Bounds on the Hermite Spectral Projection Operator

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Abstract. We study $L^p$–$L^q$ bounds on the spectral projection operator $\Pi_\lambda$ associated to the Hermite operator $H = |x|^2 - \Delta$ in $\mathbb{R}^d$. We are mainly concerned with a localized operator $\chi_E \Pi_\lambda \chi_E$ for a subset $E \subseteq \mathbb{R}^d$ and undertake the task of characterizing the sharp $L^p$–$L^q$ bounds. We obtain sharp bounds in extended ranges of $p, q$. First, we provide a complete characterization of the sharp $L^p$–$L^q$ bounds when $E$ is away from $\sqrt{\lambda} S^d - 1$. Secondly, we obtain the sharp bounds as the set $E$ gets close to $\sqrt{\lambda} S^d - 1$. Thirdly, we extend the range of $p, q$ for which the operator $\Pi_\lambda$ is uniformly bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$.

1. Introduction

Let $H$ denote the Hermite operator $-\Delta + |x|^2$ in $\mathbb{R}^d$, $d \geq 2$. The operator $H$ has a discrete spectrum $\lambda \in 2\mathbb{N}_0 + d$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\alpha \in \mathbb{N}_0^d$, let $\Phi_\alpha$ be the $L^2$–normalized Hermite function which is an eigenfunction of $H$ with eigenvalue $2(\alpha_1 + \cdots + \alpha_d) + d$. The set $\{\Phi_\alpha : \alpha \in \mathbb{N}_0^d\}$ forms an orthonormal basis in $L^2$. We consider the spectral projection operator

$$\Pi_\lambda f = \sum_{\alpha : d + 2|\alpha| = \lambda} \langle f, \Phi_\alpha \rangle \Phi_\alpha,$$

which is the orthogonal projection to the vector space spanned by eigenfunctions with the eigenvalue $\lambda$. Then, $f = \sum_{\lambda \in 2\mathbb{N}_0 + d} \Pi_\lambda f$ for $f \in L^2$.

$L^p$–$L^q$ bounds on the spectral projection operators associated to differential operators have been studied by various authors (see, for example, [33, 34, 37, 24, 28, 20]). Let $\|T\|_{p \to q}$ denote the operator norm of an operator $T$ from $L^p$ to $L^q$. Concerning the Hermite operator, the bounds

$$\|\chi_E \Pi_\lambda \chi_E\|_{p \to q} \leq B(\lambda, p, q)$$

with suitable subsets $E \subseteq \mathbb{R}^d$ has been of interest and studied by some authors. The estimates are related to Bochner-Riesz summability of the Hermite expansion [21, 41] and the unique continuation properties for the parabolic operators [10, 12]. When $E = \mathbb{R}^d$, we call (1.1) a global estimate. In such a case, the bound (1.1) with $p = 2$ and $2 \leq q \leq \infty$ was studied by Thangavelu [40], Karadzhov [21], and Koch–Tataru [25]. Especially, Koch and Tataru obtained the optimal $L^2$–$L^q$ bound for $2 \leq q \leq \infty$ except $q = 2(d + 3)/(d + 1)$. Recently, the missing endpoint estimate was proved by the authors [19] for $d \geq 3$.

In this paper, we are mainly concerned with local estimates for the projection $\Pi_\lambda$, i.e., the estimate (1.1) with bounded sets $E$. As shown in the earlier works

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by Theorem 1.2 and Proposition 5.1 below we have (e.g., see [36]). Remarkably, the estimates in Theorem 1.2 are sharp. More precisely, let \( B \) provide a complete characterization of \( L^1 \). This demonstrates validity of the aforementioned heuristics, and consequently provides a complete characterization of \( L^p \) bounds on \( \chi_B \Pi_1 \chi_B \). To state our result, we need some notations. For \( X = (a, b) \in \mathbb{R}^d \) with \( 1 < p < \infty \), we define \( Y \) by \( Y \) = \( \{ X : X \in \mathbb{R}^d \} \) for a set \( Y \subset \mathbb{R}^d \). If \( X \in Y \) and \( X \neq Y \), \( [X, Y] \) and \( (X, Y) \) denote the closed and open line segments connecting \( X \) and \( Y \), respectively. Similarly, the half open line segments \( (X, Y) \) are defined. Finally, if \( X_1, \ldots, X_k \in \mathbb{R}^d \), by \( [X_1, \ldots, X_k] \) we denote the convex hull of \( X_1, \ldots, X_k \).

**Definition 1.1.** Let \( \mathfrak{A} = \mathfrak{A}(d), \mathfrak{C} = \mathfrak{C}(d) \), and \( \mathfrak{D} = \mathfrak{D}(d) \). Let \( \beta(p, q) \) by setting\(^\text{3}\)

\[
\beta(p, q) = \begin{cases} 
-\frac{d}{2} \delta(p, q), & \left( \frac{1}{p}, \frac{1}{q} \right) \in \mathcal{R}_1, \\
\frac{d}{2} \left( \frac{1}{p} + \frac{1}{q} \right) - \frac{d+1}{2}, & \left( \frac{1}{p}, \frac{1}{q} \right) \in \mathcal{R}_2, \\
\frac{d}{2} - \frac{d}{2} \left( \frac{1}{p} + \frac{1}{q} \right), & \left( \frac{1}{p}, \frac{1}{q} \right) \in \mathcal{R}_3, \\
\frac{d}{2} \delta(p, q) - 1, & \left( \frac{1}{p}, \frac{1}{q} \right) \in \mathcal{R}_3 \cup [\mathfrak{C}, \mathfrak{D}] \cup [\mathfrak{C}', \mathfrak{D}']. 
\end{cases}
\]

Here \( \delta(p, q) = 1/p - 1/q \). For a given set \( E \subset \mathbb{R}^d \) we denote \( E_\lambda = \{ \sqrt{x} : x \in E \} \).

**Theorem 1.2.** Let \( d \geq 2 \) and \( (1/p, 1/q) \in \mathbb{D} \). Then, we have

\[
\| \chi_{\mathfrak{A}} \Pi_1 \chi_{\mathfrak{B}} \|_{L^p \to L^q} \lesssim \lambda^{\beta(p, q)}
\]

if and only if \((1/p, 1/q) \notin [\mathfrak{C}, \mathfrak{D}] \cup [\mathfrak{C}', \mathfrak{D}'] \). Moreover, we have

(i) \( \| \chi_{\mathfrak{A}} \Pi_1 \chi_{\mathfrak{B}} \|_{L^p \to L^{q, s}} \lesssim \lambda^{\beta(p, q)} \) if \((1/p, 1/q) \notin [\mathfrak{C}, \mathfrak{D}] \cup [\mathfrak{C}', \mathfrak{D}'] \);

(ii) \( \| \chi_{\mathfrak{A}} \Pi_1 \chi_{\mathfrak{B}} \|_{L^{p, 1} \to L^{q, 1}} \lesssim \lambda^{\beta(p, q)} \) if \((1/p, 1/q) = \mathfrak{C} \) or \( \mathfrak{C}' \).

Here \( \| \cdot \|_{L^p \to L^{q, s}} \) denotes the operator norm from the Lorentz space \( L^{p, r} \) to \( L^{q, s} \) (e.g., see [36]). Remarkably, the estimates in Theorem 1.2 are sharp. More precisely, by Theorem 1.2 and Proposition 5.1 below we have

\[
\| \chi_{\mathfrak{A}} \Pi_1 \chi_{\mathfrak{B}} \|_{p \to q} \sim \lambda^{\beta(p, q)}
\]

\(^1\)Note \( \beta(p, q) = \max \left( -\frac{1}{2} \delta(p, q), -1 + \frac{d}{2} \delta(p, q), -\frac{d+1}{2}, \frac{d}{2} \left( \frac{1}{p} + \frac{1}{q} \right), \frac{d-1}{2} - \frac{d}{2} \left( \frac{1}{p} + \frac{1}{q} \right) \right) \).
for \((1/p, 1/q) \in \square \setminus ([C, D] \cup [C', D'])\). When \(p = 2\) (equivalently, \(q = p'\), or \(q = 2\)) the sharp \(L^p - L^q\) (local) bounds (1.3) were previously obtained ([21, 41, 25]). However, we emphasize that the sharp bounds for other \(p, q\) are not generally accessible by mere interpolation between the previously known bounds due to change of the regimes (see Figure 1). As alluded above, there is a strong resemblance between the local estimate for \(\Pi_\lambda\) (Theorem 1.2) and the global estimate for \(\mathcal{S}_k\) (Corollary 3.3). For some special cases the local estimates in Theorem 1.2 imply those in Corollary 3.3 (see Lemma 3.4).

The implication in Lemma 3.4 remains valid while \(L^q\) is replaced by Lorentz spaces \(L^{q, \infty}\) as long as \(q > 1\). So, it is not possible to strengthen the weak type estimates in Theorem 1.2 by replacing \(L^{q, \infty}\) with the smaller space \(L^{q,r}, r < \infty\), because the same is true for \(f \to (\hat{f} |_{S^d-1})^\vee\) (see Theorem 3.2).

**Estimate near the sphere \(\sqrt{\lambda S^{d-1}}\).** As shown in [39, 23], \(\Pi_\lambda\) exhibits different behaviors when the input functions are supported near (equivalently, \(L^q\) integration is taken over) the set \(\sqrt{\lambda S^{d-1}}\). This naturally leads to considering a localization getting close to the sphere \(\sqrt{\lambda S^{d-1}}\). To do this, for \(\mu \in \{2^k : k \in \mathbb{Z}\}\), set

\[
A_\mu = \{x : (1 - |x|) \in [2^{-1}\mu, \mu]\}, \quad A_{\lambda, \mu} = \{x : \lambda^{-\frac{d}{2}} x \in A_\mu\}.
\]

We also denote

\[
\chi_\mu = \chi_{A_\mu}, \quad \chi_{\lambda, \mu} = \chi_{A_{\lambda, \mu}}.
\]

To obtain the sharp (global) \(L^2 - L^q\) estimate with \(q \geq 2\), Koch and Tataru [25] considered the localized operator \(\chi_{\lambda, \mu} \Pi_\lambda\). They showed

\[
\|\chi_{\lambda, \mu} \Pi_\lambda\|_{2 \to q} \sim \begin{cases} \lambda^{-\frac{d}{2}(2,q)} \mu^{\frac{d}{2} \delta(2,q)}, & 2 \leq q \leq \frac{2(d+1)}{d-1}, \\ (\lambda \mu)^{-\frac{d}{2} + \delta(2,q)}, & \frac{2(d+1)}{d-1} \leq q \leq \infty \end{cases}
\]

for \(\lambda^{-\frac{d}{2}} \leq \mu \leq 1/4\) (see [25, Theorem 3]). In fact, a slightly different form of weighted \(L^2\) estimate was shown but the result is essentially equivalent to (1.4).
Since the Hermite functions decay exponentially outside the ball $B(0, \sqrt{\lambda})$, the contribution from $(1 - \chi_{B(0, \sqrt{\lambda})})\Pi_\lambda$ is less significant. In fact, if $\chi_{\lambda, \mu}$ in (1.4) is replaced by the characteristic function of $A_{\lambda, \mu} := \{ \sqrt{\lambda}\mathbf{x} : |\mathbf{x}| - 1 \in [2^{-1} \lambda, \mu] \}$, similar but stronger estimates can be shown. By duality, the estimate (1.4) is equivalent to

$$\|\chi_{\lambda, \mu} \Pi_\lambda \chi_{\lambda, \mu}\|_{q \rightarrow q} \approx \begin{cases} \lambda^{-\frac{d}{2}}(q', q) \mu^{\frac{d}{2} - \frac{3}{4}d(q', q)}, & 2 \leq q \leq \frac{2(d+1)}{d-1}, \\
(\lambda \mu)^{-1 + \frac{1}{2}d(q', q)}, & \frac{2(d+1)}{d-1} \leq q \leq \infty. \end{cases}$$

(1.5) $\|\chi_{\lambda, \mu} \Pi_\lambda \chi_{\lambda, \mu}\|_{q \rightarrow q} \approx \begin{cases} \lambda^{-\frac{d}{2}}(q', q) \mu^{\frac{d}{2} - \frac{3}{4}d(q', q)}, & 2 \leq q \leq \frac{2(d+1)}{d-1}, \\
(\lambda \mu)^{-1 + \frac{1}{2}d(q', q)}, & \frac{2(d+1)}{d-1} \leq q \leq \infty. \end{cases}$

Our second result extends the estimate (1.5) to $(p, q)$ other than $(q', q)$. We set

$$\gamma(p, q) = \begin{cases} \frac{1}{2} - \frac{d+3}{4p}d(p, q), & \left(\frac{1}{p}, \frac{1}{q}\right) \in R_1, \\
d\left(\frac{1}{2p} + \frac{1}{q}\right) - \frac{3d+1}{4p}, & \left(\frac{1}{p}, \frac{1}{q}\right) \in R_2, \\
\frac{3d+1}{4p} - d\left(\frac{1}{p} + \frac{1}{2q}\right), & \left(\frac{1}{p}, \frac{1}{q}\right) \in R_2, \\
d\frac{d(p, q)}{p} - 1, & \left(\frac{1}{p}, \frac{1}{q}\right) \in R_3 \cup [C, D] \cup [C', D'], \end{cases}$$

for $(1/p, 1/q) \in \mathbb{D}$. We consider the estimate

$$\|\chi_{\lambda, \mu} \Pi_\lambda \chi_{\lambda, \mu}\|_{p \rightarrow q} \leq C \lambda^{\gamma(p, q)} \mu^{\gamma(p, q)},$$

which coincides with (1.5) when $p = q$. It is not difficult to show that the exponent in (1.6) can not be improved to any better one (see Proposition 5.1) up to a constant. It seems to be plausible to expect that the next holds true.

**Conjecture 1.3.** For $(1/p, 1/q) \in \mathbb{D}$, the estimate (1.6) holds.

We partially verify Conjecture 1.3. In order to state our result we need additional notations.

**Definition 1.4.** Let $\mathcal{B} = \left(\frac{2d^2 + 7d - 7}{2(2d - 1)(d + 1)}, \frac{2d - 3}{2(2d - 1)}\right)$. For $d \geq 2$, we set $\mathcal{L}_1 = \{1/2, 1/2, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{M}'\} \setminus \{\mathfrak{G}, \mathfrak{G}'\}$, $\mathcal{L}_2 = \{\mathfrak{A}, (1, 1/2), \mathfrak{D}\} \setminus \{\mathfrak{D}\}$, and $\mathcal{L}_3 = \{(0, 0), \mathfrak{D}, \mathfrak{G}, \mathfrak{G}', \mathfrak{D}'\} \setminus \{\mathfrak{G}, \mathfrak{D}, \mathfrak{G}', \mathfrak{D}'\}$ (Figure 2). When $d = 2$, $\mathfrak{G} = \mathfrak{G}' = (5/6, 1/6)$ (Figure 2). When $d \geq 3$, the line segment $[\mathfrak{A}, (5/6, 1/6)]$ and the line $x - y = 2/(d + 1)$ meet each other at $\mathfrak{G}$. See Figure 2 and 3.
Theorem 1.5. Let \( d \geq 2 \) and \( \lambda^{-\frac{3}{2}} \leq \mu \leq 1/4 \). If \((1/p, 1/q) \in (\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_2' \cup \mathcal{L}_3)\), (1.6) holds. Moreover, we have the following estimates:

\[
\| \chi_{\lambda, \mu} \Pi_{\lambda} \chi_{\lambda, \mu} \|_{L^{2d/(d+1)} \rightarrow L^{\infty}} \lesssim (\lambda \mu)^{1/2 d},
\]

(1.7) \[
\| \chi_{\lambda, \mu} \Pi_{\lambda} \chi_{\lambda, \mu} \|_{L^{p}, 1 \rightarrow L^{q}, \infty} \lesssim (\lambda \mu)^{-1}, \quad (1/p, 1/q) = \mathcal{G}, \mathcal{G}', \quad d \geq 3.
\]

Figure 4. The range of \( p, q \) for which (1.10) holds: \( d = 2 \) (left) and \( d \geq 3 \) (right).

Compared with the earlier results, the range where (1.6) holds is considerably extended. Among others, worth mentioning is the weak type \((1, 2)\) estimate which is equivalent to (1.7) and corresponding to the point \( \mathcal{D} \) in Figure 1. The estimate makes possible to obtain the sharp estimates for \((1/p, 1/q) \in \mathcal{L}_2\). However, the optimal bound remains unknown for \((1/p, 1/q) \in \text{int}(\mathcal{A}, \mathcal{D}, \mathcal{G}) \cup \mathcal{N}, \mathcal{D}', \mathcal{G}')\).

The proof of the estimate (1.6) is more involved. We follow the strategy developed in [19], which makes use of an explicit integral representation for the projection operator \( \Pi_{\lambda} \).

Global uniform estimate. We finally consider the (global) uniform estimates for \( \Pi_{\lambda} \), that is to say, (1.1) with \( E = \mathbb{R}^d \) and \( B \) independent of \( \lambda \). Karadzhov [21] showed

\[
\| \Pi_{\lambda} \|_{2 \rightarrow 2^{d/4}} \leq C
\]

for a constant \( C \). The bound was used to show the sharp \( L^p\)-Bochner-Riesz summability of the Hermite expansion for \( p \geq 2d/(d-2) \) and \( p \leq 2d/(d+2) \). Besides, the estimate (1.9) has applications to the strong unique continuation property for the parabolic operator. We refer the reader to [10, 12, 11, 14, 26, 8, 7] for related developments.

We obtain the uniform estimate on an extended range of \( p, q \). Let

\[
\mathcal{E} = \left( \frac{d+2}{2d}, \frac{1}{2} \right) , \quad \mathcal{F} = \left( \frac{d^2 + 2d - 4}{2d(d-1)}, \frac{d-2}{2(d-1)} \right).
\]

Theorem 1.6. Let \( d \geq 3 \) and \( \mathcal{P} = [\mathcal{E}, \mathcal{E}', \mathcal{F}, \mathcal{F}', (1/2, 1/2)] \setminus \{ \mathcal{F}, \mathcal{F}' \} \). Then,

\[
\| \Pi_{\lambda} \|_{p \rightarrow q} \leq C
\]

holds for a constant \( C \) if \((1/p, 1/q) \in \mathcal{P}\). Moreover, \( \| \Pi_{\lambda} \|_{L^{p}, 1 \rightarrow L^{q}, \infty} \leq C \) holds if \((1/p, 1/q) = \mathcal{F}, \mathcal{F}'\). When \( d = 2 \), (1.10) holds for \((1/p, 1/q) \in \square\).
Related estimates were used to show the strong unique continuation problem for the heat operator [18]. When $d = 2$, the estimate (1.10) is easy to show by duality and the $L^1-L^\infty$ estimate. In higher dimensions $d \geq 3$, uniform boundedness of $\|\Pi_\lambda\|_{p \to q}$ remains open for $(1/p, 1/q) \in \mathcal{P} \setminus \mathcal{P}$, where $\mathcal{P} := \{(a, b) \in \mathbb{D} : a - b \leq 2/d, (d - 1)/d \leq a + b \leq (d + 1)/d\}$. Indeed, (1.10) holds true only if $(1/p, 1/q) \in \mathcal{P}$ as can be seen easily by duality and the lower bounds (5.2) and (5.3) in Section 5.

