THE COMPLEX MONGE-AMPÈRE EQUATION ON COMPACT KÄHLER MANIFOLDS

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Abstract. We consider the complex Monge-Ampère equation on a compact Kähler manifold \((M, g)\) when the right hand side \(F\) has rather weak regularity. In particular we prove that estimate of \(\triangle \phi\) and the gradient estimate hold when \(F\) is in \(W^{1,p}_0\) for any \(p_0 > 2n\). As an application, we show that there exists a classical solution in \(W^{3,p}_0\) for the complex Monge-Ampère equation when \(F\) is in \(W^{1,p}_0\).

1. Introduction

Let \(M\) be a compact Kähler manifold of complex dimension \(n\) with a smooth Kähler metric \(g = g_{ij}dz^i \otimes d\bar{z}^j\). The corresponding Kähler form is given by \(\omega = \sqrt{-1}g_{ij}dz^i \wedge d\bar{z}^j\). The Kähler forms in the class \([\omega]\) can be written in terms of a Kähler potential \(\omega_\phi = \omega + \sqrt{-1}\partial \bar{\partial} \phi\). We shall use the following notations, for a function \(f\) and a holomorphic coordinate \(z = (z^1, \cdots, z^n)\),

\[
    f_{ij} = \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}, \quad \Delta f = g^{ij}f_{ij}, \quad \Delta_\phi f = g^{ij}_\phi f_{ij}.
\]

It is well known that the Ricci curvature of \(g\) is given by

\[
    R_{ij} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log(\det(g_{ij})).
\]

In particular, the Ricci form

\[
    \rho = \frac{\sqrt{-1}}{2\pi} R_{ij}dz^i \wedge d\bar{z}^j
\]

is a close \((1,1)\) form and it defines a cohomology class independent of the choice of the Kähler metric \(g\), which is the first Chern class \(c_1\) of \(M\). Therefore if a closed \((1,1)\) form is the Ricci form of some Kähler metric, then its cohomology class must represent the first Chern class \(c_1\). In the 1950s, Calabi [12] conjectured that any form in \(c_1\) can actually be written as the Ricci form of some Kähler metric. The Calabi conjecture can be reduced to solving the following complex Monge-Ampère equation

\[
    \log \left( \frac{\det(g_{ij} + \phi_{ij})}{\det(g_{ij})} \right) = F,
\]

where the function \(F\) satisfies

\[
    \int_M \exp(F)dvol_g = Vol(M).
\]
When $F$ is a smooth function on $M$, Yau [53] proved that (1.1) has a smooth solution, hence proved the Calabi conjecture; his solution also provided Kähler-Ricci flat metrics on Kähler manifolds with zero first Chern class. More generally, Calabi initiated the study of Kähler-Einstein metrics, with

$$\rho_\phi = \lambda \omega_\phi$$

for constant $\lambda$. The Kähler-Einstein equation can be deduced to

$$(1.2) \quad \log \left( \frac{\det (g_{ij} + \phi_{ij})}{\det (g_{ij})} \right) = F - \lambda \phi.$$ 

When $\lambda < 0$, Aubin [2] and Yau [53] have proved the existence of a smooth solution of (1.2). When $\lambda$ is positive, and so $M$ is a Fano manifold in algebraic-geometric language, (1.2) may or may not have a smooth solution. In this case, Tian has made enormous progress towards understanding precisely when a solution exists, see [49, 50] for example. In particular, a complete answer [49] for existence of Kähler-Einstein was given when $n = 2$; see also recently Chen-Wang’s proof [18, 19] via Kähler-Ricci flow.

Weak solution in various settings of complex Monge-Ampere equations has also been studied extensively since the pioneering work of Bedford-Taylor [3, 4]. Kolodziej [37, 38] proved that there exists a unique bounded solution of

$$(1.3) \quad \omega^n_\phi = \tilde{F} \omega^n$$

when $\tilde{F}$ is a nonnegative $L^p$ function for $p > 1$; moreover, the solution is Hölder continuous. There are further regularity, existence and uniqueness results on (1.3) when $\tilde{F}$ is less regular and/or when $\omega$ is degenerate, to mention [55, 8, 33, 21, 29, 22] to name a few. Readers are referred to the aforementioned references for a historic overview and further references.

In the early 1980s, Calabi initiated another problem [14] in Kähler geometry to seek extremal Kähler metrics, which include metrics of constant scalar curvature as a special case,

$$(1.4) \quad R_\phi = -g^{ij}_\phi \partial_i \partial_j \log \left( \det (g_{kl} + \phi_{kl}) \right) = \overline{R},$$

where the constant $\overline{R}$ is determined by $(M, [\omega])$. There is a tremendous body of work on the problems related to extremal metrics and much progress has been made in the last decade. The well known conjecture of Yau-Tian-Donaldson [54, 50, 24] asserts that if $(M, [\omega])$ is a compact Kähler manifold and $[\omega] = 2\pi c_1(L)$ for some holomorphic line bundle $L \to M$, then there is a metric of constant scalar curvature in the class $[\omega]$ if and only if $(M, L)$ is K-stable. This is a core problem in Kähler geometry. Thorough the efforts of many mathematicians, the necessary part of this conjecture is essentially done ([50, 23, 26, 17, 41, 47, 48]). The core problem is to find a way to understand existence problem.

Back to 1980s, Calabi proposed a parabolic equation, the Calabi flow,

$$(1.5) \quad \frac{\partial \phi}{\partial t} = R_\phi - \overline{R}.$$
One of the biggest difficulties to understand extremal metrics, in particular for equations (1.4) and (1.5), is how to control the metric through its scalar curvature, in contrast to its Ricci curvature. For example, it was shown [16] that all metrics in $[\omega]$ are equivalent and pre-compact in $C^{1,\alpha}$ topology when Ricci curvature and potential are both uniformly bounded; which is used to prove the extension of the Calabi flow with Ricci curvature bound. But such a result for scalar curvature is not known. Recently, Donaldson solved the constant scalar curvature equation on toric Kähler surfaces with K-stability in a series of paper [23, 25, 26, 27, 28]. The scalar curvature equation in this case can be formulated [1] as a fourth order equation for convex functions on certain convex polytope in $\mathbb{R}^2$. Roughly speaking, the key point in Donaldson’s work is to control the metric through its scalar curvature and K-stability condition; in this case, the metrics can be written in terms of the hessian of convex functions and convex analysis plays an important role.

