Obstructions for bounded shrub-depth and rank-depth

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Abstract

Shrub-depth and rank-depth are dense analogues of the tree-depth of a graph. It is well known that a graph has large tree-depth if and only if it has a long path as a subgraph. We prove an analogous statement for shrub-depth and rank-depth, which was conjectured by Hliněný, Kwon, Obdržálek, and Ordyniak [Tree-depth and vertex-minors, European J. Combin. 2016]. Namely, we prove that a graph has large rank-depth if and only if it has a vertex-minor isomorphic to a long path. This implies that for every integer $t$, the class of graphs with no vertex-minor isomorphic to the path on $t$ vertices has bounded shrub-depth.

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1 Introduction

Nešetřil and Ossona de Mendez [17] introduced the tree-depth of a graph $G$, which is defined as the minimum height of a rooted forest whose closure contains the graph $G$ as a subgraph. This concept has been proved to be very useful, in particular in the study of graph classes of bounded expansion [18]. Similar to the grid theorem for tree-width of Robertson and Seymour [24], it is known that a graph has large tree-depth if and only if it has a long path as a subgraph, see [17, Proposition 6.1]. For more information on tree-depth, the readers are referred to the surveys [20, 17] by Nešetřil and Ossona de Mendez.

There have been attempts to define an analogous concept suitable for dense graphs. For tree-width, this line of research has resulted in width parameters such as clique-width [3] and rank-width [22]. In a conference paper published in 2012, Ganian, Hliněný, Nešetřil, Obdržálek, Ossona de Mendez, and Ramadurai [8] introduced the shrub-depth of a graph class, as an extension of tree-depth for dense graphs. Recently, DeVos, Kwon, and Oum [6] introduced the rank-depth of a graph as an alternative to shrub-depth and showed that shrub-depth and rank-depth are equivalent in the following sense.

**Theorem 1.1** (DeVos, Kwon, and Oum [6]). A class of graphs has bounded rank-depth if and only if it has bounded shrub-depth.

Theorem 1.1 allows us to work exclusively with rank-depth going forward, and we omit the definition of shrub-depth. The definition of rank-depth is presented in Section 2.

One useful feature of rank-depth is that it does not increase under taking vertex-minors. In other words, if $H$ is a vertex-minor of $G$, then the rank-depth of $H$ is at most that of $G$. This allows us to consider obstructions for having small rank-depth in terms of vertex-minors. DeVos, Kwon, and Oum [6] showed that the rank-depth of the $n$-vertex path is larger than $\log n / \log(1 + 4 \log n)$ for $n \geq 2$ and thus graphs having a long path as a vertex-minor have large rank-depth. Hliněný, Kwon, Obdržálek, and Ordyniak [11] conjectured that the converse is also true. Their original conjecture was stated in terms of shrub-depth but is equivalent by Theorem 1.1. We prove their conjecture as follows.

**Theorem 1.2.** For every positive integer $t$, there exists an integer $N(t)$ such that every graph of rank-depth at least $N(t)$ contains a vertex-minor isomorphic to the path on $t$ vertices.
Courcelle and Oum [4] showed that there is a CMSO$_1$ transduction that maps a graph to its vertex-minors. Therefore, Theorem 1.2 implies that a class $G$ of graphs has bounded rank-depth if and only if for every CMSO$_1$ transduction $\tau$, there exists an integer $t$ such that $P_t \not\in \tau(G)$, which was conjectured by Ganian, Hliněný, Nešetřil, Obdržálek, and Ossona de Mendez [7].

If we apply the same proof for bipartite graphs, then we prove the following theorem on pivot-minors of graphs. Pivot-minors are more restricted in a sense that every pivot-minor of a graph is a vertex-minor but not every vertex-minor is a pivot-minor. This theorem allows us to deduce a corollary for binary matroids of large branch-depth.

**Theorem 1.3.** For every positive integer $t$, there exists an integer $N(t)$ such that every bipartite graph of rank-depth at least $N(t)$ contains a pivot-minor isomorphic to $P_t$.

The paper is organized as follows. In Section 2, we review vertex-minors and rank-depth and prove a few useful properties related to rank-depth. In Section 3, we present the proof of Theorem 1.2. In Section 4, we obtain Theorem 1.3 and discuss its consequence to binary matroids of large branch-depth. Finally, in Section 5 we conclude the paper by giving some remarks on linear $\chi$-boundedness of graphs with no $P_t$ vertex-minors.

2 Preliminaries and basic lemmas

All graphs in this paper are simple, meaning that neither loops nor parallel edges are allowed. For two sets $X$ and $Y$, we write $X \Delta Y$ for $(X \setminus Y) \cup (Y \setminus X)$.

Let $G$ be a graph. We write $V(G)$ and $E(G)$ for the vertex set and the edge set of $G$, respectively. For a vertex $v$ of $G$, we write $N_G(v)$ to denote the set of all neighbors of $v$ in $G$. For a vertex $v$ of $G$, let $G - v$ denote the graph obtained from $G$ by removing $v$ and all edges incident with $v$. For an edge $e$ of $G$, let $G - e$ denote the graph obtained from $G$ by removing $e$. For a vertex subset $S$ of $G$, we write $G[S]$ for the subgraph of $G$ induced by $S$. We write $\overline{G}$ for the complement of $G$; that is, $u$ and $v$ are adjacent in $G$ if and only if they are not adjacent in $\overline{G}$.

