THEORY OF COMPTON SCATTERING BY ANISOTROPIC ELECTRONS

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**ABSTRACT**

Compton scattering plays an important role in various astrophysical objects such as accreting black holes and neutron stars, pulsars, relativistic jets, and clusters of galaxies, as well as the early universe. In most of the calculations, it is assumed that the electrons have isotropic angular distribution in some frame. However, there are situations where the anisotropy may be significant due to the bulk motions, or where there is anisotropic cooling by synchrotron radiation or an anisotropic source of seed soft photons. Here we develop an analytical theory of Compton scattering by anisotropic distribution of electrons that can significantly simplify the calculations. Assuming that the electron angular distribution can be represented by a second-order polynomial over the cosine of some angle (dipole and quadrupole anisotropies), we integrate the exact Klein–Nishina cross section over the angles. Exact analytical and approximate formulae valid for any photon and electron energies are derived for the redistribution functions describing Compton scattering of photons with arbitrary angular distribution by anisotropic electrons. The analytical expressions for the corresponding photon scattering cross section on such electrons, as well as the mean energy of scattered photons, its dispersion, and radiation pressure force are also derived. We apply the developed formalism to the accurate calculations of the thermal and kinematic Sunyaev–Zeldovich effects for arbitrary electron distributions.

**Key words:** accretion, accretion disks – cosmic background radiation – galaxies: jets – methods: analytical – radiation mechanisms: non-thermal – scattering

**Online-only material:** color figures

1. INTRODUCTION

Compton scattering is one of the most important radiative processes that shapes the spectra of various sources: black holes and neutron stars in X-ray binaries, pulsars and pulsar wind nebulae, jets from active galactic nuclei, and the early universe. Compton scattering kernel takes a simple form if electrons are ultra-relativistic with the Lorentz factor $\gamma \gg 1$ (Blumenthal & Gould 1970). In a general case, when no restrictions are made on the energies of photon and electrons, Jones (1968) derived the kernel for isotropic electrons and photons. The formulae there contain a few misprints, but even once corrected those (see, e.g., Pe‘er & Waxman 2005) they cannot be used for calculations because of a number of numerical cancellations (see, e.g., Belmont 2009). An alternative derivation to that kernel was given by Brinkmann (1984) and Nagirner & Poutanen (1994, NP94 hereafter), who showed how to extend the numerical scheme to cover all photon and electron energies of interest in astrophysical sources.

In real astrophysical environments, the radiation field does not need to be isotropic and a more general redistribution function is required to describe angle-dependent Compton scattering. NP94 extended previous results to the situation when the photon distribution can be represented as a linear function of some polar angle cosine (Eddington approximation), deriving an analytical formula for the first moment of the kernel. Aharonian & Atoyan (1981) were the first to derive a redistribution function for arbitrary photon angular dependence (see also Prasad et al. 1986). Kershaw et al. (1986) and Kershaw (1987) have developed numerical methods to compute the kernel efficiently and with a high accuracy. All these works neglect the effect of photon polarization.

Nagirner & Poutanen (1993) have derived a general Compton scattering redistribution matrix for Stokes parameters assuming an isotropic electron distribution. A general relativistic kinetic equation incorporating the effects of induced scattering and polarization of photons as well as electron polarization and degeneracy has been derived by Nagirner & Poutanen (2001).

In this paper, we propose a method to extend previous results to the case where the electron distribution is no longer isotropic, but can have weak anisotropies which can be represented as a second-order polynomial of the cosine of some polar angle. The proposed formalism can find its application in a number of astrophysical problems. The distortion of the cosmic microwave background (CMB) caused by the hot electron gas in clusters of galaxies (i.e., the kinematic and thermal Sunyaev–Zeldovich effect) is an obvious application. The electron distribution, isotropic in the cluster frame, can be Lorentz transformed to the CMB frame, resulting in a dipole term linear in cluster velocity and a small quadrupole correction. Compton scattering then can be directly computed in the CMB frame. Another possible application concerns the models of outflowing accretion disk coronae or jets (Beloborodov 1999; Malzac et al. 2001). If the outflow velocities are non-relativistic, the radiation transport can be considered directly in the accretion disk frame following recipe by Poutanen & Svensson (1996), with the Lorentz transformed electron distribution.

Weak anisotropies in the electron distribution can arise in high-energy sources with an ordered magnetic field, because of the pitch angle dependence of the synchrotron cooling rate and/or anisotropic injection of high-energy electrons (e.g., Björnsson 1985; Roland et al. 1985; Crusius-Waetz & Lesch 1998; Schopper et al. 1998). An anisotropic electron distribution is also a very natural outcome of the photon breeding operation in relativistic jets with the toroidal magnetic field (Stern & Poutanen 2006, 2008), because the electron–positron pairs
born inside the jet by the external high-energy photons move perpendicularly to the field.

Our method is also extendable to the polarized radiation using the techniques developed in Nagirner & Poutanen (1993). It is in principle possible to calculate the scattering redistribution function in the case when the electron distribution is expressible as an arbitrary order expansion over the polar angle cosine. Unfortunately, in the latter case the analytical expressions become extremely cumbersome and the advantage over direct numerical integration becomes small.

Although here we consider only photon scattering it is also possible to apply the same method for electrons interacting with the photons in the case when photon angular distribution is expressible as an expansion of powers of the polar angle cosine. This can be of interest only in the deep Klein–Nishina regime where the electron can lose or gain a large fraction of its initial energy in one scattering and continuous energy loss approach is not applicable.

Our paper is organized as follows. In Section 2, we introduce the relativistic kinetic equation for Compton scattering and define the redistribution function and total cross section. The expressions for the total cross section, the mean energy and dispersion of scattered photons, and the radiation pressure force are given in Section 3. The exact analytical formulae for the redistribution function for mono-energetic anisotropic electrons as well as approximate formulae valid in the Thomson regime are derived in Section 4. We present the numerical examples of redistribution functions in Section 5, where we also develop the relativistic theory of the Sunyaev–Zeldovich effect. We summarize our findings in Section 6.

2. RELATIVISTIC KINETIC EQUATION

Let us define the dimensionless photon four-momentum as \( x = \{ \nu, x \} = x \{ 1, \omega \} \), where \( \omega \) is the unit vector in the photon propagation direction and \( x \equiv \nu/m_e c^2 \). The photon distribution will be described by the occupation number \( n \).

The dimensionless electron four-momentum is \( p = \{ \gamma, p \} = \{ \gamma, \vec{p} \} \), where \( \gamma \) is the unit vector along the electron momentum, \( \gamma \) and \( \nu = \sqrt{\gamma^2 - 1} \) are the electron Lorentz factor and its momentum in units of \( m_e c \), respectively, and \( p = \vec{p}/\gamma \) is the electron velocity in units of speed of light. The momentum distribution of electrons is described by the relativistically invariant distribution function \( f_e(p) \) (see Belyaev & Budker 1956; NP94).

The interaction between photons and electrons via Compton scattering (in linear approximation, i.e., ignoring induced scattering and electron degeneracy) can be described by the explicitly covariant relativistic kinetic equation for photons (Pomraning 1973; Nagirner & Poutanen 1993; NP94):

\[
\vec{x} \cdot \nabla n(x) = \frac{r_e^2}{2} \int \frac{d^3p}{\gamma} \frac{d^3p_1}{\gamma_1} \frac{d^3x_1}{x_1} \delta^4(p_1 + x_1 - p - x) F \times [n(x)f_e(p_1) - n(x)f_e(p)],
\]

where \( \nabla = \{ \partial/\partial \nu, \vec{\nabla} \} \) is the four-gradient, \( r_e \) is the classical electron radius, \( F \) is the Klein–Nishina reaction rate (Berestetskii et al. 1982)

\[
F = \left( \frac{1}{\xi} - \frac{1}{\xi_1} \right)^2 + 2 \left( \frac{1}{\xi} - \frac{1}{\xi_1} \right) + \frac{\xi}{\xi_1} + \frac{\xi_1}{\xi},
\]

and

\[
\xi = p_{1} \cdot \vec{x} = \vec{p} \cdot \vec{x}, \quad \xi_1 = p_{1} \cdot \vec{x} = \vec{p} \cdot \vec{x}_1
\]

are the four products of corresponding momenta. The second equalities in both Equations (3) arise from the four-momentum conservation law. The invariant scalar product of the photon four-momenta can be written in the laboratory frame as well as in the frame related to a specific electron

\[
q \equiv x \cdot x = xx_1(1 - \mu) = \xi \xi_1(1 - \mu_0) = \xi - \xi_1,
\]

where \( \mu = \vec{\omega} \cdot \vec{\omega}_1 \) is the cosine of the photon scattering angle in some frame and \( \mu_0 \) is the corresponding cosine in the electron rest frame.

In any frame, the kinetic equation can be also put in the usual form of the radiative transfer equation (Nagirner & Poutanen 1993)

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \vec{\omega} \cdot \vec{n} \right) n(x) = -\sigma_T N_e \xi_0(x) n(x)
\]

\[
+ \sigma_T N_e \frac{1}{x} \int_0^\infty \int_0^{2\pi} d\omega_1 R(x_1 \rightarrow x) n(x_1),
\]

where \( N_e \) is the electron density in that frame. Here we have defined the photon redistribution function

\[
R(x_1 \rightarrow x) = \frac{3}{16\pi} \frac{1}{N_e} \int \frac{d^3p}{\gamma} \frac{d^3p_1}{\gamma_1} f_e(p_1) F \delta^4(p_1 + x_1 - p - x)
\]

and the total scattering cross section (in units of Thomson cross section \( \sigma_T \))

\[
\xi_0(x) = \frac{3}{16\pi} \frac{1}{N_e} \int \frac{d^3p}{\gamma} \frac{d^3p_1}{\gamma_1} f_e(p) F \delta^4(p_1 + x_1 - p - x).
\]

2.1. Electron Distribution and Scattering Geometry

Let us now consider a specific frame \( E \). Our basic assumption is that the anisotropy of the electron distribution in this frame can be expressed as a second-order polynomial expansion in the cosine of the polar angle in some coordinate system \( (l_1, l_2, l_3) \):

\[
\frac{1}{N_e} f_e(p) = f_e(\gamma, \eta_0) = \sum_{k=0}^2 f_k(\gamma) P_k(\eta_0),
\]

where \( N_e \) is the electron density in frame \( E \), \( \eta_0 = \Omega \cdot l_3 \) is the cosine of the polar angle of the electron momentum, \( P_k(\eta_0) \) are the Legendre polynomials, and we now switched to the dimensionless distribution function \( f_k(\gamma, \eta_0) \) normalized to unity:

\[
\int d\Omega \int f_k(\gamma, \eta_0) p^2 dp = 1.
\]

The moments \( f_0, f_1, \) and \( f_2 \) determine the energy spectrum of electrons and the relative magnitudes of the isotropic and anisotropic components. The distribution function for mono-energetic electrons of energy \( \gamma_0 \) can be described by

\[
f_k(\gamma, \eta_0) = \frac{1}{4\pi \gamma} \delta(\gamma - \gamma_0) \left[ 1 + \frac{f_1}{f_0} \eta_0 + \frac{f_2}{f_0} P_2(\eta_0) \right],
\]

where the ratios \( f_1/f_0 \) and \( f_2/f_0 \) are constants.

The directions of photons in this coordinate system (see Figure 1) are given by

\[
\omega = \sqrt{1 - \eta^2} \cos \phi \, l_1 + \sqrt{1 - \eta^2} \sin \phi \, l_2 + \eta \, l_3,
\]

where \( \phi \) is the angle between the photon propagation direction and the electron momentum.

\[
\eta = \vec{\omega} \cdot \vec{l}_3.
\]
\[ \omega_1 = \sqrt{1 - \eta_1^2} \cos \phi_1 I_1 + \sqrt{1 - \eta_1^2} \sin \phi_1 I_2 + \eta_1 I_3, \]

so that the cosine of the scattering angle is

\[ \mu = \omega \cdot \omega_1 = \eta \eta_1 + \sqrt{1 - \eta^2} \sqrt{1 - \eta_1^2} \cos(\phi - \phi_1). \]

3. TOTAL CROSS SECTION AND MEAN POWERS OF ENERGIES

3.1. Total Cross Section

Let us simplify the expression for the total cross section. We follow here the approach described in NP94. We rewrite the cross section as

\[ \bar{s}_0(x) = \frac{1}{x} \int s_0(\xi) \xi f_\gamma(\gamma, \eta_\gamma) \frac{d^3p}{\gamma}, \]

where

\[ s_0(\xi) = \frac{3}{16\pi} \int \frac{d^3p}{\gamma_1} \frac{d^3x_1}{x_1} F \delta^4(p_1 + x_1 - P - x) \]

and taking the integral over \( p_1 \) in Equation (15), we get

\[ s_0(\xi) = \frac{3}{16\pi} \int \frac{d^3x_1}{x_1} F \delta(\xi_1 \cdot (P + x) - \xi \cdot P) \]

putting \( \delta(\gamma_1 + x_1 - \gamma - x) = \gamma_1 \delta(\xi_1 \cdot (P + x) - \xi \cdot P) \)

and using the identity

\[ \delta(x_1 - \gamma - x) = \gamma_1 \delta(\xi_1 \cdot (P + x) - \xi \cdot P) \]

we get

\[ s_0(\xi) = \frac{3}{8\xi} \int \frac{d^3x_1}{x_1} F d\xi_1, \]

where we used invariant \( x_1 d\xi_1 d^2\omega_1 = \xi_1 d\xi_1 d\mu_0 d\phi_0 \cdot P \delta(\xi_1 + \xi_1 - \mu_0 - \xi) \)

and the fact that \( F \) does not depend on azimuthal angle \( \phi_0 \). Substituting \( F \) from Equation (2), we get (Berestetskii et al. 1982; NP94)

\[ s_0(\xi) = \frac{3}{8\xi^2} \left[ 4 + \left( \xi - 2 - \frac{2}{\xi} \right) \ln(1 + 2\xi) + 2\xi^2 \right], \]

Putting \( \xi \to x \), we, of course, get the total Klein–Nishina cross section for a photon of energy \( x \) on electrons at rest.

