An Itô type formula for the additive stochastic heat equation

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Abstract

We use the theory of regularity structures to develop an Itô formula for $u$, the solution of the one dimensional stochastic heat equation driven by space-time white noise with periodic boundary conditions. In particular for any smooth enough function $\phi$ we can express the random distribution $(\partial_t - \partial_{xx})\phi(u)$ and the random field $\phi(u)$ in terms of the reconstruction of some modelled distributions. The resulting objects are then identified with some classical constructions of stochastic calculus.

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1 Introduction

We consider \{${u(t,x): t \in [0,T], \ x \in T = \mathbb{R}/\mathbb{Z}}$\} the solution of the additive stochastic heat equation with periodic boundary conditions and zero initial value:

$$
\begin{align*}
\partial_t u &= \partial_{xx} u + \xi \\
& \\
& u(t,0) = u(t,1) \quad t \in [0,T], \\
& u(0,x) = 0 \quad x \in T,
\end{align*}
$$

(1.1)

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where $\xi$ is a real valued space-time white noise over $\mathbb{R} \times \mathbb{T}$. This equation was originally formulated to model a one dimensional string exposed to a stochastic force (see [10]). From a theoretical point of view, the equation (1.1) represents one of the simplest examples of a stochastic PDE whose solution can be written explicitly, the so called stochastic convolution (see e.g. [29], [8]). Writing $\xi = \partial W/\partial t \partial x$, where $W$ is the Brownian sheet associated to $\xi$, one has

$$u(t, x) = \int_0^t \int_\mathbb{T} P_{t-s}(x-y) dW_{s,y}, \quad (1.2)$$

where the integral $dW_{s,y}$ is a Walsh integral taken with respect to the martingale measure associated to $W$ and $P$: $(0, +\infty) \times \mathbb{T} \to \mathbb{R}$ is the fundamental solution of the heat equation with periodic boundary conditions:

$$P_t(x) = \sum_{m \in \mathbb{Z}} G_t(x + m), \quad G_t(x) = \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{x^2}{4t}\right).$$

It is well known in the literature (see e.g. [8]) that almost surely $u$ admits a continuous modification in both variables $t$ and $x$ and it satisfies the equation (1.1) in a weak sense, that is for any smooth function $l$: $\mathbb{T} \to \mathbb{R}$ one has

$$\int_\mathbb{T} u(t, y) l(y) \, dy = \int_0^t \int_\mathbb{T} u(s, y) l''(y) \, dy \, ds + \int_0^t \int_\mathbb{T} l(y) dW_{s,y}. \quad (1.3)$$

Looking at $u$ as a process with values in an infinite dimensional space, the process $u_t = u(t, \cdot)$ is also Feller diffusion taking values in $C(\mathbb{T})$ (the space of periodic continuous functions) and $L^2(\mathbb{T})$. Its hitting properties were intensively studied in [20] by means of the Markov property, potential theory and the theory of Gaussian processes. Nevertheless some classical tools of the infinite dimensional stochastic calculus (see e.g. [8]) such as the Itô formula cannot be applied to $u_t$. Moreover for any fixed $x \in \mathbb{T}$ it has been shown in [28] that the process $t \to u(t, x)$ has an a.s. infinite quadratic variation. Therefore any attempt to apply classically the powerful theory of Itô calculus seems pointless.

Introduced in 2014 and explained through the famous “quartet” of articles ([14], [15], [7], [4]), the theory of regularity structures has provided a very general framework to prove local pathwise existence and uniqueness of a wide family of stochastic PDEs driven by space-time white noise. In this paper we will show how these new techniques allow to formulate an Itô formula for $u$. The formula itself will be expressed under a new form, reflecting the new perspective under which the stochastic PDEs are analysed. Indeed for any fixed smooth function $\varphi: \mathbb{R} \to \mathbb{R}$, we will study the quantity $(\partial_t - \partial_{xx})\varphi(u)$, interpreted as a space-time random distribution. This choice is heuristically motivated by the parabolic form of the equation (1.1) defining $u$ and it is manageable by the regularity structures, where it is possible to manipulate random distributions. Thus we are searching for a random distribution $g_\varphi$, depending on higher derivatives of $\varphi$, such that, denoting by $\langle \cdot, \cdot \rangle$ the duality bracket, one has a.s. the identity

$$\langle (\partial_t - \partial_{xx})\varphi(u), \psi \rangle = \langle g_\varphi, \psi \rangle, \quad (1.4)$$

for any test function $\psi$. We will refer to this formula as a differential Itô formula, because of the presence of a differential operator on the left hand side of (1.4). By uniqueness of the heat equation with the distributional source $g_\varphi$ (see section 2), for every $(t, x) \in [0, T] \times \mathbb{T} \to \mathbb{R}$, we can write formally

$$\varphi(u(t,x)) = \varphi(0) + \int_0^t \int_\mathbb{T} P_{t-s}(x-y) g_\varphi(s, y) \, ds \, dy \quad (1.5)$$
where for any fixed \((t, x)\) the equality (1.5) hold a.s. We call a similar identity an integral Rô formula because of the double integral on the right hand side of (1.5). This resulting formula may be one possible tool to improve our comprehension of the trajectories of \(u\), even if it is still not clear whether it will be as effective as it is for finite dimensional diffusions (see e.g. [23]).

In order to obtain these identities, we will follow the general philosophy of the regularity structure theory. Instead of working directly with the process \(u\), we will consider \(\{u_\varepsilon\}_{\varepsilon > 0}\) an approximating sequence of \(u\), solving a so called “Wong-Zakai” formulation of (1.1) (see [17] for this approximation procedure on a wider class of equations)

\[
\begin{aligned}
\frac{\partial_t u_\varepsilon}{\partial x_2} &= \partial_x u_\varepsilon + \xi_\varepsilon \\
u(t, 0) &= u_\varepsilon(t, 1) \quad t \in [0, T], \\
u(0, x) &= 0 \quad x \in \mathbb{T},
\end{aligned}
\]

(1.6)

where the random fields \(\{\xi_\varepsilon\}_{\varepsilon > 0}\) are defined by extending \(\xi\) periodically on \(\mathbb{R}^2\) and convolving it with a fixed smooth, compactly supported function \(\rho: \mathbb{R}^2 \to \mathbb{R}\) such that \(\int_{\mathbb{R}^2} \rho = 1\) and \(\rho(t, x) = \rho(-t, -x)\). That is denoting by \(*\) the convolution on \(\mathbb{R}^2\) for any \(\varepsilon > 0\) we set

\[
\rho_\varepsilon = \varepsilon^{-3} \rho(\varepsilon^{-2}t, \varepsilon^{-1}x), \quad \xi_\varepsilon(t, x) = (\rho_\varepsilon * \xi)(t, x).
\]

(1.7)

The inhomogeneous scaling in the mollification procedure is chosen accordingly to the parabolic nature of the equation (1.1). This regularisation makes \(\xi_\varepsilon\) an a.s. smooth function \(\xi_\varepsilon: [0, T] \times \mathbb{T} \to \mathbb{R}\) and the equation (1.6) admits for any \(\varepsilon > 0\) an a.s. periodic strong solution (in the analytical sense) \(u_\varepsilon: [0, T] \times \mathbb{T} \to \mathbb{R}\) which is smooth in space and time. Therefore in this case \((\partial_t - \partial_\varepsilon x)\varphi(u_\varepsilon)\) is calculated by applying the classical chain rule between \(u_\varepsilon\) and \(\varphi\), obtaining

\[
\begin{aligned}
\partial_t(\varphi(u_\varepsilon)) &= \varphi'(u_\varepsilon)\partial_t u_\varepsilon, \\
\partial_x(\varphi(u_\varepsilon)) &= \varphi'(u_\varepsilon)\partial_x u_\varepsilon,
\end{aligned}
\]

(1.8)

\[
\begin{aligned}
\partial_\varepsilon x (\varphi(u_\varepsilon)) &= \varphi''(u_\varepsilon)(\partial_\varepsilon x u_\varepsilon)^2 + \varphi'(u_\varepsilon)\partial_\varepsilon x u_\varepsilon.
\end{aligned}
\]

(1.9)

which yields:

\[
(\partial_t - \partial_\varepsilon x)\varphi(u_\varepsilon) = \varphi'(u_\varepsilon)\xi_\varepsilon - \varphi''(u_\varepsilon)(\partial_\varepsilon x u_\varepsilon)^2.
\]

(1.10)

Let us understand heuristically what happens when \(\varepsilon \to 0^+\). Since \(\rho_\varepsilon\) is an approximation of the delta function, \(u\) is a.s. continuous and the derivative is a continuous operation between distributions, we can reasonably infer that the left hand side of (1.10) converges in some sense to \((\partial_t - \partial_\varepsilon x)\varphi(u)\). Thus the right hand side of (1.10) should converge too to some limit distribution. However, written under this form, it is very hard to study this right hand side because it is possible to show

\[
\|\varphi'(u_\varepsilon)\xi_\varepsilon\| \xrightarrow{P} +\infty, \quad \|\varphi''(u_\varepsilon)(\partial_\varepsilon x u_\varepsilon)^2\| \xrightarrow{P} +\infty
\]

with respect to some norm (see the remark [22]). These two results suggest a cancellation phenomenon between two objects whose divergences compensate each other.

This simple cancellation phenomenon between two diverging random quantities, which lies at the heart of the recent study of singular SPDEs, has already been noticed in the pioneering article [30] (see also [13, 19]) and now we are able to reinterpret that result in the general context of the renormalization theory, as explained in the theory of regularity structures. By means of the notion of modelled distribution and the reconstruction theorem, we can also explain the limit as the difference of two explicit random distributions. However, these limits are only characterised by some analytical properties which cannot allow to understand immediately their probabilistic representation. Therefore the convergence is also linked with some specific identification theorems which describe their law. Summing up both these results we can state the main theorem of the paper:

3
Theorem 1.1 (Integral and differential Itô formula). Let \( \varphi \) be a function of class \( C^4_b(\mathbb{R}) \), the space of \( C^4 \) functions with all its derivatives bounded. Then for any test function \( \psi: \mathbb{R} \times \mathbb{T} \to \mathbb{R} \) with \( \text{supp} (\psi) \subset (0,T) \times \mathbb{T} \), one has

\[
\langle (\partial_t - \partial_{xx}) \varphi(u), \psi \rangle = \int_0^T \int_{\mathbb{T}} \varphi'(u_s(y)) \psi(s,y) dW_{s,y} + \frac{1}{2} \int_0^T \int_{\mathbb{T}} \psi(s,y) \varphi''(u_s(y)) C(s) dy ds \\
- \int_{[0,T]^2 \times \mathbb{T}^2} \left[ \int_{s_2 \leq s_1} \psi(s,y) \varphi''(u_s(y)) \partial_x P_{s-s_1}(y - y_1) \partial_x P_{s-s_2}(y - y_2) dy ds \right] dW_{s,y}^2.
\]

Moreover for any \((t,x) \in [0,T] \times \mathbb{T}\) we have a.s.

\[
\varphi(u_t(x)) = \varphi(0) + \int_0^t \int_{\mathbb{T}} P_{t-s}(x-y) \varphi'(u_s(y)) dW_{s,y} + \frac{1}{2} \int_0^t \int_{\mathbb{T}} P_{t-s}(x-y) \varphi''(u_s(y)) C(s) dy ds \\
- \int_{[0,t]^2 \times \mathbb{T}^2} \left[ \int_{s_2 \leq s_1} \psi(s,y) \varphi''(u_s(y)) \partial_x P_{s-s_1}(y - y_1) \partial_x P_{s-s_2}(y - y_2) dy ds \right] dW_{s,y}^2,
\]

where in both formulae the integral \( dW_{s,y}^2 \) is the multiple Skorohod integral of order two integrating the variables \( s = (s_1, s_2) \) and \( y = (y_1, y_2) \), \( u_s(y) = u(s,y) \) and \( C: (0,T) \to \mathbb{R} \) is the integrable function \( C(s) := \| P_s(\cdot) \|_{L^2(\mathbb{T})}^2 \).

Remark 1.2. It is natural to ask whether the same techniques could be applied to a generic convex function \( \varphi \), in order to establish a Tanaka formula for \( u \). In case \( \varphi \) is not a regular function, the formalism of regularity structures does not work anymore (see the section \[4\]). However even if we try to generalise the Theorem \[1.1\] using only the Malliavin calculus, the presence of a multiple Skorohod integral of order 2 in both formulae would require a priori that the random variable \( \varphi''(u(s,y)) \) ought to be twice differentiable (in the Malliavin sense). Hence the condition \( \varphi \in C^4_b(\mathbb{R}) \) in the statement appears to be optimal. Finally any Tanaka formula would require a robust theory of local times associated to \( u \), yet this notion is very ambiguous in the literature: On the one hand using some general results on Gaussian variables (such as \[11\]) we can prove the existence of a local time for any \( x \in \mathbb{T} \) of the process \( t \to u(t,x) \) with respect to its occupation measure on \( [0,T] \). On the other hand an alternative notion of a local time for \( u \) has been developed by means of distributions on the Wiener space in \[13\].

We discuss now the organization of the paper: in the section \[3\] and \[4\] we will apply the general theorems of the regularity structures theory to build the analytical and algebraic tools to study the problem: all the constructions are mostly self contained. In some cases we will also recall some previous results obtained in \[17\] and \[5\], \[7\]. Then in the section \[5\] we will combine all these tools to obtain firstly two formulae involving only objects built in the previous sections (we will refer to them as \textit{pathwise Itô formulae}) and later the identifications theorems.

We finally remark that some of the techniques presented here could potentially be used to establish an Itô formula for a non-linear perturbation of \( \langle 1.1 \rangle \), the so called generalised KPZ equation:

\[
\begin{align*}
\partial_t u &= \partial_{xx} u + g(u)(\partial_x u)^2 + h(u)(\partial_x u) + k(u) + f(u) \\
u(t,0) &= u(t,1) \quad t \in [0,T], \\
u(0,\cdot) &= u_0(\cdot)
\end{align*}
\]

Equation (1.11) where \( g, f, h, k \) are smooth functions and \( u_0 \in C(\mathbb{T}) \) is a generic initial condition. (We refer the reader to \[2\], \[10\].) Establishing such a formula in this generalized setting shall be subject to further investigations. Other possible directions of research may also take into account the
Itô formula for the solutions of other stochastic PDEs with Dirichlet boundary conditions (see [12]) and, using the reformulation in the regularity structures context of differential equations driven by fractional Brownian motion (see [1]), we could recover some classical results in the literature of fractional processes (see e.g. [9], [26]).

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2 Elements of Hölder spaces and Malliavin calculus

We recall here some preliminary notions and notations we will use throughout the paper. For any space-time variable \( z = (t, x) \in \mathbb{R}^2 \), in order to preserve the different role of time and space in the parabolic equation (1.1) we define, with an abused notation, its parabolic norm as

\[
\| z \| := \sqrt{|t| + |x|}.
\]

Moreover for any multi-index \( k = (k_1, k_2) \) the parabolic degree of \( k \) is given by \( |k| := 2k_1 + k_2 \) and we adopt the multinomial notation for monomials \( s^k = t^{k_1}x^{k_2} \) and derivatives \( \partial^k = \partial_t^{k_1} \partial_x^{k_2} \) (the derivative \( \partial_t^2f \) will be denoted in some cases by \( \partial_{xx}f \) to shorten the notation). Accordingly to the definition of \( \rho_\epsilon \) in (1.7), the parabolic rescaling of any function \( \eta : \mathbb{R}^2 \to \mathbb{R} \) of parameter \( \lambda > 0 \) and centred at \( z = (t, x) \) is given by

\[
\eta_\lambda^\epsilon(z) := \lambda^{-3} \eta\left( \frac{\bar{t} - t}{\lambda}, \frac{\bar{x} - x}{\lambda} \right), \quad \bar{z} = (\bar{t}, \bar{x}).
\]

For any non integer \( \alpha \in \mathbb{R} \), a function \( f : \mathbb{R}^2 \to \mathbb{R} \) belongs to the \( \alpha \) Hölder space \( C^\alpha \) when one of these conditions is verified:

- If \( 0 < \alpha < 1 \), \( f \) is continuous and for any compact set \( \mathcal{K} \subset \mathbb{R}^2 \)

\[
\| f \|_{C^\alpha(\mathcal{K})} := \sup_{z \in \mathcal{K}} |f(z)| + \sup_{z, w \in \mathcal{K}, z \neq w} \frac{|f(z) - f(w)|}{\| x - y \|^\alpha} < \infty.
\]

- If \( \alpha > 1 \), \( f \) has \( \lfloor \alpha \rfloor \) continuous derivative in space and \( \lfloor \alpha / 2 \rfloor \) continuous derivative in time, where \( \lfloor \cdot \rfloor \) is the integer part of a real number. Moreover for any compact set \( \mathcal{K} \subset \mathbb{R}^2 \)

\[
\| f \|_{C^\alpha(\mathcal{K})} := \sup_{z \in \mathcal{K}} \sup_{\lfloor k \rfloor \leq \lfloor \alpha \rfloor} |\partial^k f(z)| + \sup_{z, w \in \mathcal{K}, \lfloor k \rfloor = \lfloor \alpha \rfloor} \frac{\| \partial^k f(z) - \partial^k f(w) \|}{\| z - w \|^\lfloor \alpha \rfloor - \lfloor \alpha \rfloor} < \infty.
\]

- If \( \alpha < 0 \), denoting \( r = -\lfloor \alpha \rfloor + 1 \), \( f \in \mathcal{S}'(\mathbb{R}^2) \), the set of tempered distribution on \( \mathbb{R}^2 \), and the dual of the function of order \( C^\tau \) on \( \mathbb{R}^2 \). Moreover for every compact set \( \mathcal{K} \subset \mathbb{R}^2 \)

\[
\| f \|_{C^\alpha(\mathcal{K})} := \sup_{z \in \mathcal{K}} \sup_{\eta \in \mathcal{D}_r, \lambda \in (0, 1)} \frac{|\langle f, \eta_\lambda^\epsilon \rangle|}{\lambda^\alpha} < \infty,
\]
where $\mathcal{R}$ is the set of all test functions $\eta$ supported on $\{z \in \mathbb{R}^2 : \|z\| \leq 1\}$ such that all the directional derivatives up to order $r$ are bounded in the sup norm.

The spaces $\mathcal{C}^\alpha$ and the respective localised version $\mathcal{C}^\alpha(D)$, defined on a open set $D \subset \mathbb{R}^2$ are naturally a family of Fréchet spaces. Moreover for any $\alpha > 0$ and compact set $\mathcal{K} \subset \mathbb{R}^2$, defining $\mathcal{C}^\alpha(\mathcal{K})$ by restriction of $f$ on $\mathcal{K}$, we obtain a Banach space using the quantity $\|f\|_{\mathcal{C}^\alpha(\mathcal{K})}$. The elements $f \in \mathcal{C}^\alpha(\mathbb{R} \times \mathbb{T})$ are interpreted as elements of $\mathcal{C}^\alpha$ whose space variable lives in $\mathbb{T}$. Most of the classical analytical operations apply to the $\mathcal{C}^\alpha$ spaces as follows:

- **Derivation** if $f \in \mathcal{C}^\alpha$ and $k$ is a multi-index then the map $f \to \partial^k f$ is a continuous map from $\mathcal{C}^\alpha$ to $\mathcal{C}^\beta$ where $\beta = \alpha - |k|$.
- **Schauder estimates** (see [17]) if $P$ is the Heat kernel on some domain, then the space-time convolution with $P$, $f \to P * f$ is a well defined map for every $f$ supported on positive times and it sends continuously $\mathcal{C}^\alpha$ in $\mathcal{C}^{\alpha+2}$ for every real non integer $\alpha$.
- **Product** (see [14] Prop. 4.14) for any real non integer $\beta$ the map $(f,g) \to f \cdot g$ defined over smooth functions extends continuously to a bilinear map $\mathcal{B} : \mathcal{C}^\alpha \times \mathcal{C}^\beta \to \mathcal{C}^{\alpha+\beta}$ if and only if $\alpha + \beta > 0$.

The Hölder spaces and the operations defined on them provide a natural setting to formulate the deterministic PDE

$$
\begin{aligned}
\partial_t v - \partial_{xx}v &= g \\
v(t,0) &= v(t,1) \\
v(0,\cdot) &= v_0(\cdot),
\end{aligned}
$$

(2.1)

where $g \in \mathcal{C}^\beta(\mathbb{R} \times \mathbb{T})$ and $v_0 \in \mathcal{C}(\mathbb{T})$. For any $\beta > 0$, classical results on PDE theory (see e.g. [15]) imply that there exists a unique strong solution $v \in \mathcal{C}^{\beta+2}([0,T] \times \mathbb{T})$ of (2.1) which is given explicitly by the so called variation of the constant formula

$$
v(t,x) = \int_{\mathbb{T}} P_t(x-y) v_0(y) dy + (P * 1_{[0,t]} g)(t,x),
$$

(2.2)

where for any $t > 0$, $1_{[0,t]}$ is the indicator function of the interval $[0,t] \times \mathbb{T}$. Furthermore if we consider $\beta \in (-2,0)$ non-integer, the equation (2.1) admits again a unique solution $v \in \mathcal{C}^{\beta+2}([0,T] \times \mathbb{T})$ satisfying (2.1) but only in a distributional sense. This solution can be expressed again by the formula (2.2) by interpreting $1_{[0,t]}$ as a continuous linear map $1_{[0,t]} : \mathcal{C}^\beta(\mathbb{R} \times \mathbb{T}) \to \mathcal{C}^\beta(\mathbb{R} \times \mathbb{T})$ such that $(1_{[0,t]} g)(\psi) = g(\psi)$ for any smooth test function $\psi$ satisfying supp($\psi$) $\subset [0,t] \times \mathbb{T}$ and $(1_{[0,t]} g)(\psi) = 0$ if supp($\psi$) $\cap [0,t] \times \mathbb{T} = \emptyset$ (see [17] Lem. 6.1]). In particular it is possible to show (see [14] Prop. 6.9 and [12] Prop. 2.15) that for any test function $\psi$

$$
(1_{[0,t]} g)(\psi) = \lim_{N \to +\infty} g(\varphi_N \psi),
$$

(2.3)

where $\varphi_N$ is a fixed sequence of smooth cutoff functions converging a.e. to $1_{(0,t) \times \mathbb{T}}$. Thus the solution of the equation (2.1) is given by the same formula (2.2) if $g \in \mathcal{C}^\beta((0,T] \times \mathbb{T})$.

The equation (1.1) can be expressed in the context of the spaces $\mathcal{C}^\alpha$. Indeed for every $\kappa > 0$ interpreting $\xi$ as the derivative in space and time of the Brownian sheet $W$, there exists a modification of $\xi$ belonging to $\mathcal{C}^{-3/2-\kappa}(\mathbb{R} \times \mathbb{T})$ and defining $\xi_\varepsilon$ as in (1.7) one has, as $\varepsilon \to 0^+$, $\xi_\varepsilon \to \xi$ in probability for the topology of $\mathcal{C}^{-3/2-\kappa}(\mathbb{R} \times \mathbb{T})$ (see [14] Lem. 10.2). This theorem has some very deep consequences. Indeed choosing $\kappa < 1/2$ and $v_0 = 0$, we can apply the deterministic results of (2.1) with every a.s. realisation of $\xi$ and by uniqueness of the solution (1.1) we obtain the pathwise representation

$$
u(t,x) = (P * 1_{[0,t]} \xi)(t,x), \quad u_\varepsilon(t,x) = (P * 1_{[0,t]} \xi_\varepsilon)(t,x).
$$

(2.4)
Hence applying the Schauder estimates, we deduce immediately that every a.s. realisation of $u$ belongs to $C^{1/2-\epsilon}([0,T] \times \mathbb{T})$. Therefore if we want to have a pathwise notion to an object like “$u\xi$” the sum of the Hölder regularity of each factor will be $-1 - 2\kappa$ and there is no classical way to understand this product starting from the product of two smooth approximations. The same reasoning applies also for the formal object “$(\partial_x u)^2$”. We recall finally that for every distribution $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{T})$ we can define its periodic lifting $\tilde{u} \in \mathcal{S}'(\mathbb{R}^2)$ defined for every test function $\psi: \mathbb{R}^2 \to \mathbb{R}$

$$
\tilde{u}(\psi) = u(\sum_{m \in \mathbb{Z}} \psi(\cdot, m)). \quad (2.5)
$$

Thanks to this operation we obtain that $\tilde{u}$ coincides with the classical lift to $\mathbb{R}^2$ of a periodic function when $u$ is a function and we have

$$
\tilde{u}(t,x) = (G \ast 1_{[0,M]})(t,x), \quad \tilde{u}_s(t,x) = (G \ast 1_{[0,M]})(t,x).
$$

From a probabilistic point of view, $\xi$ is an isonormal Gaussian process on $H = L^2(\mathbb{R} \times \mathbb{T})$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That is we can associate with any $f \in H$ a real Gaussian random variable $\xi(f)$ such that for any couple $f, g \in H$ one has

$$
\mathbb{E}[\xi(f)\xi(g)] = \int_{\mathbb{R}} \int_{\mathbb{T}} f(s,x)g(s,x)ds dx.
$$

We denote by $I_n: H^\otimes n \to L^2(\Omega), n \geq 1$ the multiple stochastic Wiener integral with respect to $\xi$ (see [22]). $I_n$ is an isometry between the symmetric elements of $H^\otimes n$ equipped with the norm $\sqrt{m!} \cdot \|H^\otimes n$ and the Wiener chaos of order $n$, the closed linear subspace of $L^2(\Omega)$ generated by $\{H_n(\xi(h)) : \|h\|_H = 1\}$ where $H_n$ is the $n$-th Hermite polynomial. Thus we have the natural identifications $\xi(f) = I_1(f) = \int_{\mathbb{R}} \int_{\mathbb{T}} f(s,y)dW_{s,y}$. Let us introduce some elements of the Malliavin calculus with respect to $\xi$ (see [22] for a thorough introduction on this subject). We consider $\mathcal{S} \subset L^2(\Omega)$ the set of random variables $F$ of the form

$$
F = h(\xi(f_1), \cdots, \xi(f_n)),
$$

where $h: \mathbb{R}^n \to \mathbb{R}$ is a Schwartz test function and $f_1, \cdots, f_n \in H$. The Malliavin derivative with respect to $\xi$ (see [22], Def. 1.2.1) is the $H$-valued random variables $\nabla F = \{\nabla_t F : t \in \mathbb{R} \times \mathbb{T}\}$ defined by

$$
\nabla_t F = \sum_{i=1}^n \frac{\partial h}{\partial x_i}(\xi(f_1), \cdots, \xi(f_n))f_i(t).
$$

Iterating the procedure and adopting the usual convention $\nabla^0 = \text{id}$, for any $k \geq 0$ one can define the $k$-th Malliavin derivative $\nabla^k F = \{\nabla^k_{t_1, \cdots, t_k} F : t_1, \cdots, t_k \in \mathbb{R} \times \mathbb{T}\}$, which are $H^\otimes_k$ valued random variables. Moreover starting from a separable Hilbert $V$ and considering the random variables $G \in \mathcal{S}_V$ of the form

$$
G = \sum_{i=1}^n F_i v_i, \quad F_i, v_i \in \mathcal{S},
$$

we can also define the $H^\otimes_k \otimes V$ random variable $\nabla^k G$. In all of these cases the operator $\nabla^k$ is closable and its domain contains for any $1 \leq p < \infty$ the space $\mathbb{D}^{k,p}(V)$, the closure of $\mathcal{S}_V$ with respect to the norm $\| \cdot \|_{k,p,V}$ defined by

$$
\|F\|_{k,p,V}^p := \mathbb{E}[\|F\|_V^p] + \sum_{l=1}^k \mathbb{E}[\|\nabla^l F\|_{H^\otimes l \otimes V}^p]. \quad (2.7)
$$
(the space $D^{k,p}(\mathbb{R})$ is denoted by $D^{k,p}$). Trivially all variables belonging to some finite Wiener chaos are infinitely differentiable. We denote by $\delta : \text{Dom}(\delta) \subset L^2(\Omega; H) \to L^2(\Omega)$ the adjoint operator of $\nabla$ defined by duality as

$$E[\delta(u)F] = E[(u, \nabla F)_H]$$

for any $u \in \text{Dom}(\delta)$, $F \in D^{1,2}$. The operator $\delta$ is known in the literature as the Skorohod integral and for any $u \in \text{Dom}(\delta)$ we will write again $\delta(u)$ with the symbol $\int \int u(s, y)dW_{s,y}$ because $\delta$ is a proper extension of the stochastic integration over a class of non adapted integrands. Using the same procedure we define $\delta^k : \text{Dom}(\delta^k) \subset L^2(\Omega; H^{\otimes k}) \to L^2(\Omega)$, the adjoint of $\nabla^k$. Similarly to before we call the operator $\delta^k$ the multiple Skorohod integral of order $k$ and we denote it by the notation $\int_{[\mathbb{R} \times T]^k} u((t_1, x_1), \cdots , (t_k, x_k))dW_{t,x}^k$. Let us recall the main properties of $\delta^k$.

