Generalized scattering phases for asymptotically hyperbolic manifolds

J.M. Bouclet
Université de Lille 1
UMR CNRS 8524,
59655 Villeneuve d’Ascq
Jean-Marc.Bouclet@agat.univ-lille1.fr

29th March 2022

Abstract

We prove two asymptotic expansions of the generalized scattering phases. These phases are generalizations of the Birman-Krein spectral shift function associated to pairs of perturbations of the Laplacians of asymptotically hyperbolic manifolds. The first expansion, of ‘heat type’, holds for all ‘long range’ metric perturbations of the Laplacian, whereas the second one is shown under a non trapping condition.

1 Introduction and results

1.1 Introduction

In this paper, we define and study some properties of the generalized scattering phases associated to a pair of self-adjoint elliptic differential operators \((P_0, P_1)\) on an asymptotically hyperbolic manifold \(X\) of dimension \(n\). The manifolds and operators that we consider are of the same type as those considered in \([35, 18]\) or in many other related papers, especially \([30]\). Typically, \(P_0\) can be the Laplacian of the quotient of the hyperbolic space \(\mathbb{H}^n\) by some discrete group of isometries, with infinite volume, and \(P_1\) a non compactly supported perturbation of \(P_0\). The precise definition of the operators is given in subsection 1.2. Let us first explain the terminology.

The distribution \(\xi_q\) is well known: it is the spectral shift function introduced by Birman and Kreĭn \([2, 53]\). We recall that \(\xi_1\) is, in general, a measurable function which is defined if \((P_1 + i)^{-N} - (P_0 + i)^{-N} \in S_1\) for some \(N\) large enough. Here \(S_1\) is the set of trace class operators.

\[
\int_{\mathbb{R}} f(\lambda) \, d\xi_q(\lambda) = \text{tr} \left( f(P_1) - \sum_{j=0}^{q-1} \frac{d^j}{dj}\frac{d^j}{d\epsilon} f(P_\epsilon)|_{\epsilon=0} \right), \quad f \in C_0^\infty(\mathbb{R})
\]
The point of considering $\xi_q$ for $q \geq 2$ is that we can relax this trace class condition and consider operators such that $(P_1 + i)^{-N} - (P_0 + i)^{-N} \in S_q$, the Schatten class of order $q$ (for instance the Hilbert-Schmidt class if $q = 2$). We furthermore emphasize that, if $q \geq 2$, $\xi_q$ is a priori a distribution and showing that it is a (smooth or continuous or even measurable) function on the continuous spectrum is not trivial. The distributions $\xi_q$ have been introduced by Koplienko [32] for bounded or 1-dimensional Schrödinger operators and studied in the higher dimensional case by the author (see [6, 7] for more details). We also quote a similar approach considered in [28]. In [6, 7], $\xi_q$ was called spectral distribution by analogy with spectral function, however the name scattering phase is natural as well since we have shown that

$$\xi_q(\lambda) = \lim_{\delta \to 0} \frac{d}{d\lambda} \arg \det_q \left((P_1 - \lambda - i\delta)(P_0 - \lambda - i\delta)^{-1}\right).$$

The precise meaning of formula (1.2), which is well known for $q = 1$ if $(P_1 - P_0)(P_0 - z)^{-1}$ is trace class [23, 53], is explained in [7]. We just specify that $\det_q$ coincides in many cases with the standard Fredholm determinant $\text{Det}_q$ defined for perturbations of identity by elements of $S_q$ (see [28, 53]), and that we need such an extension of $\text{Det}_q$ since $(P_1 - P_0)(P_0 - z)^{-1}$ is not necessarily compact in general. We also refer to the recent paper [5] where similar determinants with $q = 1$ are studied for hyperbolic surfaces.

We now recall the definition of an asymptotically hyperbolic manifold. A complete non compact Riemannian manifold $(X, G)$, without boundary, is asymptotically hyperbolic if it is isometric, outside a compact set, to

$$((R, +\infty) \times Y, dr^2 + e^{2r}g(e^{-r}))$$

Here $Y$ is a connected compact manifold without boundary, and $g(x)$ is a family of metrics on $Y$ depending smoothly on $x \in [0, e^{-R})$. More precisely, if $S^2T^*Y$ is the vector bundle of bilinear symmetric forms on $Y$ and $\Gamma(S^2T^*Y)$ is the space of its smooth sections, we assume that $g \in C^\infty([0, e^{-R}), \Gamma(S^2T^*Y))$ and of course that $g(x)|_{T_pY \times T_pY}$ is positive definite for each $p \in Y$ and $x \in [0, e^{-R})$. We furthermore equip $Y$ with the metric $g(0)$. We could actually consider manifolds with finitely many such ends, i.e. with $(X, G)$ isometric, outside a compact set, to a finite union of manifolds like (1.3) but we restrict our attention to the one end case for notational convenience. As explained in [30, 36], such a manifold $(X, G)$ can be obtained from a compact manifold with boundary $(Z, \overline{G})$, with a boundary defining function $x$ such that $\partial Z = Y = \{x = 0\}$, by setting $X = \{x > 0\}$, the interior of $Z$, equipped with the metric $\overline{G}/x^2$. These manifolds are also called conformally compact manifolds and the most basic example is the hyperbolic space $\mathbb{H}^n$.

