phi_{10} - \frac{1}{4} \phi_{11}, \quad \text{for} \quad 2 \leq i < 9,
\]
\[
\tilde{h}_9 = -\phi_8 + \phi_9 - \frac{1}{8} \phi_{10} + \frac{3}{4} \phi_{11}, \quad \tilde{h}_{10} = -\phi_9 + \phi_{10} + \frac{7}{8} \phi_{10} + \frac{3}{4} \phi_{11}. \quad (2.3.4)
\]

### 2.3.1. Dimensionally reduced type IIA

The theory in \( d \) dimensions that results from dimensionally reducing the IIA theory on an \( n = 10 - d \) torus is found in the \( E_{11} \) non-linear realisation by decomposing \( E_{11} \) with respect to the algebra that remains when one deletes node \( d \) of the Dynkin diagram, after deletion of nodes 10 and 11, that is the subalgebra \( GL(d) \otimes GL(n) \otimes GL(1) \).

\[
\begin{array}{cccc}
1 & \ldots & d-1 & d \\
\otimes & \otimes & \otimes & \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Figure 8. The \( E_{11} \) Dynkin diagram appropriate to type IIA supergravity in \( d = 10 - n \) dimensions

The \( SL(d) \) subalgebra has the generators \( \hat{K}^a \), \( a = 1, \ldots, d \) and it is responsible in the non-linear realisation for gravity in \( d \) dimensions. The gravity line consists of nodes 1 to \( d - 1 \) inclusive. The \( GL(1) \) factor has generator \( \hat{R} \) while the \( GL(n) \) factor, associated with nodes \( d + 1 \) to 9, has the generators

\[
\hat{K}^i_j = \hat{K}^i_j - \frac{1}{8 - n} \delta^i_j \sum_{a=1}^{d} \hat{K}^a, \quad i, j = d + 1, \ldots, 10. \quad (2.3.5)
\]

One can verify that these generators commute with those of \( GL(d) \) and the space-time translations in \( d \) dimensions.

It is evident from the Dynkin diagram of figure 8 that the theory possess an \( E_{n+1} \) symmetry arising from the nodes \( d + 1 \) to 11. This contains the subalgebra \( GL(n) \otimes GL(1) \). The Chevalley generators in the Cartan subalgebra of \( E_{n+1} \) are given by

\[
\tilde{T}_i = \hat{K}^i_i - \hat{K}^{i+1}_{i+1}, \quad a = d + 1, \ldots, 9,
\]
\[
\hat{T}_{10} = -\frac{1}{8} \left( \hat{K}^{d+1}_{d+1} + \ldots + \hat{K}^9_9 \right) + \frac{7}{8} \hat{K}^{10}_{10} - \frac{3}{2} \hat{R},
\]
\[
\hat{T}_{11} = -\frac{1}{4} \left( \hat{K}^{d+1}_{d+1} + \ldots + \hat{K}^8_8 \right) + \frac{3}{4} \left( \hat{K}^9_9 + \hat{K}^{10}_{10} \right) + \hat{R}. \quad (2.3.6)
\]

Substituting equations (2.3.5) into equations (2.3.6) one finds \( H_i = \tilde{T}_i \) for \( i = d + 1, \ldots, 11 \) where \( H_i \) are Chevalley generators of \( E_{11} \).
The $E_{11}$ group element restricted to the Cartan subgroup can therefore be written in the form
\[
k_{IIA} = e \sum_{a=1}^{11} \hat{h}^a \hat{K}^a + \sum_{i=0}^{10} \hat{K}_i^i \bar{\epsilon}_i \hat{R}^i,
\] (2.3.7)

where $\hat{\varphi}_a$ are the $E_{n+1}$ Chevalley fields that parameterise the $E_{n+1}$ Cartan subalgebra.

As for the reduction of eleven dimensional supergravity, we must also find the gravity fields in the theory in $d$ dimensions corresponding to our compactification ansatz of equation (2.1.3). Following similar arguments as in that case we find the ansatz is encoded in the non-linear realisation if we take the equatation of the generators we find the physical fields in the theory in $d$ dimensions we find
\[
g = e \sum_{a=1}^{d} \hat{h}^a \hat{K}^a + \sum_{i=0}^{d} \hat{K}_i^i \bar{\epsilon}_i \hat{R}^i.
\] (2.3.8)

We are interested in the part of the ansatz of equation (2.13) that involves $< \bar{\rho} >$ and so, following the discussion around equations (2.2.11) and (2.2.13), we take $\bar{\epsilon}_1 = 0$, $\bar{\epsilon}_2 = \beta(D = 10)$, while $\bar{\epsilon}_3 = \frac{n \bar{\epsilon}_2}{8-n}$.

We now have two different way of expressing the $E_{11}$ group element; the formulation of equation (2.3.7) and that of equation (2.3.8). Equating these two and keeping only those parts associated with the compactified directions we find that
\[
e^{(\hat{h}^{d+1}_{i=1} + \hat{\epsilon}_1 \hat{\rho}) \hat{K}^{d+1}_{i=1} + \sum_{a=0}^{d} \hat{K}_a} e^2 \hat{K}^9 \bar{\epsilon}_9 \hat{K}^{10} \hat{K}_{10} e^\hat{\sigma} \hat{R}^R = e^{\hat{T}_1 \hat{\varphi}_1} = e^{\hat{\varphi}_1} \hat{R}^{d+1} \hat{K}^{d+1} + \sum_{a=0}^{d} \hat{K}_a \hat{K}^9 \hat{K}^{10} \hat{K}_{10} \hat{K}_{11} e^{\hat{\varphi}_1} \hat{R}^R
\] (2.3.9)

Equating the coefficients of the generators we find the physical fields in the $d$ dimensional type IIA theory in terms of the $E_{n+1}$ fields
\[
\hat{h}^{d+1}_{d+1} + \hat{\epsilon}_2 \hat{\rho} = \hat{\varphi}_{d+1} - \frac{1}{8} \hat{\varphi}_{10} - \frac{1}{4} \hat{\varphi}_{11},
\]
\[
\hat{h}^{d+2}_{d+2} + \hat{\epsilon}_2 \hat{\rho} = -\hat{\varphi}_{d+2} + \hat{\varphi}_{d+1} - \frac{1}{8} \hat{\varphi}_{10} - \frac{1}{4} \hat{\varphi}_{11},
\]
\[
\hat{h}^8 + \hat{\epsilon}_2 \hat{\rho} = -\hat{\varphi}_7 + \hat{\varphi}_8 - \frac{1}{8} \hat{\varphi}_{10} - \frac{1}{4} \hat{\varphi}_{11},
\]
\[
\hat{h}^9 + \hat{\epsilon}_2 \hat{\rho} = -\hat{\varphi}_8 + \hat{\varphi}_9 - \frac{1}{8} \hat{\varphi}_{10} + \frac{3}{4} \hat{\varphi}_{11},
\]
\[
\hat{h}^{10} + \hat{\epsilon}_2 \hat{\rho} = -\hat{\varphi}_9 + \frac{7}{8} \hat{\varphi}_{10} + \frac{3}{4} \hat{\varphi}_{11},
\]
\[
\hat{\sigma} = -\frac{3}{2} \hat{\varphi}_{10} + \hat{\varphi}_{11}.
\] (2.3.10)

Solving these equations for the IIA $E_{n+1}$ Chevalley fields $\hat{\varphi}_i$ in terms of the physical fields $\hat{h}^{d+1}_{d+1}, \hat{h}^{d+2}_{d+2}, ..., \hat{h}^{10}_{10}, \hat{\sigma}, \hat{\rho}$ of the theory in $d$ dimensions we find
\[
\hat{\varphi}_i = \hat{h}^{d+1}_{d+1} + \hat{h}^{d+2}_{d+2} + ... + \hat{h}^i + (n - 10 + i) \frac{8}{8-n} \hat{\epsilon}_2 \hat{\rho}, \text{ for } i = d + 1, ..., 8.
\]
\[
\hat{\varphi}_9 = \hat{h}^{d+1}_{d+1} + \hat{h}^{d+2}_{d+2} + ... + \hat{h}^9 + \frac{5n - 8}{8-n} \hat{\epsilon}_2 \hat{\rho} - \frac{1}{4} \hat{\sigma},
\]
\[
\hat{\varphi}_10 = \hat{h}^{d+1}_{d+1} + \hat{h}^{d+2}_{d+2} + ... + \hat{h}^{10} + \frac{3n - 8}{8-n} \hat{\epsilon}_2 \hat{\rho} + \frac{1}{4} \hat{\sigma}.
\]
\[ \dot{\varphi}_{10} = -\frac{1}{2} \tilde{\sigma} + \frac{2}{8-n} n \tilde{\rho}, \]
\[ \dot{\varphi}_{11} = \frac{1}{4} \tilde{\sigma} + \frac{3}{8-n} n \tilde{\rho}. \]  

\[ \text{(2.3.11)} \]

### 2.3.2. Type IIA parameters

Finally we can express the parameters \( V_{n(A)} \), \( g_d \), \( \frac{r_{d+1}}{l_d} \) and the volume \( V_j \) of the \( j \) dimensional subtorus of \( V_{n(A)} \) in terms of the expectation values, not explicitly shown, of the \( E_{n+1} \) Chevalley fields \( \dot{\varphi}_i \). For the volume of the IIA torus \( V_{n(A)} \) one finds from equations (2.1.16) and (2.3.11)

\[ V_{n(A)} = (2\pi)^n \frac{r_1 r_2 \cdots r_{d+1}}{l_{d}} = e^{\left(\frac{8}{n-8}\right) n \beta \tilde{\rho}} = e^{(\dot{\varphi}_{10} + 2 \dot{\varphi}_{11})}. \]

\[ \text{(2.3.12)} \]

Thus the IIA volume is related to the expectation value of the Chevalley fields \( \dot{\varphi}_{10} \) and \( \dot{\varphi}_{11} \) which are associated with nodes ten and eleven of the \( E_{11} \) Dynkin diagram.

The effective coupling in \( d \) dimensions is given in equation (2.2.21) and may be written

\[ g_d = e^{-\frac{8-n}{8-n} \frac{n \beta \tilde{\rho}}{2}} = e^{-2 \left(\frac{8-n}{8-n}\right) \dot{\varphi}_{10}}. \]

\[ \text{(2.3.13)} \]

the string coupling in \( d \) dimension is associated with the expectation value of the Chevalley field \( \dot{\varphi}_{10} \) and so with node ten of the \( E_{11} \) Dynkin diagram.

Equations (2.1.15), (2.1.16) and (2.1.6) may be used to show that the ratio of the radius of the circle in the \( d+1 \) direction \( r_{d+1} \) to the \( d \) dimensional Planck length \( l_d \) is given by

\[ \frac{r_{d+1}}{l_d} = \frac{\frac{l_{10}}{l_d} \frac{r_{d+1}}{l_{10}}}{\frac{l_{10}}{l_d}} = e^{\left(\frac{8}{n-8}\right) \frac{n \beta \tilde{\rho} + \tilde{h}^{d+1}_{d+1}}{2}} = e^{\dot{\varphi}_{d+1}}. \]

\[ \text{(2.3.14)} \]

Thus this last parameter is related to the expectation value of the Chevalley field \( \dot{\varphi}_{d+1} \) and so with node \( d+1 \) of the \( E_{11} \) Dynkin diagram. As in the M-theory case, nodes \( d+1 \) to 8 are associated with subtori of the \( n+1 \) torus. The dimensionless volume of the \( j \) torus \( V_j \) contained within the \( n \) torus is defined in equation (2.1.10) and may be written

\[ V_j = (2\pi)^j \left(\frac{l_{10}}{l_d}\right)^j \frac{r_{d+1} r_{d+2} \cdots r_{d+j}}{(l_{10})^j} = e^{\left(\frac{8}{n-8}\right) \frac{n \beta \tilde{\rho} + \tilde{h}^{d+1}_{d+1} + \tilde{h}^{d+2}_{d+2} + \cdots + \tilde{h}^{d+j}_{d+j}}{2}} = e^{\dot{\varphi}_{d+j}}, \]

\[ \text{(2.3.15)} \]

where we have made use of equations (2.1.15), (2.1.16), (2.1.6) and equations (2.3.11) to express the \( d \) dimensional type IIA physical fields in terms of the \( E_{n+1} \) Chevalley field. Similarly, the volume of the \( n-1 \) subtorus is given by

\[ V_{n-1} = (2\pi)^{n-1} \left(\frac{l_{10}}{l_d}\right)^{n-1} \frac{r_{d+1} r_{d+2} \cdots r_9}{(l_{10})^{n-1}} = e^{(n-1)\left(\frac{8}{n-8}\right) \frac{n \beta \tilde{\rho} + \tilde{h}^{d+1}_{d+1} + \tilde{h}^{d+2}_{d+2} + \cdots + \tilde{h}^{g}_{g}}{2}} = e^{\dot{\varphi}_g + \dot{\varphi}_{11}}. \]

\[ \text{(2.3.16)} \]
It follows that each node of the $E_{n+1}$ Dynkin diagram is associated with a specific combination of the $n+1$ parameters $V_i$, $i = 1, \ldots, n$, and $g_d$ as given below each node in figure 9.

![Dynkin diagram](image)

Figure 9. The $E_{n+1}$ Dynkin diagram labelled by the $d$ dimensional type IIA parameters

### 2.4 Parameters and fields in type IIB supergravity

The $E_{11}$ formulation of type IIB supergravity emerges from the non-linear realisation of $E_{11}$ after decomposing the $E_{11}$ algebra in terms of the algebra that results from deleting the node labelled nine in the $E_{11}$ Dynkin diagram in figure 10, namely the subalgebra $GL(10) \otimes SL(2)$.

![Dynkin diagram](image)

Figure 10. The $E_{11}$ Dynkin diagram appropriate to type IIB supergravity in 10 dimensions

The $GL(10)$ factor gives rise to ten dimensional gravity and the the gravity line in the type IIB theory that consists of nodes one to eight in addition to node 11. The $SL(2)$ factor arises from node ten and it is the $SL(2)$ symmetry of the IIB theory. The $GL(10)$ subalgebra is generated by $\hat{K}^a_b$, $a, b = 1, \ldots, 10$. Together with the generator $\hat{R}$ of the $SL(2)$ symmetry, the generators $\hat{K}^a_a$, $a = 1, \ldots, 10$ provide a basis for the Cartan subalgebra generators of $E_{11}$. The relation between the Cartan subalgebra generators of the $E_{11}$ algebra in Chevalley basis, the $H_a$, and the above generators is given by [44]

\[
H_a = \hat{K}^a_a - \hat{K}^{a+1}_{a+1}, \quad a = 1, \ldots, 8,
\]

\[
H_9 = \hat{K}^9_9 + \hat{K}^{10}_{10} - \frac{1}{4} \sum_{a=1}^{10} \hat{K}^a_a + \hat{R},
\]

\[
H_{10} = -2\hat{R},
\]

\[
H_{11} = \hat{K}^9_9 - \hat{K}^{10}_{10}.
\]

(2.4.1)
The $E_{11}$ group element restricted to the Cartan subalgebra takes the usual form, that is $e^{\sum_{a} \hat{\phi}_{a} H_{a}}$, where $\hat{\phi}$ are the fields associated with the $E_{11}$ Chevalley generators $H_{a}$ in the Cartan subalgebra which are the same no matter what theory we are considering. However, we can also express this group element in terms of the above generators as

$$g = e^{\sum_{a} h_{a}^a K_{a}^a} e^{\hat{\sigma} R}$$

(2.4.2)

where $\hat{\sigma}$ is the type IIB dilaton. Equating these two group elements we find that

$$e^{\sum_{a} \hat{\phi}_{a} H_{a}} = e^{\sum_{a} h_{a}^a K_{a}^a} e^{\hat{\sigma} R}.$$  

(2.4.3)

Comparing the coefficients of the generator $R$ we find

$$\hat{\phi}_{10} = -\frac{1}{2} \hat{\sigma},$$

(2.4.4)

one may similarly compare the coefficients of the $GL(10)$ generators to find an expression for the rest of the $E_{11}$ Chevalley fields in terms of the physical fields $\hat{h}_{a}^a$, $a = 1, ..., 10$.

### 2.4.1. Dimensionally reduced type IIB

The theory obtained from the dimensionally reduced type IIB theory on an $n = 10 - d$ torus appears if one decomposes $E_{11}$ into the sub-algebra that arises when one deletes node $d$ of the Dynkin diagram of figure 11, that is the algebra $GL(d) \otimes SL(2) \otimes GL(9 - d)$.

Figure 11. The $E_{11}$ Dynkin diagram appropriate to type IIB supergravity in $d = 10 - n$ dimensions

The $GL(d)$ algebra has the generators $\hat{K}^a_{b}$, $a, b = 1, ..., d$ and gives rise to ten dimensional gravity. The $GL(9 - d)$ algebra arises from nodes $d + 1$ to eight inclusive and node nine. The generators of $GL(9 - d)$ are given by

$$\hat{K}^i_{j} = \hat{K}^i_{j} - \frac{1}{8 - n} \delta^i_{j} \sum_{a=1}^{d} \hat{K}^a_{a}, \quad i, j = d + 1, ..., 10,$$

(2.4.5)

and one can verify that they commute with those of $GL(d)$, $SL(2)$ and the space-time translations in $d$ dimensions.
As is obvious from the Dynkin diagram of figure 11 the theory in $d$ dimensions has an $E_{n+1}$ symmetry which contains the $GL(9 - d) \otimes SL(2)$ symmetry. The generators in the Cartan subalgebra of $E_{n+1}$ are given by

$$\hat{T}_i = \hat{K}^{i+1}_{i+1} - \hat{K}^{i+2}_{i+2} \quad \text{for} \quad i = d + 1, \ldots, 8,$$

$$\hat{T}_9 = -\frac{1}{4} \left( \sum_{i=d+1}^{8} \hat{K}_i^i \right) + \frac{3}{4} \left( \hat{K}^9_9 + \hat{K}^{10}_{10} \right) + \hat{R},$$

$$\hat{T}_{10} = -2\hat{R}, \quad \hat{T}_{11} = \hat{K}^9_9 - \hat{K}^{10}_{10}. \quad (2.4.6)$$

Substituting the equation (2.4.5) into equations (2.4.6) one finds $H_i = \hat{T}_i$ for $i = d + 1, \ldots, 11$. This equality between the Cartan subalgebra elements of $E_{11}$ and those of $E_{n+1}$ is to be expected as the $E_{n+1}$ algebra is the same algebra regardless as whether the theory in $d$ dimensions is found from dimensional reduction of the eleven, IIA or IIB supergravity theories. This fact is particularly obvious once one looks at the corresponding Dynkin diagrams of figures 5, 8 and 11 that specify the subalgebras used to find the non-linearly realised theory.

As a result the group element of $E_{11}$, viewed from the IIB perspective and restricted to the Cartan subalgebra, $k_{II}B$ can be written in the form

$$k_{II}B = e^{\sum_{a=1}^{d} \hat{h}_a^a \hat{K}_a^a + \hat{\epsilon}_3 \hat{\rho} \sum_{a=1}^{d} \hat{K}^a_a \sum_{i=d+1}^{11} \phi_i \hat{T}_i}, \quad (2.4.7)$$

where $\hat{\epsilon}_3$ is a constant and $\phi_a$ are the $E_{n+1}$ Chevalley fields that parameterise the $E_{n+1}$ Cartan subalgebra.

As for the case of the dimensional reduction of the eleven dimensional and type IIA supergravity theories the gravity fields in eleven dimensions and ten dimensions, respectively, are related to those in $d$ dimensions by factors of exponentials of the scalar fields as given in the ansatz of equation (2.1.3). Following similar arguments as for these previous cases we take the group element to be given by

$$e^{\sum_{a=1}^{d} \hat{h}_a^a \hat{K}_a^a + \hat{\epsilon}_1 \hat{\rho} \sum_{a=1}^{d} \hat{K}^a_a \sum_{i=d+1}^{10} \hat{h}_i^i \hat{K}_i^i + \hat{\epsilon}_2 \hat{\rho} \sum_{i=d+1}^{10} \hat{K}_i^i \hat{\sigma} \hat{R}}. \quad (2.4.8)$$

Following the analogous discussions for the M-theory and type IIA cases we take $\hat{\epsilon}_1 = \alpha(D = 10)$, $\hat{\epsilon}_2 = \beta(D = 10)$ and $\hat{\epsilon}_3 = \frac{\hat{\epsilon}_2}{\hat{\epsilon}_3 \hat{\epsilon}}$.

