Harmonic Analysis

Change of angle in tent spaces

Changement d'angle dans les espaces de tentes

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ABSTRACT

We prove sharp bounds for the equivalence of norms in tent spaces with respect to changes of angles. Some applications are given.

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RÉSUMÉ

On propose de démontrer les comparaisons précises entre les normes dans le même espace de tentes avec deux angles différents. On donne quelques conséquences.

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Si $0 < p < \infty$ et $\alpha > 0$, l'espace des tentes $T_{p,2}^\alpha$ est défini dans [4] comme l'espace des fonctions $g$ localement de carré intégrable dans le demi-espace supérieur ouvert $\mathbb{R}^{n+1}_+$ telles que $A^{(\alpha)} g \in L^p$ où $A^{(\alpha)} g$ est définie par (1). Si $p = \infty$, on définit $T_{\mathcal{A}}^{\infty,2}$ par la condition de Carleson (3) sur les tentes $T_{\mathcal{A}} B$ et la norme est la meilleure constante $C$ dans cette inégalité. Le paramètre $\alpha$ se décrit comme le demi-angle des cônes d'appui dans la définition de la fonctionnelle d'aire $A^{(\alpha)}$. Le choix des normalisations en fonction de $\alpha$ obéit à une cohérence d'échelle pour tous les $p$.

Un des points de départ de la théorie de [4] est que pour $p$ fixé l'espace ne dépend pas de $\alpha > 0$ et que les (quasi-)normes $\|A^{(\alpha)} g\|_p$ sont deux à deux équivalentes. Ce résultat est utilisé alors pour obtenir les décompositions atomiques ($p \leq 1$), la dualité ($1 \leq p < \infty$) et l'interpolation complexe et réelle ($0 < p \leq \infty$).

Il est récemment apparu utile de connaître la dépendance précise en terme des angles dans l'équivalence des normes. Les espaces de tentes sont en effet intensivement utilisés dans la nouvelle théorie des espaces de Hardy associés aux opérateurs vérifiant des estimations de Gaffney–Davies découlant des travaux [2,7]. Cette dépendance se retrouve aussi cruciale pour étudier l'opérateur de régularité maximale dans les espaces de tentes et ses applications aux EDP paraboliques [3].

L’argument de [4], provenant de [6], ne donne pas la dépendance optimale. Dans [8], il est démontré $\|g\|_{T_{p,2}^{\alpha}} \leq C (1 + \log \alpha)^{\alpha/n+\tau} \|g\|_{T_{p,2}}$ pour $1 < p < \infty$ et $\alpha > 1$, où $\tau = \min(n, 2)$ et $C$ dépend de $n$ et $p$, dans un cadre à valeurs UMD, ce qui explique les restrictions sur $p$. La démonstration n’est pas élémentaire. Il est aussi démontré que la croissance polynomiale $\alpha^{n/\tau}$ est la meilleure possible dans le cas qui nous occupe ici.

Nous obtenons pour $0 < p \leq \infty$ et $\alpha, \beta > 0$

$$C h(p, \alpha/\beta) \|g\|_{T_{p,2}^{\alpha}} \leq \|g\|_{T_{p,2}^{\beta}} \leq C h(p, \alpha/\beta) \|g\|_{T_{p,2}^{\alpha}}$$

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où C, C' dépendent de n, p, h(p, α) = min(α−n/2, α−p/n), h(α, α) = max(α−n/2, α−n/2). Les exposants sont les meilleurs possibles. En particulier si α > 1, τ = min(2, p) et σ = max(2, p), ∥g∥_T^{α, T} ≤ Cα^{1/σ}∥g∥_{T^{α, T}} et ∥g∥_{T^{α, T}} ≤ Cα^{−n/α}∥g∥_{T^{α, T}}. Remarquer que la deuxième inégalité améliore ∥g∥_{T^{α, T}} ≤ C∥g∥_{T^{α, T}} découlant simplement de la géométrie des cônes d’appui.

La preuve de ces inégalités se résume ainsi. On se ramène à α > β = 1. Le cas p = ∞ est immédiat en utilisant des considérations élémentaires sur les tentes. Les mêmes considérations entraînent les inégalités pour les atomes et donc pour 0 < p ≤ 1. Enfin, on interpole les résultats déjà obtenus avec l’identité ∥g∥_{T^{α, T}} = α^{n/α}∥g∥_{T^{α, T}}. Les exemples montrant l’optimalité sont des indicatrices supportées dans une tente et nulles près du bord. La preuve complète se trouve plus loin.

