STABILITY OF CLOSEDNESS OF SEMI-ALGEBRAIC SETS UNDER CONTINUOUS SEMI-ALGEBRAIC MAPPINGS

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Abstract. Given a closed semi-algebraic set $X \subset \mathbb{R}^n$ and a continuous semi-algebraic mapping $G: X \to \mathbb{R}^m$, it will be shown that there exists an open dense semi-algebraic subset $\mathcal{U}$ of $L(\mathbb{R}^n, \mathbb{R}^m)$, the space of all linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$, such that for all $F \in \mathcal{U}$, the image $(F + G)(X)$ is a closed (semi-algebraic) set in $\mathbb{R}^m$. To do this, we study the tangent cone at infinity $C_\infty X$ and the set $E_\infty X \subset C_\infty X$ of (unit) exceptional directions at infinity of $X$. Specifically we show that the set $E_\infty X$ is nowhere dense in $C_\infty X \cap S^{n-1}$.

1. Introduction

Exploring generic properties of a class of objects such as sets and/or mappings, ... is a fundamental problem of Theory of Singularities. In the paper [25], considering the class of projections, i.e., the restriction of a linear surjective mapping from a vector space $V$ into a vector space $Y$ to a submanifold $X$ of $V$, John N. Mather asked what properties of mappings are true for a generic projection and he gave answers to several special cases of this question. Motivated by this work and various problems in analysis and optimization [17, 18], we consider the question if preserving closedness is a generic property of continuous semi-algebraic mappings. Namely, we study when the image of a closed semi-algebraic set under a continuous semi-algebraic mapping is closed and if the closedness is stable under small linear perturbations. The closedness of such images is of significance in analysis, since it allows one to keep lower semi-continuity of functions and to assure the existence of solutions to various extremum problems (see, for example, [1, 27]).

It is well-known that the closedness of the image of a closed convex set under a linear mapping is not preserved under small perturbations of the linear mapping. On the other hand, it was shown recently in [10] (see also [5, 6]) that for a given closed convex set $X$ in $\mathbb{R}^n$, the set

$$L(\mathbb{R}^n, \mathbb{R}^m) \setminus \text{int}(\{F \in L(\mathbb{R}^n, \mathbb{R}^m) : F(X) \text{ is closed}\})$$

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is $\sigma$-porous in $L(\mathbb{R}^n, \mathbb{R}^m)$-the space of all linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e. small with regard to both measure and category.

The aim of this paper is to prove a similar result by considering a class of sets not necessarily being convex, which is the class of semi-algebraic sets. Precisely, with the definitions in the next section, the main result of this paper is as follows.

**Main Theorem.** Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set and let $G : X \to \mathbb{R}^m$ be a continuous semi-algebraic mapping. Then the set

$$\{ F \in L(\mathbb{R}^n, \mathbb{R}^m) : (F + G)(X) \text{ is closed}\}$$

contains an open dense semi-algebraic subset of $L(\mathbb{R}^n, \mathbb{R}^m)$.

Note that in the setting of Semi-Algebraic Geometry, a semi-algebraic subset of $\mathbb{R}^N$ is open dense if and only if its complement is $\sigma$-porous. Moreover, although the results presented here still hold for sets/mappings definable in some o-minimal structure (see [29] for more on the subject), we prefer to work with semi-algebraic sets/mappings for simplicity.

The proof of Main Theorem, which is quite long and technical, is sketched as follows:

1. **Step 1.** We investigate the tangent cone at infinity $C_\infty X$ and the set $E_\infty X \subset C_\infty X$ of (unit) exceptional directions at infinity of a unbounded closed semi-algebraic set $X \subset \mathbb{R}^n$. We will show that $E_\infty X$ is nowhere dense in $C_\infty X \cap S^{n-1}$.

2. **Step 2.** We present some sufficient conditions for the closedness of the images by linear mappings of closed semi-algebraic sets. Specifically we show that if a linear mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ of maximal rank has a “good” direction in $C_\infty X \setminus E_\infty X$, which lies in the kernel of $F$, then the image $F(X)$ is closed.

3. **Step 3.** Using sufficient conditions in Step 2, we prove the theorem in the case when $G$ is a linear mapping. Finally, the general case is reduced to the case when the mapping $G$ is linear.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries of Semi-Algebraic Geometry which will be used later. The notion and an important result on exceptional directions at infinity of semi-algebraic sets (Theorem 3.7) will be given in Section 3. Some sufficient conditions for the closedness of the images by linear mappings of closed semi-algebraic sets (Theorem 4.4) will be provided in Section 4. Finally, the proof of Main Theorem will be completed in Section 5.

2. Preliminaries

2.1. **Notation.** Let $\mathbb{R}^n$ denote the Euclidean space of dimension $n$. The corresponding inner product (resp., norm) in $\mathbb{R}^n$ is denoted by $\langle x, y \rangle$ for any $x, y \in \mathbb{R}^n$ (resp., $\| x \| := \sqrt{\langle x, x \rangle}$ for any $x \in \mathbb{R}^n$). We will denote by $B^n_r(x), \overline{B}^n_r(x)$ and $S^{n-1}_r(x)$, respectively, the open ball, the
closed ball and the sphere of radius $r$ centered at $x \in \mathbb{R}^n$. For simplicity, we write $\mathbb{B}_r^n$, $\mathbb{B}_r^n$ and $\mathbb{S}^{n-1}$ if $x = 0$ and we write $\mathbb{B}_r^n$, $\mathbb{B}_r^n$ and $\mathbb{S}^{n-1}$ if $x = 0$ and $r = 1$.

For a number $t \in \mathbb{R}$ and a nonempty set $X \subset \mathbb{R}^n$, let

$$tX := \{tx : x \in X\}.$$ 

We will denote by $\operatorname{sing}(X)$ the set of singular points of $X$, i.e., the set of points at which $X$ is not a $C^1$-manifold. As usual, $\operatorname{dist}(x, X)$ stands for the Euclidean distance from $x \in \mathbb{R}^n$ to $X$, i.e.,

$$\operatorname{dist}(x, X) := \inf\{\|x - y\| : y \in X\},$$

while the closure and interior of $X \subset \mathbb{R}^n$ are denoted respectively by $\overline{X}$ and $\operatorname{int}(X)$.

If $V_1 \subseteq \mathbb{R}^n$ and $V_2 \subseteq \mathbb{R}^m$ are linear subspaces, let $L(V_1, V_2)$ denote the set of all linear mappings from $V_1$ to $V_2$. We always assume that $L(V_1, V_2)$ is equipped with the operator norm. In the case $V_1 = \mathbb{R}^n$ and $V_2 = \mathbb{R}^m$, for any linear mapping $F \in L(\mathbb{R}^n, \mathbb{R}^m)$, we may identify $F$ with the matrix of $F$ in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. Hence we can identify $L(\mathbb{R}^n, \mathbb{R}^m)$ with $\mathbb{R}^{m \times n}$.

Let $v, w \in \mathbb{R}^n$ be not equal to 0 simultaneously, denote by $\hat{v}, \hat{w}$ the angle between $v$ and $w$. For convenience, if either $v = 0$ or $w = 0$, set $\hat{v}, \hat{w} := \frac{\pi}{2}$. So $0 \leq \hat{v}, \hat{w} = \hat{w}, \hat{v} \leq \pi$. If $V \neq \{0\}$ is a linear subspace of $\mathbb{R}^n$, let $\pi_V$ be the orthogonal projection on $V$ and the angle between a non zero vector $v$ and $V$ is given by

$$\angle(v, V) = \hat{v}, \pi_V(v).$$

For two linear subspaces $V_1 \neq \{0\}$ and $V_2 \neq \{0\}$ of $\mathbb{R}^n$, let $\pi_{V_i}$ be the orthogonal projection on $V_i$ ($i = 1, 2$) and we define the angle between $V_1$ and $V_2$ by

$$\angle(V_1, V_2) := \begin{cases} \sup\{v, \pi_{V_2}(v) : v \in V_1 \setminus \{0\}\} & \text{if } \dim V_1 \leq \dim V_2 \\ \sup\{v, \pi_{V_1}(v) : v \in V_2 \setminus \{0\}\} & \text{if } \dim V_1 \geq \dim V_2 \\ \max\{v, \pi_{V_2}(v) : v \in V_1 \cap \mathbb{S}^{n-1}\} & \text{if } \dim V_1 \leq \dim V_2 \\ \max\{v, \pi_{V_1}(v) : v \in V_2 \cap \mathbb{S}^{n-1}\} & \text{if } \dim V_1 \geq \dim V_2. \end{cases}$$

Observe that if $\dim V_1 = \dim V_2$, then

$$\sup\{v, \pi_{V_2}(v) : v \in V_1 \setminus \{0\}\} = \sup\{v, \pi_{V_1}(v) : v \in V_2 \setminus \{0\}\},$$

so the definition of angle between linear subspaces makes sense. By definition,

$$0 \leq \angle(V_1, V_2) \leq \pi/2.$$

Furthermore, the equality $\angle(V_1, V_2) = 0$ implies that $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$. If $V_1$ and $V_2$ are affine subspaces of $\mathbb{R}^n$, then the angle between $V_1$ and $V_2$ are determined by the angle between the corresponding parallel linear subspaces. It is not hard to verify that $\angle(\cdot, \cdot)$ defines a metric on the Grassmannian of $d$-dimensional linear subspaces of $\mathbb{R}^n$, for $1 \leq d \leq n$. 

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2.2. **Semi-algebraic geometry.** We recall some notions and results of Semi-Algebraic Geometry, which can be found in [3, 4, 29].

**Definition 2.1.**

(i) A subset of $\mathbb{R}^n$ is called *semi-algebraic* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n : f(x) = 0; \ g_i(x) > 0, i = 1, \ldots, k\}$$

where $f$ and all $g_i$ are polynomials.

(ii) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be semi-algebraic sets. A mapping $F: X \to Y$ is said to be *semi-algebraic* if its graph

$$\{(x, y) \in X \times Y : y = F(x)\}$$

is a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$.

A major fact concerning the class of semi-algebraic sets is the following version of the Tarski–Seidenberg theorem.

**Theorem 2.2.** The image of a semi-algebraic set by a semi-algebraic mapping is semi-algebraic.

Moreover, we have the following properties:

(i) The class of semi-algebraic sets is closed with respect to Boolean operators, taking Cartesian product, closure, interior and inverse image under semi-algebraic mappings;

(ii) A composition of semi-algebraic mappings is a semi-algebraic mapping;

(iii) The inverse image of a semi-algebraic set under a semi-algebraic mapping is semi-algebraic;

(iv) The distance function to a nonempty semi-algebraic set is semi-algebraic.

The following well-known two lemmas will be of great importance to us.

**Lemma 2.3** (Monotonicity Lemma). Let $f: (a, b) \to \mathbb{R}$ be a semi-algebraic function. Then there are finitely many points $a = t_0 < t_1 < \cdots < t_k = b$ such that the restriction of $f$ to each interval $(t_i, t_{i+1})$ is analytic, and either constant or strictly monotone.

**Lemma 2.4** (Curve Selection Lemma at infinity). Let $X \subset \mathbb{R}^n$ be a semi-algebraic set, and let $f := (f_1, \ldots, f_p): \mathbb{R}^n \to \mathbb{R}^p$ be a semi-algebraic mapping. Assume that there exists a sequence $x^k \in X$ such that $\lim_{k \to \infty} \|x^k\| = \infty$ and $\lim_{k \to \infty} f(x^k) = y \in (\overline{\mathbb{R}})^p$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$. Then there exists an analytic semi-algebraic curve $\varphi: (0, \epsilon) \to \mathbb{R}^n$ such that $\varphi(t) \in X$ for all $t \in (0, \epsilon)$, we have $\lim_{t \to 0^+} \|\varphi(t)\| = \infty$ and $\lim_{t \to 0^+} f(\varphi(t)) = y$.

In the sequel we will make use of semi-algebraic Morse–Sard Theorem and Hardt’s semi-algebraic triviality Theorem.
Theorem 2.5 (Morse–Sard). Let $N$ and $M$ be $C^1$ semi-algebraic manifolds of dimensions respectively $n$ and $m$ with $n \geq m \geq 1$, and $f : N \to M$ be a $C^1$ semi-algebraic mapping. Let

$$\Sigma(f) := \{ x \in N : \text{rank } df_x < m \}.$$ 

Then $f(\Sigma(f))$ is a semi-algebraic set of dimension less than $m$.

