Growth Kinetics for a System with a Conserved Order Parameter:
Off Critical Quenches

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Abstract

The theory of growth kinetics developed previously is extended to the asymmetric case of off-critical quenches for systems with a conserved scalar order parameter. In this instance the new parameter $M$, the average global value of the order parameter, enters the theory. For $M = 0$ one has critical quenches, while for sufficiently large $M$ one approaches the coexistence curve. For all $M$ the theory supports a scaling solution for the order parameter correlation function with the Lifshitz-Slyozov-Wagner growth law $L \sim t^{1/3}$. The theoretically determined scaling function depends only on the spatial dimensionality $d$ and the parameter $M$, and is determined explicitly here in two and three dimensions. Near the coexistence curve oscillations in the scaling function are suppressed. The structure factor displays Porod’s law $Q^{-(d+1)}$ behaviour at large scaled wavenumbers $Q$, and $Q^4$ behaviour at small scaled wavenumbers, for all $M$. The peak in the structure factor widens as $M$ increases and develops a significant tail for quenches near the coexistence curve. This is in
qualitative agreement with simulations.
I. INTRODUCTION

In previous work [1] a theory based on a field-theoretic Langevin model was developed to treat the growth kinetics of a system with a conserved scalar order parameter for the case of symmetric or critical quenches. In this paper the lowest order version of this theory is extended to off-critical quenches. Quenches to a final state near the coexistence curve where the volume fraction of the minority phase is small have been studied by a variety of approaches, but theoretical studies of the Langevin model have never been extended into this regime. The techniques developed thus far are generalizations of the Lifshitz-Slyosov-Wagner (LSW) treatment [2,3] which considers one spherical droplet interacting with a mean concentration field. This approach is valid only in the limit of zero volume fraction, but other mean field theoretic and statistical mechanical techniques have been developed to incorporate the effects of the interaction of other droplets and extend the theory to slightly larger volume fractions [4–12]. Another approach to the problem is to use numerical simulations in concert with a theory describing the concentration field around spherical droplets (essentially an electrostatics problem with moving boundary conditions) [13]. Direct simulations of the Langevin equation exist for the off-critical case in two dimensions [14,15], but we are not aware of any such simulations for three dimensions [16].

The theory developed in [1] shows how one can solve some of the thorny problems associated with growth kinetics for the conserved order parameter (COP) case. The theory can be evaluated as a well-defined sequence of approximations with qualitative and quantitative improvement as one moves along this sequence [17]. In this paper we limit ourselves to the lowest order approximation in this formalism. From the work in [1] we know that there are substantial limitations associated with this approximation and these are discussed in some detail in section VI. However, it is also known from [1] that this approximation gives good results for the scaling function for correlations of the order parameter. We therefore concentrate on this quantity here.

The new element in this work compared to [1] is that the average value of the scalar...
order parameter $\psi$ is no longer zero:

$$\langle \psi(R, t) \rangle = M. \quad (1.1)$$

$M$ is independent of $R$ and $t$ because of the statistical homogeneity of the system and the conservation law, respectively.

The main results of this paper are that, as in the critical case, the theory supports a long-time scaling solution for the order parameter correlation function

$$C(R, t) = \langle \delta \psi(R, t) \delta \psi(0, t) \rangle = \psi_0^2 F\left( \frac{|R|}{L(t)} \right) \quad (1.2)$$

where $\delta \psi = \psi - M$, $\psi_0$ is the magnitude of the ordered field in equilibrium, and $L(t)$ is the characteristic length in the theory, the average size of the domains. For later convenience we define the normalized quantity $\tilde{M} = M/\psi_0$, which ranges between $-1$ and $1$. When $\tilde{M} = \pm 1$ the system is at the coexistence curve. It is found that, for all $\tilde{M}$ and for long times $t$ after the quench, the growth law $L \sim t^{1/3}$ holds. For small scaled distances $x = |R|/L(t)$ one is able to find a scaling solution of the form

$$F(x) = 1 - \tilde{M}^2 - e^{-y^2/2} \alpha x (1 + \beta_2 x + \cdots) \quad (1.3)$$

where the parameter $y$ is related to $\tilde{M}$ by

$$\tilde{M} = erf(y/\sqrt{2}). \quad (1.4)$$

Unlike the non-conserved order parameter (NCOP) case treated earlier [8], the coefficient $\beta_2$ is not zero and must be determined, along with $\alpha$, as part of a non-linear eigenvalue problem. $\beta_2$ is found to be negative for $\tilde{M} = 0$ and monotonically decreases as $\tilde{M} \rightarrow 1$. Thus, in this theory, the Tomita sum rule ($\beta_2 = 0$) [9] is strongly broken as one approaches the coexistence curve. This appears to be an important limitation of the current theory.

The scaling function, $F$, has been determined explicitly in two and three dimensions by solving the non-linear eigenvalue problem mentioned above. The dependence of $F$ on $\tilde{M}$ is
weak for small $\tilde{M}$. As $\tilde{M}$ increases the first zero of $F$ moves to larger scaled distances and the first minimum of $F$ becomes shallower. As $\tilde{M}$ increases further the oscillatory behaviour is suppressed and the predominant behaviour is that of decay, as predicted by our large $y$ asymptotic analysis. For large $\tilde{M}$ there are oscillations at large $x$ whose wavelength increases as one approaches the coexistence curve. These oscillations preserve the conservation law

$$\int d^dx F(x) = 0$$  \hspace{1cm} (1.5)$$
despite the existence of the strong decay.

The structure factor is the Fourier transform of the correlation function and one has

$$C(q, t) = \psi_0^2 L^d(t) \tilde{F}(Q)$$  \hspace{1cm} (1.6)$$
where $Q = qL$ is a scaled wavenumber. $\tilde{F}(Q)$ is characterized by five parameters. Since $\tilde{F}(Q)$ is a peaked quantity, the peak position $Q_{\text{max}}$, and height $\tilde{F}(Q_{\text{max}})$, are of interest as functions of $\tilde{M}$. The full-width at half-maximum, measured in units of $Q_{\text{max}}$, is also relevant.

