Extended Weak Coupling Limit for Friedrichs Hamiltonians

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Abstract: We study a class of self-adjoint operators defined on the direct sum of two Hilbert spaces: a finite dimensional one called sometimes a “small subsystem” and an infinite dimensional one – a “reservoir”. The operator, which we call a “Friedrichs Hamiltonian”, has a small coupling constant in front of its off-diagonal term. It is well known that, under some conditions, in the weak coupling limit the appropriately rescaled evolution in the interaction picture converges to a contractive semigroup when restricted to the subsystem. We show that in this model, the properly renormalized and rescaled evolution converges on the whole space to a new unitary evolution, which is a dilation of the above mentioned semigroup. Similar results have been studied before [AFL] in more complicated models under the name of “Stochastic Limit”.

KEY WORDS: weak coupling limit, Friedrichs model, unitary dilation

1 Introduction

1.1 Weak coupling limit

The weak coupling limit is often invoked to justify various approximations in quantum physics, at least since [VH]. It involves a dynamics depending on a small coupling constant $\lambda$. One assumes that

$$\lambda \to 0, \quad t \to \infty, \quad \lambda^2 t \text{ fixed}. \quad (1.1)$$

Usually one separates the system into two parts: a “small subsystem” and a “reservoir”. The long cumulative effect of the reservoir on the small system can in this limit lead to a Markovian dynamics (i.e., a dynamics given by a semigroup).

There exists a large literature devoted to the weak coupling limit reduced to the small subsystem. It was first put on a rigorous footing by Davies [Da1]. The setup considered by Davies, in its abstract version, consists of a dynamics generated by $H_\lambda := H_0 + \lambda W$, a projection $P$ commuting with $H_0$ and such that $PH_0P = 0$. Davies proved that under appropriate assumptions there exists the limit of the dynamics in the interaction picture restricted from the left and right by $P$, and this limit is a semigroup on Ran$P$. Perhaps, it would be appropriate to call the weak coupling limit reduced to the small subsystem the “Davies
limit”. Another name which one can use is the “reduced weak coupling limit”. (In the literature the names “weak coupling” and “van Hove limit” are used – both are rather imprecise, the latter name is especially ambiguous, since it is also used for a completely different concept in statistical physics).

Davies and a number of other authors gave applications of the above idea to physically interesting situations describing a dynamics of a composite quantum system, where $P$ is a conditional expectation onto the small system and the resulting semigroup is completely positive. Note, however, that the reduced weak coupling limit is an interesting mathematical phenomenon also in its more general version.

Some authors point out that it should be possible to use the idea of the weak coupling limit not just for the dynamics restricted to the small system, but for the whole system as well. In [AFL], Accardi, Frigerio and Lu argue that in an appropriate limit, the full unreduced dynamics of a quantum system converges to a 1-parameter group of $*$-automorphisms, which is a dilation of the completely positive semigroup obtained by Davies. They call it the “stochastic limit”. We prefer to call it the “extended weak coupling limit”, since in itself this concept does not have to involve “stochasticity”.

We believe that the above idea is interesting and worth exploring. In our next paper [DD] we would like to present our version of the extended weak coupling limit applied to quantum systems, with some improvements as compared to [AFL]. In particular, we believe that the approach of [DD] proposes a more satisfactory kind of convergence (strong*) than that of [AFL] (convergence of correlation functions) and the proofs of [DD] are considerably simpler than those of [AFL].

### 1.2 Weak coupling limit for Friedrichs Hamiltonians

In the present paper we present results of the same flavour for a class of simple operators on a Hilbert space, which we call Friedrichs Hamiltonians. We will show that for Friedrichs Hamiltonians the idea of the extended weak coupling limit works very well and yields in a rather natural fashion a unitary dilation of the semigroup $\Lambda_t$.

By a “Friedrichs Hamiltonian” we mean a self-adjoint operator $H_\lambda$ on a Hilbert space $\mathcal{H} = \mathcal{E} \oplus \mathcal{H}_R$ given by the expression

$$H_\lambda := \begin{bmatrix} E & \lambda V \\ \lambda V^* & H_R \end{bmatrix}, \quad (1.2)$$

where $E$ is a self-adjoint operator on the space $\mathcal{E}$, $V \in \mathcal{B}(\mathcal{E}, \mathcal{H}_R)$ and $H_R$ is a self-adjoint operator on $\mathcal{H}_R$. We will assume that $\mathcal{E}$ is finite-dimensional. The subscript $R$ stands for the “reservoir”.

The Friedrichs model, often under other names such as the Wigner-Weisskopf atom, is frequently used as a toy model in mathematical physics. In particular, one often considers its second quantization on the bosonic or fermionic Fock space. (Note that the latter is extensively discussed in [AIPP]).

For a large class of Friedrichs Hamiltonians, it is easy to prove that the reduced weak coupling limit exists. In this case, the reduced weak coupling limit says that under appropriate assumptions the following limit exists:

$$\lim_{\lambda \to 0} e^{itE/\lambda^2} \mathbb{1}_\mathcal{E} e^{-itH_\lambda/\lambda^2} \mathbb{1}_\mathcal{E} =: \Lambda_t, \quad (1.3)$$
and $\Lambda_t$ is a contractive semigroup on $E$.

By enlarging the space $E$ to a larger Hilbert space $Z = E \oplus \mathbb{Z}_R$, one can construct a dilation of $\Lambda_t$. This means, a unitary group $e^{-itZ}$ such that

$$1_E e^{-itZ} 1_E = \Lambda_t.$$

The operator $Z$ is actually another example of a Friedrichs Hamiltonian. We devote Section 2 to the construction of a dilation of a contractive semigroup that is well adapted to the analysis of the weak coupling limit. Note that this construction is quite different from the usual one due to Foias and Nagy [NF]. Even though it can be found in many disguises in the literature, we have never seen a systematic description of some of its curious properties. Therefore, in Section 2 we devote some space to study this construction. Note, in particular, that $Z$ is an example of a Friedrichs Hamiltonian whose definition requires a "renormalization" in the terminology of [DF2].

The main results of our paper are described in Section 4. We start from a rather arbitrary Friedrichs Hamiltonian. First we describe its Davies limit. Then we show that for an appropriate "scaling operator" $J_\lambda$ and a "renormalizing operator" $Z_{\text{ren}}$

$$\lim_{\lambda \downarrow 0} e^{it\lambda^{-2}Z_{\text{ren}}} J_\lambda^* e^{-it\lambda^{-2}H_\lambda} J_\lambda = e^{-itZ}.$$

It is this convergence of the dynamics to a dilation of the semigroup $\Lambda_t$ that we call "extended weak coupling limit". Following [DF1] [DF3], we will give two versions of these results: stationary and time-dependent.

Note that the Davies limit follows from the extended weak coupling limit, since

$$1_E e^{i\lambda^{-2}tZ_{\text{ren}}} J_\lambda^* e^{-i\lambda^{-2}tH_\lambda} J_\lambda 1_E = 1_E e^{i\lambda^{-2}tE} e^{-i\lambda^{-2}tH_\lambda} 1_E. \quad (1.4)$$

### 1.3 The case of 1-dimensional $E$

The main idea of the extended weak coupling limit can be explained already in the case of a one-dimensional small Hilbert space $E$. If $E$ has more than one eigenvalue, which is possible if $\dim E \geq 2$, then the extended weak coupling limit is more complicated to formulate and prove, which tends to obscure the whole picture. Therefore, in this subsection, we describe the main idea of our result in the case $\dim E = 1$.

Let $E = \mathbb{C}$ and $\mathcal{H}_R = L^2(\mathbb{R})$. Let $e \in \mathbb{R}$ and let $\omega$ be a function on $\mathbb{R}$. Assume that there is a unique $\hat{e} := \omega^{-1}(e)$. Let $\omega$ also stand for the corresponding multiplication operator on $\mathcal{H}_R$ respectively. Fix a function $v \in L^2(\mathbb{R})$ and denote by $\langle v \rangle$ the operator in $\mathcal{B}(\mathcal{H}_R, E)$ which acts as $\langle v \rangle := \langle v | f \rangle \in E$ and let $|v\rangle := \langle (v)\rangle^\dagger$. Consider the following Hamiltonian on $E \oplus \mathcal{H}_R$:

$$H_\lambda := \begin{bmatrix} e & \lambda \langle v \rangle \\ \lambda |v\rangle & \omega \end{bmatrix}, \quad (1.5)$$

(Note that in the literature the name "Friedrichs Hamiltonian" is usually reserved for an operator of the form (1.5). Operators of the form (1.2), should be perhaps called "generalized Friedrichs Hamiltonians").