Theorem 2.6 (Hardt). Let $X$ and $Y$ be respectively semi-algebraic sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, $f : X \to Y$ be a continuous semi-algebraic mapping. Then there exists a partition of $Y$ into finitely many semi-algebraic subsets $Y_i, i = 1, \ldots, p$, such that $f$ is semi-algebraically trivial over each $Y_i$, i.e., $f^{-1}(Y_i)$ is semi-algebraically homeomorphic to $f^{-1}(y_i) \times Y_i$ for each $i$ and any $y_i \in Y_i$.

2.3. Stratification of semi-algebraic sets.

Definition 2.7. A semi-algebraic stratification of a semi-algebraic subset $X$ of $\mathbb{R}^n$ is a finite partition $\mathcal{S} := \{ X_\alpha \}_{\alpha \in I}$ of $X$ such that:

- Each $X_\alpha$ (called a stratum of the stratification or, briefly, a stratum of $X$) is a connected semi-algebraic $C^1$-submanifold of $\mathbb{R}^n$.
- The following frontier condition holds: If $X_\alpha \cap \overline{X_\beta} \neq \emptyset$, then $X_\alpha \subset \overline{X_\beta}$.

It is well-known that every semi-algebraic set admits a semi-algebraic stratification; moreover, any finite family of semi-algebraic sets can be stratified simultaneously as follows (see, for example, [3, Proposition 2.5.1] and [4, Proposition 9.1.8]).

Proposition 2.8. Let $X$ be a semi-algebraic subset of $\mathbb{R}^n$. For any finite family of semi-algebraic subsets $\{ X_i \}_{i=1,\ldots,p}$ of $X$, there is a semi-algebraic stratification $\mathcal{S}$ of $X$ such that each $X_i$ is a union of strata in $\mathcal{S}$.

Remark 2.9. Let $X$ be a semi-algebraic subset of $\mathbb{R}^n$ and let $\{ X_i \}_{i=1,\ldots,p}$ be a finite family of semi-algebraic subsets of $X$. For any semi-algebraic stratification $\mathcal{S}$ of $X$ such that each $X_i$ is a union of strata in $\mathcal{S}$, the following conditions are equivalent:

- (i) $X_i$ is nowhere dense in $X$;
- (ii) for any stratum $X_\alpha \in \mathcal{S}$ such that $X_\alpha$ is not contained in the boundary of another stratum (i.e., there is no stratum $X_\beta$ such that $X_\alpha \subset \overline{X_\beta}$), then $X_\alpha \cap X_i = \emptyset$. In other words, for any stratum $X_\alpha \in \mathcal{S}$ such that $X_\alpha \subset X_i$, then $X_\alpha$ is contained in the boundary of another stratum.

Let $X \subset \mathbb{R}^n$ be a semi-algebraic set and let $\mathcal{S} := \{ X_\alpha \}_{\alpha \in I}$ be a semi-algebraic stratification of $X$. We define the dimension of $X$ by

$$\dim X := \max \{ \dim X_\alpha : \alpha \in I \}.$$
It is not hard to check that this definition of dimension does not depend on the stratification of \( X \). For convenience, set \( \dim \emptyset = -1 \). Let \( x \in X \), the \textit{dimension of} \( X \) \textit{at} \( x \) is defined by

\[
dim_x X := \max \{\dim X_\alpha : \alpha \in I, \ x \in \overline{X}_\alpha\}.
\]

Obviously \( \dim_x X = \dim T_x X \) if \( x \) is a non singular point of \( X \), where \( T_x X \) denotes the tangent space to \( X \) at \( x \).

We say that \( X \) has \textit{pure dimension} \( d \) if \( \dim_x X = d \) for any \( x \in X \). Moreover, for \( x \in X \), we say that \( X \) is of \textit{pure dimension} \( d \) at \( x \) if there exists an open semi-algebraic neighborhood \( U \) of \( x \) in \( \mathbb{R}^n \) such that \( \dim_y X = d \) for any \( y \in X \cap U \).

\textbf{Remark 2.10.} It can be seen that \( X \) has pure dimension \( d \) if and only if every stratum \( X_\alpha \in \mathcal{S} \) not contained in the boundary of another strata has dimension \( d \). Furthermore, \( X \) has pure dimension \( d \) at \( x \in X \) if and only if every stratum \( X_\alpha \in \mathcal{S} \) not contained in the boundary of another strata with \( x \in \overline{X}_\alpha \) has dimension \( d \).

\textbf{Lemma 2.11.} Let \( X \) be a semi-algebraic set in \( \mathbb{R}^n \). The following statements hold.

\begin{enumerate}[(i)]
  \item If \( X \neq \emptyset \) then \( \dim(\overline{X} \setminus X) < \dim X \). In particular, \( \dim \overline{X} = \dim X \).
  \item If \( X \) is contained in a semi-algebraic set \( Y \subset \mathbb{R}^n \), then \( \dim X \leq \dim Y \).
  \item If \( f : X \to \mathbb{R}^m \) is a semi-algebraic mapping, then \( \dim f(X) \leq \dim X \).
\end{enumerate}

As a consequence of Proposition 2.8 and Lemma 2.11, we get the following corollary which will be used later.

\textbf{Corollary 2.12.} With the notation and setting of Proposition 2.8, for each \( i = 1, \ldots, p \), the set \( \text{sing}(X_i) \) is also a union of strata in \( \mathcal{S} \).

\textit{Proof.} Assume for contradiction that \( \text{sing}(X_i) \) is not a union of strata in \( \mathcal{S} \) for some \( i \). Then there is a stratum \( X_\alpha \subset X_i \) such that \( X_\alpha \cap \text{sing}(X_i) \neq \emptyset \) and \( X_\alpha \not\subset \text{sing}(X_i) \). Let \( x \in X_\alpha \setminus \text{sing}(X_i) \). Clearly \( \dim_x X_i = \dim_x X_\alpha = \dim X_\alpha \).

On the other hand, let \( y \in X_\alpha \cap \text{sing}(X_i) \). It is clear that there is a stratum \( X_\beta \neq X_\alpha \) such that \( X_\beta \subset X_i \) and \( y \in \overline{X}_\beta \) since otherwise \( y \not\in \text{sing}(X_i) \), which is a contradiction. Hence \( X_\alpha \cap \overline{X}_\beta \neq \emptyset \), and so \( X_\alpha \subset \overline{X}_\beta \setminus X_\beta \). By Lemma 2.11 we have \( \dim X_\alpha < \dim X_\beta \). Therefore,

\[
\dim_x X_i = \dim X_\alpha < \dim X_\beta \leq \dim_x X_i,
\]

which is impossible. \( \square \)

\textbf{Corollary 2.13.} Let \( X \subset \mathbb{R}^n \) be a semi-algebraic set. The following statements are equivalent.

\begin{enumerate}[(i)]
  \item \( X \) is dense in \( \mathbb{R}^n \);
  \item \( X \) contains an open dense semi-algebraic subset of \( \mathbb{R}^n \);
\end{enumerate}
(iii) \( \dim(\mathbb{R}^n \setminus X) < n \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( S = \{X_\alpha\}_{\alpha \in I} \) be a semi-algebraic stratification of \( X \). Let \( Y \) be the union of strata of dimension \( n \). Then it is easy to see that \( Y \) has the desired properties.

(ii) \( \Rightarrow \) (i). Clear.

(i) \( \Rightarrow \) (iii). By contradiction, suppose that \( \dim(\mathbb{R}^n \setminus X) = n \). The set \( \mathbb{R}^n \setminus X \) is semi-algebraic (because \( X \) is semi-algebraic) and hence it contains a nonempty open set. This implies that \( X \neq \mathbb{R}^n \), which contradicts the assumption.

(iii) \( \Rightarrow \) (i). Take any \( x \not\in X \). By assumption, \( U \cap X \neq \emptyset \) for all open sets \( U \) containing \( x \). This implies that \( x \in \overline{X} \) and hence \( X \) is dense in \( \mathbb{R}^n \). \( \square \)

As a consequence of Theorem 2.6, we obtain the following useful corollaries.

**Corollary 2.14.** Let \( \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^m \) be an open dense semi-algebraic set. Then there is a set \( \mathcal{V} \subset \mathbb{R}^m \) containing an open dense semi-algebraic set in \( \mathbb{R}^m \) such that for any \( y \in \mathcal{V} \), the semi-algebraic set

\[
\mathcal{U}_y := \{x \in \mathbb{R}^n : (x, y) \in \mathcal{U}\}
\]

contains an open dense semi-algebraic set in \( \mathbb{R}^n \).

**Proof.** Let \( f : (\mathbb{R}^n \times \mathbb{R}^m) \setminus \mathcal{U} \to \mathbb{R}^m \) be the projection \( (x, y) \mapsto y \). In light of Theorem 2.6, we can write \( \mathbb{R}^m = \bigcup_{i=1}^{p} Y_i \) as a disjoint union of semi-algebraic sets \( Y_i \subset \mathbb{R}^m \) such that \( f^{-1}(Y_i) \) is semi-algebraically homeomorphic to \( f^{-1}(y_i) \times Y_i \), for each \( i \) and any \( y_i \in Y_i \). Let

\[
\mathcal{V} := \bigcup_{\dim Y_i = m} \bigcup_{i=1,\ldots,p} Y_i.
\]

Then \( \mathcal{V} \) is a semi-algebraic set in \( \mathbb{R}^m \) satisfying \( \dim(\mathbb{R}^m \setminus \mathcal{V}) < m \). By Corollary 2.13, \( \mathcal{V} \) contains an open dense semi-algebraic subset of \( \mathbb{R}^m \).

Assume for contradiction that there exists \( y \in \mathcal{V} \) such that \( \mathcal{U}_y \) does not contain an open dense semi-algebraic set in \( \mathbb{R}^n \). It follows that \( \mathbb{R}^n \setminus \mathcal{U}_y \) contains an open set in \( \mathbb{R}^n \), which yields that

\[
\dim(f^{-1}(y)) = \dim((\mathbb{R}^n \setminus \mathcal{U}_y) \times \{y\}) = n.
\]

Let \( i \in \{1, \ldots, p\} \) be such that \( y \in Y_i \). Since \( f^{-1}(Y_i) \) is semi-algebraically homeomorphic to \( f^{-1}(y) \times Y_i \), it follows that

\[
\dim(f^{-1}(Y_i)) = \dim(f^{-1}(y)) + \dim Y_i = n + m,
\]

Hence, \( f^{-1}(Y_i) \) contains an open subset of \( \mathbb{R}^n \times \mathbb{R}^m \). This contradicts the assumption that the set \( \mathcal{U} \) is dense in \( \mathbb{R}^n \times \mathbb{R}^m \). The corollary is proved. \( \square \)

**Corollary 2.15.** Let \( X \subset \mathbb{R}^n \) be an unbounded semi-algebraic set. For \( R > 0 \) large enough, the following statements hold:
2.4. Semi-algebraic transversality theorem with parameters. Let $P, X$ and $Y$ be some $C^1$-manifolds of finite dimension; $S$ and $S'$ be $C^1$-submanifolds of $Y$; and $F: X \to Y$ be a $C^1$-mapping. Denote by $d_x F: T_x X \to T_{F(x)} Y$, the differential of $F$ at $x$, where $T_x X$ and $T_{F(x)} Y$ are, respectively, the tangent space to $X$ at $x$ and the tangent space to $Y$ at $F(x)$. We write $\text{rank} F = d$ for some integer $d \geq 0$ if $\text{rank} d_x F = d$ for all $x \in X$.

**Definition 2.16.** The mapping $F$ is said to be transverse to the submanifold $S$, abbreviated by $F \pitchfork S$, if either $F(X) \cap S = \emptyset$ or for each $x \in F^{-1}(S)$, we have

$$d_x F(T_x X) + T_{F(x)} S = T_{F(x)} Y.$$  

The submanifolds $S$ and $S'$ are called transverse at $y \in S \cap S'$, denoted by $S \pitchfork_y S'$, if $T_{y} S + T_{y} S' = T_{y} Y$. Furthermore we say that $S$ is transverse to $S'$ if either $S \cap S' = \emptyset$ or $S \pitchfork_y S'$ for all $y \in S \cap S'$.

