COMPACTNESS OF THE SPACE OF FREE BOUNDARY CMC SURFACES WITH BOUNDED TOPOLOGY

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December 3, 2020

Abstract. We prove that the space of free boundary CMC surfaces of bounded topology, bounded area and bounded boundary length is compact in the $C^k$ graphical sense away from a finite set of points. This is a CMC version of a result for minimal surfaces by Fraser-Li [7].

1. Introduction

A Constant Mean Curvature (CMC) surface $\Sigma$ is a critical point of the area functional with respect to variations that preserve enclosed volume. As a consequence of the first variation of area, the scalar mean curvature has to be constant for a given choice of normal direction. If $\Sigma$ is an immersed surface with boundary immersed on $N$ with boundary and $\partial \Sigma \subset \partial N$, we say the surface is free boundary CMC if it is a critical point with respect to variations that are in addition tangent along $\partial N$.

Although most results about minimal surfaces have an equivalent version for CMC surfaces, there are many distinctions between their behaviour. For example, when the mean curvature is non-zero, the mean curvature vector defines a trivialization of the normal bundle. That is, every CMC surface on a 3-manifold is 2-sided. On the other hand, CMC surfaces may have tangential self-touching points as long as the mean curvature vector points at opposite directions on those points.

The goal of this article is to prove the CMC equivalent to Fraser-Li’s result [7] for free boundary minimal surfaces. We prove:

Theorem 5.1. Let $N$ be a compact 3-dimensional manifold with boundary. Suppose $H_{\partial N} \geq H_0$ and let $\Sigma_i$ be a sequence of free boundary embedded CMC surfaces with mean curvature $H_i$, genus $g_i$ and number of ends $r_i$ satisfying:

(a) $|H_i| \leq H_0$;
(b) $g_i \leq g_0$;
(c) $r_i \leq r_0$;
(d) area$(\Sigma_i) \leq A_0$ and
(e) length$(\partial \Sigma_i) \leq L_0$.

Then there exists a smooth properly almost embedded CMC surface $\Sigma \subset N$ and a finite set $\Gamma \subset \Sigma$ such that, up to a subsequence, $\Sigma_i$ converges to $\Sigma$ locally graphically in the $C^k$ topology on compact sets of $N \setminus \Gamma$ for all $k \geq 2$. Moreover, if $\Sigma$ is minimal then it is properly embedded.

If in addition $(N, \partial N)$ satisfies either $\text{Ric}_N > 0$ and $A_{\partial N} \geq 0$ or $\text{Ric}_N \geq 0$ and $A_{\partial N} > 0$, then:
(i) when $H_{\Sigma} = 0$ the convergence is at most 2-sheeted; 
(ii) when $H_{\Sigma} \neq 0$, then the convergence is 1-sheeted away from $\Gamma$.

Let us highlight the main differences that justify the extra hypotheses and the weaker compactness for embedded surfaces.

Firstly, we mention that there is no Steklov eigenvalue estimates for free boundary CMC surfaces. Consequently, the hypothesis of length bound remain crucial. Secondly, there is no suitable isoperimetric inequality that allows us to remove the bound on the area. One could exchange the length bound condition by stability of the surface (see [11]) but as seen on [2], in particular, stable free boundary CMC surfaces have bounded topology so this condition would be topologically restrictive. Thirdly, even under positive Ricci curvature of the ambient space a sequence of free boundary CMC surfaces may have a neck-pinching phenomenon where the norm of the second fundamental form blows-up at a point in the limit. Naturally the convergence is not smooth along these points where curvature is accumulating. Finally, the maximum principle for CMC surfaces only apply when their mean curvature vectors point in the same direction. That is, a sequence of embedded free boundary CMC surfaces may touch tangentially in the limit as long as the limiting surface is not minimal.

Despite the lack of certain useful technical results, the proof of the theorem rely on the same main ideas, each of which is proved using a less optimal technique to compensate for the above limitations. These are: $L^2$-curvature bounds from the topological bound, improvement to local pointwise curvature estimates, a removable singularity theorem for interior and boundary points and construction of positive eigenfunctions for the Jacobi operator to study convergence under curvature condition on the ambient space.

Let us briefly address our approach to each of the above tools. The integral curvature bounds follow directly from Gauss-Bonnet Theorem, from which the fact the mean curvature is non-zero introduces an extra term of area and the geodesic curvature along the boundary brings in a term of boundary length. The local pointwise curvature estimate comes from a blow-up argument as in [15] and Schauder estimates. It becomes relevant that the blow-up of a CMC surface around a point is a minimal surface in $\mathbb{R}^3$. Unlike [7, Theorem 4.1] we do not know whether a CMC surface is conformally equivalent to a punctured Riemann surface so we do not have the branch point structure to prove the removable singularity theorem. This will follow from a blow-up argument to prove that tangent cones are in fact planes (or half-planes) and a local free boundary CMC foliation argument to prove that it is also unique. These are the same ideas as in [15] together with the methods in [5] to deal with boundary singularity points. In fact, most of the calculations are done in the latter reference and only minor adaptations are necessary for the non-minimal case. Finally, the construction of positive eigenfunctions for the Jacobi operator on the limiting surface is a well known method (see [6,12,13]) and again, most of the necessary calculations are done in [5].

The extra condition on the mean curvature of $\partial N$ is only necessary to apply the maximum principle for CMC surfaces and ensure that the interior of the limiting surface is properly immersed in $N$. Removing this condition would allow for interior tangential touching points between $\partial N$ and the limit surface.
Finally, we mention that the corresponding result for closed CMC surfaces was proved by Sun in [14]. Although the core ideas are the same, the approach to certain steps is different and we focus on the behaviour at the boundary.

This article is divided as follows. Section 2 establishes notation, necessary definitions and we prove the geometric version of Schauder estimates. In section 3 we prove the local pointwise curvature estimates from uniform integral curvature bounds. The Removable Singularity Theorem is proved in section 4 and we write the proof for the specific case of a singularity along the boundary. Section 5 is dedicated to prove the Compactness Theorem.

Acknowledgements: The majority of the work on this article took place when the first author was a postdoctoral fellow at PIMS-University of British Columbia. The authors would like to thank Professors Jingyi Chen and Ailana Fraser for useful discussions and support.

2. Preliminaries

In this section we establish notation and preliminary results that will be used throughout this article.

Let \((N^3, \partial N, g)\) be a 3-dimensional manifold with non-empty boundary and a smooth Riemannian metric \(g\).

**Definition 2.1.** We say that an immersed surface \(\Sigma \subset N\) with non-empty boundary is properly almost embedded in an open set \(U \subset N\) if \((\Sigma \setminus \partial \Sigma) \cap U\) is properly immersed in \((N \setminus \partial N) \cap U\), \(\partial \Sigma \subset \partial N \cap U\) and there exists a set \(S \subset \Sigma\) such that

(a) \(\Sigma \setminus S\) is embedded;
(b) for each \(p \in S\) there exists a neighbourhood \(V\) of \(p\) in \(U\) such that \(\Sigma \cap V\) is a union of connected components \(W_j, j = 1, \ldots, l_p\), each \(W_j\) is embedded, for each \(j \neq j'\) we have \(W_j\) lying to one side of \(W_{j'}\) and \(W_j \cap W_{j'} \subset S \cap V\).

