Kaluza’s theory in generalized coordinates.

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Maxwell’s equations can be obtained in generalized coordinates by considering the electromagnetic field as an external agent. The work here presented shows how to obtain the electrodynamics for a charged particle in generalized coordinates eliminating the concept of external force. Based on Kaluza’s formalism, the one here presented extends the 5x5 metric into a 6x6 space-time giving enough room to include magnetic monopoles in a very natural way.

I. INTRODUCTION

Over seventy years ago, T. Kaluza developed a theory unifying electromagnetism and gravitation by working in a five dimensional manifold [1] [2]. In his work, influences of an electrical charge in space are treated as sources of curvature in a similar way as mass does in Einstein’s theory of general relativity. With this formalism, Maxwell’s equations can be obtained in cartesian coordinates in a rather simple way. In generalized coordinates, only four out of eight (three for each rotational and one for each divergence) Maxwell equations can be recovered correctly, those with sources. The problem resides in the asymmetrical way in which each pair of equations, with and without sources, are obtained if Kaluza’s formalism is applied. The equations with sources are obtained by means of the field equation, while the homogeneous ones have to be deduced from an identity which yields incorrect results unless the metric is cartesian. A more conventional method for obtaining Maxwell equations consists in using Bianchi’s identities [3]. Although it yields correct results using any metric, the electromagnetic field is treated as an external ”source” by including it in the stress tensor. Kaluza’s method, and consequently the one presented here, have the advantage of considering charge as a space curvature thus following the tenets of general relativity.

In this work we propose an alternative method for treating charge and mass as curvature sources. This method is not only consistent with Kaluza’s but also represents a generalization of it, by considering his 5x5 metric included in a larger one. A larger space also has enough room to establish a generalized theory in which magnetic monopoles can be introduced in a very natural way.

This paper is divided as follows. Section 2 contains a brief summary of Kaluza’s theory without specifying any metric, as presented in his original article. In section 3, through a simplified example, the cause for the incorrectness of the inhomogeneous equations is explored. Section 4 proposes an alternative to the theory making a symmetrical generalization of the metric used before, as a way of obtaining Maxwell equations using only Einstein’s field equation. Section 5 goes deeper into the interpretation of the 6x6 space-time and finally in section 6 we discuss the implications of this formalism.

II. KALUZA’S THEORY

Kaluza’s formalism generates Maxwell’s equations as well as the equations of motion for charged particles by working with a 5x5 metric defined as

\[
g_{\mu \nu} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} & A_1 \\ g_{21} & g_{22} & g_{23} & g_{24} & A_2 \\ g_{31} & g_{32} & g_{33} & g_{34} & A_3 \\ g_{41} & g_{42} & g_{43} & g_{44} & \phi \\ A_1 & A_2 & A_3 & \phi & g_{55} \end{bmatrix} \]  

(1)

The 4x4 metric consisting of \(g_{\mu \nu}\) elements with subscripts running from 1 to 4 has to be a solution of Einstein’s equations, while the fifth column and row contain the four-vector electromagnetic potential \(A_\nu\), the first three components being the ones of the usual potential vector and the fourth the scalar potential \(\phi\). The fifth element \(g_{55}\) is undefined in Kaluza’s article, but, as this extra dimension is considered as a spatial type one, \(g_{55}\) can be set equal to a constant taken to be equal to a one in the case of a Minkowski metric.
The position and velocity vectors to consider in this formalism are defined as follows

\[ x' = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ ct \\ x^5 \end{bmatrix} \]  

(2)

\[ v' = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ \frac{c}{m} \end{bmatrix} \]  

(3)

The fact that the time derivative of the fifth position coordinate is taken to be equal to the charge-mass ratio arises from comparing the equation of motion for a charged particle moving under Lorenz’s force

\[ \frac{d^2 x^\alpha}{dt^2} = \frac{q}{m} \left[ \varepsilon^\beta_{\gamma\gamma} \frac{\partial x^\alpha}{\partial t} B^\gamma + E^\gamma \right] \]  

(4)

with the one obtained by restricting the particle to move through a geodesic in this space namely,

\[ \frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \]  

(5)

Indeed Eq. (4) may be obtained from equation (5) only if \( v^5 = \frac{q}{m} \). An extra condition must be taken into account in order to recover electrodynamics within this framework. Such condition usually referred to as the cylindrical condition \[1\] namely,

\[ \frac{\partial}{\partial x^5} = 0 \]  

(6)

makes any derivative with respect to the fifth component equal to zero. The only justification for this restriction is that it leads to the correct Maxwell equations.

The structural similarity between curls and Christoffel symbols, \( \Gamma^\alpha_{\mu\nu} \), suggest a proportionality between the latter and the components of the electromagnetic field \[1\]. These symbols, taking \( g^{ij} = \delta^{ij} \) (latin indices run from 1 to 3) are computed as follows

\[ \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \left( \frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right) \]  

(7)

\[ \Gamma^5_{5\mu} = \Gamma^{5\mu}_5 = \frac{1}{2} \left( \frac{\partial g_{5\lambda}}{\partial x^\mu} - \frac{\partial g_{5\mu}}{\partial x^\lambda} \right) = F^\alpha_{\mu} \]  

(8)

where \( F^\alpha_{\mu} \), the elements of the field tensor, are defined as follows

\[ F^\alpha_{\mu} = \begin{bmatrix} 0 & B_z & -B_y & -\frac{1}{c} E_x \\ -B_z & 0 & B_x & -\frac{1}{c} E_y \\ B_y & -B_x & 0 & -\frac{1}{c} E_z \\ \frac{1}{c} E_x & \frac{1}{c} E_y & \frac{1}{c} E_z & 0 \end{bmatrix} \]  

