How to define the boundaries of a convective zone, and how extended is overshooting?

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ABSTRACT

In non-local convection theory, convection extends without limit and therefore an apparent boundary cannot be defined clearly, as in local theory. From the requirement that a similar structure for both local and non-local models has the same depth of convection zone, and taking into account the driving mechanism of turbulent convection, we argue that a proper definition of the boundary of a convective zone should be the place where the convective energy flux (i.e. the correlation of turbulent velocity and temperature) changes its sign. Therefore, it is a convectively unstable region when the flux is positive, and it is a convective overshooting zone when the flux becomes negative. The physical picture of the overshooting zone drawn by the usual non-local mixing-length theory is incorrect. In fact, convection is already subadiabatic ($\nabla < \nabla_{\text{ad}}$) long before reaching the unstable boundary, while in the overshooting zone below the convective zone, convection is subadiabatic and superradiative ($\nabla_{\text{rad}} < \nabla < \nabla_{\text{ad}}$). The transition between the adiabatic and radiative temperature gradients is continuous and smooth instead of being a sudden switch. In the unstable zone, the temperature gradient approaches a radiative temperature gradient rather than an adiabatic temperature gradient. We would like to note again that the overshooting distance is different for different physical quantities. In an overshooting zone at deep stellar interiors, the e-folding lengths of turbulent velocity and temperature are about 0.3$H_F$, whereas that of the velocity–temperature correlation is much shorter, about 0.09$H_F$. The overshooting distance in the context of stellar evolution, measured by the extent of the mixing of stellar matter, should be more extended. It is estimated to be as large as 0.25–1.7$H_F$ depending on the evolutionary time-scale. The larger the overshooting distance, the longer the time-scales. This is because of the participation of the extended overshooting tail in the mixing process.

Key words: convection – stars: evolution.

1 INTRODUCTION

As the classical treatment of convection, local theory has been used in modelling stellar structure and evolution. In the calculation of massive star evolution, Schwarzschild & Härm (1958) discovered that the hydrogen-rich radiative envelope just outside the helium-rich convective core cannot be convectively stable. This led to the paradox of so-called semiconvection. To solve this problem, the idea of semiconvection was initiated (i.e. the region outside the convective core is in a state of semiconvection). Stellar matter in this region is nearly in neutral stability ($\nabla \leq \nabla_{\text{ad}}$). Therefore, convective energy transport as a result of this mild convection can be neglected, while the mixing of chemical compositions should be important, which makes a gradient of molecular weight in this region (otherwise called the semiconvection zone). After this, there was a great debate for a long time on whether the Schwarzschild or Ledoux criteria should be applied for the neutral stability of convection, and whether the semiconvective zone should be very wide or rather narrow. Stothers (1970) commented on various establishments of semiconvection. Evolutionary scenarios for massive stars with or without semiconvection were also discussed (e.g. Chiosi & Summa 1970). It was later realized that the problem of semiconvection is in fact a result of the non-locality of stellar convection. Therefore, various non-local theories of stellar convection were worked out (Spiegel 1963; Ulrich 1970; Xiong 1977, 1981a, 1989b; Kuhfuss 1986; Grossman, Narayan & Arnett 1993; Canuto 1993; Canuto & Dubovikov 1998). Such non-local theories of stellar convection were then applied to...
the study of the structures of solar and stellar convective envelopes (Travis & Matsumiha 1973; Unno, Kondo & Xiong 1985; Xiong & Cheng 1992), stellar oscillations (Xiong 1981b; Xiong, Cheng & Deng 1998a; Xiong, Deng & Cheng 1998b; Xiong & Deng 2001, 2007) and stellar evolution (Xiong 1986a). Generally speaking, the results from a non-local theory of convection better match observations than those from the local theory. The local theory is not capable of explaining all phenomena related to convective overshooting. The non-local theory, however, can predict the main observational properties of the solar granule velocity field (Deng & Xiong 2006). It also eliminates the theoretical conflicts of so-called semi-convection during the modelling of massive star evolution. It also predicts the widening of the main-sequence band (Xiong 1986) as required by observations of supergiants (Fitzpatrick & Garmany 1990; Blaha & Humphreys 1989). However, the non-local theory of convection is complicated, and is much less straightforward to understand, much more difficult to apply and demands much more computing power than the phenomenological local or non-local mixing length theories (Maeder 1975; Bressan, Bertelli & Chiosi 1981). For these reasons, it has become general practice to use the phenomenological local or non-local mixing length treatment for stellar convection in today’s stellar evolution models. The non-local mixing of chemical compositions during the evolution of stars is dealt with by attaching a parametric overshooting zone outside the convectively unstable region. The parametric distance of convective overshooting has a great impact on the properties of stellar evolution. The goal of the present work is to discuss the calibration of the overshooting distance. The physical definition of the boundary of the convective zone is discussed in Section 2. In Section 3, we present the calibrations of the overshooting distance using numerical simulations of non-local convection and the depletion of solar lithium abundance. A summary and discussion are given in Section 4.

