Radiation from accelerated impurities in Bose-Einstein condensate

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We investigate radiation spectra arising from accelerated point-like impurities in the homogeneous Bose-Einstein condensate. A general formula for the radiation spectrum is obtained in the integral form as a function of given impurity trajectory. The Planckian spectrum is obtained for a special accelerated motion, which is shown to be unphysical. Non-Planckian spectrum is found in the case of a uniformly accelerated impurity. We compare our result with similar settings as discussed in other quantum many-body systems.

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I. INTRODUCTION

It is now a well-established experimental fact that condensation of mesoscopic number of atoms can be created and manipulated for relatively long time-scale on the order of seconds. These experimental progress enable us to probe properties of ultra-cold gases and Bose-Einstein condensate (BEC) from various different perspectives. Among them, many authors have been investigating the response properties of BECs when time-dependent perturbations are applied [1–7]. On general grounds, time-dependent disturbance on quantum many-body system will give rise to energy dissipation and hence to destroy the order and coherence in the system after some time later. However, due to the phenomenon of superfluidity of BECs, energy dissipation may or may not take place when time-dependent perturbation is applied to the condensates. An important question is to know how and when such a dissipation process occurs. This paper studies the radiation spectra accompanied by accelerated motions of an impurity in the homogeneous condensate. The motions of classical impurity is a special class of time-dependent disturbance and can be realized by using a detuned laser, a small number of different species of atoms, or many other microscopic objects in BECs. We are interested in the form of spectra obtained for various kinds of accelerated motions of the impurity.

Recently, the idea of simulating relativistic and cosmological effects in non-relativistic quantum (many-body) systems has gained great attention from various fields of researchers. There have been many activities devoted to explore this idea and how to realize it in laboratories. Several authors have studied radiation spectra arising from accelerated disturbance and responses of accelerated detectors in several different quantum many-body systems [9, 13–16]. They found the Planckian distribution for the radiation spectra and for the detector’s excitation probabilities. These results were then interpreted with a simulation of the Fulling-Davies-Unruh (FDU) effect in which any accelerated observer perceives the empty Minkowski vacuum as a finite temperature equilibrium state [17–19]. To be more precise, these studies concern simulating the DeWitt-Unruh (DU) effect and the FDU radiation. The former states that a particular accelerated detector is excited in the same way as being replaced in a thermal bath at rest and the latter concerns the phenomenon of thermal radiation from an accelerated particle in the vacuum [20, 21]. Regarding the DU effect, there exist several non-thermal response for the detector [22–24]. There also exists a debate if the original derivation of the FDU effect is mathematically correct or not [25]. It is questioned in this paper whether or not such an appearance of Planckian distribution in an analog situation can be related to the FDU effect. We briefly discuss the consequence of our result for proposed emergent theories of gravity in quantum many-body systems.

In this paper we investigate a coupling of a classical point-like impurity to the homogeneous condensate at zero temperature and study the radiation spectra accompanied by the accelerated motions of impurity. The impurity is assumed to move along a given trajectory without any back reactions. When the total number of bosons in the system is very large, the effect of point-like impurity can be represented by the time-dependent driven oscillator model, which can be diagonalized by a suitable unitary transformation. The radiation spectrum is obtained in the integral form as a function of a given impurity trajectory. In this paper, the Planckian spectrum is also obtained when the impurity is accelerated in a special manner inside the homogeneous condensate. However, this special class of impurity trajectory is shown to be unphysical and hence would not be realized to simulate such a black-body radiation. A radiation spectrum is also obtained analytically for a uniform acceleration and they are analyzed in detail.

The outline of this paper is as follows. Sec. II provides a model and its diagonalization within the Bogoliubov approximation. We obtain the radiation spectra for given impurity trajectories in Sec. III. Two accelerated motions are studied and are analyzed with their corresponding radiation spectra in Sec. IV. We discuss our result and its connection to the FDU effect in Sec. V.
II. MODEL HAMILTONIAN

The model Hamiltonian for weakly interacting bosons of mass \( M \) is

\[
H_0 = \int d^3x \left( \psi^\dagger(x) \left( -\frac{\hbar^2 \nabla^2}{2M} \right) \psi(x) \right) + \frac{g}{2} \int d^3x \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x),
\]

where the general two-body interaction term between bosons is approximated by a contact interaction \( g\delta(x - x') \) in the low energy region. The field operators \( \psi^\dagger(x) = \psi^\dagger(x,t) \) and \( \psi(x) = \psi(x,t) \) in the Heisenberg picture satisfy the equal time canonical commutation relations. We assume that the repulsive interaction \( g > 0 \) and the condition of diluteness \( na^2 \ll 1 \), where \( n \) is the number density of massive bosons. We consider the following local interaction Hamiltonian between bosons and a point-like impurity moving along a given trajectory \( \zeta(t) \):

