Hybrid models for homological projective duals and noncommutative resolutions

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Abstract

We study hybrid models arising as homological projective duals (HPD) of certain projective embeddings \( f : X \to \mathbb{P}(V) \) of Fano manifolds \( X \). More precisely, the category of B-branes of such hybrid models corresponds to the HPD category of the embedding \( f \). B-branes on these hybrid models can be seen as global matrix factorizations over some compact space \( B \) or, equivalently, as the derived category of the sheaf of \( A \)-modules on \( B \), where \( A \) is a sheaf of \( A_\infty \)-algebras. This latter interpretation corresponds to a noncommutative resolution of \( B \). We compute explicitly the algebra \( A \) by several methods, for some specific class of hybrid models. If the target space of the hybrid model is a global orbifold, \( A \) takes the form of a smash product of an \( A_\infty \)-algebra with a finite group. However, this is not the case in general because the orbifold group can only be defined locally. One needs to treat the target space as an algebraic stack in such cases. We apply our results to the HPD of \( f \) corresponding to a Veronese embedding of projective space and the projective embedding of Fano complete intersections in \( \mathbb{P}^n \).

Keywords Quantum field theory · Homological algebra · Algebraic geometry

Mathematics Subject Classification 81T30 · 18G70 · 18G80

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1 Introduction

It is known that hybrid models provide realizations of a series of two-dimensional superconformal field theories which can be obtained from certain phases of gauged linear sigma models (GLSM) [1]. Roughly speaking, a hybrid model is a two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric field theory whose target space is of the form $Y = \text{Tot}(\mathcal{E} \to B)$ for some holomorphic vector bundle (or an orbibundle, in general) $\mathcal{E}$ over the base space $B$ where the fields interact via a superpotential $W \in H^0(\mathcal{O}_Y)$ which is a holomorphic function on the total space. The hybrid model can be viewed as a family of Landau–Ginzburg (LG) models fibered over the base space. Further sufficient conditions (but not necessary) in $Y$ and $W$ guarantee that these RG models flow to SCFTs and also make them tractable as quantum field theories (QFT); see for instance [2, 3].

Recently, it is found that homological projective dual (HPD) [4] of certain projective embeddings can be described by hybrid models. This was found in mathematics [5, 6], and a physics formulation is presented in [8]: if a GLSM $\mathcal{T}_X$ for a projective morphism $f : X \to \mathbb{P}(V)$ is known, one can build up an extended GLSM $\mathcal{T}'_X$ such that the Higgs branch of one of its phases gives rise to the HPD of $f : X \to \mathbb{P}(V)$. In the abelian cases, this construction provides a very explicit characterization of the HPD of $f : X \to \mathbb{P}(V)$. Indeed, the Higgs branch of the phase of $\mathcal{T}'_X$, relevant to the question, is a hybrid model.

These hybrid models, with target space $Y = \text{Tot}(\mathcal{E} \to B)$, can be viewed as (orbifold) LG models over affine charts of $B$ in the cases that $B$ is smooth, or if it has

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1 The first appearance of HPD in the context of dynamics of GLSMs can be found in [7].
the structure of a global orbifold. This gives rise to the global structure of a non-
commutative resolution, or more generally, the B-brane category becomes the derived
category of sheaves of \(A\)-modules for some sheaf of algebras \(A\). Let us illustrate this
in a well-known example.

For a LG model with quadratic superpotential \(W_{LG}\), it was shown in [9] that
the category of B-branes (homotopy category of matrix factorizations) \(MF(W_{LG})\)
is equivalent to the derived category of finite dimensional Clifford modules, where the
Clifford algebra is defined by the Hessian of the superpotential \(W_{LG}\). In addition, if a
\(\mathbb{Z}_2\) orbifold is present that leaves \(W_{LG}\) invariant, then the category of B-branes of this
LG orbifold \(MF(W_{LG}, \mathbb{Z}_2)\) is equivalent to the derived category of finite dimensional
modules of the even subalgebra of the corresponding Clifford algebra. Consequently,
the category of matrix factorizations of a \(\mathbb{Z}_2\)-orbifold hybrid model with superpotential
quadratic along the fiber coordinates is equivalent to the derived category of the sheaf
of modules of the sheaf of even parts of a Clifford algebra over the base \(B\). This is the
case of the HPD category of the degree 2 Veronese embedding \(\mathbb{P}(V) \hookrightarrow \mathbb{P}({\text{Sym}}^2 V)\)
[10]. Thus, the hybrid model orbifold can be viewed as a noncommutative resolution
of \(B\).

In this work, we generalize the idea and construct an explicit correspondence
between hybrid models and noncommutative spaces. Consider first the case where
\(B\) can be described locally by affine charts and \(Y\) is a global orbifold such that the
orbifold group \(G\) acts trivially on \(B\). Then, at a generic point in the base \(p \in B\) of the
hybrid model, we can model its dynamics by a LG orbifold. Denote the category of
B-branes of this LG orbifold as \(MF(W, G)\), where \(G\) is the orbifold group. We first
study the \(A_\infty\)-algebra \(A_{D0} = \text{End}(B_{D0})\) associated with the endomorphism algebra
of a \(D0\)-brane \(B_{D0} \in MF(W)\) of the LG model. The algebra \(A_{D0}\) takes the form of
an \(A_\infty\) algebra with a finite number of generators (as an algebra) \(\psi_i\) that satisfy the
higher products relations given in (4.65) and (4.66) (for a homogeneous \(W\)) and for
general elements (4.69) and (4.70)\(^2\). It then sets up the equivalence between matrix
factorizations and \(A_\infty\)-modules of \(A_{D0} \sharp G\), where \(\sharp\) denotes the smash product (the
mathematical approach toward this equivalence can be found in [11]), more precisely

\[
MF(W, G) \cong D(\text{Mod} - A_{D0} \sharp G),
\]

where the appearance of the derived category is a consequence that \(MF(W, G)\) is
taken to be the homotopy category. We then use this equivalence to relate a hybrid
model to a noncommutative resolution

\[
D(Y, W) \cong D(B, A_{D0} \sharp G),
\]

i.e., the derived category of sheaf of \(A_{D0} \sharp G\)-modules over \(B\).

The more general case, for example when \(G\) acts on \(B\), or more precisely, when \(B\)
has orbifold singularities and/or it cannot be written as a global orbifold still has a
similar structure. In such a case, we have that \(D(Y, W)\) is equivalent to the derived

\(^2\) We also consider the case of inhomogeneous \(W\). The \(A_\infty\) multiplication rules are given in (B.2) and
(B.3).
category of $A$-modules for some sheaf of $A_\infty$-algebras $A$, defined over the algebraic stack $Y$. We studied this case in detail in Sect. 4.6.

These results can be used to study HPD of several spaces. As mentioned above, the GLSM construction realizes the HPDs as hybrid orbifold models, which can be identified with noncommutative resolutions, or derived categories of sheaves of $A_\infty$-modules, as the equivalence suggests. Therefore, given a projective embedding engineered by an abelian GLSM, the hybrid model describing the HPD can be read off following [8]. One can then use the correspondence discussed in this paper to give a noncommutative geometric description of the HPD. More precisely, we apply these results to the following families of examples:

1. **HPD of degree $d$ Veronese embedding of $\mathbb{P}^n$**: The HPD of the degree $d$ Veronese embedding $\mathbb{P}(V) \hookrightarrow \mathbb{P}(\text{Sym}^2 V)$, $V \cong \mathbb{C}^{n+1}$ was studied in [8, 12] and is found to be given by a hybrid LG orbifold, i.e., its target space $Y$ can be written as a global orbifold, specifically

$$Y = \text{Tot} \left( \mathcal{O} \left( -\frac{1}{d} \right)^{\oplus (n+1)} \rightarrow \mathbb{P}(\mathbb{C}^{n+1})^{-1} \right) / \mathbb{Z}_d$$

where $\mathcal{O} \left( -\frac{1}{d} \right)$ denotes an orbibundle (see “Appendix C”) and a superpotential of degree $d$ in the fiber coordinates. In Sect. 5.1, we show that the B-brane category of this hybrid model can be written as

$$D(\mathbb{P}(\mathbb{C}^{n+1})^{-1}, \mathcal{A}_0^{\#} \mathbb{Z}_d) = \langle \mathcal{A}^{(1-C_{d,n})}, \ldots, \mathcal{A}^{(-1)}, \mathcal{A}^{(0)} \cong \mathcal{A}_0 \rangle; \quad (1.3)$$

where $\mathcal{A}_0 \cong \mathcal{A}_D$ is the $A_\infty$-algebra described above. We also describe explicitly the components of the semiorthogonal decomposition (predicted in [8]), in (1.3), in terms of $\mathcal{A}_0^{\#} \mathbb{Z}_d$-modules.

2. **HPD of Fano hypersurfaces**: The HPD of the degree $d$ Fano hypersurface inside $\mathbb{P}^n$, embedded naturally, is analyzed in Sect. 5.2. This case, analyzed in [8, 12], is similar to the HPD of the degree $d$ Veronese embedding, since the target space of the HPD is also a hybrid LG orbifold with target space given by a global orbifold, namely

$$Y = \text{Tot} \left( \mathcal{O}^{\oplus (n+1)} \oplus \mathcal{O}_{\mathbb{P}^n}^{(-1)} \rightarrow \mathbb{P}^n \right) / \mathbb{Z}_d$$

and superpotential

$$W = F_d(x) + p \sum_{i=0}^n x_i y_i, \quad (1.4)$$

(see Sect. 5.2 for a detailed description of the variables). The B-brane category of this hybrid model has a (dual) Lefschetz decomposition proposed in [8]:

$$D(Y, W) = \langle \mathcal{B}_{n-1}(1-n), \mathcal{B}_{n-2}(2-n), \ldots, \mathcal{B}_2(-2), \mathcal{B}_1(-1), \mathcal{B}_0 \rangle; \quad (1.5)$$
in Sect. 5.2, we describe this decomposition explicitly using the Lefschetz decomposition of the Fano hypersurface, induced by the natural embedding. In addition, we describe $D(Y, W)$ as a noncommutative resolution

$$D(Y, W) \cong D(\tilde{\mathbb{P}}^n, A_0^\sharp \mathbb{Z}_d),$$

(1.6)
giving very explicit constructions for the cases of degrees $d = 2, 3$.

3. **HPD of Fano complete intersections**: The HPD of Fano complete intersections was studied in [8]. For an intersection of hypersurfaces of degree $d_i$, $i = 1, \ldots, k$ on $\mathbb{P}^n$, the HPD was found to be given by a hybrid model with target space

$$Y = \text{Tot}(O(-1, 0)^{\oplus (n+1)} \oplus O(1, -1) \to W \mathbb{P}(d_1, \ldots, d_k) \times \tilde{\mathbb{P}}^n)$$

and superpotential

$$W = \sum_{\alpha=1}^k p_{\alpha} F_{d_{\alpha}}(x) + p \sum_{i=0}^n x_i y_i,$$

details of the notation can be found in Sect. 5.3. In this case, the main difficulty lies on the fact that, in general, $Y$ cannot be written as a global orbifold, but as a local one. In special cases such as $d_i = d$ for all $i$, we can write it as a global orbifold. As far as we are aware this is the less studied case in the literature. In Sect. 4.6, we describe the sheaf of algebra $\mathcal{A}$ in this case, as a sheaf of algebras over the algebraic stack $Y$ and we apply it, in Sect. 5.3, to the HPD of Fano complete intersections.

This paper is organized as follows. We review the basic facts about GLSMs for HPD and $A_\infty$-algebras in Sects. 2 and 3, respectively. In Sect. 4, we set up the relationship between matrix factorizations and $A_\infty$-modules. We first find the structure of the $A_\infty$-algebra $A_{D0}$ by various means (deformation theory of $B_{D0}$ and effective superpotential, $A_\infty$-homomorphism), then we propose a functor realizing the equivalence between the category of matrix factorizations and the derived category of $A_\infty$-modules of $A_{D0}$ and sketch its generalization to the orbifold case $A_{D0}^\sharp G$. We provide checks of this proposal in “Appendix A”. We then apply this correspondence in Sect. 5 to describe the HPD of degree $d$ Veronese embedding of projective space, Fano hypersurfaces and complete intersections in projective spaces as noncommutative spaces with the structure sheaf given by the corresponding sheaf of $A_\infty$-algebras. The same result was obtained for Veronese embeddings by summing over the ribbon trees in [12], we review this method in “Appendix B”. Finally, in Sect. 6 we give some details of the functor $D(B, \text{Mod} - A) \to D(X)$, i.e., of the embedding of the noncommutative resolution category into the derived category of the universal hyperplane section of $X$. 
2 Lightning review of GLSMs for HPD

In this section, we review the construction of homological projective duals (HPD) of projective morphisms $f : X \to \mathbb{P}(S)$ (where $S \cong \mathbb{C}^{n+1}$) proposed in [8]. We refer the reader to [8] for the details of definitions and notations. The construction of [8] assumes that we have a gauged linear sigma model (GLSM) construction for $f : X \to \mathbb{P}(S)$, i.e., a GLSM having a geometric phase corresponding to a Higgs branch\(^3\) that RG flows to a nonlinear sigma model (NLSM) whose target space is the image of $f$ in $\mathbb{P}(S)$. Denote that GLSM by

$$T_X = (G, \rho_m : G \to GL(V), W, t_{\text{ren}}, R). \quad (2.1)$$

We make some remarks about the tuple $T_X$:

- The gauge group $G$ is taken to be a compact Lie group and $W \in \text{Sym}(V^\vee)^G$ denotes the superpotential.
- The FI-theta parameter $t_{\text{ren}}$ is renormalized and depends on the energy scale for the case of anomalous GLSMs. In the following, to avoid cluttering we will simply denote $t := t_{\text{ren}}$ having in mind that $t$ may depend on the energy scale.
- The (vector) R-charge assignment $R$ will not play an important role in our discussions below. Moreover, it is only well defined in the IR and depends on the phase. Hence, we will leave it unspecified.
- For simplicity, in the discussion below, we will assume that the gauge group $G$ gets classically broken to a finite subgroup in every phase. This is always true if $G$ is abelian and $\rho_m$ is faithful. In the case that $G$ is nonabelian, this is usually not true (see for example [13]); however, the GLSM phases are still well-defined theories and we expect that our analysis can be carried out. All the examples we will cover in this work correspond to $G$ abelian. Some comments and conjectures for nonabelian $G$ can be found in [8].

The morphism $f$ must be base point free; hence, $f$ defines a line bundle $\mathcal{L}$ over $X$ given by

$$\mathcal{L} = f^*O_{\mathbb{P}(S)}(1). \quad (2.2)$$

Then, since there is a corresponding character $\chi \in \text{Hom}(G, \mathbb{C}^*)$ to $\mathcal{L}$, there exists a distinguished $U(1)_\mathcal{L} \subset G$ (with associated FI-theta parameter $t_{\mathcal{L}} = \xi_{\mathcal{L}} - i\theta_{\mathcal{L}}$) associated with $\mathcal{L}$ (see [8] for more details). The components of $f$ transform under $g \in G$ by multiplication by $\chi(g)$, i.e., they have homogeneous weight under $U(1)_\mathcal{L}$ and correspond to sections of $\mathcal{L}$. In the following, we assume that the aforementioned geometric phase is a pure Higgs phase\(^4\) and its category of B-branes will be denoted by $D(X_{\xi_{\mathcal{L}} \gg 1}) := D^b\text{Coh}(X_{\xi_{\mathcal{L}} \gg 1})$, if this phase is located at $\xi_{\mathcal{L}} \gg 1$. As we vary the parameter $\xi_{\mathcal{L}}$, we find, in general, that the phase at $\xi_{\mathcal{L}} \ll -1$ has a Higgs branch whose

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\(^3\) We do not need to assume the GLSM is nonanomalous.

\(^4\) It is always possible in the cases $X$ is Fano or Calabi–Yau (CY) to set this phase being located at $\xi_{\mathcal{L}} \gg 1$. These are the cases we will cover in this work. The generalization is straightforward.
category of B-branes we denote $D(Y_{\xi_L \ll -1}, W_{\xi_L \ll -1})$ and a mixed Coulomb–Higgs branch that splits into several isolated vacua, whose categories of B-branes we denote as $E_1, \ldots, E_k$. Both categories of B-branes at the different values of $\xi_L$ are expected to be related by \[14–17\]

\[
\langle D(Y_{\xi_L \ll -1}, W_{\xi_L \ll -1}), E_1, \ldots, E_k \rangle \cong D(X_{\xi_L \gg 1}) \tag{2.3}
\]

The equivalence (2.3) is realized at the level of the GLSM via the so-called window categories. They are defined entirely via the UV datum (or GLSM datum), i.e., the tuple $T_X$. Defining a B-brane $B$ in the GLSM requires to specify a representation $\rho_M : G \to GL(M)$. If we denote $q^L$ the weight of $\rho_M$ restricted to $U(1)_L$, then we define two conditions on the weights $q^L$:

- **Small window**: $|\theta^L + 2\pi q^L| < \pi \min(N_{L, \pm})$.
- **Big window**: $|\theta^L + 2\pi q^L| < \pi \max(N_{L, \pm})$, \tag{2.4}

where $N_{L, \pm} := \sum_a (Q^L_a)^\pm$, $(x)^\pm := (|x| \pm x)/2$ and $Q^L_a$ are the weights of $\rho_m$ restricted to $U(1)_L$. Therefore, we have the definition of the window subcategories by the constraints (2.4): $W_{+, b}^L$ (resp. $W_{-, b}^L$) corresponds to the objects $B$ such that the weights $q^L$ of $\rho_M$ satisfy the big (resp. small) window constraint for $b = \lfloor \theta^L / 2\pi \rfloor$. Then, we have

\[
D(Y_{\xi_L \ll -1}, W_{\xi_L \ll -1}) \cong W_{+, b}^L \leftrightarrow D(X_{\xi_L \gg 1}) \cong W_{+, b}^L \tag{2.5}
\]

for any $b \in \mathbb{Z}$. It is straightforward to see that in the nonanomalous case, $N_{L, +} = N_{L, -}$ and small and big window categories become equivalent giving an equivalence of categories

\[
D(Y_{\xi_L \ll -1}, W_{\xi_L \ll -1}) \cong W_b^L \cong D(X_{\xi_L \gg 1}), \tag{2.6}
\]

where we dropped the $\pm$ index. This equivalence, via window categories, is known as the grade restriction rule and was originally proposed for $G$ abelian and nonanomalous GLSMs in [18] and later rigorously generalized to anomalous GLSMs and nonabelian $G$ in [14, 19], the physical aspects of these generalizations were first studied in [15, 16] for nonanomalous and nonabelian GLSMs (plus some aspects of the anomalous case only for $G = U(1)$) and for anomalous and abelian GLSMs in [17]. Our presentation of the window categories (2.4) is based on [17]. Before moving on to the construction of the GLSM containing the HPD of $f : X \to \mathbb{P}(S)$, we illustrate this construction with a few examples:

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$^5$ The space $D(Y_{\xi_L \ll -1}, W_{\xi_L \ll -1})$ denotes the category of B-branes of a hybrid Landau–Ginzburg (LG) model with target space $Y_{\xi_L \ll -1}$ and superpotential $W_{\xi_L \ll -1} := W|_{Y_{\xi_L \ll -1}}$. The details on how such category arises in the current context can be found in [8].
Consider the case \( T_X = (U(1), \rho_m : U(1) \to GL(C^{n+2}), W = p_0 F_d(x), t, R) \) where \( (p_0, x_0, \ldots, x_n) \in C^{n+2}, \rho_m \) is defined as

\[
\rho_m(\lambda) \cdot (p_0, x_0, \ldots, x_n) = (\lambda^{-d} p_0, \lambda x_0, \ldots, \lambda x_n), \quad \lambda \in U(1)
\]

and \( F_d(x) \in C[x_0, \ldots, x_n] \) is homogeneous of degree \( d \leq n + 1 \) satisfying \( d F_d^{-1}(0) = \{0\} \). Then,

\[
X_{\xi \gtrsim 1} = \{F_d = 0, p_0 = 0\} \cap Y_{\xi \gtrsim 1}, \quad Y_{\xi \gtrsim 1} = O(-d) \to \mathbb{P}^n
\]

then, the function \( f \) corresponds to the natural embedding of \( X_{\xi \gtrsim 1} \) in \( \mathbb{P}^n \) and \( \mathcal{L} = O_X(1) \). The analysis of the window categories gives [14–17]

\[
D(C^{n+1}/\mathbb{Z}_d, F_d) \cong W_{-, b}^\mathcal{L} \hookrightarrow D(X_{\xi \gtrsim 1}) \cong W_{+, b}^\mathcal{L}
\]

and the equivalence (2.3) becomes

\[
\langle D(C^{n+1}/\mathbb{Z}_d, F_d), E_1, \ldots, E_{n+1-d} \rangle \cong D(X_{\xi \gtrsim 1}).
\]

Consider the case \( T_X = (U(1), \rho_m : U(1) \to GL(V), W, t, R) \) where

\[
V = \text{Sym}^d C^{n+1} \oplus \text{Sym}^d C^{n+1} \oplus C^{n+1},
\]

denote \( (p_1, \ldots, p_{(n+d)}, y_1, \ldots, y_{(n+d)}, x_0, \ldots, x_n) \in V \), the representation \( \rho_m \) is defined by

\[
\rho_m(\lambda) \cdot (p, y, x) = (\lambda^{-d} p, \lambda^d y, \lambda x), \quad \lambda \in U(1)
\]

\[
W = \sum_{j=1}^{(n+d)} p_j (y_j - f_j(x)),
\]

where \( f_j(x) \in C[x_0, \ldots, x_n] \) are the monomials of degree \( d \) in \( x \). In this case \( X_{\xi \gtrsim 1} \cong \mathbb{P}^n \), but the function \( f : \mathbb{P}^n \to \mathbb{P}((\text{Sym}^d C^{n+1})) \) becomes the degree \( d \) Veronese embedding. However, an analysis of this GLSM shows that the small window is empty and the equivalence (2.3) becomes

\[
\langle E_1, \ldots, E_{n+1} \rangle \cong D(\mathbb{P}^n) \cong W_{+, b}^\mathcal{L}.
\]

Indeed, this GLSM is equivalent to the GLSM \( T_{\mathbb{P}^n} = (U(1), \rho_m : U(1) \to GL(C^{n+1}), W \equiv 0, t, R) \), where \( \rho_m(\lambda) \) acts with weight 1 on every variable [20], i.e., to the usual GLSM for \( \mathbb{P}^n \). However, the HPD is expected to depend on \( f \) [4]. This is reflected in the following construction of the extended GLSM \( T_X \), which is the GLSM containing the HPD. The extended GLSM \( T_X \) depends on whether it is induced from \( T_X \) or \( T_{\mathbb{P}^n} \).
Fig. 1  Higgs branches of the GLSM $T_{X}$. The theory $T_{X}$ has a geometric phase realizing the universal hyperplane section $X$ and a LG phase realizing the HPD category $\mathcal{C}$. When $D(Y, \zeta_1 \ll -1, W_1 \ll -1)$ is empty, $\mathcal{C}' \cong \mathcal{C}$, otherwise $\mathcal{C}'$ is a subcategory of $\mathcal{C}$. The dashed arrow shows the direction of RG flow.

