New truncated Milstein approximation of solution of stochastic differential equations

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Abstract
This paper presents a new explicit method, called the truncated Milstein method for the numerical solution of some stochastic differential equations. This new method was fixed under the local Lipschitz condition as well as the khasminskii-type condition. Our numerical experiments based on the obtained method demonstrate the applicability and effectiveness of the results in the evaluation of a wide range of linear and nonlinear financial problems. This new method has high convergence in comparison with the Euler, Milstein and truncated Euler methods for some linear and nonlinear stochastic differential equations.

Keywords: Stochastic differential equations, Milstein method, Euler Maruyama method, Lipschitz condition, khasminskii condition.

AMS Subject Classification: 60H10.

1 Introduction

In recent years, the study and modeling of random differential equations are rapidly increasing. Historically, it appeared as an extension of the deterministic differential modeling of overidealized situations with fluctuating behavior of the analyzed physical phenomena [14]. Actually, it is an important research area that describes important phenomena such as the spread of diseases, the motion of particles and genetic regulation [16]. There exists an obvious interest in the development of stochastic numerical methods to solve stochastic differential equations (SDEs) due to the fact that analytical solution of SDEs are, in general, not available. Simulation methods are usually based on discrete approximations of the continues solution to the SDE.
The methods of approximation are classified according to their different properties. Variance reduction techniques for SDEs can be borrowed from standard variance reduction techniques of the Monte Carlo method and clearly only apply when interest is in the functional of the process [12]. In the last decades, several works related to linear stability for the most common methods for SDE has been published [7, 6]. A linear analysis can be considered as a first step for understanding the method, but it is not an indicator of the qualitative behavior of the method in a nonlinear case [9]. Thus some theoretical work on asymptotic stability has appeared for nonlinear SDE with multiplicative noise by Bokor [3] and with additive noise by Buckwar et al. [4]. Higham, Mao, and Stuart published a very useful paper in 2002 which opened a new chapter of numerical approximations under the local Lipschitz condition [5]. Of course, the local Lipschitz condition alone is not sufficient to continuity to the exact answer. Then, we add khasminskii condition and a linear growth condition on local Lipschitz conditions, which clearly show the continuity of the analytic answer. They showed that the numerical solutions converge to the exact solution in the strong sense under the local Lipschitz condition and the linear growth condition. The classical explicit EM method has been attracting lots of attention because it has a simple algebraic structure and acceptable convergence rate under the global Lipschitz condition [7]. Recently, Mao proposes a new explicit method called the truncated EM method that focuses on those SDEs with both the drift and diffusion coefficients allowed to grow super linearly [11]. Mao proved that the strong convergence rate of the method could be arbitrarily close to a half. In this paper, we use the Milstein method in order to improve the convergence rate of estimating. So this paper is organized as follows. Section 2 presents a review of numerical methods for solving SDEs. In Section 3, we construct the explicit truncated Milstein method for numerical solution of nonlinear SDEs. Numerical results that confirm the convergence and stability properties are given in section 4. Finally, we give some conclusions in section 5.

2 A review of numerical methods

This section covers important classical methods on the subject of simulation of solutions of SDEs. The simplest approximation is the Euler scheme. This method is easy to implement and almost universally applicable, but it is not always sufficiently accurate. Then an expansion to refine the Euler scheme was presented and called the Milstein scheme. The Euler scheme has strong convergence order $\frac{1}{2}$ and weak order 1. The Milstein scheme has strong order 1, thus under the relatively modest additional conditions, expanding the diffusion term to $O(h)$ instead of $O(\sqrt{h})$, through the derivation in Milstein scheme increases the order of strong convergence, but the weak order of convergence of the Milstein is also 1 as it is for the Euler scheme.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\{\mathbb{F}_t, \sim \geq \mathbb{F}_t\}$ be an increasing right continuous family of complete sub $\sigma$ algebras of $\mathcal{F}$. Considers a d-dimensional diffusion process $X$ defined by the following Itô type of the form:

$$dX_t = f(t, X_t)dt + g(t, X_t)dB_t, X_0 = x_0. \quad (2.1)$$

where the drift coefficient $f : [t_0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and the diffusion coefficient $g : [t_0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ are differentiable functions with respect to $t$, $B = (B^1, ..., B^m)$ is an m-dimensional $\mathcal{F}_t$-adapted standard Wiener process, and $X_0$ is a $\mathcal{F}_{t_0}$ measurable random vector [12]. We examine the Euler and Milstein approximations in the next subsections.

