INTERACTIONS IN THE SL(2,IR)/U(1)
BLACK HOLE BACKGROUND

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ABSTRACT

We calculate two- and three-point tachyon amplitudes of the SL(2,IR)/U(1) two-dimensional Euclidean black hole for spherical topologies in the continuum approach proposed by Bershadsky and Kutasov. We find an interesting relation to the tachyon scattering amplitudes of standard non-critical string theory.

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1. INTRODUCTION

One of the most interesting recent problems in string theory is the study of the two-dimensional black hole solution. So far, we have not been able to solve the many conceptual problems of four-dimensional black holes, such as the breakdown of quantum coherence at the endpoint of black hole evaporation as advocated by Hawking [1], the existence of no-hair theorems or the cosmic censorship hypothesis of Penrose.

With the hope that string theory may provide a natural framework to address these problems in a simplified context, Witten [2] proposed that the exact conformal field theory that describes a black hole in two-dimensional space-time can be formulated as an SL(2,IR)/U(1) gauged Wess-Zumino-Witten (WZW) model. He argued that the standard $c = 1$ matrix model could be regarded as an analog of the extreme Reissner-Nordström black hole of four-dimensional general relativity, and that the $c = 1$ S-matrix includes the formation and evaporation of black holes in intermediate processes. However the exact solubility of this model and the existence of the $W_\infty$ symmetry may constrain the physical events [3]. Ellis et al. [4] concluded that quantum coherence would be maintained during the black hole evaporation due to these infinite number of conserved quantities.

It is therefore important to understand the relation between the 2-D black hole and the standard two-dimensional non-critical string theory, which may further shed some light on other important questions such as the background independence in 2-D string theory.

The exact background metric of Witten’s black hole has been determined by Dijkgraaf, E. Verlinde and H. Verlinde [5]. They compared this model with $c = 1$ coupled to Liouville and concluded that there might be differences, since for example the reflection coefficients of tachyons off the black hole geometry do not agree. Eguchi et al [6] have shown that the free field BRST cohomologies of both models coincide, although Distler and Nelson [7] emphasized that if one works directly in the current algebra module there appear new discrete states, which
have no counterpart in \( c = 1 \). Independently Chaudhuri and Lykken \cite{8} showed that the discrete states of the black hole form a \( W_\infty \)-type algebra, which has been analysed by Oz and Marcus in detail \cite{9}.

The connection between the 2-D black hole and the \( c = 1 \) standard non-critical string has been studied by Bershadsky and Kutasov \cite{10}. They used an equivalent formulation of the \( \text{SL}(2,\mathbb{R})/U(1) \) coset in terms of the Wakimoto free field representation of \( \text{SL}(2,\mathbb{R}) \), which makes the calculation of the \( S \)-matrix possible. They discussed the spectrum of the 2D black hole and found that the bulk correlation functions of tachyons (without interactions) agree with the \( c = 1 \) correlators without cosmological constant. The purpose of this paper is to analyse whether this equivalence still holds if the interactions of both models are taken into account. We will do this at the level of two- and three-point functions and find a remarkable analogy. We will speculate whether this equivalence holds for the \( N \)-point function, although concrete calculations will appear in a later publication.

This paper is organized as follows. In section 2 we summarize the \( \text{SL}(2,\mathbb{R})/U(1) \) coset model following closely \cite{5}. We review the form of the currents, the gauged fixed action and tachyon vertex operators in the Wakimoto free field representation \cite{10}. Furthermore, we review some facts about the cohomology of the black hole \cite{6} \cite{7}. In section 3 we calculate the three-point function of tachyon vertex operators in the black hole background. We begin with the correlation function containing one highest weight state and can obtain the general three-point function using the \( \text{SL}(2,\mathbb{R}) \) structure. In section 4 we determine the form of the two-point function of (not necessary) on shell tachyons. We compare our results with \( c = 1 \) non-critical string theory, the work of Dijkgraaf et. al. \cite{5} and the deformed matrix model of Jevicki and Yoneya \cite{11}, which has been proposed to be the discrete matrix model formulation of the black hole conformal field theory. In section 5 we give our conclusions and some general remarks about \( N \)-point functions. In appendix A we calculate the three-point function with one screening as an illustrative example. We analyse contact term interactions that arise in our computations in detail. We review the most important formulas for our calculations in appendix B.
2. THE SL(2,IR)/U(1) WZW-MODEL

In this section we review the Lagrangian formulation of the SL(2,IR)/U(1) coset in terms of the Wakimoto free field representation and some basic facts about the cohomology of the 2D black hole.

2.1. THE GAUGED WZW-MODEL

The conformal field theory that describes a black hole in two-dimensional target space-time has a Lagrangian formulation in terms of a gauged WZW-model based on the non-compact group SL(2,IR) [2]. The ungauged SL(2,IR)-WZW model has the following action:

\[ S_{WZW}(g) = \frac{k}{8\pi} \int_{\Sigma} d^2 x \sqrt{h} h^{ij} tr(g^{-1} \partial_{i} gg^{-1} \partial_{j} g) + ik \Gamma(g), \] (2.1)

where \( \Sigma \) is a Riemann surface with metric tensor \( h^{ij}, g : \Sigma \rightarrow \text{SL}(2,\mathbb{R}) \) is an SL(2,IR) valued field on \( \Sigma \) and \( k \) is a real and positive number. The Wess-Zumino term, which guarantees conformal invariance is usually represented as:

\[ \Gamma(g) = \frac{1}{12\pi} \int_{B} d^3 y \varepsilon^{abc} tr(g^{-1} \partial_{a} gg^{-1} \partial_{b} gg^{-1} \partial_{c} g), \] (2.2)

where \( B \) is a three-dimensional manifold with boundary \( \Sigma \).

The action (2.1) possesses a global \( \text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R}) \) symmetry, since it is invariant under \( g \rightarrow agb^{-1} \) with \( a, b \in \text{SL}(2,\mathbb{R}) \). The gauge-invariant generalization of (2.1) is formulated in terms of an Abelian gauge field \( A \):
\[ S_{\text{WZ}}(g, A) = S_{\text{WZ}}(g) + \frac{k}{2\pi} \int d^2z (\bar{A}t(\bar{G}g^{-1}) + Atr(\bar{G}\partial g^{-1}) + A\bar{A}(-2 + tr(\bar{G}Gg^{-1}))) \]  

(2.3)

where \( G = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) for the Euclidean theory. If we impose the Lorentz gauge condition \( \partial_\alpha A^\alpha = 0 \), the gauge slice can be parametrized as \( A^\alpha = \varepsilon^{\alpha\beta} \partial_\beta \Phi \), and the complete gauge-fixed action of the Euclidean theory is given by [5]:

\[ S^\text{gf}_{\text{WZ}} = S_{\text{WZ}}(g) + S(\Phi) + S(b, c), \]  

(2.4)

where \( \Phi \) is a free scalar field:

\[ S(\Phi) = -\frac{k}{4\pi} \int d^2z \bar{\partial} \Phi \partial \Phi, \]  

(2.5)

and \((b, c)\) is a spin \((1, 0)\) system of ghosts:

\[ S(b, c) = \int d^2(b\bar{\partial}c + \bar{b}\partial\bar{c}). \]  

(2.6)

From (2.4) we see that the gauged WZW model can be expressed through the ungauged \( \text{SL}(2, \mathbb{R}) \) WZW model and the action of the free fields \( \Phi, b \) and \( c \). As already remarked in [5] this makes the quantization straightforward.

To quantize the ungauged theory we notice that the \( \text{SL}(2, \mathbb{R}) \) symmetry gives rise to the conserved currents:

\[ J = J^a t^a = -\frac{k}{2} \partial gg^{-1}, \quad \bar{J} = \bar{J}^a t^a = -\frac{k}{2} g^{-1} \partial g, \]  

(2.7)

where \( t^1 = i\sigma_1/2 \), \( t^2 = \sigma_2/2 \) and \( t^3 = i\sigma_3/2 \), and \( \sigma_i \) are the Pauli matrices. The modes of these currents satisfy the \( \text{SL}(2, \mathbb{R}) \) current algebra of level \( k \), which is equivalent to the following operator product expansion (OPE):
\[ J_+(z)J_-(w) = \frac{k}{(z-w)^2} - \frac{2J_3(w)}{(z-w)} + \ldots \]

\[ J_3(z)J_\pm = \pm \frac{J_\pm(w)}{(z-w)} + \ldots \]

\[ J_3(z)J_3(w) = -\frac{k}{(z-w)^2} + \ldots \quad (2.8) \]

2.2. **The Wakimoto Free Field Representation of \(SL(2,\mathbb{R})\)**

Like in the case of \(SU(2)\) [12] the \(SL(2,\mathbb{R})\) current algebra can be represented in terms of a spin \((1,0)\) chiral bosonic \(\beta-\gamma\) system and a free boson \(\Phi\) satisfying [13]:

\[ \langle \gamma(z)\beta(w) \rangle = -\frac{1}{(z-w)} \quad , \quad \langle \partial\phi(z)\partial\phi(w) \rangle = -\frac{1}{(z-w)^2}, \quad (2.9) \]

and the same holds for the antiholomorphic fields \(\bar{\gamma}\) and \(\bar{\beta}\). If we introduce the notation \(\alpha_\pm^2 = 2k - 4\), the currents that satisfy the OPE (2.8) have the following form:

\[ J_+(z) = \beta(z) \quad , \quad J_3(z) = -\beta(z)\gamma(z) - \frac{\alpha_+}{2}\partial\phi(z) \]

\[ J_-(z) = \beta(z)\gamma^2(z) + \alpha_+\gamma(z)\partial\phi(z) + k\partial\gamma(z). \quad (2.10) \]

