Vanishing Shear Viscosity Limit and Boundary Layer Study on the Planar MHD system

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Abstract

We consider an initial boundary problem for the planar MHD system under the general condition on the heat conductivity $\kappa$ that may depend on both the density $\rho$ and the temperature $\theta$ satisfying $\kappa(\rho, \theta) \geq \kappa_1 \theta^q$ for some constants $\kappa_1 > 0$ and $q > 0$. Firstly, the global existence of strong solution for large initial data is obtained, and then the limit of the vanishing shear viscosity is justified. In addition, the $L^2$ convergence rate is obtained together with the estimation on the thickness of the boundary layer.

**Keywords.** MHD system, global existence, vanishing shear viscosity, boundary layer.

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1 Introduction

The planar Magnetohydrodynamics (MHD) system with constant longitudinal magnetic field is governed by the following equations:

\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + \left(\rho u^2 + p + \frac{1}{2} |b|^2\right)_x &= (\lambda u)_x, \\
(\rho w)_t + (\rho w u - b)_x &= (\mu w)_x, \\
b_t + (ub - w)_x &= (\nu b)_x, \\
(\rho e)_t + (\rho u e - \kappa e)_x + pu_x &= \lambda u^2 + \mu |w|^2 + \nu |b|^2.
\end{aligned}
\]  

(1.1)

Here $\rho$ denotes the density, $\theta$ the temperature, $u \in \mathbb{R}$ the longitudinal velocity, $w = (w_1, w_2) \in \mathbb{R}^2$ the transverse velocity, $b = (b_1, b_2) \in \mathbb{R}^2$ the transverse magnetic field, $p = p(\rho, \theta)$ the pressure, $e = e(\rho, \theta)$ the internal energy, and $\kappa = \kappa(\rho, \theta)$ the heat conductivity respectively. The coefficients $\lambda, \mu$ and $\nu$ are assumed to be positive constants, where $\lambda$
and $\mu$ are the viscosity coefficients, and $\nu$ is the magnetic diffusivity. And the state equations are
\[ p = \gamma \rho \theta, \quad e = c_v \theta, \] (1.2)
with constants $\gamma > 0$ and $c_v > 0$. Without loss of generality, set $c_v = 1$. Based on some physical models in which $\kappa$ grows like $\theta^q$, for example, $q \in [4.5, 5.5]$ for molecular diffusion in gas (see [28]), we assume that $\kappa = \kappa(\rho, \theta)$ is twice differential in $\mathbb{R}^+ \times \mathbb{R}^+$ and satisfies
\[ \kappa(\rho, \theta) \geq \kappa_1 \theta^q \quad \text{with constants } \kappa_1 > 0 \text{ and } q > 0. \] (1.3)

In this paper, we consider system (1.1) in a bounded domain $Q_T = \Omega \times (0, T)$ with \( \Omega = (0, 1) \) under the following initial and boundary conditions:
\[
\begin{aligned}
&\{ (\rho, u, \theta, w, b)(x, 0) = (\rho_0, u_0, \theta_0, w_0, b_0)(x), \\
&\{ (u, b, \theta_x)|_{x=0,1} = 0, \quad w(0, t) = w^-(t), \quad w(1, t) = w^+(t). 
\end{aligned}
\] (1.4)

We aim to study the global existence, vanishing shear viscosity limit, convergence rate and boundary layer effect of solutions to problem (1.1)-(1.4) with large initial data under the condition (1.3).

Because of its physical importance and mathematical challenge, the MHD system has been extensively studied, see [1–4, 10, 12, 15, 17, 22, 23] and the references therein. Without magnetic effect, MHD system is reduced to the compressible Navier-Stokes equations that are better understood mathematically. For example, in one space dimension, there is a seminal work by Kazhikhov and Shelukhin [14] on the global existence of strong solutions for the compressible Navier-Stokes equations with constant coefficients and large initial data. However, the corresponding result for the MHD system with constant coefficients remains well known unsolved problem.

On the other hand, as for the well-posedness theory of MHD, Vol’pert and Hudjaev [23] firstly proved the existence and uniqueness of local smooth solutions, and then the global existence of smooth solution with small initial data was established in [12]. In addition, under the following condition on $\kappa$:
\[ C^{-1}(1 + \theta^q) \leq \kappa(\rho, \theta) \leq C(1 + \theta^q), \quad q > 0, \] (1.5)
the existence of global solution to the problem (1.1)-(1.4) with large initial data was studied in [22] for $q \geq 2$, [5] for $q \geq 1$, and [6,9] for $q > 0$. In fact, the condition like (1.5) was also used in other papers, cf. [3,5,6,11,13,22] and references therein. In this paper, we will firstly show the global existence of strong solution to the problem (1.1)-(1.4) under the condition (1.3).

The problem of vanishing viscosity has been an interesting and challenging problem in many setting, in particular with boundary, for example, in the boundary layer theory (cf. [20]). And there are many mathematical results on this problem, cf. [7,8,11,19,21,23] for the work on Navier-Stokes equations, and [5,6] for the problem (1.1)-(1.4). As a second result of this paper, we will justify such limit in term of vanishing of shear viscosity and describe the convergence of $w$ and $b$ under the condition (1.3).

Now we briefly review some related works on the boundary layer theory that is one of the fundamental problems in fluid dynamics established by Prandtl in 1904. Without the magnetic effect, Frid and Shelukhin [8] investigated the boundary layer effect of the compressible isentropic Navier-Stokes equations with cylindrical symmetry, and proved the
existence of boundary layers thickness in the order of $O(\mu^\alpha)(0 < \alpha < 1/2)$. Recently, this result was investigated in a more general setting, cf. \cite{11,19} for the non-isentropic case and \cite{25} for the case with density-dependent viscosity. With the magnetic field, the authors in \cite{24} studied the problem on boundary layer for the isentropic planar MHD system with the constant initial data and obtained the same thickness of boundary layer estimate as in \cite{8,11,19}. In this paper, we extend this result to problem (1.1)-(1.4) in general setting by introducing some new analytic technique in obtaining the second derivative of the velocity field by using theories for linear parabolic equations.

In the following, some notation will be used. Firstly, denote $Q_t = \Omega \times (0, t)$ for $t \in (0, T]$. For integer $k \geq 0$, constant $p \geq 1$ and $O \subset \mathbb{R}^n$, $W^{k,p}(O)$ denote the usual Sobolev spaces. $L^p(I, B)$ is the space of all strong measurable, $p$-th-power integrable (essentially bounded if $p = \infty$) functions from $I$ to $B$, where $I \subset \mathbb{R}$ and $B$ is a Banach space. For simplicity, we also use the notation $\|\cdot\|_B$ for $f, g, \cdots$ belonging to $B$ equipped with a norm $\|\cdot\|_B$.

The initial and boundary functions are assumed to satisfy

\[
\begin{cases}
\rho_0 > 0, \; \theta_0 > 0, \; \| (\rho_0^{-1}, \theta_0^{-1}) \|_{C(\overline{\Omega})} < \infty, \; \| (w^-, w^+) \|_{C^1[0,T]} < \infty, \\
(\rho_0, w_0, \theta_0) \in W^{1,2}(\Omega), \; (b_0) \in W^{1,2}(\Omega), \; u_0 \in W^{1,2}_0(\Omega) \cap W^{2,m}(\Omega), \; m \in (1, +\infty), \\
u_0(1) = u_0(0) = 0, \; w_0(0) = w^-(0), \; b_0(1) = w^+(0).
\end{cases}
\]

Then the first result of this paper can be stated as follows.

**Theorem 1.1.** Let (1.3) and (1.6) hold. Then

(i) For any fixed $\mu > 0$, problem (1.1)-(1.4) admits a unique strong solution $(\rho, u, w, b, \theta)$ satisfying

\[
\inf_{Q_T} \rho > 0, \; \inf_{Q_T} \theta > 0, \; (\rho, u, w, b, \theta) \in L^\infty(Q_T), \; \rho \in L^\infty(0, T; W^{1,2}(\Omega)), \; \rho_t \in L^2(Q_T), \\
(u, w, b, \theta) \in L^\infty(0, T; W^{1,2}(\Omega)) \cap L^2(0, T; W^{2,2}(\Omega)), \; (u_t, w_t, b_t, \theta_t) \in L^2(Q_T).
\]

Moreover, there exists a positive constant $C$ independent of $\mu$ such that

\[
\begin{aligned}
C^{-1} \leq \rho, \theta \leq C, \; \| (u, w, b) \|_{L^\infty(Q_T)} \leq C, \\
\| (\rho_t, \rho_x, u_x, b_x, \theta_x) \|_{L^\infty(0,T;L^2(\Omega))} + \| (u_t, b_t, \theta_t, u_{xx}, \theta_{xx}) \|_{L^2(Q_T)} \leq C, \\
\| w \|_{L^\infty(0,T;L^1(\Omega))} + \| w_t \|_{L^2(Q_T)} \leq C, \\
\| \mu^{1/4} w \|_{L^\infty(0,T;L^2(\Omega))} + \| \mu^{3/4} w_{xx} \|_{L^2(Q_T)} \leq C, \\
\| \sqrt{\omega} w_x \|_{L^\infty(0,T;L^2(\Omega))} + \| \sqrt{\omega} b_{xx} \|_{L^2(Q_T)} \leq C,
\end{aligned}
\]

where $\omega : [0, 1] \rightarrow [0, 1]$ is defined by

\[
\omega(x) = \begin{cases}
x, & 0 \leq x \leq 1/2, \\
1 - x, & 1/2 \leq x \leq 1.
\end{cases}
\]
(ii) There exist functions \((\overline{\mathbf{r}}, \overline{\mathbf{u}}, \overline{\mathbf{b}}, \overline{\theta})\) in the family \(F\) defined by
\[
F : \begin{cases}
\overline{\mathbf{r}} > 0, \quad (\overline{\mathbf{r}}, \overline{\theta})_{t=0} = 0, \\
(\overline{\mathbf{r}}, 1/\overline{\mathbf{r}}, \overline{\mathbf{u}}, \overline{\mathbf{b}}, \overline{\theta}, 1/\overline{\theta}) \in L^\infty(Q_T), \quad \overline{\mathbf{w}} \in L^\infty(0, T; W^{1,1}(\Omega)), \\
(\overline{\mathbf{r}}_t, \overline{\mathbf{r}}_x, \overline{\mathbf{u}}_x, \overline{\mathbf{b}}_x, \overline{\theta}_x) \in L^\infty(0, T; L^2(\Omega)), \quad (\overline{\mathbf{u}}_x, \overline{\mathbf{b}}_x, \overline{\theta}_x) \in L^2(0, T; L^\infty(\Omega)), \\
\sqrt{\omega \overline{\mathbf{w}}_x} \in L^\infty(0, T; L^2(\Omega)), \quad \sqrt{\omega \overline{\mathbf{b}}_{xx}} \in L^2(Q_T),
\end{cases}
\]
such that as \(\mu \to 0\)
\[
(\rho, u, b, \theta) \to (\overline{\mathbf{r}}, \overline{\mathbf{u}}, \overline{\mathbf{b}}, \overline{\theta}) \quad \text{in} \quad C^\alpha(Q_T), \quad \forall \alpha \in (0, 1/4),
\]
\[
(u_x, b_x, \theta_x) \to (\overline{\mathbf{u}}_x, \overline{\mathbf{b}}_x, \overline{\theta}_x) \quad \text{strongly in} \quad L^2(Q_T),
\]
\[
(\rho_t, \rho_x) \to (\overline{\mathbf{r}}_t, \overline{\mathbf{r}}_x) \quad \text{weakly} \quad \ast \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)),
\]
\[
(u_t, b_t, \theta_t, u_{xx}, \theta_{xx}) \to (\overline{\mathbf{u}}_t, \overline{\mathbf{b}}_t, \overline{\theta}_t, \overline{\mathbf{u}}_{xx}, \overline{\theta}_{xx}) \quad \text{weakly} \quad \ast \quad \text{in} \quad L^2(Q_T),
\]
\[
b_{xx} \to \overline{\mathbf{b}}_{xx} \quad \text{weakly in} \quad L^2((\delta, 1-\delta) \times (0, T)), \quad \forall \delta \in (0, 1/2),
\]
and
\[
\mathbf{w} \to \overline{\mathbf{w}} \quad \text{in} \quad C^\alpha([\delta, 1-\delta] \times [0, T]), \quad \forall \delta \in (0, 1/2), \quad \alpha \in (0, 1/4),
\]
\[
\mathbf{w}_t \to \overline{\mathbf{w}}_t \quad \text{weakly in} \quad L^2(Q_T),
\]
\[
\mathbf{w}_x \to \overline{\mathbf{w}}_x \quad \text{weakly} \quad \ast \quad \text{in} \quad L^\infty(0, T; L^2(\delta, 1-\delta)), \quad \forall \delta \in (0, 1/2),
\]
\[
\mathbf{w} \to \overline{\mathbf{w}} \quad \text{strongly in} \quad L^r(Q_T), \quad \forall r \in [1, +\infty),
\]
\[
\sqrt{\omega} \|
\mathbf{w}_x \|
L^2(Q_T) \to 0.
\]

Moreover, \((\overline{\mathbf{r}}, \overline{\mathbf{u}}, \overline{\mathbf{b}}, \overline{\theta})\) is the unique solution of problem (1.1)-(1.4) with \(\mu = 0\) in \(F\).

(iii) Let \((\overline{\mathbf{r}}, \overline{\mathbf{u}}, \overline{\mathbf{b}}, \overline{\theta})\) be a solution for problem (1.1)-(1.4) with \(\mu = 0\). Then
\[
\|(\rho - \overline{\mathbf{r}}, u - \overline{\mathbf{u}}, \mathbf{w} - \overline{\mathbf{w}}, b - \overline{\mathbf{b}}, \theta - \overline{\theta})\|_{L^\infty(0, T; L^2(\Omega))} \\
\quad + \|(u_x - \overline{\mathbf{u}}_x, b_x - \overline{\mathbf{b}}_x, \theta_x - \overline{\theta}_x)\|_{L^2(Q_T)} = O(\mu^{1/4}).
\]

Remark 1.1. Following the argument in [22] (cf. [2]), (1.7) implies that problem (1.1)-(1.4) admits a unique classical solution if the initial data is sufficiently smooth.