The linear term in (1.3) leads to Porod’s law [20] for the large $Q$ tail of the structure factor

$$\tilde{F}(Q) = \tilde{F}(Q_{\text{max}}) A_P(\tilde{M}) \left( \frac{Q}{Q_{\text{max}}} \right)^{-(d+1)}$$  \hspace{1cm} (1.7)$$
while for small $Q$, $\tilde{F}(Q)$ behaves as [21]

$$\tilde{F}(Q) = \tilde{F}(Q_{\text{max}}) A_4(\tilde{M}) \left( \frac{Q}{Q_{\text{max}}} \right)^4$$  \hspace{1cm} (1.8)$$
This behaviour at small $Q$ is a result of a conserved diffusive field existing away from the interfaces which mediates the interaction among the interfaces. Our analysis shows that both $Q_{\text{max}}$ and $\tilde{F}(Q_{\text{max}})$ decrease to zero as $\tilde{M} \to 1$. The width of the peak increases slightly for small $\tilde{M}$, but then develops a significant tail as $\tilde{M} \to 1$ for intermediate values of $Q$ near the base of the peak. $A_P(\tilde{M})$ is a decreasing function of $\tilde{M}$, approaching zero in a cusp as $\tilde{M} \to 1$. The coefficient $A_4(\tilde{M})$ increases with increasing $\tilde{M}$, growing rapidly near $\tilde{M} = 1$. Damped oscillations are also seen in the structure factor around the $Q^{-4}$ behaviour at intermediate values of $Q$, a result also seen by other investigators [22,13].
When we compare our results with those of other workers we find good qualitative agreement for both \( F(x) \) and \( \tilde{F}(Q) \) as functions of \( \tilde{M} \). There are some quantitative differences, though. We believe that the difference in the form of \( \tilde{F}(Q) \) is due to our low estimate for the coefficient \( A_4 \), leading to a peak in \( \tilde{F}(Q) \) which is too narrow. It seems likely that the lack of quantitative agreement is associated with the breaking of the Tomita sum rule in the theory. On the other hand, this is the only theory which has led to a determination of \( A_4 \), and one can hope that using higher order approximations will give improved results.

In the next section the theory forming the basis for this work is outlined. In section III the averages are performed which are relevant to the off-critical case. The end result of these manipulations is a non-linear equation for the scaling function. Section IV looks at the various limiting cases of the theory: the small \( x \), large \( x \), small \( Q \), and large \( y \) behaviour. Section V presents the numerical study of the equation for \( F(x) \) in two and three dimensions. Comparison of the results of this paper with results from other investigators is made in section VI. The paper concludes with some comments about future areas of research and improvements to the theory.

II. THE MODEL

The dynamics are modelled using a noiseless time-dependent Ginzburg-Landau equation for a conserved scalar order parameter \( \psi \) with a non-zero average \( M \):

\[
\frac{\partial \psi(1)}{\partial t_1} = D_0 \nabla_1^2 [V'(\psi(1)) - \nabla_1^2 \psi(1)].
\]  

\( V(\psi) \) is a double-well potential with degenerate minima at \( \pm \psi_0 \), but is otherwise unspecified since, as we shall see, our results are independent of the precise form of \( V \). \(^{[23]}\) Here, the convenient notation in which \( 1 \) represents \((R_1, t_1)\) is used. The \( D_0 \nabla_1^2 \) factor in (2.1) ensures that the system has a conserved order parameter. Thermal sources of noise are neglected because it is assumed here that the quench is to zero temperature. Randomness enters the problem through the initial conditions where we assume that the initial values of \( \delta \psi = \psi - M \) are governed by a Gaussian probability distribution characterized by...
\[ \langle \psi(R, 0) \rangle = M \tag{2.2} \]
\[ \langle \delta \psi(R, 0) \delta \psi(R', 0) \rangle = \epsilon_0 \delta(R - R'). \tag{2.3} \]

Our final results are independent of the amplitude \( \epsilon_0 \) appearing in the initial distribution.

A method for extracting the correlation functions from (2.1) is described in [1]. Here we will merely outline the salient points. The order parameter is written as

\[ \psi(1) = \sigma[m(1)] - u(1) \tag{2.4} \]

where \( \sigma \) is the equilibrium interfacial profile and \( u \) represents fluctuations about this ordered value. \( m \) is assumed to be a random field whose zeros correspond to the zeros of \( \sigma \), that is, to the positions of the interfaces. The nature and interpretation of \( m \) will be discussed below. In the NCOP case the fluctuating field \( u \) can be safely ignored but in the COP case it is the field \( u \) which couples distant interfaces by permitting currents of minority phase atoms to flow through the matrix. In [1] the theory was closed by relating \( u \) back to \( \sigma \) and \( m \) via the equation

\[ u(1) = \frac{u_0}{L} \sigma(1) + \lambda \nabla^2 m(1) + \cdots \tag{2.5} \]

where \( u_0 \) and \( \lambda \) are parameters. This form satisfies the desired properties that \( u \) is conserved in bulk, odd in \( m \), and \( \mathcal{O}(1/L) \) everywhere. This last requirement comes from the fact that the interfaces are a source of \( u \) with a contribution proportional to the local curvature of an interface \( \kappa \) \( (u|_\mathcal{S} \sim \kappa) \). It can then be shown that if (2.5) holds, \( u \) satisfies the familiar form \( \nabla^2 u = 0 \) away from the interfaces. \( \sigma \) is chosen to satisfy the equation for an equilibrium interface

\[ \frac{1}{2} \sigma_2(1) = V'(\sigma(m(1))) \tag{2.6} \]

where the factor 1/2 is inserted for convenience and \( \sigma_n(m) = \partial^n \sigma(m) / \partial m^n \). The boundary condition, \( \lim_{m \to \pm \infty} \sigma = \pm \psi_0 \), guarantees that the system orders at the appropriate equilibrium value of the order parameter and results in the useful relation
As shown in [1], equations (2.5) and (2.6) can be substituted into (2.1) and the result multiplied by $\sigma(2)$ and averaged to get an equation for

$$C_{\sigma\sigma}(\mathbf{R}, t) = \langle \sigma(1)\sigma(2) \rangle. \quad (2.8)$$

At late times $1/L$ is expected to be small and one finds to leading order in $1/L$:

$$\frac{1}{2} \frac{\partial}{\partial t} C_{\sigma\sigma}(\mathbf{R}, t) = -D_0 q_0^2 \nabla^2 \left( \frac{\mu_0}{L} C_{\sigma\sigma}(\mathbf{R}, t) + \lambda \nabla^2 C_{\sigma m}(\mathbf{R}, t) \right) \quad (2.9)$$

where, $q_0^2 \equiv \langle V''(\sigma) \rangle = V''(\psi_0) + \mathcal{O}(1/L)$ and $C_{\sigma m}(\mathbf{R}, t) = \langle \sigma(1)m(2) \rangle$. Here, equal times are considered and statistical homogeneity of the system has been assumed so $t_1 = t_2 = t$ and $\mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1$. Since

$$C(\mathbf{R}, t) = \langle \delta \psi(1)\delta \psi(2) \rangle = \langle \sigma(1)\sigma(2) \rangle - M^2 + \mathcal{O}\left(\frac{1}{L}\right) \quad (2.10)$$

we see that (2.9) is an equation for $C(\mathbf{R}, t)$ to $\mathcal{O}(1/L)$ since the action of the derivatives eliminates the disconnected part of the correlation function.

