TWISTED SYMPLECTIC REFLECTION ALGEBRAS

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Abstract. In this paper we introduce the notion of twisted symplectic reflection algebras and describe the category of representations of such an algebra associated to a non-faithful $G$-action in terms of those for faithful actions of $G$.

1. Introduction

The symplectic reflection algebras $\mathcal{H}_c(G)$ corresponding to a finite subgroup $G \subset Sp(U)$ were introduced by Etingof and Ginzburg in [EG]. The representations of these algebras and of their special cases called rational Cherednik algebras were studied by many authors (see e.g. [BEG], [BEG1], [DO], [Du]).

In this paper we
- introduce a twisted version of symplectic reflection algebras;
- consider symplectic reflection algebras corresponding to a representation $G \to Sp(U)$ which is not necessarily injective.

We show that the noninjective case can be reduced to the injective case, but in this reduction, even if the cocycle we started with was trivial it might become nontrivial under reduction.

The structure of the paper is as follows. In Section 2 we recall some basic facts about twisted algebras and projective representations of finite groups. In Section 3 we define twisted reflection algebras. In Section 4 we recall some facts about module categories. In Section 5 we describe the reduction of the noninjective case to injective ones.

2. Projective representations of finite groups.

In this section we recall some standard facts about projective representations of finite groups. The main reference for this section is [K].

Let $G$ be a finite group and $\psi : G \times G \to \mathbb{C}^*$ be a two-cocycle.

Definition 2.1. The twisted group algebra of $G$ corresponding to $\psi$ is an associative algebra $\mathbb{C}_\psi G$ with a basis $g \in G$ and multiplication given by

$$g_1 \circ_\psi g_2 = \psi(g_1, g_2)g_1g_2,$$

where $g_1g_2$ is the usual product in $G$.

Definition 2.2. An element $g \in G$ is called $\psi$-regular if $\psi(g, h) = \psi(h, g)$ for all $h$ in the centralizer of $g$ in $G$.

Remark 2.3. It is known (see [K], Lemma 3.6.1), that if $g \in G$ is $\psi$-regular, so is any conjugate of $g$. 
Remark 2.4. Let ψ and ψ′ be two cohomologous cocycles. Then any element \( g \in G \) is ψ-regular if and only if it is ψ′-regular.

Let \( τ \) be a representation of \( C_ψG \). One can define the character of \( τ \) in the usual way:

\[
χ_τ(g) = \text{tr} g|_τ.
\]

The characters of projective representations are not necessarily class functions, i.e. they might be not invariant under conjugation.

Definition 2.5. A cocycle \( ψ \) is called a class-function cocycle if \( χ_τ(g) \) is a class function for every representation \( τ \) of \( C_ψG \).

Proposition 2.6 (see [K], Proposition 7.2.2). Every \( C^* \)-valued two cocycle is cohomologous to a class-function cocycle.

Analogously to the case of the trivial cocycle, the values of a twisted character on the ψ-regular elements determine the isomorphism class of \( τ \).

Proposition 2.7 (see [K], Theorem 7.2.5 (ii)). An element \( g \) is ψ-regular if and only if \( χ_τ(g) \neq 0 \) for some irreducible representation \( τ \) of \( C_ψG \).

Theorem 2.8 (see [K], Theorem 3.6.3). Let \( ψ \) be a class-function cocycle, and let \( C_i \) be ψ-regular conjugacy classes of \( G \). Let

\[
k_i = \sum_{g \in C_i} g.
\]

Then \( \{k_1, \ldots, k_l\} \) is a basis of the center of \( C_ψG \).

Theorem 2.9 (see [K], Theorem 3.6.7). The number of irreducible representations of \( C_ψG \) is equal to the number of ψ-regular conjugacy classes of \( G \).

Example 2.10 (see [K], Theorem 3.7.3). Let \( G = I_2(2m) \) be the group of symmetries of \( 2m \)-gon. It is generated by two elements \( s_1 \) and \( s_2 \) with relations

\[
s_i^2 = 1, \quad (s_1s_2)^{2m} = 1.
\]

Denote by \( ε \) a primitive 2\( m \)th root of 1. Every non-trivial cocycle \( ψ : G \times G \to C^* \) is cohomologous to \( ψ : G \times G \to C^* \) defined by

\[
ψ((s_1s_2)^i, (s_1s_2)^j) = 1, \quad ψ((s_1s_2)^i s_1, (s_1s_2)^j s_1) = ε^i
\]

for all \( 0 \leq i, j \leq 2m - 1 \) and \( δ \in \{0, 1\} \).