- **Extension of the Wiener integral** For any $h \in H^{\otimes k}$, we have $\delta^k(h) = I_k(h)$.

- **Product Formula** (see [21, Lem. 2.1]) Let $u \in \text{Dom}(\delta^k)$ be a symmetric function in the variables $t_1, \cdots , t_k$ and $F \in D^{k,2}$. If for any couple of positive integers $j, r$ such that $0 \leq j + r \leq k$ one has $\langle \nabla^j F, \delta^r u \rangle_{H^{\otimes j}} \in L^2(\Omega; H^{\otimes (k-r-j)})$ then $\langle \nabla^j F, u \rangle_{H^{\otimes j}} \in \text{Dom}(\delta^{k-r})$ and we have

$$F \delta^k(u) = \sum_{r=0}^{k} \binom{k}{r} \delta^{k-r} (\langle \nabla^j F, u \rangle_{H^{\otimes j}}).$$

(2.8)

- **Continuity property** (see [21, Pag. 8]) We have the inclusion $D^{k,2}(H^{\otimes k}) \subset \text{Dom}(\delta^k)$ and the map $\delta^2 : D^{k,2}(H^{\otimes k}) \to L^2(\Omega)$ is continuous. In other words there exists a constant $C > 0$ such that for any $u \in D^{k,2}(H^{\otimes k})$ one has

$$\|\delta^k(u)\|_{L^2(\Omega)} \leq C\|u\|_{D^{k,2}(H^{\otimes k})}.$$  

(2.9)

Extending periodically $\xi$ and the Brownian sheet $W$ to $\mathbb{R}^2$, we can transfer the Walsh integral as well as the Skorohod integral to stochastic processes $H : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ through the definition:

$$\int_{\mathbb{R}^2} H(s, y)d\tilde{W}_{s,y} := \int_{\mathbb{R} \times T} \sum_{m \in \mathbb{Z}} H(s, y + m)dW_{s,y},$$

(2.10)

as long as the right hand side above is well defined. Similar definitions hold for the multiple Skorohod integral of order $k$, mutatis mutandis. Using this notation we express

$$\xi_{\varepsilon}(t, x) = \int_{\mathbb{R}^2} \rho_{\varepsilon}(t-s, x-y)d\tilde{W}_{s,y}, \quad u(t, x) = \int_0^t \int_{\mathbb{R}} G(t-s, x-y)d\tilde{W}_{s,y}.$$ 

3 **Regularity structures**

In this part we will recall some general concepts of the theory of regularity structures to show the existence of an explicit regularity structure and a model. These objets will permit to define some analytical operations on $u$. For a quick introduction to the whole theory we refer the reader to [15]
3.1 Algebraic construction

The starting point of the theory is the notion of a regularity structure \((A, T, G)\), a triple of the following elements:

- A discrete lower bounded real subset \(A\) containing 0.
- A graded vector space \(T = \bigoplus_{\alpha \in A} T_{\alpha}\) such that each space \(T_{\alpha}\) is a Banach space with norm \(\| \cdot \|_{\alpha}\) and \(T_0\) is generated by a single element 1.
- A group \(G\) of linear operators on \(T\) such that for each \(\alpha \in A\), \(\tau \in T_{\alpha}\) and \(\Gamma \in G\), one has
  \[
  \Gamma 1 = 1 \quad \text{and} \quad \Gamma \tau - \tau \in \bigoplus_{\beta < \alpha} T_{\beta}.
  \]  

The element of a regularity structure are interpreted as a set of generalised polynomial whose elements are capable to perform a “Taylor expansion” of distributions. Recalling the equations (1.10) and (1.6), our aim is then to build a regularity structure \(\mathcal{T}\) whose elements are capable to describe for any \(\varepsilon > 0\) the systems of equations

\[
\begin{align*}
\partial_t u_\varepsilon &= \partial_{xx} u_\varepsilon + \xi_\varepsilon \\
\partial_t v_\varepsilon &= \partial_{xx} v_\varepsilon + \varphi'(u_\varepsilon)\xi_\varepsilon - \varphi''(u_\varepsilon)(\partial_x u_\varepsilon)^2.
\end{align*}
\]

(3.2)

Let us give a first description of \(\mathcal{T}\) in terms of abstract symbols. We start by considering the real polynomials on two indeterminates. For any multi-index \(k \in \mathbb{N}^2\), \(k = (k_1, k_2)\) we will write \(X^k\) as a shorthand for the monomial \(X_1^{k_1}X_2^{k_2}\) while the unit will be denoted by 1. In this way we will be able to describe smooth functions. At the same time, we introduce an additional abstract symbol \(\Xi\) to represent the space-time white noise \(\xi\) and for any symbol \(\sigma\) and \(k \in \mathbb{N}^2\) we introduce a family of symbols \(I_k(\sigma)\) (\(I_{(0,0)}(\sigma)\) is denoted by \(I(\sigma)\)) to represent formally the convolution of the \(k\)-th derivative of the heat kernel with the function associated to the symbol \(\sigma\). Since \(I_k(X^m)\) should be identified with a smooth function, we simply put it to 0 to avoid repetitions. Finally for any two symbols \(\tau_1, \tau_2\) we consider also the symbol \(\tau_1 \tau_2\), obtained by juxtaposition of \(\tau_1\) and \(\tau_2\) and identified up to some classical identifications, namely the juxtaposition with 1 does not change the symbol, \(X^i X^k = X^{i+k}\) and the iterated juxtaposition of the same symbol is denoted by an integer power. Adding all these formal rules, we denote by \(F\) the set of all possible formal expressions satisfying

- \(\{X^k\}_{k \in \mathbb{N}^2} \cup \{\Xi\} \subset F\).
- For any \(\tau_1, \tau_2 \in F\), \(\tau_1 \tau_2 \in F\).
- For any \(\sigma \in F\) and \(m \in \mathbb{N}^2\), \(I_m(\sigma) \in F\).

We write \(\mathcal{F}\) for the free vector space generated by \(F\). Similarly to polynomials, we define a homogeneity map \(\cdot|\cdot : F \to \mathbb{R}\) whose values have the properties of the degree of polynomials but in the context of the Hölder spaces. In particular we set recursively

- \(|X^k| := 2k_1 + k_2\) the parabolic degree (we imagine \(X_1\) as a time variable for the parabolic degree);
- \(|\Xi| := -3/2 - \kappa\) for some fixed parameter \(\kappa > 0\);
- \(|I_k(\sigma)| := |\tau| + 2 - 2k_1 - k_2\), \(|\tau \tau'| := |\tau| + |\tau'|\) for any \(\tau, \tau' \in F\).
Starting from the linear space $\mathcal{F}$, we introduce a subset of $F$ where we choose all reasonable products that we will need in our calculations. We write $\mathcal{J}_1(\Xi)$ as shorthand of $\mathcal{J}_{(0,1)}(\Xi)$.

**Definition 3.1.** We define the sets of symbols $T,U,U' \subset F$ as the smallest triple of sets satisfying:

- $\{\Xi\} \subset T$, $\{\mathcal{J}(\Xi)\} \cup \{X^k\}_{k \in \mathbb{N}^2} \subset U$, $\{\mathcal{J}_1(\Xi)\} \cup \{X^k\}_{k \in \mathbb{N}^2} \subset U'$;
- for every $k \geq 0$ and any finite family of elements $\tau_1,\ldots,\tau_k \in U$ and any couple of elements $\sigma_1,\sigma_2 \in U''$ then $\{\tau,\tau\Xi,\tau\sigma_1,\tau\sigma_1\sigma_2\} \subset T$ and $\tau \in U$, where $\tau = \tau_1 \ldots \tau_k$.

We denote also by $\mathcal{F}$ and $\mathcal{U}$ respectively the free vector space upon $T$ and $U$.

The definition of $T$ has an equivalent description in terms of symbols. Defining $V = \{\mathcal{J}(\Xi)^mX^l : m \in \mathbb{N}, l \in \mathbb{N}^2\}$ and for any $\sigma \in \{\Xi,\mathcal{J}_1(\Xi),\mathcal{J}_1(\Xi)^2\}$ $V_{\sigma} := \sigma V$ the set of all symbols of the form $\sigma$ times an element of $V$, it is straightforward to show the identities

$$U = V, \quad T = V_{\Xi} \sqcup V_{\mathcal{J}_1(\Xi)^2} \sqcup V_{\mathcal{J}_1(\Xi)} \sqcup V.$$

Therefore, denoting by $\mathcal{Y}_{\sigma}$ the free vector space generated upon $V_{\sigma}$, we have the decomposition $\mathcal{F} = \mathcal{Y}_{\Xi} \oplus \mathcal{Y}_{\mathcal{J}_1(\Xi)^2} \oplus \mathcal{Y}_{\mathcal{J}_1(\Xi)} \oplus \mathcal{U}$. Let us give the construction of the structure group associated to $\mathcal{F}$. For any $h \in \mathbb{R}^3$, $h = (h_1,h_2,h_3)$ we define the function $\Gamma_h : \mathcal{F} \to \mathcal{F}$ as the unique linear map such that

$$\Gamma_h(\sigma(\mathcal{J}(\Xi)^mX^l)) := \sigma[(X_1 + h_11)^l(X_2 + h_21)^2(\mathcal{J}(\Xi) + h_31)^m],$$

for any $\sigma \in \{\Xi,\mathcal{J}_1(\Xi),\mathcal{J}_1(\Xi)^2,1\}$, $m \in \mathbb{N}$, $l \in \mathbb{N}^2$. Using this explicit definition it is straightforward to show

$$\Gamma_h \Gamma_k = \Gamma_{h+k}$$

for any $h,k \in \mathbb{R}^3$ and the map $h \to \Gamma_h$ is injective. We will denote by $\mathcal{F}$ the group $\{\Gamma_h : h \in \mathbb{R}^3\}$.

**Proposition 3.2.** For any $\kappa < 1/2$, the triple $(\mathcal{A},\mathcal{F},\mathcal{G})$ where $\mathcal{A} = \{|\tau| : \tau \in T\}$ is a regularity structure.

**Proof.** To prove that $\mathcal{A}$ is a discrete lower bounded set, we show that for any $\beta \in \mathbb{R}$ the set $I := \{\tau \in T : |\tau| \leq \beta\}$ is finite. For any $\tau \in I$ by means of the identity (3.3) there exist two indices $m \in \mathbb{N}$, $n \in \mathbb{N}^2$ and $\sigma \in \{\Xi,\mathcal{J}_1(\Xi),\mathcal{J}_1(\Xi)^2\}$ such that $\tau = \sigma(\mathcal{J}(\Xi)^mX^n)$. From $|\tau| \leq \beta$ we deduce

$$n_1 + 2n_2 + (1/2 - \kappa)m \leq \beta - |\sigma|.$$

Imposing $\kappa < 1/2$, the left hand side of the inequality is strictly bigger or equal than 0 and the set $I$ is bounded. This finiteness result implies also the identity $\mathcal{F} = \bigoplus_{\gamma \in \mathcal{A}} \mathcal{F}_\gamma$ where $\mathcal{F}_\gamma = \{\tau \in T : |\tau| = \gamma\}$. Moreover there is no need to specify a norm on $\mathcal{F}_\gamma$, since it is finite dimensional. Finally the property (3.1) comes directly from Newton’s binomial formula and the positive homogeneity of the symbol $\mathcal{J}(\Xi)$.

**Remark 3.3.** The Definition 3.1 is a simplification of the vector space introduced in [16, Pag. 7] with fewer symbols. The triple $(\mathcal{A},\mathcal{F},\mathcal{G})$ is also intimately linked with $(\mathcal{A}^{HP},\mathcal{F}^{HP},\mathcal{G}^{HP})$, the regularity structure defined in [17, Pag. 13-14]. More precisely we consider $U^{HP}$ the smallest set of symbols of $F$ such that $\{X^k\}_{k \in \mathbb{N}^2} \subset U$ and satisfying the properties

$$\tau \in U^{HP} \Rightarrow \mathcal{J}(\tau), \mathcal{J}(\Xi\tau) \in U^{HP}; \quad \tau,\tau' \in U^{HP} \Rightarrow \tau\tau' \in U^{HP}.$$
\( \mathcal{F} \cap \mathcal{F}^{HP} = \mathcal{V}_2 \oplus \mathcal{U} \), it is also possible to show that the action of the group \( \mathcal{G}^{HP} \) coincides with that of \( \mathcal{G} \) on these subspaces and from the explicit definition of \( \Gamma \) one has \( \Gamma(\mathcal{V}_2) \subset \mathcal{V}_2 \) and \( \Gamma(\mathcal{U}) \subset \mathcal{U} \) for any \( \Gamma \in \mathcal{G} \) (see the Remark 3.3). Hence the subspaces \( \mathcal{V}_2 \) and \( \mathcal{U} \) are respectively a sector of regularity \(-3/2 - \kappa\) and a function-like sector of both \( \mathcal{F}^{HP} \) and \( \mathcal{F} \) (see [14, Def. 2.5]). Due to these identifications, we can transfer some results of [17] to our context.

**Remark 3.4.** As a matter of fact we can restrict our considerations once and for all to a subspace of \( \mathcal{F} \) generated by all symbols with homogeneity less than some parameter \( \zeta > 0 \). By convention we denote by \( | \cdot |_\beta \) the euclidean norm on \( \mathcal{F}_\beta \) (The euclidean norm is coherent with \( | \cdot |_\beta \) but there is no “canonical” choice since \( \mathcal{F}_\beta \) is finite dimensional). For any \( \beta \in \Lambda \), we will denote by \( \underline{\mathcal{F}}_\beta \) and \( \overline{\mathcal{F}}_\beta \) the projection operator respectively on \( \mathcal{F}_\beta \) and \( \mathcal{F}_\beta \).

**Remark 3.5.** A consequence of the definition (3.4) implies that for any \( h \in \mathbb{R}^3 \) one has \( \Gamma_h \tau \tau' = \Gamma_h \tau \Gamma_h \tau' \) for every symbol \( \tau, \tau' \in T \) such that also their product \( \tau \tau' \in T \) one has

\[
\Delta X_i = X_i \otimes 1 + 1 \otimes X_i, \quad \Delta 1 = 1 \otimes 1, \quad \Delta \sigma = \sigma \otimes 1
\]

\[
\Delta \mathcal{J}(\Xi) = \mathcal{J}(\Xi) \otimes 1 + 1 \otimes \mathcal{J}(\Xi), \quad \Delta \tau \tau' = \Delta \tau \Delta \tau'.
\]

(3.8)

Thus the group \( \mathcal{G} \) coincides with the classical way to define the structure group \( G \) of a regularity structure. Moreover comparing the relations (3.8) with the explicit definitions given in [14, Sec. 8.1] and [17, Pag. 14] we obtain that the group \( \mathcal{G} \) is simply the restriction of these general constructions when they are applied on the set of symbols defining \( \mathcal{F} \).

In order to apply some general results obtained in [5] and [7], we show how to express the regularity structure \( \mathcal{F} \) by means of the formalism of trees. Let us recall some basic notations. We start by considering labelled, rooted trees \( \tau \) (LR tree), that is \( \tau \) is a combinatorial tree (finite connected simple graph with a non-empty set of nodes \( N_\tau \) and a set of edges \( E_\tau \) without cycles and not planar) where we fixed a specific node \( \rho_\tau \in N_\tau \) called the root of \( \tau \). The trees we consider are also labelled i.e. there exists a finite set of labels \( \mathcal{L} \) and a function \( \epsilon : E_\tau \rightarrow \mathcal{L} \). These trees are the building blocks of a more general family of trees. We define a decorated tree as a triple \( \tau^n_\epsilon = (\tau, n, \epsilon) \) where \( \tau \) is a LR rooted tree and \( n : N_\tau \rightarrow \mathbb{N}^2, \epsilon : E_\tau \rightarrow \mathbb{N}^2 \) are two fixed functions. The set of decorated tree is denoted by \( \mathcal{T} \). Similarly to what we did for the set of symbols \( \mathcal{F} \) we fix a scaling function \( s : \mathcal{L} \sqcup \mathbb{N}^2 \rightarrow \mathbb{R} \) and we define a homogeneity map \( | \cdot | : \mathcal{T} \rightarrow \mathbb{R} \) as

\[
|\tau^n_\epsilon|_s := \sum_{\epsilon \in E_\tau} s(\ell(\epsilon)) - s(\epsilon(\tau)) + \sum_{x \in N_\tau} s(n(x)),
\]

(3.9)

for any \( \tau^n_\epsilon \in \mathcal{T} \). Moreover for any two elements \( \tau^n_\epsilon, \sigma^n_\epsilon \in \mathcal{T} \) we can also define a product tree \( \tau^n_\epsilon \sigma^n_\epsilon \) by simple considering \( \tau \sigma \), the tree obtained by joining the roots of \( \tau^n_\epsilon \) and \( \sigma^n_\epsilon \) and imposing \( n(\rho_\tau(\epsilon)) = n(\rho_\tau(\epsilon)) + n'(\rho_\tau(\sigma)) \). Finally for any \( m \in \mathbb{N}^2, l \in \mathcal{L} \) we define an application \( \mathcal{B}^l_m : \mathcal{T} \rightarrow \mathcal{T} \) (called grafting operation) as follows: for any \( \sigma \in \mathcal{T} \), \( \mathcal{B}^l_m \sigma \in \mathcal{T} \) is the tree with zero decoration on the root obtained by adding one more edge decorated by \( (l, m) \) to the root of \( \sigma \). The set \( \mathcal{T} \) can be constructed recursively starting from the root trees \( \{\bullet_k\}_{k \in \mathbb{N}^2} \) and applying iteratively the grafting operations and the multiplication.
Looking at the equation (3.2), we choose in this case a set of labels with three elements \( L = \{ \Xi, I, J \} \) (\( \Xi \) should represent the noise and \( I, J \) are respectively the heat kernel in the first and the second equation). Moreover, the function \( s \) is defined by

\[
s(k_1, k_2) = 2k_1 + k_2, \quad s(\Xi) = \frac{3}{2} - \kappa, \quad s(I) = s(J) = 2,
\]

where \( \kappa > 0 \) is a fixed parameter (this choice of \( s \) is done to imitate the parabolic degree and the Schauder estimates). We can easily draw a decorated tree \( \tau \) by simply putting its root at the bottom and decorating the nodes and the edges with the non-zero values of \( n, e \). For example when we write the tree

\[
\Xi \quad (1,2) \quad \Xi
\]

we have three nodes with a zero \( n \) decoration and the edge labelled with \( \Xi \) has zero \( e \) decoration.

After applying these choices, similarly to what we have done to define \( T \), we can build a regularity structure from \( T \) by choosing a suitable subset of trees \( T' \subset T \). This operation is then formalised in the context of decorated trees by the notion of a “rule” which describes what type of edges are allowed next to every label (see [5, Def. 5.7]). More precisely, denoting by \( E \) the set of all finite multisets of \( L \times \mathbb{N}^2 \), a rule is a function \( R: L \rightarrow P(E) \setminus \{\emptyset\} \) where \( P(E) \) is the power set of \( E \). Let us explain in our case what rule we choose in this context.

**Definition 3.6.** Writing \( I \) and \( I_1 \) as a shorthand for the couples \( (I, (0,0)), (I, (0,1)) \in L \times \mathbb{N}^2 \) we define

\[
R(\Xi) = \{()\}, \quad R(I) = \{() \Xi\},
\]

\[
R(J) = \{() \underbrace{I \cdots I}_{k \text{ times}}, \underbrace{I \cdots I, I_1}_{k \text{ times}}, \underbrace{I \cdots I, I_1, I_1}_{k \text{ times}}, \underbrace{I \cdots I, \Xi}_{k \text{ times}}, k \in \mathbb{N}\},
\]

where the brackets \{\} describe a subset of \( E \) and the brackets () a multiset of \( L \times \mathbb{N}^2 \) (the symbol () denotes the empty multiset).

Once we established a rule, we can consider the set of all decorated trees which strongly conforms to the rule \( R \) (see [5, Def. 5.8]), denoted by \( \Xi(R) \), that is \( \tau^n \in \Xi(R) \) if the following properties are satisfied

- Looking at the edges attached at the root \( \rho_r \), they can be expressed as \( R(l) \) for some \( l \in L \);
- for any node \( x \in N_r \setminus \{\rho_r\} \), all the edges attached at \( x \) can be written as \( R(t(e)) \), where \( e \) is the unique edge linking \( x \) to its parent.

For example we use the shorthand notation \( J = [(0,0), J] \) and we consider the two trees

\[
\Xi \quad (1) \quad \Xi \quad \Xi \quad (1) \quad \Xi
\]

the tree on the left hand side strongly conforms to the rule \( R \) but the tree on the right one does not because the multiset \( (I, J) \) is not in the image of \( R \). From the Definition 3.6 it is straightforward to see that all possible decorations of the trees in \( \Xi(R) \) are of three types. We will abbreviate them with the shorthand notations

\[
\Xi \quad I \quad I_1
\]
Thanks to this operation we can extract a specific subset of trees, but if we want to build a regularity structure it is necessary to check that the rule $R$ has two last fundamental properties. Firstly $R$ must be subcritical (see [5 Def. 5.14]), that is there exists a function $\text{reg}: \mathcal{L} \to \mathbb{R}$ such that when it is extended to $E$ as follows
\[ \text{reg}(l, k) := \text{reg}(l) - s(k), \quad \text{reg}(N) := \sum_{(l, k) \in M} \text{reg}(l, k), \]
for any $(l, k) \in \mathcal{L} \times \mathbb{N}^2$ and $M \in E$, then we have
\[ \text{reg}(l) < s(l) + \inf_{M \in R(l)} \text{reg}(M), \]
For any $l \in \mathcal{L}$. Moreover $R$ must also be normal (see [5 Def. 5.8]), that is $R(l) = \{()\}$ for every $l \in \mathcal{L}$ such that $s(l) < 0$ and for any couple of multisets $M, N \in E$ of $E$ such that $N \in R(l)$ for some $l \in \mathcal{L}$ and $M \subset N$, then $M \in R(l)$. Both properties are relatively easy to check in this specific case. Indeed the rule $R$ is normal by construction and we can verify the subcriticality using the function $\text{reg}: \mathcal{L} \to \mathbb{R}$ defined by
\[ \text{reg}(\Xi) = \frac{3}{2} - 2\kappa, \quad \text{reg}(I) = \text{reg}(J) = \frac{1}{2} - 3\kappa, \]
as long as $\kappa$ is sufficiently small. These two properties allow us to apply the results [5 Prop. 5.21], [5 Prop. 5.39] and the definition [5 Def. 6.22] to prove the following result

**Proposition 3.7.** There exists a rule $R'$ such that $R(l) \subset R'(l)$ for every $l \in \mathcal{L}$ and a group $\mathcal{G}'$ such that the triple $(\mathcal{A}', \mathcal{T}', \mathcal{E}')$ is a regularity structure, where $\mathcal{T}'$ is the free vector space generated from $\mathcal{E}(R')$ and $\mathcal{A}' = \{|\tau|_\kappa: \tau \in \mathcal{E}(R')\}$.

Even if we could give an explicit description of the group $\mathcal{G}'$ and $\mathcal{T}'$ given in the Proposition 3.7 for our purposes it is sufficient to establish a relation between the regularity structure $\mathcal{T}$ and $\mathcal{T}'$. From the explicit definition of $F$ and $\Xi$ it is possible to define recursively an injective map $\iota: F \to \Xi$ as follows:
- for any $m \in \mathbb{N}^2$ we set $\iota(\Xi) := \frac{1}{2}$, $\iota(X^m) := \cdot_m$
- For any symbol $\sigma$ such that $\iota(\sigma)$ is defined, then $\iota(J_k(\sigma)) := \mathcal{G}'(\sigma)$.
- For any couple of symbols $\sigma, \sigma'$ such that $\iota(\sigma)$ and $\iota(\sigma')$ are well defined we set $\iota(\sigma\sigma') = \iota(\sigma) \iota(\sigma')$.

We present two examples of the action of $\iota$:
\[ \iota(\Xi^2) = \quad \iota(J_1(\Xi^2) \Xi X^{(3,4)}) = \quad \iota(J_1(\Xi X^{(3,4)})) = \]

Restricting the map $\iota$ on $T$ and extending it by linearity we have the following inclusion

**Proposition 3.8.** The regularity structure $(\mathcal{A}, \mathcal{T}, \mathcal{E})$ is contained in $(\mathcal{A}', \mathcal{T}', \mathcal{E}')$ in the sense of the inclusion explained in [14 Sec. 2.1].

**Proof.** The theorem is a strict consequence of the choices done to define $\mathcal{T}'$. Firstly by definition of $R$, every decorated tree $\iota(\tau)$ for some $\tau \in T$ strongly conforms to the rule $R$, therefore it will conform to the rule $R'$. Moreover by construction of $s$ in [5, 10] we have $|\iota(\sigma)|_s = |\sigma|$ for any $\sigma \in F$. Thus $\mathcal{A} \subset \mathcal{A}'$. Finally, when we consider the groups $\mathcal{G}, \mathcal{G}'$, it has been showed in [5, Pag. 89] that the group $\mathcal{G}'$ acts on $\iota(\mathcal{T})$ in the same way as the operator $\Delta$ explained in the Remark 3.3. Therefore we obtain the inclusion of the regularity structures. $\square$
Remark 3.9. The function $\iota$ is clearly an injective map function from $F$ to $\Sigma$ but there are many trees of $\Sigma$ which do not belong to $\iota(F)$. In particular all the trees whose edges are only labelled with $I$ are not contained, because we identified all the symbols $F_k(X^m)$ to zero. Moreover none of trees labelled with $J$ belong to $\iota(F)$. The presence of two different labels $I$, $J$ to denote the heat kernel is actually done to isolate all the tree we really need for our calculation and if we identified $I = J$ in the Definition 3.9 we would obtain the rule for generalised KPZ equation as explained in [5, Sec. 5.4]. In what follows we will identify both symbols and decorated trees, without writing explicitly the map $\iota$.

3.2 Models on a regularity structure

The algebraic structure comes also with a model associated to it. In order to recall this notion and to simplify the whole exposition, we fix a parameter $\zeta \geq 2$ and with an abuse of notation we will identify all along the paper $T$ (respectively its canonical basis $T$) with the finite dimensional vector space $\mathcal{Q}_{< \zeta}T$ (resp. the finite set $\{\tau \in T : |\tau| < \zeta\}$). The same applies also for the sets $V_{\Xi}$, $V_{\mathcal{F}(\Xi)2}$, $V_{\mathcal{F}(\Xi)}$.

Definition 3.10. A model on $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ consists of a pair $(\Pi, \Gamma)$ given by:

- A map $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{G}$ such that $\Gamma_{zz} = id$ and $\Gamma_{zw} = \Gamma_{zw}$ for any $z, v, w \in \mathbb{R}^2$.
- A collection $\Pi = \{\Pi_z\}_{\varepsilon \in \mathbb{R}^2}$ of linear maps $\Pi_z : \mathcal{T} \mapsto \mathcal{G}'(\mathbb{R}^2)$ such that $\Pi_z = \Pi_0 \Gamma_{uz}$ for any $z, v \in \mathbb{R}^2$.

Furthermore, for every compact set $\mathcal{K} \subset \mathbb{R}^2$, one has

$$\|\Pi\|_{\mathcal{K}} := \sup \left\{ \frac{|(\Pi_z(\tau))|_{\beta}}{\lambda^{1\zeta}} : z \in \mathcal{K}, \lambda \in (0, 1], \tau \in T, \eta \in \mathcal{B}_2 \right\} < \infty,$$  \hspace{1cm} (3.12)

$$\|\Gamma\|_{\mathcal{K}} := \sup \left\{ \frac{|\Gamma_{zw}(\tau)|_{\beta}}{\|z - w\|^{1\zeta}} : z \neq w \in \mathcal{K}, \|z - w\| \leq 1, \tau \in T, \beta < |\tau| \right\} < \infty,$$  \hspace{1cm} (3.13)

where the set of test functions $\mathcal{B}_2$ was already introduced in the Section 2.

