QUANTUM STATE DENSITY AND CRITICAL TEMPERATURE IN M-THEORY

M.C.B. ABDALLA, A.A. BYTSENKO, AND B.M. PIMENTEL

Abstract. We discuss the asymptotic properties of quantum states density for fundamental $p-$branes which can yield a microscopic interpretation of the thermodynamic quantities in M-theory. The matching of the BPS part of spectrum for superstring and supermembrane gives the possibility of getting membrane’s results via string calculations. In the weak coupling limit of M-theory the critical behavior coincides with the first order phase transition in standard string theory at temperature less then the Hagedorn’s temperature $T_H$. The critical temperature at large coupling constant is computed by considering M-theory on manifold with topology $\mathbb{R}^9 \otimes \mathbb{T}^2$. Alternatively we argue that any finite temperature can be introduced in the framework of membrane thermodynamics.

1. Introduction

Recently deep connections between fundamental (super) membrane and (super) string theory have been found. In particular, it has been shown that the BPS spectrum of states for type IIB string on a circle is in correspondence with the BPS spectrum of fundamental compactified supermembrane [1, 2]. Membrane thermodynamics can indicate non trivial information about microscopic degrees of freedom and the behavior of quantum systems at high temperature.

Finite temperature M-theory defined on a manifold with topology $\mathbb{R}^9 \otimes \mathbb{T}^2$, at weak and strong string coupling constant regime, has been considered recently in Ref. [3]. In the small circle radius limit (radius of compactification) M-theory must recover the string thermodynamics. In type IIB superstring theory, which is associated with M-theory in the weak coupling limit, the critical temperature coincides with the Hagedorn temperature [3]. In fact, in string theory there is a first order phase transition at temperature less than $T_H$ with a large latent

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heat leading to a gravitational instability. Note that the critical behavior in type IIA superstring theory, at large coupling limit, is not well understood.

The purpose of the present paper is to consider again the above mentioned problems, comparing strong and weak coupling regimes by considering M-theory on $\mathbb{R}^9 \otimes \mathbb{T}^2$, where one of the sides of the torus $\mathbb{T}^2$ is the Euclidean time direction (fermions obey antiperiodic boundary conditions). We turned into the problem of asymptotic density of quantum states for fundamental $p$-branes already initiated in Refs. [5, 6, 7, 8].

In Section 2 we consider asymptotic expansions of generating functions and thermodynamic quantities related to fundamental $p$-branes. The light-cone Hamiltonian formalism for membranes wrapped on a torus is summarized in Section 3. The weak coupling limit of M-theory is considered in Section 4. We calculate the one-loop free energy associated to strings (and $p$-branes). The limit of strong coupling constant is analyzed in Section 5. In thermodynamics of M-theory on $\mathbb{R}^9 \otimes \mathbb{T}^2$ we find the critical temperature, which coincides with Hagedorn temperature also obtained in Ref. [3]. However we argue that a more interesting possibility allows for a finite temperature to be introduced into the membrane theory. Finally we end up with some remarks.

2. ASYMPTOTICS OF QUANTUM STATES IN $P$-BRANE THERMODYNAMICS

Let us consider multi-component versions of the classical generating functions for partition functions, namely

$$\mathcal{G}_\pm(z) = \prod_{n \in \mathbb{Z}^p / \{0\}} \left[ 1 \pm \exp \left( -z \omega_n(a, g) \right) \right]^{\pm q}, \quad (2.1)$$

where $z = y + 2\pi i x$, $\Re z > 0$, $q > 0$ and $\omega_n(a, g)$ is given by

$$\omega_n(a, g) = \left( \sum_j a_j (n_j + g_j)^2 \right)^{1/2}, \quad (2.2)$$

g_j, and $a_j$ are some real numbers.

In the context of thermodynamics of fundamental $p-$branes, classical generating functions $\mathcal{G}_\pm(z)$ can be regarded as a partition function where $z \equiv \beta$ is the inverse temperature.

In this paper we shall be working only with the temperature dependent part of the free energy $F(\beta)$. In the following we write down the
statistical free energy $F(\beta)$ (both for $p$-branes and supersymmetric $p$-branes), the internal energy $E$ and the entropy $S$

$$F_p(\beta) = -\frac{1}{\beta} \log [\mathcal{G}_-(\beta)], \quad (2.3)$$

$$F_{sp}(\beta) = -\frac{1}{\beta} \log [\mathcal{G}_+(\beta)\mathcal{G}_-(\beta)], \quad (2.4)$$

$$E = \frac{\partial}{\partial \beta} [\beta F(\beta)], \quad S = \beta^2 \frac{\partial}{\partial \beta} F(\beta). \quad (2.5)$$

However, in order to calculate the above quantities we need first to know the total number of quantum states which can be described by the quantities $\Omega_{\pm}(n)$ defined by

$$K_{\pm}(t) = \sum_{n=0}^{\infty} \Omega_{\pm}(n)t^n \equiv \mathcal{G}_{\pm}(-\log t), \quad (2.6)$$

where $t < 1$, and $n$ is a total quantum number. The Laurent inversion formula associated with the above definition has the form

$$\Omega_{\pm}(n) = \frac{1}{2\pi \sqrt{-1}} \oint K_{\pm}(t)t^{-n-1}dt, \quad (2.7)$$

where the contour integral is taken on a small circle about the origin.