There are \( m \) ψ-regular conjugacy classes in \( G \):

\[
\{s_1s_2, s_2s_1\}, \quad \{(s_1s_2)^2, (s_2s_1)^2\}, \ldots, \quad \{(s_1s_2)^m, (s_2s_1)^m\}.
\]

Example 2.11. Let \( G = S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n \) be the generalized symmetric group. It is generated by elements \( s_i \), \( 1 \leq i \leq n - 1 \) and \( w_j \), \( 1 \leq j \leq n \) with relations

\[
s_i^2 = w_j^m = 1; \quad (s_is_{i+1})^3 = 1 \text{ for } 1 \leq i \leq n - 2; \quad (s_is_j)^2 = 1 \text{ for } |i - j| \geq 2;
\]

\[
s_iw_i = w_{i+1}s_i; \quad s_iw_j = w_jw_i \text{ for } j \neq i, i + 1; \quad w_iw_j = w_jw_i.
\]

The Schur multiplier of \( G \) was computed in [DM] (see also [MJ]):

\[
H^2(G, C^*) = \begin{cases} 
\mathbb{Z}/2\mathbb{Z} = \{\gamma\}, & \text{if } m \text{ is odd, } n \geq 4; \\
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{\gamma, \lambda, \mu\}, & \text{if } m \text{ is even, } n \geq 4; \\
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{\lambda, \mu\}, & \text{if } m \text{ is even, } n = 3; \\
\mathbb{Z}/2\mathbb{Z} = \{\mu\}, & \text{if } m \text{ is even, } n = 2; \\
\{e\}, & \text{otherwise},
\end{cases}
\]
where \(\gamma, \lambda, \mu \in \{\pm 1, 1\}\).

Fix an element \((\gamma, \lambda, \mu) \in H^2(G, \mathbb{C}^*)\). There exist a cocycle \(\psi : G \times G \rightarrow \mathbb{C}^*\) whose class \([\psi]\) in \(H^2(G, \mathbb{C}^*)\) is \((\gamma, \lambda, \mu)\) such that \(C_\psi G\) is generated by elements \(t_i, 1 \leq i \leq n - 1\) and \(u_j, 1 \leq j \leq n\) with relations

\[
t_i^2 = u_j^2 = 1; \quad (t_it_{i+1})^3 = 1 \text{ for } 1 \leq i \leq n - 2; \quad (t_it_j)^2 = \gamma \cdot 1 \text{ for } |i - j| \geq 2;
\]

\[
t_iu_j = u_{i+1}t_i; \quad t_iu_j = \lambda u_jt_i \text{ for } j \neq i, i + 1; \quad u_ju_j = \mu u_ju_i \text{ for } i \neq j.
\]

If \([\psi] = (\pm 1, 1, 1)\), then the elements \(w_n^k\) are \(\psi\)-regular for all \(1 \leq k \leq m - 1\). Indeed, the centralizer of \(g = w_n^k \in G\) is

\[
C_G(w_n^k) = S_{n-1} \times (\mathbb{Z}/m\mathbb{Z})^{n-1} \times (\mathbb{Z}/m\mathbb{Z})
\]
generated by all the \(w_i\) and by \(s_i, 1 \leq i \leq i - 2\). From the relations of \(C_\psi G\) it follows that

\[
\psi(w_i, w_n) = \mu\psi(w_n, w_i) \text{ for } i \neq n \text{ and } \psi(w_n, s_j) = \lambda\psi(s_j, w_n) \text{ for } j \neq n - 1.
\]

Hence if \(\lambda = \mu = 1\), then \(\psi(w_n^k, h) = \psi(h, w_n^k)\) for all \(h\) is the centralizer of \(w_n^k\).

If \(m\) is not divisible by 4, then the elements \(s_i\) are \(\psi\)-regular only for \(\psi\) cohomologous to the trivial cocycle. If \(m\) is divisible by 4, then \(s_i\) are \(\psi\)-regular for \([\psi] = (1, 1, \pm 1)\).

Indeed, for odd \(m\) the centralizer of \(g = s_{n-1}\) is generated by \(s_j\) for \(|j - n + 1| \geq 2\) and by \(w_j\) for \(j \neq n - 1, n\). From the relations of \(C_\psi G\) it follows that

\[
\psi(s_{n-1}, s_j) = \gamma\psi(s_j, s_{n-1}).
\]

Hence \(g\) is regular only for the trivial cocycle.

For even \(m\) the centralizer of \(g = s_{n-1}\) is generated by \(s_j\) for \(|j - n + 1| \geq 2\), \(w_j\) for \(j \neq n - 1, n\), and \((w_{n-1}w_n)^{m/2}s_{n-1}\). The relations for \(C_\psi G\) give us

\[
\psi(s_{n-1}, s_j) = \gamma\psi(s_j, s_{n-1}), \quad \psi(s_{n-1}, w_j) = \lambda\psi(w_j, s_{n-1}),
\]

\[
\psi(s_{n-1}, (w_{n-1}w_n)^{m/2}s_{n-1}) = \mu\psi((w_{n-1}w_n)^{m/2}s_{n-1}, s_{n-1}).
\]

If \(m\) is not divisible by 4, \(s_{n-1}\) is regular only for trivial cocycle. Otherwise, it is regular for a cocycle \(\psi\) such that its class in \(H^2(G, \mathbb{C}^*)\) is \([\psi] = (1, 1, \pm 1)\).

