Liouville/Toda central charges from M5-branes

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Abstract: We show that the central charge of the Liouville and ADE Toda theories can be reproduced by equivariantly integrating the anomaly eight-form of the corresponding six-dimensional $\mathcal{N} = (0,2)$ theories, which describe the low-energy dynamics of M5-branes.
1. Introduction

$\mathcal{N} = 2$ supersymmetric field theories in four dimensions are very rich, both from the physical and mathematical points of view. Recently, it was observed in [1] that many $\mathcal{N} = 2$ theories can be understood in a unified manner by realizing them as a compactification of six-dimensional $\mathcal{N} = (0, 2)$ theories on a Riemann surface. Furthermore, it was noted in [2] that Nekrasov’s partition function [3] of such theories (with SU(2) gauge groups) computes the conformal blocks of the Virasoro algebra. It was also noted that the partition function on $S^4$, as given by [4], coincides with the corresponding correlation function of the Liouville theory. Soon this 2d-4d correspondence was extended in [5, 6] to the case of SU($N$) gauge groups where the Liouville theory generalizes to the $A_{N-1}$ Toda theory.

Given that these 4d theories are engineered from theories on M5-branes, one would like to understand the above correspondence in terms of string/M-theory. A step in this direction was made in [7, 8]. Hinted at by the results of [5] and [9], in [8] an interesting observation was made, namely that the anomaly eight-form of the 6d $\mathcal{N} = (0, 2)$ theory of type $A_{N-1}$ and the central charge of the Toda theory of the same type have similar structures:

$\mathcal{I}_8[A_{N-1}] = (N - 1)I_8(1) + N(N^2 - 1)\frac{p_2(N)}{24}$, \hspace{1cm} (1.1)

$c_{\text{Toda}}[A_{N-1}] = (N - 1) + N(N^2 - 1)Q^2$. \hspace{1cm} (1.2)

In this short note, we show that (1.2) with the correct value for $Q$, namely $Q = (\epsilon_1 + \epsilon_2)^2 / (\epsilon_1 \epsilon_2)$, arises from (1.1) if we consider the compactification of the 6d $\mathcal{N} = (0, 2)$ theory on $\mathbb{R}^4$ with equivariant parameters $\epsilon_{1,2}$. Furthermore, we will see that this relation works for arbitrary theories of type $A, D$ and $E$.

2. Computation

The anomaly eight-form of one M5-brane [10] is

$\mathcal{I}_8(1) = \frac{1}{48} \left[ p_2(NW) - p_2(TW) + \frac{1}{4}(p_1(TW) - p_1(NW))^2 \right]$, \hspace{1cm} (2.1)

where $NW$ and $TW$ stand for the normal and the tangent bundles of the worldvolume $W$, respectively and $p_k$ denotes the $k$-th Pontryagin class. Using this, the anomaly of

\hspace{1cm}

\hspace{1cm}Note that the Liouville theory is equivalent to the $A_1$ Toda theory.
\[
\begin{array}{|c|ccc|}
\hline
G & r_G & d_G & h_G \\
\hline
A_{N-1} & N-1 & N^2-1 & N \\
D_N & N & N(2N-1) & 2N-2 \\
E_6 & 6 & 78 & 12 \\
E_7 & 7 & 133 & 18 \\
E_8 & 8 & 248 & 30 \\
\hline
\end{array}
\]

**Table 1:** Data of the Lie algebras of type $A$, $D$, $E$. Note that $r_G(h_G + 1) = d_G$.

The $\mathcal{N} = (0,2)$ theory of type $G$ ($G = A_n, D_n, E_n$) can be written as \[11, 12, 13\]^2

\[
I_8[G] = r_G I_8(1) + d_G h_G \frac{p_2(NW)}{24}. \quad (2.2)
\]

Here $r_G$, $d_G$ and $h_G$ are the rank, the dimension, and the Coxeter number of the Lie algebra of type $G$, respectively. They are tabulated in Table 1.