On peut citer plusieurs corollaires. Le premier donne une comparaison précise avec la norme ∥Vg∥_p de la fonctionnelle de Littlewood–Paley verticale classique Vg(x) = (λ^−n |g(t, x)|^2 dt)^{1/2}. On obtient ∥Vg∥_p ≤ C∥g∥_{T^{α, T}} si p ≤ 2, et ∥Vg∥_p ≥ C∥g∥_{T^{α, T}} si p ≥ 2. Le résultat est connu par d’autres méthodes (voir [9,1]). Il se retrouve en observant que pour α → 0, α^{−n/2}A^{(α)}g(x) converge vers Vg(x) pour des fonctions g ∈ C∞ à support compact et en utilisant notre résultat. Noter que les inégalités inverses à celles-ci-dessus sont toutes fausses si p ≠ 2 (voir [1]).

Un deuxième corollaire est l’inégalité

\[ \left\| \left( \int_{\mathbb{R}^{n+1}} \frac{t}{|x-y| + t} \right)^{n/2} |g(t, y)|^2 \frac{dy dt}{t^{n+1}} \right\|_p \leq C(n, p, \lambda) \| A^{(1)}g \|_p \]

pour 0 < p < ∞ et λ > max(2/p, 1) et toute fonction g. En appliquant cette inégalité à g(t, y) = t\nabla u(t, y), u étant l’extension harmonique d’une distribution appropriée au, cela donne une démonstration simplifiée de l’inégalité de Stein (Theorem 2, Chap. IV, [9]) portant sur la grande fonction carrée une fois connue celle-sur la fonction d'aire de Lusin.

1. Introduction and main results

Let B(x, t) denote an open ball centered at x ∈ \mathbb{R}^n with radius t > 0. Define for a locally square integrable function g(t, y), (t, y) ∈ \mathbb{R}^{n+1}, for α > 0 and x ∈ \mathbb{R}^n,

\[ A^{(α)}g(x) := \left( \int_{\mathbb{R}^{n+1}} \frac{1}{|x-y|} \right)^{1/2} |g(t, y)|^2 \frac{dy dt}{t^{n+1}} \]

and for 0 < p < ∞, say that g ∈ T_{α, T}^{p, 2} if

\[ \|g\|_{T_{α, T}^{p, 2}} := \| A^{(α)}g \|_p < \infty. \]

This space was introduced in [4]. As sets the spaces for a given p are the same and the norms (or quasi-norms if p < 1) are equivalent: whenever α, β > 0, one has

\[ \| A^{(α)}g \|_p \sim \| A^{(β)}g \|_p. \]

For p = ∞, the limiting space is defined with a Carleson measure condition. Let T_{α, Tα} be the tent with aperture α above the open ball B = B(x, r), i.e., the set of (t, y) such that 0 < t < r/α and y ∈ B(x, r − αt). We define \|g\|_{T_{α, T}} as the infimum of C > 0 such that for all ball B,

\[ \int_B |g(t, y)|^2 \frac{dy dt}{t} \leq C^2 \frac{|B|}{α^n} \]

Again, the spaces T_{α, T}^{∞, 2} are the same, the norms are equivalent and the isometry property holds.

To explain the choice of the normalization in (3), we remark that for p = ∞ included, g(t, y) \mapsto h(t, y) := α^{n/2}g(t/α, y) is an isometry between T_{α, T}^{p, 2} and T_{α, T}^{p, 2} equipped with their respective (quasi-)norms.

It is shown in [4] that the spaces T_{α, T}^{p, 2}, 0 < p ≤ ∞, interpolate by the complex method and the real method. The same results hold for T_{α, Tα}^{p, 2} for fixed α, with constants (i.e., the constants in the equivalence of norms between T_{α, T}^{p, 2} and the interpolated space to which it is equal) independent of α by using the isometry property.