The weak coupling limit in this model simply states that, under some mild assumptions,

$$\lim_{\lambda \downarrow 0} 1_E e^{-i\lambda^{-2}t(H_\lambda - e)} 1_E = e^{-it}, \quad (1.6)$$
where $1_{\mathcal{E}}$ is the orthogonal projection on $\mathcal{E}$,

$$
\gamma := P \int_\mathbb{R} dx \frac{v^*(x)v(x)}{\omega(x) - e} + i\pi v^*(\hat{e})v(\hat{e}),
$$

(1.7)

and $P\frac{1}{x}$ is the principal value of $\frac{1}{x}$.

e$^{-it\hat{H}}$ is a contractive semigroup on $\mathcal{E}$. It can be dilated to a unitary group $e^{-itZ}$ on the Hilbert space on $\mathcal{E} \oplus \mathcal{H}_R$. The generator of the dilating group can be formally written in the form of a Friedrichs Hamiltonian as

$$
Z := \begin{bmatrix} \Re \gamma & v(\hat{e})\langle 1 \rangle \\ v(\hat{e})\langle 1 \rangle^* & \omega'(\hat{e})x \end{bmatrix}.
$$

(1.8)

$\omega'(\hat{e})x \in \mathbb{R}$ is the new multiplication operator on $\mathcal{H}_R = L^2(\mathbb{R})$. $1$ is the constant function with value $1$ which is of course not an element of $L^2(\mathbb{R})$. Because of this (1.8) does not make sense as an operator. Nevertheless, one can give it a precise meaning, e.g. by constructing its resolvent or its unitary group, or by imposing a cutoff and taking it away (see e.g. [DF2] and [Ku]).

To state the extended weak coupling limit, we need the unitary rescaling operator $J_\lambda \in \mathcal{B}(L^2(\mathbb{R}))$ defined as

$$
(J_\lambda f)(x) = \frac{1}{\lambda} f\left(\frac{x - \hat{e}}{\lambda^2}\right), \quad f \in L^2(\mathbb{R}).
$$

(1.9)

If the function $v$ is sufficiently regular in $\hat{e}$, we show the following results:

1. Theorem 4.5: the rescaled resolvent $J_\lambda^*(z - \lambda^{-2}(H_\lambda - e))^{-1}J_\lambda$ converges in norm to $(z - Z)^{-1}$.

2. Theorem 4.6: the rescaled unitary family $J_\lambda^* e^{-it\lambda^{-2}(H_\lambda - e)}J_\lambda$ converges strongly to $e^{-itZ}$.

1.4 Notation

We will often make the following abuse of notation. If $\mathcal{H}_0$ is a closed subspace of a Hilbert space $\mathcal{H}$, $A \in \mathcal{B}(\mathcal{H}_0)$, and $f$ is a function on the spectrum of $A$, then the expression

$$
f(A) \quad \text{stands for} \quad j_0^* f(A) j_0,
$$

(1.10)

where $j_0$ is the embedding of $\mathcal{H}_0$ into $\mathcal{H}$.

We set

$$
\mathbb{C}_+ := \{z \in \mathbb{C}, \text{Im} z > 0\}, \quad \mathbb{C}_- := \{z \in \mathbb{C}, \text{Im} z < 0\}.
$$

(1.11)

2 Dilations

Let $\mathcal{E}$ be a Hilbert space and let the family $\Lambda_{t \in \mathbb{R}^+}$ be a contractive semigroup on $\mathcal{E}$:

$$
\Lambda_t \Lambda_s = \Lambda_{t+s}, \quad \|\Lambda_t\| \leq 1, \quad t, s \in \mathbb{R}^+.
$$

(2.1)

Definition 3. (1) We say that $(Z, 1_{\mathcal{E}}, U_{t \in \mathbb{R}})$ is a unitary dilation of $\Lambda_{t \in \mathbb{R}^+}$ if

(i) $Z$ is a Hilbert space and $U_{t \in \mathbb{R}} \in \mathcal{B}(Z)$ is a 1-parameter unitary group,
(ii) $E \subset Z$ and $1_E$ is the orthogonal projection from $Z$ onto $E$,

(iii) for all $t \in \mathbb{R}^+$

$$1_E U_t 1_E = \Lambda_t.$$  \hfill (3.1)

(2) We call a dilation $(Z, 1_E, U_t \in \mathbb{R})$ minimal iff

$$\{ U_t E \mid t \in \mathbb{R} \}^c \subset Z.$$  \hfill (3.2)

We have the following theorem due to [NF]:

**Theorem 3.1.**  (1) Every contractive semigroup $\Lambda_{t \in \mathbb{R}^+}$ has a minimal unitary dilation $(Z, 1_E, U_t \in \mathbb{R})$, unique up to unitary equivalence.

(2) $1_E U_t 1_E = \Lambda_{-t}^*$ for $t < 0$.

(3) If $\Lambda_{t \in \mathbb{R}^+}$ is strongly continuous in $t$, then $U_t \in \mathbb{R}$ can be chosen to be strongly continuous in $t$.

In the following we present a construction of a unitary dilation, which is well suited for the extended weak coupling limit.

In what follows we assume that the contractive semigroup $\Lambda_t$ is norm continuous. Hence it has a generator, denoted $-i\Gamma \in \mathcal{B}(\mathcal{E})$, so that $\Lambda_t = e^{-it\Gamma}$.

Since $\Lambda_t$ is contractive, $-i\Gamma$ is dissipative:

$$\text{Im} \Gamma = \frac{1}{i2}(\Gamma - \Gamma^*) \leq 0.$$  \hfill (3.3)

Let $\mathfrak{h}$ be a Hilbert space. Set $Z_{\mathfrak{h}} = L^2(\mathbb{R}) \otimes \mathfrak{h} = L^2(\mathbb{R}, \mathfrak{h})$ and $Z = E \oplus Z_{\mathfrak{h}}$. Let $1_E$ be the orthogonal projection from $Z$ onto $E$.

We define an unbounded linear functional on $L^2(\mathbb{R})$ with the domain $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, denoted $\langle 1 \rangle$, given by the obvious prescription

$$\langle 1 | f \rangle = \int_{\mathbb{R}} f(x)dx.$$  

By $|1 \rangle$, we denote the adjoint of $\langle 1 \rangle$ in the sense of forms. (Note that the adjoint of $\langle 1 \rangle$ in the sense of forms is different from the adjoint in the sense of operators, in particular, the latter has a trivial domain).

Introduce the operator $Z_{\mathfrak{h}}$ on $Z_{\mathfrak{h}}$ as the operator of multiplication by the variable $x$:

$$(Z_{\mathfrak{h}} f)(x) = xf(x).$$

Let $\nu \in \mathcal{B}(\mathcal{E}, \mathfrak{h})$, be an operator satisfying the condition

$$\frac{1}{2i}(\Gamma - \Gamma^*) = -\pi \nu^* \nu.$$  \hfill (3.4)

Put $W = |1 \rangle \otimes \nu$ and $W^* = \langle 1 \rangle \otimes \nu^*$ and remark that the following expressions

$$W, \quad W^*, \quad WSW^*, \quad \text{with } S \in \mathcal{B}(\mathcal{E}),$$

are well-defined quadratic forms on $\mathcal{D} := \mathcal{E} \oplus ((L^1(\mathbb{R}) \cap L^2(\mathbb{R})) \otimes_{\text{al}} \mathfrak{h})$, (where $\otimes_{\text{al}}$ denotes the algebraic tensor product).
Now we combine these objects into something that is a priori a quadratic form on $\mathcal{D}$, but turns out to be a bounded operator. For clarity we will explicitly write the projections $1_\mathcal{E}$ onto $\mathcal{E}$ and $1_\mathbb{R}$ onto $\mathbb{Z}_\mathbb{R}$. For $t \geq 0$, we define
\[
U_t = 1_\mathbb{R} e^{-itZ} 1_\mathbb{R} + 1_\mathcal{E} e^{-it\Gamma} 1_\mathcal{E} \\
- i1_\mathcal{E} \int_0^t du e^{-i(t-u)\Gamma} W^* e^{-iuZ} 1_\mathbb{R} \\
- i1_\mathbb{R} \int_0^t du e^{-i(t-u)Z} W e^{-iu\Gamma} 1_\mathcal{E} \\
- 1_\mathbb{R} \int_{0 \leq u_1, u_2, u_1 + u_2 \leq t} du_1 du_2 e^{-iu_2 Z} W e^{-i(t-u_2-u_1)\Gamma} W^* e^{-iu_1 Z} 1_\mathbb{R},
\]

\[
U_{-t} = U_t^*.
\] (3.6)

For $z \in \mathbb{C}_+$, we define
\[
Q(z) := \begin{bmatrix}
0 & 0 \\
0 & (z - Z)^{-1}
\end{bmatrix} \\
+ \begin{bmatrix}
(z - \Gamma)^{-1} & (z - \Gamma)^{-1} W^* (z - Z)^{-1} \\
(z - Z)^{-1} W (z - \Gamma)^{-1} & (z - Z)^{-1} W (z - \Gamma)^{-1} W^* (z - Z)^{-1}
\end{bmatrix};
\]

\[
Q(\bar{z}) := Q(z)^*.
\] (3.7)