Now, we wrap the $(0,2)$-theory of type $G$ on a four-manifold $X_4$. The 11d theory lives on

\[
\Sigma \times X_4 \times \mathbb{R}^5, \quad (2.3)
\]

where $\Sigma$ is the worldsheet of the resulting 2d theory. We take $X_4$ to be Euclidean and $\Sigma$ to be Lorentzian. The supercharges decompose as:

\[
4_+ \times 4 \rightarrow \left(\frac{1}{2}, 2, 1, 2, \frac{1}{2}\right) + \left(\frac{1}{2}, 2, 1, 2, -\frac{1}{2}\right) + \left(-\frac{1}{2}, 1, 2, 2, \frac{1}{2}\right) + \left(-\frac{1}{2}, 1, 2, 2, -\frac{1}{2}\right), \quad (2.4)
\]

where we listed the representation contents under the decomposition

\[
SO(5, 1) \times SO(5) \rightarrow SO(1, 1) \times SU(2)_l \times SU(2)_r \times SO(3) \times SO(2). \quad (2.5)
\]

Here we have decomposed $SO(4) \simeq SU(2)_l \times SU(2)_r$ and $SO(5) \supset SO(3) \times SO(2)$. The symplectic Majorana condition acts on each factor separately.

Let us twist $\mathbb{R}^5$ over $X_4$ so that a fraction of the supersymmetry remains. We embed the spin connection of the $SU(2)_r$ factor into the $SO(3)$ factor, that is

\[
SU(2)_r \rightarrow \text{diagonal part of } [SU(2)_r \times SO(3)]. \quad (2.6)
\]

Note that the $SO(3)$ factor is the standard $SU(2)_R$ symmetry of the four-dimensional theory if we think of the setup as the compactification of the six-dimensional theory on $\Sigma$, giving an $\mathcal{N} = 2$ theory on $X_4$. Therefore this twist is the one used by \[14\].

\[\text{For } E\text{-type } \mathcal{N} = (0,2) \text{ theory, this formula is only conjectural and there has been no independent check, to our knowledge. We assume the correctness of the formula.}\]
After the twist, we get the symmetry group $\text{SO}(1, 1) \times \text{SU}(2)_l \times \text{SU}(2)_r \times \text{SO}(2)$ and supercharges
\[
\left(\frac{1}{2}, 2, 2, \frac{1}{2}\right) + \left(\frac{1}{2}, 2, 2, -\frac{1}{2}\right) + \left(-\frac{1}{2}, 1, 1 + 3, \frac{1}{2}\right) + \left(-\frac{1}{2}, 1, 1 + 3, -\frac{1}{2}\right). 
\]
(2.7)
The preserved supercharges (scalars under $\text{SU}(2)_l \times \text{SU}(2)_r$) form a two-dimensional $\mathcal{N} = (0, 2)$ superalgebra, with $\text{U}(1)$ R-symmetry.\footnote{This twist is different from the one obtained by wrapping M5-branes on a holomorphic 4-cycle in a Calabi-Yau threefold \cite{15}.}

Let us exploit this 2d $\mathcal{N} = (0, 2)$ superalgebra. We take the right-movers to be the supersymmetric side. It is known that the anomaly polynomial and the central charges are related via
\[
I_4 = \frac{c_R}{6} c_1(F)^2 + \frac{c_L - c_R}{24} p_1(T\Sigma), 
\]
(2.8)
where $F$ is the external $\text{U}(1)$ bundle which couples to the $\text{U}(1)_R$ symmetry. Let us check this formula against free multiplets. The anomaly polynomial of a right-moving complex Weyl fermion with charge $q$ is
\[
I_4 = \text{ch}(qF) \hat{A}(T\Sigma) \bigg|_4 = \frac{q^2}{2} c_1(F)^2 - \frac{p_1(T\Sigma)}{24}. 
\]
(2.9)
The right-moving chiral multiplet has one complex boson, whose anomaly is the same as that of two neutral Weyl fermions and one Weyl fermion with charge 1. In total, $I_4 = c_1(F)^2/2 - p_1(T\Sigma)/8$ with $(c_L, c_R) = (0, 3)$. On the other hand, the left-moving free real boson has $I_4 = p_1(T\Sigma)/24$ with $(c_L, c_R) = (1, 0)$. Both cases agree with (2.8).

Now let us determine $I_4$ of the compactified theory by integrating $I_8$ over $X_4$. Let us assign the Chern roots as follows: $\pm t$ for the tangent bundle of $\Sigma$; $\pm \lambda_1$, $\pm \lambda_2$ for the tangent bundle of $X_4$; and $\pm n_1$, $\pm n_2$, 0 for the normal bundle. We include the $\text{U}(1)$ R-symmetry through
\[
n_1 \rightarrow 2c_1(F),
\]
(2.10)
and the twisting \cite{26} introduces
\[
n_2 \rightarrow \lambda_1 + \lambda_2. 
\]
(2.11)
Note that the doublet of $\text{SU}(2)_r$ has the Chern roots $\pm (\lambda_1 + \lambda_2)/2$. $(n_2, 0, -n_2)$ should then be the Chern roots of the triplet, resulting in (2.11).