Motivated by an intensive usage of tent spaces in the development of new Hardy spaces associated to operators with Gaffney–Davies estimates first made in [2,7], and also by the study of maximal regularity on tent spaces towards applications for parabolic PDE’s ([3], and some more work in progress), it became interesting to know the sharp dependence of the bounds in (2) with respect to α, β. The L^2 bound is immediate by Fubini's theorem: \|g\|_{T_{α, T}^{2}} = (α/β)^{n/2}∥g∥_{T_{α, T}^{2}}. For p ≠ 2, the argument in [4], originating from [6], does not give optimal dependence. The inequality
\[ \|g\|_{L_p^}\leq C (1 + \log \alpha)^{n/\tau} \|g\|_{L_1^p}, \]

for \(1 < p < \infty\) and \(\alpha \geq 1\), where \(\tau = \min(p, 2)\) and \(C\) depends only on \(n\) and \(p\), is proved in [8] as a special case of a Banach space valued result, and, moreover, the polynomial growth \(\alpha^{n/\tau}\) is shown to be optimal for such an inequality to hold. The restriction \(p > 1\) occurs in this argument because the UMD property is required and a maximal inequality is used. Note that even the \(L^2\) bounds is not immediate in a Banach (non-Hilbert) space valued context. In discussion with T. Hytönen, we convinced ourselves that the logarithmic factor is not produced by this argument in the scalar case when \(\lambda\) results in \([6]\). At convolution with \(\alpha\)-scale invariance, all cases reduce to (4) and (5) with \(\alpha\).

Theorem 1.1. Let \(0 < p \leq \infty\) and \(\alpha, \beta > 0\). There exist constants \(C, C' > 0\) depending on \(n, p\) only, such that for any locally square integrable function \(g\),

\[ C_h(p, \alpha/\beta) \|g\|_{L_p^}\leq \|g\|_{L_p^p} \leq C_h(p, \alpha/\beta) \|g\|_{L_p^p}. \]

Moreover, the dependence in \(\alpha/\beta\) is best possible in the sense that this growth is attained.

In particular, for \(\alpha > 1\), \(\tau = \min(2, p)\) and \(\sigma = \max(2, p)\), one has

\[ \|g\|_{L_p^p} \leq C \alpha^{n/\tau} \|g\|_{L_1^p}. \quad (4) \]
\[ \|g\|_{L_p^p} \leq C \alpha^{-n/\sigma} \|g\|_{L_1^p}. \quad (5) \]

The second one improves the obvious bound \(\|g\|_{L_p^p} \leq \|g\|_{L_p^p}\). By symmetry using the relation \(h(p, \alpha)^{-1} = \tilde{h}(p, \alpha^{-1})\) and scale invariance, all cases reduce to (4) and (5) with \(\alpha > 1\).

Corollary 1.2. Let \(0 < p < \infty\). There is a constant \(0 < C < \infty\) depending on \(n, p\) only such that for any locally square integrable \(g\), if \(V g(x) = \int_0^\infty |g(t, x)|^p \frac{dt}{t}\),

\[ \|V g\|_p \leq C \|g\|_{L_1^p}, \quad \text{if } p \leq 2, \quad \|V g\|_p \geq C \|g\|_{L_1^p}, \quad \text{if } p \geq 2. \]

The corollary was proved by a different method in [1] when \(p < 2\) and the \(p > 2\) case dates back to [9]. The opposite inequalities are false if \(p \neq 2\). Starting from Theorem 1.1, the proof is a mere application of Lebesgue differentiation theorem when \(\alpha \to 0\) for \(\alpha^{-n/2} A^{(\alpha)} g(x)\) converges to \(V g(x)\) assuming \(g\) smooth with compact support. This assumption is easily removed.

Corollary 1.3. If \(0 < p < \infty\) and \(\lambda > \max(2/p, 1)\), for any locally square integrable \(g\),

\[ \left\| \left( \int_{\mathbb{R}^{n+1}} \frac{t}{|x-y|+t} \left| g(t, y) \right|^p \frac{dy \, dt}{t^n+1} \right)^{\lambda/p} \right\|_p \leq C(n, p, \lambda) \|A^{(1)} g\|_{L_p^p}. \]