Let \(m = 2\) in the example above, i.e. let \(G = BC_n\):

\[
\cdots s_1 s_2 s_3 \cdots s_{n-1} w_n
\]

The conjugacy class of the reflection \(w_n\) is \(\psi\)-regular for \([\psi] = (\pm 1, 1, 1)\), all the other reflections are regular only for the trivial cocycle.

**Proposition 2.12.** Let \(G \neq BC_n\) be a finite irreducible Coxeter group. Then all the reflections in \(G\) are regular only for the trivial cocycle.

**Proof.** Consider any two vertices \(i\) and \(j\) of the Coxeter graph connected by a simple edge. The corresponding reflections \(s_i\) and \(s_j\) generate a subgroup of \(G\) isomorphic to \(S_3\). Since \(H^2(S_3, \mathbb{C}^*) = \{e\}\), we can trivialize \(\psi\) on this subgroup.

Consider any multiple edge

\[
\begin{array}{c}
m(i, j) \\ s_i \\ s_j
\end{array}
\]

with \(m(i, j) > 3\) and \((s_is_j)^{m(i, j)} = 1\). In a finite irreducible Coxeter group there are at most 2 conjugacy classes of reflections: one corresponding to short roots and
one corresponding to long roots. So, if there is a \( \psi \)-regular reflection in \( G \), then by Remark 2.3 either \( s_i \) or \( s_j \) is also \( \psi \)-regular. The subgroup \( G(i,j) \) of \( G \) generated by \( s_i \) and \( s_j \) is isomorphic \( I_2(m(i,j)) \). From the Example 2.10 it follows that the restriction of \( \psi \) on \( G(i,j) \) can be trivialized.

One can trivialize \( \psi \) on all the edges of the Coxeter graph simultaneously, i.e. there exist a cocycle \( \psi' \) cohomologous to \( \psi \) such that in the twisted group algebra \( \mathbb{C}_\psi G \), which is generated by elements \( t_i \) corresponding to the vertices of the Coxeter graph of \( G \), the following relations are satisfied:

\[
t_i^2 = 1, \quad (t_i t_j)^{m(i,j)} = 1 \quad \text{for all } i, j \text{ such that } m(i,j) > 2.
\]

Let \( C \) denote the conjugacy class of \( \psi' \)-regular reflections in \( G \). Then \( \psi'(s_i, s_j) = \psi'(s_j, s_i) \) for all \( s_i \in C \) and \( s_j \) commuting with \( s_i \). This means that \( (t_i t_j)^2 = 1 \) for all \( (i,j) \in I \), where \( I = \{(i,j)|s_i \in C \text{ and } m(i,j) = 2\} \).

If \( G \neq BC_n \), then \( I \) is the set of all pairs of vertices which are not connected. Since \( \mathbb{C}_\psi G \) and \( \mathbb{C}G \) have the same generators and relations, they are isomorphic. Hence \( \psi' \) and \( \psi \) are cohomologous to the trivial cocycle. \( \square \)

3. Twisted symplectic reflection algebras.

Let \( G \) be a finite group acting on a symplectic vector space \( U \) with symplectic 2-form \( \omega \) and let \( \psi : G \times G \to \mathbb{C}^* \) be a class-function 2-cocycle.

**Definition 3.1.** A symplectic reflection of \( U \) is an element \( f \in Sp(U) \) such that \( \text{rank } (\text{Id} - f) = 2 \).

Let \( S = S(G,U) \subset G \) be the set of elements of \( G \) that act on \( U \) via symplectic reflections and let \( S(\psi)(G,U) \subset S(G,U) \) be its subset consisting of all \( \psi \)-regular elements. For every \( s \in S(\psi)(G,U) \) there is an \( \omega \)-orthogonal decomposition \( U = \text{Ker} (\text{Id} - s) \oplus \text{Im}(\text{Id} - s) \). Denote by \( \omega_s \) a skew-symmetric form on \( U \) which has Ker (\( \text{Id} - s \)) as the radical and coincides with \( \omega \) on Im(\( \text{Id} - s \)).

For any conjugation invariant function \( c : S(\psi)(G,U) \to \mathbb{C} \) one can define the twisted symplectic reflection algebra in the following way.

