New $\mathcal{W}_{q,p}(sl(2))$ algebras from the elliptic algebra $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$

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Abstract

We construct operators $t(z)$ in the elliptic algebra $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$. They close an exchange algebra when $p^m = q^{c+2}$ for $m \in \mathbb{Z}$. In addition they commute when $p = q^{2k}$ for $k$ integer non-zero, and they belong to the center of $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$ when $k$ is odd. The Poisson structures obtained for $t(z)$ in these classical limits are identical to the $q$-deformed Virasoro Poisson algebra, characterizing the exchange algebras at $p \neq q^{2k}$ as new $\mathcal{W}_{q,p}(sl(2))$ algebras.

Résumé

On construit des opérateurs $t(z)$ dans l’algèbre elliptique $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$ formant une algèbre d’échange quand $p^m = q^{c+2}$ où $m \in \mathbb{Z}$. De plus, ils commutent quand $p = q^{2k}$ pour $k$ entier non nul, et ils appartiennent au centre de $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$ lorsque $k$ est impair. Les structures de Poisson obtenues pour $t(z)$ dans ces limites classiques sont identiques à l’algèbre de Poisson Virasoro $q$-déformée, caractérisant les structures à $p \neq q^{2k}$ comme de nouvelles algèbres $\mathcal{W}_{q,p}(sl(2))$.

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1 Introduction

In a first paper \[1\] we started the study of classical Poisson algebra structures obtained from the elliptic algebra \( \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}(2)_c) \) defined by \[2\]. We showed the existence of a center at \( c = -2 \) generated by a trace formula identical to the case of trigonometric algebras \[3\], leading to a set of Poisson algebra structures containing the \( q \)-deformed Virasoro algebra constructed in \[4\]. They were characterized by the particular relative position of integration contours used to define the modes of the generating functions \( t(z) \) defined as formal series.

These results naturally lead us to consider three related questions:

1. Can one construct other commuting subalgebras in \( \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}(2)_c) \), defined by other conditions on \( p, q, c \) ?
2. Which classical limits (Poisson algebra structures) does one obtain from such new commuting subalgebras?
3. Can one define quantizations of these Poisson algebras, that is, closed algebraic structures with a supplementary parameter \( \hbar \) for which one can define a semi-classical limit \( \hbar \to 0 \) such that the algebra structures are abelian at \( \hbar = 0 \) and the Poisson structures defined in 2 are given by the leading \( \hbar \) order?

Question 3 is not reciprocal to 1 and 2 since nothing in 1 or 2 implies that the commuting subalgebras mentioned in 1 be obtained as “limits” of closed algebras. Indeed the Poisson algebras in \[1\] were constructed as limits of algebraic structures in \( \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}(2)_c) \) which did not close when \( c \neq -2 \); in other words the generators \( t(z) \) at \( c \neq -2 \) did not close an algebra, and the quantization procedure in \[5\] required the introduction of a new parameter as \( \hbar \).

Our starting point will be the same trace-like operator \( t(z) \) which was defined in \[1\]. In this paper we are going to show that:

a) if \( q^{c+2} = p^m \) for any \( m \in \mathbb{Z} \), the algebra \( \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}(2)_c) \) contains a quadratic subalgebra generated by \( t(z) \);

b) if in addition one has \( p = q^{2k} \) for any \( k \in \mathbb{Z}, k \neq 0 \), this quadratic subalgebra becomes abelian; moreover if \( k \) is odd, it belongs to the center of \( \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}(2)_c) \).

c) we now define \( \beta \) such that \( p^{1-\beta} = q^{2k} \): the Poisson brackets structure obtained when \( \beta \to 0 \) is isomorphic, up to a factor \( km \) for \( k \) odd and \( -km(2m - 1) \) for \( k \) even, to the Poisson brackets structure obtained in \[1\] when \( c = -2 + \beta, \beta \to 0 \). In this sense the quadratic algebras at \( q^{c+2} = p^m \) build natural quantizations of the Poisson brackets structure in \[1\] or equivalently of the \( q \)-deformed Virasoro algebra; we have thereby answered in part the question raised in \[3\] concerning the construction of other types of \( \mathcal{W}_{q,p}(\mathfrak{sl}(2)) \) - algebras (i.e. quantized \( q \)-deformed Virasoro algebras).