We shall use the results of Meinardus [9, 10, 11] that can be easily generalized to the vector-valued functions of the (2.1) type (for more detail see Ref. [3]). The $p$-dimensional Epstein zeta function $Z_p\|g\|_h(z, \varphi)$ associated with the quadratic form $\varphi[a(n+g)] = (\omega_n(a, g))^2$ for $\Re z > p$ is given by the formula

$$Z_p\|g\|_h(z, \varphi) = \sum_{n \in \mathbb{Z}^p} (\varphi[a(n+g)])^{-z/2} \times \exp \left[ 2\pi \sqrt{-1}(\mathbf{n}, \mathbf{h}) \right], \quad (2.8)$$

where $(\mathbf{n}, \mathbf{h}) = \sum_{i=1}^{p} n_i h_i$, $h_i$ are real numbers and the prime on $\sum$ means to omit the term $\mathbf{n} = -\mathbf{g}$ if all the $g_j$ are integers. For $\Re z < p$, $Z_p\|g\|_h(z, \varphi)$ is understood to be the analytic continuation of the right hand side of the Eq. (2.6). The functional equation for $Z_p\|g\|_h(z, \varphi)$ reads

$$Z_p\|g\|_h(z, \varphi) = (\det \mathbf{a})^{-1/2} \pi^{1/2} (2\pi)^{-p} \Gamma(\frac{p-z}{2}) \Gamma(\frac{z}{2}) \times \left[ \exp[-2\pi \sqrt{-1}(\mathbf{g}, \mathbf{h})] \right] \times Z_p\|\mathbf{h}\|_{-\mathbf{g}}(p-z, \varphi^*), \quad (2.9)$$
and $\varphi^*[a(n+g)] = \sum_j a_j^{-1}(n_j + g_j)^2$. Formula (2.9) gives the analytic
continuation of the zeta function. Note that $Z_p|_h^k|_0(z, \varphi)$ is an entire
function in the complex $z-$ plane except for the case when all the $h_i$
are integers. In this case $Z_p|_h^k|_0(z, \varphi)$ has a simple pole at $z = p$ with
residue

$$A(p) = \frac{2\pi p^2}{(\det a)^{1/2}\Gamma(p/2)},$$

which does not depend on the winding numbers $g_i$. Furthermore one
has $Z_p|_h^k|_0(0, \varphi) = -1$.

**Proposition 2.1.** In the half-plane $\Re z > 0$ there exists an as-

tymptotic expansion for $\mathcal{G}_\pm(z)$ uniformly in $x$ as $y \to 0$, provided
$|\arg z| \leq \pi/4$ and $|x| \leq 1/2$ and given by

$$\mathcal{G}_+(z) = \exp \left\{ q [A(p)\Gamma(p)\zeta_R(1 + p)z^{-p} - Z_p|_0^k|_0(0, \varphi)\log 2 + \mathcal{O}(y^c)] \right\},$$

(2.11)

$$\mathcal{G}_-(z) = \exp \left\{ q [A(p)\Gamma(p)\zeta_+(1 + p)z^{-p} - Z_p|_0^k|_0(0, \varphi)\log z + (d/dz)Z_p|_h^k|_0(z, \varphi)|_{z=0} + \mathcal{O}(y^{c-})] \right\},$$

(2.12)

where $0 < c_+, c_- < 1$ and $\zeta_-(s) \equiv \zeta_R(s)$ is the Riemann zeta function,
$\zeta_+(s) = (1 - 2^{1-s})\zeta_R(s)$.

By means of the asymptotic expansion of $K_\pm(t)$ for $t \to 1$, which is
equivalent to the expansion of $\mathcal{G}_\pm(z)$ for small $z$, and using formulae
(2.11) and (2.12) one arrives at a complete asymptotic limit of $\Omega_\pm(n)$:

**Theorem 2.1** (Meinardus; also see Ref. [8]). For $n \to \infty$ one has

$$\Omega_\pm(n) = C_\pm(n)\left\{ \frac{1}{p} + \frac{qA(p)\Gamma(1 + p)\zeta_\pm(1 + p)}{[2\pi(1 + p)]^{1/2}} \right\}^n [1 + \mathcal{O}(n^{-\kappa_\pm})],$$

(2.13)

$$C_\pm(p) = \frac{qA(p)\Gamma(1 + p)\zeta_\pm(1 + p)}{[2\pi(1 + p)]^{1/2}} \exp \left\{ q(d/dz)Z_p|_h^k|_0(z, \varphi)|_{z=0} \right\},$$

(2.14)

$$\kappa_\pm = \frac{p}{1 + p} \min \left( \frac{C_\pm}{p} - \delta, \frac{1}{4}, \frac{1}{2} - \delta \right),$$

(2.15)

and $0 < \delta < \frac{2}{3}$.
Using Eqs. (2.13) of the Theorem 2.1 and assuming linear Regge trajectories, i.e. the mass formula $M^2 = n$ for the number of brane states of mass $M$ to $M + dM$, one can obtain the asymptotic density for $p-$brane states as well as for super $p-$branes and they read

$$\rho(M) dM \simeq 2C_{\pm}(p) M^{(1+2p-2qZ_p^{|g|^{(0,\phi)}})/(1+p)} \exp \left[ b_{\pm}(p) M^{2p/(1+p)} \right], \quad (2.16)$$

$$b_{\pm}(p) \equiv \left(1 + \frac{1}{p} \right) [qA(p)\Gamma(1 + p)\zeta_R(1 + p)]^{\frac{1}{1+p}}. \quad (2.17)$$

$$b_{sp}(p) \equiv \left(1 + \frac{1}{p} \right) [qA(p)\Gamma(1 + p)(\zeta_+(1 + p) + \zeta_-(1 + p))]^{\frac{1}{1+p}}. \quad (2.18)$$

In the supersymmetric case we had to deal with product of generating functions $G_+^{(\beta)} \times G_-^{(\beta)}$.

This result has a universal character for all $p-$branes. With the help of Theorem 2.1, we can compute the complete $p-$brane state density (2.16), including the prefactors $C_{\pm}(p)$ and the factors $b_{\pm}(p)$, depending on the dimension of the embedding space.