Starting from $T_{X}$, we define an extension $T_{\hat{X}}$ of $T_{X}$ given by

$$T_{\hat{X}} = (\hat{G} = G \times U'(1), \hat{\rho}_m : \hat{G} \to GL(V \oplus V'), \hat{W}, \hat{R}),$$

(2.15)

where $V'$ is a representation of $U'(1) \times U(1)_{\mathcal{L}} \subseteq \hat{G}$ with weights $(-1, -Q) \oplus (1, 0)^{\oplus(n+1)}$, where $Q \in \mathbb{Z}$ is the weight of the character $\chi$ defined above. Denoting the coordinates of $V'$ as $(p, s_0, \ldots, s_n)$, the superpotential $\hat{W}$ is given by

$$\hat{W} = W + p \sum_{j=0}^{n} s_j f_j(x),$$

(2.16)

where $f_j(x)$ are the components of the image of the map $f$. The GLSM $T_{\hat{X}}$ is identified with the GLSM of the universal hyperplane section $X$ of $X$. Its Higgs branch deep in the first quadrant of the FI parameter of $U'(1) \times U(1)_{\mathcal{L}}$ corresponds to a NLSM with target space $X$. The phase space of $(\zeta', \zeta_{\mathcal{L}})$ takes generically the form specified in Fig. 1, as analyzed in [8]. In Fig. 1, we have specified the B-brane categories on the Higgs branches in every phase, which are the relevant branches for determining the HPD of $f : X \to \mathbb{P}(S)$.

Keeping $\zeta' \gg 1$ and varying $\zeta_{\mathcal{L}}$ leads to the following equivalence of categories:

$$\mathcal{C} = D(\hat{W}_{\zeta_{\mathcal{L}} \ll -1}, W_{\zeta_{\mathcal{L}} \ll -1}) \cong \hat{W}_{\zeta_{\mathcal{L}} \ll -1, b} \hookrightarrow D(X_{\zeta_{\mathcal{L}} \gg 1}) \cong \hat{W}_{\zeta_{\mathcal{L}} \gg 1, b},$$

(2.17)

where the categories $\hat{W}_{\zeta_{\mathcal{L}} \ll -1, b}$ are defined analogously to $W_{\zeta_{\mathcal{L}} \ll -1, b}$, but in the GLSM $T_{\hat{X}}$. The category $\mathcal{C}$ is identified with the HPD category of $f : X \to \mathbb{P}(S)$, i.e., the proposal of [5, 6, 8] is that the subcategory of $D(X)$ corresponding to the small window category is equivalent to $\mathcal{C}$. We proceed to illustrate $T_{\hat{X}}$ and $\mathcal{C}$ in the example of a degree $d$ Veronese embedding (the example of a Fano hypersurface is reviewed in detail in Sect. 5.2.

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5 We recall the reader that $X$ is defined as the fiber product $X \times_{\mathbb{P}(S)} H \subset X \times \mathbb{P}(S^\vee)$ where $H = \{(u, v) \in \mathbb{P}(S) \times \mathbb{P}(S^\vee) | v(u) = 0\} \subset \mathbb{P}(S) \times \mathbb{P}(S^\vee)$ is the incidence divisor.
Consider the GLSM $T_X$ corresponding the degree $d$ Veronese embedding. Then, $\hat{G} = U'(1) \times U(1)_L$. The weights of the representation $\hat{\rho}_m : \hat{G} \to GL(V \oplus V')$ are given in the following table:

| $x_0$ | $x_1$ | $\cdots$ | $x_n$ | $p$ | $s_1$ | $\cdots$ | $s_{\binom{n+d}{d}}$ |
|-------|-------|-----------|-------|-----|-------|-----------|-------------------|
| $U(1)_L$ | 1 | $\cdots$ | 1 | $-d$ | 0 | $\cdots$ | 0 |
| $U'(1)$ | 0 | $\cdots$ | 0 | $-1$ | 1 | $\cdots$ | 1 |

We remark that, here, the fields $(p, s_j)$ span the representation $V'$ and the fields $x_i$ span the representation $V$, where we simplified the model by integrating out the massive fields $(y, p)$ of the original representation. The superpotential $\hat{W}$ then becomes

$$\hat{W} = p \sum_{j=1}^{\binom{n+d}{d}} s_j f_j(x). \quad (2.18)$$

Then, the Higgs branch in the region $\zeta_L \ll -1$ and $\zeta' \gg 1$ becomes the hybrid model with target space

$$Y_{LG} = \text{Tot} \left( \mathcal{O} \left( \frac{-1}{d} \right)^{\oplus(n+1)} \rightarrow \mathbb{P}_{\binom{n+d}{d}-1} \right) / \mathbb{Z}_d \quad (2.19)$$

and superpotential

$$W_{LG} = \sum_{j=1}^{\binom{n+d}{d}} s_j f_j(x); \quad (2.20)$$

then, the category $\mathcal{C}$, corresponding to the HPD of the degree $d$ Veronese embedding of $\mathbb{P}^n$, is given by $\mathcal{C} = D(Y_{LG}, W_{LG})$. In this particular family of examples, one can show $\mathcal{C} \cong \mathcal{C}'$ [8]. We will revisit the category $\mathcal{C} = D(Y_{LG}, W_{LG})$ in Sect. 5.1

Everything can be carried over when taking linear sections of $X$, but in this work we will be mainly interested in the HPD of $X$.

Using this proposal, we can express $\mathcal{C}$ as the category of B-branes on a Higgs branch that can be described as a fibered LG model, i.e., a hybrid model, which usually will have the characteristics of a good hybrid in the sense of [2, 3], making it very tractable.

3 Lightning review of $A_\infty$ algebras and their relation to open topological strings

In this section, we present the useful definitions and results that relate $A_\infty$ to the relevant physical systems we are going to need in the subsequent sections. Let us start
with the definition of $A_\infty$ algebra (our main reference is [21] but other useful sources are [22–24]).

**Definition** An $A_\infty$ algebra over a field $\mathbb{K}$ consists of a $\mathbb{Z}$-graded $\mathbb{K}$-vector space $A$

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$  \hspace{1cm} (3.1)

endowed with homogeneous $\mathbb{K}$-linear maps $^\otimes_n : A^\otimes n \to A$, $n \geq 1$ of degree $2 - n$ satisfying the relations

$$\sum_{r+s+t=n} (-1)^{r+s+t} m_u (1^\otimes r \otimes m_s \otimes 1^\otimes t) = 0, \quad n \geq 1$$  \hspace{1cm} (3.3)

where $u = r + t + 1$ and $s \geq 1$, $r, t \geq 0$.

Let us make a few important remarks. First, note that (3.3) implies $m_1 \circ m_1 = 0$; hence, $m_1$ is a differential. Second, the maps in the tensor products, such as in (3.3), are subject to the Koszul sign rule:

$$(f \otimes g) (a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b),$$  \hspace{1cm} (3.4)

where we assume $a$ is a homogeneous element of degree $|a|$ and $|g|$ denotes the degree of the map $g$. We define next a morphism of $A_\infty$ algebras.

**Definition** A morphism $f : A \to B$ between $A_\infty$ algebras consists of a family of maps

$$f_n : A^\otimes n \to B$$  \hspace{1cm} (3.5)

of degree $1 - n$ satisfying

$$\sum_{r+s+t=n} (-1)^{r+s+t} f_u (1^\otimes r \otimes m_s^A \otimes 1^\otimes t)$$

$$= \sum_{l=1}^n \sum_{I=n} (-1)^{\epsilon(l)} m^B_I (f_{i_1} \otimes \cdots \otimes f_{i_l}), \quad n \geq 1$$  \hspace{1cm} (3.6)

where $u = r + t + 1$, $s \geq 1$, $r, t \geq 0$ and the second sum over $I = n$ means sum over all decompositions $i_1 + \ldots + i_l = n$ (with $i_k \geq 1$). The sign $\epsilon(l)$ is given by

$$\epsilon(l) = (l - 1)(i_1 - 1) + (l - 2)(i_2 - 1) + \ldots + 2(i_{l-2} - 1) + (i_{l-1} - 1).$$  \hspace{1cm} (3.7)

Finally, we denote $m_n^{A,B}$ the maps of $A$ and $B$, respectively.

---

1 The grading of $A^\otimes n$ is given by $(A^\otimes n)^p = \bigoplus_{i_1 + \ldots + i_n = p} A^1 \otimes \cdots \otimes A^n$. Springer
Note that the map $f_1$ induces a map $f_{1,*}$ between the cohomologies

$$f_{1,*} : H(A) \to H(B),$$  \hspace{1cm} (3.8)

where $H(A)$ ($H(B)$) denotes the cohomology of the differential $m_A^1$ ($m_B^1$). Then, a morphism is called quasi-isomorphism if $f_{1,*}$ is an isomorphism and is called strict if $f_i = 0$ for all $i \neq 1$.

**Definition** An $A_{\infty}$-module over $A$ is given by a $\mathbb{Z}$-graded vector space $M$ endowed with maps

$$m_n^M : M \otimes A^{\otimes n-1} \to M, \hspace{1cm} n \geq 1$$  \hspace{1cm} (3.9)

of degree $2 - n$ satisfying

$$\sum_{r+s+t=n} (-1)^{r+st} m_u^M (1^{\otimes r} \otimes \tilde{m}_s \otimes 1^{\otimes t}) = 0, \hspace{1cm} n \geq 1 \hspace{1cm} (3.10)$$

where $u = r + t + 1$, $s \geq 1$, $r, t \geq 0$ and

$$m_u^M (1^{\otimes r} \otimes \tilde{m}_s \otimes 1^{\otimes t}) = \begin{cases} m_u^M (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}), & \text{if } r > 0 \\ m_u^M (m_s^M \otimes 1^{\otimes t}), & \text{if } r = 0 \end{cases} \hspace{1cm} (3.11)$$

There is an alternative construction of the $A_{\infty}$-algebra known as the bar construction. Consider a $\mathbb{Z}$-graded $K$-vector space $V$ and the tensor algebra

$$T^*V := \bigoplus_{n \geq 1} V^{\otimes n}. \hspace{1cm} (3.12)$$

Then, any coderivation $b : T^*V \to T^*V$ can be written in terms of degree 1 maps $b_n : V^{\otimes n} \to V$. Explicitly, by denoting $b_{n,u}$ the component of $b$ mapping $V^{\otimes n}$ to $V^{\otimes u}$, we can write

$$b_{n,u} = \sum_{r+s+t=n, r+t+1=u} 1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}, \hspace{1cm} r, t \geq 1, s \geq 1. \hspace{1cm} (3.13)$$

Imposing $b^2 = 0$ is equivalent to the conditions

$$\sum_{r+s+t=n} b_u (1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}) = 0, \hspace{1cm} n \geq 1 \hspace{1cm} (3.14)$$

where $u = r + t + 1$ and $s \geq 1$, $r, t \geq 0$. Then, if we identify $V = A[1]$, where $A[1]$ is the grading shift $(A[1])^p = A^{p+1}$ and we denote the natural degree $-1$ map $h : A \to A[1]$, then if we write

$$m_n = h^{-1} \circ b_n \circ h^{\otimes n} \hspace{1cm} (3.15)$$
or equivalently\(^8\)

\[ b_n = (-1)^{\frac{n(n-1)}{2}} h \circ m_n \circ (h^{-1})^\otimes n. \quad (3.16) \]

The relations (3.14) are equivalent to (3.3). An \(A_\infty\)-algebra \(A\) is called minimal if \(m_1 \equiv 0\) and is called strictly unital if there exists a degree 0 element \(1_A \in A^0\) satisfying

\[
\begin{align*}
  m_1(1_A) &= 0, \\
  m_2(1_A \otimes a) &= m_2(a \otimes 1_A) = a, \\
  m_i(a_1 \otimes \cdots \otimes a_i) &= 0, \text{ if any } a_k = 1_A \quad i > 2
\end{align*}
\]

(3.17)

for all \(a, a_1, \ldots, a_i \in A\). Moreover, if \(A\) is equipped with a bilinear form \(\langle \cdot, \cdot \rangle : A \otimes A \rightarrow \mathbb{C}\), then \(A\) is called cyclic (w.r.t. \(\langle \cdot, \cdot \rangle\)) if it satisfies

\[
\langle a_0, b_n(a_1 \otimes \cdots \otimes a_n) \rangle = (-1)^{(|a_0|+1)(|a_1|+\cdots+|a_n|+n)} \langle a_1, b_n(a_2 \otimes \cdots \otimes a_0) \rangle,
\]

(3.18)

where \(a_i \in A\) are homogeneous elements.

We have the following important theorem [25, 26]:

**Theorem** Any \(A_\infty\)-algebra \((A, m_n)\) is \(A_\infty\)-quasi-isomorphic to a minimal \(A_\infty\)-algebra called a minimal model for \(A\). Moreover, this minimal model can be taken to be \((H(A), m_n^H)\) which is unique up to \(A_\infty\)-isomorphism and satisfies

1. The map \(f_1 : H(A) \rightarrow A\) is given by the inclusion map.
2. The map \(m_2^H\) is given by the map induced by \(m_2\).

Then, this theorem plus the conditions (3.6) for \(A_\infty\) morphisms applied to the inclusion map \(\iota : H(A) \rightarrow A\) give us a way to recursively determine the products \(m_n^H\) from the knowledge of \((A, m_n)\). Let us write some of these relations to illustrate this point (recall that \(m_1^H \equiv 0\)):

\[
\begin{align*}
  \iota \circ m_2^H &= m_2(\iota \otimes \iota) + m_1 \circ f_2, \\
  \iota \circ m_3^H &= f_2(m_2^H \otimes 1) - f_2(1 \otimes m_2^H) + m_2(\iota \otimes f_2) - m_2(f_2 \otimes \iota) + m_1 \circ f_3,
\end{align*}
\]

\[
\vdots
\]

(3.19)

so, the maps \(f_n : H(A)^\otimes n \rightarrow A\) and the higher products \(m_n^H\) can be determined recursively (see for example [27]).

In the case of topological strings, we will be interested in \(A_\infty\)-categories, which are defined as follows

\(^8\) Here, we used that the inverse of \(h^\otimes n\) is \((-1)^{\frac{n(n-1)}{2}} (h^{-1})^\otimes n\). We remark that there are different ways to define \(b_n\) in terms of \(m_n\), leading to different sign conventions.
**Definition** A $A_\infty$-category $A$ with objects $\text{Ob}(A)$ consists of the data

1. For all $A, B \in \text{Ob}(A)$, the space $\text{Hom}_A(A, B)$ is a $\mathbb{Z}$-graded vector space.
2. For all $n \geq 1$ and any set of objects $A_0, \ldots, A_n \in \text{Ob}(A)$, there exists a degree $2 - n$ map

$$m_n : \text{Hom}_A(A_{n-1}, A_n) \otimes \text{Hom}_A(A_{n-2}, A_{n-1}) \otimes \cdots \otimes \text{Hom}_A(A_0, A_1) \rightarrow \text{Hom}_A(A_0, A_n)$$

satisfying

$$\sum_{r+s+t=n} (-1)^{r+s+t} m_u (1^\otimes r \otimes m_s \otimes 1^\otimes t) = 0. \quad (3.20)$$

We also define

**Definition** An $A_\infty$-functor between $A_\infty$-categories $A_1$ and $A_2$ consists of the data

1. A map $F : \text{Ob}(A_1) \rightarrow \text{Ob}(A_2)$.
2. For all $n \geq 1$ and any set of objects $A_0, \ldots, A_n \in \text{Ob}(A_1)$ there exists a degree $1 - n$ map

$$F_n : \text{Hom}_{A_1}(A_{n-1}, A_n) \otimes \text{Hom}_{A_1}(A_{n-2}, A_{n-1}) \otimes \cdots \otimes \text{Hom}_{A_1}(A_0, A_1) \rightarrow \text{Hom}_{A_2}(F(A_0), F(A_n))$$

satisfying conditions analogous to (3.6).

