NOT-QUITE-HAMILTONIAN REDUCTION

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Abstract. The not-quite-Hamiltonian theory of singular reduction and reconstruction is described. This includes the notions of both regular and collective Hamiltonian reduction and reconstruction.

1. Introduction

Ever since the beginning of analytical mechanics, there has been an effort to understand how to reduce the equations of motion given the presence of constraints of one form or another. For example, the motion of a particle that is described by Lagrangian or Hamiltonian formalism that is constrained to move on a submanifold of configuration space will, by employing D'Alembert's principle, give a reduced system of equations of the same form. A much more sophisticated, and perhaps the most striking early example of such considerations was Jacobi's elimination of the node in the three body problem. Such examples showed that there was a relation between symmetry and conservation laws, and these were explained for variational problems by Noether in her important work [24]. Somewhat dual to this, because of the Hamilton-Jacobi theory, there was an evolving understanding of the nature of symmetry and conservation laws on the Hamiltonian side, especially in understanding non-abelian symmetry groups and the reduction of Hamilton's equations. The first serious counterpart to Noether's theorem on the Hamiltonian side was the paper of Meyer [22]. In this paper, Meyer showed that the free and proper Hamiltonian action of a connected Lie group on a symplectic manifold led to a reduced symplectic manifold and the reduced dynamics was Hamiltonian. The importance of this theorem is the realization that the structure of the equations of motion has been reproduced under reduction by symmetry. This is a recurrent theme in further work. This work was followed by a torrent of papers on reduction, all with somewhat different emphases. For example, some, such as Marsden and Weinstein [21], stressed the role of the momentum map, while others, such as Churchill, Kummer and Rod [9], looked at the relations of symmetry to averaging. During the 1980s and 1990s there was a growing awareness of the need to include singularity and the desire to discuss dynamics on the reduced space. A key observation in this time was that the dynamics on the reduced space could be described by the Poisson bracket on the invariant functions. Some of the notable works using this idea were

1 Such a judgement call is always to some extent a question of taste. The reader may have some sympathy for our point of view after rereading the earlier work of Arnol'd [2] and Smale [28].
those of Gotay and Bos [15], Arms, Cushman and Gotay [1], and Sjamaar and Lerman [27]. At this point it had become clear that the reduced space had dynamics, and that it could be described stratum by stratum using the Poisson bracket.

Since then, it is now known that the reduced space is not only a topological space, but also has a differential structure, which is completely described by an algebra of smooth functions. These smooth functions are push-forwards of functions on the original space that are invariant under the group action. Such singular spaces are described naturally by the theory of subcartesian differential spaces, and in the case under consideration, support dynamics as well because the algebra of smooth functions has a Poisson structure. It is our view that satisfying the dual requirements of describing the analytic structure of the singularities of the reduced space and defining the reduced dynamics provides a powerful justification for our use of differential spaces.

A related development in the theory of constrained Hamiltonian systems with symmetry has been the reduction of non-holonomic constraints. The regular theory for transverse linear constraints was considered by Koiller [19], and extended to the nontransverse case by Bates and Śniatycki [4]. Regular reduction of nonlinear non-holonomic constraints was given by de Leon and de Diego in [13], and singular examples involving linear constraints were considered by Bates in [3]. The singular reduction of nonlinear nonholonomic constraints was given by Bates and Nester in [7]. What is notable here is that the formulation is once again in terms of invariant functions and the Poisson bracket, the wrinkle being that the Hamiltonian operator need no longer be an invariant function, and so the reduced dynamics is given by an outer Poisson morphism.

