On the KP Hierarchy, $\hat{W}_\infty$ Algebra, and Conformal SL(2,R)/U(1) Model

I. The Classical Case

Feng Yu and Yong-Shi Wu

Department of Physics, University of Utah
Salt Lake City, Utah 84112, U.S.A.

ABSTRACT

In this paper we study the inter-relationship between the integrable KP hierarchy, nonlinear $\hat{W}_\infty$ algebra and conformal noncompact $SL(2,\mathbb{R})/U(1)$ coset model at the classical level. We first derive explicitly the Poisson brackets of the second Hamiltonian structure of the KP hierarchy, then use it to define the $\hat{W}_{1+\infty}$ algebra and its reduction $\hat{W}_\infty$. Then we show that the latter is realized in the $SL(2,\mathbb{R})/U(1)$ coset model as a hidden current algebra, through a free field realization of $\hat{W}_\infty$, in closed form for all higher-spin currents, in terms of two bosons. An immediate consequence is the existence of an infinite number of KP flows in the coset model, which preserve the $\hat{W}_\infty$ current algebra.
1 Introduction

This is the first of a two-paper series that give a unified description of our recent results [1-4] on the close relationship between the KP hierarchy and the $\hat{W}_\infty$ algebra, and their field theoretical realization in the 2d conformal non-compact $SL(2, R)/U(1)$ coset model. We will provide detailed proofs for theorems we have stated in our previous papers [1-4] and add some new results. In this paper we concentrate on the classical aspects of the problem. The generalization of the results to the quantum case and, in particular, the problem of quantizing (or quantum deforming) the KP hierarchy are left to the subsequent paper [5].

One of the most exciting recent developments in mathematical physics is the revelation of the close relationship between nonlinear integrable differential systems and 2d exactly solvable field theories, including 2d quantum gravity and string theory. The essential links between them turns out to reside in the common symmetry structure they share, which usually is described by infinite dimensional algebras and ensures the integrability of the differential systems, on one hand, and the exact solvability of the field theories, on the other. The first known example is the Virasoro algebra, the Lie algebra of the infinite dimensional conformal symmetry in two dimensions. On one hand, this algebra is known [6] to be isomorphic to the second Hamiltonian structure of the KdV (Korteweg-de Vries) equation or the KdV hierarchy. On the other hand, this algebra has been shown [7] to be the symmetry algebra, or more precisely the chiral algebra, of the so-called minimal conformal models in two dimensions, in the sense that the fields in these models form a finite number of irreducible representations (Verma modules) of this algebra. The Virasoro algebra is realized in these models as the algebra of a spin-2 current, namely the energy-momentum tensor. Similar situations occur for Zamolochikov’s extended $W_N$ algebras ($N \geq 3$) [8]. Each of them is an extension of the Virasoro algebra, which is generated by currents of spin $2 \leq s \leq N$; they are not Lie algebras, since there are nonlinear terms in the commutators of generators, but all of them contain the Virasoro algebra as a subalgebra. Again, it turns out that the extended conformal $W_N$ algebra is isomorphic to the second Hamiltonian structure of the $N$-th generalized KdV hierarchy [9], while it is shown [10] to be the chiral algebra of certain conformal models
called the $W_N$ minimal models, in the sense that fields in these models form a finite number of irreducible representations of $W_N$.

Since $W_N$ exists for any integer $N \geq 2$, Bakas [11] first raised the interesting question: Can we make sense of a large $N$ limit of the $W_N$ algebras? By now we know the answer is yes but not unique, depending on the way of defining the limit. All these “large $N$ limits” are now called the $W$ algebras of the infinite type or simply $W$-infinity algebras. Two years ago, when we entered the field, the list of known $W$-infinity algebras included: 1) a linear Lie algebra known as $w_\infty$ algebra obtained by Bakas [11]; 2) a linear deformation of Bakas’s $w_\infty$, known as $W_\infty$ algebra, proposed by Pope, Romans and Shen [12]; 3) a further linear extension of $W_\infty$ by incorporating an additional spin-1 current, again by Pope, Romans and Shen [13] who named it as $W_{1+\infty}$. All of them are (linear) Lie algebras. Though some simple free field representations of these algebras were known [14,13], but no connection to either integrable differential hierarchy or 2d conformal models had been made at that time.

It was very natural to us to try to generalize the above mentioned profound relationship between (infinite dimensional) extended conformal algebras, infinite integrable differential hierarchies and 2d conformal field theories from $W_N$ with finite $N$ to $W$-infinity algebras. We started with studying the problem of whether there is a “large $N$ limit” of $W_N$ which is related to the KP (Kadomtsev-Petviashvili) hierarchy [15] in way as $W_N$ related to the generalized KdV hierarchy. The reason we choose the KP hierarchy is twofold: First it is known [15] to contain all $N$-th generalized KdV hierarchies by reduction; in some sense one may view the KP hierarchy as a large $N$ limit of the generalized KdV hierarchies. Secondly, the latter appear in recent matrix model [16] or topological [17] approach to 2d quantum gravity and string theory in two remarkable ways: On one hand, the $N$-th KdV hierarchies can be used to formulate the so-called string equations that determine all physical correlation functions of a certain $c < 1$, $D < 2$ noncritical string [18]. On the other hand, the string partition function obeys a set of constraints satisfying the $W_N$ algebra [19]. In either way, one needs to consider an $N \rightarrow \infty$ limit to approach the famous $c = 1, D = 2$ barrier. Therefore it is generally believed that the KP hierarchy would play an significant role in the study of the $c = 1, D = 2$ string theory.
The results we have reported in our previous papers [1-4] can be summarized as follows: 1) The centerless $W_{1+\infty}$ can be identified [1] (also see [20]) as the first Hamiltonian structure of the KP hierarchy proposed by Watanabe [21]; an obvious reduction of the latter gives rise to $W_{\infty}$ [1]. 2) It is proved [2] that there exists a non-linear deformation of the $W_{\infty}$ algebra, which we call $\hat{W}_{\infty}$ and is believed to contain all $W_N$ by reduction. 3) This $\hat{W}_{\infty}$ algebra is identified [2] as the second KP Hamiltonian structure proposed by Dickey [22] (see also [23]). 4) We have found [3], in closed form for all higher spin currents, a two boson realization of the nonlinear $\hat{W}_{\infty}$ algebra. 5) Using it, we constructed a field theoretical realization of $\hat{W}_{\infty}$ as a hidden current algebra in the 2d $SL(2,R)/U(1)$ coset model. 6) In view of the well-known infinite set of KP Hamiltonians, considered as $\hat{W}_{\infty}$ charges that are in involution, the result 5) immediately implies the existence of infinitely many KP flows in the coset model, all of which preserve the $\hat{W}_{\infty}$ current algebra. In the following sections we will give a unified description and present the detailed proofs for the classical theory. The treatment of the quantum theory is left to the subsequent paper [5].

2 Bi-Hamiltonian Structure of the KP Hierarchy and $\hat{W}_{1+\infty}$ and $\hat{W}_{\infty}$

The integrable KP equation, originally proposed by Kadomtsev and Petviashvili [24], was generalized by Sato [15] to an integrable, infinite system, the so-called KP hierarchy. This set of nonlinear partial-differential equations can be nicely written in the form of a Lax pair, if one introduces a first-order pseudo-differential operator

$$L = D + \sum_{r=-1}^{\infty} u_r D^{-r-1}. \quad (2.1)$$

Here the coefficients $u_r$ are functions of $z$ and various time variables $t_m (m = 1, 2, 3, \ldots)$, and $D \equiv \partial/\partial z$ obeys the Leibniz rule $Df = fD + \partial_z f$ when acting on a regular function $f$. Recall that an arbitrary pseudo-differential operator $O$ of order $N$ has the formal
Laurent expression

\[ O = \sum_{s=-\infty}^{N} o_s D^s. \] (2.2)

The multiplication of two such operators is determined by the generalized Leibniz rule. \(D^{-1}\) is defined so that it satisfies \(DD^{-1} = D^{-1}D = 1\). It follows that, for positive \(r\),

\[ D^{-r}u = \sum_{l=0}^{\infty} \left(-\frac{r}{l}\right) u^{(l)} D^{-r-l} \] (2.3)

where \(u^{(l)} \equiv \partial^l_z u\). The KP hierarchy, written in terms of the operator (2.1), is a system of infinitely many evolution equations for the functions \(u_r(r = 0, 1, 2, \cdots)\)

\[ \frac{\partial L}{\partial t_m} = [(L^m)_+, L] \] (2.4)

where the subscript + denotes the purely differential part: i.e., for a operator (2.2), \(O_+ = \sum_{s=0}^{N} o_s D^s\), whereas \(O_-\) means the purely pseudo-differential part \(\sum_{s=-\infty}^{-1} o_s D^s\). It is easy to verify the equations in the KP hierarchy are compatible, namely different time evolutions of \(L\) commute with each other:

\[ \frac{\partial^2 L}{\partial t_m \partial t_n} = \frac{\partial^2 L}{\partial t_n \partial t_m}. \] (2.5)

Note that eq.(2.4) implies

\[ \frac{\partial L^N}{\partial t_m} = [(L^m)_+, L^N] \] (2.6)

for any positive integer \(N\). Imposing the constraint \((L^N)_- = 0\), eq.(2.6) becomes a set of differential equations in terms of an order-\(N\) differential KdV operator \(Q \equiv L^N = D^N + \sum_{i=0}^{N-1} q_i D^i\):

\[ \frac{\partial Q}{\partial t_m} = [(Q^{m/N})_+, Q] \] (2.7)

where \(Q^{m/N} \equiv (Q^{1/N})^m\) and \(Q^{1/N}\) is the unique pseudo-differential operator that satisfies \((Q^{1/N})^N = Q\). Hence the KP hierarchy (2.4) contains all the \(N\)th generalized KdV hierarchies (2.7) by reduction.