We write $A(G)$ for the adjacency matrix of $G$ over the binary field, that is, the $V(G) \times V(G)$ matrix over the binary field such that the $(x, y)$-entry is one if $x \neq y$ and $x$ is adjacent to $y$ in $G$, and zero otherwise. For an $X \times Y$ matrix $M$ and $X' \subseteq X$, $Y' \subseteq Y$, we write $M[X', Y']$ for the $X' \times Y'$ submatrix of $M$. 
Let $P_n$ denote the path on $n$ vertices, and let $K_n$ denote the complete graph on $n$ vertices. The radius of a tree is the minimum $r$ such that there is a node having distance at most $r$ from every node.

For two $n$-vertex graphs $G$ and $H$ with fixed orderings $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ on their respective vertex sets, let $G \boxtimes H$ be the graph with vertex set $V(G) \cup V(H)$ such that $(G \boxtimes H)[V(G)] = G$, $(G \boxtimes H)[V(H)] = H$, and for all $i, j \in \{1, 2, \ldots, n\}$, $v_i w_j \in E(G \boxtimes H)$ if and only if $i \geq j$. See Figure 1 for an example. An induced subgraph isomorphic to $G \boxtimes H$ for some $G$ and $H$ is called a semi-induced half-graph in [19].

2.1 Vertex-minors

For a vertex $v$ of a graph $G$, local complementation at $v$ is an operation which results in a new graph $G \hat{v}$ on $V(G)$ such that

$$E(G \hat{v}) = E(G) \Delta \{xy : x, y \in N_G(v), x \neq y\}.$$

For an edge $uv$ of a graph $G$, the operation of pivoting $uv$, denoted $G \hat{uv}$, is defined as $G \hat{uv} := G \hat{u} \hat{v} \hat{u}$. See Oum [21] for further background and properties of local complementation and pivoting. In particular, note that if $G$ is bipartite, then so is $G \hat{uv}$.

A graph $H$ is locally equivalent to $G$ if $H$ can be obtained from $G$ by a sequence of local complementations. A graph $H$ is pivot equivalent to $G$ if $H$ can be obtained from $G$ by a sequence of pivots. A graph $H$ is a vertex-minor of $G$ if $H$ is an induced subgraph of a graph that is locally equivalent to $G$. Finally, a graph $H$ is a pivot-minor of $G$ if $H$ is an induced subgraph of a graph that is pivot equivalent to $G$.

For a subset $S$ of $V(G)$, let $\rho_G(S)$ be the rank of the $S \times (V(G)\setminus S)$ submatrix of $A(G)$. This function is called the cut-rank function of $G$. It is easy to show that the cut-rank function is invariant under taking local complementations, again see Oum [21]. Thus we have the following fact.
Lemma 2.1. If $H$ is a vertex-minor of $G$ and $X \subseteq V(G)$, then

$$\rho_H(X \cap V(H)) \leq \rho_G(X).$$

The following lemmas will be used to find a long path.

Lemma 2.2 (Kim, Kwon, Oum, and Sivaraman [12, Lemma 5.6]). The graph $K_n \boxminus K_n$ has a pivot-minor isomorphic to $P_{n+1}$.

Lemma 2.3 (Kwon and Oum [13, Lemma 2.8]). The graph $K_n \boxminus K_n$ has a pivot-minor isomorphic to $P_{2n}$.

2.2 Rank-depth

We now review the notion of rank-depth, which was introduced by DeVos, Kwon, and Oum [6]. A decomposition of a graph $G$ is a pair $(T, \sigma)$ of a tree $T$ and a bijection $\sigma$ from $V(G)$ to the set of leaves of $T$. The radius of a decomposition $(T, \sigma)$ is the radius of the tree $T$. For a non-leaf node $v \in V(T)$, the components of the graph $T - v$ give rise to a partition $P_v$ of $V(G)$ by $\sigma$. The width of $v$ is defined to be

$$\max_{P' \subseteq P_v} \rho_G \left( \bigcup_{X \in P'} X \right).$$

The width of the decomposition $(T, \sigma)$ is the maximum width of a non-leaf node of $T$. We say that a decomposition $(T, \sigma)$ is a $(k, r)$-decomposition of $G$ if the width is at most $k$ and the radius is at most $r$. The rank-depth of a graph $G$ is the minimum integer $k$ such that $G$ admits a $(k, k)$-decomposition. If $|V(G)| < 2$, then there is no decomposition and the rank-depth is zero. Note that every tree in a decomposition has radius at least one and therefore the rank-depth of a graph is at least one if $|V(G)| \geq 2$.

By Lemma 2.1, it is easy to see the following.

Lemma 2.4 (DeVos, Kwon, and Oum [6]). If $H$ is a vertex-minor of $G$, then the rank-depth of $H$ is at most the rank-depth of $G$.

The next two lemmas will serve as a base case for induction in the proof of Theorem 1.2.

