NON-COMMUTATIVE RATIONAL YANG–BAXTER MAPS

ADAM DOLIWA

Abstract. Starting from multidimensional consistency of non-commutative lattice modified Gel’fand–Dikii systems we present the corresponding solutions of the functional (set-theoretic) Yang–Baxter equation, which are non-commutative versions of the maps arising from geometric crystals. Our approach works under additional condition of centrality of certain products of non-commuting variables. Then we apply such a restriction on the level of the Gel’fand–Dikii systems what allows to obtain non-autonomous (but with central non-autonomous factors) versions of the equations. In particular we recover known non-commutative version of Hirota’s lattice sine-Gordon equation, and we present an integrable non-commutative and non-autonomous lattice modified Boussinesq equation.

1. Introduction

Let \( \mathcal{X} \) be any set, a map \( R: \mathcal{X} \times \mathcal{X} \) satisfying in \( \mathcal{X} \times \mathcal{X} \) the relation
\[
R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12},
\]
where \( R_{ij} \) acts as \( R \) on the \( i \)-th and \( j \)-th factors and as identity on the third, is called Yang–Baxter map \([12, 29]\). If additionally \( R \) satisfies the relation
\[
R_{21} \circ R = \text{Id},
\]
where \( R_{21} = \tau \circ R \circ \tau \) and \( \tau \) is the transposition, then it is called reversible Yang–Baxter map.

In this paper we study properties of a non-commutative version of the maps arising from geometric crystals \([18, 13, 28]\). In particular we will demonstrate the following result.

Theorem 1.1. Given two assemblies of (non-commuting in general) variables \( x = (x_1, \ldots, x_L) \) and \( y = (y_1, \ldots, y_L) \), define polynomials
\[
\mathcal{P}_k = \sum_{a=0}^{L-1} \left( \prod_{i=0}^{a-1} y_{k+i} \prod_{i=a+1}^{L-1} x_{k+i} \right), \quad k = 1, \ldots, L,
\]
where subscripts in the formula are taken modulo \( L \). If the products \( \alpha = x_1 x_2 \ldots x_L \) and \( \beta = y_1 y_2 \ldots y_L \) are central then the map
\[
R: (x, y) \mapsto (\tilde{x}, \tilde{y}), \quad \tilde{x}_k = \mathcal{P}_k x_k \mathcal{P}_k^{-1}, \quad \tilde{y}_k = \mathcal{P}_k^{-1} y_k \mathcal{P}_k, \quad k = 1, \ldots, L,
\]
is reversible Yang–Baxter map.

It is easy to see that the products \( \alpha = x_1 \ldots x_L \) and \( \beta = y_1 \ldots y_L \) are conserved by the map \( R \). This can be used to reduce the number of variables. For example, in the simplest case \( L = 2 \) define \( x = x_1, \) \( y = y_1 \) to get a parameter dependent reversible Yang–Baxter map \( R(\alpha, \beta): (x, y) \mapsto (\tilde{x}, \tilde{y}) \)
\[
\tilde{x} = (\alpha x^{-1} + y) x (x + \beta y^{-1})^{-1}, \quad \tilde{y} = (\alpha x^{-1} + y)^{-1} y (x + \beta y^{-1}),
\]
which in the commutative case is equivalent to the \( F_{yy} \) map in the list given in \([2]\).

In recent studies on discrete integrable systems the property of multidimensional consistency \([1, 24]\) is considered as the main concept of the theory. Roughly speaking, it is the possibility of extending the number of independent variables of a given nonlinear system by adding its copies in different directions without creating this way inconsistency or multivaluedness. It is known \([2, 27]\) how to relate three dimensional consistency of integrable discrete systems with Yang–Baxter maps. There is also well known connection between Yang–Baxter maps and the braid relations.

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Non-commutative versions of integrable maps or discrete systems \cite{25, 22, 4, 26, 7} are of growing interest in mathematical physics. They may be considered as a useful platform to more thorough understanding of integrable quantum or statistical mechanics lattice systems, where the quantum Yang–Baxter equation \cite{3, 20} plays a role.