The most interesting feature of the KP hierarchy is its integrability. It is a direct consequence of its bi-Hamiltonian structure first proposed by Dickey [22]. To consider
the KP hierarchy from the Hamiltonian point of view, one starts with trying to put
\[ \frac{\partial L}{\partial t_m} = K \frac{\delta H_m}{\delta u}, \tag{2.8} \]
which is equivalent to rewriting the coefficients of
\[ [(L^m)_+, L] = \sum_{r=-1}^{\infty} K_r(m) D^{-r-1} \tag{2.9} \]
as
\[ K_r(m) = \sum_{s=1}^{\infty} k_{rs} \frac{\delta H_m}{\delta u_s} \tag{2.10} \]
where \( \delta / \delta u_s \) stands for the usual variational derivative
\[ \frac{\delta}{\delta u_s} = \sum_{k=0}^{\infty} (-\partial_z)^k \frac{\partial}{\partial u_s^{(k)}}. \tag{2.11} \]
The infinite dimensional operator matrix \( k_{rs} \) in eq.(2.10) gives rise to a Hamiltonian
structure if the associated Poisson brackets
\[ \{ u_r(z), u_s(z') \} = k_{rs} \delta (z - z') \tag{2.12} \]
form a closed algebra. Correspondingly, \( H_m \) in eq.(2.10) constitute an infinite set of
Hamiltonian functions in this Hamiltonian structure. Using eq.(2.12), we are able to
express eq.(2.8) in the desired canonical form:
\[ \frac{\partial u_r}{\partial t_m} = K_r(m) = \{ u_r(z), \oint H_m(z) dz \} \tag{2.13} \]
with appropriate boundary conditions for the integrands.

In search of Hamiltonian structures, one expands \( L^m \) as
\[ L^m = D^m + \sum_{s=-\infty}^{m-1} D^s U_s(m). \tag{2.14} \]
It follows that
\[ U_s(m) = \frac{1}{m+1} \frac{\delta \text{Res} L^{m+1}}{\delta u_s} \tag{2.15} \]
for $s \geq -1$. Here the residue, $\text{Res}O$, of a pseudo-differential operator $O$ means the coefficient of the $D^{-1}$ term in $O$. A simple calculation of the commutator $[(L^m)_+, L]$ in eq.(2.9) leads to the first Hamiltonian form of the KP hierarchy

$$K_r^{(1)}(m) = \frac{1}{m+1} \sum_{s=-1}^{\infty} k_{rs}^{(1)} \frac{\delta \text{Res}L^{m+1}}{\delta u_s}$$

$$= \{u_r, \frac{1}{m+1} \oint \text{Res}L^{m+1}(z) dz \}$$

(2.16)

where $k_{rs}^{(1)}$ is the Watanabe (or the first) Hamiltonian structure [21] and $(m+1)^{-1}\text{Res}L^{m+1}$ are corresponding Hamiltonian functions $H_m^{(1)}$. (The script 1 labels the first Hamiltonian structure, and similarly 2 will label the second one below.)

To obtain the second KP Hamiltonian form, let us express $[(L^m)_+, L]$ as

$$[(L^m)_+, L] = (LL^{m-1})_+L - L(L^{m-1}L)_+.$$ 

(2.17)

By substituting (2.14-15) for $L^{m-1}$ into the right hand side, we can recast it into the form of eq.(2.9) with

$$K_r^{(2)}(m) = \frac{1}{m} \sum_{s=-1}^{\infty} k_{rs}^{(2)} \frac{\delta \text{Res}L^m}{\delta u_s}$$

$$= \{u_r, \frac{1}{m} \oint \text{Res}L^m(z) dz \}$$

(2.18)

where $k_{rs}^{(2)}$ is the second Hamiltonian structure of Dickey [22] with $(1/m)\text{Res}L^m$ the associated Hamiltonian functions $H_m^{(2)}$.

These two KP hamiltonian structures are compactible in the sense that they can be linearly combined into a one-parameter family of Hamiltonian structures, called a bi-Hamiltonian structure [22]. To see this, for an arbitrary pseudo-differential operator $P$, we define a “Hamiltonian” operator

$$K(P) = (LP)_+L - L(PL)_+.$$ 

(2.19)

Note $K(L^{m-1})$ will give us the second Hamiltonian form (2.18). Shift $L$ by a constant:

$$\hat{L} \equiv L + c = D + \sum_{r=-1}^{\infty} \hat{u}_r D^{-r-1},$$

(2.20)
and define the corresponding

$$\hat{K}(P) = (\hat{L}P)\hat{L} - \hat{L}(P\hat{L})\hat{L}. \quad (2.21)$$

To obtain a one-parametric family of KP Hamiltonian structures labeled by $c$, let us substitute an operator $P$ of order $N$ into eq.(2.21)

$$P = \sum_{s=-\infty}^{N} D^s p_s \quad (2.22)$$

($N$ can be arbitrarily large) and rewrite (2.21) as

$$\hat{K}(P) = \sum_{r,s=-1}^{\infty} \hat{k}_{rs} p_s D^{-r-1}. \quad (2.23)$$

In the following we are going to show that for fixed pair $(r,s)$, the operator $\hat{k}_{rs}$ becomes stable (i.e. independent of $N$) when $N$ is sufficiently large. Thus, we can use these stable “entries” to form an infinite-dimensional operator matrix $\hat{k}_{rs}$, and use it to define the Poisson brackets underlying the bi-Hamiltonian structure for functions $u_r$:

$$\{u_r(z), u_s(z')\} = \hat{k}_{rs}(\hat{u}_r(z)) \delta(z - z'). \quad (2.24)$$

with $\hat{u}_{-1} = u_{-1} + c$ and $\hat{u}_r = u_r$ ($r \geq 0$) in the right side. It is easy to verify the skew-adjointness of $\hat{k}_{rs}$; the closure of $\hat{k}_{rs}$ was shown in ref. [22]. We will show that in the $c \to 0$ and $c \to \infty$ limit, one recovers the second and first KP Hamiltonian structure respectively.

After this general description, now let us derive the explicit expressions of the KP Hamiltonian structures in closed form. We start with evaluating the $\hat{k}_{rs}$ for the bi-Hamiltonian structure defined by eq.(2.24).

Proposition 1: For $r, s \geq 0$,

$$\hat{k}_{rs} = \sum_{l=0}^{s+1} \binom{s + 1}{l} D^l \hat{u}_{r+s+1-l} - \sum_{l=0}^{r+1} \binom{r + 1}{l} \hat{u}_{r+s+1-l} (-D)^l$$

$$+ \sum_{l=0}^{\infty} \sum_{t-l=0}^{l+s} (-1)^l \binom{t s}{l} \hat{u}_{t-l} - \sum_{t=r+1}^{r+s+1} \sum_{k=0}^{t-r-1} (-1)^{l+k} \binom{t-k-1}{l-k} \binom{s}{k} \hat{u}_{t-l} (-D)^l \hat{u}_{r+s-t},$$

8
\[ \hat{k}_{r-1} = - \sum_{l=1}^{r} \binom{r}{l} \hat{u}_{r-l}(-D)^l, \]
\[ \hat{k}_{-1s} = \sum_{l=1}^{s} \binom{s}{l} D^l \hat{u}_{s-l}, \quad \hat{k}_{-1} = -D. \]  \hfill (2.25)

Proof: First we observe that for a pseudo-differential operator \( P \) of order \( N \), eq. (2.21) involves only the following part of it: \( P_+ \equiv \sum_{s=0}^{\infty} D^s p_s + p_{-1}D^{-1} \). Here \( p_s \) (\( s > N \)) are formally set to zero. The advantage of this convention is that we can take \( N \) to be arbitrarily large without changing the resulting formula. Then obviously we have
\[ \hat{K}(P) = (\hat{L}P_+) \hat{L} - \hat{L}(P_+ \hat{L})_+ = \hat{L}(P_+ \hat{L})_- - (\hat{L}P_+)_- \hat{L}. \]  \hfill (2.26)

