Fuzzy Nambu-Goldstone Physics

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Abstract

In spacetime dimensions larger than 2, whenever a global symmetry $G$ is spontaneously broken to a subgroup $H$, and $G$ and $H$ are Lie groups, there are Nambu-Goldstone modes described by fields with values in $G/H$. In two-dimensional spacetimes as well, models where fields take values in $G/H$ are of considerable interest even though in that case there is no spontaneous breaking of continuous symmetries. We consider such models when the world sheet is a two-sphere and describe their fuzzy analogues for $G = SU(N+1)$, $H = S(U(N-1) \otimes U(1)) \simeq U(N)$ and $G/H = \mathbb{C}P^N$. More generally our methods give fuzzy versions of continuum models on $S^2$ when the target spaces are Grassmannians and flag manifolds described by $(N+1) \times (N+1)$ projectors of rank $\leq (N+1)/2$. These fuzzy models are finite-dimensional matrix models which nevertheless retain all the essential continuum topological features like solitonic sectors. They seem well-suited for numerical work.
1 Introduction

In spacetime dimensions larger than 2, whenever a global symmetry $G$ is spontaneously broken to a subgroup $H$, and $G$ and $H$ are Lie groups, there are massless Nambu-Goldstone modes with values in the coset space $G/H$. Being massless, they dominate low energy physics as is the case with pions in strong interactions and phonons in crystals. Their theoretical description contains new concepts because $G/H$ is not a vector space.

Such $G/H$ models have been studied extensively in 2-d physics, even though in that case there is no spontaneous breaking of continuous symmetries. A reason is that they are often tractable nonperturbatively in the two-dimensional context, and so can be used to test ideas suspected to be true in higher dimensions. A certain amount of numerical work has also been done on such 2-d models to control conjectures and develop ideas, their discrete versions having been formulated for this purpose.

Our work in this paper is on new discrete approximations to $G/H$ models. We focus on two-dimensional Euclidean quantum field theories with target space $G/H = SU(N + 1)/U(N) = \mathbb{C}P^N$. The novelty in our approach is that our discretizations are based on fuzzy physics [1] and noncommutative geometry [2]. Fuzzy physics has striking elegance because it preserves the symmetries of the continuum and because techniques of noncommutative geometry give us powerful tools to describe continuum topological features. But its numerical efficiency has not been tested [3]. We got into this investigation with this mind, our idea being to write fuzzy $G/H$ models in a form adapted to numerical work.

This is not the first paper on fuzzy $G/H$. In [4], a particular description based on projectors and their orbits was discretized. We shall refine that work considerably in this paper. Also in the continuum there is another way to approach $G/H$, namely as gauge theories with gauge invariance under $H$ and global symmetry under $G$ [5]. This approach is extended here to fuzzy physics. Such a fuzzy gauge theory involves the decomposition of projectors in terms of partial isometries [6] and brings new ideas into this field. It is also very pretty. It is equivalent to the projector method as we shall also see.

Parallel work on fuzzy $G/H$ model and their solitons is being completed by Govindarajan and Harikumar [7]. A different treatment, based on the Holstein-Primakoff realization of the $SU(2)$ algebra, has been given in [8]. A more general approach to these models on non-commutative spaces was proposed in [9].

The first two sections describe the standard $\mathbb{C}P^1$-models on $S^2$. In section 2 we discuss it using projectors, while in section 3 we reformulate the discussion in such a
manner that transition to fuzzy spaces is simple. Sections 4 and 5 adapt the previous sections to fuzzy spaces.

Long ago, general $G/H$-models on $S^2$ were written as gauge theories \[5\]. Unfortunately their fuzzification for generic $G$ and $H$ eludes us. Generalization of the considerations here to the case where $S^2 \simeq \mathbb{C}P^1$ is replaced with $\mathbb{C}P^N$, or more generally Grassmannians and flag manifolds associated with $(N+1) \times (N+1)$ projectors of rank $\leq (N+1)/2$, is easy as we briefly show in the concluding section 6. But extension to higher ranks remains a problem.

2 $\mathbb{C}P^1$ models and Projectors

Let the unit vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ describe a point of $S^2$. The field $n(x)$ in the $\mathbb{C}P^1$-model is a map from $S^2$ to $S^2$:

$$n = (n_1, n_2, n_3) : x \rightarrow n(x) \in \mathbb{R}^3, \quad n(x) \cdot n(x) := \sum_a n_a(x)^2 = 1.$$ (2.1)

These maps $n$ are classified by their winding number $\kappa \in \mathbb{Z}$:

$$\kappa = \frac{1}{8\pi} \int_{S^2} \epsilon_{abc} n_a(x) dn_b(x) dn_c(x).$$ (2.2)

That $\kappa$ is the winding of the map can be seen taking spherical coordinates $(\Theta, \Phi)$ on the target sphere ($n^2 = 1$) and using the identity $\sin \Theta d\Theta d\Phi = \frac{1}{2} \epsilon_{abc} n_a dn_b dn_c$. We omit wedge symbols in forms.

We can think of $n$ as the field at a fixed time $t$ on a (2+1)-dimensional manifold where the spatial slice is $S^2$. In that case, it can describe a field of spins, and the fields with $\kappa \neq 0$ describe solitonic sectors. We can also think of it as a field on Euclidean spacetime $S^2$. In that case, the fields with $\kappa \neq 0$ describe instantonic sectors.

Let $\tau_a$ be the Pauli matrices. Then each $n(x)$ is associated with the projector

$$P(x) = \frac{1}{2}(1 + \vec{\tau} \cdot \vec{n}(x)).$$ (2.3)

Conversely, given a $2 \times 2$ projector $P(x)$ of rank 1, we can write

$$P(x) = \frac{1}{2}(\alpha_0(x) + \vec{\alpha}(x)).$$ (2.4)

Using $\text{Tr} P(x) = 1$, $P(x)^2 = P(x)$ and $P(x)^\dagger = P(x)$, we get

$$\alpha_0(x) = 1, \quad \vec{\alpha}(x) \cdot \vec{\alpha}(x) = 1, \quad \alpha_a^*(x) = \alpha_a(x).$$ (2.5)
Thus $\mathbb{C}P^1$-fields on $S^2$ can be described either by $P$ or by $n_a = \text{Tr}(\tau_a P)$ \cite{10}.

