PARAMETRIZATIONS OF ALL WAVELET FILTERS: 
INPUT-OUTPUT AND STATE-SPACE

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Abstract. We here use notions from the theory linear shift-invariant dynamical systems to provide an explicit characterization, both practical and computable, of all rational wavelet filters. For a given $N$, ($N \geq 2$) the number of inputs, the construction is based on a factorization to an elementary wavelet filter along with of $m$ elementary unitary matrices. We shall call this $m$ the index of the filter. It turns out that the resulting wavelet filter is of McMillan degree $N \left( \frac{1}{2}(N - 1) + m \right)$.

Moreover, beyond the parameters $N$ and $m$, one confine the spectrum of the filters to lie in an open disk of radius $\rho$ (stable filters mean $\rho \in [0,1]$ and for FIR take $\rho = 0$). Then all filters can be described by a convex set of parameters $(0, \pi) \times [0, 2\pi)^2(N-1) \times [0, \rho]^m$.

Rational wavelet filters bounded at infinity, admit state space realization. The above input-output parametrization is exploited for a step-by-step construction (where in each, the index $m$ is increased by one) of state space model of wavelet filters.

1. Introduction

1.1. Problem formulation - symmetries. Over decades, filters have played a number of roles in both signal processing and applied mathematics. Of particular interest have been wavelet filters [11], [12], [30], [26], [39].

To describe them let $\mathbb{T}$, $\mathbb{D}$ be the unit circle, and the open unit disk, respectively, i.e.

$$\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \} \quad \mathbb{D} := \{ z \in \mathbb{C} : 1 > |z| \}$$

($\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$ is the closed unit disk). Roughly speaking wavelet filters are matrix valued rational functions $Sy(z)$ and $An(z)$, analytic outside $\mathbb{D}$, of a specific structure satisfying,

$$Sy(z)An(z) = I \quad z \in \mathbb{T}.$$  

In signal processing terminology, $An$ and $Sy$ are called the “analysis” and “synthesis” filters, respectively, see e.g. [46] Fig. 9.9], [49] Fig. 12.9-3], [54] Fig. 1]. The fact that $Sy$ is a (left) inverse of $An$ is referred to as the “perfect reconstruction” condition, see e.g. [46] Eq. (9.2)], [49] Section 5.6].

This work is aimed at three different communities: mathematicians interested in classical analysis, signal processing engineers and system and control engineers. Thus adopting the terminology familiar to one audience, may intimidate or even alienate the other. For example what is known to engineers as McMillan degree also

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arises in geometry of loop groups as an index. Filter banks and multi-bands from signal processing turn into representations of a Cuntz algebra, with the number of generating isometries in the Cuntz algebra equal to the number of frequency bands. Books like [9], [12] and [46] and papers like [29], [51] have made an impressive effort to be at least “bilingual”. Lack of space prevents us from providing even a concise dictionary of filter theory terms. Instead, we try to employ only basic concepts or indicate for references providing for the necessary background.

Our paper is concerned with the use of time-frequency filters in the construction of specific wavelets, for example in $L^2(\mathbb{R})$, or in $L^2(\mathbb{R}^k)$. The use of these filters offered a big boost to the list of constructive and computable approaches. The significance of this lies both on the theoretical side as well as the practical side; e.g., in adaptive wavelet approximations; as well as the building of fast computation of wavelet coefficients and reconstruction. Signal and image processing have typically been the source of time-frequency filters to be used in wavelets. Our present aim is to supplement this with tools from systems theory; as well as stressing the interdisciplinary unity of these themes. This means we here focus on studying (wavelet) filters, leaving for a future work the detailed wavelet analysis going into building bases from filters. There are several reasons for our choice of emphasis. Firstly, the process of building wavelets from filters is covered in the literature; see e.g., [9, 10, 11, 12]; but secondly, the analysis involved in this part of the subject is a separate endeavor. Indeed, it offers a variety of tools depending on the focus, for example an emphasis on the theory, as opposed to “tailoring” the basis to a specific application. We have omitted a detailed account of it as it would take us too far afield. The following sample references should help: [6, 7, 23, 24, 25, 37, 40, 41, 43, 47, 55].

To simplify presentation we next define wavelet filters by two symmetries: (1.2) and (1.4).

Denote by $U_N$, with $N$ natural, the set of $N \times N$-valued rational functions $U(z)$ unitary on the unit circle $\mathbb{T}$, i.e.

\begin{equation}
U_N = \{ U(z) : U(z)^* U(z) = I_N \quad z \in \mathbb{T} \}.
\end{equation}

Where $U^*$ denotes the complex conjugate transpose of $U$. Whenever clear from the context we shall omit the subscript $N$ and simply write $U$. The infinite-dimensional group $U$ in (1.2), is well known in literature, e.g. [1] Versions of it are studied in physics under the name loop-group [20]. In parts of the signal processing literature $U$ is referred to as the set of paraunitary matrices, see e.g. [40, Section 5.1], [49, Chapter 6]. In this work we confine the discussion to the subset of $U$ which is analytic outside $\mathbb{D}$, the open unit disk, (“Schur asymptotically stable” in some communities). In electrical networks terminology, it is sometimes referred to as “lossless”, see e.g. [49, Sections 3.5 and 14.2]. Generalizations of $U$ to the unstable case, in conjunction of wavelet filters, are addressed in [2], [3]. See also Remark 4 in Section 9.

For $N \geq 2$, we shall find it convenient to denote by $\epsilon$ the following $N$-th root of unity,

\begin{equation}
\epsilon := e^{i \frac{2\pi}{N}}.
\end{equation}
We shall say that an $N \times N$-valued ($N \geq 2$) rational function $F(z)$ belongs to $C_N$ if it is of the form,

\begin{equation}
F(z) = \begin{pmatrix}
\hat{f}_0(z) & \hat{f}_0(\epsilon z) & \cdots & \hat{f}_0(\epsilon^{N-1} z) \\
\hat{f}_1(z) & \hat{f}_1(\epsilon z) & \cdots & \hat{f}_1(\epsilon^{N-1} z) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{f}_{N-1}(z) & \hat{f}_{N-1}(\epsilon z) & \cdots & \hat{f}_{N-1}(\epsilon^{N-1} z)
\end{pmatrix},
\end{equation}

with $\epsilon$ in (1.3). In signal processing literature $F(z)$ in (1.4) is called a “filter bank” e.g. [46, Chapter 4], [49, Chapter 11]. As before, whenever clear from the context we shall omit the subscript $N$ and simply write $\mathcal{C}$.

In this work we focus our attention on wavelet filters, denoted by $\mathcal{W}_N$, that is functions within $\mathcal{U}$ (1.2) which satisfy (1.4), i.e.

\begin{equation}
\mathcal{W}_N := \mathcal{U}_N \cap C_N.
\end{equation}

In terms of (1.1) one simply takes,

$$A_n(z) = W(z) \quad \text{and} \quad S_y(z) = W \left( \frac{1}{z^*} \right)^*,$$

with $W \in \mathcal{W}$, see e.g. [49] Fig. 4.3-12 (recall that $z = \frac{1}{z^*}$ on $T$). Up to Remarks 6, 7 in Section 9 we shall not relate to (1.1) and only to (1.5). The fact that wavelet filters satisfy (1.5) has been long recognized, see e.g. [8]. In the spirit of [12], we here use (1.5) as their characterization. In signal processing literature, $N$ is referred to as “the number of bands of filter”.

The structure of this family suggests that its description is not straightforward: The set $\mathcal{U}$ is a multiplicative group, i.e. if $U_1(z)$ and $U_2(z)$ belong to $\mathcal{U}$ then so is the product $U_1U_2$. However, the set $\mathcal{U}$ is not closed under addition. In contrast, the set $C_N$ in (1.4) is closed under both addition and multiplication by a scalar. However, $C$ is not a multiplicative group, i.e. if $F_1(z)$ and $F_2(z)$ belong to $C$, the product $F_1F_2$ does not necessarily belong to this set.