The set \(S\) is called the self-touching set of \(\Sigma\). We say \(\Sigma\) is free boundary if in addition \(\Sigma\) is orthogonal to \(\partial N\).

The following lemma is a standard application of Schauder estimates in geometry. It will allow us to improve from \(C^{1, \alpha}\) to \(C^{2, \alpha}\) graphical convergence of surfaces, given pointwise curvature estimates.

**Lemma 2.2.** Let \(N\) be a three dimensional compact manifold, \(A > 0\) and \(f \in C^{0, \alpha}(N)\) with \(|f|_{0, \alpha} < H_0\). There exists \(r_0(N, A) > 0\) and \(C_0(N, A, H_0) > 0\) such that for any \(C^2\) immersed surface with free boundary \(\Sigma \subset N\) with \(H_\Sigma = f\) and \(x_0 \in \Sigma\) satisfying

\[
|A_\Sigma(x)| \leq A \text{ on } \{x \in \Sigma : d_\Sigma(x, x_0) < r_0\}
\]

there exists a function \(u\) defined on a subset of \(T_{x_0} \Sigma\) with the following properties:

(i) \(\{x \in \Sigma : d_\Sigma(x, x_0) < r_0\}\) is the graph of \(u\) over the exponential function;
(ii) the domain of \(u\) contains a ball centered at 0 and radius \(\frac{A}{2}\) and
(iii) \(|u|_{B_{r_0}(0)}|_{2, \alpha} < C_0\).

**Proof.** We may assume, that \(r_0 > 0\) is smaller than the convexity radius of \(N\) so that \(\{x \in \Sigma : d_\Sigma(x, x_0) < r_0\}\) is contained in a geodesic ball in \(N\). Hence, without loss of generality we may assume that \(\{x \in \Sigma : d_\Sigma(x, x_0) < r_0\}\) is contained in a small ball in \(\mathbb{R}^3\) with a metric \(g\), \(x_0 = 0\) and \(T_{x_0} \Sigma = \mathbb{R}^2 \times \{0\}\). We may further...
assume that $g$ is equivalent to the Euclidean metric $g_0$, that is, $c^{-1}g_0 \leq g \leq cg_0$ where the constant $c > 0$ depends only on the geometry of $N$.

**Claim 1.** If $r_0$ is sufficiently small, depending only on the geometry of $N$ and $A$, then for all $x \in \{x \in \Sigma : d_\Sigma(x, 0) < r_0\}$ we have $|\nu(x) - \nu(0)|_{g_0} \leq CAd_\Sigma(x, 0)$ for some $C > 0$ depending only on the geometry of $N$.

Indeed, let $\gamma \subset \Sigma$ be a curve parametrised by arc-length joining 0 to $x$. If we denote by $\nabla^0$ the Euclidean connection, then it follows that

$$|\nu(x) - \nu(0)|_{g_0} \leq \sup_t |\nabla^0_\gamma \nu|_{g_0} l(\gamma)$$

Since we are working in geodesic coordinates then $\nabla^0_\gamma = \nabla_\gamma + O(|\gamma|)$, hence

$$|\nabla^0_\gamma \nu|_{g_0} \leq c|\nabla^0 \nu|_g \leq c|\nabla \nu|_g + |\gamma|C \leq cA + cr_0C,$$

for some $C > 0$ depending only on the geometry of $N$. If we pick $r_0C < cA$ we obtain

$$|\nu(x) - \nu(0)|_{g_0} \leq 2cA l(\gamma).$$

Taking the limit as the length of $\gamma$ tends to $d_\Sigma(x, 0)$ we conclude the claim as desired.

Now, suppose that $CAd_\Sigma \leq 1$, then $\{x \in \Sigma : d_\Sigma(x, 0) \leq r_0\}$ is the graph of a function $u$, otherwise we would have $\nu(x)$ perpendicular to $\nu(0)$ for some $x$ hence $|
u(x) - \nu(0)| > 1$. Furthermore, it follows that $\sup |\nabla u| \leq 2CAr_0$ as long as $C^2A^2r_0^2 < \frac{3}{4}$.

Let us denote by $\Omega \subset T_0\Sigma$ the domain of $u$ and take $\delta > 0$ the largest radius such that $B_\delta(0) \subset \Omega$. In particular there exists $\bar{y} \in \partial B_\delta(0)$ such that $d_\Sigma((\bar{y}, u(\bar{y})), (0, 0)) = r_0$.

**Claim 2.** For any pair $y_1, y_2 \in B_\delta(0)$ we have $d_\Sigma((y_1, u(y_1)), (y_2, u(y_2))) \leq (1 + (2CAr_0)^2)^\frac{1}{2} |y_1 - y_2|$ and, if $4CAr_0 < 1$ then $\frac{\delta}{2} \leq \frac{1}{2} |\bar{y}|$.

We have $y_1 + t(y_2 - y_1) \in B_\delta(0) \subset \Omega$ for $t \in [0, 1]$, from which follows that $d_\Sigma((y_1, u(y_1)), (y_2, u(y_2))) \leq \int_0^1 (1 + |\nabla u(y_1 + t(y_2 - y_1))|^2)^\frac{1}{2} |y_1 - y_2| dt$ and it proves the inequality.

For the second part observe that if $4CAr_0 < 1$ then $d_\Sigma((\bar{y}, u(\bar{y})), (0, 0)) \leq \frac{\delta}{2} |\bar{y}|$. Suppose that $\delta < \frac{\bar{y}}{2}$, then $|\bar{y}| < \frac{\delta}{2}$ and $d_\Sigma((\bar{y}, u(\bar{y})), 0) \leq \frac{\delta}{2}$ which is a contradiction and it proves the claim.

**Claim 3.** If $4CAr_0 < 1$ then $\nabla u$ restricted to $B_{\frac{\delta}{2}}(0)$ is a Lipschitz function of Lipschitz constant less than $\frac{\delta}{2}A$.

The second fundamental form is a multiple of the Hessian of $u$, thus $|A\Sigma| \leq A$ implies $|\text{Hess} u| \leq (1 + |\nabla u|^2)^\frac{1}{2} A \leq 2A$ whenever $4CAr_0 < 1$. It follows that for any $y_1, y_2 \in \Omega$, $|\nabla u(y_1) - \nabla u(y_2)| \leq 2Ad_\Sigma((y_1, u(y_1)), (y_2, u(y_2)))$. From the previous claim we get that $d_\Sigma((y_1, u(y_1)), (y_2, u(y_2))) \leq (1 + (2CAr_0)^2)^\frac{1}{2} |y_1 - y_2| \leq \frac{3}{2} |y_1 - y_2|$, thus proving the claim.

