THE 3D DIMER AND ISING PROBLEMS REVISITED

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Abstract. We express the finite 3D Dimer partition function as a linear combination of determinants of oriented adjacency matrices, and the finite 3D Ising partition sum as a linear combination of products over aperiodic closed walks. The methodology we use is embedding of cubic lattice on 2D surfaces of large genus.

1. INTRODUCTION

The motivation for this work has been a renewed interest of theoretical physicists in basic statistical physics models, the Ising and the Dimer models. Let us mention two such recent promising works. The first one is a new relation (duality) of topological strings and the Dimer model, see e.g. [ORV]. The second one is a relation of the theory of discrete Riemann surfaces [M] and Theta functions with criticality of the Ising problem [AMV, CM1, CM2].

In this paper we present new formulas for the finite 3D Dimer and 3D Ising problems. The formulas are obtained in a combinatorial way. We hope that a profitable relation with the work mentioned above may be found.

A model we consider here is the Ising version of the Edwards-Anderson model. It can be described as follows. A coupling constant $J_{ij}$ is assigned to each bond $\{i,j\}$ of a given lattice graph $G$; the coupling constant characterizes the interaction between the particles represented by sites $i$ and $j$. A physical state of the system is an assignment of spin $\sigma_i \in \{+1, -1\}$ to each site $i$. The Hamiltonian (or energy function) is defined as $H(\sigma) = -\sum_{\{i,j\} \in E} J_{ij} \sigma_i \sigma_j$. The distribution of physical states over all possible energy levels is encapsulated in the partition function $Z(\beta) = \sum_\sigma e^{-\beta H(\sigma)}$ from which all fundamental physical quantities may be derived.

We may reformulate the Ising problem in graph theoretic terms as follows. A graph is a pair $G = (V,E)$ where $V$ is a set of vertices and $E$ is now the set of edges (not the energy). A graph with some regularity properties may be called a lattice graph. We associate with each edge $e$ of $G$ a weight $w(e)$ and for a subset of edges $A \subset E$, $w(A)$ will denote the sum of the weights $w(e)$ associated with the edges in $A$.

An even subgraph of a graph $G = (V,E)$ is a set of edges $U \subset E$ such that each vertex of $V$ is incident with an even number of edges from $U$. The generating function of even subgraphs $H(G,x)$ equals the sum of $\alpha^{w(U)}$ over all even subgraphs $U$ of $G$. A classic relation between the Ising partition function and the generating function of even subgraphs of the same graph states that

$$Z(\beta) = 2^n \left( \prod_{\{i,j\} \in E} \cosh(\beta J_{ij}) \right) H(G, \tanh(\beta J_{ij})).$$

A subset of edges $P \subset E$ is called a perfect matching or dimer arrangement if each vertex belongs to exactly one element of $P$. The dimer partition function on graph $G$ may be viewed as a polynomial $P(G,\alpha)$ which equals the sum of $\alpha^{w(P)}$ over all perfect matchings $P$ of $G$. This polynomial is also called the generating function of perfect matchings.

The generating functions of even subgraphs and perfect matchings may be defined in a more general way as follows: associate a variable $x_e$ with each edge $e$ of graph $G$, let $x(A) = \prod_{e \in A} x_e$ and let e.g. the generating function of perfect matchings be the sum of $x(P)$, $P$ perfect matching of $G$. All results introduced in this
paper also hold in this more general setting; however the presentation using weights rather than variables is perhaps more natural.

**Convention for the cubic lattice.** This paper studies properties of finite cubic lattices. Let us now fix some notation for them.

Let $m_1, m_2, k$ be positive integers, where $k$ is even and $m_1$ and $m_2$ odd. The cubic lattice $Q = Q(m_1m_2k)$ is the following graph:

The vertices are:

$V(xyz), x = 1, \ldots, m_1, y = 1, \ldots, m_2, z = 1, \ldots, k.$

The edges are:

- the vertical edges $v_{xyz} = \{V_{xyz}, V_{xy(z+1)}\}, z = 1, \ldots, k - 1,$
- the width edges $w_{xyz} = \{V_{xyz}, V_{x(y+1)z}\}, y = 1, \ldots, m_2 - 1,$
- the horizontal edges $h_{xyz} = \{V_{xyz}, V_{(x+1)yz}\}, x = 1, \ldots, m_1 - 1.$

**Definition 1.1.** We now fix as in Fig. 1 a linear order '$<$' on the set of the vertices of $Q$ as

$V(11), V(12), \ldots, V(1m_2), V(2m_2), V(2(m_2 - 1)), \ldots, V(m_1m_2)$

where $V(xy) = V(xy1), \ldots, V(xyk)$ and $\bar{V}_{xy}$ denotes the reversal of $V_{xy}.$