The current situation seem similar to that of the inhomogeneous Strichartz estimate as can be seen easily by duality and the lower bounds (5.2) and (5.3) in Section 5. The Theorem 1.5 in Section 5.

Theorem 1.5 in Section 5.

**Organization.** In Section 2, we formalize a form of $TT^*$ argument for $\Pi_\lambda$, by which we show the uniform estimates for $\Pi_\lambda$. Section 3 is devoted to proving the local estimates away from $\sqrt{\lambda}S^{d-1}$. In Section 4 we prove Theorem 1.5. Finally, we show lower bounds on $\|\chi_{\lambda, \mu} \Pi_\lambda \chi_{\lambda, \mu}\|_{p \to q}$ in Section 5.

**Notation.** For nonnegative quantities $A$ and $B$, $B \lesssim A$ means that there is a constant $C$, depending only on dimensions such that $B \leq CA$. Likewise, $A \sim B$ if and only if $B \lesssim A$ and $A \lesssim B$. By $B = O(A)$ we mean $|B| \lesssim A$. Additionally, we denote $A \gg B$ if $A \geq CB$ for a large constant $C > 0$.

2. $\Pi_\lambda$ and $TT^*$ Argument

We make use of an observation in [19] Section 2.1. The Hermite-Schrödinger propagator $e^{-itH}$ is given by

$$e^{-itH}f = \sum_{\lambda \in 2\mathbb{N}_0 + d} e^{-it\lambda} \Pi_\lambda f, \quad f \in S(\mathbb{R}^d).$$

Clearly, $e^{it(\lambda - H)}$ is periodic in $t$ with period $\pi$ if $\lambda \in 2\mathbb{N}_0 + d$. If $\lambda$ and $\lambda'$ are eigenvalues of $H$, $\lambda - \lambda' \in 2\mathbb{Z}$, so

$$\frac{1}{2\pi} \int I e^{i\frac{t}{2}(\lambda - \lambda')} dt = \delta(\lambda - \lambda')$$

where $I$ is an interval of length $2\pi$. It follows from (2.1) that

$$\Pi_\lambda f = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} e^{i\frac{t}{2}(\lambda - H)} f dt, \quad \forall f \in S(\mathbb{R}^d).$$

More details can be found in [19] Section 2.1.

2.1. Decomposition of $\Pi_\lambda$. Let $\eta_\circ$ be an even function in $C^\infty_c((-\pi/2 - 2^{-7}, \pi/2 + 2^{-7}))$ such that $\sum_{j \in \mathbb{Z}} \eta_\circ(t - j\pi) = 1$ for any $t \in \mathbb{R}$. Then, it follows that $\eta_\circ(t) := \eta_\circ(t + \pi) + \eta_\circ(t) + \eta_\circ(t - \pi) + \eta_\circ(t - 2\pi) = 1$ on $[-\pi/2, 3\pi/2]$. So, we can write

$$\Pi_\lambda f = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \eta_\circ(t) e^{it(\lambda - H)} f dt.$$ 

Changing variables we see

$$\Pi_\lambda f = \frac{1}{2\pi} \int_\mathbb{R} (\eta_\circ(t) + \eta_\circ(t - \pi)) e^{i\frac{t}{2}(\lambda - H)} f dt.$$

The operator $e^{-itH}$ also has an explicit kernel representation based on Mehler’s formula (e.g., see [32] and [39, p.11]). Combining the formula and (2.2), we obtain an integral representation of $\Pi_\lambda$.

**Lemma 2.1** ([19] Lemma 2.1). Let $a(t) = (2\pi i \sin t)^{-\frac{d}{4}} e^{i\pi d/4} (\eta_\circ(t) + \eta_\circ(t - \pi))$. For $\lambda \in 2\mathbb{N}_0 + d$, set

$$\phi_\lambda(x, y, t) = \frac{\lambda t}{2} + \frac{|x|^2 + |y|^2}{2} \cot t - \langle x, y \rangle \csc t.$$
Then, for all $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$
\Pi_\lambda f = \frac{1}{2\pi} \int a(t) \int e^{i\phi_\lambda(x,y,t)} f(y) \, dy \, dt.
$$

(2.3)

The function $a(t)$ has the singularities at $t = 0$ and $t = \pi$. So, we make decomposition away from them. Let $\psi \in C^\infty_c([2^{-1}, 2])$ be a nonnegative function such that $\sum_{j \geq 0} \psi(2^j t) = 1$ for $t \in (0, \pi/2 + 2^{-1})$. We set

$$
\psi_j(t) = \psi(2^j t) \eta_j(t),
$$

and

$$
\psi_j^-(t) = \psi_j(-t), \quad \psi_j^+(t) = \psi_j(\pm(t - \pi)).
$$

For a bounded function $\eta$ and $\lambda \in 2\mathbb{N}_0 + d$, we consider the operator

$$
\Pi_\lambda[\eta] = \frac{1}{2\pi} \int \eta(t) e^{i\phi_\lambda(x,y,t)} \, dt.
$$

(2.4)

Clearly, the definition makes sense for any real number $\lambda$. By (2.4) and the isometry $\|e^{-itH} f\|_2 = \|f\|_2$ it follows that

$$
\|\Pi_\lambda[\eta]\|_{2 \rightarrow 2} \leq 2\|\eta\|_1.
$$

(2.5)

Since $\sum_j (\psi_j + \psi_j^-) = \eta$, using (2.2), we now have

$$
\Pi_\lambda = \sum_{j \geq 0} \Pi_\lambda[\psi_j] + \sum_{\kappa = -, \pm} \sum_{j \geq 0} \Pi_\lambda[\psi_j^\kappa].
$$

(2.6)

The decomposition is clearly valid since the right hand side converges to $\Pi_\lambda$ as a bounded operator on $L^2$ since (2.5) gives $\|\Pi_\lambda[\psi_j]\|_{2 \rightarrow 2} \lesssim 2^{-j}$ and $\|\Pi_\lambda[\psi_j^\kappa]\|_{2 \rightarrow 2} \lesssim 2^{-j}, \kappa = -, \pm$. We recall a symmetric property observed in [19].

[Continued with the rest of the text.]
holds whenever \( \tilde{\eta} \in \mathcal{C}' \) and \( \text{supp} \tilde{\eta} \subset (2^{-1-j}, 2^{1-j}) \). Then, for \( j \geq 0 \) and \((1/p, 1/q) \in \Omega(b, r)\) we have

\[
\| \chi_E \Pi_{\lambda}[\eta] \chi_E \|_{p \to q} \lesssim q^{j(1+\delta(p, q))j}
\]

(2.8)

if \( \eta \in \mathcal{C}' \) and \( \text{supp} \eta \subset (2^{-1-j}, 2^{1-j}) \).

**Proof.** By (2.5) we have \( \| \chi_E \Pi_{\lambda}[\eta] \chi_E \|_{2 \to 2} \lesssim 2^{-j} \). The estimate (2.7) and interpolation give

\[
\| \chi_E \Pi_{\lambda}[\eta] \chi_E \|_{p \to p'} \lesssim q^{j(1+\delta(p, p'))2^{j(1+\delta(p, p'))}(\frac{2}{p} - 1)}
\]

(2.9)

whenever \( \tilde{\eta} \in \mathcal{C}' \) and \( \text{supp} \tilde{\eta} \subset (2^{-1-j}, 2^{1-j}) \). Thus, it is sufficient to show (2.8) with \( q = 2 \) and \( p = r_b \) because the other estimates follow by duality and interpolation. We claim that the estimate

\[
\| \Pi_{\lambda}[\eta] \chi_E f \|_2 \lesssim 2^{-\frac{j}{2}} q^{\frac{1}{2} - \frac{j}{2}} \| f \|_{r_b}
\]

(2.10)

holds. The inequality clearly implies (2.8) with \( q = 2 \) and \( p = r_b \).

By (2.4) we note \( \| \Pi_{\lambda}[\eta] \chi_E f \|_2^2 = \langle \chi_E \int \int \eta(t) \eta(s) e^{i\frac{t}{2r}2^{j-l}(\lambda - H)x} f \chi_E f dt \rangle \). Thus, we decompose

\[
\| \Pi_{\lambda}[\eta] \chi_E f \|_2 = \sum_{k \geq j-2} \langle \chi_E P_k \chi_E f, f \rangle,
\]

(2.11)

where

\[
P_k = \int \int \psi(2^k |t - s|) \eta(t) \eta(s) e^{i\frac{t}{2r}2^{j-l}(\lambda - H)} dt.
\]

After a simple change of variables we observe

\[
\chi_E P_k \chi_E f = \int \eta(s) \chi_E \Pi_{\lambda}[\psi(2^k \cdot |) \eta(\cdot + s)] \chi_E f ds.
\]

(2.12)

Note \( \psi(| \cdot |) = \psi + \psi(-\cdot) \). Since \( k \geq j - 2 \), \( \psi(2^k \cdot) \eta(\cdot + s) \in \mathcal{C}' \). Thus, we have (2.9) with \( \tilde{\eta} = \psi(2^k \cdot)\eta(\cdot + s) \) and \( j = k \). By the aforementioned symmetric property of the kernels (19 p. 5) the same estimate holds for \( \Pi_{\lambda}[\psi(-2^k \cdot)\eta(\cdot + s)] \). Therefore, taking integration in \( s \) and \( \lambda \), we have

\[
\| \chi_E P_k \chi_E f \|_p \lesssim q^{j(1+\delta(p, p'))2^{j(1+\delta(p, p'))}(\frac{2}{p} - 1)}
\]

(2.13)

for \( r < p \leq 2 \). This gives \( \| \langle \chi_E P_k \chi_E f, g \rangle \| \lesssim q^{j(1+\delta(p, p'))2^{j(1+\delta(p, p'))}(\frac{2}{p} - 1)} ||f||_p ||g||_p \)

by Hölder’s inequality. If we combine this and (2.11), summation over \( k \) yields

\[
\| \Pi_{\lambda}[\eta] \chi_E f \|_2 \lesssim q^{j(1+\delta(p, p'))2^{j(1+\delta(p, p'))}(\frac{2}{p} - 1)} ||f||_p
\]

for \( r_b < p \leq 2 \). (Note \( r < r_b \).) Hence, we have (2.8) when \( q = 2 \) and \( r_b < p \leq 2 \). Duality gives (2.2) for \( p = 2 \) and \( 2 \leq q < r_b' \). Thus, interpolation between those estimates and (2.9) gives (2.8) if \((1/p, 1/q) \) is contained in \( \Omega(b, r) \) but not on the line segments \(([1/2, 1/r_b'), (1/r, 1/r')], [(1/r_b, 1/2), (1/r, 1/r')]) \).

However, using the estimates above, we can obtain (2.10). Indeed, using (2.12) and (2.8), which now holds for \((1/p, 1/q) \) contained in the interior of \( \Omega(b, r) \), by Hölder’s inequality we have

\[
\| \langle \chi_E P_k \chi_E f, g \rangle \| \lesssim q^{j(1+\delta(p, p'))2^{j(1+\delta(p, p'))}(\frac{2}{p} - 1)} ||f||_p ||g||_{q'}
\]

(2.13)

for \((1/p, 1/q) \in \text{int} \Omega(b, r) \). This allows us to apply the bilinear interpolation argument (e.g., Keel and Tao [22]). Therefore, we obtain

\[
\sum_{k \geq j-2} \| \langle \chi_E P_k \chi_E f, g \rangle \| \lesssim q^{j(1+\delta(p, p'))2^{j(1+\delta(p, p'))}(\frac{2}{p} - 1)} ||f||_p ||g||_{q'}
\]

for \((1/p, 1/q) \in \Omega(b, r) \). This completes the proof.
provided that \((1/p, 1/q)\) is \(\in \Omega(b, r)\) and \((b + 1)(1/p - 1/q) = 1\). In particular, taking \(q' = p\), by (2.11) we obtain (2.10).

The following lemma is useful for obtaining some endpoint estimates.

**Lemma 2.4.** Let \(1 \leq p_0, p_1, q_0, q_1 \leq \infty\) and \(\epsilon_0, \epsilon_1 > 0\). Let \(T_j, j \in \mathbb{Z}\), be sublinear operators satisfying \(\|T_j\|_{p_0 \to q_0} \leq B_k2^{(1-k)\epsilon_k}\) for \(k = 0, 1\). Let \(\theta = \epsilon_0/(\epsilon_0 + \epsilon_1)\), \(1/p_* = \theta/p_1 + (1-\theta)/p_0\), and \(1/q_* = \theta/q_1 + (1-\theta)/q_0\). Then, the following hold:

(a) If \(p_0 = p_1 = p\) and \(q_0 \neq q_1\), then \(\|\sum_j T_jf\|_{q_*, \infty} \lesssim B_0^{1-\theta}B_1^\theta\|f\|_{p_*}\).

(b) If \(q_0 = q_1 = q\) and \(p_0 \neq p_1\), then \(\|\sum_j T_jf\|_q \lesssim B_0^{1-\theta}B_1^\theta\|f\|_{p_*}^\delta\|f\|_{p_*, 1}\).

(c) If \(p_0 \neq p_1\) and \(q_0 \neq q_1\), then \(\|\sum_j T_jf\|_{q_*, \infty} \lesssim B_0^{1-\theta}B_1^\theta\|f\|_{p_*, 1}\).

The third assertion (c) is known as 'Bourgain’s summation trick' (see [5] Section 6.2 for a formulation in abstract setting). The first (a) and the second (b) give better estimates than the restricted weak type estimate. As far as the authors are aware, this observation first appeared in [2] (see also [31, Lemma 2.3]).

**Remark 1.** Thanks to Lemma 2.4 and the estimate (2.13), which is equivalent to \(\|\chi_EF \chi_E\|_{p \to q} \lesssim \beta^{\delta(p, q)}2^{-j/2}2^{(b+1)(\delta(p, q))}\), an elementary approach to the estimate (2.10) is possible. By (c) in Lemma 2.4 we have \(\|\sum_k \chi_EF \chi_E\|_{q, \infty} \lesssim \beta^{\delta(p, q)}2^{-j/2}\|f\|_{p, 1}\), for \((1/p, 1/q)\) \(\in \Omega(b, r)\) satisfying \((b+1)\delta(p, q) = 1\). Interpolation gives, in particular, (2.8) with \(p = r_b\) and \(q = r_b',\) which is equivalent to (2.10).

By using Lemma 2.3, we can prove Theorem 1.6.

**Proof of Theorem 1.6.** To show (1.10) for \((1/p, 1/q) \in \mathcal{P}\), by interpolation it suffices to show the restricted weak type \((p,q)\) estimate for \((1/p, 1/q) = \mathfrak{F}, \mathfrak{F}'\). By duality we need only to show

\[
\|\Pi_{\lambda} f\|_{q_*, \infty} \leq C\|f\|_{p_*, 1},
\]

where \((1/p_0, 1/q_0) = \mathfrak{F}'\). Indeed, once we have (2.14), duality and interpolation give (1.10) for \((1/p, 1/q) \in (\mathfrak{F}, \mathfrak{F}')\). Note \(\Pi_{\lambda}^2 \Pi_{\lambda} = \Pi_{\lambda}\) and \((d+2, d+2) \in (\mathfrak{F}', \mathfrak{F}')\). So, we get (1.9) since \(\Pi_{\lambda}^2 = \Pi_{\lambda}\) and \(\Pi_{\lambda} \Pi_{\lambda} = C\Pi_{\lambda}\Pi_{\lambda}\). Besides, duality gives \(\|\Pi_{\lambda}\|_{2 \to 2} \leq C\). Interpolation between those estimates and \(\|\Pi_{\lambda}\|_{2 \to 2} \leq 1\) gives (1.10) for \((1/p, 1/q) \in \mathcal{P}\) (see Figure 3).

By Lemma 2.4 it is sufficient for (2.14) to show

\[
\|\sum_{j \geq 0} \Pi_{\lambda} [\psi_j] f\|_{q_*, \infty} \leq C\|f\|_{p_*, 1}.
\]

By Lemma 2.4 we have (2.7) with \(E = \mathbb{R}^d, \beta = 1, r = 1,\) and \(b = (d-2)/2\), provided that \(\tilde{\eta} \in C^J\), supp \(\tilde{\eta} \subset (2^{-1-j}, 2^{1-j})\). Using this and Lemma 2.3 we have

\[
\|\Pi_{\lambda} [\psi_j] \|_{p \to q} \lesssim 2^{j(\frac{d}{2} - \frac{j}{2} - 1)}, \quad j \geq 0,
\]

for \((1/p, 1/q)\) contained in \(\Omega((d-2)/2, 1)\) which is a quadrangle with vertices \(\{(1/2, 1/2), (1/2, (d-2)/(2d)), ((d+2)/(2d), 1/2)\}\). To show (2.15), we note that \(1/p_0 - 1/q_0 = 2/d\) and make use of the summation trick ((c) in Lemma 2.4). Using (2.16), we get restricted weak type \((p,q)\) estimate for \(\sum_{j \geq 0} \Pi_{\lambda} [\psi_j]\) if \((1/p, 1/q) \in \Omega((d-2)/2, 1)\) and \(1/p - 1/q = 2/d\). We only need to observe that \(\Omega((d-2)/2, 1) \cap \{(x,y) \in \Box : x - y = 2/d\} = [\mathfrak{F}', \mathfrak{F}']\). □
2.3. $L^2$--$L^\infty$ estimate. We now consider $L^2$--$L^\infty$ estimate for $\Pi_\lambda[\eta]$, which plays
a significant role in what follows. For a given operator $T$, by $T(x,y)$ we denote the
kernel of $T$. A simple duality argument shows

$$\|T\|_{2\to\infty} = \|T\|_{L^\infty(L^2_x)}.$$  

We also observe that

$$\Pi_\lambda[\eta]f = \frac{1}{2\pi} \sum_\lambda \tilde{\eta}(2^{-1}(\lambda' - \lambda)) \Pi_\lambda f$$

for $\eta \in C_c^\infty$, which follows from (2.1) and (2.4).