There are branes which do no wind around the torus. Those cannot be treated in the semiclassical approximation. The free energy of the zero winding sector, for infinitely small Planck length, can be associated with the energy of supergravity theory. Our goal, however, is the calculation of the temperature dependent state density of branes.

An attempt to compare the asymptotic states density of branes and the density of states of neutral black holes has been made in Ref. [5, 6].

The prefactor for the degeneracy of black hole states at mass level represents general quantum field corrections to the state density. The asymptotic behaviour of classical entropy of near-extremal black branes coincides with the asymptotic degeneracy of some weakly interacting fundamental $p$-brane excitation modes.

The asymptotic density of states is consistent with the entropy of near-extremal $p$-branes (indeed $S \sim M^{2p/(1+p)}$). In the limit $\beta \rightarrow 0$ ($T \rightarrow \infty$) the entropy of fundamental objects may be identified with $\log(\Omega_{\pm}(n))$, while the internal energy is related to $n$. Therefore, from Eqs. (2.13), (2.14) and (2.3) - (2.5) one has

$$F_p(T) \simeq -qA(p)\Gamma(p)\zeta_R(1 + p)T^{p+1}, \quad (2.19)$$

$$F_{sp}(T) \simeq -qA(p)\Gamma(p)\left[\zeta_+(1 + p) + \zeta_-(1 + p)\right]T^{p+1}, \quad (2.20)$$

$$E_p \simeq pqA(p)\Gamma(p)\zeta_R(1 + p)T^{p+1}, \quad (2.21)$$

$$E_{sp} \simeq pqA(p)\Gamma(p)\left[\zeta_+(1 + p) + \zeta_-(1 + p)\right]T^{p+1}, \quad (2.22)$$
Eliminating the quantity $T$ among equations above one gets

\[ S_p \simeq \left( 1 + p \right) q A(p) \Gamma(p) \zeta_R(1 + p) T^p, \quad (2.23) \]

\[ S_{sp} \simeq \left( 1 + p \right) q A(p) \Gamma(p) \left[ \zeta_+(1 + p) + \zeta_-(1 + p) \right] T^p. \quad (2.24) \]

Thus the behavior of the entropy can be understood in terms of the degeneracy of some interacting fundamental $p-$brane excitation modes. Generally speaking the $p-$brane approach can yield a microscopic interpretation of the entropy.

3. Toroidal membranes

In this section we will consider the light-cone Hamiltonian formalism for membranes wrapped on a torus in Minkowski space. Such a compactification of M-theory with (-,+) spin structure, having the topology $\mathbb{R}^9 \otimes \mathbb{T}^2$, assumes that the dimensions $X^{11}, X^{10}$ are compactified on a torus with radii $R_{10}, R_{11}$ and two spatial membrane directions wind around this torus.

The single-valued functions on the torus $X^{10}(\sigma, \rho), X^{11}(\sigma, \rho)$, where $\sigma, \rho \in [0, 2\pi)$ are the membrane world-volume coordinates, have the form

\[ X^{10}(\sigma, \rho) = m_0 R_{10} \sigma + \tilde{X}^{10}(\sigma, \rho), \quad X^{11}(\sigma, \rho) = R_{11} \rho + \tilde{X}^{11}(\sigma, \rho). \quad (3.1) \]

The eleven bosonic coordinates are $\{X^0, X^i, X^{10}, X^{11}\}$ and in addition the transverse coordinates $X^i(\sigma, \rho), i = 1, 2, \ldots, 8$ are all single-valued. Transverse coordinates can be expanded in a complete basis of functions on the torus, namely

\[ X^i(\sigma, \rho) = \sqrt{\alpha'} \sum_{k, \ell} X^i_{(k, \ell)} e^{ik\sigma + i\ell\rho}, \quad (3.2) \]

\[ P^i(\sigma, \rho) = \frac{1}{(2\pi)^2 \sqrt{\alpha'}} \sum_{k, \ell} P^i_{(k, \ell)} e^{ik\sigma + i\ell\rho}. \quad (3.3) \]

Here $\alpha' = (4\pi^2 R_{11} T_2)^{-1}$, while $T_2$ is the membrane tension.

The membrane Hamiltonian in light-cone formalism \[12\ \ [13\ \ [14\ [15\ [8\]

can be written as follows: $H = H_0 + H_{\text{int}}$. In particular, for bosonic modes of membrane the explicit Hamiltonian is
\[ \alpha' H_0 = 8\pi^4 \alpha'T_2^2 R_{10}^2 R_{11}^2 m^2 + \frac{1}{2} \sum_n \left[ P_{n}^i P_{-n}^i + \omega_{km}^2 X_n^i X_{-n}^i \right], \] (3.4)

\[ \alpha' H_{int} = \frac{1}{4g_A^2} \sum (n_1 \times n_2)(n_3 \times n_4) X_{n_1}^i X_{n_2}^j X_{n_3}^i X_{n_4}^j. \] (3.5)

In Eqs. (3.4) and (3.5) \( n \equiv (k, \ell), n \times n' = k\ell' - \ell k', g_A^2 \equiv R_{11}^2 (\alpha')^{-1} = 4\pi^2 R_{11}^2 T_2, \omega_{k\ell} = (k^2 + m^2 \ell^2 R_{10}^2 R_{11}^2)^{1/2} \), and \( (m, k, \ell, k', \ell') \in \mathbb{Z} \). The interaction term (3.5) depends on the type IIA string coupling \( g_A \).

