On a coupled system of a Ginzburg-Landau equation with a quasilinear conservation law

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Abstract

We study a coupled system of a complex Ginzburg–Landau equation with a quasilinear conservation law

\[
\begin{cases}
    e^{-i\theta} u_t = u_{xx} - |u|^2 u - \alpha g(v)u \\
    v_t + (f(v))_x = \alpha (g'(v)|u|^2)_x
\end{cases}, \quad x \in \mathbb{R}, \ t \geq 0
\]

which can describe the interaction between a laser beam and a fluid flow (see [Aranson, Kramer, Rev. Med. Phys. 74 (2002)]). We prove the existence of a local in time strong solution for the associated Cauchy problem and, for a certain class of flux functions, the existence of global weak solutions. Furthermore we prove the existence of standing wave solutions of the form \((u(t, x), v(t, x)) = (U(x), V(x))\) in several cases.

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1 Introduction

In [6], [7] and [8] we studied the well-posedness of several universal models describing the interaction between long and short waves. These unidimensional systems consist on the coupling between a nonlinear Schrödinger equation and a conservation law, and can be put in the general form

\[
\begin{cases}
    iu_t = u_{xx} - |u|^2 u - \alpha g(v)u \\
    v_t + (f(v))_x = \alpha (g'(v)|u|^2)_x
\end{cases}, \quad x \in \mathbb{R}, \ t \geq 0, \quad (1)
\]
where \( f \) and \( g \) are regular functions, \( u \) is complex-valued (the transverse component of a field in complex notation) and \( v \) is real-valued (a concentration). These models, originally derived by Benney (\[2\]) in the case where \( g \) is linear, have been successfully applied to several physical contexts. In water waves theory, applications include the interaction between gravity-capillary waves in a two-layer fluid, when the group velocity of the surface waves coincides with the phase velocity of the internal waves (see \[23\], \[24\], \[25\].)

In this paper we extend some results obtained in \[6\], \[7\], \[8\] to the case where the Schrödinger equation is replaced by a cubic complex Ginzburg-Landau (CGL) equation. More precisely, for 

\[
\alpha > 0 \quad \text{and} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},
\]

we consider the system

\[
\begin{aligned}
e^{-i\theta} u_t &= u_{xx} - |u|^2u - \alpha g(v)u \quad (a), \\
v_t + (f(v))_x &= \alpha(g'(v)|u|^2)_x \quad (b),
\end{aligned}
\]

for some initial data

\[
u(0, x) = u_0(x), \quad v(0, x) = v_0(x),
\]

which can be used to model the interaction between a laser beam and a fluid flow (\[27\]).

The first two sections of the present paper are devoted to the study of the Cauchy problem (3-4): in Section 2 we prove the following result concerning local strong solutions:

**Theorem 1.1 (Existence and uniqueness of local strong solutions).** Let \((u_0, v_0) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})\). Assume that \(f \in C^3(\mathbb{R})\), with \(f(0) = 0\), and \(g\) is of the form

\[ g(v) = \pm v + \rho, \quad \rho \in \mathbb{R}. \]

Then there exists \(T > 0\) and a unique strong solution

\[
(u, v) \in C^j([0, T]; H^{3-2j}(\mathbb{R})) \times C^j([0, T]; H^{2-j}(\mathbb{R})) \quad (j = 0, 1)
\]

to the Cauchy Problem (3)-(4).

Moreover, in Section 3, we study the global existence of weak solutions to the Cauchy problem for a wider class of flux functions:

**Theorem 1.2 (Existence of global weak solutions).** Let \((u_0, v_0) \in (H^1(\mathbb{R}))^2\). Assume that \(g \in C^3(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})\) with \(g \geq 0\) and \(g^{(3)}\) not affine in any open set. Moreover, assume that \(f\) is of the form

\[ f(v) = av^2 - bv^3, \quad a, b > 0. \]
Then, there exists a constant $\alpha_0 > 0$, and, for each $0 < \alpha \leq \alpha_0$, functions $u$ and $v$ with

$$u \in L^\infty_{\text{loc}}([0, +\infty[; H^1(\mathbb{R})) \cap C([0, +\infty[; L^2(\mathbb{R})),$$

$$u_t \in L^2_{\text{loc}}([0, +\infty[; L^2(\mathbb{R})),$$  \quad u(0, \cdot) = u_0(\cdot)$$

and

$$v \in L^\infty_{\text{loc}}([0, +\infty[; (L^2 \cap L^4)(\mathbb{R}))$$

such that, for all $\phi \in C^1_0(\mathbb{R} \times [0, +\infty[)$ and $\psi \in C^1_0(\mathbb{R} \times [0, +\infty[)$,

$$\int_{\mathbb{R} \times [0, +\infty[} e^{-i\theta} u_t \phi dx \, dt + \int_{\mathbb{R} \times [0, +\infty[} u_{xx} \phi_x dx \, dt$$

$$+ \int_{\mathbb{R} \times [0, +\infty[} |u|^2 u \phi dx \, dt + \alpha \int_{\mathbb{R} \times [0, +\infty[} g(v) u \phi dx \, dt = 0$$

and

$$\int_{\mathbb{R} \times [0, +\infty[} v \psi_t dx \, dt + \int_{\mathbb{R} \times [0, +\infty[} f(v) \psi_x dx \, dt$$

$$+ \int_{\mathbb{R}} v_0(x) \psi(x, 0) dx$$

$$+ \alpha \int_{\mathbb{R} \times [0, +\infty[} (g'(v)|u|^2)_x \psi dx \, dt = 0.$$

This result will be obtained by applying the $L^p$ version of the compensated compactness method of F. Murat and L. Tartar (cf. [19]) introduced by M.E. Schonbek (cf. [14]) and the vanishing viscosity method to the approximating system $(\epsilon > 0)$ with the same initial data

$$\begin{cases}
  e^{-i\theta} u^\epsilon_t = u_{xx}^\epsilon - |u^\epsilon|^2 u^\epsilon - \alpha g(v^\epsilon) u^\epsilon \quad (a) \\
  v^\epsilon_t + (f(v^\epsilon))_x = \alpha (g'(v^\epsilon)|u^\epsilon|^2)_x + \epsilon v_{xx}^\epsilon \quad (b)
\end{cases}, \quad x \in \mathbb{R}, \, t \geq 0, \quad (5)$$

In the second part of the paper (Section [4]), we study the existence of standing wave solutions for $g(v) = v + \rho, \rho > 0$ (which does not satisfy (6)) and $f(s) = as^2 - bs^3$. More precisely, we will look for standing waves of the form $u(t, x) = U(x), V(t, x) = V(x)$, with $U, V$ real solutions of

$$\begin{cases}
  U'' - U^3 - \alpha (V + \rho) U = 0 \\
  (aV^2 - bV^3)' = \alpha (U^2)' \quad \text{in } \mathbb{R}
\end{cases} \quad (6)$$

We denote by $H^1_{\text{rad}}(\mathbb{R})$ the set of functions in $H^1(\mathbb{R})$ which are even and decreasing in $|x|$.

**Theorem 1.3 (Existence of Bound States: focusing case).** Take $\rho, \alpha > 0$. Then:
1. Assume that \( a = 0 \). Then there exists \( b > 0 \) such that (1) admits a solution \((U, V)\), with
\[
U \in H^1_{rd}(\mathbb{R}) \text{ positive}, \quad V(x) = -\left(\frac{\alpha}{b}\right)^{1/3} U^{2/3}(x).
\]

2. Assume that \( b = 0 \). Then there exists \( a > 0 \) such that (1) admits a solution \((U, V)\), with
\[
U \in H^1_{rd}(\mathbb{R}) \text{ positive}, \quad V(x) = -\left(\frac{\alpha}{a}\right)^{1/2} U(x).
\]

3. There exists \( a, b > 0 \) such that (1) admits a solution \((U, V)\) with \( U > 0 \) and \( U \in H^1_{rd}(\mathbb{R}) \), and \( V < 0 \) with \( aV^2 - bV^3 = \alpha U^2 \) a.e..

**Remark 1.4.** Observe that, since \( \int ((U')^2 + \alpha(V + \rho)U^2 + U^4) = 0 \), there are no solutions \((U, V)\) with \( V \) positive and \( \rho > 0 \) in the focusing case.

We obtain stronger results in focusing case (which cannot be considered in the first part of the paper), that is, we prove existence of real solutions of
\[
\begin{cases}
U'' + U^3 - \alpha(V + \rho)U = 0 \\
(aV^2 - bV^3)' = \alpha(U'^2).
\end{cases}
\]

**Theorem 1.5 (Existence of Bound States: defocusing case).** Take \( \rho > 0 \).

1. Let \( \alpha > 0 \) and assume that \( a = 0 \) and \( b > 0 \). Then (1) admits a solution \((U, V)\), with
\[
U \in H^1_{rd}(\mathbb{R}) \text{ positive}, \quad V(x) = -\left(\frac{\alpha}{b}\right)^{1/3} U^{2/3}(x).
\]

2. Let \( \alpha > 0 \) and assume that \( a > 0 \) and \( b = 0 \). Then (1) admits two solutions \((U_1, V_1)\) and \((U_2, V_2)\), with
\[
U_1 \in H^1_{rd}(\mathbb{R}) \text{ positive}, \quad V_1(x) = -\left(\frac{\alpha}{a}\right)^{1/2} U_1(x).
\]

and
\[
U_2 \in H^1_{rd}(\mathbb{R}) \text{ positive}, \quad V_2(x) = \left(\frac{\alpha}{a}\right)^{1/2} U_2(x).
\]

3. Let \( a, b > 0 \). Then, for sufficiently small \( \alpha > 0 \), (1) admits two pairs of solutions \((U_1, V_1)\) and \((U_2, V_2)\), with
\[
U_1 > 0, \quad V_1 > 0 \quad \text{and} \quad U_2 > 0, \quad V_2 < 0;
\]
\[
U_i \in H^1(\mathbb{R}) \text{ and } aV_i^2 - bV_i^3 = \alpha U_i^2 \text{ for } i = 1, 2.
\]
These last two theorems complement some results in [4], [5] and [9]. The techniques involve variation methods for elliptic problems, and consist on either minimization with $L^p$-constraints or minimizations using Nehari-type manifolds.

The CGL equation describes (cf.[1]) a large class of phenomena like phase transitions, superconductivity, superfluidity and Bose-Einstein condensation to liquid crystals. The coupling of a CGL equation with a quasilinear conservation law can describe the interaction between a laser beam and a fluid flow. Other examples of coupling can be considered ([16] and [22]). This kind of interactions are particular cases of the general theory of the interactions between short and long waves motivated by the seminal paper of D.J.Benney ([2]) and first studied in the special case of $f$ linear, $g(v) = v$ and $\theta = \frac{\pi}{2}$ (Schrödinger equation) by M.Tsutsumi and S.Hatano (cf. [20] and [21]).

2 Existence and uniqueness of local strong solutions

The main idea to establish Theorem 1.1 is to apply a variant of T. Kato’s Theorem 6 in [11] by introducing a change of the dependent variables $(u,v)$, as done in [8] (see also [15] and [17]).