The results of this paper are two asymptotic formulas for $\xi_q$ which are similar to the heat expansion and Weyl formula for the eigenvalues counting function on a compact manifold. More precisely, these asymptotics are of the same type as those obtained in Euclidean scattering by [11, 34, 40, 47, 10, 52, 33] for $q = 1$ and [6, 7] for $q \geq 2$. See also [10, 11, 5, 19] and [20] in more geometric frameworks. The scattering phases are natural and basic tools of scattering theory in view of their relation with time delay [47], relative scattering determinants [6, 7, 43] or resonances. More specifically, their asymptotic behavior is of interest for several reasons such as relative index theory [4], trace formulas [12, 23, 6, 1] or Breit-Wigner formula [22, 42, 8].

The most popular scattering phase is Birman-Kreǐn’s function $\xi_1$ but its use leads to restrictions on the pairs of operators as already mentioned. For instance in [4], the authors are able to define the determinant $\det_1$ for a pair of operators which are, up to a unitary transform, the Laplacians associated with two metrics $G_0$ and $G_1$ as above for which $g_1(x) - g_0(x) = O(x^2)$. This last condition implies that $(P_0 + i)^{-N} - (P_1 + i)^{-N}$ is of trace class since they work in dimension 2, but
in higher dimension, the difference \((P_0 + i)^{-N} - (P_1 + i)^{-N}\) is not trace class, in general, under the sole condition \(g_1(x) - g_0(x) = \mathcal{O}(x^2)\). Our theorem \([14]\) combined with the method of \([7]\) proves directly the existence of \(\det_q ((P_1 - z)(P_0 - z)^{-1})\) for \(q \geq n\) for any \(n\) under the weaker condition that \(g_1(x) - g_0(x) = \mathcal{O}(x)\).

For the proof of theorem \([12]\), we adapt the method that we used in \([7]\), namely a refined analysis of Isozaki-Kitada’s construction \([20]\). We recall the principle of this method, which has only been used in the Euclidean context so far, in subsection \([2.3]\). We devote section \([3]\) to the relevant estimates on geodesics in the hyperbolic framework and the explicit construction is given in subsection \([5.1]\).

### 1.2 Notations

For any \(C^q\) function \(T_\epsilon\) of the real variable \(\epsilon \in [0, 1]\), scalar or vector valued, we set

\[
[T_\epsilon]_q = T_1 - \sum_{j=0}^{q-1} \frac{d^j}{d\epsilon^j} T_{\epsilon=0} = \frac{1}{(q-1)!} \int_0^1 (1 - \epsilon)^{q-1} \frac{d^q}{d\epsilon^q} T_\epsilon \, d\epsilon. \tag{1.4}
\]

We will also use the notation

\[
\{T_\epsilon\}_q = \frac{1}{(q-1)!} \int_0^1 (1 - \epsilon)^{q-1} \frac{d^q}{d\epsilon^q} T_\epsilon \, d\epsilon.
\]

Note that, if \(T_\epsilon\) is a primitive of \(T_\epsilon\), we have \([T_\epsilon]_q = \{T_\epsilon\}_q\).

We will have to consider distributions smooth with respect to a parameter. We shall say that a family of distributions \(u_\tau \in \mathcal{S}'(\mathbb{R})\) is \(C^j\) with respect to \(\tau \in J\), an interval of \(\mathbb{R}\), if for all \(f \in \mathcal{S}(\mathbb{R})\) the function \(\langle u_\tau, f \rangle\) is \(C^j\) on \(J\). In particular, \(\partial^j_\tau u_\tau\) and \(\int_J u_\tau \, d\tau\) are defined by

\[
\langle \partial^j_\tau u_\tau, f \rangle = \partial^j_\tau \langle u_\tau, f \rangle, \quad \left\langle \int_J u_\tau \, d\tau, f \right\rangle = \int_J \langle u_\tau, f \rangle \, d\tau.
\]

Another useful distributional notation is the following. If \(H\) is a self-adjoint operator on a separable Hilbert space \(\mathcal{H}\) and if \(T\) is an operator acting on (some subspace) of \(\mathcal{H}\) such that the map \(f \mapsto \text{tr}(f(H)T)\), with \(f \in \mathcal{S}(\mathbb{R})\), defines a distribution, that is if the trace is well defined and depends continuously on \(f \in \mathcal{S}(\mathbb{R})\), then we shall write this distribution

\[
\text{tr}\left(\frac{\partial E}{\partial \mu} T\right)
\]

if \(E(\mu) = E(-\infty, \mu)\) is the spectral projection of \(H\) on \((-\infty, \mu)\).

