The Geometry and Topology of 3-Manifolds and Gravity

by

J. Gegenberg

Department of Mathematics and Statistics
University of New Brunswick
Fredericton, New Brunswick
CANADA E3B 5A3

e-mail address: jack@math.unb.ca

Abstract: It is well known that one can parameterize 2-D Riemannian structures by conformal transformations and diffeomorphisms of fiducial constant curvature geometries; and that this construction has a natural setting in general relativity theory in 2-D. I will show that a similar parameterization exists for 3-D Riemannian structures, with the conformal transformations and diffeomorphisms of the 2-D case replaced by a finite dimensional group of gauge transformations. This parameterization emerges from the theory of 3-D gravity coupled to topological matter.

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1 Introduction

In the the late 1970’s Thurston\[1\] proposed a geometric classification of the topologies of closed three dimensional manifolds. In his scheme, each such manifold decomposes into a connected sum of simpler manifolds $M = M_1 \sharp M_2 \sharp \cdots$, obtained by cutting along two-spheres and tori. These simpler manifolds are conjectured to admit one and only one of eight possible geometric structures. Although this geometrization conjecture is to date unproved\[1\], it has led to significant advances in our understanding of three dimensional geometry and topology, especially hyperbolic geometry and topology. For a review see [1].

It is clear that Thurston’s scheme should be relevant to fundamental issues in theoretical physics. In particular, if the quantization of gravity does not ”fix” spatial topology, then in the functional integral approach to calculating amplitudes we should integrate over spatial topologies. Indeed, in quantum cosmology, there are interesting calculations involving such integrals. See the work of Fujiwara’s group [3] and Carlip [4]. Furthermore, it has been known for some time that locally homogeneous cosmological models -the Bianchi/Kantowski-Sachs models- are essentially based on the eight Thurston geometries refered to above [1][5].

In the following, I will show first of all, that the Thurston geometries are fiducial solutions of a three dimensional gravity theory, in the same way that the constant curvature geometries are fiducial solutions of two dimensional Einstein gravity. Secondly, it will be shown that this leads to an alternative characterization of three dimensional manifolds in terms of flat bundles with structure group SO(3) over the three manifold. The upshot is that one may

\[1\]However, an interesting possible approach to its proof, proposed by R. Hamilton and by Isenberg and Jackson[2], uses Ricci flows and the technology of Bianchi models in relativistic cosmology.
then parameterize all the Riemannian metrics on the given 3-manifold in terms of gauge transformations of the fiducial metric of the appropriate Thurston geometry.

In the remainder of this introduction we will show how two dimensional gravity theory provides a natural physical setting for the geometric structure of two dimensional manifolds. In order to accomplish this, we will first review geometric structures in general, then briefly review the classification of closed two dimensional manifolds in terms of the three geometric structures associated with the manifolds of positive, zero and negative constant curvature. In section 2 we will review Thurston’s eight geometric structures and their relation to the question of the classification of three dimensional manifolds—the “geometrization conjecture”. In section 3, after a brief explication of the properties of a relevant topological field theory of gravity interacting with topological matter, we will argue that the latter is a natural setting for Thurston’s geometrization conjecture and propose a new construction of the geometric structure of locally homogeneous three manifolds.

An \((X, G)\) structure on a manifold \(M\) is a pair \((X, G)\) where \(X\) is a manifold with \(\dim X = \dim M\) and \(G\) is a Lie group acting transitively on \(X\), such that \(M\) is covered with ”charts” \(\{(\phi_\alpha, U_\alpha)\}\) such that on the overlap \(U_\alpha \cap U_\beta \neq \emptyset\), the map \(\phi_\alpha \circ \phi_\beta^{-1} \in G\).

A locally homogeneous structure on \(M\) is an \((X, Isom(X))\) structure on \(M\), where \(X\) is a Riemannian manifold and \(Isom(X)\) is the group of isometries admitted by \(X\).

It has been known for a long time that a given closed orientable two dimensional manifold admits one and only one of the following three locally homogeneous structures:
The manifolds $S^2, T^2, H^2$ are, respectively, the closed orientable two dimensional manifolds of constant positive, zero and negative curvature - i.e. the sphere, the (flat) torus and the (closed) hyperbolic space. The groups $\text{Isom}(S^2)$, etc., are the three dimensional groups of isometries of the sphere, etc. In the sense of F. Klein, these locally homogeneous structures are called geometries.

