Hydrostatics and stability of a floating elliptical cylinder with surface tension effects for different eccentricities and Bond numbers

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Abstract. A model dealing with the conditions of equilibrium and initial stability of floating bodies in two dimensions with an arbitrary cross-section has been proposed recently (J. Fluid Mech. 847:489-519, 2018). The method is applied to investigate the hydrostatics and stability of the configuration of an elliptical cylinder floating on an infinite liquid bath for a range of two dimensionless parameters: the eccentricity of the elliptical cross-section $e$ and the Bond number $Bo$. It is shown that the prolate cylinder with a greater eccentricity $e$ can bear a greater pressing force and a greater moment and therefore is more stable. For a great Bond number, the greater the eccentricity $e$ of the cylinder, the greater the inclination angle corresponding to maximum moment. For a small Bond number, the opposite is true. However, the inclination angle of the cylinder corresponding to the maximum moment does not obey this rule. The maximum inclination angle occurs at an intermediate Bond number.

1. Introduction

Capillary floating phenomena are important in a range of small-scale applications involving interfaces. Usually, the interfaces around floating bodies are deformed due to surface tension effects, which have a significant effect on the floating phenomena when the Bond number $Bo \sim 1$ and $Bo \to 1$. This effect may lead to many unexpected behaviors of the floating bodies, such as two floating bodies tend to either attract or repel each other [1-3] and multiple equilibrium floating positions [4,5].

To investigate the hydrostatics of the floating phenomena dominated by surface tension effects, researchers have solved the Young-Laplace equation analytically and numerically under different situations, such as the axisymmetric assumptions [6] and two-dimensional assumptions [7]. In these works, the Young-Laplace equation is reduced to a second-order nonlinear ordinary differential equation, and then its asymptotic or analytical solutions are obtained. For the floating phenomena, the Young-Laplace equation is usually solved analytically with the linearization assumption due to Nicolson’s work [8]. This assumption is valid when the inclination angles of the interface are small. Therefore, it is widely applied in colloidal particles [9-11]. For an individual floating body, its force and stability analysis attracts much attention [12]. This calculation requires that the force balance equations of particles and the Young-Laplace equation for the surrounding menisci are solved simultaneously, which produces a very complex system. Even a sphere or a circular cylinder has a striking phenomenon on the water [5,13].
In this paper, we will apply a model proposed recently [12] to investigate an elliptical cylinder floating on an infinite liquid bath for a range of two parameters: the eccentricity of the elliptical cross-section \( e \) and the Bond number \( Bo \). The force and moment of the elliptical cylinder are studied. Then the inclination angle of the cylinder corresponding to the maximum moment is also provided. Finally, we characterise the stabilities of the elliptical cylinder by its force and moment profiles.

2. Model

Consider an elliptical cylinder floating horizontally on an infinite water bath in a downward gravity field (figure 1). The interfaces (menisci) on both sides of the cylinder are described by the known solution of the Young-Laplace equation in two dimension, given by [7]

\[
\frac{k u^2}{2} + \cos \psi = c,
\]

where \( c \) is a conserved quantity which arise from the integration of the Young-Laplace equation, \( u(x) \) denotes the height of the meniscus as the function of \( x \), \( \psi \) is the inclination angle of the meniscus and \( \kappa = \rho g / \sigma \) is the capillary constant, with the density difference \( \rho \) (positive) between gas and liquid, the gravitational acceleration \( g \) and the surface tension \( \sigma \) of the interface. Then we determine \( c = 1 \) from the condition at infinity:

\[
u \rightarrow 0 \text{ and } \psi \rightarrow 0 \text{ at } x \rightarrow -\infty.\]

With \( c = 1 \), (1) can be transformed to

\[
u = \frac{2}{\sqrt{\kappa}} \sin \frac{\psi}{2},\]

because the sign of the height \( u \) is the same as that of the inclination angle \( \psi \). Then the menisci on two sides of the cylinder can be easily determined by the vectors of the contact points \( p_l \) and \( p_r \) (see figure 1).