We now present a sketch of the proof of (1.7). Firstly, the uniform upper and lower bounds of the density can be obtained as in the previous literatures, cf. [5]. A key observation in the paper is to obtain a uniform bound on \(\|u_{xx}\|_{L^m(0, T; L^2(\Omega))}(m_0 > 1)\) that can be obtained by the \(L^p\)-theory of linear parabolic equations (see Lemma 2.3). In fact, by using this estimate and some delicate analysis, we then deduce the key estimates of the bounds on \(\|\omega \mathbf{w}_x\|_{L^\infty(0, T; L^2(\Omega))}\) and \(\|(u_t, b_t, \mathbf{w}_t, u_{xx}, \theta_x, \omega b_{xx})\|_{L^2(Q_T)}\) (see Lemma 2.10). In this step, the difficulty caused by the coupling between the transverse velocity and transverse magnetic field is overcome. With these uniform estimates with respect to \(\mu\), the uniform bounds on \(\sqrt{\omega} \mathbf{w}_x\|_{L^\infty(0, T; L^2(\Omega))}\) and \((\mu^{1/4})\| \mathbf{w}_x\|_{L^\infty(0, T; L^2(\Omega))} + \mu^{3/4})\| \mathbf{w}_x\|_{L^2(Q_T)}\) (see Lemma 2.13) can then be obtained that are essential to the estimation on both convergence rate and boundary layer thickness. In addition, an upper bound on \(\theta\) follows (see Lemma 2.14) together with the uniform bound on \(\|(\theta_t, \theta_{xx})\|_{L^2(Q_T)}\) (see Lemma 2.15).

The next result of this paper is about the estimation on the thickness of boundary layer. For this, we first recall the definition of a BL-thickness, cf. [8], as follows
Definition 1.2. A function $\delta(\mu)$ is called a BL-thickness for problem (1.1)-(1.4) with vanishing $\mu$ if $\delta(\mu) \downarrow 0$ as $\mu \downarrow 0$, and

$$
\lim_{\mu \to 0} \|(\rho - \bar{\rho}, u - \bar{u}, w - \bar{w}, b - \bar{b}, \theta - \bar{\theta})\|_{L^\infty(0,T;L^\infty(\delta(\mu),1-\delta(\mu)))} = 0,$$

$$
\inf \lim_{\mu \to 0} \|(\rho - \bar{\rho}, u - \bar{u}, w - \bar{w}, b - \bar{b}, \theta - \bar{\theta})\|_{L^\infty(0,T;L^\infty(\Omega))} > 0,
$$

where $(\rho, u, w, b, \theta)$ and $(\bar{\rho}, \bar{u}, \bar{w}, \bar{b}, \bar{\theta})$ are the solutions to problem (1.1)-(1.4) with $\mu > 0$ and $\mu = 0$, respectively.

The second result of this paper is

Theorem 1.3. Let the assumptions in Theorem 1.1 hold. Then any function $\delta(\mu)$ satisfying $\delta(\mu) \downarrow 0$ and $\frac{\sqrt{\mu}}{\delta(\mu)} \to 0$ as $\mu \downarrow 0$ is a BL-thickness for problem (1.1)-(1.4) such that

$$
\lim_{\mu \to 0} \|(\rho - \bar{\rho}, u - \bar{u}, b - \bar{b}, \theta - \bar{\theta})\|_{C^\alpha(\bar{Q}_T)} = 0, \quad \forall \alpha \in (0,1/4),
$$

$$
\lim_{\mu \to 0} \|w - \bar{w}\|_{L^\infty(0,T;L^\infty(\delta(\mu),1-\delta(\mu)))} = 0, \quad \inf \lim_{\mu \to 0} \|w - \bar{w}\|_{L^\infty(0,T;L^\infty(\Omega))} > 0,
$$

when $(w^-(t), w^+(t)) \neq (\bar{w}(0,t), \bar{w}(1,t))$.

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The rest of this paper is organized as follows. In Section 2, we will prove Theorem 1.1.

The proof of Theorem 1.3 will be given in Section 3.

2 Proof of Theorem 1.1

The existence and uniqueness of local solution can be obtained by using the Banach theorem and the contractivity of the operator through the linearization of the system, cf. [18,22]. Then to obtain global solution, we only need to close the a priori estimates of solutions. The next subsection is about deriving the $\mu$-uniform estimates given in (1.7). From now on, we use $C$ to denote a positive generic constant independent of $\mu$.

2.1 A priori estimates independent of $\mu$

Firstly, rewrite (1.1) as

$$
\mathcal{E}_t + \left[u(\mathcal{E} + p + \frac{1}{2}|b|^2) - w \cdot b\right]_x = \left(\lambda uu_x + \mu w_x + \nu b \cdot b_x + \kappa \theta_x\right)_x,
$$

$$
(\rho \mathcal{S})_t + (\rho u \mathcal{S})_x = \left(\frac{\kappa \theta_x}{\theta}\right)_x = \frac{\lambda u^2 + \mu |w_x|^2 + \nu |b_x|^2}{\theta} + \frac{\kappa \theta_x^2}{\theta^2},
$$

where $\mathcal{E}$ and $\mathcal{S}$ are the total energy and the entropy, respectively, given by

$$
\mathcal{E} = \rho \left[\theta + \frac{1}{2}(u^2 + |w|^2)\right] + \frac{1}{2}|b|^2, \quad \mathcal{S} = \ln \theta - \gamma \ln \rho.
$$

Lemma 2.1. Under the assumptions in Theorem 1.1, we have

$$
\int_{\Omega} \rho(x,t) dx = \int_{\Omega} \rho_0(x) dx, \quad \forall t \in (0,T),
$$

$$
\sup_{0 < t < T} \int_{\Omega} \left[\rho(\theta + u^2 + |w|^2) + |b|^2\right] dx \leq C,
$$

$$
\iint_{Q_T} \left(\frac{\lambda u^2 + \mu |w_x|^2 + \nu |b_x|^2}{\theta} + \frac{\kappa \theta_x^2}{\theta^2}\right) dx dt \leq C.
$$
Proof. Integrating (2.1) over \( Q_t = \Omega \times (0, t) \) yields

\[
\int_{\Omega} \mathcal{E} \, dx = \int_{\Omega} \mathcal{E}_{|t=0} \, dx + \mu \int_0^t (\mathbf{w} \cdot \mathbf{w}_x)|_{x=0} \, ds. \tag{2.3}
\]

To estimate the final integral on the right hand side of (2.3), we first integrate (1.1) from \( x = a \) to \( x \), where \( a = 0 \) or \( 1 \), and then integrate the resulting equation over \( \Omega \) to obtain

\[
\mu \mathbf{w}_x(a, t) = \mu (\mathbf{w}^+ - \mathbf{w}^-) - \int_{\Omega} (\rho \mathbf{w} - \mathbf{b}) \, dx - \frac{\partial}{\partial t} \left( \int_{\Omega} \int_a^x \rho \mathbf{w} \, dy \, dx \right).
\]

Taking the inner product with \( \mathbf{w}(a, t) \) and integrating over \((0, t)\) yield

\[
\mu \int_0^t (\mathbf{w} \cdot \mathbf{w}_x)(a, s) \, ds = \mu \int_0^t (\mathbf{w}^+ - \mathbf{w}^-) \cdot \mathbf{w}(a, s) \, ds - \int_0^t \mathbf{w}(a, s) \cdot \left( \int_{\Omega} (\rho \mathbf{w} - \mathbf{b}) \, dx \right) \, ds
\]

\[
- \mathbf{w}(a, t) \cdot \left( \int_{\Omega} \int_a^x \rho \mathbf{w} \, dy \, dx \right) + \mathbf{w}(a, 0) \cdot \left( \int_{\Omega} \int_a^x \rho_0 \mathbf{w}_0 \, dy \, dx \right)
\]

\[
+ \int_0^t \mathbf{w}_t(a, t) \cdot \left( \int_{\Omega} \int_a^x \rho \mathbf{w} \, dy \, dx \right) \, dt.
\]

Using Young inequality and (2.2)\(_1\), we obtain

\[
\left| \mu \int_0^t (\mathbf{w} \cdot \mathbf{w}_x)(a, s) \, ds \right| \leq C + \frac{1}{2} \int_{\Omega} \mathcal{E} \, dx + C \int_{Q_t} \mathcal{E} \, dx \, ds.
\]

Substituting it into (2.3) and using Gronwall inequality, we obtain (2.2)\(_2\). (2.2)\(_3\) follows from integrating (2.1)\(_2\) and using (2.2)\(_2\). And this completes the proof of the lemma.

The following estimates can be obtained as in [5,6,9]. For the completeness of the paper, we briefly present its proof.

**Lemma 2.2.** Under the assumptions in Theorem 1.1, we have

\[
C^{-1} \leq \rho \leq C, \quad \theta \geq C,
\]

\[
\int_0^T \|\theta\|_{L^q(\Omega)}^{q+1-\alpha} \, dt + \int_{Q_T} \frac{k \theta^2}{\theta^{1+\alpha}} \, dx \, dt \leq C, \quad \forall \alpha \in (0, \min\{1, q\}),
\]

\[
\int_0^T \|\mathbf{b}\|_{L^\infty(\Omega)} \, dt + \int_{Q_T} \left( \lambda \mathbf{u}_x^2 + \mu |\mathbf{w}_x|^2 + \nu |\mathbf{b}_x|^2 \right) \, dx \, dt \leq C,
\]

\[
\int_{Q_T} |\theta_x|^{3/2} \, dx \, dt \leq C.
\]

**Proof.** We first prove that \( \rho \leq C \). Denote

\[
\phi = \int_0^t \tilde{P}(x, s) \, ds + \int_0^x \rho_0(y) u_0(y) \, dy, \quad \tilde{P} = \lambda \mathbf{u}_x - \rho \mathbf{u}^2 - \gamma \rho \theta - \frac{1}{2} |\mathbf{b}|^2.
\]

Then

\[
(\rho \mathbf{u})_t = \tilde{P}_x, \quad \phi_t = \tilde{P}, \quad \phi|_{x=0, 1} = 0, \quad \phi|_{t=0} = \int_0^x \rho_0(y) u_0(y) \, dy.
\]
By Lemma 2.11, we have that \( \| \phi_x \|_{L^\infty(Q_T,L^1(\Omega))} + | \int_\Omega \phi dx | \leq C \), thus, \( \| \phi \|_{L^\infty(Q_T,L^\infty(\Omega))} \leq C \).

From this and the fact that the function \( F := e^{\phi/\lambda} \) satisfies

\[
D_t(\rho F) := \partial_t(\rho F) + u \partial_x(\rho F) = - \frac{1}{\lambda} \left( p + \frac{1}{2} |b|^2 \right) \rho F \leq 0,
\]

it follows that \( \rho \leq C \).

It follows from (1.1) that

\[
\theta_t + u \theta_x - \frac{1}{\rho} (\kappa \theta_x)_x \geq \frac{\lambda}{\rho} \left( u_x^2 - \frac{p}{\lambda} u_x \right) = \frac{\lambda}{\rho} \left( u_x^2 - \frac{p}{2\lambda} \right)^2 - \frac{\gamma^2}{4\lambda} \rho \theta^2.
\]

By \( \rho \leq C \), we have that \( \theta_t + u \theta_x - \frac{1}{\rho} (\kappa \theta_x)_x + K \theta^2 \geq 0 \), where \( K \) is a positive constant independent of \( \mu \). Let \( z = \theta - \theta_0 \), where \( \theta = \min_{T} \theta_0 \) with \( C = K \min_{T} \theta_0 \). Then \( z_x |_{x=0,1} = 0 \), \( z |_{t=0} \geq 0 \), and

\[
z_t + u z_x = \frac{1}{\rho} (\kappa z_x)_x + K (\theta + \theta_0) z
= \theta_t + C \frac{\min_{T} \theta_0}{(Ct + 1)^2} + u \theta_x - \frac{1}{\rho} (\kappa \theta_x)_x + K \theta^2 - K \left( \frac{\min_{T} \theta_0}{Ct + 1} \right)^2 \geq 0,
\]

thus, \( z \geq 0 \) on \( Q_T \) by the Comparison Theorem that gives \( \theta \geq C \).