A key aspect of the theory is the choice of the probability distribution $P[m]$ governing the field $m$. This point is discussed in some detail in references [1,17,24]. Here we limit ourselves to the case where $P[m]$ is given by an off-set Gaussian with

$$\bar{m}(t) = \langle m(1) \rangle_0, \quad (2.11)$$

and $\langle \ldots \rangle_0$ is over a probability distribution $P_0[\delta m]$ which is Gaussian with respect to $\delta m(1) = m(1) - \bar{m}(t)$. $P_0[\delta m]$ is then determined by the variance

$$C_0(12) = \langle \delta m(1)\delta m(2) \rangle_0. \quad (2.12)$$

Since the field $m$ can be approximately interpreted as the perpendicular distance to the nearest interface, the off-set corresponds to a greater probability to be in one phase than in the other. The effects of this non-zero average will be explored in the next section.
III. EVALUATION OF AVERAGES: THE SCALING EQUATION OF MOTION

A. Evaluation of Averages

In order for (2.9) to be a closed equation for $C_{\sigma\sigma}$, it is necessary to relate $C_{\sigma m}$ to $C_{\sigma\sigma}$. As in [1], this is done by using the Gaussian nature of $m$. Now, however, the Gaussian probability distribution must satisfy the condition $\langle m(1) \rangle_0 = \bar{m}(t) \neq 0$. Taking this into account one finds, using the standard properties of Gaussian integrals [18],

$$C_{\sigma m}(12) = \langle \sigma_1(1) \rangle_0 C_{0}(12) + M \bar{m}(t).$$

(3.1)

The spatially independent term is eliminated by the action of the Laplacian in (2.9). Since $m$ is a Gaussian random field it follows that

$$\langle \sigma_1(1) \rangle_0 = \int dx_1 \sigma_1(x_1) \Phi(x_1)$$

(3.2)

where

$$\Phi(x_1) = \frac{1}{\sqrt{2\pi S_0(t)}} e^{-\frac{1}{2}(x_1-\bar{m}(t))^2/S_0(t)}$$

(3.3)

with

$$S_0(t) = C_0(11).$$

(3.4)

Since $m$ is a measure of the distance away from an interface it is expected that in the long-time scaling regime $S_0 \sim L^2$ and $\bar{m}(t) \sim L$. Therefore the limit

$$\lim_{t \to \infty} \frac{\bar{m}(t)}{\sqrt{S_0(t)}} = y$$

(3.5)

exists. In evaluating (3.2) it is important to note that, for a wide class of potentials, $\sigma_1(x_1)$ goes exponentially to zero for large $|x_1|$. Therefore one can, after eliminating $\bar{m}$ in favour of $y\sqrt{S_0(t)}$, expand (3.2) in powers of $S_0^{-1}$ and use (2.7) to obtain

$$\langle \sigma_1(1) \rangle_0 = \frac{2\psi_0}{\sqrt{2\pi S_0(t)}} e^{-\frac{1}{2}y^2} + O(S_0^{-1}).$$

(3.6)
The parameter \( y \) can be related to \( M \) by using the derivative relation \[25\]

\[
\frac{\partial M}{\partial \bar{m}(t)} = \frac{\partial \langle \sigma \rangle_0}{\partial \bar{m}(t)} = \langle \sigma_1 \rangle_0,
\]

from which follows

\[
M = \int_0^{\bar{m}(t)} dz \sqrt{\frac{2}{\pi S_0(t)}} \psi_0 e^{-\frac{1}{2} \frac{\bar{m}^2}{S_0(t)}}.
\]

Thus, with \( \tilde{M} = M/\psi_0 \),

\[
\tilde{M} = \text{erf} \left( \frac{y}{\sqrt{2}} \right).
\]

Therefore, there is a one-to-one correspondence between \( y \) and \( \tilde{M} \). \( y = 0 \) is a critical quench and \( y \to \infty \) corresponds to a quench at the coexistence curve.

With the definition

\[
f(R, t) = \frac{C_0(R, t)}{S_0(t)}
\]

and the use of (3.1) and (3.6), the equation of motion (2.9) takes the form

\[
\frac{1}{2} \frac{\partial}{\partial t} C_{\sigma \sigma}(R, t) = -D_0 q_0^2 \nabla^2 \left( \frac{u_0}{L} C_{\sigma \sigma}(R, t) + \lambda \sqrt{\frac{2 S_0}{\pi}} \psi_0 e^{-\frac{1}{2} \frac{f^2}{S_0}} \nabla^2 f(R, t) \right).
\]

The theory can be closed by relating \( C_0(R, t) \) to \( C_{\sigma \sigma}(R, t) \) via the relation \[18\]

\[
C_{11}(R, t) = \frac{\partial C_{\sigma \sigma}(R, t)}{\partial C_0(R, t)}
\]

where \( C_{nm}(R, t) = \langle \sigma_n(1) \sigma_m(2) \rangle_0 \). The fact that \( m \) is Gaussian enables one to write

\[
C_{11}(R, t) = \int dx_1 dx_2 \sigma_1(x_1) \sigma_1(x_2) \Phi(x_1, x_2)
\]

where

\[
\Phi(x_1, x_2) = \frac{\gamma}{2\pi S_0} e^{-\frac{1}{2} \frac{\bar{m}^2}{S_0} + (x_2 - \bar{m}(t))^2 - 2f(x_1 - \bar{m}(t))(x_2 - \bar{m}(t))}
\]

with \( \gamma = (1 - f^2)^{-\frac{1}{2}} \). Again, \( \bar{m}(t) \) can be eliminated in favour of \( y \sqrt{S_0(t)} \) and the result (3.13) expanded in terms of \( S_0^{-1} \). At long times the leading order term dominates, resulting in

\[
\Phi(x_1, x_2) = \frac{\gamma}{2\pi S_0} e^{-\frac{1}{2} \frac{y^2}{S_0} + (x_2 - x_1)^2 - 2f(x_1 - \bar{m}(t))(x_2 - \bar{m}(t))}
\]
where the integral is easily found to give $4\psi_0^2$. Integration of (3.12) with the substitution (3.15), keeping in mind the definition of $f$, gives the desired relation between $C_{\sigma\sigma}$ and $f$:

$$C_{\sigma\sigma} = \frac{2\psi_0^2}{\pi} \int_0^f \frac{ds}{\sqrt{1 - s^2}} e^{-\frac{y^2}{1+s}}.$$  

(3.16)

For a critical quench $y = 0$, and this reduces to the now standard expression $C_{\sigma\sigma} = \frac{2}{\pi} \psi_0^2 \sin^{-1} f$. Equations (3.16) and (3.11) form a closed system of equations for $C_{\sigma\sigma}$.

