Quantum noncommutativity in quantum cosmology

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In the present work, we study the noncommutative version of a quantum cosmology model. The model has a Friedmann-Robertson-Walker geometry, the matter content is a radiative perfect fluid and the spatial sections have positive constant curvatures. We work in the Schutz’s variational formalism. We quantize the model and obtain the appropriate Wheeler-DeWitt equation. In this model the states are bounded. Therefore, we compute the discrete energy spectrum and the corresponding eigenfunctions. The energies depend on a noncommutative parameter. We observe that, due to the boundary conditions, the noncommutativity forces the universe to start expanding from an initial scale factor greater than zero. We also notice that, one can only construct wave-packets if the noncommutative parameter is discrete, with a well-defined mathematical expression, in a certain region of its domain.

One important arena where noncommutative (NC) ideas may play an important role is cosmology. If superstrings is the correct theory to unify all the interactions in nature, it must have played the dominant role at very early stages of our Universe. At that time, all the canonical variables and corresponding momenta describing our Universe should have obeyed a NC algebra. Inspired by these ideas some researchers have considered such NC models in quantum cosmology [1–3]. It is also possible that some residual NC contribution may have survived in later stages of our Universe. Based on these ideas some researchers have proposed some NC models in classical cosmology in order to explain some intriguing results observed by WMAP. Such as a running spectral index of the scalar fluctuations and an anomalously low quadrupole of CMB angular power spectrum [4–8]. Another relevant application of the NC ideas in classical cosmology is the attempt to explain the present accelerated expansion of our Universe [9–12].

In the present work, we study the noncommutative version of a quantum cosmology model. The model has a Friedmann-Robertson-Walker (FRW) geometry, the matter content is a radiative perfect fluid and the spatial sections have positive constant curvatures. We work in the Schutz’s variational formalism [13]. The noncommutativity that we are about to propose is not the typical noncommutativity between usual spatial coordinates. We are describing a FRW model using the Hamiltonian formalism, therefore the present model phase space is given by the canonical variables and conjugated momenta: \{a, p_a, \tau, p_\tau\}. Then, the noncommutativity, at the quantum level, we are about to propose will be between these phase space variables. Since these variables are functions of the time coordinate \(t\), this procedure is a generalization of the typical noncommutativity between usual spatial coordinates. The noncommutativity between those types of phase space variables have already been proposed in the literature. At the quantum level in Refs. [1–3], [10] and at the classical level in Refs. [9, 11, 12]. We quantize the model and obtain the appropriate Wheeler-DeWitt equation. In this model the states are bounded therefore we compute the discrete energy spectrum and the corresponding eigenfunctions. The energies depend on a noncommutative parameter. We observe that, due to the boundary conditions, the noncommutativity forces the universe to start expanding from an initial scale factor greater than zero. We also notice that, one can only construct wave-packets if the noncommutative parameter is discrete, with a well-defined mathematical expression, in a certain region of its domain.

The FRW cosmological models are characterized by
the scale factor $a(t)$ and have the following line element,

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2d\Omega^2\right),$$  \hspace{1cm} (1)

where $d\Omega^2$ is the line element of the two-dimensional sphere with unitary radius, $N(t)$ is the lapse function and $k$ gives the type of constant curvature of the spatial sections. Here, we are considering the case with positive curvature $k = 1$ and we are using the natural unit system, where $\hbar = c = G = 1$. The matter content of the model is represented by a perfect fluid with four-velocity $U^\mu = \delta^\mu_0$ in the comoving coordinate system used. The total energy-momentum tensor is given by,

$$T_{\mu, \nu} = (\rho + p)U_\mu U_\nu - pg_{\mu, \nu},$$  \hspace{1cm} (2)

where $\rho$ and $p$ are the energy density and pressure of the fluid, respectively. Here, we assume that $p = \rho/3$, which is the equation of state for radiation. This choice may be considered as a first approximation to treat the matter content of the early Universe and it was made as a matter of simplicity. It is clear that a more complete treatment should describe the radiation, present in the primordial Universe, in terms of the electromagnetic field.