Let us put, for $f$ and $g$ verifying the assumptions of Theorem 1.1,

$$F = u_t.$$  \hspace{1cm} (8)

Equation (3-a) can then be rewritten as

$$u = (\partial_{xx} - 1)^{-1}(|u|^2 u + u(\alpha g(v) - 1) + e^{-i\theta} F).$$  \hspace{1cm} (9)

Also, by differentiating (3-a) with respect to $t$ and using equation (3-b), we obtain that

$$F_t - e^{i\theta} F_{xx}$$

$$= -e^{i\theta} (2|u|^2 F + u^2 F + \alpha Fg(v) - \alpha ug'(v)f'(v)v_x + \alpha^2 ug'(v)(g'(v)|\tilde{u}|^2)_x).$$

Hence, instead of the Cauchy Problem (3)-(4), we will consider the following alternative problem, which has the advantage of not presenting derivative losses in the nonlinear term:

$$\begin{cases}
F_t - e^{i\theta} F_{xx} = K(t, F, v) \\
v_t + (f(v))_x = \alpha(g'(v)|\tilde{u}|^2)_x,
\end{cases}$$  \hspace{1cm} (10)

with

$$K(t, F, v)$$

$$= -e^{i\theta} (2|u|^2 F + u^2 F + \alpha Fg(v) - \alpha ug'(v)f'(v)v_x + \alpha^2 ug'(v)(g'(v)|\tilde{u}|^2)_x),$$
where

\[ u(t, x) = u_0(x) + \int_0^t F(s, x)ds, \]

and for initial data

\[ F(0, x) = F_0(x) := e^{i\theta}(u_{0xx}(x) - |u_0(x)|^2u_0(x)) - \alpha g(v_0(x))u_0(x) \in H^1(\mathbb{R}) \]

\[ v(0, x) = v_0(x) \in H^2(\mathbb{R}). \]

(11)

Concerning this new problem, we will show the following:

**Lemma 2.1.** Let \((F_0, v_0) \in H^1(\mathbb{R}) \times H^2(\mathbb{R})\). Then, there exists \(T > 0\) and a unique strong solution

\[ (F, v) \in C^j([0; T]; H^{1-2j}(\mathbb{R})) \times C^j([0; T]; H^j(\mathbb{R})), \quad (j = 0; 1) \]

to (10) with \(F(0, \cdot) = F_0\) and \(v(0, \cdot) = v_0\).

**Proof.** We begin by setting this Cauchy Problem in the framework of real spaces. Putting

\[ F_1 = \Re(F), F_2 = \Im(F), u_1 = \Re(u), u_2 = \Im(u), \]

\[ F_{10} = \Re(F_0) \text{ and } F_{20} = \Im(F_0), \]

with \(U = (F_1, F_2, v)\), system (10) can be rewritten as

\[ U_t + A(U)U = h(t, U) \]

(12)

for initial data

\[ U(0, x) = (F_1(0, x), F_2(0, x), v(0, x)) = (F_{10}(x), F_{20}(x), v_0(x)), \]

(13)

where

\[ A(U) = \begin{bmatrix}
-\cos(\theta) \partial_{xx} & \sin(\theta) \partial_{xx} & 0 \\
-\sin(\theta) \partial_{xx} & -\cos(\theta) \partial_{xx} & 0 \\
0 & 0 & f'(v)\partial_x
\end{bmatrix} \]

(14)

and

\[ h(t, U) = \begin{bmatrix}
\Re(K(t, F, v)) \\
\Im(K(t, F, v)) \\
\alpha(g'(v)|\tilde{u}|^2)_x
\end{bmatrix} \]

(recall that \(g'(v) = \pm 1\)).

(15)

We now note that the operator \(e^{i\theta} \partial_{xx}, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\), is the infinitesimal generator of an analytic semigroup of contractions \((T_0(t))_{t \geq 0}\) in \(L^2(\mathbb{R})\), with domain \(H^2(\mathbb{R})\), verifying the estimates (see p. 248 in [3])

\[ \|T(t)\psi\|_r \leq (\cos(\theta))^{-\frac{1}{2}(1\frac{1}{p} + \frac{1}{r})}t^{-\frac{1}{2}\left(\frac{1}{r} + \frac{1}{2}\right)}\|\psi\|_r, \forall t > 0, 1 \leq p \leq r \leq +\infty. \]

(16)
Hence, if we set $X = (H^{-1}(\mathbb{R}))^2 \times L^2(\mathbb{R})$, $Y = (H^1(\mathbb{R}))^2 \times H^2(\mathbb{R})$, then

$$A : U = (F_1, F_2, v) \in W_R \longrightarrow G(X, 1, \beta),$$

where $R > 0$, $W_R = \{U \in Y : \|U\|_Y < R\}$ and $G(X, 1, \beta)$ denotes the set of all linear operators $D$ in $X$ such that $-D$ generates a $C_0$-semigroup \{e^{-tD}\}_{t \geq 0}$ with, for all $t \geq 0$,

$$\|e^{-tD}\|_{L(X)} \leq e^\beta t, \quad \beta = \frac{1}{2}\|f''(v(x))v'(x)\|_{\infty} \leq \gamma(R),$$

where $\gamma$ is a continuous function.

Arguing as in the proof of Lemma 2.1 in [8] and by adapting the general Kato’s theory for quasilinear systems ([11]), we prove the existence of local strong solutions for the Cauchy Problem (12)–(13). □

**Proof of Theorem 1.1.** Let $(u_0, v_0) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$. For $F_0 \in H^1(\mathbb{R})$ defined by (11), we consider the solution $(F, v)$ given by Lemma 2.1. Then, putting

$$u(t, x) = u_0(x) + \int_0^t F(s, x)dx, \quad (17)$$

we deduce

$$u_{tt}(t, x) = F_t(t, x) = e^{i\vartheta}F_{xx} + K(t, F, v) =$$

$$= e^{i\vartheta}F_{xx} - e^{i\vartheta}[(2|u|^2F + u^2F + \alpha Fg(v)) - \alpha ug'(v)f'(v)v_x + \alpha^2 ug'(v)(g(v)|u|^2)_x]$$

$$= e^{i\vartheta}F_{xx} - e^{i\vartheta}[(2|u|^2F + u^2F + \alpha Fg(v)) - \alpha ug'(v)f'(v)v_x + \alpha ug'(v)v_t + (f(v)v_x)_x)]$$

$$= e^{i\vartheta}F_{xx} - e^{i\vartheta}[(2|u|^2F + u^2F + \alpha Fg(v)) - \alpha ug'(v)v_t],$$

hence

$$e^{-i\vartheta}u_{tt} = F_{xx} - 2|u|^2F - u^2F - \alpha Fg(v) - \alpha u\frac{\partial}{\partial t}(gv).$$

From $u_{txx} = F_{xx}$ and $u_t = F$ we derive

$$e^{-i\vartheta}u_{tt} = (u_{xx})_t - 2|u|^2u_t - u^2\pi_t - \alpha u_t g(v) - \alpha u\frac{\partial}{\partial t}(g(v))$$

$$= (u_{xx})_t - \alpha \frac{\partial}{\partial t}(ug(v)) - \frac{\partial}{\partial t}(|u|^2u),$$

and we obtained that

$$(e^{-i\vartheta}u_t - u_{xx} + \alpha g(v)u + |u|^2u)_t = 0.$$
We obtained that
\[ e^{-i\theta} u_t = u_{xx} - |u|^2 u - \alpha u g(v). \]
Noticing that \( u_{xx} = e^{-i\theta} u_t + |u|^2 u + \alpha u g(v), \)
\[ u = (\partial_{xx} - 1)^{-1}(|u|^2 u + u(\alpha g(v) - 1) + e^{-i\theta} F) = \tilde{u}, \]
and so
\[ v_t + (f(v))_x = \alpha (g'(v)|u|^2)_x \]
we showed that \((u,v)\) is a solution of the Cauchy Problem \((3)-(4)\). Also, from Lemma \((2.1)\) and \((17)\), we obtain that \( u \in C([0,T]; H^3(\mathbb{R})). \)

### 3 Existence of a weak solution

We begin this section by deriving an \( a \) priori estimate for the solutions of system \((5)\) which extends Lemma 1.2 in \([6]\) and Lemma 2.2 in \([7]\):

**Proposition 3.1.** Let
\[(u^\epsilon, v^\epsilon) \in C([0, +\infty[; (H^1(\mathbb{R}))^2) \cap W^{1,2}_{loc}(0, +\infty[; (L^2(\mathbb{R}))^2)\]
be a solution of system \((6)\) with initial data \((u_0^\epsilon, v_0^\epsilon) = (u_0, v_0) \in (H^1(\mathbb{R}))^2, \)
with \( f \) and \( g \) verifying the assumptions of Theorem \((1.2)\). Then, there exists a constant \( \alpha_0 > 0 \) independent of \( \epsilon \) and a positive function \( h \in C([0, +\infty[), \)
independent of \( \alpha \) and \( \epsilon \) such that for \( \alpha \leq \alpha_0, \epsilon \leq 1 \) and for all \( t \geq 0 \) we have
\[
\int |u^\epsilon|^2 + \int |u^\epsilon_x|^2 + \int (\cdot)^2 + \int (\cdot)^4 + \int_0^t \int |u^\epsilon|^2 dx d\tau \\
+ \int_0^t \int |u^\epsilon|^2 dx d\tau + \epsilon \int_0^t \int (v^\epsilon_x)^2 dx d\tau \leq h(t). \quad (18)
\]

**Proof.** For convenience, we drop the superscript \( \epsilon \). We multiply equation \((6a)\) by \( \tilde{u}_t \) and integrate in \( \mathbb{R} \) to obtain, taking the real part and denoting \( \int_\mathbb{R} \cdot dx \) simply by \( \int \cdot \),
\[
\cos \theta \left| u_t \right|^2 + \frac{1}{2} \frac{d}{dt} \left( |u|^2 + \frac{1}{2} |u|^4 + \alpha g(v)|u|^2 \right)
\]
Now, multiplying \( 5-a \) by \( \epsilon \) where
\[
\text{Finally, multiplying } (5-b) \text{ by } v \text{ and so, for } t \text{ we obtain, for } t \text{ we obtain for } t \text{ and, since }
\]
\[
and, since
\]
\[
we obtain for } t \in [0, +\infty[,
\]
\[
\frac{d}{dt} \left( \frac{1}{2} \int u^2 + \sin \theta \Im \left( \int u \overline{u}_x \right) \right) + \epsilon \int (v_x)^2 + 2 \cos \theta \Re \left( \int u \overline{u}_x \right) = 0. \quad (21)
\]
\[
\int (v_x)^2 + 2 \cos \theta \Re \left( \int u \overline{u}_x \right) = 0. \quad (20)
\]

Finally, multiplying (5-b) by \( v \) and integrating, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int v^2 = \int v \left( \alpha (g'(v)|u|^2)_x + \epsilon v_{xx} - (f(v))_x \right)
\]
\[
= -\epsilon \int (v_x)^2 + 2 \alpha \int \Re(g(v)u \overline{u}_x)
\]
\[
= -\epsilon \int (v_x)^2 + 2 \Re \left( \int e^{-i\theta} u \overline{u}_x \right)
\]
\[
= -\epsilon \int (v_x)^2 + 2 \cos \theta \Im \left( \int u \overline{u}_x \right).
\]