This notion plays a fundamental role in the whole theory, because it associates to any $\tau \in T$ an explicit distribution $\Pi_z \tau$ belonging in some way to $\mathcal{G}^{[\tau]}$. In order to compare two different models defined on the same structure, we endow $\mathcal{M}$, the set of all models on $(\mathcal{A}, \mathcal{T}, \mathcal{G})$, with the topology associated to the corresponding system of semi-distances induced by the conditions (3.12) and (3.13):

$$\|(\Pi, \Gamma), (\bar{\Pi}, \bar{\Gamma})\|_{\mathcal{M}(\mathcal{K})} := \|\Pi - \bar{\Pi}\|_{\mathcal{K}} + \|\Gamma - \bar{\Gamma}\|_{\mathcal{K}}.$$  \hspace{1cm} (3.14)

Since we want to study the processes on a finite time horizon, it is sufficient to verify the conditions (3.12) (3.13) on a fixed compact set $\mathcal{K}$ containing $[0, T] \times [0, 1]$ and we will avoid any reference of it in the notation. In this way $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ becomes a complete metric space $(\mathcal{M}$ is not a Banach space because the sum of models is not necessarily a model). In particular if a sequence $(\Pi^n, \Gamma^n)$ converges to $(\Pi, \Gamma)$, then $\Pi^n_z \tau$ converges to $\Pi_z \tau$ in the sense of tempered distributions for any $z, \tau$. To define correctly a model over a symbol of the form $F(\sigma)$, we need a technical lemmas related to a suitable decompositions of $G$, the heat kernel on $\mathbb{R}$ interpreted as a function $G : \mathbb{R}^2 \\setminus \{0\} \to \mathbb{R}$.

Lemma 3.11 (First decomposition). (see [14, Lemma 5.5]) There exists a couple of functions $K : \mathbb{R}^2 \\setminus \{0\} \to \mathbb{R}$, $R : \mathbb{R}^2 \to \mathbb{R}$ such that $G(z) = K(z) + R(z)$ in such a way that $R$ is $C^\infty(\mathbb{R}^2)$ and $K$ satisfies:
• $K$ is a smooth function on $\mathbb{R}^2 \setminus \{0\}$, supported on the set \{(t, x) \in \mathbb{R}^2 : x^2 + |t| \leq 1\} and equal to $G$ on \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : x^2 + t < 1/2, t > 0\}.

• $K(t, x) = 0$ for $t \leq 0, x \neq 0$ and $K(t, -x) = K(t, x)$.

• For every polynomial $Q : \mathbb{R}^2 \to \mathbb{R}$ of parabolic degree less than $\zeta$, one has
\[
\int_{\mathbb{R}^2} K(t, x)Q(t, x) \, dx \, dt = 0.
\] (3.15)

**Remark 3.12.** Thanks to these lemmas, it is possible to localise on a compact support the regularising action of the heat kernel. Indeed it is also possible to show (see [14, Lem 5.19]) that the map $v \to K * v$ sends continuously $\mathcal{C}^\alpha$ in $\mathcal{C}^{\alpha+2}$ for any non integer $\alpha \in \mathbb{R}$ and any distribution $v$ not necessarily compactly supported.

In what follows for any given realisation of $\xi$, the periodic lifting of $\xi$, we will provide the construction of $(\tilde{\Pi}^\varepsilon, \tilde{\Gamma}^\varepsilon)$ a sequence of models associated to $\xi^\varepsilon$ and converging to a model $(\tilde{\Pi}, \tilde{\Gamma})$ related to $\xi$. As a further simplification, we parametrise all possible models $(\Pi, \Gamma)$ on $(\mathcal{A}, \mathcal{F}, \mathcal{G})$ with a couple $(\Pi, f)$ where $\Pi : \mathcal{F} \to \mathcal{E}'(\mathbb{R}^2)$ and $f : \mathbb{R}^2 \to \mathbb{R}$. Indeed it is straightforward to check that for any given couple $(\Pi, f)$ the operators
\[
\Pi_z := \Pi f(z), \quad \Gamma_{zz'} := \Gamma_{f(z')-f(z)}.
\] (3.16)
satisfy trivially the algebraic relationships in the Definition 3.10 because of the identity (3.5). Since any realisation of $\xi$ is smooth, we firstly build a model upon any deterministic smooth function $\xi : \mathbb{R}^2 \to \mathbb{R}$ adding randomness in a second time.

**Proposition 3.13.** Let $\xi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth periodic function. If the map $\Pi$ satisfies the conditions
\[
\Pi 1 = 1, \quad \Pi X^k \tau = z^k \Pi \tau, \quad \Pi \mathcal{F}_k(\sigma) = \partial_k (K * \Pi(\sigma)), \quad \Pi \Xi = \xi, \quad \Pi \tau \tau = \tilde{\Pi} \tau \Pi \tau;
\] (3.17) (3.18) (3.19)
defined for any $k \in \mathbb{N}^d, \tau, \bar{\tau} \in T$ such that $\tau X^k \in T, \mathcal{F}_k(\tau) \in T$ and $\tau \bar{\tau} \in T$, then there exists a unique couple $(\Pi, f)$ such that, using the identifications (3.16), the associated operators $(\Pi, \Gamma)$ is a model. We call it the canonical model associated to $\xi$.

**Proof.** The hypothesis on $\xi$ and the conditions (3.17) (3.18) implies straightforwardly that $\Pi \tau$ is a smooth function for any $\tau \in T$ which is not a product of symbols. Therefore in this case the point-wise product on the right hand side of the equation (3.19) is well defined and by linearity the operator $\Pi$ exists and it is unique. In order to choose $f$ by means of (3.5), we compute explicitly
\[
\Pi_{z} (\sigma, \mathcal{F}(\Xi)^m X^k)(\tilde{z}) = \Pi_{f(z)} (\sigma, \mathcal{F}(\Xi)^m X^k)
\]
\[
= \Pi(\sigma)(\tilde{z} + (f(z))_1, z^2 + (f(z))_2 + (K * \xi)(\tilde{z}) + (f(z))_3)\]
\[
m \in m.
\] (3.20)
for any $z, \tilde{z} \in \mathbb{R}^2, \sigma \in \{\mathcal{F}_1(\Xi), \mathcal{F}_1(\Xi)^2, \Xi, 1\}$ and $k, m$ as before. Imposing the condition
\[
f(z)_i = -z_i, \quad i = 1, 2 \quad f(z)_3 = -(K * \Pi \Xi)(z).
\] (3.21)
we obtain immediately the bound $\Pi_\tau (\tilde{z}) \leq C ||\tilde{z} - z||^{\tau}$ for some constant $C > 0$ depending on $\xi$ and uniformly on $\tau \in T$. Thus the condition (3.12) is satisfied. On the other hand, using the definition (3.21) and the explicit formulae (3.16) (3.3) it is trivial to verify the property (3.13) when $\tau \in \{\mathcal{F}(\Xi), X_1, X_2\}$. Applying the multiplicative property of $\Gamma$ (see the Remark 3.5), we conclude. \[\square\]
Remark 3.14. The existence of a canonical model is a general result already proved in [14] Prop 8.27] but we repeat a simplified version of that proof to take in account the slightly different notation of this article. Looking at the definition of $f$ in (3.24), we remark that $f$ depends on $\Pi$ but in order to define it we do not need the multiplicative property of $\Pi$ (3.19), nor the smoothness of $\xi$. Therefore for any map $\Pi: \mathcal{F} \to \mathcal{S}'(\mathbb{R}^2)$, choosing $f$ as in (3.24), we can uniquely associate to it the couple $\mathcal{L}(\Pi) := (\Pi, \Gamma)$ where $\Pi$ and $\Gamma$ are given by (3.16). $\mathcal{L}(\Pi)$ is not necessarily a model but if $\Pi\Xi = \xi \in \mathcal{S}'^{-3/2-\kappa}$ for some $0 < \kappa < 1/2$ and $\Pi$ satisfies (3.17), (3.18) (these conditions can be always formulated in case when $\Pi$ takes values on distributions and if they hold we call the map $\Pi$ admissible), then the proof of the Proposition 3.13 implies also that the operators $\Gamma_{z'z}$ given by $\mathcal{L}(\Pi)$ will always satisfy the property (3.12). The choice of a kernel $K$ satisfying (3.15) is due in order to be compatible with our previous assumption on the symbols $\beta_{k}(X^m)$.

Remark 3.15. If $\xi$ is also periodic in the space variable it is straightforward to prove that the canonical model $(\Pi, \Gamma)$ associated to $\xi$ satisfies also

$$
\Pi_{(t,x+m)\tau}(t',x'+m) = \Pi_{(t,x)\tau}(t',x'), \quad \Gamma_{(t,x+m)(t',x'+m)\tau} = \Gamma_{(t,x)(t',x')\tau}
$$

(3.22)

for any couple of space-time points $z = (t, x)$, $z' = (t', x')$, $m \in \mathbb{Z}$, $\tau \in T$. Thus the canonical model is also adapted to the action of translation on $\mathbb{R}$ (for this definition see [14] Definition 3.33). Roughly speaking this property allows to apply the notion of models also for distributions periodic in space.

Remark 3.16. Recalling the inclusion of the regularity structure $\mathcal{F}$ in $\mathcal{F}'$ as explained in the Proposition 3.13, we can immediately extend the Definition 3.10 to define a model $(\Pi', \Gamma')$ over the regularity structure $(\mathcal{S}', \mathcal{F}', \mathcal{E}')$. In particular for any smooth function $\xi: \mathbb{R}^2 \to \mathbb{R}$ we can define again a canonical model in this context starting from an explicit function $\tilde{\Pi}: \Sigma(R') \to \mathcal{S}'$. Using the grafting operation the application $\tilde{\Pi}$ is defined recursively for any $k, m \in \mathbb{N}^2$, $\tau, \tau' \in \Sigma$

$$
\tilde{\Pi}(\bullet_k)(z) := z^k, \quad \tilde{\Pi}(\mathcal{S}_m^j(\tau))(z) = \tilde{\Pi}(\mathcal{S}_m^j(\tau))(z) = \partial_k^m \left( K \ast \tilde{\Pi}(\tau)(z) \right),
$$

$$
\tilde{\Pi}(\mathcal{S}_m^j(\bullet_k))(z) := \partial_k^m \left( \xi(z) z^k \right), \quad \tilde{\Pi}(\tau')(z) := \tilde{\Pi}(\tau)(z) \tilde{\Pi}(\tau')(z).
$$

These conditions are sufficient to define $\tilde{\Pi}$ without knowing in detail $R'$ and the existence of a model is provided by [1] Prop. 6.12. By construction when we restrict $\tilde{\Pi}$ on $T$ we obtain the properties (3.17) (3.18) (3.19).

### 3.3 The BPHZ renormalisation and the BPHZ model

For any $\varepsilon > 0$ we denote by $\Pi^\varepsilon$ and $\mathcal{L}(\Pi^\varepsilon) := (\Pi^\varepsilon, \Gamma^\varepsilon)$ the canonical model obtained by applying the Proposition 3.13 with a fixed a.s. realisation of $\xi_\varepsilon$. Since $\xi_\varepsilon$ converges to $\xi$ a.s. in the sense of distributions as recalled in the section 2, we would like to define a model by studying the convergence of the sequence $(\Pi^\varepsilon, \Gamma^\varepsilon)$ as $\varepsilon \to 0$. Unfortunately, it is well known from [17] that the sequence $\Pi^\varepsilon(\mathcal{F}(\Xi \Xi))$ does not converge as a distribution, implying that $\mathcal{L}(\Pi^\varepsilon)$ does not converge. A natural way to get rid of this ill-posedness and to prove a general convergence result is the main content of [3] and [7]. The main consequence of these general results will be the existence of an explicit sequence of applications $F_\varepsilon: \mathcal{M} \to \mathcal{M}$ such that the sequence $F_\varepsilon(\mathcal{L}(\Pi^\varepsilon)) := (\Pi^\varepsilon, \Gamma^\varepsilon)$ converges in probability to some random model. The model $(\Pi^\varepsilon, \Gamma^\varepsilon)$ and the limiting model are referred in the literature as the BPHZ renormalisation and the BPHZ model.

In order to satisfy the bounds (3.13) for $\hat{\Gamma}^\varepsilon$ uniformly on $\varepsilon > 0$, it is reasonable to write $(\Pi^\varepsilon, \Gamma^\varepsilon)$ as $\mathcal{L}(\Pi^\varepsilon)$, for some admissible map $\Pi^\varepsilon: \mathcal{F} \to \mathcal{S}'(\mathbb{R}^2)$ (see the Remark 3.14). This
property can be obtained by defining a sequence of linear maps \( \{ A_{\varepsilon} \}_{\varepsilon > 0} : \mathcal{T} \to \mathcal{T} \) satisfying

\[
A_{\varepsilon} \mathbf{1} = \mathbf{1} , \quad A_{\varepsilon} \mathcal{I}_k(\tau) = \mathcal{I}_k(A_{\varepsilon} \tau) , \quad A_{\varepsilon} X^k \tau = X^k A_{\varepsilon} \tau , \quad A_{\varepsilon} \Xi = \Xi ,
\]

(3.23)

for any \( k \in \mathbb{N}^d \) and \( \tau \in T \) such that \( \tau X^k \in T , \mathcal{I}_k(\tau) \in T \). Indeed combining the properties of (3.23) with the explicit definition of \( \mathcal{I}_k \) in the proposition 3.13 the application \( \mathcal{I}_k \) is automatically admissible (by analogy we call \( \{ A_{\varepsilon} \} \) an admissible renormalisation scheme) and we can define the couple \( \mathcal{L}'(\mathcal{I}_k A_{\varepsilon}) \). However the conditions (3.23) are not sufficient to prove that \( \mathcal{L}'(\mathcal{I}_k A_{\varepsilon}) \) is again a model. As a matter of fact, writing the elements of \( \mathcal{T} \) as trees and embedding \( \mathcal{T} \) in \( \mathcal{T}' \) (see the Proposition 3.13), the BPHZ renormalisation is obtained from an explicit admissible renormalisation scheme \( \{ \tilde{M}_{\varepsilon} \}_{\varepsilon > 0} : \mathcal{T} \to \mathcal{I} \) such that imposing \( \tilde{M}_\varepsilon := \mathcal{I}_k \tilde{M}_\varepsilon \) the couple \( \mathcal{L}'(\tilde{M}_\varepsilon) \) is again a model. By construction \( \tilde{M}_\varepsilon \) is a linear map \( \tilde{M}_\varepsilon : \mathcal{T}' \to \mathcal{T}' \) but an important consequence of the Theorem 3.17 will imply \( \tilde{M}_\varepsilon(\mathcal{T}) \subset \mathcal{T} \). Let us recall briefly the definition of \( \tilde{M}_\varepsilon \) in terms of decorated trees as explained in [5 Sec. 6] and [3 Sec.4] starting from \( \mathcal{T}' \) and its basis \( \mathcal{T}(R') \) (see the Proposition 3.7).

Denoting by \( \cdot \) the combinatorial operation of the disjoint union of graphs, we consider \( \hat{T}_- \), the set of all graphs \( \sigma \) such that

\[
\sigma = \tau_1 \cdot \cdots \cdot \tau_n
\]

for some \( n \geq 1 \) and \( \{ \tau_i \}_{i=1, \ldots , n} \in \mathcal{T}(R') \) union the empty graph \( \emptyset \). The elements of \( \hat{T}_- \) are called forests and we denote by \( \hat{\mathcal{F}}_- \) the free vector space generated over \( \hat{T}_- \), which is clearly a commutative algebra with \( \cdot \), whose unity is given by \( \emptyset \). For any decorated tree \( \tau^n_\varepsilon \) we say that a forest \( \gamma \in \hat{T}_- \) is a subforest of \( \tau^n_\varepsilon \) (\( \gamma \subset \tau^n_\varepsilon \)) if \( \gamma \) is an arbitrary subgraph of \( \tau^n_\varepsilon \) with no isolated vertices. For instance let us consider

\[
\gamma_1 = \begin{array}{c}
\gamma_2 = \begin{array}{c}
\gamma_3 = \begin{array}{c}
\end{array}
\end{array}
\end{array}
\]

In this case \( \gamma_2 \) and \( \gamma_3 \) are both elements of \( \hat{T}_- \) but only \( \gamma_3 \subset \gamma_1 \) because \( \gamma_2 \) has an isolated vertex. The empty forest \( \emptyset \) is always a subforest. A decorated tree \( \tau^n_\varepsilon \) and a subforest \( \gamma = \sigma_1 \cdot \cdots \cdot \sigma_n \) such that \( \gamma \subset \tau^n_\varepsilon \) are used to define the contraction tree \( \mathcal{K}_\varepsilon \gamma \tau^n_\varepsilon = (\mathcal{K}_\varepsilon \gamma , \mathcal{K}_\varepsilon \gamma \tau , \mathcal{K}_\varepsilon \gamma n) \), where

- \( \mathcal{K}_\varepsilon \gamma \tau \) is the tree obtained from \( \tau \) replacing each \( \sigma_i \) with a node.
- Denoting by \( \mathcal{K}_\varepsilon \gamma \tau , \mathcal{K}_\varepsilon \gamma n \) each node associated to the contraction of the tree \( \sigma_i \), the function \( \mathcal{K}_\varepsilon \gamma n \) is equal to \( n \) on every non contracted node of \( \mathcal{K}_\varepsilon \gamma \tau \) and for every \( i \), \( n(\mathcal{K}_\varepsilon \gamma n) = \sum_{y \in N_{\sigma_i}} n(y) \).
- \( \mathcal{K}_\varepsilon \gamma e : E \mathcal{K}_\varepsilon \gamma , \to \mathbb{N}^2 \) is equal to \( e \) on every non contracted edge of \( \mathcal{K}_\varepsilon \gamma \tau \).

In the previous example we have \( \mathcal{K}_\varepsilon \gamma_1 = \emptyset \).

Once we give \( \hat{\mathcal{F}}_- \), we define \( \hat{\mathcal{F}}_- := \hat{\mathcal{F}}_- / \mathcal{J} \) as the quotient algebra of \( \hat{\mathcal{F}}_- \) with respect to \( \mathcal{J} \), the ideal of \( \hat{\mathcal{F}}_- \) generated by the set

\[
J := \{ \tau^n_\varepsilon \in \mathcal{T}(R') : |\tau^n_\varepsilon|_0 \geq 0 \} \subset \hat{T}_- .
\]

The map \( \tilde{M}_\varepsilon \) is then defined for any \( \tau^n_\varepsilon \in \mathcal{T}(R') \) as

\[
\tilde{M}_\varepsilon \tau^n_\varepsilon := (h_\varepsilon \otimes \text{id}) \Delta \tau^n_\varepsilon .
\]

(3.24)
We will describe the objects $\Delta_-$ and $h_\varepsilon$ separately. First $\Delta_- : \mathcal{F} \rightarrow \mathcal{F}_- \otimes \mathcal{F}$ is a linear map which is explicitly given for any $\tau^n_\varepsilon \in \Xi(R^d)$ by the formula

$$\Delta_- \tau^n_\varepsilon := \sum_{\gamma \subset \tau} \sum_{\varepsilon_n, \gamma_n \leq \varepsilon_n} \frac{1}{\varepsilon_!} \binom{n}{\varepsilon_n} p(\gamma, \varepsilon_\gamma + \pi \varepsilon_{\gamma, e} \varepsilon_{\gamma, e} + \varepsilon_{\gamma, e}) \bigotimes (\mathcal{K}_{\gamma, \varepsilon}(n - \varepsilon_n), \mathcal{K}_{\varepsilon, \gamma} + \varepsilon_{\gamma, e}) \cdot (3.25)$$

Let us explain the meaning of the formula (3.25). The first sum outside is done over all subforests $\gamma \subset \tau$ and for any subforest $\gamma$, denoting by $N_{\gamma}$ and $\partial(\gamma, \tau)$ respectively the set of the nodes of $\gamma$ and the edges in $E_{\tau}$ that are adjacent to $N_{\gamma}$, the second sum is done over all functions $\varepsilon_n : N_{\gamma} \rightarrow \mathbb{N}$ and $\varepsilon_\gamma : \partial(\gamma, \tau) \rightarrow \mathbb{N}$ such that for any $x \in N_{\gamma}$ $n_\gamma(x) \leq n(x)$ with respect to the lexicographic order. Furthermore the operation $p : \mathcal{F}_- \rightarrow \mathcal{F}_-$ is the projection on the quotient and for any $e : \partial(\gamma, \tau) \rightarrow \mathbb{N}$ the function $\pi e_\gamma : N_{\gamma} \rightarrow \mathbb{N}$ is given by

$$\pi e_\gamma(x) := \sum_{e \in \partial(\gamma, \tau)} e_\gamma(e).$$

The remaining combinatorial coefficients are finally interpreted in a multinomial sense, that is for any function $l : S \rightarrow \mathbb{N}$ where $S$ is a finite set we have $l! = \prod_{y \in S} l(y)!^1 l(y)^2$ and similarly for the binomial coefficients. In principle the summations over $n_\gamma$ and $e_\gamma$ are done over an infinite set of values but the projection $p$ makes the sum finite.

On the other hand the map $h_\varepsilon$ has the explicit form

$$h_\varepsilon := g_\varepsilon(\Pi)\hat{\Delta}_-. \hspace{1cm} (3.26)$$

The first object in (3.26) is given by a map $\hat{\Delta}_- : \mathcal{F}_- \rightarrow \mathcal{F}_-$. Its name is twisted-antipode and it is characterised as the only homomorphism (so then $\hat{\Delta}_-(\emptyset) = \emptyset$) such that for any tree $\tau^n_\varepsilon \neq \emptyset$, denoting by $M^\varepsilon$ the forest product, one has the identity

$$\hat{\Delta}_- \tau^n_\varepsilon := -M^\varepsilon(\hat{\Delta}_- \otimes \text{id})(\Delta_- \tau^n_\varepsilon - \tau^n_\varepsilon \otimes 1). \hspace{1cm} (3.27)$$

Finally the last object $g_\varepsilon(\Pi) : \mathcal{F}_- \rightarrow \mathbb{R}$ is the only real character on the algebra $\mathcal{F}_-$ such that for any tree $\tau^n_\varepsilon \in \Xi(R^d)$

$$g_\varepsilon(\Pi)(\tau^n_\varepsilon) := \mathbb{E}\hat{\Pi}_\varepsilon(\tau^n_\varepsilon)(0), \hspace{1cm} (3.28)$$

where $\hat{\Pi}_\varepsilon$ is the extension of $\Pi^\varepsilon$ over all the decorated trees as explained in the Remark 3.16. We combine all these definitions to obtain the explicit form of the application $\hat{M}_\varepsilon$.

**Theorem 3.17.** By fixing $\kappa > 0$ sufficiently small and restricting the map $\hat{M}_\varepsilon$ defined in (3.21) on $\mathcal{F}$, we have $\hat{M}_\varepsilon = M_\varepsilon$, where $M_\varepsilon : \mathcal{F} \rightarrow \mathcal{F}$ is the unique linear map satisfying $M_\varepsilon = \text{id}$ on $\mathcal{F}_1(\Xi) \oplus \mathcal{U}$ and for any integer $m$ and $k \in \mathbb{N}^2$

$$M_\varepsilon(\mathcal{F}_1(\Xi)^m \mathcal{F}_1(\Xi)^k) = \Xi \mathcal{F}_1(\Xi)^m \mathcal{F}_1(\Xi)^k - (mC^1_\varepsilon) \mathcal{F}_1(\Xi)^m \mathcal{F}_1(\Xi)^k, \hspace{1cm} (3.29)$$

where the constants $C^1_\varepsilon$ and $C^2_\varepsilon$ are given by

$$C^1_\varepsilon := \mathbb{E}[\hat{\Pi}_\varepsilon(\Xi \mathcal{F}_1(\Xi))(0)] = \int_{\mathbb{R}^2} \rho_\varepsilon(z)(K \ast \rho_\varepsilon)(z)dz, \hspace{1cm} (3.30)$$

$$C^2_\varepsilon := \mathbb{E}[\hat{\Pi}_\varepsilon(\mathcal{F}_1(\Xi)^2)(0)] = \int_{\mathbb{R}^2} (K_x \ast \rho_\varepsilon)^2(z)dz. \hspace{1cm} (3.31)$$
Proof. Thanks to the result [5, Theorem 6.17], for any \( k \in \mathbb{N}^d, \tau \in T \) such that \( \tau \mathbf{X}^k \in T \), \( \mathcal{I}_k(\tau) \in T \) the map \( \widetilde{M}_e \) always satisfies

\[
\widetilde{M}_e \mathbf{1} = \mathbf{1}, \quad \widetilde{M}_e \mathcal{I}_k(\tau) = \mathcal{I}_k(\widetilde{M}_e \tau), \quad \widetilde{M}_e \mathbf{X}^k \tau = \mathbf{X}^k \widetilde{M}_e \tau.
\]

Therefore to prove the theorem it is sufficient to show for any \( m \) the identities

\[
\begin{align*}
\widetilde{M}_e(\mathcal{I}_m(\Xi)\mathcal{I}(\Xi)^m) &= \mathcal{I}_m(\Xi)\mathcal{I}(\Xi)^m, \quad \widetilde{M}_e(\mathcal{I}(\Xi)^m) = \mathcal{I}(\Xi)^m \\
\widetilde{M}_e(\Xi\mathcal{I}(\Xi)^m) &= \Xi\mathcal{I}(\Xi)^m - (mC_e^{1}\mathcal{I}(\Xi)^{m-1})1_{m=1}, \\
\widetilde{M}_e(\mathcal{I}_m(\Xi)^2\mathcal{I}(\Xi)^m) &= \mathcal{I}_m(\Xi)^2\mathcal{I}(\Xi)^m - C_e^{2}\mathcal{I}(\Xi)^m.
\end{align*}
\] (3.32)

Denoting by \( W \) the set of symbols

\[
W := \{\mathcal{I}_m(\Xi)^2\mathcal{I}(\Xi)^m, \mathcal{I}_m(\Xi)\mathcal{I}(\Xi)^m, \Xi\mathcal{I}(\Xi)^m, \mathcal{I}(\Xi)^m : m \in \mathbb{N}\},
\]

we have to calculate the operator \( \Delta_\gamma \) and \( b_\gamma \) over the elements of \( W \). In order to do that we need to know for any \( w \in W \) what are the subforests \( \gamma \subset w \) and in principle we should know the explicit form of the forests in \( \tilde{\mathcal{F}}_\gamma \) and the rule \( R' \) which defines \( \tilde{\mathcal{F}}_\gamma \). However we remark that every subgraph \( \gamma \) included in \( w \) with no isolated vertices can be expressed as a disjoint union of trees belonging to \( \Xi(R) \). Thus the knowledge of \( R' \) is unnecessary and in the definition of \( \Delta_\gamma \) in \( \[3.25\] \), we can restrict the first sum over this set. Secondly we fix \( \kappa > 0 \) sufficiently small such that the only trees of \( \Xi(R) \) with strictly negative homogeneity that are included in \( W \) are the following

\[
\left\{ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \right\}.
\]

Denoting by \( \tau_m = \Xi(\mathcal{I}(\Xi)^m) \), we calculate the quantity \( \widetilde{M}_e(\tau_m) \) in case \( m = 0, 1 \) explaining all the passages. Firstly we can apply the simplified version of \( \Delta_\gamma \) in \( [3.25] \) and the recursive definition of \( \hat{\Delta}_\gamma \) in \( \[3.27\] \) to obtain immediately

\[
\Delta_\gamma \mathbf{1} = \mathbf{0} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1}, \quad \hat{\Delta}_\gamma \mathbf{1} = -\mathbf{1},
\]

\[
\begin{align*}
\Delta_\gamma \mathbf{1} &= \mathbf{0} \otimes \mathbf{1}^\vee + \mathbf{1}^\vee \otimes \mathbf{1} + l_{(0,1)} \otimes \mathbf{1}^\vee + \mathbf{1}^\vee \otimes \mathbf{1} + l_{(0,1)} \otimes \mathbf{1}^\vee + \mathbf{1} \otimes \mathbf{1}, \\
\hat{\Delta}_\gamma \mathbf{1} &= -\mathbf{1}^\vee + \mathbf{1}^\vee \otimes \hat{\Delta}_\gamma (l_{(0,1)}) + \mathbf{1} \otimes \hat{\Delta}_\gamma (l_{(0,1)}) - \hat{\Delta}_\gamma (l_{(0,1)}) \mathbf{1}.
\end{align*}
\]