We first recall the most important notations used in \[3\] and the results obtained in \[1\].

1.1 The elliptic quantum algebra \( \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}(2)_c) \)

The elliptic quantum algebra \( \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}(2)_c) \) \[4\] \[3\] was defined as follows. \( \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}(2)_c) \) is an algebra of operators \( L_{\varepsilon \varepsilon', n} \) such that \( L_{\varepsilon \varepsilon', n} = 0 \) if \( \varepsilon \varepsilon' \neq (-1)^n (\varepsilon, \varepsilon' = + \text{ or } -) \) and one sets \( L_{\varepsilon \varepsilon'}(z) = \sum_{n \in \mathbb{Z}} L_{\varepsilon \varepsilon', n} z^n \) (in the sense of formal series) which is encapsulated into a \( 2 \times 2 \) matrix

\[
L = \begin{pmatrix} L_{++} & L_{+-} \\ L_{-+} & L_{--} \end{pmatrix}
\]  

(1.1)
(therefore $L_{+}(z)$ and $L_{-}(z)$ are even while $L_{+}(z)$ and $L_{-}(z)$ are odd functions of $z$).

One then defines $A_{q,p}(\hat{g}(2)c)$ by imposing the following constraints on $L_{\epsilon e'}(z)$:

$$R_{12}^{+}(z/w) L_{1}(z) L_{2}(w) = L_{2}(w) L_{1}(z) R_{12}^{+}(z/w), \quad (1.2)$$

where $L_{1}(z) \equiv L(z) \otimes \mathbb{I}$, $L_{2}(z) \equiv \mathbb{I} \otimes L(z)$ and $R_{12}^{+}(x)$ is given by the (suitably normalized) $R$-matrix of the eight vertex model found by Baxter [7]:

$$R_{12}^{+}(x) = \tau(q^{1/2}x^{-1}) \frac{1}{\mu(x)} \begin{pmatrix} a(u) & 0 & 0 & d(u) \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{pmatrix} \quad (1.3)$$

The functions $a(u), b(u), c(u), d(u)$ are given by

$$a(u) = \frac{\text{snh}(\lambda - u)}{\text{snh}(\lambda)}, \quad b(u) = \frac{\text{snh}(u)}{\text{snh}(\lambda)}, \quad c(u) = 1, \quad d(u) = k \text{snh}(\lambda - u) \text{snh}(u). \quad (1.4)$$

The function $\text{snh}(u)$ is defined by $\text{snh}(u) = -\text{sn}(iu)$ where $\text{sn}(u)$ is Jacobi’s elliptic function with modulus $k$. If the elliptic integrals are denoted by $K, K'$ (let $k^2 = 1 - k^2$),

$$K = \int_{0}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{and} \quad K' = \int_{0}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}}, \quad (1.5)$$

the functions $a(u), b(u), c(u), d(u)$ become functions of the variables

$$p = \exp \left(-\frac{\pi K'}{K}\right), \quad q = -\exp \left(-\frac{\pi \lambda}{2K}\right), \quad x = \exp \left(\frac{\pi u}{2K}\right). \quad (1.6)$$

The normalization factors in (1.3) are chosen as follows [3]:

$$\tau(x) = x^{-1} (q^{2}x^{-2}; q^{4})_{\infty} (q^{3}x^{-2}; q^{4})_{\infty} = x^{-1} \frac{\vartheta_{q^{2}}(x^2q)}{\vartheta_{q^{2}}(x^{-2}q)}, \quad (1.7)$$

$$\frac{1}{\mu(x)} = \frac{1}{\kappa(x^2)} \frac{(p^2; p^2)_{\infty}}{(p; p^2)_{\infty}} \frac{\vartheta_{p^2}(px^2) \vartheta_{p^2}(q^2)}{\vartheta_{p^2}(q^2x^2)}, \quad (1.8)$$