Lemma 2.5. Let $G$ be a graph of rank-depth $m$. Then $G$ has a connected component of rank-depth at least $m - 1$. 
Proof. If $m < 2$, then it is trivial, as the one-vertex graph has rank-depth zero. Thus, we may assume that $m \geq 2$.

Suppose for contradiction that every connected component of $G$ has rank-depth at most $m - 2$. Let $C_1, C_2, \ldots, C_t$ be the connected components of $G$. For each $i \in \{1, 2, \ldots, t\}$,

- if $C_i$ contains at least two vertices, then we take an $(m - 2, m - 2)$-decomposition $(T_i, \sigma_i)$ where $r_i$ is a node of $T_i$ having distance at most $m - 2$ to every node of $T_i$, and

- if $C_i$ consists of one vertex, then let $T_i$ be the one-node graph on $\{r_i\}$ and let $\sigma_i : V(C_i) \to \{r_i\}$ be the uniquely possible function.

We obtain a new decomposition $(T, \sigma)$ of $G$ by taking the disjoint union of $T_i$’s and adding a new node $r$ and adding edges $rr_i$ for all $i \in \{1, 2, \ldots, t\}$. For every vertex $v$ of $G$, define $\sigma(v) = \sigma_i(v)$ if $v$ is a vertex of $C_i$. Then $(T, \sigma)$ has depth at most $m - 1$ and width at most $m - 2$. This contradicts the assumption that $G$ has rank-depth $m$.

We conclude that $G$ has a connected component of rank-depth at least $m - 1$. \hfill \Box

The following lemma can be proven similarly to Lemma 2.5. For a graph $G$ of rank-depth $m$ and a non-empty vertex set $A$, it is easy to check that $G - A$ has rank-depth at least $m - |A|$, and by Lemma 2.5 $G - A$ has a connected component of rank-depth at least $m - |A| - 1$. But, by a direct argument, we can guarantee that there is a connected component of $G - A$ of rank-depth at least $m - |A|$. We include the full proof for completeness.

**Lemma 2.6.** Let $G$ be a graph of rank-depth $m$ and $A$ be a non-empty proper subset of $V(G)$. Then $G - A$ has a connected component of rank-depth at least $m - |A|$.

**Proof.** If $|A| \geq m$, then any connected component has rank-depth at least zero. Thus, we may assume that $|A| < m$. This implies that $m \geq 2$ as $A$ is non-empty.

Suppose for contradiction that every connected component of $G - A$ has rank-depth at most $m - |A| - 1$. Let $C_1, C_2, \ldots, C_t$ be the connected components of $G - A$. For each $i \in \{1, 2, \ldots, t\}$,

- if $C_i$ contains at least two vertices, then we take an $(m - |A| - 1, m - |A| - 1)$-decomposition $(T_i, \sigma_i)$ where $r_i$ is a node of $T_i$ having distance at most $m - |A| - 1$ to every node of $T_i$, and
• if \( C_i \) consists of one vertex, we set \( T_i \) to be the one-node graph on \( \{r_i\} \) and let \( \sigma_i : V(C_i) \rightarrow \{r_i\} \) be the uniquely possible function.

We obtain a new decomposition \((T, \sigma)\) of \( G \) by taking the disjoint union of \( T_i \)'s and adding a new node \( r \) and adding edges \( rr_i \) for all \( i \in \{1, 2, \ldots, t\} \), and additionally appending \(|A|\) leaves to \( r \) and assigning each vertex of \( A \) to a distinct leaf with the map \( \sigma \). For every vertex \( v \) of \( G - A \), define \( \sigma(v) = \sigma_i(v) \) if \( v \) is a vertex of \( C_i \). Then \((T, \sigma)\) has depth at most \( m - |A| \) and width at most \( m - 1 \). Because \(|A| \geq 1\), this contradicts the assumption that \( G \) has rank-depth \( m \).

We conclude that \( G - A \) has a connected component of rank-depth at least \( m - |A| \).

\( \square \)

**Lemma 2.7.** Let \( m \) and \( d \) be positive integers. Let \( G \) be a graph with a vertex partition \((A, B)\) such that connected components of \( G[A] \) and \( G[B] \) have rank-depth at most \( m \) and \( \rho_G(A) \leq d \). Then \( G \) has rank-depth at most \( m + d + 1 \).

**Proof.** Let \( C_1, \ldots, C_p \) be the connected components of \( G[A] \), and \( D_1, \ldots, D_q \) be the connected components of \( G[B] \). For each \( i \in \{1, 2, \ldots, p\} \),

• if \( C_i \) contains at least two vertices, then we take an \((m, m)\)-decomposition \((T_i, \sigma_i)\) where \( r_i \) is a node of \( T_i \) having distance at most \( m \) to every node of \( T_i \), and

• if \( C_i \) consists of one vertex, then set \( T_i \) as the one-node graph on \( \{r_i\} \) and \( \sigma_i : V(C_i) \rightarrow \{r_i\} \) as the uniquely possible function.

Similarly, we define \((F_j, \mu_j)\) for each \( D_j \) where \( f_j \) is a node of \( F_j \) having distance at most \( m \) to every node of \( F_j \).