Mode operators, related to basic functions \( X^i(\sigma, \rho), P^i(\sigma, \rho) \), are

\[ X^i_{(k,\ell)} = \frac{1}{\sqrt{-2\omega_{(k,\ell)}}} \left[ \alpha^i_{(k,\ell)} + \tilde{\alpha}^i_{(-k,-\ell)} \right], \quad P^i_{(k,\ell)} = \frac{1}{\sqrt{2}} \left[ \alpha^i_{(k,\ell)} - \tilde{\alpha}^i_{(-k,-\ell)} \right], \] (3.6)

\[ (X^i_{(k,\ell)})^\dagger = X^i_{(-k,-\ell)}, \quad (P^i_{(k,\ell)})^\dagger = P^i_{(-k,-\ell)}, \] (3.7)

and \( \omega_{(k,\ell)} \equiv \text{sign}(k) \omega_{k\ell} \). The canonical commutation relations read

\[ [X^i_{(k,\ell)}, P^j_{(k',\ell')}^\dagger] = \sqrt{-1} \delta_{k+k'} \delta_{\ell+\ell'} \delta^{ij}, \] (3.8)

\[ [\alpha^i_{(k,\ell)}, \alpha^j_{(k',\ell')}^\dagger] = \omega_{(k,\ell)} \delta_{k+k'} \delta_{\ell+\ell'} \delta^{ij}. \] (3.9)

We have similar relations for the \( \tilde{\alpha}^i_{(k,\ell)} \).

Finally the mass operator takes the form

\[ M^2 = 2p^+ p^- - (p^i)^2 - p_{10}^2 = 2(H_0 + H_{int}) - (p^i)^2 - p_{10}^2. \] (3.10)

The Hamiltonian of membrane is non linear. But there are two situations where one can simplify this Hamiltonian:

(i) The limit \( g_A \to 0 \).

(ii) The other limit of large \( g_A \).

We shall consider these two cases in the next two sections.

4. Zero torus area limit of M-theory \( (g_A \to 0) \)

The zero torus area limit of M-theory on \( T^2 \) is related to the asymptotic \( g_A \to 0 \) at fixed \( (R_{10}/R_{11}) \). Such a limit in M-theory leads to a ten-dimensional type IIB string theory. More precisely, it has been shown \cite{II} \cite{I5} that quantum states of M-theory describe the \((p, q)\) strings bound states of type IIB superstring theory. We will consider in this section string theory at finite temperature associated with the \( g_A \to 0 \) limit of M-theory.
4.1. **Critical temperature in type II string theory.** Let us consider string theory in Euclidean space (time coordinate $X^0$ is compactified on a circle of circumference $\beta$). The presence of coordinates compactified on circles gives rise to winding string states. The string single-valued function $X^0(\sigma, \tau)$ admits an expansion:

$$X^0(\sigma, \tau) = x^0 + 2\alpha'p^0\tau + 2R_0w_0\sigma + \tilde{X}(\sigma, \tau),$$

where $p^0 = \ell_0(R_0)^{-1}$, $\ell_0, m_0 \in \mathbb{Z}$. The Hamiltonian and the level matching constraints becomes

$$H = \alpha'p_i^2 + \frac{m_0^2R_0^2}{\alpha'} + \alpha'\frac{\ell_0^2}{R_0^2} + 2(N_L + N_R - a_L - a_R) = 0,$$

$$N_L - N_R = \ell_0m_0,$$

where $a_L, a_R$ are the normal ordering constants, which represent the vacuum energy of the 1+1 dimensional field theory. In the case of type II superstring the number operators in the $m_0 = \pm 1$ sector read

$$N_L = \sum_{n=1}^{\infty} \left[ \alpha_i^i \alpha_n^i + (n-\frac{1}{2})S_n^a S_n^a \right], \quad N_R = \sum_{n=1}^{\infty} \left[ \tilde{\alpha}_i^i \tilde{\alpha}_n^i + (n-\frac{1}{2})\tilde{S}_n^a \tilde{S}_n^a \right],$$

where $i = 1, ..., 8$, $a = 1, ..., 8$. The normal-ordering constants are the same as in the NS sector of the NSR formulation, i.e. $a_L = a_R = 1/2$.

The critical temperature of string can be obtained as usual by determining the radius $R_0$ at which appear infrared instabilities. In the following we reproduce the critical Hagedorn temperature using the general state density formulation (2.16) - (2.18), which is more suitable for generalizing to membrane theory. Indeed, in the string case $p = 1$ the inverse critical temperature becomes:

$$\beta_{cr} = \frac{1}{2}b_{sp}(1) = [12\zeta_R(2)A(1)]^{1/2} = \left[ 2\pi^2 A(1) \right]^{1/2},$$

where the the factor 1/2 is related to the right- and left- moving modes ($N_L = N_R$) of closed string. Taking into account that $A(1) = 2$ and restoring the $\alpha'$ dependence one finds the critical Hagedorn’s temperature

$$T_H = \frac{1}{2\pi \sqrt{\alpha'}}.$$
This result is well known. In general the Hagedorn's temperature depends on the normal-ordering constants $a_L$, $a_R$ and has the form $T_H^{-1} = 2\pi \sqrt{2\alpha'(a_L + a_R)}$.

