The local dynamics of streamlines in turbulence

Philip Schaefer, Markus Gampert and Norbert Peters

Institute for Combustion Technology, RWTH Aachen
Templergraben 64, 52056 Aachen, Germany
E-mail: pschaefer@itv.rwth-aachen.de

Abstract. Based on a coordinate transformation, similar to the one used in the flamelet approach in non-premixed combustion, we introduce a coordinate locally tangent to streamlines and transform the Navier-Stokes equations to an equation for the variation of the absolute value of the velocity \( u_i \). Different from previous approaches, the unsteady term splits into two terms, one of which contains all information about the time- and space-dependence of the streamline coordinate. Based on the DNS of homogeneous isotropic decaying turbulence at a Taylor based Reynolds number of \( Re_\lambda = 71 \) we first discuss the temporal variation of the streamline coordinate field, reflecting non-local fast and slow changes of the latter. Then, based on the transformed equation we discuss the balance of the different terms along streamlines and discern terms of different orders of magnitude.

1. Introduction

In the present work we pick up the idea first presented by Wang (2010) who introduced the concept of streamlines and streamline segments in turbulent flows. While other works on streamlines mainly focus on their geometrical aspects, cf. Rao (1978), Yannacopoulos et al. (2002), Braun et al. (2006) and Brons (2007), Wang derives an equation for the variation of the absolute value \( u \) along streamlines, based on the Navier-Stokes equations. We show, that the introduction of a formal coordinate transformation yields the same equation as the one presented by Wang. The new coordinate is locally aligned with the streamline and thus is space and time dependent which yields a formal split up of the unsteady term and introduces the dynamical properties of the new coordinate into the equation. This equation can then be interpreted as a quasi one-dimensional equation along streamlines. In chapter 2 we perform the formal change of the coordinate system and derive the equation along streamlines. In chapter 3 we present our DNS, followed by a discussion of the dynamics of the new coordinate in chapter 4. In chapter 5, we analyse the balance of the equation along streamlines based on the DNS data before we conclude the paper in chapter 6.

2. Analysis

We start from the Navier-Stokes equations for constant density flows,

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2},
\]

(1)
where \( u_i \) denotes the velocity field, \( p \) the pressure, \( \nu \) the constant viscosity and summation over repeated indices, according to Einstein’s summation rule is implied. Wang (2010) introduced the vector

\[
t_i = \frac{u_i}{u},
\]

locally tangential to the streamlines, where \( u = (u_i u_i)^{1/2} \) is the absolute value of the velocity. Note that \( u \) is related to the turbulent kinetic energy \( k = 1/2 u^2 \). Introducing (2) into the Navier-Stokes equations, the following, local equation for the variation of \( u \) can be derived,

\[
\frac{\partial u}{\partial t} + u_i \frac{\partial u}{\partial x_i} = -t_i \frac{\partial p}{\partial x_i} + \frac{1}{u} [\chi - \varepsilon] + \nu \frac{\partial^2 u}{\partial x_j^2}.
\]

In (3) the instantaneous scalar dissipation rate and dissipation are defined as

\[
\chi = \nu (\partial u/\partial x_i)^2, \\
\varepsilon = \nu (\partial u_i/\partial x_j)^2.
\]

Wang (2010) introduced the curvilinear coordinate, \( s \), along streamlines as

\[
\frac{\partial}{\partial s} = t_i \frac{\partial}{\partial x_i},
\]

yielding

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial s} = - \frac{\partial p}{\partial s} + \frac{1}{u} [\chi - \varepsilon] + \nu \frac{\partial^2 u}{\partial x_j^2}.
\]

One could be let to the conclusion at this point, that (6) can be interpreted as an equation for the temporal evolution of \( u \) along the coordinate \( s \). This however is not true, as no formale change of the coordinate system has been performed, which becomes especially evident for the unsteady term, as it has to be interpreted as a partial derivative with respect to the time \( t \) at a fixed spatial position \( x_i \). We will show however, that the introduction of a new coordinate system will allow us to recover (6) but with a new term that reflects the space and time dependent character of a coordinate locally attached to a streamline. To this end we define the new coordinate system

\[
x_1 \rightarrow S_1(x_i, t), \\
x_2 \rightarrow S_2(x_i, t), \\
x_3 \rightarrow S_3(x_i, t), \\
t \rightarrow \tau.
\]

From (7) we deduce the gradient operator in the new coordinate system

\[
\vec{\nabla} = \vec{e}_1 h_1 \frac{\partial}{\partial S_1} + \vec{e}_2 h_2 \frac{\partial}{\partial S_2} + \vec{e}_3 h_3 \frac{\partial}{\partial S_3},
\]

where \( e_i \) are the unit vectors pointing in directions \( S_i \) and \( h_i \) are the Lame coefficients,
Figure 1. Coordinate system locally aligned with a streamline

\[ h_1 = \frac{\partial S_1}{\partial s_1}, \quad h_2 = \frac{\partial S_2}{\partial s_2}, \quad h_3 = \frac{\partial S_3}{\partial s_3}. \]  

