Asymptotic behaviour in the time synchronization model

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Abstract

There are two types $i = 1, 2$ of particles on the line $\mathbb{R}$, with $N_i$ particles of type $i$. Each particle of type $i = 1, 2$ moves with constant velocity $v_i$. Moreover, any particle of type $i = 1, 2$ jumps to any particle of type $j = 1, 2$ with rates $N_i^{-1} \alpha_{ij}$. We discuss in details the initial desynchronization of this particle system, namely, we are interested in behaviour of the process when the total number of particles $N_1 + N_2$ tends to infinity, $N_1/N_2 \to \text{const}$ and the time $t > 0$ is fixed.

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1 The Model

The simplest formulation of the model, we consider here, is in terms of the particle system. On the real line there are $N_1$ particles of type 1 and $N_2$ particles of type 2, $N = N_1 + N_2$. Each particle of type $i = 1, 2$ performs two independent movements. First of all, it moves with constant speed $v_i$ in the positive direction. We assume further that $v_i$ are constant and different, thus we can assume without loss of generality that $0 \leq v_1 < v_2$. The degenerate case $v_1 = v_2$ is different and will be considered separately.

Secondly, at any time interval $[t, t + dt]$ each particle of type $i$ independently of the others with probability $\alpha_{ij} dt$ decides to make a jump to some particle of type $j$ and chooses the coordinate of the $j$-type particle, where to jump, among the particles of type $j$, with probability $1/N_j$. Here $\alpha_{ij}$ are given nonnegative parameters for $i, j = 1, 2$. Further on, unless otherwise stated, we assume that $\alpha_{11} = \alpha_{22} = 0$, $\alpha_{12}, \alpha_{21} > 0$.

After such instantaneous jump the particle of type $i$ continues the movement with the same velocity $v_i$. This defines continuous time Markov chain $\{x_k^{(i)}(t)\}, i = 1, 2; k = 1, \ldots, N_i$, where $x_k^{(i)}(t)$ is the coordinate of $k$-th particle of type $i$ at time $t$. We assume that the initial coordinates $x_k^{(i)}(0)$ of the particles at time 0 are given. We are interested in the long time evolution of this system on various scales with $N \to \infty$, $t = t(N) \to \infty$.

In different terms, this can be interpreted as the time synchronization problem. In general, time synchronization problem can be presented as follows. There are $N$ systems (processors,
units, persons etc.) There is an absolute (physical) time \( t \), but each processor \( j \) fulfills a homogeneous job in its own proper time \( t_j = v_j t \), \( v_j > 0 \). Proper time is measured by the amount \( v \) of the job, accomplished by the processor for the unit of the physical time, if it is disjoint from other processors. However, there is a communication between each pair of processors, which should lead to drastic change of their proper times. In our case the coordinates \( x_k^{(i)}(t) \) can be interpreted as the modified proper times of the particles-processors, the nonmodified proper time being \( x_k^{(i)}(0) + v t \).

There can be many variants of exact formulation of such problem, see \([1, 7, 10, 2, 15]\). We will call the model considered here the basic model, because there are no restrictions on the jump process. Many other problems include such restrictions, for example, only jumps to the left are allowed. Due to absence of restrictions, this problem, as we will see below, is a "linear problem" in the sense that after scalings it leads to linear equations. In despite of this it has nontrivial behaviour, one sees different picture on different time scales.

There are, however, other interesting interpretations of this model, related to psychology, biology and physics; For example, in social psychology perception of time and life tempo strongly depends on the social contacts and intercourse. We will not enter the details here.

## 2 Main results

We show that the process consists of three consecutive stages: initial desynchronization up to the critical scale, critical slow down of desynchronization and final stabilization.

**Final stabilization** The first theorem shows that for \( N_i \) fixed and \( t \to \infty \) there is a synchronization: all particles asymptotically, as \( t \to \infty \), move with the same constant velocity \( v \), that is like \( vt \). However it does not say how fluctuations depend on \( N_i \).

Put
\[
m(t) = \min_{i,k} x_k^{(i)}(t)
\]

**Theorem 1** For any fixed \( N_1, N_2 \) there exists \( v = v(N_1, N_2) > 0 \) such that for any \( i = 1, 2 \) and any \( k = 1, \ldots, N_i \) a.s.

\[
\lim_{t \to \infty} \frac{x_k^{(i)}(t)}{t} = v
\]

Moreover, the distribution of the vector \( \{ x_k^{(i)}(t) - m(t), i = 1, 2; k = 1, \ldots, N_i \} \) tends to a stationary distribution.

The velocity \( v \) will be written down explicitely in terms of this distribution, it depends of course on \( \alpha_{ij} \) and \( v_i \). Note that both the velocity and the distribution do not depend on the initial coordinates.

**Initial desynchronization** Now we consider the case when \( N \to \infty \) but \( t \) is fixed. More exactly, we consider a sequence of pairs \( (N_1, N_2) \) such that \( N_1, N_2 \to \infty \) so that \( \frac{N_1}{N_2} \to c_i \), where \( c_1 + c_2 = 1, c_i > 0 \). It is convenient here to consider positive measures or generalized functions
\[
m^{(N_i)}(t, x) = \frac{1}{N_i} \sum_k \delta(x - x_k^{(i)}(t)), \quad x \in \mathbb{R}_+
\]
defined by the coordinates of \( N_i \) particles of type \( i \) at time \( t \). We assume that at time \( t = 0 \) for any bounded \( C^1 \)-functions \( \phi_i(x) \) on \( \mathbb{R} \) the sequence \( (m_i^{(N_i)}(0, \cdot), \phi_i) \) converges to some number.

