Corrigendum: Quasi-exact-solvability of the $A_2/G_2$ Elliptic model: algebraic forms, $sl(3)/g^{(2)}$ hidden algebra, polynomial eigenfunctions, (2005 J. Phys. A: Math. Theor. 48 155201) 

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- The first two lines of equation (11) should be
  \[ \Delta_\nu(x, y; \tau, \mu) = 3 \left( \frac{x}{3} + \tau x^2 + \mu x^3 + (\mu - \tau^2) y^2 - \mu \tau xy^2 - \mu^2 x^2 y^2 \right) \frac{\partial^2}{\partial x^2} \]
  \[ + y \left( 3 + 8 \tau x + 7 \mu x^2 - 3 \mu \tau y^2 - 6 \mu^2 xy^2 \right) \frac{\partial^2}{\partial x \partial y} + \ldots \]

- The first line of equation (24) should be
  \[ k_{A_2}(x, y) = 2 \nu (1 + 3 \nu)(2 + 3 \nu) \mu y \left( 2 \tau + 3 \mu x - 3 \mu^2 y^2 \right) + . \]

- In the next four lines after equation (24)
  This operator is anti-invariant with respect to $y \rightarrow -y$,
  \[ k_{A_2}(x, y) = -k_{A_2}(x, -y), \]
  ...Thus, after the change of variables $(x, y) \rightarrow (u = x, v = y^2)$ the operator $k_{A_2}^2(u, v)$ remains algebraic.

- Page 10, equation (25):
  line 4: 1st term $\rightarrow J_3^2$
  line 9: instead of ’+.’ $\rightarrow -$.
  line 11: $-4/3 \rightarrow +4/3$
  the last line: no ’$y^2$’ in the first term (it must be dropped)
  The correct form of equation (25) looks as
  \[ k_{A_2} = J_1^2 J_4 + 3 (2 + 3 \nu) \tau J_1 J_3 J_4 - \frac{2}{9} (1 + 3 \nu)(2 + 3 \nu) J_1 J_3 J_5 \]
  \[ + 3 \tau J_1 J_3 J_6 + \nu (2 + 3 \nu) J_1 J_3 J_5 - 3 \nu J_1 J_6 J_5 - (1 + 9 \nu) \tau J_3 J_5 J_6 \]
  \[ + \frac{1}{3} (12 \mu + 12 \tau^2 - (1 + 3 \nu)(11 \mu + 16 \tau^2) + (1 + 3 \nu)^2 (\mu + 8 \tau^2)) J_5^2 J_4 \]
\[-\frac{8}{9}(1 + 3\nu)(2 + 3\nu)\tau J_5^2 J_8 \]
\[+ 4(2 + 3\nu)(1 - 3\nu)(\mu \tau J_5^2 J_8) \]
\[+ \frac{2}{3}(3\tau^2 + (1 + 3\nu)(5\mu + 4\tau^2) - (1 + 3\nu)^2(\mu + 8\tau^2))J_5 J_4 J_3 \]
\[+ \left(\mu + 8\tau^2 + 2(1 + 3\nu)(\mu - 4\tau^2)\right)J_5 J_6 J_6 \]
\[+ \frac{2}{9}(1 + 36\nu + 72\nu^2)\tau J_5 J_5 J_6 - (1 - 3\nu)J_5 J_6 J_2 - \frac{4}{3}(1 + 6\nu)\tau J_5 J_6 J_5 \]
\[+ 2(2 + 3\nu)\mu^2 J_3 J_7 J_8 \]
\[- 4(1 + 3\nu)\mu^2 J_6 J_6 J_6 + \frac{1}{3}(1 + 3\nu)(2 + 3\nu)(\mu + 8\tau^2)J_4 J_3^2 \]
\[-(\mu(1 + 6\nu) - 2(5 + 12\nu)\tau^2)J_5 J_3 J_6 \]
\[+ \frac{4}{3}(1 + 3\nu)(2 + 3\nu)\mu^2 J_4 J_4 J_7 - \tau(3\mu - 2\tau^2)J_4^3 \]
\[-3\mu(2\mu - \tau^2)J_5^2 J_8 - 3(\mu - 2\tau^2)J_6 J_6^2 \]
\[+ 2(7 + 6\nu)\mu\tau J_4 J_6 J_7 - 3\mu^2\tau J_3 J_4^2 \]
\[\frac{1}{9}(2 + 9\nu^2)J_5 J_5 J_1 - \frac{4}{9}(1 + 18\nu^2)\tau J_5 J_3^2 \]
\[\frac{-4}{3}(2 + 3\nu)\mu J_3 J_5 J_7 - \frac{2}{27}J_5^3 + \frac{2}{3}(1 + 6\nu)\mu J_5 J_7 J_5 - J_6 J_3 J_6 - 2(1 - 4\nu)\tau J_6 J_6 J_6 \]
\[-2\tau J_6 J_6 J_6 - \frac{5}{3}\mu J_6 J_5 J_7 - \frac{1}{3}\mu^2(5 - 72\nu^2)J_7 J_3 J_4 - \mu^2(1 + 6\nu)J_7 J_3 J_8 \]
\[+ 4\mu^2 J_7 J_6 J_6 + 12\mu\tau J_6 J_6^2 - 9\mu\tau J_6 J_6 J_6 - 2\mu^3 J_6^3 \].

- Page 11, equation (29), third line, second term: \( t \rightarrow u \), the expression should be \( T_1 = \nu \partial_u J_{n}(u) \).
Quasi-exact-solvability of the $A_2/G_2$ elliptic model: algebraic forms, $sl(3)/g^{(2)}$ hidden algebra, and polynomial eigenfunctions

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Abstract
The potential of the $A_2$ quantum elliptic model (three-body Calogero–Moser elliptic model) is defined by the pairwise three-body interaction through the Weierstrass $℘$-function and has a single coupling constant. A change of variables has been found, which are $A_2$ elliptic invariants, such that the potential becomes a rational function, while the flat space metric, as well as its associated vector, are polynomials in two variables. It is shown that the model possesses the hidden $sl(3)$ algebra—the Hamiltonian is an element of the universal enveloping algebra $U_{sl(3)}$ for the arbitrary coupling constant—thus, it is equivalent to the $sl(3)$-quantum Euler–Arnold top. The integral, in a form of the third order differential operator with polynomial coefficients, is constructed explicitly, being also an element of $U_{sl(3)}$. It is shown that there exists a discrete sequence of the coupling constants for which a finite number of polynomial eigenfunctions, up to a (non-singular) gauge factor, occurs. For these values of the coupling constants there exists a particular integral: it commutes with the Hamiltonian in action on the space of polynomial eigenfunctions, and the Hamiltonian is invariant with respect to two-dimensional projective transformations. It is shown that the $A_2$ model has another hidden algebra $g^{(2)}$ introduced in Rosenbaum et al (1998 Int. J. Mod. Phys. A 13 3885). The potential of the $G_2$ quantum elliptic model (three-body Wolfes elliptic model) is defined by the pairwise and three-body interactions through the Weierstrass $℘$-function and has two coupling constants. A change of variables has been found, which are $G_2$ elliptic invariants, such that the potential becomes a rational function, while the flat space metric, as well as its associated vector, are polynomials in two variables. It is shown the model...
possesses the hidden $\mathfrak{g}^{(2)}$ algebra. It is shown that there exists a discrete family of the coupling constants for which a finite number of polynomial eigenfunctions up to a (non-singular) gauge factor occurs. For these values of the coupling constants, there exists a particular integral.