For a given geometry, the allowed topologies are determined by the set of finite subgroups of the corresponding isometry group.

Furthermore, there is a fiducial constant curvature Riemannian metric $\hat{g}$ for a given geometry and topology. This fiducial metric can be written in terms of the modular parameters associated with the complex structure of the Riemann surface form of the manifold. Locally, a Riemannian metric is determined by three smooth functions of the coordinates. Hence a given Riemannian metric $g$ on the manifold can be written:

$$g = e^{2\sigma} \Phi^*(\hat{g}),$$

(1)

where $\sigma$ is a function on the manifold and $\Phi$ is a diffeomorphism. We may view, at least formally, the fiducial metric $\hat{g}$ as the equivalence class of the Riemannian metrics $g$ modulo the action of multiplication by a conformal factor $e^{2\sigma}$ and the diffeomorphism group.

At first sight this seems wrong: it would seem to lump into the same equivalence class two distinct geometries, for example a flat metric $g_{(0)} = \delta_{ij} dx^i \otimes dx^j$ and a metric of constant positive curvature $g_{(+)} = (1 - r^2/4)^{-2} g_{(0)}$, where $r^2 := \delta_{ij} x^i x^j$. It is true that in a given coordinate patch $U$ these two metrics are conformally related, but this cannot be extended to the whole
manifold—the conformal factor \((1 - r^2/4)^{-2}\) is not defined at the points where \(r^2 = 4\).

This emerges quite naturally from two dimensional general relativity theory. We write the action functional for that theory in first order ("Palatini") form. The fields are a dyad \(e^a_\mu\) and an SO(2) connection \(\omega^a_{b\mu}\). The action functional is:

\[
S^{(2)} = \frac{1}{2\pi} \int_{M^2} d^2x \epsilon_{ab} e_\mu^a e_\nu^b R^{\mu\nu}(\omega).
\]

In the above, \(e\) is the determinant of \(e^a_\mu\) and \(R^{\mu\nu}(\omega)\) is the Ricci tensor constructed from the connection \(\omega^a_{b\mu}\). The equations of motion, obtained by requiring that \(S^{(2)}\) is stationary under variations of \(e\) and \(\omega\), fix the compatibility of the connection \(\omega^a_{b\mu}\) with an arbitrary smooth dyad \(e^a_\nu\). In fact, \(S^{(2)}\) is a topological invariant of the manifold \(M^2\)—it is the Euler number \(\chi(M^2)\). This fact effectively fixes the way that the locally arbitrary dyads "glue together" in overlapping patches so that the geometry is "compatible" with a fiducial geometry appropriate to the topology of \(M^2\). By "compatible" here, I mean that the metric is of the form of Eqn. (1).

The topological invariance of \(S^{(2)}\) can be viewed as a consequence of the "gauge invariance" of \(S^{(2)}\) under conformal transformations and diffeomorphisms. By conformal transformations, I mean transformations of the dyad field components of the form:

\[
e_\mu^a(x) \rightarrow e^{\sigma(x)} e_\mu^a(x).
\]

Under such transformations, the integrand in the action is mapped to itself plus a total derivative, and since \(M^2\) is compact, this term vanishes by the smoothness of the conformal factor \(e^a\).

\[^2\text{One might think that this argument could also apply to } S^{(2)}\text{ itself, since the latter can}\]
2 Three-Manifold Geometries

The situation for closed orientable 3-manifolds is complicated first of all by the fact that there are locally homogeneous structures on 3-manifolds that are not isometric to one of the three constant curvature spaces: $S^3$, $E^3$, $H^3$. In Table 1. below, the eight 3-manifold geometries are listed, specifying the underlying manifold, the isometry group and a fiducial Riemannian metric admitting the corresponding isometry group [10][11][3].
Table 1.