Then we evaluate the anomaly polynomial. Notice that $\lambda_1$ and $\lambda_2$ will be integrated over $X_4$. Since the 2d spacetime effectively behaves as four dimensional inside the
anomaly polynomial, forms whose degree along $T\Sigma$ is higher than four automatically vanish. We get:

$$I_4 = \left[\frac{r_G + 2d_G h_G}{12}\right] \int (\lambda_1^2 + \lambda_2^2) + \left[\frac{3r_G + 4d_G h_G}{12}\right] \int \lambda_1 \lambda_2 c_1(F)^2 - \left[\frac{r_G}{48}\right] \int (\lambda_1^2 + \lambda_2^2) + \frac{r_G}{48} \int \lambda_1 \lambda_2 \frac{p_1(T\Sigma)}{2}. \quad (2.12)$$

Translating to $c_{L,R}$ using (2.8), we find

$$c_L = \frac{1}{2} (P_1(X_4) + 3 \chi(X_4)) r_G + (P_1(X_4) + 2 \chi(X_4)) d_G h_G,$$

$$c_R = \chi(X_4) r_G + (P_1(X_4) + 2 \chi(X_4)) d_G h_G. \quad (2.13)$$

Here, $\chi(X_4) = \int_{X_4} e(X_4)$ is the Euler number of $X_4$, and $P_1(X_4) = \int_{X_4} p_1(X_4)$ is the integrated first Pontryagin class which is three times the signature of $X_4$.

For example, let us wrap one M5-brane on $X_4 = K3$, in which case there is effectively no twisting. We start from $I_8(1)$ instead of $I_8[G]$, which effectively means using $r_G = 1$ and $d_G h_G = 0$ in (2.13). Using $P_1(K3) = -48$ and $\chi(K3) = 24$, we obtain

$$c_L = 24, \quad c_R = 12 \quad (2.14)$$

which is the value for the heterotic string, as it should be.

The case we are most interested in is $X_4 = \mathbb{R}^4$, considering the characteristic classes in the equivariant sense\(^4\). We take the action of $U(1)^2$ to rotate two orthogonal two-planes in $\mathbb{R}^4$, and call the equivariant parameters $\epsilon_{1,2}$ respectively. The Chern classes of the two two-planes are $\epsilon_{1,2}$. Thus we have $p_1(T\mathbb{R}^4) = \epsilon_1^2 + \epsilon_2^2$ and $e(T\mathbb{R}^4) = \epsilon_1 \epsilon_2$. We then use the localization formula, in the case where the fixed points are isolated:

$$\int_M \alpha = \sum_p \frac{\alpha|_p}{e(N_p)}.$$\(^4\)

\(^4\)Equivariant cohomology is a cohomology theory which also captures the action of a group on a space. For simplicity we only consider the abelian case $U(1)^n$. Consider the space of differential forms on $M$ valued in the polynomial of the formal parameters $\epsilon_a$, ($a = 1, \ldots, n$), and consider the deformed differential $D_\epsilon = d + \epsilon_a \kappa^a$. Here $\kappa$ is the interior product and $\kappa^a$ is the Killing vector of the $a$-th $U(1)$. Then $D_\epsilon^2 = \epsilon_a \mathcal{L}_{k_a}$ where $\mathcal{L}_{k_a}$ is the Lie derivative by $k_a$. We define the equivariant cohomology $H_{U(1)^n}(M)$ to be the cohomology of $D_\epsilon$ on the space of differential forms invariant under $U(1)^n$. Note that the formal parameters $\epsilon_a$ have degree 2. Equivariant characteristic classes are elements of the equivariant cohomology. For example, consider $\mathbb{C}$ acted on by $U(1)$ which rotates the phase, and let the equivariant parameter be $\epsilon$. The Chern class $c_1(T\mathbb{C})$ in the standard sense is of course trivial, but the equivariant Chern class is given by $c_1(T\mathbb{C}) = \epsilon$. For more details, see e.g. [10].
The summation is over the fixed points $p$, and $e(N_p)$ is the equivariant Euler class of the normal bundle of $p$ inside $M$. In our case the only fixed point is the origin. Therefore we have

$$P_1(\mathbb{R}^4) = \frac{\epsilon_1^2 + \epsilon_2^2}{\epsilon_1 \epsilon_2}, \quad \chi(\mathbb{R}^4) = 1. \quad (2.15)$$