Keywords: three-body quantum elliptic system, hidden algebra, polynomial eigenfunctions, Euler–Arnold quantum top

The $A_2$ elliptic model (three-body elliptic Calogero–Moser model, see, e.g., [1]) describes three particles on the real line with the pairwise interaction given by the Weierstrass $\wp$-function. It is characterized by the Hamiltonian

$$
\mathcal{H}_{A_2}^{(e)} = \frac{1}{2} \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + \nu (\nu - 1) \left( \wp(x_1 - x_2) + \wp(x_2 - x_3) + \wp(x_3 - x_1) \right)
$$

$$
\equiv - \frac{1}{2} \Delta^{(3)} + V,
$$

(1)

where $\Delta^{(3)}$ is the three-dimensional Laplace operator and $\kappa \equiv \nu (\nu - 1)$ is the coupling constant. The Weierstrass function $\wp(x) \equiv \wp(x|g_2, g_3)$ (see, e.g., [2]) is defined as

$$
(\wp'(x))^2 = 4 \wp^3(x) - g_2 \wp(x) - g_3 = 4(\wp(x) - e_1)(\wp(x) - e_2)(\wp(x) - e_3),
$$

(2)

where $g_{2,3}$ are its invariants and $e_{1,2,3}$ are its roots; usually, $e \equiv e_1 + e_2 + e_3 = 0$ is chosen. As was indicated in [3] the whole symmetry of (1) is the central and co-central extended loop group $\tilde{L}(SL(3))$. Note that the spectrum of the quasi-periodic eigenfunctions is usually treated in the Bethe Anzatz formalism; see, e.g., [4, 5] and references therein. It is worth mentioning that the spectrum was treated perturbatively in [6] and [7].

If in (2) the trigonometric limit is taken, $\Delta \equiv g_2^3 + 27 g_3^3 = 0$, with one of the periods going to infinity, the Hamiltonian of the $A_2$ trigonometric/hyperbolic Calogero–Moser–Sutherland model (three-body Sutherland model) occurs. If both invariants $g_2 = g_3 = 0$, we arrive at the $A_2$-rational (or, saying it differently, at the three-body Calogero–Moser) model. For future convenience, we parameterize the invariants as follows:

$$
g_2 = 12 (\tau^2 - \mu), \quad g_3 = 4\tau(2\tau^2 - 3\mu),
$$

(3)

where $\tau$ and $\mu$ are the parameters.

The Hamiltonian (1) is translation invariant; thus, it makes sense to introduce center-of-mass coordinates

$$
Y = \sum_{i=1}^{3} x_i, \quad y_i = x_i - \frac{1}{3} Y,
$$

(4)

with the condition $\sum_{i=1}^{3} y_i = 0$. The Laplacian $\Delta^{(3)} \equiv \sum_{i=1}^{3} \frac{\partial^2}{\partial y_i^2}$ in these coordinates takes the form

$$
\Delta^{(3)} = 3 \partial_Y^2 + \frac{2}{3} \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2} \right).
$$
Separating out the center-of-mass coordinate \( Y \), a two-dimensional Hamiltonian arises

\[
\mathcal{H}_{A_2} = -\frac{1}{3} \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2} \right) + \nu (\nu - 1) \left( \varphi(y_1 - y_2) + \varphi(2y_1 + y_2) + \varphi(y_1 + 2y_2) \right). \tag{5}
\]

Since we will be interested in the general properties of the operator \( \mathcal{H}_{A_2} \) without a loss of generality, we can assume that the operator (5) is defined on the real plane, \( y_{1,2} \in \mathbb{R}^2 \), while the fundamental domain of the Weierstrass function \( \wp(x) \) is not fixed. The whole discrete symmetry of the Hamiltonian (5) is \( S^2 \oplus \mathbb{Z}_2 \times ((T_2)^2 \oplus (T_2)^2) \). It consists of permutation \( S^2(y_1 \leftrightarrow y_2) \), reflection \( \mathbb{Z}_2(y_{1,2} \leftrightarrow -y_{1,2}) \) and four translations \( T_{r,1(2)}: y_{1(2)} \rightarrow y_{1(2)} + 1 \) and \( T_{r,1(2)}: y_{1(2)} \rightarrow y_{1(2)} + i \tau \) (periodicity). Perhaps, \( S^2 \oplus (T_2)^2 \oplus (T_2)^2 \) can make sense as a double-affine \( A_2 \) Weyl group.

Let us consider a formal eigenvalue problem

\[
\mathcal{H}_{A_2} \Psi = E \Psi, \tag{6}
\]

without posing concrete boundary conditions. Assume \( f(x) \) to be the non-constant solution of the equation

\[
f(x)^2 = 4f(x)^3 - 12\tau f(x)^2 + 12\mu f(x). \tag{7}
\]

Thus, it can be written as

\[ f(x) = \varphi(x|g_2, g_3) + \tau, \]

cf (2), (3). Now let us introduce the new variables

\[
x = \frac{f(y_1) - f(y_2)}{f(y_1)f'(y_2) - f(y_2)f'(y_1)}, \quad y = \frac{2(f(y_1) - f(y_2))}{f(y_1)f'(y_2) - f(y_2)f'(y_1)}, \tag{8}
\]

which have the property

\[ x(-y_1, -y_2) = x(y_1, y_2), \quad y(-y_1, -y_2) = -y(y_1, y_2). \]

They are invariant with respect to the partial discrete symmetry of the Hamiltonian (5): \( S^2 \oplus (T_2)^2 \oplus (T_2)^2 \). It can be shown that in the rational limit \( \tau = \mu = 0 \), where the three-body Calogero–Moser model emerges, the variables \( x, y \) coincide with those found in Rühl–Turbiner [8]

\[
x = -\left( y_1^2 + y_2^2 + y_1y_2 \right), \quad y = -y_1y_2 \left( y_1 + y_2 \right). \tag{8.1}
\]

In the trigonometric limit \( \mu = 0 \) the three-body Sutherland Hamiltonian emerges in a form of the algebraic operator [8]

\[
x = \frac{1}{\alpha^2} \left[ \cos (\alpha y_1) + \cos (\alpha y_2) + \cos \left( \alpha \left( y_1 + y_2 \right) \right) - 3 \right],
\]

\[
y = \frac{2}{\alpha^3} \left[ \sin (\alpha y_1) + \sin (\alpha y_2) - \sin \left( \alpha \left( y_1 + y_2 \right) \right) \right]. \tag{8.2}
\]

Here \( \alpha \) is a parameter such that \( \tau = \alpha^2/12 \). It is worth noting that the variables (8.1) and (8.2), being \( A_2 \) Weyl invariants, were obtained, making the averaging over some orbits in \( A_2 \) root space; see, e.g., [15]. It is an interesting open question whether (8) can be obtained
as a result of averaging over some orbits, in particular, orbits generated by fundamental weights.