| Manifold   | IsometryGroup | Metric                                      |
|------------|---------------|---------------------------------------------|
| $S^3$      | $SO(4)$       | $\cos^2 y dx^2 + dy^2 + (dz - \sin ydx)^2$ |
| $E^3$      | $R^3 \times SO(3)$ | $dx^2 + dy^2 + dz^2$                     |
| $H^3$      | $PSL(2, \mathbb{C})$ | $dx^2 + e^{2x}(dy^2 + dz^2)$             |
| $S^2 \times E^1$ | $(Isom(S^2) \times Isom(E^1))^+$ | $dx^2 + dy^2 + \sin^2 ydz^2$             |
| $H^2 \times E^1$ | $(Isom(H^2) \times Isom(E^2))^+$ | $dx^2 + dy^2 + e^{2x}dz^2$               |
| $SL(2, \mathbb{R})$ | $Isom(H^2) \times R$ | $\cosh^2 y dx^2 + dy^2 + (dz + \sinh ydx)^2$ |
| Nil        | $Isom(E^2) \times R$ | $dx^2 + dy^2 + (dz - xdy)^2$               |
| Sol        | $Sol \times (\mathbb{Z}_2)^2$ | $dx^2 + e^{-2x}dy^2 + e^{2x}dz^2$         |

In the table, the seventh and eighth geometries are the most obscure. The 3-manifold $Nil$ is a twisted product of $E^1$ with $E^2$. Alternatively, $Nil$ is the manifold of the Heisenberg group, i.e., the group of matrices:

\[
\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}.
\]

The 3-manifold $Sol$ is the solvable Lie group. It is the only geometry whose isometry group is three dimensional-its isotropy subgroup is trivial. The metrics in Table 1. were obtained by Fagundes, and are the standard metrics for the three dimensional space sections of the Bianchi/ Kantowski-Sachs models associated with the given geometries.

The further complication is the fact, contrary to the 2-D case, that direct sums of two or more of these geometries may not admit one of the eight geometries. Thurston has conjectured that any closed orientable 3-manifold can be decomposed into components:

\[ M = M_1 \# M_2 \# \cdots, \]
such that each of the \( M_1, M_2, \) etc. admit one and only one of the eight geometries. This decomposition consists of cutting \( M \) along 2-spheres and tori and gluing 3-balls to the resulting boundary spheres on each piece. This conjecture has not been proved to date. However large classes of 3-manifolds have been shown to obey it \([11]\).

### 3 Three Dimensional Gravity

In Chapter 1, we saw that two dimensional Einstein gravity theory, while essentially a trivial theory physically, nevertheless encodes the structure of two dimensional geometry and topology. It would be nice if this were to be the case in three dimensions, but it is not. The reason for this is as follows.

The Einstein-Hilbert action functional in three dimensions is:

\[
S^{(3)}_0 = \int_{M^3} d^3x \sqrt{g} g^{\mu\nu} R_{\mu\nu}.
\]  

(4)

Its stationary points are the flat Riemannian geometries on \( M^3 \). One of the eight 3-manifold geometries, namely \((E^3, R^3 \times SO(3))\), is a stationary point. The other two maximally symmetric geometries, namely \((S^3, SO(4))\) and \((H^3, PSL(2, C))\) are the stationary points of action functionals obtained from \( S^{(3)}_0 \) above by adding “cosmological constant terms” of the form \( \Lambda \sqrt{g} \) to the integrand. There does not appear to be any simple prescriptions for constructing action functionals whose stationary points are, respectively, the remaining five anisotropic geometries. This is in contrast to the two dimensional case where essentially all Riemannian geometries are stationary points of some simple action functional.

The action functional for three dimensional gravity originates from a topological invariant- the Chern-Simons invariant- as was shown by Achucarro and
Townsend \cite{13} and rediscovered later by E. Witten \cite{14}. However, the Chern-Simons invariant does not encode the same topological information as does the Einstein-Hilbert action functional in two dimensions. In fact the latter, as was mentioned above, is the Euler number of the manifold; in the case of closed compact three dimensional manifolds, the Euler number is zero.\footnote{In two dimensions, there is a functional analogous to the Chern-Simons form in three dimensions \cite{15} \cite{16} \cite{17} \cite{18}. It is the action functional for a two-dimensional topological field theory of the so-called BF type.}

