Fixing the conformal window in QCD*

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ABSTRACT: A physical characterization of Landau singularities is emphasized, which should trace the lower boundary \( N_f^\ast \) of the conformal window in QCD and supersymmetric QCD. A natural way to disentangle “perturbative” from “non-perturbative” contributions below \( N_f^\ast \) is suggested. Assuming an infrared fixed point persists in the perturbative part of the QCD coupling in some range below \( N_f^\ast \) leads to the condition \( \gamma(N_f^\ast) = 1 \), where \( \gamma \) is the critical exponent. Using the Banks-Zaks expansion, one gets \( 4 \leq N_f^\ast \leq 6 \). This result is incompatible with the existence of an analogue of Seiberg duality in QCD. The presence of a negative ultraviolet fixed point is required both in QCD and in supersymmetric QCD to preserve causality within the conformal window. Evidence for the existence of such a fixed point in QCD is provided.

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1. Introduction

The notion of an infrared (IR) finite coupling has been used extensively in recent years, especially in connection [1] with the phenomenology of power corrections in QCD. The present investigation is motivated by the desire to understand better the theoretical background behind such an assumption. In particular, given an IR finite coupling $\alpha$, does it remain finite within perturbation theory itself (such as the two-loop coupling with opposite signs one and two loop beta function coefficients), or does one need a non-perturbative contribution $\delta \alpha$ to cancel $\alpha = \alpha_{PT} + \delta \alpha$? The answer I shall suggest is a mixed one: the perturbative part of the QCD coupling is always IR finite but, below the so called “conformal window” (the range of $N_f$ values where the theory is scale invariant at large distances and flows to a non-trivial IR fixed point), one still needs a $\delta \alpha$ term since the perturbative coupling is no more causal there, despite being IR finite. As the main outcome, one obtains an equation to determine the lower boundary of the conformal window in QCD. The plan of the paper is as follows. In section 2 I review the evidence and present a formal argument for the existence of Landau singularities in the perturbative coupling. A more physical argument,
relating Landau singularities to the very existence of the conformal window and a
two-phase structure of QCD is given in section 3, which also suggests a clean way to
disentangle “perturbative” from “non-perturbative” below the conformal window. In
section 4, two scenarios for causality breaking are described. In section 5, an equation
to determine the bottom of the conformal window in QCD is suggested, and is solved
through the Banks-Zaks expansion in section 6. Section 7 gives evidence, through
a modified Banks-Zaks expansion, for the existence of a negative ultraviolet (UV)
fixed point in QCD, necessary for the consistency of the present approach. Section
8 contains the conclusions.

2. Evidence for Landau singularities in the perturbative coupling

The only present evidence for a Landau singularity in the perturbative renormalized\(^1\)
coupling is still the old Landau-Pomeranchuk leading log QED calculation, now re-
formulated in QCD as a \(N_f \to -\infty\) (“large \(\beta_0\)”) limit. In this limit, the perturbative
coupling is one-loop: \(\alpha_{PT}(k^2) = \frac{1}{\beta_0} \log(k^2/\Lambda^2)\), where \(\Lambda\) is the Landau pole. The
question is whether there is a singularity at finite \(N_f\). Some light on this problem
is provided by further considering the \(N_f\) dependence. Indeed, another (conflicting)
boundary value is available around the value \(N_f = N_f^0 = 16.5\) (I consider \(N_c = 3\)) where the one loop coefficient \(\beta_0 = \frac{1}{4}(11 - \frac{2}{3}N_f)\)
of the beta function vanishes (“small \(\beta_0\)” limit). For \(N_f\) slightly below 16.5 a weak
coupling (Banks-Zaks) IR fixed point develops \[\text{2, 3, 4}\] and the perturbative coupling
is causal, i.e. there are no Landau singularities in the whole first sheet of the complex
\(k^2\) plane. Can then the perturbative coupling remain causal down to \(N_f = -\infty\)?
I shall assume that a Landau singularity cannot arise “spontaneously” in the limit,
i.e. that “the limit of a sequence of causal couplings must itself be causal”. In such
a case, the existence of a Landau pole at \(N_f \to -\infty\) implies the existence of a finite
value \(N_f^*\) below which Landau singularities appear on the first sheet of the complex
\(k^2\) plane and perturbative causality is lost, which is the common wisdom (at \(N_f^*\)
itself, according to the above philosophy, the coupling must still be causal). The
range \(N_f^* < N_f < N_f^0\) where the perturbative coupling is causal and flows to a finite
IR fixed point is taken as the definition of the “conformal window” for the sake of
the present discussion. Let us now refine this definition, and give a more physical
argument for the existence of Landau singularities, which illuminates their physical
meaning.

