Moduli Spaces of Curves with Homology Chains and $c = 1$ Matrix Models

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Abstract

We show that introducing a periodic time coordinate in the models of Penner-Kontsevich type generalizes the corresponding constructions to the case of the moduli space $S_{g,n}^k$ of curves $C$ with homology chains $\gamma \in H_1(C, \mathbb{Z}_k)$. We make a minimal extension of the resulting models by adding a kinetic term, and we get a new matrix model which realizes a simple dynamics of $\mathbb{Z}_k$-chains on surfaces. This gives a representation of $c = 1$ matter coupled to two-dimensional quantum gravity with the target space being a circle of finite radius, as studied by Gross and Klebanov.
1 Introduction

The models of Penner-Kontsevich type \[1, 2\] make use of a cellular decomposition of the moduli space \(\mathcal{M}_{g,n}\) of algebraic curves which relies on the following fact \[7\]: smooth algebraic curves \(C\) of genus \(g\) with \(n\) marked points \(x_1, \ldots, x_n\) and positive numbers \(p_1, \ldots, p_n\) are in 1-1 correspondence with ribbon graphs with metric \(1\). This combinatorial description greatly simplifies the sum over the space of isomorphism classes of surfaces, which becomes substituted by a sum over isomorphism classes of graphs.

From the point of view of two-dimensional quantum gravity it would be interesting to get an analogous description not only for curves, but also for curves carrying additional structures. As a matter of fact, the sum over surfaces corresponds to a theory of pure two-dimensional quantum gravity, while matter fields should be described by “spin” structures. Three years ago, E. Witten \[3\] has proposed a model of 2D-gravity interacting with matter fields which involves a sum over the moduli space of curves with \(k\)-th roots of the canonical bundle. The model we discuss in this paper was inspired by Ref. \[3\] but differs from it on several aspects. First of all, we consider here the moduli space of curves with fixed \(\mathbb{Z}_k\)-chains, which do not coincide with \(k\)-th roots of the canonical bundle since the correspondence between the two objects is non-canonical and depends on the choice of a reference \(k\)-th root. Secondly, we were not able at this stage to describe combinatorially the Euler class which is an essential ingredient of Witten’s paper. On the other hand, we were able to give a natural combinatorial description of \(\mathbb{Z}_k\)-chains which fits very well in the Penner-Kontsevich picture. We have shown that it is possible to give a cellular decomposition of the moduli space \(\mathcal{S}_{g,n}^k\) of couples \((C, \gamma)\) with \(\gamma \in H_1(C, \mathbb{Z}_k)\) and that there exist a matrix model realization generating this cellular decomposition. This is obtained by simply substituting the usual hermitian matrix by a hermitian matrix field depending on a time variable defined on a circle of finite radius \(R\). This gives a general method for extending models of the Penner-Kontsevich type to the moduli space of curves with spin structures. However, this is a non-interacting picture, since the new partition function factorizes in a product of the old ones. But now we are in a position to make a minimal modification to these models by introducing a kinetic \((\dot{M}(\phi))^2\) term. The resulting model is a representation of \(c = 1\) matter coupled to two-dimensional quantum gravity with the target space being the circle of radius \(R\). This model has been studied by Gross and Klebanov \[4\]. It presents different...

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1 A ribbon graph with metric is a graph with a cyclic ordering of the edges around each vertex and with assigned positive lengths of the edges. These graphs arise as set of separatrices of the closed geodesics of certain minimal area metrics on \(C\) and carry all the information about the complex structure of the curve.
behaviours for $R > R_c$ and $R < R_c$, and a Kosterlitz-Thouless phase transition for $R = R_c = 1$. For $R \to +\infty$ the non-singlet states of the model decouple. These states are identified with vortices on the world sheet, and they become relevant for $R < R_c$. In our construction the parameter $R$ interpolates between the topologically trivial situation $R = +\infty$ and a new topological phase which has yet to be investigated. We conjecture that the model at the critical point $R = R_c = 1$ should describe some non-trivial topological characteristics of the spaces $S_{g,n}$.