I will argue here that the three dimensional theory of gravity interacting with topological matter, developed in collaboration with S. Carlip \cite{12} in 1991, encodes the geometry and topology of 3-manifolds in manner strongly analogous to the situation of Einstein gravity and 2-manifolds. We suppose that $M^3$ is a smooth orientable closed compact 3-manifold. The cotangent bundle to $M^3$, $T^*M^3$, is a fiber bundle over $M^3$ with structure group SO(3). The fibers are three dimensional vector spaces which come equipped with a “natural” metric $\delta_{ab}$ and volume element $\epsilon^{abc}$. A smooth frame field on $M^3$ is a set of three independent 1-form fields $E_a$ on $M^3$. A spin connection $A_a$ on $M^3$ is an SO(3) connection on $M^3$. The cotangent bundle $T^*M^3$ has fibers isomorphic to the Lie algebra $\mathcal{L}_{SO(3)}$ of SO(3). A spin connection $A_a$ is compatible with a frame field $E^a$ if:

$$D_{(A)} E^a := dE^a + \frac{1}{2} \epsilon^{ab}_{\ c} A_b E^c = 0.$$  \hfill (5)

In the above, $D_{(A)}$ is the covariant exterior derivative with respect to the connection $A_a$. The $\mathcal{L}_{SO(3)}$ indices $a, b, ...$ are raised and lowered by the metric $\delta_{ab}$. A spin connection and compatible frame field determine a Riemannian metric $g := \delta_{ab} E^a \otimes E^b$ on $T M^3$.

Let $E^a, B^a, C_a$ be three sets of 1-form fields over $M^3$. The fields need not be non-degenerate nor mutually linearly independent, nor compatible with
connection $A^a$. The action is a functional of the spin connection $A_a$ and the 1-form fields $B^a, C_a, E^a$:

$$S^{(3)} = \frac{1}{2} \int_{M^3} E^a \wedge F_a(A) + B^a \wedge D_{(A)} C_a.$$  \hfill (6)

The curvature $F_a(A)$ of the connection $A_a$ is

$$F_a(A) := dA_a + \frac{1}{4} \epsilon_a^{bc} A_b A_c.$$

The stationary points of $S^{(3)}$ are given by the solutions of the following set of first order partial differential equations:

$$F_a(A) = 0,$$  \hfill (7)
$$D_{(A)} B^a = 0,$$  \hfill (8)
$$D_{(A)} C_a = 0,$$  \hfill (9)
$$D_{(A)} E^a + \frac{1}{2} \epsilon_a^{abc} B_b \wedge C_c = 0.$$  \hfill (10)

In general, the three 1-form fields $E^a$ are not a frame field compatible with the spin connection $A_a$. This is because of the term in $B_b \wedge C_c$ in the last equation of motion.

Nevertheless, the equations of motion above determine a Riemannian geometry on $TM^3$ as follows. Consider the 1-form field $Q_a$ satisfying

$$\epsilon_a^{abc} (Q_b \wedge E_c - B_b \wedge C_c) = 0.$$  \hfill (11)

Then the equation of motion for the $E^a$ can be written as:

$$dE^a + \frac{1}{2} \epsilon_a^{abc} (A_b + Q_b) \wedge E_c = 0.$$  \hfill (12)

This is of the form of the condition that the frame field $E^a$ is compatible with the connection $\omega_a$ defined by:

$$\omega_a := A_a + Q_a.$$  \hfill (13)
The following Theorem is straightforward to prove:

**Theorem:** Each of the eight geometries of Table 1. is a stationary point of the action functional $S^{(3)}$.

**Proof:** We prove the theorem by constructing solutions of Eqns.(7) on a coordinate patch $U$ of $M^3$. Since $A_n$ is flat, we can choose a gauge so that $A_n = 0$ on $U$. In Table 2. below I display for each of the eight 3-manifold geometries the corresponding frame field $E^a$ (obtained by simply factoring the fiducial metric given in Table 1.) and closed 1-form fields $B^a, C_n$ such that the last of the equations of motion in Eqns(7) is satisfied. The compatible spin connection for each geometry is precisely that given by Eqns(8)-(10).
The action functional $S^{(3)}$ is invariant under the group whose infinitesimal generators are given by the following \[12\]:

$$\delta B^a = D_{(A)} \rho^a + \frac{1}{2} \epsilon^{abc} B_b \tau_c,$$  
$$\delta C^a = D_{(A)} \lambda^a + \frac{1}{2} \epsilon^{abc} C_b \tau_c,$$  
$$\delta E^a = D_{(A)} \xi^a + \frac{1}{2} \epsilon^{abc} (E_b \tau_c + B_b \lambda_c + C_b \rho_c),$$  
$$\delta A^a = D_{(A)} \tau^a,$$

where the twelve quantities $\tau^a, \lambda^a, \rho^a, \xi^a$ are infinitesimal parameters generating the transformations. The group generated by these infinitesimal transformations is $I(\text{ISO}(3))$. The notation $IG$ denotes the group obtained by taking the semi-direct product of the Lie Group $G$ with its own Lie algebra $L_G$.

I now conjecture that given the topology of a prime manifold $M^3$, most Riemannian metrics on $M^3$ can be parameterized by the gauge parameters of a gauge transformation of the form of Eqns.(16). We first note the following \[12\]: (i.) The action Eq. (6) is invariant up to a total divergence under gauge transformations with finite values of the gauge parameters $\xi^a, \rho^a, \lambda^a$ as long as $\tau^a = 0$; (ii.) For any topology, the equations of motion admit the trivial configuration $A^a = B^a = C^a = E^a = 0$.

We can obtain the following configuration, which is a solution of the equations of motion, by performing the following finite gauge transformations in succession on the trivial configuration \[12\]:

$$\delta_1 (B^a, C^a, E^a) = (d\rho^a, 0, d\xi^a),$$  
$$\delta_2 (B^a, C^a, E^a) = \left(0, d\lambda^a, \frac{1}{2} \epsilon^{abc} \delta_1 B_b \lambda_c\right),$$

(18)

to get

$$E^a = d\xi^a + \frac{1}{2} \epsilon^{abc} d\rho_b \lambda_c,$$

(19)
\[ B^a = d\rho^a, \quad (20) \]
\[ C^a = d\lambda^a. \quad (21) \]

By the appropriate choice of gauge parameters, we can get the fiducial \( E^a \) in Table 2. In a sense to be enlarged on below, the triad \( E^a \) defined by Eq.(19) is general. Consider the choice of parameters \( \xi^A = 0, \rho^1 = 0, \rho^A = x^A \) with \( A = 2, 3 \). Then the metric \( g_{\mu\nu} = \delta_{ab}E^a_\mu E^b_\nu \) can be written in ”ADM” form:

\[ ds^2 = N^2(dx^1)^2 + h_{AB}(dx^A + N^A dx^1)(dx^B + N^B dx^1), \quad (22) \]

where

\[ N^2 = \frac{\lambda_3}{2} \partial_1 \xi^1, \quad N^3 = -\frac{\lambda_2}{2} \partial_1 \xi^1, \quad N = \frac{\lambda_1}{\lambda} \partial_1 \xi^1, \]

with

\[ \tilde{\lambda}_A := \lambda_A + 2\epsilon_{AB} \partial_B \xi^1, \]
\[ \tilde{\lambda} := \sqrt{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}, \]

and

\[ h_{AB} dx^A dx^B = e^{2\phi}|d\omega + \mu d\bar{\omega}|^2, \quad (23) \]

with

\[ e^{\phi} := \frac{1}{4}(\lambda_1 + \tilde{\lambda}), \]
\[ \mu := \frac{e^{-2\phi}(\tilde{\lambda}_2 + i\tilde{\lambda}_3)^2}{16}. \quad (24) \]

The \( E^3, H^3, S^2 \times E^1, H^2 \times E^1 \) and \( Sol \) geometries can be so parameaterized. The others likely can, though not with the particular metric triad given in Table 1. and 2. However, at least \( S^3 \) with metric/triad as in these tables can be parameterized by Eq.(19) but now with the gauge parameters chosen as:

\[ (\rho^a) = (x, \ln|\csc y - \cot y|, x \cot y), \]

13
\[(\lambda^a) = (0, -2\sin y, 2x),\]
\[(\xi^a) = (0, y + \frac{x^2}{2}, z).\]

In the case of \(S^2 \times E^1\), with \(x^2, x^3\), i.e., \(\omega, \overline{\omega}\), coordinates on \(S^2\)-as argued in [12]- all metrics on \(S^2 \times E^1\) are gauge equivalent to the one which is gauged from the trivial configuration. Hence all metrics on \(S^2 \times E^1\) are gauge equivalent to the fiducial metric given in Table 1. This is because all “Beltrami differentials” \(\mu\) on \(S^2\) are equivalent up to diffeomorphisms [13].