**Lemma 2.5.** Let $\lambda^{-\frac{2}{3}} \lesssim \mu \leq 1/4$ and $2^{-\gamma} \gtrsim (\lambda \mu)^{-1}$. If $\eta \in C^1$, then there is a constant $C$, independent of $\lambda$ and $\mu$, such that

$$\|\chi_{\lambda,\mu} \Pi_\lambda[\eta]\|_{1\to\infty} \lesssim 2^{-\frac{4}{9} \frac{\lambda}{\mu}}$$

and

$$\|\Pi_\lambda[\eta]\|_{1\to\infty} \lesssim 2^{-\frac{4}{9} \lambda^{-\frac{2}{3}}}.$$ 

**Proof.** We first show (2.19). The proof follows an argument similar to that of
Lemma 2.9 in [19].

By orthogonality, $\|\Pi_\lambda[\eta(x,.)]\|^2 = \pi^{-2 \lambda} \sum_\lambda \tilde{\eta}(2^{-1}(\lambda') - \lambda)^2 \|\Pi_\lambda (x,.x)\|$. Also note that $|\tilde{\eta}(\tau)| \lesssim 2^{-\gamma}(1 + 2^{-\gamma} |\tau|)^{-N}$ for any $N$. Thus, by (2.18) and (2.17), we have

$$\|\chi_{\lambda,\mu} \Pi_\lambda[\eta]\|^2 \lesssim \sup_{x \in A_{\lambda,\mu}} \sum_\lambda 2^{-2\gamma} (1 + 2^{-\gamma} |\lambda - \lambda'|)^{-N} \|\Pi_\lambda(x,.x)\|.$$ 

Since $\|T\|_{1\to\infty} = \|T\|_{L^\infty_x}$ for an operator $T$, this reduces the proof of (2.19) to showing

$$\sum_\lambda 2^{-2\gamma} (1 + 2^{-\gamma} |\lambda - \lambda'|)^{-N} \|\chi_{\lambda,\mu} \Pi_\lambda \chi_{\lambda,\mu}\|_{1\to\infty} \lesssim 2^{-\frac{4}{9} \frac{\lambda}{\mu}}$$

for $2^{-\gamma} \gtrsim (\lambda \mu)^{-1}$. Using $\Pi_{\lambda'} = \Pi_\lambda$ and duality, we note $\|\chi_{\lambda,\mu} \Pi_{\lambda'} \chi_{\lambda,\mu}\|_{1\to\infty} \leq \|\chi_{\lambda,\mu} \Pi_\lambda\|^2_{1\to\infty}$. Thus, the estimate (2.21) follows from a stronger estimate

$$\sum_\lambda (1 + 2^{-\gamma} |\lambda - \lambda'|)^{-N} \|\chi_{\lambda,\mu} \Pi_\lambda\|^2 \lesssim \|\chi_{\lambda,\mu} \Pi_{\lambda'}\|^2 \lesssim C 2^{-\gamma} (\lambda \mu)^{-\frac{2}{3}},$$

where $\chi_{\lambda,\mu} = \chi_{\lambda,\mu}$ and $A_{\lambda,\mu} = \{x : |x| \geq \lambda^{1/2}(1 - \mu)\}$.

To handle the sum above, we use the estimates

$$\|\chi_{\lambda,\mu} \Pi_\lambda\|_{1\to\infty} \lesssim (\lambda \mu)^{-\frac{2}{3}},$$

$$\|\chi_{\lambda,\mu} \Pi_\lambda\|_{1\to\infty} \lesssim (\lambda \mu)^{-\frac{2}{3}},$$

and

$$\|\chi_{\lambda,\mu} \Pi_\lambda\|_{1\to\infty} \lesssim (\lambda \mu)^{-\frac{2}{3}}.$$ 

We define $\ell(\mu) = \ell(\rho,\lambda,\lambda')$ by setting $\ell(\rho) = (\lambda/\lambda')^{\frac{2}{3}}\rho$ if $\lambda \geq \lambda'$ and $\ell(\rho) = (\lambda' - \lambda)/\lambda + \rho$ otherwise. Since $\lambda^{-\frac{2}{3}} \lesssim \mu$, we have $\ell(\mu) \gtrsim (\lambda')^{-\frac{2}{3}}$. Note that $(\lambda')^{\frac{2}{3}}(1 - \ell(\mu)) \lesssim \lambda^{\frac{2}{3}}(1 - \mu)$. Thus, it follows that $\|\chi_{\lambda,\mu} \Pi_\lambda\|_{1\to\infty} \lesssim \|\chi_{\lambda',\ell(\rho)} \Pi_{\lambda'}\|_{1\to\infty}$. By (2.24), the left hand side of (2.22) is bounded above by

$$C \sum_\lambda (1 + 2^{-\gamma} |\lambda - \lambda'|)^{-N} (\lambda' \ell(\mu))^{-\frac{2}{3}}.$$
To prove \((2.22)\) it is sufficient to show the above sum is bounded by \(C2^j(\lambda\mu)^{(d/2-j)/2}\). Indeed, considering separately the cases \(\lambda \geq \lambda'\) and \(\lambda < \lambda'\), we only have to show
\[
\sum_{\lambda' < \lambda} \left(1 + 2^{-j}(\lambda - \lambda')\right)^{-N}(\lambda'/\lambda)^{d-2} \lesssim 2^j
\]
and
\[
\sum_{\lambda' > \lambda} \left(1 + 2^{-j}(\lambda' - \lambda)\right)^{-N}(\lambda'/\lambda)^{d-2} \left((\lambda' - \lambda)/(\lambda\mu) + 1\right)^{d-2} \lesssim 2^j.
\]
Both follow from a simple computation. Particularly, we use \(C\) to prove \((2.22)\) it is sufficient to show the above sum is bounded by the second inequality.

One can easily show the estimate \((2.20)\) in the same manner using \(\|\Pi_\lambda\|_{2\to\infty} \lesssim \lambda^{d-2}\). So, we omit the detail.

**The marginal case** \(2^j \geq \lambda\mu\). Making use of the previous estimates, we obtain estimates for \(\sum_{2^j \geq \lambda\mu} \chi_{\lambda\mu}\Pi_\lambda[\psi_j]|_{\lambda\mu}\), whose contribution turns out to be less significant.

**Lemma 2.6.** Let \(\mu \in \mathbb{D}^{-1} := \{2^{-k} : -k \in \mathbb{N}\}\) and \(2^{j-1} \leq \lambda\mu < 2^j\). Suppose \(\eta \in C^\infty\). Then, for \((1/p, 1/q) \in \varnothing\), we have
\[
\|\chi_{\lambda\mu}\Pi_\lambda[\eta]\|_{p\to q} \lesssim \lambda^{(p,q)}\mu^{\gamma(p,q)}.
\]

**Proof.** Note \((\lambda\mu)^{-1+4\delta(p,q)} \lesssim \lambda^{(p,q)}\mu^{\gamma(p,q)}\). So, \((2.20)\) follows once we have
\[
\|\chi_{\lambda\mu}\Pi_\lambda[\eta]\|_{p\to q} \lesssim (\lambda\mu)^{-1+4\delta(p,q)}.
\]

In view of interpolation and duality it is sufficient to show \((2.26)\) for \((p, q) = (2, 2), (2, \infty), (1, \infty)\). The estimate \((2.26)\) for \((p, q) = (2, 2)\) is clear from \((2.5)\) since \(\|\eta\|_1 \lesssim 2^{-j}\). Since \(2^{j-1} \sim \lambda\mu\), \((2.26)\) for \((p, q) = (2, \infty)\) follows from \((2.19)\).

Note that \(\tilde{\eta}(\tau) = \mathcal{O}(2^{-j}\infty (1 + 2^{-j}\infty |\tau|)^{-N})\) for any \(N \in \mathbb{N}\). Using \((2.18)\), we obtain
\[
\|\chi_{\lambda\mu}\Pi_\lambda[\eta]\|_{1\to \infty} \lesssim \sum_{\lambda' \sim \lambda\mu} 2^{2j} \left(1 + 2^{-j}\infty |\lambda' - \lambda|\right)^{-N}\|\chi_{\lambda\mu}\Pi_\lambda[\eta]\|_{1\to \infty}.
\]

Since \(2^{j} \sim \lambda\mu\), \((2.26)\) for \((p, q) = (1, \infty)\) follows from \((2.21)\).

### 2.4. Estimates for the kernel of \(\Pi_\lambda\)

In this subsection we are concerned with estimates for the kernel of \(\Pi_\lambda[\psi_j]\). To do so, it is more convenient to consider a rescaled operator. For \(\eta \in C^\infty_\varnothing(\mathbb{R})\), by \(\mathfrak{P}_\lambda[\eta]\) we denote the operator whose kernel is given by
\[
\mathfrak{P}_\lambda[\eta](x, y) = \Pi_\lambda[\eta](\sqrt{x\lambda}, \sqrt{y\lambda}).
\]

As before, by Mehler’s formula (cf. \((2.3)\)) and scaling one can see
\[
\mathfrak{P}_\lambda[\eta](x, y) = \frac{1}{2\pi} \int \eta(t)a(t)e^{t\lambda\phi_1(x,y,t)} dt.
\]

To obtain estimates for \(\mathfrak{P}_\lambda[\psi_j]\) we examine the phase function of the oscillatory integral. A calculation shows
\[
\partial_s\phi_1(x, y, s) = -\frac{Q(x, y, \cos s)}{2\sin^2 s},
\]
where
\[
Q(x, y, \tau) := (\tau - \langle x, y \rangle)^2 - \mathcal{D}(x, y),
\]
\[
\mathcal{D}(x, y) := 1 + (x, y)^2 - |x|^2 - |y|^2.
\]

The stationary point of \(\phi_1(x, y, \cdot)\) is given by the zeros of \(Q(x, y, \cos \cdot)\). \(\mathcal{D}(x, y)\), which is the discriminant of the quadratic equation \(Q(x, y, \tau) = 0\), regulates the
nature of stationary point of the phase function $\phi_1(x, y, \cdot)$. In fact, we can obtain bounds on $\Phi_\lambda[\psi_j](x, y)$ in terms of $|D(x, y)|$, which are to be used later.

**Lemma 2.7.** Let $\mu \in \mathbb{R}^d$ and let $x, y \in \mathbb{R}^d$ satisfy that $1 - C_1 \mu \leq |x|, |y| \leq 1 - c_1 \mu$ for some $0 < c_1 < C_1$. If $1 + (x, y) \geq 10^{-2}$, then for any $N > 0$ there exists a constant $B = B(c_1, C_1, N, d)$ such that the following hold for $j \geq 0$:

(a) If $-D(x, y) \gtrsim \mu^2$, then

$$|\Phi_\lambda[\psi_j](x, y)| \lesssim B \left(2^{\frac{d - 2j}{2}}(\lambda 2^j |D(x, y)| + 1)^{-N}, \ 2^{-j} \lesssim |D(x, y)| \right)^{\frac{1}{4}},$$

(b) If $|D(x, y)| \ll \mu^2$, then

$$|\Phi_\lambda[\psi_j](x, y)| \lesssim B \left(2^{\frac{d - 2j}{2}}(\lambda 2^j |D(x, y)| + 1)^{-N}, \ 2^{-j} \ll \mu^{\frac{1}{4}}\right),$$

(c) If $D(x, y) \sim \mu^2$, then

$$|\Phi_\lambda[\psi_j](x, y)| \lesssim B \left(2^{\frac{d - 2j}{2}}(\lambda 2^j |D(x, y)| + 1)^{-N}, \ 2^{-j} \approx \mu^{\frac{1}{4}}\right).$$

If $1 + (x, y) < 10^{-2}$, then $|\Phi_\lambda[\psi_j](x, y)| \lesssim 2^{\frac{d - 2j}{2}}(\lambda 2^j)^{-N}$ for $j \geq 0$.

**Proof.** We first consider the case $1 + (x, y) \geq 10^{-2}$ and show (a)–(c). The estimates (a) and (b) can be shown in a similar way. We begin with observing that $0 < 1 - (x, y) \sim 1 - (x, y)^2$ since $1 + (x, y) \geq 10^{-2}$. So, we have

$$(2.31) \quad 1 - (x, y) \sim 2 - |x|^2 - |y|^2 - D(x, y) \sim \mu$$

if $-D(x, y) \geq \mu^2$ or $|D(x, y)| \ll \mu^2$.

Note that $1 - \cos s \sim 2^{-2j}$ if $s \in \text{supp} \psi_j$. When $|D(x, y)| \gtrsim \mu^2$, recalling (2.24) and using (2.31) we see $Q(x, y, \cos s) \gtrsim 2^{-4j}$ for $2^{-j} \gg |D(x, y)|^{1/4}$ and $s \in \text{supp} \psi_j$. Thus, from (2.28) we have

$$|\partial_s \phi_1(x, y, s)| \gtrsim \left\{ \begin{array}{ll} 2^{2j} |D(x, y)|, & 2^{-j} \lesssim |D(x, y)|^{1/4} \\ 2^{-2j}, & 2^{-j} \gg |D(x, y)|^{1/4} \end{array} \right.$$

for $s \in \text{supp} \psi_j$ if $-D(x, y) \gtrsim \mu^2$. We also note that $|(d/ds)^n((\sin s)^{-2}))| \lesssim 2^{(2+n)j}$ and $|\partial^n_\mu(\cos s - (x, y))| \lesssim 2^{nj} \max(\mu, 2^{-2j})$ for any $n \in \mathbb{N}_0$. Being combined with (2.28), these bounds yield

$$|\partial^n_\mu \phi_1(x, y, s)| \gtrsim \left\{ \begin{array}{ll} 2^{(1+n)j} |D(x, y)|, & 2^{-j} \lesssim |D(x, y)|^{1/4} \\ 2^{-(n-3)j}, & 2^{-j} \gg |D(x, y)|^{1/4} \end{array} \right., \quad s \in \text{supp} \psi_j$$

if $D(x, y) \lesssim |\mu^2|$. Also, we have $|(d/ds)^n(2^{-dj/2}a_\psi_j)| \lesssim 2^{nj}$, $n \in \mathbb{N}_0$. Thus, integration by parts for the integral $\Phi_\lambda[\psi_j](x, y)$ (recall (2.27)) gives the estimate in (a) (see, for example, [19] Lemma 2.5).

We can show (b) in the same manner as above. Since $1 - \cos s \sim 2^{-2j}$ and $|D(x, y)| \ll \mu^2$, we have $Q(x, y, \cos s) \sim \mu^2$ if $2^{-j} \ll \mu^{1/4}$ and $Q(x, y, \cos s) \sim 2^{-4j}$ if $2^{-j} \gg \mu^{1/4}$. Thus, using (2.28) we have

$$|\partial_s \phi_1(x, y, s)| \gtrsim \left\{ \begin{array}{ll} 2^{2j} \mu^2, & 2^{-j} \ll \mu^{1/4} \\ 2^{-2j}, & 2^{-j} \gg \mu^{1/4} \end{array} \right., \quad s \in \text{supp} \psi_j.$$
Similarly as in the proof of (a), we get
\[
|\partial_s^2 \phi_1(x, y, s)| \lesssim \begin{cases} 
2((1+n)j)\mu^2, & 2^{-j} \ll \mu^{\frac{1}{2}}, \\
2(n-3j), & 2^{-j} \approx \mu^{\frac{1}{2}}, \\
2(n-j), & s \in \text{supp } \psi_j
\end{cases}
\]
for any \( n \in \mathbb{N} \). Combining those and \( |(d/ds)^n(2^{-j/2}a\psi_j)| \lesssim 2^{nj} \), by routine integration by parts we obtain the desired estimate in (b) \([19 \text{ Lemma 2.5}]\).

We now prove (c). From 2.31, we note \( 1 - (x, y) \lesssim \mu \) since \( D(x, y) \sim \mu^2 \). Using this, we have \( Q(x, y, \cos s) \sim 2^{-4j} \) if \( 2^{-j} \gg \mu^{\frac{1}{2}} \). Thus, the estimate for the case \( 2^{-j} \gg \mu^{1/2} \) can be obtained in the same manner as in the proof of (b). Therefore, we only need to show (c) assuming that
\[
2^{-j} \lesssim \mu^{\frac{1}{2}}.
\]
In this case \( \text{supp } \psi_j \) may contain at least one of stationary points of \( \phi_1(x, y, \cdot) \). The equation \( Q(x, y, \cdot) = 0 \) has two roots \( \tau_{\pm} = (x, y) \pm \sqrt{D(x, y)} \). Thus, by the van der Corput lemma we get
\[
\eta_k^* = \psi(2^j |s - s_k|), \quad \kappa = \pm,
\]
and \( k_0 \) be an integer such that \( \mu^{1/2}/(2C) \leq 2^{-k_0} < \mu^{1/2}/C \) for a large constant \( C > 0 \). We also set
\[
\eta^*(s) = 1 - \sum_{k > k_0} \eta_k^+(s) - \sum_{k > k_0} \eta_k^-(s)
\]
so that \( \eta^* + \sum_{k > k_0} \eta_k^+ + \sum_{k > k_0} \eta_k^- = 1 \).

We first estimate the sum \( \sum_{k > k_0} \mathcal{P}_\lambda[\psi_j \eta_k^+](x, y) \). From 2.28, note that
\[
\partial_s \phi_1(x, y, s) = -\frac{(\cos s - \cos s_-)}{2 \sin^2 s} \int_{s_+}^s \sin u du.
\]
Since \( \cos s_+ - \cos s_- \sim \mu \), we have \( |\cos s - \cos s_-| \sim \mu \) on \( \text{supp}(\psi_j \eta_k^+) \). We also note \( \int_{s_+}^s \sin u du \sim 2^{-2-2k} \) for \( s \in \text{supp}(\psi_j \eta_k^+) \). Hence, \( |\partial_s \phi_1(x, y, s)| \gtrsim 2^{-2-2k} \mu \)

\[
|\sum_{k > k_0} \mathcal{P}_\lambda[\psi_j \eta_k^+](x, y)| \lesssim \sum_{k > k_0} 2^{\frac{k}{2}} \min(2^k \lambda^{-1} 2^{-j} \mu, 2^{-k}) \lesssim 2^{\frac{k}{2}} \lambda^{-1} 2^{-j} \mu^{\frac{1}{2}}.
\]
In the same manner, one can show \( |\sum_{k > k_0} \mathcal{P}_\lambda[\psi_j \eta_k^-](x, y)| \lesssim 2^{\frac{k}{2}} \lambda^{-1} 2^{-j} \mu^{\frac{1}{2}} \).

To complete the proof of (c), it remains to show \( |\mathcal{P}_\lambda[\psi_j \eta^*](x, y)| \lesssim 2^{\frac{k}{2}} \lambda^{-1} 2^{-j} \mu^{\frac{1}{2}} \). Note that \( Q(x, y, \cos s) \sim \mu^2 \) for \( s \in \text{supp}(\psi_j \eta^*) \). This gives \( |\partial_s \phi_1(x, y, s)| \gtrsim 2^{\frac{k}{2}} \mu^2 \).