**Remark 2.17.** (i) By definition, it easy to see that if $\dim X \geq \dim Y$ and $S = \{s\}$, then $F \pitchfork S$ if and only if either $F^{-1}(s) = \emptyset$ or rank $d_x F = \dim Y$ for all $x \in F^{-1}(s)$. Furthermore, in the case $\dim X < \dim Y$, we have $F \pitchfork S$ if and only if $F^{-1}(S) = \emptyset$.

(ii) Let $y \in S \cap S'$ be such that the manifolds $S$ and $S'$ are transverse at $y$. Then $y$ is a non singular point of $S \cap S'$ and it holds that

$$T_y (S \cap S') = T_y S \cap T_y S'.$$

The following result [12, 13] will be useful in our study.

**Theorem 2.18** (Semi-algebraic transversality theorem with parameters). Let $P, X, Y$, and $S$ are semi-algebraic sets and let $F: P \times X \to Y$ be a semi-algebraic $C^1$-mapping. For each $p \in P$, consider the mapping $F_p: X \to Y$ defined by $F_p(x) := F(p, x)$. If $F \pitchfork S$, then the set

$$Q := \{p \in P : F_p \pitchfork S\}$$

contains an open dense semi-algebraic set in $P$.

**Proof.** The proof for density of $Q$ is done in [12, 13]. Then the desired conclusion follows from Corollary 2.13.
3. Tangent cones and exceptional directions at infinity

In this section, we define and investigate the tangent cone $C_\infty X$ and the set $E_\infty X$ of exceptional directions at infinity of an unbounded semi-algebraic set $X$ in $\mathbb{R}^n$.

First of all, let us agree to call a set $C \subset \mathbb{R}^n$ a cone if whenever $x \in C$, then $tx \in C$ for all $t > 0$ (that is, $C \setminus \{0\}$ is a union of rays). The term “ray” means “open ray emanating from the origin $0 \in \mathbb{R}^n$”, i.e., we consider only “rays” with the endpoint 0 but 0 is not included.

The following lemma is simple but useful whose proof is left to the reader.

**Lemma 3.1.** Let $C \subset \mathbb{R}^n$ be a semi-algebraic cone and let $D := C \cap S^{n-1}$. The following statements hold true:

(i) $D$ is a semi-algebraic set in $\mathbb{R}^n$.

(ii) $v$ is a singular point of $C$ if and only if the ray through $v$ contains only singular points of $C$. In particular, $\text{sing}(C)$ is also a cone and we have $\text{sing}(C) \cap S^{n-1} = \text{sing}(D)$.

(iii) If $v \in D \setminus \text{sing}(D)$, then for all $t > 0$ we have $tv \in C \setminus \text{sing}(C)$ and $T_{tv}C \cong T_vD \oplus \mathbb{R}v$.

(iv) $C \setminus \{0\}$ is homeomorphic to $D \times (0, +\infty)$. In particular, $\dim_v C = \dim_v D + 1$ for all $v \in D$ and so $\dim C = \dim D + 1$.

(v) Assume that $C$ has the form $C := \{tv : t > 0, v \in B\}$ where $B$ is a path connected set in $\mathbb{R}^n$ such that $0 \notin \overline{B}$. Then for all $R > 0$, the set $C \setminus \overline{B}_R$ is path connected.

**Definition 3.2.** By the tangent cone at infinity (known also as the asymptotic cone) of a subset $X$ of $\mathbb{R}^n$ we mean the set $C_\infty X := \left\{ v \in \mathbb{R}^n : \text{there exists sequences } x^k \in X \text{ and } t_k \in (0, +\infty) \text{ such that } x^k \to \infty \text{ and } t_k x^k \to v \text{ as } k \to \infty \right\}$.

By definition, $C_\infty X$ is nonempty if and only if the set $X$ is unbounded. Moreover, we have the following lemma, which is a version at infinity of [21, Lemma 1.2] (see also [11, Corollary 2.18]).

**Lemma 3.3.** Let $X$ be an unbounded semi-algebraic subset of $\mathbb{R}^n$. Then $C_\infty X$ is a nonempty closed semi-algebraic cone of dimension at most $\dim_\infty X$, where $\dim_\infty X$ is defined in Corollary [2,15].

**Proof.** By definition, it is easy to check that $C_\infty X$ is a nonempty closed cone, and it is a semi-algebraic set in light of Theorem [2,2].
It remains to show \( \dim C_\infty X \leq \dim_\infty X \). To this end, for each \( R > 0 \), define
\[
A_R := \{ (tx,t) \in \mathbb{R}^n \times (0, +\infty) : \ x \in X \setminus \mathbb{B}^n_R \}.
\]
Clearly, \( A_R \) is a nonempty semi-algebraic set, which is homeomorphic to \((X \setminus \mathbb{B}^n_R) \times (0, +\infty)\). Hence, for all \( R \) large enough, we have
\[
\dim A_R = \dim_\infty X + 1.
\]
Note that
\[
C_\infty X \times \{0\} \subset \overline{A}_R \cap (\mathbb{R}^n \times \{0\}) \subset \overline{A}_R \setminus A_R.
\]
By Lemma 2.11, therefore
\[
\dim C_\infty X \leq \dim (\overline{A}_R \setminus A_R) < \dim A_R = \dim_\infty X + 1,
\]
which completes the proof of the lemma.

The following lemma strengthens Lemma 3.3.

**Lemma 3.4.** Let \( X \) be an unbounded semi-algebraic subset of \( \mathbb{R}^n \). For each \( v \in C_\infty X \cap S^{n-1} \), there is a sequence \( x^k \in X \setminus \text{sing}(X) \) such that
\[
x^k \to \infty, \quad \frac{x^k}{\|x^k\|} \to v, \quad \text{and} \quad \dim_{x^k} X \geq \dim_v C_\infty X \quad \text{for all} \quad k.
\]

**Proof.** Let \( v \in C_\infty X \cap S^{n-1} \). Since \( \text{sing}(X) \) is nowhere dense in \( X \), it suffices to show that there is a sequence \( x^k \in X \) satisfying the conclusion of the lemma. To this end, let \( d := \dim_v C_\infty X \) and
\[
A := \left\{ u \in C_\infty X \cap S^{n-1} : \text{there exists a sequence} \ x^k \in X \text{ such that} \ x^k \to \infty, \ \frac{x^k}{\|x^k\|} \to u, \ \text{and} \ \dim_{x^k} X \geq d \ \text{for all} \ k \right\}.
\]
Clearly \( A \) is a closed set. Assume for contradiction that \( v \notin A \). Then there is \( \delta > 0 \) such that \( \mathbb{B}_\delta^m(v) \cap A = \emptyset \). Consider the closed cone
\[
C := \{ tx \in \mathbb{R}^n : t \geq 0 \ \text{and} \ x \in \mathbb{B}_\delta^m(v) \cap S^{n-1} \}.
\]
By construction, there exists a constant \( R > 0 \) such that \( \dim((X \cap C) \setminus \mathbb{B}^n_R) < d \), and so \( \dim_\infty (X \cap C) < d \).

On the other hand, it is not hard to see that \( C_\infty (X \cap C) = C_\infty X \cap C \). Therefore
\[
\dim(C_\infty (X \cap C)) = \dim(C_\infty X \cap C) \geq \dim_v C_\infty X = d > \dim_\infty (X \cap C),
\]
which contradicts Lemma 3.3. \( \square \)
We are interested in tangent limits at infinity of an unbounded semi-algebraic subset \( X \) of \( \mathbb{R}^n \). Here, by definition, for \( 1 \leq d \leq \dim X \), a \( d \)-dimensional linear subspace \( P \subset \mathbb{R}^n \) is called a tangent limit at infinity of \( X \) if there exists a sequence \( x^k \in X \setminus \text{sing}(X) \) such that
\[
x^k \to \infty, \quad \dim_{x^k} X = d \quad \text{for all} \quad k, \quad \text{and} \quad T_{x^k} X \to P.
\]
If, in addition, \( \frac{x^k}{\|x^k\|} \to v \), then \( P \) is called a tangent limit at infinity of \( X \) along \( v \).

**Lemma 3.5** (see also [8, Lemma 3.6], [26, Lemma 4], [30, Theorem 11.8, Theorem 22.1]). Let \( X \) be an unbounded semi-algebraic subset of \( \mathbb{R}^n \). Let \( P \) be a tangent limit at infinity of \( X \) along a direction \( v \in \mathbb{S}^{n-1} \). Then there exists a \( C^1 \) semi-algebraic curve \( \varphi: (0, \varepsilon) \to X \setminus \text{sing}(X) \) such that
\[
\lim_{t \to 0^+} \varphi(t) = \infty, \quad \lim_{t \to 0^+} \frac{\varphi(t)}{\|\varphi(t)\|} = v, \quad \text{and} \quad \lim_{t \to 0^+} T_{\varphi(t)} X = P.
\]
Moreover \( v \in P \).

**Proof.** The existence of the curve \( \varphi \) with the desired properties follows immediately from Lemma 2.4. It remains to prove \( v \in P \). To this end, consider the semi-algebraic function \( r: (0, \varepsilon) \to \mathbb{R}, \ t \mapsto \|\varphi(t)\| \). By Lemma 2.3 (perhaps after reducing \( \varepsilon \)), we may assume that \( r(t) \) is of class \( C^1 \) and strictly decreasing on \( (0, \varepsilon) \); moreover, the inverse function \( t(r) \) is also of class \( C^1 \). Set \( u(t) := \frac{\varphi(t)}{\|\varphi(t)\|} \). Now for \( r > 0 \) sufficiently large, the point \( ru(t(r)) = \varphi(t(r)) \) belongs to \( X \setminus \text{sing}(X) \), and so \( (ru(t(r)))' \in T_{ru(t(r))} X \). When \( r \to \infty \), l’Hospital’s Rule gives:
\[
v = \lim_{r \to \infty} \frac{ru(t(r))}{r} = \lim_{r \to \infty} (ru(t(r)))' \in \lim_{r \to \infty} T_{ru(t(r))} X = \lim_{t \to 0^+} T_{\varphi(t)} X = P.
\]
Therefore \( v \in P \), which completes the proof of the lemma. \( \square \)

**Definition 3.6.** By the set of exceptional directions at infinity of an unbounded semi-algebraic set \( X \subset \mathbb{R}^n \) we mean the set
\[
E_\infty X := \left\{ v \in (C_\infty X \setminus \text{sing}(C_\infty X)) \cap \mathbb{S}^{n-1} : \sup_P \angle(P, T_v(C_\infty X)) > 0 \right\},
\]
where the supremum is taken over all tangent limits \( P \) at infinity of \( X \) along \( v \).

Note that exceptional directions are studied first in the complex case in [16, 22, 23] and later in the real case in [8, 26].

As will be shown in the next sections, to obtain the closedness for the image of \( X \) by linear mappings, \( E_\infty X \) is the set that should be avoided. Fortunately, the following result, which is a version at infinity of [8, Theorem 1.1] (see also [26, Theorem 3]), states that \( E_\infty X \) is “small” in \( C_\infty X \cap \mathbb{S}^{n-1} \)-the set of directions in \( C_\infty X \).
Theorem 3.7. Let $X$ be an unbounded semi-algebraic subset of $\mathbb{R}^n$. Then the set $E_\infty X$ is a nowhere dense semi-algebraic set in $D := C_\infty X \cap \mathbb{S}^{n-1}$. In particular,

$$\dim E_\infty X < \dim X - 1.$$ 

Proof. As $X$ is unbounded, $D$ is non empty. By Theorem 2.2, $E_\infty X$ is a semi-algebraic set, so it remains to prove that $E_\infty X$ is nowhere dense in $D$. We consider two cases.

Case 1: $X$ has pure dimension $d := \dim X$. (cf. [8, Theorem 1.1]).