4.2. The one-loop free energy of strings. Let us consider the semi-classical free energy associated with fundamental compactified (super) $p$-branes (which is known to be divergent) embedded in flat $D$-dimensional manifolds with topologies $\mathcal{M} = \mathbb{S}^1 \otimes \mathbb{T}^p \otimes \mathbb{R}^{D-p-1}$. For the simplest quantum field model the free energy associated with bosonic and fermionic degrees of freedom has the form (see for example Refs. [16, 7, 8])

$$F^{(b,f)}(\beta) = -\pi^p (\det \mathcal{A})^{1/2} \int_0^\infty ds (2s)^{-(D-p+2)/2} \Xi^{(b,f)}(s, \beta) \times \Theta \left[ \begin{array}{c} g \\ 0 \end{array} \right] (0|\Omega) \exp \left( -\frac{sM_0^2}{2\pi} \right),$$

$$\Xi^{(b)}(s, \beta) = \theta_3 \left( 0 \middle| \frac{\sqrt{-1}\beta^2}{2s} \right) - 1, \quad \Xi^{(f)}(s, \beta) = 1 - \theta_4 \left( 0 \middle| \frac{\sqrt{-1}\beta^2}{2s} \right),$$

(4.7)

and $\theta_3(\nu|\tau)$ and $\theta_4(\nu|\tau) = \theta_3(\nu + i/2|\tau)$ are the Jacobi theta functions. Here $\mathcal{A} = \text{diag}(R_1^{-2}, \ldots, R_p^{-2})$ is a $p \times p$ matrix. The global parameters $R_j$ characterizing the non-trivial topology of $\mathcal{M}$ appear in the theory due to the fact that coordinates $x_j (j = 1, \ldots, p)$ obey the conditions $0 \leq x_j < 2\pi R_j$. The number of topological configurations of quantum fields is equal to the number of elements in group $H^1(\mathcal{M}; \mathbb{Z}_2)$, that is, the first cohomology group with coefficients in $\mathbb{Z}_2$. The multiplet $g = (g_1, \ldots, g_p)$ defines the topological type of field (i.e., the corresponding twist), and depends on the field type chosen in $\mathcal{M}$, $g_j = 0$ or $1/2$. In our case $H^1(\mathcal{M}; \mathbb{Z}_2) = \mathbb{Z}_2^p$ and so the number of topological configurations of real scalars (spinors) is $2^p$.

We follow the notations and treatment of Ref. [17] and introduce the theta function with characteristics $\mathbf{a}, \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_p$,

$$\Theta \left[ \begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \right] (\mathbf{z}|\Omega) = \sum_{\mathbf{n} \in \mathbb{Z}_p} \exp \left[ \pi \sqrt{-1}(\mathbf{n} + \mathbf{a})\Omega(\mathbf{n} + \mathbf{a}) + 2\pi \sqrt{-1}(\mathbf{n} + \mathbf{a})(\mathbf{z} + \mathbf{b}) \right].$$

(4.9)

In this connection $\Omega = (s\sqrt{-1}/2\pi^2)\text{diag}(R_1^2, \ldots, R_p^2)$.

The above mentioned method of the free energy calculation admits a subsequent development for extended objects. We will assume that
the free energy is equivalent to a sum of the free energies of quantum fields which are present in the modes of a \( p \)-brane. The factor \( \exp(-sM_0^2/2\pi) \) in Eq. (4.7) should be understood as \( \text{Tr} \exp(-sM_0^2/2\pi) \), where \( M \) is the mass operator of the brane and the trace is taken over an infinite set of Bose-Fermi oscillators \( N_n^{(b)}, N_n^{(f)} \).

The one-loop free energy of fields contained in a (super) \( p \)-brane can be evaluated making use the Mellin-Barnes representation for the energy integral \( F(\beta) = \int_{\mathbb{R}^+} ds \tilde{\gamma}(s, D) \text{Tr}[M^2] - s \) (see Ref. [18]) and the mass operators

\[
\text{Tr} \left[ e^{-tM^2} \right] = \frac{1}{2\pi\sqrt{-1}} \int_{S = s_0} ds \Gamma(s) \text{Tr}[tM^2]^{-s}, \quad (4.10)
\]

\[
\text{Tr}[M^2]^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} \left[ e^{-tM^2} \right], \quad (4.11)
\]

\[
\text{Tr} \left[ e^{-tM^2} \right] = \mathcal{G}_-(t), \quad \text{Tr} \left[ e^{-tM_{2p}^2} \right] = \mathcal{G}_+(t) \mathcal{G}_-(t), \quad (4.12)
\]

\[
\mathcal{G}_\pm(t) = \prod_{n \in \mathbb{Z}^p \setminus \{0\}} \left[ 1 \pm \exp(-t\omega_n) \right]^{\pm(D-p-1)}.
\]

One can use some substraction procedure for the divergent terms in \( \mathcal{G}_-(t), \mathcal{G}_+(t) \) in order to proceed regularization (see for detail Ref. [18]). To simplify the calculation we put \( a_j = R_j = 1 \) and the final result for the free energy [19, 20] is:

\[
F_p(\beta) \simeq -Q(D, p) \sum_{k=1}^\infty \frac{\Gamma(pk + \frac{1-p}{2})}{\Gamma(k)} \zeta_R(2pk + 1 - p)x^{1+p(2k-1)}, \quad (4.14)
\]

\[
F_{sp}(\beta) \simeq -Q(D, p) \sum_{k=1}^\infty \frac{\Gamma(pk + \frac{1-p}{2})}{\Gamma(k)} \zeta_R(2pk + 1 - p)x^{1+p(2k-1)}, \quad (4.15)
\]

where \( Q(D, p) \) is an integer function and

\[
x = \beta^{-1} y(p) \equiv \beta^{-1} \left[ (D - p - 1)2^{3p-2-\frac{p-1}{2}} \pi \frac{1}{2} \Gamma \left( \frac{p + 1}{2} \right) \zeta_R(p + 1) \right]^{\frac{1}{2p}}.
\]

The first terms of the leading behavior of the series (4.14), (4.15) (\( k = 1 \)) have to coincide with formulae (2.19) and (2.20) at \( q = D - p - 1 \). Therefore we have

\[
Q(D, p) = (D - p - 1)A(p)\Gamma(p)[y(p)]^{-1-p}. \quad (4.17)
\]
The asymptotic expansion of $\Gamma(s)$ for large value of $|s|$ has the form

$$\Gamma(s) = s^{s-\frac{1}{2}}e^{-s\sqrt{2\pi}} (1 + \mathcal{O}(s^{-1})) , \quad \arg s < \pi,$$  \hspace{1cm} (4.18)

and for $p > 1$ the power series (4.14) and (4.15) are divergent for any $x > 0$.