Thus, by the van der Corput lemma we get \( |\mathcal{P}_\lambda[\psi_j \eta^*](x, y)| \lesssim 2^{\frac{k}{2}} \lambda^{-1} 2^{-j} \mu^{\frac{1}{2}} \). Combining this and a trivial bound \( |\mathcal{P}_\lambda[\psi_j \eta^*](x, y)| \lesssim 2^{\frac{k}{2}} \mu^{1/2} \), we obtain the desired estimate since \( 2^{-j} \lesssim \mu^{1/2} \).

To prove the last assertion, we observe that \( |\cos s - (x, y)| \geq 10^{-1} \) for \( s \in \text{supp } \psi_j \) and \( D(x, y) = (1 + (x, y))^2 - |x + y|^2 \leq 10^{-4} \) if \( 1 + (x, y) < 10^{-2} \). Thus,
\( Q(x, y, \cos s) \sim 1 \) for \( s \in \text{supp } \psi_j \). Combining this and (2.23), we have \( |\partial_s \phi_1| \gtrsim 2^j \). Therefore, repeated integration by parts, as before, gives the desired estimate. \( \square \)

**Corollary 2.8.** Let \( E \subset \mathbb{R}^d \). Suppose that \( |D(x, y)| \geq c_0 \) for a constant \( c_0 > 0 \) whenever \( x, y \in E \). Then, there is a constant \( C = C(c_0) \) such that

\[
\| \chi_{E} \Pi_{\lambda}[\psi_j] \chi_{E} \|_{1 \to \infty} \leq C 2^{\frac{d+1}{2}j} \lambda^{-\frac{j}{2}}, \quad j \geq 0.
\]

**Proof.** By rescaling \( (x, y) \to (\sqrt{x}, \sqrt{y}) \), (2.32) is equivalent to

\[
|\mathfrak{P}_{\lambda}[\psi_j](x, y)| \lesssim C 2^{\frac{d+1}{2}j} \lambda^{-\frac{j}{2}}
\]

for \( x, y \in E \) and \( j \geq 0 \). Since \( D(x, y) = (1 - |x|^2)^2 \), we have \(|x| \leq 1 - c \) or \(|x| \geq 1 + c \) for a small constant \( c = c(c_0) > 0 \) if \( x \in E \).

Let \( x, y \in E \). Then, by symmetry we need only to consider the cases \(|x|, |y| \leq 1 - c; |x| \leq 1 - c, 1 + c \leq |y|; \) and \( 1 + c \leq |x|, |y| \). For the first case, by choosing \( C_1 = 2 \) and \( c_1 = 2c \) in Lemma 2.7, we may assume that \( 1 - C_1 \mu \leq |x|, \mu \leq 1 - c_1 \mu \) with \( \mu = 1/2 \). Thus, (2.23) follows from (a) or (c) in Lemma 2.7. When \(|x| \leq 1 - c \) and \( 1 + c \leq |y| \), we have \( D(x, y) \leq (1 - |x|^2)(1 - |y|^2) \leq -c^2 \). By (2.28) and (2.29) this gives \( |\partial_s \phi_1(x, y, s)| \gtrsim 2^j \) for \( s \in \text{supp } \psi_j \). Using van der Corput’s lemma, we get

\[
|\mathfrak{P}_{\lambda}[\psi_j](x, y)| \lesssim 2^{\frac{d}{2}j} \min(\lambda^{-1}2^{-j}, 2^{-j}) \lesssim 2^{\frac{d+1}{2}j} \lambda^{-\frac{j}{2}}.
\]

For the third case \( 1 + c \leq |x|, |y| \), we may assume \( D(x, y) \geq c_0 \) since the estimate (2.23) follows by the same argument as above if \( D(x, y) \leq -c_0 \). Thus, we have \( 1 + c < (x, y)^2 \). If \( (x, y) > (1 + c_0)^{1/2} \), the two distinct roots \( r_1 < r_2 \) of the equation \( Q(x, y, \tau) = 0 \) are bigger than or equal to 1 because \( Q(x, y, 1) = |x - y|^2 \geq 0 \). Since \( r_2 > (1 + c_0)^{1/2}, |Q(x, y, \cos s)| = |(r_1 - \cos s)(r_2 - \cos s)| \gtrsim (1 - \cos s) \). Using (2.28), we see

\[
|\partial_s \phi_1(x, y, s)| \gtrsim 1.
\]

The same lower bound holds if \( (x, y) < (1 + c_0)^{1/2} \). In fact, \( r_1, r_2 \leq -1 \) because \( Q(x, y, -1) = |x + y|^2 \geq 0 \). Therefore, van der Corput lemma gives \( |\mathfrak{P}_{\lambda}[\psi_j](x, y)| \lesssim 2^{\frac{d}{2}j} \min(\lambda^{-1}, 2^{-j}) \lesssim 2^{\frac{d+1}{2}j} \lambda^{-\frac{j}{2}} \). This completes the proof. \( \square \)

### 3. Estimate away from \( \sqrt{\lambda} \mathbb{S}^{d-1} \): Proof of Theorem 1.2

In this section we prove Theorem 1.2 and show the failure of the estimate (1.3) for \((1/p, 1/q) \in [\mathcal{C}, \mathcal{D}] \cup [\mathcal{C}', \mathcal{D}'] \). This and the lower bounds on \( \| \chi_{B_{\lambda}} \Pi_{\lambda} \|_{p-q} \) in Proposition 5.1 below show that the bounds in Theorem 1.2 are sharp.

#### 3.1. Proof of Theorem 1.2

Making use of Lemma 2.2 and 2.6, we see that it is sufficient to consider

\[
\tilde{\Pi}_{\lambda} := \sum_{0 \leq j \leq j_0} \Pi_{\lambda}[\psi_j]
\]

in place of \( \Pi_{\lambda} \) where \( 2^{j_0} \sim \lambda \). That is to say, the same estimates hold for \( \tilde{\Pi}_{\lambda} \) as those for \( \Pi_{\lambda} \) in Theorem 1.2.

Since \( |D(x, y)| \geq 1/2 \) for \( x, y \in B \), we have the estimate (2.32). Applying Lemma 2.3, we get

\[
\| \chi_{B_{\lambda}} \Pi_{\lambda}[\psi_j] \chi_{B_{\lambda}} \|_{p-q} \lesssim \lambda^{-\frac{j}{2}d(p,q)} 2^{\frac{1}{2}j} \lambda^{-\frac{1}{2}(d(p,q)-1)j},
\]

provided that \((1/p, 1/q) \) is contained in the close quadrangle with vertices \((1/2, 1/2), \mathfrak{A}, \mathfrak{B}' \) and \((1, 0) \). Thus, summation over \( j \) gives

\[
\| \chi_{B_{\lambda}} \tilde{\Pi}_{\lambda} \chi_{B_{\lambda}} \|_{p-q} \lesssim \lambda^{\delta(p,q)},
\]
for $(1/p, 1/q) \in \mathcal{R}_1 \setminus [\mathcal{C}, \mathcal{C}']$ (see Figure 1). It is convenient for our purpose to note 
that for $(p, q) = \left(\frac{d-2}{2}, \frac{d-1}{2}\right)$ if $(1/p, 1/q) \in (\mathcal{C}, \mathcal{D})$, and for $(p, q) = (\infty, 1)$ if $(1/p, 1/q) = \mathcal{C}$. By (c) in Lemma 2.4 and (3.1) we have

$$
\|\chi_{B_{\lambda}} \Pi_{\lambda} \chi_{B_{\lambda}}\|_{L^{p-1} \rightarrow L^{p, \infty}} \lesssim \lambda^{-\frac{d}{2}} \delta(p, q),
$$

for $(1/p, 1/q) = \mathcal{C}, \mathcal{C}'$, which satisfies $\delta(p, q) = 2/(d + 1)$. Interpolation yields (3.2) for $(1/p, 1/q) \in \mathcal{R}_1$. Using the estimate (3.1) and taking sum over $0 \leq j \leq j_0$, we get (3.2) for $(p, q) = (1, \infty)$.

Now, in view of interpolation, to complete the proof we need only to show (3.2) with $(p, q) = (1, 2)$ and the weak type estimate

$$
\|\Pi_{\lambda} |\psi_j| f\|_2 \lesssim 2^{-\frac{d}{2}} \lambda^{-\frac{d}{2}} \|f\|_1, \quad 2^{j} \lesssim \lambda.
$$

Summation over $j$ clearly yields (3.2) with $(p, q) = (1, 2)$. It now remains to show (3.4). Interpolation of the estimates (3.5) and (3.1) with $(1/p, 1/q) = (1, 0), \mathcal{C}$ gives

$$
\|\chi_{B_{\lambda}} \Pi_{\lambda} |\psi_j| \chi_{B_{\lambda}}\|_{p \rightarrow q} \lesssim 2^{jd(\frac{p-1}{2} + \frac{d}{2})} \lambda^{\frac{d}{2}(\frac{1}{2} - \frac{d}{2}) - \frac{d}{2} q - 1},
$$

for $(1/p, 1/q)$ contained in the closed triangle $\mathcal{T}$ with vertices $(1, 1/2), \mathcal{C}, (1, 0)$. Now, fixing $p \in [1, 2d/(d+1)/(d^2 + 4d - 1)]$ and choosing two $q_0, q_1$ such that $q_0 < 2d/(d+1) < q_1$ and $(1/p, 1/q_0), (1/p, 1/q_1) \in \mathcal{T}$, we have the estimates (3.6) with $p = p_0 = p_1$ and $q = q_0, q_1$. Then, we apply (a) in Lemma 2.4 to two these two estimates to get the weak type estimate for $p, q$ such that $(1/p, 1/q) \in (\mathcal{C}, \mathcal{D})$. (Figure 1 is helpful here.) Hence, for $1 \leq p < 2d/(d+1)/(d^2 + 4d - 1)$, we obtain the estimate (3.4). This completes the proof of Theorem 1.2.

**Remark 2.** As can be easily seen from the proof, the same bounds remain to hold on $\chi_{E, \Pi_{\lambda} \chi_{E, \lambda}}$ (in place of $\chi_{B_{\lambda}} \Pi_{\lambda} \chi_{B_{\lambda}}$) as long as $E$ is a measurable set $\subseteq \mathbb{R}^d$ such that $E = -E$ and

$$
\mathcal{D}(x, y) \geq c_0, \quad \forall x, y \in E
$$

for some constant $c_0 > 0$. The same condition was used in [30] to study $L^p$ boundedness of Bochner-Riesz means of the Hermite expansion.

### 3.2. Boundedness of the operator $\varphi_k$ and a transplantation result

In this section we consider the estimate (1.3) for $(1/p, 1/q) \in \mathcal{R}_3$ and discuss how it is related to its counterpart to the Laplacian, that is to say, the estimate for $\varphi_k$.

To put our discussion in a proper context, recalling (1.2), we consider the estimate

$$
\|\varphi_k\|_{p \rightarrow q} \lesssim k^{-1 + \frac{d}{2} (p, q)}
$$

for $k \geq 1$. The following lemma shows that (3.7) is equivalent to the estimate

$$
\|\hat{f}_{[2^{j}-1)'}\|_q \lesssim \|f\|_p.
$$

**Lemma 3.1.** The estimate (3.7) holds for all $k \geq 1$ if and only if the estimate (3.8) holds for all $f \in \mathcal{S}(\mathbb{R}^d)$. The equivalence remains valid with $L^p$, $L^q$ spaces replaced, respectively, by the Lorentz spaces $L^{p, r}, L^{q, s}$ if $1 < p, q \leq \infty$ and $1 \leq r, s \leq \infty$.
Proof. We consider

\[ \widetilde{\varphi}_k f = \frac{k}{(2\pi)^d} \int_{|\xi| \leq 1} e^{ix\xi} \hat{f}(\xi) d\xi. \]

By scaling we note that \( \|\tilde{\varphi}_k\|_{p \to q} = k^{\frac{d}{2}(p,q)-1} \|\varphi_k\|_{p \to q}. \) Thus, the estimate (3.7) is equivalent to

(3.9) \[ \|\varphi_k\|_{p \to q} \lesssim 1. \]

Letting \( k \to \infty \) gives (3.8). Conversely, making use of the spherical coordinates and Minkowski’s inequality, one can easily see that (3.8) implies (3.9) and then (3.7) via scaling. Extension to the Lorentz spaces is clear since \( L^{p,r} \) is a Banach space if \( 1 < p \leq \infty \) and \( 1 \leq r \leq \infty. \) We omit the detail. \( \square \)

The operator \( f \to (\hat{f}|_{\mathbb{S}^{d-1}})^{\vee} \) is imbedded in a family of operators which are called the Bochner-Riesz operators of negative order:

\[ S^{\alpha} f(x) = (2\pi)^{-d} \int e^{ix\xi}(1-|\xi|^2)_{+}^{\alpha} \hat{f}(\xi) d\xi, \]

where \( \Gamma \) is the gamma function. For \( \alpha \geq -1, \) the operator is defined by analytic continuation of the distribution \( (1-|\xi|^2)_{+}^{\alpha}/\Gamma(\alpha+1). \) In fact, \( S^{-1} f = (\hat{f}|_{\mathbb{S}^{d-1}})^{\vee} \) if and only if \( (1/p,1/q) \in \mathcal{R}_3. \) Furthermore, we have \( \| (\hat{f}|_{\mathbb{S}^{d-1}})^{\vee} \|_{q,\infty} \lesssim \| f \|_p \) if \( (1/p,1/q) \in (\mathcal{C},\mathcal{D}) \] and \( \| (\hat{f}|_{\mathbb{S}^{d-1}})^{\vee} \|_{q,\infty} \lesssim \| f \|_{p,1} \) if \( (1/p,1/q) \in (\mathcal{C},\mathcal{C}'). \)

Using Theorem 3.2, Lemma 3.1, and the Stein-Tomas theorem, we can obtain the sharp \( L^{p,L^{q}} \) boundedness of the operator \( S^{-1}. \)

**Theorem 3.2** ([4, 3, 2, 10]). The operator \( f \to (\hat{f}|_{\mathbb{S}^{d-1}})^{\vee} \) is bounded from \( L^p \) to \( L^q \) if and only if \( (1/p,1/q) \in \mathcal{R}_3. \) Furthermore, we have \( \| (\hat{f}|_{\mathbb{S}^{d-1}})^{\vee} \|_{q,\infty} \lesssim \| f \|_p \) if \( (1/p,1/q) \in (\mathcal{C},\mathcal{D}) \] and \( \| (\hat{f}|_{\mathbb{S}^{d-1}})^{\vee} \|_{q,\infty} \lesssim \| f \|_{p,1} \) if \( (1/p,1/q) \in (\mathcal{C},\mathcal{C}'). \)

**Corollary 3.3.** Let \( d \geq 2 \) and \( (1/p,1/q) \in \mathbb{D}. \) Then, we have

(3.10) \[ \|\tilde{\varphi}_k\|_{p \to q} \sim k^{\beta(p,q)}, \quad (1/p,1/q) \notin [\mathcal{C},\mathcal{D}] \cup [\mathcal{C}',\mathcal{D}'], \]

(3.11) \[ \|\tilde{\varphi}_k\|_{L^{p} \to L^{q,\infty}} \leq Ck^{\beta(p,q)}, \quad (1/p,1/q) \in (\mathcal{C},\mathcal{D}), \]

(3.12) \[ \|\tilde{\varphi}_k\|_{L^{p,1} \to L^{q,\infty}} \leq Ck^{\beta(p,q)}, \quad (1/p,1/q) = (\mathcal{C}',\mathcal{C}'). \]

**Proof.** Combining Theorem 3.2 and Lemma 3.1, we get the estimate (3.7) for \( (1/p,1/q) \in \mathcal{R}_3 \), including the weak type estimate (3.10) and the restricted weak type estimate (3.12) for \( (1/p,1/q) \in (\mathcal{C},\mathcal{D}) \) and \( (1/p,1/q) = (\mathcal{C}',\mathcal{C}'), \) respectively. So, we need only to show (3.11).

We have \( \|\tilde{\varphi}_k\|_{2 \to q} \lesssim k^{\beta(2,q)} \) for \( q = 2(d+1)/(d-1) \), which can be shown similarly as before, using the spherical coordinates, the Stein-Tomas theorem, and Plancherel’s theorem. The estimate \( \|\tilde{\varphi}_k\|_{2 \to \infty} \lesssim k^{\beta(2,\infty)} \) follows from the Cauchy-Schwarz inequality and Plancherel’s theorem. These two estimates respectively correspond to the points \( \mathbb{A}' \) and \( (1/2,0) \) in Figure 1 and then duality gives the estimates

\footnotetext[2]{We write \( \int_{|\xi| \leq 1} e^{ix\xi} \hat{f}(\xi) d\xi = C_d \int_{r \leq 1} \int_{|\eta| \leq 1} e^{ixr} r^{d-1} d\eta dr. \)}

\footnotetext[3]{Duality gives \( \| (\hat{f}|_{\mathbb{S}^{d-1}})^{\vee} \|_q \lesssim \| f \|_{p,1} \) if \( (1/p,1/q) \in (\mathcal{C}',\mathcal{D}'). \)}
for $(1/p, 1/q) = \mathfrak{A}$, and $(1, 1/2)$. Since we have (3.10) for $(1/p, 1/q) \in \mathcal{R}_3$, and (3.11) and (3.12) together with $\|\psi_k\|_{2 \rightarrow 2} \lesssim 1$, interpolation and duality give

$$\|\tilde{\psi}_k\|_{p \rightarrow q} \lesssim k^{\delta(p,q)}, \quad (1/p, 1/q) \notin [\mathcal{C}, \mathcal{D}] \cup [\mathcal{C}', \mathcal{D}'].$$

The opposite inequality can be easily shown. Since $\|\tilde{\psi}_k\|_{p \rightarrow q} = k^{\delta(p,q)-1} \|\tilde{\psi}_k\|_{p \rightarrow q}$, by duality we need only to show

$$\|\tilde{\psi}_k\|_{p \rightarrow q} \gtrsim \max(k^{1-\frac{1}{2(p,q)}}, 1, k^{\frac{d}{2(p,q)}}).$$

The second lower bound is trivial. Since the multiplier of the operator $\tilde{\psi}_k$ is radial and supported in $O(k^{-1})$-neighborhood of the sphere $S^{d-1}$, the first and the third lower bounds can be shown by using, respectively, a Knapp type example and the asymptotic expansion of the Bessel function (for example, see [4]). This completes the proof of (3.10). $\square$

The following shows that the estimate (1.3) implies (3.8) when $(1/p, 1/q) \in \mathcal{R}_3$. Lemma 3.4.

Let $B$ be a ball of small radius centered at the origin. Suppose

$$(3.13) \quad \|\chi_B \Pi \chi_B\|_{p \rightarrow q} \lesssim \chi^{\frac{d}{2}\delta(p,q)-1}$$

holds. Then we have the estimate (3.8).

By this and Theorem 3.2 it follows that (1.3) holds if and only if $(1/p, 1/q) \in \mathcal{R}_3$.

