Correlation functions of integrable $O(n)$ spin chains

G.A.P. Ribeiro*

School of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3010, Australia

May 20, 2020

Abstract

We study the correlation functions of the integrable $O(n)$ spin chain in the thermodynamic limit. We derived functional equations of the quantum Knizhnik Zamolodchikov type for the density matrix. We give the explicit solution for two-site correlations for the $O(n)$ for the $n = 3, 4, \ldots, 8$ cases at zero temperature.

Keywords: Integrability, spin chains, correlation functions

*E-mail: pavan@df.ufscar.br; On leave of absence from Departamento de Física, Universidade Federal de São Carlos, PO Box 676, 13565-905, São Carlos-SP, Brazil.
1 Introduction

The correlation functions of integrable quantum model have been largely studied over the last decades \cite{1, 2}. The prominent case is the $SU(2)$ spin-1/2 chain, which was successfully studied in the thermodynamic limit by many different viewpoints, ranging from multiple integrals, algebraic Bethe ansatz methods, hidden Grassmann structure to functional equations of the quantum Knizhnik Zamolodchikov type \cite{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}. There are also results for its higher-spin generalization \cite{15, 16, 17, 18, 19, 20, 21}, where explicit results were obtained by functional equations \cite{20, 21}.

Nevertheless, the natural generalization of these correlation functions studies for high rank algebras $SU(n)$ for $n > 2$ remained open for decades. This is mainly due to the intricate structure of the Bethe states, which are still being unveiled in the context of e.g algebraic Bethe ansatz \cite{22, 23} and separation of variables \cite{24} with the aim of developing a manageable approach to evaluate correlations functions. However, very recently the first explicit results for short-distance correlations in the thermodynamic limit for the $SU(3)$ case were evaluated via the functional equations of quantum Knizhnik Zamolodchikov type \cite{25, 26}.

On the other hand, the high rank Lie algebras $B_n$, $C_n$ and $D_n$ \cite{27, 28} and superalgebras $OSp(n|2m)$ \cite{29, 30, 31} are less studied. These cases have been treated via analytical Bethe ansatz \cite{27} and algebraic Bethe ansatz (nested Bethe ansatz) \cite{30, 31} and its spectral and critical properties have been studied \cite{27, 32, 33}. Nevertheless, there are no results for correlation functions.

In this work, we start the investigation of correlation functions of integrable high rank spin chains (other than $SU(n)$) via the approach of functional equations of quantum Knizhnik Zamolodchikov type. We derived a
functional equation for the reduced density matrix for the isotropic integrable $O(n)$ spin chain. The functional equation plus a number of other properties like intertwining symmetry, asymptotic, analyticity and normalization allow for the fully determination of the density matrix. We provide the explicit solutions for the two-sites density matrix for the the $O(n)$ for the $n = 3, 4, \ldots, 8$ cases.

This paper is organized as follows. In section 2, we outline the integrable structure of the model. In section 3 we introduce the reduced density matrix and its functional equations and symmetry properties. In the section 4 we present the zero temperature solution for two-site correlation functions for the cases $O(3)$, $O(4)$ and $O(5)$. In the appendix we present additional results for the cases $O(6)$, $O(7)$ and $O(8)$. Our conclusions are given in section 6.

2 The model

The Hamiltonian of the integrable $O(n)$ spin chain can be written as [27, 28, 29, 31],

$$
\mathcal{H} = - \sum_{i=1}^{L} \left( I_{i,i+1} - P_{n,i+1} + \frac{1}{\Delta} E_{i,i+1} \right),
$$

(1)

where $L$ is number of sites, $\Delta = (n - 2)/2$ and $I_{i,i+1}$, $P_{n,i+1}$ and $E_{i,i+1}$ are the identity, permutation and Temperley-Lieb operator acting on the sites $i$ and $i+1$. The Hilbert space is $V^\otimes L$, where $V = \mathbb{C}^n$.

Here, we recall that the Hamiltonian $\mathcal{H}$ is obtained from the logarithmic derivative of the row-to-row transfer matrix $T(\lambda) = \text{Tr}_{A} [R_{A,L}(\lambda) \ldots R_{A,1}(\lambda)]$, such that $\mathcal{H} = \frac{d}{d\lambda} \log T(\lambda) \bigg|_{\lambda=0} = \sum_{i=1}^{L} h_{i,i+1}$, where $h_{i,i+1} = P_{n,i+1} \frac{d}{d\lambda} R_{i,i+1} \bigg|_{\lambda=0}$ and that the $R$-matrix is a solution the well-known Yang-Baxter equation

$$
R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu).
$$

(2)
For the case of $O(n)$ integrable chain, the $R$-matrix can be conveniently written as \[27, 28, 29, 31\]

$$R_{12}(\lambda) = \frac{\lambda}{\lambda + 1} I_{12} + \frac{1}{\lambda + 1} P_{12} - \frac{\lambda}{(\lambda + 1)(\lambda + \Delta)} E_{12}. \quad (3)$$

This matrix has the important properties of regularity, unitarity and crossing as given below,

$$R_{12}(0) = P_{12}, \quad (4)$$
$$R_{12}(\lambda) R_{21}(-\lambda) = I_{12}, \quad (5)$$
$$R_{12}(\lambda) = g(\lambda)(V \otimes I) R_{12}^{t_2}(-\lambda - \rho)(V \otimes I), \quad (6)$$

where $t_2$ is transposition in the second space, the crossing parameter is given by $\rho = \Delta$ and the crossing matrix $V$ is a unity anti-diagonal matrix $V = \text{anti-diagonal}(1, 1, \ldots, 1)$ and $g(\lambda) = -\lambda(1 - \lambda - \Delta)/((\lambda + 1)(\lambda + \Delta))$.

