A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations

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Abstract

Let $S(n)$ be the symmetric group on $n$ points. A subset $S$ of $S(n)$ is intersecting if for any pair of permutations $\pi, \sigma$ in $S$ there is a point $i \in \{1, \ldots, n\}$ such that $\pi(i) = \sigma(i)$. Deza and Frankl [9] proved that if $S \subseteq S(n)$ is intersecting then $|S| \leq (n-1)!$. Further, Cameron and Ku [4] show that the only sets that meet this bound are the cosets of a stabilizer of a point. In this paper we give a very different proof of this same result.

1 Introduction

Cameron and Ku [4] proved a version of the Erdős-Ko-Rado theorem for permutations. In this paper we give an alternate proof to this theorem which is substantially different from the one given by Ku and Cameron.

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The Erdős-Ko-Rado theorem \cite{erdos1961intersection} is a central result in extremal combinatorics. There are many interesting proofs and extensions of this theorem, for a summary see \cite{erdos1968intersections}. The Erdős-Ko-Rado theorem gives a bound on the size of a family of intersecting $k$-subsets of a set and describes exactly which families meet this bound.

1.1 Theorem. (Erdős, Ko and Rado \cite{erdos1961intersection}) Let $k, n$ be positive integers with $n > 2k$. Let $A$ be a family of $k$-subsets of $\{1, \ldots, n\}$ such that any two sets from $A$ have non-trivial intersection, then $|A| \leq (n-1)!$. Moreover, $|A| = (n-1)!$ if and only if $A$ is the collection of all $k$-subsets that contain a fixed $i \in \{1, \ldots, n\}$.

The Erdős-Ko-Rado theorem has been extended to objects other than subsets of a set. For example, Hsieh \cite{hsieh1994extensions} and Frankl and Wilson \cite{frankl1992intersection} give a version for intersecting subspaces of a vector space over a finite field, Berge \cite{berge1978hypergraphs} proves it for intersecting integer sequences, Rands \cite{rands1979extensions} extends it to intersecting blocks in a design and Meagher and Moura \cite{meagher2003extensions} prove a version for partitions.

The extension we give here is to intersecting permutations. Let $S(n)$ be the symmetric group on $\{1, \ldots, n\}$. Permutations $\pi, \sigma \in S(n)$ are said to be intersecting if $\pi(i) = \sigma(i)$ for some $i \in \{1, \ldots, n\}$. Similar to the case for subsets of a set, there are obvious candidates for maximum intersecting systems of permutations, these are the sets

$$S_{i,j} = \{\pi \in S(n) : \pi(i) = j\}, \quad i, j \in \{1, \ldots, n\}. \quad (1.1)$$

These sets are the cosets of a stabiliser of a point.

1.2 Theorem. (Cameron and Ku \cite{cameron1992extensions}) Let $n \geq 2$. If $S \subseteq S(n)$ is an intersecting family of permutations then:

(a) $|S| \leq (n-1)!$.

(b) If $|S| = (n-1)!$ then $S$ is a coset of a stabiliser of a point.

The proof given by Cameron and Ku uses an operation called fixing which is similar to the shifting operation used in the original proof of Erdős-Ko-Rado. They show that a maximum intersecting family of permutations is closed under this fixing operation. Assuming that the family contains the identity permutation, and thus each permutation in the family has a fixed point, they next consider the set system formed by the sets of fixed points for each permutation in the family. Cameron and Ku prove that if the family of permutations is closed
under the fixing operation, then this set system is an intersecting set system. Finally, they prove the result by showing that if a family of intersecting permutations has size \((n - 1)!\), then the sets in the intersecting set system must all intersect in the same point.

Our proof uses a graph called the permutation graph which appears in the paper by Cameron and Ku. This graph is a union of graphs in an association scheme, we use properties of this association scheme together with information about the group representation of the symmetric group to get the result.

This approach has been used to prove the standard Erdős-Ko-Rado theorem for sets \([20, \text{Section 5.4}]\) and also to prove versions of the Erdős-Ko-Rado theorem for other objects such as the 3×3 uniform partitions and vector spaces over a finite field \([14]\). It is interesting that this method also works for permutations and hoped that this method can be generalized to other objects.

The proof we give only applies for \(n > 6\), for smaller \(n\) the result can be verified using GAP \([11]\).

2 The Clique-Coclique Bound

In this section we give a proof of the clique-coclique bound for the union of graphs in an association scheme. Although this bound is not new, it was originally proven by Delsarte \([5]\), and an alternate proof for vertex-transitive graphs is given by Cameron and Ku \([4]\), the proof given here is new.