**Definition 1.2.** Let us now define a set of pairs of edges $R(Q)$.

We let $H$ be the set of all horizontal edges. For each $x \leq m_1$ we let $W_x$ consist of all width edges with their x-coordinate equal to $x.$ We call sets $H$ and $W_x$ layers of $Q.$ It is convenient to depict the edges of a layer as the horizontal edges of a square grid, where lower left corner is the smallest in the fixed ordering of vertices. The first column of the layer $H$ is ordered according to the fixed ordering of vertices. Size of layer $H$ is $m_1 \times km_2,$ sizes of all $W_x$ are $m_2 \times k.$

If $e$ is an edge of such a layer $L$ then let $h(L, e)$ ($v(L, e)$ respectively) denote the horizontal (vertical respectively) coordinate of the first vertex of $e$ in $L.$ For a layer $L$ we let $R(L)$ be the set of all pairs $(e, e')$ of edges of $L$ such that $h(L, e') < h(L, e), v(L, e') \leq v(L, e) \leq v(L, e) + 1,$ and $h(L, e)$ is even while $h(L, e')$ is odd, example on Fig. 2. Finally we let $R(Q)$ to be the union of $R(L)$ over all layers of $Q.$

2. **Statement of the main results: The Dimer Problem**

Our result here is a simplification of the main result of [1]. The reader is heartily encouraged to check the original statement for a comparison. Before we write the main theorems let us introduce some necessary notation.
Determinants. $Q$ is a bipartite graph, which means that its vertices may be partitioned into two sets $V_1, V_2$ such that if $e$ is an edge of $Q$ then $|e \cap V_1| = |e \cap V_2| = 1$. We have $|V_1| = |V_2| = m_1 m_2 k/2$. Let $Z$ be the square $([V_1] \times |V_2|)$ matrix defined by $Z_{ij} = 0$ if $\{i, j\}$ is not an edge of $Q$. If $e = \{ij\}$ is an edge of $Q$ ($i \in V_1, j \in V_2$) then $Z_{ij} = \alpha^{w(e)}$ if $i < j$ (in fixed linear order) and $Z_{ij} = -\alpha^{w(e)}$ if $i > j$.

We will consider matrix $Z$ with its rows and columns ordered in agreement with the fixed ordering and we will assume that $V_{111} \in V_1$. A signing of a matrix is obtained by multiplying some of the entries of the matrix by $-1$.

An orientation of a graph $G = (V, E)$ is a digraph $D = (V, A)$ obtained from $G$ by assigning an orientation to each edge of $G$, i.e. by ordering the elements of each edge of $G$. The elements of $A$ are called arcs. We say that signing $Z$ of $Z$ corresponds to orientation $D$ of $Q$ if $Z_{ij} = -Z_{ij}$ for $(ij) \in D$ and $i > j$. For arc $(ij)$ of orientation $D$ of $Q$ we let $\text{sign}(D, (ij)) = 1$ if $i < j$, and $\text{sign}(D, (ij)) = -1$ otherwise.

**Theorem 1.** The dimer partition function $\mathcal{P}(Q, \alpha)$ is

$$\mathcal{P}(Q, \alpha) = 2^{-C} \sum_D (-1)^{|\{(e, e') \in R(Q); \text{sign}(D, e) = \text{sign}(D, e') = -1\}|} \det(Z(D)),$$

where the sum is over all orientations of $Q$ with all vertical edges positive, and $C = km_1 (m_2 - 1)/2 + km_2 (m_1 - 1)/2$.

Knowing the proof Theorem 1 can be easily rewritten as

**Theorem 2.**

$$\mathcal{P}(Q, \alpha) = 2^C \alpha^{w(M)} - (2^C + 1)E,$$

where $M$ is the unique perfect matching of $Q$ consisting of vertical edges only and $E$ equals the average of $\det(Z(D))$ over all orientations $D$ of $Q$ satisfying: $\{|(e, e') \in R(Q); \text{sign}(D, e) = \text{sign}(D, e') = -1\}$ is even and (again all vertical edges are positive).

There are $2^C$ orientations to be sum over in Theorem 1 and $2^{C-1}(2^C + 1)$ terms in Theorem 2 to be averaged over. Therefore the exact numerical analysis is probably not possible. We just tested Theorem 1 for small planar lattices. However it could be interesting to study statistical properties of determinants $\det(Z(D))$. Let us remark here also that Theorem 1 for a planar graph (2D Ising model, $m_1 = 1$) is still nontrivial, a question remains how to reduce it to the Kasteleyn’s one determinant. And would it be possible to apply similar reduction even for the non-planar graph?

