Rotational subsets of the circle

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Abstract

A rotational subset, relative to a continuous transformation $T : T \to T$ on $T = \mathbb{R}/\mathbb{Z}$, is a closed, invariant subset of $T$ that is minimal and on which $T$ respects the standard orientation of the unit circle. In the case where $T$ is the standard angle doubling map, such subsets were studied by Bullet and Sentenac. The case where $T$ multiplies angles by an integer $d > 2$ was studied by Goldberg and Tresser, and Blokh, Malaga, Mayer, Oversteegen, and Parris. These authors prove that infinite rotational subsets arise as extensions of irrational rotations of the unit circle. This paper studies the extent to which such results hold for general continuous maps of the circle. In particular, we prove the structure theorem mentioned above holds for the wider class of continuous transformations $T$ with finite fibers. Our methods are more analytic in nature than the works mentioned. The paper concludes with a construction of infinite rotational sets for a class of continuous maps that includes examples that are not equivalent to the model cases treated previously.

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Introduction

In what follows, \( \mathbb{T} \) denotes the unit circle with the standard orientation.

**Definition 1.** Let \( X \subset \mathbb{T} \) and \( f : X \to X \) be a continuous transformation. The map \( f \) *preserves cyclic order* if, for any \( P, Q, R \in X \) with distinct images, the arcs \( PQR \) and \( f(P)f(Q)f(R) \) have the same orientation.

Now consider a continuous transformation \( T : \mathbb{T} \to \mathbb{T} \) and a compact set \( X \subseteq \mathbb{T} \).

**Definition 2.** The subset \( X \) is *rotational* if

- \( X \) is invariant, i.e. \( TX \subseteq X \),
- \( X \) is minimal, and
- \( T|_X \) preserves cyclic order.

Our objective is to study the structure of infinite, rotational, proper subsets under fairly general assumptions about the nature of \( T \). The main result is:

**Theorem 3.** Let \( T : \mathbb{T} \to \mathbb{T} \) be a continuous function with finite fibers and \( X \subset \mathbb{T} \) an infinite, rotational, proper subset of \( \mathbb{T} \) with respect to this transformation. Then:

i. The dynamical system \( (X, T) \) is an extension of an irrational rotation of the circle.

ii. The function \( \phi : X \to \mathbb{T} \) that realizes this extension has singleton fibers except at countably many points of \( \mathbb{T} \). Over these exceptional points, the fibers have cardinality two, corresponding to endpoints of gaps of the set \( X \) in \( \mathbb{T} \).

iii. \( (X, T) \) has a unique ergodic measure \( \mu \) and \( \phi_*\mu \) is the standard Lebesgue measure on \( \mathbb{T} \).

The above theorem was proved in the case when \( T : \mathbb{T} \to \mathbb{T} \) is the angle doubling map by Bullet and Sentenac [2]. It was established in the case where \( T \) is the standard \( d \)-fold cover of the unit circle by Goldberg and Tresser [3] and by Blokh, Malalgh, Mayer, Oversteegen and Parris [1]. These works were motivated by the study of the action of quadratic dynamical systems in the complement of the Julia set.

The proof of theorem will be accomplished over the next two sections. We then revisit the \( d \)-fold cover case from our point of view. The last section presents a class of rotational subsets that include examples which are not conjugate to the previously known cases. On the basis of theorem and the examples of the last section, it is natural to ask if rotational subsets exist for any continuous transformation of \( \mathbb{T} \) with degree larger than 1.

**Structure of rotational subsets**

*Henceforth*, unless otherwise stated, we will work on the case where

- \( T \) has finite fibers, and

- the rotational set \( X \) is an infinite, proper subset of \( \mathbb{T} \).
Suppose $x_0 \in X$ is an isolated point of such a dynamical system. Minimality implies that the forward orbit, $\{T^n x_0 : n > 0\}$, is dense in $X$. As a consequence, we have that $T^n x_0 = x_0$ for some positive integer $n$. The forward orbit $O$ must then be finite as well as dense. Therefore, $X = \{T^n x_0 : n \geq 0\}$. This contradiction implies that $X$ cannot have any isolated points. Therefore $X$ is a perfect subset of $T$.

By conjugating with the appropriate rotation, we can assume that $0 \notin X$. Parameterize $T$ by the unit interval $[0,1]$ and note that

\[0 < \alpha = \text{inf} \, X < \beta = \text{sup} \, X < 1.\]

**Lemma 4.** Suppose $a, b \in X$ and $Ta = b$. Then, there are strictly monotone sequences $a_n$ and $b_n$ such that $\lim_{n \to \infty} a_n = a$, $\lim_{n \to \infty} b_n = b$ and $T a_n = b_n$, for all $n \in \mathbb{N}$.