\(^1\)In QED, the well established “triviality” property gives only direct evidence \[\text{5}\] for a singularity
in the bare coupling constant.
3. Landau singularities and conformal window

I assume the existence of a two-phase structure in QCD as the number of flavors $N_f$ is varied.

i) For $N_f^* < N_f < N_f^0$ (the conformal window) the theory is scale invariant at large distances, and the vacuum is “perturbative”, in the sense there is no confinement nor chiral symmetry breaking. Conformal window amplitudes (generically noted as $D_{PT}(Q^2)$, where $Q$ stands for an external scale) are in this generalized sense “perturbative”, i.e. could in principle be determined from information contained in perturbation theory to all orders (although they should also include contributions from all instanton sectors): this motivates the subscript $PT$.

ii) For $0 < N_f < N_f^*$ there is a phase transition to a non-trivial vacuum, with confinement and chiral symmetry breaking, as expected in standard QCD.

A direct, physical motivation for Landau singularities can now be given: they trace the lower boundary $N_f = N_f^*$ of the conformal window. This statement is implied from the following two postulates:

1) Conformal window amplitudes $D_{PT}(Q^2)$ can be analytically continued in $N_f$ below the bottom $N_f^*$ of the conformal window.

2) For $N_f < N_f^*$, the (analytically continued) conformal window amplitudes $D_{PT}(Q^2)$ must differ from the full QCD amplitude $D(Q^2)$, since one enters a new phase, i.e. we have

$$D(Q^2) = D_{PT}(Q^2) + D_{NP}(Q^2)$$  \hspace{1cm} (3.1)

(wheras $D(Q^2) \equiv D_{PT}(Q^2)$ within the conformal window). Assuming QCD to be a unique theory at given $N_f$, $D_{PT}(Q^2)$ cannot provide a consistent solution if $N_f < N_f^*$: this must be signalled by the appearance of unphysical Landau singularities in $D_{PT}(Q^2)$. $N_f^*$ should thus coincide with the value of $N_f$ below which (first sheet) Landau singularities appear in $D_{PT}(Q^2)$. The occurrence of a “genuine” non-pertubative component $D_{NP}(Q^2)$ is then necessary below $N_f^*$ in order to cancel the Landau singularities present in $D_{PT}(Q^2)$. If these assumptions are correct, they provide an interesting connection between information contained in principle in “perturbation theory” (over all instanton sectors), which fix the structure of the conformal window amplitudes and “genuine” non-perturbative phenomena, which fix the bottom of the conformal window. In addition, eq.(3.1) provide a neat way to disentangle the “perturbative” from the genuine “non-perturbative” part of an amplitude, for instance the part of the gluon condensate related to renormalons from the one reflecting the presence of the non-trivial vacuum. Note also $D_{PT}(Q^2)$ and $D_{NP}(Q^2)$ are separately free of renormalon ambiguities, but contain Landau singularities below $N_f^*$, so the renormalon and Landau singularity problems are also disentangled. In order to get a precise condition to determine $N_f^*$, one needs to look in more details how causality can be broken in the perturbative coupling.
4. Scenarios for causality breaking

There are two main scenarios:

i) The “standard” one where the IR fixed point present within the conformal window just disappears when \( N_f < N_f^* \) while a real, space-like Landau singularity is generated in the perturbative coupling. For instance, two simple zeroes of the beta function can merge into a double zero when \( N_f = N_f^* \) before moving to the complex plane (a plausible scenario \([8]\) in supersymmetric QCD (SQCD)).

ii) Alternatively, it is possible for the fixed point to be still present\(^2\) in the perturbative part of the coupling in some range of \( N_f \) below \( N_f^* \). The motivation behind this assumption is the observation \([7, 8, 6]\) that for QCD effective charges associated to Euclidean correlators (the only ones for which the notion of \( k^2 \) plane analyticity makes sense) the Banks-Zaks expansion in QCD (as opposed to SQCD \([6]\)) seems to converge down to fairly small values of \( N_f \). In this case there can be no space-like Landau singularity, and causality must be violated by the appearance of complex Landau singularities on the first sheet of the \( k^2 \) plane. The example below suggests that they arise as the result of the continuous migration to the first sheet, through the time-like cut, of some second sheet singularities already present when \( N_f > N_f^* \).