Lemma 3.4 may be compared with the known fact [30, 23] that a local $L^p$ bound on the Hermite Bochner-Riesz means implies an $L^p$ bound on the classical Bochner-Riesz means. Our proof below is similar to that in [23], where transplantation of $L^p$ bounds for differential operators was proved. However, unlike $L^p$ bound, $L^p-L^q$ estimate ($p \neq q$) is not scaling invariant. The particular form of the bound (3.13) plays a crucial role. Our argument also extends to general second order elliptic operators without difficulty as long as the associated spectral projection operator satisfies the same form of bound.

In order to prove Lemma 3.4 we recall a special case of Hörmander’s result [17, Theorem 5.1]. Theorem 3.5. Let $P$ be a self-adjoint elliptic differential operator of order 2 with $C^\infty$-coefficients on $\mathbb{R}^d$ and $p$ be its principal part. Then, for $x, y$ in a compact subset and sufficiently close to each other, we have

$$|e(x, y, \lambda) - (2\pi)^{-d} \int_{p(y, \xi) < \lambda} e^{i\psi(x, y, \xi)} d\xi| \leq C(1 + |\lambda|)^{\frac{d}{2}} \lambda,$$

with $C$ independent of $\lambda$ where $e(x, y, \lambda)$ is the spectral function of $P$, i.e., the kernel of the spectral projection operator $\Pi_{[0, \lambda]}$ and $\psi$ is a function homogeneous in $\xi$ of degree 1 which satisfies $p(x, \nabla_x \psi) = p(y, \xi)$ and

$$\psi(x, y, \xi) = \langle x - y, \xi \rangle + O(|x - y|^2|\xi|).$$

Proof of Lemma 3.4. Let $k, \nu$ be large positive integers. Consider an auxiliary projection operator

$$\bar{\Pi} = \sum_{k\nu < \lambda \leq (k+1)\nu} \Pi_{\lambda}.$$

By the triangle inequality and the assumption (3.13) we have

$$\|\chi_B \Pi \chi_B\|_{p \rightarrow q} \lesssim \sum_{k\nu < \lambda \leq (k+1)\nu} \|\chi_B \Pi \chi_B\|_{p \rightarrow q} \lesssim k^{-1}(k\nu)^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}\footnote{Here, the operator $\Pi_{[0, \lambda]}$ is defined by the typical spectral resolution.}.$$

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Let $f, g$ be nontrivial functions in $C^\infty_c(\mathbb{R}^d)$ such that $\text{supp } f$, $\text{supp } g \subset B$. Since $\langle \Pi f, g \rangle = \langle \chi_B \Pi \chi_B f, g \rangle$, we have

$$\left| \int \int \Pi(x, y)f(x)g(y)dxdy \right| \lesssim k^{-1}(k\nu)^{\frac{d}{2}}(\frac{1}{\nu} - \frac{1}{4})\|f\|_p\|g\|_{q'}$$

Rescaling $(x, y) \to (\nu^{-1/2}x, \nu^{-1/2}y)$ gives the equivalent estimate

$$(3.15) \quad \left| \nu^{-\frac{d}{2}} \int \int (\nu^{-\frac{d}{2}}x, \nu^{-\frac{d}{2}}y)F(x)G(y)dxdy \right| \lesssim k^{-1+\frac{d}{2}}(\frac{1}{\nu} - \frac{1}{4})\|F\|_p\|G\|_{q'}$$

provided that $F$ and $G$ are supported in $\sqrt{\nu}B$. Taking the radius of $B$ small enough, we may apply Theorem [3.5]. Since $\Pi(\nu^{-1/2}x, \nu^{-1/2}y) = e(\nu^{-1/2}x, \nu^{-1/2}y, (k+1)\nu) - e(\nu^{-1/2}x, \nu^{-1/2}y, \nu k)$, by Theorem [3.5] we have

$$\tilde{\Pi}(\nu^{-\frac{d}{2}}x, \nu^{-\frac{d}{2}}y) = (2\pi)^{-d} \int_{|\xi| < (k+1)\nu} e^{i\psi(\nu^{-\frac{d}{2}}x, \nu^{-\frac{d}{2}}y, \xi)}d\xi + R_{v,k}(x, y),$$

where $R_{v,k}(x, y) = O(|k\nu|^{\frac{d}{2}})$. Changing variables $\xi \to \nu^{1/2}\xi$ gives

$$\tilde{\Pi}(\nu^{-\frac{d}{2}}x, \nu^{-\frac{d}{2}}y) = (2\pi)^{-d}\nu^{\frac{d}{2}} \int_{|\xi| < (k+1)\nu} e^{i\psi(\nu^{-\frac{d}{2}}(x, y), \nu^{1/2}\xi)}d\xi + R_{v,k}(x, y).$$

Combining this and (3.15) yields

$$\left| \int \int \left( \int_{k \leq |\xi| < k+1} e^{i\psi(\nu^{-\frac{d}{2}}(x, y), \nu^{1/2}\xi)}d\xi \right) f(x)g(y)dxdy \right| \lesssim k^{-1+\frac{d}{2}}(\frac{1}{\nu} - \frac{1}{4})$$

whenever $f$ and $g$ are supported in $\sqrt{\nu}B$ and $\|f\|_p = \|g\|_{q'} = 1$. Here $R_{v,k} = O(\nu^{\frac{d}{2}}(k\nu)^{\ell})$. From (3.14), note that the phase function $\psi(\nu^{-1/2}(x, y), \nu^{1/2}\xi) \to \langle x - y, \xi \rangle$ as $\nu \to \infty$. Thus, taking $\nu \to \infty$, we obtain

$$\left| \int \int \left( \int_{k \leq |\xi| < k+1} e^{i(x - y, \xi)}d\xi \right) f(x)g(y)dxdy \right| \lesssim k^{-1+\frac{d}{2}}(\frac{1}{\nu} - \frac{1}{4})$$

if $\|f\|_p = \|g\|_{q'} = 1$. This gives (3.7), which is equivalent to (3.8) as seen above. □

4. Estimate near $\sqrt{\lambda}S^{d-1}$: Proof of Theorem 1.5.2

In this section we prove Theorem 1.5.2. To this end, it is more convenient to consider the rescaled operator $\mathcal{P}_\lambda$ instead of $\Pi_\lambda$. We note that

$$(4.1) \quad \|\chi_E\mathcal{P}_\lambda[\eta]\chi_E\|_{L^{p,r} \to L^{q,s}} = \lambda^{\frac{d}{2}(\frac{1}{p} - \frac{1}{4})} \|\chi_{\sqrt{\lambda}E}\Pi_\lambda[\eta]\chi_{\sqrt{\lambda}E}\|_{L^{p,r} \to L^{q,s}},$$

for any measurable set $E \subset \mathbb{R}^d$ where $\sqrt{\lambda}E := \{x : \lambda^{-1/2}x \in E\}$. The key part of the proof is to show the following.

**Proposition 4.1.** Let $\lambda^{-2/3} \leq \mu \leq 1/4$ and $2j < \lambda\mu$. Then, we have

$$(4.2) \quad \|\sum_{2j < \lambda\mu} \chi_{\mu}\mathcal{P}_\lambda[\psi_j]\chi_{\mu}\|_{L^{\infty} \to L^{\infty}} \lesssim \lambda^{-\frac{1}{4}}\mu^{\frac{d}{4}},$$

$$(4.3) \quad \|\chi_{\mu}\mathcal{P}_\lambda[\psi_j]\chi_{\mu}\|_{L^{6/5,1} \to L^{6,\infty}} \lesssim \lambda^{-\frac{5}{6}}\mu^{1/2j}, \quad j \geq 0.$$

Once we have the above estimates, the assertions in Theorem 1.5.2 can easily be verified.
Proof of Theorem 1.5. To prove Theorem 1.5 it suffices to show (1.7) and (1.8). Indeed, note that\(\|\chi_{\lambda,\mu} \Pi_\lambda \chi_{\lambda,\mu}\|_{2 \to \infty} \leq C \lambda^{(d-2)/4} \mu^{(d-1)/4}\). This follows by (1.4) since \(\|\chi_{\lambda,\mu} \Pi_\lambda \chi_{\lambda,\mu}\|_{2 \to \infty} \leq \|\chi_{\lambda,\mu} \Pi_\lambda\|_{2 \to 2}\). The desired estimate (1.6) for \((1/p, 1/q) \in (\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_2' \cup \mathcal{Q}_3)\) follows from those estimates and the previously known estimate (1.4) (equivalently, (1.5)) via interpolation and duality (see Figure 2 and 3).

Thanks to (4.1), (1.7) follows from (4.2) and Lemma 2.6. Similarly, for (1.8) it is enough to show
\begin{equation}
\|\sum_{2^j < \lambda} \chi_{\lambda,\mu} \Pi_\lambda \psi_j \chi_{\lambda,\mu}\|_{L^{p,1} \to L^{q,\infty}} \lesssim \mu^{-\frac{1}{2} \cdot \frac{1}{q}} \cdot j = 0.
\end{equation}

When \(d = 2\), there is nothing to prove since \(L^p-L^q\) estimate holds by (1.5). Thus, we may assume \(d \geq 3\). To do this, we use Lemma 2.3 and Lemma 2.4. By scaling, i.e., (4.1), the estimate (4.3) is equivalent to
\begin{equation}
\|\chi_{\lambda,\mu} \Pi_\lambda \psi_j \|_{L^{p,1} \to L^{q,\infty}} \lesssim \mu^{-\frac{d}{2} \cdot \frac{1}{q}} \cdot j = 0.
\end{equation}

Since \(d \geq 3\), the exponent of \(2^j\) is positive. So, we can apply Lemma 2.3 with \(r = 6/5\), \(b = (\lambda \mu)^{-1/2}\), and \(b = (d-1)/2\) to get
\begin{equation}
\|\chi_{\lambda,\mu} \Pi_\lambda \psi_j \|_{p \to q} \lesssim \mu^{-\frac{1}{2} \cdot \frac{1}{q}} \cdot (1/p, 1/q) = \mathcal{S}, \mathcal{S}'.
\end{equation}

4.1. Reduction via sectorial decomposition. We prove the estimates (4.2) and (4.3) while assuming under the assumption that
\(\mu \ll 1\).

The case \(\mu \sim 1\) can be handled in a similar way but much easier (see Remark 4).

We make use of a decomposition of \(A_\mu \times A_\mu\), which was used in [19]. Note that
\begin{equation}
D(x, y) = -|x|^2 |y|^2 \sin^2 \theta(x, y) + (1 - |x|^2)(1 - |y|^2),
\end{equation}
where \(\theta(x, y)\) denotes the angle between \(x\) and \(y\). Since \(|(1 - |x|^2)(1 - |y|^2)| \sim \mu^2\) for \((x, y) \in A_\mu \times A_\mu\), relative size of \(\theta(x, y)\) against \(\mu\) is efficient to control \(D\). This can be exploited by a Whitney type decomposition of \(S^{d-1} \times S^{d-1}\) away from its diagonal (see [19] Section 2.4).

Adopting the typical dyadic decomposition process, for each integer \(\nu \geq 0\) we partition \(S^{d-1}\) into spherical caps \(\Theta^\nu_k\) such that \(\Theta^\nu_k \subset \Theta^\nu_k'\) for some \(k\) whenever \(\nu \geq \nu'\), and \(c_d 2^{-\nu} \leq \text{diam}(\Theta^\nu_k) \leq C_d 2^{-\nu}\) for some constants \(c_d, C_d > 0\). Let \(\nu_\circ := \nu_\circ(\mu)\) denote the integer \(\nu_\circ\) such that
\begin{equation}
\mu/2 < C 2^{-\nu_\circ} \leq \mu
\end{equation}
for a large positive constant \(C\). By a Whitney type decomposition of \(S^{d-1} \times S^{d-1}\) away from its diagonal, we may write
\begin{equation}
S^{d-1} \times S^{d-1} = \bigcup_{\nu_\circ \in \mathbb{N}_0: 2^{-\nu_\circ} \leq \mu} \bigcup_{k \approx \nu, k'} \Theta^\nu_k \times \Theta^\nu_k',
\end{equation}
where \(k \sim \nu\) if \(k' \sim \nu\) if \(\text{dist}(\Theta^\nu_k, \Theta^\nu_k') \sim 2^{-\nu}\) if \(\nu > \nu_\circ\) and \(\text{dist}(\Theta^\nu_k, \Theta^\nu_k') \lesssim 2^{-\nu}\) if \(\nu = \nu_\circ\) (for example, see [38] p.971). It should be noted that the sets \(\Theta^\nu_k\) and \(\Theta^\nu_k'\) are not necessarily separated at \(\nu = \nu_\circ\). For a fixed \(\mu\) we define
\begin{equation}
A^\nu_k = \{ x \in A_\mu : |x|^{-1} x \in \Theta^\nu_k \}.
\end{equation}
and set $\chi_k^\nu = \chi_k A_k^{\nu}$. Thus we can write

$$\chi_k \mathcal{P}_\lambda [\psi_j] \chi_\mu = \sum_{2^{-\nu} \leq 2^{-\nu} \leq k} \sum_{k \sim k'} \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k'.'$$

(4.6)

The following simple lemma basically reduces the estimate for $\sum_{k \sim k'} \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k'$ to a uniform estimate for $\chi_k \mathcal{P}_\lambda [\psi_j] \chi_k'$ with $k \sim k'$.

**Lemma 4.2.** Let $1 \leq p \leq q \leq \infty$ and let $T$ be an operator from $L^p$ to $L^q$. Suppose we have the estimate $\|\chi_k^\nu T_{\mathcal{P}_\lambda} |^{\nu} \|_{L^p} \leq B$ whenever $k \sim k'$. Then, with $C$ only depending on $T$, we have $\| \sum_{k \sim k'} \chi_k^\nu T_{\mathcal{P}_\lambda} |^{\nu} \|_{L^p} \leq CB$. The same also holds when $L^p$ and $L^q$ are replaced by $L^P$ and $L^Q$.

**Proof.** The last assertion is clear. We only provide the proof for $L^p$ and $L^q$. It is enough to show $\| \sum_{k \sim k'} \chi_k^\nu T_{\mathcal{P}_\lambda} |^{\nu} \|_{L^p} \leq CB \| \mathcal{P}_\lambda \|_p$ for any measurable set $\mathcal{P}$ (e.g., see Stein [36, p.195]). Besides, note that $\| \sum_{k \sim k'} f_k \|_{L^q} \leq (\sum_{k} \| f_k \|_p)^{1/q}$ if $f_k$ are disjoint. By combining those factors, one can easily see the desired inequality since $\chi_k^\nu$ are boundedly overlapping.

To obtain the desired estimates, we separately consider the cases $2^{-\nu} \gg \mu$ and $2^{-\nu} \leq \mu$ for which we have $|D(x, y)| \geq 2^{-2\nu}$ and $|D(x, y)| \leq 2^\nu$, respectively.

When $2^{-\nu} \gg \mu$. Since $-D(x, y) \sim 2^{-\nu}$ on $A_k^{\nu} \times A_k^{\nu'}$, substituting $N = 1/2$ in (a) (or the last assertion if $2^{-\nu} \sim 1$) of Lemma 2.4, we have

$$\| \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k' \|_{1 \to \infty} \lesssim \lambda^{-1/2} \mu^{1/2} \mu^{1/2}.$$  

(4.7)

By (2.19) and (4.1) it follows that $\| \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k' \|_{L^2} \lesssim 2^{-1} \lambda^{-1} \mu \mu^{1/2}$. Combining this estimate and (4.7), we obtain

$$\| \sum_{j \geq 0} \lambda \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k' \|_{L^2} \lesssim \lambda^{-1} \mu^{1/2} \mu^{1/2}.$$  

By (4.9) and summation over $\nu : \mu \ll 2^{-\nu} \lesssim 1$ give

$$\| \sum_{j \geq 0} \lambda \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k' \|_{L^2} \lesssim \lambda^{-1/2} \mu^{1/2} \mu^{1/2}.$$  

By (4.10) and (4.11), we have $\| \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k' \|_{L^2} \lesssim \lambda^{-1} \mu^{1/2}$. Interpolation with (4.12) gives the estimate $\| \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k' \|_{L^6} \lesssim \lambda^{-1} \mu^{1/2} \mu^{1/2}$. Using Lemma 4.2 and taking sum over $\nu : \mu \ll 2^{-\nu} \lesssim 1$, we obtain

$$\sum_{j \geq 0} \lambda \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k' \|_{L^6} \lesssim \lambda^{-1} \mu^{1/2} \mu^{1/2}.$$  

When $2^{-\nu} \lesssim \mu$. We now consider the case $2^{-\nu} \sim \mu$. From (4.9) and (4.8) we note that the contributions in this case $2^{-\nu} \gg \mu$ are acceptable to the estimates (4.2) and (4.3). Since there are only $O(1)$, to show (4.2) and (4.3) it is sufficient to consider a single $\nu$ such that $2^{-\nu} \lesssim \mu$. By Lemma 4.2 we need only show the estimates

$$\| \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k' \|_{L^6} \lesssim \lambda^{-1/2} \mu^{1/2} \mu^{1/2}.$$  

for $k \sim \nu$, $k'$. Note that $A_k^\nu$ and $A_k^\nu'$ are contained in a set of diameter $\sim \mu$ if $k \sim \nu$.

Thus, for $(x, y) \in A_k^\nu \times A_k^\nu'$, $k \sim \nu$, we have

$$\| \chi_k \mathcal{P}_\lambda [\psi_j] \chi_k' \|_{L^6} \lesssim \lambda^{-1/2} \mu^{1/2} \mu^{1/2}.$$  

(4.10)
Let \( \varepsilon, c \ll 1 \) be positive constants which are to be specified later. For further reduction we cover \( A_k^r \) and \( A_{k'}^r \) by collections of essentially disjoint cubes \( \{Q\} \) and \( \{Q'\} \) of side length \( c\varepsilon, \mu \), respectively, so that

\[
A_k^r \subset \bigcup Q, \quad A_{k'}^r \subset \bigcup Q'.
\]

Note that \( \partial_x D(x, y) = 2((x, y) - 1)y + 2(y - x) \) and \( \partial_y D(x, y) = 2((x, y) - 1)x + 2(x - y) \). Since \( 1 - \langle x, y \rangle \sim \mu \) and \( |x - y| \ll \mu \) for \( (x, y) \in A_k^r \times A_{k'}^r \), we have \( \partial_x D \) and \( \partial_y D \) are \( O(\mu) \). Thus, taking \( c \) small enough, we have one of the following hold for each \( Q \times Q' \):

\[
|D(x, y)| \gtrsim \varepsilon_0 \mu^2, \quad \forall (x, y) \in \tilde{Q} \times \tilde{Q}',
\]

\[
|D(x, y)| \ll \varepsilon_0 \mu^2, \quad \forall (x, y) \in \tilde{Q} \times \tilde{Q}',
\]

where \( \tilde{Q} \) and \( \tilde{Q}' \) denote \( c^2 \varepsilon, \mu \)-neighborhoods of \( Q \) and \( Q' \), respectively.