$$\frac{1}{\kappa(x^2)} = \frac{(q^4x^2; p, q^4)_{\infty}}{(q^4x^2; p, q^4)_{\infty}} \frac{(q^2x^{-2}; p, q^4)_{\infty}}{(q^2x^{-2}; p, q^4)_{\infty}} \frac{(pq^2x^{-2}; p, q^4)_{\infty}}{(pq^2x^{-2}; p, q^4)_{\infty}}, \quad (1.9)$$

where one defines the infinite multiple products as usual by

$$(x; p_1, \ldots, p_m)_{\infty} = \prod_{n_i \geq 0} (1-x p_1^{n_1} \ldots p_m^{n_m}) \quad (1.10)$$

and $\vartheta$ is the Jacobi Theta function:

$$\vartheta_{a}(x) = (x; a)_{\infty} (ax^{-1}; a)_{\infty} (a; a)_{\infty}. \quad (1.11)$$
The \( \vartheta \) function satisfies in particular the following identities:

\[
\begin{align*}
\vartheta_a(ax) &= \vartheta_a(x^{-1}) = -x^{-1} \vartheta_a(x), \\
\vartheta_a(a^s x) &= (-a^{(s-1)/2} x^{-s}) \vartheta_a(x).
\end{align*}
\]  
\tag{1.12a}
\tag{1.12b}

In eq. (1.2) \( R_{12}^+ \) is defined by \( R_{12}^+(x, q, p) \equiv R_{12}^+(x, q, pq^{-2}c) \).

The \( q \)-determinant \( q \)-det \( L(z) \) is defined by

\[
R_{12}^+ - \det L(z) = R_{12}^+(q^{c/2} z) L(z)^{-1}.
\]

From eqs. (1.4), (1.6)–(1.11) it follows that \( A_{q,p}(\hat{sl}(2)_c) \) is well defined if \(|q| \) and \(|p| \) are strictly smaller than 1, and we shall restrict ourselves to this sector of the parameter space.

### 1.2 The center of the elliptic quantum algebra \( A_{q,p}(\hat{sl}(2)_c) \)

For convenience, we introduce the following two matrices:

\[
L^+(z) \equiv L(q^{c/2} z), \quad L^-(z) \equiv \sigma^1 L(- p^{1/2} z) \sigma^1,
\]

and define the operators generated by

\[
t(z) = \text{Tr}(L(z)) = \text{Tr}\left( L^+(q^{c/2} z)L^-(z)^{-1} \right)
\]

In ref. [4], we proved the following results:

**Theorem 1** For all values of \( p, q, c \) the operators \( t(z), t(w) \) satisfy an exchange relation of the type

\[
t(z)t(w) = \mathcal{Y}(z/w)^{i_1j_1}_{j_2i_2} L(w)^{j_2}_{i_2} L(z)^{j_1}_{i_1},
\]

where the matrix \( \mathcal{Y}(z/w) \) is given by

\[
\mathcal{Y}(z/w) = \left( \left( R_{12}^+(w/z) R_{12}^+(q^{c+2} w/z)^{-1} R_{12}^+(z/w)^{-1} \right)^{t_2} R_{12}^+(q^c z/w)^{t_2} \right)^{t_2}.
\]

**Theorem 2** When \( c = -2 \), the operators \( t(z) \) lie in the center of the algebra \( A_{q,p}(\hat{sl}(2)_{-2}) \). In particular the matrix \( \mathcal{Y} \) is equal to the 4 \( \times \) 4 unit matrix, that is \(|t(z), t(w)| = 0\).

**Theorem 3** There exists a natural Poisson structure on the center of \( A_{q,p}(\hat{sl}(2)_{-2}) \) given by

\[
\{t(z), t(w)\} = -(\ln q) \left( (w/z) \frac{d}{d(w/z)} \ln \tau(q^{1/2} w/z) - (z/w) \frac{d}{d(z/w)} \ln \tau(q^{1/2} z/w) \right) t(z)t(w),
\]

and leading to a whole set of Poisson structures for the modes \( t_n = \int_C \frac{dz}{2\pi i z} z^{-n} t(z) \).