(9)

In this scheme, Maxwell inhomogeneous equations may be obtained through Einstein’s field equation

\[ G^{\alpha\beta} = KT^{\alpha\beta} \]  

(10)

since the energy momentum tensor \( T^{\alpha\beta} \) has now a fifth column and row containing the electric current
\[
T_{\alpha 5} = T^{5\alpha} = \rho_0 v^\alpha v^5 = \begin{bmatrix}
\rho_0 v^1 \\
\rho_0 v^2 \\
\rho_0 v^3 \\
\rho_0 c^2/m^2
\end{bmatrix} = J^\alpha
\]

and \(K\) is the coupling constant. The homogeneous equations can be derived, as suggested by Kaluza, from the identity

\[
F_{\lambda \alpha, \beta} + F_{\alpha \beta, \lambda} + F_{\beta \lambda, \alpha} = 0
\]

which arises from applying the cylindrical condition (eq. 6), to the identity

\[
\left( \Gamma_{\alpha \beta \lambda} + \Gamma_{\beta \lambda \alpha} + \Gamma_{\lambda \alpha \beta} \right)_{,\mu} = \Gamma_{\mu \lambda}^{\alpha \beta} + \Gamma_{\mu \alpha}^{\beta \lambda} + \Gamma_{\mu \beta}^{\lambda \alpha}
\]

taking \(\mu = 5\).

In cartesian coordinates this mechanism yields the complete set of Maxwell equations. However, if one works in generalized coordinates, Eq. (13) does not correspond to the correct sourceless electromagnetic equations, a fact which will be shown in the next section. A strong objection can be primarily made to Eq. (13). Christoffel symbols by themselves are not tensors and, although some combinations of them are, Eq. (13) is not tensorial, in contrast with Eq. (10) which has tensors in both sides.

Summarizing, the theory has its weak point in the way sourceless equations are obtained. On the other hand, the mechanism by which inhomogeneous equations are derived, Einstein’s field equation, is irrefutable and far more elegant.

### III. FIVE DIMENSIONS AND GENERALIZED COORDINATES

In this section we give the necessary arguments to circumvent the objection raised in the preceding section regarding the derivation of Maxwell’s sourceless equations following Kaluza’s procedure. To do so and for the sake of simplicity, we shall take as an example a Minkowski space with spherical symmetry. The covariant and contravariant metric tensors are thus given by,

\[
g_{\mu \nu} = \begin{bmatrix}
1 & 0 & 0 & 0 & A_1 \\
0 & r^2 & 0 & 0 & r^2 A_2 \\
0 & 0 & r^2 \sin^2 \theta & 0 & r^2 \sin^2 \theta A_3 \\
0 & 0 & 0 & -1 & -\frac{1}{c^2} \phi \\
A_1 & r^2 A_2 & r^2 \sin^2 \theta A_3 & -\frac{1}{c^2} \phi & 1
\end{bmatrix}
\]

and

\[
g^{\mu \nu} = \begin{bmatrix}
1 & 0 & 0 & 0 & -A_1 \\
0 & \frac{1}{r^2} & 0 & 0 & -A_2 \\
0 & 0 & \frac{1}{r^2 \sin^2 \theta} & 0 & -A_3 \\
0 & 0 & 0 & -1 & -\frac{1}{c^2} \phi \\
-A_1 & -A_2 & -A_3 & -\frac{1}{c^2} \phi & 1
\end{bmatrix}
\]

In this metric which is similar to the one used in Kaluza’s article, the two metric tensors are inverse discarding quadratic terms. This means that (14) is the inverse of (13) only if second order terms are eliminated. This approximation seems to be correct since it was also used in Kaluza’s article with a cartesian metric. Geometric coefficients must be added to the components of the electromagnetic potential which simply implies working in physical coordinates in order to obtain geometric terms and coefficients in the equations. If \(A_\nu^{\text{ten}}\) are tensorial components of a vector, its physical components are computed as follows

\[
A_\nu^{\text{phys}} = \sqrt{g_{\nu \kappa}} A_\nu^{\text{ten}}
\]

It must be pointed out that, with these metric coefficients \(g^{ij}\), permutations of indices in the Christoffel symbols of Eq. (13) do not only yield a change in sign as with a cartesian metric in which \(g^{ij} = \delta^{ij}\) with \(i\) and \(j\) running from 1 to 3, but also metric coefficients relate these permutations. Moreover, the equation
\[ \Gamma^\alpha_{\beta\gamma} = - \frac{g^{\alpha\delta}}{g^{\beta\delta}} \Gamma^\delta_{\beta\gamma} \]  

(17)

gives a relation between permutations of indices in Christoffel symbols which makes equation (13) dependent on the value given to the indices unless the metric is cartesian, resulting in more than four incorrect equations since each permutation of indices leads to an equation with different coefficients. To clarify these statements let us take, for example, the third component of the following equation

\[ \varepsilon^\nu_{\beta\gamma} \left( \frac{\partial E_\beta}{\partial x^\nu} - \frac{\partial E_\gamma}{\partial x^\nu} \right) = - \frac{1}{c} \frac{\partial B^\nu}{\partial t} \]  

(18)

In the space we are working in, both sides of this equation in terms of electromagnetic potentials should read

\[ -2 \frac{\partial A_2}{r \sin \theta} \frac{\partial\phi}{\partial t} + \frac{1}{\sin \theta} \frac{\partial^2 A_2}{\partial t \partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial^2 A_1}{\partial \theta \partial \phi} \]  