More precisely, in topological string theory we encounter cyclic, unital and minimal $A_\infty$-categories\(^9\) and we take the field $\mathbb{K} = \mathbb{C}$ from now on. It is easy to see that the $A_\infty$-category of a single object is equivalent to an $A_\infty$-algebra. Next, we move on to explain how these structures arise in topological strings. For simplicity, we consider a worldsheet with disk topology and boundary conditions characterized by a single D-brane $D$. Upon topological twist, this configuration has a single scalar nilpotent supercharge $Q$. The “off-shell” space of open strings stretching from $D$ to itself is given by a graded vector space, which we denote $V_D$ and there is an action of $Q$ in this vector space; hence, we can take the cohomology

$$\text{End}(D) := \text{H}_Q(V_D), \quad (3.21)$$

which is the space of physical states of the topological strings stretching between $D$ and itself. If we denote $\psi_a$ the elements of $\text{End}(D)$, their disk correlators encode the Stasheff conditions (3.3). More precisely, the disk correlator of two elements (the

\(^9\) We say an $A_\infty$-category has strict identities if, for every $A \in \text{Ob}(A)$ there is a degree 0 element $1_A \in \text{Hom}_A(A, A)$ satisfying the conditions (3.17), whenever a map $m_n$ can be consistently inserted as defined in (3.20).
boundary topological metric), denoted \( \langle \psi_a, \psi_b \rangle \), equips \( \text{End}(\mathcal{D}) \) with an (nondegenerate) inner product. In [28, 29], it is found that the relation between the disk correlators with more than two insertions and the maps \( b_k \)

\[
B_{i_0 i_1 \cdots i_k} := (-1)^{|a_1|+\cdots+|a_{k-1}|+k-k-1} \langle \psi_{i_0} \psi_{i_1} P \int \psi^{(1)}_{i_2} \cdots \int \psi^{(1)}_{i_{k-1}} \psi_{i_k} \rangle = \langle \psi_{i_0}, b_k(\psi_{i_1}, \cdots, \psi_{i_k}) \rangle,
\]

(3.22)

where \( \psi_a^{(1)} \) denotes the 1-form descendants of \( \psi_a \). The correlators (3.22) are defined using an appropriate regulator [29], and they satisfy a cyclicity condition:

\[
B_{i_0 i_1 \cdots i_k} = (-1)^{|a_m|+1+|a_0|+\cdots+|a_{k-1}|+k} B_{i_k i_0 \cdots i_{k-1}}. 
\]

(3.23)

It is important to remark that on the right-hand side of (3.22), the operators \( \psi_a \) should be considered in the space \( \text{End}(\mathcal{D}) \). In other words, the graded space \( A \) is identified with \( \text{End}(\mathcal{D}) \). Hence, up to a sign that, in general depends on the degree of the insertions, we can identify

\[
B_{i_0 i_1 \cdots i_k} \sim \langle m_k(\psi_{i_0}, \cdots, \psi_{i_{k-1}}), \psi_{i_k} \rangle.
\]

(3.24)

In general, for a SCFT we can define a trace function\(^{10}\)

\[
\gamma : A \to \mathbb{C}
\]

(3.25)

of degree \( -\hat{c} = -\frac{c}{3} \), where \( c \) is the central charge of the SCFT. Then, the inner product can be written as

\[
\langle \cdot, \cdot \rangle : A \otimes A \to \mathbb{C}, \quad (\psi_a, \psi_b) \mapsto \langle \psi_a, \psi_b \rangle = \gamma(m_2(\psi_a, \psi_b)).
\]

(3.26)

Then, we simply write the relation:

\[
B_{i_0 i_1 \cdots i_k} = \gamma \left( m_2(m_k(\psi_{i_0}, \cdots, \psi_{i_{k-1}}), \psi_{i_k}) \right),
\]

(3.27)

where the sign is hidden in \( \gamma \). When considering multiple branes, this structure becomes an \( A_\infty \)-category. For instance, in the case of a SCFT defined by the NLSM with a CY target space \( X \), the category of B-branes (topological open strings in the B-model) is equivalent to \( D\text{Coh}(X) \), the derived category of coherent sheaves on \( X \) [31–34] and an \( A_\infty \) structure on this category has been derived from physics and mathematical point of view [35–38]. Analogous results also exist in the case of \( G \)-equivariant categories of matrix factorizations \( MF_G(W) \), when \( G \) is a finite abelian group and \( W \) is a quasi-homogeneous polynomial [29, 39, 40].

\(^{10}\) This is just the (twisted) correlator on the sphere. See for example the review [30].
In this section, we will apply the results reviewed in Sect. 3 to the specific case of LG orbifolds. We begin by reviewing the physics approach of categories of matrix factorizations, arising as B-branes on LG orbifolds. Fix a vector space $V$ of rank $N$ with coordinates denoted by $x_i$, $i = 1, \ldots, N$. We specify a left $R$-symmetry given by a $\mathbb{C}^*$ action on $V$ with weights $q_i \in \mathbb{Q} \cap (0, 1)$. The orbifold group will be specified by a finite abelian group $G$ and a representation $\rho_{orb} : G \to GL(V)$. We specify a superpotential, that is a holomorphic, $G$-invariant function $W : \mathbb{C}^N \to \mathbb{C}$, $W \in \mathbb{C}[x_1, \ldots, x_N]$. As an $\mathcal{N} = (2, 2)$ theory, the LG orbifold is specified by the data

\[ (W, G, \rho_{orb}, \mathbb{C}_L^*), \tag{4.1} \]

but we impose some extra requirement on (4.1). In order for the vector $R$-symmetry to be nonanomalous, we require $W$ to be quasi-homogeneous, of weight 1 under the $\mathbb{C}_L^*$ action, i.e., $W(\lambda^q \phi_i) = \lambda W(\phi_i)$ [41]. (This implies that $W$ has charge 2 under the vector $R$-symmetry.) Moreover, $W$ being quasi-homogeneous implies $dW^{-1}(0) = \{0\}$, i.e., $W$ is compact, in the sense that it defines a compact SCFT in the IR. Quasi-homogeneity of $W$ guarantees that we always have the symmetry $x_j \to e^{2\pi i q_j} x_j$. If $d$ denotes the lowest nonzero integer such that $dq_i \in \mathbb{Z}$ for all $i$, then this specifies a $\mathbb{Z}_d$ action generated by $J = \text{diag}(e^{2\pi i q_1}, \ldots, e^{2\pi i q_N})$. Denote by $\text{Aut}(W)$ the group of diagonal automorphisms of $W$, i.e.,

\[ \text{Aut}(W) = \left\{ \text{diag}(e^{2\pi i \lambda_1}, \ldots, e^{2\pi i \lambda_N}) \in U(1)^N : W(e^{2\pi i \lambda_i} x_i) = W(x_i) \right\}. \tag{4.2} \]

We then say an orbifold group $G$ is admissible if it satisfies

\[ \langle J \rangle \subseteq G \subseteq \text{Aut}(W), \tag{4.3} \]

and we will require this condition. B-type D-branes $\mathcal{B}$ in LG orbifolds are characterized in terms of matrix factorizations of $W$ [9, 42]. More precisely, $\mathcal{B}$ consists of the data

\[ \mathcal{B} = (M, \sigma, Q, R_M, \rho_M), \tag{4.4} \]

where $M$ (the Chan–Paton space) is a free $\mathbb{C}[x_1, \ldots, x_N]$-module, $\sigma$ is an involution on $M$, inducing a $\mathbb{Z}_2$-grading (so we can write $M = M_0 \oplus M_1$, with $\sigma M_i = (-1)^i M_i$) and $Q(x)$ is a $\mathbb{Z}_2$-odd endomorphism on $M$ satisfying

\[ Q^2 = W \cdot \text{id}_M. \tag{4.5} \]

Under the vector $R$-charge, $W$ has charge 2: $W(\lambda^{2q_i} x_i) = \lambda^2 W(x_i)$ with the charges $q_i$ of the left $R$-symmetry. Therefore, by (4.5), $Q$ must have vector $R$-charge 1. This defines a compatible representation $R_M : U(1)_V \to GL(M)$ of the vector
R-symmetry, satisfying:

$$R_M(\lambda) Q(\lambda^{2q_i} x_i) R_M^{-1}(\lambda) = \lambda Q(x_i),$$  \hspace{1cm} (4.6)

as well as another compatible representation of $G$, $\rho_M : G \to GL(M)$ satisfying

$$\rho_M(g)^{-1} Q(\rho_{orb}(g) \cdot x_j) \rho_M(g) = Q(x_j).$$  \hspace{1cm} (4.7)

Given a pair of B-branes $B_i = (M^{(i)}, \sigma_i, Q_i, R_i^{(i)}, \rho_M^{(i)}), i = 1, 2$, we can define the space of morphisms between them, $\text{Hom}(B_1, B_2)$ as graded morphisms

$$\Psi : M^{(1)} \to M^{(2)},$$  \hspace{1cm} (4.8)

i.e., $\Psi \in V_{r_1,r_2} := \text{Mat}_{r_1,r_2}(\mathbb{C}[x_1, \ldots, x_N])$, the space of $r_1 \times r_2$ matrices with coefficients in $\mathbb{C}[x_1, \ldots, x_N]$, where $r_i = \text{rk}(M^{(i)})$ satisfying

$$D_{12} \circ \Psi := Q_2 \Psi - \sigma_2 \Psi \sigma_1 Q_1 = 0$$  \hspace{1cm} (4.9)

modulo $D_{12}$-exact morphisms. The differential $D_{12}$ can be identified with the conserved supercharge $Q$ of the worldsheet theory on the open string stretching between $B_1$ and $B_2$. Therefore, we can denote

$$\text{Hom}(B_1, B_2) = H_{D_{12}}(V_{r_1,r_2}).$$  \hspace{1cm} (4.10)

The space $\text{Hom}(B_1, B_2)$ is $\mathbb{Z}_2$-graded and we denote its homogeneous components, and elements, as

$$\text{Hom}(B_1, B_2) = H^0(B_1, B_2) \oplus H^1(B_1, B_2), \hspace{0.5cm} \phi_i \in H^0(B_1, B_2), \hspace{0.5cm} \psi_i \in H^1(B_1, B_2).$$  \hspace{1cm} (4.11)

The category $MF(W, G)$ of objects $B$ with morphisms defined as

$$\text{Hom}_{MF(W,G)}(B_1, B_2) := \text{Hom}(B_1, B_2)^G,$$  \hspace{1cm} (4.12)

i.e., $\Psi \in \text{Hom}_{MF(W,G)}(B_1, B_2)$ satisfies

$$\rho_M^{(2)}(g)^{-1} \Psi(\rho_{orb}(g) \cdot x_i) \rho_M^{(1)}(g) = \Psi(x_i),$$  \hspace{1cm} (4.13)

will be referred as the category of B-branes on the LG orbifold. This category also has a grading that we will review next.
4.1 Gradings

The category $MF(W, G)$ defined above has a natural $\mathbb{Q}$-grading given by the R-charge. More precisely, it is the fact that the superpotential $W$ is quasi-homogeneous that guarantees the existence of this $\mathbb{Q}$-grading (because then the vector R-charge is conserved) [43]. The orbifold by $G$ satisfying (4.3) guarantees that the physical states will have integer R-charges [44], and hence, we can put an integer grading on open string states. For a reduced and irreducible matrix factorization $\mathcal{B} \in MF(W, G)$, the map $\rho_M$ satisfies [43]

$$\rho_M(J) = \sigma \circ R_M(e^{i\pi}) e^{-i\pi \varphi}$$

(4.14)

for some $\varphi \in \frac{2}{d}\mathbb{Z}$. The morphism $\Psi \in H\text{om}_{MF(W, G)}(\mathcal{B}_1, \mathcal{B}_2)$ has R-charge $q_\Psi \in \mathbb{Q}$ defined by

$$R_{M(2)}(\lambda) \Psi(\lambda^{2q_i}x_i) R_{M(1)}(\lambda)^{-1} = \lambda^{q_\Psi} \Psi(x_i).$$

(4.15)

Then, a $\mathbb{Z}$-grading on $\Psi$ is defined by

$$\deg(\Psi) = \varphi_2 - \varphi_1 + q_\Psi.$$

(4.16)

The category $MF(W, G)$ with this additional grading is known in the mathematics literature as the category of graded, $G$-equivariant matrix factorizations [45].

4.2 Effective superpotential, deformations and $A_\infty$ structures

The category $MF(W, G)$ can be given an $A_\infty$ structure [46], and the higher-order products can be read off from the computation of the unobstructed deformations of the objects $\mathcal{B} \in MF(W, G)$, as we will explain in this section, and will become useful later. However, it is very convenient to use a description of $MF(W, G)$ that follows very closely [9]. Consider first the case of a trivial orbifold

$$G = 1,$$

(4.17)

then we denote the category just as $MF(W)$. Then, in the case $dW^{-1}(0) = \{0\}$, this category has a single generator [47] given by the matrix factorization $B_{D0} = (M, \sigma, Q_{D0}, R_M)$ with

$$Q_{D0} = \sum_{i=1}^{N} \left( x_i \bar{\eta}_i + q_i \frac{\partial W}{\partial x_i} \eta_i \right),$$

(4.18)

where the subscript $D0$ is because this matrix factorization is reminiscent to the $D0$-brane in [9]. The objects $\bar{\eta}_i, \eta_i, i = 1, \ldots, N$ are generators of a Clifford algebra of
rank \(2N\), namely they satisfy the relations
\[
\{\bar{\eta}_i, \eta_j\} = \delta_{i,j} \mathbf{1} \quad \{\bar{\eta}_i, \bar{\eta}_j\} = \{\eta_i, \eta_j\} = 0.
\] (4.19)

Then, we can consider
\[
A_{D0} := \text{Hom}_{MF}(W)(B_{D0}, B_{D0}),
\] (4.20)
which has the structure of an \(A_\infty\)-algebra\(^{11}\) [47] and moreover we have the equivalence
\[
MF(W) \cong D(\text{Mod} - A_{D0})
\] (4.21)
where \(D(\text{Mod} - A_{D0})\) stands for the derived category of \(A_\infty\)-modules over \(A_{D0}\). Given an object \(B \in MF(W)\), the module associated with \(B\) is given by \(M_B := \text{Hom}_{MF}(W)(B_{D0}, B_{D0})\), where the maps
\[
m_B^B : M_B \otimes A_{D0}^{\otimes n-1} \rightarrow M_B
\] (4.22)
come from the \(A_\infty\) structure of the category \(MF(W)\), in particular
\[
m_2^B : M_B \otimes A_{D0} \rightarrow M_B, \quad m_2^B(\psi^B, \psi) = \psi^B \circ \psi.
\] (4.23)

When we add an orbifold, we expect the following equivalence:
\[
MF(W, G) \cong D(\text{Mod} - A_{D0}\sharp G),
\] (4.24)
where \(A_{D0}\sharp G\) is the smash product between \(A_{D0}\) and the group algebra \(\mathbb{C}[G]\), the product in \(A_{D0}\sharp G\) is given by [11]
\[
(a \sharp g_1) \cdot (b \sharp g_2) = (a \cdot g_1 b g_1^{-1}) \sharp g_1 g_2;
\] (4.25)
hence, studying the algebra \(A_{D0}\) is crucial. The higher-order products \(m_n : A_{D0}^{\otimes n} \rightarrow A_{D0}\) can be read off from the effective superpotential \(W_{\text{eff}}\) defined by
\[
W_{\text{eff}} = \text{Tr} \left( \sum_{k=2}^{\infty} \sum_{i_0, i_1, \ldots, i_k} B_{i_0 \cdots i_k} \frac{Z_{i_0} Z_{i_1} \cdots Z_{i_k}}{k+1} \right),
\] (4.26)
the function \(W_{\text{eff}}\) encodes obstructions to the boundary deformations of the SCFT, and can be computed as follows. Consider the matrix factorization \(Q_{D0}\), then our objective is to find a deformed matrix factorization
\[
Q_{D0}^{\text{def}} = Q_{D0} + \sum_{\tilde{m} \in B} \alpha_{\tilde{m}} \tilde{m}, \quad \tilde{m} := \prod_{i=1}^{n} u_i^{m_i}
\] (4.27)
\(^{11}\) In the particular case that \(W\) is homogeneous of degree 2, then \(A_{D0}\) becomes simply the (complex) Clifford algebra \(Cl(q)\) associated with the quadratic form \(q_{ij} := \partial_i \partial_j W\) [9, 48].
where $B \subset \mathbb{N}^n$, $n = \dim H^1(B_{D0}, B_{D0})$, $\alpha_{\vec{m}}$ are fermionic operators and $u_i$, $i = 1, \ldots, n$ are commutative parameters. The matrix factorization satisfies

$$
(Q_{D0}^{\text{def}})^2 = W \cdot \text{id}_M + \sum_{i=1}^n f_i(u)\phi_i, \quad (4.28)
$$

where, using the same notation as (4.11), $\phi_i \in H^0(B_{D0}, B_{D0})$. Then, the critical locus of $W_{\text{eff}}$ coincides, as a set, with $f_1 = f_2 = \cdots = f_n = 0$. More precisely, if we identify the variables $Z_i$ with the parameters $u_i$, $Z_i \equiv u_i$ in $W_{\text{eff}}$, then $dW_{\text{eff}}^{-1}(0)$ coincides with the solutions to the equations $f_1 = f_2 = \cdots = f_n = 0$, i.e., we can integrate the equations

$$
\frac{\partial \tilde{W}_{\text{eff}}}{\partial u_i} = f_i(u) \quad (4.29)
$$

and $\tilde{W}_{\text{eff}}$ coincides with $W_{\text{eff}}$ up to a nonlinear redefinition of the parameters $u_i$. The operators $\alpha_{\vec{m}}$ are computed iteratively. We can summarize this process as follows. Define $|\vec{m}| := \sum m_i$. We start by defining

$$
\alpha_{e_i} := \psi_i \quad i = 1, \ldots, n \quad (4.30)
$$

with $e_i$, $i = 1, \ldots, n$ the canonical basis of $\mathbb{N}^n$. Then, in the first step we write

$$
Q_{D0}^{\text{def},(1)} = Q_{D0} + \sum_{i=1}^n u_i\alpha_{e_i} \quad (4.31)
$$

and we look at the terms of order $|\vec{m}| = 2$ in $(Q_{D0}^{\text{def},(1)})^2$, denote them $\sum |\vec{m}|=2 y_{\vec{m}} u_{\vec{m}}$. Then, if $y_{\vec{m}}$ is $Q_{D0}$-exact, then we can define an operator $\alpha_{\vec{m}}$ (with $|\vec{m}| = 2$) such that

$$
\beta_{\vec{m}} := -y_{\vec{m}} = [Q_{D0}, \alpha_{\vec{m}}]. \quad (4.32)
$$

Then, if we denote $B_2$ the set of all vectors $\vec{m}$ with $|\vec{m}| = 2$ such that $y_{\vec{m}}$ is exact, we can write

$$
Q_{D0}^{\text{def},(2)} = Q_{D0} + \sum_{i=1}^n u_i\alpha_{e_i} + \sum_{\vec{m} \in B_2} \alpha_{\vec{m}} u_{\vec{m}} \quad (4.33)
$$

and repeat the process to find the operators $\alpha_{\vec{m}}$ with $|\vec{m}| > 2$. The process ends when none of the $y_{\vec{m}}$ are $Q_{D0}$-exact.

### 4.3 LG model with homogeneous superpotential

In this section, we consider the case of a LG model with chiral superfields $x_i$, $i = 1, \cdots, n$. The superpotential $W$ is a homogeneous polynomial in $x_i$ with degree $d \geq 2$. 

\[ \text{Springer} \]
We will set the orbifold $G$ to be trivial in this section; hence, the relevant category of B-branes will be $MF(W)$. The $D0$-brane of this model is the matrix factorization $B_{D0}$ described in Sect. 4.18; therefore,

$$Q_{D0} = \sum_{i=1}^{n} \left( x_i \bar{\eta}_i + \frac{1}{d} \frac{\partial W}{\partial x_i} \eta_i \right). \quad (4.34)$$

We want to deduce the multiplication rule of the $A_\infty$-algebra

$$A_{D0} = \text{End}(B_{D0}). \quad (4.35)$$

The generators of the ring $H^1(B_{D0}, B_{D0})$ are straightforward to compute and given by\(^{12}\)

$$\psi_i = i \sqrt{\frac{d(d-1)}{2}} \bar{\eta}_i - \frac{i}{\sqrt{2d(d-1)}} \sum_{j=1}^{n} \frac{\partial^2 W}{\partial x_i \partial x_j} \eta_j, \quad 1 \leq i \leq n, \quad (4.36)$$

which satisfy \{ $Q_{D0}$, $\psi_i$ \} = 0. Note that

$$\{\psi_i, \psi_j\} = \frac{\partial^2 W}{\partial x_i \partial x_j} \textbf{1}, \quad (4.37)$$

where $\textbf{1} \in H^0(B_{D0}, B_{D0})$ is the identity operator and \{ $\psi_i, \psi_j$ \} $\simeq$ 0, for $d > 2$, i.e., (4.37) says that \{ $\psi_i, \psi_j$ \} is $Q_{D0}$-exact, because \{ $Q_{D0}$, $\eta_i$ \} $= x_i$. Hence, any monomial in $x_i$ with positive degree is $Q_{D0}$-exact. We remark that the operators (4.36) generate the whole $H^1(B_{D0}, B_{D0})$ as a ring (not necessarily as a vector space). Indeed, they generate the whole space $\text{End}(B_{D0})$. This can be shown, for instance, using the explicit form of $Q_{D0}$ and the fact that the Dirac matrices $\eta_i, \bar{\eta}_i, i = 1, \ldots, n$ plus the identity $\textbf{1}$ generate the off-shell dg algebra.