In terms of $P$, $\kappa$ is
\[
\kappa = \frac{1}{2\pi i} \int_{S^2} \text{Tr} P (dP) (dP) .
\] (2.6)

There is a family of projectors, called Bott projectors \cite{11, 12} which play a central role in our approach. Let
\[ z = (z_1, z_2), \quad |z|^2 := |z_1|^2 + |z_2|^2 = 1 . \] (2.7)
The $z$'s are points on $S^3$. We can write $x \in S^2$ in terms of $z$:
\[ x_i(z) = z^\dagger \tau_i z \] (2.8)
The Bott projectors are
\[
P_\kappa(x) = v_\kappa(x) v_\kappa^\dagger(x), \quad v_\kappa(z) = \begin{bmatrix} z_1^\kappa \\ z_2^\kappa \end{bmatrix} \frac{1}{\sqrt{Z_\kappa}} \quad \text{if } \kappa \geq 0 , \]
\[
Z_\kappa \equiv |z_1|^{2|\kappa|} + |z_2|^{2|\kappa|} , \]
\[
v_\kappa(z) = \begin{bmatrix} z_1^{\ast|\kappa|} \\ z_2^{\ast|\kappa|} \end{bmatrix} \frac{1}{\sqrt{Z_\kappa}} \quad \text{if } \kappa < 0 . \] (2.9)
The field $n^{(\kappa)}$ associated with $P_\kappa$ is given by
\[
n^{(\kappa)}_a(x) = \text{Tr} \tau_a P_\kappa(x) = v_\kappa^\dagger(z) \tau_a v_\kappa(z) . \] (2.10)
Under the phase change $z \to ze^{i\theta}$, $v_\kappa(z)$ changes $v_\kappa(z) \to v_\kappa(z)e^{i\kappa \theta}$, whereas $x$ is invariant. As this phase cancels in $v_\kappa(z)v_\kappa^\dagger(z)$, $P_\kappa$ is a function of $x$ as written.

The $\kappa$ that appears in eqs. (2.9) (2.10) is the winding number as the explicit calculation of section 3 will show. But there is also the following argument.

In the map $z \to v_\kappa(z)$, for $\kappa = 0$, all of $S^3$ and $S^2$ get mapped to a point, giving zero winding number. So, consider $\kappa > 0$. Then the points
\[
\left( z_1 e^{i\frac{2\pi l}{\kappa}}, z_2 e^{i\frac{2\pi m}{\kappa}} \right), \quad l, m \in \{0, 1, \ldots, \kappa - 1\}
\]
have the same image. But the overall phase $e^{i\frac{2\pi}{\kappa} m}$ of $z$ cancels out in $x$. Thus, generically $\kappa$ points of $S^2$ (labeled by $l$) have the same projector $P_\kappa(x)$, giving winding number $\kappa$. As for $\kappa < 0$, we get $|\kappa|$ points of $S^2$ mapped to the same $P_\kappa(x)$. But because of the complex conjugation in eq. (2.9), there is an orientation-reversal in map giving $-|\kappa| = \kappa$ as winding numbers. One way to see this is to use
\[
P_{-|\kappa|}(x) = P_{|\kappa|}(x)^T \] (2.11)
Substituting this in (2.6), we can see that \( P_{\pm|\kappa|} \) have opposite winding numbers.

The general projector \( \mathcal{P}_\kappa(x) \) is the gauge transform of \( P_{\kappa}(x) \):

\[
\mathcal{P}_\kappa(x) = U(x)P_{\kappa}(x)U(x)\dagger
\]  

(2.12)

where \( U(x) \) is a unitary \( 2 \times 2 \) matrix. Its \( n(\kappa) \) is also given by (2.10), with \( P_{\kappa} \) replaced by \( \mathcal{P}_\kappa \). The winding number is unaffected by the gauge transformation. That is because \( U \) is a map from \( S^2 \) to \( U(2) \) and all such maps can be deformed to identity since \( \pi_2(U(2)) = \{ \text{identity} e \} \).

The identity

\[
\mathcal{P}_\kappa(d\mathcal{P}_\kappa) = (d\mathcal{P}_\kappa)(\mathbb{I} - \mathcal{P}_\kappa)
\]  

(2.13)

which follows from \( \mathcal{P}_\kappa^2 = \mathcal{P}_\kappa \), is valuable when working with projectors.

\section{An Action}

Let \( \mathcal{L}_i = -i(x \wedge \nabla)_i \) be the angular momentum operator. Then a Euclidean action in the \( \kappa \)-th topological sector for \( n(\kappa)(x) \) (or a static Hamiltonian in the (2+1) picture) is

\[
S_\kappa = -\frac{c}{2} \int_{S^2} d\Omega (\mathcal{L}_i n_b^{(\kappa)})(\mathcal{L}_i n_b^{(\kappa)}) , \quad c = \text{a positive constant},
\]  

(3.1)

where \( d\Omega \) is the \( S^2 \) volume form \( d\cos\theta d\varphi \). We can also write

\[
S_\kappa = -c \int_{S^2} d\Omega \text{Tr}(\mathcal{L}_i \mathcal{P}_\kappa)(\mathcal{L}_i \mathcal{P}_\kappa).
\]  

(3.2)

The following identities, based on (2.13), are also useful:

\[
\text{Tr} \mathcal{P}_\kappa(\mathcal{L}_i \mathcal{P}_\kappa)^2 = \text{Tr}(\mathcal{L}_i \mathcal{P}_\kappa)(\mathbb{I} - \mathcal{P}_\kappa)(\mathcal{L}_i \mathcal{P}_\kappa) = \text{Tr}(\mathbb{I} - \mathcal{P}_\kappa)(\mathcal{L}_i \mathcal{P}_\kappa)^2 = \frac{1}{2} \text{Tr}(\mathcal{L}_i \mathcal{P}_\kappa)^2
\]  

(3.3)

Hence

\[
S_\kappa = -2c \int_{S^2} d\Omega \text{Tr} \mathcal{P}_\kappa \mathcal{L}_i \mathcal{P}_\kappa \mathcal{L}_i \mathcal{P}_\kappa
\]  

(3.4)

The Euclidean functional integral for the actions \( S_\kappa \) is

\[
Z(\psi) = \sum_\kappa e^{i\kappa\psi} \int \mathcal{D}\mathcal{P}_\kappa e^{-S_\kappa}
\]  