Now, multiplying \( 5-a \) by \( e^{i\theta} \overline{u} \), integrating in \( \mathbb{R} \) and taking the real part, we obtain, for } t \in [0, +\infty[,
\[
\frac{d}{dt} \int |u|^2 + 2 \cos \theta \int \left( |u_x|^2 + |u|^4 + \alpha g(v)|u|^2 \right) = 0.
\]

where
\[
F(v) = \int_0^v f(\xi) d\xi.
\]
From (21) we easily derive, for $t \in [0, +\infty[$,
\[
\epsilon \int_0^t \int (v_x)^2 \, dx \, d\tau + \frac{1}{2} \int v^2 \leq |\sin \theta| \|u\|_2 \|u_x\|_2 \\
+ 2 \cos \theta \left( \int_0^t \|u_x\|_2^2 \, d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|u_t\|_2^2 \, d\tau \right)^{\frac{1}{2}} + M_0,
\] (22)
with
\[
M_0 = \frac{1}{2} \int v_0^2 + \sin \theta \Im \left( \int u_0 u_{0x} \right).
\] (23)

Now, recall that
\[
g(\xi) \geq 0, \, \xi \in \mathbb{R}.
\] (24)

Since $\alpha > 0$, we derive from (20), for $t \in [0, +\infty[$,
\[
\int |u|^2 + 2 \cos \theta \left( \int_0^t \int |u_x|^2 \, dx \, d\tau + \int_0^t \int |u|^4 \, dx \, d\tau \right) \leq \int |u_0|^2.
\] (25)

**Remark 3.2.** If the support of $g'$ is contained in $[0, +\infty[$ it is not difficult, for $v_0 \geq 0$ a.e., to deduce, from (24-b) and for a fixed $\epsilon > 0$, that $v(t, x) \geq 0$ a.e. in $[0, +\infty[ \times \mathbb{R}$. Indeed, putting $v_- := -\min(v, 0)$,
\[
v_t v_- = (v_-)_t v_-, \quad v_x v_- = (v_-)_x v_- \\
\text{and} \quad v_x (v_-)_x = ((v_-)_x)^2 \quad (\text{cf. [10], chap. II}).
\]

Multiplying (24-b) by $v_-$ and integrating in space and in the time interval $[0, t]$ yields
\[
\frac{1}{2} \int (v_-)^2 - \frac{1}{2} \int (v_{0-})^2 + \epsilon \int_0^t \int ((v_-)_x)^2 \, dx \, d\tau \\
\leq \frac{\epsilon}{2} \int_0^t \int ((v_-)_x)^2 \, dx \, d\tau + \frac{1}{2\epsilon} \int_0^t \int (f'(v))^2 (v_-)^2 \, dx \, d\tau,
\]
from where we deduce that
\[
\int (v_-)^2 \leq \frac{1}{2\epsilon} \int_0^t \int (f'(v))^2 (v_-)^2 \, dx \, d\tau, \, t \geq 0,
\]
which implies, since $f'(v) \in L^\infty$, that $v_- = 0$ a.e. (by Gronwall’s inequality).
In this case, and for $v_0 \geq 0$ a.e., we can replace (24) by $g(\xi) \geq 0$ for $\xi \geq 0$.

From (22) and (25) we derive, for $t \in [0, +\infty[$,
\[
\epsilon \int_0^t \int (v_x)^2 \, dx \, d\tau + \frac{1}{2} \int v^2 \leq c \left( 1 + \|u_x\|_2^2 + \int_0^t \|u_t\|_2^2 \, d\tau \right)^{\frac{1}{2}},
\] (26)
with $c > 0$ independent of $\alpha$ and $\epsilon$. 

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**3 Existence of a weak solution**
To simplify, we take $a = b = 1$, so that $f(v) = v^2 - v^3$. We have $F(v) = \frac{1}{4}v^3 - \frac{1}{4}v^4$ and $f'(v) = 2v - 3v^2$, and so, since $2v \leq 3v^2 + \frac{1}{4}$,

$$-\epsilon \int f'(v)(v_x)^2 dx d\tau \geq -\frac{1}{3}\epsilon \int (v_x)^2. \quad (27)$$

For positive constants $c_0$ and $c_1$,

$$-\int F(v) \geq -c_0 \int v^2 + c_1 \int v^4. \quad (28)$$

Moreover, by integrating in $[0,t]$ equation (18), and using (24), (27) and (28), we deduce that

$$2 \cos \theta \int_0^t \int |u_t|^2 + \int |u_x|^2 + \frac{1}{2} \int |u|^4 + c_1 \int v^4 - \frac{\epsilon}{3} \int_0^t \int (v_x)^2

\leq \int (|u_0|e^2 + \frac{1}{2} |u_0|^4 + \alpha g(v_0)|u_0|^2 - F(v_0))

- \epsilon \alpha \int_0^t \int g'(v) |u|^2 v_x - \epsilon \alpha \int_0^t \int g''(v) |u|^2 (v_x)^2.

Combining this with (25) and (26) yields, for $t \in [0, +\infty[$,

$$\int |u|^2 + \int |u|^4 + \int v^2 + \int v^4 + \int_0^t \int |u_x|^2 dx d\tau

+ \int_0^t \int |u|^4 dx d\tau + \int |u_x|^2 + \int |u_t|^2 dx d\tau + \epsilon \int_0^t \int (v_x)^2 dx d\tau

\leq c \left(1 + \alpha \epsilon \int_0^t \int |uu_xv_x| dx d\tau + \alpha \epsilon \int_0^t \int |u|^2 (v_x)^2 dx d\tau\right)

+ c \left(1 + \|u_x\|^2 + \int_0^t \|u_t\|^2 d\tau\right)^{\frac{1}{2}}. \quad (29)$$

where $c > 0$ is a constant independent of $\alpha \leq \alpha_0$ (for some $\alpha_0$) and $\epsilon$. Let us set

$$q(t) = 1 + \|u_x\|^2 + \int_0^t \|u_t\|^2 d\tau. \quad (30)$$

We deduce from (29) and the Gagliardo-Nirenberg inequality

$$\|u\|_{\infty} \leq \|u\|^\frac{1}{2} \|u_x\|^\frac{1}{2} \leq \|u_0\|^\frac{1}{2} \|u_x\|^\frac{1}{2}$$

(recall (20)) that, for $u \in H^1(\mathbb{R})$,

$$q(t) + \epsilon \int_0^t \int (v_x)^2 dx d\tau \leq c \left(1 + \alpha \epsilon \int_0^t \|u_x\|^\frac{3}{2} \|v_x\|_2 d\tau\right)

+ \alpha \epsilon \int_0^t \|u_x\|^2 \|v_x\|^2_2 d\tau + \frac{1}{2} q(t), \quad (31)$$
hence
\[
q(t) \leq \psi(t) := \kappa \left(1 + \alpha \epsilon \int_0^t \|u_x\|_2^2 \|v_x\|_2 \, d\tau + \alpha \epsilon \int_0^t \|u_x\|_2 \|v_x\|_2^2 \, d\tau\right), \tag{32}
\]
Now,
\[
\psi'(t) = \kappa \alpha \epsilon \left(\|u_x\|_2^2 \|v_x\|_2 + \|u_x\|_2 \|v_x\|_2^2\right) \leq \kappa \alpha \epsilon \left(\psi(t) \|v_x\|_2 + \psi(t) \|v_x\|_2^2\right),
\]
and
\[
\psi'(t) \psi^{-\frac{1}{2}}(t) \leq \kappa \alpha \epsilon \left(\psi(t) \|v_x\|_2 + \psi(t) \|v_x\|_2^2\right),
\]
and
\[
-2 \int_0^t \psi^\frac{1}{2}(\tau) e^\tau \, d\tau + 2 \left[\psi^\frac{1}{2}(\tau) e^\tau \right]_0^t \leq \kappa \left[\alpha \epsilon \psi(t) \left(\int_0^t e^{2\tau} \, d\tau\right)^\frac{1}{2} \left(\int_0^t \epsilon \|v_x\|_2^2 \, d\tau\right)^\frac{1}{2}
\right.
\]
\[
+ \alpha \epsilon \int_0^t \epsilon \|v_x\|_2^2 \, d\tau
\]
\[
\leq \kappa \alpha \epsilon \psi(t) e^t + \kappa \alpha \epsilon \psi(t),
\]
therefore
\[
-2 \int_0^t \psi^\frac{1}{2}(\tau) e^\tau \, d\tau + 2 \left[\psi^\frac{1}{2}(\tau) e^\tau \right]_0^t \leq \kappa \alpha \epsilon \psi(t) e^t + \kappa \alpha \epsilon \psi(t),
\]
by (26) and (32). Hence, for \(\epsilon \leq 1\),
\[
\phi^\frac{1}{2}(t) e^t \leq c \phi^\frac{1}{2}(t) e^t + c \int_0^t \phi^\frac{1}{2} e^\tau \, d\tau + c,
\]
and, for \(\alpha \leq \alpha_0\) such that \(1 - \alpha \epsilon > 0\), we derive, by Gronwall's inequality,
\[
\phi(t) \leq h_1(t), \quad h_1 \in C(\mathbb{R}^+), \tag{33}
\]
hence, in view of (32),
\[
q(t) = 1 + \|u_x\|_2^2 + \int_0^t \|u_t\|_2^2 \, d\tau \leq \phi(t) \leq h_1(t).
\]
Finally, combining this inequality with (25), (26) and (29), we deduce the inequality stated in Proposition 3.1.

Next, this \(a\ priori\) estimate will allow us to show the existence of a global unique strong solution to the approximated system (5):
Theorem 3.3. Let $f$ and $g$ verifying the assumptions of Theorem 1.2 and $(u_0, v_0) \in (H^1(\mathbb{R}))^2$. Then there exists a unique solution

$$(u^e, v^e) \in C([0, +\infty[; (H^1(\mathbb{R}))^2) \cap W^{1,2}_{loc}(0, +\infty[; (L^2(\mathbb{R}))^2)$$

to system (2) with initial data $(u_0^e, v_0^e) = (u_0, v_0) \in (H^1(\mathbb{R}))^2$.

Proof. Once again, we drop the superscript $\epsilon$.

For the (local) existence in $C([0, T']; (H^1(\mathbb{R}))^2)$, for a certain $T' < \infty$, since $e^{i\theta} \partial_{xx} - \frac{\pi}{2} < \theta < \frac{\pi}{2}$, is the infinitesimal generator of an analytic semigroup of contractions $\{T_\theta(t)\}_{t \geq 0}$ in $L^2(\mathbb{R})$, with domain $H^2(\mathbb{R})$ and verifying the estimate (16), the result follows by considering the Duhamel formulas of the auxiliary system

$$
\begin{cases}
\begin{aligned}
e^{-i\theta} u_t - u_{xx} &= -|\tilde{u}|^2 \tilde{u} - \alpha g(\tilde{v}) \tilde{u} \\
v_t - v_{xx} &= -(f(\tilde{v}))_x + \alpha (g'(\tilde{v}) |\tilde{u}|^2)_x
\end{aligned}
\end{cases}
$$

and a convenient Banach fixed-point technique (cf. [6], Proposition 2.1, for the Schrödinger equation case).