Einstein’s equations for the metric \([1]\) and the energy momentum tensor \([2]\) are equivalent to the Hamilton equations generated by the super-hamiltonian constraint,

$$\mathcal{H} = -\frac{p_a^2}{12a} - 3a + \frac{p_T}{a},$$  \hspace{1cm} (3)

where $p_a$ and $p_T$ are the momenta canonically conjugated to $a$ and $T$ the latter being the canonical variable associated to the fluid \([1]\). The total Hamiltonian is given by $N\mathcal{H}$ and we shall work in the conformal gauge, where $N = a$. The commutative version of the present model was first treated in Ref. \([13]\).

We wish to quantize the model following the Dirac formalism for quantizing constrained systems \([10]\). First we introduce a wave-function which is a function of the canonical variables $\hat{a}$ and $\hat{T}$,

$$\Psi = \Psi(\hat{a}, \hat{T}).$$  \hspace{1cm} (4)

Then, we impose the appropriate commutators between the operators $\hat{a}$ and $\hat{T}$ and their conjugate momenta $\hat{P}_a$ and $\hat{P}_T$. Working in the Schrödinger picture, the operators $\hat{a}$ and $\hat{T}$ are simply multiplication operators, while their conjugate momenta are represented by the differential operators,

$$p_a \to -\frac{\partial}{\partial \hat{a}}, \hspace{1cm} p_T \to -i\frac{\partial}{\partial \hat{T}}.$$  \hspace{1cm} (5)

Finally, we demand that the operator corresponding to $N\mathcal{H}$ annihilate the wave-function $\Psi$, which leads to the Wheeler-DeWitt equation,

$$\left(\frac{1}{12} \frac{\partial^2}{\partial \hat{a}^2} - 3\hat{a}^2\right)\Psi(a, \tau) = -i \frac{\partial}{\partial \tau} \Psi(a, \tau),$$  \hspace{1cm} (6)

where the new variable $\tau = -T$ has been introduced. The operator $N\mathcal{H}$ is self-adjoint \([15]\) with respect to the internal product,

$$(\Psi, \Phi) = \int_0^\infty da \Psi(a, \tau)^* \Phi(a, \tau),$$  \hspace{1cm} (7)

if the wave functions are restricted to the set of those satisfying either $\Psi(0, \tau) = 0$ or $\Psi'(0, \tau) = 0$, where the prime $'$ means the partial derivative with respect to $a$. Here, we consider wave functions satisfying the former type of boundary condition and we also demand that they vanish when $a$ goes to $\infty$.

In order to introduce the noncommutativity in the present model, we shall follow the prescription used in Refs. \([1][3]\). In the present model, the noncommutativity will be between the two operators $\hat{a}$ and $\hat{\tau}$,

$$[\hat{a}, \hat{\tau}] = i\theta,$$  \hspace{1cm} (8)

where $\hat{a}$ and $\hat{\tau}$ are the noncommutative version of the operators. This noncommutativity between those operators can be taken to functions that depend on the noncommutative version of those operators with the aid of the Moyal product. Consider two functions of $\hat{a}$ and $\hat{\tau}$, let’s say, $f$ and $g$. Then, the Moyal product between those two function is given by:

$$f(\hat{a}, \hat{\tau}) \ast g(\hat{a}, \hat{\tau}) = f(\hat{a}, \hat{\tau}) \exp \left[i\theta/2\left(\hat{a}\hat{\tau} - \hat{\tau}\hat{a}\right)\right]g(\hat{a}, \hat{\tau}).$$

Using the Moyal product, we may adopt the following Wheeler-DeWitt equation for the noncommutative version of the present model,

$$\left[\frac{\hat{a}^2}{12} + \frac{\hat{\tau}^2}{2}\right] \Psi(\hat{a}, \hat{\tau}) + 3\hat{a}^2 \Psi(\hat{a}, \hat{\tau}) = 0.$$  \hspace{1cm} (9)