(3.5)

where the angle \( \psi \) is induced by the instanton sectors as in QCD.
Using the identity \( dP = -\epsilon_{ijk} dx_i x_j i\mathcal{L}_k P \), we can rewrite the definition of the winding number as

\[
\kappa = \frac{1}{8\pi} \int_{S^2} d\Omega \epsilon_{ijk} x_i \epsilon_{abc} n^{(k)}_a i\mathcal{L}_j n^{(k)}_b i\mathcal{L}_k n^{(k)}_c =
\]

\[
= \frac{1}{2\pi i} \int_{S^2} d\Omega \text{Tr} P_\kappa \epsilon_{ijk} x_i i\mathcal{L}_j P_\kappa i\mathcal{L}_k P_\kappa .
\]

The Belavin-Polyakov bound \[13\]

\[
S_\kappa \geq 4\pi c |\kappa|
\]

follows from (3.6) on integration of

\[
(i\mathcal{L}_i n^{(k)}_a \pm \epsilon_{ijk} x_j \epsilon_{abc} n^{(k)}_b i\mathcal{L}_k n^{(k)}_c)^2 \geq 0,
\]

or from (3.7) on integration of

\[
\text{Tr} (P_\kappa (i\mathcal{L}_j P_\kappa) \pm i\epsilon_{ijk} x_j P_\kappa (i\mathcal{L}_k P_\kappa)) \geq 0 .
\]

From this last form it is easy to rederive the bound in a way better adapted to fuzzification. Using Pauli matrices \( \{\sigma_i\} \) we first rewrite (3.4) and (3.7) as

\[
S_\kappa = c \int_{S^2} d\Omega \text{Tr} P_\kappa (i\sigma \cdot \mathcal{L} P_\kappa) ,
\]

\[
\kappa = -\frac{1}{4\pi} \int_{S^2} d\Omega \text{Tr} (\sigma \cdot x P_\kappa (i\sigma \cdot \mathcal{L} P_\kappa)) .
\]

The trace is now over \( \mathbb{C}^2 \times \mathbb{C}^2 = \mathbb{C}^4 \), where \( \tau_a \) acts on the first \( \mathbb{C}^2 \) and \( \sigma_i \) on the second \( \mathbb{C}^2 \) (so they are really \( \tau_a \otimes 1 \) and \( 1 \otimes \sigma_i \)). Then, with \( \epsilon_1, \epsilon_2 = \pm 1 \),

\[
\frac{1 + \epsilon_2 \tau \cdot n^{(k)}_a}{2} \sigma_i ((i\mathcal{L}_i P_\kappa) + \epsilon_1 i\epsilon_{ijk} x_j (i\mathcal{L}_k P_\kappa)) = (1 + \epsilon_1 \sigma \cdot x) \frac{1 + \epsilon_2 \tau \cdot n^{(k)}_a}{2} (i\sigma \cdot \mathcal{L} P_\kappa) ,
\]

since \( x \cdot \mathcal{L} = 0 \). The inequality (3.10) is equivalent to

\[
\text{Tr} \left[ \frac{1 + \epsilon_1 \sigma \cdot x}{2} \frac{1 + \epsilon_2 \tau \cdot n^{(k)}_a}{2} (i\sigma \cdot \mathcal{L} P_\kappa) \right]^\dagger \cdot \left[ \frac{1 + \epsilon_1 \sigma \cdot x}{2} \frac{1 + \epsilon_2 \tau \cdot n^{(k)}_a}{2} (i\sigma \cdot \mathcal{L} P_\kappa) \right] \geq 0 ,
\]

from which (3.8) follows by integration.

## 4 \( \mathbb{C}P^1 \)-models and Partial Isometries

If \( \mathcal{P}(x) \) is a rank 1 projector at each \( x \), we can find its normalized eigenvector \( u(z) \):

\[
\mathcal{P}(x) u(z) = u(z) , \quad u^\dagger(z) u(z) = 1 .
\]
Then
\[ \mathcal{P}(x) = u(z)u^\dagger(z) \]  
(4.2)
If \( \mathcal{P} = \mathcal{P}_\kappa \) an example of \( u \) is \( v_\kappa \). \( u \) can be a function of \( z \), changing by a phase under \( z \to ze^{i\theta} \). Still, \( \mathcal{P} \) will depend only on \( x \).

We can regard \( u(z)^\dagger \) (or a slight generalization of it) as an example of a partial isometry \([6]\) in the algebra \( \mathcal{A} = \mathcal{C}^{\infty}(S^3) \otimes_\mathbb{C} \text{Mat}_{2\times 2}(\mathbb{C}) \) of \( 2 \times 2 \) matrices with coefficients in \( \mathcal{C}^{\infty}(S^3) \). A partial isometry in a \(*\)-algebra \( A \) is an element \( U^\dagger U \in A \) such that \( UU^\dagger \) is a projector; \( UU^\dagger \) is the support projector of \( U^\dagger \). It is an isometry if \( U^\dagger U = 1 \). With

\[ U = \begin{pmatrix} u_1 & 0 \\ u_2 & 0 \end{pmatrix} \in \mathcal{A} \]  
(4.3)
we have

\[ \mathcal{P} = UU^\dagger \]  
(4.4)
so that \( U^\dagger \) is a partial isometry.

We will be free with language and also call \( u^\dagger \) as a partial isometry.

The partial isometry for \( P_\kappa \) is \( v_\kappa^\dagger \).

Now consider the one-form

\[ A_\kappa = v_\kappa^\dagger dv_\kappa . \]  
(4.5)
Under \( z_i \to z_i e^{i\theta(x)} \), \( A_\kappa \) transforms like a connection:

\[ A_\kappa \to A_\kappa + i\kappa d\theta \]

(\( A_\kappa \) are connections for \( U(1) \) bundles on \( S^2 \) for Chern numbers \( \kappa \), see later.) Therefore

\[ D_\kappa = d + A_\kappa \]  
(4.6)
is a covariant differential, transforming under \( z \to ze^{i\theta} \) as

\[ D_\kappa \to e^{i\kappa \theta} D_\kappa e^{-i\kappa \theta} \]  
(4.7)
and

\[ D^2_\kappa = dA_\kappa \]  
(4.8)
is its curvature.

At each \( z \), there is a unit vector \( w_\kappa(z) \) perpendicular to \( v_\kappa(z) \). An explicit realization of \( w_\kappa(z) \) is given by

\[ w_{\kappa,\alpha} = i \tau_{2\alpha\beta} v_{\kappa,\beta}^* := \epsilon_{\alpha\beta} v_{\kappa,\beta}^* \]  
(4.9)
Since $w^\dagger_\kappa v_\kappa = 0$,

$$B_\kappa = w^\dagger_\kappa dv_\kappa, \quad B^*_\kappa = (dv^\dagger_\kappa)w_\kappa = -v^\dagger_\kappa dw_\kappa$$  \hspace{1cm} (4.10)

are gauge covariant,

$$B_\kappa(z) \rightarrow e^{i\theta(x)}B_\kappa e^{i\theta(x)}, \quad B_\kappa(z)^* \rightarrow e^{-i\theta(x)}B^*_\kappa e^{-i\theta(x)}$$  \hspace{1cm} (4.11)

under $z \rightarrow ze^{i\theta}$.