Multiplying (1.1) by \( \theta^{-\alpha} \) with \( \alpha \in (0, \min\{1, q\}) \) and integrating over \( Q_T \), we have

\[
\int \int_{Q_T} \frac{\lambda u_x^2}{\theta^\alpha} dx dt + \alpha \int \int_{Q_T} \frac{\kappa \theta_x^2}{\theta^1+\alpha} dx dt \leq \int \int_{Q_T} [(\rho \theta)_t + (\rho u \theta)_x + \rho u_x] \theta^{-\alpha} dx dt. \tag{2.5}
\]

From Lemma 2.11, \( \theta \geq C \) and Hölder inequality, we obtain

\[
\int \int_{Q_T} [(\rho \theta)_t + (\rho u \theta)_x] \theta^{-\alpha} dx dt = \frac{1}{1 - \alpha} \int \Omega \rho \theta^{1-\alpha} dx - \frac{1}{1 - \alpha} \int \Omega \rho \theta_0^{1-\alpha} dx \leq C,
\]

and

\[
\int \int_{Q_T} \rho u_x \theta^{-\alpha} dx dt \leq \frac{1}{2} \int \int_{Q_T} \frac{u_x^2}{\theta^\alpha} dx dt + C \int \int_{Q_T} \rho \theta^{2-\alpha} dx dt
\leq \frac{1}{2} \int \int_{Q_T} \frac{u_x^2}{\theta^\alpha} dx dt + C \int_0^T \| \theta \|_{L^\infty(\Omega)}^{1-\alpha} ds.
\]

By the embedding theorem, Young inequality and \( \theta \geq C \), we have that if \( q \geq 1 - \alpha \), then

\[
\int_0^T \| \theta \|_{L^\infty(\Omega)}^{1-\alpha} ds \leq C + C \int \int_{Q_T} | \theta^{-\alpha} \theta_x | dx ds
\]

\[
\leq C + C \int_0^T \left( \int \Omega \frac{| \theta_x |^2 \theta^{1-\alpha}}{\theta^1+\alpha} dx \right)^{1/2} dt
\]

\[
\leq C + \epsilon \int \int_{Q_T} \frac{\kappa \theta_x^2}{\theta^1+\alpha} dx dt, \quad \forall \epsilon \in (0, 1).
\]
If $0 < q < 1 - \alpha$, then
\[
\int_0^T \| \theta \|_{L^\infty(\Omega)}^{1-\alpha} dt \leq C + C \int_{Q_T} \theta^{-\alpha} \theta_x dx dt
\]
\[
\leq C + C \int_0^T \left( \int_{\Omega} \frac{\theta^q |\theta_x|^2}{\theta^{1+\alpha}} \theta^{-q} dx \right)^{1/2} dt
\]
\[
\leq C + \epsilon \int_{Q_T} \kappa \theta_x^2 \frac{1}{\theta^{1+\alpha}} dx dt + \frac{C}{\epsilon} \int_0^T \| \theta \|_{L^\infty(\Omega)}^{1-q} dt
\]
\[
\leq C(\epsilon) + \epsilon \int_{Q_T} \kappa \theta_x^2 \frac{1}{\theta^{1+\alpha}} dx dt + \frac{1}{2} \int_0^T \| \theta \|_{L^\infty(\Omega)}^{1-\alpha} dt.
\]
Substituting them into (2.5) and taking a small $\epsilon$, we obtain that $\int_{Q_T} \kappa \theta_x^2 \frac{1}{\theta^{1+\alpha}} dx dt \leq C$. As a consequence, $\int_0^T \| \theta \|^{q+1-\alpha}_{L^\infty(\Omega)} dt \leq C$.

Integrating (1.1) over $Q_T$ and using Lemma 2.1, we have
\[
\int_{Q_T} \left( \lambda u_x^2 + \mu |w_x|^2 + \nu |b_x|^2 \right) dx dt = \frac{1}{2} \int_{\Omega} \rho \theta dx - \frac{1}{2} \int_{\Omega} \rho_0 \theta dx + \int_{Q_T} pu_x dx dt
\]
\[
\leq C + \frac{\lambda}{2} \int_{Q_T} u_x^2 dx dt + C \int_0^T \| \theta \|_{L^\infty(\Omega)} \int_{\Omega} \rho \theta dx dt
\]
\[
\leq C + \frac{\lambda}{2} \int_{Q_T} u_x^2 dx dt.
\]
Hence, $\int_{Q_T} \left( \lambda u_x^2 + \mu |w_x|^2 + \nu |b_x|^2 \right) dx dt \leq C$. Consequently, $\int_0^T \| b \|_{L^\infty(\Omega)}^2 dt \leq C$.

Simple calculation yields
\[
D_t \left( \frac{1}{\rho F} \right) = \frac{1}{\lambda} \left( p + \frac{1}{2} |b|^2 \right) \frac{1}{\rho F}.
\]
Thus, $\| (\rho F)^{-1} \|_{L^\infty(Q_T)} \leq C$ by using $\int_0^T \left( \| \theta \|_{L^\infty(\Omega)} + \| b \|_{L^\infty(\Omega)}^2 \right) dt \leq C$, so $\rho \geq C$.

It remains to show (2.4). By (2.4) and (2.4), we have
\[
\int_{Q_T} \kappa \theta_x^2 \frac{1}{\theta^{1+\alpha}} dx dt \leq C.
\]
Then, from Lemma 2.1 (2.4) and the Hölder inequality, we have
\[
\theta \leq \int_{\Omega} \theta dx + \int_{\Omega} |\theta_x| dx \leq C + C \left( \int_{\Omega} \theta_x^2 dx \right)^{1/2} \left( \int_{\Omega} \theta dx \right)^{1/2}.
\]
Thus, (2.6) yields
\[
\int_0^T \| \theta \|_{L^\infty(\Omega)}^2 dt \leq C.
\]
It follows from the Hölder inequality, Lemma 2.1 and (2.7) that
\[
\int_{Q_T} |\theta_x|^{3/2} dx dt \leq \left( \int_{Q_T} \theta_x^2 \frac{1}{\theta} dx dt \right)^{3/4} \left( \int_{Q_T} \theta^2 dx dt \right)^{1/4}
\]
\[
\leq C \left( \int_0^T \| \theta \|_{L^\infty(\Omega)} \int_{\Omega} \theta dx dt \right)^{1/4} \leq C.
\]
And this completes the proof of the lemma.
Lemma 2.3. Under the assumptions in Theorem 1.1, we have

\[ \sup_{0 < t < T} \int_{\Omega} |b|^4 \, dx + \int_{Q_T} |b|^2 |b_x|^2 \, dx \, dt \leq C. \]

Proof. Taking the inner product of (1.1) with \(4|b|^2 b\) and integrating over \(Q_t\), we obtain

\[ \int_{\Omega} |b|^4 \, dx + 4\nu \int_{Q_t} |b|^2 |b_x|^2 \, dx + 8\nu \int_{Q_t} |b \cdot b_x|^2 \, dx = \int_{\Omega} |b|^4 \, dx + 4 \int_{Q_t} w_x \cdot (|b|^2 b) \, dx - 4 \int_{Q_t} (ub)_x \cdot (|b|^2 b) \, dx. \]  

(2.8)

Using the Young inequality, we have

\[ \int_{Q_t} w_x \cdot (|b|^2 b) \, dx = -\int_{Q_t} w_x \cdot (b_x |b|^2) \, dx - 2 \int_{Q_t} (w \cdot b)(b \cdot b_x) \, dx \]

\[ \leq \frac{\nu}{4} \int_{Q_t} |b|^2 |b_x|^2 \, dx + C \int_{Q_t} |w|^2 |b|^2 \, dx \]

\[ \leq \frac{\nu}{4} \int_{Q_t} |b|^2 |b_x|^2 \, dx + C \int_0^t \int_\Omega |\nabla b|^2 \, dx \, dt \]

\[ \leq \frac{\nu}{4} \int_{Q_t} |b|^2 |b_x|^2 \, dx + C, \]

(2.9)

where we have used (2.2) and (2.3). On the other hand, we have

\[ -\int_{Q_t} (ub)_x \cdot |b|^2 b \, dx = 3 \int_{Q_t} u(b_x \cdot b)|b|^2 \, dx \]

\[ \leq \frac{\nu}{4} \int_{Q_t} |b|^2 |b_x|^2 \, dx + C \int_{Q_t} u^2 |b|^4 \, dx \]

\[ \leq \frac{\nu}{4} \int_{Q_t} |b|^2 |b_x|^2 \, dx + C \int_0^t \int_{\Omega} |u|^2 \, \Omega \, d\Omega \int_\Omega |b|^4 \, dx \].

(2.10)

Plugging (2.9) and (2.10) into (2.8) and using the Gronwall inequality, we complete the proof of the lemma by noticing \(\int_0^T \|u^2\|_{L^\infty(\Omega)} \, dt \leq C \int_{Q_T} u_x^2 \, dx \, dt \leq C\). \(\square\)

Lemma 2.4. Under the assumptions in Theorem 1.1, we have

\[ \sup_{0 < t < T} \int_{\Omega} \rho_x^2 \, dx + \int_{Q_T} (\rho_x^2 + \theta \rho_x) \, dx \, dt \leq C, \]

(2.11)

\[ |\rho(x, t) - \rho(y, s)| \leq C \left( |x - y|^{1/2} + |s - t|^{1/4} \right), \quad \forall (x, t), (y, s) \in Q_T. \]

Proof. Set \(\eta = 1/\rho\). It follows from the equation (1.1) that \(u_x = \rho(\eta_t + \eta u_x)\). Substituting it into (1.1) yields

\[ [\rho(u - \lambda \eta_x)]_t + [\rho u(u - \lambda \eta_x)]_x = \gamma \rho^2 (\theta \eta_x - \eta \theta_x) - b \cdot b_x. \]
Multiplying it by \((u - \lambda \eta_x)\) and integrating over \(Q_t\), we have
\[
\frac{1}{2} \int_{\Omega} \rho(u - \lambda \eta_x)^2 \, dx + \gamma \int_{Q_t} \theta \rho \eta_x^2 \, dxds
= \frac{1}{2} \int_{\Omega} \rho_0(u_0 + \lambda \rho_0^2 \rho \eta_x^2) \, dx + \gamma \int_{Q_t} \rho^2 \theta \eta_x \, dxds
- \gamma \int_{Q_t} \rho^2 \eta_x (u - \lambda \eta_x) \, dxds - \int_{Q_t} \mathbf{b} \cdot \mathbf{z} (u - \lambda \eta_x) \, dxds.
\]
(2.12)

Using the Young inequality and Lemmas 2.1-2.2, we obtain
\[
\gamma \int_{Q_t} \rho^2 \eta_x \, dxds \leq \frac{\gamma \lambda}{2} \int_{Q_t} \theta \rho \eta_x^2 \, dxds + C \int_{Q_t} \theta \eta_x \, dxds
\leq \frac{\gamma \lambda}{2} \int_{Q_t} \theta \rho \eta_x^2 \, dxds + C \int_0^t \|\theta\|_{L^\infty(\Omega)} \int_{\Omega} u^2 \, dxds
\]
(2.13)

By the Cauchy inequality, (2.6) and Lemma 2.3, we have
\[
- \gamma \int_{Q_t} \rho^2 \eta_x (u - \lambda \eta_x) \, dxds - \int_{Q_t} \mathbf{b} \cdot \mathbf{z} (u - \lambda \eta_x) \, dxds
\leq C + C \int_0^t (1 + \|\theta\|_{L^\infty(\Omega)}) \int_{\Omega} \rho(u - \lambda \eta_x)^2 \, dxds.
\]
(2.14)

Substituting (2.13) and (2.14) into (2.12) and using the Gronwall inequality, we obtain
\[
\sup_{0 < t < T} \int_{\Omega} \rho_x^2 \, dx + \int_{Q_T} \theta \rho_x^2 \, dxds \leq C.
\]

Using this estimate and Lemma 2.2, one can derive from the equation (1.1) that
\[
\int_{Q_T} \rho_t^2 \, dxdt \leq C \int_0^T \|u_x\|_{L^\infty(\Omega)} \int_{\Omega} \rho_x^2 \, dxdt + C \int_{Q_T} u_x^2 \, dxdt \leq C,
\]
that implies (2.11)_1.

We now turn to (2.11)_2. Let \(\beta(x) = \rho(x, t) - \rho(x, s)\) for any \(x \in [0, 1]\) and \(s, t \in [0, T]\) with \(s \neq t\). Then for any \(x \in [0, 1]\) and \(\delta \in (0, 1/2]\), there exist some \(y \in [0, 1]\) and \(\xi\) between \(x\) and \(y\) such that \(\delta = |y - x|\) and \(\beta(\xi) = \frac{1}{x-y} \int_y^x \beta(z) \, dz\), and
\[
\beta(x) = \frac{1}{x-y} \int_y^x \beta(z) \, dz + \int_\xi^x \beta'(z) \, dz.
\]
Thus, from the Hölder inequality and (2.11), we have

\[|\beta(x)| \leq \frac{1}{\delta} \left| \int_y^x \beta(z)dz \right| + \left| \int_\xi^x \beta'(z)dz \right| \]

\[\leq \frac{1}{\delta} \left| \int_y^x \int_s^t \rho \tau d\tau dz \right| + \left| \int_\xi^x [\rho_z(z, t) - \rho_z(z, s)] dz \right| \]

\[\leq \frac{1}{\delta} \left( \int_{QT} \rho^2 d\tau d\rho \right)^{1/2} \left| x - y \right|^{1/2} |s - t|^{1/2} + \left( 2 \sup_{0 < t < T} \int_0^1 \left| \rho_z(z, t) \right|^2 dz \right)^{1/2} \left| x - \xi \right|^{1/2} \]

\[\leq C\delta^{-1/2} |s - t|^{1/2} + C\delta^{1/2} \]

If \(0 < |s - t|^{1/2} < 1/2\), taking \(\delta = |s - t|^{1/2}\) yields

\[|\rho(x, s) - \rho(x, t)| \leq C |s - t|^{1/4}. \tag{2.15}\]

If \(|s - t|^{1/2} \geq 1/2\), then (2.15) holds because \(\rho\) is uniformly bounded in \(\mu\).

From (2.11), we have that \(|\rho(x, t) - \rho(y, t)| = \left| \int_y^x \rho_zdz \right| \leq C|x - y|^{1/2}\). Thus, (2.11) is proved and this completes the proof of the lemma.

The following is the key lemma in this paper that leads to a new approach for the estimation on the uniform bounds in the general setting presented in this paper.

**Lemma 2.5.** Under the assumptions in Theorem 1.1, we have

\[
\int_{QT} |u_{xx}|^{m_0} dx dt \leq C, \quad m_0 = \min\{m, 4/3\}. \tag{2.16}
\]

In particular,

\[
\int_0^T \|u_x\|_{L^\infty(O)}^m dt \leq C. \tag{2.17}
\]

**Proof.** We will apply the \(L^p\) estimates of linear parabolic equations (see [16, Theorem 7.17]) to obtain (2.16). Firstly, rewrite the equation (1.1) as

\[
\frac{u_t - \lambda}{\rho} u_{xx} = -uu_x - \gamma \theta - \frac{\gamma}{\rho} \rho_x \theta - \frac{1}{\rho} \mathbf{b} \cdot \mathbf{b}_x =: f. \tag{2.18}
\]

From (2.11), the coefficient \(a(x, t) := \lambda/\rho\) is uniformly bounded in \(C^{1/2, 1/4}(\overline{Q}_T)\). By noticing the condition \(u_0 \in W^{2,m}(\Omega)\) for some \(m > 1\) in (1.6), it suffices to give a uniform bound of \(f\) in \(L^{4/3}(Q_T)\).