2 HOW DO WE DEFINE THE BOUNDARIES OF A CONVECTIVE ZONE?

Normally, in a local theory of convection, the boundary of the convective zone is given by the so-called Schwarzschild criterion

$$\nabla = \nabla_{ad},$$  \hspace{1cm} (1)

which is derived by analysis of the local convective stability. However, from a strictly hydrodynamics point of view, all hydrodynamic phenomena including convective motions in stars are non-local, and therefore there should be no well-defined boundary for convective motion in an extended medium. In a certain sense, forcing a definition of a boundary for a convective zone is always an artefact. Defining a boundary for stellar convection is necessary in practice for stellar evolution calculations, but this cannot be done arbitrarily, and instead some objective standards should be respected. These standards should at least include the following.

(i) In fact, most calculations for stellar structure and evolution still use the local theory of convection. Therefore, the definition of a boundary given by a non-local convection theory should be kept as close as possible to that given by local theory. In other words, local and non-local convection models with the same depth of convective zone should be made to have structures as similar as possible.

(ii) The definition of a boundary should be physically pronounced in any case; that is, the unstable convective zone should be the driving (excitation) region of convective motion, and the adjacent overshooting zone should be the dissipation region of convective motion.

Stellar convection occurs because of some internal instability of the thermal structure in gravitationally stratified fluid. Therefore, a study of the resulting convective motions should be based on the dynamical equations of fluid. The complete dynamic equations of time-dependent non-local convection theory can be found in our previous work (Xiong 1981, 1989a). For the sake of clarity, and to make it easier to read this paper, we present the derivation of the dynamic equations of turbulent convection in steady fluid. The conservation of the momentum and energy of fluid dynamics can be expressed as

$$\frac{\partial (\rho u^i)}{\partial t} + \nabla_i (\rho u^l u^k + \rho u^l u^k \rho u^l u^k + \rho g^{ik} \nabla_k \psi) = \nabla_i \sigma^{ik} (u),$$  \hspace{1cm} (2)

$$\frac{\partial (\rho H)}{\partial t} + \nabla_i (\rho u^l H) - \frac{\partial P}{\partial t} - u^k \nabla_k P + \nabla_k F_k^i = \rho \epsilon_N + \sigma^{ik} (u) \nabla_i u_i.$$  \hspace{1cm} (3)

Here, $\rho$, $P$ and $\rho$ are the commonly used labels for density, temperature and pressure (including radiative pressure) of gas, $H$ and $\epsilon_N$ are the enthalpy and nuclear energy generation rate per unit mass, $u^l$ is the $l$th component of the fluid motion vector, $\sigma^{ik} (u)$ is the viscous stress tensor and $F_k^i$ is the $i$th component of the radiative flux vector. The explicit summation rule of tensor calculations is used (i.e. a pair of subscript and superscript indices mean summation from 1 to 3 for that index). When convection occurs, any physical quantity can be written as the sum of averaged and turbulent fluctuated components as

$$X = \bar{X} + X'.$$  \hspace{1cm} (4)

Putting the expressions of all the quantities in the form of equation (4) into equations (2) and (3), and averaging all the equations, the dynamic equations for the mean flow can be derived as

$$\frac{\partial (\bar{\rho} u^i)}{\partial t} + \nabla_i (\bar{\rho} \bar{u} u^k + \bar{\rho} u^l u^k + \bar{g}^{ik} P) = \nabla_i \bar{\sigma}^{ik} (\bar{u}),$$  \hspace{1cm} (5)

$$\frac{\partial (\bar{\rho} H)}{\partial t} + \nabla_i (\bar{\rho} \bar{u} H) - \frac{\partial \bar{P}}{\partial t} - \bar{u}^k \nabla_k P + \nabla_k \bar{F}_k^i = \bar{\rho} \epsilon_N + \bar{\sigma}^{ik} (\bar{u}) \nabla_i \bar{u}_i.$$  \hspace{1cm} (6)

Subtracting the corresponding equations of the mean motion equations (5) and (6) from equations (2) and (3), and considering the static state of the flow, that is

$$\bar{u} = \frac{\partial \bar{P}}{\partial \bar{t}} = \frac{\partial \bar{\rho}}{\partial \bar{t}} = \frac{\partial \bar{H}}{\partial \bar{t}} = 0,$$  \hspace{1cm} (7)

the dynamic equations for the fluctuation quantities can be derived as

$$\frac{\partial u^i}{\partial t} + \frac{1}{\bar{\rho}} \left( \bar{g}^{ik} P' + \bar{p} u^l u^k - \bar{g}^{ik} u_k' \right) + \bar{g}^{ik} \left( \frac{\bar{p}'}{\bar{\rho}} \nabla_k \bar{\Phi} + \nabla_k \Phi' \right) = \frac{1}{\bar{\rho}} \nabla_i \bar{\sigma}^{ik} (u'),$$  \hspace{1cm} (8)