I shall assume that this is the scenario which prevails in QCD. As the simplest example, consider the two-loop coupling, which satisfies the renormalization group (RG) equation
\[
d\alpha/d\log k^2 = -\beta_0 \alpha^2 - \beta_1 \alpha^3.
\]
If \( \beta_0 > 0 \) but \( \beta_1 < 0 \), there is an IR fixed point at \( \alpha_{IR} = -\beta_0/\beta_1 \). It has been shown \([4, 10, 6]\) that this coupling has a pair of complex conjugate Landau singularities on the second (or higher) sheet if
\[
0 < \gamma_{2-loop} = -\frac{\beta_0^2}{\beta_1} < 1 \tag{4.1}
\]
where \( \gamma_{2-loop} \) is the 2-loop critical exponent (see below). For \( \gamma_{2-loop} > 1 \), the second sheet singularities move to the first sheet through the time-like cut, which is reached when \( \gamma_{2-loop} = 1 \). The latter condition thus determines the bottom of the conformal window in this model. Note that in the limit \( \beta_1 \to 0^- \) where \( \gamma_{2-loop} = +\infty \), one gets the one loop coupling and the complex conjugate singularities collapse to a space-like Landau pole. This limit is thus the analogue of the \( N_f \to -\infty \) limit in QCD.

5. An equation to determine the bottom of the conformal window in QCD

Assuming from now on that the second scenario described above applies, i.e. that the IR fixed point persists in some range of \( N_f \) below \( N_f^* \), one needs some information on

\(^2\)This assumption is consistent with the suggestion \([3]\) that the perturbative coupling has a non-trivial IR fixed point down to \( N_f = 2 \) in QCD. However the full non-perturbative coupling must still differ by a \( \delta \alpha \) term, since the perturbative coupling is non-causal below \( N_f = N_f^* \).
the location of Landau singularities in coupling constant space to derive an equation for \( N_f^* \). I shall assume throughout that there are no complex (in \( \alpha \) space) Landau singularities (such as complex poles in the beta function), i.e. that the Landau singularities originate only from the \( \alpha < 0 \) or from the \( \alpha > \alpha_{IR} \) regions, and argue that the condition

\[
0 < \gamma < 1
\]  

is both necessary and sufficient for causality in QCD. Consequently, the lower boundary of the conformal window is obtained from the equation

\[
\gamma(N_f = N_f^*) = 1
\]  

where \( \gamma \) is the critical exponent defined as the derivative of the beta function at the fixed point

\[
\gamma = \left. \frac{d\beta(\alpha)}{d\alpha} \right|_{\alpha = \alpha_{IR}}
\]  

As is well known, this is a universal quantity, independant of the definition of the coupling, and eq.(5.2) is a renormalization scheme invariant condition, as it should. The argument proceeds in two steps.

1) Let us first assume (this will be justified below) there is an \( \alpha > \alpha_{IR} \) UV Landau singularity (for instance a pole in the beta function at \( \alpha_P > \alpha_{IR} \)), in the domain of attraction of \( \alpha_{IR} \). One can then show \[6\] that eq.(5.1) is a necessary condition for causality. I give an improved version of the argument of \[6\]. Solving the RG equation \( d\alpha/d\log k^2 = \beta(\alpha) \) around \( \alpha = \alpha_{IR} \), one gets

\[
\alpha(k^2) = \alpha_{IR} - \left( \frac{k^2}{\Lambda^2} \right)^\gamma + ...
\]  