**Theorem 2** Then for any \( t \) there are weak deterministic limits

\[
\lim_{N \to \infty} \frac{1}{N} m_i^{(N_i)}(t, x) = m_i(t, x)
\]

where \( m_i(t, x) \) satisfy the following equations

\[
\frac{\partial m_1}{\partial t} + v_1 \frac{\partial m_1}{\partial x} = \alpha_{12} (m_2 - m_1) \\
\frac{\partial m_2}{\partial t} + v_2 \frac{\partial m_2}{\partial x} = \alpha_{21} (m_1 - m_2)
\]

(1) (2)

Now we want to study the asymptotic behaviour of \( m_i(x, t) \) for \( t \to \infty \). Denote

\[
a_i(t) = \int x m_i(x, t)dx, \quad d_i(t) = \int (x - a_i(t))^2 m_i(t, x) \, dx
\]

**Theorem 3** There exist constants \( v, d > 0 \) such that as \( t \to \infty \)

\[
a_i(t) = vt + a_{i0} + o(1) \\
d_i(t) = dt + d_{i0} + o(1)
\]

for some constants \( a_{i0}, d_{i0} \). Moreover,

\[
\Delta_i(x, t) = \frac{m_i(x, t) - a_i(t)}{\sqrt{d_i(t)}}
\]

tends to \( \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) \) pointwise as \( t \to \infty \).

**Critical point and uniform estimates** Here we assume that \( N_1 = [c_1 N] \), \( N_2 = [c_2 N] \) for some \( c_i > 0, c_1 + c_2 = 1 \). Introduce the empirical means (mass centres) for types 1 and 2

\[
\bar{x}^{(i)}(t) = \frac{1}{N_i} \sum_{k=1}^{N_i} x_k^{(i)}(t),
\]

the empirical variances

\[
S_i^2(t) = \frac{1}{N_i} \sum_{k=1}^{N_i} \left( x_k^{(i)}(t) - \bar{x}^{(i)}(t) \right)^2
\]

and their means

\[
\mu_i(t) = \mathbb{E}x^{(i)}(t), \quad l_{12}(t) = \mu_1(t) - \mu_2(t), \quad R_i(t) = \mathbb{E}S_i^2(t)
\]

The following asymptotic results hold for any sequence of pairs \( (N, t) \) with \( N \to \infty \) and \( t = t(N) \to \infty \).
Theorem 4 We have the following asymptotical results as $t \to \infty$:

$$l_{12}(t) \to \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}}, \quad \frac{\mu_i(t)}{t} \to \frac{\alpha_{12} v_2 + \alpha_{21} v_1}{\alpha_{12} + \alpha_{21}}$$

Assume now that $N_i = c_i N$, where $c_i > 0, c_1 + c_2 = 1$.

Theorem 5 There are the following three regions of asymptotic behaviour, uniform in $t(N)$ for sufficiently large $N$:

- if $\frac{t(N)}{N} \to 0$, then $R_i(t(N)) \sim h \varphi_2 t(N)$,
- if $t = t(N) = sN$ for some $s > 0$, then $R_i(t(N)) \sim h (1 - e^{-\varphi_2 s}) N$,
- if $\frac{t(N)}{N} \to \infty$, then $R_i(t(N)) \sim h N$,

where the constant $\varphi_2 > 0$ can be explicitly calculated and

$$h = \frac{2 \alpha_{12} \alpha_{21} (v_1 - v_2)^2}{\varphi_2 (\alpha_{12} + \alpha_{21})^3}.$$ 

The proofs of Theorems 4 and 5 are given in [13, 14].

3 Limit $t \to \infty$

In this section we will prove Theorem [1].

Two particles. It is useful to consider first the case when $N_1 = N_2 = 1$. Thus consider the process $(x^{(1)}(t), x^{(2)}(t))$. We will prove that there exist deterministic limits

$$\lim_{t \to \infty} \frac{x^{(i)}(t)}{t} = v$$

for $i = 1, 2$ and some $v > 0$, moreover the distribution of the random variable $\rho(t) = x^{(2)}(t) - x^{(1)}(t)$ tends to some distribution on $\mathbf{R}_+$.

We can assume that $v_1 = 0, v_2 > 0$. The Markov chain $\rho(t) = x^{(2)}(t) - x^{(1)}(t)$ on $\mathbf{R}_+$ satisfies the Doeblin condition, that is from any $x \in \mathbf{R}_+$ there is a jump rate to 0, bounded away from zero, here it equals $\alpha_{12} + \alpha_{21}$. It follows that $\rho(t)$ is ergodic. Then as $t \to \infty$ there exists the limiting (invariant) distribution $F(x)$ for $\rho(t)$. Let

$$t_1 < t_2 < \cdots$$

time moments when $x^{(1)}(t) = x^{(2)}(t)$. It is clear that $t_k - t_{k-1}$ are independent random variables, exponentially distributed with parameter $\alpha_{12} + \alpha_{21}$. It follows that $F(x)$ is exponential with the density

$$p(x) = \lambda \exp(-\lambda x), \quad \lambda = \frac{\alpha_{12} + \alpha_{21}}{v_2 - v_1}$$
Thus, if the limits $\lim_{t \to \infty} \frac{x_i(t)}{t}$ exist, then they are equal. Let us prove that they exist and
\[
v = v_1 + \alpha_{12} \int x p(x) dx
\]
(3)
In fact, the particle 1 moves with constant speed $v_1$ and performs on the time interval $[0,T]$ independent exponentially distributed jumps in the positive direction. As $T \to \infty$, the number of these jumps asymptotically equals $\alpha_{12} T$, and the mean jump asymptotically is $\int x p(x) dx$.