We can now equate the two ways of writing the $E_{11}$ group element, restricted to the Cartan subalgebra and for $d < 9$ dimensions; that of equations (2.4.7) and (2.4.8). Keeping only parts not involving the $SL(d)$ generators we find the relations

$$e^{(\hat{h}^{d+1}_{d+1} + \hat{\epsilon}_2 \hat{\rho}) \hat{K}^{d+1}_{d+1} + \hat{\epsilon} (\hat{h}^9_9 + \hat{\epsilon}_2 \hat{\rho}) \hat{K}^9_9 e^{(\hat{h}^{10}_{10} + \hat{\epsilon}_2 \hat{\rho}) \hat{K}^{10}_{10}} e^{\hat{\sigma} \hat{R}} = e^{\hat{\phi}_i \hat{T}_i}$$

$$= e^{\hat{\phi}_d+1 (\hat{K}^{d+1}_{d+1} - \hat{K}^{d+2}_{d+2}) \ldots e^{(\hat{h}^9_9 + \hat{\epsilon} (\hat{h}^{10}_{10} + \hat{\epsilon}_2 \hat{\rho})) \hat{K}^{10}_{10}} e^{-2\hat{\phi}_{10} \hat{K}^9_9 e^{\hat{\phi}_{11} (\hat{K}^9_9 - \hat{K}^{10}_{10})}}. \quad (2.4.9)$$
Using the relations of equation (2.4.6) and equating the coefficients of the generators gives the relations between the $d$ dimensional physical fields and the $E_{n+1}$ Chevalley fields

$$
\hat{h}^{d+1}_{d+1} + \hat{e}_2 \hat{\rho} = \hat{\phi}_{d+1} - \frac{1}{4} \hat{\phi}_9, \quad \hat{h}^{d+2}_{d+2} + \hat{e}_2 \hat{\rho} = -\hat{\phi}_{d+1} + \hat{\phi}_{d+2} - \frac{1}{4} \hat{\phi}_9, \ldots
$$

$$
\hat{h}^8 + \hat{e}_2 \hat{\rho} = -\hat{\phi}_7 + \hat{\phi}_8 - \frac{1}{4} \hat{\phi}_9, \quad \hat{h}^9 + \hat{e}_2 \hat{\rho} = -\hat{\phi}_8 + \frac{3}{4} \hat{\phi}_9 + \hat{\phi}_{11},
$$

$$
\hat{h}^{10} + \hat{e}_2 \hat{\rho} = \frac{3}{4} \hat{\phi}_9 - \hat{\phi}_{11}, \quad \hat{\sigma} = -2\hat{\phi}_{10}.
$$

(S.4.10)

Solving these equations for the $E_{n+1}$ fields in terms of the scalars $\hat{\sigma}$, $\hat{h}^{d+1}_{d+1}$, ..., $\hat{h}^{10} + \hat{e}_2 \hat{\rho}$ found upon dimensional reduction yields

$$
\hat{\phi}_i = \hat{h}^{d+1}_{d+1} + \hat{h}^{d+2}_{d+2} + \ldots + \hat{h}^i + (n - 10 + i) \frac{8}{8 - n} \hat{e}_2 \hat{\rho}, \quad d + 1 \leq i < 8.
$$

$$
\hat{\phi}_9 = \frac{4n}{8 - n} \hat{e}_2 \hat{\rho}, \quad \hat{\phi}_{10} = -\frac{1}{2} \hat{\sigma} + \frac{2}{8 - n} n \hat{e}_2 \hat{\rho}, \quad \hat{\phi}_{11} = \sum_{a=d+1}^{9} h^a_a + \frac{4(n - 2)}{8 - n} \hat{\rho}.
$$

(2.4.11)

### 2.4.2. Type IIB parameters

The Volume of the torus in the type IIB theory $V_{n(B)}$, the $d$ dimensional effective coupling $g_d$, the ratio of the radius $r_{d+1}$ to the Planck length $l_d$ and the volume $V_j$ of the $j$ dimensional subtorus of $V_{n(B)}$ are expressible in terms of the $E_{n+1}$ Chevalley fields $\hat{\phi}_i$. For the volume of the IIB torus $V_{n(B)}$ one finds from equations (2.1.16) and (2.4.11),

$$
V_{n(B)} = (2\pi)^n \frac{r_{10} r_9 \ldots r_{d+1}}{l_d^n} = e^{\left(\frac{\hat{\sigma}}{8 - n}\right) n \hat{\rho}} = e^{2\hat{\phi}_9}.
$$

(2.4.12)

Thus the volume on which the IIB theory is compactified is related to the expectation value of the Chevalley field $\hat{\phi}_9$ which is associated with node nine of the $E_{11}$ Dynkin diagram.

The effective coupling in $d$ dimensions, given in equation (2.1.24), may be written

$$
g_d = e^{\frac{\hat{\sigma}}{8 - n} - \frac{n \hat{\rho}}{8 - n}} = e^{-2\left(\frac{\hat{\sigma}}{8 - n}\right) \hat{\phi}_{10}}.
$$

(2.4.13)

thus the string coupling constant of the IIB string theory in $d$ dimensions is related to the expectation value of the Chevalley field $\hat{\phi}_9$ which is associated with node ten of the Dynkin diagram.

Equations (2.1.15),(2.1.16) and (2.1.6) may be used to rewrite the ratio of the radius of the circle in the $d + 1$ direction $r_{d+1}$ to the $d$ dimensional Planck length $l_d$ as

$$
\frac{r_{d+1}}{l_d} = \frac{l_{10} r_{d+1}}{l_d} = e^{\frac{\hat{\sigma}}{8 - n} \hat{\rho} + \hat{h}^{d+1}_{d+1}} = e^{\hat{\phi}_{d+1}}.
$$

(2.4.14)

Thus the above ratio is related to the expectation value of the Chevalley fields $\hat{\phi}_{d+1}$ which is associated with node $d + 1$ of the Dynkin diagram.
Again, nodes $d+1$ to 8 are associated with subtori of the $n+1$ torus. The dimensionless volume of the $j$ torus $V_j$ contained within the $n$ torus is defined in equation (2.1.10) and may be written

$$V_j = (2\pi)^j \left(\frac{l_1}{l_d}\right)^j \frac{r_{d+1}r_{d+2}\ldots r_{d+j}}{(l_1)^j} = e^{j\left(\frac{s}{8-n}\right)\beta\hat{\rho} + h^{d+1} + h^{d+2} + \ldots + h^{d+j}} = e^{\hat{d}+j},$$

where we have made use of equations (2.1.15), (2.1.16), (2.1.6) and equations (2.4.11) to express the $d$ dimensional type IIB physical fields in terms of the $E_{n+1}$ Chevalley field.

Similarly, the volume of the $n-1$ subtorus is given by

$$V_{n-1} = (2\pi)^{n-1} \left(\frac{l_1}{l_d}\right)^{n-1} \frac{r_{d+1}r_{d+2}\ldots r_9}{(l_1)^{n-1}} = e^{(n-1)\left(\frac{s}{8-n}\right)\beta\hat{\rho} + h^{d+1} + h^{d+2} + \ldots + h^9} = e^{\hat{d}+\hat{\rho}},$$

(2.4.16)

It follows that each node of the $E_{n+1}$ Dynkin diagram is associated with a specific combination of the $n+1$ parameters $V_i$, $i = 1, \ldots, n$, and $g_d$ as given below each node in figure 12.

$$V_{n-1}V_{n(B)}^{-\frac{1}{2}}$$

Figure 12. The $E_{n+1}$ Dynkin diagram labelled by the $d$ dimensional type IIB parameters

3. Relations between the fields and parameters in the different theories

The maximal string theory in $d$ dimensions is unique, however, it can be derived by dimensional reduction on a torus from the type IIA theory or the IIB string theories or even the eleven dimensional M-theory. The parameters of the theory in $d$ dimensions can be thought of arising from the volume of the torus and all its subvolumes, and in the case of the ten dimensional theories also the expectation value of the dilaton. As such for dimensional reduction from M-theory, say, one finds a set of parameters that are related in a one to one way with those found by dimensional reduction from the IIB theory. In the previous section we derived the relations between the parameters and the expectation values of the fields associated with the Cartan subalgebra. In this section we will first find the relations between the fields in the $d$-dimensional theory, as they arise from the different dimensional reductions from eleven dimensions and the IIA and IIB theories. This is straightforward to do using the $E_{11}$ formulation of the theories as the fields appear in the $E_{11}$ group element and, as the group element is the same no matter what dimensional reduction is used, we can simply equate the different group elements to find the desired relations between the fields. We will then find how the parameters in the theory in $d$ dimensions are given in terms of the expectation values of the fields occurring.
in the different possible dimensional reductions. As a result we can then find the relations between the different descriptions of the parameters in the $d$-dimensional theory arising from the different dimensional reductions.

### 3.1. M-theory and IIA

In the $E_{11}$ non-linear realisation of eleven dimensional supergravity dimensionally reduced on an $m = n + 1$ torus the group element restricted to the Cartan subalgebra and written in terms of the physical fields was given in equation (2.2.9), that is,

$$g_M = e^\sum_{a=1}^d h_a^a K_a^a + e_1 \rho \sum_{a=1}^d K_a^a e^\sum_{i=d+1}^{11} h^i_i K^i_i + e_2 \rho \sum_{i=d+1}^{11} K^i_i. \quad (3.1.1)$$

On the other hand the $d$ dimensional theory obtained from the ten dimensional IIA supergravity by dimensionally reducing on an $n$ torus possesses a group element, which when restricted to the Cartan subalgebra and written in terms of the physical fields, take the form given in (2.3.8), that is,

$$g_{IIA} = e^\sum_{a=1}^d \tilde{h}_a^a K_a^a + \tilde{e}_1 \tilde{\rho} \sum_{a=1}^d \tilde{K}_a^a e^\sum_{i=d+1}^{10} \tilde{h}^i_i \tilde{K}^i_i + \tilde{e}_2 \tilde{\rho} \sum_{i=d+1}^{10} \tilde{K}^i_i e^\tilde{\sigma} \tilde{R}. \quad (3.1.2)$$

The relationship between the M-theory generators in $d$ dimensions $K_a^a$, $a = 1, ..., d$, $K^i_i$, $i = d + 1, ..., 11$ and the type IIA physical generators in $d$ dimensions $\tilde{K}_a^a$, $a = 1, ..., d$, $\tilde{K}^i_i$, $a = d + 1, ..., 10$, $\tilde{R}$ may be found by equating the Cartan subalgebra generators $H_a$, $a = 1, ..., 11$ of each theory. These expressions were given in equations (2.2.1) and (2.3.2) and using these one finds that [52]

$$K_a^a = \tilde{K}_a^a, \quad a = 1, ..., d, \quad K^i_i = \tilde{K}^i_i, \quad i = d + 1, ..., 10, \quad K^{11}_{11} = \frac{1}{8} \sum_{i=1}^{10} \tilde{K}^i_i + \frac{3}{2} \tilde{R}. \quad (3.1.3)$$

The theory in $d$ dimensions is unique as is evident from viewing the $E_{11}$ Dynkin diagrams of figures 2 and 4 for M-theory and type IIA respectively. As such we may equate the M-theory and type IIA group elements,

$$g_M = g_{IIA}. \quad (3.1.4)$$

One may use equations (3.1.3) to substitute for either the M-theory, or type IIA physical generators, in equation (3.1.4) and read off the relationship between the physical fields of these theories by equating the coefficients of the generators. This gives

$$\tilde{h}_a^a + \tilde{e}_2 \tilde{\rho} = h_a^a + e_2 \rho + \frac{1}{8} (h_{11}^{11} + e_2 \rho), \quad a = 1, ..., d,$$

$$\tilde{h}^i_i + \tilde{e}_2 \tilde{\rho} = h^i_i + \frac{1}{8} h_{11}^{11} + \frac{9}{8} e_2 \rho, \quad i = d + 1, ..., 10,$$

$$\tilde{\sigma} = \frac{3}{2} (h_{11}^{11} + e_2 \rho). \quad (3.1.5)$$
The volume of the type IIA torus \(V_{n(A)}\) is given as a function of the type IIA physical field \(\tilde{\rho}\) and the \(E_{n+1}\) Chevalley fields in equation (2.3.12). In the following equation we recall these results, but express the result when written in terms of physical IIA fields as the physical M-theory fields using equation (3.1.5) and the \(d\) dimensional parameters \(V_j(M), j = 1, \ldots, m\) arising from the dimensional reduction of M-theory on an \(m\) torus

\[
V_{n(A)} = e^{\left(\frac{\pi}{\sqrt{n}}\right)n\tilde{e}_2\tilde{\rho}} = e^{(\tilde{\phi}_{10}+2\tilde{\phi}_{11})} = e^{\sum_{i=d+1}^{10} h^i_1+\frac{2}{\sqrt{n}}n\tilde{e}_2\tilde{\rho}} = V_{m-1(M)}. \tag{3.1.6}
\]

One can verify that one finds the same result if one expresses the Chevalley fields in terms of the physical M-theory fields using equation (3.1.5) and the type IIA coupling \(g_{s(A)}\), that is, arising from the dimensional reduction of M-theory on an \(n\) torus.

\[
V_{m(M)} = e^{\left(\frac{\pi}{\sqrt{n}}\right)(n+1)e_2\rho} = e^{3\tilde{\phi}_{11}} = e^{\left(\frac{\pi}{\sqrt{n}}\right)n\tilde{e}_2\tilde{\rho}+\tilde{\phi}\tilde{\sigma}} = V_{n(A)}^3\frac{e_6}{n(A)g_d^6}. \tag{3.1.7}
\]

In the final equalities we have converted the volume of the M-theory torus into the IIA physical fields using equation (3.1.5) and the \(d\) dimensional parameters \(V_{n(A)}, V_j, j = 1, \ldots, n-1\) and \(g_d\) arising from the dimensional reduction of type IIA on an \(n\) torus. One can easily verify that one finds the same result using equation (2.3.11) to convert the Chevalley fields into the type IIA physical fields.

We note that the last equation of (3.1.5) implies the standard relationship between the IIA coupling \(g_{s(A)} = e^\tilde{\rho}\) in ten dimensions, the eleven dimensional Planck length \(l_{11}\) and the radius of the compactified eleventh dimension \(r_{11}\) given in equation (2.1.27); indeed one finds the well known result

\[
g_{s(A)}^3 = e^{2\tilde{\phi}\tilde{\sigma}} = e^{(h^i_{11}+e_2\rho)} = \frac{r_{11}}{l_{11}}. \tag{3.1.8}
\]

### 3.2. IIA and IIB

In the \(E_{11}\) non-linear realisation of type IIA supergravity dimensionally reduced on an \(n\) torus the group element restricted to the Cartan subalgebra and written in terms of the physical fields was given in equation (2.3.8), that is,

\[
g_{IIA} = e^{\sum_{a=1}^{d} \tilde{h}^a_1 \tilde{K}^a_1 + \tilde{e}_1 i \tilde{R}} \sum_{a=1}^{d} \tilde{K}^a_1 e^{\sum_{i=d+1}^{10} \tilde{h}^i_1 \tilde{K}^i_1 + \tilde{e}_2 i \tilde{R}}. \tag{3.2.1}
\]

On the other hand the \(d\) dimensional theory obtained from ten dimensional type IIB supergravity by dimensionally reducing on an \(n\) torus possesses a group element, which when restricted to the Cartan subalgebra and written in terms of the physical fields, that takes the form given in (2.4.8), that is,

\[
g_{IIB} = e^{\sum_{a=1}^{d} \tilde{h}^a_1 \tilde{K}^a_1 + \tilde{e}_1 i \tilde{R}} \sum_{a=1}^{d} \tilde{K}^a_1 e^{\sum_{i=d+1}^{10} \tilde{h}^i_1 \tilde{K}^i_1 + \tilde{e}_2 i \tilde{R}}. \tag{3.2.2}
\]
The relationship between the type IIA generators in \( d \) dimensions \( \hat{R}, \hat{K}^a_a, a = 1, \ldots, d, \hat{K}^i_i \), \( i = d + 1, \ldots, 10 \), and the type IIB physical generators in \( d \) dimensions \( \hat{R}, \hat{K}^a_a, a = 1, \ldots, d, \hat{K}^i_i, i = d + 1, \ldots, 10 \), \( R \) may be found by equating the Cartan subalgebra generators \( H_a, a = 1, \ldots, 11 \) of each theory. These expression were given in equations (2.3.2) and (2.4.1) and using these one finds that [52]

\[
\hat{K}^a_a = \hat{K}^a_a, \quad a = 1, \ldots, d, \quad \hat{K}^i_i = \hat{K}^i_i, \quad i = d + 1, \ldots, 10, \quad \hat{K}^{10}_{10} = \frac{1}{4} \sum_{i=1}^9 \hat{K}^i_i - \frac{3}{4} \hat{K}^{10}_{10} - \hat{R},
\]

\[
\hat{R} = \frac{1}{16} \sum_{i=1}^9 \hat{K}^i_i - \frac{7}{16} \hat{K}^{10}_{10} + \frac{3}{4} \hat{R}
\]  

(3.2.3)

The theory in \( d \) dimensions is unique as is evident from viewing the \( E_{11} \) Dynkin diagrams of figures 8 and 11 for the type IIA and type IIB supergravity theories respectively. As such we may equate the type IIA and type IIB group elements,

\[
g_{\text{IIA}} = g_{\text{IIB}},
\]  

(3.2.4)

One may use equations (3.2.3) to substitute for either the type IIA, or type IIB physical generators, in equation (3.2.4) and read off the relationship between the physical fields of these theories by equating the coefficients of the generators. This gives

\[
\tilde{h}^a_a + \tilde{\epsilon}_2 \tilde{\rho} = \hat{h}^a_a + \hat{\epsilon}_2 \hat{\rho} + \frac{1}{4} \left( \hat{h}^{10}_{10} + \hat{\epsilon}_2 \hat{\rho} \right), \quad a = 1, \ldots, d,
\]

\[
\tilde{h}^i_i + \tilde{\epsilon}_2 \tilde{\rho} = \hat{h}^i_i + \frac{1}{4} \hat{h}^{10}_{10} + \frac{5}{4} \hat{\epsilon}_2 \hat{\rho} + \frac{1}{16} \tilde{\sigma}, \quad i = d + 1, \ldots, 9,
\]

\[
\hat{h}^{10}_{10} + \hat{\epsilon}_2 \hat{\rho} = -\frac{3}{4} \left( \hat{h}^{10}_{10} + \hat{\epsilon}_2 \hat{\rho} \right) - \frac{7}{16} \tilde{\sigma}
\]

\[
\tilde{\sigma} = -\left( \hat{h}^{10}_{10} + \hat{\epsilon}_2 \hat{\rho} \right) + \frac{3}{4} \tilde{\sigma}.
\]  

(3.2.5)

The volume of the type IIA torus \( V_{n(A)} \) is given as a function of the type IIA physical field \( \rho \) and the \( E_{n+1} \) Chevalley fields in equation (2.3.12). In the following equation we recall these results, but express the result when written in terms of physical type IIA fields as the physical type IIB fields using equation (3.2.5) and the \( d \) dimensional parameters \( V_{n(B)}, V_{j}, j = 1, \ldots, n - 1 \) and \( g_d \) arising from the dimensional reduction of type IIB on an \( n \) torus

\[
V_{n(A)} = e^{\left( \frac{s}{s - n} \right) n \tilde{\epsilon}_2 \tilde{\rho}} = e^{(\phi_{10} + 2 \phi_{11})} = e^{2 \sum_{i=d+1}^9 \hat{i}^i_i + 2 \left( \frac{b_{n+2}}{b_{n}} \right) n \hat{\epsilon}_2 \hat{\rho} - \frac{1}{2} \tilde{\sigma}} = V_{n-1(B)} V_{n(B)}^{-1} g_d^{-\frac{4}{n}}
\]  

(3.2.6)

One can verify that one finds the same result if one express the Chevalley fields in terms of the type IIB theory physical fields using equation (2.4.11). Thus we see that the volume of the type IIA torus \( V_{n(A)} \) has the same expression in terms of the \( E_{n+1} \) Chevalley fields from
both the type IIA and type IIB perspectives, as must be the case through the uniqueness of the Chevalley fields.

A similar story applies to the volume of the type IIB torus which was given in terms of the type IIB theory physical fields and Chevalley fields in equation (2.4.12), that is

\[ V_{n(B)} = e^{\left(\frac{s}{2}\right)n^2\rho} = e^{2\rho} = e^{2\sum_{i=d+1}^{n} \hat{h}_i^2 + 2\left(\frac{2n-s}{8-n}\right)\hat{e}_2\hat{\rho} - \hat{\sigma}} = V_{n-1(A)}^2 V_{n(A)} g_d^{\frac{4}{8-n}}. \] (3.2.7)

In the final equalities we have converted volume of the type IIB torus into the type IIA physical fields using equation (3.1.5) and the \( d \) dimensional parameters \( V_{n(A)}, V_j, j = 1, \ldots, n - 1 \) and \( g_d \) arising from the dimensional reduction of type IIA on an \( n \) torus. One can easily verify that one finds the same result using equation (2.3.11) to convert the Chevalley fields into the IIA physical fields.

One may note that the last two equations of (3.2.5) give the T-duality correspondence between the radius of the circle upon which the IIA theory is compactified \( \hat{r}_{10} \) and the radius of the circle upon which the IIB theory is compactified \( \hat{\rho}_{10} \) and the T-duality correspondence between the coupling constants \( g_s(A) \) and \( g_s(B) \). The T-duality correspondence between the radii is given by

\[ \hat{r}_{10} = e^{\hat{h}_{10}^2 + \hat{e}_2\hat{\rho}_{10}(A)} = e^{\hat{h}_{10}^2 + \hat{e}_2\hat{\rho}_{10}} l_s e^{\frac{4}{2}} = e^{-\frac{4}{2}(\hat{h}_{10}^2 + \hat{e}_2\hat{\rho}) - \frac{7}{15}\hat{\sigma}} e^{-\frac{4}{2}(\hat{h}_{10}^2 + \hat{e}_2\hat{\rho}) + \frac{3}{38}\hat{\sigma}} l_s \]

\[ = e^{-\left(\hat{h}_{10}^2 + \hat{e}_2\hat{\rho}\right)} e^{-\frac{4}{2}\hat{\sigma}} l_s = \frac{l_{10(B)}}{r_{10}} e^{-\frac{4}{2}\hat{\sigma}} l_s = \frac{l_s}{r_{10}}. \] (3.2.8)

While the T-duality correspondence between the coupling constants is given by

\[ g_s(A) = e^{\hat{\sigma}} = e^{-\left(\hat{h}_{10}^2 + \hat{e}_2\hat{\rho}\right) + \frac{4}{2}\hat{\sigma}} = \frac{l_{10(B)}}{r_{10}} e^{-\frac{4}{2}\hat{\sigma}} l_s = \frac{l_s}{r_{10}} g_s(B). \] (3.2.9)

These relations are the Buscher rules for T-duality that relate the string coupling \( g_s(A) \) and radius \( \hat{r} \) of the type IIA string to the string coupling \( g_s(B) \) and radius \( \hat{\rho} \) of the type IIB string.