3. **Statement of the main results: The Ising Problem**

The theory described in section 1 combined with the results of 11 yields an expression of the 3D Ising partition function as a linear combination of products over aperiodic closed walks on $Q$. We get a remarkably simple formula, analogous to the statement for the dimer partition function.

**Products over aperiodic closed walks.** Let $G = (V, E)$ be a planar graph embedded in the plane and for each edge $e$ let $x_e$ be an associate variable. Let $A = (V, A(G))$ be an arbitrary orientation of $G$. If $e \in E$ then $a_e$ will denote the orientation of $e$ in $A(G)$ and $a^{-1}$ will be the reversed directed edge to $a$. We let $x_e = x_a^{-1} = x_e$. A circular sequence $p = \v_1, a_1, v_2, a_2, ..., a_n, (v_{n+1} = v_1)$ is called non-periodic closed walk if the following conditions are satisfied: $a_i \in \{a_e, a_e^{-1} : e \in E\}$, $a_i \neq a_i^{-1}$ and $(a_1, ..., a_n) \neq Z^n$ for some sequence $Z$ and $m > 1$. We let $X(p) = \prod_{i=1}^n x_{a_i}$. We further let $\text{sign}(p) = (-1)^{1+n(p)}$, where $n(p)$ is a rotation number of $p$, i.e. the number of integral revolutions of the tangent vector. Finally let $W(p) = \text{sign}(p) X(p)$.

Now assume $p$ is a closed aperiodic walk in 3D cubic lattice $Q$. Can we define its rotation number? The following construction provides a solution.

Assume some vertices of planar graph $G$ embedded in the plane of degree 4 are marked. Then we call a walk $p$ correct if it satisfies the 'crossover condition' at each marked vertex, i.e. it never enters and exits a marked vertex in a pair of neighbouring edges along the vertex.

We naturally get a planar graph from $Q$ if we consider $Q$ drawn as bold $Q(3,3,2)$ in Figure 3 and put a new vertex to each edge-crossing. These new vertices will form the set of the marked vertices. Clearly there is a natural bijection which associates to each walk $p$ of $Q$ the corresponding correct walk $p'$ in the new
graph and we define rotation of \( p \) to be equal to rotation of \( p' \). This defines \( W(p) \) for each closed aperiodic walk \( p \) in \( Q \).

There is a natural equivalence on non-periodic closed walks: \( p \) is equivalent with reversed \( p \). Each equivalence class has two elements and will be denoted by \([p]\). We let \( W([p]) = W(p) \) and note that this definition is correct since equivalent walks have the same sign.

We denote by \( \prod(1 + W([p])) \) the formal infinite product of \((1 + W([p]))\) over all equivalence classes of non-periodic closed walks of \( G \).

Let \( D \) be an orientation of \( Q \). We further let

\[
\prod_D (1 + W([p])) = \prod_D (1 + W([p]))|_{x_e = \text{sign}(D, e)\alpha^{w_e}}.
\]

**Theorem 3. Generating function of even subgraphs is**

\[
\mathcal{H}(Q, \alpha) = 2^{-C} \sum_D (-1)^{|\{(e, e') \in R(Q); \text{sign}(D, e) = \text{sign}(D, e') = -1\}|} \prod_D (1 + W([p])),
\]

where the sum is over the orientations of \( Q \) with all the vertical edges positive.

**Remark.** Theorem 3 expresses the 3D Ising partition function as a linear combination of infinite products over aperiodic closed walks. For planar lattice this reduces to just one infinite product (conjectured by Feynman, proved in [S]) which provides an equivalence of planar Ising model with the quantum field theory of free fermion. There have been several attempts to generalize this equivalence for 3D. Earlier papers (e.g. [D]) attempt to replace closed aperiodic walks by 2D surfaces. Recent promising development mentioned in the introduction deals with discrete Riemann surfaces and the theta function:

It is known that the planar Ising partition function in the thermodynamical limit behaves like a 'common term' times a Riemann theta function. For torus, [AMV] asserts that in the thermodynamic limit and near criticality, the Ising partition function behaves like a 'common term' times a linear combination of four Riemann theta functions corresponding to torus. In [M] and [CM1, CM2], there is evidence that for the critical Ising model, the dependence of the determinants of adjacency matrices on the Kasteleyn orientations is exactly the same as the dependence of the determinants of the Dirac operator, of the corresponding conformal field theory, on the spin structures of the 2D Riemann surface; this is given in terms of theta functions of half-integer characteristics.

We believe that theorem 3 may provide a new insight into these efforts.

4. **Drawing of cubic lattices.**

In this section we describe how we draw cubic lattices on 2-dimensional surfaces.

