Hodge structures in topological quantum mechanics

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Abstract. We parametrize some quantum numbers in two-dimensional quantum mechanics by means of a Hodge structure. For an anyon system the statistical parameter leads to a deformation of the Hodge structure.

1. Introduction
In this paper we consider applications of Hodge structures to quantum mechanics. We consider some function spaces and show that they admit Hodge structures or their deformations. Especially the space of harmonic functions in quantum mechanics admits the Hodge structure; azimuthal and magnetic quantum numbers are weights of the Hodge structure. In the three dimensional quantum mechanics the Hodge structure is almost algebraic.

We consider in the paper theory of anyons as a special case of topological quantum mechanics (it means that wave functions are not univalent). In the case of anyons on the plane, the Hodge structure is topologically deformed.

The paper is organized as follows. In section 2 we give a notion of a Hodge structure. In section 3 we consider some examples of Hodge structures in polynomial and harmonic polynomial spaces. In section 4 there are given forms of wave function for anyons with fractional statistical parameter on a plane. In the section 5 we consider anyonic harmonic functions and we show this space admits the topologically deformed Hodge structure. In section 6 we give some concluding remarks.

2. Preliminaria
Let $C^* = C \setminus \{0\}$ be the multiplicative group of complex numbers not equals zero. The general form of irreducible representations of the group is given by

\[ \gamma_{l,m} : C^* \rightarrow C^*, \ \gamma(re^{i\phi}) = r^le^{im\phi}, \ l \in \mathbb{R}, m \in \mathbb{Z}. \]  \hspace{1cm} (1)

We define weights of the representation $\gamma_{l,m}$ as follows:

\[ w_r = l \text{ the radial weight} \] \hspace{1cm} (2)

\[ w_a = m \text{ the angular weight}. \] \hspace{1cm} (3)
A complex one-dimensional representation of the form

\[ \Gamma_{p,q}(\zeta) = \zeta^p \zeta^q, \quad p, q \in \mathbb{Z} \]  

will be called an irreducible algebraic representation. Let us mention that in the case \( q = 0 \) the representation \( \Gamma_{p,0} \) is an algebraic group representation.

The weights of the irreducible algebraic representation \( \Gamma_{p,q} \) have values:

\[ w_r = p + q \quad \text{the radial weight,} \quad w_a = p - q \quad \text{the angular weight.} \]  

Observe that the difference of these weights is an even number:

\[ w_r - w_a = 2q. \]  

The irreducible representation \( \gamma_{l,m} \) is an algebraic one iff \( w_r = l \in \mathbb{Z} \) and \( w_r - w_a = l - m \in 2\mathbb{Z} \).

If the weights \( w_r \) and \( w_a \) are integral numbers then the representation \( \gamma_{l,m} \) is called almost algebraic one.

**Example 1.** The representation \( \gamma \) given by \( \gamma(\zeta) = |\zeta| \) is an almost algebraic one (the radial weight equals 1 and angular equals 0). The square \( \gamma^2 \) of the representation \( \gamma \) is an algebraic representation:

\[ \gamma^2(\zeta) = |\zeta|^2 = \zeta \bar{\zeta}. \]  

Now let us define the notion of the Hodge structure. For an excellent survey of Hodge structures see [1].

**Definition of Hodge structure.** Let \( V \) be a real vector space. A Hodge structure is a representation \( \Gamma \) of the group \( \mathbb{C}^* \) in \( \text{GL}(V) \) which splits into one dimensional complex irreducible algebraic representations of the form

\[ \Gamma = \bigoplus_{pq} m_{p,q} \Gamma_{p,q}, \]  

where

\[ m_{p,q} = m_{q,p}. \]  

Hodge structure is strictly connected with the Hodge decomposition

\[ V = \bigoplus V_{p,q}, \quad \dim V_{p,q} = m_{p,q}, \]  

where \( V_{p,q} \) are called Hodge subspaces.

If decomposition

\[ \Gamma = \bigoplus n_{l,m} \gamma_{l,m} \]

has the property \( n_{l,m} = n_{l,-m} \) then we say that this representation is almost algebraic Hodge structure. Complexification of Hodge (almost algebraic Hodge resp.) structure we also call Hodge (almost algebraic Hodge resp.) structure.
3. Examples of Hodge structures

Example 2. Polynomials of three variables.