Of course, constraints in mechanics do not have to have anything to do with symmetry. There is a less mature, but somewhat parallel stream of development that tries to understand the nature of the constraints that show up in systems where the Lagrangian is degenerate in the sense that the Legendre transformation does not define a local diffeomorphism. This theory, inaugurated by Dirac in [14], (giving what is now called the Dirac constraint algorithm), describes a way to produce a Hamiltonian on a submanifold of the phase space. The constraint algorithm has been geometrized by Gotay, Nester and Hinds [16] and Lusanna [20]. However, the nature of such constraints in the Lagrangian is such that the initial data set, which is the subset of the original space on which the Lagrangian is defined actually has local solutions of the Euler-Lagrange equations, can be a singular space. Our experience is that the best way to deal with such constraints and their singularities, as well as the related constructions of reduced spaces, first class functions, etc., is to employ the theory of differential spaces [5].

This note generalizes the singular reduction and reconstruction of a Hamiltonian dynamical system to the case in which the Hamiltonian is not necessarily invariant under the proper Hamiltonian action of a connected Lie group on a symplectic manifold, but nevertheless still manages to have reduced dynamics. Consistent

\footnote{It is our contention that this is about as far as the theory can be developed without the notion of differential spaces. The results of this stage of the development of singular reduction are completely described in the monograph of Ratiu and Ortega [25].}
with the previous cases, the singular reduced dynamics is given in terms of a Poisson bracket on the invariant functions. The main difference in the not-quite Hamiltonian case with the singular Hamiltonian case is that the reduced dynamics is not given by the Poisson bracket of an invariant function with an invariant Hamiltonian, as now the bracket of the Hamiltonian with an invariant function is an outer Poisson morphism on the invariant functions. Furthermore, in a manner similar to Hamiltonian reconstruction, integration of the Hamiltonian dynamics is given by integration of an equation on the dual of the Lie algebra, after which the original dynamics is reconstructed from the reduced dynamics via integration with respect to a moving isotropy subgroup of the original group.

2. Preliminaries

Denote a Hamiltonian system by \((P, \omega, h)\). Here \(P\) is the phase space, \(\omega\) the symplectic form, and \(h\) is the Hamiltonian. The Hamiltonian vector field \(X_h\) satisfies Hamilton’s equations \(X_h \omega = dh\). Denote by \(G\) a connected Lie group, and by \(\phi\) its action on \(P\). A blanket assumption in this paper is that the action \(\phi\) is proper and Hamiltonian. Denote the momentum map for the action \(\phi\) by \(j: P \to g^*\).

The quotient space \(\bar{P} := P/G\), the space of \(G\)-orbits, is given the quotient topology. Because the action of the group \(G\) is proper, \(\bar{P}\) has a much richer structure than merely that of a topological space. In fact, \(\bar{P}\) is known to be a stratified subcartesian differential space (see [10] or [29]). In particular, this means that the ring of continuous functions on \(\bar{P}\), denoted \(C^\infty(\bar{P})\) (declared to be the smooth functions), which are push-forwards of smooth \(G\)-invariant functions on \(P\), satisfy the conditions

1. The family \(\{f^{-1}(I) \mid f \in C^\infty(\bar{P}) \text{ and } I \text{ is an open interval in } \mathbb{R}\}\)

is a subbasis for the topology of \(\bar{P}\).

2. If \(f_1, \ldots, f_n \in C^\infty(\bar{P})\) and \(F \in C^\infty(\mathbb{R}^n)\), then \(F(f_1, \ldots, f_n) \in C^\infty(\bar{P})\).

3. If \(f : \bar{P} \to \mathbb{R}\) is a function such that for each \(p \in \bar{P}\), there is an open neighbourhood \(U\) of \(p\) and a function \(f_p \in C^\infty(\bar{P})\) satisfying \(f_p|_U = f|_U\), then \(f \in C^\infty(\bar{P})\).

\(\bar{P}\) is subcartesian means that it is Hausdorff and each point \(p \in \bar{P}\) has a neighbourhood \(U\) diffeomorphic to a subset \(V\) of \(\mathbb{R}^n\). The stratification of \(\bar{P}\) is given by orbit type. Since the many technical details in the proof of this would lead us too far astray, we refer the reader to the discussions in [8], [10], or [29]. The reader should also note that because the group action is Hamiltonian, the stratification of the quotient space \(\bar{P}\) is determined by the Poisson bracket on the invariant functions. However, we state below definitions and results that are essential for this paper.