In Section 2 we use three dimensional consistency of non-commutative Kadomtsev–Petviashvili (KP) map to construct corresponding Yang–Baxter maps following ideas of \cite{18, 19} applied there in the commutative case. It turns out that we can construct the solutions under periodicity and centrality (of certain products of the variables) assumptions. Then in Section 3 we consider implication of the centrality assumption on the level of the non-commutative modified lattice Gel’fand–Dikii equations. In the simplest case we recover non-autonomous version of non-commutative Hirota’s sine-Gordon equation \cite{4}. We present also an integrable non-commutative and non-autonomous lattice modified Boussinesq equation.

Remark. Throughout the paper we will work with division rings of (non-commutative) rational functions in a finite number of (non-commuting) variables. This approach is intuitively accessible, see however \cite{5} for formal definitions.

2. Non-commutative rational realization of the symmetric group

2.1. KP maps. Consider the linear problem of the non-commutative KP hierarchy \cite{19, 26, 9}
\begin{equation}
\phi_{k+1}(n) - \phi_k(n + \varepsilon_i) = \phi_k(n)u_{i,k}(n), \quad k \in \mathbb{Z}, \quad n \in \mathbb{Z}^N, \quad i = 1, \ldots, N,
\end{equation}
here $\phi_k: \mathbb{Z}^N \to \mathbb{D}^M$, and $\mathbb{D}$ is a division ring, and $\varepsilon_i \in \mathbb{Z}^N$ has 1 at $i$-th place and all other zeros. The potentials $u_{i,k}: \mathbb{Z}^N \to \mathbb{D}$ satisfy then the compatibility conditions
\begin{equation}
u_{j,k}u_{i,k}(j) = u_{i,k}u_{j,k}(i), \quad u_{i,k}(j) + u_{j,k+1} = u_{j,k(i)} + u_{i,k+1}, \quad 1 \leq i \neq j \leq N,
\end{equation}
where we write $u_{i,k(j)}(n)$ instead of $u_{i,k}(n + \varepsilon_j)$, and we skip the argument $n$. In consequence we obtain the transformation rule
\begin{equation}
u_{i,k(j)} = (u_{i,k} - u_{j,k})^{-1}u_{i,k}(u_{i,k+1} - u_{j,k+1}), \quad i \neq j,
\end{equation}
which can be written as a non-commutative discrete KP map
\[
(u_i, u_j) \mapsto (u_{i(j)}, u_{j(i)}), \quad u_i = (u_{i,k}), \quad k \in \mathbb{Z}.
\]

Proposition 2.1. \cite{9} \cite{19} The non-commutative discrete KP map is three-dimensionally consistent, i.e. both ways to calculate $u_{i(j)}$ give the same result, see Figure 1.

Remark. Three dimensional consistency of the non-commutative discrete KP map is a consequence \cite{9} of the four dimensional consistency of the so called Desargues maps \cite{8}.

To make connection with the Yang–Baxter maps consider $N$-cube graph, whose vertices are identified with binary sequences of length $N$ with two vertices connected by an edge if their sequences differ at one place only. The shortest paths from the initial vertex $(0,0,\ldots,0)$ to the terminal one $(1,1,\ldots,1)$ can
be identified with permutations: a permutation $\sigma \in S_N$ corresponds to the path with subsequent steps in directions $(\varepsilon_{\sigma(1)}, \varepsilon_{\sigma(2)}, \ldots, \varepsilon_{\sigma(N)})$. The symmetric group acts then on paths by the left natural action $\rho_{\sigma} = \pi_{\sigma \rho}$. Given initial weights $u_i$, $i = 1, \ldots, N$, on edges connecting the initial vertex $(0,0,\ldots,0)$ with the vertex $\varepsilon_i$, by the KP map we attach a weight to each edge of the cube graph. Each such path gives then a sequence of weights $w^0$, for example $w^{1d} = (u_1, u_{2(1)}, \ldots, u_{N(1,2,\ldots,N-1)})$. We are interested in maps $r_\sigma$ from the reference weights $w^{1d}$ to weights $w^\sigma$. In particular, we study maps $r_i$, $i = 1, \ldots, N - 1$, which correspond to transpositions $\sigma_i = (i, i + 1)$ generating the symmetric group $S_N$ and satisfying the Coxeter relations [17]

\[
\sigma_j^2 = \text{Id}, \quad \text{involutivity}
\]

\[
\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}, \quad \text{braid relations}
\]

