Variational Formulation for Quaternionic Quantum Mechanics

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In honour of the 70th birthday of Prof. J. A. C. Alcarás

Abstract. A quaternionic version of Quantum Mechanics is constructed using the Schwinger’s formulation based on measurements and a Variational Principle. Commutation relations and evolution equations are provided, and the results are compared with other formulations.

Keywords. Quaternionic Quantum Mechanics, Variational Principle.

1. Introduction

In 1936 Birkhoff and von Neumann [1] have shown the existence of a propositional calculus as fundamental ingredient of Quantum Mechanics (QM), which could be written using only the outputs of measures. It does not assume any set of numbers or even a particular vectorial space, but contains the essentials of QM such as uncertainty relations and complementary properties. Of course, the authors showed that there are three different realizations for this propositional calculus, corresponding to the real or complex numbers or still quaternions. Octonions and higher dimensional extensions of the complex numbers are discarded, since they can not have a conservation law for the probability current [2].

We can ask: which of these three realizations of the “general” QM of Birkhoff and von Neumann is present in Nature? Here it is implicit the hypothesis that
the set of numbers of a given theory reflects part of the physical information about the system. While the differences between the real and complex QM are relatively simple and well known [3], the quaternionic version has many new and rich characteristics. Therefore, it sounds strange that such possibility is not much explored, but there are very good reasons for this. First, the problem of writing a quaternionic Schrödinger equation is not trivial since it involves the explicit use of imaginary unit. Second, the representation of composite systems by a direct product is more difficult due to the noncommutativity of the quaternionic valued wave functions.

Here, we implement a quaternionic version of Schwinger’s Measurement Algebra and build the dynamics based on the Action Principle. In each step, the analogy with the usual QM is used as inspiration, but the peculiarities emerging from the quaternionic noncommutativity are always emphasized.

The theory constructed by this means is quite distinct from Adler’s approach [2], having similarities with the work of Finkelstein, Jauch, Schiminovich and Speiser, [4, 5, 6].

2. Measurement Symbols

The classical theory of physical measurements is based on the concept that the interaction between the system under observation and the measurement apparatus can be done arbitrarily small or, at least, precisely compensated, in such way to specify an idealized measurement which does not disturb any other property of the system. However, the experiment had demonstrated that the interaction can not be done arbitrarily small neither the disturb produced can be precisely compensated since it is uncontrollable and unpredictable. The fact that the interaction can not be arbitrarily small is expressed by the finite size of the Planck constant, while the uncontrollable character of the interaction is given by the uncertainty principle. Therefore, the measurement of a given property can produce a significant change in the value of another previously measured property, and then there is no sense in speaking about an microscopic system with definite values for all its attributes. This is in contradiction with the classical representation of physical quantities by numbers. The laws of a microscopic physical system must then be expressed in a non-classical mathematical language constituting a symbolic expression of the properties of microscopic measurements.

In what follows, we will develop the general lines of such mathematical structure discussing about simplified physical systems where any physical quantity $A$ can have only a finite number of different values $a^1, a^2, a^3, \ldots$. The most simple measurement consider an ensemble of similar independent systems which is divided by the apparatus of measurement in sub-ensembles distinguished by the defined values of the physical quantity under measurement. Let us denote $M_a$ the selective measurement accepting any system having value $a$ for the property $A$ and rejecting any other. The addition of such symbols is defined as implying a less
specific measure, resulting in a sub-ensemble associated with any value under the sum, none of them being distinguished of the others by the measurement.

The multiplication of measurement symbols implies the sequence of measurements reading from right to left. From the physical meaning of such operations, we learn that addition is commutative and associative while multiplication is only associative. Using $\hat{1}$ and $\hat{0}$ to represent respectively the measures which accept and reject all systems, the properties of the elementary selective measurement are given by:

$$
\begin{align*}
\hat{M}_a \hat{M}_a &= \hat{M}_a & (2.1a) \\
\hat{M}_a \hat{M}_{a'} &= \hat{0} & (2.1b) \\
\sum_a \hat{M}_a &= \hat{1} & (2.1c)
\end{align*}
$$

From the meaning of the measurements represented by $\hat{1}$ and $\hat{0}$ we directly read the following algebraic properties:

$$
\begin{align*}
\hat{1} \hat{M}_a &= \hat{M}_a \hat{1} = \hat{M}_a \\
\hat{0} \hat{M}_a &= \hat{M}_a \hat{0} = \hat{0} \\
\hat{M}_a + \hat{0} &= \hat{M}_a 
\end{align*}
$$

what justifies the adopted notation. The algebraic properties of $\hat{1}$, $\hat{0}$ and $\hat{M}_a$ are consistent provided that the multiplication be distributive,

$$
\sum_a \left( \hat{M}_a \hat{M}_{a'} \right) = \hat{M}_{a'} = \hat{M}_{a'} \hat{1} = \hat{M}_{a'} \sum_a \hat{M}_a
$$

All laws of multiplication for measurement symbols given above can be combined in a single expression,

$$
\hat{M}_a \hat{M}_{a'} = \delta_{aa'} \hat{M}_a
$$

with the introduction of the symbol

$$
\delta_{aa'} = \begin{cases} 
\hat{1}, & a = a' \\
\hat{0}, & a \neq a'
\end{cases}
$$

known as *Kronecker’s delta*.

From these definitions one sees that the measurement symbols belong to a noncommutative ring [7].