\[
H_1(t) = \lambda \int d^3x \delta(x - \zeta(t)) \psi^\dagger(x) \psi(x),
\]

where \( \lambda \) represents the coupling constant between bosons and the impurity. As we discuss later, \( \lambda \) can be time-dependent in general when the impurity coupling is switched on and off in experiments. For the homogeneous Bose gas under consideration, the field operators are expanded in terms of the plane wave basis with periodic boundary conditions in a finite size box \( V = L^3 \) to obtain the total Hamiltonian \( H = H_0 + H_1 \) as

\[
H = \sum_k \epsilon_k a_k^\dagger a_k + \frac{g}{2V} \sum_{k,k',q} a_{k+q}^\dagger a_{k'}^\dagger a_k a_{k'} + \frac{\lambda}{V} \sum_{k,k'} e^{-i\mathbf{k} \cdot \mathbf{\zeta}(t)} a_{k+q}^\dagger a_k a_{k'},
\]

where \( \epsilon_k = \hbar^2 k^2 / 2M \) is the free kinetic energy of bosons and the summation is taken over integers \( n_i \) for \( k = 2\pi(n_x, n_y, n_z)/L \). In the following we are interested in the effects of impurities to the homogeneous condensate in the large \( N \) limit. We will investigate the first order correction on the homogeneous condensate.

We follow Bogoliubov’s treatment to simplify the full Hamiltonian within the number-fixed framework \([28]\). Define the operator \( \beta_0 = (N_0 + 1)^{-1/2} a_0 \) where \( N_0 = a_0^\dagger a_0 \) is the number operator of condensed bosons. This operator and its Hermite conjugate satisfy \( \beta_0 \beta_0^\dagger = 1 \) and \( \beta_0^\dagger \beta_0 = 1 - \Pi_0 \), where \( \Pi_0 = |N_0 = 0 \rangle \langle N_0 = 0 \rangle \) is the projection operator onto the state \( N_0 = 0 \). In the presence of the homogeneous BEC for the zero mode, we can exclude the state \( N_0 = 0 \) to approximate \( \beta_0^\dagger \beta_0 \approx 1 \), i.e., \( \beta_0 \) is an almost unitary operator, and \( [\beta_0, \beta_0^\dagger] \approx 0 \). More precisely this approximation holds between matrix elements when we compute expectation values. Then \( a_0^\dagger = \sqrt{N_0} \beta_0^\dagger \) and \( a_0 = \beta_0 \sqrt{N_0} \) allow us to eliminate the bare zero mode operators \( a_0^\dagger \) and \( a_0 \). We then introduce a new set of the creation and the annihilation operators \( \alpha_0^\dagger \) and \( \alpha_0 \) for non-zero momentum: \( \alpha_0^\dagger = a_k^\dagger \beta_0 \), \( \alpha_0 = \beta_0^\dagger a_k \) for \( k \neq 0 \). They satisfy the equal time canonical commutation relations: \( [\alpha_0^\dagger(t), \alpha_0(t)] = \delta_{k,k'} - \delta_{k,k'} \Pi_0 \approx \delta_{k,k'} \). \( \alpha_0^\dagger(t), \alpha_0(t) \approx [\alpha_0^\dagger(t), \alpha_0^\dagger(t)] = 0 \). The vacuum state of the Fock space \( \mathcal{H}_0 \) for the new set of the creation and the annihilation operators is the same as the plane wave basis, i.e., \( \alpha_0(0) = 0 \).

The total number operator \( N = N_0 + \sum_k \alpha_0^\dagger \alpha_0 \) commutes with the total Hamiltonian, where the prime over the summation is used to indicate the omission of the zero mode for the summation. Therefore we can choose the basis in which \( N \) is diagonalized with given value \( N \). This gives the condition \( N_0 + \sum_k \alpha_0^\dagger \alpha_0 = N \) which allows us to eliminate the zero mode occupation number operator \( N_0 \). Using an expansion for the square root function we can proceed to do a systematic expansion in \( 1/\sqrt{N} \). The Bogoliubov transformation to diagonalize the Hamiltonian without an impurity term is \( U_B = \exp(iG_B) \):

\[
G_B = \frac{i}{2} \sum_k \theta_k \alpha_0^\dagger \alpha_k + \text{h.c.},
\]

\[
\theta_k = \tanh^{-1}(\frac{gn}{\hbar \omega_k + \epsilon_k + gn}),
\]

\[
\hbar \omega_k = \sqrt{\epsilon_k (\epsilon_k + 2gm)}.
\]