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| > 1, \quad \text{commutativity.}
\]

In order to find such maps $r_i$ we have to find the so called first companion map

\[
(u_i, u_{j(1)}) \mapsto (u_j, u_{i(j)}),
\]

where we use variation of the terminology of [2] [27] where Yang–Baxter maps were studied in relation to multidimensionally consistent edge-field maps.

2.2. The first companion map and the centrality assumption. We will concentrate on deriving the first companion map, which we temporarily denote by $r : (x, y) \mapsto (x', y')$, where by (2.2)

\[
x'_i y'_i = x_i y_i, \quad y'_i + x'_{i+1} = y_i + x_{i+1}.
\]

For $\ell \in \mathbb{Z}_+$ define polynomials

\[
\mathcal{P}_k^{(\ell)} = \sum_{a=0}^{\ell} \left( \prod_{i=0}^{a-1} y_{k+i} \prod_{i=a+1}^{\ell} x_{k+i} \right) = \sum_{a=0}^{\ell} \left( \prod_{i=0}^{a-1} y_{k+i} \prod_{i=a+1}^{\ell} x_{k+i} \right)
\]

\[
x_{k+1} x_{k+2} \ldots x_{k+\ell} + y_k x_{k+2} \ldots x_{k+\ell-1} x_{k+\ell} + \cdots + y_k y_{k+1} \ldots y_{k+\ell-2} x_{k+\ell} + y_k \ldots y_{k+\ell-2} y_{k+\ell-1},
\]

which satisfy the recurrence relations

\[
\mathcal{P}_k^{(\ell)} = \mathcal{P}_k^{(\ell-1)} x_{k+\ell} + \prod_{i=0}^{\ell-1} y_{k+i} = \prod_{i=1}^{\ell} x_{k+i} + y_k \mathcal{P}_k^{(\ell-1)}
\]

where by definition $\mathcal{P}_k^{(0)} = 1$. By $\mathcal{P}_k^{(\ell)}'$ denote analogous polynomials for primed variables.

Lemma 2.2. Assume that $x'_i, y'_i$ satisfy equations (2.4) then $\mathcal{P}_k^{(\ell)} = \mathcal{P}_k^{(\ell)}$.

Proof. For $\ell = 1$ we have just the second of equations (2.4). For $\ell \geq 1$ notice that by (2.4) the product \[ \prod_{i=0}^{\ell-1} (y_{k+i} + x_{k+i+1}) \] is equal to its primed version. It splits into the sum of $\mathcal{P}_k^{(\ell)}$ and the part with summands containing the factors \[ x_{k+p} y_{k+p} \ldots \] with possible $p = 1, \ldots, \ell - 1$. We group such unwanted terms into (disjoned) parts depending on the smallest $p$. Such a part has the structure

\[
\mathcal{P}_k^{(p-1)} x_{k+p} y_{k+p} \prod_{i=p+1}^{\ell-1} (y_{k+i} + x_{k+i+1}),
\]

which due to the induction assumption and equations (2.4) is equal to its primed version, therefore both cancel out.