### 3 Density matrix and functional equations

In \[34\] it was first developed a scheme to deal with thermal correlation functions of integrable spin-1/2 spin chains, which was later extended to the case of higher-spin integrable $SU(2)$ chains \[19, 20, 21\] and also more recently it was generalized to the case of integrable $SU(n)$ spin chains \[25, 26\].

Within this approach the main object of concern is inhomogeneous reduced density matrix defined in the thermodynamic limit $L \to \infty$ and finite Trotter number $N$ \[34\] (see Figure \(\text{II}\)),

$$D_m(\lambda_1, \lambda_2, \ldots, \lambda_m) = \frac{\langle \Phi_0 | T_1(\lambda_1) T_2(\lambda_2) \cdots T_m(\lambda_m) | \Phi_0 \rangle}{\Lambda_0(\lambda_1) \Lambda_0(\lambda_2) \cdots \Lambda_0(\lambda_m)}, \quad (7)$$

where $T_j(x)$ is the monodromy matrix $T_j(x) = R_{j,N}(x - u_N) \cdots R_{j,2}(x - u_2)R_{j,1}(x-u_1)$ associated to the quantum transfer matrix $t_j^{QT\!M}(x) = \text{Tr}[T_j(x)]$
\[ D_m(\lambda_1, \lambda_2, \ldots, \lambda_m) = \Phi_0 \]

\[ \Phi_0 \]

Figure 1: Graphical illustration of the un-normalized density matrix \( D_m(\lambda_1, \lambda_2, \ldots, \lambda_m) \). The infinitely many column-to-column transfer matrices to the left and to the right are replaced by the boundary states they project onto.

and \( |\Phi_0\rangle \) is the eigenstate associated to the leading eigenvalue \( \Lambda_0(x) \) of the quantum transfer matrix.

It is worth to note that the connection with the physical density matrix for \( m \)-sites is obtained from the inhomogeneous reduced density matrix via the homogeneous limit \( \lambda_j \to 0 \) and the Trotter limit \( N \to \infty \),

\[ \hat{D}_{[1,m]} = \lim_{N \to \infty} \lim_{\lambda_1, \ldots, \lambda_m \to 0} D_m(\lambda_1, \lambda_2, \ldots, \lambda_m). \]  

This poses the question of how to efficiently compute the referred inhomogeneous reduced density so that these limit can be taken. In the standard case of \( SU(2) \) density matrix, this is e.g addressed by the derivation of a set of discrete functional equations by use of the integrability structure plus the crossing symmetry \([14]\). The solution of these functional equations provides the complete determination of the reduced density matrix at finite temperature and zero temperature.
Similarly for the case of integrable $O(n)$ chains, there exist the necessary integrable structure and the crossing symmetry, which allow us to proceed along the same lines as \[14\] in deriving the discrete functional equation for the density matrix. In this case, the equation reads,

$$D_m(\lambda_1, \lambda_2, \cdots, \lambda_m - \Delta) = A_m(\lambda_1, \cdots, \lambda_m)[D_m(\lambda_1, \cdots, \lambda_m)], \quad (9)$$

where the linear operator $A_m$ can be written as,

$$A_m(\lambda_1, \cdots, \lambda_m)[B] := \text{Tr}_m \left[ R_{1,2}(-\lambda_{1,m}) \cdots R_{m-1,m}(-\lambda_{m-1,m})(P_s)_{m,m+1} \right. \times \left. (B \otimes I_{m+1}) R_{m-1,m}(\lambda_{m-1,m}) \cdots R_{1,2}(\lambda_{1,n}) \right], \quad (10)$$

where $\lambda_{i,j} = \lambda_i - \lambda_j$ and $P_s$ is a (not normalized) projector onto the two-site singlet and the partial trace is taken in the $m$-th vertical space. The graphical depiction of the functional equation \((9)\) is given in Figure \(2\).

Moreover the analyticity properties of the density matrix are clear, since its matrix elements can be written as,

$$D_m(\lambda_1, \cdots, \lambda_m)_{\xi_1, \cdots, \xi_m}^{\xi_1', \cdots, \xi_m'} = \frac{Q(\lambda_1, \cdots, \lambda_m)}{\Lambda_0(\lambda_1) \cdots \Lambda_0(\lambda_m)}, \quad (11)$$

where $Q(\lambda_1, \cdots, \lambda_m)$ is a multivariate polynomial of degree up to $2N$ in each variable (for suitable normalization of the $R$-matrix).
Besides, we have the properties of normalization \( \text{Tr}[D_m(\lambda_1, \cdots, \lambda_2)] = 1 \), asymptotic condition,
\[
\lim_{\lambda_m \to \infty} D_m(\lambda_1, \cdots, \lambda_m) = D_{m-1}(\lambda_1, \cdots, \lambda_{m-1}) \otimes \text{Id}, \tag{12}
\]
and intertwining symmetry relations
\[
(R_{k,k+1}(\lambda_k - \lambda_{k+1}))^{-1} D_m(\lambda_1, \cdots, \lambda_k, \lambda_{k+1}, \cdots, \lambda_m) R_{k,k+1}(\lambda_k - \lambda_{k+1}) = D_m(\lambda_1, \cdots, \lambda_{k+1}, \lambda_k, \cdots, \lambda_m), \tag{13}
\]
which altogether resolves the under-determinacy of the functional equations. This means that with the use of the functional equation (9), analyticity (11), normalization, asymptotic (12) and intertwining (13) properties we can fully determine the density matrix.

