CHAMBER STRUCTURE AND WALLCROSSING IN THE ADHM
THEORY OF CURVES I

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ABSTRACT. ADHM invariants are equivariant virtual invariants of moduli spaces
of twisted cyclic representations of the ADHM quiver in the abelian category
of coherent sheaves of a smooth complex projective curve X. The goal of
the present paper is to present a generalization of this construction employing
a more general stability condition which depends on a real parameter. This
yields a chamber structure in the ADHM theory of curves, residual ADHM
invariants being defined by equivariant virtual integration in each chamber.
Wallcrossing results and applications to local stable pair invariants will be
presented in the second part of this work.

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1. Introduction

1.1. Overview. Let $X$ be a smooth projective variety over $\mathbb{C}$ equipped with a very ample line bundle $\mathcal{O}_X(1)$. Let $M_1, M_2$ be fixed invertible sheaves on $X$ and let $E_\infty$ be a fixed coherent locally free $\mathcal{O}_X$-modules.

**Definition 1.1.** An ADHM sheaf $E$ on $X$ with twisting data $(M_1, M_2)$ and framing data $E_\infty$ is a coherent $\mathcal{O}_X$-module $E$ decorated by morphisms

$$
\Phi : E \otimes_X M_i \to E, \quad \phi : E \otimes_X M_1 \otimes_X M_2 \to E_\infty, \quad \psi : E_\infty \to E
$$

with $i = 1, 2$ satisfying the ADHM relation

$$(1.1) \quad \Phi_1 \circ (\Phi_2 \otimes 1_{M_1}) - \Phi_2 \circ (\Phi_1 \otimes 1_{M_2}) + \psi \circ \phi = 0.$$ 

An ADHM sheaf $E$ will be said to have Hilbert polynomial $P$ if $E$ has Hilbert polynomial $P$. If $X$ is a curve, an ADHM sheaf $E$ will be said to be of type $(r, e) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ if $E$ has rank $r \in \mathbb{Z}_{\geq 0}$ and degree $e \in \mathbb{Z}$.

Motivated by string theoretic questions, the moduli problem for such decorated sheaves has been considered in detail in [20]. Very briefly, the main results of [20] can be summarized as follows

- There exists a stability condition which yields a separated algebraic moduli space of finite type over $\mathbb{C}$ of stable ADHM sheaves with fixed data $X = (X, M_1, M_2, E_\infty)$ and Hilbert polynomial $P$. The stability condition in question requires $E$ to be torsion free, $\psi$ to be nontrivial, and forbids the existence of nontrivial saturated proper subsheaves $0 \subset E' \subset E$ preserved by $\Phi_1, \Phi_2$ and containing the image of $\psi$. This is essentially a cyclicity condition.
- If $X$ is a smooth projective curve over $\mathbb{C}$, the above moduli space is equipped with a torus equivariant perfect tangent-obstruction theory which yields residual theory by virtual integration on the fixed loci. In particular the torus fixed loci have been shown to be proper of finite type over $\mathbb{C}$.
- If $X$ is a smooth projective curve over $\mathbb{C}$ and $E_\infty = \mathcal{O}_X$, the resulting residual theory is equivalent to a local version of the stable pair theory constructed by Pandharipande and Thomas [71]. The proof of this equivalence relies on the relative Beilinson spectral sequence [70] for projective bundles. This was shown in [20] to yield a torus equivariant isomorphism of moduli spaces equipped with perfect tangent obstruction theories between the moduli space of ADHM sheaves on $X$ and a moduli space of coherent systems on a projective bundle over $X$. The projective bundle $\pi : Y \to X$ is determined by the data $(M_1, M_2)$, that is $Y = \text{Proj}(\mathcal{O}_X \oplus M_1 \oplus M_2)$.

The main goal of the present paper is to construct a chamber structure in the ADHM theory of curves employing a more general stability condition which depends on a real stability parameter. Equivariant virtual invariants will be defined in each chamber and a wallcrossing result at the origin will be proven. Before summarizing the main results in more detail note that there at least two main reasons for this construction.

First note the presence of stability parameters is very natural in moduli problems for decorated sheaves [86, 12, 10, 6, 87, 7, 53, 54, 34, 33, 32, 13, 11, 15, 77, 79, 78]. Recall that variations of the stability parameter have played a crucial role in the proof of the Verlinde formula [86, 87], birational transformations of moduli spaces
the quantum cohomology of the Grassmannian \([7]\), as well as the topology of moduli spaces of sheaves of surfaces \([25, 32, 29, 72, 59, 31, 62, 63]\). From this point of view, the stability condition employed in \([20]\) lacks a proper conceptual framework. Indeed in this paper it will be shown that the stability condition in question is in fact an asymptotic form of a more general \(\delta\)-stability condition, where \(\delta \in \mathbb{R}_{>0}\) is a stability parameter.

Moreover, moduli spaces of coherent systems have been recently related to curve counting problems by Pandharipande and Thomas \([71]\) and to generalized Donaldson-Thomas invariants by Joyce and Song \([43]\). Again, the stability condition employed in these cases is an asymptotic form of the stability conditions for coherent systems defined in \([53, 54, 32]\). Therefore a natural question raised in the introduction of \([71]\) is whether such virtual invariants admit deformations corresponding to variations of the stability parameter for coherent systems. This would require in principle the construction of a virtual cycle on the moduli space of stable coherent systems on smooth projective threefolds for arbitrary values of the stability parameter. As observed in the introduction of \([71]\), such a cycle is not expected to exist in general, therefore the question seems to have a negative answer. However this paper will provide a positive answer for local stable pair theory, which was related to the ADHM theory of curves in \([20]\). It will be proven that such deformations of the local invariants exist, but the moduli spaces employed in the construction are not isomorphic to moduli spaces of coherent systems for generic values of the stability parameter (see remark \((1.3)\) below).

A similar chamber structure in the stable pair theory of smooth projective Calabi-Yau threefolds has been previously constructed by Toda in \([88]\), employing variations of stability conditions in the derived category \([6]\). The main results of \([88]\) are a wallcrossing formula \([88, \text{Thm. 1.3}]\) for the generating functional of stable pair invariants defined via the stack theoretic topological Euler character \([41, 39]\) which implies \([88, \text{Cor. 1.4}]\) the BPS rationality conjecture \([71]\) for such invariants.

The chamber structure constructed in the present paper is obtained by variations of stability conditions in a suitable abelian category, properness and virtual smoothness results being proven in each chamber. As a result, invariants are defined by equivariant virtual integration for any generic value of the stability parameter. One of the main applications of this construction are wallcrossing formulas for these invariants which imply the BPS rationality conjecture for local stable pair invariants, as in \([88]\). These results will be derived from Joyce’s Ringel-Hall algebra theory \([38, 39, 40, 42, 41]\), as well as the theory of generalized Donaldson-Thomas invariants of Joyce and Song \([43]\) in the second part of the present work \([17]\).

A different motivation for this paper resides in string theoretic questions concerning wallcrossing phenomena for BPS states in Calabi-Yau compactifications and black holes \([19, 21, 23, 36]\). More specifically, Jafferis and Moore \([36]\) have shown that the spectrum of BPS states in a conifold compactification depends on an extra real parameter in addition to the expected complexified Kähler moduli. This extra parameter is expected to be related to the real stability parameter introduced in this paper although the precise connection will be left for future work.

Finally note that similar constructions for Donaldson-Thomas invariants of toric resolutions of crepant threefold singularities have been carried out in \([38, 73, 59, 61, 60, 48]\) as well as the physics literature \([18, 69]\). These invariants are constructed

\(^{1}\) I thank A. Bayer for pointing this out.
in terms of moduli spaces of quiver representations, using the stability conditions defined in [46].

The present work consists of two parts, the first being focused on existence and construction results as described below. The second part is concerned with wallcrossing formulas and applications to local stable pair and Gopakumar-Vafa invariants.

1.2. Construction results. This paper will consider ADHM sheaves on a smooth projective curve \( X \) over \( \mathbb{C} \) with fixed twisting data \((M_1, M_2)\) and fixed framing data \( E_\infty = \mathcal{O}_X \), in which case the ADHM theory is equivalent to the local Pandharipande-Thomas theory for the total space of the rank two bundle \( M_1^{-1} \oplus M_2^{-1} \). Moreover, \( M_1, M_2 \) will be chosen so that there is an isomorphism \( M_1 \otimes_X M_2 \cong K_X^{-1} \), which implies that the total space in question is \( K \)-trivial. This condition will be needed in the construction of a symmetric perfect obstruction theory for generic values of the stability parameter.

The \( \delta \)-stability condition for ADHM sheaves is introduced in section (2.1), definition (2.1). Note that lemma (2.3) allows us to restrict our treatment to positive values of the stability parameter. Several basic properties as well as preliminary boundedness results are proven in section (2.2).

Sections (3.1), (3.2) reformulate \( \delta \)-stability for ADHM sheaves as a slope stability condition in an abelian subcategory of quiver sheaves. This condition belongs to the class of slope stability conditions considered in [2, 3, 68] in the context of Hitchin-Kobayashi correspondence for quiver sheaves. These results will be needed in the analysis of wallcrossing behavior in later sections. Some homological algebra results are proven in section (3.3).

Section (4) consists of a detailed analysis of variations of the stability parameter \( \delta \in \mathbb{R}_{>0} \). It is proven in section (4.1), lemmas (4.4), (4.7), (4.8), that for fixed rank \( r \in \mathbb{Z}_{\geq 1} \) and fixed degree \( e \in \mathbb{Z} \) there are finitely many critical values \( \delta_i \in \mathbb{R}_{>0} \), \( i = 1, \ldots, N \) dividing the positive real axis into stability chambers. The set of \( \delta \)-stable ADHM sheaves of type \((r,e)\) is constant within each chamber, and strictly semistable objects may exist only if \( \delta \) takes a critical value. Moreover, for sufficiently large \( \delta \), \( \delta \)-stability is equivalent to the stability condition used in [20]. Section (4.2) is focused on the wallcrossing behavior of \( \delta \)-stable objects when \( \delta \) specializes to a critical value. In particular lemmas (4.10), (4.13), respectively (4.14), (4.15) examine the interaction between generic \( \delta \)-stability and semistability at a critical value, respectively the origin.

The moduli problem for \( \delta \)-semistable ADHM sheaves is the main subject of section (5). The first result proven in this section is the following.

**Theorem 1.2.** The groupoid \( \mathcal{M}^\alpha_{\delta}(\mathcal{X}, r, e) \), where \( \mathcal{X} \) denotes the triple \((X, M_1, M_2)\), is an Artin stack of finite type over \( \mathbb{C} \) for any \((r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \) and any \( \delta \in \mathbb{R}_{>0} \). If \( \delta \in \mathbb{R}_{>0} \) is noncritical of type \((r,e)\), the moduli stack \( \mathcal{M}^\alpha_{\delta}(\mathcal{X}, r, e) \) is a separated algebraic space of finite type over \( \mathbb{C} \).

Other moduli stacks needed in the second part of this paper are similarly constructed in section (5.2).

**Remark 1.3.** (i) The GIT construction of moduli spaces of ADHM sheaves has been carried out by Jardim [37] over projective spaces and Schmitt [75] over arbitrary smooth projective varieties. In particular, using the results of [75] one can prove that
the moduli space constructed in theorem (1.3) for noncritical stability parameter is a quasi-projective scheme over \( \mathbb{C} \).

(ii) According to lemma (4.7), the moduli space of \( \delta \)-semistable ADHM sheaves is isomorphic to the moduli space constructed in \([20, \text{Thm. 1.1}]\) for sufficiently large \( \delta \). In particular it is also isomorphic to a quasi-projective moduli space of asymptotically stable coherent systems as proven in \([20, \text{Thm. 1.11}]\). The proof of this theorem relies on a vanishing result \([20, \text{Lemma 2.5}]\) for the morphism \( \phi : E \otimes_X M_1 \otimes_X M_2 \to O_X \) for asymptotically stable ADHM sheaves \( E = (E, \Phi_1, \Phi_2, \phi, \psi) \). However, such a vanishing result no longer necessarily holds when \( \delta \) lies in other chambers on the positive real axis. Therefore in this case, the moduli space of \( \delta \)-semistable ADHM sheaves is not expected to be isomorphic to a moduli space of \( \delta \)-semistable coherent systems.

(iii) Since \( \mathcal{M}_\delta^{ss}(X, r, e) \) is a separated algebraic space of finite type over \( \mathbb{C} \) for noncritical \( \delta \in \mathbb{R}_{>0} \), it has a coarse moduli space \([45]\), which is furthermore isomorphic to \( \mathcal{M}_\delta^{ss}(X, r, e) \). In the following we will implicitly identify \( \mathcal{M}_\delta^{ss}(X, r, e) \) with its coarse moduli space.

(iv) Lemma (2.3) implies that theorem (1.2) also holds for stability parameters \( \delta \in \mathbb{R}_{<0} \).

Next note that there is a natural torus \( T = \mathbb{C}^\times \times \mathbb{C}^\times \) action on the moduli space of \( \delta \)-stable ADHM sheaves, defined by scaling the ADHM data as follows

\[
(t_1, t_2) \times (E, \Phi_1, \Phi_2, \phi, \psi) \to (E, t_1 \Phi_1, t_2 \Phi_2, t_1 t_2 \phi, t_1 t_2 \psi).
\]

Let \( S \simeq \mathbb{C}^\times \subset T \) denote the antidiagonal torus defined by the embedding \( t \to (t^{-1}, t) \). The next result proven in section (5.4) establishes the existence of a torus equivariant perfect tangent-obstruction theory for noncritical values of the stability parameter.

**Theorem 1.4.** Let \( \delta \in \mathbb{R}_{>0} \) be a noncritical stability parameter of type \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\). Then the algebraic moduli space \( \mathcal{M}_\delta^{ss}(X, r, e) \) has a \( T \)-equivariant as well as \( S \)-equivariant perfect tangent-obstruction theory. Moreover, the perfect tangent-obstruction theory of \( \mathcal{M}_\delta^{ss}(X, r, e) \) is \( S \)-equivariant symmetric.

Using this structure, one would like to define a residual \( \delta \)-ADHM theory of curves by virtual integration on the torus fixed loci. In particular, a properness result for the fixed loci is needed. Properness of the \( T \)-fixed loci can be proven by analogy with \([20, \text{Prop. 3.15}]\). However, properness of the \( S \)-fixed loci is more difficult, and will require a more elaborate proof, which is presented in section (5.3). The proof relies on an inductive argument involving in particular a generalization of Langton’s properness results \([51]\) to flat families of ADHM sheaves. This requires several preliminary results concerning the structure of fixed loci in the moduli space of semistable Higgs sheaves – section (6.1), and framed Hitchin pairs – section (6.2), as well as elementary modifications of flat families of ADHM sheaves – section (6.4). Similar extensions of Langton’s proof to moduli of decorated sheaves have been carried out in \([65]\) for Hitchin pairs and \([80]\) for \( D \)-modules. The final result proven in section (6.3) is

**Theorem 1.5.** Let \( \delta \in \mathbb{R}_{>0} \) be a noncritical stability parameter of type \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\). Then the fixed locus \( \mathcal{M}_\delta^{ss}(X, r, e)^S \) is a proper algebraic space of finite type over \( \mathbb{C} \).

An immediate corollary of theorem (1.5) is
Corollary 1.6. Under the conditions of theorem (1.5), the fixed locus $\mathcal{M}_\delta^{ss}(X, r, e)^T$ is a proper algebraic space of finite type over $\mathbb{C}$.

Theorems (1.4) and (1.5) and corollary (1.6) imply the following

Corollary 1.7. Suppose the conditions of theorem (1.5) are satisfied. Then the following statements hold.

(i) The $T$-fixed locus $\mathcal{M}_\delta^{ss}(X, r, e)^T$ is equipped with an induced perfect tangent-obstruction theory, which yields a virtual fundamental cycle $[\mathcal{M}_\delta^{ss}(X, r, e)^T] \in A_\bullet(\mathcal{M}_\delta^{ss}(X, r, e)^T)$ and a virtual normal bundle $N_{\mathcal{M}_\delta^{ss}(X, r, e)^T/\mathcal{M}_\delta^{ss}(X, r, e)^T} \in K_0^\text{equiv}($\mathcal{M}_\delta^{ss}(X, r, e)^T)$ in the equivariant $K$-theory of locally free sheaves of the fixed locus.

(ii) Analogous results hold for the $S$-fixed loci. Moreover, in this case, the induced perfect tangent-obstruction theory is symmetric and the resulting virtual fundamental cycle is a 0-cycle.

Using corollary (1.7), the residual ADHM theory of the data $\mathcal{X} = (X, M_1, M_2)$ is defined as follows

Definition 1.8. Let $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, and $\delta \in \mathbb{R}_{>0}$ be a noncritical stability parameter. Then the $T$-equivariant $\delta$-ADHM invariant of type $(r, e)$ is defined by

$$A_T^{\delta}(r, e) = \int_{[\mathcal{M}_\delta^{ss}(X, r, e)^T]} e_T^{-1}(N_{\mathcal{M}_\delta^{ss}(X, r, e)^T/\mathcal{M}_\delta^{ss}(X, r, e)}).$$

The $S$-equivariant $\delta$-ADHM invariant of type $(r, e)$ $A_S^{\delta}(r, e)$ is defined analogously.

Finally, the goal of section (7) is to prove that the deformation results obtained by Joyce and Song for coherent sheaves on Calabi-Yau threefolds also hold for locally free ADHM sheaves on curves. More precisely, theorems (7.1), (7.2) are entirely analogous to theorems [43, Thm. 5.2], [43, Thm. 5.3] and follow by the same type of deformation theory arguments applied to locally free ADHM sheaves. Moreover, it is proven that the Behrend function identities proven in [43, Thm. 5.9] also hold with appropriate modifications in the present case. Using these statements, the main results in the theory of generalized Donaldson-Thomas invariants of [43] apply to ADHM sheaf invariants, allowing one to derive explicit wallcrossing formulas. This will be presented in detail in the second part of this paper.

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Notation and Conventions. Throughout this paper, we will denote by $\mathcal{S}$ the category of schemes of finite type over $\mathbb{C}$. For any such schemes $X, S$ we set $X_S = S \times X$ and $X_s = \text{Spec}(k(s)) \times_S X$ for any point $s \in S$, where $k(s)$ is the residual field of $s$. Let also $\pi_S : X_S \to S, \pi_X : X_S \to X$ denote the canonical projections. We will also set $F_S = \pi_X^* F$ for any $O_X$-module $F$. Given a morphism
f : S' → S, we will denote by f_X = f × 1_X : X_{S'} → X_S. Any morphism f : S' → S, yields a commutative diagram of the form

\[
\begin{array}{c}
X_{S'} \xrightarrow{π_X} X \\
\downarrow f_X \quad \quad \quad \downarrow 1_X \\
X_S \xrightarrow{π_X} X
\end{array}
\]

Then for any \( \mathcal{O}_X \)-module \( F \) there is a canonical isomorphism \( F_{S'} \cong f_X^* F_S \) which will be implicit in the following.

2. \( δ \)-stability for ADHM sheaves

2.1. Definition and basic properties. Let \( X \) be a smooth projective curve over an infinite field \( K \) of characteristic 0 equipped with a very ample line bundle \( \mathcal{O}_X(1) \). Let \( M_1, M_2 \) be fixed line bundles on \( X \) so that \( M_1 \otimes_X M_2 \simeq K_X^{-1} \). Such an isomorphism will be fixed throughout this paper, and we will also set \( M = M_1 \otimes_X M_2 \). As stated in the beginning of section 1, we will consider ADHM sheaves \( 2.1 \) \( \mathcal{E} = (E, Φ_1, Φ_2, φ, ψ) \) on \( X \) with twisting data \( (M_1, M_2) \) and framing data \( E_∞ = \mathcal{O}_X \). Given a coherent locally free \( \mathcal{O}_X \)-module \( E \) we will denote by \( r(E), d(E), μ(E) \) the rank, degree, respectively slope of \( E \). An ADHM sheaf \( \mathcal{E} \) will be called locally free if \( E \) is a coherent locally free \( \mathcal{O}_X \)-module. If \( \mathcal{E} \) is a locally free ADHM sheaf the pair \( (r(E), d(E)) \) will be called the type of \( \mathcal{E} \).

Let \( δ \in \mathbb{R} \setminus \{0\} \) be a stability parameter. For a nontrivial locally free ADHM sheaf \( \mathcal{E} \) we define the \( δ \)-slope of \( \mathcal{E} \) to be

\[
μ_δ(\mathcal{E}) = μ(E) + \frac{δ}{r(E)}.
\]

Moreover a subsheaf \( 0 \subset E' \subset E \) will be called \( \Phi \)-invariant if \( Φ_i(E' \otimes M_i) \subset E' \) for \( i = 1, 2 \).

Definition 2.1. Let \( δ \in \mathbb{R} \setminus \{0\} \) be a stability parameter. A nontrivial locally free ADHM sheaf \( \mathcal{E} = (E, Φ_1, Φ_2, φ, ψ) \) is \( δ \)-(semi)stable if the following conditions are satisfied

(i) \( ψ \) is nontrivial if \( δ > 0 \) respectively \( φ \) is nontrivial if \( δ < 0 \).

(ii) Any \( \Phi \)-invariant nontrivial proper saturated subsheaf \( 0 \subset E' \subset E \) so that \( \text{Im}(ψ) \subset E' \) satisfies

\[
μ(E') + \frac{δ}{r(E')} (≤) μ_δ(\mathcal{E}).
\]

(iii) Any \( \Phi \)-invariant nontrivial proper saturated subsheaf \( 0 \subset E' \subset E \) so that \( E' \otimes_X M \subset \text{Ker}(φ) \) satisfies

\[
μ(E') (≤) μ_δ(\mathcal{E}).
\]

We also define (semi)stability at \( δ = 0 \) as follows.

Definition 2.2. A nontrivial locally free ADHM sheaf \( \mathcal{E} = (E, Φ_{1,2}, φ, ψ) \) be on \( X \) is \( 0 \)-(semi)stable if any \( \Phi \)-invariant proper nontrivial saturated subsheaf \( 0 \subset E' \subset E \) so that \( \text{Im}(ψ) \subset E' \) or \( E' \otimes_X M \subset \text{Ker}(φ) \) satisfies

\[
μ(E') (≤) μ(E).
\]
Let $\mathcal{E} = (E, \Phi_1, \Phi_2, \phi, \psi)$ be a locally free ADHM sheaf on $X$ of type $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Then the data
\begin{equation}
\tilde{E} = E^\vee \otimes_X M^{-1}
\end{equation}
\begin{equation}
\tilde{\Phi}_i = (\Phi_i^\vee \otimes 1_{M_i}) \otimes 1_{M^{-1}} : \tilde{E} \otimes M_i \to \tilde{E}
\end{equation}
\begin{equation}
\tilde{\phi} = \psi^\vee \otimes 1_{M^{-1}} : \tilde{E} \otimes_X M \to \mathcal{O}_X
\end{equation}
\begin{equation}
\tilde{\psi} = \phi^\vee : \mathcal{O}_X \to \tilde{E}
\end{equation}
with $i = 1, 2$, determines a locally free ADHM sheaf $\tilde{\mathcal{E}}$ of type
\begin{equation}
\tilde{r} = r \quad \tilde{e} = -e + 2r(g - 1)
\end{equation}
where $g \in \mathbb{Z}_{\geq 0}$ is the genus of $X$. $\tilde{\mathcal{E}}$ will be called the dual of $\mathcal{E}$ in the following. Note that $\tilde{\mathcal{E}}$ has the same twisting and framing data as $\mathcal{E}$.

**Lemma 2.3.** Let $\delta \in \mathbb{R}_{>0}$ be a positive stability parameter and let $\mathcal{E}$ be a locally free ADHM sheaf on $X$. Then $\mathcal{E}$ is $\delta$-(semi)stable if and only if $\tilde{\mathcal{E}}$ is $(-\delta)$-(semi)stable.

**Proof.** Straightforward verification of stability conditions.

Using lemma [23] it suffices to consider only positive stability parameters $\delta \in \mathbb{R}_{>0}$ from this point on.

### 2.2. Boundedness results

This subsection consists of several boundedness results for semistable ADHM sheaves required at later stages in the paper. As in the previous subsection, $X$ is a smooth projective curve over an infinite field $K$ of characteristic 0 and $M_1, M_2$ are fixed line bundles on $X$ equipped with an isomorphism $M_1 \otimes_X M_2 \simeq K_{X}^{-1}$.

**Lemma 2.4.** Let $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ be a fixed type. Then the set of isomorphism classes of locally free sheaves $E$ of type $(r, e)$ on $X$ so that $(E, \Phi_{1,2}, \phi, \psi)$ is a $\delta$-semistable ADHM sheaf for some $\delta \in \mathbb{R}_{\geq 0}$ and some morphisms $(\Phi_{1,2}, \phi, \psi)$ is bounded.