To conclude that the solution $(u, v)$ is in $W^{1,2}(0, T'; (L^2(\mathbb{R}))^2)$ we notice that, since

$$\frac{\partial}{\partial x} \left( \alpha g'(v)|u|^2 - f(v) \right) \in L^2(0, T', L^2(\mathbb{R})),$$

$$v \in L^2(0, T'; H^2(\mathbb{R}))$$

and

$$\frac{\partial v}{\partial t} \in L^2(0, T'; L^2(\mathbb{R})).$$

Moreover, since

$$|u|^2 u + \alpha g(v) u, \frac{\partial}{\partial x} \left( |u|^2 u + \alpha g(v) u \right) \in L^2(0, T'; L^2(\mathbb{R})),
$$

we also conclude, by the properties of the semigroup $\{T_\theta(t)\}_{t \geq 0}$, that

$$u \in L^2(0, T'; H^2(\mathbb{R}))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0, T'; L^2(\mathbb{R})).$$

To obtain the estimates which yield the global in time existence of $(u, v)$, we apply (18) and the Duhamel formula for the heat equation

$$v(t) = e^{\Delta t} v_0 + \int_0^t e^{\Delta (t-\tau)} \frac{\partial}{\partial x} \left[ \alpha g'(v)|u|^2 - f(v) \right](\tau) d\tau.$$
and the well-known estimate
\[
\left\| \frac{\partial}{\partial x} e^{\Delta t} \psi \right\|_2 \leq \frac{C}{t^{\frac{3}{4}}} \| \psi \|_1, \quad t > 0
\]
to obtain
\[
\| v_x(t) \|_2 \leq \| v_{0x} \|_2 + c \int_0^t \frac{1}{(t - \tau)^{\frac{3}{4}}} \left[ \| u \|_2 \| v \|_2 + (\| u \|^2_4 + \| v \|^2_4 + \| v \|_2) \| v_x \|_2 \right] d\tau
\]
\[
\leq \| v_{0x} \|_2 + c \int_0^t \frac{d\tau}{(t - \tau)^{\frac{3}{4}}} + c \int_0^t \| v_x(\tau) \|_2 \frac{d\tau}{(t - \tau)^{\frac{3}{4}}}
\]
by (18). We then obtain an estimate for \( \int_0^t \| v_{xx}(\tau) \|_2^2 \) and for \( \int_0^t \| v_t(\tau) \|_2^2 \), which achieves the proof.

We are now in condition to show the existence of a global weak solution to the initial Ginzburg Landau System:

**Proof of Theorem 1.2.** Let \((u_0, v_0) \in (H^1(\mathbb{R}))^2\) and
\[
(u^\epsilon, v^\epsilon) \in C([0, +\infty]; (H^1(\mathbb{R}))^2 \cap W^{1,2}_{\text{loc}}([0, +\infty]; (L^2(\mathbb{R}))^2)
\]
the corresponding solution of (13) for initial data \((u_0, v_0)\). Furthermore, we assume that \( \epsilon \leq 1 \) and \( \alpha \leq \alpha_0 \) (see Proposition 3.1).

Let us fix \( T > 0 \). We have, by (18), that \((u^\epsilon)_\epsilon\) is bounded in
\[
L^\infty([0, T]; H^1(\mathbb{R}))
\] and \((u^\epsilon)_\epsilon\) is bounded in
\[
L^2([0, T]; L^2(\mathbb{R})) \cap L^\infty([0, T]; H^{-1}(\mathbb{R})).
\]
By applying Aubin’s Lemma (for each interval \([R, R]| - R, R[| - T, T[\) and a standard diagonal extraction, there exists \( u \in L^\infty([0, T]; H^1(\mathbb{R})) \) such that, \( u^\epsilon \in L^2([0, T]; L^2(\mathbb{R})) \), for each \( T > 0 \), and a subsequence still denoted \((u^\epsilon)_\epsilon\) such that
\[
u^\epsilon \rightharpoonup u \text{ in } L^\infty([0, T]; H^1(\mathbb{R}))\text{ weakly-* and a.e. in } [0, +\infty[ \times \mathbb{R},
\]
\[
u^\epsilon_t \rightharpoonup u_t \text{ in } L^2([0, T]; L^2(\mathbb{R}))\text{ weakly.} \tag{34}
\]
Hence, \( u \in C([0, +\infty[; L^2(\mathbb{R})) \) and \( u(0) = u_0 \).

We can also deduce, by (13), that there exists \( v \in L^\infty([0, T]; (L^4 \cap L^2)(\mathbb{R})) \) and \( w \in L^\infty([0, T]; L^\frac{4}{3}(\mathbb{R})) \) such that
\[
v^\epsilon \rightharpoonup v \text{ in } L^\infty([0, T]; L^4 \cap L^2(\mathbb{R}))\text{ weakly-*},
\]
\[
f(v^\epsilon) - \alpha |u^\epsilon|^2 g(v^\epsilon) \rightharpoonup w \text{ in } L^\infty([0, T]; L^\frac{4}{3}(\mathbb{R}))\text{ weakly-*}. \tag{35}
\]
Moreover we have, with $p = 3$,

$$ |f'(\xi)| \leq c(1 + |\xi|^{p-1}), \xi \in \mathbb{R} $$

and for each real convex $C^2$ entropy function $\eta$ with compact support, we deduce from (5) with $q_1$ and $q_2 \in C^2(\mathbb{R})$ such that $q'_1 = \eta'f'$ and $q'_2 = \alpha \eta'g''$, we have

$$
\frac{\partial}{\partial t}\eta(v') + \frac{\partial}{\partial x}(q_1(v') - |u'|^2 q_2(v')) = (\alpha \eta'(v') g'(v') - q_2(v'))(|u'|^2)_x + \epsilon \eta'(v') \frac{\partial^2 v}{\partial x^2}
$$

$$
= (\alpha \eta'(v') g'(v') - q_2(v'))(|u'|^2)_x + \epsilon \frac{\partial^2}{\partial x^2} \eta(v') - \epsilon \eta''(v') \left( \frac{\partial v}{\partial x} \right)^2.
$$

Hence, by (18), we derive (see [7], theorem 2.1, for a similar deduction and argument) that if $\Omega$ is an open bounded subset of $]0, +\infty[ \times \mathbb{R}$, we have

$$
\frac{\partial}{\partial t}\eta(v') + \frac{\partial}{\partial x}(q_1(v') - |u'|^2 q_2(v')) \in K(\Omega) + B(\Omega),
$$

where $K(\Omega)$ is a compact set of $H^{-1}(\Omega)$ and $B(\Omega)$ is a bounded set of finite measures in $\Omega$.

Because of (32) (and the $L^2$ strong convergence of $(u^\varepsilon)_\varepsilon$ in each interval $]-R, R[\times] - T, T[$, we can now apply a variant of the Corollary 3.1 of Theorem 3.2 in [14] and, by a suitable diagonal extraction, we can deduce that

$$
f(v') - \alpha |u'|^2 g'(v') \rightharpoonup f(v) - \alpha |u|^2 g'(v) \text{ in } D'(]0, +\infty[ \times \mathbb{R}).
$$

Hence, by (35),

$$
f(v') - \alpha |u'|^2 g'(v') \rightharpoonup f(v) - \alpha |u|^2 g'(v) \text{ in } L^\infty([0, +\infty[ ; L^4(\mathbb{R}))\text{weakly-}*}. \quad \square
$$

**Remark 3.4.** In the framework of Theorem 1.2, if $v_0 \geq 0$ a.e., we can apply Remark 3.3 (if the support of $g'$ is contained in $[0, +\infty[ )$ and, since $v^\varepsilon \rightharpoonup v$ in $L^\infty([0, T] ; L^1 \cap L^2(\mathbb{R}))$ weakly-*, for each $T > 0$, we can conclude, since $v^\varepsilon \geq 0$ a.e. in $\mathbb{R} \times [0, +\infty[$, that

$$
v \geq 0 \text{ a.e. in } \mathbb{R} \times [0, +\infty[.
$$

Moreover, in this case, we can replace the condition $g \geq 0$ in $\mathbb{R}$ by the weaker condition $g \geq 0$ in $[0, +\infty[$.
4 Existence of standing waves

In this section we will study, by different techniques, the existence of bound states (more precisely, of standing waves) for (3) in the defocusing $(-|u|^2 u)$ and focusing $(+|u|^2 u)$ cases, for $\alpha > 0$, $g(v) = v + \rho$ with $\rho > 0$, $f(s) = a s^2 - b s^3$ with $a, b \geq 0$. In the focusing case, we will consider a minimization problem with an $L^p$ constraint. In the defocusing case, the special structure of the action functional will allow us to consider a minimization problem on a Nehari manifold (thus, solutions will actually be ground-states).

Observe that the embedding $\mathcal{H}^1_r(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ is not compact, where $\mathcal{H}^1_r(\mathbb{R})$ denotes the space of radially symmetric functions (i.e. even functions) of $\mathcal{H}^1(\mathbb{R})$. However, if $u \in L^2(\mathbb{R})$ is even and decreasing in $|x|$, it is easy to establish $|u(x)| \leq |x|^{-\frac{1}{2}} \|u\|_{L^2(\mathbb{R})}$ (see [12, p. 341]). Hence, by Strauss’ compactness lemma [18],

$$H^1_r(\mathbb{R}) \hookrightarrow L^q(\mathbb{R}) \ (q > 2),$$

where $H^1_r(\mathbb{R}) = \{u \in H^1(\mathbb{R}) : u \text{ is decreasing with respect to } |x| \}$.  

4.1 The defocusing case: Proof of Theorem 1.3

We will look for $(U, V)$ real solution of (3) with $U \in H^1(\mathbb{R})$. We solve the following (equivalent) problem, where a differential equation is coupled with a pointwise identity:

$$\begin{cases} U'' - U^3 - \alpha (V + \rho) U = 0 & \text{in } \mathbb{R} \\ a V^2 - b V^3 = \alpha U^2 & \text{in } \mathbb{R} \\ U(x), V(x) \to 0 \text{ as } |x| \to \infty. \end{cases}$$

We split the proof of Theorem 1.3 in three cases: $a = 0$ and some $b > 0$, $a > 0$ and $b = 0$, and $a, b > 0$.