Let \(\mathcal{A} = \{A_0, \ldots, A_d\}\) be an association scheme with \(d\) classes on \(v\) vertices and let \(v_i\) be the valency of the \(i\)-th graph. Denote the principal matrix idempotents of the association scheme by \(E_0, \ldots, E_d\) and let \(m_i\) be the dimension of the eigenspace belonging to \(E_i\). We note that

\[
E_0 = \frac{1}{v} J
\]

where \(J\) is the all-ones matrix.

2.1 Theorem. (Delsarte \([5, \text{Theorem 3.9}]\)) Let \(\mathcal{A}\) be an association scheme on \(v\) vertices and let \(X\) be the union of some of the graphs in the scheme. If \(C\) is a clique and \(S\) is an independent set in \(X\), then

\[
|C|, |S| \leq v.
\]  

(2.1)
If equality holds and $x$ and $y$ are the respective characteristic vectors of $C$ and $S$, then

$$x^T E_j x y^T E_j y = 0 \quad \text{for all } j > 0.$$ 

Proof. We have the following fundamental identity (see [12, Section 12.6]):

$$\sum_{i=0}^{d} \frac{1}{vv_i} x^T A_i x A_i = \sum_{j=0}^{d} \frac{1}{m_j} x^T E_j x E_j$$

from which it follows that

$$\sum_{i=0}^{d} \frac{1}{vv_i} x^T A_i x y^T A_i y = \sum_{j=0}^{d} \frac{1}{m_j} x^T E_j x y^T E_j y. \quad (2.2)$$

Now suppose $C$ is a clique and $S$ is an independent set in $X$, and let $x$ and $y$ be their respective characteristic vectors. The graph $X$ is a union of graphs in the scheme, if $A_i$ is one of the graphs in this union then $A_i y = 0$ otherwise $A_i x = 0$. So for all $i > 0$,

$$x^T A_i x y^T A_i y = 0,$$

and hence the left side of Equation (2.2) is

$$\frac{1}{v} x^T x y^T y = \frac{|C| |S|}{v}. \quad (2.3)$$

For all $j$, the matrix $E_j$ is positive semidefinite and therefore

$$x^T E_j x y^T E_j y \geq 0.$$ 

Consequently the right side of Equation (2.2) is bounded below by its first term:

$$x^T E_0 x y^T E_0 y = \frac{1}{v^2} x^T J x y^T J y = \frac{|C|^2 |S|^2}{v^2}. \quad (2.4)$$

It follows from (2.3) and (2.4) that $|C| |S| \leq v$, as required. If equality holds the remaining condition follows immediately. \qed

We will prove a simple, but useful corollary of this result.

2.2 Corollary. Let $X$ be a union of graphs in an association scheme with the property that the clique-coclique bound holds with equality. Assume that $C$ is a maximum clique and $S$ is a maximum independent set in $X$ with characteristic vectors $x$ and $y$ respectively. If $E_j$ are the idempotents of the association scheme, then for $j > 0$ at most one of the vectors $E_j x$ and $E_j y$ is not zero.
Proof. If \( j > 0 \), then
\[
x^T E_j x y^T E_j y = 0.
\]
Since \( E_j \) is positive semidefinite, \( z^T E_j z = 0 \) if and only if \( E_j z = 0 \). \( \square \)

3 The Permutation Graph

For a positive integer \( n \) define the permutation graph \( P(n) \) to be the graph whose vertex set is the set of all permutations of an \( n \)-set and vertices \( \pi \) and \( \sigma \) are adjacent if and only if they are not intersecting, that is \( \pi(i) \neq \sigma(i) \) for all \( i \in \{1, \ldots, n\} \). The intersecting families of permutations are exactly the independent sets in \( P(n) \). We will show that the size of the maximum independent set in \( P(n) \) is \( (n - 1)! \) and the only sets that meet this bound are the sets \( S_{i,j} \) from Equation 1.1.

Let \( d(n) \) be the number of derangements of an \( n \)-set (that is the number permutation with no fixed points), then the graph \( P(n) \) is \( d(n) \)-regular. The number of derangements of a set of size \( n \) is defined by the following recursive formula
\[
d(n) = (n - 1) (d(n - 1) + d(n - 2))
\] (3.1)
with \( d(1) = 0 \) and \( d(2) = 1 \).

The permutation graph is a vertex-transitive graph, in fact, \( P(n) \) is a Cayley graph whose connection set is the set of all derangements. Since this set is closed under conjugation, \( P(n) \) is a normal Cayley graph (for more on normal Cayley graphs see [15, Section 5.2]).

Further, the graph \( P(n) \) is a union of graphs in the association scheme known as the conjugacy class scheme on \( S(n) \). The conjugacy class scheme can be constructed for any group \( G \) and is an association scheme on the elements of \( G \). Using the regular representation each element of \( G \) can be expressed as a \( |G| \times |G| \) permutation matrix. For any conjugacy class \( C \) in \( G \) define \( A_C \) to be the sum of the permutation matrices for all the elements in the conjugacy class. Then
\[
\mathcal{A} = \{ A_C : C \text{ a conjugacy class in } G \}
\]
is the conjugacy class scheme on \( G \) (for more on the conjugacy class scheme see [21, page 54]).

