Polytropic gas spheres: An approximate analytic solution of the Lane-Emden equation

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ABSTRACT

Polytropic models play a very important role in galactic dynamics and in the theory of stellar structure and evolution. However, in general, the solution of the Lane-Emden equation can not be given analytically but only numerically. In this paper we give a good analytic approximate solution of the Lane-Emden equation. This approximation is very good for any finite polytropic index $n$ and for the isothermal case at a level $<1\%$. We also give analytic expressions of the mass, pressure, temperature, and potential energy as a function of radius.

Key words: star:evolution, star:neutron, star:white dwarfs, Galaxy:kinematics and dynamics, galaxy: structure

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1. Introduction

As stellar encounters are little important for galaxies, clusters of galaxies, or globular clusters (Binney & Tremaine 1987), the fundamental dynamics describing these systems is that of a collisionless system. In the general case the collisionless Boltzmann equation cannot be solved because it involves too many independent variables. However, we can get certain exact solutions of the collisionless Boltzmann equation for a subset of possible stellar-dynamical equilibria. The system with a polytropic state equation, corresponding to isotropic velocity dispersion tensors, is one of them. In stellar structure and evolution theory, the polytropic model also plays an important role (Chandrasekhar 1939).

For a polytropic system, the relation of pressure $P$ and density $\rho$ is given by

$$ P = K \rho^n \equiv K \rho^{1+\frac{1}{n}} , $$

where $K$ is the polytropic constant, $\gamma$ is the adiabatic index, and $n$ the polytropic index. $K$ is fixed in the degenerate gas sphere (e.g. in a white dwarf or in a neutron star) and free in a non-degenerate system. In galactic dynamics $n$ is larger than $\frac{1}{2}$ (Binney & Tremaine 1987) which means that no polytropic stellar system can be homogeneous. $n$ ranges from 0 to $\infty$ in the case of the theory of stellar structure and evolution (Kippenhahn & Weigert 1989, Chandrasekhar 1939). With the polytropic relation (1), hydrostatic equilibrium, and Poisson’s equation for gravitational potential field, we derive the Lane-Emden equation (see, for instance, Binney & Tremaine 1987, Kippenhahn & Weigert 1989) for spherical symmetry:

$$ \frac{d^2 \omega}{d \xi^2} + \frac{2}{\xi} \frac{d \omega}{d \xi} = -\omega^n . $$

The dimensionless variables $\omega$ and $\xi$ are defined by

$$ \xi = Ar , \quad A^2 = \frac{4\pi G}{(n+1)K} \rho_c \frac{n-1}{n} , $$

$$ \omega = \left( \frac{\rho}{\rho_c} \right)^{1/n} , $$

where $\rho_c$ is the density at the center of the sphere and $G$ the Newtonian gravitational constant. $\omega$ corresponds to the dimensionless gravitational potential. We have excluded the isothermal case $n = \infty$, which we will discuss in detail in a later section.

The Lane-Emden equation must be solved with the original central conditions:

$$ \omega(0) = 1 , \quad \left( \frac{d \omega}{d \xi} \right)_{\xi=0} = 0 , $$

which will ensure the regularity of the solution at the center. The solution gives $\omega$ as a function of $\xi$. From $\omega$ we can get the density profile $\rho$. Only for the three values of $n = 0, 1, \text{and} 5$ can the solution be given in analytic form. Apart from these three cases the Lane-Emden equation has to be solved numerically. As the Lane-Emden equation has a regular singularity at $\xi = 0$, we can expand $\omega(\xi)$ in a power series as

$$ \omega(\xi) = 1 - \frac{1}{6} \xi^2 + \frac{n}{120} \xi^4 + \ldots $$
From (2) and (4) we have \( \frac{2}{\xi} \frac{d \omega}{d \xi} = -\frac{2}{3} = 2 \delta \frac{d^2 \omega}{d \xi^2} \) at \( \xi = 0 \). In this paper, we make the approximation of taking the second derivative term as \( \delta \frac{d^2 \omega}{d \xi^2} \) to obtain a good approximate analytic solution of the Lane-Emden equation. We give our analysis for the isothermal Lane-Emden equation in section 2, comparing our results with numerical solution. The general case is dealt with in section 3 followed by discussion and conclusion in section 4.