Applying (2.13), we find

$$c_R = \frac{\epsilon_1^2 + 3\epsilon_1 \epsilon_2 + \epsilon_2^2}{2\epsilon_1 \epsilon_2} r_G + \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} d_G h_G,$$

$$c_L = r_G + \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} d_G h_G. \quad (2.16)$$

Upon the identification $\epsilon_1/\epsilon_2 = b^2$ advocated in [2], $c_L$ perfectly agrees with the central charge of the conformal Toda theory of type $G$ [17]:

$$c_{\text{Toda}}[G] = r_G + \left(b + \frac{1}{b}\right)^2 d_G h_G. \quad (2.17)$$

### 3. Discussion

A couple of comments are in order. First, recall that in the construction of [1] the $\mathcal{N} = 2$ theories are obtained by wrapping M5-branes on $\mathbb{R}^4 \times \Sigma$, with a suitable twist on $\Sigma$ which preserves one half of the supersymmetry. So far, we have not taken this twist into account. When we perform it, the right-moving sector, which was the supersymmetric part, becomes topological and so $c_R \to 0$, while $c_L$ is untouched and agrees with the central charge of the Liouville/Toda theories. This is consistent with the fact that Nekrasov’s partition function computes the chiral half of the Liouville/Toda correlation functions.

Second, notice that Nekrasov’s partition function was computed after introducing an equivariant deformation of $\mathbb{R}^4$ by a $U(1)^2$ action with parameters $\epsilon_{1,2}$. More precisely, the symmetry of the 4d theory is

$$\text{SO}(4) \times \text{SU}(2)_R \simeq \text{SU}(2)_l \times \text{SU}(2)_r \times \text{SU}(2)_R. \quad (3.1)$$

The topological theory has a modified Lorentz group

$$\text{SO}(4)' \simeq \text{SU}(2)_l \times \text{SU}(2)_{r'}, \quad (3.2)$$

where $\text{SU}(2)_{r'}$ is the diagonal subgroup of $\text{SU}(2)_r \times \text{SU}(2)_R$. The $U(1)^2$ used in the equivariant deformation is the Cartan subgroup of this modified $\text{SO}(4)'$. This motivated
our choice in (2.6). In view of this, it is also reasonable to evaluate the anomaly polynomial in the same equivariant sense \(^5\). It would be nice to have a better understanding of this point.

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**A. Central charges of Sicilian gauge theories of type A, D, E**

In \(^2\) the central charges \(a\) and \(c\) of the 4d superconformal Sicilian theories of \(A\) type (obtained by wrapping M5-branes on a genus-\(g\) Riemann surface), both in the \(\mathcal{N} = 2\) and \(\mathcal{N} = 1\) case, were computed from the 6d anomaly polynomial. We observe that from (2.2) the computation can be performed for the whole ADE series.

Let us start with the \(\mathcal{N} = 2\) case. Using the same Chern roots as in section 2, the line bundle of the \(\mathcal{N} = 1\) R-symmetry is incorporated by:

\[
\begin{align*}
    n_1 &\to n_1 + \frac{4}{3}c_1(F), \\
    n_2 &\to n_2 + \frac{4}{3}c_1(F).
\end{align*}
\]

\(\mathcal{N} = 2\) SUSY requires \(n_1 + t = 0, n_2 = 0\). The integral over the Riemann surface is \(\int_{\Sigma} t = 2 - 2g\).

The 4d ‘t Hooft anomalies of \(U(1)_R\) are read from the formula:

\[
I_6 = \frac{\text{tr } R^3}{6} c_1(F)^3 - \frac{\text{tr } R}{24} c_1(F)p_1(T_4). \tag{A.1}
\]

Comparing this with the integral of \(I_8\), we get:

\[
\text{tr } R^3 = \frac{2}{27}(g-1)(13r_G + 16d_G h_G) \quad \text{tr } R = \frac{2}{3}(g-1)r_G. \tag{A.2}
\]

