DETERMINANT BUNDLE IN A FAMILY OF CURVES, AFTER A. BEILINSON AND V. SCHECHTMAN

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Abstract. Let \( \pi : X \to S \) be a smooth projective family of curves over a smooth base \( S \) over a field of characteristic 0, together with a bundle \( E \) on \( X \). Then A. Beilinson and V. Schechtman define in [1] a beautiful “trace complex” \( \text{tr} A^*_E \) on \( X \), the 0-th relative cohomology of which describes the Atiyah algebra of the determinant bundle of \( E \) on \( S \). Their proof reduces the general case to the acyclic one. In particular, one needs a comparison of \( R\pi_* (\text{tr} A^*_F) \) for \( F = E \) and \( F = E(D) \) where \( D \) is étale over \( S \) (see Theorem 2.3.1, reduction ii) in [1]). In this note, we analyze this reduction in more details and correct a point.

1. Introduction

Let \( \pi : X \to S \) be a smooth projective morphism of relative dimension 1 over a smooth base \( S \) over a field \( k \) of characteristic 0. One denotes by \( T_X \) and the tangent sheaf over \( k \), by \( T_X/S \) the relative tangent sheaf, and by \( \omega_{X/S} \) the relative dualizing sheaf. For an algebraic vector bundle \( E \) on \( X \), one writes \( E^\circ = E^* \otimes_{\mathcal{O}_X} \omega_{X/S} \). Let \( \text{Diff}(E, E) \) (resp. \( \text{Diff}(E/S, E/S) \subset \text{Diff}(E, E) \)) be the sheaf of first order (resp. relative) differential operators on \( E \) and \( \epsilon : \text{Diff}(E, E) \to \mathcal{E}nd(E) \otimes_{\mathcal{O}_X} T_X \) be the symbol map. The Atiyah algebra \( \mathcal{A}_E := \{ a \in \text{Diff}(E, E) | \epsilon(a) \in \text{id}_E \otimes_{\mathcal{O}_X} T_X \} \) of \( E \) is the subalgebra of \( \text{Diff}(E, E) \) consisting of the differential operators for which the symbolic part is a homothety. Similarly the relative Atiyah algebra \( \mathcal{A}_{E/S} \subset \mathcal{A}_E \) of \( E \) consists of those differential operators with symbol in \( \text{id}_E \otimes_{\mathcal{O}_X} T_{X/S} \) and \( \mathcal{A}_{E, \pi} \subset \mathcal{A}_E \) with symbols in \( T_\pi = d\pi^{-1}(\pi^{-1}T_S) \subset T_X \). Let \( \Delta \subset X \times_S X \) be the diagonal. Then there is a canonical sheaf isomorphism \( \text{Diff}(E/S, E/S) \cong \frac{E \otimes E^\circ (2\Delta)}{E \otimes E^\circ} \) (see [1], section 2) which is locally written as follows. Let \( x \) be a local coordinate of \( X \) at a point \( p \), and \( (x, y) \) be the induced local coordinates on \( X \times_S X \) at \( (p, p) \), such that the equation of \( \Delta \) becomes \( x = y \). Let \( e_i \) be a local basis of \( E \), \( e_i^* \)

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be its local dual basis. Then the action of
\[
P = \sum_{i,j} e_i \otimes e_j^* P_{ij}(x,y) \, dy
\]
on \begin{equation}
\begin{aligned}
s = \sum_{\ell} s_\ell(y) e_\ell
\end{aligned}
\end{equation}
is
\[
\begin{aligned}
P(s) &= \sum_i e_i \sum_j (P_{ij}^{(1)}(x,0)s_j(x) + P_{ij}(x,x)s_j^{(1)}(x,0)),
\end{aligned}
\]
where
\[
\begin{aligned}
P_{ij}(x,y) &= P_{ij}(x,x) + (y-x)P_{ij}^{(1)}(x,y-x) \\
s_j(y) &= s_j(x) + (y-x)s_j^{(1)}(x,y-x).
\end{aligned}
\]
Beginning with
\[
\begin{aligned}
0 &\to \frac{E \boxtimes E^o}{E \boxtimes E^o(-\Delta)} \to \frac{E \boxtimes E^o(2\Delta)}{E \boxtimes E^o(-\Delta)} \to \mathcal{D}iff(E/S, E/S) \to 0
\end{aligned}
\]
restricting to \( \mathcal{A}_{E/S} \subset \mathcal{D}iff(E,E) \), and pushing forward by the trace map \( \frac{E \boxtimes E^o}{E \boxtimes E^o(-\Delta)} \to \omega_\Delta \cong \omega_{X/S} \), yields an exact sequence
\[
\begin{aligned}
0 &\to \omega_{X/S} \to^{tr} \mathcal{A}_E^{-1} \xrightarrow{\gamma_E} \mathcal{A}_{E/S} \to 0
\end{aligned}
\]
One defines the trace complex \( ^{tr}\mathcal{A}^\bullet \) by \( \mathcal{A}_{E,\pi}^i \) for \( i = 0, ^{tr}\mathcal{A}_E^{-1} \) for \( i = -1 \), \( \mathcal{O}_X \) for \( i = -2 \) and 0 else, with differentials \( d^{-1} := \gamma_E \) and \( d^{-2} \) equal to the relative Kähler differential (see [1], section 2).