While this progress is indeed impressive, the general existence problem is very elusive and beyond us. In retrospect, the memorable feature of Yau’s solution to Calabi conjecture is as follows: given a Kähler manifold with null first Chern class, does the $C^2$ estimate of potential follow from $C^0$ estimate of potential immediately?

**Guiding Problem:** Given the cscK equation, can we control $C^2$ and higher derivative estimates in terms of $C^0$ and/or $C^1$ estimates? Is this realistic?

Donaldson’s work is very inspirational in the sense that this is true for cscK metrics on toric Kähler surfaces. In other words, for cscK metric on a toric surface, once Kähler potential is bounded, then all higher derivatives will be bounded a priori. One of course would expect the more challenge case for general Kähler manifolds. Nevertheless this is a key problem in Kähler geometry we wish to consider.

We can view the scalar curvature equations as two coupled second order equations:

$$\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = e^F \det g,$$

and

$$R_\phi = -\Delta_\phi F - g^i_{\bar{j}}(R_{ij}(g)).$$

We hope to get some weak regularity of the volume form through its scalar curvature (the second equation), and then try to control the metric/potential by its volume form from the first equation. To make this strategy successful, we want to impose as weak regularity on $F$ as possible, but still try to get a upper bound control on $n + \Delta \phi$, hence control the metric.

Motivated in part by these problems, we are interested in the classical solution of (1.1) when $F$ has rather weak regularity; in particular, we assume $F \in W^{1,p_0}$ for $p_0 > 2n$. We believe that such results would be important, in particular for problems related to scalar curvature in Kähler geometry.

One of the key steps towards solving (1.1) is the estimate of $\Delta \phi$, which is obtained by Aubin [2] and Yau [53]. Roughly speaking, such an estimate asserts that, for a
smooth solution of (1.1),
\begin{equation}
0 < n + \Delta \phi \leq Ce^{C_1(\phi - \inf \phi)},
\end{equation}
where $C$ depends on $F$ up to its second derivatives. To deal with the case when $F$ has weaker regularity, we use integral method and Moser’s iteration (see [42]) instead of the maximum principle. This allows us to obtain estimate of $\Delta \phi$ and gradient estimate of $\phi$ when $F \in W^{1,p_0}$ for $p_0 > 2n$. In particular, we have

**Theorem 1.1.** Let $M$ be a compact Kähler manifold of complex dimension $n$ with a smooth metric $g = g_{i\bar{j}}dz^i \otimes d\bar{z}^j$. Let $F$ be a function in $W^{1,p_0}$ for some $p_0 > 2n$, then (1.1) has a classical solution $\phi \in W^{3,p_0}$. To prove Theorem 1.1, we shall derive the following two a priori estimates. First we have the estimate of $\Delta \phi$, which depends in addition on the Lipschitz norm of $\phi$.

**Theorem 1.2.** If $\phi$ is a smooth solution of (1.1), then
\begin{equation}
0 < n + \Delta \phi \leq C = C(\|\phi\|_{L^\infty}, \|\nabla \phi\|_{L^\infty}, \|F\|_{W^{1,p_0}, p_0, M, g, n}).
\end{equation}

**Remark 1.3.** Note that in Theorem 1.2, $p_0 > 2n$ is optimal in the sense that when $p_0 \leq 2n$, $F \in W^{1,p_0}(M, g)$ does not imply $F \in L^\infty$ by the Sobolev embedding. Actually in general this is not true; hence (1.7) cannot hold in general when $F \in W^{1,p_0}$ for $p_0 \leq 2n$. In particular, when $p_0 \leq 2n$, one cannot expect a classical solution in Theorem 1.1.

The $L^\infty$ estimate of $\phi$ was originally derived by Yau [53] after obtaining estimate of $\Delta \phi$ in (1.6). Kolodziej [37] proved that there exists a unique bounded solution $\phi$ of (1.3) $\tilde{F} \in L^p$ and $\tilde{F}$ is nonnegative, $p > 1$; note that his result holds for the degenerate case. Later on he proved the Hölder estimate of $\phi$ in [38]. Hence we shall assume that $\|\phi\|_{L^\infty}$ is bounded and derive further regularity of $\phi$.

The gradient estimate of $\phi$ was not required in [2, 53] to derive the estimate of $\Delta \phi$. But we shall need this estimate. Such an estimate, which also holds for (1.3) and the Dirichlet problem of the complex Monge-Ampère equation on manifolds with boundary, is derived by [34, 15, 9, 32, 31, 44] in various settings; in particular, Blocki [9] established the gradient estimate of (1.1) when $F$ is Lipschitz (we should mention that his results also holds for (1.3) when). We shall prove, when $F$ in $W^{1,p_0}, p_0 > 2n$, that

**Theorem 1.4.** If $\phi$ is a smooth solution of (1.1), then
\begin{equation}
|\nabla \phi| \leq C(\|\phi\|_{L^\infty}, \|F\|_{W^{1,p_0}, p_0, M, g, n}).
\end{equation}

With a slight modification, the proof of Theorem 1.2 and Theorem 1.4 holds for (1.2), see Remark 2.2 and Remark 3.1. In particular, the $L^\infty$ bound of $\phi$ follows from the maximum principle when $\lambda = -1$; hence Theorem 1.1 also holds for (1.2) when $\lambda = -1$.

We still use integral method and iteration argument to prove Theorem 1.4; the new ingredient is that we obtain some inequalities, see (3.11) and (3.15) below, which are essential for the iteration process. Similar inequalities are derived in [9], but mainly
with an emphasis on the point of local maximum of the barrier functions, which is sufficient for the maximum principle argument, but not for our case.