(19)

while Eq. (13), taking \( \alpha = 4, \beta = 2, \lambda = 1 \) and \( \mu = 5 \) turns out to be as follows

\[ -r \frac{\partial A_2}{\partial t} - \frac{1}{cr^3} \frac{\partial \phi}{\partial \theta} + \frac{\partial^2 A_1}{\partial \theta \partial \phi} + \frac{1}{2} \left( 1 - r^2 \right) \frac{\partial^2 A_2}{\partial r \partial \phi} + \frac{1}{2c} \left( 1 + \frac{1}{r^2} \right) \frac{\partial^2 \phi}{\partial r \partial \phi} = 0 \]  

(20)

which can be written as

\[ \frac{1}{2r} \frac{\partial^2 A_2}{\partial r \partial t} - \frac{1}{cr^2} \frac{\partial \phi}{\partial \theta} + \frac{1}{2cr} \left( 1 + \frac{1}{r^2} \right) \frac{\partial^2 \phi}{\partial r \partial \phi} = \frac{\partial A_1}{\partial t} + \frac{r}{2} \frac{\partial A_2}{\partial r} - \frac{1}{2r} \frac{\partial^2 A_1}{\partial \theta \partial \phi} \]  

(21)

Eq. (21) is obviously wrong. In Eq. (13), no derivatives of the scalar potential should appear since its right hand side is a time derivative of \( B \) which is only in terms of the vector potential. The left hand side contains the rotational of the gradient of \( \phi \) which is equal to zero. Similar results are obtained by changing the indices for the rest of the equations. These equations are derived in Appendix A.

On the other hand, the inhomogeneous Maxwell equations are correctly recovered using Einstein’s field equation as was mentioned. This induces us to reformulate the theory in order to be able to obtain electromagnetic equations, within Kaluza’s formalism, based only in Einstein’s field equation without having to invoke the cylindrical condition to work on the geometrical identity given by Eq. (13).

**IV. KALUZA’S FORMALISM IN 6 DIMENSIONS.**

The alternative here presented for Kaluza’s formalism is based on a 6x6 metric. This space-time can thus be considered as an extension of Kaluza’s case. To proceed we define the following metric tensors with components accounting for spherical symmetry namely,

\[ g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 & A_1 & Z_1 \\ 0 & r^2 & 0 & 0 & r^2 A_2 & r^2 Z_2 \\ 0 & 0 & r^2 \sin^2 \theta & 0 & r^2 \sin^2 \theta A_3 & r^2 \sin^2 \theta Z_3 \\ 0 & 0 & 0 & -1 & -\frac{1}{c} \phi & -\frac{1}{c} \eta \\ A_1 & r^2 A_2 & r^2 \sin^2 \theta A_3 & \frac{1}{c} \phi & 1 & g_{56} \\ Z_1 & r^2 Z_2 & r^2 \sin^2 \theta Z_3 & -\frac{1}{c} \eta & g_{65} & 1 \end{bmatrix} \]  

(22)

\[ g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 & -A_1 & -Z_1 \\ 0 & \frac{1}{r} & 0 & 0 & -A_2 & -Z_2 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} & 0 & -A_3 & -Z_3 \\ 0 & 0 & 0 & -1 & \frac{1}{c} \phi & \frac{1}{c} \eta \\ -A_1 & -A_2 & -A_3 & \frac{1}{c} \phi & 1 & g_{56} \\ -Z_1 & -Z_2 & -Z_3 & \frac{1}{c} \eta & g_{65} & 1 \end{bmatrix} \]  

(23)

The structure of these tensors is proposed to maintain symmetry between the fifth and sixth dimensions. The quantities \( Z_n \) and \( \eta \) are left unspecified for the time being but will be interpreted later when the role that each one plays in the equations becomes clear. Also \( g_{56} = g_{65} \) is left unsettled but it has to be proposed as time independent.
in order to recover the conventional definitions for the fields (see Appendix B). The position and velocity vectors are proposed, just following Kaluza, as follows

\[
x^{\nu} = \begin{bmatrix} r \\ \theta \\ \varphi \\ ct \\ x^5 \\ x^6 \end{bmatrix}
\]  

(24)

\[
v^{\nu} = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ c \\ \frac{q}{m} \frac{\partial x}{\partial t} \end{bmatrix}
\]  

(25)

The equation of motion to be considered in this formalism is that of a particle under the influence of a generalized Lorentz’s force namely,

\[
\frac{d^2x^\alpha}{dt^2} = \frac{q}{m} \left[ \varepsilon^{\alpha}_{\beta \gamma} \frac{\partial x^\beta}{\partial t} B^\gamma + E^\alpha \right] + \frac{g}{m} \left[ \varepsilon^{\alpha}_{\beta \gamma} \frac{\partial x^\beta}{\partial t} E^\gamma - B^\alpha \right]
\]  

(26)

This space is a more general one and has enough room to work in a generalized scheme as will be soon shown. Here \( g \) stands for the magnetic charge. The Christoffel symbols that appear in the equation of a geodesic (Eq. 5) in this space are to be compared with the coefficients of velocity components and charge in Eq. (26).