**Definition 3.2.** (cf. [EG], Theorem 1.3) The twisted symplectic reflection algebra \( \mathcal{H}_c(G,U,\psi) \) is the quotient of the semidirect product \( (TU) \rtimes \mathbb{C}_\psi G \) by the two-sided ideal \( I \) generated by elements

\[
y \otimes x - x \otimes y - \omega(x,y) \cdot 1 - \sum_{s \in S(\psi)(G,U)} c(s) \cdot \omega_s(x,y) \cdot s, \quad \forall x, y \in U.
\]

**Remark 3.3.** In the case of the trivial cocycle \( \psi \) and a faithful representation \( U \) symplectic reflection algebras were introduced by Etingof and Ginzburg in [EG].

**Remark 3.4.** If \( c(s) = 0 \) for all \( \psi \)-regular reflections \( s \in S(\psi)(G,U) \), then

\[
\mathcal{H}_c(G,U,\psi) = (SU) \rtimes \mathbb{C}_\psi G,
\]

where \( SU \) is the symmetric algebra of \( U \).

**Remark 3.5.** Analogously to the untwisted case one can define twisted rational Cherednik algebras.
Example 3.6. By Proposition 2.12 and Remark 3.4, the twisted rational Cherednik algebra \( \mathcal{H}_c(G, U, \psi) \) for \( G \neq BC_n \) is isomorphic to \((SU) \ltimes C \psi G\).

Let \( G = BC_n, \psi \) be a nontrivial two-cocycle, and \( G' \) be the subgroup of \( G \) generated by the unique \( \psi \)-regular conjugacy class. Since the cocycle \( \psi \) trivializes on \( G' \)
\[
\mathcal{H}_c(G, U, \psi) = C \psi G \otimes_{C G} H_c(G').
\]

The algebra \( \mathcal{H}_c(G, U, \psi) \) has a natural filtration obtained by placing \( C \psi G \) in grade degree 0 and \( U \) in grade degree 1. In the associated graded algebra \( \text{gr}(\mathcal{H}_c(G, U, \psi)) \) any two elements \( x, y \in U \) commute. Hence the tautological imbedding \( U \hookrightarrow \text{gr}(\mathcal{H}_c(G, U, \psi)) \) extends to a surjective graded algebra homomorphism:
\[
\phi : (SU) \ltimes C \psi G \to \text{gr}(\mathcal{H}_c(G, U, \psi)).
\]

Theorem 3.7. (Poincaré-Birkhoff-Witt theorem) The homomorphism \( \phi \) defined above is an isomorphism of graded algebras.

Proof. Proof is analogous to the untwisted case (see [EG], Theorem 1.3). \( \square \)

Definition 3.8. Two algebras are called Morita equivalent if the categories of modules over these algebras are equivalent.

In this paper we study the representations of \( \mathcal{H}_c(G, U, \psi) \) when the action of \( G \) on \( U \) is not faithful. Denote by \( K \) the kernel of the action. Then we have the following exact sequence
\[
1 \to K \hookrightarrow G \xrightarrow{\pi} W \to 1,
\]
where the action of \( W \) on \( U \) is faithful. We will show that the twisted symplectic reflection algebra \( \mathcal{H}_c(G, U, \psi) \) decomposes into a direct sum of subalgebras \( \mathcal{H}_i \) and each of \( \mathcal{H}_i \) is Morita equivalent to an algebra \( \mathcal{H}_c(H, U, \zeta) \) for some subgroup \( H \subset W \) and some 2-cocycle \( \zeta : H \times H \to \mathbb{C}^\times \). To formulate the statement more precisely we need to introduce the notion of module categories.

4. Module categories over \( \text{Rep} W \)

In what follows we will reproduce the main definitions and facts about module categories. For more details and proofs see [O].

4.1. Module categories and module functors. Let \( \mathcal{C} \) be a monoidal category with tensor product \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \), associativity isomorphism \( a_{X,Y,Z} \), and functorial isomorphisms \( r_X : X \otimes 1 \to X \) and \( l_X : 1 \otimes X \to X \) (see [O] or [BK] for more details).

Definition 4.1. (see [BK] Definitions 2.1.1 and 2.1.2) A monoidal category \( \mathcal{C} \) is called rigid if every object in \( \mathcal{C} \) has right and left duals.

Definition 4.2. (cf. [ENO]) A fusion category \( \mathcal{C} \) is a \( \mathbb{C} \)-linear semisimple rigid monoidal category with finitely many simple objects and finite dimensional spaces of morphisms, such that the endomorphism algebra of the neutral object is \( \mathbb{C} \).