Bogoliubov’s excitation (the “bogolon”) is created and annihilated by the operators

\[
b_0^\dagger = U_B \alpha_0^\dagger U_B^\dagger = \alpha_k^\dagger \cosh \theta_k + \alpha_{-k}^\dagger \sinh \theta_k,
\]

\[
b_k = U_B \alpha_k U_B^\dagger = \alpha_k^\dagger \cosh \theta_k + \alpha_{-k}^\dagger \sinh \theta_k,
\]

respectively. Keeping up to the second order in the new set of creation and annihilation operators, we obtain the effective Hamiltonian for the impurity motion in the homogeneous BEC as

\[
H = E_0 + \sum_k \hbar \omega_k b_k^\dagger b_k + \sum_k (f_k(t)b_k^\dagger + \text{h.c.}).
\]

Here \( E_0 = gnN/2 - gm/2 + \lambda n + \sum_k (\hbar \omega_k - \epsilon_k - gn)/2 \) is the ground state energy without impurity and

\[
f_k(t) = \frac{n \lambda}{N \hbar \omega_k} e^{-i \mathbf{k} \cdot \mathbf{\zeta}(t)}.
\]

The homogeneous condensate is defined by the new vacuum state of the Fock space \( \mathcal{H}_B; b_k^\dagger \mathcal{BEC} = 0 \).

Thus, the impurity effects in our model are described by a time-dependent driven oscillator model, or the time-dependent van Hove model. It is noted that the effective interaction Hamiltonian, the third term in eq. (9), is changed from the density coupling (2) to the linear coupling: \( \sqrt{n} \lambda \int d^3x \delta(x - \zeta(t)) (\psi^\dagger \hat{\psi} + \hat{\psi} \psi) \), where \( \psi \) is the bosonic field operator excluding zero mode. Note that the factor \( 1/\sqrt{N} \) in Eq. (10) is due to the condensation.
which introduces the significant difference and gives rise to the macroscopic effect.

We can solve the above Hamiltonian (9) exactly with the initial condition such that the operators evolve without impurity asymptotically at the infinitely remote past \( t = -\infty \). The solution is

\[
\begin{align*}
\hat{b}_k(t) &= \hat{b}^\text{ini}_k(t) + \phi_k(t), \\
\hat{b}^\dagger_k(t) &= \hat{b}^{\text{ini}}_k(t) + \phi_k(t),
\end{align*}
\]

where \( \phi_k(t) \) is a \( c \)-number function

\[
\phi_k(t) = -\frac{i n \lambda}{\hbar} \sqrt{\frac{\epsilon_k}{N\hbar\omega_k}} I_k(t) e^{-i\omega_k t},
\]

\[
I_k(t) = \int_{-\infty}^t dt' e^{i\omega_k t' - ik\zeta(t')},
\]

When the impurity is injected at some time \( t_i \), the solution is given instead by the similar formulae (11,12,13,14) with the replacement of the time integral \( \int_{-\infty}^t \rightarrow \int_{t_i}^t \).

The asymptotic creation and annihilation operators denoted by \( \hat{b}^{\text{ini}}_k \) and \( \hat{b}^\dagger_k \) are governed by the Heisenberg equation with the diagonalized Hamiltonian \( \hat{H}_\text{in} = \hat{E}_0 + \sum_k \hbar\omega_k \hat{b}^{\text{ini}}_k \hat{b}^\dagger_k \). In the diagonalized Hamiltonian, the new ground state energy is defined by \( \hat{E}_0(t) = \hat{E}_0 + \sum_k \text{Re}(f_k(t)\phi_k(t)) \). The energy spectrum \( \hbar\omega_k \) is that of the gapless excitations characterized by \( \omega_k \approx kc \) for a small \( k \), where \( c = \sqrt{gn/M} \) is the speed of sound of the condensate. This is when \( k\xi \ll 1 \) holds, where \( \xi = h/(2Mc) \) is the coherence length of the condensate. For large \( k \), on the other hand, \( \hbar\omega_k \approx \epsilon_k \) which is the free particle energy spectrum. We remark that the spectrum \( \omega_k \) and the speed of sound \( c \) are the same as in the original Bogoliubov model without impurities. Therefore the motions of impurity do not affect neither \( \omega_k \) nor \( c \) within our approximation.

