Zeta functions and dynamical systems

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Abstract

In this brief paper we present a very simple strategy to investigate dynamical determinants for uniformly hyperbolic systems. The construction builds on the recent introduction of suitable functional spaces which allow us to transform simple heuristic arguments into rigorous ones. Although the results so obtained are not exactly optimal, the straightforwardness of the argument makes them noticeable.

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1. Introduction

The goal of this paper is to investigate the properties of the dynamical Fredholm determinants of uniformly hyperbolic systems and to relate them to the statistical properties of such systems. This subject has been widely investigated and there exists a large literature where many partial results are obtained. We refer the reader to [1] for references and an introduction to the subject, to [3, 8] for a more recent account of the situation and to [4] for an in-depth discussion of the physical relevance of these issues.

The basic idea presented in this paper is to study the action of the dynamics on an appropriate singular functional kernel, as suggested in Dmitry Dolgopyat’s thesis (Princeton 1997), to obtain results on the radius of convergence of the dynamical Fredholm determinant, its relation to the spectral properties of the transfer operator and the Ruelle resonances. The new ingredient allowing us to carry out such a programme is the possibility, after [6] and [2], of introducing spaces in which such singular kernels are legal objects. To clarify matters we start with a folklore explanation.

Let \( X \) be a \( d \)-dimensional \( C^{r+1} \) Riemannian manifold and \( T : X \to X \) a \( C^{r+1} \) diffeomorphism which satisfies some hyperbolicity condition. (We assume at least that all the periodic points of \( T \) are hyperbolic.) For each \( g \in C^r(X, \mathbb{C}) \) we define the Ruelle transfer
operator $T_g : C'(X, \mathbb{C}) \to C'(X, \mathbb{C})$ by
$$T_g h := g \cdot h \circ T.$$ (1.1)
The dynamical Fredholm determinant of this operator $T_g$ is formally defined by
$$d_{T,g}^0(z) = \exp \left[ - \sum_{n \geq 1} \frac{z^n}{n} \sum_{x \in \text{Fix} T^n} \frac{g_n(x)}{|\det(Id - DT^n(x))|} \right],$$ (1.2)
where $g_n(x) := \prod_{i=0}^{n-1} g(T^i(x))$.

Let us first note that this can be heuristically regarded as the determinant $\det(Id - z \cdot T_g)$. Indeed, let $\delta$ be the distribution on $X^2 = X \times X$ defined by
$$\delta(h) = \int_X h(x, x) \, dx.$$ Then the ‘kernel’ of the operator $T_g^n$ is given by $(Id \otimes T_g)^n(\delta)$. Thus, as in the case of operators with smooth kernel, it would be natural to define
$$\text{Tr} T_g = \langle \delta, (Id \otimes T_g)^n(\delta) \rangle.$$ Though the product of two distributions is not defined in general, we will be able to give an appropriate meaning to the right-hand side above since the singular supports (in $C'$ sense) of $\delta$ and $(Id \otimes T_g)(\delta)$ do not intersect. We find
$$\langle \delta, (Id \otimes T_g)^n(\delta) \rangle = \sum_{x \in \text{Fix} T^n} \frac{g_n(x)}{|\det(Id - DT^n(x))|}.$$ The definition (1.2) of the dynamical Fredholm determinant follows then via the formal relation $\det(Id - zA) = \exp(-\sum_{n \geq 1}(z^n\text{Tr}A^n)/n)$.

It is thus natural to expect that the properties of the dynamical Fredholm determinant as a holomorphic function are closely related to the spectral properties of the operator $T_g$. In this paper, we present an argument providing exactly such a relation, although in a slightly more restrictive setting. The argument is rigorous, yet it follows the above simple ideas very closely.