The stress tensor follows from the Sugawara construction and is given by the following expression:

\[ T_{\text{SL}(2,\mathbb{R})} = -\frac{\Delta}{(k-2)} \quad , \quad \Delta = -\frac{1}{2} : J_+J_- + J_-J_+ : + : J_3J_3 : \quad (2.11) \]

which after inserting the currents leads us to:
\[ T_{\text{SL}(2,\mathbb{R})} = \beta \partial \gamma - \frac{1}{2} (\partial \phi)^2 - \frac{1}{\alpha_+} \partial^2 \phi. \]  

(2.12)

This energy momentum tensor has a central charge as a function of the level \( k \) of the Kac-Moody algebra:

\[ c = \frac{3k}{k - 2}. \]  

(2.13)

To determine the form of the primary fields in the Wakimoto representation, we need to introduce some basic facts about the representation theory of \( \text{SL}(2,\mathbb{R})^\star \). The basic fields from which we can build all the other states are the Kac-Moody primaries that satisfy:

\[ J_n^\pm |j, m\rangle = J_n^3 |j, m\rangle = 0 \quad \text{for} \quad n > 0, \]  

(2.14)

and are characterized by the zero mode Casimir eigenvalue \( j \) and by the eigenvalue of \( J^3_0 \):

\[ \Delta_0 |j, m\rangle = j(j + 1)|j, m\rangle, \quad J^3_0 |j, m\rangle = m |j, m\rangle, \]  

(2.15)

where \( \Delta_0 = -\frac{1}{2}(J^+_0 J^-_0 + J^-_0 J^+_0) + (J^3_0)^2 \). We can construct the representation by acting with raising and lowering operators \( J^+_0 \) and \( J^-_0 \). A solution of (2.15) is

\[ J^+_0 |j, m\rangle = (-m + j) |j, m + 1\rangle \]  

(2.16)

\[ J^-_0 |j, m\rangle = (-m - j) |j, m - 1\rangle. \]

If \( j \in \mathbb{R} \) it is standard in the representation theory of \( \text{SL}(2,\mathbb{R}) \) [14] to introduce new states \( |j, m\rangle \) that satisfy

\* We thank L. Alvarez-Gaumé for his notes on representation theory of \( \text{SL}(2,\mathbb{R}) \).
\[ J_0^+ |j, m\rangle = \sqrt{(j-m)(j+m+1)}|j, m+1\rangle \]
\[ J_0^- |j, m\rangle = \sqrt{(j+m)(j-m+1)}|j, m-1\rangle \]
\[ J_0^3 |j, m\rangle = m|j, m\rangle \] (2.17)

These recursion relations are satisfied up to a function depending only on \( j \), if we normalize the states as:
\[ |j, m\rangle = \sqrt{\Gamma(j+m+1)\Gamma(j-m+1)}|j, m\rangle. \] (2.18)

We will work with the states that satisfy (2.16) unless otherwise stated. These Kac-Moody primaries define an irreducible representation of SL(2,IR) on which we can impose two types of constraints [7]:

1. Hermiticity constraints. We demand that \( \Delta_0 \) and \( J_0^3 \) should have real eigenvalues. This means \( m \in \mathbb{R} \) and \( j = -\frac{1}{2} + i\lambda \) or \( j \in \mathbb{R} \). The types of Hermitian representations of SL(2,IR) can be classified as follows [15]:
   - Principal discrete series: Highest-weight or lowest-weight representation. Contain a state annihilated by \( J_0^+ \) and \( J_0^- \) respectively. They are one-sided and infinite dimensional. From (2.16) we see that these modules satisfy either \((j+m)\) or \((j-m)\) is an integer and:
     \[ |\text{HWS module}\rangle = |j, m\rangle \quad m = j, j-1, \ldots \]
     \[ |\text{LWS module}\rangle = |j, m\rangle \quad m = -j, -j+1, \ldots \] (2.19)
   - If in addition \( 2j \) is an integer, the representation is double sided.
   - Principal continuous series. Satisfies \( j = -\frac{1}{2} + i\lambda \) with \( \lambda, m \in \mathbb{R} \).
• Supplementary series. In this case $j \in \mathbb{R}$, but neither $j + m$ nor $j - m$ is an integer.

2. **Unitarity constraints.** The states are constrained to have positive norm, i.e. $J_0^+ J_0^-$ and $J_0^- J_0^+$ should have positive eigenvalues. This imposes restrictions on the allowed values of $j$. We will not impose any constraints of unitarity on our states.

The Kac-Moody primaries $|j, m\rangle$ are created, in the Wakimoto representation, by the action of the following (not normalized) “tachyon” vertex operator on the SL(2)-vacuum:

$$V_{j \, m}(z) = : e^{i j} e^{\alpha^a (z) \phi_a(z)} ,$$

and the same expression for the antiholomorphic part. These vertex operators have the following OPE with the energy momentum tensor and the currents:

$$T(z) V_{j \, m}(w) = \frac{h_{j,m}}{(z-w)^2} V_{j \, m}(w) + \frac{1}{(z-w)} \partial V_{j \, m}(w) + \ldots$$

$$J_a(z) V_{j \, m}(w) = \frac{t^{a}_{j,m}}{(z-w)} V_{j \, m}(w) + \ldots$$

Using the free field representation of the currents (2.10), it is easy to check that $V_{j \, m}$ satisfies the definition of a Kac-Moody primary (2.14), as well as (2.15) and (2.16). The conformal dimension of $V_{j \, m}$, obtained from the OPE with the energy momentum tensor (2.12), is:

$$h_{j,m} = - \frac{j(j+1)}{k-2} .$$

The action associated with the energy momentum tensor (2.12) is:
If we would like to calculate correlation functions of the vertex operators $V_{j,m}$ we would need a screening charge, in order to guarantee for the charge conservation arising from the zero mode integration. This screening charge can be obtained directly from the WZW model (2.1), if it is mapped with the Gauss-decomposition to a Wakimoto free field theory [16]. It can be constructed independently [10] if we take into account that the ungauged model has the SL(2,IR)symmetry, which leads to Ward identities for the correlators:

\begin{equation}
\langle T(z)V_1(z_1,\bar{z}_1)\ldots V_N(z_N,\bar{z}_N) \rangle = \sum_{i=1}^{N} \left( \frac{h_i}{(z-z_i)^2} + \frac{1}{(z-z_i)} \frac{\partial}{\partial z_i} \right) \langle V_1(z_1,\bar{z}_1)\ldots V_N(z_N,\bar{z}_N) \rangle
\end{equation}

\begin{equation}
\langle J^a(z)V_1(z_1,\bar{z}_1)\ldots V_N(z_N,\bar{z}_N) \rangle = \sum_{i=1}^{N} \frac{t^a_i}{(z-z_i)} \langle V_1(z_1,\bar{z}_1)\ldots V_N(z_N,\bar{z}_N) \rangle. \quad (2.24)
\end{equation}

Since these identities should be satisfied in the free field representation, the screening charge must have a regular OPE with the stress tensor and the currents. We must also take into account that only positive integer powers of $\beta$ are well defined through bosonization, as we will see in a moment. The screening that satisfies these conditions can be represented as the following surface integral:

\begin{equation}
Q = \int d^2 z J(z,\bar{z}), \quad J(z,\bar{z}) = \beta(z)\bar{\beta}(\bar{z}) e^{-2\alpha_+\phi(z,\bar{z})}. \quad (2.25)
\end{equation}

One can easily check the identities [12]
\[ J_+(z)J(w, \bar{w}) \sim \text{reg.}, \quad J_3(z)J(w, \bar{w}) \sim \text{reg.} \]

\[ J_-(z)J(w, \bar{w}) \sim \partial_\omega \left( \frac{e^{-\frac{\omega}{\alpha_+} \phi(w, \bar{w})}}{z-w} \right). \quad (2.26) \]

The total derivative appearing in the last OPE requires a careful treatment of contact terms \(*\), as we will do later. This comes from the fact that we are working with a surface integral and not with a contour integral, where this contribution generically vanishes. As is known to be correct for a Coulomb gas model [17] or Liouville theory [18] [19] [20], the screening charge will be added to the action and considered as the interaction of the model [10]:

\[ S = \frac{1}{2\pi} \int \partial \phi \bar{\partial} \phi - \frac{2}{\alpha_+} R^{(2)} \phi + \beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma} + 2\pi M \beta \bar{\beta} e^{-\frac{\alpha_+}{\alpha} \phi}. \quad (2.27) \]

The parameter \( M \) has been identified with the mass of the black hole [10] [6] that is responsible for the non-trivial background of the theory.

To evaluate correlation functions, it will be useful to bosonize the \( \beta-\gamma \) system as follows [21]:

\[ \beta = -i \bar{\partial} v e^{iv-u}, \quad \gamma = e^{u-iv}, \quad (2.28) \]

where \( u \) and \( v \) are ordinary bosons

\[ \langle u(z)u(w) \rangle = \langle v(z)v(w) \rangle = -\log(z-w). \quad (2.29) \]

This allows us to make sense of fractional and negative powers of \( \gamma \), while \( \beta^\alpha \) is well defined only for \( \alpha \) positive integer.