2 Combinatorial description of $H_1(C, \mathbb{Z}_k)$

Let us introduce some notation. With $C_{g,n}$ we will denote a smooth algebraic curve of genus $g$ with $n$ punctures $x_1, \ldots, x_n$ and $n$ positive numbers $p_1, \ldots, p_n$ associated to them. $X$ will be the corresponding fat graph with metric, obtained from the Strebel-Kontsevich construction. Let $X$ be the set of oriented edges of $X$, $s_0$ be the permutation of $X$ which clockwise exchanges edges with a common source, $s_1$ the orientation-exchanging permutation, and $s_2 = s_0s_1$. Finally we set $X_j = X/s_j$, $j = 0, 1, 2$ and denote with $[x]_j$ the equivalence class of $x \in X$ in $X_j$.

Let $C'_{g,n} = C_{g,n} \setminus \{x_1, \ldots, x_n\}$ and consider the group $H_1(C'_{g,n}, \mathbb{Z}_k)$ of homology chains with coefficients in $\mathbb{Z}_k$ (the cyclic group of order $k$). We will show that $H_1(C'_{g,n}, \mathbb{Z}_k)$ is canonically isomorphic to the group $G_X$ of colorings $c : X_1 \to \mathbb{Z}_k$ such that

$$\sum_{x \in [y]_0} c(x) \equiv 0 \pmod{k} \quad \text{for all } y \in X_1, \quad (2.1)$$

$$\sum_{x \in [y]_1} c(x) \equiv 0 \pmod{k} \quad \text{for all } y \in X_1. \quad (2.2)$$

As a matter of fact, let us consider the application $\phi_y : X \to X'$ which cancels the edge $y \in X_1$ and identifies the corresponding vertices. It is easily seen that $\phi_y$ induces an isomorphism $G_X \to G_{X'}$. Therefore, we can reduce to the case of a graph $\tilde{X}$ with $2g - 1 + n$ edges and only one vertex, homotopically equivalent to $X$. The edges $x_j$, $j = 1, \ldots, 2g - 1 + n$ are now closed cycles and generate $H_1(\tilde{X}, \mathbb{Z}_k)$. We can therefore identify $c(x_j) \in \mathbb{Z}_k$ with the coefficient of the cycle $x_j$, thus getting $G_X \simeq G_{\tilde{X}} \simeq H_1(\tilde{X}, \mathbb{Z}_k) \simeq H(X, \mathbb{Z}_k)$.

Let us now define $S_{g,n}^k$ as the moduli space of couples

$$(C_{g,n}, \gamma) \quad (2.3)$$

where $\gamma \in H_1(C'_{g,n}, \mathbb{Z}_k)$ is a $\mathbb{Z}_k$-chain on the smooth algebraic curve $C'_{g,n}$ deprived of the $n$ marked points $x_1, \ldots, x_n$. The datum (2.3) can be combinatorially realized...
using the colored graphs described in (2.1) and (2.2). These graphs provide a cell decomposition of $S^k_{g,n,\text{dec}}$ in the same way as the usual Strebel graphs provide a cell decomposition of $M^\text{dec}_{g,n}$.

3 Vertices

Let $\{M_a\}_{a \in \mathbb{Z}_k}$ be a collection of matrices $M_a \in gl(N)$. Let us consider terms of the form $\text{Tr} (M_{a_1} \ldots M_{a_n})$. These terms are conveniently represented as colored fat vertices $v = [(a_1, \ldots, a_n)] \in V_n$, where

$$V_n = \{(a_1, \ldots, a_n) \in (\mathbb{Z}_k)^n\}/\sim$$

and $\sim$ represents equivalence with respect to cyclic rotations of the indices. We have the formula

$$\text{Tr} \left( \sum_{a \in \mathbb{Z}_k} M_a \right)^n = \sum_{v \in V_n} \frac{n}{\#\text{Aut} v} \text{Tr} M_{a_1(v)} \ldots M_{a_n(v)}. \quad (3.2)$$

As a matter of fact,

$$\text{Tr} \left( \sum_{a \in \mathbb{Z}_k} M_a \right)^n = \sum_{(a_i) \in (\mathbb{Z}_k)^n} \text{Tr} M_{a_1} \ldots M_{a_n} \quad (3.3)$$

and the group $\mathbb{Z}_k$ of cyclic rotations acts on the set of $\text{Tr} M_{a_1} \ldots M_{a_n}$ terms with fixed points of multiplicity $\#\text{Aut} v$; moreover, all the $n/\#\text{Aut} v$ terms belonging to a given orbit give the same contribution to the sum.