This set of Poisson structures is parametrized by the relative positions of the contours \( C_z \) and \( C_w \) around the origin, used to extract \( \{t_n, t_m\} \) from (1.18). The initial Poisson bracket structure (1.18) must be understood in the sense of formal series for \( t(z), t(w) \). Hence, depending on the choice of relative positions of the contours for \( z \) and \( w \), or equivalently on the choice of a formal series expansion for the meromorphic structure functions according to whether \(|z/w| \in ]q^k, q^{k+1}[\) for some \( k \) in \( \mathbb{Z} \), one gets distinct Poisson structures for the modes \( t_n \) labeled by \( k \). This fact was also hinted at in the study of the quantized version [3].
2 Quadratic subalgebras in $A_{q,p}(\hat{sl}(2)_c)$

We now turn to the task of identifying possible closed (eventually abelian) algebras of trace-like generators in $A_{q,p}(\hat{sl}(2)_c)$. Since the generic problem is far too vast we shall simply ask the question whether the generators $t(z)$ already defined in (1.13) may close an exchange algebra. We first prove:

**Theorem 4** For any integer $m$, if $p$, $q$, $c$ are connected by the relation $p^m = q^{c+2}$, the generators $t(z)$ realize an exchange algebra with all generators $L(w)$ of $A_{q,p}(\hat{sl}(2)_c)$:

\[ t(z)L(w) = F\left(m, \frac{w}{z}\right) L(w)t(z) \quad (2.1) \]

where

\[ F(m, x) = \prod_{s=1}^{2m} q^{-\frac{(2q^s p^s - s)}{q^s (x^2 - q^s p^s)} - \frac{(2q^s p^s - s)}{q^s (x^2 - q^s p^s)}} \quad \text{for } m > 0 , \quad (2.2a) \]

\[ F(m, x) = \prod_{s=0}^{2|m|-1} q^{-\frac{(2q^s p^s - s)}{q^s (x^2 - q^s p^s)} - \frac{(2q^s p^s - s)}{q^s (x^2 - q^s p^s)}} \quad \text{for } m < 0 . \quad (2.2b) \]

**Proof:** The proof runs much along the lines of the commutativity proof in [1]. It is easier to formulate it in terms of $L^+(w)$:

\[ t(z) L^+_2(w) = \text{Tr}_1 \left( L^+_1(zq^2)^{t_1} (L^-_1(z)^{-1})^{t_1} L^+_2(w) \right) \]

\[ = \text{Tr}_1 \left( L^+_1(zq^2)^{t_1} (L^-_1(z)^{-1})^{t_1} L^+_2(w) \right) \]

\[ = \text{Tr}_1 \left( L^+_1(zq^2)^{t_1} (R^+_{21}(q^2 w/z)^{t_1})^{-1} L^+_2(w) (L^-_1(z)^{-1})^{t_1} R^+_{21}(q^{-2} w/z)^{t_1} \right) . \quad (2.3) \]

(by the exchange algebra for $L^-$ and $L^+$, see eq. (3.3) of [1])

From the exchange algebra between $L^+_1(z)$ and $L^-_2(w)$, redefining $z \rightarrow q^2 z$ and using the crossing symmetry property $(R^+_{21}(x)^{-1})^{t_1} = (R^+_{21}(x^{-2})^{t_1})^{-1}$ one also has:

\[ L^+_1(zq^2)^{t_1} (R^+_{21}(q^2 w/z)^{t_1})^{-1} L^+_2(w) = L^+_2(w) (R^+_{21}(q^{-2} w/z)^{-1})^{t_1} L^+_1(zq^2)^{t_1} . \quad (2.4) \]

In order to use (2.4) so as to reexpress the first three factors in (2.3) one needs to set $q^{-c-2} = p^{-m}$ and to use the $p$-shift property of $R^+_{21}$ as:

\[ R^+_{21}(xp) = \left( \tau(xq^2)\tau(x^{-1}q^2)\tau(xq^2 p^2)\tau(x^{-1}q^2 p^{-2}) \right)^{-1} R^+_{21}(x) \]

\[ \equiv F^{-1}(x) R^+_{21}(x) . \quad (2.5) \]