Next we define the following quadratic form on $\mathcal{D}$, using the matrix notation with respect to the decomposition $\mathcal{Z} = \mathcal{E} \oplus \mathbb{Z}_\mathbb{R}$:
\[
Z^+ = \begin{bmatrix}
\Gamma & W^* \\
W & \mathbb{Z}_\mathbb{R}
\end{bmatrix}, \quad Z^- = \begin{bmatrix}
\Gamma^* & W^* \\
W & \mathbb{Z}_\mathbb{R}
\end{bmatrix}.
\] (3.8)

Last, for $k \in \mathbb{N}$, we define approximants $W_k \in \mathcal{B}(\mathcal{E}, L^2(\mathbb{R}, \mathfrak{h}))$ to the form $W$
\[
W_k u := |1_{[-k,k]} \rangle \otimes \nu,
\] (3.9)

and approximants $Z_{\mathbb{R},k} \in \mathcal{B}(\mathbb{Z}_\mathbb{R})$ for $Z_R$
\[
Z_{\mathbb{R},k} := 1_{[-k,k]} (Z_R) Z_R,
\] (3.10)

where $1_{[-k,k]}$ denotes the characteristic function of $[-k,k]$. We set
\[
Z_k = \begin{bmatrix}
\operatorname{Re} \Gamma & W_k^* \\
W_k & Z_{\mathbb{R},k}
\end{bmatrix}.
\] (3.11)

We have

\textbf{Theorem 3.2}. Let $U_t$ be as in (3.6), $Q(z)$ as in (3.7), $Z^\pm$ as in (3.8), and $Z_k$ as in (3.11), with $\Gamma$ satisfying condition (3.4).

(1) The family $Q(z)$ is the resolvent of a self-adjoint operator $Z$, that is, there exists a unique self-adjoint operator $Z$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$
\[
Q(z) = (z - Z)^{-1}.
\] (3.12)
(2) $U_t$ extends to a unitary, strongly continuous one-parameter group in $B(Z)$ and

$$U_t = e^{-itZ}. \quad (3.13)$$

(3) Fix $\text{Im}z_0 > 0$. Dom$Z$ consists of vectors $\psi$ of the following form:

$$\psi = \begin{bmatrix} u \\ (z_0 - Z_R)^{-1}Wu + g \end{bmatrix}, \quad u \in \mathcal{E}, \ g \in \text{Dom}Z_R, \quad (3.14)$$

and $Z$ transforms $\psi$ into

$$Z\psi = \begin{bmatrix} \Gamma u + W^*g \\ z_0(z_0 - Z_R)^{-1}Wu + Z_Rg \end{bmatrix}. \quad (3.15)$$

(4) For $\psi \in \text{Dom}Z$, we have

$$Z\psi \equiv \lim_{k \to \infty} Z_k\psi. \quad (3.16)$$

(5) For $\psi, \psi' \in \mathcal{D}$, the function $\mathbb{R} \ni t \mapsto \langle \psi|U_t\psi' \rangle$ is differentiable away from $t = 0$, its derivative $t \mapsto \frac{d}{dt}\langle \psi|U_t\psi' \rangle$ is continuous away from 0 and at $t = 0$ it has the left and the right limit equal respectively to

$$-i\langle \psi|Z^+\psi' \rangle = \lim_{t \downarrow 0} t^{-1}\langle \psi|(U_t - 1)\psi' \rangle, \quad (3.17)$$

$$-i\langle \psi|Z^-\psi' \rangle = \lim_{t \uparrow 0} t^{-1}\langle \psi|(U_t - 1)\psi' \rangle. \quad (3.18)$$

(6) The group $U_t$ dilates the semigroup generated by $-i\Gamma$, that is, for $t \geq 0$,

$$1_{\mathcal{E}}U_t1_{\mathcal{E}} = e^{-it\Gamma}. \quad (3.19)$$

(7) This dilation is minimal iff $\mathfrak{h} = \text{Ran}\nu$.

Remark 3.3. Naturally, every densely defined operator gives rise to a quadratic form on its domain. However, $Z^+$ and $Z^-$ are not derived from $Z$ in this way. This is seen from the explicit description of these domains, as well as from the fact that for $\psi \in \text{Dom}Z$ we have $\frac{d}{dt}U(t)\psi \big|_{t=0} = -iZ\psi$, which should be compared with (3.17) and (3.18).

Remark 3.4. Motivated by (3.11) and (3.16), we can say that in some sense the operator $Z$ is given by the matrix

$$Z = \begin{bmatrix} \text{Re}\Gamma & W^* \\ W & Z_R \end{bmatrix}. \quad (3.20)$$

One should however remember, that strictly speaking the expression (3.20) does not define an operator. To define it an appropriate “renormalization” is needed: one needs to impose a symmetric cutoff and then remove it. The precise meaning of this renormalization is described by (3.14) and (3.15), or by (3.16). Nevertheless, in the sequel, we will freely use expressions of the form (3.20) remembering that its meaning is given by Theorem 3.2.
Remark 3.5. For $\lambda \in \mathbb{R}$, introduce the following unitary operator on $Z$

$$j_\lambda u = u, \quad u \in E; \quad j_\lambda g(y) := \lambda^{-1} g(\lambda^{-2} y), \quad g \in Z_R.$$ 

Note that

$$j_\lambda^* Z_R j_\lambda = \lambda^2 Z_R, \quad j_\lambda^* |1\rangle = \lambda |1\rangle.$$ 

Therefore, the operator $Z$ enjoys the following scaling property, which plays an important role in the extended weak coupling limit:

$$\lambda^{-2} j_\lambda^* \begin{bmatrix} \lambda^2 \text{Re} \Gamma \\ \lambda W \\ Z_R \end{bmatrix} j_\lambda = \begin{bmatrix} \text{Re} \Gamma & W^* \\ W & Z_R \end{bmatrix}. $$

4 Weak coupling limit

4.1 Notation and Assumptions

Let $E$ and $H_R$ be Hilbert spaces. We assume that $E$ is finite dimensional. We set $H = E \oplus H_R$.

Fix a self-adjoint operator $H_R$ on $H_R$ and $E$ on $E$. Let the free Hamiltonian $H_0$ on $H$ be given as

$$H_0 = E \oplus H_R.$$ 

Let $V \in B(E, H_R)$. By a slight abuse of notation we denote by $V$ the corresponding operator on $H$. For $\lambda \in \mathbb{R}$, let the interacting Friedrichs Hamiltonian be

$$H_\lambda = H_0 + \lambda (V + V^*). \quad (4.1)$$

We write $E = \sum_{e \in \text{sp} E} \epsilon_1(E)$ where $\epsilon, \text{sp}(E)$ are the eigenvalues and spectral projections of $E$. The spectral subspace of $E$ for $e$ is denoted $E_e$. Let us list the assumptions that we will use in our construction.

A1: Let $h_0, h_1, h_2, \ldots, h_\infty$ denote the Hilbert spaces of dimension $0, 1, 2, \ldots, \infty$. We assume that there exists a partition of $\mathbb{R}$ into measurable sets $I_0, I_1, I_2, \ldots, I_\infty$ and a unitary identification

$$H_R \simeq \int_{\mathbb{R}} h(x)dx \simeq \bigoplus_{n=0}^{\infty} L^2(I_n) \otimes h_n, \quad (4.2)$$

where $h(x) := h_n$ for $x \in I_n$, and $H_R$ is the operator of the multiplication by the variable $x$. Thus, if $f = \int_{\mathbb{R}} f(x)dx \in H_R$, then

$$(H_R f)(x) = xf(x),$$

for Lebesgue almost all $x$. Moreover, there exists a measurable function

$$\mathbb{R} \ni x \mapsto v(x) \in B(E, h(x))$$

such that for Lebesgue a.a. $x \in \mathbb{R}$ and all $u \in E$

$$(Vu)(x) = v(x)u. \quad (4.3)$$
In what follows, the identification (4.2) is fixed and will be used to define the scaling operator $J_\lambda$.

**A2:** For any $e \in \text{sp}E$, there exists $n(e) \in \{0, 1, 2, \ldots, \infty\}$ such that $e$ belongs to the interior of $I_{n(e)}$. We will write $h_e$ for $h_{n(e)}$. Moreover, we assume that $v$ is continuous at $\text{sp}E$, so that for $e \in \text{sp}E$, we can unambiguously define $v(e) \in B(E, h_e)$.

**A3:** There is $\delta > 0$, such that for a certain $c > 0$ and for all $e \in \text{sp}E$,\[ \|v^*(x)v(x) - v^*(e)v(e)\| \leq c|x - e|^\delta. \] (4.4)

We also assume that $x \mapsto \|v(x)\|$ is bounded.

### 4.2 The reduced weak coupling limit

In this subsection we describe the reduced weak coupling limit (or the Davies limit) for Friedrichs Hamiltonians. The Davies limit is usually given in its time-dependent version described in Theorem 4.2. Its stationary form, which comes from [DF1, DF3], has some technical advantages over the time dependent version.