Next, we propose an explicit expression for the functor from $MF(W)$ to the category of modules of $A_{D0}$, for the case at hand. For any matrix factorization $B = (M, \sigma_M, Q_M, R_M)$, the corresponding $A_\infty$-module is given by $Hom(B_{D0}, B)$. The $A_\infty$-module structure is given by the $A_\infty$-multiplications of the $A_\infty$-category $MF(W)$ as described in (4.22). Conversely, given any $\mathbb{Z}_2$-graded $A_\infty$-module $N$ of $A_{D0}$, we propose that the corresponding matrix factorization is given by $M = N \otimes \mathbb{C}[x_1, \ldots, x_n]$ and

$$Q_M(\phi) = \sum_{k=1}^{d-1} \sum_{i_1, i_2, \ldots, i_k} m^N_{k+1}(\phi, \psi_{i_1}, \ldots, \psi_{i_k}) x_{i_1} x_{i_2} \cdots x_{i_k} \quad (4.38)$$

for $\phi \in M$. We provide several consistency checks for (4.38) in “Appendix A.”

\(^{12}\) Equation (4.36) denotes a particular representative of the $Q_{D0}$-class of $\psi_i$. All the computations where the explicit matrix form of $\psi_i$ is used do not depend on the choice of representative.
The $A_\infty$ structure of $A_{D0}$ was constructed explicitly in [12], in the case $W$ homogeneous of degree $d$. The $A_\infty$-algebra relations were found to be given by (A.1) when $d = 2$ or (A.2) and (A.3) when $d > 2$. In [12], this $A_\infty$ structure was proved by summing over the ribbon trees. In the remainder of this subsection, we give some alternative derivation of (A.1), (A.2) and (A.3) and we give explicit expressions for the higher-order products $m_d$, when acting on arbitrary elements of $A_{D0}$. For this purpose, we analyze separately the case $d = 2$ and $d > 2$. In the following, we write the map

$$\iota : A_{D0} = \text{End}(B_{D0}) \rightarrow V_{M_{D0}},$$

where $V_{M_{D0}}$ is the space of rank $(M)$-square matrices with values in $\mathbb{C}[x_1, \ldots, x_n]$, i.e., the space of endomorphisms of $B_{D0}$ without taking the homology. We can always give to the algebra $V_{M_{D0}}$ a dg algebra structure and will not spoil the $A_\infty$-relations of $A_{D0}$. In other words, $V_{M_{D0}}$ is the off-shell algebra of open strings and we can always find a dg algebra that is $A_\infty$-quasi-isomorphic to it (this fact is true for any $A_\infty$-algebra [24]). We will denote the image under $\iota$ of $\psi_i$ in $V_{M_{D0}}$ by $v_i$:

$$\iota(\psi_i) = v_i.$$  

In the following, we will also make use of the open disk one-point function correlators on the disk, for LG models. This was computed in [49] (see [50, 51] for a mathematical treatment) and is given by

$$\langle \Phi \rangle = \frac{1}{(2\pi i)^n} \oint_{x_i = 0} \text{Str} \left( \frac{\partial Q_{D0}}{\partial x_1} \wedge \cdots \wedge \frac{\partial Q_{D0}}{\partial x_n} \Phi \right) \frac{\partial W}{\partial x_1} \cdots \frac{\partial W}{\partial x_n} dx_1 \cdots dx_n.$$  

This corresponds to the (B-model) correlator in $D^2$ with a single boundary insertion $\Phi \in A_{D0}$ and boundary conditions defined by the brane $B_{D0}$.

### 4.3.1 $A_{D0}$ for $d = 2$

From the relations (3.6) and (3.19) (i.e., $f_2 : A_{D0}^\otimes 2 \rightarrow V_{M_{D0}}$):

$$\iota(m_2(\psi_i, \psi_j)) = v_i v_j + \{Q_{D0}, f_2(\psi_i, \psi_j)\}.$$  

When $d = 2$, $v_i v_j$ has no $Q_{D0}$-exact terms; therefore, we can choose $f_2(\psi_i, \psi_j) = 0$. Consequently,

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W}{\partial x_i \partial x_j},$$

i.e., under the multiplication $m_2$, $A_{D0}$ is the same as the Clifford algebra $Cl(n, \mathbb{C})$ with the quadratic form given by the Hessian of $W$. (4.43) can also be obtained by
computing the correlation functions, using (4.41). We illustrate this case with a simple example:

4.3.2 Example: \( W = x_1 x_2 \)

The D0-brane is

\[
Q = x_1 \bar{\eta}_1 + x_2 \bar{\eta}_2 + \frac{x_2}{2} \eta_1 + \frac{x_1}{2} \eta_2.
\]  

(4.44)

The fermionic open string states are

\[
\psi_1 = \bar{\eta}_1 - \frac{1}{2} \eta_2, \quad \psi_2 = \bar{\eta}_2 - \frac{1}{2} \eta_1.
\]  

(4.45)

The bosonic open string states can be taken to be\(^{13}\) \( e = 1 \) and \( \phi \) such that \( \langle e, \phi \rangle = 1 \) and \( \langle e, e \rangle = \langle \phi, \phi \rangle = 0 \). One can compute

\[
\gamma(m_2(\psi_1, \psi_2)) = \langle \psi_1 \psi_2 \rangle = \frac{1}{(2\pi i)^2} \oint_{x_1 = x_2 = 0} \text{Str} \left( \frac{\partial Q}{\partial x_1} \frac{\partial Q}{\partial x_2} \psi_1 \psi_2 \right) = 1,
\]  

(4.46)

from which we also deduce

\[
\gamma(m_2(m_2(\psi_1, \psi_2), e)) = 1.
\]

Furthermore,

\[
\gamma(m_2(m_2(\psi_1, \psi_2), \psi_1 \psi_2)) = \langle \psi_1 \psi_2 \psi_1 \psi_2 \rangle
\]

\[
= \frac{1}{(2\pi i)^2} \oint_{x_1 = x_2 = 0} \text{Str} \left( \frac{\partial Q}{\partial x_1} \frac{\partial Q}{\partial x_2} \psi_1 \psi_2 \psi_1 \psi_2 \right) = 1,
\]

thus

\[
m_2(\psi_1, \psi_2) = e + \phi.
\]

The same computation shows

\[
m_2(\psi_1, \psi_1) = m_2(\psi_2, \psi_2) = 0.
\]

\(^{13}\) A representative for \( \phi \) can be taken to be, for instance, \( \bar{\eta}_1 \bar{\eta}_2 - \frac{1}{2} \bar{\eta}_1 \eta_1 - \frac{1}{2} \bar{\eta}_2 \eta_2 + \frac{1}{4} \eta_1 \eta_2 \).
4.3.3 $d > 2$

Now assume the degree of $W$ is greater than 2. Because \( \{ v_i, v_j \} = \frac{\partial^2 W}{\partial x_i \partial x_j} \) is $QD_0$-exact, we can take $f_2$ such that

\[
\iota(m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i)) = \{ v_i, v_j \} + \{ QD_0, f_2(\psi_i, \psi_j) + f_2(\psi_j, \psi_i) \} = 0,
\]

and therefore

\[
m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = 0,
\]

which means that $A_{D0}$ is the exterior algebra $\wedge \cdot \mathbb{C}^n$ under the multiplication $m_2$.

Alternatively, $m_2(\psi_i, \psi_j)$ can be determined by the correlation functions (4.41). First note that the one-point correlator

\[
\langle \psi_{i_1} \cdots \psi_{i_m} \rangle = 0 \text{ if } i_s = i_t
\]

for any pair of indices $i_s$ and $i_t$. This is simply because $\{ \psi_i, \psi_j \}$ and $\psi_i^2$ are $QD_0$-exact and as a consequence the correlation function can be rewritten as a sum of correlation functions, each involving a $QD_0$-exact operator. A corollary of this observation is that

\[
\langle \psi_{i_1} \cdots \psi_{i_m} \rangle = 0 \text{ if } m > n.
\]

Now assume that $m \leq n$. The formula (4.41) implies

\[
\langle \psi_{i_1} \cdots \psi_{i_m} \rangle = \frac{1}{(2\pi i)^n} \oint_{x_i=0} \text{Str} \left( \frac{\partial^2 QD_0}{\partial x_1} \wedge \cdots \wedge \frac{\partial^2 QD_0}{\partial x_n} \psi_{i_1} \cdots \psi_{i_m} \right) dx_1 \cdots dx_n.
\]

The degree of the denominator of the integrand is $nd - n$. In order to have a nonzero result, the numerator of the integrand must have degree $nd - 2n$, which can only result from the term in $\prod \frac{\partial QD_0}{\partial x_1} \wedge \cdots \wedge \frac{\partial QD_0}{\partial x_m} \psi_{i_1} \cdots \psi_{i_m}$ proportional to $\eta_1 \cdots \eta_n$. But in order to make nonzero contribution to the supertrace, there should also be $n \eta'$s to contract with the $\eta'$s, this is possible only when $m = n$. In conclusion,

\[
\langle \psi_{i_1} \cdots \psi_{i_m} \rangle = 0 \text{ if } m < n,
\]

and it is straightforward to compute from (4.41) that

\[
\langle \psi_1 \psi_2 \cdots \psi_n \rangle = 1
\]

up to a normalization factor, which implies $\psi_{i_1} \cdots \psi_{i_m}$ is dual to $\pm \prod_{j \neq i_s, 1 \leq s \leq m} \psi_j$. Combining (4.48), (4.49), and (4.51), we see that for a monomial $f$ in $\psi_i$,

\[
\gamma(m_2(m_2(\psi_i, \psi_j), f(\psi_1, \cdots, \psi_n))) = \pm \langle \psi_i \psi_j f(\psi_1, \cdots, \psi_n) \rangle
\]
vanishes unless \( f(\psi_1, \ldots, \psi_n) = \pm \prod_{k \neq i,j} \psi_k \). This allows us to conclude that \( t(m_2(\psi_i, \psi_j)) \) only contains the term dual to \( \prod_{k \neq i, j} v_k \), i.e.,

\[
m_2(\psi_i, \psi_j) = \frac{1}{2} \pi (v_i, v_j).
\] (4.54)

where \( \pi : V_{\text{MD}0} \to A_{\text{D}0} \) is the projection onto \( Q_{\text{D}0} \)-classes.

Now, we compute the higher-order multiplications \( m_k, k > 2 \). In principle, this can be done by performing the algorithm described in Sect. 3 using (3.6). Here, we take the physical perspective and determine the multiplications by studying the deformations of \( Q_{\text{D}0} \) as reviewed in Sect. 4.2.

Assume that we deform \( Q_{\text{D}0} \) using the fermionic generators \( \psi_i \):

\[
Q_{\text{D}0}^{\text{def}} = Q_{\text{D}0} + \sum_{\vec{m} : |\vec{m}| > 0} \alpha_{\vec{m}} u^{\vec{m}}.
\] (4.55)

where \( \alpha_{e_i} = \psi_i / (\sqrt{d(d-1)}) \). In principle, we can consider further deformations by other elements of \( H^1(B_{\text{D}0}, B_{\text{D}0}) \) (and even elements of \( H^0(B_{\text{D}0}, B_{\text{D}0}) \)), but if we are interested in extracting the higher-order products involving only \( \psi_i \)'s, this suffices. Indeed, if the most general first-order deformation (i.e., \( |m| = 1 \)) has the form

\[
\sum_i u_i \psi_i + \sum_\mu u_\mu \Lambda_\mu
\] (4.56)

where \( \Lambda_\mu \) denote the operators in \( H^1(B_{\text{D}0}, B_{\text{D}0}) \) that are not \( \psi_i \)'s, then if we set \( u_\mu = 0 \), after running the algorithm outlined in Sect. 4.2 we will get that \( (Q_{\text{D}0}^{\text{def}})^2 \) has the form

\[
(Q_{\text{D}0}^{\text{def}})^2 = W \cdot 1 + \sum_a f_a(u_i) \phi_a
\] (4.57)

where \( f_a \)'s will have the following interpretation:

\[
f_a(u_i) = \left. \frac{\partial \tilde{W}_{\text{eff}}}{\partial u_a} \right|_{u_\mu = 0} = \sum_{k \geq 2} \sum_{i_1, \ldots, i_k} (m_k(\psi_{i_1}, \ldots, \psi_{i_k}), \Lambda_a D) u_{i_1} \cdots u_{i_k}
\]

(4.58)

where \( a \) runs over all operators in \( A_{\text{D}0} \) and \( \Lambda_a D \) denotes the operator dual to \( \Lambda_a \in \{ \psi_i, \Lambda_\mu \} \). Hence, (4.57) will contain all the information we need about the higher products \( m_k(\psi_{i_1}, \ldots, \psi_{i_k}) \), when we set \( u_\mu = 0 \).

Define \( \alpha_{i_1 \cdots i_s} := \alpha_{e_{i_1} + \cdots + e_{i_s}}, \beta_{i_1 \cdots i_s} := \beta_{e_{i_1} + \cdots + e_{i_s}} \) and \( W_{i_1 \cdots i_s} := \partial_{i_1} \cdots \partial_{i_s} W \). Then, we have

\[
\beta_{ij} = \{ \alpha_i, \alpha_j \} = \frac{1}{d(d-1)} W_{ij}.
\]
As \( d > 2 \), \( \beta_{ij} \) is \( Q_{D0} \)-exact: \( d(d-1)\beta_{ij} = \mathcal{W}_{ij} = \{Q_{D0}, \sum_k \mathcal{W}_{ijk} \eta_k\}/(d-2) \). Then, we can take \( \alpha_{ij} = -1 \sum_k \mathcal{W}_{ijk} \eta_k/(d(d-1)(d-2)) \) to cancel \( \beta_{ij} \) in \( (Q_{D0}^{\text{def}})^2 \). Then, at degree 3, one computes \( \beta_{ijk} = -\mathcal{W}_{ijk}/(d(d-1)(d-2)) \) and \( \alpha_{ijk} = \sum_l \mathcal{W}_{ijk l} \eta_l/(d(d-1)(d-2)) \). This process continues and at degree \( m \) we have

\[
\beta_{i_1\cdots i_m} = (-1)^m \frac{\mathcal{W}_{i_1\cdots i_m}}{d(d-1)\cdots(d-m+1)}
\]

and

\[
\alpha_{i_1\cdots i_m} = (-1)^{m-1} \frac{\sum_j \mathcal{W}_{i_1\cdots i_m j} \eta_j}{d(d-1)\cdots(d-m+1)}
\]

for \( 2 \leq m \leq d \). In particular, \( \beta_{i_1\cdots i_d} \) is not \( Q_{D0} \)-exact and cannot be canceled by a choice of \( \alpha_{i_1\cdots i_d} \). As a result,

\[
(Q_{D0}^{\text{def}})^2 = Q_{D0}^2 + \frac{(-1)^d}{d!} \sum_{r_1+\cdots+r_n=d} \frac{\partial^d \mathcal{W}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} u_1^{r_1} u_2^{r_2} \cdots u_n^{r_n} \mathbf{1}.
\]

This means the obstruction to the deformation is given by the identity operator. Therefore, \( \tilde{\mathcal{W}}_{\text{eff}} \) takes the form

\[
\tilde{\mathcal{W}}_{\text{eff}} = \sum_{i_1,\ldots,i_d} (m_d(\psi_{i_1}, \ldots, \psi_{i_d}), \Lambda) u_{i_1} \cdots u_{i_d} u_0 + \mathcal{O}(u_0^2), \quad \Lambda = \psi_1 \cdots \psi_n,
\]

where \( \Lambda \) is the dual to the identity operator. Due to the correlation function (4.52) and (4.61) together with (4.62), we conclude that

\[
m_d(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_d}) + \text{cyclic permutations} = \frac{(-1)^d}{d!} \mathcal{W}_{i_1\cdots i_d},
\]

and

\[
m_k(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_k}) + \text{cyclic permutations} = 0, \quad k \neq d,
\]

where the \( i_l \)'s are not necessarily distinct. Using directly the algorithm outlined in 3, and (3.6)\(^{14}\), we can further determine exactly all the higher-order products \( m_k \). This computation ends up determining the \( A_{\infty} \) relations as

- If \( d = 2 \), \( A_{D0} \) is a family of Clifford algebras:

\[
m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 \mathcal{W}}{\partial x_i \partial x_j}.
\]

\(^{14}\) [12] used a different approach, namely summing the ribbon trees. We review this idea in the “Appendix B.”
If \( d > 2 \), \( \mathcal{A}_0 \) is an \( A_{\infty} \)-algebra, where \( m_2 \) is the wedge product, \( m_k = 0 \) for \( k = 1, 3, 4, \ldots, d - 1 \) and

\[
m_d(\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_d}) = \frac{1}{d!} \frac{\partial^d W}{\partial x_{i_1} \cdots \partial x_{i_d}}.
\]

(4.66)

We illustrate with an example the computation of \( m_k \), exactly, for \( d = 3 \) (using the relations (3.6) and (3.19)):

\[
\{v_i, v_j\} = W_{ij} = \left\{ Q, \sum_k W_{ijk} \eta_k \right\}.
\]

Because

\[
v_i v_j = \frac{1}{2} \{v_i, v_j\} + \frac{1}{2} [v_i, v_j] = \frac{1}{2} W_{ij} + \frac{1}{2} [v_i, v_j],
\]

we have

\[
m_2(\psi_i, \psi_j) = \frac{1}{2} \pi \left( [v_i, v_j] \right), \quad f_2(\psi_i, \psi_j) = -\frac{1}{2} \sum_k W_{ijk} \eta_k.
\]

Thus, \( m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = 0 \).

\[
v_j v_i v_k = \{v_i, v_j\} v_k - v_i v_j v_k = W_{ij} v_k - W_{jk} v_i + v_i v_k v_j
\]

yields

\[
v_i [v_j, v_k] - [v_i, v_j] v_k = \left\{ Q, - \sum_l W_{jkl} \eta_l v_i - \sum_l W_{ijl} v_k \eta_l \right\}.
\]

Therefore,

\[
f_2(\psi_i, m_2(\psi_j, \psi_k)) - f_2(m_2(\psi_i, \psi_j), \psi_k)
\]

\[
= \frac{1}{2} \left( \sum_l W_{ijl} v_k \eta_l + \sum_l W_{jkl} \eta_l v_i \right).
\]

\[
\iota(m_3(\psi_i, \psi_j, \psi_k)) = f_2(\psi_i, m_2(\psi_j, \psi_k)) - f_2(m_2(\psi_i, \psi_j), \psi_k) - v_i f_2(\psi_j, \psi_k)
\]

\[
- f_2(\psi_i, \psi_j) v_k
\]

\[
= \frac{1}{2} \sum_l \left( W_{ijl} v_k \eta_l + W_{jkl} \eta_l v_i + W_{jkl} v_i \eta_l + W_{ijl} \eta_l v_k \right)
\]

\[
= \frac{1}{2} \left( W_{ijl} \delta_{lk} + W_{jkl} \delta_{li} \right) = W_{ijk}.
\]

In conclusion, \( m_3(\psi_i, \psi_j, \psi_k) = W_{ijk} \).
Therefore, in general, we can say that all the elements of $A_{D0}$ can be written as linear combinations of the form $(d > 2)$

$$\Lambda_i = \psi_1 \wedge \cdots \wedge \psi_{i_r}, \quad \deg(\Lambda_i) := r$$

(4.67)

where we defined $\deg(\Lambda_i)$ for later convenience and $\wedge$ denotes the usual skew-symmetric wedge product. The fact that all the elements can be written as in the formula (4.67) is just a consequence of (4.47). Now, we can determine $m_d(\Lambda_1, \Lambda_2, \cdots, \Lambda_d)$ for $\Lambda_i = \psi_{i_1} \wedge \cdots \wedge \psi_{i_{\deg(\Lambda_i)}}$. Since $m_k = 0$ for $k \neq 2, d$, the relation (3.3) can be solved by the rule

$$m_k(\Lambda_1, \Lambda_2, \cdots, m_2(\Lambda_i, \Lambda_{i+1}), \cdots, \Lambda_{k+1})$$

$$= (-1)^{\deg(\Lambda_i)(\deg(\Lambda_{i+1}) + \cdots + \deg(\Lambda_{k+1}))} m_2(m_k(\Lambda_1, \cdots, \Lambda_{i-1}, \Lambda_{i+1}, \cdots, \Lambda_{k+1}), \Lambda_i)$$

$$+ (-1)^{\deg(\Lambda_{i+1})(\deg(\Lambda_{i+2}) + \cdots + \deg(\Lambda_{k+1}))} m_2(m_k(\Lambda_1, \cdots, \Lambda_i, \Lambda_{i+2}, \cdots, \Lambda_{k+1}), \Lambda_{i+1}).$$

(4.68)

By repeated use of (4.68), we are led to the conclusion that

$$m_d(\psi_{i_0} \wedge \cdots \wedge \psi_{i_{t_1}}, \psi_{j_0} \wedge \cdots \wedge \psi_{j_{t_2}}, \cdots, \psi_{k_0} \wedge \cdots \wedge \psi_{k_{t_d}})$$

(4.69)

is equal to the sum

$$\sum_{a_1=0}^{t_1} \sum_{a_2=0}^{t_2} \cdots \sum_{a_d=0}^{t_d} m_d(\psi_{i_{a_1}}, \psi_{j_{a_2}}, \cdots, \psi_{k_{a_d}})(-1)^{a_1+\cdots+d_d}.$$

(4.70)

$$\psi_{i_0} \wedge \cdots \hat{\psi}_{i_{a_1}} \wedge \cdots \wedge \psi_{i_{t_1}} \wedge \psi_{j_0} \wedge \cdots \hat{\psi}_{j_{a_2}} \wedge \cdots \wedge \psi_{j_{t_2}} \wedge \cdots \wedge \hat{\psi}_{k_0} \wedge \cdots \wedge \psi_{k_{a_d}} \wedge \cdots \wedge \psi_{k_{t_d}}$$

up to an overall sign.