### 3.3. M-theory and IIB

In the \( E_{11} \) non-linear realisation of eleven dimensional supergravity dimensionally reduced on an \( m = n + 1 \) torus the group element restricted to the Cartan subalgebra and written in terms of the physical fields was given in equation (2.2.10), that is,

\[ g_M = e^{\sum_{a=1}^{d} h_{a}^a K_{a} + \hat{e}_1 \hat{\rho} \sum_{a=1}^{d} K_{a}^a \rho \sum_{i=d+1}^{11} \hat{h}_i^i K_{i} + \hat{e}_2 \hat{\rho} \sum_{i=d+1}^{11} K_{i}^i}. \] (3.3.1)

On the other hand the \( d \) dimensional theory obtained from the ten dimensional IIB supergravity by dimensionally reducing on an \( n \) torus possess a group element, which when restricted to the Cartan subalgebra and written in terms of the physical fields, that take the form given in (2.4.8), that is,

\[ g_{IIB} = e^{\sum_{a=1}^{d} \hat{h}_{a}^a K_{a} + \hat{e}_1 \hat{\rho} \sum_{a=1}^{d} K_{a}^a \rho \sum_{i=d+1}^{10} \hat{h}_i^i K_{i} + \hat{e}_2 \hat{\rho} \sum_{i=d+1}^{10} K_{i}^i e^{\hat{\sigma}} \hat{R}}. \] (3.3.2)
The relationship between the M-theory generators in \( d \) dimensions \( K^a_a, a = 1, ..., d, K^i_i, i = d + 1, ..., 11 \) and the type IIB physical generators in \( d \) dimensions \( \hat{K}^a_a, a = 1, ..., d, \hat{K}^i_i, i = d + 1, ..., 10 \), \( \hat{R} \) may be found by equating the Cartan subalgebra generators \( H_a, a = 1, ..., 11 \) of each theory. These expression were given in equations (2.2.1) and (2.4.1) and using these one finds that [52]

\[
\hat{K}^a_a = K^a_a, \quad a = 1, ..., d, \quad \hat{K}^i_i = K^i_i, \quad i = d + 1, ..., 10,
\]

\[
\hat{K}^{10}_{10} = \frac{1}{3} \sum_{i=1}^{9} K^i_i - \frac{2}{3} (K^{10}_{10} + K^{11}_{11}), \quad \hat{R} = -\frac{1}{2} (K^{10}_{10} - K^{11}_{11}) \tag{3.3.3}
\]

The theory in \( d \) dimensions is unique as is evident from viewing the \( E_{11} \) Dynkin diagrams of figures 5 and 11 for the M-theory and type IIB theories respectively. As such we may equate the M-theory and type IIB group elements,

\[
g_M = g_{IIB}, \tag{3.3.4}
\]

One may use equations (3.3.3) to substitute for either the M-theory or type IIB physical generators in equation (3.3.4) and read off the relationship between the physical fields of these theories by equating the coefficients of the generators. This gives

\[
h^a_a + e_1 \rho = \hat{h}^a_a + \hat{e}_1 \hat{\rho} + \frac{1}{3} (\hat{h}^{10}_{10} + \hat{e}_2 \hat{\rho}), \quad a = 1, ..., d,
\]

\[
h^i_i + e_2 \rho = \hat{h}^i_i + \frac{4}{3} \hat{e}_2 \hat{\rho} + \frac{1}{3} \hat{h}^{10}_{10}, \quad i = d + 1, ..., 9,
\]

\[
h^{10}_{10} + e_2 \rho = -\frac{2}{3} \hat{h}^{10}_{10} - \frac{2}{3} \hat{e}_2 \hat{\rho} - \frac{1}{2} \hat{\sigma},
\]

\[
h^{11}_{11} + e_2 \rho = -\frac{2}{3} \hat{h}^{10}_{10} - \frac{2}{3} \hat{e}_2 \hat{\rho} + \frac{1}{2} \hat{\sigma}. \tag{3.3.5}
\]

The volume of the type IIB torus \( V_{n(B)} \) is given as a function of the type IIB physical field \( \hat{\rho} \) and the \( E_{n+1} \) Chevalley fields in equation (2.4.12). In the following equation we recall these results, but in the final equalities we express the result when written in terms of physical type IIB fields as the physical fields of M-theory using equation (3.3.5) and the \( d \) dimensional parameters \( V_{j(M)}, j = 1, ..., m, \) arising from the dimensional reduction of M-theory on an \( m \) torus

\[
V_{n(B)} = e^{\left(\frac{8}{8-n}\right)n \hat{e}_2 \hat{\rho}} = e^{2 \hat{\phi}_9} = e^2 \sum_{i=d+1}^{9} h^i_i + \frac{12(n-2)}{8-n} e_2 \rho = V_{m-2(M)}^2 V_{m(M)}^{-2}. \tag{3.3.6}
\]

One can verify that one finds the same result if one expresses the Chevalley fields in terms of the M-theory physical fields using equation (2.2.16). Thus we see that the volume of the type IIB torus \( V_{n(B)} \) has the same expression in terms of the \( E_{n+1} \) Chevalley fields from both the type IIB and M-theory perspectives, as must be the case.
A similar story applies to the volume of the M-theory torus which was given in terms of the M-theory physical fields and Chevalley fields in equation (2.2.17), that is

\[ V_{m(M)} = e^\left(\frac{n^2}{2n}\right)e_{2\rho} = e^{3\varphi_{11}} = e^{12\left(\frac{n-r}{2n}\right)\hat{e}_2\hat{\rho} - 3\hat{h}_{10}} = V_n^{-\frac{3}{2}} V_{n(B)}^{-\frac{2}{3}}. \]  

(3.3.7)

In the final equalities we have converted the volume of the M-theory torus into the IIB physical fields using equation (3.3.5) and the d dimensional parameters \( V_{n(B)}, V_j, j = 1, ..., n - 1 \) and \( g_d \) arising from the dimensional reduction of type IIB on an n torus. One can easily verify that one finds the same result using equation (2.4.11) to convert the Chevalley fields into the IIB physical fields.

### 4. Limits in automorphic forms

In this chapter we are interested in the behaviour of automorphic forms as the parameters are taken to their limits. The \( E_{n+1} \) automorphic forms appearing as the coefficient functions of the higher derivative terms in the type II string effective action in \( d = 10 - n \) dimensions are built from representations of \( E_{n+1} \) whose states are denoted by \( |\psi_{E_{n+1}}\rangle \).

The \( E_{n+1} \) automorphic forms can be written as functions of the state \( |\varphi_{E_{n+1}}\rangle \) which is given by

\[ |\varphi_{E_{n+1}}\rangle = L(g_{E_{n+1}}^{-1})|\psi_{E_{n+1}}\rangle. \]  

(4.0.1)

Here the group element \( g_{E_{n+1}} \) belongs to the coset \( E_{n+1}/H \) where \( H \) is the maximal compact subgroup which is the same as the Cartan involution invariant subgroup. The symbol \( L \) denotes that the group element is in the representation to which \( |\psi_{E_{n+1}}\rangle \) belongs. Using the Iwasawa decomposition, and the local subgroup, we may write the coset representative \( L(g_{E_{n+1}}^{-1}) \) in terms of the Cartan subalgebra elements \( \hat{H} \) and the positive root generators \( \hat{E}_\alpha \) of \( E_{n+1} \) and so write the state \( |\varphi_{E_{n+1}}\rangle \) in the form

\[ |\varphi_{E_{n+1}}\rangle = e^{\frac{1}{\sqrt{2}} \sum_{a=0}^{n+1} \hat{\varphi}_a \hat{H}_a} e^{-\sum_{\alpha > 0} c_\alpha \hat{E}_\alpha} |\psi_{E_{n+1}}\rangle. \]  

(4.0.2)

We note that we may write the Cartan subalgebra part in terms of Weyl basis generators rather than Chevalley generators as \( \sum_{a=1}^{n+1} \hat{\varphi}_a \hat{H}_a = \hat{\varphi} \cdot \hat{H} \); the relation being determined by the equation \( H_a = \hat{\alpha} \cdot \hat{H} \). Further discussion of the theory of non-linear realisations can be found in appendix B. We are interested in the behaviour of the automorphic forms in the limits of the parameters, but as the parameters are related to the Chevalley fields \( \hat{\varphi}_a \), the limit can first be carried out on the state \( |\varphi_{E_{n+1}}\rangle \).

This is most easily explained by giving an example from which the general case is apparent. Given a basis of states of the representation we take the states of the representation to be a sum of the basis states with integer coefficients. For example for the case of seven dimensions we have the group \( SL(5) \) and we are interested in the \( 5 \) dimensional representation of \( SL(5) \) and in this case the state \( |\psi_{SL(5)}\rangle \) is given by

\[ |\psi_{SL(5)}\rangle = m_1|\mu_i\rangle + m_2|\mu_i\rangle + ... + m_5|\mu_5\rangle \]  

(4.0.3)

where \( |\mu_i\rangle, i = 1, ..., 5 \) are the states in the \( 5 \) of \( SL(5) \) with weight \( \mu_i \) and \( m_i, i = 1, ..., 5 \) are the corresponding integer coefficients.
The representations that occur in the automorphic forms are those that the charges of strings belong to. As such the states $|\psi_{\text{SL}(5)}\rangle$ can be thought of as carrying the string charges and the presence of the integer coefficients corresponds to the well known charge quantisation. By adopting integers in the states we have in effect taken a discrete group and it turns out that this corresponds to taking a discretisation of the group $E_{n+1}$ found by using the Chevalley method [55].

4.1. The general case

We will now formulate the constraints placed on an arbitrary higher derivative term in $d$ dimensions by taking the type IIA, type IIB, M-theory and single dimension decompactification limits along with the $j$ dimensional subtorus volume limit and the $d$ dimensional perturbative limit.

4.1.1. Perturbative Limit

String perturbation theory in $d = 10 - n$ dimensions is an expansion in powers of the string coupling $g_d$. As we found earlier in this paper the coupling $g_d$ in $d = 10 - n$ dimensions may be expressed in terms of the $E_{n+1}$ Chevalley fields and is given by

$$g_d = e^{-\frac{g_s}{4}}\dot{\phi}_{10},$$

where we have made use of equations (2.1.24) and (2.4.11). The dependence of the $d$ dimensional string coupling is independent of the perspective of the $d$ dimensional theory, i.e. whether we choose to express the $d$ dimensional theory in terms of the dimensionally reduced fields of type IIA/B string theory or M-theory. When deriving the parameters in terms of the Chevalley fields earlier in this paper we deleted node $d$ in the $E_{11}$ Dynkin diagram and we labeled the nodes of the resulting Dynkin diagram of $E_{n+1}$ by $11 - n, 10 - n, \ldots, 11$. However, in this chapter it is more logical to use the labelling $1, \ldots, n + 1$. In this relabelling the node $n$ becomes node 10 and so $\dot{\phi}_{10}$ is now $\dot{\phi}_n$. We note that in $d < 10$ dimensions the effective actions of type IIA and IIB string and M-theory are equivalent and so there is only one string coupling in $d$ dimensions. Examining equation (4.1.1), we find that the perturbative limit $g_d \to 0$ in $d$ dimensions is equivalent to $\dot{\phi}_n \to \infty$. Taking $\dot{\phi}_n \to \infty$ corresponds to deleting node $n$ in the $E_{n+1}$ Dynkin diagram, as shown in figure 13.

$$
\begin{array}{cccccccc}
\cdot & - & \ldots - & \cdot & - & \cdot & - & \cdot & - & \times \\
1 & n-3 & n-2 & n-1 & n & \hline & & & & & \\
& n+1 & \cdot & & & & & & \\
\end{array}
$$

Figure 13. The $E_{n+1}$ Dynkin diagram with node $n$ deleted.

The algebra remaining after this deletion is the $GL(1) \times SO(n, n)$ subalgebra of $E_{n+1}$. Let us denote the generator of the $GL(1)$ by $X$ which we may write as

$$X = \sum_{a=1}^{n+1} c_a H_a.$$ 

Demanding that it commute with $SO(n, n)$ and in particular the Chevalley generators $E_a$, $a = 1, \ldots, n-1, n+1$ implies that

$$X = H_1 + 2H_2 + \ldots + (n-2)H_{n-2} + \frac{n}{2}H_{n-1} + \frac{(n-2)}{2}H_{n+1} + 2H_n.$$ (4.1.2)
Using the relation between the Chevalley $H_a$ and Weyl $\vec{H}$ description of the generators in the Cartan subalgebra, given by $H_a = \vec{a} \cdot \vec{H}$, and the decomposition of the $E_{n+1}$ roots given in appendix A, which we recall here with the appropriate labelling $\vec{a}_a = (0, a_x, a_y, a_z)$, $a = 1, \ldots, n-1, n+1$ and $\alpha_n = (x, -\lambda_{n-1})$ and $x^2 = \frac{8-n}{4}$, we find that

$$X = x(1,0) \cdot \vec{H} \equiv x(\vec{H})_1.$$  (4.1.3)

In deriving this equation we have used that the roots and fundamental weights of $SO(n, n)$ obey the equation $\alpha_1 + 2\alpha_2 + \ldots + (n-2)\alpha_{n-1} + \frac{(n-2)}{2}\alpha_{n-1} + 2\lambda_{n-1} = 0$. As explained in appendix B, the group element that appears in the automorphic form, see equation (4.0.2), contains the expression $e^{-\sum_{a=1}^{n+1} \phi_a H_a}$, which in terms of the $GL(1) \times SO(n, n)$ decomposition becomes

$$\exp(-\sum_{a=1}^{n+1} \phi_a H_a) = \exp(-x\phi_n(\vec{H})_1) \exp(-\sum_{a=1}^{n-1} \phi_a \omega_a - \phi_n \bar{\lambda}_{n-1} + \phi_{n+1} \omega_n).$$  (4.1.4)

where $\vec{H}$ is a vector consisting of the last $n$ components of $\vec{H}$, the first term is in $GL(1)$ and the second term in $SO(n, n)$. We have identified the coefficient of the first term as $\phi_n$ by examining the first component of the left and right hand sides of this vector equation when written in terms of the Weyl generators $\vec{H}$. Thus the $GL(1)$ factor corresponds to the factor $e^{-x\phi_n(\vec{H})}$ in the $E_{n+1}$ group element and, from equation (4.1.1), to the powers of the effective coupling $g_d$. In taking the $g_d \to 0$ limit we must fix the $n$ quantities $\sum_{a=1}^{n-1} \phi_a \omega_a - \phi_n \bar{\lambda}_{n-1} + \phi_{n+1} \omega_n$, to preserve the $SO(n, n)$ symmetry.

The decomposition of the full $E_{n+1}$ algebra into representations of its $GL(1) \times SO(n, n)$ subalgebra can be classified into a level [56-58]. The level is just the number of times the simple root $\vec{a}_n$ occurs in the corresponding root when decomposed in terms of simple roots. Clearly, the level zero part of the decomposition is just $GL(1) \times SO(n, n)$, as is clear from figure 13. The decomposition of the representations of $E_{n+1}$ into representations of $GL(1) \times SO(n, n)$ can similarly be classified according to the level, the level in this case is the number of times the simple root $\vec{a}_n$ occurs in the root string constructed from the highest weight of the representation [59].

As we discussed above the perturbative limit $g_d \to 0$ corresponds to deleting node $n$ in the $E_{n+1}$ Dynkin diagram. In this limit an $E_{n+1}$ automorphic form has an expansion in powers of the string coupling in $d$ dimensions $g_d$, which is controlled by the $GL(1)$ factor. The coefficient functions in this expansion are automorphic forms of $SO(n, n)$ built from the representations of $SO(n, n)$ that occur in the decomposition of the representation of $E_{n+1}$, from which the original representation is built, into those of $SO(n, n)$. These latter $SO(n, n)$ automorphic forms can be labelled by the level, discussed above. Using equation (4.1.4) and (4.0.2) we can write the state $|\varphi_{E_{n+1}}\rangle$ from which the automorphic form is built as

$$|\varphi_{E_{n+1}}\rangle = e^{-x\phi_n(\vec{H})}|\varphi_{SO(n,n)}^{(0)}\rangle + \ldots = e^{-x\phi_n w_1}|\varphi_{SO(n,n)}^{(0)}\rangle + \ldots = g_d^{\frac{4xw_1}{8-n}}|\varphi_{SO(n,n)}^{(0)}\rangle + \ldots$$  (4.1.5)
where $\varphi_{SO(n,n)}^{(0)}$ is the level zero contribution and so is built from the level zero representation, with highest weight $w$ in the decomposition and $w_1$ is the first component of $w$. In this equation $+$... denotes the states formed from the higher level representations in the decomposition. Clearly the $g_d$ dependence of the level $l$ contributions is given by $g_d^{4 - 4d^2 \Delta}$.

Generic $E_{n+1}$ automorphic forms $\Phi$ constructed from $|\varphi_{E_{n+1}}\rangle$ are expected to be homogeneous functions which should satisfy the relation

$$\Phi_{E_{n+1}} (a | \varphi_{E_{n+1}} \rangle) = a^c \Phi_{E_{n+1}} (|\varphi_{E_{n+1}}\rangle),$$

where $a$ is a real number and $c$ is a scale factor that depends on the particular structure of the automorphic form. Using this homogeneity property of $E_{n+1}$ automorphic forms, and equation (4.1.5), one may write

$$\Phi_{E_{n+1}} (|\varphi_{E_{n+1}}\rangle) = g_d^{4c w_1} \Phi_{E_{n+1}} (|\varphi_{SO(n,n)}^{(0)}\rangle) + \ldots$$

where $+$... are terms that contain higher order contributions in $g_d$ and level. In the $g_d \to 0$ limit $\Phi_{E_{n+1}}$ becomes a sum of $SO(n,n)$ automorphic forms with coefficients that are powers of the $d$ dimensional effective coupling $g_d$.

We require that the perturbative terms are consistent with a perturbative expansion in $g_d$. In string frame this implies that each term has a $g_d$ dependence that is of the form $g_d^{2q-2}$ where $q$ is the genus. String frame in $d$ dimensions is related to Einstein frame by $g_{E\mu\nu} = g_d^{-\frac{4}{d-2}} g_{S\mu\nu}$. Upon rescaling to string frame we find

$$\int d^d x \sqrt{-g_S} g_d^{\frac{4\Delta-2d}{d-2}} \Phi_{E_{n+1}} O_S,$$

where $O$ is some polynomial in the $d$ dimensional curvature $R$, Cartan forms $P$ or field strengths $F$, the subscript $S$ denotes string frame quantities and $\Delta$ is the number of space time metrics minus the number of inverse space time metrics in $O_S$. Demanding that the perturbative limit of this generic higher derivative term exists from a string theory perspective means that in the limit $g_d \to 0$ equation (4.1.8) agrees with a perturbative expansion in $g_d$, for this one requires that each term is multiplied by a factor of the form $g_d^{-2+2n}$, where $n$ is either zero or a positive integer. Considering the $E_{n+1}$ automorphic form as a function of the state $|\varphi_{E_{n+1}}\rangle$ we see from equations (4.1.7) and (4.1.8) that this condition is given by

$$\lim_{g_d \to 0} g_d^{\frac{4\Delta-2d}{d-2}} \left( g_d^{\frac{4c w_1}{d-2}} \Phi_{E_{n+1}} (|\varphi_{SO(n,n)}^{(0)}\rangle) + \ldots \right) = g_d^{-2+2n_0} \Phi_{SO(n,n)}^{(0)} + \ldots$$

where $n_0$ is a non-negative integer.

4.1.2. Type IIB volume Limit

Type IIB string theory in $d = 10$ dimensions exhibits an $SL(2,Z)$ symmetry. So an arbitrary higher derivative term in $d = 10 - n$ dimensions should, in the large volume limit
where we have made use of equation (2.4.12). The dependence of the volume of the type IIB torus $V_{n(B)}$ on the Chevalley field $\dot{\varphi}_g$ is independent of the perspective of the $d$ dimensional theory, i.e. whether we choose to express the $d$ dimensional theory in terms of the dimensionally reduced fields of type IIA/B string theory or M-theory. Relabelling the $E_{n+1}$ part of the Dynkin diagram as in section 4.1.1, node 9 becomes node $n-1$ and so $\dot{\varphi}_g$ is now $\dot{\varphi}_{n-1}$. Examining equation (4.1.10), we find that the type IIB volume limit $V_{n(B)} \to \infty$ in $d$ dimensions is equivalent to $\dot{\varphi}_{n-1} \to \infty$. Taking $\dot{\varphi}_{n-1} \to \infty$ corresponds to deleting node $n-1$ in the $E_{n+1}$ Dynkin diagram, as shown in figure 14.

\[
\begin{array}{cccccc}
 & & & & & \\
 & \bullet & & & & \\
 & & \cdots & & & \\
1 & & \bullet & & \otimes & n \quad n-1 \quad n-2 \quad n-3 \quad n+1 \\
\end{array}
\]

Figure 14. The $E_{n+1}$ Dynkin diagram with node $n-1$ deleted

The algebra remaining after this deletion is the $GL(1) \times SL(2) \times SL(n)$ subalgebra of $E_{n+1}$. Let us denote the generator of the $GL(1)$ by $X$ which we may write as $X = \sum_{a=1}^{n+1} c_a H_a$. Demanding that it commute with $SL(2) \times SL(n)$ and in particular the Chevalley generators $E_a$, $a = 1, \ldots, n-2, n, n+1$ implies that

\[
X = H_1 + 2H_2 + \ldots + (n-2)H_{n-2} + \frac{n}{2} H_{n-1} + \frac{(n-2)}{2} H_{n+1} + \frac{n}{4} H_n.
\]  

(4.1.11)

Using the relation between the Chevalley $H_a$ and Weyl $\tilde{H}$ description of the generators in the Cartan subalgebra, given by $H_a = \tilde{a}_a \cdot \tilde{H}$, and the decomposition of the $E_{n+1}$ roots given in appendix A, which we recall here with the appropriate labelling $\tilde{a}_a = (0, \alpha_a)$, $a = 1, \ldots, n-2$, $\tilde{a}_{n+1} = (0, \alpha_{n+1})$, $\tilde{a}_n = (0, \beta,\underline{0})$ and $\tilde{a}_{n-1} = (x, -\mu, -\lambda_{n-1})$ and $x^2 = \frac{8-n}{2n}$, we find that

\[
X = x(1,0) \cdot \tilde{H} \equiv x(\tilde{H})_1
\]  

(4.1.12)

up to an overall scale factor. In deriving this equation we have used that the simple root $\beta$ and fundamental weight $\mu$ of $SL(2)$ satisfy $2\mu - \beta = 0$ and the simple roots $\alpha_i$ and fundamental weight $\lambda_i$ of $SL(n)$ obey the equation $\alpha_1 + 2\alpha_2 + \ldots + (n-2)\alpha_{n-2} + \frac{(n-2)}{2}\alpha_{n-1} - \frac{n}{2}\lambda_{n-2} = 0$. As explained in appendix B, the group element that appears in the automorphic form, see equation (4.0.2), contains the expression $e^{-\sum_{a=1}^{n+1} \dot{\varphi}_a H_a}$ which in terms of the $GL(1) \times SL(2) \times SL(n)$ decomposition becomes

\[
\exp\left(-\sum_{a=1}^{n+1} \dot{\varphi}_a H_a\right) = \exp(-x\dot{\varphi}_{n-1}(\tilde{H})_1) \exp((\mu\dot{\varphi}_{n-1} - \beta\dot{\varphi}_n) (\tilde{H})_2)
\]
\[
\times \exp\left(- \left( \sum_{a=1}^{n-2} \hat{\varphi}_a \hat{\omega}_a + \hat{\varphi}_{n+1} \hat{\omega}_{n-1} - \hat{\varphi}_{n-1} \Delta_{n-2} \right) \cdot \bar{H} \right), \tag{4.1.13}
\]

where \( \bar{H} \) is a vector consisting of the last \( n-1 \) components of \( \bar{H} \), the first term is in \( GL(1) \), the second term in \( SL(2) \) and the third term is in \( SL(n) \). We have identified the coefficient of the first term as \( \hat{\varphi}_{n-1} \) by examining the first component of the left and right hand sides of this vector equation when written in terms of the generator in Weyl generators \( \bar{H} \). Thus the \( GL(1) \) factor corresponds to the factor \( e^{-x \hat{\varphi}_{n-1} (\bar{H})} \) in the \( E_{n+1} \) group element and, from equation (4.1.10), to the powers of the volume of the type IIB torus \( V_{n(B)} \). In taking the \( V_{n(B)} \rightarrow \infty \) limit we must fix the quantities \( \left( \sum_{a=1}^{n-2} \hat{\varphi}_a \hat{\omega}_a + \hat{\varphi}_{n+1} \hat{\omega}_{n-1} - \hat{\varphi}_{n-1} \Delta_{n-2} \right) \) and \( \mu \hat{\varphi}_{n-1} - \beta \hat{\varphi}_n \) to preserve the \( SL(2) \times SL(n) \) symmetry.