After tedious calculations it can be found that the $A_2$ elliptic Calogero–Moser potential (see (1), (5)) in the new variables (8) takes a rational form,

$$V(x, y) = \frac{3\nu(\nu - 1)}{4} \left(\frac{x + 2\tau x^2 + \mu x^3 - 6(\mu - \tau^2)y^2 + 3\mu xy^2}{D}\right)^2,$$

where

$$12D(x, y) = 9\nu^2x^4y^2 + 54\tau^2\nu^2x^2y^4 + 27\mu^2(3\tau^2 - 4\mu)y^6 - 12\mu x^5 - 72\tau\mu x^3y^2$$

$$- 108\mu^3\tau^2x^4y^4 - 18(4\tau^2 + 5\mu)x^4y^2$$

$$- 54\tau(2\tau^2 - 3\mu)y^4 - 4x^3 - 108\tau xy^2 - 27y^2.$$ 

It is worth noting that the potential (9) is symmetric in $V(x, y) = V(x, -y)$, as well as $D(x, y) = D(x, -y)$. Furthermore, the two-dimensional Laplacian in (5) becomes the Laplace–Beltrami operator

$$\Delta_\tau(x, y) = g^{-1/2} \frac{\partial}{\partial z_i} g^{\frac{1}{2}}g^{ij} \frac{\partial}{\partial z_j} = g^{ij} \frac{\partial^2}{\partial z_i \partial z_j} + \left(\frac{g_{ij}g^{ij}}{2} + g^{ij}\right) \frac{\partial}{\partial z_j},$$

where $g_{ij} \equiv \frac{\partial g}{\partial z_i} z_j$, $g_{ij} \equiv \frac{\partial g}{\partial z_j} z_i$, and $\text{Det}(g_{ij}) = g$, which in (x,y)-coordinates looks explicitly like

$$\Delta_\tau(x, y; \tau, \mu) = 3\left(\frac{x}{3} + \tau x^2 + \mu x^3 + (\mu - \tau^2)y^2 - \mu xy^2 - \mu^2 x^2 y^2\right)\frac{\partial^2}{\partial x^2} y$$

$$\times \left(3 + 8\tau x + 7\mu x^2 - 3\mu y^2 - 6\mu^2 xy\right)\frac{\partial^2}{\partial y} y$$

$$+ \left(-\frac{x^2}{3} + 3\tau y^2 + 4\mu xy^2 - 3\mu^2 y^2\right)\frac{\partial^2}{\partial y^2}$$

$$+ \left(1 + 4\tau x + 5\mu x^2 - 3\mu xy^2 - 6\mu^2 xy^2\right)\frac{\partial}{\partial x}$$

$$+ 2y\left(2\tau + 3\mu x - 3\mu^2 y^2\right)\frac{\partial}{\partial y},$$

(11)

Thus, the flat contravariant metric, defined by the symbol of the Laplace–Beltrami operator in these coordinates, becomes a two-parametric polynomial in $x, y$. The Hamiltonian is the sum of the Laplace–Beltrami operator (11) with polynomial coefficients and the rational potential (9). Taking in the Laplace–Beltrami operator (11), the rational limit $\tau = \mu = 0$, we arrive at the Laplace–Beltrami operator $\Delta_\tau^{(rat)}$ of the three-body Calogero–Moser model [8]. If the trigonometric limit $\mu = 0$ is taken, the Laplace–Beltrami operator $\Delta_\tau^{(trig)}$ of the three-body Sutherland model emerges [8].

The denominator $D$ in (9) turns out to be equal to the determinant of the contravariant metric $D = \text{Det}(g^{ij}) = \frac{1}{g}$. It is worth noting some properties of the determinant $D$: in the rational case, $D^{1/2}$ is the zero mode of the Laplace–Beltrami operator

$$\Delta_\tau^{(rat)} D^{1/2} = 0.$$


In the trigonometric case
\[ \Delta_g^{(\text{trig})} D^{1/2} = -12 \pi D^{1/2}, \]
and in the general case,
\[ \Delta_g (x, y; \tau, \mu) D^{1/2} = -12 \tau \left( 1 - \mu \left( 2x - 3 \mu y^2 \right) \right) D^{1/2}. \]

It is easy to verify that the determinant \( D(x, y) \) given by formula (10) can be written as
\[ D(x, y) = \frac{1}{12} W^2, \] (12)
where the function
\[ W = \frac{\partial y}{\partial y_2} \frac{\partial x}{\partial y_1} - \frac{\partial x}{\partial y_2} \frac{\partial y}{\partial y_1}, \] (13)
is the Jacobian associated with the change of variables \((y_1, y_2) \rightarrow (x, y)\). The equation \( w^2 = 12 D(x, y) \) can be considered the equation for the elliptic surface\(^4\). One can verify that the Jacobian \( W \) admits a representation in factorized form,
\[ W(y_1, y_2) = \frac{\sigma(y_1 - y_2) \sigma(y_1 + 2y_2) \sigma(y_2 + 2y_1)}{\sigma^3(y_1) \sigma^3(y_2) \sigma^3(y_1 + y_2)}. \] (14)

Here the Weierstrass \( \sigma \)-function [2] has the parameters \( g_i \) given by (3) and \( e = -\tau \) is a root of the \( \wp \)-Weierstrass function, \( \wp'(-\tau) = 0 \). The function \( \sigma_1 \) is the \( \sigma \)-function associated with the half-period \( \omega \) corresponding to the root \(-\tau\), thus, \( \wp'(\omega) = -\tau \). By definition (see [2]),
\[ \sigma_1(x) = \frac{\sigma(x + \omega)}{\sigma(\omega)} \exp \left( -\frac{\sigma'(\omega)}{\sigma(\omega)} x \right). \]

Note that in the one-dimensional case \( n = 1 \) the Jacobian becomes
\[ W(y_i) = - \wp'(y_i) = \frac{\sigma(2y_i)}{\sigma_1(y_i)^2}. \]
see, [2], chapter 20, problem 24.