Finally, on $B_{\frac{\delta}{2}}(0)$ the function $u$ satisfies the equation $\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = f$. The coefficients of the equation have $C^{2,\alpha}$ norm depending on the metric $g$ and the Lipschitz constant of $\nabla u$. In particular all coefficients have $C^{2,\alpha}$ norm depending only on $N$ and $A$. 
Let us consider the case without boundary components on \( \{ x \in \Sigma : d_{\Sigma}(x, x_0) < r_0 \} \). Then Schauder estimates [8 Corollary 6.3] implies that there exists a constant 
\( C = C(N, A) \) such that 
\[ \| u \|_{B_{2\alpha}^s(0)} \leq C(\| u \|_{B_{2\alpha}^s(0)} + \| f \|_{0, \alpha}). \]
Note that \( u(0) = 0 \) and \( u(0) \leq 2Ar_0 \). By the same inequality with an extra term depending on the \( C^{1, \alpha} \) norm of the inward normal derivative along the boundary. Since \( \Sigma \) is free boundary, \( u \) satisfies homogenous Neumann boundary conditions in which case the extra term vanishes and we obtain the same result. \( \square \)

In the following we make precise the notion of graphical convergence of surfaces (see also [12]).

**Definition 2.3.** Let \( N \) be a 3-manifold, with or without boundary, \( \Sigma_i \) a sequence of smooth surfaces in \( N \) and \( \Sigma \) a smooth surface in \( N \). Pick \( p \in \Sigma \) and \( r > 0 \) sufficiently small so that we may identify \( B^N_r(p) \) with an Euclidean ball (or half-ball if \( p \in \partial N \)) with the same metric as \( N \). We say that \( \Sigma_i \) converges locally graphically in the \( C^k \) \((C^{k, \alpha})\) topology to \( \Sigma \) at \( p \) if for \( r > 0 \) sufficiently small we have:

(a) \( \Sigma \cap B^N_r(p) \) is the graph of a \( C^k \) \((C^{k, \alpha})\) function \( u \) defined on \( B_r(0) \subset T_p\Sigma \);
(b) \( \Sigma_i \cap B^N_r(p) \) is the graph of \( C^k \) \((C^{k, \alpha})\) functions \( u^1_i, \ldots, u^L_i \) defined on \( B_r(0) \subset T_p\Sigma \) for \( i \) sufficiently large and
(c) \( u^1_i \) converges to \( u \) in the \( C^k \) \((C^{k, \alpha})\) topology as \( i \to \infty \).

If \( L \) is constant for sufficiently large \( i \), we say that the convergence is \( L \)-sheeted.

**Remark 2.4.** Let \( U \subset N \) be an open set. If \( \Sigma_i \) is a sequence of embedded surfaces converging locally graphically with \( L \)-sheets to a two-sided surface \( \Sigma \) on a compacts sets of \( U \) then for each compact set \( \Omega \subset \Sigma \cap U \), \( r > 0 \) sufficiently small and \( i \) sufficiently large we may write \( \Sigma_i \cap B^N_r(\Omega) \) as the graph of functions \( u_{i,j} : \Omega \to \mathbb{R}, j = 1, \ldots, L \), under the exponential map in the normal direction of \( \Sigma \). In other words, each connected component \( \Sigma_{i,j} \) of \( \Sigma_i \cap B^N_r(\Omega) \) can be written as \( \{ \exp_x(u_{i,j}(x)N_\Sigma(x)) : x \in \Omega \} \) for each \( j = 1, \ldots, L \). In addition, \( u_{i,j} \) must tend to \( 0 \) for each \( j = 1, \ldots, L \).

3. Curvature estimate

In this section we prove an improvement from uniformly small total curvature to uniform local pointwise curvature estimate. The proof is inspired by [15 Theorem 1] and we focus on the local estimates around a boundary point. We point out as well that the same proof holds even if the surface is not CMC but has uniformly bounded \( C^{0, \alpha} \) mean curvature.

**Theorem 3.1.** Let \( N \) be a compact 3-manifold with boundary. There exists a small enough \( r_0 > 0 \) such that the following holds: whenever \( \Sigma \) is a properly immersed CMC surface in \( N \), \( Q \in \Sigma \), \( \partial \Sigma \cap B^N_{r_0}(Q) \) is either empty or free boundary in \( \partial N \cap B^N_{r_0}(Q) \) and the mean curvature of \( \Sigma \) satisfies \( H_\Sigma \leq H_0 \). Then there exists
where the constant \( C_0 \) only depends on geometry of \( B_{r_0}^N(Q) \) and \( H_0 \).

Proof. Suppose false, that is, for \( r_n \to 0 \) and \( \varepsilon_n \to 0 \) there exist free boundary CMC surfaces \( \Sigma_n \subset N \) and \( Q_n \in \Sigma_n \) satisfying:

(i) \( H_n \leq H_0 \);
(ii) \( \int_{\Sigma_n \cap B_{r_n}^N(Q_n)} |A_n|^2 \leq \varepsilon_n \) and
(iii) \( \max_{0 \leq \sigma \leq r_n} \left( \sigma^2 \sup_{\Sigma \cap B_{r_n}^N(Q_n)} |A_n|^2 \right) > n. \)

Pick \( 0 < \sigma_n < r_n \) such that

\[
\sigma_n^2 \sup_{\Sigma \cap B_{r_n}^N(Q_n)} |A_n|^2 = \max_{0 \leq \sigma \leq r_n} \left( \sigma^2 \sup_{\Sigma \cap B_{r_n}^N(Q_n)} |A_n|^2 \right)
\]

and write \( \lambda_n^2 = \sup_{\Sigma \cap B_{r_n}^N(Q_n)} |A_n|^2 \). For each \( n \) there exists \( z_n \in \Sigma \cap B_{r_n}(Q_n) \) such that \( |A_n(z_n)| > \frac{1}{\lambda_n^2} \).

By taking a subsequence we have \( Q_n \to Q \) and \( B_{r_n}^N(Q_n) \) contained in a geodesic ball of \( N \). Without loss of generality we may assume that \( \Sigma_n \cap B_{r_n}(Q_n) \subset \mathbb{R}^3 \) with a metric \( g \). Henceforth we denote by \( B_r(p) \) the ball in \( \mathbb{R}^3 \) with respect to the metric \( g \).

