CHEBYSHEV POLYNOMIALS ON GENERALIZED JULIA SETS

GÖKALP ALPAN

Abstract. Let \((f_n)_{n=1}^{\infty}\) be a sequence of nonlinear polynomials satisfying some mild conditions. Furthermore, let \(F_m(z) = (f_m \circ f_{m-1} \ldots \circ f_1)(z)\) and \(\rho_m\) be the leading coefficient for \(F_m\). It is shown that on the Julia set \(J(f_n)\), the Chebyshev polynomial of the degree \(\deg F_m\) is of the form \(F_m(z)/\rho_m - \tau_m\) for all \(m \in \mathbb{N}\) where \(\tau_m \in \mathbb{C}\). This generalizes the result obtained for autonomous Julia sets in [10].

1. Introduction

Let \((f_n)_{n=1}^{\infty}\) be a sequence of rational functions in \(\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\). Let us define the associated compositions by \(F_m(z) := f_m \circ \ldots \circ f_1(z)\) for each \(m \in \mathbb{N}\). Then the set of points in \(\overline{\mathbb{C}}\) for which \((F_n)_{n=1}^{\infty}\) is normal in the sense of Montel is called the Fatou set for \((f_n)_{n=1}^{\infty}\). The complement of the Fatou set is called the Julia set for \((f_n)_{n=1}^{\infty}\) and is denoted by \(J(f_n)\). The metric considered here is the chordal metric. Julia sets corresponding to a sequence of rational functions, to our knowledge, were considered first in [8]. There are several papers appeared in the literature (see e.g. [4, 6, 7, 16]) which show the possibility for adapting the results on autonomous Julia sets to this more general setting with some minor changes. By an autonomous Julia set, we mean the set \(J(f_n)\) with \(f_n(z) = f(z)\) for all \(n \in \mathbb{N}\) where \(f\) is a rational function.

The Julia set \(J(f_n)\) is never empty provided that \(\deg f_n \geq 2\) for all \(n\). Moreover, \(F_k^{-1}(F_k(J(f_n))) = J(f_n)\) for all positive \(k\). If, in addition, we assume that \(f_n = f\) for all \(n\) then \(f(J(f)) = f^{-1}(J(f)) = J(f)\) where \(J(f) := J(f_n)\). But without the last assumption, we do not have \(g(J(f_n)) = J(f_n)\) or \(g^{-1}(J(f_n)) = J(f_n)\) for a rational function \(g\) with \(\deg g \geq 2\), in general. That is the main reason why further techniques are needed in this framework.

If \(f\) is a nonlinear complex polynomial then \(J(f) = \partial \{z \in \mathbb{C} : f^{(n)}(z) \to \infty\}\) and \(J(f)\) is an infinite compact subset of \(\mathbb{C}\) where \(f^{(n)}\) is the \(n\)-th iteration of \(f\). The next result is due to Kamo-Borodin [10]:

**Theorem 1.1.** Let \(f(z) = z^m + a_{m-1}z^{m-1} + \ldots + a_0\) be a nonlinear complex polynomial and \(T_k(z)\) be a Chebyshev polynomial on \(J(f)\). Then \((T_k \circ f^{(n)})(z)\) is also a Chebyshev polynomial on \(J(f)\) for each \(n \in \mathbb{N}\). Moreover, there exists a complex number \(\tau\) such that \(f^{(n)}(z) = \tau\) is a Chebyshev polynomial on \(J(f)\) for all \(n \in \mathbb{N}\).

Let \(K \subset \mathbb{C}\) be a compact set with \(\text{Card}K \geq m\) for some \(m \in \mathbb{N}\). Recall that, for every \(n \in \mathbb{N}\) with \(n \leq m\), the unique monic polynomial \(P_n\) of degree \(n\) satisfying

\[
\|P_n\|_K = \inf_{Q_{n-1} \in \mathcal{P}_{n-1}} \|z^n - Q_{n-1}(z)\|_K,
\]

is called the \(n\)-th Chebyshev polynomial on \(K\) where \(\| \cdot \|_K\) is the sup-norm on \(K\) and \(\mathcal{P}_{n-1}\) is the space of all polynomials of degree less than or equal to \(n - 1\).
In Section 2, we review some standard facts about the generalized Julia sets and the Chebyshev polynomials. In the last section, we present a result which can be seen as a generalization of Theorem 1.1. Polynomials considered in these sections are always nonlinear complex polynomials unless stated otherwise. For a deeper discussion of Chebyshev polynomials, we refer the reader to [13, 14, 17]. For different aspects of the theory of Julia sets, see [11] among others.