Since there are at most \( O((c\varepsilon_0)^{-d}) \) many \( Q \) and \( Q' \), the matter is reduced to showing the following estimates for each \( Q \times Q' \):

\[
\|\chi_Q \mathcal{P}_\lambda[\psi_j] \chi_{Q'}\|_{L^{6/(5, 1)} \rightarrow L^{6, \infty}} \lesssim \lambda^{-\frac{d+2}{2}} \mu^{-\frac{3}{2}} \mu^{-\frac{d}{2}},
\]

\[
\|\sum_{j \geq 0} \chi_Q \mathcal{P}_\lambda[\psi_j] \chi_{Q'}\|_{L^{6/(5, 1)} \rightarrow L^{6, \infty}} \lesssim \lambda^{-\frac{d}{2}} \mu^{-\frac{3}{2}}.
\]

If (4.11) holds, one can easily obtain the desired estimates (4.13) and (4.14) by the same argument as above. Indeed, (c) with \( N = 1/2 \) in Lemma 2.7 Lemma 2.5 and (2.9), respectively, give

\[
\|\chi_Q \mathcal{P}_\lambda[\psi_j] \chi_{Q'}\|_{L^{1} \rightarrow L^{1}} \lesssim 2^{\frac{d+1}{2}} \lambda^{-\frac{d}{2}} \mu^{-\frac{3}{2}},
\]

\[
\|\chi_Q \mathcal{P}_\lambda[\psi_j] \chi_{Q'}\|_{L^{2} \rightarrow L^{2}} \lesssim 2^{-\frac{d}{2}} \lambda^{-\frac{d}{2}} \mu^{-\frac{3}{2}}, \quad 2^j \leq 2 \lambda \mu,
\]

\[
\|\chi_Q \mathcal{P}_\lambda[\psi_j] \chi_{Q'}\|_{L^{2} \rightarrow L^{2}} \lesssim \lambda^{-\frac{d}{2}} 2^{-j}.
\]

Applying Lemma 2.4 to (4.15) and (4.16), we obtain (4.14). In the same manner, the estimate (4.13) follows by (4.15) and (4.17).

We now consider the case (4.12). In this case, the estimates (4.16) and (4.17) remains valid. However, (4.14) holds only for \( j \) such that \( 2^{-j} \ll \mu^{1/2} \) or \( 2^{-j} \gg \mu^{1/2} \) as can be seen by taking \( N = 1/2 \) in (b) of Lemma 2.7. Thus, repeating the same argument above, we obtain (4.13) for \( 2^{-j} \ll \mu^{1/2} \) and

\[
\|\sum_{2^{-j} \ll \mu^{1/2}} \chi_Q \mathcal{P}_\lambda[\psi_j] \chi_{Q'}\|_{L^{6/(5, 1)} \rightarrow L^{6, \infty}} \lesssim \lambda^{-\frac{d}{2}} \mu^{\frac{d}{2}}.
\]

Therefore, the proof of (4.13) and (4.14) is reduced to showing

\[
\|\chi_Q \mathcal{P}_\lambda[\psi_j] \chi_{Q'}\|_{L^{6/(5, 1)} \rightarrow L^{6, \infty}} \lesssim \lambda^{-\frac{d}{2}} \mu^{-\frac{d}{2}},
\]

\[
\|\chi_Q \mathcal{P}_\lambda[\psi_j] \chi_{Q'}\|_{L^{6/(5, 1)} \rightarrow L^{6, \infty}} \lesssim \lambda^{-\frac{d}{2}} \mu^{\frac{d}{2}}
\]

for \( j \) satisfying \( 2^{-j} \sim \mu^{1/2} \) while assuming (4.12).

Let \( c_Q \) and \( c_{Q'} \) denote the centers of the cubes \( Q \) and \( Q' \), respectively. By \( s \in (0, \pi/2) \) we denote the number such that \( \cos s = \langle c_Q, c_{Q'} \rangle \), and set

\[
\psi_e(s) = \eta_e \left( \frac{s - s_e}{\sqrt{c_\mu}} \right)
\]

where \( \eta_e \in C_c^\infty((-2, 2)) \) such that \( \eta_e = 1 \) on \([-1, 1]\). Then, we decompose

\[
\chi_Q \mathcal{P}_\lambda[\psi_j] \chi_{Q'} = \chi_Q \mathcal{P}_\lambda[\psi_e \psi_j] \chi_{Q'} + \chi_Q \mathcal{P}_\lambda[(1 - \psi_e) \psi_j] \chi_{Q'}.
\]
It is easy to show that $\chi_Q \mathcal{P}_\lambda [(1 - \psi_c) \psi_j] \chi_{Q'}$ has acceptable bounds. To see this, we recall (4.10) and note that $(x, y) = \cos s_c + O(\varepsilon \mu)$ for $(x, y) \in Q \times Q'$. Since $|\mathcal{D}(x, y)| \ll \varepsilon \mu^2$ and $c, \mu < 1$, using (2.20), we see that

$$Q(x, y, \cos s) \gtrsim \varepsilon \mu^2, \quad \forall (x, y) \in Q \times Q'$$

if $s \in \text{supp}((1 - \psi_c) \psi_j)$. Via (2.25) this lower bound gives $|\partial_s \phi_1 (x, y, s)| \gtrsim \mu$ for $(x, y, s) \in Q \times Q' \times \text{supp}((1 - \psi_c) \psi_j)$. Thus, recalling (2.27) and applying van der Corput’s lemma, we get $|\mathcal{P}_\lambda [(1 - \psi_c) \psi_j] (x, y)| \lesssim (\lambda \mu)^{-\frac{4}{2}}$ for $(x, y) \in Q \times Q'$. Here we also use $|\partial_s^\alpha (a(1 - \psi_c) \psi_j)| \lesssim \mu^{-(n+d)/2}$. Since $\mu \gtrsim \lambda^{-\frac{2}{3}}$, by the above bound we obtain

$$\| \chi_Q \mathcal{P}_\lambda [(1 - \psi_c) \psi_j] \chi_{Q'} \|_{1 \to \infty} \lesssim 2^{\frac{d-1}{2}} (\lambda \mu)^{-\frac{1}{2}} + 2^{-j} \sim \sqrt{\mu}.$$  

Meanwhile, by (2.19) and (2.5) we have the estimates (4.17) and (4.18) with $\psi_j$ replaced by $(1 - \psi_c) \psi_j$ when $2^{-j} \sim \sqrt{\mu}$. Interpolation shows that $\chi_Q \mathcal{P}_\lambda [(1 - \psi_c) \psi_j] \chi_{Q'}$ has the acceptable bounds.

Now, the proof of Proposition 4.1 reduces to proving (4.18) and (4.19) with $\psi_j$ replaced by $\psi_c \psi_j$ assuming (4.12). Before proceeding further, we replace $\chi_Q$ and $\chi_{Q'}$ with smooth functions $\tilde{\chi}_Q$ and $\tilde{\chi}_{Q'}$, respectively, which are adapted to $Q$ and $Q'$. More precisely, $\tilde{\chi}_Q$ and $\tilde{\chi}_{Q'}$ satisfy that $\tilde{\chi}_Q \chi_Q = \chi_Q$, $\tilde{\chi}_{Q'} \chi_{Q'} = \chi_{Q'}$, $\partial^\alpha \tilde{\chi}_Q$, $\partial^\alpha \tilde{\chi}_{Q'} = O(\mu^{-|\alpha|})$, and $\tilde{\chi}_Q$ and $\tilde{\chi}_{Q'}$ are supported in $Q$ and $Q'$ ($c^2 \varepsilon \mu$-neighborhoods of $Q$ and $Q'$), respectively. Now, the desired estimates follow from the next proposition.

**Proposition 4.3.** Let $j$ satisfy $2^{-j} \sim \mu^{1/2}$ and let $Q$ and $Q'$ be the cubes of side length $c \varepsilon \mu$ given as above. Suppose (4.12) holds. Then, we have

$$\| \tilde{\chi}_Q \mathcal{P}_\lambda [\psi_c \psi_j] \chi_{Q'} \|_{L^{6/5,1} \to L^{6,\infty}} \lesssim \lambda^{-\frac{d-3}{2}} \mu^{-\frac{1}{2}};$$  

$$\| \tilde{\chi}_Q \mathcal{P}_\lambda [\psi_c \psi_j] \chi_{Q'} \|_{L^{3,1} \to L^\infty} \lesssim \lambda^{-\frac{1}{2}} \mu^{\frac{d-2}{2}}.$$  

We make some observations, which are to be useful in what follows:

$$|\sin \theta (x, y)| \sim \mu, \quad \forall (x, y) \in \tilde{Q} \times \tilde{Q'},$$

$$|x - y| \sim \mu, \quad \forall (x, y) \in \tilde{Q} \times \tilde{Q'}.$$  

The first (4.22) follows by (4.5) and (4.12) since $(1 - |x|^2)(1 - |y|^2) \sim \mu^2$ if $x, y \in A_\mu$. To see the second (4.23), note that $|x - y|^2 = (1 - \langle x, y \rangle)^2 - \mathcal{D}(x, y)$. Thus, by (4.12) and (4.11) we have (4.23).

4.2. **2nd-order derivative of $\phi_1$.** To prove Proposition 4.1, we can no longer rely only on the first order derivative of $\phi_1$. When the discriminant $\mathcal{D}(x, y)$ vanishes, the equation $Q(x, y, \cdot) = 0$ has a zero of order 2. Furthermore, the stationary point of $\phi_1$ and the zero of $\partial^2 \phi_1$ converge to each other as $\mathcal{D}$ approaches to zero. Thus, van der Corput’s lemma gives a decay estimate of $O(\lambda^{-1/3})$ when $\mathcal{D}(x, y) = 0$. However, such a bound is not sufficient for us to obtain the sharp bound since we need $L^1 - L^\infty$ bound of $O(\lambda^{-1/2})$ to make our argument work. To overcome this problem, we break the integral dyadically away from the zero of $\partial^2 \phi_1$. Before doing so, we need to take a close look at $\partial^2 \phi_1$.

A computation shows

$$\partial^2 \phi_1 (x, y, s) = -\frac{R(x, y, \cos s)}{\sin^3 s},$$

where $R(x, y, s)$ is a certain function of $(x, y, s)$.
where
\[ \mathcal{R}(x, y, \tau) = (x, y)^2 \tau^2 - (|x|^2 + |y|^2)\tau + (x, y). \]

From (4.22) we note that \( x \neq y \) and \( x \neq -y \). Thus, \( \mathcal{R}(x, y, \cdot) \) has two distinct roots
\[ \tau^\pm(x, y) = \frac{|x|^2 + |y|^2 \pm |x + y||x - y|}{2(x, y)}. \]

It is easy to see \( \tau^+(x, y) > 1 > \tau^-(x, y) \), and hence the role of \( \tau^- \) is more important. \( \partial_x^2 \phi_1(x, y, \cdot) \) has a unique zero on \((0, \pi/2)\) for \((x, y) \in \tilde{Q} \times \tilde{Q}'\), which we denote by \( S_c(x, y) \). That is to say,
\[ \cos S_c(x, y) = \tau^-(x, y). \]

As clear from (4.10) and (4.25), \( S_c \) is smooth on \( \tilde{Q} \times \tilde{Q}' \). Using \(|x + y|^2 - |x - y|^2 = 4(x, y)|\), we also have
\[ 1 - \cos S_c(x, y) = 2|x - y|/(|x + y| + |x - y|). \]

Since \(|x + y| \sim 1\) for \((x, y) \in \tilde{Q} \times \tilde{Q}'\), \(1 - \cos S_c(x, y) \sim |x - y| \sim \mu \) and \( S_c(x, y) \sim \mu^{1/2} \).

Let \( s_c(x, y) \in (0, \pi/2) \) denote the point such that \( \cos s_c(x, y) = (x, y) \). As mentioned before, \( S_c(x, y) \) converges to \( s_c(x, y) \) as \( D(x, y) \to 0 \). Indeed, note that \( \mathcal{R}(x, y, \tau) = (\tau - (x, y))(\tau - \mu) + D(x, y) \tau \). Since \( \mathcal{R}(x, y, \cos S_c) = 0 \), this gives
\[ \cos S_c - (x, y) = D(x, y) \cos S_c (1 - (x, y) \cos S_c)^{-1}. \]

Hereafter, we occasionally drop the variables \( x, y \) to simplify the notation as long as no ambiguity arises. Since \( 1 - \cos S_c \sim \mu \) for \((x, y) \in \tilde{Q} \times \tilde{Q}'\), by (4.10) we have \( 1 - (x, y) \cos S_c \gtrsim \mu \). Hence, it follows that
\[ |\cos S_c - (x, y)| \lesssim |D(x, y)|\mu^{-1} \ll \varepsilon_0 \mu \]
for \((x, y) \in \tilde{Q} \times \tilde{Q}'\). From (4.10) we see \( s_c(x, y) \sim \mu^{1/2} \). Thus, it follows that \(|S_c(x, y) - s_c(x, y)| \lesssim |D(x, y)|\mu^{-3/2} \) for \((x, y) \in \tilde{Q} \times \tilde{Q}'\).

**Decomposition away from \( S_c \).** We now break
\[ \tilde{\chi}_Q \mathcal{P}_x[\psi \psi] \tilde{\chi}_Q' = \sum_l \mathcal{P}^l : = \sum_l \tilde{\chi}_Q \mathcal{P}_x[\psi \psi](2^l \cdot - S_c)|\tilde{\chi}_Q' . \]

Note that \( \mathcal{P}^l \neq 0 \) only if \( 2^{-l} \lesssim (\varepsilon_0 \mu)^{1/2} \). To handle \( \mathcal{P}^l \), changing variables
\[ s \to S^l_c(x, y, s) := 2^{-l} s + S_c(x, y), \]
we write
\[ \mathcal{P}^l(x, y) = \mu^{-\frac{d}{2}} 2^{-l} \int e^{i\lambda \Phi(x, y, s)} A(x, y, s) ds, \]
where
\[ \Phi(x, y, s) = \phi_1(x, y, S^l_c(x, y, s)), \]
\[ A(x, y, s) = \tilde{\chi}_Q(x) \tilde{\chi}_Q'(y) (\mu^\frac{d}{2} a \psi \psi)(S^l_c(x, y, s)) \psi(|s|). \]

Note that \( \partial_x^2 \Phi = 2^{-2l}(\partial_x^2 \phi_1)(x, y, S^l_c) \). Since \( (x, y) \sim 1 \) and \( \sin S^l_c \sim \mu^{1/2} \) on \( \text{supp } A \), (4.24) and \( \mathcal{R}(x, y, \cos s) = (x, y)(\cos S_c - \cos s)(\tau^+ - \cos s) \) give
\[ |\partial_x^2 \phi_1(x, y, S^l_c)| \sim \mu^{-\frac{d}{2}} \cos S_c - \cos S^l_c |\tau^+ - \cos S^l_c|. \]
Note that $1 - \cos S_{t}^{l} \sim \mu$ and $|\cos S_{c}^{l} - \cos S_{c}| \sim 2^{-l} \mu^{1/2}$ on $\text{supp} \ A$. Since $\tau^{+}(x, y) - 1 = ((|x| + |y|)/2(x, y) \sim \mu$ (see (4.28)), we also have $\tau^{+} - \cos S_{c}^{l} \sim \mu$ on $\text{supp} \ A$. Thus, we have

$$\|\partial^{2}_{x} \phi_{1}(x, y, S_{c}^{l})\| \sim 2^{-l}$$

on $\text{supp} \ A$. Note $\partial^{n}_{x} A(x, y, s) = O(1)$ for $n \in \mathbb{N}_{0}$. By van der Corput’s lemma we get

$$\|\Psi^{*}_{\tau} \|_{1 \rightarrow \infty} \lesssim \lambda^{-\frac{d}{2}} \mu^{-\frac{d-2}{2}}$$

We also have the following estimates:

**Proposition 4.4.** Let $2^{-l} \lesssim (\varepsilon \mu)^{1/2}$. Then, the following estimates hold:

$$\|\Psi^{*}_{\tau} \|_{2 \rightarrow \infty} \lesssim \lambda^{-\frac{d}{2}} \mu^{-\frac{d-2}{2}}$$

$$\|\Psi^{*}_{\tau} \|_{2 \rightarrow 2} \lesssim \lambda^{-\frac{d}{2}} 2^{-l}.$$  

Using these estimates, one can easily verify the desired estimates (4.20) and (4.21). Indeed, applying Lemma 2.4 to the estimates (4.30) and (4.32), we get the restricted weak type estimate (4.20). The restricted type estimate in (4.21) can be obtained similarly using (4.30) and (4.31) when $d = 2$. If $d \geq 3$, interpolating (4.30) and (4.31) and then taking sum over $l$ give the desired estimate, a strong type estimate for $(p, q) = (\frac{2d}{d-1}, \infty)$.

To complete the proof, it remains to prove the estimates (4.31) and (4.32).

#### 4.3. Proof of (4.31) and (4.32)

We begin with recalling the following bounds on derivatives of $\Phi$ and $A$, which was proved in [19]. In fact, the estimate (4.34) below was shown only for $\beta = 0$ in [19] Lemma 4.4. However, one can easily show (4.34) using (4.33) and following the argument there.

**Lemma 4.5.** [19] Lemma 4.4, 4.5] Let $2^{-l} \lesssim (\varepsilon \mu)^{1/2}$ and $s \in \text{supp} \psi(| \cdot |)$. If $(x, y) \in \tilde{Q} \times Q'$, then for any $\alpha, \beta \in \mathbb{N}_{0}^{d}$

$$|\partial^{2}_{x} \partial^{\beta}_{y} S_{c}(x, y, s)| \lesssim \mu^{\frac{d}{2} - |\alpha| - |\beta|},$$

(4.33)

$$|\partial^{\alpha}_{x} \partial^{\beta}_{y} A(x, y, s)| \lesssim \mu^{-|\alpha| - |\beta|},$$

(4.34)

$$|\partial^{\alpha}_{x} \partial^{\beta}_{y} \Phi(x, y, s)| \lesssim \mu^{\frac{d}{2} - |\alpha| - |\beta|}.$$  

By (2.17), to prove (4.31) it is sufficient to show

$$\int |\Psi^{*}_{\tau} \!(x, y)|^{2} \, dy \lesssim 2^{-l} \lambda^{-1} \mu^{-\frac{d-2}{2}}.$$  

We write

$$\int |\Psi^{*}_{\tau} \!(x, y)|^{2} \, dy = 2^{-2l} \mu^{-\frac{d}{2}} \int \left( \int \tilde{A}(x, y, s, t)e^{i\lambda \Psi(x, y, s, t)} \, dy \right) \, dt ds,$$  

where

$$\Psi(x, y, s, t) = \phi_{1}(x, y, S_{c}^{l}(x, y, s)) - \phi_{1}(x, y, S_{c}^{l}(x, y, t))$$

and

$$\tilde{A}(x, y, s, t) = A(x, y, s)A(x, y, t)$$

We now claim that

$$|\partial_{y} \Psi(x, y, s, t)| \gtrsim 2^{-l}|s - t|, \quad \forall (x, y) \in \tilde{Q} \times \tilde{Q}'.$$  


holds if we take $\varepsilon_0$ sufficiently small. From (4.34) and (4.35), it follows that $\partial_s^\alpha \tilde{A} = O(\mu^{-|\alpha|})$ and $\partial_s^\alpha \Psi = O(\mu^{-|\alpha|+1}2^{-l}|s-t|)$ for $(x, y) \in \tilde{Q} \times \tilde{Q}'$. Thus, by using (4.40), routine integration by parts gives
\[
\left| \int \tilde{A}(x, y, s, t)e^{\i \lambda \Psi(x, y, s, t)}dy \right| \lesssim \mu^d (1 + \lambda \mu 2^{-l}|s-t|)^{-N}.
\]
Combining this with (4.37) and integrating in $s$, we get (4.36).