The space $K_{CR}^C(x, y, z)$ of complex polynomials of three real variables $(x, y, z)$. Let us make the following substitution of variables:

$$P(u, \bar{u}, z) = Q(x, y, z), \text{ where } u := x + iy.$$ 

So any polynomial can be presented in the form:

$$P(u, \bar{u}, z) = \sum P_{n,m} u^n \bar{u}^m z^k,$$

where coefficients are uniquely determined by the polynomial $Q$ (or equivalently by $P$). The representation $\Gamma$ defined in the following way

$$\Gamma(\zeta)P = P_\zeta, \text{ where } P_\zeta(u, \bar{u}, |\zeta| z)$$

(12)

is not algebraic, but is almost algebraic:

$$\Gamma = \bigoplus_{l \in \mathbb{N}_0} \bigoplus_{m=-l}^{m} n_{l,m} \gamma_{l,m}.$$ 

(13)

Monomials of the form

$$\phi_{n,m,k}(u, \bar{u}, z) = u^n \bar{u}^m z^k, \text{ } n, m, k \in \mathbb{N}_0$$

constitute a basis in $K_{CR}^C(x, y, z)$. Weights of the basis monomials

$$w_r = n + m + k, \text{ } w_a = n - m$$

are integers so the representation $\Gamma$ in the polynomial space is an almost algebraic Hodge structure. Observe that the radial weight is the homogeneous degree of a homogenous polynomial.

Example 3. Spherical harmonics in $R^3$.

The space $H_{CR}^C(x, y, z)$ of harmonic polynomials is a subspace of $K_{CR}^C(x, y, z)$ and consists of polynomials which satisfy the Laplace equation

$$\Delta P = 0.$$ 

(14)

A basis of this space is a family of spherical harmonics $Y_{m}^{l}$, $l \in \mathbb{N}_0, m = -l, -l+1, \ldots, l$ given by the following formula [2]:

$$Y_{m}^{l} = \left[\frac{2l+1}{4\pi}(l+m)!(l-m)!\right]^\frac{1}{2} \sum_{k} \frac{(-u)^{m+k} \bar{u}^k z^{l-m-2k}}{2^{m+2k} (m+k)! k!(l-m-2k)!},$$

(14)

where $u = x + iy$ is the complex variable and $z$ is the real coordinate. The summation is over integer numbers $k$ such that $k + m, k, l - m - 2k$ are nonnegative. The space of harmonic polynomials is equivariant with respect to the representation $\Gamma$ from the previous example. The restriction $\Gamma_{H}$ has a decomposition

$$\Gamma_{H} = \bigoplus_{l \in \mathbb{N}_0} \bigoplus_{m=-l}^{m} \gamma_{l,m}.$$ 

(15)
Hence the space of spherical harmonics admits almost algebraic Hodge structure. The weights of the harmonic $Y_{lm}^n$ are $w_r = l$ and $w_m = m$. Physical interpretation of the weights of spherical harmonics in quantum mechanics: $l$- azimuthal quantum number, $m$- magnetic quantum number.

**Example 4. Polynomials of even number of variables.**

Let $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$ for $x_k, y_k \in \mathbb{R}$, and $z = (z_1, \ldots, z_N)$, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_N)$ for $z_k \in \mathbb{C}$, where $k = 1, \ldots, N$. The space $K^C(x, y)$ of complex polynomials of $2N$ real variables.

Let us make the following substitution of variables: $z_k = x_k + iy_k$. So any polynomial of variables $(x, y)$ can be expressed as a polynomial of the variables $(z, \bar{z})$. The algebraic representation

$$
\Gamma_{2N} : \mathbb{C}^* \longrightarrow GL_C(K^C(x, y))
$$

is given in a natural way: $\Gamma_{2N}(\zeta)P = P(\zeta z, \bar{\zeta} z)$,

$$
P_\zeta(z, \bar{\zeta}z) = P(z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N), \quad (16)
$$

where $\zeta = (\zeta_1, \ldots, \zeta_N)$ The representation $\Gamma$ has a Hodge decomposition:

$$
\Gamma_{2N} = \bigoplus_{p, q \in \mathbb{N}_0} m_{pq}\Gamma_{p,q}. \quad (17)
$$