4.1. **Theory of generalised g-graphs.**

**Definition 4.1.** A surface polygonal representation \( S_g \) of an orientable 2D surface of genus \( g \) consists of a base \( B_0 \) and \( 2g \) bridges \( B_j^G \), \( i = 1, \ldots, g \) and \( j = 1, 2 \), where

i) \( B_0 \) is a convex 4\( g \)-gon with vertices \( a_1, \ldots, a_{4g} \) numbered clockwise;

ii) \( B_1^i, i = 1, \ldots, g \), is a 4-gon with vertices \( x_1^i, x_2^i, x_3^i, x_4^i \) numbered clockwise. It is glued with \( B_0 \) so that the edge \([x_1^i, x_2^i]\) of \( B_1^i \) is identified with the edge \([a_{4(i-1)+1}, a_{4(i-1)+2}]\) of \( B_0 \) and the edge \([x_3^i, x_4^i]\) of \( B_1^i \) is identified with the edge \([a_{4(i-1)+3}, a_{4(i-1)+4}]\) of \( B_0 \);

iii) \( B_2^i, i = 1, \ldots, g \), is a 4-gon with vertices \( y_1^i, y_2^i, y_3^i, y_4^i \) numbered clockwise. It is glued with \( B_0 \) so that the edge \([y_1^i, y_2^i]\) of \( B_2^i \) is identified with the edge \([a_{4(i-1)+2}, a_{4(i-1)+3}]\) of \( B_0 \) and the edge \([y_3^i, y_4^i]\) of \( B_2^i \) is identified with the edge \([a_{4(i-1)+4}, a_{4(i-1)+5}]\) of \( B_0 \). Indexing is mod \( 4g \).

**Definition 4.2.** A graph \( G \) is called a \( g \)-graph if it is embedded on \( S_g \) so that all the vertices belong to the base \( B_0 \), and the embedding of each edge uses at most one bridge. We denote the set of the edges embedded entirely on the base by \( E_0 \) and the set of the edges embedded on each bridge \( B_j^G \) by \( E_j, i = 1, \ldots, g \). \( j = 1, 2 \). Moreover the following conditions need to be satisfied.

1. the outer face of \( G_0 = (V, E_0) \) is a cycle, and it is embedded on the boundary of \( B_0 \),
2. if \( e \in E_1 \) then \( e \) is embedded entirely on \( B_1^i \) with one end-vertex belonging to \([x_1^i, x_2^i]\) and the other one to \([x_3^i, x_4^i]\). Analogously for \( e \in E_2 \).
We need a generalization of the notion of a g-graph.

**Definition 4.3.** Any graph \( G \) obtained by the following construction will be called *generalized g-graph*.

1. Let \( g = g_1 + \ldots + g_n \) be a partition of \( g \) into positive integers.
2. Let \( S_{g_i} \) be a polygonal representation of a surface of genus \( g_i \), \( i = 1, \ldots, n \). Let us denote the basis and the bridges of \( S_{g_i} \) by \( B_0^i \) and \( B_{1,k}^i \), \( i = 1, \ldots, n, j = 1, \ldots, g_i \) and \( k = 1, 2 \).
3. For \( i = 1, \ldots, n \) let \( H_i \) be a \( g_i \)-graph with the property that the subgraph of \( H_i \) embedded on \( B_0^i \) is a cycle, embedded on the boundary of \( B_0^i \). Let us denote it by \( C_i \).
4. Let \( G_0 \) be a 2-connected graph properly embedded on the plane and let \( F_1, \ldots, F_n \) be a subset of the faces of \( G_0 \). Let \( G_i \) be the cycle bounding \( F_i \), \( i = 1, \ldots, n \). Let each \( K_i \) be isomorphic to \( C_i \).
5. Then \( G \) is obtained by gluing the \( H_i \)'s into \( G_0 \) so that each \( K_i \) is identified with \( C_i \).

**Orientations.** Let \( G \) be a g-graph and let \( G_j^i = (V, E_0 \cup E_j^i) \). An orientation \( D_0 \) of \( G_0 \) such that each inner face of each 2-connected component of \( G_0 \) is clockwise odd in \( D_0 \) is called a *basic orientation* of \( G_0 \). Note that a basic orientation always exists for a planar graph. Further we define the orientation \( D_j^i \) of each \( G_j^i \) as follows: We consider \( G_j^i \) embedded on the plane by the planar projection of \( E_j^i \) outside \( B_0 \), and complete the basic orientation \( D_0 \) of \( G_0 \) to an orientation of \( D_j^i \) so that each inner face of each 2-connected component of \( G_j^i \) is clockwise odd. The orientation \( -D_j^i \) is defined by complete reversing of the orientation \( D_j^i \) of \( G_j^i \).

Observe that after fixing a basic orientation \( D_0 \), the orientation \( D_j^i \) is uniquely determined for each \( i, j \).