2. Lane-Emden equation for isothermal sphere

The Lane-Emden equation for an isothermal sphere (see Kippenhahn & Weigert 1989, Binney & Tremaine 1987) can be written as

\[
\frac{d^2 \omega}{d \xi^2} + \frac{2 \omega}{\xi} = e^{-\omega},
\]

(6)

where \( \omega \) is related to the mass density at dimensionless radius \( \xi \) by

\[
\varrho = \varrho_c e^{-\omega},
\]

(7)

and \( \varrho_c \) is the density at the origin. It is easy to find (see also Binney & Tremaine 1987) that

\[
\omega = -\ln \left( \frac{2}{\xi^2} \right),
\]

(8)

is one of the solutions of Eq. (6). This solution describes a model known as the singular isothermal sphere. Unfortunately, the singular isothermal sphere has infinite density at \( \xi = 0 \). To obtain a solution that is well behaved at the origin, equation (6) has to be integrated with the central conditions

\[
\omega(0) = 0, \quad \left( \frac{d \omega}{d \xi} \right)_{\xi=0} = 0.
\]

(9)

Its solution can not be given by analytic expressions but only numerical computation. The Lane-Emden equation has a regular singularity at \( \xi = 0 \). In order to understand the behaviour of the solutions there, a power series expansion similar to (5) can be derived and has to be used. For large radius where the effect of the central conditions is very weak the solution should asymptotically approach the singular isothermal solution. The isothermal sphere consisting of an ideal gas has an infinite radius as well as an infinite mass.

At small radius, a useful approximation to \( \varrho(\xi) \) is the modified Hubble law (Binney & Tremaine 1987), which was introduced empirically by King (1962),

\[
\left( \frac{\varrho}{\varrho_c} \right) = \frac{1}{\left( 1 + \frac{\xi^2}{\eta} \right)^{\frac{3}{2}}},
\]

(10)

Comparing this relation with a numerical solution of (6) one can say that for \( \xi \lesssim 5 \) the relative error is less than 5 %. Expression (10) does not fit the isothermal profile well at \( \xi \)
\( \geq 9 \) as it approaches asymptotically to a logarithmic slope \(-3\) and not \(-2\), as is required by the isothermal profile.

Since \( \left( \frac{d^2 \omega}{d\xi^2} \right)_{\xi=0} = 1/3 = \frac{1}{2} \left( \frac{2 d\omega}{\xi d\xi} \right)_{\xi=0} \) at \( \xi = 0 \) from (6), it is reasonable to approximate the second derivative term in (6) with

\[
\frac{1}{2} \left( \frac{2 d\omega}{\xi d\xi} \right)
\]
as our first approximation and to get a first differential equation

\[
\frac{3}{2} \frac{2 d\omega}{\xi d\xi} = e^{-\omega}, \quad (11)
\]

which can be integrated immediately, giving

\[
\omega_1 = \ln \left( 1 + \frac{\xi^2}{6} \right), \quad (12)
\]

and

\[
\frac{\theta_1}{\theta_c} = e^{-\omega_1} = \frac{1}{1 + \frac{\xi^2}{6}}, \quad (13)
\]

where we have used the condition (9) to determine the integration constant. The subscript 1 indicates a first approximation.

When \( \xi \) is very large, Eq. (12) becomes

\[
\omega_1 \sim \ln \left( \frac{\xi^2}{6} \right), \quad (14)
\]

which has a very similar profile to (8) and the same tangent as the numerical solution. Figure 1 shows the numerical solution of equation (6) and our first approximation (13), which shifted by 1/3 approximately reaches the singular solution for large \( \xi \).
The first order (dashed line indicated by $\frac{\rho_1}{\rho_c}$) and the second order approximation (dashed line indicated by $\frac{\rho_2}{\rho_c}$) of the density for the isothermal case. The first order approximation has been multiplied by $1/3$ to show its same tangent as the numerical result (solid line) for large $\xi$.