**Proof.** Since $X$ is perfect, one can construct a sequence $a_n \in X$ such that $a_n \neq a$ for all $n \in \mathbb{N}$ and $a_n \to a$ as $n \to \infty$. By passing to a subsequence, one can further arrange the $a_n$ to be strictly monotonic. Since $T$ has finite fibers, $Ta_n \neq b$ for all but finitely many $n$. Moreover, $Ta_n \to Ta$. By once again passing to a subsequence, if necessary, we may arrange that $Ta_n$ is strictly monotone. \ 

**Proposition 5.** The fibers of $T|X$ have cardinality at most two.

**Proof.** Suppose, to the contrary, that $x_0, x_1, x_2 \in X$ are three distinct points arranged in increasing order with the same image under $T$. So $Tx_i = y$ for $i = 0, 1$ and 2. By lemma 4, there are strictly monotone sequences $a_n^{(i)}$, $i = 0, 1, 2$ such that $a_n^{(i)} \to x_i$. Moreover, $b_n^{(i)} = T a_n^{(i)}$ are strictly monotone and approach $y$.

The proof proceeds according to how $b_n^{(0)}$ and $b_n^{(2)}$ approach $y$.

Case $b_n^{(0)}$ and $b_n^{(2)}$ approach $y$ from the same side: Without loss of generality, assume that both sequences approach $y$ from below. In this case, we can use the properties of the sequence $b_n^{(i)}$ to find indices $k$ and $l$ so that

\[b_k^{(0)} < b_l^{(2)} < y\]

and $a_k^{(0)}$, $x_1$ and $a_l^{(2)}$ are in increasing order.

This means that $T$ changes the cyclic order of the points $a_k^{(0)}$, $x_1$, and $a_l^{(2)}$.

Case $b_n^{(0)} \searrow y$ and $b_n^{(2)} \nearrow y$:

Choose $n$ sufficiently large so that

\[a_n^{(0)} < x_1 < a_n^{(2)}\]

But $T a_n^{(2)} < y < T a_n^{(0)}$, so $T$ doesn’t preserve cyclic order.

Case $b_n^{(0)} \nearrow y$ and $b_n^{(2)} \searrow y$: By invoking lemma 4 again, we construct a small perturbation of $x_1$, say $a'$, with the property that $Ta' \neq y$. We treat the case where $Ta' < y$ — the other case is handled similarly.

In this situation, there is a sufficiently large $n$ with the property that

\[Ta' < T a_n^{(0)} < y < T a_n^{(2)}\]

Observe that $T$ doesn’t preserve the cyclic order of $a_n^{(0)}$, $a'$, and $a_n^{(2)}$.

Thus, in all three cases, we have a contradiction. \ 

The minimality condition insures that $T$ is surjective. Consequently, we may put $\alpha' = \max \{x : Tx = \beta\}$ and $\beta' = \min \{x : Tx = \alpha\}$.  

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Proposition 6. The inequality
\[ \alpha < \alpha' < \beta < \beta' \]
holds. Moreover, \([\alpha', \beta']\) is a gap for \(X\), i.e. \(X \cap [\alpha', \beta'] = \emptyset\).

Proof. Suppose that \(\alpha' > \beta'\). Minimality ensures that \(\beta' > \alpha\). By using the perturbation result (lemma 4), we can find an \(x\) near \(\alpha\) such that \(\alpha < Tx < \beta\). From this we have that \(T\) reverses the cyclic order of the triple \(x, \beta', \alpha'\).

Now, consider an \(x \in X\) with \(\alpha' < x < \beta'\). By the definition of \(\alpha'\) and \(\beta'\), \(Tx\) must be distinct from \(\alpha\) or \(\beta\). Hence \(Tx\) must be strictly between \(\alpha\) and \(\beta\). This means that \(T\) reverses the cyclic order of \(\alpha', x, \beta'\). Contradiction.

Finally, since \(X\) has no isolated points, \(\alpha' \neq \alpha\) and \(\beta' \neq \beta\).

Lemma 7. Let \(x, y \in X\) If \(x < y\) and \(Tx < Ty\), then for any \(z \in X\) with \(x < z < y\), we must have \(Tx \leq Tz \leq Ty\).

Proof. Failure of the conclusion clearly entails that \(T\) reverses the cyclic order of the triple \(x, z, y\).

Proposition 8. \(\alpha < T\beta \leq T\alpha < \beta\).

Proof. Note that minimality rules out the possibilities that \(T\alpha = \alpha\) and \(T\beta = \beta\).

If \(T\beta = \alpha\), then by lemma 4 there is an \(x \in X\) near \(\beta\) such that \(T\alpha > Tx > \alpha\). This means that \(T\) reverses the cyclic order of the triple \(\alpha, x, \beta\). In a similar way, \(T\alpha = \beta\) can also be ruled out.