Now, we obtain a new decomposition \((T, \sigma)\) of \( G \) as follows. Let \( T \) be the tree obtained by taking the disjoint union of all of \( T_i \)'s and \( F_j \)'s, adding new vertices \( x \) and \( y \), an edge \( xy \), edges \( xr_i \) for all \( i \in \{1, \ldots, p\} \), and edges \( yf_j \) for all \( j \in \{1, \ldots, q\} \). Define \( \sigma(v) = \sigma_i(v) \) if \( v \) is a vertex of \( C_i \), and \( \sigma(v) = \mu_j(v) \) if \( v \) is a vertex of \( D_j \). Then \((T, \sigma)\) has depth at most \( m + 2 \) and width at most \( m + d \). Because \( d \geq 1 \), \( G \) has rank-depth at most \( \max\{m + 2, m + d\} \leq m + d + 1 \). \( \square \)

### 2.3 Rank-width

We now review the definition of rank-width. A rank-decomposition of a graph \( G \) is a pair \((T, L)\) of a tree \( T \) whose vertices each have degree either
one or three, and a bijection \( L \) from \( V(G) \) to the set of leaves of \( T \). The \textit{width} of an edge \( e \) of \( T \) is the cut-rank in \( G \) of the set of all leaves assigned to one of the components of \( T - e \). The \textit{width} of the rank-decomposition \((T, L)\) is the maximum width of an edge of \( T \). Finally, the \textit{rank-width} of \( G \) is the minimum width over all rank-decompositions of \( G \). Graphs with at most one vertex do not admit rank-decompositions and we define their rank-width to be zero.

### 3 The proof

We write \( R(n; k) \) to denote the minimum number \( N \) such that every coloring of the edges of \( K_N \) with \( k \) colors induces a monochromatic complete subgraph on \( n \) vertices. The classical theorem of Ramsey \cite{ramsey1930problem} implies that \( R(n; k) \) exists.

The following lemma is well known. We include its proof for the sake of completeness.

**Lemma 3.1.** Let \( G \) be a graph of rank-width at most \( q \) and let \( M \subseteq V(G) \). If \( |M| \geq 3 + 1 \) for a positive integer \( k \), then there is a vertex partition \((X, Y)\) of \( G \) such that \( \rho_H(X) \leq q \) and \( \min(|M \cap X|, |M \cap Y|) > k \).

**Proof.** Suppose that there is no such vertex partition. Let \((T, L)\) be a rank-decomposition of width at most \( q \). For each edge \( uv \) of \( T \), let us orient \( e \) towards \( v \) if the component of \( T - e \) containing \( u \) has at most \( k \) vertices in \( L(M) \). By the assumption, every edge is oriented. Since \( T \) is acyclic, there is a node \( w \) of \( T \) such that all edges of \( T \) incident with \( w \) are oriented towards \( w \). But this implies that \( |M| \leq 3k \), a contradiction. \( \square \)

For a path \( P \) with an endpoint \( x \) and a graph \( H \) and a non-empty subset of vertices \( S \subseteq V(H) \), we denote by \((P, x) + (H, S)\) the graph obtained from the disjoint union of \( P \) and \( H \) by adding all edges between \( x \) and \( S \). We now prove our main proposition; Theorem 1.2 will follow quickly after.

**Proposition 3.2.** For all positive integers \( a, b, t, q \), there exists an integer \( f(a, b, t, q) \) such that every graph of rank-width at most \( q \) and rank-depth at least \( f(a, b, t, q) \) has a vertex-minor isomorphic to either \( P_t \) or \((P_a, x) + (H, S)\) where \( x \) is an endpoint of \( P_a \), \( H \) is a connected graph of rank-depth at least \( b \), and \( S \) is a non-empty subset of \( V(H) \).

**Proof.** For all positive integers \( b, t, q \), we set

\[
f(1, b, t, q) := b + 2,
\]
and for $a \geq 2$, we set
\begin{align*}
u &:= \max(3 \cdot (2^a - 1) + 1, t - 1), \\
r &:= R(u + 1; 2^{a-1}), \\
g_i &:= \begin{cases} b + q + 2 & \text{if } i = r, \\
f(a - 1, g_{i+1}, t, q) & \text{if } i \in \{0, 1, 2, \ldots, r - 1\}, \end{cases} \\
f(a, b, t, q) &:= g_0.
\end{align*}

We prove the proposition by induction on $a$. Let $G$ be a graph whose rank-depth is at least $f(a, b, t, q)$ and rank-width is at most $q$. If $a = 1$, then it has a component $G'$ of rank-depth at least $b + 1$ by Lemma 2.5. Let $v \in V(G')$. By Lemma 2.6, $G' - v$ has a connected component $H$ of rank-depth at least $b$. So, $(G'[\{v\}], v) + (H, N_G(v) \cap V(H))$ is the second outcome.

Thus, we may assume that $a \geq 2$. Suppose that $G$ has no vertex-minor isomorphic to $P_t$. We claim that $G$ contains the second outcome.

Let $H_0 := G$. Observe that $H_0$ has rank-depth at least \( f(a, b, t, q) = g_0 \).