The integral method uses the ideas in the well-known theory of De Giorgi [20], Nash [43] and Moser [42]. Roughly stated, such a theory deals with the regularity problem of the elliptic operator of the form

$$Lu = D_i(a^{ij} D_j u) = f$$

with measurable coefficients $a^{ij}$ such that $\lambda |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq \Lambda |\xi|^2$ for some positive constants $\lambda, \Lambda$. Uniform ellipticity is essential in De Giorgi-Nash-Moser theory.

One observation in the present paper is that the upper bound for $a^{ij}$ is not necessary on compact manifolds (without boundary), which is required essentially to deal with the terms from cut-off functions. To overcome the absence of the apriori lower bound for $g_i^\phi$ (or the absence of the apriori uniform Sobolev constant for $\omega_\phi$), we use $a^{ij} = (n + \Delta \phi) g_i^\phi$ instead when estimating $\Delta \phi$. For example, to estimate $\Delta \phi$, we compute

$$\int_M u g_i^\phi (u^p)_i (u^p)_j d\text{vol}_\phi,$$

where $u = \exp(-C\phi)(n + \Delta \phi)$ is the barrier function used in [53]. Combining Yau’s computation [53], this gives the starting point of iteration process. The iteration process for the gradient estimate is much more complicated than (1.9) technically, which depends on the computations as in (3.11)–(3.20) in an essential way. But module these technical details, we believe that these ideas can be applied to other nonlinear equations on compact manifolds.

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2. Estimates of $\Delta \phi$

We shall always assume a normalization condition

$$\int_M \phi d\text{vol}_g = 0.$$

We shall also assume that $\phi$ and $F$ are both smooth and derive a priori estimates of $\Delta \phi$ depending on $\|\phi\|_{L^\infty}, \|\nabla \phi\|_{L^\infty}, \|F\|_{W^{1,p_0}}, p_0, M, g, n$. For simplicity, we shall not emphasize the dependence on $M, g$; while the dependence on the geometry of $(M, g)$ can be made quite explicit from the computation. We shall then prove Theorem 1.2.

**Proof.** We start with Yau’s computation [53],

$$\Delta \phi \left( \exp(-C_1 \phi)(n + \Delta \phi) \right)$$

$$\geq \exp(-C_1 \phi) \left[ \Delta F - n^2 \inf_{i \neq l} R_{ii\bar{l}l} - C_1 n(n + \Delta \phi) \right] + \exp \left( -C_1 \phi - \frac{F}{n - 1} \right) \left( C_1 + \inf_{i \neq l} R_{ii\bar{l}l} \right) (n + \Delta \phi)^{\frac{n}{n-1}}.$$  

(2.2)
Note that for any $\varepsilon > 0$, by Young’s inequality, there is a constant $C = C(\varepsilon, n)$ such that

\[ n + \Delta \phi \leq \varepsilon(n + \Delta \phi)^{\frac{n}{n-1}} + C(\varepsilon, n). \]

Denote $u = \exp(-C_1\phi)(n + \Delta \phi)$. We can rewrite (2.2) as

\[ (2.3) \quad C_2(n + \Delta \phi)^{\frac{n}{n-1}} + \exp(-C_1\phi)\Delta F - C_3 \leq \Delta \phi u, \]

where $C_2, C_3$ depend on $C_1, \|\phi\|_{L^\infty}, \sup F, n$. Note that $u > 0$ is bounded from below by a fixed positive number. We use $dvol_g$ to denote the volume form of $g$, and $dvol_\phi$ the volume form of $g_{ij} + \phi_{ij}$. In particular,

\[ dvol_\phi = \exp(F)dvol_g. \]

We compute, for $p > 0$,

\[ (2.4) \quad \Delta \phi(u^p) = pu^{p-1}\Delta \phi u + p(p-1)u^{p-2}|\nabla u|^2_\phi. \]

Integration by parts, we compute

\[ (2.5) \quad \int_M u^{p+1}\Delta \phi(u^p)dvol_\phi = -\frac{p+1}{p} \int_M u|\nabla (u^p)|^2_\phi dvol_\phi. \]

On the other hand by (2.4), we compute

\[ (2.6) \quad \int_M u^{p+1}\Delta \phi(u^p)dvol_\phi = p \int_M u^{2p}(\Delta \phi u)dvol_\phi + p(p-1) \int_M u^{2p-1}|\nabla u|^2_\phi dvol_\phi \]

\[ = p \int_M u^{2p}(\Delta \phi u)dvol_\phi + \frac{p-1}{p} \int_M u|\nabla (u^p)|^2_\phi dvol_\phi. \]

It follows from (2.5) and (2.6) that

\[ (2.7) \quad -2 \int_M u|\nabla (u^p)|^2_\phi dvol_\phi = p \int_M u^{2p}(\Delta \phi u)dvol_\phi. \]

We then compute, by (2.3),

\[ (2.8) \quad \int_M u^{2p}(\Delta \phi u)dvol_\phi \geq \int_M u^{2p} \left(C_2(n + \Delta \phi)^{\frac{n}{n-1}} + \exp(-C_1\phi)\Delta F - C_3\right) dvol_\phi. \]

To deal with the term involved with $\Delta F$, we compute,

\[ \int_M u^{2p} \exp(-C_1\phi)\Delta F dvol_\phi \]

\[ = \int_M u^{2p} \exp(-C_1\phi)\Delta F \exp(F)dvol_g \]

\[ = -\int_M \nabla F \nabla \left(\exp(F - C_1\phi)u^{2p}\right) dvol_g \]

\[ = -\int_M (|\nabla F|^2 u^{2p} - C_1 \nabla F \nabla \phi u^{2p} + 2\nabla (u^p)\nabla F u^p) \exp(F - C_1\phi)dvol_g \]

\[ = -\int_M (|\nabla F|^2 u^{2p} - C_1 \nabla F \nabla \phi u^{2p} + 2\nabla (u^p)\nabla F u^p) \exp(-C_1\phi)dvol_\phi. \]
By (2.7), (2.8) and (2.9), we compute