We can account for $U(x)$ by considering

$$\mathcal{V}_\kappa = Uv_\kappa, \quad \mathcal{A}_\kappa = \mathcal{V}_\kappa^\dagger d\mathcal{V}_\kappa, \quad \mathcal{D}_\kappa = d + \mathcal{A}_\kappa, \quad \mathcal{D}_\kappa^2 = d\mathcal{A}_\kappa$$

\hspace{1cm} (4.12)

$$\mathcal{W}_\kappa = (\tau_2 U^\ast \tau_2)w_\kappa, \quad B_\kappa = \mathcal{W}_\kappa^\dagger d\mathcal{V}_\kappa.$$  \hspace{1cm} (4.13)

$\mathcal{A}_\kappa$ is still a connection, and the properties (4.11) are not affected by $U$. $\mathcal{P}_\kappa$ is the support projector of $\mathcal{V}_\kappa^\dagger$, and

$$\mathcal{W}_\kappa \mathcal{W}_\kappa^\dagger = 1 - \mathcal{P}_\kappa, \quad (1 - \mathcal{P}_\kappa)\mathcal{V}_\kappa = 0.$$  \hspace{1cm} (4.14)

Gauge invariant quantities being functions on $S^2$, we can contemplate a formulation of the $\mathbb{CP}^1$-model as a gauge theory. Let $\mathcal{J}_i$ be the lift of $L_i$ to angular momentum generators appropriate for functions of $z$,

$$(e^{i\theta_iJ_i}f)(z) = f(e^{-i\theta_i\tau_i/2}z),$$  \hspace{1cm} (4.15)

and let

$$B_{\kappa,i} = \mathcal{W}_\kappa^\dagger \mathcal{J}_i \mathcal{V}_\kappa.$$  \hspace{1cm} (4.16)

Now, $\mathcal{W}_\kappa B_{\kappa,i} \mathcal{V}_\kappa^\dagger$ is gauge invariant, and should have an expression in terms of $\mathcal{P}_\kappa$. Indeed it is, in view of (4.13),

$$\mathcal{W}_\kappa B_{\kappa,i} \mathcal{V}_\kappa^\dagger = \mathcal{W}_\kappa \mathcal{W}_\kappa^\dagger (\mathcal{J}_i \mathcal{V}_\kappa) \mathcal{V}_\kappa^\dagger = (1 - \mathcal{P}_\kappa)\mathcal{J}_i (\mathcal{V}_\kappa^\dagger \mathcal{V}_\kappa^\dagger) = (1 - \mathcal{P}_\kappa)(L_i \mathcal{P}_\kappa) = (L_i \mathcal{P}_\kappa) \mathcal{P}_\kappa.$$  \hspace{1cm} (4.17)

Therefore we can write the action (3.2, 3.4) in terms of the $B_{\kappa,i}$:

$$S_\kappa = -2c \int_{S^2} d\Omega \text{ Tr } \mathcal{P}_\kappa (L_i \mathcal{P}_\kappa)(L_i \mathcal{P}_\kappa) = 2c \int_{S^2} d\Omega \text{ Tr } ((L_i \mathcal{P}_\kappa) \mathcal{P}_\kappa)^\dagger ((L_i \mathcal{P}_\kappa) \mathcal{P}_\kappa) =$$

$$= 2c \int_{S^2} d\Omega \text{ Tr } (\mathcal{W}_\kappa B_{\kappa,i} \mathcal{V}_\kappa^\dagger) (\mathcal{W}_\kappa B_{\kappa,i} \mathcal{V}_\kappa^\dagger) = 2c \int_{S^2} d\Omega B^*_{\kappa,i} B_{\kappa,i}.$$  \hspace{1cm} (4.18)

It is instructive also to write the gauge invariant $(d\mathcal{A}_\kappa)$ in terms of $\mathcal{P}_\kappa$ and relate its integral to the winding number (2.6). The matrix of forms

$$\mathcal{V}_\kappa(d + \mathcal{A}_\kappa)$$  \hspace{1cm} (4.19)
is gauge invariant. Here
\[ dV^\dagger_\kappa = (dV^\dagger_\kappa) + V^\dagger_\kappa d \]
where \( d \) in the first term differentiates only \( V^\dagger_\kappa \). Now
\[ V_\kappa (d + V^\dagger_\kappa (dV_\kappa)) V^\dagger_\kappa \]
and
\[ \mathcal{P}_\kappa d\mathcal{P}_\kappa = V_\kappa V^\dagger_\kappa d (V_\kappa V^\dagger_\kappa) = V_\kappa (dV_\kappa) V^\dagger_\kappa + V_\kappa (dV^\dagger_\kappa) + V_\kappa V^\dagger_\kappa d \]  (4.19)
are equal. Hence, squaring
\[ V_\kappa (d + A_\kappa)^2 V^\dagger_\kappa = V_\kappa (dA_\kappa) V^\dagger_\kappa = \mathcal{P}_\kappa (d\mathcal{P}_\kappa) (d\mathcal{P}_\kappa) \]  (4.20)
on using \( d^2 = 0 \), eq.(4.19) and \( \mathcal{P}_\kappa (d\mathcal{P}_\kappa)\mathcal{P}_\kappa = 0 \). Thus
\[ \int_{S^2} (dA_\kappa) = \int_{S^2} \text{Tr} V_\kappa (dA_\kappa) V^\dagger_\kappa = \int_{S^2} \text{Tr} \mathcal{P}_\kappa (d\mathcal{P}_\kappa) (d\mathcal{P}_\kappa) . \]  (4.21)
We can integrate the LHS. For this we write (taking a section of the bundle \( U(1) \rightarrow S^3 \rightarrow S^2 \) over \( S^2 \setminus \{ \text{north pole}(0,0,1) \} \)),
\[ z(x) = e^{-i\tau_3 \varphi/2} e^{-i\tau_2 \theta/2} e^{-i\tau_3 \varphi/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix} . \]  (4.22)
Taking into account the fact that \( U(\vec{x}) \) is independent of \( \varphi \) at \( \theta = 0 \), we get
\[ \int_{S^2} (dA_\kappa) = -\int e^{i\kappa \varphi} de^{-i\kappa \varphi} = 2\pi i\kappa . \]  (4.23)
This and eq.(4.21) reproduce eq.(2.6).

