A strictly monotone measure on tame sets that corresponds to a numerosity

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July 17, 2020

Abstract

Adapting standard methods from geometric measure theory, we provide an example of a polynomial-valued measure $\mu$ on tame sets in $\mathbb{R}^d$ which satisfies many desirable properties. Among these is strict monotonicity: the measure of a proper subset is strictly less than the measure of the whole set. Using techniques from non-standard analysis, we display that the domain of $\mu$ can be extended to all subsets of $\mathbb{R}^d$ (up to equivalence modulo infinitesimals). The resulting extension is a numerosity function that encodes the $i$-dimensional Hausdorff measure for all $i \in \mathbb{N}$, as well as the $i$-th intrinsic volume functions.

1 Introduction

Among the most influential texts in human history, Euclid’s *Elements* contains axioms, postulates, and theorems which have been studied for over 2000 years. Among these is Euclid’s fifth common notion—or fifth axiom, which states that “the whole is greater than the part”. In so many modern words: the size of a set ought to be larger than that of any of its proper subsets. While the principle is philosophically intuitive, this axiom does not hold in the infinite case for either Cantorian cardinality or real-valued measure: both include sets which have the same size as their proper subsets.

However, recent work in nonstandard analysis has developed a Euclidean notion of size known as “numerosity” (see Definition 6), for which the numerosity of a proper subset is strictly less than that of the whole set. Nonstandard analysis lends itself well to this possibility, since it ascribes a finely-grained arithmetic to infinite(small) numbers. Numerosity functions which are compatible with measures have been shown to exist. However, proofs of their existence give little information on how to compute them, even on well-behaved sets in Euclidean space.

To alleviate this issue, this paper focuses on using classical geometric measure theory to satisfy Euclid’s fifth axiom. As a main result, we give an example of a polynomial-valued measure which satisfies the fifth axiom. We show the function has a simple definition in terms of intrinsic volumes and that it is unique under some basic assumptions. We then prove that
this extends (modulo infinitesimals) to a numerosity function on \( \mathcal{P}(\mathbb{R}^d) \) that encodes all Hausdorff \( i \)-measures for \( i \in \mathbb{N} \) on tame sets.

## 2 Background

Within geometric measure theory, this paper focuses on the concept of intrinsic volumes defined in an o-minimal structure. This underlies all of our results, and thus, the following section includes all relevant definitions. Readers who are familiar with the topic should feel free to read only the comments between the definitions in section 2.1

While most of our results do not require nonstandard analysis, our final key theorem does. However, to grasp the main result, no background in nonstandard analysis is needed except the definitions provided here. The proofs require background which can be found in [1].

### 2.1 Geometric Measure Theory

Originally defined in model theory, we will consider the concept of a structure in its set-theoretic, o-minimal form as defined by van den Dries [2]. We adapt this definition specifically for \( \mathbb{R} \), instead of an arbitrary densely ordered set.

**Definition 1.** An o-minimal structure, or simply structure on \( \mathbb{R} \) is a sequence \( S = (S_i)_{i \in \mathbb{N}} \) such that for each \( i \geq 0 \):

- \( S_1 \) is a boolean algebra of subsets of \( \mathbb{R}^i \). That is, \( \emptyset \in S_1 \) and \( S_1 \) is closed under unions and complements.
- \( S_2 \) If \( A \in S_i \), then \( \mathbb{R} \times A \) and \( A \times \mathbb{R} \) are elements of \( S_{i+1} \).
- \( S_3 \) The diagonal element \( \{(x_1, \ldots, x_i) \in \mathbb{R}^i : x_1 = x_i\} \) belongs to \( S_i \).
- \( S_4 \) If \( A \in S_{i+1} \) then \( \pi(A) \) is in \( S_i \). Here \( \pi : \mathbb{R}^{i+1} \to \mathbb{R}^i \) denotes the projection map on the first \( i \) coordinates.
- \( S_5 \) The \( < \) relation is in \( S_2 \); that is, \( \{(x, y) \in \mathbb{R}^2 : x < y\} \in S_2 \).
- \( S_6 \) \( S_1 \) consists of exactly all finite unions of points and open intervals.

A set is called tame or definable (in an o-minimal structure) if it is an element of some \( S_i \). A map is definable if its graph is definable.

For the remainder of this paper, we assume we are working within some predetermined o-minimal structure \( S = (S_i)_{i \in \mathbb{N}} \). We additionally stipulate that \( S \) includes all singletons of \( \mathbb{R} \) as well as the addition and multiplication operations. This implies it contains all semialgebraic sets, as shown in [2].
Definition 2. We denote the collection of all definable sets by

\[ U = \bigcup_{i \in \mathbb{N}} S_i. \]

Based on \( S \), there is a well-defined mapping \( \chi : U \to \mathbb{Z} \) called the Euler Characteristic.

Definition 3. The (o-minimal) Euler Characteristic \( \chi \) is defined so that \( \chi(\sigma) = (-1)^i \) for any open \( i \)-simplex \( \sigma \). Moreover, on tame sets it satisfies the valuation property:

\[ \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B). \]

There exists a unique function satisfying the above definition \[2\].