Now, define \( \hat{\Sigma}_n = \lambda_n(\Sigma_n - z_n) \), it satisfies:

(a) \( |\hat{A}_n(\hat{x}_n)| \leq 2 \) for all \( \hat{x}_n \in \hat{\Sigma}_n \cap B_1(0) \) and \( n \) sufficiently large;
(b) \( |\hat{A}_n(0)| > \frac{1}{2} \) and
(c) \( \int_{\hat{\Sigma}_n} |\hat{A}_n|^2 \leq \varepsilon_n \)

Indeed, if \( \hat{x}_n \in B_1(0) \) then \( \hat{x}_n = \lambda_n(x_n - z_n) \) with \( x_n \in \Sigma_n \cap B_{1/n}(z_n) \). It follows that \( x_n \in \Sigma \cap B_{r_n}^N(Q_n) \).

Since

\[
\left( \sigma_n - \frac{1}{\lambda_n} \right)^2 \sup_{\Sigma \cap B_{r_n}^N(Q_n)} |A_n|^2 \leq \max_{0 \leq \sigma \leq r_n} \left( \sigma^2 \sup_{\Sigma \cap B_{r_n}^N(Q_n)} |A_n|^2 \right) = \sigma_n^2 \lambda_n^2,
\]

it implies that

\[
|A_n(x_n)|^2 < \left( \frac{1}{1 - \frac{1}{\sigma_n \lambda_n}} \right)^2 \lambda_n^2.
\]

We know that \( \sigma_n^2 \lambda_n^2 > n \) thus \( |A_n(x_n)| < 2 \lambda_n \) for \( n \) sufficiently large, which proves (a). Property (b) follows from rescaling and (c) holds because the total curvature is scale invariant.

Let \( \tilde{\Sigma}_n \) denote the connected component of \( \hat{\Sigma}_n \cap B_1(0) \) containing \( 0 \). We may further assume, after an ambient rotation and translation that \( T_0 \tilde{\Sigma}_n = \{ x_3 = 0 \} \).

Using property (a), it follows from Lemma 2.2 that there exists \( \tilde{r}_0 \) independent of \( n \) such that \( \Sigma_n \cap B_{\tilde{r}_0}(0) \) is the graph of a function \( u_n \) satisfying \( \| u_n \|_{B_{\tilde{r}_0}(0)} \|_{2, \alpha} < \tilde{C}_0 \)

where \( \tilde{C}_0 \) is independent of \( n \). If \( 0 \) is a boundary point then the domain of \( u_n \) is a half ball but Lemma 2.2 remains true.
Finally, $u_\kappa$ converges, up to a subsequence, in the $C^2$ topology to a function $u_\infty$. If we denote its graph by $\Sigma_\infty$ then it satisfies $|\hat{A}_\infty(0)| \geq \frac{1}{2}$ from (b) and $\int_{\Sigma_\infty} |\hat{A}_\infty|^2 = 0$ from (c), which is a contradiction and completes the proof of the Theorem. □

4. Removable singularities

As we will see later, the compactness result does not give us smooth convergence everywhere. The points in which we do not have sufficient curvature estimates are potential singularities either because of a neckpinching phenomenon where the curvature may blow up, or self-touching points where the convergence is not single sheeted. However, we are still able to prove that these are removable singularities so the limiting surface is still a smooth object.

We are going to prove that if the total curvature is bounded on a CMC surface, then isolated singularities are removable. This is an adaptation of [15, Theorem 2] and the arguments are the same except for the foliation argument to prove uniqueness of the tangent cone. We are going to focus on the case in which the singularity is along the boundary, but the same result holds for interior singularities and the proof follows the exact same arguments.

The idea of the proof is to improve the integral curvature bound to a pointwise curvature decay near the singularity to show that the tangent cones are totally geodesic. By adapting the foliation argument of White [15] we prove that the tangent cone is unique from which we can show that near the singularity the surface is indeed the graph of a $C^1$ function. We can then improve it further using elliptic regularity.

Firstly, let us prove the existence of a local CMC foliation with free boundary which will be needed later. This is a straightforward adaptation of [5, Section 3] together with White’s approach to deal with a family of functionals [15, Appendix].

Let $\theta \in (0, \frac{\pi}{4})$ and define $D_\theta = \{ x \in \mathbb{R}^2 : (x_1 + a)^2 + x_2^2 \leq 1 \text{ and } x_1 \geq 0 \}$, where $a = \cos^{-1}(\theta) \in (\frac{1}{\sqrt{2}}, 1)$. This is the part of the disk of radius 1 centered on the $x_1$-axis that intersects the line $x_1 = 0$ at angle $\theta$. Its boundary components are denoted by $\partial_0 D_\theta = \partial D_\theta \cap \{ x \in \mathbb{R}^2 : x_1 = 0 \}$ and $\partial_+ D_\theta = \partial D_\theta \setminus \partial_0 D_\theta$. The regular cylinder over $D_\theta$ in $\mathbb{R}^3$ is denoted by $C_\theta = D_\theta \times \mathbb{R}$, with corresponding boundary components $\partial_0 C_\theta = \partial_0 D_\theta \times \mathbb{R}$ and $\partial_+ C_\theta = \partial_+ D_\theta \times \mathbb{R}$.

Given a function $f \in C^{2,\alpha}(D_\theta)$, we define $N_g^+(f)$ to be the normal vector over graph$(f)$ with respect to $g$ pointing in the positive direction of the $x_3$-axis, that is,
sequence $\Sigma$.

Let us denote by $X$ the space of $C^{2, \alpha}$ metrics on $C_0$ and define the map

$$\Phi : \mathbb{R} \times \mathbb{R} \times X \times C^{2, \alpha}(\partial_+ D_0) \times C^{2, \alpha}(D_0) \to C^{0, \alpha}(D_0) \times C^{1, \alpha}(\partial_0 D_0) \times C^{2, \alpha}(\partial_+ D_0)$$

by

$$\Phi(h, t, g, u, w) = \left( H_g^+(t + u) - h, \frac{\partial}{\partial \eta_g}(t + u), u_{|_{\partial_+ D_0}} - w \right),$$

where $\eta_g$ is the inward conormal vector along $\partial_0 D_0$.

**Proposition 4.1 ([5] Proposition 21).** For every $t_0 \in \mathbb{R}$, there exist a neighbourhood $\delta_0$ of the Euclidean metric $\delta$ in $X$, $\varepsilon_0 > 0$ and $u : (-\varepsilon_0, \varepsilon_0) \times (t_0 - \varepsilon_0, t_0 + \varepsilon_0) \times U_{t_0} \times B^{C^{2, \alpha}(\partial_+ D_0)}(0) \to C^{2, \alpha}(D_0)$ so that $t \mapsto \text{graph}(t + u(h, t, g, w))$ defines a $C^{2, \alpha}$ foliation of $D_0 \times [t_0 - \varepsilon_0, t_0 + \varepsilon_0]$ by surfaces with constant mean curvature $h$ with respect to the metric $g$, free boundary along $\partial_0 C_0$ and $(t + u)|_{\partial_0 D_0} = t + w$.

Furthermore, if $h > 0$ then $\widehat{H}_g(t + u)$ points in the positive direction of the $x_3$-axis and in the negative direction when $h < 0$.