(The extra decoration with \((0,1)\) comes from the operation \( \pi \) in the definition of \( \Delta_\gamma \).) Using the general definition of \( \Delta_\gamma \) in \( \[3.25\] \) and the recursive identity \( \[3.27\] \) the calculation in case of the symbol \( \Xi \mathbf{X}(0,1) \) are given by

\[
\Delta_\gamma l_{(0,1)} = \mathbf{0} \otimes l_{(0,1)} + l_{(0,1)} \otimes \mathbf{1} + l_{(0,1)} \otimes \mathbf{1}, \quad \hat{\Delta}_\gamma (l_{(0,1)}) = -l_{(0,1)} + \mathbf{1} \otimes l_{(0,1)},
\]

To complete the calculation of \( \widetilde{M}_e \), we need to apply \( g_e(\Pi) \) on the images of the pseudo antipode. By definition of \( \tilde{\Pi}^e \) one has

\[
\begin{align*}
g_e(\Pi)(\mathbf{1}) &= E \left[ \int_{\mathbb{R}^2} \rho_e(-z_1)d\tilde{W}_{z_1} \right] = 0, \quad g_e(\Pi) \left( l_{(0,1)} \right) = 0, \\
g_e(\Pi) \left( \mathbf{1}^\vee \right) &= E \left[ \int_{\mathbb{R}^2} \rho_e(-z_1)d\tilde{W}_{z_1} \int_{\mathbb{R}^2} K \ast \rho_e(-z_2)d\tilde{W}_{z_2} \right] = C_e^1.
\end{align*}
\] (3.33)
Hence we conclude firstly

\[ h_ε(1) = h_ε\left(\int_{(0,1)}\right) = 0, \quad h_ε\left(\chi_\cdot\right) = -C_ε^1. \]  

(3.34)

Plugging the formulae (3.34) in the sums of \( \Delta_- \) we obtain the right identities of (3.33) for \( \tilde{M}_ετ_m \) \( m = 0, 1 \). Let us pass to the calculation of \( \tilde{M}_ετ_m \) \( m = 2, 3 \). Writing \( \Delta_-τ_m \) and \( \tilde{A}_-τ_m \), a deep consequence of (3.34) and (3.33) is then all the subforests containing the trees \( \Xi \) or \( \Xi^{(0,1)} \) between the connected components will become zero after applying \( h_ε \) or \( g_ε(\Pi) \) thereby not giving any contribution for \( \tilde{M}_ε \). Denoting by \((\cdots)\) all these terms we have

\[
\Delta_\varepsilon \left( \begin{array}{c}
\chi_0
\end{array} \right) = 0 \otimes \chi_0 \otimes 1 + (\cdots),
\]

\[
\Delta_\varepsilon \left( \begin{array}{c}
\omega_0
\end{array} \right) = 0 \otimes \omega_0 \otimes 1 + (\cdots),
\]

\[
\tilde{A}_\varepsilon \left( \begin{array}{c}
\chi_0
\end{array} \right) = \tilde{A}_\varepsilon \left( \begin{array}{c}
\chi_0
\end{array} \right) \otimes 1 + (\cdots),
\]

\[
\tilde{A}_\varepsilon \left( \begin{array}{c}
\omega_0
\end{array} \right) = \tilde{A}_\varepsilon \left( \begin{array}{c}
\omega_0
\end{array} \right) \otimes 1 + (\cdots).
\]

Similarly we also have

\[
\Delta_\varepsilon \left( \chi_{(0,1)} \right) = 0 \otimes \chi_{(0,1)} + \chi_{(0,1)} \otimes 1 + (\cdots), \quad \tilde{A}_\varepsilon \left( \chi_{(0,1)} \right) = \tilde{A}_\varepsilon \left( \chi_{(0,1)} \right) + (\cdots).
\]

Therefore the calculation of \( \tilde{M}_ετ_m \) is obtained once we know the constants

\[ g_ε(\Pi) \left( \begin{array}{c}
\chi_{(0,1)}
\end{array} \right), \quad g_ε(\Pi) \left( \begin{array}{c}
\omega_{(0,1)}
\end{array} \right), \quad g_ε(\Pi) \left( \begin{array}{c}
\chi
\end{array} \right), \quad g_ε(\Pi) \left( \begin{array}{c}
\omega
\end{array} \right). \]

The first two constants from the left are zero by definition of \( \Pi^ε \) and because we are taking the expectations over a product of an odd number of centred Gaussian variables. On the other hand using the shorthand notation \( K_ε = K * ρ_ε \) we have

\[ g_ε(\Pi) \left( \begin{array}{c}
\chi
\end{array} \right) = \mathbb{E} \left[ \int_{\mathbb{R}^2} K_ε(-z_1)d\tilde{W}_{z_1} \right]^2 = \int_{\mathbb{R}^2} (K_ε(z))^2dz \]

and applying the Wick’s formula for the product of four Gaussian random variables we obtain

\[
g_ε(\Pi) \left( \begin{array}{c}
\chi_0 \chi_1 \chi_2 \chi_3
\end{array} \right) = \]

\[
= \mathbb{E} \left[ \int_{\mathbb{R}^2} K_ε(-z_1)d\tilde{W}_{z_1} \int_{\mathbb{R}^2} K_ε(-z_2)d\tilde{W}_{z_2} \int_{\mathbb{R}^2} K_ε(-z_3)d\tilde{W}_{z_3} \int_{\mathbb{R}^2} ρ_ε(-z_4)d\tilde{W}_{z_4} \right]
\]

\[
= 3 \mathbb{E} \left[ \int_{\mathbb{R}^2} K_ε(-z_1)d\tilde{W}_{z_1} \int_{\mathbb{R}^2} K_ε(-z_2)d\tilde{W}_{z_2} \right] \mathbb{E} \left[ \int_{\mathbb{R}^2} K_ε(-z_3)d\tilde{W}_{z_3} \int_{\mathbb{R}^2} ρ_ε(-z_4)d\tilde{W}_{z_4} \right]
\]

\[
= 3 \int_{\mathbb{R}^2} (K_ε(z))^2dz \int_{\mathbb{R}^2} (K_ε(z))ρ_ε(z)dz = 3 g_ε(\Pi) \left( \begin{array}{c}
\chi
\end{array} \right) C_ε^1.
\]

By replacing the values of \( g_ε(\Pi) \) in the above calculations of \( \tilde{A}_-τ_m \) one has

\[ h_ε \left( \begin{array}{c}
\chi_{(0,1)}
\end{array} \right) = h_ε \left( \begin{array}{c}
\omega_{(0,1)}
\end{array} \right) = h_ε \left( \begin{array}{c}
\chi \omega_0
\end{array} \right) = 0. \]  

(3.35)
Moreover the values of \( \tilde{M}_\varepsilon \tau_m \) coincide with (3.32) for \( m \leq 3 \). Looking at \( \tilde{M}_\varepsilon \tau_m \) if \( m > 3 \) and \( \tilde{M}_\varepsilon (\mathcal{S}(\Xi)^m) \), the terms in the left factor of the sum \( \Delta_\varepsilon \tau_m \) and \( \mathcal{S}(\Xi)^m \) will contain an arbitrary subforest obtained respectively from the forest product of these trees

\[
\left\{ i_1, i_2, \cdots, i_{\ell(0,1)}, i_{(0,1)} \right\}, \quad \left\{ i_1, i_{(0,1)} \right\}.
\]

and denoting by \( (\cdots) \) the forest in the left factor which becomes zero after applying the values of \( h_\varepsilon \) in (3.31) (3.33) we obtain

\[
\Delta_\varepsilon \tau_m = \emptyset \otimes \tau_m + m \left( I^{\otimes} - \otimes \right) + (\cdots), \quad \Delta_\varepsilon \tau_{m-1} = \emptyset \otimes I^{\otimes} + (\cdots).
\]

(The factor \( m \) appears because the tree associated to \( \Xi \mathcal{S}(\Xi) \) appears \( m \) times inside \( \tau_m \)) Therefore we prove the first part of the equations (3.32). We pass to the terms of the form \( m \cdot \cdots \) and denoting by \( (\cdots) \) the terms that do not give any effective contribution in calculations after applying \( h_\varepsilon \) as before to denote the terms that do not give any effective contribution in calculations after

\[
\Delta_\varepsilon \hat{1} = \emptyset \otimes \hat{1} + \hat{1} \otimes \mathbf{1} + (\cdots), \quad \Delta_\varepsilon \hat{\mathbf{1}} = \emptyset \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \mathbf{1} + (\cdots),
\]

\[
\hat{\Delta}_\varepsilon \hat{1} = -\hat{1} + (\cdots), \quad \hat{\Delta}_\varepsilon \hat{\mathbf{1}} = -\hat{\mathbf{1}} + \hat{\mathbf{1}} + (\cdots),
\]

Applying the map \( \hat{\Pi}^\varepsilon \) we obtain also

\[
g_{\varepsilon}(\mathbf{1}) = 0, \quad g_{\varepsilon}(\hat{\mathbf{1}}) = \mathbb{E} \left[ \int_{\mathbb{R}^2} \partial_z K_\varepsilon(-z_1) d\tilde{W}_{z_1} \right]^2 = C_{\varepsilon}^2,
\]

\[
g_{\varepsilon}(\hat{\mathbf{1}}) = \mathbb{E} \left[ \int_{\mathbb{R}^2} K_\varepsilon(-z_1) d\tilde{W}_{z_1} \int_{\mathbb{R}^2} \partial_z K_\varepsilon(-z_2) d\tilde{W}_{z_2} \right] = \int_{\mathbb{R}^2} K_\varepsilon(z) \partial_z K_\varepsilon(z) dz = 0,
\]

where first and the last identity of (3.36) are obtained because we take the expectation of a centred Gaussian variable and the function \( x \to K_\varepsilon(t,x) \partial_z K_\varepsilon(t,x) \) is odd in \( x \) for any \( t > 0 \). Then we obtain

\[
h_{\varepsilon}(\mathbf{1}) = h_{\varepsilon}(\hat{\mathbf{1}}) = 0, \quad h_{\varepsilon}(\hat{\mathbf{1}}) = -C_{\varepsilon}^2, \quad (3.37)
\]

and consequently the right identities of (3.32) for \( \tilde{M}_\varepsilon \sigma_m \) and \( \tilde{M}_\varepsilon \eta_k \). Passing to the calculation of \( \tilde{M}_\varepsilon \sigma_m \) for \( m = 2, 3 \) we have

\[
\Delta_\varepsilon \hat{\mathbf{1}} = \emptyset \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \mathbf{1} + (\cdots), \quad \Delta_\varepsilon \hat{\mathbf{1}} = \emptyset \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \mathbf{1} + (\cdots),
\]

\[
\hat{\Delta}_\varepsilon \hat{1} = -\hat{1} + (\cdots), \quad \hat{\Delta}_\varepsilon \hat{\mathbf{1}} = -\hat{\mathbf{1}} + \hat{\mathbf{1}} + (\cdots),
\]

\[
\hat{\Delta}_\varepsilon \hat{1} = -\hat{1} + (\cdots), \quad \hat{\Delta}_\varepsilon \hat{\mathbf{1}} = -\hat{\mathbf{1}} + \hat{\mathbf{1}} + (\cdots),
\]

\[
\hat{\Delta}_\varepsilon \hat{1} = -\hat{1} + (\cdots), \quad \hat{\Delta}_\varepsilon \hat{\mathbf{1}} = -\hat{\mathbf{1}} + \hat{\mathbf{1}} + (\cdots),
\]

\[
\hat{\Delta}_\varepsilon \hat{1} = -\hat{1} + (\cdots), \quad \hat{\Delta}_\varepsilon \hat{\mathbf{1}} = -\hat{\mathbf{1}} + \hat{\mathbf{1}} + (\cdots).
\]

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Moreover using again the same algebraic notations, the Wick’s formula and the definition of \( \hat{\Pi}^\varepsilon \) we have
\[
\Delta_\varepsilon \left( \hat{\omega}_{0,1} \right) = 0 \otimes \hat{\omega}_{0,1} + \hat{\omega}_{0,1} \otimes 1 + (\cdots), \quad \hat{\Delta}_\varepsilon \left( \hat{\omega}_{0,1} \right) = -\hat{\omega}_{0,1} + (\cdots),
\]
\[
g_\varepsilon (\Pi) \left( \hat{\omega}_{0,1} \right) = 0, \quad g_\varepsilon (\Pi) \left( \hat{\omega}_{0,1} \right) = 2 \left( g_\varepsilon (\Pi) \left( \hat{\omega}_{0,1} \right) \right)^2 + g_\varepsilon (\Pi) \left( \hat{\omega}_{0,1} \right) g_\varepsilon (\Pi) \left( \hat{\omega}_{0,1} \right).
\]
Thus yielding finally
\[
h_\varepsilon \left( \hat{\omega}_{0,1} \right) = h_\varepsilon \left( \hat{\omega}_{0,1} \right) = 0, \quad (3.38)
\]
and \((3.32)\) when \( m \leq 2, k \leq 1 \). In case \( m > 2 \) or \( k > 1 \), the terms in the left factor of \( \Delta_\varepsilon \sigma_m \) and \( \Delta_\varepsilon \eta_k \) are respectively forests composed by the trees
\[
\left\{ \hat{\varnothing}, \hat{\omega}_{0,1}, \hat{\varnothing}, \hat{\omega}_{0,1}, \hat{\varnothing}, \hat{\omega}_{0,1}, \hat{\varnothing}, \hat{\omega}_{0,1} \right\} \quad \text{or} \quad \left\{ \hat{\varnothing}, \hat{\omega}_{0,1}, \hat{\varnothing}, \hat{\omega}_{0,1} \right\}.
\]
Applying the identities \((3.33)\), \((3.37)\) and \((3.38)\) the only relevant terms in the sums become
\[
\Delta_\varepsilon \sigma_m = 0 \otimes \sigma_m + \hat{\omega}_{0,1} \otimes \hat{\omega}_{0,1} + (\cdots), \quad \Delta_\varepsilon \eta_k = 0 \otimes \eta_k + (\cdots).
\]
Thus we obtain the final part of the identities \((3.32)\) and we conclude. \(\square\)

We will henceforth fix the parameter \( \kappa \) in order to keep the Theorem \((3.17)\) true. By construction of the BPHZ renormalisation (see [5 Sec.6]) the application \( \hat{M}_\varepsilon \) is an admissible renormalisation scheme and, denoting by \( \Pi^\varepsilon = \Pi \hat{M}_\varepsilon \), the couple \( \hat{K}^\varepsilon = (\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon) \) obtained from the Remark \((3.14)\) is always a model for any \( \varepsilon > 0 \). The explicit form of the map \( \hat{M}_\varepsilon \) obtained in the Theorem \((3.17)\) allows us to write explicitly also \((\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)\).

**Proposition 3.18.** For any \( z \in \mathbb{R}^2 \) and \( z', z'' \in \mathbb{R}^2 \) one has
\[
\hat{\Pi}^\varepsilon_z = \Pi^\varepsilon \hat{M}_\varepsilon, \quad \hat{\Gamma}^\varepsilon_{z'z''} = \Gamma^\varepsilon_{z'z''}.
\]
Furthermore the model \((\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)\) is also adapted to the action of translation on \( \mathbb{R} \).

**Proof.** By definition of \( \hat{K}^\varepsilon \), the model \((\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)\) can be represented as the couple \((\hat{\Pi}^\varepsilon, \hat{f})\), where the function \( \hat{f}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^3 \) is defined as
\[
\hat{f}_\varepsilon (z)_i = -z_i, \quad i = 1, 2 \quad \hat{f}_\varepsilon (z)_3 = -(K \ast \hat{\Pi}^\varepsilon \Xi)(z) = -(K \ast \Pi^\varepsilon \Xi)(z).
\]
Thus the function \( \hat{f}_\varepsilon \) coincides with \( f_\varepsilon \), the same function obtained from the decomposition of the canonical model \((\Pi^\varepsilon, \Gamma^\varepsilon)\) as \((\Pi^\varepsilon, f_\varepsilon)\). By definition of \( \Gamma \) we have straightforwardly \( \hat{\Gamma}^\varepsilon_{z'z''} = \Gamma^\varepsilon_{z'z''} \).

In case of \( \hat{\Pi}^\varepsilon \) we can apply immediately the identity \((3.15)\) with the previous result to obtain
\[
\hat{\Pi}^\varepsilon_z = \hat{\Pi}^\varepsilon \Gamma_{f_\varepsilon (z)} = \Pi^\varepsilon \hat{M}_\varepsilon \Gamma_{f_\varepsilon (z)} = \Pi^\varepsilon \hat{M}_\varepsilon \Gamma_{f_\varepsilon (z)}.
\]
Then the formula \((3.39)\) holds as long as for any \( h \in \mathbb{R}^3 \) and \( \varepsilon > 0 \) one has
\[
M_\varepsilon \Gamma_h = \Gamma_h M_\varepsilon.
\]
Let us verify the identity \((3.41)\) for any \( \tau \in V_\Xi \cup V_{f_\varepsilon (\Xi)} \cup V_{\mathcal{X}_1 (\Xi)} \cup U \). In case \( \tau \in V_{\mathcal{X}_1 (\Xi)} \cup U \), this identity holds trivially because \( \Gamma_h \) leaves invariant the subspace \( \mathcal{Y}_{\mathcal{X}_1 (\Xi)} \cup \mathcal{Z} \) and \( M_\varepsilon \) is the
identity when it is restricted to this subspace. On the other hand if \( \tau \in V_\Xi \cup V_{\mathcal{J}_1}(\Xi)^2 \), the multiplicative property of \( \Gamma_h \) (see the Remark 3.3) and the behaviour of \( M_\varepsilon \) on the polynomials in (3.29) reduces to verify (3.41) over the symbols \( \mathcal{J}_1(\Xi)^2 \mathcal{J}(\Xi)^m \) and \( \Xi \mathcal{J}(\Xi)^m \) for any \( m \geq 1 \). Writing \( \hat{h} = (h_1, h_2, h_3) \) we obtain

\[
M_\varepsilon \Gamma_h(\mathcal{J}_1(\Xi)^2 \mathcal{J}(\Xi)^m) = (\mathcal{J}_1(\Xi)^2 - C_\varepsilon^2) \sum_{n=0}^{m} \binom{m}{n} \mathcal{J}(\Xi)^n h_3^{m-n} = \Gamma_h M_\varepsilon(\mathcal{J}_1(\Xi)^2 \mathcal{J}(\Xi)^m),
\]

\[
M_\varepsilon \Gamma_h(\Xi \mathcal{J}(\Xi)^m) = \sum_{n=0}^{m} \binom{m}{n} (\Xi \mathcal{J}(\Xi)^n) = n C_\varepsilon \mathcal{J}(\Xi)^{n-1}) h_3^{m-n} = \sum_{n=0}^{m} \binom{m}{n} \Xi \mathcal{J}(\Xi)^m h_3^{m-n} - mC_\varepsilon \sum_{n'=0}^{m-1} \binom{m-1}{n'} \mathcal{J}(\Xi)^{m-1-n'} = \Xi(\mathcal{J}(\Xi) + h_3 1)^m - mC_\varepsilon (\mathcal{J}(\Xi) + h_3 1)^{m-1} = \Gamma_h M_\varepsilon(\Xi \mathcal{J}(\Xi)^m).
\]

Thus yielding the result. The identity (3.39) implies immediately the properties in the identity (3.22). Therefore (\( \hat{\Pi}', \hat{\Gamma} \)) is adapted to the action of translations.

We study the convergence of \( \mathcal{L}(\hat{\Pi}') \) in the space of models. Embedding the regularity structure \( \mathcal{L} \) into \( \mathcal{L}' \) as explained in the Proposition 3.8, it is possible to prove the convergence of \( \mathcal{L}(\hat{\Pi}') \) using on \( T \) the general criterion exposed in [7 Thm. 2.15]. To apply this statement we introduce some notation. Representing all the elements of \( \tau \in T \) as decorated trees, we denote by \( E_\Xi(\tau) \) the set of edges labelled by \( \Xi \). By construction every element \( e \in E_\Xi(\tau) \) is written uniquely as \( e = (e_\Xi, e_{\Xi}) \), where \( e_\Xi \) is one of the terminal nodes of \( \tau \). This decomposition allows to define the sets

\[
N_\Xi(\tau) := \{ e_\Xi : e \in E_\Xi(\tau) \}, \quad N_{\Xi}(\tau) := \{ e_{\Xi} : e \in E_\Xi(\tau) \}, \quad N(\tau) := N_{\tau} \setminus N_{\Xi}(\tau).
\]

Moreover, expressing \( \tau \) as \( \tau^n \) for some decoration \( n \), we write \( \tau^0 \) to denote the decorated tree whose decoration \( n \) is replaced by zero in every node. Let us express the convergence theorem in this context.

**Theorem 3.19.** There exists a random model \( (\hat{\Pi}, \hat{\Gamma}) \) such that

\[
(\hat{\Pi}_e, \hat{\Gamma}_e) \xrightarrow{p} (\hat{\Pi}, \hat{\Gamma})
\]

with respect to the metric \( \| \cdot \|_{\mathcal{L}} \). We call \( (\hat{\Pi}, \hat{\Gamma}) \) the BPHZ model.

**Proof.** This theorem is a direct consequence of [7 Thm. 2.15]. Expressing the hypothesis of this theorem in our context, we obtain the thesis after checking the following property: for any \( \tau^0_n \in T \) and every subtree \( \sigma^n_\varepsilon \) included in \( \tau^0_n \) such that \( \sharp(N(\sigma)) \geq 2 \) one has

1. For any non-empty subset \( A \subset E_\Xi(\tau) \) such that \( \sharp(A) \geq 2 \), \( \sharp(N_\sigma(\tau)) \) is even, one has

\[
|\sigma^0_\varepsilon|_s + \sum_{\epsilon \in A} s(\ell(\epsilon)) + 3\sharp(A) > 0,
\]

2. For any \( e \in E_\Xi, |\sigma^0_\varepsilon|_s - s(\ell(e)) > 0 \) and \( |\sigma^0_\varepsilon|_s > -3/2 \).

Since \( s(\Xi) = -3/2 - \kappa \), it is sufficient to prove the condition \( |\sigma^0_\varepsilon|_s > -\frac{3}{2} \) for any \( \sigma^n_\varepsilon \) and the inequality (3.43) becomes

\[
|\sigma^0_\varepsilon|_s + \left(\frac{3}{2} - \kappa\right) \sharp(A) > 0,
\]

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or,

describe the condition f. For any given parameters

Definition 4.1. in our general context.

The main function of a regularity structure and a model upon that is to provide a coherent smooth functions via Taylor’s formula. Since for any function

framework to approximate random distributions in a way similar polynomials approximate

(4.1) holds only when the test function is supported in the future. Otherwise the right hand side integrand will not be an adapted integrand and we cannot interpret \( \tilde{\Pi}_z \Xi \tau(\psi) \) neither as a Skorohod integral. An explicit formula to describe the law of \( \tilde{\Pi}_z \Xi \tau(\psi) \) is given by

\[
\tilde{f}(z)_i = -z_i, \quad i = 1, 2, \quad \tilde{f}(z)_3 = -(K * \tilde{\xi})(z).
\]

The model \((\tilde{\Pi}, \tilde{\Gamma})\) is also adapted to the action of translation on \( \mathbb{R} \), as a consequence of the Proposition 3.18 on the converging sequence \((\Pi^\epsilon, \Gamma^\epsilon)\).

4 Calculus on regularity structures

In this section we will show how the models \((\Pi^\epsilon, \Gamma^\epsilon)\) and \((\tilde{\Pi}, \tilde{\Gamma})\) can be used to describe respectively \( u_\epsilon \) and \( u \) and, more generally, what kind of analytical operations we can define on a the regularity structure \( \mathcal{T} \).

4.1 Modelled distributions

The main function of a regularity structure and a model upon that is to provide a coherent framework to approximate random distributions in a way similar polynomials approximate smooth functions via Taylor’s formula. Since for any function \( f : \mathbb{R} \to \mathbb{R} \) it is possible to describe the condition \( f \in C^\gamma \) in terms of \( F : \mathbb{R}^2 \to \mathbb{R}^{\gamma} \), the vector of its derivatives, for any fixed model we introduce an equivalent version of this space in our general context.

Definition 4.1. For any given parameters \( \gamma > 0, \eta \in (-2, \gamma) \) and \((\Pi, \Gamma)\) a model upon \((\mathcal{A}, \mathcal{T}, \mathcal{G})\), we define \( \mathcal{D}^{\gamma; \eta} \) as the set of all function \( U : \mathbb{R}^2 \to \mathcal{L}_{\leq \gamma, \mathcal{T}} \) such that for every compact set \( \mathcal{K} \subset \mathbb{R}^2 \), one has

\[
|U|_{\gamma, \eta} := \sup_{z \in \mathcal{K}} \sup_{\alpha < \gamma} \frac{|U(z)|_\alpha}{|t|(\frac{|z|}{|t|})^{\alpha \wedge 0}} + \sup_{(z', z) \in \mathcal{K}(\gamma)} \frac{|U(z) - \Gamma z U(z')|_\alpha}{|z - z'|^{\gamma - \alpha}} < +\infty,
\]
where $\mathcal{H}^{(2)}$ denotes the set of pairs of points $(z, z') \in \mathcal{H}^2$ such that $|z - z'| \leq 1/2 \sqrt{|t| \wedge |t'|}$.

The elements of $D_{\gamma}^\eta$ are called modelled distributions.

**Remark 4.2.** The definition of the set $D_{\gamma}^\eta$ does depend depend in a crucial way on the underlying model $(\Pi, \Gamma)$. To remark this dependency we will adopt for the same set the alternative notation $D_{\gamma}^\eta(\Pi)$. Similarly we recall that the quantities $\|U\|_{\gamma, \eta}$ depend on the compact set $\mathcal{H}$ but we avoid to put the symbol $\mathcal{H}$ in the notation because for our finite time horizon setting, we will henceforth prove the results on a fixed compact set $\mathcal{H} \subset \mathbb{R}^2$ containing $[0, T] \times [0, 1]$. The presence of an extra parameter $\eta$ allows more freedom than the classical $C^\gamma$ spaces. In this way the coordinates of $U$ are to blow at rate $\eta$ near the set $P = \{(t, x) \in \mathbb{R}^2: t = 0\}$ and the condition $\eta > -2$ is put to keep this singularity integrable. By definition of $D_{\gamma}^\eta$, for any value $\gamma \geq \gamma' > 0$ and $U \in D_{\gamma}^\eta$ the projection $\mathcal{Q}_{\gamma} U \in D_{\gamma'}^\eta$.

For any given model the couple $(D_{\gamma}^\eta(\Pi), \| \cdot \|_{\gamma, \eta})$ is clearly a Banach space. Since we will consider modelled distributions belonging to different models, for any couple of models $(\Pi, \Gamma)$ and $(\Pi', \Gamma')$ and modelled distributions $U \in D_{\gamma}^\eta(\Pi)$, $U' \in D_{\gamma}^\eta(\Pi')$ we define the quantity

$$\|U, U'\|_{\gamma, \eta} := \sup_{z, w, \alpha} \frac{|U(z) - U'(z) - \Gamma_{zw} U(w) + \Gamma_{zw}' U'(w)|_{\alpha}}{(|t| \wedge |t'|)^{\frac{\gamma}{2} - 1} |z - z'|^{\gamma - \alpha}} + \sup_{z, \alpha} \frac{|U'(z) - U'(z)|_{\alpha}}{|t|^{\frac{2\alpha - \gamma}{2} \wedge 0}},$$

where the parameters $z, w, \alpha$ belong to the same sets as the quantity (4.1). This function together with the norm $\| \cdot \|_{\mathcal{M}}$ on models endows the fibred space

$$\mathcal{M} \times D_{\gamma}^\eta := \{(\Pi, \Gamma), U): (\Pi, \Gamma) \in \mathcal{M}, U \in D_{\gamma}^\eta(\Pi)\}$$

of a complete metric structure (e.g. we can use the distance $\| \cdot \|_{\gamma, \eta} + \| \cdot \|_{\mathcal{M}}$). Combining the knowledge of a model $(\Pi, \Gamma) \in \mathcal{M}$ and $U \in D_{\gamma}^\eta(\Pi)$, it is possible to define uniquely a distribution such that the coordinates of $U$ has the same role of the derivatives of a function in the Taylor's formula. This association takes the name of reconstruction theorem and it is one of the main theorems in the theory of regularity structures (for its proof see [14, Sec. 3, Sec. 6]).