**Conjecture.** For arbitrary \( n \) the Jacobian
\[ W = \frac{\prod_{i>j}^n \sigma(y_i - y_j)}{\prod_{i=1}^{n+1} \sigma_i^{e+1}(y_i)}, \]
where \( y_1 + y_2 + \cdots + y_{n+1} = 0. \)

There are two essentially different degenerations of the \( \wp \)-Weierstrass function in the trigonometric case: (I) when \( e = -\tau \) is a double root; thus, \( e = 2\tau \) is the simple root and then \( \mu = 0 \), and (II) when \( e = -\tau \) is a simple root and \( \mu = -\frac{3}{4} \tau^2 \). In both cases
\[ 4 \text{ In the case of the } A_1 \text{ elliptic Calogero model, the variable } x, \text{ which is the invariant with respect to the symmetry of the } A_1 \text{ Hamiltonian } Z_2 \oplus (T_2) \oplus (T_2), \text{ is equal to Weierstrass function, } x = \wp(x) \text{ (see [9]), the function } W = \frac{d}{dx} \text{ is the Jacobian and the determinant } D(x) \text{ is a cubic polynomial } D = P_3(x); \text{ the equation analogous to (12) defines the elliptic curve, } w^2 = P_3(x). \]
\[ \phi(x) = \frac{\alpha^2}{4 \sin^2 \frac{\alpha}{2}} - \frac{\alpha^2}{12} \]

but in case (I) \( \tau = \frac{\alpha^2}{12} \), whereas in case (II) \( \tau = -\frac{\alpha^2}{6} \). For the first degeneration the Jacobian is

\[ W(y_1, y_2) = \frac{8}{\alpha^3} \sin \frac{\alpha (y_1 - y_2)}{2} \sin \frac{\alpha (y_1 + 2y_2)}{2} \sin \frac{\alpha (2y_1 + y_2)}{2} \]

and for the second one the Jacobian is factorized as follows

\[ W(y_1, y_2) = \frac{8}{\alpha^3} \sin \frac{\alpha (y_1 - y_2)}{2} \sin \frac{\alpha (y_1 + 2y_2)}{2} \sin \frac{\alpha (2y_1 + y_2)}{2} \]

where \( \alpha \) is a parameter such that \( \tau = \alpha^2/12 \). The factorization of the case (I) cannot be generalized to the elliptic case where, in general, we have no multiple roots.

Surprisingly, the gauge rotation of (5) with determinant \( D(10) \) as a gauge factor

\[ h(x, y) = -3D^{-\frac{1}{3}}(H_{\Lambda_z} - E_0)D^{\frac{1}{3}}, \]

where \( E_0 = 3u(3\nu + 1)\tau \), transforms the Hamiltonian \( H_{\Lambda_z} - E_0 \) into the algebraic operator (16).

\[ h(x, y) = \left( x + 3rx^2 + 3\mu x^3 + 3(\mu - \tau^2)y^2 - 3\mu xy^2 - 3\mu^2 x^2 y^2 \right) \frac{\partial^2}{\partial x^2} \]

\[ + \left( 3 + 8tx + 7\mu x^2 - 3\mu ry^2 - 6\mu^2 xy^2 \right) \frac{\partial^2}{\partial x \partial y} \]

\[ + \frac{1}{3} \left( -x^2 + 9ry^2 + 12\mu xy^2 - 9\mu^2 y^4 \right) \frac{\partial^2}{\partial y^2} \]

\[ + (1 + 3\nu)(1 + 4tx + 5\mu x^2 - 3\mu ry^2 - 6\mu^2 xy^2) \frac{\partial}{\partial x} \]

\[ + 2(1 + 3\nu)y\left( 2x + 3\mu x - 3\mu^2 y^2 \right) \frac{\partial}{\partial y} \]

\[ + 3\nu(1 + 3\nu)\mu \left( 2x - 3\mu y^2 \right). \]

Note the important \( \mathbb{Z}_2 \) symmetry property of this gauge-rotated Hamiltonian,

\[ h(x, y) = h(x, -y). \]

Thus, it follows that in the variables \((u = x, v = y^2)\) the operator \( h \) remains algebraic,

\[ h(u, v) = \left( u + 3\tau u^2 + 3\mu u^3 + 3(\mu - \tau^2)v - 3\mu uv - 3\mu^2 u^2 v \right) \frac{\partial^2}{\partial u^2} \]

\[ + 2v \left( 3 + 8\tau u + 7\mu u^2 - 3\mu rv - 6\mu^2 uv \right) \frac{\partial^2}{\partial u \partial v} \]

\[ + 4v \left( -\frac{u^2}{3} + 3\tau v + 4\mu uv - 3\mu^2 v^2 \right) \frac{\partial^2}{\partial v^2} \]

\[ + (1 + 3\nu)(1 + 4\tau u + 5\mu u^2 - 3\mu rv - 6\mu^2 uv) \frac{\partial}{\partial u} \]
It is an alternative algebraic form of the gauge-rotated operator (15). Note that the variables \((u, v)\) are invariants with respect to the whole discrete symmetry of the Hamiltonian (5): \(S^3 \oplus \mathbb{Z}_2 \ltimes ((T_1)^2 \oplus (T_2)^2)\), unlike the variables \((x, y)\).

The operator \(h(x,y)\) also has a property of self-similarity: the gauge-rotated operator \(\tilde{h} = D^{-m}hD^m\) with \(m = \left(\frac{1}{2} - \nu\right)\) has polynomial coefficients, as does the corresponding gauge-rotated operator \(\tilde{k}_A = D^{-m}k_A D^m\) (see later). It is easy to verify that

\[
\tilde{h}_\nu = h_{4-3\nu} = 12(1 - 2\nu) \tau.
\]

Evidently, the operator \(\tilde{h}_\nu\) has the same functional form of the potential (9) as for the operator \(h_{\nu}\).

Let

\[
J_1 = \frac{\partial}{\partial x}, \quad J_2 = \frac{\partial}{\partial y}, \quad J_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad J_4 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \quad J_5 = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}, \quad J_6 = y \frac{\partial}{\partial y},
\]

\[
J_7 = x \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 3\nu \right), \quad J_8 = y \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 3\nu \right).
\]

Notice that these formulas define a representation \((-3\nu, 0)\) of the Lie algebra \(sl(3)\) in differential operators of the first order (see, e.g., [8]). If the spin (mark) of representation

\[
-3\nu = n
\]

takes an integer value, a finite-dimensional representation appears: the space of polynomials

\[
P_n = \left\{ x^p y^q \mid 0 \leq p + q \leq n \right\}, \quad \text{dim} P_n = \frac{(n + 2)(n + 1)}{2},
\]

is preserved by \(J\). It is worth mentioning that the space (19) is invariant under linear transformations,

\[
x \rightarrow a_1 x + b_1 y + c_1, \quad y \rightarrow a_2 x + b_2 y + c_2,
\]

where \(a, b,\) and \(c\) are the parameters. It can be easily shown by direct calculation that for any \(\nu\) the operator \(h\) (16) can be rewritten in terms of \(sl(3)\) generators,

\[
h = (1 + 3\nu) J_1 J_2 - 3\nu J_3 J_1 + 3 J_1 J_6 + 3 \tau J_5^2 + 6 \tau (1 - 4\nu) J_3 J_6
\]

\[
+ 3 \left( \mu - \tau^2 \right) J_4^2
\]

\[
+ \tau (1 + 12\nu)(J_1 J_3 + J_3 J_4) + 2(1 + 3\nu) \mu J_3 J_5 - 3 \mu \tau J_3 J_5 - \left( \frac{1}{3} \right) J_5^2 + 3 \tau J_6^2
\]

\[
+ 4 \mu J_6 J_7 + \mu (1 - 6\nu) J_7 J_3 - 3 \mu^2 J_5^2.
\]