Substituting the explicit expressions of \( \hat{L} \) and \( P_+ \) into eq. (2.26), it becomes
\[ \hat{K}(P) = (D + \sum_{r=1}^{\infty} \hat{u}_r D^{-r-1}) \left( \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} D^s p_s \hat{u}_l D^{s-l} \right) + \sum_{t=1}^{\infty} p_{-1} D^{-t-1} \left( D + \sum_{l=1}^{\infty} \hat{u}_l D^{-l-1} \right) \]
\[ - (\sum_{r=1}^{\infty} \sum_{s=0}^{\infty} \hat{u}_r D^{-r-1} + \sum_{l=1}^{\infty} \hat{u}_l D^{-l-1}) \left( D + \sum_{t=1}^{\infty} \hat{u}_t D^{-t-1} \right) \]
\[ = (D + \sum_{r=1}^{\infty} \hat{u}_r D^{-r-1}) \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{t=s-k}^{\infty} \binom{s}{k} (\hat{u}_t p_s)^{(k)} D^{s-k-t-1} \]
\[ - \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hat{u}_r D^{-r-1+s} p_s (D + \sum_{t=1}^{\infty} \hat{u}_t D^{-t-1}) \]
\[ + \sum_{t=0}^{\infty} p_{-1} \hat{u}_t D^{-t-1} - \sum_{r=0}^{\infty} \hat{u}_r D^{-r-1} p_{-1} - p'_{-1} \]
where \( p'_{-1} \equiv \partial_2 p_{-1} \) and several terms involving \( \hat{u}_{-1} \) or \( p_{-1} \) have been cancelled. After moving every \( D \) to most left, it follows that
\[ \hat{K}(P) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{t=s-k}^{\infty} \binom{s}{k} \left[ (\hat{u}_t p_s)^{(k+1)} D^{s-k-t-1} + (\hat{u}_t p_s)^{(k)} D^{s-k-t} \right] \]
\[ - \sum_{s=0}^{\infty} \sum_{r=1}^{\infty} \sum_{l=0}^{\infty} \hat{u}_r p_s^{(l)} D^{-r-l+s} \]
\[ + \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \sum_{l=0}^{\infty} \sum_{t=s-k}^{\infty} \binom{-r-1}{l} \binom{s}{k} \hat{u}_r p_s^{(k+l)} D^{-r-l+s-k-t-2} \]
\[ - \sum_{s=0}^{\infty} \sum_{r=1}^{\infty} \sum_{l=0}^{\infty} \binom{-r-1}{l} \binom{s}{k} \hat{u}_r p_s^{(l)} D^{-r-l+s-t-2} \]
\[ + \sum_{r=0}^{\infty} \hat{u}_r p_{-1} D^{-r-1} - \sum_{r=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{-r-1}{l} \right) \hat{u}_r p_{-1}^{(l)} D^{-r-1} - p'_{-1} \]

Now we rearrange the ordering and dummy indices of the multiple summations so as to cast \( \hat{K}(P) \) into the form of eq.(2.23). We finally obtain

\[ \hat{K}(P) = \sum_{r,s=0}^{\infty} \sum_{l=0}^{s+1} \left( \frac{s+1}{l} \right) (\hat{u}_{r+s+1-l} p_{-1}^{(l)}) - \sum_{r=0}^{\infty} \sum_{l=0}^{r+1} (-1)^l \left( \frac{r+1}{l} \right) \hat{u}_{r+s+1-l} p_{-1}^{(l)} \]

\[ + \sum_{l=0}^{\infty} \left[ \sum_{t=0}^{l+s} (-1)^t \left( \frac{t-s-1}{l} \right) - \sum_{t=r+1}^{r+s+1} \sum_{k=0}^{t-r-1} (-1)^{t+k} \left( \frac{t-k-1}{l-k} \right) \left( \frac{s}{k} \right) \right] \cdot \hat{u}_{l-1} (\hat{u}_{r+s-l} p_{-1}^{(l)}) \cdot D^{-r-1} \]

\[ - \sum_{r=0}^{\infty} \sum_{l=0}^{s} \left( \frac{r}{l} \right) \hat{u}_{l-p_{-1}^{(l)}} D^{-r-1} + \sum_{s=0}^{\infty} \sum_{l=0}^{s} \left( \frac{s}{l} \right) (\hat{u}_{s-l} p_{-1}^{(l)}) - p'_{-1}. \] (2.27)

One can easily read eq.(2.25) off. (QED)

We note that eq.(2.25) is, though not manifestly, indeed skew-adjoint as required by the antisymmetry of the Poisson bracket, and is independent of \( N \).

Proposition 2:

\[ (\hat{k}_{rs}(z) + \hat{k}_{sr}(w)) \delta(z-w) = 0. \] (2.28)

We skip the proof which is straightforward.

Let us show that the KP bi-Hamiltonian structure (2.24) incorporates both the first and second Hamiltonian structures.

Proposition 3:

\[ \{ u_r(z), u_s(z') \}_2 = \lim_{c \to 0} \hat{k}_{rs}(z) \delta(z-z') \quad r, s \geq -1; \] (2.29)

\[ \{ u_r(z), u_s(z') \}_1 = \lim_{c \to \infty} \left( \frac{1}{c} \hat{k}_{rs}(z) \right) \delta(z-z') \quad r, s \geq 0. \] (2.30)

Proof: The first limit is expected, if we note that by taking it, eq.(2.21) coincides with (2.19) which leads to the second Hamiltonian structure. Therefore, \( k_{rs}^{(2)} = \hat{k}_{rs} \mid_{c=0} \).

For the second limit, we have

\[ \lim_{c \to \infty} \left( \frac{1}{c} \hat{K}(P) \right) = [P_+, L] \] (2.31)

The right-hand side with \( P = L^m \) gives rise to the first Hamiltonian structure. To obtain the explicit expressions from eq.(2.25), we use the fact that when \( c \to \infty \), only terms
which are bilinear in \( \hat{u} \)'s and involve the \( \hat{u}^{−1} = u^{−1} + c \) current survive:

\[
\lim_{c \to \infty} \left( \frac{1}{c} \hat{k}_{rs} \right) = \sum_{l=0}^{s} \binom{s}{l} D^l u_{r+s-l} - \sum_{l=0}^{r} \binom{r}{l} u_{r+s-l}(-D)^l. \tag{2.32}
\]

It is exactly the same as derived directly from eq.(2.31) [1]. (The \( u^{−1} \) current becomes decoupled from the algebra.) (QED)

Therefore, the linear first KP Hamiltonian structure is related to the nonlinear second KP Hamiltonian structure by contraction. In ref.[1], we have identified the former with the \( W_{1+\infty} \) algebra. When compared to usual generators of \( W_{1+\infty} \) we should assign spin \( r + 1 \) to the current \( u_r \) (\( r \geq 0 \)). We will call the second Hamiltonian structure (2.25) (with \( c = 0 \)) as the \( \hat{W}_{1+\infty} \) algebra, in which the current \( u_r \) is considered to have spin \( r + 2 \) (\( r \geq −1 \)). Despite the shift in spin, both algebras have the same spectrum for the spin of the currents: \( s \geq 1 \), since \( u^{−1} \) decouples and is not included in \( W_{1+\infty} \).

It has been convenient to keep \( u_{−1} \neq 0 \) in the discussion of the KP bi-Hamiltonian structure. However, from the dynamical point of view, \( u_{−1} \) is trivial. It does not evolve at all in accordance to the KP hierarchy (2.4): \( \partial u_{−1}/\partial t_m = 0 \). Furthermore, consider the action of \( L \) on the space of functions: \( (D + \sum_{r=−1}^{\infty} u_r D^{−r−1})(z)f(z) \). With a change in function \( f(z) \to \exp(-\int^z u_{−1}(z')dz')f(z) \), one can always remove \( u_{−1} \). Hence, without loss of generality, we may deal with the case with \( u_{−1} = 0 \). In this case, there appears no modification to the first Hamiltonian structure, as \( u_{−1} \) decouples from \( W_{1+\infty} \). Nevertheless, \( u_{−1} = 0 \) is a second class constraint in \( \hat{W}_{1+\infty} \). One needs the Dirac brackets to handle it. This results in

\[
\{u_r(z), u_s(z')\}_2 = k^{(2D)}_{rs}(z)\delta(z − z') \tag{2.33}
\]

Alternatively we may consider the modification in the second KP Hamiltonian structure induced by \( u_{−1} = 0 \). Eqs.(2.8)-(2.10) become

\[
\frac{\partial L}{\partial t_m} = (LL^{m−1})_+ L - L(L^{m−1}L)_+ \\
= \frac{1}{m} \sum_{r=−1}^{\infty} \sum_{s=0}^{\infty} k^{(2)}_{rs} \frac{\delta \text{Res} L^m}{\delta u_s} D^{−r−1} + [\text{Res} L^{m−1}, L]. \tag{2.34}
\]
The $D^0$ term in the left side vanishes, so does that in the right side. This gives

$$(Res L^{m-1})' = \frac{1}{m} \sum_{s=0}^{\infty} k_{-1s}^{(2)} \frac{\delta Res L^m}{\delta u_s}. \quad (2.35)$$

It allows us to express eq.(2.34) in the modified second Hamiltonian form

$$K_r^{(2D)}(m) = \frac{1}{m} \sum_{s=0}^{\infty} k_{rs}^{(2D)} \frac{\delta Res L^m}{\delta u_s} \quad r \geq 0 \quad (2.36)$$

with

$$k_{rs}^{(2D)} = k_{rs}^{(2)} - \sum_{l=1}^{r} (-1)^l \binom{r}{l} u_{r-l} D^{l-1} k_{-1s}^{(2)}. \quad (2.37)$$

Proposition 4: Eq.(2.37) is equivalent to (2.33).