For $i \in \{1, 2, \ldots, r\}$, we recursively find tuples $(A_i, x_i, H_i, S_i)$ from $H_{i-1}$ such that
\begin{itemize}
  \item $A_i$ is isomorphic to $P_{a-1}$ and $x_i$ is an endpoint of $A_i$,
  \item $H_i$ is a connected graph of rank-depth at least $g_i$,
  \item $S_i$ is a non-empty subset of $V(H_i)$, and
  \item $(A_i, x_i) + (H_i, S_i)$ is a vertex-minor of $H_{i-1}$.
\end{itemize}

Let $i \in \{1, 2, \ldots, r\}$ and assume that $H_{i-1}$ is a given graph of rank-depth at least $g_{i-1}$. Then by the induction hypothesis, $H_{i-1}$ has a vertex-minor $(A_i, x_i) + (H_i, S_i)$ where $A_i$ is isomorphic to $P_{a-1}$, $x_i$ is an endpoint of $A_i$, and

Figure 2: The graph $G_1[A_1 \cup \cdots \cup A_r \cup V(H_r)]$, when $r = 3$ and $a = 5$. 
Thus, as the latter is connected, 

\[ \text{H}_i \] is a connected graph of rank-depth at least \( g_i \), and \( S_i \) is a non-empty subset of \( V(H_i) \). By the choice of functions \( g_0, g_1, \ldots, g_r \), we can obtain the tuples for all \( i \in \{1, 2, \ldots, r\} \).

Observe that for \( i < j \), no vertex in \( V(A_i) \setminus \{x_i\} \) has a neighbor in \( H_i \), and therefore, the sequence of local complementations to obtain \( (A_j, x_j) + (H_j, S_j) \) from \( H_{j-1} \) does not change previous paths \( A_1, \ldots, A_{j-1} \), but may change the edges between \( x_1, x_2, \ldots, x_{j-1} \).

By definition, \( H_r \) is connected and has rank-depth at least \( g_r \). Let \( G_1 \) be the graph obtained from \( G \) by following the sequence of local complementations to obtain \( (A_1, x_1) + (H_1, S_1) \), \ldots, \( (A_r, x_r) + (H_r, S_r) \). See Figure 2 for a depiction. Note that \( G_1[V(H_i) \cup \{x_i\}] \) is connected for each \( i \), as \( H_i \) is connected, \( S_i \) is non-empty, and we apply local complementations only inside \( H_i \) to obtain later \( H_j \)’s.

If for some \( i \in \{1, 2, \ldots, r\} \), \( N_{G_1}(x_i) \cap V(H_r) = \emptyset \), then by taking a shortest path from \( x_i \) to \( V(H_r) \) in the graph \( G_1[V(H_i) \cup \{x_i\}] \), we can directly obtain the second outcome. So, we may assume that each of \( \{x_1, x_2, \ldots, x_r\} \) has a neighbor in \( V(H_r) \).

Note that for \( i < j \), only the endpoint \( x_i \) in \( A_i \) can have a neighbor in \( A_j \) in \( G_1 \), and therefore, there are \( 2^{a-1} \) possible ways of having edges between \( A_i \) and \( A_j \) in \( G_1 \). Since \( r = R(u+1; 2^{a-1}) \), by applying the theorem of Ramsey, we deduce that there exists a subset \( W \subseteq \{1, 2, \ldots, r\} \) of size \( u+1 \) such that for all \( i < j \) with \( i, j \in W \), \( \ell \) : the \( \ell \)-th vertex of \( A_j \) is adjacent to \( x_i \) in \( G_1 \) are identical.

If \( x_i \) has a neighbor in \( V(A_j - x_j) \) in \( G_1 \) for some \( i < j \) with \( i, j \in W \), then \( G_1 \) has \( \overline{K_u} \boxtimes \overline{K_u} \) or \( \overline{K_u} \boxtimes K_u \) as an induced subgraph. Since \( u \geq t - 1 \), by Lemmas \( 2.2 \) and \( 2.3 \), \( G_1 \) contains a pivot-minor isomorphic to \( P_t \), contradicting the assumption. So, for all \( i < j \) with \( i, j \in W \), \( x_i \) has no neighbors in \( V(A_j - x_j) \).

Note that \( \{x_i : i \in W\} \) is an independent set or a clique in \( G_1 \). If it is an independent set, then for some \( i' \in W \), we set

\[
\text{G}_2 := G_1 \text{ and } W' := W \setminus \{i'\}.
\]

If \( \{x_i : i \in W\} \) is a clique, then we choose a vertex \( x_{i'} \) for some \( i' \in W \) and locally complement at \( x_{i'} \). Then \( \{x_i : i \in W \setminus \{i'\}\} \) becomes an independent set. We set

\[
\text{G}_2 := G_1 * x_{i'} \text{ and } W' := W \setminus \{i'\}.
\]

Let \( M := \{x_i : i \in W'\} \) and \( H := G_2[V(H_r) \cup M \cup \{x_{i'}\}] \).