Using the standard relations between \(a\), \(c\) and \(\text{tr } R\), \(\text{tr } R^3\), we get:

\[
\begin{align*}
a &= (g-1) \frac{5r_G + 8d_G h_G}{24} \\
c &= (g-1) \frac{r_G + 2d_G h_G}{6}.
\end{align*} \tag{A.3}
\]

This agrees with \(^{18}\) for the \(A\) series, and with \(^{19}\) for the \(D\) series. Similar formulas can be obtained in the \(\mathcal{N} = 1\) case. The R-symmetry bundle is given by \(n_1 \to n_1 + c_1(F)\) and \(n_2 \to n_2 + c_1(F)\), while \(\mathcal{N} = 1\) SUSY requires \(n_1 + n_2 + t = 0\). We get:

\[
\begin{align*}
a &= (g-1) \frac{6r_G + 9d_G h_G}{32} \\
c &= (g-1) \frac{4r_G + 9d_G h_G}{32}. \tag{A.4}
\end{align*}
\]

\(^5\)Note that Nekrasov’s partition function itself can be computed as an equivariant integral over the instanton moduli space.
References

[1] D. Gaiotto, “$\mathcal{N}=2$ Dualities,” arXiv:0904.2715 [hep-th].

[2] L. F. Alday, D. Gaiotto, and Y. Tachikawa, “Liouville Correlation Functions from Four-Dimensional Gauge Theories,” arXiv:0906.3219 [hep-th].

[3] N. A. Nekrasov, “Seiberg-Witten Prepotential from Instanton Counting,” Adv. Theor. Math. Phys. 7 (2004) 831–864, arXiv:hep-th/0206161.

[4] V. Pestun, “Localization of Gauge Theory on a Four-Sphere and Supersymmetric Wilson Loops,” arXiv:0712.2824 [hep-th].

[5] N. Wyllard, “$A_{N-1}$ Conformal Toda Field Theory Correlation Functions from Conformal $\mathcal{N}=2$ SU$(N)$ Quiver Gauge Theories,” arXiv:0907.2189 [hep-th].

[6] A. Mironov and A. Morozov, “On AGT Relation in the Case of U$(3)$,” arXiv:0908.2569 [hep-th].

[7] R. Dijkgraaf and C. Vafa, “Toda Theories, Matrix Models, Topological Strings, and $\mathcal{N}=2$ Gauge Systems,” arXiv:0909.2453 [hep-th].

[8] G. Bonelli and A. Tanzini, “Hitchin Systems, $\mathcal{N}=2$ Gauge Theories and W-Gravity,” arXiv:0909.4031 [hep-th].

[9] F. Benini, Y. Tachikawa, and B. Wecht, “Sicilian Gauge Theories and $\mathcal{N}=1$ Dualities,” arXiv:0909.1327 [hep-th].

[10] E. Witten, “Five-Brane Effective Action in M-Theory,” J. Geom. Phys. 22 (1997) 103–133, arXiv:hep-th/9610234.

[11] J. A. Harvey, R. Minasian, and G. W. Moore, “Non-Abelian Tensor-Multiplet Anomalies,” JHEP 09 (1998) 004, arXiv:hep-th/9808060.

[12] K. A. Intriligator, “Anomaly Matching and a Hopf-Wess-Zumino Term in 6D, $N=(2,0)$ Field Theories,” Nucl. Phys. B581 (2000) 257–273, arXiv:hep-th/0001205.

[13] P. Yi, “Anomaly of (2,0) Theories,” Phys. Rev. D64 (2001) 106006, arXiv:hep-th/0106165.

[14] E. Witten, “Topological Quantum Field Theory,” Commun. Math. Phys. 117 (1988) 353.

[15] J. M. Maldacena, A. Strominger, and E. Witten, “Black Hole Entropy in M-Theory,” JHEP 12 (1997) 002, arXiv:hep-th/9711053.

[16] M. Libnei, “Lecture notes on equivariant cohomology,” arXiv:0709.3615 [math].
[17] T. J. Hollowood and P. Mansfield, “Quantum Group Structure of Quantum Toda Conformal Field Theories. 1,” *Nucl. Phys. B330* (1990) 720.

[18] D. Gaiotto and J. Maldacena, “The Gravity Duals of \( \mathcal{N} = 2 \) Superconformal Field Theories,” arXiv:0904.4466 [hep-th].

[19] Y. Tachikawa, “Six-Dimensional \( D_N \) Theory and Four-Dimensional SO-USp Quivers,” arXiv:0905.4074 [hep-th].