Taking the second derivative of $\omega_1$ as our approximation to the second derivative term and replacing the right hand side term in equation (6) with (13), we obtain the second approximation to the solution of equation (6)

$$\omega_2 = 2 \ln \left( 1 + \frac{\xi^2}{6} \right) - \left( \frac{\xi^2}{6} \right) \frac{1}{1 + \frac{\xi^2}{6}} + \text{cons},$$

or

$$\frac{\rho_2}{\rho_c} = e^{-\omega_2} = \text{cons} \left( 1 + \frac{\xi^2}{6} \right)^{-2} e^{\left( \frac{\xi^2}{6}\right)/\left( 1 + \frac{\xi^2}{6} \right)},$$

where subscript 2 indicates the second approximation. Integration constant $\text{cons}$ in (15) is determined by the central condition (9) and is zero. Equations (16) as an approximation to the solution of (6) is good for small $\xi$. For large $\xi$, the approximation has an asymptotic behavior, $\frac{\rho}{\rho_c} \propto \xi^{-4}$, which does not fit the isothermal profile as the same reason for the modified Hubble law discussed above. We also show the second approximation in figure 1.
As the first order approximation (12) is good for large $\xi$ and the second order approximation (15) is good for small $\xi$, we combine them and obtain a more general approximation

$$
\omega_0 = \alpha \omega_2 + (1 - \alpha) \omega_1 = (1 + \alpha) \ln \left(1 + \frac{\xi^2}{6}\right) - \frac{\xi^2}{6} \frac{\alpha}{1 + \frac{\xi^2}{6}} + \text{cons} \quad .
$$

(17)

where $0 \leq \alpha \leq 1$, coming from the condition (9), which requires $\omega = 0$ at $\xi = 0$. For very large $\xi$, equation (17) asymptotically approaches

$$(1 + \alpha) \ln \left(\frac{\xi^2}{6}\right) - \alpha + \text{cons},$$

but not equation (8). So, we change the constant term in (17) to

$$-\alpha \ln \left(1 + \frac{\xi^2}{A}\right)$$

to make equation (17) approach (8) for very large $\xi$. From (8) and (17) $A$ takes $3^{1/\alpha} 12e$. Defining the general approximation as $\omega = \omega_0 + \Delta \omega$ and substituting it in equation (6), we get the final analytic approximation

$$
\omega = (1 + \alpha) \ln \left(1 + \frac{\xi^2}{6}\right) - \frac{\xi^2}{6} \frac{\alpha}{1 + \frac{\xi^2}{6}}
- \alpha \ln \left(1 + \frac{\xi^2}{3^{1/\alpha} 12e}\right) + \ln \left(1 + (2^{-\alpha} - 1) \frac{D\xi^2}{1 + D\xi^2}\right),
$$

(18)

and

$$
\frac{\varrho}{\varrho_c} = \frac{\left(1 + \frac{\xi^2}{6}\right)^{\alpha}}{1 + (2^{-\alpha} - 1) \frac{D\xi^2}{1 + D\xi^2}} e^{\alpha(\xi^2/6)/(1 + \xi^2/6)} ,
$$

(19)

where $e$ is the natural constant, and $D$ a constant depending on $\alpha$. Equation (19) reaches the singular solution for $\xi \to \infty$. The discrepancy of (19) to the numerical solution is therefore at its largest value when $\xi$ is intermediate. Adjusting $\alpha$ and $D$, we can modify the fitness for intermediate $\xi$. One good combination is $\alpha = 0.551$ and $D = 3.84 \times 10^{-4}$, which makes the largest relative error of equation (19) to the numerical solution of (6) be 0.72%. In figure 2, we compare the structure of densities obtained both numerically and analytically and show the relative error $(\varrho_n - \varrho)/\varrho_n$, where $\varrho_n$ is the numerical computation solution of (6). For $\xi \to \infty$, the relative error approaches zero. As in the isothermal case $\omega$ has no physical meaning, we give only the result for density $\varrho$. The largest error for $\omega$ is also about 0.72%.
Fig. 2 The combination of the first and second approximations with $\alpha = 0.551$ and $D = 3.84 \times 10^{-4}$ (dashed line) with its relative error to the numerical result (solid line) for the isothermal case. It’s difficult to distinguish the two lines by eyes. The relative error is less than 0.72%.

3. Lane-Emden equation for general case

We can apply a similar method for the general case. We first discuss the $n \neq 1$ case. The limit case $n = 1$ will be discussed later. For $n \neq 1$, equation (2) also has a singular solution as in the isothermal case

$$\omega = \left( \frac{(1-n)^2}{2(n-3)} \xi^2 \right)^{1/(1-n)}.$$  \hspace{1cm} (20)

