A remark on weaken restricted isometry property in compressed sensing

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Abstract

The restricted isometry property (RIP) has become well-known in the compressed sensing community. Recently, a weaken version of RIP was proposed for exact sparse recovery under weak moment assumptions. In this note, we prove that the weaken RIP is also sufficient for stable and robust sparse recovery by linking it with a recently introduced robust width property in compressed sensing. Moreover, we show that it can be widely apply to other compressed sensing instances as well.

1 Introduction

The concept of restricted isometry property (RIP), which was introduce by Candès and Tao in [3], has become well-known in the compressed sensing community. Recently, a weaken version of RIP appeared in [7] as an important tool for exact sparse recovery under weak moment assumptions. In this note, we prove that the weaken RIP is also sufficient for stable and robust sparse recovery.

A key observation is that the weaken RIP implies the robust width property, which was proposed in [2] and allows uniformly stable and robust recovery for many instances in compressed sensing [2, 9]. We establish our discovery in a very general form by introducing the concepts of atom norm and simple subset, and hence other instances in compressed sensing such as low-rank recovery are commonly covered. The results in this study indicate that the weaken RIP plays a vital role similar to the original RIP.

2 Robust width and weaken-RIP

2.1 Robust width property

The robust width property was formally defined in [2]. Before stating its definition, we first introduce some notations. Let $\mathcal{H}$ be a finite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the norm induced by the inner product over $\mathcal{H}$. We denote the Euclidean norm by $\| \cdot \|_2$. Let $\Phi : \mathcal{H} \rightarrow \mathbb{F}^m$ denote some known linear operator, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1 (robust width property, [2]). We say a linear operator $\Phi : \mathcal{H} \rightarrow \mathbb{F}^m$ satisfies the $(\rho, \alpha)$-robust width property over $B_2$ if

$$\|x\| \leq \rho \|x\|_2$$

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for every $x \in H$ such that $\|\Phi x\|_2 < \alpha \|x\|$; or equivalently if

$$\|\Phi x\|_2 \geq \alpha \|x\|$$

for every $x \in H$ such that $\|x\| > \rho \|x\|^\#$. Here, $\|\cdot\|^\#$ is certain norm used to promote certain structured solutions to underdetermined systems of linear equations.

### 2.2 Weaken-RIP property

The RIP was originally introduced in [3]. In the latter, many researchers have contributed to the topic of RIP; for more information on its development, one could refer to [5]. Here, we write down the definition of RIP formulated in [5].

**Definition 2 (RIP, [3, 5]).** Let $\Sigma_k := \{x \in \mathbb{R}^n, \|x\|_0 \leq k\}$ be the $k$-sparse vector set, where $\|x\|_0$ stands for the number of nonzero components of vector $x$. A matrix $\Gamma \in \mathbb{R}^{m \times n}$ is said to have $(k, \delta)$-RIP property for sparsity $k \in [n] := \{1, 2, \ldots, n\}$ and distortion $0 < \delta < 1$ if for every $k$-sparse vector in $\Sigma_k$, it holds

$$(1 - \delta)\|x\|_2 \leq \|\Gamma x\|_2 \leq (1 + \delta)\|x\|_2.$$ 

The following is the definition of the weaken RIP appeared in [7].

**Definition 3 (weaken-RIP, [7]).** Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^n$. A matrix $\Gamma \in \mathbb{R}^{m \times n}$ is said to have $(k, \alpha, \beta)$-weaken-RIP property for sparsity $k \in [n]$ and distortions $\alpha, \beta > 0$ if

a) for every $k$-sparse vector in $\Sigma_k$, $\|\Gamma x\|_2 \geq \alpha \|x\|_2$ and

b) for every $i \in [n]$, $\|\Gamma e_i\|_2 \leq \beta$.

Compared with the original RIP, the weaken-RIP are strictly weaker, as it suffices to verify the right-hand side of the RIP just for 1-sparse vectors and not for all $s$-sparse vectors [7]. Because of such relaxed restriction of the right-hand side of the RIP, the weaken-RIP contributes as a main reason for weakening moment assumptions in exact sparse recovery. In this note, we will show that the weaken-RIP also ensures stable and robust sparse recovery. Moreover, the idea behind of the weaken-RIP can widely apply to other instances in compressed sensing. In what follows, we try to state a generalized version of the weaken-RIP. To this end, let $\mathcal{B}$ be a collection of atoms such that the following holds

$$\mathcal{H} = \{x \in \mathcal{H} : x = \sum_{b \in \mathcal{B}} \sigma_b b, \sigma_b \geq 0, \forall b \in \mathcal{B}\}.$$ 