To obtain the total scattering cross section on an anisotropic electron distribution, we have to calculate the angular integrals over incoming electron directions in Equation (14). We introduce the cosines between electron momentum and photons:

\[ \xi = \Omega \cdot \omega, \quad \xi_1 = \Omega \cdot \omega_1 \]

so that

\[ \xi = x(\gamma - \rho \xi), \quad \xi_1 = x_1(\gamma - \rho \xi_1). \]

We choose the spherical coordinate system and measure the polar angle from the direction of the initial photon \( \omega \), we get

\[ \bar{s}_0(x, \eta) = \frac{1}{x} \int_0^{2\pi} \frac{d^2p}{\gamma} \int_{-1}^{1} d\xi s_0(\xi) \int_0^{\pi} d^2\Phi f_\gamma(\gamma, \eta_\gamma). \]

Azimuth \( \Phi \) is now defined as the difference between the azimuths of the electron momentum direction and the vector \( I_3 \)

in a frame with \( z \)-axis along \( \omega \). The angular variable \( \eta_\gamma \) in the expansion (8) is expressed in this frame as

\[ \eta_\gamma = \eta \xi + \sqrt{1 - \eta^2} \sqrt{1 - \xi^2} \cos \Phi. \]

The physical meaning of Equation (21) can also be understood if we consider a monoenergetic beam of electrons along the \( I_3 \) axis: \( f_\gamma(\gamma, \eta_\gamma) = \delta(\eta_\gamma - 1)\delta(\gamma - \eta_\gamma)/(2\pi p \gamma) \). Then

\[ \bar{s}_0(x, \eta) = (1 - \beta \eta) s_0(x'), \]

where \( x' = x \gamma(1 - \beta \eta) \) is the photon energy in the electron rest frame (we omitted subscript 0 in \( \gamma \) and \( \beta \)). The factor \( 1 - \beta \eta \) in Equation (23) accounts for the reduced number of collision per unit length.

When calculating the azimuthal integral in Equation (21) we just have to integrate \( P_k(\eta_\gamma) \) with \( \eta_\gamma \) given by Equation (22). The properties of the Legendre polynomials give us the average

\[ \bar{P}_k(\eta_\gamma) = P_k(\eta) \bar{P}_k(\xi), \]

so that the averaged distribution function becomes

\[ \bar{f}_\gamma(\gamma, \eta_\gamma) = \frac{1}{2\pi} \int_0^{2\pi} d\Phi f_\gamma(\gamma, \eta_\gamma) = \sum_{k=0}^{2} P_k(\gamma) P_k(\eta), \]

The cross section now can be written as

\[ \bar{s}_0(x, \eta) = 4\pi \sum_{k=0}^{2} P_k(\eta) \int_1^{\infty} d\gamma d\gamma' f_k(\xi(\gamma, \gamma'), \gamma), \]

where

\[ \Delta_{\delta k}(x, \gamma) = \frac{1}{2\gamma \gamma'} \int_{-1}^{1} P_k(\xi) \xi s_0(\xi) d\xi. \]

Changing the integration variable and expressing \( P_k \) through Equation (20), we get

\[ \Delta_{\delta k}(x, \gamma) = \sum_{n=0}^{k} b_{nk} \chi_{0n}, \]

where

\[ \chi_{0n}(x, \gamma) = \frac{1}{2\gamma \gamma' x^{2n}} \int_{\gamma(\gamma - p)}^{\gamma(\gamma + p)} \xi^{n+1} s_0(\xi) d\xi \]

and

\[ b_{00} = 1, \quad b_{01} = \gamma/p, \quad b_{12} = -1/p, \quad b_{02} = 1/2p^2(2\gamma^2 + 1), \quad b_{11} = 3\gamma/p^2, \quad b_{22} = 3/2p^2. \]
The zeroth function $\Delta_{00} = \chi_{00}$ coincides with the function $\Psi_{0}(x, \gamma)$ from NP94. When electron distribution is isotropic (i.e., $f_1 = f_2 = 0$), expression (26) for the total cross section is reduced to Equation (3.4.1) from NP94 and the dependence on $\eta$ obviously disappears. Functions $\chi_{0n}$ of two arguments can be presented through the functions of one argument:

$$\chi_{0n}(x, \gamma) = \frac{1}{2 \gamma p} \frac{u^{2n}}{2 + n} \psi_{n+1,0}(x u) \bigg|_{u = \gamma + p}^{u = \gamma - p},$$

where

$$\psi_{10}(\xi) = \int_{0}^{\xi} t^{n} s_{0}(t) dt.$$  \hspace{1cm} (32)

The explicit expressions for the functions $\psi_{i0}(\xi)$ ($i = 1, 2, 3$) can be found in Appendix B (see also NP94). Thus the total cross section is given by a single integral over the electron energy (26) with all functions under the integral given by numerical expressions (31) for $\chi_{0n}$.

For mono-energetic electron distribution of the Lorentz factor $\gamma$ given by Equation (10), we can introduce the cross section analogously to Equation (26):

$$\overline{\sigma_{0}}(x, \gamma, \eta) = \sum_{k=0}^{2} \frac{f_k}{f_0} P_k(\eta) \Delta_{0k}(x, \gamma).$$  \hspace{1cm} (33)

For isotropic mono-energetic electrons, the total cross section is shown in Figure 2(a). The relative corrections arising due to the dipole and quadrupole terms in the electron distribution are shown in Figures 2(b) and 2(c), respectively. These have to be multiplied by the angle- and, possibly, the energy-dependent factor $P_k(\eta)f_k/f_0$ to obtain the final correction. In the Thomson limit, at small $x \gamma^2 \ll 1$, the cross section takes the form (see Appendix D)

$$\overline{\sigma_{0}}(x, \gamma, \eta) \approx 1 - \frac{1}{3} \frac{f_1}{f_0} \eta \beta,$$  \hspace{1cm} (34)

where the correction to unity term can be easily obtained by averaging the transport cross section over electron directions (i.e., integrating $1 - \beta \zeta$ $d\zeta$ over the angles). This corresponds to the flattening in Figure 2(b) at $\Delta_{01}/\Delta_{00} = -\beta/3$. The correction from the quadrupole term in this regime as well as for non-relativistic electrons becomes negligible:

$$\Delta_{02}/\Delta_{00} \approx -\frac{4}{15} \beta^2 (x \gamma).$$  \hspace{1cm} (35)

3.2. Mean Powers of Scattered Photon Energy

In some situations, the full relativistic kinetic equations can be substituted by the approximate one obtained in the Fokker–Planck approximation. This requires knowledge of various moments of the redistribution function, such as total cross section, the mean energy and dispersion of the scattered photons (see NP94, Vurm & Poutanen 2009). It is often time consuming to numerically compute the integrals of the redistribution function and instead direct calculations of the moments are preferable. Below we obtain analytical expressions for the mean energy and dispersion of the energy of scattered photons in frame $E$ as a function of the initial photon energy $x$ and the direction of its propagation relative to a symmetry axis of the electron distribution $I_3$. 

Figure 2. (a) Total cross section $\chi_{00} = \Delta_{00}$ for isotropic mono-energetic electrons of various momenta $p = 0.1, 1, 10, 10^2, 10^3$ (from top to bottom, dot-dashed, solid, dotted, dashed, and triple-dot-dashed curves) as a function of photon energy $x$. (b) Relative correction to the cross section arising from the dipole term in the electron distribution (8) for the same $p$ as in panel (a). At small $x$ the curves approach the asymptotic value in the Thomson limit $-\beta/3$. (c) Relative correction to the cross section arising from the quadrupole term $\Delta_{02}/\Delta_{00}$. (A color version of this figure is available in the online journal.)
Following NP94, we define the mean of powers of energy of scattered photons:

$$\overline{x_j^i s_0(x)} = \frac{1}{x} \int \langle x_j^i \rangle s_0(\xi) f_\xi(\gamma, \eta_\xi) \frac{d^3p}{\gamma},$$  \hspace{1cm} (36)

where now

$$\langle x_j^i \rangle s_0(\xi) = \frac{3}{16 \pi \xi} \int \frac{d^3p_1 \, d^3x_1}{\gamma_1} F x_j^i \delta^4(p_1 + x_1 - p - \xi)$$

$$= \frac{3}{16 \pi \xi} \int x_j^{i+1} d^2\omega_1 F \delta \times \{ x_1 [\gamma + x - \omega_1 \cdot (p\Omega + x\omega)] - \xi \}. \hspace{1cm} (37)$$

### 3.2.1. Averaging Over Photon Directions

Quantities (37) are not invariants (except for $j = 0$), and we have to compute the scattered photon energy in a certain frame, which we choose to be frame $E$. Because of the additional term $x_j^i$ under the integral, a simple change of variables to the electron rest frame as in Equation (17) is not possible. Instead, we use the $\delta$-function to take the integral over $x_1$:

$$\langle x_j^i \rangle s_0(\xi) = \frac{3}{16 \pi \xi^2} \int x_j^{i+2} d^2\omega_1 F. \hspace{1cm} (38)$$

Now we change the variables to those in the electron rest frame (with subscript 0). We choose the coordinate system with the polar axis along the direction of the incoming photon $\omega_0$. In this frame, the cosine of the angle between the electron momentu and the incoming photon is $\xi_0$. The cosine of the angle between the outgoing photon momentum and the electron is then $\xi_0 = \zeta_0 \mu_0 + \sqrt{1 - \zeta_0^2} \sqrt{1 - \mu_0^2} \cos \phi_0$.

We use invariants $x_j^2 d^2\omega_1 = \xi_0^2 d\mu d\phi_0$ and the energy conservation law in the electron rest frame $\xi_1 = \xi / (1 + \xi [1 - \mu_0])$, to get (see NP94)

$$x_j^2 d^2\omega_1 = d\xi_1 d\phi_0. \hspace{1cm} (39)$$

Finally, we have

$$\langle x_j^i \rangle s_0(\xi) = \frac{3}{16 \pi \xi^2} \int \xi_1^i F d\xi_1 \int_0^{2\pi} x_j^i d\phi_0. \hspace{1cm} (40)$$

Because $\xi_1$ is the energy of scattered photon in the electron rest frame, the Doppler effect gives us

$$x_1 = \xi_1 (\gamma + p\xi_0)$$

$$= \xi_1 \left( \gamma + p\xi_0 \mu_0 + p\sqrt{1 - \xi_0^2} \sqrt{1 - \mu_0^2} \cos \phi_0 \right). \hspace{1cm} (41)$$

where we now can substitute

$$\mu_0 = 1 + \frac{1}{\xi} - \frac{1}{\xi_1}, \hspace{1cm} p\xi_0 = \frac{x}{\xi} - \gamma,$$

which are consequences of the conservation law and of the Lorentz transformation $x = \xi (\gamma + p\xi_0)$, respectively. The terms containing a linear combination of square roots and $\cos \phi_0$ will disappear after averaging over $\phi_0$.

We now introduce moments of the invariant cross section

$$s_j(\xi) = \frac{3}{8 \xi^{2j}} \int \xi_1^i F d\xi_1. \hspace{1cm} (43)$$

For $j = 0$, we get of course the total cross section $s_0$ given by Equation (18). NP94 derived the corresponding expressions for $j = 1, 2$:

$$s_1(\xi) = \frac{3}{8 \xi^2} \left( \frac{3}{2} \xi^2 - \frac{3}{2} \xi - \frac{\xi}{2} \frac{R_1}{R_0} - \frac{\xi^2}{3} \frac{R_0}{R_0} \right) \hspace{1cm} (44)$$

$$s_2(\xi) = \frac{R_1}{16} (9 + R_2 + 3 R_2^2 + 3 R_2^3), \hspace{1cm} (45)$$

where $l_\xi = \ln (1 + 2 \xi)$, and $R_\xi = 1/(1 + 2 \xi)$.

For the mean energy of the scattered photon we have then

$$\langle x_1 \rangle s_0(\xi) = \xi_0 [\gamma S_1(\xi) + x S_2(\xi)]. \hspace{1cm} (46)$$

and for the mean square of energy

$$\langle x_1^2 \rangle s_0(\xi) = \gamma \xi^2 S_3(\xi) - \gamma x \xi S_2(\xi) + x^2 S_0(\xi) - \xi^2 S_1(\xi). \hspace{1cm} (47)$$

where

$$S_1(\xi) = [s_0(\xi) - s_1(\xi)] / \xi; \hspace{1cm} S_2(\xi) = [s_1(\xi) - s_2(\xi)] / \xi; \hspace{1cm} S_3(\xi) = \xi [S_1(\xi) - S_2(\xi)] / \xi; \hspace{1cm} S_4(\xi) = 3 S_1(\xi) - 4 S_2(\xi); \hspace{1cm} S_7(\xi) = S_1(\xi) - S_4(\xi)/2; \hspace{1cm} S_8(\xi) = S_2(\xi) - 3 S_4(\xi). \hspace{1cm} (48)$$

All functions $S_j(\xi)$ are elementary. In addition, they are defined in such a way so that not to become zero at $\xi = 0$. The series expansion of functions $s$ and $S$ for small arguments are presented in Appendix A.

### 3.2.2. Averaging Over Electron Directions

We need to integrate in Equation (36) over anisotropic electron distribution. We follow the derivation of the total cross section that lead from Equation (21) to Equation (26). Representing integral over electron momentum $d^3p = p^2 dp d\zeta d\Phi$ we get

$$\overline{x_j^i s_0(x, \eta)} = 4\pi \int \sum_{k=0}^2 P_k(\eta) \int_1^{\infty} p_\gamma d\gamma \int f_k \Delta_\eta(x, \gamma), \hspace{1cm} (49)$$

where

$$\Delta_\eta(x, \gamma) = \frac{1}{2\gamma x^{1+j}} \int_{-1}^1 P_k(\xi) \xi \langle x_j^i \rangle s_0(\xi) d\xi = \sum_{n=0}^k b_{nk} \chi_{jn} \hspace{1cm} (50)$$

and

$$\chi_{jn}(x, \gamma) = \frac{1}{2\gamma p x^{2+j+n}} \int_{x^{-1}}^x \left( \langle x_j^i \rangle s_0(\xi) \xi^{1+n} \right) d\xi. \hspace{1cm} (51)$$

Functions $\chi_{j0} = \Delta_{j0}$ coincide with functions $\Psi_{j0}(x, \gamma)$ introduced by NP94, while functions $\chi_{jn}$ are given by Equation (31). The explicit expressions for the function $\chi_{jn}$ for $j = 1, 2$ (which are analogous to functions $\Psi_{j1}$ and $\Psi_{j2}$ from NP94) can be obtained using expression for mean powers of energies (46) or (47):

$$\chi_{1n}(x, \gamma) = \frac{1}{2\gamma p} \frac{\xi^{3+n}}{3 + n} [\gamma \Psi_{2+n,1}(x\xi) + x \Psi_{2+n,0}(x\xi)] \bigg|_{\xi = \gamma - p} \bigg|_{\xi = \gamma + p} \hspace{1cm} (52)$$
\( \chi_{\text{an}}(x, \gamma) = \frac{1}{2 \gamma p} \left[ \gamma^2 \frac{u_{4+n}}{4+n} \Psi_{3+n,4}(xu) - \gamma \frac{u_{2+n}}{3+n} \Psi_{2+n,5}(xu) + \frac{u_{2+n}}{2+n} \Psi_{1+n,6}(xu) - \frac{u_{4+n}}{4+n} \Psi_{3+n,7}(xu) \right] \bigg|_{u=\gamma-p}, \)

where

\[ \Psi_{ij}(\xi) = \int_0^{\xi} t^i S_j(t) dt. \]  

These are related to functions

\[ \Psi_{ij}(\xi) = \int_0^{\xi} t^i s_j(t) dt, \]

because functions \( S_j \) are expressed through \( s_j \). The explicit expressions for both types of these functions as well as their series expansions for small arguments are given in Appendix B.