\[
\int_M u|\nabla (u^p)|_\phi^2 \, dv_{\phi} \leq \frac{p}{2} \int_M u^{2p} \left( C_3 - C_2 (n + \Delta \phi)^{\frac{\alpha}{n-1}} \right) \, dv_{\phi}
\]
\[
+ \frac{p}{2} \int_M \left( |\nabla F|^2 u^{2p} - C_1 \nabla F \nabla \phi u^{2p} + 2 \nabla (u^p) \nabla F u^p \right) \exp(-C_1 \phi) \, dv_{\phi}
\]
\[
\leq pC_4 \int_M (1 + |\nabla F|^2 + |\nabla F||\nabla \phi|) u^{2p} \, dv_{\phi}
\]
\[
+ pC_4 \int_M |\nabla F| \nabla (u^p) \, u^p \, dv_{\phi} - \frac{p}{2}C_2 \int_M u^{2p} (n + \Delta \phi)^{\frac{\alpha}{n-1}} \, dv_{\phi},
\]  
(2.10)

where \( C_4 = C_4(C_1, C_3, \|\phi\|_{L^\infty}) \). Note that \( (n + \Delta \phi) g^{ij} \xi_i \xi_j \geq |\xi|^2 \) for any vector \( \xi = (\xi_1, \cdots, \xi_n) \). Then we compute

\[
\int_M u|\nabla (u^p)|_\phi^2 \, dv_{\phi} = \int_M \exp(-C_1 \phi) (n + \Delta \phi) g^{ij} \partial_i (u^p) \partial_j (u^p) \, dv_{\phi}
\]
\[
\geq a \int_M |\nabla (u^p)|^2 \exp(F) \, dv_{g},
\]  
(2.11)

where \( a \) is a constant depending only on \( C_1, \|\phi\|_{L^\infty} \), and

\[
pC_4 \int_M |\nabla (u^p)| \, |\nabla F| \, u^p \, dv_{\phi} \leq \frac{a}{2} \int_M |\nabla (u^p)|^2 \exp(F) \, dv_{g}
\]
\[
+ \frac{p^2 C_4^2}{2a} \int_M |\nabla F|^2 u^{2p} \exp(F) \, dv_{g}.
\]  
(2.12)

Combine (2.10), (2.11) and (2.12), we compute

\[
\int_M \left( |\nabla (u^p)|^2 + pC_5 u^{2p+\frac{\alpha}{n-1}} \right) \, dv_{\phi}
\]
\[
\leq p^2 C_6 \int_M u^{2p} |\nabla F|^2 \, dv_{\phi} + pC_6 \int_M (1 + |\nabla F|^2 + |\nabla F||\nabla \phi|) u^{2p} \, dv_{\phi},
\]  
(2.13)

where both positive constants \( C_5, C_6 \) depend on \( \|\phi\|_{L^\infty}, \sup F, n \). We shall assume \( p \geq 1/4 \) from now on. Then we can get that, by (2.13),

\[
\int_M \left( |\nabla (u^p)|^2 + pC_7 u^{2p+\frac{\alpha}{n-1}} \right) \, dv_{g} \leq p^2 C_8 \int_M u^{2p} (|\nabla F|^2 + 1) \, dv_{g},
\]  
(2.14)

where \( C_7 = C_7(\|\phi\|_{L^\infty}, \|F\|_{L^\infty}, n) \) and \( C_8 = C_8(\|\phi\|_{L^\infty}, \|\nabla \phi\|_{L^\infty}, \|F\|_{L^\infty}) \). To get \( L^\infty \) bound of \( u \), we use the iteration method (see [42]). Recall the Sobolev inequality for \( (M, g) \); there is a constant \( C_s = C_s(n, g) \) such that,

\[
\left( \int_M f^{\frac{2n}{n-1}} \, dv_{g} \right)^{\frac{n-1}{n}} \leq C_s \left( \int_M |\nabla f|^2 \, dv_{g} + \text{Vol}^{-n}(M, g) \int_M f^2 \, dv_{g} \right).
\]  
(2.15)
Note that the above Sobolev inequality is scaling invariant. Let \( f = u^p \); it follows from (2.14) and (2.15) that

\[
\|u\|_{L^\frac{2pn}{n-1}} \leq (pC)^{1/p} \left( \int_M u^{2p}(|\nabla F|^2 + 1) dvol_g \right)^{\frac{1}{2p}}.
\]

By the Hölder inequality, we get that

\[
\int_M u^{2p}(|\nabla F|^2 + 1) dvol_g \leq \left( \int_M u^{2pq_0} \right)^{1/q_0} \left( \int_M (|\nabla F|^2 + 1)^{\frac{q_0}{p_0}} \right)^{\frac{2}{p_0}},
\]

where \( 1/q_0 + 2/p_0 = 1 \). When \(|\nabla F| \in L^{p_0} \), it then follows that

\[
\|u\|_{L^\frac{2pn}{n-1}} \leq (pC)^{1/p} \|u\|_{L^{2p_0}},
\]

where \( C = C(\|\phi\|_{L^\infty}, \|\nabla \phi\|_{L^\infty}, \|F\|_{W^{1,p_0}}, p_0, M, g) \). When \( 1 < q_0 < \frac{n}{n-1} \), let

\[
b = \frac{n}{(n-1)q_0} > 1.
\]

We can rewrite (2.17) as, for \( p \geq 1 \),

\[
\|u\|_{L^{pb_k}} \leq (pC)^{\frac{2q_0}{p}} \|u\|_{L^p}.
\]

We can then apply the standard iteration argument; let \( p = b^k \) for \( k \geq 0 \) in (2.18), then we compute from (2.18) that, for \( k \geq 1 \),

\[
\log \|u\|_{L^{b^k}} \leq \frac{2q_0}{b^{k-1}} (\log b^{k-1} + \log C) + \log \|u\|_{L^{b^{k-1}}}.
\]

It follows that

\[
\log \|u\|_{L^{b^{k+1}}} \leq \sum_{i=0}^{k} \frac{2q_0}{b^i} (\log b^i + \log C) + \log \|u\|_{L^1}.
\]

Since \( b > 1 \), it is clear that, when \( k \to \infty \),

\[
\sum_{i=0}^{\infty} \frac{2q_0}{b^i} (\log b^i + \log C) \leq C.
\]