If \( \mathcal{A} \) is the conjugacy class scheme for the symmetric group \( S(n) \), then the adjacency matrix of \( P(n) \) is the sum of \( A_C \) over all conjugacy classes \( C \) of derangements. Since \( P(n) \) is the sum of graphs in an
association scheme the clique-coclique bound (Inequality 2.1) holds. With this bound, it is straightforward to get the first statement of Theorem 1.2. This proof of the bound in Theorem 1.2 is included in [4, Theorem 5] and it was also shown by Deza and Frankl [9].

3.1 Theorem. The size of a maximum clique in $P(n)$ is $n$.

Proof. A clique in $P(n)$ can have no more than $n$ vertices. This is clear since the image of 1 (or any other element in $\{1, \ldots, n\}$) must be distinct for each permutation in the clique. Further, each row of a Latin square of order $n$ is a permutation in $S_n$ and the set of all rows in a Latin square of order $n$ is a clique of size $n$ in $P(n)$. Since a Latin square of order $n$ exists for every $n$ the theorem holds. \qed

3.2 Theorem. The size of a maximum independent set in $P(n)$ is $(n - 1)!$.

Proof. Since the graph $P(n)$ is a union of graphs in an association scheme the clique-coclique bound holds for $P(n)$, that is

$$\alpha(P(n)) \leq \frac{|V(P(n))|}{\omega(P(n))}.$$ 

From Theorem 3.1, $\omega(P(n)) = n$ and hence

$$\alpha(P(n)) \leq (n - 1)!.$$ 

Finally, the sets $S_{i,j}$ from Equation 1.1 are independent sets of size $(n - 1)!$. \qed

4 Eigenvalues of $P(n)$

In this section we will find two eigenvalues of the adjacency matrix of $P(n)$. Eigenvalues of the adjacency matrix of $P(n)$ will simply be refer to as the eigenvalues of $P(n)$.

4.1 Lemma. For all positive integers $n$

$$d(n) \quad \text{and} \quad -\frac{d(n)}{n-1}$$

are eigenvalues for $P(n)$. 

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Proof. Consider the independent set $S_{n,n}$ as defined in Equation 1.1. The partition

$$\{S_{n,n}, V(P(n)) \setminus S_{n,n}\}$$

is the orbit partition of $S(1) \times S(n-1)$ acting on the vertices of $P(n)$, hence it is an equitable partition. The quotient graph of $P(n)$ with respect to this partition is

$$
\begin{pmatrix}
0 & d(n) \\
\frac{d(n)}{n-1} & d(n) - \frac{d(n)}{n-1}
\end{pmatrix}.
$$

The eigenvalues of this quotient graph are $d(n)$ and $-\frac{d(n)}{n-1}$. Since the partition is equitable these are also eigenvalues for the graph $P(n)$. □

Since $P(n)$ is a $d(n)$-regular graph, $d(n)$ is the largest eigenvalue of $P(n)$. By Equation 3.1

$$\frac{d(n)}{n-1} = -(d(n-1) + d(n-2))$$

so this eigenvalue is also an integer.

The eigenvalues of a graph can be used to find bounds on the size of the maximum independent sets. In particular, if $X$ is a $d$-regular vertex-transitive graph with least eigenvalue $\tau$ then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}.$$ 

This is known as the ratio bound for independent sets (see [13, Lemma 9.6.2] for a proof). Ku [18] has conjectured that the least eigenvalue of $P(n)$ is $-\frac{d(n)}{n-1}$. If this is true, then the ratio bound gives the first part of Theorem 1.2.

The eigenvalues of a graph in a conjugacy class scheme and the idempotents of the conjugacy class scheme can be determined by the character table of the group. We will state these formulas for the conjugacy class scheme on the symmetric group.

It is well-known that each irreducible character of $S(n)$ corresponds to an integer partition of $n$. To denote that $\lambda$ is an integer partition of $n$, we write $\lambda \vdash n$. If $\lambda \vdash n$, we will represent the character of $S_n$ corresponding to $\lambda$ by $\chi_\lambda$. Each partition $\lambda$ of $n$ corresponds to a module, we will call this the $\lambda$-module. For more on the representation theory of the symmetric group see [10, Chapter 4].
For each \( \lambda \vdash n \) there is a principal idempotent in the scheme. This idempotent is the \( n! \times n! \) matrix whose entries are given by

\[
(E_\lambda)_{\pi, \sigma} = \frac{\chi_\lambda(1)}{n!} \chi_\lambda(\pi^{-1} \sigma)
\]

where \( \pi, \sigma \in S(n) \).

