Densities of Lévy walks and the corresponding fractional equations

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Abstract

In this paper we derive explicit formulas for the densities of Lévy walks. Our results cover both jump-first and wait-first scenarios. The obtained densities solve certain fractional differential equations involving fractional material derivative operators. In the particular case, when the stability index is rational, the densities can be represented as an integral of Meijer G function. This allows to efficiently evaluate them numerically. Our results show perfect agreement with the Monte Carlo simulations.

1 Introduction

The Continuous Time Random Walk (CTRW) is a stochastic process determined uniquely by i.i.d. random variables $J_1, J_2, \ldots$ representing consecutive jumps of the random walker, and i.i.d. positive random variables $T_1, T_2, \ldots$ representing waiting times between jumps, [12]. The trajectories of CTRW are step functions with intervals $T_i$ and jumps $J_i$.

Lévy walk is a particular case of CTRW satisfying the following additional condition $|J_i| = T_i$ for every $i \in \mathbb{N}$ (length of the jump equal to the length of the preceding waiting time). Lévy walks were defined for the first time in [22, 11] in the framework of generalized master equations. Since then, they became one of the most popular models in statistical physics with large number of important applications. Lévy walks have been found to be excellent models in the description of various real-life phenomena and complex anomalous systems. The most striking examples are: transport of light in optical materials [1], foraging patterns of animals [2, 3, 6], epidemic spreading [4, 7], human travel [3, 8], blinking nano-crystals [16], and fluid flow in a rotating annulus [24].

The main idea underlying the Lévy walk is the spatial-temporal coupling, which is manifested by the condition $|J_i| = T_i$. Thus, even if we assume that the distribution of jumps $Y_i$ is heavy-tailed with diverging moments, the Lévy walk itself has finite moments of all orders (intuitively, long jumps are penalized by requiring more time to be performed). This is very different from $\alpha$-stable Lévy processes with $\alpha < 2$, which have infinite second moment. These desirable properties make Lévy walk particularly attractive for physical applications.

Let us now recall the formal definition of Lévy walk. Let $T_i$, $i = 1, 2, \ldots$, be the sequence of i.i.d. positive random variables, representing the waiting times of the walker. Assume that they belong to the domain of attraction of one-sided $\alpha$-stable law, i.e.

$$n^{-1/\alpha} \sum_{i=1}^{[nt]} T_i \overset{d}{\to} S_\alpha(t)$$

as $n \to \infty$. Here, $S_\alpha(t)$ is the $\alpha$-stable subordinator with the Laplace transform [9]

$$\mathbb{E}(\exp\{-sS_\alpha(t)\}) = \exp\{-ts^\alpha\}, \quad 0 < \alpha < 1.$$
Denote by
\[ N(t) = \max\{k \geq 0 : \sum_{i=1}^{k} T_i \leq t\} \]
the corresponding counting process. Next, define the jumps of the walker as
\[ J_i = I_i T_i, \]
where \( I_1, I_2, \ldots \) are i.i.d. random variables, which are assumed independent of the sequence of waiting times \( T_i \). Each \( I_i \) governs the direction of the jump, i.e.
\[ \mathbb{P}(I_i = 1) = p, \quad \mathbb{P}(I_i = -1) = 1 - p, \]
\( 0 \leq p \leq 1 \). Thus, the walker jumps up with probability \( p \) and down with probability \( 1 - p \). Clearly, the condition \( |J_i| = T_i \) is satisfied. The sequence of jumps \( J_i \) defined above belongs to the domain of attraction of \( \alpha \)-stable distribution \[ n^{-1/\alpha} \sum_{i=1}^{[nt]} J_i \xrightarrow{d} L_\alpha(t) \]
as \( n \to \infty \). The process \( L_\alpha(t) \) is the \( \alpha \)-stable Lévy motion with the corresponding Fourier transform
\[ \mathbb{E}(\exp\{ikL_\alpha(t)\}) = \exp\{-|k|^\alpha \cos(\pi \alpha/2)(1 - i(2p - 1) \tan(\pi \alpha/2) \text{sgn}(k))\}. \]
Later on we will also use the corresponding left-limit process
\[ L^-_\alpha(t) \overset{def}{=} \lim_{s \searrow t} L_\alpha(s). \]
Finally, the process
\[ R(t) = \sum_{i=1}^{N(t)} J_i \]
is called Lévy walk. \( R(t) \) is also known as \textit{wait-first Lévy walk} in the literature \[25\], since the walker at the beginning of its motion \( (t = 0) \) first waits and then performs the jump.