As in the case of functions \( \Delta_{\text{ax}} \), for calculating \( \Delta_{\beta} \), we consider three regimes: (1) \( x\gamma \ll 1 \), when we use the series expansion (see Appendix D); (2) for \( p \ll 1 \) we numerically take the integral in Equation (50) using Gaussian quadrature; and (3) in other cases, we use the sum in Equation (50) and analytical expressions for \( \chi_{\text{an}} \).

For mono-energetic electron distribution (10) of the Lorentz factor \( \gamma \), we can introduce the mean powers of photon energy analogously to Equation (49):

\[ \bar{x}^j \bar{\sigma}(x, \gamma, \eta) = x^j \sum_{k=0}^{2} \frac{f_k}{f_0} \bar{P}_k(\eta) \Delta_{jk}. \]

The mean energy of scattered photons for such electrons for a scattering act is given by the ratio of Equations (56) and (33). It is shown in Figure 3(a). In the low-energy (Thomson) limit, the energy gain factor is given by a well-known expression \( \bar{x}^{\text{iso}} / x = 1 + 4p^2 / 3 \), which translated to \( \frac{4}{3} \gamma^2 \) at large \( \gamma \). The relative corrections arising due to the dipole and quadrupole terms in the electron distribution are shown in Figures 3(b) and 3(c), respectively. Using asymptotic expansions of \( \Delta_{jk} \) in the Thomson limit (see Appendix D), we get the asymptotic value

\[ \bar{x}(x, \gamma, \eta) = \gamma^2 \left( 1 + \frac{1}{2} \beta^2 - \frac{3}{2} \beta \gamma \frac{\gamma}{f_0} \eta + \frac{15}{16} \beta^2 \gamma^2 \frac{\gamma}{f_0} \eta \right). \]

Thus in non-relativistic limit \( \beta \ll 1 \), the correction is negligible. In the relativistic limit \( \gamma \gg 1 \), the relative corrections arising from the two terms are

\[ \frac{\bar{x}(x, \gamma, \eta)}{\bar{x}^{\text{iso}}(x, \gamma)} = 1 - \frac{1}{2} \frac{f_2}{f_0} \eta, \]

\[ \frac{\bar{x}(x, \gamma, \eta)}{\bar{x}^{\text{iso}}(x, \gamma)} = 1 + \frac{1}{2} \frac{f_2}{f_0} \eta. \]

3.3. Energy Exchange and Dispersion

The difference of the photon energies before and after scattering \( x - x_1 \) is of course just the energy transfer to the electron gas. For the fixed angle between electrons and incident photons \( \zeta \) (and fixed electron energy \( \gamma \)), the energy loss averaged over the directions of scattered photons is \( x - \langle x_1 \rangle \).

![Figure 3](image-url)

Figure 3. (a) Mean energy of scattered photons in units of the incident photon energy \( \bar{x}^{\text{iso}}(x, \gamma) / x = \Delta_{\text{ax}} / \Delta_{\text{an}} \) as a function of \( x \) for isotropic mono-energetic electrons of various momenta \( p = 0, 1, 1, 10, 10^2, 10^3 \) (from bottom to top, dot-dashed, solid, dotted, dashed, and triple-dot-dashed curves). The asymptotic value at small \( x \) in the Thomson approximation is \( 1 + \frac{1}{2} \gamma^2 \). (b) A correction to the mean energy (in units of \( \bar{x}^{\text{iso}} \)) arising from the linear term in the electron distribution (8) with \( f_1 / f_0 = 1 \). Solid, dotted, and dashed curves correspond for \( p = 1, 10, 100 \), respectively. The curves from top to bottom correspond to \( \eta = -1, -0.5, 0, 0.5, 1 \). At small \( x \), the curves approach the limiting value given by Equation (58). (c) Same as (b), but for the quadrupole term in the electron distribution (8) with \( f_2 / f_0 = 1 \). These are even functions of \( \eta \), the curves from the bottom to the top correspond to \( \eta = 0, 0.5, 1 \). At small \( x \), the curves approach the limiting value given by Equation (59).

(A color version of this figure is available in the online journal.)

The product \( (x - \langle x_1 \rangle) N_e \sigma_T s_0(\xi) \) is then the energy loss on a unit length. From Equation (46) we can easily get (see also NP94)

\[ (x - \langle x_1 \rangle) s_0(\xi) = x s_0(\xi) - \gamma \xi S_1(\xi) - x \xi S_2(\xi) = (x + x \xi - \gamma \xi) S(\xi). \]

The corresponding energy loss (per unit length and in units \( N_e \sigma_T \)) averaged over the electron directions (and integrated over
electron energies) becomes (see Equations (26) and (49))
\[(x - x_i)\tilde{s}_0(x, \eta) = 4\pi x^2 \sum_{k=0}^{2} P_k(\eta) \int_{1}^{\infty} p d\gamma f_k (\Delta_0 k - \Delta_1 k) .\]

The heating rate per unit volume is then
\[\dot{E} = N_e \sigma T \int dx \int d^2 \omega I(x, \omega) (1 - \frac{x_i}{x}) \tilde{s}_0(x, \eta),\]

where \(I(x, \omega)\) is the specific intensity of radiation in a given direction. This expression can be positive (the so-called Compton heating) when the photons typically have larger energies than the electron gas, or negative (Compton cooling) when one considers cooling of the relativistic electron gas by soft radiation.

The dispersion of the scattered photon energy is given by the usual expression \(\overline{D}(x) = x_i^2 - x^2\), which of course depends on the electron momentum distribution. For mono-energetic electrons we can define the dispersion as
\[\overline{D}(x, \gamma, \eta) = x_i^2(x, \gamma, \eta) - x^2(x, \gamma, \eta),\]

where \(x_i(x, \gamma, \eta)\) are given by Equation (56). The dispersion for isotropic electrons is shown in Figure 3(a). The low-energy (Thomson) limit for \(x \ll 1/\gamma\) is (see NP94)
\[\overline{D}(x, \gamma) = x^2 \frac{2}{45} (23\gamma^2 - 8) \eta^2.\]

The relative corrections arising due to the dipole and quadrupole terms in the electron distribution reach about 50% and are shown in Figures 4(b) and 4(c), respectively.

### 3.4. Radiation Force

Now we would like to derive analytic expression for the radiation force acting on the electron gas. NP94 have developed a formalism appropriate for isotropic electron distribution, when the averaged transferred momentum is along the momentum of the incoming photons because of the symmetry. For the electron distribution described by Equation (8), the momentum is transferred in the plane containing the initial photon momentum and the symmetry axis \(I_3\). If the incident photons are axially symmetric around \(I_3\), then obviously, the total momentum transferred to the electrons has to be parallel to \(I_3\) by symmetry. Here we derive more general formulae for the total momentum transferred by a beam of photons propagating along direction \(\omega\) (such as \(\omega \cdot \hat{I}_3 = \eta\)), as well as its projections to \(\hat{I}_3\) and the perpendicular direction.

Let us introduce the vector basis
\[e_1(\omega, I_3) = \frac{I_3 - \eta \omega}{\sqrt{1 - \eta^2}}, \quad e_2(\omega, I_3) = \frac{\omega \times I_3}{\sqrt{1 - \eta^2}}, \quad e_3 = \omega.\]

In a single scattering act the momentum transferred is
\[Q = x\omega - x_i\omega.\]

The components of the momentum transferred to the electron gas along and perpendicular to \(\omega\) are
\[Q_3 = x - x_i \omega_1 \cdot e_3 = x - x_i \mu,\]
\[Q_1 = -x_i \omega_1 \cdot e_1 = -x_i \frac{\eta_1 - \eta \mu}{\sqrt{1 - \eta^2}}.\]

Analogously to Equation (36), we define the mean transferred momentum as
\[\overline{Q}\tilde{s}_0(x) = \frac{1}{x} \int (Q)\tilde{s}_0(x) \xi f(\gamma, \eta) \frac{d^3p}{\gamma},\]
where
\[
\langle Q \rangle_{s0}(\xi) = \frac{3}{16\pi \xi} \int \frac{d^3p_1 d^3x_1}{\gamma_1 x_1} F Q \, \delta^4(p_1 + x_1 - p - \xi)
\]
\[
= \frac{3}{16\pi \xi} \int x_1 d^3x_1 d^2\omega_1 F Q \, \delta^4(p_1 + x_1 - p - \xi)
\]
\[
\times \{x_1[\gamma + x - \omega_1 \cdot (p\Omega + x\omega)] - \xi\}. \tag{70}
\]

3.4.1. Averaging Over Photon Directions

Let us introduce a vector basis defined by the photon and electron momenta:
\[
e_1(\omega, \Omega) = \frac{\Omega - \xi}{\sqrt{1 - \xi^2}}, \quad e_2(\omega, \Omega) = \frac{\omega \times \Omega}{\sqrt{1 - \xi^2}}, \quad e_3 = \omega,
\]
where \(\xi = \omega \cdot \Omega\), therefore \(\Omega = \sqrt{1 - \xi^2} e_1(\omega, \Omega) + \xi e_3\).

Fixing the angle \(\arccos \frac{\xi}{\omega}\) along the electron momentum: \(\frac{\xi}{\omega} = \cos\frac{\Omega_0}{\omega}\), we can get the mean momentum transmitted in the direction \(\omega\):
\[
\langle Q \rangle_{s0}(\xi) = (x - x_1) s_0(\xi) + (x_1 (1 - \mu)) s_0(\xi). \tag{72}
\]

The first term is given by Equation (60), the second term is
\[
\langle x_1 (1 - \mu) \rangle_{s0}(\xi) = \frac{3}{16\pi \xi^2} \int \xi_1 (1 - \mu) F d^2\omega_1
\]
\[
= \frac{3}{16\pi \xi} \int \xi_1 (1 - \mu_0) F d\xi_1 d\phi_0
\]
\[
= \frac{3}{8\pi \xi} \int_0^\xi F (\xi - \xi_1) d\xi_1
\]
\[
= \frac{\xi}{x} [s_0(\xi) - s_1(\xi)] = \frac{\xi^2}{x} S_1(\xi), \tag{73}
\]

where we have used the invariant given by Equation (4) and changed the variables according to Equation (39). Thus, for the fixed electron and photon energies and the angle between their momenta, the mean momentum transmitted to the electron gas in the direction of the initial photon propagation \(\omega\), in accordance with Equations (48), (46), and (73), is (NP94)
\[
\langle Q \rangle_{s0}(\xi) = (x - x_1 \mu) s_0(\xi) = \left(x + x_1 \xi - \gamma \xi \xi + \frac{\xi^2}{x}\right) S_1(\xi). \tag{74}
\]

In contrast to NP94, we are now interested to know the total momentum transfer. Obviously, by symmetry, it has to lie in the \((\Omega, \omega)\) plane. Averaging expression (68) for \(Q\) over angles is not easy, but we can compute the momentum transferred along the electron momentum: \(Q_{\Omega} \equiv x_1 - x_1 \xi_1\). Using identity \(x_1 \xi_1 = (\gamma x_1 - x_1) / p\) and substituting Equations (41) and (42), similarly to Equation (40), we get
\[
\langle x_1 \xi_1 \rangle_{s0}(\xi) = \frac{3}{16\pi \xi^2} \int_0^\xi F d\xi_1 \int_0^{2\pi} x_1 \xi_1 d\phi_0
\]
\[
= \frac{3}{8\xi^2} \int_0^\xi F d\xi_1 \frac{1}{p}
\]
\[
\times \left[\left(\frac{\gamma x}{\xi} + \frac{\gamma x}{\xi^2} - \frac{\gamma^2}{\xi} - 1\right) \xi_1 + \gamma \left(\gamma - \frac{x}{\xi}\right)\right]
\]
\[
= \frac{1}{p} \left(\gamma x + \frac{\gamma x}{\xi} - \gamma^2 - \xi\right) s_1(\xi).
\]

\[
\langle Q\rangle_{s0}(\xi) = \frac{1}{p} \left[\gamma^2 \xi s_1(\xi) + \gamma x \xi S_1(\xi) - \xi s_1(\xi)\right]. \tag{75}
\]

and using identity \(x_1 = (\gamma x - x_1) / p\), we finally obtain
\[
\langle Q \rangle_{s0}(\xi) = \langle x_1 \xi_1 \rangle_{s0}(\xi) = \frac{1}{p} \left(\gamma x + \gamma x \xi - \gamma^2 \xi - \xi^2\right) S_1(\xi). \tag{76}
\]

The momentum along \(e_1(\omega, \Omega)\) is then simply
\[
\langle Q \rangle_{s0}(\xi) = \frac{\langle Q \rangle_{s0}(\xi) - \langle Q \rangle_{s0}(\xi)}{\sqrt{1 - \xi^2}} s_0(\xi)
\]
\[
= \frac{S_1(\xi)}{p\sqrt{1 - \xi^2}} \left(\xi - 2\gamma^2 \xi + \xi^3\right)
\]
\[
= -p\sqrt{1 - \xi^2} S_1(\xi). \tag{77}
\]

Thus the total transferred momentum \(Q\) averaged over directions of scattered photons can be decomposed into two components along basis vectors:
\[
\langle Q \rangle = \langle Q \rangle_1 e_1(\omega, \Omega) + \langle Q \rangle_3 e_3.
\]

3.4.2. Averaging Over Electron Directions

As in previous sections, we choose to measure the azimuth of the electron momentum \(\Phi\) in the frame defined by Equations (65) from the projection of vector \(I_3\) onto the plane perpendicular to \(\omega\). Therefore,
\[
\langle Q \rangle = \langle Q \rangle_1 [\cos \Phi e_1(\omega, I_3) + \sin \Phi e_2(\omega, I_3)] + \langle Q \rangle_3 e_3.
\]