In what follows we assume that \( \mathcal{C} \) is a fusion category over \( \mathbb{C} \).
**Definition 4.3.** (cf. [O], Definition 6) A module category over C is a semisimple category \( \mathcal{M} \) together with a bifunctor \( \otimes : C \times \mathcal{M} \to \mathcal{M} \) and functorial associativity \( m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M) \), and unit isomorphisms \( l_M : 1 \otimes M \to M \) for any \( X, Y \in C \) and \( M \in \mathcal{M} \) such that the following two diagrams commute:

\[
\begin{array}{ccc}
(X \otimes 1) \otimes M & \xrightarrow{m_{X,1,M}} & X \otimes (1 \otimes M) \\
\downarrow r_X \otimes id & & \downarrow id \otimes l_M \\
X \otimes M & & \\
\end{array}
\]

and

\[
\begin{array}{ccc}
((X \otimes Y) \otimes Z) \otimes M & \xrightarrow{a_{X,Y,Z} \otimes id} & (X \otimes (Y \otimes Z)) \otimes M \\
\downarrow m_{X,Y,Z,M} & & \downarrow m_{X,Y,Z,M} \\
X \otimes ((Y \otimes Z) \otimes M) & \xrightarrow{id \otimes m_{Y,Z,M}} & X \otimes (Y \otimes (Z \otimes M)) \\
\end{array}
\]

**Proposition 4.4** ([O], Lemma 2). Let \( \mathcal{M} \) be a module category over a fusion category \( C \). Then for any \( X \in C \) and \( M_1, M_2 \in \mathcal{M} \) we have canonical isomorphisms

\[
\text{Hom}(X \otimes M_1, M_2) \cong \text{Hom}(M_1, X \otimes M_2) \quad \text{and} \quad \text{Hom}(M_1, X \otimes M_2) \cong \text{Hom}(X^* \otimes M_1, M_2).
\]

**Definition 4.5.** Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be two module categories over \( C \). A module functor from \( \mathcal{M}_1 \) to \( \mathcal{M}_2 \) is a functor \( F : \mathcal{M}_1 \to \mathcal{M}_2 \) together with functorial isomorphisms \( c_{X,M} : F(X \otimes M) \to X \otimes F(M) \) for any \( X \in C \) and \( M \in \mathcal{M}_1 \) such that the following two diagrams commute:

\[
\begin{array}{ccc}
F((X \otimes Y) \otimes M) & \xrightarrow{c_{X,Y,M}} & (X \otimes Y) \otimes F(M) \\
\downarrow Fm_{X,Y,M} & & \downarrow m_{X,Y,F(M)} \\
F(X \otimes (Y \otimes M)) & \xrightarrow{id \otimes c_{Y,M}} & X \otimes (Y \otimes F(M)) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
F(1 \otimes M) & \xrightarrow{c_{1,M}} & 1 \otimes F(M) \\
\downarrow F1_M & & \downarrow l_{F(M)} \\
F(M) & \xrightarrow{id} & F(M) \\
\end{array}
\]

**Definition 4.6.** Two module categories \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) over \( C \) are equivalent if there exists a module functor from \( \mathcal{M}_1 \) to \( \mathcal{M}_2 \) which is an equivalence of categories.

**Definition 4.7.** Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be two module categories over a fusion category \( C \). Their direct sum is the category \( \mathcal{M}_1 \times \mathcal{M}_2 \) with additive and module structures defined coordinatewise.
Definition 4.8. A module category $\mathcal{M}$ is called indecomposable if it is not equivalent to a direct sum of two nontrivial module categories.

4.2. Module categories over $\text{Rep}(W)$. In this section we will give a description of all the indecomposable module categories over the category $\text{Rep}(W)$ of finite dimensional representations of a finite group $W$. These module categories are in one-to-one correspondence with conjugacy classes of pairs $(H,[\zeta])$, where $H \subset W$ is a subgroup and $[\zeta] \in H^2(H,\mathbb{C}^*)$.

Given a pair $(H,\zeta)$ where $\zeta$ is a two-cocycle on $H$, denote by $\text{Rep}(H,\zeta)$ the category of finite dimensional representations of $\mathbb{C}Z H$. The usual tensor multiplication by elements of $\text{Rep}(W)$ defines the structure of a module category on $\text{Rep}(H,\zeta)$. Note that if $\zeta$ and $\zeta'$ are cohomologous, then the corresponding module categories $\text{Rep}(H,\zeta)$ and $\text{Rep}(H,\zeta')$ are equivalent.

Theorem 4.9 (see [O], Theorem 2). Let $\mathcal{M}$ be an indecomposable module category over $\text{Rep}(W)$. Then there exist $H \subset W$ and a two-cocycle $\zeta : H \times H \to \mathbb{C}^*$ such that $\mathcal{M}$ is equivalent to $\text{Rep}(H,\zeta)$. Two module categories $\text{Rep}(H_1,\zeta_1)$ and $\text{Rep}(H_2,\zeta_2)$ are equivalent if and only if pairs $(H_1,[\zeta_1])$ and $(H_2,[\zeta_2])$ are conjugate under the adjoint action of $W$.