Definition 4. The \( i \)th intrinsic volume is denoted \( \mu_i : U \to \mathbb{R} \cup \{\pm \infty\} \). Given \( A \in U \) such that \( A \subset \mathbb{R}^d \) we define its value as

\[ \mu_i(A) = \int_{G_{d,d-i}} \int_{L^\perp} \chi(A \cap (L + x)) \, dx \, d\gamma(L) \]

where \( L^\perp \) denotes the \( i \)-subspace perpendicular to \( L \) and \( \gamma \) denotes the Haar measure on the Grassmannian \( G_{d,d-i} \). Here, \( \gamma \) is scaled such that

\[ \gamma(G_{n,m}) = \binom{n}{m} \frac{\beta_n}{\beta_m \beta_{n-m}} \]

where \( \beta_i \) is the volume of the unit ball in \( i \) dimensions.

It is well known that, when normalized as such, the intrinsic volumes are independent of the dimension of the ambient space in which \( A \) is embedded. This dimension independence implies that two congruent sets in \( U \) always have the same intrinsic volumes, regardless of whether they are embedded in \( \mathbb{R}^d \) or \( \mathbb{R}^{d'} \).

Definition 5. For a nonempty tame set \( A \subset \mathbb{R}^d \) we define \( \dim A \) to be the Hausdorff dimension of \( A \). Equivalently, \( \dim A \) is the o-minimal dimension of \( A \). For sets in \( U \), these familiar notions of dimension coincide on nonempty sets (see appendix). By convention, \( \dim \emptyset = -\infty \).

2.2 Nonstandard Analysis

Most results in this paper do not require nonstandard analysis at all. However, Theorem \[22\] is the culminating result, and is fundamentally nonstandard. The reader interested in details of this theorem is assumed to be familiar with basic nonstandard analysis and the property of \( \kappa \)-enlargement. For reference, see \[1\]. The only other concept needed is:
Definition 6. An elementary numerosity or just numerosity on a set \( \Omega \) is a function
\[
n : \mathcal{P}(\Omega) \to [0, \infty)_F
\]
(where \( F \) denotes any ordered field containing \( \mathbb{R} \)) with the properties that
- \( n(\{x\}) = 1 \) for all \( x \in \Omega \)
- \( n(A \cup B) = n(A) + n(B) \) for all \( A \) and \( B \) such that \( A \cap B = \emptyset \).

As a consequence of this definition, elementary numerosities are strictly monotone on the power set of \( \Omega \). That is, \( A \subsetneq B \) implies \( n(A) < n(B) \). Furthermore, the empty set is the only set with numerosity 0. For infinite \( \Omega \), this means that \( F \) must contain infinite (and thus also infinitesimal) numbers [3].

A goal of this paper is to approximate these properties of numerosity functions using only polynomial-valued measures which are not defined on all of \( \mathcal{P}(\Omega) \). That is, we are looking for a classical geometric measure theory result which achieves some of the goals of Definition 6 but without using nonstandard analysis. After proving such a result, we extend it to the nonstandard universe and show the existence of a numerosity function which approximately equals our more classical polynomial-valued measure.

3 Definitions

The following definitions are original to this paper. They describe the classical context in which we will try to approximate the properties of a numerosity. Definition 8 creates the main character \( \mu \) – the polynomial-valued measure which all our results concern.

Definition 7. A polynomial-valued measure is a function \( \nu : U \to \mathbb{R}[x] \) where \( U \) is a boolean algebra of sets such that \( \nu(A \cup B) = \nu(A) + \nu(B) \) when \( A \cap B = \emptyset \). Here \( \mathbb{R} = \mathbb{R} \cup \{\pm \infty\} \). In the case where
\[
\nu(A) = a_0 + \cdots + (+\infty)x^k + \cdots + a_dx^d \quad \text{and} \quad \nu(B) = b_0 + \cdots + (-\infty)x^k + \cdots + b_dx^d,
\]
we place no restrictions on the \( x^k \) coefficient of \( \nu(A \cup B) \), but all indices where the sum is defined must be additive.

Definition 8. Let \( \mathbb{K} = \mathbb{R} \cup \{\pm \infty\} \). Define \( \mu : U \to \mathbb{K}[x] \) by
\[
\mu_A(x) = \mu_0(A) + \mu_1(A)x + \cdots + \mu_d(A)x^d
\]
where \( d = \dim A \) and \( \mu_i \) is the \( i \)th intrinsic volume (Definition 4). Note \( \mu_k(A) = 0 \) for \( k > d \).

Definition 9. We order \( \mathbb{K}[x] \) lexicographically. Let \( f(x) = a_0 + \cdots + a_px^p \) and let \( g(x) = b_0 + \cdots + b_qx^q \). Let \( i \) be the largest index for which \( a_i \neq b_i \). Then \( f < g \) if and only if \( a_i < b_i \).

This is a total ordering on \( \mathbb{K}[x] \).

*This is the ordering considered on polynomials for the remainder of the paper.
Definition 10. Recall we defined $\mathcal{U}$ to be the collection of all tame sets in an o-minimal structure over $\mathbb{R}$. Now, we define $\mathcal{U}_f$ to be the collection of $A \in \mathcal{U}$ such that $|\mu_i(A)| < \infty$ for all $i \in \mathbb{N}$. Note that $\mu_i(A) = 0$ for $i > \dim A$.