From Lemmas 2.2 and 2.3, the second term and the forth term on the right hand side of (2.18) are uniformly bounded in \(L^{3/2}(Q_T)\) and \(L^2(Q_T)\), respectively.

Using the Hölder inequality and Lemma 2.1, we obtain

\[
u^2 \leq 2 \int \Omega |uu_x| dx \leq 2 \left( \int \Omega u^2 dx \right)^{1/2} \left( \int \Omega u_x^2 dx \right)^{1/2} \leq C \left( \int \Omega u_x^2 dx \right)^{1/2}. \]
It follows from Lemma 2.2 that \( \int_0^T \| u \|_{L^\infty(\Omega)}^4 \, dt \leq C \). The Young inequality gives

\[
\iint_{Q_T} |w_x|^{3/2} \, dx \, dt \leq C \iint_{Q_T} u_x^2 \, dx \, dt + C \iint_{Q_T} u^6 \, dx \, dt \\
\leq C + C \int_0^T \| u \|_{L^\infty(\Omega)}^4 \int_\Omega u^2 \, dx \, dt \leq C.
\]

For the third term on the right hand side of (2.18), we use (2.11) and (2.7) to obtain

\[
\iint_{Q_T} |\rho \theta|^{4/3} \, dx \, dt \leq C \iint_{Q_T} \rho^2 \theta \, dx \, dt + C \iint_{Q_T} \theta^2 \, dx \, dt \leq C.
\]

Consequently, \( \| f \|_{L^{4/3}(Q_T)} \leq C \). Thus (2.16) is proved. (2.17) is an immediate consequence of (2.16), and this completes the proof of the lemma.

As a direct application of Lemma 2.5, we have

**Lemma 2.6.** Under the assumptions in Theorem 1.1, we have

\[
\mu \sup_{0 < t < T} \int_\Omega |w_x|^2 \, dx + \mu^2 \iint_{Q_T} |w_{xx}|^2 \, dx \, dt \leq C.
\]

**Proof.** Rewrite (1.1) as

\[ -w_t + \frac{\mu}{\rho} w_{xx} = u w_x - \frac{1}{\rho} b_x. \]  

(2.19)

Taking the inner product with \( \mu w_{xx} \) and integrating over \( Q_t \) yield

\[
\frac{\mu}{2} \int_\Omega |w_x|^2 \, dx + \mu^2 \iint_{Q_t} \frac{1}{\rho} |w_{xx}|^2 \, dx \, ds \\
= \frac{\mu}{2} \int_\Omega |w_0x|^2 \, dx - \mu \iint_{Q_t} \frac{1}{\rho} b_x \cdot w_{xx} \, dx \, ds \\
- \frac{\mu}{2} \iint_{Q_t} u_x |w_x|^2 \, dx \, ds + \mu \int_0^t w_t \cdot w_x \bigg|_{x=1}^x \, ds \\
\leq C \mu + \frac{\mu^2}{4} \iint_{Q_t} \frac{1}{\rho} |w_{xx}|^2 \, dx \, ds + C \iint_{Q_t} |b_x|^2 \, dx \, ds \\
+ C \int_0^t \| u_x \|_{L^\infty(\Omega)} \left( \mu \int_\Omega |w_x|^2 \, dx \right) \, ds + C \mu \int_0^t \| w_x \|_{L^\infty(\Omega)} \, ds.
\]

By the embedding theorem and Hölder inequality, we obtain

\[
|w_x|^2 \leq C \left( \int_\Omega |w_x|^2 \, dx + \iint_{Q_T} |w_x| |w_{xx}| \, dx \right) \\
\leq C \int_\Omega |w_x|^2 \, dx + C \left( \int_\Omega |w_x|^2 \, dx \right)^{1/2} \left( \int_\Omega |w_{xx}|^2 \, dx \right)^{1/2}.
\]  

(2.21)
Thus, using the Young inequality yields

\[
\mu \int_0^t \|w_x\|_{L^\infty(\Omega)} ds \\
\leq C \mu \int_0^t \left( \int_\Omega |w_x|^2 dx \right)^{1/2} ds + C \int_0^t \mu^{1/4} \left( \mu \int_\Omega |w_x|^2 dx \right)^{1/4} \left( \mu^2 \int_\Omega |w_{xx}|^2 dx \right)^{1/4} ds \\
\leq C \sqrt{\mu} + \frac{C \mu}{\epsilon} \int_\Omega |w_x|^2 dxds + \epsilon \mu^2 \int_{Q_t} \frac{1}{\rho} |w_{xx}|^2 dxds, \quad \forall \epsilon \in (0, 1).
\]

Plugging it into (2.20) and taking a small \(\epsilon > 0\), we have

\[
\mu \int_\Omega |w_x|^2 dx + \mu^2 \int_{Q_t} \frac{1}{\rho} |w_{xx}|^2 dxds \leq C + C \int_0^t \left( 1 + \|u_x\|_{L^\infty} \right) \left( \mu \int_\Omega |w_x|^2 dx \right) ds.
\]

Thus, the lemma follows from the Gronwall inequality and (2.17).

We now turn to prove the other estimates in Theorem 1.1. For this, we need the following three lemmas.

**Lemma 2.7.** Under the assumptions in Theorem 1.1, we have

\[
\int_\Omega |b_x|^2 \omega^2 dx + \int_{Q_t} |b_{xx}|^2 \omega^2 dxds \leq C \int_{Q_t} |w_x|^2 \omega^2 dxds + C \int_{Q_t} \left( \int_{Q_t} u_{xx}^2 dxds \right)^{1/2},
\]

where \(\omega\) is the same as the one defined in Theorem 1.1.

**Proof.** Taking the inner product of (1.1) with \(b_{xx}\omega^2(x)\) and integrating over \(Q_t\) give

\[
- \int_{Q_t} b_t \cdot b_{xx} \omega^2 dxdt + \nu \int_{Q_t} |b_{xx}|^2 \omega^2 dxds = \int_{Q_t} (u_b)_x \cdot b_{xx} \omega^2 dxds - \int_{Q_t} w_x \cdot b_{xx} \omega^2 dxds.
\]

To estimate the first integral on the left hand side of (2.22), we use integration by parts and (1.1) to obtain

\[
\int_{Q_t} b_t \cdot b_{xx} \omega^2 dxds = -\frac{1}{2} \int_\Omega |b_x|^2 \omega^2 dx + \frac{1}{2} \int_\Omega |b_{0x}|^2 \omega^2 dx - 2 \int_{Q_t} b_t \cdot b_x \omega' dxds
\]

\[
= -\frac{1}{2} \int_\Omega |b_x|^2 \omega^2 dx + \frac{1}{2} \int_\Omega |b_{0x}|^2 \omega^2 dx - \frac{1}{2} \int_\Omega |b_{0x}|^2 \omega^2 dx
\]

\[
= -2 \int_{Q_t} (u_bxx + w_x - ub_x - u_x b) \cdot b_x \omega' dxds.
\]

For the third term on right hand side of (2.23), by the Cauchy inequality and Lemmas 2.2 and 2.3 we have

\[
- 2 \int_{Q_t} (u_bxx + w_x - u_x b) \cdot b_x \omega' dxds
\]

\[
\leq \frac{\nu}{4} \int_{Q_t} |b_{xx}|^2 \omega^2 dxds + C \int_{Q_t} |b_x|^2 dxds + C \int_{Q_t} |w_x|^2 \omega^2 dxds
\]

\[
+ C \int_{Q_t} u_{xx}^2 dxds + C \int_{Q_t} |b \cdot b_x|^2 dxds
\]

\[
\leq C + \frac{\nu}{4} \int_{Q_t} |b_{xx}|^2 \omega^2 dxds + C \int_{Q_t} |w_x|^2 \omega^2 dxds.
\]
By noticing $u(1, t) = u(0, t) = 0$, we have
\[ |u(x, t)| \leq \|u_x\|_{L^\infty(\Omega)} \omega(x), \] (2.24)
thus,
\[ 2 \iint_{Q_t} u |b_x|^2 \omega' dx ds \leq C \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_\Omega |b_x|^2 \omega^2 dx ds. \]

Substituting them into (2.23) yields
\[ \int_{Q_t} b_t \cdot b_{xx} \omega^2 dx dt \leq C - \frac{1}{2} \int_\Omega |b_x|^2 \omega^2 dx + \frac{\nu}{4} \iint_{Q_t} |b_{xx}|^2 \omega^2 dx ds \\
+ C \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_\Omega |b_x|^2 \omega^2 dx ds + C \iint_{Q_t} |w_x|^2 \omega^2 dx ds. \] (2.25)

As to the two terms on the right hand side of (2.22), we use the Young inequality to obtain
\[ \iint_{Q_t} (u b)_x \cdot b_{xx} \omega^2 dx ds - \iint_{Q_t} w_x \cdot b_{xx} \omega^2 dx ds \\
\leq \frac{\nu}{4} \iint_{Q_t} |b_{xx}|^2 \omega^2 dx ds + C \iint_{Q_t} |(u b)_x|^2 \omega^2 dx ds + C \iint_{Q_t} |w_x|^2 \omega^2 dx ds. \] (2.26)

It remains to treat the second term on the right hand side of (2.26). By Lemma 2.2 we obtain
\[ \int_0^t \|u_x\|^2_{L^\infty(\Omega)} ds \leq C \iint_{Q_t} |u_x u_{xx}| dx ds \leq C \left( \iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}. \] (2.27)

It follows from Lemma 2.1 that
\[ \iint_{Q_t} |(u b)_x|^2 \omega^2 dx ds \leq C \iint_{Q_t} u^2 |b_x|^2 \omega^2 dx ds + C \iint_{Q_t} u_x^2 |b|^2 \omega^2 dx ds \\
\leq C \int_0^t \|u_x\|^2_{L^\infty(\Omega)} \int_\Omega |b_x|^2 \omega^2 dx ds + C \left( \iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}. \]

Substituting it into (2.26) and then substituting the resulting inequality and (2.25) into (2.22), then the Gronwall inequality completes the proof of the lemma.

**Lemma 2.8.** Under the assumptions in Theorem 1.1, we have
\[ \int_\Omega |w_x|^2 \omega^2 dx + \mu \iint_{Q_t} |w_{xx}|^2 \omega^2 dx ds \leq C + C \left( \iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}, \] (2.28)
\[ \iint_{Q_t} (|w|^2 + u^2 |w_x|^2) dx dt \leq C + C \iint_{Q_t} u_{xx}^2 dx ds. \]

**Proof.** Taking the inner product of (2.19) with $w_{xx} \omega^2(x)$ and integrating over $Q_t$ give
\[ - \iint_{Q_t} w_t \cdot w_{xx} \omega^2 dx dt + \mu \iint_{Q_t} |w_{xx}|^2 \omega^2 dx ds \\
= \iint_{Q_t} u w_{xx} \cdot w_{xx} \omega^2 dx ds - \iint_{Q_t} b_x \cdot w_{xx} \omega^2 dx ds. \] (2.29)
Thus, from Lemmas 2.2 and 2.6, we obtain

$$\int\int_{Q_t} w_t \cdot w_{xx} \omega^2 dxdt$$

$$= -\frac{1}{2} \int \int_{\Omega} |w_x|^2 \omega^2 dx + \frac{1}{2} \int \int_{\Omega} |w_{tx}|^2 \omega^2 dx - 2 \int \int_{Q_t} w_t \cdot w_x \omega' dxdt$$

$$= -\frac{1}{2} \int \int_{\Omega} |w_x|^2 \omega^2 dx + \frac{1}{2} \int \int_{\Omega} |w_{tx}|^2 \omega^2 dx - 2 \int \int_{Q_t} \left( \frac{\mu}{\rho} w_{xx} - uw_x + \frac{b_x}{\rho} \right) \cdot w_x \omega' dxds$$

$$\leq C - \frac{1}{2} \int \int_{\Omega} |w_x|^2 \omega^2 dx + C \int \int_{Q_t} |w_{xx}|^2 dxds + C \int \int_{Q_t} |w_x|^2 \omega^2 dxds$$

$$+ C \int \int_{Q_t} |b_x|^2 dxds + C \int \int_{Q_t} |u||w_x|^2 \omega dxds.$$

From (2.24), we have

$$\int \int_{Q_t} |u||w_x|^2 \omega dxds \leq \int_0^t \|u_x\|_{L^\infty(\Omega)} \int \Omega |w_x|^2 \omega^2 dxds.$$

Thus, from Lemmas 2.2 and 2.6, we obtain

$$\int \int_{Q_t} w_t \cdot w_{xx} \omega^2 dxdt \leq C - \frac{1}{2} \int \int_{\Omega} |w_x|^2 \omega^2 dx + C \int_0^t \|u_x\| L^\infty(\Omega) \int \Omega |w_x|^2 \omega^2 dxds.$$

To estimate the right hand side of (2.29), we use (2.24) to obtain

$$\int \int_{Q_t} u w_x \cdot w_{xx} \omega^2 dxds = -\frac{1}{2} \int \int_{Q_t} |w_x|^2 [u_x \omega^2 + 2u\omega'] dxds$$

$$\leq C \int_0^t \|u_x\| L^\infty(\Omega) \int \Omega |w_x|^2 \omega^2 dxds,$$

and

$$-\int \int_{Q_t} b_x \cdot w_{xx} \frac{\omega^2}{\rho} dxds = \int \int_{Q_t} \left( w_x \cdot b_{xx} \frac{\omega^2}{\rho} dxds + 2w_x \cdot b_x \frac{\omega \omega'}{\rho} - w_x \cdot b_x \frac{\omega^2 \rho_x}{\rho^2} dxds \right)$$

$$\leq C \int \int_{Q_t} \left(|b_{xx}|^2 \omega^2 + |w_x|^2 \omega^2 + |b_x|^2 + |b_x|^2 \omega^2 \rho_x^2 \right) dxds$$

$$\leq C + C \int \int_{Q_t} |w_x|^2 \omega^2 dxds + C \int \int_{Q_t} |b_{xx}|^2 \omega^2 dxds,$$

where we have used the fact that by using Lemmas 2.2 and 2.4 it holds

$$\int \int_{Q_t} |b_x|^2 \omega^2 \rho_x^2 dxds \leq C \int_0^t \||b_x|^2 \omega^2\|_{L^\infty(\Omega)} ds \leq C \int \int_{Q_t} |(b_x|^2 \omega^2)| dxds$$

$$\leq C \int \int_{Q_t} |b_x|^2 |\omega\omega'| dxds + C \int \int_{Q_t} |b_x \cdot b_{xx}| \omega^2 dxds$$

$$\leq C + C \int \int_{Q_t} |b_{xx}|^2 \omega^2 dxds.$$
Substituting the above estimates into (2.29) and using Lemma 2.7, we have
\[ \int_{\Omega} |w_x|^2 \omega^2 dx + \mu \int_{Q_t} |w_{xx}|^2 \omega^2 dx ds \]
\[ \leq C + C \int_0^t (1 + \|u_x\|_{L^\infty(\Omega)}) \int_{\Omega} |w_x|^2 \omega^2 dx ds + C \left( \int_{Q_t} u_{xx}^2 dx ds \right)^{1/2}. \]

Thus, the first estimate in the lemma follows from the Gronwall inequality and (2.17).