The vacuum state of the Fock space \( \mathcal{H}_\text{in} \) for \( \hat{b}^{\text{ini}}_k \) and \( \hat{b}^\dagger_k \) is defined by \( \hat{b}^\dagger_k |0\rangle_{\text{in}} = 0 \). At remote past \( t = -\infty \), the vacuum \( |0\rangle_{\text{in}} \) coincides with the homogeneous condensate \( |\text{BEC}\rangle \). The physical interpretation of the above result is quite simple. The motions of impurity creates the time-dependent coherent states. The dressed Bogoliubov’s excitations do not annihilate the homogeneous condensate vacuum:

\[
\hat{b}^\dagger_k |\text{BEC}\rangle = -\phi_k(t)|\text{BEC}\rangle.
\]

**III. RADIATION SPECTRUM**

One of main effects described by our model is the effect of energy radiation due to the motions of impurity in the homogeneous condensate. From our model we find that even though the systems is in the ground state where no elementary excitation is present, there is energy transferred from the motions of impurity to massive bosons.

In other words, the motions of impurity will dress bogolons in such a way that bogolons will not see the homogeneous condensate as the vacuum. Since we do not take into account any back reaction on the impurities, this transferred energy is the energy required in order for the impurities to move along given trajectories. It means that we need to feed this amount of energy to keep the impurities moving. More realistic model for the radiation effects in BEC should be done by considering quantum impurities and taking into account the dynamics of them.

The occupation number for bogolons with respect to the homogeneous condensate

\[
\hat{n}_k(t) \equiv \langle \text{BEC} | \hat{b}^\dagger_k \hat{b}^\dagger_k | \text{BEC} \rangle = \frac{n^2 \lambda^2 \epsilon_k |I_k(t)|^2}{N\hbar^3 \omega_k |I_k(t)|^2},
\]

counting the emitted bogolons accompanying with the motions of impurity in BEC.

We next evaluate the expectation value for the number operator \( \hat{N}_k(t) = \hat{a}^\dagger_k(t)\hat{a}_k(t) = \hat{a}^\dagger_k(t)\hat{a}_k(t) \) for the excited particles with respect to the vacuum \( \langle 0|\hat{N}_k(t)|0\rangle \). This number counts the number of non-condensated particles. The depletion of the condensate \( d(t) \) due to the quantum fluctuation can be evaluated by summing over all modes and divided by the total particle number \( N \):

\[
d(t) = \frac{1}{N} \sum \langle 0|\hat{N}_k(t)|0\rangle \approx \frac{8}{3} \sqrt{\frac{gn^3}{\pi}} + \frac{1}{N} \sum (\frac{\epsilon_k}{\hbar\omega_k}) |\phi_k(t)|^2
\]

\[
+ \frac{gn}{2\hbar\omega_k} |\phi^*_k(t) - \phi_{-k}(t)|^2.
\]

Since \( |\phi_k|^2 \) has an additional factor \( 1/N \), the motions of impurity do not contribute to the depletion of the homogeneous condensate in the thermodynamic limit. In contrast, real experiments have always a finite number of particles and size. Therefore, there are effects on the depletion of condensates due to the motions of the impurity.

The dissipated energy \( \mathcal{E}_k(t) \) for a given mode \( k \) is obtained by multiplying \( \hat{n}_k(t) \) by the excitation energy \( \hbar\omega_k \), i.e., \( \mathcal{E}_k(t) = \hbar\omega_k \hat{n}_k(t) \) and the total dissipated energy \( \mathcal{E}(t) \) is given by summing over all modes. Therefore, for the \( t \rightarrow \infty \) limit which takes into account the complete trajectory of impurity motion, we obtain rather a simple expression the total dissipated energy in the thermodynamic limit as

\[
\mathcal{E}_\text{tot} = \lim_{t \rightarrow \infty} \sum \mathcal{E}_k(t) \approx \frac{n^2 \lambda^2}{\hbar^3} \int \frac{d^3k}{(2\pi)^3} \epsilon_k \int_{-\infty}^\infty dt e^{i\omega_k t - ik\zeta(t)} \int_{-\infty}^\infty dt e^{i\omega_k t - ik\zeta(t)} |^2.
\]

When an impurity is moving with a constant velocity, i.e., \( \zeta(t) = vt \), the total dissipated energy per unit time is proportional to \( \int d^3k \epsilon_k \delta(\omega_k - k \cdot v) \). This clearly shows that the speed of impurity \( v = |v| \) needs to exceed the
speed of sound to create finite amount of radiation, otherwise there is no radiation from the impurity motion [2, 3, 6, 7]. When the impurity acquires some acceleration, on the other hand, the integral (19) produces a non-zero value, that is a finite amount of energy radiation.