Proof: By definition, \{u_r(z), u_s(z')\}_2D = k_{rs}^{(2D)}(z) \delta(z - z'). We need to show

$$k_{rs}^{(2D)} = k_{rs}^{(2)} - k_{r-1}^{(2)} k_{s-1-1}^{(2)} \quad (2.38)$$

From eq.(2.25), $k_{-1-1}^{(2)} = -D$, $k_{r-1}^{(2)} = -\sum_{l=1}^{r} (-1)^l \binom{r}{l} u_{r-l} D^l$. Eq.(2.38) is therefore identical to (2.37). (QED)

Explicitly for the modified second Hamiltonian structure we have

Proposition 5:

$$k_{rs}^{(2D)} = \sum_{l=0}^{s+1} \binom{s+1}{l} D^l u_{r+s+1-l} - \sum_{l=0}^{r+1} \binom{r+1}{l} u_{r+s+1-l} (-D)^l$$

$$+ \sum_{l=0}^{\infty} \sum_{t=l+1}^{l+s} (-1)^t \binom{t-s-1}{l} u_{t-l-1} D^l u_{r+s-t}$$

$$- \sum_{t=r+1}^{r+s} \sum_{k=0}^{t-r-1} (-1)^{l+k} \binom{t-k-1}{l-k} \binom{s}{k} u_{t-l-1} D^l u_{r+s-t}$$

$$- \sum_{l=1}^{r} \sum_{k=1}^{s} (-1)^l \binom{r}{l} \binom{s}{k} u_{r-l} D^{l+k-1} u_{s-k}. \quad (2.39)$$

Proof: The first four terms of eq.(2.39) rewrite $k_{rs}^{(2)}$ with $r, s \geq 0$ and $u_{-1} = 0$. The last term is obtained by substituting $k_{-1s}^{(2)} = \sum_{k=1}^{s} \binom{s}{k} D^k u_{s-k}$ into eq.(2.37). (QED)

As shown in ref.[2], the modified second KP Hamiltonian structure (2.33) can be viewed as the unique nonlinear, centerless deformation of $W_\infty$ (and $w_\infty$) under certain natural homogeneity requirements. Therefore we called it as the $\hat{W}_\infty$ algebra. It is a
nonlinear current algebra formed by all currents of spin \( s \geq 2 \). In this way we have seen that the \( \hat{W}_{1+\infty} \) is a universal \( W \)-algebra, in the sense that it contains all known \( W \)-algebras of the infinite type either by contraction or by reduction. Also it is conjectured \([2, 23]\) to contain all \( W_N \)-algebras of the finite type by reduction: Since \( W_N \) is known to be isomorphic to the second Hamiltonian structure of the \( N \)-th generalized KdV hierarchy (also modified by a condition similar to \( u_{-1} = 0 \)), and the KP hierarchy contains all generalized KdV hierarchies, it is natural to expect that the nonlinear \( \hat{W}_\infty \) reduces to \( W_N \) by imposing the second class constraints \( (L^N)_- = 0 \). (Technically it would not be easy to carry out this reduction because of the infinite number of constraints.) In view of this, \( \hat{W}_\infty \) seems physically more interesting than \( \hat{W}_{1+\infty} \). In the following we will focus on the former, and will suppress the scripts “2D” for the Poisson brackets associated with it when no confusion arises.

We note that with \( u_{-1} = 0 \), there is an interesting relation between \( \hat{W}_\infty \) and \( W_{1+\infty} \) as follows \([2]\):

\[
\oint \left\{ \hat{W}_{r+1}(z), \hat{W}_{s+1}(z') \right\}_{2D} dz' = \oint \left\{ W_r(z), W_s(z') \right\}_{1} dz',
\]

(2.40)

where the \( \hat{W}_\infty \)-current \( \hat{W}_{r+1} \) and \( W_{1+\infty} \)-current \( W_r \) \((r \geq 1)\) are respectively the Hamiltonian functions

\[
\hat{W}_{r+1} \equiv H_{r}^{(2D)} = \frac{1}{r} \text{Res} L_r; \quad W_r \equiv H_{r-1}^{(1)} = \frac{1}{r} \text{Res} L_r.
\]

(2.41)

This relation is a consequence of the coexistence of two KP Hamiltonian structures, which implies the generalized Lenard recursion relation in the KP hierarchy:

\[
\frac{\partial u_r}{\partial t_m} = K_{r}^{(2D)}(m) = K_{r}^{(1)}(m),
\]

(2.42)

or canonically, from eq.(2.13),

\[
\{ u_r, \oint H_{m}^{(2D)}(z) dz \}_{2D} = \{ u_r, \oint H_{m}^{(1)}(z) dz \}_{1}.
\]

(2.43)

By multiplying both sides of (2.43) by \( \delta \hat{W}_{s+1}/\delta u_r (= \delta W_s/\delta u_r) \), it is clear that (2.43) is identical to (2.40).

In conclusion of this section, we mention that the supersymmetric version of KP bi-Hamiltonian structure thus the complete integrability of a super KP hierarchy, and its relation to the super \( W \)-algebras of the infinite type have been worked out in refs.\([25,26]\)
3 Free Field Realization of $\hat{W}_\infty$

The main issue of this section is to construct a two boson realization of the nonlinear $\hat{W}_\infty$ algebra in closed form, which will play a crucial role in our later discussions.

To begin with, let us briefly review some known results of free field representations of $W$-algebras of both finite and infinite type, in terms of (pseudo-)differential operators. We will make extensive use of the techniques in operator calculus [15]. First, $W_{1+\infty}$ can be realized with a pair of fermions [13] $\bar{\psi}(z)$ and $\psi(z)$ satisfying the Poisson brackets

$$\{\bar{\psi}(z), \psi(z')\} = \delta(z - z'),$$
$$\{\bar{\psi}(z), \bar{\psi}(z')\} = \{\psi(z), \psi(z')\} = 0. \quad (3.1)$$

In the KP basis [1], the $W_{1+\infty}$ generators $u_r(z)$ are simply summarized by

$$L \equiv D + \sum_{r=0}^{\infty} u_r D^{-r-1} = D - \bar{\psi}D^{-1}\psi. \quad (3.2)$$

The Poisson brackets among $u_r$ from eq.(3.1) exactly yield eq.(2.30) plus (2.32). Now considering only the spin $s \geq 2$ currents in eq.(3.2), the $W_\infty$ generators $v_r$ naturally have a free fermion realization too:

$$(LD)_- \equiv \sum_{r=0}^{\infty} v_r D^{-r-1} = -(\bar{\psi}D^{-1}\psi)_-. \quad (3.3)$$

Meanwhile, $W_\infty$ can be realized in terms of a pair of free bosons [14] $\bar{\phi}(z)$ and $\phi(z)$, with their currents $\bar{j}(z) = \bar{\phi}'(z)$, $j(z) = \phi'(z)$ satisfying

$$\{\bar{j}(z), j(z')\} = \partial_z \delta(z - z');$$
$$\{\bar{j}(z), \bar{j}(z')\} = \{j(z), j(z')\} = 0. \quad (3.4)$$

The expression of $v_r(z)$ is given by

$$L \equiv D + \sum_{r=0}^{\infty} v_r D^{-r-1} = D + \bar{j}D^{-1}j. \quad (3.5)$$

The Poisson brackets among $v_r$ either from eqs.(3.4)-(3.5) or from (3.1) and (3.3) precisely lead to those of $W_\infty$, or the linear part of eq.(2.33) plus (2.39).
On the other hand, recall that for finite \( N \), the differential KdV operator of order \( N \) can be expressed via the Miura transformation [27] in terms of \( N \) free bosons \( \phi_i \):

\[
Q = D^N + \sum_{i=0}^{N-1} q_i D^i = \prod_{i=1}^{N} (D + j_i)
\]

(3.6)

where \( j_i = \phi'_i \). The second Hamiltonian structure of the \( N \)-th generalized KdV hierarchy (2.7) with \( q_i \) in eq.(3.6) as generators then has a free boson realization through the Poisson brackets among \( j_i \):

\[
\{ \hat{j}_i(z), \hat{j}_k(z') \} = \delta_{i,k} \partial_z \delta(z - z').
\]

(3.7)

The simple reduction of this Hamiltonian structure with the (second class) constraint \( q_{N-1} = 0 \) gives rise to the \( N - 1 \) free boson representation of the \( W_N \) algebra.

In turn, for the desired \( \hat{W}_\infty \) case, we notice the fact that the nonlinearities of \( \hat{W}_\infty \) emerge as deformation of \( W_\infty \), which is nicely realized by eq.(3.5) with only two free bosons. This motivates us to search for a free field realization of \( \hat{W}_\infty \) in terms of two scalars by adding higher nonlinear terms to (3.5). Our main result is summarized in the following KP operator (with \( u_{-1} = 0 \))

\[
L = D + \sum_{r=0}^{\infty} u_r D^{-r-1} = D + \frac{1}{\bar{j}D - (j + \bar{j})j}
\]

\[
= D + \bar{j}D^{-1}j + \bar{j}D^{-1}(\bar{j} + j)D^{-1}j + \cdots.
\]

(3.8)

written in terms of the chiral currents \( j(z) \) and \( \bar{j}(z) \).