(9)

\( s_i \) denotes the physical dimensions in \( S_i \) direction. We choose \( \vec{e}_1 = \vec{t} \) and \( \vec{e}_2 \cdot \vec{t} = 0, \vec{e}_3 \cdot \vec{t} = 0 \), so that the system \( (\vec{t}, \vec{e}_2, \vec{e}_3) \) forms an orthogonal basis in which \( \vec{e}_1 \) points locally in direction of the streamline, cf. figure 1.

We now need to specify the properties of the new coordinates, especially \( S_1 \). We set

\[ t_i \frac{\partial S_1}{\partial x_i} = 1. \]  

(10)

Replacing the gradient operator in (10) with the one in (8) yields for the Lame coefficient \( h_1 \)

\[ h_1 = \frac{\partial S_1}{\partial s_1} = 1. \]  

(11)

From (11) we deduce, that \( S_1 \) increases linearly (with unity slope) along the physical coordinate \( s_1 \), meaning that \( S_1 \) corresponds to the arclength along a streamline. Introducing (11) into (8) and projecting the gradient along the \( t_i \) direction we find

\[ t_i \frac{\partial}{\partial x_i} \rightarrow \frac{\partial}{\partial S_1}. \]  

(12)

On the other hand the time derivative becomes,

\[ \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \tau} + \frac{\partial S_1}{\partial t} \frac{\partial}{\partial S_1} + \frac{\partial S_2}{\partial t} \frac{\partial}{\partial S_2} + \frac{\partial S_3}{\partial t} \frac{\partial}{\partial S_3}. \]  

(13)

Assuming the gradients of \( u \) in directions perpendicular to the streamline \( (S_2 \text{ and } S_3) \) to be small as compared to those in streamline direction \( (S_1) \), we neglect the last two terms on
the r.h.s. of (13) so that going back to (3), we obtain the same equation as has already been obtained by Wang (2010), but with the unsteady term split up into two terms

\[
\frac{\partial u}{\partial \tau} + \frac{\partial S_1}{\partial t} \frac{\partial u}{\partial S_1} + u \frac{\partial u}{\partial S_1} = -\frac{\partial p}{\partial S_1} + \frac{1}{u} [\chi - \varepsilon] + \nu \frac{\partial^2 u}{\partial x_j^2}.
\]  

(14)

The coordinate \(S_1\) is thus identical to the coordinate \(s\) originally used by Wang. Note that with (7) we could also transform the Laplacian \(\partial^2/\partial x_j^2\) in the viscous term, however in the present study we refrain from such an attempt and leave the term unclosed (in \(S_1\) as it will turn out to be small in the local balance.

In (14) the first term on the l.h.s. represents the change of \(u\) at a fixed value of \(S_1\), while the second term proportional to the time rate of change of \(S_1\) represents the time- and space dependence of the new coordinate system attached to a streamline.

3. Numerics

We will analyse the balance of (14) along streamlines based on the direct numerical simulation (DNS) of homogeneous isotropic decaying turbulence. The Navier-Stokes equations were solved numerically in a cubic box of size \(2\pi\) with periodic boundary conditions employing a pseudo-spectral method in space and a second-order Adam-Bashforth method in time. Aliasing errors are eliminated by isotropic truncation. Computations were performed on an IBM BlueGene/P machine at the research center Jülich and the calculation used 16,384 processors. The initial velocity field is random and isotropic and is generated so that it satisfies a prescribed energy spectrum. The initial energy spectrum is taken from Mansour & Wray (1993) and has the form

\[
E(\kappa) = \frac{3}{2A} \frac{\kappa^\sigma}{\kappa_p^{\sigma+1}} \exp\left(-\frac{\sigma}{2} \left(\frac{\kappa}{\kappa_p}\right)^2\right),
\]  

(15)

where

\[
A = \int_0^\infty \kappa^\sigma \exp(-\sigma \kappa^2/2)d\kappa.
\]  

(16)

The constant \(\kappa_p\) is the wave number at which \(E(\kappa)\) has its maximum and is set to \(\kappa_p = 10\). We use \(\sigma = 2\) in eq. (15). The Reynolds number is \(Re_\lambda = 71\). An overview of the most relevant parameters of the DNS is given in table 1.

