Bound State Internal Interactions as a Mechanism for Exponential Decay

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We hypothesize that the uncontrolled interactions among the various components of quantum mechanical bound states and the background fields, sometimes known as virtual particle exchange, affect the state of the quantum system as do typical scattering interactions. Then with the assumption that the interior environment of unstable particles is disordered, we derive in the limit of continuous internal interactions an exactly exponential decay probability at all times and Fermi’s Golden Rule for the decay rates. Our result offers a resolution to the long-standing problems with the standard theoretical treatments, such as the lack of exponential time evolution for Hilbert Space vectors and energy spectra unbounded from below.

With the advent of quantum engineering and quantum information processing has come experimental and theoretical interest in frequent or continuous interactions between quantum systems and their environments [1–3]. The theoretical framework developed for such systems has been applied to bound states representing unstable particles [4,5], with interesting results. Here we will take a similar approach, but instead of interactions with an external environment or measuring device, we will model a bound state’s internal interactions.

A quantum mechanical treatment of the internal dynamics of unstable particles is complicated by the continual interactions between constituents and the background quantum fields, sometimes called virtual particle exchange. We will ignore dynamics to start, however, and focus instead on the kinematics of continuous, internal scattering interactions. With a simple hypothesis about the nature of such interactions, we derive for all excited bound states an exponential decay probability. When we include dynamics, we derive Fermi’s Golden Rule for the decay rates.

Our model is based on the following two assumptions for bound states representing unstable particles:

**Assumption 1: Kinematics**

The interactions among the components of bound states, and with the background fields, are no different from scattering interactions or interactions with measuring devices. If component systems are not already in the space of preparable bound states, upon interacting they will be projected onto such states.

**Assumption 2: Disordered Internal Environment**

The first assumption implies a sequence of projections onto preparable states, corresponding to a sequence of scattering interactions among the state’s components occurring over time. We assume a stochastic sequence to model the presumed disorder of the internal environment.

To illustrate the consequences of Assumption 2, consider an analogous sequence in a controlled environment: an experiment in which an electron’s spin is measured along a sequence of axes, where at each measurement in the sequence the axis is chosen to point in a random direction. In such a sequence of measurements, the probability is nearly zero that consecutive measurements will leave the electron in states that commute. Similar sequences with non-commuting observables have been studied [11,12], but by focusing on decay rather than state dynamics, we will find a different result.

In the context of a bound state, represent one of its components at preparation by \( \rho_n \), which is one operator in a direct product of density operators representing the entire bound state. Let \( n \) label the state of the system immediately after the \( n \)th interaction in the sequence of internal interactions; \( \rho_{n+1} \). Then as a consequence of the uncontrolled and disordered interior of an unstable particle, we will assume \( [\rho_n, \rho_{n+1}] \neq 0 \) for almost all interactions. This is easy to imagine as the consequence of a component interacting, say, with a sequence of background fields, each having a total angular momentum randomly aligned in space.

**EXPONENTIAL DECAY**

We will study our model in the limit of continuous internal interactions. A framework for treating the kinematics of continuous interactions has been developed already for the Quantum Zeno Effect (QZE) [13], for which it was predicted and verified experimentally [13,14] that frequent, controlled measurements into a fixed state or subspace can inhibit a quantum system from leaving that state or subspace. By Assumption 2, however, we will need to generalize for uncontrolled interactions that leave a bound state in a sequence of differing spaces.

As done for the QZE, we model the constant internal interactions as the continuous limit of a sequence of \( N \) measurement interactions, each following the previous by a duration \( \delta \), which we will take to zero in the limit. Let
a projection operator $\Lambda_u$ span a possible state space of
the components of a preparable, excited bound state, so
that $\Lambda_u$ represents the observable that the bound state is
undecayed.

By Assumption 1, if the system is found to be in an
undecayed state upon the $n$th interaction, corresponding
to the observable $\Lambda_u^n$, we will represent it by a density op-
erator $\rho(t=0)$ from the space of undecayed states. By
assumption 2, undecayed states in the sequence will differ
from each other internally, but we can drop the index $n$
because we are concerned only with decay and not with
the internal states themselves. Returning to Assumption
1, at the subsequent interaction from the sequence, oc-
curring at a time $\delta$ after the previous, the probability
that the system remains undecayed is $\text{Tr} (\Lambda_u \rho(\delta))$.