**Theorem 4.3 (Reconstruction theorem).** For any $(\Pi, \Gamma) \in \mathcal{M}$ there exists a unique map $\mathcal{R}: D_{\gamma}^\eta(\Pi) \to \mathcal{S}'(\mathbb{R}^2)$, called the reconstruction operator, satisfying the following properties:

- **(Generalised Taylor expansion)** for any compact set $\mathcal{H} \subset \mathbb{R}^2$ there exists a constant $C > 0$ such that
  $$\langle (\mathcal{R}U - \Pi_z U(z)) \eta^2 \rangle \leq C \lambda^\gamma$$
  uniformly over $\eta \in B_2$, $\lambda \in (0, 1]$ and $z \in \mathcal{H}$;

- the distribution $\mathcal{R}U \in C^{\alpha_\eta \wedge \eta}$ where $\alpha_\eta := \min\{\alpha \in \mathcal{A}: \mathcal{Q}_\eta U \neq 0\}$ and in case $\alpha \wedge \eta = 0$ we set by convention $C^0$ the space of locally bounded functions;

- **(Local Lipschitz property)** for any fixed $R > 0$ and any couples $(\Pi', \Gamma'), (\Pi, \Gamma) \in \mathcal{M}$, $U \in D_{\gamma}^\eta(\Pi)$, $U' \in D_{\gamma}^\eta(\Pi')$ such that $\|U, U'\|_{\gamma, \eta} + \|((\Pi, \Gamma), (\Pi', \Gamma'))\|_{\mathcal{M}} < R$ and $\alpha_{U'} = \alpha_U = \alpha$, denoting by $\mathcal{R}$ and $\mathcal{R}'$ the respective reconstruction operators, there exists a constant $C > 0$ depending on $R$ such that
  $$\|\mathcal{R}'U' - \mathcal{R}U\|_{\alpha_{U'} \wedge \eta} \leq C \left(\|U, U'\|_{\gamma, \eta} + \|((\Pi, \Gamma), (\Pi', \Gamma'))\|_{\mathcal{M}}\right).$$

**Remark 4.4.** The reconstruction map has in some rare cases an explicit expression. For instance if $\Pi, \tau$ is a continuous function for every $\tau \in T$ (like the model $\mathcal{L}(\Pi)$ or $\mathcal{L}(\Pi')$ for any $\varepsilon > 0$) and $U \in D_{\gamma}^\eta(\Pi)$, then $\mathcal{R}U$ is a continuous function given explicitly by

$$\mathcal{R}(U)(z) = \Pi_z U(z))(z).$$
Introducing the space $\mathcal{D}^{\gamma,\eta}$ of all modelled distributions taking values in $\mathcal{U}$, the identity (4.3) holds also if $\gamma(\Pi, \Gamma)$ is a generic model and $U \in \mathcal{D}^{\gamma,\eta}(\Pi)$, because the elements of the canonical basis of $\mathcal{U}$ have all non negative homogeneity (see for further details in [13, Sec. 3.4]).

Concerning the regularity of $\mathcal{A}U$, the result stated in the Theorem 4.3 is optimal because of the presence the parameter $\eta$ in the definition and the possible explosion of the components of $U$. However if we forget the behaviour at 0 it is also possible to prove $\mathcal{R}U \in \mathcal{C}^{\beta_U}(\mathbb{R}^2 \setminus P)$ where $\beta_U := \min\{a \in \mathbb{R} \setminus \mathbb{N} : Q_\alpha U \neq 0\}$ (see [14, Sec. 6]) and the local Lipschitz property (4.3) holds on the same space $\mathcal{C}^{\beta_U}(\mathbb{R}^2 \setminus P)$. We finally conclude that for any value $\gamma \geq \gamma' > 0$ and $U \in \mathcal{D}^{\gamma,\eta}$ we have the identity $\mathcal{R}\mathcal{Q}_\gamma U = \mathcal{R}U$, therefore to define correctly the distribution $\mathcal{R}U$ is sufficient to fix $\gamma > 0$ such that $\mathcal{Q}_\gamma$ is generated by the set $\{\tau \in T : |\tau| \leq 0\}$. In what follows we will denote by $\mathcal{R}_\varepsilon$ (respectively $\mathcal{R}$) the reconstruction operator associated to the space $\mathcal{D}^{\gamma,\eta}(\Pi^\varepsilon, \Gamma^\varepsilon)$ (resp. $\mathcal{D}^{\gamma,\eta}(\Pi, \Gamma)$).

Remark 4.5. If a model $(\Pi, \Gamma)$ is adapted to the action of the translations (see the equations (3.22)) we and the function $U$ is periodic in the space variable on $\mathbb{R}^2$, then using the general result [14, Prop. 3.38] we obtain also $\mathcal{R}U = \hat{u}$ for some $u \in \mathcal{C}^{\alpha_U,\beta_U}(\mathbb{R} \times T)$, with an abuse of notation we can identify $\mathcal{R}U$ with $u$.

Using the shorthand notation $1_{(0,T)} = 1_{(0,T) \times \mathbb{R}}$, we introduce $1_{(0,T)}\Xi : \mathbb{R}^2 \to \mathcal{T}$, defined for any $z = (t, x) \in \mathbb{R}^2$ by:

$$
(1_{(0,T)}\Xi)(z) := 1_{(0,T)}(z)\Xi = \left\{ \begin{array}{ll} \Xi & \text{if } t \in (0, T), \\ 0 & \text{Otherwise}. \end{array} \right.
$$

For any fixed realisation of $\xi$ and any choice of the parameters $\gamma > 0$ and $-2 < \eta < \gamma$, the definitions of $\hat{\gamma}$ and $\hat{\Gamma}$ implies immediately $1_{(0,T)}\Xi \in \mathcal{D}^{\gamma,\eta}(\Pi^\varepsilon)$ for all $\varepsilon > 0$ and $1_{(0,T)}\Xi \in \mathcal{D}^{\gamma,\eta}(\hat{\Pi})$. The reconstruction of $1_{(0,T)}\Xi$ in both cases can be explicitly calculated.

Proposition 4.6. for any $z \in [0, T] \times T$ one has

$$
\mathcal{R}_\varepsilon(1_{(0,T)}\Xi)(z) = 1_{(0,T) \times T}(z)\xi(\varepsilon)(z), \quad \mathcal{R}(1_{(0,T)}\Xi) = 1_{[0,T]}\xi.
$$

where the second identity holds a.s. as distributions.

Proof. As we recalled in the Remark 4.4 to prove the first part (4.5) we can apply directly the identity (4.4) obtaining the result trivially. Using the Theorem 3.19 related to the convergence of models and the local Lipschitz continuity of the reconstruction map, the distribution $\mathcal{R}_\varepsilon(1_{(0,T)}\Xi)$ converges in probability to $\mathcal{R}(1_{(0,T)}\Xi)$ with respect to the topology of $\mathcal{C}^{-3/2-\kappa}(D \times T)$, where $D$ is an open set containing $[0, T]$. Since $\xi$ converges in probability to $\xi$ with respect to the topology $\mathcal{C}^{-3/2-\kappa}(\mathbb{R} \times T)$ (see [14, Lem 10.2]) and the multiplication for $1_{(0,T)}$ extends in a continuously with the operator $1_{[0,T]}$ introduced in the section 2, $1_{(0,T) \times T}(z)\xi(\varepsilon)$ converges in probability to $1_{[0,T]}\xi$ with respect to the same topology. We conclude by uniqueness of the limit.

Remark 4.7. Denoting by $\mathcal{R}_\varepsilon$ the reconstruction operator with respect to the canonical model $(\Pi^\varepsilon, \Gamma^\varepsilon)$ we have also $1_{(0,T)}\Xi \in \mathcal{D}^{\gamma,\eta}(\Pi^\varepsilon)$ and $\mathcal{R}_\varepsilon 1_{(0,T)}\Xi = \mathcal{R}_\varepsilon 1_{(0,T)}\Xi$, because $1_{\Pi^\varepsilon}1_{\Xi} = 1_{\Pi^\varepsilon}1_{\Xi}$ for any $\varepsilon > 0$. Using the same argument above one has $\mathcal{R}_\varepsilon 1_{(0,T)}\Xi$ converges in probability to $1_{[0,T]}\xi$ as before. Nevertheless the sequence $(\Pi^\varepsilon, \Gamma^\varepsilon)$ does not converge and we cannot interpret $1_{[0,T]}\xi$ as the reconstruction of some modelled distribution, unless we study the model $(\hat{\Pi}, \hat{\Gamma})$.  

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4.2 Operations with the stochastic heat equation

Although modelled distributions look very unusual, the reconstruction theorem associates to them a distribution, which is a classical analytical object. Under this identification it is possible to lift up some operations on the $C^{\gamma}$ spaces directly at the level of the modelled distributions as it was explained in detail in [14, Sec. 4, 5, 6]. Moreover this “lifting” procedure is also continuous with respects to the intrinsic topology of the modelled distributions. In what follows we will briefly recall them to put them in relation with the stochastic heat equation.

**Convolution**

The first operation to define is the convolution with $G$, the heat kernel on $\mathbb{R}$. In other terms, we analyse under which conditions we can associate continuously to any $((\Pi, \Gamma), V) \in \mathcal{M} \times \mathcal{D}^{\gamma, \eta}$ one modelled distribution $\mathcal{R}(V) \in \mathcal{D}^{\tilde{\gamma}, \tilde{\eta}}(\Pi)$ such that

$$\mathcal{R}(\mathcal{P}(V)) = G * \mathcal{R}V. \quad (4.6)$$

For our purposes we are not interested to describe this operation in its full generality. Indeed recalling the formulae (2.4) and (4.5) it is sufficient to define $\mathcal{P}$ only in the case of the modelled distribution $V = 1_{(0, T)} \Xi$ to have an expression of $u_\epsilon$ and $u$, the solution of (1.6) and (1.1), as the reconstruction of some modelled distributions. In this case we can restate the convolution with $G$ with the convolution with two other kernels thanks to this technical lemma (its proof is a direct consequence of [14, Lemma 7.7]).

**Lemma 4.8** (Second decomposition). For any fixed $T > 0$, there exists a function $\tilde{R}: \mathbb{R}^2 \to \mathbb{R}$ such that

- For every distribution $v \in C^\beta(\mathbb{R} \times \mathbb{T})$ with $\beta > -2$ non integer and supported on $[0, +\infty)$ one has
  $$ (G * \tilde{v})(z) = (K * \tilde{v})(z) + (\tilde{R} * \tilde{v})(z), \quad (4.7) $$
  where $K$ is the function introduced in the Lemma [3.11], $z \in (-\infty, T + 1] \times \mathbb{R}$ and $\tilde{v}$ is the periodic lifting of $v$.

- $\tilde{R}$ is smooth, $\tilde{R}(t, x) = 0$ for $t \leq 0$ and it is compactly supported.

Thanks to this decomposition it is sufficient to write $\mathcal{P} = \mathcal{R} + \mathcal{R}$, for some operators $\mathcal{R}$ and $\tilde{R}$ satisfying

$$\mathcal{R}(\mathcal{P}(V)) = K * \mathcal{R}V, \quad \mathcal{R}(\mathcal{R}(V)) = \tilde{R} * \mathcal{R}V. \quad (4.8)$$

Considering the case of $\mathcal{R}$, we remark that for any distribution $v$ supported on positive times the distribution $(\tilde{R} * v)$ will always be a smooth function on $\mathbb{R}^2$ by hypothesis on $\tilde{R}$. Thus for any fixed couple $((\Pi, \Gamma), V) \in \mathcal{M} \times \mathcal{D}^{\gamma, \eta}$ such that $\mathcal{R}V$ is supported on $\mathbb{R}_+ \times \mathbb{R}$ and the model $(\Pi, \Gamma)$ is of the form $\mathcal{L}(\Pi)$ for some admissible map $\Pi$ (see the Remark [3.14]), the operator $\mathcal{R}$ can be easily defined for any $\tilde{\gamma} > 0$ as the lifting of the $\tilde{\gamma}$-th order Taylor polynomial of $(\tilde{R} * \mathcal{R}V)$, that is:

$$\mathcal{R}(V)(z) := \sum_{|k| < \tilde{\gamma}} (\partial^k \tilde{R} * (\mathcal{R}V))(z) \frac{X_k}{k!}. \quad (4.9)$$

From this definition it is straightforward to check that $\mathcal{R}(V) \in \mathcal{D}^{\tilde{\gamma}, \tilde{\eta}}(\Pi)$ for any $\tilde{\gamma} > 0$ and $-2 < \tilde{\eta} < \gamma$ and that it satisfies the second identity of (4.8). Moreover the application $\mathcal{R}: \mathcal{M} \times \mathcal{D}^{\gamma, \eta} \to \mathcal{D}^{\tilde{\gamma}, \tilde{\eta}}$ is also continuous with respect to the topology of $\mathcal{M}$, as a consequence of [14, Lem. 7.3]. This continuity property is a consequence of the compact support of $\tilde{R}$ and it is the
main reason behind the choice of a decomposition of $G$ different from the Lemma 3.11. On the other hand the kernel $K$ is not a smooth and the definition of $\hat{R}$ depends on the model and it is given as a consequence of this general result (for its proof see the “Extension theorem” [13 Thm. 5.14] and the “Multi-level Schauder estimates” [14 Thm. 5.14, Prop 6.16]).

**Proposition 4.9.** For any couple $((\Pi, \Gamma), V) \in \mathcal{M} \times \mathbb{D}^{\gamma \eta}$ where $(\Pi, \Gamma)$ is of the form $\mathcal{S}(\Pi)$ for some admissible map $\Pi$ and $\gamma > 0, -2 < \eta < \gamma$ are not integers, there exists a regularity structure $(\mathcal{A}_2, \mathcal{T}_2, \mathcal{E}_2)$ including $(\mathcal{A}, \mathcal{T}, \mathcal{E})$, a linear map $\tilde{I}: \mathcal{T} \to \mathcal{T}_2$ satisfying $\tilde{I}(\Xi) = \mathcal{I}(\Xi)$ and a model $(\Pi^{\delta}, \Gamma^{\delta})$ extending $(\Pi, \Gamma)$ on $\mathcal{T}_2$ such that, imposing $\gamma = \gamma + 2, \eta = \alpha \nu \wedge \eta + 2$, the applications $\mathcal{N}: \mathcal{M} \times \mathbb{D}^{\gamma \eta} \to \mathbb{D}^{\gamma \eta \delta}_2, J: \mathcal{T} \to \mathbb{D}^{\gamma \eta \delta}_2$ given by

$$
\mathcal{N}(V)(z) := \sum_{|k| < \gamma + 2} \left( (\partial^k K) * (\mathcal{R}V - \Pi_2 V(z)) \right)(z) \frac{X_k}{k!},
$$

$$
J(z) := \sum_{|k| < \tau + 2} \left( (\partial^k K) * \Pi_2 \tau \right)(z) \frac{X_k}{k!},
$$

are well defined and the application

$$
\mathcal{R}(V)(z) := \mathcal{Q}_{\gamma}(\tilde{I}(V))(z) + J(z)V(z) + \mathcal{N}(V)(z),
$$

is a map $\mathcal{R}: \mathcal{M} \times \mathbb{D}^{\gamma \eta} \to \mathbb{D}^{\gamma \eta \delta}(\Pi_2)$ satisfying the first identity of (4.8) without any restriction on the support of $\mathcal{R}(V)$. Moreover $\mathcal{R}$ is also continuous with respect to the topology of $\mathcal{M}$.

Choosing in the definition of $\mathcal{R}$ the same parameters $\gamma$ and $\eta$ of $\hat{R}$, the application $\mathcal{P} = \mathcal{R} + \hat{R}$ is a well defined map $\mathcal{P}: \mathcal{M} \times \mathbb{D}^{\gamma \eta} \to \mathbb{D}^{\gamma \eta}$ which depends continuously on the topology of the models. We will denote by $\mathcal{R}, \mathcal{R}_e, \mathcal{P}_e$ (resp. $\hat{R}, \hat{R}_e, \hat{P}_e$) the operators $\mathcal{R}, \mathcal{R}$ and $\mathcal{P}$ associated to the model $(\Pi^{\delta}, \Gamma^{\delta})$ (resp. $(\Pi, \Gamma)$). Let us calculate $\mathcal{P}_e(1_{(0,T)}\Xi)$ and $\hat{P}_e(1_{(0,T)}\Xi)$ in this case.

**Proposition 4.10.** For any $\gamma > 0$ and every $-3/2 < \eta < \gamma$ non integer, using the shorthand notation $\gamma = \gamma + 2$, the modelled distribution $U_e := \mathcal{P}_e(1_{(0,T)}\Xi)$ and $U := \hat{P}(1_{(0,T)}\Xi)$ belong respectively to $\mathbb{D}^{\gamma,1/2-\kappa}_e(\Pi^e)$ and $\mathbb{D}^{\gamma,1/2-\kappa}_d(\Pi)$ and they are both given explicitly for any $z \in [0, T] \times \mathbb{R}$ by the formulae

$$
U_e(z) = \tilde{u}_e(z)1 + 1_{(0,T)}(z)\hat{1} + \sum_{0 < |k| < \gamma} v^k_e(z) \frac{X_k}{k!},
$$

$$
U(z) = \tilde{u}(z)1 + 1_{(0,T)}(z)\hat{1} + \sum_{0 < |k| < \gamma} v^k(z) \frac{X_k}{k!},
$$

where $v^k_e(z) = (\partial_k \hat{R} * 1_{[0,T]}(z))\hat{\xi}(z)$ and $v^k(z) = (\partial_k \hat{R} * 1_{[0,T]}(z))\hat{\xi}(z)$. Moreover we have for any $z \in [0, T] \times \mathbb{R}$, $\mathcal{R}_e(U_e)(z) = u_e$ and $\hat{R}(U)(z) = u(z)$.

**Proof.** The proposition is a direct consequence of the definition of $\hat{R}$ in the Proposition 4.9. In particular we have immediately $\hat{R}_e(1_{(0,T)}\Xi) \in \mathbb{D}^{\gamma,1/2-\kappa}_e(\Pi^e)$ and $\hat{R}(1_{(0,T)}\Xi) \in \mathbb{D}^{\gamma,1/2-\kappa}_d(\Pi)$ because $\alpha 1_{(0,T)}\Xi = -3/2 - \kappa$. Considering the explicit formula (4.12), which defines $\hat{R}$, by definition of $1_{(0,T)}\Xi$ we have for any $z \in \mathbb{R}^2$

$$
\hat{R}_e(1_{(0,T)}\Xi)(z) = 1_{(0,T)}\hat{\Pi}_e^e(\Xi)(z), \quad \hat{R}(1_{(0,T)}\Xi) = 1_{(0,T)}\hat{\Pi}^e_{d}(\Xi).
$$
Hence the function $\mathcal{N}(1_{(0,T)}\Xi)$ defined in \((4.10)\) is constantly equal to zero in case of $\hat{\mathcal{R}}_e$ and $\hat{\mathcal{R}}$.

Summing up the definition of $\hat{I}$, the definition of $J$ and the identity \((4.5)\), we obtain
\[
\hat{\mathcal{R}}_e(1_{(0,T)}\Xi)(z) = (K \ast 1_{[0,T]}\xi_e)(z)1 + 1_{(0,T)}(z)1, \\
\hat{\mathcal{R}}(1_{(0,T)}\Xi)(z) = (K \ast 1_{[0,T]}\xi)(z)1 + 1_{(0,T)}(z)1.
\]

Applying again the identity \((4.5)\) and the definition of $\mathcal{R}$ in \((4.9)\), the formulae \((4.13)\) \((4.14)\) follows from the distributional identity \((1.7)\) and \((2.6)\). The last identities on the reconstruction follow straightforwardly from the general identity \((1.6)\).

\begin{remark}
For any $\varepsilon > 0$ it is also possible to consider $\mathcal{R}_e$, the convolution operator associated the canonical model $(\Pi^e, \Gamma^e)$. Following the Remark \((4.4)\) related to the modelled distribution $1_{(0,T)}\Xi$ in the case of the canonical model and the proof of the Proposition \((4.10)\) we obtain also that $\mathcal{R}_e(1_{(0,T)}) \in \mathcal{D}^{\gamma,1/2-\kappa}(\Pi^e)$ and the identity $\mathcal{R}_e(1_{(0,T)})(z) = U_e(z)$ for any $z \in \mathbb{R}^2$, implying $\mathcal{R}_eU_e = u_e$. Another consequence of the Proposition \((4.10)\) is then $\tilde{\gamma} > 2$. However in order to reconstruct $u_e$ and $u$ from $U_e$ and $U$, as explained in the Remark \((4.4)\) we can relax this condition by writing $U_e$ and $U$ as elements of $\mathcal{D}^{\gamma',1/2-\kappa}$ for some $0 < \gamma' \leq \gamma$.

Writing $u_e$ and $u$ as the reconstruction of some modelled distribution, we obtain immediately the following convergence.

\begin{proposition}
Let $u_e$ and $u$ be the solutions respectively of the equations \((1.6)\) and \((1.1)\). Then as $\varepsilon \rightarrow 0^+$
\[
\sup_{(t,x) \in [0,T] \times \mathbb{T}} |u_e(t,x) - u(t,x)| \stackrel{P}{\rightarrow} 0.
\]
Moreover $u_e \rightarrow u$ in probability with respect to the topology of $\mathcal{C}^{1/2-\kappa}((0,T) \times \mathbb{T})$.
\end{proposition}

\begin{proof}
Thanks to the Proposition \((4.10)\) the Proposition \((4.9)\) and the local Lipschitz property of the reconstruction map, there exists a continuous map $\Psi : \mathcal{M} \rightarrow \mathcal{C}^0([0,T] \times \mathbb{T})$ such that $u_e = \Psi((\Pi^e, \Gamma^e))$ and $u = \Psi((\tilde{\Pi}, \tilde{\Gamma}))$. Thus the limit \((4.15)\) is a direct consequence of the Theorem \((3.19)\). Restricting $u_e$ and $u$ on $(0,T) \times \mathbb{T}$ and following the Remark \((3.4)\) on the regularity of the reconstruction operator outside the origin, we obtain that $\Psi$ is also a continuous map $\Psi : \mathcal{M} \rightarrow \mathcal{C}^{1/2-\kappa}((0,T) \times \mathbb{T})$, concluding in the same way.
\end{proof}

\begin{composition}
For any $((\Pi, \Gamma), V) \in \mathcal{M} \times \mathcal{D}_y^{\gamma,n}$, the general property of the reconstruction operator ensures us that $\mathcal{R}V$ is a function (see the Remark \((1.4)\)). In particular for any function $h : \mathbb{R} \rightarrow \mathbb{R}$ sufficiently smooth we can find a modelled distribution $H(V)$ such that
\[
\mathcal{R}(H(V)) = h \circ \mathcal{R}V.
\]
We call this operation the lifting of $h$ and we write it as a linear map $H : \mathcal{D}_y^{\gamma,n}(\Pi) \rightarrow \mathcal{D}_y^{\gamma,n}(\Pi)$ (the lifting of a function $f$ will always be denoted in capital letters $F$). For any smooth function $h$ the function $H(V) : \mathbb{R}^2 \rightarrow \mathcal{U}$ is given by
\[
H(V)(z) := \mathcal{Q}_{<\gamma} \sum_{k \geq 0} \frac{h^{(k)}(v(z))}{k!} (V(z) - \mathcal{R}(V)(z)1)^k,
\]
where the power $k$ is done with respect to the product of symbols. Denoting by $C^n_b(\mathbb{R})$ the space of $C^n$ functions with all bounded derivatives up to the $n$-th order, we apply the general theory to deduce a sufficient condition to define the lifting $H(V)$ when $h$ is not necessarily smooth.

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Proposition 4.13. For any \( \gamma > 0 \), \( 0 \leq \eta < \gamma \), recalling the notation \( \beta_V := \min\{ a \in \mathcal{A} \mid \mathbb{N} : \mathcal{Q}_a V \neq 0 \} \) the lifting of \( h \) in (4.17) is well defined and it depends continuously on the topology of \( \mathcal{M} \times \mathcal{D}_y^{\gamma,\eta} \) if \( h \in C^\beta_y(\mathbb{R}) \) where \( \beta \) is the smallest integer \( \beta \geq ((\gamma/\beta_V) \vee 1) + 1 \).

Proof. Following the general results [14, Thm. 6.13], [17, Prop. 3.11], the map \( H \to H(V) \) is local Lipschitz with respect to the metric \( \| \cdot \|_\gamma,\eta + \| \cdot \|_\mathcal{M} \) as long as \( h \) is a \( \lambda \)-Hölder function where \( \lambda \geq ((\gamma/\beta_V) \vee 1) + 1 \). Thus we obtain the thesis.

Remark 4.14. Applying this proposition in case of \( U_z \) and \( U \) we obtain easily \( \beta_{U_z} = \beta_U = |\mathcal{F}(\Xi)| = 1/2 - \kappa \). Thus when we consider for any \( \gamma' > 0 \) the projection on \( \mathcal{D}_y^{\gamma,1/2-\kappa} \) of the modelled distributions \( U_z, U \) introduced in (4.13) and (4.14), the theorem applies for any \( h \in C^\beta_y(\mathbb{R}) \) where \( \beta \) is the smallest integer \( \beta \geq ((2\gamma'/1 - 2\kappa) \vee 1) + 1 \). Since this operation depends only on the algebraic structure, we have also the same result on \( U_z \) interpreted as a modelled distribution with respect to the canonical model \( (\Pi^c, \Gamma^c) \).

Space derivative

Thanks to its definition, the regularity structure \( \mathcal{F} \) allows us to define easily a linear map \( D_x : \mathcal{U} \to \mathcal{F} \) which behaves like a space derivative on abstract symbols. Indeed it is sufficient to characterise \( D_x \) as the unique linear map satisfying

\[
D_x 1 = 0,
D_x X_1 = 0,
D_x X_2 = 1
D_x (\mathcal{F}(\Xi)) = \mathcal{F}_1(\Xi),
\]

\[
D_x (\tau \sigma) = (D_x \tau) \sigma + (D_x \sigma) \tau.
\]

for any couple \( \tau, \sigma \) such that \( \sigma \tau \in U \). Thus by composition we can define for any couple \((\Pi, \Gamma), V) \in \mathcal{M} \times \mathcal{D}_y^{\gamma,\eta}\) the function \( D_x V : \mathbb{R}^2 \to \mathcal{F} \). This abstract operation which is defined at the level of \( \mathcal{U} \) can pass directly at the level of the reconstruction, thanks to the explicit structure of the models we are considering.

Proposition 4.15. For any model \((\Pi, \Gamma)\) of the form \( \mathcal{L}^c(\Pi) \) for some admissible map \( \Pi \), the operator \( D_x \) is an abstract gradient which is compatible with \((\Pi, \Gamma)\), as explained in the definitions [14, Def. 5.25, Def. 5.26]. Moreover for any \( V \in \mathcal{D}_y^{\gamma,\eta} \) such that \( \gamma > 1, 0 \leq \eta < \gamma \) the application \( V \to D_x V \) is an application \( D_x : \mathcal{D}_y^{\gamma,\eta}(\Pi) \to \mathcal{D}^{\gamma-1,\eta-1}(\Pi) \) depending continuously on the topology of \( \mathcal{M} \times \mathcal{D}_y^{\gamma,\eta} \) such that

\[
\mathcal{R}(D_x V) = \partial_x(\mathcal{R} V),
\]

where the equality is interpreted in the sense of distributions.

Proof. By construction of the application \( D_x \) and using the multiplicative property of \( \Gamma_h \) (see the Remark 3.5) it is straightforward to prove recursively for any \( \beta \in \mathcal{A} \) and all \( h \in \mathbb{F}^3 \) the following identities

\[
D_x (\mathcal{Q}_\beta \mathcal{U}) \subset \mathcal{Q}_{\beta - 1} \mathcal{U},
D_x \Gamma_h = \Gamma_h D_x.
\]

Hence \( D_x \) is an abstract gradient operator following [14, Def. 5.25]. Let us fix a model \((\Pi, \Gamma)\) of the form \( \mathcal{L}^c(\Pi) \) for some admissible map \( \Pi \), then the conditions (3.17) (3.18) defining an admissible map imply that \( \Pi : \mathcal{U} \to \mathcal{D}^c(\mathbb{R}^2) \) is well defined and for any \( u \in U \)

\[
\Pi D_x u = \partial_x \Pi u,
\]

where the derivative \( \partial_x \) is interpreted in the sense of distributions. Summing up the properties (4.20) (4.21) and recalling the definition of \( \Pi_z \) in (3.16), for any \( z \in \mathbb{R}^2 \) we obtain for any \( u \in U \)

\[
\Pi_z D_x u = \Pi f(z) D_x u = \Pi D_x f(z) u = (\partial_x \Pi) f(z) = \partial_x (\Pi f(z) u) = \partial_x \Pi_z,
\]

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where the equality \((\partial_z \Pi) \Gamma f(z) = \partial_z (\Pi \Gamma f(z))\) holds because the operator \(\Gamma_h\) acts as a translation. Therefore \(D_x\) is compatible with \((\Pi, \Gamma)\), as explained \([14, \text{Def. 5.26}]\). The remaining part of the statement follows directly from \([14, \text{Prop. 6.15}]\) and the linearity of \(D_x\).