In both theorems about the reduced weak coupling limit we do not suppose Assumptions A1, A2 and A3.

**Theorem 4.1 (Stationary reduced weak coupling limit).** Suppose that for $e \in \text{sp}E$ and $z \in \mathbb{C}_+$\[ \lim_{\epsilon \downarrow 0} V^*(e + \epsilon z - H_R)^{-1}V \] exists and is independent of $z$. Set\[ \Gamma^*_e := \lim_{\epsilon \downarrow 0} 1_{E_e} V^*(e + \epsilon z - H_R)^{-1}V 1_{E_e}, \]
\[ \Gamma^* := \sum_{e \in \text{sp}E} \Gamma^*_e. \]

Then

(1) for $z \in \mathbb{C}_+$,
\[ \lim_{\lambda \to 0} 1_{E}(z - \lambda^{-2}(H_{\lambda} - e))^{-1}1_{E} = (z - \Gamma^*_e)^{-1}1_{E}; \] (4.5)

(2) for all continuous functions with compact support $f \in C_c([0, +\infty])$,
\[ \lim_{\lambda \downarrow 0} \int_{\mathbb{R}^+} dt f(t)e^{i\lambda^{-2}t}E_{1\mathcal{E}}e^{-i\lambda^{-2}tH_s}1_{E} = \int_{\mathbb{R}^+} dt f(t)e^{-it\Gamma^*}, \] (4.6)

where all limits are in operator norm.

**Theorem 4.2 (Time-dependent reduced weak coupling limit).** Assume that
\[ \lim_{t \to \infty} \int_{0}^{t} e^{isE}V^*e^{-isH_0}V ds \]
exists. Set
\[ \Gamma^\text{dyn}_e := \lim_{t \to \infty} \int_0^t 1_{\mathcal{E}_e} V^* e^{-i\lambda(H_R - e)V} 1_{\mathcal{E}_e} ds, \]
\[ \Gamma^\text{dyn} := \sum_{e \in \text{sp} \mathcal{E}} \Gamma^\text{dyn}_e. \]
Then
\[ \lim_{\lambda \to 0} \sup_{0 < t < T} \left\| e^{i\lambda^{-2}tE} 1_{\mathcal{E}_e} e^{-i\lambda^{-2}tH_R} - e^{-i\Gamma^\text{dyn}_e} \right\| = 0. \] (4.7)

In practice, \( \Gamma^\text{st} \) and \( \Gamma^\text{dyn} \) coincide. They will be denoted simply by \( \Gamma \) and called the Davies generator:

**Theorem 4.3 (Formula for the Davies generator).** Suppose that Assumptions A1, A2 and A3 are true. Then the assumptions of Theorems 4.1 and 4.2 are true. Moreover, for \( e \in \text{sp} \mathcal{E} \) and \( z \in \mathbb{C}^+ \),
\[ -i \lim_{t \to +\infty} \int_0^t ds V^* e^{-i\lambda(H_R - e)V} = \lim_{\epsilon \downarrow 0} \frac{V - e + \epsilon z - H_R}{\epsilon} V. \]
\[ = P \int_{\mathbb{R}} dx \frac{v^*(x)v(x)}{x - e} + i\pi v^*(e)v(e). \]
where \( P \) denotes the principal value. Consequently, the stationary and time dependent Davies generator coincide:
\[ \Gamma_e := \Gamma^\text{dyn}_e = \Gamma^\text{st}_e \]
\[ = 1_{\mathcal{E}_e} \left( P \int_{\mathbb{R}} dx \frac{v^*(x)v(x)}{x - e} + i\pi v^*(e)v(e) \right) 1_{\mathcal{E}_e}. \] (4.8)

### 4.3 Asymptotic space and dynamics

Let \( e \in \text{sp} \mathcal{E} \). The asymptotic reservoir space and “total” space corresponding to \( e \) is
\[ \mathcal{Z}_{R e} := \mathcal{L}^2(\mathbb{R}) \otimes \mathfrak{h}_e = \mathcal{L}^2(\mathbb{R}, \mathfrak{h}_e), \]
\[ \mathcal{Z}_e := \mathcal{E}_e \oplus \mathcal{Z}_{R e}. \]

We have the projections
\[ 1_{\mathcal{E}_e} : \mathcal{Z}_e \to \mathcal{E}_e, \quad 1_{\mathcal{Z}_{R e}} : \mathcal{Z}_e \to \mathcal{Z}_{R e}. \]

Let \( \mathcal{Z}_{R e} \) be the operator of multiplication by the variable in \( \mathbb{R} \) on \( \mathcal{Z}_{R e} \). We define the map \( \nu_e : \mathcal{E}_e \to \mathfrak{h}_e \)
\[ \nu_e := v(e)1_{\mathcal{E}_e}. \]
Under the assumptions A1, A2, A3, we define the operator \( \Gamma_e \) on \( \mathcal{E}_e \), as in (4.8).

Note that \( -\pi v^* \nu_e = \frac{1}{\pi}(\Gamma_e - \Gamma) \), which is the analog of the condition (3.4) for \( \mathcal{Z}_{R e}, \Gamma_e \) and \( \nu_e \) for the space \( \mathcal{Z}_e = \mathcal{E}_e \oplus \mathcal{Z}_{R e} \). One can thus apply the
procedure of Section 2 and construct a unitary dilation of the semigroup $e^{-it\Gamma_e}$, as defined in (3.6). We will denote this dilation by $e^{-itZ_e}$.

We construct the full asymptotic space as a direct sum of independent reservoirs, for each eigenvalue of $E$:

$$
\mathcal{h} := \bigoplus_{e \in \text{sp}E} \mathcal{h}_e,
$$

$$
Z_R := \bigoplus_{e \in \text{sp}E} Z_{R_e} = L^2(\mathbb{R}, \mathcal{h}),
$$

$$
Z := \bigoplus_{e \in \text{sp}E} Z_e = \mathcal{E} \oplus Z_R.
$$

We have the asymptotic reservoir Hamiltonian

$$
Z_R = \bigoplus_{e \in \text{sp}E} Z_{R_e}.
$$

We define the map $\nu : \mathcal{E} \to \mathcal{h}$,

$$
\nu := \bigoplus_{e \in \text{sp}E} \nu_e,
$$

where we used the decomposition $\mathcal{E} = \bigoplus_{e \in \text{sp}E} \mathcal{E}_e$ and $\mathcal{h} = \bigoplus_{e \in \text{sp}E} \mathcal{h}_e$. We also have the operator $\Gamma$ on $\mathcal{E}$ as defined in Section 2.

Clearly, $Z_R$, $\Gamma$ and $\nu$ satisfy the condition (3.4). One can thus apply the procedure of Section 2 and construct a unitary dilation $e^{-itZ}$ of the semigroup $e^{-it\Gamma}$ on $Z = \mathcal{E} \oplus Z_R$.

Obviously, everything we constructed commutes with the orthogonal projections $1_e : Z \to Z_e$, and we have

$$
Z = \bigoplus_{e \in \text{sp}E} Z_e.
$$

We define the renormalizing hamiltonian $Z_{\text{ren}}$ on $Z$:

$$
Z_{\text{ren}} := \sum_{e \in \text{sp}E} e1_e = E + \sum_{e \in \text{sp}E} e1_{R_e}.
$$

4.4 Scaling

For any $e \in \text{sp}E$, we choose an open set $\bar{I}_e$ such that $e \in \bar{I}_e \subset I_e$ and $\bar{I}_e$ are mutually disjoint. For $\lambda \in \mathbb{R}^+$, define the family of contractions $J_{\lambda,e} : \mathcal{E}_e \oplus Z_{R_e} \to \mathcal{E} \oplus L^2(\bar{I}_e, \mathcal{h}_e)$, which on $g_e \in Z_{R_e}$ act as

$$
(J_{\lambda,e}g_e)(y) = \begin{cases} 
\frac{1}{\sqrt{\lambda}}g_e\left(\frac{y-e}{\sqrt{\lambda}}\right), & \text{if } y \in \bar{I}_e; \\
0, & \text{if } y \in \mathbb{R}\setminus\bar{I}_e.
\end{cases}
$$

and on $\mathcal{E}_e$ equals $1_{\mathcal{E}_e}$. Note that

$$
J_{\lambda,e}^*J_{\lambda,e} = 1_{\mathcal{E}_e} \oplus 1_{\mathcal{E}_{-2}(I_e-e)}(Z_{R_e}), \quad J_{\lambda,e}J_{\lambda,e}^* = 1_{\mathcal{E}_e} \oplus 1_{\bar{I}_e}(H_R).
$$

For $\psi = \bigoplus_{e \in \text{sp}E} \psi_e$ we set

$$
J_{\lambda}\psi := \bigoplus_{e \in \text{sp}(E)} J_{\lambda,e}\psi_e.
$$

Note that $J_{\lambda}$ is a partial isometry from $Z$ to $\mathcal{H}$. 11
Remark 4.4. The precise form of $J_\lambda$ only matters in a neighbourhood of $\text{sp}E$. For instance, let $\tilde{I}_e \ni y \mapsto \eta_e(y)$ be increasing functions differentiable at $e$ and such that $\frac{d\eta_e}{dy}(e) = 1$ for $e \in \text{sp}(E)$. Set
\[
(J^n_{\lambda,e}g_e)(y) = \left\{ \begin{array}{ll}
\frac{1}{\lambda^2}g_e\left(\frac{\eta_{\lambda}(y)-\eta_e(e)}{\lambda^2}\right) & \text{if } y \in \tilde{I}_e, \\
0 & \text{if } y \in \mathbb{R}\setminus\tilde{I}_e.
\end{array} \right.
\] (4.12)

Then all the statements in this paper remain true if one replaces $J_\lambda$ by $J^n_{\lambda,e}$.