From now on we assume $L$-periodicity condition: $x_k + L = x_k$, $y_k + L = y_k$. Define $\mathcal{P}_k = \mathcal{P}_k^{(L-1)}$, then Lemma 2.2 and recurrence relations (2.5) imply

\[
\mathcal{P}_k x_k + \prod_{i=0}^{L-1} y_{k+i} = \prod_{i=1}^{L} x'_{k+i} + y'_k \mathcal{P}_k^{(L-1)} + \prod_{i=1}^{L} x_{k+i} + y_k \mathcal{P}_k^{(L-1)} = \mathcal{P}_k x'_k + \prod_{i=0}^{L-1} y'_{k+i}.
\]

Notice that if we would impose the additional normalization condition

\[
\prod_{i=1}^{L} x'_{k+i} = \prod_{i=0}^{L-1} y_{k+i}, \quad \prod_{i=0}^{L-1} y'_{k+i} = \prod_{i=1}^{L} x_{k+i},
\]
then equations (2.4) could be solved as

\[(2.7) \quad x'_k = \mathcal{P}_k^{-1} y_k \mathcal{P}_{k+1}, \quad y'_k = \mathcal{P}_k x_k \mathcal{P}_{k+1}^{-1}.\]

However, equations (2.7) and condition (2.6) are not compatible for general non-commuting variables. The above procedure of getting solutions works if we make additional centrality assumptions which state that \(\alpha = \prod_{i=1}^n x_i\) and \(\beta = \prod_{i=1}^n y_i\) commute with other elements of the division ring.

**Lemma 2.3.** Under the centrality assumptions the products \(\prod_{i=1}^L x_{k+i}\) and \(\prod_{i=1}^L y_{k+i}\) do not depend on the index \(k\). Moreover

\[(2.8) \quad \mathcal{P}_k x_k - y_k \mathcal{P}_{k+1} = \alpha - \beta,\]

which means that the above expression is central and independent of index \(k\) as well. In particular \(\mathcal{P}_k x_k\) commutes with \(y_k \mathcal{P}_{k+1}\).

**Proof.** The first part follows from identities

\[\prod_{i=1}^L x_{k+i} = (x_1 \ldots x_{k-1})^{-1} \alpha (x_1 \ldots x_{k-1}), \quad \prod_{i=1}^L y_{k+i} = (y_1 \ldots y_{k-1})^{-1} \beta (y_1 \ldots y_{k-1}),\]

where we used also the periodicity assumption. The second part is implied by equations (2.5). \(\square\)

**Proposition 2.4.** Under the centrality assumption the expressions for \(x'_k\) and \(y'_k\) given by (2.7) provide the unique solution of equations (2.4) supplemented by the normalization conditions \(\alpha' = \beta\) and \(\beta' = \alpha\).

**Proof.** Notice that by Lemma 2.3 \(x'_k\) and \(y'_k\) given by (2.7) satisfy the normalization condition. Then also both expressions

\[y'_k + x'_{k+1} - y_k - x_{k+1} = (\mathcal{P}_k x_k - y_k \mathcal{P}_{k+1}) \mathcal{P}_{k+1}^{-1} + \mathcal{P}_{k+1}^{-1} (y_{k+1} \mathcal{P}_{k+2} - \mathcal{P}_{k+1} x_{k+1})\]

and

\[x'_k y'_k - x_k y_k = \mathcal{P}_{k+1}^{-1} (y_k \mathcal{P}_{k+1} \mathcal{P}_k x_k - \mathcal{P}_k x_k y_k \mathcal{P}_{k+1}) \mathcal{P}_{k+1}^{-1}\]

vanish due to Lemma 2.3. \(\square\)

**Corollary 2.5.** The first companion map \((x, y) \mapsto (x', y')\) given above is involutory.

**Corollary 2.6.** The problem of finding the first companion of the KP map in the periodic reduction can be considered as a refactorization problem \(A(x)A(y) = A(x')A(y')\), where the matrix

\[(2.9) \quad A(x) = \begin{pmatrix} -x_1 & 0 & \ldots & 0 & \lambda \\ 1 & -x_2 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -x_{L-1} & 0 \\ 0 & 0 & \ldots & 1 & -x_L \end{pmatrix}\]

with the central spectral parameter \(\lambda\) is the \(L\)-periodic reduction of the discrete non-commutative KP hierarchy (Gelfand–Dikii system) linear problem (2.1) studied in [9].