We will also use extensively Schatten classes \(\mathcal{S}_q\) of real order \(q \geq 1\). We simply recall that, by definition, a bounded operator \(A\) on a separable Hilbert space \(\mathcal{H}\) belongs to \(\mathcal{S}_q = \mathcal{S}_q(\mathcal{H})\) if \(|A|^q = (A^*A)^{q/2}\) is trace class and that the norm \(|.|_q\) on \(\mathcal{S}_q\) is defined by \(|A|^q = \text{tr}(|A|^q)\). We will need the following Hölder type estimates

\[
|AB|_q \leq |A|_q |B|_q, \quad q^{-1} = q_1^{-1} + q_2^{-1} \tag{1.5}
\]

for all \(A \in \mathcal{S}_{q_1}\) and \(B \in \mathcal{S}_{q_2}\). This estimate still holds if \(A\) (resp. \(B\)) is bounded with \(q_1 = \infty\) (resp. \(q_2 = \infty\)), which is consistent with the notation \(|.|_\infty\) for the operator norm on \(\mathcal{H}\). For a more general presentation of Schatten classes, we refer to \([24]\) and \([53]\).

On the manifold \(X\), we will mainly work near infinity. If \(y = (y_1, \ldots, y_{n-1})\) are coordinates on \(Y\) defined on \(U_{n-1} \subset Y\), such that \(y: U_{n-1} \rightarrow \Omega \subset \mathbb{R}^{n-1}\) is a diffeomorphism, then \(r, y_1, \ldots, y_{n-1}\)
are coordinates on an subset \( U_n \subset X \) diffeomorphic to \((R, \infty) \times \Omega\). We call \((R, \infty) \times \Omega\) a chart at infinity. We can write \( Y \) as the finite union of open sets \( U_{n-1} \) and thus we get an atlas of a neighborhood of infinity on \( X \) given by a finite number of charts at infinity. If necessary, we can choose each \( \Omega \) to be convex, since we fix the coordinates once for all in each chart. We assume that, in each chart at infinity, the volume density giving the \( L^2 \) structure on \( X \) can be written

\[
d\text{vol} = a(r, y)[dr \wedge dy_1 \wedge \cdots \wedge dy_{n-1}]
\]

with a smooth and bounded on \((R, \infty) \times \Omega\). In particular, this implies that the pullback on \( X \) of any function \( u \in L^2((R, \infty) \times \Omega, d\text{rd}y)\), supported in the chart, belongs to \( L^2(X) = L^2(X, d\text{vol})\).

We now describe the operators \( P_0 \) and \( P_1 \). Let us start with an example. The expression of the Laplace Beltrami operator associated to \((1.3)\) in a chart at infinity is

\[
\Delta^G = -\partial_r^2 - e^{-2r} \Delta^g(e^{-r}) - \left(n - 1 - e^{-r} \frac{\partial_r \det g(x)}{2\det g(x)}\right) \partial_r
\]

where \( \Delta^g(x) \) is the expression of the (negative) Laplacian of \( Y \) associated to the metric \( g(x) \). Furthermore \( d\text{vol}(G) = e^{(n-1)r/2} \sqrt{\det g(e^{-r}, y)} [dr \wedge dy_1 \wedge \cdots \wedge dy_{n-1}] \) is the volume form induced by \( G \), hence

\[
e^{(n-1)r/2} \Delta^G e^{-(n-1)r/2} = -\partial_r^2 - e^{-2r} \Delta^g(e^{-r}) + c_n^2 + e^{-r} \frac{\partial_r \det g(x)}{2\det g(x)}|_{x=e^{-r}} (\partial_r - c_n),
\]

with \( c_n = (n-1)/2 \), is selfadjoint w.r.t the density \( \sqrt{\det g(e^{-r}, y)[dr \wedge dy_1 \wedge \cdots \wedge dy_{n-1}]} \) which is of the form \((1.6)\). Guided by \((1.7)\), we shall consider operators \( P_0, P_1 \) which are both second order elliptic differential operators, symmetric w.r.t to \( d\text{vol} \), whose expressions in each chart at infinity are

\[
P_j = -\partial_r^2 - g_j(e^{-r}, y, e^{-r} \partial_y) + e^{-r} \sum_{|\alpha|+|l| \leq 1} v_j^{\alpha,l}(e^{-r}, y)(e^{-r} \partial_y)^\alpha \partial_r^l, \quad j = 0, 1.
\]