The irreducible subspaces are one dimensional vector spaces over $\mathbb{C}$ and the family of complex monomials of the form

$$
\phi_{n_1,\ldots,n_N,m_1,\ldots,m_N}(z, \bar{z}) = z_1^{n_1} \ldots z_N^{n_N} \bar{z}_1^{m_1} \ldots \bar{z}_N^{m_N},
$$

where $p = \sum_{i=1}^N n_i$, $q = \sum_{j=1}^N m_j$ gives a basis consistent with this decomposition. In this case the partial weight of a homogeneous polynomial also has the meaning of the homogeneity degree.

**Example 5. Harmonic polynomials of even numbers of variables.**

Consider the space $H^C(x, y)$ of complex harmonic polynomials of $2N$ real variables. This space is equivariant with respect to the representation $\Gamma_{2N}$. For $n = 1$ the angular weights for harmonic polynomial are $w_a = \pm w_r$, and for $N > 1$

$$
w_a = -w_r, -w_r + 2, \ldots, w_r - 2, w_r
$$

The space is a model of the space of harmonics for a quantum system of $n$ identical but distinguishable softcore particles. The weights for a harmonic can be treated as quantum numbers: the angular weight $w_a$ is the plane angular momentum $m$ and radial weight $w_r$ is an analog of azimuthal quantum number.

Let us introduce the following operators:

$$
\hat{n}_h = \sum z_k \frac{\partial}{\partial z_k}, \quad \hat{n}_a = \sum \bar{z}_k \frac{\partial}{\partial \bar{z}_k}.
$$

Hodge subspaces are eigenspaces of these operators and the eigenvalues are indices of irreducible algebraic representations.

Operators corresponding to quantum numbers $l, m$ are

$$
\hat{l} = \hat{n}_h + \hat{n}_a, \quad \hat{m} = \hat{n}_h - \hat{n}_a.
$$

The number $l$ is the homogeneity degree of a harmonic and $m$ is the planar angular momentum.
4. Wave functions for anyons on a plane

Anyons are identical indistinguishable hardcore particles. Wave functions for the system with
the statistical parameter \( \nu \in (0, 1) \cup (1, 2) \) can be considered as not univalent functions on \( \mathbb{C}^N \)
of the form [3]-[6]:

\[
\prod_{k<l} (z_l - z_k)^{\nu} F(z, \bar{z}) \quad (18)
\]

or

\[
\prod_{k<l} (\bar{z}_l - \bar{z}_k)^{2-\nu} F(z, \bar{z}), \quad (19)
\]

where \( F \) is a symmetric function on \( \mathbb{C}^N \). In the following we assume that \( F \) is a power series of
\((z, \bar{z})\) variables.

The fractional powers of the Vandermonde determinant

\[
\prod_{k<l} (z_l - z_k)^{\nu}, \quad \prod_{k<l} (\bar{z}_l - \bar{z}_k)^{2-\nu}
\]

are the topological factors and they are no univalent factors. The theory of anyon is an example
of topological quantum mechanics, because the wave function are no univalent.

Now let us decompose the space \( \mathbb{C}^N \) on two mutually orthogonal spaces: the space of mass
centre motion and the relative position space. The complex orthonormal mass centre coordinate is

\[
u_0 = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} z_k. \quad (20)
\]

We deal only with the relative motion space, because the topological factors depend only on
relative positions. Let

\[ u = (u_1, \ldots, u_M), \text{ where } M = N - 1 \quad (21)\]

be an orthonormal system of relative coordinates. There are such a types of relative parts of
anyon wave functions

\[
\Psi_I = V^\nu(u) F_I(u, \bar{\nu}) \text{ and } \Psi_{II} = V^{2-\nu}(\bar{\nu}) F_{II}(u, \bar{\nu}), \quad (22)
\]

where \( F_I(u, \bar{\nu}), F_{II}(u, \bar{\nu}) \) are univalent, symmetric functions in relation to transpositions of
particles, it means that the composed function \( F(u(z), \bar{\nu}(z^*)) \) is symmetric or antisymmetric
in relation to \( z \)-coordinates, and \( V(u) \) (\( V(\bar{\nu}) \) resp.) is the Vandermonde determinant of
holomorphic positions \( z \) (antiholomorphic \( z^* \) resp.).