One has an exact sequence of complexes
\[
\begin{aligned}
0 &\to \Omega^\bullet_{X/S}[2] \to^{tr} \mathcal{A}_E^\bullet \to (T_{X/S} \to T_\pi)[1] \to 0,
\end{aligned}
\]
where \( \Omega^\bullet_{X/S} \) is the relative de Rham complex of \( \pi \). Taking relative cohomology, one obtains the exact sequence
\[
\begin{aligned}
0 &\to \mathcal{O}_S \to R^0\pi_*(^{tr}\mathcal{A}_E^\bullet) \to T_S \to 0.
\end{aligned}
\]
Furthermore \( R^0\pi_*(^{tr}\mathcal{A}_E^\bullet) \) is a sheaf of algebras ([1], 1.2.3). One denotes by \( \pi(^{tr}\mathcal{A}_E^\bullet) \) the sheaf on \( S \) together with its algebra structure.

Finally, let \( \mathcal{B}_i, i = 1, 2 \) be two sheaves of algebras on \( S \), with an exact sequence of sheaves of algebras
\[
\begin{aligned}
0 &\to \mathcal{O}_S \to \mathcal{B}_i \to T_S \to 0.
\end{aligned}
\]
One defines \( \mathcal{B}_1 + \mathcal{B}_2 \) by taking the subalgebra of \( \mathcal{B}_1 \oplus \mathcal{B}_2 \), inverse image \( \mathcal{B}_1 \times_{T_S} \mathcal{B}_2 \) of the diagonal embedding \( T_S \to T_S \oplus T_S \), and its push out via the trace map \( \mathcal{O}_S \oplus \mathcal{O}_S \to \mathcal{O}_S \).

The aim of this note is to prove
Theorem 1.1. Let $D \subset X$ be a divisor, étale over $S$. One has a canonical isomorphism
\[
\pi^!(\mathcal{A}_E^\bullet) \cong \pi^!(\mathcal{A}_{E(-D)}^\bullet) + \mathcal{A}_{\det \pi_*(E|_D)}
\]
This is [1] Theorem 2.3.1, ii). We explain in more details the proof given there and correct a point in it.

2. Proof of Theorem 1.1

The proof uses the construction of a complex $\mathcal{L}^\bullet$, together with maps $\mathcal{L}^\bullet \to \pi^! \mathcal{A}_E^\bullet$ and $\mathcal{L}^\bullet \to \pi^! \mathcal{A}_{E(-D)}^\bullet \oplus i_{D*} \mathcal{A}_{E|_D}$ inducing isomorphisms from $R^0 \pi_* \mathcal{L}^\bullet$ with the left and the right hand side of theorem 1.1. We make the construction of $\mathcal{L}^\bullet$ and the maps explicit, and show that the induced morphisms are surjective, with the same (non-vanishing) kernel.

We first recall the definition of the sub-complex $\mathcal{L}^\bullet \subset \pi^! \mathcal{A}_E^\bullet$ (see [1], theorem 2.3.1, ii)): $\mathcal{L}^0 \subset \pi^! \mathcal{A}_E^0$ consists of the differential operators $P$ with $\epsilon(P) \in T_\pi < -D >$, where $T_\pi < -D > = T_\pi \cap T_X < -D >$ and $T_X < -D > = \text{Hom}_{\mathcal{O}_X}(\Omega^1_X < D >, \mathcal{O}_X)$ where $\Omega^1_X < D >$ denotes the sheaf of 1-forms with log poles along $D$. In particular, $\pi^! i_{D*} \mathcal{A}_{E|_D}$ is defined as the kernel. Then $\mathcal{L}^{-2} = \mathcal{O}_X$. The product structure on $\pi^! \mathcal{A}_E^\bullet$ is defined in [1], 2.1.1.2, and coincides with the Lie algebra structure on $\pi^! \mathcal{A}_E^0 = \mathcal{A}_{E,\pi}$. Since $\mathcal{L}^{-2} = \pi^! \mathcal{A}_E^{-2}$, to see that the product structure stabilizes $\mathcal{L}^\bullet$, one just has to see that $\mathcal{L}^0 \subset \pi^! \mathcal{A}_E^0$ is a subalgebra, which is obvious, and that $\mathcal{L}^0 \times \mathcal{L}^{-1} \to \pi^! \mathcal{A}_E^{-1}$ takes values in $\mathcal{L}^{-1}$, which is a consequence of proposition 2.2.