For \( C \) a conjugacy class in \( S(n) \) the eigenvalues of \( A_C \) are

\[
p^\lambda_C = \frac{|C|}{\chi_\lambda(1)} \chi_\lambda(c), \quad c \in C
\]

where \( \lambda \) ranges over all partitions of \( n \) (for a proof of this see [2, Chapter II, Section 2.7]).

It follows from this that the eigenvalues of \( P(n) \) are

\[
\sum_C p^\lambda_C, \quad \lambda \vdash n
\]

where the sum is taken over all conjugacy classes of derangements.

For the partition \( \lambda = [n] \) the value of \( p^\lambda_C \) is \( |C| \) and thus

\[
\sum_C p^{[n]}_C = \sum_C |C| = d(n)
\]

where the sum is taken over all conjugacy classes of derangements.

For any \( x \in S(n) \) the value of \( \chi_{[n-1,1]}(x) \) is one less than the number of fixed points in \( x \), so for \( C \) any conjugacy class of derangements \( p^{[n-1,1]}_C = -\frac{|C|}{n-1} \). Thus

\[
\sum_C p^{[n-1,1]}_C = \sum_C -\frac{|C|}{n-1} = -\frac{d(n)}{n-1}
\]

again, the sum is taken over all conjugacy classes of derangements.

5 The \((n-1)\)-module

For a subset \( S \subseteq S(n) \) let \( v_S \) be the characteristic vector of \( S \) and if \( S \) is one of the independent sets \( S_{i,j} \) defined in Equation 1.1 then we will simply denote \( v_S \) by \( v_{i,j} \). Throughout this section 1 will denote the all-ones vector, the length of 1 will be clear from context.

We will first show that for any maximum independent set \( S \) the vector \( v_S - \frac{1}{n} 1 \) is in the module corresponding to the representation
The next step will be to prove that the vectors $v_{i,j} - \frac{1}{n} \mathbf{1}$ span the $[n-1,1]$-module. Finally we show that any characteristic vector of a maximum independent set that is in this span must be one of $v_{i,j}$ for $i,j \in \{1,\ldots,n\}$.

5.1 Lemma. Let $n$ be an integer with $n > 6$. Let $S$ be a maximum independent set in $P(n)$ and $v_S$ be the characteristic vector of $S$. Then the vector $v_S - \frac{1}{n} \mathbf{1}$ is in the $[n-1,1]$-module.

Proof. First, a simple calculation shows that $v_S - \frac{1}{n} \mathbf{1}$ is orthogonal to $\mathbf{1}$, so this vector is not in the $[n]$-module.

For $\lambda$ an integer partition of $n$ let $\chi_\lambda$ be the character of $S_n$ corresponding to $\lambda$. For $C$ a maximum clique in $P(n)$ define $\chi_\lambda(C) = \sum_{x \in C} \chi_\lambda(x)$.

If $\chi_\lambda(C) \neq 0$, then by Equation 4.1 $E_\lambda v_C \neq 0$. By Corollary 2.2 this implies at $E_\lambda v_S = 0$ which in turn implies that $E_\lambda(v_S - \frac{1}{n} \mathbf{1}) = 0$ for all partitions $\lambda \neq [n]$. This means that the vector $v_S - \frac{1}{n} \mathbf{1}$ is orthogonal to the $\lambda$-module. If this is true for every partition $\lambda \vdash n$ except $[n-1,1]$, then for every maximum independent set $S$ the vector $v_S - \frac{1}{n} \mathbf{1}$ is in the $[n-1,1]$-module. To prove this theorem, we will show for every $\lambda \vdash n$ with $\lambda \neq [n-1,1]$ there is a maximum clique $C$ such that $\chi_\lambda(C) \neq 0$.

For $n > 6$ there is a decomposition of the complete digraph on $n$ vertices into $n-1$ directed cycles [1]. Each of these directed cycles is a cycle of length $n$ in $S_n$. Moreover, no two cycles in the decomposition share an edge so these cycles are adjacent in $P(n)$. Let $T$ be the $n$-clique whose elements are the $n$-cycles in this decomposition together with the identity of $S(n)$.

Since every $x \in T$, except the identity, is an $n$-cycle for every $\lambda \vdash n$ the value of $\chi_\lambda(x)$ is the same. Thus

$$\chi_\lambda(T) = \sum_{x \in T} \chi_\lambda(x) = \chi_\lambda(1) + (n-1)\chi_\lambda(x) \quad x \text{ an } n\text{-cycle.}$$

Further, $\chi_\lambda(x) = \pm 1$ for every character $\chi_\lambda$ (for a proof of this see [22]). Since $\chi_\lambda(1)$ is positive, if $\chi_\lambda(C) = 0$, then $\chi_\lambda(x) = -1$ and $\chi_\lambda(1) = n-1$. For $n > 6$ the only partitions of $n$ with $\chi_\lambda(1) = n-1$ are $[n-1,1]$ and $[2,1^{n-2}]$. 9
If \( n \) is even, then for \( x \) an \( n \)-cycle \( \chi_{[2,1\ldots 2]}(x) = 1 \) so \( \lambda \) must be \([n-1, 1]\).