2. Preliminaries

Autonomous polynomial Julia sets enjoy plenty of nice properties. These sets are non-polar compact sets which are regular with respect to the Dirichlet problem. Moreover, there are a couple of equivalent ways to describe these sets. For further details, see [11]. In order to have similar features for the generalized case, we need to put some restrictions on the given polynomials. The conditions used in the following definition are from Section 4 in [5].

Definition 2.1. Let \( f_n(z) = \sum_{j=0}^{d_n} a_{n,j} \cdot z^j \) where \( d_n \geq 2 \) and \( a_{n,d_n} \neq 0 \) for all \( n \in \mathbb{N} \). We say that \( \{f_n\} \) is a regular polynomial sequence if the following properties are satisfied:

- There exists a real number \( A_1 > 0 \) such that \( |a_{n,d_n}| \geq A_1 \), for all \( n \in \mathbb{N} \).
- There exists a real number \( A_2 \geq 0 \) such that \( |a_{n,j}| \leq A_2 |a_{n,d_n}| \) for \( j = 0, 1, \ldots, d_n - 1 \) and \( n \in \mathbb{N} \).
- There exists a real number \( A_3 \) such that
  \[
  \log |a_{n,d_n}| \leq A_3 \cdot d_n,
  \]
  for all \( n \in \mathbb{N} \).

If \( \{f_n\} \) is a regular polynomial sequence then we use the notation \( \{f_n\} \in \mathcal{R} \). Here and in the sequel, \( F_l(z) := (f_1 \circ \cdots \circ f_l)(z) \) and \( \rho_l \) is the leading coefficient of \( F_l \). Let \( \mathcal{A}_{\{f_n\}}(\infty) := \{ z \in \mathbb{C} : (F_n(z))_{n=1}^{\infty} \text{ goes locally uniformly to } \infty \} \) and \( \mathcal{K}_{\{f_n\}} := \{ z \in \mathbb{C} : (F_n(z))_{n=1}^{\infty} \text{ is bounded} \} \). In the next theorem, we list some facts that will be necessary for the subsequent results.

Theorem 2.2. \([5]\) Let \( \{f_n\} \in \mathcal{R} \). Then the following hold:

(a) \( J_{\{f_n\}} \) is a compact set in \( \mathbb{C} \) with positive logarithmic capacity.

(b) For each \( R > 1 \) satisfying

\[
A_1 R \left(1 - \frac{A_2}{R-1}\right) > 2,
\]

we have \( \mathcal{A}_{\{f_n\}}(\infty) = \bigcup_{k=1}^{\infty} F_k^{-1}(\Delta_R) \) and \( f_n(\overline{\Delta_R}) \subset \Delta_R \) where \( \Delta_R = \{ z \in \mathbb{C} : |z| > R \} \). Furthermore, \( \mathcal{A}_{\{f_n\}}(\infty) \) is a domain in \( \overline{\mathbb{C}} \) containing \( \Delta_R \).

(c) \( \Delta_R \subset F_k^{-1}(\Delta_R) \subset F_{k+1}^{-1}(\Delta_R) \subset \mathcal{A}_{\{f_n\}}(\infty) \) for all \( k \in \mathbb{N} \) and each \( R > 1 \) satisfying (2.1).

(d) \( \partial \mathcal{A}_{\{f_n\}}(\infty) = \partial J_{\{f_n\}} = \partial \mathcal{K}_{\{f_n\}} \) and \( \mathcal{K}_{\{f_n\}} = \mathbb{C}\setminus \mathcal{A}_{\{f_n\}}(\infty) \). Thus, \( \mathcal{K}_{\{f_n\}} \) is a compact subset of \( \mathbb{C} \) and \( J_{\{f_n\}} \) has no interior points.