On the other hand, with the proposed metric, the Christoffel symbols can also be computed from definition (7). The symbols to be proportional to the electromagnetic field components are shown here as they arise directly by introducing the elements of the metric tensors (23) and (22) in definition (7). These symbols, in terms of potentials become

\[
\Gamma^\alpha_{\beta 5} = g^{\alpha \alpha} (A_{\beta, \alpha} - A_{\alpha, \beta} + Z_{\alpha g56, \beta})
\]  

(27)

\[
\Gamma^\alpha_{\beta 6} = g^{\alpha \alpha} (Z_{\beta, \alpha} - Z_{\alpha, \beta} + A_{\alpha g56, \beta})
\]  

(28)

Comparing the symbols obtained from the geodesic and Eq. (26) with the ones obtained directly from their definition (7), the expressions for electric and magnetic fields arise. Since we now have two sets of Christoffel symbols, there are two expressions for each field (see Appendix B).

\[
E^{\nu} = - \frac{\partial \phi}{\partial x^{\nu}} - \frac{\partial A_{\nu}}{\partial t}
\]  

(29)

\[
E^{\nu} = \varepsilon^{\nu}_{\beta \gamma} \left( \frac{\partial Z_\beta}{\partial x^{\gamma}} - \frac{\partial Z_\gamma}{\partial x^{\beta}} \right) + M^{\nu}
\]  

(30)

\[
B^{\nu} = - \frac{\partial \eta}{\partial x^{\nu}} - \frac{\partial Z_\nu}{\partial t}
\]  

(31)

\[
B^{\nu} = \varepsilon^{\nu}_{\beta \gamma} \left( \frac{\partial A_\beta}{\partial x^{\gamma}} - \frac{\partial A_\gamma}{\partial x^{\beta}} \right) + Q^{\nu}
\]  

(32)

where the vectors \( M^{\nu} \) and \( Q^{\nu} \) are defined as

\[
M^{\nu} = \begin{bmatrix} Z_2 \frac{\partial \phi}{\partial x^2} \\ Z_3 \frac{\partial \phi}{\partial x^3} \\ Z_1 \frac{\partial \phi}{\partial x^1} \end{bmatrix}, \quad Q^{\nu} = \begin{bmatrix} A_2 \frac{\partial \phi}{\partial x^2} \\ A_3 \frac{\partial \phi}{\partial x^3} \\ A_1 \frac{\partial \phi}{\partial x^1} \end{bmatrix}
\]  

(33)
These vectors are irrotational and since curls are solenoidal vectors, expressions (30) and (32) are in accordance with a generalized Helmholtz theorem for tensors [4]. The duality in Faraday’s tensor has been previously exhibited [4] as based in this theorem but dual expressions for the fields are now introduced. Vectorial fields can always be decomposed as a sum of a solenoidal vector field and an irrotational one which validates expressions (30) and (32). Thus two sets of symmetrical expressions for the fields are obtained one of these sets being the expressions of the decomposition mentioned in the previous line. The implications of Eqs. (29) to (32) will be discussed in section 5.

Within this framework, the complete set of Maxwell equations can be derived (Appendix C). The inhomogeneous equations are obtained in the same way as in the previous section. The new quantity $Q^\nu$ is irrotational and doesn’t affect the structure of $\varepsilon^\nu_{\beta\gamma} \left( \frac{\partial B^\gamma}{\partial x^\beta} - \frac{\partial B^\beta}{\partial x^\gamma} \right)$ while the other equation with sources is identically obtained provided the same definition for $E$ is used. The sourceless equations are obtained once again by means of Einstein’s field equation $R^{6\beta} = KT^{6\beta}$ (34) since the energy momentum tensor has now one additional column and row

$$T^{6\beta} = T^{6\beta\alpha} = \rho_0 v^\beta v^6 = \left[ \begin{array}{c} \rho_0 v^1 \frac{\partial x^6}{\partial t} \\ \rho_0 v^2 \frac{\partial x^6}{\partial t} \\ \rho_0 v^3 \frac{\partial x^6}{\partial t} \\ \rho_0 \frac{\partial x^6}{\partial t} \\ \rho_0 \frac{\partial x^6}{\partial t} \end{array} \right] = K^\beta$$ (35)

In Appendix C it is proved that Eq. (34) leads to the four missing Maxwell equations with some extra terms that make the complete set of equations symmetric. This new terms will be interpreted in next section.

Thus, the purpose of this section is accomplished. The mechanism for obtaining both homogeneous and inhomogeneous equations is the same. This makes the theory completely symmetric and the cylindrical condition not so fundamental for the recovery of Maxwell’s equations.

V. IMPLICATIONS OF THE SIXTH DIMENSION

In Kaluza’s theory, the fifth dimension is associated with electric charge. As electric charge generates curvature in space, it becomes intuitive that the sixth dimension is a curvature source similar to it. The complete set of equations (the equation of motion and Maxwell relations) as is obtained in this formalism is given by

$$\frac{d^2 x^\alpha}{dt^2} = \frac{q}{m} \left[ \varepsilon^\alpha_{\beta\gamma} \frac{\partial x^\beta}{\partial t} B^\gamma + E^\alpha \right] + \frac{g}{m} \left[ \varepsilon^\alpha_{\beta\gamma} \frac{\partial x^\beta}{\partial t} E^\gamma - B^\alpha \right]$$ (36)

$$\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} - K$$ (37)

$$\nabla \times B = \frac{\partial E}{\partial t} + \mu_0 J$$ (38)

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0 m}$$ (39)

$$\nabla \cdot B = \mu_0 \frac{q}{m}$$ (40)

This finally confirms that the sixth dimension is associated with a magnetic charge $g$. Eqs. (37) and (40) show the presence of magnetic charge and it’s associated magnetic current $K$. These equations are completely symmetric as the new definitions (29) to (32) for electric and magnetic fields result in this formalism. The new potentials $Z_\alpha$ and $\eta$ can be interpreted as electromagnetic potentials. $Z_\alpha$ plays the role of an electric vector potential and $\eta$ a scalar magnetic potential. Both appear in these definitions but do not affect the behavior of charged particles since they do not introduce extra effects in the equations. The magnetic charge and current terms in the equations arise exclusively
from the extra column in the energy-momentum tensor $T^{6\beta}$. To recover the equations without this extra terms, it is sufficient to make the magnetic charge equal to zero, $\partial x^\nu = 0$, just as was done with the fifth component of the position vector to recover the equations of motion for a particle without electrical charge.