### B. The Scaling Regime

At long times the correlation function is expected to have a scaling solution. The ansatz

$$C_{\sigma\sigma}(R, t) = \psi_0^2 F(x),$$

(3.17)

with $x = |R|/L(t)$ being the scaled distance, can be substituted into equations (3.11) and (3.16) to give

$$x\bar{F}' = \nabla^2 (\bar{u}\bar{F} + \nabla^2 f)$$

(3.18)

$$\bar{F} = \frac{2}{\pi} \int_0^f \frac{ds}{\sqrt{1 - s^2}} e^{-\frac{y^2}{1+s}}$$

(3.19)

where

$$\bar{F}(x) = e^{y^2/2} F(x)$$

(3.20)

$$\bar{u} = \frac{2u_0 D_0 \psi_0^2}{L^3 L}$$

(3.21)

and we have chosen $L = (\lambda \bar{u}/\psi_0 u_0) \sqrt{2S_0/\pi}$. Note that, for scaling we require that $\bar{u}$ is a constant and therefore $L \sim t^{1/3}$. One expects $\bar{F}$ to have a weaker $y$ dependence than $F$ for large $y$. It is useful to express $f$ in terms of

$$\Phi = \sqrt{\frac{1-f}{1+f}}$$

(3.22)

in equation (3.19) to obtain the well-behaved integral representation
\[
\bar{F} = \frac{4}{\pi} \int_{1}^{\Phi} \frac{ds}{1 + s^2 e^{-\frac{y^2}{2}}}.
\]  \hspace{1cm} (3.23)

For future reference, the spatial derivative of (3.23) is

\[
\frac{\partial \bar{F}}{\partial x} = -\frac{4}{\pi} \frac{e^{-y^2 \Phi^2/2}}{1 + \Phi^2} \frac{\partial \Phi}{\partial x}.
\]  \hspace{1cm} (3.24)

It is noteworthy that \(\bar{F}\) has a lower bound. Let \(\Phi \to \infty\) in (3.23) and write

\[
\bar{F}_{\text{min}} = \frac{4}{\pi} e^{y^2/2} \int_{\infty}^{1} \frac{ds}{1 + s^2} e^{-\frac{s^2}{2}} (1 + s^2) = \frac{4}{\pi} e^{y^2/2} I(y).
\]  \hspace{1cm} (3.25)

\(I(y)\) has the following properties:

\[
I(0) = -\frac{\pi}{4},
\]

\[
\frac{d}{dy} I(y) = \frac{\pi}{2} \left[ 1 - \text{erf} \left( \frac{y}{\sqrt{2}} \right) \right] \frac{d}{dy} \text{erf} \left( \frac{y}{\sqrt{2}} \right).
\]  \hspace{1cm} (3.26)

Integrating these equations yields

\[
I(y) = -\frac{\pi}{4} \left( 1 - \text{erf} \left( \frac{y}{\sqrt{2}} \right) \right)^2 = -\frac{\pi}{4} (1 - |\tilde{M}|)^2.
\]  \hspace{1cm} (3.27)

Thus, the lower bound on \(\bar{F}\) is

\[
\bar{F}_{\text{min}} = -e^{y^2/2} (1 - |\tilde{M}|)^2.
\]  \hspace{1cm} (3.28)

As \(\tilde{M} \to 1\) this lower bound approaches zero.

Equations (3.18) and (3.23) constitute a non-linear eigenvalue problem for \(\bar{F}\), with a unique solution determined by the boundary conditions at small \(x\) and the physical condition \(\bar{F} \to 0\) exponentially as \(x \to \infty\). We will see that \(\bar{u}\) is determined as part of the solution. The only parameters entering into the determination of \(\bar{F}\) are \(y\) and the dimension \(d\), which appears in the spherically symmetric Laplacian.

\textbf{IV. LIMITING CASES}

\textbf{A. Small \(x\) behaviour}

The small \(x\) behaviour of \(\bar{F}\) can be determined analytically. We find that \(\bar{F}\) has the form
\[ \bar{F} = \bar{F}(0) - \alpha x(1 + \beta_2 x^2 + \beta_3 x^3 + \cdots). \]  

(4.1)

Expanding \( \Phi \) for small \( x \)

\[ \Phi = \Phi_1 x + \Phi_2 x^2 + \cdots \]  

(4.2)

and using (3.24) to connect the power series for \( \bar{F} \) and \( \Phi \) gives

\[ \Phi_1 = \frac{\pi \alpha}{4} \]  

(4.3)

\[ \Phi_2 = \frac{\pi \alpha \beta_2}{4}. \]  

(4.4)

Substitution of these results into (3.18) give, at \( \mathcal{O}(1/x) \)

\[ \bar{u} = -\frac{3\pi^2 \alpha \beta_2 (d + 1)}{4}. \]  

(4.5)

\( \bar{F}(0) \) can be determined from (3.28) by noting that \( \Phi(0) = 0 \) and then using the same technique that was used to derive (3.28). The result is

\[ \bar{F}(0) = (1 - \tilde{M}^2)e^{\sigma^2/2}. \]  

(4.6)

The equation of motion (3.18) can be partially integrated using the Green’s function for the Laplacian \([25]\). For \( d > 2 \) the result is

\[ \bar{u}\bar{F} + \nabla^2 f = \frac{1}{d - 2} \left[ \frac{d}{x^{d-2}} I_d(x) - 2(I_2(x) - I_2(\infty)) \right] \]  

(4.7)

with

\[ I_d(x) = \int_0^x z^{d-1} \bar{F}(z)dz. \]  

(4.8)

The results (1.3) and (4.4) can be substituted into (4.7). Since one can show that

\[ \lim_{x \to 0} \frac{I_d(x)}{x^{d-2}} = \lim_{x \to 0} I_2(x) = 0 \]  

(4.9)

one has at \( \mathcal{O}(1) \)

\[ \frac{2}{d - 2} I_2(\infty) = -\frac{\pi^2 \alpha^2 d}{4} + \bar{u}\bar{F}(0). \]  

(4.10)
For $d = 2$ the analogous results are

$$
\bar{u} \bar{F} + \nabla^2 f = (1 - 2 \ln x)I_2(x) + 2(J_2(x) - J_2(\infty)) \tag{4.11}
$$

with

$$
J_2(x) = \int_0^x dz \, z \bar{F}(z) \ln z. \tag{4.12}
$$

Since $\lim_{x \to 0} J_2(x) = 0$ substitution of (4.3) and (4.4) into (4.11) give, at $O(1)$

$$
-2J_2(\infty) = -\frac{\pi^2 \alpha^2}{2} + \bar{u} \bar{F}. \tag{4.13}
$$

Knowledge of the parameters $\alpha$ and $\beta_2$ allows one to determine the constants $\bar{u}$ and $I_2(\infty)$ (or $J_2(\infty)$) appearing in the equation of motion for a given value of $y$. Numerically, what this means is that values for $\alpha$ and $\beta_2$ are chosen and equations (4.7) (or (4.11)) and (3.23) integrated forward in $x$. $\alpha$ and $\beta_2$ are adjusted until $\bar{F}$ satisfies both the conservation law (1.5) and the boundary condition $\bar{F} \to 0$ exponentially as $x \to \infty$.