The left-hand side equals the grand square function of Stein when \(g(t, y) = t \nabla u(t, y)\), \(u\) being the harmonic extension of a suitable distribution \(u_0\) on \(\mathbb{R}^n\). Hence, for all \(0 < p < \infty\) and \(\lambda > \max(2/p, 1)\), it is dominated in \(L^p\) by \(\|A^{(1)} g\|_{L_p}\) which is the \(L^p\) norm of the area functional of Lusin defined from \(u_0\). However, it is known from Stein–Weiss’ theory that \(A^{(1)} g \in L^p(\mathbb{R}^n)\) if and only if \(\alpha^{-n-1} < p < \infty\) and \(u_0\) belongs to the Hardy space \(H^p(\mathbb{R}^n)\) (see [9]). This gives a simple proof of Theorem 2, Chap. IV in [9]. The lower exponent \(\alpha^{-n-1}\) is only due to the choice of the extension. Using an extension by convolution with \(r^{-n} \varphi(x/r)\) with \(\varphi \in C_0^\infty(\mathbb{R}^n)\) having all vanishing moments but the one of order 0, \(\alpha^{-n-1}\) becomes 0 by the results in [6]. At \(\lambda = 2/p\) and \(p < 2\), a weak type inequality is plausible, the Lorentz norm \(L^{p, \infty}\) replacing the Lebesgue norm \(L^p\) in the left-hand side. It would give a simple proof of the weak type \((p, p)\) result of Fefferman [5] for Stein’s grand square function. We leave this open.

The proof of the corollary is easy by splitting the upper half space according to \(|x-y|/t\) compared to powers \(2^k, k \geq 0\), and one obtains

\[ \left( \int_{\mathbb{R}^{n+1}} \frac{t}{|x-y|+t} \left| g(t, y) \right|^p \frac{dy \, dt}{t^n+1} \right)^{\lambda/p} \leq C(n, \lambda) \sum_{k \geq 0} 2^{-kn\lambda/2} A^{(2^k+1)} g(x). \]
It remains to use \( \|A^{(\alpha)}g\|_p \leq C\alpha^{n/p}\|A^{(1)}g\|_p \) for \( \alpha = 2^{k+1} \) in appropriate arguments for \( p \geq 1 \) or \( p \leq 1 \).

The proof of Theorem 1.1 is an easy matter using in part atomic theory for tent spaces, again proved in [4]. Recall that for \( 0 < p \leq 1 \), the tent space \( T^{p,2}_p \) has an atomic decomposition: A \( T^{p,2}_p \) atom is a function \( a(t, x) \) supported in a tent \( T_B \) with the estimate

\[
\int_{T_B} |a(t, x)| \frac{2 \, dx \, dt}{t} \leq |B|^{-\left(\frac{2}{p} - 1\right)}.
\]

There is a constant \( C(C(n, p) > 0 \) such that \( \|a\|_{T^{p,2}_p} \leq C \). Any \( T^{p,2}_p \) function \( g \) can be represented as a series \( g = \sum \lambda_j a_j \) where \( a_j \) is a \( T^{p,2}_p \) atom and \( \sum |\lambda_j|^p \sim \|g\|_{T^{p,2}_p}^p \). By the isometry property, a \( T^{p,2}_a \) atom is a function \( A(u, x) \) supported a tent \( T_B \) with the estimate

\[
\int_{T_B} |A(u, x)| \frac{2 \, du \, dx}{u} \leq \alpha^{-n} |B|^{-\left(\frac{2}{p} - 1\right)}
\]

and the decomposition theorem holds in \( T^{p,2}_a \).

2. Proof of Theorem 1.1

Recall that we may assume \( \alpha > \beta = 1 \) and it is enough to prove (4) and (5). Fix \( p = \infty \) first. Let \( \|g\|_{T^{\infty,2}_1} = 1 \) and \( B \) be a ball. As \( \mathcal{T}_0 B \subset T_1 B \),

\[
\int_{\mathcal{T}_0 B} |g(t, x)| \frac{2 \, dx \, dt}{t} \leq \int_{T_1 B} |g(t, x)| \frac{2 \, dx \, dt}{t} \leq |B| = \frac{\alpha^n |B|}{\alpha^p}.
\]

Hence, \( \|g\|_{T^{\infty,2}_1} \leq \alpha^{n/2} \). This shows \( \|g\|_{T^{n,2}_1} \leq \alpha^{n/2} \|g\|_{T^{\infty,2}_1} \) for all \( g \).