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1 Of course, such properties characterize the measurement symbols as *projectors* on the space of physical states. The projective geometry originated from this complete set of projectors can be explored to construct a pair of dual vector spaces of creation and annihilation operators representing the *out* and *in* stages of an elementary measurement.
3. Compatible Properties

Two quantities $A_1$ and $A_2$ are compatible when the measurement of one of them does not destroy the knowledgement of a previous measurement of the other. The selective measures $\hat{M}_{a_1}$ and $\hat{M}_{a_2}$, taken in this order, produce an ensemble where it is possible, simultaneously\(^2\), to attribute the values $a_1$ to $A_1$ and $a_2$ to $A_2$. The symbol for such composite measurement is

$$\hat{M}_{a_1a_2} = \hat{M}_{a_1} \hat{M}_{a_2} = \hat{M}_{a_2} \hat{M}_{a_1}$$

From such definition it is easy to see that the compatibility is an equivalence relation.

A complete set $A$ of compatible quantities $A_1, \ldots, A_r$ means that any pair of such properties is compatible and there is no other compatible quantity outside the set, except the functions constructed from the set $A$. In fact, $A$ is an equivalence class. The measurement symbol

$$\hat{M}_a = \prod_r \hat{M}_{a_r}$$

describes a complete measurement where the selected systems have definite values for the maximum number of possible attributes. Any tentative for determining the value of another independent physical quantity will produce uncontrollable changes on the previously measured values. Therefore, the optimum information about a given system is achieved making a complete selective measurement. The systems accepted by the complete selective measurement $\hat{M}_a$ are known to being in the state $a$. The symbolic properties for the complete measures are the same as for the elementary selective measurements, i.e., (2.1a), (2.1b) and (2.1c).

4. Changing States Measurements

A more general kind of measure incorporates a change on the state of the system. The symbol $\hat{M}^a_{a'}$ represents a complete selective measurement which accepts systems in the $a_1$ state and let out systems in the state $a$. The measurement process $\hat{M}_a$ is the special case when no change on the state occurs,

$$\hat{M}_a = \hat{M}^a_a$$

The properties of successive measurements of this specie are given by

$$\hat{M}^a_{a_1} \hat{M}^{a_3}_{a_2} = \delta^{a_2}_{a_3} M^a_{a_1}$$

(4.1)

since if $a_3 \neq a_2$ the second stage of the apparatus does not select any system emerging from the first one, and if $a_3 = a_2$ all systems coming from the first

\(^2\)Note that the use of the word *simultaneously* is made without any reference to a definition of *simultaneity* and also without reference to the concept of *time*. Here, we are presuming that in an intuitive way it is clear to the reader the sense in which these words are been used. The concept of temporal evolution is associated with the notion of dynamics which will be investigated below based on the Action Principle.
stage are accepted by the second, being the composite measurement a selection of systems in the state $a_4$ and letting it out in the state $a_1$. Observe that if we interchange both stages, then
\[ \hat{M}_{a_3}^{a_4} \hat{M}_{a_3}^{a_2} = \delta_{a_4}^{a_1} M_{a_3}^{a_2} \]
what is not the same as $\text{in}$ 1. Therefore, we realize that the multiplication of complete measurements symbols is noncommutative.

The physical quantities belonging to a complete set do not exhaust the totality of physical attributes in a system. One can form others complete sets $B, C, \ldots$, which are mutually incompatible and, for each choice of non-interfering physical characteristics, there is a set of selective measurements concerning to systems in the appropriate states $\hat{M}_{b_1}^{b_2}, \hat{M}_{c_1}^{c_2}, \ldots$. The most general selective measurement links two complete sets of incompatible properties. Let $\hat{M}_b^a$ be the measurement process rejecting all systems which are not in the state $b$ and allowing to emerge only systems in the state $a$. The composite measurement $\hat{M}_a^b \hat{M}_c^d$ will select systems in the state $d$ and let them in the state $a$, so it should be proportional to the selective measurement $\hat{M}_a^d$.

The examples considered until now include the passing of all or none system through both stages, as realized by the symbols $\hat{1}$ and $\hat{0}$. Notwithstanding, in general we can just admit that measures of the property $B$ upon a system in the state $c$, which belongs to a complete set incompatible with $B$, will furnishes an statistical distribution of all possible results. So, only a fraction of the systems emerging from the first stage is accepted by the second one. We can express this by the general multiplication law:
\[ \hat{M}_b^a \hat{M}_c^d = |a\rangle \langle b| c\rangle \langle d| = M_a^d (\langle b| c\rangle) \] (4.2)
where $\langle b| c\rangle$ is a number characterizing the statistical relationship between the states $b$ and $c$. In particular,
\[ \langle a| a\rangle = \delta_{a'}^a, \quad a, a' \sqsubset A \]
where $\sqsubset$ means that $a$ and $a'$ are defined sets of values for the complete set $A$. Since that the numbers $\langle a| b\rangle$ link the states $a$ and $b$ they are called transformation function.

The measurement symbols $M_a^b$ equipped with addition and multiplication as defined above and together with the scalar ring $\langle b| c\rangle$ form an algebra, which we call the Measurement Algebra. Observe that nothing was said about the particular set of numbers $\langle b| c\rangle$ to be adopted. In fact, as matter for mathematical and physical meaning consistency, it is enough that $\langle b| c\rangle$ belongs to an scalar ring.

Of course, the order in which the scalars $\langle a| b\rangle$ appear in the product (4.2) is very important, since it reflects on the ring multiplication law, allowing the definition of different measurement algebras. Therefore, the most general form to indicate the multiplication rule for measurement symbols is $\hat{M}_a^b \hat{M}_c^d = M_a^d (\langle b| c\rangle)$, since it does not make any reference to the order of the scalar on the product. However, we will maintain the scalars on a preferable central position on the
product. Our main interest here is to suppose that the scalars are quaternions and investigate what are the physical implications of such assumption.