Thus, the gauge-rotated Hamiltonian \(h(x, y)\) describes an \(sl(3)\)-quantum Euler–Arnold top. Hence, the three-body elliptic Calogero–Moser model with an arbitrary coupling constant is equivalent to the \(sl(3)\)-quantum Euler–Arnold top. If the coupling constant in (1) takes discrete values
the Hamiltonian $h(x, y)$ and the Hamiltonian (5) both have a finite-dimensional invariant subspace $P_n$. Hence, there may exist a finite number of analytic eigenfunctions of the form

$$Ψ_{n,i} = P_{n,i}(x, y) \cdot D^{-2}, \quad i = 1, \ldots, \frac{(n + 2)(n + 1)}{2},$$

where polynomial $P_{n,i}(x, y) \in P_n$, see (19). For example, for $n = 0$ (at zero coupling),

$$E_{0,1} = 0, \quad P_{0,1} = 1.$$

For $n = 1$ at coupling

$$κ = \frac{4}{9},$$

the operator $h$ has a three-dimensional kernel (three zero modes) of the type $(a_1 x + a_2 y + b)$. The first nontrivial solutions appear for $n = 2$ and

$$κ = \frac{10}{9}.$$

There exist six polynomial eigenstates. Eigenvalues are given by the roots of the algebraic equation of degree 6,

$$\left(E^2 + 4πE + 4μ\right)\left(E^2 + 8πE + 4μ + 12π^2\right)\left(E^2 + 12πE + 4μ + 16π^2\right) = 0,$$

given by

$$E_{±}^{(1)} = -2\left(π ± \sqrt{π^2 - μ}\right), \quad E_{±}^{(2)} = -2\left(2π ± \sqrt{4π^2 - μ}\right), \quad E_{±}^{(3)} = -2\left(3π ± \sqrt{9π^2 - μ}\right).$$

The corresponding eigenfunctions are of the form $(a_1 x^2 + a_2 xy + a_3 y^2 + b_1 x + b_2 y + c)$. Using formulas (8) and (15), one can construct the corresponding eigenfunctions for the original Hamiltonian (1) in an explicit form.

Observation 1. Let us construct the operator

$$i_{\text{par}}^{(n)}(x, y) = \prod_{j=0}^{n} (J^0(n) + j),$$

where

$$J^0(n) = x \frac{∂}{∂x} + y \frac{∂}{∂y} - n,$$

is the Euler–Cartan generator of the algebra $sλ(3)$ (18). It can be immediately seen that the algebraic operator $h(x, y)$ (16) at integer $n$ commutes with $i_{\text{par}}^{(n)}(x, y)$,

$$\left[h(x, y), i_{\text{par}}^{(n)}(x, y)\right] \cdot P_n \rightarrow 0.$$

Hence, $i_{\text{par}}^{(n)}(x, y)$ is the particular integral [10] of the $A_2$ elliptic model (5).

It is known (see [1]) that the $A_2$ elliptic model is (completely) integrable, having a certain third-order differential operator $k_{A_2}$ as the integral. Perhaps the easiest way to find this integral is to look for it in a form of an algebraic differential operator of the third order, $[h(x, y), k_{A_2}(x, y)] = 0$. In the explicit form it is given by the following
expression:

\[ k_{A_{2}}(x, y) = -2\nu(1 + 3\nu)(2 + 3\nu)\mu y \left( 2\tau + 3\mu x - 3\mu^{2}y^{2} \right) \]
\[ + \frac{1}{3}(1 + 3\nu)(2 + 3\nu)y \left( \mu + 8\tau^{2} + 28\mu\tau x + 21\mu^{2}x^{2} - 9\mu^{2}\tau y^{2} - 18\mu^{3}xy^{2} \right) \frac{\partial}{\partial x} \]
\[ - \frac{2}{9}(1 + 3\nu)(2 + 3\nu)\mu \left( 1 + 4\tau x + 6\mu x^{2} - 24\mu\tau y^{2} - 36\mu^{2}xy^{2} + 27\mu^{3}y^{4} \right) \frac{\partial}{\partial y} \]
\[ + (2 + 3\nu)\mu \left( 3\tau + 4\left( 2\tau^{2} + \mu \right)x + 17\mu\tau x^{2} + 8\mu^{2}x^{3} \right) \]
\[ + 3\mu \left( (2 + 3\nu)\left( x + 4\tau x^{2} + 5\mu x^{3} + 3\left( \mu - 4\tau^{2} \right)y^{2} - 27\mu^{2}x^{2}y^{2} \right) \right) \frac{\partial^{2}}{\partial x^{2}} \]
\[ - \frac{2}{3}(2 + 3\nu)\left( 1 + \frac{8}{3}\tau x + 3\mu x^{2} - 7\mu\tau y^{2} - 10\mu^{2}xy^{2} + 6\mu^{3}y^{4} \right) \frac{\partial^{2}}{\partial y^{2}} \]
\[ + \left( 1 + 5\tau x + 2\left( 2\mu + 3\tau^{2} \right)x^{2} + 3\mu \left( \tau^{2} - 2\mu \right)xy^{2} + 9\mu\tau x^{3} \right) \frac{\partial^{3}}{\partial x^{3}} \]
\[ - \frac{2}{3}(2 + 3\nu)\left( 1 + \frac{8}{3}\tau x + 3\mu x^{2} - 7\mu\tau y^{2} - 10\mu^{2}xy^{2} + 6\mu^{3}y^{4} \right) \frac{\partial^{3}}{\partial y^{3}} \]
\[ + 3\mu \left( \tau^{2} - 2\mu \right)y^{4} + 19\mu\tau x^{2}y^{2} - 6\mu^{3}x^{2}y^{4} + 10\mu^{2}x^{3}y^{2} - 6\mu^{2}\tau xy^{4} \right) \frac{\partial^{3}}{\partial x^{2}\partial y} \]
\[ + 3\mu \left( (2 + 3\nu)\left( x + 10\tau x^{2} + 11\mu x^{3} - 13\mu\tau xy^{2} + 3\left( \mu - 2\tau^{2} \right)y^{2} \right) \right) \frac{\partial^{3}}{\partial x^{2}\partial y} \]
\[ + 3\mu \left( (2 + 3\nu)\left( x + 10\tau x^{2} + 11\mu x^{3} - 13\mu\tau xy^{2} + 3\left( \mu - 2\tau^{2} \right)y^{2} \right) \right) \frac{\partial^{3}}{\partial y} \]
\[ - \frac{2}{27}(2 + 3\nu)\left( 1 + \frac{8}{3}\tau x + 3\mu x^{2} - 7\mu\tau y^{2} - 10\mu^{2}xy^{2} + 6\mu^{3}y^{4} \right) \frac{\partial^{3}}{\partial x\partial y^{2}} \]
\[ - \frac{2}{27}(2 + 3\nu)\left( x + 10\tau x^{2} + 11\mu x^{3} - 13\mu\tau xy^{2} + 3\left( \mu - 2\tau^{2} \right)y^{2} \right) \frac{\partial^{3}}{\partial y^{3}} \]
\[ + 3\mu \left( (2 + 3\nu)\left( x + 10\tau x^{2} + 11\mu x^{3} - 13\mu\tau xy^{2} + 3\left( \mu - 2\tau^{2} \right)y^{2} \right) \right) \frac{\partial^{3}}{\partial y^{3}} \]