Streamlines are identified within the flow fields from a given initial point in ascending and descending velocity direction using a third order Runge Kutta scheme for the spatial integration and a linear interpolation scheme in space. The different terms in (14) are treated as scalar fields in the cartesian grid, which were calculated during the DNS calculation and then interpolated.
along the streamline. Special care was taken when calculating spatial derivatives of the absolute value \( u \), which a-priori cannot be transformed to Fourier space. In order to calculate the derivatives in Fourier space however, which allows the highest possible accuracy, all terms were rewritten in such a way, that only derivatives with respect to components of the velocity field \( u_i \), appear.

4. Dynamics of the \( S_1 \)-field

(10) is a first order steady partial differential field equation for the unknown \( S_1 \) field. In order to obtain the field \( S_1 \) for a given velocity field \( u_i \) from (10) however, appropriate boundary conditions need to be defined. (10) can be interpreted as a pure convection equation, where the tangent vector to the streamline \( t_i \) plays the role of the convecting velocity. As a diffusive term is absent, information is only transported in \( t_i \) direction, meaning along a single streamline. Thus setting the boundary condition can be interpreted as setting the origin for the measure field of the arclength distance along a streamline. Wang (2010) introduced the concept of streamline segments, which are finite segments along a streamline, whose boundaries are defined by a pair of adjacent local minima and maxima of \( u \) along the streamline, thus points where \( \partial u/\partial s = 0 \). Interpreting \( \partial u/\partial s \) as a scalar field in the cartesian coordinate system, \( \partial u/\partial s = 0 \) defines an iso-surface, whose coordinates will be called \( x_i,0 \). (10) counts the \( S_1 \) field positive in \( t_i \) direction and at points where the streamline intersects the iso-surface, we will denote the "entrance" side of the surface with + and the "exit" side of the surface" with −, so that the following boundary condition applies

\[
S_1(x^+_i,0) = 0. \tag{17}
\]

The value of \( S_1 \) at a given point then corresponds to the arclength distance of that point along the streamline passing through the point to the beginning of the corresponding segment.

However, different from for example the flamelet approach in non-premixed combustion, where the mass fraction \( Z \) plays a similar role as the \( S_1 \) field does in the present study, see Peters (2000), a transport equation which explicitly contains \( \partial S_1/\partial t \) is missing. The time rate of change can thus only be obtained numerically from two consecutive time steps. We approximate the temporal derivative as

\[
\frac{\partial S_1}{\partial t} \approx \frac{S_1(t + \Delta t) - S_1(t)}{\Delta t}, \tag{18}
\]

where we use the DNS data fields from two consecutive time steps. The \( S_1 \) field could be obtained from (10) by numerically solving the field equation with the above described boundary condition. However, it is difficult to enforce the boundary condition on the implicitly defined iso-surface \( \partial u/\partial s = 0 \). In addition, any discretisation of the partial derivatives would introduce numerical diffusion, which would lead to a non physical transport of information across streamlines. Also, calculating the \( S_1 \) field in the entire box and then interpolating it on the streamline leads to non-negligible errors during the interpolation, as one inherent feature of the field is, that it is not necessarily steady in space. Thus any interpolation of the field would lead to erranous results.

Instead, as we are only interested in the \( S_1 \) field along the streamline, we use a direct approach, cf. figure 2. A streamline (bold line) is identified at time \( t \) which passes through the iso-surface at that time (thin line), thus defining the boundaries of the streamline segments at the points of intersection. \( S_1 \) at time \( t \) is simply calculated as the arclength along the streamline, periodically
set to zero at the boundaries of segments. To obtain \( S_1 \) at \( t + \Delta t \) however, we start new streamlines (bold dotted lines) beginning on the old streamline based on the velocity field at \( t + \Delta t \). These are traced back (in the opposite direction of the velocity field) until they hit the new iso-surface (thin dotted line). Their arclength is then the value of \( S_1 \) at time \( t + \Delta t \) on their starting point on the old streamline. As can already be inferred from the illustration, the temporal evolution of the \( S_1 \) field is complicated and subject to different physical effects. First, the iso-surface will move in space, thus taking on a different shape as well as position on the streamline. Second, points that lie on the streamline at instance \( t \) do not lie on one single streamline anymore at \( t + \Delta t \). This together with the motion of the boundary yields a non-local rate of change of the \( S_1 \) field in time. Especially, we expect two different types of changes to act on the \( S_1 \) field, namely slow and fast changes. Slow changes occur, when locally the directional field changes little and the shape and position of the iso-surfaces also only changes little. In this case, one can say that the streamline segment, although it does not possess a lagrangian identity, still "exists" at \( t + \Delta t \). However, if the streamline segment "ceases" to exist, meaning that either the new streamlines will reach a completely different position on the iso-surface, or that the iso-surface has changed its shape in such a dramatic form, that new intersections with the streamline are formed, then a fast change in \( S_1 \) will occur. Note, that these fast changes are not an artefact of the finite time step \( \Delta t \), but rather an inherent feature of the \( S_1 \) field and its connection to the iso-surface. We expect however, that such fast changes only affect limited, small parts of the flow field and for the purpose of this paper, we only analyze the slow changes. Special attention has to be paid to the temporal separation \( \Delta t \), which if it is chosen too small, will yield a large degree of numerical errors in the calculation of (18), while if it is chosen too large, too many changes will have taken place so that the physical effects on \( S_1 \) will be obscured. A good compromise of the above constraints has been attained for \( \Delta t \approx 0.2 t_\eta \), with the Kolmorogov time \( t_\eta = (\nu/\varepsilon)^{1/2} \).