Only the middle inequality remains. Suppose, to the contrary, that \(T\beta > T\alpha\). The previous proposition implies that \(T\alpha \leq Tx \leq T\beta\), for any \(x \in X\) that is strictly between \(\alpha\) and \(\beta\). Thus, \(\text{Im} T\) is a proper subset of \(X\). This violates the minimality assumption.

For any \(x_0, x_1 \in [0, 1]\), set \(X_{x_0, x_1} = X \cap [x_0, x_1]\).

Proposition 9. \(T|X\) is monotone increasing on the sets \(X_{\alpha, \alpha'}\) and \(X_{\beta', \beta}\). Moreover,
\[ Tx > x \quad \forall x \in X_{\alpha, \alpha'} \]
and
\[ Tx < x \quad \forall x \in X_{\beta, \beta'}. \]

Proof. Let \(x, y \in X \cap [\alpha, \alpha']\) with \(x < y\). Suppose \(Ty < Tx\). Since \(y < \beta',\) the definition of \(\beta'\) forces \(\alpha = T\beta' < Ty\). Thus, \(T\) inverts the cyclic order of \(x, y, \beta'\). This contradiction shows that \(T\) is monotonic increasing on \([\alpha, \alpha']\). A similar argument applies to \(X \cap [\beta', \beta]\).

Next, let \(x \in X \cap [\alpha, \alpha']\) and suppose \(Tx \leq x\). Since \(T\) has no fixed points in \(X\), \(Tx < x\). As neither \(\alpha\) nor \(\alpha'\) have this property, \(\alpha < x < \alpha'\) and hence \(\alpha < T\alpha \leq Tx < x\). Lemma 7 then implies that \(X \cap [\alpha, x]\) is a nonempty, proper, closed invariant subset of \(X\). Thus \(Tx > x\) for \(x \in X \cap [\alpha, \alpha']\). The last inequality can be verified similarly.
Coding by irrational rotations of the circle

By the Krylov-Bogolioubov theorem, there is a Borel probability measure on $X$ that is invariant under $T$. Fix one such, $\mu$. Regard $\mu$ as a measure on $[0, 1)$ and write $\tilde{\Phi}$ for its cumulative distribution function. Since every point of the infinite set $X$ has dense orbit, the invariant probability measure $\mu$ cannot include any point masses. Thus $\tilde{\Phi}$ is continuous. As a consequence, $\Phi = \tilde{\Phi}|_X$ is a continuous, monotone increasing map from $X$ to $[0, 1]$. Finally, because $\tilde{\Phi}(\alpha) = 0$ and $\tilde{\Phi}(\beta) = 1$
and the fact that $\tilde{\Phi}$ is locally constant on the complement of $X$, $\Phi : X \to [0, 1]$ is surjective.

**Proposition 10.**

- If $x_0, x_1 \in X$ and $X \cap (x_0, x_1) = \emptyset$, then $\Phi(x_0) = \Phi(x_1)$.

- On the other hand, if $X \cap (x_0, x_1) \neq \emptyset$, then $\Phi(x_0) > \Phi(x_1)$.

**Proof.** The first statement follows directly from the observation that $\mu$ contains no point masses.

To prove the second statement, first put $U = X \cap (x_0, x_1)$ and

$$\nu_n(U, x) = |\{k : k = 0, \ldots, n - 1 \text{ and } T^k x \in U\}|.$$

Finally, write $\mathcal{P} : L^2(X, \mu) \to L^2(X, \mu)$ for the projection onto the subspace of function left invariant by $T$. By Mean Ergodic Theorem,

$$\lim_{n \to \infty} \frac{\nu_n(U, x)}{n}$$

converges in $L^2(X, \mu)$ to the projection $\mathcal{P}1_U$. On the other hand, since $U$ is a non-empty open set and $(X, T)$ is a minimal dynamical system, there is an $\epsilon > 0$ such that

$$\liminf_{n \to \infty} \frac{\nu_n(U, x)}{n} > \epsilon \quad \text{for all } x \in X.$$

(See, for example, proposition 4.7 in [4].) Hence,

$$\Phi(x_1) - \Phi(x_0) = \mu(U) \geq \|\mathcal{P}1_U\| \geq \epsilon > 0.$$

A corollary of this is that any non-empty open subset of $X$ has positive $\mu$-measure.

**Proposition 11.**

- There is an irrational number $\theta_0 \in (0, 1)$ such that

$$\Phi(Tx) = \theta_0 + \Phi(x) \mod 1$$

for all $x \in X$.