In this way, the \( \hat{W}_\infty \) generators \( u_r \) are expressed as functions of \( \bar{j} \) and \( j \). Notice the first two terms, \( D + \bar{j}D^{-1}j \), realizes the linear \( W_\infty \) in the KP basis (3.5), while the remaining terms in \( \bar{j} \) and \( j \) represent the nonlinear deformation of \( \hat{W}_\infty \). Explicitly, the first a few \( u_r \) have the following expression:

\[
u_0 = \bar{j}j,
\]

\[
u_1 = -\bar{j}j' + \bar{j}j^2 + \bar{j}^2j,
\]

\[
u_2 = \bar{j}j'' - 3\bar{j}jj' - 2\bar{j}j^2j' - \bar{j}j'j + \bar{j}j^3 + 2\bar{j}^2j^2 + \bar{j}^3j,
\]

\[
u_3 = -\bar{j}j''' + 4\bar{j}jj'' + 3\bar{j}jj'^2 + 3\bar{j}^2j'' + 3\bar{j}^2j'j + \bar{j}j''j
\]

\[-6\bar{j}j^2j' - 9\bar{j}^2jj' - 3\bar{j}j'^2j - 3\bar{j}^3j' - 3\bar{j}^2j'j
\]

\[+\bar{j}j^4 + 3\bar{j}^2j^3 + 3\bar{j}^3j^2 + \bar{j}^4j.
\]

(3.9)
By checking the Poisson brackets among them, we find eq.(3.9) is in fact the unique two boson realization of these \( \hat{W}_\infty \) generators up to isomorphisms. Nevertheless the key issue is to prove the realization (3.9) for all generators \( u_r \). This amounts to showing that the Poisson brackets among \( u_r \), evaluated according to eq.(3.4), not only form a closed algebra, but also are identical to those of \( \hat{W}_\infty \) (2.33), the modified second Hamiltonian structure of the KP hierarchy. Using operator calculus, this is equivalent to proving the brackets between two functionals \( \oint f(u_r(z))dz \) and \( \oint g(u_s(z))dz \) satisfy

\[
\{ \oint f(u_r(z))dz, \oint g(u_s(z'))dz' \} = \oint \sum_{r,s=0}^{\infty} \frac{\delta f}{\delta u_r} k_{rs}^{(2D)}(z) \frac{\delta g}{\delta u_s}(z)dz.
\] (3.10)

First, the variation of the functional \( \oint fdz \) of \( u_r \) and their derivatives is given by

\[
\delta \oint fdz = \sum_{r=0}^{\infty} \oint \delta u_r \frac{\delta f}{\delta u_r}dz = \oint (\delta \overline{j} \frac{\delta f}{\delta j} + \delta j \frac{\delta f}{\delta \overline{j}})dz
\] (3.11)

where \( \delta f/\delta u_r \), etc, are the usual variational derivatives defined by eq.(2.11). Let us introduce the variational derivative with respect to the KP operator \( L \) to summarize the variations \( \delta f/\delta u_r \):

\[
\frac{\delta f}{\delta L} = \sum_{r=0}^{\infty} D^r \frac{\delta f}{\delta u_r}.
\] (3.12)

Then eq.(3.11) can be put into a neat form

\[
\delta \oint fdz = \oint \text{Res}(\frac{\delta f}{\delta L})dz
\] (3.13)

in which

\[
\delta L = \sum_{r=0}^{\infty} \delta u_r D^{-r-1}
\]

\[
= (1 + \overline{j} \frac{1}{D - (\overline{j} + j)}) \delta \overline{j} \frac{1}{D - (\overline{j} + j)} j
\]

\[
+ j \frac{1}{D - (\overline{j} + j)} \delta j (1 + \frac{1}{D - (\overline{j} + j)}).
\] (3.14)

Therefore we obtain the following expressions of \( \delta f/\delta \overline{j} \) and \( \delta f/\delta j \) by comparing eqs.(3.11) with (3.13):

\[
\frac{\delta f}{\delta \overline{j}} = \text{Res}(\frac{1}{D - (\overline{j} + j)} \frac{\delta f}{\delta L} (1 + \overline{j} \frac{1}{D - (\overline{j} + j)})) ,
\]

\[
\frac{\delta f}{\delta j} = \text{Res}((1 + \frac{1}{D - (\overline{j} + j)} j) \frac{\delta f}{\delta L} \overline{j} \frac{1}{D - (\overline{j} + j)}).
\] (3.15)
Now we proceed to evaluate the Poisson bracket

$$\{ \oint f \, dz, \oint g \, dz' \} = \oint \left( \frac{\delta f}{\delta j} \left( \frac{\delta g}{\delta j} \right)' + \frac{\delta f}{\delta j} \left( \frac{\delta g}{\delta j} \right)' \right) \, dz. \quad (3.16)$$

Replacing \((\delta g/\delta j)'\) by the commutator \([D - (\bar{j} + j), \delta g/\delta j]\), similarly for \((\delta g/\delta j)'\), and substituting eq.(3.15) for \(\delta f/\delta j\), \(\delta g/\delta j\), etc., into eq.(3.16), we have

$$\{ \oint f \, dz, \oint g \, dz' \} = \oint Res\{ \frac{1}{D - (\bar{j} + j)} \cdot \delta j \left( 1 + \bar{j} \cdot \frac{1}{D - (\bar{j} + j)} \cdot \delta g \right) \}

\begin{equation}
[(D - (\bar{j} + j))Res((1 + \frac{1}{D - (\bar{j} + j)} \cdot \delta g) \cdot \frac{1}{D - (j + j)}) - Res((1 + \frac{1}{D - (j + j)} \cdot \delta j) \cdot \frac{1}{D - (j + j)})]

+ (1 + \frac{1}{D - (j + j)} \cdot \delta j \cdot \frac{1}{D - (j + j)})\]

\begin{equation}
[(D - (\bar{j} + j))Res(\frac{1}{D - (j + j)} \cdot \delta g) \cdot \frac{1}{D - (j + j)} \cdot (1 + \bar{j} \cdot \frac{1}{D - (j + j)})] - Res(\frac{1}{D - (j + j)} \cdot \delta g \cdot \frac{1}{D - (j + j)} \cdot (1 + \bar{j} \cdot \frac{1}{D - (j + j)}) \cdot (D - (j + j)))]dz. \quad (3.17)
\end{equation}

It follows, by using the identities \(ResP = (P_-(D - (\bar{j} + j)))_+ = ((D - (\bar{j} + j))P_+)_+\),

$$\{ \oint f \, dz, \oint g \, dz' \}

= \oint Res\{ \frac{1}{D - (j + j)} \cdot \delta j \left( 1 + \bar{j} \cdot \frac{1}{D - (j + j)} \cdot \delta g \right) \}

\begin{equation}
[(D - (\bar{j} + j))((1 + \frac{1}{D - (j + j)} \cdot \delta j) \cdot \frac{1}{D - (j + j)}) \cdot (D - (\bar{j} + j)) +

- ((D - (\bar{j} + j))(1 + \frac{1}{D - (j + j)} \cdot \delta j) \cdot \frac{1}{D - (j + j)}) \cdot (D - (\bar{j} + j))] 

+ (1 + \frac{1}{D - (j + j)} \cdot \delta j \cdot \frac{1}{D - (j + j)})\]

\begin{equation}
[(D - (\bar{j} + j))((1 + \frac{1}{D - (j + j)} \cdot \delta j) \cdot \frac{1}{D - (j + j)}) \cdot (D - (\bar{j} + j)) +

- ((D - (\bar{j} + j))(1 + \frac{1}{D - (j + j)} \cdot \delta j) \cdot \frac{1}{D - (j + j)}) \cdot (D - (\bar{j} + j))]dz. \quad (3.18)
\end{equation}

A direct calculation gives

$$\{ \oint f \, dz, \oint g \, dz' \} = \oint Res[\bar{j} \cdot \frac{1}{D - (j + j)} \cdot \delta j \left( D - j \right) \cdot ((1 + \frac{1}{D - (j + j)} \cdot \delta j) \cdot \frac{1}{D - (j + j)})]$$
\[-\frac{\delta f}{\delta L}(1 + \bar{j}\frac{1}{D - (j + j)})((D - j)\frac{\delta g}{\delta L}\bar{j}\frac{1}{D - (j + j)}j+) + (1 + \bar{j}\frac{1}{D - (j + j)}j)\frac{\delta f}{\delta L}\bar{j}\frac{1}{D - (j + j)}j\frac{\delta g}{\delta L}(D - j) + (D - j)\frac{\delta f}{\delta L}\bar{j}\frac{1}{D - (j + j)}j\frac{\delta g}{\delta L}(1 + \bar{j}\frac{1}{D - (j + j)}j+)dz\]

\[= \oint Res[\frac{\delta f}{\delta L}(L\frac{\delta g}{\delta L}) + L - \frac{\delta f}{\delta L}L\frac{\delta g}{\delta L}(L - D)\frac{\delta g}{\delta L} + \frac{\delta f}{\delta L}((L - D)\frac{\delta g}{\delta L}) + L - \frac{\delta f}{\delta L}(L - D)]dz\]

where we have frequently applied the theorem

\[\oint Res[P, Q]dz = 0\]  

for arbitrary two operators \(P\) and \(Q\).