**Definition 4.4.** Let \( G \) be a g-graph, \( g \geq 1 \). An orientation \( D \) of \( G \) which equals the basic orientation \( D_0 \) on \( G_0 \) and which equals \( D_j^i \) or \( -D_j^i \) on \( E_j^i \) is called relevant. We define its type \( r(D) \in \{-1, 1\}^{2g} \) as follows: For \( i = 0, \ldots, g - 1 \) and \( j = 1, 2 \), \( r(D)_{2i+j} \) equals \(+1\) or \(-1\) according to the sign of \( D_j^i \) in \( D \).

Moreover we let \( c(r(D)) \) equal the product of \( c_i \), \( i = 0, \ldots, g - 1 \), where \( c_i = c(r_{2i+1}, r_{2i+2}) \) and \( c(1, 1) = c(1, -1) = c(-1, 1) = 1/2 \) and \( c(-1, -1) = -1/2 \).

Observe that \( c(r(D)) = (-1)^n2^{-g} \), where \( n = \{|i; r_{2i+1} = r_{2i+2} = -1\}| \).

For each generalized g-graph \( G \) we can define \( 4^g \) relevant orientations \( D_1, \ldots, D_{4^g} \) with respect to a fixed basic orientation of \( G_0 \), and coefficients \( c(r(D_i)) \), \( i = 1, \ldots, n \) in the same way as for a g-graph. Now we write theorem proved in [GL] which will be essential in proof of Theorem [I] in the future.
4.2. Cubic lattices as generalized g-graphs. In this subsection we will describe how to realize $Q(m_1, m_2, k)$ as a subgraph of a generalized g-graph. Let us denote by $Q^L$ the larger lattice $Q(m_1, m_2, 2k + 1).

**How to draw $Q^L$ on the plane.** First draw the paths $V_{xy}$ along a cycle in the linear order as in Def. 4.2. Next, draw the horizontal edges inside this cycle, and the width edges outside of this cycle as depicted in Fig. 3 below where $Q^L = Q(3, 3, 5)$ is properly drawn. The figure also shows how $Q = Q(3, 3, 2)$ is embedded in $Q^L = (3, 3, 5)$.

We keep the following rule: the interiors of the curves representing $h_{xyz}$ and $h_{(x+1)yz}$ ($w_{xyz}$ and $w_{x(y+1)z}$ respectively) intersect if and only if $z$ is even. We denote by $C(e)$ the curve representing edge $e$.

![Figure 3](image-url)  

**Figure 3.** Drawing of a cube $Q^L = Q(3, 3, 3)$ in a plane.

Now we modify $Q^L$ into a generalized g-graph $Q'$. We introduce new vertices to some edge-crossings of $Q^L$ and delete and subdivide some edges. In this way we obtain a generalised g-graph $Q'$ with an even subdivision of $Q(m_1, m_2, k)$ as its subgraph. The construction is analogous and independent for each plane, and so we describe it only for a plane $W_z$.

The construction is described by Fig. 4 for the edges between $V_{x(y-1)}, V_{xy}$ and between $V_{xy}, V_{x(y+1)}$, for $x$ odd and $y < m_2 - 1$ even.

![Figure 4](image-url)  

**Figure 4.** Construction of $Q'$ for part of the plane $W_z$ ($x$ odd), $V_{x(y-1)}, V_{xy}, V_{x(y+1)}$ ($y$ even).

(1) For each $y$ even let $Aux_1 = \{w_{xyz} : z \text{ odd}\}$. For each edge $e$ of $Aux_1$ introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of $W_{x(y-1)} \cup W$, where $W = W_{x(y+1)}$ in case $y < m - 1$ and $W = \emptyset$ otherwise. By this operation, each $e \in Aux_1$ is replaced by a path. Call each edge of this path *auxiliary*. 

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(2) For each \( y \) even let \( Aux_2 = \{ w_{x(y-1)}, w_{x(y-1)n} \} \cup A \), where \( A = \{ w_{x,y} \} \) in case \( y < m - 1 \) and \( A = \emptyset \) otherwise. For each edge \( e \) of \( Aux_2 \) introduce a new vertex to each intersection of \( C(e) \) with the curves representing the edges of \( W_{xy} \). Hence each edge \( e \in Aux_2 \) is replaced by a path. Call each edge of this path auxiliary. For each \( y \) even the edges \( v_{x,y}, v_{y,x(n-1)} \) and also \( v_{x,y+1}, v_{y,x(n-1)} \) will also be called auxiliary. In Figure 4, the auxiliary edges are represented by dashed lines.