In the next section we are going to illustrate the feasibility of this approach by the computation of the two-sites \((m = 2)\) density matrix \(D_2(\lambda_1, \lambda_2)\). It is worth to emphasize that the two-site density matrix for the case of integrable \(O(n)\) spin chains is already a non-trivial result, similarly to the case of higher-spin \(SU(2)\) chains [20, 21].

4 Computation of the two-site density matrix

Due to the \(O(n)\) invariance, the two-sites reduced density matrix can be written as a combination the identity, permutation and Temperley-Lieb operator, likewise the \(R\)-matrix (3), therefore we have that
\[
D_2(\lambda_1, \lambda_2) = \rho_1(\lambda_1, \lambda_2)I_{12} + \rho_2(\lambda_1, \lambda_2)P_{12} + \rho_3(\lambda_1, \lambda_2)E_{12}, \tag{14}
\]
which leaves us with the unknown functions \(\rho_i(\lambda_1, \lambda_2)\) to be determined.

Therefore, one has to work out the functional equations for the expansion coefficients \(\rho_i(\lambda_1, \lambda_2)\). This is done by replacing the above expression for the
density matrix (14) into the functional equation (9), which results into the following set of functional equations,

\[
\begin{pmatrix}
\rho_1(\lambda_1 - \Delta, \lambda_2) \\
\rho_2(\lambda_1 - \Delta, \lambda_2) \\
\rho_3(\lambda_1 - \Delta, \lambda_2)
\end{pmatrix} = \mathcal{A}(\lambda) \cdot \begin{pmatrix}
\rho_1(\lambda_1, \lambda_2) \\
\rho_2(\lambda_1, \lambda_2) \\
\rho_3(\lambda_1, \lambda_2)
\end{pmatrix}, \quad \lambda_1 = u_i, \quad (15)
\]

where \( \lambda = \lambda_1 - \lambda_2 \) and the matrix \( \mathcal{A}(\lambda) \) is given by,

\[
\mathcal{A}(\lambda) = \begin{pmatrix}
\frac{\lambda^2(\lambda^2-(\Delta^2+1))}{(\lambda^2-1)(\lambda^2-\Delta^2)} & -\frac{\lambda(\lambda^2+\lambda^2-\Delta)}{(\lambda^2-1)(\lambda^2-\Delta^2)} & \frac{\lambda}{(\lambda+1)(\lambda+\Delta)} \\
\frac{2\Delta\lambda^2}{(\lambda^2-1)(\lambda^2-\Delta^2)} & \frac{\lambda(\Delta+\lambda\Delta-\lambda^2)}{(\lambda^2-1)(\lambda^2-\Delta^2)} & \frac{\lambda(\Delta-\lambda)}{(\lambda+1)(\lambda+\Delta)} \\
\frac{2\Delta((1+\Delta)\Delta-\lambda^2)}{(\lambda^2-1)(\lambda^2-\Delta^2)} & \frac{\Delta^2+\Delta(-1+2\Delta(1+\Delta))\lambda_1-(1+\Delta)(\Delta\lambda^2-(1+2\Delta))\lambda^2+\lambda^4}{(\lambda^2-1)(\lambda^2-\Delta^2)} & \frac{\Delta-\lambda}{\Delta+(1+\Delta)\lambda+\lambda^2}
\end{pmatrix}
\]

For convenience, we define the intermediate auxiliary functions \( \Omega_0(\lambda_1, \lambda_2) = 1 = \text{Tr}[D_2(\lambda_1, \lambda_2)], \Omega_1(\lambda_1, \lambda_2) = \text{Tr}[P_{12}D_2(\lambda_1, \lambda_2)] \) and \( \Omega_2(\lambda_1, \lambda_2) = \text{Tr}[E_{12}D_2(\lambda_1, \lambda_2)] \) such that,

\[
\begin{pmatrix}
1 \\
\Omega_1(\lambda_1, \lambda_2) \\
\Omega_2(\lambda_1, \lambda_2)
\end{pmatrix} = \begin{pmatrix}
n^2 & n & n \\
n & n^2 & n \\
n & n & n^2
\end{pmatrix} \cdot \begin{pmatrix}
\rho_1(\lambda_1, \lambda_2) \\
\rho_2(\lambda_1, \lambda_2) \\
\rho_3(\lambda_1, \lambda_2)
\end{pmatrix}, \quad (16)
\]

where it is worth to recall that \( \Delta = (n - 2)/2 \) (or conversely \( n = 2(\Delta + 1) \)).

Therefore, we can chose suitable new functions

\[
1 = \Omega_0(\lambda_1, \lambda_2), \quad (17)
\]

\[
\omega_1(\lambda_1, \lambda_2) = \frac{1}{\lambda^2 - 1} \left[ 1 - \Omega_1(\lambda_1, \lambda_2) + \frac{(\lambda^2 - \Delta)}{(\lambda^2 - \Delta^2)} \Omega_2(\lambda_1, \lambda_2) \right],
\]

\[
\omega_2(\lambda_1, \lambda_2) = \frac{\lambda}{(\lambda + 1)(\lambda^2 - \Delta^2)} \Omega_2(\lambda_1, \lambda_2),
\]

which can almost completely decouple the set of equations.
The set of equation (15) in terms of the new functions are written as,