### 2.3. Realization of Coxeter relation under the centrality assumption

Consider again a sequence \((w_1, w_2, \ldots, w_N)\) of weights along a shortest path in \(N\)-cube from the initial to the terminal vertex, each weight is a sequence \(w_j = (w_{j,1}, \ldots, w_{j,L})\) of non-commuting variables satisfying the centrality assumption that the product \(\alpha_j = w_{j,1} w_{j,2} \ldots w_{j,L}\) commutes with all \(w_{j,k}\). As we already have mentioned the symmetric group \(S_N\) acts in natural way on the paths and thus on the weights. To make use of results of Section 2.2 define polynomials

\[(2.10) \quad \mathcal{P}_{j,k} = \sum_{a=0}^{L-1} \left( \prod_{i=0}^{a-1} w_{j+1,i,k+i} \prod_{i=a+1}^{L-1} w_{j,k+i} \right),\]

where the second index should be considered modulo \(L\).
If we assume centrality of the products
\[ \alpha_j = \prod_{i=1}^{L} w_{j,i} \]
then the maps \( r_j \) satisfy the Coxeter relations
\[ r_j^2 = \text{Id}, \quad r_j \circ r_{j+1} \circ r_j = r_{j+1} \circ r_j \circ r_{j+1}, \quad r_i \circ r_j = r_j \circ r_i \quad \text{for} \quad |i-j| > 1. \]

\textbf{Proof.} The commutativity part is clear from definition of the maps. Involutivity part comes from Corollary 2.6 and the braid relations follow from the path-interpretation and uniqueness of the companion map subject to normalization conditions (2.6). Equivalently, we can use the unique refactorization interpretation given in Corollary 2.5 and follow the argumentation presented in [29]. \( \square \)

\textbf{Corollary 2.8.} The action of \( r_j \) on the central elements \( \alpha_i \) is
\[ r_j(\alpha_j) = \alpha_{j+1}, \quad r_j(\alpha_{j+1}) = \alpha_j, \quad r_j(\alpha_i) = \alpha_i \quad \text{for} \quad i \neq j, j + 1. \]

\textbf{Corollary 2.9.} We can consider the division ring \( \mathbb{D} \) as a division algebra over a fixed subfield \( \mathbb{k} \) of its center. Therefore we can state the centrality condition as \( \alpha_j \in \mathbb{k} \).

\textbf{Corollary 2.10.} Define second companion map \( R = \tau \circ r \), i.e. \( R : (x,y) \mapsto (\tilde{x},\tilde{y}) = (y',x') \) then by (2.7) we obtain formulas (1.3). Due to well known relation [29] between realizations of the Coxeter relations and reversible Yang–Baxter maps we proved in this way Theorem 1.1.

3. Non-commutative Gel’fand–Dikii systems with the centrality condition

In [9] [10] we studied periodic reductions of the discrete KP hierarchy under two extreme assumptions about non-commutativity/commutativity of dependent variables. Results of Section 2.2 suggest to consider an analogous centrality condition on the level of equations (2.2). By simple calculation we obtain the following result.

\textbf{Proposition 3.1.} In the \( L \)-periodic reduction \( u_{i,k+L} = u_{i,k} \) of the non-commutative KP system (2.2) assume centrality of the products \( U_i = u_{i,1}u_{i,2} \cdots u_{i,L} \). Then the products \( u_{i,k}u_{i,k+1} \cdots u_{i,k+L-1} \) do not depend on index \( k \), and \( U_i \) is a function of \( u_i \) only
\[ U_{i(j)} = U_{i}, \quad j \neq i. \]

Using the above result one can obtain non-autonomous non-commutative discrete equations of the modified Gel’fand–Dikii type. It is known [9] that the first part of equations (2.2) implies existence of potentials \( \rho_k \) such that \( u_{i,k} = \rho_k^{-1} \rho_{k(i)} \), while the second part gives the corresponding vertex form of the non-commutative discrete KP hierarchy
\[ (\rho_{k(j)}^{-1} - \rho_{k(i)}^{-1}) \rho_{k(i)} = \rho_{k+1}^{-1} (\rho_{k+1(i)} - \rho_{k+1(j)}), \quad k \in \mathbb{Z}/(L\mathbb{Z}), \quad i \neq j. \]
Then we replace one of the functions $\rho_i$ by others and the central non-autonomous factors. To make connection with known results it is convenient to define central functions $F_i = (U_i)^{1/L}$ of the corresponding single variables $u_i$, and then consider the central function $G$ defined by compatible system $G^{(i)} = F_iG$. We remark that such $G$ is a product of functions of single variables.