**Proof.** Observe that $\Phi$ defined above is a $C^1$ function and $D_5 \Phi(0, t_0, \delta, 0, 0)$ defines the same isomorphism as in [5] Appendix B. The result then follows from the Implicit Function Theorem.

Let us denote by $B_1^+ = \{x \in \mathbb{R}^3 : \|x\| \leq 1, x_1 \geq 0\}$ the upper half ball in $\mathbb{R}^3$ and $\partial_0 B_1^+ = \partial B_1^+ \cap \{x \in \mathbb{R}^3 : x_1 = 0\}$.

**Theorem 4.2.** Let $g$ be a Riemannian metric on $B_1^+$ and $\Sigma$ be a smooth, properly embedded, CMC surface in $B_1^+ \setminus \{0\}$, $\partial \Sigma \subset \partial B_1^+$, free boundary on $\partial_0 B_1^+ \setminus \{0\}$ and $0 \in \partial \Sigma$. Suppose $\int_{\Sigma} |A_S|^2 \leq C$ then $\Sigma \cup \{0\}$ is a smooth properly embedded CMC surface in $B_1^+$.

**Proof.** Let $r_0 > 0$ and $\varepsilon_0 > 0$ be as in Theorem 3.4. Pick $\delta > 0$ sufficiently small so that $\int_{\Sigma \cap B_1^+} |A_S|^2 \leq \varepsilon_0$. It follows from Theorem 3.4 that $|A_S(x)| d_g(x, 0) \leq C_0,$ whenever $d_g(x, 0) < \delta$.

**Claim 1.** Every tangent cone of $\Sigma$ at 0 is a union of half-planes in $\mathbb{R}^3 \setminus \{0\}$.

Let $r_i \to \infty$ be any sequence and $\Sigma_i = r_i \Sigma$ its corresponding blow-up around 0.

Observe that the curvature estimate above is scale invariant, so $\Sigma_i$ satisfies the same curvature bounds whenever $d_g(x, 0) < r_i \delta$. It follows that, up to a subsequence, $\Sigma_i$ converges locally graphically in the $C^{1, \alpha}$ topology on compact sets to a complete surface $\Sigma_\infty$ in $T_0 N$, which we identify with $\mathbb{R}^3$ with the Euclidean metric. Lemma 2.2 implies that $\Sigma_i$ in fact converges locally graphically in the $C^{2, \alpha}$ topology on compact sets of $\mathbb{R}^3 \setminus \{0\}$. In particular, $\Sigma_\infty$ has free boundary on $\{x \in \mathbb{R}^3 : x_1 = 0\} \setminus \{0\}$ and for any compact set $K \subset \mathbb{R}^3 \setminus \{0\}$ we have

$$\int_{K \cap \Sigma_\infty} |A_S|^2 = \lim_{i \to \infty} \int_{K \cap \Sigma_i} |A_i|^2 = \lim_{i \to \infty} \int_{(r_i^{-1} K) \cap \Sigma} |A_S|^2 = 0.$$
That is, \( A_\infty = 0 \). Hence \( \Sigma_\infty \) is an union of half-planes perpendicular to \( \{ x \in \mathbb{R}^3 : x_1 = 0 \} \).

**Claim 2.** If \( \delta > 0 \) is sufficiently small then \( \Sigma \cap B^+_{\frac{\delta}{2}} \setminus \{0\} \) is topologically a finite and disjoint union of disks, half disks or half-disks punctured at 0 with free boundary on \( \partial_0 B^+_{\frac{\delta}{2}} \setminus \{0\} \).

Firstly, we improve the curvature estimates. Fix \( y \in \Sigma_i \cap B^+_{r_i \delta} \) and put \( x = r_i^{-1} y \in \Sigma \cap B^+_{\delta} \). Then \( |A_\Sigma(x)|d_g(x,0) = |A_i(y)|d_g(y,0) \to 0 \) as \( i \to \infty \) from the previous claim. Thus \( \lim_{x \to 0} |A_\Sigma(x)|d_g(x,0) = 0 \).

Secondly, we use a standard Morse Theory argument. Let \( f : \Sigma \cap B^+_{\hat{\delta}} \to \mathbb{R} \) be defined by \( f(x) = \frac{1}{2}d_g(x,0)^2 \). We can write its Hessian at a point \( x \) and direction \( v \) as \( \text{Hess} f(v,v)_x = A_\Sigma(v,v)_x g(N_\Sigma(x), \hat{\gamma}(1))_x + Q(x,v,v) \), where \( N_\Sigma \) is the normal vector field on \( \Sigma \), \( \gamma \) is the minimizing geodesic in \( B^+_{\hat{\delta}} \) from 0 to \( x \) and \( Q \) is a quadratic form satisfying \( Q \geq \frac{1}{2}g \) as long as \( d_g(x,0) < \delta \) is sufficiently small. Hence, any critical point of \( f \) in a small neighbourhood of 0 is a strict local minimum, even if the critical point is along the boundary. It follows from Morse Theory for manifolds with boundary \([10]\) that every connected component of \( \Sigma \cap B^+_{\hat{\delta}} \) is a disk with a single critical point, a free boundary half disk with a single critical point or a free boundary half disk punctured at 0 without critical points. Since critical points cannot accumulate, it must be finite for \( \delta \) sufficiently small. This proves the claim.

Next, we shall prove that for each punctured half-disk its tangent cone is unique. Pick \( \hat{\Sigma} \subset \Sigma \cap B^+_{\delta} \setminus \{0\} \) a connected component that corresponds to a punctured half-disk. Let \( \hat{\Sigma}_i \) be the corresponding dilation by \( r_i \to \infty \) and \( \hat{\Sigma}_\infty \subset \Sigma_\infty \) its tangent cone at 0. As we have seen above, \( \Sigma_\infty \) is a half-plane perpendicular to \( \{ x \in \mathbb{R}^3 : x_1 = 0 \} \).

**Claim 3.** The tangent cone \( \hat{\Sigma}_\infty \) is unique, that is, it is independent of the blow-up sequence \( r_i \to \infty \).

Without loss of generality let us identify \( \hat{\Sigma}_\infty \) with the half-plane \( P_+ = \{ x \in \mathbb{R}^3 : x_3 = 0 \} \cap \{ x \in \mathbb{R}^3 : x_1 \geq 0 \} \). We may also assume, possibly taking another subsequence, that the mean curvature vector of \( \hat{\Sigma}_i \) points in the positive direction of the \( x_3 \)-axis for \( i \) sufficiently large.

Let \( D_\theta \) and \( C_\theta \) be as in Proposition \([4.4]\) Since \( \hat{\Sigma}_i \) is embedded, then for \( i \) sufficiently large \( \hat{\Sigma}_i \cap \partial_+ C_\theta \) is the graph of a function \( w_i = w_{r_i} \) over \( \partial_+ D_\theta \). We know that \( \hat{\Sigma}_i \) converges in \( C^{2,\alpha} \) graphical sense to \( P_+ \) away from zero, from which follows that \( \|w_i\|_{2,\alpha} \to 0 \).