\[
\begin{pmatrix}
1 \\
\omega_1(\lambda_1 - \Delta, \lambda_2) \\
\tilde{\omega}_2(\lambda_1 - \Delta, \lambda_2)
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
\frac{1}{\lambda-1} - \frac{1}{\lambda} + \frac{1}{\lambda-\Delta} - \frac{1}{\lambda-\Delta+1} & -1 & 0 \\
\alpha(\lambda)(\frac{1}{\lambda-1} - \frac{1}{\lambda}) & -\alpha(\lambda) & 1
\end{pmatrix}
\times
\begin{pmatrix}
1 \\
\omega_1(\lambda_1, \lambda_2) \\
\tilde{\omega}_2(\lambda_1, \lambda_2)
\end{pmatrix},
\]

where we conveniently introduce the function \(\tilde{\omega}_2(\lambda_1, \lambda_2) = \alpha(\lambda)\omega_2(\lambda_1, \lambda_2)\) and \(\alpha(\lambda) = \frac{\Gamma(\frac{\lambda}{\Delta})}{\Gamma(\frac{\Delta}{\lambda})}\) such that \(\frac{\alpha(\lambda)}{\alpha(\lambda-\Delta)} = (\frac{\lambda}{\lambda-\Delta} - 1)\).

Besides, one has to impose the intertwining symmetry which implies that the density matrix is symmetric \(D_2(\lambda_1, \lambda_2) = D_2(\lambda_2, \lambda_1)\) under the exchange of the arguments and therefore the same applies to its expansion coefficients \(\rho_i(\lambda_1, \lambda_2)\).

In addition, the asymptotic condition implies that

\[
\lim_{\lambda_j \to \infty} \rho_k(\lambda_1, \lambda_2) = \begin{cases} 
\frac{1}{n^2}, & k = 1 \\
0, & k \neq 1
\end{cases}
\]

which implies that \(\lim_{\lambda_j \to \infty} \omega_k(\lambda_1, \lambda_2) = 0\).

### 4.1 Zero temperature solution

At zero temperature, the functional equations (18) hold for arbitrary values \(\lambda_1\). This is due to the fact that at zero temperature one has to take the Trotter limit \((N \to \infty)\) and therefore the horizontal spectral parameters \(u_i\) can take an infinite number of continuous values. Besides, at zero temperature the \(\omega_i\) functions depend on the difference of the arguments.

The equation for \(\omega_1(\lambda_1, \lambda_2) = \omega_1(\lambda)\) is fully decoupled as given by,

\[
\omega_1(\lambda - \Delta) + \omega_1(\lambda) = \frac{1}{\lambda-1} - \frac{1}{\lambda} + \frac{1}{\lambda-\Delta} - \frac{1}{\lambda-\Delta+1},
\]

### 8
which can be solved via Fourier transform resulting in,

\[
\omega_1(\lambda) = -\frac{d}{d\lambda} \log \left[ \frac{\Gamma(1 + \frac{\lambda}{2\Delta})\Gamma(\frac{1}{2} - \frac{\lambda}{2\Delta})\Gamma(\frac{1}{2} + \frac{\lambda}{2\Delta})\Gamma(\frac{1}{2} + \frac{1}{2\Delta})\Gamma(\frac{1}{2} + \frac{1}{2\Delta} + \frac{\lambda}{2\Delta})\Gamma(\frac{1}{2} + \frac{1}{2\Delta} - \frac{\lambda}{2\Delta})\Gamma(\frac{1}{2} + \frac{1}{2\Delta} + \frac{\lambda}{2\Delta})}{\Gamma(1 - \frac{\lambda}{2\Delta})\Gamma(\frac{1}{2} + \frac{\lambda}{2\Delta})\Gamma(\frac{1}{2} + \frac{1}{2\Delta} - \frac{\lambda}{2\Delta})\Gamma(\frac{1}{2} + \frac{1}{2\Delta} + \frac{\lambda}{2\Delta})\Gamma(\frac{1}{2} + \frac{1}{2\Delta})\Gamma(\frac{1}{2} - \frac{\lambda}{2\Delta})\Gamma(\frac{1}{2} + \frac{1}{2\Delta})\Gamma(\frac{1}{2} - \frac{\lambda}{2\Delta})} \right],
\]

(21)

where \(\Gamma(z)\) is the gamma function.

Taking the homogeneous limit \(\lambda_i = \lambda = 0\), result in the ground state energy of the \(O(n)\) spin chain \([27, 33]\),

\[
E_{gs} = \omega_1(0) = -\frac{1}{\Delta} \left[ 2 \log(2) - \psi_0 \left( \frac{1}{2\Delta} \right) + \psi_0 \left( \frac{1}{2} + \frac{1}{2\Delta} \right) \right],
\]

(22)

where \(\psi_0(z)\) is the digamma function \(\psi_0(z) = \frac{d}{dz} \log \Gamma(z)\).

On the one hand the equation (20) for \(\omega_1(\lambda)\) is an inhomogeneous equation, on the other hand its inhomogeneity is a rational function, which allows for closed analytically solution (21). Nevertheless, the equation for the \(\tilde{\omega}_2(\lambda)\) is also an inhomogeneous equation given as,

\[
\tilde{\omega}_2(\lambda - \Delta) - \tilde{\omega}_2(\lambda) = \alpha(\lambda) \left[ \frac{1}{\lambda - 1} - \frac{1}{\lambda} - \omega_1(\lambda) \right],
\]

(23)

however, in this case the functions \(\alpha(\lambda)\) and \(\omega_1(\lambda)\) (21) appear in the inhomogeneous function. Therefore, this equation is in general very hard to solve analytically. The exception is the case \(n = 3\) (or conversely \(\Delta = 1/2\)), where both functions \(\alpha(\lambda)\) and \(\omega_1(\lambda)\) becomes rational functions themselves, which allows for an analytical solution.