It remains to show that eq. (3.19) really leads to (3.10). Notice that the expression \((\frac{\delta g}{\delta L})_+ + L - (\frac{\delta g}{\delta L})_+\) in the first two terms of (3.19) is actually the (unmodified) second KP Hamiltonian form (2.19) or (2.21) with the operator \(P\) there being \(\delta g/\delta L\). So we need to identify the last two terms of (3.19) to be the modification induced by the constraint \(u_{-1} = 0\).

On one hand, the last two terms of (3.19) are equal to

\[\oint Res[\frac{\delta f}{\delta L}(\bar{j}\frac{1}{D - (j + j)})\bar{j}\frac{\delta g}{\delta L}\bar{j}\frac{1}{D - (j + j)}j - \bar{j}\frac{1}{D - (j + j)}j\frac{\delta g}{\delta L}\bar{j}\frac{1}{D - (j + j)}j+]dz\]

\[= \oint Res[\frac{\delta f}{\delta L}\bar{j}\frac{1}{D - (j + j)}Res(\frac{\delta g}{\delta L}L - L\frac{\delta g}{\delta L})\frac{1}{D - (j + j)}j]dz. \]  

(3.21)

On the other hand, parallel to the analysis in the last section, if we denote the unmodified second KP Hamiltonian form as

\[K(P) = (LP)_+ - L(PL)_+ \]  

(3.22)

where \(P_+ \equiv \frac{\delta g}{\delta L}\) and \(P_- = (ResP)D^{-1}\), the corresponding modified Hamiltonian form (with \(u_{-1} = 0\)) appears to be

\[K(P_+) = ((LP)_+ - L(PL)_+)_-\]

\[= ((L\frac{\delta g}{\delta L})_+ - L(\frac{\delta g}{\delta L})_+ + [ResP, L])_-\]  

(3.23)

18
Note that

\[(ResP)' = Res(\frac{\delta g}{\delta L} L - L \frac{\delta g}{\delta L})\]  

(3.24)

Moreover, in the realization (3.8), we simply have

\[[ResP, L]_- = \bar{j}(ResP \frac{1}{D - (j + j)} - \frac{1}{D - (j + j)} ResP)\] 

(3.25)

Thus, by substituting eq.(3.24) into (3.25) and comparing it with (3.21), we obtain

\[\{ \oint fdz, \oint gdz' \} = \oint Res[\frac{\delta f}{\delta L} ((L\frac{\delta g}{\delta L})_+ L - L(\frac{\delta g}{\delta L})_+ + [ResP, L])]dz\] 

(3.26)

\[= \oint Res(\frac{\delta f}{\delta L} K(\frac{\delta g}{\delta L}))dz\]

as desired.

Finally we come up with

**Proposition 6:** Eq.(3.10) holds true.

**Proof:** The modified second Hamiltonian form (3.23) can be rewritten as (recall \(\delta g/\delta L\) is defined by eq.(3.12))

\[K(\frac{\delta g}{\delta L}) = \sum_{r,s=0}^{\infty} k_{rs} \frac{\delta g}{\delta u_s} D^{-r-1}.\]

(3.27)

Substituting eqs.(3.27) and (3.12) into (3.26), we have

\[\{ \oint fdz, \oint gdz' \} = \oint Res(\sum_{r,s,t=0}^{\infty} D^t \frac{\delta f}{\delta u_t} k_{rs} \frac{\delta g}{\delta u_s} D^{-r-1})dz\]

(3.28)

\[= \oint \sum_{r,s=0}^{\infty} \frac{\delta f}{\delta u_t} k_{rs} \frac{\delta g}{\delta u_s} dz.\]

From Propositions 4 and 5 in last section, the infinite dimensional matrix \(k_{rs}\) in eqs.(3.27) and (3.28) is none but the \(\hat{W}_\infty\) structure (2.39). (QED)

This proves our two free boson realization of the \(\hat{W}_\infty\) algebra (3.8) to all orders.
4 $\hat{W}_\infty$ and $SL(2, R)/U(1)$ Coset Model

As is well known, infinite dimensional algebras, such as the Virasoro algebra [6], Kac-Moody algebra [28] and extended $W_N$ algebras [8], have shown up in several large classes of 2d conformal field models, such as minimal models [7], Wess-Zumino-Witten models [29] and some compact coset models [30], and have played a central role in solving them. It is natural to explore the connection between the $\hat{W}_\infty$ algebra and more general conformal models. In this section, we are going to show that $\hat{W}_\infty$ indeed arises in the noncompact $SL(2, R)/U(1)$ coset model [31], and to present a construction of all $\hat{W}_\infty$ currents in closed form, which exploits the known free boson realization of the coset model [32,33].

Recall the boson realization of the $SL(2, R)_k$ current algebra at the classical level

$$J_\pm = \sqrt{\frac{k}{2}} e^{\pm \sqrt{\frac{k}{2}} \phi_3 (\phi_1' \mp i \phi_2')} e^{\pm \sqrt{k} \phi_1},$$

$$J_3 = -\sqrt{\frac{k}{2}} \phi_3; \quad \text{\textnormal{(4.1)}}$$

with $\phi_i$ denoting three free bosons [32]. When taking the coset $SL(2, R)_k/U(1)$ or gauging $U(1)$, one considers only vertex operators that commute with the $U(1)$ current $J_3$. This is equivalent to imposing the restriction $J_3 = 0$ or simply $\phi_3 = 0$. Thus we are interested only in the $J_\pm$ part of eq.(4.1), which now depend only on two bosons $\phi_1 = (1/\sqrt{2})(\phi + \bar{\phi})$ and $\phi_2 = (1/\sqrt{2i})(\phi - \bar{\phi})$. They become just the $SL(2, R)_k/U(1)$ parafermion currents [33]:

$$\psi_+ = je^{\bar{\phi} + \phi}, \quad \psi_- = je^{-\bar{\phi} - \phi}. \quad \text{\textnormal{(4.2)}}$$

Here and below, we set the level parameter, $k$, to unity. This does not lose any generality at the classical level. To generate higher-spin $\hat{W}_\infty$ currents from the $SL(2, R)/U(1)$ currents (4.2), we propose to study their ordinary bilocal product $\psi_+(z)\psi_-(z')$, and expand this product in powers of $z - z' \equiv \epsilon$ to all orders:

$$je^{\bar{\phi} + \phi}(z)je^{-\bar{\phi} - \phi}(z') = \sum_{r=0}^{\infty} c_r(z) \frac{\epsilon^r}{r!}. \quad \text{\textnormal{(4.3)}}$$
This expansion is the classical counterpart of the operator product expansion. Our main result is that the bilocal product (4.3) is a generating function of $\hat{W}_\infty$, in the sense that the coefficient functions $c_r(z)$ in (4.3), as functions of $\bar{j}$ and $\hat{j}$, are nothing but the $\hat{W}_\infty$ generators $u_r(z)$ realized by eq.(3.8) or (3.9):

$$c_r(z) = u_r(z). \quad (4.4)$$

To give a compact notation, it is convenient to introduce the generating function of a pseudo-differential operator $P(z) = \sum_{r=0}^\infty p_r(z)D^{-r-1}$ as a bilocal function $F(z, z') = \sum_{r=0}^\infty p_r(z)(z - z')^r/r!$, and denote this by $P(z) \iff F(z, z')$.

Then eq.(4.4) can be restated as

Proposition 7:

$$\bar{j} \frac{1}{D - (\bar{j} + j)} j(z) \iff \bar{j} e^{\tilde{\phi} + \phi}(z) j e^{-\tilde{\phi} - \phi}(z'). \quad (4.5)$$

Proof: Let us write down the expansion of the right hand side

$$\bar{j} e^{\tilde{\phi} + \phi}(z) j e^{-\tilde{\phi} - \phi}(z')$$

$$= \bar{j} e^{\tilde{\phi} + \phi} \sum_{n=0}^\infty \frac{(-1)^n}{n!} j^{(n)} \epsilon^n e^{-\sum_{m=0}^\infty \frac{(-1)^m}{m!} (\tilde{\phi} + \phi)^{(m)} \epsilon^m}$$

$$= \sum_{n=0}^\infty \sum_{k=0}^n \frac{(-1)^n}{n!k!} \bar{j} j^{(n)} \epsilon^n \sum_{m=0}^\infty \frac{(-1)^m}{(m + 1)!} (\bar{j} + j)^{(m)} \epsilon^{m+1})^k \quad (4.6)$$

where $\epsilon = z - z'$. As anticipated, eq.(4.6) only depends on the chiral currents $\bar{j}$ and $j$ (not on $\tilde{\phi}$ and $\phi$ themselves). The expansion of the left side of eq.(4.5) is

$$\bar{j} \frac{1}{D - (\bar{j} + j)} j(z) = \sum_{k=0}^\infty \bar{j} (D^{-1}(\bar{j} + j))^k D^{-1} j. \quad (4.7)$$