Here \( g_j(x, y, \eta) \) is the principal symbol of \( \Delta^g(x) \), \( j = 0, 1 \), that is the expression of the metric \( g_j \) on the fibers of \( T^*Y \) with coordinates \( \eta_1, \cdots, \eta_{n-1} \) dual of \( y_1, \cdots, y_{n-1} \), and the functions \( v_j^{\alpha,l}(x, y) \) are smooth and bounded on \([0, e^{-R}] \times \Omega\). The fact that \( P_1 \) is a perturbation of \( P_0 \) is reflected by the assumption that

\[
g_1|_{x=0} = g_0|_{x=0}.
\]

We will use extensively the principal symbol of \( P_0 + \epsilon(P_1 - P_0) \), denoted by \( p_\epsilon \), which has the form

\[
p_\epsilon(r, y, \rho, \eta) = \rho^2 + e^{-2r} g_\epsilon(r, y, \eta)
\]

where \( g_\epsilon(r, y, \eta) = g_0(e^{-r}, y, \eta) + \epsilon(g_1 - g_0)(e^{-r}, y, \eta) \) and \( \rho \) is dual variable of \( r \). Note that \( g_1(x, y, \eta) - g_0(x, y, \eta) = O(1)|\eta|^2 \) by \((1.9)\).

1.3 Results

In the next two theorems, \( P_0 \) and \( P_1 \) are two operators as described above such that \((1.9)\) holds. We recall that \( P_\epsilon = P_0 + \epsilon(P_1 - P_0) \).
Theorem 1.1. i) For all $q \geq n$ and $f \in \mathcal{S}(\mathbb{R})$, $[f(P_{t})]_{q}$ is trace class and there exists a unique $\xi_{q} \in \mathcal{S}'(\mathbb{R})$ which vanishes below $\inf(\sigma(P_{0}) \cup \sigma(P_{t}))$ such that

$$\int_{\mathbb{R}} f(\lambda) \, d\xi_{q}(\lambda) = \text{tr} [f(P_{t})]_{q}.$$ 

ii) The Laplace transform of $\xi_{q}$ has a complete asymptotic expansion as $t \downarrow 0+$, namely

$$\text{tr} [e^{-tP_{t}}]_{q} \sim t^{-n/2} \sum_{k \geq 0} a_{k} t^{k}, \quad \text{with} \quad a_{0} = \Gamma \left( \frac{n}{2} + 1 \right) (2\pi)^{-n} \omega_{n} \int_{X} [d\text{vol}]_{q}. \quad (1.10)$$

Here $d\text{vol}$ is volume density obtained naturally from $p_{t}$, that is in local coordinates

$$d\text{vol} = e^{(n-1)r} \det \left( \partial^{2}_{r} g_{0}(r, y, \eta)/2 \right)^{-1/2} |dr \wedge dy_{1} \wedge \cdots \wedge dy_{n-1}|.$$ 

The other coefficients $a_{1}, a_{2}, \cdots$ can be expressed as integrals of functions of the symbols of $P_{t}$.

Example. If $(X, G_{0})$ and $(X, G_{1})$ are of the form (1.2), outside a compact set, associated respectively to $g_{0}$ and $g_{1}$ satisfying (1.3), then the operators $P_{0} = e^{(n-1)r/2} \Delta G_{0} e^{-(n-1)r/2} - c_{n}^{2}$ and $P_{1} = U^{1/2} e^{(n-1)r/2} \Delta G_{1} e^{-(n-1)r/2} U^{-1/2} - c_{n}^{2}$, with $U = d\text{vol}(G_{1})/d\text{vol}(G_{0})$, satisfy the assumptions of the theorem.

The leading term of (1.10) involves the regularized volume term $a_{0}$. Other kinds of regularization have been considered for similar purposes, for instance the $0-$volume used in [20] or those used in [10] [24].

If we knew that $\xi_{q}$ was a monotone function, this result would yield immediately an equivalent for $\xi_{q}(\lambda)$ as $\lambda \uparrow \infty$ by Karamata’s Tauberian theorem. Unfortunately we don’t know that it is a function neither that it is monotone (it is not the case in general). Nevertheless, we have the following result.

Theorem 1.2. i) $\xi_{q}$ is a continuous function on $(0, \infty)$.

ii) Assume that $G$ is of the form (1.3) with $q = g_{1}$. Assume furthermore that $G$ is non trapping (see below) and that the principal symbols of $P_{0}$ and $P_{1}$ coincide outside the region $\{r > r_{0}\}$, for some $r_{0}$ large enough, then we have the complete asymptotic expansion

$$\xi_{q}(\lambda) \sim \lambda^{n/2} \sum_{k \geq 0} b_{k} \lambda^{-k}, \quad \lambda \uparrow +\infty. \quad (1.11)$$

The coefficients $b_{k}$ can be deduced from (1.10) and in particular $b_{0} = (2\pi)^{-n} \omega_{n} \int_{X} [d\text{vol}]_{q}$. 