It remains to show (4.40). We write
\[
\partial_y \Psi(x, y, s, t) = E + F,
\]
where
\[
E = (\partial_s \phi_1(x, y, S_0(x, y, s)) - \partial_s \phi_1(x, y, S_0(x, y, t)))\partial_y S_c(x, y),
\]
\[
F = \partial_y \phi_1(x, y, S_c(x, y, s)) - \partial_y \phi_1(x, y, S_c(x, y, t)).
\]
The mean value theorem gives $E = \partial_s^2 \phi_1(x, y, S_0(x, y, s^*)) 2^{-l} (s - t) \partial_y S_c$ for some $s^* \in (s, t)$. Since $2^{-l} \lesssim (\varepsilon_0, \mu)^{1/2}$, using (4.29) and (4.33), we see $E = O(2^{-2l} \mu^{-1/2} |s-t|) = O((\varepsilon_0 2^{-l} |s-t|)$. Therefore, to show (4.40), it is enough to verify
\[
\text{(4.41)} \quad |F| \gtrsim 2^{-l} |s-t|
\]
taking $\varepsilon_0 > 0$ small enough. To show (4.41), we exploit the form of $\phi_1$. For simplicity, fixing $x, y$, we denote $S_0(s) = 2^{-l} s + S_c(x, y)$. By a direct computation we get
\[
F = ay - bx - y.
\]
where
\[
a = \left( \frac{\cos S(t) - 1}{\sin S(t)} - \frac{\cos S(t) - 1}{\sin S(t)} \right), \quad b = \left( \frac{1}{\sin S(t)} - \frac{1}{\sin S(t)} \right).
\]
By the mean value theorem it is clear that $|a| \sim 2^{-l} |s-t|$ and, similarly, $|b| \sim 2^{-l} |s-t|$. Since $|x-y| \sim \mu$, (4.41) follows once we show
\[
\text{(4.42)} \quad \sin^2 \theta(y, x-y) \sim 1, \quad \forall (x, y) \in \tilde{Q} \times \tilde{Q}'.
\]
Now, we recall (4.22), so (4.42) follows from (4.23) since $|y|^2 |x-y|^2 \sin^2 \theta(y, x-y) = |x|^2 |y|^2 \sin^2 \theta(x, y)$. \hfill \Box

4.4. Proof of (4.32). Let us define an oscillatory integral operator $I^\lambda_s(\Phi, A)$ by
\[
I^\lambda_s(\Phi, A) f(x) = \int e^{\i \lambda \Phi(x, y, s)} A(x, y, s) f(y)dy.
\]
We observe that $\mathfrak{P}_s^\lambda f = \mu^{-\frac{d}{2}} 2^{-l} \int I^\lambda_s(\Phi, A) f ds$. By Minkowski’s inequality we see $\|\mathfrak{P}_s^\lambda f\|_{L^2} \leq \mu^{-\frac{d}{2}} 2^{-l} \sup_s \|I^\lambda_s(\Phi, A)\|_{L^2 \to L^2}$. Thus, (4.32) follows from
\[
\text{(4.43)} \quad \|I^\lambda_s(\Phi, A) f\|_2 \lesssim \lambda^{-\frac{d}{2}} \mu^\frac{d}{2} \|f\|_2, \quad \forall s.
\]
Note that $I^\lambda_s(\Phi, A) \neq 0$ only if $s \in \text{supp } \check{\psi}$. To obtain (4.43), we make use of the following well known lemma.

---

5 One may rescale, that is to say, $y \rightarrow \mu y + c_{Q'}$ where $c_{Q'}$ is the center of $Q'$. 
We begin with claiming that $\cos \langle I \cdot \rangle$ is invertible for any $s$, that is, $\det \partial_x \partial_y \tilde{\Phi}(x, y, s) \neq 0$.

By Lemma 4.7, it follows that $\det \partial_x \partial_y \tilde{\Phi}(x, y, s) \sim \mu^{-d/2}$ for any $(x, y, s) \in \tilde{Q} \times \tilde{Q'} \times \supp \psi(\cdot \cdot \cdot)$.

Assuming this for the moment, we prove (4.43).

**Proof of (4.43).** Recall that $c_Q$ denotes the center of a cube $Q$. We denote $l_{\mu}(x, y) = (\mu x, \mu y) + (c_Q, c_Q')$ and set

$$\bar{\Phi}(x, y, s) = \mu^{-\frac{d}{2}} \Phi(l_{\mu}(x, y), s),$$

$$\bar{A}(x, y, s) = A(l_{\mu}(x, y), s).$$

Then, changing variables $(x, y) \to l_{\mu}(x, y)$, we have

$$\| I^\lambda_s(\Phi, A) \|_{2 \to 2} = \mu^d \| I^\lambda^\mu \frac{3}{2}(\tilde{\Phi}, \tilde{A}) \|_{2 \to 2}.$$ 

We apply Lemma 4.6 to $I^\lambda^\mu \frac{3}{2}(\tilde{\Phi}, \tilde{A})$. Using (4.34) and (4.35), one can easily see

$$| \partial^\alpha_x \partial^\beta_y \tilde{\Phi}(x, y, s) | \lesssim 1, \quad | \partial^\alpha_x \partial^\beta_y \tilde{A}(x, y, s) | \lesssim 1, \quad \forall (x, y, s) \in \supp \tilde{A}.$$ 

By Lemma 4.7, it follows that $\det \partial_x \partial_y \tilde{\Phi} \sim 1$. Applying Lemma 4.6 gives the estimate $\| I^\lambda^\mu \frac{3}{2}(\tilde{\Phi}, \tilde{A}) \|_{2 \to 2} \lesssim \lambda^{-d/2} \mu^{-3d/4}$. Therefore, we get (4.43) as desired. \qed

**Proof of Lemma 4.7.** We begin with claiming that

$$\partial_x \partial^T_y \Phi(x, y, s) = \mathbf{H} + O(\varepsilon_0^{1/2} \mu^{-1/2})$$

for $(x, y, s) \in \tilde{Q} \times \tilde{Q'} \times \supp \psi(\cdot \cdot \cdot)$ where\n
$$\mathbf{H} = -(\sin S_c)^{-1} I + \partial_x S_c \partial^T_y \partial_x \phi_1(x, y, S_c) + \partial_x \partial_x \phi_1(x, y, S_c) \partial^T_y S_c.$$ 

Here $I$ denotes the $d \times d$ identity matrix. Using the chain rule, we write

$$\partial_x \partial_y \Phi(x, y, s) = \partial_x \partial^T_y \phi_1(x, y, S_c^1) + \partial_x S_c \partial^T_y \partial_x \phi_1(x, y, S_c^1) + \partial_x \partial_x \phi_1(x, y, S_c^1) \partial^T_y S_c + \partial_x \partial_y \phi_1(x, y, S_c^1) \partial^T_y S_c + \partial_x \partial_y \phi_1(x, y, S_c^1) \partial^T_y S_c.$$ 

By (4.29) and (4.33), we see $\partial^2_x \phi_1(x, y, S_c^1) \partial_x S_c \partial^T_y S_c = O(2^{-1} \mu^{-1}) = O(\varepsilon_0^{1/2} \mu^{-1/2})$. Since $2^{-1} \lesssim (\varepsilon_0 \mu)^{1/2}$ and $| \cos S_c^1 - \cos S_c | \sim 2^{-1} \mu^{1/2}$, by (4.27), it follows that $| \cos S_c^1 - (x, y) | = O(\varepsilon_0^{1/2} \mu)$. Thus, by (4.29), we see $\partial_x \phi_1(x, y, S_c^1) = O(\varepsilon_0 \mu)$ because $| D | \ll \varepsilon_0 \mu$. Combining this and (4.33), we have $\partial_x \partial^T_y \phi_1(x, y, S_c^1) \partial_x \partial^T_y S_c = O(\varepsilon_0 \mu^{-1/2})$. Therefore, we need only to consider the other terms in the right hand side of (4.40). Therefore, to show (4.44), we note $\partial_x \partial^T_y \phi_1(x, y, S_c) = -(\sin S_c)^{-1} I$, and

$$\partial_x \partial_x \phi_1(x, y, S_c) = \frac{y \cos S_c - x}{\sin^2 S_c}, \quad \partial_y \partial_x \phi_1(x, y, S_c) = \frac{x \cos S_c - y}{\sin^2 S_c}.$$
Since $|\cos S_c^i - \cos S_c| \sim 2^{-i}\mu^{1/2}$ and $|\sin S_c^i - \sin S_c| \sim 2^{-i}$, from the above identities we see that $S_c^i$ appearing in the first to third terms on the right hand side of (4.46) can be replaced by $S_c$ allowing an error of $O(\varepsilon_1\mu^{-1/2})$. Thus, we get (4.43).

Thanks to (4.43), the matter reduces to showing that $\det H \sim \mu^{-d/2}$. To this end, we need to obtain precise expressions for $\partial_x S_c$, $\partial_y S_c$. We set

$$v(x, y) = y \cos S_c - x, \quad w(x, y) = x \cos S_c - y.$$  

From (4.20), (4.23), and (4.22), we note that the two vectors $(\cos S_c - 1)x$ and $x - y$ are of size $\sim \mu$ and are separated by $\sim \mu$. Thus, writing $w = (\cos S_c - 1)x + (x - y)$, we see $|w| \sim \mu$. In the same manner, it follows that $|v| \sim \mu$. So, we have

$$|v(x, y)| \sim \mu, \quad |w(x, y)| \sim \mu.$$  

We now observe that $v^\top w = R(x, y, \cos S_c)$. This gives

$$v^\top w = 0.$$  

Differentiating in $x$, we have $\partial_x (v^\top w) = -\sin S_c \partial_x S_c (x^\top v + y^\top w) + \cos S_c v \cdot w - w$. Combining this and (4.50) yields

$$\partial_x S_c = - (\sin S_c)^{-1} (x^\top v + y^\top w)^{-1} (w - \cos S_c v).$$  

Similarly, we have $\partial_y S_c = - (\sin S_c)^{-1} (x^\top v + y^\top w)^{-1} (v^\top - \cos S_c w)\cdot w$, which can also be shown using symmetry of $S_c$ and $v(y, x) = w(x, y)$. From (4.48), we also note that $\partial_x \partial_y \phi_1 (x, y, S_c) = (\sin S_c)^{-2} v$ and $\partial_y \partial_x \phi_1 (x, y, S_c) = (\sin S_c)^{-2} w$. Applying these identities to (4.45), we obtain

$$H = - \frac{\sin^2 S_c (x^\top v + y^\top w) I + v v^\top + w w^\top - 2 \cos S_c v^\top w}{\sin^3 S_c (x^\top v + y^\top w)}.$$  

By (4.50) and (4.48), it follows that $x^\top v \cos S_c = y^\top v$ and $y^\top w \cos S_c = x^\top w$. Using those and $\sin^2 S_c = 1 - \cos^2 S_c$, we see

$$\sin^2 S_c (x^\top v + y^\top w) = x^\top v + y^\top w - \cos S_c (y^\top v + x^\top w) = -|v|^2 - |w|^2.$$  

For the last equality we use (4.48) again. Thus, we obtain

$$H = - \frac{1}{\sin S_c} L,$$

where

$$L = I - \frac{1}{(|v|^2 + |w|^2)} (v v^\top + w w^\top - 2 \cos S_c v^\top w).$$

To complete the proof, it remains to show that $\det L \sim 1$ since $\sin S_c \sim \mu^{1/2}$.

Let $V = \text{span}(v, w)$. Then, it is easy to see that $V$ is an invariant subspace of $L$. Let $M$ denote the matrix of the linear map $L|_V$ (restricted to $V$) with respect to bases $v, w$. Using (4.50), we see $L v = (|v|^2 + |w|^2)^{-1} |v|^2 v$ and $L w = (|v|^2 + |w|^2)^{-1} |w|^2 w$. Thus, we obtain

$$M = \frac{1}{|v|^2 + |w|^2} \begin{pmatrix} |w|^2 & 2 |w|^2 \cos S_c \\ 0 & |v|^2 \end{pmatrix}.$$  

Note that $V^\perp$ is also an invariant subspace and $L u = u$ for all $u \in V^\perp$. So, $\det L = \det M$ and $\det L = (|v|^2 + |w|^2)^{-2} |v|^2 |w|^2$. By (4.49) we conclude $\det L \sim 1$. □
Remark 3. When $\mu \sim 1$, to prove (4.2) and (4.3) is much simpler. Especially, we don’t need the sectorial decomposition. One may just cover $A_\mu \times A_\mu$ with disjoint cubes $\{Q \times Q^\prime\}$ of small side length $c_0 \leq 1$ so that either (4.11) or (4.12) holds. We only need to consider the latter case since the first can be handled easily as before. Since (4.12) holds and $\mu \leq 1/4$, taking $c, z_0$ small enough, we have
\[ |(x, y)| \geq c_0, \quad (x, y) \in Q \times Q^\prime \]
for a constant $c_0 > 0$. So, the estimates for $\chi_Q \Pi_\lambda \psi_j \chi_{Q^\prime}$ are easy to show since $Q(x, y, \cos s) \sim 1$ for $(x, y, s) \in Q \times Q^\prime \times \supp \psi_j$ when $2^{-j} \ll 1$. Thus, we may assume $2^{-j} \sim 1$. We now note that (4.12) and (4.23) holds with $\mu \sim 1$. Thus, the previous proofs of (4.11) and (4.32) work without modification.

5. Lower bounds on $\|\chi_{\lambda, \mu} \Pi_{\lambda} \chi_{\lambda, \mu}\|_{p \to q}$

In this section, we show that the bound (1.6) can not be improved, that is to say, there is a constant $C > 0$ such that
\[ \|\chi_{\lambda, \mu} \Pi_{\lambda} \chi_{\lambda, \mu}\|_{p \to q} \geq C \lambda^{\beta(p, q) \mu \gamma(p, q)} \]

For the purpose, it is sufficient to show the lower bounds (5.2)–(5.4) in Proposition 5.1 below. Comparing those lower bounds immediately yields (1.1).

Proposition 5.1. Let $d \geq 2$, $\lambda \gg 1$, and $\lambda^{-2/3} \leq \mu \leq 2^{-1}$. Then, we have
\[ \|\chi_{\lambda, \mu} \Pi_{\lambda} \chi_{\lambda, \mu}\|_{p \to q} \gtrsim \lambda^{-\gamma(p, q) \mu \gamma(p, q)} \]
\[ \|\chi_{\lambda, \mu} \Pi_{\lambda} \chi_{\lambda, \mu}\|_{p \to q} \gtrsim \lambda^{-\gamma(p, q) \mu \gamma(p, q)} \]
\[ \|\chi_{\lambda, \mu} \Pi_{\lambda} \chi_{\lambda, \mu}\|_{p \to q} \gtrsim \lambda^{-\gamma(p, q) \mu \gamma(p, q)} \]

In particular, the lower bounds in Proposition 5.1 with $\mu = 2^{-1}$ also shows sharpness of the estimate (1.3).

The lower bounds (5.2) and (5.3) which yield (5.1) for $(1/p, 1/q) \in \mathcal{R}_1 \cup \mathcal{R}_3$ can be shown by using the known lower bound on $\|\chi_{\lambda, \mu} \Pi_{\lambda} \chi_{\lambda, \mu}\|_{q \to q}$. We recall (1.4) of which sharpness was shown in [25]. By the $TT^*$ argument we have
\[ \|\chi_{\lambda, \mu} \Pi_{\lambda} \chi_{\lambda, \mu}\|_{p \to p'} \geq C \lambda^{\beta(p, p') \mu \gamma(p, p')} \]
for $1 \leq p \leq 2$ with $C > 0$ depending only on $d$. Suppose that (5.1) fails for some $(1/p_0, 1/q_0) \in \mathcal{R}_1 \cup \mathcal{R}_3$, $p'_0 \neq q_0$, that is to say, there are sequences $\lambda_k, \mu_k$ such that
\[ C_k := \lambda_k^{-\beta(p_0, q_0) \mu_k \gamma(p_0, q_0)} \|\chi_{\lambda_k, \mu_k} \Pi_{\lambda} \chi_{\lambda_k, \mu_k}\|_{p_0 \to q_0} \to 0 \]
as $k \to \infty$. By duality we have $\|\chi_{\lambda_k, \mu_k} \Pi_{\lambda} \chi_{\lambda_k, \mu_k}\|_{q_0 \to p'_0} \leq C_k \lambda_k^{\beta(p_0, q_0) \mu_k \gamma(p_0, q_0)} \|\chi_{\lambda_k, \mu_k} \Pi_{\lambda} \chi_{\lambda_k, \mu_k}\|_{p_0 \to q_0} \to 0$ since $p'_0 \neq q_0$, interpolation with the estimate $\|\chi_{\lambda_k, \mu_k} \Pi_{\lambda} \chi_{\lambda_k, \mu_k}\|_{p_0 \to q_0} \leq C_k \lambda_k^{\beta(p_0, q_0) \mu_k \gamma(p_0, q_0)}$ gives $\|\chi_{\lambda_k, \mu_k} \Pi_{\lambda} \chi_{\lambda_k, \mu_k}\|_{p_0 \to p'_0} \leq C k \lambda_k^{\beta(p_0, q_0) \mu_k \gamma(p_0, q_0)}$ where $1/p_* - 1/p'_* = 1/p_0 - 1/q_0$ because $\beta(p_0, q_0) = \beta(p_*, p'_*)$ and $\gamma(p_0, q_0) = \gamma(p_*, p'_*)$. This contradicts to the lower bound (5.5) if we let $k \to \infty$. 