- If $P, Q$ and $R$ are distinct points of $X \subset \mathbb{T}$ with distinct images under $\Phi$ then the arcs $PQR$ and $\Phi(P)\Phi(Q)\Phi(R)$ both have the same orientation in $\mathbb{T}$. 

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Proof. Put \( \theta_0 = \mu([\beta', \beta]) \). If \( x \in X \cap [\alpha, \alpha'] \), then the \( T \) invariance of the measure \( \mu \) implies that

\[
\Phi(Tx) = \mu([\alpha, Tx]) \\
= \mu([\alpha, T\beta]) + \mu([T\alpha, Tx]) \\
= \mu([T\beta', T\beta]) + \mu([T\alpha, Tx]) \\
= \theta_0 + \mu([\alpha, x]) \\
= \theta_0 + \Phi(x).
\]

If \( x \in X \cap [\beta', \beta] \), then

\[
\Phi(Tx) = \mu([\alpha, Tx]) \\
= \mu([T\beta', Tx]) \\
= \mu([\beta', x]) \\
= \Phi(x) - \mu([\alpha, \alpha']) \\
= \Phi(x) + \mu([\beta', \beta]) - 1 \\
= \Phi(x) + \theta_0 \mod 1.
\]

Moreover, \( \theta_0 \) is irrational. Otherwise, any finite (closed) orbit in \( T \) under translation by \( \theta_0 \) would have a non-dense preimage under \( \Phi \) that is \( T \) invariant.

The second statement follows directly from the definition of \( \Phi \).

If we define \( \phi : X \to \mathbb{T} \) by \( \phi(x) = \exp(2\pi i \Phi(x)) \) and \( \tau : \mathbb{T} \to \mathbb{T} \) by \( \tau(z) = \exp(2\pi i \theta_0)z \), the above considerations show that \( \phi \) is a factor map from \((X, T)\) to \((\mathbb{T}, \tau)\).

**Lemma 12.** Let \( X_0 \) be those points \( x = X \) with the property that for every \( \delta > 0 \), the intervals \( (x, x + \delta) \) and \( (x, x - \delta) \) contain infinitely many points of \( X \). Then: \( X_0 \) is a dense, uncountable subset of \( X \).

**Proof.** A point is in the complement of \( X_0 \) in \( X \) precisely when it is the endpoint of a gap, i.e. a maximal open interval contained in \( \mathbb{T} \setminus X \). Since there are at most countably many such open intervals, we conclude that \( X \setminus X_0 \) is countable. As the perfect set \( X \) is uncountable, \( X_0 \) is uncountable as well. Furthermore, each point of \( X \setminus X_0 \) is nowhere dense. Therefore, \( X \setminus X_0 \) is nowhere dense. In other words, \( X_0 \) is dense.

**Theorem 13.** The dynamical system \((X, T)\) is uniquely ergodic.

**Proof.** Let \( x_0 \in X_0 \). By proposition 10 and lemma 12

- \( x \in X \) and \( x < x_0 \Rightarrow \Phi(x) < \Phi(x_0) \), and
- \( x \in X \) and \( x > x_0 \Rightarrow \Phi(x) > \Phi(x_0) \).

Therefore, for any \( y \in X \) and \( n \geq 0 \),

\[
T^ny \leq x_0 \iff \{\Phi(y) + n\theta_0\} \leq \Phi(x_0).
\]
(Here the braces denote fractional part.) Consequently,
\[
\lim_{n \to \infty} \frac{\# \{k : T^k y \leq x_0 \text{ and } 0 \leq k < n\}}{n} = \lim_{n \to \infty} \frac{\# \{k : \Phi(y) + n\theta_0 \leq \Phi(x_0) \text{ and } 0 \leq k < n\}}{n} = \Phi(x_0) = \mu(X \cap [\alpha, x_0]).
\]

The second limit has been evaluated by invoking the Weyl’s equidistribution theorem. Since such \(x_0\) are dense in \(X\), the cumulative distribution of the measure \(\mu\) is uniquely determined by \((X, T)\). In other words, \((X, T)\) is uniquely ergodic.

With this last result, we have completed the proof of theorem 3.

The \(z \mapsto z^d\) case

We now specialize to the case where \(T : \mathbb{T} \to \mathbb{T}\) is given by
\[
T x = d \cdot x \mod 1.
\]
In this situation, the inverse image of 0 consists of the \(d\) points
\[
\xi_k = \frac{k}{d} \mod 1 \quad k = 0, \ldots, d - 1.
\]
In addition, \(T\) has \(d - 1\) fixed points:
\[
\eta_k = \frac{k}{d - 1} \mod 1 \quad k = 0, \ldots, d - 2.
\]
We also set \(\xi_d = \eta_d = 1\). Given our blanket conditions on \(X\), none of the \(\xi_k\) or \(\eta_k\) lie in \(X\).