4.3. Examples. 1. Consider $W = \mathbb{Z}/2\mathbb{Z}$. There are two indecomposable module categories over $\text{Rep}(\mathbb{Z}/2\mathbb{Z})$: $\mathcal{M}_1$ corresponding to the subgroup $e \subset W$ with trivial cocycle and $\mathcal{M}_2$ corresponding to the group $W$ itself with trivial cocycle. Category $\mathcal{M}_1$ has one irreducible object and category $\mathcal{M}_2$ has two irreducible objects, which are permuted by multiplication by $\text{sgn} \in \text{Rep}(W)$.

2. More generally, let $W = \mathbb{Z}/n\mathbb{Z} = \langle a \rangle$. Then for every integer $d$ dividing $n$ there exists a unique indecomposable module category $\mathcal{M}_d$ over $\text{Rep}(W)$ with $d$ irreducible objects. This category corresponds to the subgroup $H \subset W$ of order $d$ with trivial cocycle.

3. Let $G = S_4$ be the symmetric group in 4 elements. The map $S_4 \to S_4 = W$ defines on $\text{Rep}(S_4)$ a structure of a module category over $\text{Rep}(S_3)$. The action of $V \in \text{Rep}(S_3)$ is by tensor product. The simple objects of $\text{Rep}(S_4)$ are $\text{triv}$, $\text{sgn}$, the two-dimensional representation $V^2$ and two three-dimensional representations $V^3_{(1)}$ and $V^3_{(2)}$. The representations $\text{triv}$, $\text{sgn}$ and $V^2$ generate a module category equivalent to $\text{Rep}(S_4)$, and $V^3_{(1)}$, $V^3_{(2)}$ generate a module category equivalent to $\text{Rep}(\mathbb{Z}/2\mathbb{Z})$ with the trivial cocycle.

5. Representations of $\mathcal{H}_c(G,U,\psi)$.

5.1. Notations. Let $G$ be a finite group and $\pi : G \to W \subset \text{Sp}(U)$ be its representation. For any two-cocycle $\psi : G \times G \to \mathbb{C}^*$ the category $\text{Rep}(G,\psi)$ of representations of $\mathbb{C}G$ is a module category over $\text{Rep}(W)$ with the action given by tensor product. Let $\text{Rep}(G,\psi) = \bigoplus \mathbb{C}_i$ be its decomposition into a direct sum of indecomposable module categories. By Theorem 4.9 each of $\mathbb{C}_i$ is equivalent to a category $\text{Rep}(H,\zeta)$ for some subgroup $H \subset W$ and some two-cocycle $\zeta : H \times H \to \mathbb{C}^*$.

Let $C_1,\ldots,C_n$ be conjugacy classes in $W$ such that the set $C_i \cap H$ is not empty and contains some $\zeta$-regular elements. Each set $(C_i \cap H)^{(\zeta)}$ consisting of $\zeta$-regular elements in $C_i \cap H$ is a union of several conjugacy classes in $H$:

$$(C_i \cap H)^{(\zeta)} = \bigcup_{j=1}^{m_i} C_j^{i_j}.$$
5.2. Main Lemma.

**Lemma 5.1.** Let $\mathcal{M}$ be a subcategory of $\text{Rep}(G, \psi)$ equivalent to $\text{Rep}(H, \zeta)$ as a module category over $\text{Rep}(W)$. Denote by $F : \mathcal{M} \to \text{Rep}(H, \zeta)$ the equivalence of these two module categories. Fix $g \in G$ and let $\pi(g)$ be its projection on $W$.

(i) If $\pi(g) \notin C_i$ for any $i$, then for every $M \in \mathcal{M}$

$$\chi_M(g) = 0.$$ 

(ii) If $\pi(g) \in C_i$, then for every simple object $M \in \mathcal{M}$

$$\chi_M(g) = \sum_{j=1}^{m_i} \alpha(C_i^j) \chi_{F(M)}(C_i^j)$$

where the coefficients $\alpha(C_i^j)$ do not depend on $M$.

**Proof of the lemma 5.1.**

(i) Let $K(\mathcal{M})$ be the Grothendieck group of the module category $\mathcal{M}$. Denote by $\mathcal{C}$ the algebra of class-functions on the set of $\zeta$-regular elements in $H$. Then we have a map $\text{Rep}(W) \to \mathcal{C}$, and by Proposition 2.7 the action of $\text{Rep}(W)$ in $K(\mathcal{M})$ factors through $\mathcal{C}$.

Let $f$ be the class function on $W$ which is equal to 1 at the conjugacy class of $\pi(g)$ and is zero elsewhere. Then $f$ maps to 0 in $\mathcal{C}$ since $\pi(g)$ is not conjugate to a $\zeta$-regular element of $H$. Let $A_f \in \text{Rep}(W)$ be the virtual representation corresponding to $f$. Then $A_f \otimes V = 0$. Taking characters at $g$ we get $f(\pi(g)) \chi_M(g) = 0$. Since $f(\pi(g)) = 1$, we get $\chi_M(g) = 0$.