The decomposition of the full \( E_{n+1} \) algebra into representations of its \( GL(1) \times SL(2) \times SL(n) \) subalgebra can be classified into a level [56-58]. The level is just the number of times the simple root \( \vec{\alpha}_{n-1} \) occurs in the corresponding root when decomposed in terms of simple roots. Clearly, the level zero part of the decomposition is just \( GL(1) \times SL(2) \times SL(n) \), as is clear from figure 14. The decomposition of the representations of \( E_{n+1} \) into representations of \( GL(1) \times SL(2) \times SL(n) \) can similarly be classified according to the level, the level in this case is the number of times the simple root \( \vec{\alpha}_{n-1} \) occurs in the root string constructed from the highest weight of the representation [59].

As we discussed above, the large volume limit of the type IIB torus \( V_{n(B)} \rightarrow \infty \) corresponds to deleting node \( n-1 \) in the \( E_{n+1} \) Dynkin diagram. In this limit an \( E_{n+1} \) automorphic form has an expansion in powers of the volume of the type IIB torus \( V_{n(B)} \), which is controlled by the \( GL(1) \) factor. The coefficient functions in this expansion are automorphic forms of \( SL(2) \times SL(n) \) built from the representations of \( SL(2) \times SL(n) \) that occur in the the decomposition of the representation of \( E_{n+1} \), from which the original representation is built, into those of \( SL(2) \times SL(n) \). These latter \( SL(2) \times SL(n) \) automorphic forms can be labelled by the level, discussed above. We note that since the type IIB theory in ten dimensions can not depend on the moduli of the torus it is necessary for the automorphic forms found after taking the \( V_{n(B)} \rightarrow \infty \) limit to be constructed from the trivial representation of \( SL(n) \). Using equation (4.1.13) and (4.0.2) we can write the state \( |\varphi_{E_{n+1}}\rangle \) from which the automorphic form is built as

\[
|\varphi_{E_{n+1}}\rangle = e^{-x \hat{\varphi}_n (\bar{H})} |\varphi^{(0)}_{SL(2) \times SL(n)} \rangle + \ldots
\]

\[
= e^{-x \hat{\varphi}_n w_1} |\varphi^{(0)}_{SL(2) \times SL(n)} \rangle + \ldots
\]

\[
= V_{n(B)}^{-\frac{x w_1}{2}} |\varphi^{(0)}_{SL(2) \times SL(n)} \rangle + \ldots \tag{4.1.14}
\]

where \( \varphi^{(0)}_{SL(2) \times SL(n)} \) is the level zero contribution and so is built from the level zero representation, with highest weight \( w \) in the decomposition and \( w_1 \) is the first component of \( w \). In this equation \( + \ldots \) denotes the states formed from the higher level representations in the decomposition. Clearly the \( V_{n(B)} \) dependence of the level \( l \) contributions is given by

\[
\frac{V_{n(B)}^{-\frac{x w_1 - l x^2}{2}}}{V_{n(B)}}
\]

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Generic $E_{n+1}$ automorphic forms $\Phi$ constructed from $|\varphi_{E_{n+1}}\rangle$ are expected to be homogeneous functions which should satisfy the relation

$$\Phi_{E_{n+1}}(a|\varphi_{E_{n+1}}\rangle) = a^c \Phi_{E_{n+1}}(|\varphi_{E_{n+1}}\rangle),$$

(4.1.15)

where $a$ is a real number and $c$ is a scale factor that depends on the particular structure of the automorphic form. Using this homogeneity property of $E_{n+1}$ automorphic forms, and equation (4.1.14), one may write

$$\Phi_{E_{n+1}}(|\varphi_{E_{n+1}}\rangle) = V_{n(B)}^{\frac{e_{E_{n+1}}}{8}} \Phi_{E_{n+1}}(|\varphi_{SL(2) \times SL(n)}^{(0)}\rangle) + \ldots$$

(4.1.16)

where $+\ldots$ are terms that contain higher order contributions in $V_{n(B)}$.

We require that the terms remaining in the large volume limit of the type IIB torus match the known coefficient functions of the higher derivative terms in the type IIB effective action in ten dimensions. By dimensional analysis one sees that an arbitrary $d$ dimensional higher derivative term in Einstein frame takes the form

$$l_d^{k-d} \int d^d x \sqrt{-g} \Phi_{E_{n+1}} \mathcal{O} = l_d^{k-(10-n)} V_{10(B)}^{-\frac{k-(10-n)}{8}} \int d^d x \sqrt{-g} \Phi_{E_{n+1}} \mathcal{O},$$

(4.1.17)

where $\mathcal{O}$ is a $k$ derivative polynomial in the $d$ dimensional curvature $R$, Cartan forms $P$ or field strengths $F$ and we have used equation (2.1.37) to express the $d$ dimensional Planck length in terms of the ten dimensional type IIB Planck length $l_{10(B)}$ and the volume of the type IIB torus $V_{n(B)}$. From equation (2.1.17) we see that in the large volume limit of the type IIB torus one has

$$\lim_{V_{n(B)} \to \infty} l_{10(B)}^{n} \int d^{d} x \sqrt{-g} V_{n(B)}^{\frac{8-n}{8}} = \int d^{10} x \sqrt{-g},$$

(4.1.18)

where the $V_{n(B)}$ factor is that found upon dimensional reduction of type IIB string theory to $d = 10 - n$ dimensions and the power $\frac{8-n}{8}$ associated with this factor is a consequence of the definition of $V_{n(B)}$ in equation (2.1.16). Therefore any term with $V_{n(B)}$ dependence $V_{n(B)}^{\frac{8-n}{8}}$ is preserved in the limit while any term with a lesser power of $V_{n(B)}$ vanishes in the limit. Note that one must be careful when considering non-analytic terms in the action that appear divergent in the $V_{n(B)} \to \infty$ limit. Demanding that the large volume limit $V_{n(B)} \to \infty$ of this generic higher derivative term exists from a string theory perspective means that the limit in equation (4.1.17) exists and that the resulting terms are ten dimensional higher derivative terms with a coefficient function that is a sum of $SL(2)$ automorphic forms. Examining equation (4.1.17) and substituting the $E_{n+1}$ automorphic form as a function of the state $|\varphi_{E_{n+1}}\rangle$ in (4.1.16) one finds that in the $V_{n(B)} \to \infty$ limit

$$\lim_{V_{n(B)} \to \infty} l_d^{k-(10-n)} V_{n(B)}^{-\frac{k-(10-n)}{8}} \int d^d x \sqrt{-g} V_{n(B)}^{\frac{8-n}{8}} \Phi_{E_{n+1}}(|\varphi_{SL(2) \times SL(n)}^{(0)}\rangle) + \ldots \mathcal{O}$$

$$= \lim_{V_{n(B)} \to \infty} l_d^{k-(10-n)} V_{n}^{-\frac{k-(10-n)}{8}} \int d^d x \sqrt{-g} V_{n}^{\frac{8-n}{8}}$$
where we have made use of equation (4.1.18) and denoted the generic \( d = 10 \) type IIB polynomials in the ten dimensional curvature \( \hat{R} \), Cartan form \( \hat{P} \) and field strengths \( \hat{F} \) that arise in the decompactification of the \( d \) dimensional polynomial in the curvature \( R \), Cartan forms \( P \) and field strengths \( F \) by \( \hat{O} \). The object \( \hat{O} \) contains fields with dependence on the compactified directions. Although the automorphic form possesses integer sums associated with the Kaluza-Klein modes of the compactification it is not entirely clear if this is enough to reinstate this coordinate dependence. In the above equation we have simply restored by hand this dependence in a general coordinate manner.

Therefore as a function of the state \( |\varphi_{E_{n+1}}⟩ \) the type IIB volume limit condition is given by

\[
\lim_{V_{n(B)} \to \infty} V_{n(B)}^{2-k} \left( V_{n(B)}^{E_{n+1}} \left( |\varphi_{SL(2)}⟩ × SL(n)⟩ \right) + \ldots \right) = a_0 \Phi_{SL(2)}^{(0)} + \ldots
\]  

(4.1.20)

where \( a_0 \) is a constant that depends on the \( E_{n+1} \) automorphic form. Any higher derivative term in \( d \) dimensions that converges to a higher derivative term that is not compatible with type IIB string theory in \( d = 10 \) dimensions, must be rejected as a possible higher derivative term in \( d \) dimensions. As we noted earlier, any non-vanishing term in the \( V_{n(B)} \to \infty \) limit must have a coefficient function that is constructed from the trivial representation of \( SL(n) \) so that the ten dimensional theory does not depend on the moduli of the torus.

### 4.1.3. Decompactification of a single dimension limit

Type II string theory in \( d = 10 - n \) dimensions exhibits an \( E(n+1, Z) \) symmetry. So, in the decompactification of a single dimension limit \( \frac{r_{d+1}}{l_d} \to \infty \) an arbitrary higher derivative term in \( d = 10 - n \) dimensions should become a linear combination of \( d + 1 \) dimensional higher derivative terms with coefficient functions that transform as \( E_n(Z) \) automorphic forms. As we found earlier, the ratio of the radius in the compact \( d + 1 \) direction to the \( d \) dimensional Planck length \( l_d \), in \( d = 10 - n \) dimensions, may be expressed in terms of the \( E_{n+1} \) Chevalley fields and is given by

\[
\frac{r_{d+1}}{l_d} = \frac{r_{10}}{l_{10}} = e^{\varphi_{d+1}} = \frac{l_{10}}{l_{d}},
\]  

(4.1.21)

where we have made use of equations (2.1.6), (2.1.15), (2.1.16) and (2.4.11). The dependence of the ratio of the radius in the compact \( d + 1 \) direction \( r_{d+1} \) to the \( d \) dimensional Planck length \( l_d \) on the Chevalley field \( \varphi_{d+1} \) is independent of the perspective of the \( d \) dimensional theory, i.e. whether we choose to express the \( d \) dimensional theory in term of the dimensionally reduced fields of type IIA/B string theory or M-theory. Relabelling the \( E_{n+1} \) part of the Dynkin diagram as in section 4.1.1, node \( d+1 \) becomes node 1 and so \( \varphi_{d+1} \) is now \( \varphi_1 \). Examining equation (4.1.21), we find that the decompactification of a single dimension limit \( \frac{r_{d+1}}{l_d} \to \infty \) in \( d \) dimensions is equivalent to \( \varphi_1 \to \infty \). Taking
Figure 15. The $E_{n+1}$ Dynkin diagram with node 1 deleted.

The algebra remaining after this deletion is the $GL(1) \times E_n$ subalgebra of $E_{n+1}$. Let us denote the generator of the $GL(1)$ by $X$ which we may write as $X = \sum_{a=1}^{n+1} c_a H_a$. Demanding that it commute with $E_n$ and in particular the Chevalley generators $E_a$, $a = 2, \ldots, n+1$ implies that

$$X = \sum_{a=1}^{n-2} \left( \frac{8-n+a}{2} \right) H_a + 2H_{n-1} + H_n + \frac{3}{2} H_{n+1}. \quad (4.1.22)$$

Using the relation between the Chevalley $H_a$ and Weyl $\vec{H}$ description of the generators in the Cartan subalgebra, given by $H_a = \vec{a}_a \cdot \vec{H}$, and the decomposition of the $E_{n+1}$ roots given in appendix A, which we recall here with the appropriate labelling $\vec{a}_1 = (x, -\Delta_1)$, $\vec{a}_a = (0, \omega_{a-1})$, $a = 2, \ldots, n+1$, and $x^2 = \frac{8-n}{9-n}$, we find that

$$X = x(1,0) \cdot \vec{H} \equiv x(\vec{H})_1 \quad (4.1.23)$$

up to an overall scale factor. In deriving this equation we have used that the simple roots $\vec{\alpha}_i$ and fundamental weight $\lambda_1$ of $E_n$ obey the equation $\sum_{a=2}^{n-2} \left( \frac{8-n+a}{2} \right) \vec{\alpha}_{a-1} + 2\vec{\alpha}_{n-2} + \vec{\omega}_{n-1} + \frac{3}{2} \vec{\omega}_n - \left( \frac{9-n}{2} \right) \Delta_1 = 0$. As explained in appendix B, the group element that appears in the automorphic form, see equation (4.0.2), contains the expression $e^{-\sum_{a=1}^{n+1} \hat{\varphi}_a H_a}$ which in terms of the $GL(1) \times E_n$ decomposition becomes

$$\exp(-\sum_{a=1}^{n+1} \hat{\varphi}_a H_a) = \exp(-x\hat{\varphi}_1 (\vec{H}_1)) \exp(-\left( \sum_{a=2}^{n+1} \hat{\varphi}_a \omega_{a-1} - \hat{\varphi}_1 \Delta_1 \right) \cdot \vec{H}) \quad (4.1.24)$$

where $\vec{H}$ is a vector consisting of the last $n$ components of $\vec{H}$, the first term is in $GL(1)$ and the second term in $E_n$. We have identified the coefficient of the first term as $\hat{\varphi}_1$ by examining the first component of the left and right hand sides of this vector equation when written in terms of the generator in Weyl generators $\vec{H}$. Thus the $GL(1)$ factor corresponds to the factor $e^{-x\hat{\varphi}_1 (\vec{H})_1}$ in the $E_{n+1}$ group element and, from equation (4.1.20), to the ratio of the radius in the compact $d+1$ direction to the $d$ dimensional Planck length $l_d$. In taking the $\frac{r_{d+1}}{l_d} \to \infty$ limit we must fix the $n$ quantities $\left( \sum_{a=2}^{n+1} \hat{\varphi}_a \omega_{a-1} - \hat{\varphi}_1 \Delta_1 \right)$ to preserve the $E_n$ symmetry.

The decomposition of the full $E_{n+1}$ algebra into representations of its $GL(1) \times E_n$ subalgebra can be classified into a level [56-58]. The level is just the number of times the
simple root $\tilde{\alpha}_1$ occurs in the corresponding root when decomposed in terms of simple roots. Clearly, the level zero part of the decomposition is just $GL(1) \times E_n$, as is clear from figure 9. The decomposition of the representations of $E_{n+1}$ into representations of $GL(1) \times E_n$ can similarly be classified according to the level, the level in this case is the number of times the simple root $\tilde{\alpha}_1$ occurs in the root string constructed from the highest weight of the representation [59].

As we discussed above, the decompactification of a single dimension limit $r_{d+1}/l_d \to \infty$ corresponds to deleting node 1 in the $E_{n+1}$ Dynkin diagram. In this limit an $E_{n+1}$ automorphic form has an expansion in powers of the ratio of the radius in the compact $d+1$ direction to the $d$ dimensional Planck length $l_d$, which is controlled by the $GL(1)$ factor. The coefficient functions in this expansion are automorphic forms of $E_n$ that occur in the decomposition of the representation of $E_{n+1}$, from which the original representation is built, into those of $E_n$. These latter $E_n$ automorphic forms can be labelled by the level, discussed above. Using equation (4.0.2) and (4.1.23) we can write the state $|\varphi_{E_{n+1}}\rangle$ from which the automorphic form is built as

$$|\varphi_{E_{n+1}}\rangle = e^{-x\dot{\varphi}_1(\vec{H})_1} |\varphi^{(0)}_{E_n}\rangle + ...$$

$$= e^{-x\dot{\varphi}_1 w_1} |\varphi^{(0)}_{E_n}\rangle + ...$$

$$= \left(\frac{r_{d+1}}{l_d}\right)^{-xw_2} |\varphi^{(0)}_{E_n}\rangle + ...$$

(4.1.25)

where $\varphi^{(0)}_{E_n}$ is the level zero contribution and so is built from the level zero representation, with highest weight $w$, in the decomposition and $w_1$ is the first component of $w$. In this equation $+...$ denotes the states formed from the higher level representations in the decomposition. Clearly the $r_{d+1}/l_d$ dependence of the level $l$ contributions is given by

$$\left(\frac{r_{d+1}}{l_d}\right)^{-xw_1 - lx^2}.

Generic $E_{n+1}$ automorphic forms $\Phi$ constructed from $|\varphi_{E_{n+1}}\rangle$ are expected to be homogeneous functions which should satisfy the relation

$$\Phi_{E_{n+1}} (a|\varphi_{E_{n+1}}\rangle) = a^c \Phi_{E_{n+1}} (|\varphi_{E_{n+1}}\rangle),$$

(4.1.26)

where $a$ is a real number and $c$ is a scale factor that depends on the particular structure of the automorphic form. Using this homogeneity property of $E_{n+1}$ automorphic forms, and equation (4.1.25), one may write

$$\Phi_{E_{n+1}} (|\varphi_{E_{n+1}}\rangle) = \left(\frac{r_{d+1}}{l_d}\right)^{-cxw_1} \Phi_{E_{n+1}} (|\varphi^{(0)}_{E_n}\rangle) + ...$$

$$= \left(\frac{-\left(\frac{2-n}{d+1}\right)}{r_{d+1}}\right)^{-cxw_1} \Phi_{E_{n+1}} (|\varphi^{(0)}_{E_n}\rangle) + ...$$

(4.1.27)

where $+...$ are terms that contain higher order contributions in $r_{d+1}$ and we have used equation (2.1.35) to express the state in terms of the $d+1$ dimensional Planck length $l_{d+1}$. 

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We require that the terms remaining in the decompactification of a single dimension limit match the known coefficient functions of the higher derivative terms in the type II effective action in \(d + 1\) dimensions. By dimensional analysis one sees that an arbitrary \(d\) dimensional higher derivative term in Einstein frame takes the form

\[
l^k_{d-d} \int d^d x \sqrt{-g} \Phi_{E_{n+1}} \mathcal{O} = \int d^d x \sqrt{-g} \frac{l^d}{d+1} \frac{(d-1)(k-d)}{d-2} \frac{k-d}{d-2} \Phi_{E_{n+1}} \mathcal{O} \tag{4.1.28}
\]

where we have used equation (2.1.35) to express \(l_d\) in terms of \(r_{d+1}\) and \(l_{d+1}\) and \(\mathcal{O}\) is a \(k\) derivative polynomial in the \(d\) dimensional curvature \(R\), Cartan forms \(P\) or field strengths \(F\). From equation (2.1.11) we see that in the decompactification of a single dimension limit one has

\[
\lim_{r_{d+1} \to \infty} l_{d+1} \int d^d x \sqrt{-g} \frac{r_{d+1}}{l_{d+1}} = \int d^{d+1} x \sqrt{-\hat{g}}; \tag{4.1.29}
\]

where the \(\frac{r_{d+1}}{l_{d+1}}\) factor is that found upon dimensional reduction of \(d+1\) dimensional type II string theory to \(d\) dimensions. Therefore any term with \(\frac{r_{d+1}}{l_{d+1}}\) dependence \(\frac{r_{d+1}}{l_{d+1}}\) is preserved in the limit while any term with a lesser power of \(\frac{r_{d+1}}{l_{d+1}}\) vanishes in the limit. Note that one must be careful when considering non-analytic terms in the action that appear divergent in the \(\frac{r_{d+1}}{l_{d+1}} \to \infty\) limit. Demanding that the decompactification of a single dimension limit \(\frac{r_{d+1}}{l_{d+1}}\) of this generic higher derivative term exists from a string theory perspective means that the \(\frac{r_{d+1}}{l_{d+1}} \to \infty\) limit of equation (4.1.27) exists and that the resulting terms are \(d+1\) dimensional higher derivative terms with a coefficient function that is a sum of \(E_n\) automorphic forms (or zero). Examining equation (4.1.28) and substituting the \(E_{n+1}\) automorphic form as a function of the state \(|\psi_{E_{n+1}}\rangle\) in (4.1.27) one finds that in the \(\frac{r_{d+1}}{l_{d+1}} \to \infty\) limit an arbitrary \(d\) dimensional higher derivative term becomes

\[
\lim_{r_{d+1} \to \infty} \int d^d x \sqrt{-g} \frac{l_{d+1}}{r_{d+1}} \frac{(d-1)(k-d)}{d-2} \frac{k-d}{d-2} \Phi_{E_{n+1}} \mathcal{O} = \int d^{d+1} x \sqrt{-\hat{g}} \lim_{r_{d+1} \to \infty} l_{d+1} \frac{r_{d+1}}{l_{d+1}} \frac{(d-1)(k-d)}{d-2} \frac{k-d}{d-2} \Phi_{E_{n+1}} \mathcal{O}, \tag{4.1.30}
\]

where \(\hat{\mathcal{O}}\) labels the different \(d+1\) dimensional type II polynomials in the \(d+1\) dimensional curvature \(\hat{R}\), Cartan form \(\hat{P}\) and field strengths \(\hat{F}\) that arise in the decompactification of the \(d\) dimensional polynomial in the curvature \(R\), Cartan forms \(P\) and field strengths \(F\). The fields \(\hat{\mathcal{O}}\) are the \(d+1\) dimensional analogues of the decompactified type IIB \(\hat{\mathcal{O}}\) fields discussed around equation (4.1.19). Therefore as a function of the state \(|\psi_{E_{n+1}}\rangle\) the decompactification of a single dimension condition is given by

\[
\lim_{r_{d+1} \to \infty} l_{d+1} \frac{(d-1)(k-d)}{d-2} \frac{k-d}{d-2} \left(\frac{l_{d+1}}{r_{d+1}} \frac{r_{d+1}}{l_{d+1}} \right) \Phi_{E_{n+1}} \mathcal{O}, \tag{4.1.31}
\]

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\[ a_{0} \hat{\Phi}_{E_{n}^{(0)}}^{(0)} + \ldots \]  

where \( a_{0} \) is constant that depends on the \( E_{n+1} \) automorphic form. Any higher derivative term in \( d \) dimensions that converges to a higher derivative term that is not compatible with type II string theory in \( d + 1 \) dimensions, must be rejected as a possible higher derivative term in \( d \) dimensions.