Let $T : X \to X$ be an Anosov diffeomorphism, i.e. there exists a $DT$-invariant decomposition $TM = E^u \oplus E^s$ and constants $\lambda \in (0, 1)$ and $C > 0$ such that
$$\|DT^n|_{E^u}\| \leq C\lambda^n, \quad \|DT^{-n}|_{E^s}\| \leq C\lambda^n \quad \text{for all } n \geq 0.$$ In [6,2], Banach spaces $B$ of distributions on $X$ are defined so that the operator $T_g$ extends to a bounded operator $T_g : B \to B$ whose essential spectral radius is bounded by $\|g\|_{L^\infty} \cdot \lambda^{a_r}$, where $a_r := \min\{\lfloor r/2 \rfloor, r - \lfloor r/2 \rfloor\}$, $\lfloor a \rfloor$ being the closest integer to $a \in \mathbb{R}$. In addition, it is shown that the eigenvalues outside the essential spectral radius have a well-defined dynamical meaning (Ruelle resonances). Let $\rho_\ast = \|g\|_{L^\infty} \cdot \lambda^{a_r}/2$. Our result is as follows.

**Theorem 1.** $d_{T,g}^0(z)$ extends holomorphically to $D(\rho_\ast^{-1}) = \{|z| < \rho_\ast^{-1}\}$ and the zeros of such an extension are in one–one correspondence, with multiplicity, to the inverse of the eigenvalues of $T_g : B \to B$ in the region $\{|z| > \rho_\ast\}$.

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3 Note that if the number of periodic points does not grow more than exponentially, then $d_{T,g}^0(z)$ is well defined and holomorphic in a sufficiently small disc.

4 Formally, $\delta$ is $\delta(x - y)$ (where now $\delta$ is the physicists delta function) and the kernel is given by $g(x)\delta(T_n x - y)$.

5 As we shall see later, the action of $Id \otimes T_g$ can in fact be extended to an operator on the space of distributions.

6 Actually, [6] deals explicitly only with the adjoint of $T_g$ in the case $g \equiv 1$ (SRB measures), yet the extension to the present setting is straightforward.
This result is not new or optimal. Kitaev [7] has given a stronger result for the extendibility part of the former claim\(^7\), while the spectral interpretation, albeit for a smaller radius, appeared already in [8], and, more recently, in [3] it has been obtained for a domain corresponding to the result of Kitaev. Nevertheless, the proofs yielding such sharper results are far more complex than the present argument.

The non-optimality of the above theorem is the price for considering, in the following, the operator \(T^* \otimes T^*\) instead of the operator \(Id \otimes T\) used in the previous heuristic argument. Unfortunately, we do not know how to treat the transfer operator \(Id \otimes T\) directly as the mapping \(Id \times T\) is not hyperbolic and the extension of the results in [2, 6] to the partially hyperbolic setting is far from trivial (if possible at all).

2. Basic definitions

For \(g \in C' (X, \mathbb{C})\) we define the transfer operator \(T_g : C' (X, \mathbb{C}) \rightarrow C' (X, \mathbb{C})\) by
\[
T_g h := g \cdot (h \circ T).
\]
Its formal adjoint (in fact, dual) \(T^*_g : C' (X, \mathbb{C}) \xrightarrow{\circ} C' (X, \mathbb{C})\) is given by the transfer operator
\[
T^*_g f := | \det DT^{-1}| \cdot (g \cdot f) \circ T^{-1}.
\]
That is, \((T^*_g h, f)_{L^2(X)} = (\tilde{h}, T^*_g f)_{L^2(X)}\). Let \(D'_r(X)\) be the space of distributions on \(X\) of order \(r\). Using the above formal relation, we can extend the operators \(T_g\) and \(T^*_g\) to continuous operators \(T_g : D'_r(X) \rightarrow D'_r(X)\) and \(T^*_g : D'_r(X) \rightarrow D'_r(X)\), respectively.