Set \(I_k = [\xi_k, \xi_{k+1}]\) for \(k = 0, \ldots, d - 1\) and note that the interior of \(I_k\) are precisely those points in \(\mathbb{T}\) with a unique \(d\)-adic expansion that starts with the digit \(k\). Note also that \(TI_k = [0, 1]\), since \(T\) is just the shift map on the \(d\)-adic expansion. Moreover, \(T\) is monotonic increasing on the interior of each \(I_k\). Each closed interval \(I_k\) contains a unique fixed point \(\eta_k\). The behavior of \(T\) at these fixed points can be readily determined. In particular, one checks that:
\[
T x < x \quad \text{for } \xi_k < x < \eta_k
\]
and
\[
T x > x \quad \text{for } \eta_k < x < \xi_{k+1}.
\]
Let \(X_1, \ldots, X_\ell\) be the non-empty sets in the list
\[
X \cap I_0, X \cap I_1, \ldots, X \cap I_{d-1}.
\]
The indexing can be arranged so \(X_i \subset I_{k_i}\) for some \(0 \leq k_i < d\) with
\[
k_1 < k_2 < \cdots < k_\ell.
\]
Set
\[ \alpha_i = \inf X_i \quad \text{and} \quad \beta_i = \sup X_i. \]

Since \( X \) is perfect, \( \alpha_i < \beta_i \) for \( i = 1, \ldots, \ell \) and \( X \cap (\alpha_i, \beta_i) \) is non-empty. Consequently, \( \Phi(X_i) \) is the closed interval \([\Phi(\alpha_i), \Phi(\beta_i)]\) and has positive length (see proposition [10]). Because \( \mu \) has no mass on the open interval \((\beta_i, \alpha_{i+1})\), \( \Phi(\beta_i) = \Phi(\alpha_{i+1}) \). Set \( t_0 = 0 \) and \( t_i = \Phi(\beta_i) \) for \( i = 1, \ldots, \ell \). Then,
\[
0 = t_0 < t_1 < \cdots < t_\ell = 1
\]
and
\[
\Phi(X_i) = [t_{i-1}, t_i] \quad \text{for } i = 1, \ldots, \ell.
\]

Recall a consequence of our previous analysis: every point \( x \in X \) with \( Tx > x \) must lie to the left of every point \( y \in X \) with \( Ty < y \). Thus, each \( X_i \) must lie completely to one side of the unique fixed point in \( I_{k_i} \). Moreover, if \( \sup X_i < \eta_{k_i} \) and \( \inf X_j = \eta_{k_j} \) then \( i > j \) and \( k_i > k_j \). Let \( m \) be the last index, \( i \) between 1 and \( \ell \) with \( \sup X_i < \eta_{k_i} \). Clearly, \( m < \ell \) and \( \sup X_m = \alpha' \) and \( \inf X_{m+1} = \beta' \). Hence, \( t_m = \Phi(\alpha_{m+1}) = \Phi(\beta_m) = -\theta_0 \).

We next seek to understand the fibers of \( \Phi \). Set
\[
D_0 = \{ \omega \in [0, 1] : \omega + n\theta_0 \neq t_i \text{ for any } n \geq 0 \text{ and } i = 1, \ldots, \ell \}.
\]
(Note that almost every \( \omega \in [0, 1] \) is in \( D_0 \).) The sequence of points \( \omega + n\theta_0 \) with \( n \geq 0 \) determines a sequence of intervals of the form \([t_k, t_{k+1}]\). This, in turn, implies that the base \( d \) expansion of any point in the fiber of \( \omega \) is uniquely determined. Therefore, \( \Phi^{-1}(\omega) \) is a singleton.

In summary, this discussion shows how any rotational subset \( X \) must arise from the symbolic flow of an irrational rotation of \( \mathbb{T} \) relative to an appropriate partition. The next section shows that this process can be reversed.

**The Inverse Process**

Let \( \theta_0 \) be an irrational number in \( \mathbb{T} \) and let \( \tau : \mathbb{T} \rightarrow \mathbb{T} \) denote rotation by \( \theta_0 \). Consider a partition of \([0, 1]\) into \( \ell \leq d \) subintervals with the requirement that one of the interior nodes is \( -\theta_0 \):
\[
0 = t_0 < t_1 < \cdots < t_m < t_{m+1} < \cdots < t_\ell = 1
\]
and \( t_m = -\theta_0 \). Set \( J_k = [t_k, t_{k+1}] \) for \( k = 0, \ldots, \ell - 1 \). Next, select a coding that maps \( \{0, \ldots, \ell - 1\} \) to the set of digits \( \{0, \ldots, d - 1\} \). More precisely, choose integers \( k_0, \ldots, k_{\ell-1} \) that satisfy
\[
0 \leq k_0 < k_1 < \cdots < k_{\ell-1} \leq d - 1.
\]

We will show that this data, determines a rotational subset of \( \mathbb{T} \) that inverts the process described in previous section.