Consider the semi-algebraic function

$$\rho: X \setminus \text{sing}(X) \to [0, +\infty), \quad x \mapsto \|x\|.$$ 

By Theorem 2.5, the set of critical values of $\rho$ is finite. Hence there is $R > 0$ such that the interval $(R, +\infty)$ contains no critical values of $\rho$. Fix $\varepsilon \in (0, \frac{1}{R})$ and let

$$B := \left\{ (u, r) \in \mathbb{S}^{n-1} \times (0, \varepsilon) : \frac{u}{r} \in X \setminus \text{sing}(X) \right\}.$$ 

It is clear that $B$ is semi-algebraic. In addition, $B$ is non singular since it is the inverse image of the non singular set $(X \setminus \text{sing}(X)) \setminus \mathbb{B}_{1/\varepsilon}$ by the diffeomorphism

$$\phi: \mathbb{S}^{n-1} \times (0, +\infty) \to \mathbb{R}^n \setminus \{0\}, \quad (u, r) \mapsto \frac{u}{r}.$$ 

Consider the non empty and non singular semi-algebraic set $\widetilde{B} := (D \setminus \text{sing}(D)) \times \{0\}$ and the semi-algebraic function

$$f: \widetilde{B} \cup B \to \mathbb{R}, \quad (u, r) \mapsto r.$$ 

Note that $\widetilde{B} \subset \overline{B} \setminus B$. Furthermore, we have

$$\rank(f|_{\widetilde{B}}) = 0 \quad \text{and} \quad \rank(f|_{B}) = 1.$$ 

Indeed, the first equation is clear. To prove the second equation, by contradiction, suppose that $d_{(u,r)}f = 0$ for some $(u, r) \in B$. We have $\|\frac{u}{r}\| = \frac{1}{r} > \frac{1}{\varepsilon} > R$ and so $\frac{u}{r}$ is not a critical point of $\rho$. Furthermore,

$$T_{(u,r)}B = \ker d_{(u,r)}f \subset T_u\mathbb{S}^{n-1} \times \{0\}.$$ 

Therefore

$$T_{\frac{u}{r}}(X \setminus \text{sing}(X)) = d_{(u,r)}\phi(T_{(u,r)}B) \subset d_{(u,r)}\phi(T_u\mathbb{S}^{n-1} \times \{0\}) = T_{\frac{u}{r}}\mathbb{S}^{n-1}_{1/\varepsilon},$$ 

and so $\frac{u}{r}$ is a critical point of $\rho$, which is a contradiction.

Let

$$Z := \left\{ v \in D \setminus \text{sing}(D) : \text{the pair } (B, \widetilde{B}) \text{ does not satisfy} \right.$$ 

$$\text{the Thom's } a_f \text{ condition at } (v, 0) \}.$$ 

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Recall that the pair \((B, \tilde{B})\) satisfies the Thom’s\( a_f\) condition at \((v, 0) \in \tilde{B}\) if and only if for any sequence \(w^k = (u^k, r_k) \in B\) converging to \((v, 0) \in \tilde{B}\), we have 

\[
\angle(T_{w^k} f^{-1}(r_k), T_v D \times \{0\}) \to 0 \quad \text{as} \quad k \to +\infty.
\]

By [2, Lemma 15] (see also [20, Lemma 1.2] and [24, Lemma 2]), it is not hard to see that \(Z\) is nowhere dense in \(D \setminus \text{sing}(D)\).

Now to prove the theorem, it is enough to show that \(E_\infty X \subset Z\), or, equivalently,

\[
D \setminus (Z \cup \text{sing}(D)) \subset D \setminus (E_\infty X \cup \text{sing}(D)).
\]

To this end, take arbitrarily \(v \in D \setminus (Z \cup \text{sing}(D))\). We need to show that \(v \not\in E_\infty X\). In fact, let \(P\) be a tangent limit at infinity of \(X\) along \(v\). By definition, there is a sequence \(x^k \in X \setminus \text{sing}(X)\) such that 

\[
x^k \to \infty, \quad \frac{x^k}{\|x^k\|} \to v, \quad \text{and} \quad T_{x^k} X \to P.
\]

Set \(r_k := \frac{1}{\|x^k\|}\) and \(w^k := \frac{x^k}{\|x^k\|}\). Then the sequence \(w^k := (u^k, r_k) \in B\) tends to \((v, 0) \in \tilde{B}\). Since \(v \not\in Z \cup \text{sing}(D)\), it holds that

\[
\angle(T_{w^k} f^{-1}(r_k), T_v D \times \{0\}) \to 0 \quad \text{as} \quad k \to \infty.
\]

Equivalently,

\[
\angle(T_{w^k} [(r_k(X \setminus \text{sing}(X)) \cap S^{n-1}) \times \{r_k\}], T_v D \times \{0\}) \to 0 \quad \text{as} \quad k \to \infty,
\]

where

\[
r_k(X \setminus \text{sing}(X)) := \{r_k x : x \in X \setminus \text{sing}(X)\}.
\]

Consequently,

\[
\angle(T_{w^k} [r_k(X \setminus \text{sing}(X)) \cap S^{n-1}], T_v D) \to 0 \quad \text{as} \quad k \to \infty. \quad (2)
\]

On the other hand, \(\|x^k\| > R\) for all large \(k\). For all such \(k\), \(x^k\) is a regular point of the function \(\rho\) and so it is a non singular point of \(X \cap S^{n-1}_{\frac{r_k}{r_k^\infty}}\). This, combined with the assumption that \(X\) has pure dimension \(d\), Lemma [3.1 iv) and Lemma [3.3] yields

\[
\dim_{x^k}(X \cap S^{n-1}_{\frac{r_k}{r_k^\infty}}) = \dim X - 1 = \dim_{\infty} X - 1 \geq \dim C_\infty X - 1 = \dim D \geq \dim_v D = \dim T_v D.
\]
By definition, we obtain
\[
\angle(T_{x^k}X, T_vD) = \sup \{ z, \pi_{T_{x^k}X}(z) : z \in T_vD \setminus \{0\} \} \\
\leq \sup \{ z, \pi_{T_{x^k}(X \cap S^n_{\frac{1}{r_k}})}(z) : z \in T_vD \setminus \{0\} \} \\
= \angle(T_{x^k}(X \cap S^n_{\frac{1}{r_k}}), T_vD) \\
= \angle(T_{x^k}((X \setminus \text{sing}(X)) \cap S^n_{\frac{1}{r_k}})), T_vD),
\]
where \( \pi_{T_{x^k}X} \) and \( \pi_{T_{x^k}(X \cap S^n_{\frac{1}{r_k}})} \) are the orthogonal projections on \( T_{x^k}X \) and \( T_{x^k}(X \cap S^n_{\frac{1}{r_k}}) \) respectively. Observe that the homothety
\[
(X \setminus \text{sing}(X)) \cap S^n_{\frac{1}{r_k}} \rightarrow r_k(X \setminus \text{sing}(X)) \cap S^n_{\frac{1}{r_k}}, \; x \mapsto r_k x,
\]
is a diffeomorphism. In particular, the linear subspaces \( T_{x^k}((X \setminus \text{sing}(X)) \cap S^n_{\frac{1}{r_k}}) \) and \( T_{x^k}((X \setminus \text{sing}(X)) \cap S^n_{\frac{1}{r_k}}) \) determine the same plane in the Grassmannian of \((d-1)\)-dimensional linear subspaces of \( \mathbb{R}^n \). Therefore
\[
\angle(T_{x^k}X, T_vD) \leq \angle(T_{x^k}(r_k(X \setminus \text{sing}(X)) \cap S^n_{\frac{1}{r_k}}), T_vD).
\]
This, together with (2), gives us \( \angle(P, T_vD) = 0 \). Note that \( T_v(C_\infty X) = T_vD \oplus \mathbb{R} v \) (by Lemma 3.1(iii)) and \( v \in P \) (by Lemma 3.5). Hence, \( \angle(P, T_v(C_\infty X)) = 0 \). This yields \( v \notin E_\infty X \), which ends the proof for the case \( X \) has pure dimension \( d \).

**Case 2: \( X \) is an arbitrary semi-algebraic set.**

For \( i = 0, \ldots, \dim X \), set
\[
X_i := \{ x \in X : \dim_x X = i \}.
\]
Then \( X_i \) is either empty or of pure dimension \( i \). Moreover \( X = \bigcup_{i=0}^{\dim X} X_i \). Let \( C_\infty X_i \) denote the tangent cone at infinity of \( X_i \). In view of Proposition 2.8 there is a semi-algebraic stratification \( \mathcal{S} = \{ D_\alpha \}_{\alpha \in \mathcal{I}} \) of \( D \) such that, for \( i = 1, \ldots, \dim X \), each of the sets \( C_\infty X_i \cap S^{n-1} \) and \( E_\infty X_i \) is a finite union of strata in \( \mathcal{S} \). From Remark 2.9 to prove that \( E_\infty X \) is nowhere dense in \( D \), it is enough to show that for any stratum \( D_\alpha \) not contained in the boundary of another strata (i.e., there is no stratum \( D_\beta \) such that \( D_\alpha \subset D_\beta \)), we have \( D_\alpha \cap E_\infty X = \emptyset \).

Suppose for contradiction that \( D_\alpha \cap E_\infty X \neq \emptyset \). Then there is an integer \( 0 < d \leq \dim X \), a \( d \)-dimensional linear subspace \( P \subset \mathbb{R}^n \) and a sequence \( x^k \in X \setminus \text{sing}(X) \) such that \( \dim_{x^k} X = d \) for all \( k \); moreover, we have
\[
x^k \to \infty, \; \frac{x^k}{\|x^k\|} \to v \in D_\alpha \cap E_\infty X, \; T_{x^k}X \to P, \; \text{and} \; \angle(P, T_v(C_\infty X)) > 0.
\]
Clearly \( v \in C_\infty X_d \). By the construction, \( C_\infty X_d \cap S^{n-1} \) is a finite union of strata in \( \mathcal{S} \) including \( D_\alpha \). This, together with the fact that \( D_\alpha \) is not contained in the boundary of another strata,
implies that \( v \notin \text{sing}(C^\infty X_d \cap S^{n-1}) \). So \( v \notin \text{sing}(C^\infty X_d) \) in view of Lemma 3.1(ii). On the other hand, since \( v \in D_\alpha \) and \( D_\alpha \) is not contained in the boundary of another strata, it follows that \( \dim_v D = \dim_v(C^\infty X_d \cap S^{n-1}) \), i.e., \( \dim_v C^\infty X = \dim_v C^\infty X_d \), so
\[
T_v(C^\infty X) = T_v(C^\infty X_d).
\]
Consequently
\[
\angle(P, T_v(C^\infty X_d)) > 0,
\]
which yields \( v \in E_\infty X_d \). In particular, \( D_\alpha \cap E_\infty X_d \neq \emptyset \). By the construction, \( C^\infty X_d \cap S^{n-1} \) and \( E_\infty X_d \) are unions of strata in \( S \), thus
\[
D_\alpha \subset E_\infty X_d \subset C^\infty X_d \cap S^{n-1}.
\]
Since \( D_\alpha \) is not contained in the boundary of another strata in \( S \), it follows that \( D_\alpha \) is not nowhere dense in \( C^\infty X_d \cap S^{n-1} \), and so neither is \( E_\infty X_d \). This contradicts Case 1. Hence \( E_\infty X \) is nowhere dense in \( D \) and the theorem is proved.

\[\square\]

4. Sufficient conditions for closedness

In this section, we provide some sufficient conditions for the closedness of the images by linear mappings of semi-algebraic sets. Let us start with the following notation.

**Definition 4.1** (cf. [28, 19]). Given a linear mapping \( F : \mathbb{R}^n \to \mathbb{R}^m \), we set
\[
\nu(F) := \inf_{y \in \mathbb{R}^m, \|y\|=1} \|F^* y\|,
\]
where \( F^* \) stands for the adjoint operator of \( F \).

By definition, \( \nu(F) > 0 \) if and only if \( F \) is surjective. Furthermore, we have the following simple observation.