Next we define \(T^*_g \otimes T^*_g : C' (X^2, \mathbb{C}) \rightarrow C' (X^2, \mathbb{C})\) as the unique extension of \(T^*_g \otimes T^*_g : C' (X, \mathbb{C}) \otimes C' (X, \mathbb{C})\). The latter operator reads
\[
T^*_g \otimes T^*_g (\psi)(x, y) := g(T^{-1} x) \cdot | \det DT^{-1}| \cdot g(y) \cdot \psi(T^{-1} x, T y).
\]
So it can be interpreted as the transfer operator associated to the hyperbolic mapping \(T^{-1} \times T\) with the weight \(\tilde{g}(x, y) = g(T^{-1} x) \cdot | \det DT^{-1}| \cdot g(y)\). Note that the formal adjoint of \(T^*_g \otimes T^*_g\) is \(T_g \otimes T^*_g\). As above, we can extend these operators to
\[
T_g \otimes T^*_g (\psi) : D'_r(X^2) \rightarrow D'_r(X^2) \quad \text{and} \quad T^*_g \otimes T_g (\psi) : D'_r(X^2) \rightarrow D'_r(X^2).
\]

3. The Banach spaces of distributions

The spaces of distributions on which we have defined the transfer operators are not appropriate for a study of the dynamics. Many recent works (e.g. [2, 5, 6]) have focused on the problem of finding adapted functional spaces and different choices have different advantages. Accordingly, it may be better not to focus on a particular choice but to discover which properties are needed to carry out the study of the Zeta functions. We will list the properties of the function spaces that suffice for our argument. Yet, for definiteness of exposition we will comment explicitly the Banach spaces introduced in [6] and remark that they indeed satisfy such properties.

\(^7\) Essentially, instead of the bound \(\rho_{-1}^{-1}\), Kitaev has the more natural bound \((\|g\|_{L^\infty} \cdot \lambda_{-2}^{-1})^{-1} \sim (\|g\|_{L^\infty} \cdot \lambda_{-1})^{-1}\), that is the inverse of the bound for the essential spectral radius of \(T_g : B \rightarrow B\).

\(^8\) Here, by \(D'_r(X)\) we mean the dual of the space \(C'_r(X)\) defined as follows. For \(r \geq 0\), let \(\lfloor r \rfloor\) be its integer part. We denote by \(\mathcal{C}^\circ\) the set of functions which are \(\lfloor r \rfloor\) times continuously differentiable and whose \(|r|\)th derivative is Holder continuous of exponent \(r - \lfloor r \rfloor\) if \(r\) is not an integer. To fix the notation, in this paper we choose, for each \(r \in \mathbb{R}_+\), a norm on \(\mathcal{C}^\circ\) functions so that \(|f_1 f_2|_{\lfloor r \rfloor} \leq |f_1|_{\lfloor r \rfloor} |f_2|_{\lfloor r \rfloor}\). We will denote by \(\mathcal{C}^\circ\) the closure in \(\mathcal{C}^\circ\) of the set of \(\mathcal{C}^\circ\) functions. It coincides with \(\mathcal{C}^\circ\) if \(r\) is an integer, but is strictly included in it otherwise. In any case, it contains \(\mathcal{C}^\circ\) for all \(r' > r\).
In [6], Guillé and Liverani introduced a scale of Banach spaces $B^{p,q}$ with $q \in \mathbb{R}_+$, $p \in \mathbb{N}$ and $p + q < r$ adapted to $C^{r+1}$ Anosov diffeomorphisms $T : X \to X$. The parameters $p$ and $q$ will be fixed at the end of the argument. We denote $B = B^{p,q}$ and set

$$\rho = \rho_{p,q} = \lambda_{\min[p,q]}\|g\|_{L^\infty}, \quad \tilde{\rho} = \rho\|g\|_{L^\infty}.$$  

The basic properties of $B$ are the following (see [6] for a proof):

(P1) $C^r(X)$ is continuously embedded in $B$ and its image is a dense subset,

(P2) $\tilde{B}$ is continuously embedded in $D'_r(X)$,

(P3) $T_g : B \to B$ is a bounded operator with essential spectral radius bounded by $\rho^9$.