There are thus rays

\[
k^2 = |k^2| \exp \left( \pm \frac{i\pi}{\gamma} \right)
\]  

in the complex \( k^2 \) plane, which in the infrared limit \( |k^2| \to 0 \) are mapped by eq.(5.4) to positive real values of the coupling larger than \( \alpha_{IR} \). Assuming an expansion

\[
\beta(\alpha) = \gamma(\alpha - \alpha_{IR}) + \gamma_1(\alpha - \alpha_{IR})^2 + ...
\]  

the corrections to eq.(5.4) are given by a series

\[
\log(k^2/\Lambda^2) = \frac{1}{\gamma} \log(\alpha_{IR} - \alpha) + \frac{\gamma_1}{\gamma^2}(\alpha_{IR} - \alpha) + ...
\]  

with real coefficients, showing that the only contribution to the phase for \( \alpha > \alpha_{IR} \) comes from the logarithm on the right hand side of eq.(5.7). The trajectories in the \( k^2 \) plane which map to the \( \alpha > \alpha_{IR} \) region are thus straight lines to all orders of
perturbation theory around $\alpha_{\text{IR}}$. This fact suggests that even away from the infrared limit, these trajectories are given by the rays eq.(5.5). As $|k^2|$ is increased along these rays, the coupling will flow to the assumed UV Landau singularities, reached at some finite value of $|k^2|$. If $\gamma > 1$ the rays, hence also the singularities, are located on the first sheet of the $k^2$ plane, showing that eq.(5.1) is a necessary condition for causality.

2) To assert whether eq.(5.1) is also sufficient for causality, one has to make sure that no other sources of (first sheet) Landau singularities are present, but the one arising from the $\alpha > \alpha_{\text{IR}}$ region. Barring Landau singularities at complex $\alpha$, a potential problem can still arise from an eventual UV Landau singularity at $\alpha < 0$, in the domain of attraction of the trivial IR fixed point $\alpha = 0^-$. Indeed at weak coupling the solution of the RG equation is controlled by the 2-loop beta function

$$\log(k^2/\Lambda^2) = \frac{1}{\beta_0 \alpha} + \frac{\beta_1}{\beta_0^2} \log \alpha + \text{const} + ...$$  \hspace{1cm} (5.8)

where the $\text{const}$ is real. For $\alpha < 0$ the right hand side of eq.(5.8) acquires a $\pm i\pi \beta_1/\beta_0^2$ imaginary part, which implies the rays

$$k^2 = |k^2| \exp\left(\pm i\pi \frac{\beta_1}{\beta_0^2}\right)$$  \hspace{1cm} (5.9)

map to the $\alpha < 0$ region. Along the rays eq.(5.9), we are effectively in a QED like situation: increasing $|k^2|$, the coupling is either attracted to a non-trivial UV fixed point, or reaches an UV Landau singularity at some finite $|k^2|$. In the latter case, one must require that the condition:

$$|\frac{\beta_2^2}{\beta_1}| < 1$$  \hspace{1cm} (5.10)

is satisfied in the whole $N_f$ range where eq.(5.1) is valid, which will confine the rays, hence the singularities to the second (or higher) sheet.

However in QCD condition eq.(5.10) can be satisfied only if $\beta_1 < 0$, and coincides with the 2-loop causality condition eq.(4.1), which requires $N_f > 9.7$. Therefore, to preserve causality within the conformal window as determined by eq.(5.1), $\gamma$ should reach 1 in the region $N_f > 9.7$, which is clearly excluded (see Fig. 1 below). In order that eq.(5.1) be also a sufficient condition for causality one must thus check that a non-trivial (finite or infinite) UV fixed point $\alpha_{UV}$ is present at negative $\alpha$!

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3Eq.(6.10) is however a necessary condition for causality for any beta function which admits an UV Landau singularity at negative $\alpha$ (in the domain of attraction of the $0^-$ trivial IR fixed point), and applies also if $\beta_1$ is positive in a general theory!

4It is a priori possible in QCD to have an $\alpha < 0$ UV Landau singularity rather then an $\alpha < 0$ UV fixed point. In such a case the bottom of the conformal window would be given as in the two-loop model by the condition $-\frac{\beta_2^2}{\beta_1} = 1$, yielding $N_f^* \approx 9.7$. Indeed, at such large $N_f$, any eventual UV Landau singularity from the $\alpha > \alpha_{\text{IR}}$ region has not yet reached the first sheet since $\gamma < 0.4$ as shown by Fig. 1. This possibility is however disfavored as explained in the text.
A minimal example satisfying this requirement is the 3-loop beta function
\[ \beta(\alpha) = -\beta_0 \alpha^2 - \beta_1 \alpha^3 - \beta_2 \alpha^4 \] with \( \beta_0 > 0 \) and \( \beta_2 < 0 \) (\( \beta_1 \) can have any sign).