**Definition 2.1.** A differential space \(M\) is a topological space endowed with the ring \(C^\infty(M)\) of continuous functions that satisfy the three conditions above.
DEFINITION 2.2. A map \( \psi : M \to N \) between differential spaces \( M \) and \( N \) is smooth if \( \psi^* f \in C^\infty(M) \) for \( f \in C^\infty(N) \). A smooth map between differential spaces is a local diffeomorphism if it is a local homeomorphism with a smooth inverse.

THEOREM 2.3. For every derivation \( X \) of the ring of smooth functions on a subcartesian differential space \( M \), and each point \( p \in M \), there exists a unique maximal integral curve of \( X \) through \( p \).

Proof. The proof may be found in [29]. q.e.d.

The domain of the unique integral curve of \( X \) through \( p \), denote by \( (\exp tX)p \) the point on the integral curve of \( X \) through \( p \) corresponding to \( t \). This gives a local one-parameter group \( \exp tX \) of local transformations of \( M \).

DEFINITION 2.4. A derivation \( X \) of \( C^\infty(M) \) is a vector field on \( M \) if \( \exp tX \) is a local one-parameter group of local diffeomorphisms of \( M \).

THEOREM 2.5. Orbits of a family of vector fields on \( M \) are smooth manifolds immersed in \( M \).

Proof. The proof may be found in [29]. q.e.d.

THEOREM 2.6. If \( M = \bar{P} \) is the space of orbits of a proper action of a connected Lie group \( G \) on a manifold \( P \), then orbits of the family of all vector fields on \( M \) coincide with the strata of the orbit type stratification of \( P \).

Proof. The proof may be found in [29]. q.e.d.

A main concern of this paper is when the connected Lie group \( G \) has a proper Hamiltonian action \( \phi \) on \( P \) and this action can be divided out to produce a reduced space \( \bar{P} \) that also has reduced dynamics. As a first step we extend a well-known theorem for free and proper actions to the case of merely proper actions. Let \( \phi : G \times P \to P : (g, p) \mapsto \phi(g, p) =: \phi_g(p) \) be a proper action of the connected Lie group \( G \) on the manifold \( P \) and let \( \rho : P \to \bar{P} \) be the orbit map. Then \( \rho^*(C^\infty(\bar{P})) = C^\infty(P)^G \). For a vector field \( X \) on \( P \),

\[
\phi_{g*}X(p) = T\phi_g(X(\phi_{g^{-1}}(p))),
\]

and for a function \( f \in C^\infty(P) \),

\[
(\phi_{g*}X) \cdot f = \phi_{g^{-1}}^*(X \cdot \phi_g^*f).
\]

PROPOSITION 2.7. If \( X \) is a vector field on \( P \) such that \( \phi_{g*}X - X \) is tangent to orbits of the action of \( G \), then \( X \) descends to a vector field \( \bar{X} = \rho_*X \) on \( \bar{P} \).

Proof. For a \( G \)-invariant function \( f \) on \( P \) and \( g \in G \),

\[
\phi_{g^{-1}}^*(X \cdot f) = (\phi_{(g^{-1})*}X) \cdot f = (\phi_{(g^{-1})*}X) \cdot \phi_g^*f = (\phi_gX) \cdot f = (\phi_gX - X) \cdot f + X \cdot f = X \cdot f,
\]
because \(\phi_g X - X\) is tangent to orbits of the action of \(G\) and \(f\) is \(G\)-invariant. Hence, \(X \cdot f\) is \(G\)-invariant. Thus \(X\) is a derivation of \(C^\infty(P)^G\), which implies that it descends to a derivation \(\bar{X} = \rho_* X\) of \(C^\infty(\bar{P})\). Integration of the derivation \(\bar{X}\) gives rise to a maximal integral curve \(\bar{c}\) of \(\bar{X}\) through \(\bar{p}\) such that \(\bar{c}(t) = \rho \circ c(t)\), where \(c\) is the maximal integral curve of \(X\) through \(p\).