4.5 Main results

In this subsection we state the two main results of our paper. They say that the dynamics generated by $H_\lambda$ after an appropriate rescaling and renormalization, for a small coupling approaches the asymptotic dynamics. Again, we present two versions of the result: stationary and time-dependent.

Theorem 4.5 (Stationary extended weak coupling limit). Assume A1, A2, A3. Let $Z_e$ and $Z$ be as defined in Section 4.3 and let $J_\lambda$ be as defined in Section 4.4. For all $\psi, \psi' \in \mathcal{C}$, we have
\[
\lim_{\lambda \downarrow 0} J^*_\lambda (z - \lambda^{-2}(H_\lambda - e))^{-1} J_\lambda = (z - Z_e)^{-1} J_e.
\] (4.13)

(2) For all continuous functions with compact support $f \in \mathcal{C}([0, +\infty])$,
\[
\lim_{\lambda \downarrow 0} \int_{\mathbb{R}^+} dt f(t)e^{it\lambda^{-2}tZ_{ren}J^*_\lambda e^{-i\lambda^{-2}tH_\lambda}}J_\lambda = \int_{\mathbb{R}^+} dt f(t)e^{-itZ},
\] (4.14)

where all limits are in operator norm.

Theorem 4.6 (Time-dependent extended weak coupling limit). Assume A1, A2, A3. Let $Z_e$ and $Z$ be as defined in Section 4.3 and let $J_\lambda$ be as defined in Section 4.4. For all $\psi \in Z$ and $t \in \mathbb{R}$,
\[
\lim_{\lambda \downarrow 0} e^{it\lambda^{-2}tZ_{ren}J^*_\lambda e^{-i\lambda^{-2}tH_\lambda}}J_\lambda \psi = e^{-itZ} \psi.
\] (4.15)

Remark 4.7. From the proof of Theorem 4.6, it follows immediately that (4.15) can be stated uniformly in $t$ on compact intervals, but in weak operator topology. For all $\psi, \psi' \in Z$ and $0 < T < \infty$,
\[
\lim_{\lambda \downarrow 0} \sup_{0 \leq |t| \leq T} \left| \langle \psi' | e^{it\lambda^{-2}tZ_{ren}J^*_\lambda e^{-i\lambda^{-2}tH_\lambda}}J_\lambda \psi - e^{-itZ} \psi \rangle \right| = 0.
\] (4.16)

Remark 4.8. One can also state (4.15) in the interaction picture, avoiding the renormalizing Hamiltonian $Z_{ren}$. For all $t \in \mathbb{R}$ and $\psi \in Z$,
\[
\lim_{\lambda \downarrow 0} J^*_\lambda e^{i\lambda^{-2}tH_0}e^{-i\lambda^{-2}tH_\lambda}J_\lambda \psi = e^{itZ_e}e^{-itZ} \psi.
\] (4.17)

This is seen most easily by remarking that for all $t \in \mathbb{R}$ and $\psi \in Z$,
\[
\lim_{\lambda \downarrow 0} J^*_\lambda e^{i\lambda^{-2}tH_0}J_\lambda e^{-i\lambda^{-2}tZ_{ren}} \psi = e^{itZ_e} \psi.
\] (4.18)
5 Proofs

5.1 Proof of Theorem 3.2

Statement (1) of Theorem 3.2 follows by the arguments described in a slightly different context in [DF2] (Theorem 2.1). One can take over the proof of [DF2] almost verbatim. For completeness, we reproduce an adjusted proof.

Let \( W_k \in B(\mathcal{E}, L^2(\mathbb{R}, h)) \) for \( k \in \mathbb{N} \) be defined as in (3.9). Put

\[
\Gamma_k(z) := \text{Re}\Gamma + W_k^*(z - Z_R)^{-1}W_k. \tag{5.1}
\]

Obviously, the operator

\[
Z_k := \text{Re}\Gamma + Z_R + W_k^* + W_k \tag{5.2}
\]

is a well defined self-adjoint operator on \( \text{Dom}Z_R \) (since it is a bounded perturbation of \( Z_R \)). By the Feshbach formula (see (5.36)), one checks that the resolvent \( (z - Z_k)^{-1} \) is norm convergent to \( Q(z) \): It suffices to remark that for all \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\lim_{k \to \infty} \Gamma_k(z) = \Gamma(z), \tag{5.3}
\]

\[
\lim_{k \to \infty} W_k^*(z - Z_R)^{-1} = W^*(z - Z_R)^{-1} \tag{5.4}
\]

in norm. It follows that \( Q(z) \) satisfies the resolvent formula. To obtain that \( Q(z) \) is actually the resolvent of a (uniquely defined) self-adjoint operator, it suffices (see [Ka]) to establish for all \( z \in \mathbb{C} \setminus \mathbb{R} \),

(1) \( \text{Ker}Q(z) = \{0\} \),

(2) \( \text{Ran}Q(z) \) is dense in \( Z \),

(3) \( Q^*(z) = Q(\overline{z}) \).

(3) is obvious. To prove (1), we let \( u \oplus g \in \mathcal{E} \oplus Z_R \) and we assume \( Q(z)u \oplus g = 0 \). Suppose that e.g. \( z \in \mathbb{C}_+ \). Then

\[
(z - Z_R)^{-1}(u + W^*(z - Z_R)^{-1}g) = 0, \tag{5.5}
\]

\[
(z - Z_R)^{-1}W(z - \Gamma)^{-1}(u + W^*(z - Z_R)^{-1}g) + (z - Z_R)^{-1}g = 0. \tag{5.6}
\]

Inserting (5.5) into (5.6) yields \( (z - Z_R)^{-1}g = 0 \) and hence \( g = 0 \). Combined with (5.5), the latter implies \( u \oplus g = 0 \).

Using (1) and (3), we get (2), since

\[
\text{Ran}Q(z)^\perp = \ker Q(z)^* = \ker Q(\overline{z}) = \{0\}. \tag{5.7}
\]

Hence, statement 1 of Theorem 3.2 is proven.

To prove Statement (2) we take \( \psi, \psi' \in \mathcal{D} \) and compute the following Laplace transform:

\[
-i \int_0^{+\infty} dt e^{izt} \langle \psi | U_t \psi' \rangle = \langle \psi | Q(z) \psi' \rangle. \tag{5.8}
\]

By functional calculus and the fact that \( Q(z) = (z - Z)^{-1} \),

\[
-i \int_0^{+\infty} dt e^{izt} \langle \psi | e^{-itZ} \psi' \rangle = \langle \psi | Q(z) \psi' \rangle. \tag{5.9}
\]
Both \( t \mapsto \langle \psi | U_t \psi' \rangle \) and \( t \mapsto \langle \psi | e^{-itZ} \psi' \rangle \) are continuous functions and we can apply the inverse Laplace transform to (5.8) and (5.9), which yields \( \langle \psi | U_t \psi' \rangle = \langle \psi | e^{-itZ} \psi' \rangle \). By the density of \( \mathcal{D} \) we obtain \( U_t = e^{-itZ} \). This in particular proves that \( U_t \) satisfies the group property.

To prove Statement (3) we note that any vector in \( Z_\mathbb{R} \) can be written as \( (z_0 - Z_\mathbb{R})g \) for some \( g \in \text{Dom} Z_\mathbb{R} \). Given such \( g \), any vector in \( \mathcal{E} \) can be written as \( (z_0 - \Gamma)u - W^*g \) (here we use the invertibility of \( z_0 - \Gamma \)). Set

\[
\phi := \begin{bmatrix} (z_0 - \Gamma)u - W^*g \\ (z_0 - Z_\mathbb{R})g \end{bmatrix}.
\]

Then \( \psi = Q(z_0)\phi \) equals (3.14) and \( Z\psi = -\phi + z_0Q(z_0)\phi \) equals (3.15).

Statements (4)-(6) follow by straightforward calculations.