It then follows that, when \( k \to \infty \),

\[
\|u\|_{L^\infty} \leq C \|u\|_{L^1} \leq C,
\]

where \( C = C(\|\phi\|_{L^\infty}, \|\nabla \phi\|_{L^\infty}, \|F\|_{W^{1,p_0}}, p_0, M, g, n) \). In other words,

\[
0 < n + \Delta \phi \leq C.
\]

**Remark 2.1.** When \((M, g)\) has nonnegative bisectional curvature, one can get

\[
\Delta \phi(\Delta \phi) \geq \Delta F.
\]

By taking \( u = n + \Delta \phi \) as in the arguments above, one can derive the following estimate of \( \Delta \phi \) directly, without even assuming \( \|\phi\|_{L^\infty} \) bound,

\[
0 < n + \Delta \phi \leq C(\|F\|_{W^{1,p_0}}, p_0, M, g, n).
\]
Remark 2.2. For simplicity let $\lambda = 1$ or $-1$ in (1.2). Then (2.3) still holds (with different $C_2$ and $C_3$). Hence Theorem 1.2 holds for (1.2) by the same argument.

3. The gradient estimate

We shall prove Theorem 1.4 in this section.

Proof. We shall assume that $F$ and $\phi$ are both smooth and derive estimate of $\|\nabla \phi\|_{L^\infty}$ depending on $\|F\|_{W^{1,p_0}, p_0}, \|\phi\|_{L^\infty}$, $M$, $g$, $n$. We compute

$$\Delta_\phi(\|\nabla \phi\|^2) = g_\phi^{i\bar{j}} \partial_i \partial_j \left( g^{k\bar{l}} \phi_k \phi_{\bar{l}} \right)$$

(3.1)

$$= g_\phi^{i\bar{j}} \partial_i (\partial_j g^{k\bar{l}} \phi_k \phi_{\bar{l}} + g^{k\bar{l}} \phi_k \partial_j \phi_{\bar{l}})$$

$$= g_\phi^{i\bar{j}} (\partial_j g^{k\bar{l}} \phi_k \phi_{\bar{l}} + \partial_k g^{k\bar{l}} \partial_j (\phi_k \phi_{\bar{l}}) + g^{k\bar{l}} \partial_j (\phi_k \phi_{\bar{l}} + \phi_{\bar{l}} \phi_k))$$

$$+ g_\phi^{i\bar{j}} g^{k\bar{l}} \partial_i (\partial_j \phi_k \phi_{\bar{l}} + \phi_k \partial_j \phi_{\bar{l}}).$$

For simplicity, we can pick up a coordinate system such that at one point, $g_{ij} = \delta_{ij}, \partial_k g_{ij} = \partial_k g_{i\bar{j}} = 0$. We take derivative of (1.1), at the given point,

$$\partial_k \phi_i = \delta_k^i \phi - \Gamma_k^{l\bar{i}} \phi_l; \partial_k g_{ij} = \delta_k^i \phi_j - \Gamma_k^{j\bar{l}} \phi_l.$$

Let $A(t) : \mathbb{R} \rightarrow \mathbb{R}$ be an auxiliary function which will be specified later. We compute

$$\triangle_\phi A(\phi) = g_\phi^{i\bar{j}} \partial_i (A^{'} \phi_{\bar{j}}) = g_\phi^{i\bar{j}} (A^{''} \phi_i \phi_{\bar{j}} + A^{'} \phi_{i\bar{j}}),$$

and

$$\triangle_\phi e^{-A(\phi)} = e^{-A(\phi)} ((A^{'})^2 - A^{''}) g_\phi^{i\bar{j}} \phi_i \phi_{\bar{j}} - e^{-A(\phi)} A^{'} g_\phi^{i\bar{j}} \phi_{i\bar{j}}$$

(3.4)

$$= e^{-A(\phi)} ((A^{'})^2 - A^{''}) g_\phi^{i\bar{j}} \phi_i \phi_{\bar{j}} + e^{-A(\phi)} A^{'} g_\phi^{i\bar{j}} \phi_{i\bar{j}} - ne^{-A(\phi)} A^{'}.$$

Then by (3.3) and (3.5), we compute

$$\triangle_\phi (e^{-A(\phi)} |\nabla \phi|^2) = \triangle_\phi e^{-A(\phi)} |\nabla \phi|^2 + e^{-A(\phi)} \triangle_\phi (|\nabla \phi|^2)$$

$$+ g_\phi^{i\bar{j}} (\partial_i (e^{-A(\phi)}) \partial_j (|\nabla \phi|^2) + \partial_j (|\nabla \phi|^2) \partial_i (e^{-A(\phi)}))$$

$$= e^{-A(\phi)} ((A^{'})^2 - A^{''}) g_\phi^{i\bar{j}} \phi_i \phi_{\bar{j}} |\nabla \phi|^2 + e^{-A(\phi)} A^{'} g_\phi^{i\bar{j}} \phi_{i\bar{j}} |\nabla \phi|^2$$

$$- ne^{-A(\phi)} A^{'} |\nabla \phi|^2 + e^{-A(\phi)} g_\phi^{i\bar{j}} g^{k\bar{l}} R_{i\bar{j}pq} \phi_k \phi_{\bar{l}}$$

$$+ e^{-A(\phi)} g^{k\bar{l}} (F_k \phi_{\bar{l}} + F_{\bar{l}} \phi_k) + e^{-A(\phi)} g_\phi^{i\bar{j}} g^{k\bar{l}} (\phi_k \phi_{i\bar{l}} + \phi_{i\bar{l}} \phi_k)$$