The purpose of this work is to provide an easy-to-compute characterization of all wavelet filters described by (1.5) in both presentations: input-output and state-space. To this end we introduce the notion of an elementary wavelet filter, i.e. a minimal McMillan degree element in $\mathcal{W}_N$. This degree turns to be $\frac{1}{2}N(N-1)$.

A wavelet filter is then characterized as an elementary filter multiplied from the left by an element of $\mathcal{U}_N$ of the form $U_N(z)$ (in signal processing terminology, $N$-decimated $N$-expanded $U$, see e.g. [49] Subsection 4.1.1). In turn, this $U(z^N)$ can be factorized into $m$ elementary members of $\mathcal{U}$. We call this $m$ the index of the filter. The McMillan degree of the resulting wavelet filter is $N \left( \frac{1}{2}(N-1) + m \right)$.

1.2. Background motivation. As already pointed out, from application point of view wavelet filters are of interest [2], [3], [12] and the references cited there. In their simplest form, for discrete-time signals, the study of filters makes use of dual frequency variables, leading to functions of a complex variable.

Aside from applications, the uses of filters in mathematics are manifold: In duality theories in harmonic analysis, see e.g., [20], [34], [19], [20], [21]; in wavelets from the fundamental work of Daubechies in [18] to [39], [4], [5], [45], [46], [53], and other
families of frames and orthogonal systems [51], [54]; in operator theory [1]; in operator algebras [12], [20], in representation theory (for examples the notion of filter bank we discuss is closely tied to certain representations of the Cuntz algebras), in geometry (for example what we introduce below as McMillan degree also arises in geometry as an index); the theory of matrix functions and their factorization, see e.g., [42], [52], [28], [29], [30], [31], [32], [33] and the references cited there; as well as in mathematical physics. All of these instances are in addition related fields in engineering, e.g., in [8], [12], [13], [14], [15], [16], [22] the fundamental ideas have been influential; as well as in neighboring fields; see for example [2], [3], [10], [11], [48], [49]. Even within harmonic analysis, there are several viewpoints. Some authors, for example, (see e.g., [17] study the geometry of inner functions with the use of the Cuntz relations built from composition operators. By contrast, our present work takes as its point of departure de Branges spaces and applications to systems theory, see e.g., [36], [1]; and control theory, see [44], [50].

As already mentioned, this paper addresses different audiences: Engineers from the fields of system and control or from signal processing and mathematician interested in classical analysis. Thus, some of this work is of background nature (e.g. sections 3, 5, 10). Each section covers a theme and begins with a brief summary of the main ideas.

The outline is as follows. In Section 2 we address the set $\mathcal{C}$ described in (1.4), while in Section 3 the set $\mathcal{U}$ (1.2). In Section 4 wavelet filters are characterized in the input-output framework. The rest of the work is devoted to state-space realization. Relevant known background is given for general systems in Section 5 and for the set $\mathcal{U}$ in subsection 10.2. Realizations of elementary: Wavelet filters and unitary functions are given in Sections 6 and 7 respectively. The combination of both yields the realization of wavelet filters in 8. The powerful parametrization of wavelet filters introduced here, opens the door for future research on both sides Engineering and Mathematics. Sample future research topics are given in Section 9. Additional background is relegated to the appendix in section 11.

Many of the ideas here are familiar to some research communities. Nevertheless, as far as we know, none of the results below, designated as original, appeared before.

## 2. The family $\mathcal{C}_N$

Here we outline (see Lemma 2.1) a practical and geometric characterization of the set $\mathcal{C}$ defined through the symmetry condition we introduced in (1.4).

To this end, we denote by $\hat{P}$ the following $N \times N$ permutation matrix,

$$
\hat{P} = \begin{pmatrix}
0_{1 \times (N-1)} & 1 \\
I_{N-1} & 0_{(N-1) \times 1}
\end{pmatrix}.
$$

Using (1.3) note that,

$$
\hat{P} = U \text{diag}\{1, \epsilon, \epsilon^2, \ldots, \epsilon^{N-1}\} U^*,
$$

for some unitary $U$, i.e. $U^*U = I$. Thus, in particular $\hat{P}_k^{k_{k \in [0, N-1]} \neq I}$ and $\hat{P}_k^{k_{k \in N}} = I$. In addition $\hat{P}^* \hat{P} = I$.

**Lemma 2.1.** I. A rational function $F(z)$ is in $\mathcal{C}_N$ if and only if it satisfies

$$
F(\epsilon z) = F(z)\hat{P}.
$$
II. Let \( F_a(z) \) and \( F_b(z) \) be in \( \mathcal{C}_N \) then,

\[
F_b(\epsilon z) F_a(\epsilon z) = F_b(z) F_a(z)^*.
\]

If in addition \( F_a(z) \) is invertible then,

\[
F_b(\epsilon z) F_a(\epsilon z)^{-1} = F_b(z) F_a(z)^{-1}.
\]

**Proof:** I. One direction is clear. If \( F \) and so \( F_1 \) satisfy

\[
\text{Conversely, if (2.2) holds, (1.4)}, \text{ i.e. it belongs to} \mathcal{C}_N \text{, then also (2.2) holds.}
\]

Note that the relation in (2.3) holds whenever \( G \) is of the form (1.4), i.e. it is of the form (1.4), then the columns of \( F(\epsilon z) \) are cyclically shifted to the left by one, namely \( f_j(z) := \begin{pmatrix} f_0(\epsilon^j z) \\ \vdots \\ f_{N-1}(\epsilon^j z) \end{pmatrix} \) with \( j = 0, \ldots, N-1, \) so indeed (2.2) holds.

Conversely, if (2.2) holds, \( f_0(z), \ldots, f_{N-1}(z) \) the columns of \( F(z) \), see (2.4), satisfy

\[
f_0(\epsilon z) = f_1(z) \\
f_1(\epsilon z) = f_2(z) \\
\vdots \\
f_{N-2}(\epsilon z) = f_{N-1}(z) \\
f_{N-1}(\epsilon z) = f_0(z)
\]

and so \( f_2(z) = f_0(\epsilon^2 z), \ldots, f_{N-1}(z) = f_0(\epsilon^{N-1} z) \). Namely, \( F(z) \) is of the form (1.4), i.e. it belongs to \( \mathcal{C}_N \).

II. Substituting in (2.2) and using (2.1) yields

\[
F_b(\epsilon z) F_a(\epsilon z)^* = F_b(z) \hat{P} \left( F_a(z) \hat{P} \right)^* = F_b(z) \hat{P} \hat{P}^* F_a(z)^* = F_b(z) F_a(z)^*,
\]

so the first part is established. For the second part, use (2.2) to write \( \hat{P} = F_a(z)^{-1} F_a(\epsilon z) \).

Substituting in (2.2) for \( F_b(z) \) yields

\[
F_b(\epsilon z) = F_b(z) \left|_{P=F_a(z)^{-1} F_a(\epsilon z)} \right. = F_b(z) F_a(z)^{-1} F_a(\epsilon z).
\]

By assumption \( F_a(\epsilon z) \) is invertible. Multiplying by \( F_a(\epsilon z)^{-1} \) from the right completes the proof.

Note that the relation in (2.3) holds whenever

\[
F_b(z) = G(z) F_a(z),
\]

for arbitrary invertible matrix valued rational function \( G(z) \) satisfying \( G(\epsilon z) = G(z) \).

Note also that this is the case if one takes \( G \) to be \( G(z^N) \) (the passage from \( G(z) \) to \( G(z^N) \) is addressed in part 10.1). It turns out that this sufficient condition is also necessary.