It is worth mentioning eq.(5.10) is always violated \([\text{I}]\) in the lower part of the
conformal window in SQCD as determined by duality \([\text{II}]\), and the previous argument thus
implies the existence of a negative UV fixed point in this theory. In fact the
“exact” NSVZ \([\text{III}]\) beta function for \( N_f = 0 \) does exhibit an (infinite) UV fixed
point as \( \alpha \to -\infty \), which might be the parent of a similar one present within Seiberg
conformal window.

There is some evidence that a negative UV fixed point is indeed also present
in QCD. At the three-loop level, QCD effective charges associated to Euclidean
correlators have typically \( \beta_2 < 0 \), and appear \([\text{IV}]\) to be causal and admit a negative
UV fixed point, even somewhat below the two-loop causality boundary \( N_f = 9.7 \).
This evidence could be washed out in yet higher orders (e.g. the Pade improved
three-loop beta functions \([\text{V}]\)). However, a different and more systematic derivation
is given in section 7. The plausible existence of a nearby negative UV fixed point\(^5\)
in QCD, hence the absence of an UV Landau singularity at negative \( \alpha \), justifies a
posteriori (barring complex \( \alpha \) singularities) the above assumption that there must
be an \( \alpha > \alpha_{IR} \) singularity, to provide the necessary causality violation below the
conformal window.

6. Computing \( N_f^* \) through the Banks-Zaks expansion

One can try to use the Banks-Zaks expansion to compute \( \gamma \) and determine \( N_f^* \), the
lower boundary of the conformal window. The Banks-Zaks expansion \([\text{VI}, \text{VII}, \text{VIII}]\) is an
expansion of the fixed point in powers of the distance \( N_f^0 - N_f = 16.5 - N_f \) from
the top of the conformal window, which is proportional to \( \beta_0 \). The solution of the
equation
\[ \beta(\alpha) = -\beta_0 \alpha^2 - \beta_1 \alpha^3 - \beta_2 \alpha^4 + ... = 0 \] (6.1)
in the limit \( \beta_0 \to 0 \), with \( \beta_i (i \geq 1) \) finite is obtained as a power series

\(^5\)The non-trivial UV fixed point is actually not relevant to the proper analytic continuation of
the coupling at complex \( k^2 \), which must be consistent \([\text{IX}]\) with (UV) asymptotic freedom. This
means that in presence of this fixed point, the correct analytic continuation must involve complex
rather than negative values of \( \alpha \) along the rays eq. (5.9). One should approach the non-trivial
(rather than the trivial) IR fixed point as \( |k^2| \to 0 \), and the trivial (rather then the non-trivial) UV
fixed point as \( |k^2| \to \infty \). This is possible since the solution of eq.(5.8) is not unique for a given
(complex) \( k^2 \). For the same reason, any eventual IR Landau singularity arising from the region
\( \alpha < \alpha_{UV} \), in the domain of attraction of the non-trivial UV fixed point, is not relevant to the correct
analytic continuation. On the other hand, any UV Landau singularity in the domain of attraction
of either the trivial or the non-trivial IR fixed points as considered above is relevant to the proper
analytic continuation, since the coupling will flow to the only UV fixed point available, namely the
trivial one, once the UV Landau singularity is passed.
\[ \alpha_{IR} = a + \mathcal{O}(a^2) \]  

(6.2)

where the expansion parameter \( a \equiv \frac{8}{321}(16.5 - N_f) = \frac{16}{17} \beta_0 \). The Banks-Zaks expansion for the critical exponent eq.(5.3) is presently known \([4, 8, 6]\) up to next-to-next-to leading order:

\[ \gamma = \frac{107}{16} a^2 (1 + 4.75a - 8.89a^2 + ...) \]  

(6.3)