We finish this section with a very simple example to illustrate the consistency between the $\tilde{\mathcal{W}}_{\text{eff}}$ computation and the relations (4.65) and (4.66):

4.3.4 Example: $W = x^d$

Let’s consider the LG model with a single chiral superfield $x$ and a superpotential $W = x^d, d \geq 2$. The D0-brane is given by

$$Q = x\bar{\eta} + x^{d-1}\eta.$$ 

(4.71)

The bosonic open string state is $e = 1$ and the fermionic open string state is $\psi = \bar{\eta} - \eta$. One can use the Kapustin–Li formula [49] to compute the three-point correlation function

$$\langle \psi \psi \bar{\psi} \rangle = \frac{1}{2\pi i} \oint_{x=0} \text{Str} \left( \frac{dQ}{dx} \psi \bar{\psi} \bar{\psi} \right) = \left\{ \begin{array}{ll} 1, & d = 2, \\ 0, & 0, d > 2. \end{array} \right.$$
From the relation

$$
\langle \psi \psi \psi \rangle = \gamma (m_2(m_2(\psi, \psi), \psi))
$$

and

$$
\gamma(e) = 0, \quad \gamma(\psi) = 1,
$$

we see

$$
m_2(\psi, \psi) = \begin{cases} 
    e, & d = 2, \\
    0, & d > 2.
\end{cases}
$$

It was shown in [52] that the effective superpotential, or disk partition function, of the LG model with Dirichlet boundary condition, which is equivalent to (4.71), is

$$
W_{\text{eff}} = \text{Tr} \left( \frac{Z^{d+1}}{d+1} \right)
$$

up to a rescaling, where $Z$ is the world volume field dual to $\psi$. From this effective superpotential, we conclude

$$
\gamma \left( m_2(m_s(\psi \otimes^s), \psi) \right) = \begin{cases} 
    1, & s = d, \\
    0, & s \neq d.
\end{cases}
$$

or equivalently

$$
m_s(\psi \otimes^s) = \begin{cases} 
    e, & s = d, \\
    0, & s \neq d.
\end{cases}
$$

Finally, we remark that in principle we can study the correspondence between $MF(W)$ and $D(\text{Mod} - A_{D0})$ from the point of view of tensor products of minimal models. A homogeneous superpotential $W \in \mathbb{C}[x_1, \ldots, x_n]$ of degree $d$ at a special point in complex structure moduli can be seen as the tensor product of $n A_{d-1}$ minimal models. This relates to the well-known structure of tensor products in matrix factorization categories. It will be interesting to study further how this tensor product structure translates to the category $D(\text{Mod} - A_{D0})$ as it is well known that tensor products of $A_\infty$-algebras are rather nontrivial [53–55].

### 4.4 Inhomogeneous superpotential

Consider now a quasi-homogeneous superpotential $W \in \mathbb{C}[x_1, \ldots, x_N]$. Then, we write the superpotential as a sum of homogeneous polynomial-degree terms:

$$
W(x) = \sum_{l=2}^{d} W^{(l)}(x),
$$

(4.72)
where each term $W^{(l)}(x)$ has polynomial-degree $l$, i.e., where we assign degree 1 to each variable $x_i$. The brane $B_{D0}$ and the fermionic generators of the open string states are still given by (4.34) and (4.36), respectively. If we turn on deformations as in (4.55), we can use the same argument\textsuperscript{16} to deduce that the obstruction is given by

$$\frac{(-1)^l}{l!} \sum_{i_1 + \cdots + i_n = l} \frac{\partial^l W^{(l)}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} u_1^{i_1} u_2^{i_2} \cdots u_n^{i_n}$$

(4.73)

at degree $l$. Therefore, we have the following multiplications:

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W^{(2)}}{\partial x_i \partial x_j}$$

(4.74)

and

$$m_l(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_l}) = \frac{1}{l!} \frac{\partial^l W^{(l)}}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}}$$

(4.75)

for $3 \leq l \leq d$ and $m_k = 0$ for $k > d$.

### 4.5 Landau–Ginzburg orbifold

So far we have considered LG models with trivial orbifold group. For a LG orbifold, there is a finite abelian group $G$ acting on the field space, and $W \in \mathbb{C}[x_1, \ldots, x_n]^G$. As such, the open string states are those invariant under the action of $G$. Let $\psi_1, \ldots, \psi_n$ be the degree-one fermionic generators (4.36) in End$_{MF}(W)(B_{D0})$ of the LG model without orbifolding. Note that each $\psi_1$ transforms in a definite representation, when we take the $G$ action into account. When we incorporate the orbifold the brane defined by the matrix factorization $Q_{D0}$ in (4.34) requires the specification of a representation $\rho_M$, compatible with $\rho_{orb}$. It is easy to see that $\rho_M : G \to GL(M)$ is almost completely fixed by its action, via conjugation, over $\eta_j, \bar{\eta}_j$, in the definition of $Q_{D0}$. Then, $\rho_M$ is fixed up to its action on the Clifford vacuum $|0\rangle$ (defined by $\eta_j |0\rangle = 0$ for all $j$).

Hence, we can label $B_{D0}^{(a)}$ in the category $MF(W, G)$ by a single (one-dimensional) irreducible representation\textsuperscript{17}: the representation of $|0\rangle$. Then, if $G = \mathbb{Z}_d$, we have $a = 0, \ldots, d - 1$ and we denote

$$\mathcal{B}_{orb} := \bigoplus_{a=0}^{d-1} B_{D0}^{(a)} \in MF(W, G),$$

(4.76)

\textsuperscript{16} Also, one can apply an argument based on ribbon trees as reviewed in “Appendix B.”

\textsuperscript{17} These are sometimes called the orbit branes related to $B_{D0}$ [56, 57].
where the set of branes $B_{D0}^{(a)}$, $a = 0, \ldots, d - 1$ form a set of generators of $MF(G, W)$ [11] and the algebra $A_{D0}$ must be replaced by

$$A_{orb} = \text{End}_{MF(W, G)}(B_{orb}).$$

(4.77)

The correspondence $MF(W, G) \cong D(\text{Mod} - A_{orb})$ was studied in [56, 58], for homogeneous potentials. Moreover, the results in [11] implies the following isomorphism of $A_{\infty}$ algebras:

$$A_{orb} \cong A_{D0} \# G,$$

(4.78)

where $A_{D0} \# G$, the smash product of $A_{D0}$ and $\mathbb{C}[G]$, is regarded as an $A_{\infty}$-algebra with $m_2(g_1, g_2) = g_1 \cdot g_2$ and $m_k = 0$ for $k \neq 2$ $(g_1, g_2 \in G)$. Then, $A_{orb}$ can be regarded as the product of two $A_{\infty}$-algebras. We can still use the construction introduced in Sect. 4.3 to set up the correspondence between objects of $MF(W, G)$ and the $A_{\infty}$-modules over $A_{D0} \# G$. The difference is that the module is not only an $A_{\infty}$-module of $A_{D0}$, but also a $\mathbb{C}[G]$-module, this corresponds to the fact that the Chan–Paton spaces of the matrix factorizations of LG orbifold all carries a $G$-representation.

Specifically for the case $G = \mathbb{Z}_d$, because the multiplication $m_2 A_{D0} \# \mathbb{Z}_d$ of $A_{D0} \# \mathbb{Z}_d$ satisfies

$$m_2 A_{D0} \# \mathbb{Z}_d(a \# g_1, b \# g_2) = m_2 A_{D0}(a, g_1 b g_1^{-1}) \# (g_1 \cdot g_2),$$

(4.79)

an $A_{D0} \# \mathbb{Z}_d$-module is of the form $\bigoplus_{\ell=0}^{d-1} M_i \otimes \rho_{\ell+i}$, where $\bigoplus_{i=0}^{d-1} M_i$ is a $\mathbb{Z}_d$-graded $A_{D0}$-module and $\rho_{i}$ denotes the one-dimensional representation of $\mathbb{Z}_d$ with weight $2\pi i l/d$.

For example, when $d = 2$, $A_{D0}$ is a Clifford algebra $Cl(n, \mathbb{C})$; hence, if $G = \mathbb{Z}_2$, we have

$$MF(W, \mathbb{Z}_2) \cong D(\text{Mod} - A_{D0} \# \mathbb{Z}_2).$$

(4.80)

The category $MF(W, \mathbb{Z}_2)$ is very similar to the graded category $MF(W)$ but its morphisms are different. Because of the $\mathbb{Z}_2$ orbifold, all the morphisms between irreducible objects are either even or odd, but not both. This is exactly the category studied in [59] and so, we can use the results in [59] to conclude

$$MF(W, \mathbb{Z}_2) \cong D(\text{Mod}_{\mathbb{Z}_2} - Cl(n, \mathbb{C})),

(4.81)

where $D(\text{Mod}_{\mathbb{Z}_2} - Cl(n, \mathbb{C}))$ denotes the derived category of graded modules over $Cl(n, \mathbb{C})$. A classical result of Atiyah-Bott-Shapiro [60] (see also [59]) establishes

$$MF(W, \mathbb{Z}_2) \cong D(\text{Mod}_{\mathbb{Z}_2} - Cl(n, \mathbb{C})) \cong D(\text{Mod} - Cl_0(n, \mathbb{C})),

(4.82)

where $Cl_0(n, \mathbb{C})$ denotes the even part of the Clifford algebra $Cl(n, \mathbb{C})$. We remark that the category considered in [9] is the category $MF(W)$ where the morphisms are
odd and even and hence is equivalent to $D(\text{Mod} - Cl(n, \mathbb{C}))$, where the modules are not graded. Finally, we illustrate the correspondence (4.80) with an example. Set $W = \sum_{i=1}^{2m} x_i^2$ and each $x_i$ is $\mathbb{Z}_2$-odd. Let $S_+$ and $S_-$ be the spinor representation with left and right chirality, respectively. The $A_{D0} \# \mathbb{Z}_2$ module $M := S_+ \otimes \rho_0 \oplus S_- \otimes \rho_1$ corresponds to the matrix factorization with $Q_M = \sum_{i=1}^{m} (x_{2i-1} + ix_{2i}) \eta_i + \sum_{i=1}^{m} (x_{2i-1} - ix_{2i}) \eta_i$ and the vacuum being $\mathbb{Z}_2$-even. The $A_{D0} \# \mathbb{Z}_2$ module $S_+ \otimes \rho_1 \oplus S_- \otimes \rho_0$ corresponds to the matrix factorization with the same $Q_M$ but the vacuum being $\mathbb{Z}_2$-odd ($\overline{\eta}_i$ and $\eta_i$ are $\mathbb{Z}_2$-odd in both cases).

### 4.6 Hybrid model

Finally, let us consider the $A_\infty$ structure of the matrix factorizations of hybrid models. Start with the trivial fibration. In this case, the theory under consideration is defined on the target space

$$(V_f \times V_b)/G,$$

where $V_b = \mathbb{C}^m$ and $V_f = \mathbb{C}^n$ are regarded as the base space and the fiber, respectively, and $G$ is the orbifold group acting on $V_b$ and $V_f$. Suppose that the base coordinates are $z_i, i = 1, \cdots, m$ and the fiber coordinates are $x_j, j = 1, \cdots, n$. The superpotential $W(x, z)$ is a $G$-invariant holomorphic function in $x_j$ and $z_i$.

The analog of the $D0$-brane for the LG models we discussed in previous sections is point-like along each fiber, i.e., it localizes to the base space. We call this brane the reference brane and denote it by $B_0$. The superpotential can be written as

$$W = \sum_l W^{(l)}(x, z),$$

where $W^{(l)}$ is homogeneous in $x_i$ with degree $l$, and therefore the endomorphism $Q_0 \in B_0$ (see (4.4)) is

$$Q_0 = \sum_{i=1}^{n} x_i \overline{\eta}_i + \sum_{i=1}^{n} \sum_l \frac{1}{l!} \frac{\partial^{l} W^{(l)}(x, z)}{\partial x_i} \overline{\eta}_i.$$

Then, as discussed in Sect. 4.5, the $A_\infty$-algebra of the hybrid model on (4.83) is

$$A_0 G,$$

where $A_0$ is generated by $\psi_i(z) \in \text{End}_{MF(W)}(B_0)$ satisfying

$$m_2(\psi_i(z), \psi_j(z)) = \frac{\partial^2 W^{(2)}(x, z)}{\partial x_i \partial x_j},$$

$$m_l(\psi_{i_1}(z), \psi_{i_2}(z), \cdots, \psi_{i_l}(z)) = \frac{1}{l!} \frac{\partial^l W^{(l)}(x, z)}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}}, l > 2.$$
For a hybrid model (for details on the precise definition of hybrid models see [3]), defined on a space of the form

\[ Y := \text{Tot}(\mathcal{V} \xrightarrow{\pi} B) \]

(4.84)

with superpotential \( W \in H^0(\mathcal{O}_Y) \), where \( \mathcal{V} \) is a vector bundle over the base space \( B \), we can decompose \( Y \) into a set of coordinate patches

\[ Y = \bigcup_i U_i, \]

such that every patch has the form

\[ U_i = \left( \mathcal{V}_f \times \mathcal{V}_b \right) / G_i = \widetilde{U}_i / G_i, \]

and \( \dim \mathcal{V}_f = \text{rank} \mathcal{V}_b = \dim B \). We use the notation \( \widetilde{U}_i \) to denote the affine space \( \mathcal{V}_f \times \mathcal{V}_b / G_i \).

The reference brane \( B_0 \) is of the form

\[
\begin{aligned}
\mathcal{O} \xrightarrow{X} \pi^* \mathcal{V} \xrightarrow{f(X, P)} \pi^* \wedge^2 \mathcal{V} \xrightarrow{f(X, P)} \cdots \xrightarrow{f(X, P)} \pi^* \wedge^{n-1} \mathcal{V} \xrightarrow{f(X, P)} \pi^* \wedge^n \mathcal{V},
\end{aligned}
\]

where \( X \) denotes collectively the coordinates along the fiber of \( \mathcal{V} \) and \( f(X, P) \) is a map that depends also on the base coordinate \( P \) such that \( f(X, P) \cdot X = X \cdot f(X, P) = W(X, P) \). Then, from the discussion above, the reference brane \( B_0 \) that is point-like along each fiber gives rise to an \( A_\infty \)-algebra \( A_0(G) \) within each coordinate patch, we can define a sheaf of \( A_\infty \)-algebra \( \mathcal{A} \) by

\[ \mathcal{A}(U_i) = A_0(U_i)G_i. \]

If we denote by \( x^{(i)} \) and \( z^{(i)} \) the local fiber and base coordinates in \( U_i \), then \( A_0(U_i) \) is generated by \( \psi_s^{(i)}, s = 1, \ldots, n \), which satisfy

\[
\begin{aligned}
m_2(\psi_s^{(i)}, \psi_t^{(i)}) + m_2(\psi_t^{(i)}, \psi_s^{(i)}) &= \frac{\partial^2 W^{(2)}(x^{(i)}, z^{(i)})}{\partial x_s^{(i)} \partial x_t^{(i)}}, \\
m_l(\psi_s^{(i)}, \psi_{s_2}^{(i)}, \ldots, \psi_{s_l}^{(i)}) &= \frac{1}{l!} \frac{\partial^l W^{(l)}(x^{(i)}, z^{(i)})}{\partial x_{s_1}^{(i)} \partial x_{s_2}^{(i)} \cdots \partial x_{s_l}^{(i)}}. l > 2.
\end{aligned}
\]

In other words, the algebra \( A_0(U_i) = A_0(\widetilde{U}_i) \), i.e., it is constructed by ignoring the orbifold structure. In order to understand how these algebras are glued together as we change charts, we have to be careful with the treatment of the orbifold singularities of \( B \). It has been proposed that the correct mathematical framework to study this problem
is to view the GLSM as an algebraic stack\textsuperscript{18}. In this context, the intersection of two patches \( U_i \cap U_j \) is given by the fibered product over \( Y \)
\[
U_i \cap U_j = U_i \times_Y U_j \cong (V_f \times \mathcal{V}_{ij})/G_{ij},
\]
where \( G_{ij} \) is a subgroup of \( G_i \times G_j \) and \( \mathcal{V}_{ij} \) is a quasiprojective variety. Therefore, in order to define \( \mathcal{A}(U_i \cap U_j) \) we have to be more careful, because \( V_f \times \mathcal{V}_{ij} \) is not necessarily an affine space. The main difference with the usual LG orbifold case is that in this case the branes \( B_0^{(g)} \in MF(W, G_{ij}), g \in G_{ij} \) are not necessarily all inequivalent objects. There can be a subgroup \( H_{ij} \subseteq G_{ij} \) such that
\[
B_0^{(0)} \cong B_0^{(h)} \quad \text{for all } h \in H_{ij},
\]
then the algebra \( \mathcal{A}(U_i \cap U_j) \) is defined by
\[
\mathcal{A}(U_i \cap U_j) = \text{End}_{MF(W, G_{ij})} \left( \bigoplus_{g \in G_{ij}/H_{ij}} B_0^{(g)} \right).
\]

The subgroup \( H_{ij} \) depends on the specific model we study. We illustrate it in an example below. However, we expect that \( \mathcal{A} \) has the structure of a sheaf of algebras over \( Y \); hence, we must have the isomorphism
\[
\mathcal{A}(U_i)|_{U_i \cap U_j} \cong \mathcal{A}(U_j)|_{U_i \cap U_j}
\]
and the inclusions
\[
\mathcal{A}(U_i)|_{U_i \cap U_j} \hookrightarrow \mathcal{A}(U_i \cap U_j) \hookleftarrow \mathcal{A}(U_j)|_{U_i \cap U_j}.
\]

The category of matrix factorizations of the hybrid model defined on \( Y \) is thus equivalent to the derived category of sheaves of \( \mathcal{A} \)-modules. In the next section, we apply these results to homological projective duality.