### 4.1.3. M-theory volume Limit

Type II string theory in \( d \) dimensions may be decompactified to eleven dimensional supergravity on an \((n + 1)\) torus by taking the limit \( V_{(M)} \to \infty \). As we found earlier, the volume of the M-theory torus \( V_{(M)} \) may be expressed in terms of the \( E_{n+1} \) Chevalley fields and is given by

\[
V_{(M)} = (2\pi)^{m} \frac{r_{11} r_{10} r_{9} \ldots r_{d+1}}{l_{d}^{m}} = e^{\frac{n}{m} m \beta \rho} = e^{3 \hat{\phi}_{11}},
\]  

where we have made use of equation (2.2.17). The dependence of the volume of the M-theory torus on the Chevalley field \( \hat{\phi}_{11} \) is independent of the perspective of the \( d \) dimensional theory, i.e. whether we choose to express the \( d \) dimensional theory in terms of the dimensionally reduced fields of type IIA/B string theory or M-theory. Relabelling the \( E_{n+1} \) part of the Dynkin diagram as in section 4.1.1, node 11 becomes node \( n + 1 \) and so \( \hat{\phi}_{11} \) is now \( \hat{\phi}_{n+1} \). Examining equation (4.1.32), we find that the large volume limit of the M-theory torus \( V_{(M)} \to \infty \) in \( d \) dimensions is equivalent to \( \hat{\phi}_{n+1} \to \infty \). Taking \( \hat{\phi}_{n+1} \to \infty \) corresponds to deleting node \( n + 1 \) in the \( E_{n+1} \) Dynkin diagram, as shown in figure 16.

![Figure 16. The \( E_{n+1} \) Dynkin diagram with node \( n + 1 \) deleted](image)

The algebra remaining after this deletion is the \( GL(1) \times SL(n+1) \) subalgebra of \( E_{n+1} \). Let us denote the generator of the \( GL(1) \) by \( X \) which we may write as \( X = \sum_{a=1}^{n+1} c_{a} H_{a} \). Demanding that it commute with \( SL(n + 1) \) and in particular the Chevalley generators \( E_{a} \), \( a = 1, \ldots, n \) implies that

\[
X = \sum_{a=1}^{n+1} a H_{a} + \frac{2}{3} (n - 2) H_{n-1} + \ldots + \frac{n - 2}{3} H_{n} + \frac{n + 1}{3} H_{n+1}.
\]  

Using the relation between the Chevalley \( H_{a} \) and Weyl \( \tilde{H} \) description of the generators in the Cartan subalgebra, given by \( H_{a} = \tilde{\alpha}_{a} \cdot \tilde{H} \), and the decomposition of the \( E_{n+1} \) roots given in appendix A, which we recall here with the appropriate labelling \( \tilde{\alpha}_{a} = (0, \alpha_{a}) \), \( a = 1, \ldots, n \), \( \tilde{\alpha}_{n+1} = (x, \alpha_{n-2}) \) and \( x^{2} = \frac{8 - n}{n+1} \), we find that

\[
X = x(1, 0) \cdot \tilde{H} \equiv x(\tilde{H})_{1},
\]  

(4.1.34)
up to an overall scale factor. In deriving this equation we have used that the simple roots \( \alpha_i \)
and fundamental weight \( \lambda_{n-2} \) of \( SL(n+1) \) obey the equation 
\[
\sum_{a=1}^{n-2} a \alpha_a + \frac{2}{3} (n - 2) \alpha_{n-1} + \frac{n-2}{3} \alpha_n - \frac{n+1}{3} \lambda_{n-2} = 0.
\]
As explained in appendix B, the group element that appears in the automorphic form, see equation (4.0.2), contains the expression 
\[
e^{-\sum_{a=1}^{n+1} \varphi_a H_a}
\]
which in terms of the \( GL(1) \times SL(n+1) \) decomposition becomes
\[
\exp(-\sum_{a=1}^{n+1} \varphi_a H_a) = \exp(-x \varphi_{n+1}(H)_1) \exp(-\sum_{a=1}^{n} \varphi_a \alpha_a - \varphi_{n+1} \lambda_{n-2}) \cdot H)
\]
(4.1.35)

where \( H \) is a vector consisting of the last \( n \) components of \( \vec{H} \), the first term is in \( GL(1) \) and
the second term is in \( SL(n+1) \). We have identified the coefficient of the first term as \( \varphi_{n+1} \)
by examining the first component of the left and right hand sides of this vector equation
when written in terms of the Weyl generators \( \vec{H} \). Thus the \( GL(1) \) factor corresponds to the
factor \( e^{-x \varphi_{n+1}(H)_1} \) in the \( E_{n+1} \) group element and, from equation (4.1.32), to the powers
of the volume of the M-theory torus \( V_{m(M)} \). In taking the \( V_{m(M)} \rightarrow \infty \) limit we must fix
the \( n \) quantities \( \sum_{a=1}^{n} \varphi_a \alpha_a - \varphi_{n+1} \lambda_{n-2} \) to preserve the \( SL(n+1) \) symmetry.

The decomposition of the full \( E_{n+1} \) algebra into representations of its \( GL(1) \times SL(n+1) \)
subalgebra can be classified into a level \([56-58]\). The level is just the number of times the
simple root \( \vec{\alpha}_{n+1} \) occurs in the corresponding root when decomposed in terms of simple
roots. Clearly, the level zero part of the decomposition is just \( GL(1) \times SL(n+1) \), as is clear
from figure 16. The decomposition of the representations of \( E_{n+1} \) into representations of
\( GL(1) \times SL(n+1) \) can similarly be classified according to the level, the level in this case
is the number of times the simple root \( \vec{\alpha}_{n+1} \) occurs in the root string constructed from the
highest weight of the representation \([59]\).

As we discussed above, the large volume limit of the M-theory torus \( V_{m(M)} \rightarrow \infty \)
corresponds to deleting node \( n+1 \) in the \( E_{n+1} \) Dynkin diagram. In this limit an \( E_{n+1} \)
automorphic form has an expansion in powers of the volume of the M-theory torus \( V_{m(M)} \),
which is controlled by the \( GL(1) \) factor. The coefficient functions in this expansion are
automorphic forms of \( SL(n+1) \) built from the representations of \( SL(n+1) \) that occur in
the decomposition of the representation of \( E_{n+1} \), from which the original representation
is built, into those of \( SL(n+1) \). These latter \( SL(n+1) \) automorphic forms can be labelled
by the level, discussed above. We note that since M-theory can not depend on the moduli
of the \( m \) torus it is necessary for the automorphic forms found after taking the \( V_{m(M)} \rightarrow \infty \)
limit to be constructed from the trivial representation of \( SL(n+1) \). Using equation (4.0.2)
and (4.1.35) we can write the state \( |\varphi_{E_{n+1}} \rangle \) from which the automorphic form is built as
\[
|\varphi_{E_{n+1}} \rangle = e^{-x \varphi_{n+1} \lambda_1} |\varphi_{SL(n+1)}^{(0)} \rangle + ... \\
= e^{-x \varphi_{n+1} \lambda_1} |\varphi_{SL(n+1)}^{(0)} \rangle + ... \\
= V_{m(M)}^{-\frac{x w}{3}} |\varphi_{SL(n+1)}^{(0)} \rangle + ... 
\]
(4.1.36)

where \( \varphi_{SL(n+1)}^{(0)} \) is the level zero contribution and so is built from the level zero represent-
tation, with highest weight \( w \) in the decomposition and \( w_1 \) is the first component of \( w \).
In this equation $+ \ldots$ denotes the states formed from the higher level representations in the decomposition. Clearly the $V_m(M)$ dependence of the level $l$ contributions is given by

\[
\frac{x_{m+1} - t_{x^2}}{V_m(M)}.
\]

Generic $E_{n+1}$ automorphic forms $\Phi$ constructed from $|\varphi_{E_{n+1}}\rangle$ are expected to be homogeneous functions which should satisfy the relation

\[
\Phi_{E_{n+1}}\left(a|\varphi_{E_{n+1}}\rangle\right) = a^c \Phi_{E_{n+1}}\left(|\varphi_{E_{n+1}}\rangle\right),
\]

where $a$ is a real number and $c$ is a scale factor that depends on the particular structure of the automorphic form. Using this homogeneity property of $E_{n+1}$ automorphic forms, and equation (4.1.36), one may write

\[
\Phi_{E_{n+1}}\left(|\varphi_{E_{n+1}}\rangle\right) = V_{m(M)}^{-\frac{a_{m}}{m(M)}} \Phi_{E_{n+1}}\left(|\varphi^{(0)}_{SL(n+1)}\rangle\right) + \ldots
\]

where $+ \ldots$ are terms that contain higher order contributions in $V_m(M)$.

We require that the terms remaining in the large volume limit of the M-theory torus match the known coefficient functions of the higher derivative terms in the M-theory effective action in eleven dimensions. By dimensional analysis one sees that an arbitrary $d$ dimensional higher derivative term in Einstein frame takes the form

\[
l_k^{-d} \int d^d x \sqrt{-g} \Phi_{E_{n+1}} \mathcal{O} = l_k^{-d} V_{m(M)}^{-\frac{k-d}{11}} \int d^d x \sqrt{-g} \Phi_{E_{n+1}} \mathcal{O},
\]

where $\mathcal{O}$ is a $k$ derivative polynomial in the $d$ dimensional curvature $R$, Cartan forms $P$ or field strengths $F$. From equation (2.1.17) we see that in the large volume limit of the M-theory torus one has

\[
\lim_{V_m(M) \to \infty} l_k^{-d} \int d^d x \sqrt{-g} V_{m(M)}^{-\frac{a_{m}}{m(M)}} = \int d^{11} x \sqrt{-\hat{g}},
\]

therefore any term with $V_m(M)$ dependence $V_{m(M)}^{-\frac{a}{m(M)}}$ is preserved in the limit while any term with a lesser power of $V_m(M)$ vanishes in the limit. Note that one must be careful when considering non-analytic terms in the action that appear divergent in the $V_m(M) \to \infty$ limit. Demanding that the large volume limit $V_m(M)$ of this generic higher derivative term exists from an M-theory perspective means that the large volume limit $V_m(M) \to \infty$ of equation (4.1.39) exists and that the resulting terms are eleven dimensional higher derivative terms with coefficient functions that are $SL(n+1)$ automorphic forms that can only be constructed from the trivial representation of $SL(n+1)$ since the M-theory effective action can not depend on the moduli of the torus. Examining equation (4.1.39) and substituting the $E_{n+1}$ automorphic form as a function of the state $|\varphi_{E_{n+1}}\rangle$ in (4.1.38) one finds that in the $V_m(M) \to \infty$ limit

\[
\lim_{V_m(M) \to \infty} l_k^{-d} V_{m(M)}^{-\frac{k-d}{11}} \int d^d x \sqrt{-g} \Phi_{E_{n+1}}\left(|\varphi^{(0)}_{SL(n+1)}\rangle\right) + \ldots \mathcal{O}
\]
where the higher derivative terms in the effective action of the type IIA theory in $d$ dimensions in string frame be multiplied by a factor of ...

\[
\lim_{V_{m(M)} \rightarrow \infty} \int d^{11} x \sqrt{-g} \left( V_{m(M)} \frac{2-k}{2} \Phi_{E_{n+1}} (|\varphi_{SL(n+1)}^{(0)}|) + \ldots \right) \hat{O}, \quad (4.1.41)
\]

where $\hat{O}$ labels the different $d = 11$ M-theory polynomials in the eleven dimensional curvature $\hat{R}$, and field strengths $\hat{F}$ that arise in the decompactification of the $d$ dimensional polynomial in the curvature $R$, Cartan forms $P$ and field strengths $F$. The fields $\hat{O}$ are the eleven dimensional M-theory analogues of the decompactified type IIB $\hat{O}$ fields discussed around equation (4.1.19). Therefore, as a function of the state $|\varphi_{E_{n+1}}\rangle$, the M-theory volume limit condition is given by

\[
\lim_{V_{m(M)} \rightarrow \infty} \left( V_{m(M)} \frac{2-k}{2} \Phi_{E_{n+1}} (|\varphi_{SL(n+1)}^{(0)}|) + \ldots \right) = a_0 + \ldots \quad (4.1.42)
\]

where $a_0$ is a constant arising from the level zero contribution that depends on the $E_{n+1}$ automorphic form and ... denote contributions at higher levels. Any higher derivative term in $d$ dimensions that converges to a higher derivative term that is not compatible with the effective action of M-theory in $d = 11$ dimensions, must be rejected as a possible higher derivative term in $d$ dimensions. As we noted earlier, any non-vanishing term in the $V_{m(M)} \rightarrow \infty$ limit must have a coefficient function that is constructed from the trivial representation of $SL(n+1)$ so that the M-theory effective action in $d = 11$ dimensions does not depend on the moduli of the torus.

### 4.1.3. Type IIA volume Limit

Type IIA string theory in $d = 10$ dimensions possesses a global $GL(1, R)$ symmetry. In this case the scalar sector that parameterises the coset associated with the global $GL(1, R)$ symmetry is trivial. However, in the large volume limit $V_{n(A)} \rightarrow \infty$, one still requires that the higher derivative terms in the effective action of the type IIA theory in $d = 10$ dimensions in string frame be multiplied by a factor of $e^{(-2+2g)\phi}$ where $g$ is the genus of the $d = 10$ type IIA perturbative contribution.

The volume of the $n$ torus in the type IIA theory may be expressed in terms of the $E_{n+1}$ Chevalley fields and is given by

\[
V_{n(A)} = (2\pi)^n r_1 r_9 \cdots r_{d+1} \frac{1}{l_n^{\delta}} = e^{\frac{8}{n} \beta \rho} = e^{\phi_{10} + 2 \phi_{11}}, \quad (4.1.43)
\]

where we have made use of equations (2.3.12). The dependence of the volume of the type IIA torus $V_{n(A)}$ on the Chevalley fields $\phi_{10}$ and $\phi_{11}$ is independent of the perspective of the $d$ dimensional theory, i.e. whether we choose to express the $d$ dimensional theory in terms of the dimensionally reduced fields of type IIA/B string theory or M-theory. Similarly, from equations (2.3.11), one finds that the type IIA string coupling in ten dimensions $g_{s(A)}$ may be written

\[
g_{s(A)} = e^{-\frac{1}{2} \phi_{10} + \phi_{11}}. \quad (4.1.44)
\]
Relabelling the $E_{n+1}$ part of the Dynkin diagram as in section 4.1.1, nodes 10 and 11 become node $n$ and $n+1$ and so $\dot{\varphi}_{10}$ is now $\dot{\varphi}_n$ and $\dot{\varphi}_{11}$ is now $\dot{\varphi}_{n+1}$. Examining equation (4.1.43), we find that the type IIA volume limit $V_{n(A)} \to \infty$ in $d$ dimensions is equivalent to $\dot{\varphi}_n + 2\dot{\varphi}_{n+1} \to \infty$. Taking $\dot{\varphi}_n + 2\dot{\varphi}_{n+1} \to \infty$ corresponds to deleting nodes $n$ and $n+1$ in the $E_{n+1}$ Dynkin diagram, as shown in figure 17.

\[
\begin{array}{cccccc}
\cdot & - & \ldots - & \cdot & - & \cdot & - & \cdot & - & \cdot & - & \cdot \\
1 & & n-3 & n-2 & n-1 & n
\end{array}
\]

Figure 17. The $E_{n+1}$ Dynkin diagram with nodes $n$ and $n+1$ deleted

The algebra remaining after this deletion is the $GL(1) \times GL(1) \times SL(n)$ subalgebra of $E_{n+1}$. Let us denote the generators of the $GL(1) \times GL(1)$ part of the subalgebra by $X$ which we may write as $X = \sum_{a=1}^{n+1} c_a H_a$. Demanding that $X$ commutes with $SL(n)$ and in particular the Chevalley generators $E_a$, $a = 1, \ldots, n-1$ implies that

\[
X = \sum_{a=1}^{n-2} c a H_a + ((n-1)c-d) H_{n-1} + (nc-2d) H_n + dH_{n+1}
\]  

(4.1.45)

where $c$ and $d$ are real numbers. Using the relation between the Chevalley $H_a$ and Weyl $\tilde{H}$ description of the generators in the Cartan subalgebra, given by $H_a = \tilde{a}_a \cdot \tilde{H}$, and the decomposition of the $E_{n+1}$ roots given in appendix A, which we recall here with the appropriate labelling $\tilde{a}_a = (0, \alpha_a)$, $a = 1, \ldots, n-1$, $\tilde{a}_n = (x, -\frac{\Delta_n-2\Delta_{n-1}}{y}, -\Delta_{n-1})$, $\tilde{a}_{n+1} = (0, y, -\Delta_{n-2})$ and $x^2 = \frac{8-n}{4}$, $y^2 = \frac{n}{4}$ we find that

\[
X = (Ax, By, 0) \cdot \tilde{H} \equiv Ax(\tilde{H})_1 + By(\tilde{H})_2,
\]  

(4.1.46)

up to scale factors $A$ and $B$. In deriving this equation we have used that the simple roots $\alpha_a$ and fundamental weights $\Delta_{n-1}, \Delta_{n-2}$ of $SL(n)$ obey the equations $\alpha_1 + 2\alpha_2 + \ldots + (n-2)\alpha_{n-2} + (n-1)\alpha_{n-1} - n\Delta_{n-1} = 0$ and $-\alpha_{n-1} + 2\Delta_{n-1} - \Delta_{n-2} = 0$. As explained in appendix B, the group element that appears in the automorphic form, see equation (4.0.2), contains the expression $e^{-\sum_{a=1}^{n+1} \dot{\varphi}_a H_a}$ which in terms of the $GL(1) \times GL(1) \times SL(n)$ decomposition becomes

\[
\exp\left(-\sum_{a=1}^{n+1} \dot{\varphi}_a H_a\right) = \exp(-x\dot{\varphi}_n (\tilde{H})_1) \exp(-y(\dot{\varphi}_{n+1} - \frac{\Delta_n-2\Delta_{n-1}}{y^2} \dot{\varphi}_n)(\tilde{H})_2)
\]

\[
\times \exp(-\sum_{a=1}^{n-1} \dot{\varphi}_a \alpha_a - \dot{\varphi}_n \Delta_{n-1} - \dot{\varphi}_{n+1} \Delta_{n-2})(\tilde{H}),
\]  

(4.1.47)

50
where $\vec H$ is a vector consisting of the last $n-1$ components of $\vec H$, the first term and second term correspond to the two $GL(1)$ factors and the third term is in $SL(n)$. We have identified the coefficient of the first term as $\phi_n$ and the second term as a linear combination of $\phi_n$ and $\phi_{n+1}$ by examining the first component of the left and right hand sides of this vector equation when written in terms of the generator in Weyl generators $\vec H$. Thus the two $GL(1)$ factors correspond to $e^{-x\phi_n}\vec H_1$ and $e^{-y(\phi_{n+1} - \frac{\phi_n - 2\phi_{n-1}}{y}\phi_n)}(\vec H)_2$ in the $E_{n+1}$ group element and, from equations (4.1.43) and (4.1.44), to products of the type IIA volume $V_{n(A)}$ and ten dimensional type IIA string coupling $g_{s(A)}$. In taking the $V_{n(A)} \to \infty$ limit we must fix the $n-1$ quantities $\sum_{a=1}^{n-1} \phi_a \Delta_a - \phi_n \Delta_{n-1} - \phi_{n+1} \Delta_{n-2}$ to preserve the $SL(n)$ symmetry and $-\frac{3}{2} \phi_{10} + \phi_{11}$ to preserve the type IIA string coupling as given in equation (4.1.44).

The decomposition of the full $E_{n+1}$ algebra into representations of its $GL(1) \times GL(1) \times SL(n)$ subalgebra can be classified into a level [56-58]. The level is indexed by the number of times the simple root $\alpha_n$ and the simple root $\alpha_{n+1}$ occur in the corresponding root when decomposed in terms of simple roots. Clearly, the level zero part of the decomposition is just $GL(1) \times GL(1) \times SL(n)$, as is clear from figure 17. The decomposition of the representations of $E_{n+1}$ into representations of $GL(1) \times GL(1) \times SL(n)$ can similarly be classified according to the level, the level in this case is the number of times the simple root $\alpha_n$ and the simple root $\alpha_{n+1}$ occur in the root string constructed from the highest weight of the representation.