We assume that $\mathcal{B}$ is centrally symmetric about the origin and consider the atom norm $\|\cdot\|_{\mathcal{B}}$ induced by $\mathcal{B}$. Recall that $\|\cdot\|_{\mathcal{B}}$ has the following expression [4]:

$$\|x\|_{\mathcal{B}} = \inf \{\sum_{b \in \mathcal{B}} \sigma_b : x = \sum_{b \in \mathcal{B}} \sigma_b b, \sigma_b \geq 0, \forall b \in \mathcal{B}\}.$$ 

Let $k < n$ and define $k$-simple subset relative to $\mathcal{B}$ as follows:

$$\mathcal{A} := \{x \in \mathcal{H} : x = \sum_{i=1}^{k} \sigma_i b_i, \sigma_i \geq 0, b_i \in \mathcal{B}\}.$$ 

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Before going forward, we give two well-known instances in compressed sensing. The first one is the sparse vector recovery, where we can take $B = \{ \pm e_1, \pm e_2, \ldots, \pm e_n \}$, and then the corresponding $k$-simple subset $A$ relative to $B$ is just the $k$-sparse vector set $\Sigma_k$, and the induced norm by $B$ is the $\ell_1$-norm. The second one is the low-rank matrix recovery, where we can take $B = \{ uv^T \in \mathbb{R}^{n \times n} : \| u \|_2 = \| v \|_2 = 1 \}$, and then the corresponding $k$-simple subset $A$ relative to $B$ is $k$-rank matrix set, and the induced norm by $B$ is the Schatten $\ell_1$-norm. For more cases, please refer to [4].

**Definition 4** (Generalized-weaken-RIP). Let $A$ be a $k$-simple subset relative to $B$. A linear operator $\Phi : H \rightarrow \mathbb{F}^m$ is said to have $(k, \alpha, \beta)$-generalized-weaken-RIP property for sparsity $k \in [n]$ and distortions $\alpha, \beta > 0$ if

1. for every element in $A$, $\| \Phi x \|_2 \geq \alpha \| x \|_2$ and
2. for every element in $B$, $\| \Phi x \|_2 \leq \beta \| x \|_2$.

Note that the above definition on one hand is a generalization of the weaken-RIP, and on the other hand can also be viewed as a weaken version of the generalized RIP in [1].

**Definition 5** (Generalized-RIP, [1]). Let $E$ be some subset of $H$. A linear operator $\Phi : H \rightarrow \mathbb{F}^m$ is said to have $(k, \alpha, \beta)$-generalized-RIP property for sparsity $k \in [n]$ and distortions $\alpha, \beta > 0$ if

1. for every element in $E$, $\| \Phi x \|_F \geq \alpha \| x \|_G$ and
2. for every element in $E$, $\| \Phi x \|_F \leq \beta \| x \|_G$

where $\| \cdot \|_F$ and $\| \cdot \|_G$ are certain norms.

### 3 Main results

Now, we state our main result:

**Theorem 1** (weaken-RIP implies robust width property). Let $A$ be a $k$-simple subset relative to $B$, assume that each element $b \in B$ satisfies $\| b \| = 1$, and take $\| \cdot \|_2 = \| \cdot \|_B$. If linear operator $\Phi : H \rightarrow \mathbb{F}^m$ has the $(k, \alpha, \beta)$-generalized-weaken-RIP property for sparsity $1 < k \in [n]$ and distortions $\alpha, \beta > 0$, then it must have the $(\rho, \tilde{\alpha})$-robust width property with

$$\tilde{\alpha} = \sqrt{\alpha^2 - \frac{\beta^2 - \alpha^2}{\rho^2 (k - 1)}}.$$  

**Proof.** We divide the proof into two steps.

**Step 1:** Let $x$ be any nonzero element in $H$; then it has representations via the elements in $B$. Take one of its representations, say $x = \sum_{b \in B} \sigma_b b$, $\sigma \geq 0$, $b \in B$. Let $\gamma = \sum_{b \in B} \sigma_b$; it must be positive since $x$ is a nonzero element. Now, we prove the following inequality via the Maurey empirical method, which was employed to prove similar results in [8, 7]:

$$\| \Phi x \|_2^2 \geq \alpha^2 \| x \|_2^2 - \frac{\gamma}{k - 1} \left( \sum_{b \in B} \sigma_b \| \Phi b \|_2^2 - \alpha^2 \sum_{b \in B} \sigma_b \| b \|_2^2 \right).$$  