Applying the Proposition 4.19 to \(U_\zeta\) and \(U\), we can write \(\partial_z u_\zeta\) and \(\partial_z u\) as the reconstruction of some modelled distributions.

**Corollary 4.16.** For any \(\gamma' > 1\) let \(U_\zeta, U\) be the projection on \(\mathcal{D}^{\gamma',1/2-\kappa}\) of the modelled distributions introduced in \((4.13)\) and \((4.14)\) for any fixed realisation of \(\xi\). Then the modelled distributions \(D_x U_\zeta\) and \(D_x U\) belong respectively to \(\mathcal{D}^{\gamma'-1,1/2-\kappa}\) and for any \(\varepsilon > 0\) one has

\[
\mathcal{R}_\varepsilon(D_x U) = \hat{\mathcal{R}}_\varepsilon(D_x U) = \partial_z u_\zeta, \quad \hat{\mathcal{R}}(D_x U) = \partial_z u
\]

(4.22)

where the second identity holds on \(\mathcal{C}^{-1/2-\kappa}((0,T) \times \mathbb{T})\).

**Product**

We conclude the list of operation on modelled distributions with the notion of product between modelled distribution. Even if \(\mathcal{F}\) is not an algebra with respect to the juxtaposition product \(m\) introduced in the section 3 we can still consider \(m\) as a well defined bilinear map on some subspaces of \(\mathcal{F}\) such as \(m : \mathcal{M} \times \mathcal{F} \to \mathcal{F}\) or \(m : (\mathcal{V}_f(\Xi) \oplus \mathcal{U}) \times (\mathcal{V}_f(\Xi) \oplus \mathcal{U}) \to \mathcal{F}\). Therefore for any couple of modelled distribution \(V_1, V_2\) and \(\gamma > 0\) we define the function \(V_1 V_2 : \mathbb{R}^2 \to \mathcal{F}\) as

\[
V_1 V_2(z) := \mathcal{G}_\gamma(V_1(z) V_2(z))
\]

(4.23)
as long as the point-wise product on the right hand side of \((4.23)\) is well defined. The behaviour of this operation is described in \([14, \text{Proposition 6.12}]\), which we recall here.

**Proposition 4.17.** Let \((\Pi, \Gamma) \in \mathcal{M}\) and \(V_1 \in \mathcal{D}^{\gamma_1,\eta_1}(\Pi), \ V_2 \in \mathcal{D}^{\gamma_2,\eta_2}(\Pi)\) be a couple of modelled distributions such that the point-wise product is well defined. If the parameters

\[
\gamma = (\gamma_1 + \alpha_2) \land (\gamma_2 + \alpha_1), \quad \eta = (\eta_1 + \eta_2) \land (\eta_1 + \alpha_2) \land (\eta_2 + \alpha_1),
\]

(4.24)

where for \(i = 1, 2\) \(\alpha_i = \alpha_{V_i}\) satisfy the conditions \(\gamma > 0\) and \(-2 < \eta < \gamma\), then the function \(V_1 V_2\) is a well defined element of \(\mathcal{D}^{\gamma,\eta}\). Moreover this operation is continuous with respect to the topology of \(\mathcal{M} \times \mathcal{D}^{\gamma,\eta}\).

**Remark 4.18.** Differently to the other operations we defined before, where we related the reconstruction operator to some classical operations on distribution, the reconstruction \(\mathcal{R}(V_1 V_2)\) cannot be defined directly as an analytical operation between \(\mathcal{R}(V_1)\) and \(\mathcal{R}(V_2)\), even if we interpret the product as the operation between Hölder spaces explained in the section 2 (See in the Proposition 4.19 an example of a well defined product of modelled distributions such that the respective reconstruction of the factors cannot be multiplied classically). However in case of the canonical model \((\Pi^\varepsilon, \Gamma^\varepsilon)\) for any \(\varepsilon > 0\) we can apply the multiplicative property of \(\Pi^\varepsilon\) on symbols and the explicit form of the reconstruction operator in \((4.4)\) to obtain for any couple of \(V_1, V_2 \in \mathcal{D}^{\gamma,\eta}(\Pi^\varepsilon)\) the general identity

\[
\mathcal{R}_\varepsilon(V_1 V_2) = \mathcal{R}_\varepsilon V_1 \mathcal{R}_\varepsilon V_2.
\]

(4.25)

We conclude the section by applying directly the Proposition 4.17 to verify the existence of two specific modelled distribution related to \(U_\zeta\) and \(U\).
Proposition 4.19. Let $U_\varepsilon$, $U$ be the projection on $\mathcal{D}^{\gamma,1/2-\kappa}$ of the modelled distributions introduced in (4.13) and (4.14) for any fixed realisation of $\xi$ and $\gamma' > 0$. Choosing $\gamma' = 3/2 + 2\kappa$ for any $\varphi \in C^0_0(\mathbb{R})$ the modelled distributions $\Phi'(U_\varepsilon)\Xi$, $\Phi''(U_\varepsilon)(D_xU_\varepsilon)^2$ and $\Phi'(U)\Xi$, $\Phi''(U)(D_xU)^2$ are respectively well defined element of $\mathcal{D}^{\kappa,-1-2\kappa}(\hat{\Pi}^\varepsilon)$ for any fixed $\varepsilon > 0$ and $\mathcal{D}^{\kappa,-1-2\kappa}(\hat{\Pi})$. Moreover as $\varepsilon \to 0$ we have

$$\|\Phi'(U_\varepsilon)\Xi, \Phi'(U)\Xi\|_{\kappa,-1-2\kappa} \overset{\mathbb{P}}{\to} 0, \quad \|\Phi''(U_\varepsilon)(D_xU_\varepsilon)^2, \Phi''(U)(D_xU)^2\|_{\kappa,-1-2\kappa} \overset{\mathbb{P}}{\to} 0. \quad (4.26)$$

Proof. Using the Proposition 4.13 and the Remark 4.14 to $\varphi'$ and $\varphi''$, the modelled distributions $\Phi''(U_\varepsilon)$ $\Phi'(U_\varepsilon)$ are well defined if $\varphi'$, $\varphi'' \in C^0_0(\mathbb{R})$ where $\beta$ is the smallest integer such that $\beta \geq ((2\gamma'/1 - 2\kappa) \vee 1) + 1$. Choosing $\gamma' = 3/2 + 2\kappa$ we have $((2\gamma'/1 - 2\kappa) \vee 1) + 1 > 4$ and $\beta = 5$. Thus by hypothesis on $\varphi$ we can lift the functions $\varphi'$, $\varphi''$ to modelled distributions. By construction of $\mathcal{F}$ the point-wise product of $\Phi''(U_\varepsilon)(D_xU_\varepsilon)^2$, $\Phi'(U_\varepsilon)\Xi$, $\Phi''(U)(D_xU)^2$ and $\Phi'(U)\Xi$ are well defined. Thus we recover the thesis by application the Proposition 4.19. In case of $\Phi''(U_\varepsilon)\Xi$ and $\Phi''(U)\Xi$ supposing that $\Xi \in \mathcal{D}^{2\gamma_2,\eta_2}$ for some $\gamma_2 > 0$ and $\eta_2$ sufficiently big the parameters $\gamma$ and $\eta$ in (4.24) become

$$\gamma = \gamma' - 3/2 - \kappa = \kappa, \quad \eta = 1/2 - \kappa - \alpha_{\Xi} = -1 - 2\kappa. \quad (4.27)$$

On the other hand when we consider $\Phi''(U_\varepsilon)(D_xU_\varepsilon)^2$ and $\Phi''(U)(D_xU)^2$ we are doing two products. Choosing $\gamma' > 1$ we start again from the functions $D_xU_\varepsilon$, $D_xU$ given from the Corollary 4.16. Even in this case the parameters $\gamma$ and $\eta$ given by (4.24) for the product $(D_xU_\varepsilon)^2$, $(D_xU)^2$ are given by (4.27) because $\alpha_{D_xU_\varepsilon} = \alpha_{D_xU} = -1/2 - \kappa$. Multiplying then for $\Phi''(U)$ and $\Phi'(U_\varepsilon)$ the parameters $\gamma$ and $\eta$ of this last product become

$$\gamma = (\gamma' - 3/2 - \kappa) \wedge (\gamma' - 1 - 2\kappa) = \kappa, \quad \eta = (-1 - 2\kappa) \wedge (-1/2 - 3\kappa) \quad (4.28)$$

which becomes equal to (4.27) because $\kappa$ was fixed sufficiently small. Thus the modelled distribution are well defined and the convergence property (4.26) is direct consequence of the Theorem 4.19 and the continuous nature of the operations involved.

Remark 4.20. Following the proof of the Proposition (4.19), the choice of the parameter $\gamma'$ and $\varphi$ in the statement could be replaced by a generic value $\gamma' > 3/2 + \kappa$ and a function $\varphi$ with the right number of bounded derivatives. The value $3/2 + 2\kappa$ was simply chosen in order to find the smallest subspace where the modelled distributions $\Phi'(U_\varepsilon)\Xi$, $\Phi''(U_\varepsilon)(D_xU_\varepsilon)^2$, $\Phi'(U)\Xi$, $\Phi''(U)(D_xU)^2$ are well defined.

5 Itô formula

We combine the explicit knowledge of the sequence $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ with the operations on the modelled distributions defined in the section 4 to describe the random distribution $(\partial_t - \partial_{xx})\varphi(u)$ and $\varphi(u)$, when $u$ is the solution of (1.1) and $\varphi$ is a sufficiently smooth function, as explained in the introduction. The resulting formulae will be called differential and integral Itô formula, in accordance to the formal definitions given in the equations (1.4) and (1.5).

5.1 Pathwise Itô formulae

The first type of identities we show are called pathwise differential Itô Formula and pathwise integral Itô Formula. The additional adjective in their denomination is chosen because these identities involve in their terms the reconstruction of some modelled distribution, an object which is defined pathwise.
Theorem 5.1 (Pathwise differential Itô Formula). Let $u$ be the solution of \((1.1)\) and $\varphi \in C_b^7(\mathbb{R})$. Then we have the identity

$$
(\partial_t - \partial_{xx})(\varphi(u)) = \hat{R}(\Phi'(U)\Xi) - \hat{R}(\Phi''(U)(D_x U)^2),
$$

where the equality holds a.s. as elements of $C^{-3/2-\kappa}((0, T) \times \mathbb{T})$.

Proof. The identity \((5.1)\) will be obtained by rearranging the equality \((1.10)\) in terms of modelled distributions and sending $\varepsilon \rightarrow 0$. Recalling the Proposition \(4.10\) and \(4.19\) we write $u_\varepsilon = \hat{R}_\varepsilon u_\varepsilon$ where $U_\varepsilon$ is the projection on $\mathcal{D}^{3/2+2\kappa, 1/2-\kappa}(\mathbb{R}^\varepsilon)$ of the modelled distributions introduced in \(4.13\). The hypothesis on $\varphi$ and the definition of $U_\varepsilon$ allow to lift $\varphi'$ and $\varphi''$ at the level of the modelled distributions and we can rewrite the identity \((1.10)\) as

$$
(\partial_t - \partial_{xx})\varphi(u_\varepsilon) = (\hat{R}_\varepsilon \Phi'(U_\varepsilon))(\hat{R}_\varepsilon \Xi) - (\hat{R}_\varepsilon \Phi''(U_\varepsilon))(\hat{R}_\varepsilon D_x U_\varepsilon)^2. 
$$

On the other hand the same Proposition \(4.19\) implies the modelled distributions $\Phi'(U_\varepsilon)\Xi$ and $\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2$ belong to $\mathcal{D}^{-1/2+2\kappa}(\mathbb{R}^\varepsilon)$. Consequently the functions $\hat{R}_\varepsilon(\Phi'(U_\varepsilon)(D_x U_\varepsilon)^2)$ and $\hat{R}_\varepsilon(\Phi''(U_\varepsilon)\Xi)$ are well defined and they are given explicitly by

$$
\hat{R}_\varepsilon(\Phi'(U_\varepsilon)\Xi)(z) = \Pi_x^\varepsilon (M\Phi'(U_\varepsilon)\Xi(z))(z),
$$

$$
\hat{R}_\varepsilon(\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2)(z) = \Pi_x^\varepsilon (M\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2(z))(z),
$$

for any $z \in (0, T) \times \mathbb{T}$ as a consequence of the equation \((4.4)\) and the Proposition \(3.18\). From these equalities we deduce an explicit relation between the functions on the left hand side of \((5.3)\) and the right hand side of \((5.2)\). In order to lighten the notation we write down $\Phi'(U_\varepsilon)\Xi$ and $\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2$ on the canonical basis of $\mathcal{D}_{<\kappa}\mathcal{T}$ and $(D_x U_\varepsilon)^2$ on the canonical bases $\mathcal{D}_{<1/2+2\kappa}\mathcal{T}$ without referring explicitly to $z \in (0, T) \times \mathbb{R}$ the indicator $1_{[0, T]}$ and the periodic lifting obtaining

$$
\Phi'(U_\varepsilon)\Xi = \varphi'(u_\varepsilon)\mathbbm{1} + \varphi''(u_\varepsilon)\mathbbm{1} + \varphi''(u_\varepsilon)\mathbbm{1},
$$

$$
\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2 = \varphi''(u_\varepsilon)(D_x U_\varepsilon)^2 + \varphi'''(u_\varepsilon)\mathbbm{1} + \varphi''(u_\varepsilon)\mathbbm{1},
$$

Then we apply of the renormalisation map $M_\varepsilon$

$$
M_\varepsilon(\Phi'(U_\varepsilon)\Xi) = \varphi'(u_\varepsilon)\mathbbm{1} + \varphi''(u_\varepsilon)\mathbbm{1} - C_\varepsilon^1 \mathbbm{1} + \varphi''(u_\varepsilon)\mathbbm{1} - C_\varepsilon^1 \mathbbm{1} - 2C_\varepsilon^1 \mathbbm{1} + \varphi''(u_\varepsilon)\mathbbm{1} - C_\varepsilon^1 \mathbbm{1} - 2C_\varepsilon^1 \mathbbm{1} - 3C_\varepsilon^1 \mathbbm{1},
$$

$$
M_\varepsilon((D_x U_\varepsilon)^2) = \mathbbm{1} + 2 \mathbbm{1} + \mathbbm{1} - C_\varepsilon^2 \mathbbm{1} - (D_x U_\varepsilon)^2 - C_\varepsilon^2 \mathbbm{1},
$$

$$
M_\varepsilon(\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2) = \varphi''(u_\varepsilon)((D_x U_\varepsilon)^2 - C_\varepsilon^2 \mathbbm{1}) + 2\varphi'''(u_\varepsilon)\mathbbm{1} + \varphi''(u_\varepsilon)\mathbbm{1} - C_\varepsilon^2 \mathbbm{1} + \varphi''(u_\varepsilon)\mathbbm{1} - C_\varepsilon^2 \mathbbm{1} + \varphi''(u_\varepsilon)\mathbbm{1} - C_\varepsilon^2 \mathbbm{1} + \varphi''(u_\varepsilon)\mathbbm{1} - C_\varepsilon^2 \mathbbm{1}.
$$

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To conclude the calculation we apply the operator $\Pi_\varepsilon \cdot (z)$ on both sides of the above equations. As a consequence of the notion of model we have $\Pi_\varepsilon \tau(z) = 0$ for every $\tau \in T$ of the form $\sigma_1 \sigma_2$ with $|\sigma_1| > 0$. Hence one has the identities

$$\hat{R}_\varepsilon(\Phi(U_\varepsilon)\Xi) = \varphi'(u_\varepsilon)\xi_\varepsilon - \varphi''(u_\varepsilon)C^1_\varepsilon = (\hat{R}_\varepsilon\Phi(U_\varepsilon))(\hat{R}_\varepsilon\Xi) - \varphi''(u_\varepsilon)C^1_\varepsilon, \quad (5.4)$$

$$\hat{R}_\varepsilon(\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2) = \varphi''(u_\varepsilon) \left( \Pi_\varepsilon(D_x U_\varepsilon)^2(z) - C^2_\varepsilon \right) = \hat{R}_\varepsilon \Phi''(U_\varepsilon) \left( \Pi_\varepsilon(D_x U_\varepsilon)^2(z) - C^2_\varepsilon \right). \quad (5.5)$$

Writing $\Pi_\varepsilon(D_x U_\varepsilon)^2(z) = \hat{R}_\varepsilon(D_x U_\varepsilon)^2$, the multiplicative property of $\hat{R}_\varepsilon$ in $[4.22]$ and the identity $[4.22]$ imply that the equality $[5.3]$ becomes

$$\hat{R}_\varepsilon(\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2) = (\hat{R}_\varepsilon \Phi''(U_\varepsilon))(\hat{R}_\varepsilon D_x U_\varepsilon)^2 - \varphi''(u_\varepsilon)C^2_\varepsilon. \quad (5.6)$$

Resuming up the equations $(5.4)$ and $(5.6)$, we obtain the final rearrangement

$$(\partial_t - \partial_{xx})\varphi(u_\varepsilon) = (\hat{R}_\varepsilon(\Phi(U_\varepsilon)\Xi) - \hat{R}_\varepsilon(\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2) + \varphi''(u_\varepsilon) \left( C^1_\varepsilon - C^2_\varepsilon \right). \quad (5.7)$$

Let us now send $\varepsilon \to 0^+$. The left hand side of $(5.7)$ converges in probability to $(\partial_t - \partial_{xx})\varphi(u)$ thanks to the Proposition $[4.12]$ and the fact that the derivative is a continuous operation between Hölder spaces. On the other hand, the local Lipschitz property of the reconstruction operator $\hat{R}$ in $[4.3]$ and the convergence $[4.26]$ imply

$$\hat{R}_\varepsilon(\Phi(U_\varepsilon)\Xi) \xrightarrow{P} \hat{R}(\Phi(U)\Xi), \quad \hat{R}_\varepsilon(\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2) \xrightarrow{P} \hat{R}(\Phi''(U)(D_x U)^2), \quad (5.8)$$

with respect to the topology of $\mathcal{C}^{-(3/2-\kappa)}((0, T) \times T)$. Thus the theorem holds as long as the deterministic sequence $C^1_\varepsilon - C^2_\varepsilon \to 0$, which is the main consequence of the Lemma $[A.2]$.

**Remark 5.2.** Looking at the identities $(5.4)$ and $(5.6)$ separately and the convergence result $(5.8)$, we obtain the existence of two sequences of random variables $X^1_\varepsilon, X^2_\varepsilon \in \mathcal{C}^{-(3/2-\kappa)}((0, T) \times T)$ converging in probability such that

$$\varphi'(u_\varepsilon)\xi_\varepsilon = X^1_\varepsilon + \varphi''(u_\varepsilon)C^1_\varepsilon, \quad \varphi''(u_\varepsilon)(\partial_x u_\varepsilon)^2 = X^2_\varepsilon + \varphi''(u_\varepsilon)C^2_\varepsilon.$$

Since we know from the Lemma $[A.2]$ that the deterministic sequences $C^1_\varepsilon$ and $C^2_\varepsilon$ are both diverging, we obtain easily

$$\|\varphi'(u_\varepsilon)\xi_\varepsilon\|_{\mathcal{C}^{-(3/2-\kappa)}((0, T) \times T)} \xrightarrow{P} +\infty, \quad \|\varphi''(u_\varepsilon)(\partial_x u_\varepsilon)^2\|_{\mathcal{C}^{-(3/2-\kappa)}((0, T) \times T)} \xrightarrow{P} +\infty.$$

Thus we can justify rigorously the calculations done in the introduction.

From the formula $(5.1)$ we can identify $\varphi(u)$ with the solution of the following equation

$$\begin{cases}
\partial_t v - \partial_{xx} v = \hat{R}(\Phi(U)\Xi) - \hat{R}(\Phi''(U)(D_x U)^2) \\
v(0, \cdot) = \varphi(0).
\end{cases}$$

Using the general results in the section $[2]$ we obtain immediately.

**Corollary 5.3** (Pathwise integral Itô Formula). For any $\varphi \in C^2_\varepsilon(\mathbb{R})$ and $(t, x) \in [0, T] \times T$ we have

$$\varphi(u(t, x)) \equiv \varphi(0) + (P * 1_{[0, \beta]} \hat{R}(\Phi(U)\Xi))(t, x) - (P * 1_{[0, \beta]} \hat{R}(\Phi''(U)(D_x U)^2))(t, x). \quad (5.9)$$
5.2 Identification of the differential formula

Thanks to the explicit Gaussian structure involving the definition of $u$ in (1.2), in order to obtain the Theorem 1.1 we can identify the random distributions $\hat{\mathcal{R}}(\Phi'(U)\Xi)$ and $\hat{\mathcal{R}}(\Phi''(U)(D_x U)^2)$ appearing in the formula (5.1) with some explicit classical operations of stochastic calculus (the so called identification theorems of the introduction). In what follows we will denote by $(\mathcal{F}_t)_{t\in\mathbb{R}}$ the natural filtration of $\xi$, that is $\mathcal{F}_t := \sigma\{\{\xi(s) : \psi|_{(t,\infty)\times T} = 0 ; \psi \in L^2(\mathbb{R} \times T)\}\}$. In case of $\hat{\mathcal{R}}(\Phi'(U)\Xi))$ this identification is done by means of a general result contained in [17].

**Proposition 5.4.** Let $\hat{(\Pi, \Gamma)}$ be the BPHZ model and $\varphi \in C^2_b(\mathbb{R})$. Then for any smooth function $\psi : \mathbb{R} \times T \to \mathbb{R}$ with $\text{supp} \, \psi \subset (0, +\infty) \times T$, one has for any $t \in (0, T]$\\
\[
(1_{[0,t]}\hat{\mathcal{R}}(\Phi'(U)\Xi))(\psi) = \int_0^t \int_T \varphi'(u(s,y))\psi(s,y)dW_{s,y}.
\] (5.10)

**Proof.** Thanks to the inclusion of $\mathcal{F}_\Xi$ and $\mathcal{U}$ into the regularity structure $\mathcal{S}_{\Pi, \Gamma}$ and the identification of the BPHZ model $(\Pi, \Gamma)$ with the Itô model, both defined in [17] (see the Remark 3.3 and 3.20), the identity (5.10) is a consequence of [17] Theorem 6.2 applied to the modelled distribution $\Phi'(U) \in \mathcal{D}_{\mathcal{U}}^{\varphi'}$, with $\gamma' = 3/2 + 2\kappa$, $\eta' = 1/2 - \kappa$ given the Proposition 5.4. Let us check that $\Phi'(U)$ satisfy the hypothesis of this theorem. For any $z \in [0, T] \times \mathbb{R}$, $z = (t, x)$ applying the definition (4.17) to the explicit form of $U$ in (4.14) we have $\Phi'(U)(z) = \varphi'((\tilde{u}(z))1 + \varphi''((\tilde{u}(z)))v^{(0,1)}(z)X^{(0,1)} + \frac{\varphi'''((\tilde{u}(z)))}{2}1_{(0,T)}(z) + \frac{\varphi'((\tilde{u}(z)))}{6}1_{(0,T)}(z)$.$

Since $\{u(t, x)\}_{(t, x)}$ and $\{v^{(0,1)}(t, x)\}_{(t, x)}$ are adapted to the filtration $\mathcal{F}_t$, so is the process $\{\Phi'(U)(t, x)\}_{(t, x)}$. Moreover $\mathbb{E}|\Phi'(U)|_{\varphi', \eta'}^p < +\infty$ for any $p > 2$ thanks to the local Lipschitz property of $\Phi'$, the lifting of $\varphi'$, and the bounds $\mathbb{E} \sup_{z \in [0, T] \times T} |u(z)|^p < +\infty, \mathbb{E} \sup_{z \in [0, T] \times T} |v^{(0,1)}(z)|^p < +\infty$, (5.11)

that are true because $u$ and $v^{(0,1)}$ are centred Gaussian processes whose variance is uniformly bounded on $[0, T] \times T$. \hfill \Box

We pass to the identification of $\hat{\mathcal{R}}(\Phi''(U)(D_x U)^2)$. In this case no general result can be applied and following the same procedure of [30], we can identify this random distribution by means of a different approximation of the process $u$ using the heat semigroup on $u$. For any $\varepsilon > 0$ we define the process

\[
u^\varepsilon_t(x) := \int_T P_\varepsilon(x - y)u(t, y)dy.
\] (5.12)

**Lemma 5.5.** For any $\varepsilon > 0$ the process $\nu^\varepsilon$ satisfies the following properties:

- for any $t > 0$ the process $\{\nu^\varepsilon_t(x)\}_{x \in \mathbb{T}}$ has a.s. smooth trajectories, satisfying for any integer $m \geq 0$ the a.s. identity

\[
\partial_x^m \nu^\varepsilon_t(x) = \int_0^t \int_T \partial_x^m P_{t-s}(x-y)dW_{s,y}.
\] (5.13)
for any \( x \in \mathbb{T} \) the process \( \{u_{t}^\varepsilon(x)\}_{t \in [0,T]} \) is the strong solution of the equation

\[
\begin{aligned}
& \begin{cases}
    du_t^\varepsilon(x) = \partial_{xx}(u_t^\varepsilon)(x)dt + dW_t^\varepsilon(x) \\
    u_0^\varepsilon(x) = 0,
\end{cases}
\end{aligned}
\tag{5.14}
\]

where \( W_t^\varepsilon(x) \) is the \((\mathcal{F}_t)\)-martingale

\[
W_t^\varepsilon(x) = \int_0^t \int_{\mathbb{T}} P_\varepsilon(x-y) dW_{s,y}, \quad \langle W_t^\varepsilon(x) \rangle_t = t \int_{\mathbb{T}} P(\varepsilon, x-y)^2dy =: tC_\varepsilon(x).
\]

By sending \( \varepsilon \to 0 \) one has

\[
\sup_{(t,x) \in [0,T] \times \mathbb{T}} |u_t^\varepsilon(t, x) - u(t, x)| \to 0 \quad \text{a.s.} \tag{5.15}
\]

Proof. We start by considering the trajectories of \( x \to u_t^\varepsilon(x) \) for any fixed \( t > 0 \). Since \( u \) is a.s. a continuous function, the regularisation property of the heat kernel \( P \) implies the desired property on its trajectories. Moreover for any integer \( m \geq 0 \) we can pass the derivative under the Lebesgue integral to obtain

\[
\partial_x^m u_t^\varepsilon(x) = \int_{\mathbb{T}} \partial_x^m P_\varepsilon(x-y) u(t,y)dy \quad \text{a.s.}
\]

Using the straightforward bound

\[
\int_{\mathbb{T}} \int_0^1 \int_{\mathbb{T}} (\partial_x^m P_\varepsilon(x-y) P_{1-s}(y-v))^2 ds dv dy < +\infty,
\]

we can obtain the formula \((5.13)\) by writing the stochastic integral in \((5.13)\) as a Wiener integral and applying the stochastic Fubini theorem for Wiener integral, as explained in [24, Thm. 5.13.1]. For any fixed \( x \in \mathbb{T} \) we study the process \( t \to u_t^\varepsilon(x) \). By definition of mild solution of the equation \((1.1)\) \( u \) satisfies the equality \((1.3)\) for any smooth function \( l : \mathbb{T} \to \mathbb{R} \), thus the identity \((5.14)\) follows by simply setting \( l(y) = P_\varepsilon(x-y) \) in \((1.3)\). Finally for any \( (t,x) \in [0,T] \times \mathbb{T} \) the a.s. Hölder continuity of \( u \) in the space and time implies the convergence \((5.15)\), using the classical property of the heat semigroup on continuous functions. \( \square \)

**Theorem 5.6.** Let \((\hat{\Pi}, \hat{\Gamma})\) be the BPHZ model and \( \varphi \in C_b^1(\mathbb{R}) \). Then for any smooth function \( \psi : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \) with \( \text{supp} \ (\psi) \subset (0, +\infty) \times \mathbb{T} \), one has for any \( t \in (0,T] \)

\[
1_{[0,t]}(\mathcal{H} (\Phi''(U)(D_x U)^2) (\psi)) = -\frac{1}{2} \int_0^t \int_{\mathbb{T}} \psi(s,y) \varphi''(u(s,y)) C(s)dy ds \tag{5.16}
\]

\[
+ \int_{[0,t]^2 \times \mathbb{T}^2} \left[ \int_{s_2 \times s_1} \int_{\mathbb{T}} \psi(s,y) \varphi''(u(s,y)) \partial_x P_{s-s_1} (y-y_1) \partial_x P_{s-s_2} (y-y_2) dy ds dy ds \right] dW_{s,y}^2,
\]

where \( C : (0,T) \to \mathbb{R} \) is the deterministic integrable function \( C(s) := \|P_s(\cdot)\|_{L^2(\mathbb{T})}^2 \).