We also let $\mathcal{U}_b$ be the set of all bounded sets in $\mathcal{U}$. Notice that since the Grassmannian is compact, $\mathcal{U}_b \subset \mathcal{U}_f$. This inclusion is proper. Furthermore, $\mathcal{U}_b$ is closed under finite unions, intersections, and relative complements, while $\mathcal{U}_f$ is not.

There are several notions which would be ideal for a notion of size in Euclidean space to satisfy. They are described in the following definitions, which will allow us to succinctly state Theorem 19.

Definition 11. The definable sets $\mathcal{U}$ come equipped with a notion of convergence known as flat convergence. For a reference, see [4]. Correspondingly, we call a function $f : \mathcal{U} \to \mathbb{R}[x]$ flat-continuous or just continuous if for every flat-convergent sequence $A_n \to A$ of sets in $\mathcal{U}$, $f_{A_n} \to f_A$.

Definition 12. $f : \mathcal{U} \to \mathbb{R}[x]$ is called homogeneous if for every $\beta \in \mathbb{R}$ and $A \in \mathcal{U}$, it holds that $f_A(\beta x) = f_{\beta A}(x)$, where $\beta A = \{\beta a : a \in A\}$. Equivalently, $f$ is homogeneous of degree $i$ in each of its $x^i$ components.

Definition 13. $f : \mathcal{U} \to \mathbb{R}[x]$ is called Euclidean-invariant if for every rigid motion $\phi$ and every $A \in \mathcal{U}$, $f_A = f_{\phi A}$.

Definition 14. $f : \mathcal{U} \to \mathbb{R}[x]$ is called intrinsic if for all $A \in \mathcal{U}$, $f_A = f_{A \times \{(0)\}}$. Here $(0)$ is a 1-tuple in $\mathbb{R}^1 = \mathbb{R}$. In other words, $f_A$ is independent of the ambient space in which $A$ is measured.

4 Main Results

4.1 Geometric Measure Theory

Our first main result is a consequence of the following lemma. Its proof is not particularly illuminating, and is left to the appendix for the inclined reader.

Lemma 15. Let $A$ be a definable subset of $\mathbb{R}^d$ of dimension $m$. Then $\mu_m(A) = \mathcal{H}^m(A) > 0$.

Following immediately from this lemma is a key theorem:

Theorem 16. $\mu$ is strictly monotone on $\mathcal{U}_f$. More specifically, if $A \subseteq B$ with $A \in \mathcal{U}_f$ and $B \in \mathcal{U}$, then $\mu_A < \mu_B$.

Proof. First, note that $C = B \setminus A$ is definable and nonempty. Therefore, $k = \dim C > 0$, and so $\mu_k(C) > 0$. Since the order is lexicographical, $\mu_C = \mu_k(C)x^k + \cdots + \mu_0(C) > 0$. Since $\mu_A + \mu_C = \mu_B$ and $\mu_A$ has only finite coefficients, it follows $\mu_A < \mu_B$. \qed

†Here, $f$ is viewed as a tuple, and convergence indicates that the $x^i$ coefficients of $f_{A_n}$ converges to the $x^i$ coefficient of $f_A$ for all $i \in \mathbb{N}$. 

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**Theorem 17** (Polynomial Product Formula). Let \( A, B \in U \). Then 
\[ \mu_{A \times B} = \mu_A \cdot \mu_B. \]

**Proof.** This is a direct consequence of the product theorem for intrinsic volumes [4]:
\[ \mu_k(A \times B) = \sum_{i=1}^{k} \mu_i(A)\mu_{k-i}(B). \]

**Lemma 18.** If a polynomial-valued measure \( \nu : U \to \mathbb{R}[x] \) is homogeneous and Euclidean-invariant, then \( \nu_{[0,1]} = \alpha x \) for some \( \alpha \in \mathbb{R} \). We call \( \alpha \) the **scale factor** of \( \nu \).

**Proof.**
\[ \nu_{[0,1]} = f(x) \implies \nu_{[0,2]} = 2f(x) \quad \text{(Euclidean invariance)} \]
\[ = f(2x) \quad \text{(Homogeneity)} \]

Letting \( f(x) = a_0 + \cdots + a_kx^k \), we get the \( k \) equations:
\[ 2a_i = 2^ia_i \quad (i = 0, \ldots, k) \]

which implies \( i = 1 \) or \( a_i = 0 \). The lemma follows, letting \( \alpha = a_1 \).

We now show an analogue of Hadwiger’s Theorem stated in terms of \( \mu \). It shows that our polynomial-valued measure is, up to a constant, the only one satisfying several conditions similar to those of Hadwiger’s Theorem.

**Theorem 19.** All functions \( \nu \) satisfying the following conditions:

1. \( \nu : U \to \mathbb{R}[x] \) is continuous, homogeneous, Euclidean-invariant, and intrinsic.
2. \( \nu \) satisfies the Polynomial Product Formula [Theorem 17] on \([0,1]^i\) for \( i \geq 0 \).

are of the form \( \nu_X = \mu_{\alpha X} \), where \( \alpha \) is the scale factor of \( \nu \).