Similarly one can get
\[
v = v_2 - \alpha_{21} \int x p(x) dx
\]
(4)
From this and (3) we have
\[
v = \frac{\alpha_{21} v_1 + \alpha_{12} v_2}{\alpha_{21} + \alpha_{12}}
\]

General case. Let us prove first the second statement of the theorem. We can put $v_1 = 0$ and change the coordinate system putting $m(t) = 0$. Consider a configuration of particles at time $t$. Denote the particle, which has coordinate $m(t) = 0$ at time $t$, as particle 0. Let $p(t+2)$ be the probability that at time $t+2$ each particle will be inside the interval $[0,2v_2]$. This probability can be (very roughly) estimated from below as
\[
p(t + 2) \geq \min(p_{01}p_{22}p_1, p_{02}p_3p_4)
\]
To prove this consider first the case when the particle 0 has type 1. Under this condition $p(t+2)$ can be estimated from below as $p_{01}p_{22}p_1$, where $p_{01}$ is the probability that particle 0 does not do any jumps in the time interval $(t,t+2)$, $p_2$ is the probability that each particle of type 2 jumps at least once to the particle 0 in the time interval $(t,t+1)$ and does not do any more jumps in the time interval $(t,t+2)$, $p_1$ is the probability that each particle of type 1 jumps to some particle of type 2 in the time interval $(t+1,t+2)$. Similarly, under the condition that the particle 0 has type 2, $p(t+2)$ can be estimated from below as $p_{02}p_3p_4$, where $p_{02}$ is the probability that the particle 0 does not do any jumps in the time interval $(t,t+2)$, $p_3$ is the probability that each particle of type 1 jumps at least once to the particle 0 in the time interval $(t,t+1)$ and does not do any more jumps in the time interval $(t,t+2)$, $p_4$ is the probability that each particle of type 2 jumps to some particle of type 1 in the time interval $(t+1,t+2)$.

This means that the Markov chain $L = \left\{ x_k^{(1)}(t) - m(t), i = 1, 2; k = 1, ..., N_i \right\}$ satisfies the Doeblin condition. Then it is ergodic and has some stationary distribution. We will write now formula for $v$, assuming however that $\alpha_{ii} = 0$. For this we need some marginals of this stationary distribution.

Let $A_i(t)$ be the event that at time $t$ at the point $m(t)$ there is a particle of type $i$, and $q_i = \lim_{t \to \infty} P(A_i(t))$ be the stationary (limiting) probability of $A_i$. Let $p_i(y)$ be the stationary conditional (under the condition $A_i$) probability density of the distance from $m$ to the nearest particle. In the time interval $[T,T+dt]$ the particle in $m(t)$ moves with the speed $v_i$, and moreover can make one jump. This gives, for example under the condition $A_1$, constant movement $v_1 dt$ of $m$, and the jump of $m$ to the nearest point with rate $\alpha_{12} dt$. Thus as $T \to \infty$ we have
\[
E(m(T+dt) | m(T)) - m(T) = q_1 \left( v_1 + \alpha_{12} \int y p_1(y) dy \right) dt +
\]
\[ q_2 \left( v_2 + \alpha_{21} \int y p_2(y) dy \right) dt + o(1) \]

and then
\[ v = q_1 \left( v_1 + \alpha_{12} \int y p_1(y) dy \right) + q_2 \left( v_2 + \alpha_{21} \int y p_2(y) dy \right) \]

**About Doeblin chains.** In the standard theory of Doeblin chains, see [3], it is assumed that transition probabilities are absolutely continuous with respect to some positive measure \( \mu \) on the state space.

If at time 0 all \( x_k^{(i)} \) are different, then for any \( t \) it is true that all \( x_k^{(i)} \) are different a.s. Thus transition probabilities (for example, for the embedded chain at times 0, 1, 2, \ldots) are absolutely continuous with respect to Lebesgue measure on \((R_+^{N_1 - 1} \times R_+^{N_2}) \cup (R_+^{N_1} \times R_+^{N_2 - 1})\).

If at time 0 some coordinates coincide, then a.s. in finite time \( \tau \) they become all different.

4 Limit \( N \to \infty \)

It is very intuitive to introduce the following continuous model. Let \( m_i(0, x) \), \( x \in \mathbb{R} \), \( i = 1, 2 \), be positive smooth functions, \( M_i = \int m_i(0, x) dx = 1 \). We call them continuous mass distributions of type \( i \) at time \( t = 0 \). The dynamics of the masses is deterministic — during time \( dt \) from each element \( dm_{1} \) of the mass the part \( \alpha_{12} dt dm_{1} \) goes out and distributes correspondingly to the mass \( m_2(x) \), namely it becomes the mass distribution with density \( m_2(x) \alpha_{12} dt dm_{1} \), and vice-versa, interchanging 1 and 2. Moreover each mass element moves with velocities \( v_1 \) and \( v_2 \) correspondingly. From this we easily get linear equations (1)–(2) for mass distribution \( m_i(t, x) \) at time \( t \) with the initial conditions
\[ m_i(0, x) = f_i(x) \]

Now we will prove convergence of \( N \) particle model to the continuous model.

4.1 Convergence: the martingale problem

Here we prove Theorem [2]. We consider continuous time Markov process
\[ \xi_{N_1, N_2}(t) = (x_1^{(1)}(t), \ldots, x_{N_1}^{(1)}(t); x_1^{(2)}(t), \ldots, x_{N_2}^{(2)}(t)) \tag{5} \]

with the state space \( \mathbb{R}^{N_1+N_2} \). Its generator
\[
(L_{N_1, N_2} f) \left( x^{(1)}; x^{(2)} \right) = \sum_{i=1}^{N_1} \frac{\partial}{\partial x_i^{(1)}} f \left( x^{(1)}; x^{(2)} \right) + \sum_{j=1}^{N_2} \frac{\partial}{\partial x_j^{(2)}} f \left( x^{(1)}; x^{(2)} \right) + \\
+ \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{\alpha_{12}}{N_2} \left[ f \left( x^{(1)}; x^{(2)} \right)_{i \to j} - f \left( x^{(1)}; x^{(2)} \right) \right] + \\
+ \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \frac{\alpha_{21}}{N_1} \left[ f \left( x^{(1)}; x^{(2)} \right)_{i \to j} - f \left( x^{(1)}; x^{(2)} \right) \right],
\]
where the following notation is used

\[
(x^{(1)}; x^{(2)}) = (x_1^{(1)}, \ldots, x_{N_1}^{(1)}; x_1^{(2)}, \ldots, x_{N_2}^{(2)}),
\]

\[
(x^{(1)}; x^{(2)})_{i \to j} = (x_1^{(1)}, \ldots, x_{i-1}^{(2)}, x_j^{(1)}; x_i^{(1)}, \ldots, x_{N_1}^{(1)}; x_1^{(2)}, \ldots, x_{N_2}^{(2)}),
\]

\[
(x^{(1)}; x^{(2)})_{i \to j} = (x_1^{(1)}, \ldots, x_{i-1}^{(2)}, x_{i+1}^{(1)}, \ldots, x_{N_1}^{(1)}; x_1^{(2)}, \ldots, x_{N_2}^{(2)}),
\]

is defined on bounded \(C^1\)-functions.