B. Large $x$ behaviour

For large $x$ both $\bar{F}$ and $f$ are small and $\Phi$ can be expanded about its asymptotic value

$$
\Phi(x) = 1 + \eta(x). \tag{4.14}
$$

Substitution into (3.24) and integration yields a relation between $\bar{F}$ and $\eta$

$$
\bar{F}(x) = -\frac{2}{\pi} e^{-y^2/2} \eta(x). \tag{4.15}
$$

Also, from (3.22) we have $f(x) = -\eta(x)$ so we may rewrite (3.18) in the large-\(x\) limit as

$$
x \bar{F}' = \nabla^2 (\bar{u} \bar{F} + \frac{\pi}{2} e^{y^2/2} \nabla^2 \bar{F}). \tag{4.16}
$$

This equation supports damped oscillatory solutions of the form

$$
\bar{F} \sim \bar{F}_0 x^{-2d/3} \exp \left( -\frac{3 e^{-y^2/6}}{2^{8/3} \pi^{1/3}} x^{4/3} \right) \cos \left( \frac{3^{3/2} e^{-y^2/6}}{2^{8/3} \pi^{1/3}} x^{4/3} + \phi \right) \tag{4.17}
$$

where $\bar{F}_0$ and $\phi$ are constants. Note that as $y \to \infty$ the wavelength of the oscillations increases and the exponential term goes to 1. This means that in this limit one must go to progressively larger values of $x$ before one sees this asymptotic behaviour.
C. Small Q behaviour

It is the Fourier transform of the order parameter correlation function,

\[ C(q, t) = \int d^d R e^{i q R} \langle \psi(R, t) \psi(0, t) \rangle \]

\[ = \psi_0^2 L^d(t) \tilde{F}(Q) + (2\pi)^d M^2 \delta(q) \]  \hspace{1cm} (4.18)

with \( Q = qL(t) \) and it is

\[ \tilde{F}(Q) = \int d^d x e^{i Q \cdot x} F(x), \]  \hspace{1cm} (4.19)

that is measured in a scattering experiment. In the total scattering cross-section we expect that at long times there is a dynamic contribution to the forward Bragg peak,

\[ \lim_{t \to \infty} \psi_0^2 L^d(t) \tilde{F}(Q) = A \delta(q), \]  \hspace{1cm} (4.20)

in addition to the static contribution \((2\pi)^d M^2 \delta(q)\). Since

\[ A = \int d^d q \lim_{t \to \infty} \psi_0^2 L^d(t) \tilde{F}(Q) \]

\[ = (2\pi)^d \psi_0^2 F(0) = (2\pi)^d \psi_0^2 (1 - \tilde{M}^2), \]  \hspace{1cm} (4.21)

the total contribution to the forward Bragg peak at late times is \((2\pi)^d \psi_0^2 \delta(q)\), as expected.

To examine the small Q behaviour of the structure factor \( \tilde{F}(Q) \), it is useful to consider the moments of \( F(x) \)

\[ W_p = \int d^d x x^p F(x) \]  \hspace{1cm} (4.22)

which can be found by multiplying \((3.18)\) by \( x^p \) and integrating. The result is that \( W_0 = W_2 = 0 \) while

\[ W_4 = -\frac{8d(d + 2)}{d + 4} e^{-\frac{1}{\tilde{y}^2}} \int d^d x f(x). \]  \hspace{1cm} (4.23)

Thus we have

\[ \tilde{F}(Q) \sim AQ^d \]  \hspace{1cm} (4.24)
for small $Q$ where

$$A = -\frac{e^{-y^2/2}}{d + 4} \int d^d x f(x). \quad (4.25)$$

The $Q^4$ behaviour of $\tilde{F}(Q)$ at small $Q$ is a consequence of the fact that in the theory $u$ is conserved away from the interfaces, and this behaviour does not depend on the specific ansatz for $u$. In order for $\tilde{F}(Q)$ to be positive definite (4.25) implies that

$$\int d^d x f(x) < 0. \quad (4.26)$$

This means that $\lim_{q \to 0} \langle |m_q(t)|^2 \rangle / S_0 < 0$. As pointed out by Yeung et al. [26] this is a shortcoming of the fact that $m$ is a Gaussian variable. This problem is resolved when non-Gaussian corrections to the probabilty distribution for the field $m$ are considered [1].

D. Large $y$ behaviour

An analytic result for the limit as one approaches the coexistence curve, $\tilde{M} \to 1$, $y \to \infty$ is of interest because it allows one to make comparisions with other theories developed for this regime. From the numerical analysis in the next section the following facts emerge. The first is that, as $y$ increases, the scaled length $x$ over which the correlation function takes significant values decreases. This suggests that $x$ should be rescaled as

$$x = y^p z \quad (4.27)$$

with $p < 0$. Second, it appears that $\bar{u}$ grows as some power of $y$ for large $y$. We are led to assume the form

$$\bar{u} = \bar{u}_\infty y^n \quad (4.28)$$

with $n > 1$. Finally, since the interesting behaviour occurs near the origin where the quantity $\Phi$ is small, we can rescale $\Phi$

$$\Phi = \frac{\phi}{y} \ll 1 \quad (4.29)$$
where $\phi$ grows slowly but does not break the bound near the origin. Using this definition in equation (3.22) leads to

$$f \approx 1 - 2\frac{\phi^2}{y^2}. \tag{4.30}$$

Armed with these results we proceed to re-examine the theory. Using (4.29) in equation (3.23) and letting $t = y s$ in the integrand gives

$$\bar{F}(z) = \frac{4}{\pi y} \int_{\phi(z)}^{y} \frac{dt}{1 + (t/y)^2} e^{-t^2/2}. \tag{4.31}$$

To leading order in $y^{-1}$ this is

$$\bar{F}(z) = \frac{4}{\pi y} \int_{\phi(z)}^{\infty} dt e^{-t^2/2} = \frac{1}{y} \bar{F}_\infty(z) \tag{4.32}$$

where

$$\bar{F}_\infty(z) = 2\sqrt{\frac{2}{\pi}} \left(1 - erf\left(\frac{\phi(z)}{\sqrt{2}}\right)\right). \tag{4.33}$$