The reason to take the scalars on a central multiplicative position comes from the recognition that measurement symbols are in fact projectors on the several possible states of two different complete sets of observables. To reinforce such character, we adopt the notation

$$\hat{M}_b^a = |a\rangle \langle b|$$

Then, the most general way in which a measurement symbol can appear together an scalar is

$$\hat{M}_b^a (q) = |a\rangle q \langle b|$$

being $q$ any element of the ring under which the measurement algebra is defined. As stated before, we will assume that the numbers $q$ are *quaternions*, defined by

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3, \quad e_i e_j = -\delta_{ij} + \sum_{k=1}^{3} \varepsilon_{ijk} e_k, \quad q_n \in \mathbb{R} \quad \forall \ n \in \{0, ..., 3\}$$

When $q = 1$ we simply denote $\hat{M}_b^a (1) = \hat{M}_b^a$. This notation is useful because it maintains separated in an explicitly way the two parts of the measurement symbol corresponding to the physical Hilbert space of states $\mathcal{H}$ and its dual $\mathcal{H}^\dagger$. In the language of second quantization, this notation directly alludes to the annihilation (right) and creation (left) processes of particles or field fluctuations involved in a measurement act. It is important to stand out that since the products of vector by scalars are defined over a noncommutative ring, these products have sense only a definite order, which we take as right for the kets ($|a\rangle q, \forall |a\rangle \in \mathcal{H} (\mathbb{H}), \forall q \in \mathbb{H}$) and left for the bras ($q \langle b|, \forall \langle b| \in \mathcal{H}^\dagger (\mathbb{H}), \forall q \in \mathbb{H}$), where $\mathcal{H} (\mathbb{H})$ is the Hilbert space of eigenstates of a given complete set of observables.

Quaternions are a particular realization of a *Clifford algebra* $\mathbb{R}$, so a even more general theory can be recognized.

### 5. Transformation Functions

The fundamental transformation law for the measurement symbols is essentially unaffected by the specific choice of the scalar ring. Actually, using the notation of the previous section, measurement symbols of one kind can be transformed in symbols of another kind:

$$\hat{M}_d^c = |c\rangle \langle d| = \sum_{a,b} \hat{M}_a \hat{M}_d^a \hat{M}_b = \sum_{a,b} |a\rangle \langle a|c\rangle \langle d|b\rangle \langle b| \quad (5.1)$$

Carefully preserving the composition of products, one can interpret this relation as a double mapping of vectors $|c\rangle$ and covectors $\langle d|$ on the linear combinations $\sum_a |a\rangle \langle a|c\rangle$ and $\sum_b \langle d|b\rangle \langle b|$ respectively. Therefore, the composition law for
transformation functions in a quaternionic ring is
\[ \sum_b \langle a|b \rangle \langle b|c \rangle = \langle a|c \rangle \]
from which we obtain the completeness relations
\[ \sum_a^N \sum_b^N \langle a|b \rangle \langle b|a \rangle = \sum_a^N 1 = N \]
\[ \sum_b^N \sum_a^N \langle b|a \rangle \langle a|b \rangle = \sum_b^N 1 = N' \]

However, since quaternions do not commute, the preservation of the number of degrees of freedom imply that
\[ \sum_a^N \sum_b^N \langle a|b \rangle \langle b|a \rangle = \sum_b^N \sum_a^N \langle b|a \rangle \langle a|b \rangle \] (5.2)
Except for systems with only one degree of freedom, this does not mean that
\[ \langle a|b \rangle \langle b|a \rangle = \langle b|a \rangle \langle a|b \rangle \]
for any pair of quaternionic transformation functions. The n, the relation (5.2) implies a restriction, but its interpretations is not easy.

6. The Trace Functional and the Statistical Interpretation

One of the most important actions over the measurement algebra is the trace functional, which associates each element of the algebra to one scalar. Since here the scalar ring is noncommutative, there are three kinds of trace functional called respectively left, right and central trace:
\[ Tr_a^M \hat{b}^a(q) \equiv q \langle b|a \rangle \]
\[ Tr_R^M \hat{b}^a(q) \equiv \langle b|a \rangle q \]
\[ Tr_C^M \hat{b}^a(q) \equiv \sum\langle |a| q \langle b|c \rangle \]

In the standard complex case, the trace functional is related to the statistical interpretation of quantum mechanics. Here we have a more complicated situation since none of the above trace functionals has an invariant law of transformation. Nevertheless, the multiplication law is invariant under the following mapping:
\[ \hat{M}_a^b = |a\rangle \langle b| \rightarrow |a\rangle \lambda_a^{-1} \lambda_b \langle b| = \hat{M}_a^b (\lambda_a^{-1} \lambda_b) \] (6.1a)
\[ \langle a|b \rangle \rightarrow \lambda_a \langle a|b \rangle \lambda_b^{-1} \] (6.1b)
where quaternions \( \lambda_a, \lambda_b \) are not null. Therefore, the transformation function \( \langle a|b \rangle \) can not itself have a direct physical interpretation, and shall configure in a combination invariant under \( \text{(6.1)} \).

The appropriate basement for the statistical interpretation of the transformation function can be inferred from a sequence of elementary measurement,
\( \hat{M}_b \hat{M}_a \hat{M}_b \), which differs from \( \hat{M}_b \) only by virtue of the disturbance caused by the intermediary measurement of the attribute \( A \). Only a fraction of the systems selected by the initial measurement of \( B \) is transmitted through the complete set. Hence, we obtain the following symbolic statement:

\[
\hat{M}_b \hat{M}_a \hat{M}_b = \hat{M}_b (p (a|b))
\]

where the number

\[
p (a|b) = \langle b|a \rangle \langle a|b \rangle \quad (6.2)
\]

should be invariant under \( \text{Id} \). It means that

\[
\lambda_b \langle b|a \rangle \langle a|b \rangle \lambda_b^{-1} = \langle b|a \rangle \langle a|b \rangle
\]

Now, if one considers a measurement of the property \( A \) which does not distinguish between two states, one arrives on the additivity of \( p (a|b) \),

\[
\hat{M}_b \left( \hat{M}_a + \hat{M}_{a'} \right) \hat{M}_b = (p (a|b) + p (a'|b)) \hat{M}_b
\]