When $n > 5$, the solution of equation (2) under central condition (4) should asymptotically approach (20) for $\xi \to \infty$. We approximately solve Eq. (2) with the same technique as for the isothermal sphere. We approximate the second derivative term in Eq. (2) with $\frac{\delta^2}{\xi^2} \frac{d\omega}{d\xi}$ and get

$$\frac{2 + \delta}{2} \frac{d\omega}{\xi} = -\omega^n,$$  \hspace{1cm} (21)

where $\delta$ is a constant. $\delta$ can affect the fitness of the approximation strongly for $n < 1$ but not for $n > 1$. We can however get very good approximation if fixing $\delta$ and adjusting
other parameters (see below) for any \( n \). Therefore, we will always take \( \delta = 1 \). Integrating equation (21), we get
\[
\omega_1 = (1 + A_n \xi^2)^{1/(1-n)} ,
\]
where \( A_n \) is \((n - 1)/6\). Here, we have used the central condition (4) to determine the integration constant. However, this first analytic approximation does not have the right behaviour for large \( \xi \) compared to the numerical solution of equation (2). Equations (22) for any \( n \geq 1 \) can not reach zero at finite \( \xi \) and give an infinite radius. Integrating Eq. (2) gives
\[
\omega = -\int \frac{\xi}{2} \omega^n d\xi - \int \frac{\xi}{2} \frac{d^2 \omega}{d\xi^2} d\xi .
\]
Substituting (22) in the right-hand-side of Eq. (23) and integrating, we get our second approximate analytic solution:
\[
\omega_2 = \text{cons} + 2 \left( 1 + A_n \xi^2 \right)^{1-n} + \frac{\xi^2}{6} \left( 1 + A_n \xi^2 \right)^{n/(1-n)} ,
\]
where the constant \( \text{cons} \) is determined by the central condition (4) as \(-1\). This equation, however, reaches zero at finite \( \xi \) for any finite \( n \) and gives a radius smaller than the one obtained from numerical computation.

Since solution (22) gives too large values and (24) gives too small values for large \( \xi \), we construct a more general approximation for \( \omega \) as a linear combination of equations (22) and (24)
\[
\omega_0 = \alpha \omega_2 + (1 - \alpha) \omega_1 \\
= -\alpha + (1 + \alpha) \left( 1 + A_n \xi^2 \right)^{1/(1-n)} + \frac{\alpha \xi^2}{6} \left( 1 + A_n \xi^2 \right)^{n/(1-n)} .
\]
We would like to point out that if we keep \( \delta \) in (21) as a free parameter, then equation (25) gives a good approximation to the solution of (2) at level \(< 1\%\) with proper \( \delta \) and \( \alpha \) for \( n < 1 \). In this case \( \delta \) varies from 0 to 1. We notice that (25) asymptotically reaches the constant \(-\alpha\) not zero when \( \xi \rightarrow \infty \) for \( n \geq 5 \). As (25) should approaches the singular solution at very large \( \xi \) (see also, Eggleton, 1995) as in the isothermal case, we change the constant \(-\alpha\) in Eq. (25) to \(-\alpha \left( 1 + B_n \xi^2 \right)^{1/(1-n)}\) and let (25) for \( \xi \rightarrow \infty \) equal to the singular solution (20) when \( n \gg 5 \) and to zero when \( n = 5 \). We get
\[
B_n = \begin{cases} 
A_n \alpha^{n-1} \left( 1 + \alpha \frac{n}{n-1} \left( \frac{2(n-5)}{9(n+1)} \right)^{1/(n-1)} \right)^{1-n} , & \text{for } n \geq 5 \\
\frac{n(n-1)}{(n+1)^2} \frac{6}{5} \left( \frac{4\alpha}{4+5\alpha} \right)^4 , & \text{for } n < 5 
\end{cases}
\]
where we have considered the conditions that \( B_n \) should equal zero at \( n = 0 \) and 1 and continue at \( n = 5 \) to get the expression for \( n < 5 \). With \( B_n \) given by (26), (25) gives a good approximation with relative error less than 5\% for \( n \lesssim 5 \) and about 1\% for \( n \gg 5 \).
Fig. 3 The analytic approximation before a $\Delta \omega$ modification given by equation (29) is fairly good. Here is the result for $n = 3/2$ as an example. For $n = 3/2$ case, the relative error is less than about 1% when $\alpha$ is taken as 0.44.

Figure 3 gives the results for $n = 1.5$ as an example. For $n = 1.5$, the relative error is less than about 1% when $\alpha = 0.44$.