\subsection*{4.6.1 Example: hybrid model on \( \text{Tot}(\mathcal{O}(-1) \to W \mathbb{P}(2, 3)) \)}

As an illustrative example, we derive the sheaf of \( A_{\infty} \)-algebra associated with a hybrid model on \( Y = \text{Tot}(\mathcal{O}(-1) \to W \mathbb{P}(2, 3)) \). This is a simple example where we cannot write \( Y \) as a global orbifold. Let \( p_1 \) and \( p_2 \) be the homogeneous coordinates of the weighted projective space, i.e., \( (p_1, p_2) \sim (\lambda^2 p_1, \lambda^3 p_2) \). Assume that the hybrid model is defined by the superpotential \( W = P_1 X^2 + P_2 X^3 \), where \( x \) is the fiber

\textsuperscript{18} For an introduction to algebraic stacks, see for example [61] and for the specific case of smooth toric Deligne–Mumford stacks, relevant for the examples in this work, see [62]. A relation between GLSMs and stacks can be found in [63].
coordinate. The reference brane \( B_0 \) is given by the following matrix factorization:

\[
\mathcal{O} \xrightarrow{X} \mathcal{O}(-1).
\]

(4.90)

Let \( U_i \) be the open set defined by \( p_i \neq 0 \), then \( U_1 \cong \mathbb{C}^2/\mathbb{Z}_2 \) and \( U_2 \cong \mathbb{C}^2/\mathbb{Z}_3 \). The local coordinates are \((x_1, z_1)\) in \( U_1 \) and \((x_2, z_2)\) in \( U_2 \), where \( x_1 \) and \( x_2 \) denote the fiber coordinates. Then, \( z_1 \) and \( z_2 \) satisfy \( z_1^2 z_2^3 = 1 \) on \( U_1 \cap U_2 \). In \( U_1 \), the superpotential reads \( W = x_1^2 + z_1 x_1^3 \) and the generator of \( \mathbb{Z}_2 \) acts as \((x_1, z_1) \mapsto (-x_1, -z_1) \). In \( U_2 \), the superpotential reads \( W = z_2 x_2^2 + x_2^3 \) and the generator of \( \mathbb{Z}_3 \) acts as \((x_2, z_2) \mapsto (\exp(-2\pi i/3) x_2, \exp(-2\pi i/3) z_2) \). In \( U_1 \), the reference brane (4.90) can be written as \( B_{01} \):

\[
\mathbb{C}_+ \xrightarrow{x_1} \mathbb{C}_-, \quad \frac{x_1}{x_1 + z_1 x_1^2}.
\]

where the subscript \( \pm \) indicates whether the complex plane is \( \mathbb{Z}_2 \)-even or \( \mathbb{Z}_2 \)-odd. As discussed before, the \( A_{\infty} \)-algebra \( A(U_1) \) is the \( \mathbb{Z}_2 \)-invariant subspace of \( \text{End}(B_{01}^{(0)} \oplus B_{01}^{(1)}) \), where \( B_{01}^{(0)} = B_{01} \) and \( B_{01}^{(1)} \) is obtained from \( B_{01} \) by a \( \mathbb{Z}_2 \)-twist, i.e., \( B_{01}^{(1)} \) is the matrix factorization

\[
\mathbb{C}_- \xrightarrow{x_1} \mathbb{C}_+, \quad \frac{x_1}{x_1 + z_1 x_1^2}.
\]

Therefore, we have four independent \( \mathbb{Z}_2 \)-invariant endomorphisms in \( \text{End}(B_{01}^{(0)} \oplus B_{01}^{(1)}) \), namely \( \text{id}_0^+ \pm 1 \) and \( \psi_1^0 \pm 1 \), where \( \text{id}_0^+ + 1(\text{id}_0^+ - 1) \) is the identity map on \( B_{01}^{(0)}(B_{01}^{(1)}) \) and \( \psi_1^0 + 1(\psi_1^0 - 1) \) is the homomorphism in \( \text{Hom}(B_{01}^{(0)} , B_{01}^{(1)})\)\( \text{Hom}(B_{01}^{(1)} , B_{01}^{(0)}) \) given by the matrix

\[
\begin{pmatrix}
0 & -i(1 + z_1 x_1) \\
i & 0
\end{pmatrix}.
\]

The structure of the \( A_{\infty} \)-algebra is given by

\[
m_2(\psi_1, \psi_1) = 2, \quad m_3(\psi_1, \psi_1, \psi_1) = z_1.
\]

Similarly, in \( U_2 \), the reference brane (4.90) can be written as \( B_{02} \):

\[
\mathbb{C}_0 \xrightarrow{x_2} \mathbb{C}_2, \quad \frac{x_2}{z_2 x_2 + x_2^2}.
\]

where the subscript \( a \) indicates that the complex plane has \( \mathbb{Z}_3 \)-weight \( \exp(2\pi i a/3) \) for \( a = 0, 1, 2 \) mod 3. The \( A_{\infty} \)-algebra \( A(U_2) \) is the \( \mathbb{Z}_3 \)-invariant subspace of
where \( B^{(0)}_{02} = B_{02} \) and \( B^{(a)}_{02} \) is obtained from \( B_{02} \) by a \( \mathbb{Z}_3 \)-twist, i.e., \( B^{(a)}_{02} \) is the matrix factorization

\[
\mathbb{C}_a \xrightarrow{x_2} \mathbb{C}_{a+2}.
\]

We have six independent \( \mathbb{Z}_3 \)-invariant endomorphisms in \( \text{End}(\oplus_{a=0,1,2} B^{(a)}_{02}) \), namely id\( \hat{x} \) exp\( (2\pi ia/3) \) and \( \psi_2 \hat{x} \) exp\( (2\pi ia/3) \) for \( a = 0, 1, 2 \) mod 3, where id\( \hat{x} \) exp\( (2\pi ia/3) \) is the identity map on \( B^{(a)}_{01} \) and \( \psi_2 \hat{x} \) exp\( (2\pi ia/3) \) is the homomorphism in \( \text{Hom}(B^{(a)}_{02}, B^{(a+2)}_{02}) \) given by the matrix

\[
\begin{pmatrix}
0 & -i(z_2 + x_2) \\
i & 0
\end{pmatrix}.
\]

The structure of the \( A_\infty \)-algebra is given by

\[ m_2(\psi_2, \psi_2) = 2z_2, \quad m_3(\psi_2, \psi_2, \psi_2) = 1. \]

It is easy to check that \( \psi_1 \) and \( \psi_2 \) constitute a section of \( \mathcal{O} \oplus \mathcal{O}(2) \). We see that \( \mathcal{A}(U_1) = \mathcal{A}_0(U_1) \mathbb{Z}_2 \) and \( \mathcal{A}(U_2) = \mathcal{A}_0(U_2) \mathbb{Z}_3 \), where \( \mathcal{A}_0(U_1) \) is generated by \( \psi_1 \).

The intersection \( U_1 \cap U_2 \) is given by the fibered product, as indicated above, then we find

\[ U_1 \cap U_2 = U_1 \times_Y U_2 \cong (\mathbb{C} \times \mathbb{C}^*)/\mathbb{Z}_2 \times \mathbb{Z}_3. \]

We denote the coordinates of \( U_1 \cap U_2 \) as \((x_{12}, z_{12}) \in \mathbb{C} \times \mathbb{C}^* \). The generator of \( \mathbb{Z}_2 \) acts as \((x_{12}, z_{12}) \rightarrow (-x_{12}, -z_{12}) \), and the generator of \( \mathbb{Z}_3 \) acts as \((x_{12}, z_{12}) \rightarrow (x_{12}, \exp(-2\pi i/3)z_{12}) \). The relation between \((-x_{12}, -z_{12}) \) and the coordinates in the charts \( U_i \) is given by

\[ z_1 = z_{12}^3, \quad z_2 = z_{12}^{-2}, \quad x_{12} = x_1 = x_2 z_{12}^{-1}. \]

The superpotential in the intersection can be written as \( W = x_{12}^2 + z_{12}^3 x_{12}^3 \). Given any matrix factorization of \( W \) in the intersection \( U_1 \cap U_2 \), i.e., an object of \( MF(W, \mathbb{Z}_2 \times \mathbb{Z}_3) \), defined over \( \mathbb{C} \times \mathbb{C}^* \), we can define a similarity transformation \cite{43} \( z_{12}^s \mathbf{1} \), where \( \mathbf{1} \) is the identity on the Chan–Paton space and \( s \in \mathbb{Z} \). In \( U_1 \cap U_2 \) clearly \( z_{12}^s \mathbf{1} \) is invertible for any \( s \) and it leaves invariant the endomorphism \( Q \) but it changes \( \rho_M \), shifting all its weights simultaneously by \((-1)^s \exp(-2\pi i s/3) \). So, it is clear from the argument above, the matrix factorization \( B^{(0)}_{01} \mid_{U_1 \cap U_2} \), given by:

\[
\mathbb{C}_{(+,0)} \xrightarrow{x_{12}} \mathbb{C}_{(-,0)}.
\]
where the subscript $C_{(a,b)}$ labels the weights of the $\mathbb{Z}_2 \times \mathbb{Z}_3$ representation, is equivalent to $\mathcal{B}^{(1)}_{01}\mid_{U_1 \cap U_2}$, given by:

$$
\begin{array}{ccc}
  C_{(\cdot,0)} & \xrightarrow{x_{12}} & C_{(+,0)} \\
  x_{12} & \xrightarrow{x_{12} + \zeta_{12}^3 x_{12}^2} & x_{12}.
\end{array}
$$

Moreover, we can also show $\mathcal{B}^{(0)}_{01}\mid_{U_1 \cap U_2} \cong \mathcal{B}^{(a)}_{02}\mid_{U_1 \cap U_2}$, for any $a$ using the similarity transformation

$$
\begin{pmatrix}
  0 & x_{12} + \zeta_{12}^3 x_{12}^2 \\
  x_{12} & 0
\end{pmatrix}
\begin{pmatrix}
  z_{12} & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  0 & \zeta_{12}^{-1} x_{12} + \zeta_{12}^2 x_{12} \\
  z_{12} x_{12} & 0
\end{pmatrix}
\begin{pmatrix}
  z_{12}^{-1} & 0 \\
  0 & 1
\end{pmatrix}
$$

composed with the transformation $\zeta_{12}^s \mathbf{1}$ for appropriately chosen $s$. The induced transformation on the open string morphism, namely

$$
\begin{pmatrix}
  0 & -i(1 + z_1 x_1) \\
  i & 0
\end{pmatrix}
\begin{pmatrix}
  z_{2}^{-\frac{1}{2}} & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  0 & -i(z_2 + x_2) \\
  i & 0
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  0 & z_{2}^{-\frac{1}{2}}
\end{pmatrix}
$$

let us conclude that

$$
\mathcal{A}(U_1)\mid_{U_1 \cap U_2} \cong \mathcal{A}(U_2)\mid_{U_1 \cap U_2}
$$

as expected. We also conclude that $\mathcal{A}(U_1 \cap U_2)$ is thus generated by a single element $\psi_{12}$ and the restriction map of the sheaf $\mathcal{A}$ is given by

$$
\begin{array}{ccc}
  \mathcal{A}(U_1) & \to & \mathcal{A}(U_1 \cap U_2) \\
  \psi_1 & \mapsto & \psi_{12} \\
  \psi_2 & \mapsto & \psi_{12}
\end{array}
$$

5 Examples of categories of B-branes on HPD phases

In this section, we apply the results from the previous section to HPD constructed from GLSMs introduced in [8]. The Higgs branch category $\mathcal{C}$ defined in (2.17) takes the form

$$
\mathcal{C} = D(\hat{Y}_{\xi\zeta} \ll -1, \hat{W}_{\xi\zeta} \ll -1),
$$

which generically corresponds to a hybrid model as the ones reviewed in Sect. 4.6. We analyze the following examples in detail:

- Degree $d$ Veronese embeddings.
- Fano complete intersections in $\mathbb{P}^n$. 

\( \mathcal{C} \) Springer
5.1 HPD of veronese embedding

As reviewed in Sect. 2, the HPD category (2.17) of degree-$d$ Veronese embedding of $\mathbb{P}^n = \mathbb{P}(V)$ (dim $V = n + 1$) can be described by the category of B-branes on hybrid model with target space

$$\text{Tot} \left( \mathcal{O} \left( -\frac{1}{d} \right)^{\oplus(n+1)} \rightarrow \mathbb{P}^{(n+d)/d-1} \right) / \mathbb{Z}_d$$

with superpotential

$$W = \sum_{a=1}^{(n+d)/d} s_a f_a(x),$$

where $f_a$'s are the degree $d$ monomials in $x_i$, $i = 0, \cdots, n$ and $s_a$ are homogeneous coordinates in $\mathbb{P}^{(n+d)/d-1}$. One can interpret the category of matrix factorizations of this LG model as the derived category of a noncommutative space $D(\mathbb{P}^{(n+d)/d-1}, A_0^{\ast}\mathbb{Z}_d)$, i.e., the category of sheaves of modules over the sheaf of $A_\infty$-algebras $A_0^{\ast}\mathbb{Z}_d$. Here, when restricted to a single fiber, the sheaf of $A_\infty$-algebra $A_0$ is the algebra of endomorphisms of the $D0$-brane we defined previously, so $A_0$ localizes to the base projective space. Let us denote by $Q_0$ the matrix factorization corresponding to the reference brane $\mathcal{B}_0^{(0)}$ given by

$$Q_0 = \sum_i \left( x_i \eta_i + \frac{1}{d} \frac{\partial W}{\partial x_i} \eta_i \right),$$

where we have chosen the trivial $\mathbb{Z}_d$ representation for the Clifford vacuum. Then, at a generic point $p \in \mathbb{P}^{(n+d)/d-1}$, $A_0,p$ is given by the $A_\infty$-algebra with relations (4.65) and (4.66).

For completeness, let us write the generators of $D(\mathbb{P}^{(n+d)/d-1}, A_0^{\ast}\mathbb{Z}_d)$ as $A_0^{\ast}\mathbb{Z}_d$-modules. The B-brane $\mathcal{B}_0^{(0)}$ can be represented as the curved complex

$$\mathcal{O} \xrightarrow{x^{-\frac{1}{d}}} \mathcal{O}(-\frac{1}{d}) \otimes V \xrightarrow{x^{-\frac{1}{d}}} \mathcal{O}(-\frac{1}{d}) \otimes \wedge^2 V \xrightarrow{x^{-\frac{1}{d}}} \cdots \xrightarrow{x^{-\frac{1}{d}}} \mathcal{O}(-\frac{1}{d}) \otimes \wedge^n V \xrightarrow{x^{-\frac{1}{d}}} \mathcal{O}(-\frac{n+1}{d}) \otimes \wedge^{n+1} V,$$

where $\mathcal{O}(m)$ denotes orbibundles over $\mathbb{P}^{(n+d)/d-1}$. Define $\mathcal{B}_0^{(l)}$ to be the matrix factorization (5.5) twisted by $\mathcal{O}(l/d)$ for $l \in \mathbb{Z}$ and therefore the sheaf of $A_\infty$ modules

---

19 The notation $\mathcal{O}(m)$ with $m \in \mathbb{Q}$ denotes an orbibundle over $\mathbb{P}^{(n+d)/d-1}$. See [8] or “Appendix C” for details.
corresponding to $B_0^{(l)}$ is given by\textsuperscript{20}

\[
A^{(l)} := \text{Hom} \left( \bigoplus_{a=0}^{d-1} B_0^{(a)}, B_0^{(l)} \right) = \bigoplus_{a=0}^{d-1} \bigoplus_{k \geq 0} \mathcal{O} \left( \frac{l-a}{d} - k \right) \otimes \wedge^{kd+a} V, \tag{5.6}
\]

where the sum over $k$ is such that $kd + a \leq n + 1$, then we can simplify (5.6) to

\[
A^{(l)} = \bigoplus_{i=0}^{n+1} \mathcal{O} \left( \frac{l-i}{d} \right) \otimes \wedge^i V. \tag{5.7}
\]

We can therefore write

\[
D(\mathbb{P}^{(n+d)-1}, A_0 \# \mathbb{Z}_d) = \langle A^{(1-C_{d,n})}, \ldots, A^{(-1)}, A^{(0)} \cong A_0 \rangle, \tag{5.8}
\]

where $C_{d,n} = d \left( \frac{n+d}{d} \right) - (n+1)$ is the expected number of factors obtained from the analysis of the Coulomb vacua in [8].

### 5.2 HPD of Fano hypersurface in projective space

A degree $d \leq n$ Fano hypersurface in $\mathbb{P}^n$, denoted $\mathbb{P}^n[d]$, can be described by a GLSM with $U(1)$ gauge group, $n + 1$ chiral multiplets $x_i$ with gauge charge 1, one chiral multiplet $p$ with gauge charge $-d$ and a superpotential

\[
W_{\text{Fano}} = p_0 F_d(x), \tag{5.9}
\]

where the polynomial $F_d(x)$, of degree $d$, is the defining equation of the hypersurface (which we assume to be smooth). We consider $x_i$ to be the coordinates on a complex vector space $V$ ($\dim V = n + 1$); hence, $\mathbb{P}^n = \mathbb{P}(V)$. In this case, the equivalence (2.3) takes the form

\[
D^b \text{Coh} (\mathbb{P}^n[d]) = \langle M F (F_d, \mathbb{Z}_d), \mathcal{O}_{\mathbb{P}^n[d]}, \mathcal{O}_{\mathbb{P}^n[d]}(1), \ldots, \mathcal{O}_{\mathbb{P}^n[d]}(n-d) \rangle. \tag{5.10}
\]

The associated Lefschetz decomposition is

\[
D^b (\mathbb{P}^n[d]) = \langle A_0, A_1, \ldots, A_{n-d} \rangle, \tag{5.11}
\]

where $A_0 := \langle M F (F_d, \mathbb{Z}_d), \mathcal{O}_{\mathbb{P}^n[d]} \rangle, A_i := \langle \mathcal{O}_{\mathbb{P}^n[d]} \rangle, i > 0$. The small window category $\mathcal{W}^{\mathcal{L}}_{c,b}$ of this GLSM, defined in (2.4) consists of B-branes with charges $q$ satisfying

\[
\left| q + \frac{\theta}{2\pi} \right| < \frac{d}{2}. \tag{5.12}
\]

\textsuperscript{20} Note that the $\text{Hom}$’s are taken over the orbifold category of the hybrid model defined by (5.2), (5.3).
We choose the theta angle (i.e., the integer $b$) as $\theta = \pi d - \varepsilon$, with $0 < \varepsilon \ll 1$, so $q$ takes values $-(d - 1), -(d - 2), \cdots, -2, -1, 0$.

As shown in [8], the universal hyperplane section can be described by the geometric phase (with FI parameters lying in the first quadrant) of the GLSM $T_X$, this corresponds to a GLSM with gauge group $U(1)^L \times U(1)^R$ and with matter content

\[
x_0 \quad x_1 \cdots \quad x_n \quad p_0 \quad p \cdots \quad y_0 \cdots \quad y_n
\]

$U(1)^L \quad 1 \quad 1 \cdots 1 \quad -d \quad -1 \cdots 0 \cdots 0$

$U(1)^R \quad 0 \quad 0 \cdots 0 \quad 0 \quad -1 \cdots 1 \cdots 1$

and superpotential

\[
\hat{W} = p_0 F_d(x) + p \sum_{i=0}^{n} x_i y_i.
\]

The HPD of $\mathbb{P}^n[d]$ with Lefschetz decomposition given by (5.11) can be described by the Higgs branch of the phase of $T_X$ corresponding to the FI parameters lying in the second quadrant: $(\zeta_L \ll -1, \zeta_1 \gg 1)$. This is a hybrid model with target space

\[
Y := \text{Tot} \left( O_{\mathbb{P}^n}^{\oplus(n+1)} \oplus O_{\mathbb{P}^n}(-1) \to \mathbb{P}^n \right) / \mathbb{Z}_d
\]

and superpotential

\[
W = F_d(x) + p \sum_{i=0}^{n} x_i y_i, \quad (5.13)
\]

where $\mathbb{P}^n := \mathbb{P}(V^\vee)$, $x_i$’s are fiber coordinates of $O_{\mathbb{P}^n}^{\oplus(n+1)}$, $p$ is the fiber coordinate of $O_{\mathbb{P}^n}(-1)$ and the $y_i$’s are homogeneous coordinates on the base $\mathbb{P}^n$. The $\mathbb{Z}_d$ orbifold acts with weight 1 on the $x_i$’s and $-1$ on $p$. Denote the category of B-branes of this hybrid model as $D(Y, W)$. Then, $D(Y, W)$ has a (dual) Lefschetz decomposition that takes the form (as proposed in [8]):

\[
D(Y, W) = \langle B_{n-1}(1-n), B_{n-2}(2-n), \cdots, B_2(-2), B_1(-1), B_0 \rangle, \quad (5.14)
\]

where we have the equivalence of categories $B_0 \cong A_0$. Denote the functor implementing this equivalence by $\mathcal{F}$:

\[
\mathcal{F} : A_0 \to B_0.
\]

Then, $B_i = \langle \mathcal{F}(MF(F_d, \mathbb{Z}_d)), \mathcal{F}(O) \rangle$ for $0 \leq i \leq d$, $B_j = \langle \mathcal{F}(MF(F_d, \mathbb{Z}_d)) \rangle$ for $d + 1 \leq j \leq n - 1$. The relationship between the Lefschetz decomposition and its dual decomposition is illustrated in Fig. 2.
Next, we describe the functor $F$ explicitly. Define the matrix factorization $Q'$ of $p \sum_{i=0}^{n} x_i y_i$ as

$$O_{\mathbb{P}^n}(1) \xleftarrow{\sum_{i=0}^{n} x_i y_i} \xrightarrow{p} O_{\mathbb{P}^n}.$$ 

Any matrix factorization $\mathcal{M} \in MF(F_d, \mathbb{Z}_d)$ can be lifted to a GLSM B-brane with $U(1)_\mathcal{L}$ charges in the small window (5.12). On the other hand, the category $\hat{\mathcal{W}}_{L-b} \cong D(Y, W)$ in (2.17) can be chosen (by adjusting $b$) such that the $U(1)_\mathcal{L}$ charges $q'$ in the $T_X$ model satisfy $q' \in \{-d-1, -(d-2), \ldots, -1, 0, 1\}$. Then, the tensor product $\mathcal{M} \otimes Q'$ has $U(1)_\mathcal{L}$ charges belonging to $\hat{\mathcal{W}}_{L-b}$. The same is true for the B-brane $O_{\mathbb{P}^n[d]} \otimes Q'$. We conclude that the functor is given by

$$\mathcal{F}(\mathcal{M}) = \mathcal{M} \otimes Q'.$$

For every fixed point on the base, the superpotential along the fiber is given by (5.13) with fixed $y_i$. The reference brane $B_0^{(0)}$ can be written as

$$Q_0 = \sum_{i=0}^{n} (x_i \eta_i) + p \eta_{n+1} + \sum_{i=0}^{n} \left( \frac{1}{d} \frac{\partial F_d}{\partial x_i} + \frac{1}{2} p y_i \right) \eta_i + \frac{1}{2} \left( \sum_{i=0}^{n} x_i y_i \right) \eta_{n+1},$$

where the trivial representation of $\mathbb{Z}_d$ is chosen for the Clifford vacuum. Because the superpotential (5.13) has a quadratic term $p \sum_{i=0}^{n} x_i y_i$ and a degree-$d$ term $F_d(x)$, the structure of the $A_0$ factor in the sheaf of $A_{\infty}$-algebras $A_0^* \mathbb{Z}_d$ is determined by the

\[21\] Here, we should think of $O_{\mathbb{P}^n[d]}$ as its lift to a matrix factorization for the GLSM $T_X$, with theta angle $\theta = \pi d - \epsilon$ and $U(1)_\mathcal{L}$ charges $q \in \{-d, 0\}$. So, it does not belong to $\mathcal{W}_{L-b}$, but it does belong to $\hat{\mathcal{W}}_{L-b}$ upon tensoring with $Q'$. 