So, taking a measurement of \( A \) unable to select any of such states, one obtains

\[
\hat{M}_b \left( \sum_a \hat{M}_a \right) \hat{M}_b = \hat{M}_b
\]

what implies:

\[
\sum_a p (a|b) = 1
\]

Such properties characterize \( p (a|b) \) as a probability measure \( \mathbb{P} \) of observing the state \( a \) in a measurement made over a system known to be in the state \( b \). However, probability measures are positive real numbers, then we must impose a restriction on the the numbers which figure in the measurement algebra. Until now, all we have made can be applied equally to quaternions or complex numbers. In fact, no physical information was used to select the nature of such numbers, being only necessary they form a scalar ring in order to obtain an algebra from the elementary selective measurements. Therefore, any field, as \( \mathbb{R} \) or \( \mathbb{C} \), for instance, is candidate to figure as scalars in this construction of the quantum theory, but also a ring which is not a field, as quaternions or octonions, could be used. The extension of Quantum Mechanics that we want to do here is to get quaternions as the scalar ring used to construct the measurement algebra.

So, the probability measure \( p (a|b) \) must satisfy \( p (a|b) \geq 0 \). Besides, the arbitrary reading convention in the multiplicative law implies that such probability shall be symmetric. The simplest way to accomplish all these properties is to demand \( Q = \lambda_b \langle b|a \rangle \) and \( \bar{Q} = \langle a|b \rangle \lambda_b^{-1} \) to be a conjugated pair. Of course, in such case one obtains

\[
Q \bar{Q} = \bar{Q} Q = |Q|^2 \geq 0
\]

\[
\lambda_b \langle b|a \rangle \langle a|b \rangle \lambda_b^{-1} = \langle a|b \rangle \lambda_b^{-1} \lambda_b \langle b|a \rangle = \langle a|b \rangle \langle b|a \rangle
\]

On the other hand,

\[
\bar{Q} = \overline{\langle \lambda_b \langle b|a \rangle \rangle} = \overline{\langle b|a \rangle \lambda_b} = \langle a|b \rangle \lambda_b^{-1}
\]
This let us with the following statements:
\[
\langle b|a \rangle \langle a|b \rangle = \langle a|b \rangle \langle b|a \rangle \geq 0
\]
\[
\langle b|a \rangle \bar{\lambda}_b = \langle a|b \rangle \lambda_b^{-1}
\]

Again, the simplest way to solve this system is taking
\[
\bar{\lambda}_b = \lambda_b^{-1}
\]
\[
\langle b|a \rangle = \langle a|b \rangle
\]

With this choice one is able to recover all the properties of the probability measure \( p(a|b) \).

Using an exponential representation for \( \lambda_a \) we see that the first condition above can be written in the form
\[
\lambda_a = A e^{e_{\lambda} \varphi(a)} \rightarrow A^{-1} e^{-e_{\lambda} \varphi(a)} \rightarrow A^2 = 1 \rightarrow A = \pm 1
\]
where
\[
|A| = |\lambda_a| = \left[ (\lambda_0^0)^2 + (\lambda_1^1)^2 + (\lambda_2^2)^2 + (\lambda_3^3)^2 \right]^{1/2}
\]
\[
e_{\lambda} = \frac{\lambda_1^1 e_1 + \lambda_2^2 e_2 + \lambda_3^3 e_3}{\left[ (\lambda_1^1)^2 + (\lambda_2^2)^2 + (\lambda_3^3)^2 \right]^{1/2}}
\]
\[
\varphi(a) = \arctan \left( \frac{\lambda_0^0}{|\lambda_a|} \right) \quad \varphi(a) \in [0, \pi]
\]

the choice for the signal in \( A \) is arbitrary and no physical effect can be distinguished by one particular choice. Therefore we will take the positive signal. Since \( \lambda_a \) is a unitary arbitrary number its phase \( \varphi(a) \) can be an arbitrary real number.

Thus, besides the problems concerning about the definition of the trace functional one is still able to construct an statistical interpretation for the Quaternionic Quantum Mechanics. In fact, such result indicates that the roots for the statistical interpretation are in the propositional calculus\(^3\) of Birkhoff and von Neumann [1], and not in the particular system of numbers adopted to construct the theory.

Another very important piece for the construction of the statistical interpretation was the automorphism \( \langle a|b \rangle \rightarrow \lambda_a \langle a|b \rangle \lambda_b^{-1} \) for the scalar ring \( \mathbb{H} \). But, physically, what means such identification? We know that the elements of the scalar ring represent logical relations between the possible physical states of the system under consideration. Clearly, it is even possible to say when two of such relation are “the same thing” for states taken in distinct physical systems without departing the traditional concepts of pure logic\(^4\), i.e., without using the concepts of structured networks introduced by Birkhoff and von Neumann [1]. However, this defines such numbers modulo automorphisms [4]. In the case of a quantum theory with only real numbers this is sufficient to determine completely such numbers.

\(^3\)Or, in our construction, in the Measurement Algebra relations.

\(^4\)The role for the abstract mathematical logic in Physics is discussed in a very interesting way in [10].
In the complex case, it still stands an ambiguity, which is manifested under the existence of a conjugated algebra. In Quaternionic Quantum Mechanics such ambiguousness is infinitely bigger. This requires the introduction of more structure elements in the theory\(^5\). In the following we will delimitate what are such structures.