**Proof.** This is the generalization of [20, Prop. 2.7] to $\delta$-stable ADHM sheaves. The proof will be based on Maruyama’s theorem [57]

**Theorem 2.5.** (Maruyama). A family of sheaves $E$ with fixed Hilbert polynomial $P$ and $\mu_{\text{max}}(E) \leq C$ for a fixed constant $C$ is bounded.

and the following standard technical lemma (used for example in the proof of [63 Prop. 3.2], [76 Thm. 3.1]).

**Lemma 2.6.** Let $E$ be a torsion-free sheaf of rank $r \geq 2$ on $X$. Suppose $E$ is not semistable, and let
\begin{equation}
0 = HN_0(E) \subset HN_1(E) \subset \ldots \subset HN_h(E) = E
\end{equation}
be the Harder-Narasimhan filtration of $E$. Then
\begin{equation}
\mu(HN_1(E)) + (r - 1)\mu(E/HN_{h-1}(E)) \leq r\mu(E).
\end{equation}

According to Maruyama’s theorem it suffices to prove that there exists a constant $C$ independent on $\delta$ so that $\mu_{\text{max}}(E) \leq C$ for all $\delta$-semistable ADHM sheaves $\mathcal{E} = (E, \Phi_{1,2}, \phi, \psi)$ on $X$ of type $(r, e)$ and all $\delta \in \mathbb{R}_{\geq 0}$. 
If $E$ is semistable $\mu_{\text{max}}(E) = \mu(E)$ is clearly bounded. In particular this is the case if $r = 1$, hence we will assume $r \geq 2$ from now on in this proof. Suppose $E$ is not semistable of rank $r \geq 2$, and let
\begin{equation}
0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_h(E) = E, \tag{2.6}
\end{equation}
h $\geq 2$, be the Harder-Narasimhan filtration of $E$. Note that the successive quotients are locally free and semistable. In particular this implies that $h \leq r$.

Suppose first $E$ is $\delta$-semistable for some $\delta \in \mathbb{R}_{>0}$ or that $E$ is 0-semistable and $\psi : E_{\infty} \to E$ is not identically zero. Let $j_\psi \in \{1, \ldots, h-1\}$ be the index determined by
\begin{align*}
\text{Im}(\psi) &\not\subset HN_{j}(E), \quad \text{for } j \leq j_\psi, \\
\text{Im}(\psi) &\subset HN_{j}(E), \quad \text{for } j \geq j_\psi + 1.
\end{align*}
Note that the morphism $\overline{\psi} : E_{\infty} \to E/HN_{j_\psi}(E)$ is nontrivial and [35] Lemma 1.3.3 implies that
\begin{equation}
\mu_{\text{min}}(E_{\infty}) \leq \mu_{\text{max}}(E/HN_{j_\psi}(E)). \tag{2.7}
\end{equation}
By construction (see the proof of [35] Thm. 1.3.4) we have $\mu_{\text{max}}(E/HN_{j_\psi}(E)) = \mu(HN_{j_\psi+1}(E)/HN_{j_\psi}(E))$, therefore we obtain
\begin{equation}
\mu_{\text{min}}(E_{\infty}) \leq \mu(HN_{j_\psi+1}(E)/HN_{j_\psi}(E)). \tag{2.8}
\end{equation}
Moreover, if $j_\psi = h - 1$, inequality (2.7) specializes to
\begin{equation}
\mu_{\text{min}}(E_{\infty}) \leq \mu(E/HN_{h-1}(E)). \tag{2.9}
\end{equation}
which yields
\begin{equation}
- \mu(E/HN_{h-1}(E)) \leq -\mu_{\text{min}}(E_{\infty}). \tag{2.10}
\end{equation}
If $j_\psi < h - 1$, we claim $HN_j(E)$ cannot be $\Phi$-invariant, for any $j_\psi \leq j \leq h - 1$. According to the $\delta$-stability condition, if $HN_j(E)$ is $\Phi$-invariant, $i = 1, 2$ for some $j_\psi \leq j \leq h - 1$, it follows that
\begin{equation}
\mu(HN_j(E)) + \frac{\delta j}{r(HN_j(E))} \leq \mu(E) + \frac{\delta j}{r}. \tag{2.11}
\end{equation}
Since $\delta \geq 0$ and $r(HN_j(E)) < r$, this yields a contradiction because $\mu(HN_j(E)) > \mu(E)$ for all $j = 1, \ldots, h - 1$. This proves the claim.

Therefore for each $j \in \{j_\psi + 1, \ldots, h - 1\}$ there exists $i_j \in \{1, 2\}$ so that $\Phi_{i_j}(HN_j(E) \otimes M_{i_j}) \not\subset HN_j(E)$. Then the same argument as in the proof of [35] Prop 3.2 and [76] Thm. 3.1] shows that
\begin{equation}
\mu(HN_j(E)/HN_{j-1}(E)) \leq \mu(HN_{j+1}(E)/HN_j(E)) - \deg(M_{i_j}). \tag{2.12}
\end{equation}
Summing inequalities (2.12) from $j = j_\psi + 1$ to $j = h - 1$ we obtain
\begin{equation}
\mu(HN_{j_\psi+1}(E)/HN_{j_\psi}(E)) \leq \mu(E/HN_{h-1}(E)) - \sum_{j=j_\psi+1}^{h-1} \deg(M_{i_j}). \tag{2.13}
\end{equation}
Then using inequality (2.7) we obtain
\begin{equation}
\mu_{\text{min}}(E_{\infty}) + \sum_{j=j_\psi+1}^{h-1} \deg(M_{i_j}) \leq \mu(E/HN_{h-1}(E)). \tag{2.14}
\end{equation}
which further yields
\[ \mu_{\text{min}}(E_\infty) - (h - 1)\max\{|\deg(M_1)|, |\deg(M_2)|\} \leq \mu(E/HN_{h-1}(E)). \]

Since we have established above that \( h \leq r \), we finally obtain
\[ (2.10) \quad - \mu(E/HN_{h-1}(E)) \leq -\mu_{\text{min}}(E_\infty) + (r - 1)\max\{|\deg(M_1)|, |\deg(M_2)|\} \]

Taking into account (2.8), (2.10), inequality (2.5) implies the existence of the required upper bound for \( \mu(HN_1(E)) = \mu_{\text{max}}(E) \).

Next suppose \( E \) is 0-semistable and \( \psi \) is identically zero. If \( \phi \) is also trivial, definition (2.2) implies that the data \((E, \Phi_1, \Phi_2)\) is a semistable Higgs sheaf on \( X \) as defined in (A.1). If \( \phi \) is not trivial, definition (2.2) implies that the data \((E, \Phi_1, \Phi_2, \phi)\) is a framed Hitchin pair on \( X \) as defined in (A.3). Then boundedness has been proven in [65, 81, 76, 79]. This concludes the proof of lemma (2.4).

Lemma (2.4) implies the following corollary by a standard argument.

**Corollary 2.7.** The set of isomorphism classes of ADHM sheaves of type \((r, e)\) on \( X \) which are \( \delta \)-semistable for at least one value \( \delta \in \mathbb{R}_{>0} \) is bounded.

The proof of lemma (2.4) also implies the following.

**Corollary 2.8.** Let \( r \in \mathbb{Z}_{\geq 1} \) be a fixed rank. Then there exists \( c \in \mathbb{Z} \) depending only on \( r \) so that for any \( e \in \mathbb{Z} \), \( e < c \) and any \( \delta \in \mathbb{R}_{>0} \) there are no \( \delta \)-semistable ADHM sheaves of type \((r, e)\) on \( X \).

**Proof.** Suppose \( E = (E, \Phi_1, \Phi_2, \psi) \) is a \( \delta \)-semistable ADHM sheaf of type \((r, e)\). If \( E \) is semistable, it follows that \( \mu(E) \geq \mu_{\text{min}}(E_\infty) \) since there must exist a nontrivial morphism \( \psi : E_\infty \to E \). If \( E \) is not semistable, equation (2.10) implies that
\[ \mu(E) > \mu_{\text{min}}(E) = \mu(E/HN_{h-1}(E)) \geq \mu_{\text{min}}(E_\infty) - (r - 1)\max\{|\deg(M_1)|, |\deg(M_2)|\}. \]

This proves the claim.

\[ \square \]

### 3. Categorical Formulation

In this subsection we reformulate \( \delta \)-stability of ADHM sheaves as a stability condition in a certain abelian category. This will enable us to study the behavior of the moduli spaces \( \mathcal{M}_s^\delta(X, r, e) \) as a function of the stability parameter \( \delta \). Similar constructions have been carried out for example in [32, 47] for moduli spaces of coherent systems. The abelian category in question will be constructed as a subcategory of an abelian category of twisted quiver sheaves on \( X \). Then the slope stability condition defined below belongs to the class of stability conditions studied in [2, 8].

#### 3.1. An abelian subcategory of ADHM quiver sheaves.

Let \( X \) be a scheme over an infinite field \( K \) of characteristic 0 and let \( (M_1, M_2) \) be fixed invertible sheaves on \( X \). Set \( M = M_1 \otimes_X M_2 \) and suppose there is a fixed isomorphism \( M \simeq K_X^{-1} \) as in the previous section. ADHM quiver sheaves on \( X \) are \((M_1, M_2)\)-twisted representations of an ADHM quiver in the abelian category of quasi-coherent sheaves on \( X \). Such objects have been considered in the literature in [2, 8, 28, 34, 78]. Basically ADHM quiver sheaves are defined by the same data as ADHM sheaves except that the framing data \( E_\infty \) is not fixed. More precisely we have
Definition 3.1. (i) An ADHM quiver sheaf on $X$ is a collection $\mathcal{E} = (E, E_\infty, \Phi, \phi, \psi)$ where $E, E_\infty$ are coherent $O_X$-modules and

$$\Phi_i : E \otimes_X M_i \to E, \quad \phi : E \otimes_X M \to E_\infty, \quad \psi : E_\infty \to E$$

are morphisms of $O_X$-modules satisfying the ADHM relation.

(ii) A morphism between two ADHM quiver sheaves $\mathcal{E}, \mathcal{E}'$ is a pair $(\xi, \xi_\infty)$ of morphisms of $O_X$-modules

$$\xi : E \to E', \quad \xi_\infty : E_\infty \to E'_\infty$$

satisfying the obvious compatibility conditions with the data $(\Phi, \phi, \psi), (\Phi', \phi', \psi')$.

(iii) An ADHM quiver sheaf $\mathcal{E}$ will be called locally free if $E, E_\infty$ are locally free $O_X$-modules.

According to [2, 3, 28, 84, 78], ADHM quiver sheaves on $X$ form an abelian category $\mathcal{Q}_X$. Now define a subcategory $\mathcal{C}_X$ of the abelian category $\mathcal{Q}_X$ as follows

- The objects of $\mathcal{C}_X$ are coherent ADHM quiver sheaves with $E_\infty = V \otimes O_X$ where $V$ is a finite dimensional vector space over $K$ (possibly trivial). We will denote by $v \in \mathbb{Z}_{\geq 0}$ the dimension of $V$.
- Given two objects $\mathcal{E}, \mathcal{E}'$ of $\mathcal{C}_X$ a morphism from $\mathcal{E}$ to $\mathcal{E}'$ is a morphism $(\xi, \xi_\infty)$ of ADHM quiver sheaves so that

$$\xi_\infty = f \otimes 1_{O_X}$$

where $f : V \to V'$ is a $K$-linear map.

Remark 3.2. (i) Note that there is an obvious one-to-one correspondence between ADHM sheaves on $X$ and objects of $O_X$ with $v = 1$. Similarly, there is a obvious one-to-one correspondence between Higgs sheaves, as defined in [A, 1], and objects of $\mathcal{C}_X$ with $v = 0$.

(ii) Given two ADHM sheaves $\mathcal{E}, \mathcal{E}'$ on $X$ a morphism $(\xi, \lambda) : \mathcal{E} \to \mathcal{E}'$ in $\mathcal{C}_X$ is a morphism of ADHM sheaves as defined in [20, Def 2.1] if and only if $\lambda = 1$. This distinction is important in the construction of moduli spaces. Note also that if $(\xi, \lambda) : \mathcal{E} \to \mathcal{E}'$ is an morphism in $\mathcal{C}_X$, with $\lambda \neq 0$, then $(\lambda^{-1} \xi, 1) : \mathcal{E} \to \mathcal{E}'$ is a morphism of ADHM sheaves.

Lemma 3.3. The category $\mathcal{C}_X$ is abelian.

Proof. Reduces to a straightforward verification that $\mathcal{C}_X$ contains all kernel, images and cokernels of its morphisms, as well as direct sums. This is easily done by standard diagram chasing. We will omit the details.

Remark 3.4. Note that if $X$ is connected and proper over $K$ the subcategory $\mathcal{C}_X$ is a full subcategory of the abelian category of ADHM quiver sheaves $\mathcal{Q}_X$.

3.2. $\delta$-stability in the abelian category. In this subsection let $X$ be a smooth projective curve over an infinite field $K$ of characteristic 0. Let $\delta \in \mathbb{R}$. Given an object $\mathcal{E}$ of $\mathcal{C}_X$, the type of $\mathcal{E}$ is the triple $(r(E), d(E), v) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. If $r(E) > 0$, the $\delta$-slope of $\mathcal{E}$ is defined by

$$\mu_\delta(\mathcal{E}) = \mu(E) + \frac{v \delta}{r(E)}$$
Note that if $v = 0$, $E$ is a Higgs sheaf and $\mu_\delta(E) = \mu(E)$ is the usual slope of $E$ for any value of $\delta \in \mathbb{R}$.

Let $\delta \in \mathbb{R}$. Then we define $\delta$-(semi)stability for objects of $C_X$ as follows.

**Definition 3.5.** An object $E$ of $C_X$ of type $(r, e, v) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ will be called $\delta$-(semi)stable if the following inequality holds for any proper nontrivial subobject $0 \subset E' \subset E$ in $C_X$ of type $(r', e', v')$

$$r (e' + v' \delta) \leq r' (e + v \delta)$$

By analogy with [11] Sect. 2, for any subobject $0 \subset E' \subset E$ set

$$\beta_\delta(E', E) = re' - r'e + (re' - r'v) \delta.$$  

Then condition (3.2) is equivalent to $\beta_\delta(E', E) \leq 0$. Note that if $r, r' > 0$ this condition becomes $\mu_\delta(E') \leq \mu_\delta(E)$.

**Remark 3.6.** Note that the object $O = (0, C, 0, 0, 0, 0)$ is stable according to definition (3.5) for any value of $\delta \in \mathbb{R}$.

**Lemma 3.7.** Let $E$ be a $\delta$-semistable object of $C_X$ of rank $r \in \mathbb{Z}_{\geq 1}$ for some $\delta \in \mathbb{R}_{> 0}$. Then $E$ is a locally free $O_X$-module. Moreover, if in addition $v(E) > 0$, the morphism $\psi : O_X \to O_X$ must be nontrivial.

**Proof.** Suppose $E$ is not locally free and let $T_0(E)$ be the torsion submodule of $E$ i.e. we have an exact sequence of $O_X$-modules

$$0 \to T_0(E) \to E \to F \to 0$$

with $T_0(E)$ torsion and $F$ torsion-free. Then it immediately follows that

$$\Phi_i(T_0(E) \otimes_X M_i) \subseteq T_0(E) \quad \phi(T_0(E) \otimes_X M) = 0$$

for $i = 1, 2$. Therefore the data

$$T_0(E) = (T_0(E), 0, \Phi_1|_{T_0(E)}, 0, 0)$$

is a subobject of $E$. Then $\delta$-(semi)stability implies that $d(T_0(E)) \leq 0$. Since $X$ is a smooth projective curve, this implies in turn $T_0(E) = 0$.

Suppose $E = (E, V, \Phi_{1, 2}, \phi, \psi)$ is $\delta$-semistable for some $\delta \in \mathbb{R}_{> 0}$, $v = \dim(V) > 0$, and $\psi = 0$. Then note that $E' = (E, \Phi_{1, 2})$ is a quotient of $E$ in $C_X$ and $\mu_\delta(E') < \mu_\delta(E)$. This contradicts the $\delta$-semistability of $E$.

**Lemma 3.8.** Let $E$ be a $\delta$-stable object of $C_X$ for some $\delta \in \mathbb{R}_{> 0}$. Then any nontrivial endomorphism $(\xi, f) : E \to E$ in $C_X$ must be an isomorphism. If in addition the ground field $K$ is algebraically closed, the endomorphism ring of $E$ is isomorphic to $K$.

**Proof.** Follows by standard arguments analogous for example to the proof of [20] Lemma 2.3.

Given remark 3.2, an immediate consequence of lemma 3.8 is

**Corollary 3.9.** Suppose $K$ is algebraically closed and let $E$ be a nontrivial $\delta$-stable ADHM sheaf on $X$ for some $\delta \in \mathbb{R}_{> 0}$. Then the automorphism group of $E$ is trivial.
Lemma 3.10. Let \( \mathcal{E} = (E, V, \Phi_{1,2}, \phi, \psi) \) be an object of \( \mathcal{C}_X \) of rank \( r(\mathcal{E}) \geq 1 \) and let \( \mathcal{E}' = (E', V', \Phi_{1,2}', \phi', \psi') \) be a subobject of \( \mathcal{E} \). Let \( \overline{E}' \) be the saturation of \( E' \) in \( E \). Then the data

\[
(3.3) \quad \overline{E}' = (E', V', \Phi_i|_{\overline{E}' \otimes_X M_i}, \phi|_{\overline{E}' \otimes_X M_i}, \psi|_V \otimes O_X)
\]

with \( i = 1, 2 \), is a subobject of \( \mathcal{E} \).

Proof. If \( E' \) is saturated there is nothing to prove. Therefore we will assume that \( E' \subset \overline{E}' \) is a non-saturated proper subsheaf.

First we will prove that \( \overline{E}' \) is \( \Phi_i \)-invariant for \( i = 1, 2 \). Suppose this is not the case. Then the induced morphism

\[
\overline{\Phi}_i : \overline{E}' \otimes_X M_i \to E/\overline{E}'
\]

is non-trivial for some value of \( i = 0, 1 \). Note that \( E' \otimes M_i \subset \ker(\overline{\Phi}_i) \) since \( E' \) is \( \Phi_i \)-invariant and \( E' \subset \overline{E}' \). Therefore \( \overline{\Phi}_i \) factors through a morphism

\[
\overline{\Phi}_i : (E'/\overline{E}') \otimes_X M_i \to E/\overline{E}'.
\]

Since \( \overline{E}' \) is the saturation of \( E' \) the domain of \( \overline{\Phi}_i \) is a torsion coherent sheaf on \( X \) while the target of \( \overline{\Phi}_i \) is a torsion free \( O_X \)-module. Therefore \( \overline{\Phi}_i = 0 \), and \( \overline{\Phi}_i = 0 \) as well. We have reached a contradiction.

An identical argument proves that

\[
\text{Im}(\phi|_{\overline{E}' \otimes X M}) \subset V' \otimes O_X
\]

which concludes the proof of lemma (3.10).

The subobject \( \overline{E}' \) constructed in lemma (3.10) will be called the saturation of \( \mathcal{E}' \) in \( \mathcal{E} \).

Lemma 3.11. Let \( \delta \in \mathbb{R} \). Let \( \mathcal{E} \) be a locally free object of \( \mathcal{C}_X \) of rank \( r(\mathcal{E}) \geq 1 \) and \( v(\mathcal{E}) = 1 \). Then \( \mathcal{E} \) is \( \delta \)-semistable if and only if it is \( \delta \)-(semi)stable as an ADHM sheaf.

Proof. It suffices to consider \( \delta > 0 \). The proof for \( \delta \leq 0 \) is completely analogous.

\((\Rightarrow)\) Suppose \( \mathcal{E} \) is a \( \delta \)-semistable object of \( \mathcal{C}_X \) with \( v(\mathcal{E}) = 1 \). Then lemma (3.7) implies that \( E \) is a locally free \( O_X \)-module and \( \psi \neq 0 \). Let \( 0 \subset E' \subset E \) be a non-trivial proper saturated \( \Phi \)-invariant subsheaf. If \( \text{Im}(\psi) \subset E' \), the data \( \mathcal{E}' = (E', K, \Phi_{1,2}|_{E'}, \phi|_{E' \otimes X M}, \psi) \) defines a subobject of \( \mathcal{E} \) with \( v(\mathcal{E}') = 1 \). If \( E' \otimes X M \subset \ker(\phi) \), the data \( \mathcal{E}' = (E', \Phi_{1,2}|_{E'}) \) defines a subobject of \( \mathcal{E} \) with \( v(\mathcal{E}') = 0 \). Then the stability condition (3.1) implies the stability condition (2.1).

\((\Leftarrow)\) Suppose \( \mathcal{E} \) is a \( \delta \)-stable ADHM sheaf on \( X \). Obviously, \( \mathcal{E} \) is an object of \( \mathcal{C}_X \) with \( v(\mathcal{E}) = 1 \) and \( \psi \neq 0 \). Let \( 0 \subset E' \subset \mathcal{E} \) be a non-trivial proper subsheaf of \( \mathcal{E} \) in \( \mathcal{C}_X \). Note that either \( v(\mathcal{E}') = 1 \), in which case \( \text{Im}(\psi) \subset E' \), or \( v(\mathcal{E}') = 0 \), in which case \( E' \otimes X M \subset \ker(\phi) \). Let \( \overline{E}' \) be the saturation of \( \mathcal{E} \) constructed in lemma (3.10). Note that \( \mu(\mathcal{E}') = \mu(\mathcal{E}) \geq \mu(\mathcal{E}) \). If \( \overline{E}' = E \) we have \( \mu(E') = \mu(E) \) and condition (4.1) is automatically satisfied. If \( \overline{E}' \subset \mathcal{E} \) is a proper (non-trivial) subsheaf, it is straightforward to check that condition (2.1) applied to the saturated subsheaf \( 0 \subset \overline{E}' \subset \mathcal{E} \) implies (3.1).

We also have the following obvious lemma.
Lemma 3.12. Let $\delta \in \mathbb{R}$ and let $E$ be a locally free object of $\mathcal{C}_X$ or rank $r(E) \geq 1$ and $v(E) = 0$. Then $E$ is $\delta$-stable if and only if $E$ is (semi)stable as a Higgs sheaf.

To conclude this subsection, note that since the category $\mathcal{C}_X$ is noetherian and artinian the standard properties of $\delta$-(semi)stable objects hold. That is we have

Proposition 3.13. (i) Harder-Narasimhan filtrations in $\mathcal{C}_X$ exist and satisfy the same properties as Harder-Narasimhan filtrations of coherent sheaves on smooth projective varieties.

(ii) The subcategory of $\delta$-(semi)stable objects of $\mathcal{C}_X$ with fixed $\delta$-slope is noetherian and artinian. The simple objects in this subcategory are precisely the $\delta$-stable objects. In particular Jordan-Hölder filtrations exist and satisfy the same properties as Jordan-Hölder filtrations of semistable coherent sheaves on smooth projective varieties.

Furthermore, the following result analogous to [21 Sect. 2] also holds

Lemma 3.14. Let $\delta \in \mathbb{R}_{>0}$ and suppose $E$ is an object of $\mathcal{C}_X$ which is not $\delta$-semistable. Then there exists a unique $\delta$-destablizing proper subobject $0 \subset F \subset E$ so that $\beta_\delta(F,E) > 0$ and

(i) If $0 \subset F' \subset F$ is a proper subobject of $F$ then $\beta_\delta(F',E) < \beta_\delta(F,E)$.

(ii) If $0 \subset F' \subset E$ is any other proper subobject of $E$ then $\beta_\delta(F',E) \leq \beta_\delta(F,F)$ and equality holds only if $F \subseteq F'$.

Moreover, $F$ is maximal among all proper $\delta$-destablizing subobjects of $E$ satisfying condition (i) above.

By analogy with [21 Sect. 2] the subobject $0 \subset F \subset E$ found in lemma 3.14 will be called the $\beta_\delta$-subobject of $E$.

3.3. Extensions of ADHM quiver sheaves. Next we prove some basic homological algebra results for ADHM quiver sheaves. The homological algebra of quiver sheaves without relations has been treated in detail in [28]. Our task is to generalize some of the results of [25] to ADHM quiver sheaves, which have quadratic relations. In this subsection we will take $X$ to be a separated scheme of finite type over an infinite field $K$ of characteristic zero, and will employ Čech resolutions rather than injective resolutions as in [28]. Let $M_1, M_2$ be fixed invertible sheaves on $X$.