As shown in \[13\], \( R(t) \) obeys the following scaling limit in distribution
\[ \frac{R(nt)}{n} \xrightarrow{d} X(t) \]
as \( n \to \infty \). Here \( X(t) \) is the right-continuous version of the process \( L^-_\alpha(S^-_\alpha(t)) \). Moreover, \( S^-_\alpha(t) \) is the inverse of \( S_\alpha(t) \), i.e. \( S^-_\alpha(t) = \inf\{\tau \geq 0 : S_\alpha(\tau) > t\} \). Additionally, the instants of jumps as well as the respective jump lengths of the processes \( L_\alpha(t) \) and \( S_\alpha(t) \) are exactly the same. Also, the probability density function (PDF) \( p_t(x) \) of \( X(t) \) satisfies the following fractional equation \[10, 13\]
\[ \left[ p \left( \frac{\partial}{\partial t} \mp \frac{\partial}{\partial y} \right)^\alpha + (1 - p) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^\alpha \right] p_t(x) = \delta_0(x) \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}. \] (1.3)
Here, the operators \( \left( \frac{\partial}{\partial t} \mp \frac{\partial}{\partial y} \right)^\alpha \) are the fractional material derivatives introduced in \[23\]. In the Fourier-Laplace space they are given by
\[ \mathcal{F}_y \mathcal{L}_t \left\{ \left( \frac{\partial}{\partial t} \mp \frac{\partial}{\partial y} \right)^\alpha f(y,t) \right\} = (s \mp ik)^\alpha f(k,s). \]
In what follows, we will find the explicit solution of (1.3) by determining the PDF \( p_t(x) \) of the limit process \( X(t) \) given in (1.2).

We will also consider the so-called jump-first Lévy walk \([15]\)

\[
\tilde{R}(t) = \sum_{i=1}^{N(t)+1} J_i.
\]

It has the following scaling limit in distribution \([14, 15]\)

\[
\frac{\tilde{R}(nt)}{n} \overset{d}{\to} Y(t) = L_\alpha(S_\alpha^{-1}(t))
\]

as \( n \to \infty \). Moreover, the PDF \( w_t(y) \) of \( Y(t) \) satisfies the fractional equation of the form \([10, 14]\)

\[
\left[p \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial y} \right)^\alpha + (1-p) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial y} \right)^\alpha \right] w_t(y) = \frac{\alpha}{\Gamma(1-\alpha)} \int_t^\infty \delta_0(y-u) u^{-\alpha-1} du.
\]

In the next two sections we will derive explicit formulas for the PDFs of the limit processes \( X(t) \) and \( Y(t) \), respectively. This way we will obtain solutions of fractional equations (1.3) and (1.5).

2 Densities of wait-first Lévy walks

Let us start with the wait-first scenario. In the result below we determine the PDF of the process \( X(t) = L_\alpha(S_\alpha^{-1}(t)) \).

**Theorem 2.1** If \( p \in (0,1) \) and \( \alpha \in (0,1) \) then the PDF \( p_t(x) \) of the process \( X(t) = L_\alpha(S_\alpha^{-1}(t)) \) equals

\[
p_t(x) = \frac{1}{2p^{\frac{\alpha}{2}}(1-p)^{\frac{\alpha}{2}}} \frac{\alpha}{\Gamma(1-\alpha)} \int_t^\infty \int_{|x|}^\infty \frac{z^{1-\alpha}}{(t-w)^\alpha} r_\alpha \left( \frac{w+x}{2p^{1/\alpha}} \right) \left( \frac{w-x}{2(1-p)^{1/\alpha}} \right) \int_0^\infty \delta_0(w-u) u^{-\alpha-1} du \, dz \, dw,
\]

where \( r_\alpha(x) \) is a density of a positive \( \alpha \)-stable random variable \( Z_\alpha \) with the Laplace transform

\[
E e^{-uZ_\alpha} = e^{-u^\alpha}.
\]