$$\frac{\partial (\rho H'}{\partial t} + \bar{p} \bar{H}') + \nabla_i \left( \rho \bar{u}^l \bar{H}' + \bar{p} \bar{u}^l \bar{H} - \bar{g}^{ik} \bar{H} \right) - \frac{\partial P'}{\partial t} - \bar{u}^k \nabla_k P + \nabla_k \bar{F}_k^i = \bar{\rho} \epsilon_N + \bar{\rho} \epsilon_N - \bar{\sigma}^{ik} \nabla_i \bar{u}_i + \bar{\sigma}^{ik} \nabla_i u_i.'$$  \hspace{1cm} (9)
By using certain thermodynamic relations, and following some
deductions and simplifications, equation (9) can be written as
\[
\frac{\partial}{\partial t} \left( \frac{T'}{T} \right) + u' \langle \nabla \ln T - \nabla_{ad} \ln \tilde{P} \rangle + \frac{1}{\bar{\rho} \tilde{C}_p T} \times \left\{ u' \nabla \bar{P} + \nabla_\perp \left[ \bar{\rho} \tilde{C}_p T \left( \frac{w^2}{T} - \frac{w^2}{T} \right) \right] \right\} = \frac{1}{\bar{\rho} \tilde{C}_p T} \left[ \nabla_\perp F_{\sigma}^2 + \sigma^2 (u) \nabla_\perp u \right].
\] (10)

Here, \( w' \) is the density weighted fluctuation of turbulent velocity:
\[ w^k = \frac{\bar{\rho} w^2}{\bar{\rho}}. \]

Starting from equations (8) and (10), we have the following dynamic
equations for the auto- and cross-correlations of turbulent velocity
and temperature fluctuations:
\[
\frac{3}{2} \frac{\partial x^2}{\partial t} = \frac{1}{B \tilde{C}_p r^2} V + \frac{1}{\bar{\rho} \tilde{C}_p \bar{M}_i} \left( 4\pi \nu \overline{w'u'} w'^{\eta/\mu} \right) - \frac{1}{\bar{\rho} \tilde{C}_p \bar{M}_i} \frac{1}{c_i^3} \left( x + c_i \right) Z,
\]
\[
\frac{\partial Z}{\partial t} = \frac{2}{B \tilde{C}_p r^2} \left( \nabla - \nabla_{ad} \right) V + \frac{1}{\bar{\rho} \tilde{C}_p \bar{M}_i} \frac{1}{c_i^3} \left( x + c_i \right) Z
\]
\[
\times \left[ \frac{B \tilde{M}_i}{c_i r^2} \left( \frac{T'}{T} \right) - 1.56 \frac{B \tilde{M}_i}{c_i r^2 \tilde{P}} \left( x + c_i \right) Z, \right.
\]
\[
\frac{\partial V}{\partial t} = \frac{1}{B \tilde{C}_p r^2} \left( \nabla - \nabla_{ad} \right) x^2 + \frac{1}{\bar{\rho} \tilde{C}_p \bar{M}_i} \frac{1}{c_i^3} \left( x + c_i \right) Z
\]
\[
\times \left[ \frac{4\pi \nu \overline{w'u'} w'^{\eta/\mu}}{B \tilde{C}_p r^2} \left( \frac{T'}{T} \right) - 0.78 \frac{B \tilde{M}_i}{c_i r^2 \tilde{P}} (3x + c_i) V \right. \] (14)

Here, \( x^2, Z \) and \( V \) are, respectively, the auto- and cross-correlations of turbulent velocity \( u' \) and the relative temperature fluctuation \( T'/\bar{T} \),
de fined as follows:
\[
x^2 = \frac{\overline{w^2} w^2}{3}, \]
\[
Z = \frac{\overline{T'} T}{T}, \]
\[
V = \frac{\overline{w'T'} T}{T}. \]
\( \nabla_{ad} \) is the adiabatic temperature gradient, \( \nabla = \partial \ln \bar{T}/\partial \ln \bar{P} \) is the temperature gradient, and \( c_i \) is a variable related to the effect of thermal
conductivity:
\[
x_i = \frac{3a \bar{C}_p M_i}{c_i \bar{C}_p P^3}. \]
\( P_e = x/x_i \) is the effective Peclet number of turbulent convection. \( B = -\langle \nabla \rho \overline{\rho} / \partial \overline{T} \rangle_T \) is the expansion coefficient of gas. A detailed
derivation of the dynamic equations of correlations can be found in
our previous work (Xiong 1978, 1981, 1989a). Equations (12)–(14)
are the dynamic equations of turbulent convection in steady fluid,
and these have very clear physical meanings. Equation (12), for
instance, is for the conservation of turbulent kinetic energy. The
left-hand side is the rate of variations of turbulent energy per unit
volume, which is equal to the sum of the three terms on the right-
hand side. The first term is the work carried out by buoyant force,
\[
W_{\text{buo}} = \frac{GM_i \bar{P}}{r^2} BV. \] (19)
length parameters $\alpha$ and $\gamma$, respectively, it is clear that the two expressions are the same. The stability condition for convection is the Schwarzschild criterion (equation 1) in a chemically homogeneous medium. When there is a molecular weight gradient, the neutral stability condition should be the Ledoux criterion (Xiong 1981a):