\* We thank G. Moore for pointing this out to us.
As already emphasized in [10] the realization of the sigma model (2.1) with a \( \beta-\gamma \) system, where the left and right scheme are completely decoupled, corresponds to a particular choice of contact terms. Different choices of contact terms correspond to different choices of coordinates in the coupling constant space [22] [23], i.e. the space of metrics.

2.3. The Cohomology of the Gauged Model

To construct the conformal field theory of the Euclidean black hole we are interested in the coset SL(2,\( \mathbb{R} \))/U(1). To gauge the U(1) subgroup, we introduce a gauge boson \( X \) and a pair of fermionic ghosts \( B, C \) of spin 1,0 respectively:

\[
\langle \partial X(z) \partial X(w) \rangle = -\frac{1}{(z-w)^2} \quad , \quad \langle C(z) B(w) \rangle = \frac{1}{(z-w)}. \tag{2.30}
\]

For the Euclidean theory the boson \( X \) is compact with radius \( R = \sqrt{k} \) in units of the self dual radius. The nilpotent BRST-charge of this symmetry is:

\[
Q^{U(1)} = \oint C(z) \left( J^3 - i \sqrt{\frac{k}{2}} \partial X \right) dz. \tag{2.31}
\]

The tachyon states of the gauged theory are the dressed ghost number zero primary fields, which are invariant under \( Q^{U(1)} \)†:

\[
\mathcal{V}_{j \, m} =: \gamma^j \, e^{\frac{i}{2} \phi} \, e^{i \frac{m}{\sqrt{\pi}} \int} \sqrt{k} X :. \tag{2.32}
\]

The conformal weight of these fields is

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† Since \( j \) and \( m \) are arbitrary at this moment, these operators will be called tachyons.
\[ h_{j,m} = -\frac{j(j + 1)}{k - 2} + \frac{m^2}{k}, \quad (2.33) \]

obtained from the OPE with the energy momentum tensor of the U(1) gauged theory:

\[ T_{\text{SL}(2, \mathbb{R})/U(1)} = \beta \partial \gamma - \frac{1}{2} (\partial \phi)^2 - \frac{1}{\alpha_+} \partial^2 \phi - \frac{1}{2} (\partial X)^2 - B \partial C. \quad (2.34) \]

The central charge of the gauged model:

\[ c = \frac{3k}{k - 2} - 1 \quad (2.35) \]

is \( c = 26 \) for \( k = 9/4 \). The total anomaly cancels if we take into account the pair of fermionic diffeomorphism ghosts \( b, c \) of spin 2, -1, respectively, with central charge \( c_{gh} = -26 \) and energy momentum tensor:

\[ T_{gh} = -2b \partial c - \partial bc. \quad (2.36) \]

The total stress tensor is given by \( T_{\text{total}} = T_{\text{SL}(2, \mathbb{R})/U(1)} + T_{gh} \). For \( k = 9/4 \) the additional bosonic Liouville field decouples. We will see later however, that it will be important in order to define a consistent regularization of the correlation functions.

The BRST operator coming from the diffeomorphism invariance

\[ Q^{\text{Diff}} = \oint c(z) \left( T_{\text{SL}(2, \mathbb{R})/U(1)} + \frac{1}{2} T_{gh} \right), \quad (2.37) \]

satisfies

\[ (Q^{\text{Diff}})^2 = 0, \quad \{Q^{U(1)}, Q^{\text{Diff}}\} = 0. \quad (2.38) \]
Invariance under $Q^{Diff}$ up to a total derivative implies the on shell condition

$$\frac{j(j+1)}{k-2} + \frac{m^2}{k} = 1,$$

which is, for $k = 9/4$:

$$2j + 1 = \pm \frac{2}{3} m.$$  \hspace{1cm} (2.40)

Since we are dealing with two BRST charges, one can consider the cohomology of $Q^{total} = Q^{U(1)} + Q^{Diff}$ or the iterated cohomology, where the states are annihilated by each of the BRST operators [7], as done in [10]. The complete cohomology of the gauged WZW model has been computed in [7] [6].

Using the Wakimoto free field representation (without screening charge), Eguchi et al. [6] have determined the cohomology of $Q^{total}$, which is believed to agree with the iterated cohomology. It coincides with the cohomology of $c = 1$ Liouville theory without cosmological constant [24]. From physical reasons one could expect that (at least for tachyons) a similar correspondence may exist if the interactions are taken into account [25], although to our knowledge this has not yet been proven.

Distler and Nelson [7] calculated both cohomologies directly in the current algebra module. In addition to the tachyon and $c = 1$ discrete states, they obtained new discrete states, which have no counterpart in Liouville theory coupled to $c = 1$ matter. The discrete states classified in [7] are:

$$\bar{D}^\pm: \quad m = \pm \frac{3(2s - 4r - 1)}{8}, \quad j = \frac{2s + 4r - 5}{8}$$

$$D^\mp: \quad m = \pm \frac{3(s - 2r + 1)}{4}, \quad j = \frac{s + 2r - 3}{4}$$  \hspace{1cm} (2.41)

$$C: \quad m = \frac{3(s - r)}{2}, \quad j = \frac{s + r - 1}{2},$$
where $r, s$ are positive integers. The states $\tilde{\mathcal{D}}^\pm$ are the new discrete states, while $\mathcal{D}^\pm$ appear in $c = 1$ Liouville theory as well as $\mathcal{C}$ and belong to the discrete and supplementary series of $\text{SL}(2,\mathbb{R})$ respectively. We will see later how part of this spectrum appears as poles in tachyon correlation functions, as it is known to hold for minimal models or $c = 1$ matter coupled to gravity [18] [19] [20].

As a prescription to “glue” together the left and right representation, we will consider in analogy to [5] only spinless $\text{SL}(2,\mathbb{R})$ primaries i.e. with $m = \bar{m}$. For the compact boson $X$ we have furthermore the restriction:

$$m = \frac{1}{2}(n_1 + n_2 k), \quad \bar{m} = -\frac{1}{2}(n_1 - n_2 k), \quad n_1, n_2 \in \mathbb{Z}.$$ (2.42)

States with $n_1 = 0$ are called winding modes. The momentum modes which satisfy $n_2 = 0$ are out of the theory, since they have spin.

3. THE THREE-POINT FUNCTION

In this section we calculate the three-point function of tachyons in the black hole geometry. We begin with the correlator containing one highest weight and will see that the $\text{SL}(2,\mathbb{R})$ structure fixes the correlators involving three generic tachyons.

3.1. THE CONSERVATION LAWS FOR $N$-POINT CORRELATORS

The free field approach is appropriate to compute the amplitudes, which obey a special energy sum rule, where the number of screenings $s$ is a positive integer. As in the case of the tachyon background, the desired correlators are determined indirectly by an analytic continuation in $s$ [18] [19] [20]. In general we wish to calculate the following scattering amplitude:
\[ A_{m_1 \ldots m_N}^{j_1 \ldots j_N} = \langle V_{j_1 m_1} \ldots V_{j_N m_N} \left( \int d^2z \beta \bar{\beta} e^{-\frac{x}{\alpha_+} \phi} \right)^s \rangle_{M=0} \] (3.1)

containing \( N \) tachyons \( V_{j m} \) and \( s \) screening charges. We will use a shorthand notation for the vertex operators, where the antiholomorphic dependence with \( m = \bar{m} \) will be understood throughout.

The conservation laws for an \( N \)-point amplitude are obtained from the zero mode integration of the fields [10]. Integrating over the zero mode \( \phi_0 = \phi - \tilde{\phi} \) à la Goulian and Li [18] gives:

\[ \langle \prod_{i=1}^N V_{j_i m_i} \rangle = M^s \Gamma(-s) \prod_{i=1}^N \bar{V}_{\tilde{j_i} \bar{m}_i} \left( \int \beta \bar{\beta} e^{-\frac{x}{\alpha_+} \Phi} d^2z \right)^s \rangle_{M=0}, \] (3.2)

where

\[ s = \sum_{i=1}^N j_i + 1 \] (3.3)

and we have absorbed a factor \(-\alpha_+/2\) into the definition of the path integral. From the zero mode of \( X \) it follows that

\[ \sum_{i=1}^N m_i = 0. \] (3.4)

The number of zero modes of the \( \beta-\gamma \) system, determined by the Riemann-Roch theorem, leads for spherical topologies to the condition:

\[ \#\beta - \#\gamma = 1. \] (3.5)

This constraint is equivalent to (3.3) and (3.4) for states of the form (2.32).
For an arbitrary correlator the number of screenings will not be a positive integer. We will determine the correlator on the r.h.s. of (3.2) for integer screenings, and more general amplitudes are obtained by an analytic continuation in $s$. We begin with the three-point functions, since we can use similar tricks to calculate two-point functions in a simple way.