The Belavin-Polyakov bound [13] for \( S_\kappa \) can now be got from the inequality
\[ \text{Tr} C^\dagger_{\kappa,i} C_{\kappa,i} \geq 0 , \quad C_{\kappa,i} = \mathcal{W}_\kappa \mathcal{B}_{\kappa,i} V^\dagger_\kappa \pm \mathcal{W}_\kappa (\epsilon_{ij} x_j B_{\kappa,l}) V^\dagger_\kappa . \]  (4.24)

4.1 Connection to an earlier paper.

In a previous paper [4], for \( \kappa > 0 \), the fuzzy \( \sigma \)-model was based on the continuum projector
\[ P^{(\kappa)}(x) = P_1(x) \otimes ... \otimes P_1(x) = \prod_{i=1}^{\kappa} \frac{1}{2}(1 + \tau^{(i)} \cdot x) \]  (4.25)
and its unitary transform
\[ \mathcal{P}^{(\kappa)}(x) = U^{(\kappa)}(x) P^{(\kappa)}(x) U^{(\kappa)}(x)^{-1} , \quad U^{(\kappa)}(x) = U(x) \otimes ... \otimes U(x) \quad (\kappa \text{ factors}) . \]  (4.26)
At each $x$, the stability group of $P^{(\kappa)}(x)$ is $U(1)$ with generator $\frac{1}{2} \sum_{i=1}^{\kappa} \tau^{(i)} \cdot x$, and we get a sphere $S^2$ as $U(x)$ is varied. Thus $U^{(\kappa)}(x)$ gives a section of a sphere bundle over a sphere, leading us to identify $P^{(\kappa)}$ with a $\mathbb{C}P^1$-field. Furthermore, the R.H.S. of eq.(4.21) (with $P^{(\kappa)}$ replacing $P_\kappa$) gives $\kappa$ as the invariant associated with $P^{(\kappa)}$, suggesting a correspondence between $\kappa$ and winding number.

We can write $P^{(\kappa)} = V^{(\kappa)} V^{(\kappa)\dagger}$, $V^{(\kappa)} = V_1 \otimes \ldots \otimes V_1$ ($\kappa$ factors),

$$\mathcal{A}^{(\kappa)} = \frac{i}{2\pi} \int dA^{(\kappa)} = \kappa. \quad (4.28)$$

So $\kappa$ is the Chern invariant of the projective module associated with $P^{(\kappa)}$.

For $\kappa < 0$, we must change $x$ to $-x$ in (4.25), and accordingly change other expressions.

But we missed the fact that $\kappa$ cannot be identified with the winding number of the map $x \to P_\kappa(x)$. To see this, say for $\kappa > 0$, we show that there is a winding number $\kappa$ map from $P^{(\kappa)}$ to $P_\kappa(x)$. As that is also the winding number of the map $x \to P_\kappa(x)$, the map $x \to P^{(\kappa)}(x)$ must have winding number 1.

The map $P^{(\kappa)} \to P_\kappa(x)$ is induced from the map

$$\mathcal{V}^{(\kappa)} \to \mathcal{V}_\kappa = \begin{pmatrix} \mathcal{V}^{(\kappa)}_{11} \\ \mathcal{V}^{(\kappa)}_{22} \end{pmatrix} \quad (4.29)$$

and their expressions in terms of $\mathcal{V}^{(\kappa)}$ and $\mathcal{V}_\kappa$. In (4.29) all the points $\mathcal{V}^{(\kappa)}(z_1 e^{2\pi ij/\kappa}, z_2 e^{2\pi il/\kappa})$, $j, l \in \{0, 1, \ldots, \kappa - 1\}$, have the same image, but in the passage to $P^{(\kappa)}$ and $P_\kappa$ the overall phase of $z$ is immaterial. However, the projectors for $\mathcal{V}^{(\kappa)}_\kappa(z_1, z_2)$ and $\mathcal{V}^{(\kappa)}_\kappa(z_1, z_2 e^{2\pi ij/\kappa})$ are distinct and map to the same $P_\kappa$, giving winding number $\kappa$.

We have not understood the relation between the models based on $P^{(\kappa)}$ and $P_\kappa$.

### 5 Fuzzy $\mathbb{C}P^1$-models

The advantage of the preceding formulation using $\{z_\alpha\}$ is that the passage to fuzzy models is relatively transparent. Thus let $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2 \setminus \{0\}$. We can then identify $z$ and $x$ as

$$z = \frac{\xi}{|\xi|}, \quad |\xi| = \sqrt{|\xi_1|^2 + |\xi_2|^2}, \quad x_i = z^\dagger \tau_i z. \quad (5.1)$$
Quantization of the $\xi$’s and $\xi^*$’s consists in replacing $\xi_\alpha$ by annihilation operators $a_\alpha$ and $\xi^*_\alpha$ by $a^\dagger_\alpha$. $|\xi|$ is then the square root of the number operator:

\[
\hat{N} = \hat{N}_1 + \hat{N}_2, \quad \hat{N}_1 = a_1^\dagger a_1, \quad \hat{N}_2 = a_2^\dagger a_2, \\
\hat{\xi}^\dagger_\alpha = \frac{1}{\sqrt{\hat{N}}} a_\alpha^\dagger = \frac{1}{\sqrt{\hat{N} + 1}} a_\alpha, \quad \hat{\xi}_\alpha = \frac{1}{\sqrt{\hat{N} + 1}} a_\alpha = a_\alpha \frac{1}{\sqrt{\hat{N}}}, \\
\hat{x}_i = \frac{1}{\sqrt{\hat{N}}} a_\tau^\dagger a .
\] (5.2)

(We have used hats on some symbols to distinguish them as fuzzy operators).

We will apply these operators only on the subspace of the Fock space with eigenvalue $n$ of $\hat{N}$, $\geq 1$, where $\frac{1}{\sqrt{\hat{N}}}$ is well defined. This restriction is natural and reflects the fact that $\xi$ cannot be zero.