Under the rescaling outlined above the equation of motion (4.7) for $d > 2$ becomes

$$\bar{u}_\infty y^{n-1} \bar{F}_\infty(z) - 2y^{-2(p+1)} \nabla^2 \phi^2 = \frac{y^{2p-1}}{d-2} \left[\frac{d}{z^{d-2}} I_d(z) - 2(I_2(z) - I_2(\infty))\right] \tag{4.34}$$

with

$$I_d(z) = \int_{0}^{z} d s s^{d-1} \bar{F}_\infty(s). \tag{4.35}$$

Since $p < 0$ the right hand side does not contribute as $y \to \infty$. The integrals in the $d = 2$ case also do not contribute in this limit so the following results are valid for $d \geq 2$. Balancing powers of $y$ on the left hand side of (4.34) gives

$$p = -\frac{n + 1}{2} \tag{4.36}$$

and

$$\bar{u}_\infty \bar{F}_\infty - 2 \nabla^2 \phi^2 = 0. \tag{4.37}$$

Using (4.33) this becomes a simple equation for $\phi$

$$\bar{u}_\infty \sqrt{\frac{2}{\pi}} \left(1 - erf\left(\frac{\phi}{\sqrt{2}}\right)\right) = \nabla^2 \phi^2. \tag{4.38}$$

In the numerical solution of (4.38), $\bar{u}_\infty$ is a parameter which is found from a fit of the numerically determined $\bar{u}$ to the form (4.28) for large $y$. 

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E. Small and large \( z \) behaviour for large \( y \)

An examination of (4.38) in the limit of large and small \( z \) is instructive. When \( z \) is small \( \phi \) is expected to be small so \( \phi \) can be expanded as a power series in \( z \)

\[
\phi = \phi_1 z + \phi_2 z^2 + \cdots. \tag{4.39}
\]

Also, in this limit, (4.38) simplifies to the form

\[
\bar{u}_\infty \sqrt{\frac{2}{\pi}} \left( 1 - \sqrt{\frac{2}{\pi}} \phi \right) = \nabla^2 \phi^2. \tag{4.40}
\]

Matching powers of \( z \) to leading order gives

\[
\phi_1 = \left( \frac{2}{\pi} \right)^{1/4} \left( \frac{\bar{u}_\infty}{2d} \right)^{1/2}. \tag{4.41}
\]

This result allows one to make predictions about the asymptotic behaviour of \( \alpha \) and \( \beta_2 \) when \( y \) is large. Using (4.27) and (4.33) one can write

\[
\bar{F}(x) = \frac{2}{y} \frac{2}{\sqrt{\pi}} - \frac{4}{\pi} \phi_1 y^{-p-1} x + \cdots \tag{4.42}
\]

giving

\[
\alpha = \alpha_\infty y^{p-1} \tag{4.43}
\]

for large \( y \) with \( \alpha_\infty = 4\phi_1/\pi \). If we also write

\[
\beta_2 = \beta_{2\infty} y^m \tag{4.44}
\]

then

\[
\bar{u} = -\frac{3\pi^2 \alpha_\infty \beta_{2\infty} (d+1)}{4} y^{m-p-1} = \bar{u}_\infty y^n. \tag{4.45}
\]

Matching the coefficient and the exponent leads, using (4.36), to the following relationships:

\[
\alpha = \alpha_\infty y^{(n-1)/2} \tag{4.46}
\]

\[
\beta_2 = \beta_{2\infty} y^{(n+1)/2} \tag{4.47}
\]

\[
\alpha_\infty = -\frac{12}{\sqrt{2\pi}} \left( \frac{d+1}{d} \right) \beta_{2\infty}. \tag{4.48}
\]
Thus, a graph of $\beta_2/\alpha$ vs. $y$ for large $y$ will be linear with a slope that depends only on the dimensionality of the system.

At large $z$, $\phi$ is large and $(4.38)$ is well approximated by

$$u_\infty \frac{2}{\pi \phi} e^{-\phi^2/2} = \nabla^2 \phi^2.$$  

(4.49)

For $d > 2$, standard asymptotic analysis yields, at next to leading order:

$$\phi(z) = \ln^{1/2} \left( \frac{z^4}{\ln z} \left( \frac{u_\infty}{2\pi(d-2)} \right)^2 \right).$$

(4.50)

This implies that

$$F_\infty(z) = \frac{4(d-2)}{u_\infty} \frac{1}{z^2}$$

(4.51)

for large $z$ and $d > 2$. For $d = 2$ one has

$$F_\infty(z) = \frac{F_0}{(1 + \frac{u_\infty F_0}{16} z^2)^2}.$$  

(4.52)

It is clear that, near the coexistence curve, the oscillations in the scaling form become insignificant and are dominated by a strong decay.

V. NUMERICAL SOLUTION OF THE NON-LINEAR EIGENVALUE PROBLEM

A. $\alpha$ and $\beta_2$ as a function of $y$

In this section the numerical solution of $(4.7)$ (or $(4.11)$) coupled with $(3.23)$ in two and three dimensions will be discussed. The equations are integrated forward from $x = 0$ using a fourth order Runge-Kutta integrator with step size $\delta x = 0.001$, subject to the initial conditions $F(0) = (1 - \bar{M}^2)e^{y^2/2}$, $F'(0) = -\alpha$, $F''(0) = -2\alpha\beta_2$, and $f'(0) = 0$. This method of integration seems numerically stable and insensitive to the choice of $\delta x$. The search for the eigenvalues $\alpha$ and $\beta_2$ involves requiring that the solution $\bar{F}$ obey the conservation law $(1.5)$ and have the physically acceptable behaviour $\bar{F} \rightarrow 0$ exponentially as $x \rightarrow \infty$. This search is performed by fixing $\alpha$ and then searching for the value of $\beta_2$ which pushes the diverging,
unphysical solution to larger values of $x$. The value of $\alpha$ is then adjusted so that the flat region of $\bar{F}$ at large $x$ is properly zeroed. The procedure is repeated with the new value of $\alpha$ until the exponentially growing solution is pushed as far from the origin as possible and until $\bar{F}$ is zeroed as well as possible. The degree to which the conservation law is satisfied naturally depends on how well the function is zeroed. The convergence of the eigenvalues is fast and $\bar{F}$ can be zeroed to better than $10^{-6}$ using this method.