IV. RADIATION FROM ACCELERATED IMPURITIES

We consider three examples to illustrate our model and its general solution. We are interested in the occupation number (16) and the total dissipated energy (19) for the \( t \to \infty \) limit, i.e., in evaluating the integral:

\[
I_k = \lim_{t \to \infty} I_k(t) = \int_{-\infty}^{\infty} dt \ e^{\imath \omega_k t - \imath k \zeta(t)}. \tag{20}
\]

This limit, of course, should be understood as an idealization to obtain analytical results, and one needs to calculate a finite interval to compare with experiments. In the following, we also calculate the numbers of emitted bogolons \( d\tilde{n}_k \) between \( k \) and \( k + dk \) in the thermodynamic volume limit. Similarly, the energy radiated between \( k \) and \( k + dk \) is \( d\tilde{E}_k = \hbar \omega_k d\tilde{n}_k \). We note that a constant translation to the impurity trajectory \( \zeta(t) \to \zeta(t) + \zeta_0 \) gives rise to a phase factor which vanishes upon taking the absolute square. Thus any translation of the impurity trajectory will not produce any physical effect.

A. Black-body-like radiation

We first seek a trajectory which leads to the Planckian distribution for the occupation number (16). With a simple analysis, the trajectory with two parameters \( \zeta_0 \) and \( \Gamma_0 > 0 \)

\[
\zeta(t) = \zeta_0 e^{-\Gamma_0 t} (0, 0, 1), \tag{21}
\]

leads to the integral

\[
I_k = \int_{-\infty}^{\infty} dt \ \exp(\imath \omega_k t - \imath k \zeta_0 e^{-\Gamma_0 t}). \tag{22}
\]

Here \( k_z \) is the \( z \) component of the wave number vector of the emitted bogolons. The parameter \( \Gamma_0 \) can be expressed in terms of the acceleration constant \( a \) as \( \Gamma_0 = a/c \) and we refer it to as the acceleration parameter. The case of negative acceleration \( \Gamma_0 < 0 \) can be analyzed similarly and we only consider the positive case in this paper. Changing the variable from \( t \) to \( \eta = e^{-\Gamma_0 t} \) reads

\[
I_k = \frac{1}{\Gamma_0} \int_0^{\infty} d\eta \ \eta^{-i\omega_k/\Gamma_0 - 1} e^{-ik_z \zeta_0 \eta}. \tag{23}
\]

To carry out this integral, one formally introduces the regularization \( \omega_k/\Gamma_0 \to \omega_k/\Gamma_0 + \imath \epsilon \) where \( \epsilon \) is a small positive number which is set to zero at the end of calculations. Such a regularization, however, needs to be justified and this will be discussed in the next section. It is important to note that the \( z \) component of the wave vector \( k_z \) can be both positive or negative. With the above mentioned regularization, we obtain the integral as

\[
I_k = \frac{1}{\Gamma_0} |k_z \zeta_0|^2 \frac{\Gamma_0}{\imath \omega_k} \Gamma \left( -i \frac{\omega_k}{\Gamma_0} \right) e^{-\text{sgn}(k_z \zeta_0) \pi \omega_k/\Gamma_0}, \tag{24}
\]

where \( \text{sgn}(x) \) is the sign function and \( \Gamma(x) \) is the Gamma function.

The appearance of the above integrals (22) is rather surprising when comparing with an informal derivation of the FDU effect using time-dependent Doppler shift, see in particular eq. (7) of Ref. [32]. Similar formulae have been used by many authors to discuss FDU-like effects in non-relativistic systems [9, 13, 14]. However our calculation exhibits a different result as discussed in the next section.

The occupation number is

\[
\tilde{n}_k = \frac{n^2 \lambda^2 \epsilon_k}{N \hbar^2 \omega_k \omega_k \Gamma_0} \frac{2\pi}{\lambda \omega_k \Gamma_0} \times \left\{ \begin{array}{ll} \left[ e^{2\pi \omega_k/\Gamma_0} - 1 \right]^{-1} & (k_z \zeta_0 > 0) \\ \left[ 1 - e^{-2\pi \omega_k/\Gamma_0} \right]^{-1} & (k_z \zeta_0 < 0) \end{array} \right. \tag{25}
\]

When the direction of emitted bogolon is on the \( xy \) plane, i.e., \( k_z = 0 \), the above integral is proportional to the Dirac delta distribution \( \delta(\omega_k) \). Using the formula \( x\delta(x) = 0 \) together with the factor \( \epsilon_k/\omega_k \), we get zero radiation for \( k_z \zeta_0 = 0 \). We now examine this result (25) for the infrared and the ultraviolet region in detail. From now on, we choose the impurity trajectory to be confined within the positive \( z \) space, i.e., \( \zeta_0 > 0 \), and hence the impurity is accelerated from a positive infinity point to \( +\infty \) toward to the origin as the time changes from \( -\infty \) to \( +\infty \). First of all, we observe that the distribution of emitted bogolons does not depend on the direction within the upper half volume \( (z > 0) \) and the lower half volume \( (z < 0) \), respectively. Secondly, the radiation is independent of a scaling with respect to the parameter of the trajectory \( \zeta_0 \) and is determined solely from the acceleration parameter \( \Gamma_0 \). Lastly, the mathematical expression of the emission is the black-body radiation with the temperature \( k_B T_U = \hbar \Gamma_0/2\pi \) in the upper half volume and is completely different from thermal radiation in the lower half volume. We note a similar result was obtained and discussed in Ref. [13].