**Remark 3.1.** The meaning of (P3) is that we use (P2) to identify $B$ with a subspace of $D'_r(X)$ and consider the restriction of $T_g$ to $B$. With such an identification the embedding in (P1) is required to be the standard embedding of $C^\infty$ in $D'_r(X)$. In the following we will use the embeddings (P2) and (P5) to identify the elements of $B$ (and $\tilde{B}$) with distributions without making further remarks.

As already noted in (2.1) the operator $T^*_g \otimes T_g$ is a transfer operator for the Anosov diffeomorphism $T^{-1} \times T : X^2 \to X^2$ with the same hyperbolicity constant $\lambda$ of $T$. So, as above, we can introduce a Banach space $\tilde{B}$ with the following properties:

(P4) $C^r(X^2)$ is continuously embedded in $\tilde{B}$ and its image is a dense subset,

(P5) $\tilde{B}$ is continuously embedded in $D'_r(X^2)$,

(P6) $T^*_g \otimes T_g : \tilde{B} \to \tilde{B}$ is a bounded operator with essential spectral radius bounded by $\tilde{\rho}$.

The reader should be aware that usually there is some freedom in the definition of the Banach spaces. For example, in [6] they depend on a family $\Sigma$ of admissible leafs, that is, $C^{r+1}$ embedded compact $d + 1$ dimensional submanifolds with boundary close to local stable manifolds. By taking the family appropriately, we can ensure that $\tilde{B}$ enjoys the following extra properties:

(P7) $(T^*_g \otimes T_g)^{n_0}(\delta)$ is contained in $\tilde{B}$ for some $n_0 \in \mathbb{N},$

(P8) the functional $\delta : C^\infty(X^2) \to \mathbb{C}$, $\delta(\phi) = \delta(\phi)$, extends to a bounded functional $\tilde{\delta} : \tilde{B} \to \mathbb{C}$.

Finally, as should be apparent from the previous heuristic argument, we need some control on how to approximate singular kernels by smooth ones. Let $\{(U_i, \Psi_i : U \to \mathbb{R}^d)\}_{i=1}^k$ be a $C^{\infty}$ atlas of $X$, and let $\{\rho_i\}_{i=1}^k$ be a $C^{\infty}$ partition of unity subordinated to such an atlas. Next, define the functions $j_i \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$ so that $\int_{\mathbb{R}^d} j_i = 1$ and $\text{supp}(j_i) \subset \{x \in \mathbb{R}^d : \|x\| \leq 1\}$. We then define

$$J_\varepsilon f(x) := \sum_i \int_{\mathbb{R}^d} \rho_i \circ \Psi_i^{-1}(y) j_i(\Psi_i(x) - y) f \circ \Psi_i^{-1}(y) \, dy \quad \text{for } f \in C^r(X).$$

Let $J_\varepsilon^*$ be the formal adjoint of $J_\varepsilon$. Clearly these extend to bounded operators

$$J_\varepsilon : D'_r(X) \to C^\infty(X) \quad \text{and} \quad J_\varepsilon^* : D'_r(X) \to C^\infty(X).$$

9 As already mentioned in footnote 6, strictly speaking [6] does not apply. To prove (P3) it suffices to follow step by step the computations in [6, lemma 6.3], only keeping track of the weight $g$. Note that [6, lemma 6.2] is not needed here since there is no factor $\det DT$ to deal with.

10 If we would simply apply (P3) to $T_g^* \otimes T_g$, we would have an additional factor related to $\det DT$ in the expression of $\tilde{\rho}$ in (P6). This would result in a smaller bound than $\rho_+$ in theorem 1. To obtain the bound $\tilde{\rho}$ used in (P6), we must again go back to the argument in [6]: as before, following verbatim the computations in [6, lemma 6.3], but keeping track of the weight, and then using a trivial generalization of [6, lemma 6.2].

11 In the definition of $\tilde{B}$, one can take $\Sigma$ so that the diagonal in $X^2$ is covered by finitely many elements in $\Sigma$. This immediately implies (P8).