The main result of this section is

Proposition 3.15. Let $E' = (E', E'_\infty, \Phi'_1, \ldots, \phi', \psi')$, $E'' = (E'', E''_\infty, \Phi''_1, \ldots, \phi'', \psi'')$ be coherent locally free ADHM quiver sheaves on $X$. Consider the following complex $\mathcal{C}(E'', E')$ of coherent locally free $\mathcal{O}_X$-modules

$$\begin{array}{c}
\text{H}om_X(E'' \otimes_X M_1, E') \\
\oplus \\
\text{H}om_X(E'' \otimes_X M_2, E') \\
\text{H}om_X(E''_\infty, E'_\infty) \\
\oplus \\
\text{H}om_X(E'' \otimes_X E'_\infty) \\
\oplus \\
\text{H}om_X(E'') \\
\text{H}om_X(E'_\infty, E')
\end{array}
\oplus
\begin{array}{c}
d_1 \\
d_2\text{H}om_X(E'' \otimes_X M, E') \rightarrow 0
\end{array}
\begin{array}{c}
0 \\
\text{H}om_X(E'' \otimes_X E'_\infty)
\end{array}
\begin{array}{c}
\text{H}om_X(E'_\infty, E')
\end{array}
$$

where

$$d_1(\alpha, \alpha_\infty) = ( - \alpha \circ \Phi''_1 + \Phi'_1 \circ (\alpha \otimes 1_{M_1}), - \alpha \circ \Phi''_2 + \Phi'_2 \circ (\alpha \otimes 1_{M_2}), - \alpha_\infty \circ \phi'' + \phi' \circ (\alpha \otimes 1_{M}), - \alpha \circ \psi'' + \psi' \circ \alpha_\infty)$$
for any local sections \((\alpha, \alpha_\infty)\) of the first term and
\[
d_2(\beta_1, \beta_2, \gamma, \delta) = \beta_1 \circ (\Phi_2^\dagger \otimes 1_{M_1}) - \Phi_2^\dagger \circ (\beta_1 \otimes 1_{M_2}) - \beta_2 \circ (\Phi_1^\dagger \otimes 1_{M_2}) + \Phi_1^\dagger \circ (\beta_2 \otimes 1_{M_1}) + \psi' \circ \gamma + \delta \circ \phi''
\]
for any local sections \((\beta_1, \beta_2, \gamma, \delta)\) of the middle term. The degrees of the three terms in (3.4) are 0, 1, 2 respectively.

Then there are group isomorphisms
\[
(3.5) \quad \text{Ext}^k_{\mathcal{Q}_X}(\mathcal{E}'', \mathcal{E}') \simeq H^k(X, \mathcal{C}(\mathcal{E}'', \mathcal{E}'))
\]
for \(k = 0, 1\), where \(\text{Ext}^k_{\mathcal{Q}_X}(\mathcal{E}'', \mathcal{E}')\) denote extension groups in the abelian category of ADHM quiver sheaves on \(X\).

**Proof.** Since \(X\) is a separated scheme of finite type over \(\mathbb{C}\), it admits affine open covers and we can employ Čech resolutions in the construction of the hypercohomology double complex associated to \(\mathcal{C}(\mathcal{E}'', \mathcal{E}')\). Then the correspondence stated in lemma (3.15) follows by repeating the proof of [20, Prop. 4.5] based on Čech cochain computations in the present context. We will omit the details. \(\square\)

**Remark 3.16.** Proposition (3.15) can be given an alternative proof employing methods analogous to [28]. Then one can prove that the isomorphisms (3.5) hold for all values of \(k \in \mathbb{Z}\). We will not present the details here because the cases \(k = 0, 1\) suffice for our purposes.

Proposition (3.15) implies

**Corollary 3.17.** Suppose \(X\) is connected and proper over \(K\). Let \(\mathcal{E}', \mathcal{E}''\) be locally free objects of \(\mathcal{C}_X\) with \(v(\mathcal{E}') + v(\mathcal{E}'') \leq 1\). Then there are group isomorphisms
\[
(3.6) \quad \text{Ext}^k_{\mathcal{C}_X}(\mathcal{E}'', \mathcal{E}') \simeq H^k(X, \mathcal{C}(\mathcal{E}'', \mathcal{E}'))
\]
for \(k = 0, 1\), where \(\text{Ext}^k_{\mathcal{C}_X}(\mathcal{E}'', \mathcal{E}')\) denote extension groups in the abelian category \(\mathcal{C}_X\).

**Proof.** The case \(k = 0\) follows from the fact that \(\mathcal{C}_X\) is a full subcategory of \(\mathcal{Q}_X\) if \(X\) is connected and proper over \(K\). For \(k = 1\) note that there is a natural injective homomorphism
\[
\text{Ext}^1_{\mathcal{C}_X}(\mathcal{E}'', \mathcal{E}') \hookrightarrow \text{Ext}^1_{\mathcal{Q}_X}(\mathcal{E}'', \mathcal{E}').
\]
If \(v(\mathcal{E}') + v(\mathcal{E}'') \leq 1\), it follows that at least one of \(v(\mathcal{E}')\), \(v(\mathcal{E}'')\) vanishes. Then it is straightforward to check that the above homomorphism is also surjective. \(\square\)

4. Chamber Structure

The goal of this section is to study the behavior of the \(\delta\)-stability condition on the parameter \(\delta \in \mathbb{R}_{>0}\) keeping the data \(\mathcal{X} = (X, M_1, M_2), E_\infty = O_X\) as well as the type \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) fixed. The ground field will be an infinite field \(K\) of characteristic 0 as in section [2].
4.1. Critical stability parameters. This section establishes the existence of a
chamber structure of the positive real axis so that the set of \( \delta \)-stable ADHM sheaves is constant in each chamber.

First consider the case of ADHM sheaves of rank \( r = 1 \).

Lemma 4.1. Let \( \mathcal{E} = (E, \Phi_1, \Phi_2, \phi, \psi) \) be a locally free ADHM sheaf of type \((1, e)\),
e \( e \in \mathbb{Z}_r \) on \( X \) so that \( \psi \) is nontrivial. Then \( \mathcal{E} \) is \( \delta \)-stable for any stability parameter
\( \delta \in \mathbb{R}_{>0} \). In particular the moduli space \( \mathcal{M}(X, 1, e) \) is independent of \( \delta \in \mathbb{R}_{>0} \).

Proof. Since \( r = 1 \), \( E \) has no nontrivial proper saturated subsheaves \( 0 \subset E' \subset E \).
Hence, since \( \psi \) is nontrivial the \( \delta \)-stability conditions are automatically satisfied for
any \( \delta \in \mathbb{R}_{>0} \).

Remark 4.2. Given lemma \[4.1\], a rank one locally free ADHM sheaf on \( X \) with
\( \psi \neq 0 \) will be called in the following stable, without any reference to a stability
parameter.

Next let \( r \geq 2 \). Let \( \delta \in \mathbb{R}_{>0} \) be stability parameter. Suppose there exists a \( \delta \)-
semistable ADHM sheaf \( \mathcal{E} \) of type \((r, e)\) on \( X \) which is not \( \delta \)-stable. Then definition \[4.1\] implies that \( \delta \) must be of the form

\[
\delta = \frac{re' - er'}{r'} \quad \text{or} \quad \delta = \frac{er' - re'}{r - r'}
\]

for some \( 1 \leq r' \leq r - 1 \), \( e' \in \mathbb{Z} \).

Definition 4.3. A stability parameter \( \delta \in \mathbb{R}_{>0} \) is called numerically critical of type
\((r, e)\) in \( \mathbb{Z}_{\geq 2} \times \mathbb{Z} \) if it is of the form \[4.1\].

Let \( \Delta_{(r, e)} \subset \mathbb{R}_{>0} \) denote the set of numerically critical parameter of fixed type
\((r, e) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z} \). Since all such parameters are rational numbers with denominators
in the finite set \{1, \ldots, r - 1\} it follows that there exists an isomorphism \( \Delta_{(r, e)} \cong \mathbb{Z}_{>0} \). For each \( n \in \mathbb{Z}_{>0} \) let \( \delta_n \in \Delta_{(r, e)} \) denote the corresponding numerically critical parameter. We will also set \( \delta_0 = 0 \) in order to simplify the exposition.

Lemma 4.4. Let \((r, e) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z} \) be a fixed type. Then the following hold

(i) For any \( n \in \mathbb{Z}_{\geq 0} \) and any \( \delta \in (\delta_n, \delta_{n+1}) \) an ADHM sheaf of type \((r, e)\) is
\( \delta \)-semistable if and only if it is \( \delta \)-stable.

(ii) For any \( n \in \mathbb{Z}_{>0} \), the set of \( \delta \)-stable ADHM sheaves is constant for \( \delta \in
(\delta_n, \delta_{n+1}) \)

Proof. Statement (i) follows immediately from the definition of numerically critical stability parameters.

In order to prove (ii) suppose there exist two values \( \delta_n \prec \gamma_1 \prec \gamma_2 \prec \delta_{n+1} \) for
some \( n \in \mathbb{Z}_{>0} \) so that (ii) fails for \( \gamma_1, \gamma_2 \).

This implies at least one of the following statements

(a) There exists a \( \gamma_1 \)-stable ADHM sheaf of type \((r, e)\) which is not \( \gamma_2 \)-stable.

(b) There exists a \( \gamma_2 \)-stable ADHM sheaf of type \((r, e)\) which is not \( \gamma_1 \)-stable.

Suppose (a) holds and let \( \mathcal{E} \) be such an ADHM sheaf. Then there must exist a
\( \gamma_2 \)-destabilizing \( \Phi \)-invariant proper saturated subsheaf \( 0 \subset E' \subset E \). Since \( \gamma_1 \prec \gamma_2 \)
it is straightforward to check that \( E \) must violate condition (ii) of definition \[4.1\];
condition (iii) cannot be violated by any subsheaf of \( E \). Therefore, \( \text{Im}(\psi) \subset E \) and

\[
\mu(E') + \frac{\gamma_1}{r(E')} < \mu(E) + \frac{\gamma_1}{r}, \quad \mu(E') + \frac{\gamma_2}{r(E')} \geq \mu(E) + \frac{\gamma_2}{r}.
\]
Let
\[ \delta = (\mu(E) - \mu(E')) \frac{rr(E')}{r(E') - r}. \]
Then inequalities (4.2) imply that \( \gamma_1 < \delta \leq \gamma_2 \). By construction we also have
\[ \mu(E') + \delta \frac{r(E')}{r(E')} = \mu(E) + \frac{\delta}{r} \]
which implies that \( E \) is strictly \( \delta \)-semistable. This leads to a contradiction since by assumption there cannot exist any numerically critical values in the interval \( (\gamma_1, \gamma_2) \).

Case \((b)\) is analogous except \( E' \) will be required to be a destabilizing subsheaf so that \( E' \otimes_X M \subseteq \text{Ker}(\phi) \) since \( \gamma_1 < \gamma_2 \).

\[ \square \]

**Definition 4.5.** A locally free ADHM sheaf \( E \) on \( X \) is asymptotically stable if \( \psi \) is not identically zero and there is no nontrivial \( \Phi \)-invariant proper saturated subsheaf \( 0 \subset E' \subset E \) so that \( \text{Im}(\psi) \subseteq E' \).

**Lemma 4.6.** The set of isomorphism classes of locally free coherent sheaves \( E \) of type \( (r, e) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z} \) on \( X \) with the property that \( E \) is the underlying sheaf of an asymptotically stable ADHM sheaf \( E \) is bounded.

**Proof.** This result has been proven in [20, Prop. 2.7] for \( K = \mathbb{C} \). The proof for an arbitrary infinite ground field \( K \) of characteristic 0 is identical.

\[ \square \]

**Lemma 4.7.** Let \( (r, e) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z} \) be fixed as above. Then there exists \( c_1 \in \mathbb{R}_{>0} \) depending only on \( (r, e) \) so that the following statements hold for any \( \delta > c_1 \):

(i) An ADHM sheaf \( E \) of type \( (r, e) \) on \( X \) is \( \delta \)-semistable if and only if it is asymptotically stable.

(ii) An ADHM sheaf \( E \) of type \( (r, e) \) on \( X \) is \( \delta \)-semistable if and only if it is \( \delta \)-stable.

**Proof.** First we prove that there exists \( c_1 \in \mathbb{R}_{>0} \) depending only on \( (r, e) \) so that any \( \delta \)-semistable ADHM sheaf on \( X \) of type \( (r, e) \) with \( \delta > c_1 \) is asymptotically stable. Recall [20, Lemma 2.4] that given any ADHM sheaf \( E = (E, \Phi_{1,2}, \phi, \psi) \) with \( \psi \neq 0 \) there is a canonical nontrivial \( \Phi \)-invariant saturated subsheaf \( E_0 \subset E \) so that \( \text{Im}(\psi) \subset E_0 \). Moreover, by construction \( E_0 \) is a subsheaf of any \( \Phi \)-invariant saturated subsheaf \( E' \subset E \) so that \( \text{Im}(\psi) \subset E' \).

Since \( E_0 \) is canonically constructed in terms of the data \( E \), it follows that the set of the sheaves \( E_0 \) associated to all \( \delta \)-semistable ADHM sheaves \( E \) of type \( (r, e) \), with \( \delta \in \mathbb{R}_{>0} \), is bounded. Therefore the numerical invariants \( (r(E_0), d(E_0)) \) can take only a finite set of values. This implies that there exists \( c_1 \in \mathbb{R}_{>0} \) so that for all \( \delta > c_1 \) we have

\[ \mu(E_0) + \frac{\delta}{r(E_0)} > \mu_\delta(E) \tag{4.3} \]

whenever \( E_0 \) is a proper subsheaf of \( E \).

Suppose there exists a \( \delta \)-semistable ADHM sheaf \( E_0, \delta > c_1 \), which is not asymptotically stable. Therefore there exists a nontrivial \( \Phi \)-invariant proper saturated subsheaf \( 0 \subset E' \subset E \) so that \( \text{Im}(\psi) \subseteq E' \). As observed above, by construction \( E_0 \) must be a subsheaf of any such subsheaf, hence in particular \( E_0 \) will be a proper
isomorphism classes of locally free sheaves $E$. This yields a contradiction since then inequality (4.3) implies that $E$ is not $\delta$-semistable.

Next we prove that there exists $c_2 \in \mathbb{R}_{>0}$ so that any asymptotically stable ADHM sheaf $E$ is $\delta$-stable for all $\delta > c_2$. According to lemma 4.1, the set of isomorphism classes of locally free sheaves $E$ of type $(r,e)$ so that $E$ is the underlying sheaf of an asymptotically stable ADHM sheaf is bounded. This implies that there exists a positive constant $C' \in \mathbb{R}_{>0}$ depending only on $(r,e)$ so that $\mu_{\max}(E) < C'$ for any such locally free sheaf $E$. It follows that there exists $c_2 \in \mathbb{R}_{>0}$ so that for any $\delta > c_2$, condition (2.2) of definition (2.1) is automatically satisfied for any asymptotically stable ADHM sheaf $E$, and any $\Phi$-invariant nontrivial proper saturated subsheaf $0 \subset E' \subset E$ so that $E' \otimes_X M \subset \ker(\phi)$. Since the stability condition (ii) of definition (2.1) is trivially satisfied for asymptotically stable ADHM sheaves, the claim follows.

In order to conclude the proof of proposition (4.7), take $\delta_\infty = \max\{c_1, c_2\}$. 

\[ \text{Lemma 4.8.} \] Let $(r,e) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}$ be a fixed type. Then there exists $\delta_N \in \Delta_{(r,e)} \cup \{0\}$, $N \geq 0$, depending only on $(r,e)$ so that for all $\delta > \delta_N$, any $\delta$-semistable ADHM sheaf of type $(r,e)$ is asymptotically stable.

\[ \text{Proof.} \] Follows directly from lemmas (4.4) and (4.7).

\[ \text{Definition 4.9.} \] Suppose the integer $N$ found in lemma (4.8) is nonzero. Then the stability parameters $\delta_i \in \mathbb{R}_{>0}$, $i = 1, \ldots, N$ will be called critical values of type $(r,e)$. Moreover, a parameter $\delta \in \mathbb{R}_{>0}$ will be called noncritical of type $(r,e)$ if $\delta \notin \{\delta_1, \ldots, \delta_N\}$.

4.2. Wallcrossing behavior. This subsection analyzes the behavior of $\delta$-stable ADHM sheaves as $\delta$ specializes to a critical value. In order to simply the exposition we will formally set $\delta_0 = 0$ as above, and $\delta_{N+1} = +\infty$.

\[ \text{Lemma 4.10.} \] Let $(r,e) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}$ be a fixed type so that the integer $N$ in proposition (4.7) is nonzero. Then the following hold

(i) Let $E$ be a $\delta$-stable ADHM sheaf on $X$ of type $(r,e)$, with $\delta \in (\delta_i, \delta_{i+1})$ for some $i = 1, \ldots, N$. Then $E$ is $\delta_i$-semistable and it has a Jordan-Hölder filtration in the abelian category $\mathcal{C}_0$ of the form

\[ 0 = JH_0(E) \subset JH_1(E) \subset JH_2(E) \subset \cdots \subset JH_{j-1}(E) \subset JH_j(E) = E, \quad j \geq 1 \]

so that $r(JH_l(E)) \geq 1$ for any $1 \leq l \leq j$ and $r(JH_l(E)) < r$, $\nu(JH_l(E)) = 0$ for $0 \leq l \leq j-1$. In particular $E$ is either $\delta_i$-stable or there is a nontrivial extension in the abelian category $\mathcal{C}_X$ of the form

\[ 0 \to E' \to E \to E'' \to 0 \]

where $E'$ is a semistable Higgs sheaf of rank $r(E') \geq 1$ and $E''$ is a $\delta_i$-stable ADHM sheaf of rank $r(E'') \geq 1$ and $\mu_{\delta_i}(E') = \mu_{\delta_i}(E'') = \mu_{\delta_i}(E)$.

(ii) Let $E$ be a $\delta$-stable ADHM sheaf on $X$ of type $(r,e)$, with $\delta \in (\delta_{i-1}, \delta_i)$ for some $i = 1, \ldots, N$. Then $E$ is $\delta_i$-semistable and it has a Jordan-Hölder filtration in the abelian category $\mathcal{C}_X$ of the form (4.4) where $r(JH_l(E)) \geq 1$, $\nu(JH_l(E)) = 1$ for all $1 \leq l \leq j$, and $r(JH_l(E)) < r$ for all $0 \leq l \leq j-1$. In particular $E$ is $\delta_i$-stable.
or there is a nontrivial extension in the abelian category \( \mathcal{C}_X \) of the form

\[
0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0
\]

where \( \mathcal{E}' \) is a \( \delta_i \)-stable ADHM sheaf of rank \( r(\mathcal{E}') \geq 1 \), \( \mathcal{E}'' \) is a semistable Higgs sheaf of rank \( r(\mathcal{E}'') \geq 1 \), and \( \mu_{\delta_i}(\mathcal{E}') = \mu_{\delta_i}(\mathcal{E}'') = \mu_{\delta_i}(\mathcal{E}) \).

(iii) Let \( \mathcal{E} \) be a \( \delta_i \)-stable ADHM sheaf on \( X \) of type \( (r, e) \) for some \( i = 1, \ldots, N \). Then \( \mathcal{E} \) is \( \delta \)-stable for any \( \delta \in (\delta_{i-1}, \delta_{i+1}) \).

**Proof.** We will prove the first and third statements. The second statement is analogous to the first one.

Lemma (4.4) implies that \( \mathcal{E} \) is \( \gamma \)-stable for any \( \gamma \in (\delta_i, \delta_{i+1}) \). Then any \( \Phi \)-invariant nontrivial proper saturated subsheaf \( \text{Im}(\psi) \subseteq \mathcal{E}' \subset \mathcal{E} \) must satisfy

\[
\mu(\mathcal{E}') + \frac{\gamma}{r(\mathcal{E}')} < \mu(\mathcal{E}) + \frac{\gamma}{r}
\]

for any \( \gamma \in (\delta_i, \delta_{i+1}) \). Similarly, any \( \Phi \)-invariant nontrivial proper saturated subsheaf \( 0 \subset \mathcal{E}' \otimes_X M \subseteq \text{Ker}(\phi) \) must satisfy

\[
\mu(\mathcal{E}') < \mu(\mathcal{E}) + \frac{\gamma}{r}
\]

for any \( \gamma \in (\delta_i, \delta_{i+1}) \).

In the first case, it follows that

\[
\mu(\mathcal{E}') + \frac{\delta_i}{r(\mathcal{E}')} \leq \mu(\mathcal{E}) + \frac{\delta_i}{r}
\]

since \( r(\mathcal{E}') < r \).

In the second case, suppose there exists such a subsheaf \( \mathcal{E}' \) so that

\[
\mu(\mathcal{E}') > \mu(\mathcal{E}) + \frac{\delta_i}{r}
\]

Then Grothendieck’s lemma [55, Lemma 1.7.9] implies that for fixed \( \mathcal{E} \) the set of isomorphism classes of such subsheaves is bounded. Then it follows that there exists \( \gamma \in (\delta_i, \delta_{i+1}) \) so that

\[
\mu(\mathcal{E}') > \mu(\mathcal{E}) + \frac{\gamma}{r}
\]

This would contradict \( \gamma \)-stability. Therefore \( \mathcal{E} \) must be \( \delta_i \)-semistable, and it has a Jordan-Hölder filtration of the form \([133]\) according to proposition (3.13). It is straightforward to check that none of the objects \( JH_l(\mathcal{E}) \), \( 1 \leq l \leq j \) may have rank zero and none of the objects \( JH_l(\mathcal{E}) \), \( 0 \leq l \leq j - 1 \) may have rank equal to \( r \). If the length of the filtration is \( j = 1 \), it follows that \( \mathcal{E} \) is \( \delta_i \)-stable.

Suppose \( j \geq 2 \). By the general properties of Jordan-Hölder filtrations, all subobjects \( JH_l(\mathcal{E}) \subseteq \mathcal{E}, l = 1, \ldots, j \) must have the same \( \delta_i \)-slope as \( \mathcal{E} \). Let us denote by \( E_l \) the underlying locally free \( \mathcal{O}_X \)-module of the ADHM sheaf \( JH_l(\mathcal{E}) \), \( l = 1, \ldots, j \). Let \( v_l = v(JH(E_l)), l = 1, \ldots, j \). Then

\[
\mu(E_l) + \frac{v_l \delta_i}{r(E_l)} = \mu(\mathcal{E}) + \frac{\delta_i}{r}
\]

for all \( l = 1, \ldots, j \). However, since \( \mathcal{E} \) is \( \delta \)-stable,

\[
\mu(E_l) + \frac{v_l \delta}{r(E_l)} < \mu(\mathcal{E}) + \frac{\delta}{r}
\]
must also hold for all \( l = 1, \ldots, j - 1 \). These inequalities imply
\[
\frac{v_l(\delta - \delta_i)}{r(E_i)} < \frac{\delta - \delta_i}{r}
\]
for all \( l = 1, \ldots, j - 1 \). Since \( r(E_l) < r \) for \( l \neq j \), it follows that \( v_l = 0 \) for all \( l = 1, \ldots, j - 1 \). This implies that the last quotient \( JH_j(E)/JH_{j-1}(E) \) is a \( \delta_i \)-stable ADHM sheaf on \( X \). Then the exact sequence (4.5) is obtained by setting \( E' = JH_{j-1}(E), E'' = JH_j(E) \). The extension (4.5) has to be nontrivial because \( E \) is \( \delta \)-stable, hence indecomposable.

Next let us prove the third statement. Suppose \( E \) is a \( \delta_i \)-stable ADHM sheaf on \( X \) of type \((r,e)\) and suppose there exists \( \delta \in (\delta_{i-1}, \delta_{i+1}) \) so that \( E \) is not \( \delta \)-stable. This means that there exists subsheaf \( 0 \subset E' \subset E \) which does not satisfy the conditions of definition (2.1). Since \( E \) is \( \delta_i \)-stable, only the following cases may occur

1. \( \delta < \delta_i \) and \( E' \) is a \( \Phi \)-invariant nontrivial proper saturated subsheaf \( 0 \subset E' \otimes_X M \subset \text{Ker}(\phi) \) so that
\[
\mu(E') \geq \mu(E) + \frac{\delta}{r}
\]
2. \( \delta > \delta_i \) and \( E' \) is a \( \Phi \)-invariant nontrivial proper saturated subsheaf \( \text{Im}(\psi) \subset E' \subset E \) so that
\[
\mu(E') + \frac{\delta}{r(E')} \geq \mu(E) + \frac{\delta}{r}
\]
Since \( E \) is \( \delta_i \)-stable, conditions (2.2), respectively (2.1), yield an upper bound for \( \mu(E') \). Therefore in both cases Grothendieck’s lemma implies that the set of isomorphism classes of \( \delta \)-destabilizing subsheaves \( E' \) is bounded for fixed \( \delta \).

Suppose case (1) holds. Then boundedness implies that it exists \( \gamma \in (\delta, \delta_i) \) so that
\[
\mu(E') < \mu(E) + \frac{\gamma}{r}
\]
for all \( \delta \)-destabilizing subsheaves \( E' \). Moreover, if \( 0 \subset E' \subset E \) is a \( \Phi \)-invariant nontrivial proper subsheaf so that \( 0 \subset E' \otimes_X M \subset \text{Ker}(\phi) \) and
\[
\mu(E') < \mu(E) + \frac{\delta}{r}
\]
it follows that \( E' \) satisfies condition (4.9) since \( \delta < \gamma \). Therefore condition (4.9) is satisfied by all \( \Phi \)-invariant proper nontrivial saturated subsheaves \( 0 \subset E' \otimes_X M \subset \text{Ker}(\phi) \).