This operator is invariant with respect to \( y \rightarrow -y \),

\[ k_{A_{2}}(x, y) = k_{A_{2}}(x, -y) \]

similarly to the gauge-rotated Hamiltonian \( h(x, y) \) (see (16)). Thus, after the change of variables \( (x, y) \rightarrow (u = x, v = y) \) the operator \( k_{A_{2}}(u, v) \) remains algebraic. Let us note for \( (2 + 3\nu) = 0 \) or, saying it differently, for \( n = 2 \) the operator \( k_{A_{2}} \) becomes a third-order homogeneous differential operator and contains third derivatives only. This operator can be rewritten in terms of \( sl(3) \)-generators,
It is evident that if \( \nu = n/3 \) the operator (24) has the space \( \mathbb{P}_n \) as a finite-dimensional invariant subspace. It seems natural to assume that the gauge-rotated integral \( k_{A_2} \), written in variables \( x_{12}, x_2, x_3 \),

\[
k_{A_2} = J_2^2 J_4 + 3(2 + 3\nu)\tau J_3 J_4 - \frac{2}{9}(1 + 3\nu)(2 + 3\nu)J_3 J_5
\]

\[
+ 3\tau J_4 J_6 + 12\nu(1 + 3\nu)J_3 J_7 - 3\nu J_6 J_8 - (1 + 9\nu)\tau J_5 J_4
\]

\[
+ \frac{1}{3}(2\mu + 12\tau^2 - (1 + 3\nu)(11\mu + 16\tau^2) + (1 + 3\nu)^2(\mu + 8\tau^2))J_4^3 J_4
\]

\[
- \frac{2}{3}(3\tau^2 + (1 + 3\nu)(5\mu + 4\tau^2) - (1 + 3\nu)^2(\mu + 8\tau^2))J_4 J_4 J_5
\]

\[
+ (\mu + 8\tau^2 + 2(1 + 3\nu)(\mu - 4\tau^2))J_3 J_6 J_6
\]

\[
+ \frac{2}{9}(1 + 36\nu + 72\nu^2)\tau J_3 J_5 J_5 = - (1 - 3\nu)J_3 J_6 J_2
\]

\[
= - \frac{4}{3}(1 + 6\nu)\tau J_6 J_7 J_7 + 2(2 + 3\nu)\mu^2 J_3 J_7 J_8
\]

\[
+ - 4(1 + 3\nu)\mu^2 J_3 J_8 J_8 + \frac{1}{3}(1 + 3\nu)(2 + 3\nu)(\mu + 8\tau^2)J_4 J_5^3
\]

\[
- (\mu(1 + 6\nu) - 2(5 + 12\nu)\tau^2)J_4 J_5 J_5
\]

\[
= \frac{4}{3}(1 + 3\nu)(2 + 3\nu)\mu^2 J_4 J_5 J_4 J_7 - \tau(3\mu - 2\tau^2)J_4^3
\]

\[
- 3\mu(2\mu - \tau^2)J_4^2 J_8 - 3(1 + 2\tau^2)J_4 J_6^2
\]

\[
+ 2(7 + 6\nu)\mu^2 J_4 J_4 J_7 - 3\mu^2 \tau J_4 J_8^2 - \frac{1}{3}(2 + 9\nu^2)J_3 J_5 J_5
\]

\[
\frac{4}{9}(1 + 18\nu^2)\tau J_3 J_5 J_7^2 - \frac{4}{3}(2 + 3\nu)\mu J_3 J_7 J_7 - \frac{2}{27}J_5^3
\]

\[
+ \frac{2}{3}(1 + 6\nu)\mu J_3 J_3 J_7 = - \frac{5}{3}\mu J_3 J_3 J_7 - \frac{1}{3}\mu(5 + 72\nu^2)J_3 J_3 J_4
\]

\[
- \mu^2(1 + 6\nu)\mu^2 J_3 J_3 J_4 + 4\mu^2 J_3 J_3 J_5 + 12\mu^2 J_3 J_3 J_6 - 9\mu J_3 J_3 J_6 - 2\mu^3 J_8^3.
\]

(25)

It is evident that if \(-3\nu = n\) the operator (24) has the space \( \mathbb{P}_n \) as a finite-dimensional invariant subspace. It seems natural to assume that the gauge-rotated integral \( k_{A_2} \), written in variables \( x_1, x_2, x_3 \),

\[
K_{A_2} = D^3 k_{A_2} D^{-3},
\]

should coincide with the integral found recently by Oshima [11].

An important observation should be made about a connection of the determinant (10) \( D \equiv D(\tau, \mu) \) with discriminants. It can be shown that \( D \) being written in Cartesian coordinates has the factorized form,

\[
D(0, 0) = 4x^3 + 27y^2 \sim (y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2,
\]

so, it is the discriminant of the cubic equation;

\[
D(\tau, \mu) = 12\tau x^4 + 4\tau x^3 + 72\tau^2 x^2 y^2 + 108\tau xy^2 + 27y^2 + 108\tau^4 y^4 \sim \times \sin^2 \alpha(y_1 - y_2) \sin^2 \alpha(y_1 - y_3) \sin^2 \alpha(y_2 - y_3)
\]

(26)
is a trigonometric discriminant where \( \tau = \frac{\alpha^2}{3} \). In general, \( D(\tau, \mu) = \frac{W^2(\tau, \mu)}{12} \), where (cf (14))

\[
W(\tau, \mu) \sim \frac{\sigma(y_1 - y_2)\sigma(y_2 - y_3)\sigma(y_3 - y_1)}{\sigma^3(y_1)\sigma^3(y_2)\sigma^3(y_3)},
\] (27)

and \( \sigma(x) \) and \( \sigma_i(x) \) are the Weierstrass \( \sigma \) functions (see [2]), which might be an elliptic discriminant.