2.1 Euler scheme

One of the most used scheme of approximation is the Euler method. Let the Itô process $\{X_t, 0 \leq t \leq T\}$ is the solution of SDE (2.1), with initial deterministic value $X_0 = x_0$ and the discretization $\Pi_N$ of the interval $[0, T]$ , $0 = t_0 < t_1 < \ldots < t_N = T$. The Euler approximation of $X$ is a continuous stochastic process $Y$ satisfying the iterative scheme,

$$Y_{i+1} = Y_i + f(t_i, Y_i)(t_{i+1} - t_i) + g(t_i, Y_i)(B_{t_{i+1}} - B_{t_i}), \quad (2.2)$$

for $i = 0, 1, \ldots, N - 1$, with $Y_0 = X_0, Y(t_i) = Y_i$. Usually the time increment $\Delta t = t_{i+1} - t_i$ is taken to be constant ($\Delta t = \frac{T}{N}$) and $B_{t_{i+1}} - B_{t_i} \sim N(0, \Delta t)$ [10].
2.2 Milstein scheme

The Milstein scheme makes use of Ito’s lemma to increase the accuracy of the approximation by adding the second-order term. Denoting by $g_x$ the partial derivative of $g(t,x)$ with respect to $x$, this approximation look like

$$Y_{t+1} = Y_t + f(t,Y_t)\Delta t + g(t,Y_t)\Delta B_t + \frac{1}{2}g_x(t,Y_t)(g_x(t,Y_t))^2 \Delta t$$

or in more symbolic form,

$$Y_{t+1} = Y_t + f \, \Delta t + g \, \Delta B_t + \frac{1}{2}gg_x (\{\Delta B_t\}^2 - \Delta t)$$

The new term $\frac{1}{2}gg_x (\{\Delta B_t\}^2 - \Delta t)$ has mean zero and is uncorrelated with the Euler terms.

3 The truncated Milstein method

We extend the idea presented in Mao to earn the truncated Milstein method [11]. We impose two hypotheses in this paper.

**Assumption 3.1.** Assume that the coefficients $f$ and $g$ satisfy the local Lipschitz condition: for any $R > 0$, there is a $k_R > 0$ such that

$$|f(x) - f(y)| + |g(x) - g(y)| \leq k_R \, |x - y|$$

for all $x, y \in \mathbb{R}^d$ and $\vee$ is the maximum of two real numbers.

**Assumption 3.2.** We also assume that the coefficients satisfy the Khasminskii-type condition: there is a pair of constants $p > 0$ and $k > 0$ such that

$$x^T f(x) + \frac{p-1}{2} |g(x)|^2 \leq k(1 + |x|^2)$$

for all $x \in \mathbb{R}^d$ and $x^T$ is the transpose of $x$ [8].

**Lemma 3.1.** Under Assumptions 3.1 and 3.2, the SDE(1) has a unique global solution $x(t)$ and moreover, $\sup_{0 \leq t \leq T} |x(t)|^p < \infty$ for all $T > 0$.

To define the truncated Milstein numerical approximation choose a strictly increasing function $\mu : R \to R$ such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{|x| \leq u} (|f(x)| + |g(x)|) \leq \mu(u) \, \forall u \geq 1$$

and choose a number $\Delta^* \in (0, 1]$ and a strictly decreasing function $h : (0, \Delta^*) \to (0, \infty)$ such that $h(\Delta^*) \geq \mu(2)$, $\lim_{\Delta \to 0} h(\Delta) = \infty$ and $\Delta^2 h(\Delta) \leq 1$, $\forall \Delta \in (0, \Delta^*]$. Now, we define the truncated functions:

$$f_x(\Delta) = f\left(\left\{x \, | \, \mu^{-1}(h(\Delta))\right\}^x\right)$$

and

$$g_x(\Delta) = g\left(\left\{x \, | \, \mu^{-1}(h(\Delta))\right\}^x\right).$$

These truncated functions preserve the Khasminskii-type condition [11]. The discrete-time truncated Milstein solution $X_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ are formed by setting $X_\Delta(0) = x_0$ and computing

$$X_\Delta(t_{k+1}) = X_\Delta(t_k) + f_x(\Delta) + g_x(\Delta) \Delta B_k + \frac{1}{2}gg_x(\Delta)^2 - \Delta,$$

where $g_x(\Delta) = \frac{\partial g}{\partial x}\left(\left\{x \, | \, \mu^{-1}(h(\Delta))\right\}^x\right).$
4 Numerical Simulation

This section covers some numerical examples. Our numerical experiments based on the obtained method demonstrate the applicability and effectiveness of the results in the evaluation of a wide range of stochastic problems. Now we explain how to apply the method. It is obvious that the choices of functions $\mu(u)$ and $h(\Delta)$ are essential in order to use the method. The forms of these two functions are highly related to the structures of the drift and diffusion coefficients of the SDE (2.1). We illustrate how to choose these functions by the following examples.