The total momentum transfer averaged over the electron distribution is obtained from Equation (69):
\[
\overline{Q_{s0}(x, \eta)} = \frac{1}{x} \int_0^\infty p \, d\gamma \int_{-1}^1 d\xi \xi \int_0^{2\pi} d\Phi \langle Q \rangle_{s0}(\xi) f_0(\gamma, \eta_0).
\]

\[
\overline{Q_{s0}(x, \eta)} = \frac{2}{k(k + 1)} f_k \Pi_k^1(\eta) \Pi_k^1(\xi), \tag{81}
\]

where \(\Pi_k^1\) are the associated Legendre functions:
\[
\Pi_k^1(u) = \sqrt{1 - u^2}, \quad \Pi_k^2(u) = 3u \sqrt{1 - u^2}. \tag{82}
\]

It is worth mentioning that the isotropic component of the electron distribution \(f_0\) does not contribute to the momentum transferred perpendicular to \(\omega\) by symmetry. Substituting expression (77) into Equation (80), we get the first component of the vector
\[
\overline{Q_{s0}(x, \eta)} = 4\pi x \sum_{k=1}^2 \Pi_k^1(\eta) \int_0^\infty p \, d\gamma \, f_k \Delta^1_{\alpha}(x, \gamma). \tag{83}
\]
where
\[ \Delta_{ik}^1(x, \gamma) = \frac{1}{k(k+1)} \frac{1}{2} \gamma x^2 \int_{-1}^{1} P_k^1(\xi) \xi \langle Q_1 \rangle s_0(\xi) d\xi. \quad (84) \]

Changing the integration variable to \( \xi \), and introducing a set of functions
\[ \chi_{1n}^1(x, \gamma) = \frac{1}{2\gamma p} \int_{x(y-p)}^{x(y+p)} \sqrt{1 - \xi^2} \langle Q_1 \rangle s_0(\xi) \xi^n d\xi \]
\[ = \frac{1}{2\gamma p} \left[ \frac{u^{+n}}{3+n} \Psi_{2+n,1}(xu) - 2\gamma \frac{u^{+n}}{4+n} \Psi_{3+n,1}(xu) \right]_{\xi=\gamma} + \frac{u^{+n}}{5+n} \Psi_{4+n,1}(xu) \bigg|_{\xi=\gamma}, \quad n = 0, 1, \]
we get
\[ \Delta_{11}^1 = 1/2 \chi_{10}^1, \quad \Delta_{12}^1 = 1/2p \left( \gamma \chi_{10}^1 - \chi_{11}^1 \right). \quad (86) \]

Now let us evaluate the component of vector (80) along \( e_3 \). Because \( \langle Q_3 \rangle \) does not depend on azimuth \( \Phi \), the azimuthal integration just gives the averaged electron distribution given by Equation (25). Thus, we get
\[ \overrightarrow{Q}_3 \overline{s}_0(x, \eta) = 4\pi x \sum_{k=0}^{2} P_k(\eta) \int_{1}^{\infty} p \gamma d\gamma f_k(\Delta_{0k} - \Delta_{1k} + \Delta_{1k}^*), \quad (87) \]
where
\[ \Delta_{ik}^*(x, \gamma) = \frac{1}{2\gamma p} \int_{-1}^{1} P_k(\xi) \xi (x_1(1-\mu)) s_0(\xi) d\xi \]
\[ = \sum_{n=0}^{\infty} b_{nk} \chi_{1n}^*, \quad (88) \]
and
\[ \chi_{1n}^*(x, \gamma) = \frac{1}{2\gamma p} \int_{x(y-p)}^{x(y+p)} (x_1(1-\mu)) s_0(\xi) \xi^n d\xi \]
\[ = \frac{1}{2\gamma p} \left[ \frac{u^{+n}}{4+n} \Psi_{3+n,1}(xu) \bigg|_{\xi=\gamma} \right], \quad n = 0, 1, 2. \quad (89) \]

For isotropic electron distribution, the only function of interest is \( \Delta_{11}^* \), which coincides with the function \( \Psi_1^*(x, \gamma) \) introduced by NP94. Now combining Equations (83) and (87), we get the momentum transfer along the symmetry axis of the electron distribution \( I_3 \) and perpendicular to it:
\[ \overrightarrow{Q}_3 = \sqrt{1 - \eta^2} \overrightarrow{Q}_3 + \eta \overrightarrow{Q}_3, \quad (90) \]
\[ \overrightarrow{Q}_3 = -\eta \overrightarrow{Q}_3 + \sqrt{1 - \eta^2} \overrightarrow{Q}_3. \quad (91) \]
Expressions (87), (83), and (90) give the momentum transferred to the electron gas (in terms of one integral over the electron energy) along \( \omega \), perpendicular to that direction as well as along vector \( I_3 \) and perpendicular to it.

Similarly to Equation (62), we can also get the two components of the momentum transfer rate per unit volume:
\[ P_{1,3} = \frac{N_e \sigma_T}{c} \int \frac{d^3x}{x} \int d^2\omega I(x, \eta) \overrightarrow{Q}_{1,3} \overline{s}_0(x, \eta). \quad (92) \]

For mono-energetic electron distribution of the Lorentz factor \( \gamma \) given by Equation (10), the momenta transferred along \( \omega \) and perpendicular to it are
\[ \overrightarrow{Q}_3 \overline{s}_0(x, \gamma, \eta) = x \sum_{k=0}^{2} \frac{f_k(\eta)}{f_0} \left( \Delta_{0k} - \Delta_{1k} + \Delta_{1k}^* \right), \quad (93) \]
\[ \overrightarrow{Q}_1 \overline{s}_0(x, \gamma, \eta) = x \sum_{k=1}^{2} \frac{f_k(\eta)}{f_0} P_k(\eta)(\Delta_{1k}^*), \quad (94) \]
where we kept the notations for the functions \( \overrightarrow{Q}_3 \) and \( \overrightarrow{Q}_1 \), but added the argument \( \eta \). To get the average momentum transferred in a single scattering act, one needs to divide these expression by the total cross section \( \overline{s}_0(x, \gamma, \eta) \). The \( \omega \) component of the transferred momentum for isotropic electrons is shown in Figure 5(a). As shown by NP94, the low-energy (Thomson) limit is given by \( x(1 + 2p^2/3) \). The angular-dependence corrections arising due to the dipole and quadrupole terms in the electron distribution are shown in Figures 5(b) and 5(c), respectively. While the component perpendicular to \( \omega \) is zero for isotropic electrons, a substantial momentum component arises in the anisotropic case. For a large linear term of the electron distribution with \( f_1/f_0 = 1 \), the momentum transferred in that direction is shown in Figure 6(a). Similar results in the case of the quadrupole term with \( f_2/f_0 = 1 \) are shown in Figure 6(b). In the Thomson limit, we get (see Appendix D)
\[ \overrightarrow{Q}_3 \overline{s}_0(x, \gamma, \eta) = x \left[ 1 + \frac{2}{3} \beta^2 - \frac{f_1}{f_0} \beta \frac{4}{15} (4\gamma^2 + 1) \right. \]
\[ \left. + \frac{f_2}{f_0} P_2(\eta) \frac{4}{15} \right] \left( \frac{p^2}{5} \right). \quad (95) \]
\[ \overrightarrow{Q}_1 \overline{s}_0(x, \gamma, \eta) = x \gamma^2 \beta \sqrt{1 - \eta^2} \times \left[ -f_1 \frac{1}{f_0} 3 \left( 1 + \beta^2 \right) + \frac{f_2}{f_0} \eta \beta \right]. \quad (96) \]

4. REDISTRIBUTION FUNCTIONS FOR ANISOTROPIC ELECTRONS

We would like to reduce the expression for the redistribution function (6) to a form suitable for calculations. For the electron distribution of the form (8), this function should depend on the energies of incoming and scattered photons \( x_1 \) and \( x \), the corresponding (cosines of) polar angles \( \eta_1 \) and \( \eta \) as well as the difference in azimuth \( \phi - \phi_1 \) (or cosine of the scattering angle \( \mu \)).

4.1. Integration Over Electron Directions

The three-dimensional integral over \( p \) in Equation (6) disappears due to the \( \delta \)-function. For further simplifications we can also use the identity
\[ \delta(y_1 + x_1 - y - x) = \gamma \delta(x_1 \cdot (P_1 + \Sigma_1) - x_1 \cdot P_1). \quad (97) \]

At this stage, we drop subscript 1 with the electron quantities and get
\[ R(x_1 \rightarrow x) = \frac{3}{16 \pi} \int \frac{d\gamma}{\gamma} \delta(\Gamma) f_e(\gamma, \eta_e) F, \quad (98) \]
\[ Q = \eta \] where
\[ \eta = -\eta \]

\( f \) is the isotropic quantity \( \omega = p \) for anisotropic electrons with \( f_1/f_0 = 1 \) in units of the isotropic quantity \( T^{\text{iso}} \). Solid, dotted, and dashed curves correspond to \( \eta = -1, -0.5, 0, 0.5, 1 \) (pink, green, black, red, and blue curves, respectively).

(c) Same as (b), but for the electron distribution \( g \) with the quadrupole term \( f_2/f_0 = 1 \) for the same \( \eta \) as in panel (b). These are even functions of \( \eta \); the curves from the bottom to the top correspond to \( \eta = 0, 0.5, 1 \). The flat parts of the curves correspond to the Thomson limit given by Equation (95).

Figure 6. (a) Momentum transferred in the direction perpendicular to \( \omega \) (in units of the momentum along \( \omega \) for isotropic distribution) arising from the linear term in the electron distribution \( g \) with \( f_1/f_0 = 1 \). For \( \eta = \pm 1 \), the momentum is zero by symmetry. From the top to the bottom curves correspond to \( \eta = -0.5 \) (green), 0.5 (red), 0 (black). Solid, dotted, and dashed curves correspond to \( p = 1, 10, 100 \), respectively. (b) Momentum transferred in the direction perpendicular to \( \omega \) arising from the quadrupole term in the electron distribution \( g \) with \( f_2/f_0 = 1 \) for the same \( p \) as in panel (a). The curves from bottom to top correspond to \( \eta = \pm 0.25, 0.5, 0.75 \) (black, red, blue curves). The momentum is zero for \( \eta = -1, 0, 1 \) because of the symmetry. The flat parts of the curves correspond to the Thomson limit given by Equation (96).

(A color version of this figure is available in the online journal.)

The transparent way is to choose the polar axis along the direction of the transferred momentum as was proposed by Aharonian & Atoyan (1981; see also Prasad et al. 1986):

\[ n \equiv (x_1 \omega_1 - x \omega) / Q, \tag{100} \]

where

\[ Q^2 = (x_1 \omega_1 - x \omega)^2 = x^2 + x_1^2 - 2x_1x \mu = (x-x_1)^2 + 2q. \tag{101} \]

With this definition we get

\[ \cos \kappa \equiv n \cdot \omega = (x_1 \mu - x) / Q, \quad \sin \kappa = x_1 \sqrt{1 - \mu^2 / Q} \tag{102} \]

and

\[ \cos \alpha \equiv n \cdot l_3 = (x_1 \eta_1 - x \eta) / Q. \tag{103} \]

Thus one of the integration variables becomes \( \cos \theta = \Omega \cdot n \) and another is azimuth \( \Phi \). The redistribution function (98) then can be written as

\[ R(x_1 \rightarrow x) = \frac{3}{16 \pi} \int_1^{\infty} p d \gamma \int_{-1}^{1} d \cos \theta \]
\[ \times \int_0^{2 \pi} d \Phi f_\delta(\gamma, \eta, \Phi) F(\Gamma), \tag{104} \]
where now
\[ \Gamma = \gamma (x_1 - x) - q - pQ \cos \theta. \] (105)

Integrating first over \( \cos \theta \) using the \( \delta \)-function we get
\[ R(x_1 \to x) = \frac{3}{16 \pi} \int_{\gamma_0}^{\infty} d\gamma \frac{1}{Q} \int_0^{2\pi} d\Phi f_e(\gamma, \eta_e) F, \] (106)
where we need to substitute
\[ \cos \theta = \frac{\gamma (x_1 - x) - q}{pQ} \] (107)
into the expressions for \( \eta_e \) and \( F \) (see below). This yields
\[ \sin \theta = \frac{b}{\sqrt{r \, pQ}}, \] (108)
where
\[ b = \sqrt{r \, p^2 Q^2 - [\gamma (x_1 - x) - q]^2}, \quad r = \frac{1 + \mu}{1 - \mu}. \] (109)

The lower limit for the integral over \( \gamma \) comes from the requirement that \( |\cos \theta| \leq 1 \):
\[ \gamma \geq \gamma_s(x, x_1, \mu) = \frac{1}{2} (x - x_1 + Q \sqrt{1 + 2/q}). \] (110)

4.2. Integration Over the Azimuth

In order to calculate the azimuthal integral in Equation (106) we have to express \( \xi \) and \( \xi_1 \) (that enter the expression for \( F \)) and \( \eta_e \) in terms of the integration variable \( \Phi \). We measure the azimuth \( \Phi \) from the projection of \( \omega \) onto the plane normal to \( n \), so that in this system
\[ \omega = (\sin \kappa, 0, \cos \kappa) \] (111)
and the unit vector along the electron momentum is
\[ \Omega = (\sin \theta \cos \Phi, \sin \theta \sin \Phi, \cos \theta). \] (112)

Thus we can express the angle between the electron momentum and \( I_3 \) (see Figure 1) through \( \Phi \):
\[ \eta_e \equiv \Omega \cdot I_3 = \cos \theta \cos \alpha + \sin \theta \sin \alpha \cos (\chi - \Phi), \] (113)
where \( \chi \) is the azimuth of the vector \( I_3 \) in the \( n \) frame. We can also write
\[ \eta = \omega \cdot I_3 = \cos \kappa \cos \alpha + \sin \kappa \sin \alpha \cos \chi, \] (114)
and use this expression to obtain \( \cos \chi \). Substituting it to Equation (113) we thus express the electron polar angle \( \eta_e \) in Equation (8) through the integration variable \( \Phi \).