It also must be noted that the operator \( h(u, v) \) (see (17)) can be rewritten in terms of the generators of the algebra \( g^{(2)} \); the infinite-dimensional, eleven-generated algebra of differential operators introduced in [16] (see for a discussion [15]). It is spanned by the Euler–Cartan generator

\[
\tilde{J}_0(n) = u \partial_u + 2v \partial_v - n,
\] (28)

(cf the Euler–Cartan generator \( J_0(n) \) of \( sl(3) \)-algebra), and

\[
J^1 = \partial_u, \ J^2_n = u \partial_u - \frac{n}{3}, \ J^3_n = 2v \partial_v - \frac{n}{3},
\]

\[
J^4_n = u^2 \partial_u + 2uv \partial_v - nu = u\tilde{J}_0(n),
\] (29)

\[
R_0 = \partial_v, \ R_1 = u \partial_u, \ R_2 = u^2 \partial_v,
\]

\[
\mathcal{T}_0 = v \partial_v^2, \ \mathcal{T}_1 = v \partial_v \tilde{J}_0(n), \ \mathcal{T}_2 = v \tilde{J}_0(n) \left( \tilde{J}_0(n) + 1 \right) = v \tilde{J}_0(n) \tilde{J}_0(n - 1),
\]

where \( n \) is a parameter. If \( n \) takes an integer value, the algebra \( g^{(2)} \) has a common invariant subspace (finite-dimensional representation space),

\[
Q_n = \langle u^p v^q \mid 0 \leq p + 2q \leq n \rangle,
\] (30)

where it acts irreducibly. The space (30) is invariant with respect to the polynomial transformations

\[
u \rightarrow \nu + A_u, \quad v \rightarrow v + A_i u^2 + Bu + C,
\]

where \( A_{u,v}, B, \) and \( C \) are constants.

Note that if in (21) the parameter \( n \) is an integer and thus, \( -3\nu = n \), the operator (17) at \( \nu = -n/3 \) has the finite-dimensional invariant subspace (30). This operator can be rewritten in terms of generators (29). Hence, the algebra \( g^{(2)} \) is a hidden algebra of the \( A_2 \) elliptic Calogero–Moser model, as well as being an alternative to the hidden algebra \( sl(3) \). Let us note that it was already known that the algebra \( g^{(2)} \) is the hidden algebra of the \( G_2 \) rational and trigonometric models [16]. Next we will show that it remains the hidden algebra of the \( G_2 \) elliptic model! Note that it can be shown that the square of operator (24), \( k^2_{\chi} \), written in the variables \( u, v \) is the algebraic operator and it commutes with the Hamiltonian (17), \([h(u, v), k^2_{\chi}(u, v)] = 0\).

It turns out that if we add to the operator \( h(u, v) \) (17) the operator

\[
h_m(u, v) = 6\left(1 + 2\mu u + \mu v^2\right)\frac{\partial}{\partial u} + 4\left(-u^2 + 3\tau v + 3\mu\nu\right)\frac{\partial}{\partial v} + 18 \nu \mu \ u
\] (31)

\[
= 6J^1 - 4R_2 + 6\tau J^2_{-3} + 6\tau J^3_{-3} + 6\mu J^4_{-3} - 12\nu,
\]

the resulting operator

\[
h_G(\lambda, u, v) = h(u, v) + \lambda h_m(u, v),
\] (32)

where \( \lambda \) is an arbitrary parameter, is an algebraic form for the \( G_2 \) elliptic model. Let us denote by \( \tilde{D}(u, v) \) the right side of (10) written in the variables \( u = x, v = y^2 \). Then the gauge
transformation
\[ \tilde{h}_{G_2}(u, v) = p \left( h_{G_2}(u, v) + 3v(3\nu + 6\lambda + 1) \right)p^{-1}, \]
where \( p = \sqrt{v^2 D^{-\frac{1}{2}}} \), brings the operator \( h_{G_2} \) to the Schrödinger operator form
\[ \tilde{h}_{G_2} = \Delta_g + \lambda (3\lambda - 1) \frac{u^2}{v} + 9(\nu - \lambda)(\nu - \lambda - 1) \frac{u + 2\nu u^2 + \mu u^3 - 6\nu + 6\nu^2 v + 3\mu v^2}{D} . \]

Now the transformation
\[ u = \frac{f'(y_1) - f'(y_2)}{f(y_1)f'(y_2) - f(y_2)f'(y_1)}, \quad v = \frac{2(f(y_1) - f(y_2))}{f(y_1)f'(y_2) - f(y_2)f'(y_1)}, \]
where \( f \) is defined by (7), relates the Hamiltonian \( \tilde{h}_{G_2} \) with the Hamiltonian of the \( G_2 \) elliptic model [1]
\[ H_{G_2} = -\frac{1}{3} \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_1 \partial y_2} \right) + (\nu - \lambda)(\nu - \lambda - 1) \left( \varphi(y_1 - y_2) + \varphi(2y_1 + y_2) + \varphi(y_1 + 2y_2) \right) + \frac{\lambda (3\lambda - 1)}{3} \left( \varphi(y_1) + \varphi(y_2) + \varphi(y_1 + y_2) \right). \]

The \( G_2 \) elliptic Hamiltonian is characterized by two coupling constants, which can be parameterized as \( \kappa = (\nu - \lambda)(\nu - \lambda - 1) \) (see (1)) and \( \kappa_2 = \lambda (3\lambda - 1) \). If \( \kappa_2 = 0 \), the \( A_2 \) elliptic model occurs. If \( \nu = -\frac{n}{3}, n = 0, 1, 2, \ldots \) the \( G_2 \) elliptic Hamiltonian
\[ H_{G_2} = -\frac{1}{3} \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_1 \partial y_2} \right) + \left( \frac{n}{3} + \lambda \right) \left( \frac{n}{3} + \lambda + 1 \right) \left( \varphi(y_1 - y_2) + \varphi(2y_1 + y_2) + \varphi(y_1 + 2y_2) \right) + \frac{\lambda (3\lambda - 1)}{3} \left( \varphi(y_1) + \varphi(y_2) + \varphi(y_1 + y_2) \right) \]
has a number of polynomial eigenfunctions. They have the form
\[ \Psi_{n,i} = Q_{n,i}(u, v) \sqrt{\frac{u^2}{D}} \varphi_{n,i} \], \( i = 1, \ldots, \text{dim}Q_n \), (35)
where polynomial \( Q_{n,i}(u, v) \in Q_n \), see (30). If \( \lambda = \frac{1}{3} \), the coupling constant \( \kappa_2 = 0 \), the \( G_2 \) elliptic Hamiltonian degenerates to the \( A_2 \) elliptic Hamiltonian (5), which has polynomial eigenfunctions
\[ \Psi_{n,i} = Q_{n,i}(u, v) \sqrt{\frac{u^2}{D}} \varphi_{n,i} \], \( i = 1, \ldots, \text{dim}Q_n \), (36)
(cf (35)) at coupling constant,
\[ \kappa = \frac{n + 1}{9} (n + 4), n = 0, 1, 2, \ldots \], (37)
(cf (21)).
It is known that the $G_2$ elliptic model has the integral in the form of a sixth-order differential operator (see [1]). It can be easily shown that there must exist a differential operator $k_{G_2}(u, v)$ of degrees less than six such that

$$k_{G_2} = k_{A_1}(u, v) + zk_m(u, v),$$

commutes with the $G_2$ elliptic Hamiltonian (32). It is evident that $k_0(u, v)$ can be rewritten in terms of the generators of the algebra $g^{2(1)}$. This will be calculated elsewhere.