Note that the condition on the principal symbol means that the terms of order 2 of $P_{1} - P_{0}$ are supported near infinity. For instance $P_{1}$ can be a perturbation of $P_{0}$ with $P_{0} = -\partial^{2}_{r} - e^{-2r} \Delta g_{0}(0)$ near infinity (the ‘product case’) or with $P_{0}$ associated to a metric with constant curvature near infinity. The latter can be of special interest in view of the recent results of [13].

We recall that $G$ is a non trapping metric, if for any compact subset $K$ of $T^{*}X \setminus 0$ there exists $T_{K} \geq 0$ such that $\phi^{t}(K) \cap K$ is empty for all $|t| \geq T_{K}$, if $\phi^{t}$ is the geodesic flow on $T^{*}X$. The results already obtained on $\mathbb{R}^{n}$ let us hope that the non trapping condition in theorem 1.2 can be relaxed to get a Weyl formula, i.e. an equivalent for $\xi_{q}$. However this is an open question.

We shall use methods of semi-classical analysis to prove these theorems and we will consider

$$H_{\epsilon} = h^{2} P_{\epsilon} = H_{0} + \epsilon V, \quad h \in (0, 1]. \quad (1.12)$$
We will use the notations $E_\epsilon(\mu)$ for the spectral projection of $H_\epsilon$ on $(-\infty, \mu)$ and

\[ R_\epsilon(z) = (H_\epsilon - z)^{-1}, \quad U_\epsilon(t) = e^{-itH_\epsilon/h}. \]

In theorem 1.1 we will choose $h = t^{1/2}$ and in theorem 1.2 we shall set $h = \lambda^{-1/2}$ and consider the rescaled scattering phases $\xi_q(\mu, h)$ associated to $H_0, H_1$, near the energy $\mu = 1$, since one has clearly $\xi_q(\mu h^{-2}) = \xi_q(\mu, h)$.

The non trapping condition will be used to show that

\[ \int_\mathbb{R} \| \langle r \rangle^{-M} f(H_\epsilon) U_\epsilon(t) \langle r \rangle^{-M} \|_\infty dt \leq C_{M, f, h_0, w} \tag{1.13} \]

for some $M > 0$, all $f \in C_0^\infty$ supported close to 1 and uniformly w.r.t. $h \in (0, h_0]$ and $\epsilon \in [0, 1]$. Such estimates are essentially well known under the non trapping condition. They have been proved in the Euclidean case by Robert-Tamura [48] and simplified by Gérard-Martinez [20] (see also [45]), using the theory of Mourre [37] with a conjugate operator (see appendix B) defined as a suitable perturbation of generator of dilations $r.hD_r + hD_r.r$. As explained by Hislop and Froese in [15], the generator of dilations does not fit the hyperbolic framework and they built explicitly another conjugate operator which makes Mourre theory applicable. In appendix B, we sketch the proof leading to (1.13) by combination of the ideas of [20, 45] and [15], which is necessary since (1.13) is only known for fixed $h$ in the asymptotically hyperbolic case [18, 14].

Note finally that if (1.13) can be improved so that $\| \langle r \rangle^{-M} f(H_\epsilon) U_\epsilon(t) \langle r \rangle^{-M} \|_\infty$ has a polynomial bound in $h^{-1}$, then our method would show that $\xi^{(p)}_q$ has a complete expansion, obtained by differentiating (1.11).

Acknowledgments: I want to thank Peter Hislop for helpful discussions as well as Gilles Carron and Didier Robert for their interest and useful remarks.

## 2 The basic tools of the proof

The purpose of this section, which is of pedagogic nature, is to describe the main tools of the proof of theorem 1.2. The formulas that we are going to display hold for a much wider class of operators than those defined on asymptotically hyperbolic manifolds and we want to separate the general ideas, sketched in this section, from the specific analysis of the hyperbolic context given in sections 3 and 4.

### 2.1 Representation formula of $\xi_q$

Let us first assume that $V$, defined by (1.12), is compactly supported so that the Birman-Krein spectral shift function $\xi_1,\epsilon(\mu, h)$ is well defined for the pair $H_0, H_\epsilon$ (this is essentially standard but is anyway a consequence of lemma 4.10). Then we can use the well known Birman-Solomyak formula [3]

\[ \text{tr} (f(H_\epsilon) - f(H_0)) = \int_0^1 \text{tr} (f'(H_\epsilon_\epsilon) \epsilon V) \, ds. \tag{2.1} \]