Existence for $a = 0$ and some $b > 0$ (Theorem 1.3-1.) In this situation, the second equation in (37) is equivalent to

$$V(x) = -\left(\frac{\alpha}{b}\right)^{1/3} U^{2/3}(x),$$

hence we aim at solving

$$- U'' + \alpha \rho U - \frac{\alpha^{4/3}}{b^{1/3}} U^{5/3} + U^3 = 0 \ \text{ in } \mathbb{R}. \quad (38)$$
4 Existence of standing waves

Consider the $C^1$ functional $J : H^1(\mathbb{R}) \to \mathbb{R}$ defined as

$$J(U) = \frac{1}{2} \int_\mathbb{R} ((U')^2 + \alpha \rho U^2) + \frac{1}{4} \int U^4,$$

constrained to the manifold

$$\mathcal{M}_1 = \left\{ U \in H^1(\mathbb{R}) : \int |U|^{8/3} = 1 \right\}.$$

Let us check that $\inf_{\mathcal{M}_1} J$ is achieved. In fact, $J \geq 0$, and we can take a minimizing sequence $(U_n) \subset H^1(\mathbb{R})$:

$$U_n \in \mathcal{M}_1, \quad J(U_n) \to \inf_{\mathcal{M}_1} J.$$

By eventually replacing $U_n$ by $|U_n|^*$, the Schwarz symmetrization of its absolute value, we can assume that $U_n \in H^1_{rd}(\mathbb{R})$ and $U_n \geq 0$. Moreover,

$$\frac{1}{2} \min\{1, \alpha \rho\} \|U_n\|_{H^1}^2 \leq J(U_n) \leq C$$

and, since $\rho, \alpha > 0$, $(U_n)_n$ is bounded in $H^1$-norm. Thus there exists $U \in H^1_{rd}(\mathbb{R})$ such that, up to a subsequence,

$$U_n \rightharpoonup U \text{ weakly in } H^1(\mathbb{R}),$$

$$U_n \to U \text{ strongly in } L^p(\mathbb{R}), \forall p > 2.$$

(taking into account the compact embedding (36)). So $U \geq 0$ and $U \in \mathcal{M}_1$; since $J$ is lower-semicontinuous, $J(U) = \min_{\mathcal{M}_1} J$. Thus, there exists $\lambda \in \mathbb{R}$ (a Lagrange multiplier) such that

$$-U'' + \alpha \rho U - \lambda U^{5/3} + U^3 = 0.$$  \hspace{1cm} (40)

Since $U \not\equiv 0$ ($U \in \mathcal{M}_1$) and $U \geq 0$, by the strong maximum principle we have $U > 0$ in $\mathbb{R}$. Testing (40) by $U$ itself, we have

$$\lambda = \int ((U')^2 + \alpha \rho U^2 + U^4) > 0.$$

Therefore, we can choose $b > 0$ in such a way that $\frac{\alpha^{4/3}}{b^{1/3}} = \lambda$. Defining $V(x) := -(\frac{\alpha}{b})^{1/3} U^{2/3}$ we have that $(U, V)$ solves (37), hence is a solution to (6). This proves Theorem 1.3-1.

**Existence for some $a > 0$ and $b = 0$ (Theorem 1.3-2.)** This case is very similar to the previous one, hence we just stress the differences. Since there are no solutions with $V > 0$ (recall Remark [1.4]), we are lead to take

$$V(x) = -\left(\frac{a}{\alpha}\right)^{1/2} U(x),$$

\hspace{1cm} (41)
and solve
\[-U'' + \alpha \rho U - \frac{\alpha^{3/2}}{a^{1/2}} U^2 + U^3 = 0\]
by minimizing the functional \(J\) defined in (39), this time on the manifold
\[\mathcal{M}_2 = \left\{ U \in H^1(\mathbb{R}) : \int |U|^3 = 1 \right\} .\]

The constrained minimization problem leads to the existence of \(U \in H^1(\mathbb{R})\) positive solution
\[-U'' + \alpha \rho U - \lambda U^2 + U^3 = 0 \tag{42}\]
for some \(\lambda > 0\). Since \(\alpha > 0\) is fixed, we can choose \(a > 0\) in such a way that \(\frac{\alpha^{3/2}}{a^{1/2}} = \lambda\). Therefore \((U, V)\), with \(U\) solution of (42) and \(V\) solution of (41), solves (37). This proves Theorem 1.3-2.

**Existence for some \(a, b > 0\) (Theorem 1.3-3.)** This is the most challenging case. Observe that the polynomial \(P(s) := s^2 - s^3\) vanishes only at \(s = 0\) and \(s = 1\), being negative if and only if \(s > 1\). Moreover, \(P\) achieves a local maximum at \(P \left( \frac{2}{3} \right) = \frac{4}{27}\). We will obtain a solution \((U, V)\) with \(U > 0\) and \(V < 0\).

Consider the restriction \(\tilde{P} := P|_{-\infty, 0}\) (which is invertible), and the continuous function
\[g(s) = \begin{cases} \tilde{P}^{-1}(s) & s > 0 \\ 0 & s \leq 0 \end{cases}\]
which is negative for \(s > 0\). Observe that, if \(V := g(\alpha U^2)\), then \(V^2 - V^3 = \alpha U^2\). Moreover,
\[\lim_{s \to +0^+} \frac{g(s)}{s^{1/2}} = \lim_{s \to +\infty} \frac{g(s)}{s^{1/3}} = -1. \tag{43}\]

We aim at solving the equation:
\[-U'' + \alpha (g(\alpha U^2) + \rho) U + U^3 = 0\]
and we will succeed to do it, up to a Lagrange multiplier. Define \(G(s) := \int_0^s g(\xi) \, d\xi\), which is negative for \(s > 0\) and satisfies
\[\lim_{s \to +0^+} \frac{G(s)}{s^{3/2}} = -\frac{2}{3}, \quad \lim_{s \to +\infty} \frac{G(s)}{s^{4/3}} = -\frac{3}{4}. \tag{44}\]

**Lemma 4.1.** The following minimization problem has a nonnegative solution:
\[\inf_{U \in \mathcal{M}_3} J(U), \quad \text{with} \quad \mathcal{M}_3 = \left\{ U \in H^1(\mathbb{R}) : \int G(\alpha U^2) = -1 \right\} .\]
Proof. 1) We start by checking that $\mathcal{M}_3 \neq \emptyset$. Fix $w \in H^1(\mathbb{R})$ a positive, radially decreasing function, and take:

$$\varphi(t) := \int_{\mathbb{R}} G(\alpha tw^2).$$

Since $\varphi$ is continuous and $\varphi(0) = 0$, the claim follows if we prove that $\varphi(t) \to -\infty$ as $t \to +\infty$. From (44), there exists $A > 0$ such that $G(s) \leq -As^{4/3}$ for all $s \geq \alpha w^2(1)$. Thus, for $x \in [-1, 1]$ and $t \geq 1$, $\alpha tw^2(x) \geq \alpha w^2(1)$ and so $G(\alpha tw^2(x)) \leq -A\alpha^{4/3}t^{4/3}w^{8/3}(x)$. Therefore, since $G \leq 0$,

$$\varphi(t) \leq \int_{-1}^1 G(\alpha tw^2(x)) \leq -At^{4/3} \int_{-1}^1 \alpha^{4/3}w^{8/3}(x) \to -\infty,$$

as $t \to +\infty$.

2) Reasoning exactly as in the proof of the case $a = 0$, $b > 0$, we can take a minimizing sequence of nonnegative, radially decreasing functions: $(U_n) \in H^1_{rd}(\mathbb{R})$ such that $U_n \in \mathcal{M}_3$ and $J(U_n) \to \inf_{\mathcal{M}_3} J$. This sequence is bounded in $H^1(\mathbb{R})$, thus there exists $U \in H^1_{rd}(\mathbb{R})$, nonnegative, such that, up to a subsequence, $U_n \to U$ weakly in $H^1(\mathbb{R})$, strongly in $L^p(\mathbb{R})$, for $p > 2$. From the strong convergence in $L^3(\mathbb{R})$ and $L^{8/3}(\mathbb{R})$, there exist $h_1 \in L^3(\mathbb{R})$, $h_2 \in L^{8/3}(\mathbb{R})$ such that $|U_n| \leq h_1, h_2$ for every $n \in \mathbb{N}$. Moreover, since $|G(s)| \leq C(|s|^{3/2} + |s|^{4/3})$, then

$$|G(\alpha U_n^2)| \leq C'(|U_n|^3 + |U_n|^{8/3}) \leq C'(h_1^3 + h_2^{8/3}) \in L^1(\mathbb{R}),$$

and by Lebesgue’s Dominated Convergence Theorem we have

$$-1 = \lim \int G(\alpha U_n^2) = \int G(\alpha U^2).$$

Therefore the proof of this lemma is complete, by observing that

$$\inf_{\mathcal{M}_3} J \leq J(U) \leq \lim \inf J(U_n) = \inf_{\mathcal{M}_3} J.$$

After the previous lemma, we are ready to prove the existence result for some $a, b > 0$.

Proof of Theorem 4.3. The previous lemma yields the existence of a non-trivial $H^1(\mathbb{R})$–solution (which is nonnegative and radially decreasing) to the problem

$$-U'' + \alpha(\lambda g(\alpha U^2) + \rho) U + U^3 = 0 \text{ in } \mathbb{R}.$$
for some $\lambda \in \mathbb{R}$. The strong maximum principle yields that $U > 0$. By testing the equation by $U$ itself, we see that
\[ \int_{\mathbb{R}} (\alpha \rho U^2 + U^4) + \lambda \int_{\mathbb{R}} \alpha g(U^2) U^2 = 0, \]
and since $g(\alpha U^2) < 0$, then actually $\lambda > 0$. Take $V(x) := \lambda g(\alpha U^2) < 0$.

Then, by definition of $g$,
\[ \frac{1}{\lambda^2} V^2 - \frac{1}{\lambda^3} V^3 = P(\frac{1}{\lambda} V) = \alpha U^2. \]

In conclusion, $(U, V)$ solves (7) with the choice $a := \frac{1}{\lambda^2} > 0$, $b = \frac{1}{\lambda^3} > 0$. \[ \square \]

### 4.2 The focusing case: Proof of Theorem 1.5

We will now look for $(U, V)$ real solutions of (7), solving instead the following (equivalent) problem:
\[
\begin{aligned}
U'' + U^3 - \alpha (V + \rho) U &= 0 \quad \text{in } \mathbb{R} \\
\alpha V^2 - b V^3 &= \alpha U^2 \quad \text{in } \mathbb{R} \\
U(x), V(x) &\to 0 \text{ as } |x| \to \infty.
\end{aligned}
\]

(45)

The results in this case are more complete, since we can use a Nehari manifold/Mountain pass approach instead of an $L^p$ constraint. As in the defocusing case, we split the proof of Theorem 1.5 in three cases.