**Observation 2.** Let us construct the operator

$$i_{\text{par}}^{(n)}(u, v) = \prod_{j=0}^{n} \left( J^0(n) + j \right),$$

(38)

where

$$J^0(n) = u \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial y} - n,$$

is the Euler–Cartan generator of the algebra $g^{2(2)}$ (28). It can be immediately seen that the algebraic operator $h(u, v)$ (32) at integer $n$ commutes with $i_{\text{par}}^{(n)}(u, v)$,

$$\left[ h(u, v), i_{\text{par}}^{(n)}(u, v) \right]: Q_n \rightarrow 0.$$

Hence, $i_{\text{par}}^{(n)}(u, v)$ is the particular integral [10] of the $G_2$ elliptic model (5).

In this paper we demonstrate that both $A_2$ and $G_2$ quantum elliptic models belong to two-dimensional, quasi-exactly-solvable (QES) problems [12, 14]. We show the existence of an algebraic form of the $A_2$ elliptic Hamiltonian, which is the second-order polynomial element of the universal enveloping algebra $U_{sl(3)}$ and an algebraic form of the $G_2$ elliptic Hamiltonian, which is the element of the algebra $g^{2(2)}$. We construct explicitly the integral for the $A_2$ case—commuting with the Hamiltonian—as the third-order polynomial element of the universal enveloping algebra $U_{sl(3)}$. If the algebra $U_{sl(3)}$ appears in a finite-dimensional representation, those elements possess a finite-dimensional invariant subspace. This phenomenon happens for a discrete sequence of coupling constants (21) for which both polynomial eigenfunctions and a particular integral occur. In a similar way, if the algebra $g^{2(2)}$ appears in a finite-dimensional representation those elements possess a finite-dimensional invariant subspace. This also happens for the $G_2$ elliptic model: for the one-parametric family of coupling constants, a number of polynomial eigenfunctions occurs, as well as a particular integral.

The situation looks very similar to the case of the $A_1$ elliptic model (the Lame Hamiltonian, see, e.g., [9] and references therein), where the new variable, which transforms the $A_1$ elliptic Hamiltonian to the algebraic operator, is $x = \frac{1}{\varphi_{\nu_3}}$. A generalization to $A_N$ elliptic models for $N > 2$ seems straightforward. It is worth noting that a certain algebraic form for a general $BC_N$ elliptic model was found some time ago in [17, 18] (see also [9]). The existence of the $sl(N+1)$ hidden algebra structure was also shown, and is equivalent to the $sl(N+1)$ quantum Euler–Arnold top. Such generalizations can be regarded as a multivariate generalization of the Lame Hamiltonian and the Lame polynomials.

**Note added.** After the present study was completed, based on the transformation (8), the following has been formulated.

**Conjecture (M Matushko, August 2014).**

The analog of transformation (8) for arbitrary $n$ is given by the solution of the linear system
\[ M \mathbf{u} = \mathbf{e}, \]

where \( \mathbf{u} = (u_1, \ldots, u_N)^T, \mathbf{e} = (1, 1, \ldots, 1)^T \) with

\[ M_j' = \frac{d^{j-1} \varphi_j(y_j)}{dy_j^{j-1}}. \]

It is evidently correct for \( n = 1 \). The validity of this conjecture will be checked elsewhere.

It is worth commenting that the determinant of this linear system is the elliptic generalization of the Van der Monde determinant (see, e.g., [2]). The vector \( \mathbf{e} \) looks like the highest root among \( A_N \) roots in the basis of simple roots.

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Note added in proof. We were informed [13] that, with a direct calculation, the conjecture by Matushko was checked and confirmed at \( N = 4 \) for the \( A_3 \) four-body quantum elliptic Calogero model. This model possesses the hidden algebra \( sl(4) \) and is equivalent to \( sl(4) \) Euler–Arnold quantum top. If the coupling constant \( \nu = \nu(\nu - 1) \) is such that \( -4\nu = n \), where \( n \) is positive integer, the \( A_3 \) quantum elliptic Calogero Hamiltonian has a number of polynomial eigenfunctions written in terms of \( sl(4) \) elliptic invariants.

References

[1] Olshanetsky M A and Perelomov A M 1983 Quantum integrable systems related to Lie algebras Phys. Rep. 94 313–93
[2] Whittaker E T and Watson G N 1927 A Course in Modern Analysis 4th edn (Cambridge: Cambridge University Press)
[3] Etingof P and Kirillov A 1994 Representations of affine Lie algebras, parabolic differential equations, and Lame functions Duke Math. J. 74 585–614
[4] Felder G and Varchenko A 1997 Three formulas for eigenfunctions of integrable Schrödinger operators Composito Mathematica 107 143–75
[5] Nekrasov N A and Shatashvili S 2010 Quantization of integrable systems and four dimensional gauge theories 16th Int. Congress on Mathematical Physics, (Prague, August 2009) ed P Exner (Singapore: World Scientific) pp 265–289 (arXiv: 0908.4052 [hep-th])
[6] Nunez J F, Fuertes W G and Perelomov A M 2003 A perturbative approach to the quantum elliptic Calogero–Sutherland model Phys. Lett. A 307 233–8
[7] Langmann E 2004 An explicit solution of the (quantum) elliptic Calogero-Sutherland model arXiv:math-ph/0407050
[8] Rühl W and Turbiner A V 1995 Exact solvability of the Calogero and Sutherland models Mod. Phys. Lett. A10 2213–22
[9] Turbiner A V 1989 Lame equation, \( sl(2) \) and isospectral deformation J. Phys. A: Math. Gen. 22 1–3

Turbiner A V 2015 The \( BC_1 \) Elliptic model: algebraic forms, hidden algebra \( sl(2) \), polynomial eigenfunctions J. Phys. A: Math. Theor. 48 192002
[10] Turbiner A V 2013 Particular integrability and (quasi)-exact-solvability J. Phys. A: Math. Theor. 46 025203
[11] Oshima T 2007 Completely integrable systems associated with classical root systems SIGMA 3 061
[12] Turbiner A V 1988 Quasi-exactly-solvable problems and the SL(2, R) group Comm. Math. Phys. 118 467–74
[13] Matushko M 2015 Private communication
[14] Turbiner A V 1994 Lie-algebras and linear operators with invariant subspaces Lie Algebras, Cohomology and New Applications to Quantum Mechanics (vol 160) Contemporary Mathematics (Providence, RI: American Mathematical Society) pp 263–310
[15] Turbiner A V 2013 From quantum $A_N$ (Sutherland) to $E_8$ trigonometric model: space-of-orbits view SIGMA 9 003
[16] Rosenbaum M, Turbiner A V and Capella A 1998 Solvability of $G(2)$ integrable system Int. J. Mod. Phys. A 13 3885–904
[17] Gomez-Ullate D, Gonzalez-Lopez A and Rodriguez M A 2001 Exact solutions of a new elliptic Calogero–Sutherland model Phys. Lett. B 511 112–8
[18] Brihaye Y and Hartmann B 2003 Multiple algebraisations on elliptic Calogero–Sutherland model J. Math. Phys. 44 1576–83