In the simplest case $L = 2$ define, like in [9], a function $x$ by $\rho_1 = xG$. Then

$$u_{i,1} = \rho_1^{-1} \rho_{1(i)} = x^{-1} x_{(i)} F_i, \quad \text{and} \quad u_{i,2} = \rho_2^{-1} \rho_{2(i)} = x^{-1} x_{(i)} F_i,$$

which inserted in equations (3.2) produces the non-commutative Hirota (or discrete sine-Gordon or lattice modified Korteweg–de Vries) equation studied in [14, 15, 4]

$$(x_{(i)} F_i - x_{(i)} F_j) x_{(ij)} = (x_{(i)} F_i - x_{(j)} F_j) x.$$

Remark. To recover the equation in the form studied in [4] notice that after extracting $x_{(j)}^{-1}$ the expressions in brackets commute, and use inverses of the non-autonomous factors $F_i$.

For $L = 3$ define unknown functions $x$ and $y$ by equations

$$\rho_1 = xG, \quad \rho_3 = y^{-1}G, \quad G^{(i)} = F_iG$$

which allows to find

$$\rho_2^{-1} \rho_{2(i)} = u_{i,2} = x_{(i)}^{-1} y y^{-1} F_i.$$ 

Making such substitution in (3.2) for $k = 1$ and $k = 3$ we obtain the following non-commutative integrable two-component system (equation for $k = 2$ is then its consequence)

$$(x_{(i)}^{-1} F_i - x_{(i)}^{-1} F_j) x_{(ij)} = (x_{(i)}^{-1} x y_{(i)} F_i - x_{(j)}^{-1} x y_{(j)} F_j) y^{-1},$$

$$(y_{(i)} F_i - y_{(i)} F_j) y_{(ij)}^{-1} = x^{-1} (x_{(i)} F_i - x_{(j)} F_j).$$

Next, by elimination of the field $x$ we obtain integrable non-commutative and non-autonomous (with central non-autonomous coefficients $F_i$) version of the lattice modified Boussinesq [20] equation

$$\left[ (y_{(i)} F_i - y_{(i)} F_j)^{-1} y_{(ij)}^{-1} \right]_{(ij)} y (y_{(i)} F_i - y_{(j)} F_j) =$$

$$\left[ y_{(ij)} (y_{(i)} F_i - y_{(i)} F_j)^{-1} \right]_{(i)} y (y_{(i)} F_i - y_{(j)} F_j)^{-1} (y_{(i)} F_i^2 - y_{(j)} F_j^2) y^{-1}.$$ 

4. CONCLUDING REMARKS

We presented a non-commutative rational Yang–Baxter map obtained from the non-commutative discrete KP hierarchy subject to periodicity and centrality constraints. The corresponding integrable systems, which generalize the non-commutative non-autonomous Hirota’s sine-Gordon equation [4] have been also considered. In particular we have obtained an integrable non-commutative and non-autonomous lattice modified Boussinesq equation. We remark, see [9, 10], that three dimensional consistency of the equations considered here is a consequence of four dimensional compatibility of the non-commutative Hirota’s discrete KP system [8], where the counterpart of the functional Yang–Baxter equation is the functional pentagon equation [11]. Since the solutions of the pentagon equation presented in [11] allow for quantization (understood as a reduction from the non-commutative case by adding certain commutation relations preserved by the integrable evolution), we expect that also the non-commutative rational Yang–Baxter map obtained above can be quantized in such a way also. It would be instructive to understand various applications of the Hirota discrete KP systems and its reductions reviewed in [21] from that perspective.

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