In general, we can write an integral expression as the solution. In order to do so, we use analyticity in the variable \(\lambda\) and Fourier transform the above equations. The resulting equations are algebraically solved for the Fourier coefficients and yield product expressions. Then, we Fourier transform back and find integrals of convolution type

\[
\tilde{\omega}_2(\lambda) = \int_{-\infty}^{\infty} K(\lambda - \mu) \varphi(\mu) \frac{d\mu}{2\pi},
\]

(24)
where
\[ K(z) = \int_{\mathbb{R}+i0} \frac{e^{ikz}}{1 - e^{-\Delta k}} dk, \]  
(25)

and
\[ \varphi(\lambda) = \alpha(\lambda) \left[ \frac{1}{\lambda - 1} - \frac{1}{\lambda} - \omega_1(\lambda) \right]. \]  
(26)

The integral expression can be evaluated numerically. In what follows we show the results for the cases \( n = 3, 4, 5 \) and the cases \( n = 6, 7, 8 \) are presented in the appendix.

4.1.1 The \( O(3) \) case

We have evaluated numerically the convolution integral (24) at the homogeneous point \( \lambda = 0 \) and obtained the value of \( \tilde{\omega}_2(0) \). In Table 1, we show the results from exact diagonalization for the ground state energy \( \omega_1(0) \) and the correlation function \( \tilde{\omega}_2(0) \). The numerical results for finite lattices show agreement with the infinite lattice result obtained from the solution of the functional equations.

| Length | \( \omega_1(0) \)       | \( \tilde{\omega}_2(0) \)       |
|--------|-------------------------|---------------------------------|
| \( L = 2 \) | -6.00000000000000000000 | -3.00000000000000000000 |
| \( L = 4 \) | -4.350781059358212       | -1.937042571331636           |
| \( L = 6 \) | -4.146234978548967       | -1.809210082780898           |
| \( L \to \infty \) | -4.00000000000000000000 | -1.719824178261938           |

Table 1: Comparison of numerical results from exact diagonalization for \( L = 2, 4, 6 \) sites with the analytical result in the thermodynamic limit for correlation \( \omega_i(0) \) for the \( O(3) \) case.

Having the results for \( \omega_i(0) \), we can use the relations (16-17) to evaluate...
the expansion coefficient of the density matrix (see Table 2). We also show
the results obtained from the exact diagonalization.

| Length | $\rho_1(0, 0)$ | $\rho_2(0, 0)$ | $\rho_3(0, 0)$ |
|--------|----------------|----------------|----------------|
| $L = 2$ | 0.0000000000000000 | 0.0000000000000000 | 0.3333333333333333 |
| $L = 4$ | 0.051321778178776 | -0.028127541270379 | 0.20749554006738 |
| $L = 6$ | 0.057286824340209 | -0.030682311157652 | 0.19215517147035 |
| $L \to \infty$ | 0.061350915507139 | -0.032041025072214 | 0.18132161188412 |

Table 2: Comparison of numerical results from exact diagonalization for $L = 2, 4, 6$ sites with the analytical result in the thermodynamic limit for the $\rho_i(0, 0)$ for the $O(3)$ case.

As previously mentioned, the case $n = 3$ can be analytically solved since both functions $\alpha(\lambda)$ and $\omega_1(\lambda)$ becomes rational functions themselves,

$$
\alpha(\lambda) = \frac{1}{2\lambda(2\lambda + 1)(2\lambda - 1)(2\lambda - 2)}, \\
\omega_1(\lambda) = \frac{4}{(2\lambda + 1)(2\lambda - 1)}. 
$$

(27)

In this case, the solution for $\tilde{\omega}_2(\lambda)$ is obtained via Fourier transform and can be written as,

$$
\tilde{\omega}_2(\lambda) = -\frac{(\lambda^2(4\lambda^2 - 1)^2 - 9)}{27\lambda^2(4\lambda^2 - 1)^2} + \frac{c_1}{\sin^2(2\pi\lambda)} + c_2, 
$$

(28)

where one has to add some suitable periodic function to ensure the function $\tilde{\omega}_2(\lambda)$ is free of pole at $\lambda = 0$ which fixes $c_1 = -\frac{4}{3}\pi^2$ and it has zero asymptotic which fixes $c_2 = \frac{1}{27}$.

In the homogeneous limit one has,

$$
\omega_1(0) = -4, \\
\tilde{\omega}_2(0) = \frac{8}{3} - \frac{4}{9}\pi^2 = -1.719824178261937 \ldots,,
$$

(29)
which is in full agreement with the numerical result in Table 1.

Using the previous result, we can also obtain analytically the values of the coefficients \( \rho_i(0, 0) \) \[20\],

\[
\begin{align*}
\rho_1(0, 0) &= \frac{1}{2} - \frac{2}{45} \pi^2 = 0.061350915507139 \ldots, \\
\rho_2(0, 0) &= -\frac{19}{18} + \frac{14}{135} \pi^2 = -0.032041025072214 \ldots, \\
\rho_3(0, 0) &= -\frac{1}{9} + \frac{4}{135} \pi^2 = 0.1813216188412 \ldots,
\end{align*}
\]

which again are in agreement with the numerical results in Table 2.