We will consider the limiting behaviour of this process when \(t = \text{const}, N_1, N_2 \to \infty\). It is not convenient to deal with the sequence \(\xi_{N_1,N_2}(t)\) of processes because the dimension of the state space changes with \(N_1, N_2\).

Denote

\[
M_{N_1,N_2}(t) = \left( \frac{1}{N_1} \sum_{i=1}^{N_1} \delta(\cdot - x_i^{(1)}(t)), \frac{1}{N_2} \sum_{j=1}^{N_2} \delta(\cdot - x_j^{(2)}(t)) \right),
\]

where \(\delta(x), x \in \mathbb{R}\), is the \(\delta\)-function. One can see that the generalized functions

\[
\frac{1}{N_1} \sum_{i=1}^{N_1} \delta(\cdot - x_i^{(1)}(t)), \quad \frac{1}{N_2} \sum_{j=1}^{N_2} \delta(\cdot - x_j^{(2)}(t))
\]

represent empirical "densities" or masses of (type 1 and 2 correspondingly) particles at time \(t\). Thus, if \(\phi(x) = (\phi_1(x), \phi_2(x))\), where \(\phi_i \in S'(\mathbb{R})\), then for fixed particle positions \(x_1^{(1)}(t), \ldots, x_{N_1}^{(1)}(t)\) and \(x_1^{(2)}(t), \ldots, x_{N_2}^{(2)}(t)\) the vector function \(M_{N_1,N_2}(t)\) is a linear functional on the vector test functions \(\phi\), that is

\[
\langle M_{N_1,N_2}(t), \phi \rangle = \frac{1}{N_1} \sum_{i=1}^{N_1} \phi_1(x_i^{(1)}(t)) + \frac{1}{N_2} \sum_{j=1}^{N_2} \phi_2(x_j^{(2)}(t)).
\]

Fix some \(T > 0\). Then \((M_{N_1,N_2}(t), 0 \leq t \leq T)\) can be considered as a Markov process taking its values in the space of tempered distributions \(S'(\mathbb{R}) \times S'(\mathbb{R})\). In the sequel we consider \(S'(\mathbb{R}) \times S'(\mathbb{R})\) as a topological space equipped with the strong topology (see Subsection 5.2). Without loss of generality one can assume that the trajectories of the process \(M_{N_1,N_2}(t)\) are right continuous functions with left limits. So it is natural to consider the Skorohod space \(\Pi^T = D([0,T], S'(\mathbb{R}) \times S'(\mathbb{R}))\) of functions on \([0,T]\) with values in \(S'(\mathbb{R}) \times S'(\mathbb{R})\) as a coordinate space of the process \(M_{N_1,N_2}(t)\). Subsection 5.2 explains how to introduce topology on this space. Let \(\mathcal{B}(\Pi^T)\) be the corresponding Borel \(\sigma\)-algebra. Denote \(P_{N_1,N_2}^T\) the probability measure on \((\Pi^T, \mathcal{B}(\Pi^T))\), induced by the process \((M_{N_1,N_2}(t), 0 \leq t \leq T)\).

Our assumption for the theorem is that for any test function \(\phi(x)\) the sequence \(\langle M_{N_1,N_2}(0), \phi \rangle\) weakly converges as \(N_1, N_2 \to \infty\).

We want to prove that as \(N_1, N_2 \to \infty\) the sequence of probability distributions \(P_{N_1,N_2}^T\) has a weak limit, and this limit is a one-point measure, that is the only trajectory \((m_1(t), m_2(t))\), \(0 \leq t \leq T\), which is the classical solution of the system \((1)-(2)\). We split a proof of this result into the next two propositions.

**Proposition 6** The family of probability distributions \(\{P_{N_1,N_2}^T\}_{N_1,N_2} \) on \((\Pi^T, \mathcal{B}(\Pi^T))\) is tight.

**Proposition 7** Limit points of the family of distributions \(P_{N_1,N_2}^T\) are concentrated on the weak solutions of the system \((1)-(2)\).
4.1.1 Tightness

Before proving Proposition 10 we start with some preliminary lemmas. We want to prove that the family of distributions \( P_{N_1,N_2}^T \) of the random process \( (M_{N_1,N_2}(t), 0 \leq t \leq T) \), with values in the space of generalized functions, is tight. By the theorem 4.1 of [9] (see also Subsection 5.2), it is sufficient to prove that for any test function \( \psi = (\psi_1(x), \psi_2(x)) \) the family of random processes \( (M_{N_1,N_2}(t), \psi), 0 \leq t \leq T \), with values in \( \mathbb{R}^1 \), is tight. This will be done in the Proposition 10 below.

Fix some test function \( \psi = (\psi_1(x), \psi_2(x)) \) and consider the random process

\[
F_{\psi,N_1,N_2} (x^{(1)}(t); x^{(2)}(t)) = (M_{N_1,N_2}(t), \psi) = \frac{1}{N_1} \sum_{i=1}^{N_1} \psi_1(x_i^{(1)}(t)) + \frac{1}{N_2} \sum_{j=1}^{N_2} \psi_2(x_j^{(2)}(t))
\]

This is a function of the Markov process \( \xi_{N_1,N_2}(t) \), thus (see [6, Lemma 5.1, p. 330], for example) the following two processes are martingales:

\[
W_{\psi,N_1,N_2}(t) = F_{\psi,N_1,N_2} (x^{(1)}(t); x^{(2)}(t)) - F_{\psi,N_1,N_2} (x^{(1)}(0); x^{(2)}(0)) - \int_0^t L_{N_1,N_2} F_{\psi,N_1,N_2} (x^{(1)}(s); x^{(2)}(s)) \, ds \tag{6}
\]

\[
V_{\psi,N_1,N_2}(t) = (W_{\psi,N_1,N_2}(t))^2 - \int_0^t L_{N_1,N_2} F_{\psi,N_1,N_2}^2 (x^{(1)}(s); x^{(2)}(s)) \, ds + 2 \int_0^t F_{\psi,N_1,N_2} (x^{(1)}(s); x^{(2)}(s)) L_{N_1,N_2} F_{\psi,N_1,N_2} (x^{(1)}(s); x^{(2)}(s)) \, ds.
\]

For shortness we will write \( F(x^{(1)}; x^{(2)}) \) instead of \( F_{\psi,N_1,N_2}(x^{(1)}; x^{(2)}) \).