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relations\footnote{At a generic point $y \in \mathbb{P}^n$ the superpotential $W|_y$ satisfies $dW|_y^{-1}(0) = \{0\}$; hence, the assumed condition on the potential of LG orbifold is fulfilled.}

\begin{equation}
\begin{aligned}
m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) &= \left. \frac{\partial^2 W}{\partial x_i \partial x_j} \right|_{x_i=0}, \\
m_d(\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_d}) &= \left. \frac{\partial^d W}{\partial x_{i_1} \cdots \partial x_{i_d}} \right|_{x_i=0}
\end{aligned}
\end{equation}

at each point of the base, where we have identified $p$ with $x_{n+1}$.

As in the case of Veronese embedding, the global sheaf structure of $\mathcal{A}_0 \sharp \mathbb{Z}_d$ can also be read off from the global behavior of $x_i$ and $p$; we present two examples for illustration.

\subsection*{5.2.1 Example: quadrics}

In the case $d = 2$, at each point of the base, the superpotential is quadratic in the fiber coordinates. Therefore, the sheaf of algebra is the Clifford algebra associated with the quadratic form given by \(\frac{\partial^2 W}{\partial x_i \partial x_j}, i = 0, \ldots, n+1, x_{n+1} := p\) at fixed $y_i$. We can take the reference brane $\mathcal{B}_0^{(0)}$ to be given by the matrix factorization

\begin{equation}
Q_0 = \sum_{i=0}^{n} (x_i \eta_i) + p \eta_{n+1} + \sum_{i=0}^{n} \left( \frac{1}{2} \frac{\partial F_2}{\partial x_i} + \frac{1}{2} p y_i \right) \eta_i + \frac{1}{2} \left( \sum_{i=0}^{n} y_i x_i \right) \eta_{n+1};
\end{equation}

hence, the curved complex associated with $\mathcal{B}_0^{(0)}$ is given by

\begin{equation}
\begin{aligned}
\mathcal{O}_+ &\to \mathcal{O}_- \otimes \mathcal{V} &\to \cdots &\to \mathcal{O}_{(-p-1)} \otimes \mathcal{V}^{n+1} \\
\mathcal{O}_{(-1)} &\to \mathcal{O}_{(-p-1)} \otimes \mathcal{V}^{n} &\to \mathcal{O}_{(-p)} \otimes \mathcal{V}^{n+1}
\end{aligned}
\end{equation}

where all the sheaves $\mathcal{O}_{(-j)}^a$ in (5.17) denote sheaves over $\mathbb{P}^n$ and the subindex $\pm$ indicates the $\mathbb{Z}_2$-weight of the sheaf. A similar computation as the one in Sect. 5.1 let us conclude that

\begin{equation}
\mathcal{A}_0 \cong \left( \bigoplus_{i=0}^{n+1} \mathcal{O}_{(-j)} \otimes \wedge^i \mathcal{V} \right) \oplus \left( \bigoplus_{i=0}^{n+1} \mathcal{O}_{(-j+1)} \otimes \wedge^i \mathcal{V} \right)
\end{equation}

globally. So, we can write

\begin{equation}
D(Y, W) \cong D(\mathbb{P}^n, \mathcal{A}_0 \sharp \mathbb{Z}_2), \tag{5.19}
\end{equation}

i.e., the hybrid model B-brane category $D(Y, W)$ is equivalent to the derived category of sheaves of $\mathcal{A}_\infty \mathcal{A}_0 \sharp \mathbb{Z}_2$-modules.
5.2.2 Example: cubic hypersurfaces

The $\mathcal{B}_0^{(0)}$-brane is given by

$$Q_0 = \sum_{i=0}^{n} (x_i \bar{n}_i) + p \bar{n}_{n+1} + \sum_{i=0}^{n} \left( \frac{1}{3} \frac{\partial F_3}{\partial x_i} + \frac{1}{2} p y_i \right) \eta_i + \frac{1}{2} \left( \sum_{i=0}^{n} x_i y_i \right) \eta_{n+1},$$

and its associated curved complex is

$$\mathcal{O}_0 \rightarrow \mathcal{O}_1 \otimes V \rightarrow \cdots \rightarrow \mathcal{O}_{n+1} \otimes \wedge^{n+1} V \rightarrow \mathcal{O}_{n-1} \otimes V \rightarrow \mathcal{O}_n \otimes \wedge^n V,$$

where the subscripts of the line bundles are the $\mathbb{Z}_3$-weights. At each point of the base, the $A_\infty$-algebra is given by

$$m_2(\psi_i, \psi_{n+1}) + m_2(\psi_{n+1}, \psi_i) = y_i,$$

$$m_3(\psi_i, \psi_j, \psi_k) = \frac{\partial^3 F_3}{\partial x_i \partial x_j \partial x_k}.$$

Globally, the sheaf of $A_\infty$-algebra $\mathcal{A}_{D0}$ is

$$\mathcal{A}_{D0} = \left( \bigoplus_{i=0}^{n+1} \mathcal{O}_i \otimes \wedge^i V \right) \oplus \left( \bigoplus_{i=1}^{n-1} \mathcal{O}_i \otimes \wedge^{i+1} V \right) \oplus \left( \mathcal{O}_n \otimes \wedge^n V \right).$$

Therefore, the HPD of $\mathbb{P}^n[3]$ is the noncommutative space $(\mathbb{P}^n, \mathcal{A}_{0}^{\sharp} \mathbb{Z}_3)$.

5.3 HPD of complete intersections

The method can also be applied to the HPD of Fano complete intersections of the form $\mathbb{P}^n[d_1, d_2, \cdots, d_k]$, $\sum_{\alpha=1}^{k} d_\alpha < n + 1$. From the GLSM construction, it is straightforward to see that the HPD can be described by the hybrid model on the space

$$Y = \text{Tot} \left( \mathcal{O}(-1, 0)^{\oplus(n+1)} \oplus \mathcal{O}(1, -1) \rightarrow W_{\mathbb{P}}(d_1, \cdots, d_k) \times \mathbb{P}^n \right),$$

with superpotential

$$W = \sum_{\alpha=1}^{k} p_\alpha F_d(x) + p \sum_{i=0}^{n} x_i y_i,$$
where \( p_\alpha \) are homogeneous coordinates of the weighted projective space \( \mathbb{W}^P(d_1, \ldots, d_k) \), \( y_i \) are homogeneous coordinates of \( \mathbb{P}^n \), \( x_i \) and \( p \) are coordinates along the fibers of \( \mathcal{O}(-1,0)^{\oplus(n+1)} \) and \( \mathcal{O}(1,-1) \), respectively.

As in the case of hypersurfaces, \( \mathcal{A}_0 \) is spanned by \( \psi_0, \ldots, \psi_{n+1} \) and the \( A_\infty \)-products of \( \mathcal{A}_0 \) are determined by

\[
m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W}{\partial x_i \partial x_j} \bigg|_{x_i=0},
\]

\[
m_{d_\alpha}(\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_d}) = \frac{\partial^d W}{\partial x_{i_1} \cdots \partial x_{i_d}} \bigg|_{x_i=0}
\]

at each point of the base \( \mathbb{W}^P[d_1, \ldots, d_k] \times \mathbb{P}^n \), where we have identified \( x_{n+1} := p \). However, \( \mathcal{A}_0 \) is not the sheaf of algebra corresponding to the hybrid model under consideration, because we cannot ignore the orbifold singularity coming from the affine patches of \( \mathbb{W}^P[d_1, \ldots, d_k] \) and, for generic \( d_\alpha \), we cannot write the space \( Y \) as a global orbifold. Exceptions are, for instance if \( d_\alpha = d \) for all \( \alpha \), then we have a noncommutative resolution of \( \mathbb{P}^{k-1} \times \mathbb{P}^n \). Otherwise, we have to deal with a sheaf of algebras over a singular space. Therefore, we must resort to the general framework outlined in Sect. 4.6. Here, we just illustrate the idea by a case study, which can be easily generalized.

### 5.3.1 Example: \( \mathbb{P}^n[2, 3] \)

We study the HPD of the complete intersection of a quadric hypersurface and a cubic hypersurface in \( \mathbb{P}^n \), denoted by \( X \). The GLSM for this complete intersection is a \( U(1) \) gauge theory with the following matter content and gauge charges:

\[
\begin{array}{cccccc}
X_0 & X_1 & \cdots & X_n & P_1 & P_2 \\
1 & 1 & \cdots & 1 & -2 & -3
\end{array}
\]

and the superpotential is

\[
W_0 = P_1 G_2(X) + P_2 G_3(X),
\]

where \( G_2 \) and \( G_3 \) are quadric and cubic polynomials in \( X_i \), respectively. We assume \( n \geq 4 \). Suppose that \( \zeta \) is the FI parameter of this model. Then, the geometric phase (\( \zeta \gg 1 \)) is a nonlinear sigma model with target space of the complete intersection defined by \( G_2 = G_3 = 0 \) in \( \mathbb{P}^n \).

For \( \zeta \ll -1 \) and generic \( G_2, G_3 \), the Higgs branch of the GLSM above is a hybrid model on

\[
Y_0 \equiv \text{Tot} \left( \mathcal{O}(-1)^{\oplus(n+1)} \rightarrow \mathbb{W}^P(2, 3) \right)
\]

with the superpotential given by (5.23), where \( P_1, P_2 \) are homogeneous coordinates on the weighted projective space \( \mathbb{W}^P(2, 3) \) and \( X_i \)'s are the fiber coordinates. This hybrid
model can be viewed as two LG orbifold theories, each defined on a coordinate patch of \( WP(2, 3) \), glued together in a consistent way. Let’s denote by \( U_i \) the coordinate patch where \( P_i \neq 0 \). Then, we have

\[
U_1 \cong \mathbb{C}/\mathbb{Z}_2, \quad U_2 \cong \mathbb{C}/\mathbb{Z}_3.
\]

Assume that \( Z_1 \) and \( Z_2 \) are local coordinates of \( U_1 \) and \( U_2 \), respectively. It is easy to see that when \( P_1 \neq 0 \), the hybrid model reduces to a LG model with \( Z_2 \) orbifold on \( \mathbb{C}^{\oplus(n+2)} \), with the \( \mathbb{Z}_2 \)-action:

\[
e^{\pi i} \cdot (Z_1, X_i) = (e^{3\pi i} Z_1, e^{-\pi i} X_i) = (-Z_1, -X_i)
\]

and superpotential

\[
W_{01} = G_2(X) + Z_1 G_3(X).
\]

Similarly, when \( P_2 \neq 0 \), the hybrid model reduces to a LG model with \( Z_3 \) orbifold on \( \mathbb{C}^{\oplus(n+2)} \), with the \( \mathbb{Z}_3 \)-action:

\[
e^{\frac{2\pi i}{3}} \cdot (Z_2, X_i) = (e^{\frac{4\pi i}{3}} Z_2, e^{-\frac{2\pi i}{3}} X_i) = (e^{-\frac{2\pi i}{3}} Z_2, e^{-\frac{2\pi i}{3}} X_i)
\]

and superpotential

\[
W_{02} = Z_2 G_2(X) + G_3(X).
\]

Matrix factorizations of the hybrid model are of the form \((\mathcal{E}, Q)\), where \( \mathcal{E} \) is a coherent sheaf with \( \mathbb{Z}_2 \)-grading on \( Y_0 \) defined in (5.24), and \( Q \) is an odd endomorphism of \( \mathcal{E} \) that squares to \( W_0 \cdot \text{id} \). When restricted to a coordinate patch, the orbifold action on \( \mathcal{E} \) can be read off from the sheaf structure. For example, sections of \( \mathcal{O}(a) \) transform as \( \exp(\pi i) \cdot \lambda = \exp(a\pi i) \lambda \) under \( \mathbb{Z}_2 \) in \( U_1 \) while they transform as \( \exp(2\pi i/3) \cdot \lambda = \exp(a2\pi i/3) \lambda \) under \( \mathbb{Z}_3 \) in \( U_2 \). Also, \( Q(X, P_1, P_2) \) is replaced by \( Q(X, 1, Z_1) \) in \( U_1 \) and is replaced by \( Q(X, Z_2, 1) \) in \( U_2 \).

There is a semiorthogonal decomposition of the derived category of coherent sheaves of the complete intersection \( X \):

\[
D(X) = \langle MF(Y_0, W_0), \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(n-4) \rangle,
\]

therefore the Lefschetz decomposition associated with the embedding \( X \hookrightarrow \mathbb{P}^n \) is\(^{23}\)

\[
D(X) = \langle A_0, A(1), \ldots, A(n-5) \rangle,
\]

(5.25)

where \( A_0 = \langle MF(Y_0, W_0), \mathcal{O}_X \rangle \) if \( n \geq 5 \), \( A_0 = MF(Y_0, W_0) \) if \( n = 4 \), and \( A_i = \langle \mathcal{O}_X \rangle \) for \( i > 0 \).

\(^{23}\) Note that \( n \geq 5 \) in the Fano case. In the Calabi–Yau case, \( n = 4 \), and \( D(X) \cong MF(Y_0, W_0) \).
From our general construction, the GLSM for the universal hyperplane section \( X \) of the embedding \( X \hookrightarrow \mathbb{P}^n \) is a \( U(1) \times U(1) \) GLSM with the following matter content and gauge charges:

\[
\begin{array}{cccccccc}
X_0 & X_1 & \cdots & X_n & P_1 & P_2 & P & Y_0 & \cdots & Y_n \\
U(1)_1 & 1 & 1 & \cdots & 1 & -2 & -3 & -1 & 0 & \cdots & 0 \\
U(1)_2 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 1 & \cdots & 1 \\
\end{array}
\]

together with the superpotential

\[
W = P_1 G_2(X) + P_2 G_3(X) + P \sum_{i=0}^{n} X_i Y_i. \quad (5.26)
\]

The geometric phase \( (\xi_1 \gg 1, \xi_2 \gg 1) \) realizes the universal hyperplane section \( X \).

The semiorthogonal decomposition of \( D(X) \) corresponding to (5.25) is

\[
D(X) = \langle \mathcal{C}, \mathcal{A}_1(1) \boxtimes D(\tilde{\mathbb{P}}^n), \cdots, \mathcal{A}_{n-5}(n-5) \boxtimes D(\tilde{\mathbb{P}}^n) \rangle,
\]

where \( \mathcal{C} \) is the HPD category.

From the general argument, \( \mathcal{C} \) consists of the B-branes of the Higgs branch on the phase with \( \xi_1 \ll -1, \xi_2 \gg 1 \). This Higgs branch is described by a hybrid model on

\[
Y \equiv \text{Tot} \left( \mathcal{O}(-1, 0)^{\oplus (n+1)} \oplus \mathcal{O}(1, -1) \rightarrow \mathbb{W}\mathbb{P}(2, 3) \times \tilde{\mathbb{P}}^n \right)
\]

with superpotential (5.26), where \( P_1, P_2 \) are homogeneous coordinates of the weighted projective space \( \mathbb{W}\mathbb{P}(2, 3) \), \( Y_i \) are homogeneous coordinates of \( \mathbb{P}^n \), \( X_i \) and \( P \) are coordinates along the fibers of \( \mathcal{O}(-1, 0)^{\oplus (n+1)} \) and \( \mathcal{O}(1, -1) \), respectively. Again, we can think of this hybrid model as a family of LG orbifolds defined on coordinate patches of \( \mathbb{W}\mathbb{P}(2, 3) \times \tilde{\mathbb{P}}^n \) glued together consistently. The reference brane \( \mathcal{B}_0 \) has the tachyon profile

\[
Q_0 = \sum_{i=0}^{n} X_i \tilde{\eta}_i + P \tilde{\eta}_{n+1} + \sum_{i=0}^{n} \left( \frac{1}{2} P_1 \frac{\partial G_2}{\partial X_i} + \frac{1}{3} P_2 \frac{\partial G_3}{\partial X_i} \right) \eta_i + \left( \sum_{i=0}^{n} X_i Y_i \right) \eta_{n+1},
\]

where the vacuum state of the Clifford algebra is identified with \( \mathcal{O} \) on \( \mathbb{W}\mathbb{P}(2, 3) \times \tilde{\mathbb{P}}^n \). Notice that \( X_i \)'s are sections of \( \mathcal{O}(-1, 0) \) and \( P \) is a section of \( \mathcal{O}(1, -1) \), we see that the sheaf of algebras \( \mathcal{A} = \text{End}_{MF(Y, W)}(\mathcal{B}_0) \) can be written globally as

\[
\mathcal{A} = \bigoplus_{m=0}^{n+1} \left( \mathcal{O}(-m, 0) \oplus \mathcal{O}(1-m, -1) \right) \otimes \wedge^m V,
\]

where \( V = \mathbb{C}^{n+1} \). The \( A_\infty \) structure of \( \mathcal{A} \) can be analyzed in the same way as the example shown in Sect. 4.6.
6 The functor $D(B, \text{Mod} - \mathcal{A}) \to D(\mathcal{X})$

In this section, we give a description of the functor from the category $D(B, \text{Mod} - \mathcal{A}) := D(Y, W)$ used to describe the B-brane category of the hybrid models on $Y = \text{Tot}(\mathcal{V} \to B)$ that arises as HPD of Fano embeddings. Given a projective embedding, the HPD category $\mathcal{C}$ is a subcategory of $D(\mathcal{X})$ by definition, where $\mathcal{X}$ is the universal hyperplane section. Therefore, there exists a fully faithful functor from the proposed HPD category $D(B, \text{Mod} - \mathcal{A})$ to $D(\mathcal{X})$, realizing the equivalence $D(B, \text{Mod} - \mathcal{A}) \cong \mathcal{C}$, where $D(B, \text{Mod} - \mathcal{A})$ is the derived category of $A_\infty \mathcal{A}$-modules. The GLSM construction gives a simple description of this functor.