7. The Adjoint

Other important aspect of the probabilistic interpretation for (6.2) is the symmetry

\[ p(a|b) = p(b|a) \]

Remember the arbitrary convention for reading the measurement symbols and their products: the order of the events is read from right to left. But any equation involving the measurement symbols is equally valid if interpreted in the opposite sense and none physical result can depend of what is the convention adopted. Introducing the right-handed interpretation, \( \langle a|b \rangle \) acquire the same meaning of \( \langle b|a \rangle \) in the left-handed convention. We can conclude that the probability connecting the states \( a \) and \( b \) in a given sequence must be constructed symmetrically from \( \langle a|b \rangle \) and \( \langle b|a \rangle \). Of course, this is the reason why \( p(a|b) \) should be symmetric. The introduction of the opposite convention for the measurement symbols will be called the *adjoint* operation and will denoted by \(^\dagger\). Therefore,

\[
\hat{M}_a^b = \hat{M}_b^a
\]

and

\[
M_a^a^\dagger = M_a^a
\]

in particular,

\[
M_a^a = M_a
\]

what means that \( \hat{M}_a \) is a self-adjoint operator. For the product of measurements symbols we have

\[
\left( \hat{M}_a^b \hat{M}_c^d \right)^\dagger = \hat{M}_d^c \hat{M}_b^a = \hat{M}_c^d \hat{M}_a^b
\]

The meaning of addition is not changed by the adjoint operation what permits to extend these properties for all element in the measurement algebra:

\[
\left( \hat{X} + \hat{Y} \right)^\dagger = \hat{X}^\dagger + \hat{Y}^\dagger \quad \left( \hat{X}\hat{Y} \right)^\dagger = \hat{Y}^\dagger \hat{X}^\dagger \quad \left( \lambda \hat{Y} \right)^\dagger = \hat{\lambda}^\dagger \hat{Y}
\]

where \( \lambda \in \mathbb{H} \).

\(^5\)Of course, these observations are crucial to construct the representation for systems with many particles.
8. Infinitesimal Variation of Transformation Functions

Taking infinitesimal variations of the two fundamental properties of the transformations functions, we find

\[ \sum_b \left[ \delta \langle a | b \rangle \left( \langle b | c \rangle \right) + \langle a | b \rangle \delta \langle b | c \rangle \right] = \delta \langle a | c \rangle \]  

(8.1)

\[ \delta \langle a | b \rangle = \delta \langle b | a \rangle \]

In the ordinary complex case \[11\] the numbers \( \delta \langle a | b \rangle \) are interpreted as representing the matrix elements of an infinitesimal operator,

\[ \delta \langle a | b \rangle = i \langle a | \delta \hat{W}_{ab} | b \rangle \]

where the constant \( i \) was chosen in order to assure that the operator \( \delta \hat{W}_{ab} \) is self-adjoint.

Here, it is an open question what constant should be chosen since actually we have tree imaginary unities. The most general case is let the imaginary unity to be an operator \( \hat{\iota} \) where \( i \hat{\iota} = \hat{\iota} \) can be considered as a particular case for \( \mathbb{C} \).

Let it be so, defining

\[ \delta \langle a | b \rangle = \langle a | i \delta \hat{W}_{ab} \rangle \]

(8.2)

where \( i \) is a quaternionic valued operator that we will be fixed later under the requirement of \( \delta \hat{W}_{ab} \) be a self-adjoint operator. Using this definition it is easy to see that the additivity and the skewsymmetry in ordering infinitesimal operators are the same as in the complex case \[11\],

\[ \delta \hat{W}_{ac} = \delta \hat{W}_{ab} + \delta \hat{W}_{bc} \]

\[ \delta \hat{W}_{ba} = -\delta \hat{W}_{ab} \]

On the other hand,

\[ \delta \langle a | b \rangle = \langle b | \delta \hat{W}_{ab}^{\dagger} \hat{\iota}^{\dagger} | a \rangle = \langle b | i \delta \hat{W}_{ba} | a \rangle \]

what let us to the operatorial identity,

\[ \delta \hat{W}_{ab}^{\dagger} \hat{\iota}^{\dagger} + i \delta \hat{W}_{ab} = \hat{0} \]

If we impose

\[ \delta \hat{W}_{ab} = \delta \hat{W}_{ab}^{\dagger} \]

(8.3)

\[ [i, \delta \hat{W}_{ab}] = \hat{0} \]

(8.4)

we obtain:

\[ \hat{\iota} = -i \hat{\iota}^{\dagger} \]

This identity can be interpreted as a generalization of the complex conjugation over \( \mathbb{C} \), and shows that the operator \( \hat{\iota} \) behaves like an “imaginary unit”. The condition (8.3) assures the reality of the spectrum associated to infinitesimal operators. The condition (8.4) can be satisfied in several ways:

1. demanding that all infinitesimal operator commutes with the imaginary unity;
2. letting the imaginary unity to commute with any operator;
3. claiming that an infinitesimal operator commutes with any other operator.

In the standard quantum mechanics Schwinger choose the last option [11], which was subsequently extended to more general variations by several authors [12]. Here, we can see no reason to discard the other two options. In fact, in their work on quaternionic quantum theory, Finkelstein, Jauch Schiminovich and Speiser [4] have adopted a particular case of the second option in the list above interpreting it as a superselection rule [6]. For while, we will require that at least one of the three conditions above is satisfied, i.e., we will work directly assuming only the general statement (8.4).

With these choices, unitary infinitesimal operators can be expressed as
\[ \hat{U} = \hat{1} + \hat{G}, \quad \hat{U}^\dagger = \hat{U}^{-1} = \hat{1} - \hat{G}, \quad \hat{G} = -\hat{G}^\dagger = i\hat{W} \]
and infinitesimal variations of operators are induced by the commutator with the generator
\[ \delta \hat{X} = - \left[ \hat{X}, \hat{G} \right] = \left[ \hat{G}, \hat{X} \right] \] (8.5)

These are all ingredients necessary to describe completely the one particle physical states. We will not approach here the problem of representing composite systems, but it is clear that such extension is possible. Now we are ready to analyse the dynamic characteristics which are changed by the use of quaternions.