**Lemma 8** The following estimates hold:

i) \( |L_{N_1,N_2} F (x^{(1)}; x^{(2)})| \leq C_1(\psi, \psi_1, \psi_2, \alpha_{12}, \alpha_{21}) \) uniformly in \( N_1, N_2 \) and \( (x^{(1)}; x^{(2)}) \);

ii) uniformly in \( x^{(1)}, x^{(2)} \)

\[
|L_{N_1,N_2} F^2 (x^{(1)}; x^{(2)}) - F(x^{(1)}; x^{(2)}) L_{N_1,N_2} F(x^{(1)}; x^{(2)})| \leq \frac{C_{12}(\alpha_{12}, \psi_1)}{N_1} + \frac{C_{21}(\alpha_{21}, \psi_2)}{N_2}. \tag{7}
\]

**Proof of the lemma.** Note that

\[
F \left( (x^{(1)}; x^{(2)}) \right) - F \left( x^{(1)}; x^{(2)} \right) = \frac{1}{N_1} \left( \psi_1 \left( x_j^{(2)} \right) - \psi_1 \left( x_i^{(1)} \right) \right),
\]

\[
F \left( (x^{(1)}; x^{(2)}) \right) - F \left( x^{(1)}; x^{(2)} \right) = \frac{1}{N_2} \left( \psi_2 \left( x_i^{(1)} \right) - \psi_2 \left( x_j^{(2)} \right) \right).
\]

Thus

\[
L_{N_1,N_2} F (x^{(1)}; x^{(2)}) = \frac{\psi_1}{N_1} \sum_{i=1}^{N_1} \psi_1 \left( x_i^{(1)} \right) + \frac{\psi_2}{N_2} \sum_{j=1}^{N_2} \psi_2 \left( x_j^{(2)} \right) + \]

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Then
\[
\left| L_{N_1,N_2} F \left( x^{(1)}; x^{(2)} \right) \right| \leq |v_1| \| \psi'_1 \|_C + |v_2| \| \psi'_2 \|_C + 2 \alpha_{12} \| \psi_1 \|_C + 2 \alpha_{21} \| \psi_2 \|_C
\]

and the assertion i) of the lemma is proved. To prove assertion ii) it is convenient to represent 
\( L_{N_1,N_2} = L_{N_1,N_2}^0 + L_{N_1,N_2}^1 \) as the sum of "differential" \( L_{N_1,N_2}^0 \) and "jump" \( L_{N_1,N_2}^1 \) parts.

It is easy to see that
\[
L_{N_1,N_2}^0 F^2(x^{(1)}; x^{(2)}) - 2F(x^{(1)}; x^{(2)}) L_{N_1,N_2}^0 F(x^{(1)}; x^{(2)}) = 0.
\]

Let us prove that uniformly in \( x^{(1)}, x^{(2)} \)
\[
\left| L_{N_1,N_2} F^2 \left( x^{(1)}; x^{(2)} \right) - F(x^{(1)}; x^{(2)}) L_{N_1,N_2} F(x^{(1)}; x^{(2)}) \right| \leq \frac{4 \alpha_{12} \| \psi_1 \|_{C}^2}{N_1} + \frac{4 \alpha_{21} \| \psi_2 \|_{C}^2}{N_2}.
\]

In fact
\[
F^2 \left( \left( x^{(1)}; x^{(2)} \right) \right)_{i \rightarrow j} - F^2 \left( x^{(1)}; x^{(2)} \right) = \left( F \left( \left( x^{(1)}; x^{(2)} \right) \right)_{i \rightarrow j} - F \left( x^{(1)}; x^{(2)} \right) \right) \times
\]
\[
\times \left( 2 F \left( x^{(1)}; x^{(2)} \right) + \frac{1}{N_1} \left( \psi_1 \left( x^{(2)} \right) - \psi_1 \left( x^{(1)} \right) \right) \right)
\]
\[
= 2 F \left( x^{(1)}; x^{(2)} \right) \left[ F \left( \left( x^{(1)}; x^{(2)} \right) \right)_{i \rightarrow j} - F \left( x^{(1)}; x^{(2)} \right) \right] +
\]
\[
+ \left[ \frac{1}{N_1} \left( \psi_1 \left( x^{(2)} \right) - \psi_1 \left( x^{(1)} \right) \right) \right]^2.
\]

and similarly for expressions with \( \left( x^{(1)}; x^{(2)} \right)_{i \rightarrow j} \). Thus
\[
L_{N_1,N_2}^1 F^2(x^{(1)}; x^{(2)}) = 2F \left( x^{(1)}; x^{(2)} \right) \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{\alpha_{12}}{N_2} \left[ F \left( \left( x^{(1)}; x^{(2)} \right) \right)_{i \rightarrow j} - F \left( x^{(1)}; x^{(2)} \right) \right]
\]
\[
+ \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{\alpha_{12}}{N_2} \left[ \frac{1}{N_1} \left( \psi_1 \left( x^{(2)} \right) - \psi_1 \left( x^{(1)} \right) \right) \right]^2
\]
\[
+ 2F \left( x^{(1)}; x^{(2)} \right) \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \frac{\alpha_{21}}{N_1} \left[ F \left( \left( x^{(1)}; x^{(2)} \right) \right)_{i \rightarrow j} - F \left( x^{(1)}; x^{(2)} \right) \right]
\]
\[
+ \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \frac{\alpha_{21}}{N_1} \left[ \frac{1}{N_2} \left( \psi_2 \left( x^{(1)} \right) - \psi_2 \left( x^{(2)} \right) \right) \right]^2
\]
\[
\begin{align*}
&= 2FL_{N_1,N_2}^1 F + \frac{\alpha_{12}}{N_1} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{1}{N_2N_1} \left( \psi_1 \left(x_j^{(2)}\right) - \psi_1 \left(x_i^{(1)}\right)\right)^2 - \\
&+ \frac{\alpha_{21}}{N_2} \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \frac{1}{N_1N_2} \left( \psi_2 \left(x_i^{(1)}\right) - \psi_2 \left(x_j^{(2)}\right)\right)^2.
\end{align*}
\]

the estimate (9) follows from this. Lemma is proved.