$$\nabla = \nabla_{ad} + \nabla_{\mu}. \quad (29)$$

Hence, from a hydrodynamics point of view, the local mixing length theory is only a special simplified case of our statistical theory of correlations for turbulent convection. The third-order correlation terms in equations (12)–(14) represent the non-local effect of turbulent convection. By neglecting these, the equations become the local expression equations (23)–(25) or their explicit form equations (26)–(28). In this case, a convective zone will have a clearly defined boundary given by the Schwarzschild (or Ledoux) criterion; $\nabla > \nabla_{ad}$ is convectively unstable, while $\nabla < \nabla_{ad}$ is stable (radiative). As we show later, within the convectively unstable ($V > 0$) zone far from the boundary, the third-order correlation terms in equations (12)–(14) can be safely neglected compared with other terms. This means that the local expression of convection is a good first approximation at the deep interior of an unstable zone. This is exactly the reason why mixing length theory is still widely applied in the calculations of stellar structures. However, when studying the entirety of a convective zone, especially near the boundary of a convective zone and in the overshooting region, the third correlation terms cannot be neglected. Instead, these are the true reasons for the existence of convective overshooting. In non-local convection theory, it is clear from equations (12)–(14) that the turbulent velocity and temperature fluctuations are different from zero everywhere:

$$x > 0; \quad Z > 0. \quad (30)$$

These conditions make it difficult and uncertain to define the boundary of a convective zone. The Schwarzschild criterion is overwhelmingly used to fix a boundary. Within the framework of non-local mixing length theory, the temperature gradient is thought of as very near and slightly higher than the adiabatic temperature gradient in the unstable region. It is also near but slightly lower than the adiabatic temperature gradient within the overshooting zone (Zahn 1991; Monteiro, Christensen-Dalsgaard & Thompson 2000). We show that such a picture of convective overshooting is incorrect following the dynamical theory of turbulent convection. The reason for such a mistake is that there is an implicit hypothesis that has been applied in phenomenological mixing length theory, that is, turbulent velocity is fully correlated (either positively or inversely) with temperature (see, for example, Xiong & Cheng 1992; Petrovay & Marik 1995). In fact, when convection is very effective (the effective Peclet number $x/X_c \gg 1$), the correlation between turbulent velocity and temperature fluctuations near the boundary and in the overshooting zone decreases very quickly and eventually vanishes (Xiong & Cheng 1992).

Equations (12)–(14) are derived under general conditions, among which are two important assumptions as follows.

(i) Convection is subsonic, and the relative fluctuations of temperature and density are both far less than unity:

$$|\rho'/\rho| \ll 1; \quad |T'/\bar{T}| \ll 1. \quad (31)$$

(ii) The inelastic approximation, which actually filters out all acoustic waves, is not important for energy transfer in subsonic convection.

It is well known that the dynamic equations of turbulent correlations have no closure because of the non-linearity of hydrodynamics. This means that the third-order correlations must be present in the dynamic equations of the second-order correlations, while the fourth-order correlations must turn up in the equations of the third-order correlations, etc. Some hypothesis must be used in order to obtain closure for the dynamic equations of the correlations. Obviously, the closure cannot be unique. A few methods have been adopted so far (Xiong 1981a, 1989b; Canuto 1993; Grossman et al. 1993; Canuto & Dubovikov 1998). In our opinion, good closure should meet the following conditions:

(i) The solutions of the resulting equations must be physically sensible. For instance, the standard quasi-normal approximation seems to be better in terms of mathematics; however, it gives solutions such as $x^2 < 0$ or $Z < 0$, which are physically nonsense (Grossman 1996). Such a seemingly reasonable assumption, if not modified somehow, cannot be used for the closure of the dynamical equations of the third-order correlations.

(ii) The solutions presented should not be in contradiction with observations. For instance, it should reproduce the main observational properties of the solar granular velocity field, it should be able to explain the pulsation instabilities of low-temperature stars having an extended convective envelope, and it should be able to model the observed lithium abundance patterns in the atmospheres of the Sun and solar-type stars, and so on.

(iii) The solutions provided should be comparable with those of direct hydrodynamical simulations.

Our non-local theory of convection (Xiong 1981a, 1989a,b) has been tested against the above standards, and satisfactory results have been reached. Therefore, we have good reason to believe that it has expressed well the dynamic behaviours of stellar turbulent convection.

The solid line in Fig. 1 shows the fractional convective flux $L_c/L$ versus the depth $\log P$ for a model of the solar convective zone calculated using our non-local convection theory. The dashed line is that of a local convection model, which has the same depth of convective zone. It follows from the figure that there is almost no difference between the local and non-local models, except for some sizable deviations near the boundary of the convective zone. Fig. 2 depicts the
relative squared sound speed and density difference between the local and non-local solar convection zone models with the same depth of convective zone versus depth. The relative difference between the two models is mostly below 1 per cent, excluding the solar surface region. In this case, the boundary of the (non-local) convective zone is set at the place where the turbulent velocity–temperature correlation vanishes:

\[ V = 0. \] (32)

Passing through the boundary, \( V \) changes its sign: within the convective zone

\[ V > 0, \] (33)

and in the overshooting zone

\[ V < 0. \] (34)

It is clear from Figs 1 and 2 that if the boundary is defined as such, the structures of the local and non-local convection models with the same depth of convective zone should be similar. This can be understood by the fact that, within the stellar interior, the turbulent kinetic energy flux \( (L_t) \) is generally much less than that of thermal convection \( (L_c) \), and the turbulent pressure \( (P_t = \rho \Delta v^2) \) is much less than that of gas \( (P_g) \). It can easily be shown that \( P_t/P_g \sim x^2/C_s^2 = Ma^2 \), where \( C_s \) is the local sound speed and \( Ma \) is the Mach number of turbulence; \( L_t/L_c \) is of the same order of magnitude as \( P_t/P_g \). Except for the top of the convective zone, we have \( Ma \ll 1 \). It follows from Fig. 3 that, for the Sun, \( L_t/L_c < 1 \) per cent, and therefore the thermal convection \( L_c \) dominates the pressure–temperature \((P-T)\) structure. It is then clear that when defining the boundary of convective zone by \( V \) changing its sign, the structures of the local and non-local models having the same depth of convective zone will be similar.