Example 4.1. Consider the Geometric Brownian Motion (GBM) which follows the SDE,

$$dX_t = \mu X_t dt + \sigma X_t dB_t,$$

(4.6)

where $\mu$ and $\sigma$ are constants.

The analytical solution of this equation is the form,

$$X_t = X_0 \exp \left( (\mu - \frac{\sigma^2}{2}) t + \sigma B_t \right).$$

(4.7)

The truncated Milstein approximation for this example is the form,

$$\sup_{|x| \leq u} (|f(x)| \lor |g(x)|) \leq \frac{1}{2} u \quad \forall u \geq 1$$

$$\mu^{-1}(u) = 2u, h(\Delta) = \Delta^{\frac{3}{2}}, \epsilon = \frac{1}{2}$$

$$f_{\Delta}(x) = f(\left( \frac{|x| \lor 2\Delta^{\frac{1}{2}}}{|x|} \right),$$

$$g_{\Delta}(x) = g(\left( \frac{|x| \lor 2\Delta^{\frac{1}{2}}}{|x|} \right).$$

In order to illustrate the convergence of the method, we have simulated sample trajectories with step size $\Delta t = 0.01$. Figures (1), (2) and (3) shows the computer simulations of the sample paths of $X(t)$ with exact the solution is given by Eq.(4.7) versus the truncated Euler Maruyama and truncated Milstein approximations with 1000, 10000 and 100000 sample paths. The criterion of mean square error (MSE) was used to show the efficiency of the proposed model. Tables 1 and 2 shows the amounts of MSE with truncated EM and truncated Milstein schemes with various step size $\Delta t$ for this example.

Example 4.2. Consider the Ornstein-Uhlenbeck equation (OU) which follows the SDE,

$$dX_t = \mu X_t dt + \sigma dB_t,$$

(4.8)

where $\mu$ and $\sigma$ are constants.

The analytical solution of this example is the form,

$$X_t = e^{\mu t} X_0 + \int_0^t e^{\mu (t-s)} dB_s.$$ 

(4.9)

The truncated Milstein approximation for this example is the form:

$$\sup_{|x| \leq u} (|f(x)| \lor |g(x)|) \leq \frac{1}{2} u, \quad \forall u \geq 1$$

$$\mu^{-1}(u) = 2u, h(\Delta) = \Delta^{\frac{3}{2}}, \epsilon = \frac{1}{2}$$
\[ f_\Delta(x) = f\left( \left\lfloor \frac{x}{2\Delta^{\frac{1}{2}}} \right\rfloor \frac{x}{|x|} \right), \]
\[ g_\Delta(x) = g\left( \left\lfloor \frac{x}{2\Delta^{\frac{1}{2}}} \right\rfloor \frac{x}{|x|} \right). \]

Since the analytical solution has not the close form, we use \( E(X_t) \) instead of \( X(X) \) for numerical simulation. Figures (4), (5) and (6) shows the comparison between numerical simulations with truncated EM and truncated Milstein schemes with 100, 1000 and 10000 sample paths. Tables 3 and 4 shows the amounts of MSE with truncated EM and truncated Milstein schemes with various step size \( \Delta t \) for this example.

**Example 4.3.** Consider the nonlinear SDE:
\[ dX_t = aX_t(b - X_t^2)dt + cX_tdB_t. \] (4.10)

That is a famous SDE in financial mathematics.

In order to earn the truncated Milstein approximation for this example, let us consider:
\[ a = 0.1, b = 1, c = 0.2, \mu(u) = 0.2u^2, h(\Delta) = \Delta \left( \frac{\varepsilon}{2} \right), \varepsilon = 0.5 \]
\[ \mu^{-1}(h(\Delta)) = \mu^{-1}(\Delta^{\frac{\varepsilon}{2}}) = \mu^{-1}(\Delta^{\frac{1}{2}}) = \sqrt{3}\Delta^{\frac{1}{2}} \]
\[ f_\Delta(x) = f\left( \left\lfloor \frac{x}{\sqrt{3}\Delta^{\frac{1}{2}}} \right\rfloor \frac{x}{|x|} \right), \]
\[ g_\Delta(x) = g\left( \left\lfloor \frac{x}{\sqrt{3}\Delta^{\frac{1}{2}}} \right\rfloor \frac{x}{|x|} \right). \]

It was proved in [11] that the truncated EM method applied to the SDE (4.10) has convergence. It is therefore sufficient to compare our new truncated Milstein method with this model. Figures (7), (8) and (9) shows the computer simulation of the sample paths of \( X(t) \) respectively with \( 10^4, 10^5 \) and \( 10^6 \) trajectories. We observe that the new method has a good efficiency. Since this example has not analytical solution, then the mean square differences between truncated EM and truncated Milstein are shown in Table 5 for various \( \Delta t \).