**Corollary 9**

\[
\sup_{t \leq T} \mathbb{E} \left( W_{\psi,N_1,N_2}(t) \right)^2 \to 0, \quad N_1, N_2 \to \infty.
\]

**Proof.** As \(V_{\psi,N_1,N_2}\) is a martingale with mean zero, it is sufficient to prove that the expectation of

\[
\int_0^t \left[ L_{N_1,N_2} F^2 \left(x^{(1)}(s);x^{(2)}(s)\right) - 2F \left(x^{(1)}(s);x^{(2)}(s)\right) L_{N_1,N_2} F \left(x^{(1)}(s);x^{(2)}(s)\right) \right] ds
\]

tends to zero. This follows from the estimate (7) of the lemma.

**Proposition 10** The sequence of distributions of real valued random processes

\[
F_{\psi,N_1,N_2} \left(x^{(1)}(t);x^{(2)}(t)\right), \quad t \in [0,T],
\]

is tight.

**Proof of Proposition 10** Remind that the following representation holds

\[
F_{\psi,N_1,N_2} \left(x^{(1)}(t);x^{(2)}(t)\right) = F_{\psi,N_1,N_2} \left(x^{(1)}(0);x^{(2)}(0)\right) + W_{\psi,N_1,N_2}(t) + \\
+ \int_0^t L_{N_1,N_2} F_{\psi,N_1,N_2} \left(x^{(1)}(s);x^{(2)}(s)\right) ds
\]

Note that our initial assumption is that the sequence \(F_{\psi,N_1,N_2} \left(x^{(1)}(0);x^{(2)}(0)\right)\) weakly converges as \(N_1,N_2 \to \infty\).

Prove now that the sequence

\[
\left\{ \eta_{N_1,N_2}(t) = \int_0^t L_{N_1,N_2} F \left(x^{(1)}(s);x^{(2)}(s)\right) ds, \quad t \in [0,T] \right\}_{N_1,N_2}
\]

is tight. We use subsection 6.1 of the Appendix. By assertion i) of the lemma

\[
\left| \int_0^t L_{N_1,N_2} F \left(x^{(1)}(s);x^{(2)}(s)\right) ds \right| \leq C_1(\psi,v_1,v_2,\alpha_{12},\alpha_{21}) \cdot T,
\]

thus, the condition 1) of the Appendix holds. The condition 2) also holds, as one can prove that

\[
w'(\eta_{N_1,N_2},\gamma) \leq 2\gamma \cdot C_1(\psi,v_1,v_2,\alpha_{12},\alpha_{21}).
\]

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Estimate, uniform in \( n \equiv N \), equality for submartingales with right continuous trajectories (see [3]), we have the following estimate, uniform in \( N_1, N_2 \),

\[
P \left( \sup_{t \leq T} |W_{\psi, N_1, N_2}(t)| > C \right) \leq \frac{\sup_{t \leq T} E(W_{\psi, N_1, N_2}(t))^2}{C^2}
\]

Then from the corollary [9] the condition 1) of Appendix holds. Thus

\[
P \left( |W_{\psi, N_1, N_2}(\tau + \theta) - W_{\psi, N_1, N_2}(\tau)| > \varepsilon \right) \leq \frac{E(W_{\psi, N_1, N_2}(\tau + \theta) - W_{\psi, N_1, N_2}(\tau))^2}{\varepsilon^2} = \frac{E \int_{\tau}^{\tau + \theta} V_{\psi, N_1, N_2}(s) ds}{\varepsilon^2} \leq \frac{\theta \cdot (C_{12}(\alpha_{12}, \psi_1)/N_1 + C_{21}(\alpha_{21}, \psi_2)/N_2)}{\varepsilon^2}
\]

Using this estimate one can check the sufficient condition of Aldous. Then Proposition [10] is proved.

This concludes also the proof of Proposition [6].

4.1.2 Weak solutions

Definition 11 We say that the pair of functions \( M(t) = (m_1(t, x), m_2(t, x)) \) is a weak solution of the system [7] - [2], if for any pair \( \phi_1(x), \phi_2(x) \in S(\mathbb{R}) \) the following identities hold

\[
\langle M(t), \phi \rangle = \langle M(0), \phi \rangle + \int_0^t \langle M(s), (v_1 \phi'_1 - \alpha_{12} \phi_1 + \alpha_{21} \phi_2, v_2 \phi'_2 - \alpha_{12} \phi_1 + \alpha_{21} \phi_2) \rangle ds,
\]

where \( \phi(x) = (\phi_1(x), \phi_2(x)) \), and the action of \( G(x) = (g_1(x), g_2(x)) \) on the test function \( \phi(x) \) can be written as

\[
\langle G, \phi \rangle = \int g_1(x) \phi_1(x) dx + \int g_2(x) \phi_2(x) dx
\]

Note that from the representation [6] and the identity [3] it follows that

\[
\langle M_{N_1, N_2}(t), \phi \rangle = W_{\psi, N_1, N_2}(t) + \sum_{i=0}^{N_2-1} \sum_{j=0}^{N_1-1} \int_{j/N_1}^{(j+1)/N_1} \langle M_{N_1, N_2}(s), (v_1 \phi'_1 - \alpha_{12} \phi_1 + \alpha_{21} \phi_2, v_2 \phi'_2 - \alpha_{12} \phi_1 + \alpha_{21} \phi_2) \rangle ds,
\]