2.1. First variations of force and moment.

With above considerations, we deal with the conditions of forces and moments acting on the cylinder, including the weight force, the surface tension forces, the buoyancy and the compensating pressure force due to line pressure and the corresponding moments (see the details in [12]). Then we establish the first variations of the force and moment as follows:

\[
\delta f_{\text{net}} = \delta f_i^l + \delta f_i^r + \delta f_p + \delta B + \delta G,
\]
\[ \delta \mathbf{M} = \delta \mathbf{M}_t + \delta \mathbf{M}_p + \delta \mathbf{M}_B + \delta \mathbf{M}_G, \]  

(5)

where \( \mathbf{f}_{\text{net}} \) and \( \mathbf{M} \) denote the resultant force and the resultant moment acting on the elliptical cylinder, respectively. The resultant force consists of the surface tension forces \( f'_t + f'_r \), the press force \( f'_p \), the buoyance \( B \) and the weight \( G \). The resultant moment consists of the moments due to the surface tension forces, the press force, the buoyance and the weight. It is noted that these variations are considered in the context of the principle of virtual work. Therefore the menisci are determined as equilibrium configurations (see equation 3). Then we just need to connect these above variations and the infinitesimal displacements during movement and rotation. For the force model, only the vertical net force is considered, because the horizontal equilibrium is satisfied automatically [12]. These relations for the force model are shown as follows:

\[ \delta f'_{\text{net}} = -\frac{\rho g}{2} \left[ (\delta x_r - \delta x_l)(u_r + u_l) + (x_r - x_l)(u_r + u_l) \right] \delta h, \]  

\[ \delta f'_{\text{net}} = \sigma (\sin \phi_1 \delta \phi_1 + \sin \phi_2 \delta \phi_2) + \rho g \delta V_g, \]  

(6)

with the relations

\[ \delta x_l = -\frac{\sqrt{\kappa}}{R} \delta h, \]  

(7)

\[ \delta u_l = \frac{\sin \left( \frac{\phi_l + \theta}{2} + \frac{\pi}{4} \right)}{\sqrt{\kappa}} \delta h, \]  

(8)

\[ \delta \phi_l = -\frac{\sqrt{\kappa}}{\sqrt{\kappa} R \cos \phi_1 + \sin \left( \frac{\phi_l + \theta}{2} + \frac{\pi}{4} \right)} \delta h, \]  

(9)

\[ \delta V_g = \frac{1}{2} \left[ (u_l - u_r)(\delta x_r + \delta x_l) + (x_r - x_l)(\delta u_l + \delta u_r - 2 \delta h) \right], \]  

(10)

where \( f'_{\text{net}} \) denotes the vertical net force, \( \delta h \) denotes the infinitesimal vertical movement of the cylinder, \( R \) denotes the curvature radius of the solid at the contact point, \( \phi \) is the inclination angle of the solid, the subscripts \( l \) and \( r \) denote the quantities measured at the left and right sides of the cylinder, respectively, and \( \kappa = \rho g / \sigma \) is the capillary constant, with the density difference \( \rho \) (positive) between gas and liquid, the gravitational acceleration \( g \) and the surface tension \( \sigma \) of the interface. The variations on the right side can be determined by replacing \( l \) with \( r \) in (6-9) and changing the sign of the right hand of (6).

Then the force model is formulated as:
where the omissions are the ODEs for the basic geometric quantities not shown and the corresponding initial conditions. For a more detailed derivations of (5-9) reader is referred to [12].

Similar to the force model, we can formulate the moment model by three steps. First, we write the first order variation of the moment. Second, we make a connection between the variations involved in (5) and the infinitesimal rotation angle $\delta\xi$. Third, we derive the system of ODEs for the resultant moment profile with the above relations. Different from the force model, it is noted that $\delta f_{\text{net}} = 0$ is required for the moment model, because the moment model requires that the vertical equilibrium is keeping during the rotation. This condition gives

$$x_{\text{ro}} = \frac{\rho g \eta/2 + \xi_i (x_r + \cos \phi, R_r) + \xi_r (x_r + \cos \phi, R_r)}{\rho g (x_r - x_l + u_l \tan \phi_l + u_l \tan \phi_l) + \xi_i + \xi_r}$$

(12)

with

$$\eta = x_r^2 - x_l^2 - u_l^2 + 2 \tan \phi_l x_l u_l + 2 \tan \phi_l x_l u_l,$$

(13)

$$\xi_i = \frac{\sqrt{k} \sigma \sin (\phi_l + \theta) - \rho g u_l \sin \left(\frac{\phi_l + \theta}{2} + \frac{\pi}{4}\right) \tan \phi_l}{\sqrt{k} \cos \phi_l R_l + \sin \left(\frac{\phi_l + \theta}{2} + \frac{\pi}{4}\right)},$$

(14)

$$\xi_r = \frac{\sqrt{k} \sigma \sin (\phi_r + \theta) - \rho g u_r \sin \left(\frac{\phi_r + \theta}{2} + \frac{\pi}{4}\right) \tan \phi_r}{\sqrt{k} \cos \phi_r R_r + \sin \left(\frac{\phi_r + \theta}{2} + \frac{\pi}{4}\right)},$$

(15)

which imposes a restriction on the abscissa of the rotation center. For the moment model, reader is referred to [12].