As in section 1, we denote by $\mathcal{A}_{E/S}$ the relative Atiyah algebra of $E$, with symbolic part $T_{X/S}$ and by $\mathcal{A}_{E,\pi}$ Beilinson’s subalgebra of the global Atiyah algebra with symbolic part $T_\pi$. If $\iota : F \subset E$ is a vector bundle, isomorphic to $E$ away of $D$, then one has an injection of differential operators
\[
\text{Diff}(E, F) \xrightarrow{i} \text{Diff}(E, E)
\]
induced by $\iota$ on the second argument, and an injection
\[
\text{Diff}(E, F) \xrightarrow{j} \text{Diff}(F, F)
\]
induced by $\iota$ on the first argument. One has

Definition 2.1.

$\mathcal{A}_{(E/S,F/S)} := \mathcal{A}_{E/S} \cap_i \text{Diff}(E, F) \cong \mathcal{A}_{F/S} \cap_j \text{Diff}(F, F)$

Recall $\gamma_E : \pi^! \mathcal{A}_E^{-1} \to \mathcal{A}_{E/S}$ denotes the map coming from the filtration by the order of poles of $\mathcal{O}_{X\times X}(*\Delta)$ on $\pi^! \mathcal{A}_E^{-1}$.

One has
Proposition 2.2.  
\[ \gamma_E^{-1}(A_{E/S,E(-D)/S}) \cong \gamma_{E(-D)}^{-1}(A_{E/S,E(-D)/S}) \cong \mathcal{L}^{-1}. \]

Proof. One considers 
\[ \begin{align*}
\frac{E(-D) \boxtimes E^\circ(2\Delta) + E \boxtimes E^\circ}{E \boxtimes E^\circ(-\Delta)} &= \left[ \frac{E(-D) \boxtimes E^\circ(2\Delta)}{E(-D) \boxtimes E^\circ(-\Delta)} \oplus \frac{E \boxtimes E^\circ}{E \boxtimes E^\circ(-\Delta)} \right]/\left[ \frac{E(-D) \boxtimes E^\circ}{E \boxtimes E^\circ(-\Delta)} \right] \\
\end{align*} \]
which, via the natural inclusion to 
\[ \frac{E(-D) \boxtimes E^\circ(2\Delta)}{E \boxtimes E^\circ(-\Delta)} \]
is the inverse image \( \gamma_E^{-1}\left( \text{Diff}(E, E(-D)) \right) \) (here we abuse of notation, still denoting by \( \gamma_E \) the map coming from the filtration), and via the map coming from the natural inclusion 
\[ \frac{E(-D) \boxtimes E^\circ(2\Delta)}{E(-D) \boxtimes E^\circ(-\Delta)} \rightarrow \frac{E(-D) \boxtimes E^\circ(D)(2\Delta)}{E(-D) \boxtimes E^\circ(D)(-\Delta)} \]
and the identification with the first term of the filtration on 
\[ \frac{E(-D) \boxtimes E^\circ(D)(2\Delta)}{E(-D) \boxtimes E^\circ(D)(-\Delta)} \]
is the inverse image \( \gamma_{E(-D)}^{-1}\left( \text{Diff}(E, E(-D)) \right) \).

The filtration induced by the order of poles of \( \mathcal{O}_{X \times X}(*D) \) induces the exact sequences 
\[ \begin{align*}
0 &\rightarrow \mathcal{H}om(E, E(-D)) \rightarrow A_{(E/S,E(-D)/S)} \rightarrow T_{X/S}(-D) \rightarrow 0 \\
0 &\rightarrow \mathcal{E}nd(E) \rightarrow A_{E/S} \rightarrow T_{X/S} \rightarrow 0 \\
0 &\rightarrow \mathcal{E}nd(E) \rightarrow A_{E(-D)/S} \rightarrow T_{X/S} \rightarrow 0.
\end{align*} \]

Now, as one has an injection \( \mathcal{L}^* \subset \mathcal{A}_E^* \) with cokernel \( Q \), and again by looking at the filtration by the order of poles on the sheaf in degree (-1), one obtains 
\[ Q \cong \mathcal{E}nd(E)|_D[1] \]
and