It is worth noting that these results for \( O(3) \) coincides with the findings for the spin-1 \( SU(2) \) chain \[20\]. This is due to the fact its \( R \)-matrices are exactly same due to the isomorphism between these groups.

### 4.1.2 The \( O(4) \) case

Next we show in Table 3 the result for the numerical evaluation of the convolution integral \[24\] at the homogeneous point \( \lambda = 0 \) for \( n = 4 \) along with the comparison with the exact diagonalization for finite lattices.

| Length | \( \omega_1(0) \) | \( \tilde{\omega}_2(0) \) |
|--------|-------------------|-------------------|
| \( L = 2 \) | -4.0000000000000000 | 4.0000000000000000 |
| \( L = 4 \) | -3.0000000000000000 | 2.2500000000000000 |
| \( L = 6 \) | -2.868517091819053 | 2.057097576517726 |
| \( L \to \infty \) | -2.772588722239781 | 1.921812055672715 |

Table 3: Comparison of numerical results from exact diagonalization for \( L = 2, 4, 6 \) sites with the analytical result in the thermodynamic limit for correlation \( \omega_i(0) \) for the \( O(4) \) case.

In addition, we evaluate the coefficients \( \rho_i(0, 0) \) of the density matrix which are given in Table 4.
Again, the exact diagonalization for finite lattices are in accordance with the infinite lattice result.

Due to the isomorphism \( O(4) \sim SU(2) \times SU(2) \), one can write the analytical solution for the case \( n = 4 \) in terms of the solution for the \( SU(2) \) spin-1/2. The above isomorphism means that the \( R \)-matrix (3) for \( n = 4 \) can be decomposed as [31],

\[
R^{[O(4)]}(\lambda) = R^{[SU(2)]}(\lambda) \otimes R^{[SU(2)]}(\lambda),
\]

where

\[
R^{[SU(2)]}(\lambda) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\lambda}{\lambda+1} & \frac{1}{\lambda+1} & 0 \\
0 & \frac{1}{\lambda+1} & \frac{\lambda}{\lambda+1} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

This also implies that the two-sites density matrix for \( O(4) \) can be written as

\[
D_2^{[O(4)]}(\lambda_1, \lambda_2) = D_2^{[SU(2)]}(\lambda_1, \lambda_2) \otimes D_2^{[SU(2)]}(\lambda_1, \lambda_2),
\]

where the two-sites density matrix for the \( SU(2) \) spin-1/2 chain [9] is given

| Length | \( \rho_1(0,0) \) | \( \rho_2(0,0) \) | \( \rho_3(0,0) \) |
|--------|----------------|----------------|----------------|
| \( L = 2 \) | 0.0000000000000000 | 0.0000000000000000 | 0.2500000000000000 |
| \( L = 4 \) | 0.0347222222222222 | -0.0277777777777777 | 0.1388888888888889 |
| \( L = 6 \) | 0.038254471371995 | -0.029363821569783 | 0.126345936081805 |
| \( L \to \infty \) | 0.040680064040199 | -0.030217991507056 | 0.117497735346250 |

Table 4: Comparison of numerical results from exact diagonalization for \( L = 2, 4, 6 \) sites with the analytical result in the thermodynamic limit for the \( \rho_i(0,0) \) for the \( O(4) \) case.
by,

\[
D_2^{SU(2)}(\lambda_1, \lambda_2) = \begin{pmatrix}
\frac{1}{4} + \frac{\omega(\lambda)}{6} & 0 & 0 & 0 \\
0 & \frac{1}{4} - \frac{\omega(\lambda)}{6} & \frac{\omega(\lambda)}{3} & 0 \\
0 & \frac{\omega(\lambda)}{3} & \frac{1}{4} - \frac{\omega(\lambda)}{6} & 0 \\
0 & 0 & 0 & \frac{1}{4} + \frac{\omega(\lambda)}{6}
\end{pmatrix},
\]  

(34)

where \(\omega(\lambda) = (\lambda^2 - 1) \frac{d}{d\lambda} \log \left[ \frac{\Gamma(1 + \frac{1}{4})\Gamma(\frac{1}{2} - \frac{1}{4})}{\Gamma(1 - \frac{1}{4})\Gamma(\frac{1}{2} + \frac{1}{4})} \right] + \frac{1}{2}.

This implies that

\[
\omega_1(\lambda) = \frac{1 - 2\omega(\lambda)}{\lambda^2 - 1}
\]  

(35)

\[
\tilde{\omega}_2(\lambda) = \frac{1}{4} [\omega_1(\lambda)]^2
\]  

(36)

In the homogeneous limit one has,

\[
\omega_1(0) = -4 \log 2 = 2.77258872239781\ldots,
\]

\[
\tilde{\omega}_2(0) = 4(\log 2)^2 = 1.921812055672806\ldots,
\]  

(37)

which is in full agreement with the numerical results in Table 3.

Using the previous result, we can also obtain analytically the values of the coefficients \(\rho_i(0, 0)\),

\[
\rho_1(0, 0) = \frac{1}{18} + \frac{1}{18} \log 2 - \frac{1}{9} (\log 2)^2 = 0.040680064040196\ldots,
\]

\[
\rho_2(0, 0) = \frac{1}{18} - \frac{5}{18} \log 2 + \frac{2}{9} (\log 2)^2 = -0.030217991507051\ldots,
\]  

(38)

\[
\rho_3(0, 0) = -\frac{1}{36} + \frac{1}{18} \log 2 + \frac{2}{9} (\log 2)^2 = 0.117497735346264\ldots,
\]

which again are in agreement with the numerical results in Table 4.