Finally, if \( n \) is odd we need to prove that \( \lambda \) is \([n-1, 1]\). To do this we construct a clique \( T \) with \( \chi_{[2,1\ldots 2]}(T) \neq 0 \). Consider an \( n \times n \) Latin square with the first row \((1,2,\ldots,n)\) and the second row \((2,1,n,3,4,\ldots,n-1)\). Such a Latin square exists since any Latin rectangle can be extended to a Latin square \([16]\). The rows of this Latin square will then be permutations in our clique. The first row corresponds to the identity permutation, the second to an odd permutation. The first row will contribute \( n-1 \) to the sum \( \chi_{[2,1\ldots 2]}(T) \) and the second row will contribute 1. Each of the last \( n-2 \) permutations will contribute no less than \(-1\) to the sum so the sum cannot be 0. 

Next we give a basis for the \([n-1, 1]\)-module.

5.2 Lemma. For any \( i, j \in \{1, \ldots, n-1\} \) let \( v_{i,j} \) denote the characteristic vector of the independent set \( S_{i,j} = \{ \pi \in S(n) : \pi(i) = j \} \). The vectors \( v_{i,j} - \frac{1}{n} \mathbf{1} \) form a basis for the \([n-1, 1]\)-module.

Proof. From Lemma 5.1 the vectors \( v_{i,j} - \frac{1}{n} \mathbf{1} \) are elements in the \([n-1, 1]\)-module. The dimension of the \([n-1, 1]\)-module is \((n-1)^2\), so we only need to show that these vectors are linearly independent. Since \( \mathbf{1} \not\in \text{span}\{v_{i,j} : i, j \in \{1, \ldots, n-1\}\} \), it is enough to show that the vectors \( v_{i,j} \) are linearly independent.

Order the pairs in \( \{1, \ldots, n-1\} \) so that pair \((i, j)\) occurs before \((k, \ell)\) if \( i < k \) or if \( i = k \) and \( j < \ell \). Let \( H \) be a 01-matrix with size \( n! \times (n-1)^2 \) defined as follows: the columns are indexed by the pairs from the \((n-1)\)-set in the above ordering and the rows are indexed by all the permutations of an \( n \)-set. The \((\pi,(i,j))\)-entry of \( H \) is 1 if and only if \( \pi(i) = j \).

Let \( I_n \) be the \( n \times n \) identity matrix and \( J_n \) the \( n \times n \) all-ones matrix. The adjacency matrix of the complete graph on \( n \) vertices is \( K_n = J_n - I_n \). It is not hard to see with the given ordering on the pairs that

\[
H^T H = (n-1)!I_{(n-1)^2} + (n-2)!(K_{n-1} \otimes K_{n-1}).
\]

Since 0 is not an eigenvalue of this matrix, it has rank \((n-1)^2\). Finally, the rank of \( H \) is equal to the rank of \( H^T H \) and the result holds. 

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6 Proof of Theorem 1.2

Let $H$ be the $n! \times (n - 1)^2$ matrix whose rows are the elements of the symmetric group on $n$ points and columns are the ordered pairs from $\{1, \ldots, n - 1\}$ with the $(\pi, (i, j))$ position of $H$ equal to 1 if $\pi(i) = j$ and zero otherwise.

Denote the columns of $H$ by $h_{i,j}$. By Lemma 5.2, the vectors

$$h_{i,j} - \frac{1}{n} 1, \quad i, j \leq n - 1$$

are a basis for the $[n - 1, 1]$-module. By Lemma 5.1, for any independent set $S$, the vector $v_S - \frac{1}{n} 1$ is in the $[n - 1, 1]$-module. In particular, it is in

$$\text{span} \left\{ h_{i,j} - \frac{1}{n} 1 : i, j \in \{1, \ldots, n - 1\} \right\}.$$

This implies that the characteristic vector of any maximal independent set is in the span of column space of $H$ and $1$.

Let $\sigma$ be the identity permutation on the $n$-set and let $N(\sigma)$ denote the set of permutations adjacent to $\sigma$ in $P(n)$ (these are the derangements). Consider three submatrices of $H$:

(a) $N$ the submatrix whose rows are the permutations in $N(\sigma)$,

(b) $M$ the submatrix of $N$ whose columns are all the pairs $(i, j)$ with $i, j \in \{1, \ldots, n - 1\}$ and $i \neq j$,

(c) $W$ the submatrix of $H$ whose columns are all the pairs $(i, i)$ with $i \in \{1, \ldots, n - 1\}$.