4 Perturbative expansion

Let $d\mu_a(M_a)$ be a collection of gaussian measures and let $d\mu(M) = \Pi_{a \in \mathbb{Z}_k} d\mu_a(M_a)$. Let us introduce the average

$$\langle \cdot \rangle = \frac{\int_{(\mathcal{H}_N)^t} \left( \cdot \right) d\mu(M)}{\int_{(\mathcal{H}_N)^t} d\mu(M)}, \quad (4.1)$$

where $\mathcal{H}_N$ is the space of $N \times N$ hermitian matrices. The function

$$Z(\mu, c) = \left\langle \exp \text{Tr} \sum_{k=1}^K \sum_{\{a^k\}} \frac{c_k(a_1^k, \ldots, a_{l(k)}^k)}{\#\text{Aut} v_k} M_{a_1^k} \ldots M_{a_{l(k)}^k} \right\rangle, \quad (4.2)$$

\footnote{The suffix “dec” (“decorated”) refers to the fact that each curve carries the additional datum of $n$ positive numbers $p_1, \ldots, p_n$ associated to the punctures.}
where the $c_k$ are arbitrary coefficients, can be expanded in a (formal) power series as

$$Z(\mu, c) = \sum_{m_1, \ldots, m_K} \sum_{\{a^{k,r}\}} \left( \prod_{k=1}^{K} \frac{1}{n_k! \# \text{Aut } v_k^{n_k}} \right) \cdot c_k(a_1^k, \ldots, a_{n_k}^k)^{n_k} \cdot \left\langle \prod_{k=1}^{K} \prod_{r_k=1}^{n_k} \text{Tr} \left( M_{a_1}^{a_{r_k}} \ldots M_{a_{n_k}}^{a_{r_k}} \right) \right\rangle.$$  

By Wick’s theorem this decomposes in a sum over fat graphs. We get $\rho(\mathcal{X})$ identical contributions for each isomorphism class $[\mathcal{X}]$. In view of the identity

$$\rho(\mathcal{X}) \cdot \# \text{Aut } \mathcal{X} = \prod_{k=1}^{K} n_k! \cdot (\# \text{Aut } v_k)^{n_k}$$

which can be easily verified [3], we finally get

$$Z(\mu, c) = \sum_{[\mathcal{X}]} \frac{C(\mathcal{X})}{\# \text{Aut } \mathcal{X}},$$

where $\text{Aut } \mathcal{X}$ is the group of automorphisms of the graph $\mathcal{X}$ with coloring $c : X_1 \rightarrow Z_k$ such as in (2.1), (2.2) and $C(\mathcal{X})$ is computed according to the usual Feynman rules as a product of contributions from the edges and vertices of the graph $\mathcal{X}$. Eq. (4.5) has the form needed in order to represent the combinatorial version of the integration of a volume form over the orbifold $S_{g,n}^k$.

5 Matrix integral

We will now build up a matrix integral whose perturbative expansion contains terms corresponding to colored graphs verifying (2.1) and (2.2). The action $S$ must satisfy the following conditions:

i) terms of the form $\text{Tr} \left( M_{a_1} \ldots M_{a_n} \right)$ must come with a weight $1/\# \text{Aut } v$ in order to implement (4.5);

ii) terms of the form $\text{Tr} \left( M_{a_1} \ldots M_{a_n} \right)$ with $\sum_{i=1}^{n} a_i \not\equiv 0 \pmod{k}$ should not appear in order to verify (2.1);

iii) only propagators of the form $\langle M_{a} M_{-a} \rangle$ must be different from zero in order to verify (2.2).

The three conditions are easily realized:

i) From (5.2) we see that the weight $1/\# \text{Aut } v$ naturally comes from terms of the form $\text{Tr} \left( \sum c_a M_a \right)^n$;

ii) terms with $\sum_{j=1}^{n} a_j \not\equiv 0 \pmod{k}$ can be canceled by choosing $c_a = \exp \left( \frac{ia\phi}{R} \right)$ and taking residues: we will then have a sum of terms of the form

$$\int_{0}^{2\pi R} \frac{d\phi}{2\pi R} \left( \sum_a \exp \left( \frac{ia\phi}{R} \right) M_a \right)^n,$$  

(5.1)
with $R$ a new positive real parameter.