**Proof.** We start with reminding the following equality of distributions (see \([13]\)):

\[
(L_\alpha(t), S_\alpha(t)) \overset{d}{=} \left( p^{\frac{1}{\alpha}} S_\alpha^{(1)}(t) - (1-p)^{\frac{1}{\alpha}} S_\alpha^{(2)}(t), p^{\frac{1}{\alpha}} S_\alpha^{(1)}(t) + (1-p)^{\frac{1}{\alpha}} S_\alpha^{(2)}(t) \right),
\]

where \( S_\alpha^{(1)}(t) \) and \( S_\alpha^{(2)}(t) \) are independent copies of \( S_\alpha(t) \). Using this fact it was shown in \([13]\) that the Lévy measure \( \nu_{(L_\alpha,S_\alpha)} \) of \((L_\alpha(t), S_\alpha(t))\) equals

\[
\nu(dx,ds) = p\delta_s(dx)\nu_{S_\alpha}(ds) + (1-p)\delta_{-s}(dx)\nu_{S_\alpha}(ds),
\]

3
where \( \nu_{S_{\alpha}}(ds) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha} ds \). Hence, an infinitesimal generator \( A \) of the process \( (L_{\alpha}(t), S_{\alpha}(t)) \) equals

\[
Af(x,s) = \int_{\mathbb{R}^2} [f(x+y,s+w) - f(x,s)] \nu_{(L_{\alpha},S_{\alpha})}(dy,dw) = \int_{\mathbb{R}^2} [f(x+y,s+w) - f(x,s)] (p\delta_w(dy)\nu_{S_{\alpha}}(dw) + (1-p)\delta_{-w}(dy)\nu_{S_{\alpha}}(dw)).
\]

In [17] the authors consider (among other processes) the 2-dimensional process \((X(t), V(t-))\), where \( V(t) \) is the age process counting time that passed since the last jump of \( X(t) \). Equation (2.4) together with Remark 4.2 in the mentioned paper provides us with the joint distribution of \((X(t), V(t-))\)

\[
P(X(t) = dx, V(t-) = dv) = \nu_{(L_{\alpha},S_{\alpha})}(\mathbb{R} \times [v, \infty))U(dx, t-dv)1_{0 \leq v \leq t}, \quad (2.5)
\]

where \( U(dx, ds) \) is the 0-potential measure of the process \((L_{\alpha}(t), S_{\alpha}(t))\), defined as

\[
U(dx, ds) = \int_{0}^{\infty} P(L_{\alpha}(u) = dx, S_{\alpha}(u) = ds) \, du.
\]

The process \( V(t) \) is beyond our interest, but we will later obtain the distribution of \( X(t) \) as a marginal distribution of \((X(t), V(t-))\). This is the reason why we turn now our attention to Eq. (2.5). We have

\[
\nu_{(L_{\alpha},S_{\alpha})}(\mathbb{R} \times [v, \infty)) = p\nu_{S_{\alpha}}([v, \infty)) + (1-p)\nu_{S_{\alpha}}([v, \infty)) = \nu_{S_{\alpha}}([v, \infty)) \quad (2.6)
\]

Furthermore, a density \( u(x,s) \) of the potential measure \( U \) can be calculated as

\[
u_{S_{\alpha}}([v, \infty)) = \int_v^\infty \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha} dx = \frac{v^{-\alpha}}{\Gamma(1-\alpha)}, \quad (2.6)
\]

where \( w_t(x,s) \) is the PDF of the process \((L_{\alpha}(t), S_{\alpha}(t))\). From Eq. (2.2) we get

\[
P(L_{\alpha}(t) = dx, S_{\alpha}(t) = ds) = P \left( S_{\alpha}^{(1)} = \frac{ds + dx}{2p^\frac{1}{\alpha}}, S_{\alpha}^{(2)} = \frac{ds - dx}{2(1-p)^{\frac{1}{\alpha}}} \right).
\]

The Jacobian determinant in the above formula for the linear transformation of \((dx, ds)\) equals \( \frac{1}{2p^{1/\alpha}(1-p)^{1/\alpha}} \). Taking into account the independence of \( S_{\alpha}^{(1)}(t) \) and \( S_{\alpha}^{(2)}(t) \) we express \( w_t(x,s) \) in terms of \( q_t(s) \) - the PDF of \( S_{\alpha}(t) \):

\[
w_t(x,s) = \frac{1}{2p^{\frac{1}{\alpha}}(1-p)^{\frac{1}{\alpha}}} q_t \left( \frac{s + x}{2p^{\frac{1}{\alpha}}} \right) q_t \left( \frac{s - x}{2(1-p)^{\frac{1}{\alpha}}} \right).
\]