To prove Statement (7), we observe that

\[
\text{Span}\{e^{-itZ}\mathcal{E}, \ t \in \mathbb{R}\}^{\text{cl}} = \text{Span}\{(z - Z)^{-1}\mathcal{E}, \ z \in \mathbb{C} \setminus \mathbb{R}\}^{\text{cl}}.
\]

(5.10)

Since \( \text{Span}\{x \mapsto (z - x)^{-1}, \ z \in \mathbb{C} \setminus \mathbb{R}\} \) is dense in \( L^2(\mathbb{R}) \), and using the fact that \( (z - \Gamma)^{-1}, z \in \mathbb{C}_+ \) and \( (z - \Gamma^*)^{-1}, z \in \mathbb{C}_- \), are invertible, we have

\[
\text{Span}\{(z - Z)^{-1}\mathcal{E}, \ z \in \mathbb{C} \setminus \mathbb{R}\}^{\text{cl}} = \{u \oplus (L^2(\mathbb{R}) \otimes \nu u), \ u \in \mathcal{E}\}.
\]

(5.11)

This easily implies Statement (7).

\[\blacksquare\]

### 5.2 Proof of Theorem 4.1

Theorem 4.1 is essentially a special case of Theorem 3.2 from [DF3] (see also [DF1]). For the convenience of the reader, and because the case we consider allows for some simplifications, we sketch the proof below.

Let

\[
G^{-1}(e, \lambda, z) := 1_{\mathcal{E}} \left(z - \lambda^{-2}(H_{\lambda} - e)\right)^{-1} 1_{\mathcal{E}},
\]

(5.12)

which yields immediately the bound

\[
\|G^{-1}(e, \lambda, z)\| \leq |\text{Im} z|^{-1}.
\]

(5.13)

In the following we simplify the notation \( G(e, \lambda, z) \) into \( G \) (hence, we fix a certain \( e \in \text{sp} E \)) and we put

\[
G_d = \sum_{c' \in \text{sp} E} 1_{\mathcal{E}_{c'}} G_{1_{\mathcal{E}_{c'}}}, \quad G_\alpha := G - G_d
\]

(5.14)

and \( 1_{\mathcal{E}_c} := 1_{\mathcal{E}} - 1_{\mathcal{E}_c} \).

By the Feshbach formula (see further: (5.36)), we have

\[
G = z - \lambda^{-2}(E - e) - \lambda^{-2}1_{\mathcal{E}}V^*(z - \lambda^{-2}(H_{R} - e))^{-1} V1_{\mathcal{E}}.
\]

By the assumption of Theorem 4.1, it is immediate that

\[
\lim_{\lambda \downarrow 0} 1_{\mathcal{E}_c} G^{-1}_d = (z - \Gamma e)^{-1}.
\]

(5.15)

By the Neumann expansion and the assumption of Theorem 4.1 one has for small enough \( \lambda \) and some \( c > 0 \),

\[
\|1_{\mathcal{E}_c} G^{-1}_d\| \leq c \lambda^2, \quad \|G_\alpha\| < c.
\]

(5.16)
From $G = G_d + G_o$, we deduce
\[
G^{-1} = G_d^{-1} - G_d^{-1}G_oG_d^{-1} + G_d^{-1}G_oG_d^{-1}G_oG^{-1},
\]
from which
\[
1_{\mathcal{E}_e}(G^{-1} - G_d^{-1}) = -1_{\mathcal{E}_e}G_oG_d^{-1}1_{\mathcal{E}_d}G_o(1 - G_oG^{-1}).
\]
Using (5.13), (5.15) and (5.16), we see that the right hand side of (5.18) vanishes, yielding
\[
\lim_{\lambda \to 0} 1_{\mathcal{E}_e}G^{-1} = \lim_{\lambda \to 0} 1_{\mathcal{E}_d}G_o^{-1} = \left(\frac{z}{z} - \Gamma_{st}^{\alpha e}\right)^{-1}.
\]
(5.19)

Writing
\[
1_{\mathcal{E}_e}G^{-1} = 1_{\mathcal{E}_d}G_o^{-1} - 1_{\mathcal{E}_d}G_o^{-1}G_oG^{-1}
\]
and using (5.16), one sees that
\[
\lim_{\lambda \to 0} 1_{\mathcal{E}_d}G^{-1} = 0.
\]
(5.21)

Together, (5.19) and (5.21) end the proof of (1).

(2) follows from (1) as in [DF1].

5.3 Proof of Theorem 4.2

Theorem 4.2 is a special case of a well known result of Davies [Da1], reproduced e.g. in [DF1]. For the convenience of the reader, and because some simplifications are possible, we sketch the proof below.

We start from the following representation for $\Lambda_{t,\lambda} := e^{it\lambda^{-2}E_1e^{-it\lambda^{-2}H_\lambda}1_{\mathcal{E}_e}}$:
\[
\Lambda_{t,\lambda} = 1 + \int_0^t D_{\lambda,t}(u)\Lambda_{u,\lambda}du,
\]
with
\[
D_{\lambda,t}(u) = \lambda^{-2} \int_u^t e^{i\lambda^{-2}vE_1V^*e^{-i\lambda^{-2}(v-u)H_\lambda}V}e^{-i\lambda^{-2}uE}dv
\]
\[
= \sum_{e,e' \in \text{sp}E} \int_0^{\lambda^{-2}(t-u)} 1_{\mathcal{E}_e}V^*e^{-is(H_\lambda-e)}V1_{\mathcal{E}_e}e^{-i\lambda^{-2}u(e'-e)}ds.
\]

Let for $T > 0$, $Q := C_0([0,T])$ be the Banach space of continuous functions, equipped with the supremum norm. Define the operators $K_\lambda$ and $K$ on $Q$ by (for $0 \leq t \leq T$)
\[
(K_\lambda f)(t) = \int_0^t D_{\lambda,t}(s)f(s)ds, \quad (Kf)(t) = -i\Gamma_{\text{dyn}} \int_0^t f(s)ds.
\]
(5.24)

We will prove that
\[
s - \lim_{\lambda \to 0} K_\lambda = K.
\]
(5.25)

Let
\[
\Gamma := -i \lim_{t \to +\infty} \int_0^t V^*e^{-is(H_\lambda-e)}V ds,
\]
(5.26)
whose existence was proven in Theorem 4.3.

One checks that for all \( t \in [0, T] \)
\[
\lim_{\lambda \downarrow 0} \left| (K_{\lambda}f)(t) + i \sum_{e,e'} \int_0^t 1_{\varepsilon_e} \hat{1}_{\varepsilon_{e'}} e^{-i\lambda^{-2}s(e'-e)} f(s)\,ds \right| = 0, \tag{5.27}
\]
which follows by the assumption of Theorem 4.2 and dominated convergence. Since \( f \) is (bounded and continuous, hence) integrable, the Riemann-Lesbegue lemma yields, for \( e, e' \in \text{sp}E \),
\[
\lim_{\lambda \downarrow 0} i \int_0^t 1_{\varepsilon_e} \hat{1}_{\varepsilon_{e'}} e^{-i\lambda^{-2}s(e'-e')} f(s)\,ds = \delta_{e,e'} \int_0^t \Gamma_{e\text{ dyn}} f(s)\,ds, \tag{5.28}
\]
and hence (5.27) proves (5.25). Note that \( \Lambda_{t,\lambda} \) and \( \Lambda_t := e^{-it\Gamma_{\text{ dyn}}} \) satisfy the equations.
\[
\Lambda_\lambda = \Lambda_0 + K_{\lambda} \Lambda_\lambda, \quad \Lambda = \Lambda_0 + K\Lambda, \tag{5.29}
\]
where \( \Lambda_0 \) is the constant function with value \( \Lambda_0 = \Lambda_{0,\lambda} = 1 \). Remark that by the assumption of Theorem 4.2, there exists a constant \( c \) and a \( \lambda_0 \) such that for all \( \lambda \leq \lambda_0 \) and for all \( n \in \mathbb{N} \), we have
\[
\|K^n_{\lambda}\| \leq \frac{(ct)^n}{n!}, \quad \|K^n\| \leq \frac{(ct)^n}{n!}. \tag{5.30}
\]
This means that both
\[
(1 - K_{\lambda})^{-1} = \sum_{n=0}^{+\infty} K^n_{\lambda}, \quad (1 - K)^{-1} = \sum_{n=0}^{+\infty} K^n \tag{5.31}
\]
exist and that for each \( n \in \mathbb{N} \), \( s - \lim_{\lambda \downarrow 0} K^n_{\lambda} = K \). By (5.29), we thus have
\[
\Lambda_\lambda - \Lambda = ((1 - K_{\lambda})^{-1} - (1 - K)^{-1})\Lambda_0 = \sum_{n=0}^{+\infty} (K^n_{\lambda} - K^n)\Lambda_0. \tag{5.32}
\]
Since each term in the right-hand side vanishes as \( \lambda \downarrow 0 \) and the sequence is absolutely convergent by (5.30), Theorem 4.2 follows.

### 5.4 Proof of Theorem 4.3

Let us first state a general lemma about the principal value.