Proof. We prove firstly the result when \( \psi = h \otimes l \) where \( h : [0,t] \to \mathbb{R} \) is a compactly supported smooth function and \( l : \mathbb{T} \to \mathbb{R} \). \( \psi \) is compactly supported up to time \( t \). Therefore we can forget the operator \( 1_{[0,t]} \) on the right hand side of \((5.16)\) and we can apply the Theorem \((5.1)\) and the Proposition \((5.3)\) obtaining

\[
\mathcal{H} (\Phi''(U)(D_x U)^2) (\psi) = \left( -\partial_t (\varphi(u)) + \partial_x^2 (\varphi(u)) + \mathcal{H} (\Phi'(U) \Xi) \right) (\psi) =
\]

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\[ \int_0^t \int_\Omega \left( \varphi(u(s,y))h'(s)(l(y)+\varphi(u(s,y))h(s)l''(y) \right) dy \, ds + \int_0^t \int_\Omega \varphi'(u(s,y))h(s)(l(y))dW_{s,y}. \quad (5.17) \]

Let us recover the right hand side of (5.17) via a different approximation. Using the process \( u^\varepsilon \) defined in (5.12), we can apply the classical Itô formula to the semimartingale \( h(s)\varphi(u^\varepsilon_s(x)) \) and we obtain

\[ h(t)\varphi(u^\varepsilon_t(y)) - h(0)\varphi(u^\varepsilon_0(y)) = \int_0^t h'(s)\varphi(u^\varepsilon_s(y)) \, ds + \int_0^t h(s)\partial_{xx}(u^\varepsilon_s(y))\varphi'(u^\varepsilon_s(y)) \, ds + \int_0^t h(s)\varphi'(u^\varepsilon_s(y))dW_{s,y}^\varepsilon(y) + \frac{1}{2}C_\varepsilon(y) \int_0^t h(s)\varphi''(u^\varepsilon_s(y)) \, ds. \quad (5.18) \]

The left hand side of (5.18) is a.s. equal to zero by hypothesis on \( h \) and we can still apply the formula (1.9) with \( u^\varepsilon \) instead of \( u_e \). Hence we can rewrite the equation (5.18) as

\[ \int_0^t \left( \varphi(u^\varepsilon_s(y))h'(s) + \partial_{xx}(u^\varepsilon_s(y))h(s)l''(y) \right) dy + \int_0^t \varphi'(u^\varepsilon_s(y))h(s)(l(y))dW_{s,y}^\varepsilon(y) = \int_0^t \left[ (\partial_{xx}u^\varepsilon_s)^2(y) - \frac{C_\varepsilon(y)}{2} \right] h(s)\varphi''(u^\varepsilon_s(y)) \, ds. \quad (5.19) \]

By multiplying both sides of (5.19) with \( l \) and integrating by part over \( \Omega \) to transfer the second derivative on \( l \), the equation (5.19) becomes

\[ \int_\Omega \int_0^t \left( \varphi(u^\varepsilon_s(y))h'(s)(l(y) + \varphi(u^\varepsilon_s(y))h(s)l''(y) \right) dy + \int_0^t \varphi'(u^\varepsilon_s(y))h(s)(l(y))dW_{s,y}^\varepsilon(y) dy = \int_0^t \int_\Omega (\partial_{xx}u^\varepsilon_s)^2(y) - \frac{C_\varepsilon(y)}{2} \right) h(s)\varphi''(u^\varepsilon_s(y)) \, ds. \quad (5.20) \]

Writing the integral with respect to \( dW_{s,y}^\varepsilon(x) \) as a Walsh integral, we can apply the boundedness of \( \varphi' \) and \( \varphi \) to apply a stochastic Fubini’s theorem on \( dW_{s,y} \) (see [6, Thm. 65])

\[ \int_\Omega \int_0^t \varphi'(u^\varepsilon_s(y))h(s)(l(y))dW_{s,y}^\varepsilon(y) dy = \int_\Omega \left( \int_0^t \int_\Omega P_\varepsilon(z-y)\varphi'(u^\varepsilon_s(z))h(s)(l(y))dW_{s,z} \right) dy = \int_0^t \int_\Omega \left( \int_\Omega P_\varepsilon(z-y)\varphi'(u^\varepsilon_s(y))h(s)(l(y))dy \right) dW_{s,z}. \quad (5.21) \]

Let us prove that the left hand side of (5.21) converges in \( L^2(\Omega) \) to the right hand side of (5.17). From the uniform convergence (5.15) of \( u^\varepsilon \), it is straightforward to show as \( \varepsilon \to 0 \)

\[ \int_0^t \int_\Omega \varphi(u^\varepsilon_s(y))h'(s)(l(y))dyds \to \int_0^t \int_\Omega \varphi(u_s(y))h'(s)(l(y))dyds \quad \text{a.s.} \]

\[ \int_0^t \int_\Omega \varphi(u^\varepsilon_s(y))h(s)l''(y)dyds \to \int_0^t \int_\Omega \varphi(u_s(y))h(s)l''(y)dyds \quad \text{a.s.} \]

and the convergence holds also in \( L^2(\Omega) \) because these random variables are also uniformly bounded. In case of the stochastic integral in (5.21), the same uniform convergence of \( u^\varepsilon \) in (5.21) implies that

\[ \sup_{(s,z)\in[0,T] \times \Omega} \left| \int_\Omega P_\varepsilon(z-y)\varphi'(u^\varepsilon_s(y))h(s)(l(y))dy - \varphi'(u_s(z))h(s)(l(z) \right| \to 0 \quad \text{a.s.} \]
and bounding these quantity by some constant we obtain by dominated convergence

$$E \left[ \int_0^t \int_T \left( \int_T P_\varepsilon (z - y) \varphi'(u_s^\varepsilon (y)) h(s) l(y) dy - \varphi'(u_s(z)) h(s) l(z) \right)^2 ds dz \right] \to 0.$$  

Hence the proof is complete as long as the right hand side of (5.20) converges in $L^2(\Omega)$ to the right hand side of (5.4). Using the shorthand notations $P^\varepsilon_s(y) = \partial_s P_{\varepsilon+s}(y)$, $P^\varepsilon_s(y) := P_{\varepsilon+s}(y)$, the formula (5.13) on $u^\varepsilon$ when $m = 1$ becomes

$$\partial_z u^\varepsilon_s(y) = \int_0^s \int_T P^\varepsilon_s r(y - z) dW_{r,z}.$$  

Thus writing it as a Wiener integral, we can express $(\partial_z u^\varepsilon_s(y))^2$ using the Wiener chaos decomposition of a product (see [22, Prop. 1.1.2]) obtaining

$$(\partial_z u^\varepsilon_s(y))^2 = I_2 \left( 1_{[0,s]^2 \times T^2} P^\varepsilon_{s-}(y - \cdot) P^\varepsilon_{s-}(y - \cdot) \right) + \int_0^s \int_T [p^\varepsilon_{s-r}(y - z)]^2 dz dr.$$  

Hence, recalling the definition of $C_\varepsilon(y)$, the right hand side of (5.20) equals $A^\varepsilon_1 + A^\varepsilon_2$ where

$$A^\varepsilon_1 := \int_0^t \int_T l(y) h(s) \varphi''(u^\varepsilon_s(y)) \left( \int_0^s \int_T [p^\varepsilon_{s-r}(y - z)]^2 dr dz - \frac{1}{2} \int T P^\varepsilon_s(x - y)^2 dy ds \right) dy ds,$$

$$A^\varepsilon_2 := \int_0^t \int_T l(y) h(s) \varphi''(u^\varepsilon_s(y)) I_2 \left( 1_{[0,s]^2} P^\varepsilon_{s-}(y - \cdot) P^\varepsilon_{s-}(y - \cdot) \right) dy ds.$$  

We treat both terms separately. In case of $A^\varepsilon_1$ a simple integration by parts on $T$ and the smoothness of $P$ outside the origin imply

$$\int_0^s \int_T [p^\varepsilon_{s-r}(y - z)]^2 dr dz = - \int_0^{s+\varepsilon} \int_T P_{s-r}(y - z) \partial_t P_{s-r}(y - z) dr dz =$$

$$= - \int_\varepsilon^{s+\varepsilon} \partial_t \left( \int_T \frac{P_r(y - z)^2}{2} dy \right) dr = \int_T \frac{P_s(y - z)^2}{2} dy - \int_T \frac{P_{s+\varepsilon}(y - z)^2}{2} dy.$$  

Using the invariance by translations of the function $\zeta \to \int_T P(s, \zeta - y)^2 dy$, we can rewrite

$$A^\varepsilon_1 = - \frac{1}{2} \int_0^t \int_T l(y) h(s) \varphi''(u^\varepsilon_s(y)) \left( \int_T P_{s+\varepsilon}(y - z)^2 dz \right) ds dx$$

$$= - \frac{1}{2} \int_0^t \int_T l(y) h(s) \varphi''(u^\varepsilon_s(y)) \| P_{s+\varepsilon}(-\cdot) \|_{L^2(T)}^2 ds dx.$$  

Therefore from the convergence (5.15) we obtain

$$A^\varepsilon_1 \to - \frac{1}{2} \int_0^t \int_T \psi(s, y) \varphi''(u(s, y)) C(s) dy ds \quad a.s.$$  

and the convergence holds also in $L^2(\Omega)$ because the sequence $A^\varepsilon_1$ is uniformly bounded. We pass to the treatment of $A^\varepsilon_2$. In order to identify its limit, we interpret the double Wiener integral $I_2$ as a multiple Skorohod integral of order 2. Then we want to rewrite the quantity

$$l(y) h(s) \varphi''(u^\varepsilon_s(y)) I_2 \left( 1_{[0,s]^2} P^\varepsilon_{s-}(y - \cdot) P^\varepsilon_{s-}(y - \cdot) \right)$$

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using the product formula (2.8) for $\delta^2$ to commute the deterministic integral in $ds \, dy$ with the stochastic integration. Looking at the random variable $U^\varepsilon_s(y) := l(y) h(s) \varphi''(u^\varepsilon_s(y))$, one has $U^\varepsilon_s(y) \in \mathbb{D}^{2,2}$ because $u^\varepsilon \in \mathbb{D}^{2,2}$ belongs to some fixed Wiener chaos and $\varphi''$ has both two derivatives bounded. Applying the classical chain rule formula for the Malliavin derivative (see [22], Proposition 1.2.3) we have
\[
\nabla_{s_1,y} U^\varepsilon_s(y) = l(y) h(s) \varphi''(u^\varepsilon_s(y)) P^\varepsilon_{s-s_1}(y - y_1),
\]
\[
\nabla^2_{s_1,y_1,s_2,y_2} U^\varepsilon_s = l(y) h(s) \varphi''(u^\varepsilon_s(y)) P^\varepsilon_{s-s_1}(y - y_1) P^\varepsilon_{s-s_2}(y - y_2).
\]

For any $\varepsilon > 0$ it is straightforward to check that the hypothesis of the product formula (2.8) are satisfied, therefore we can write
\[
U^\varepsilon_s(y) I_2 \left( 1_{[0,s]^2 \times T^2} p^\varepsilon_{s-\cdot}(y - \cdot) p^\varepsilon_{s-\cdot}(y - \cdot) \right) =
\int_{[0,s]^2} U^\varepsilon_s(y) p^\varepsilon_{s-s_1}(y - y_1) p^\varepsilon_{s-s_2}(y - y_2) dW^2_{s_1,y_1,s_2,y_2}
\]
\[
+ 2 \int_{[0,s] \times T} \left( \int_{[0,s] \times T} \nabla_{s_1,y_1} U^\varepsilon_s(y) p^\varepsilon_{s-s_1}(y - y_1) p^\varepsilon_{s-s_2}(y - y_2) ds_1 dy_1 \right) dW_{s_2,y_2}
\]
\[
+ \int_{[0,s]^2 \times T^2} \nabla^2_{s_1,y_1,s_2,y_2} U^\varepsilon_s(y) p^\varepsilon_{s-s_1}(y - y_1) p^\varepsilon_{s-s_2}(y - y_2) ds_1 dy_1 ds_2 dy_2.
\]

Looking at the deterministic deterministic integrals in the right hand side of (5.23), they are both zero as a consequence of the trivial identity
\[
\int_T P^\varepsilon_{s-r}(y - z) p^\varepsilon_{s-\cdot}(y - z) dz = \int_T \frac{\partial_x (P^\varepsilon_{s-r}(y - z))^2}{2} dz = 0.
\]

Thus we can interchange the product of $U^\varepsilon_s(y)$ with the multiple Skorokod integral of order 2. For any $\varepsilon > 0$ the stochastic integrand inside $dW^2_{s_1,y_1,s_2,y_2}$ is a smooth function in all its variables $s_1,y_1,s_2,y_2,s,y$, then it is square integrable when we integrate it on its referring domain. Therefore we can apply a Fubini type theorem for Skorohod integrals (see e.g. [23]) to finally obtain
\[
A^2_2 = \int_{[0,t]^2 \times T^2} \left( \int_{s_1 \vee s_2}^t \int_T h(s) l(y) \varphi''(u^\varepsilon_{s-s_1}(y - y_1)) p^\varepsilon_{s-s_2}(y - y_2) dy ds \right) dW^2_{s,y}. 
\]

Let us explain the convergence of $A^2_2$ to the multiple Skorohod integral of order two in the final formula (5.16). On one hand we proved that all the previous terms in the formula converge in $L^2(\Omega)$. Then if the sequence of functions
\[
F^\varepsilon(s,y) := \int_{s_1 \vee s_2 + \varepsilon}^{t+\varepsilon} \int_T h(s - \varepsilon) l(y) \varphi''(u^\varepsilon_{s-s_1}(y - y_1)) p^\varepsilon_{s-s_2}(y - y_2) dy ds,
\]
where $s = (s_1, s_2)$ and $y = (y_1, y_2)$, converges in $L^2(\Omega \times [0,t]^2 \times T^2)$ to the function
\[
F(s,y) := \int_T \int_{s_1 \vee s_2}^t h(s) l(y) \varphi''(u_{s-s}(x)) p^0_{s-s_1}(y - y_1) p^0_{s-s_2}(y - y_2) dy ds,
\]
the theorem will follow because the multiple Skorohod integral is a closed operator. From the a.s. convergence of $u^\varepsilon$ in (5.15) it is straightforward to prove that $F^\varepsilon$ converges to $F$ a.s. and a.e. Then we conclude by dominated convergence by proving that $\|F^\varepsilon\|_{L^2}$, the square norm of

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same argument as before to prove strong and writing $\hat{\psi}$

\[ \|F^\varepsilon\|_{L^2}^2 = 2 \int_{\Delta_{2,t}} \int_{T^2} F^\varepsilon(s,y)^2 ds dy = \]
\[ = 2 \int_{\varepsilon}^{t+\varepsilon} ds \int_{\varepsilon}^{t+\varepsilon} dr \int_{T} dx \int_{T} dy \int_{T} dx h(s-\varepsilon)h(r-\varepsilon)l(y)l(x)\varphi''(u_{s-\varepsilon}(x))\varphi''(u_{r-\varepsilon}(y)) \quad (5.25) \]
\[ \times \int_{0}^{s/r} \int_{0}^{t/r} p_{s-r}(x-y_1)p_{r-s}(y-y_2)p_{r-s}(y-y_2)ds dy, \]

where we adopted the shorthand notation $ds = ds_1 ds_2$, $dy = dy_1 dy_2$ and we applied the classical Fubini theorem. Integrating by parts with respect to $y_1$ and $y_2$ and applying the semigroup property of $P$ we obtain

\[ \int_{0}^{s/r} \int_{0}^{t/r} p_{s-r}(x-y_1)p_{r-s}(y-y_2)p_{r-s}(y-y_2)ds dy = \]
\[ = \int_{0}^{s/r} \int_{0}^{t/r} \int_{T} p_{s-r}(x-y_1)p_{r-s}(y-y_2)p_{r-s}(y-y_2)ds dy = \]
\[ \int_{0}^{s/r} \int_{0}^{t/r} \int_{T} \partial_t p_{s-r}(x-y_1)p_{r-s}(y-y_2)\partial_{y_2} p_{r-s}(y-y_2)ds dy \quad (5.26) \]
\[ = \int_{0}^{s/r} \int_{0}^{t/r} \partial_t p_{s-r-2s_2}(y-y_2)\partial_{y_2} p_{s+r-2s_2}(y-y_2)ds dy_1 ds_2 = \frac{1}{8} \left( \frac{P_0(y-x)}{\alpha=s+r} \right)^2 \]

Bounding $\varphi l h$ with a deterministic constant and applying the rough estimate

\[ (P_{s-r}(y-x) - P_{s+r}(y-x))^2 \leq (P_{s-r}(y-x))^2 + (P_{s+r}(y-x))^2, \]

there exists a constant $M > 0$ such that for any $\varepsilon > 0$ one has

\[ \|F^\varepsilon\|_{L^2}^2 \leq M \int_{0}^{T} \int_{T^2} (P^0(y-x))^2 ds dy dx = M \int_{0}^{T} C(s) ds < +\infty. \]

Thereby obtaining the thesis. To conclude the result when $\psi$ is a generic smooth function supported on $(0,t) \times \mathbb{T}$, we apply the formula (5.16) with a sequence of test functions $h_N \otimes l_N: (0,t) \times \mathbb{T} \to \mathbb{R}$ converging to $\psi$ as rapidly decreasing functions. This convergence is very strong and writing $\hat{\varphi}(\Phi''(U)(D_x U)^2)(h_N \otimes l_N)$ as the right hand side of (5.17) we can use the same argument as before to prove

\[ \hat{\varphi}(\Phi''(U)(D_x U)^2)(h_N \otimes l_N) \xrightarrow{L^2(\Omega)} \hat{\varphi}(\Phi''(U)(D_x U)^2)(\psi). \]

and then we can repeat the same argument above to prove that the double Skorohod integral converges to the respective quantity. When $\psi$ is a generic test function defined on $(0, +\infty) \times \mathbb{T}$ we repeat the same calculations with the sequence of tests function $\varphi_N \psi$ where $\varphi_N$ is introduced in (2.3) and it converges a.e. to the indicator function $1_{(0,t) \times \mathbb{T}}$. 

**Remark 5.7.** The Proposition 5.3 and the Theorem 5.6 are formulated when the test function $\psi: \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ is supported on positive times in order to be coherent with the statement of the Theorem 1.1 and [17] Theorem 6.2. However for any generic test function $\psi$, we can apply the identification theorems to the sequence $\varphi_N \psi$ given in (2.3) and the explicit definition of the indicator operator $1_{(0,t)}$ to obtain that the same result holds without any restriction on the support of the test function.
Remark 5.8. The approximating procedure we used to prove this result is very different compared to the proof of [17, Theorem 6.2]. In that case the result is obtained by studying of the approximating sequence which is introduced to describe the reconstruction map in the proof of the Theorem 4.3. That is $1_{[0,t]}\hat{P}(\Phi(U)\Xi)$ is a.s. the limit in the $\mathcal{C}^{-3/2-\kappa}$ topology of the smooth random fields

$$\hat{P}^n(\Phi(U)\Xi)(z) := 1_{[0,t]}(s) \sum_{z \in \Lambda^n([0,t])} \hat{P}(\Phi(U)\Xi(z))(\varphi^n_s(z),$$

where $\Lambda^n([0,T])$ denotes the dyadic grid on $[0,T] \times \mathbb{T}$ of order $n$ and the functions $\varphi^n_s(z)$ are obtained by rescaling of a specific compactly supported function $\varphi: \mathbb{R} \times \mathbb{T} \to \mathbb{R}$. It turns out that when we study the sequence (5.27) in the $L^2(\Omega)$ topology the behaviour of this sequence is completely determined by knowing only the terms $\hat{P}(\Phi(U)\Xi)(\varphi^n_s(z))$ for $t \in U$, thus we can apply the identity (5.34) and conclude. However considering the same approximations of $1_{[0,t]}\hat{P}(\Phi^n(U)(D_x U)^2)$, we do not have the same simplification. In particular, the splitting of the heat kernel $G$ as a sum $K + R$ as explained in the Lemma 5.11 make all the calculations very indirect and it does not allow to use directly some nice properties of $P$ such as the semigroup property. A general methodology to describe the stochastic properties of the reconstruction operator with respect to the BPHZ model is still missing.

Remark 5.9. From the formulae (5.10) and (5.14) we can easily write the periodic lifting of the reconstruction defined above. Indeed for any smooth function $\psi: \mathbb{R}^2 \to \mathbb{R}$ with supp $(\psi) \subset (0, +\infty) \times \mathbb{R}$ we have the identities

$$1_{[0,t]}\hat{P}(\Phi(U)\Xi)(\psi) = \int_0^t \int_{\mathbb{R}} \psi(s, y) \varphi'(\tilde{u}_s(y))d\tilde{W}_{s,y},$$

$$1_{[0,t]}\hat{P}(\Phi(U)(D_x U)^2)(\psi) = -\frac{1}{2} \int_0^t \int_{\mathbb{R}} \psi(s, y) \varphi''(\tilde{u}_s(y))C(s)dy ds + \int_{[0,t]^2} \int_{\mathbb{R}^2} \psi(s, y) \varphi''(\tilde{u}_s(y)) \partial_x G_{s-s_1}(y-y_1) \partial_x G_{s-s_2}(y-y_2)dy_1 dy_2 ds d\tilde{W}_{s,Y}.$$  

And the indicator operator on the right hand side tell us that these identities hold for any smooth function $\psi$ (see the Remark 5.7).

5.3 Identification of the integral formula

We pass to the identification of the terms involving the convolution with $P$. In principle this operation is deterministic and it should be obtained by applying the previous results to the deterministic test function $\psi: \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ given by $\psi(s, y) = P_{t-s}(x - y)$ for some $(t, x) \in [0,T] \in \mathbb{T}$. However the function $\psi$ is not smooth because $\psi$ has a singularity at $(t, x)$. In order to skip this obstacle we recall an additional property of the function $K: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, introduced in the Lemma 3.11 and the Lemma 4.3.

Lemma 5.10. There exists a sequence of smooth positive function $K_n: \mathbb{R}^2 \to \mathbb{R}$, $n \geq 0$ satisfying supp($K_n$) = \{z = (t, x) \in \mathbb{R}^2: ||z|| \leq 2^{-n}, t > 0\} such that for any $z \in \mathbb{R}^2 \setminus \{0\}$

$$K(z) = \sum_{n \geq 0} K_n(z).$$

Moreover for every distribution $u \in \mathcal{C}^\alpha$ with $-2 < \alpha < 0$ non integer one has for any $z \in \mathbb{R}^2$

$$(K \ast u)(z) = \sum_{n \geq 0} (K_n \ast u)(z).$$
Proof. The Kernel $K$ satisfies automatically the property (5.30) by construction, as expressed in [14] Ass. 5.1. Moreover for all test functions $\psi$ and $N \geq 0$ we have the identity

$$\left(\sum_{n=0}^{N} K_n \ast u\right)(\psi) = \sum_{n=0}^{N} (K_n \ast u)(\psi). \tag{5.32}$$

Following [14, Lem. 5.19], the right hand side sequence of (5.32) is a Cauchy sequence with respect to the topology of $C^{\alpha+2}$. Thus by uniqueness of the limit we obtain the equality

$$(K \ast u) = \sum_{n \geq 0} (K_n \ast u), \tag{5.33}$$

as elements of $C^{\alpha+2}$. Since $\alpha + 2 > 0$ (5.33) is an equality between functions, thereby obtaining the thesis.

**Theorem 5.11.** Let $\varphi \in C^1_0(\mathbb{R})$. Then for any $(t, x) \in [0, T] \times \mathbb{T}$ one has

$$P * (1_{[0,t]} \hat{\mathcal{R}}(\Phi'(U)\Xi))(t, x) = \int_{0}^{t} \int_{\mathbb{T}} P_{t-s}(x - y)\varphi'(u_s(y))dW_{s,y}, \tag{5.34}$$

$$P * (1_{[0,t]} \hat{\mathcal{R}}(\Phi''(U)(D_x U)^2))(t, x) = -\frac{1}{2} \int_{0}^{t} \int_{\mathbb{R}} \psi(s,y)\varphi''(\tilde{u}_s(y))C(s)dy\,ds$$

$$+ \int_{[0,t]^2 \times \mathbb{R}} \left[ \int_{s_2 \vee s_1} \int_{\mathbb{R}} \psi(s,y)\varphi''(\tilde{u}_s(y))\partial_x P_{s-s_1}(y-y_1)\partial_x P_{s-s_2}(y-y_2)dy\,ds \right] dW_{s,y}^2. \tag{5.35}$$

**Proof.** We will prove equivalently the identities (5.34) and (5.35) on the periodic lifting. Using the Lemma 3.11 on the distribution $v = \hat{\mathcal{R}}(\Phi''(U)(D_x U)^2)$, $\hat{\mathcal{R}}(\Phi'(U)\Xi)$, for any $(t, x) \in [0, T] \times \mathbb{R}$ we have the general identity

$$(P * 1_{[0,t]}v)(t, x) = (G * 1_{[0,t]}v)(t, x) = (K \ast 1_{[0,t]}v)(t, x) + (R \ast 1_{[0,t]}v)(t, x). \tag{5.36}$$

Since the function $R$ is smooth we can apply directly the formulae (5.28), (5.29). On the other hand the Lemma 5.10 implies

$$K \ast (1_{[0,t]}v)(t, x) = \sum_{n \geq 0} K_n \ast (1_{[0,t]}v)(t, x) = \lim_{N \to +\infty} 1_{[0,t]}v(\eta_N),$$

where $\eta_N: \mathbb{R}^2 \to \mathbb{R}$ is the sequence of compactly smooth functions

$$\eta_N(s, y) := \sum_{n=0}^{N} K_n(t - s, x - y).$$

Thus we will recover the result by applying the Proposition 5.4 and the Theorem 5.6 to the function $\psi = \eta_N$ and studying the convergence of this sequence with respect to the topology of $L^2(\Omega)$. In case $v = \hat{\mathcal{R}}(\Phi'(U)\Xi)$ one has trivially

$$1_{[0,t]}v(\eta_N) = \int_{0}^{t} \int_{\mathbb{R}} \eta_N(s,y)\varphi'(\tilde{u}(s,y))d\tilde{W}_{s,y}.$$

Since $\varphi'$ is bounded, there exists a constant $M > 0$ such that for any $(s,y) \in [0,t] \times \mathbb{R}$ and $N \geq 0$ one has

$$|\eta_N(s,y)\varphi'(\tilde{u}(s,y))| \leq MG_{t-s}(x - y).$$
The function \( (s, y) \to G_{t-s}(x - y) \) is \( L^2 \) integrable on \([0, t] \times \mathbb{R}\) and \( \eta_N(s, y) \) converges a.e. to \( K \). By using the Itô isometry and the dominated convergence we can straightforwardly prove

\[
\frac{1}{[0, t]} \varphi(\eta_N)_{L^2(\Omega)} \rightarrow \int_0^t \int_{\mathbb{R}} K(t - s, x - y) \varphi'(\tilde{u}_s(y))d\tilde{W}_{s,y},
\]

Thereby obtaining the identity \([5.34]\) summing this term with respective containing the function \( R \). Let us pass to the case of \( g_{s-r}(x - y) := \partial_x G_{s-r}(x - y) \) and \( O_t = [0, t] \times \mathbb{R} \). Looking again at the equation \([5.29]\) we have \( \frac{1}{[0, t]} \varphi(\eta_N) = A^1_N + A^2_N \) where

\[
A^1_N = -\frac{1}{2} \int_{O_t} \varphi''(\tilde{u}_s(y))C(s)dy ds,
\]

\[
A^2_N = \int_{O_t \times O_{t'}} \int_{s_2 \lor s_1} \int_{\mathbb{R}} \varphi''(\tilde{u}_s(y))g_{s-s_1}(y - y_1)g_{s-s_2}(y - y_2)dy ds
\]

\[
d\tilde{W}_{s,y}. \]

From the definition of \( \eta_N \) one has a.e. and a.s.