3.2. THE THREE-POINT FUNCTION WITH ONE HIGHEST-WEIGHT STATE

It turns out that the simplest way to address a general three-point tachyon amplitude is to take one state belonging to the discrete representation of $\text{SL}(2,\mathbb{R})$. We choose in this section one of the tachyons as a highest-weight state, for example $j_1 = m_1$. We will see later, that an arbitrary three-point function, where no restriction to the representation to which the tachyons belong is made, can be expressed as a function of this one. The reason for this is that the $\text{SL}(2,\mathbb{R})$ Clebsch-Gordan coefficients can be analytically continued from one representation of $\text{SL}(2,\mathbb{R})$ to the others [15]. For the time being, the level of the Kac-Moody algebra is still arbitrary and we will need this restriction only when we want to make the analytic continuation in the number of screenings at the end.

Using $\text{SL}(2,\mathbb{C})$ transformations on the integrand, we can fix the three-tachyon vertex operators at $(z_1, z_2, z_3) = (0, 1, \infty)^*$:

$$\tilde{\mathcal{A}}_{j_1 \, m_2 \, m_3} = \int \prod_{i=1}^{s} d^2 z_i \langle \gamma^\frac{2}{n+1} j_1 \phi (0) \gamma^\frac{2}{n+1} j_2 \phi (1) \gamma^\frac{2}{n+1} j_3 \phi (\infty) \beta \bar{\beta} (z_i) \rangle$$

$$\langle \gamma^j j_2 - m_2 (\infty) \gamma^j j_3 - m_3 (\infty) \beta (z_i) \rangle \langle \gamma^j j_2 - m_2 (1) \gamma^j j_3 - m_3 (\infty) \bar{\beta} (z_i) \rangle. \quad (3.6)$$

This correlator has the following form after bosonizing the $\beta-\gamma$ system:

* We will drop the zero mode in the next formulas.
\[ \mathcal{A}_{j_1 \ m_2 \ m_3}^{j_1 \ j_2 \ j_3} = \int \prod_{i=1}^{s} d^2 z_i |z_i|^{-4 \rho_j} |1-z_i|^{-4 \rho_j} \prod_{i<j} |z_i-z_j|^{4 \rho_j} - \bar{\mathcal{P}}^{-1} \frac{\partial \mathcal{P}}{\partial z_1} \ldots \frac{\partial \mathcal{P}}{\partial z_s} \bar{\mathcal{P}}^{-1} \frac{\partial \bar{\mathcal{P}}}{\partial \bar{z}_1} \ldots \frac{\partial \bar{\mathcal{P}}}{\partial \bar{z}_s} \] 

where:

\[ \mathcal{P} = \prod_{i=1}^{s} (1-z_i)^{m_2-j_2} \prod_{i<j} (z_i-z_j), \]

(3.7)

The derivatives of the above expression come from the bosonization of \( \beta \) (2.28).

We can show that the following identity holds:

\[ \mathcal{P}^{-1} \frac{\partial^s \mathcal{P}}{\partial z_1 \ldots \partial z_s} = \frac{\Gamma(-j_2 + m_2 + s)}{\Gamma(-j_2 + m_2)} \prod_{i=1}^{s} (1-z_i)^{-1}. \]

(3.9)

The proof uses the definition of the Vandermonde:

\[ \prod_{i=1}^{s} (z_i - z_j) = \sum_{\sigma(p(1), \ldots, p(s))} \text{sign}(p) z_{p(1)}^{s-1} \ldots z_{p(s)}^{s-1}, \]

(3.10)

where the sum goes over all permutations of the indices. Inserting this expression for the Vandermonde in the definition of \( \mathcal{P} \) lead after a simple calculation, to (3.9).

We introduce the notation \( \Delta(x) = \Gamma(x)/\Gamma(1-x) \) and \( \rho = -2/\alpha_+^2 = -1/(k-2) \). The complete expression for the amplitude is:

\[ \mathcal{A}_{j_1 \ m_2 \ m_3}^{j_1 \ j_2 \ j_3} = (-)^s \Delta(j_2 - m_2 + 1) \Delta(j_3 - m_3 + 1) \mathcal{I}(j_1, j_2, j_3, k). \]

(3.11)

We have used the identity:
\[
\left( \frac{\Gamma(-j_2 + m_2 + s)}{\Gamma(-j_2 + m_2)} \right)^2 = (-)^s \Delta(j_2 - m_2 + 1) \Delta(j_3 - m_3 + 1) \quad (3.12)
\]

which holds for integer \( s \). The remaining integral:

\[
\mathcal{I}(j_1, j_2, j_3, k) = M^s \Gamma(-s) \int \prod_{i=1}^s d^2 z_i |z_i|^{-4 \rho j_1} |1 - z_i|^{-4 \rho j_2} \prod_{i < j} |z_i - z_j|^{-4 \rho} \quad (3.13)
\]

can be solved using the Dotsenko-Fateev (B.9) formula of [17]. A careful treatment of the regularization for the case \( k = 9/4 \) is needed. We will discuss this solution and the analytic continuation to arbitrary \( s \) later. From the above simple result, we can already see that \( SL(2, \mathbb{R}) \) fusion rules appear (see for example the appendix of [8]). If the second tachyon is also in the highest-weight module, i.e. \( m_2 = j_2 - \mathbb{N} \), then the result vanishes unless the conjugate of \( j_3 \) is also in the discrete representation i.e. \( M = J - \mathbb{N} \), where \( (J, M) = (-1 - j_3, -m_3) \). We will now see how this generalizes for an arbitrary three-point correlator, where a proportionality to a Clebsch-Gordan coefficient appears.

### 3.3. Three-Point Function of Arbitrary Tachyons

After getting an expression for the three-point function containing one highest-weight state and two generic tachyons, we would like to see how we can obtain the amplitude of three generic tachyons. Acting with the lowering operator \( J_0^- = \oint J^{-}(z) dz \) we compute the amplitude \( \mathcal{A}_{j_1 - k_1, m_1}^{j_2, j_3} \), where the holomorphic \( m_1 = j_1 - k_1 \) dependence has been changed by an integer \( k_1 \). We will make an analytic continuation in \( k_1 \) to non-integer values, while for the time being \( s \) will be taken as an integer. Our computation shows that the general three-point function of (not necessarily on shell) tachyons has the form:
where $C$ is essentially the SL(2, R) Clebsch-Gordan coefficient, whose expression we now calculate. The square takes the antiholomorphic $\bar{m}$ dependence into account. We will use the Baker-Campbell-Hausdorff formula:

$$e^{\alpha_J} V_{j m} e^{-\alpha_J} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} [J^0, V_{j m}]_k,$$

where we have defined

$$[J^0, V_{j m}]_0 = V_{j m}, \quad [J^0, V_{j m}]_k = [J^0, [J^0, V_{j m}]_{k-1}].$$

The lowering operator $J^0$ acts on the holomorphic part of the vertex operators:

$$[J^0, V_{j m}] = -(j+m) V_{j m-1}, \quad [J^0, V_{j m}]_k = (-)^k \frac{\Gamma(j+m+1)}{\Gamma(j+m-k+1)} V_{j m-k}.$$

We are going to use the fact that $J^0$ commutes with the screening charge $Q$, which is actually only true up to a total derivative. The surface terms that appear are discussed in appendix A. It is shown that they can be neglected in a particular region of $(j_i, m_i)$, and the other regions can be obtained by analytic continuation [26]. With the above formulas and the fact that $J^0$ annihilates the vacuum, we get an identity for the general amplitude as a function of the one with one highest weight state, if we identify in powers of $\alpha$ the r.h.s. and l.h.s. of:

$$\mathcal{A}_{m_1, m_2, m_3}^{j_1, j_2, j_3} = C^2 \mathcal{I}(j_1, j_2, j_3, k),$$

(3.14)
\[
\sum_{k_2, k_3 = 0}^{\infty} \frac{\alpha^{k_2 + k_3}}{k_2! k_3!} \frac{\Gamma(j_2 + m_2 + 1)}{\Gamma(j_2 + m_2 + 1 - k_2)} \frac{\Gamma(j_3 + m_3 + 1)}{\Gamma(j_3 + m_3 + 1 - k_3)} A_{j_1 - k_1, m_1, m_1}^{j_2, j_3} = \\
\sum_{k_1 = 0}^{\infty} \frac{(-\alpha)^{k_1}}{k_1!} \frac{\Gamma(2j_1 + 1)}{\Gamma(2j_1 + 1 - k_1)} A_{j_1, m_2, m_3}^{j_2, j_3}.
\]

(3.18)

Taking into account the antiholomorphic \(\bar{m}\) dependence and formula (3.11) we get:

\[
A_{j_1 - k_1, m_2, m_3}^{j_2, j_3} = \left( \frac{\Gamma(-2j_1)}{\Gamma(-j_1 - m_1)} \sum_{n=0}^{k_1} \frac{\Gamma(k_1 + 1)}{n! \Gamma(k_1 + 1 - n)} \frac{\Gamma(j_2 + m_2 + 1)}{\Gamma(j_2 + m_2 + 1 - n)} \right)^2 \mathcal{I}(j_1, j_2, j_3, k)
\]

(3.19)

Our aim is to give up the condition that \(k_1\) is an integer. First we notice that the above sum can be extended to \(\infty\) and written in terms of the generalized hypergeometric function \(3F_2\) (B.5), which has a definition in terms of the Pochhammer double-loop contour integral that possesses a unique analytic continuation to the whole complex plane of all its indices [15]. Comparing (3.14) and (3.19):

\[
\mathcal{C} = \frac{\Gamma(-2j_1)}{\Gamma(-j_1 - m_1)} \frac{\Gamma(j_3 + m_3 + 1)}{\Gamma(-j_2 + m_2)} \frac{\Gamma(j_1 + j_3 + m_2 + 1)}{\Gamma(-j_1 + j_3 - m_2 + 1)}
\]

\[
\lim_{x \to 1} 3F_2(j_2 - m_2 + 1, m_1 - j_1, -j_2 - m_2; -j_1 - j_3 - m_2, -j_1 + j_3 - m_2 + 1 | x).
\]

(3.20)

The appearance of the generalized hypergeometric function in our result is natural, since this function is always present in the theory of \(\text{SL}(2, \mathbb{R})\)-Clebsch-Gordan
coefficients [15]. It can be expanded in $s$, which will be useful to check the analytic
continuation in $k_1$ in simple examples, as we do in appendix A. In general $3F_2$ has a
complicated expansion as a sum of $\Gamma$ functions. Fortunately, for on shell tachyons,
which is the case we are interested in, we get a simple result.