12 Note that $J_\varepsilon$ is well defined for $\varepsilon$ small enough.
We also define
\[ \tilde{J}_\epsilon : \mathcal{D}'(X^2) \to C^\infty(X^2) \]
as the unique extension of \( J_\epsilon \otimes J_\epsilon : C'(X) \otimes C'(X) \to C^\infty(X^2) \). Then we have\(^{13}\)

(P9) for \( u \in B \), \( J_\epsilon^* J_\epsilon u \to u \) in \( B \), as \( \epsilon \to +0 \),

(P10) for \( u \in \tilde{B} \), we have \( \tilde{J}_\epsilon u \to u \) in \( \tilde{B} \), as \( \epsilon \to +0 \).

**Remark 3.2.** From now on we will use only the above properties, regardless of the way the spaces are actually constructed.

Here are two consequences of properties (P1)–(P10) that show the relevance of the above objects for the problem at hand.

**Lemma 3.3.** Let \( \{\ell_i\} \subset B' \) and \( \{e_i\} \subset B \). If \( \sum_{i=1}^k \ell_i \otimes e_i \) belongs to \( \tilde{B} \), then we have
\[ \bar{\delta} \left( \sum_{i=1}^k \ell_i \otimes e_i \right) = \sum_{i=1}^k \ell_i(e_i). \]

**Proof.** Define \( J'_\epsilon : B' \to C^\infty \) by the duality relation \( J'_\epsilon(h) := \ell(J_\epsilon^* h) \). By (P10), \( \tilde{J}_\epsilon(\sum_{i=1}^k \ell_i \otimes e_i) = \sum_{i=1}^k J'_\epsilon(\ell_i) \otimes J_\epsilon(e_i) \in C^\infty(X^2) \) converges to \( \sum_{i=1}^k \ell_i \otimes e_i \) in \( \tilde{B} \), as \( \epsilon \to +0 \). Since
\[ \bar{\delta} \left( \sum_{i=1}^k J'_\epsilon(\ell_i) \otimes J_\epsilon(e_i) \right) = \sum_{i=1}^k (J'_\epsilon(\ell_i), J_\epsilon(e_i))_{L^2(X)} = \sum_{i=1}^k J'_\epsilon(\ell_i)(J_\epsilon(e_i)) \]
\[ = \sum_{i=1}^k \ell_i(J_\epsilon^* J_\epsilon(e_i)), \]
the claim of the lemma follows by taking the limit \( \epsilon \to +0 \) in the above equality owing to properties (P8) and (P9). \( \square \)

**Lemma 3.4.** Set \( \delta_T = (T_g^* \otimes \text{Id})\delta \). For each \( n \geq n_0 \), we have
\[ \bar{\delta}((T_g^* \otimes T_g)^n \delta) = \sum_{x \in \text{Fix } T_g^{2n}} \frac{g_{2n}(x)}{\det(\text{Id} - DT_g^{2n}(x))}, \]
\[ \bar{\delta}((T_g^* \otimes T_g)^n \delta_T) = \sum_{x \in \text{Fix } T_g^{2n+1}} \frac{g_{2n+1}(x)}{\det(\text{Id} - DT_g^{2n+1}(x))}. \]

**Proof.** Let \( \delta_\epsilon = \tilde{J}_\epsilon((T_g^* \otimes T_g)^n \delta) \). By (P6), (P7) and (P10) we have \( (T_g^* \otimes T_g)^n \delta, \delta_T \subset (T_g^* \otimes T_g)^n \delta \) in \( \tilde{B} \). We leave it to the reader to check that the right-hand side of the first equality above is obtained as the limit \( \lim_{\epsilon \to +0} \bar{\delta}((T_g^* \otimes T_g)^n \delta_\epsilon) \)^{14}. The second equality is obtained in a parallel manner. \( \square \)

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\(^{13}\) These properties are essentially proven in [6, section 7], or see [8].