**Lemma 5.1.** Let \( f \) be a bounded function on \( \mathbb{R} \) such that \( \frac{f(x)}{1 + |x|^\delta} \in L^1(\mathbb{R}) \), \( f \) is continuous at 0 and there exist \( \delta, C > 0 \) such that for \( |x| < C \Rightarrow |f(x) - f(0)| \leq |x|^{\delta} \). Then, for \( z \in \mathbb{C}_+ \),
\[
-i \lim_{T \to +\infty} \int_0^T dt \int_{\mathbb{R}} dx f(x)e^{-itx} = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} f(x)(\epsilon z - x)^{-1} \,dx \tag{5.33}
\]
\[
= -P \int_{\mathbb{R}} \frac{f(x)}{x} \,dx + \pi f(0).
\]
Proof. For the first expression of (5.33), we write

\[-i \lim_{T \to +\infty} \int_0^T dt \int_\mathbb{R} dx f(x)e^{-itx} \]

\[= \int_\mathbb{R} \frac{(-1 + e^{-iT})}{x} f(x) dx \]

\[= f(0) \int_{|x| \leq C} \frac{-1 + e^{-iT}}{x} dx + \int_{|x| \leq C} \frac{(f(x) - f(0))e^{-iT}}{x} dx \]

\[-\int_{|x| > C} \frac{f(x)}{x} dx \int_{|x| > C} \frac{x f(x)}{x} dx. \quad (5.34)\]

The first term, by the residue calculus goes to $f(0)i\pi$. By the Riemann-Lebesgue Lemma, the third and the fifth term on the right of (5.34) go to zero. The second and fourth term yield $P \int \frac{f(x)}{x} dx$.

To get the second equality in (5.33), we write $z = a + ib$ and compute:

\[\int_\mathbb{R} f(x) (\epsilon z - x)^{-1} dx \]

\[= \int_\mathbb{R} \frac{cibf(x)}{(\epsilon a - x)^2 + (\epsilon b)^2} dx \]

\[+ \int_{|x| < \mu} f(x) \frac{(\epsilon a - x)}{(\epsilon a - x)^2 + (\epsilon b)^2} dx \]

\[+ \int_{|x| > \mu} \frac{x f(x)}{(\epsilon a - x)^2 + (\epsilon b)^2} dx, \quad (5.35)\]

where $0 < \mu < 1$ is fixed. The sum of the last two terms converges to $-P \int \frac{f(x)}{x} dx$. The second term can be estimated by

\[sup |f| \int_{|x| < \mu} \left| \frac{|\epsilon a - x|}{(x - \epsilon a)^2 + \epsilon^2 b^2} - \frac{|x|}{x^2 + \epsilon^2 b^2} \right| dx \]

\[= \frac{sup |f|}{2} \left| \log \frac{(\mu + \epsilon a)^2 + \epsilon^2 b^2}{(\mu - \epsilon a)^2 + \epsilon^2 b^2} \right| \rightarrow 0. \]

To apply this lemma, it suffices to note that $f(x) := v^*(x)v(x)$ is a bounded $L^1$ function, continuous and Hölder at $spE$.

5.5 Proof of Theorem 4.5

Lemma 5.2. Let $e, e' \in spE$ and $z \in \mathbb{C}_+$. Then

\[\lim_{\lambda \downarrow 0} \frac{1}{\lambda} V^*(z - \lambda^{-2}(H_R - e))^{-1} J_{\lambda, e'} = (1 \otimes v^*(e) (z - Z_{R_e})^{-1} \delta_{e, e'}. \]
Proof. Let \( g_{e'} \in Z_{R,e'} \). Then

\[
\frac{1}{\lambda} V^*(z - \lambda^{-2}(H_R - e))^{-1} J_{\lambda,e'}\]

\[
= \frac{1}{\lambda^2} \int_{I_{e'}} v^*(y) \left( z - \frac{y - e}{\lambda^2} \right)^{-1} g_{e'}(\frac{y - e'}{\lambda^2}) dy
\]

\[
\lambda^{-2}(I_{e'} - e') \int v^*(e' + \lambda^2 x)(z - x + \lambda^{-2}(e - e'))^{-1} g_{e'}(x) dx.
\]

For \( e \neq e' \), we estimate the square of the norm by

\[
\lambda^{-2}(I_{e'} - e') \int \| v^*(e' + \lambda^2 x)(z - x + \lambda^{-2}(e - e'))^{-1} \|^2 dx \int \| g_{e'}(x) \|^2 dx
\]

\[
\leq \sup_{y \in \mathbb{R}} \| v(y) \|^2 \int \| (z - x + \lambda^{-2}(e - e'))^{-1} \|^2 dx \int \| g_{e'}(x) \|^2 dx \rightarrow 0.
\]

The first integral in the last line vanishes by Lebesgue dominated convergence since \( e \notin (I_{e'} - e') \). For \( e = e' \),

\[
\lambda^{-2}(I_{e} - e) \int \| v^*(e + \lambda^2 x) - v^*(e) \| (z - x)^{-1} g_{e}(x) dx
\]

\[
\leq \int \| v^*(e + \lambda^2 x) - v^*(e) \| (z - x)^{-1} \|^2 dx \int \| g_{e}(x) \|^2 dx \rightarrow 0,
\]

by the Lebesgue dominated convergence theorem, since \( v \) is bounded and continuous in \( e \). Since \( g_{e'} \) enters the above estimates only via \( \| g_{e'} \|^2 = \int \| g_{e'}(x) \|^2 dx \), the convergence is in norm.

The proof of Theorem 4.5 is based on the formula

\[
(z - H_\lambda)^{-1} = (z - H_R)^{-1} + (1 - \lambda(z - H_R)^{-1} V) G_\lambda(z)
\]

\[
\times (1 - \lambda V^*(z - H_R)^{-1}),
\]

(5.36)

where \( G_\lambda(z) := 1 \mathbb{E}(z - H_\lambda)^{-1} 1 \mathbb{E} \). After appropriate rescaling and sandwiching with \( J_{\lambda,e'}^{*} \) and \( J_{\lambda,e''}^{*} \), (5.36) becomes

\[
J_{\lambda,e'}^{*}(z - \lambda^{-2}(H_\lambda - e))^{-1} J_{\lambda,e''}
\]

\[
= \delta_{e',e''} 1_{\lambda^{-2}(I_{e'} - e')} (z - Z_{R,e'} - \lambda^{-2}(e' - e))^{-1}
\]

\[
+ \left( 1_{\mathbb{E},e'} + J_{\lambda,e'}^{*} \frac{1}{\lambda}(z - \lambda^{-2}(H_\lambda - e))^{-1} V \right)
\]

\[
\times G_\lambda(z,e)
\]

\[
\times \left( V^*(z - \lambda^{-2}(H_\lambda - e))^{-1} \frac{1}{\lambda} J_{\lambda,e''} + 1_{\mathbb{E},e''} \right),
\]

(5.37)

where \( G_\lambda(z,e) := 1 \mathbb{E}(z - \lambda^{-2}(H_\lambda - e))^{-1} 1 \mathbb{E} \).
The first term of (5.37) has $\delta_{e',e''}$, because $\tilde{I}_e$ and $\tilde{I}_{e''}$ are disjoint. It converges to
\[ \delta_{e,e'}\delta_{e,e''}(z - Z_{R_e})^{-1}. \]
By the stationary Davies limit (Theorem 4.1 (1)),
\[ G_\lambda(z,e) \to 1_{e_e}(z - \Gamma_e)^{-1}1_{e_e}. \]
Finally, by application of Lemma 5.2, the second term of the rhs of (5.37) converges to
\[ \delta_{e,e'}\delta_{e,e''}(1_{e_e} + (z - Z_{R_e})^{-1}|1 \otimes v(e)) \times 1_{e_e}(z - \Gamma_e)^{-1}1_{e_e} \times (|1 \otimes v^*(e) (z - Z_{R_e})^{-1} + 1_{e_e}). \]

5.6 Proof of Theorem 4.6

We start with the time dependent analog of Lemma 5.2.