Theorem 2.3. One has an exact sequence 
\[ 0 \rightarrow R^0\pi_*(\mathcal{E}nd(E)|_D) \rightarrow R^0\pi_*(\mathcal{L}^*) \rightarrow R^0\pi_*(^{tr}\mathcal{A}_E^*) \rightarrow 0. \]
On the other hand, one has an injection $\mathcal{L}^* \subset \mathcal{A}^*_{E(-D)} \oplus i_D^* \mathcal{A}_{E[D]}$ with cokernel $\mathcal{P}$, and, as $\mathcal{L}^*$ injects into $\mathcal{A}^*_{E(-D)}$, one has an exact sequence
\[(2.11) \quad 0 \rightarrow i_D^* \mathcal{A}_{E[D]}[0] \rightarrow \mathcal{P} \rightarrow [\mathcal{A}^*_{E(-D)}/\mathcal{L}^*] \rightarrow 0,
\]
where $i_D : D \rightarrow X$ is the closed embedding. We see that the induced filtration on the sheaf in degree (-1) of $\mathcal{P}$ has graded pieces $(0, T\mathcal{L}^*)$ for any $m$. If $D$ is irreducible, one has graded pieces $(0, \mathcal{E}nd(E|D), T_{X/S}|D)$, whereas the filtration on the sheaf in degree (0) has graded pieces $(0, T_\pi/T\pi < -D \geq T_{X/S}|D)$. This last point comes from the obvious

**Lemma 2.4.**

\[
\{P \in \text{Diff}(E, E), P(E(-D)) \subset E(-D)\} \cong \{P \in \text{Diff}(E(-mD), E(-mD)), \epsilon(P) \in \mathcal{E}nd(E) \otimes T < -D\}
\]
for any $m \in \mathbb{Z}$, where $\epsilon$ is the symbol map.

So

**Lemma 2.5.** $[\mathcal{A}^*_{E(-D)}/\mathcal{L}^*]$ is quasiisomorphic to $\mathcal{E}nd(E|D)[1]$.

The connecting morphism $R^{-1}\pi_*[\mathcal{A}^*_{E(-D)}/\mathcal{L}^*] \rightarrow R^0\pi_*(i_D^* \mathcal{A}_{E[D]})[0]$ is just the natural embedding $\pi_*(\mathcal{E}nd(E|D)) \rightarrow \pi_*(i_D^* \mathcal{A}_{E[D]})$ with cokernel $\pi_*\pi|_D^{-1}T_S$. If $D$ is irreducible, one has $\pi_*\pi|_D^{-1}T_S \cong T_S$, and therefore

**Proposition 2.6.** If $D$ is irreducible, one has an exact sequence
\[0 \rightarrow R^0\pi_*\mathcal{L}^* \rightarrow R^0\pi_*[\mathcal{A}^*_{E(-D)}] \oplus R^0\pi_*(i_D^* \mathcal{A}_{E[D]}) \rightarrow T_S \rightarrow 0\]
and the image of $R^0\pi_*\mathcal{L}^*$ is obtained from the direct sum by taking the pull back under the diagonal embedding $T_S \rightarrow T_S \oplus T_S$.

On the other hand, still assuming $D$ irreducible, one has the exact sequence
\[(2.12) \quad 0 \rightarrow \pi_*\mathcal{E}nd(E|D) \rightarrow R^0\pi_*(i_D^* \mathcal{A}_{E[D]}) \rightarrow T_S \cong \pi_*\pi|_D^{-1}T_S \rightarrow 0\]
and the Atiyah algebra $\mathcal{A}_{\det(\pi_*E|D)}$ is the push out of $R^0\pi_*[i_D^* \mathcal{A}_{E[D]}]$ by the trace map $\pi_*\mathcal{E}nd(E|D) \rightarrow \mathcal{O}_S$.

Defining
\[(2.13) \quad \mathcal{K} := \text{Ker}(\mathcal{O}_S \oplus \pi_*\mathcal{E}nd(E|D) \xrightarrow{\text{id} \oplus \text{Tr}} \mathcal{O}_S) \cong \pi_*\mathcal{E}nd(E|D),\]
one thus obtains

**Theorem 2.7.** If $D$ is irreducible, one has an exact sequence
\[0 \rightarrow \mathcal{K} \rightarrow R^0\pi_*\mathcal{L}^* \rightarrow \pi_*[\mathcal{A}^*_{E(-D)}] \oplus \mathcal{A}_{\det(\pi_*E|D)} \rightarrow 0.\]
It can be easily shown that the embedding \( \pi_* \mathcal{E}nd(E|_D) \subset R^0 \pi_* \mathcal{L}^* \) in theorems 2.3 and 2.7 is the same embedding of a subsheaf of ideals. It finishes the proof of theorem 1.1 when \( D \) is irreducible. In general, since \( D \) is \( \acute{e}tale \) over \( S \), its irreducible components are disjoint, thus one proves theorem 1.1 by adding one component at a time.

References

[1] A. Beilinson, V. Schechtman: Determinant Bundles and Virasoro Algebras, Commun. Math. Phys. 118 (1988), 651-701.