The mean curvature of \( \hat{\Sigma}_i \) is given by \( \hat{H}_i = r_{i}^{-1} H_\Sigma \) which is constant at each \( i \) and tends to 0 as \( i \) tends to infinity. If we denote by \( g_i \) the corresponding blow up of the metric \( g \) in a neighbourhood of 0 in \( B^+_{\hat{\delta}} \), it follows from Proposition \([4.1]\) that for all \( i \) sufficiently large there exists \( \varepsilon_0 > 0 \) and a unique function \( u_{i,t} = u(\hat{H}_i, t, g_i, w_i) : D_\theta \to \mathbb{R} \) for each \( |t| < \varepsilon_0 \) such that \( u_{i,t} \to t+w_i \) on \( \partial_+ D_\theta \), the graph of \( u_{i,t} \) over \( D_\theta \) meets \( \partial_\theta C_\theta \) orthogonally, it has constant mean curvature equal to \( \hat{H}_i \) and its mean curvature vector points in the same direction as the mean curvature vector of \( \hat{\Sigma}_i \). Furthermore, \( u_{i,t} \) varies smoothly on \( t \) and it defines a foliation of a region \( D_\theta \times [-c, c] \) for some \( c > 0 \) independent of \( i \).

Let \( t_i \in (\varepsilon_0, \varepsilon_0) \) be such that \( u_{i,t_i}(0) = 0 \). Since their mean curvature vector points in the same direction, we may apply the maximum principle to the graph
of $u_{i,t}$ and $\hat{\Sigma}_i$ to conclude that $\hat{\Sigma}_i$ must be contained entirely to one side of the graph of $u_{i,t}$. The side itself will depend on whether $t_i$ is positive or negative. In case the mean curvature vector of $\hat{\Sigma}_i$ were to point in the negative direction of the $x_3$-axis, we would have used the same argument with $u(\hat{-H}_i, t, g_i, w_i)$ instead.

Finally, pick another sequence $r'_i \to \infty$ and let $\hat{\Sigma}'_i, P'_+$ be its corresponding blow-up and tangent cone respectively. For each $k$ we pick $i_k$ so that $r'_i > kr_k$. Observe that $\hat{\Sigma}_{i_k} = \frac{r'_i}{r_k} \hat{\Sigma}_k$ which is contained to one side of the graph of $\frac{r'_i}{r_k} u_{k,t_k}$. Since each $u_{k,t_k}$ is regular at 0, $\frac{r'_i}{r_k} u_{k,t_k}$ must converge to a unique tangent cone, that is, $P_+$. If $P'_+$ were different from $P_+$, then it would imply that $\frac{r'_i}{r_k} \hat{\Sigma}_k$ contains points on both sides of the graph of $\frac{r'_i}{r_k} u_{k,t_k}$. Notice that $P'_+$ must also be a half plane with free boundary on $\{ x \in \mathbb{R}^3 : x_1 = 0 \}$. This concludes the claim.

It follows from uniqueness of the tangent cone that $\hat{\Sigma} \cup \{ 0 \}$ is the graph of a $C^1$ function $u$ around a neighbourhood of 0 over the tangent cone. Since $u$ is also a solution to the CMC equation with smooth coefficients, then by elliptic regularity it must also be smooth.

To conclude the proof, now that we know that $\hat{\Sigma} \cup \{ 0 \}$ is regular, it follows from the maximum principle that it is the unique connected component containing 0. Thus $\hat{\Sigma} \cup \{ 0 \}$ is properly embedded. \[ \square \]

We now state the equivalent result for removable interior singularities without repeating the proof.

**Theorem 4.3.** Let $g$ be a Riemannian metric on $B_1$ and $\Sigma$ be a smooth, properly embedded, CMC surface in $B_1 \setminus \{ 0 \}$ with $H_\Sigma \leq H_0$, and $0 \in \Sigma$. Suppose $\int_\Sigma |A_\Sigma|^2 \leq C$ then $\Sigma \cup \{ 0 \}$ is a smooth properly embedded CMC surface in $B_1$.

**Remark 4.4.** The proof is exactly the same with a minor modification on the construction of the foliation. That is, by dropping the Neumann component on the definition of $\Phi$ in Proposition 4.1.

5. **Compactness Theorem**

In this section we will prove our main theorem for free boundary embedded CMC surfaces. As we shall see, the limiting surface may not be embedded because CMC surfaces may have tangential self-intersection as long as the normal vector points at opposite directions.

**Theorem 5.1.** Let $N$ be a compact 3-dimensional manifold with boundary. Suppose $H_{\partial N} \geq H_0$ and let $\Sigma_i$ be a sequence of free boundary embedded CMC surfaces with mean curvature $H_i$, genus $g_i$ and number of ends $r_i$ satisfying:

(a) $|H_i| \leq H_0$;
(b) $g_i \leq g_0$;
(c) $r_i \leq r_0$;
(d) area($\Sigma_i$) $\leq A_0$ and
(e) length($\partial \Sigma_i$) $\leq L_0$.

Then there exists a smooth properly almost embedded CMC surface $\Sigma \subset N$ and a finite set $\Gamma \subset \Sigma$ such that, up to a subsequence, $\Sigma_i$ converges to $\Sigma$ locally graphically in the $C^k$ topology on compact sets of $N \setminus \Gamma$ for all $k \geq 2$. Moreover, if $\Sigma$ is minimal then it is properly embedded.
If in addition \((N, \partial N)\) satisfies either \(\text{Ric}_N > 0\) and \(A_{\partial N} \geq 0\) or \(\text{Ric}_N \geq 0\) and \(A_{\partial N} > 0\), then:

(i) when \(H_N = 0\) the convergence is at most 2-sheeted;

(ii) when \(H_N \neq 0\), then the convergence is 1-sheeted away from \(\Gamma\).

Proof. Let us denote by \(A_i\) the second fundamental form of \(\Sigma_i\). Given \(x \in \Sigma\), it follows from Gauss equation that \(|A_i|^2(x) = H_i^2 + 2K_N(T_x\Sigma) - 2K_i(x)\), where \(K_N, K_i\) are the sectional curvatures of \(N\) and \(\Sigma\), respectively. From Gauss-Bonnet theorem we have

\[
\int_{\Sigma_i} |A_i|^2 = H_i^2 \text{area}(\Sigma_i) + 2 \int_{\Sigma_i} K_N(T_xN) + 2 \int_{\partial \Sigma_i} \kappa_g + 4\pi(2g_i + r_i - 2),
\]

where \(\kappa_g\) denotes the geodesic curvature of \(\partial \Sigma_i\). Because \(\Sigma_i\) is free boundary, we have that \(\kappa_g = A_{\partial N}(\tau_{\partial \Sigma_i}, \tau_{\partial \Sigma})\), where \(A_{\partial N}\) is the second fundamental form of \(\partial N\) with respect to the inward unit normal vector and \(\tau_{\partial \Sigma}\) is the unit tangent vector of \(\partial \Sigma\). Since \(N\) is compact, there exists a constant \(C = C(N) > 0\) such that

\[
\int_{\Sigma_i} |A_i|^2 \leq C(H_i^2 \text{area}(\Sigma_i) + g_i + r_i + \text{area}(\Sigma_i) + \text{length}(\partial \Sigma_i)).
\]

Hence, from hypotheses (a)-(e) we have that the total curvature is uniformly bounded by a constant \(C_0 = C_0(N, H_0, g_0, r_0, A_0, L_0) > 0\).