First, we have a functor $\mathcal{F} : D(B, \text{Mod} - \mathcal{A}) \to MF(\text{Tot}(\mathcal{V} \to B), W)$. This functor can be defined by generalizing the functor from $D(\text{Mod} - A_0)$ to the category of matrix factorizations of LG models described in Sect. 4.3 (cf. eq. (4.38)). Suppose we have an object $\mathcal{N}$ in $D(B, \text{Mod} - \mathcal{A})$, i.e., $\mathcal{N}$ is a sheaf of $A_\infty$ module of $\mathcal{A}$ with $A_\infty$ actions given by $m_i^N$, $i \geq 1$. The matrix factorization $\mathcal{F}(\mathcal{N})$ can be defined as follows. The Chan–Paton sheaf of $\mathcal{F}(\mathcal{N})$ is

$$\mathcal{M} = \mathcal{N} \otimes \text{Sym} \mathcal{V}^\vee.$$ 

Let $x_i, i = 1, \ldots, \text{rank}(\mathcal{V})$ be a set of linearly independent sections of $\mathcal{V}^\vee$,\footnote{If $B = \cup_i U_i$ is an open cover, one can choose a basis $\{e_1^{(i)}, \ldots, e_{\text{rank}(\mathcal{V})}^{(i)}\}$ of $\mathcal{V}|_{U_i}$ for every open patch $U_i$, then $x_j$ can be defined in $U_i$ as the linear function such that $x_j(e_k^{(i)}) = \delta_{jk}$.} then $x_1 x_2 \cdots x_k$ is a section of $\text{Sym} \mathcal{V}^\vee$ for any set of indices $i_1, \ldots, i_k$. Consequently, the Tachyon profile $Q_\mathcal{M}$ of $\mathcal{F}(\mathcal{N})$ can be defined by

$$Q_\mathcal{M}(\phi) = \sum_k \sum_{i_1, \ldots, i_k} m_{i_{k+1}}^N(\phi, \psi_{i_1}, \ldots, \psi_{i_k}) x_{i_1} x_{i_2} \cdots x_{i_k},$$

where $\phi$ is a section of $\mathcal{N}$. This functor is an equivalence with the inverse functor given by $B \to \text{Hom}_{MF(\text{Tot}(\mathcal{V} \to B), W)}(\mathcal{B}_0, B)$, where $\mathcal{B}_0$ is the reference brane that is point-like along each fiber of $\mathcal{V}$. In particular, $\mathcal{A} = \text{End}_{MF(\text{Tot}(\mathcal{V} \to B), W)}(\mathcal{B}_0)$.

Now, we can use the GLSM $T_\mathcal{X}$ defined in eq.(2.15) to construct a fully faithful functor $\mathcal{G}$:

$$\mathcal{G} : MF(\text{Tot}(\mathcal{V} \to B), W) \to D(\mathcal{X}).$$

Recall that the hybrid model on $\text{Tot}(\mathcal{V} \to B)$ with superpotential $W$ is the Higgs branch of the GLSM $T_\mathcal{X}$ in the phase with $\zeta_\mathcal{L} \ll -1, \zeta' \gg 1$ (LG phase), while the Higgs branch of the phase with $\zeta_\mathcal{L} \gg 1, \zeta' \gg 1$ (geometric phase) is the nonlinear sigma model with target space $\mathcal{X}$. Consequently, the functor $\mathcal{G}$ can be implemented by the brane transport across the classical phase boundary at $\zeta_\mathcal{L} = 0, \zeta' \gg 1$. More precisely, for each matrix factorization in the category $MF(\text{Tot}(\mathcal{V} \to B), W)$, we first lift it to a matrix factorization of the GLSM $T_\mathcal{X}$. Then, after possibly combining this matrix factorization with several branes that are empty in the LG phase, we can transport it to the geometric phase and project it to an object in $D(\mathcal{X})$. Here, the grade
restriction rule states that the combined brane must be in the small window category of the local model associated with the phase boundary. Thus, we see the brane transport realizes the functor $\mathcal{G}$. There are examples in [8] demonstrating how this procedure works.

In conclusion, the fully faithful functor from $D(B, \text{Mod} - \mathcal{A})$ to $D(\mathcal{X})$ is given by $\mathcal{G} \circ \mathcal{F}$, which is an equivalence between $D(B, \text{Mod} - \mathcal{A})$ and the HPD category $\mathcal{C}$.

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Declarations

Conflict of interest statement On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix A: Checkings on the $(\text{Mod} - \mathcal{A}_{D0}) \to \text{MF}(W)$ functor

Here, we provide several checks for the proposed functor (4.38).

Check 1. $d = 2$ Case. When $d = 2$, we have $\mathcal{A}_{D0} = Cl(n, \mathbb{C})$, the Clifford algebra defined by the quadratic form $\frac{\partial^2 W}{\partial x_i \partial x_j}$. The correspondence between the matrix factorization for quadratic superpotentials and Clifford modules is given in [9, 59], which matches (4.38).

Check 2. $N = \mathcal{A}_{D0}$. It is straightforward to check (4.38) for the case the module is $\mathcal{A}_{D0}$ itself. Then, (4.38) corresponds to $Q_{D0}$. In fact, as shown in [12], the fermionic generators $\psi_i, i = 1, \cdots, n$, satisfy

- If $d = 2$, $\mathcal{A}_{D0}$ is the Clifford algebra:

  $$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W}{\partial x_i \partial x_j}.$$  
(A.1)

- If $d > 2$, $\mathcal{A}_{D0}$ is an $A_\infty$-algebra, where $m_2$ satisfies

  $$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = 0,$$  
(A.2)

  $$m_k = 0 \text{ for } k = 1, \cdots, d - 1 \text{ and }$$

  $$m_d(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_d}) = \frac{1}{d!} \frac{\partial^d W}{\partial x_{i_1} \cdots \partial x_{i_d}}.$$  
(A.3)
Thus, when \( d > 2 \), if we identify \( m_2(\cdot, v_i) \) with \( \{\eta_i, \cdot\} \) according to (A.2), then (A.3) tells us that \( m_d(\cdot, \psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_{d-1}}) \) should be identified with

\[
\frac{1}{d!} \sum_{i=1}^{n} \frac{\partial^d W}{\partial x_i \partial x_{i_1} \cdots \partial x_{i_d-1}} \{\eta_i, \cdot\}.
\]

Because \( W \) is homogeneous with degree \( d \), we see that \( Q_M \) defined by (4.38) is exactly \( Q_{D_0} \) defined by (4.34) in this case.

**Check 3.** \( Q_M^2 = W \cdot \text{id} \). Here, we will show that the object \( Q_M^2 \) is indeed a matrix factorization of \( W \). We will make the assumption that \( m_s^N = 0 \) for \( s > d \) (which can be shown below to be true for the case \( n = 1 \)), it can be shown that \( Q_M^2 = W \cdot \text{id} \). For example, when \( d = 3 \)

\[
Q_M(\phi) = \sum_{ij} m_3^N(\phi, \psi_i, \psi_j)x_ix_j + \sum_{i} m_2^N(\phi, \psi_i) x_i.
\]

Therefore,

\[
Q_M^2(\phi) = \sum_{ijkl} m_3^N(m_3^N(\phi, \psi_i, \psi_j), \psi_k) x_i x_j x_k x_l
\]

\[
+ \sum_{ijkl} m_3^N(m_2^N(\phi, \psi_i), \psi_j, \psi_k) x_i x_j x_k
\]

\[
- \sum_{ijkl} m_2^N(m_3^N(\phi, \psi_i, \psi_j), \psi_k) x_i x_j x_k + \sum_{ij} m_2^N(m_2^N(\phi, \psi_i), \psi_j) x_i x_j.
\]

(A.4)

From (3.3), we get

\[
m_2^N(m_2^N(\phi, \psi_i), \psi_j) = m_2^N(\phi, m_2(\psi_i, \psi_j)),
\]

then the last term of (A.4) vanishes due to (A.2). The first term of (A.4) also vanishes because of (3.3), (A.3) and \( m_k(\cdots, 1, \cdots) = 0 \) for \( k > 2 \). Also from (3.3) and (A.3), the second and third terms of (A.4) yield

\[
\sum_{ijkl} \left( m_3^N((m_2^N(\phi, \psi_i), \psi_j, \psi_k) - m_2^N((m_3^N(\phi, \psi_i, \psi_j), \psi_k)) \right) x_i x_j x_k
\]

\[
= \sum_{ijkl} m_2^N((\phi, m_3(\psi_i, \psi_j, \psi_k)) x_i x_j x_k = \frac{1}{3!} \phi \sum_{ijkl} \frac{\partial^3 W}{\partial x_i \partial x_j \partial x_k} x_i x_j x_k = W \cdot \phi,
\]

which shows \( Q_M^2 = W \cdot \text{id} \).

**Check 4.** \( n = 1 \) case. Finally, we show that the functor reproduces the matrix factorizations for the case \( n = 1 \), i.e., \( W = x^d \). In this case, the \( D0 \)-brane is given by
the matrix factorization

\[ Q_{D0} = x\eta + x^{d-1}\eta. \tag{A.5} \]

The fermionic generator of \( A_{D0} = Hom(B_{D0}, B_{D0}) \) is \( \psi = \eta - x^{d-2}\eta \). Let \( B_l \) be the matrix factorization with

\[ Q_M = x^l\overline{\pi} + x^{d-l}\pi \tag{A.6} \]

where \( 1 < l < d \) and \( \{\pi, \pi\} = \{\overline{\pi}, \overline{\pi}\} = 0 \). Next, we will show that (4.38) recovers \( Q_M \). Start by considering the bosonic state \( \phi_0 \in Hom_0(B_{D0}, B_l) \) and the fermionic state \( \phi_1 \in Hom_1(B_{D0}, B_l) \). If the vacuum state of \( M \) is denoted as \( |\Omega\rangle \), then

\[ \phi_0|0\rangle = |\Omega\rangle, \quad \phi_0\eta|0\rangle = x^{l-1}\overline{\pi}|\Omega\rangle, \]

and

\[ \phi_1|0\rangle = \overline{\eta}|\Omega\rangle, \quad \phi_1\eta|0\rangle = -x^{d-l-1}|\Omega\rangle. \]

In matrix form,

\[ \psi = \begin{pmatrix} 0 & -x^{d-2} \\ 1 & 0 \end{pmatrix}, \quad \phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & x^{l-1} \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 0 & -x^{d-l-1} \\ 1 & 0 \end{pmatrix}. \]

Using the algorithm reviewed in Sect. 3, one can compute \((\iota\psi = v)^{25}\)

\[ m_k(\psi^\otimes k) = 0, \quad f_k(\psi^\otimes k) = (-1)^k x^{d-k-1}\eta, \quad 1 < k < d, \]

and

\[ m_d(\psi^\otimes d) = 1, \quad f_d(\psi^\otimes d) = 0. \]

By composing the homomorphisms, one gets

\[ \phi_0 \circ \psi = x^{l-1}\phi_1, \quad \phi_1 \circ \psi = -x^{d-l-1}\phi_0. \]

Therefore,

\[ \phi_0 \circ \psi = d\tilde{\phi}_0^{(1)} = \tilde{\phi}_0^{(1)} Q_{D0} - Q_M\tilde{\phi}_0^{(1)}, \]

\[ \phi_1 \circ \psi = -d\tilde{\phi}_1^{(d-l-1)} = -\tilde{\phi}_1^{(d-l-1)} Q_{D0} - Q_M\tilde{\phi}_1^{(d-l-1)}, \]

where

\[ \tilde{\phi}_0^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & x^{l-2} \end{pmatrix}, \quad \tilde{\phi}_1^{(d-l-1)} = \begin{pmatrix} 0 & x^{d-l-2} \\ 0 & 0 \end{pmatrix}. \]

---

\[ ^{25} \text{Another computation for this result can be found in the example at the end of Sect. 4.3.} \]
from which one can deduce that

\[ m_2^N(\phi_0, v) = 0, \quad f_2^N(\phi_0, v) = -\tilde{\phi}_0^{(1)}. \]

and

\[ m_2^N(\phi_1, \psi) = 0, \quad f_2^N(\phi_1, \psi) = \tilde{\phi}_1^{(d-l-1)}. \]

It can be shown by induction that

\[ m_{k+1}^N(\phi_0, \psi \otimes k) = 0, \quad f_{k+1}^N(\phi_0, \psi \otimes k) = -\tilde{\phi}_0^{(k)}, \quad 1 < k < l, \]

where

\[ \tilde{\phi}_0^{(k)} = \begin{pmatrix} 0 & 0 \\ 0 & x^{l-k-1} \end{pmatrix}. \]

Similarly,

\[ m_{k+1}^N(\phi_1, \psi \otimes k) = 0, \quad f_{k+1}^N(\phi_1, \psi \otimes k) = \tilde{\phi}_1^{(d-l-k)}, \quad 1 < k < d-l, \]

where

\[ \tilde{\phi}_1^{(k)} = \begin{pmatrix} 0 & x^{k-1} \\ 0 & 0 \end{pmatrix}. \]

Now, one can compute

\[ u_m m_{l+1}^N(\phi_0, \psi \otimes l) = -f_1^N(\phi_0, \psi \otimes (l-1)) \circ \psi - \phi_0 \circ f_1(\psi \otimes l) = \phi_1, \]

similarly \( u_{m+d-l+1}^N(\phi_1, \psi \otimes (d-l)) = \phi_0, \) and all the higher-order multiplications vanish. Therefore, in the basis \( \{\phi_0, \phi_1\}, (4.38) \) yields

\[ Q_M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x^l + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x^{d-l} = \begin{pmatrix} 0 & x^d-l \\ x^l & 0 \end{pmatrix}, \]

which is exactly the matrix factorization (A.6) we started with.

**Appendix B: \( A_\infty \)-algebras defined by ribbon trees**

The structure of the \( A_\infty \)-algebra \( A = \text{End}(B_{D_0}) \) corresponding to a Landau–Ginzburg model with homogeneous superpotential was derived in [12] using the method of summing over ribbon trees. In this “Appendix,” we review the idea of [12] and generalize it to LG models with inhomogeneous superpotentials.

---

26 We use \( \phi_i \) to denote the cohomology class and the representative; the meaning should be clear from the context.
Let $\iota$ be an embedding of $H(A) := H_{m_1}(A)$ into $A$. If we define the projection $\pi : A \to H(A)$ such that $\pi \circ \iota = 1$ and there is a map $h : A \to A$ of degree $-1$ such that $1 - \iota \circ \pi = m_1^A \circ h + h \circ m_1^A$ and $h^2 = \pi h = h 1 = 0$, then the $A_\infty$ products $m_k : H(A)^{\otimes k} \to H(A)$, $k \geq 2$ can be defined by summing over the contributions from ribbon trees [65]:

$$m_k = \sum_T m_{k,T}. \tag{B.1}$$

For a LG model with degree-$d$ superpotential, the ribbon trees contributing to the sum have one root and $d$ leaves such that the valency of any vertex is 2 or 3 [12]. (B.1) is a solution to the defining relations (3.3).

In our convention, $\iota(\psi_i) = v_i$ defined by (4.36), consequently $h$ can be defined to be $h = \sum_i \eta_i \frac{\partial}{\partial x_i}$ where $\eta_i$ acts by multiplication in the Clifford algebra.

By definition, a ribbon tree is a tree $T$ with a collection of vertices, external edges and internal edges such that: (a) Each external edge is incident to a single vertex. (b) Each internal edge is incident to exactly two vertices. (c) One of the external edge is the root, the other external edges are the leaves. Every ribbon tree $T$ with one root and $k$ leaves determines a term $m_{k,T}$ in (B.1).

Given a tree $T$, to compute $m_{k,T}(\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_k})$ we put $\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_k}$ on the leaves from left to right and then act on them a series of maps as follows:

- Each leaf gives a map $\iota$;
- Each bivalent vertex gives a map $f$;
- Each internal edge gives a map $h$;
- Each trivalent vertex corresponds to the multiplication in $A$;
- The root gives the map $\pi$

while reading from the top to the bottom. Here $f$ is defined by

$$\left\{ \sum_i \frac{\partial W}{\partial x_i} \eta_i, \ldots \right\} - \sum_i \frac{\partial W}{\partial x_i} \eta_i,$$

where $\{\}$ denotes commutator/anticommutator depending on whether the second argument is of even/odd degree and the second term is the usual multiplication of the Clifford algebra.

It is shown in [12] that there is always a tree given by Fig. 3a making a nontrivial contribution to $m_2$. For $d > 2$, this is the only contribution and it makes $m_2$ to satisfy $m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = 0$. When $d = 2$, there is another nontrivial contribution from the tree given by Fig. 3b. The effect of Fig. 3b is to modify $m_2$ such that $m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W}{\partial x_i \partial x_j}$.

In general, other than the tree in Fig. 3a, the only ribbon tree that can make a nonzero contribution is the one in Fig. 4. If the input of the tree is $\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_k}$, then before hitting the root, the image of the set of maps encoded in the tree is $\frac{1}{k!} \frac{\partial^k W}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}}$ plus some $Q$-exact terms. When $k \neq d$, this image is $Q$-exact and annihilated by the
Fig. 3  a Ribbon tree contributing to $m_2$.  b Another contribution to $m_2$ when $d = 2$

projection $\pi$, so the output of the tree is zero. When $k = d$, the output is $\frac{1}{k!} \frac{\partial^k W}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}$ because it is not $Q$-exact.

In summary, we have

- If $d = 2$, $\mathcal{A}$ is a Clifford algebra given by:

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W}{\partial x_i \partial x_j}.$$
• If \( d > 2 \), \( m_2 \) is the wedge product, \( m_k = 0 \) for \( k \neq 2 \) and \( k \neq d \).

\[
m_d(\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_d}) = \frac{\partial^d W}{\partial x_{i_1} \cdots \partial x_{i_d}}.
\]

Now assume we have a inhomogeneous superpotential of the form

\[
W = \sum_{l=2}^{d} W^{(l)},
\]

where \( \text{deg} \ W^{(l)} = l \). Because now every nonvanishing derivative \( \frac{\partial^l W^{(l)}}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}} \) is not \( Q \)-exact, we can use the same argument to deduce that

\[
m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W^{(2)}}{\partial x_i \partial x_j} \tag{B.2}
\]

and

\[
m_l(\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_l}) = \frac{1}{l!} \frac{\partial^l W^{(l)}}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}} \tag{B.3}
\]

for \( 3 \leq l \leq d \).

**Appendix C: Orbibundle**

Let \( X \) be a smooth manifold admitting a \( G \)-action, where \( G \) is a group. An orbibundle on the quotient stack \([X/G]\) is a fiber bundle \( E \rightarrow X/G \) with each fiber an orbifold. Explicitly, let \( V \) be a vector space admitting a representation of \( G \):

\[
\rho : G \rightarrow GL(V).
\]

The fiber of \( E \) is \( V / \rho(G) \). If \( \{U_\alpha : \alpha \in I\} \) is an open cover of \( X/G \) and

\[
\phi_\alpha : U_\alpha \times V / \rho(G) \rightarrow \pi^{-1}(U_\alpha)
\]

are the corresponding local trivializations. Then, the transition functions \( g_{\alpha\beta} = \phi^{-1}_\alpha \circ \phi_\beta \) take values in \( GL(V) / \rho(G) \). A local section of \( E \) is given by a \( \rho(G) \)-invariant function \( s_\alpha : U_\alpha \rightarrow V \) so the relation

\[
s_\alpha = g_{\alpha\beta} \cdot s_\beta
\]

is well defined on \( U_\alpha \cap U_\beta \). Given a representation of \( G \) as above, the orbibundles on \([X/G]\) are classified by \( H^1(X, GL(V) / \rho(G)) \). When the representation is trivial,
the orbibundle is just an ordinary vector bundle. When \( \dim_C V = 1 \), we call it a line bundle.

A morphism between two orbibundles \( E_1 \xrightarrow{\pi_1} X/G \) and \( E_2 \xrightarrow{\pi_2} X/G \) is a bundle map \( f : E_1 \rightarrow E_2 \), i.e., \( \pi_2 \circ f = \pi_1 \). Given local trivializations of \( E_1 \) and \( E_2 \) in an open set \( U \):

\[
\phi_1 : U \times V_1/\rho_1(G) \rightarrow \pi_1^{-1}(U), \\
\phi_2 : U \times V_2/\rho_2(G) \rightarrow \pi_2^{-1}(U),
\]

and for each \( x \in U \), \( f_U(x) := \phi_2^{-1} \circ f \circ \phi_1|_x \) is a linear map from \( V_1 \) to \( V_2 \) satisfying

\[
f_U(x) \circ \rho_1(g) = \rho_2(g) \circ f_U(x)
\]

for all \( g \in G \).

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