**Lemma 4.2.** Let \( L \subset \mathbb{R}^n \) be a linear subspace and let \( F : \mathbb{R}^n \to L \) be a linear mapping such that \( F|_L \) is the identity mapping. Let \( L_k \subset \mathbb{R}^n \) be a sequence of linear subspaces of same dimension such that \( \dim L_k \geq \dim L \). If \( \lim_{k \to \infty} \angle(L_k, L) = 0 \), then
\[
\liminf_{k \to \infty} \nu(F|_{L_k}) \geq 1.
\]

**Proof.** Without loss of generality, we may suppose that the sequence \( L_k \) is convergent, so let \( L' := \lim_{k \to \infty} L_k \). Since \( \dim L_k \geq \dim L \) and \( \angle(L_k, L) \to 0 \) by the assumption, we get \( L \subseteq L' \). This, together with the assumption that \( F|_L \) is the identity mapping, implies that
\[
\mathbb{B}^n \cap L \subset F(\mathbb{B}^n \cap L').
\]
By [19, Proposition 2.3], we have
\[
\nu(F|_{L'}) = \sup\{r \geq 0 : \mathbb{B}^n \cap L \subset F(\mathbb{B}^n \cap L')\}.
\]
Hence $\nu(F|_{L'}) \geq 1$. By combining this with [9, Claim 3.8], we obtain
\[
\lim_{k \to \infty} \nu(F|_{L_k}) = \nu(F|_{L'}) \geq 1,
\]
which completes the proof of the lemma. \qed

We provide a sufficient condition for the restriction of a linear mapping on a cone to be surjective.

**Lemma 4.3.** Let $C \subset \mathbb{R}^n$ be a semi-algebraic cone and let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. If there exists $v \in C \setminus \text{sing}(C)$ such that $d_v F|_{T_v(C \cap S^{n-1})}$ is surjective, then
\[
F(C \setminus \overline{E_R}) = \mathbb{R}^m \quad \text{for any } R > 0.
\]
In particular, $F(C) = \mathbb{R}^m$.

**Proof.** By the assumption, there exists a constant $\delta > 0$ such that
\[
\mathbb{B}_\delta^m \subset F(C \cap S^{n-1}).
\]
Since $C$ is a cone with vertex at the origin, we have for all $R > 0$,
\[
F(C \setminus \overline{E_R}) = \bigcup_{r > R} F[r(C \cap S^{n-1})] = \bigcup_{r > R} r F(C \cap S^{n-1}) \supset \bigcup_{r > R} r \mathbb{B}_\delta^m = \bigcup_{r > R} \mathbb{B}_{r\delta}^m = \mathbb{R}^m,
\]
which yields $F(C \setminus \overline{E_R}) = \mathbb{R}^m$. \qed

The main result of this section is as follows.

**Theorem 4.4.** Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set and let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping of maximal rank. Set $D := C_\infty X \cap S^{n-1}$ and assume that one of the following conditions holds:
1. $n \leq m$;
2. $\ker F \cap D = \emptyset$;
3. there exists a point $v \in (\ker F \cap D) \setminus (\overline{E_\infty X} \cup C_\infty(\text{sing}(X)) \cup \text{sing}(D))$ such that the linear mapping $d_v F|_{T_v D}$ is surjective. Recall that the set $E_\infty X$ is defined in [1].

Then $F(X)$ is closed. Moreover, if Condition (c) is satisfied, then $F(X) = \mathbb{R}^m$.

**Proof.** (a) Observe that the case $n \leq m$ is trivial since in this case, $F$ is a linear isomorphism from $\mathbb{R}^n$ onto $F(\mathbb{R}^n)$. So for the remainder of the proof, we suppose that $n > m$.

(b) Let $y \in \overline{F(X)}$. Then there is a sequence $x^k \in X$ such that $\lim_{k \to \infty} F(x^k) = y$. We will show that $x^k$ has a convergent subsequence. For contradiction, assume that $x^k \to \infty$. 

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Passing to a subsequence if necessary, we may suppose that \( \lim_{k \to \infty} \frac{x^k}{\|x^k\|} = u \in D \). Since \( D \) is compact and \( \ker F \cap D = \emptyset \), it holds that \( \|F(u)\| \geq \min_{x \in D} \|F(x)\| > 0 \) and so

\[
\|y\| = \lim_{k \to \infty} \|F(x^k)\| = \lim_{k \to \infty} \|x^k\| \left\| F \left( \frac{x^k}{\|x^k\|} \right) \right\| = +\infty,
\]

which is impossible. Therefore the sequence \( x^k \) has a cluster point, say \( x \). Clearly \( x \in X \) and \( F(x) = y \), so \( y \in F(X) \).

(c) Take any \( x \in \mathbb{R}^n \). Since \( d_v F|_{T_vD} \) is a surjective mapping from \( T_v D \) onto \( \mathbb{R}^m \), there is \( y \in T_v D \) such that \( d_v F(y) = F(x) \). Observe that \( d_v F(y) = F(y) \) because \( F \) is a linear mapping. Thus, \( F(x) = F(y) \), and so

\[
x = (x - y) + y \in \ker F + T_v D.
\]

Since \( x \) is arbitrary in \( \mathbb{R}^n \), this implies that \( \ker F + T_v D = \mathbb{R}^n \). Consequently,

\[
\ker F + T_v(C_\infty X) = \mathbb{R}^n \tag{3}
\]

as it is clear that \( T_v D \subset T_v(C_\infty X) \subset \mathbb{R}^n \). Let \( W \) be the orthogonal complement of the linear subspace \( \ker F \cap T_v(C_\infty X) \) in \( \ker F \). Then \( W \cap T_v(C_\infty X) = \{0\} \). In addition, from (3) we have

\[
W \oplus T_v(C_\infty X) = \mathbb{R}^n.
\]

Let

\[
\pi : W \oplus T_v(C_\infty X) \to T_v(C_\infty X)
\]

be the natural projection on \( T_v(C_\infty X) \). Clearly \( \pi \) is a linear mapping. Furthermore, for any \( x \in \mathbb{R}^n \), we have \( \pi(x) - x \in W \subset \ker F \), so \( F(\pi(x)) = F(x) \). Consequently, \( F(\pi(X)) = F(X) \).

For \( \delta > 0 \), let

\[
C_\delta(v) := \{ tx : t > 0 \text{ and } x \in \mathbb{B}^n_\delta(v) \cap \{v\} + T_v D \}, \tag{4}
\]

which is a semi-algebraic cone and is an open set in the linear subspace \( T_v(C_\infty X) \). Clearly,

\[
sing(C_\delta(v)) = \emptyset, \quad v \in C_\delta(v), \quad \text{and} \quad T_v(C_\delta(v)) \cap S^{n-1} = T_v D.
\]

By our assumption (c), the linear mapping \( d_v F|_{T_v(C_\delta(v))\cap S^{n-1}} \) is surjective. Hence in light of Lemma 4.3 \( \mathbb{R}^m = F(C_\delta(v) \setminus \mathbb{B}^n_R) \) for all \( R > 0 \).

Assume that we have proved the following inclusion:

\[
C_\delta(v) \setminus \mathbb{B}^m_R \subset \pi(X) \tag{5}
\]

for some \( \delta > 0 \) and \( R > 0 \). This, of course, implies that

\[
\mathbb{R}^m = F(C_\delta(v) \setminus \mathbb{B}^m_R) \subset F(\pi(X)) = F(X),
\]

which yields \( F(X) = \mathbb{R}^m \), and so the image \( F(X) \) is closed.
Therefore, we are left with proving the inclusion (5). We distinguish two cases.

Case 1: $\dim T_v(C_\infty X) = n$. In this case $\pi$ is the identity mapping; in particular, $\pi(X) = X$.

Observe that the cone $C_\infty(sing(X))$ is a (possibly empty) closed set (by Lemma 3.3) and does not contain the point $v$ (by our assumption (c)). Hence we can find a constant $\delta > 0$ such that
\[ B_\delta^n(v) \cap C_\infty(sing(X)) = \emptyset. \] (6)
We will show that there is $R > 0$ such that the inclusion (5) holds, or equivalently,
\[ C_\delta(v) \setminus B_R^n \subset X. \]
Indeed, if this is not the case, then there exists a sequence $x^k \in C_\delta(v) \setminus X$ such that $x^k \to \infty$. Passing to a subsequence if necessary, suppose that $x^k / \|x^k\| \to u$. Clearly, $u \in B_\delta^n(v)$. On the other hand, by Lemma 3.4 we can find a sequence $y^k \in X \setminus sing(X)$ such that $y^k \to \infty$, $y^k / \|y^k\| \to u$, and $\dim y^k X = n$ for all $k$.

Observe that $y^k$ is an interior point of $X$. Thus there exists a point $z^k$ in the segment $[x^k, y^k]$ such that $z^k$ is a boundary point of $X$. Clearly, $z^k \in sing(X)$. Moreover, it is not hard to check that $z^k / \|z^k\| \to u$, which yields $u \in C_\infty(sing(X))$. Thus
\[ u \in B_\delta^n(v) \cap C_\infty(sing(X)), \]
which contradicts (6). Therefore, the inclusion (5) holds in this case.

Case 2: $\dim T_v(C_\infty X) < n$. We have
\[ \dim W = \dim(\ker F) - \dim(\ker F \cap T_v(C_\infty X)) \]
\[ = \dim(\ker F) - [\dim(\ker F) + \dim T_v(C_\infty X) - n] \]
\[ = n - \dim(T_v(C_\infty X)) > 0, \]
where the second equality follows from (13). Therefore $W \neq \{0\}$. Since $W \oplus T_v(C_\infty X) = \mathbb{R}^n$, it follows from Lemma 3.1(iii) that $W \oplus T_vD = v^\perp$, here and in the following $v^\perp$ stands for the hyperplane perpendicular to $v$ through the origin.

For $\delta > 0$, set
\[ Q_\delta(v) := \{ x + y : x \in B_\delta^n(v) \cap (\{v\} + T_vD), y \in W, \|y\| < \delta \}. \]
The boundary of $Q_\delta(v)$ can be written as the union $H \cup V$ where
\[ H := \{ x + y : x \in B_\delta^n(v) \cap (\{v\} + T_vD), y \in W, \|y\| = \delta \}, \]
\[ V := \{ x + y : x \in S_\delta^{n-1}(v) \cap (\{v\} + T_vD), y \in W, \|y\| \leq \delta \}. \]
Let $Q^*_\delta(v)$, $H^*$ and $V^*$ be, respectively, the union of all rays from the origin in $\mathbb{R}^n$ through points of $Q_\delta(v)$, $H$ and $V$. The construction can be seen in Figures 1 and 2.

Let $X_\delta := X \cap Q^*_\delta(v)$, which is an unbounded semi-algebraic set. Inspired by [26, Lemma 5], we have the following technical lemma.

**Lemma 4.5.** The following statements hold:

(i) We have $Q^*_\delta(v) \subset \{v\} + v^\perp$ and moreover, $Q^*_\delta(v)$ is an open neighborhood of $v$ in $\{v\} + v^\perp$ for the induced topology. In particular, $Q^*_\delta(v)$ is an open set in $\mathbb{R}^n$.

(ii) For any $w \in Q^*_\delta(v)$, we have

$$\|w - u\| \leq 2\delta\|u\| \quad \text{and} \quad \|u\| \leq \|\pi(w)\|,$$

where $u$ is the orthogonal projection of $w$ on $\mathbb{R}v$.

(iii) For $\delta > 0$ small enough, the intersection $C_\infty X \cap Q^*_\delta(v)$ contains only non singular points of $C_\infty X \cap (\{v\} + v^\perp)$. In particular, $C_\infty X \cap Q^*_\delta(v)$ is non singular.

(iv) For $\delta > 0$ small enough, $C_\infty X \cap H = \emptyset$.

(v) For $\delta > 0$ small enough, there exists $r > 0$ such that $(X_\delta \setminus \mathbb{B}^n_r) \cap H^* = \emptyset$.

(vi) For $\delta > 0$ small enough, there exists $r > 0$ such that $X_\delta \setminus \mathbb{B}^n_r$ is non singular and each connected component of $X_\delta \setminus \mathbb{B}^n_r$ is unbounded. Moreover, $\dim(X_\delta \setminus \mathbb{B}^n_r) \geq \dim_v(C_\infty X)$.