The left hand side of (2.1) is $-\langle \xi_1,\epsilon, f' \rangle$ so, with the notations of subsection 1.2, we get directly

\[ \xi_{1,\epsilon}(\mu, h) = -\int_0^1 \text{tr} \left( \frac{\partial E_{\epsilon\mu}}{\partial \mu} V \right) \, ds = -\int_0^\epsilon \text{tr} \left( \frac{\partial E_{\mu}}{\partial \mu} V \right) \, ds, \tag{2.2} \]
since both sides of (2.2) have the same derivative (in the distributions sense w.r.t. $\mu$) and vanish for $\mu \ll 0$. For the general case, i.e. $V$ non compactly supported, we consider the family

$$V^{(\kappa)} = \theta(\kappa r)V\theta(\kappa r), \quad \kappa \in [0,1]$$  \hspace{1cm} (2.3)

with $\theta \in C^\infty_0$, $\theta = 1$ near 0, and thus we can define the associated family of spectral shift functions $\xi^{(\kappa)}_1$. We obtain the following representation formula for $\xi_q$:

Lemma 2.1. In the distributions sense on $\mathbb{R}_\mu$, we have

$$\xi_q(\mu, h) = -\lim_{\kappa \downarrow 0} \left\{ \text{tr} \left( \frac{\partial E^{(\kappa)}_\mu}{\partial \mu} V^{(\kappa)}(\cdot) \right) \right\}_q$$  \hspace{1cm} (2.4)

Here $E^{(\kappa)}_\mu$ is the spectral resolution associated with $H^{(\kappa)}_\mu = H_0 + \epsilon V^{(\kappa)}$.

The proof of this lemma (given in section 4) is not very hard and formally obvious from (2.2) and the definition of $\xi_q$. Formula (2.4) leads obviously to the study of distributions of the form

$$\text{tr} \left( \frac{\partial E^{(\kappa)}_\mu}{\partial \mu} f(H^{(\kappa)}_\mu) V^{(\kappa)}(\cdot) \right)$$  \hspace{1cm} (2.5)

with $f \in C^\infty_0$, $f \equiv 1$ close to 1. Note that such a function $f$ can be added for free since we only consider $\mu$ close to 1. In order to simplify our notations, we drop the index $\kappa$ in the sequel but the reader must keep in mind that we work with a perturbation of the form (2.3).

As usual, we consider the (semi-classical) Fourier transform of (2.5) which is $\text{tr} (U_\epsilon(t)f(H_\epsilon)V)$. It is thus natural to consider distributions of the more general form

$$\text{tr} (U_\epsilon(t)f(H_\epsilon)K_\epsilon)$$  \hspace{1cm} (2.6)

with $K_\epsilon$ trace class, for instance $K_\epsilon = \hat{f}(H_\epsilon)\hat{V}$ with $\hat{f} \in C^\infty_0$, $\hat{f}f = f$. Recall that we consider perturbations of the form (2.3), so that $f(H^{(\kappa)}_\mu)V^{(\kappa)}$ is trace class for each $\kappa > 0$, but for $\kappa = 0$ we only have $K_\epsilon \in S_q$ in which case the trace makes sense only once $\{\}$ has been taken.

We will have to consider the $\epsilon$ derivatives of $f(H_\epsilon)U_\epsilon(t)$, that is why we quote quote the formula

$$\partial_\epsilon U_\epsilon(t) = \frac{i}{\hbar} \int_0^t U_\epsilon(t-s)VVU_\epsilon(s) \, ds$$  \hspace{1cm} (2.7)

which holds, for instance, in the strong sense on the domain of the operators $H_\epsilon$ (which is independent of $\epsilon$, see proposition 1.13). Since we want to use (1.13), it is important to keep a spectral cutoff in front of each $U_\epsilon(t)$. To that end, we use a very simple trick. Let us introduce the notations

$$U_\epsilon^f(t) = f(H_\epsilon)U_\epsilon(t), \quad S_\epsilon^f = f(H_\epsilon).$$  \hspace{1cm} (2.8)

Then for any small neighborhood $\tilde{I}$ of supp $f$, we can choose $\tilde{f}$ smooth, supported in $\tilde{I}$, such that $\tilde{f}f = f$ and we obtain

$$\partial_\epsilon U_\epsilon^f(t) = \partial_\epsilon \left( S_\epsilon^f U_\epsilon(t) S_\epsilon^f \right) = (\partial_\epsilon S_\epsilon^f) U_\epsilon(t) S_\epsilon^f + S_\epsilon^f (\partial_\epsilon U_\epsilon(t)) S_\epsilon^f + S_\epsilon^f U_\epsilon(t) (\partial_\epsilon S_\epsilon^f).$$

and using (2.4) we get the formula

$$\partial_\epsilon U_\epsilon^f(t) = \partial_\epsilon S_\epsilon^f U_\epsilon^f(t) + \frac{i}{\hbar} \int_0^t U_\epsilon^f(t-s)VVU_\epsilon^f(s) \, ds + U_\epsilon^f(t) \partial_\epsilon S_\epsilon^f.$$  \hspace{1cm} (2.9)