Fig. 1 clearly demonstrates that convective motions near both the upper and lower boundaries of the convective zone are very different. This is because, in the atmosphere, the density is very low and \( P_c < x/x_c < 1 \); therefore, convective energy transfer is inefficient. As a result, there exists a thin superadiabatic layer on top of the convective zone. Passing through the upper boundary, the turbulent velocity–temperature correlation \( R_{VT} = V/\sqrt{\Delta T/\Delta P} \) drops quickly from \( \sim 1 \) to \( \sim -1 \). This theoretical prediction agrees with the observations of the solar granular velocity field (Leighton, Neyes & Simon 1962; Salucci et al. 1994) and the results of hydrodynamic simulations (Kupka 2003). Contrary to the situations in the solar atmosphere, convection is highly efficient in terms of energy transfer \( (P_t \gg 1) \) in the deep interiors of the Sun. Towards the lower boundary of the convective zone, the turbulent velocity–temperature correlation \( R_{VT} \) decreases abruptly and approaches zero \( (|R_{VT}| \ll 1) \). What makes it so different at the two boundaries is the distinct effective Peclet number.

Fig. 3(a) shows the work carried out by buoyant force \( \rho \omega_{buo} \), the net gain \((>0)\) or loss \((-<0)\) of turbulent kinetic energy as a result of the non-locality of turbulence \( -\rho \partial \Delta L_t/\partial M_t \) and the turbulent viscous dissipation rate \( \rho \epsilon \) as functions of depth in the solar convective zone. As shown in Fig. 3(a), the contribution from non-local convection is much less than the other two quantities in the convective zone, except in the narrow regions near the boundary of the convective zone and in the overshooting zone. The energy balance in the bulk of the convective zone is a result of the interplay of the work carried out by the buoyant force \( \omega_{buo} \) and the viscous dissipation rate \( \rho \epsilon \). Turbulence retrieves energy from the buoyant force, while at the same time it dissipates energy due to viscosity. The former factor, as the source, originates primarily from large eddies, while the latter occurs in the viscous dissipation range of the highest end of the turbulent spectrum. Turbulence gains energy from the buoyant force, which is then cascaded from the lowest to higher and higher wavenumbers of the turbulent spectrum, and is eventually converted into thermal energy as a result of molecular viscosity. It follows from Fig. 3(a)
Because of the complexity of non-local convection theory, almost all the models of stellar structure and evolution are still using local convection theory. Convective overshooting is defined as the penetration of convective motion through the classical boundary of the convectively unstable zone into the adjacent stable region. The extent of overshooting is not the same for different physical quantities following our dynamic theory of convection, and this leads to some problems in understanding the true nature and estimation of the extension of overshooting. Followed by the great success of helioseismology, people are expecting to draw a firm conclusion to the long-debated overshooting distance at the bottom of the solar convective zone using the helioseismology method. Gough & Sekii (1993) reported that they could not find any definite evidence for the existence of overshooting in the Sun, while others have given an upper limit of 0.05–0.25 $H_P$ (Monteiro, Christensen-Dalsgaard & Thompson 1994; Roxburgh & Vorontsov 1994; Christensen-Dalsgaard, Monteiro & Thompson 1995; Basu & Antia 1994; Basu 1997). Such results are understandable. Indeed, what the technique of helioseismology tests is the (adiabatic) sound speed in the Sun, while the sound speed is determined by the $P$–$T$ structures. In the overshooting zone, however, convective flux is negligible. This is why the overshooting below the bottom of the solar convective zone has not been detected by helioseismic diagnosis. From Fig. 2, it is clear that the relative difference in sound speed between the local and non-local solar models is less than 1 per cent. Such tiny differences were indeed detected by the inversion of the adiabatic sound speed in helioseismology (Basu 1997). However, the observed abrupt increase of the adiabatic sound speed at the bottom of the solar convective zone was not correctly attributed to non-local overshooting. In fact, the convective flux changes its sign when crossing the boundary of the convective zone, becoming negative in the overshooting zone ($L_a < 0$). There, the radiative flux $L_r$ will be even larger than the total flux of the Sun ($L_\odot$). In the overshooting zone, the temperature gradient will overtake the radiative counterpart $\nabla > \nabla_{rad}$ (see Fig. 4). As a result, the temperature at the bottom of the solar convective zone will increase, just as has been detected using the helioseismology technique.