**Proposition 2.2.** Let \( F_a(z) \) and \( F_b(z) \) be in \( \mathcal{C}_N \) with \( F_a \) invertible. Then,

I. There exists a matrix valued rational function \( G(z) \) such that

\[
F_b(z) F_a(z)^{-1} = G(z^N).
\]

II. If in addition \( G \hat{P} = \hat{P} G \) then also

\[
F_a(z)^{-1} F_b(z) = G(z^N).
\]
Proof: I. Denoting \( F := F_a F_b^{-1} \), from (2.3) it follows that
\[
F(\epsilon z) = F(z).
\]
The function \( F \) has a Laurent expansion
\[
F(z) = \sum_{k=-k_0}^{\infty} F_k z^k
\]
converging in a punctured disk \( 0 < |z| < r \) for some \( r > 0 \). Equation (2.6) implies that
\[
\sum_{k=-k_0}^{\infty} F_k z^k = \sum_{k=-k_0}^{\infty} F_k \epsilon^k z^k.
\]
By uniqueness of the Laurent expansion we get that
\[
F_k = 0, \quad \text{for} \quad k \notin \mathbb{N} Z.
\]
Thus, if \( k_0 > 0 \), we may assume without loss of generality that \( k_0 = Nn_0 \) for some \( n_0 \in \mathbb{N} \). Thus, one can write,
\[
F(z) = G(z^N),
\]
so part I is established.

For part II denote \( \hat{F} = F_a^{-1} F_b \) and note that
\[
\hat{F}(\epsilon z) = F_a(\epsilon z)^{-1} F_b(\epsilon z) = \left( F_a(z) \hat{P} \right)^{-1} F_b(z) \hat{P}
\]
\[
= P^* \hat{F}(z) P_{|F_a=\hat{P}} = \hat{P}^* \hat{P} \hat{F}(z) = \hat{F}(z).
\]
In a way similar to part I one obtains \( \hat{F} = \hat{G}(z^N) \). \( \square \)

The proof of Proposition 2.2 can be mimicked to obtain the following result:

**Proposition 2.3.** Let \( F_a(z) \) and \( F_b(z) \) be in \( \mathcal{C}_N \). Then the products
\[
F_b(z) (F_a(z^*))^* \quad \text{and} \quad F_b(z) (F_a(1/z^*))^*
\]
are matrix valued rational functions of \( z^N \).

**Proof:** Here we make use of (2.1) to see that for
\[
G(z) = F_b(z) (F_a(z^*))^* \quad \text{or} \quad G(z) = F_b(z) (F_a(1/z^*))^*
\]
we have (2.0) i.e. \( G(z) = G(\epsilon z) \). \( \square \)

We now introduce the notion of elementary wavelet filters. To this end we need some preliminaries. For \( N \geq 2 \) let \( Q_N \) be the following (constant) \( N \times N \) unitary matrix,
\[
Q_N := \frac{1}{\sqrt{N}} \begin{pmatrix}
\epsilon^{-(0-0)} & \epsilon^{-(0-1)} & \epsilon^{-(0-2)} & \ldots & \epsilon^{-(0-(N-1))} \\
\epsilon^{-(1-0)} & \epsilon^{-(1-1)} & \epsilon^{-(1-2)} & \ldots & \epsilon^{-(1-(N-1))} \\
\epsilon^{-(2-0)} & \epsilon^{-(2-1)} & \epsilon^{-(2-2)} & \ldots & \epsilon^{-(2-(N-1))} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\epsilon^{-(N-1-0)} & \epsilon^{-(N-1-1)} & \epsilon^{-(N-1-2)} & \ldots & \epsilon^{-(N-1-(N-1))}
\end{pmatrix}.
\]

For example,
Recall that $Q_N$ is called the Discrete Fourier Transform matrix, see e.g. [9, Subsection 3.1.1], [49, page 794]. Namely, if $\hat{x} \in \mathbb{C}^N$ is the discrete Fourier transform of $x \in \mathbb{C}^N$, then they are related through

$$\hat{x} = \sqrt{N} Q_N x \quad x = \frac{1}{\sqrt{N}} Q_N^* \hat{x}.$$ 

Now, we denote by $\hat{U}(z)$ the following special element of $U_N$,

$$\hat{U}_N(z) := \text{diag} \left\{ 1, \frac{1}{z}, \ldots, \frac{1}{z^{N-1}} \right\}.$$ 

Note that $\hat{U}_N(z)$ is of McMillan degree $\frac{1}{2}N(N-1)$.

Next, using (2.7) and (2.8) for $N \geq 2$, we define the elementary wavelet filter,

$$\hat{W}_N(z) := \hat{U}_N(z) Q_N .$$

Note that the McMillan degree of $\hat{W}_N$ is as of $\hat{U}_N$, i.e. $\frac{1}{2}N(N-1)$.

**Corollary 2.4.** I. Let $F \in \mathcal{C}_N$ be given and let $L(z)$, $R(z)$ be arbitrary $N \times N$ valued rational functions, almost everywhere invertible. Then

$$\{L(z^N)F(z)\} \quad \text{and} \quad \{F(z)R(z^N) : R(z^N)\hat{P} = \hat{P}R(z^N)\},$$

are families in $\mathcal{C}_N$.

II. For given $N \geq 2$ let $\hat{W}_N(z)$ be the elementary wavelet filter in (2.9). Then,

$$\hat{W} \in \mathcal{C}.$$ 

Moreover, all elements in $\mathcal{C}_N$ are given by

$$\mathcal{C}_N = \{ L(z^N)\hat{W}_N(z) : L \text{ rational, invertible} \}.$$ 

Item I follows from Proposition 2.2.

II. To see that $\hat{W}$ in (2.9) is indeed in $\mathcal{C}_N$, note that

$$\hat{W}(\epsilon z) = \hat{U}(\epsilon z) Q = \hat{U}(z) \left( \hat{U}(z)_{|z=\epsilon} \right) Q = \hat{U}(z) Q P = \hat{W}(z) \hat{P}.$$ 

Now (2.10) follows from item I since $\hat{W}_N(z)$ is invertible.

### 3. The group $U$

As already mentioned the infinite-dimensional group $U$ is well known in literature see e.g. [11], [13], [20], [33] and the references cited there. We here review properties of this set, necessary to our construction in the sequel.

Recall that if $U(z)$ is in the set $U$, then so are $(U(z^*))^*$ and if stability is compromised also $U(\frac{1}{z})$.

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1Below, $L$ and $R$ stand for “left” and “right” respectively.
To construct the set $\mathcal{U}$ first recall that (up to multiplication by a constant of a unity modulus) a unity degree, scalar, stable rational function $\phi_\alpha(z)$ mapping $\mathbb{T}$ to itself can always be written as,

$$
\phi_\alpha(z) = \frac{1 - \alpha^* z}{z - \alpha} \quad \alpha \in \mathbb{D}.
$$

We shall call $\phi_\alpha(z)$ in (3.1) an elementary scalar unitary function. Every scalar rational function mapping $\mathbb{T}$ to itself of McMillan degree $p$, may be factored to elementary functions of the form (3.1), namely

$$
\prod_{j=1}^{p} \frac{1 - \alpha_j^* z}{z - \alpha_j} \quad \alpha_j \in \mathbb{D},
$$

where the $\alpha_j$’s are not necessarily distinct. As in this rational function the poles are within $\mathbb{D}$ and the zeros are outside $\mathbb{D}$, no cancellation can occur. Thus the issue of minimality of realization, central to system theory see e.g. [36, [44, Section 5.5]], can here be avoided.