So, if our general solution is

$$\omega = \omega_0 + \Delta \omega,$$

$\Delta \omega$ would be very small. From Eqs.(2) and (25) as well as (26), we get at very large $\xi$ for $n = 5$ and at $\xi \to \xi_n$ for $n < 5$,

$$\Delta \omega \sim \frac{1}{\xi},$$

as where $\omega_0 \sim 0$. In order to avoid the singularity of $\Delta \omega$ at $\xi = 0$, we set

$$\Delta \omega = \frac{C_n \xi^{2\beta-1}}{(1 + D_n \xi^{\beta})^2},$$

where $C_n$ and $D_n$ are small constants and the index $\beta$ determined from Eq.(2) is

$$\beta = 6.47 - 7.01 \beta_1 + 5.53 \beta_1^2 - 25.63 \beta_2 + 49.42 \beta_2^2 - 26.88 \beta_2^3,$$
where $\beta_1 = \frac{1}{1 + (n - 5)^2}$ and $\beta_2 = \frac{1}{1 + (n - 3)^2}$. With the same procedure and the same reason as for $n \leq 5$, we have for $n > 5$

$$\Delta \omega \simeq \frac{E_n F_n \xi^{\eta + \frac{2}{n}}}{1 + F_n \xi^{\eta}} + \frac{C_n \xi^{2\beta - 1}}{(1 + D_n \xi^\beta)^2}, \quad (31)$$

where the index $\eta$ is $(n - 2) / (n - 4)$ and the coefficient $E_n$ determined from equation (2) is

$$E_n \simeq \frac{1}{n - 4} A_n^{1-n} \left( \frac{2(n - 5)}{9(n + 1)} \right)^{\frac{n-3}{n-1}} \left( \left( \frac{n - 3}{3(n - 1)} \right)^{\frac{n-4}{n-1}} - \left( \frac{2(n - 5)}{9(n + 1)} \right)^{\frac{n-4}{n-1}} \right), \quad (32)$$

with which our approximation always asymptotically approaches the singular solution (20) at large $\xi$ for $n > 5$. To fit the solution for intermediate $\xi$, the parameter $F_n$ can be given as

$$F_n \simeq \begin{cases} 
\frac{(n-5)^6}{(n-3)^6(4n+50)}, & \text{for } n \geq 5 \\
0, & \text{for } n < 5 
\end{cases}, \quad (33)$$

Finally, we configure a general result for any $n$

$$\omega = -\alpha \left( 1 + B_n \xi^2 \right)^{1/(1-n)} + (1 + \alpha) \left( 1 + A_n \xi^2 \right)^{1/(1-n)} + \frac{\alpha}{6} \xi^2 \left( 1 + A_n \xi^2 \right)^{n/(1-n)} + \frac{E_n F_n \xi^{\eta + \frac{2}{n}}}{1 + F_n \xi^{\eta}} + \frac{C_n \xi^{2\beta - 1}}{(1 + D_n \xi^\beta)^2}, \quad (34)$$

where $\eta$ is $(n - 2) / (n - 4)$, $A_n$ is $(n - 1)/6$, $B_n$ is given by (26), $E_n$ by (32) and $\beta$ by (30). The free parameters $\alpha$, $F_n$, $C_n$ and $D_n$ should be determined by fitting the numerical solution of equation (2). (34) with proper values of the parameters $\alpha$, $C_n$, $D_n$ and $F_n$ can give a very good approximation solution of equation (2) under the central condition (4).

For $n >> 5$, equation (34) tends to the singular solution as $\xi \to \infty$. In this case, (34) always approximates the solution of Eq.(2) with $< 0.1\%$. Especially when $n > 10$, the last term and when $n \gtrsim 500$ the last two terms in equation (34) are not important and can be ignored and the approximation is within the level $0.1\%$. As $n$ decreases, (34) slowly deviates the singular solution to finally reach $\omega = 1/(1 + \xi^2/3)^{1/2}$ at $\xi$ very large for $n = 5$ where the sum of the first three terms in (34) approach zero in accordance with (25) and (26). (34) reaches zero at infinity for $n \geq 5$ and at finite $\xi_n$ for $n < 5$. All the properties of (34) discussed above are also the properties of the exact solution of (2) (a detail discussion to the general properties of the solution of equation (2), see Binney & Tremaine (1987), Kippenhahn & Weigert (1989)).
Again for \( n = \frac{3}{2} \) but after the modification given by equation (29). The figure shows the variation of potential \( \omega \) and density \( \rho \) with dimensionless radius \( \xi \). The relative approximation error of \( \omega \) to numerical computation \( \omega_n \) is also shown. As in figure 2, it is difficult to distinguish the solid and dashed lines. The largest error is 0.12\%.