Let \( \|g\|_{T^{n,2}_1} = 1 \) and \( B \) be a ball. As \( T_1 B \subset T_0 (\alpha B) \) where \( \alpha B \) is the ball concentric with \( B \) dilated by \( \alpha \),

\[
\int_{T_0 B} |g(t, x)| \frac{2 \, dx \, dt}{t} \leq \int_{T_0 (\alpha B)} |g(t, x)| \frac{2 \, dx \, dt}{t} \leq \frac{|\alpha B|}{\alpha^n} = |B|.
\]

Hence, \( \|g\|_{T^{n,2}_1} \leq 1 \) and \( \|g\|_{T^{n,2}_1} \leq \|g\|_{T^{n,2}_1} \) for all \( g \).

Fix now \( p < \infty \). Let \( B \) be a ball and \( a \) be a \( T^{p,2}_p \) atom supported in a tent \( T_1 B \). As \( \mathcal{T}_0 B \subset T_0 (\alpha B) \), we have \( a \) is supported in \( \mathcal{T}_0 (\alpha B) \) and

\[
\int_{\mathcal{T}_0 (\alpha B)} |a(t, x)| \frac{2 \, dx \, dt}{t} = \int_{T_1 B} |a(t, x)| \frac{2 \, dx \, dt}{t} \leq |B|^{-\left(\frac{2}{p} - 1\right)} = \left(\frac{\alpha^n |a|}{|\alpha B|}\right)^{\left(\frac{2}{p} - 1\right)}.
\]

Thus \( \alpha^{-n/p} a \) is a \( T^{p,2}_a \) atom. An atomic decomposition of any element of \( T^{p,2}_p \) is up to multiplication by \( \alpha^{-n/p} \) an atomic decomposition in \( T^{p,2}_a \), proving \( \|g\|_{T^{p,2}_a} \leq C(n, p) \alpha^{n/p} \|g\|_{T^{p,2}_1} \).

Next, let \( B \) be a ball and \( a \) be a \( T^{p,2}_a \) atom supported in a tent \( T_0 B \). As \( \mathcal{T}_0 B \subset T_1 B \), we have \( a \) is supported in \( T_1 B \) and

\[
\int_{T_1 B} |a(t, x)| \frac{2 \, dx \, dt}{t} = \int_{T_0 B} |a(t, x)| \frac{2 \, dx \, dt}{t} \leq \alpha^{-n} |B|^{-\left(\frac{2}{p} - 1\right)}.
\]

Thus \( \alpha^{n/2} a \) is a \( T^{p,2}_1 \) atom. As above, we conclude that \( \|g\|_{T^{p,2}_a} \leq C(n, p) \alpha^{-n/2} \|g\|_{T^{p,2}_1} \).

For \( 1 < p < 2 \) and \( 2 < p < \infty \), we conclude by interpolation with the \( p = 2 \) equality \( \|g\|_{T^{p,2}_a} = \alpha^{n/2} \|g\|_{T^{2,2}_a} \). We have shown (4) and (5).

The sharpness of the bounds is seen by saturating these inequalities. Fix \( \alpha > 1 \) large. Let \( B \) be the unit ball. Set \( a_1(t, y) = 1_{T_1 B} 1_{T(t, y)_{1/2}}(t, y) \). It is easy to see that \( \|a_1\|_{T^{1,2}_1} \approx 1 \). Now, we have that \( A^{(\alpha)} a_1 \) has support equal to \( B(0, \alpha) \), is bounded by a constant \( c(n) > 0 \) and equal to that constant on the ball \( B(0, \frac{\alpha + 1}{2}) \). Thus \( \|a_1\|_{T^{p,2}_1} \sim \alpha^{n/p} \). This proves that (4) is optimal when \( p \leq 2 \) and (5) is optimal when \( 2 < p \). Next, let \( a_2(t, y) = \alpha_1(\alpha t, y) \). By scaling \( \|a_2\|_{T^{p,2}_a} = \alpha^{n/2} \|a_1\|_{T^{p,2}_a} \sim \alpha^{n/2} \). Now, \( A^{(1)} a_2 \) is supported in \( B \), bounded by a constant \( c(n) > 0 \) and equal to that constant on \( B(0, \frac{\alpha + 1}{2\alpha}) \). Thus \( \|a_2\|_{T^{p,2}_1} \sim 1 \). This proves that (4) is optimal when \( p \geq 2 \) and (5) is optimal when \( p \leq 2 \).
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