If \( E' \) is a \( \Phi \)-invariant proper nontrivial saturated subsheaf of \( E \) so that \( \text{Im}(\psi) \subset E' \subset E \), \( \delta_i \)-stability implies that
\[
\mu(E') + \frac{\delta_i}{r(E')} < \mu(E) + \frac{\delta_i}{r}
\]
Since \( \gamma < \delta_i \) and \( r(E') < r \) it follows that
\[
\mu(E') + \frac{\gamma}{r(E')} < \mu(E) + \frac{\gamma}{r}
\]
Therefore we conclude that \( E \) is \( \gamma \)-stable. Since \( \delta, \gamma \) belong to the same chamber we have reached a contradiction.

Case (2) leads to a similar contradiction by an analogous argument.
Remark 4.11. Note that the morphism \( E \rightarrow E'' \) in equation (4.5), respectively \( E' \rightarrow E \) in equation (4.6), may be assumed to be a morphism of ADHM sheaves according to remark (3.2).

The following is an immediate consequence of lemma (4.10).

Corollary 4.12. Under the assumptions of lemma (4.10), let \( \delta_i, i = 1, \ldots, N \) be a critical stability parameter of type \((r,e)\) and let \( \delta_i \in (\delta_i-1, \delta_i), \delta_\pm \in (\delta_i, \delta_{i+1}) \) be noncritical stability parameters.

(i) Suppose \( E \) is a \( \delta_+ \)-stable ADHM sheaf on \( X \) of type \((r,e)\) which is not \( \delta_- \)-stable. Then \( E \) is strictly \( \delta_\pm \)-semistable, and in particular it fits in a nontrivial extension of the form (4.6). Moreover, the one step filtration \( 0 \subset E' \subset E \) determined by (4.6) is a Harder-Narasimhan filtration for \( E \) with respect with \( \delta_- \)-stability.

(ii) Suppose \( E \) is a \( \delta_- \)-stable ADHM sheaf on \( X \) of type \((r,e)\) which is not \( \delta_+ \)-stable. Then \( E \) is strictly \( \delta_\pm \)-semistable, and in particular it fits in a nontrivial extension of the form (4.6). Moreover, the one step filtration \( 0 \subset E' \subset E \) determined by (4.6) is a Harder-Narasimhan filtration for \( E \) with respect with \( \delta_\pm \)-stability.

For future reference, let us record the following partial converse to lemma (4.10).

Lemma 4.13. Under the assumptions of lemma (4.10) let \( \delta_i \in \mathbb{R}_{>0} \) be a critical stability parameter of type \((r,e)\) and let \( \delta_i \in \mathbb{Z}_{\geq 2} \times \mathbb{Z} \). Then the following hold.

(i) There exists \( 0 < \epsilon_+ < \delta_{i+1} - \delta_i \), so that the following holds for any \( \delta_+ \in (\delta_i, \delta_i + \epsilon_+) \). A locally free ADHM sheaf \( E \) of type \((r,e)\) on \( X \) is \( \delta_\pm \)-semistable if and only if it is either \( \delta_+ \)-stable or there exists a unique filtration of the form (4.10)

\[
0 \subset E' \subset E
\]
so that \( E' \) is a \( \delta_+ \)-stable ADHM sheaf of rank \( r(E') \geq 1 \), and \( E'' = E/E' \) is a semistable Higgs sheaf of rank \( r(E'') \geq 1 \) satisfying

\[
\mu_{\delta_+}(E') > \mu(E'') \quad \mu_(E') = \mu(E'').
\]

In particular, if \( E \) is not \( \delta_+ \)-stable, any \( \delta_+ \)-destabilizing subobject \( 0 \subset F \subset E \) must be a \( \delta_\pm \)-semistable ADHM sheaf of rank \( r(F') \geq 1 \) satisfying conditions (4.11).

(ii) There exists \( 0 < \epsilon_- < \delta_i - \delta_{i-1} \) so that the following holds for any \( \delta_- \in (\delta_i - \epsilon_-, \delta_i) \). A locally free ADHM sheaf \( E \) of type \((r,e)\) on \( X \) is \( \delta_- \)-stable if and only if it is either \( \delta_- \)-stable or there exists a unique filtration of the form (4.12)

\[
0 \subset E' \subset E
\]
so that \( E' \) is a semistable Higgs sheaf of rank \( r(E') \geq 1 \), and \( E'' = E/E' \) is a \( \delta_- \)-stable ADHM sheaf of rank \( r(E'') \geq 1 \) satisfying

\[
\mu(E') > \mu_{\delta_-}(E'') \quad \mu(E') = \mu(E'').
\]

In particular, if \( E \) is not \( \delta_- \)-stable, any \( \delta_- \)-destabilizing subobject \( 0 \subset F \subset E \) must be a semistable Higgs sheaf of rank \( r(F') \geq 1 \) satisfying conditions (4.11).

Proof. It suffices to prove statement (i) since the proof of (ii) is analogous.

Let \( \delta_\pm \in (\delta_i, \delta_{i+1}) \) be an arbitrary noncritical stability parameter of type \((r,e)\). Suppose \( E \) is a \( \delta_\pm \)-semistable ADHM sheaf on \( X \). Then \( E \) is either \( \delta_+ \)-stable or there is a Harder-Narasimhan filtration of \( E \) with respect to \( \delta_- \)-stability

(4.14)

\[
0 \subset E_1 \subset \cdots \subset E_h = E
\]
where \( h \geq 2 \). It is straightforward to check that \( \mathcal{E}_l, 1 \leq l \leq h \) must have rank \( r(\mathcal{E}_l) \geq 1 \) and the successive quotients \( \mathcal{E}_{l+1}/\mathcal{E}_l, 0 \leq l \leq h-1 \) must also have rank \( r(\mathcal{E}_{l+1}/\mathcal{E}_l) \geq 1 \). Then by the general properties of Harder-Narasimhan filtrations

\[
\mu_{\delta_+}(\mathcal{E}_1) > \mu_{\delta_+}(\mathcal{E}_2/\mathcal{E}_1) > \cdots > \mu_{\delta_+}(\mathcal{E}_{h-1}/\mathcal{E}_{h-2})
\]

and

\[
\mu_{\delta_+}(\mathcal{E}_1) > \mu_{\delta_+}(\mathcal{E})
\]

for all \( 1 \leq l \leq h-1 \). Since \( \mathcal{E} \) is \( \delta_- \)-semistable by assumption, inequalities \( (4.10) \) imply that \( v(\mathcal{E}_l) = 1 \) for \( l = 1, \ldots, h \). Therefore all quotients \( \mathcal{E}_{l+1}/\mathcal{E}_l, 0 \leq l \leq h-1 \) are semistable Higgs sheaves on \( X \). Moreover, using the \( \delta_- \)-semistability condition and inequalities \( (4.16) \) we have

\[
\delta_+ \left( \frac{1}{r} - \frac{1}{r} \right) < \mu(\mathcal{E}_l) - \mu(\mathcal{E}) \leq \delta_1 \left( \frac{1}{r} - \frac{1}{r} \right)
\]

for all \( l = 1, \ldots, h \).

Now let \( \gamma \in (\delta_i, \delta_{i+1}) \) be a fixed stability parameter. Then we claim that the set of isomorphism classes of locally free ADHM sheaves \( \mathcal{E}' \) on \( X \) satisfying condition

(\*) There exists a \( \delta_- \)-semistable ADHM sheaf \( \mathcal{E} \) of type \( (r,e) \) and a stability parameter \( \delta_+ \in (\delta_i, \gamma] \) so that \( \mathcal{E}' \simeq \mathcal{E}_l \) for some \( l \in \{0, \ldots, h\} \), where \( 0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_h = \mathcal{E}, \ h \geq 1, \) is the Harder-Narasimhan filtration of \( \mathcal{E} \) with respect to \( \delta_- \)-stability.

is bounded. In order to prove this claim note that under the current conditions, inequalities \( (4.17) \) imply

\[
-\gamma \leq \mu(\mathcal{E}_l) - \mu(\mathcal{E}) < \frac{\delta_1}{r}.
\]

Moreover the set of isomorphism classes of \( \delta_- \)-semistable ADHM sheaves of type \( (r,e) \) is bounded according to lemma \( (4.4) \). Therefore the above claim follows from Grothendieck’s lemma.

Then it follows that the set of numerical types \( (r', e') \) of locally free ADHM sheaves \( \mathcal{E}' \) satisfying property (\*) is finite. This implies that there exists \( 0 < \epsilon_+ < \gamma - \delta_i \) so that for any \( \delta_+ \in (\delta_i, \delta_i + \epsilon_+) \), and any \( \delta_- \)-semistable ADHM sheaf \( \mathcal{E} \) of type \( (r,e) \) inequalities \( (4.17) \) can be satisfied only if

\[
\mu_{\delta_+}(\mathcal{E}_l) = \mu_{\delta_+}(\mathcal{E})
\]

for all \( l = 1, \ldots, h \). Hence also

\[
\mu(\mathcal{E}_l/\mathcal{E}_{l-1}) = \mu_{\delta_+}(\mathcal{E})
\]

for all \( l = 2, \ldots, h \). Then \( (4.15) \) implies that we must have \( h = 2 \). Therefore for all \( \delta_+ \in (\delta_i, \delta_i + \epsilon_+) \), any locally free \( \delta_- \)-semistable ADHM sheaf \( \mathcal{E} \) of type \( (r,e) \) is either \( \delta_- \)-stable or has a Harder-Narasimhan filtration with respect to \( \delta_- \)-stability of the form \( (4.10) \) so that \( \mathcal{E}', \mathcal{E}'' = \mathcal{E}/\mathcal{E}' \) satisfy conditions \( (4.11) \).

Next note that the set of numerical types

\[
\text{Sat}_{\delta_+}(r,e) = \{ (r', e') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} | r' \leq r, r(e' + \delta_i) = r'(e + \delta_i) \}
\]

is finite. Therefore \( 0 < \epsilon_+ < \gamma - \delta_i \) above may be chosen so that there are no critical stability parameters of type \( (r', e') \) in the interval \( (\delta_i, \delta_i + \epsilon_+) \) for any \( (r', e') \in \text{Sat}_{\delta_+}(r,e) \). This implies in particular that for all \( \delta_+ \in (\delta_i, \delta_i + \epsilon_+) \), and any locally free \( \delta_- \)-semistable \( \delta_- \)-unstable ADHM sheaf \( \mathcal{E} \) of type \( (r,e) \), the first
step $\mathcal{E}'$ in the Harder-Narasimhan filtration of $\mathcal{E}$ must be $\delta_+$-stable rather than $\delta_+$-semistable.

Conversely, suppose $\mathcal{E}$ is a locally free ADHM sheaf of type $(r, e)$ on $X$ which has a filtration of the form (4.10) with $\mathcal{E}'$ $\delta_+$-stable and satisfying conditions (4.11) for some $\delta_+ \in (\delta_1, \delta_1 + \epsilon_+)$. By the above choice of $\epsilon_+$, there are no critical stability parameters of type $(\delta, \delta_1 + \epsilon_+)$ since $(\delta, \delta_1 + \epsilon_+) \in \text{Sat}_{\delta_1}(r, e)$. Then lemma (4.10) implies that $\mathcal{E}$ is $\delta_1$-semistable. This further implies that $\mathcal{E}$ is $\delta_1$-semistable since it is an extension of semistable objects of equal $\delta_1$-slope. □

Next we consider the behavior of $\delta$-semistable ADHM sheaves as $\delta$ specializes to 0. The following results hold by analogy with lemmas (4.10) and (4.13). Since the proofs are very similar, they will be omitted.

**Lemma 4.14.** Let $X$ be a smooth projective curve over an infinite field $K$ of characteristic zero. Let $\delta \in \mathbb{R} \setminus \{0\}$ be a noncritical stability parameter of type $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ so that there are no critical stability parameters of type $(r, e)$ in the interval $(0, \delta)$ if $\delta > 0$, respectively $(\delta, 0)$ if $\delta < 0$. Then any $\delta$-stable ADHM sheaf $\mathcal{E}$ of type $(r, e)$ on $X$ is 0-semistable.

Conversely,

**Lemma 4.15.** Under the same conditions let $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Then there exist $\epsilon_+ > 0$, $\epsilon_- < 0$ so that the following hold.

(i) For any stability parameter $0 < \delta_+ < \epsilon_+$ an ADHM sheaf $\mathcal{E} = (E, \Phi_1, \Phi_2, \phi, \psi)$ of type $(r, e)$ with $\psi$ nontrivial is 0-semistable if and only if it is either $\delta_+$-stable or there exists a unique filtration $0 \subset \mathcal{E}' \subset \mathcal{E}$ where $\mathcal{E}'$ is a $\delta_+$-stable ADHM sheaf of rank $r(\mathcal{E}') \geq 1$ and $\mathcal{E}'' = \mathcal{E}/\mathcal{E}'$ a semistable Higgs sheaf of rank $r(\mathcal{E}'') \geq 1$ satisfying

$$\mu_{\delta_+}(\mathcal{E}') > \mu(\mathcal{E}''), \quad \mu_0(\mathcal{E}') = \mu(\mathcal{E}'') = \mu_0(\mathcal{E})$$

Moreover an ADHM sheaf $\mathcal{E} = (E, \Phi_1, \Phi_2, \phi, 0)$ of type $(r, e)$ is 0-semistable if and only if there exists a unique filtration $0 \subset \mathcal{E}' \subset \mathcal{E}$ where $\mathcal{E}' \simeq O = (0, \mathbb{C}, 0, 0, 0, 0)$ and $\mathcal{E}'' = \mathcal{E}/\mathcal{E}'$ is a semistable Higgs sheaf of type $(r, e)$.

(ii) For any stability parameter $\epsilon_- < \delta_- < 0$ an ADHM sheaf $\mathcal{E} = (E, \Phi_1, \Phi_2, \phi, \psi)$ of type $(r, e)$ with $\phi$ nontrivial is 0-semistable if and only if it is either $\delta_-$-stable or there exists a unique filtration $0 \subset \mathcal{E}' \subset \mathcal{E}$ where $\mathcal{E}'$ is a semistable Higgs sheaf of rank $r(\mathcal{E}') \geq 1$ and $\mathcal{E}'' = \mathcal{E}/\mathcal{E}'$ a semistable ADHM sheaf of rank $r(\mathcal{E}'') \geq 1$ satisfying

$$\mu(\mathcal{E}') > \mu_{\delta_-}(\mathcal{E}''), \quad \mu(\mathcal{E}') = \mu_0(\mathcal{E}'') = \mu_0(\mathcal{E})$$

Moreover an ADHM sheaf $\mathcal{E} = (E, \Phi_1, \Phi_2, 0, \psi)$ of type $(r, e)$ is 0-semistable if and only if there exists a unique filtration $0 \subset \mathcal{E}' \subset \mathcal{E}$ where $\mathcal{E}'$ is a semistable Higgs sheaf of type $(r, e)$ and $\mathcal{E}'' = \mathcal{E}/\mathcal{E}' \simeq O = (0, \mathbb{C}, 0, 0, 0, 0)$.

5. **Moduli Stacks and Torus Actions**

The main goal of this section is to prove theorems (1.2) and (1.4). Other moduli stacks needed in the second part of this paper will be constructed as well. Natural torus actions on these stacks will be defined and some structure results for the fixed loci will be proven. In the following $X$ is a smooth projective curve over $\mathbb{C} M_1, M_2$ are fixed line bundles on $X$ equipped with a fixed isomorphism $M_1 \otimes X M_2 \simeq K_X^{-1}$ as in section (2) and $E_{\infty} = \mathcal{O}_X$. The triple $(X, M_1, M_2)$ will be denoted by $\mathcal{X}$. 
5.1. Moduli stacks of \(\delta\)-semistable ADHM sheaves. Let \(S\) be the category of schemes of finite type over \(\mathbb{C}\). Let \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) be a fixed type and \(\delta \in \mathbb{R}_{>0}\) be a fixed stability parameter. Standard arguments show that flat families of \(\delta\)-semistable ADHM sheaves parameterized by complex schemes of finite type form a groupoid \(\mathcal{M}_{\delta}^S(\mathcal{X}, r, e)\) over \(S\). For completeness recall, [20, Def. 3.1], that a flat family of ADHM sheaves on \(X\) parameterized by a scheme \(S\) of finite type over \(\mathbb{C}\) is an ADHM sheaf \(E_S = (E_S, \Phi_S, \phi_S, \psi_S)\) on \(X_S\) with twisting data \((M_1)_S\), \((M_2)_S\) and framing data \(E_{S, \infty} = \mathcal{O}_{X_S}\) so that \(E_S\) is flat over \(S\). An isomorphism of flat families of ADHM sheaves parameterized by \(S\) is an isomorphism of ADHM sheaves on \(X_S\).

Proof of theorem (1.2). Given corollary (3.9) and lemma (2.4) the proof of this theorem is entirely analogous to the proof of theorem [20, Thm. 1.1]. Namely, repeating the basic constructions for decorated sheaves [65, 34, 33, 77, 78, 27] in the present framework, one obtains a quasi-projective parameter space \(R\) for \(\delta\)-semistable ADHM sheaves. Then the groupoid \(\mathcal{M}_{\delta}^S(\mathcal{X}, r, e)\) is shown to be isomorphic to a quotient stack \([R/G]\) for some affine algebraic group \(G\).

\(\square\)

5.2. Other moduli stacks. For future reference we next construct several moduli stacks of objects of the abelian category \(\mathcal{C}_X\), where \(X\) is a smooth projective curve over \(\mathbb{C}\) as in the previous subsection.

Let \(\delta \in \mathbb{R} \setminus \{0\}\) be a stability parameter. Again, using standard arguments we construct the following groupoids over \(S\):

- \(\mathbf{Ob}(\mathcal{X})\): the groupoid of flat families of locally free objects of \(\mathcal{C}_X\). An object of \(\mathbf{Ob}(\mathcal{X})\) over a \(\mathbb{C}\)-scheme \(S\) of finite type is a flat family \(E_S = (E_S, \Phi_S, \phi_S, \psi_S)\) of locally free ADHM quiver sheaves of \(X_S\), where \(E_{S, \infty} = \pi_S^* F_S\) for some locally free \(\mathcal{O}_S\)-module \(F_S\) and the restriction \(E_S|_X\) of any point \(s \in S\) is a locally free object of \(\mathcal{C}_X\). The definition of isomorphisms is standard.
- \(\mathbf{Ob}(\mathcal{X})_{\leq 1}\): the groupoid of all flat families of locally free objects of \(\mathcal{C}_X\) with \(0 \leq v \leq 1\). The construction is the same as above, except that \(v\) may take only values in \(\{0, 1\}\). Therefore the \(\mathcal{O}_S\)-module \(F_S\) in the previous definition is either the zero module or an invertible sheaf on \(S\).
- \(\mathbf{Ob}(\mathcal{X}), \mathbf{Ob}(\mathcal{X}, r, e, v)\): the groupoid of flat families of locally free objects of \(\mathcal{C}_X\) with fixed \(v \in \mathbb{Z}_{\geq 0}\), respectively the groupoid of flat families of locally free objects of \(\mathcal{C}_X\) with fixed type \((r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}\).
- \(\mathbf{Ob}_{\delta}^S(\mathcal{X}, r, e, v)\): the groupoid of flat families of \(\delta\)-semistable objects of \(\mathcal{C}_X\) with fixed type \((r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \{0, 1\}\), with \(\delta \in \mathbb{R}\).
- \(\mathpzc{Ex}(\mathcal{X})\): the groupoid of three term exact sequences of locally free objects of \(\mathcal{C}_X\) constructed by analogy with [38, Def. 7.2]. Note that there are canonical forgetful morphisms \(p, p', p'' : \mathpzc{Ex}(\mathcal{X}) \to \mathbf{Ob}(\mathcal{X})\).
- \(\mathpzc{Ex}(\mathcal{X}, \alpha, \alpha', \alpha'')\): the groupoid of three term exact sequences of locally free objects of \(\mathcal{C}_X\) with fixed type \(\alpha = (r, e, v), \alpha' = (r', e', v'), \alpha'' = (r'', e'', v'')\) in \(\mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}\), \(\alpha = \alpha' + \alpha''\).

Remark 5.1. Note that the stack \(\mathbf{Ob}(\mathcal{X})_0\) is canonically isomorphic to the stack \(\mathfrak{higgs}(\mathcal{X})\) of all locally free Higgs sheaves on \(X\) as defined in (4.1). The stack \(\mathbf{Ob}_{\delta}^S(\mathcal{X}, r, e, 0)\) is canonically isomorphic to the moduli stack \(\mathfrak{higgs}_{\delta}^S(\mathcal{X}, r, e)\) of
semistable Higgs sheaves of type \((r,e)\) on \(X\) for any value of \(\delta \in \mathbb{R} \setminus \{0\}\). Both notations will be used interchangeably from now on.

In order to formulate the next result, note that for any \(\delta \in \mathbb{R}\) and any \((r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) there is a torus action \(\mathbb{C}^\times \times \mathcal{M}_{\delta}^{ss}(X,r,e) \to \mathcal{M}_{\delta}^{ss}(X,r,e)\) defined as follows. For any flat family of \(\delta\)-semistable ADHM sheaves of type \((r,e)\) on \(X\) parameterized by a scheme \(S\) of finite type over \(\mathbb{C}\), and any \(z : S \to \mathbb{C}\) we set

\[
(5.1) \quad \mathcal{E}_S^z = (E_S, \Phi_{S,1,2}, z\phi_S, z^{-1}\psi_S).
\]

Then the following holds.

**Lemma 5.2.** (i) For any \(\delta \in \mathbb{R}\) and any \((r,e,v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \{0,1\}\), \(\mathcal{M}_{\delta}^{ss}(X,r,e,v)\) is an Artin stack of finite type over \(\mathbb{C}\).

(ii) The groupoids \(\mathcal{Ob}(X), \mathcal{Ob}(X)_{\leq 1}, \mathcal{Ob}(X)_v, \mathcal{Ob}(X,r,e,v), r,e,v \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}\) are Artin stacks of locally finite type over \(\mathbb{C}\) and there are open and closed immersion of Artin stacks

\[
(5.2) \quad \mathcal{Ob}(X,r,e,v) \hookrightarrow \mathcal{Ob}(X)_v \hookrightarrow \mathcal{Ob}(X)_{\leq 1} \hookrightarrow \mathcal{Ob}(X)
\]

for any \(r,e,v \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}\).

(iii) For any stability parameter \(\delta \in \mathbb{R}\) and any \((r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) there are open immersions of Artin stacks

\[
(5.3) \quad \mathcal{Ob}_{\delta}^{ss}(X,r,e,1) \hookrightarrow \mathcal{Ob}(X)_1 \hookrightarrow \mathcal{Ob}(X)_{\leq 1},
\]

\[
\mathcal{Ob}_{\delta}^{ss}(X,r,e,0) \hookrightarrow \mathcal{Ob}(X)_0 \hookrightarrow \mathcal{Ob}(X)_{\leq 1}.
\]

(iv) Suppose \(\delta \in \mathbb{R}\) is a noncritical stability parameter of type \((r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\). Then there is an isomorphism of Artin stacks

\[
(5.4) \quad \mathcal{Ob}_{\delta}^{ss}(X,r,e,1) \simeq \mathcal{M}_{\delta}^{ss}(X,r,e)/\mathbb{C}^\times
\]

where the right hand side of equation \((5.4)\) is the quotient stack of the algebraic space \(\mathcal{M}_{\delta}^{ss}(X,r,e)\) by the torus action \((5.1)\). In particular, the automorphism of group of any \(\delta\)-semistable \(\mathcal{E}\) of \(\mathbb{C}^\times\) type \((r,e,1)\) is isomorphic to \(\mathbb{C}^\times\).

(v) The groupoids \(\mathcal{Ex}(X), \mathcal{Ex}(X,\alpha,\alpha',\alpha'')\), \(\alpha,\alpha',\alpha'' \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}\) are algebraic stacks of locally finite type over \(\mathbb{C}\) and the canonical forgetful morphisms \(p,p',p'' : \mathcal{Ex}(X) \to \mathcal{Ob}(X)\) are of finite type over \(\mathbb{C}\).

**Proof.** The proof of lemma \((5.2)\) (i) follows from the boundedness lemma \((2.4)\), again in complete analogy with \([20\text{ Thm }1.1]\).

Given the construction of the parameter spaces for Higgs sheaves \([85\text{ Sec }6.4]\) and for ADHM sheaves \([20\text{ Thm }4.6.2.1]\), statements \((5.2)\)(ii) and \((5.2)\)(v) follow by analogy with \([92\text{ Thm }9.4]\), \([33\text{ Thm }9.4]\). The fact that the natural morphisms \((5.3)\) are open immersions follows from the fact that Higgs sheaf semistability as well as \(\delta\)-semistability are open conditions in flat families.

Lemma \((5.2)\)(iv) follows from lemmas \((3.1)\) and \((3.2)\) taking into account remark \((3.2)\).