(34) is the exact solution for \( n = 0 \). For general \( n \), we have to adjust \( \alpha, C_n, D_n \) and \( F_n \) to get a good approximation. Equation (34) gives a good approximation to the solution of equation (2) for small and large \( \xi \), which is insensitive to the parameters within proper ranges of values. For intermediate \( \xi \), however, special \( \alpha, C_n, D_n \) and \( F_n \) are needed to reduce the error of the approximation. In figures 4 and 5, we show the variations of potential \( \omega \), density \( \rho \) and relative error \( (\omega_n - \omega)/\omega_n \) with radius \( \xi \) for \( n = 3/2 \) and for \( n = 5 \), respectively. For \( n = 3/2 \), the relative error is larger near the finite radius \( \xi_n \) and oscillates around the mean value zero. Such oscillation behavior also exists in figure 5 for \( n = 5 \). In fact, the oscillation always exists for any \( n \). We give the values of the parameters used to get figures 4 and 5 and the largest relative fitting errors \( \epsilon_{\text{max}} \) in table 1. In table 1, we give the values of the parameters for several polytropic indices \( n \) and the largest fitting errors \( \epsilon_{\text{max}} \). We also give the best fitting value for \( F_n \) in the table. With the values of \( \alpha, C_n \) and \( D_n \) and the value of \( F_n \) given by equation (33), equation (34) can also give an approximation at level \( \epsilon_{\text{max}} < 1\% \).
Fig. 5 Results for $n = 5$. For $n \geq 5$, gas sphere has an infinite radius and approaches its singular solution when $\xi$ is very large. For $n = 5$, the equation has an analytic solution. The figure compares the approximate and the exact analytic solutions and shows their relative difference. For $\xi \to 0$ or $\xi \to \infty$, the difference approaches zero. The same as in the cases for $n = \infty$ and for $n = 3/2$, it is very difficult to distinguish the solid and dashed lines. The largest difference is 0.34%.

For $n < 5$, the relative error at $\xi \to \xi_n$ is larger as the numerical solution $\omega_n$ approaches zero. (34) with $B_n$ given by (26) reaches zero at finite $\xi$ and is the exact solution for $n = 0$. For $n \geq 5$, the error is at its maximum when $\xi$ is in the range of around 1 to 20, depending on index $n$. When $\xi$ is very large, the approximation approaches the singular solution. With the increasing distance to the center of the gas sphere, the effect of the central boundary (4) on the structure of the sphere decreases. Therefore, the solution under the central boundary condition (4) approaches the singular solution. The approximation is very good even for $\xi \to \infty$. For $n > 10$, $C_n$ can be set to zero. For $n > 500$, $E_n$ is less than $10^{-3}$ and $\Delta \omega$ given by equation (31) becomes unimportant.

For $n = 1$, integrating (21) and considering the central boundary condition (4) we get the first approximation

$$\omega_1 = e^{-\xi^2/6}.$$  

Then, with the same procedure as for general $n$, we get our second approximation
Table 1. Parameters for several polytropic indices $n$

$n$ polytropic index, $\alpha$ mixing parameter of the first and the second approximations, $\epsilon_{max}$ (given in percent) maximum relative error of analytic approximation to numerical calculation result. For other parameters, see the text.

