The exact solutions in verified cosmological models based on generalized scalar-tensor gravity

I V Fomin and S V Chervon

1 Bauman Moscow State Technical University, Moscow, Russia
2 Kazan Federal University, Kazan, Russia
3 Ulyanovsk State Pedagogical University, Ulyanovsk, Russia

E-mail: ingvor@inbox.ru, chervon.sergey@gmail.com

Abstract. We consider the cosmological models built on the basis of scalar-tensor gravity, in which the non-minimal coupling of a scalar field and the Ricci scalar leads to the evolution of the field, the deviation of its potential from the flat shape and the difference in the dynamics of the early universe from pure exponential expansion. It is shown that such models are in a good agreement with modern observational data. As an example of the application of the proposed approach, the exact solutions for the different models of cosmological inflation were obtained.

1. Introduction

At present time, the most consistent approach to explain the physical processes occurring in the early universe, which solves the problems of the Big Bang Theory, is the theory of cosmological inflation, which postulates the existence of a stage of accelerated expansion of the universe at times close to the Planck time [1–5]. As a source of accelerated expansion in inflationary models, as a rule, the evolution of a scalar field (inflaton) is considered. After the discovery of the Higgs boson in experiments at the Large Hadron Collider [6], an additional justification was obtained for the possibility of using a scalar field to describe the evolution of the early universe. The scalar field with the Higgs potential can be considered as an inflaton, leading to early acceleration in the expansion of the universe and the subsequent formation of elementary particles [3].

However, in the case of Einstein gravity, the model of cosmological inflation with the Higgs potential does not correspond to modern observational restrictions on the values of cosmological perturbation parameters [7]. Thus, the use of the Higgs mechanism for constructing verifiable inflationary models requires the modification of the gravity theory. Also, some inflationary models with other physical potentials do not correspond to modern observational constraints [7] what can be corrected by using, for example, the scalar-tensor modifications of general relativity (GR).

This paper outlines the principle of constructing the cosmological inflation models based on scalar-tensor gravity theories (STG) [8-12], in which the non-minimal coupling of a scalar field and curvature is the source of the deviation from the pure exponential expansion, the source of a scalar field evolution, which is constant in GR, and changes the form of its potential as well [12]. In the framework of the proposed approach, we consider the relation between the Hubble parameter $H(t)$ and the function $F(\phi)$ determining the non-minimal coupling of a scalar field and curvature in the following form $H = \lambda \sqrt{F}$. The consequence of this approach is the possibility of
verifying the models of cosmological inflation constructed with this condition taken into account. Based on this method, the model of cosmological inflation with the Higgs potential and the other types of the potentials is proposed.

2. The special class of the cosmological models based on scalar-tensor gravity

Let us consider the inflationary models based on the evolution of a scalar field \( \phi \) which is coupled to the Ricci scalar \( R \) in the system of units \( 8\pi G = c = 1 \) on the basis of the action [8-12]

\[
S_{STG} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} F(\phi) R - \frac{\omega(\phi)}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right],
\]

(1)

where \( V(\phi) \) is the potential of a scalar field, \( F(\phi) \) is the function determining the character of the non-minimal coupling of the field and scalar curvature, \( \omega(\phi) \) is the kinetic function.

For the case of a spatially flat universe with the geometry determined by the Friedmann-Robertson-Walker metric

\[
ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right),
\]

(2)

the following equations of dynamics correspond to the action (1) [8-12]

\[
E_1 \equiv 3F\dot{H}^2 + 3H\ddot{F} - \frac{\omega}{2} \dot{\phi}^2 - V(\phi) = 0,
\]

(3)

\[
E_2 \equiv 3F\dot{H}^2 + 2F\ddot{H} + \ddot{F} + \frac{\omega}{2} \dot{\phi}^2 - V(\phi) = 0,
\]

(4)

\[
E_3 \equiv \omega \dot{\phi}^2 + 3\omega H \dot{\phi} + \frac{1}{2} \dot{\phi}^2 \omega' + V'_\phi - 6H^2 F'_\phi - 3\dot{H} F'_\phi = 0,
\]

(5)

where a dot means time derivative and \( V'_\phi = dV/d\phi \). It needs to note, also, that among the equations (3)-(5) are independent two only taking into account the additional expression [12]

\[
\dot{\phi} E_3 + \dot{E}_1 + 3H (E_1 - E_2) = 0.
\]

(6)