The origin of the condition $p^m = q^{c+2}$ is precisely in that it is the only one allowing for a substitution of (2.4) in (2.3).
One then extracts from (2.3) the prefactor generated by the use of (2.5) inside (2.4). This prefactor reads:

\[
F(m, x) \equiv \frac{R_{21}^+(x p^{-m})}{R_{21}^+(x)} = \begin{cases} 
\prod_{s=1}^{m} F(x p^{-s}) & \text{for } m > 0, \\
\prod_{s=0}^{m-1} F(x p^{-s})^{-1} & \text{for } m < 0.
\end{cases}
\]  

(2.6)

Therefore one has:

\[
t(z) L_2^+(w) = F(m, q^{2} w/z) L_2^+(w) \text{Tr}_1 \left( (R_{21}^{+*}(q^{-2} w/z)^{-1})^{t_1} L_1^+(z q^{2})^{t_1} (L_1^{-}(z)^{-1})^{t_1} R_{21}^{+*}(q^{-2} w/z)^{t_1} \right),
\]

(2.7)

and the two \(R\)-matrices cancel due to the same mechanism as in (2.9):

\[
\text{Tr}_1 \left( R_{21} Q_1 R'_{21} \right) = \text{Tr}_1 \left( Q_1 R_{21}^{t_2} R_{21}^{t_2} \right)^{t_2},
\]

(2.8)

if \(R\) and \(R'\) are \(c\)-number matrices.

Finally, (2.4) can be computed using (1.7):

\[
F(m, x) = \prod_{s=1}^{2m} q^{-1} \frac{\partial_{q^4}(x^2 q^2 p^{-s}) \partial_{q^4}(x^{-2} q^2 p^{s})}{\partial_{q^4}(x^2 p^{-s}) \partial_{q^4}(x^{-2} p^{s})} \quad \text{for } m > 0,
\]

and

\[
F(m, x) = F(|m|, x^{-1} p^{2})^{-1} \quad \text{for } m < 0,
\]

which is formula (2.2).

Then, recalling that \(L^+(w) = L(q^2 w)\), one gets (2.1) as stated. \(\blacksquare\)

Remark 1: For \(m = 0\), the relation can be realized in two ways: either \(c = -2\), which is the case studied in [1] and leads directly to a center \(t(z)\) \((F(m, x) = 1)\); or \(q = \exp \left( \frac{2i\pi z}{c+2} \right)\), hence \(|q| = 1\), which we have decided not to consider here owing to the potential singularities in the elliptic functions. Hence \(m = 0\) will be disregarded from now on.

Remark 2: Equation (2.1) can be interpreted as meaning that \(t(z)\) act in a uniform way as a sort of derivation on \(A_{q,p}(\hat{sl}(2)\varepsilon)\). We shall comment more extensively on this fact in the conclusion.

Remark 3: The function \(F(m, x)\) is invariant under the shift \(p \rightarrow pq^4\) due to the periodicity properties of \(\partial_{q^4}\). Hence one can restrict the parameter space of our algebras to any set \(p \in ]p_0, p_0 q^4[\).

An immediate corollary is:

**Theorem 5** When \(p^m = q^{m+2}\), \(t(z)\) closes a quadratic subalgebra:

\[
t(z) t(w) = \mathcal{Y}_{p,q,m}(\frac{w}{z}) t(w) t(z)
\]

(2.9)

where

\[
\mathcal{Y}_{p,q,m}(x) = \begin{cases} 
\left[ \prod_{s=1}^{2m-1} x^2 \frac{\partial_{q^4}(x^{-2} p^{-s}) \partial_{q^4}(x^2 q^2 p^{s})}{\partial_{q^4}(x^2 p^{s}) \partial_{q^4}(x^{-2} q^2 p^{-s})} \right]^2 & \text{for } m > 0, \\
\left[ \prod_{s=1}^{2|m|} x^2 \frac{\partial_{q^4}(x^{-2} p^{-s}) \partial_{q^4}(x^2 q^2 p^{s})}{\partial_{q^4}(x^2 p^{s}) \partial_{q^4}(x^{-2} q^2 p^{-s})} \right]^2 & \text{for } m < 0.
\end{cases}
\]