\(\square\)

### 5.3. Torus actions and fixed foci.

Next we define natural torus actions on the above moduli stacks. We will employ the definition of group actions on stacks given in \([73\text{ Def. }1.3, \text{ Def. }2.1]\). Let \(T = \mathbb{C}^\times \times \mathbb{C}^\times\). Then there is a torus action \(T^\times \times \mathcal{Ob}(X) \to \mathcal{Ob}(X)\) defined as follows. Given any flat family \(\mathcal{E}_S = \)
\((E_S, (\pi_S^* F_S)^{\oplus v}, \Phi_{S,1}, \Phi_{S,2}, \phi_S, \psi_S)\) parameterized by a scheme \(S\) of finite type over \(\mathbb{C}\), and \(t_1, t_2 : S \to \mathbb{T}\), set
\[
(t_1, t_2) \times E_S \to E_S^{(t_1, t_2)}
\]
where
\[
(5.5) \quad E_S^{(t_1, t_2)} = (E_S, (\pi_S^* L_S)^{\oplus v}, t_1 \Phi_{S,1}, t_2 \Phi_{S,2}, t_1 t_2 \phi_S, \psi_S).
\]
Moreover, if \(\xi : E_S \xrightarrow{\sim} E_S^t\) is an isomorphism of flat families over \(S\), the isomorphism \(\xi^{(t_1, t_2)} : E_S^{(t_1, t_2)} \xrightarrow{\sim} E_S^{(t_1, t_2)}\) is given by the same isomorphism \(\xi : E_S \to E_S\) of coherent \(\mathcal{O}_{X_S}\)-modules, since \(\mathbb{T}\) acts linearly on the data \((\Phi_{S,1,2}, \phi_S, \psi_S)\) leaving the underlying coherent sheaf \(E_S\) unchanged.

Let \(S \simeq \mathbb{C}^* \subset \mathbb{T}\) be the antidiagonal subtorus defined by the embedding \(t \to (t^{-1}, t)\). Then the action (5.3) induces an \(S\)-action on \(\text{Db}(\mathcal{X})_{\leq 1}\). In this case we will use the notation \(E_S = E_S^{(t^{-1}, t)}\), \(\xi^t = \xi(t^{-1}, t)\).

Obviously, there are analogous actions on any substack of the form \(\text{Db}(\mathcal{X})_{\leq 1}\), \(\text{Db}(\mathcal{X})_v\), \(\text{Db}(\mathcal{X}, r, e, v)\), \(\text{Db}_{\text{ss}}^S(\mathcal{X}, r, e, v)\), with \((r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}\) so that the open immersions (5.2), (5.3) are equivariant. Moreover, there are analogous torus actions \(\mathbb{T} \times \text{Mod}_{\mathcal{X}}^S(\mathcal{X}, r, e) \to \text{Mod}_{\mathcal{X}}^S(\mathcal{X}, r, e), \mathbb{T} \times \text{Mod}_{\mathcal{X}}^S(\mathcal{X}, r, e) \to \text{Mod}_{\mathcal{X}}^S(\mathcal{X}, r, e)\) obtained by setting \(v = 1\) and \(F_S = \mathcal{O}_S\) in (5.3).

Finally note that since the \(\mathbb{T}\) and \(\mathbb{C}\) actions defined above are linear on the ADHM data, it is straightforward to check that they can be canonically lifted to torus actions on the stack \(\mathcal{C}(\mathcal{X})\) so that the forgetful morphisms \(\mathbb{p}, \mathbb{p}', \mathbb{p}''\) are equivariant. A flat family
\[
0 \to E_S' \xrightarrow{\xi_S} E_S^{n_S} \xrightarrow{\eta_S} E_S'' \to 0
\]
of three term exact sequences is mapped by \((t_1, t_2) : S \to \mathbb{T}\) to
\[
0 \to E_S^{(t_1, t_2)} \xrightarrow{\xi_S^{(t_1, t_2)}} E_S^{(t_1, t_2)} \xrightarrow{\eta_S^{(t_1, t_2)}} E_S^{(t_1, t_2)} \to 0
\]
where \(\xi_S^{(t_1, t_2)}\), \(\eta_S^{(t_1, t_2)}\) have the same underling morphisms of \(\mathcal{O}_{X_S}\)-modules as \(\xi_S\), \(\eta_S\). Obviously, this torus action preserves the substacks of fixed type \((\alpha, \alpha', \alpha'')\).

Let \(\mathbb{M}\) denote one of the stacks in lemma (5.2(i) - (5.2(iv)). Let \(\mathbb{M}^S\) be the stack theoretic fixed locus as defined in [4] Prop 2.5]. Recall that a morphism \(S \to \mathbb{M}^S\), where \(S\) is a scheme of finite type over \(\mathbb{C}\), is determined by the data \(\{\xi_S(t), \xi_S(t) t : S \to S\}\) where \(\xi_S\) is a flat family of locally free objects of \(\mathcal{C}_X\) parameterized by \(S\) and for any morphism \(t : S \to \mathbb{S}\)
\[
(5.6) \quad \xi_S(t) : E_S \xrightarrow{\sim} E_S^t
\]
is an isomorphism of flat families over \(S\) satisfying the identity
\[
(5.7) \quad \xi_S(t') = \xi_S(t') \circ \xi_S(t)
\]
for all \(t, t' : S \to \mathbb{S}\) (see [4] Prop 2.5]). In the following a flat family satisfying this property will be called \(S\)-fixed up to isomorphism.

**Lemma 5.3.** Let \(E_S = (E_S, (\pi_S^* F_S, \Phi_{S,1,2}, \phi_S, \psi_S)\) be a \(\mathbb{S}\)-fixed flat family of locally free objects of \(\mathcal{C}_X\) parameterized by a connected scheme \(S\) of finite type over \(\mathbb{C}\). Then there are direct sum decompositions
\[
(5.8) \quad E_S \simeq \bigoplus_{n \in I} E_S(n), \quad F_S = \bigoplus_{n \in J} F_S(n)
\]
with \(I, J \subset \mathbb{Z}\) finite subsets, satisfying the following conditions
If $E_S$, respectively $F_S$, are not the zero sheaf, no direct summand in $(5.8)$ is the zero sheaf.

- All components $\Phi_S, i(n, m) : E_S(n) \otimes X_S (M_i)_S \rightarrow E_S(m)$, $n, m \in I$, with respect to the direct sum decomposition of $X_S$ are identically zero if $m \neq n + (-1)^{i-1}$ for $i = 1, 2$.
- All components $\phi_S(n, m) : E_S(n) \otimes X_S M_S \rightarrow \pi_S^* F_S(m)$, $n \in I$, $m \in J$, with respect to the direct sum decomposition of $X_S$ are identically zero if $m \neq n$.
- All components $\psi_S(n, m) : \pi_S^* F_S(n) \rightarrow E_S(m)$, $n \in I$, $m \in J$, with respect to the direct sum decomposition of $X_S$ are identically zero if $m \neq n$.

Proof. Note that $\xi_S(t)^t$ in equation $(5.7)$ is identical to $\xi_S(t)$ as an isomorphism of $O_{X_S}$-modules, as observed below $(5.6)$. This implies that the underlying coherent $O_{X_S}$-module $E_S$ of a flat family $E_S$ fixed by $S$ up to isomorphism has a $S$-linearization. Then lemma $(5.3)$ follows from a standard analysis of the fixed locus conditions analogous to [20, Prop. 3.15]. Details will be omitted.

5.4. Virtual smoothness for noncritical $\delta$. In this subsection we prove theorem [4.1]. By analogy with the proof of theorem [20, Thm. 1.5] we first need a vanishing result in the deformation theory of $\delta$-stable ADHM sheaves. According to [20, Prop 4.5, 4.9] the deformation complex $C(\mathcal{E})$ of a locally free ADHM sheaf $\mathcal{E}$ on $X$ is the following complex of $O_X$-modules

$$(5.9)$$

$$0 \rightarrow \text{Hom}_X(E \otimes X M_1, E) \oplus \text{Hom}_X(E \otimes X M_2, E) \oplus \text{Hom}_X(E \otimes X M, O_X) \oplus \text{Hom}_X(O_X, E)$$

where

$$d_1(\alpha) = (\alpha \circ \Phi_1 + \Phi_1 \circ (\alpha \otimes 1_{M_1}), \quad - \alpha \circ \Phi_2 + \Phi_2 \circ (\alpha \otimes 1_{M_2}),$$

$$\phi \circ (\alpha \otimes 1_M), \quad - \alpha \circ \psi)$$

for any local section $\alpha$ of $\text{Hom}_X(E, E)$, and

$$d_2(\beta_1, \beta_2, \gamma, \delta) = \beta_1 \circ (\Phi_2 \otimes 1_{M_1}) - \Phi_2 \circ (\beta_1 \otimes 1_{M_2})$$

$$- \beta_2 \circ (\Phi_1 \otimes 1_{M_2}) + \Phi_1 \circ (\beta_2 \otimes 1_{M_1})$$

$$+ \psi \circ \gamma + \delta \circ \phi$$

for any local sections $(\beta_1, \beta_2, \gamma, \delta)$ of the middle term of $(5.9)$. The degrees of the terms of $(5.9)$ are 0, 1, 2 respectively.

Then the main technical element in the proof of virtual smoothness is the following vanishing result for the hypercohomology groups of the complex $(5.9)$.

Lemma 5.4. Let $\mathcal{E}$ be a $\delta$-stable ADHM sheaf on $X$. Then $\mathbb{H}^i(X, C(\mathcal{E})) = 0$, for all $i \geq 3$ and for all $i \leq 0$.

Proof. The proof is similar to the proof of lemma [20, Lemma 4.10]. We will present the detailed computation for $i \geq 3$. The proof for $i \leq 0$ is entirely analogous. Since the degrees of the three terms in $(5.9)$ are 0, 1, 2 respectively, it follows that
all terms $E^{p,q}_i, p + q \geq 4$ in the standard hypercohomology spectral sequence are trivially zero. Therefore $H^i(X, \mathcal{C}(E)) = 0$ for $i \geq 4$. Moreover, the only nonzero term on the diagonal $p + q = 3$ is

$$E^{2,1}_1 = H^1(X, \mathcal{H}om_X(E \otimes_X M, E)).$$

The differential $d^{1,1}_1 : E^{1,1}_1 \to E^{2,1}_1$ is the map

$$H^1(X, \mathcal{H}om_X(E \otimes_X M_1, E))$$
$$\oplus$$
$$H^1(X, \mathcal{H}om_X(E \otimes_X M_2, E))$$
$$\oplus$$
$$d^{1,1}_1$$
$$H^1(X, \mathcal{H}om_X(E \otimes_X M, \mathcal{O}_X))$$
$$\oplus$$
$$H^1(X, \mathcal{H}om_X(\mathcal{O}_X, E))$$

induced by $d_2$.

We claim that this map is surjective if $E$ is $\delta$-stable. In order to prove this claim we will prove that the dual map $(d^{1,1}_1)^{\vee} : (E^{2,1}_1)^{\vee} \to (E^{1,1}_1)^{\vee}$ is injective if $E$ is $\delta$-stable. Using Serre duality and the condition $M \simeq K_X^{-1}$, the dual differential is (up to isomorphism) a linear map of the form

$$\text{Hom}_X(E, E \otimes_X M_{-1})$$
$$\oplus$$
$$\text{Hom}_X(E, E \otimes_X M_{-1})$$
$$\oplus$$
$$\text{Hom}_X(E, E \otimes_X M_{-1})$$
$$\oplus$$
$$\text{Hom}_X(E, \mathcal{O}_X, E)$$
$$\oplus$$
$$\text{Hom}_X(\mathcal{O}_X, E)$$

which maps a global homomorphism $\alpha \in \text{Hom}_X(E, E)$ to

$$-(\alpha \circ \Phi_2) \otimes 1_{M_{-1}^{-1}} + (\Phi_2 \otimes 1_{M_{-1}^{-1}}) \circ \alpha$$
$$\alpha \circ \Phi_1 \otimes 1_{M_{-1}^{-1}} - (\Phi_1 \otimes 1_{M_{-1}^{-1}}) \circ \alpha$$
$$\phi \otimes 1_{M_{-1}} \circ \alpha$$

Suppose $\alpha \in \text{Ker}((d^{1,1}_1)^{\vee})$, is nonzero. It follows that $\text{Ker}(\alpha)$ is $\Phi$-invariant and $\text{Im}(\psi) \subseteq \text{Ker}(\alpha)$. Since $\alpha$ is a morphism of locally free sheaves, $\text{Ker}(\alpha)$ must be a saturated subsheaf of $E$. Moreover, $\text{Im}(\alpha)$ is $\Phi$-invariant and $\text{Im}(\alpha) \otimes_X M \subseteq \text{Ker}(\phi)$. Then the stability conditions in definition (2.1) immediately lead to a contradiction.

Let $T = \mathbb{C}^\times \times \mathbb{C}^\times$ and $S \subset T$ act on $\mathfrak{M}^{\delta}(\mathcal{X}, r, e)$ as in (5.5). These are the torus actions employed in theorem (1.4) whose proof will be presented below.

**Proof of theorem (1.4).** If $\delta \in \mathbb{R}_{>0}$, is a noncritical stability parameter of type $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, any $\delta$-semistable ADHM sheaf of type $(r, e)$ on $\mathcal{X}$ is $\delta$-stable. Then, given lemma (5.4), the existence of a $T$-equivariant perfect tangent-obstruction theory follows by repeating the steps in (20, Sect 5.) in the present setting. Details will be omitted. Obviously, the resulting perfect tangent-obstruction theory is also $S$-equivariant.
In order to prove the second part of theorem \[1.3\], note that there is a universal ADHM locally free ADHM sheaf on \[\mathcal{M}_3^\mathrm{ss} (\mathcal{X}, r, e) \times X\] since all stable objects have trivial automorphisms. Let \(p : \mathcal{M}_3^\mathrm{ss} (\mathcal{X}, r, e) \times X \to \mathcal{M}_3^\mathrm{ss} (\mathcal{X}, r, e)\), \(\pi_X : \mathcal{M}_3^\mathrm{ss} (\mathcal{X}, r, e) \times X \to X\) be the canonical projections. Let \(\mathcal{C}(\mathcal{E})\) be the locally free complex \[5.9\] associated to \(\mathcal{E}\). Then Grothendieck duality \[6.1\] for the projection morphism \(p\) yields an isomorphism

\[(\mathcal{R}p_*(\mathcal{C}(\mathcal{E})))^\vee \simeq \mathcal{R}p_* \mathcal{R}\mathcal{H}\text{om}(\mathcal{C}(\mathcal{E}), \pi_X^* K_X [1]).\]

This isomorphism is compatible with the induced \(T\) as well as \(S\) actions.

Taking into account the isomorphism \(M \simeq K_X^{-1}\) and the fact that \(\mathcal{C}(\mathcal{E})\) is locally free, a straightforward computation shows that

\[\mathcal{R}\mathcal{H}\text{om}(\mathcal{C}(\mathcal{E}), \pi_X^* K_X [1]) \simeq \mathcal{C}(\mathcal{E})[1]\]

as \(S\)-equivariant complexes. Therefore we obtain an isomorphism of \(S\)-equivariant complexes

\[(\mathcal{R}p_*(\mathcal{C}(\mathcal{E})))^\vee \simeq \mathcal{R}p_* (\mathcal{C}(\mathcal{E}))[1].\]

This yields the required \(S\)-equivariant nondegenerate pairing on the perfect tangent-obstruction theory of \(\mathcal{M}_3(\mathcal{X}, r, e)\). \(\square\)

Note that a direct consequence of lemma \[4.4\] is

**Corollary 5.5.** Under the assumptions of lemma \[4.4\], for any \(n \in \mathbb{Z}_{\geq 0}\) and any \(\gamma_1, \gamma_2 \in (\delta_n, \delta_{n+1})\), there is a canonical isomorphism of algebraic moduli spaces

\[\mathcal{M}_3^{\ast\ast} (\mathcal{X}, r, e) \simeq \mathcal{M}_3^{\ast\ast} (\mathcal{X}, r, e)\]

equipped with \(T\)-equivariant as well as \(S\)-equivariant symmetric perfect tangent-obstruction theories.

### 6. Properness of Torus Fixed Loci

The main goal of this section is to prove theorem \[1.5\] working under the same conditions as in section \[3\]. The proof will rely on several preliminary results.

#### 6.1. \(S\)-fixed semistable Higgs sheaves

Let \(K\) be a field over \(\mathbb{C}\) and let \(\mathcal{E}_K\) be a semistable Higgs sheaf of type \((r, e)\) on \(X_K = X \times \text{Spec}(K)\) fixed by \(S\) up to isomorphism. In particular, \(\mathcal{E}_K\) determines a morphism \(\epsilon_K : \text{Spec}(K) \to \mathcal{Higgs}^{\ast\ast}(\mathcal{X}, r, e)\) to the moduli stack of semistable Higgs sheaves of type \((r, e)\) on \(X\). Recall – theorem \[4.3\] – that there is a surjective universally closed morphism \(\varphi : \mathcal{Higgs}^{\ast\ast}(\mathcal{X}, r, e) \to \mathcal{Higgs}^{\ast\ast}(\mathcal{X}, r, e)\) which satisfies the properties listed in \[1\] Thm. 4.14]. There is also a proper Hitchin morphism \(h : \mathcal{Higgs}^{\ast\ast}(\mathcal{X}, r, e) \to \mathbb{H}\) where \(\mathbb{H} = \oplus_{p=0}^r H^0(X, \text{Sym}^p(M_1^{-1} \oplus M_2^{-1}))\). Then we claim

**Lemma 6.1.** \(h \circ \varphi \circ \epsilon_K = 0\).

**Proof.** Let \(\mathcal{E}_K = (E_K, \Phi_{K, 1}, \Phi_{K, 2})\). Given the construction of the Hitchin map, the morphism \(h \circ \varphi \circ \epsilon_K\) is determined by the data

\[(\text{tr}(\Phi_{K, 1}^{n_1} \Phi_{K, 2}^{n_2})), \quad n_1, n_2 \leq 0, \quad n_1 + n_2 \leq r\]

where we have used the conventions in remark \[4.4\] (ii).

In order to prove lemma \[6.1\] it suffices to show that the data \[6.1\] vanishes for any semistable \(\mathcal{E}_K\) fixed by \(S\). Any such Higgs sheaf must satisfy conditions (i) and (ii) of lemma \[4.3\] where \(S = \text{Spec}(K)\).
Note that tr(Φ^n_1K,1, Φ^n_2K,2) is a homogeneous element of H_K for the action of S_K with weight n_1 - n_2. Therefore if (E_K, Φ_1K,1, Φ_2K,2) is fixed by the S_K-action, we must have

\[(6.2) \quad tr(Φ^n_1K,1, Φ^n_2K,2) = 0\]

for all n_1, n_2 ≥ 0, 0 < n_1 + n_2 ≤ r, n_1 ≠ n_2. In order to prove the claim we have to show that the vanishing result (6.2) holds for n_1 = n_2 as well.

Note that only finitely many terms in character decomposition of E_K are non-trivial. Hence

\[(6.3) \quad E_K = \bigoplus_{s_1 ≤ n ≤ s_2} E_K(n)\]

for some s_1, s_2 ∈ Z, s_1 ≤ s_2. If s_1 = s_2, Φ_1K,1, Φ_2K,2 are trivial according to lemma (6.3) and there is nothing to prove. Therefore we will assume s_1 < s_2. Then lemma (6.3) implies that only components of the form

\[Φ_{K,1}(n) : E_K(n) → E_K(n+1) \quad Φ_{K,2}(n-1) : E_K(n) → E_K(n-1)\]

are allowed to be non-trivial. This implies that the monomials (Φ_1K,1Φ_2K,2)^n, n ≥ 1 have the following block form with respect to the decomposition (6.3)

\[(Φ_{K,1}Φ_{K,2})^n = \text{diag}(0, (Φ_{K,1}(s_1)Φ_{K,2}(s_1))^n, \ldots, (Φ_{K,1}(s_2 - 1)Φ_{K,2}(s_2 - 1))^n).\]

Using the structure results proven in lemma (6.3), condition (A.1) in definition (A.1) is equivalent to the following relations

\[(6.4) \quad Φ_{K,1}(n)Φ_{K,2}(n) = Φ_{K,2}(n+1)Φ_{K,1}(n+1)\]

for all s_1 ≤ n ≤ s_2 - 1. In particular,

\[(6.5) \quad Φ_{K,1}(s_2 - 1)Φ_{K,2}(s_2 - 1) = 0, \quad Φ_{K,2}(s_1)Φ_{K,1}(s_1) = 0.\]

If s_2 = s_1 + 1, the required vanishing result follows immediately from relations (6.5). Hence we will assume s_1 ≤ s_2 - 2 in the following. Then we will prove by an inductive argument that

\[(6.6) \quad tr(Φ_{K,1}(s_2 - k)Φ_{K,2}(s_2 - k))^n = 0\]

for all 1 ≤ k ≤ (s_2 - s_1) and all n ≥ 1. The case k = 1 follows immediately from relations (6.5). Then note that for any 2 ≤ k ≤ (s_2 - s_1) and any n ≥ 1 we have

\[tr(Φ_{K,1}(s_2 - k)Φ_{K,2}(s_2 - k))^n = tr(Φ_{K,2}(s_2 - k + 1)Φ_{K,1}(s_2 - k + 1))^n\]

using invariance of the trace under cyclic permutations of the arguments. This proves the inductive step, hence the required vanishing result follows.

Next we prove a result concerning extensions of flat families of semistable Higgs sheaves. Let R be a discrete valuation ring over C with fraction field K. Let p ∈ Spec(R) denote the closed point. Let E_K be a semistable Higgs sheaf of type (r, e) ∈ Z_{≥ 1} × Z on X_K and let ε_K : Spec(K) → η\text{Higgs}^{\text{ss}}(X, r, e) be the corresponding morphism to the algebraic stack of semistable Higgs bundles of type (r, e) on X.

**Lemma 6.2.** Suppose the morphism

\[(6.7) \quad h \circ φ \circ ε_K : \text{Spec}(K) → \mathbb{H}\]

is trivial. Then there exists a flat family E_R of semistable Higgs sheaves of type (r, e) on X parameterized by Spec(R) so that E_R|_{X_K} ≃ E_K.
Proof. Let
\[ 0 = JH_0(E_K) \subset JH_1(E_K) \subset \cdots \subset JH_j(E_K) = E_K \]
be the Jordan-Hölder filtration of \( E_K \) in the abelian category \( \mathcal{C}_{X_K} \). Let \( G_{K,i} = JH_i(E_K)/JH_{i-1}(E_K) \), \( i = 1, \ldots, j \) denote the successive quotients of the above filtration; \( G_{K,i} \) are projective schemes, it follows that the morphisms \( \eta \) are projective, it follows that there exist flat families \( \varphi_i : \text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{Z}) \) so that the Hitchin maps for all \( i = 1, \ldots, j \).

Let \((r_i, e_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) denote the type of \( G_{K,i} \), \( i = 1, \ldots, j \). Note that each successive quotient \( G_{K,i} \) determines a morphism \( \epsilon_{K,i} : \text{Spec}(\mathbb{C}) \to \text{Higgs}^{ss}(\mathcal{X}, r_i, e_i) \) for all \( i = 1, \ldots, r \). Let \( h_i : \text{Higgs}^{ss}(\mathcal{X}, r_i, e_i) \to \mathbb{H}_i \) denote the Hitchin maps and \( \varphi_i : \text{Higgs}^{ss}(\mathcal{X}, r_i, e_i) \to \text{Higgs}^{ss}(\mathcal{X}, r_i, e_i) \) denote the morphisms provided by \([1] \text{ Thm 4.14}\) for each \( i = 1, \ldots, j \). Note that under the assumptions of lemma \([6,2]\), \( h_i \circ \varphi_i \circ \epsilon_{K,i} = 0 \) by the construction of the Hitchin map. Therefore \( \varphi_i \circ \epsilon_{K,i} : \text{Spec}(\mathbb{C}) \to \text{Higgs}^{ss}(\mathcal{X}, r_i, e_i) \) factors through a morphism \( \eta_{K,i} : \text{Spec}(\mathbb{C}) \to \text{Higgs}^{ss}(\mathcal{X}, r_i, e_i) \) where \( \text{Higgs}^{ss}(\mathcal{X}, r_i, e_i) = h_i^{-1}(0) \) is the central fiber of the Hitchin map, for all \( i = 1, \ldots, j \). Since \( \text{Higgs}^{ss}(\mathcal{X}, r_i, e_i) \) is a projective scheme, it follows that the morphisms \( \eta_{K,i} \) admit extensions \( \eta_{R,j} : \text{Spec}(\mathbb{R}) \to \text{Higgs}^{ss}(\mathcal{X}, r_i, e_i) \) for all \( i = 1, \ldots, j \). Since the morphisms \( \varphi_i \) are surjective, it follows that there exist flat families \( E_{R,i} \), \( i = 1, \ldots, j \), of semistable Higgs sheaves on \( X_R \) parameterized by \( \text{Spec}(\mathbb{R}) \) so that \( G_{R,i}|_{X_K} \) are S-equivalent to \( G_{K,i} \) for all \( i = 1, \ldots, j \). However since \( G_{K,i} \) are stable Higgs sheaves, it follows that \( G_{R,i}|_{X_K} \) must be also stable and isomorphic to \( G_{K,i} \) for all \( i = 1, \ldots, j \). Therefore we have established so far that the successive quotients \( G_{K,i} \), \( i = 1, \ldots, j \) extend to \( X_R \) up to isomorphism.