The next result is an immediate consequence of Theorem 2.2.
Proposition 2.3. Let \((f_n) \in \mathcal{R}\). Then
\[
\lim_{k \to \infty} \left( \sup_{a \in \mathbb{C} \setminus F_k^{-1}(\Delta_R)} \operatorname{dist}(a, K_{f_n}) \right) = 0,
\]
where \(\operatorname{dist}(\cdot)\) is the distance function and \(R\) be a real number satisfying (2.1).

Proof. Since the Euclidean metric and the chordal metric are strongly equivalent on the compact subsets of \(\mathbb{C}\), we consider here the Euclidean metric. By using parts (b), (c) and (d) of Theorem 2.2 we have \(\mathbb{C} \setminus F_k^{-1}(\Delta_R) \subset \mathbb{C} \setminus F_k^{-1}(\Delta_R)\) which implies that
\[
ak_k := \sup_{a \in \mathbb{C} \setminus F_k^{-1}(\Delta_R)} \operatorname{dist}(a, K_{f_n})
\]
is a decreasing sequence.

Suppose that \(a_k \to \epsilon\) as \(k \to \infty\) for some \(\epsilon > 0\). Then, by compactness of the set \(\mathbb{C} \setminus F_k^{-1}(\Delta_R)\), there exists \(b_k \in \mathbb{C} \setminus F_k^{-1}(\Delta_R)\) for each \(k\) such that \(\operatorname{dist}(b_k, K_{f_n}) \geq \epsilon\). But since \(\cap_{k=1}^{\infty} \mathbb{C} \setminus F_k^{-1}(\Delta_R) = K_{f_n}\) by parts (b) and (d) of Theorem 2.2, \(b_k\) should have an accumulation point \(b\) in \(K_{f_n}\) with \(\operatorname{dist}(b, K_{f_n}) > \epsilon/2\) which is clearly impossible. This completes the proof. \(\square\)

For a compact set \(K \subset \mathbb{C}\), the smallest closed disc \(B(a, r)\) containing \(K\) is called the Chebyshev disk for \(K\). The center \(a\) of this disk is called the Chebyshev center of \(K\). These concepts were crucial and widely used in the paper [10]. The next result which is vital for the proof of Lemma 3.1 is from [12].

Theorem 2.4. Let \(L \subset \mathbb{C}\) be a compact set with \(\operatorname{card} L \geq 2\) having the origin as its Chebyshev center. Let \(L_p = p^{-1}(L)\) for some monic complex polynomial \(p\) with \(\deg p = n\). Then \(p\) is the unique Chebyshev polynomial of degree \(n\) on \(L_p\).

3. Results

First, we begin with a lemma which is also interesting in its own right.

Lemma 3.1. Let \(f\) and \(g\) be two non-constant complex polynomials and \(K\) be a compact subset of \(\mathbb{C}\) with \(\operatorname{card} K \geq 2\). Furthermore, let \(\alpha\) be the leading coefficient of \(f\). Then the following propositions hold.

(a) The Chebyshev polynomial of the degree \(\deg f\) on the set \((g \circ f)^{-1}(K)\) is of the form \(f(z)/\alpha - \tau\) where \(\tau \in \mathbb{C}\).

(b) If \(g\) is given as a linear combination of monomials of even degree and \(K = D(0, R)\) for some \(R > 0\) then the \(\deg f\)-th Chebyshev polynomial on the set \((g \circ f)^{-1}(K)\) is \(f(z)/\alpha\).

Proof. Let \(K_1 := g^{-1}(K)\). Then \((g \circ f)^{-1}(K) = f^{-1}(K_1) = (f/\alpha)^{-1}(K_1/\alpha)\) where \(K_1/\alpha = \{z : z = z_1/\alpha \text{ for some } z_1 \in K_1\}\). By the fundamental theorem of algebra, \(\operatorname{card}(K_1/\alpha) = \operatorname{card}K_1 \geq \operatorname{card}K\) and \(K_1\) is compact by the continuity of \(g(z)\). The set \(K_1/\alpha\) is also compact since the compactness of a set is preserved under a linear transformation. Let \(\tau\) be the Chebyshev center for \(K_1/\alpha\). Then \(K_1/\alpha - \tau\) is a compact set with the Chebyshev center as the origin where \(K_1/\alpha - \tau := \{z : z = z_1 - \tau \text{ for some } z_1 \in K_1/\alpha\}\). Note that, \(\operatorname{card}(K_1/\alpha - \tau) = \operatorname{card}(K_1/\alpha)\) and \((f/\alpha)^{-1}(K_1/\alpha) = (f/\alpha - \tau)^{-1}(K_1/\alpha - \tau)\). Using Theorem 2.4 for \(p(z) = f(z)/\alpha - \tau\)
and $L = K_1/\alpha - \tau$, we see that $p(z)$ is the deg $f$-th Chebyshev polynomial on the set $L_p = (g \circ f)^{-1}(K)$. This proves the first part of the theorem.