9. The Variational Principle

The quantum dynamics for the system will be obtained from the Schwinger Action Principle [11] here expressed as
\[ \delta \langle a_{t_2} | b_{t_1} \rangle = \langle a_{t_2} | i\delta \hat{S}_{t_1, t_2} | b_{t_1} \rangle \]
\[ \delta \hat{S}_{t_1, t_2} = \left[ \hat{p} \cdot \delta \hat{q} - \hat{H} \delta t \right] \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \delta \hat{L} \cdot \left( \delta \hat{q} - \hat{q} \dot{\delta} t \right) = \hat{G}_2 - \hat{G}_1 \]
\[ \hat{p} = \frac{\partial \hat{L}}{\partial \dot{\hat{q}}}, \quad \hat{H} = \hat{p} \cdot \hat{q} - \hat{L} \]

The Hamiltonian \( \hat{H} \) and Lagrangian \( \hat{L} \) operators are self-adjoints.

Schwinger Action Principle is the quantum counterpart of the classical Weiss Principle [13], which can be considered the most general variational principle for classical fields. Schwinger Principle has been successfully applied in Minkowski [14], curved [15] or torsioned spaces [16], as well as to describe quantum gauge transformations [17] and many other problems. Here, we will apply the Action Principle to extract dynamic and kinematic information from a canonical formulation for Quaternionic Quantum Mechanics.

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6In [4] the imaginary unity operator is denoted by \( \hat{η} \).
10. Commutation Relations and Time Evolution for Operators

The canonical (anti)commutation relations can be obtained from the action using the canonical infinitesimal generator

\[ \hat{G} = i\hat{p}_r\delta\hat{q}^r \]

from which we extract the following set of functional relationships:

\[
\begin{align*}
\delta\hat{q}^s &= -[\hat{q}^s, \hat{p}_r]_\mp \hat{p}_r\delta\hat{q}^r - i[\hat{q}^s, \hat{p}_r]_\mp \delta\hat{q}^r \mp i\hat{p}_r[\hat{q}^s, \delta\hat{q}^r]_\mp \\
\hat{0} &= -[\hat{p}_s, \hat{i}]_\mp \hat{p}_r\delta\hat{q}^r - i[\hat{p}_s, \hat{p}_r]_\mp \delta\hat{q}^r \mp i\hat{p}_r[\hat{p}_s, \delta\hat{q}^r]_\mp \\
\hat{0} &= [\hat{q}^s, \hat{i}] \delta\hat{p}_r\hat{q}^r + i[\hat{q}^s, \delta\hat{p}_r]_\mp \hat{q}^r \pm i\delta\hat{p}_r[\hat{q}^s, \hat{q}^r]_\mp \\
\delta\hat{p}_s &= [\hat{p}_s, \hat{i}] \delta\hat{p}_r\hat{q}^r + i[\hat{p}_s, \delta\hat{p}_r]_\mp \hat{q}^r \pm i\delta\hat{p}_r[\hat{p}_s, \hat{q}^r]_\mp
\end{align*}
\]

This gives a system of equations between the canonical variables and their variations whose formal solution is unknown. One possible solution is to choose infinitesimal variations in order that

\[
\begin{align*}
[q^s, \delta q^r]_\mp &= [p^s, \delta q^r]_\mp = \hat{0} \\
[p^s, \delta p^r]_\mp &= [p^s, \delta p^r]_\mp = \hat{0} \\
\end{align*}
\]

However, terms involving the (anti)commutator of \(\hat{i}\) still remain, which could imply in “deviations” from the canonical commutation relations. That is why in \(\text{[4]}\) is adopted the superselection rule

\[
[\hat{q}^s, \hat{i}] = [\hat{p}_s, \hat{i}] = \hat{0}
\]

which conduct to

\[
\begin{align*}
[p_s, \hat{p}_r]_\mp &= \hat{0} \\
[q^s, \hat{q}^r]_\mp &= \hat{0} \\
-\hat{i}[\hat{q}^s, \hat{p}_r]_\mp &= \delta^s_r
\end{align*}
\]

To obtain an expression closer to the complex case, let us to suppose that the anti-hermitean operator \(\hat{i}\) is also unitary. By this way,

\[
\begin{align*}
[p_s, \hat{p}_r]_\mp &= \hat{0} \\
[q^s, \hat{q}^r]_\mp &= \hat{0} \\
[q^s, \hat{p}_r]_\mp &= \hat{i}\delta^s_r
\end{align*}
\]

This means that to obtain the standard form of the Heisenberg algebra for the canonical variables \(\hat{q}\) and \(\hat{p}\) one shall to demand both conditions 2 and 3 from section \(\text{[4]}\).

The equation of motion for operators can also be obtained from the variational principle doing variations only in the temporal parameter,

\[
\frac{d\hat{A}}{dt} = i\left[\hat{A}, \hat{H}\right] + \frac{\delta \hat{A}}{\delta t}
\]

\footnote{We are adopting the sum convention.}
10.1. Application: The Quaternionic Harmonic Oscillator

Assuming that a quaternionic harmonic oscillator is described by the following Lagrangian operator

\[ L = \frac{1}{2} \left( \dot{\hat{q}}^3 \hat{q} - \omega^2 \hat{q} \right) , \quad \hat{q} = \sum_{\alpha=0}^{3} q^\alpha e_\alpha \]

Taking functional variations of this operator, we find

\[ \delta L = \frac{1}{2} \left( (\delta \dot{\hat{q}}^3) \hat{q} + (\delta \hat{q}^3) \dot{\hat{q}} - \omega^2 \left[ (\delta \hat{q}^3) q + \hat{q}^{1 \delta} q \right] \right) = \]

\[ = \frac{1}{2} \left( \frac{d (\delta \hat{q}^3 + \hat{q}^{1 \delta} \delta q)}{dt} - [\delta \hat{q}^3 (\dot{\hat{q}}^3 + \omega^2 q) + (\hat{q}^{3 \delta} + \omega^2 q) \delta \hat{q}^3] \right) \]