The proof of lemma \([6,2]\) will be concluded by an inductive argument. Suppose there exists a flat family \( E_{R,i} \) of semistable Higgs sheaves on \( X \) parameterized by \( \text{Spec}(\mathbb{R}) \) so that \( E_{R,i}|_{X_K} \simeq JH_i(E_K) \) for some value of \( i = 1, \ldots, j-1 \). In particular this holds for \( i = 1 \) according to the previous paragraph. Then we claim that there exists a flat family \( E_{R,i+1} \) of semistable Higgs sheaves on \( X \) parameterized by \( \text{Spec}(\mathbb{R}) \) so that \( E_{R,i+1}|_{X_K} \simeq JH_{i+1}(E_K) \). Note that since \( E_{R,i}|_{X_K} \simeq JH_i(E_K) \) there is an exact sequence of Higgs sheaves on \( X_K \)
\[ 0 \to E_{R,i}|_{X_K} \xrightarrow{\eta_{R,i+1}} JH_{i+1}(E_K) \xrightarrow{\eta_{R,i+1}} G_{R,i+1}|_{X_K} \to 0 \]
which determines an extension class \( e_K \in \mathbb{H}^1(X_K, \mathcal{C}(G_{R,i+1}|_{X_K}, E_{R,i}|_{X_K})) \) of objects in \( \mathcal{C}_{X_K} \) according to proposition \([3,4]\). Set \( X_R = X \times \text{Spec}(\mathbb{R}) \). Let \( \mathcal{C}(G_{R,i+1}, \mathcal{E}_{R,i}) \) be the complex \([3,4]\) of coherent locally free \( \mathcal{O}_{X_R} \)-modules corresponding to the pair \( (G_{R,i+1}, \mathcal{E}_{R,i}) \).

By construction we have a canonical isomorphism
\[ \mathcal{C}(G_{R,i+1}, \mathcal{E}_{R,i})|_{X_K} \simeq \mathcal{C}(G_{R,i+1}|_{X_K}, E_{R,i}|_{X_K}) \]
of complexes of \( \mathcal{O}_{X_K} \)-modules. Since \( X_K \subset X_R \) is a Zariski open subset, this isomorphism yields a well defined restriction map
\[ r : \mathbb{H}^1(X_R, \mathcal{C}(G_{R,i+1}, \mathcal{E}_{R,i})) \to \mathbb{H}^1(X_K, \mathcal{C}(G_{R,i+1}|_{X_K}, E_{R,i}|_{X_K})) \]
which is a morphism of \( \mathbb{R} \)-modules. This follows easily for example from the construction of hypercohomology complexes in terms of Čech resolutions.

Let \( t \in R \) be a uniformizing parameter. Since \( R \) is a discrete valuation ring, there exists \( n \in \mathbb{Z} \) and an extension class \( e_R \in \mathbb{H}^1(X_R, \mathcal{C}(G_{R,i+1}, JH_i(\mathcal{E}_R))) \), so
that
\[ r(\epsilon_R) = t^n \epsilon_K. \]

According to proposition (3.15) and corollary (3.17), \( \epsilon_R \) determines an extension of locally free Higgs sheaves on \( X_R \) of the form

\[ 0 \longrightarrow \mathcal{E}_{R,i} \xrightarrow{f_{R,i}} \mathcal{E}_{R,i+1} \xrightarrow{g_{R,i}} \mathcal{G}_{R,i+1} \longrightarrow 0 \]

up to isomorphism of extensions. By construction, any such extension is isomorphic to \( \mathcal{J}_{H_i+1}(\mathcal{E}_K) \) when restricted to \( X_K, \mathcal{E}_{R,i+1} |_{X_K} \approx \mathcal{J}_{H_{i+1}}(\mathcal{E}_K) \). Moreover, since \( \mathcal{E}_{R,i}, \mathcal{G}_{R,i+1} \) are flat families of semistable Higgs sheaves on \( X \) of slope equal to \( r/e \) it follows that \( \mathcal{E}_{R,i} \) is also a flat family of semistable Higgs sheaves on \( X \) of the same slope. This concludes the inductive argument.

\[ \square \]

6.2. Asymptotic Fixed Loci and Framed Hitchin Pairs. Next we prove that a similar result holds for asymptotically stable ADHM sheaves of type \((r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) on \( X \).

**Lemma 6.3.** Suppose \( \delta > \delta_N \) is an asymptotic stability parameter. Then the \( S \)-fixed locus \( \mathcal{M}_{\delta}^a(X, r, e)^S \) is isomorphic to a projective scheme over \( \mathbb{C} \).

**Proof.** According to [20, Lemma 2.5] any asymptotically stable ADHM sheaf on \( X \) with \( E_\infty = \mathcal{O}_X \) has \( \phi = 0 \). Then note that if \( \mathcal{E} = (E, \Phi_1, \Phi_2, \psi) \) is a locally free ADHM sheaf with \( \phi = 0 \), it follows that the dual ADHM sheaf \( \tilde{\mathcal{E}} \) constructed in equation (2.4) determines a framed Hitchin pair as definition (A.5). Moreover, it is straightforward to check that \( \mathcal{E} \) is an asymptotically stable ADHM sheaf if and only if \( \tilde{\mathcal{E}} \) is an asymptotically stable framed Hitchin pair.

Since this correspondence is also holds for flat families of asymptotically stable objects, it follows that we have an isomorphism of algebraic moduli spaces \( \mathcal{M}_{\delta}^a(X, r, e)^S \approx FH_{\delta}^a(X, r, e) \). Moreover there is a natural \( S \)-action on \( FH_{\delta}^a(X, r, e) \) so that this is an equivariant isomorphism. Therefore the \( S \)-fixed loci will also be isomorphic.

Now one can prove that the fixed locus \( FH_{\delta}^a(X, r, e)^S \) is a proper scheme over \( \mathbb{C} \) repeating the proof of lemmas (6.1), (6.2) since theorem (A.8) provides a proper Hitchin morphism for asymptotically stable framed Higgs pairs. Since \( FH_{\delta}^a(X, r, e)^S \) is also a closed subscheme of \( FH_{\delta}^a(X, r, e) \), which is quasi-projective, it follows that \( FH_{\delta}^a(X, r, e)^S \) is a projective scheme over \( \mathbb{C} \).

\[ \square \]

6.3. \( S \)-fixed loci and Harder-Narasimhan filtrations. Another technical lemma required in the proof of theorem (1.5) is the following. Let \((r,e) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}\) be a fixed type so that the set of critical stability parameters of type \((r,e)\) is nonempty. Let \( \delta_i \in \mathbb{R}_{>0}, i = 1, \ldots, N \) be a critical stability parameter of type \((r,e)\) and let \( \delta_- \in (\delta_{i-1}, \delta_i), \delta_+ \in (\delta_i, \delta_{i+1}) \). Again we formally set \( \delta_0 = 0 \) and \( \delta_{N+1} = +\infty \) in order to simplify the notation. Let \( K \) be a field over \( \mathbb{C} \) and let \( \mathcal{E}_K \) be a locally free ADHM sheaf of type \((r,e)\) on \( X_K = X \times \text{Spec}(K) \). Consider the following situations

(a) \( \mathcal{E}_K \) is \( \delta_+ \)-stable and \( \delta_- \)-unstable, or
(b) \( \mathcal{E}_K \) is \( \delta_- \)-stable and \( \delta_+ \)-unstable.
Recall that according to corollary [11,2], if case (a) above holds, \( E_K \) has a one-step Harder-Narasimhan filtration \( 0 \subset E'_K \subset E_K \) with respect to \( \delta \)-stability so that \( E'_K \) is a semistable Higgs sheaf on \( X_K \) and \( E''_K = E_K/E'_K \) is a \( \delta \)-stable ADHM sheaf on \( X_K \). Similarly, if case (b) above holds, \( E_K \) has a one-step Harder-Narasimhan filtration \( 0 \subset E'_K \subset E_K \) with respect to \( \delta \)-stability so that \( E'_K \) is a \( \delta \)-stable ADHM sheaf on \( X_K \) and \( E''_K = E_K/E'_K \) is a semistable Higgs sheaf on \( X_K \). Let \( (r', e') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}, (r'', e'') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \) denote the type of \( E'_K, E''_K \) respectively. Then we have the following

**Lemma 6.4.** Suppose \( E_K \) satisfies either condition (a) or condition (b) above and is fixed by \( S \) up to isomorphism. Then \( E'_K, E''_K \) are also fixed by \( S \) up to isomorphism.

**Proof.** It suffices to consider only one case since the proof of other one is identical. Let \( E_K \) be an ADHM sheaf on \( X_K \) satisfying condition (a). Then for each morphism \( t : K \to S \) there is an isomorphism \( \xi_K(t) : E_K \to E'_K \) as in equation (5.6). Moreover it straightforward to check that the torus action (5.3) preserves the Harder-Narasimhan filtrations i.e.

\[
(E_K^t) = (E'_K)^t.
\]

Since Harder-Narasimhan filtrations are unique, they must also be preserved by isomorphisms, hence lemma (6.4) follows.

### 6.4. Elementary Modifications of Families of ADHM Sheaves

Let \( X \) be a smooth projective curve over \( \mathbb{C} \) and \( M_1, M_2 \) fixed line bundles on \( X \) as in the previous section. Let \( R \) be a discrete valuation ring over \( \mathbb{C} \) with fraction field \( K \). Let \( m \in R \) be the maximal ideal of \( R \) and \( t \in m \) a uniformizing parameter. Let \( X_R = X \times \text{Spec}(R) \) and let \( \pi_X : X_R \to X, \pi_R : X_R \to \text{Spec}(R) \) denote the canonical projections. Let \( p \in \text{Spec}(R) \) denote the closed point and \( X_p = X \times \{p\} \). Note that the residual field of \( p \) is \( k \cong \mathbb{C} \) and \( X_p \cong X \). Let \( (M_1)_R = \pi_X^*M_1, (M_2)_R = \pi_X^*M_2 \).

For all \( n \in \mathbb{Z}_{\geq 0} \) set \( X_n = X \times \text{Spec}(R/m^n) \). Note that \( X_n \) is a closed subscheme of \( X_R \) and let \( \iota_n : X_n \hookrightarrow X_R \) denote the closed embedding. Obviously, \( X_n \) is a Cartier divisor on \( X_R \) defined by the function \( t^n \in \Gamma_{X_R}(\mathcal{O}_{X_R}) \).

Let \( E_R = (E_R, \Phi_{R, 1}, \Phi_{R, 2}, \phi_R, \psi_R) \) be a locally free ADHM quiver sheaf on \( X_R \) with twisting data \( ((M_1)_R, (M_2)_R) \) and framing data \( (E_R)_\infty = \mathcal{O}_{X_R} \). For any \( n \in \mathbb{Z}_{\geq 1} \), let \( E_n = E_R \otimes_{X_R} \mathcal{O}_{X_n} \) denote the ADHM quiver sheaf obtained by restricting the data of \( E_R \) to \( X_n \). Suppose for some \( n \in \mathbb{Z}_{\geq 1} \) there is an exact sequence in the abelian category \( \mathcal{E}_{X_n} \) of the form

\[
0 \to E''_n \to E'_n \to E''_n \to 0
\]

where

\[
E'_n = (E'_n, \mathcal{O}_{X_n}^{e'_n}, \Phi'_{n, 1}, \Phi'_{n, 2}, \psi'_n), \quad E''_n = (E''_n, \mathcal{O}_{X_n}^{e''_n}, \Phi''_{n, 1}, \Phi''_{n, 2}, \psi''_n)
\]

are locally free ADHM quiver sheaves on \( X_n \). In particular, \( v'_n, v''_n \in \{0, 1\}, v'_n + v''_n = 1 \), and the morphisms

\[
\mathcal{O}_{X_n}^{e'_n} \to \mathcal{O}_{X_n}, \quad \mathcal{O}_{X_n} \to \mathcal{O}_{X_n}^{e''_n}
\]

are given by multiplication by complex numbers. Note that by pushing forward the data of \( E''_n \) via the closed embedding \( \iota_n : X_n \hookrightarrow X_R \) we obtain a torsion ADHM quiver sheaf \( \iota_{n*}E''_n \) on \( X_R \) and (6.8) yields a surjective morphism \( E_R \to \iota_{n*}E''_n \to 0 \) in the abelian category \( \mathcal{Q}_{X_R} \). Let \( F_R \) be the kernel of the projection \( E_R \to \iota_{n*}E''_n \to 0 \).
Lemma 6.5. (i) \( F_R \) is a locally free ADHM quiver sheaf on \( X_R \) with twisting data \(((M_1)_R, (M_2)_R)\) and framing data \((F_R)_\infty = \mathcal{O}_{X_R}\).

(ii) The restriction \( \iota_n^* F_R \) fits in an exact sequence in the abelian category \( C_{X_n} \) of the form

\[
0 \to \mathcal{E}_n^\prime \to \iota_n^* F_R \to \mathcal{E}_n^\prime \to 0.
\]

(iii) There is an isomorphism of ADHM sheaves on \( X_K \) \( \xi_K : F_R|_{X_K} \to \sim E_R|_{X_K} \).

Proof. Let \( F_R = (F_R, (F_R)_\infty, \rho_R, \eta_R) \) be the ADHM data of \( F_R \). By construction, \( F_R \) fits in an exact sequence of \( \mathcal{O}_{X_R} \)-modules

\[
0 \to F_R \to E_R \to \iota_n^* E_n^\prime \to 0.
\]

Since \( X_n \) is a Cartier divisor on \( X_R \) and \( E_n^\prime \) is a locally free \( \mathcal{O}_{X_n} \)-module, [56, Thm 1.3] implies that \( F_R \) is a coherent locally free \( \mathcal{O}_{X_R} \)-module.

The morphisms \((\Psi_{R,1}, \rho_R, \eta_R)\) fit in commutative diagrams of \( \mathcal{O}_{X_R} \)-modules with exact rows of the form

\[
0 \to F_R \otimes_{X_R} (M_i)_R \to E_R \otimes_{X_R} (M_i)_R \to \iota_n^* E_n^\prime \otimes_{X_R} (M_i)_R \to 0
\]

where \( i = 1, 2, \)

\[
0 \to \mathcal{O}_{X_R} \to \iota_n^* \mathcal{O}_{X_n} \to \mathcal{O}_{X_R} \to 0
\]

and

\[
0 \to F_R \otimes_{X_R} M_R \to E_R \otimes_{X_R} M_R \to \iota_n^* E_n^\prime \otimes_{X_R} M_R \to 0
\]

where by convention \( \mathcal{O}_{X_n}^{\oplus v_n''} = 0 \) if \( v_n'' = 0 \).

Let \( U = X_R \setminus X_n \) be the Zariski open subset complementary to \( X_n \). By construction, the restriction \( F_R|_U \) is isomorphic to \( E_R|_U \). Therefore the morphisms \( (\Psi_{R,1,2}, \rho_R, \eta_R)|_U \) satisfy the ADHM relation over \( U \). Since \( F_R \) is locally free, it follows that they must satisfy the ADHM relation over \( X_R \). Hence \( F_R \) is indeed a locally free ADHM sheaf on \( X_R \).
According to [56, Thm 1.3] there is an exact commutative diagram of $\mathcal{O}_{X_R}$-modules

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & & \\
E_R(-X_n) & 1 & E_R(-X_n) & \\
\downarrow & & & \\
0 & F_R & E_R & t_n \ast E''_n & 0 \\
\downarrow & & & \downarrow 1 & \\
0 & t_n \ast E'_n & t_n \ast E_n & t_n \ast E''_n & 0 \\
\downarrow & & & & \\
0 & 0 & 0 & 0 & \\
\end{array}
\]

where $E_R(-X_n) = E_R \otimes_{X_R} \mathcal{O}_{X_R}(-X_n)$. This yields a filtration of $\mathcal{O}_{X_R}$-modules

\[
0 \subset F_R(-X_n) \subset E_R(-X_n) \subset F_R \subset E_R.
\]

Note that we also have isomorphisms of $\mathcal{O}_{X_R}$-modules

\[
E_R \xrightarrow{\iota^m} F_R(-X_n), \quad F_R \xrightarrow{\iota^m} F_R(-X_n)
\]

Applying $\otimes_{X_R} \mathcal{O}_{X_n}$ to the commutative diagram (6.12) we obtain a commutative diagram of $\mathcal{O}_{X_R}$-modules with exact rows

\[
F_R \otimes_{X_R} (M_i)_R \otimes_{X_R} \mathcal{O}_{X_n} \xrightarrow{f_i} E_R \otimes_{X_R} (M_i)_R \otimes_{X_R} \mathcal{O}_{X_n} \xrightarrow{t_n \ast E''_n \otimes_{X_R} (M_i)_R} 0
\]

\[
\Phi_{R,i} \otimes 1_{\mathcal{O}_{X_n}} \xrightarrow{g} \Phi_{R,i} \otimes 1_{\mathcal{O}_{X_n}} \xrightarrow{t_n \ast \Phi''_{n,i}}
\]

\[
F_R \otimes_{X_R} \mathcal{O}_{X_n} \xrightarrow{g} E_R \otimes_{X_R} \mathcal{O}_{X_n} \xrightarrow{t_n \ast E''_n} 0
\]

for $i = 1, 2$. Using the filtration (6.16) and the isomorphisms (6.17) we obtain the following isomorphisms of $\mathcal{O}_{X_R}$-modules

\[
\text{Ker}(f_i) \simeq E_R(-X_n)/F_R(-X_n) \otimes_{X_R} (M_i)_R \simeq E_R/F_R \otimes_{X_R} (M_i)_R \simeq t_n \ast E''_n \otimes_{X_R} (M_i)_R
\]

\[
\text{Ker}(g) \simeq E_R(-X_n)/F_R(-X_n) \simeq E_R/F_R \simeq t_n \ast E''_n
\]

for $i = 1, 2$. Moreover by construction the following diagram is commutative

\[
\begin{array}{cccc}
E_R \otimes_{X_R} (M_i)_R & F_R \otimes_{X_R} (M_i)_R \\
\downarrow \Phi_{R,i} & \downarrow \Phi_{R,i} & \\
E_R \otimes_{X_R} (M_i)_R & F_R \otimes_{X_R} (M_i)_R \\
\downarrow \Phi_{R,i}(-X_n) & \downarrow \Phi_{R,i}(-X_n) & \\
E_R(-X_n) & F_R \\
\end{array}
\]
where $\Phi_{R,i}(-X_n) = \Phi_{R,i} \otimes 1_{\mathcal{O}_{X_R}(-X_n)}$, $i = 1, 2$. Then using again isomorphisms \eqref{6.17}, the commutative diagram \eqref{6.18} yields a commutative diagram of $\mathcal{O}_{X,n}$-modules with exact rows \eqref{6.19}

$$
\begin{array}{cccccccc}
0 & \longrightarrow & E''_n \otimes_{X_n} \iota^*_n(M_i)_R & \longrightarrow & \iota^*_n F_R \otimes_{X_n} \iota^*_n(M_i)_R & \longrightarrow & E'_n \otimes_{X_n} \iota^*_n(M_i)_R & \longrightarrow & 0 \\
\phi''_{n,i} & & \iota^*_n \Phi_{R,i} & & \iota^*_n \phi_{n,i} & & \phi_{n,i} & & \\
0 & \longrightarrow & E''_n & \longrightarrow & \iota^*_n F_R & \longrightarrow & \iota^*_n M_R & \longrightarrow & 0.
\end{array}
$$

for $i = 1, 2$.

Similarly, applying $\otimes_{X_R} \mathcal{O}_{X,n}$ to the diagram \eqref{6.13} yields the following commutative diagram of $\mathcal{O}_{X,R}$-modules with exact rows \eqref{6.20}

$$
\begin{array}{cccccccc}
F_R \otimes_{X_R} M_R \otimes_{X_R} \mathcal{O}_{X,n} & \overset{f}{\longrightarrow} & E_R \otimes_{X_R} M_R \otimes_{X_R} \mathcal{O}_{X,n} & \longrightarrow & \iota_n E''_n \otimes_{X_R} M_R & \longrightarrow & 0 \\
\rho_R \otimes 1_{\mathcal{O}_{X,n}} & & \phi_R \otimes 1_{\mathcal{O}_{X,n}} & & \iota_n \phi''_{n} & & \iota_n \phi''_{n} & & \\
\mathcal{O}_{X,n} & \overset{(1-v''_{n})}{\longrightarrow} & \mathcal{O}_{X,n} & \longrightarrow & \mathcal{O}_{X,n} & \longrightarrow & 0.
\end{array}
$$

Using again the filtration \eqref{6.10} and the isomorphisms \eqref{6.17} we obtain the following isomorphism of $\mathcal{O}_{X,R}$-modules

$$
\text{Ker}(f) \simeq E_R(-X_n)/F_R(-X_n) \otimes_{X_R} M_R \simeq E_R/F_R \otimes_{X_R} M_R \simeq \iota_n E''_n \otimes_{X_R} M_R
$$

Note that if $v''_{n} = 0$, the restriction of the vertical morphism $\rho \otimes 1_{\mathcal{O}_{X,n}}$ to Ker($f$) is trivial in the diagram \eqref{6.20}. If $v''_{n} = 1$, there is by construction a commutative diagram of the form

$$
\begin{array}{cccccccc}
E_R \otimes_{X_R} M_R & \overset{\tau^n}{\longrightarrow} & E_R(-X_n) \otimes_{X_R} M_R & \longrightarrow & F_R \otimes_{X_R} M_R \\
\phi_R & & \phi_R(-X_n) & & \rho_R & & \\
\mathcal{O}_{X_R} & \overset{\tau^n}{\longrightarrow} & \mathcal{O}_{X_R}(-X_n) & \longrightarrow & \mathcal{O}_{X_R}
\end{array}
$$

where $\phi_R(-X_n) = \phi_R \otimes 1_{\mathcal{O}_{X_R}(-X_n)}$ and both morphisms in the bottom row are canonical isomorphisms. Therefore, if $v''_{n} = 1$, the diagram \eqref{6.20} yields again a commutative diagram of $\mathcal{O}_{X,n}$-modules with exact rows analogous to \eqref{6.19}.

Entirely analogous arguments applied to the diagram \eqref{6.14} allow us to conclude that $\iota^*_n F_R$ is indeed an extension of the form \eqref{6.10}.

The third statement of lemma \eqref{6.5} follows easily from the commutative diagrams \eqref{6.12} - \eqref{6.14} and remark \eqref{3.2}.

Next let $n = 1$ in lemma \eqref{6.5}, hence $X_n = X_1 \simeq X$. Suppose that $\delta \in \mathbb{R}_{>0}$ is a stability parameter so that $E_1 = \iota^*_1 E_R$ is not $\delta$-semistable, and $E'_1$ is the destabilizing $\beta_{\delta}$-subobject of $E_1$ determined by lemma \eqref{3.14}. Let $F_1 = \iota^*_1 F_R$. Then we have the following lemma by analogy with \cite{51} Lemma 5.1.

**Lemma 6.6.** If $0 \subset \mathcal{G} \subset F_1$ is a nontrivial subobject of $F_1$, then $\beta_{\delta}(\mathcal{G}, F_1) \leq \beta_{\delta}(E'_1, E_1)$ and equality holds only if the morphism

$$
\mathcal{G} \hookrightarrow F_1 \twoheadrightarrow E'_1
$$

is surjective.
Proof. There are two cases. First suppose \( \mathcal{G} \) is a subobject of \( \mathcal{E}''' \). Note that the exact sequence (6.3) implies that \( \mathcal{F}_1, \mathcal{E}_1 \) have the same type \((r, e) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z} \) and
\[
\beta_\delta(\mathcal{E}_1', \mathcal{E}_1) = 0.
\]
Hence \( \beta_\delta(\mathcal{E}'', \mathcal{F}_1) < 0 \) since \( \beta_\delta(\mathcal{E}_1', \mathcal{E}_1) > 0 \) by assumption. Now let \( \mathcal{H} \subseteq \mathcal{E}_1 \) be the inverse image of \( \mathcal{G} \) in \( \mathcal{E}_1 \), that is there is an exact sequence
\[
0 \to \mathcal{E}_1' \to \mathcal{H} \to \mathcal{G} \to 0
\]
in \( \mathcal{C}_X \). This yields \( \beta_\delta(\mathcal{G}, \mathcal{F}_1) = \beta_\delta(\mathcal{H}, \mathcal{E}_1) - \beta_\delta(\mathcal{E}_1', \mathcal{E}_1) \leq 0 \) and lemma (6.6) follows.