(1)
Let $Y$ be a random element in $\mathcal{H}$ defined by
$$P(Y = \gamma b) = \frac{\sigma_b}{\gamma},$$
where $b \in B$. Then,
$$EY = \sum_{b \in B} \gamma b \frac{\sigma_b}{\gamma} = \sum_{b \in B} \sigma_b = x.$$

Let $Y_1, Y_2, \ldots, Y_k$ be independent copies of $Y$ and set $z = \frac{1}{k} \sum_{i=1}^k Y_i$. Then, $z \in A$ for every realization of $Y_1, Y_2, \ldots, Y_k$. By the definition of the $(k, \alpha, \beta)$-generalized-weaken-RIP property, we have $\|\Phi z\|_2^2 \geq \alpha^2 \|z\|^2$ and hence
$$E\|\Phi z\|_2^2 \geq \alpha^2 E\|z\|^2.$$

It is straightforward to verify the following relationships:

i) $E\langle Y, Y \rangle = \gamma \sum_{b \in B} \sigma_b \|b\|^2$,

ii) for every $1 \leq i \leq k$,
$$E\langle \Phi Y_i, \Phi Y_i \rangle = \gamma \sum_{b \in B} \sigma_b \|\Phi b\|^2,$$

iii) and for each pair $(i, j)$ satisfying $1 \leq i \neq j \leq k$,
$$E\langle \Phi Y_i, \Phi Y_j \rangle = \|\Phi x\|_2^2.$$

Therefore, we have
$$E\|\Phi z\|_2^2 = \frac{1}{k^2} \sum_{i,j=1}^k E\langle \Phi Y_i, \Phi Y_j \rangle = \frac{k-1}{k} \|\Phi x\|_2^2 + \frac{\gamma}{k} \sum_{b \in B} \sigma_b \|\Phi b\|^2.$$

Similarly, it holds that
$$E\|z\|^2 = \frac{1}{k^2} \sum_{i,j=1}^k E\langle Y_i, Y_j \rangle = \frac{k-1}{k} \|x\|_2^2 + \frac{\gamma}{k} \sum_{b \in B} \sigma_b \|b\|^2.$$

Substituting these two relationships to $E\|\Phi z\|_2^2 \geq \alpha^2 E\|z\|^2$ and rearranging terms, we obtain the desired inequality.

**Step 2:** Apply the upper bound estimation in weaken-RIP and the robust width property to finish the proof. Notice that $\|b\| = 1$ and $\|\Phi b\|^2 \leq \beta^2 \|b\|^2 = \beta^2$, the inequality (1) can be simplified into
$$\|\Phi x\|_2^2 \geq \alpha^2 \|x\|^2 - \gamma^2 (\beta^2 - \alpha^2) \frac{1}{k-1}.$$  

Since the expression of $x$ is taken arbitrarily, we derive that
$$\|\Phi x\|_2^2 \geq \sup_{x=\sum_{b \in B} \sigma_b b, \sigma_b \geq 0} \left( \alpha^2 \|x\|^2 - \frac{\gamma^2 (\beta^2 - \alpha^2)}{k-1} \right)$$
$$= \alpha^2 \|x\|^2 - \frac{\beta^2 - \alpha^2}{k-1} \sup_{x=\sum_{b \in B} \sigma_b b, \sigma_b \geq 0} (\sum_{b \in B} \sigma_b)^2$$
$$= \alpha^2 \|x\|^2 - \frac{\|x\|_2^2 (\beta^2 - \alpha^2)}{k-1},$$
where the last relationship follows from the expression of atom norm \( \| \cdot \|_B \). Therefore, for every \( x \in \mathcal{H} \) satisfying \( \| x \| > \rho \| x \|_B \) we get

\[
\| \Phi x \|_2^2 \geq \left[ \alpha^2 - \frac{\beta^2 - \alpha^2}{\rho^2(k-1)} \right] \| x \|_2^2,
\]

which implies the \((\rho, \tilde{\alpha})\)-robust width property. This completes the proof. \( \square \)

It has been shown that the \((\rho, \alpha)\)-robust width property is sufficient (and necessary up to constants) for uniformly stable and robust sparse recovery by convex minimization. Under the same setting of Theorem 1, the \((k, \alpha, \beta)\)-generalized-weaken-RIP property is a stronger property and hence ensures uniformly stable and robust sparse recovery as well. In the following, we first introduce the concept of compressed sensing space, and then state a group of uniformly stable and robust sparse recovery results without proof details; they can be obtained directly by combining Theorem 1 with the stable and robust results in \([2, 9]\).