![Figure 1: Comparison exact solution versus truncated Em and truncated Milstein approximations with 1000 sample path.](image-url)
Figure 2: Comparison exact solution versus truncated Euler and truncated Milstein approximations with 10000 sample path

Figure 3: Comparison exact solution versus truncated Euler and truncated Milstein approximations with 100000 sample path

Figure 4: Comparison exact solution versus truncated Euler and truncated Milstein approximations with 100 sample path
Figure 5: Comparison exact solution versus truncated Em and truncated Milstein approximations with 1000 sample path

Figure 6: Comparison exact solution versus truncated Em and truncated Milstein approximations with 10000 sample path

Figure 7: Comparison exact solution versus truncated Em and truncated Milstein approximations with $10^4$ sample path
Figure 8: Comparison exact solution versus truncated Em and truncated Milstein approximations with $10^5$ sample path.

Figure 9: Comparison exact solution versus truncated Em and truncated Milstein approximations with $10^6$ sample path.
| $\Delta t$ | MSE with 100 sample paths | MSE with 1000 sample paths | MSE with 10000 sample paths |
|----------|--------------------------|---------------------------|-----------------------------|
| 0.1      | 0.0438                   | 0.0275                    | 0.0048                      |
| 0.01     | 0.1392                   | 0.0073                    | 9.5075e-04                  |
| 0.001    | 0.0839                   | 0.0415                    | 0.0079                      |
| 0.0001   | 0.0824                   | 0.0078                    | 9.4438e-04                  |

Table 1: Values of mean square error (MSE) using truncated EM approximation for example (1).

| $\Delta t$ | MSE with 100 sample paths | MSE with 1000 sample paths | MSE with 10000 sample paths |
|----------|--------------------------|---------------------------|-----------------------------|
| 0.1      | 0.0365                   | 0.0100                    | 0.0048                      |
| 0.01     | 0.0515                   | 0.0065                    | 9.5075e-04                  |
| 0.001    | 0.0415                   | 0.0062                    | 4.8360e-04                  |
| 0.0001   | 0.0650                   | 0.0034                    | 9.4438e-04                  |

Table 2: Values of mean square error (MSE) using truncated Milstein approximation for example (1).

| $\Delta t$ | MSE with 100 sample paths | MSE with 1000 sample paths | MSE with 10000 sample paths |
|----------|--------------------------|---------------------------|-----------------------------|
| 0.1      | 0.0394                   | 0.0115                    | 0.0048                      |
| 0.01     | 0.0161                   | 8.8949e-04                | 6.5799e-04                  |
| 0.001    | 0.0342                   | 0.0026                    | 2.8768e-04                  |
| 0.0001   | 0.0035                   | 1.6213e-04                | 1.2316e-04                  |

Table 3: Values of mean square error (MSE) using truncated EM approximation for example (2).

| $\Delta t$ | MSE with 100 sample paths | MSE with 1000 sample paths | MSE with 10000 sample paths |
|----------|--------------------------|---------------------------|-----------------------------|
| 0.1      | 0.0083                   | 0.0017                    | 0.0034                      |
| 0.01     | 0.0087                   | 7.5227e-04                | 3.4404e-04                  |
| 0.001    | 0.0031                   | 1.5584e-04                | 2.4794e-04                  |
| 0.0001   | 0.0011                   | 3.3372e-04                | 1.1117e-04                  |

Table 4: Values of mean square error (MSE) using truncated Milstein approximation for example (2).

| $\Delta t$ | MSE with 100 sample paths | MSE with 1000 sample paths | MSE with 10000 sample paths |
|----------|--------------------------|---------------------------|-----------------------------|
| 0.1      | 5.9927e-05               | 1.7230e-05                | 6.0656e-07                  |
| 0.01     | 4.8153e-05               | 3.7470e-05                | 6.8611e-07                  |
| 0.001    | 2.0487e-05               | 1.1700e-05                | 2.030e-07                   |
| 0.0001   | 1.0524e-05               | 6.3219e-06                | 3.2510e-07                  |

Table 5: Values of mean square error (MSE) for example (3).

5 Conclusions

In this paper, we have constructed a new method, called the truncated Milstein scheme, for the approximation of nonlinear stochastic differential equations. At first, we define the discrete time truncated functions and then using
the Milstein method, we define the truncated Milstein method. Numerical simulations have successfully shown the convergence of the proposed model in comparison with the EM and truncated EM methods. Obtaining the order of strong convergence for the truncated Milstein method under the local Lipschitz condition and Khasminskii condition is the future goal of this paper.

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