The kernel \( F \) depends on the four-products \( \xi \) and \( \xi_1 \), which can be rewritten as
\[ \xi_1 = x(\gamma - p \xi), \quad \xi = q + \xi_1, \] (115)
where
\[ \zeta \equiv \Omega \cdot \omega = \cos \theta \cos \kappa + \sin \theta \sin \kappa \cos \Phi. \] (116)

Equation (115) then can be transformed to
\[ \xi_1 = \frac{q}{Q^2}(d_+ - b \cos \Phi), \quad \xi = \frac{q}{Q^2}(d_+ - b \cos \Phi), \] (117)
where we defined
\[ d_+ = \gamma (x + x_1) - x(x - x_1 \mu), \quad d_- = \gamma (x + x_1) + x_1(x - x_1 \mu) = d_+ + Q^2, \] (118)
which have the following property:
\[ (d_+ - b^2)/Q^2 = (\gamma - x)^2 + r \equiv a_1^2, \quad (d_- - b^2)/Q^2 = (\gamma + x_1)^2 + r \equiv a_2^2. \] (119)

Function \( F \) in the azimuthal integral in Equation (106) is an even function of \( \Phi \). Therefore the terms in \( f_e \) containing \( \sin \Phi \) give zero contribution. Neglecting these terms we can express the azimuthal integral as
\[ \int_0^{2\pi} \eta_e F d\Phi = \int_0^{2\pi} \eta_e F d\Phi, \] (120)
\[ \int_0^{2\pi} \eta_e F d\Phi = \int_0^{2\pi} \eta_e F d\Phi, \] (121)
where
\[ \eta_e = \cos \theta \cos \alpha + \sin \theta \sin \alpha \cos \chi \cos \Phi, \] (122)
\[ \eta_e^2 = \cos^2 \theta \cos^2 \alpha + \sin^2 \theta \sin^2 \alpha \sin^2 \chi + 2 \sin \theta \sin \alpha \cos \theta \cos \alpha \cos \chi \cos \Phi + \sin^2 \theta \sin^2 \alpha \cos \chi \cos \Phi. \] (123)

Thus expansion (8) (with \( \eta_e \) and \( \eta_e^2 \) substituted by \( \eta_e \) and \( \eta_e^2 \), respectively) is a quadratic function of \( \cos \Phi \). Expressing
\[ \cos \Phi = -\frac{Q^2 \xi + \xi_1}{2b} q + \frac{d_+ + d_-}{2b}, \] (124)
and using the identity \( \xi = \xi_1 + q \), we obtain an expansion of \( f_e \) that is symmetric in \( \xi \) and \( \xi_1 \):
\[ f_e(\gamma) = c_0 + c_{\Sigma} \xi + \xi_1 q + c_{t1} \xi_1 \xi_1. \] (125)

The coefficients \( c_0, c_{\Sigma}, \) and \( c_{t1} \) can be represented in the form
\[ c_0 = f_0 + c_{01} f_1 + c_{02} f_2, \]
\[ c_{\Sigma} = c_{11} f_1 + c_{12} f_2, \]
\[ c_{t1} = c_{22} f_2, \] (126)
where the coefficients in front of \( f_{0,1,2} \) can easily be derived after lengthy but straightforward calculation:
\[ c_{01} = \frac{2 \rho - \epsilon + \epsilon_1}{2 \rho (1 + \mu)}, \]
\[ c_{11} = -\frac{\epsilon + \epsilon_1}{2 \rho (1 + \mu)}, \]
\[ c_{02} = \frac{3}{4 \rho^2 (1 + \mu)^2} [(\epsilon - \rho^2) + (\epsilon_1 + \rho^2) + \lambda (a_2^2 + a_1^2)] - \frac{1}{2}, \]
\[ c_{12} = -\frac{3}{4 \rho^2 (1 + \mu)^2} [(\epsilon + \epsilon_1) (2 \rho - \epsilon + \epsilon_1) + \lambda (d_+ + d_-)], \]
\[ c_{22} = \frac{3}{2 \rho^2 (1 + \mu)^2} [(\epsilon + \epsilon_1)^2 + \lambda Q^2]. \] (127)
Here, we defined
\[ \epsilon \equiv x(\eta_1 - \eta), \quad \epsilon_1 \equiv x_1(\eta - \eta_1), \quad \rho = \gamma(\eta + \eta_1), \]
\[ \lambda = \mu^2 + \eta^2 + \eta_1^2 - 2\mu\eta\eta_1 - 1. \] 
(128)

The redistribution function (106) is then expressed as
\[ R(x_1, \omega_1 \to x, \omega) = \frac{3}{8} \int_{\gamma(x_1, \omega_1, \omega)}^{\infty} dy \gamma [c_0 R_0 + c_2 R_2 + c_{11} R_{11}], \] 
(129)
where we have introduced three functions
\[ R_0(x_1, \omega_1, \mu, \gamma) = \frac{1}{\pi Q} \int_{0}^{\pi} F d\Phi, \] 
(130)
\[ R_2(x_1, \omega_1, \mu, \gamma) = \frac{1}{\pi Q^2} q \int_{0}^{\pi} (\xi + \eta_1) F d\Phi, \] 
(131)
\[ R_{11}(x_1, \omega_1, \mu, \gamma) = \frac{1}{\pi Q^2 q^2} \int_{0}^{\pi} \xi_1 F d\Phi. \] 
(132)

Alternatively, we can represent the redistribution function as a sum of three terms arising from the corresponding three terms in the electron distribution:
\[ R(x_1, \omega_1 \to x, \omega) = \frac{3}{8} \int_{\gamma(x_1, \omega_1, \omega)}^{\infty} dy \gamma [f_0 R_0 + f_1 R_1 + f_2 R_2], \] 
(133)
where
\[ R_1(x_1, \omega_1, \omega_1, \mu, \gamma) = c_{01} R_0 + c_{11} R_2 \]
\[ R_2(x_1, \omega_1, \omega_1, \mu, \gamma) = c_{02} R_0 + c_{12} R_2 + c_{22} R_{11}. \] 
(134)

Using the Klein–Nishina cross section (2) in the form
\[ F = 2 + q^2 - 2q - 2 \left( \frac{1}{\xi_1} - \frac{1}{\xi} \right) + \frac{1}{\xi^2}, \] 
(135)
and remembering that \( \xi = \xi_1 + q \), we see that the integrals (130)–(132) involve integrals of types
\[ \int_{0}^{\pi} \xi^s d\Phi, \quad \int_{0}^{\pi} \xi_1^s d\Phi, \] 
(166)
where \( s = -2, ..., 2 \). The integrals over non-negative powers of \( \xi \) and \( \xi_1 \) are trivial. For the negative powers, using Equations (117) and (119) we get (see Nagirner & Poutanen 1993, for details)
\[ \int_{0}^{\pi} \frac{d\Phi}{\xi_1} = \frac{\pi Q}{q} \frac{1}{a_-}, \quad \int_{0}^{\pi} \frac{d\Phi}{\xi^s} = \frac{\pi Q}{q^2} \frac{d}{a_-^s} \] 
(137)
and similar equations for \( \xi \) which we get by substituting \( \xi, a_+ \), and \( d_+ \) for \( \xi_1, a_- \), and \( d_- \), respectively.

After some straightforward algebra we get the expressions for \( R_0, R_2 \) and \( R_{11} \):
\[ R_0 = \frac{2}{Q} \left( \frac{q^2 - 2q}{q^2} + \left( \frac{1}{a_-} - \frac{1}{a_+} \right) + \frac{d_-}{a_-^2} \right), \] 
(138)
which was obtained by Aharonian & Atoyan (1981) (see also Nagirner & Poutanen 1993),
\[ R_2 = \frac{2}{Q^3} (d_- + d_+) + \left( \frac{1 - 2}{q} \right) \left( \frac{1}{a_-} + \frac{1}{a_+} \right) + \frac{d_-}{a_-^2}, \] 
(139)
and
\[ R_{11} = \frac{2}{Q^3} \left( d_- + d_+ \right) + \left( 1 - \frac{2}{q} \right) \left( \frac{1}{a_-} + \frac{1}{a_+} \right) + \frac{d_-}{a_-^2} \] 
(140)

Equation (129) (or alternatively Equations (133) and (134)) together with our computed redistribution functions (138)–(140) and the coefficients (126)–(128) give the full analytical solution for the redistribution function describing scattering of arbitrary photons from the electron gas which anisotropy can be described by Equation (8).

### 4.3. Alternative Redistribution Functions

An alternative expression for the redistribution function \( R(x_1 \to x) \) can be obtained if we compute the moments
\[ R_\phi(x_1, \omega_1, \mu, \gamma) = \frac{1}{\pi Q} \int_{0}^{\pi} \cos \Phi \right) F d\Phi \]
\[ = \frac{q^2 - 2q - 2 \frac{1}{q} \left( \frac{d_-}{a_-} - \frac{d_+}{a_+} \right) + q^2 \left( \frac{1}{a_-^2} + \frac{1}{a_+^2} \right)}{q^2 b^2}, \] 
(141)
\[ R_{\phi\phi}(x_1, \omega_1, \mu, \gamma) = \frac{1}{\pi Q} \int_{0}^{\pi} \cos^2 \Phi \right) F d\Phi \]
\[ = \frac{1}{q^2 b^2} \left( \frac{d_-}{a_-} + \frac{d_+}{a_+} \right) + \frac{1}{q^2} \left( \frac{d_-}{a_-^2} + \frac{d_+}{a_+^2} \right). \] 
(142)

The expressions for \( R_1 \) and \( R_2 \) then take the form
\[ R_1 = d_{01} R_0 + d_{11} R_\phi \]
\[ R_2 = d_{02} R_0 + d_{12} R_\phi + d_{22} R_{\phi\phi} \] 
(143)
where
\[ d_{01} = \cos \theta \right) \cos \alpha, \]
\[ d_{11} = \sin \theta \right) \sin \alpha \right) \cos \chi = \frac{b}{p(1 + \mu)^2}, \]
\[ d_{02} = \frac{2}{q} \left( \frac{\cos^2 \theta \right) \cos^2 \alpha + \sin^2 \theta \right) \sin^2 \alpha \right) \sin^2 \chi - \frac{1}{2}, \]
\[ d_{12} = \frac{3}{2} \left( d_{01} \right)^2 + \sin^2 \theta \right) \cos^2 \alpha \right) - \frac{1}{2}, \]
\[ d_{22} = \frac{3}{2} \sin^2 \theta \right) \sin^2 \alpha \right) \cos 2\chi \]
\[ = \frac{3}{2} \left( d_{11} \right)^2 - \sin^2 \theta \right) \cos^2 \alpha \right). \] 
(144)

### 4.4. Approximate Redistribution Functions

Approximate forms of Equations (138)–(140) can be obtained by making certain simplifying assumptions about the scattering. For example, in the Thomson regime in the electron rest frame...
the Klein–Nishina kernel $F$ is just $1 + \mu_\perp^2$. Assuming further isotropic scattering in that frame and substituting $F$ by $4/3$, we now get for the integrals (130)–(132):

$$R_0 \approx \frac{4}{3Q},$$

(145)

$$R_2 \approx \frac{4}{3Q^3}(d_+ + d_-),$$

(146)

$$R_{11} \approx \frac{4}{3Q^3}(d_- + \frac{b^2}{2}).$$

(147)

The expression for $R_0$ was derived by Arutyunyan & Nikogosyan (1980). For the alternative functions (141), (142), we then have

$$R_\phi = 0, \quad R_{\phi\phi} = \frac{2}{3Q}.$$  

(148)

These then give

$$R_1 \approx d_{01}R_0 = \cos \theta \cos \alpha R_0,$$

(149)

$$R_2 \approx \left(d_{02} + \frac{1}{2}d_{22}\right)R_0 = P_2(\cos \theta)P_2(\cos \alpha)R_0,$$

(150)

with $\cos \theta$ and $\cos \alpha$ given by Equations (107) and (103), respectively. The approximate expressions are better than 50% accurate in the Thomson regime for $x_1\gamma < 0.1$ at all scattered photon energies.

### 4.5. Relation to the Mean Powers of Photon Energies

The relation between the redistribution function averaged over any electron distribution and the mean powers of photon energies follows directly from their definitions (6), (7), and (36):

$$\overline{x_1^2} R_0(x, \eta) = \frac{1}{x} \int x_1^{r1}dx_1 \int d^2\omega_1 R(x \rightarrow x_1).$$

(151)

This relation is valid for any electron distribution. Comparing Equations (133) and (49), we get a relation between the functions depending on the electron energy:

$$\Delta_{jk}(x, \gamma)P_k(\eta) = \frac{3}{32\pi\gamma^r px^mp_{r1}x_1^{r1}}\int_{x^{r1}(x, \gamma)} x_1^{r1}dx_1 \times \int R_k(x_1, x; x_1; x; \mu; \gamma) d^2\omega_1,$$

(152)

where $\eta_1 = \eta_\mu + \sqrt{1 - \eta^2}\sqrt{1 - \mu^2} \cos \Phi$ and $R_0$ depends only on the scattering angle $\mu$, but not $\eta, \eta_1$. The integrals over the solid angle can be represented as the integrals over $dx_1$ and $d\Phi$, and the limits on $\mu, \mu(x_1, x, \gamma)$, and $\mu^*(x_1, x, \gamma)$ are given by Equations (F6)–(F8) with the arguments $x$ and $x_1$ reversed. Using Equations (87) and (84), we also get

$$\Delta_{ik}^*(x, \gamma)P_\phi(\eta) = \frac{3}{32\pi^r px^m} \int x_1^2 dx_1 \int (1 - \mu) R_k d^2\omega_1,$$

(153)

$$\Delta_{ik}^*(x, \gamma)P_\phi^1(\eta) = \frac{3}{32\pi^r px^m} \int x_1^2 dx_1 \int \sqrt{1 - \mu^2} \cos \Phi R_k d^2\omega_1.$$  

(154)

In order to check the accuracy of our derivations we compared the left-hand sides of Equations (152)–(154) to the right-hand sides, where the integrals were performed numerically and obtained consistent results.