Once prepared initially, the probability for the system
to continue to be found in the excited state throughout
the interaction sequence is $\text{Tr} (\Lambda_u \rho(\delta))^N$. As noted by
the authors of [12], this product must be distinguished
from each other internally, but we can drop the index $n$
because we are concerned only with decay and not with
the internal states themselves. Returning to Assumption
1, at the subsequent interaction from the sequence, oc-
curring at a time $\delta$ after the previous, the probability
that the system remains undecayed is $\text{Tr} (\Lambda_u \rho(\delta))$.

To model the uncontrolled internal interactions of an
unstable particle, however, we appeal to Assumption 2
above, with the consequence that $[\rho(\delta = 0), \Lambda_u] \neq 0$
throughout the sequence. With a nonzero $\tau^{-1}$ in equa-
tion (3), our model predicts exponential decay:
\[ P_u(t) = \lim_{N \to \infty} \text{Tr} (\Lambda_u \rho(\delta))^N = e^{-t/\tau}. \] (5)

Note that exponential decay follows from the kinemat-
ics of constant internal interactions and is therefore gen-
eral and independent of a bound state’s dynamics, as
observed in experiments with unstable particles.

**DECAY RATE**

A straightforward calculation of the decay rate in equa-
tion (3) results in Fermi’s Golden Rule (FGR). A partic-
ular strength of the present derivation is that it illumi-
mates the assumptions required to apply FGR. To con-
nect with this well-known formula and to compare with
standard textbook derivations, we will change to bra-ket
otation for the derivation and treat a non-relativistic
system. This is of course not required.

According to Assumption 1 above, we would have a
different $\Lambda_u$ for each interaction in the sequence. Given
this and the fact that the internal workings of unstable
particles are presently not well understood, writing the
$\Lambda_u$ will be difficult. Fortunately we can work instead with
$\Lambda_d$, which we define to project onto decayed subspaces of
the system. Then $\text{Tr} (\Lambda_d \rho(\delta))$ is the probability to find
the system decayed, and it is related to the undecayed
probability by $\text{Tr} (\Lambda_u \rho(\delta)) = 1 - \text{Tr} (\Lambda_d \rho(\delta))$. Let
\[ \Lambda_d = \sum_\eta \int d^2 \Omega dE \lambda(E, \Omega, \eta) \Lambda_{E, \Omega, \eta}, \] (6)

where $\Lambda_{E, \Omega, \eta} = \lvert \psi_{E, \Omega, \eta} \rangle \langle \psi_{E, \Omega, \eta} \rvert$ are projection operators
onto the infinitesimal subspaces representing the states of
the decayed system, and $\lambda(E, \Omega, \eta)$ is the density of these
states. Here $E$ is the energy, and $\eta$ represents the excita-
tion level of the system, as well as any other quantum
numbers required to specify its state. $\Omega$ describes the
direction of momentum of the decay products. To con-
nect with the simplest form of FGR, we assume spherical
symmetry and an isotropic density of final states. After
integrating over $\Omega$, we are left with
\[ \Lambda_d = \sum_\eta \int dE \lambda(E, \eta) \Lambda_{E, \eta}, \] (7)

where
\[ \Lambda_{E, \eta} = \lvert \psi_{E, \eta} \rangle \langle \psi_{E, \eta} \rvert. \] (8)

With the shift in focus to $\Lambda_d$ and the decayed state,
we will consider the asymptotically free decay products
and use the standard machinery of scattering theory and a perturbation approach to proceed. Let $H = H_0 + V$ be the total Hamiltonian for the system. As in scattering theory, $V$ represents all internal interactions, and $H_0$ is the free Hamiltonian for all components of the system after decay. $\Lambda_d$ projects onto the lower energy states of the bound system as well as any emitted decay products, so $|\psi_{E,\eta}\rangle$ are dimensionless eigenkets of $H_0$. Energy eigenkets of $H_0$ are $|E,\eta\rangle$, and eigenkets of $H$ are the Lippmann-Schwinger kets, $|E,\eta^+\rangle$, where $\langle E,\eta^+| = |E,\eta\rangle + \lim_{\epsilon \to 0} \frac{1}{(E-H_0)+i\epsilon} V|E,\eta\rangle$.