We next extend this idea to factorize elements in $\mathcal{U}_N$. To this end, we need some preliminaries. The unit sphere in $\mathbb{C}^N$ will be denoted by,

$$
S_{N-1} := \{ v \in \mathbb{C}^N : v^* v = 1 \}.
$$

We can now introduce an $N$-dimensional elementary unitary matrix of McMillan degree one,

$$
V(z) := I_N + \left( \frac{1 - \alpha^* z}{z - \alpha} - 1 \right) v v^*, \quad v \in S_{N-1} \quad \alpha \in \mathbb{D}.
$$

Recall that $V(z)$ is a Finite Impulse Response (FIR) filter if and only if $\alpha = 0$.

For $j$ natural, we shall find convenient to denote,

$$
V_j(z) := V(z, v_j, \alpha_j),
$$

with $v_j \in S_{N-1}$ and $\alpha_j \in \mathbb{D}$, parameters.

We can cite an adapted version of [1, Theorem 3.11], see also [49, Eq. (14.9.2)]

**Theorem 3.1.** Using (3.3) (up to multiplication by constant unitary matrices from the left and from the right) $U(z)$ in $\mathcal{U}_N$ of McMillan degree $p$, can always be written as

$$
U(z) = \prod_{j=1}^{p} V_j(z).
$$

We now illustrate the richness associated with $U(z)$ in (3.4) already for $p = 2$,

$$
U(z) = V(z, v_1, \alpha_1) V(z, v_2, \alpha_2)
$$

Consider two “extreme” cases:

$$
U(z)_{v_1=v_2=\nu} = I_N + \left( \frac{(1-\alpha_1^* \nu) (1-\alpha_2^* \nu)}{(z-\alpha_1)(z-\alpha_2)} - 1 \right) v v^*_{\alpha_1=-\alpha_2=\sqrt{\nu}} = I_N + \left( \frac{1-\alpha^* \nu}{z^2-\alpha} - 1 \right) v v^* = V(z^2, v, \alpha),
$$
see (3.6) below. In contrast,

\[ U(z) \big|_{v_1^2 v_2 = 0, \alpha_1 = \alpha_2 = 0} = I_N + \left( \frac{1 - \alpha^* z}{z - \alpha} - 1 \right) (v_1 v_1^* + v_2 v_2^*) \big|_{N=2} \]

\[ = \frac{1 - \alpha^* z}{z - \alpha} I_2. \]

In particular one can use Theorem 3.1 to construct \( \hat{U}(z) \) in (2.8) by taking in (3.5) the parameters \( \alpha_j = 0 \) and \( v_j = e_j \), the unit vectors with the appropriate multiplicity. Recall that the resulting \( \hat{U}_N(z) \) is of McMillan degree \( \frac{1}{2} N(N - 1) \) (=the number of elementary unitary matrices involved).

Before proceeding, we find it convenient to introduce the \( \rho \)-scaled disk

\[ D_\rho := \rho \cdot D \quad \rho \in [0, 1]. \]

We shall let \( \alpha \), the pole of \( V(z) \) in (3.2) to lie within \( D_\rho \) and thus to quantify the stability of the unitary matrices at hand: From arbitrary asymptotic (in fact exponential) stability corresponding to \( \rho = 1 \), to finite impulse response, when \( \rho = 0 \). We shall thus call \( \rho \) the spectral radius of the unitary matrix \( V(z) \). The spectral radius, \( \rho \), is associated with concept of spectrum of an operator, for further motivation and details see (5.3). A word of caution, this is not to be confused with notion of the frequency spectrum of signals, commonly used in electrical engineering.

Following (3.2) and (3.4) we find it convenient to introduce an \( N \times N \), \( N \)-decimated \( N \)-expanded elementary unitary matrix of McMillan degree \( N \),

\[ V(z^N) := I_N + \left( \frac{1 - \alpha^* z}{z^N - \alpha} + 1 \right) vv^*, \]

with \( v_j \in S_{N-1} \) and \( \alpha_j \in D_\rho \) parameters, for some prescribed \( \rho \in [0, 1] \).

For future reference we now formulate a version of Theorem 3.1 for \( N \)-decimated \( N \)-expanded unitary matrix.

**Corollary 3.2.** Let \( U \in U_N \) be of the form \( U(z^N) \). Then (up to multiplication by constant unitary matrices from the left and from the right) using (3.6) it can always be written as,

\[ U(z^N) = \prod_{j=1}^m V_j(z^N). \]

We shall call \( m \) the index of \( U(z^N) \). Moreover, \( U(z^N) \) is of McMillan degree \( mN \).

With a slight abuse of terminology we shall refer to \( V(z^N) \) in (3.6) as an elementary unitary matrix (omitting the “\( N \)-decimated \( N \)-expanded” qualifier).

In the next section we combine the above properties of the set \( U \) along with the set \( C \) from Section 2 to characterize \( W \), the set of wavelet filters.

**4. The set of Wavelet filters \( W_N \)**

Our first observation on the structure of \( W_N \), the set of \( N \times N \) wavelet filters (1.5), is an immediate consequence of Proposition 2.2.

**Proposition 4.1.** Let \( W_a(z) \) and \( W_b(z) \) be in \( W_N \). Then there exists \( U \in U_N \) such that

\[ W_b(z) = U(z^N) W_a(z). \]
Indeed, substitute unitary matrices in Proposition 2.2 so that, \( F_a(z) = W_a(z) \), \( F_b(z) = W_b(z) \), and \( G(z^N) = U(z^N) \).

We next exploit Proposition 4.1 and Corollaries 2.4, 3.2 to provide an easy-to-compute characterization of all wavelet filters in \( W_N \) (\( N \) prescribed) taking the McMillan degree as a parameter. Strictly speaking the parameter will be \( m \), the index of the filter, yet to be defined. The exact relation between \( m \) and the McMillan degree is given in Corollary 4.4 below.

One can refine this parametrization and for a prescribed \( m \), to limit the spectral radius of the wavelet filter to \( \rho \) for some \( \rho \in [0, 1) \).

**Theorem 4.2.** I. Let \( W \in W_N \) be a given wavelet filter, and let \( L(z^N) \) and \( R(z^N) \) be in \( U_N \). Then,

\[
\{L(z^N)W(z)\} \quad \text{and} \quad \{W(z)R(z^N) : R(z^N)\hat{P} = \hat{P}R(z^N)\},
\]

are families of wavelet filters.

II. For given \( N \geq 2 \) let \( \hat{W}_N(z) \) be the elementary Wavelet filter in (2.9). Then,

\( \hat{W}_N \in W_N \).

III. All wavelet filters are given by,

\[
W_N = \left\{ \left( \prod_{j=1}^{m} V_j(z^N) \right) \hat{W}_N(z) : v_j \in \mathbb{S}_{N-1}, \alpha_j \in \mathbb{D}_\rho \right\},
\]

where \( \rho \in [0, 1] \), the spectral radius of the filter, is prescribed.

We shall call \( m \) in (4.1), the index of the filter.

For an arbitrary given \( W(z) \) in \( W_N \) we consider equivalent all wavelet filters of the form

\[ QW(z), \]

where \( Q \) varies over all constant \( N \times N \) unitary matrices.

From (4.1) we have the following.

**Corollary 4.3.** Let \( W_1(z), W_2(z), W_3(z) \) be in \( W_N \), then

\[ W_1W_2^{-1}W_3 \in W_N. \]

The characterization of wavelet filters in (4.1) implies that these are matrix valued functions of “quantized” McMillan degrees.

**Corollary 4.4.** An index \( m \) wavelet filter of dimension \( N \) is of McMillan degree, \( N \left( \frac{1}{2}(N-1)+m \right) \).

Indeed, as already mentioned, the McMillan degree of \( \hat{W}_N(z) \) is \( \frac{1}{2}N(N-1) \). From the construction in (3.6) and (4.1), each of the \( m \) elementary unitary matrices \( V_j(z^N) \) contributes \( N \) to the overall McMillan degree.