Gough & Sekii (1993) measured the extension of overshooting at the bottom of the solar convective zone following the picture of the overshooting zone given by phenomenological non-local mixing length theory (as illustrated in Fig. 5). In the unstable zone, the
temperature gradient is slightly higher than the adiabatic temperature gradient, while being slightly lower in the overshooting zone. After an overshooting distance $d_{ov}$, the temperature gradient switches suddenly from adiabatic to radiative, creating a discontinuity in the temperature gradient at the bottom of the overshooting zone. The size of this jump in the temperature gradient is proportional to the overshooting distance $d_{ov}$. Such a discontinuity in the temperature gradient is exactly what Gough & Sekii used to detect the overshooting distance $d_{ov}$. It is this implicit assumption of a full (either positive or negative) correlation between turbulent velocity and temperature fluctuations that creates the misunderstandings about the overshooting zone in non-local mixing length theory (Xiong 1985; Petrovay & Marik 1995). In reality, however, the turbulent velocity–temperature correlation decreases very quickly and approaches zero near the lower boundary of the solar convective zone, where convective energy transfer is very efficient, as demonstrated in Fig. 1. Therefore, there is no similarity in the structure of the overshooting zone between the phenomenological non-local mixing length theory and the dynamic theory of non-local convection.

In our view, the temperature gradient is already smaller than the adiabatic temperature gradient $\nabla < \nabla_{ad} < \nabla_{rad}$ before reaching the lower boundary of the convective zone. The convective flux becomes negative when passing through the boundary, and therefore the temperature gradient $\nabla$ is smaller than the adiabatic temperature gradient $\nabla_{ad}$ but higher than the radiative temperature gradient $\nabla_{rad} (\nabla_{ad} < \nabla < \nabla_{rad})$. The temperature gradient changes continuously instead of abruptly from $\nabla_{ad}$ to $\nabla_{rad}$. The structure of the overshooting zone in our dynamic theory of non-local convection is shown in Fig. 4. In the overshooting zone under the convective zone, there is a narrow ($\sim 0.25 H_{P}$) and weakly super-radiative region. Actually, the overshooting zone is nearly radiative rather than nearly adiabatic. Therefore, it is no surprise that Gough & Sekii (1993) could not find any firm evidence for the existence of overshooting under the bottom of the solar convective zone. We can further justify that the methods based on stellar thermal ($P-T$) structure will all underestimate the true overshooting distance. In the lower overshooting zone, the turbulent velocity–temperature correlation is very small, and convective energy transfer there is negligible. The overshooting distance detected by the convective energy flux will be far smaller than that represented by the turbulent velocity and temperature fields. It follows from Fig. 6 that, in the overshooting zones at either the surface or the bottom of the solar convective zone, the overshooting distances of turbulent velocity and temperature are both very extended; their e-folding lengths of overshooting are given in Table 1. Our theoretical e-folding length agrees well with those derived from observations of the solar granular field (Keil & Canfield 1978; Nesis & Mattig 1989; Komm, Mattig & Neiss 1991).

The overshooting distance that is important to stellar evolution is the extension of the non-local convective mixing of chemical elements. Obviously, it is neither that of convective energy transfer nor those of turbulent velocity and temperature fluctuations. Calculations of massive star evolution under our complete non-local theory of convection show that the non-local convective mixing overshoots by a very extended distance (Xiong 1986).

Although the overshooting at the bottom of the solar convective zone cannot be observed directly, we fortunately have another excellent indicator for the extension of overshooting: the lithium abundance of the Sun and solar-type stars. No matter how disputed the mechanism of lithium depletion in the atmospheres of solar-type stars is, it can provide, at least, an upper limit for the extension of the overshooting zone in these stars. As is well known, $^7$Li becomes destroyed as a result of the reaction $^7$Li$(P, a)^4$He at a temperature of $2.5 \times 10^6$ K. The depth of the solar convective zone, as given by helioseismology, is $r_s/R_\odot \approx 0.713$, and the temperature at the bottom of the zone is $T_s \approx 2.26 \times 10^6$ K (Christensen-Dalsgaard, Gough & Thompson 1991; Basu & Antia 1997). This temperature is not high enough to burn lithium. Without overshooting (bringing lithium deeper to higher temperatures) there will be virtually no depletion of lithium in the Sun. It is the overshooting that brings lithium down to the burning region at a higher temperature, and causes the depletion. Figs 7(a)–(d) show the lithium abundance depletions as a result of overshooting for $M = 0.90, 0.95, 1.00$ and $1.05 M_\odot$ stellar models as functions of depth ($\log P$), when the effect of evolution is not considered. The dashed lines in the plots indicate the boundary of the convective zone. As seen from Fig. 7(c), there is a gradually accelerating reduction of lithium abundance in the overshooting zone. In the upper part of the overshooting zone, the mixing caused by non-local convection is very efficient. The abundance remains the same as in the convectively unstable zone for about 0.4 $H_P$ in length downwards. It is then followed by a partially mixed region where lithium abundance is reduced more quickly towards the centre and vanishes suddenly. Such a partial mixing region is about 0.5 $H_P$ in depth. Even deeper is the non-mixing zone. If the overshooting...