We will consider three tachyons on the right branch satisfying $j \geq -1/2$. To
fulfil the $m$ conservation law (3.4), we take without loss of generality $m_2 \leq 0$. If
we choose $k = 9/4$ the on shell conditions are (2.40):

$$2j_2 + 1 = -\frac{2}{3}m_2, \quad 2j_i + 1 = \frac{2}{3}m_i \quad \text{for} \quad i = 1, 3. \quad (3.21)$$

This fixes $j_2$ as a function of the screening,

$$j_2 = \frac{s}{2} - \frac{1}{4}, \quad m_2 = \frac{3}{2}s - \frac{3}{4}. \quad (3.22)$$

With this kinematic relations it is easy to check that the generalized hypergeometric
function is well-poised and we can apply Dixon’s theorem [27] to simplify $3F_2$. With
(3.21) it is straightforward to obtain:

$$C^2 = \rho^{2s} \left( \frac{\Gamma(-2j_1)}{\Gamma(-2j_1 + s)} \right)^2 \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \prod_{i=1}^{3} \Delta \left( 2j_i + \frac{5}{4} \right). \quad (3.23)$$

We now have to evaluate the remaining part of the amplitude (3.14). The integral
$\mathcal{I}(j_i, k)$ can be solved using the Dotsenko-Fateev (B.9) formula [17]:

$$\mathcal{I}(j_1, j_2, j_3, k) = M^s \Gamma(-s) \int \prod_{i=1}^{s} d^2 z_i |z_i|^{-4\rho j_1} |1 - z_i|^{-4\rho j_2 - 2} \prod_{i<j} |z_i - z_j|^{4\rho} \quad (3.24)$$

$$= \rho^{2s} (-\pi \Delta(1 - \rho)M)^s \left( \frac{\Gamma(-2j_1 + s)}{\Gamma(-2j_1)} \right)^{\frac{1}{2}} \mathcal{Y}_1 \mathcal{Y}_2,$$
where
\[ Y_{13} = \prod_{i=0}^{s-1} \Delta(-2j_1\rho+i\rho)\Delta(-2j_3\rho+i\rho) \quad , \quad Y_2 = \Gamma(-s)\Gamma(s+1) \prod_{i=0}^{s-1} \Delta((i+1)\rho)\Delta(-2j_2\rho+i\rho). \] (3.25)

This result holds for arbitrary level \( k \) but integer screenings, so that it has to be transformed in order to obtain an expression valid for a non-integer \( s \). This analytic continuation will be done à la Di Francesco and Kutasov [19], using the on shell condition and taking the kinematics into account. From the above definition we notice that \( Y_2 \) has dangerous singularities for \( k = 9/4 \), i.e. \( \rho = -4 \), while \( Y_{13} \) is well defined since \( j_1 \) and \( j_3 \) could be chosen arbitrarily.

We first consider \( Y_{13} \). Choosing \( j_1, j_3 \notin \mathbb{Z}/8 \) and the kinematic relations for \( \rho = -4 \), we obtain:
\[ Y_{13} = \rho^{2s-2} \prod_{i=1,3} \Delta(-8j_i - 2)\Delta\left(2j_i + \frac{3}{4}\right). \] (3.26)

The product \( Y_2 \) is more subtle because we have to find a regularization that preserves the symmetries of the theory (see [28] for related issues to this subject). We will shift the level \( k \) away from the critical value and introduce a small parameter \( \varepsilon \rightarrow 0 \), that can be set to zero for \( Y_{13} \). This can be done by setting \( \rho = -4 + 16\varepsilon \), i.e. \( k = 9/4 + \varepsilon \), and taking the limit \( \varepsilon \rightarrow 0 \). With this modification of the level, the total central charge will be of order \( \varepsilon \) and the Liouville field \( e^{\gamma\varphi_L} \) has to be taken into account in order to get an anomaly-free theory. For the on shell condition we make the general ansatz:
\[ 2j + 1 = \pm \frac{2}{3}m + \varepsilon r(j, \gamma), \] (3.27)

which in the limit \( \varepsilon \rightarrow 0 \) reduces to our previous result. Here \( r(j, \gamma) \) could be, in
principle, an arbitrary function of the \( j \)'s and the Liouville dressing \( \gamma \). With the kinematics (3.27), we get:

\[
s = 2j_2 + \frac{1}{2} + \frac{\varepsilon}{2} \tilde{r}
\]  

(3.28)

with \( \tilde{r} = r(j_1, \gamma_1) - r(j_2, \gamma_2) + r(j_3, \gamma_3) \). After simple transformations we obtain:

\[
\mathcal{Y}_2 = (-)^{s+1} \rho^{2s+1} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{4}{8})} \Delta(-8j_2 - 2) \Delta\left(2j_2 + \frac{3}{4}\right).
\]  

(3.29)

In the above formula there appears a multiplicative factor:

\[
\mathcal{R} = \frac{\Gamma(s + \frac{\tilde{r} + 4}{8})}{\Gamma(\frac{\tilde{r} + 4}{8}) \Gamma(s + 1)},
\]  

(3.30)

which comes from the regularization. We have no further constraint on the parameter \( \tilde{r} \) that appears in the above expression. The best we can do is to fix it by physical arguments. Choosing \( \tilde{r} = 4 \) will imply that the three-point function factorizes in leg factors and the four-point function will have obvious symmetry properties. This imposes strong constraints on \( \tilde{r} \). We will have to set \( \tilde{r} = 4 \) and the contribution of the renormalization factor is one. We will come back to this point when we discuss the \( N \)-point functions in our next publication.

Our result for the on shell tachyon three-point amplitude is obtained from (3.14), (3.24), (3.26) and (3.29):

\[
\mathcal{A}(j_1, j_2, j_3) = -\rho^{6s-1}(-\pi M \Delta(-\rho))^s \prod_{i=1}^{3} \Delta(-8j_i - 2) \Delta\left(2j_i + \frac{3}{4}\right) \Delta\left(2j_i + \frac{5}{4}\right).
\]  

(3.31)
which can be transformed finally to *:

\[ A(j_1, j_2, j_3) = \tilde{M}^s \prod_{i=1}^{3} \Delta(-4j_i - 1) \]  

(3.32)

where \( \tilde{M} = -\pi M \Delta(\rho) \rho^{-2} \).

We can translate our notation to the one used in \( c = 1 \). It has been shown [6] [10] that the black hole mass operator corresponds to a discrete state of \( c = 1 \) up to BRST exact terms. In this language the \( \beta-\gamma \) contribution of the correlators cancels out and the vertex operators can be written as:

\[ V_{p}^\pm = e^{(-\sqrt{2} \pm |p|)\phi + ipX}. \]  

(3.33)

Here \( \pm \) denotes the tachyon vertex operators on the right (wrong) branch, which represent the incoming (outgoing) wave at infinity [29]. We can write the result for the three-point function of on shell tachyons in the black hole background as:

\[ A(p_1, p_2, p_3) = \tilde{M}^s \prod_{i=1}^{3} \Delta(1 - \sqrt{2}|p_i|) \]  

(3.34)

Which can be compared with the \( c = 1 \) three-point function, with tachyonic background [19]:

\[ A(p_1, p_2, p_3) = [\mu \Delta(\rho)]^s \prod_{i=1}^{3} (-\pi) \Delta(1 - \sqrt{2}|p_i|) \]  

(3.35)

There appear several remarkable features in our result:

* We absorb a factor of two in the definition of the path integral
• In order to get finite correlation functions, the black hole mass has been infinitely renormalized, as is done for the cosmological constant $\mu$ in Liouville. Here it is known to be equivalent to the replacing $e^{-\sqrt{2}\varphi} \rightarrow \varphi e^{-\sqrt{2}\varphi}$, which may have physical interesting consequences. Perhaps this will be the case for the black hole model as well.

• From the zero mode integration of the $N$-point function in both theories, we see that the number of screenings for the black hole is twice the value of the $c = 1$ model.

• The amplitude can be factorized in leg poles, which (with this normalization) have resonance poles where the $c = 1$ discrete states are placed. The new discrete states of Distler and Nelson do not appear. The explanation of this is that these extra states are BRST trivial in the Wakimoto representation. This has been shown by Bershadsky and Kutasov for the first examples [10].