Also, from the combination of Eqs. (29) to (32) and Maxwell’s equations one can easily obtain four wave equations for potentials, providing a Lorentz’s gauge for the new four vector potential is considered. In the procedure, continuity relations for electric and magnetic charge are obtained.

The role of the vectors $M^\nu$ and $Q^\nu$ (Eq. 33) becomes clear by taking a closer look to definitions (30) and (32). In these expressions, the fields are decomposed as a curl plus a vector which is irrotational. In order to obtain both Maxwell equations that feature charge density as a source, one has to calculate the divergence of both fields. In the procedure, the divergence of the rotational term vanishes since rotationals are solenoidal vectors leaving alone the divergence of $M^\nu$ and $Q^\nu$ which should then be proportional to the charge densities. If these vectors do not appear in the definitions, the result would be solenoidal magnetic and electric fields meaning that neither magnetic nor electric charges exist. The spatial derivatives of the metric elements $g_{56} = g_{65}$ appear in $M^\nu$ and $Q^\nu$ which makes its value fundamental since if it is proposed as a constant, these vectors would be equal to zero with the consequences mentioned before.

VI. CONCLUSIONS

Theodore Kaluza proposed many decades ago a way to unify electromagnetic and gravitational theories. His formalism, although very useful and elegant, shows a weak point in the way a pair of Maxwell’s equations are obtained. Moreover, this mechanism doesn’t give correct results when working in generalized coordinates which are necessary to get the correct physics of most systems.

The alternative here presented is a similar formalism which consists in working in a larger space-time. The new extra dimension makes it possible to use Einstein’s field equation twice and all the equations can be correctly recovered. Even in the absence of this curvature source, the extra dimension remains necessary.

Additional to the fact of leading to the complete and correct set of Maxwell equations, this formalism allows the introduction of magnetic charge in a very elegant and natural way. The metric and ensuing procedure are proposed based only in a symmetrical way of treating electric and magnetic fields. As a result of this, magnetic charge effects arise in the theory in the same way electric effects do. Also, following the same procedure as in conventional electromagnetism, wave equations for potentials in the presence of magnetic charge, can be obtained without having to introduce any singularity in space, as done in other works.

Some approximations have been made in order to verify the equations that emerge from the classical treatment of electromagnetism. First of all, the metric used neglects quadratic terms. These terms may introduce additional measurable effects. In Kaluza’s work, the basic hypothesis of a small electric charge-mass ratio has to be considered. In the formalism here presented a similar assumption has to be made about the magnetic charge-mass ratio. This is of course a mere conjecture since no magnetic monopoles have yet been detected.

The correct set of equations in generalized coordinates can also be obtained using Bianchi’s identities. We consider the formalism here presented more elegant since charges and electromagnetic potentials are included in the geometrical description of space-time and no external forces or fields have to be introduced. Electromagnetic effects are consequences of space configuration. Kaluza’s theory has been found to be very useful for the formulation of magnetohydrodynamics equations in the context of irreversible process thermodynamics and in the development of gauge theories so that we consider fundamental the reformulation of the theory in generalized coordinates to treat problems with different spatial symmetries.

VII. APPENDIX A

Sourceless Maxwell’s equations in vectorial notation read:

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad (41)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (42)$$

To write Eq. (41) in terms of potentials, first we express the vector potential $A_\nu$ and the gradient of the scalar potential $\phi$ in physical coordinates:
\[ A_{\nu}^{\text{phys}} = \begin{bmatrix} \frac{A_1}{r} \\ r A_2 \\ r\sin\theta A_3 \end{bmatrix} \] (43)

\[ \nabla \phi^{\text{phys}} = \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{1}{r} \frac{\partial \phi}{\partial \theta} \\ \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \end{bmatrix} \] (44)

The electric field is then

\[ E' = \begin{bmatrix} -\frac{\partial A_1}{\partial t} - \frac{\partial \phi}{\partial r} - \frac{1}{r} \frac{\partial r A_2}{\partial t} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} - \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \end{bmatrix} \] (45)

To calculate the curl of \( E \), the following determinant has to be expanded

\[ \nabla \times E = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \nabla \cdot E' & r \frac{\partial \theta}{\partial x^1} & r \sin \theta \frac{\partial \varphi}{\partial x^1} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ E^1 & E^2 & r \sin \theta E^3 \end{vmatrix} \] (46)

The third component of the \( \nabla \times E \) is then

\[ \frac{r \sin \theta}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} \left( r \left( -\frac{\partial r A_2}{\partial t} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) \right) - \frac{\partial}{\partial \theta} \left( -\frac{\partial A_1}{\partial t} - \frac{\partial \phi}{\partial \varphi} \right) \right] \frac{\partial \varphi}{\partial \varphi} \] (47)

After some algebra Eq. (47) reduces to

\[ -2 \frac{\partial A_2}{\partial t} - r \frac{\partial^2 A_2}{\partial \theta \partial r} + \frac{1}{r} \frac{\partial^2 A_1}{\partial r \partial \theta} \] (48)

Eq. (48) has to be divided by \( \sqrt{g_{33}} \) to return the expression to its tensorial components which finally results in Eq. (41). Note that in equation 48 the scalar potential does not appear, which agrees with the property of the gradient being irrotational.