In the string case ($p = 1$) the corresponding series in Eqs. (4.14) and (4.15) can be resummed into trigonometrical form using the identities

$$\sum_{k=1}^{\infty} \zeta_R(2k)x^{2k} = \frac{1}{2} - \frac{1}{2} \pi x \cot(\pi x),$$  \hspace{1cm} (4.19)

$$\sum_{k=1}^{\infty} \zeta_R(2k)(1 - 2^{-2k}) x^{2k} = \frac{\pi x}{4} \tan\left(\frac{\pi x}{2}\right).$$  \hspace{1cm} (4.20)

The finite radius of Laurent series convergence $|x| < 1$ corresponds to the Hagedorn temperature in string thermodynamics (see for detail Ref. [18]). Using trigonometric relations, formulae (4.14) and (4.15) display a certain periodicity in temperature. The physical meaning of that behaviour is still obscure.

5. LARGE STRING COUPLING LIMIT ($g_A \to \infty$)

We now focus on the case (ii) mentioned in the end of section 3. by letting the constant $g_A$ be large. In this limit $R_{10}, R_{11}$ are large with fixed ($R_{10}/R_{11}$) and the non linear interacting Hamiltonian is multiplied by the small constant $g^{-2}_A$ so that it can be considered perturbatively. In the leading order of perturbative theory in $g^{-2}_A$ the interaction term can be dropped. The solution to the membrane equations of motion has the form [3]

$$X^i(\sigma, \rho, \tau) = x^i + \alpha'^i p^i \tau + \sqrt{-\alpha'}/2 \sum_{n \neq (0,0)} \omega_n^{-1} \times \left[\alpha_n^i e^{ik\sigma + il\rho} + \tilde{\alpha}_n^i e^{-ik\sigma - il\rho}\right] e^{i\omega_n \tau}.$$  \hspace{1cm} (5.1)

The momentum components in the directions $X^{10}$ and $X^{11}$ are given by $p_{10} = (\ell_{10}/R_{10})$, and $p_{11} = (\ell_{11}/R_{11})$, where $\ell_{10}, \ell_{11} \in \mathbb{Z}$. The nine-dimensional mass operator reads $M^2 = \mathcal{S}$, where

$$\mathcal{S} = \frac{\ell_{10}^2}{R_{10}^2} + \frac{\ell_{11}^2}{R_{11}^2} + \frac{m_0^2 R_{10}^2}{\alpha'^2} + \frac{1}{\alpha'} H;$$  \hspace{1cm} (5.2)

$$H = \sum_{k, \ell} \left(\alpha^i_{(-k, \ell)} \alpha^j_{(k, \ell)} + \tilde{\alpha}^i_{(-k, \ell)} \tilde{\alpha}^j_{(k, \ell)}\right).$$  \hspace{1cm} (5.3)

The level-matching conditions are [21 3]
Here

\[ N^+_\sigma = \sum_{\ell=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{k}{\omega_{kl}} \alpha^{i}_{(-k,-\ell)} \alpha^{\dagger i}_{(k,m)}, \quad (5.5) \]

\[ N^-_\sigma = \sum_{\ell=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{k}{\omega_{kl}} \tilde{\alpha}^{i}_{(-k,-\ell)} \tilde{\alpha}^{\dagger i}_{(k,\ell)}, \quad (5.6) \]

\[ N^+_\rho = \sum_{\ell=1}^{\infty} \sum_{k=0}^{\infty} \frac{\ell}{\omega_{kl}} \left[ \alpha^{i}_{(-k,-\ell)} \alpha^{\dagger i}_{(k,\ell)} + \tilde{\alpha}^{i}_{(-k,-\ell)} \tilde{\alpha}^{\dagger i}_{(k,\ell)} \right], \quad (5.7) \]

\[ N^-_\rho = \sum_{\ell=1}^{\infty} \sum_{k=0}^{\infty} \frac{\ell}{\omega_{kl}} \left[ \alpha^{i}_{(-k,-\ell)} \alpha^{\dagger i}_{(k,\ell)} + \tilde{\alpha}^{i}_{(-k,-\ell)} \tilde{\alpha}^{\dagger i}_{(k,\ell)} \right]. \quad (5.8) \]

Let us now define the quantum operator \( \hat{H} \) as

\[ \hat{H} = \sum_n \left( : \alpha^{i}_{(-k,-\ell)} \alpha^{\dagger i}_{(k,\ell)} : + : \tilde{\alpha}^{i}_{(-k,-\ell)} \tilde{\alpha}^{\dagger i}_{(k,\ell)} : \right), \quad (5.9) \]

where the annihilation operators \( \alpha^{i}_{(k,\ell)} \), \( \tilde{\alpha}^{i}_{(k,\ell)} \) are determined for \( k > 0 \) and \( \ell \in \mathbb{Z} \), and \( k = 0, \ell > 0 \). In Eq. (5.9) the normal ordering means taking the annihilation operators to the right. One can find the relation (see Ref. [3])

\[ H = \hat{H} + 2(D-3)E, \quad E = \frac{1}{2} \sum_{k,\ell} \omega_{kl}. \quad (5.10) \]

The constant energy shift \( 2(D-3)E \) (\( E \) is the Casimir energy) represents the purely bosonic contribution to the vacuum energy of the \((2+1)\) dimensional field theory. In the case of supersymmetry-preserving boundary conditions for fermions the contributions to the vacuum energy coming from bosonic and fermionic fields cancel out [22, 21]. This result also holds in a supersymmetric theory when non-linear terms are included.