Suppose further that $g(z) = \sum_{n=0}^{\infty} a_n \cdot z^{2n}$ for some $n \geq 1$ and $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ with $a_n \neq 0$. Let $K = \overline{D(0, R)}$ for some $R > 0$. Then the Chebyshev center for $K_1/\alpha = g^{-1}(K)/\alpha = g^{-1}(D(0, R))/\alpha$ is the origin since $g(z)/\alpha = g(-z)/\alpha$ for all $z \in \mathbb{C}$. Thus, $f(z)/\alpha$ is the deg $f$-th Chebyshev polynomial for $(g \circ f)^{-1}(K)$ under these extra assumptions.

□

The next theorem shows that it is possible to obtain similar results to Theorem 1.1 in a richer setting.

**Theorem 3.2.** Let $(f_n) \in \mathcal{R}$. Then the following hold:

(a) For each $m \in \mathbb{N}$, the deg $F_m$-th Chebyshev polynomial on $J_{(f_n)}$ is of the form $F_m(z)/\rho_m - \tau_m$ where $\tau_m \in \mathbb{C}$.

(b) If, in addition, each $f_n$ is given as a linear combination of monomials of even degree then $F_m(z)/\rho_m$ is the deg $F_m$-th Chebyshev polynomial on $J_{(f_n)}$ for all $m$.

*Proof.* Let $m \in \mathbb{N}$ be given and $R > 1$ satisfy (2.1). For each natural number $l > m$, define $g_l := f_l \circ \ldots \circ f_{m+1}$. Then $F_l = g_l \circ F_m$ for each such $l$. Using part (a) of Lemma 3.1 for $g = g_l$, $f = F_m$ and $K = \overline{D(0, R)}$, we see that the $(d_1 \cdots d_m)$-th Chebyshev polynomial on $(g_l \circ F_m)^{-1}(\overline{D(0, R)})$ is of the form $F_m(z)/\rho_m - \tau_l$ where $\tau_l \in \mathbb{C}$. Let $C_l := ||F_m(z)/\rho_m - \tau_l||_{(g_l \circ F_m)^{-1}(K)}$. Note that, by part (c) of Theorem 2.2 $F_s^{-1}(\overline{D(0, R)}) \subset F_{s+1}^{-1}(\overline{D(0, R)})$ provided that $s < t$. This implies that $(C_j)_{j=m+1}^{\infty}$ is a decreasing sequence of positive numbers and hence has a limit $C$. The last follows from the observation that the norms of the Chebyshev polynomials of the same degree on a nested sequence of compact sets constitute a decreasing sequence.

Let $P_{d_1 \cdots d_m}(z) = \sum_{j=0}^{d_1 \cdots d_m} a_j z^j$ be the $(d_1 \cdots d_m)$-th Chebyshev polynomial on $\mathcal{K}_{(f_n)}$. Since $\mathcal{K}_{(f_n)} \subset (g_l \circ F_m)^{-1}(\overline{D(0, R)})$ for each $l$, $||P_{d_1 \cdots d_m}||_{\mathcal{K}_{(f_n)}} := C_0 \leq C$. Suppose that $C_0 < C$.