(3.6)

$$- e^{-A(\phi)} A^{'} g_\phi^{i\bar{j}} g^{k\bar{l}} (\phi_i (\phi_k \phi_{\bar{l}} + \phi_{\bar{l}} \phi_k) + \phi_{i\bar{l}} (\phi_k \phi_{i\bar{l}} + \phi_{i\bar{l}} \phi_k)).$$
To estimate the right hand side of (3.6), we can pick up a coordinate system such that, at the given point, \( g_{ij} = \delta_{ij}, \phi_{ij} = \delta_{ij} \phi_i = \lambda_i \). Note that
\[
g^g_{ij} g_{ij} = \sum_i \frac{1}{1 + \lambda_i}, g^g_{ij} \phi_i \phi_j = \sum_i \frac{\phi_i \phi_i}{1 + \lambda_i}.
\]
Then we compute
\[
(A')^2 g^g_{ij} \phi_i \phi_j |\nabla \phi|^2 + g^g_{ij} g^{kl} \phi_k \phi_{jl} - A' g^g_{ij} g^{kl} (\phi_i \phi_k \phi_{jl} + \phi_j \phi_l \phi_k)
\]
(3.7)
\[
= \sum_{k, i} \left( (A')^2 \frac{\phi_i \phi_i}{1 + \lambda_i} \phi_k \phi_k + \frac{\phi_{ki} \phi_{ki}}{1 + \lambda_i} - A' \frac{1}{1 + \lambda_i} (\phi_i \phi_k \phi_{ki} + \phi_l \phi_k \phi_{ki}) \right)
\]
\[
= \sum_{k, i} \frac{1}{1 + \lambda_i} \left( A' \phi_i \phi_k - \phi_{ki} \right) \left( A' \phi_k \phi_l - \phi_{ki} \right) \geq 0.
\]
We shall assume that the bisectional curvature of \( g \) is bounded from below, namely for some \( B \geq 0 \),
\[
R_{ijkl} \geq -B (g_{ij} g_{kl} + g_{il} g_{kj}).
\]
We can then compute
\[
g^g_{ij} g^{kl} g^{pq} R_{ijkl} \phi_k \phi_l = \sum_{k, i, l} \frac{R_{ijkl}}{1 + \lambda_i} \phi_k \phi_l
\]
(3.8)
\[
\geq -B \left( \sum_i \frac{1}{1 + \lambda_i} \right) |\nabla \phi|^2 - B \sum_i \frac{\phi_i \phi_i}{1 + \lambda_i}.
\]
We can also compute
\[
g^g_{ij} g^{kl} (\phi_i \phi_k \phi_j \phi_l) = \sum_i \frac{\lambda_i^2}{1 + \lambda_i} = \Delta \phi - n + \sum_i \frac{1}{1 + \lambda_i},
\]
(3.9)
and
\[
g^g_{ij} g^{kl} (\phi_i \phi_k \phi_j + \phi_j \phi_k \phi_i) = 2 \sum_i \frac{\lambda_i \phi_i \phi_i}{1 + \lambda_i} = 2|\nabla \phi|^2 - \sum_i \frac{2 \phi_i \phi_i}{1 + \lambda_i}.
\]
(3.10)
We can then estimate (3.6), taking (3.7), (3.8), (3.9) and (3.10) into account, that
\[
\triangle \phi \left( e^{-A(\phi)} |\nabla \phi|^2 \right) \geq - A'' e^{-A(\phi)} g^g_{ij} \phi_i \phi_j |\nabla \phi|^2 + (A' - B) e^{-A(\phi)} g^g_{ij} g_{ij} |\nabla \phi|^2 + (2A' - B) e^{-A(\phi)} \phi_i \phi_j - (n + 2) A' e^{-A(\phi)} |\nabla \phi|^2 + e^{-A(\phi)} (\triangle \phi - n + g^g_{ij} g_{ij}) - 2 e^{-A(\phi)} |\nabla F| |\nabla \phi|.
\]
(3.11)
Now we shall specify the function \( A \). Recall that \( \| \phi \|_{L^\infty}, \| F \|_{L^\infty} \) are bounded and let \( C_0 \) be a fixed positive constant such that \( C_0 = 1 + \| \phi \|_{L^\infty} \). We then choose
\[
A(t) = (B + 2)t - \frac{t^2}{2C_0}.
\]
It then follows that
\[
B + 1 \leq A'(\phi) = B + 2 - \frac{\phi}{C_0} \leq B + 3, \quad A''(\phi) = -C_0^{-1}.
\]

It is also easy to see that (for example see [53]),
\[
(3.12) \quad g_{ij}^\phi g_{ij} \geq \sum_i \frac{1}{1 + \lambda_i} \geq (n + \Delta \phi)^{1/(n-1)} \exp(-F/(n-1)).
\]

For simplicity, we then use $\varepsilon$ to denote a fixed positive small number, and $C$ a fixed positive large number, if it is not specified, which depend only on $\|\phi\|_{L^\infty}, \|F\|_{L^\infty}, n, B, M, g$. These constants can vary line by line. We then compute, by (3.12),
\[
-A'' e^{-A(\phi)} g_{ij}^\phi \phi_i \phi_j |\nabla \phi|^2 + (A' - B)e^{-A(\phi)} g_{ij}^\phi g_{ij} |\nabla \phi|^2 \geq e^{-A(\phi)} |\nabla \phi|^2 \left( \frac{1}{C_0} \sum \frac{\phi_i \phi_j}{1 + \lambda_i} + \sum \frac{1}{1 + \lambda_i} \right) \geq e^{-A(\phi)} |\nabla \phi|^2 \left( \frac{1}{C_0} \sum \frac{\phi_i \phi_j}{1 + \lambda_i} + (n + \Delta \phi)^{1/(n-1)} e^{-F/(n-1)} \right) \geq 2\varepsilon_0 |\nabla \phi|^{2+2/n},
\]
where we have applied the inequality, for each $i$,
\[
\frac{\phi_i \phi_j}{1 + \lambda_i} + (n + \Delta \phi)^{1/(n-1)} \geq n(n - 1)^{1-n/n} |\phi_i|^{2/n},
\]
and $\varepsilon_0 = \varepsilon_0(\|\phi\|_{L^\infty}, \|F\|_{L^\infty}, n)$ is a fixed positive constant. Note that for any $\varepsilon > 0$, by Young’s inequality, there exists a constant $C = C(\varepsilon, n)$ such that
\[
(3.14) \quad |\nabla \phi|^2 \leq \varepsilon |\nabla \phi|^{2+2/n} + C(\varepsilon, n).
\]