The results for the eigenvalues $\alpha$ and $\beta_2$ are shown in Fig. 1. One sees that $\alpha$ initially decreases reaching a minimum at $y \sim 1$ and then rapidly becomes large and positive as the coexistence curve is approached. The eigenvalue $\beta_2$ is negative at $y = 0$ and monotonically decreases as $y$ increases, decreasing rapidly as $y$ becomes large ($\bar{M} \to 1$). Equation (4.48) predicts that a graph of $\beta_2/\alpha$ will be linear for large $y$ with a slope -0.157 for $d = 3$ and -0.139 for $d = 2$. We see this linear behaviour and have measured slopes of about -0.143 and -0.129 for $d = 2$ and 3 respectively. The exponents $n$ and $p$, and the coefficient $\bar{u}_\infty$ defined in (4.27) and (4.28) can be found by fitting the large $y$ behaviour of $\alpha$ and $\beta_2$ to the forms (4.46) and (4.47). For three dimensions one finds $n \sim 8$, $p \sim -4.5$ and $\bar{u}_\infty \sim 0.0033$. In two dimensions one has $n \sim 6$, $p \sim -3.5$ and $\bar{u}_\infty \sim 0.024$. When considering these results it should be kept in mind that only a few values of $y$ around $y = 4$ were used to obtain these values. In principle, both the exponents and the coefficient can be accurately obtained by extending the numerical analysis to larger values of $y$. In practice, this is difficult due to reasons that are discussed below.

**B. Scaling Function as a function of $\bar{M}$**

The dependence of the scaling function $F(x)$ on $\bar{M}$ is shown in Fig. 3 for two dimensions and in Fig. 4 for three dimensions. In these plots $F(x)$ is normalized so that $F(0) = 1$. Both figures show that $F(x)$ depends only weakly on $\bar{M}$ for values of $\bar{M} < 0.4$. The scaling function has a prominent oscillatory component which is necessary to satisfy the conservation law. At intermediate values of $\bar{M}$, the position of the first minimum of $F(x)$ occurs at larger
values of $x$ and the depth of this minimum decreases as $\tilde{M}$ increases. The depth of the oscillations is greater in two dimensions than in three. These stronger oscillations make the presence of the lower bound on $F(x)$ noticeable, and near the coexistence curve the minima in the scaling function are very flat in order to be consistent with this bound.

As $\tilde{M} \rightarrow 1$ the scaling function approaches its asymptotic form (4.33) which can be determined by numerically solving (4.38) using the values of $\bar{u}_\infty$ found in the previous section. Since we know the exponent $p$, we can rescale the distance $x$ using (4.27) and plot $\tilde{F}_\infty(z) = y\tilde{F}(xy^{-p})$ for large values of $y$. This is done for two and three dimensions in Fig. 4. The asymptotic forms obtained by solving (4.38) are also shown in this figure. We see that $F(x)$ decays very rapidly when the system is near the coexistence curve. Oscillations do occur for these values of $y$, but they occur at large $x$ and have a small amplitude and large wavelength. For $d = 3$ the curves appear to approach the asymptotic form as $y$ increases. However, for $d = 2$ the asymptotic form is not approached if one uses $\bar{u}_\infty = 0.024$. A value of $\bar{u}_\infty = 0.036$ gives a better fit and the form obtained using this value is the one shown in Fig. 4. Matching the asymptotic form to the rescaled large $y$ scaling function is another way to determine $\bar{u}_\infty$. We believe that the two methods for finding $\bar{u}_\infty$ give different values because in the fit of $\alpha$ and $\beta_2$ to the forms (4.46) and (4.47) we do not have values of $y$ which are large enough to be in the asymptotic regime. Larger values of $y$ are difficult to reach because one runs into numerical problems as the theoretical lower bound on $F(x)$ approaches zero. These numerical problems are especially significant in two dimensions since the oscillations in the correlation function are stronger than in three dimensions.

C. Scaling of the Structure Factor

The structure factor,

$$\tilde{F}(Q) = \int d^d x e^{iQ \cdot x} F(x)$$

was calculated by taking the Fourier transform of our numerically determined $F(x)$. We find that as $\tilde{M}$ increases the height of the peak decreases and the peak position moves to
smaller values of $Q$. Graphs of the normalized structure factor for various $\tilde{M}$ are shown in Fig. 5 for $d = 2$ and Fig. 6 for $d = 3$. Logarithmic plots reveal the power-law dependence of $\tilde{F}(Q)$ for large and small $Q$ (Fig. 7). For small $Q$, $\tilde{F}(Q) \sim Q^4$ in both $d = 2$ and 3, for all $\tilde{M}$. Small deviations from the $Q^4$ behaviour can be seen, but we attribute these to the unreliability of the numerical determination of $F(x)$ for extremely large $x$. For large $Q$, one observes Porod’s law, $\tilde{F}(Q) \sim Q^{-(d+1)}$, for all $\tilde{M}$. The coefficients $A_4$ and $A_P$ defined in the introduction are determined and plotted in Figs. 8 and 9 respectively as functions of $\tilde{M}$. $A_4$ increases with increasing $\tilde{M}$ and $A_P$ is a decreasing function of $\tilde{M}$, approaching zero like a cusp at the coexistence curve.

Figures 5 and 6 show that the width of the peak increases slightly, but is rather insensitive to changes in $\tilde{M}$ until very near the coexistence curve. In the logarithmic plots there appear to be damped oscillations in $\tilde{F}(Q)$ at intermediate $Q$ before the onset of the $Q^{-(d+1)}$ behaviour. In two dimensions, as one approaches the coexistence curve, the main peak decreases in amplitude until it is comparable to these oscillations, which show up as a shoulder to the main peak. In three dimensions there is a tail on the large $Q$ side of the peak in the structure factor, which also grows as $\tilde{M}$ increases. Both the secondary peak and the tail may be related to the fact that the Tomita sum rule is strongly broken as $\tilde{M} \to 1$. The large coefficient of $x^2$ in the small $x$ expansion of $\bar{F}(x)$ will lead to corrections to Porod’s law for the medium $Q$ behaviour of the structure factor.

VI. COMPARISONS

In order to test the validity of the assumptions made in this paper the results for $F(x)$ and $\tilde{F}(Q)$ will be compared with the relevant results of other investigators. Experiments involving neutron scattering off of a binary alloy have been done for a fixed $\tilde{M}$ [27], but we have been unable to find any experimental study of the dependence of $\tilde{F}(Q)$ on $\tilde{M}$. One problem with doing experiments near the coexistence curve is that the small volume fraction of the minority phase causes the structure factor to have a small amplitude, thus
making it difficult to measure. Another problem that arises when comparing experiment to theory is that it is unclear what volume fraction was used in a given experiment, making a straightforward comparison difficult.