It remains to prove that translations along integral curves of \(\bar{X}\) gives rise to a maximal integral curve \(\bar{c}\) of \(\bar{X}\) through \(\bar{p}\) such that \(\bar{c}(t) = \rho \circ c(t)\), where \(c\) is the maximal integral curve of \(X\) through \(p\).

It is important to note that the assertion of mapping \(G\)-orbits to \(G\)-orbits does not require the flow of \(X\) to be complete, because \(\exp tX\) is interpreted in the sense that if two points \(p\) and \(q\) are in the same \(G\)-orbit and \(\exp tX(p)\) and \(\exp tX(q)\) are both defined, then \(\exp tX(p)\) and \(\exp tX(q)\) are in the same \(G\)-orbit. It is in this way that the reduced local flow \(\exp t\bar{X}\) is defined. Note that the reduced local flow may be defined for all time even though the original vector field may be incomplete everywhere, and have no positive minimum time of existence on any \(G\)-orbit.

Therefore, \(\exp t\bar{X}(\bar{p}) = (\exp t\bar{X})(\rho(p)) = \rho \circ (\exp tX)(p)\). In other words,

\[
\rho \circ \exp tX = (\exp t\bar{X}) \circ \rho.
\]

Since \(\exp tX\) is a local one-parameter group of local transformations of \(P\), it follows that \(\exp t\bar{X}\) is a local one-parameter group of local transformations of \(\bar{P}\). Since \(\bar{f} = \rho^* \bar{f} \in C^\infty(\bar{P})\) implies that the pull-back \(\bar{f} \circ \rho^* \bar{f} \in C^\infty(P)^G\), it follows that

\[
\rho^* (\exp t\bar{X})^* \bar{f} = (\exp tX)^* \rho^* \bar{f} = (\exp tX)^* f
\]

is \(G\)-invariant. This implies that \((\exp t\bar{X})^* \bar{f} \in C^\infty(\bar{P})\), and ensures that \(\exp t\bar{X}\) is a local diffeomorphism of \(\bar{P}\).

Since the projection of the orbit type stratification of \(P\) to \(\bar{P}\) is a stratification of \(\bar{P}\), and these strata coincide with the orbits of the family of all vector fields on \(\bar{P}\), it follows that

**Corollary 2.8.** The reduced vector field \(\bar{X}\) preserves the stratification of \(\bar{P}\) by orbit type.

### 3. Not-quite-Hamiltonian reduction

Suppose that the Hamiltonian system \((P, \omega, h)\) has a proper Hamiltonian action \(\phi\) of the connected Lie group \(G\) with momentum map \(j = (j_1, \ldots, j_n)\). Let the infinitesimal generators of the \(G\)-action be \(X_a\), and \(X_{a, \omega} \omega = d j_a\). Suppose that the Hamiltonian \(h\) is not \(G\)-invariant, but the Poisson brackets satisfy

\[
\{j_a, h\} = f_a(j_1, \ldots, j_n)
\]

\(^3\) On \(\{(x, y) | y > 0\}\) consider the vector field \(X = \partial_x + y^2 \partial_y\), with group action generated by \(y \partial_y\).
for some functions $f_a$ of the momenta. This implies that the variation in the Hamiltonian vector field under the group action is tangent to the group orbits,

$$\phi_* v - v \equiv 0 \pmod{X_a},$$

and consequently the vector field $X_v$ descends to the reduced vector field $\bar{X} := \rho_* X_v$ on the quotient $\bar{P}$. Furthermore, it implies that the Poisson bracket of a $G$-invariant function with the Hamiltonian is still a $G$-invariant function. Said slightly differently, the mapping