The exact solutions of the equations (3)-(5) can be constructed on the basis of some initially specified relations between the parameters of cosmological models, i.e. by using the different ansatzes. As such an ansatz we consider the relation between the Hubble parameter \( H(t) \) and the function of non-minimal coupling \( F(\phi(t)) = F(\phi) \) in the following form

\[
H = \lambda \sqrt{F}, \quad \lambda = const.
\]

(7)

Substituting the expression (7) into the equations of cosmological dynamics (3)-(4), we obtain

\[
V(\phi) = 3\lambda^2 F^2 + 3\lambda \sqrt{F} \dot{F} + \frac{1}{2} \dot{F},
\]

(8)

\[
\omega(\phi) \dot{\phi}^2 = -\dot{F}.
\]

(9)

Further, we define the conditions of slow-roll regime for inflation based on the scalar-tensor gravity in the following form [12,13]

\[
\left| \dot{\phi} \right| \ll H \left| \dot{\phi} \right| \ll H^2 |\phi|, \quad \frac{1}{2} \left| \omega(\phi) \right| |\dot{\phi}|^2 \ll |V(\phi)|,
\]

(10)

\[
\delta_F = \frac{\dot{F}}{HF} \ll 1.
\]

(11)
After substituting the relation (7) into definitions of the slow-roll parameters [13] one has
\[ \delta_F = 2\dot{H}/H^2 = -2\epsilon. \]
Thus, the condition of slow-roll regime in these models is similar to one in the case of Einstein gravity \( \epsilon \ll 1 \).

An important criterion for verifiability of cosmological inflation models is the correspondence of the spectral parameters of cosmological perturbations induced by small quantum fluctuations of a scalar field at the stage of cosmological inflation to observational constraints.

The following formulas for calculating the spectral parameters of cosmological perturbations for cosmological inflation models based on scalar-tensor gravity for the case of the dependence \( H = \lambda \sqrt{F} \) at the crossing of the Hubble radius \( k = aH \) were obtained in [12]:

1. Power spectra of scalar and tensor perturbations
\[
\mathcal{P}_S = \frac{\lambda^2}{16\pi^2\epsilon^2} \left( \frac{1}{(\epsilon - \delta)} \right) \simeq \frac{\lambda^2}{16\pi^2\epsilon (\epsilon - \delta)}, \quad \mathcal{P}_T = \frac{2\lambda^2}{\pi^2} = \text{const.} \tag{12}
\]

2. Spectral indices of scalar and tensor perturbations
\[
n_S - 1 \equiv \frac{d\ln \mathcal{P}_S}{d\ln k} \approx 2\epsilon\delta - 4\epsilon + 2\delta + (1 - \epsilon) \left( \frac{\epsilon\delta - \xi}{\epsilon - \delta} \right), \quad n_T \equiv \frac{d\ln \mathcal{P}_T}{d\ln k} = 0. \tag{13}
\]

3. Tensor-to-scalar ratio
\[
r = 16\epsilon \left( \frac{2(\epsilon - \delta)}{(1 - \epsilon)^2} \right) \approx 32\epsilon (\epsilon - \delta), \tag{14}
\]

where the slow-roll parameters
\[
\epsilon = -\frac{\dot{H}}{H^2}, \quad \delta = \epsilon - \frac{\dot{\epsilon}}{2H\epsilon} = -\frac{\ddot{H}}{2HH}, \quad \xi = \epsilon\delta - \frac{1}{\bar{H}}, \tag{15}
\]
at the inflationary stage \( \epsilon \ll 1, \delta \ll 1, \xi \ll 1, \)

We also write down the expressions for the spectral parameters of cosmological perturbations in terms of “flow parameters”, which are also often used to analyze cosmological inflationary models and are defined as follows [8]
\[
\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_{n+1} = \frac{\dot{\epsilon}_n}{H\epsilon_n}. \tag{16}
\]

Now, we can write down the relationship between slow-roll and flow parameters
\[
\epsilon = \epsilon_1, \quad \epsilon_1 = \epsilon. \tag{17}
\]
\[
\delta = \epsilon_1 - \frac{1}{2}\epsilon_2, \quad \epsilon_2 = 2(\epsilon - \delta).\tag{18}
\]
\[
\xi = \frac{1}{2}\epsilon_2\epsilon_3, \quad \epsilon_3 = \frac{\xi}{\epsilon - \delta}. \tag{19}
\]