Moreover the self-similarity of \( S_{\alpha}(t) \) provides us with the equation

\[
q_u(x) = \frac{1}{u^{\frac{1}{\alpha}}} q_1 \left( \frac{x}{u^{\frac{1}{\alpha}}} \right) = \frac{1}{u^{\frac{1}{\alpha}}} r_{\alpha} \left( \frac{x}{u^{\frac{1}{\alpha}}} \right),
\]
here $r_\alpha(x)$ is the density of the random variable $S_\alpha(t)$. We take into account the two above equations in Eq. (2.7) and substitute $u = z^{-\alpha}$ to calculate the integral:

$$u(x, s) = \int_0^\infty \frac{1}{2p^{\frac{1}{\alpha}}(1 - p)^{\frac{1}{\alpha}}} q_u \left( \frac{s + x}{2p^{\frac{1}{\alpha}}} \right) q_u \left( \frac{s - x}{2(1 - p)^{\frac{1}{\alpha}}} \right) du$$

$$= \int_0^\infty \frac{1}{2p^{\frac{1}{\alpha}}(1 - p)^{\frac{1}{\alpha}}} \frac{u^{\frac{2}{\alpha}}}{r_\alpha} \left( \frac{s + x}{2p^{\frac{1}{\alpha}}} u^{\frac{\alpha}{\alpha}} \right) r_\alpha \left( \frac{s - x}{2(1 - p)^{\frac{1}{\alpha}}} u^{\frac{\alpha}{\alpha}} \right) du$$

$$= \int_0^\infty \frac{\alpha}{2p^{\frac{1}{\alpha}}(1 - p)^{\frac{1}{\alpha}}} z^{1 - \alpha} r_\alpha \left( \frac{s + x}{2p^{\frac{1}{\alpha}}} z \right) r_\alpha \left( \frac{s - x}{2(1 - p)^{\frac{1}{\alpha}}} z \right) dz.$$  \hspace{1cm} (2.8)

Finally we combine eqs. (2.5), (2.6), (2.8) and integrate with respect to $dv$ to obtain

$$p_t(x) = \frac{\alpha}{\Gamma(1 - \alpha) 2p^{\frac{1}{\alpha}}(1 - p)^{\frac{1}{\alpha}}} \int^t_{|x|} \int_0^\infty v^{-\alpha} z^{1 - \alpha} r_\alpha \left( \frac{t - v + x}{2p^{\frac{1}{\alpha}}} z \right) r_\alpha \left( \frac{t - v - x}{2(1 - p)^{\frac{1}{\alpha}}} z \right) dz dv$$

$$= \frac{\alpha}{\Gamma(1 - \alpha) 2p^{\frac{1}{\alpha}}(1 - p)^{\frac{1}{\alpha}}} \int^t_{|x|} \int_0^\infty \frac{z^{1 - \alpha}}{(t - w)^{\alpha}} r_\alpha \left( \frac{w + x}{2p^{\frac{1}{\alpha}}} z \right) r_\alpha \left( \frac{w - x}{2(1 - p)^{\frac{1}{\alpha}}} z \right) dz dw$$

We used here the fact that $r_\alpha(x)$ vanishes outside the positive half-line and substituted $v = t - w$.

One should mention here that the PDF of wait-first Lévy walk in the extreme cases $p = 0$ and $p = 1$ was already derived in \[13\].

For the special case $\alpha = \frac{1}{2}$ we get the following simple expression for $p_t(x)$:

**Corollary 2.1** When $\alpha = \frac{1}{2}$ and $p \in (0, 1)$ the PDF $p_t(x)$ can be expressed as

$$p_t(x) = \frac{\sqrt{\pi}}{\pi} p(1 - p) \frac{(t - |x|)^{\frac{3}{2}}}{(2p^2t + (1 - 2p)(t + x))(2p^2|x| + (1 - 2p)(x + |x|))^{\frac{3}{2}}} 1_{(0,t)}(|x|)$$

**Proof.** We have (see \[21\])

$$r_{1/2}(x) = \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{2}} \exp \left( \frac{-1}{4x} \right).$$