**Existence for $a = 0$ and $b > 0$ (Theorem 1.5.1.)** In this situation, the second equation in (45) is equivalent to
\[ V(x) = -\left(\frac{\alpha}{b}\right)^{1/3} U^{2/3}(x), \]
hence we aim at solving
\[ -U'' + \alpha \rho U = \frac{\alpha^{4/3}}{b^{1/3}} U^{5/3} + U^3. \]

(46)

Weak solutions of (46) correspond to critical points of the $C^1$-action functional $A : H^1(\mathbb{R}) \to \mathbb{R}$ defined by
\[ A(U) = \frac{1}{2} \int \left( (U')^2 + \rho \alpha U^2 \right) - \frac{3\alpha^{4/3}}{8b^{1/3}} \int |U|^{8/3} - \frac{1}{4} \int U^4. \]

We introduce the Nehari set
\[ \mathcal{N} = \{ U \in H^1(\mathbb{R}) : U \neq 0, A'(U)[U] = 0 \} = \left\{ U \in H^1(\mathbb{R}) : U \neq 0, \int ((U')^2 + \rho \alpha U^2) = \frac{\alpha^{4/3}}{b^{1/3}} \int |U|^{8/3} + \int U^4 \right\}. \]
Proof of Theorem 1.2.1. We will prove this result by showing that the quantity
\[ \inf_{U \in \mathcal{N}} A(U) \]

is a critical level of $A$, being achieved by a positive solution of (46). Although this fact follows from standard arguments, we sketch the proof since we are dealing with an unbounded set $\mathbb{R}$ and working with one space dimension. Since the proof is long, we split it in several steps.

1) Given $U \in H^1(\mathbb{R}) \setminus \{0\}$, let us check the existence of $t > 0$ such that $tU \in \mathcal{N}$. Consider the map $\varphi_U : [0, +\infty) \to \mathbb{R}$ defined as
\[ \varphi_U(t) := A(tU) = \frac{t^2}{2} \int ((U')^2 + \rho \alpha U^2) - \frac{3\alpha^{4/3}}{8b^{1/3}} \int |U|^{8/3} - \frac{t^4}{4} \int U^4. \]

We have $\varphi(0) = 0$, $\varphi(t) \to -\infty$ as $t \to +\infty$, and $\varphi(t) > 0$ for $t > 0$ sufficiently small. Then $\varphi_U(t)$ admits a critical point at $t^* > 0$, which corresponds to a point $t^*U \in \mathcal{N}$ (it is actually simple to see that $t^*$ is the unique positive critical point of $\varphi_U$, corresponding to its global maximum). An important observation that we use ahead is that, if in addition $A'(U)[U] \leq 0$, then $\varphi_U(1) \leq 0$, and so $t^* \leq 1$.

2) The set $\mathcal{N}$ is a $C^1$-manifold. In fact, for $F(U) := A'(U)[U]$ with $U \in \mathcal{N}$, we have
\[ F'(U)[U] = 2 \int ((U')^2 + \rho \alpha U^2) - \frac{8\alpha^{4/3}}{3b^{1/3}} \int |U|^{8/3} - 4 \int U^4 \]
\[ = -\frac{2\alpha^{4/3}}{3b^{1/3}} \int |U|^{8/3} - 2 \int |U|^4 < 0. \]

Moreover, this implies that constrained critical points are free critical points: for $U \in \mathcal{N}$ such that $A|_{\mathcal{N}}(U) = 0$, there exists $\lambda \in \mathbb{R}$, a Lagrange multiplier, such that $A'(U) = \lambda F'(U)$. Using $U$ as test function, we see that $\lambda = 0$, thus $A'(U) = 0$.

3) Combining the Sobolev embeddings $H^1(\mathbb{R}) \hookrightarrow L^{8/3}(\mathbb{R})$, $L^4(\mathbb{R})$ with the definition of $\mathcal{N}$, we deduce the existence of $C_1, C_2 > 0$ such that
\[ (\|U\|_{8/3}^2 + \|U\|^2_4) \leq C_1 \|U\|_{H^1}^2 \leq C_2 (\|U\|_{8/3}^{8/3} + \|U\|^4_4) \quad \forall U \in \mathcal{N}. \]

Since $2 < 8/3 < 4$, there exists $\delta > 0$ such that
\[ \|U\|_{8/3} + \|U\|_4 \geq \delta \quad \forall U \in H^1(\mathbb{R}). \quad (47) \]

4) For $U \in \mathcal{N}$, we have
\[ A(U) = \frac{\alpha^{4/3}}{8b^{1/3}} \int |U|^{8/3} + \frac{1}{4} \int |U|^4 > 0, \quad (48) \]
thus $\inf_{\mathcal{N}} A \geq 0$. 

5) We are now ready to prove the existence of a minimizer using direct methods. Take a minimizing sequence \((U_n) \subset H^1(\mathbb{R})\): \(U_n \in \mathcal{N}\) such that \(A(U_n) \to \inf_{\mathcal{N}} A\). From (43), this sequence is bounded in \(L^{8/3}(\mathbb{R})\) and \(L^4(\mathbb{R})\); since \(U_n \in \mathcal{N}\), then the sequence is also bounded in \(H^1(\mathbb{R})\). Take the Schwarz symmetrization \([U_n]^*\), and let \(t_n > 0\) be such that \(t_n[U_n]^* \in \mathcal{N}\) (recall Step 1). Since \(|[U_n]^*|_{H^1} \leq \|[U_n]\|_{H^1}\) and \(|[U_n]^*|_p = \|[U_n]\|_p\) for \(p \geq 1\), then \(A'([U_n]^*)[U_n]^* \leq A'(U_n)[U_n] = 0\), and \(t_n \leq 1\) by Step 1. From (48), we see directly that \(J(t_n[U_n]^*) \leq J(U_n)\). So, \((t_n[U_n]^*)\) is also a minimizing sequence, being radially decreasing, nonnegative, and bounded in \(H^1(\mathbb{R})\). We denote this new sequence again by \(U_n\).

In conclusion, there exists \(U \in H^1_{rd}(\mathbb{R})\), nonnegative, such that (up to a subsequence) \(U_n \rightharpoonup U\) weakly in \(H^1(\mathbb{R})\). From (49), the converge is strong in \(L^p(\mathbb{R})\), for every \(p > 2\). Step 3 yields that \(U \neq 0\).

Finally, since \(A(U)[U] \leq 0\), we may take \(0 < t \leq 1\) such that \(tU \in \mathcal{N}\), and

\[
\inf_{\mathcal{N}} A \leq A(tU) = \frac{\alpha^{4/3} 8/3}{81/3} \int |U|^8/3 + \frac{t^4}{4} \int |U|^4 \leq \frac{\alpha^{4/3} 8/3}{81/3} \int |U|^8/3 + \frac{1}{4} \int |U|^4 = \lim_n \frac{\alpha^{4/3} 8/3}{81/3} \int |U_n|^8/3 + \frac{1}{4} \int |U_n|^4 = \lim_n A(U_n) = \inf_{\mathcal{N}} A.
\]

In particular \(t = 1\), \(U \in \mathcal{N}\) and \(A(U) = \inf_{\mathcal{N}} A\). By Step 2 we deduce that \(A'(U) = 0\), that is, \(U\) solves (10). Since \(U \neq 0\), then \(U > 0\) by the strong maximum principle.

6) In conclusion, the pair \((U, V)\), for \(U\) positive solution of (10) and \(V = -(\frac{a}{b})^{1/3} U^{2/3}\) solve (45), which proves Theorem 1.5-1.

**Existence for \(a > 0\) and \(b = 0\) (Theorem 1.5-2.)**

In this case the second equation in (15) yields

\[
V(x) = \pm \left(\frac{\alpha}{a}\right)^{1/2} U(x),
\]

and for this reason we obtain two pairs of solutions.

The proof of Theorem 1.5-2 follows the lines of the previous case \(a = 0\), \(b > 0\), with very few changes. We are lead this time to the problems

\[
-U'' + \alpha \rho U = \pm \left(\frac{\alpha}{a}\right)^{1/2} U^2 + U^3, \quad U \in H^1(\mathbb{R}),
\]

with associated action functionals

\[
A_{\pm}(U) := \frac{1}{2} \int ((U')^2 + \alpha \rho U^2) \pm \int \frac{1}{3} \left(\frac{\alpha}{a}\right)^{1/2} U^3 - \frac{1}{4} \int |U|^4.
\]

Unlike the sign of the cubic term, the sign of the quadratic term \(\left(\frac{\alpha}{a}\right)^{1/2} s^2\) in (19) is not important: since it is an \(o(s)\) as \(s \to 0\), and is dominated in
absolute value by $C(1 + |s|^3)$ for all $s \in \mathbb{R}$, the proof is analogous to the one of Theorem 1.5-1. Solutions are critical points associated to the critical levels

$$c_\pm = \{ U \in H^1_{\text{rad}}(\mathbb{R}) : U \neq 0, \ A'_\pm(U)[U] = 0 \}.$$  

### Existence for $a, b > 0$ and $\alpha > 0$ small

The remainder of the paper is devoted to the proof of Theorem 1.5-3. This result follows directly from Theorem 4.2 and Theorem 4.6 below. First, we prove the existence of a solution whose components have different signs.

**Theorem 4.2.** Take $\rho, a, b > 0$. Then, for sufficiently small $\alpha > 0$, $\text{admits a solution } (U, V), \text{ with } U \in H^1(\mathbb{R}) \text{ and } U > 0, V < 0 \text{ in } \mathbb{R}.$

In order to prove this result, let $\tilde{f} := f|_{-\infty,0]},$ where $f(t) = at^2 - bt^3,$ and take the function

$$g(s) = \begin{cases} \tilde{f}^{-1}(s) & s > 0 \\ 0 & s \leq 0. \end{cases} \quad (50)$$

which is negative for $s > 0$. The asymptotic behavior at the origin and at plus infinity is

$$\lim_{s \to 0^+} \frac{g(s)}{s^{1/2}} = -\frac{1}{\sqrt{a}}, \quad \lim_{s \to +\infty} \frac{g(s)}{s^{1/3}} = -\frac{1}{\sqrt[3]{b}}. \quad (51)$$

In particular, there exists $c_1, c_2 > 0,$ depending only on $a, b,$ such that

$$|g(s)| \leq c_1(s^{1/2} + s^{1/3}) \quad (52)$$
$$|G(s)| \leq c_2(s^{3/2} + s^{4/3}) \quad (53)$$

for every $s > 0.$ Consider the problem

$$-U'' + \alpha \rho U = -g(\alpha U^2)\alpha U + U^3, \quad U \in H^1(\mathbb{R}), \quad (54)$$

and the associated functional

$$A(U) := \frac{1}{2} \int ((U')^2 + \alpha U^2) + \frac{1}{2} \int G(\alpha(U^+)^2) - \frac{1}{4} \int (U^+)^4, \quad U \in H^1(\mathbb{R}),$$

where $G(t) = \int_0^t g(\xi) \, d\xi.$

**Lemma 4.3.** Nontrivial critical points of $A$ are positive solutions of (54).

**Proof.** If $A'(U) = 0$ with $U \neq 0,$ then

$$-U'' + \alpha \rho U = -g(\alpha(U^+)^2)\alpha U^+ + (U^+)^3.$$
Multiplying this equation by $U^-$ and integrating by parts yields

$$\int ((U')^2 + \rho \alpha(U^-)^2) = 0,$$

so $U^- \equiv 0$ and $U \geq 0$. Since $U \not\equiv 0$, the strong maximum principle implies that $U > 0$, hence $U$ is a positive solution of (54).