It is possible to rewrite the Wheeler-DeWitt equation \([9]\) in terms of the commutative version of the operators $\hat{a}$ and $\hat{\tau}$ and the ordinary product of functions. In order to do that, we must initially introduce the following transformation between the noncommutative and the commutative operators,

$$\hat{a} = \tilde{a} - \frac{\theta}{2}\hat{\tau},$$  \hspace{1cm} (10)

$$\hat{\tau} = \tilde{\tau} - \frac{\theta}{2}\hat{a},$$

and the momenta remain the same. Then, using the properties of the Moyal product it is possible to write the potential term in Eq. \([9]\) in the following way,

$$3\tilde{a}^2 \Psi(\tilde{a}, \tilde{\tau}) = 3\left(\tilde{a} - \frac{\theta}{2}\hat{\tau}\right)^2 \Psi(\hat{a}, \hat{\tau}).$$  \hspace{1cm} (11)

Finally, we may write the commutative version of the Wheeler-DeWitt equation \([10]\), to first order in the commutative parameter $\theta$, in the Schrödinger picture as,
\[
\frac{1}{12} \frac{\partial^2 \Psi(a, \tau)}{\partial a^2} - 3a^2 \Psi(a, \tau) = -i(1 - 3\theta a) \frac{\partial \Psi(a, \tau)}{\partial \tau}. \tag{12}
\]

For a vanishing \(\theta\) this equation reduces to the Schrödinger equation of an one dimensional harmonic oscillator restricted to the positive domain of the variable \(\theta\).

In order to solve this equation, we start imposing that the wave function \(\Psi(a, \tau)\) has the following form,

\[
\Psi(a, \tau) = A(a)e^{-iE\tau}. \tag{13}
\]

Introducing this ansatz in Eq. (14), we obtain the eigenvalue equation,

\[
\frac{d^2 A}{da^2} - 36a^2 A + (12 - 36\theta a)EA = 0, \tag{14}
\]

where \(E\) is the eigenvalue and it is associated with the fluid energy. It is possible to rewrite Eq. (14) such that it becomes similar to a one dimensional, quantum mechanical, harmonic oscillator eigenvalue equation. In order to do that one has to perform the following transformations,

\[
x = \sqrt{6a + 3\theta E}/\sqrt{6}, \quad \lambda = 3\theta E^2/2 + 2E. \tag{15}
\]

Introducing these transformations in Eq. (14), we obtain the new eigenvalue equation,

\[
\frac{d^2 A}{dx^2} + (\lambda - x^2)A = 0. \tag{16}
\]

This equation is the one dimensional, quantum mechanical, harmonic oscillator eigenvalue equation and has solutions for the following discrete values of \(\lambda\),

\[
\lambda = 2n + 1, \tag{17}
\]

where \(n = 0, 1, 2, 3, \ldots\). As a consequence of the second transformation of Eq. (15) combined with the result from Eq. (17), we obtain that the fluid energy is discrete. It has the following values,

\[
E(n, \theta) = \frac{2}{3\theta^2} \left( -1 + \sqrt{1 + \frac{3}{2}(2n + 1)\theta^2} \right). \tag{18}
\]

It is important to notice that \(E(n, \theta)\) is always positive for any value of \(n\) and \(\theta\), including negative values of \(\theta\). Another important property of \(E(n, \theta)\) is that, when \(\theta \to 0\), in Eq. (18), we have that \(E(n, \theta) \to n + 1/2\). Which is the correct expression for the one dimensional, quantum mechanical, harmonic oscillator energies. For a fixed value of \(\theta\), \(E(n, \theta)\) increases when \(n\) increases. On the other hand, for a fixed value of \(n\), \(E(n, \theta)\) decreases when \(\theta\) increases.