\(^{14}\) If in trouble, see [8] for details.
4. The proof

Take $\sigma > \max\{\rho, \beta\}$ arbitrarily. Then properties (P3) and (P6) imply

$$T_g^n = P + R : \mathcal{B} \to \mathcal{B}, \quad T_g^n \otimes T_g^n = \tilde{P} + \tilde{R} : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}},$$

where $P$ and $\tilde{P}$ are of finite rank, the spectral radius of $R$ and $\tilde{R}$ are bounded by $\sigma$ and $PR = RP = 0$, $\tilde{P} \tilde{R} = \tilde{R} \tilde{P} = 0$. Note that, for each $h, f \in \mathcal{C}^\infty(X), n \in \mathbb{N}$,

$$(T_g^n \otimes T_g^n)^\sigma h \otimes f) = \delta((T_g^n h \otimes (T_g^n) f) = \langle (T_g^n) h, (T_g^n) f \rangle_{L^2(X)}$$

$$= \langle \tilde{h}, (T_g^n h) \rangle_{L^2(X)} $$

where, in the last expression, $h$ is interpreted as an element of $\mathcal{D}_T$ (and hence $\mathcal{B}$) via the natural embedding. It follows, using the Neumann series for $|z|$ small enough,

$$(\text{Id} - z(T_g^n \otimes T_g^n))^{-1}(T_g^n \otimes T_g^n)^{n0} h \otimes f) = ((\text{Id} - z(T_g^n)^2)^{-1} T_g^{2n} h)(f),$$

where $n_0$ is the integer in the condition (P7). Note that, from (4.1) and the arbitrariness of $\sigma$, both sides of (4.3) have meromorphic extensions to a closed disc $\{z \in \mathbb{C} : |z| \leq \sigma^{-1}\}$, and the equality must hold for those extensions. Since $R^{2n}$ and $\tilde{R}^n$ can be written as

$$R^{2n} = \frac{1}{2\pi i} \int_{|z| = \sigma} z^n (\text{Id} - (T_g)^2)^{-1} dz,$$

$$\tilde{R}^n = \frac{1}{2\pi i} \int_{|z| = \sigma} z^n (\text{Id} - (T_g \otimes T_g)^{-1} (T_g^n \otimes T_g)^n h \otimes f)),$$

we have, for all $n > n_0$ and each $h, f \in \mathcal{C}^\infty(X)$,

$$\tilde{R}^n \delta(h \otimes f) := \tilde{R}^n \delta(h \otimes f) = \tilde{\rho}^{n, n_0} \delta(h \otimes f) = (\tilde{R}^{2n} h)(f),$$

$$\tilde{R}^n \delta(h \otimes f) := \tilde{R}^n \delta(h \otimes f) = \tilde{\rho}^{n, n_0} \delta(h \otimes f) = (\tilde{R}^{2n} h)(f).$$

Next, write the finite rank operator $P^{2n}$ as $P^{2n} = \sum_{i=1}^k e_i^{(n)} \otimes e_i^{(n)}$, where $e_i^{(n)} \in \mathcal{B}'$ and $e_i^{(n)} \in \mathcal{B}$. By (4.4), for each $h, f \in \mathcal{C}^\infty(X)$ and $n > n_0$,

$$\tilde{R}^n \delta(h \otimes f) = \sum_{i=1}^k e_i^{(n)}(h) \cdot e_i^{(n)}(f) = \sum_{i=1}^k e_i^{(n)} \otimes e_i^{(n)}(h \otimes f).$$

Since $\mathcal{C}^\infty(X) \otimes \mathcal{C}^\infty(X)$ is dense in $\mathcal{C}^p(X^2)$ in the $\mathcal{C}^p$ topology, and since the elements of $\tilde{\mathcal{B}}$ are distributions of order $p$ by hypothesis, it follows, for each $n > n_0$,

$$\tilde{R}^n \delta = \sum_{i=1}^k e_i^{(n)} \otimes e_i^{(n)} \in \mathcal{B}' \times \mathcal{B}. $$

We can thus apply lemma 3.3 obtaining

$$\tilde{\delta}(\tilde{R}^n \delta) = \sum_{i=1}^k e_i^{(n)}(e_i^{(n)}(h) \cdot e_i^{(n)}(f) = \text{Tr} P^{2n}.$$