(vii) For every $\epsilon > 0$, there are $\delta > 0$ and $r > 0$ such that

$$\angle(T_x X_\delta, T_v(C_\infty X)) < \epsilon \quad \text{for all} \quad x \in X_\delta \setminus \mathbb{B}^n_r.$$
Figure 2: Picture in $Q_\delta^*(v)$.

Proof. (i) Straightforward from the construction.

(ii) Take arbitrarily $w \in Q_\delta^*(v)$. Let $w'$ be the intersection of $Q_\delta(v)$ and the ray through $w$, then

$$\frac{\|w' - v\|}{\|w - u\|} = \frac{\|v\|}{\|u\|} = \frac{1}{\|u\|}.$$  

So

$$\|w - u\| = \|w' - v\|\|u\| \leq 2\delta\|u\|.$$  

Finally, by the construction, we have $\pi(w) - w \in W \subset v^\perp$ and $u - w \in v^\perp$. Hence, $\pi(w) - u \in v^\perp = u^\perp$, and so $\pi(w) - u \perp u$. By the Pythagorean theorem, $\|\pi(w)\|^2 = \|\pi(w) - u\|^2 + \|u\|^2$, whence $\|\pi(w)\| \geq \|u\|$. (The calculations can be seen from Figure 3.)
(iii) Since $v$ is a non singular point of $D$, it follows from Lemma 3.1(iii) that $v$ is also a non singular point of $C\infty X$ and $T_v(C\infty X) \supset \mathbb{R}^n$. Hence, by definition, $C\infty X \cap w (\{v\} + v^\perp)$. Consequently, for any $u \in C\infty X \cap (\{v\} + v^\perp)$ close enough to $v$, it holds that $u \not\in \text{sing}(C\infty X)$ and $C\infty X \cap u (\{v\} + v^\perp)$. In particular, $C\infty X \cap (\{v\} + v^\perp)$ is non singular at $u$. Observe that

$$C\infty X \cap Q_\delta(v) \subset C\infty X \cap (\{v\} + v^\perp)$$

and $\|u - v\| < 2\delta$

for any $u \in C\infty X \cap Q_\delta(v)$. Hence, by taking $\delta > 0$ sufficiently small, item (iii) follows.

(iv) Let $\delta > 0$ be sufficiently small so that $C\infty X \cap Q_\delta(v)$ contains only non singular points of $C\infty X \cap (\{v\} + v^\perp)$ in view of item (iii). Clearly $H \subset Q_\delta(v)$, so $C\infty X \cap H \subset C\infty X \cap Q_\delta(v)$. For $u \in C\infty X \cap Q_\delta(v)$, we have $\|u - v\| \leq 2\delta$. Moreover, since $v$ is a non singular point of $C\infty X \cap Q_\delta(v)$, it holds that

$$\angle(u - v, T_v(C\infty X \cap Q_\delta(v))) \to 0$$

as $u \to v$ with $u \in C\infty X \cap Q_\delta(v)$. Shrinking $\delta$ if necessary, we may assume that

$$\sin(\angle(u - v, T_v(C\infty X \cap Q_\delta(v)))) < \frac{c}{2}$$

(7)
for any \( u \in C_\infty X \cap \overline{Q_\delta(v)} \), where
\[
c := \inf_{w \in W; \|w\|=1} \sin(\angle(w, T_v D)) > 0.
\]
Note that from the proof of item (iii), we have \( C_\infty X \cap v \) \( \{v\} + v^\perp \). So \( C_\infty X \cap v \) \( \mathbb{S}^{n-1} \) and moreover, \( C_\infty X \cap Q_\delta(v) \) in view of item (i). Combining these facts with Remark 2.17(ii), we get
\[
T_v(C_\infty X \cap Q_\delta(v)) = T_v(C_\infty X) \cap T_v(Q_\delta(v)) = T_v(C_\infty X) \cap v^\perp = T_v(C_\infty X) \cap \mathbb{S}^{n-1} = T_v D.
\]
This, together with (7), implies that for any \( u \in C_\infty X \cap Q_\delta(v) \), we have
\[
\sin(\angle(u - v, T_v D)) < \frac{c}{2}. \tag{8}
\]
Now take arbitrarily \( u \in H \). By definition, there exist \( x \in \overline{E_\delta(v)} \cap \{v\} + T_v D \) and \( y \in \mathcal{W} \) with \( \|y\| = \delta \) such that \( u = x + y \). Then \( x - v \in T_v D \) and
\[
\text{dist}(u - v, T_v D) = \text{dist}(u - x + x - v, T_v D) = \text{dist}(y + (x - v), T_v D)
\]
\[
= \text{dist}(y, T_v D) = \sin\left(\angle\left(\frac{y}{\|y\|}, T_v D\right)\right) \|y\| \geq c\|y\| = c\delta.
\]
This implies that
\[
\sin(\angle(u - v, T_v D)) = \frac{\text{dist}(u - v, T_v D)}{\|u - v\|} \geq \frac{\text{dist}(u - v, T_v D)}{\|u - x\| + \|x - v\|} \geq \frac{c}{2},
\]
which combined with (8) gives the desired conclusion.

(v) Let \( \delta > 0 \) be sufficiently small as in item (iv). Assume for contradiction that there exists a sequence \( x^k \in X_\delta \cap H^* \) such that \( x^k \to \infty \). Let \( \ell_k \) be the ray through \( x^k \). Passing to a subsequence if necessary, we may suppose that the sequence of rays \( \ell_k \) converges to a ray \( \ell \). Clearly, \( \ell \subset C_\infty X \cap H^* \). Hence \( C_\infty X \cap H \neq \emptyset \), which contradicts item (iv).

(vi) We know that \( C_\infty (\text{sing}(X)) \) is a closed set (by Lemma 3.3) and does not contain the point \( v \) (by our assumption (c)). Hence for all sufficiently small \( \delta > 0 \) we have
\[
\overline{E_\delta(v)} \cap C_\infty (\text{sing}(X)) = \emptyset.
\]
For each such \( \delta \), there exists \( r = r(\delta) > 0 \) such that \( Q_\delta^*(v) \setminus \overline{E_\delta(v)} \) does not contain any singular point of \( X \). Observe that \( Q_\delta^*(v) \) is an open set, so for \( \delta > 0 \) small enough, \( X_\delta \setminus \overline{E_\delta(v)} \) is non singular.

Since \( X_\delta \setminus \overline{E_\delta(v)} \) is a semi-algebraic set, it has only a finite number of connected components. Hence, by increasing \( r \) if necessary we may assume that each connected component of \( X_\delta \setminus \overline{E_\delta(v)} \) is unbounded. Moreover, in view of Lemma 3.3, we have
\[
\dim(X_\delta \setminus \overline{E_\delta(v)}) \geq \dim(C_\infty (X_\delta \setminus \overline{E_\delta(v)})) \geq \dim_v(C_\infty (X_\delta \setminus \overline{E_\delta(v)})) = \dim_v(C_\infty X).
\]
(vii) Let $\delta > 0$ be sufficiently small and $r > 0$ be sufficiently large such that items (iii) and (vi) hold.

Note that $D = C_\infty X \cap S^{n-1}$ is a closed set because the cone $C_\infty X$ is closed (in view of Lemma 3.3). By the definition of differentiable manifolds, it is not hard to see that $\text{sing}(D)$ is also a closed set. Since $v \notin \overline{E_\infty X} \cup \text{sing}(D)$ (by our assumption (c)), by shrinking $\delta$ if necessary, we get

$$Q_\delta^*(v) \cap (\overline{E_\infty X} \cup \text{sing}(D)) = \emptyset.$$ 

On the other hand, it is clear that

$$C_\infty X_\delta \subset C_\infty X \cap Q_\delta^*(v) \subset Q_\delta^*(v).$$

Therefore

$$C_\infty X_\delta \subset (\overline{E_\infty X} \cup \text{sing}(D)) = \emptyset. \quad (9)$$

Let $\epsilon > 0$. We claim that for all $r$ large enough,

$$\sup_{x \in X_\delta \setminus \mathbb{E}_r} \left[ \inf_{u \in C_\infty X \cap Q_\delta^*(v)} \angle(T_x X_\delta, T_u(C_\infty X)) \right] < \frac{\epsilon}{2}. \quad (10)$$

In fact, if this is not the case, then there exists a sequence $x^k \in X_\delta$ with $x^k \to \infty$ such that

$$\theta := \lim_{k \to \infty} \inf_{u \in C_\infty X \cap Q_\delta^*(v)} \angle(T_{x^k} X_\delta, T_u(C_\infty X)) \geq \frac{\epsilon}{2}. \quad (11)$$

Passing to a subsequence if necessary, we can suppose that $\frac{x^k}{\|x^k\|} \to \omega \in C_\infty X_\delta$. Let $\ell_\omega$ be the ray through $\omega$. Observe that $\ell_\omega \subset Q_\delta^*(v)$, so let $\tilde{\omega}$ be the intersection point of $\ell_\omega$ and $Q_\delta(v)$. It is clear that $\tilde{\omega} \in C_\infty X \cap Q_\delta(v)$ and so

$$\inf_{u \in C_\infty X \cap Q_\delta^*(v)} \angle(T_{x^k} X_\delta, T_u(C_\infty X)) \leq \angle(T_{x^k} X_\delta, T_{\tilde{\omega}}(C_\infty X)).$$

On the other hand, we deduce from (9) that $\omega \notin \overline{E_\infty X} \cup \text{sing}(D)$. By Lemma 3.3 ii), $\tilde{\omega} \notin \text{sing}(C_\infty X)$ and so $T_{\tilde{\omega}}(C_\infty X) = T_{\omega}(C_\infty X)$. Thus,

$$\angle(T_{x^k} X_\delta, T_{\tilde{\omega}}(C_\infty X)) = \angle(T_{x^k} X_\delta, T_{\omega}(C_\infty X)) \to 0.$$ 

Therefore, $\theta = 0$, which contradicts (11). Hence, (10) holds for all $r$ large enough.

We now let

$$\varphi(\delta) := \sup_{u \in C_\infty X \cap Q_\delta(v)} \angle(T_u(C_\infty X), T_v(C_\infty X)).$$

Since $T_u(C_\infty X) \to T_v(C_\infty X)$ as $u \to v$ with $u \in C_\infty X$ and since $\|u - v\| < 2\delta$ for $u \in C_\infty X \cap Q_\delta(v)$, it follows that $\varphi(\delta) \to 0$ as $\delta \to 0$. So for $\delta > 0$ small enough, we have $\varphi(\delta) < \frac{\epsilon}{2}$. In combining this with (10), we obtain for every $x \in X_\delta \setminus \overline{\mathbb{E}_r}$ that

$$\angle(T_x X_\delta, T_v(C_\infty X)) \leq \inf_{u \in C_\infty X \cap Q_\delta^*(v)} \left( \angle(T_x X_\delta, T_u(C_\infty X)) + \angle(T_u(C_\infty X), T_v(C_\infty X)) \right) < \epsilon.$$ 

The lemma is proved. \qed
We are now in position to finish the proof of the inclusion (5) and hence that of the theorem. To this end, let \( \delta \in \left(0, \frac{1}{2}\right) \) be sufficiently small and \( r > 0 \) be sufficiently large such that the conclusions of Lemma 4.5 hold. Let \( \Gamma \) be a connected (unbounded) component of \( X_\delta \setminus \overbar{B}_r \) of dimension at least \( \dim_v(C_\infty X) \). By Lemma 4.2 and Lemma 4.5(vii), we can see that
\[
\nu(\pi|_{T_x \Gamma}) \geq \frac{1}{2} \quad \text{for all} \quad x \in \Gamma,
\tag{12}
\]
(perhaps after decreasing \( \delta \) and increasing \( r \)). Let \( R > 0 \) be large enough so that \( \pi(\overbar{B}_r) \subset \overbar{B}_R \).