Consequently, from (2.24), the first estimate of the lemma and (2.27), we obtain
\[ \int_{Q_T} u^2 |w_x|^2 dx dt \leq C + \int_{Q_T} u^2 |w_{xx}|^2 dx dt \leq C + C \int_{Q_T} u_{xx}^2 dx dt. \]

Furthermore, by Lemmas 2.2 and 2.6, we derive from (2.19) that
\[ \int_{Q_T} |w_t|^2 dx dt \leq C + \int_{Q_T} u^2 |w_x|^2 dx dt \leq C + C \int_{Q_T} u_{xx}^2 dx dt. \]

And this completes the proof of the lemma.

**Lemma 2.9.** Under the assumptions in Theorem 1.1, we have
\[ \int_{\Omega} |b_x|^2 dx + \int_{Q_t} |b_t|^2 dx dt \leq C + C \left( \int_{Q_t} u_{xx}^2 dx ds \right)^{1/2}. \]

**Proof.** Taking the inner product of (1.1) with \( b_t \) and integrating over \( Q_t \) yield
\[ \frac{\nu}{2} \int_{\Omega} |b_x|^2 dx + \int_{Q_t} |b_t|^2 dx dt = \frac{\nu}{2} \int_{\Omega} |b_0|^2 dx + \int_{Q_t} \left[ w_x - (u b)_x \right] \cdot b_t dx dt. \] (2.30)

Note that
\[ \int_{Q_t} w_x \cdot b_t dx dt = \int_{\Omega} w_x \cdot b dx - \int_{\Omega} w_{0x} \cdot b_0 dx - \int_{Q_t} (w_t)_x \cdot b dx dt \]
\[ \quad = - \int_{\Omega} w_{0x} \cdot b_0 dx - \int_{\Omega} w \cdot b_x dx + \int_{Q_t} w_t \cdot b_x dx dt. \]

Thus, from Lemmas 2.1 and 2.2 and (2.28), we obtain
\[ \int_{Q_t} w_x \cdot b_t dx dt \leq C + \frac{\nu}{4} \int_{\Omega} |b_x|^2 dx + \left( \int_{Q_t} |b_x|^2 dx dt \right)^{1/2} \left( \int_{Q_t} |b_t|^2 dx dt \right)^{1/2} \]
\[ \leq C + \frac{\nu}{4} \int_{\Omega} |b_x|^2 dx + C \left( \int_{Q_t} |w_t|^2 dx dt \right)^{1/2} \]
\[ \leq C + \frac{\nu}{4} \int_{\Omega} |b_x|^2 dx + C \left( \int_{Q_t} u_{xx}^2 dx ds \right)^{1/2}. \]
From the Cauchy inequality, Lemma 2.1 and (2.27), we have
\[- \iint_{Q_t} (ub)_x \cdot b_x dx dt \leq \frac{1}{2} \iint_{Q_t} |b_t|^2 dx dt + \frac{1}{2} \iint_{Q_t} |(ub)_x|^2 dx ds \leq \frac{1}{2} \iint_{Q_t} |b_t|^2 dx dt + C \int_t^t \|u^2\|_{L^\infty(\Omega)} \int_{\Omega} |b_x|^2 dx ds + C \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_{\Omega} |b|^2 dx ds \leq \frac{1}{2} \iint_{Q_t} |b_t|^2 dx dt + C \int_0^t \|u^2\|_{L^\infty(\Omega)} \int_{\Omega} |b_x|^2 dx ds + C \left( \iint_{Q_t} u_x^2 dx ds \right)^{1/2}. \]

Substituting these estimates into (2.30), the Gronwall inequality yields the proof of the lemma.

We are now ready to prove the following estimates.

**Lemma 2.10.** Let the assumptions in Theorem 1.1 hold. Then \( \|(u, b)\|_{L^\infty(Q_T)} \leq C \), and
\[
\sup_{0 < t < T} \int_{\Omega} \left( \rho_t^2 + u_x^2 + \theta^2 \right) dx + \int_{Q_T} \left( u_t^2 + u_{xx}^2 + \kappa \theta_x^2 \right) dx dt \leq C, \tag{2.31}
\]
\[
\sup_{0 < t < T} \int_{\Omega} |w_x|^2 \omega_x^2 dx + \int_{Q_T} \left( u^2 |w_x|^2 + |w_t|^2 \right) dx dt \leq C, \tag{2.32}
\]
\[
\sup_{0 < t < T} \int_{\Omega} |b_x|^2 dx + \int_{Q_T} \left( |b_t|^2 + |b_{xx}|^2 \omega_x^2 \right) dx dt \leq C.
\]

**Proof.** Rewrite (1.1) as \( \sqrt{\rho} u_t - \frac{\lambda}{\sqrt{\rho}} u_{xx} = -\sqrt{\rho} u_x - \gamma \sqrt{\rho} \theta_x - \frac{\gamma}{\sqrt{\rho}} \rho_x \theta - \frac{1}{\sqrt{\rho}} b \cdot b_x \) to obtain
\[
\frac{\lambda}{2} \int_{\Omega} u_x^2 dx + \int_{Q_T} \left( \rho_t^2 u_t + \lambda^2 \rho_x^{-1} u_{xx}^2 \right) dx dt \leq \frac{\lambda}{2} \int_{\Omega} u_{0x}^2 dx + C \int_{Q_T} \left( u_{xx}^2 + \theta_x^2 + \rho_x^2 \theta^2 + |b \cdot b_x|^2 \right) dx ds \leq C + C \int_0^t \|u^2\|_{L^\infty(\Omega)} \int_{\Omega} u_x^2 dx ds + C \int_{Q_T} \theta_x^2 dx ds + C \int_0^t \|\theta^2\|_{L^\infty(\Omega)} \int_{\Omega} \rho_x^2 dx ds \leq C + C \int_0^t \|u^2\|_{L^\infty(\Omega)} \int_{\Omega} u_x^2 dx ds + C \int_{Q_T} \left( \theta_x^2 + \theta^2 \right) dx ds,
\]
where we have used (2.11) and \( \int_0^t \|\theta^2\|_{L^\infty(\Omega)} ds \leq C \int_{Q_t} (\theta^2 + \theta_x^2) dx ds. \)
Then multiplying (1.1), by \( \theta \) and integrating over \( Q_t \) give
\[
\frac{1}{2} \int_{\Omega} \rho \theta^2 dx + \int_{Q_t} \kappa \theta_x^2 dx ds = \frac{1}{2} \int_{\Omega} \rho_0 \theta_0^2 dx - \int_{Q_t} pu_x \theta dx ds
+ \int_{Q_t} \theta \left( \lambda u_x^2 + \mu |w_x|^2 dx + |b_x|^2 \right) dx ds.
\]
Note that
\[
- \iint_{Q_t} pu_x \theta dx ds = -\gamma \int_{Q_t} \rho \theta^2 u_x dx ds \leq C \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_{\Omega} \rho \theta^2 dx ds.
\]

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On the other hand, by using Lemmas 2.2, 2.6 and 2.9 and (2.27), we obtain
\[
\int Q_t \theta (\lambda u^2 + \mu |w_x|^2 + |b_x|^2) dx ds 
\leq C \int_0^t \|u_x\|_{L^\infty(\Omega)}^2 \int_\Omega \theta dx + C \int_0^t \|\theta\|_{L^\infty(\Omega)} \left\{ \int_\Omega (\mu |w_x|^2 + |b_x|^2) dx \right\} ds 
\leq C + C \left( \int_{Q_t} u^2_{xx} dx ds \right)^{1/2}.
\]
Plugging these estimates into (2.33), the Gronwall inequality implies
\[
\int_\Omega \theta^2 dx + \int_{Q_t} \kappa \theta^2_x dx ds \leq C + C \left( \int_{Q_t} u^2_{xx} dx ds \right)^{1/2}.
\]
(2.34)
Then (2.32) gives
\[
\int_\Omega u^2_x dx + \int_{Q_t} (u^2_t + u^2_{xx}) dx dt 
\leq C + C \int_0^t \|u^2_x\|_{L^\infty(\Omega)} \int_\Omega u^2_x dx ds + C \left( \int_{Q_t} u^2_{xx} dx ds \right)^{1/2} 
\leq C + C \int_0^t \|u^2_x\|_{L^\infty(\Omega)} \int_\Omega u^2_x dx ds + \frac{1}{2} \int_{Q_t} u^2_{xx} dx ds.
\]
Thus, the Gronwall inequality yields
\[
\int_\Omega u^2_x dx + \int_{Q_t} (u^2_t + u^2_{xx}) dx dt \leq C.
\]
Consequently, (2.31) follows from (2.34) and Lemmas 2.7-2.9. And the proof of the lemma is completed.

Lemma 2.11. Under the assumptions in Theorem 1.1, we have
\[
\sup_{0 < t < T} \int_\Omega |w_x| dx \leq C.
\]
In particular, \( \|w\|_{L^\infty(Q_T)} \leq C. \)

Proof. Denote \( z = w_x. \) Differentiating (2.19) in \( x \) gives
\[
z_t = \left( \frac{\mu}{\rho} z_x \right)_x - (uz)_x + \left( \frac{b_x}{\rho} \right)_x.
\]
(2.35)
Denote \( \Phi_\epsilon(\cdot) : \mathbb{R}^2 \to \mathbb{R}^+ \) for \( \epsilon \in (0, 1) \) by \( \Phi_\epsilon(\xi) = \sqrt{\epsilon^2 + |\xi|^2}, \forall \xi \in \mathbb{R}^2. \) Observe that \( \Phi_\epsilon \) has the following properties
\[
\left\{ \begin{array}{l}
|\xi| \leq |\Phi_\epsilon(\xi)| \leq |\xi| + \epsilon, \\ |\nabla_\xi \Phi_\epsilon(\xi)| \leq 1, \\ \lim_{\epsilon \to 0^+} \Phi_\epsilon(\xi) = |\xi|, \\ 0 \leq \xi \cdot \nabla_\xi \Phi_\epsilon(\xi) \leq \Phi_\epsilon(\xi), \\ \eta D^2 \Phi_\epsilon(\xi) \eta^\top \geq 0, \forall \xi, \eta \in \mathbb{R}^2,
\end{array} \right.
\]
(2.36)
where $\xi^T$ stands for the transpose of the vector $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, and $D^2_\xi g$ is the Hessian matrix of the function $g : \mathbb{R}^2 \to \mathbb{R}$.

Taking the inner product of (2.35) with $\nabla_\xi \Phi_\epsilon(z)$ and integrating over $Q_t$, we have

$$
\int_\Omega \Phi_\epsilon(z) dx - \int_\Omega \Phi_\epsilon(w_{0x}) dx
= -\mu \int_0^T \int_{Q_t} \frac{1}{\rho} z_x D^2_\xi \Phi_\epsilon(z)(z_x)^T dx ds
d - \int_0^T \int_{Q_t} (uz)_x \cdot \nabla_\xi \Phi_\epsilon(z) dx ds
+ \int_0^T \int_{Q_t} \left( \frac{b_x}{\rho} \right)_x \cdot \nabla_\xi \Phi_\epsilon(z) dx ds
+ \int_0^T \int_{Q_t} \mu \frac{z_x \cdot \nabla_\xi \Phi_\epsilon(z)}{\rho} \left| \frac{x=1}{x=0} \right| ds = \sum_{j=1}^4 E_j.
$$

(2.37)

From (2.36), it follows that $E_1 \leq 0$, and

$$
E_2 = - \int_0^T \int_{Q_t} (uz_x + u_x z) \cdot \nabla_\xi \Phi_\epsilon(z) dx ds
= \int_0^T \int_{Q_t} (u_x \Phi_\epsilon(z) - u_x z \cdot \nabla_\xi \Phi_\epsilon(z)) dx ds
\leq C \int_0^T \|u_x\| L^\infty(\Omega) \int_\Omega \Phi_\epsilon(z) dx ds.
$$

For $E_3$, using the equation (1.1)_4 yields

$$
E_3 = \int_0^T \int_{Q_t} \frac{b_x}{\rho} \cdot \nabla_\xi \Phi_\epsilon(z) dx ds
- \int_0^T \int_{Q_t} \frac{b_x}{\rho^2} \rho_x dx ds
= \frac{1}{\nu} \int_0^T \int_{Q_t} \left[ \frac{b_t + (ub)_x - z}{\rho} \right] \cdot \nabla_\xi \Phi_\epsilon(z) dx ds
- \int_0^T \int_{Q_t} \frac{b_x}{\rho^2} \rho_x dx ds
\leq C \int_0^T [||b_t|| + |(ub)_x| + |\rho_x| |b_x|] dx ds \leq C,
$$

where we have used (2.36) and Lemmas 2.4 and 2.10.