We now wish to “count” how many wavelet filters are there.

**Observation 4.5. I.** An index \( m \) wavelet filter of dimension \( N \) and spectral radius \( \rho \), can always be parameterized as a point in the real set,

\[
\left( [0, \pi] \times [0, 2\pi] \right)^{2(N-1)} \times [0, \rho]^m \quad \rho \in (0, 1].
\]
The set of Finite Impulse Response (FIR) filters (zero spectral radius) may be parameterized by,

\[ ([0, \pi] \times [0, 2\pi])^{2N-3} \]

II. In the above parametrization, the set of wavelet filters is convex.

**Proof** I. First, since the structure of \( \hat{W}_N \) is fixed, from (2.9) it is clear that the number of free parameters in \( W \) is given by the number of parameters in a unitary matrix \( U(z^N) \) in (3.7). In turn, recall that each of the \( m \) elementary unitary matrices \( V_j \) in (3.7) may be identified with a point in \( S^N_{N-1} \times [0, \rho) \). Thus, the number of parameters in a unitary matrix \( U(z^N) \) is given by

\[ (S^N_{N-1} \times [0, \rho))^m, \]

and out of them, the Finite Impulse Response (FIR) filters are given by the factor,

\[ S^m_{N-1} \].

Recall now that a point on \( v \in S_{N-1} \), can be equivalently parameterized as a point in the real “box” \([0, 2\pi]^{2N-1}\). For example, for \( N = 3 \) one has \( v = \begin{pmatrix} \cos(\delta)e^{i\alpha} \\ \cos(\eta)\sin(\delta)e^{i\beta} \\ \sin(\eta)\sin(\delta)e^{i\gamma} \end{pmatrix} \) with \( \alpha, \beta, \gamma, \delta, \eta \in [0, 2\pi) \).

As we are actually interested only in elements of the form \( vv^* \), the number of parameters is reduced to \([0, \pi] \times [0, 2\pi]^{2N-3} \). For example, in the above case of \( N = 3 \), without loss of generality one can take \( \alpha = 0 \) and \( \delta \in [0, \pi) \).

To complete the construction use the polar presentation of elements within a disk \( \mathbb{D}_\rho \) in \( \mathbb{C} \) as points in \([0, 2\pi] \times [0, \rho) \). Thus, an index 1 filter is parameterized by

\[ [0, \pi] \times [0, 2\pi]^{2N-3} \times [0, 2\pi] \times [0, \rho) = [0, \pi] \times [0, 2\pi]^{2(N-1)} \times [0, \rho) \]

so the construction is complete.

II. Is an immediate consequence of part I. \( \square \)

The favorable properties of FIR filters are well known, see e.g. Sections 2.4.2, beginning of 3.3. In particular, they dramatically attenuate any noise. However, Observation 4.5 suggests that whenever the noise level is sufficiently low, it is not recommended to restrict the discussion to FIR wavelet filters: For \( 1 >> \rho > 0 \) the filter is “almost FIR” but there are already “many more” filters than FIR’s. For a related discussion, see also the end of Section III in [27].

The convex parametrization of all wavelet filters introduced in Observation 4.5 may be in particular convenient for optimization, see e.g. [16], [27], [48] and [50].

Following (4.1) by multiplying from the left a given \( W_a \in W_N \) of McMillan degree \( N \left( \frac{1}{2}(N-1) + m \right) \), by an elementary unitary matrix \( V(z^N) \), i.e. increasing the index by 1, one can construct another wavelet filter \( W_b(z) \) of McMillan degree \( N \left( \frac{1}{2}(N+1) + m \right) \). Indeed,

\[ W_b(z) = V(z^N)W_a(z), \]

where \( V \) is as in (3.6) with \( v \in S_{N-1} \) and \( \alpha \in \mathbb{D}_\rho \), parameters. This is illustrated next.
Example 4.6. Following (2.9) let $\hat{W}_2(z)$ be the two dimensional elementary wavelet filter,

$$\hat{W}_2(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ z & -\frac{1}{z} \end{pmatrix}. \quad (4.2)$$

Let $V_\alpha(z^2)$ in $U_2$ be an elementary unitary matrix of the form of (3.6) with $v = (0 \ 1^T)$ and a parameter $\alpha \in \mathbb{D}_\rho$,

$$V_\alpha(z^2) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-\alpha^*z^2} \end{pmatrix}. \quad (4.3)$$

Clearly, it is of McMillan degree 2.

Multiplying $\hat{W}_2(z)$ from the left by $U_\alpha(z^2)$ yields $W_a(z)$, a wavelet filter with $N = 2$ and $m = 1$, i.e. of McMillan degree three,

$$W_a(z) = V_\alpha(z^2)\hat{W}_2(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \frac{1-\alpha^*z^2}{z(z^2-\alpha)} & \frac{1-\alpha^*z^2}{z(z^2-\alpha)} \end{pmatrix}. \quad (4.4)$$

Next, let $V_\beta(z^2)$ be a member of $U_2$ of index two (i.e. a product of a pair of elementary matrices of the form of (3.6) both with $v = (0 \ 1^T)$) where $\beta \in \mathbb{D}_\rho$ is a parameter,

$$V_\beta(z^2) = \begin{pmatrix} 1 & 0 \\ \frac{1-\beta^*z^4}{z^2} & \frac{1}{z^2} \end{pmatrix}. \quad (4.5)$$

Clearly, it is of McMillan degree 4.

In turn, multiplying $W_a(z)$ in (4.4) from the left by $U_\beta(z^2)$ from (4.5), yields $W_b(z)$, a wavelet filter with $N = 2$ and $m = 3$, i.e. of McMillan degree seven (see Corollary 4.4),

$$W_b(z) = V_\beta(z^2)W_a(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-\beta^*z^4 & 1-\beta^*z^4 \\ \frac{1-\alpha^*z^2}{z^2} & \frac{1-\alpha^*z^2}{z^2} \end{pmatrix}. \quad (4.6)$$

Note that it is only for $\alpha = \beta = 0$ that $W_b(z)$ is a FIR filter, see Observation 4.5.

State space realization of $\hat{W}_2$ in equation (4.2), is given in item 1 of Example 6.1 below. State space realizations of $W_a(z)$ and $W_b(z)$ in equations (4.4) and (4.6), respectively are given in Example 8.1 below. □

The rest of this work is devoted to state-space realization of wavelet filters. This topic has gained popularity in the last two decades, see e.g. [13], [14], [15], [16], [22], [48], [49 Section 13.4], [50] and [51]. In Sections 5, 6 below we review the background necessary to translate the above presented construction procedure of wavelet filters from matrix valued rational functions setting (input-output in engineering terminology) to state-space framework.

5. State space realization - preliminaries

Every $N \times N$ valued rational function $F(z)$ analytic at infinity ( $\lim_{z \to \infty} F(z)$ exists) can be written as

$$F(z) = C(zI_p - A)^{-1}B + D,$$
where $B \in \mathbb{C}^{p \times N}$ and $C \in \mathbb{C}^{N \times p}$. This may be viewed as the $Z$-transform of

\[(5.1) \quad x(n + 1) = Ax(n) + Bu(n) \quad y(n) = Cx(n) + Du(n),\]

where $u, y$ are $N$-dimensional input and output respectively and $x$ is $p$-dimensional (state in system theory terminology). Namely, $F(z)$ maps the $Z$-transform of $u$ to the $Z$-transform of $y$. We find it convenient to write (5.1) in the $(p + N) \times (p + N)$ system matrix form,

\[(5.2) \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.\]

In principle, rectangular system matrices, (i.e. the number of inputs differs from the number of outputs) are of interest, but they are mostly beyond the scope of this work, see Remark 6 in Section 9. As already mentioned, the issue of minimality of realization is here trivially satisfied and thus can be avoided.