5.1. Finite temperature quantum states in M-theory. Here we consider membrane excitation states with non-trivial winding numbers around the target space torus. It can be shown that the spectrum of the light-cone membrane Hamiltonian is discrete [22, 21, 3]. Let the Euclidean time coordinate \( X^0 \) plays the role of \( X^{10} \). Then fermions will obey antiperiodic boundary conditions around \( X^0 \) but periodic boundary conditions around \( X^{11} \). In the sector \( m_0 = \pm 1 \) fermions are
antiperiodic under the replacement $\sigma \to \sigma + 2\pi$ (while periodic under $\rho \to \rho + 2\pi$). The Hamiltonian operator becomes

$$H = \ell_0^2 + \ell_{11}^2 + R_0^2 \alpha'^2 + \frac{1}{\alpha'} (\tilde{H} + 2(D-3)E)$$

where

$$\tilde{H} = \sum_n \left[ : \alpha_{-n} \alpha_n^i : + : \alpha_{-n} \alpha_{n}^i : + \omega_{k+\frac{1}{2}m} \left( : S_{-n}^a n_S_n^a : + : \tilde{S}_{-n}^a \tilde{n} S_{n}^a : \right) \right]$$

and

$$E = E_B + E_F = \frac{1}{2} \sum_{k,m} (\omega_{km} - \omega_{k+\frac{1}{2}m})$$

In the case of supersymmetric membrane Eq. (2.18) gives

$$b_{sp}(2) = \frac{3}{2} \left[ 28 \zeta(3) A(2) \right]^{1/3}$$

Eq. (2.16) shows that for linear Regge-like trajectories the thermal partition function always diverges ($\int d \tilde{M} \exp(\tilde{M}^{3/4} - \beta M)$ is divergent). This IR divergence in the free energy might be regularized by some effects of brane theory, for example, like imposing U-duality or choosing non-linear behavior of Regge trajectory (let say $M^{(1+p)/p}$ or something similar). The U-duality properties of the membrane, considered in this paper, has been discussed in Ref. [23]. Even in the divergent case the factor $b_{sp}(p)$, associated with the regularized partition function, gives a correct value of the brane critical temperature. The statistical mechanical density of states (degeneracies) is given in Eqs. (2.13), (2.16) - (2.18). In the supersymmetric case the prefactors $C(2) = C_{-2}(2) C_{+2}(2)$ ($C_{-2}(2), C_{+2}(2)$ given by Eq. (2.14)) represent general quantum field theoretical correction to the state density.

Following the lines of paper [3] and taking into account that $\mathbf{a} = \text{diag}(1, g_{\text{eff}}^{-2})$ and $A(2) = 2\pi g_{\text{eff}}$, we havethe critical value $\beta_{cr} \simeq b_{sp}(2) = (3/2) \left[ 56 \pi \zeta(3) g_{\text{eff}} \right]^{1/3}$. Finally, restoring the dependence on parameter $\alpha'$ we have

$$T_{cr} \simeq \frac{2}{3\sqrt{\alpha'}} [56 \pi \zeta(3) g_{\text{eff}}]^{-1/3}.$$

We would like remark that in this approach the following condition for coupling constants holds: $g_{\text{eff}}^2 = 4\pi^2 T_{cr}^2 e_A^2 = 8\pi^3 \alpha' T_{cr}^2 R_{11}^3 (\ell_P)^{-1}$. Thus the critical temperature depends on the cutt-off parameter which is proportional to the Planck length $\ell_P$ [24, 3].
A more interesting possibility allows for a finite temperature to be introduced into the quantized fundamental (super) \( p \)-brane theory. We proof this statement in the next section.

5.2. Field thermodynamics presented in M-theory. In fact, the series (4.14), (4.15) are divergent, nevertheless one can construct an analytic continuation of these expressions. Let us define for \( |z| < \infty \)

\[
\mathcal{W}_\pm(z) = \sum_{k=0}^{\infty} \frac{\sqrt{\pi}}{\Gamma(k+1)\Gamma(pk + \frac{p+2}{2})} \nu_\pm(k; p) \left( \frac{z}{2} \right)^{p(2k+1)+1},
\]

where the factors \( \nu_\pm(k; p) \) have the form

\[
\nu_-(k; p) = (-1)^{pk+1},
\]

\[
\nu_+(k; p) = \nu_-(k; p) \left[ 1 - 2^{-p(2k+1) - 1} \right].
\]

For finite variable \( z \) these series converge and the convergence improves rapidly with the increasing of the integer number \( p \). Let \( z = j \cdot 2\pi x \), then the series

\[
\sum_{j=1}^{\infty} \mathcal{W}_\pm(j \cdot 2\pi x) = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \frac{\sqrt{\pi} \nu_\pm(k; p)}{\Gamma(k+1)\Gamma(pk + \frac{p+2}{2})} (j\pi x)^{p(2k+1)+1}. \tag{5.19}
\]

Now, if we commute the (now divergent) sum \( \Sigma_j \) with \( \Sigma_k \) extra terms of the type \( x^{-1}W_\pm(p) \) wi1 be generated on the right hand side of equation (5.19). Thus the result is

\[
\sum_{j=1}^{\infty} \mathcal{W}_\pm(j 2\pi x) + x^{-1}W_\pm(p)
\]

\[
= \sum_{k=0}^{\infty} \frac{\pi}{\Gamma(k+1)\Gamma(pk + \frac{p+2}{2})} \nu_\pm(k; p) \zeta_R[-p(2k + 1)] x^{1+p(2k+1)}
\]

\[
= \sin \left( \frac{\pi p}{2} \right) \sum_{k=1}^{\infty} \frac{\Gamma(pk + \frac{p}{2})}{\Gamma(k)} \zeta_\pm(2pk + 1 - p) x^{1+p(2k-1)}, \tag{5.20}
\]

where \( W_\pm(p) \) is an integer function of \( p \) (see for example Ref. [19]). In the second equality the functional equation for \( \zeta_R(s) \) has been used.