As we discussed above, the large volume limit of the type IIA torus $V_{n(A)} \to \infty$ corresponds to deleting nodes $n$ and $n+1$ in the $E_{n+1}$ Dynkin diagram. In this limit an $E_{n+1}$ automorphic form has a simultaneous expansion in powers of the volume of the type IIA volume $V_{n(A)}$ and the type IIA string coupling in ten dimensions $g_{s(A)}$ which are controlled by the $GL(1) \times GL(1)$ part of the $E_{n+1}$ group element. The coefficient functions in this expansion are automorphic forms of $SL(n)$ built from the representations of $SL(n)$ that occur in the decomposition of the representation of $E_{n+1}$, from which the original representation is built, into those of $SL(n)$. These latter $SL(n)$ automorphic forms can be labelled by the level, discussed above. We note that since the type IIA theory in ten dimensions can not depend on the moduli of the $n$ torus it is necessary for the automorphic forms found after taking the $V_{n(A)} \to \infty$ limit to be constructed from the trivial representation of $SL(n)$. Using equations (4.0.2), (4.1.43) and (4.1.44) we can write the state $|\varphi_{E_{n+1}}\rangle$ from which the automorphic form is built as

$$|\varphi_{E_{n+1}}\rangle = e^{-x\phi_n}(\vec H_1) e^{-y(\phi_{n+1} - \frac{\phi_n - 2\phi_{n-1}}{y} \phi_n)}(\vec H)_2 |\varphi_{SL(n)}^{(0,0)}\rangle + ...$$

$$= e^{-x\phi_n w_1} e^{-y(\phi_{n+1} - \frac{\phi_n - 2\phi_{n-1}}{y} \phi_n) w_2} |\varphi_{SL(n)}^{(0,0)}\rangle + ...$$

$$= V_{n(A)}^{\frac{x w_1}{y}} + \frac{\Delta_{n-2}^+}{2y} |\varphi_{SL(n)}^{(0,0)}\rangle + ... (4.1.48)$$

where $\varphi_{SL(n)}^{(0,0)}$ is the level $(0,0)$ contribution and so is built from the level $l_1 = l_2 = 0$ representation, with highest weight $w$ in the decomposition and $w_1$, $w_2$ are the first and
second components of $w$ respectively. In this equation $\ldots$ denotes the states formed from the higher level representations in the decomposition. Clearly the $V_{n(A)}$ dependence of the level $(l_1, l_2)$ contributions is given by

$$
-V_{n(A)}\left(\frac{xw_1-l_1x^2}{4} + \left(\frac{\Delta_n-2\Delta_n-1}{4y} - \frac{3}{y}\right)(w_2+l_1\frac{\Delta_n-2\Delta_n-1}{y} - l_2y)\right)
$$

while the $g_{s(A)}$ dependence of the level $(l_1, l_2)$ contributions is given by

$$
g_{s(A)}\left(\frac{xw_1-l_1x^2}{2} - \left(\frac{\Delta_n-2\Delta_n-1}{2y} + \frac{1}{y}\right)(w_2+l_1\frac{\Delta_n-2\Delta_n-1}{y} - l_2y)\right).
$$

Generic $E_{n+1}$ automorphic forms $\Phi$ constructed from $|\varphi_{E_{n+1}}\rangle$ are expected to be homogeneous functions which should satisfy the relation

$$
\Phi_{E_{n+1}}(a|\varphi_{E_{n+1}}\rangle) = a^c\Phi_{E_{n+1}}(|\varphi_{E_{n+1}}\rangle),
$$

where $a$ is a real number and $c$ is a scale factor that depends on the particular structure of the automorphic form. Using this homogeneity property of $E_{n+1}$ automorphic forms, and equation (4.1.48), one may write

$$
\Phi_{E_{n+1}}(|\varphi_{E_{n+1}}\rangle) = V_{n(A)}^{-\frac{xw_1}{4}}\left(\frac{\Delta_n-2\Delta_n-1}{4y} - \frac{3}{y}\right)\frac{cw_2}{g_{s(A)}}\left(\frac{\Delta_n-2\Delta_n-1}{2y} + \frac{1}{y}\right)\frac{cw_2}{g_{s(A)}}
$$

$$
\times\Phi_{E_{n+1}}(|SL(n)^{(0,0)}\rangle) + \ldots
$$

where $\ldots$ are terms that contain higher order contributions in $V_{n(A)}$ and $g_{s(A)}$ at levels $l_1 > 0$ or $l_2 > 0$.

We require that the terms remaining in the large volume limit of the type IIA torus match the known coefficient functions of the higher derivative terms in the type IIA effective action in ten dimensions. By dimensional analysis one sees that an arbitrary $d$ dimensional higher derivative term in Einstein frame takes the form

$$
l_d^{k-d}\int d^d x \sqrt{-g}\Phi_{E_{n+1}} O = l_{10}^{k-d} V_{n(A)}^{-\frac{k-d}{8}}\int d^d x \sqrt{-g}\Phi_{E_{n+1}} O,
$$

where $O$ is a $k$ derivative polynomial in the $d$ dimensional curvature $R$, Cartan forms $P$ or field strengths $F$. In the large volume limit of the type IIA torus one has

$$
\lim_{V_{n(A)} \to \infty} l_{10}^n \int d^d x \sqrt{-g} V_{n(A)}^{\frac{8-n}{4}} = \int d^{10} x \sqrt{-g},
$$

therefore any term with $V_{n(A)}$ dependence $V_{n(A)}^{\frac{8-n}{4}}$ is preserved in the limit while any term with a lesser power of $V_{n(A)}$ vanishes in the limit. Note that one must be careful when considering non-analytic terms in the action that appear divergent in the $V_{n(A)} \to \infty$ limit. Demanding that the large volume limit $V_{n(A)}$ of this generic higher derivative term exists
from a type IIA perspective means that the limit $V_{n(A)} \to \infty$ of equation (4.1.53) exists and that the resulting terms are ten dimensional type IIA higher derivative terms with coefficient functions that are $SL(n)$ automorphic forms that can only be constructed from the trivial representation of $SL(n)$ since the ten dimensional type IIA effective action can not depend on the moduli of the torus. Examining equation (4.1.53) and substituting the constructed from the trivial representation of $SL$ in $d$ automorphic form and ...

any non-vanishing term in the $V$ from a type IIA perspective means that the limit $V_{n(A)} \to \infty$ of equation (4.1.53) exists and that the resulting terms are ten dimensional type IIA higher derivative terms with coefficient functions that are $SL(n)$ automorphic forms that can only be constructed from the trivial representation of $SL(n)$ since the ten dimensional type IIA effective action can not depend on the moduli of the torus. Examining equation (4.1.53) and substituting the $E_{n+1}$ automorphic form as a function of the state $|\varphi_{E_{n+1}}\rangle$ in (4.1.52) one finds that in the $V_{n(A)} \to \infty$ limit

$$
\lim_{V_{n(A)} \to \infty} l_{10}^{k-d} V_{n(A)}^\frac{\Delta - d}{2} \int d^d x \sqrt{-g} (V_{n(A)} \Phi_{E_{n+1}}^{(0)}(|SL(n)\rangle) + \ldots) \mathcal{O}
$$

$$
\times g_s(A) = \lim_{V_{n(A)} \to \infty} l_{10}^{k-d} \int d^d x \sqrt{-g} V_{n(A)}^\frac{8-n}{2} \Phi_{E_{n+1}}^{(0)}(|SL(n)\rangle) + \ldots) \mathcal{O}
$$

where we have made use of equation (4.1.54) and denoted the different polynomials in the type IIA ten dimensional curvature $\tilde{R}$, field strengths $\tilde{F}$ and derivatives of the type IIA dilaton that arise in the decompactification of the $d$ dimensional polynomial in the curvature $R$, Cartan forms $P$ and field strengths $F$ by $\hat{O}$. The fields $\hat{O}$ are the ten dimensional type IIA analogues of the decompactified type IIB $\hat{O}$ fields discussed around equation (4.1.19). Therefore, as a function of the state $|\varphi_{E_{n+1}}\rangle$, the type IIA volume limit condition is given by

$$
\lim_{V_{n(A)} \to \infty} \frac{\Delta - d}{2} \frac{\Delta - d}{2} + \frac{\Delta - d}{2} \frac{\Delta - d}{2} - \frac{3}{2} y) w_2
$$

$$
\times g_s(A) - c(\frac{\Delta - d}{2} \frac{\Delta - d}{2} + \frac{3}{2} y) w_2 \Phi_{E_{n+1}}^{(0)}(|\varphi_{SL(n)}\rangle) + \ldots) \mathcal{O}
$$

$$
= a_0 g_s(A) - c(\frac{\Delta - d}{2} \frac{\Delta - d}{2} + \frac{3}{2} y) w_2 \mathcal{O} + \ldots
$$

where $a_0$ is a constant arising from the level zero contribution that depends on the $E_{n+1}$ automorphic form and ... denote contributions at higher levels. As we noted earlier, any non-vanishing term in the $V_{n(A)} \to \infty$ limit must have a coefficient function that is constructed from the trivial representation of $SL(n)$ so that the type IIA effective action in $d = 10$ dimensions does not depend on the moduli of the torus.
In addition, the perturbative terms remaining after taking the limit must agree with a perturbative expansion in the ten dimensional type IIA string coupling $g_s(A)$. In string frame this implies that each term has a $g_s(A)$ dependence of the form $g_s^{\Delta - 2 + 2g}$, where $g$ is the genus. String frame in ten dimensions is related to Einstein frame by $g_{E\mu\nu} = e^{-\frac{1}{2}w}g_{S\mu\nu}$. Upon rescaling to Einstein frame in the type IIA ten dimensional theory we find

$$
\int d^{10}x \sqrt{-g} g_s(A) O_S,
$$

where $O$ is some polynomial in the ten dimensional curvature $R$, fields strengths $F$ or derivatives of the type IIA dilaton, $S$ denotes string frame quantities and $\Delta$ is the number of ten dimensional type IIA space time metrics minus the number of inverse space time metrics.

Considering the $E_{n+1}$ automorphic form as a function of the state $|\varphi_{E_{n+1}}\rangle$ we see from equations (4.1.56) and (4.1.57) that for the terms remaining in the type IIA volume limit to agree with a perturbative expansion in $g_s(A)$ we require

$$
\frac{\Delta - 2}{g_s(A)} \left( \frac{c_x w_1}{g_s(A)} - c(\frac{\Delta - 2}{2g} + \frac{1}{2}w)w_2 \Phi_{E_{n+1}}(|\varphi_{SL(n)}^{(0,0)}\rangle) + \ldots \right) \hat{O} = g_s^{-2 + 2n_0} \hat{O} + \ldots
$$

where $n_0$ is a non-negative integer.

### 4.1.6. Large volume Limit of a $j$ dimensional subtorus

Type II string theory in $k < 10$ dimensions exhibits an $E_{11-k}(Z)$ symmetry. So in the large volume limit of a $j$ dimensional subtorus $V_j \to \infty$ an arbitrary higher derivative term in $d = 10 - n$ dimensions should give a sum of $d + j$ dimensional higher derivative terms whose coefficient functions are $E_{11-(d+j)}(Z)$ automorphic forms. As we found earlier, the volume of a $j$ dimensional subtorus $V_j$ may be expressed in terms of the $E_{n+1}$ Chevalley fields and is given by

$$
V_j = (2\pi)^j \frac{T_{d+1}T_{d+2}\ldots T_{d+j}}{(l_d)^j} = e^{\dot{\varphi}_{d+j}}.
$$

where we have made use of equations (2.2.19), (2.3.15) and (2.4.15). The dependence of the volume of the $j$ dimensional subtorus $V_j$ on the Chevalley field $\dot{\varphi}_{d+j}$ is independent of the perspective of the $d$ dimensional theory, i.e. whether we choose to express the $d$ dimensional theory in terms of the dimensionally reduced fields of type IIA/B string theory or M-theory. Relabelling the $E_{n+1}$ part of the Dynkin diagram as in section 4.1.1, node $d+j$ becomes node $j$ and so $\dot{\varphi}_{d+j}$ is now $\dot{\varphi}_j$. Examining equation (4.1.59), we find that the large volume limit of the $j$ dimensional subtorus $V_j \to \infty$ in $d$ dimensions is equivalent to $\dot{\varphi}_j \to \infty$. Taking $\dot{\varphi}_j \to \infty$ corresponds to deleting node $j$ in the $E_{n+1}$ Dynkin diagram, as shown in figure 18.
The algebra remaining after this deletion is the $GL(1) \times SL(j) \times E_{n+1-j}$ subalgebra of $E_{n+1}$. Let us denote the generator of the $GL(1)$ by $X$ which we may write as $X = \sum_{a=1}^{n+1} c_a H_a$. Demanding that it commute with $SL(j) \times E_{n+1-j}$ and in particular the Chevalley generators $E_a$, $a = 1, \ldots, j-1, j+1, \ldots, n+1$ implies that

$$X = \sum_{a=1}^{j} aH_a + \left(\frac{j-n+9}{j-n+8}\right) jH_{j+1} + \left(\frac{j-n+10}{j-n+8}\right) jH_{j+2} + \ldots + \left(\frac{6}{j-n+8}\right) jH_{n-2}$$

$$+ \left(\frac{4}{j-n+8}\right) jH_{n-1} + \left(\frac{2}{j-n+8}\right) jH_{n} + \left(\frac{3}{j-n+8}\right) jH_{n+1}.$$  \hfill (4.1.60)

Using the relation between the Chevalley $H_a$ and Weyl $\tilde{H}$ description of the generators in the Cartan subalgebra, given by $H_a = \tilde{\alpha}_a \cdot \tilde{H}$, and the decomposition of the $E_{n+1}$ roots given in appendix A, which we recall here with the appropriate labelling $\tilde{\alpha}_a = (0, \alpha_a)$, $a = 1, \ldots, j-1$, $\tilde{\alpha}_j = (x, -\lambda_{j-1}, \ldots, -\lambda_1)$, $\tilde{\alpha}_b = (0, 0, \hat{\alpha}_{k-j})$, $b = j+1, \ldots, n+1$ and $x^2 = \frac{(n+1)(8-n-j) - 9j}{j(n+1-j)(8-n+j)}$, we find that

$$X = x(1,0) \cdot \tilde{H} \equiv x(\tilde{H})_1,$$  \hfill (4.1.61)

up to an overall scale factor. In deriving this equation we have used that the simple roots $\alpha_i$ and fundamental weight $\lambda_{j-1}$ of $SL(j)$ obey the equation $\sum_{i=1}^{j-1} i\alpha_i - j\lambda_{j-1} = 0$ and the simple roots $\hat{\alpha}_i$ and fundamental weights $\hat{\lambda}_i$ of $E_{n+1-j}$ obey the equation $(j-n+9)\hat{\alpha}_1 + (j-n+10)\hat{\alpha}_2 + \ldots + 6\hat{\alpha}_{n-2-j} + 4\hat{\alpha}_{n-1-j} + 2\hat{\alpha}_{n-j} + 3\hat{\alpha}_{n+1-j} - j(j-n+8)\hat{\lambda}_{j-1} = 0$. As explained in appendix B, the group element that appears in the automorphic form, see equation (4.0.2), contains the expression $e^{-\sum_{a=1}^{n+1} \varphi_a H_a}$ which in terms of the $GL(1) \times SL(j) \times E(n+1-j)$ decomposition becomes

$$\exp(-\sum_{a=1}^{n+1} \varphi_a H_a) = \exp(-x\varphi_j (\tilde{H})_1) \exp\left(\sum_{i=1}^{j-1} \varphi_i \alpha_i - \varphi_j \lambda_{j-1}\right) (\tilde{H})$$  

$$\times \exp(-\sum_{a=j+1}^{n+1} \varphi_a \hat{\alpha}_{a-j} - \varphi_j \hat{\lambda}_1) (\tilde{H}).$$  \hfill (4.1.62)

where $\tilde{H}$ is a vector consisting of the $j-1$ components of the $SL(j)$ part of $\tilde{H}$ and $\bar{H}$ is a vector consisting of the $n+1-j$ components of the $E_{n+1-j}$ part of $\tilde{H}$, the first term is in $GL(1)$, the second term in $SL(j)$ and the third term is in $E_{n+1-j}$. We have identified the coefficient of the first term as $\varphi_j$ by examining the first component of the left and right hand sides of this vector equation when written in terms of the Weyl generators $\tilde{H}$. Thus the $GL(1)$ factor corresponds to the factor $e^{-x\varphi_j (\tilde{H})_1}$ in the $E_{n+1}$ group element and, from equation (4.1.59), to the powers of the $j$ dimensional subtorus volume $V_j$. In taking the
$V_j \to \infty$ limit we must fix the $j-1$ quantities $\sum_{i=1}^{j-1} \hat{\varphi}_i \hat{\alpha}_i - \hat{\varphi}_j \hat{\lambda}_j$ and $\sum_{a=j+1}^{n+1} \hat{\varphi}_a \hat{\alpha}_{a-j} - \hat{\varphi}_j \hat{\lambda}_j$ to preserve the $SL(j) \times E_{n+1-j}$ symmetry.

The decomposition of the full $E_{n+1}$ algebra into representations of its $GL(1) \times SL(j) \times E_{n+1-j}$ subalgebra can be classified into a level [56-58]. The level is just the number of times the simple root $\hat{\alpha}_j$ occurs in the corresponding root when decomposed in terms of simple roots. Clearly, the level zero part of the decomposition is just $GL(1) \times SL(j) \times E_{n+1-j}$, as is clear from figure 18. The decomposition of the representations of $E_{n+1}$ into representations of $GL(1) \times SL(j) \times E_{n+1-j}$ can similarly be classified according to the level, the level in this case is the number of times the simple root $\hat{\alpha}_j$ occurs in the root string constructed from the highest weight of the representation [59].

As we discussed above, the large volume limit of the $j$ dimensional subtorus $V_j \to \infty$ corresponds to deleting node $j$ in the $E_{n+1}$ Dynkin diagram. In this limit an $E_{n+1}$ automorphic form has an expansion in powers of the volume of the $j$ dimensional subtorus $V_j$, which is controlled by the $GL(1)$ factor. The coefficient functions in this expansion are automorphic forms of $SL(j) \times E_{n+1-j}$ built from the representations of $SL(j) \times E_{n+1-j}$ that occur in the decomposition of the representation of $E_{n+1}$, from which the original representation is built, into those of $SL(j) \times E_{n+1-j}$. These latter $SL(j) \times E_{n+1-j}$ automorphic forms can be labelled by the level, discussed above. We note that since the type II theory in $d+j$ dimensions can not depend on the moduli of the $j$ torus it is necessary for the automorphic forms found after taking the $V_j \to \infty$ limit to be constructed from the trivial representation of $SL(j)$. Using equation (4.1.62) and (4.0.2) we can write the state $|\varphi_{E_{n+1}}\rangle$ from which the automorphic form is built as

$$|\varphi_{E_{n+1}}\rangle = e^{-x\varphi_j(\vec{H})_1}|\varphi_{SL(j)\times E_{n+1-j}}^{(0)}\rangle + \ldots$$

$$= e^{-x\varphi_j w_1}|\varphi_{SL(j)\times E_{n+1-j}}^{(0)}\rangle + \ldots$$

$$= V_j^{-xw_1}|\varphi_{SL(j)\times E_{n+1-j}}^{(0)}\rangle + \ldots$$

(4.1.63)

where $\varphi_{SL(j)\times E_{n+1-j}}^{(0)}$ is the level zero contribution and so is built from the level zero representation, with highest weight $w$ in the decomposition and $w_1$ is the first component of $w$. In this equation $+\ldots$ denotes the states formed from the higher level representations in the decomposition. Clearly the $V_j$ dependence of the level $l$ contributions is given by $V_j^{-x(w_1-lx^2)}$.

Generic $E_{n+1}$ automorphic forms $\Phi$ constructed from $|\varphi_{E_{n+1}}\rangle$ are expected to be homogeneous functions which should satisfy the relation

$$\Phi_{E_{n+1}}(a|\varphi_{E_{n+1}}\rangle) = a^c \Phi_{E_{n+1}}(|\varphi_{E_{n+1}}\rangle),$$

(4.1.64)

where $a$ is a real number and $c$ is a scale factor that depends on the particular structure of the automorphic form. Using this homogeneity property of $E_{n+1}$ automorphic forms, and equation (4.1.63), one may write

$$\Phi_{E_{n+1}}(|\varphi_{E_{n+1}}\rangle) = V_j^{-cxw_1} \Phi_{E_{n+1}}(|\varphi_{SL(j)\times E_{n+1-j}}^{(0)}\rangle) + \ldots$$

(4.1.65)
where $+\ldots$ are terms that contain higher order contributions in $V_j$ at higher levels.