In conclusion we have, for $n > n_0$,

$$\tilde{\delta}(T_g^n \otimes T_g^n)^{\sigma} h \otimes f) = \delta(\tilde{R}^n \delta) \quad \text{Tr} P^{2n} + O(\sigma^n).$$

By replacing $\delta$ by $\delta_T = (T_g^n \otimes \text{Id})\delta$ in the above argument, we obtain also

$$\tilde{\delta}(T_g^n \otimes T_g^n)^{\sigma} h \otimes f) = \delta(\tilde{R}^n \delta_T) + \delta(\tilde{R}^n \delta_T) = \text{Tr} P^{2n+1} + O(\sigma^n).$$

15 Here one has to choose $\sigma$ so that no eigenvalues belong to the circle $\{z \in \mathbb{C} : |z| = \sigma\}$.

16 This is a direct consequence of the Stone–Weierstrass theorem.
Using lemma 3.4 we can conclude

\[
d^{♭}_{T,g}(z) = \exp \left( -\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \text{Fix}T^n} \frac{g_0(x)}{|\det(\text{Id} - D_xT^n)|} \right)
\]

\[
= \exp \left( -\sum_{n=1}^{\infty} \frac{z^n}{n} \text{Tr} P^n + p_0(z) + \sum_{n=1}^{\infty} \mathcal{O}(\sigma^{n/2})z^n \right)
\]

\[
= \det(\text{Id} - zP)\exp \left( -p_0(z) - \sum_{n=1}^{\infty} \mathcal{O}(\sigma^{n/2})z^n \right), \quad (4.6)
\]

where \(p_0(z)\) is a polynomial of order \(2n_0 + 1\).

Since the last series is convergent for \(|z| < \sigma^{-1/2}\), we have that the \(d^{♭}_{T,g}(z)\) is holomorphic in such a disc and the zeros correspond to the eigenvalues of \(P\), that is to the eigenvalues of \(T_g : B \rightarrow B\) in the region \(|z| > \sigma^{1/2}\).

Applying the above argument to the case where \(p = \lfloor r/2 \rfloor\) and \(q\) is arbitrarily close to \(r - \lfloor r/2 \rfloor\) we obtain theorem 1. Let us conclude by reiterating the generality of the approach.

**Remark 4.1.** Given any Banach spaces \(B\) and \(\tilde{B}\) satisfying the properties (P1)–(P10) the above proof applies, hence we obtain theorem 1 with \(\rho_* = \max\{\rho^{-1}, \tilde{\rho}^{-1/2}\}\).

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**References**

[1] Baladi V 2000 Positive transfer operators and decay of correlations *Advanced Series in Nonlinear Dynamics* vol 16 (Singapore: World Scientific)

[2] Baladi V and Tsujii M Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms *Ann. Inst. Fourier* at press (Preprint math.DS/0505015)

[3] Baladi V and Tsujii M *Dynamical determinants and spectrum for hyperbolic diffeomorphisms* in preparation (Preprint math.DS/0606434)

[4] Cvitanović P, Artuso R, Mainieri R, Tanner G and Vattay G 2003 *Chaos: Classical and Quantum* (Copenhagen: Niels Bohr Institute) http://chaosbook.org

[5] Faure F and Roy N 2006 Ruelle–Pollicott resonances for real analytic hyperbolic maps *Nonlinearity* 19 1233–52

[6] Gouëzel S and Liverani C 2006 Banach spaces adapted to Anosov systems *Ergod. Theory Dyn. Syst.* 26 189–217

[7] Kitaev A Yu 1999 Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness *Nonlinearity* 12 141–79

Kitaev A Yu 1999 Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness *Nonlinearity* 12 1717–19 (Corrigendum)

[8] Liverani C 2005 Fredholm determinants, Anosov maps and Ruelle resonances *Discrete Contin. Dyn. Syst.* 13 1203–15