By definition, \( H \) is locally equivalent to the graph \( G_1[V(H_r) \cup M \cup \{x_{i'}\}] \). Thus, as the latter is connected, \( H \) is also connected. Similarly, since \( H_r = \)
$G_1[V(H_r)]$ has rank-depth at least $g_r$, $H$ has rank-depth at least $g_r$. Also, note that $H$ has rank-width at most $q$ and $M$ is an independent set of size $u \geq 3 \cdot (2^q - 1) + 1$ in $H$. Thus, by Lemma 3.1, $H$ admits a vertex partition $(X, Y)$ such that $|M \cap X| > 2^q - 1$, $|M \cap Y| > 2^q - 1$, and $\rho_H(X) \leq q$.

Since $H$ has rank-depth at least $g_r = b + q + 2$ and $\rho_H(X) \leq q$, by Lemma 2.7, $H[X]$ or $H[Y]$ has a connected component of rank-depth at least $b + 1$. Without loss of generality, we assume that $H[X]$ has a connected component $Q$ of rank-depth at least $b + 1$.

Now, if $M \cap Y$ has a vertex $x_i$ that has no neighbor in $Q$, then by taking a shortest path from $x_i$ to $Q$ in $H$, along with $A_i$, we can find the second outcome.

Thus, we may assume that in $H$, all vertices in $M \cap Y$ have a neighbor in $Q$. Since $\rho_H(X) \leq q$, there are at most $2^q - 1$ distinct non-zero rows in the matrix $A(H)[M \cap Y, V(Q)]$. As $|M \cap Y| \geq 2^q$, by the pigeon-hole principle, $H$ has two vertices $x_{i_1}$ and $x_{i_2}$ in $M \cap Y$ for some $i_1, i_2 \in W'$ that have the same neighborhood in $Q$.

First assume that $x_{i_1}$ has exactly one neighbor in $Q$, say $w$. As $Q$ has rank-depth at least $b + 1$, $Q - w$ has a connected component $Q'$ having rank-depth at least $b$ by Lemma 2.6. Then

$$(G_2[V(A_{i_1}) \cup \{w\}], w) + (Q', N_{G_2}(w) \cap V(Q'))$$

is the required second outcome. So, we may assume that $x_{i_1}$ has at least two neighbors in $Q$. Let $w$ be a neighbor of $x_{i_1}$ in $Q$.

Since $x_{i_1}$ and $x_{i_2}$ have the same neighborhood in $Q$ and they are not adjacent, if we pivot $x_{i_2}w$, then the edges between $x_{i_1}$ and $N_H(x_{i_1}) \cap V(Q)$ are removed and $x_{i_2}$ becomes the unique neighbor of $x_{i_1}$ in $V(Q) \cup \{x_{i_2}\}$. Note that $G_2[V(Q) \cup \{x_{i_2}\}]$ is connected, and thus $(G_2 \wedge x_{i_2}w)[V(Q) \cup \{x_{i_2}\}]$ is also connected. As $Q$ has rank-depth at least $b + 1$, $(G_2 \wedge x_{i_2}w)[V(Q) \cup \{x_{i_2}\}] - x_{i_2}$ has a connected component $Q'$ that has rank-depth at least $b$. Then

$$(G_2 \wedge x_{i_2}w)[V(A_{i_1}) \cup \{x_{i_2}\}], x_{i_2}) + (Q', N_{G_2 \wedge x_{i_2}w}(x_{i_2}) \cap V(Q'))$$

is the second outcome. This proves the proposition.

Proposition 3.2 implies the following result.

**Theorem 3.3.** For all positive integers $t$ and $q$, there exists an integer $F(t, q)$ such that every graph of rank-width at most $q$ and rank-depth at least $F(t, q)$ contains a vertex-minor isomorphic to $P_t$. 

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Proof. We take $F(t, q) := f(t - 1, 1, t, q)$ where $f$ is the function in Proposition 3.2. \qed

A circle graph is the intersection graph of chords on a circle. It is easy to see that $P_t$ is a circle graph. We can derive Theorem 1.2 by taking $q = \beta(P_t)$ from the following recent theorem.

**Theorem 3.4** (Geelen, Kwon, McCarty, and Wollan [10]). For every circle graph $H$, there exists an integer $\beta(H)$ such that every graph of rank-width more than $\beta(H)$ contains a vertex-minor isomorphic to $H$.

**Theorem 1.2.** For every positive integer $t$, there exists an integer $N(t)$ such that every graph of rank-depth at least $N(t)$ contains a vertex-minor isomorphic to $P_t$.

*Proof.* We take $N(t) := F(t, \beta(P_t))$ where $\beta$ is the function given in Theorem 3.4 and $F$ is the function from Theorem 3.3. If a graph has rank-width more than $\beta(P_t)$, then by Theorem 3.4, it contains a vertex-minor isomorphic to $P_t$. So, we may assume that a graph has rank-width at most $\beta(P_t)$. Then by Theorem 3.3, it contains a vertex-minor isomorphic to $P_t$. \qed

4  **Pivot-minors**

We can prove a stronger result on bipartite graphs, by slightly modifying the proof of Proposition 3.2. Suppose that a given graph $G$ is bipartite in the proof of Proposition 3.2. The only place that we have to apply local complementation instead of pivoting is when the set $\{x_i : i \in W\}$ is a clique, and we want to change it into an independent set. But if $G$ is bipartite, then the obtained set $\{x_i : i \in W\}$ has no triangle, and so it is an independent set since $|W| \geq 3$. Therefore, we can proceed only with pivoting. For bipartite graphs, we can use the following theorem due to Oum [21], obtained as a consequence of the grid theorem for binary matroids [9].