We require that the terms remaining in the large volume limit of the $j$ dimensional subtorus match the known coefficient functions of the higher derivative terms in the type II string effective action in $d + j$ dimensions. By dimensional analysis one sees that an arbitrary $d$ dimensional higher derivative term in Einstein frame takes the form

$$
t_d^{k-d} \int d^d x \sqrt{-g} \Phi_{E_{n+1}} \mathcal{O} = \int d^d x \sqrt{-g} \Phi_{E_{n+1}} \mathcal{O},
$$

(4.1.66)

where $\mathcal{O}$ is a $k$ derivative polynomial in the $d$ dimensional curvature $R$, Cartan forms $P$ or field strengths $F$. From iterating equation (2.1.11) we see that in the large volume limit of the $j$ dimensional subtorus one has

$$
\lim_{V_j \to \infty} l_{d+j}^j \int d^d x \sqrt{-g} V_j^{\frac{d-2}{d-2+j}} = \int d^d x \sqrt{-g},
$$

(4.1.67)

therefore any term with $V_j$ dependence $V_j^{\frac{d-2}{d-2+j}}$ is preserved in the limit while any term with a lesser power of $V_j$ vanishes in the limit. Note that one must be careful when considering non-analytic terms in the action that appear divergent in the $V_j \to \infty$ limit. Demanding that the large volume limit $V_j \to \infty$ of this generic higher derivative term exists from a string theory perspective means that the $V_j \to \infty$ limit of equation (4.1.66) exists and that the resulting terms are $d + j$ dimensional higher derivative terms with a coefficient function that is a sum of $SL(j) \times E_{n+1-j}$ automorphic forms. Examining equation (4.1.64) and substituting the $E_{n+1}$ automorphic form as a function of the state $|\varphi_{E_{n+1}}\rangle$ in (4.1.65) one finds that in the $V_j \to \infty$ limit

$$
\lim_{V_j \to \infty} l_d^{k-d} V_j^{\frac{d}{8+j-n}} \int d^d x \sqrt{-g} (V_j^{-\frac{d-2}{d-2+j}} \Phi_{E_{n+1}} (|\varphi_{SL(j) \times E_{n+1-j}}^{(0)}\rangle + \ldots) \mathcal{O})
$$

$$
= \lim_{V_j \to \infty} l_d^{k-d} V_j^{\frac{d}{8+j-n}} \int d^d x \sqrt{-g} (V_j^{-\frac{d-2}{d-2+j}} \Phi_{E_{n+1}} (|\varphi_{SL(j) \times E_{n+1-j}}^{(0)}\rangle + \ldots) \mathcal{O})
$$

$$
= l_d^{k-(d+j)} \int d^{d+j} x \sqrt{-g} \lim_{V_j \to \infty} \left( V_j^{\frac{d}{8+j-n}} (V_j^{-\frac{d-2}{d-2+j}} \Phi_{E_{n+1}} (|\varphi_{SL(j) \times E_{n+1-j}}^{(0)}\rangle + \ldots) \mathcal{O}) \right),
$$

(4.1.68)

where $\mathcal{O}$ denotes the different $d + j$ dimensional type II string theory polynomials in the $d + j$ dimensional curvature $R$, Cartan forms $P$ and field strengths $F$ that arise in the decompactification of the $d$ dimensional polynomial in the curvature $R$, Cartan forms $P$ and field strengths $F$. The fields $\mathcal{O}$ are the $d + j$ dimensional analogues of the decompactified type IIB $\mathcal{O}$ fields discussed around equation (4.1.19). Therefore as a function of the state $|\varphi_{E_{n+1}}\rangle$ the large volume limit of the $j$ dimensional subtorus condition is given by

$$
\lim_{V_j \to \infty} \left( V_j^{\frac{d}{8+j-n}} (V_j^{-\frac{d-2}{d-2+j}} \Phi_{E_{n+1}} (|\varphi_{SL(j) \times E_{n+1-j}}^{(0)}\rangle + \ldots) \mathcal{O}) \right) = a_0 \Phi_{E_{n+1-j}}^{\varphi_{SL(j) \times E_{n+1-j}}} \mathcal{O} + \ldots
$$

(4.1.69)
where \( a_0 \) is a constant that depends on the \( E_{n+1} \) automorphic form. Any higher derivative term in \( d \) dimensions that converges to a higher derivative term that is not compatible with type II string theory in \( d + j \) dimensions, must be rejected as a possible higher derivative term in \( d \) dimensions. As we noted earlier, any non-vanishing term in the \( V_j \to \infty \) limit must have a coefficient function that is constructed from the trivial representation of \( SL(j) \) so that the \( d + j \) dimensional theory does not depend on the moduli of the \( j \) dimensional subtorus.

### 4.2 The Eisenstein-like automorphic form constructed from the 5 of \( SL(5) \) with highest weight \( \tilde{\Lambda}_{n+1} \)

The automorphic form that appears as the coefficient function of the \( R^4 \) higher derivative terms in \( d = 7 \) dimensions is the unconstrained Eisenstein-like automorphic form constructed from the 5 of \( SL(5) \). Through taking the limits, discussed in the previous section, we will find conditions under which this automorphic form could exist as a coefficient function for a higher derivative term in the \( d = 7 \) dimensional effective action of type II string theory.

\[
\begin{align*}
\bullet & \quad 3 \\
\big| & \\
\bullet & \quad 2 \\
\big| & \\
\bullet & \quad - \quad \bullet \\
1 & \quad 4
\end{align*}
\]

Figure 19. The \( SL(5) \) Dynkin diagram

The representation of \( SL(5) \) with highest weight \( \tilde{\Lambda}^3 \) is the 5 of \( SL(5) \). The five weights in the root string of this representation are

\[
\tilde{\Lambda}^3_{0000}, \quad \tilde{\Lambda}^3_{0010}, \quad \tilde{\Lambda}^3_{0110}, \quad \tilde{\Lambda}^3_{1110}, \quad \tilde{\Lambda}^3_{1111},
\]

where we have adopted notation such that

\[
\tilde{\Lambda}^k_{a_1 a_2 a_3 a_4} = \tilde{\Lambda}^k - \sum_{i=1}^{4} a_i \tilde{\alpha}_i.
\]

The \( SL(5) \) lattice state \( |\psi\rangle \) transforming under the 5 of \( SL(5) \) may be written in terms of these weights as

\[
|\psi\rangle = m_1 |\tilde{\Lambda}^3_{0000}\rangle + m_2 |\tilde{\Lambda}^3_{0010}\rangle + m_3 |\tilde{\Lambda}^3_{0110}\rangle + m_4 |\tilde{\Lambda}^3_{1110}\rangle + m_5 |\tilde{\Lambda}^3_{1111}\rangle.
\]

where \( m_i \in \mathbb{Z}, i = 1, 2, ..., 5 \). The \( SL(5) \) group element is given by

\[
L(g^{-1}) = e^{\frac{1}{\sqrt{2}} \tilde{q} \tilde{h}} e^{-\sum_{\alpha > 0} \chi_{\alpha} E_{\alpha}},
\]

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where the \( SL(5) \) algebra is in Weyl basis. Comparing the Cartan subalgebra part of the group element \( L(g^{-1}) \) with the Cartan subalgebra part of the internal part of the \( E_{11} \) group element \( e^{-\vec{\phi} \cdot \vec{H}} \), where \( \vec{\phi} = (\tilde{\phi}_{d+1}, \tilde{\phi}_{d+2}, ..., \tilde{\phi}_{11}) \) and \( \tilde{\phi}_i \) are the \( E_{n+1} \) Cartan subalgebra fields in the \( E_{11} \) group element in Weyl basis, one finds

\[
\phi_i = -\sqrt{2} \tilde{\phi}_{d+i}.
\]

Therefore the \( SL(5) \) group element may be written

\[
L(g^{-1}) = e^{-\vec{\phi} \cdot \vec{H}} e^{-\sum_{\alpha > 0} \chi_{E_{\alpha}}}. \tag{4.2.5}
\]

The non-linearly realised lattice state \( |\phi\rangle \) is defined by

\[
|\phi\rangle = L(g^{-1}) |\psi\rangle = e^{-\vec{\phi} \cdot \vec{H}} e^{-\sum_{\alpha > 0} \chi_{E_{\alpha}} (m_1 |\vec{\Lambda}_{0000}\rangle + m_2 |\vec{\Lambda}_{0010}\rangle)} + e^{-\vec{\phi} \cdot \vec{H}} e^{-\sum_{\alpha > 0} \chi_{E_{\alpha}} (m_3 |\vec{\Lambda}_{0110}\rangle + m_4 |\vec{\Lambda}_{1110}\rangle + m_5 |\vec{\Lambda}_{1011}\rangle)}. \tag{4.2.6}
\]

The invariant unconstrained Eisentein-like automorphic form constructed from \( |\phi\rangle \) is defined by

\[
\Phi(|\phi\rangle) = \frac{1}{\Lambda} \sum_{\Lambda} u^s, \tag{4.2.7}
\]

where \( \Lambda \) is the lattice spanned by \( m_1, m_2, ..., m_5 \) and

\[
u = \langle \phi_{D}\phi |\phi\rangle. \tag{4.2.8}
\]

From the above equations we see that the scale factor associated with \( \Phi \) is \( c = -2s \).

**4.2.1. Perturbative Limit**

The perturbative limit is found by deleting node 3 in the \( SL(5) \) Dynkin diagram given in figure 19. Deleting node 3 decomposes \( SL(5) \) into a \( GL(1) \times SO(3,3) \) subalgebra. The simple roots and fundamental weights of \( SL(5) \) under this decomposition are given in appendix A.1 with \( n = 3 \). The highest weight \( w \) of the automorphic form in this case is \( w = \vec{\Lambda}_3 = (\frac{1}{x}, 0) \). From equation (4.1.9) with \( c = -2s \), \( d = 7 \), \( n = 3 \) one finds that the \( d = 7 \) dimensional perturbative limit condition is given by

\[
\lim_{g_d \to 0} \frac{4s^2 - 14}{5} \left( g_d^{\frac{8sxw_1}{5}} \Phi_{E_{n+1}} (|\vec{\phi}_{SO(n,n)}\rangle) + \ldots \right) = g_d^{-2+2n_0} \Phi_{SO(n,n)}^{(0)} + \ldots . \tag{4.2.9}
\]

where \( \Delta \) is the number of inverse spacetime metrics minus the number of spacetime metrics in the higher derivative term for which \( \Phi_{SL(5)} \) appears as the coefficient function and \( n_0 \) is a non-negative integer. Since \( w_1x = 1 \), this condition is equivalent to

\[
\frac{4s^2 - 14 - 8s^2}{5} = g_d^{-2+2n_0}. \tag{4.2.10}
\]
In other words, one finds that for Φ to appear as the coefficient function of an arbitrary higher derivative term in the $d = 7$ dimensional effective action one requires

$$n_0 = 4\Delta - 4 - 8s. \quad (4.2.11)$$

We note that the well known $R^4$ higher derivative term with $\Delta = 4$ has the coefficient function $Φ$ with $s = \frac{3}{2}$ [30,31] and contains a perturbative contribution at tree level $n_0 = 0$. Similarly, the $\partial^4 R^4$ higher derivative term with $\Delta = 6$ has the coefficient function $Φ$ with $s = \frac{5}{2}$ [30,31] and contains a perturbative contribution at tree level $n_0 = 0$.

### 4.2.2. Type IIB volume limit

The type IIB volume limit is found by deleting node 2 in the $SL(5)$ Dynkin diagram given in figure 19. Deleting node 2 decomposes $SL(5)$ into a $GL(1) \times SL(2) \times SL(3)$ subalgebra. The simple roots and fundamental weights of $SL(5)$ under this decomposition are given in appendix A.3 with $n = 3$. The highest weight $w$ of the automorphic form in this case is $w = \tilde{\Lambda}_3 = (\frac{1}{2x}, \mu, 0)$. From equation (4.1.20) with $c = -2s$, $d = 7$, $n = 3$ one finds that the large volume limit of the type IIB torus condition is given by

$$\lim_{V_{\alpha(B)} \to \infty} V_{\alpha(B)}^{2-k} \left( V_{\alpha(B)}^{\Sigma_1^2} \Phi_{E_{n+1}}(\varphi_{SL(2) \times SL(n)}) + \ldots \right) = a_0 \hat{Φ}^{(0)}_{SL(2)} + \ldots \quad (4.2.12)$$

Since the $SL(3)$ weight of the level zero contribution is 0 we find that the level zero part of the automorphic form does not depend on the moduli of the torus and therefore may be preserved in the type IIB volume limit. Since $w_1 x = \frac{1}{2}$, for the level zero part of $Φ$ to be preserved in the large volume limit of the type IIB torus we require

$$\frac{2 - k}{8} + \frac{s}{2} = 0. \quad (4.2.13)$$

The dependence of $s$ on the number derivatives $k$ of the higher derivative term agrees with the prediction of [28] as expected. The $R^4$ higher derivative term has $k = 8$ and coefficient function $Φ$ with $s = \frac{3}{2}$ [30,31]. Thus, in the large volume limit of the type IIB torus the level zero part of $Φ$ is preserved and is constructed from the representation of $SL(2)$ with highest weight $\mu$. Similarly, the $\partial^4 R^4$ higher derivative term has $k = 12$ and coefficient function $Φ$ with $s = \frac{5}{2}$ [30,31]. Again, in the large volume limit of the type IIB torus the level zero part of $Φ$ is preserved and is constructed from the representation of $SL(2)$ with highest weight $\mu$.

### 4.2.3. M-theory volume limit

The M-theory volume limit is found by deleting node 4 in the $SL(5)$ Dynkin diagram given in figure 19. Deleting node 4 decomposes $SL(5)$ into a $GL(1) \times SL(4)$ subalgebra. The simple roots and fundamental weights of $SL(5)$ under this decomposition are given in appendix A.2 with $n = 3$. The highest weight $w$ of the automorphic form in this case is $w = \tilde{\Lambda}_3 = (\frac{1}{2x}, \lambda_3, \lambda_3)$. From equation (4.1.42) with $c = -2s$, $d = 7$, $n = 3$ one finds that the large volume limit of the M-theory torus condition is given by

$$\lim_{V_{\alpha(M)} \to \infty} V_{\alpha(M)}^{2-k} \left( V_{\alpha(M)}^{\Sigma_1^2} \Phi_{E_{n+1}}(\varphi_{SL(n+1)}) + \ldots \right) = a_0 + \ldots, \quad (4.2.14)$$
where $a_0$ is a constant. Since the $SL(4)$ weight of the level zero contribution is $\lambda_3$, we observe that the level zero part of the automorphic form depends on the moduli of the torus and therefore cannot be preserved in the M-theory volume limit. Since $w_1 x = \lambda_3 \lambda_1 = \frac{1}{2}$, for the level zero part of $\Phi$ to be vanish in the large volume limit of the M-theory torus we require
\[
\frac{2 - k}{9} + \frac{s}{6} = 0. \tag{4.2.15}
\]
The $R^4$ higher derivative term has $k = 8$ and coefficient function $\Phi$ with $s = \frac{3}{2}$ [30,31]. Thus, in the large volume limit of the M-theory torus the level zero part of $\Phi$ vanishes as expected. Similarly, the $\partial^4 R^4$ higher derivative term has $k = 12$ and coefficient function $\Phi$ with $s = \frac{5}{2}$ [30,31]. Again, in the large volume limit of the M-theory torus the level zero part of $\Phi$ vanishes. Further analysis of the automorphic form $\Phi$ with $s = \frac{3}{2}$ demonstrates that in the $V_{m(M)} \to \infty$ limit the level one contribution of $\Phi$ provides an automorphic form constructed from the trivial representation of $SL(4)$. For $s = \frac{5}{2}$ this level one contribution converges to the coefficient of the eleven dimensional $R^4$ higher derivative term in the $V_{m(M)} \to \infty$ limit.

### 4.2.4. Type IIA volume limit

The type IIA volume limit is found by deleting nodes 3 and 4 in the $SL(5)$ Dynkin diagram given in figure 19. Deleting nodes 3 and 4 decomposes $SL(5)$ into a $GL(1) \times GL(1) \times SL(3)$ subalgebra. The simple roots and fundamental weights of $SL(5)$ under this decomposition are given in appendix A.5 with $n = 3$. The highest weight $w$ of the automorphic form in this case is $w = \vec{\Lambda}_3 = (\frac{1}{2}, 0, 0)$. From equation (4.1.56) with $c = -2s$, $d = 7$, $n = 3$ one finds that the large volume limit of the type IIA torus condition is given by
\[
\lim_{V_{n(A)} \to \infty} (V_{n(A)}^{\frac{2 - k}{9}} V_{n(A)}^{\frac{s}{6}} g_{s(A)}^{-s_{xw_1}} \Phi \Phi_{E_{n+1}} (|\varphi_{SL(n)}^{(0,0)}|)) + \ldots
\]
\[
= a_0 g_{s(A)}^{-s_{xw_1}} \Phi \Phi_{E_{n+1}} + \ldots \tag{4.2.16}
\]
where $a_0$ is a constant. Since the $SL(3)$ weight of the level zero contribution is 0 we observe that the level zero part of the automorphic form does not depend on the moduli of the torus and therefore may be preserved in the type IIA volume limit. Since $w_1 x = 1$, for the level zero part of $\Phi$ to be preserved in the large volume limit of the type IIA torus we require
\[
\frac{2 - k}{8} + \frac{s}{2} = 0. \tag{4.2.17}
\]

The dependence of $s$ on the number derivatives $k$ of the higher derivative term agrees with the prediction of [29] as expected. The $R^4$ higher derivative term has $k = 8$ and coefficient function $\Phi$ with $s = \frac{3}{2}$ [30,31]. Thus, in the large volume limit of the type IIA torus the level zero part of $\Phi$ is preserved and contains a type IIA string coupling factor $g_s^{-\frac{s}{2}}$. Similarly, the $\partial^4 R^4$ higher derivative term has $k = 12$ and coefficient function $\Phi$ with $s = \frac{5}{2}$ [30,31]. Again, in the large volume limit of the type IIB torus the level zero part of $\Phi$ is preserved and contains a type IIA string coupling factor $g_s^{-\frac{s}{2}}$.  

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From equation (4.1.58), the type IIA perturbative condition is
\[
g_{s(A)} \left( g_{s(A)} \right) \left( g_{s(A)} + \ldots \right) = g_{s(A)}^{-2+2n_0} + \ldots
\]
where \( \Delta \) is the number of inverse \( d = 10 \) spacetime metrics minus the number of \( d = 10 \) spacetime metrics in the decompactified type IIA higher derivative term and \( n_0 \) is a non-negative integer. Therefore if the level zero part of \( \Phi \) is preserved in the large volume limit of the type IIA torus, i.e. equation (4.2.16) holds, then for \( \Phi \) to appear as the coefficient function of an arbitrary higher derivative term in the \( d = 7 \) dimensional effective action one requires
\[
n_0 = \frac{\Delta - 1 - 2s}{4}.
\]
We note the that the well known \( R^4 \) higher derivative term in the \( d = 7 \) effective action has the coefficient function \( \Phi \) with \( s = \frac{3}{2} \) [30,31] gives rise to a \( d = 10 \) type IIA \( R^4 \) term with \( \Delta = 4 \) containing a perturbative contribution at tree level \( n_0 = 0 \). Similarly, the \( \partial^4 R^4 \) higher derivative term in the \( d = 7 \) effective action contains the coefficient function \( \Phi \) with \( s = \frac{5}{2} \) [30,31] gives rise to a \( d = 10 \) type IIA \( \partial^4 R^4 \) term with \( \Delta = 6 \) from a perturbative contribution at tree level \( n_0 = 0 \).

4.2.5. Decompactification of a single dimension limit

The decompactification of a single dimension limit is found by deleting node 1 in the \( SL(5) \) Dynkin diagram given in figure 19. Deleting node 1 decomposes \( SL(5) \) into a \( GL(1) \times SL(2) \times SL(3) \) subalgebra. The simple roots and fundamental weights of \( SL(5) \) under this decomposition are given in appendix A.4 with \( n = 3 \). The highest weight \( w \) of the automorphic form in this case is \( w = \tilde{\lambda}_3 = \left( \frac{\lambda_2 \lambda_1}{2}, \lambda_2 \right) \). From equation (4.1.31) with \( c = -2s, d = 7, n = 3 \) one finds that the \( d = 7 \) decompactification of a single dimension limit condition is given by
\[
\lim_{\frac{r_{d+1}}{l_{d+1}} \to \infty} \left( \frac{6(k-7)}{l_{d+1}} \right)^{\frac{2-k}{2}} \Phi_{E_{n+1}} \left( \left| \varphi_{E_n}^{(0)} \right| \right) + \ldots
\]
where \( a_0 \) is a constant. Since \( w_1 x = \lambda_2, \lambda_1 = \frac{1}{3} \), for the level zero part of \( \Phi \) to be preserved in the decompactification of a single dimension limit we require
\[
2 - k + 4s = 0.
\]
The \( R^4 \) higher derivative term has \( k = 8 \) and coefficient function \( \Phi \) with \( s = \frac{3}{2} \) [30,31]. Thus, in the decompactification of a single dimension limit the level zero part of \( \Phi \) is preserved and is constructed from the representation of \( E_3 = SL(2) \times SL(3) \) with highest weight \( \lambda_2 \) which is equivalent to the \( 3 \) of \( SL(3) \) with highest weight \( \lambda_2 \). Similarly, the \( \partial^4 R^4 \) higher derivative term has \( k = 12 \) and coefficient function \( \Phi \) with \( s = \frac{5}{2} \) [30,31]. Again, in the decompactification of a single dimension limit the level zero part of \( \Phi \) is
preserved and is constructed from the representation of $E_3 = SL(2) \times SL(3)$ with highest weight $\lambda_2$ which is equivalent to the 3 of $SL(3)$ with highest weight $\Lambda_2$.

**Conclusion**

The maximal string theory in $d = 11 - n$ dimensions has $n$ parameters. These parameters are in one to one relation with the volumes of the tori and subtori that arise in the dimensional reduction from eleven dimensions and from the viewpoint of the IIA and IIB theories also the expectation value of the dilaton field. The volumes of the tori, and subtori, are encoded in the vacuum expectation values of the diagonal components of the metric. Indeed, the parameters can be thought of as the expectation values of the scalars in the part of the scalar coset in the $d$ dimensional theory that belong to the Cartan subalgebra of $E_n$. In this paper we have found the precise relationship between these parameters and the just mentioned scalar fields. In doing so we have also found the correspondence between the nodes of the $E_n$ Dynkin diagram and the parameters. Thus the results in this paper provide a precise way of implementing the possible limits of the parameters in terms of the scalar fields that belong to the Cartan subalgebra.

As explained in the introduction there has been a recent interest in the higher derivative terms in the string effective action and the automorphic forms of $E_n$ that they contain. However, the considerations of these papers have, with two exceptions, been confined to terms in the effective action that have less than 14 space-time derivatives. However, if one knew all the automorphic forms that arise then one would know all possible string corrections. Thus it is desirable to develop some systematic understanding of the automorphic forms that do arise in the string effective action. One method of investigating if conjectured automorphic forms are acceptable is to study their limits as the parameters are varied. Particularly instructive has been the study of the limit of small string coupling as the result must be consistent with the form of perturbation theory that string theory predicts. In this paper we use the earlier results to investigate the behaviour of generic automorphic forms in the possible limits of the parameters. The consequences one might draw from these calculations are left to a future paper.