4. THE TWO-POINT FUNCTION

In this section we will calculate the two-point function and compare our results with standard non-critical string theory, the positions of the resonance poles that follow from the quantum mechanical analysis carried out by Dijkgraaf et. al. [5] and the correlators of the deformed matrix model of Jevicki and Yoneya [11]. If we have $c \leq 1$ matter coupled to Liouville theory the simplest way to construct a two-point function of on shell tachyons is to use the three-point function. One of the operators is then set to be the dressed identity and this is the derivative with respect to the cosmological constant $\mu$ of the two-point function [19]. In this case the situation is different because the interaction is not a tachyon but a discrete state of $c = 1$ [10] [6]. Fortunately we can construct the two-point function of (not necessarily on shell) tachyons in the black hole background. We will do that with two independent methods, as a double check of our computation.
4.1. Two-point Function with $s - 1$ Integrated Screenings

We can perform a direct computation, fixing the position of one of the screenings at $z_s = \infty$ and evaluating the remaining $(s - 1)$-integrals:

$$\langle V_{j_1 m_1 (0)} V_{j_2 m_2 (1)} \rangle = \lim_{z_s \to \infty} \langle V_{j_1 m_1 (0)} V_{j_2 m_2 (1)} \beta(z_s) \bar{\beta}(\bar{z}_s) e^{-\frac{2}{s+1} \phi(z_s, \bar{z}_s)} Q^{s-1} \rangle. \quad (4.1)$$

We can follow closely the steps of the previous computation, although in this case it will be much simpler. We use the representation (2.20) for the vertex operators and the bosonization formulas for the $\beta$-$\gamma$ system. The zero mode integrations give:

$$s = j_1 + j_2 + 1, \quad m_1 + m_2 = 0. \quad (4.2)$$

Due to the kinematic relations satisfied by the two-point amplitude, the part of the integrand coming from the $\beta$-$\gamma$ system can be written as:

$$P^{-1} \frac{\partial^s P}{\partial z_1 \ldots \partial z_s} P^{-1} \frac{\partial^s \bar{P}}{\partial \bar{z}_1 \ldots \partial \bar{z}_s} = (-)^s \Delta (1+j_1-m_1) \Delta (1+j_2-m_2) \prod_{i=1}^{s} |z_i|^{-2} |1-z_i|^{-2}. \quad (4.3)$$

where

$$P = \prod_{i=1}^{s} z_i^{m_1-j_i} (1-z_i)^{m_2-j_j} \prod_{i<j} (z_i - z_j). \quad (4.4)$$

This can easily be seen with the substitution $z_i = 1/y_i$. After evaluating the $\phi$ contractions and taking $z_s \to \infty$, the complete $1 \to 1$ amplitude is reduced to the evaluation of the following integral:
\[ \langle V_{j_1 m_1}(0)V_{j_2 m_2}(1) \rangle = (-)^s M^s \Gamma(-s) \Delta(1 + j_1 - m_1) \Delta(1 + j_2 - m_2) \]

\[ \int \prod_{i=1}^{s-1} d^2 z_i |z_i|^{-4j_1-i-2} |1 - z_i|^{-4j_2-2} \prod_{i < j}^{s-1} |z_i - z_j|^{4\rho}. \quad (4.5) \]

The result is well defined for \( \rho \neq -4 \) and can be obtained from (B.6):

\[ \langle V_{j_1 m_1}(0)V_{j_2 m_2}(1) \rangle = (-)^s M^s \Gamma(-s) \Delta(1 + j_1 - m_1) \Delta(1 + j_2 - m_2) \Gamma(s) \]

\[ (\pi \Delta(1 - \rho))^{s-1} \prod_{i=1}^{s-1} \Delta(i\rho) \prod_{i=0}^{s-2} \Delta(-2j_1\rho + i\rho) \Delta(-2j_2\rho + i\rho) \Delta(1 + \rho(s - i)). \quad (4.6) \]

In the next section we show that the two-point function of (not necessarily) on shell tachyons is only different from zero for \( j_1 = j_2 = j \), so that we set \( s = 2j + 1 \) and, from the conservation law \( m_1 = -m_2 = m \geq 0 \), we obtain for an arbitrary level:

\[ \langle V_{j m}(0)V_{j -m}(1) \rangle = (-\pi M \Delta(-\rho))^s \Delta(1+j-m) \Delta(1+j+m) s \Delta(1-s) \Delta(\rho s), \quad (4.7) \]

If \( k \neq 9/4 \) and the tachyons do not belong to a discrete representation of \( \text{SL}(2, \mathbb{R}) \), the above amplitude has one divergence, which appears for \( s \) integer and comes from the zero mode integration. For \( k = 9/4 \) we demand \( j \notin \mathbb{Z}/2 \), which implies that \( s \) is non-integer. This can always be done, since the above expression is well defined in this case. The final expression for on shell tachyons is:
\[ \langle V_j m(0) V_j -m(1) \rangle = \frac{\tilde{M}^{2j+1}}{2j+1} (\Delta(-4j-1))^2, \quad (4.8) \]

where \( \tilde{M} \) is the renormalized black hole mass previously defined. We will comment on this result in section (4.3).

4.2. TWO-POINT FUNCTION WITH S INTEGRATED SCREENINGS

We can obtain the answer from the three-point tachyon amplitude that contains one highest-weight state \( V_{j_1 j_1} \), taking the limit \( j_1 = i\varepsilon \to 0 \). In this way we are fixing only two points of the SL(2, \( \mathbb{C} \)) invariant 1 \( \to \) 1 amplitude. The result will contain a divergence coming from the volume of the dilation group [30] [25].

First we can show that this amplitude is diagonal by pushing \( J_0^+ \) and \( J_0^- \) through the correlator [15]. Here again we will use that the screening charge commutes with the currents. We obtain the relations

\[ \frac{\langle V_{j_2 m_2} V_{j_3 m_3+1} \rangle}{\langle V_{j_2 m_2+1} V_{j_3 m_3} \rangle} = \frac{j_2 - m_2}{j_3 - m_3} = \frac{j_3 + m_3 + 1}{j_2 + m_2 + 1}. \quad (4.9) \]

From the \( X \) zero mode integration we get \( m_2 = m \) and \( m_3 = -1 - m \), so that this equation has two solutions, one with \( j_2 = -1 - j_3 \), which contains no screenings, and another one with \( j_2 = j_3 \), which is a two-point function with \( s = 2j_2 + 1 \) screenings. The first one is normalized to one up to the divergence \( \Gamma(0) \) coming from the zero mode integral. We now compute the two-point function with screenings.

We first consider the case of arbitrary \( k \) and take the limit \( k \to 9/4 \) at the end of the calculation. From (3.11), we obtain for \( j_1 = i\varepsilon \):
\[
A^{j_2 \ j_3}_{m_2 \ m_3} = (-)^s \Delta(1 + j_2 - m_2) \Delta(1 + j_3 - m_3) I(j_2, j_3, \rho), \quad (4.10)
\]

where

\[
I(j_2, j_3, k) = M^s \Gamma(-s) \lim_{\varepsilon \to 0} \int \prod_{n=1}^s d^2 z_n |z_n|^{-4\rho \varepsilon i} |1 - z_n|^{-4\rho j_2 - 2} \prod_{n<m} |z_n - z_m|^{4\rho} =
\]

\[
-(\pi M \Delta(-\rho))^s \Delta(1 - s) \Delta(\rho s) \lim_{\varepsilon \to 0} \Delta(1 - 2\rho \varepsilon i) \Delta((\varepsilon i - j_2 + j_3)\rho) \Delta((\varepsilon i + j_2 - j_3)\rho).
\]

Here we have simplified the products of well defined \(\Delta\)-functions. To evaluate the limit we take into account (B.4) and the following representation of the delta distribution:

\[
\delta(j_2 - j_3) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + (j_2 - j_3)^2}.
\]

In total we obtain for the two-point function of generic (in general off shell) tachyons:

\[
A^{j_2 \ j_3}_{m_2 \ m_3} = 2\pi i \rho \delta(j_2 - j_3)(-\pi M \Delta(-\rho))^s \Delta(1 - s) \Delta(\rho s) \Delta(1 + j_2 - m_2) \Delta(1 + j_3 - m_3)
\]

\[
(4.13)
\]

where \(s = 2j_2 + 1\) and the level is arbitrary. This agrees with (4.7) up to a factor \(s\) and the volume of the dilation group \(\delta(j_2 - j_3)\).
4.3. Comparison to other Approaches

Using the $c = 1$ language, we obtain for the two-point function (4.8) of two Seiberg on shell tachyons in the black hole background\(^\star\),

\[
\langle \mathcal{V}_p^+(0)\mathcal{V}_{-p}^+(1) \rangle = \frac{\widetilde{M}^{\frac{|p|}{\sqrt{2}|p|}}}{(\Delta(1 - \sqrt{2}|p|))^2},
\]

which we now compare with different models.

The two-point function of $c = 1$ with cosmological constant is:

\[
\langle \mathcal{V}_p^+(0)\mathcal{V}_{-p}^+(1) \rangle = \frac{\mu^{\frac{\sqrt{2}|p|}{\sqrt{2}|p|}}}{(\Delta(1 - \sqrt{2}|p|))^2}.
\]

Comparing with the two-point function in the black hole background we find the same features as for the three-point function. The pole structure is the same as the one of the two-point function of $c = 1$, while the screenings differ by a factor of two.