The new form of the free energy is:

\[
F_p(\beta) \sim \frac{Q(D, p)}{\sin \left( \frac{\pi p}{2} \right)} \sum_{k=0}^{\infty} \frac{(-1)^{pk} \pi}{\Gamma(k+1)\Gamma(pk + \frac{p+2}{2})}
\]
\begin{align}
\times \zeta_R[-p(2k + 1)] \left( \frac{\beta}{y(p)} \right)^{-1-p(2k+1)}, \quad (5.21)
\end{align}

\begin{align}
F_{sp}(\beta) \approx \frac{Q(D, p)}{\sin \left( \frac{\pi}{2} \right)} \sum_{k=0}^{\infty} \frac{(-1)^{pk} \left[ 2 - 2^{1-p(2k+1)} \right] \pi}{\Gamma(k + 1) \Gamma \left( pk + \frac{p+2}{2} \right)} \\
\times \zeta_R[-p(2k + 1)] \left( \frac{\beta}{y(p)} \right)^{-1-p(2k+1)}. \quad (5.22)
\end{align}

The divergent series in Eqs. (4.14) and (4.15) for the $p$-branes free energy, when reexpressed on the left hand side of Eq. (5.20), remain well-defined for finite temperature and have a smooth $\beta \to \infty \ (T \to 0)$ limit. The statistical internal energy and the entropy of finite temperature field theories (5.21), (5.22) can be easy calculated using Eq. (2.5).

6. Conclusions

It has been demonstrated recently that the BPS part of spectrum of type IIB string on a circle does match with the BPS part of supermembrane spectrum of states. In this paper we had dealt with the same discrete supermembrane spectrum as it has been used in the membrane-string correspondence. We calculated the critical temperature in the strong string coupling limit by considering M-theory on $\mathbb{R}^9 \otimes \mathbb{T}^2$ (one of the sides of the torus is the Euclidean time direction, and fermions obey antiperiodic boundary conditions). Yet a finite temperature can be introduced in membrane thermodynamics; we have proved this statement. It means physically that a membrane (if it can be quantized semi-classicaly) behaves like an ideal gas of quantum modes, which corresponds to a field theory at finite temperature (zero critical temperature).

There are deep connections between strings and membranes; at least they should be considered as different limits of a more general M-theory. Indeed, string results may be obtained via membrane-string correspondence and vice versa. Therefore, even being not a fundamental theory of (super) $p-$ branes may provide new deep insights in the understanding of string theory and consistent formulation of M-theory.

References

[1] J.H. Schwarz, Phys. Lett. B 360, 13 (1995).
[2] J.G. Russo and A.A. Tseytlin, Nucl. Phys. B 490, 121 (1997).
[3] J.G. Russo, “Free Energy and Critical Temperature in Eleven Dimensions”, hep-th/0101132.
[4] J.J. Atick and E. Witten, Nucl. Phys. B 310, 291 (1988).
[5] A.A. Bytsenko, K. Kirsten and S. Zerbini, Phys. Lett. B 304, 235 (1993).
[6] A.A. Bytsenko, K. Kirsten and S. Zerbini, Mod. Phys. Lett. A 9, 1569 (1994).
[7] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, "Zeta Regularization Techniques with Applications", World Sci., Singapore (1994).
[8] A.A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini, Phys. Reports 266, 1 (1996).
[9] G. Meinardus, Math. Z. 59, 338 (1954).
[10] G. Meinardus, Math. Z. 61, 289 (1954).
[11] G.E. Andrews, "The Theory of Partitions". In Encyclopedia of Mathematics and its Applications, Ed. Gian-Carlo Rota. Addison-Wesley Publishing Company (1976).
[12] E. Bergshoeff, E. Sezgin and Y. Tanii, Nucl. Phys. B 298, 187 (1988).
[13] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B 305 [FS 23], 545 (1988).
[14] J.G. Russo, in: Proceedings of the APCTP Winter School “Dualities in Gauge and String Theories” (Eds. Y.M. Cho and S. Nam), World Scientific Pub. Co. (1997); hep-th/9703118
[15] J.G. Russo, Nucl. Phys. B 535, 116 (1998).
[16] S.D. Odintsov, Rivista Nuovo Cim. 15, 1 (1992).
[17] D. Mumford, "Tata Lectures on Theta I, II", Birkhäuser (1983, 1984).
[18] A.A. Bytsenko, E. Elizalde, S.D. Odintsov and S. Zerbini, Nucl. Phys. B 394, 423 (1993).
[19] A.A. Actor and A.A. Bytsenko, Phys. Lett. B 315, 74 (1993).
[20] A.A. Bytsenko and S.D. Odintsov, Progr. Theor. Phys. 98, 987 (1997).
[21] M.J. Duff, T. Inami, C.N. Pope, E. Sezgin and K.S. Stelle, Nucl. Phys. B 297, 515 (1988).
[22] E. Bergshoeff, E. Sezgin and P.K. Townsend, Ann. Phys. 185, 330 (1987).
[23] A.A. Bytsenko and S.D. Odintsov, Förlchr. Phys. 41, No.3, 325 (1989).
[24] A.A. Bytsenko and S.A. Ktitorov, Phys. Lett. B 225, 325 (1989).

Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-900 - São Paulo, SP, Brazil E-mail address: mabdalla@ift.unesp.br

Departamento de Física, Universidade Estadual de Londrina, Caixa Postal 6001, Londrina-Parana, Brazil; on leave from Sant-Petersburg State Technical University, Russia E-mail address: abyts@uel.br

Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-900 - São Paulo, SP, Brazil E-mail address: pimentel@ift.unesp.br