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**Appendix A: Decompositions of the $E_{n+1}$ algebra**

The $d$ dimensional parameters of the effective actions of type IIA/B string theory and M-theory compactified on an $n$ torus are associated with specific nodes of the $E_{n+1}$ Dynkin diagram. In taking particular limits in these parameters the resulting theory is expected to possess a symmetry given by the deletion of the node $E_{n+1}$ Dynkin diagram node relevant to that parameter.
A.1. Perturbative Limit

To investigate the properties of our automorphic form in the $g_d \to 0$ limit it will be expedient to decompose the $E_{n+1}$ algebra into a $GL(1) \times SO(n, n)$ subalgebra. To do this we delete node $n$ of the Dynkin diagram, the simple roots $\tilde{\alpha}$ of $E_{n+1}$ then decompose as

$$\tilde{\alpha}_i = (0, \tilde{\alpha}_i), \quad i = 1, ..., n - 1, \quad (A.1)$$

$$\tilde{\alpha}_n = \left( x, -\tilde{\lambda}_{n-1} \right), \quad (A.2)$$

$$\tilde{\alpha}_{n+1} = (0, \tilde{\alpha}_n), \quad (A.3)$$

where the tilde denotes $SO(n, n)$ simple roots and fundamental weights. The variable $x$ is fixed by the condition on the length of the simple roots, $\tilde{\alpha}_{n+1}^2 = 2 = x^2 + \tilde{\lambda}_{n-1}^2$, this leads to

$$x^2 = \frac{8 - n}{4}. \quad (A.4)$$

The corresponding fundamental weights are

$$\tilde{\Lambda}_i = \left( \frac{\tilde{\lambda}_i \tilde{\lambda}_{n-1}}{x}, \tilde{\lambda}_i \right), \quad i = 1, ..., n - 1, \quad (A.5)$$

$$\tilde{\Lambda}_n = \left( \frac{1}{x}, 0 \right), \quad (A.6)$$

$$\tilde{\Lambda}_{n+1} = \left( \frac{\tilde{\lambda}_n \tilde{\lambda}_{n-1}}{x}, \tilde{\lambda}_n \right). \quad (A.7)$$

A.2 M-theory Limit

To investigate the properties of our automorphic form in the $V_m(M) \to \infty$ limit we decompose the $E_{n+1}$ algebra into a $GL(1) \times SL(n+1)$ subalgebra. To do this we delete node $n + 1$ of the Dynkin diagram, the simple roots $\tilde{\alpha}$ of $E_{n+1}$ then decompose as

$$\tilde{\alpha}_i = (0, \tilde{\alpha}_i), \quad i = 1, ..., n, \quad (A.8)$$

$$\tilde{\alpha}_{n+1} = (x, -\underline{\lambda}_{n-2}), \quad (A.9)$$

where the underline denotes $SL(n+1)$ simple roots and fundamental weights. The variable $x$ is fixed by the condition on the length of the simple roots, $\tilde{\alpha}_{n+1}^2 = 2 = x^2 + \underline{\lambda}_{n-2}^2$, this leads to $x^2 = \frac{8 - n}{n + 1}$. The corresponding fundamental weights are

$$\tilde{\Lambda}_i = \left( \frac{\underline{\lambda}_i \underline{\lambda}_{n-2}}{x}, \underline{\lambda}_i \right), \quad i = 1, ..., n, \quad (A.10)$$

$$\tilde{\Lambda}_{n+1} = \left( \frac{1}{x}, 0 \right). \quad (A.11)$$
A.3 IIB Volume Limit

To investigate the properties of our automorphic form in the $V_{n(B)} \rightarrow \infty$ limit we decompose the $E_{n+1}$ algebra into a $GL(1) \times SL(2) \times SL(n)$ subalgebra. To do this we delete node $n-1$ of the Dynkin diagram, the simple roots $\tilde{\alpha}$ of $E_{n+1}$ then decompose as

\[ \tilde{\alpha}_i = (0, 0, \alpha_i), \quad i = 1, \ldots, n-2, \]  
\[ \tilde{\alpha}_{n-1} = (x, -\mu_1, -\Delta_{n-2}), \]  
\[ \tilde{\alpha}_n = (0, \beta_1, 0), \]  
\[ \tilde{\alpha}_{n+1} = (0, 0, \underline{\alpha}_{n-1}), \]

where the underline denotes $SL(n)$ simple roots and fundamental weights and $\mu_1, \beta_1$ are the fundamental weight and simple root of $SL(2)$ respectively. The variable $x$ is fixed by the condition on the length of the simple roots, $\tilde{\alpha}_n^2 = 2 = x^2 + \lambda_{n-2}^2 + \mu_1^2$, this leads to $x^2 = \frac{8-n}{2n}$. The corresponding fundamental weights are

\[ \tilde{\Lambda}^i = \left( \frac{\Delta_{n-2}}{x} \lambda_i, 0, \lambda_i \right), \quad i = 1, \ldots, n-2, \]  
\[ \tilde{\Lambda}^{n-1} = \left( \frac{1}{x}, 0 \right), \]  
\[ \tilde{\Lambda}^n = \left( \frac{1}{2x}, \mu_1, 0 \right), \]  
\[ \tilde{\Lambda}^{n+1} = \left( \frac{\lambda_{n-1} \lambda_{n-2}}{x}, 0, \lambda_{n-1} \right). \]

A.4 Decompactification of a Single Dimension Limit

To investigate the properties of our automorphic form in the $\frac{ r_{d+1}}{l_s} \rightarrow \infty$ limit we decompose the $E_{n+1}$ algebra into a $GL(1) \times E_n$ subalgebra. To do this we delete node 1 of the Dynkin diagram, the simple roots $\tilde{\alpha}$ of $E_{n+1}$ then decompose as

\[ \tilde{\alpha}_1 = \left( x, -\lambda_1 \right), \]  
\[ \tilde{\alpha}_i = (0, \hat{\alpha}_{i-1}), \quad i = 2, \ldots, n+1, \]

where the hat denotes $E_n$ simple roots and fundamental weights. The variable $x$ is fixed by the condition on the length of the simple roots, $\tilde{\alpha}_1^2 = 2 = x^2 + \hat{\lambda}_1^2$. The corresponding fundamental weights are

\[ \tilde{\Lambda}^1 = \left( \frac{1}{x}, 0 \right), \]
\[ \tilde{\Lambda}^i = \left( \frac{\lambda_{i-1} \hat{\lambda}_1}{x}, \hat{\lambda}_{i-1} \right), \quad i = 2, \ldots, n+1. \]
We now proceed to calculate the inner products of the $E_n$ fundamental weights. To do this we decompose the $E_n$ algebra into a $GL(1) \times SL(n)$ subalgebra by deleting node $n + 1$, one finds
\[ \hat{\alpha}_i = (0, \tilde{\alpha}_i), \quad i = 1, \ldots, n - 1, \] (A.24)
\[ \hat{\alpha}_n = (y, -\Delta_{n-3}), \] (A.25)
with fundamental weights
\[ \hat{\lambda}_i = \left( \frac{\lambda_i}{y}, \Delta_i \right), \quad i = 1, \ldots, n - 1, \] (A.26)
\[ \hat{\lambda}_n = \left( \frac{1}{y}, 0 \right). \] (A.27)

The variable $y$ is fixed by the condition $\hat{\alpha}^2_{n-2} = 2$, this gives $y^2 = \frac{9-n}{n}$. We then have
\[ \hat{\lambda}_1, \hat{\lambda}_1 = \left( \frac{3}{ny}, \Delta_i \right), \left( \frac{3}{ny}, \Delta_i \right) \]
\[ = \frac{9}{n^2y^2} + \frac{n-1}{n} \]
\[ = \frac{10 - n}{9 - n}, \] (A.28)
where we have made use of the expression $\lambda_i, \lambda_j = \frac{i(n-j)}{n}$ for $i \leq j$. We may now substitute this back into $\bar{\alpha}_1, \bar{\alpha}_1$ to fix the variable $x$,
\[ x^2 = 2 - \hat{\lambda}_1, \hat{\lambda}_1 \]
\[ = \frac{8 - n}{9 - n}. \] (A.29)

A.5 IIA Volume Limit

The decomposition of representations of $E_{n+1}$ into those of $GL(1) \times GL(1) \times SL(n)$ is given by deleting nodes $n$ and $n + 1$ of the Dynkin diagram appropriate to the type IIA theory. In this section we will find how the roots and weights of $E_{n+1}$ decompose in terms of those of $GL(1) \times GL(1) \times SL(n)$.

Let us carry out the decomposition by first deleting node $n$ to find the roots and fundamental weights of $D_n$ and then delete node $n + 1$ to find the algebra $SL(n)$. Using the methods given in reference [56], the simple roots of $E_{n+1}$ can be expressed as
\[ \tilde{\alpha}_i = (0, \tilde{\alpha}_i), \quad i = 1, \ldots, n - 1, n + 1 \quad \tilde{\alpha}_n = (x, -\tilde{\lambda}_{n-1}). \] (A.30)
Here $\tilde{\alpha}_i, i = 1, \ldots, n$ are the roots of $D_n$ and $\tilde{\lambda}_i$ are its fundamental weights which are given by

$$\tilde{\lambda}_i = \left( \frac{\tilde{\lambda}_i \cdot \tilde{\lambda}_{n-1}}{x}, \tilde{\lambda}_i \right), \quad i = 1, \ldots, n-1, n+1 \quad \tilde{\lambda}_n = \left( \frac{1}{x}, \hat{0} \right). \quad (A.31)$$

The variable $x$ is fixed by demanding that $\tilde{\alpha}_i^2 = 2 = x^2 + \tilde{\lambda}_{n-1}^2$.

We now delete node $n$ to find the $A_{n-1}$ algebra. The roots of $E_{n+1}$ are found from the above roots by substituting the corresponding decomposition of the $D_n$ roots and weights into those of $A_{n-1}$. The roots of $D_n$ in terms of those of $A_{n-1}$ are given by $\tilde{\alpha}_i = (0, \underline{\alpha}_j), \ i = 1, \ldots, n-1$ and $\tilde{\alpha}_n = (y, -\underline{\lambda}_{n-2})$ while the fundamental weights are given by $\tilde{\lambda}_i = \left( \frac{\lambda_{n-2} \lambda_i}{y}, \lambda_i \right), \ i = 1, \ldots, n-1$ and $\tilde{\lambda}_{n+1} = \left( \frac{1}{y}, \hat{0} \right)$. Requiring $\tilde{\alpha}_{n+1}^2 = 2$ gives $y^2 = \frac{4}{n}$. We then find that the roots of $E_{n+1}$ are given by

$$\tilde{\alpha}_i = (0, \underline{\alpha}_j), \quad i = 1, \ldots, n-1, \tilde{\alpha}_n = \left( x, -\frac{\lambda_{n-2} \cdot \lambda_{n-1}}{y}, -\underline{\lambda}_{n-1} \right), \tilde{\alpha}_{n+1} = (0, y, -\underline{\lambda}_{n-2}). \quad (A.32)$$

The fundamental weights of $E_{n+1}$ are found in the same way to be

$$\tilde{\lambda}_i = \left( \frac{c_i}{x}, \frac{\lambda_{n-2} \cdot \lambda_i}{y}, \underline{\lambda}_i \right), \quad i = 1, \ldots, n-1, \quad (A.33)$$

$$\tilde{\lambda}_n = \left( \frac{1}{x}, 0, \hat{0} \right), \quad (A.34)$$

$$\tilde{\lambda}_{n+1} = \left( \frac{n-2}{4x}, \frac{1}{y}, \hat{0} \right), \quad (A.35)$$

where $c_i = \frac{i}{2}, \ i = 1, \ldots, n-2$ and $c_{n-1} = \frac{n}{4}$. As $\tilde{\lambda}_{n-1}^2 = \frac{n}{4}$ we find that $x^2 = \frac{8-n}{4}$.

A.6 $j$ dimensional subtorus limit

To investigate the properties of our automorphic form in the $V_j \to \infty$ limit we decompose the $E_{n+1}$ algebra into a $GL(1) \times SL(j) \times E_{n+1-j}$ subalgebra. To do this we delete node $j$ of the Dynkin diagram, the simple roots $\tilde{\alpha}_i$ of $E_{n+1}$ then decompose as

$$\tilde{\alpha}_i = (0, \underline{\alpha}_j, \hat{0}), \quad i = 1, \ldots, j-1, \quad (A.36)$$

$$\tilde{\alpha}_j = \left( x, -\underline{\lambda}_{j-1}, -\tilde{\lambda}_1 \right), \quad (A.37)$$

$$\tilde{\alpha}_k = (0, \hat{\alpha}_{k-j}), \quad k = j+1, \ldots, n+1, \quad (A.38)$$

where the underline and the hat denote $SL(j)$ and $E_{n+1-j}$ quantities and $\alpha, \lambda$ are the respective simple root and fundamental weights of the corresponding algebra. The corresponding fundamental weights are

$$\tilde{\lambda}_i = \left( \frac{\hat{\alpha}_i \lambda_{j-1}}{x}, \lambda_i, \hat{0} \right), \quad i = 1, \ldots, j-1, \quad (A.39)$$
\[ \vec{X}^j = \left( \frac{1}{x}, 0, \hat{\lambda}_0 \right), \]  
\[ \vec{X}^k = \left( \frac{\hat{\lambda}_{k-j} \hat{\lambda}_1}{x}, 0, \hat{\lambda}_{k-j} \right), \quad k = j + 1, \ldots, n + 1. \] 

The variable \( x \) is fixed by the condition on the length of the simple roots, \( \vec{\alpha}_j^2 = 2 = x^2 + \hat{\lambda}_1 \hat{\lambda}_1 + \Delta_j - 1 + \Delta_{j-1} \). After some work one finds
\[ x^2 = \frac{(n+1)(8-n+j) - 9j}{j(n+1-j)(8-n+j)}. \]  

**Appendix B: Automorphic forms and non-linearly realised lattice states**

Generic higher derivative corrections in the effective action of Type II string theory, compactified on an \( n \)-torus to \( d = 10-n \) dimensions, are polynomials in the curvature \( R \), Cartan forms \( P \) and degree \( k \) field strengths \( F_k \) multiplied by an automorphic form \( \Phi_{E_{n+1}} \) transforming under the \( E_{n+1} \) U-duality group. One may construct an \( E_{n+1} \) automorphic form \( \Phi_{E_{n+1}} \) from the function \( |\varphi_{E_{n+1}}\rangle \), which is defined by,
\[ |\varphi_{E_{n+1}}\rangle = L(g^{-1})|\psi\rangle \]  

where \( L(g^{-1}) \) is a representation of the coset element \( g \in E_{n+1}/K \), \( K \) being the maximal compact sub-group of \( E_{n+1} \) and \( |\psi\rangle \) is a linear representation of \( E_{n+1}(Z) \). Using the Iwasawa decomposition and fixing the local group element \( h \in K \) to be the identity, we may write
\[ L(g^{-1}) = e^{-\sum_{\alpha>0} \chi_{\alpha} E_{\alpha}} \]  

where \( H \) are the generators in the Cartan sub-algebra of \( E_{n+1} \), in Weyl basis, and \( E_{\alpha} \) are the positive root generators, while \( \chi_{\alpha} \) are the axions and \( \phi \) is a vector whose components are linear combinations of the physical fields of type IIA/B string theory or M-theory, namely, the type IIA/B dilaton, the \( n \)-torus volume modulus \( \rho \) and the remaining \( n \) or \( n-1 \) moduli \( \phi \). Instead of writing the coset element \( g \in E_{n+1}/K \) in terms of the type IIA, type IIB or M-theory physical fields we may write it as a function of the \( E_{n+1} \) Chevalley fields \( \hat{\varphi}_i \), \( i = d+1, \ldots, 11 \) parameterising the \( E_{n+1} \) symmetry. The fields in the \( E_{n+1} \) part of the \( E_{11} \) group element \( \hat{\varphi}_i \) are equal to those in the physical field parameterisation of the group element \( L(g^{-1}) \) used to construct the automorphic form, up to a numerical factor. We find, through comparing the normalisations of the fields associated with the Cartan subalgebra in the \( E_{n+1} \) part of the \( E_{11} \) group element \( e^{\vec{\varphi} \cdot \vec{H}} \) and those in the automorphic form group element \( e^{-\frac{1}{2} \vec{\varphi} \cdot \vec{H}} \) that \( \phi_i = -\sqrt{2} \vec{\varphi}_i \), where \( \vec{\varphi}_i \) are the \( E_{n+1} \) fields in Weyl basis. So the coset element \( g \in E_{n+1}/K \) as a function of the \( E_{n+1} \) fields \( \hat{\varphi}_i \) is
\[ L(g^{-1}) = e^{-\vec{\varphi} \cdot \vec{H}} e^{-\sum_{\alpha>0} \chi_{\alpha} E_{\alpha}}, \]
where $\vec{\varphi} = (\varphi_1, \varphi_2, ..., \varphi_{n+1})$. In Weyl basis, where the commutator of the Cartan subalgebra elements $H_i$ with the positive root generators is $[H_i, E_\alpha] = \tilde{\alpha}_i E_\alpha$, the action of the Cartan subalgebra $\tilde{H}$ on $|\psi_{\Lambda_k}\rangle$ is

$$|\varphi_{\Lambda_k}\rangle = e^{-\vec{\varphi}.\tilde{H}}|\psi_{\Lambda_k}\rangle$$

where $[\Lambda_k]$ is the set of weights in the representation of $E_{n+1}$ with highest weight $\Lambda_k$. The Weyl basis of $E_{n+1}$ fields $\tilde{\varphi}_i$ are related to the $E_{n+1}$ Chevalley basis fields $\varphi_a$ by $\tilde{\varphi}_i = \varphi_a \alpha_i^a$, where $\alpha_i^a$ is the $i$'th component of the $a$'th simple root. We will denote the automorphic form that is a function of the above non-linearly realised $\Lambda_k$ representation of $E_{n+1}$ by $\Phi_{E_{n+1}}(|\varphi_{\Lambda_k}\rangle)$.

The large volume limit of the type IIA/B and M-theory torus, along with the perturbative limit are associated with a single node of the $E_{n+1}$ Dynkin diagram. To evaluate an $E_{n+1}$ automorphic form in these limits it is expedient to delete the relevant node giving a decomposition of the $E_{n+1}$ algebra in terms of a $GL(1)$ factor, that corresponds to the parameter of interest, and a rank $n$ subalgebra. Splitting the set of positive roots $\tilde{\alpha} > 0$ into the set of positive roots $\tilde{\alpha}^* > 0$ which contain the simple root $\tilde{\alpha}_l$, where $l$ is the deleted node, and the remaining roots $\tilde{\alpha}$ that do not contain $\tilde{\alpha}_l$, we have

$$e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha} = e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha} e^{-\sum_{\alpha^* > 0} \chi_{\alpha^*} \tilde{E}_{\alpha^*}}. \quad (B.4)$$

By the Baker-Campbell-Hausdorff lemma we may write this as

$$e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha} = e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha - \sum_{\alpha^* > 0} \chi_{\alpha^*} \tilde{E}_{\alpha^*}} e^{-\frac{1}{2} \left[ \sum_{\alpha > 0} \chi_\alpha E_\alpha, \sum_{\alpha^* > 0} \chi_{\alpha^*} \tilde{E}_{\alpha^*} \right]} \cdots, \quad (B.6)$$

where ... denotes the higher order commutators of $\sum_{\alpha > 0} \chi_\alpha E_\alpha$ and $\sum_{\alpha^* > 0} \chi_{\alpha^*} \tilde{E}_{\alpha^*}$. The action of $e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha}$ on a set of states $|\mu_i\rangle$ of weight $\mu_i$ can be expressed as

$$e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha} |\mu_i\rangle = e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha} e^{-\sum_{\alpha^* > 0} \chi_{\alpha^*} \tilde{E}_{\alpha^*}} e^{-\frac{1}{2} \left[ \sum_{\alpha > 0} \chi_\alpha E_\alpha, \sum_{\alpha^* > 0} \chi_{\alpha^*} \tilde{E}_{\alpha^*} \right]} \cdots |\mu_i\rangle$$

$$= e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha} |\mu_i\rangle - e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha} \sum_j \tilde{\chi}_{ij} |\mu_j\rangle \quad (B.7)$$

where $\tilde{\chi}_{ij}$ is a polynomial in the fields $\chi_{\tilde{\alpha}^*}$, $\chi_{\alpha^*}$. Note that $|\mu_j\rangle$ is at a lower level in $\tilde{\alpha}_l$ than $|\mu_i\rangle$.

Defining $|\psi_{A}^{p,j}\rangle$ to be the state carrying the linear representation $j$ of the subalgebra at level $p$. The non-linearly realised lattice state may be written

$$|\varphi_{\Lambda_k}\rangle = e^{-x \varphi_{d+1}(\tilde{H})} e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha}$$
\[ \times e^{-\sum_{\alpha^* > 0} \chi_{\alpha^*} E_{\alpha^*}} e^{-\frac{1}{2} \left[ \sum_{\alpha > 0} \chi_{\alpha} E_{\alpha} \sum_{\alpha^* > 0} \chi_{\alpha^*} E_{\alpha^*} \right]} \ldots \sum_{p,j} \left| \psi^{p,j}_A \right\rangle \]

\[ = e^{-x \dot{\varphi}_{d+1}(\bar{H})_{1} - \dot{\varphi}_{d+1} H} e^{-\sum_{\alpha > 0} \chi_{\alpha} E_{\alpha}} \sum_{p,j} \left| \tilde{\psi}^{p,j}_A \right\rangle \]

\[ = e^{-x \dot{\varphi}_{d+1}(\Lambda_k)_{1}} \sum_{p,j} e^{\pi x^2 \dot{\varphi}_{d+1}} \left| \tilde{\varphi}^{p,j}_A \right\rangle, \] (B.8)

where \( \left| \tilde{\psi}^{p,j}_A \right\rangle \) and \( \left| \tilde{\varphi}^{p,j}_A \right\rangle \) are the shifted, by the action of all positive root generators associated with the deleted node, linearly realised and non-linearly realised lattice states respectively.

One may then use the homogeneity property of a generic \( E_{n+1} \) automorphic form to write

\[ \Phi_{E_{n+1}} \left( \left| \varphi_{\Lambda_k} \right\rangle \right) = e^{-cx \dot{\varphi}_{d+1}(\Lambda_k)_{1}} \Phi_{E_{n+1}} \left( \sum_{p,j} e^{\pi x^2 \dot{\varphi}_{d+1}} \left| \tilde{\varphi}^{p,j}_A \right\rangle \right) \] (B.9)

where \( c \) is a constant that depends on the structure of the particular automorphic form under consideration.

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