Here we have dropped another $\Omega$ because we will work in the rest frame of the excited system.

Using equation (11) and the fact that $\frac{d}{dt}\rho_a(t) = -\frac{i}{\hbar}[H_a,\rho_a(t)]$, and writing the density operator of the excited system at preparation, or at $\delta = 0$, as $\rho(0) = |\phi\rangle\langle\phi|$, the decay rate from (3) becomes

$$\tau^{-1} = \frac{2}{\hbar} \sum_{\eta_f} \int dE \frac{dE_f}{| \langle \psi_{E,\eta}\rangle \rangle V| \langle \phi| \rangle \rangle} \langle \phi| \langle \psi_{E,\eta}\rangle \rangle \lambda(E_f,\eta_f),$$

where we have used $[\Lambda_d, H_0] = 0$. We have added the subscript $f$ to distinguish the energy and quantum numbers of the final state. $|\phi\rangle$ is the bound state vector, so we expand it in terms of the total Hamiltonian eigenkets, $|E,\eta^+\rangle$. Using (9) once, we have for the decay rate,

$$\tau^{-1} = \lim_{\epsilon \to 0} \frac{2}{\hbar} \sum_{\eta_f} \int dE \frac{dE_f}{| \langle \psi_{E,\eta}\rangle \rangle V| \langle \phi| \rangle \rangle} \langle \phi| \langle \psi_{E,\eta}\rangle \rangle \lambda(E_f,\eta_f)$$

$$= \int dE \frac{dE_f}{| \langle \psi_{E,\eta}\rangle \rangle V| \langle \phi| \rangle \rangle} \langle \phi| \langle \psi_{E,\eta}\rangle \rangle \lambda(E_f,\eta_f).$$

The first term on the right hand side of (11) contains $\langle E,\eta|\psi_{E,\eta}\rangle$, which is proportional to the Kronecker delta $\delta_{\eta}\eta_f$, where $\eta_f$ represents, among other things, the final excitation level of the system after decay. Because $|\phi|\rangle = \psi_{E,\eta^+}$ is the wave function of the excited system, the first term vanishes. We are then left with

$$\tau^{-1} = \frac{2}{\hbar} \sum_{\eta_f} \int dE \frac{dE_f}{| \langle \psi_{E,\eta}\rangle \rangle V| \langle \phi| \rangle \rangle} \langle \phi| \langle \psi_{E,\eta}\rangle \rangle \lambda(E_f,\eta_f)$$

$$\sum_{\eta} \int dE \frac{dE_f}{| \langle \psi_{E,\eta}\rangle \rangle V| \langle \phi| \rangle \rangle} \langle \phi| \langle \psi_{E,\eta}\rangle \rangle \lambda(E_f,\eta_f)\delta(E-E_f)$$

$$\tau^{-1} = \frac{2}{\hbar} \sum_{\eta_f} \int dE \frac{dE_f}{| \langle \psi_{E,\eta}\rangle \rangle V| \langle \phi| \rangle \rangle} \lambda(E_f,\eta_f)\delta(E-E_f).$$

It is clear from (12) that any contribution to the decay rate will come only from interactions coupling the excited state to the decay products, as expected. The delta function in (12) enforces energy conservation. The final energy, $E_f$, includes the energy of all decay products. One can write, for instance, $E_f = E_{\eta_f} + h\omega_f$, where $\omega_f$ is the angular frequency of an outgoing photon. Equation (12) is our most general result for the decay rate, $\tau^{-1}$, of our model system. Recall that, for brevity, we did assume an isotropic density of final states above for equation (7), and for a non-isotropic density equation (12) would change in straightforward way.