| $n$ | $\alpha$ | $C_n$          | $D_n$          | $F_n$ | $\epsilon_{max}$(%) |
|-----|----------|----------------|----------------|-------|---------------------|
| 0.5 | 0.5      | $1.30659 \times 10^{-4}$ | $2.3 \times 10^{-4}$ | 0     | 0.37                |
| 1   | 0.455    | $1.27746 \times 10^{-4}$ | 0              | 0     | 0.15                |
| 1.5 | 0.481    | $4.61841 \times 10^{-4}$ | $1.1 \times 10^{-3}$ | 0     | 0.12                |
| 2   | 0.512    | $8.3218 \times 10^{-4}$ | $2.56 \times 10^{-2}$ | 0     | 0.28                |
| 3   | 0.53     | $5.56215 \times 10^{-4}$ | $2.745 \times 10^{-2}$ | 0     | 0.60                |
| 5   | 0.545    | $1.9415 \times 10^{-3}$ | $3.348 \times 10^{-2}$ | 0     | 0.34                |
| 6   | 0.56     | $2.0 \times 10^{-4}$ | $3.0 \times 10^{-2}$ | $5.3 \times 10^{-5}$ | 0.50    |
| 7   | 0.526    | $1.53 \times 10^{-7}$ | $1.1 \times 10^{-3}$ | $7.40 \times 10^{-4}$ | 0.39     |
| 8   | 0.540    | $6.1 \times 10^{-8}$ | $1.0 \times 10^{-3}$ | $2.7 \times 10^{-3}$ | 0.35     |
| 9   | 0.546    | $3 \times 10^{-8}$ | $9 \times 10^{-4}$ | $5.4 \times 10^{-3}$ | 0.26     |
| 10  | 0.538    | $1.12 \times 10^{-8}$ | $9 \times 10^{-4}$ | $8.91 \times 10^{-3}$ | 0.24     |
| 20  | 0.613    | 0              | ...            | $4.2 \times 10^{-2}$ | 0.12     |
| 50  | 0.65     | 0              | ...            | 0.101 | 0.060               |
| 500 | 0.58     | 0              | ...            | 0.19  | 0.021               |

$$\omega_2 = -1 + 2e^{-\xi^2/6} + \frac{\xi^2}{6}e^{-\xi^2/6}. \quad (36)$$

(35) and (36) are also the limit of expressions (22) and (24) for $n \to 1$, so the $\omega$ is continuous at $n = 1$. From the limit case of equations (26) and (34) for $n = 1$, we therefore get the final result for the $n = 1$

$$\omega = -\alpha e^{-\frac{\alpha}{10}(\frac{4}{10})^4\xi^2}+(1+\alpha) e^{-\frac{\xi^2}{6}} + \frac{\alpha}{6} \xi^2 e^{-\frac{\xi^2}{6}} + \frac{C_n \xi^{2\beta-1}}{(1 + D_n \xi^\beta)^2}, \quad (37)$$

where $\beta$ is given by equation (30), $C_n$ and $D_n$ are constant. In table 1, we also give a set values of $C_n$ and $D_n$ and the largest approximate error of equation (37). For $n = 1$, the largest relative error of the approximation is only 0.15%.

4. Discussion and conclusion

We gave our analytic approximation solutions to the Lane-Emden equations for isothermal sphere in section 2 and for general polytropic cases in section 3. From the analytic results we can derive some useful relations.

From (3) and (2) for the general case and from (7) and (6) for the isothermal case, the mass contained in a sphere of radius $r$

$$m = \int_0^r 4\pi r^2 gdr \quad \Rightarrow \quad 4\pi g_c r^3 \left( \pm \frac{1}{\xi} \frac{d\omega}{d\xi} \right), \quad (38)$$
where $-$ sign is for the general case and $+$ for the isothermal case. Dimensional $r$ and non-dimensional $\xi$ are related for the general case by equation (3) and for the isothermal case by

$$\xi = \left(\frac{4\pi G \rho_c}{K}\right)^{1/2} r = Ar \quad .$$

(39)

For the general case and $n \neq 1$, it follows from (34) that

$$m(r) = 4\pi \rho_c r^3 \left(\frac{2\alpha B_n}{1-n} (1 + B_n \xi^2)^{\frac{n}{1-n}} + \frac{1}{3} \left(1 + A_n \xi^2\right)^{\frac{n}{1-n}} + \frac{n\alpha}{18} \xi^2 \left(1 + A_n \xi^2\right)^{\frac{n}{1-n} - 1}\right)$$

$$- E_n F_n \xi^n + \frac{2}{1-n} \ln \left(1 - \frac{F_n \xi^n}{(1 + F_n \xi^n)^2}\right) - C_n \xi^{2\beta - 3} 2\beta - 1 - D_n \xi^\beta \right) .$$

(40)

For $n < 5$, the star has a finite radius $R$ or $\xi_n$, which is estimated from (34) by setting $\omega$ to zero, and a finite mass $M$. From (40) we can get the expression for mass $M$ from which the central density $\rho_c$ is determined. However, for $n \geq 5$ the polytropic gas sphere has infinite radius and infinite mass. In this case we cannot estimate the central density directly. From equations (1)and (3) we have

$$P = K \rho_c^{\frac{n+1}{n}} \omega^{n+1} \quad ,$$

(41)

where $\omega$ is given by (34) and $K$ is determined by equation (3) as $A$ is given by $A = \xi_n / R$ for $n < 5$. For an ideal gas the temperature is given as

$$T = \frac{\mu m_p}{k} \frac{P}{\rho} = \frac{\mu m_p}{k} K \rho_c^{1/n} \omega \quad ,$$