5.1 The fuzzy projectors for $\kappa > 0$. On referring to (2.9), we see that if $\kappa > 0$, for the quantized versions $\hat{v}_\kappa$, $\hat{v}_\kappa^\dagger$ of $v_\kappa$, $v_\kappa^*$, we have

\[
\hat{v}_\kappa = \begin{bmatrix} a_1^\kappa \\ a_2^\kappa \end{bmatrix} \frac{1}{\sqrt{Z_\kappa}}, \quad \hat{v}_\kappa^\dagger = \frac{1}{\sqrt{Z_\kappa}} \begin{bmatrix} (a_1^\dagger)^\kappa & (a_2^\dagger)^\kappa \end{bmatrix}, \quad \hat{v}_\kappa^\dagger \hat{v}_\kappa = \mathbb{1}, \\
\hat{Z}_\kappa = \hat{Z}_\kappa^{(1)} + \hat{Z}_\kappa^{(2)}, \quad \hat{Z}_\kappa^{(\alpha)} = \hat{N}_\alpha (\hat{N}_\alpha - 1) ... (\hat{N}_\alpha - \kappa + 1) .
\] (5.3)

The fuzzy analogue of $U$ is a $2 \times 2$ unitary matrix $\hat{U}$ whose entries $\hat{U}_{ij}$ are polynomials in $a_\alpha^\dagger a_b$. As for $\hat{v}_\kappa$, the quantized version of $v_\kappa$, it is just

\[
\hat{v}_\kappa = \hat{U} \hat{v}_\kappa
\] (5.4)

and fulfills

\[
\hat{v}_\kappa^\dagger \hat{v}_\kappa = \mathbb{1},
\] (5.5)

$\hat{v}_\kappa^\dagger$ being the quantized version of $v_\kappa^\dagger$. We thus have the fuzzy projectors

\[
\hat{P}_\kappa = \hat{v}_\kappa^\dagger \hat{v}_\kappa, \quad \hat{P}_\kappa = \hat{v}_\kappa \hat{v}_\kappa^\dagger.
\] (5.6)

Unlike $\hat{v}_\kappa, \hat{v}_\kappa$ and their adjoints, $\hat{P}_\kappa$ and $\hat{P}_\kappa$ commute with the number operator $\hat{N}$. So we can formulate a finite-dimensional matrix model for these projectors as follows. Let $F_n$ be the subspace of the Fock space where $\hat{N} = n$. It is of dimension $n + 1$, and carries the $SU(2)$ representation with angular momentum $n/2$, the $SU(2)$ generators being

\[
L_i = \frac{1}{2} a_\tau^\dagger a .
\] (5.7)
Its standard orthonormal basis is $|\frac{n}{2}, m\rangle$, $m = -\frac{n}{2}, -\frac{n}{2} + 1, \ldots, \frac{n}{2}$. Now consider $F_n \otimes \mathbb{C}^2 := F_n^{(2)}$, with elements $f = (f_1, f_2)$, $f_a \in F_n$. Then $\hat{P}_\kappa$, $\hat{P}_\kappa$ act on $F_n^{(2)}$ in the natural way. For example

$$f \rightarrow \hat{P}_\kappa f, \quad (\hat{P}_\kappa f)_a = (\hat{P}_\kappa)_{ab} f_b = (\hat{V}_{\kappa,a} \hat{V}_{\kappa,b}^\dagger) f_b . \quad (5.8)$$

We now can write explicit matrices for $\hat{P}_\kappa$ and $\hat{P}_\kappa$. We have:

$$\hat{P}_\kappa = \begin{pmatrix}
a_1^\kappa & a_2^\kappa \\
a_1^\dagger \frac{1}{Z_\kappa} & a_2^\dagger \frac{1}{Z_\kappa}
\end{pmatrix},$$

$$a_1^\kappa \frac{1}{Z_\kappa} = \frac{1}{(\hat{N}_1 + \kappa)(\hat{N}_1 + 1)} + \frac{\hat{Z}_\kappa^{(2)}}{2} a_1^\kappa, \quad a_1^\kappa a_1^\dagger = (\hat{N}_1 + \kappa)(\hat{N}_1 + 1) ,$$

from which its matrix $\hat{P}_\kappa(n)$ for $\hat{N} = n$ can be obtained.

The matrix $\hat{P}_\kappa$ is the unitary transform $\hat{U} \hat{P}_\kappa(n) \hat{U}^\dagger$ where $\hat{U}$ is a $2 \times 2$ matrix and $\hat{U}_{ab}$ is itself an $(n + 1) \times (n + 1)$ matrix. As for the fuzzy analogue of $\mathcal{L}_i$, we define it by

$$\mathcal{L}_i \hat{P}_\kappa = [L_i, \hat{P}_\kappa] . \quad (5.10)$$

The fuzzy action

$$S_{F,\kappa}(n) = \frac{c}{2(n+1)} \text{Tr}_{\hat{N} = n} (\mathcal{L}_i \hat{P}_\kappa)^\dagger (\mathcal{L}_i \hat{P}_\kappa) , \quad c = \text{constant} , \quad (5.11)$$

follows, the trace being over the space $F_n^{(2)}$.

### 5.2 The Fuzzy Projector for $\kappa < 0$.

For $\kappa < 0$, following an early indication, we must exchange the roles of $a_a$ and $a_a^\dagger$.

### 5.3 Fuzzy Winding Number.

In the literature [14], there are suggestions on how to extend (2.6) to the fuzzy case. They do not lead to an integer value for this number except in the limit $n \rightarrow \infty$.

There is also an approach to topological invariants using Dirac operator and cyclic cohomology. Elsewhere this approach was applied to the fuzzy case [14] [15] and gave integer values, and even a fuzzy analogue of the Belavin-Polyakov bound. However they were not for the action $S_{F,\kappa}$, but for an action which approaches it as $n \rightarrow \infty$. Below, in section 5.4, we present an alternative approach to this bound which works for $S_{F,\kappa}$. It looks like (3.8), except that $\kappa$ becomes an integer only in the limit $n \rightarrow \infty$. 

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There is also a very simple way to associate an integer to $\hat{V}_\kappa$ [16, 3]. It is equivalent to the Dirac operator approach. We can assume that the domain of $\hat{V}_\kappa$ are vectors with a fixed value $n$ of $\hat{N}$. Then after applying $\hat{V}_\kappa$, $n$ becomes $n - \kappa$ if $\kappa > 0$ and $n + |\kappa|$ is $\kappa < 0$. Thus $\kappa$ is just the difference in the value of $\hat{N}$, or equivalently twice the difference in the value of the angular momentum, between its domain and its range.

We conclude this section by deriving the bound for $S_{F,\kappa}(n)$.