For the other topologies the situation is rather subtle. Consider the case of \(E^3\)-i.e., \(T^3\) since we are considering closed compact topologies. In this case \(\mu\) fixes a point in Teichmüller space, up to diffeomorphisms. Hence a given geometry \(S^1 \times T^2[\mu]\), with the subscript \([\mu]\) denoting the Teichmüller parameters of the torus, is gauge equivalent to the trivial configuration with the gauge parameters constrained by Eq.(24). All such geometries are then gauge equivalent to the fiducial \(T^3\) geometry with appropriate Teichmüller parameters. A more comprehensive analysis is currently underway.

Furthermore, I conjecture that each of the 3-manifold geometries is characterized by a flat \(SO(3)\) connection \(A_a\) modulo gauge equivalence under the group \(I(ISO(3))\), and two \(\mathcal{L}_{SO(3)}\)-valued 1-form fields \(B^a, C_a\) closed with respect to the flat connection \(A_a\).

Finally, it is worth noting here that this structure somewhat resembles Thurston’s characterization of geometric structures in terms of a flat bundle equipped with transverse foliation and canonical section [7].

The proof of these conjectures is currently under investigation.

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References

[1] W. Thurston, *The Geometry and Topology of Three Manifolds*, Princeton Un. Lecture Notes, 1978.

[2] R. Hamilton, J. Diff. Geom. 17, 255 (1982); J. Isenberg and M. Jackson, J. Diff. Geom. 35, 723 (1992).

[3] Y. Fujiwara et. al. Phys. Rev. D 44, 1756 (1991); 1763 (1991).

[4] S. Carlip, ”The Sum Over Topologies in Three-Dimensional Euclidean Quantum Gravity”, UCD preprint, UCD-92-16 (1992); ”Entropy vs. Action in the (2+1)-Dimensional Hartle-Hawking Wave Function”, UCD preprint, UCD-92-8 (1992).

[5] H. V. Fagundes, Phys. Rev. Letts. 54, 1200 (1985); Gen. Rel. and Grav. 24, 199 (1992).

[6] A. Ashtekar and J. Samuel, Class. Quant. Grav. 8, 2191 (1991); Y. Fujiwara, H. Ishihara, H. Kodama, ”Comments on Closed Bianchi Models”, Kyoto Univ. preprint, KUCP-55 (1993).

[7] W.M. Goldman, ”Geometric structures on manifolds and varieties of representations”, in *Geometry of Group Representations*, ed. by W.M Goldman and A.R. Magid, AMS (Contemporary Mathematics, Vol. 74), 1988.
[8] See for example, M. H. Freedman and F. Luo, *Selected Applications of Geometry to Low-Dimensional Topology*, Am. Math. Soc. (University Lecture Series, V.1) 1989.

[9] W. Thurston, L’Enseignment Mathematique **29**, 15 (1983).

[10] W.P Thurston, Bull. Am. Math. Soc. **6**, 357 (1982).

[11] P. Scott, Bull. London Math. Soc. **15**, 401 (1983).

[12] S. Carlip and J. Gegenberg, Phys. Rev. D**44**, 424 (1991).

[13] A. Achucarro and P.K. Townsend, Phys. Lett. B **180**, 89 (1986).

[14] E. Witten, Nuc. Phys. B**311**, 46 (1988).

[15] R. Jackiw in *Quantum Theory of Gravity*, ed. S. Christensen, Adam Hilger Press, 1984.

[16] C. Teitelboim in *Quantum Theory of Gravity*, ed. S. Christensen, Adam Hilger Press, 1984.

[17] A. H. Chamseddine and D. Wyler, Phys. Lett. B**228**, 75 (1989).

[18] K. Isler and C. Trugenberger, Phys. Rev. Lett. **63**, 834 (1989).

[19] L. Baulieu and M. Bellon, Phys. Lett. B**196**, 142 (1987).