\[
\eta_N(s, y) \varphi''(\tilde{u}_s(y))C(s) \to K(t - s, x - y)\varphi''(\tilde{u}(s, y))C(s).
\]

Moreover there exists a constant \( M > 0 \) such that for every \( N \geq 0 \)

\[
|A^1_N| \leq M \int_{O_t} G_{t-s}(x - y)C(s)dy ds = M \int_0^t C(s) ds < \infty
\]

(the last equality comes by integrating on \( \mathbb{R} \) the density function of a Gaussian random variable). Therefore by dominated convergence we obtain

\[
A^1_N \xrightarrow{L^2(\Omega)} -\frac{1}{2} \int_0^t \int_{\mathbb{R}} K(t - s, x - y)\varphi''(\tilde{u}_s(y))C(s)dy ds.
\]

Let us pass to the convergence of the sequence \( A^2_N \). Introducing the functions \( \{\Phi^N\}_{N \geq 0}, \Phi_K, \{F^N\}_{N \geq 0} \) and \( F_K \) defined by

\[
\Phi^N(s, y, s_1, s_2, y) := \eta_N(s, y)\varphi''(\tilde{u}_s(y))g_{s-s_1}(y - y_1)g_{s-s_2}(y - y_2),
\]

\[
\Phi_K(s, y, s_1, s_2, y) := K(t - s, x - y)\varphi''(\tilde{u}_s(y))g_{s-s_1}(y - y_1)g_{s-s_2}(y - y_2),
\]

\[
F^N(s, y) := \int_{s_2 \lor s_1} \int_{\mathbb{R}} \Phi^N(s, y, s, y)dy ds,
\]

\[
F_K(s, y) := \int_{s_2 \lor s_1} \int_{\mathbb{R}} \Phi_K(s, y, s, y)dy ds,
\]

(a usual \( s = (s_1, s_2) \) and \( y = (y_1, y_2) \) and \( s_1 \lor s_2 \leq t \), we will prove the last convergence

\[
A^2_N \xrightarrow{L^2(\Omega)} \int_{O_t \times O_{t'}} F_K(s, y) d\tilde{W}_{s,y}.
\]

The multiple Skorohod integral is a continuous map from \( H = D^{2,2}(L^2(O_t \times O_t)) \) to \( L^2(\Omega) \) (see the definition of \( D^{2,2}(V) \) and the inequality \([2.4]\) in the section \( 2 \)). Then the result \([5.44]\) will follow by proving that \( F^N \) and \( F_K \) belong to \( H \) and \( F^N \to F_K \) in \( H \). In order to prove these results we calculate the first and second Malliavin derivative of \( F^N \) and \( F_K \) thanks to chain rule formula of the Malliavin derivative (see \([23 \text{ Proposition 1.2.3}]\)). In particular we have

\[
\nabla_{t_1, z_1} F^N(s, y) = \int_{s_2 \lor s_1} \int_{t_1} \int_{\mathbb{R}} \eta_N(s, y)\varphi^{(3)}(\tilde{u}_s(y))G_{s-t_1}(y - z_1)g_{s-s_1}(y - y_1)g_{s-s_2}(y - y_2)dy ds,
\]

\[
\tag{5.38}
\nabla_{t, z} F^N(s, y) = \int_{s_1 \lor s_2} \int_{\mathbb{R}} \eta_N(s, y)\varphi''(\tilde{u}_s(y))G_{s-t}(y - z)g_{s-s_1}(y - y_1)g_{s-s_2}(y - y_2)dy ds,
\]

\[
\tag{5.37}
\int_{O_t \times O_{t'}} F_K(s, y) d\tilde{W}_{s,y}.
\]

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\[ \nabla^2_{t_1z_1t_2z_2} F^N(s,y) = \int_{s_2<s_1<s_1<z_2} \eta_N(s,y) \varphi^{(4)}(\tilde{u}_s(y)) G_{s-t_2}(y-z_2) G_{s-t_1}(y-z_1) \]

and similarly for \( F_K \) by replacing \( \eta_N \) with \( K(t - s, x - y) \). Bounding uniformly \( \eta_N \) and \( K \) by \( G_t(x - y) \) we have trivially \( \|F^N\|^2_K \leq \|F_G\|^2_K \) for every \( N \geq 0 \) where

\[ \Phi_G(s,y,z,s) := G_{t-s}(x-y) \varphi^{(r)}(\tilde{u}_s(y)) g_{s-s_1}(y-y_1) g_{s-s_2}(y-y_2), \]

\[ F_G(s,y) := \int_{s_2<s_1} \int_R \Phi_G(s,y,z,s) ds dy, \]

(The Malliavin derivatives of \( F_G \) are given by (5.38) and (5.39) where \( \eta_N \) is replaced by \( G \).) Then in order to show \( F^N \) and \( F_K \) are elements of \( H \), it is sufficient to prove that the random variables

\[ \alpha_1 := \int_{(O_t)^2} (F_G(s,y))^2 ds dy, \quad \alpha_2 := \int_{(O_t)^3} (\nabla_{t_1z_1} F_G(s,y))^2 ds dy dt dz, \]

\[ \alpha_3 := \int_{(O_t)^4} (\nabla^2_{t_1z_1t_2z_2} F_G(s,y))^2 ds dy dz dt dz, \]

are uniformly bounded. Let us analyse them separately. Looking at the expression of \( \alpha_1 \), it is possible to express the term \( g_{s-s_1}(y-y_1) g_{s-s_2}(y-y_2) \) in the definition of \( F_G \) in the same way as in the equations (5.25) where the kernels \( p \) and \( P \) are replaced by \( g \) and \( G \), thereby obtaining

\[ \alpha_1 = \int_{(O_t)^2} G_{t-r}(x-z) G_{t-s}(x-y) \varphi^{(r)}(\tilde{u}_s(y)) \varphi^{(r)}(\tilde{u}_r(z)) \]

\[ \times (G_{|s-r|}(y-z) - G_{s+r}(y-z))^2 dr ds dy dz. \]

By hypothesis on \( \varphi \) and bounding roughly the difference of a square there exists a constant \( M > 0 \) such that for every \( N \geq 0 \)

\[ \alpha_1 \leq M \int_{O_t \times O_t} G_{t-s}(x-y) G_{t-r}(x-z) (G_{|s-r|}(z-y))^2 + (G_{s+r}(z-y))^2 dr ds dy dz. \] (5.40)

Let us show that the deterministic integral in the right hand side of (5.40) is finite. By definition of \( G \) one has

\[ G_{|s-r|}(z-y)^2 + G_{s+r}(z-y)^2 = \frac{G_{|s-r|}/2(z-y)}{\sqrt{8\pi|s-r|}} + \frac{G_{s+r}/2(z-y)}{\sqrt{8\pi(s+r)}}. \] (5.41)

Plugging this identity in the deterministic integral in the right hand side of (5.40) we can apply the semigroup property of \( G \) and using a rough estimate on the heat kernel there exist some constants \( C, C' > 0 \) such that

\[ \int_{O_t \times O_t} G_{t-s}(x-y) G_{t-r}(x-z) \left( \frac{G_{|s-r|}/2(z-y)}{\sqrt{8\pi|s-r|}} + \frac{G_{s+r}/2(z-y)}{\sqrt{8\pi(s+r)}} \right) dr ds dy dz \]

\[ = \int_0^t \int_{O_t} G_{t-s}(x-y) \left( \frac{G_{t-r+|s-r|}/2(x-y)}{\sqrt{8\pi|s-r|}} + \frac{G_{t+(s-r)}/2(x-y)}{\sqrt{8\pi(s+r)}} \right) ds dr dy \]

\[ \leq C \int_0^t \int_{O_t} \frac{1}{\sqrt{t-s}} \left( \frac{G_{t-r+|s-r|}/2(x-y)}{\sqrt{8\pi|s-r|}} + \frac{G_{t+(s-r)}/2(x-y)}{\sqrt{8\pi(s+r)}} \right) ds dr dy \]

\[ \leq C' \int_0^t \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{|s-r|}} + \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s+r}} ds dr < +\infty. \]
Let us consider the term $\Gamma$ where the function $\Gamma^3_{s,r}(z, y)$ of $\alpha$.

Working exactly in the same way on the other integrals and bounding roughly the functions and writing explicitly the integral on the right hand side there exists a constant $M > D$ (we are again integrating on the density function of a Gaussian random variable). Passing to $\alpha_2$, we rewrite $(\nabla_{t_1 z_1} F(s, y))^2$ as a double integral and applying again the semigroup property of $G$ we have

$$\alpha_2 = 2 \int_{(\Delta_{t_1} \times \mathbb{R}^2) \times O_2} (\nabla_{t_1 z_1} F^N(s, y))^2 ds dy dt_1 dz_1$$

$$= 2 \int_{(O_1)^2} G_{t-s}(x-y)G_{t-r}(x-z)\varphi(\tilde{u}_s(y))\varphi(\tilde{u}_r(z))\Gamma^3_{s,r}(z, y) dr ds dy dz,$$

where the function $\Gamma^3_{s,r}(z, y)$ is defined through the identities

$$\Gamma^3_{s,r}(z, y) := \int_0^{s/r} \int_0^{t_1} \int_0^{s_2} \Gamma^3_{s,r,s,t_1}(z, y) ds dt_1 ds_2 + \int_0^{s/r} \int_0^{t_1} \int_0^{s_2} \Gamma^3_{s,r,s,t_1}(z, y) ds dt_1 ds_2,$$

$$\Gamma^3_{s,r,s,t_1}(z, y) := G_{s+r-2t_1}(y-z)g_{s+r-2s_1}(y-z)g_{s+r-2s_2}(y-z).$$

Let us consider the term $\Gamma^3_{s,r}(z, y)$. Using the elementary estimates

$$|g_t(y)| \leq \sup_{u \in \mathbb{R}} \left( \frac{u}{\sqrt{4\pi t}} \exp^{-u^2} \right) \frac{1}{t^{1/4}}, \quad |G_t(y)| \leq \frac{1}{\sqrt{t}}, \quad (5.42)$$

we can bound each term of the sum defining $\Gamma^3_{s,r}(z, y)$ by some integrable functions depending only on $r$ and $s$. For example in case of the first term in the sum defining $\Gamma^3_{s,r}(z, y)$ there exists a constant $C > 0$ such that

$$\left| \int_0^{s/r} \int_0^{t_1} \int_0^{s_2} \Gamma^3_{s,r,s,t_1}(z, y) ds dt_1 ds_2 \right| \leq C \int_0^{s/r} \int_0^{t_1} \int_0^{s_2} \frac{1}{s + r - 2s_1} ds_2 \frac{1}{s + r - 2s_2} ds_1 \frac{1}{\sqrt{s + r - 2t_1}} dt_1,$$

and writing explicitly the integral on the right hand side there exists a constant $C' > 0$ such that this integral is bounded by

$$C' \left[ \ln(s + r)^2 (\sqrt{s + r} - \sqrt{|s - r|}) + \ln(s + r) \int_{|s - r|}^{s+r} \frac{\ln(y)}{\sqrt{y}} dy + \int_{|s - r|}^{s+r} \frac{\ln(y)^2}{\sqrt{y}} dy \right].$$

Working exactly in the same way on the other integrals and bounding roughly the functions without singularity by some constant depending on the finite time parameter $T$, it is possible to show that there exists a constant $D_T > 0$ depending on $T$ such that

$$|\Gamma^3_{s,r}(z, y)| \leq D_T \left( 1 + \int_{|s - r|}^{s+r} \frac{\ln(y)^2}{\sqrt{y} / \sqrt{1}} dy + \int_{|s - r|}^{s+r} \frac{\ln(y)}{\sqrt{y} / \sqrt{1}} dy \right). \quad (5.43)$$

let us denote the right hand side of (5.43) by $C_T(s, r)$. This function is clearly integrable on $[0, T]^2$, therefore integrating on the remaining components and bounding the derivatives there exists a constant $M > 0$ such that

$$\alpha_2 \leq M \int_{(O_1)^2} G_{t-s}(x-y)G_{t-r}(x-z)C_T(s, r) dr ds dy dz = M \int_{[0, T]^2} C_T(s, r) dr ds < +\infty.$$
Considering $\alpha_3$, we write $(\nabla_{t,z}^2 F(s,y))^2$ using the same technique to express $\alpha_2$ and we obtain

$$\alpha_3 = 8 \int_{(\Delta_{2,t} \times \mathbb{R})^2} (\nabla_{t,z}^2 F_N(s,y))^2 ds \, dy \, dt \, dz$$

$$= 8 \int_{(O)^2} \eta_N(s,y)\eta_N(r,z)\varphi(\tilde{u}_s(y))\varphi(\tilde{u}_r(z)) \Gamma_{s,r}^4(z,y) dr ds dy dz .$$

(the factor 8 comes out because the function $(\nabla_{t,z}^2 F_N(s,y))^2$ is symmetric under the change of coordinates $s_1 \rightarrow s_2$, $t_1 \rightarrow t_2$ and $s \rightarrow t$). The function $\Gamma_{s,r}^4(z,y)$ is defined through the new identities

$$\Gamma_{s,r}^4(z,y) := \int_0^{s \wedge r} \int_0^{t_1} \int_0^{s_2} \Gamma_{s,r,s,t}^4(z,y) ds dt + \int_0^{s \wedge r} \int_0^{t_2} \int_0^{s_2} \Gamma_{s,r,s,t}^4(z,y) dt_1 ds dt_2$$

$$+ \int_0^{s \wedge r} \int_0^{t_2} \int_0^{s_2} \Gamma_{s,r,s,t}^4(z,y) dt_1 ds dt_2 ,$$

$$\Gamma_{s,r,s,t}^4(z,y) := G_{s+r-2t_1}(y-z)G_{s+r-2t_2}(y-z)g_{s+r-2s_1}(y-z)g_{s+r-2s_2}(y-z) .$$

Recalling the elementary estimates in (5.42), we can similarly bound every single integral appearing in $\Gamma_{s,r}^4(z,y)$ in the same way implying there exists an integrable function $B_T(r,s)$ such that $|\Gamma_{s,r}^4(z,y)| \leq B_T(s,t)$. bounding $\varphi^{(4)}$ we conclude there exists a constant $M > 0$ such that

$$\alpha_3 \leq M \int_{(O)^2} G_{t-s}(x-y)G_{t-r}(x-z)B_T(r,s) dr ds dy dz = M \int_{[0,\bar{t}]^2} B_T(s,r) dr ds < +\infty .$$

Thus we conclude that the sequence $\alpha$ is uniformly bounded and $F^N, F_K \in H$. As a matter of fact the previous estimates have a stronger consequence because they imply that the functions $\Phi^N(s,y,s,y)$ and $\Phi_K(s,y,s,y)$ defined above are a.e. on $s,y,s,y$ and a.s. dominated by some integrable functions. Therefore looking at the quantity

$$\|F^N - F_K\|_H^2 = \mathbb{E} \int_{(O)^2} \left( F^N(s,y) - F_K(s,y) \right)^2 ds \, dy$$

$$+ \mathbb{E} \int_{(O)^3} (\nabla_{t,z} F^N(s,y) - \nabla_{t,z} F_K(s,y))^2 ds \, dy \, dt_1 \, dz_1$$

$$+ \mathbb{E} \int_{(O)^4} (\nabla_{t,z}^2 F^N(s,y) - \nabla_{t,z}^2 F_K(s,y))^2 ds \, dy \, dt_1 \, dz_1$$

(5.44)

since we have trivially the a.e. a.s. convergence of the functions

$$\Phi^N(s,y,s,y) \to \Phi_K(s,y,s,y), \quad \nabla_{t,z} F^N(s,y) \to \nabla_{t,z} F_K(s,y),$$

$$\nabla_{t,z}^2 F^N(s,y) \to \nabla_{t,z}^2 F_K(s,y) ,$$

we obtain $\|F^N - F_K\|_H^2 \to 0$ by dominated convergence and the theorem is proved. \Box

**Proof of the Theorem** \[\Box\] For any $\varphi \in C_b^4(\mathbb{R})$ the differential and the integral formula are obtained applying straightforwardly the previous results. Looking at their proofs we realise that the Skorohod and the Wiener integrals and their convolution with $P$, differently from the reconstructions, are well defined if the derivatives of $\varphi$ are bounded up to the order 4. Thus for any fixed $\varphi \in C_b^4(\mathbb{R})$ the formula holds using a classical density argument of the $C^\infty$ functions with all bounded derivatives and repeating the same calculations. \Box
Remark 5.12. Using the integrability of the random field $u$ in (5.11) and looking carefully at the proof of the identity (5.35), we could actually lower down slightly the hypothesis on $\varphi$ in the Theorem 1.1 supposing that $\varphi$ has only the second, the third and the fourth derivative bounded. Indeed the function $\varphi'(u)$ will have linear growth and the right hand side of (5.34) will be always well defined. In this way the same density argument should provide to extend the Theorem 1.1 even in this case. These slight modifications should allow us to obtain a differential and an integral formula even for the random field $u^2$, giving an interesting decomposition of this random field.

Appendix A Renormalisation constants

We calculate the asymptotic behaviour of the renormalisations constants defined in (3.30), (3.31). A preliminary result to analyse them lies on a remarkable identity on $G$, the heat kernel on $\mathbb{R}$, interpreted as a function $G: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$.

Lemma A.1. For any $z \in \mathbb{R}^2 \setminus \{0\}$ one has

\[ 2 \int_{\mathbb{R}^2} G_x(z - \bar{z}) G_x(-\bar{z}) d\bar{z} = G(z) + G(-z) \]  

(A.1)

Proof. We verify this identity by calculating the space-time Fourier transform

\[ f \to \hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i (\xi \cdot z)} f(z) dz \]

of both sides. In order to do that, we recall the elementary identity

\[ \hat{G}(\xi) = \frac{1}{2\pi i \xi_1 + 4\pi^2 \xi_2^2}. \]

Using the notation $\bar{u}(z) = u(-z)$, for any function $u: \mathbb{R}^2 \to \mathbb{R}$, we rewrite the left hand side of (A.1) as $2G_x * \overline{G_x}(z)$. Applying the Fourier transform, we then obtain

\[ 2\overline{G_x} * \overline{G_x}(\xi) = 2\overline{G_x}(\xi) \overline{G_x}(\xi) = 2(2\pi i \xi_2 \hat{G}(\xi))(2\pi i \xi_2 \hat{G}(-\xi)) = \frac{8\pi^2 \xi_2^2}{4\pi^2 \xi_1^2 + (4\pi^2 \xi_2^2)^2}. \]

On the other hand, the same operation on the right hand side of (A.1) implies

\[ \hat{G}(\xi) + \hat{G}(-\xi) = \frac{8\pi^2 \xi_2^2}{(2\pi i \xi_1 + 4\pi^2 \xi_2^2)(-2\pi i \xi_1 + 4\pi^2 \xi_2^2)} = \frac{8\pi^2 \xi_2^2}{4\pi^2 \xi_1^2 + (4\pi^2 \xi_2^2)^2}. \]

By uniqueness of the Fourier transform, we conclude.

Lemma A.2. Let $C_\varepsilon^1$, $C_\varepsilon^2$ be the constants introduced in (3.30), (3.31). Then the following estimates hold as $\varepsilon \to 0^+$

\[ C_\varepsilon^1 = \frac{1}{\varepsilon} \int_{\mathbb{R}^2} G(s,y)\rho^2(s,y) dsdy + o(1); \]  

(A.2)

\[ C_\varepsilon^2 = \frac{1}{\varepsilon} \int_{\mathbb{R}^2} (G_x * \rho)^2(s,y) dsdy + o(1); \]  

(A.3)

\[ C_\varepsilon^1 = C_\varepsilon^2 + o(1). \]  

(A.4)
**Proof.** All the integrals we consider in the proof will be taken on the whole space $\mathbb{R}^2$, therefore we will not write it explicitly. Moreover for any function $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, any integer $m$ and $\epsilon > 0$, we introduce the shorthand notation

$$S^m_{\epsilon}(F)(t, x) := \epsilon^m F(\epsilon^2 t, \epsilon x).$$

Using the definition of $C^1_{\epsilon}$, together with the hypothesis $\rho(-z) = \rho(z)$ one has

$$C^1_{\epsilon} = \int \int K(w)\rho_{\epsilon}(z)\rho_{\epsilon}(z - w) dwdz$$

$$= \int K(w) \int \rho_{\epsilon}(z)\rho_{\epsilon}(w - z) dzdw = \int K(w)(\rho_{\epsilon})^2(w) dw.$$

A simple change of variable tells us that $(\rho_{\epsilon})^2(w) = (\rho^2_{\epsilon})(w)$. Therefore we deduce that

$$C^1_{\epsilon} = \int K(t, x)\epsilon^{-3} \rho^2 \left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}\right) dt dx = \int K(\epsilon^2 t, \epsilon x)\rho^2(t, x) dt dx$$

$$= \frac{1}{\epsilon} \int S^1_{\epsilon}(K)(t, x)\rho^2(t, x) dt dx.$$

Since $S^1_{\epsilon}(K)$ is equal to $S^1_{\epsilon}(G)$ as $\epsilon \to 0^+$ and $G$ satisfies $S^1_{\epsilon}(G) = G$, one has

$$S^1_{\epsilon}(K)(t, x)\rho^2(t, x) \to G(t, x)\rho^2(t, x) \text{ a.e.}$$

Moreover, it is straightforward to show that the function $G\rho^2$ is integrable and it dominates $S^1_{\epsilon}(K)^2$, therefore we obtain

$$\int S^1_{\epsilon}(K)(t, x)\rho^2(t, x) dt dx \to \int G(t, x)\rho^2(t, x) dt dx,$$

by dominated convergence. We recover the identity (A.2), by using the decomposition $G = K + R$, as explained in the Lemma 3.11. Writing again $S^1_{\epsilon}(G) = G$, we obtain

$$\frac{1}{\epsilon} \int [G(t, x) - S^1_{\epsilon}(K)(t, x)] \rho^2(t, x) dt dx = \int S^0_{\epsilon}(R)(t, x)\rho^2(t, x) dt dx.$$

Since the function $R$ is smooth, bounded and $R(0, 0) = 0$ and $\rho^2$ is compactly supported, by sending $\epsilon \to 0^+$ it is straightforward to conclude

$$\int S^0_{\epsilon}(R)(t, x)\rho^2(t, x) dt dx \to R(0, 0) \int \rho^2(t, x) dt dx = 0.$$

Thus the estimate (A.2) holds. Passing to the identity (A.3), we rewrite $C^2_{\epsilon}$ as

$$C^2_{\epsilon} = \int_{\mathbb{R}^2} (K_x * \rho_{\epsilon})^2(z) dz = \frac{1}{\epsilon} \int \epsilon(K_x * \rho_{\epsilon})^2(z) dz.$$

Let us express $\int \epsilon(K_x * \rho_{\epsilon})^2(z) dz$ in terms of the operator $S^2_{\epsilon}$ and of $\rho$. We apply a change of variable to get

$$(K_x * \rho_{\epsilon})(\epsilon^2 t, \epsilon x) = \int K_x(\epsilon^2 t - \epsilon^2 s, \epsilon x - \epsilon y)\rho(s, y) ds dy = (S^0_{\epsilon}(K_x) * \rho)(t, x).$$

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Therefore for any \( z = (t, x) \) we write
\[
(K_x * \rho_\varepsilon)(t, x) = (S^0_\varepsilon(K_x) * \rho) \left( \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon} \right).
\]

Applying again the same change of variable we obtain
\[
\int \varepsilon(K_x * \rho_\varepsilon)^2(z)dz = \varepsilon^4 \int (S^0_\varepsilon(K_x) * \rho)^2(t, x)dtdx = \int (S^2_\varepsilon(K_x) * \rho)^2(t, x)dtdx.
\]

By sending \( \varepsilon \to 0^+ \) the function \( S^2_\varepsilon(K_x) \) becomes equal to \( S^2_\varepsilon(G_x) \) and, using the scaling relation \( S^2_\varepsilon(G_x) = G_z \), for a.e. couple of points \( (t, x), (s, y) \) one has
\[
S^2_\varepsilon(K_x)(t-s, x-y)\rho(s, y) \to G_z(t-s, x-y)\rho(s, y).
\]
The function \( G_z(t-s, x-y)\rho(s, y) \) is clearly integrable in both variables \( (t, x) \) \( (s, y) \). Therefore as a consequence of Fubini’s theorem we get
\[
(S^2_\varepsilon(K_x) * \rho)(t, x) \to (G_x * \rho)(t, x) \quad \text{a.e.}
\]
which implies trivially
\[
(S^2_\varepsilon(K_x) * \rho)^2(t, x) \to (G_x * \rho)^2(t, x) \quad \text{a.e.}
\]
The function \( (G_x * \rho)^2 \) is also integrable and it dominates \( (S^2_\varepsilon(K_x) * \rho)^2 \), then
\[
\int (S^2_\varepsilon(K_x) * \rho)^2(t, x)dtdx \to \int (G_x * \rho)^2(z)dz.
\]
Writing again the decomposition \( G = K + R \) as explained in the Lemma 3.11 and then using Cauchy-Schwarz inequality we deduce
\[
\frac{1}{\varepsilon} \int (G_x * \rho)^2(z)dz - \frac{1}{\varepsilon} \int \varepsilon(K_x * \rho_\varepsilon)^2(z)dz = \frac{1}{\varepsilon} \int (S^2_\varepsilon(G_x) * \rho)^2(z) - (S^2_\varepsilon(K_x) * \rho)^2(z)dz
= \frac{1}{\varepsilon} \int 2(S^2_\varepsilon(K_x) * \rho)(z)(S^2_\varepsilon(R_x) * \rho)(z)dz
= \frac{1}{\varepsilon} \int 2(S^2_\varepsilon(K_x) * \rho)(z)S^2_\varepsilon(R_x) * \rho)(z)dz + \frac{1}{\varepsilon} \int (S^2_\varepsilon(R_x) * \rho)^2(z)dz
\leq 2 \left( \int (S^2_\varepsilon(K_x) * \rho)^2(z)dz \right)^{1/2} \left( \int (S^2_\varepsilon(R_x) * \rho)^2(z)dz \right)^{1/2} + \varepsilon \int (S^2_\varepsilon(R_x) * \rho)^2(z)dz
\]
Now, the function \( R \) is smooth, bounded and it satisfies \( R(0,0) = 0 \). Using the identity \( S^1_\varepsilon(R_x) * \rho = S^0_\varepsilon(R) * \rho_x \), it is straightforward to show that
\[
(S^1_\varepsilon(R_x) * \rho)(z) \to 0 \quad \text{a.e.}
\]
Moreover we can bound \( (S^0_\varepsilon(R) * \rho_x)^2 \) by an integrable function. Thus we obtain
\[
\int (S^1_\varepsilon(R_x) * \rho)^2(z)dz \to 0,
\]
and the identity (A.3) follows. To finally prove the identity (A.4), it is sufficient to show
\[
\int G(s, y)\rho_x^2(s, y)dxdy = \int (G_x * \rho)^2(s, y)dxdy.
\]
Starting from the identity (A.1), we convolve both sides of the equation with the function $\rho^{*2} = \rho * \rho$. Therefore for any $u \in \mathbb{R}^2$ the left hand side of (A.1) becomes

$$2(G_x * G_x) * \rho^{*2}(u) = \int \int \int 2G_x(u - v - w)G_x(-w)\rho(v - x)\rho(x)dx dv dw.$$ 

Choosing the following change of variable

$$\begin{cases} x' = x \\
v' = v - x \\
w' = w + x \end{cases} \quad \begin{cases} x = x' \\
v = v' + x' \\
w = w' - x' \end{cases} \quad dx dv dw = dx' dv' dw'.$$

the integral becomes

$$\int \int \int 2G_x(u - v' - w')G_x(-w' + x')\rho(v')\rho(x')dx' dv' dw'.$$

Using the identity $\rho(x') = \rho(-x')$ the above integral equals

$$2 \int \int \int G_x(u - v' - w')G_x(-w' - x')\rho(v')\rho(x')dx' dv' dw' = 2 \int (G_x * \rho)(u - w')(G_x * \rho)(-w')dw'.$$

On the other hand, the right hand side of (A.1) convolved with $\rho^{*2}$ gives

$$\int G(u - w)\rho^{*2}(w)dw + \int G(w - u)\rho^{*2}(w)dw = \int G(u - w)\rho^{*2}(w)dw + \int G(-u - w)\rho^{*2}(w)dw = (G * \rho^{*2})(u) + (G * \rho^{*2})(-u).$$

Evaluating both sides in $u = 0$, we finally conclude.

\[ \square \]

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