**Theorem 4.1** (Oum [21]). For every bipartite circle graph $H$, there exists an integer $\gamma(H)$ such that every bipartite graph of rank-width more than $\gamma(H)$ contains a pivot-minor isomorphic to $H$.

Thus we deduce the following theorem for bipartite graphs.

**Theorem 1.3.** For every positive integer $t$, there exists an integer $N(t)$ such that every bipartite graph of rank-depth at least $N(t)$ contains a pivot-minor isomorphic to $P_t$. 

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Theorem 1.3 allows us to obtain the following corollary for binary matroids, solving a special case of a conjecture of DeVos, Kwon, and Oum [6] on general matroids. We need a few terms to state the corollary. The branch-depth of a matroid is defined analogously to the definition of the rank-depth obtained by replacing the cut-rank function with the matroid connectivity function [6]. Let $F_t$ be the fan graph, that is the union of $P_t$ with one vertex adjacent to all vertices of $P_t$, see Figure 3. As usual, $M(F_t)$ denotes the cycle matroid of $F_t$.

**Corollary 4.2.** For every positive integer $t$, there exists an integer $N(t)$ such that every binary matroid of branch-depth at least $N(t)$ contains a minor isomorphic to $M(F_t)$.

**Proof.** It is known [2, 21] that the connectivity function of a binary matroid is equal to the cut-rank function of a corresponding bipartite graph, called a fundamental graph. Furthermore for two binary matroids $M$ and $N$, if $N$ is connected, and a fundamental graph of $N$ is a pivot-minor of a fundamental graph of $M$, then either $N$ or $N^*$ is a minor of $M$, see Oum [21, Corollary 3.6]. Since $(M(F_t))^*$ has a minor isomorphic to $M(F_{t-1})$, we deduce the corollary from Theorem 1.3 because the path graph $P_{2t-1}$ is a fundamental graph of $M(F_t)$. \qed

We show that the class $\{K_n \boxtimes K_n : n \geq 1\}$ has unbounded rank-depth, while for every positive integer $n$, $K_n \boxtimes K_n$ has no pivot-minor isomorphic to $P_5$. It implies that contrary to Theorem 1.3, the class of graphs having no $P_n$ pivot-minor has unbounded rank-depth for each $n \geq 5$.

Kwon and Oum [16, Lemma 6.5] showed that for every integer $n \geq 2$, $K_n \boxtimes K_n$ contains a vertex-minor isomorphic to $P_{2n-2}$. Thus, $\{K_n \boxtimes K_n : n \geq 1\}$ has unbounded rank-depth.

Now, we show that for $n \geq 1$, $K_n \boxtimes K_n$ has no pivot-minor isomorphic to $P_5$. We prove a stronger statement that $K_n \boxtimes K_n$ has no pivot-minor isomorphic to $K_{1,3}$. Dabrowski et al. [5] characterized the class of graphs having no pivot-minor isomorphic to $K_{1,3}$ in terms of forbidden induced subgraphs. See Figure 4 for bull, $W_4$, and $BW_3$.  

![Figure 3: The fan graph $F_5$.](image-url)
Theorem 4.3 (Dabrowski et al. [5]). A graph has a pivot-minor isomorphic to $K_{1,3}$ if and only if it has an induced subgraph isomorphic to one of $K_{1,3}$, $P_5$, bull, $W_4$, and $BW_3$.

Lemma 4.4. For $n \geq 1$, $K_n \boxtimes K_n$ has no induced subgraph isomorphic to one of $K_{1,3}$, $P_5$, bull, $W_4$, and $BW_3$.

Proof. As the maximum size of an independent set in $K_n \boxtimes K_n$ is 2, $K_n \boxtimes K_n$ has no induced subgraph isomorphic to one of $K_{1,3}$, $P_5$, and bull.

Also $K_n \boxtimes K_n$ has no induced cycle of length 4 because such a cycle should contain two vertices in each $K_n$ but the edges between two $K_n$'s have no induced matching of size 2. Therefore, it has no induced subgraph isomorphic to $W_4$ or $BW_3$. 

By Theorem 4.3 and Lemma 4.4, $K_n \boxtimes K_n$ has no pivot-minor isomorphic to $K_{1,3}$, and to $P_5$. Thus, for all $n \geq 5$, the class of graphs having no $P_n$ pivot-minor includes $\{K_n \boxtimes K_n : n \geq 1\}$, which has unbounded rank-depth. It may be interesting to see whether every graph with sufficiently large rank-depth contains either $P_n$ or $K_n \boxtimes K_n$ as a pivot-minor. We leave it as an open question.

Question 1. Does there exist a function $f$ such that for every $n$, every graph with rank-depth at least $f(n)$ contains a pivot-minor isomorphic to $P_n$ or $K_n \boxtimes K_n$?