(3) The edges \( w_{xyz}, y \) even and \( z \) even will be called relevant for \( Q \). If \( y < m - 1 \) then the relevant edges are subdivided by two vertices (added in 2.) into three edges of \( Q' \). The middle one will be called special and the other two long. If \( y = m - 1 \) then the relevant edge \( w_{xyz} \) is subdivided by one vertex into two edges of \( Q' \). The one incident to \( V_{xm} \) will be called special and the other one long. If \( e \) is a relevant edge of \( Q \), then we choose a initial long edge \( f \) and we let \( w(e) = w(f) \). We let the weight of the special edge and of the remaining long edge be equal to 0.

(4) The edges of \( W_{x(y-1)} \cup W \) also got subdivided by new vertices introduced in step 1 and step 2.

(5) We delete all edges of the paths obtained from \( w_{x(y-1)}, w_{x(y+1)} \), \( 1 < z < n \) odd, as well as we delete all vertexes in intersections of those edges with auxiliary edges. In Figure 4, the deleted edges are represented by dotted lines.

(6) Each edge \( e \in \{ w_{x(y-1)}, w_{x(y+1)} : z \) even } \), is subdivided by new vertices introduced in step 1 into a path. We let the weights assigned to the edges of the path equal 0 except of one initial edge whose weight is let equal \( w(e) \). The edge \( e \) of this path such that the interior of \( C(e) \) does not intersect interior of any curve representing a long edge will also be special. The others are called short.

(7) All vertical edges which are not auxiliary (see Fig 4) will be called special. In Figure 4, the special edges are represented by normal lines.

(8) Let \( Aux \) denote the set of all auxiliary edges. Then deletion of \( Aux \) results in a subdivision of \( Q_{m_1, m_2} \). We subdivide some special edges (vertical and border width ones) so that the graph \( Q' = Q - Aux \) is an even subdivision of \( Q_{m_1, m_2} \). All these new edges will be special, and we set their weights equal 0.

This finishes the construction. In Figure 4, the long and short edges are represented by fat lines.

\( Q' \) is a generalised g-graph. Its planar part \( Q_p \) is set of all the auxiliary and special edges. Other edges (i.e. the short and long) are drawn on a face of \( Q' \) and they may be drawn onto a pair of bridges above this face. One bridge contains one long edge, and the other bridge contains all the short edges.

Now let us set appropriate weights to all the auxiliary edges. In order to do that first subdivide and add some auxiliary edges and vertices to \( Q_p \) in such a way that there exists a matching \( M_{Aux} \) of \( V(Q') \) - \( V(Q) \) consisting of auxiliary edges, and a perfect matching \( M_0 \) of \( Q_p \). We set weights of edges from \( M_{Aux} \) as \( w(e) = 0 \). Weights of other auxiliary edges \( w(e) = -\infty \). With such properties of \( Q' \) we have an important result: \( \mathcal{P}(Q',\alpha) = \mathcal{P}(Q(m_1, m_2, k), \alpha) \).

Now we introduce a basic orientation \( D^p \) of \( Q^p \) in order to prepare our stage to use the Theorem 4 for the generalised g-graph \( Q' \). Orientation \( D^p \) has the following properties:

1. \( D^p \) on special edges is in agreement with the natural ordering (Def. 4).
2. All the signs \( s(D^p, P) \) in Pfaffian \( Pf_{Q^p}(D^p, \alpha) \) (Def. 4) are positive.
3. The orientation of edges on a bridge has positive sign if and only if it is in agreement with the natural ordering (Def. 1).

The construction of such a \( D^p \) is possible according to Kasteleyn [K].

5. The proofs

5.1. Proof of Theorem 4 We use Theorem 4 for the graph \( Q' \). Then we may write

\[
\mathcal{P}(Q(m_1, m_2, k), \alpha) = \mathcal{P}(Q', \alpha) = \sum_{i=1}^{4^9} c(r(D_i)) Pf_{Q^p}(D_i, \alpha) = \sum_{i=1}^{4^9} c(r(D_i)) Pf_{Q^p}(D_i^{\alpha}, \alpha) = \sum_{i=1}^{4^9} c(r(D_i)) Pf_{Q^p}(D_i^{\alpha}, \alpha),
\]
where the first equality is the result of previous section. The second equality holds because of the Theorem 4. The third equality holds because of the setting of the weights of edges in $Q'$ in the previous section. And finally, orientation $D_1^*$ being induced on $Q$ by $D_1$ on $Q$, the fourth equality follows from the three facts: setting of the weights of edges, $\tilde{Q}$ is even subdivision of $Q$, it is possible to permute vertexes covered by $M_{Aux}$ so that the sign is correct.

In Section 9 we have defined for a bipartite graph $G$ a matrix $Z(D)$ depending on a orientation $D$. It is easy to check that for $Q$ holds $Pf_Q(D_1^*, \alpha) = det(Z(D_1^*))$.