When the emission is restricted only for the low momenta, i.e., for the linear dispersion regime \( \omega_k \approx kc \) and \( kc \ll \Gamma_0 \), the above two cases show the same asymptotic behavior. The number of emitted bogolons between \( k \) and \( k + dk \) in the thermodynamic volume limit is given by

\[
d\tilde{n}_k \simeq \frac{\lambda^2}{2g hc} k \ dk \ d\Omega \ \text{for} \ k \ll \xi^{-1}, \Gamma_0/c, \tag{26}
\]
where \( d\Omega \) is the element of solid angle. Although this formula resembles the Rayleigh-Einstein-Jeans law for the black-body radiation, it cannot be interpreted as a signature of the thermal radiation. The reason is simply because the effective “temperature” \( T_U \propto \Gamma_0 \) does not show up for the long wavelength approximation.

A serious difficulty arises for the ultraviolet region where the emitted bogolons within an infinitesimal wave number \( dk \) is

\[
\tilde{d}\tilde{n}_k \approx \frac{4\pi g n^2 \lambda^2}{\hbar^3 \Gamma_0 c^2} dkd\Omega \times \begin{cases} e^{-2\pi \omega_k / \Gamma_0} (k \zeta_0 > 0) \\ 1 + e^{-2\pi \omega_k / \Gamma_0} (k \zeta_0 < 0) \end{cases},
\]

for \( k \gg \xi^{-1}, \Gamma_0 / c \). This becomes a constant and is independent of the excitation energy of bogolons for the lower half volume as \( k \) approaches infinity. Therefore, the ultraviolet divergent for the total dissipated energy occurs for this half volume when integrating all over the wave numbers.

One way to understand the origin of the above mentioned ultraviolet divergence is to look at the form of potential corresponding to the trajectory (21). With a simple analysis of classical mechanics, the potential is \( V(\xi) = -M_{\text{imp}} T_0 \xi^2 / 2 \) with \( M_{\text{imp}} \) the mass of impurity, which is an inverted harmonic potential. It is clear that the speed of impurity is not bounded and can become infinite for the asymptotic past and/or future. In reality, it is impossible to construct such a potential for the entire space since it requires an infinite amount of energy. Thus, this type of trajectories is only possible within a finite region and hence is limited by a given experimental situation, and one should not integrate the whole trajectory in the integral (20).

For a finite integral interval specified with \( (t_i, t_f) \), the integral of interest is calculated by the same regularization procedure as

\[
I_k(t_i, t_f) = \frac{\la k \xi_0 \ra^{2\pi \omega_k / \Gamma_0}}{\Gamma_0} e^{-\text{sgn}(k_s) \pi \omega_k / 2 \Gamma_0} \left[ \gamma(-i \frac{\omega_k}{\Gamma_0}, i k \xi_0 e^{-\Gamma_0 t_i}) - \gamma(-i \frac{\omega_k}{\Gamma_0}, i k \xi_0 e^{-\Gamma_0 t_f}) \right],
\]

where \( \gamma(x, z) \) is the incomplete gamma function of the first kind. We note that in the limit \( t_i \to -\infty \), the first term in (28) converges to the previous result (24). However, the second term oscillates with the angular frequency \( \omega_k \) as the final time \( t_f \) approaches \( \infty \). This behavior indicates that the regularization and infinite time integral limit do not commute each other and physics for finite time interval may be different from the above idealized infinite time interval limit. To analyze further, we consider a special case \( t_i = 0 \) and set \( t_f = T \). Using the asymptotic expansions of the incomplete gamma function, the number of emitted bogolons takes in the long wavelength regime as

\[
d\tilde{n}_k \sim \frac{\lambda^2 c T^2}{2\hbar^3} k^3 \, dk \, d\Omega \quad \text{for} \quad k \ll \xi^{-1} / \Gamma_0 / c,
\]

whereas it behaves as in the short wavelength regime

\[
d\tilde{n}_k \sim \frac{4n^3 g^2 \lambda^2}{\hbar^4 c^4} \left[ 1 - \cos k \xi_0 (1 - e^{-\Gamma_0 T}) \right] k^{-2} dk d\Omega,
\]

for \( k \gg \xi^{-1}, \Gamma_0 / c \).