The eigenfunctions to the eigenvalue equation (16) are given by,

\[
A_n(x) = C_n H_n(x)e^{-x^2/2}, \tag{19}
\]

where \(H(x)\) are the Hermite polynomials of degree \(n\) and \(C_n\) are constants. Since we want to consider solutions that vanish at the origin, we shall take only the odd degree Hermite polynomials: \(n = 1, 3, 5, \ldots\). Besides that, those solutions will be normalized, in the variable domain \((0, \infty)\), only if the constants \(C_n\) are equal to:

\[
C_n = 2^{n-1/2}(n!)^{-1/2}(2)^{-1/4}. \tag{20}
\]

We may, now, write the eigenfunctions that are solutions to the initial eigenvalue equation (14), with the aid of the first transformation of Eq. (16) and Eq. (19),

\[
A_n(a) = C_nH_n(\sqrt{6a + 3\theta E}/\sqrt{6})e^{-(\sqrt{6a + 3\theta E}/\sqrt{6})^2/2}. \tag{21}
\]

For a fixed \(n\), the wavefunction \(\Psi(a, \tau)\) Eq. (13), obtained with the corresponding eigenfunction \(A_n(a)\) Eq. (20), can only satisfy the boundary condition \(\Psi(0, \tau) = 0\), when the Hermite polynomial \(H_n(\sqrt{6a + 3\theta E}/\sqrt{6})\) vanishes. This happens only, when its argument vanishes, which means that, \(a = -\theta E/2\). Since \(a\) and \(E\) are positive, we learn from that result that \(\theta\) has to be negative. More importantly, we learn also that in the present noncommutative model the scalar factor cannot ever vanishes. \(a\) can only vanishes when \(\theta\) vanishes, since \(E\) is always positive. Therefore, the noncommutativity seems to prevent the Universe to start from a zero scalar factor state. The greater the absolute value of \(\theta\), the greater the initial value of \(a\).

The most general expression of \(\Psi(a, \tau)\) Eq. (13), which is a solution to Eq. (12), is a linear combination of the eigenfunctions \(A_n(a)\), Eq. (20), combined with the exponential factor present in Eq. (13),

\[
\Psi(a, \tau) = \sum_{n=1, 3, 5, \ldots}^{\infty} B_n A_n(a)e^{-iE(n, \theta)\tau}, \tag{21}
\]

where the \(B_n\)'s are constants and \(E(n, \theta)\) is given in Eq. (18). Again, the above wavefunction has to satisfy the boundary condition \(\Psi(0, \tau) = 0\). Observing Eq. (21), we notice that it will only happen if several Hermite polynomials, of different degrees \(n\), vanish when their arguments vanish. For fixed \(\theta\), we notice that for different eigenfunctions \(A_n(a)\), their arguments vanish for different values of \(a\), \(a_n = -\theta E(n, \theta)/2\). Therefore, in principle, the \(\Psi(a, \tau)\) Eq. (21) cannot satisfy the desired boundary condition. In order to obtain a wavefunction that satisfies the above boundary condition, we shall make the assumption that, in a certain region of the \(\theta\) domain, this parameter is quantized and is given by: \(\theta_n = \alpha/E(n, \theta)\), where \(\alpha\) is a negative constant. Now, all Hermite polynomials, of different degree \(n\), vanish at the same value: \(a = -\alpha/2\). Then, \(\Psi(a, \tau)\) Eq. (21) satisfies the desired boundary condition.

We may, now, find the explicit expression of \(\theta_n\) by combining the above assumption with the expression of \(E(n, \theta)\) Eq. (18),

\[
\theta_n = \frac{2\alpha}{(2n + 1) - \frac{3\alpha^2}{2}}. \tag{22}
\]
Since, \( \theta_n \) and \( \alpha \) are negative, the denominator of the above expression must be positive. From this condition, we learn that \( \theta \) has to be in the following range: 
\[-\sqrt{2/(2n + 1)/3} < \alpha \leq 0 \]. As we have seen, above, in order for \( \Psi(a, \tau) \) Eq. (21) to satisfy the desired boundary condition, \( \alpha \) has to have a single well defined value. We must choose this value, such that the, denominator of Eq. (22) is always positive, for any value of \( n \). Supposing that the eigenfunction of degree one (\( n = 1 \)) will always be present in the linear combination Eq. (21), we must take the value of \( \alpha \) to be in the range: 
\[-\sqrt{2} < \alpha \leq 0 \]. From Eq. (22), we learn that for a fixed value of \( \alpha \), \( \theta_n \) increases when \( n \) increases. In the same way, for a fixed value of \( n \), \( \theta_n \) increases when \( \alpha \) increases.