Denote by \(\mu_i\) the Radon measure on \(N\) defined by \(\mu_i(U) = \int_{\Sigma_i \cap U} |A_i|^2\), for a subset \(U \subset N\). It follows from the above that there exists a Radon measure \(\mu\) in \(N\) such that, up to a subsequence, \(\mu_i\) converges weakly to \(\mu\). Furthermore, the set \(\Gamma = \{p \in N : \mu(\{p\}) \geq 1\}\) has at most \(C_0\) elements.

For each \(x \in N \setminus \Gamma\) there exists \(r > 0\) such that \(\mu(B_r^N(x)) < 1\). Hence, for each \(i\) sufficiently large \(\mu_i(B_r^N(x)) < 1\), that is, \(\int_{\Sigma_i \cap B_r^N(x)} |A_i|^2 < 1\). By possibly choosing a smaller value of \(r\), it follows from Theorem [3.1] that

\[
\sup_{\Sigma_i \cap B_r^N} |A_i|^2 \leq C,
\]

for some constant \(C > 0\) independent of \(i\).

Let \(r < r_0\) as in Lemma [2.2] and suppose that, up to a subsequence, \(\Sigma_i \cap B_r^N(x)\) is non-empty for all \(i\) sufficiently large. Since \(\Sigma_i\) is embedded then \(\Sigma_i \cap B_r^N(x)\) is the union of disjoint embedded connected components \(\Sigma_{i,1}, \ldots, \Sigma_{i,L}\) each of which is the graph of a function defined on an open ball of fixed radius on \(T_{y_i}\Sigma_{i,j}\) for some \(y_i, j \in \Sigma_{i,j}\), \(j = 1, \ldots, L\). Because \(\Sigma_i\) is compact, the number of sheets \(L\) must be finite and thus constant for \(i\) sufficiently large. Hence, under the appropriate identifications we may further assume that \(\Sigma_{i,j} \cap B_{r'}^N(x)\) is the graph of a function \(u_{i,j}\) defined on \(B_{r'}(0) \subset T_{y_i}\Sigma_{i,j}\), for some \(0 < r' < \frac{r}{\delta}\) depending only on \(L\) and a fixed \(y_i \in \Sigma_i \cap B_{r'}^N(x)\). Furthermore \(u_{i,j}\) has uniform \(C^{2,\alpha}\) bounds as in Lemma [2.2].

We may assume that, up to a subsequence, \(y_i\) converges to \(y'\) and \(T_{y_i}\Sigma_i\) converges to a plane \(P \subset T_{y'}N\). In which case, under further identifications, we have that \(\Sigma_{i,j} \cap B_{r'}^N(y')\) is the graph of a function \(u'_{i,j}\) defined on an open ball on \(P\) (or half-ball in case \(y'\) is on the boundary of \(N\)) and uniform \(C^{2,\alpha}\) estimates for all \(i\) sufficiently large, and each \(j = 1, \ldots, L\). Hence, up to a subsequence \(u'_{i,j}\) converges to a function \(u_j\) in the \(C^{2,\beta}\) topology for all \(\beta < \alpha\) and \(\Sigma_i \cap B_{r'}^N(y')\) converges to
Consider the variation $\Sigma'$ for $u_i$ least 2 connected components $\Sigma' \subset N \setminus \Gamma$ locally graphically in the $C^{2,\beta}$ topology on $K$. Taking a countable exhaustion by compact sets and using a diagonal argument we have that, up to a subsequence, $\Sigma_i$ converges to $\Sigma'$ locally graphically in the $C^{2,\beta}$ topology on compact sets of $N \setminus \Gamma$. Smooth convergence away from $\Gamma$ follows from Allard’s regularity Theorem [3, 4].

Since $\Sigma_i$ is a properly almost embedded CMC surface with free boundary along $\partial N \setminus \Gamma$, then $\Sigma'$ is a properly almost embedded CMC surface in $N \setminus \Gamma$ with free boundary along $\partial N \setminus \Gamma$. Observe that on a neighbourhood of any self-touching point of $\Sigma'$ the surface can be written as connected components that lie to one side of one another. A transversal self-intersection is an open condition so it would contradict the fact that $\Sigma_i$ is embedded.

Define $\Sigma$ to be the closure of $\Sigma'$, so $\Gamma = \Sigma \setminus \Sigma'$. Since $\Gamma$ is finite, there exists $r > 0$ so that $B_r^N(p) \setminus \{p\}$ contains no points of $\Gamma$. From graphical convergence on compact sets of $B_r^N(p) \setminus \{p\}$, each sheet must converge to an embedded component of $\Sigma \cap B_r^N(p) \setminus \{p\}$. Since there are only finitely many sheets, we may pick $r > 0$ sufficiently small so that every component of $\Sigma \cap B_r^N(p) \setminus \{p\}$ contains $p$ in its closure.

Claim 1. The limit surface $\Sigma$ is regular.

We know that $\int_{\Sigma \cap (B_r^N(p) \setminus \{p\})} |A_{\Sigma}|^2 \leq C_0$. Thus we may apply the Removable Singularity Theorems [1, 2] or [3, 4, 9] to each embedded component of $\Sigma \cap (B_r^N(p) \setminus \{p\})$ depending on whether $p$ belongs to the boundary of $N$ or to the interior. Hence $\Sigma = \Sigma' \cup \Gamma$ is a regular surface.

Now, suppose that $H_\Sigma = 0$ then the Maximum Principle for minimal surfaces implies that there are no self-touching points so $\Sigma$ is embedded.

We now prove properties (i) and (ii). Henceforth, let us assume that either $\text{Ric}_N > 0$ and $\text{A}_{\partial N} > 0$ or $\text{Ric}_N > 0$ and $\text{A}_{\partial N} > 0$.

The following argument is the same as in [12] and [5] and we include the main idea without repeating the calculations.

Claim 2. If $H_\Sigma \neq 0$ then the convergence is 1-sheeted away from $\Gamma$.