It remains to estimate $E_4$. From (2.19), we have

$$
\left| \frac{\mu}{\rho(a, t)} z_x(a, t) \right| = \left| w_t(a, t) - \frac{b_x(a, t)}{\rho(a, t)} \right| \leq C + C |b_x(a, t)|, \quad \text{where } a = 0 \text{ or } 1.
$$

(2.38)

On the other hand, we can first integrate (1.1)_4 from $a$ to $y \in [0, 1]$ in $x$ and then integrate the resulting equation over $(0, 1)$ in $y$ to obtain

$$
b_x(a, t) = -\frac{1}{\nu} \left\{ \int_0^1 \int_a^y b_t(x, t) dx dy + \int_0^1 (ub - w)(y, t) dy + w(a, t) \right\}.
$$

Thus, from Lemmas 2.1 and 2.10, we have that $\int_0^T |b_x(a, t)|^2 dt \leq C$. Then, one can derive from (2.38) that $\int_0^T \left| \frac{\mu}{\rho(a, t)} z_x(a, t) \right| dt \leq C + C \int_0^T |b_x(a, t)| dt \leq C$. Hence

$$
E_4 \leq C \int_0^T \left\{ \left| \frac{\mu}{\rho(1, t)} z_x(1, t) \right| + \left| \frac{\mu}{\rho(0, t)} z_x(0, t) \right| \right\} dt \leq C.
$$
Substituting the above estimates in (2.37) and using the Gronwall inequality, we obtain

$$\int_{\Omega} \Phi_\epsilon(z) dx \leq C + \int_{\Omega} \Phi_\epsilon(w_0) dx.$$  

Taking the limit $\epsilon \to 0$ yields $\int_{\Omega} |w_x| dx \leq C$. This and $\int_{\Omega} |w|^2 dx \leq C$ imply that $|w| \leq C$, and it completes the proof of the lemma.

Lemma 2.12. Under the assumptions in Theorem 1.4, we have

$$\int_{\Omega} |w_x|^2 dx + \mu \int_{Q_T} (|w|^4 + |w_x|^2) dx dt \leq C.$$  

Proof. For any fixed $z \in [0, 1]$, we first integrate (1.1) from $z$ to $y \in [0, 1]$ in $x$, and then integrate the resulting equation over $(0, 1)$ in $y$ to have

$$b_x(z, t) = -\frac{1}{\nu} \left\{ \int_0^1 \int_z^y b_t(x, t) dx dy + \int_0^1 (ub - w)(y, t) dy - (ub - w)(z, t) \right\},$$

which together with Lemmas 2.10 and 2.11 imply that $\int_0^T \|b_x\|^2_{L^\infty(\Omega)} dt \leq C$.

Combining Lemmas 2.10 and 2.11 gives

$$\int_{Q_T} |b_x|^4 dx dt \leq C \int_0^T \|b_x\|^2_{L^\infty(\Omega)} \int_{\Omega} |w_x|^2 dx dt \leq C.$$  

The proof of the lemma is then completed.

Now some estimates given in Lemmas 2.6 and 2.10 can be improved as follows.

Lemma 2.13. Under the assumptions in Theorem 1.4, we have

$$\sqrt{\mu} \sup_{0 < t < T} \int_{\Omega} |w_x|^2 dx + \mu^{3/2} \int_{Q_T} |w_{xx}|^2 dx dt \leq C,$$

$$\sup_{0 < t < T} \int_{\Omega} |w_x|^2 dx + \int_{Q_T} (\mu |w_{xx}|^2 + |b_{xx}|^2) \omega dx dt \leq C.$$  

Proof. For the first estimate, we can use an argument similar to Lemma 2.6. The key point
is to estimate the term \(-\mu \int_{Q_t} \frac{1}{\rho} b_x \cdot w_{xx} dxds\) in (2.20). Indeed, we have

\[
-\mu \int_{Q_t} \frac{1}{\rho} b_x \cdot w_{xx} dxds = \mu \int_{Q_t} \frac{b_{xx} \cdot w_x}{\rho} dxds - \mu \int_{Q_t} \frac{b_x \cdot w_x}{\rho^2} \rho \cdot w_{xx} dxds - \mu \int_0^t \frac{b_x \cdot w_x}{\rho} \bigg|_{x=1}^1 dxds
\]

\[
\leq C\mu \int_{Q_t} |b_{xx}|^2 dxds + C\mu \int_{Q_t} |w_x|^2 dxds + \mu \int_{Q_t} |b_x|^2 \rho^2 dxds + C\mu \int_0^t \|b_x\|^2_{L^\infty(\Omega)} ds
\]

\[
\leq C\mu + C\mu \int_{Q_t} |b_{xx}|^2 dxds + C\mu \int_{Q_t} |w_x|^2 dxds + C\mu \int_0^t \|w_x\|^2_{L^\infty(\Omega)} ds
\]

where we have used the fact that by Lemmas 2.4 and 2.12, it holds

\[
\int_{Q_t} |b_x|^2 dxds \leq \int_0^t \|b_x\|^2_{L^\infty(\Omega)} ds
\]

By (2.21) and Lemma 2.6, we have

\[
\mu \left( \int_0^t \|w_x\|^2_{L^\infty(\Omega)} ds \right)^{1/2} \leq C\sqrt{\mu} + C\mu^{1/4} \left( \mu \int_{Q_t} |w_x|^2 dxds \right)^{1/4} \left( \mu^2 \int_{Q_t} |w_{xx}|^2 dxds \right)^{1/4}
\]

\[
\leq C\sqrt{\mu} + \epsilon \mu \int_{Q_t} |w_x|^2 dxds + \epsilon \mu^2 \int_{Q_t} \frac{1}{\rho} |w_{xx}|^2 dxds, \quad \forall \epsilon \in (0, 1).
\]

From Lemma 2.10, we have

\[
\int_{Q_t} |b_{xx}|^2 dxds \leq C + C \int_{Q_t} |w_x|^2 dxds.
\]

Thus, (2.41) follows from the Gronwall inequality.

Inserting the above estimates into (2.39) and taking a small \(\epsilon > 0\), we have

\[
-\mu \int_{Q_t} \frac{1}{\rho} b_x \cdot w_{xx} dxdt \leq C\sqrt{\mu} + C\mu \int_{Q_t} |w_x|^2 dxdt + \frac{\mu^2}{4} \int_{Q_t} \frac{1}{\rho} |w_{xx}|^2 dxds.
\]

Then, an argument similar to Lemma 2.6 leads to

\[
\mu \int_{\Omega} |w_x|^2 dx + \mu^2 \int_{Q_t} |w_{xx}|^2 dxds \leq C\sqrt{\mu} + C \int_0^t \left( 1 + \|u_x\|_{L^\infty(\Omega)} \right) \left( \mu \int_{\Omega} |w_x|^2 dx \right) ds.
\]

Thus, the first estimate of this lemma follows from the Gronwall inequality and (2.17).
The second estimate can be proved by the arguments similar to Lemma 2.7 and (2.28) and by the first estimate and Lemmas 2.10, 2.12. In fact, this can be done by using \( \omega \) instead of \( \omega^2 \) in (2.22) and (2.28) and noticing the following facts:

\[
\mu \iint_{Q_T} |w_x \cdot w_{xx}| dxdt \leq C\sqrt{\mu} \iint_{Q_T} |w_x|^2 dx dt + C \mu^{3/2} \iint_{Q_T} |w_{xx}|^2 dxdt \leq C,
\]

\[
\iint_{Q_T} |b_x \cdot w_x| dxdt \leq C \int_0^T \|b_x\|_{L^\infty(\Omega)} \iint_{\Omega} |w_x| dxdt \leq C.
\]

The proof is completed. \( \square \)

As a consequence of Lemma 2.13 and (2.40), we have

\[
\mu^{3/2} \iint_{Q_T} |w_x|^4 dxdt \leq C \mu \int_0^T \|w_x\|_{L^\infty(\Omega)}^2 \left( \sqrt{\mu} \iint_{Q_T} |w_x|^2 dx \right) dt \leq C. \tag{2.42}
\]

With the above estimates, we are now ready to show the upper bound estimate on \( \theta \).

**Lemma 2.14.** Let the assumptions in Theorem 1.1 hold. Then \( \theta \leq C \).

**Proof.** Rewrite the equation (1.1) into

\[
\theta_t = a(x,t)\theta_{xx} + b(x,t)\theta_x + c(x,t)\theta + f(x,t), \tag{2.43}
\]

where \( a = \rho^{-1} \kappa, b = \rho^{-1} \kappa_x - u, c = -\gamma u_x, f = \rho^{-1}(\lambda u_x^2 + \mu|w_x|^2 + \nu|b_x|^2) \).

Set \( z = \theta_x \). Differentiating the equation (2.43) in \( x \) yields

\[
z_t = (az_x)_x + (bz)_x + (c \theta)_x + f_x. \tag{2.44}
\]

For \( \epsilon \in (0, 1) \), denote \( \varphi_\epsilon : \mathbb{R} \to \mathbb{R}^+ \) by \( \varphi_\epsilon(s) = \sqrt{s^2 + \epsilon^2} \), satisfying

\[
\varphi'_\epsilon(0) = 0, \quad |\varphi'_\epsilon(s)| \leq 1, \quad 0 \leq s^2 \varphi''_\epsilon(s) \leq \epsilon, \quad \lim_{\epsilon \to 0} \varphi_\epsilon(s) = |s|.
\]

Multiplying (2.44) by \( \varphi'_\epsilon(z) \), integrating over \( Q_t \), and noticing \( \varphi'_\epsilon(z)|_{x=0.1} = \varphi'_\epsilon(\theta_x)|_{x=0.1} = 0 \), \( \varphi''_\epsilon(s) \geq 0 \) and \( |\varphi'_\epsilon(s)| \leq 1 \), we have

\[
\int_{\Omega} \varphi_\epsilon(z) dx - \int_{\Omega} \varphi_\epsilon(\theta_{0x}) dx ds = -\iint_{Q_t} (az_x^2 + bzz_x)\varphi''_\epsilon(z) dx ds + \iint_{Q_t} [(\epsilon \theta)_x + f_x] \varphi'_\epsilon(z) dx ds,
\]

\[
= -\iint_{Q_t} \left[ (z_x + \frac{b}{2a})^2 \varphi''_\epsilon(z) + \frac{b^2}{4a} z^2 \varphi'_\epsilon(z) \right] dx ds + \iint_{Q_t} [(\epsilon \theta)_x + f_x] \varphi'_\epsilon(z) dx ds,
\]

\[
\leq \int_{Q_t} \frac{b^2}{4a} z^2 \varphi''_\epsilon(z) dx ds + \iint_{Q_t} (|\epsilon \theta_x| + |f_x|) dx ds. \tag{2.45}
\]

Thus, letting \( \epsilon \to 0 \) in (2.45) and using \( 0 \leq s^2 \varphi''_\epsilon(s) \leq \epsilon \) and \( \lim_{\epsilon \to 0} \varphi_\epsilon(s) = |s| \), we have

\[
\int_{\Omega} |\theta_x| dx \leq \int_{\Omega} |\theta_{0x}| dx + \iint_{Q_t} (|\epsilon \theta_x| + |c \theta_x| + |f_x|) dx ds. \tag{2.46}
\]
By Lemma 2.10 we have
\[ \int Q T |c x \theta| dx dt \leq C \left( \int Q T u_{xx}^2 dx dt \right)^{1/2} \left( \int Q T \theta^2 dx dt \right)^{1/2} \leq C, \]
\[ \int Q T |c \theta_x| dx dt \leq C \left( \int Q T u_{xx}^2 dx dt \right)^{1/2} \left( \int Q T \theta_x^2 dx dt \right)^{1/2} \leq C. \]

By Lemmas 2.4, 2.10 and 2.12, and (2.42), we obtain
\[ \int Q T |f_x| dx dt \leq C \int Q T (|u_x| u_{xx} + \mu |w_x| w_{xx} + |b_x| b_{xx}) dx dt + C \int Q T \left( u_x^2 + \mu |w_x|^2 + |b_x|^2 \right) |\rho_x| dx dt \leq C. \]

Substituting the above estimates into (2.46) yields that \( \int \Omega |\theta_x| dx \leq C \), which yields the upper bound of \( \theta \) because \( \int \Omega \theta dx \leq C \).

The following estimates are about the derivatives of \( \theta \).