One may understand why the wavefunction must vanishes at \( a = -\alpha/2 \) instead of \( a = 0 \) by studying the potential energy, from the eigenvalue equation (14). If we introduce the assumption that \( \theta_n E(n, \theta \rightarrow \alpha, \tau) = \alpha \), in Eq. (14), we may interpret the following term as the potential energy \( V(a) \): 
\( V(a) = 36a^2 + 36\alpha a \). Now, studying this potential we learn that it has a minimum not in \( a = 0 \), but in \( a = -\alpha/2 \). The minimum value of \( V(a) \) is equal to: 
\( V(a = -\alpha/2) = -9\alpha^2 \). Then, the transformations Eq. (18) reduced the present problem with potential \( V(a) \) to an equivalent harmonic oscillator in the variable \( x \). The origin of that harmonic oscillator is located exactly in the minimum value of \( V(a) \). Therefore, the description of the present problem in terms of \( x \) must start from \( x = 0 \) or, in terms of \( a \), must start from \( a = -\alpha/2 \).

Now, we would like to verify that the wavefunction and quantities computed with it are well defined, with respect to the boundary conditions \( \Psi(a = -\alpha/2, \tau) = \Psi(a \rightarrow \infty, \tau) = 0 \) and the assumption \( \theta_n E(n, \theta \rightarrow \alpha) = \alpha \). In order to do that, we shall compute, for particular choices, the scale factor expected value, \( \langle a \rangle \). First of all, we must introduce \( \theta_n \) Eq. (22) in \( E(n, \theta) \) Eq. (18) and obtain an expression for the energy as a function of \( n \) and \( \alpha \), \( E(n, \alpha) \). Then, we must define the wavepacket by choosing a finite number of eigenfunctions contributing in the linear combination Eq. (21). Let us choose the first twenty odd eigenfunctions, in other words, \( n = 1, 3, 5, ..., 39 \).

Next, we compute the eigenvalues, \( E(n, \alpha) \) for those 20 eigenfunctions, with the aid of Eq. (18). In order to do that we must choose the values of \( \alpha \) and the \( B_n \). In the present situation, we shall choose \( \alpha = -0.7064965809 \), which is in the range \( -\sqrt{2} < \alpha \leq 0 \). \( B_1 = B_2 = B_3 = ... = B_{37} = 1 \) and \( B_3 = B_7 = B_{11} = ... = B_{39} = -1 \).

Finally, we must compute the scale factor expected value, using the following expression,

\[
\langle a \rangle (\tau) = \frac{\int_{-\alpha/2}^{\infty} a |\Psi(a, \tau)|^2 da}{\int_{-\alpha/2}^{\infty} |\Psi(a, \tau)|^2 da}.
\]

FIG. 1. \( \langle a \rangle (\tau) \) for the time interval \( 0 \leq \tau \leq 50 \).

Here, we had to modify the internal product Eq. (7), due to the fact that the wavefunctions no longer vanish at \( a = 0 \). Figure 1 shows \( \langle a \rangle (\tau) \) for the time interval \( 0 \leq \tau \leq 50 \). It is clear from that figure that \( \langle a \rangle (\tau) \) oscillates between a maximum and a minimum value and never vanishes. Therefore, the boundary conditions and the assumption we considered give rise to well defined wavefunctions and quantities computed with it.

The fact that \( \langle a \rangle (\tau) \) never vanishes is expected because the integration domain in Eq. (24) extends over positive values of \( a \) and does not include the zero. In the same way, if we compute the Ricci and Kretschmann scalars expect values [7], we would find that they never diverge. They will be always finite. In this way, the noncommutativity acts in a way that prevents, the present model, to have an initial singularity, at the quantum level.

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