If the columns of $H$ are arranged so that the first $n - 1$ columns correspond to the pairs $(i, i)$ for $i = 1, \ldots, n - 1$, and the rows are arranged so the first row corresponds to the permutation $\sigma$ and the next $d(n)$ rows correspond to the neighbours of $\sigma$, then $H$ has the following block structure:

$$
\begin{array}{cc}
1 & 0 \\
0 & M \\
H_1 & H_2
\end{array}
$$

and the first $n - 1$ columns form the matrix $W$.

6.1 Lemma. For all $n$ the rank of $M$ is $(n - 1)(n - 2)$. 


\[(a, b) \quad \pi_{a,b} \quad 1\rightarrow 2 \quad 1\rightarrow 3 \quad 2\rightarrow 3 \quad 2\rightarrow 1 \quad 3\rightarrow 1 \quad 3\rightarrow 2 \]

\begin{tabular}{|c|c|c|c|c|c|}
\hline
(1,1) & (1,4,2,3) & 0 & 0 & 1 & 0 & 1 \\
\hline
(1,2) & (1,4,3,2) & 0 & 0 & 0 & 1 & 0 \\
\hline
(2,1) & (1,2,4,3) & 1 & 0 & 0 & 0 & 1 \\
\hline
(2,2) & (1,3,2,4) & 0 & 1 & 0 & 0 & 1 \\
\hline
(3,1) & (1,2,3,4) & 1 & 0 & 1 & 0 & 0 \\
\hline
(3,2) & (1,3,4,2) & 0 & 1 & 0 & 1 & 0 \\
\hline
\end{tabular}

Table 1: The submatrix of \(M\) for \(n = 4\).

**Proof.** The matrix \(K_{n-1} \otimes I_{n-2}\) has rank \((n - 1)(n - 1)\); we show it is a submatrix of \(H\). To find this submatrix, we reorder the rows and columns of \(H\).

Order the pairs from \{1, \ldots, n-1\} so that the pair \((i, i + j \mod (n-1))\) occurs before \((k, k + \ell \mod (n-1))\) if \(i < k\) or \(i = k\) and \(j < \ell\). Order the columns of \(M\) with this ordering.

Next we define an ordering on a subset of derangements. Let \(a \in \{1, \ldots, n-1\}\) and \(b \in \{1, \ldots, n-2\}\). Define a permutation of \(\{1, \ldots, n\}\) as follows:

\[
\pi_{a,b}(i) = \begin{cases} 
    n & \text{if } a = i; \\
    i + b & \text{if } a \neq i \text{ and } i + b < n; \\
    i + b + 1 \pmod{n} & \text{if } a \neq i \text{ and } i + b \geq n.
\end{cases}
\]

Note that the value of \(\pi_{a,b}(n)\) is forced.

Order these permutations so that \(\pi_{a_1,b_1}\) occurs before \(\pi_{a_2,b_2}\) if \(a_1 < a_2\) or \(a_1 = a_2\) and \(b_1 < b_2\). Consider the submatrix of \(M\) induced by the rows corresponding to the permutations \(\pi_{a,b}\) for \(a \in \{1, \ldots, n-1\}\) and \(b \in \{1, \ldots, n-2\}\). This submatrix of \(M\) is \(K_{n-1} \otimes I_{n-2}\).

**6.2 Lemma.** If \(y\) is in the kernel of \(N\), then \(Hy\) lies in the column space of \(W\).

**Proof.** Assume \(y\) is in the kernel of \(N\). Let \(y_M\) denote the vector of length \((n - 1)(n - 2)\) formed by taking the final \((n - 1)(n - 2)\) entries of \(y\). Then

\[0 = Ny = [0|M]y = My_M.\]

Since \(M\) has rank \((n - 1)(n - 2)\), the last \((n - 1)(n - 2)\) entries of \(y\) are all 0. Thus \(Hy\) is in the column space of \(W\).
Let \( [N|1] \) be the \( d(n) \times ((n-1)^2 + 1) \) matrix with a column of ones added to \( N \), and \( [M|1] \) the \( d(n) \times ((n-1)(n-2) + 1) \) matrix with a column of ones added to \( M \). As above, for a length \( (n-1)^2 + 1 \) vector \( y \), the vector formed by the last \( (n-1)(n-2) + 1 \) entries of \( y \) will be denoted by \( y_{[M|1]} \).

**6.3 Lemma.** If \( y \) is in the kernel of \( [N|1] \), then \( y_{[M|1]} \) is a scalar multiple of \( (1, 1, \ldots, 1, -(n-2)) \).