5. Anyonic harmonics

The very important problem in the theory of anyons is to determine anyonic harmonics [7].

General forms of anyonic harmonics in the relative motion space are given by the following
equations:

\[
H_I = P(\bar{\nu}_k \frac{\partial}{\partial u_l} - \nu_l \frac{\partial}{\partial u_k}) V^{n+\nu}(u) f(u), \quad (23)
\]

\[
H_{II} = P(u_k \frac{\partial}{\partial u_l} - u_l \frac{\partial}{\partial u_k}) V^{n+2-\nu}(\bar{\nu}) f(\bar{\nu}), \quad (24)
\]

where \( P \) is a symmetric (antisymmetric resp.) polynomial for \( n \) even (odd resp.), \( f \) is a symmetric
holomorphic polynomial.
Let us denote by $l_I$ and $l_{II}$ homogeneity degrees of functions $H_I$ and $H_{II}$. The admissible values of planar angular momentum $m_I$ and $m_{II}$ are [7]:

$$m_I = l_I - 2(n_I)_{\text{max}}, \quad l_I - 2(n_I)_{\text{max}} + 2, \ldots, l_I - 2, \quad l_I,$$  \hspace{1cm} (25)

and

$$m_{II} = -(l_{II} - 2(n_{II})_{\text{max}}), \quad -(l_I - 2(n_{II})_{\text{max}} + 2), \ldots, -(l_{II} - 2), \quad -l_{II},$$  \hspace{1cm} (26)

where

$$(n_I)_{\text{max}} = \left\lfloor \frac{l_I}{2} - \nu \right\rfloor, \quad \text{and} \quad (n_{II})_{\text{max}} = \left\lfloor \frac{l_{II}}{2} - (2 - \nu) \right\rfloor.$$  \hspace{1cm} (27)

and the square bracket denotes the integer part.

For $P$ and $f$ homogeneous the harmonics are bihomogeneous ones. The total homogeneity

$$l = l_f + \left( \frac{M}{2} \right) \mu.$$

The total plane angular momentum

$$m = \pm (l - 2l_P).$$

In general quantum numbers $l, m$ may not be integral. Hence representation corresponding to these numbers is the representation of the group $\tilde{C}^*$ which is the universal cover of $C^*$:

$$\Gamma(r, \phi) H(u, \pi) = H(re^{i\phi}u, re^{i\phi}u).$$  \hspace{1cm} (28)

The universal cover of $C^*$ is the group $\tilde{C}^* := \mathbb{R}_+ \times \mathbb{R}$ with the multiplicative law

$$(r_1, \phi_1)(r_2, \phi_2) := (r_1r_2, \phi_1 + \phi_2).$$

The representation $\Gamma$ splits into the representations of the type $\Gamma_{\alpha,q}$ and $\Gamma_{p,\beta}$ with $\alpha = \left( \frac{N}{2} \right) \nu + n, \beta = \left( \frac{N}{2} \right) (2 - \nu) + n, \quad p, q \in \mathbb{N}$ given by

$$\Gamma_{\alpha,q}(r, \phi) = r^{\alpha+q}e^{2\pi i (\alpha-q)} \quad \text{and} \quad \Gamma_{p,\beta}(r, \phi) = r^{p+\beta}e^{2\pi i (p+\beta)}.$$

We say that on the space of anyonic harmonics there exists a topologically deformed Hodge structure.

6. Conclusions

We have shown in this paper that the space of anyonic harmonics admits a topological deformation of a Hodge structure. The deformed Hodge structure is a complex representation of the universal covering of the multiplicative group $\tilde{C}^*$. The representation in general is not a complexification of any real representation in contrary to the not deformed Hodge structure.

This structure determines only two quantum numbers $l, m$ which is not a complete system of quantum numbers. There is an open problem how to introduce additional quantum numbers to obtain complete description of the space of quantum states for the system of anyons.

Acknowledgments

The paper was partially supported by the grant NN 201 373236.
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