$$\{ \cdot, h \} : C^\infty(\bar{P}) \to C^\infty(\bar{P}) : f \mapsto \{ f, h \}$$

is an outer Poisson morphism (it is an outer morphism because $h \notin C^\infty(\bar{P})_G$.) This gives (singular) reduced dynamics on the quotient $\bar{P}$ in the Poisson form

$$\dot{f} = \{ f, h \} \quad f \in C^\infty(\bar{P})^G,$$

as $C^\infty(\bar{P})^G$ is identified with the smooth functions on $\bar{P}$.

**Example 3.1.** Consider a particle moving in linear gravity. This system has the Hamiltonian description

$$h = \frac{1}{2} p^2 + q.$$

Then the vector field $Y = \delta_q$ is a symmetry of the symplectic form, has momentum $j = p$, and even though the Hamiltonian is not $Y$-invariant, $\mathcal{L}_Y h \neq 0$, the derivative $\mathcal{L}_Y dh = 0$. Since an invariant function is just a function of the variable $p$, the Poisson bracket of an invariant function with the Hamiltonian is an invariant function. Note that $\mathcal{L}_Y dh = 0$ yields, by the magic formula,

$$\mathcal{L}_Y dh = d(Y \cdot dh) = Y \cdot (ddh) = d(Y \cdot dh) = 0,$$

which implies that $Y \cdot dh = c$, $c$ a constant. It follows that the time evolution of the moment $j$ is of the form $j(t) = ct + d$, for some constants $c$ and $d$.

**Example 3.2.** The classical particle with spin. One may reduce the spinning charged rigid body in a magnetic field with the nonlinear constraint of constant length of angular momentum to get Souriau’s model of a classical particle with spin (see [7] and [12].)

**Example 3.3.** Collective Hamiltonians [17]. If the Hamiltonian is a pullback of a function on the dual algebra by the momentum map, $h = j^* f$, $f : g^* \to \mathbb{R}$, then the Casimirs play the role of the invariant functions.

**Example 3.4.** Rotating coordinates. Let $G = SO(2)$ be the group of rotations about the $x^3$ axis in $\mathbb{R}^3$, and $j$ the momentum for the lifted action on the cotangent bundle $T^* \mathbb{R}^3$. Since $G$ acts isometrically for the standard metric on $\mathbb{R}^3$, one sees the addition of the momentum $j$ to the Hamiltonian in the $G$-moving coordinate system. This is a specific case of the general phenomenon of the addition of collective terms

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This global formulation has been given for the sake of cleanliness. Locally, it seems that it is a question of to what extent the condition $\phi_* X_v - X_v \equiv 0 \pmod{X_a}$ implies the existence of functions $k_b^a$ defined by $k_b^a d j_b = -d(j_a, h)$ which are functionally dependent on the $j_a$s. Some of the subtleties involved are discussed in [23], and related global issues are discussed in [19].
in Hamiltonians viewed in co-moving coordinates along one-parameter isometry groups.

4. Reconstruction (1)

A general way to reconstruct an integral curve \( c(t) \) from an integral curve \( \tilde{c}(t) = \rho(c(t)) \) in the reduced space is to first pull back the \( G \)-action to any lift \( b(t) \) of \( \tilde{c}(t) \). The reconstruction problem is to find a curve \( g(t) \) in the group \( G \) so that

\[
    c(t) = \phi(g(t), b(t))
\]

satisfies the dynamical equation \( \dot{c} = X_h(c) \). Differentiation of equation (1) with respect to the parameter \( t \) yields a non-autonomous differential equation for the group variable:

\[
    D_1\phi(g, b) \cdot \dot{g} + D_2\phi(g, b) \cdot \dot{b} = X_h.
\]

However, this approach neglects a key component of the Hamiltonian structure of the system, namely the time dependence of the momenta. A refinement of the reconstruction procedure that is not only adapted to the Hamiltonian structure, but furthermore reduces to the usual reconstruction procedure in the case when the Hamiltonian is \( G \)-invariant, runs as follows.