### 5. APPLICATIONS

#### 5.1. Examples of Redistribution Functions

Now we demonstrate the properties of the derived redistribution functions. We consider a volume filled by electrons with the angular distribution given by Equation (8). The emissivity in a direction $\omega$ at energy $x$ can be obtained from the radiative transfer Equation (5) and is given by the integral over the redistribution function

$$\epsilon(x) = \sigma T N_e x^2 \int_0^{\infty} \frac{dx_1}{x_1^2} \int d^2\omega_1 I(x_1) R(x_1, \omega_1 \rightarrow x, \omega),$$

(155)

where $I(x_1) = 2m_e(c^2/h^2)\chi_1^2 n(x_1)$ is the specific intensity of the incident radiation normalized to the photon density as

$$N_{ph} = \frac{1}{m_e c^3} \int d^2\omega \int I(x) \frac{dx}{x}.$$  

(156)

Let us consider mono-energetic (with energy $\gamma$) electron distribution (10). Consider also a monochromatic source of isotropic seed photons at energy $x_1$ with total photon number density $N_{ph}$. According to Equation (133) we can write the emissivity at an observer direction $\eta$ for a given scattering angle as

$$\overline{\epsilon}(x, \eta, \mu) = \frac{3}{32\pi m_e c^3} \sigma T N_e N_{ph} x_1^2 \frac{1}{x_1^2} \gamma^2 \int R_0 + \frac{f_1}{f_0} R_1 + \frac{f_2}{f_0} R_2,$$

(157)

which is related to the scattering angle-averaged emissivity as $\epsilon(x) = \frac{1}{x} \int \overline{\epsilon}(x, \eta, \mu) d\mu$, and where (for $k = 1, 2$)

$$\overline{R}_k(x, \eta; x_1; \mu; \gamma) = \frac{1}{2\pi} \int_0^{2\pi} d\Phi R_k(x, \eta; x_1; \eta_1; \mu; \gamma).$$

(158)

and $\eta_1 = \eta_\mu + \sqrt{1 - \eta^2}\sqrt{1 - \mu^2} \cos \Phi$. These functions obviously possess symmetry properties:

$$\overline{R}_1(x, -\eta; x_1; \mu; \gamma) = -\overline{R}_1(x, \eta; x_1; \mu; \gamma),$$

(159)

$$\overline{R}_2(x, -\eta; x_1; \mu; \gamma) = R_2(x, x_1; \mu; \gamma).$$

(160)

We compute separately the emissivities resulting from three terms in the electron distribution, i.e., functions $R_0, \overline{R}_1, \overline{R}_2$ (see Equation (157)), and show in Figures 7–10 the function $R_0$ multiplied by $x$ (i.e., quantity proportional to the photon number emissivity) for better visibility as well as the ratios $\overline{R}_1/R_0$ and $\overline{R}_2/R_0$. The main behavior of the functions can easily be understood using formulae (149)–(150) derived in the Thomson limit and isotropic scattering approximation. Averaging them over the azimuth and using relation $\overline{R}_k(\eta_\mu) = P_k(\eta)\overline{R}_k(\mu)$, we get

$$\overline{R}_1 \approx \frac{x_1^2}{Q} \frac{1}{x_1^2} \frac{x_1 \mu - x}{Q},$$

(161)

$$\overline{R}_2 \approx \frac{x_1^2}{Q} \frac{P_2(\mu) - 2x_1 \mu + x^2}{Q^2} P_2(\eta) P_2(\cos \theta).$$

(162)

These approximate expressions become extremely accurate for high $p$ (i.e., accuracy is about $10^{-3}$ at $p = 100$).

For a small electron momentum $p = 0.1$ and low photon energies $x_1 = 10^{-2}$, the exact redistribution functions are shown...
in Figure 7. In this regime, scattering is nearly coherent with the scattered photon energies bounded by (see Equation (F2)) \( x^\pm/x_1 \approx 1 \pm p\sqrt{2(1-\mu)} \). In this regime, \( x-x_1 \ll q \ll x, x_1 \) and \( \cos \theta \approx 1 - x_1 x/pQ \) is a nearly linear function of \( x/x_1 \), because \( Q/x_1 \approx \sqrt{2(1-\mu)} \). For \( \mu \) not too close to 1, the azimuth averaging of \( \cos \alpha \) gives \( -\eta \sqrt{1-\mu}/2 \) and thus \( \mathcal{R}_1/R_0 \approx (1-x/x_1)\eta/(2p) \). For \( \eta = 0 \), the function is always zero, because of the symmetry. Similarly, the nearly quadratic dependence of \( \mathcal{R}_2/R_0 \) on energy results from the \( \cos^2 \theta \) term, while at \( \mu \approx 1/3 \) the function becomes more complicated because of the cancellation in the \( P_5(\cos \alpha) \) term.

In the opposite limit of the relativistic electrons (see Figures 9 and 10), the approximation (145) for the function \( R_0 \) works fine up to \( x_1/x \lesssim 0.1 \), while as said above the ratios \( \mathcal{R}_1/R_0 \) and \( \mathcal{R}_2/R_0 \) are very close to those given by Equations (161) and (162) for any photon and electron energies. At small scattered photon energies \( x \ll x_1 \), \( x R_0 \propto x/x_1 \) and

\[
\mathcal{R}_1/R_0 \approx \eta \mu \cos \theta, \quad \mathcal{R}_2/R_0 \approx P_2(\eta) P_2(\mu) P_2(\cos \theta),
\]

with \( \cos \theta \approx 1 - x/x_1 \). At high scattered photon energies \( x \gg x_1 \), the photons are scattered at large angles in the electron rest frame and therefore they are beamed in the direction of the incoming electrons. In that case, the angular distribution of the scattered photons resemble that of the electrons. In this regime \( x R_0 \propto \text{const} \) and \( \mathcal{R}_1/R_0 \approx \eta \) and \( \mathcal{R}_2/R_0 \approx P_2(\eta) \), which gives the flat dependences clearly seen in Figures 9 and 10, and \( \epsilon(x) \propto f_\epsilon(\gamma, \eta) x/x_1 \).

**5.2. Sunyaev–Zeldovich Effect**

Let consider a cloud of isotropic Maxwellian electrons of temperature \( \Theta \equiv kT/e^2 \), which moves with velocity \( c \beta \) (corresponding Lorentz factor \( \Gamma_\beta \)) through the isotropic CMB of temperature \( \Theta_{\text{cmb}} \equiv kT_{\text{cmb}}/m_\epsilon e^2 \). We compute the thermal and kinematic Sunyaev–Zeldovich effects (Zeldovich & Sunyaev 1969; Sunyaev & Zeldovich 1972), i.e., the spectrum of the scattered radiation (and resulting deviations from the blackbody) as a function of \( \Theta \) and the angle between the line of sight and the direction of motion.

One approach would be to make a Lorentz transformation of the incident radiation to the comoving frame, compute the Compton scattered radiation using the kernel corresponding to isotropic electron distribution, and then to Lorentz transform it back to the observer frame. Another way is to compute the electron distribution in the observer frame, approximate it by expansion (8) and directly compute the Compton scattered radiation in the observer frame. The second approach might be favorable from numerical point of view if the object velocity is variable in space and/or time, as allows to pre-compute redistribution functions at a fixed grid of angles and photon energies.
where $\tau$ is the Lorentz invariant optical depth for Thomson scattering, in the limit of small optical depth, we get the correction to the blackbody spectrum:

$$S(x, \eta) = \frac{n_{bb}(x)}{\exp(x_\eta) - 1},$$  \hspace{1cm} (165)

where $x_\eta = x/\Theta_{\text{emb}} = h\nu/kT_{\text{emb}}$. From the radiative transfer equation (5), in the limit of small optical depth, we get the correction to the blackbody spectrum:

$$\Delta n(x, \eta) = n(x, \eta) - n_{bb}(x) = -\tau_T \bar{n}_{o}(x') n_{bb}(x) + S(x, \eta),$$  \hspace{1cm} (166)

where $\tau_T$ is the Lorentz invariant optical depth for Thomson scattering.

\begin{align*}
S(x, \eta) &= \frac{1}{\eta} \int_0^\infty dx_1' \int d^2\omega_1' \mathcal{R}^{o}(x_1' \rightarrow x') n_{bb}(x_1) \exp(-\gamma'/\Theta) \frac{4\pi}{\Gamma_2} \frac{1}{1 - \beta_b \eta'} \\
&= \frac{1}{\eta} \int_0^\infty dx_1' \int d^2\omega_1' \mathcal{R}^{o}(x_1' \rightarrow x') n_{bb}(x_1) \exp(-\gamma'/\Theta) \frac{4\pi}{\Gamma_2} \frac{1}{1 - \beta_b \eta'} \\
&= \frac{N_e}{4\pi} \frac{1}{\Gamma_2} \frac{1}{1 - \beta_b \eta'} \exp(-\gamma'/\Theta) \exp\left[p \Gamma_b \eta_c / \Theta\right].
\end{align*}

is the source function, and we used here the fact that the photon occupation number is Lorentz invariant. The energy transformation is given by Doppler shift $x = x'\mathcal{D}$ and $x_1 = x_1'\mathcal{D}_1$ with the Doppler factors

$$\mathcal{D} = \frac{1}{\Gamma_b(1 - \beta_b \eta')}, \quad \mathcal{D}_1 = \Gamma_b(1 + \beta_b \eta').$$  \hspace{1cm} (168)

The relation between the angles is given by the aberration formula

$$\eta' = \frac{\eta - \beta_b \eta_c}{1 - \beta_b \eta_c}.$$  \hspace{1cm} (169)

We note here that $\bar{n}_{o}$ is equal to unity with high accuracy, because scattering is in the deep Thomson regime. The calculation of the redistribution function $R$ involves numerical integration over the Maxwellian distribution (see Equation (133); note that $f_1 = f_2 = 0$) $f_0 = f'_0(\gamma) = f'_0(p) / N_e$ given by Equation (164). Thus the source function (167) involves four-dimensional integral to be taken numerically, which is rather time consuming.

5.2.2. Scattering in the External Frame

We can also compute the same effect directly in the external frame. The electron Lorentz factor in the comoving frame is

$$\gamma = \frac{\gamma - \beta_b \eta_c}{1 - \beta_b \eta_c},$$  \hspace{1cm} (170)

where $\eta_c$ is the cosine of the angle between the electron momentum and the direction of cloud motion. Because the distribution function is Lorentz invariant, we easily get the electron distribution in the external frame:

$$f_e(p) = f'_e(p') = \frac{N_e}{4\pi} \frac{1}{\Gamma_2} \frac{1}{1 - \beta_b \eta_c} \exp(-\gamma') \exp(p \Gamma_b \eta_c / \Theta) \exp(p \Gamma_b \eta_c / \Theta).$$  \hspace{1cm} (171)

We note that $\bar{n}_{e}$ is equal to unity with high accuracy, because scattering is in the deep Thomson regime. The calculation of the redistribution function $R$ involves numerical integration over the Maxwellian distribution (see Equation (133); note that $f_1 = f_2 = 0$) $f_0 = f'_0(\gamma) = f'_0(p) / N_e$ given by Equation (164). Thus the source function (167) involves four-dimensional integral to be taken numerically, which is rather time consuming.

5.2.2. Scattering in the External Frame

We can also compute the same effect directly in the external frame. The electron Lorentz factor in the comoving frame is related to the electron four-momentum in the external frame as

$$\gamma = \frac{\gamma - \beta_b \eta_c}{1 - \beta_b \eta_c},$$  \hspace{1cm} (170)

where $\eta_c$ is the cosine of the angle between the electron momentum and the direction of cloud motion. Because the distribution function is Lorentz invariant, we easily get the electron distribution in the external frame:

$$f_e(p) = f'_e(p') = \frac{N_e}{4\pi} \frac{1}{\Gamma_2} \frac{1}{1 - \beta_b \eta_c} \exp(-\gamma') \exp(p \Gamma_b \eta_c / \Theta) \exp(p \Gamma_b \eta_c / \Theta).$$  \hspace{1cm} (171)
can be taken analytically:

\[
\Delta n(x, \eta) = -\tau_T \bar{\Delta} n(x, \eta) + S(x, \eta).
\]

The scattering cross section is given by Equation (26) and in the Thomson limit it is just \(S(x, \eta, \gamma) \approx 1 - \beta_n \gamma\). The source function is now

\[
S(x, \eta) = \tau_T \frac{1}{x} \int_0^\infty x_1 n_{bb}(x_1) d\omega \int d^3p R(x_1 \to x),
\]

where the redistribution function \(R\) given by Equation (133) is averaged over directions of incident photons, but still depends on the scattered photon direction \(\eta\). This form of the source function is more favorable compared to Equation (167) from numerical point of view, as it can be tabulated in advance at a given grid of photon energies and angles. Computed directly it still involves numerical calculations of four-dimensional integrals.

5.2.3. Isotropic Scattering in the Thomson Regime in the Electron Rest Frame

In the Thomson limit (as in the case of the Sunyaev–Zeldovich effect), the calculations in the external frame can be dramatically simplified. We can use the azimuthally averaged approximate expression (145), (161), and (162) for the redistribution functions

\[
\int d^2\omega_1 R(x_1 \to x) = 2\pi \int_{-1}^{1} d\mu \frac{3}{8} f^\infty \gamma \int_{-1}^{1} d\gamma \frac{f_{BB}(\gamma)}{\gamma},
\]

where the proportionality coefficient \(C = R_0 \exp(-\gamma_e/\Theta)/[4\pi K_2(1/\Theta)]\). The zeroth order term in \(\beta_n\) was derived by Poutanen (1994; see also Poutanen & Svensson 1996).

Evaluation of the source function (174) now involves only two numerical integrations over the photon energy \(x_1\) and cosine of the scattering angle \(\mu\), reducing the computational time by 2–3 orders of magnitude.

For all three methods we numerically compute the correction function for the blackbody intensity

\[
\Delta I(x) = \frac{1}{\tau_T} x_1^3 \Delta n(x, \eta),
\]

and compare the results of calculations in Figure 11. The three different methods give nearly identical results.

6. CONCLUSIONS

We have developed the exact analytical theory of Compton scattering by anisotropic distribution of electrons that can be represented by a second-order polynomial over the cosine of some angle (dipole and quadrupole anisotropies). For the total cross section, we reduce the nine-dimensional integral to a single integral over the electron energy. Analogous expressions have been derived for the mean energy of the scattered photons and its dispersion. We also obtained analytical expressions for the radiation pressure force acting on the electron gas. These moments can be used for analytical estimations as well as for the numerical solutions of the kinetic equations in the Fokker–Planck approximation (see, e.g., Vurm & Poutanen 2009).

Furthermore, the expression for the redistribution function describing angle-dependent Compton scattering by anisotropic
The total cross section and mean powers of energy of scattered photons are expressed through the functions of one variable. Exact analytical formulae valid for any photon and electron energies are derived in the case of monoenergetic electrons. We have also derived approximate expressions for the redistribution function, assuming isotropic scattering in the electron rest frame, which are very accurate in the case of relativistic electrons interacting with soft photons in the Thomson regime.

We applied the developed formalism to the accurate calculations of the thermal and kinematic Sunyaev–Zeldovich effects for arbitrary electron distributions. A very similar problem arises in outflowing coronae around accreting black holes and neutron stars, where the bulk motion causes electron anisotropy. Another application could be a computation of the radiative transport in the synchrotron self-Compton sources with ordered magnetic field, where the electron distribution can have strong deviations from the isotropy because of pitch angle-dependent cooling. These problems will be considered in future publications.