Now let
\[
u = \exp(-A(\phi))(|\nabla \phi|^2 + 1).
\]
We can then compute, taking (3.11), (3.13) and (3.14) into account, that
\[
\Delta \phi \nu \geq \varepsilon_0 |\nabla \phi|^{2+2/n} + e^{-A(\phi)}(n + \Delta \phi) - C |\nabla F| |\nabla \phi| - C,
\]
where we have used that, by (3.5),
\[
\Delta \phi (e^{-A(\phi)}) \geq -C.
\]

We then compute, for $p > 0$, that
\[
(3.16) \quad \Delta \phi (u^p) = pu^{p-1} \Delta \phi u + p(p - 1)u^{p-2}|\nabla u|_\phi^2.
\]

We then compute, by (3.15),
\[
\int_M u^{p-1} \Delta \phi u dvol_\phi \geq \int_M u^{p-1}(\varepsilon_0 |\nabla \phi|^{2+2/n} + e^{-A(\phi)}(n + \Delta \phi) - C |\nabla F| |\nabla \phi| - C) dvol_\phi.
\]
It then follows, taking (3.16) into account, that
\[
\int_M p(p - 1)u^{p-2}|\nabla u|^2_{\phi} + pu^{p-1}(\varepsilon)|\nabla \phi|^{2+2/n} + e^{-A(\phi)}(n + \Delta \phi)) \leq \int_M pu^{p-1}(C|\nabla F||\nabla \phi| + C)dvol_{\phi}.
\] (3.17)

When \( p \geq 1 \), we can compute that
\[
p(p - 1)u^{p-2}|\nabla u|^2_{\phi} + pu^{p-1}e^{-A(\phi)}(n + \Delta \phi) \geq \varepsilon p\sqrt{p - 1}u^{p-3/2}|\nabla u|.
\] (3.18)

Note that \( u \) is bounded away from 0, and \( |\nabla \phi| \leq Cu^{1/2} \). It then follows from (3.17), (3.18) that
\[
\int_M \sqrt{p - 1}u^{p-3/2} |\nabla u|dvol_g \leq C \int_M u^{p-1/2}(|\nabla F| + 1)dvol_g.
\] (3.19)

We can rewrite (3.19) as, for \( p \geq 3/4 \), that
\[
\int_M |\nabla (u^p)|dvol_g \leq C \sqrt{p} \int_M u^p(|\nabla F| + 1)dvol_g.
\] (3.20)

One can actually get the following,
\[
\int_M (|\nabla (u^p)| + C_1\sqrt{p}u^{p+1/2+1/n})dvol_g \leq C \sqrt{p} \int_M u^p(|\nabla F| + 1)dvol_g,
\]
for some positive constant \( C_1 \); but we shall not need this. To get \( L^\infty \) bound of \( u \), we use the iteration method. Recall the following Sobolev inequality; there exists a positive constant \( c = c(M, g, n) \) such that for \( f \in W^{1,1}(M, g) \), we have
\[
\|f\|_{L^{2n/2n-1}} \leq c \int_M (|\nabla f| + Vol(M, g)^{-1/2n}|f|)dvol_g.
\] (3.21)

Taking \( f = u^p \), it then follows from (3.20) and (3.21) that
\[
\|u^p\|_{L^{2n/2n-1}} \leq C \sqrt{p} \int_M u^p(|\nabla F| + 1).
\] (3.22)

Now if \( \|\nabla F\|_{L^{p_0}} \) is bounded, then by the Hölder inequality, we can get
\[
\int_M u^p(|\nabla F| + 1)dvol_g \leq \left(\int_M u^{p_0}dvol_g\right)^{1/q_0} \left(\int_M (|\nabla F| + 1)^{p_0}dvol_g\right)^{1/p_0},
\] (3.23)

where \( 1/p_0 + 1/q_0 = 1 \). So we can get that, from (3.22) and (3.23),
\[
\|u\|_{L^{2n/2n-1}} \leq (pC_2)^{1/2p}\|u\|_{L^{p_0}},
\] (3.24)

where \( C_2 = C_2(\|\phi\|_{L^\infty}, \|F\|_{W^{1,p_0}(M, g, n)}) \). When \( p_0 > 2n \), then \( 1 < q_0 < 2n/(2n - 1) \). Let \( b = q_0^{-1}2n/(2n - 1) > 1 \), and take \( p = q_0^{-1}b^k \) in (3.24) for \( k \geq 0, k \in \mathbb{N} \), it then follows that
\[
\log \|u\|_{L^\infty} \leq \sum_{k=0}^{\infty} \frac{q_0}{b^k} (\log(q_0^{-1}b^k) + \log C_2) + \log \|u\|_{L^1}.
\] (3.25)
THE COMPLEX MONGE-AMPERE EQUATION

It is clear that
\[ \|u\|_{L^1} \leq C \int_M |\nabla \phi|^2 dvol_g + C = -C \int_M \phi \Delta \phi dvol_g + C \leq C. \]

It then follows that
\[ (3.26) \quad \|u\|_{L^\infty} \leq C_3 = C_3(\|\phi\|_{L^\infty}, \|F\|_{W^{1,p}_0}, p_0, M, g, n). \]

\[ \square \]

**Remark 3.1.** Let \( \lambda = 1 \) or \(-1\) in (1.2), then (3.15) still holds. Hence Theorem 1.4 holds for (1.2).

### 4. Hölder Estimates of Second Order and Solve the Equation

To apply the method of continuity to solve (1.1), one needs to obtain Hölder estimates of second order derivatives of \( \phi \) and then higher order regularity follows from Schauder theory. For the Monge-Ampère equations, \( C^3 \) estimates date back to Calabi’s seminal third order estimates [13]; the idea is used in [53] to obtain \( C^3 \) estimates of \( \phi \) for (1.1). Later on Evans [30], Krylov [39, 40] proved that Hölder estimates of second order hold for fully nonlinear concave uniform elliptic operators. All these results are originally stated for \( F \) with second derivatives or higher. But the Hölder estimates of second order derivatives are studied for uniform elliptic operators when right hand side has weaker regularity [46, 10, 6, 52]. In particular, for the complex Monge-Ampère equation, Blocki [6] proved that the Hölder estimates hold when \( F \) is Lipschitz and \( \Delta \phi \) is bounded. With slight modifications of his argument, one can show that such an estimate holds when \( F \in W^{1,p_0} \). The estimates can be localized; find a local potential \( \Phi_0 \) in a open domain \( \Omega \subset M \) such that \( g_{ij} = \Phi_0 \delta_{ij} \). We can rewrite (1.1) in \( \Omega \) as
\[ (4.1) \quad \det(v_{ij}) = f, \]
where \( v = \Phi_0 + \phi, f = \exp(F) \det(g_{ij}) \). In particular, we can assume that \( 0 < \lambda \leq \Delta v \leq \Lambda \) and \( f \in W^{1,p_0} \). Suppose \( \|v\|_{L^\infty}, \|f\|_{W^{1,p_0}} \leq K \), then we have