**Proof.** As in the previous lemma,

\[
0 = [N|1]y = [0|M|1]y = [M|1]y_{[M|1]}.
\]

Since \( M \) has full column rank, the dimension of the kernel of \( [M|1] \) is at most 1. Each row of \( M \) has exactly \( n-2 \) entries equal to one and all other entries zero, so the vector \( (1, 1, \ldots, 1, -(n-2)) \) is in the kernel of \( [M|1] \) and is a basis for the kernel of \( [M|1] \).

We now have all the tools to prove the second statement of Theorem 1.2.

**Proof of Theorem 1.2.** Let \( S \) be an independent set of size \( (n-1)! \) in \( P(n) \). Assume that the identity permutation \( \sigma \) is in \( S \) and let \( v_S \) be the characteristic vector of \( S \).

By Lemma 5.1, \( v_S \) is in the span \{1, \( h_{i,j} : i, j \leq n-1 \} \). We consider two cases, first when \( v_S \) is in span \{1, \( h_{i,j} : i, j \leq n-1 \} \) and second when it is not.

**case 1.** Assume \( v_S \in \text{span}\{h_{i,j} : i, j = 1, \ldots, n-1\} \), or, equivalently, that \( v_S = Hy \) for some vector \( y \).

Since \( S \) is an independent set no neighbours of \( \sigma \) can be in \( S \) and \( Ny = 0 \). By Lemma 6.2, \( v_S = Wx \) for some vector \( x \).

For any \( i \in \{1, \ldots, n-1\} \) assume the \( i \)-th entry of the vector \( x \) is non-zero. As \( n \geq 3 \), there is a permutation \( \pi \) with \( \pi(i) = i \) and no other fixed points. This means that the entry in the row corresponding to \( \pi \) of \( v_S \) must be equal to the \( i \)-th entry of \( x \). Since \( v_S \) is a 01-vector, \( x \) must also be a 01-vector.

Further, since \( n \geq 4 \), for every pair of distinct \( i, j \in \{1, \ldots, n-1\} \) there is a permutation \( \pi \) that fixes \( i \) and \( j \) but no other points. If the \( i \)-th and \( j \)-th entries of \( x \) are both non-zero then the entry in the row corresponding to \( \pi \) of \( v_S \) is 2. Since \( v_S \) must be a 01-vector, there is only one non-zero entry in \( x \). Thus \( v_S \) is one of the columns of \( W \) and \( S = S_{i,i} \) for some \( i \in \{1, \ldots, n-1\} \).
**case 2.** Assume $v_S$ is not in the column space of $H$. Equivalently, there is some vector $y$ such that $v_S = [H|1]y = Hy_H + c1$ where $y_H$ denotes the vector formed from the first $(n-1)^2$ entries of $y$ and $c$ is a non-zero constant.

As in case 1, no neighbours of $\sigma$ are in $S$ so $[N|1]y = 0$. By Lemma 6.3 there is a non-zero $c$ such that

$$y[M|1] = -\frac{c}{(n-2)}(1,1,\ldots,1,-(n-2)).$$

This determines all entries, up to multiplication by a constant, of $y$ except the first $n-1$.

For each $i \leq n-1$ there is a permutation $\pi$ with $\pi(i) = i$ and no other fixed points. If $y_i$ is the $i$-th entry of $y$ then the entry in $v_S$ corresponding to $\pi$ is

$$y_i + (n-3)\left(-\frac{c}{n-2}\right) + c$$

which must be either 0 or 1. This implies that

$$y_i = -\frac{c}{n-2} \text{ or } y_i = 1 - \frac{c}{n-2}.$$  

Since $n \geq 4$ for any distinct pair $i,j \in \{1,\ldots,n-1\}$ there is a permutation that fixes both $i$ and $j$ and no other points. If both $y_i$ and $y_j$ are equal to $1 - \frac{c}{n-2}$, then the entry in the vector $v_S$ which corresponds to this permutation is

$$2\left(1 - \frac{c}{n-2}\right) + (n-4)\left(-\frac{c}{n-2}\right) + c = 2$$

which is a contradiction since $v_S$ is a 01-vector. Thus at most one of the first $n-1$ entries of $y$ is $1 - \frac{c}{n-2}$.

Next, assume that exactly one of the first $n-1$ entries is $1 - \frac{c}{n-2}$. Since $\sigma \in S$ the sum of the first $n-1$ entries of $y$ is 1. But this means that

$$1 - \frac{c}{n-2} + (n-2)\left(-\frac{c}{n-2}\right) + c = 1,$$

which implies that $c = 0$, a contradiction. Hence, all the entries of $y$, except that last, are $-\frac{c}{n-2}$.

Using the fact that the sum of the first $n-1$ entries of $y$ is 1

$$(n-1)\frac{c}{(n-2)} + c = 1$$
which implies that $c = -(n-2)$.