In order to prove the inclusion (5), it is enough to show that \( \pi \) is surjective. Consequently, \( \pi \) theorem. To this end, let \( \Gamma \) be the orthogonal projection of \( x \) such that \( \gamma \) be the orthogonal projection of \( x \) such that \( \pi \) is surjective. Consequently, \( \pi \) is a submersion at \( a \). Hence \( \pi(\Gamma) \) holds. Let \( \delta \) be such that \( \pi(\overbar{B}_r \cup \pi(\Gamma)) \). By the construction, \( \Gamma \subset Q_\delta(v) \), and so
\[
\pi(\Gamma) \subset \pi(Q_\delta^*(v)) = C.
\]
On the other hand, since \( \Gamma \) is unbounded, there exists \( w \in \Gamma \) with \( \|w\| > 2R \). Denote by \( u \) the orthogonal projection of \( w \) on \( \mathbb{R}v \). In light of Lemma 1.5(ii), we have
\[
\|\pi(w)\| \geq \|u\| \geq \|w\| - \|w - u\| \geq \|w\| - 2\delta\|u\| \geq (1 - 2\delta)\|w\| > \frac{\|w\|}{2} > R.
\]
Therefore, \( y := \pi(w) \in \pi(\Gamma) \cap (C \setminus \overbar{B}_R) \). In view of (12), the linear mapping
\[
d_w(\pi|_\Gamma) : T_w \Gamma \to T_y(T_v(C_\infty X))
\]
is surjective. Consequently, \( \pi|_\Gamma \) is a submersion at \( w \). Since \( C \setminus \overbar{B}_R \) is open in the linear subspace \( T_v(C_\infty X) \), it follows that \( y \) in an interior point of \( \pi(\Gamma) \cap (C \setminus \overbar{B}_R) \) for the induced topology on \( T_v(C_\infty X) \). As \( C \setminus \overbar{B}_R \) is path connected (by Lemma 3.1(i)), there is a continuous curve
\[
\gamma : [0, 1] \to C \setminus \overbar{B}_R
\]
such that \( \gamma(0) = x \) and \( \gamma(1) = y \). Let \( t_0 \in [0, 1] \) be such that
\[
\gamma(0, t_0) \cap \pi(\Gamma) = \emptyset \quad \text{and} \quad z := \gamma(t_0) \in \overbar{\pi(\Gamma)}.
\]
Obviously, \( z \) is a boundary point of \( \overbar{\pi(\Gamma)} \) in \( C \setminus \overbar{B}_R \). So we can find a sequence \( x^k \in \Gamma \) such that \( \pi(x^k) \to z \). With no loss of generality, we may assume that \( \|\pi(x^k)\| \leq \|z\| + 1 \). Let \( u^k \) be the orthogonal projection of \( x^k \) on \( \mathbb{R}v \). In light of Lemma 4.5(ii), we have
\[
\|x^k\| \leq \|x^k - u^k\| + \|u^k\| \leq (2\delta + 1)\|u^k\| \leq (2\delta + 1)\|\pi(x^k)\| \leq (2\delta + 1)(\|z\| + 1),
\]
which yields that the sequence \( x^k \) is bounded. Passing to a subsequence if necessary, we may assume that the sequence \( x^k \) converges to a limit \( a \in \overbar{\Gamma} \). Since \( \Gamma \subset X_\delta \setminus \overbar{B}_r \), it follows from Lemma 4.5(v) that \( a \notin H^* \). Furthermore, since \( C \cap V^* = \emptyset \) and \( z \in C \), we have \( \pi(a) = z \notin V^* \). This, together with the definition of \( \pi \), implies that \( a \notin V^* \). Hence \( a \in Q_\delta^*(v) \) and so \( a \in \Gamma \). Consequently, from (12), the mapping \( \pi|_\Gamma \) is a submersion at \( a \). Note that the
point \( z = \pi(a) \) belongs to the set \( C \setminus \overline{B}_R \), which is open in the linear subspace \( T_v(C_\infty X) \). Therefore, \( z \) cannot be a boundary point of \( \overline{\pi(Γ)} \) in \( C \setminus \overline{B}_R \). This contradiction ends the proof of the theorem.

In the condition (c) of Theorem [4.4] if we do not exclude \( v \) from the set \( E_\infty X \cup C_\infty (\text{sing}(X)) \), then the condition that \( d_v F|_{T_v D} \) is surjective does not necessarily imply that \( F(X) \) is closed. Let us give two illustrating examples below.

**Example 4.6.** Let

\[
X := \{(x, y) \in \mathbb{R}^2 : x^2 y^2 - 1 \geq 0\}
\]

and

\[
F: \mathbb{R}^2 \to \mathbb{R}, \ (x, y) \mapsto x.
\]

A simple computation shows that \( C_\infty X = \mathbb{R}^2 \) and so \( D := C_\infty X \cap S^1 = S^1 \). Moreover we have

\[
E_\infty X = \emptyset, \quad \text{sing}(X) = \{x^2 y^2 - 1 = 0\}
\]

and

\[
C_\infty (\text{sing}(X)) = \{x = 0\} \cup \{y = 0\}.
\]

Let \( v := (0, 1) \in \ker F \cap C_\infty (\text{sing}(X)) \). Clearly, the restriction of \( d_v F \) on \( T_v D = \{y = 0\} \) is surjective. However \( F(X) = \mathbb{R} \setminus \{0\} \) is not closed.

**Example 4.7.** Let

\[
Y := \{(x, y, z) \in \mathbb{R}^3 : z^4 - x^4 - x^2 - y^6 = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 : x^4 - z^4 - z^2 - y^6 = 0\},
\]

\( X := Y \setminus \mathbb{B}^3 \) and

\[
F: \mathbb{R}^3 \to \mathbb{R}, \ (x, y, z) \mapsto x - z.
\]

A simple computation shows that

\[
C_\infty X = \{y = 0\} \quad \text{and} \quad D := C_\infty X \cap S^2 = \{x^2 + z^2 = 1, \ y = 0\}.
\]

It can be also shown that

\[
E_\infty X = \left\{x = \pm \frac{1}{\sqrt{2}}, \ y = 0, \ z = \pm \frac{1}{\sqrt{2}}\right\}.
\]

For example, let \( v := \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \). We will show that \( v \) is an exceptional direction at infinity of \( X \). For the other directions in \( E_\infty X \), the situation is similar.
Let 
\[ f(x, y, z) := z^4 - x^4 - x^2 - y^6 \]
and
\[ \gamma(t) := (t, 0, (t^4 + t^2)^\frac{1}{2}) \in f^{-1}(0) \subset X \text{ for } t > 0. \]
Then \( \nabla f(x, y, z) = (-4x^3 - 2x, -6y^5, 4x^3) \) and so
\[ \nabla f(\gamma(t)) = (-4t^3 - 2t, 0, 4(t^4 + t^2)^\frac{3}{2}). \]
Hence \( \frac{\nabla f(\gamma(t))}{\|\nabla f(\gamma(t))\|} \to \left( -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \) as \( t \to +\infty. \) Consequently
\[ P := \lim_{t \to +\infty} T_{\gamma(t)} X = \left( -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)^\perp. \]
Obviously
\[ P = \{x = z\} \neq \{y = 0\} = T_v(C_\infty X), \]
so \( \angle(P, T_v(C_\infty X)) > 0 \) (see Figure 4). Thus \( v \) is an exceptional direction at infinity of \( X \).
(In fact, it is not hard to see that every 2-dimensional linear subspace of \( \mathbb{R}^3 \) containing \( v \) is a tangent limit at infinity of \( X \) along \( v \).)

Note that \( v \in \ker F \) and the restriction of \( d_v F \) on \( T_v D = \{ x = -z, y = 0 \} \) is surjective but \( F(X) = \mathbb{R} \setminus \{ 0 \} \) is not closed.

5. PROOF OF THE MAIN THEOREM

The goal of this section is to give a proof of Main Theorem. First of all, we need some preparation. Let

\[ A := \{ F \in L(\mathbb{R}^n, \mathbb{R}^m) : F \text{ has maximal rank} \} \]

which is an open dense semi-algebraic set in \( L(\mathbb{R}^n, \mathbb{R}^m) \).

**Lemma 5.1.** Let \( D \subset \mathbb{S}^{n-1} \) be a semi-algebraic set and let \( F \in A \). For any sequence \( v^k \in D \) such that \( \lim_{k \to \infty} v^k = v \in \ker F \), there is a sequence of linear mappings \( F_k \in A \) such that \( v^k \in \ker F_k \) and \( F_k \to F \). Moreover if \( v^k \notin \text{sing}(D) \), then \( F_k \) can be chosen such that \( F_k|_{T_{v_k}D} \) has maximal rank.

**Proof.** Let \( \pi^1 : \mathbb{R}^n \to v^\perp \) designate the orthogonal projection on \( v^\perp \). By definition, then \( F(\pi^1(x)) = F(x) \) for all \( x \in \mathbb{R}^n \).

For \( x \in \mathbb{R}^n \), denote by \( \ell_k(x) \) the line through \( x \) and orthogonal to \( (v^k)^\perp \), i.e., \( \ell_k(x) \) is parallel to \( v^k \), and let \( \pi^2_k(x) \) be the (unique) point of the intersection \( \ell_k(x) \cap v^\perp \) (see Figure 5). It is not hard to check that \( \pi^2_k : \mathbb{R}^n \to v^\perp \) is a linear mapping satisfying \( \pi^2_k(v^k) = 0 \). Set \( G_k := F \circ \pi^2_k \in L(\mathbb{R}^n, \mathbb{R}^m) \). Then \( v^k \in \ker G_k \). Furthermore, we have

\[
\| G_k - F \| = \max_{\| x \|=1} \| G_k(x) - F(x) \|
= \max_{\| x \|=1} \| (F \circ \pi^2_k)(x) - F(x) \|
= \max_{\| x \|=1} \| F(\pi^2_k(x)) - F(\pi^1(x)) \|
\leq \| F \| \max_{\| x \|=1} \| \pi^2_k(x) - \pi^1(x) \|.
\]

On the other hand, it is clear that

\[
\| \pi^2_k(x) - \pi^1(x) \| = \| x - \pi^1(x) \| \tan \widehat{v^k, v}
\leq \| x \| \tan \widehat{v^k, v}.
\]

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Therefore
\[ \| G_k - F \| \leq \| F \| \tan \hat{v}, v, \]
which yields \( \| G_k - F \| \to 0 \) as \( k \to +\infty \). Since \( A \) is open and \( F \in A \), we have \( G_k \in A \) for all \( k \) sufficiently large. Passing to a subsequence if necessary, we may assume that \( G_k \in A \) for any \( k \) and the first statement follows.

Now assume that \( v^k \not\in \text{sing}(D) \). Let
\[ A_k := \{ G \in L((v^k)^\perp, \mathbb{R}^m) : \text{G}|_{T_{\nu_{v^k}D}} \text{ has maximal rank} \}. \]
Clearly \( A_k \) is dense in \( L((v^k)^\perp, \mathbb{R}^m) \). Hence, for each \( k \), there is a sequence \( G_{kl} \in A_k \) such that \( G_{kl} \to G_k|_{(v^k)^\perp} \) as \( l \to \infty \). For each \( l \), define the linear mapping \( F_{kl} : \mathbb{R}^n \to \mathbb{R}^m \) by
\[ F_{kl}|_{(v^k)^\perp} = G_{kl} \quad \text{and} \quad F_{kl}(v^k) = 0. \]
Then it is not hard to check that \( F_{kl} \to G_k \) as \( l \to \infty \) and \( F_{kl}|_{T_{\nu_{v^k}D}} \) has maximal rank. Furthermore, since \( A \) is open and \( G_k \in A \), for all \( k \) sufficiently large, we have \( F_{kl} \in A \) for \( k, l \) large enough. Without loss of generality, assume that \( \| F_{kk} - G_k \| < \frac{1}{k} \) and let \( F_k := F_{kk} \), then \( F_k(v^k) = 0 \) and \( F_k|_{T_{\nu_{v^k}D}} \) has maximal rank. Moreover,
\[ \| F_k - F \| \leq \| F_k - G_k \| + \| G_k - F \| \to 0 \]
as \( k \to \infty \) so \( F_k \to F \). The lemma is proved. \( \square \)
Now we are in position to prove Main Theorem.

**Proof of Main Theorem.** The proof is divided into two cases. First of all, we consider the case \( G \equiv 0 \).

**Case 1: Linear case.** If \( n \leq m \), then in view of Theorem 4.4, \( F(X) \) is closed for any \( F \in A \) and we are done, recall that \( A \) is defined by (13). So from now on, suppose that \( n > m \).