Using the truncated expansion eq.(6.3), one finds that \( \gamma < 1 \) for \( N_f \geq 5 \), with \( \gamma = 1 \) reached for \( N_f = N_f^* \simeq 4 \). To assess whether it is reasonable to use perturbation theory down to \( N_f = N_f^* \), let us look at the magnitude of the successive terms within the parenthesis in eq.(6.3). They are given by: 1, 1.44, -0.82. Although the next to leading term gives a very large correction, and the series seem at best poorly converging at \( N_f = N_f^* \), one can observe that the next-to next to leading term still gives a moderate correction to the sum of the first two terms, which might be considered together \([3]\) as building the “leading” contribution, since they are both derived from information contained \([4, 6]\) in the minimal 2-loop beta function necessary to get a non-trivial fixed point. Indeed, keeping only the first two terms in eq.(6.3), one finds that \( \gamma = 1 \) is reached for \( N_f = N_f^* \simeq 6 \). On the other hand, using a [1,1] Pade approximant as a model\(^6\) for extrapolation of the perturbative series, one gets

\[ \gamma = \frac{107}{16} a^2 \frac{1 + 6.62a}{1 + 1.87a} \]  

(6.4)

which yields \( \gamma = 1 \) for \( N_f = N_f^* \simeq 5 \). Fig. 1 shows \( \gamma \) as a function of \( N_f \):

Note that in the obtained range of \( N_f^* \) values (4 < \( N_f^* < 6 \), \( \beta_1 \) is still positive (\( \beta_1 \) changes sign for \( N_f = 8.05 \)) and of the same sign as \( \beta_0 \). The fixed point must thus arise from the contributions of higher then 2 loop beta function corrections, although I am assuming the Banks-Zaks expansion is still converging there. This is consistent with the previously mentioned fact that QCD effective charges have negative 3-loop beta function coefficients in the above range of \( N_f \) values.

### 7. Further evidence for a negative UV fixed point in QCD

Additional evidence for the existence of a couple of (negative-positive) UV-IR fixed points is provided by the following modified Banks-Zaks argument. Assume \( \beta_1 = 0 \), i.e. \( N_f = 8.05 \). Then a real fixed point can still exist at the three-loop level if \( \beta_2 < 0 \), and actually one gets a pair of opposite signs zeroes at \( \tilde{\alpha} = \pm (-\beta_0/\beta_2)^{1/2} \), an IR and an UV fixed point. If \( \beta_0 \) is small enough, they are weakly coupled, and calculable

\(^6\)The alternative [0,2] Pade yields a result (\( \gamma < 0.26 \) for any \( N_f \)) inconsistent with the present framework. It also predicts a not very plausible \( \mathcal{O}(a^5) \) coefficient of \( \simeq -192 \).
through a modified Banks-Zaks expansion around $N_f = 16.5$, applied to the auxiliary function $\tilde{\beta}(\alpha) \equiv \beta(\alpha) + \beta_1 \alpha^3$ with the two-loop term removed. One gets

$$\tilde{\alpha} = \pm \tilde{a} \left( 1 \mp \frac{1}{2} \frac{\beta_{3,0}}{\beta_{2,0}} \tilde{a} + \ldots \right)$$  \hspace{1cm} (7.1)$$

where the expansion parameter $\tilde{a} \equiv (-\beta_0/\beta_{2,0})^{1/2}$, and $\beta_{i,0}$, the $N_f = 16.5$ values of $\beta_i (i = 2, 3)$, are scheme dependent and can be obtained \cite{footnote} from the knowledge of the coefficients in eq.(6.3). For effective charges associated to Euclidean correlators, the correction in eq.(7.1) ranges from 0.1 to 0.7 at $N_f = 8.05$ (where $\tilde{\beta}(\alpha)$ coincides with $\beta(\alpha)$), which is encouraging evidence for a couple of UV and IR fixed points around this value of $N_f$ (which is within the alleged conformal window, but below the 2-loop causality region). Additional support is given by the calculation of the auxiliary critical exponent $\tilde{\gamma} \equiv d\tilde{\beta}(\alpha)/d\alpha|_{\alpha=\tilde{\alpha}_{IR}}$, whose modified Banks-Zaks expansion is

$$\tilde{\gamma} = 2 \beta_0 \tilde{a}(1 + \mathcal{O}(\tilde{a}^2))$$  \hspace{1cm} (7.2)$$

(no $\mathcal{O}(\tilde{a})$ correction!), which yields at $N_f = 8.05$ (where it coincides with $\gamma$) $0.6 < \tilde{\gamma} < 0.7$ for effective charges associated to Euclidean correlators, in reasonable agreement with the standard Banks-Zaks result (Fig. 1) $0.5 < \gamma < 0.6$.