5 Concluding remarks

5.1 Linear $\chi$-boundedness

We define linear rank-width. For an ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set of a graph $G$, its width is defined as the maximum of $\rho_G(\{v_i, \ldots, v_1\})$ for all $i \in \{1, 2, \ldots, n - 1\}$, and the linear rank-width of $G$ is defined as the minimum width of all orderings of $G$. If $|V(G)| < 2$, then the linear rank-width of $G$ is defined as 0.
Graphs of bounded rank-depth have bounded linear rank-width, which was already known through the notions of shrub-depth and linear clique-width [7]. Kwon and Oum [15] proved it directly as follows.

**Proposition 5.1** (Kwon and Oum [15]). *Every graph of rank-depth $k$ has linear rank-width at most $k^2$.***

We write $\chi(G)$ to denote the chromatic number of $G$ and $\omega(G)$ to denote the maximum size of a clique of $G$. A class $\mathcal{C}$ of graphs is $\chi$-bounded if there is a function $f$ such that $\chi(H) \leq f(\omega(H))$ for all induced subgraphs $H$ of a graph in $\mathcal{C}$. In addition, if $f$ can be taken as a polynomial function, then $\mathcal{C}$ is polynomially $\chi$-bounded. If $f$ can be taken as a linear function, then $\mathcal{C}$ is linearly $\chi$-bounded.

**Proposition 5.2** (Nešetřil, Ossona de Mendez, Rabinovich, and Siebertz [19]). *For every positive integer $r$, there exists an integer $c(r)$ such that for every graph $G$ of linear rank-width at most $r$,*

$$\chi(G) \leq c(r) \omega(G).$$

By combining Proposition 5.1 and Proposition 5.2, we can prove the following, which answers a previous question by Kim, Kwon, Oum, and Sivaraman [12].

**Theorem 5.3.** *For every positive integer $t$, the class of graphs with no vertex-minor isomorphic to $P_t$ is linearly $\chi$-bounded.*

We remark that there is an alternative way to prove Theorem 5.3 without using linear rank-width. First, DeVos, Kwon, and Oum [6, Lemma 4.10] showed that if a graph has rank-depth $k$, then it has an $(a, k)$-shrubbery where

$$a = (1 + o(1))2^{(2^{2k+1}(2^{2k+2} - 1) + 1)k/2}.$$

(Please see [6] for the definition of an $(a, k)$-shrubbery.) Lemma 2.16 of Nešetřil, Ossona de Mendez, Rabinovich, and Siebertz [19] states that every class of bounded shrub-depth can be partitioned into bounded number of vertex-disjoint induced subgraphs, each of which is a cograph. Its (short and easy) proof shows that a graph with an $(a, k)$-shrubbery can be partitioned into at most $a$ vertex-disjoint induced subgraphs, each of which is a cograph. Since cographs are perfect, we deduce that if $G$ has rank-depth at most $k$, then $\chi(G) \leq \omega(G)(1 + o(1))2^{(2^{2k+1}(2^{2k+2} - 1) + 1)k/2}$. 
5.2 When does the class of $H$-vertex-minor-free graphs have bounded rank-depth?

For a set $H$ of graphs, we say that $G$ is $H$-minor-free if $G$ has no minor isomorphic to a graph in $H$, and $G$ is $H$-vertex-minor-free if $G$ has no vertex-minor isomorphic to a graph in $H$. Robertson and Seymour [24] showed that $H$-minor-free graphs have bounded tree-width if and only if $H$ contains a planar graph. As an analogue, Geelen, Kwon, McCarty, and Wollan [10] showed that $H$-vertex-minor-free graphs have bounded rank-width if and only if $H$ contains a circle graph. Interestingly, Theorem 1.2 allows us to characterize the classes $H$ such that $H$-vertex-minor-free graphs have bounded rank-depth. This is due to the following theorem; the equivalence between (a) and (b) was shown by Kwon and Oum [13] and the equivalence between (a) and (c) was shown by Adler, Farley, and Proskurowski [1].

**Theorem 5.4** (Kwon and Oum [13]; Adler, Farley, and Proskurowski [1]). Let $H$ be a graph. The following are equivalent.

(a) $H$ has linear rank-width at most one.

(b) $H$ is a vertex-minor of a path.

(c) $H$ has no vertex-minor isomorphic to $C_5$, $N$, or $Q$ in Figure 5.

We define linear rank-width in the next subsection. Here, we only need the fact that linear rank-width does not increase when we take vertex-minors and that paths have linear rank-width 1 and arbitrary large rank-depth to deduce the following corollary from Theorems 5.4 and 1.2.

**Corollary 5.5.** Let $H$ be a set of graphs. Then the following are equivalent.

(a) The class of $H$-vertex-minor-free graphs has bounded rank-depth.

(b) $H$ contains a graph of linear rank-width at most one.

(c) $H$ contains a graph with no vertex-minor isomorphic to $C_5$, $N$, or $Q$.

\[\begin{array}{ccc}
C_5 & N & Q \\
\end{array}\]

Figure 5: Obstructions for being a vertex-minor of a path.
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