**Proof.** Suppose that some function \( \nu \) satisfies the above properties. Then \( \nu_{[0]} = \nu_{[0,0]} = (\nu_{[0])}^2 \) by the intrinsic property and condition 2. We can show that \( \nu_{[0]} \) is not identically zero: in fact, since the scale factor is 1, \( A = [0,1] \) has nonzero measure, and hence \( 0 \neq \nu_A = \nu_{A \times [0]} = \nu_A \cdot \nu_{[0]} \).

Since the polynomial \( \nu_{[0]} \) is not the zero polynomial, it has an interval \((a,b)\) on which it is nonzero. On this interval, we can divide our first equation by \( \nu_{[0]} \) to get \( \nu'_{[0]} = 1 \). Since our polynomial \( \nu_{[0]} \) is 1 on the interval \((a,b)\), it must be identically 1 on all of \( \mathbb{R} \). Combining this with Euclidean invariance already gives that \( \nu_X = \#(X) = \mu_0(X) \) on finite sets.

Consider the projection function \( P_i : \mathbb{R}[x] \to \mathbb{R} \) defined by
\[ P_i(a_0 + \cdots + a_ix^i + \cdots + a_kx^k) = a_i. \]

Notice that requirement (1) gives each \( P_i\nu \) the following characteristics:
• $P_i \nu$ is homogeneous of degree $i$.
• $P_i \nu$ is Euclidean-invariant
• $P_i \nu$ is flat-continuous.

Furthermore, $P_i \nu$ is a valuation. Here, we can apply Hadwiger’s Theorem for definable sets: any flat-continuous and Euclidean-invariant valuation on tame sets in $\mathbb{R}^d$ is a linear combination of the intrinsic volumes. This simple corollary of Hadwiger’s Theorem for constructible functions [5] is proved in the appendix. From this, plus homogeneity, it follows that $P_i \nu$ is a constant multiple of $\mu_i$ for each $i$. Finally, notice that the product formula applied inductively to $\nu_{[0,1)}(x) = \alpha x$ gives us that $P_i \nu_{[0,1)} = \alpha^i$ for all $i > 0$. The fact that $\nu_X = \mu_0(X)$ on finite sets means this also holds for $i = 0$. But this sets each constant of proportionality, so $P_i \nu_X = \alpha^i \mu_i(X)$ for all $i \in \mathbb{N}$ and $X \in \mathcal{U}$. Hence $\nu_X = \mu_\alpha x$.

4.2 Nonstandard Analysis

From here, we extend $\mu$ to a numerosity. The following lemmas are key to the third main result – Theorem 22. Since their proofs are not illuminating for our purposes, they are left to the appendix, but a brief intuition is given here.

Lemma 20 shows that all polynomials $p$ with integer constant term have arbitrarily large values of $N \in \mathbb{N}$ for which $p(N)$ is nearly an integer. Lemma 21 expands on this to show that if we have a finite collection of tame sets $A_i$, we can pick a finite sample $\lambda \subset \mathbb{R}^d$ for which $\mu_{A_i}(N)$ is not only close to an integer, but is in fact almost exactly the cardinality of $\lambda \cap A_i$. In Theorem 21, we apply this result to show the existence of a hyperfinite sample $F$ for which $\#(F \cap A)$ approximates $\mu_{A_i}$.

**Lemma 20.** Let $||\alpha||$ denote the distance to the nearest integer to $\alpha \in \mathbb{R}$. Fix $0 < \epsilon < 1$. For a finite collection of polynomials $\{p_i(x)\}_i$ with $p_i(0) \in \mathbb{N}$ there exist infinitely many $N \in \mathbb{N}$ such that $||p_i(N)|| < \epsilon$ for all $i$.

**Lemma 21.** Fix $\epsilon = 1/m$ for integer $m > 0$, $A_1, ..., A_v \in \mathcal{U}_b$, and $x_1, ..., x_k \in \mathbb{R}^d$. Then there exists a finite set $\lambda \subset \mathbb{R}^d$ such that $x_1, ..., x_k \in \lambda$ and for all $i = 1, ..., v$,

$$\left|\#(\lambda \cap A_i) - \mu_{A_i}(\#(\lambda \cap [0, 1]))\right| < \epsilon.$$ 

These two lemmas, as well as the following proof were inspired by Theorem 2.2 in [3], which was a significant source of inspiration for this paper.

**Theorem 22.** In any model of nonstandard analysis satisfying the property of $(2^{#(\mathbb{R}^d)})^+$-enlargement, there exists a numerosity function $n : \mathcal{P}(\mathbb{R}^d) \to *\mathbb{N}$ and a hyperreal $\omega \in *\mathbb{N}$ such that $n(A) \approx \mu_A(\omega)$ for any $A \in \mathcal{U}_b$. Moreover, $n(A) = [\mu_A(\omega)]$ on $\mathcal{U}_b$.