\( \text{iii) in order to implement the third condition we will take in the quadratic part only terms of the form } \text{Tr } M_a M_{-a} \text{ and extend integration to the space of matrices } M_a \text{ such that } M_a = M_{-a}^\dagger. \)

More precisely, let us define the space of matrix trigonometric polynomials

$$\mathcal{T}_{2k+1}^+ = \{ M(\phi) : S_R^2 \to gl(N) ; M(\phi) = \sum_{a=-k}^{k} \exp\left(\frac{i a \phi}{R}\right) M_a, M_a = M_{-a}^\dagger, M(\phi) > 0 \}. \quad (5.2)$$

(In the following we will restrict for notational convenience to the $Z_{2k+1}$-case). Take

$$DM(\phi) = \prod_{a=-k}^{k} dM_a \quad (5.3)$$

and define

$$Z_{2k+1} = \frac{\int_{\mathcal{T}_{2k+1}^+} DM(\phi) \exp(-S[M(\phi)])}{\int_{\mathcal{T}_{2k+1}^+} DM(\phi) \exp(-\frac{1}{2}Q[M(\phi)])}, \quad (5.4)$$

where $S$ can be chosen e.g. to be given by (5.6) or (5.9) and

$$\frac{Q}{2} = \frac{1}{2} \int_0^{2\pi R} \frac{d\phi}{2\pi R} \text{Tr} [M(\phi)]^2 = \text{Tr} \left( \frac{M_0^2}{2} + \sum_{a=1}^{k} M_a M_{-a} \right), \quad (5.5)$$

which gives $\langle (M_a)_{ij} (M_b)_{kl} \rangle = \delta_{b,-a} \delta_{il} \delta_{jk}$ in the case (5.6) and the obvious generalization $Q_A$ in the case (5.3).

**Extension of the Penner model:** Take

$$S = -Nt \int_0^{2\pi R} \frac{d\phi}{2\pi R} \text{Tr} [M(\phi) + \log(1 - M(\phi))] \quad (5.6)$$

$$= \frac{Q}{2} + Nt \sum_{v \in V'} \frac{1}{\# \text{Aut } v} \text{Tr} (M_{a_1(v)} \ldots M_{a_n(v)}),$$

where $V = \bigcup_{n=3}^{+\infty} V_n$ and

$$V'_n = \{ [(a_1, \ldots, a_n)] \in V_n : \sum_{j=1}^{n} a_j \equiv 0 \pmod{k} \}. \quad (5.7)$$

Repeating Penner’s computation we finally get

$$\log Z_{2k+1}(N, t) = \sum_{g,n} \chi(S_{g,n}^{2k+1}) \cdot N^{2-2g} t^{2-2g-n}, \quad (5.8)$$

where $\chi(S_{g,n}^{2k+1})$ is now the virtual Euler characteristic of the space $S_{g,n}^{2k+1}$. \[3\]

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3 The virtual Euler characteristic is the generalization to orbispaces of the usual Euler characteristic, and it is obtained summing the weight function $(-1)^{\text{dim } c}/\# \text{Aut } c$ over all the orbicells $c$ of the orbispace (in the case of ordinary spaces it would be just $(-1)^{\text{dim } c}$).
Extension of the Kontsevich model: Take

\[ S = - \int_0^{2\pi R} \frac{d\phi}{2\pi R} \text{Tr} \left( -\frac{\Lambda}{2} [M(\phi)]^2 + \frac{i}{6} [M(\phi)]^3 \right) \]  
\[ = \frac{QA}{2} - \frac{i}{2} \sum_{v \in V_3} \frac{1}{\# \text{Aut} \ v} \text{Tr} (M_{a_1(v)} M_{a_2(v)} M_{a_3(v)}). \]  

(5.9)

Repeating Kontsevich computations we get

\[ \log Z_{2k+1} = \sum_k \left( \tau_0^{k_0} \tau_1^{k_1} \ldots \right) s_{2k+1}^{2g,n} \frac{t_{\tau_j}^{k_j}}{k_j!} \]  
\[ = \left( \frac{2\pi i}{2k+1} \right)^n \int_0^{2\pi R} \frac{d\phi}{2\pi R} \text{Tr} \left( \hat{M} \left( \sum_{a=-k}^k \epsilon^a M_a \right) \right)^n. \]  

(5.10)

where the subscript indicates that the Chern forms \( \tau_0, \tau_1, \ldots \) are now integrated over \( S_{2k+1}^{2g,n} \) instead than over \( M_{g,n} \).