Before proceeding, recall that $\rho(A)$, the spectral radius of a square matrix $A$, is the radius of the smallest disk, centered at the origin of the complex plane, containing the eigenvalues (=spectrum) of $A$. For some details, see e.g. [28] items 5.6.8-5.6.14.

In particular, recall that for in (5.1) one has that there exists $\beta \geq 1$ so that for all natural $n$,

\[(5.3) \quad \|x(n)\|_{\nu(n)\in\mathbb{Z}} \leq \beta \|x(0)\| \rho^n, \quad \forall x(0)\]

where $\rho = \rho(A)$, the spectral radius of the matrix $A$, which in turn coincides with the spectral radius defined in Sections 3, 4.

For future reference we now recall in the state space realization of a composition of a pair of rational functions (series connection or cascade of systems in Electrical Engineering terminology). Namely $F_\Delta(z), F_a(z)$ are of compatible dimensions and $F_b(z)$ is obtained,

\[(5.4) \quad F_b(z) = F_\Delta(z)F_a(z).\]

Assuming the state-space realization of each $F_a(z)$ and $F_\Delta(z)$ is known, one can construct a realization of the resulting $F_b(z)$, see e.g. [36] Subsection 8.3.3, [51] Eq. (4.15).

**Observation 5.1.** Given $l \times q$ and $q \times r$ valued rational functions $F_\Delta(z), F_a(z)$, respectively, admitting state space realization

\[
F_a(z) = C_a(zI - A_a)^{-1}B_a + D_a \quad A_a \in \mathbb{C}^{p_a \times p_a} \\
B_a \in \mathbb{C}^{p_a \times r} \quad C_a \in \mathbb{C}^{q \times p_a} \quad D_a \in \mathbb{C}^{q \times r} \\
F_\Delta(z) = C_\Delta(zI - A_\Delta)^{-1}B_\Delta + D_\Delta \quad A_\Delta \in \mathbb{C}^{p_\Delta \times p_\Delta} \\
B_\Delta \in \mathbb{C}^{p_\Delta \times q} \quad C_\Delta \in \mathbb{C}^{l \times p_\Delta} \quad D_\Delta \in \mathbb{C}^{l \times q}.
\]

The system matrix $M_b$ associated with the realization of $F_b(z)$ in (5.4) is given by,

\[
M_b = \begin{pmatrix} A_\Delta & B_\Delta C_a & B_\Delta D_a \\ 0 & A_a & B_a \\ C_\Delta & D_\Delta C_a & D_\Delta D_a \end{pmatrix}.
\]
Although not central to our discussion, we comment that even when the realizations of $F_a(z)$ and $F_\Delta(z)$ are minimal, the realization in $M_b$ is not necessarily minimal.

We now point out that whenever $A_a$ and $A_\Delta$ are upper triangular, the resulting $A_b$ is upper triangular as well. Fortunately, this is indeed the case with all wavelet filters, see e.g. (8.2), the system matrix associated with $W_b(z)$ from (4.6). See also Remark 2 in Section 9.

State space realization of unitary functions are briefly discussed in part 10.2.

We now outline the rest of this work: From the characterization of wavelet filters in (4.1) it follows that every wavelet filter of dimension $N$ and index $m$ is a product of the form

$$\prod_{j=1}^m V_j(z^N) \hat{W}(z).$$

State space realizations of elementary wavelet filters $\hat{W}_N(z)$ and of elementary unitary matrices $V_j(z^N)$ are given in sections 6 and 7 respectively. In section 8 we combine them.

6. STATE SPACE REALIZATION OF ELEMENTARY WAVELET FILTERS

Recall that $\hat{W}_N(z)$, the elementary wavelet filter in (2.9) can be described as the minimal McMillan degree ($= \frac{1}{2}N(N-1)$) filter in $\mathcal{W}_N$. In addition, it is a FIR filter. Thus, there is no surprise that there is a two stages recipe for the construction of $\hat{M}_N$, the associated system matrix.

1. We start by constructing an auxiliary permutation matrix, e.g. $[28, 0.9.5]$, denoted by $\tilde{M}_N$ so that

$$\tilde{M}_N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 

First, $A$ is the following $\frac{1}{2}N(N-1) \times \frac{1}{2}N(N-1)$ matrix,

$$A = \text{diag}\{ J_1(0), \ldots, J_{N-1}(0) \},$$

where $J_k(0)$ is a $k$-dimensional Jordan block associated with a zero eigenvalue, e.g. $[28, 3.1.1]$. Next, $D$ is a $N \times N$ matrix with zeroes everywhere, except a single 1 in the upper left corner.

Now, $C$ and $\hat{B}$ are constructed so that (i) the 1’s in each are in a descending staircase order, e.g.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and (ii) the resulting $\hat{M}_N$ is a permutation matrix.

2. Using $Q_N$ from (2.7),

$$\hat{M}_N = \tilde{M}_N \text{diag}\{ I_p, Q_N \} = \begin{pmatrix} A & \hat{B}Q_N \\ C & D \end{pmatrix}.$$ 

This is illustrated by the following.

**Example 6.1.** We here consider the system matrix $\hat{M}_N$ (5.2) associated with the elementary wavelet filter $\hat{W}_N(z)$ in (2.9).
1. For \( N = 2 \), it is given by,
\[
\tilde{M}_N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]
\[
\hat{M}_N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]

2. For \( N = 4 \), the system matrix is given by,
\[
\tilde{M}_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}
\]
\[
\hat{M}_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{i}{2} & \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} & \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} & \frac{1}{2} & \frac{i}{2} \end{pmatrix}.
\]

7. State space realization of elementary unitary matrices

We start by showing that the system matrix associated with an elementary unitary matrix-valued function see [3.0], may be obtained from the system matrix associated with an elementary scalar function.

**Observation 7.1.** Let \( V(z^N) \), be an elementary unitary function as in [3.0], i.e.
\[
(7.1) \quad V(z^N) = I_N + \left( \frac{1 - \alpha^* z^N}{z^N - \alpha} - 1 \right) vv^* \quad v \in \mathbb{S}_{N-1} \quad \alpha \in \mathbb{D}_p.
\]

With the same \( \alpha \), let \( \psi_\alpha(z) \) be the scalar unitary function,
\[
(7.2) \quad \psi_\alpha(z^N) := 1 - \frac{|\alpha|^2}{z^N - \alpha} = c(zi_N - A)^{-1}b,
\]

for some \( A \in \mathbb{C}^{N \times N} \), \( b \in \mathbb{C}^{N \times 1} \) and \( c \in \mathbb{C}^{1 \times N} \).

Then the \( 2N \times 2N \) system matrix \( M \) associated with \( V(z^N) \) in (7.1) is given by,
\[
M = \begin{pmatrix} A & bv^* \\ vc & I_N - (1 + \alpha^*)vv^* \end{pmatrix}.
\]

Indeed, it is straightforward to verify that one can equivalently write \( V(z^N) \) in (7.1) as,
\[
V(z^N) = v \psi_\alpha(z)v^* + I_N - (1 + \alpha^*)vv^* = vc(zi_N - A)^{-1}bv^* + I_N - (1 + \alpha^*)vv^*.
\]

This is illustrated next.