**Proof.** Let $\Lambda$ be the collection of all finite subsets of $\mathbb{R}^d$. Define the following sets:

• For all $x \in \mathbb{R}^d$, let $\hat{x} = \{\lambda \in \Lambda : x \in \lambda\}$.
• For all \( A \in \mathcal{U}_b \) and \( m \in \mathbb{N} \), let
\[
\Gamma(A, m) = \{ \lambda \in \Lambda : |\#(\lambda \cap A) - \mu_A(\#(\lambda \cap [0, 1]))| < \varepsilon \}.
\]

Now, let \( \mathcal{G} := \{ \hat{x} : x \in \mathbb{R}^d \} \cup \{ \Gamma(A, m) : A \in \mathcal{U}_b \text{ and } m \in \mathbb{N} \} \).

By the lemma we just showed, \( \mathcal{G} \) has the finite intersection property: if the collection is \( \hat{x}_1, ..., \hat{x}_k, \Gamma(A_1, m_1), ..., \Gamma(A_v, m_v) \), then simply choose the \( \lambda \) given in the lemma by \( m = \max(m_1, ..., m_v), A_1, ..., A_v \) and \( x_1, ..., x_k \).

Because \( \#(\mathcal{G}) \leq 2\#(\mathbb{R}^d) \), the enlarging property allows us to choose a set \( F \in \bigcap_{G \in \mathcal{G}} G^* \).

Then the hyperfinite sample \( F \) has the following properties:

1. For all \( x \in \mathbb{R}^d \), we have \( F \in ^* \hat{x} \), and therefore \( ^* x \in F \).

2. For every \( A \in \mathcal{U}_b \), we have \( F \in ^* \Gamma(A, m) \) for all natural numbers \( m \), and thus
\[
^* \#(F \cap A) \approx \mu_A(^* \#(F \cap [0, 1])).
\]

The result follows by letting \( n(A) := ^* \#(F \cap A) \) and \( \omega := ^* \#(F \cap [0, 1]) \). It can be verified using property 1 and the transfer principal on cardinality that \( n \) is a numerosity. Property 2 then gives us the main thesis. The fact that \( n(A) = [\mu_A(\omega)] \) for \( A \in \mathcal{U}_b \) follows because \( n(A) \) is hyperinteger-valued and every hyperreal number is infinitely close to at most one hyperinteger. \( \qed \)

**Corollary 23.** For tame sets \( A \in \mathcal{U}_f \),
\[
\text{st} \left( \frac{n(A)}{\omega^i} \right) \geq \mathcal{H}^i(A)
\]
with equality holding if \( A \in \mathcal{U}_b \), or \( \dim A \neq i \).

**Proof.** Let \( A \subset \mathbb{R}^d \) be tame. If \( \dim A > i \), then \( \dim A \geq i + 1 \), which means \( n(A) > \omega^{i+1/2} \). Thus
\[
\text{st} \left( \frac{n(A)}{\omega^i} \right) \geq \text{st}(\omega^{1/2}) = \infty = \mathcal{H}^i(A).
\]

Equality follows. If \( \dim A < i \), then \( \dim A \leq i - 1 \), so \( n(A) < \omega^{i-1/2} \). Hence
\[
\text{st} \left( \frac{n(A)}{\omega^i} \right) \leq \text{st}(\omega^{-1/2}) = 0 = \mathcal{H}^i(A)
\]
as desired. Finally, if \( \dim A = i \), then suppose \( A \in \mathcal{U}_b \). In that case,
\[
\text{st} \left( \frac{n(A)}{\omega^i} \right) = \text{st} \left( \frac{[\mu_A(\omega)]}{\omega^i} \right) = \text{st} \left( \frac{\mu_i(A)\omega^i + O(\omega^{i-1})}{\omega^i} \right) = \mu_i(A) = \mathcal{H}^i(A).
\]
The last equality follows by Lemma 13. On the other hand, suppose \( A \notin U_b \). Note that for any \( n \in \mathbb{N} \), \( A_n = \{ a \in A : d(a, 0) < n \} \) is bounded. Since \( A_n \to A \) as \( n \to \infty \), continuity implies \( \mu_i(A_n) \to \mu_i(A) \). Therefore,

\[
\text{st}\left( \frac{n(A_n)}{\omega_i} \right) = \mu_i(A_n) = \mathcal{H}^i(A_n) \to \mu_i(A).
\]

Since \( \mathcal{H}^i \) is a measure, it is continuous from below, so \( \mathcal{H}^i(A) = \mu_i(A) \).

\( A \) contains subsets \( A_n \) of arbitrarily measure \( \text{st}\left( \frac{n(A_n)}{\omega_i} \right) \) arbitrarily close to \( \mu_i(A) \). By monotonicity, \( \text{st}\left( \frac{n(A)}{\omega_i} \right) \geq \mu_i(A) = \mathcal{H}^i(A) \).

\[ \square \]

5 Appendix

Proof that Hausdorff and o-minimal dimension coincide on nonempty sets. Denote Hausdorff and o-minimal dimension by hDim and oDim, respectively. Let \( A \in U \) be nonempty. \( A \) is the disjoint union of a finite number of cells \( C_1 \cup \cdots \cup C_k \). Assume these are ordered by increasing dimension.

By the maximum property for Hausdorff dimension, \( \text{hDim}(A) = \text{hDim}(C_k) \).