Taking into account this formula and using the property of Gamma function $\Gamma(3/2) = \frac{1}{2} \Gamma(1/2) = \sqrt{\pi}$ we can calculate the following integral:

$$\int_0^\infty z^{\frac{1}{2}} r_{1/2} \left( \frac{w + x}{2p^2z} \right) r_{1/2} \left( \frac{w - x}{2(1 - p)^2z} \right) dz$$

$$= \frac{1}{4\pi} \left( \frac{w + x}{4p^2(1 - p)^2} \right)^{\frac{1}{2}} \int_0^\infty z^{\frac{1}{2}} \exp \left( - \frac{p^2(w - x) + (1 - p)^2(w + x)}{2(w + x)(w - x)} \frac{1}{z} \right) dz$$

$$= \frac{2^{\frac{3}{2}}}{\sqrt{\pi}} \frac{p^3(1 - p)^3}{(p^2(w - x) + (1 - p)^2(w + x))^{\frac{3}{2}}}.$$
Substituting Eq. (2.9) into Eq. (2.1) we arrive at

\[ p_t(x) = \frac{1}{\sqrt{2\pi}} p(1-p) \int_{|x|}^t (t-w)^{-\frac{1}{2}} \left( (1-p)^2 + p^2 \right) w + \left( (1-p)^2 - p^2 \right) x^{-\frac{3}{2}} dw 1_{(0,t)}(|x|). \]

One can notice that (below \(a, b\) and \(c\) are constants)

\[ \frac{d}{dw} \left( \frac{-2(c-w)^{\frac{1}{2}}}{(a+bc)(a+bw)^{\frac{1}{2}}} \right) = (c-w)^{-\frac{1}{2}}(a+bw)^{-\frac{3}{2}}, \]

which finally implies

\[ p_t(x) = \frac{\sqrt{2}}{\pi} p(1-p) \frac{(t-|x|)^{\frac{1}{2}}}{(2p^2 t + (1-2p)(t+x))(2p^2|x| + (1-2p)(x+|x|))^{\frac{3}{2}}} 1_{(0,t)}(|x|). \]

It is worth to mention here, that in a special case \(\alpha = \frac{1}{2}\) and \(p = \frac{1}{2}\) we recover the known result (13)

\[ p_t(x) = \frac{1}{2\pi} \int_{|x|}^t (t-w)^{-\frac{1}{2}} w^{-\frac{3}{2}} dw 1_{(0,t)}(|x|) = \frac{(t-|x|)^{\frac{1}{2}}}{\pi t|x|^2} 1_{(0,t)}(|x|). \]

Figures 1 and 2 present the densities obtained from the above Corollary.

![Figure 1: Plot of PDF \(p_t(x)\) of the process \(L_\alpha^{-1}(S_\alpha(t))\) for different values of \(t\), \(\alpha = 0.5\), \(p = 0.1\).](image)

The drawback of using Theorem 2.1 to calculate \(p_t(x)\) in the general case is the necessity of knowing values of \(r_\alpha(x)\). One can use some known algorithms to approximate \(r_\alpha(x)\), however results are not perfect. Computations are time-consuming and inaccurate. They differ from those obtained via Monte-Carlo methods. However, as the next Corollary shows, we can express \(p_t(x)\) in the form of an integral from Meijer G function (see [19]). This special function is implemented in most of numerical packages, including Mathematica and Matlab. This representation is valid for rational \(\alpha\) and can be used to approximate irrational \(\alpha\) with a rational one.
Figure 2: For $\alpha = 0.5$ we compare the obtained densities $p_1(x)$ for different values of $p$ (red solid lines) with densities estimated using Monte Carlo methods (blue pluses).

**Corollary 2.2** If $p \in (0, 1)$ and $\alpha = \frac{l}{k}$, where $l, k \in \mathbb{N}$, then the PDF $p_t(x)$ of the process $X(t) = L_\alpha^{-1}(S_\alpha^{-1}(t))$ equals

$$
p_t(x) = \frac{2^{1-\alpha-k+l}}{p^{k-l}} \frac{\pi^{\alpha}}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-w)^{\alpha}} \frac{(w+x)^{\alpha-1}}{w-x} G_{l+k,l+k}^{k,k} \left( \left( \frac{p}{1-p} \right)^{\frac{1}{\alpha}} \left( \frac{w+x}{w-x} \right) \right| \Delta(k,k(\frac{\alpha}{k}-1)), \Delta(l,0), \Delta(k,0), -\Delta(l,(\frac{\alpha}{k}-1)) \right) \right] \text{d}w \mathbf{1}_{(0,t)}(|x|),
$$

where $G_{l_1,k_2}^{k_1,k_2}(x \mid \Delta(k,a), \Delta(l,b))$ is the Meijer G function (see [17]) and $\Delta(k,a) = \left\{ \frac{a+1}{k}, \frac{a+2}{k}, \ldots, \frac{a+k-1}{k} \right\}$ is a special list of $k$ elements.