For case 2 there is only one possibility for $y$, this is

$$y = (1, 1, \ldots, -(n-2)).$$

Every row in $[H|1]$ that corresponds to a permutation that maps $n$ to $n$ has exactly $(n-1)$ entries equal to one and all other entries equal to zero. All the other rows has exactly $(n-2)$ entries equal to one and all other entries equal to zero. From this it follows that $[H|1]y = v_S$ is the characteristic vector of the set $S_{n,n}$. \hfill \Box

## 7 Further Work

We have only considered the simplest version of the Erdős-Ko-Rado theorem. The full version of the Erdős-Ko-Rado theorem is concerned with $t$-intersecting subsets. For an integer $t$, subsets $A, B \subseteq \{1, \ldots, n\}$ are $t$-intersecting if $|A \cap B| \geq t$.

### 7.1 Theorem. (Erdős-Ko-Rado [7])

Let $t \leq k \leq n$ be positive integers. Let $A$ be a family of pairwise $t$-intersecting $k$-subsets of $\{1, \ldots, n\}$. There exist a function $f(k, t)$ such that for $n \geq f(n, k)

$$|A| \leq \binom{n-t}{k-t}.$$\n
Moreover, a $t$-intersecting family $A$ meets this bound if and only if $A$ is the collection of all $k$-subsets that contain a fixed $t$-subset.

Permutations $\pi, \sigma \in S(n)$ are $t$-intersecting if

$$|\{i \in \{1, \ldots, n\} : \pi(i) = \sigma(i)\}| \geq t.$$\n
Again, there is an obvious family of candidates for the maximum system of $t$-intersecting permutations. Assume

$$A = \{(x_i, y_i) : i = 1, \ldots, t \text{ and } x_i, y_i \in \{1, \ldots, n\}\}$$

with $x_i \neq x_j$ and $y_i \neq y_j$ for all $i \neq j$. Then the family

$$S_A = \{\pi : \pi(x_i) = y_i \text{ for all } (x_i, y_i) \in A\}.$$

is $t$-intersecting and $|S_A| = (n-t)!$.

Deza and Frankl [9] conjecture that Theorem 7.1 can also be extended to families of $t$-intersecting permutations.
7.2 Conjecture. (Deza and Frankl [9]) For \( n \) sufficiently large, the size of the maximum set of permutations of an \( n \)-set that are pairwise \( t \)-intersecting is \( (n - t)! \).

Cameron and Ku note that their method cannot be extended to \( t \)-intersecting permutations. It is possible that the proof presented in this paper may be extended as follows.

Define a graph \( P_t(n) \) whose vertices are the permutations of an \( n \)-set, where two vertices are adjacent if they agree on no more than \( t \) points. Note that \( P(n) = P_0(n) \).

The graph \( P_t(n) \) is a sum of all \( A_C \) where \( C \) is a conjugacy classes in which the elements have no more than \( t \) fixed points.

The graph \( P_t(n) \) is vertex transitive so we have that

\[
\alpha(P_t(n))\omega(P_t(n)) \leq n!
\]

This is Equation 3 in Deza and Frankl [9]. They also note that if there exists a sharply 2-transitive set of permutations of \( \{1, \ldots, n\} \) (say \( PGL(2, n) \)) then there is a clique of size \( n(n - 1) \) and we have the bound on the 2-intersecting permutations.

We conjecture that the shifted characteristic vector of a 2-intersecting permutation family lies in a union of modules. Define the depth of a partition \( \lambda \vdash n \) with \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) to be \( n - \lambda_1 \).

7.3 Conjecture. Let \( v_S \) be the characteristic vector of a maximum independent set in \( P_1(n) \). Then the vector \( v_S - \frac{|S|}{n!}1 \) lies in the sum of the modules whose partitions have depth no more than 2. That is the sum of the following modules

\[
\begin{align*}
[n], & \quad [n - 1, 1], & \quad [n - 2, 2], & \quad [n - 2, 1, 1].
\end{align*}
\]

The dimensions of the sum of these modules and the dimension of the span of \( v_A - \frac{|S|}{n!}1 \) agree for \( n = 4, 5, 6 \) where \( A = \{(i, j), (k, \ell)\} \).

This conjecture can be generalized to \( t \)-intersecting permutation systems.

7.4 Conjecture. Let \( v_S \) be the characteristic vector of a maximum independent set in \( P_t(n) \). Then the vector \( v_S - \frac{|S|}{n!}1 \) lies in the sum of the modules whose partitions have depth no more than \( t \).

Finally, the proof of the Erdős-Ko-Rado theorem for permutations given in this paper is an application of a method that has been used
to prove the Erdős-Ko-Rado theorem for set systems and its analogue for intersecting vector spaces over a finite field. Another direction for this work is to apply this method to other objects such as perfect matchings and uniform partitions with a plan of developing a more general theory of Erdős-Ko-Rado theorems.

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