5.4 The Fuzzy Bound.

A proper generalization of the Belavin-Polyakov bound to its fuzzy version involves a slightly more elaborate approach. This is because the straightforward fuzzification of $\vec{\sigma} \cdot \vec{x}$ and $\vec{\tau} \cdot \vec{\tau}^{(\kappa)}$ and their corresponding projectors do not commute, and the product of such fuzzy projectors is not a projector. We use this elaborated approach only in this section. It is not needed elsewhere. In any case, what is there in other sections is trivially adapted to this formalism.

The approach taken here is not new. It is essential, and has been widely used, for example for the study of the fuzzy Dirac operator [17].

The operators $a_\alpha^\dagger a_\beta$ acting on the vector space with $\hat{N} = n$ generate the algebra $\text{Mat}(n+1)$ of $(n+1) \times (n+1)$ matrices. The extra structure comes from regarding them not as observables, but as a Hilbert space of matrices $m, m', \ldots$ with scalar product $(m', m) = \frac{1}{n+1} \text{Tr}_{\mathbb{C}^{n+1}} m'^\dagger m$, with the observables acting thereon.

To each $\alpha \in \text{Mat}(n+1)$, we can associate two linear operators $\alpha^{L,R}$ on $\text{Mat}(n+1)$ according to

$$\alpha^L m = \alpha m, \quad \alpha^R m = m \alpha, \quad m \in \text{Mat}(n+1).$$

(5.12)

$\alpha^L - \alpha^R$ has a smooth commutative limit for operators of interest. It actually vanishes, and $\alpha^{L,R} \to 0$ if $\alpha$ remains bounded during this limit.

Consider the angular momentum operators $L_i \in \text{Mat}(n+1)$. The associated ‘left’ and ‘right’ angular momenta $L_i^{L,R}$ fulfill

$$(L_i^L)^2 = (L_i^R)^2 = \frac{n}{2} \left( \frac{n}{2} + 1 \right).$$

(5.13)

We now regard $a_\alpha, a_\alpha^\dagger$ of section 5 as left operators $a_\alpha^L$ and $a_\alpha^R$. $\hat{P}_\kappa^L$ thus becomes a $2 \times 2$ matrix with each entry being a left multiplication operator. It is the linear operator $\hat{P}_\kappa^L$ on $\text{Mat}(n+1) \otimes \mathbb{C}^2$. We tensor this vector space with another $\mathbb{C}^2$ as before to get $\mathcal{H} = \text{Mat}(n+1) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, with $\sigma_i$ acting on the last $\mathbb{C}^2$, and $\sigma \cdot L^\dagger \hat{P}_\kappa^L$ denoting the operator $\sigma_i (L_i \hat{P}_\kappa)^L$. 

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We can repeat the previous steps if there are fuzzy analogues $\gamma$ and $\Gamma$ of continuum ‘world volume’ and ‘target space’ chiralities $\vec{\sigma} \cdot \vec{x}$ and $\vec{\tau} \cdot \vec{n}(\kappa)$ which mutually commute. Then $\frac{1}{2}(1 \pm \gamma), \frac{1}{2}(1 \pm \Gamma)$ are commuting projectors and the expressions derived at the end of Section 3 generalize, as we shall see.

There is such a $\gamma$, due to Watamura[18], and discussed further by [4]. Following [4], we take

$$\gamma \equiv \gamma^L = \frac{2\sigma \cdot L^L + 1}{n + 1}.$$  (5.14)

The index $L$ has been put to emphasize its left action on $\text{Mat}(n + 1)$.

As for $\Gamma$, we can do the following. $\hat{P}_\kappa$ acts on the left on $\text{Mat}(n + 1)$, let us call it $\hat{P}^L_\kappa$. It has a $\hat{P}^R_\kappa$ acting on the right and an associated $\Gamma \equiv \Gamma^R_\kappa = 2\hat{P}^R_\kappa - 1, \quad (\Gamma^R_\kappa)^2 = 1$.  (5.15)

As it acts on the right and involves $\tau$’s while $\gamma$ acts on the left and involves $\sigma$’s,

$$\Gamma^L \Gamma^R_\kappa = \Gamma^R_\kappa \gamma^L.$$  (5.16)

The bound for (5.11) now follows from

$$\text{Tr}_H \left( \frac{1 + \epsilon_1 \gamma^L}{2} \frac{1 + \epsilon_2 \Gamma^R_\kappa \sigma}{2} \right)^\dagger \left( \frac{1 + \epsilon_1 \gamma^L}{2} \frac{1 + \epsilon_2 \Gamma^R_\kappa \sigma}{2} \right) \geq 0$$  (5.17)

($\epsilon_1, \epsilon_2 = \pm 1$), and reads

$$S_{F,\kappa} = \frac{c}{4(n + 1)} \text{Tr}_H(\sigma \cdot \mathcal{L}\hat{P}^L_\kappa)^\dagger(\sigma \cdot \mathcal{L}\hat{P}^L_\kappa)$$

$$\geq \frac{c}{4(n + 1)} \text{Tr}_H \left( (\epsilon_1 \gamma^L + \epsilon_2 \Gamma^R_\kappa)(\sigma \cdot \mathcal{L}\hat{P}^L_\kappa)(\sigma \cdot \mathcal{L}\hat{P}^L_\kappa) \right)$$

$$+ \frac{c}{4(n + 1)} \text{Tr}_H \left( \epsilon_1 \epsilon_2 \gamma^L \Gamma^R_\kappa(\sigma \cdot \mathcal{L}\hat{P}^L_\kappa)(\sigma \cdot \mathcal{L}\hat{P}^L_\kappa) \right).$$  (5.18)

The analogue of the first term on the R.H.S. is zero in the continuum, being absent in (5.11), but not so now. As $n \to \infty$, (5.18) reproduces (3.8) to leading order $n$, but has corrections which vanish in the large $n$ limit.

A minor clarification: if $\tau$’s are substituted by $\sigma$’s in $2\hat{P}^L_\kappa - 1$, then it is $\gamma^L$. The different projectors are thus being constructed using the same principles.

6 $\mathbb{C}P^N$-Models

We need a generalization of the Bott projectors to adapt the previous approach to all $\mathbb{C}P^N$. 