So, by counting the powers in $\bar{j}$ and $j$ in eqs.(4.6) and (4.7), it is sufficient to show

$$\bar{j} (D^{-1}(\bar{j} + j))^k D^{-1} j$$

$$\iff \sum_{n=0}^\infty \frac{(-1)^n}{n!k!} \bar{j} j^{(n)} \epsilon^n \sum_{m=0}^\infty \frac{(-1)^m}{(m + 1)!} (\bar{j} + j)^{(m)} \epsilon^{m+1})^k. \quad (4.8)$$

Indeed, by moving all factors $D^{-1}$ in the left hand side to the most right, we have

$$\bar{j} (D^{-1}(\bar{j} + j))^k D^{-1} j$$

$$= \sum_{m_1, m_2, \ldots, m_k, n=0}^\infty \left( -1 \right)^{m_1} \binom{\ldots}{m_1 \ldots \frac{m_2}{n}} \left( \ldots \frac{-m_1 - m_2 - \cdots - m_k - k - 1}{n} \right) \bar{j}(\bar{j} + j)^{(m_1)} (\bar{j} + j)^{(m_2)} \ldots (\bar{j} + j)^{(m_k)} j^{(n)} D^{-m_1 - m_2 - \cdots - m_k - n - k - 1}. \quad (4.9)$$
By applying the identity 
\[
\begin{pmatrix}
-\frac{a}{b}
\end{pmatrix} = (-1)^b \begin{pmatrix}
a + b - 1
\end{pmatrix}
\]
and then totally symmetrizing the indices \(m_i\), it follows that

\[
\bar{j}(D^{-1}(\bar{j} + j))^k D^{-1} j = \sum_{m_1, m_2, \ldots, m_k, n=0}^{\infty} (-1)^{m_1+m_2+\cdots+m_k+n} \frac{(m_1 + m_2 + \cdots + m_k + n + k)!}{(m_1 + 1)!(m_2 + 1)! \cdots (m_k + 1)! n! k!} \]

\[
\bar{j}(\bar{j} + j)(m_1)(\bar{j} + j)(m_2) \cdots (\bar{j} + j)(m_k) j^{(n)} D^{m_1-m_2-\cdots-m_k-n-k-1}
\]

\[
\Leftrightarrow \sum_{m_1, m_2, \ldots, m_k, n=0}^{\infty} (-1)^{m_1+m_2+\cdots+m_k+n} \frac{(m_1 + 1)!(m_2 + 1)! \cdots (m_k + 1)! n! k!}{(m_1 + 1)! (m_2 + 1)! \cdots (m_k + 1)! n! k!} \]

\[
\bar{j}(\bar{j} + j)(m_1)(\bar{j} + j)(m_2) \cdots (\bar{j} + j)(m_k) j^{(n)} e^{m_1+m_2+\cdots+m_k+k+n}. \quad (4.10)
\]

On the right side all \(m_i\) sums factorize, and we obtain the right side of eq.(4.8). (QED)

We can rewrite eqs.(4.3) and (4.4) as

\[
u_r(z) = \psi_+(z)(-\partial_z)^r \psi_-(z), \quad (4.11)
\]

or according to the KP operator

\[
L = D + \sum_{r=0}^{\infty} u_r D^{-r-1} = D + \psi_+ D^{-1} \psi_- . \quad (4.12)
\]

It gives us a construction of all the \(\hat{W}_\infty\) currents (in the KP basis), in a very compact form, in terms of the basic \(SL(2, R)/U(1)\) parafermion or coset currents. Combining Propositions 7 and 6, these currents must satisfy the \(\hat{W}_\infty\) algebra, or their Poisson brackets must be isomorphic to those of the (modified) second KP Hamiltonian structure. This is the first time in the literature that a current algebra with infinitely many independent currents is shown to exist in a noncompact coset model, in closed form. It also follows

\[
\{ j e^{\tilde{\phi}+\phi}(z) j e^{-\tilde{\phi}-\phi}(z - \epsilon), j e^{\tilde{\phi}+\phi}(w) j e^{-\tilde{\phi}-\phi}(w - \sigma) \} = \sum_{r,s}^{\infty} k_{rs}(z) \delta(z - w) \frac{\epsilon^r \sigma^s}{r!s!}. \quad (4.13)
\]

It is possible to verify this by direct calculation of the left side.
5 Involutive KP charges and $\hat{W}_\infty$ Symmetry in the $SL(2, R)/U(1)$ Coset Model

We have shown that the $\hat{W}_\infty$ algebra appears in both the integrable KP hierarchy and the conformal $SL(2, R)/U(1)$ model. This actually implies a close connection between the KP hierarchy and the coset model. In particular, KP flows are expected to be realizable by two free boson fields in the coset model.

Let us start with the integrability of the KP hierarchy. First, in the KP Hamiltonian form (2.13), the Hamiltonian functions

$$H_m \equiv H_m^{(2)} = H_m^{(1)} = \frac{1}{m} \text{Res} L^m$$

(5.1)

are conserved charge densities. Namely their integrals are invariant under the KP flows:

$$\frac{\partial}{\partial t_n} \oint H_m(z, t_k) dz = 0.$$  (5.2)

Secondly, these infinitely many charges

$$Q_m \equiv \oint H_m(z) dz$$

(5.3)

are in involution - they are independent, commuting charges - with respect to both the first and second Hamiltonian structures (regardless of whether $u_{-1} = 0$ or $u_{-1} \neq 0$, though we will present the formulas only for the former case)

$$\{Q_n^{(2)}, Q_m^{(2)}\}_2 = \{Q_n^{(1)}, Q_m^{(1)}\}_1 = 0.$$  (5.4)

This ensures the compatibility and complete integrability of the KP hierarchy (2.4).

One can prove either of eqs.(5.2) and (5.4) first, then the other follows. For example, with the substitution $f = H_n$ and $g = H_m$ in eq.(3.26), it becomes

$$\{Q_n^{(2)}, Q_m^{(2)}\}_2 = \oint \text{Res}(L^{n-1}K(L^{m-1})) dz = \oint \text{Res}(L^{n-1}([L^m]_+ + [\text{Res} P, L])) dz = 0$$

where we have used $\delta((1/m) \text{Res} L^m)/\delta L = L^{m-1}$ and the identity (3.20). Then the recursion relations (2.40) tells $\{Q_n^{(1)}, Q_m^{(1)}\}_1 = 0$ and eq.(5.2) follows by applying eq.(2.13).
The charge densities (5.1) are just the $\hat{W}_\infty$-currents, $\hat{W}_{r+1}$, in the Hamiltonian basis (2.41). The existence of the infinite set of corresponding involutive KP charges (5.3) implies the existence of as many commuting $\hat{W}_\infty$ charges in the $SL(2,R)/U(1)$ coset model. These $\hat{W}_\infty$ charges give rise to a huge infinite dimensional symmetry in this noncompact conformal model, which is generated by the multi-time KP flows

$$\frac{\partial u_r}{\partial t_m} = \{u_r, Q_m\}. \quad (5.5)$$

Or, infinitesimally under a $\hat{W}_\infty$ transformation we have

$$\delta_m u_r = \epsilon_m \{u_r, Q_m\} \quad (5.6)$$

with $\epsilon_m$ the infinitesimal parameters. An important property of these symmetry transformations is that the $\hat{W}_\infty$ current algebra (2.33) is preserved under these flows: We have, infinitesimally,

$$\frac{\partial}{\partial t_m} \left( \{u_r(z), u_s(z')\} - k_{rs}(z)\delta(z - z') \right) = \{\{u_r(z), Q_m\}, u_s(z')\} - \{k_{rs}(z)\delta(z - z'), Q_m\}$$

$$= \{\{u_r(z), u_s(z')\}, Q_m\} - \{k_{rs}(z)\delta(z - z'), Q_m\} = 0. \quad (5.7)$$

We also notice that these symmetries are abelian to each other, in the sense that the KP flow in one charge direction is invariant under the transformation generated by another $\hat{W}_\infty$ charge: i.e.,

$$\delta_m \left( \frac{\partial u_r}{\partial t_n} - \{u_r, Q_n\} \right) = \epsilon_m \left( \frac{\partial u_r}{\partial t_n}, Q_m \right) - \{\delta_m u_r, Q_n\}$$

$$= \epsilon_m \{\{u_r, Q_n\}, Q_m\} - \epsilon_m \{\{u_r, Q_m\}, Q_n\} = \epsilon_m \{u_r, \{Q_n, Q_m\}\} = 0. \quad (5.8)$$

These properties are consequences of eqs.(5.2) and (5.4); especially, eqs.(5.7) and (5.8) verify the compatibility of the KP flows. We note that these results hold for the $W_{1+\infty}$ charges as well, simply because of the recursion relations (2.43) or (2.40).