On the other hand, the left hand side of equation 13, with \( \alpha = 4 \), \( \beta = 2 \), \( \lambda = 1 \) and \( \mu = 5 \) should be equal to zero because of the cylindrical condition (Eq. 6):

\[ (\Gamma^4_{21} + \Gamma^2_{14} + \Gamma^1_{42})_5 = 0 \] (49)

Which means

\[ \Gamma^4_{51} + \Gamma^2_{54} + \Gamma^1_{52} = 0 \] (50)

The Christoffel symbols involved in Eq. (49) are computed as follows:

\[ \Gamma^4_{51} = \frac{1}{2} g^{44} \left( \frac{\partial g_{54}}{\partial x^1} + \frac{\partial g_{14}}{\partial x^5} - \frac{\partial g_{51}}{\partial x^4} \right) = \frac{1}{2c} \frac{\partial A_1}{\partial t} + \frac{1}{2c} \frac{\partial \phi}{\partial r} \] (51)

\[ \Gamma^2_{54} = \frac{1}{2} g^{22} \left( \frac{\partial g_{52}}{\partial x^4} + \frac{\partial g_{24}}{\partial x^5} - \frac{\partial g_{54}}{\partial x^2} \right) = \frac{1}{2c r^2} \frac{\partial r A_2}{\partial t} + \frac{1}{2c r^2} \frac{\partial \phi}{\partial \theta} \] (52)

\[ \Gamma^1_{52} = \frac{1}{2} g^{11} \left( \frac{\partial g_{51}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^5} - \frac{\partial g_{52}}{\partial x^1} \right) = \frac{1}{2} \frac{\partial A_1}{\partial \theta} - r^2 \frac{\partial A_2}{\partial r} - 2r A_2 \] (53)

Introducing the previous results in Eq. (49) we obtain an incorrect expression for Eq. (41), namely

\[ -\frac{r}{c} \frac{\partial A_2}{\partial t} - \frac{1}{2c} \frac{\partial \phi}{\partial \theta} + \frac{1}{c} \frac{\partial^2 A_1}{\partial \theta \partial t} + \frac{1}{2c} (1 - r^2) \frac{\partial^2 A_2}{\partial r \partial t} + \frac{1}{2c} \left( 1 + \frac{1}{r^2} \right) \frac{\partial^2 \phi}{\partial r \partial \theta} = 0 \] (54)
VIII. APPENDIX B

The equation for the geodesic is:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0 \quad (55)$$

In the six dimensional space proposed the first of the three equations in the geodesic turns out as follows:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + \Gamma^1_{5\nu} \frac{1}{m} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} + \Gamma^1_{6\nu} \frac{g}{m} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0 \quad (56)$$

which after expanding the sums over \(\nu\) reads

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + \Gamma^1_{51} \frac{1}{m} \frac{\mathrm{d}x^1}{\mathrm{d}s} + \Gamma^1_{52} \frac{q}{m} \frac{\mathrm{d}x^2}{\mathrm{d}s} + \Gamma^1_{53} \frac{q}{m} \frac{\mathrm{d}x^3}{\mathrm{d}s} + \Gamma^1_{54} \frac{g}{m} \frac{\mathrm{d}x^4}{\mathrm{d}s} + \Gamma^1_{61} \frac{q}{m} \frac{\mathrm{d}x^1}{\mathrm{d}s} + \Gamma^1_{62} \frac{q}{m} \frac{\mathrm{d}x^2}{\mathrm{d}s} + \Gamma^1_{63} \frac{q}{m} \frac{\mathrm{d}x^3}{\mathrm{d}s} + \Gamma^1_{64} \frac{g}{m} \frac{\mathrm{d}x^4}{\mathrm{d}s} = 0 \quad (57)$$

On the other hand, the first component of the equation of motion under a generalized Lorentz’s (Eq. 26) force is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{q}{m} \frac{\partial x^2}{\partial t} B_3 - \frac{q}{m} \frac{\partial x^3}{\partial t} B_2 + \frac{g}{m} \frac{\partial x^2}{\partial t} E_3 - \frac{g}{m} \frac{\partial x^3}{\partial t} E_2 - \frac{g}{m} B_1 \quad (58)$$

Then the relationship between the Christoffel symbols and the components of the electromagnetic field should be, for this equation:

$$\Gamma^1_{51} = 0 \quad (59)$$

$$\Gamma^1_{52} = B_3 \quad (60)$$

$$\Gamma^1_{53} = -B_2 \quad (61)$$

$$\Gamma^1_{54} = \frac{1}{c} E_1 \quad (62)$$

and

$$\Gamma^1_{61} = 0 \quad (63)$$

$$\Gamma^1_{62} = E_3 \quad (64)$$

$$\Gamma^1_{63} = -E_2 \quad (65)$$

$$\Gamma^1_{64} = -\frac{1}{c} B_1 \quad (66)$$

Calculating the symbols \(\Gamma^1_{52}\) and \(\Gamma^1_{64}\) directly from the definition we have

$$\Gamma^1_{52} = \frac{1}{2} g^{11} \left( \frac{\partial g_{51}}{\partial x^2} - \frac{\partial g_{52}}{\partial x^1} \right) + \frac{1}{2} g^{16} \frac{\partial g_{56}}{\partial x^2} = \frac{1}{2} \left( \frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1} \right) + \frac{1}{2} Z_1 \frac{\partial g_{56}}{\partial x^2} = B_3 \quad (67)$$

$$\Gamma^1_{64} = \frac{1}{2} g^{11} \left( \frac{\partial g_{61}}{\partial x^4} - \frac{\partial g_{64}}{\partial x^1} \right) + \frac{1}{2} g^{15} \frac{\partial g_{65}}{\partial x^4} = \frac{1}{2} \left( \frac{\partial Z_1}{\partial x^4} + \frac{1}{c} \frac{\partial \eta}{\partial x} \right) = -\frac{1}{c} B_1 \quad (68)$$