Proof of (5.4). It remains to prove (5.4). To do so, we mainly rely on asymptotic properties of the Hermite functions. Let $h_k(t)$ denote the $L^2$-normalized $k$-th Hermite function of which eigenvalue is $2k$. We make use of the following lemma from [25]. Also see [1] and [13].
Lemma 5.2. \textbf{[25] Lemma 5.1} Let \( \nu = \sqrt{2k+1} \). We set
\[ s^-(t) = \int_0^t \sqrt{\tau^2 - \nu^2} d\tau \quad \text{and} \quad s^+(t) = \int_0^t \sqrt{\nu^2 - \nu^2} d\tau. \]
Then, the following hold:
\[
h_{2k}(t) = \begin{cases} a_{2k}(\nu^2 - t^2)^{-\frac{1}{4}} (\cos s^-_\nu(t) + \mathcal{E}), & |t| < \nu - \frac{1}{4}, \\ O(\nu^{-\frac{1}{2}}), & \nu - \frac{1}{4} < |t| < \nu + \frac{1}{4}, \\ a_{2k}e^{-s^-_\nu(|t|)}(t^2 - \nu^2)^{-\frac{1}{4}}(1 + \mathcal{E}), & \nu + \frac{1}{4} < |t|, \end{cases}
\]
\[
h_{2k+1}(t) = \begin{cases} a_{2k+1}(\nu^2 - t^2)^{-\frac{1}{4}} (\sin s^-_\nu(t) + \mathcal{E}), & |t| < \nu - \frac{1}{4}, \\ O(\nu^{-\frac{1}{2}}), & \nu - \frac{1}{4} < |t| < \nu + \frac{1}{4}, \\ a_{2k+1}e^{-s^-_\nu(|t|)}(t^2 - \nu^2)^{-\frac{1}{4}}(1 + \mathcal{E}), & \nu + \frac{1}{4} < |t|, \end{cases}
\]
where \(|a^\pm_{2k}| \sim 1\) and \(\mathcal{E} = O(|t^2 - \nu^2|^{-\frac{1}{2}}|t| - \nu|^{-1})\).

We also need the following lemma.

Lemma 5.3. Let \( 1 \leq p \leq 2 \) and \( x_0 \in A_{\lambda, \mu} \). Then, if \( \lambda^{d/2} \lesssim \mu \leq 1 \), we have
\[
\|\Pi_\lambda(x_0, \cdot)\|_{L^p(A_{\lambda, \mu})} \lesssim \lambda^{-\frac{d}{4} + \frac{d}{2}} \mu^{-\frac{1}{4} + d\left(\frac{d}{2} - \frac{1}{4}\right)}.
\]

Proof. If \( \mu \sim 1 \), the estimate follows by Hölder’s inequality and the bound \(\|\Pi_\lambda\|_{1 \to \infty} \lesssim \lambda^{d-2/2}\). Indeed, since \(|A_{\lambda, \mu}| \sim \lambda^d/2\) and \(\|\Pi_\lambda(x_0, \cdot)\|_2^2 = \Pi_\lambda(x_0, x_0)\), we see
\[
\|\Pi_\lambda(x_0, \cdot)\|_{L^p(A_{\lambda, \mu})} \leq |A_{\lambda, \mu}|^{\frac{1}{2}-\frac{1}{4}} \|\Pi_\lambda(x_0, x_0)\|^{\frac{1}{2}} \lesssim \lambda^{-\frac{d}{4} + \frac{d}{2}}.
\]

Thus, we may assume \(\mu \ll 1\). Let \( S_0 = (B(x_0, C\lambda^{1/2} \mu) \cap B(-x_0, C\lambda^{1/2} \mu)) \cap A_{\lambda, \mu} \) for a large constant \(C > 0\). We note \(\|\Pi_\lambda(x_0, \cdot)\|_{L^q(S_0)} \lesssim \|\chi_{\lambda, \mu} \Pi \chi_{\lambda, \mu}\|_{2 \to \infty}\) from (2.47). Recalling (1.0) for \(p = 2\) and \(q = \infty\), we have \(\|\Pi_\lambda(x_0, \cdot)\|_{L^2(S_0)} \lesssim C\lambda^{(d-2)/4} \mu^{(d-1)/4}\). Thus, by Hölder’s inequality we have
\[
\|\Pi_\lambda(x_0, \cdot)\|_{L^q(S_0)} \lesssim |S_0|^{\frac{1}{2} - \frac{d}{4}} \|\Pi_\lambda(x_0, \cdot)\|_{L^2(S_0)} \lesssim \lambda^{-\frac{d}{4} + \frac{d}{2}} \mu^{-\frac{1}{4} + d\left(\frac{d}{2} - \frac{1}{4}\right)}.
\]
So, it is sufficient to show
\[
\|\Pi_\lambda(x_0, \cdot)\|_{L^p(S_0)} \lesssim \lambda^{-\frac{d}{4} + \frac{d}{2}} \mu^{-\frac{1}{4} + d\left(\frac{d}{2} - \frac{1}{4}\right)}.
\]

By the symmetric property of the kernels of \(\Pi_\lambda|\psi_j|\), \(\Pi_\lambda|\psi_j^\dagger|\), \(\kappa = -, \pm \pi\), it follows that \(\|\sum_{j \geq 0} \Pi_\lambda|\psi_j|\sum_{0} \Pi_\lambda|\psi_j^\dagger||_{L^p(A_{\lambda, \mu}, S_0)} = \|\sum_{j \geq 0} \Pi_\lambda|\psi_j^\dagger|\sum_{0} \Pi_\lambda|\psi_j||_{L^p(A_{\lambda, \mu}, S_0)}\), \(\kappa = -, \pm \pi\). For the desired estimate, by (2.6) we need only to show that
\[
(5.6) \quad \|\sum_{0} \Pi_\lambda|\psi_j||_{L^p(A_{\lambda, \mu}, S_0)} \lesssim \lambda^{-\frac{d}{4} + \frac{d}{2}} \mu^{-\frac{1}{4} + d\left(\frac{d}{2} - \frac{1}{4}\right)}.
\]

For \(l \geq 1\), set \(S_l = \{x \in A_{\lambda, \mu} : |x - x_0| \in C\lambda^\frac{d}{2} \mu^{2l-1}, 2^l\}\). To prove the above estimate, it is enough to show
\[
\|\sum_{0} \Pi_\lambda|\psi_j||_{L^p(S_0)} \lesssim \left(\mu 2^l\right)^{-\frac{d-2}{2}} \left(\lambda^\frac{d}{2} \mu 2^dl\right)^{\frac{d}{2} - N}, \quad l \geq 1.
\]

Summation over \(l\) gives the estimate (1.5.0) because \(\lambda^3/2 \gtrsim 1\). Using (4.0) and scaling, we observe
\[
\|\sum_{0} \Pi_\lambda|\psi_j||_{L^p(S_0)} \lesssim \sum_{k \geq 0} \sup_{k'} \|\chi_k^\psi_\lambda \Pi_\lambda|\psi_j|\chi_{k'}^\psi\|_{L^p, \nu} |S_1|^{1/p},
\]
where \(\nu\) satisfies \(2^{-\frac{d}{2}} \sim C\mu 2^l\). Note that \(D \sim -(\mu 2^l)^2\) if \((x, y) \in \text{ supp } \chi_k \times \text{ supp } \chi_{k'}, k \sim \nu k'.\) Thus, using (4.0) in Lemma 2.7 we obtain
\[
\sum_{k \geq 0} \sup_{k' \sim \nu k'} \|\chi_k^\psi_\lambda \Pi_\lambda|\psi_j|\chi_{k'}^\psi\|_{L^p, \nu} \lesssim \left(\mu 2^l\right)^{-\frac{d-2}{2}} \left(\lambda(\mu 2^l)^{\frac{d}{2}}\right)^{-N},
\]
for any $N \in \mathbb{N}$. Therefore, we get the desired estimate. \hfill \square

To show (5.4), we first claim that there is a point $x_0 \in A_{\lambda, \mu}$ such that
\begin{equation}
(5.7) \quad \int_{A_{\lambda, \mu}} \Pi_\lambda(x_0, y)^2 dy \gtrsim \mu^{\frac{d}{2}} (\lambda \mu)^{\frac{d-2}{2}}.
\end{equation}
Combined with (2.17), this shows sharpness of the bound (1.6) for $p = 2$ and $q = \infty$. Assuming (5.7) for the moment we prove (5.4). Let us set
$$f(x) = \Pi_\lambda(x, x) \chi_{A_{\lambda, \mu}}(x).$$
By (5.7) we have $\Pi_\lambda f(x_0) \gtrsim \mu^{\frac{d}{2}} (\lambda \mu)^{\frac{d-2}{2}}$. We now recall the following lemma.

**Lemma 5.4.** [30] Lemma 4.5] Let $\lambda \in 2\mathbb{N}_0 + d$ and $\mu \in [\lambda^{-\frac{d}{2}}, \frac{1}{2}]$. Suppose that $h \in \mathcal{S}(\mathbb{R}^d)$ is an eigenfunction of $H$ with eigenvalue $\lambda$, i.e., $Hh(x) = \lambda h(x)$. If $y_0 \in A_{\lambda, \mu}$, then for any $\alpha \in \mathbb{N}_0^d$ we have
\begin{equation}
(5.8) \quad |\partial^\alpha h(y_0)| \leq C(\lambda \mu)^{\frac{1}{2} |\alpha|} \|h\|_{L^\infty(B(y_0, 2(\lambda \mu)^{-\frac{1}{2}}))}
\end{equation}
with $C$ independent of $\lambda$, $\mu$ and $h$.

By this lemma we also have $\|\nabla \Pi_\lambda f\|_{L^\infty(A_{\lambda, \mu})} \lesssim \mu^{1/2} (\lambda \mu)^{(d-1)/2}$. By the mean value theorem we see that $\Pi_\lambda f(x) \gtrsim \mu^{1/2} (\lambda \mu)^{(d-2)/2}$ if $x \in B(x_0, c(\lambda \mu)^{-1/2})$ for a constant $c > 0$ small enough. Thus, we have
$$\|\chi_{\lambda \mu} \Pi_\lambda \chi_{\lambda \mu} f\|_q \gtrsim \|\Pi_\lambda f\|_{L^q(B(x_0, c(\lambda \mu)^{-1/2}) \cap A_{\lambda, \mu})} \gtrsim \mu^{\frac{d}{2}} (\lambda \mu)^{\frac{d-2}{2}}.$$
Combining this and the estimate $\|f\|_p \lesssim \lambda^{-\frac{d}{2} + \frac{d}{2} \mu - \frac{d}{4} + d(\mu - \frac{1}{2})}$, which follows from Lemma 5.3 we obtain (5.3). It remains to show (5.7).

**Proof of (5.7).** Let $\lambda = 2N + d$. We set
$$J = \{\alpha \in \mathbb{N}_0^d : |\alpha| = N, N \mu / (2^{10} d) \leq \alpha_j \leq N \mu / (2^9 d), \quad 2 \leq j \leq d\},$$
$$\ell = (2\sqrt{\lambda} - 1)^{-1} \sqrt{\lambda \mu},$$
and $Q_\ell = [\sqrt{\lambda}(1 - 2\mu), \sqrt{\lambda}(1 - 3\mu/2)] \times [-\ell, \ell]^{d-1}$. Noting that $Q_\ell \subset A_{\lambda, \mu}$ and $|Q_\ell| \sim (\lambda \mu)^{d-1}$, we have
$$\sum_{\alpha \in J} \int_{B_{d-1}(0, (\lambda \mu)^{-\frac{1}{2}})} \frac{1}{\sqrt{\lambda}(1 - 2\mu)} \int_{\sqrt{\lambda}(1 - 3\mu/2)} |\Phi_\alpha(x, x)| \, dx \, dx \geq \lambda^{\frac{d}{2}} \mu^{\frac{d-2}{2}}.$$
This is an easy consequence of Lemma 5.2. Thus, there exists $x_0 \in [\sqrt{\lambda}(1 - 2\mu), \sqrt{\lambda}(1 - 3\mu/2)]$ such that $\sum_{\alpha \in J} |\Phi_\alpha(x_0)| \gtrsim (\lambda \mu)^{d-1}$. We consider
$$g(x) = \chi_{Q_\ell}(x) \sum_{\alpha \in J} c_\alpha \Phi_\alpha(x),$$
where $c_\alpha \in \{-1, 1\}$ such that $c_\alpha \Phi_\alpha(x_0) = |\Phi_\alpha(x_0)|$. By (1.3) with $q = 2$, we have
$$\|g\|_2 \leq \|\sum_{\alpha \in J} c_\alpha \Phi_\alpha(x)\|_{L^2(A_{\lambda, \mu})} \lesssim \mu^{\frac{d}{2}} \|\sum_{\alpha \in J} c_\alpha \Phi_\alpha(x)\|_2 \lesssim \mu^{\frac{d}{2}} (\lambda \mu)^{\frac{d-2}{2}}.$$
We now set
$$a_{u,v} := \int_{-\ell}^\ell h_u(t)h_v(t)dt, \quad a_{u,v}^* := \int_{\sqrt{\lambda}(1 - 3\mu/2)} h_u(t)h_v(t)dt,$$
and \(A_{\alpha,\beta} = \int_{Q_{\ell}} \Phi_{\alpha}(x)\Phi_{\beta}(x)dx\). Note that \(A_{\alpha,\beta} = a^{*}_{\alpha,\beta} \prod_{i=2}^{d} a_{\alpha_i,\beta_i}\) and \(\Pi_\lambda g(x) = \sum_{\alpha \in I} \sum_{\beta : |\beta| = N} c_\alpha A_{\alpha,\beta} \Phi_{\beta}(x)\). Thus, we write

\[
(5.9) \quad \Pi_\lambda g(x) = I(x) + I(x),
\]

where

\[
I(x) = \sum_{\alpha \in I} c_\alpha A_{\alpha,\alpha} \Phi_{\alpha}(x), \quad I(x) = \sum_{\alpha \in I} \sum_{\beta : |\beta| = N, \alpha \neq \beta} c_\alpha A_{\alpha,\beta} \Phi_{\beta}(x).
\]

Using Lemma 5.2 it is easy to see that \(a_{u,u} \sim 1\) and \(a^{*}_{u,u} \sim \mu^{1/2}\) if \(u \sim N\). Consequently, it follows that \(A_{\alpha,\alpha} \sim \mu^{1/2}\) if \(\alpha \in J\). However, if \(u \neq v\), \(a_{u,v}\) is exponentially decaying, thus we may regard \(I\) as a minor error. More precisely,

\[
(5.10) \quad A_{\alpha,\beta} \lesssim e^{-c\lambda \mu}
\]

if \(\alpha \in J\), \(|\beta| = N\), and \(\alpha \neq \beta\). Assuming this for the moment, we prove (5.7).

By (5.10) it follows that \(I(x) = O((\lambda \mu)^a e^{-b\lambda \mu})\) for some \(a, b > 0\). Hence, our choices of \(c_\alpha\) and \(x_0\) ensures that there exists \(C > 0\) such that \(I(x_0) = \sum_{\alpha \in J} c_\alpha A_{\alpha,\alpha} \Phi_{\alpha}(x_0) \geq C \mu^{1/2}(\lambda \mu)^{3d/4-1}\). Since \(Q_{\ell} \subset A_{\lambda,\mu}\) and \(x_0 \in A_{\lambda,\mu}\), recalling (5.9), we obtain

\[
\langle \chi_{\lambda,\mu}(x_0) \Pi_\lambda(x_0, \cdot) \chi_{\lambda,\mu}, g \rangle = \Pi_\lambda g(x_0) \geq C \mu^{1/2}(\lambda \mu)^{3d/4-1} - O((\lambda \mu)^a e^{-b\lambda \mu}).
\]

Since \(\|g\|_2 \lesssim \mu^{1/4}(\lambda \mu)^{(d-1)/2}\), by duality we get (5.7) as desired.

We now show (5.10). Recalling the identity \(2u \cdot v)h_u h_v = h_u h'_v - h'_u h_v\) (for example, see [10] p. 2), we have

\[
a_{u,v} = \frac{1}{2(u-v)} \int_{-\ell}^{\ell} h_u(s)h_v'(s) - h'_u(s)h_v(s)ds.
\]

Thus, integration by parts gives

\[
a_{u,v} = \frac{1}{2(u-v)} \left( h_u(\ell)h'_v(\ell) - h_u(-\ell)h'_v(-\ell) - h'_u(\ell)h_v(\ell) + h'_u(-\ell)h_v(-\ell) \right)
\]

if \(u \neq v\). Note that \(h_u\) is odd if \(u\) is odd and \(h_u\) is even otherwise. So, \(h_u h'_v\) is even if \(u + v\) and odd if \(u\) or \(v\) is odd otherwise. Hence, \(a_{u,v} = 0\) if \(u + v\) is odd. Using the identity \(h_u'(s) = sh_u(s) - \sqrt{u + 2} h_{u+1}(s)\) [10] p. 5], we obtain

\[
(5.11) \quad a_{u,v} = \frac{1 + (-1)^{u+v}}{\sqrt{2(u-v)}} \left( \sqrt{u + 1} h_{u+1}(\ell)h_v(\ell) - \sqrt{v + 1} h_u(\ell)h_{v+1}(\ell) \right).
\]

Note that \(\ell - \sqrt{2u+1} \geq \sqrt{\lambda \mu} \sim \ell\) for \(2^{-10} N \mu / d \leq u \leq 2^{-9} N \mu / d\). Thus, we have \(s^{+}_{\sqrt{2u+1}}(\ell) \geq \lambda \mu\) as long as \(2^{-10} N \mu / d \leq u \leq 2^{-9} N \mu / d\). By Lemma 5.2 it now follows that

\[
|h_u(\ell)| \lesssim e^{-c\sqrt{\lambda \mu} \ell} \lesssim e^{-c\lambda \mu}
\]

for some \(c > 0\) if \(2^{-10} N \mu / d \leq u \leq 2^{-9} N \mu / d\). Combining this with (5.11) and [10] Lemma 1.5.2, we have \(|a_{u,v}| \lesssim e^{-c\lambda \mu}\) for \(2^{-10} N \mu / d \leq u \leq 2^{-9} N \mu / d\) and \(v \leq N\) if \(u \neq v\). Note that \(|a_{u,v}| \lesssim 1\) and \(|a^{*}_{u,v}| \lesssim \mu^{1/2}\) for any \(u, v\). We also note that there is at least one \(j \in \{2, \ldots, d\}\) such that \(a_{\alpha,\beta} = 0\) if \(\alpha \neq \beta\), \(\alpha \in J\), and \(|\beta| = N\). Therefore, we get (5.10) because \(A_{\alpha,\beta} = a^{*}_{\alpha,\beta} \prod_{i=2}^{d} a_{\alpha_i,\beta_i}\). \(\square\)
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