Furthermore, the free boson realization (3.8)-(3.9) or the bosonized parafermion generalization (4.3)-(4.4) of the $\hat{W}_\infty$ currents allows us to have a natural representation of the Hamiltonian functions (5.1) in terms of $\bar{j}$ and $j$. The first few currents read

$$H_1 = u_0 = \bar{j}j,$$
\[ H_2 = u_1 + \frac{1}{2}u'_0 = \frac{1}{2}(\bar{j}'j - \bar{j}j') + \bar{j}^2 j + \bar{j}j^2, \]
\[ H_3 = u_2 + u'_1 + \frac{1}{3}u''_0 + u'_0 = \frac{1}{3}(\bar{j}''j - \bar{j}'j' + \bar{j}j'') \]
\[ \bar{j}''j + \bar{j}'j^2 - \bar{j}^2 j' - \bar{j}jj' + \bar{j}^3 j + 3\bar{j}'j^2 + \bar{j}j^3, \]
where \( H_1 = \hat{W}_2 \) is the energy-momentum tensor in the classical \( SL(2, R)/U(1) \) model. Using the corresponding expressions of the charges \( Q_m \) in terms of \( \bar{j} \) and \( j \), one can easily summarize the KP flows (5.5) for \( u_r \) by the following flows of \( j \) and \( \bar{j} \):
\[ \frac{\partial \bar{j}}{\partial t_m} = \{ \bar{j}, Q_m \}, \quad \frac{\partial j}{\partial t_m} = \{ j, Q_m \} \quad (5.10) \]
with \( Q_m \) here the integrals of eq.(5.9). The first few equations in this hierarchy read
\[ \frac{\partial \bar{j}}{\partial t_1} = \bar{j}', \quad \frac{\partial j}{\partial t_1} = j', \]
\[ \frac{\partial \bar{j}}{\partial t_2} = 2(\bar{j}j)' + (\bar{j}'j' + \bar{j}''), \]
\[ \frac{\partial j}{\partial t_2} = 2(\bar{j}j)' + (j'^2)' - j'''. \quad (5.11) \]
One may use the equations (5.10) to define an integrable hierarchy whose hamiltonian structure is simply the Poisson brackets (3.4) for \( \bar{j} \) and \( j \). From the hierarchy (5.10), one can obtain all the composite KP evolutions (5.5) for \( u_r \). We may call this hierarchy as the two-boson reduced KP hierarchy. (See also ref.[34].)

Similarly, the \( \bar{j}-j \) hierarchy (5.10) is invariant under the \( \hat{W}_\infty \) transformations
\[ \delta_{m}\bar{j} = \epsilon_m \{ \bar{j}, Q_m \}, \quad \delta_m j = \epsilon_m \{ j, Q_m \}; \quad (5.12) \]
and the fundamental brackets (3.4) (and further the composite \( \hat{W}_\infty \) brackets) are invariant under the \( \bar{j}-j \) flows: e.g.,
\[ \frac{\partial \{ \bar{j}, j \}}{\partial t_m} = \{ \{ \bar{j}, Q_m \}, j \} + \{ \bar{j}, \{ j, Q_m \} \} = \{ \{ \bar{j}, j \}, Q_m \} = 0. \quad (5.13) \]
Incidentally we point out that based on the free fermion realization (3.2) of \( W_{1+\infty} \), we have alternative free field expressions for the KP Hamiltonians (5.1); e.g.,
\[ H_1 = \bar{\psi}\psi, \quad H_2 = \frac{1}{2}(\bar{\psi}'\psi - \bar{\psi}\psi'), \]
\[ H_3 = \frac{1}{3}(\bar{\psi}''\psi - \bar{\psi}'\psi' + \bar{\psi}\psi''). \quad (5.14) \]
\footnote{Some recent discussions on this issue based on eq.(3.8) are given in ref.[34].}
Their charges can also be used to generate a compatible and integrable $\bar{\psi}$-$\psi$ hierarchy

$$\frac{\partial \bar{\psi}}{\partial t_m} = \{\bar{\psi}, Q_m\}, \quad \frac{\partial \psi}{\partial t_m} = \{\psi, Q_m\}.$$  \hspace{1cm} (5.15)

It has similar invariance properties to those of the $\bar{j}$-$j$ hierarchy. Exactly the same way, we may also have a parafermion reduced KP hierarchy $-\psi_+ - \psi_-$ flows according to eq.(4.12).

To conclude, we make two remarks. First, our results in this paper may have interesting applications to physics, in particular to string theory. By now it is well-known that the noncompact $SL(2, R)/U(1)$ model provides us a world-sheet sigma model for the 2D string theory with black hole interpretation [35]. It is conceivable that the infinite $\hat{W}_\infty$ symmetry or charges would be useful in the discussion on the spectrum of 2D black holes [36]. Secondly, though our treatment in this paper is restricted to the classical theory, it gives us some crucial hints, as we will see in the subsequent paper [5], about how to construct a quantum deformation of the KP hierarchy.

Acknowledgement

We thank I. Bakas for useful discussions. The work was supported in part by U.S. NSF-grant PHY-9008452.

REFERENCES

1. F. Yu and Y.-S. Wu, Phys. Lett. B263 (1991) 220.
2. F. Yu and Y.-S. Wu, Nucl. Phys. B373 (1992) 713.
3. F. Yu and Y.-S. Wu, Phys. Rev. Lett. 68 (1992) 2996.
4. F. Yu and Y.-S. Wu, Utah preprint UU-HEP-92/11, May 1992, to be published in Phys. Lett. B.
5. F. Yu and Y.-S. Wu, Utah preprint UU-HEP-92/12, August 1992.

6. J. L. Gervais, Phys. Lett. B160 (1985) 277; B. A. Kuperschmidt, Phys. Lett. A109 (1985) 417.

7. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B241 (1984) 333; Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. B240 [FS 12] (1984) 312; B. L. Feigin and D. B. Fuchs, Func. Aual. Appl. 16 (1982).

8. A. B. Zamolodchikov, Theor. Math. Phys. 65 (1985) 1205.

9. I. M. Gelfand and L. A. Dickey, Russ. Math. Surv. 30 (1975) 77; Funct. Anal. Appl. 10 (1976) 259; I. M. Gelfand and I. Dorfman, Funct. Anal. Appl. 15 (1981) 173; B. A. Kupershmidt and G. Wilson, Invent. Math. 62 (1981) 403.

10. V. A. Fateev and S. L. Lykyanov, Int. J. Mod. Phys. A3 (1988) 507; A. A. Belavin, Adv. Study Pure Math. 19 (1989) 117.

11. I. Bakas, Phys. Lett. B228 (1989) 57; Comm. Math. Phys. 134 (1990) 487.

12. C. Pope, L. Romans and X. Shen, Phys. Lett. B236 (1990) 173; Nucl. Phys. B339 (1990) 191.

13. C. Pope, L. Romans and X. Shen, Phys. Lett. B242 (1990) 401.

14. I. Bakas and E. Kiritsis, Nucl. Phys. B343 (1990) 185.

15. M. Sato, RIMS Kokyuroku 439 (1981) 30; E. Date, M. Jimbo, M. Kashiwara and T. Miwa, in Proc. of RIMS Symposium on Nonlinear Integrable Systems, eds. M. Jimbo and T. Miwa, (World Scientific, Singapore, 1983); G. Segal and G. Wilson, Publ. IHES 61 (1985) 1.

16. D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127; M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635; E. Brezin and V. Kazakov, Phys. Lett. B236 (1990) 144.
17. E. Witten, Nucl. Phys. B340 (1990) 281; R. Dijkgraaf and E. Witten, Nucl. Phys. B342 (1990) 281.

18. M. Douglas, Phys. Lett. B238 (1990) 176; T. Banks, M. Douglas, N. Seiberg and S. Shenker, Phys. Lett. B238 (1990) 279.

19. E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 457; J. Goeree, Nucl. Phys. B358 (1991) 737.

20. K. Yamagishi, Phys. Lett. B259 (1991) 436.

21. Y. Watanabe, Ann. di Mat. Pura Appl. 86 (1984) 77.

22. L. A. Dickey, Annals New York Academy of Sciences, 491 (1987) 131.

23. J. M. Figueroa-O’Farrill, J. Mas and E. Ramos, Phys. Lett. B266 (1991) 298; preprint BONN-HE-92/20, US-FT-92/7 or KUL-TF-92/20.

24. B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Doklady 15 (1970) 539.

25. F. Yu, Nucl. Phys. B375 (1992) 173.

26. F. Yu, J. Math. Phys. 33 (1992) 3180.

27. V. Drinfel’d and V. Sokolov, Sov. Probl. Mat. 24 (1984) 81.

28. V. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B247 (1984) 83; D. Gepner and E. Witten, Nucl. Phys. B278 (1986) 493.

29. E. Witten, Comm. Math. Phys. 92 (1986) 455.

30. P. Goddard, A. Kent and D. Olive, Phys. Lett. B152 (1985) 88; Comm. Math. Phys. 103 (1986) 105.

31. L. Dixon, J. Lykken and M. Peskin, Nucl. Phys. B235 (1989) 215.

32. A. Gerasimov, A. Marshakov and A. Morozov, Nucl. Phys. B328 (1989) 664.

33. O. Hernández, Phys. Lett. B233 (1989) 355.
34. D. A. Depireux, Laval preprint LAVAL-PHY-21-92; J. M. Figueroa-O’Farrill, J. Mas and E. Ramos, preprint BONN-HE-92/17, US-FT-92/4 or KUL-TF-92/26.

35. E. Witten, Phys. Rev. D44 (1991) 314.

36. J. Ellis, N. Mavromatos and D. Nanopoulos, Phys. Lett. B272 (1991) 261; B276 (1992) 56; B284 (1992) 27.