Observe that the Poisson brackets \( \{f, h\} = f(a, j_1, \ldots, j_n) \) define a vector field on the dual of the Lie algebra \( g^* \) given by the differential equations

\[
    \frac{dj}{dt} = f_a(j_1, \ldots, j_n).
\]

This differential equation is the first reconstruction equation. Denote by \( \mu(t) \) an integral curve of this vector field. To find the integral curve \( c(t) \) with initial condition \( p = c(0) \) then requires two curves: the first is the integral curve \( \tilde{c}(t) \) of the reduced dynamics with initial condition \( \rho(p) = \tilde{c}(0) \), and the second is the integral curve \( \mu(t) \) with initial condition \( j(p) = \mu(0) \). The curve \( \tilde{c}(t) \) is then lifted to a curve \( b(t) \) that satisfies the two constraints \( \rho(b(t)) = \tilde{c}(t) \) and \( j(b(t)) = \mu(t) \). Once again, the dynamical reconstruction problem is to find a curve \( g(t) \) in the group \( G \) so that \( c(t) = \phi(g(t), b(t)) \) satisfies the dynamical equation \( \dot{c} = X_h(c) \), but now the curve \( g(t) \) lies in the stability group \( G_{\mu(t)} \). Differentiation of this condition with respect to \( t \) yields the second reconstruction equation:

\[
    D_1\phi(g, b) \cdot \dot{g} + D_2\phi(g, b) \cdot \dot{b} = X_h.
\]

This version of the equation is chosen rather than (1) because the group is smaller, even though it is varying in time.\footnote{A special case of the preceding is in some sense quite typical. Suppose that the stability group \( \mu(0) \) is abelian and of minimal dimension, and that this is stable in the sense that the stability group \( G_{\mu(t)} \) is also abelian and connected, and so of this is a general method because it applies to any reduced dynamics, Hamiltonian or otherwise. This is the approach taken, for example in \([11]\).}

\footnote{In the special case of Hamiltonian reduction, the functions \( f_a \equiv 0 \) because the momenta are constants of motion, so the curve \( \mu(t) \) is a constant, and hence the stability group \( G_{\mu} \) does not depend on \( t \).}

\footnote{An analogous construction may be found in \([26]\), in the case of commuting invariance groups.}
the form $\mathbf{R}^r \times \mathbf{T}^s$ for some integers $r$ and some $s$. This allows us to choose an interval $I := (-\epsilon, \epsilon)$ about $t = 0$, an identification of $G_{\mu(t)}$ with $G_{\mu(0)}$ and thus realize a trivialization of $\phi(G_{\mu(t)}, b(t))$ as $I \times G_{\mu(0)}$ where the lift $b(t)$ is the product $I \times e$, where $e$ is the identity in the group. To see this might work in practice, consider the following example.

**Example 4.1.** (The elliptic particle.) Consider the Hamiltonian system $(P, \omega, h)$ with configuration space $Q = \mathbf{R}^2$, phase space $P = T^*Q = T^* \mathbf{R}^2$, projection $\pi : P \rightarrow Q$, symplectic form $\omega = dx \wedge dp_x + dy \wedge dp_y$, and Hamiltonian

$$h = \frac{1}{2}((1 + k^2/2 + y^2)p_x^2 - 2xyp_xp_y + (1 - k^2/2 + x^2)p_y^2),$$

with $0 < k < 1$. The Euclidian group $G = \text{SE}(2, \mathbf{R})$ acts on the configuration space $Q = \mathbf{R}^2$, as the group of matrices with affine action

$$\phi : G \times Q \rightarrow Q : \begin{pmatrix} \cos \theta & -\sin \theta & u \\ \sin \theta & \cos \theta & v \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta x - \sin \theta y + u \\ \sin \theta x - \cos \theta y + v \end{pmatrix}.$$ 