The two-point function of the deformed matrix model is given by the expression [11]:

\[
\langle \mathcal{V}_p^+(0)\mathcal{V}_{-p}^+(1) \rangle \sim \frac{M^{\frac{|p|}{\sqrt{2}|p|}}}{\Delta \left(1 - \frac{|p|}{\sqrt{2}}\right)^2}.
\]

The position of the resonance poles are in agreement with the quantum mechanical analysis of Dijkgraaf et. al. [5]. Here only half of the states of $c = 1$ (the supplementary series) appear as poles in the leg factors. Both two-point functions can be reconciled if we take into account that we can renormalize the tachyon vertex operators in a different way according to the $\text{SL}(2,\mathbb{R})$ representation theory. If in the normalization (2.18) we use the on shell condition and take into account the antiholomorphic piece, we renormalize our operators as follows:

\[\text{\textsuperscript{*} We have absorbed a factor of 2 as before.}\]
\[ \mathcal{V}_p^+ \to \frac{\mathcal{V}_p^+}{\Delta \left( \frac{1}{2} - \frac{|p|}{\sqrt{2}} \right)} \]  

(4.17)

Here a regular function depending on \( p \) has been dropped, which is not determined by (2.18). The two-point function of these operators in the black hole background agrees with (4.16). It is simple to see how the \( N \)-point function of these differently normalized tachyons behaves. The \( N \)-point function with chirality \((+\ldots,+,-)\), will be divided by a factor \( \Delta(-s - (N - 1)/2) \), coming from the state with the opposite chirality. This will imply, that \((+\ldots,+,-)\) odd-point functions of these operators vanish for positive integer screenings, if they were previously finite. For even point functions this factor is of course irrelevant for integer \( s \) so that they are finite in this case. Which one is the correct physical normalization is to be clarified. For related problems see [25].

We can compare the result that we obtain for this three-point function with the deformed matrix model where these odd-point functions generically vanish. While the zero of our three-point function holds for integer screenings and has it’s origin in leg factors, this does not seem to be the case for the deformed matrix model, where the nonfactorizable part of the three-point function coming from the collective field theory has zeros for arbitrary odd-point functions [11]. We think it must be found a way out of that if we would like to reconcile both approaches.

5. CONCLUSIONS AND OUTLOOK

Using the Wakimoto free field representation of the \( \text{SL}(2,\mathbb{R})/\text{U}(1) \) Euclidean black hole, we have found that tachyon two- and three-point correlation functions share a remarkable analogy with the amplitudes of \( c = 1 \) coupled to Liouville at non-vanishing cosmological constant. This observation was made by Bershadsky and Kutasov for the bulk amplitudes, i.e. those amplitudes where no screening charge is needed to satisfy the total charge balance. In order to have non-vanishing
correlation functions, we have infinitely renormalized the black hole mass. This is a well-known phenomenon for \( c = 1 \) and in this case it has interesting physical consequences. Perhaps this is the case for the black hole mass as well, further investigation is desirable.

The amplitudes factorize in leg factors, which have poles at all the discrete states of \( c = 1 \). The new discrete states of Distler and Nelson [7] do not appear, because they are BRST trivial in the Wakimoto representation, as checked in [10] for the first examples. With our renormalization of the operators, we do not reproduce the pole structure of the correlators of the deformed matrix model [11]. Whether this discrepancy could be merely a normalization of the operators was discussed. The scaling of the correlators is different from \( c = 1 \), but can be reproduced with the substitution \( \mu^2 = M \).

Of course, there are a lot of interesting questions, suggested by these observations. An important question is to see whether this relation to \( c = 1 \) persists for the \( N \)-point function. Our preliminary analysis of the pole structure indicates that again all the poles of \( c = 1 \) appear. Further work is needed to obtain a closed expression for all \( N \)-point functions on the sphere. In particular, it will be nice to see if we are able to obtain the result using the Ward identities of \( c = 1 \). We will report on that in a later publication.

Furthermore it will be interesting to see whether it is possible to find a relation to the model of Vafa and Mukhi [31] and to use the powerful methods of topological field theory to evaluate correlators on higher genus in the continuum approach.

As an important application of the obtained two point function we consider the determination of the exact Penrose diagram for the quantum theory in the Minkowski case. Work in this direction is in progress.

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APPENDIX A

As an illustrative example we will consider in more detail the three-point function with one screening. In this case we have:

\[ m_1 + m_2 + m_3 = j_1 + j_2 + j_3 = 0 \]  
(A.1)

and the correlator is given by:

\[
\mathcal{A}_{j_1, j_2, j_3}^{m_1, m_2, m_3} = \int d^2 z \langle V_{j_1 m_1}(0) V_{j_2 m_2}(1) V_{j_3 m_3}(\infty) \beta(z) \overline{\beta}(\overline{z}) e^{-\frac{\alpha_+}{\alpha_+} \phi(z, \overline{z})} \rangle_{M=0} 
\]

\[ = M \Gamma(-s) \int d^2 z \left( \frac{j_1 - m_1}{z} - \frac{j_2 - m_2}{1 - z} \right) \left( \frac{j_1 - m_1}{z} - \frac{j_2 - m_2}{1 - z} \right) |z|^{-4\rho j_1} |1 - z|^{-4\rho j_2}. \]  
(A.2)

To evaluate this integral directly, without any restriction on the \( m \) dependence of the three vertex operators, we have to use partial integration. We get:

\[
\mathcal{A}_{m_1, m_2, m_3}^{j_1, j_2, j_3} = M \Gamma(-s) \left( \frac{-m_1 j_2 + m_2 j_1}{j_1} \right)^2 \int d^2 z |z|^{-4\rho j_1} |1 - z|^{-4\rho j_2 - 2} + \mathcal{B}(j_1, j_2) + \mathcal{B}^*(j_1, j_2) \]  
(A.3)

where \( \mathcal{B}(j_1, j_2) \) is the part coming from the boundaries of the region of integration.
It is proportional to:

\[ B(j_1, j_2) \sim \int d^2z \frac{\partial}{\partial \bar{z}} \left( |z|^{-4\rho j_1} |1 - z|^{-4\rho j_2 - 2} (1 - \bar{z}) \right). \]  

(A.4)

This integral can be evaluated using (see appendix A of [23]):

\[ \int_{\Sigma} d^2z \frac{\partial}{\partial \bar{z}} (f(z, \bar{z})) = -i \frac{1}{2} \oint_{\partial \Sigma} dz f. \]  

(A.5)

We obtain:

\[ B(j_1, j_2) \sim -\frac{i}{2} \lim_{\varepsilon \to 0} \oint_{\varepsilon} dz |z|^{-4\rho j_1} |1 - z|^{-4\rho j_2 - 2} (1 - \bar{z}) = \pi \lim_{\varepsilon \to 0} \varepsilon^{-8\rho j_2}, \]  

(A.6)

where the integral is around a small circle of radius \( \varepsilon \) around 1. The contribution of the surface term is zero for \( j_2 > 0 \), finite for \( j_2 = 0 \) and diverges for \( j_2 < 0 \). As argued by Green and Seiberg [26], one has to add a finite contact term in the case where the boundary terms are finite and an infinite term if they diverge in order to render the amplitude analytic. These contact terms can be avoided by calculating amplitudes in an appropriate kinematic configuration, where the contact terms are not needed, and then analytically continuing to the desired kinematics. For \( s > 1 \), our argumentation will be the same as for \( s = 1 \), and we will restrict the values of \( j \) to the regions where the boundary terms vanish. This also means that we will restrict to the kinematic regions where \( J_0^- \) commutes with the screening charge.

We can compare this result with the one, following from (3.14) which is based on the analytic continuation in \( k_1 = j_1 - m_1 \) to non-integer values. We obtain \( C \) for integer screenings expanding (3.20):
\[ C = (-)^s \frac{\Gamma(-2j_1)\Gamma(s + 1)}{\Gamma(-2j_1 + s)} \sum_{i=0}^{s} \frac{\Gamma(m_1 - j_1 + i)}{\Gamma(m_1 - j_1)} \Gamma(-m_2 - j_2 + i) \frac{\Gamma(-m_1 - j_1 + s - i)}{\Gamma(-m_1 - j_1)} \frac{\Gamma(m_2 - j_2 + s - i)}{\Gamma(m_2 - j_2)} \frac{(-)^i}{(s-i)!i!} \]  

(A.7)

which for \( s = 1 \) gives:

\[ C_{s=1} = \frac{j_1 m_2 - m_1 j_2}{j_1} \]  

(A.8)

With (3.14) and (3.13) the amplitude becomes (A.3), where \( B(j_1, j_2) = 0 \). With this explicit example, one can already see that the analytic continuation in \( k_1 \) is correct. The integral can be solved using (B.7) for \( m = 1 \) and the result is (3.32).
APPENDIX B

For convenience we will collect the identities of $\Gamma$ functions that we have used in our computations:

$$\Gamma(1 + z - n) = (-1)^n \frac{\Gamma(1 + z) \Gamma(-z)}{\Gamma(n - z)}, \quad n \in \mathbb{N}, \quad (B.1)$$

$$\prod_{i=0}^{n-1} (i+x) = \frac{\Gamma(n+x)}{\Gamma(x)}, \quad \Delta(x)\Delta(-x) = -\frac{1}{x^2}, \quad \Delta(x)\Delta(1-x) = 1, \quad \Delta(1+x) = -x^2\Delta(x) \quad (B.2)$$