**Lemma 4.1.** For any \( \Omega' \subset \subset \Omega \),
\[ \|v\|_{C^{2,\alpha}(\Omega')} \leq C(\Omega, \Omega', \lambda, \Lambda, K), \]
where \( \alpha = \alpha(\Omega, \Omega', \lambda, \Lambda, K) \).

**Proof.** The result is proved in [6] (see Theorem 3.1 in [6]) when \( F \) is Lipschitz. Our proof here is a slight modification. The idea is the same as in Evans-Krylov theory; but to deal with the case when \( F \) has weaker regularity, one needs to use the Harnack inequality as in [36] Theorem 8.18. We use notations as in [6]. We observe that (3.2) in [6] holds with \( f^* \in L^{p_0} \) if \( F \in W^{1,p_0} \). Hence the Harnack inequality (Theorem 8.18 in [36]) still applies. By the Sobolev embedding, \( f \in C^\alpha \) for \( \alpha = 1 - 2n/p_0 \), then (3.3) in [6] holds with \( |z - w| \) replaced by \( |z - w|^{\alpha/n} \). Then the argument as in ([36] Section 17.4) or ([6] Theorem 3.1) applies. \( \square \)
Since \((M, g)\) is a smooth compact manifold, a standard covering argument applies and one can get global Hölder estimate of second order derivatives of \(\phi\) by using Lemma 4.1. Now we are in the position to prove Theorem 1.1.

**Proof.** If \(F \in W^{1,p_0}\) on \(M\) such that \(\|F\|_{W^{1,p_0}} \leq K\) for some positive constant \(K\). Let \(F^k\) be a sequence of smooth functions such that \(F_k \to F\) in \(W^{1,p_0}\); in particular, We can assume \(\|F_k\|_{W^{1,p_0}} \leq K + 1\) for any \(k\). By Yau’s result [53], there is a smooth solution \(\phi^k\) which solves

\[
\log \left( \frac{\det(g_{ij} + \phi^k_{ij})}{\det(g_{ij})} \right) = F^k
\]

such that \((g_{ij} + \phi^k_{ij}) > 0\) with normalized condition \(\int_M \phi^k dg = 0\). Since \(\|\phi^k\|_{L^\infty}\) is uniformly bounded [38] independent of \(k\), we can get that \(|\nabla \phi^k|, |\Delta \phi^k|\) are both bounded by Theorem 1.2 and Theorem 1.4, independent of \(k\). We can then get uniform Hölder estimate of second order by Lemma 4.1. To get \(W^{3,p_0}\) estimate, we can localize the estimate as follows. Let \(\partial\) denote an arbitrary first order differential operator in a domain \(\Omega \subset M\). Once Hölder estimate of second order is proved, we compute in \(\Omega\)

\[
\Delta_{\phi^k}(\partial \phi^k) = \partial(F^k - \log(\det(g_{ij})) - g_{ij} \partial g_{ij}).
\]

It then follows from \(L^p\) theory, for example see [36] Chapter 9, for any \(\Omega' \subset \subset \Omega\),

\[
\|\partial \phi^k\|_{W^{2,p_0}(\Omega')} \leq C(\Omega, \Omega', p_0, K).
\]

In particular we get,

\[
\|\phi^k\|_{W^{3,p_0}(M, g)} \leq C(p_0, K).
\]

Then there is a subsequence of \{\(\phi^k\)\} which converges to \(\phi\) and \(\phi\) is in \(W^{3,p_0}\) such that

\[
(g_{ij} + \phi_{ij}) > 0
\]

defines a \(W^{1,p_0}\) (and \(C^\alpha, \alpha = 1 - 2n/p_0\)) Kähler metric; hence \(\phi\) is a classical solution of (1.1). The solution of (1.1) is actually unique [12, 53]. \(\square\)

The Sobolev embedding theorem asserts that \(F \in C^\alpha, \alpha = 1 - 2n/p_0\) when \(F \in W^{1,p_0}\); while \(p_0 > 2n\) is exactly on the border line, namely when \(p_0 \leq 2n\), \(F\) does not have to be in \(L^\infty\). We might then believe that the similar results hold for \(F \in C^\alpha\), namely when \(F \in C^\alpha, \alpha \in (0, 1)\), (1.1) has a classical solution \(\phi \in C^{2,\alpha}\).

The gradient estimate and estimate of \(\Delta \phi\) in the present paper are both global; actually such estimates cannot be made purely local, for example see [5, 35]. But one might ask whether such an estimate holds or not for the Dirichlet problem of the complex Monge-Ampère equations in a bounded strongly pseudoconvex domain in \(\mathbb{C}^n\). In particular, one might consider the complex Monge-Ampère equation in a bounded strongly pseudoconvex domain \(\Omega \subset \mathbb{C}^n\),

\[
\log \det(\phi_{ij}) = f,
\]

where \(f\) is Lipschitz (or \(C^\alpha\)), and \(\phi|_{\partial \Omega} = 0\); is \(\phi \in C^{2,\alpha}\)? This problem was studied in [45] when \(f\) is Lipschitz; however the proof in [45] has a gap [7, 35]. In essence, one would like to understand if the results for the real Monge-Ampère equation (}
see Caffarelli [11], Trudinger-Wang [51] for example) hold or not for the complex Monge-Ampère equation.

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