**Lemma 4.4.** There exists $(U_n) \subset H^1(\mathbb{R})$ and $c > 0$ such that

$$A(U_n) \to c, \quad A'(U_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}).$$

**Proof.** Recalling (51), we get the existence of $c_3, c_4 > 0$, depending on $a$ and $b$, such that for every $s > 0$

$$g(s) \geq -c_3(s^{1/2} + s^{1/3}), \quad G(s) \geq -c_4(s^{3/2} + s^{4/3}).$$

Let us check that the functional $A$ satisfies all the assumptions of the Mountain Pass Lemma (we will use the version from \[26, Theorem 1.15\], which does not require that $A$ satisfies the Palais-Smale condition, and whose conclusion is precisely (55)):

- $A(0) = 0$
- We have, denoting by $S_p$ the best Sobolev constant of the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ and using (56):

  $$A(U) \geq \min\left\{\frac{1}{2}, \frac{\alpha}{2}\right\} \|U\|_{H^1}^2 - \frac{c_3}{2} \int (\alpha^{3/2}|U|^3 + \alpha^{4/3}|U|^{8/3}) - \frac{1}{4} \int U^4$$

  $$\geq \min\left\{\frac{1}{2}, \frac{\alpha}{2}\right\} \|U\|_{H^1}^2$$

  $$- \frac{c_3}{2} \left(\alpha^{3/2} S_3^3 \|U\|_{H^1}^3 + \alpha^{4/3} S_{8/3}^6 \|U\|_{H^1}^{8/3}\right) - \frac{S_4^4}{4} \|U\|_{H^1}^4$$

  and thus there exists $\varepsilon > 0$ small (depending on $a$), we have

  $$\inf_{\|U\|_{H^1}=\varepsilon} A(U) > 0.$$  

- Let $w \in H^1(\mathbb{R})$ be a positive function. Then, by reasoning exactly as in point 1) of the proof of Lemma 4.1 we deduce that

  $$A(tw) \to -\infty \quad \text{as } t \to +\infty.$$  

  In conclusion, there exists $\bar{U} \in H^1(\mathbb{R})$ with $\|\bar{U}\|_{H^1} > \varepsilon$ such that $A(\bar{U}) < 0$.

Thus, \[26, Theorem 1.15\] applies, yielding the existence of a sequence $(U_n) \subset H^1(\mathbb{R})$ satisfying (55).
Lemma 4.5. Equation $\text{(54)}$ admits a positive solution $U \in H^1(\mathbb{R})$.

Proof. Let $U_n$ be the sequence given by Lemma 4.4.

1) Let us check that $(U_n)$ is bounded in $H^1(\mathbb{R})$. We have

$$\frac{1}{2} \int ((U'_n)^2 + \alpha \rho U_n^2) + \frac{1}{2} \int G(\alpha (U_n^+)^2) - \frac{1}{4} \int (U_n^+)^4 \leq C$$

and

$$\int ((U'_n)^2 + \alpha \rho U_n^2) + \int g(\alpha (U_n^+)^2) \alpha (U_n^+)^2 - \int (U_n^+)^4 = o(\|U_n\|).$$

By multiplying the second equation by $3/8$ and subtracting it from the first inequality, we have, using also $\text{(52)}$ and $\text{(53)}$

$$\frac{1}{8} \int ((U'_n)^2 + \alpha \rho U_n^2) \leq C + o(\|U_n\|) - \frac{1}{8} \int (U_n^+)^4 + \int \left(\frac{3}{8} g(\alpha (U_n^+)^2) \alpha (U_n^+)^2 - \frac{1}{2} G(\alpha (U_n^+)^2)\right) \leq C + o(\|U_n\|_{H^1(\mathbb{R})}) - \frac{1}{8} \int (U_n^+)^4 + C \int (\alpha^{3/2} + \alpha^{4/3}) ((U_n^+)^2 + (U_n^+)^4)$$

for some $C$ depending only on $c_3$ (and, thus, only on $a$ and $b$). Therefore,

$$\frac{\min\{1, \rho a\}}{8} \|U_n\|^2_{H^1} \leq C + C'\|U_n\|_{H^1} + C'' \int (\alpha^{3/2} + \alpha^{4/3}) \|U_n\|^2_{H^1}$$

$$+ \left(C(\alpha^{3/2} + \alpha^{4/3}) - \frac{1}{8}\right) \int U_n^4.$$

Choosing $\alpha$ sufficiently small such that $C''(\alpha^{3/2} + \alpha^{4/3}) \leq \frac{\min\{1, \rho a\}}{16}$ and $C(\alpha^{3/2} + \alpha^{4/3}) - \frac{1}{8} \leq 0$, we have

$$\frac{\min\{1, \rho a\}}{16} \int ((U'_n)^2 + \alpha \rho U_n^2) \leq C + C'\|U_n\|_{H^1},$$

hence $(U_n)$ is a bounded sequence in $H^1(\mathbb{R})$.

2) From the previous step, there exists $U \in H^1(\mathbb{R})$ such that, up to a subsequence,

$$U_n \rightharpoonup U \text{ weakly in } H^1(\mathbb{R}) \quad U_n \rightarrow U \text{ strongly in } L^p_{\text{loc}}(\mathbb{R}), \quad \forall p \geq 1.$$ 

Then, for every $\varphi \in C_0^\infty(\mathbb{R})$, we have $A'(U)[\varphi] = \lim_n A(U_n)[\varphi] = 0$, and $U$ is a critical point of $A$, i.e., $A'(U) = 0$.

3) To conclude, let us show that without loss of generality we can assume $U \neq 0$. If this is true, Lemma 4.3 follows directly from Lemma 4.3.
The first observation is that, since \( U_n \) is bounded in \( H^1(\mathbb{R}) \) and \( A'(U_n)[U_n] = o(\|U_n\|_{H^1}) \), then actually
\[
A'(U_n)[U_n] = o(1). \tag{57}
\]
This implies that we cannot have \( U_n \to 0 \) in \( L^4(\mathbb{R}) \), otherwise (57) combined with (52) would yield \( U_n \to 0 \) in \( H^1(\mathbb{R}) \) and \( A(U_n) \to 0 \), contradicting the positivity of \( c > 0 \) in Lemma 4.4. Therefore \( U_n \not\to U \) in \( L^4(\mathbb{R}) \), and since \( U_n \) is bounded in \( H^1(\mathbb{R}) \) there exists \( R, x_n \in \mathbb{R} \) and \( l > 0 \) such that
\[
\int_{B_R(x_n)} U_n^4 \geq l > 0
\]
(check for instance Lemma 1.21 in [26]). Thus, defining \( V_n(x) = U_n(x - x_n) \), we have
\[
A(V_n) = A(U_n) \to c, \quad A'(V_n) \to 0, \quad \int_{B_R(0)} V_n^4 \geq l > 0.
\]
By repeating the previous arguments we obtain the existence of \( V \in H^1(\mathbb{R}) \) such that \( V_n \to V \) weakly in \( H^1(\mathbb{R}) \), strongly in \( L^p_{\text{loc}}(\mathbb{R}) \) (up to a subsequence). Moreover, \( A'(V) = 0 \) and \( \int_{B_R(0)} V^4 \geq l > 0 \), hence \( V \) is a nontrivial critical point of \( A \), hence a positive solution of (54) by Lemma 4.3.

**Conclusion of the proof of Theorem 4.2.** Let \( U \in H^1(\mathbb{R}) \) be the positive solution of (54) provided by the previous lemma. Let \( V := g(\alpha U^2) \). Then, by definition of \( g \) (recall (50)), we have \( V < 0 \) and \( V^2 - bV^3 = \alpha U^2 \). In particular, \( (U, V) \) is a solution of (45).

Having concluded the proof of Theorem 4.2, we turn to the existence of a solution pair with both components positive.

**Theorem 4.6.** Take \( \rho, a, b > 0 \). Then, for sufficiently small \( \alpha > 0 \), (45) admits a solution pair \((U, V)\), with \( U \in H^1(\mathbb{R}) \) and \( U > 0, V > 0 \) in \( \mathbb{R} \).

In order to prove this last result, consider this time \( \tilde{f} := f|_{[0, \frac{4a}{27b^2}]} \), where we recall that \( f(s) = as^2 - bs^3 \). This function is strictly increasing in \( [0, \frac{4a}{27b^2}] \), hence invertible. Since \( f(\frac{4a}{27b^2}) = \frac{4a^3}{27b^2 \rho} \), we can take the continuous, nonnegative functions
\[
\tilde{h}(s) = \begin{cases} 
0 & s \leq 0 \\
\tilde{f}^{-1}(s) & 0 \leq s < \frac{4a^3}{27b^2 \rho} \\
\frac{4a^3}{27b^2 \rho} & s \geq \frac{4a^3}{27b^2 \rho} 
\end{cases}, \quad h(s) = \min\{\tilde{h}(s), \frac{\rho}{2}\}. \tag{58}
\]
We will solve
\[
-U'' + \alpha \rho U + h(\alpha U^2)\alpha U = U^3, \tag{59}
\]
obtaining positive solutions as critical points of the functional $A : H^1(\mathbb{R}) \to \mathbb{R}$ which now is defined by

$$A(U) = \frac{1}{2} \int ((U')^2 + \rho \alpha U^2) + \frac{1}{2} \int H(\alpha U^2) - \frac{1}{4} \int (U^+)^4, \quad (60)$$

where $H(t) := \int_0^t h(\xi) \, d\xi$. Since $0 \leq h(t) \leq \min\{\frac{2a}{\alpha^2}, \frac{2b}{\alpha^2}\}$, then $0 \leq H(\alpha t^2) \leq \min\{\frac{2a}{\alpha^2}, \frac{2b}{\alpha^2}\} \alpha t^2$ and the functional is well defined. Observe also that, if $U > 0$ is such that $\alpha U^2 \leq \frac{4a^3}{27b^2}$, then $V := h(\alpha U^2) > 0$ and $aV^2 - bV^3 = \alpha U^2$.

**Lemma 4.7.** Nontrivial critical points of $A$ are positive solutions of (54).

**Proof.** If $A'(U) = 0$ with $U \neq 0$, then

$$-U'' + \alpha \rho U + h(\alpha U^2)\alpha U = (U^+)^3.$$

Multiplying this equation by $U^-$ and integrating by parts, we obtain

$$\int ((U')^2 + \rho \alpha (U^-)^2) + \int h(\alpha U^2)\alpha U^2 = 0.$$

Since $h \geq 0$, we have $U^- \equiv 0$ and $U \geq 0$, and the conclusion follows from the strong maximum principle. $\square$

**Lemma 4.8.** There exists $U > 0$, a critical point of $A$, such that

$$A(U) \leq c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} A(\gamma(t)),$$

where

$$\Gamma := \{ \gamma \in C([0,1], H^1(\mathbb{R})) : \gamma(0) = 0, \ A(\gamma(1)) < 0 \}.$$

**Proof.** Similarly to the proof of Lemma 4.4, let us check that the functional (60) satisfies the assumptions of [26, Theorem 1.15]:

- $A(0) = 0$
- Since $H \geq 0$, we have

$$A(U) \geq \frac{\min\{1, \alpha \rho\}}{2} \|U\|_{H^1}^2 - C\|U\|_{H^1}^4$$

and thus, for $\varepsilon > 0$ small,

$$\inf_{\|U\|_{H^1} = \varepsilon} A(U) > 0.$$

In particular, since $c \geq \inf_{\mathcal{N}} A$, this implies that $c > 0$. 