Let \( h = h(t) \in \Pi^T = D([0, T], S'(\mathbb{R}) \times S'(\mathbb{R})) \). For fixed \( \phi \) define the functional

\[
J_{\phi, T}(h) = \sup_{t \leq T} \left| \langle h(t), \phi \rangle - \langle h(0), \phi \rangle - \int_0^t \langle h(s), (v_1 \phi'_1 - \alpha_{12} \phi_1 + \alpha_{21} \phi_2, v_2 \phi'_2 - \alpha_{12} \phi_1 + \alpha_{21} \phi_2) \rangle ds \right|.
\]

In particular,

\[
\sup_{t \leq T} |W_{\phi, N_1, N_2}(t)| = J_{\phi, T}(M_{N_1, N_2}).
\]

The rest of the proof is standard (see [3]) and consists of three steps.
Step 1. From the definition of topology on $\Pi T$ it follows that $J_{\phi,T}(\cdot): \Pi T \to \mathbb{R}_+$ is a continuous functional.

Step 2. Note that

$$\forall \varepsilon > 0 \; P \{ J_{\phi,T}(M_{N_1,N_2}) > \varepsilon \} \equiv P^T_{N_1,N_2} \{ h: J_{\phi,T}(h) > \varepsilon \} \to 0 \quad (N_1, N_2 \to \infty)$$

by Kolmogorov inequality and Corollary [9].

Step 3. As $J_{\phi,T}(\cdot)$ is continuous, then the set $\{ h: J_{\phi,T}(h) > 0 \}$ is open in $\Pi T$. It follows now that for any limiting point $P^T_\infty$ of the family $\{ P^T_{N_1,N_2} \}_{N_1,N_2}$ we have

$$P^T_\infty \{ h: J_{\phi,T}(h) > \varepsilon \} \leq \limsup_{N_1,N_2} P^T_{N_1,N_2} \{ h: J_{\phi,T}(h) > \varepsilon \}.$$

That is, for any $\varepsilon > 0$ we have $P^T_\infty \{ h: J_{\phi,T}(h) > \varepsilon \} = 0$. In other words, all limiting points $P^T_\infty$ of the family $\{ P^T_{N_1,N_2} \}_{N_1,N_2}$ have support on the set $\{ h: J_{\phi,T}(h) = 0 \}$, which consists of weak solutions of (1)-(2).

This completes proof of Proposition 7.

The problem of uniqueness of the weak solution of (1)-(2) is quite simple because the system (1)-(2) is linear. In the Subsection 4.2 we shall see that this system of first order differential equations has a unique classical solution which can be obtained in explicit way.

4.2 Time asymptotics for the continuous model

We prove here Theorem 8.

Define the means (mass centrum) $a_i(t) = \int x m_i(t,x) dx$ and variance (momentum of inertia) $d_i(t) = \int (x - a_i(t))^2 m_i(t,x) dx$.

From (1)-(2) we get the following equations for the means

$$\dot{a}_1 = v_1 + \alpha_{12} (a_2 - a_1)$$
$$\dot{a}_2 = v_2 + \alpha_{21} (a_1 - a_2)$$

It follows that equation for $a_2(t) - a_1(t)$ is closed and has the following solution

$$a_2(t) - a_1(t) = \frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}} \left( 1 - e^{-(\alpha_{12} + \alpha_{21})t} \right) + (a_2(0) - a_1(0)) e^{-(\alpha_{12} + \alpha_{21})t}.$$ 

Thus

$$a_2(t) - a_1(t) \to \frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}} \quad (t \to +\infty)$$

and similarly

$$\frac{d}{dt} a_i(t) \to \frac{\alpha_{21} v_1 + \alpha_{12} v_2}{\alpha_{12} + \alpha_{21}} \quad (t \to +\infty)$$

The equations for variances are

$$\dot{d}_1 = \alpha_{12} (d_2 - d_1) + \alpha_{12} (a_2(t) - a_1(t))^2$$
$$\dot{d}_2 = \alpha_{21} (d_1 - d_2) + \alpha_{21} (a_1(t) - a_2(t))^2$$
Or, equivalently
\[
\frac{d}{dt} (\alpha_{21}d_1 + \alpha_{12}d_2) = 2\alpha_{12}\alpha_{21} (a_2(t) - a_1(t))^2
\]
\[
\frac{d}{dt} (d_2 - d_1) = -(\alpha_{12} + \alpha_{21}) (d_2 - d_1) + (\alpha_{21} - \alpha_{12}) (a_2(t) - a_1(t))^2
\]

From this we get
\[
d_2(t) - d_1(t) \to \text{const} = \left(\frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}}\right)^2 \cdot \frac{\alpha_{21} - \alpha_{12}}{\alpha_{12} + \alpha_{21}}
\]
and
\[
\frac{d}{dt} (\alpha_{21}d_1 + \alpha_{12}d_2) \to 2\alpha_{12}\alpha_{21} \left(\frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}}\right)^2
\]

Thus the growth of variances is asymptotically linear. Moreover, both are asymptotically equal.

Now we come to the solution of the equations. Define the Fourier transforms
\[
m_i(x,t) = \int \exp(ixp)g_i(p,t)dp
\]
We get
\[
\frac{\partial g_1}{\partial t} + v_1ipg_1 = \alpha_{12}(g_2 - g_1)
\]
\[
\frac{\partial g_2}{\partial t} + v_2ipg_2 = \alpha_{21}(g_1 - g_2)
\]
with initial conditions \(m_i(0, x) = m_i(x), i = 1, 2\). We write this system in the vector form
\[
\frac{dg}{dt} = Ag
\]
where
\[
A = \begin{pmatrix}
-i(v_1p - \alpha_{12}) & \alpha_{12} \\
\alpha_{21} & -i(v_2p - \alpha_{21})
\end{pmatrix}
\]
For eigenvalues we have
\[
\lambda_\pm = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}
\]
where
\[
a = i(v_1 + v_2)p + \alpha_{12} + \alpha_{21}, \quad b = -v_1v_2p^2 + ip(v_1\alpha_{21} + v_2\alpha_{12})
\]
One can write the solution as
\[
g = C_+ \phi_+ \exp(t\lambda_+) + C_- \phi_- \exp(t\lambda_-)
\]
where \(\phi_\pm\) are eigenfunctions. Note that for small \(p\) there are two roots. One has \(\text{Re} \lambda_- < 0\), thus strongly decreasing term. Another is
\[
\lambda_+ = c_1p + c_2p^2 + O(p^3), c_2 \neq 0
\]
for small \( p \).