Lemma 5.3. Let $g_e \in L^1(\mathbb{R}, h_e) \cap L^2(\mathbb{R}, h_e) = D \cap Z_{R,e}$. Then, uniformly for $|t| < T$, we have the convergence
\[ \lambda^{-1}V^*e^{i\lambda^{-2}H_{R_e}}J_{\lambda,e}g_e \to |1 \otimes v^*(e)e^{-itZ_{R_e}}g_e|. \]

Proof.
\[
\frac{1}{\lambda}V^*e^{-it\lambda^{-2}(H_{R_e} - c)}J_{\lambda,e}g_e = \frac{1}{\lambda^2} \int_{I_e} v^*(y)e^{-it\lambda^{-2}(y - c)}g_e\left(\frac{y - e'}{\lambda^2}\right)dy \\
= \int_{\lambda^{-2}(I_e - c)} v^*(e + \lambda^2x)e^{itx}g_e(x)dx \\
\to v^*(e) \int e^{-itx}g(x)dx.
\]

The proof of Theorem 4.6 is based on the time dependent analog of the formula (5.36):
\[ e^{-itH_{\lambda}} = e^{-itH_{R_e}} + T_{\lambda}(t) \]
\[ +i\lambda \int_0^t T_{\lambda}(t - s) e^{isH_{R_e}}ds + i\lambda \int_0^t e^{isH_{R_e}}VT_{\lambda}(t - s)ds \\
-\lambda^2 \int_0^t \int_{s_1 + s_2 \leq t} e^{-is_1H_{R_e}}VT_{\lambda}(t - s_1 - s_2) e^{-is_2H_{R_e}}ds_1ds_2, \]
where
\[ T_{\lambda}(t) := 1_{e_e}e^{-itH_{\lambda}}1_{e_e}. \]
Rescaling, multiplying from the left by $e^{it\lambda^{-2}Z_{ren}J_{\lambda,e}^*}$ and from the right by $J_{\lambda,e'}$, we obtain

\begin{align*}
1 & e^{it\lambda^{-2}tZ_{ren}J_{\lambda,e}^*} e^{-i\lambda^{-2}tH_{\lambda}} J_{\lambda, e'} \\
& = J_{\lambda,e} e^{-i\lambda^{-2}t(H_{\lambda} - e)} J_{\lambda,e'} + e^{it\lambda^{-2}e} 1_{\mathcal{E}} e^{-itH_{\lambda}} J_{\lambda,e'} \\
& + \int_0^t T_{\lambda}(t-s) V^* e^{-i\lambda^{-2}sH_{\lambda}} J_{\lambda,e'} ds \\
& - \lambda^{-2} e^{it\lambda^{-2}e} \int_0^t \int_{0 \leq s_1, s_2, s_1 + s_2 \leq t} J_{\lambda,e}^* e^{-is_1 \lambda^{-2}H_{\lambda}} e^{i(t-s_2)H_{\lambda}} \\
& \times V T_{\lambda}(t-s_1 - s_2) V^* e^{-i\lambda^{-2}s_2H_{\lambda}} J_{\lambda,e'} ds_1 ds_2. \tag{5.38}
\end{align*}

where

\begin{equation}
T_{\lambda}(t) := 1_{\mathcal{E}} e^{-i\lambda^{-2}tH_{\lambda}} 1_{\mathcal{E}}. \tag{5.38}
\end{equation}

The first term of (5.38) converges to

\begin{equation}
\delta_{e,e'} e^{-itZ_{\lambda}} 1_{\mathcal{R}_e}. \tag{5.39}
\end{equation}

To handle the next terms we use repeatedly the fact that

\begin{equation}
\| T_{\lambda}(s) - e^{is\lambda^{-2}E} e^{-is\Gamma} \| \to 0
\end{equation}

uniformly for $0 \leq s \leq t$. The second term converges to

\begin{equation}
e^{it\lambda^{-2}e} 1_{\mathcal{E}} e^{-it\lambda^{-2}E} e^{-it\Gamma} 1_{\mathcal{E}} e^{i\lambda^{-2}e}, \tag{5.40}
\end{equation}

The third term acting on $g_{e'} \in L^1(\mathbb{R}, h_e) \cap L^2(\mathbb{R}, h_e)$, converges to

\begin{align*}
& i e^{it\lambda^{-2}e} 1_{\mathcal{E}} e^{-i(t-s)\lambda^{-2}E} e^{-i(t-s)\Gamma} (1|v'(e') e^{-is(Z_{\lambda} + \lambda^{-2}e')} g_{e'} ds \\
& = i 1_{\mathcal{E}} e^{-i(t-s)\Gamma} (1|v'(e') e^{-isZ_{\lambda}} g_{e'} e^{i\lambda^{-2}(e-e')} ds. \tag{5.40}
\end{align*}

If $e - e' \neq 0$, this goes to zero by the Lebesgue-Riemann Lemma. Therefore, (5.40) equals

\begin{equation}
\delta_{e,e'} 1_{\mathcal{E}} e^{-i(t-s)\Gamma} (1|v'(e') e^{-isZ_{\lambda}} g_{e'} ds. \tag{5.41}
\end{equation}

The fourth term sandwiched between $g_{e} \in L^1(\mathbb{R}, h_e) \cap L^2(\mathbb{R}, h_e)$ and $u \in \mathcal{E}$ converges to

\begin{align*}
i & \int_0^t \langle g_{e'} e^{-isZ_{\lambda}} (1|v(e) e^{-i(t-s)\Gamma} e^{-i\lambda^{-2}(t-s)E} 1_{\mathcal{E}} u \rangle e^{i(t-s)\lambda^{-2}e} ds \\
& = i \int_0^t \langle g_{e'} e^{-isZ_{\lambda}} (1|v(e) e^{-i(t-s)\Gamma} 1_{\mathcal{E}} u \rangle e^{i(t-s)\lambda^{-2}(e-e')} ds. \tag{5.42}
\end{align*}
Again, if \( e - e' \neq 0 \), this goes to zero by the Lebesgue-Riemann Lemma. Therefore, (5.42) equals

\[
\delta_{e,e'} \int_0^t \left\langle g_e | e^{-is_1 Z_R} |1 \rangle \otimes v(e) e^{-i(t-s)R} 1_{E_e} u \right\rangle \, ds. \tag{5.43}
\]

The fifth term sandwiched between \( g_e \in L^1(\mathbb{R}, h_e) \cap L^2(\mathbb{R}, h_e) \) and \( g_{e'} \in L^1(\mathbb{R}, h_{e'}) \cap L^2(\mathbb{R}, h_{e'}) \) converges to

\[
- \int \int_{0 \leq s_1, s_2, s_1 + s_2 \leq t} e^{i(t-s_1)\lambda^{-2}e} \left\langle g_e | e^{-is_1 Z_R} v(e) \otimes |1\rangle \right\rangle \, ds_1 ds_2 \times e^{-i(t-s_1-s_2)\lambda^{-2}e} e^{i(t-s_1-s_2)R} \langle v(e')^* \otimes \langle 1 | e^{-is_2 (Z_R + \lambda^{-2}e')} g_{e'} \rangle \rangle \, ds_1 ds_2
\]

\[
= - \sum_{e'' \in \sigma(e)} \int \int_{0 \leq s_1, s_2, s_1 + s_2 \leq t} e^{i(t-s_1)\lambda^{-2}(e-e'')-s_2 \lambda^{-2}(e'-e'')} \left\langle g_e | e^{-is_1 Z_R} v(e) \otimes |1\rangle \right\rangle \, ds_1 ds_2 \times e^{-i(t-s_1-s_2)R} 1_{E_{e''}} v(e')^* \otimes \langle 1 | e^{-is_2 Z_R} g_{e'} \rangle \, ds_1 ds_2. \tag{5.44}
\]

By the Riemann-Lebesgue Lemma the terms with \( e - e'' \neq 0 \) or \( e' - e'' \neq 0 \) go to zero. Thus (5.44) equals

\[
- \delta_{e,e'} \int \int_{0 \leq s_1, s_2, s_1 + s_2 \leq t} \left\langle g_e | e^{-is_1 Z_R} v(e) \otimes |1\rangle \right\rangle \times e^{-i(t-s_1-s_2)R} 1_{E_{e''}} v(e')^* \otimes \langle 1 | e^{-is_2 Z_R} g_{e'} \rangle \, ds_1 ds_2.
\]

Thus we proved that for \( \psi, \psi' \in \mathcal{D} \) we have

\[
\sup_{0 \leq t \leq T} \left| \left\langle \psi | e^{it\lambda^{-2}Z_{ren}} J_\lambda^* e^{-it\lambda^{-2}H_\lambda} J_\lambda \psi' \right\rangle - \left\langle \psi | e^{-itZ} \psi' \right\rangle \right| \to 0, \quad \lambda \to 0.
\]

By density, this can be extended to the whole \( \mathcal{Z} \). Using the fact that \( e^{-itZ} \) is unitary and \( e^{it\lambda^{-2}Z_{ren}} J_\lambda^* e^{-it\lambda^{-2}H_\lambda} J_\lambda \) contractive we obtain that for \( \psi \in \mathcal{Z} \)

\[
\lim_{\lambda \to 0} e^{it\lambda^{-2}Z_{ren}} J_\lambda^* e^{-it\lambda^{-2}H_\lambda} J_\lambda \psi = e^{-itZ} \psi.
\]

**Acknowledgments.** The research of J. D. was partly supported by the Postdoctoral Training Program HPRN-CT-2002-0277 and the Polish grants SPUB127 and 2 P03A 027 25. Part of the work was done when both authors visited the Erwin Schrödinger Institute (J. D. as a Senior Research Fellow), as well as during a visit of J. D. at K. U. Leuven supported by a grant of the ESF. W. D. R. is an Aspirant of the FWO-Vlaanderen.

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