By Proposition 2.5 of [6], \( \text{hDim}(C_k) = \text{oDim}(C_k) \).

But \( \text{oDim}(C_k) = \text{oDim}(A) \) by definition, so we are done.

\[ \square \]

Lemma 24 (Hadwiger’s Theorem for definable sets). Let \( v \) be a valuation on definable sets in \( \mathbb{R}^d \) which is flat-continuous and Euclidean-invariant. Then \( v \) is a linear combination of the intrinsic volumes of dimension at most \( d \).

Proof. Fix \( v \) as in the hypothesis. Then \( v \) defines a valuation \( V \) on constructible functions (a function with discrete image and tame level sets [5]) by

\[
V(f) = \sum v(A_i) f(A_i)
\]

where the sum is taken over all level sets \( A_i \) of \( f \) and \( f(A_i) \) denotes the only value \( f \) takes on all of \( A_i \). It is easily verified that \( V \) is continuous and Euclidean-invariant. Hadwiger’s theorem for constructible functions [5] gives that

\[
V(f) = \sum_{i=0}^{d} \int_{\mathbb{R}^d} (c_i \circ f) d\mu_i,
\]

where \( c_i: \mathbb{R} \to \mathbb{R} \) with \( c_i(0) = 0 \). But if \( f \) is a characteristic function of some tame set \( S \), then \( V(f) = v(S) \), and our equation becomes

\[
v(S) = \sum_{i=0}^{d} c_i(1) \mu_i(S).
\]

\[ \square \]
Proof of Lemma 15. Set \( n = d - m \). For \( x \in \mathbb{R}^m \), define the set \( A_x = \{ y \in \mathbb{R}^n : (x, y) \in A \} \). Also, for any integer \( i \), define \( X_i := \{ x \in \mathbb{R}^m \mid \dim(A_x) = i \} \). These definitions and theorems concerning them can be found in §3.3 of Michel Coste’s intro to o-minimal geometry [6].

According to Coste’s Theorem 3.18, we have \( \dim(\bigcap (X_i \times \mathbb{R}^n)) = \dim(X_i) + i \) for any nonnegative integer \( i \leq n \). Therefore, \( \dim(A) \geq \dim(X_i) + i \) and so for \( i > 0 \), we have the strict inequality \( m = \dim(A) > \dim(X_i) \).

Furthermore, as shown on page 14 of [3] we have that
\[
\mu_m(A) = \int_{G_{d,m}} \int_L \chi(\pi^{-1}(x)) \, dx \, \gamma(L)
\]
where \( \pi : A \to L \) is the orthogonal projection mapping (to avoid confusion, we will sometimes explicitly write the domain and codomain as a subscript like \( \pi_{X \to Y} : X \to Y \)).

Focusing on the inner integral, we can partition \( L \). Note that there is a rigid motion \( \phi_L : \mathbb{R}^m \to L \), and so we can partition \( L \) as \( L = \phi_L(X_0) \cup \cdots \cup \phi_L(X_n) \). However, by our prior inequality, \( \dim(X_i) < m \) for \( i \neq 0 \). But the integral is over \( L \approx \mathbb{R}^m \), so for \( i \neq 0 \),
\[
\int_{\phi_L(X_i)} \chi(\pi^{-1}(x)) \, dx = 0.
\]

So our equation becomes
\[
\mu_m(A) = \int_{G_{d,m}} \int_{\phi_L(X_0)} \chi(\pi^{-1}(x)) \, dx \, \gamma(L),
\]
but the zero-dimensional Euler characteristic is just cardinality. Hence, we have \( \chi(\pi^{-1}(x)) = \#(\pi^{-1}(x)) \geq 1 \) on \( \phi_L(X_0) \). Since the integrand is nonnegative, positivity will follow if any sub-integral is positive.

For almost every \( L \in G_{d,m} \) the projection \( \pi_{A \to L}(A) \) will be \( m \)-dimensional [3]. We know \( \pi(A) \) is definable because it the image of a definable set under a semialgebraic map. This implies \( \pi(A) \) contains an open \( m \)-cell and must therefore have positive \( m \)-Lebesgue measure (denoted \( \lambda \)). Fixing some \( L_0 \) on which this holds, we have \( \int_{L_0} \chi(\pi^{-1}(x)) \, dx \geq \lambda(\pi_{A \to L_0}(A)) \).

Since the function \( L \mapsto \int_L \chi(\pi^{-1}(x)) \, dx \) is continuous, we have that in a \( G_{d,m} \)-neighborhood of some fixed \( L_0 \),
\[
\int_L \chi(\pi_{A \to L}(x)) \, dx > \frac{\lambda(\pi_{A \to L_0}(A))}{2} > 0,
\]
which shows that \( \mu_m(A) > 0 \).