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Fortunately this can be easily done. The space \( \mathbb{C}P^N \) is the space of \((N+1) \times (N+1)\) rank 1 projectors. The important point is the rank. So we can write
\[
\mathbb{C}P^N = \langle U^{(N+1)} P_0 U^{(N+1)\dagger} : P_0 = \text{diag} \begin{pmatrix} 0, \ldots, 0, 1 \end{pmatrix}, U^{(N+1)} \in U(N+1) \rangle . \quad (6.1)
\]

As before, let \( z = (z_1, z_2), \ |z_1|^2 + |z_2|^2 = 1, \) and \( x_i = z_i^* \tau_i z \). Then we define
\[
\begin{pmatrix} z^\kappa_1 \\ z^\kappa_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \kappa > 0; \quad \begin{pmatrix} z^{*\kappa}_1 \\ z^{*\kappa}_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \kappa < 0 . \quad (6.2)
\]

Since
\[
\begin{align*}
\langle v^{(N)}(z) | v^{(N)}(z) \rangle &= 1 , \\
P^{(N)}(x) &= v^{(N)}(z) v^{(N)}(z)^\dagger \in \mathbb{C}P^N .
\end{align*}
\]

We can now easily generalize the previous discussion, using \( P^{(N)}_\kappa \) for \( P_\kappa \) and \( U^{(N+1)} \) for \( U \), and subsequently quantizing \( z_\alpha, z_\alpha^* \). In that way we get fuzzy \( \mathbb{C}P^N \)-models.

\( \mathbb{C}P^N \) models can be generalized by replacing the target space by a general Grassmannian or a flag manifold. They can also be elegantly formulated as gauge theories \[5\]. But we are able to formulate only a limited class of such manifolds in such a way that they can be made fuzzy. The natural idea would be to look for several vectors
\[
v^{(N)(i)}_{k_i}(z), \quad i = 1, \ldots, N \quad (6.4)
\]
in \((N+1)\) dimensions which are normalized and orthogonal,
\[
v^{(N)(i)\dagger}_{k_i}(z) v^{(N)(j)}_{k_j}(z) = \delta_{ij} \quad (6.5)
\]
and have the equivariance property
\[
v^{(N)(i)}_{k_i}(z e^{i\theta}) = v^{(N)(i)}_{k_i}(z) e^{i k_i \theta} \quad (6.6)
\]
The orbit of the projector \( \sum_{i=1}^M v^{(N)(i)}_{k_i}(z) v^{(N)(i)\dagger}_{k_i}(z) \) under \( U^{(N+1)} \) will then be a Grassmannian for each \( M \leq N \), while the orbit of \( \sum_i \lambda_i v^{(N)(i)}_{k_i}(z) v^{(N)(i)\dagger}_{k_i}(z) \) with possibly unequal \( \lambda_i \) under \( U^{(N+1)} \) will be a flag manifold.

But we can find such \( v^{(N)(i)}_{k_i} \) only for \( i = 1, 2, \ldots, M \leq \frac{N+1}{2} \).
For instance in an \((N+1) = 2L\)-dimensional vector space, for integer \(L\), we can form the vectors

\[
v_{k_1}^{(N)(1)}(z) = \begin{pmatrix} z_1^{k_1} \\ z_1^{k_1} \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{Z_{k_1}}} , \quad v_{k_2}^{(N)(2)}(z) = \begin{pmatrix} 0 \\ 0 \\ z_2^{k_2} \\ z_2^{k_2} \end{pmatrix} \frac{1}{\sqrt{Z_{k_2}}} , \quad \ldots , \quad v_{k_L}^{(N)(L)}(z) = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ z_2^{k_L} \\ z_2^{k_L} \end{pmatrix} \frac{1}{\sqrt{Z_{k_L}}}
\]

for \(k_i > 0\). For those \(k_i\) which are negative, we replace \(v_{k_i}^{(N)(i)}(z)\) here by \(v_{|k_i|}^{(N)(i)}(z)^*\):

\[
v_{k_i}^{(N)(i)}(z) = v_{|k_i|}^{(N)(i)}(z)^* , \quad k_i < 0 .
\]

These \(v_{k_i}^{(N)(i)}\) are orthonormal for all \(z\) with \(\sum_{\alpha} |z_{\alpha}|^2 = 1\), so that we can handle Grassmannians and flag manifolds involving projectors up to rank \(L\).

If \(N\) instead is \(2L\), we can write

\[
v_{k_1}^{(N)(1)}(z) = \begin{pmatrix} z_1^{k_1} \\ z_1^{k_1} \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{Z_{k_1}}} , \quad v_{k_2}^{(N)(2)}(z) = \begin{pmatrix} 0 \\ 0 \\ z_2^{k_2} \\ z_2^{k_2} \end{pmatrix} \frac{1}{\sqrt{Z_{k_2}}} , \quad \ldots , \quad v_{k_L}^{(N)(L)}(z) = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ z_2^{k_L} \\ z_2^{k_L} \end{pmatrix} \frac{1}{\sqrt{Z_{k_L}}}
\]

for \(k_i > 0\), and use \((6.8)\) for \(k_i < 0\).

But we can find no vector \(v_{k_{L+1}}^{(N)(L+1)}(z)\) fulfilling

\[
v_{k_i}^{(N)(i)}(z)^+ v_{k_{L+1}}^{(N)(L+1)}(z) = \delta_{i,L+1} , \quad i = 1, 2, \ldots, L + 1 , \quad v_{k_{L+1}}^{(N)(L+1)}(ze^{i\theta}) = v_{k_{L+1}}^{(N)(L+1)}(z)e^{ik_{L+1}\theta} .
\]

The quantization or fuzzification of these models can be done as before.

But lacking suitable \(v_{k_i}^{(i)}\) for \(i > L\), the method fails if the target flag manifold involves projectors of rank \(> \frac{N+1}{2}\).
Note that we cannot consider vectors like

\[
v'(z) = \begin{pmatrix}
0 \\
\vdots \\
0 \\
\frac{1}{|z_i|^k} \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad k > 0, \ i = 1 \text{ or } 2
\]  \hspace{1cm} (6.11)

and \(v'(z)^*\). That is because \(z_i\) can vanish compatibly with the constraint \(|z_1|^2 + |z_2|^2 = 1\), and \(v'(z), v'(z)^*\) are ill-defined when \(z_i = 0\).

As mentioned before, the flag manifolds are coset spaces \(M = SU(K)/S(U(k_1) \otimes U(k_2) \otimes \ldots \otimes U(k_\sigma)), \sum k_i = K\). Since \(\pi_2(M) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}\), a soliton on \(M\) is now characterized by \(\sigma\) winding numbers, with each number allowed to take either sign. The two possible signs for \(k_i\) in \(v^{(i)}_{k_i}\) reflect this freedom.

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