Let \( \mathcal{D}_0 \) be the set of \( \omega \in [0, 1] \) satisfying \( \tau^n(\omega) = \omega + n\theta_0 \neq t_i \mod 1 \) for all \( n \in \mathbb{N}_0 \) and \( i = 0, \ldots, \ell \). In other words, \( \mathcal{D}_0 \) consists of those points of \( \mathbb{T} \setminus \{t_0, t_1, \ldots, t_{\ell-1}\} \) whose forward orbit doesn’t contain any of the nodes \( t_i \).

**Proposition 14.**  
\( i. \) The complement of \( \mathcal{D}_0 \) in \( \mathbb{T} \) is countable.
ii. For every \( t \in \mathbb{T} \), there is an integer \( n \geq 0 \) with the property that \( T^n t \in \mathcal{D}_0 \).

**Proof.** The map \( \tau \) is invertible. The complement of \( \mathcal{D}_0 \) is just the countable set

\[
\{ \tau^{-k} t_i : i = 1, \ldots, \ell \text{ and } k \geq 0 \}.
\]

For the proof of the second claim, fix \( t \in \mathbb{T} \). If the orbit of \( t \) hits the \( \{ t_i : i = 0, \ldots, \ell \} \) infinitely often, then there must be an index \( i \) such that

\[
t + n \theta_0 = t_i \mod 1
\]

for infinitely many \( n \in \mathbb{N} \). This means that there are two distinct, positive integers \( n_0, n_1 \) with the property that

\[
(n_1 - n_0) \theta_0 = 0 \mod 1.
\]

But this contradicts the condition that \( \theta_0 \) is irrational. This argument proves that the forward orbit of such a \( t \) must eventually lie completely in \( \mathcal{D}_0 \). \( \square \)

The trajectory of any point \( \omega \in \mathcal{D}_0 \) can be encoded by an infinite string,

\[
E(\omega) = a_0 a_1 a_2, ...
\]

where \( a_n = k_i \) precisely when \( \omega + n \theta_0 \mod 1 \) is in \( J_i \). Note that \( E(\omega + \theta_0) \) is the shift \( a_1 a_2 \ldots \). The Kronecker approximation theorem implies that each of the digits \( k_i, i = 0, \ldots, \ell - 1 \) occurs infinitely often. In particular, we may unambiguously interpret \( E(\omega) \) as the \( d \)-adic expansion of a unique real number in the open unit interval \([0, 1[\).

**Proposition 15.** The map \( E : \mathcal{D}_0 \to \mathbb{T} \) is a continuous, injective, monotonic increasing and

\[
E(\tau(\omega)) = T(E(\omega)).
\]

**Proof.** Kronecker’s theorem also implies injectivity of \( E \). Let \( \omega \) and \( \omega' \) be distinct points in \( \mathcal{D}_0 \). Set \( \delta = \omega - \omega' \) and note that there must be an \( s \) with the property that \( s \) and \( s' = s + \delta \) lie in the interior of different \( J_i \)'s. We may then choose \( n \in \mathbb{N}_0 \) with the property that \( \tau^n \omega \) and \( \tau^n \omega' = \tau^n \omega + \delta \) approximate \( s \) and \( s' \), respectively. If the error is sufficiently small then \( \tau^n \omega \) and \( \tau^n \omega' \) will be in different \( J_i \)'s. Hence the \( d \)-adic encoding of \( \omega \) and \( \omega' \) are different. Since neither of these encodings can end with an infinite string of \( d - 1 \)'s, \( E(\omega) \neq E(\omega') \).

Equation 2 follows directly from equation 1 and the ensuing discussion.

Let \( \omega \in \mathcal{D}_0 \) and \( \omega_n \) a sequence in \( \mathcal{D}_0 \) that converges to \( \omega \). Then, since \( \tau^k \omega \) is in the interior of one of the \( J_i \), \( \tau^k \omega_n \) must eventually have this property as well. Thus, \( E \) must be continuous.