8. Conclusions

1) I have suggested a direct, physical motivation for Landau singularities, assuming a two-phase structure of QCD: they should trace the lower boundary $N_f^*$ of the conformal window. This approach avoids the notoriously tricky disentangling of the “perturbative” from the “non-perturbative” part of the QCD amplitudes within the
conformal window, since they are by definition entirely “perturbative” there. On the other hand, such a disentangling is naturally achieved below the conformal window, by introducing the analytic continuation of the conformal window amplitudes to the $N_f < N_f^\ast$ region.

2) Assuming that the perturbative QCD coupling has a non-trivial IR fixed point $\alpha_{IR}$ even in some range below $N_f^\ast$ leads to the equation $\gamma(N_f = N_f^\ast) = 1$ to determine $N_f^\ast$ from the critical exponent $\gamma$ at the IR fixed point. Using the available terms in the Banks-Zaks expansion, this equation yields $4 \leq N_f^\ast \leq 6$. It would clearly be desirable to have more terms to better control the accuracy of the Banks-Zaks expansion. Note that this condition is inconsistent with the existence of an analogue of Seiberg duality in QCD, which would rather imply $\gamma(N_f = N_f^\ast) = 0$.

3) Some conditions on the QCD beta function are required: the only source of (UV) Landau singularities must come from the $\alpha > \alpha_{IR}$ region. One needs in particular a negative UV fixed point $\alpha_{UV}$. There is indeed some evidence for such a fixed point in QCD.

4) A negative UV fixed point is also required in the SQCD case, where duality fixes the conformal window.

5) It is possible the finite IR fixed point persists in the perturbative QCD coupling down to the $N_f \rightarrow -\infty$ one-loop limit. A simple example is provided by a beta function with one positive pole $\alpha_P$ (the required Landau singularity) and two opposite sign zeroes $\alpha_{IR}$ and $\alpha_{UV}$: $\beta(\alpha) = -\beta_0 \alpha^2 (1 - \alpha/\alpha_{IR})(1 - \alpha/\alpha_{UV})/(1 - \alpha/\alpha_P)$ where $\alpha_{UV} < 0$ and $0 < \alpha_{IR} < \alpha_P$. The one-loop limit is achieved for $\alpha_{IR} = \alpha_P$ and $\alpha_{UV} = -\infty$. A paper developing further these issues is under preparation.

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References

[1] See e.g. Yu.L. Dokshitzer, in International Conference “Frontiers of Matter”, Blois, France, June 1999 [hep-ph/9911299].

[2] T. Banks and A. Zaks, Nucl. Phys. B196 189 (1982).

[3] A.R. White, Phys.Rev. D29 1435 (1984) ; Int. J. Mod. Phys. A8 4755 (1993) [hep-th/9303053].

[4] G. Grunberg, Phys.Rev. D46 2228 (1992).

[5] G. Grunberg, Phys.Lett. B349 469 (1995).

[6] E. Gardi and G. Grunberg, JHEP 03 024 (1999) [hep-th/9810192].
[7] P.M. Stevenson, *Phys. Lett.* **B331** 187 (1994) [hep-ph/9402276]; S.A. Caveny and P.M. Stevenson, [hep-ph/9705319].

[8] E. Gardi and M. Karliner, *Nucl. Phys.* **B529** 383 (1998) [hep-ph/9802218].

[9] N.G. Uraltsev, private communication; G. Grunberg, [hep-ph/9705290].

[10] E. Gardi, G. Grunberg and M. Karliner, *JHEP* **07** 007 (1998) [hep-ph/9806462].

[11] N. Seiberg, *Nucl. Phys.* **B435** 129 (1995) [hep-th/9411149].

[12] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys.* **B229** 381 (1983).