While there are high quality numerical simulations for critical quenches [28–30], there has been far less work on off-critical quenches. One example is the direct numerical simulation of the Cahn-Hilliard equation in two dimensions performed by Chakrabarti et al. [15]. Here, we compare their result for the correlation function with ours. Their functions are scaled so that the first zero of the correlation function occurs at \( x = 1 \), and we have adjusted our length scale to correspond to this. The comparisons for volume fractions \( \phi = 0.5, 0.21 \) and \( 0.05 \) are shown in Fig. 10. The relationship between \( \tilde{M} \) and the volume fraction \( \phi \) is

\[
\phi = \frac{1}{2}(1 - \tilde{M})
\]  

(6.1)

which is valid for quenches to \( T = 0 \). The quantitative agreement is poor. In particular, the theory predicts that at large \( x \) the oscillations in \( F(x) \) have much larger amplitude than seen in the simulations. Nevertheless, the positions of peaks and troughs of the oscillations are in qualitative agreement. In addition, we agree on the observation that oscillations in the correlation function become weaker and have longer wavelength as the coexistence curve is approached. In summary, the qualitative agreement is reasonable. We are unaware of any direct simulation of the Cahn-Hilliard equation in three dimensions for the off-critical case. Such simulations are difficult because large system sizes are required to give a statistically meaningful distribution of droplets when the volume fraction is small.

One can also make comparisons with generalizations of the LSW theory [11,12]. This is in the regime of Ostwald ripening [31]. While much of the analysis in this case has focussed on the droplet distribution function, more recently a number of authors have determined \( \tilde{F}(Q) \). In particular, here we will compare our three dimensional results with those of Akaiwa and Voorhees [13]. They assume that the droplets are spherical and interacting essentially electrostatically through a concentration field with both monopole and dipole contributions. Both the droplet size distribution function and the structure factor can be extracted by
numerical simulation of the equations produced by the theory. Our structure factors and
those of [13] are compared in Fig. 11. Both results agree and give Porod’s law at large
$Q$. At small $Q$ both results exhibit $Q^4$ behaviour, although our results seem to have a
smaller coefficient of $Q^4$ than theirs. This may also be why the widths of our peaks are
consistently smaller than those of [13]. There is also significant disagreement on the shape
of the structure factor for values of $Q$ just above the peak. In the theory presented here this
regime of $Q$ may be strongly affected by the breaking of the Tomita sum rule.

VII. CONCLUSION

In this paper it has been shown that the theory developed in [1] can be extended to the
case of off-critical quenches. The LSW $t^{1/3}$ law and the associated scaling behaviour are
determined for the entire concentration range. The scaling function is a function only of
the parameters $d$ and $\tilde{M}$, changing significantly only close to the coexistence curve where
the oscillations observed in the critical case are damped out. The structure factor exhibits
Porod’s law for large $Q$ and $Q^4$ behaviour at small $Q$. This is the first theory which is
capable of sensibly treating spinodal decomposition over the entire concentration range.

As discussed above, there are a number of virtues of this theory. However, there are also
important limitations. First, we have not been able to make contact with the LSW theory in
the $\tilde{M} \to 1$ limit. This will require extending the current theory (or some improved version)
to treat the droplet distribution in the dilute limit. This is a difficult but, to us, interesting
challenge. Secondly, it is clear that we must extend the theory developed here to include non-
Gaussian corrections if we are to remedy the problem of $C_0(q,t)$ going negative for small
$q$. Since one expects this quantity to enter the determination of the droplet distribution
function in an important way it is crucial to include non-Gaussian corrections if one is to
make progress in this area. Non-Gaussian corrections have already been used to treat the
critical COP case and this is discussed in [1]. Finally, it seems reasonable to assume that
the primary reason that we do not obtain good quantitative agreement for $\tilde{F}(Q)$ and $F(x)$
is that we do not satisfy the Tomita sum rule. We speculate that in order to satisfy the Tomita sum rule an improved treatment of the gradient term in the constitutive relation (2.5) is required.

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FIGURES

FIG. 1. The eigenvalues $\alpha$ and $\beta_2$ as a function of $y$. (a) $\alpha$ for two dimensions (upper curve) and three dimensions (lower curve). (b) $\beta_2$ for two dimensions (lower curve at $y = 0$) and three dimensions (upper curve at $y = 0$).

FIG. 2. The normalized scaling function $F(x)$ in two dimensions for various $\tilde{M}$. In terms of decreasing depth of the first minimum the curves correspond to (a) $\tilde{M} = 0, 0.2$, and $0.4$. (b) $\tilde{M} = 0.6, 0.8$, and $0.9$.

FIG. 3. The normalized scaling function $F(x)$ in three dimensions for various $\tilde{M}$. In terms of decreasing depth of the first minimum the curves correspond to (a) $\tilde{M} = 0, 0.2$, and $0.4$. (b) $\tilde{M} = 0.6, 0.8$, and $0.9$.

FIG. 4. The large $y$ asymptotic scaling function $\tilde{F}_\infty(z)$. From lowest to uppermost the solid curves correspond to $y = 2.5, 3, 3.5$, and $4$ ($\tilde{M} = 0.9876, 0.9973, .9995, \text{ and } .9999$ respectively). The infinite $y$ form ($\tilde{M} = 1$), obtained from our asymptotic analysis is shown as a dashed line. (a) two dimensions. (b) three dimensions.

FIG. 5. The normalized structure factor in two dimensions. From lowest to uppermost the curves correspond to $\tilde{M} = 0, 0.4, 0.6, 0.8$, and $0.9$.

FIG. 6. The normalized structure factor in three dimensions. From lowest to uppermost the curves correspond to $\tilde{M} = 0, 0.4, 0.6, 0.8$, and $0.9$.

FIG. 7. Logarithmic plots of $e^{y^2/2} \tilde{F}(Q)$. At $\ln(Q) = 0$, from lowest to uppermost the curves correspond to $\tilde{M} = 0, 0.4, 0.6, 0.8$, and $0.9$. (a) two dimensions. (b) three dimensions. In both graphs $Q^4$ behaviour is seen at small $Q$ and Porod’s law $Q^{-(d+1)}$ behaviour occurs at large $Q$ (dotted lines).

FIG. 8. The coefficient $A_4(\tilde{M})$ appearing in the small $Q$ behaviour of $\tilde{F}(Q)$ for two (upper curve) and three (lower curve) dimensions.
FIG. 9. The Porod’s law coefficient $A_P(\tilde{M})$ for two (lower curve) and three (upper curve) dimensions.

FIG. 10. A comparison of our scaling forms for the correlation function (solid lines) in two dimensions with those of [15] (dashed lines). The horizontal axis has been chosen so that the first zero of $F(x)$ occurs at $x = 1$ for both functions. (a) $\phi = 0.5$. (b) $\phi = 0.21$. (c) $\phi = 0.05$.

FIG. 11. A comparison of our structure factors (solid lines) in three dimensions with those of [13] (dashed lines). (a) $\phi = 0.3$. (b) $\phi = 0.2$. (c) $\phi = 0.1$. (d) $\phi = 0.05$. 