Next suppose \( \mathcal{G} \) is not a subobject of \( \mathcal{E}''' \) and let \( \mathcal{G}' \neq 0 \) be the image of the morphism (6.21). Then we have an exact sequence in \( \mathcal{C}_X \)
\[
0 \to \mathcal{G}'' \to \mathcal{G} \to \mathcal{G}' \to 0
\]
which yields \( \beta_\delta(\mathcal{G}, \mathcal{F}_1) = \beta_\delta(\mathcal{G}', \mathcal{E}) + \beta_\delta(\mathcal{G}'', \mathcal{F}_1) \). Note that \( \beta_\delta(\mathcal{G}', \mathcal{E}_1) \leq \beta_\delta(\mathcal{E}_1', \mathcal{E}_1) \) and equality holds only if \( \mathcal{G}' = \mathcal{E}_1' \). Moreover, according to the first case \( \beta_\delta(\mathcal{G}'', \mathcal{F}_1) \leq 0 \). This implies lemma (6.6).

\[\square\]

6.5. **Properness.** Using the preliminary results proven so far we conclude this section with the proof of theorem (1.5).

Let \( \delta \in \mathbb{R}_{>0} \) be a noncritical stability parameter of type \((r, e) \). According to theorem (1.2), there is a separated algebraic moduli space \( \mathcal{M}_{\delta}^*(\mathcal{X}, r, e) \) of finite type over \( \mathbb{C} \) parameterizing isomorphism classes of \( \delta \)-stable objects. Since the moduli space is separated over \( \mathbb{C} \), the \( \mathcal{S} \)-fixed locus \( \mathcal{M}_{\delta}^*(\mathcal{X}, r, e)^{\mathcal{S}} \) is a closed algebraic subspace of \( \mathcal{M}_{\delta}^*(\mathcal{X}, r, e) \). In particular it is separated of finite type over \( \mathbb{C} \). Therefore it suffices to prove universal closedness by checking the valuative criterion [52, Thm. 7.10]. Since the fixed loci are separated algebraic stacks of finite type over \( \mathbb{C} \), it suffices to check the valuative criterion for complete discrete valuation rings \( R \) over \( \mathbb{C} \).

Let \( R \) be such a ring, let \( K \) be the fraction field of \( R \) and let \( k \simeq \mathbb{C} \) be the fraction field of \( k \). Let \( p \in \text{Spec}(R) \) denote the closed point of \( \text{Spec}(R) \) and \( t \in R \) denote a uniformizing parameter. In the following we will denote by \( X_K = X \times \text{Spec}(K), X_R = X \times \text{Spec}(R), X_p = X \times \{p\} \). The structure sheaves of \( \text{Spec}(K) \), \( \text{Spec}(R) \) will be denoted by \( \mathcal{O}_K, \mathcal{O}_R \) respectively. Similarly the tensor product of \( \mathcal{O}_K \), respectively \( \mathcal{O}_R \)-modules will be denoted by \( \otimes_K, \otimes_R \) and the global section functors for \( \text{Spec}(R) \), \( \text{Spec}(K) \) will be denoted by \( \Gamma_R, \Gamma_K \).

Let \( \mathcal{E}_K \) be a \( \delta \)-stable ADHM sheaf of type \((r, e) \) on \( X_K \) fixed by \( \mathcal{S} \) up to isomorphism. Since the fixed locus \( \mathcal{M}_{\delta}^*(\mathcal{X}, r, e)^{\mathcal{S}} \) is a closed algebraic subspace of the moduli space \( \mathcal{M}_{\delta}^*(\mathcal{X}, r, e) \) it suffices to prove that there exists a flat family \( \mathcal{E}_R \) of \( \delta \)-stable ADHM sheaves on \( X \) (not necessarily fixed by \( \mathcal{S} \) up to isomorphism) so that \( \mathcal{E}_R|_{X_K} \simeq \mathcal{E}_K \) as ADHM sheaves on \( X_K \). Then the valuative criterion will then be satisfied since the closed embedding \( \mathcal{M}_{\delta}^*(\mathcal{X}, r, e)^{\mathcal{S}} \to \mathcal{M}_{\delta}^*(\mathcal{X}, r, e) \) is proper. The proof will proceed by induction on \( r \geq 1 \). First consider \( r = 1 \).

**Lemma 6.7.** Let \( \mathcal{E}_K = (E_K, \Phi_{K,1}, \Phi_{K,2}, \phi_K, \psi_K) \) be a stable ADHM sheaf of type \((1, e) \), \( e \in \mathbb{Z} \), on \( X_K \) fixed by the \( \mathcal{S} \) up to isomorphism. Then there exists a flat family \( \mathcal{E}_R \) of locally free ADHM sheaves of type \((1, e) \) on \( X \) so that \( \mathcal{E}_R|_{X_K} \) is isomorphic to \( \mathcal{E}_K \) and the restriction \( \mathcal{E}_R|_{X_p} \) is stable.
Proof. Follows from lemmas [4.11], [6.3].

Let \( r \geq 2 \). Suppose the fixed locus \( \mathfrak{M}^*_\delta(X, r', e)^S \) is proper over \( \mathbb{C} \) for all types \((r', e)\) with \( 1 \leq r' < r \), \( e \in \mathbb{Z} \) and, for a fixed type \((r', e)\), for all noncritical stability parameters of type \((r', e)\). Then we have to prove the analogous statement holds for each fixed type \((r, e)\), \( e \in \mathbb{Z} \). This will be done by a chamber inductive argument starting with the asymptotic chamber \( \delta > \delta_N \). Lemma [6.3] implies that the statement is holds for \( \delta > \delta_N \). If \( N = 0 \), there is nothing to prove, so we will assume \( N \geq 1 \) in the following. Set \( \delta_0 = 0 \) and \( \delta_{N+1} = +\infty \).

Suppose \( \mathfrak{M}^*_\delta(X, r, e)^S \) is proper over \( \mathbb{C} \) for \( \delta \in (\delta_i, \delta_{i+1}) \), for some \( i = 1, \ldots, N \). Then we will prove that the same holds for \( \delta \in (\delta_{i-1}, \delta_i) \). Let \( R \) be a complete discrete valuation ring over \( \mathbb{C} \) as above. Let \( \mathcal{E}_K \) be a \( \delta_- \)-stable ADHM sheaf of type \((r, e)\) on \( X_K \) fixed by \( S \) up to isomorphism, for some \( \delta_- \in (\delta_{i-1}, \delta_i) \).

Lemma 6.8. There exists a flat family \( \mathcal{E}_R \) of locally free ADHM sheaves on \( X \) parameterized by \( \text{Spec}(R) \) so that \( \mathcal{E}_R|_{X_K} \) and \( \mathcal{E}_K \) are isomorphic ADHM sheaves on \( X_K \).

Proof. Since \( \mathcal{E}_K \) is \( \delta_- \)-stable, lemma [4.10] implies that one of the following cases must hold:

\begin{enumerate}
  \item \( \mathcal{E}_K \) is \( \delta_- \)-stable, hence also \( \delta_+ \)-stable for any \( \delta_+ \in (\delta_i, \delta_{i+1}) \), or
  \item \( \mathcal{E}_K \) is strictly \( \delta_- \)-semistable and there is a nontrivial extension
  \begin{equation}
  0 \to \mathcal{E}'_K \to \mathcal{E}_K \to \mathcal{E}''_K \to 0
  \end{equation}
  in the abelian category \( \mathcal{C}_{X_K} \), where \( \mathcal{E}'_K \) is a \( \delta_- \)-stable ADHM sheaf on \( X_K \), \( \mathcal{E}''_K \) is a semistable Higgs sheaf on \( X_K \) and \( \mu_{\delta_-}(\mathcal{E}_K) = \mu_{\delta_-}(\mathcal{E}''_K) \). Moreover, according to remark [4.11], we may assume that \( \mathcal{E}'_K \to \mathcal{E}_K \) is a morphism of ADHM sheaves on \( X_K \).
\end{enumerate}

Suppose case (1) holds. Recall that \( \mathfrak{M}^*_\delta(X, r, e)^S \) is a proper algebraic space of finite type over \( \mathbb{C} \) according to the chamber induction hypothesis. Then [52] Rem. 7.4 implies that there exists a flat family \( \mathcal{E}_R \) of \( \delta_+ \)-stable ADHM sheaves on \( X \) parameterized by \( \text{Spec}(R) \) and an isomorphism \( \mathcal{E}_R|_{X_K} \cong \mathcal{E}_K \) of ADHM sheaves on \( X_K \).

Suppose case (2) holds. Let \( r' = \text{rk}(\mathcal{E}'_K), e' = \text{deg}(\mathcal{E}'_K) \), respectively \( r'' = \text{rk}(\mathcal{E}''_K), e'' = \text{deg}(\mathcal{E}''_K) \). Note that \( r' < r \). According to corollary [4.12], the filtration \( 0 \subset \mathcal{E}'_K \subset \mathcal{E}_K \) is a Harder-Narasimhan filtration of \( \mathcal{E}_K \) with respect to \( \delta_+ \)-stability. Then lemma [6.3] implies that \( \mathcal{E}'_K, \mathcal{E}''_K \) are fixed by \( S \) up to isomorphism.

Moreover, since \( \mathcal{E}'_K \) is \( \delta_- \)-stable, according to lemma [4.10] (iii) there exists \( \epsilon > 0 \) so that \( \mathcal{E}'_K \) is \( \gamma \)-stable for any \( \gamma \in (\delta_i - \epsilon, \delta_i) \). In particular there exists a noncritical value \( \gamma \in (\delta_i - \epsilon, \delta_i) \) of type \((r', e')\) so that \( \mathcal{E}'_K \) is \( \gamma \)-stable and there are no critical stability parameters of type \((r', e')\) in the interval \([\gamma, \delta_i] \). Since \( r' < r \), the rank induction hypothesis implies according to [52] Rem. 7.4 that there exists a flat family \( \mathcal{E}'_R \) of \( \gamma \)-stable ADHM sheaves on \( X \) parameterized by \( \text{Spec}(R) \), so that \( \mathcal{E}'_R|_{X_K} \cong \mathcal{E}'_K \).

According to lemmas [6.1], [6.2], the same holds for the semistable Higgs sheaf \( \mathcal{E}''_K \). That is there exists a flat family of \( \mathcal{E}''_R \) of semistable Higgs sheaves on \( X \) parameterized by \( \text{Spec}(R) \) so that \( \mathcal{E}''_R|_{X_K} \cong \mathcal{E}''_K \).

Moreover, using proposition [3.15], the extension (6.22) determines an extension class \( e_K \in H^1(X_K, \mathcal{C}(\mathcal{E}''_K, \mathcal{E}'_K)) \), where \( \mathcal{C}(\mathcal{E}''_K, \mathcal{E}'_K) \) is the complex (3.3) of coherent
locally free $\mathcal{O}_{X_K}$-modules associated to the pair $(\mathcal{E}'_R, \mathcal{E}_L)$. Then we claim that there exists a locally free extension

$$0 \to \mathcal{E}'_R \to \mathcal{E}_R \to \mathcal{E}''_R \to 0$$

of ADHM quiver sheaves on $X_R$ so that $\mathcal{E}_R|_{X_K} \simeq \mathcal{E}_K$. The argument is essentially identical to the one employed in the second part of the proof of lemma (6.2).

Let $\mathcal{C}(\mathcal{E}'_R, \mathcal{E}_L)$ be the complex (3.3) of coherent locally free $\mathcal{O}_{X_R}$-modules corresponding to the pair $(\mathcal{E}'_R, \mathcal{E}_L)$. By construction we have a canonical isomorphism

$$\mathcal{C}(\mathcal{E}'_R, \mathcal{E}_L)|_{X_K} \simeq \mathcal{C}(\mathcal{E}'_K, \mathcal{E}_K)$$

of complexes of $\mathcal{O}_{X_K}$-modules. Since $X_K \subset X_R$ is a Zariski open subset, this isomorphism yields a well defined restriction map

$$r : \mathbb{H}^1(X_R, \mathcal{C}(\mathcal{E}'_R, \mathcal{E}_L)) \to \mathbb{H}^1(X_K, \mathcal{C}(\mathcal{E}'_K, \mathcal{E}_K))$$

which is a morphism of $R$-modules.

Let $t \in R$ be a uniformizing parameter. Since $R$ is a discrete valuation ring, there exists $n \in \mathbb{Z}$ and a hypercohomology class $\epsilon_R \in \mathbb{H}^1(X_R, \mathcal{C}(\mathcal{E}'_R, \mathcal{E}_L))$ so that

$$r(\epsilon_R) = t^n \epsilon_K.$$

According to proposition (3.15), $\epsilon_R$ determines an extension of ADHM quiver sheaves on $X_R$

$$0 \to \mathcal{E}'_R \to \mathcal{E}_R \to \mathcal{E}''_R \to 0$$

up to isomorphism of extensions. By construction there is a commutative diagram of extensions

$$\begin{array}{cccccc}
0 & \to & \mathcal{E}'_R|_{X_K} & \xrightarrow{f_R|_{X_K}} & \mathcal{E}_R|_{X_K} & \xrightarrow{t^n g_R|_{X_K}} & \mathcal{E}''_R|_{X_K} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E}'_K & \xrightarrow{f_K} & \mathcal{E}_K & \xrightarrow{g_K} & \mathcal{E}''_K & \to & 0
\end{array}$$

where the vertical arrows are isomorphisms in $\mathcal{C}_{X_K}$. Moreover, by eventually changing the representative $\mathcal{E}_R$ within the isomorphism class of extensions determined by $\epsilon_R$, we may assume that $\mathcal{E}'_R \to \mathcal{E}_R$ is a morphism of ADHM sheaves on $X_R$. Since the morphism $\mathcal{E}'_K \to \mathcal{E}_K$ is also a morphism of ADHM sheaves on $X_K$ according to the current working hypothesis (see condition (2) above) it follows that central vertical arrow in the above diagram is an isomorphism of ADHM sheaves $\mathcal{E}_R|_{X_K} \simeq \mathcal{E}_K$.

In order to complete the proof of theorem (1.5) it suffices to prove

**Lemma 6.9.** Given the flat family $\mathcal{E}_R$ determined by lemma (6.8) there exists another flat family $\mathcal{E}'_R$ of $\delta_-$-stable locally free ADHM sheaves on $X$ parameterized by $\text{Spec}(R)$ so that $\mathcal{E}_R|_{X_K} \simeq \mathcal{E}'_R|_{X_K}$ as ADHM sheaves on $X_K$.

**Proof.** Given lemmas (6.14), (6.15), (6.16), the proof of lemma (6.9) is identical to the proof of the second main theorem of [51 Sect. 3]. More specifically, suppose $\mathcal{E}_1 = \mathcal{E}_R|_{X_K}$ is not $\delta_-$-stable. Then, using lemmas (6.13), (6.16) one constructs inductively a sequence

$$\ldots \hookrightarrow \mathcal{E}_R^{(n)} \hookrightarrow \mathcal{E}_R^{(n-1)} \hookrightarrow \cdots \hookrightarrow \mathcal{E}_R^{(1)} = \mathcal{E}_R$$
of locally free ADHM sheaves on \(X_R\) as follows.

Set \(E^{(0)}_R = E_R\). Let \(E^{(n)}_1 = E^{(n)}_R |_{X_1}\) be the restriction to the closed fiber, let \(F^{(n)} \subset E^{(n)}_1\) be the \(\delta_-\)-subobject of \(E^{(n)}_1\) defined in lemma \(6.14\) and let \(G^{(n)} = E^{(n)}_1 / F^{(n)}\). Then \(E^{(n+1)}\) is the elementary modification of \(E^{(n)}\) along \(X_1 \subset X_R\) determined by the exact sequence of \(O_{X_1}\)-modules

\[
0 \to F^{(n)} \to E^{(n)}_1 \to G^{(n)} \to 0
\]

as in lemma \(6.3\). In particular there is a commutative diagram of \(O_{X_1}\)-modules

\[
\begin{array}{ccccccccc}
0 & & & & & & & & 0 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

According to lemma \(6.6\) we have

\[
\beta_{\delta_-}(F^{(n+1)}, E^{(n+1)}_1) \leq \beta_{\delta_-}(F^{(n)}, E^{(n)}_1)
\]

for all \(n \in \mathbb{Z}_{\leq 0}\). Note that \(\beta_{\delta_-}(F^{(0)}, E^{(0)}_1) > 0\) by assumption. If there exists a value of \(n \in \mathbb{Z}_{\geq 0}\) so that \(\beta_{\delta_-}(F^{(n)}, E^{(n)}_1) < 0\), lemma \(6.9\) follows.

Suppose there is no such value of \(n \in \mathbb{Z}_{\geq 0}\). Then proceeding by analogy with the proof of Lemma 2 [51 Sect. 5], lemma \(6.6\) implies that the sequence \(6.27\) is split and the morphisms \(F^{(n+1)} \to F^{(n)}\), \(G^{(n)} \to G^{(n+1)}\) are isomorphisms for sufficiently large \(n \in \mathbb{Z}_{\geq 0}\). Moreover, lemma \(4.13\) implies that \(E^{(n)}_1\) is \(\delta_-\)-semistable and \(F^{(n)}\) is a semistable Higgs sheaf on \(X_1\) with \(\mu_{\delta_-}(F^{(n)}) = \mu_{\delta_-}(E^{(n)}_1)\) for all \(n \in \mathbb{Z}_{\geq 0}\).

For any \(n \in \mathbb{Z}_{\geq 0}\), \(m \in \mathbb{Z}_{\geq 1}\) set \(E^{(n)}_R |_{X_m} = E^{(n)}_m\) using the notation of section \(6.4\); \(E^{(n)}_m\) is a locally free ADHM sheaf on \(X_m \subset X_R\). Note that there is a morphism of \(O_{X_m}\)-modules \(E^{(n)}_m \to E^{(0)}_m\) determined by the inclusion morphisms in \(6.26\). Let \(F_m = \text{Im}(E^{(m)}_m \to E^{(0)}_m)\) and \(G_m = \text{Coker}(E^{(m)}_m \to E^{(0)}_m)\) for all \(m \in \mathbb{Z}_{m \geq 1}\); \(F_m\) is a locally free Higgs sheaf on \(X_m\).

Then, using lemma \(6.6\), all arguments in the proof of Lemma 2 [51 Sect. 5] carry over to the sequence of locally free ADHM sheaves \(6.26\) since \(R\) is assumed to be complete. This proves that there exists a \(\delta_-\)-destabilizing subobject \(0 \subset F_K \subset E_K\), which contradicts the inductive hypothesis.

In conclusion, there exists \(n \in \mathbb{Z}_{\geq 1}\) so that \(E^{(n)}_R\) is a flat family of \(\delta_-\)-stable objects of \(O_X\) of type \((r, e)\). Lemma \(6.3\, iii)\) implies that there exists an isomorphism \(E^{(n)}_R |_{X_K} \simeq E_K\) of ADHM sheaves on \(X_K\). Lemma \(6.11\) implies that the restriction \(E^{(n)}_R |_{X_p}\) is a \(\delta_-\)-stable ADHM sheaf on \(X\), concluding the proof of lemma \(6.9\).
7. Versal Deformations and Holomorphic Chern-Simons Theory

The main goal of this section is to show that the recent results [43, 44] of Joyce and Song on Behrend functions for algebraic moduli stacks of coherent sheaves on smooth projective Calabi-Yau threefolds also hold for the moduli stacks constructed in section [52]. This will be needed in the second part of this paper, which is concerned with wallcrossing formulas for ADHM invariants.

Let $\mathcal{Ob}(\mathcal{X})_{\leq 1}$ be the algebraic moduli stacks constructed in lemma [52]. Let $\mathcal{M}(\mathcal{X})_{\leq 1}$ denote the coarse algebraic moduli space of simple objects in $\mathcal{Ob}(\mathcal{X})_{\leq 1}$. The first goal of this section is to prove that the statements of [43, Thm 5.2] and [43, Thm 5.3] hold for $\mathcal{M}(\mathcal{X})_{\leq 1}$, $\mathcal{Ob}(\mathcal{X})_{\leq 1}$, respectively $\mathcal{Ob}(\mathcal{X})_{0}$, $\mathcal{M}(\mathcal{X})_{0}$. For simplicity, we will denote extensions groups in the abelian category $C\mathcal{X}$ by $\text{Ext}(\cdot, \cdot)$ in the following. Then by analogy with [43] we claim

**Theorem 7.1.** Let $\mathcal{M}$ denote either $\mathcal{M}(\mathcal{X})_{\leq 1}$ or $\mathcal{M}(\mathcal{X})_{0}$. Then for each $\mathbb{C}$-valued point $[\mathcal{E}] \in \mathcal{M}(\mathbb{C})$ there exists a finite-dimensional complex manifold $U$, a holomorphic function $\omega: U \to \mathbb{C}$ and a point $u \in U$ so that $\omega(u) = d\omega(u) = 0$ and $\mathcal{M}(\mathbb{C})$ is locally isomorphic as a complex analytic space to $\text{Crt}(\omega)$ near $u$. Moreover, $U$ can be taken to be isomorphic to an open neighborhood of $u = 0$ in the vector space $\text{Ext}^1(\mathcal{E}, \mathcal{E})$.

Similarly, let $\mathfrak{M}$ denote either $\mathcal{Ob}(\mathcal{X})_{\leq 1}$ or $\mathcal{Ob}(\mathcal{X})_{0}$. According to [43 Thm 5.3], the general theory of Artin stacks implies that for any $\mathbb{C}$-valued point $[\mathcal{E}] \in \mathfrak{M}(\mathbb{C})$, there exists an $\text{Aut}(\mathcal{E})$-invariant subscheme $S \subset \text{Ext}^1(\mathcal{E}, \mathcal{E})$ over $\mathbb{C}$ parameterizing an $\text{Aut}(\mathcal{E})$-equivariant versal family $\mathcal{E}_S$ of locally free objects of $\mathcal{C}\mathcal{X}$ with $v \in \{0, 1\}$ so that $\mathcal{E}_S|_{X_0} \simeq \mathcal{E}$. Moreover there exists an étale morphism of Artin stacks $\Phi : [S/\text{Aut}(\mathcal{E})] \to \mathfrak{M}$ so that $\Phi([0]) = [\mathcal{E}]$.

\[
\Phi_* : \text{Stab}([0]) \simeq \text{Aut}(\mathcal{E}) \to \text{Stab}([\mathcal{E}])
\]

and

\[
d\Phi : T_{[0]}[S/\text{Aut}(\mathcal{E})] \simeq \text{Ext}^1(\mathcal{E}, \mathcal{E}) \rightarrow T_{[\mathcal{E}], \mathfrak{M}}
\]

are natural isomorphisms. Then the following holds.

**Theorem 7.2.** For each $\mathbb{C}$-valued point $[\mathcal{E}] \in \mathfrak{M}(\mathbb{C})$ there exists an open neighborhood of $0$ $U \subset \text{Ext}^1(\mathcal{E}, \mathcal{E})$ in the analytic topology, a holomorphic function $\omega: U \to \mathbb{C}$ so that $\omega(0) = d\omega(0) = 0$ and an open neighborhood of $0$ $V \subset S_{an}$ so that there is an isomorphism of complex analytic spaces $\Xi: \text{Crt}(\omega) \to V$ satisfying $\Xi(0) = 0$ and $d\Xi_0 = \text{Id}_{\text{Ext}^1(\mathcal{E}, \mathcal{E})}$. Moreover, if $G$ is a maximal compact subgroup of $\text{Aut}(\mathcal{E})$, $U, v, V$ can be chosen to be $G^\mathbb{C}$-equivariant.

**Proof of Theorems (7.1), (7.2).** Since we have restricted ourselves to locally free objects, theorems (7.1), (7.2) can be proven using gauge theoretic methods in complete analogy with [43 Thm 5.2], [43 Thm 5.3]. One has to check that the main arguments in [58], [43] carry over to the current decorated bundle moduli problem. A gauge theoretic approach to the wallcrossing formulas for ADHM invariants has been previously employed for example in [12, 10, 11, 66, 7, 13, 85, 67, 50, 16, 3, 15, 14, 55, 8, 9]. The main steps will be outlined below.
**Step 1.** Since $X$ is a smooth projective curve over $\mathbb{C}$, it has a complex manifold structure, which will be denoted by $\hat{X}$. Let $M_1^n, M_2^n$ denote the complex holomorphic line bundles on $\hat{X}$ corresponding to the invertible sheaves $M_1, M_2$ on $X$. Let $\hat{M}_1, \hat{M}_2$ denote the underlying $C^\infty$-vector bundles of the holomorphic line bundles $M_1^n, M_2^n$ and let $\overline{\vartheta}_1, \overline{\vartheta}_2$ denote the corresponding Dolbeault operators. Set $\hat{M} = \hat{M}_1 \otimes_{\mathbb{C}} \hat{M}_2$ and note that there is a fixed isomorphism $\hat{M}^\vee \cong \Lambda_X^{1,0}$. Let $\overline{\vartheta}_0$ denote the canonical Dolbeault operator acting on sections of the trivial complex line bundle on $\hat{X}$.