**Lemma 2.15.** Under the assumptions in Theorem 1.1, we have
\[ \sup_{0<t<T} \int_{\Omega} \theta_x^2 dx + \int Q T (\theta_t^2 + \theta_{xx}^2) dx dt \leq C. \]

**Proof.** Rewrite the equation (1.1) as
\[ \rho \theta_t - (\kappa \theta_x)_x = \lambda u_x^2 + \mu |w_x|^2 + \nu |b_x|^2 - \rho u \theta_x - \gamma \rho \theta u_x =: f. \]  

We first estimate \( \|f\|_{L^2(Q_T)} \). By (2.1), (2.42) and Lemmas 2.10, 2.14 and 2.12, we have
\[ \int Q T f^2 dx dt \leq C \int Q T \left( u_x^4 + \mu^2 |w_x|^4 + \nu^2 |b_x|^4 + \rho^2 u^2 \theta_x^2 + \rho^2 u_x^2 \theta_x^2 \right) dx dt \leq C. \]  

Multiplying (2.47) by \( \kappa \theta_t \) and integrating over \( Q_t \), we have
\[ \int Q_t \rho k \theta_t^2 dx dt + \int Q_t \kappa \theta_x (\kappa \theta_t)_x dx dt = \int Q_t f \kappa \theta_t dx dt. \]  

Observe that \( (\kappa \theta_t)_x = (\kappa \theta_x)_t + \kappa \rho \rho_x \theta_t + \kappa \rho \theta_x (\rho_x u + \rho u_x) \), then
\[ \int Q_t \kappa \theta_x (\kappa \theta_t)_x dx dt = \frac{1}{2} \int_\Omega \kappa^2 \theta_x^2 dx - \frac{1}{2} \int_\Omega \kappa^2 (\rho_0, \theta_0) \theta_0^2 dx + \int Q_t \left[ \kappa \kappa \rho \rho_x \theta_x \theta_t + \kappa \kappa \rho \theta_x^2 (\rho_x u + \rho u_x) \right] dx dt. \]

Then substituting this into (2.49) yields
\[ \int Q_t \rho k \theta_t^2 dx dt + \int \Omega \kappa^2 \theta_x^2 dx \]
\[ \leq C - 2 \int Q_t \left[ \kappa \kappa \rho \rho_x \theta_x \theta_t + \kappa \kappa \rho \theta_x^2 (\rho_x u + \rho u_x) - f \kappa \theta_t \right] dx dt. \]
By \( C^{-1} \leq \rho, \theta \leq C \) and (1.3), we have that \( \kappa_1 \leq \kappa \leq C \) and \( |\kappa_\rho| \leq C \). By Young inequality, (2.11), (2.48) and Lemma 2.10, we obtain

\[
-2 \int_{Q_t} \left[ \kappa \kappa_\rho \rho_x \theta_t + \kappa \kappa_\rho \theta_x^2 (\rho_x u + \rho u_x) - f_\kappa \theta_t \right] dx dt \\
\leq C + \frac{1}{4} \int_{Q_t} \rho \kappa \theta_t^2 dx dt + C \int_{Q_t} (\kappa \theta_x)^2 (\rho_x^2 + |\rho_x| + |u_x|) dx ds \\
\leq C + \frac{1}{4} \int_{Q_t} \rho \kappa \theta_t^2 dx dt + C \int_0^t \|\kappa \theta_x\|_{L^\infty(\Omega)}^2 ds.
\]

(2.51)

Now we estimate the second integral on the right hand side of (2.51). By the embedding theorem and Young inequality, we have

\[
\int_0^t \|\kappa \theta_x\|_{L^\infty(\Omega)}^2 ds \leq \int \int_{Q_t} |\kappa \theta_x|^2 dx ds + 2 \int \int_{Q_t} |\kappa \theta_x| (|\kappa \theta_x|) dx ds \\
\leq C + \frac{\epsilon}{\epsilon} + \frac{1}{2} \int \int_{Q_t} (\kappa \theta_x)^2 dx ds, \quad \forall \epsilon > 0.
\]

Then, from (2.47), we obtain

\[
\int_0^t \|\kappa \theta_x\|_{L^\infty(\Omega)}^2 ds \leq C + \epsilon \int \int_{Q_t} (\rho \kappa \theta_t^2 + f_\omega^2) dx dt.
\]

Plugging it into (2.51), taking a small \( \epsilon > 0 \) and using (2.48), we obtain

\[
-2 \int_{Q_t} \left[ \kappa \kappa_\rho \rho_x \theta_t + \kappa \kappa_\rho \theta_x^2 (\rho_x u + \rho u_x) - f_\kappa \theta_t \right] dx dt \leq C + \frac{1}{2} \int \int_{Q_t} \rho \kappa \theta_t^2 dx dt.
\]

This together with (1.3) and (2.50) give

\[
\sup_{0 < t < T} \int_\Omega \theta_x^2 dx + \int_\Omega \theta_t^2 dx \leq C.
\]

(2.52)

By (2.52) and Lemma 2.14 it follows from (1.1) that \( \|\theta_{xx}\|_{L^2(\Omega_T)} \leq C \), and this completes the proof of the lemma.

In summary, all the estimates in (1.7) have been proved.

### 2.2 Proof of Theorem 1.1(ii)

Similar to (2.11), one can show that

\[
\|(u, b, \theta)\|_{C^{1/2,1/4}(\Omega_T)} \leq C, \quad \|w\|_{C^{1/2,1/4}([\delta, 1-\delta] \times [0, T])} \leq C, \quad \forall \delta \in (0, 1/2).
\]

(2.53)

From (2.11), (2.1), (2.53) and Lemmas 2.2, 2.4 and 2.10, 2.15 it follows that there exist a subsequence \( \mu_j \to 0 \) such that the corresponding solution to the problem (1.1)-(1.4) with \( \mu = \mu_j \), still denoted by \( (\rho, u, w, b, \theta) \), converges to \( (\overline{\rho}, \overline{u}, \overline{w}, \overline{b}, \overline{\theta}) \in \mathbb{F} \) in the following sense:

\[
(\rho, u, b, \theta) \rightarrow (\overline{\rho}, \overline{u}, \overline{b}, \overline{\theta}) \quad \text{in} \quad C^\alpha(\overline{Q}_T), \quad \forall \alpha \in (0, 1/4),
\]

\[
(\rho_t, \rho_x, u_x, b_x, \theta_x) \rightarrow (\overline{\rho}_t, \overline{\rho}_x, \overline{u}_x, \overline{b}_x, \overline{\theta}_x) \quad \text{weakly} \quad - \ast \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)),
\]

\[
(u_t, b_t, \theta_t, u_{xx}, \theta_{xx}) \rightarrow (\overline{u}_t, \overline{b}_t, \overline{\theta}_t, \overline{u}_{xx}, \overline{\theta}_{xx}) \quad \text{weakly in} \quad L^2(Q_T),
\]

\[
b_{xx} \rightarrow \overline{b}_{xx} \quad \text{weakly in} \quad L^2((\delta, 1-\delta) \times (0, T)), \quad \forall \delta \in (0, 1/2),
\]

\[
\overline{b}_{xx} \rightarrow \overline{b}_{xx} \quad \text{weakly in} \quad L^2((\delta, 1-\delta) \times (0, T)), \quad \forall \delta \in (0, 1/2),
\]

\[
\overline{b}_{xx} \rightarrow \overline{b}_{xx} \quad \text{weakly in} \quad L^2((\delta, 1-\delta) \times (0, T)), \quad \forall \delta \in (0, 1/2),
\]

\[
\overline{b}_{xx} \rightarrow \overline{b}_{xx} \quad \text{weakly in} \quad L^2((\delta, 1-\delta) \times (0, T)), \quad \forall \delta \in (0, 1/2),
\]
and

\[ \mathbf{w} \to \overrightarrow{\omega} \quad \text{in} \quad C^\alpha([\delta, 1 - \delta] \times [0, T]), \quad \forall \delta \in (0, 1/2), \quad \alpha \in (0, 1/4), \]
\[ \mathbf{w}_t \to \mathbf{w}_t \quad \text{weakly in} \quad L^2(Q_T), \]
\[ \mathbf{w}_x \to \overrightarrow{\omega}_x \quad \text{weakly} - * \quad \text{in} \quad L^\infty(0, T; L^2(\delta, 1 - \delta)), \quad \forall \delta \in (0, 1/2), \]
\[ \mathbf{w} \to \overrightarrow{\omega} \quad \text{strongly in} \quad L^r(Q_T), \quad \forall r \in [1, +\infty), \]
\[ \sqrt{\mu} \|\mathbf{w}_x\|_{L^2(Q_T)} \to 0. \]

We now show the strong convergence of \((u_x, b_x, \theta_x)\) in \(L^2(Q_T)\). Multiplying \((1.1)_2\) with \(\mu = \mu_j\) by \((u - \overline{\theta})\) and integrating over \(Q_T\), we have

\[
\lambda \int_{Q_T} (u_x - \overline{\omega}_x)^2 \, dxdt + \lambda \int_{Q_T} \overline{\omega}_x (u_x - \overline{\omega}_x) \, dxdt \\
= -\int_{Q_T} [\rho u_t + (\rho u^2 + \gamma \rho \theta + \frac{1}{2}|b|^2_x)] (u - \overline{\omega}) \, dxdt.
\]

Then, from Lemmas 2.4, 2.10 and 2.14 we obtain

\[ u_x \to \overline{\omega}_x \quad \text{strongly in} \quad L^2(Q_T) \quad \text{as} \quad \mu_j \to 0. \]

Similarly, one has

\[ (b_x, \theta_x) \to (\overline{\mathbf{b}}_x, \overline{\theta}_x) \quad \text{strongly in} \quad L^2(Q_T) \quad \text{as} \quad \mu_j \to 0. \]

Thus, one can check that \((\mathbf{r}, \overline{\omega}, \mathbf{w}, \overline{\mathbf{b}}, \overline{\theta})\) is a solution to the problem \((1.1)-(1.4)\) with \(\mu = 0\) in the sense of distribution. On the other hand, one can see from Theorem 1.1(iii) that problem \((1.1)-(1.4)\) with \(\mu = 0\) admits at most one solution in \(\mathcal{F}\). Therefore, the above convergence holds for any \(\mu_j \to 0\). The proof of Theorem 1.1(ii) is then completed.

### 2.3 Proof of Theorem 1.1(iii)

The proof is divided into several steps. For convenience, set

\[ \tilde{\rho} = \rho - \bar{\rho}, \quad \tilde{u} = u - \overline{\omega}, \quad \tilde{w} = w - \overrightarrow{\omega}, \quad \tilde{b} = b - \overline{\mathbf{b}}, \quad \tilde{\theta} = \theta - \overline{\theta}, \]
\[ \mathcal{H}(t) = \| (\tilde{\rho}, \tilde{u}, \tilde{w}, \tilde{b}, \tilde{\theta}) \|^2_{L^2(\Omega)}, \quad D(t) = 1 + \| (u_x, b_x, \overline{\omega}_x, \overline{\mathbf{b}}_x, \overline{\theta}_x) \|^2_{L^\infty(\Omega)} + \| (\overline{\omega}_t, \overline{\theta}_t, \overline{\omega}_x, \overline{\mathbf{b}}_x, \overline{\theta}_x) \|^2_{L^2(\Omega)}. \]

Clearly, \(D(t) \in L^1(0, T)\).

**Step 1** Claim that

\[
\int_{Q_t} \tilde{\rho}^2 \, dx \leq \epsilon \int_{Q_t} \tilde{u}_x^2 \, dx ds + \frac{C}{\epsilon} \int_0^t D(s) \mathcal{H}(s) \, ds, \quad \forall \epsilon > 0. \tag{2.54}
\]

From the equations of \(\rho\) and \(\mathbf{r}\), we have that \(\tilde{\rho}_t = - (\rho \tilde{u}_x + \rho \tilde{\omega})_x\). Multiplying it by \(\tilde{\rho}\) and integrating over \(Q_t\), and using the Young inequality, we obtain

\[
\frac{1}{2} \int_{\Omega} \tilde{\rho}^2 \, dx = -\int_{Q_t} (\rho \tilde{u}_x \tilde{\rho} + \rho_x \tilde{\omega} \rho \tilde{\rho}) \, dxds \leq \frac{1}{2} \int_{Q_t} \tilde{u}_x \rho^2 \, dxds
\]
\[
\leq \frac{\epsilon}{4} \int_{Q_t} \tilde{u}_x^2 \, dxds + C \int_{Q_t} \tilde{u}^2 \rho_x^2 \, dxds + \frac{C}{\epsilon} \int_0^t (1 + \|\tilde{u}_x\|_{L^\infty(\Omega)}) \int_{\Omega} \tilde{\rho}^2 \, dxds, \quad \forall \epsilon > 0.
\]
From (2.11), we have, for the above $\epsilon$,
\[
C \int_Q \tilde{\varphi}^2 \rho_x^2 dx ds \leq C \int_0^t \|\tilde{u}\|_{L^\infty(\Omega)}^2 ds \leq \frac{\epsilon}{4} \int_Q \tilde{\varphi}^2 dx ds + \frac{C}{\epsilon} \int_Q \tilde{u}^2 dx ds.
\]
Thus, the claim (2.54) holds.

**Step 2** Claim that
\[
\int_q \tilde{u}^2 dx + \int_q \tilde{\varphi}_x^2 dx ds \leq C \int_0^t D(s)\mathcal{H}(s) ds.
\]
From the equations of $u$ and $\bar{\varphi}$, we have
\[
(\rho \tilde{u})_t + (\rho u \tilde{\varphi})_x + \rho \bar{\varphi}_t + (\rho u - \bar{\rho} \bar{\varphi})_x + \gamma (\rho \theta - \bar{\rho} \bar{\varphi})_x + \frac{1}{2} (|b|^2 - |\bar{b}|^2)_x = \lambda \tilde{u}_{xx}.
\]
Multiplying it by $\tilde{u}$ and integrating over $Q_t$ give
\[
\frac{1}{2} \int_Q \rho \tilde{u}^2 dx + \lambda \int_Q \tilde{\varphi}_x^2 dx ds
= -\int_Q \tilde{\varphi}_x u \tilde{u} dx ds - \int_Q (\rho u - \bar{\rho} \bar{\varphi}) \bar{\varphi}_x \tilde{u} dx ds + \gamma \int_Q (\rho \theta - \bar{\rho} \bar{\varphi}) \bar{\varphi}_x \tilde{u} dx ds
+ \frac{1}{2} \int_Q (|b|^2 - |\bar{b}|^2) \tilde{u}_x dx ds =: \sum_{i=1}^{4} I_i.
\]
Observe that $\rho u - \bar{\rho} \bar{\varphi} = \rho \tilde{u} + \bar{\varphi} \bar{\rho}$ and $\rho \theta - \bar{\rho} \bar{\varphi} = \rho \tilde{\varphi} + \bar{\varphi} \bar{\rho}$. Then
\[
I_1 \leq C \int_0^t \left( \int_Q \tilde{\varphi}_x^2 dx ds \right)^{1/2} \left( \int_Q \tilde{u}_x^2 dx ds \right)^{1/2} \|\tilde{u}\|_{L^\infty(\Omega)} dt
\leq C \int_0^t \left( \int_Q \tilde{\varphi}_x^2 dx ds \right) \left( \int_Q \tilde{u}_x^2 dx ds \right) dt + C \int_0^t \|\tilde{u}\|_{L^\infty(\Omega)} ds
\leq C \int_0^t D(s)\mathcal{H}(s) ds + \frac{\lambda}{4} \int_Q \tilde{\varphi}_x^2 dx ds,
\]
\[
I_2 \leq C \int_Q \left( |\tilde{\varphi}_x^2| + |\tilde{\varphi}_x| |\tilde{\varphi}_x| \right) dx ds
\leq C \int_Q \left( |\tilde{\varphi}_x^2| + 1 \right) \tilde{u}_x^2 + \tilde{\varphi}_x^2 dx ds \leq C \int_0^t D(s)\mathcal{H}(s) ds,
\]
and
\[
I_3 \leq \frac{\lambda}{4} \int_Q \tilde{u}_x^2 dx ds + C \int_Q (\tilde{\varphi}_x^2 + \tilde{\varphi}_x^2) dx ds \leq \frac{\lambda}{4} \int_Q \tilde{u}_x^2 dx ds + C \int_0^t D(s)\mathcal{H}(s) ds.
\]
Using $\|b, \bar{b}\|_{L^\infty(Q_T)} \leq C$, we obtain
\[
I_4 \leq \frac{\lambda}{4} \int_Q \tilde{u}_x^2 dx ds + C \int_Q |\tilde{b}|^2 dx ds \leq \frac{\lambda}{4} \int_Q \tilde{u}_x^2 dx ds + C \int_0^t D(s)\mathcal{H}(s) ds.
\]
Substituting (2.57)-(2.60) into (2.56) completes the proof of (2.55).