• Taking a positive function \( w \in H^1(\mathbb{R}) \) we have, since \( H(\alpha t^2) \leq \frac{\rho}{2} \alpha t^2 \),
\[
A(tw) \leq \frac{t^2}{2} \int ((w')^2 + \frac{3\rho}{2} \alpha w^2) - \frac{t^4}{4} \int w^4 \rightarrow -\infty
\]
as \( t \rightarrow +\infty \).

Thus, \[26, Theorem 1.15] implies the existence of a sequence \((U_n) \subseteq H^1(\mathbb{R})\) such that
\[
A(U_n) \rightarrow c > 0, \quad A'(U_n) \rightarrow 0.
\]
This sequence is bounded in \( H^1(\mathbb{R}) \), since
\[
C + o(\|U_n\|_{H^1}) \geq A(U_n) - \frac{1}{4} A'(U_n)[U_n]
= \frac{1}{4} \int ((U_n')^2 + \rho \alpha U_n^2) + \frac{1}{2} \int H(\alpha U_n^2) - \frac{1}{4} \int h(\alpha U_n^2) \alpha U_n^2
\geq \frac{1}{4} \int ((U_n')^2 + \alpha (\rho - h(\alpha U_n^2)) U_n^2).
\]

Thus \( U_n \rightharpoonup U \) weakly in \( H^1(\mathbb{R}) \) (up to a subsequence), and since \( A(U_n) \rightarrow c > 0 \), then \( U_n \not\rightarrow 0 \) in \( L^4(\mathbb{R}) \). Reasoning exactly as in the proof of Lemma \[4.5\] we can assume without loss of generality that \( U \neq 0 \), and \( A'(U) = 0 \). Therefore \( U \) is positive as a consequence of Lemma \[4.7\].

Finally, since
\[
A(U_n) = A(U_n) - \frac{3}{8} A'(U_n)[U_n] + o(1)
= \frac{1}{8} \int ((U_n')^2 + (\rho - \frac{3}{4} h(\alpha U_n^2)) \alpha U_n^2)
+ \frac{1}{2} \int H(\alpha U_n^2)
+ \frac{1}{8} \int (U_n^+)^4 + o(1)
\]
and all integrands are nonnegative for \( \alpha \) sufficiently small, from Fatou’s lemma conclude that
\[
c = \lim_n A(U_n) \geq \liminf \left( A(U_n) - \frac{3}{8} A'(U_n)[U_n] + o(1) \right)
\geq \liminf (A(U) - \frac{3}{8} A'(U)[U]) = A(U). \tag*{\blacksquare}
\]

Up to this point, we have obtained a positive solution of the equation \[59\]. In order to conclude the proof of Theorem \[1.6\] we need to show that \( \alpha U^2 \leq \frac{4b^3}{27b^2} \) for \( \alpha \) small (observe that \( U \) depends on \( \alpha \), so this is a delicate step). Havind that in mind, consider the auxiliary functional
\[
\tilde{A}(U) = \frac{1}{2} \int ((U')^2 + \frac{3\rho}{2} \alpha U^2) - \frac{1}{4} \int (U^+)^4,
\]
which satisfies:

\[ A(U) \leq \tilde{A}(U), \quad \forall U \in H^1(\mathbb{R}). \]  

(61)

It is classical to see (see for e.g. [26]) that \( \tilde{A} \) admits the following (least action) critical level in \( H^1(\mathbb{R}) \):

\[ c_{\tilde{A}} = \inf_{u \in H^1(\mathbb{R})} \sup_{t > 0} \tilde{A}(tu) = \inf_{\mathcal{N}_\tilde{A}} \tilde{A}, \]

where \( \mathcal{N}_\tilde{A} = \{ u \in H^1(\mathbb{R}) : u \neq 0, \ \tilde{A}'(u)u = 0 \} \),

which is achieved by a unique (up to translation) radial positive solution of

\[-W'' + \frac{3\rho}{2}W = W^3 \text{ in } \mathbb{R}, \quad W(x) \to 0 \text{ as } |x| \to \infty.\]

This solution is explicitly known to be

\[ W(x) = \sqrt{2} \sqrt{\frac{3\rho}{2}} \operatorname{sech} (\sqrt{\frac{3\rho}{2}} x), \]

so that there exists \( \kappa > 0 \) independent of \( \rho \) and \( \alpha \) such that

\[ 0 < c_{\tilde{A}} = A(W) = \frac{1}{4} \int ((W')^2 + \frac{3\rho}{2} \alpha W^2) = \frac{1}{4} \int W^4 \leq \kappa \left( \frac{3\rho}{2} \alpha \right)^{3/2} \]  

(62)

Lemma 4.9. Let \( U \) be the critical point obtained in Lemma 4.8. Then we have \( A(U) \leq c_{\tilde{A}} \).

Proof. We have, by (61),

\[ A(U) \leq c \leq \sup_{t > 0} A(tW) \leq \sup_{t > 0} \tilde{A}(tW) = \tilde{A}(W) = c_{\tilde{A}}, \]

and the conclusion follows. \( \Box \)

Conclusion of the proof of Theorem 4.6. Let \( U \) be the critical point of \( A \) obtained in Lemma 4.8. Then

\[ A(U) = A(U) - \frac{1}{4} A'(U)[U] \geq \frac{1}{4} \int ((U')^2 + \frac{\rho}{2} \alpha U^2) \]

and, by combining (62) with Lemma 4.9 and since the embedding \( H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \) is continuous,

\[ \|U\|^2_{L^\infty} \leq C_1 \|U\|^2_{H^1} \leq \frac{C_1}{\sqrt{\rho \alpha}} \int ((U')^2 + \rho \alpha U^2) \leq \frac{C_2}{\sqrt{\rho \alpha}} A(U) \leq C_3 \rho \alpha, \]

for \( C_1, C_2, C_3 \) independent of \( a, b, \rho, \alpha \). Now choose \( \alpha \) small so that

\[ \alpha \|U\|^2_{L^\infty} \leq C_3^2 \rho \alpha \leq \frac{4\alpha^3}{27b^2} \]

and \( \tilde{h}(\alpha U^2) \leq \frac{b}{7} \) (recall (58)). Then \( V := h(\alpha U^2) = \tilde{f}^{-1}(\alpha U^2) \) satisfies

\[ aV^2 - bV^3 = \alpha U^2, \]

and \( (U, V) \) is a solution of (45). \( \Box \)
Conclusion of the proof of Theorem 1.3. This result is a direct consequence of Theorem 4.2 and Theorem 1.2.

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References

[1] I.S. Aranson, L. Kramer; The world of the complex Ginzburg-Landau equation, Rev. Modern Phys. 74 (2002), no. 1, 99–143.

[2] D.J. Benney; A general theory for interactions between short and long waves, Studies in Appl. Math., 56 (1977), 81–94.

[3] T. Cazenave, F. Dickstein, F.B. Weissler; Finite-time blowup for a complex Ginzburg-Landau equation, SIAM J. Math. Anal. 45 (2013), no. 1, 244–266.

[4] T. Cazenave, F. Dickstein, F.B. Weissler; Standing waves of the complex Ginzburg-Landau equation, Nonlinear Anal. 103 (2014), 26–32.

[5] R. Cipolatti, F. Dickstein, J.-P. Puel; Existence of standing waves for the complex Ginzburg-Landau equation, J. Math. Anal. Appl. 422 (2015), no. 1, 579–593.

[6] J.-P. Dias, M. Figueira; Existence of weak solutions for a quasilinear version of Benney equations, J. Hyperbolic Diff. Eq. 4 (2007), 555–563.

[7] J.-P. Dias, M. Figueira, H. Frid; Vanishing viscosity with short wave long wave interactions for systems of conservation laws, Arch. Rational Mech. Anal. 196 (2010) 981–1010.

[8] J.-P. Dias, M. Figueira, F. Oliveira; Existence of local strong solutions for a quasilinear Benney system, C. R. Math. Acad. Sci. Paris 344 (2007), no. 8, 493–496.
[9] J.-P. Dias, M. Figueira, F. Oliveira; *Existence and linearized stability of solitary waves for a quasilinear Benney system*, Proc. Royal Soc. Edinburgh, Section A, 146 (2016), 547-564.

[10] E. Godlewski, P.A. Raviart; Hyperbolic systems of conservation laws, Ellipses, Paris, 1991.

[11] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, Lecture Notes in Mathematics, Springer-Verlag, New-York, 448 (1975), 25-70.

[12] P. L. Lions, The concentration-compactness principle in the calculus of variations, Part 1, Ann. Inst. H. Poincaré 1 (1984), 109–145.

[13] F. Murat; *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa 5 (1978), 489–507.

[14] M.E. Schonbek; *Convergence of solutions to nonlinear dispersive equations*, Comm. Partial Diff. Eq. 7 (1982), 959–1000.

[15] F. Oliveira; *Stability of the solitons for the one-dimensional Zakharov-Rubenchik equation*, Phys. D 175 (2003), no. 3-4, 220–240.

[16] S. Snoussi; *On the local and global existence of solution for a general Ginzburg-Landau like equation coupled with a Poisson equation in $L^p(\mathbb{R}^d)$*, Differential Integral Equations 13 (2000), no. 1-3, 61–98.

[17] Y. Shibata, Y. Tsutsumi; *Local existence of solution for the initial-boundary value problem of fully nonlinear wave equation*, Nonlinear Anal. 11 (1987), no. 3, 335–365.

[18] W. A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977), 149–162.

[19] L. Tartar; *Compensated compactness and applications to partial differential equations*, Research Notes in Math. 39, pg.136–212, Pitman, 1979.

[20] M. Tsutsumi, S. Hatano; *Well-posedness of the Cauchy problem for the long wave - short wave resonance equations*, Nonlinear Anal. 22 (1994), 155-171.

[21] M. Tsutsumi, S. Hatano; *Well-posedness of the Cauchy problem for Benney’s first equations of long wave short wave interactions*, Funkcialaj Ekvacioj 37 (1994), 289–316.
4 Existence of standing waves

[22] M. Tsutsumi, H. Kasai; *The time-dependent Ginzburg-Landau Maxwell equations*, Nonlinear Anal. 37 (1999), no. 2, 187–216.

[23] M. Funakoshi and M. Oikawa, The resonant interaction between a long internal gravity wave and a surface gravity wave packet, *J. Phys. Soc. Japan* 52 (1983), 1982-1995.

[24] R. H. J. Grimshaw, The modulation of an internal gravity-wave packet and the resonance with the mean motion, *Stud. Appl. Math.* 56 (1977), 241-266.

[25] V.D. Djordjevic, L.G. Redekopp, On the two-dimensional packets of capillary-gravity waves, *J. Fluid. Mech.* 79 (1977) 703-714

[26] M. Willem; *Minimax Theorems*, Birkhäuser 1996.

[27] Aranson, Kramer, *Rev. Med. Phys.* 74 (2002)).