Finally, let us show that \( \mu_m(A) \) is equal to the Hausdorff \( m \)-measure of \( A \). Since we have shown that the integrand is the cardinality on all but a negligible set, we have
\[
\mu_m(A) = \int_{G_{m+n,m}} \int_L \#(\pi^{-1}(x)) \, dx \, \gamma(L).
\]

But since \( \pi \) is an orthogonal projection, we have \( \pi^{-1}(x) = A \cap [L^+ + x] \) where \( L^+ \) is the \( n \)-plane through the origin which is perpendicular to \( L \). Change of variables is simple
since \( \gamma_{m+n,m}(\{K\}) = \gamma_{m+n,n}(\{K^\perp\}) \) gives a Jacobian determinant of 1 for \( K \mapsto K^\perp \). Performing this reparameterization with \( M = L^\perp \) then shows that
\[
\mu_m(A) = \int_{G_{d,n}} \int_{M^\perp} \#(A \cap [M + x]) \, d\mathbf{x} \, d\gamma(M).
\]

But this is an integral over all affine planes of codimension \( m \), so by the Cauchy-Crofton Formula \( \cite{9} \), \( \mu_m(A) = \mathcal{H}^m(A) \).

We will use the following to prove Lemma 20.

**Lemma 25.** Let \( T = \mathbb{R}/\mathbb{Z} \). A sequence \( x : \mathbb{N} \to T^d \) is equidistributed if and only if \( h \cdot x : \mathbb{N} \to T \) is equidistributed for all \( h \in \mathbb{Z}^d \setminus \{0\} \). Here \( \cdot \) is the standard dot product modulo 1.

**Proof.** As proven by Tao in \( \cite{10} \), \( x \) is equidistributed if and only if
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \exp(h \cdot x(n)) = 0 \quad \text{for all } h \in \mathbb{Z}^d \setminus \{0\}.
\]

Clearly, this holds if and only if
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \exp(lh \cdot x(n)) = 0 \quad \text{for all } h \in \mathbb{Z}^d \setminus \{0\}, l \in \mathbb{Z} \setminus \{0\}.
\]

Applying Weyl’s criterion in one dimension \( \cite{11} \), this holds precisely when \( h \cdot x : \mathbb{N} \to T \) is equidistributed for all \( h \in \mathbb{Z}^d \setminus \{0\} \).

**Proof of Lemma 20.** We proceed by induction on \( k \), the number of polynomials. For \( k = 1 \), either \( p_1 \) has only rational coefficients or it has at least one irrational. If it has an irrational one, Weyl’s criterion shows that \( p_1(n) \) is an equidistributed sequence modulo 1 \( \cite{12} \), and thus has infinitely many \( N \) for which \( \|p_1(N)\| < \epsilon \). If \( p_1 \) has only rational coefficients, we can simply choose \( N \) to be the product of every coefficient’s denominator to get \( \|p_1(N)\| = 0 \).

Now suppose the theorem holds for some fixed \( k \). Consider a polynomial list \( p_0, p_1, \ldots, p_k \) of length \( k + 1 \). If \( \{p_0(x), p_1(x), \ldots, p_k(x), 1, x, x^2, \ldots\} \) is \( \mathbb{Q} \)-linearly independent, then for all \( h \in \mathbb{Z}^d \setminus \{0\} \), the polynomial
\[
q = \sum_{i=0}^{N} h_i p_i
\]
has an irrational coefficient, and is thus equidistributed modulo 1. Moreover, notice that \( q(n) = h \cdot (p_0(n), \ldots, p_k(n)) \), so Lemma 25 applies and equidistribution gives the existence of infinitely many \( N \) for which \( \|p_i(N)\| < \epsilon \) for all \( i = 0, \ldots, k \).

If, on the other hand, the aforementioned collection is \( \mathbb{Q} \)-linearly dependent, Lemma 25 does not apply. However,
\[
p_0(x) = q_1 p_1(x) + \cdots + q_k p_k(x) + r_0 + r_1 x + \cdots + r_s x^s
\]
for \( q_i, r_j \in \mathbb{Q} \). If we let \( q \) be the product of all denominators of the \( r_j \), we have that at any natural number \( n \),

\[
\|p_0(qn)\| = \|p_1(qn) + \cdots + p_k(qn)\|.
\]

Apply the inductive hypothesis to \( p_1(qx), \ldots, p_k(qx) \). Choose \( M \) so that \( \|p_i(qM)\| < \epsilon/2k \) for all \( i = 1, \ldots, k \). At this small scale, the \( \| \cdot \| \) function is subadditive in \( k \) arguments, so

\[
\|p_0(qM)\| \leq \|p_1(qM)\| + \cdots + \|p_k(qM)\| < k(\epsilon/2k) < \epsilon.
\]

Therefore, at \( N = qM \), \( \|p_i(N)\| < \epsilon \) for all \( i = 0, \ldots, k \). By induction, the proof is complete. \( \square \)

**Proof of Lemma 21.** Let \( A = A_1 \cup \cdots \cup A_v \cup \{x_1, \ldots, x_k\} \). Define \( B_1, \ldots, B_u \) as the partition of \([0, 1)\) generated by \( A_1 \cap [0, 1), \ldots, A_v \cap [0, 1) \). To complement, define \( C_1, \ldots, C_w \) as the partition of \( A \setminus [0, 1) \) generated by the collection \( A_1 \setminus [0, 1), \ldots, A_v \setminus [0, 1) \). Note \( B_i, C_i \in \mathcal{U}_b \) for all \( i \).