**Step 3** Claim that

$$
\int_{\Omega} \tilde{\theta}^2 dx + \int_{Q_t} \tilde{\theta}_t^2 dx ds \\
\leq C \sqrt{\mu} + \epsilon \int_{Q_t} (\tilde{u}_x^2 + |\tilde{b}_x|^2) dx ds + \frac{C}{\epsilon} \int_0^t D(s)\mathbb{H}(s) ds, \quad \forall \epsilon > 0.
$$

(2.61)

From the equations of $\theta$ and $\tilde{\theta}$, we have

$$
(\rho \tilde{\theta})_t + (\rho \tilde{u})_x + \tilde{\rho} \tilde{\theta}_t + (\rho \tilde{u} + \tilde{\theta}) \tilde{\theta}_x + \gamma \rho \tilde{u} \tilde{\theta}_x + \gamma (\rho \tilde{\theta} + \tilde{\rho} \tilde{u}) \tilde{\theta}_x = [\kappa(\rho, \theta) \tilde{\theta}_x]_x + [(\kappa(\rho, \theta) - \kappa(\overline{\rho}, \overline{\theta})) \tilde{\theta}_x]_x + \lambda (u_x^2 - \overline{\theta}_x^2) + \mu |\tilde{w}_x|^2 + \nu (|\tilde{b}_x|^2 - |\overline{\mathbb{B}}_x|^2).
$$

Multiplying it by $\tilde{\theta}$ and integrating over $Q_t$ give

$$
\frac{1}{2} \int_{\Omega} \rho \tilde{\theta}^2 dx + \int_{Q_t} \kappa \tilde{\theta}_x^2 dx ds \\
= -\int_{Q_t} \tilde{\rho} \tilde{\theta}_x \tilde{\theta}_x dx ds - \int_{Q_t} (\tilde{\rho} \tilde{u} + \tilde{\theta}) \tilde{\theta}_x dx ds - \int_{Q_t} \rho \tilde{u} \tilde{\theta}_x dx ds \\
- \int_{Q_t} \rho \tilde{u} \tilde{\theta}_x dx ds - \int_{Q_t} \tilde{\theta}_x \kappa(\rho, \theta) - \kappa(\overline{\rho}, \overline{\theta}) \tilde{\theta}_x dx ds \\
+ \lambda \int_{Q_t} (u_x + \overline{\theta} \tilde{\theta}_x dx ds + \mu \int_{Q_t} |\tilde{w}_x|^2 \tilde{\theta} dx ds \\
+ \nu \int_{Q_t} (|\tilde{b}_x|^2 - |\overline{\mathbb{B}}_x|^2) \tilde{\theta} dx ds =: \sum_{i=1}^9 E_i.
$$

By the Hölder inequality and Young inequality, we have

$$
E_1 + E_2 + E_3 \leq C \int_0^t \left( \int_{\Omega} (\tilde{\rho}^2 + \tilde{u}^2) dx \right)^{1/2} \left( \int_{\Omega} \tilde{\theta}^2 (\tilde{\theta}_t^2 + \tilde{\theta}_x^2 + \overline{\theta}_x^2) dx \right)^{1/2} dt \\
\leq C \int_0^t \left( \int_{\Omega} (\tilde{\theta}_t^2 + \overline{\theta}_x^2 + \overline{\theta}_x^2) dx \right)^{1/2} \left( \int_{\Omega} (\tilde{\rho}^2 + \tilde{u}^2) dx \right)^{1/2} \|\tilde{\theta}\|_{L^\infty(\Omega)} dt \\
\leq C \int_0^t \left( \int_{\Omega} (\tilde{\theta}_t^2 + \overline{\theta}_x^2 + \overline{\theta}_x^2) dx \right) \left( \int_{\Omega} (\tilde{\rho}^2 + \tilde{u}^2) dx \right) dt + \int_0^t \|\tilde{\theta}\|^2_{L^\infty(\Omega)} ds \\
\leq C \int_0^t D(s)\mathbb{H}(s) ds + \frac{\nu_1}{4} \int_{Q_t} \tilde{\theta}_x^2 dx ds.
$$

Hence

$$
E_5 + E_6 + E_7 \leq \epsilon \int_{Q_t} \tilde{u}_x^2 dx ds + \frac{C}{\epsilon} \int_0^t \left( 1 + \|u_x\|^2_{L^\infty(\Omega)} + \|\overline{\theta}_x\|^2_{L^\infty(\Omega)} \right) \int_{\Omega} \tilde{\theta}^2 dx ds \\
\leq \epsilon \int_{Q_t} \tilde{u}_x^2 dx ds + \frac{C}{\epsilon} \int_0^t D(s)\mathbb{H}(s) ds, \quad \forall \epsilon \in (0, 1).
$$

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By $C^{-1} \leq \rho, \overline{\rho}, \theta, \overline{\theta} \leq C$, we obtain that $|\kappa(\rho, \theta) - \kappa(\overline{\rho}, \overline{\theta})| \leq C(|\overline{\rho}| + |\overline{\theta}|)$. Thus,

$$E_0 \leq \frac{K_1}{4} \int_{Q_t} \rho \, dx \, ds + C \int_{Q_t} \rho \overline{\theta} \, (\overline{\rho}^2 + \overline{\theta}^2) \, dx \, ds$$

$$\leq \frac{K_1}{4} \int_{Q_t} \rho \, dx \, ds + C \int_{0}^{t} \|\overline{\theta}\|_{L^\infty(\Omega)} \int_{\Omega} (\overline{\rho}^2 + \overline{\theta}^2) \, dx \, ds$$

$$\leq \frac{K_1}{4} \int_{Q_t} \rho \, dx \, ds + C \int_{0}^{t} D(s) \mathbb{H}(s) \, ds.$$

By (2.42), we have

$$E_8 \leq C \sqrt{\mu}.$$

For $E_9$, by noticing $|b_x|^2 - |\overline{b}_x|^2 = (b_x + \overline{b}_x) \cdot \overline{b}_x$, we obtain

$$E_9 \leq \varepsilon \int_{Q_t} |\overline{b}_x|^2 \, dx \, ds + \frac{C}{\varepsilon} \int_{0}^{t} \left( \|b_x\|^2_{L^\infty(\Omega)} + \|\overline{b}_x\|^2_{L^\infty(\Omega)} \right) \int_{\Omega} \overline{\theta}^2 \, dx \, ds$$

$$\leq \varepsilon \int_{Q_t} |\overline{b}_x|^2 \, dx \, ds + \frac{C}{\varepsilon} \int_{0}^{t} D(s) \mathbb{H}(s) \, ds, \quad \forall \varepsilon > 0.$$

Substituting these estimates into (2.62) completes the proof of (2.61).

**Step 4** Claim that

$$\int_{\Omega} |\overline{\bar{w}}|^2 \, dx \leq C \sqrt{\mu} + \varepsilon \int_{Q_t} |\overline{b}_x|^2 \, dx \, ds + \frac{C}{\varepsilon} \int_{0}^{t} D(s) \mathbb{H}(s) \, ds, \quad \forall \varepsilon > 0. \quad (2.63)$$

From the equations for $w$ and $\bar{w}$, we have $\rho \overline{\bar{w}_t} + \rho u \overline{\bar{w}_x} + \rho \overline{\bar{w}} \overline{\bar{w}_x} - \overline{b}_x + \frac{\rho}{\rho} \overline{\bar{b}_x} = \mu w_{xx}$. Taking the inner product of the identity with $\overline{\bar{w}}$ and integrating over $Q_t$ gives

$$\frac{1}{2} \int_{\Omega} \rho |\overline{\bar{w}}|^2 \, dx = \int_{Q_t} (\mu w_{xx} \cdot \overline{\bar{w}} - \rho u \overline{\bar{w}_x} \cdot \overline{\bar{w}} + \overline{b}_x \cdot \overline{\bar{w}} - \frac{\rho}{\rho} \overline{\bar{b}_x} \cdot \overline{\bar{w}}) \, dx \, ds$$

$$\leq C \mu^2 \int_{Q_t} |w_{xx}|^2 \, dx \, ds + C \int_{Q_t} \overline{\bar{u}}^2 |\overline{\bar{w}_x}|^2 \, dx \, ds$$

$$+ \frac{C}{\varepsilon} \int_{0}^{t} D(s) \mathbb{H}(s) \, ds + \varepsilon \int_{Q_t} |\overline{b}_x|^2 \, dx \, ds, \quad \forall \varepsilon > 0. \quad (2.64)$$

Since

$$|\overline{u}(x, t)| = |u(x, t) - \overline{u}(x, t)| = \left| \int_{0}^{x} \overline{u}_x \, dx \right| \leq \left( \int_{0}^{1} \overline{u}_x^2 \, dx \right)^{1/2} \omega^{1/2}(x), \forall x \in [0, 1/2],$$

$$|\overline{u}(x, t)| = |u(x, t) - \overline{u}(x, t)| = \left| \int_{x}^{1} \overline{u}_x \, dx \right| \leq \left( \int_{0}^{1} \overline{u}_x^2 \, dx \right)^{1/2} \omega^{1/2}(x), \forall x \in [1/2, 1],$$

we have $|\overline{u}(x, t)|^2 \leq \left( \int_{0}^{1} \overline{u}_x^2 \, dx \right) \omega(x)$. Thus, from Lemma 2.13 and (2.55), we obtain

$$\int_{Q_t} \overline{\bar{u}}^2 |\overline{\bar{w}_x}|^2 \, dx \, ds \leq \int_{0}^{t} \left( \int_{0}^{1} \overline{u}_x^2 \, dx \right) \left( \int_{\Omega} |\overline{\bar{w}_x}|^2 \omega \, dx \right) \, ds \leq C \int_{0}^{t} D(s) \mathbb{H}(s) \, ds.$$
Substituting this into (2.64) completes the proof of (2.63).

**Step 5** Claim that

\[
\int_{\Omega} |\tilde{b}|^2 dx + \int_{Q_t} |\tilde{b}_x|^2 dx ds \leq C \int_0^t D(s) H(s) ds. \tag{2.65}
\]

From the equations of \( b \) and \( \tilde{b} \), we have that \( \tilde{b}_t + (u \tilde{b})_x + (\tilde{u} b)_x - \tilde{w}_x - \nu \tilde{b}_{xx} = 0 \). Taking the inner product of the identity with \( \tilde{b} \) and integrating over \( Q_t \) gives

\[
\frac{1}{2} \int_{\Omega} |\tilde{b}|^2 dx + \nu \int_{Q_t} |\tilde{b}_x|^2 dx ds = -\frac{1}{2} \int_{Q_t} u_x |\tilde{b}|^2 dx ds + \int_{Q_t} (\tilde{u} \tilde{b} - \tilde{b} \cdot \tilde{w}) dx ds
\]

\[
\leq \frac{\nu}{2} \int_{Q_t} |\tilde{b}_x|^2 dx ds + C \int_0^t D(s) H(s) ds.
\]

Thus, the claim (2.65) follows.

Combining the above five steps and taking a small constant \( \epsilon > 0 \), we complete the proof of Theorem 1.1(iii) by the Gronwall inequality. Therefore, the proof of Theorem 1.1 is completed.

### 3 Proof of Theorem 1.3

From \( \sup_{0 < t < T} \int_{\Omega} (|w_x|^2 + |\overline{w}_x|^2) \omega dx \leq C \) obtained in Theorem 1.1, we have

\[
\sup_{0 < t < T} \left( \int_{\delta}^{1-\delta} (|w_x|^2 + |\overline{w}_x|^2) dx \right) \leq \frac{C}{\delta}, \quad \forall \delta \in (0, 1/2).
\]

Using the embedding theorem, Theorem 1.1(iii) and the Hölder inequality, we have

\[
\|w - \overline{w}\|_{L^\infty(\delta, 1-\delta)} \leq C \left( \int_{\Omega} |w - \overline{w}|^2 dx + \int_{\delta}^{1-\delta} |w - \overline{w}| \|w_x - \overline{w}_x\| dx \right)
\]

\[
\leq C \sqrt{\mu} + C \left( \int_{\delta}^{1-\delta} |w - \overline{w}|^2 dx \int_{\delta}^{1-\delta} |w_x - \overline{w}_x|^2 dx \right)^{1/2}
\]

\[
\leq C \sqrt{\mu} + C \left( \frac{\sqrt{\mu}}{\delta} \right)^{1/2}, \quad \forall \delta \in (0, 1/2).
\]

This completes the proof of Theorem 1.3.

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