Until now we have avoided specifying the state of the excited system at preparation, but when the excited state is prepared with sharp values, $E_i$ and $\eta_i$, such that

$$\sum_{\eta_f} \int dE \frac{dE_f}{| \langle \psi_{E,\eta}\rangle \rangle V| \langle \phi| \rangle \rangle} \lambda(E_f,\eta_f)\delta(E-E_f)$$

$$\tau^{-1} = \frac{2}{\hbar} \sum_{\eta_f} \int dE \frac{dE_f}{| \langle \psi_{E,\eta}\rangle \rangle V| \langle \phi| \rangle \rangle} \lambda(E_f,\eta_f)\delta(E-E_f).$$

For a channel labeled by the initial and final internal quantum numbers $\eta_i$ and $\eta_f$, we have for the decay rate

$$\tau^{-1}_{\eta_i,\eta_f} = \frac{2}{\hbar} \sum_{\eta_f} \int dE \frac{dE_f}{| \langle \psi_{E,\eta}\rangle \rangle V| \langle \phi| \rangle \rangle} \lambda(E,f)\delta(E-E_f).$$

where the notation $E_i = E_f \equiv E$ reflects energy conservation. Thus we have derived Fermi’s Golden Rule for the time-independent decay rate. Though it is obscured in the notation of equation (13), $|\psi_{E,\eta}\rangle$ is proportional to a Lippmann-Schwinger ket, which can be iteratively expanded to obtain the Born series.

In the definition of $\Lambda_d$, we sum over $\eta_f$ and implicitly over different decay channels. The derivation does suggest a natural definition for decay channels, however. Partition the interaction potential, $V$, into two parts:

$$V = V_k + V_\ell.$$

If there is a final state, $|\psi_{E_k,\eta_k}\rangle$, such that

$$\int dE \langle \phi|\langle E,\eta|\psi_{E_k,\eta_k}\rangle\rangle (\langle E,\eta|\psi_{E_k,\eta_k}\rangle\rangle (\langle E,\eta|\psi_{E_k,\eta_k}\rangle\rangle \delta(E-E_{k}) \neq 0,$$

and

$$\int dE \langle \phi|\langle E,\eta|\psi_{E_k,\eta_k}\rangle\rangle (\langle E,\eta|\psi_{E_k,\eta_k}\rangle\rangle (\langle E,\eta|\psi_{E_k,\eta_k}\rangle\rangle \delta(E-E_{k}) = 0,$$

then the decay rate for a channel labeled $k$ is

$$\tau^{-1}_k = \frac{2}{\hbar} \sum_{\eta_f} \int dE \frac{dE_f}{| \langle \psi_{E,\eta}\rangle \rangle V| \langle \phi| \rangle \rangle} \lambda(E_f,\eta_f)\delta(E-E_f).$$

and similar for $\ell$. This generalizes to any number of partitions into channels, and it is straightforward to show that the total decay rate is the sum of the rates for various channels, as expected. When there are two available decay channels, as indicated in (13), the non-decay probability as a result of continuous internal interactions is

$$P_\alpha(t) = e^{-t/\tau_\alpha} e^{-t/\tau_\alpha},$$

as verified by experiments.
CONCLUSION

In the same continuous limit as taken for the QZE, the kinematical effect of uncontrolled internal scattering interactions is an exponentially decreasing non-decay probability for unstable particles at all times. Upon specifying a system’s dynamics, calculation of the decay rate results in a straightforward derivation of Fermi’s Golden Rule. If continuous internal interactions over a finite $t = N \delta$ are realized in nature, then they are a candidate to explain quantum mechanically the normally empirically or approximately [19] derived exponential decay law.

Regarding our use of projective measurements, quantum systems undergoing continuous, non-demolition probes of non-commuting observables have been observed [20], and sequences of unitary interactions with varying ancillary systems have been studied [21]. These would provide interesting avenues to extend our model to non-projective internal interactions, especially as engineered quantum systems become more complex.

Specific to Fermi’s Golden Rule, while we believe our derivation is particularly transparent and informative, there have been operationally similar derivations based also on a time-dependent perturbation approach (for example, [22]). Such derivations are complicated by justifying a time-independent decay rate, however. As seen in equation (3), here time independence is a consequence of continuous interactions.

Finally, there are reasons related to the stability of matter to expect that, if a spontaneously decaying state has a Lorentzian energy wave function and is represented by a state vector from a Hilbert space, then its survival probability cannot be exponential at large times [23]. Any deviation from exponential decay, however, has never been observed for unstable particles. With our model, such arguments do not apply, because the system is not represented by a continuously evolving state vector throughout the decay process. In the absence of interactions, $\phi(E)$ need not be a Lorentzian, as required for a system to be bounded from below in energy. Rather, the Lorentzian line shape emerges as a consequence of continuous interactions.

[1] T. J. Elliott and V. Vedral, Physical Review A 94, 012118 (2016).