3. Numerical results and discussion

The elliptical cross-section is given by the parametric equation (see figure 1)

$$r(t) = \left(a \cos t, b \sin t + h^*\right),$$

(16)

where $h^*$ denotes the initial height of the center of the cross-section. And the radius of curvature at the contact points is given by

$$R = \frac{\left(a^2 \sin^2 t + b^2 \cos^2 t\right)^{3/2}}{ab}.$$  

(17)

We investigate the configuration by varying two dimensionless parameters: the eccentricity $e = \sqrt{1 - b^2/a^2}$ and the Bond number $Bo = \kappa ab$. There are two orientations for the same eccentricity: the prolate and oblate elliptical cylinders. We will discuss them separately. These physical parameters are used in the following paper: $\theta = \pi/2$ and $\rho_s = 0.5 \rho$. 

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Figure 2. Force profiles of the prolate and oblate elliptical cylinders for different eccentricities.

Figure 2 shows the force profiles of the prolate and oblate elliptical cylinders for different eccentricities. For the same Bond number, the prolate cylinders can bear a greater force than that of the oblate cylinders. However, the maximum depth is smaller. For the prolate cylinders, the greater the eccentricity, the greater the restoring force. For the oblate cylinders, the opposite is true. The force profiles of the elliptical cylinders have been investigated theoretically in [4], and the conclusions are consistent.

Figure 3. Force profiles of the prolate and oblate elliptical cylinders for different Bond numbers.

However, the size effect on the force profiles is not considered in [4]. Figure 3 shows the force profiles for different Bond numbers. We can see that the greater the Bond number, the greater the restoring force with the eccentricity $e = 0.9$. However, the relative maximum depth is smaller. This depth is scaled by $\sqrt{ab}$.

We are more concerned about the moment profiles because few literatures have studied them. More importantly, the rotational stability can also be determined from the moment profiles: $\frac{\delta M}{\delta \varepsilon} > 0$ corresponds to an unstable equilibrium while $\frac{\delta M}{\delta \varepsilon} < 0$ corresponds to a stable equilibrium. Figure 4a shows the moment profiles with $Bo = 1$ for different eccentricities. The greater eccentricities, the greater the maximum moment. A similar configuration (spheroid) has been investigated theoretically in [14], and results show a prolate spheroid is rotationally stable whereas an oblate spheroid is unstable. These results are consistent with our result: the prolate elliptical cylinders are rotationally stable and the oblate elliptical cylinders are unstable. The configuration with the maximum moment corresponds to the critical point with $\frac{\delta M}{\delta \varepsilon} = 0$. Figure 4b shows these configurations for different eccentricities. The inclination angle for these configurations is denoted by $e_m$. It is found that $e_m$ increases with the increase of $e$ with $Bo = 1$, see figure 4c.
Figure 4. (a) Moment profiles and (b) the configurations with the maximum moment for different eccentricities, and (c) the relationship between inclination angle corresponding to the maximum moment and the eccentricity.

Figure 5a shows moment profiles for different Bond numbers. We can see that the greater the Bond numbers, the greater the maximum moment. However, the inclination angle does not seem to obey this rule. Then we plot the inclination angle corresponding to the maximum moment vs. the Bond number. There are two interesting observations. The first one is that the maximum inclination angle corresponding to maximum moment occurs at an intermediate Bond number (see the bulges of the lines in figure 6). The second one is that the two inclination angle corresponding to the maximum moment at a small and a large Bond numbers may be complementary.
4. Conclusions

Based on the above results, we can conclude two conclusions. The first is that the prolate cylinder with a greater eccentricity \( e \) can bear a greater pressing force and a greater moment and therefore is more stable. The second is that for a great Bond number, the greater the eccentricity \( e \) of the cylinder, the greater the inclination angle corresponding to maximum moment. For a small Bond number, the opposite is true. However, the maximum inclination angle corresponding to maximum moment occurs at an intermediate bond number. Additionally, there is an interesting observation that the two inclination angles corresponding to the maximum moment at a small and a large Bond numbers may be complementary. This interesting results may be proved theoretically in the future.

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