Define a basis for the Lie algebra $\mathfrak{g} = \text{se}(2, \mathbf{R})$ by setting

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

The matrices $f^1 = 2e_1^\dagger$, $f^2 = 2e_2^\dagger$, and $f^3 = e_3^\dagger$ form a dual basis of the dual of the Lie algebra $\mathfrak{g}^\ast = \text{se}(2, \mathbf{R})^\ast$ in the sense that if the natural pairing $\langle \mu, X \rangle = \frac{1}{2} \text{tr}(\mu X)$, the pairings

$$\langle f^k, e_l \rangle = \delta^k_l.$$ 

The action lifts to a Hamiltonian action on phase space with momentum map $j : P \rightarrow \mathfrak{g}^\ast$ with components $j = (j_1, j_2, j_3)$, where $j_1 = p_x, j_2 = p_y, j_3 = yp_x - xp_y$. In matrices,

$$j : P \rightarrow \mathfrak{g}^\ast : (x, y, p_x, p_y) \rightarrow \begin{pmatrix} 0 & j_3 & 0 \\ -j_3 & 0 & 0 \\ 2j_1 & 2j_2 & 0 \end{pmatrix}.$$ 

The components of the momentum map satisfy the Poisson bracket relations

$$\{j_1, j_2\} = 0, \quad \{j_2, j_3\} = -j_1, \quad \{j_3, j_1\} = -j_2.$$ 

The Hamiltonian is not invariant under the $G$-action on the phase space, as

$$\{j_1, h\} = j_2j_3, \quad \{j_2, h\} = -j_1j_3, \quad \{j_3, h\} = -k^2j_1j_2.$$ 

The $G$-invariant functions on $P$ are all functions of $\sigma = |p|^2 = p_x^2 + p_y^2$. Thus the equation on the reduced space is given by the Poisson bracket

$$\sigma = \{\sigma, h\} = 0,$$
which immediately integrates to \( \sigma = \text{constant} \). The first reconstruction equation is the differential equation in the dual algebra

\[
\frac{dj_1}{dt} = j_2 j_3, \quad \frac{dj_2}{dt} = -j_3 j_1, \quad \frac{dj_3}{dt} = -k^2 j_2 j_3.
\]

For simplicity, the reconstruction will be given for the integral curve with initial condition \((x_0, y_0, p_{x0}, p_{y0}) = (-1, 0, 0, 1)\), so the initial values \((\sigma_0, j_{10}, j_{20}, j_{30}) = (1, 0, 1, 1)\).

Denote by \( \mu(t) = (\mu_1(t), \mu_2(t), \mu_3(t)) \) the solution of this initial value problem in \( g^* \).

This implies that

\[
\sigma(t) = 1, \quad \mu_1(t) = \text{sn}(t; k), \quad \mu_2(t) = \text{cn}(t; k), \quad \mu_3(t) = \text{dn}(t; k),
\]

where \( \text{sn}(t; k), \text{cn}(t; k) \) and \( \text{dn}(t; k) \) are the Jacobi elliptic functions.

Now we should examine the second reconstruction equation and the isotropy subgroup \( G_{\mu(t)} \), (the subgroup of \( G \) that fixes \( \mu(t) \) under the coadjoint action), which is the one-parameter subgroup

\[
G_{\mu(t)} = \{ \exp sX \mid s \in \mathbb{R} \}
\]

where

\[
X = 2\mu_1(t)e_1 - 2\mu_2(t)e_2 + \mu_3(t)e_3 \in g.
\]