(2.10)
Proof: From (2.11) one has:

\[ t(z) L^+ (w) = F \left( m, q^{\frac{w}{z}} \right) L^+ (w) t(z), \]  
(2.11a)

\[ t(z) (L^- (w))^{-1} = F^{-1} \left( m, -p^{\frac{1}{z}} \right) (L^- (w))^{-1} t(z), \]  
(2.11b)

hence (recalling that \( t(z) = \text{Tr}(L^+ (q^{\frac{1}{z}}) L^- (z)^{-1}) \))

\[ t(z) t(w) = \frac{F \left( m, q^{-\frac{w}{z}} \right)}{F \left( m, -p^{\frac{1}{z}} \right)} t(w) t(z). \]  
(2.12)

The explicit expression for \( F \) in (2.2) gives the result after extensive use of the two “periodicity” properties (1.12).

Remark 4: When \( m = 1 \) the exchange function in (2.10) is exactly the square of the exchange function in the quantization of the \( q \)-deformed Virasoro algebra proposed in [5], once the replacements \( q^2 \to p, p \to q, x^2 \to x \) are done. This is a first indication in our context that the elliptic algebra \( \mathcal{A}_{q,p}(\widehat{sl}(2)_c) \) appears to be the natural setting to define quantized \( q \)-deformed Virasoro algebras.

Remark 5: As an additional connection we notice that all exchange functions \( Y_{p,q,m}(x) \) obey the typical identities for the Feigin-Frenkel function:

\[ Y(xq^2) = Y(x), \]
\[ Y(xq) = Y(x^{-1}). \]  
(2.13)

Our exchange algebras then appear as natural generalizations of the \( W_{q,p}(sl(2)) \) algebra in [4]. This interpretation will be reinforced by the next results.

3 Commuting subalgebras and Poisson structures

We now show:

Theorem 6 For \( p = q^{2k}, k \in \mathbb{Z} \setminus \{0\} \), one has

\[ F(m, x) = 1 \quad \text{for } k \text{ odd,} \]  
(3.1)

\[ F(m, x) = q^{-2m x^4} \left[ \frac{\partial_q t(x^2 q^2)}{\partial_q t(x^2)} \right]^{4m} \quad \text{for } k \text{ even.} \]  
(3.2)

Hence when \( k \) is odd \( t(z) \) is in the center of the algebra \( \mathcal{A}_{q,p}(\widehat{sl}(2)_c) \), while when \( k \) is even \( t(z) \) is not in a (hypothetical) center of \( \mathcal{A}_{q,p}(\widehat{sl}(2)_c) \). However in both cases, one has \([t(z), t(w)] = 0\).

Proof: Theorem 3 is easily proved using the explicit expression for \( F(m, x) \) and the periodicity properties of \( \partial \)-functions in (1.12). Due to Remark 3 one needs only to consider the cases \( k = 1 \) and \( k = 2 \). \( k = 0 \) is excluded since it would lead to \( p = 1 \) and potential singularities.
This now allows us to define Poisson structures on the corresponding abelian algebras even though $t(z)$ is not in the center of $A_{q,p}(\hat{sl}(2)_c)$ for $k$ even. They are obtained as limits of the exchange algebra (2.9). Reciprocally this time, since the initial non-abelian structure for $t(z)$ is closed, the exchange algebras (2.9) are a natural quantization of the Poisson algebras which we obtain. This was not the case in [1], where no intermediate closure condition such as $p^n = q^{c+2}$ could be obtained.

**Theorem 7** Setting $q^{2k} = p^{1 - \frac{2}{k}}$ for any integer $k \neq 0$, one defines the $k$-labeled Poisson structure as:

$$\left\{ t(z), t(w) \right\}_k = \lim_{\beta \to 0} \frac{1}{\beta} \left[ t(z)t(w) - t(w)t(z) \right]$$

$$= 2km \ln q \left\{ \frac{x^2}{1 - x^2} - \frac{x^{-2}}{1 - x^{-2}} + \sum_{n=0}^{\infty} \left[ -\frac{2x^2q^{4n}}{1 - x^2q^{4n}} + \frac{2x^2q^{4n+2}}{1 - x^2q^{4n+2}} \right] \right\} \text{ for } k \text{ odd}, \tag{3.3a}$$

$$= -2km(2m - 1) \ln q \left\{ \frac{x^2}{1 - x^2} - \frac{x^{-2}}{1 - x^{-2}} + \sum_{n=0}^{\infty} \left[ -\frac{2x^2q^{4n}}{1 - x^2q^{4n}} + \frac{2x^2q^{4n+2}}{1 - x^2q^{4n+2}} \right] \right\} \text{ for } k \text{ even}. \tag{3.3b}$$