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APPENDIX A

FUNCTIONS $s_j$ AND $S_j$

All function $s_j$ and $S_j$ can be expanded to the series, which converge in the region $\xi < 1/2$. It is easy to show that

$$s_j(\xi) = \sum_{n=0}^{\infty} a_{jn} (-2 \xi)^n, \quad S_j(\xi) = \sum_{n=0}^{\infty} A_{jn} (-2 \xi)^n,$$

(A1)

where

$$a_{0n} = \frac{3}{8} \left[ n + 2 + \frac{2}{n+1} + \frac{8}{n+2} - \frac{16}{n+3} \right],$$

$$a_{1n} = \frac{1}{8} \left[ n(n+5) + \frac{24}{n+3} \right], \quad a_{2n} = \frac{1}{32} (n^3 + 9n^2 + 22n + 32).$$

(A2)

Using Equations (48), one can obtain the expressions for the coefficients $A$ through $a$:

$$A_{1n} = 2(a_{1n+1} - a_{0n+1}), \quad A_{2n} = 2(A_{1n+1} - a_{1n+1}), \quad A_{3n} = 2(a_{2n+1} - a_{1n+1}),$$

$$A_{4n} = 2(A_{3n+1} - A_{1n+1}), \quad A_{5n} = 3A_{4n} - 4A_{3n}, \quad A_{7n} = A_{3n} - A_{4n}/2, \quad A_{6n} = a_{2n} - 3A_{7n}.$$

(A3)

The coefficients for $n = 0, 1, 2$ are presented in Table 1.

**Table 1**

| $n$ | $a_{0n}$ | $a_{1n}$ | $a_{2n}$ |
|-----|----------|----------|----------|
| 0   | 1        | 1        | 13/10    |
| 1   | 1        | 3/2      | 47/20    |
| 2   | 1        | 2        | 15/4     |

APPENDIX B

AUXILIARY FUNCTIONS $\psi_{ij}$ AND $\Psi_{ij}$

The total cross section and mean powers of energy of scattered photons are expressed through the functions of one variable

$$\psi_{ij}(\xi) = \frac{i + 1}{\xi^{i+1}} \int_{0}^{\xi} x^i s_j(x) dx, \quad \psi_{-ij}(\xi) = \frac{1}{\xi} \int_{0}^{\xi} \left[ 1 - s_j(x) \right] \frac{dx}{x}, \quad \Psi_{ij}(\xi) = \frac{i + 1}{\xi^{i+1}} \int_{0}^{\xi} x^i S_j(x) dx.$$

(B1)

Calculations of functions $\psi_{ij}$ involve integrals of the following types:

$$\int dx \ x^n \ln(1 + 2x), \quad \int dx \ \frac{x^n}{(1+2x)^\ell}, \quad g(\xi) = \int_{0}^{\xi} \ln(1 + 2x) \ \frac{dx}{x},$$

(B2)

where $m = -1, 0, 1, 2$ and $n, l = 1, 2, 3, 4$. All integrals are elementary except $g(\xi)$, which is described in details in Appendix C. The explicit expressions for the functions $\psi_{ij}$ are the following:

$$\psi_{-10}(\xi) = \frac{1}{8\xi} \left[ \frac{4}{\xi^2} + \frac{2}{\xi^3} + \left( \frac{8 + 3}{\xi^2} - \frac{2}{\xi^3} \right) l_\xi - 3 L_\xi - \frac{11}{3} \right],$$

$$\psi_{-11}(\xi) = \frac{1}{8\xi} \left[ \frac{7}{3} + \frac{2}{\xi^2} + \left( \frac{8 + 1}{\xi^3} \right) l_\xi - 4 L_\xi - R_\xi \right].$$
\[ \psi_{-2}(\xi) = \frac{1}{16 \xi^4} \left[ 11 + 16 \xi - 7 \xi R_e - 3 \xi^2 - R_e^2 \right], \]
\[ \psi_0(\xi) = \frac{3}{8 \xi^4} \left[ g(\xi) - \frac{2}{\xi} + \left( \frac{1}{2} + \frac{3}{2 \xi^2} + \frac{1}{\xi^2} \right) \xi \right] - \frac{3 \xi}{2}, \]
\[ \psi_1(\xi) = \frac{3}{8 \xi^2} \left[ \frac{1}{\xi} - \frac{2}{3} + \left( \frac{4}{3} - \frac{1}{2 \xi^2} \right) \xi - \frac{1}{3} R_e - \frac{1}{6} R_e^2 \right], \]
\[ \psi_2(\xi) = \frac{1}{32 \xi^3} \left[ \frac{7}{2} + 9 \xi - R_e - \frac{3}{2} R_e^2 - R_e^3 \right], \]
\[ \psi_{10}(\xi) = \frac{3}{4 \xi^2} \left[ \left( \xi + \frac{9}{2} + \frac{2}{\xi} \right) \xi - 4 - \xi + \xi^2 R_e - 2 g(\xi) \right], \]
\[ \psi_{11}(\xi) = \frac{1}{4 \xi^2} \left[ 6 + 4 \xi - 3 \left( \frac{3}{2} + \frac{1}{\xi} \right) \xi - \xi (1 + \xi) R_e^2 \right], \]
\[ \psi_{12}(\xi) = \frac{1}{16 \xi^2} \left[ \frac{1}{2} + 9 \xi - 4 \xi - R_e + \frac{1}{2} R_e^3 \right], \]
\[ \psi_{20}(\xi) = \frac{9}{32 \xi^3} \left[ 25 \xi + (2 \xi^2 - 8 \xi - 5) \xi + \xi R_e - 8 g(\xi) \right], \]
\[ \psi_{21}(\xi) = \frac{9}{8 \xi^3} \left[ g(\xi) + \frac{2}{3} \xi - \frac{3}{2} \xi - \frac{1}{4} \xi - \xi^2 + \frac{1}{R_e^2} \right], \]
\[ \psi_{22}(\xi) = \frac{3}{64 \xi^3} \left[ 9 \xi^2 - 8 \xi + 5 \xi - \xi (2 + 11 \xi + 10 \xi^2) \right], \]
\[ \psi_{30}(\xi) = \frac{3}{12 \xi^2} \left[ \xi^3 + \frac{31}{2} \xi^2 + \frac{31}{4} \xi + \frac{2 \xi^3 - 6 \xi^2 - 12 \xi - 7}{2} \right] \xi - \frac{3}{4} \xi R_e, \]
\[ \psi_{31}(\xi) = \frac{1}{24 \xi^2} \left[ 16 \xi^3 - 27 \xi^2 - 45 \xi + 3(7 + 12 \xi) \xi + 3 \xi (1 + 3 \xi) R_e^2, \right], \]
\[ \psi_{32}(\xi) = \frac{1}{64 \xi^2} \left[ 6 \xi^3 - 4 \xi^2 + 5 \xi - 3 \xi + \xi (1 + 4 \xi + 2 \xi^2) \right] R_e^3, \]
\[ \psi_{40}(\xi) = \frac{15}{128 \xi^2} \left[ \xi^4 + \frac{230}{9} \xi^3 + \frac{23}{6} \xi^2 - \frac{17}{12} \xi + \left( 4 \xi^4 - \frac{32}{3} \xi^3 - 16 \xi^2 + \frac{11}{12} \right) \xi + \xi R_e \right], \]
\[ \psi_{41}(\xi) = \frac{5}{64 \xi^2} \left[ 18 \xi^4 - 12 \xi^3 - 9 \xi^2 + 8 \xi + 3 (4 \xi^2 - 1) \xi - (2 + 5 \xi) R_e^2, \right], \]
\[ \psi_{42}(\xi) = \frac{5}{256 \xi^2} \left[ 18 \xi^4 - \frac{32}{3} \xi^3 + 10 \xi^2 - 12 \xi + \frac{11}{2} \xi + (1 + 7 \xi + 14 \xi^2) \right] R_e^3 \right]. \] (B3)

where \( \xi = \ln(1 + 2 \xi) \) and \( R_e = 1/(1 + 2 \xi) \).

The explicit expressions for \( \Psi_{ij} \) can be obtained using definitions (48) for \( S_j \):

\[
\begin{align*}
\Psi_{11} &= 2 \left( \psi_{00} - \psi_{01} \right) / \xi, \\
\Psi_{16} &= 3 \Psi_{13} + 3 \Psi_{14} / 2, \\
\Psi_{23} &= 3 \left( \psi_{11} - \psi_{12} \right) / 2 \xi, \\
\Psi_{26} &= 2 \Psi_{23} + 3 \Psi_{24} / 2, \\
\Psi_{33} &= 4 \left( \psi_{21} - \psi_{22} \right) / 3 \xi, \\
\Psi_{37} &= \Psi_{33} - \Psi_{34}, \\
\Psi_{42} &= 5 \left( \psi_{31} - \psi_{33} \right) / 4 \xi, \\
\Psi_{45} &= 3 \Psi_{44} - 4 \Psi_{43}, \\
\Psi_{54} &= 6 \left( \psi_{41} - \psi_{43} \right) / 5 \xi.
\end{align*}
\]
\[
\begin{align*}
\Psi_{14} &= 2 \left( \psi_{11} - \psi_{10} - \psi_{12} \right) / \xi, \\
\Psi_{22} &= 3 \left( \psi_{11} - \psi_{11} \right) / 2 \xi, \\
\Psi_{25} &= 3 \Psi_{24} - 4 \Psi_{23}, \\
\Psi_{33} &= 3 \Psi_{34} - 4 \Psi_{33}, \\
\Psi_{41} &= 5 \left( \psi_{30} - \psi_{31} \right) / 4 \xi, \\
\Psi_{44} &= 5 \left( \psi_{31} - \psi_{33} \right) / 4 \xi, \\
\Psi_{51} &= 6 \left( \psi_{40} - \psi_{41} \right) / 5 \xi.
\end{align*}
\] (B4)

In these formulae the argument \( \xi \) is omitted. For complete evaluation of these functions we need to compute 18 different functions \( \psi_{ij} \) given above.

To prevent the loss of accuracy if \( \xi \) is very small, we can use the series expansions (see NP94) that directly follow from the definitions (B1) and Taylor expansions (A1):

\[ \psi_{ij}(\xi) = \sum_{n=0}^{\infty} a_{jn} (-2 \xi)^n \frac{i + 1}{i + n + 1}, \quad i > -1, \quad \psi_{-1j}(\xi) = \sum_{n=0}^{\infty} a_{jn+1} (-2 \xi)^n \frac{2}{n + 1}, \] (B5)
\[ \Psi_{ij}(\xi) = \sum_{n=0}^{\infty} A_{jn} (-2 \xi)^n \frac{i + 1}{i + n + 1}, \]  
\text{(B6)}

with \( A_{jn} \) and \( A_{jn} \) given by Equations (A2) and (A3), respectively.

**APPENDIX C**

**AUXILIARY FUNCTION \( g(\xi) \)**

Calculations of function \( \psi_{ij} \) from Appendix B involve integral

\[ g(\xi) = \int_{0}^{\xi} \ln(1 + 2x) \frac{dx}{x}. \]  
\text{(C1)}

We repeat here for completeness the method of calculations of this integral from NP94. It is possible to write a relation between the values of this function on \( \xi < \frac{1}{2} \) and \( \xi > \frac{1}{2} \). Let us define for that the auxiliary function for \( \xi \leq \frac{1}{2} \)

\[ g^*(\xi) = g\left(\frac{\xi}{2}\right) = \int_{0}^{\xi} \ln \left(1 + \frac{x}{2}\right) \frac{dx}{x}. \]  
\text{(C2)}

It can be presented by series

\[ g^*(\xi) = \begin{cases} 
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\xi^n}{n^2} & \text{if } \xi \leq \xi_a < 1, \\
\frac{\pi^2}{12} + \ln 2 \ln \xi + \sum_{k=0}^{\infty} \frac{(1 - \xi)^{k+2}}{k+2} \sum_{m=1}^{k+1} \frac{1}{2^m m} & \text{if } \xi_a \leq \xi \leq 1.
\end{cases} \]  
\text{(C3)}

As \( \xi_a \) we can take 0.8–0.9. Then,

\[ g(\xi) = \begin{cases} 
g^*(2 \xi) & \text{if } 0 \leq \xi \leq 1/2, \\
\frac{\pi^2}{12} & \text{if } \xi = 1/2, \\
\frac{\pi^2}{6} + \frac{1}{2} \ln^2(2 \xi) - g^*(1/2 \xi) & \text{if } \xi > 1/2.
\end{cases} \]  
\text{(C4)}

**APPENDIX D**

**ASYMPTOTIC EXPANSIONS OF FUNCTIONS \( \chi_{jn} \) AND \( \Delta_{jn} \) IN THE THOMSON LIMIT**

Using Taylor expansion (B5) of functions \( \psi_{0n} \) for small arguments, it is easy to get an expansion of functions \( \chi_{0n} \) in the Thomson limit \( x \gamma \ll 1 \):

\[ \chi_{0n}(x, \gamma) = \gamma^n \sum_{l=0}^{\infty} (-2x \gamma)^l a_{0l} \kappa_{n+l+1}, \]  
\text{(D1)}

where

\[ \kappa_l = \frac{(1 + \beta)^{l+1} - (1 - \beta)^{l+1}}{2\beta(l+1)} = \sum_{k=0}^{\text{int}(l/2)} \frac{l! \beta^{2k}}{(2k+1)! (l-2k)!} = \frac{(1 + \beta)^l}{l+1} \sum_{k=0}^{l} \frac{1}{[\gamma(1 + \beta)]^{2k}} \]  
\text{(D2)}

and \( \text{int}(x) \) is the integer part of \( x \). A few first functions are

\[ \kappa_1 = 1, \quad \kappa_2 = 1 + \frac{1}{3} \beta^2, \quad \kappa_3 = 1 + \beta^2, \quad \kappa_4 = 1 + 2 \beta^2 + \frac{1}{3} \beta^4, \quad \kappa_5 = 1 + \frac{10}{3} \beta^2 + \beta^4, \]
\[ \kappa_6 = 1 + 5 \beta^2 + 3 \beta^4 + \frac{1}{7} \beta^6, \quad \kappa_7 = 1 + 7 \beta^2 + 7 \beta^4 + \beta^6. \]  
\text{(D3)}