6 The kinetic term

The extensions (5.6) and (5.9) respectively of the Penner and Kontsevich model are trivial since \( S_{2k+1}^{2g,n} \) is just a non-branched covering of \( M_{g,n} \) made-up of \((2k+1)^{2g+n-1}\) sheets homeomorphic to \( M_{g,n} \). This can also be seen directly by discrete Fourier transforming, \( \hat{\text{M}}_j = \sum_{a=-k}^k \epsilon^a M_a \) with \( \epsilon = \exp \frac{2\pi i}{2k+1}, j = -k, \ldots, k \), and noticing that

\[ \sum_{j=-k}^k (\hat{\text{M}}_j)^n = \sum_{j=-k}^k \left( \sum_{a=-k}^k \epsilon^a M_a \right)^n = \int_0^{2\pi R} \frac{d\phi}{2\pi R} \left( \sum_{a=-k}^k \exp \left( \frac{2\pi i a \phi}{R} \right) M_a \right)^n. \]  

(6.1)

This means that each \( Z_{S_{2k+1}^{2g,n}} \) factorizes in a \((2k+1)\)-fold product of \( Z_{M_{g,n}} \).

However, we are now in a position to make a minimal extension to these trivial models by adding to \( Q/2 \) a kinetic term of the form

\[ \frac{1}{2} \int_0^{2\pi R} \frac{d\phi}{2\pi R} \text{Tr} [\hat{M}(\phi)]^2 = \sum_{a=1}^k \frac{a^2}{\mathcal{R}} M_{a} M_{-a}. \]  

(6.2)

This gives the propagator

\[ \langle (M_a)_{ij} (M_b)_{kl} \rangle = \delta_{a,-b} \frac{\delta_{il} \delta_{jk}}{1 + a^2 / \mathcal{R}^2}, \]  

(6.3)

meaning that high values of \( a \) are disfavoured, \( i.e. \), cycles on the surface repels each other. This gives a simple dynamics of cycles on surfaces.

The new partition function, which is now defined as an integral over \( S_{g,n}^{2k+1,\text{dec}} \) does no more project to a linear combination of integrals over \( S_{g,n}^{2k+1} \), since the
product $\prod_{x} \frac{1}{1 + a^2(x)/R^2}$ now depends (as an almost everywhere constant function) on the lengths $p_i$ of the boundary components of the graphs.

The matrix models that are obtained by adding the term (6.2) to (5.6) or to (5.9) are representations of $c = 1$ matter coupled to two-dimensional quantum gravity with target space being a circle of radius $R$. This model has been studied by Gross and Klebanov [4], using results of Marchesini and Onofri [5]. For $R \to +\infty$ the non-singlet sector of the matrix model decouples, since the non-singlet state energies are logarithmically divergent in the double scaling limit. The non-singlet degrees of freedom are identified with vortices on the world sheet, which are dynamically irrelevant for large radius, but condense at a critical value of the radius, giving rise to a Kosterlitz-Thouless phase transition. The complete free energy is given by a sum

$$F = F_s + F_{\text{ns}},$$

where the non-singlet correction to the free energy has the simple form

$$F_{\text{ns}} = -\frac{1}{2\pi R} N^2 \exp\left(-2\pi R \delta(\mu)\right)$$

with $\delta(\mu) \sim |\log \mu|$ in the continuum limit $\mu \to +\infty$, $N\mu = \text{const}$. In [4] $\mu$ plays the role of an UV-cutoff, and it has to be identified with $k_0 = k/R$ in our context. For $k_0 \to +\infty$ we have thus $F_{\text{ns}} \sim N^2/Rk_0^2\pi R$. Letting $k \to +\infty$ with fixed $R$ the groups $H_1(C,\mathbb{Z}_k)$ give the group $H_1(C,\mathbb{Z})$. If we instead let $k \to +\infty$, $R \to +\infty$ with $k_0 = \text{const}$ the groups $H_1(C,\mathbb{Z}_k)$ go over to $H_1(C,Rk_0)$ with $Rk_0 = R/k_0\mathbb{Z}$.

In our construction the parameter $R$ interpolates between the topologically trivial situation $R = +\infty$ and a new topological phase which has yet to be investigated. We conjecture that the model at the critical point $R = R_c = 1$ should describe some non-trivial topological characteristics of the space $S_{g,n}$.

**Acknowledgments**

We thank E. Arbarello, M. Cornalba, R. Dijkgraaf and C. Procesi for inspiring discussions, and E. Witten for his comments about the relation of our work with the model described in Ref. [3]. One of us (A.G.) is pleased to thank B. Dubrovin for long discussions had on the subject and for his warm support.
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