In equation (68) the quantity \(g_{56}\) is supposed as time independent in order to recover the conventional definition of the magnetic field. Eqs (67) and (68) exhibit how the magnetic field can be represented in two different ways (Eqs. (23) and (31)). Similar results, now verifying equations (29) and (30), are obtained by computing \(\Gamma^1_{54}\) and \(\Gamma^1_{62}\) from definition namely,

$$\Gamma^1_{54} = \frac{1}{2} g^{11} \left( \frac{\partial g_{51}}{\partial x^4} - \frac{\partial g_{54}}{\partial x^1} \right) + \frac{1}{2} g^{16} \frac{\partial g_{56}}{\partial x^4} = \frac{1}{2} \left( \frac{\partial A_1}{\partial x^4} + \frac{1}{c} \frac{\partial \phi}{\partial x} \right) = \frac{1}{c} E_1 \quad (69)$$

$$\Gamma^1_{62} = \frac{1}{2} g^{11} \left( \frac{\partial g_{61}}{\partial x^4} - \frac{\partial g_{62}}{\partial x^1} \right) + \frac{1}{2} g^{15} \frac{\partial g_{65}}{\partial x^4} = \frac{1}{2} \left( \frac{\partial Z_1}{\partial x^4} - \frac{\partial Z_2}{\partial x^1} \right) + \frac{1}{2} A_1 \frac{\partial g_{65}}{\partial x^4} = E_3 \quad (70)$$
In this appendix it is shown how the complete set of correct Maxwell equations is obtained only by means of Einstein’s equation. In the first part, equations with sources are obtained. This procedure can be carried out in a 5x5 space-time giving the same results, since the Christoffel symbols required for the following operations are the ones with one index equal to 5.

Einstein’s field equation relates Ricci’s tensor with the mass-energy tensor. If we assume the curvature scalar to be zero, the field equation turns out as follows:

\[ R_{\mu\nu} = KT_{\mu\nu} \]  

(71)

where Ricci’s tensor is defined as [7]

\[ R_{\mu\nu} = \Gamma_{\mu\nu,\rho}^{\rho} + \Gamma_{\mu\rho,\nu}^{\rho} - \Gamma_{\mu\nu}^{\rho} \Gamma_{\rho\sigma}^{\sigma} - \Gamma_{\mu\nu}^{\rho} \Gamma_{\rho\sigma}^{\sigma} \]  

(72)

In order to obtain the inhomogeneous equations, let’s take \( \mu = 5 \) and let \( \nu \) run from 1 to 4.

\[ R_{51} = \frac{\cot \theta}{2} A_2 + \frac{1}{2} \frac{\partial Z_1}{\partial r} \frac{\partial A_3}{\partial \varphi} + \frac{1}{2} \frac{\partial Z_2}{\partial r} \frac{\partial A_3}{\partial \varphi} - \frac{\csc^2 \theta}{2} \frac{\partial Z_3}{\partial r} \frac{\partial A_3}{\partial \varphi} - \frac{\cot \theta}{2r} \frac{\partial A_3}{\partial \varphi} + \frac{1}{2} \frac{\partial A_3}{\partial r} - \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial \varphi^2} \]  

(73)

\[ R_{52} = -A_2 + \frac{r^2}{2} \frac{\partial^2 A_3}{\partial r^2} \frac{\partial A_3}{\partial \varphi} + \frac{\cot \theta}{2r} \frac{\partial A_3}{\partial \varphi} - \frac{\csc^2 \theta}{2} \frac{\partial A_3}{\partial \varphi} - \frac{1}{2} \frac{\partial^2 A_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial \varphi^2} \]  

(74)

\[ R_{53} = \frac{r^2}{2} \sin^2 \theta \frac{\partial^2 A_3}{\partial \varphi^2} - \frac{\cot \theta}{2} \frac{\partial A_3}{\partial \varphi} + \frac{1}{2r^2} \frac{\partial A_3}{\partial \varphi} + \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial \varphi^2} - \frac{1}{2} \frac{\partial^2 A_3}{\partial r^2} + \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial r^2} \]  

(75)

\[ R_{54} = \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 A_3}{\partial \varphi^2} \]  

(76)

If each one of these equations is equaled with its corresponding element of the energy-momentum tensor \( T_{\mu5} \) (Eq. [11]), Maxwell equations with sources are correctly recovered.

To obtain the homogeneous equations, the sixth row of Ricci’s tensor has to be calculated. The procedure is exactly the same as the one to obtain the inhomogeneous equations and the results are:

\[ R_{61} = \frac{\cot \theta}{2} \frac{\partial Z_3}{\partial r} + \frac{1}{2} \frac{\partial^2 Z_3}{\partial r^2} \frac{\partial r}{\partial \varphi} + \frac{1}{2} \frac{\partial^2 Z_3}{\partial r^2} \frac{\partial r}{\partial \varphi} - \frac{\csc^2 \theta}{2} \frac{\partial^2 Z_3}{\partial r^2} \frac{\partial r}{\partial \varphi} - \frac{\cot \theta}{2r} \frac{\partial Z_3}{\partial \varphi} + \frac{1}{2} \frac{\partial Z_3}{\partial r} - \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} \]  