Clearly, in light of Corollary 2.13, in order to prove Main Theorem, it is enough to show that the semi-algebraic set \( \{ F \in L(\mathbb{R}^n, \mathbb{R}^m) : F(X) \text{ is closed} \} \) is dense in \( L(\mathbb{R}^n, \mathbb{R}^m) \). Since \( A \) is open dense in \( L(\mathbb{R}^n, \mathbb{R}^m) \), it is sufficient to show that the semi-algebraic set

\[
B := \{ F \in A : F(X) \text{ is closed} \}
\]

is dense in \( A \). Let \( F \in A \setminus B \). We need to show that there is a sequence \( F_k \in B \) such that \( F_k \to F \). In view of Theorem 4.4(b), we have \( \ker F \cap D \neq \emptyset \).

Consider the filtration

\[
X^0 := X \supseteq X^1 := \text{sing}(X) \supseteq X^2 := \text{sing}(X^1) \supseteq \cdots \supseteq X^p := \text{sing}(X^{p-1}) \neq \emptyset,
\]

where \( X^p \) is non singular. Set

\[
D := D_0 := C_\infty X \cap S^{n-1} \supset D_1 := C_\infty X^1 \cap S^{n-1} \supset \cdots \supset D_p := C_\infty X^p \cap S^{n-1}.
\]

Let \( S := \{ D_\alpha \}_{\alpha \in I} \) be a semi-algebraic stratification of \( D \) such that for \( i = 1, \ldots, p \), each of the sets \( D_i \) and \( E_\infty X^i \) is a union of strata in \( S \). Note that, in view of Corollary 2.12, \( \text{sing}(D_i) \) is also a finite union of strata in \( S \) for each \( i \). For any stratum \( D_\alpha \) which is not contained in the boundary of another strata in \( S \) (i.e., there is no stratum \( D_\beta \) such that \( D_\alpha \subset \overline{D_\beta} \)) and for \( i = 0, \ldots, p \), we have

- \( D_\alpha \cap \text{sing}(D_i) = \emptyset \); and
- \( D_\alpha \cap E_\infty X^i = \emptyset \) (because \( E_\infty X^i \) is nowhere dense in \( D_i \) in view of Theorem 3.7).

We consider three sub-cases.

**Case 1.1: There is a stratum \( D_\alpha \) with \( \dim D_\alpha \geq m \) not contained in the boundary of another strata such that \( \ker F \cap D_\alpha \setminus C_\infty(\text{sing}(X)) \neq \emptyset \).**

Since \( D_\alpha \cap \text{sing}(D) = \emptyset \) and \( D_\alpha \cap E_\infty X = \emptyset \), we have

\[
\ker F \cap D_\alpha \setminus (E_\infty X \cup C_\infty(\text{sing}(X)) \cup \text{sing}(D)) = \ker F \cap D_\alpha \setminus C_\infty(\text{sing}(X)) \neq \emptyset.
\]

Accordingly, there is \( v \in \ker F \) and a sequence \( v^k \in D_\alpha \setminus (\overline{E_\infty X \cup C_\infty(\text{sing}(X)) \cup \text{sing}(D)}) \) such that \( v^k \to v \). By Lemma 5.1 there is a sequence \( F_k \in A \) such that

\[
v^k \in \ker F_k, \ F_k \to F \quad \text{and} \quad F_k|_{T_{v^k}D} \text{ has maximal rank}.
\]

Since \( \dim D_\alpha \geq m \), it follows that \( \dim T_{v^k}D \geq m \), and so \( F_k|_{T_{v^k}D} \) is surjective. In light of Theorem 4.4(c), \( F_k(X) = \mathbb{R}^m \), so \( F_k \in B \) and Case 1.1 is solved.
Case 1.2: There is a stratum \( D_\alpha \) with \( \dim D_\alpha \geq m \) not contained in the boundary of another strata such that \( \ker F \cap D_\alpha \setminus C_\infty(\operatorname{sing}(X)) = \emptyset \) and that \( \ker F \cap \overline{D} \neq \emptyset \).

Since \( D_1 := C_\infty(\operatorname{sing}(X)) \cap \mathbb{S}^{n-1} \) is a union of strata in \( S \) and since \( D_\alpha \) is a stratum in \( S \), the conditions \( \ker F \cap D_\alpha \setminus C_\infty(\operatorname{sing}(X)) = \emptyset \) and \( \ker F \cap \overline{D} \neq \emptyset \) together imply that

\[
D_\alpha \cap C_\infty(\operatorname{sing}(X)) \neq \emptyset,
\]

and so

\[
D_\alpha \subset C_\infty(\operatorname{sing}(X)) \cap S^{n-1} = C_\infty(X^1) \cap S^{n-1} = D_1
\]

because \( D_1 \) is a union of strata. Note that \( C_\infty(\operatorname{sing}(X^p)) = C_\infty(\emptyset) = \emptyset \), so

\[
D_\alpha \not\subset C_\infty(\operatorname{sing}(X^p)).
\]

Hence there is an index \( i \in \{1, \ldots, p\} \) such that

\[
D_\alpha \subset C_\infty(\operatorname{sing}(X^i)) \quad \text{and} \quad D_\alpha \not\subset C_\infty(\operatorname{sing}(X^{i+1})).
\]

By repeating the argument in Case 1.1 for \( X^{i+1} \), it follows that there is a sequence \( F_k \in \mathcal{A} \) with \( F_k(X^i) = \mathbb{R}^m \) such that \( F_k \to F \). Thus we also have \( F_k(X) = \mathbb{R}^m \). So \( F_k \in \mathcal{B} \) and Case 1.2 is solved.

Note that if Case 1.1 or Case 1.2 holds, then there is a stratum \( D_\alpha \) with \( \dim D_\alpha \geq m \) such that \( \ker F \cap \overline{D}_\alpha \neq \emptyset \). Hence it remains only one following case to consider.

Case 1.3: \( \ker F \cap \overline{D}_\alpha = \emptyset \) for any stratum \( D_\alpha \) with \( \dim D_\alpha \geq m \).

Let

\[
\mathcal{D}_1 := \bigcup_{\dim D_\alpha \geq m} D_\alpha \quad \text{and} \quad \mathcal{D}_2 := \bigcup_{\dim D_\alpha < m} D_\alpha.
\]

By construction, \( \ker F \cap \overline{\mathcal{D}_1} = \emptyset \). Since \( \ker F \cap D \neq \emptyset \), it follows that \( \mathcal{D}_2 \neq \emptyset \).

Recall that we identify \( L(\mathbb{R}^n, \mathbb{R}^m) \) with \( \mathbb{R}^{m \times n} \). For each stratum \( D_\alpha \) with \( \dim D_\alpha < m \), define the semi-algebraic mapping

\[
\Phi_\alpha : \mathbb{R}^{m \times n} \times D_\alpha \to \mathbb{R}^m, \quad (A, v) \mapsto Av.
\]

Write \( A = (a_{ij})_{i=1, \ldots, m} \). Since \( D_\alpha \subset D \subset \mathbb{S}^{n-1} \), for each \( v \in D_\alpha \) there exists an index \( i_0, 1 \leq i_0 \leq n \), such that \( v_{i_0} \neq 0 \), and so the Jacobian matrix \( D\Phi_\alpha \) of \( \Phi_\alpha \) at \( (A, v) \) contains the following diagonal matrix

\[
\frac{\partial \Phi_\alpha}{\partial (a_{i_0 j}, \ldots, a_{i_0 q})} = \begin{pmatrix}
v_{i_0} & 0 & \cdots & 0 \\
0 & v_{i_0} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & v_{i_0}
\end{pmatrix}.
\]
which has rank $m$. Hence $D\Phi_\alpha$ is of rank $m$ at every point $(A, v) \in \mathbb{R}^{m \times n} \times D_\alpha$. Consequently, $\Phi_\alpha \cap \{0\}$ (in $\mathbb{R}^m$). By Theorem 2.18

$$\mathcal{U}_\alpha := \{A \in \mathbb{R}^{m \times n} : \Phi_\alpha(A, \cdot) \cap \{0\}\}$$

contains an open dense semi-algebraic set in $\mathbb{R}^{m \times n}$. On the other hand, since $\dim D_\alpha < m$, for each fixed $A \in \mathbb{R}^{m \times n}$, the mapping $\Phi_\alpha(A, \cdot) : D_\alpha \to \mathbb{R}^m$ is transverse to $\{0\}$ if and only if

$$\{v \in D_\alpha : \Phi_\alpha(A, v) = 0\} = \emptyset.$$

Therefore, we can write

$$\mathcal{U}_\alpha = \{G \in L(\mathbb{R}^n, \mathbb{R}^m) : \ker G \cap D_\alpha = \emptyset\}.$$

Let

$$\mathcal{A}' := \bigcap_{\dim D_\alpha < m} \mathcal{U}_\alpha.$$

Then $\mathcal{A}'$ contains an open dense semi-algebraic set in $L(\mathbb{R}^n, \mathbb{R}^m)$, and so there exists a sequence $F_k \in \mathcal{A} \cap \mathcal{A}'$ such that $F_k \to F$ as $k \to +\infty$. It is clear that $\ker F_k \cap D_2 = \emptyset$ and for $k$ large enough, we have $\ker F_k \cap \overline{\mathcal{A}_1} = \emptyset$. Therefore, $\ker F_k \cap D = \emptyset$. In view of Theorem 4.4(b), $F_k(X)$ is closed. This ends the proof for the linear case.

Case 2: General case. Since the set $X$ is closed and the mapping $G$ is continuous, the semi-algebraic set

$$Z := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in X, y = G(x)\}$$

is closed. Applying Case 1 to $Z$, we can see that the set

$$\{F \in L(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m) : F(Z) \text{ is closed}\}$$

contains an open dense semi-algebraic set, say $\mathcal{U} \subset L(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)$. Consider the mapping

$$\Phi : L(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m) \to L(\mathbb{R}^n, \mathbb{R}^m) \times L(\mathbb{R}^m, \mathbb{R}^m), \Phi(F) \mapsto (F_1, F_2),$$

where $F_1 \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $F_2 \in L(\mathbb{R}^m, \mathbb{R}^m)$ are linear mappings defined as follows

$$F_1(x) := F(x, 0) \quad \text{and} \quad F_2(y) := F(0, y) \quad \text{for all} \quad x \in \mathbb{R}^n, \ y \in \mathbb{R}^m.$$

A simple calculation shows that $\Phi$ is a linear isomorphism. Consequently, $\Phi(\mathcal{U})$ is an open dense semi-algebraic set in $L(\mathbb{R}^n, \mathbb{R}^m) \times L(\mathbb{R}^m, \mathbb{R}^m)$. In view of Corollary 2.14 there exists a linear isomorphism $F_2 \in L(\mathbb{R}^m, \mathbb{R}^m)$ such that the set

$$\{F \in L(\mathbb{R}^n, \mathbb{R}^m) : (F, F_2) \in \Phi(\mathcal{U})\}$$

contains an open dense semi-algebraic set in $L(\mathbb{R}^n, \mathbb{R}^m)$. Or, equivalently, the set

$$\{(F_2)^{-1} \circ F \in L(\mathbb{R}^n, \mathbb{R}^m) : (F, F_2) \in \Phi(\mathcal{U})\}$$

contains an open dense semi-algebraic set in $L(\mathbb{R}^n, \mathbb{R}^m)$.
Take arbitrarily \((F, F_2^*) \in \Phi(Z)\). Then the image \([\Phi^{-1}(F, F_2^*)](Z)\) is closed, and so is the set \((F_2^*)^{-1} \circ [\Phi^{-1}(F, F_2^*)](Z)\). Observe that
\[
(F_2^*)^{-1} \circ [\Phi^{-1}(F, F_2^*)](Z) = \{(F_2^*)^{-1}(F(x) + F_2^*(y)) : x \in X, y = G(x)\}
\]
\[
= \{[(F_2^*)^{-1} \circ F](x) + y : x \in X, y = G(x)\}
\]
\[
= \{[(F_2^*)^{-1} \circ F](x) + G(x) : x \in X\}.
\]
Hence the last set is also closed. The theorem is proved. \(\square\)

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