This can be obviously iterated and proves the following more general result.
Lemma 2.2. For all $k \geq 1$, $\partial_t^k U^j(t)$ is a linear combination with universal coefficients of

$$h^{-j} \int_{F^j_t} \partial_t^l S^{j0}_t U^0(t_0) \partial_t^j S^j_t V \cdots V \partial_t^{j-l} S^{j-l}_t(t_j) U^j(t_j) \partial_t^j S^j_t d^j(t_0, \cdots, t_j), \tag{2.10}$$

with $1 \leq j \leq k$, $t_0 + \cdots + t_{j+1} = k - j$, $f_0, \cdots, f_j, \tilde{f}_0, \cdots, \tilde{f}_{j+1} \in C_0^\infty(\hat{I})$ and of $\partial_t^j S^{j0}_t U^0(t) \partial_t^{k-j} S^j_t$, for $0 \leq l \leq k$. In (2.10), we have used the notations

$$F^j_t = \{(t_0, \cdots, t_j) \in [0, t]^{j+1} | t_0 + \cdots + t_j = t\}$$

and $d^j(t_0, \cdots, t_j)$ for the ($j$-dimensional) Lebesgue measure on the hyperplane $t_0 + \cdots + t_j = t$.

In the applications, we will get estimates on (2.10) using (2.8) combined with the following easy estimate

$$\int_\mathbb{R} \left| \int_{F^j_t} \psi_0(t_0) \psi_1(t_1) \cdots \psi_j(t_j) \, d^j(t_0, \cdots, t_j) \right| \, dt \leq \int_\mathbb{R} |\psi_0| * |\psi_1| * \cdots * |\psi_j| \, dt \tag{2.11}$$

valid for all integrable functions $\psi_0, \cdots, \psi_j$.

2.2 Two microlocal tools

The operators $K_\epsilon$ in (2.10) will essentially be pseudo-differential operators. Sometimes we will need to shift the support of their symbols by the classical Hamilton flow $\phi^r_t$. To that end, we will use the fact that for any $t_0$ we have

$$\text{tr} (U_\epsilon(t) f(H_\epsilon) K_\epsilon) = \text{tr} (U_\epsilon(t) f(H_\epsilon) U_\epsilon(t_0) K_\epsilon(-t_0)). \tag{2.12}$$

This remark was used by Robert in (2.10) and follows trivially by centrality of the trace. In general, we cannot obtain an explicit formula for $U_\epsilon(s) K_\epsilon(-s)$ and rather get an approximation. This implies that we have to study an error term and this is why we display the following explicit formulas. If $K_\epsilon^s$ is $C^1$ w.r.t. $s$, satisfying $K_\epsilon^0 = K_\epsilon$, we have

$$U_\epsilon(t_0) K_\epsilon U_\epsilon(-t_0) = K_\epsilon^{t_0} + \frac{i}{\hbar} \int_0^{t_0} U_\epsilon(t_0 - s) \left( \hbar \frac{\partial}{\partial s} K_\epsilon^s - [H_\epsilon, K_\epsilon^s] \right) U_\epsilon(s - t_0) \, ds \tag{2.13}$$

and this leads to the following exact formula

$$\text{tr} (U_\epsilon(t) f(H_\epsilon) K_\epsilon) = \text{tr} (U_\epsilon(t) f(H_\epsilon) K_\epsilon^{t_0}) + \frac{i}{\hbar} \int_0^{t_0} \text{tr} \left( U_\epsilon(t) f(H_\epsilon) \left( \hbar \frac{\partial}{\partial s} K_\epsilon^s - [H_\epsilon, K_\epsilon^s] \right) \right) U_\epsilon(s - t_0) \, ds \tag{2.14}$$

since the terms $U_\epsilon(t_0 - s)$ and $U_\epsilon(s - t_0)$ cancel out by centrality. Thus, if we are able to find such a $K_\epsilon^s$ with $\hbar \partial_s K_\epsilon^s - [H_\epsilon, K_\epsilon^s]$ small in a certain sense, we see that the study of $\text{tr} (U_\epsilon(t) f(H_\epsilon) K_\epsilon)$ reduces to the one of $\text{tr} (U_\epsilon(t) f(H_\epsilon) K_\epsilon^{t_0})$, up to a remainder which is given explicitly by (2.11). The method leading to the calculation of $K_\epsilon^s$ is the usual one given by the Egorov theorem. We refer to (2.10) for a proof of this theorem. The main point is that the symbol (in each chart) of $K_\epsilon^s$ has an explicit expression in term of the symbol of $K_\epsilon$ and of the Hamiltonian flow of $p_\epsilon$.