**Proof.** For $\alpha = l/k$ where $k, l$ are positive integers with $k > l$ we can express $r_{\alpha}(x)$ in terms of the Meijer G function (see [18]):

$$
r_{l/k}(x) = \sqrt{kl} \frac{1}{(2\pi)^{(k-l)/2}} \frac{\pi^{\alpha}}{\Gamma(1-\alpha)} \frac{1}{x} \frac{G_{l,k}^{k,l} \left( \frac{l}{k} x^{-\frac{1}{\alpha}} \left| \Delta(l,0), \Delta(k,0) \right. \right)}{\Delta(l,0), \Delta(k,0)}.
$$
This gives us
\[
\int_0^\infty z^{1-\alpha} q_\alpha \left( \frac{w + x}{2 p^\frac{1}{\alpha}} \right) z \alpha \left( \frac{w - x}{2(1 - p)^\frac{1}{\alpha}} \right) \, dz
\]
\[
= \frac{kl}{(2\pi)^{(k-l)}} (w - x)(w + x) \int_0^\infty z^{1-\alpha} G_{t,k}^{k,0} \left( \frac{t}{k^k} \left( \frac{w + x}{2 p^\frac{1}{\alpha}} \right) \right) \Delta(t,0) \Delta(k,0) \, dz
\]
\[
= \frac{k}{(2\pi)^{(k-l)}} (w - x)(w + x) \int_0^\infty u^{\alpha-1} G_{t,k}^{k,0} \left( \frac{t}{k^k} \left( \frac{w + x}{2 p^\frac{1}{\alpha}} \right) \right) \Delta(t,0) \Delta(k,0) \, du
\]
\[
= \frac{k}{(2\pi)^{(k-l)}} (w - x)(w + x) \left( \frac{t}{k^k} \right)^{-\alpha/l} \left( \frac{w + x}{2 p^\frac{1}{\alpha}} \right) G_{t+k,l+k}^{k,k} \left( \frac{p}{1 - p} \right)^{\frac{1}{\alpha}} \left( \frac{w + x}{w - x} \right) \Delta(t,k(k-1)) \Delta(k,0) \Delta(l,l-1),
\]

We substituted in the above equations $z^{-l} = u$. Moreover we used properties of Meijer G functions (see [19]) to calculate the integral from multiplication of these functions. To end the proof we substitute the calculated above integral into Eq. (2.1).

\[
\text{Figure 3 presents densities calculated here. Another method for calculating } r_{\alpha}, \text{ where is not necessarily rational, was presented in [20].}
\]

\[
\text{Figure 3: For } p = 0.25 \text{ we compare the densities } p_1(x) \text{ obtained in Corollary 2.2 for different values of } \alpha \text{ (red solid lines) with densities estimated using Monte Carlo methods (blue pluses).}
\]

\section{Densities of first-jump Lévy walks}

This section is devoted to the first-jump scenario. Below we find the PDF of the process $Y(t)$ from (1.4). At the same time this PDF solves the fractional equation (1.5).
\textbf{Theorem 3.1} If \( p \in (0,1) \) and \( \alpha \in (0,1) \) then the PDF \( w_t(y) \) of the process \( Y(t) = L_\alpha(S_\alpha^{-1}(t)) \) equals:

(i) if \( |y| < t \), then

\[
w_t(y) = \frac{\alpha^2 p^{1-\frac{1}{\alpha}}}{2(1-p)^{\frac{1}{\alpha}\Gamma(1-\alpha)}} \int_{-t}^{t} \int_{|y|+(t-y)x}^{y} \int_{0}^{\infty} z^{1-\alpha} \frac{r_\alpha(\frac{w+x}{2p^{\frac{1}{\alpha}}})}{r_\alpha(\frac{w-x}{2(1-p)^{\frac{1}{\alpha}}})} dy \, dz \, dx,
\]

(ii) if \( y \geq t \), then

\[
w_t(y) = \frac{\alpha^2 p^{1-\frac{1}{\alpha}}}{2(1-p)^{\frac{1}{\alpha}\Gamma(1-\alpha)}} \int_{-t}^{t} \int_{|y|+(t+y-x)}^{y} \int_{0}^{\infty} z^{1-\alpha} \frac{r_\alpha(\frac{w+x}{2p^{\frac{1}{\alpha}}})}{r_\alpha(\frac{w-x}{2(1-p)^{\frac{1}{\alpha}}})} dy \, dz \, dx,
\]