| Table 1. The e-folding lengths measured in local pressure scaleheight. |
|--------------------------------------------------------------|
| M/M_\odot | x | z | V | lower overshooting zone |
|-----------|---|---|---|--------------------------|
| 0.800     | 0.47| 0.36| 0.20| 0.25| 0.25| 0.080| 0.26| 0.30 |
| 0.850     | 0.48| 0.36| 0.21| 0.25| 0.25| 0.082| 0.42| 0.54 |
| 0.900     | 0.50| 0.35| 0.20| 0.25| 0.25| 0.069| 0.50| 0.58 |
| 0.925     | 0.50| 0.36| 0.21| 0.25| 0.25| 0.104| 0.67| 0.74 |
| 0.950     | 0.50| 0.33| 0.20| 0.26| 0.25| 0.076| 0.60| 0.69 |
| 0.975     | 0.49| 0.33| 0.20| 0.31| 0.31| 0.093| 0.85| 0.91 |
| 1.000     | 0.50| 0.32| 0.20| 0.29| 0.29| 0.096| 1.05| 1.11 |
| 1.025     | 0.63| 0.30| 0.21| 0.29| 0.29| 0.096| 1.05| 1.11 |
| 1.050     | 0.52| 0.30| 0.19| 0.30| 0.36| 0.092| 1.25| 1.33 |
| 1.075     | 0.60| 0.30| 0.21| 0.38| 0.38| 0.114| 1.64| 1.69 |

Figure 6. The auto- and cross-correlations of turbulent velocity and temperature $x, z^{1/2}$ and $V$ versus depth ($\log P$) for a non-local convection model of the Sun. The vertical dashed lines indicate the upper and lower boundaries of the convective zone.

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Figure 7. The lithium abundance versus depth (log $T$) and age (labelled on the curves) for main-sequence stars. The dashed vertical line indicates the location of the lower convective boundary. Evolution is not considered in the calculations. (a) $M = 0.90 \, M_{\odot}$; (b) $M = 0.95 \, M_{\odot}$; (c) $M = 1.0 \, M_{\odot}$; (d) $M = 1.05 \, M_{\odot}$.

The equation for the conservation of lithium abundance can be written as

$$\frac{1}{C} \frac{\partial C}{\partial t} = -\frac{1}{C} \frac{\partial}{\partial M_r} (4\pi r^2 \rho U) - q,$$

(35)

where $C$ is the lithium abundance by mass and $q$ is the burning rate of lithium. $U$ is the correlation of the radial component of turbulent velocity $w'_r$ and the turbulent fluctuations of lithium abundance defined as

$$U = \overline{w'_r C}.$$

(36)

Therefore, $4\pi r^2 \rho U$ is the total flux of the convective mixing of lithium passing through the sphere of radius $r$. The first term on the right-hand side of equation (35) is the rate of variation of lithium abundance as a result of non-local convective mixing. It follows from Fig. 6 that, in the overshooting zone, the turbulent velocity decreases nearly exponentially. The non-mixed region does not mean there is no mixing at all. In our picture, mixing is always there; the only difference is the degree of mixing. When the non-local mixing time-scale $\tau_{\text{mix}}$ becomes much longer than the nuclear timescale $\tau_{\text{nuc}}$ of depletion (i.e. the non-local mixing cannot feed fresh lithium into the burning zone), lithium abundance vanishes abruptly. Figs 8(a)–(d) give the two terms on the right-hand side of equation (35) and their sum (the depletion rate of lithium) versus depth near the lower boundary of the convective zones for 0.90-, 0.95-, 1.00- and 1.05-$M_{\odot}$, main-sequence stellar models, respectively. The units for these quantities are all yr$^{-1}$. The vertical dashed line shows the lower boundaries in these models. As is clearly shown in Fig. 8, the depletion of lithium starts at $T \sim 2.5 \times 10^6$ K, and it goes up very quickly as temperature increases (being proportional to the 21st power of temperature). In all the envelopes of these models, $U < 0$, this means that convection continues to feed lithium from the outer layers into the inner layers in order to supply the depletion at the burning zone. In the convectively unstable zone (where $q$ is extremely small), convective mixing is very efficient, and $d(4\pi r^2 \rho U)/dM_r$ is nearly constant (see Fig. 8). Towards the deep interior, the nuclear burning rate $q$ increases abruptly. In the upper
part of the overshooting zone, mixing as a result of convective overshooting is still efficient enough to compensate for the depletion due to nuclear burning. Therefore, the lithium abundance profile is still horizontal (see Figs 7 and 8). At the lower part of the overshooting zone, however, the convective overshooting mixing (the dotted lines in Fig. 8) can no longer support the balance between mixing and depletion (long dashed lines in Fig. 8). \(-\frac{1}{C} \frac{d(4\pi\rho U)}{dM_r}\) decreases abruptly towards the centre, corresponding to lithium abundance dropping off in Fig. 7. This means that the boundary of the overshooting zone is reached at this place. For the \(M = 0.90\, M_\odot\) star, the convective zone is very deep with its bottom already at the burning region of lithium. This may lead to a shallower overshooting zone. Toward higher masses, convection becomes shallower, and the bottom of the zone moves farther away from the region of burning (see Figs 8c and d). The overshooting zone then becomes more extended. Fig. 9 presents the overshooting distance \(d_{ov}\) measured by lithium depletion as a function of stellar mass. It is clear that the overshooting distance defined by the dropping off of lithium abundance by a factor of \(e\) is shown in the eighth column of Table 1, while that defined by the distance from the boundary of the convective zone to where lithium becomes zero is given in the last column of Table 1. It is clear from Figs 7(a)–(d) that lithium abundance vanishes very quickly after the e-folding depletion; the distance between any pair of models is less than 0.1 \(H_p\). Completely different from the lithium abundance profile in Fig. 7, the turbulent velocity \(v\), the temperature fluctuation \(Z_{1/2}\) and the velocity–temperature correlation \(V\) decrease exponentially with depth in the overshooting zone. The e-folding distances determined from the curves are given in the columns 2–4 (for the upper part of the overshooting zone), and in columns 4–7 (the lower part of the overshooting zone). The e-folding distances given by the turbulent velocity and temperature fields are very close to the analytical asymptote in our theory (Xiong 1989b). These hardly change with stellar mass, and are rather different from the e-folding distances defined by lithium depletion. The upper and lower overshooting zones are slightly different in terms of these e-folding distances. This is because, in the overshooting zone above
the convective zone, the gas density is low so that the effective Peclet number $P_e \ll 1$, and the convective energy transfer is inefficient. In contrast, in the overshooting zone at the bottom of the convective zone, $P_e \gg 1$, and convection is highly efficient. These arguments mark the distinct properties of the velocity–temperature correlation in the upper and lower overshooting zones. In the surface overshooting zone, $R_{VT} \sim -1$, while in the bottom overshooting zone, $R_{VT} \sim -0.0$. When plotting Fig. 1, $R_{VT}$ has been magnified in order to show the detail. In fact, we should have $-10^3 < R_{VT} < 0$ in the lower overshooting zone, so that it is completely buried in the solid line of $(L_e/L)$.