(ii) Let $f_i$ be a class function on $W$ which is equal to 1 on $C_i$ and is zero elsewhere. Denote by $A_i \in \text{Rep}(W)$ the corresponding virtual representations. Let $B_1, \ldots, B_k$ be all the irreducible representations of $\mathbb{C}_\zeta H$. The action of $\text{Rep}(W)$ on $\text{Rep}(H, \zeta)$ is given by

$$A_i \otimes B_j = \sum_{r=1}^{k} \xi_{i,j}^r B_r.$$ 

Consider vectors $v^{(t)} \in \mathbb{C}^k$ defined by $v_j^{(t)} = \chi_{B_j}(h_t)$, where $h_t$ are representatives of $\zeta$-regular conjugacy classes in $H$. These vectors form a basis in $\mathbb{C}^k$, since the characters of irreducible representations form a basis in the space of class functions. Moreover they are eigenvectors of the matrices $\Xi_i = (\xi_{i,j}^r)$:

$$\Xi_i(v^{(t)}) = \begin{cases} v^{(t)}, & \text{if } h_t \in C_i; \\ 0, & \text{otherwise}. \end{cases}$$

Let $M_l$ for $1 \leq l \leq k$ be the irreducible representation of $\mathbb{C}_\psi G$ such that $F(M_l) = B_l$. Let $g \in G$ and $\pi(g) \in C_i$. Then the vector $u \in \mathbb{C}^k$ with coordinates $u_j = \chi_{M_l}(g)$ satisfies the equations $\Xi_i(u) = \delta_{ij}u$ and hence it is a linear combination of vectors $v^{(t)}$ such that $h_t$ is in $C_i$.

□
5.3. **Main Theorem.** The category $\text{Rep}(G, \psi)$ decomposes into a direct sum of indecomposable module categories in the following way:

$$\text{Rep}(G, \psi) = \bigoplus_i C_i.$$ 

In particular, the regular representation $C_0 G$ of $G$ decomposes into a direct sum $C_0 G = \bigoplus_i C_0 G^{(i)}$, where $C_0 G^{(i)} \in \text{Ob}(C_i)$. Hence $1 \in C_0 G$ can be written as $1 = \bigoplus_i e_i$, where $e_i \in C_0 G^{(i)}$ are idempotents.

Fix a conjugation invariant function $c : S_\psi(G, U) \to \mathbb{C}^*$. Let $\mathcal{H}_c(G, U, \psi)$ be the corresponding symplectic reflection algebra. The categories $C_i$ are module categories over $\text{Rep}(W)$, in particular they are closed under tensor multiplication by $U$. This means that the constructed above idempotents $e_i$ are central in $\mathcal{H}_c(G, U, \psi)$. Moreover they are minimal with such property since $C_i$ are indecomposable. Hence the algebra $\mathcal{H}_c(G, U, \psi)$ can be decomposed in a direct sum of subalgebras in the following way:

$$\mathcal{H}_c(G, U, \psi) = \bigoplus_i \mathcal{H}_i,$$

where $\mathcal{H}_i = e_i \mathcal{H}_c(G, U, \psi)$.

**Theorem 5.2.** If the module category $C_i$ is equivalent to a category $\text{Rep}(H, \zeta)$, then the algebra $e_i \mathcal{H}_c(G, U, \psi)$ is Morita equivalent to the algebra $\mathcal{H}_{c'}(H, U, \zeta)$ for a conjugation invariant function $c' : S_\zeta(H, U) \to \mathbb{C}^*$ defined by

$$c'(C^j) = \frac{\alpha(C^j)}{\alpha(e)|C^j|} \sum_{\pi(C)|C^j_i} c(|C^j|),$$

where $C^j_i$ is a conjugacy class of $\zeta$-regular reflections in $H$ and $|C^j|$ and $|C^j|$ stand for the number of elements in the conjugacy class. (See Lemma 5.1 for definition of $\alpha(C^j)$ and $\alpha(e)$.)

**Proof.** Let $\mathcal{A}$ and $\mathcal{B}$ be the categories of representations of $e_i \mathcal{H}_c(G, U, \psi)$ and of $\mathcal{H}_{c'}(H, U, \zeta)$ respectively. Denote by $\overline{F}$ the equivalence $\mathcal{L}_i \to \text{Rep}(H, \zeta)$.