5 The \(O(5)\) case

Finally, we show in Table 5 and 6 the result for the numerical evaluation of the convolution integral (24) at the homogeneous point \(\lambda = 0\) for \(n = 5\) along
with the comparison with the exact diagonalization results for small lattices. Here the calculation is more subtle since by definition \( \tilde{\omega}_2(0) \) must be zero for non-integers values of \( 1/\Delta \). Therefore we had to evaluate its derivative \( \tilde{\omega}_2'(0) \).

\[
\text{Table 5: Comparison of numerical results from exact diagonalization for } L = 2, 4 \text{ and Lanczos for } L = 6 \text{ sites with the analytical result in the thermodynamic limit for correlation } \omega_i(0) \text{ for the } O(5) \text{ case.}
\]

| Length | \( \omega_1(0) \) | \( \tilde{\omega}_2(0) \) |
|--------|------------------|------------------|
| \( L = 2 \) | -3.333333333333333 | 6.594547532155967 |
| \( L = 4 \) | -2.603912563829967 | 3.329844337362441 |
| \( L = 6 \) | -2.497610064125614 | 2.955489567911210 |
| \( L \to \infty \) | -2.418399152312290 | 2.697303095421986 |

\[
\text{Table 6: Comparison of numerical results from exact diagonalization for } L = 2, 4 \text{ and Lanczos for } L = 6 \text{ sites with the analytical result in the thermodynamic limit for the } \rho_i(0,0) \text{ for the } O(5) \text{ case.}
\]

| Length | \( \rho_1(0,0) \) | \( \rho_2(0,0) \) | \( \rho_3(0,0) \) |
|--------|------------------|------------------|------------------|
| \( L = 2 \) | 0.0000000000000000 | 0.0000000000000000 | 0.2000000000000000 |
| \( L = 4 \) | 0.024257767401215 | -0.021781358237085 | 0.10049251231013 |
| \( L = 6 \) | 0.026877470789173 | -0.023307749344340 | 0.088920395398474 |
| \( L \to \infty \) | 0.028642125394723 | 0.024107800279869 | 0.080897173306257 |

Nevertheless, in the \( O(5) \) case we could not obtain an analytical solution of the integral \([23]\), since the inhomogeneous function is given in terms of digamma functions. The analytical evaluation of integrals of this kind has eluded us so far. Integral of similar type also appears in the evaluation of
three-sites correlation for the $SU(3)$ integrable spin chain [20], which are also short of analytical solution. Therefore, it is important to obtain analytically the solution of integral of the above kind, since these appear in the evaluation of correlation function of integrable chains of different symmetries.

6 Conclusion

We derived functional equations of quantum Knizhnik-Zamolodchikov type for the reduced density matrix for integrable $O(n)$ spin chains.

By the use of the functional equations and additional properties like analyticity, intertwining symmetry, asymptotic and normalization the density matrix elements are fully determined. We illustrate that by solving the equations for two-site correlation functions for the $O(n)$ for the $n = 3, 4, \ldots, 8$ cases at zero temperature.

Although we have shown that this approach can be fruitful for explicit computation of correlation functions, we still need to find an analytical solution for the function $\omega_2(\lambda_1, \lambda_2)$ in order to proceed for correlation at longer distances. We expect more naturally that this approach holds for other integrable systems, e.g like the for $Sp(2m)$ spin chain. It seems also feasible, although more challenging, to extend it for the case of $OSp(n|2m)$ spin chains. We hope to come back to these issues in the future.

Acknowledgments

The author thanks the University of Melbourne for support and hospitality and the São Paulo Research Foundation (FAPESP) for financial support through the grant 2018/25824-0.
Appendix: Results for $O(6)$, $O(7)$ and $O(8)$

For completeness, in this appendix we present in Tables 7-9 the results for the cases $O(6)$, $O(7)$ and $O(8)$, whose evaluation goes along the same lines as the $O(5)$ case.

### Table 7: Results for the $O(6)$ case.

| Length | $\omega_1(0)$ | $\tilde{\omega}_2(0)$ | $\rho_1(0,0)$ | $\rho_2(0,0)$ | $\rho_3(0,0)$ |
|--------|---------------|------------------------|--------------|--------------|--------------|
| $L = 2$ | -3.000000 | 3.000000 | 0.00000000 | 0.00000000 | 0.16666670 |
| $L = 4$ | -2.425390 | 1.402090 | 0.01757970 | -0.01653040 | 0.07771880 |
| $L = 6$ | -2.334340 | 1.198740 | 0.01974220 | -0.01811120 | 0.06632480 |
| $L \to \infty$ | -2.263940 | 1.064260 | 0.02112990 | -0.01885960 | 0.05874710 |

### Table 8: Results for the $O(7)$ case.

| Length | $\omega_1(0)$ | $\tilde{\omega}_2(0)$ | $\rho_1(0,0)$ | $\rho_2(0,0)$ | $\rho_3(0,0)$ |
|--------|---------------|------------------------|--------------|--------------|--------------|
| $L = 2$ | -2.800000 | 1.879820 | 0.00000000 | 0.00000000 | 0.14285700 |
| $L = 4$ | -2.326210 | 0.831166 | 0.01320930 | -0.01269980 | 0.06309200 |
| $L = 6$ | -2.246650 | 0.684007 | 0.01502840 | -0.01420530 | 0.05186360 |
| $L \to \infty$ | -2.182120 | 0.585933 | 0.01621030 | -0.01496530 | 0.04435020 |