Comparing with the previous expressions (26, 27), the finite interval case is totally different from the infinite case both for the long and the short wavelength regimes. Importantly, \( d\tilde{n}_k \) does not decay exponentially for large value of wave numbers but decays algebraically. Therefore, the trajectory (21) cannot be used to observe the temperature-like radiation unless its acceleration covers an infinite time period. This is, however, impossible in real experiments and we conclude that the appearance of mathematical expression for the Planckian distribution (25) is an artifact of invalid extrapolation for the physical parameters.

**B. Relativistic uniform acceleration**

The relativistic trajectory of a uniformly accelerated particle along the \( z \)-direction is

\[
\xi(t) = \frac{c^2}{a} \sqrt{1 + \left(\frac{at}{c}\right)^2} (0, 0, 1),
\]

with a given proper acceleration \( a \). To discuss a simulation of the relativistic effect in BECs, we replace the speed of light by the speed of sound \( c \) in the condensate. Note that this trajectory can be realized with a non-relativistic potential of the form \( V(\xi) = \frac{1}{2} M_{\text{imp}} (c^2 / a \xi)^2 \). Importantly, the speed of the impurity is always bounded by that of sound, \( |\xi(t)| < c \). In the limit \( c \to \infty \) or within the short time period \( at / c \ll 1 \), the trajectory becomes a non-relativistic quadratic one such as \( at^2 / 2 + (\text{constant}) \). To evaluate the above integral (20), we change the integration variable \( t \) to a dimensionless variable \( s \) through \( t = (c / a) \sinh s \), which is called a proper time in relativistic and, we define

\[
\mu_k = \frac{c}{a} \sqrt{\omega_k^2 - (ck_z)^2} = \frac{kc}{a} \sqrt{\sin^2 \theta + k^2 \xi^2},
\]

\[
\sigma_k = \tanh^{-1}\left(\frac{ck_z}{\omega_k}\right),
\]

where \( \theta \) is an angle between \( k \) and the \( z \) axis. Then,

\[
I_k = \frac{2c}{a} \left[ \pi \delta(\mu_k) \sinh \sigma_k + i K_1(\mu_k) \sin \sigma_k \right],
\]

where \( K_j(x) \) is the \( j \)-th modified Bessel function of the second kind. The first term in (34) will not contribute to the occupation number for the following reasons. The first is that \( \mu_k = 0 \) if and only if \( k = 0 \). The second reason follows from two properties of the Dirac delta distribution; \( \delta(f(x)) = \sum_i \delta(x - x_i) / |f'(x_i)| \), where \( x_i \) are the roots of \( f(x) = 0 \), and \( x \delta(x) = 0 \). The final reason
is the presence of the factor $\epsilon_k/\omega_k$ in (16). Therefore the occupation number in the thermodynamic limit is

$$d\tilde{n}_k = \frac{2n\lambda^2 k^3 \cos^2 \theta}{(2\pi)^3 M c^3 \hbar \omega_k} \left( \frac{K_1(\mu_k)}{\mu_k} \right)^2 \,dk \,d\Omega. \quad (35)$$

Let us now examine the result for the infrared and the ultraviolet region using the asymptotic behaviors of $K_1(x)$. The $\mu_k$ becomes very small in the infrared region, and hence the leading term of (35) becomes

$$d\tilde{n}_k \simeq \frac{2n\lambda^2 k^3 \cos^2 \theta}{(2\pi)^3 M c^3 \hbar (\sin^2 \theta + k^2 \xi^2)^2} \,dk \,d\Omega. \quad (36)$$

Therefore, the leading term has a singularity around $k \simeq 0$ at the angles $\theta = 0, \pi$, i.e., along the direction of trajectory, which results in the divergent result for the total number of emitted bogolons. This infrared singularity due to bogolons has the same origin as the famous infrared catastrophe in quantum electrodynamics [33]. There, the total number of emitted photons due to a motion of accelerated charged particle diverges as $k \to 0$. However, the physical measured quantity, the radiated energy, is still finite.

We next use the asymptotic form of $K_1$ for large $\mu_k$ to get the leading term:

$$d\tilde{n}_k \simeq \frac{n\lambda^2 (2M)^3 \cos^2 \theta}{(2\pi)^2 \hbar^3 c^2} \exp\left( -\frac{\hbar^2 c}{Mc} \right) \,dk \,d\Omega. \quad (37)$$

Hence the dissipated energy in the ultraviolet region after integrating over the range of solid angles is

$$dE_k \simeq \frac{4n\lambda^2 \omega^3}{3\pi \hbar^3 a} \exp( -\frac{\hbar^2 c}{Ma} ) \,dk. \quad (38)$$

Thus $dE_k$ asymptotically depends on $k$ through the exponential factor $\exp[-\hbar^2 c/(Ma)]$. We note that this factor can also be obtained by evaluating the integral using the stationary phase method. This high momentum behavior could be used to measure the number $hc/Ma$.