As a result, from the relations (16)-(18), after neglecting the term \( 2\epsilon_1^2 \approx 0 \), we get
\[
\mathcal{P}_S \simeq \frac{\lambda^2}{8\pi^2\epsilon_1\epsilon_2}, \quad \mathcal{P}_T = \frac{2\lambda^2}{\pi^2} = \text{const.} \tag{20}
\]
\[
n_S \simeq 1 - 3\epsilon_1 - \epsilon_2 - \epsilon_1\epsilon_2 - \epsilon_3 + \frac{2\epsilon_1^2}{\epsilon_2}, \quad n_T = 0. \tag{21}
\]
\[
r \approx 16\epsilon_1\epsilon_2. \tag{22}
\]

Let us consider some properties of the inflationary models based on the relation \( H = \lambda\sqrt{F} \) which follows from the expressions (20)-(22):
1. Since the power spectrum of tensor perturbations is scale-invariant, the amplitude of the relic gravitational waves of arbitrary frequencies is 
   \[ A_T = \sqrt{2\lambda/\pi} = \text{const} \] (\( A_T^2 = P_T \)), i.e. it depends on the value of the constant \( \lambda \) only at the crossing of the Hubble radius.

2. For the case of accelerated expansion of the universe according to the power-law inflation with the Hubble parameter
   \[ H = n/t \] from definitions (19) we obtain \( \epsilon_1 = 1/n \) and \( \epsilon_2 = 0 \), which leads to a divergence \( P_S \to \infty, n_S \to \infty \). Thus, the power-law inflation in these models is eliminated, and the power-law expansion without acceleration occurs after the completion of the inflationary stage, which corresponds to Friedman’s cosmological solutions corresponding to the further dynamics of the universe [3].

The observational constraints on the values of the spectral parameters of cosmological perturbations, according to the observations of PLANCK, BICEP2 and Keck-Array, are determined as follows [14]

\[ P_S = 2.1 \times 10^{-9}, \quad n_S = 0.9663 \pm 0.0041, \quad r < 0.065, \quad (23) \]

and the values of flow parameters that satisfy the constraints (23) are estimated as

\[ \epsilon_1 < 0.0097, \quad \epsilon_2 = 0.032^{+0.009}_{-0.008}, \quad \epsilon_3 = 0.19^{+0.55}_{-0.55}. \quad (24) \]

Let us estimate the values of the parameters of cosmological perturbations in the inflationary models based on the connection (7) with the flow parameters (24). Substituting the expressions (24) into formula (21) gives the correspondence to the observational constraint (23) on the value of the spectral index of scalar perturbations \( n_S \). The value of tensor-to-scalar ratio is \( r \approx 16\epsilon_1\epsilon_2 < 0.005 \), which also corresponds to observational constraints. Further, based on expressions (20) and (23)-(24), we obtain the following parameter \( \lambda < 7 \times 10^{-6} \), where we consider the positive value of this parameter, since for the case of real function of non-minimal coupling \( F(\phi) \) the Hubble parameter \( H > 0 \) for \( \lambda > 0 \) only. The amplitude of the relic gravitational waves in the system of units \( 8\pi G = c = 1 \) is \( A_T < 3.1 \times 10^{-6} \) and the power spectrum \( P_T < 10^{-11} \). Thus, in the general case, the models with the connection \( H = \lambda \sqrt{F} \) correspond to the observational constraints.

3. The inflationary models with quasi de Sitter dynamics based on STG

Now, we consider application the method of generating function for analysis of background dynamic equations (8)-(9) for quasi de Sitter model with linear deviation from exponential expansion. On the basis of this method one can obtain the exact solutions and reconstruct the parameters of inflationary models from known Hubble parameter and given coupling function. The choice of the non-minimal coupling function is based on the known types of scalar-tensor gravity theories [8,9].

Firstly, we represent the Hubble parameter in terms of some generating function \( f(t) \) which characterize the deviation from de Sitter expansion

\[ H(t) = f(t) + \lambda. \quad (25) \]

In terms of this function, from the equations (7)-(9), one has

\[ F(t) = \left( 1 + \frac{f(t)}{\lambda} \right)^2, \quad (26) \]

\[ V(\phi) = \frac{1}{\lambda^2} \left[ 3(f(t) + \lambda)^4 + 6(f(t) + \lambda)^2 f^2 + f^2 + (f(t) + \lambda)^2 \right], \quad (27) \]

\[ \omega(\phi) \dot{\phi}^2 = -\frac{2}{\lambda^2} \left[ f^2 + (f(t) + \lambda)^2 \right]. \quad (28) \]
If \( f(t) = 0 \) the model is reduced to the pure exponential de Sitter expansion based on the Einstein gravity \((F = 1)\) with the constant potential, \( V = 3\lambda^2 = \text{const} \), corresponding to the constant scalar filed, \( \phi = \text{const} \).