Therefore, the infinitesimal generator for the functional variations in the fundamental operator is

\[ G = \frac{1}{2} i \left( \delta \hat{q}^3 \hat{q} + \hat{q}^{1 \delta} \delta q \right) \]

\[ \tilde{G} = -\frac{1}{2} i \left( \delta \hat{q}^3 \hat{q} + \hat{q}^{1 \delta} \delta q \right) \]

whose induced variations are\(^8\)

\[ \delta q^\beta = \frac{1}{2} \left( \left[ q^\beta , i \delta q^{\alpha \dagger} q_\alpha \right] + \left[ q^\beta \tilde{q}^{\alpha \dagger} , i \delta q^{\alpha \dagger} q_\alpha \right] \right) = \]

\[ = -\frac{1}{2} i \left( \delta q^{\alpha \dagger} \left[ q^\beta , \tilde{q}_\beta \right] + \tilde{q}^{\alpha \dagger} \delta q^\beta \right) \]

\[ \delta q^{\beta \dagger} = -\frac{1}{2} i \left( \delta q^{\alpha \dagger} \left[ q^\beta \tilde{q}_\beta , q^\alpha \right] + \left[ q^{\alpha \dagger} , \tilde{q}^{\beta \dagger} \right] \delta q^\beta \right) \]

\[ \delta q^{\beta} = -\frac{1}{2} i \left( \delta q_{\alpha} \left[ q^\beta , \tilde{q}_\alpha \right] + \left[ q^\beta \tilde{q}_{\alpha} , q^\alpha \right] \delta \tilde{q}^\beta \right) \]

\[ \delta q^{\beta \dagger} = -\frac{1}{2} i \left( \delta q_{\alpha} \left[ q^{\beta \dagger} , \tilde{q}^{\alpha \dagger} \right] + \left[ q^{\beta \dagger} \tilde{q}^{\alpha \dagger} , q^\alpha \right] \delta \tilde{q}^{\beta \dagger} \right) \]

Assuming that the operators \(q^\beta \), \(\tilde{q}^{\beta \dagger}\) and \(q^{\beta \dagger}\) are kinematically independent, we have the canonical commutation relations,

\[ \left[ q^{\beta \dagger} , q_{\alpha} \right] = \left[ q^\beta , \tilde{q}_\alpha \right] = \left[ q^\beta , q^{\alpha \dagger} \right] = 0 \]

\[ \left[ q^{\beta \dagger} , q_{\alpha} \right] = \left[ q^\beta , q^{\alpha \dagger} \right] = i \delta^\beta_{\alpha} \]

\(^8\)Simplifying notation we will omit the symbol \(\hat{q}\) from the operator in this section. We maintain it only over the imaginary unity in order to reinforce that here it is an operator.

\(^9\)The position of the indices is completely arbitrary here since we are dealing with a cartesian space.
11. Schrödinger Equation and the Coordinate Representation

Taking variations only over the final state in a given transition,
\[
\delta |b_{1}\rangle = 0 \rightarrow \delta \hat{q}(t_1) = \hat{0} \quad \delta t_1 = 0
\]
\[
\delta \langle a_{2} | \neq 0 \rightarrow \delta \hat{q}(t_2) \neq \hat{0} \quad \delta t_2 \neq 0
\]
we have
\[
\delta (\langle a_{2} | b_{1} \rangle) = \langle a_{2} | \hat{P}_2 \cdot \delta \hat{q}_2 - \hat{H} \delta t_2 | b_{1} \rangle
\]
Now, let us identify the description \(a\) as the generalized coordinates, i.e., the description where the operators \(\hat{q}\) are diagonal, and the state \(|b_{1}\rangle\) as an arbitrary state \(|\Psi\rangle\). From the commutation relations deduced before we have
\[
\delta (\langle q_{2} | \hat{q}_2 \cdot \hat{P}_2 \hat{q}_2 | \hat{q}_2 | \hat{q}_2 \rangle - \langle q_{2} | \hat{q}_2 \hat{q}_2 \hat{H} \delta t_2 ) = \delta q_2 \cdot (\langle q_{2} | \hat{P}_2 \hat{q}_2 | \hat{q}_2 | \hat{q}_2 \rangle - \langle q_{2} | \hat{q}_2 \hat{H} | \hat{q}_2 \rangle
\]
But,\[
\delta (\langle q_{2} | \hat{q}_2 \rangle) = \delta q_2 \cdot \frac{\partial \langle q_{2} | \hat{H} | \hat{q}_2 \rangle}{\partial q_2} + \delta t_2 \frac{\partial \langle q_{2} | \hat{H} | \hat{q}_2 \rangle}{\partial t_2}
\]
then,
\[
\frac{\partial \langle q_{2} | \hat{q}_2 \rangle}{\partial q_2} = \langle q_{2} | \hat{P}_2 \hat{q}_2 | \hat{q}_2 \rangle
\]
\[
\frac{\partial \langle q_{2} | \hat{q}_2 \rangle}{\partial t_2} = -\langle q_{2} | \hat{H} | \hat{q}_2 \rangle
\]
Inserting a completeness relation for the coordinate eigenstates, we find
\[
\frac{\partial \langle q_{2} | \hat{q}_2 \rangle}{\partial q_2} = \int d\tilde{q} \langle q_{2} | \hat{q}_2 \rangle \langle \tilde{q}_2 | \hat{P}_2 | \hat{q}_2 \rangle
\]
\[
\frac{\partial \langle q_{2} | \hat{q}_2 \rangle}{\partial t_2} = -\int d\tilde{q} \langle q_{2} | \hat{q}_2 \rangle \langle \tilde{q}_2 | \hat{H} | \hat{q}_2 \rangle
\]
The first equation gives the representation of the momentum operator in the coordinate representation assuming that the spectrum of \(\hat{q}\) is known, while the second is the Schrödinger equation.