4 SUMMARY AND DISCUSSION

In this paper, we have presented a detailed discussion of the definition of the boundary of the convective zone, and the distance of convective overshooting. The main results can be summarized as follows.

(i) Choosing as the boundary the place where the convective flux (or equivalently the turbulent velocity–temperature correlation) changes its sign is the most proper and convenient method. Where the convective flux is greater than zero is the convectively unstable zone, while the overshooting zone is when the convective flux is smaller than zero. To define the convective zone as such means that the local and non-local convection models with the same depth of convection zone not only have similar structures, but also have very clear physical meanings. A convective zone is the buoyant force driving zone for convective motion, while an overshooting zone is the dissipation zone against convective motion, which can only be supported by non-local convective diffusion.

(ii) It is incorrect to talk about a general overshooting distance for stellar convection. The overshooting distance is different for different physical quantities. The effect of overshooting for convective energy transfer, for instance, is not important. This is because the density in a stellar atmosphere is very low and convection there is not efficient. In stellar interiors, however, the correlation between turbulent velocity and temperature drops off quickly towards the boundary of the convective zone. Therefore, convection is always inefficient for either the surface or interior overshooting zones. However, the overshooting distances of turbulent velocity and temperature fluctuations are extended, and the e-folding lengths can reach 0.25–0.5 $H_F$. The overshooting distance that is important to stellar evolution is the extent of the convective mixing of chemical elements. From the example set by the depletion of lithium in solar-type stars, it is found that convective mixing of matter occurs on stellar evolutionary (nuclear) time-scales, and is very efficient. Very extended and weak overshooting can still induce fairly efficient mixing on a very long time-scale of evolution. Therefore, we anticipate a very extended overshooting for mixing of matters, the e-folding length of which is generally larger than that of turbulent velocity and temperature. In the massive stellar model, for instance, it can reach one pressure scaleheight (Xiong 1986).

The problem of lithium depletion in solar-type stars is a special case of overshooting mixing that moves lithium to the burning region from the surface. Under such circumstances, the overshooting distance depends on the location of the convective zone relative to that of the lithium burning region. Obviously, such a conclusion cannot be simply generalized to the case of the core nuclear reaction. For lithium depletion in solar-type stars, the convective zone becomes shallower for higher masses, and lithium depletion becomes slower (with a longer time-scale). This makes the extended overshooting tail efficient in mixing, and the mixing range becomes more extended. As a result, the overshooting distance detected by lithium depletion in solar-type stars becomes larger for increasing stellar mass. In fact, this indicates that the mixing of matter becomes more extended for increasing nuclear burning time-scales.

Using a similar non-local convection theory, we have calculated the early stages of the post-main-sequence evolution for massive stars. The overshooting distances from the cores are extended, typically $1–2 H_F$, and the fractional mass within the overshooting zone $M_{overshoot}/M_0 \sim 0.1–0.5$ (Xiong 1986). A quasi-isotropy of turbulence was assumed. By considering the non-isotropy of turbulent convective motion, the overshooting distance will be reduced accordingly. Very recently, Meakin & Arnett (2007) studied the overshooting distances from the convective core and the burning convective shell. Because of the high consumption of computational resources, only two and eight convective turnovers for the convective core and burning convective shell, respectively, were carried out in their three-dimensional simulations. Besides, the convective overshooting time-scale is even longer than the convective turnover (Xiong 1989b). The three-dimensional simulations of Meakin & Arnett (2007) are far from quasi-steady state, and therefore it is difficult to compare quantitatively their results with those of our non-local convection theory.

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