A $V$-framed holomorphic ADHM bundle on $\hat{X}$ is defined by the data $\hat{E} = (\hat{E}, \overline{\vartheta}, V, \hat{\Phi}_{1,2}, \hat{\phi}, \hat{\psi})$ where $\hat{E}$ is a $C^\infty$ complex vector bundle on $X$,

$$\overline{\vartheta} : C^\infty(\hat{E}) \to C^\infty(\hat{E} \otimes_{\mathbb{C}} \Lambda_X^{0,1})$$

is a semiconnection (or Dolbeault operator) on $\hat{E}$ as defined in [43, Def. 9.1], $V$ is a complex vector space of dimension $v \in \{0, 1\}$ and

$$\begin{align*}
\hat{\Phi}_i &\in C^\infty(Hom(\hat{E} \otimes_{\mathbb{C}} \hat{M}_i, \hat{E})), \\
\hat{\phi} &\in C^\infty(Hom(\hat{E} \otimes_{\mathbb{C}} \hat{M}, \Lambda_X^{0,0} \otimes_{\mathbb{C}} V)) \\
\hat{\psi} &\in C^\infty(\hat{E} \otimes_{\mathbb{C}} V^\vee)
\end{align*}$$

(7.3)

are $C^\infty$-morphisms of complex bundles on $\hat{X}$ satisfying the ADHM relation. In addition note that the Dolbeault operators $\overline{\vartheta}, \overline{\vartheta}_0, \overline{\vartheta}_1, \overline{\vartheta}_2$ determine similar differential operator on each space of section in (7.3), and each morphism $(\hat{\Phi}_{1,2}, \hat{\phi}, \hat{\psi})$ is required to lie in the kernel of the corresponding Dolbeault operator. Obviously, $\hat{\phi}$ and $\hat{\psi}$ are identically zero if $v = 0$. Note that using the fixed isomorphism $\hat{M}^\vee \cong \Lambda_X^{1,0}$, the morphism $\hat{\phi}$ is identified with a section in $C^\infty(Hom(\hat{E}, \Lambda_X^{1,0} \otimes_{\mathbb{C}} V))$. Such an identification will be implicit from now on.

For future reference a $V$-framed $C^\infty$ ADHM bundle is defined by data $\hat{E}, \hat{\Phi}_{1,2}, \hat{\phi}, \hat{\psi}$ as above, except that no holomorphy condition is imposed. The definition of isomorphisms of $C^\infty$ of $V$-framed ADHM bundles is obvious.

Given any $C^\infty$ complex vector bundle $\hat{E}$, consider the following infinite dimensional affine space

$$\mathcal{A}_{ADHM} = \mathcal{A} \times C^\infty(Hom(\hat{E} \otimes_{\mathbb{C}} (\hat{M}_1 \oplus \hat{M}_2), \hat{E})) \times \times C^\infty(Hom(\hat{E}, \Lambda_X^{1,0} \otimes_{\mathbb{C}} V)) \times C^\infty(\hat{E} \otimes_{\mathbb{C}} V^\vee)$$

(7.4)

where $\mathcal{A}$ is the affine space of semiconnections on $\hat{E}$.

The group of gauge transformations acting $\mathcal{A}_{ADHM}$ is the infinite dimensional Lie group $\mathcal{G} = C^\infty(\text{Aut}(\hat{E})) \times (\mathbb{C}^\times)^{\times v}$ where $\text{Aut}(\hat{E}) \subset \text{End}(\hat{E})$ is the subbundle of invertible endomorphisms of $\hat{E}$. Note that the second factor $(\mathbb{C}^\times)^{\times v}$ represents the stabilizer of the canonical Dolbeault operator $\overline{\vartheta}_0$ on the trivial line bundle $\Lambda_X^{0,0} \times_{\mathbb{C}} V$. The later has to be kept fixed in this construction since our goal is to obtain a local presentation of moduli stacks of objects in the abelian category $\mathcal{C}_X$ rather than the larger abelian category $\mathcal{Q}_X$ of ADHM quiver sheaves on $X$.

The data $(\overline{\vartheta}, \hat{\Phi}_{1,2}, \hat{\phi}, \hat{\psi})$ will be called simple if its stabilizer in $\mathcal{G}$ is isomorphic to the canonical $\mathbb{C}^\times$ subgroup. The subspace of simple data will be denoted by $\mathcal{A}_{ADHM}^\text{simple}$.

Suppose $\hat{E} = (\hat{E}, \hat{\Phi}_{1,2}, \hat{\phi}, \hat{\psi})$ is a $V$-framed holomorphic ADHM bundle on $\hat{X}$ and let $(\overline{\vartheta} + A, \Psi_{1,2}, \rho, \eta) \in \mathcal{A}_{ADHM}$, where $A \in C^\infty(\text{End}(\hat{E}) \otimes_{\mathbb{C}} \Lambda_X^{0,1})$. Note that there
are natural cup-products
\begin{equation}
(7.5)
\mathcal{C}^\infty(\text{End}(\hat{E}) \otimes C^0_X) \otimes \mathcal{C}^\infty(\text{Hom}(\hat{E} \otimes \hat{M}_1, \hat{E})) \to \mathcal{C}^\infty(\text{Hom}(\hat{E} \otimes \hat{M}_1, \hat{E}) \otimes C^0_X)
\end{equation}
\begin{equation}
(7.6)
\mathcal{C}^\infty(\text{End}(\hat{E}) \otimes C^0_X) \otimes \mathcal{C}^\infty(\hat{E} \otimes V) \to \mathcal{C}^\infty(\hat{E} \otimes V) \otimes \mathcal{C}^0_X
\end{equation}
\begin{equation}
C^\infty(\text{Hom}(\hat{E}, (\Lambda^1_X \otimes C) V)) \otimes C^\infty(\text{End}(\hat{E}) \otimes C) \to C^\infty(\text{Hom}(\hat{E}, (\Lambda^1_X \otimes C) V))
\end{equation}
where \(i = 1, 2\). Then the data
\begin{equation}
(\hat{E}, \mathcal{F} + A, \Phi_1 + \Psi_1, \Phi_2 + \Psi_2, \hat{\phi} + \rho, \hat{\psi} + \eta)
\end{equation}
defines a \(V\)-framed holomorphic ADHM sheaf structure on \(\hat{E}\) if the following conditions are satisfied
\begin{equation}
(7.6)
\begin{align*}
\overline{\partial} \psi_1 + [A, \Phi_1 + \Psi_1] &= 0 \\
\overline{\partial} \psi_2 + [A, \Phi_2 + \Psi_2] &= 0 \\
\overline{\partial} \rho - (\phi + \rho)A &= 0 \\
\overline{\partial} \eta + A(\psi + \eta) &= 0
\end{align*}
\end{equation}
where all products are the natural cup-products (7.5) and the commutators are commutators of cup-products.

Finally, note that there is an obvious one-to-one correspondence between \(V\)-framed holomorphic ADHM bundles \(\hat{E}\) on the complex manifold \(X\) and locally free objects \(\mathcal{E}\) of \(C_X\) with \(v \in \{0, 1\}\). Moreover, for any \(V\)-framed holomorphic ADHM bundle \(\hat{E}\) on \(X\) there is a three term complex \(\mathcal{C}(\hat{E})\) of \(C^\infty\) complex vector bundles on \(X\) constructed by analogy with the deformation complex \([5, 2]\). Given the Dolbeault operator \(\overline{\partial}, \mathcal{C}(\hat{E})\) admits a canonical Dolbeault resolution, which yields in turn a hypercohomology double complex. Let \(\mathcal{C}_{\text{Db}}(\hat{E})\) be the total complex of the resulting double complex, and let \(\mathcal{C}_{\text{Db}}(\hat{E})\) denote the complex obtained by taking global \(C^\infty\) sections of the terms of \(\mathcal{C}_{\text{Db}}(\hat{X})\). Let \(\mathbb{H}^k_{\text{Db}}(\hat{X}, \mathcal{C}(\hat{E}))\), \(k \in \mathbb{Z}_{\geq 0}\), be the cohomology groups of \(\mathcal{C}_{\text{Db}}(\hat{E})\). Then there are isomorphisms of complex vector spaces
\begin{equation}
\mathbb{H}^k_{\text{Db}}(\hat{X}, \mathcal{C}(\hat{E})) \simeq \mathbb{H}^k(X, \mathcal{C}(\mathcal{E}))
\end{equation}
for all \(k \in \mathbb{Z}_{\geq 0}\).

Choosing hermitian structures on \(X, \hat{E}\), the complex \(\mathcal{C}_{\text{Db}}(\hat{E})\) will be elliptic, and the hypercohomology groups \(\mathbb{H}^k_{\text{Db}}(\hat{X}, \mathcal{C}(\hat{E}))\) are identified with spaces of harmonic bundle valued differential forms using Hodge theoretic methods. Analogous constructions have been carried in \([3\text{, Sect. 3}]\) for Higgs bundles, respectively \([9\text{, Sect. 5}]\) for triples, hence details will be omitted.

We can also construct complex Banach manifolds by taking appropriate Sobolev completions of the spaces \(\mathcal{A}_{\text{ADHM}}, \mathcal{A}_{\text{ADHM}}^0\), respectively a complex Banach group by taking a Sobolev completion of \(\mathcal{G}\) as in \([58\text{, Sect. 1}]\), \([43\text{, Sect. 9.1}]\).

Given a \(C^\infty\) complex vector bundle \(\hat{E}\) on \(X\), families of \(V\)-framed holomorphic ADHM structures on \(\hat{E}\) are defined by analogy with \([58\text{, Def. 1.5}], [44\text{, Def. 9.2}, or [50\text{, Def. 2.3}]. Then the existence of a versal deformation family extending a
given $V$-framed holomorphic ADHM bundle $\hat{E}$ follows from [49, Thm. 1.1] by an argument analogous to [50, Thm 2.4]. Alternatively, one can check that the proof of [58, Thm 1.1] carries over to the present situation with appropriate modifications.

**Step 2.** Next note that $X$ also has a structure of complex analytic space $X^{an}$, and one can obviously construct a category of analytic ADHM quiver sheaves by analogy with the abelian category $\mathcal{C}_X$. This category will be denoted by $\mathcal{C}_X^{an}$. Then given a locally free object $\mathcal{E}^{an}$ of $\mathcal{C}_X^{an}$ with $v \in \{0, 1\}$, one has to establish the existence of a versal deformation family of analytic objects of $\mathcal{C}_X^{an}$ which extends $\mathcal{E}^{an}$. The analogous result for complex analytic vector bundles has been proven in [24, 82]. As observed for example in [50] the extension to families of decorated analytic bundles follows from the complex analytic version of the standard representability for Hom functors presented for example in [22, Thm 5.8]. The complex analytic version of this result follows from [23].

Now let $\hat{E}$ be a $V$-framed holomorphic ADHM bundle on $\hat{X}$ and let $T$ denote the base of the versal deformation family of $V$-framed holomorphic ADHM bundles extending $\hat{E}$; $T$ is a finite dimensional complex analytic space. Let $\pi_{X^{an}} : X^{an} \times T \to T$ denote the canonical projection. Let $\mathcal{E}^{an}$ the locally free object of $\mathcal{C}_X^{an}$ corresponding to $\hat{E}$. Then by analogy with [58 Prop 2.3], [43 Prop. 9.5], or [50 Thm. 2.5] there exists a versal deformation family $\mathcal{E}^{an}_T$ with base $T$ extending $\mathcal{E}^{an}$ and an isomorphism $\pi_{X^{an}}^{*} \mathcal{E}^{an}_T \cong \pi_{X^{an}}^{*} \hat{E}$ of $C^\infty$ $V$-framed ADHM sheaves which induces the versal deformation family of $\hat{E}$. Moreover, $\mathcal{E}^{an}_T$ is universal if $\hat{E}$ is simple.

**Step 3.** Next let $E$ be a locally free object of $\mathcal{C}_X$ and let $E^{an}$ be the corresponding complex analytic object. Using the standard representability result [22 Thm 5.8] for Hom functors, and the existence of an algebraic versal deformation family extending $E$ proven in [43 Prop. 9.8], it follows that there exists an algebraic versal deformation family of objects of $\mathcal{C}_X$ extending $E$. Moreover, [43 Prop. 9.9] proves that the algebraic and the analytic versal deformation families associated to a given holomorphic vector bundle are locally isomorphic with respect to the complex analytic topology. The extension of this result to ADHM sheaves is straightforward.

**Step 4.** In order to conclude the proof of theorems (7.1, 7.2) it suffices to prove the existence of a holomorphic functional on the Sobolev completion of the space $\mathcal{A}_{ADHM}$, for any $V$-framed holomorphic ADHM sheaf $\hat{E}$ on $\check{X}$, with the same properties as the holomorphic Chern-Simons functional employed in [43 Sect. 9.5].

As observed in step 1 above, the Dolbeault operators $\partial$, $\partial_0$, $\partial_1$, $\partial_2$ determine similar Dolbeault operators on all spaces of sections in the right hand side of (7.4). In order to keep the notation short all the resulting operators will be denoted by the same symbol $\bar{\partial}$, the distinction being clear once the argument of the operator is specified.

Then the holomorphic Chern-Simons functional for $V$-framed ADHM sheaves is defined by

$$CS : (A, \Psi_1, 2, \rho, \eta) \to \int_X \text{Tr}(\Psi_2 \bar{\partial} \Psi_1 + \rho \bar{\partial} \eta + A[\hat{\Phi}_1, \Psi_2] + A[\Psi_1, \hat{\Phi}_2] + A[\Psi_1, \Psi_2] + A \psi \rho + A \eta \phi + A \eta \rho)$$
where all products are the natural cup-products \([7.5]\) and the commutators are commutators of cup-products. Note that given the cup-products \([7.5]\) it is straightforward to check that the integrand in the right hand side of equation \([7.7]\) is a section of \(\Lambda_{X}^{1,1}\). It is also straightforward to check that the functional \([7.7]\) is gauge invariant, and the critical points of the functional \([7.7]\) are determined by the equations \([7.6]\).

\[\Phi\]

**Lemma 7.3.** Let \(\mathcal{E}_1, \mathcal{E}_2\) be two locally free ADHM sheaves on \(X\) of numerical types \((r_1, e_1, 1), (r_2, e_2, 0)\). Let
\[
\chi(\mathcal{E}_1, \mathcal{E}_2) = \dim \text{Ext}^0(\mathcal{E}_1, \mathcal{E}_2) - \dim \text{Ext}^1(\mathcal{E}_1, \mathcal{E}_2)
- \dim \text{Ext}^0(\mathcal{E}_2, \mathcal{E}_1) + \dim \text{Ext}^1(\mathcal{E}_2, \mathcal{E}_1).
\]

Then \(\chi(\mathcal{E}_1, \mathcal{E}_2) = e_2 - r_2(g - 1)\) depends only on the numerical types of \(\mathcal{E}_1, \mathcal{E}_2\).

**Proof.** Recall that corollary \((3.17)\) proves that the extension groups \(\text{Ext}^1(\mathcal{E}_1, \mathcal{E}_2)\) are the hypercohomology groups of the three term locally free complex \(\mathcal{C}(\mathcal{E}_1, \mathcal{E}_2)\) written in equation \((3.4)\). Then lemma \((7.3)\) follows by a straightforward application of the Riemann-Roch theorem.

\[\square\]

Next let \(\nu\) denote Behrend’s constructible function for the Artin stack \(\mathcal{Ob}(\mathcal{X})\) constructed in \([43, \text{Prop. 4.4}]\). Then the following theorem holds by analogy with \([43, \text{Thm. 5.9}]\).

**Theorem 7.4.** Let \(\mathcal{E}_1, \mathcal{E}_2\) be locally free objects of \(\mathcal{C}_X\) with \(\nu(\mathcal{E}_1) + \nu(\mathcal{E}_2) \leq 1\). Then the following identities hold
\[
(7.8) \quad \nu([\mathcal{E}_1 \oplus \mathcal{E}_2]) = (-1)^{\chi(\mathcal{E}_1, \mathcal{E}_2)} \nu([\mathcal{E}_1])\nu([\mathcal{E}_2])
\]
\[
(7.9) \quad \int_{[\mathcal{E}] \in \mathbb{P}(\text{Ext}^1(\mathcal{E}_2, \mathcal{E}_1))} \nu(\mathcal{E})d\chi = \int_{[\mathcal{E}] \in \mathbb{P}(\text{Ext}^1(\mathcal{E}_1, \mathcal{E}_2))} \nu(\mathcal{E})d\chi
\]
\[
\dim \text{Ext}^1(\mathcal{E}_2, \mathcal{E}_1) - \dim \text{Ext}^1(\mathcal{E}_1, \mathcal{E}_2)
\]

**Appendix A. Higgs Sheaves and Framed Hitchin Pairs**

For completeness we summarize the main results on moduli of Higgs sheaves and framed Hitchin pairs used in this paper. Let \(X\) be a smooth projective curve over an infinite field \(K\) of characteristic 0. Let \((M_1, M_2)\) be fixed line bundles on \(X\).

**Definition A.1.** (i) A Higgs sheaf of type \((r, e)\) on \(X\) with coefficient sheaves \((M_1, M_2)\) is a collection \(\mathcal{E} = (E, \Phi_1, \Phi_2)\) where \(E\) is a coherent sheaf of type \((r, e)\) on \(X\) and
\[
\Phi_i : E \otimes_X M_i \to E
\]
are morphisms of \(\mathcal{O}_X\)-modules satisfying
\[
(A.1) \quad \Phi_1 \circ (\Phi_2 \otimes 1_{M_1}) - \Phi_2 \circ (\Phi_1 \otimes 1_{M_2}) = 0.
\]

(ii) A morphism of Higgs sheaves \(\mathcal{E}, \mathcal{E}'\) is a morphism \(\xi : E \to E'\) of coherent sheaves on \(X\) satisfying the obvious compatibility conditions with the data \(\Phi_{1,2}, \Phi'_{1,2}\).
(iii) A Higgs sheaf $E = (E, \Phi_1, \Phi_2)$ of type $(r, e)$ on $X$ is called (semi)stable if any nontrivial $\Phi$-invariant proper saturated subsheaf $0 \subset E' \subset E$ satisfies

$$\mu(E') \leq \mu(E).$$

(A.2)

The following theorem summarizes the properties of moduli of Higgs sheaves following [65, 81, 76].

**Theorem A.2.** Suppose $X$ is a smooth projective curve over $\mathbb{C}$ and let $M_1, M_2$ be fixed line bundles on $X$ as in the main text. Let $X = (X, M_1, M_2)$. Then

(i) For $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ there is a quasi-projective coarse moduli scheme $\text{Higgs}^{ss}(X, r, e)$ over $\mathbb{C}$ parameterizing $S$-equivalence classes of Higgs sheaves of type $(r, e)$ on $X$ with coefficient sheaves $(M_1, M_2)$. The scheme $\text{Higgs}^{ss}(X, r, e)$ contains an open subscheme $\text{Higgs}^s(X, r, e)$ which parameterizes isomorphism classes of stable Higgs sheaves.

(ii) There is a proper Hitchin morphism $h : \text{Higgs}^{ss}(X, r, e) \to \mathbb{H}$ where $\mathbb{H} = \bigoplus_{i=0}^\infty H^0(X, \text{Symm}^n(M_1^{−1} \oplus M_2^{−1}))$ mapping a polystable Higgs sheaf of type $(r, e)$ to its characteristic polynomial.

Moreover, the boundedness results and the construction of parameter spaces presented in [65, 81, 76] imply the following theorem.

**Theorem A.3.** For fixed $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ there is an algebraic moduli stack of finite type over $\mathbb{C}$ $\mathcal{H}^{ss}(X, r, e)$ of semistable Higgs sheaves of type $(r, e)$ on $X$. The coarse moduli scheme $\text{Higgs}^{ss}(X, r, e)$ is a good moduli space for the stack $\mathcal{H}^{ss}(r, e)$ in the sense of [1]. In particular there is a surjective universally closed morphism $\varphi : \mathcal{H}^{ss}(X, r, e) \to \text{Higgs}^{ss}(X, r, e)$.

**Proof.** Using the results of [65, 81, 76] the algebraic stack in question is in fact a quotient stack of the form $[R/GL(M, \mathbb{C})]$ for some quasi-projective scheme $R$ and some $M > 0$. The second statement is entirely analogous to [1, Ex. 7.7].

**Remark A.4.** (i) Note that for certain values of $(r, e)$ any semistable Higgs sheaf of type $(r, e)$ must be stable. For example this holds if $(r, e)$ are coprime.

(ii) In order to simply the exposition, we will adopt the following notation conventions for Higgs sheaves. Suppose $E = (E, \Phi_1, \Phi_2)$ is a Higgs sheaf on $X$ with coefficient sheaves $M_1, M_2$. Given a finite sequence $\Phi_{i_n} : E \otimes_X M_{i_n} \to E$, where $n \geq 1$ and $(i_1, \ldots, i_n) \in \{1, 2\}^n$, the composition

$$\Phi_{i_1} \circ (\Phi_{i_{n-1}} \otimes 1_{M_{i_n}}) \circ \cdots \circ (\Phi_{i_1} \otimes 1_{M_{i_n}}) : E \otimes_X M_{i_1} \otimes_X \cdots \otimes_X M_{i_n} \to E$$

will be denoted by

$$\Phi_{i_1} \Phi_{i_{n-1}} \cdots \Phi_{i_1} : E \otimes_X M_{i_1} \otimes_X \cdots \otimes_X M_{i_n} \to E.$$

**Definition A.5.** A framed Hitchin pair of type $(r, e)$ on $X$ is defined by the data $(E, \Phi_1, \Phi_2, \chi)$ where $(E, \Phi_1, \Phi_2)$ is a Higgs sheaf of type $(r, e)$ on $X$ with coefficient sheaves $M_1, M_2$ as defined in (A.2) and $\chi : E \to \mathcal{O}_X$ is a morphism of sheaves.

**Remark A.6.** (i) Note that in [79], framed Hitchin pairs are augmented by adding an arbitrary $\epsilon \in \mathbb{C}$ to the data $\mathcal{H}$ of definition (A.5). This is needed in order to obtain a projective coarse moduli space of semistable objects. For the purposes of the present paper, it suffices to consider such objects with $\epsilon \neq 0$, in which case it can be set to $\epsilon = 1$. Therefore $\epsilon$ can be omitted altogether. As a result, the moduli spaces will be only quasi-projective.
The objects defined in (A.5) should be called symmetric Hitchin pairs with twisting data $M_1 \oplus M_2$ and framing data $O_X$ in the terminology of [79]. For simplicity we will call them framed Hitchin pairs in this paper since there will be no risk of confusion.

Let us next recall $\delta$-stability for framed Hitchin pairs [79]. Let $\delta \in \mathbb{R}_{>0}$ be a stability parameter.

**Definition A.7.** A framed Hitchin pair $(E, \Phi_1, \Phi_2, \chi)$ is $\delta$-(semi)stable if $\chi$ is not identically zero and the following conditions hold

(i) For any $\Phi$-invariant nontrivial proper saturated subsheaf $0 \subset E' \subset E$ we have

\begin{equation}
\mu(E') - \frac{\delta}{\text{rk}(E')} \leq \mu(E) - \frac{\delta}{r}.
\end{equation}

(ii) For any $\Phi$-invariant nontrivial proper saturated subsheaf $0 \subset E' \subset E$ so that $E' \subseteq \text{Ker}(\chi)$ we have

\begin{equation}
\mu(E') \leq \mu(E) - \frac{\delta}{r}.
\end{equation}

A framed Hitchin pair $(E, \Phi_1, \Phi_2, \chi)$ of type $(r, e)$ on $X$ is asymptotically stable if $\chi$ is nontrivial and there exists no $\Phi$-invariant nontrivial proper saturated subsheaf $0 \subset E' \subset E$ so that $E' \subseteq \text{Ker}(\chi)$.

Then [79, Thm 1.7] and [79, Prop. 2.9] imply the following

**Theorem A.8.** (i) There is a quasi-projective moduli scheme $FH^s_{\delta}(X, r, e)$ parameterizing $S$-equivalence classes of $\delta$-semistable framed Hitchin pairs of type $(r, e)$ on $X$. There is an open subscheme $FH^s_{\delta}(X, r, e) \subseteq FH^s_{\delta}(X, r, e)$ which is a fine moduli space for isomorphism classes of $\delta$-stable framed Hitchin pairs of type $(r, e)$ on $X$.

(iii) Let $H = \oplus_{a=1}^{r_1} H^0(X, S^a(M_x^{-1} \oplus M_y^{-1}))$. Then there is a proper morphism $fh : FH^s_{\delta}(X, r, e) \to H$ mapping a $\delta$-polystable framed Hitchin pair to its characteristic polynomial.

(iii) For fixed $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ there exists $\delta_\infty \in \mathbb{R}_{>0}$ so that for any $\delta > \delta_\infty$, a framed Hitchin pair of type $(r, e)$ is $\delta$-semistable if and only if it is $\delta$-stable and if and only if it is asymptotically stable.

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