### Table 9: Results for the $O(8)$ case.

| Length | $\omega_1(0)$ | $\tilde{\omega}_2(0)$ | $\rho_1(0,0)$ | $\rho_2(0,0)$ | $\rho_3(0,0)$ |
|--------|---------------|------------------------|--------------|--------------|--------------|
| $L = 2$ | -2.666670 | 1.347915 | 0.00000000 | 0.00000000 | 0.12500000 |
| $L = 4$ | -2.263760 | 0.572129 | 0.01024330 | -0.00996890 | 0.05302260 |
| $L = 6$ | -2.193380 | 0.454788 | 0.01177580 | -0.01132500 | 0.04211880 |
| $L \to \infty$ | -2.133390 | 0.374522 | 0.01280290 | -0.01206230 | 0.03463910 |
References

[1] V.E. Korepin, N.M. Bogoliubov, A.G. Izergin *Quantum inverse scattering method and correlation functions* (CUP, Cambridge, 1993).

[2] M. Jimbo and T. Miwa, *Algebraic Analysis of Solvable Lattice Models* (AMS, Rhode Island, 1995).

[3] M. Takahashi, J. Phys. C: Solid State Phys., 10 (1977) 1289.

[4] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, Phys. Lett. A, 168 (1992) 256.

[5] M. Jimbo and T. Miwa, J. Phys. A, 29 (1996) 2923.

[6] N. Kitanine, J. M. Maillet and V. Terras, Nucl. Phys. B, 567 (2000) 554.

[7] F. Göhmann, A. Klümper and A. Seel, J. Phys. A, 38 (2005) 1833.

[8] H. E. Boos and V. E. Korepin, J. Phys. A, 34 (2001) 5311.

[9] H. Boos, M. Jimbo, T. Miwa, F. Smirnov and Y. Takeyama, Algebra and Analysis, 17 (2005) 115; H. Boos, M. Jimbo, T. Miwa, F. Smirnov and Y. Takeyama, St Petersburg Math. J., 17 (2006) 85.

[10] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, Y. Takeyama, Comm. Math. Phys. 261 (2006) 245.

[11] H. Boos, F. Göhmann, A. Klümper and J. Suzuki, J. Stat. Mech. (2006) P04001.

[12] J. Damerau, F. Göhmann, N. P. Hasenclever and A. Klümper, J. Phys. A, 40 (2007) 4439.
[13] N. Kitanine, K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, J. Stat. Mech. (2009) P04003.

[14] Britta Aufgebauer and Andreas Klümper, J. Phys. A: Math. Theor. 45 (2012) 345203.

[15] A. H. Bougourzi and R. A. Weston, Nucl. Phys. B, 417 (1994) 439.

[16] M. Idzumi, Int. J. Mod. Phys. A, 9 (1994) 4449.

[17] N. Kitanine, J. Phys. A, 34 (2001) 8151.

[18] T. Deguchi and C. Matsui, Nucl. Phys. B, 831 (2010) 359.

[19] F. Göhmann, A. Seel and J. Suzuki, J. Stat. Mech. (2010) P11011.

[20] A. Klümper, D. Nawrath and J. Suzuki, J. Stat. Mech. (2013) P08009.

[21] G.A.P. Ribeiro and A. Klümper, J. Phys. A: Math. Theor. 49 (2016) 254001.

[22] A. Liashyk, S. Z. Pakuliak, E. Ragoucy, N. A. Slavnov, J. Stat. Mech. (2019) 044001; S. Pakuliak, E. Ragoucy, N. A. Slavnov, Nucl. Phys. B, 881 (2014) 343; S. Belliard, S. Pakuliak, E. Ragoucy, N. A. Slavnov, J. Stat. Mech. (2013) P02020.

[23] M. Wheeler, Nucl. Phys. B, 875 (2013) 186.

[24] J. M. Maillet, G. Niccoli, J. Math. Phys., 59 (2018) 091417. J. M. Maillet, G. Niccoli, L. Vignoli, On Scalar Products in Higher Rank Quantum Separation of Variables, [arXiv:2003.04281] [math-ph].

[25] H.E. Boos, A. Hutsalyuk, Kh. S. Nirov, J. Phys. A: Math. Theor. 51 (2018), [arXiv:1804.09756] [hep-th].
[26] G.A.P. Ribeiro and A. Klümper, J. Stat. Mech.: Theor. Exp. (2019) P013103, arXiv:1804.10169 [math-ph].

[27] N. Yu. Reshetikhin, Lett. Math. Phys., 7 (1983) 205; N. Yu. Reshetikhin, Theor. Math. Fiz. 63 (1985) 347; N. Yu. Reshetikhin, Lett. Math. Phys, 14 (1987) 235.

[28] A. Kuniba and J. Suzuki, Comm.Math.Phys. 173 (1995) 225.

[29] P.P.Kulish, J.Sov.Math.35 (1986) 2648.

[30] M.J. Martins, Phys. Rev. Lett., 74 (1995) 3316; Nucl. Phys. B, 450 (1995) 768; Phys. Lett. B, 359 (1995) 334;

[31] M. J. Martins, P. B. Ramos, Nucl. Phys. B, 500 (1997) 579.

[32] A. Klümper, J. Phys. A: Math. Gen., 23 (1990) 809.

[33] M.J. Martins, B. Nienhuis, R. Rietman, Phys. Rev. Lett. 81, (1998) 504; H. Frahm, M.J. Martins, Nucl. Phys. B, 930 (2018) 545.

[34] F. Göhmann, A. Klümper and A. Seel, J. Phys. A, 37 (2004) 7625.