**Proof:** We note that

$$\left\{ t(z), t(w) \right\}_k = \left. \frac{dY_{p,q,m}}{d\beta} \right|_{\beta = 0} t(z)t(w) = \left. \frac{d\ln Y_{p,q,m}}{d\beta} \right|_{\beta = 0} t(z)t(w), \tag{3.4}$$

the two equalities coming from the fact that $Y_{p,q,m} = 1$ when $q^{2k} = p$. The proof is then obvious from (2.10) and the definition of $\vartheta$-functions (1.11) as absolutely convergent products (for $|q| < 1$), hence as in [1], the series in (3.3) are convergent and define univocally a structure function for $t(z)$.

This formula coincides exactly with the Poisson structure of the center of $A_{q,p}(\hat{sl}(2)_c)$, provided one reabsorbs $km$ and $-km(2m - 1)$ into the definition of the classical limit as $\beta \to km\beta$ for $k$ odd and $\beta \to -km(2m - 1)\beta$ for $k$ even. By the same mechanism as [1], it leads to a rich set of Poisson brackets for the modes of $t(z)$ defined as $t_n = \oint_C \frac{dz}{2\pi i z^n} z^{-n} t(z)$ due to the poles of the structure function at $z/w = q^n, n \in \mathbb{Z}$.

4 Concluding remarks

Theorem 7 now provides us with an immediate interpretation of the quadratic structures (2.3). Since we have seen in [1] that the Poisson structures derived from (3.3) contained in particular the $q$-deformed Virasoro algebra (up to the delicate point of the central extension which is not explicit in (3.3)), the quadratic algebras (2.3) are inequivalent (for different values of $m!$) quantizations of the $q$-deformed Virasoro algebra, globally defined on the $\mathbb{Z}$-labeled 2-dimensional subsets of parameters.
defined by $p^m = q^{c+2}$. They are thus generalized $W_{q,p}(sl(2))$ algebras at $c = -2 + \frac{\ln p}{\ln q}$. Moreover the closed algebraic relation (2.1) may, in such a frame, acquire a crucial importance as a $q$-deformation of the Virasoro-current commutations relations. This would then provide us with the full $q$-deformed structure required to construct possible generalizations of quantum Ruijsenaars-Schneider models, following the original derivation of $q$-deformed Virasoro algebras [3, 4].

A better understanding of the undeformed limit $q \to 1$ would help us to clarify this interpretation if one could indeed identify the standard Virasoro-Kac Moody structure in such a limit. The difficulty lies in the correct definition of this limit for the generators $L(z)$ and $t(z)$ which should be consistent with such an interpretation.

As in [3] the help could come from an explicit bosonization of the elliptic algebra as was done for $U_q(\hat{sl}(N)_c)$ in [11]. This would also provide us with a solution of the previously mentioned central-extension problem. At this time a bosonized version of the elliptic algebra $A_{q,p}(\hat{sl}(2)_c)$ is available only at $c = 1$ [13] using bosonized vertex operators constructed in [14].

It is important to note also that such vertex operators were interpreted as $q$-analogs of primary fields for $q$-deformed $W_N$ algebras [11, 13]. Again this establishes a connection between $q$-Virasoro and $q$-$W_N$ algebras and the elliptic algebra $A_{q,p}(\hat{sl}(2)_c)$ or its very recent generalization to arbitrary $N$ $A_{q,p}(\hat{sl}(N)_c)$ [13], but only by using explicit bosonized forms. Our connection on the other hand, where $q$-Virasoro algebras are directly constructed as subalgebras of $A_{q,p}(\hat{sl}(2)_c)$, uses only the abstract algebraic structure of the elliptic algebra. Extensions of such a construction may be worth looking for, providing the framework to understand this vertex operator-primary field connection at a purely algebraic level.

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