(77)

\[ R_{62} = -Z_2 + \frac{r^2}{2} \frac{\partial^2 Z_2}{\partial \varphi^2} \frac{\partial A_3}{\partial \varphi} + \frac{1}{2} \frac{\partial^2 Z_2}{\partial \varphi^2} \frac{\partial A_3}{\partial \varphi} - \frac{\csc^2 \theta}{2} \frac{\partial^2 Z_2}{\partial \varphi^2} \frac{\partial A_3}{\partial \varphi} - \frac{1}{2r^2} \frac{\partial^2 Z_2}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 Z_2}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 Z_2}{\partial \varphi^2} \]  

(78)

\[ R_{63} = \frac{r^2}{2} \sin^2 \theta \frac{\partial^2 Z_3}{\partial \varphi^2} - \frac{\cot \theta}{2} \frac{\partial Z_3}{\partial \varphi} + \frac{1}{2r^2} \frac{\partial Z_3}{\partial \varphi} + \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} - \frac{1}{2} \frac{\partial^2 Z_3}{\partial \varphi^2} \]  

(79)

\[ R_{64} = \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} + \frac{1}{2r^2} \frac{\partial^2 Z_3}{\partial \varphi^2} \]  

(80)

If we introduce expressions [30] and [31] as definitions for \( E \) and \( B \) respectively, and set each of the previous equations (77) to (80) equal to its corresponding element in the energy-momentum tensor \( T_{\mu5} \), Eqs. (77) and (80) are obtained. For example, to prove equation (79) corresponds to the third component of equation (77) we will calculate...
the third component or the curl of the electric field using the alternative expression in Eq. (30). First, the vector potential $Z_\beta$ has to be expressed in its physical components as follows

$$Z_{\nu}^{\text{phys}} = \begin{bmatrix} Z_1 \\ rZ_2 \\ rsin\theta Z_3 \end{bmatrix}$$

(81)

The curl of $Z_{\nu}^{\text{phys}}$ is calculated by expanding following the determinat

$$\nabla \times Z = \frac{1}{r^2 sin\theta} \begin{vmatrix} \vec{r} & \vec{\theta} & r sin\theta \vec{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ Z_1 & r (rZ_2) & rsin\theta (rsin\theta Z_3) \end{vmatrix} = \frac{1}{r^2 sin\theta} \begin{bmatrix} r^2 \frac{\partial sin^{2} \theta Z_3}{\partial \varphi} - r^2 \frac{\partial Z_2}{\partial \varphi} \\ r \left( \frac{\partial Z_1}{\partial \theta} - sin^{2} \theta \frac{\partial^{2} Z_3}{\partial \varphi^{2}} \right) \\ rsin\theta \left( \frac{\partial^{2} Z_3}{\partial \theta \partial \varphi} - \frac{\partial Z_2}{\partial \theta} \right) \end{bmatrix}$$

(82)

Since $M^\nu$ is irrotational, $\nabla \times E = \nabla \times (\nabla \times Z)$ which is calculated as follows

$$\nabla \times E = \frac{1}{r^2 sin\theta} \begin{vmatrix} \vec{r} & \vec{\theta} & r sin\theta \vec{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ [\nabla \times Z]^1 & r [\nabla \times Z]^2 & rsin\theta [\nabla \times Z]^3 \end{vmatrix}$$

(83)

The third component of the last equation reads

$$\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r sin\theta} \frac{\partial Z_1}{\partial \varphi} - r sin\theta \frac{\partial Z_3}{\partial \varphi} - 2sin\theta Z_3 \right) - \frac{\partial}{\partial \theta} \left( sin\theta \frac{\partial Z_3}{\partial \theta} + 2 cos\theta Z_3 - csc \theta \frac{\partial Z_2}{\partial \varphi} \right)$$

(84)

which after some algebraic manipulation turns out as follows

$$\frac{1}{r sin \theta} \frac{\partial^2 Z_1}{\partial r \partial \varphi} - r sin \theta \frac{\partial^2 Z_3}{\partial r^2} - 4 sin \theta \frac{\partial^2 Z_3}{\partial r \partial \theta} - sin \theta \frac{\partial^2 Z_3}{\partial \theta^2} - \frac{3 cos \theta \partial Z_3}{\partial \theta} - \frac{cos \theta \partial Z_2}{r \partial \theta} + csc \theta \frac{\partial^2 Z_2}{r \partial \theta \partial \varphi}$$

(85)

Expression (83) is the third physical component of the curl of the electric field. To obtain the third component of Eq. (37), also $\frac{1}{r} \frac{\partial^2 \phi}{\partial r^2}$ - K has to be in physical components which implies multiplying by $r sin \theta$. Then, to make it clear that equation (85) corresponds to (37), it has to be expressed in the following way:

$$\frac{1}{r sin \theta} \left( \frac{\partial^2 Z_1}{\partial r \partial \varphi} - r^2 sin^2 \theta \frac{\partial^2 Z_3}{\partial r^2} - 4r sin^2 \theta \frac{\partial Z_3}{\partial r} - sin^2 \theta \frac{\partial^2 Z_3}{\partial \theta^2} - \frac{3}{2} sin \theta \frac{\partial Z_3}{\partial \theta} - \frac{cos \theta \partial Z_2}{sin \theta} + \frac{\partial^2 Z_2}{sin \theta \partial \varphi} \right)$$

(86)

This finally proves that Eqs. (37) to (40) correspond to the missing Maxwell’s equations (37) and (40). These equations include magnetic sources. It is important to point out, once again, that to obtain Maxwell’s equations without magnetic sources it is sufficient to make $\frac{\partial B^k}{\partial t} = 0$ which implies $T_{\mu \theta} = 0$.