(iii) if \( y \leq (-t) \), then

\[
w_t(y) = \frac{\alpha^2 (1-p)^{1-\frac{1}{\alpha}}}{2p^\frac{1}{\alpha}\Gamma(1-\alpha)} \int_{-t}^{t} \int_{|y|+(t+y-x)}^{y} \int_{0}^{\infty} z^{1-\alpha} \frac{r_\alpha(\frac{w+x}{2p^\frac{1}{\alpha}})}{r_\alpha(\frac{w-x}{2(1-p)^{\frac{1}{\alpha}}})} dy \, dz \, dx,
\]

Here \( r_\alpha(y) \) is a density of a positive \( \alpha \)-stable random variable \( Z_\alpha \) with the Laplace transform

\[ Ee^{-uZ_\alpha} = e^{-u^\alpha} \]

and \( a \lor b = \max(a,b) \)

\textbf{Proof.} From Remark 4.2 in [17] we get

\[ P(Y(t) = dy, R(t) = dr) = \int_{x \in \mathbb{R}} \int_{w \in [0,t]} U(dx, dw) \nu_{L_\alpha,S_\alpha}(dy - x, dr + t - w) \]

where \( R(t) \) is a process that counts how long the process \( Y(t) \) remains constant from the moment \( t \). After applying Eq. (2.3) the above equation reads

\[ P(Y(t) = dy, R(t) = dr) = \int_{x \in \mathbb{R}} \int_{w \in [0,t]} U(dx, dw) \left[ p\delta_{dr+t-w}(dy - x) \nu_{S_\alpha}(dr + t - w) ight. \\
+ (1-p)\delta_{-dr-t+w}(dy - x) \nu_{S_\alpha}(dr + t - w) \right] \]
Now we can recover the distribution of $Y(t)$ as a marginal distribution. We calculate the density $w_t(y)$ applying Tonelli’s theorem to change the order of integration:

$$w_t(y) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_{x \in [0,1]} \int_{w \in [0,t]} u(x, w) \left[ p \delta_{y-w}(y-x)(dr + t-w)^{-1-\alpha} + (1-p) \delta_{dr-t+w}(y-x)(dr + t-w)^{-1-\alpha} \right]dw dx dr$$

$$= \frac{\alpha}{\Gamma(1 - \alpha)} \int_{-t}^{t} \int_{|x|}^{t} u(x, w) \left[ p 1_{(-\infty,y)}(x) 1_{(t-y+x,\infty)}(w) (y-x)^{-1-\alpha} + (1-p) 1_{(y,\infty)}(x) 1_{(t+y-x,\infty)}(w) (x-y)^{-1-\alpha} \right]dw dx.$$  

The area of integration in the above integral depends on the relation of $y$ with $t$ - we have three cases. The first case is described by the condition $|y| < t$. Then

$$w_t(y) = \frac{p \alpha}{\Gamma(1 - \alpha)} \int_{-t}^{y} \int_{|x| \vee (t-y+x)}^{t} u(x, w) (y-x)^{-1-\alpha} dw dx$$

$$+ \frac{(1-p) \alpha}{\Gamma(1 - \alpha)} \int_{y}^{t} \int_{|x| \vee (t+y-x)}^{t} u(x, w) (x-y)^{-1-\alpha} dw dx.$$

If $y \geq t$ then

$$w_t(y) = \frac{p \alpha}{\Gamma(1 - \alpha)} \int_{-t}^{t} \int_{|x| \vee (t-y+x)}^{t} u(x, w) (y-x)^{-1-\alpha} dw dx$$

and finally if $y \leq (-t)$ then

$$w_t(y) = \frac{(1-p) \alpha}{\Gamma(1 - \alpha)} \int_{y}^{t} \int_{|x| \vee (t+y-x)}^{t} u(x, w) (x-y)^{-1-\alpha} dw dx.$$

Substituting Eq. (2.3) for $u(x, w)$ in all three cases ends the proof. \hfill \square

In the special case $\alpha = \frac{1}{2}$ we get the following simple expression for $w_t(y)$:

**Corollary 3.1** When $\alpha = \frac{1}{2}$ the PDF $w_t(y)$ can be expressed as

$$w_t(y) = \frac{p(1-p)t^3}{\pi} \frac{t}{y(t-y)^{\frac{1}{2}} + t + y} \left[ (t-y)^{\frac{1}{2}} - (t+y)^{\frac{1}{2}} + y (t-y)^{\frac{1}{2}} + (t+y)^{\frac{1}{2}} \right]$$

if $|y| < t$ and as

$$w_t(y) = \frac{p}{\pi} \frac{t^{\frac{1}{2}}}{y(p-t)^{\frac{1}{2}} + (1-p)(y+t)^{\frac{1}{2}}}$$

if $y \geq t$ and as

$$w_t(y) = \frac{p-1}{\pi} \frac{t^{\frac{1}{2}}}{y(p-t)^{\frac{1}{2}} + (1-p)(-y-t)^{\frac{1}{2}}}$$

if $y \leq (-t)$.
Proof. The density \( w_t(y) \) is symmetric in a following sense:
\[
w_t(y)_p = w_t(-y)_{1-p},
\]
where \( w_t(y), \) denotes the PDF of the process \( Y(t) \) with parameter \( p = c. \) Thus it is enough to calculate \( w_t(y) \) for \( y > 0 - \) we assume now \( y > 0. \) There are two cases: \( y < t \) and \( y \geq t. \) In the first case from Theorem 3.1 and Eq. (2.9) we have
\[
w_t(y) = \frac{p^2(1-p)}{2^{3/2}\pi} \int_{-y}^{y} \int_{t-y-x}^{t} (p^2(w-y) + (1-p)^2(w+y))^{-3/2}(y-x)^{-3/2}dwdx + \frac{p(1-p)^2}{2^{3/2}\pi} \int_{y}^{t} \int_{y}^{-y-x} (p^2(w-y) + (1-p)^2(w+y))^{-3/2}(x-y)^{-3/2}dwdx.
\]
Notice that in the above integrals \( |x| \vee (t-y+x) = t-y+x \) when \( x \geq \frac{y-t}{2} \) and \( |x| \vee (t-y+x) = |x| = -x \) if \( x \leq \frac{y-t}{2}. \) Similarly \( |x| \vee (t+y-x) = t+y-x \) when \( x \leq \frac{y+t}{2} \) and \( |x| \vee (t+y-x) = |x| = x \) if \( x \geq \frac{y+t}{2}. \) Thus \( w_t(y) \) can be expressed as a sum of four integrals, each of them having a simple region of integration. We can calculate all of them using standard integration techniques. After tedious computations we finally get the desired result for \( w_t(y) \) in case when \( y \in (0, t). \) In the second case, that is \( y \geq t, \) from Theorem 3.1 and Eq. (2.9) we have
\[
w_t(y) = \frac{p^2(1-p)^2}{2^{3/2}\pi} \left( \int_{-t}^{0} \int_{-x}^{t} (p^2(w-y) + (1-p)^2(w+y))^{-3/2}(y-x)^{-3/2}dwdx + \int_{0}^{t} \int_{x}^{t} (p^2(w-y) + (1-p)^2(w+y))^{-3/2}(y-x)^{-3/2}dwdx \right)
\]
Again, applying standard integration techniques we obtain the desired result for \( w_t(y) \) when \( y > t. \)

In a special case \( \alpha = \frac{1}{2} \) and \( p = \frac{1}{2} \) the PDF \( w_t(y) \) has the form
\[
w_t(y) = \frac{1}{2\pi} t^{\frac{1}{2}} \left( t \left( t - \frac{1}{2} (t - y)^{\frac{1}{2}} - (t - y)^{\frac{1}{2}} \right) - y \left( (t - y)^{\frac{1}{2}} + (t + y)^{\frac{1}{2}} \right) \right)
\]
if \( |y| < t \) and
\[
w_t(y) = \frac{1}{\pi |y|} \left( t^{\frac{1}{2}} (|y| - t^{\frac{1}{2}})^2 + |y| + t^{\frac{1}{2}} \right)
\]
if \( |y| \geq t. \) Figure [3] presents \( w_1(y) \) for different values of \( p. \) Notice that in all cases we have a characteristic sharp peak at \( y = t \) and \( y = -t. \) However when \( p \to 0 \) the left peak gets bigger and the right one diminishes. The opposite situation appears when \( p \to 1. \)
Figure 4: For $\alpha = 0.5$ we compare the densities $w_1(y)$ obtained in Corollary 3.1 for different values of $p$ (red solid lines) with densities estimated using Monte Carlo methods (blue pluses).

Figure 5: Plot of PDF $w_t(y)$ of the process $L_\alpha(S_{\alpha^{-1}}(t))$ for different values of $t$, $\alpha = 0.5$, $p = 0.5$.

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