If, by hypothesis, the operator \(\hat{q}\) has always the same value in any point of the coordinate space and at any instant of time, then
\[
\frac{\partial \langle q_{2} | \hat{q}_2 \rangle}{\partial q_2} = \iota \langle q_{2} | \hat{P}_2 \hat{q}_2 | \hat{q}_2 \rangle \quad (11.1a)
\]
\[
\frac{\partial \langle q_{2} | \hat{q}_2 \rangle}{\partial t_2} = -\iota \langle q_{2} | \hat{H} \hat{q}_2 | \hat{q}_2 \rangle \quad (11.1b)
\]
where \(\iota\) is the expected value of \(\hat{q}\). Of course, this last hypothesis is contained in the statement 2 of the section and it imply that the operator \(\hat{q}\) is actually a constant imaginary pure quaternion.

\[\text{Note that we have made use of the fact that the spectrum of the coordinates is real.}\]
12. Final Remarks

The Schwinger Measurement Algebra formulation for quantum kinematics is a powerful tool to disconnect the physical contents in quantum measurements from the mathematical requirements of consistence. At same time, it provides a natural way to achieve generalizations of standard Quantum Mechanics and provide a clear view of the price paid for such generalizations.

In particular, besides we have found difficulties to construct a linear functional relating operators and values in the quaternionic ring, it was still possible to achieve a well defined statistical interpretation for Quaternionic Quantum Mechanics. The essential elements for such construction are the noncompatibility of successive measurements, providing the fundamental law of multiplication for measurement symbols, and the automorphism \( \langle a|b \rangle \rightarrow \lambda_\alpha \langle a|b \rangle \lambda_\alpha^{-1} \) of the scalar ring. In principle, any theory with these basic characteristics can also have an statistical interpretation. Notwithstanding, for an appropriate interpretation, some additional properties are required for the probability measure \( p(a|b) \), such as the conservation of their associated current in a closed system. In fact, it is the essential feature that Adler used to prove the non-extensivity of Quantum Mechanics for octonions or higher dimension hypercomplex numbers [2].

It must be stressed that there are several problems which are not investigated above, such as the effect of the superselection rules \((1111)\) over representations of the canonical variables \( \hat{p} \) and \( \hat{q} \), or the physical effects of the new quaternionic degrees of freedom.

Although we have not treated composite systems (i.e., many particle systems) it is possible to advance some characteristics which should originate from the quaternionic noncommutativity. It is well known that in the classical physics there are no phase relations to be considered among subsystems of a bigger system (non-interacting particles) if we sum or multiply (by Cartesian product) their phase spaces. In Complex Quantum Mechanics there are phase relations between states which are important if we sum their state spaces, but they are not important for the product of such spaces (understood as a tensorial product). In Quaternionic Quantum Mechanics these phase relations should be important whatever one is dealing with sum or product of spaces, since the phase factor now is a quaternion.

This new feature of Quaternionic Quantum Mechanics can be interpreted in terms of a complementarity argument. In classical physics does exist complementarity relations because all measurements can, in principle, have an infinity precision. In the real and complex quantum mechanics there are complementarity relations among physical properties of the same system, but not between properties of different non-interacting systems. In quaternionic quantum mechanics there are complementarity between some properties for any pair of physical systems or subsystems. This is because the phase factor \( e^{i\phi(a)} \) can not be additively composed when multiplying quaternions, as specified in the section 6. Therefore, there is no reasonable way to form composite systems in order to have all observables associated in one system to be compatible with all the observables in any other
systems or, in other words, to commute with all the others observables in different systems. Actually, one can expect this to be greater difficulty to describe many particle systems in quaternionic quantum theory.

It is important to observe that besides the notion of a quaternionic Hilbert space has been a little vague here it is possible to develop the concepts the Geometry of States, as done by Schwinger [11], for the quaternionic ring. The idea and properties of such vectorial space emerge naturally in the Geometry of States. This was not done here simply by matters of space and convenience since that we were interested not only on the kinematical side, but also in the dynamic aspects of the quaternionic theory. For those interested in the spectral theory of quaternionic Hilbert spaces is interesting to check [5] where the main theorems and ideas are introduced with a pedagogical explanation of how to perform the calculations in a vectorial space of scalars in $\mathbb{H}$.

With respect to quaternionic quantum mechanics of a single particle one can observe that the points where the operator $\hat{\imath}$ appears are essentially the same where the Planck constant $\hbar$ should be. Of course, using a different system of units, one realizes that the operator $\hat{\imath}$ takes the place of the combination $i/\hbar$ accordingly the analogy applied here. By this way, the introduction of operators which fail to commute with $\hat{\imath}$ can be understood as to promote the Planck “constant” to a new dynamic variable, being interesting to investigate the fluctuations in the quantum of action in such case. On the other hand, the superselection rule expressed by the second condition in the section 8 together with the hypothesis made in the final of the section 11 gave a classical meaning to $\imath$ excluding the interference between their different states. This is equivalent to “freeze” the actual value of the imaginary unity operator suppressing this new possibilities. Therefore, we find a natural extension of the equations (11.1) admitting that the operator $\hat{\imath}$ actually is a new fundamental field, i.e., a new dynamic variable which depends from the space-time point where it is observed. This idea was partially developed in [6] where it is proposed a quaternionic general covariance principle, which means a theory for the parallel transport of quaternions over a manifold, and a field equation for the operator $\iota$. One of the most surprising results of this theory is that the field equations obtained are very similar to the electromagnetic ones but with three fundamental vectorial bosons, one neutral and massless and two others massive and charged. So, Quaternionic Quantum Mechanics could be considered one of the first attempts to construct an unified theory for the electroweak interactions (1963) and perhaps could model at least a sector of the complete electroweak interactions.

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