**Example 7.2.** 1. Consider the system matrix \( \tilde{M}_\alpha \) associated with the scalar function \( \psi_\alpha(z^N) \) in (7.2) with \( N = 2 \). It depends on the parameter \( \alpha \in \mathbb{D}_p \) and is given by,
\[
\tilde{M}_{\alpha,2} = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\sqrt{1 - |\alpha|^2}} \\ \frac{1 - |\alpha|^2}{\sqrt{1 - |\alpha|^2}} & 0 \end{pmatrix}.
\]
2. Now, using Observation 7.1 with \( v = (1) \), the system matrix associated with the \( 2 \times 2 \) valued \( V_\alpha(z^2) \) in (4.3) is again depending on \( \alpha \) and given by,

\[
M_\alpha = \begin{pmatrix}
\sqrt{\alpha} & 1 & 0 & 0 \\
0 & -\sqrt{\alpha} & 0 & \sqrt{1 - |\alpha|^2} \\
0 & 0 & 1 & 0 \\
\sqrt{1 - |\alpha|^2} & 0 & 0 & -\alpha^*
\end{pmatrix}
\]

(7.3)

3. Now consider the system matrix \( \tilde{M}_\alpha \) associated with the scalar function \( \psi_\alpha(z^N) \) in (7.2) with \( N = 4 \). It depends on the parameter \( \alpha \in \mathbb{D}_\rho \) and is given by,

\[
\tilde{M}_{\alpha,4} = \begin{pmatrix}
\alpha^* & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\alpha^* & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i\alpha^* & 1 & 0 & 0 & 0 & 0 \\
\sqrt{1 - |\alpha|^2} & 0 & 0 & 0 & -\alpha^* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

(7.4)

4. Next, substituting \( \alpha = \beta \) in the above \( \tilde{M}_{\alpha,4} \) and using Observation 7.1 with \( v = (1) \), yields the system matrix associated with \( V_\beta(z^2) \) in (4.5). It depends on the parameter \( \beta \in \mathbb{D}_\rho \) and is given by,

\[
M_{\beta,4} = \begin{pmatrix}
\beta^* & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta^* & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i\beta^* & 1 & 0 & 0 & 0 & 0 \\
\sqrt{1 - |\beta|^2} & 0 & 0 & 0 & -\beta^* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

(7.4)

8. A Recipe for State Space Realization of Wavelet Filters

Recall that in Proposition 4.1 we introduced the following construction scheme of wavelet filters. By multiplying from the left a given \( W_\alpha \in \mathcal{W}_N \), of McMillan degree \( N (\frac{1}{2}(N - 1) + m) \), by an elementary unitary matrix \( V(z^N) \), one can construct another wavelet filter \( W_\beta(z) \) of McMillan degree \( N (\frac{1}{2}(N + 1) + m) \). Indeed,

\[
W_\beta(z) = V(z^N)W_\alpha(z),
\]

where \( V \in \mathcal{U} \) is elementary with parameters \( v \in \mathbb{S}_{N-1} \) and \( \alpha \in \mathbb{D}_\rho \), see (3.6).

Assuming the state space realization of \( W_\alpha(z) \) is already known, the state space realization of \( V(z^N) \) is given in Example 7.2 and thus \( M_b \), the system matrix associated with \( W_\beta(z) \) can be obtained through Observation 5.1.

A somewhat similar idea appeared in [51, Section 4].

**Example 8.1.** We here construct, in two steps, the system matrix \( M_b \) associated with \( W_b(z) \) in (4.6), a wavelet filter of dimension \( N = 2 \) and index \( m = 3 \).

Recall that \( W_a(z) \) in (4.4) is of McMillan degree three. Using Observation 5.1 from the realizations in item 1 of Example 6.1 and in (7.3) (item 2 in Example 7.2), we
construct $M_a$, the system matrix associated with $W_a(z)$,

$$M_a = \begin{pmatrix}
\sqrt{\alpha} & 1 - |\alpha|^2 & 0 & 0 & 0 \\
0 & -\sqrt{\alpha} & 1 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}} & -\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}} \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & -\alpha^* & 0 & 0
\end{pmatrix}. $$

Next, recall that $W_b(z)$ in (4.6) is of McMillan degree seven. Using Observation 5.1, again, from the realizations in (8.1) and in (7.4) (item 3 in Example 7.2), we construct $M_b$, the system matrix associated with $W_b(z)$,

$$M_b = \begin{pmatrix}
\beta^\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta^\frac{1}{4} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i\beta^\frac{1}{4} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{1 - |\beta|^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{\alpha} & 1 - |\alpha|^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{1 - |\beta|^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \sqrt{1 - |\alpha|^2} & 0 & -\alpha^* & 0 \\
\sqrt{1 - |\beta|^2} & 0 & 0 & 0 & 0 & 0 & \sqrt{1 - |\beta|^2} & -\beta^* \\
0 & 0 & 0 & 0 & \sqrt{1 - |\alpha|^2} & 0 & -\alpha^* & 0 \\
\sqrt{1 - |\beta|^2} & 0 & 0 & 0 & 0 & 0 & \sqrt{1 - |\beta|^2} & -\beta^* \\
0 & 0 & 0 & 0 & \sqrt{1 - |\alpha|^2} & 0 & -\alpha^* & 0
\end{pmatrix}. $$


9. **Concluding remarks**

1. The parameterization of all wavelet filters in Observation 4.6 has several advantages:
   - One can use the index $m$ as a design parameter.
   - One has a clear view on the trade-off involved in increasing the index $m$: Additional degrees of freedom on the expense of computational burden, which is even more apparent from the state-space model.
   - The spectral radius $\rho$, is another design parameter. This point is extended in item 4 below.
   - $\rho$ offers another the trade-off. For $1 >> \rho > 0$ one obtains “almost FIR” filters, which is a much larger family than FIR’s. However, for prescribed $m$, increasing $\rho$, weakens the noise attenuation and the slows down the rate of convergence.
   - The set of design parameters is convex and thus convenient for design through optimization, see e.g. [16], [27], [50] and [48].

2. The stage-by-stage construction of state space realization introduced in Sections 7, 8 is not only easy to employ, but also reveals the appealing structure of the realization of wavelet filters, see e.g. [8, 2]. This is significant in facilitating the computations involved when one wishes to employ Matlab based algorithms such as LMI toolbox as proposed in e.g. [14], [16], [50], or solvers of the Riccati equation, see e.g. [13], [15].

3. Further properties of rational wavelet filters
In the present paper we concentrated on a systems-theoretic approach to a parameterization of the family of all wavelet filters as matrix functions. There are important applications which we only sketch briefly here, for example:

(a) Within the entire family, identify and analyze those wavelet filters that have vanishing moments; (the vanishing-moment wavelets have especially attractive algorithms for approximation!);

(b) Give numerical recipes for how to translate our parameterization into wavelet functions on the real line or on more general ambient spaces, so obtaining scaling functions, wavelet generators (father/mother functions) and associated wavelet bases from our filter matrix-functions.

Both (a) and (b) are major topics in wavelet theory, see e.g. [6, 7, 23, 24, 25, 37, 40, 41, 43, 47, 55].

4. Relaxing Schur stability (taking $\rho > 1$)

In the description of the set $\mathcal{U}$ in (1.2) we confined the discussion to Schur asymptotically stable functions, i.e. analytic outside $\mathbb{D}_\rho$ for some prescribed $\rho \in [0, 1]$. Consequently, in the stage-by-stage construction of $\mathcal{U}$ in Section 3 we take the poles to be within $\mathbb{D}_\rho$.

However, in some applications this requirement is not imperative, e.g. where the filters are not associated with dynamical systems. Then, one can extend the discussion and allow the poles of a function in $\mathcal{U}$ to be in $\mathbb{D}_\rho \setminus \mathbb{T}$, for some $\rho > 1$, see e.g. [1, Section 3] and in the framework of wavelet filters see [2], [3].

5. Not necessarily rational wavelet filters.

Equation (3.6) may be substituted by,

$$\tilde{V}(z^N) := I_N + (\psi(z^N) - 1) vv^*,$$

where $v \in \mathbb{S}_{N-1}$ is a parameter and the scalar function $\psi(z)$, mapping $\mathbb{T}$ to itself, is with the appropriate analyticity conditions, e.g. meromorphic outside $\mathbb{D}$. For example, one can take

$$\psi(z^N) = e^{\frac{N+1}{z}},$$

which maps the open unit disk to itself.