To proceed in proof of Theorem 1 we have to establish the dependence of $c(r(D_1))$ on the orientation $D_1^*$ of $Q$ and specify the orientations we sum over in the language of $D$. We let $D_1^r$ denote the sign of the bridge containing $e$, and $D_1^s$ denotes the sign of the other bridge containing the set of short edges.

The relevant edges $w_{x(m_2-1)z}(Q(m_1 m_2))$ and $h_{(m_1-1)y z}(Q(m_1 m_2))$ are associated with only one long edge of $Q'$. If $e$ is such relevant edge of $Q$, we will call it border edge, and we denote by $e_1$ the corresponding long edge. We let $D_1(e) = (d_{11}^1(e_1), d_{12}^1(e_1), +, +)$. Each relevant non-border edge $e$ of $Q$ has two long edges $e_1, e_2$ associated with it. We let $D_1(e) = (d_{11}^1(e_1), d_{12}^1(e_1), d_{11}^2(e_2), d_{12}^2(e_2))$. A relevant vector is any element $r$ of $\{[+, -]^4\}^R$ such that $r(e)_3 = r(e)_4 = +$ for each relevant border edge $e$ of $Q$.

From the definition of $c(r(D_1))$ and from properties of the generalized g-graph $Q'$ of genus $g$ follows that $c(r(D_1)) = 2^{-g}(-1)^n$, where $n$ is number of faces of $Q'$ above them edges on both bridges are oriented in negative sense, i.e. against fixed ordering. There is a natural bijection between relevant orientations of $Q'$ and relevant vectors. If $r$ is a relevant vector, then let $D_1(r)$ denote the corresponding relevant orientation of $Q'$ and let sgn$(r)$ of a relevant vector $r$ be calculated as follows: $\text{sgn}(r) = 2^g c(r(D_1)) = (-1)\{(t(e)_i): i = 0, 1; r(e)_{2i+1} = r(e)_{2i+2} = -1\}$.

There is $C_r = k m_1 (m_2 - 1)/2 + k m_2 (m_1 - 1)/2$ relevant edges and $C_b = k (m_1 + m_2)$ border edges. Genus of $Q'$ is therefore $g = C_r - C_b$. There are $2^{C_r - C_b}$ relevant vectors.

Let us consider a set $D$ of relevant orientations $D_1$ such that there is at least one relevant non-border edge $e$ with $r(e)_2 \neq r(e)_4$, i.e. orientation of bridges with short edges incident to $e$ are different.

**Proposition 5.1.** Sum of contribution of all orientations from $D$ to $P(Q, \alpha)$ is zero.

**Proof:** For given orientation $D \in D$ define a set $A_D = \{e \text{ relevant, } r(e)_2 \neq r(e)_4\}$. Now let fix a set of relevant edges $A_f$ and consider all orientations $D_f$ such that $A_{D_f} = A_f$. Fix also one relevant edge $e_f \in A_f$.

Remark: $c(r(D_1)) + c(r(D_2)) = 0$ if $r(D_1)$ and $r(D_2)$ differs only in one relevant edge $e$ in such a way that $r(D_1)(e)_2 = r(D_2)(e)_2 \neq r(D_1)(e)_4 = r(D_2)(e)_4$ and $r(D_1)(e)_1 = -r(D_2)(e)_1$, $r(D_1)(e)_3 = -r(D_2)(e)_3$. It is also that induced orientations $D_1^r = D_2^s$.

Using this remark for $e = e_f$ we have for every $A_f$: $\sum_{D_f \in D_f} c(r(D_1)) Pf_Q(D_1^*, \alpha) = 0$.

The claim in Proposition 5.1 follows directly. $\square$

Let us call all $D_1 \notin D$ useful orientations and from now we sum only over such orientations. There is $2^{C_r - 2C_b}$ of useful orientations. Similarly we define useful relevant vectors.

**Proposition 5.2.** Each orientation $D$ of $Q$, which agrees with the fixed order on vertical edges, corresponds to $2^{C_r - C_b}$ of useful orientations $D_1$ of $Q'$ and all those $D_1$ have the same sign $c(r(D_1))$.

**Proof:** Let $r$ be a useful vector. Then $D(r)$ determines uniquely $r(e)_2$ and $r(e)_4$ for each relevant edge $e$ and also $r(f)_1$ for each relevant border edge $f$. Hence $D(r)$ determines uniquely $r(f)$ for each relevant border edge $f$. Moreover $D(r)$ determines uniquely the product $r(e)_1 \times r(e)_3$ for each relevant non-border edge $e$. Since there are $C_r - C_b$ relevant non-border edges, there is $2^{C_r - C_b}$ orientations $D_1$ corresponding to it.