Let $\epsilon = \min\{(C - C_0)/2, 1\}$ and

$$\delta = \frac{\epsilon}{\max\{|a_1|, |a_2|, \ldots, |a_{d_1 \cdots d_m}|\} (8R)^{d_1 \cdots d_m (d_1 \cdots d_m)^2}}.$$

By Proposition 2.3 there exists a real number $N_0 > m$ such that $N > N_0$ with $N \in \mathbb{N}$ implies that

$$\sup_{z \in \mathbb{C} \setminus F_{N_0+1}^{-1}(\Delta_R)} \text{dist}(z, \mathcal{K}_{(f_n)}) < \delta.$$

Therefore, for any $z \in F_{N_0+1}^{-1}(\overline{D(0, R)})$, there exists a number $z' \in \mathcal{K}_{(f_n)}$ with $|z - z'| < \delta$. Hence, for each $z \in F_{N_0+1}^{-1}(\overline{D(0, R)})$,

$$|P_{d_1 \cdots d_m}(z)| < |P_{d_1 \cdots d_m}(z')| + \frac{\epsilon}{2} < C \leq \left| \frac{F_m(z)}{\rho_m - \tau_{N_0+1}} \right|_{F_{N_0+1}^{-1}(\overline{D(0, R)})},$$

which contradicts with the fact that $F_m(z)/\rho_m + \tau_{N_0+1}$ is the $(d_1 \cdots d_m)$-th Chebyshev polynomial on $F_{N_0+1}^{-1}(\overline{D(0, R)})$. Thus $C_0 = C$. 

Since \((C_j)_{j=m+1}^\infty\) is decreasing we have \(|\tau_l| < 2C_m + |\tau_m|\) provided that \(l > m\). Thus \((\tau_l)_{l=m+1}^\infty\) has at least one convergent subsequence \((\tau_k)_{k=1}^\infty\) with a limit \(\tau_m\). Therefore,

\[
C \leq \lim_{k \to \infty} \left| \frac{F_m(z)}{\rho_m} - \tau_m \right|_{l_k^{-1}(D(0,R))} \leq \lim_{k \to \infty} (C_k + |\tau_k - \tau_m|) = C.
\]

By uniqueness of the Chebyshev polynomials and (3.1), \(F_m(z)/\rho_m - \tau_m\) is \((d_1 \cdots d_m)\)-th Chebyshev polynomial on \(\mathcal{K}_{(f_n)}\). By the maximum principle, for any polynomial \(Q\), we have

\[
||Q||_{\mathcal{K}_{(f_n)}} = ||Q||_{\partial \mathcal{K}_{(f_n)}} = ||Q||_{J_{(f_n)}}.
\]

Hence the Chebyshev polynomials on \(\mathcal{K}_{(f_n)}\) and \(J_{(f_n)}\) should coincide. This proves the first assertion.

Suppose further that, the assumption given in part (b) is satisfied. Then by part (b) of Lemma 3.1 for \(g = g_l\), \(f = F_m\) and \(K = D(0,R)\), the \((d_1 \cdots d_m)\)-th Chebyshev polynomial on \((g_l \circ F_m)^{-1}(D(0,R))\) is of the form \(F_m(z)/\rho_m - \tau_l\) where \(\tau_l = 0\) for \(l > m\). Any subsequence of \((\tau_l)_{l=m+1}^\infty\) converges to 0. Thus, arguing as above, we can reach the conclusion that \(F_m(z)/\rho_m\) is the \((d_1 \cdots d_m)\)-th Chebyshev polynomial for \(J_{(f_n)}\) under this extra assumption. This completes the proof. \(\square\)

This theorem gives the total description of \(2^n\) degree Chebyshev polynomials for the most studied case, i.e., \(f_n(z) = z^2 + c_n\) with \(c_n \in \mathbb{C}\) for all \(n\). If \((c_n)_{n=1}^\infty\) is bounded then the logarithmic capacity of \(J_{(f_n)}\) is 1. Moreover, if \(|c_n| \leq 1/4\) for all \(n\), \(J_{(f_n)}\) is connected. If \(|c_n| < c < 1/4\) for some positive \(c\), then \(J_{(f_n)}\) is quasicircle. See [4], for the definition of a quasicircle and proofs of the above facts.

For a non-polar infinite compact set \(K \subset \mathbb{C}\), let us define the sequence \((W_n(K))_{n=1}^\infty\) by \(W_n(K) = ||P_n||/(\text{Cap}(K))^n\) for all \(n \in \mathbb{N}\). There are recent studies on the asymptotic behaviour of these sets on several occasions. See e.g. [2, 9, 18].