Figure 3 shows the variation of \( u \) along a representative streamline identified in the flow field. Starting from the beginning of every element, the \( S_1 \) field is displayed at \( t \) and \( t + \Delta t \). From the two times the term \( \partial S_1 / \partial t \) follows based on (18). As can be observed this term is discontinous and only defined within one segment. In addition, the term is negative most of the time and within one element of the same order of magnitude as the absolute value of \( u \). Figure 3 illustrates only the slow changes which act on the \( S_1 \) field: the negative value of \( \partial S_1 / \partial t \) is due to the motion of the entrance boundary defined by the iso-surface. It’s motion is complicated, though dominated by convection. This explains, why \( O(\partial S_1 / \partial t) = -O(u) \), as the convection along the streamline is always positive. In addition, the non-local change of the directional field \( t_1 \) as well as the change in shape of the iso-surface lead to a superimposed fluctuation.

5. Analysis of the balance along streamlines

Based on the above made analysis we will in the following examine the different terms of (14) to discern balances of different orders in the local dynamical equation. Figure 4 shows the split of the unsteady term as derived in (13), where the first term on the r.h.s. has been calculated from the balance of the three terms and the term on the l.h.s. has been calculated from the DNS using a finite difference scheme and two consecutive time steps. The vertical dotted lines indicate the position of the streamline segments. As can be observed, different from figure 3, the new term proportional to the temporal rate of change of the \( S_1 \) field is not discontinous any more, but a steady term. Discontinuities arising at the boundaries of segments are eliminated when multiplied by \( \partial u / \partial S_1 \) as this term by definition vanishes at the boundaries, which are local extreme points of \( u \) along the streamline. In addition, as can be observed all terms stay of the same order of magnitude.

Figure 5 shows the full balance of all terms in (14). As expected, all terms proportional to
the viscosity $\nu$ turn out to be small as compared to the other terms in the local instantaneous balance.
Figure 4. Split of the unsteady term (13)

Figure 5. Balance of all terms in (14)

Figure 6 shows the leading order balance along a representative streamline from the DNS. As can be observed, the two convective terms show an opposite pattern and seem to follow each other (with opposite signs) closely, while the unsteady term rather follows the remaining pressure gradient term. The streamline segments are implied by the zero crossings of the convective term $u \partial u/\partial S_1 = 0$. In positive segments, i.e. where the convective term is positive, fluid particles are accelerated, while in negative segments the opposite is the case. As can be observed, the pressure gradient is most of the time positive in negative segments, while it is negative in positive ones.
so that it can be assumed responsible for the acceleration, i.e. decceleration of fluid particles along streamlines.

![Figure 6. Balance of the leading order terms in (14)](image)

6. Conclusion

The introduction of a new coordinate system locally aligned with the direction of streamlines in turbulent flow fields has yielded an equation for the variation of the absolute value of the velocity field $u$ along streamlines, which contains, different from previous work, a new term that accounts for the temporal and spatial dependence of the new coordinate attached to streamlines. This coordinate is subject to fast and slow changes in time, of which only the latter were analysed in the course of the present work. As streamlines are segmented based on local extreme points of $u$, the dynamics of $S_1$ reflect the dynamics of moving boundaries as well as the dynamics of the underlying directional field $t_i$ which at every instance determines the streamline patterns. While the latter turns out to be rather small, the first effect introduces a term which, to some extend, counteracts the regular convective term in the equation. While the term $\partial S_1/\partial t$ is discontinuous at the boundaries of segments, the new term in the balance equation stays continuous. Based on the balance of the different terms, we conclude that locally all terms proportional to the viscosity $\nu$ are of lower order, while the leading order balance only contains the unsteady, convective and pressure gradient term. As the two convective terms tend to follow each other rather closely (with opposite signs), the unsteady term balances the pressure gradient term, thus giving rise to the classical picture that in regions of positive pressure gradient (i.e. negative streamline segments) fluid particles are decceleration due to the opposing pressure, while in regions of negative pressure gradient (i.e. positive streamline segments) fluid particles are accelerated.

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