To apply the prior lemma, we let our finite collection \( \{p_1, \ldots, p_z\} \) of polynomials be the collection of all \( \mu_{B_i} \) and \( \mu_{C_i} \) defined above. Without loss of generality, \( p_1, \ldots, p_y \) are constant while \( p_{y+1}, \ldots, p_z \) are non-constant.

Define \( \lambda'_0 = \{x_1, \ldots, x_k\} \cup \{B_i : B_i \text{ is finite}\} \). Choose the first \( N \in \mathbb{N} \) large enough so that

- \( N > \#(\lambda'_0) \),
- \( p_i(N) > k + 1 \) for \( i > y \),
- \( p_i(x) \) for \( i > y \) are strictly increasing for all \( x > N \) (unnecessary), and
- for all \( i \),
  \[
  \|\mu_{B_i}(N)\| < \frac{\epsilon}{2\max(u, w)} \quad \text{and} \quad \|\mu_{C_i}(N)\| < \frac{\epsilon}{2\max(u, w)}.
  \]

Considering those \( B_i \) which are infinite, pick exactly \( \#(\mu_{B_i}(N)) \) points in \( B_i \) and add them to \( \lambda'_0 \). Do this for every \( i \) and define the resulting set as \( \lambda_0 \). For the \( B_i \) which are finite, \( \mu_{B_i}(N) = \#(B_i) = \#(\lambda'_0 \cap B_i) = \#(\lambda_0 \cap B_i) \) regardless of \( N \).

Because the collection of \( B_i \)'s is a partition of \([0, 1)\), we then have

\[
\sum_{i=1}^{u} \#(\lambda_0 \cap B_i) = \sum_{i=1}^{u} \#(\mu_{B_i}(N)) = \sum_{i=1}^{u} (\mu_{B_i}(N) \pm \|\mu_{B_i}(N)\|)
\]

\[
= \mu_{[0,1)}(N) + \sum_{i=1}^{u} \pm \|\mu_{B_i}(N)\|
\]

\[
= N + \sum_{i=1}^{u} \pm \|\mu_{B_i}(N)\|.
\]

But \( \|\mu_{B_i}(N)\| < \epsilon/u \), and thus

\[
\sum_{i=1}^{u} \pm \|\mu_{B_i}(N)\| < \epsilon \leq 1.
\]
Since $\sum_{i=1}^{u} \#(\lambda_0 \cap B_i)$ must be an integer, it follows that $\sum_{i=1}^{u} \pm ||\mu_{B_i}(N)|| = 0$ and so

$$\sum_{i=1}^{u} \#(\lambda_0 \cap B_i) = N$$

and thus $\#(\lambda_0 \cap [0, 1)) = N$.

To add points outside $[0, 1)$, let $\lambda = \lambda_0 \cup \lambda_1$, where $\lambda_1 \cap \lambda_0 = \emptyset$ and $\lambda_1$ has exactly $[\mu_{C_i}(N)] - \#(C_i \cap \lambda_0)$ points from each $C_i$. For finite $C_i$, this can be done by including every point, because $[\mu_{C_i}(N)] = \#(C_i)$. For other $C_i$, this can be done because they are infinite and $\mu_{C_i}(N) > k$.

Next, notice that by this definition,

$$\left| \#(\lambda \cap C_i) - \mu_{C_i}(N) \right| = ||\mu_{C_i}(N)|| < \epsilon/2w.$$  

Furthermore, since $\lambda \cap \lambda_0 = \lambda_0$, 

$$\left| \#(\lambda \cap B_i) - \mu_{B_i}(N) \right| = ||\mu_{B_i}(N)|| < \epsilon/2u.$$

Finally, consider some $A_i$. Some subset of $B_1, \ldots, B_u, C_1, \ldots, C_w$ forms a partition of $A_i$. Relabel this partition as $A_i = X_1 \cup \cdots \cup X_l$. By finite additivity of cardinality and $\mu$, it follows that

$$\left| \#(\lambda \cap A_i) - \mu_{A_i}(N) \right| = \left| \sum_{i=1}^{l} \left[ \#(\lambda \cap X_i) - \mu_{X_i}(N) \right] \right| < \sum_{i=1}^{u} \epsilon/2u + \sum_{i=1}^{w} \epsilon/2w = \epsilon.$$

Since $x_1, \ldots, x_k \in \lambda_0 \subset \lambda$ and $N = \#(\lambda \cap [0, 1))$, this concludes the proof. □

6 Acknowledgments

I would like to acknowledge the financial support of the University of Chicago Directed Reading Program in mathematics. Without the program, this work would likely not have happened. I would also like to thank Hana Jia Kong, my advisor in the program, who helped and encouraged me in the early stages of learning about this topic.

I also thank the MathOverflow and Math Stack Exchange communities for their significant contributions to this paper, in both concept and proof. In particular, I would like to thank Jakub Konieczny for making his proof of Lemma 25 publicly available. That proof is paraphrased in this paper. I would also like to thank Noam D. Elkies for pointing me towards Weyl’s criterion.

Finally, I would like to thank my parents for always encouraging me to follow my passions, even if they have no idea what I am doing.

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