Now, we consider the case of the generating function \( f(t) = -\xi t \) corresponding to the quasi-Sitter dynamics \( H(t) = -\xi t + \lambda \) [3], where \( \xi \) is the constant of non-minimal coupling of a scalar field and the curvature. In this case, the equations (26)-(28) can be represented as

\[
F(\phi) = \left(1 - \frac{\xi t}{\lambda}\right)^2, \tag{29}
\]

\[
V(\phi) = 3\lambda^2 \left( F(\phi) - \frac{\xi}{\lambda^2} \right)^2 - 2 \left( \frac{\xi}{\lambda} \right)^2, \tag{30}
\]

\[
\omega(\phi) \dot{\phi}^2 = -\frac{2\xi^2}{\lambda^2}. \tag{31}
\]

Therefore, by setting the function of non-minimal coupling \( F(\phi) \), from the equations (29)-(31) it is possible to obtain the exact solutions of the inflationary models and afterward to calculate cosmological parameters.

4. The examples of the exact solutions in inflationary models with quasi de Sitter dynamics

As the example of proposed approach, we firstly consider the inflationary model based on the Brans-Dicke gravity [8,9] with the coupling function \( F(\phi) = \phi \). For this case, from the equations (29)-(31) we obtain the following exact solutions

\[
\phi(t) = \left(1 - \frac{\xi t}{\lambda}\right)^2, \tag{32}
\]

\[
\omega(\phi) = -\frac{1}{2\phi}, \tag{33}
\]

\[
V(\phi) = 3\lambda^2 \left( \phi - \frac{\xi}{\lambda^2} \right)^2 - 2 \frac{\xi^2}{\lambda^2}, \tag{34}
\]

corresponding to the quadratic potential with the additional constant term [3].

Secondly, for the case of the exponential coupling function [8,9]

\[
F(\phi) = \exp \left(-\frac{2}{3\phi}\right), \tag{35}
\]

we obtain the solutions for the potential in Starobinsky inflation [1,7] with the additional constant term

\[
\phi(t) = -\sqrt{6} \ln \left(1 - \frac{\xi t}{\lambda}\right), \tag{36}
\]

\[
\omega(\phi) = -\frac{1}{3} \exp \left(-\sqrt{\frac{2}{3}}\phi\right), \tag{37}
\]

\[
V(\phi) = 3\lambda^2 \left( \frac{\xi}{\lambda^2} - e^{-\sqrt{2/3}\phi} \right)^2 - 2 \frac{\xi^2}{\lambda^2}, \tag{38}
\]

for the following relation between constant parameters \( \xi = \lambda^2 \). Also we note that cosmological inflation model based on the Starobinsky potential has good agreement with observational data for the case of Einstein gravity [7].
Further, for induced gravity $F(\phi) = \phi^2$ [8,9] one has the solutions for the Higgs potential [3] with additional constant term

$$\phi(t) = \pm \left(1 - \frac{\xi t}{\lambda}\right)$$  \hspace{1cm} \text{(39)}

$$\omega(\phi) = -\frac{1}{2};$$  \hspace{1cm} \text{(40)}

$$V(\phi) = 3\lambda^2 \left(\phi^2 - \frac{\xi}{\lambda^2}\right)^2 - 2\frac{\xi^2}{\lambda^2}. $$  \hspace{1cm} \text{(41)}

Finally, for the case of non-minimal coupling $F(\phi) = 1 - \xi\phi^2$ we obtain

$$\phi(t) = \pm \frac{1}{\lambda} \sqrt{(2\lambda - \xi t)} t;$$  \hspace{1cm} \text{(42)}

$$\omega(\phi) = \frac{2\xi^2 \phi^2}{\xi \phi^2 - 1};$$  \hspace{1cm} \text{(43)}

$$V(\phi) = 3\lambda^2 \xi^2 \phi^4 - 6\xi (\lambda^2 - \xi) \phi^2 + 3\lambda^2 - 6\xi + \frac{\xi^2}{\lambda^2}, $$  \hspace{1cm} \text{(44)}

the solutions for the Higgs potential (44) with the additional constant term as well.