In order \(W_n(K)\) to be bounded, there are sufficient conditions given in terms of the smoothness of the outer boundary of \(K\) in [2, 18]. There is also an old and open question proposed by Ch.Pommerenke which is in the inverse direction: Find (if it is possible) a continuum \(K\) with \(\text{Cap}(K) = 1\) such that \((W_n(K))_{n=1}^\infty\) is unbounded. To answer this question positively, it is very natural to consider a continuum with a nonrectifiable outer boundary. Thus, we make the following conjecture:

**Conjecture 3.3.** Let \(f(z) = z^2 + 1/4\). Then, \((W_n(J(f)))_{n=1}^\infty\) is unbounded.

As discussed in [2], for \(f(z) = z^2 + 1/4\), \(J(f)\) has dimension 2 and in this case \(J(f)\) is not a quasicircle. Hence, Theorem 2 of [2] is not applicable since it requires even stronger assumptions on the outer boundary. But, unfortunately, proving Conjecture 3.3 is not an easy task. Theorem 1.1 is not sufficient to find all Chebyshev polynomials on \(J(f)\) explicitly. Nonetheless, one can claim, further, that for \(f_n(z) = z^2 + 1/4 - \epsilon_n\), with a fastly decreasing positive sequence \((\epsilon_n)_{n=1}^\infty\) (for example if the terms satisfy \(\sum_{n=1}^\infty \sqrt{\epsilon_n} < \infty\) with \(0 < 1/4 < \epsilon_n\)), the sequence \((W_n(J_{(f_n)}))_{n=1}^\infty\) is also unbounded.

**References**

[1] Alpan, G., Goncharov, A.: Orthogonal polynomials on generalized Julia sets. Manuscript submitted for publication.
[2] Andrievskii, V.V.: Chebyshev Polynomials on a System of Continua. Constr. Approx. doi: 10.1007/s00365-015-9280-8
[3] Brolin, H.: Invariant sets under iteration of rational functions. Ark. Mat. 6(2), 103–144 (1965)
[4] Brück, R.: Geometric properties of Julia sets of the composition of polynomials of the form $z^2 + c_n$. Pac. J. Math. 198, 347–372 (2001)
[5] Brück, R., Büger, M.: Generalized Iteration. Comput. Methods Funct. Theory 3, 201–252 (2003)
[6] Büger, M.: Self-similarity of Julia sets of the composition of polynomials. Ergodic Theory Dyn. Syst. 17, 1289–1297 (1997)
[7] Comerford, M.: Hyperbolic non-autonomous Julia sets. Ergodic Theory Dyn. Syst. 26, 353–377 (2006)
[8] Fornæss, J.E., Sibony, N.: Random iterations of rational functions. Ergodic Theory Dyn. Syst. 11, 687–708 (1991)
[9] Goncharov, A., Hatinoğlu, B.: Widom Factors. Potential Anal. 42, 671-680 (2015)
[10] Kamo, S.O., Borodin, P.A.: Chebyshev polynomials for Julia sets. Mosc. Univ. Math. Bull. 49, 44-45 (1994)
[11] Milnor, J.: Dynamics in one complex variables. Princeton University Press, Annals of Mathematics Studies, 160, Princeton University Press, Princeton, NJ, (2006)
[12] Ostrovskii, I.V., Pakovitch, F., Zaidenberg, M.G.: A remark on complex polynomials of least deviation. Internat. Math. Res. Notices. 14, 699–703 (1996)
[13] Peherstorfer, F., Schiefermayr, K.: Description of extremal polynomials on several intervals and their computation I, II. Acta Math. Hungar. 83, 27–58, 59–83 (1999)
[14] Peherstorfer, F., Steinbauer, R.: Orthogonal and $L_q$-extremal polynomials on inverse images of polynomial mappings. J. Comput. Appl. Math. 127, 297–315 (2001)
[15] Pommerenke, Ch.: Problems in Complex Function Theory. Bull. London Math. Soc. 4, 354-366 (1972)
[16] Rugh, H.H.: On the dimensions of conformal repellers. Randomness and parameter dependency. Ann. Math. 168(3), 695–748 (2008)
[17] Sodin, M., Yuditskii, P.: Functions deviating least from zero on closed subsets of the real axis. St. Petersbg. Math. J. 4, 201–249 (1993)
[18] Totik, V., Varga, T.: Chebyshev and fast decreasing polynomials. Proc. London Math. Soc. doi:10.1112/plms/pdv014