Let \( \xi_t \) be a random variable with density \( m(x, t) \), \( g(k) \) - its characteristic function. We are interested in \( \frac{1}{\sqrt{t}}(\xi_t - a) \), \( a = \mathbb{E}\xi_t \), its characteristic function is

\[
\exp(-ia\frac{k}{\sqrt{t}})g(\frac{k}{\sqrt{t}})
\]

Using (10) we get the result.

**Remark 12** One can see that there is no solution of the type

\[
m_i(t, x) = f_i(x - vt)
\]

as then \( f_i \) would be exponents.

**Remark 13** For the singular initial conditions, that is when \( x_{k}^{(i)}(0) = 0 \) for \( k = 1, \ldots, N_i; i = 1, 2 \), one can get the same asymptotic results.

5 Appendix

5.1 Probability measures on the Skorohod space: tightness

Let \( \{ (\xi^n_t, t \in [0, T]) \}_{n \in \mathbb{N}} \) be a sequence of real random processes which trajectories are right-continuous and admit left-hand limits for every \( 0 < t \leq T \). We will consider \( \xi^n_t \) as random elements with values in the Skorohod space \( D_T(\mathbb{R}) := D([0,T],\mathbb{R}^1) \) with the standard topology. Denote \( P^n_T \) the distribution of \( \xi^n_t \), defined on the measurable space \( (D_T(\mathbb{R}), B(D_T(\mathbb{R}))) \).

The following result can be found in [1].

**Theorem 14** The sequence of probability measures \( \{ P^n_T \}_{n \in \mathbb{N}} \) is tight iff the following two conditions hold:

1) for any \( \varepsilon > 0 \) there is \( C(\varepsilon) > 0 \) such that

\[
\sup_n P^n_T \left( \sup_{0 \leq t \leq T} |\xi^n_t| > C(\varepsilon) \right) \leq \varepsilon;
\]

2) for any \( \varepsilon > 0 \)

\[
\lim_{\gamma \to 0} \limsup_n P^n_T (\xi: w'(\xi; \gamma) > \varepsilon) = 0,
\]

where for any function \( f : [0,T] \to \mathbb{R} \) and any \( \gamma > 0 \) we define

\[
w'(f; \gamma) = \inf \max \sup_{t_1 \leq s < t < t_{i+1}} |f(t) - f(s)|,
\]

moreover the inf is over all partitions of the interval \( [0,T] \) such that

\[
0 = t_0 < t_1 < \cdots < t_r = T, \quad t_i - t_{i-1} > \gamma, \quad i = 1, \ldots, r.
\]

The following theorem is known as the sufficient condition of Aldous [6].

**Theorem 15** Condition 2) of the previous theorem follows from the following condition

\[
\forall \varepsilon > 0 \lim_{\gamma \to 0} \limsup_n P^n_T (|\xi_{t+\theta} - \xi_t| > \varepsilon) = 0,
\]

where \( \mathcal{R}_T \) is the set of Markov moments (stopping times) not exceeding \( T \).
5.2 Strong topology on the Skorohod space. Mitoma theorem

Remind that Schwartz space $S(\mathbb{R})$ is a Frechet space (complete locally convex space, the topology of which is generated by countable family of seminorms, that implies metrizability, see [11]). In the dual space $S'(\mathbb{R})$ of tempered distributions there are at least two ways to define topology (both not metrizable):

1) **weak topology** on $S'(\mathbb{R})$, where all functionals

$$\langle \cdot, \phi \rangle, \quad \phi \in S(\mathbb{R})$$

are continuous.

2) **strong topology** on $S'(\mathbb{R})$, which is generated by the set of seminorms

$$\left\{ \rho_A(M) = \sup_{\phi \in A} |\langle M, \phi \rangle| : \ A \subset S(\mathbb{R}) - \text{bounded} \right\}.$$

We shall consider $S'(\mathbb{R})$ as equipped with the strong topology. Details can be found in [11].

The problem of introducing the Skorohod topology on the space $D_T(S') := D([0, T], S'(\mathbb{R}))$ was studied in [9] and [12]. The topology on this space is defined as follows. Let $\{\rho_A\}$ be a family of seminorms, which generates strong topology in $S'(\mathbb{R})$. For each seminorm $\rho_A$ define a pseudometrics

$$d_A(y, z) = \inf_{\lambda \in A} \left\{ \sup_t |y_t - z_{\lambda(t)}| + \sup_{t \neq s} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \right\}, \quad y, z \in D_T(S'),$$

where the inf is over the set $A = \{\lambda = \lambda(t), \ t \in [0, T]\}$ of all strictly increasing maps of the interval $[0, T]$ into itself. Equipped with the topology of the projective limit for the family $\{d_A\}$ the set $D_T(S')$ becomes a completely regular topological space.

Let $B(D_T(S'))$ be the corresponding Borel $\sigma$-algebra. Let $\{P_n\}$ be a sequence of probability measures on $(D_T(S'), B(D_T(S'))).$ For each $\phi \in S(\mathbb{R})$ consider a map $I_\phi : y \in D_T(S') \to y(\phi) \in D_T(\mathbb{R}).$ The following result belongs to I. Mitoma [9].

**Theorem 16** Suppose that for any $\phi \in S(\mathbb{R})$ the sequence $\{P_nI^{-1}_\phi\}$ is tight in $D_T(\mathbb{R})$. Then the sequence $\{P_n\}$ itself is tight in $D_T(S')$.

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