It only remains to prove that \( E \) is monotone increasing. Let \( \omega, \omega' \in \mathcal{D}_0 \) with \( \omega < \omega' \). Write \( E(\omega) = a_0 a_1 a_2 \ldots \) and \( E(\omega') = a_0' a_1' a_2' \ldots \). Because \( E \) is injective, there is a first index \( i \geq 0 \) for which \( a_i \neq a_i' \). If \( i = 0 \), \( \omega < \omega' \) implies \( a_0 < a_0' \). This in turn yields \( E(\omega) < E(\omega') \). If \( i > 0 \), then \( a_{i-1} = a_{i-1}' \) entails that both \( \tau^{i-1} \omega \) and \( \tau^{i-1} \omega' \) are in interior of the same \( J_n \). Since \( \tau \) is increasing on the interior of any \( J_n \), we must have \( a_i < a_i' \). As a consequence, \( E(\omega) < E(\omega') \) in this case as well. \( \square \)
Write $\mu = E_\ast \mathcal{L}$ for the image of Lebesgue measure on $\mathbb{T}$ under $E$. By equation $2$, $\mu$ is invariant under $T$. Since, $E$ is injective and $\mathcal{L}$ is absolutely continuous, $\mu$ has no point masses and hence is absolutely continuous.

Write $X_0$ for the image of $\mathcal{D}_0$ under $E$ and set $X = \overline{X_0}$. Since $X_0$ is invariant under $T : \mathbb{T} \to \mathbb{T}$, so is its closure $X$. The Borel measure $\mu$ is supported on $X$. Define a map $\Phi : X \to \mathbb{T}$ by

$$\Phi(x) = \mu([\alpha, x])$$

where $\alpha = \min X$. If $x = Et$ for some $t \in \mathcal{D}_0$,

$$\Phi(x) = \mu([\alpha, x]) = \mathcal{L}(E^{-1}[\alpha, x]) = \mathcal{L}([0, t]) = t$$

It follows that $E : \mathcal{D}_0 \to X_0$ and $\Phi|_{X_0}$ are inverses. Thus $\Phi$ is an invariant map from $X_0$ to $\mathcal{D}_0$. The continuity of $\Phi$ then implies that it is invariant on $X$ as well.

**Theorem 16.** The dynamical system $(X, T)$ is rotational.

**Proof.** We can now prove the minimality of $(X, T)$. Let $x \in X$ and write $t = \Phi(x)$. We would like to prove that the orbit of $x$ is dense in $X$. We know that there is a positive integer $N$ with the property that for all $n \geq N$, $\Phi(T^n x) = t + n\theta_0 \mod 1$ is in $\mathcal{D}_0$. The continuity of $E$ together with the density of the set \{ $t + n\theta_0 : n \geq N$ \} in $\mathbb{T}$ implies that $\{ T^n x : n \geq N \}$ is dense in $X_0$ and hence in $X$. Consequently, $(X, T)$ is minimal. Finally, since $(T, \tau)$ preserves cyclic order and $\Phi$ is monotonic, it is easy to check that $(X, T)$ preserves cyclic order. It follows that $(X, T)$ is rotational and that $(T, \tau)$ is its canonical factor.

**Examples for a class of continuous maps**

In this section, we show that infinite rotational sets exist in a certain class of continuous transformations on $\mathbb{T}$. Members of this class can have arbitrarily many fixed points and hence will not, in general, be conjugate to the transformations treated in the previous two sections.

Here as before, we parametrize $\mathbb{T}$ by the half-open interval $[0, 1]$. Fix an integer $d > 1$, and let

$$0 = x_0 < x_1 < \cdots < x_d = 1$$

be a partition of the unit interval. Consider a continuous transformation $T : \mathbb{T} \to \mathbb{T}$ that, for each $k = 0, ..., d - 1$, satisfies

i. monotone increasing function on each half-open interval $[x_k, x_{k+1})$, and

ii. $T(x_k) = 0$ and $\lim_{x \to x_{k+1}} T(x) = 1$.

We set up a standard symbolic encoding for $T$ in terms of infinite strings in the alphabet $\mathcal{A} = \{0, 1, ..., d - 1\}$. In particular, for each finite (non-empty) word $I = i_1 i_2 ... i_n$, let

$$A_I = \{ x \in [0, 1] : \forall k = 1, ... n, f^{k-1}(x) \in [x_{i_k}, x_{i_k}+1] \}.$$ 

We collect some useful remarks that are easy to check.
Remark 17.  

i. The \( A_I \) are half-open intervals that are closed on the left.

ii. If the finite word \( I \) is a prefix for the word \( J \), then \( A_J \subset A_I \).

iii. For any fixed \( I \) and \( n \geq |I| \), the collection

\[
\{ A_J : |J| = n \text{ and } I \text{ is a prefix for } J \}
\]

is a partition of \( A_I \).

iv. If \( I = i_1i_2...i_n \) and \( I' = i_2...i_n \), then

\[ x \in A_I \implies Tx \in A_{I'} . \]

v. If \( I \) precedes \( J \) in lexicographical order, then

\[ x \in A_I \text{ and } y \in A_J \implies x < y . \]

Each point \( x \in \mathbb{T} \) determines a unique infinite word \( \iota(x) \in \mathbb{A}^\mathbb{N} \). Since, by construction,

\[
\iota \circ T = S \circ \iota
\]

we have that \( \mathbb{A}^\mathbb{N} \) is a Borel measurable factor of \( (\mathbb{T}, T) \).