Let $r, s$ be useful and both lead to the same $D$ of $Q$. Then $r(e)_2 = s(e)_2 = s(e)_4 = r(e)_4$ for each relevant non-border edge $e$ and $r(e)_2 = s(e)_2$ and $s(e)_1 = r(e)_1$ for each relevant border edge. This implies that $\text{sgn}(r) = \text{sgn}(s)$. $\square$
Concluding from what we did till now we can write the dimer partition function as

\[ P(Q, \alpha) = 2^{-C_r} \sum_D \text{sgn}(D) \det(Z(D)), \]

where the sum is over all orientations of Q with all vertical edges positive. At this point the only thing missing in proof of Theorem 1 is to concretize \( \text{sgn}(D) \).

**Proposition 5.3.** \( \text{sgn}(D) = (-1)^{|\{(e, e') \in R(Q); \text{sign}(D,e)=\text{sign}(D,e')=-1\}|} \), where the set of edges \( R(Q) \) is defined in 1.2.

**Proof:** Let us remind that \( \text{sgn}(D) = (-1)^n \) where \( n \) is number of faces with both bridges above them negative. Each non-border relevant edge \( e \) corresponds to two faces with bridges. The contribution of the two corresponding faces to sign could be negative only if orientation of \( e \) is negative (\( r(e)_1 \neq r(e)_3 \)). And moreover if orientation of corresponding shot edges is negative (\( r(e)_2 = r(e)_4 = -1 \)). It follows from the construction that this happen only if product of orientation of all \( e' \) (as defined in 1.2) is negative. Considering all the relevant edges we get the claim of Proposition 5.3. \( \Box \)

### 5.2. Proof of Theorem 2

**Proposition 5.4.** The average of \( \det(Z(D)) \) over all orientations of \( Q \) with all vertical edges fixed positive equals \( \alpha w(M) \), where \( M \) is perfect matching of \( Q \) consisting only from vertical edges.

**Proof:** By the linearity of expectation the contribution of other than vertical edges cancel out when we calculate the average of \( \det(Z(D)) \). Since \( Q(m_1m_2k) \) has exactly one perfect matching consisting of vertical edges only, Proposition 5.4 follows. \( \Box \)

There is \( 2^{2C} \) orientation \( D \) in 1 to be summed over. And \( 2^{C-1}(2^C + 1) \) of them have positive sign. Proof: Looking at Figure 2 we can see that as soon as there is at least one relevant (shaded) edge negative there is so many orientation with positive as with negative sign. Only if all the relevant edges are positive the sign is also positive. It means there is \( 2^C \) more positive orientation then negative.

By Theorem 1 and Proposition 5.4 we have then

\[ P(Q, \alpha) = 2^{-C} [-2^{2C} \alpha w(M) + 2E(2^{C-1}(2^C + 1))] = -2^{2C} \alpha w(M) + E(2^C + 1), \]

where \( E \) equals the average of \( \det(Z(D)) \) over all \( D \) of \( Q \) with positive sign. And Theorem 2 is proved. \( \Box \)

### 5.3. Proof of Theorem 3

We use a straightforward reformulation of a theorem of [L2]:

**Theorem 5.** Let \( G \) be a generalized g-graph. Then

\[ \mathcal{H}(G, \alpha) = \sum_{i=1}^{4^g} c(r(D_i)) \prod_{D_i} (1 + W([p])), \]

where \( D_i \) are all the relevant orientations of \( G \), and rotation of \( p \) and \( \prod_{D_i} (1 + W([p])) \) are defined in the same way as described for \( Q \) in section 3.

Proof of this theorem could be found in [L2] and for planar graphs in [S].

Let \( D_i, i = 1, \ldots, 4^g \) be the relevant orientations of generalised g-graph \( Q' \). If we apply theorem 5 to \( Q' \), from the choice of weights we get

\[ \mathcal{H}(Q, \alpha) = \sum_{i=1}^{4^g} c(r(D_i)) \prod_{D_i} (1 + W([p])). \]

The same analysis as in the previous section finishes the proof of Theorem 3.
6. Conclusions

We express the finite 3D Dimer partition function as a linear combination of determinants of oriented adjacency matrices in Theorem H and the finite 3D Ising partition function as a linear combination of products over aperiodic closed walks in Theorem K.

The methodology we use is embedding of cubic lattice on 2D surfaces of large genus. By this embedding cubic lattice is transformed to a generalized g-graph. Then we use theorems from GL [L1, L2], which generalize approach of K.

We believe this approach may relate to recent developments in conformal field theory [M, CM1, CM2] and topological string theory [ORV], and ultimately teach us something about 3D basic statistical physics problems.

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