(42)

where $\mu$ is the mean molecular weight, $m_p$ is the mass of atomic Hydrogen, and $k$ is the Boltzmann constant. From the definition of potential energy within radius $r$ of the sphere

$$E_g \equiv - \int_0^m Gm' \frac{dm'}{r'} \quad ,$$

(43)

the polytropic relation (1), and the hydrostatic equilibrium equation, we get for $n \neq 5$

$$E_g = - \frac{3}{5 - n} \frac{G m^2}{r} - \frac{3}{5 - n} \Phi m - \frac{n + 1}{5 - n} 4\pi r^3 P \quad ,$$

(44)

where $\Phi$ is gravitational potential, given by

$$\Phi = -(n + 1) K \rho_c^{\frac{1}{n}} \quad .$$

(45)

This means that for $n < 5$ the polytropic sphere has finite radius and the potential is set to zero at the surface. For $n \geq 5$ the potential becomes zero at infinity.
In polytropic models, $n = 3$ and $n = 3/2$ are two important cases. We have given $n = 3/2$ as one example before. Here, as another example, we construct a polytropic model of index 3 of the sun ($M = 1.989 \times 10^{33} g$ and $R = 6.96 \times 10^{10} cm$) and compare our results with that from numerical computation. For $n = 3$, from table 1 and equations (26) and (30), we have

\begin{align*}
\alpha &= 0.53, \\
\beta &= 2.199, \\
B_n &= 4.648 \times 10^{-3}, \\
C_n &= 5.56215 \times 10^{-4} and D_n = 2.745 \times 10^{-2}.
\end{align*}

Then we get $\xi_3 = 6.897$ from (34) and $A = \xi_3/R = 9.895 \times 10^{-11}$. Substituting these in equation (40) we get central density $\rho_c = 76.96 g cm^{-3}$, $K = 3.87 \times 10^{14}$ from equation (3) and consequently, $P_c = 1.26 \times 10^{17} dyn/cm^2$ from equation (41) as $\xi = 0$ at the center. For the ideal gas $\mu = 0.62$ we get from equation (42) the central temperature $T_c = 1.2 \times 10^7 K$. From Kippenhahn and Weigert (1990), numerical computation gives $\xi_3 = 6.897$, $\rho_c = 76.39 g cm^{-3}$, $P_c = 1.24 \times 10^{17} dyn/cm^2$ and $T_c = 1.2 \times 10^7 K$. We see that our approximation for $n = 3$ gives a quite good result at a level 0.7% for density, 1.6% for pressure and 0% for temperature.

For the isothermal sphere, the mass from equations (18) and (38) is

\begin{equation}
m = 8\pi \rho_c r^3 \left( \frac{1 + \alpha}{6 + \xi^2} - \frac{6\alpha}{(6 + \xi^2)^2} - \frac{\alpha}{3^{1/\alpha} 12e + \xi^2} + \frac{D(2^{-\alpha} - 1)}{(1 + 2^{-\alpha} D\xi^2)(1 + D\xi^2)} \right) ,
\end{equation}

where $\alpha = 0.551$ and $D = 3.84 \times 10^{-4}$. When $\xi >> \sqrt{6}$ the mass changes with radius as $m \propto r$. The pressure is given by polytropic relation (1) for $n = \infty$. The constant $K$ is related to the central density $\rho_c$ by (39). However, as the isothermal sphere does not have a finite radius, the constant $A$ cannot be determined as in the case $n < 5$. In the isothermal sphere the temperature is constant everywhere. From definition (43) and the hydrostatic equilibrium equation, we find the potential energy within radius $r$ for the isothermal sphere

\begin{equation}
E_g = 4\pi r^3 K \rho - 3Km ,
\end{equation}

where $\rho$ is given by (19) and $m$ by (46).

In this paper we have given a good analytic approximate solution of the Lane-Emden equation with which we have obtained analytic expressions for the mass contained in radius $r$, the pressure, temperature, and gravitational potential energy within radius $r$. It would be interesting to apply the approximation in modeling the structures of stars and galaxies.

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