As we can see, various non-minimal couplings of the scalar field and curvature lead to solutions that have different types of potentials and scalar field evolutions.

Now, we consider the correspondence of the parameters of cosmological perturbations for the models under consideration to the observational data. As one can see from the expressions (12)-(14), for the case of a definite connection between the Hubble parameter and the non-minimum coupling function, the values of the cosmological perturbation parameters at the crossing of Hubble radius depend on the type of expansion of the early universe only. For this reason, for all the above models with quasi de Sitter dynamics $H(t) = -\xi t + \lambda$, we obtain the same values of the parameters of cosmological perturbations.

For these models one has the following slow-roll parameters $\epsilon = \xi (\lambda^2 - 2\xi t)^{-2}$, $\delta = 0$, $\xi = 0$. The e-folds number as the function of cosmic time is

$$N(t) = \int H(t) dt = -\frac{1}{2} \xi t^2 + \lambda t.$$  \hspace{1cm} \text{(45)}

From the inverse relationship $t = t(N)$ we find the expression of the first slow-roll parameter in terms of the e-folds number $\epsilon = \xi (\lambda^2 - 2\xi N)^{-1}$. We also note that the value of the e-folds number at the end of the inflation stage is estimated as $N = 50 - 60$ [14].

Taking into account the condition $\epsilon \ll 1$ and considering the minimal allowable value of the spectral index of scalar perturbations $n_S = 0.9622$ (corresponding to the maximum value of the slow-roll parameter $\epsilon_*$ and, accordingly, the tensor-to-scalar ratio $r_*$), from the relations (12)-(14) we get

$$P_S \simeq \frac{\lambda^2}{16\pi^2 c_s^2} = \left[\frac{\lambda}{4\pi} \frac{(\lambda^2 - 120\xi)}{4\pi}\right]^2 = 2.1 \times 10^{-9},$$  \hspace{1cm} \text{(46)}

$$n_S \simeq 1 - 4\epsilon_* = 1 - 4\xi (\lambda^2 - 120\xi)^{-1} = 0.9622.$$  \hspace{1cm} \text{(47)}

From the equations (46)-(47) one can obtain the following value of the slow-roll parameter $\epsilon_* = 0.0095$ and estimate the parameter $\lambda < 5.4 \times 10^{-6}$ and the constant of non-minimal coupling of a scalar field and curvature in the considered models $\xi < 1.3 \times 10^{-13}$. The maximum value of the tensor-to-scalar ratio is determined from the expression $r_* \simeq 32\epsilon_*^2 = 0.003 < 0.065$ which corresponds to observational constraint (23).
5. Conclusion
In this paper, we considered a method for constructing the exact solutions for cosmological inflationary models based on the special choice of the relationship between the Hubble parameter and the function of the non-minimal interaction of the scalar field and curvature $H = \lambda \sqrt{F}$.

Based on the choice of the non-minimal coupling function corresponding to various theories of scalar-tensor gravity, the corresponding physical potentials of the scalar field and kinetic functions for the case of quasi de Sitter expansion of the early universe were obtained. It is shown that this class of inflationary models corresponds to observational constraints on the values of the spectral parameters of cosmological perturbations.

In spite of the fact that we restricted ourselves in investigations of quasi-de Sitter model with $H(t) = -\xi t + \lambda$, this approach of constructing the exact solutions for cosmological inflationary models based on scalar-tensor gravity can be generalized to arbitrary dynamics of accelerated expansion of the early universe based on the expressions (26)-(28).

Also we note that the successful detection of gravitational waves from the black holes and neutron stars merger [15,16] defines the additional restrictions on the parameters of modified gravity theories. The data on the detection of gravitational waves during the merging of black holes have confirmed that the observed gravitational waves are consistent with the prediction of General Relativity for binary systems [15]. In the considered gravity models with condition $H = \lambda \sqrt{F}$, the exponential accelerated expansion of the universe in the present era $H = \lambda$ gives the correspondence to Einstein gravity, $F = 1$, for CDM-like model with a constant scalar field which is the source of the second accelerated expansion of the universe.

It should be noted that to limit the possible types of cosmological inflationary models, it is necessary to directly measure the characteristics of relic gravitational waves. At present, various methods of carrying out such experiments [17-24] are being considered, which makes it possible to count on the possibility of direct experimental verification of theoretical models of the early universe.

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