Remark 18. The lexicographical ordering on \( \mathbb{A}^\mathbb{N} \) and the usual order on \([0, 1]\) are also compatible with the factor map \( \iota \). In particular, for any \( x, y \in [0, 1] \),

i. \( x < y \) implies \( \iota(x) \leq \iota(y) \), and

ii. \( \iota(x) < \iota(y) \) implies \( x < y \).

Lemma 19. If \( I \in \mathbb{A}^\mathbb{N} \) doesn’t terminate with an infinite sequence of \( d-1 \)’s, there is an \( x \in [0, 1] \) with the property that \( \iota(x) = I \).

Proof. Let \( I = i_1i_2... \) and set \( I_n = i_1...i_n \) for each \( n \in \mathbb{N} \). It is enough to show that

\[
\bigcap_{n=1}^{\infty} A_{I_n} \neq \emptyset .
\]

The hypothesis entails that for any \( I_n \) there is an \( m > n \) with the property that \( \overline{A_{I_m}} \subset A_{I_n} \). This in turn means that

\[
\bigcap_{n=1}^{\infty} A_{I_n} = \bigcap_{n=1}^{\infty} \overline{A_{I_n}}.
\]

The claim follows since a decreasing sequence of bounded, closed intervals must have a non-empty intersection.

Lemma 20. Let \( x_0 \in \mathbb{T} \) be a point with the property that \( \iota(x_0) \) doesn’t terminate with an infinite sequence of 0’s. Then \( x_0 \) is a point of continuity for the map \( \iota : \mathbb{T} \to \mathbb{A}^\mathbb{N} \).
Proof. Write \( I = i_1i_2\ldots \) for \( \iota(x_0) \) and let \( I_n = i_1\ldots i_n \). The hypothesis entails that \( x_0 \) does not lie on the left endpoint of any of the \( A_{I_n} \). In other words, \( x_0 \) is in the interior of all the \( A_{I_n} \). Suppose \( x_k \to x_0 \) as \( k \to \infty \). For any \( n \in \mathbb{N} \), \( x_k \in A_{I_n} \) for all sufficiently large \( k \). For such \( k \), \( \iota(x_k) \) will have \( I_n \) as a prefix.

In view of the analysis of the previous section, we may select an infinite rotational subset \( X \subset \mathbb{T} \) for the transformation \( T_0 : x \mapsto d \cdot x \mod 1 \) that is an extension for an irrational rotation of \( \mathbb{T} \) by \( \theta_0 \in \mathbb{R}/\mathbb{Z} \). By mapping each point of \( X \) to its \( d \)-adic expansion, we may embed \( X \) continuously in \( \mathcal{A}^\mathbb{N} \). (The transformation \( T_0 \) is just the restriction of the standard shift map on \( \mathcal{A}^\mathbb{N} \) to \( X \).

Since no point of \( X \) has a \( d \)-adic expansion that terminates in an infinite sequence of \( d \)’s, there is an \( a \in \mathbb{T} \) with the property that \( \iota(a) \in X \). The orbit of \( a \),

\[ O(a) = \{ T^i a : i \geq 0 \} \]

is invariant under \( T \). By remark 13 and equation 3, \( T \) preserves cyclic order. One can check that the same facts extend to the closure \( Y = \overline{O(a)} \).

**Lemma 21.** Every point of \( Y \) is a point of continuity for \( \iota \).

**Proof.** By lemma 20 it suffices to show that for every \( y \in Y \), \( \iota(y) \) does not end with an infinite sequence of 0’s. If this is not the case then by applying \( T \) sufficiently many times we obtain a point \( y \in Y \) for which \( \iota(y) = 000\ldots \). In particular, the orbit of \( y \) lies in the set \( \iota^{-1}(000\ldots) \). The analysis in the previous section implies that there exist two points \( u, v \in X \) whose order is reversed by \( S \). Since \( \iota(a) \) has dense orbit in \( X \), there are integers \( n, m \in \mathbb{N} \) such that \( \iota(T^n a) \) and \( \iota(T^m a) \) are so close to \( u \) and \( v \) that their order is reversed by \( S \) as well. By remark 13, \( T^n a \) and \( T^m a \) have their orders reversed by \( T \). Consequently, \( T \) does not preserve the cyclic order of \( y, T^n a \) and \( T^m a \). This contradiction proves that every point of the closed set \( Y \) is a point of continuity of \( \iota \).
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