Suppose by contradiction that $\Sigma_i$ converges to $\Sigma$ with at least 2 sheets. Since the convergence is graphical over compact sets of $N \setminus \Gamma$, for any compact set $\Omega \subset \Sigma \setminus \Gamma$ and a sufficiently small small tubular neighbourhood $V_r(\Omega)$ of $\Omega$, $\Sigma_i \cap V_r(\Omega)$ contains at least 2 connected components $\Sigma_i^0$ and $\Sigma_i^1$ each of which can be written as follows, for $i$ sufficiently large:

$$\Sigma_i^\nu = \{\exp_p(u_i^\nu(p)N_\Sigma(p)) : p \in \Omega\},$$

for $u_i^\nu : \Omega \to \mathbb{R}$, $\nu = 0, 1$.

We may assume without loss of generality that $u_i^1 > u_i^0$ for all $i$ sufficiently large.

Let us denote $v_i(t) = u_i^0 + t(u_i^1 - u_i^0)$ and $\Phi_i(p, t) = \exp_p(v_i(t)(p)N_\Sigma(p))$ for $p \in \Omega$. Consider the variation $\Sigma_i(t) = \{\Phi_i(p, t) : p \in \Omega\}$ so that $\Sigma_i(t) = \Sigma_i^\nu$, $\nu = 0, 1$. Take a function $w \in C_c^{\infty}(\Omega)$ with compact support and define $w^\nu \in C_c^{\infty}(\Sigma_i^\nu)$ by $w^\nu(\Phi(p, \nu)) = u(p)$ for each $\nu = 0, 1$. Now, consider any compactly supported
vector field $X$ in $V_r(\Omega)$ such that $g(X, N_{\Sigma^\nu}) = w^\nu$ along $\Sigma^\nu_i$, for each $\nu = 0, 1$, and its associated flow $\Psi(q, s)$ on $V_r(\Omega)$. Finally we consider the following 2-parameter variation:

$$\Sigma_i(t, s) = (\Psi_s)_t \Sigma_i(t).$$

From the first variation formula it follows that

$$\frac{\partial}{\partial s}|_{s=0} area(\Sigma_i(\nu, s)) = -H_i \int_{\Sigma_i^\nu} w^\nu,$$

for each $\nu = 0, 1$. Since the mean curvature vectors of each sheet converge to $\bar{H}_\Sigma$, we have that their normal vectors point in the same direction for $i$ sufficiently large. That is, their scalar mean curvature has the same sign, so we have

$$\frac{\partial}{\partial s}|_{s=0} area(\Sigma_i(1, s)) = -H_i \int_{\Sigma} (J\Phi_i^1 - J\Phi_i^0)w,$$

where for each $\nu = 0, 1$, $\Phi_i^\nu(p) = \Phi_i(p, \nu)$ and $J\Phi_i^\nu$ its corresponding Jacobian.

Following the same idea and calculations as in [5], using the Mean Value Theorem and taking the second variation of area for some $\nu$,

$$-\int_\Omega w \left(L_i\tilde{h}_i + H_i F_i(\tilde{h}_i)\right) + \int_{\partial N \cap \Omega} B_i(w, \tilde{h}_i) = 0.$$

Where $L_i$ is an elliptic operator in divergence form, $L_i$ converges uniformly to the Jacobian operator $L_\Sigma$, of $\Sigma$ as $i \to \infty$, $F_i(\tilde{h}_i) = (J\Phi_i^1 - J\Phi_i^0)$ and $B_i(w_1, w_2)$ converges uniformly to $w_1(\frac{\partial w_2}{\partial \eta} + A_{\partial N}(N_\Sigma, N_\Sigma)w_2)$, where $\eta$ denotes the outward co-normal vector on $\partial \Sigma$ and $A_{\partial N}$ the second fundamental form with respect to the outward co-normal of $\partial N$. Since $w^\nu_i$ tends to 0 at least $C^2$, then $\Phi_i^\nu$ tends to the identity uniformly, that is, $F_i(\tilde{h}_i)$ tends to 0.

Fix a point $q_0 \in \Omega$ and define $h_i(p) = \frac{\tilde{h}_i(p)}{\tilde{h}_i(q_0)}$. Using Harnack estimates and a blow-up contradiction argument we have that $\tilde{h}_i$ converges smoothly to $h$ on $\Omega$ satisfying

$$\begin{cases} L_\Sigma h = 0, & \text{on } \Omega; \\ \frac{\partial h}{\partial \eta} + A_{\partial N}(N_\Sigma, N_\Sigma)h = 0, & \text{on } \partial \Sigma \cap \Omega, \end{cases}$$

Since the blow up of the elliptic equation for $h_i$ is the one above, the proof is the same as in [5] Claim 1, p.20].

It follows from the Maximum Principle for elliptic equations that $h > 0$ on $\Omega$. By taking an exhaustion of compact sets over $\Sigma \setminus \Gamma$ and using a diagonal sequence argument, we have that $\tilde{h}_i$ converges to $h > 0$ on compact sets of $\Sigma \setminus \Gamma$. Again, it follows from the exact same argument as in [5] that $h$ is uniformly bounded on $\Sigma \setminus \Gamma$ thus it extends to a smooth function over $\Sigma$, which is positive by the maximum principle.

Finally, we observe that the conditions $\text{Ric}_N > 0$ and $A_{\partial N} \geq 0$ or $\text{Ric}_N \geq 0$ and $A_{\partial N} > 0$ imply that the quadratic form

$$Q_\Sigma(w_1, w_2) = -\int_\Sigma w_1 L_\Sigma w_2 + \int_{\partial \Sigma} w_1 \left(\frac{\partial w_2}{\partial \eta} + A_{\partial N}(N_\Sigma, N_\Sigma)w_2\right)$$

is positive definite. Hence its first eigenvalue is positive. However, $h$ is a positive eigenfunction so it must correspond to the first eigenvalue, which is a contradiction.
since the corresponding eigenvalue is 0. We conclude that the convergence must be
1-sheeted away from Γ.

The case \( H_\Sigma = 0 \) follows the exact same arguments as above. Unlike the previous case, the mean curvature vector of each graph converges to 0 so they may have opposite orientation. As a consequence, the scalar mean curvature may have opposite sign, hence the function \( F_i \) would not tend to 0 as desired.

Claim 3. If \( H_\Sigma = 0 \) then the convergence is at most 2-sheeted.

Suppose the convergence is at least 3-sheeted. Then at least 2 of which, say \( u_1^j, u_0^j \) have the same orientation, that is, the normal vector along their graphs points in the same direction. Following the same argument as in the previous case, the function \( F_i \) will tend to 0 so we can construct a positive eigenfunction \( h > 0 \) for the Jacobi operator \( L_\Sigma \) on \( \Sigma \), corresponding to the eigenvalue 0. This is a contradiction because \( L_\Sigma \) is a positive operator under either of the conditions
\[ \text{Ric}_N > 0 \text{ and } A_{\partial N} \geq 0 \text{ or } \text{Ric}_N \geq 0 \text{ and } A_{\partial N} > 0. \]

This completes the proof.

\[ \square \]

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