The first three terms of the expansion (D1) are as follows:

\[ \chi_{0n}(x, \gamma) \approx \gamma^n \left[ \kappa_{1+1n} - 2x \gamma \kappa_{2+1n} + \frac{26}{5} (x \gamma)^2 \kappa_{3+1n} \right]. \]  
\text{(D4)}
Function $\Delta_{00}$ coincides with $\chi_{00}$, and functions $\Delta_{01}$ and $\Delta_{02}$ can be obtained using definitions (27) and expansion (D1). For $\Delta_{01}$, we get
\[
\Delta_{01} = -\beta \sum_{l=0}^{\infty} (-2xy)^l a_{0l} \zeta_{l+1} \approx -\frac{\beta}{3} \left[ 1 - 4xy + \frac{78}{5} (xy)^2 \left( 1 + \frac{\beta^2}{5} \right) \right],
\]
where
\[
\zeta_l = \frac{\kappa_i - \kappa_l}{\beta^2} = \sum_{k=0}^{\text{int}(l-1/2)} \frac{2k + 2}{(2k + 3)!} \beta^{2k} \frac{l!}{(l - 1 - 2k)!},
\]
\[
\zeta_1 = \frac{1}{3}, \quad \zeta_2 = \frac{2}{3}, \quad \zeta_3 = 1 + \frac{\beta}{5}, \quad \zeta_4 = \frac{4}{3} + \frac{4}{5} \beta^2, \quad \zeta_5 = \frac{5}{3} + 2\beta^2 + \frac{1}{2} \beta^4, \quad \zeta_6 = 2 + 4\beta^2 + \frac{6}{5} \beta^4.
\]
Respectively for $\Delta_{02}$, we have
\[
\Delta_{02} = \beta^2 \sum_{l=0}^{\infty} (-2xy)^l a_{0l} \Lambda_{l+1} \approx -\frac{4}{15} \beta^2 (xy) \left[ 1 - \frac{39}{5} xy \right],
\]
where
\[
\Lambda_l = \frac{1}{2\beta^2} \left[ 3 \kappa_l - 2\kappa_{l+1} + \kappa_{l+2} - \kappa_l \right] = \sum_{k=0}^{\text{int}(l/2)} \frac{l!}{(l - 2k)!} \frac{2k}{(2k + 3)(2k + 1)(2k + 3)} \beta^{2k},
\]
\[
\Lambda_1 = 0, \quad \Lambda_2 = \frac{2}{15}, \quad \Lambda_3 = \frac{2}{5}, \quad \Lambda_4 = \frac{4}{5} + \frac{4}{35} \beta^2.
\]
Similarly, for functions $\chi_{1n}$, using expansions (B6) we get
\[
\chi_{1n}(x, y) = y^n \sum_{l=0}^{\infty} (-2xy)^l \left( \gamma A_{1l} x A_{2l} \right) \kappa_{n+l+2} = y^n \left[ \gamma^2 \kappa_{2n} + \sum_{l=1}^{\infty} (-2xy)^l \left( \gamma^2 A_{1l} \kappa_{n+l+2} - \frac{A_{2l-1}}{2} \kappa_{n+l+1} \right) \right],
\]
Functions $\Delta_{1n}$ can then be obtained using definitions (50):
\[
\Delta_{10} = \chi_{10} \approx 1 + \frac{4}{3} \beta^2 - \frac{x y}{5} \left( 4y^2 - 27 - 2\beta^2 \right),
\]
\[
\Delta_{11} = -p \sum_{l=0}^{\infty} (-2xy)^l \left( \gamma A_{1l} x A_{2l} \right) \zeta_{l+2} \approx -\beta \left[ \frac{2}{3} y^2 - \frac{x y}{5} \left[ 21y^2 (1 + \beta^2/5) - 4 \right] \right],
\]
\[
\Delta_{12} = p\beta \sum_{l=0}^{\infty} (-2xy)^l \left( \gamma A_{1l} x A_{2l} \right) \Lambda_{l+2} \approx \frac{2}{15} \beta^2 \left[ y^2 - \frac{3xy}{5} \left( 21y^2 - 2 \right) \right],
\]
For functions $\chi_{2n}$, we can write the expansion
\[
\chi_{2n}(x, y) = y^n \sum_{l=0}^{\infty} (-2xy)^l \left[ \left( \gamma^4 A_{4l} - \gamma^2 A_{2l} \right) \kappa_{n+l+3} - \gamma^2 A_{Sl} \kappa_{n+l+2} + A_{6l} \kappa_{n+l+1} \right]
\]
\[
\approx y^n \left[ \gamma^2 \kappa_{3n+3} + \frac{1}{10} \kappa_{1n} \right],
\]
where we kept only the zeroth term in $xy$ of the series. Expansions for $\Delta_{2n}$ can then be obtained using definitions (50):
\[
\Delta_{20} = \chi_{20} \approx 1 + \frac{2}{15} \beta^2 (21y^2 + 4),
\]
\[
\Delta_{21} = -\beta \sum_{l=0}^{\infty} (-2xy)^l \left[ \left( \gamma^4 A_{4l} - \gamma^2 A_{2l} \right) \zeta_{l+3} - \gamma^2 A_{Sl} \zeta_{l+2} + A_{6l} \zeta_{l+1} \right] \approx -\beta \left[ 1 + \frac{2}{75} \beta^2 (63y^2 + 34) \right],
\]
\[
\Delta_{22} = \beta^2 \sum_{l=0}^{\infty} (-2xy)^l \left[ \left( \gamma^4 A_{4l} - \gamma^2 A_{2l} \right) \Lambda_{l+3} - \gamma^2 A_{Sl} \Lambda_{l+2} + A_{6l} \Lambda_{l+1} \right] \approx \beta^2 \frac{1}{75} (42y^2 - 11).
\]
Let us now discuss the properties of functions $\chi_{ln}^\ast$. The series expansion can be easily obtained from the definition (89) and series (B6):

$$\chi_{ln}^\ast(x, \gamma) = \sum_{i=0}^{\infty} (-2x\gamma) i A_{il} \kappa_{n+i}. \quad \text{(D13)}$$

For $\Delta_{lk}^\ast$ we get

$$\Delta_{l0}^\ast = \chi_{l0}^\ast \approx \gamma^2 (1 + \beta^2),$$

$$\Delta_{l1}^\ast = -\gamma p \sum_{i=0}^{\infty} (-2x\gamma) i A_{il} \zeta_{i+3} \approx -\gamma p (1 + \beta^2/5),$$

$$\Delta_{l2}^\ast = p^2 \sum_{i=0}^{\infty} (-2x\gamma) i A_{il} \Lambda_{i+3} \approx \frac{2}{5} p^2. \quad \text{(D14)}$$

The series expansion for functions $\chi_{ln}^\perp$ is

$$\chi_{ln}^\perp(x, \gamma) = \frac{\gamma^{n+1}}{p} \sum_{i=0}^{\infty} (-2x\gamma) i A_{il} \left( \kappa_{n+i+2} - 2\gamma^2 \kappa_{n+i+3} + \gamma^2 \kappa_{n+i+4} \right)$$

$$= \frac{2}{3} \gamma^{n+1} p \sum_{i=0}^{\infty} (-2x\gamma) i A_{il} \left( \beta^2 \Lambda_{n+i+2} - \kappa_{n+i+2} \right). \quad \text{(D15)}$$

For $\Delta_{lk}^\perp$ we get

$$\Delta_{l1}^\perp = \frac{1}{2} \chi_{l0}^\perp \approx -\frac{1}{3} \gamma p (1 + \beta^2/5),$$

$$\Delta_{l2}^\perp = \frac{1}{2p} (\gamma \chi_{l0}^\perp - \chi_{l1}^\perp) = \frac{1}{3} p^2 \sum_{i=0}^{\infty} (-2x\gamma) i A_{il} \left[ \Lambda_{i+2} - \Lambda_{i+3} + \zeta_{i+2} \right] \approx \frac{2}{15} p^2. \quad \text{(D16)}$$

**APPENDIX E**

**ELIMINATING CANCELLATIONS IN REDISTRIBUTION FUNCTIONS**

If formulae (138), (139), and (140) are used as they stand, numerical cancellations appear at certain regions of parameter space. For example if $x$ and $x_1$ are small, the quantities $a_- a_*$ and $1/a_- 1/a_*$ are close to each other. Also a combination containing a sum of $d_-/a_\perp$ and $d_+/a_\perp$ minus double the difference $1/a_-$ and $1/a_*$ has a cancellation. Therefore it is useful to rewrite the expressions in a form not containing those cancellations. The cancellations appearing in (138) were dealt with in Nagirner & Poutanen (1993).

Defining

$$u = a_+ - a_- = \frac{(x + x_1)(2\gamma + x_1 - x)}{a_- + a_+}, \quad v = a_- a_+, \quad \text{(E1)}$$

they got

$$R_0 = \frac{2}{Q} + \frac{u}{v} \left[ 1 - \frac{2}{q} \right] + u \frac{(u^2 - Q^2)(u^2 + 5v)}{2q^2v^3} + \frac{Q^2}{q^2v^2}. \quad \text{(E2)}$$

Using definitions (E1), we get from (139) and (140)

$$R_\Sigma = (a_- + a_*) \left[ \frac{2u}{Q} + \frac{1}{v} \left( 1 - \frac{2}{q} \right) + \frac{(u^2 - Q^2)(u^2 + 3v)}{2q^2v^3} + \frac{Q^2}{q^2v^2} \right], \quad \text{(E3)}$$

$$R_{ll} = \frac{1}{2Q^2} \left[ u^2(u^2 + 4v) + 2b^2 \right] + \frac{1}{2Q} \left[ 1 - \frac{4}{q} \right] + \frac{u}{q^2v} \quad \text{(E4)}$$

Another loss of accuracy occurs in the $u^2 - Q^2$ term, when $\gamma$ is close to $\gamma_s(x, x_1, \mu)$. We can use the following formulae (Nagirner & Poutanen 1993):

$$u^2 - Q^2 = 2rqCD_u, \quad D_u = (\gamma + x_1 - x + \gamma_\ast)(\gamma - \gamma_\ast), \quad C = 2/[\gamma(\gamma + x_1 - x) + r + x_1\mu + v]. \quad \text{(E5)}$$
The redistribution functions \( R \) by Equation (110), them here for completeness. For fixed photon energies and scattering angles satisfying the relation \( |\cos \theta| \leq 1 \), where \( \cos \theta \) is given by Equation (107). These limits were discussed in NP94, but we repeat them here for completeness. For fixed photon energies and scattering angle, we already got the limits on the electron energies given by Equation (110), \( \gamma \geq \gamma_s(x, x_1, \mu) \). If we are interested in the interval of scattered photon energies for the fixed \( x_1, \gamma, \) and \( \mu \), we have then \( x^- \leq x \leq x^+ \), where

\[
x^\pm(x_1, \gamma, \mu) = x_1 \pm \frac{\mu + \gamma(\gamma + x_1)(1 - \mu) \pm p(1 - \mu)a_+}{1 - 2\gamma x_1 (1 - \mu) + x_1^2 (1 - \mu)^2}.
\]  

(F1)

If the energy of the scattered photon \( x \) is fixed, the initial photon \( x_1 \) lies in the interval

\[
\begin{align*}
x_1^- & \leq x_1 \leq x_1^+ & \text{if } 0 \leq x (1 - \mu) \leq \gamma - p, \\
x_1 > x_1^- & \text{if } \gamma - p \leq x (1 - \mu) \leq \gamma + p,
\end{align*}
\]  

(F2)

where

\[
x_1^\pm(x, \gamma, \mu) = x \pm \frac{\mu + \gamma(\gamma - x)(1 - \mu) \pm p(1 - \mu)a_-}{1 - 2\gamma x (1 - \mu) + x^2 (1 - \mu)^2}.
\]  

(F3)

In Equations (F1) and (F3), the quantities \( a_\pm \) are defined by Equations (119). If \( |x - x_1| \leq 2xx_1 \), the quantity \( \gamma_s(x, x_1, \mu) \) as a function of \( \mu \) has a minimum

\[
\gamma_{\min} = 1 + (x - x_1 + |x - x_1|)/2
\]  

(F4)

at \( \mu = \mu_{\min} = 1 - |x - x_1|/xx_1 \), while in the opposite case, \( |x - x_1| \geq 2xx_1 \), the function is monotonic with the minimum reached at the boundary \( \mu = -1 \) (see Figure 12). Correspondingly, the limits of variations of \( \mu \) depend on the photon energies \( x, x_1 \) and the electron energy \( \gamma \) and are given by

\[
\mu_m \leq \mu \leq \mu_+,
\]  

(F5)

where

\[
\mu_m(x, x_1, \gamma) = \begin{cases} 
-1 & \text{if } |x - x_1| \geq 2xx_1, \\
-1 & \text{if } |x - x_1| \leq 2xx_1 \text{ and } \gamma \geq \gamma_s(x, x_1, -1), \\
-1 & \text{if } |x - x_1| \leq 2xx_1 \text{ and } \gamma_{\min} \leq \gamma \leq \gamma_s(x, x_1, -1), \\
1 - \frac{q_+}{xx_1} & \text{if } |x - x_1| \geq 2xx_1.
\end{cases}
\]  

(F6)

\[
\mu_+(x, x_1, \gamma) = 1 - \frac{q_-}{xx_1} = 1 - \frac{(x - x_1)^2}{xx_1 q_+}.
\]  

and

\[
\gamma_s(x, x_1, -1) = [x - x_1 + (x + x_1)\sqrt{1 + 1/xx_1}]/2,
\]  

(F7)

\[
q_{\pm} = \pm \frac{p^2 + \gamma(x_1 - x) \pm p\sqrt{(\gamma + x_1 - x)^2 - 1}}{2},
\]  

(F8)
For the angle-averaged redistribution function, the lower limit on the electron energy is

\[
\gamma_\ast(x, x_1) = \begin{cases} 
\gamma_\ast(x, x_1, -1) & \text{if } |x - x_1| \geq 2 x x_1, \\
\gamma_{\min} & \text{if } |x - x_1| \leq 2 x x_1.
\end{cases}
\]  

(F9)

The limits of variation of the scattered photon energy \(x\) as a function of incident photon energy \(x_1\) and \(\gamma\) can be found by inverting Equation (F9). We obtain

\[
x^{-}(x_1, \gamma) \leq x \leq x_{m}(x_1, \gamma),
\]  

(F10)

where

\[
x_{m}(x_1, \gamma) = \begin{cases} 
\gamma + x_1 - 1 & \text{if } 1 \leq \gamma \leq 1 + \frac{2x_1^2}{1 - 2x_1} \text{ and } x_1 < 1/2, \\
x'(x_1, \gamma) & \text{if } 1 + \frac{2x_1^2}{1 - 2x_1} \leq \gamma \text{ and } x_1 < 1/2, \\
\gamma + x_1 - 1 & \text{if } x_1 \geq 1/2,
\end{cases}
\]

and

\[
x^{\pm}(x_1, \gamma) = x_1 (\gamma \mp p) / (\gamma \mp p + 2 x_1).
\]  

(F11)