Let us define a functor $F : \mathcal{A} \to \mathcal{B}$ in the following way. Every $e_i \mathcal{H}_c(G, U, \psi)$-module $M$ is in particular a $C_0 G$-module which lies in $C_i$. The module $\overline{F}(M)$ has a natural action of $U$ coming from the corresponding action on $M$ via the functorial isomorphisms

$$\overline{F}(U \otimes M) \to U \otimes \overline{F}(M).$$

**Claim 5.3.** The action of $U$ defines a structure of $\mathcal{H}_{c'}(H, U, \zeta)$-module on $\overline{F}(M)$.

**Proof.** We have to check that if the element

$$y \otimes x - x \otimes y - \omega(x, y) \cdot 1 - \sum_{s \in S_\zeta(G, U)} c(s) \cdot \omega_s(x, y) \cdot s$$

acts by 0 on $M$ for all $x, y \in U$, then

$$y \otimes x - x \otimes y - \omega(x, y) \cdot 1 - \sum_{s \in S_\zeta(H, U)} c'(s) \cdot \omega_s(x, y) \cdot s$$

acts by 0 on $\overline{F}(M)$ for all $x, y \in U$. 


Let $\rho(x, y) = \omega(x, y) \cdot 1 + \sum_{s \in S_{\psi}(G, U)} c(s) \cdot \omega_s(x, y) \cdot s$. This element can be viewed as a homomorphism $\rho$ from $U \otimes U \otimes M$ to $M$. By Proposition 4.4, since the category $\text{Rep} W$ is a fusion category

$$\text{Hom}(U \otimes U \otimes M, M) \cong \text{Hom}(U \otimes M, U^* \otimes M).$$

Denote by $\rho$ the map from $U \otimes M$ to $U^* \otimes M$ corresponding to $\rho$. Let $\omega: U^* \to U$ be the map given by the symplectic form $\omega$. Then

$$(\psi \otimes \text{Id}) \circ \rho = \text{Id} \otimes \text{Id} - \sum_{s \in S_{\psi}(G, U)} c(s)(\text{Id} \otimes s \otimes s) =$$

$$= \text{Id} \otimes \text{Id} - \text{Id} \otimes \left( \sum_{s \in S_{\psi}(G, U)} c(s)s \right) + \sum_{s \in S_{\psi}(G, U)} c(s)s \otimes s$$
as a map from $U \otimes M$ to $U \otimes M$.

To prove the claim, it is enough to check that the element $\sum_{s \in S_{\psi}(G, U)} c(s)s$ acts in any irreducible representation $\tau$ of $G$ by the same constant as the element $\sum_{s \in S_{\psi}(G, U)} c'(s)s$ acts in $F(\tau)$.

Let $\tau$ be an irreducible representation of $G$ and $\tilde{C}$ be a conjugacy class of $\psi$-regular reflections in $G$, which projects onto a conjugacy class $C$ in $W$. Then the element $\sum_{s \in \tilde{C}} s$ acts in $\tau$ by

$$\lambda = \frac{\vert \tilde{C} \vert \chi_{\tau}(\tilde{C})}{\dim \tau},$$

where $\vert \tilde{C} \vert$ is the number of elements in $\tilde{C}$.

By lemma 5.1 $\lambda$ is nonzero only if $C = C_i$ for some $i$, i.e. if $C \cap H$ contains a $\zeta$-regular element. In this case

$$\lambda = \sum_{j=1}^{m_i} \frac{\alpha(C_i^j) \chi_{F(\tau)}(C_i^j)}{\alpha(e) \dim F(\tau)}.$$
Let 
\[ c'(C_{ij}) = \frac{\alpha(C_{ij})}{\alpha(e) \mid C_{ij} \mid} \sum_{\pi(C) = C_{ij}} c(\tilde{C}) \mid \tilde{C} \mid. \]

Since \( \sum_{s \in C_{ij}} s \) acts by \( \frac{|C_{ij}|}{\dim F(\tau)} \chi_{F}(\tau)(C_{ij}) \) in \( F(\tau) \), the element \( \sum_{s \in S_{C}(H,U)} c'(s) s \) acts in \( F(\tau) \) by \( \Lambda \).

\[ \square \]

**Claim 5.4.** The functor \( F \) is an equivalence of categories.

**Proof.**

1. The functor \( F \) is surjective on isomorphism classes of objects in \( B \). Indeed, any object \( N \in B \) is a \( \text{Rep}(H, \zeta) \)-module, and hence is equal to \( F(M) \) for some \( \text{Rep}(G, \psi) \)-module \( M \). Since \( F \) is an equivalence of module categories over \( \text{Rep} W \), it respects the \( U \)-action, so \( F(M) = N \) as \( B \)-module.

2. The map \( F : \text{Hom}_{A}(M_{1}, M_{2}) \to \text{Hom}_{B}(F(M_{1}), F(M_{2})) \) is an isomorphism for any \( M_{1}, M_{2} \in A \), since \( F \) is an equivalence of module categories.

\[ \square \]

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