Thus, if $W(z)$ is a wavelet filter satisfying (1.3), then so is the product

$$\tilde{V}(z^N)W(z).$$

A thorough study of non rational generalizations of wavelet filters can be found in [2].

6. Non-square wavelet filters

The property $\mathcal{C}$ defined in (1.4) can in fact be generalized to having

$$F(z) = \begin{pmatrix} \hat{f}_0(z) & \hat{f}_0(\epsilon z) & \cdots & \hat{f}_0(\epsilon^{N-1} z) \\ \hat{f}_1(z) & \hat{f}_1(\epsilon z) & \cdots & \hat{f}_1(\epsilon^{N-1} z) \\ \vdots \\ \hat{f}_{l-1}(z) & \hat{f}_{l-1}(\epsilon z) & \cdots & \hat{f}_{l-1}(\epsilon^{N-1} z) \end{pmatrix},$$

satisfying the condition (2.2),

$$F(\epsilon z) = F(z)\hat{P},$$
where \( \hat{P} \) is as before.

Now for the requirement on \( \mathbb{T} \) we return to (1.1). For \( N \geq l \), this requirement is
generalized to coisometry \( FF^* = I_l \).

If however \( l > N \) (in signal processing literature this is referred to as the “over-
sampling” case. In other places “frames” are discussed) we return to (1.1) and take,

\[
An(z) = F(z) \quad \quad Sy(z) = \left( F \left( \frac{1}{z^*} \right)^* F(z) \right)^{-1} \cdot F \left( \frac{1}{z^*} \right)^*.
\]

In particular, \( Sy(z) \) is a left inverse of \( An(z) \), on \( \mathbb{T} \). For mathematical analysis
of this case see e.g. \([2, 33]\). From engineering point of view this case is addressed in
a series of papers \([13, 14, 15, 16]\).

7. Relaxing the condition (1.1) on all \( \mathbb{T} \).
One can require that \( SyAn \approx I \) only on part of \( \mathbb{T} \). This leads to “wavelets on
fractals”, see e.g. \([19, 20, 21]\). See also \([27]\).

10. Appendix -additional background

10.1. The Z-transform of N-decimated N-expanded sequence. We next address
the passage from \( G(z) \) to \( G(z^N) \) and recall in the following background. Let
\( \{a_k\}_{k=-\infty}^{\infty} \) be a sequence. For a natural \( N \), its decimated version is the sequence,
\( \{b_k\}_{k=-\infty}^{\infty} \) with \( b_k = a_{Nk} \). The N-expanded version of \( \{b_k\}_{k=-\infty}^{\infty} \) is the sequence
\( \{c_k\}_{k=-\infty}^{\infty} \) where all elements are zero, except \( c_{Nk} = b_k \). Alternatively, one can
write that all elements in \( \{c_k\}_{k=-\infty}^{\infty} \) are zero, except \( c_{Nk} = a_{Nk} \), see e.g. \([40]\)
Chapter 3, \([19]\) Subsection 4.1.1.

Now, we recall that if the Z-transform of \( \{a_k\}_{k=-\infty}^{\infty} \) exists and denoted by \( F(z) \),
the Z-transform of \( \{c_k\}_{k=-\infty}^{\infty} \) exists and is given by \( F(z^N) \). Indeed, assume
that \( \{a_k\}_{k=-\infty}^{\infty} \) is so that all elements to the left of \( a_{-k_0} \), for some positive finite \( k_0 \),
vanish. Thus, \( \{a_k\} \) can be written as a sum \( A_+ + A_- \) with \( A_+ = \{a_k\}_{k=-\infty}^{0} \)
and \( A_- = \{a_k\}_{1}^{\infty} \). Clearly, the Z-transform of \( A_- \), denoted by \( F_-(z) \) is well defined
and is rational. Assume in addition that the Z-transform of \( A_+ \), denoted by \( F_+(z) \),
is rational. Then, the fact that the Z-transform of \( \{c_k\}_{k=-k_0}^{0} \) exists and is rational,
is obvious. Now to see that also Z-transform of \( \{c_k\}_{k=1}^{\infty} \) exists and is rational we examine
the associated Hankel matrices. Let \( H_a \) and \( H_c \) be the Hankel matrices
associated with \( \{a_k\}_{k=-\infty}^{\infty} \) and \( \{c_k\}_{k=1}^{\infty} \), respectively, e.g.

\[
H_a = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\
a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{28} \\
a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{28} & a_{38} \\
a_{14} & a_{15} & a_{16} & a_{17} & a_{28} & a_{38} & a_{48} \\
a_{15} & a_{16} & a_{17} & a_{28} & a_{38} & a_{48} & a_{58} \\
a_{16} & a_{17} & a_{28} & a_{38} & a_{48} & a_{58} & a_{68} \\
a_{17} & a_{28} & a_{38} & a_{48} & a_{58} & a_{68} & a_{78}
\end{pmatrix},
\]

\[
H_c = \begin{pmatrix}
a_{11} & 0 & 0 & a_{14} & 0 & 0 & a_{17} \\
0 & 0 & a_{14} & 0 & 0 & a_{17} & 0 \\
a_{14} & 0 & 0 & a_{17} & 0 & 0 & 0 \\
0 & a_{17} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{17} & 0 & 0 & a_{48} & 0 \\
0 & 0 & a_{17} & 0 & 0 & a_{48} & 0 \\
a_{17} & 0 & 0 & a_{48} & 0 & 0 & a_{78}
\end{pmatrix}.
\]

Taking

\[
E := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

one can always write,

\[
\hat{h} = E H_a E^* = E H_c E^* \quad \quad H_c = E^* \hat{h} E \quad \quad \hat{h} = \begin{pmatrix}
\hat{a}_{11} & \hat{a}_{14} & \hat{a}_{17} \\
\hat{a}_{14} & \hat{a}_{17} & \hat{a}_{48} \\
\hat{a}_{17} & \hat{a}_{48} & \hat{a}_{78}
\end{pmatrix}.
\]

Namely, \( \hat{h} \) is another Hankel matrix. This implies that

\[
\text{rank}(H_c) = \text{rank}(E^* \hat{h} E) = \text{rank}(\hat{h}) \leq \text{rank}(H_a).
\]
By assumption the rank of \( H_a \) is finite then so is \( \text{rank}(H_a) \), which implies that both \( \{a_k\}_{k=1}^{\infty} \) and \( \{c_k\}_{k=1}^{\infty} \) admit state space realization, see e.g. [36, Lemma 6.5-7], [44, Lemma 5.5.5] and thus the \( Z \)-transform of each sequence exists and is rational.

10.2. Functions in \( \mathcal{U} \) admitting state space realization. We first cites an adapted version of [1, Theorem 3.9] (a closely related result appeared in [22, Theorem 3])

**Theorem 10.1.** Let \( U(z) \) analytic at infinity, be of McMillan degree \( p \), and let

\[ U(z) = C(zI - A)^{-1}B + D \]

be a minimal realization of \( U \). Then,

\[ U \in \mathcal{U}_N \]

if and only if there exists a \( p \times p \) non-singular Hermitian matrix \( H \) (uniquely determined from the given realization) such that

\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & I_N \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & I_N \end{pmatrix} \].

This \( H \) is called the **Hermitian matrix associated** (with the given minimal realization).

Using the system matrix \( M \) notation from [62], the Stein Equation in Theorem 10.1 can be compactly written as

\[ M^* \hat{H} M = \hat{H} \]

\[ \hat{H} := \text{diag}\{H, I_N\} \].

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