Lastly we notice that two length scales in this example have different orders of magnitudes; the coherence length $\xi$ and the acceleration length $l_a = c^2/a$. In most real experimental situations $\xi \ll l_a$, since accelerations available in laboratories are very restricted. Then we again use the asymptotic form of $K_1$ to estimate the total dissipated energy for the weak acceleration limit as

$$E_{tot} \simeq \frac{n\lambda^2 M^2 \omega}{10\hbar^3} \sqrt{ \frac{\hbar \alpha}{\pi Mc^3} }. \quad (39)$$

Although the speed of impurities never exceed the speed of sound we expect a small finite amount of radiation due to the uniformly accelerated impurity.

V. DISCUSSION

In this paper, we have investigated a simple model for the accelerated motions of classical impurity in the homogeneous condensate and radiations accompanied by them. It is shown that the impurity couples linearly to the field of the Bogoliubov excitations as shown in (9). Since the effective theory of the low energy excitations are known to resemble a relativistic mass-less field theory [34], one naively expects that this model can be used to simulate the FDU radiation for a suitably chosen accelerated motion of impurity. Indeed, our model also shares similar mathematical formulas with other proposed models as discussed in Refs. [9, 13-15, 32].

We have, however, pointed out that the appearance of the Planckian distribution for a radiation spectra corresponding to the impurity trajectory (21) is valid only in the mathematical limit in which the impurity is accelerated from the infinitely past to the infinitely remote future. This limit requires infinite amount of energy and hence one can hardly expect it to be simulated in laboratories. To check whether or not any trace of such an extreme situation can be observed in real experiment, we have examined the finite acceleration time case in detail and have shown that the results are completely different from the idealized limit. Obtaining the Planckian distribution also requires an appropriate regularization, otherwise the integral does not converge to the expression (24). This regularization can be understood, for example, as the adiabatic switching off of the impurity coupling, i.e., $\lambda \to \lambda \exp(-\epsilon t)$ with $\epsilon \to 0$ at the end of calculation. We, however, emphasize that the usage of other adiabatic factors will result in different formulæ and hence the appearance of the Planckian distribution is regularization dependent. Our result indicates that similar difficulties may exist for observing the Planckian distribution in other mentioned proposals.

We have also examined a relativistic trajectory for a uniformly accelerated impurity and have analyzed the radiation spectrum in detail. The result (38) shows an exponential decay of the emitted excitations for short wavelength behavior of the radiation spectrum. This exponential factor is sometimes referred to as a Boltzmann factor of the Bose-Einstein distribution and is related to the effective temperature of the FDU effect. With this correspondence, the effective temperature here is $k_B T = \hbar a/2c$ which differs by a factor $\pi$ from the original FDU temperature $T_U = \hbar \Omega_0/2\pi$. It is questionable if we can relate an appearance of the Boltzmann factor to the black-body radiation which is an incoherent radiation from the thermal equilibrium state. This is because radiations from the motions of impurity in condensate are coherent processes. The same statement also holds for the FDU effect. The thermal nature of reservoir is a consequence of a statistical ensemble average. From this averaging procedure, the expectation value of occupation number for bosons results in the Planckian distribution, but not the other way around. In other words, an apparent mathematical expression does not mean that the system is in a thermal equilibrium at certain temperature. This point was originally discussed in Ref. [35].

Along the same line of analysis, we gave a preliminary study for a radiation spectrum from a constant circular
motion [7, 8]. It is expected that this kind of impurity motion can be realized in the current experimental techniques. In future, we intend to study the circular motion in more detail based on realistic settings such as including a trapping potential, finite temperature effects, the back reaction to the impurity, and so on. It is also interesting to examine a connection to the so-called the circular FDU radiation and to discuss a possibility to simulate such an effect in BECs.

Finally, we briefly comment on a possible connection to the emergent theories of gravity. Several authors have proposed that gravitation is considered as an emergent phenomenon arising from collective excitations in quantum many-body systems [36–45]. In these proposals a physical vacuum is considered as a real condensed state which contains a huge number of heavy bosons, and the Minkowski vacuum is assumed to be the homogeneous Minkowski vacuum as a thermal state, and to clarify the physical mechanism leading to this effect if it exists.

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