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma(x + \frac{1}{2}), \quad \Delta(2x) = 2^{4x-1}\Delta(x)\Delta\left(x + \frac{1}{2}\right). \quad (B.3)$$

To regularize the result of the singular integrals we use:

$$\lim_{\varepsilon \to 0} \Gamma(-n + \varepsilon) = \frac{(-)^n}{\varepsilon \Gamma(n+1)} + \mathcal{O}(1) \quad \text{for} \quad n \in \mathbb{N}. \quad (B.4)$$

The definition of the hypergeometric function, which we need in section 3 to compute the three-point function of generic tachyons, is:

$$\begin{align*}
{}_{3}F_{2}(\alpha, \beta, \gamma; \rho, \sigma|x) &= \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \sum_{\nu=0}^{\infty} \frac{\Gamma(\alpha + \nu)\Gamma(\beta + \nu)\Gamma(\gamma + \nu)}{\Gamma(\rho + \nu)\Gamma(\sigma + \nu)} \frac{x^\nu}{\nu!}. \quad (B.5)
\end{align*}$$

The following identity, known as “Dixon’s theorem”, is useful to evaluate the three-point function of on shell tachyons:
3F2(a, b, c; 1 + a − b, 1 + a − c|1) = \frac{\Gamma(1 + \frac{a}{2})\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 - b - c + \frac{a}{2})}{\Gamma(1 + a)\Gamma(1 - b + \frac{a}{2})\Gamma(1 - c + \frac{a}{2})\Gamma(1 + a - b - c)}.

(B.6)

Finally we have used the Dotsenko-Fateev (B.9) formula to evaluate the integrals:

\frac{1}{m!} \int \prod_{i=1}^{m} \left( \frac{1}{2} i dz_i d\bar{z}_i \right) \prod_{i=1}^{m} |z_i|^{2\alpha} |1 - z_i|^{2\beta} \prod_{i<j}^{m} |z_i - z_j|^{4\rho} = \pi^{m} (\Delta(1 - \rho))^{m}

\prod_{i=1}^{m} \Delta(ip) \prod_{i=0}^{m-1} \Delta(1 + \alpha + ip)\Delta(1 + \beta + ip)\Delta(-1 - \alpha - \beta - (m - 1 + i)\rho). \quad (B.7)

REFERENCES

1. S. Hawking, “Particle Creation by Black Holes”, Commun. Math. Phys. 43 (1975) 199.
2. E. Witten, “String Theory and Black Holes”, Phys. Rev. D 44 (1991) 44.
3. E. Witten, “Two-dimensional String Theory and Black Holes”, Lecture given at the Conference on Topics in Quantum Gravity, Cincinnati, OH, Apr 3-4, 1992, hep-th/9206069.
4. J. Ellis, N. E. Mavromatos and D. V. Nanopoulos, “Quantum Coherence and Two-Dimensional Black Holes”, Phys. Lett. B 267(1991) 465.
5. R. Dijkgraaf, E. Verlinde and H. Verlinde, “String Propagation in Black Hole Geometry”, Nucl. Phys. B 371 (1992) 269.
6. T. Eguchi, H. Kanno and S. Yang, “W∞ Algebra in Two-Dimensional Black Hole”, Phys. Lett. B 298 (1993) 73.
7. J. Distler and P. Nelson, “New Discrete States of Strings Near a Black Hole”, *Nucl. Phys. B* **374** (1992) 123.

8. S. Chaudhuri and J. Lykken, “String Theory, Black Holes and SL(2,\(\mathbb{I}\R\)) Current Algebra”, *Nucl. Phys. B* **396** (1993) 270.

9. N. Marcus and Y. Oz, “The Spectrum of 2-D Black Hole or does the 2-D Black Hole have Tachyonic W Hair?”, Tel Aviv University preprint, TAVP-2046-93, [hep-th/9305003](https://arxiv.org/abs/hep-th/9305003).

10. M. Bershadsky and D. Kutasov, “Comment on Gauged WZW Theory”, *Phys. Lett B* **266** (1991) 345.

11. A. Jevicki and T. Yoneya, “A Deformed Matrix Model for the Black Hole Background in Two-Dimensional String Theory”, Santa Barbara Preprint NSF-ITP-93-67, [hep-th/9305109](https://arxiv.org/abs/hep-th/9305109); K. Demeterfi and J. P. Rodrigues, “States and Quantum Effects in the Collective Field Theory of the Deformed Matrix Model” Princeton Preprint PUPT-1407, [hep-th/9306141](https://arxiv.org/abs/hep-th/9306141); U. Danielsson, “A Matrix Model Black Hole”, CERN-TH-6916, [hep-th/9306063](https://arxiv.org/abs/hep-th/9306063); K. Demeterfi, I. Klebanov and J. P. Rodrigues, “The S Matrix of the Deformed c = 1 Matrix Model”, Princeton Preprint PUPT-1416, hep/th-9308036.

12. V. S. Dotsenko, “The Free Field Representation of the SU(2) CFT”, *Nucl. Phys. B* **338** (1990) 747; “Solving the SU(2) CFT with the Wakimoto Free Field Representation”, *Nucl. Phys. B* **358** (1991) 547.

13. M. Wakimoto, “Fock Representations of the Affine Lie Algebra \(A_1^{(1)}\)”, *Commun. Math. Phys.* **104** (1986) 605.

14. N. Ja. Vilenkin and A. U. Klimyk, “Representation of Lie Groups and Special Function”, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, 1992.

15. W. J. Holman and L. C. Biedenharn, “A general Study of the Wigner Coefficients of SU(1,1)”, *Ann. Phys* **47** (1968) 205; V. Bargman, “Irreducible unitary representations of the Lorentz group”, *Ann. Math.* **48** (1947) 568.
16. A. Gerasimov, A. Morozov, M. Olshanetsky and A. Marshakov, “Wess-Zumino-Witten Model as a Theory of Free Fields”, *Int. J. Mod. Phys.* A 5 (1990) 2495.

17. V. S. Dotsenko and V. A. Fateev, “Four point correlation function and the operator algebra in the 2D conformal invariant theories with central charge $c < 1$”, *Nucl. Phys.* B 251 (1985) 691; V. S. Dotsenko and V. A. Fateev, “Conformal algebra and multipoint correlation functions in 2D statistical models”, *Nucl. Phys.* B 240 (1984) 312.

18. M. Goulian and M. Li, “Correlation Functions in Liouville Theory”, *Phys. Rev. Lett.* 66 (1991) 2051.

19. P. Di Francesco and D. Kutasov, “World Sheet and Space Time Physics in Two Dimensional (Super-)String Theory”, *Nucl. Phys.* B 375 (1992) 119; “Correlation Functions in 2-D String Theory”, *Phys. Lett.* B 261 (1991) 385.

20. V. S. Dotsenko, “Correlation Functions of Local Operators in 2-D Gravity Coupled to Minimal Matter”, Summer School Cargèse 1991, hep-th/9110030; “Three Point Correlation Functions of Minimal CFT Coupled to 2-D Gravity, *Mod. Phys. Lett.* A 6 (1991) 3601.

21. D. Friedan, E. Martinec and S. Shenker, “Conformal Invariance, Supersymmetry and String Theory”, *Nucl. Phys.* B 271 (1986) 93.

22. D. Kutasov, “Geometry On the Space of Conformal Field Theories and Contact Terms”, *Phys. Lett.* B 220 (1989) 153.

23. G. Moore, “Finite in All Directions”, Yale Preprint YCTP-P12-93, hep-th/9305139

24. B. H. Lian and G. J. Zuckerman, “2D Gravity with $c = 1$ Matter”, *Phys. Lett.* B 266 (1991) 21.

25. P. Ginsparg and G. Moore, “Lectures in 2D Gravity and 2D String Theory”, Yale Preprint YCTP-P23-92, LA-UR-92-3479, hep-th/9304011.
26. M. B. Green and N. Seiberg, “Contact Interactions in Superstring Theory”, *Nucl. Phys.* B 299 (1988) 559.

27. A. Erdélyi, “Higher Trascendental Functions”, p. 188-189, New York: McGraw-Hill, 1953.

28. E. Martinec and S. L. Shatashvili, “Black Hole Physics and Liouville Theory”, *Nucl. Phys.* B368 (1992) 368.

29. T. Eguchi, “$c = 1$ Liouville Theory Perturbed by the Black Hole Mass Operator”, Tokyo Preprint UT 650, July 1993, [hep-th/9307185](http://arxiv.org/abs/hep-th/9307185).

30. N. Seiberg, “Notes on Quantum Liouville Theory and Quantum Gravity”, in Common Trends in Mathematics and Quantum Field Theory, Proc. of the 1990 Yukawa International Seminar, *Prog. Theor. Phys. Suppl.* 102 (1990) 319, and in Random Surfaces and Quantum Gravity, proceedings of 1990 Cargése Workshop, edited by O. Alvarez, E. Marinari and P. Windey.

31. S. Mukhi and C. Vafa, “Two Dimensional Black Hole as a Topological Coset of $c = 1$ String Theory”, Harvard preprint HUTP-93-4002, [hep-th/9301083](http://arxiv.org/abs/hep-th/9301083).