**Definition 6.** ([2]) A compressed sensing space \((\mathcal{H}, \mathcal{A}, \| \cdot \|_2)\) with bound \( L \) consists of a finite-dimensional Hilbert space \( \mathcal{H} \), a subset \( \mathcal{A} \subseteq \mathcal{H} \), and a norm \( \| \cdot \|_2 \) on \( \mathcal{H} \) with following properties:

(i) \( 0 \in \mathcal{A} \).

(ii) For every \( a \in \mathcal{A} \) and \( v \in \mathcal{H} \), there exists a decomposition \( v = z_1 + z_2 \) such that

\[
\| a + z_1 \|_2 = \| a \|_2 + \| z_1 \|_2, \quad \| z_2 \|_2 \leq L \| v \|_2.
\]

**Theorem 2** (weaken-RIP implies stable and robust recovery). Let \( \mathcal{A} \) be a \( k \)-simple subset relative to \( \mathcal{B} \) and assume that each element \( b \in \mathcal{B} \) satisfies \( \| b \| = 1 \). Let \( \| \cdot \|_B \) be the induced norm by \( \mathcal{B} \) and \( \| \cdot \|_\infty \) be its dual norm. Suppose that there exists a bound \( L \) such that the triple \((\mathcal{H}, \mathcal{A}, \| \cdot \|_B)\) is a compressed sensing space. If linear operator \( \Phi : \mathcal{H} \rightarrow \mathbb{F}^m \) has the \((k, \alpha, \beta)\)-generalized-weaken-RIP property for sparsity \( 1 < k \leq [n] \) and distortions \( \alpha, \beta > 0 \), then

a) for every \( x^o \in \mathcal{H}, \theta \in (0, 1), \lambda > 0, \sigma > 0 \) and \( w \in \mathbb{F}^M \) satisfying \( \| \Phi^T w \|_\infty \leq \theta \lambda \sigma \), any solution \( x^* \) to the Lasso model

\[
\text{minimize } \frac{1}{2} \| \Phi x - (\Phi x^o + w) \|_2^2 + \lambda \sigma \| x \|_B
\]

satisfies \( \| x^* - x^o \| \leq C_0 \| x^o - a \|_B + C_1 \cdot \sigma \) for every \( a \in \mathcal{A} \). Here, \( \lambda \) is some turning parameter and \( \sigma \) is a measurement of the noise level, and

\[
C_0 = \left( \frac{1 - \theta}{2 \rho} - L \right)^{-1}, \quad C_1 = \frac{(1 + \theta) \lambda}{\tilde{\alpha}^2 \rho}
\]

where \( \tilde{\alpha} \) is given by Theorem 1 and the parameter \( \rho \) satisfies

\[
\sqrt{\frac{\beta^2 - \alpha^2}{\alpha^2(k-1)}} < \rho < \frac{1 - \theta}{2L}.
\]

b) for every \( x^o \in \mathcal{H} \) and \( e \in \mathbb{F}^M \) with \( \| e \|_2 \leq \epsilon \), any solution \( x^* \) to the basis pursuit model

\[
\text{minimize } \| x \|_B, \text{ subject to } \| \Phi x - (\Phi x^o + w) \|_2 \leq \epsilon
\]

satisfies \( \| x^* - x^o \| \leq C_0 \| x^o - a \|_B + C_1 \cdot \sigma \) for every \( a \in \mathcal{A} \).
satisfies $\|x^* - x^2\| \leq C_2\|x^2 - a\|_B + C_3 \cdot \sigma$ for every $a \in A$. Here,
\[ C_2 = 2\rho, \quad C_3 = \frac{2}{\alpha} \]
where the parameter $\rho$ satisfies
\[ \sqrt{\frac{\beta^2 - \alpha^2}{\alpha^2(k - 1)}} < \rho < \frac{1}{4L}. \]

c) for every $x^2 \in \mathcal{H}, \lambda > 0, \sigma > 0$ and $w \in \mathbb{F}^M$ satisfying $\|\Phi^T w\|_\diamond \leq \lambda \sigma$, any solution $x^*$ to the Dantzig selector model

\[
\text{minimize } \|x\|_B, \text{ subject to } \|\Phi^T (\Phi x - (\Phi x^2 + w))\|_\diamond \leq \lambda \sigma
\]
satisfies $\|x^* - x^2\| \leq C_4\|x^2 - a\|_z + C_5 \cdot \sigma$ for every $a \in A$. Here,
\[ C_4 = \left( \frac{1}{2\rho} - L \right)^{-1}, \quad C_5 = \frac{2\lambda}{\alpha^2 \rho} \]
where the parameter $\rho$ satisfies
\[ \sqrt{\frac{\beta^2 - \alpha^2}{\alpha^2(k - 1)}} < \rho < \frac{1}{2L}. \]

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