However, we gain a somewhat different insight if we proceed a little differently than a direct application of the theory suggests. The component \( j_3 \) of the momentum map implies

\[
\mu_3(t) = y\mu_1(t) - x\mu_2(t),
\]

which is the equation of a moving line in the \( xy \)-plane. Picking the point \( q_0 = \mu_3(-\mu_2, \mu_1) \) to be the point on the line nearest the origin, parametrize the moving line as

\[
q(s) = q_0 + s(\mu_1, \mu_2),
\]

where \( s \) is an arc length parameter on the line. A differential equation for the parameter \( s \) yielding the reconstruction of the desired integral curve is

\[
\frac{d}{dt}[q(s(t), t)] = \pi_* X_h.
\]

Taking the inner product of this with the unit vector \((\mu_1, \mu_2)\) gives

\[
\dot{s} = \mu_1 \dot{x} + \mu_2 \dot{y} = (1 + k^2/2 + y^2)p_x^2 - xy p_x p_y +
+ (1 + k^2/2 + x^2)p_y^2 - xy p_x p_y
\]

\[
= 1 + \mu_3^2 + \frac{k^2}{2} (\mu_1^2 - \mu_2^2)
\]

\[
= 2 - k^2/2, \quad \text{a constant!}
\]

The reconstructed integral curve \( c(t) \) immediately follows.

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\(^8\)The reader will immediately observe that the relation \( \{\sigma, h\} = 0 \) implies the system is completely integrable in the sense of Liouville. However, the construction of action-angle variables is more involved than our subsequent analysis.
5. Reconstruction (2)

That the previous example had such a pretty solution suggests that something deeper is at work. Our preferred view is to see the Hamiltonian as a sum of two commuting Hamiltonians, symbolically written as \( h = h_\sigma + h_j \), \( \{ h_\sigma, h_j \} = 0 \), thinking of \( h_\sigma \) as the invariant part and \( h_j \) as the collective part. This implies that the flow of the Hamiltonian \( h \) may be found as the composition of the flows of \( h_\sigma \) and \( h_j \). Hence, an alternative reconstruction procedure presents itself: reconstruct the dynamics of the invariant part \( h_\sigma \) using the fixed subgroup \( G_{\mu(t)} \) as well as that of \( h_j \) and then compose. Note, however, that there is no free lunch here. The flow of \( h_\sigma \) must be reconstructed treating each point along the flow of \( h_j \) as a new initial condition. In other words, instead of integrating a time-dependent differential equation in which \( \mu(t) \) varies, the integration is over a one-parameter family of equations, each of which have constant \( \mu \).

6. Continuations

Our discussion has left many avenues unexplored. Possible further explorations include the following

(1) Assumptions on the group action. For the sake of brevity, only proper group actions were considered, as the theory is now well established. However, in some cases of interest, such as the coadjoint action of a Lie group on the dual of its Lie algebra, the action need not be proper. There are more general types of group actions, such as polite actions (see [6]) that allow reduction by symmetry and reconstruction in terms of differential equations on manifolds. It would be very interesting to see to what extent the theory presented here extends to more general group actions.

(2) Functional dependence. This paper avoided all issues of functional dependence by assuming that the Poisson bracket \( \{ j_\sigma, h \} \) was a globally defined function of the momenta. It would be interesting to be able to weaken this to the condition that the Hamiltonian vector field \( X_h \) is tangent to the group orbits, as that suffices for the existence of reduced dynamics. The global issues involved are somewhat subtle. For an example see theorem 4 of [18].

(3) Complete integrability. In the theory of completely integrable Hamiltonian systems, the flow is seen to be linearized on tori. This means that there is a local action of a torus group under which the flow is invariant. The generalization of this theory to not-quite-Hamiltonian systems is unclear, as there is presently no precise notion of what a completely integrable not-quite-Hamiltonian system should be.

(4) Geometric quantization. It is of interest to understand to what extent quantization and reduction commute in the case under consideration.

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The reader might suspect that this is due to the Hamiltonian being collective. While correct, our view is that this is not the best answer, because it places misleading attention on the invariant functions being Casimirs. See [17] for more details.
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