If $K_\epsilon$ is a pseudo-differential operator with a symbol supported in a suitable region of $T^*X$ (and this can be achieved by replacing $K_\epsilon$ by $K_\epsilon^{t_0}$ thanks to the above trick), we shall see that it can be factorized as

$$K_\epsilon = A_\epsilon B_\epsilon^s, \quad \text{with} \ A_\epsilon, B_\epsilon : L^2(\mathbb{R}) \to L^2(X). \tag{2.15}$$
In practice such a factorization will only be obtained approximately, i.e. $K_{\varepsilon} - A_{\varepsilon}B_{\varepsilon}^*$ negligible (in a sense defined rigorously in section [3]). The operator $A_{\varepsilon}$ will be used to intertwine $U_{\epsilon}(t)$ with a free dynamic given by $U(t) = \exp(-itP/h)$, i.e. make $U_{\epsilon}(t)A_{\varepsilon} - A_{\varepsilon}U(t)$ small, in some sense, with $P = p(hD)$ differential operator with constant coefficients on $\mathbb{R}^n$. The explicit formula for $U_{\epsilon}(t)A_{\varepsilon} - A_{\varepsilon}U(t)$ is easily seen to be

$$U_{\epsilon}(t)A_{\varepsilon} - A_{\varepsilon}U(t) = \frac{1}{i\hbar} \int_0^t U_{\epsilon}(t-s)(H_{\varepsilon}A_{\varepsilon} - A_{\varepsilon}P)U(s)\, ds$$

(2.16)

and this shows that, if (2.15) holds, then we have

$$\text{tr} (U_{\epsilon}(t)f(H_{\varepsilon})K_{\varepsilon}) - \text{tr} (f(H_{\varepsilon})A_{\varepsilon}U(t)B_{\varepsilon}^*) =$$

$$\frac{1}{i\hbar} \int_0^t \text{tr} (f(H_{\varepsilon})U_{\epsilon}(t-s)(H_{\varepsilon}A_{\varepsilon} - A_{\varepsilon}P)U(s)B_{\varepsilon}^*)\, ds.$$  

(2.17)

Thus if we are able to show that $(H_{\varepsilon}A_{\varepsilon} - A_{\varepsilon}P)U(s)B_{\varepsilon}^*$ is small (see lemma 5.4), we see that we are left with the study of $\text{tr} (f(H_{\varepsilon})A_{\varepsilon}U(t)B_{\varepsilon}^*)$. This trace is easy to study since we shall have explicit expressions for the operators $A_{\varepsilon}, B_{\varepsilon}$ and $U(t)$. The construction of the operators $A_{\varepsilon}$ and $B_{\varepsilon}$ will follow the scheme of Isozaki-Kitada’s method explained in the next subsection.

### 2.3 The method of Isozaki-Kitada

In this part, we recall the principle of the construction of the operators $A_{\varepsilon}, B_{\varepsilon}$ by the method of Isozaki-Kitada introduced in [29]. This method has only been used on $\mathbb{R}^n$ for Euclidean scattering [21, 44, 47, 15, 6, 7] but it turns out that it can be used in our framework as well, with some changes on which we shall put emphasize in section 5.

We first recall the algebraic formulas which enters into the game for a general differential operator of order 2 (on a general manifold $X$), which we still denote $H_{\varepsilon} = h^2P_{\varepsilon}(x,Dx)$ with

$$P_{\varepsilon}(x,Dx) = p_{\varepsilon}(x,Dx) + p_{\varepsilon}^{(1)}(x,Dx) + p_{\varepsilon}^{(2)}(x)$$

in coordinates $x = (x_1,\ldots,x_n)$. Here $p_{\varepsilon}(x,\xi)$ is the principal symbol, i.e. homogeneous of degree 2 w.r.t. to the dual coordinates $\xi = (\xi_1,\ldots,\xi_1)$, and $p_{\varepsilon}^{(j)}$ are homogeneous of degree $2 - j$ for $j = 1,2$. We look for $A_{\varepsilon}, B_{\varepsilon}$ defined as operators of the form $J(\varphi_{\varepsilon}, a_{\varepsilon})$ and $J(\varphi_{\varepsilon}, b_{\varepsilon})$ where

$$J(\varphi_{\varepsilon}, a_{\varepsilon})u(x) = (2\pi h)^{-n} \int \int e^{i\varphi_{\varepsilon}(x,\xi)-x'\cdot\xi}a_{\varepsilon}(x,\xi,h)u(x')\, dx'\, d\xi.$$  

(2.18)

Note that (2.18) defines actually an operator from $L^2(\mathbb{R}^n)$ into itself (under suitable conditions on $\varphi_{\varepsilon}$ and $a_{\varepsilon}$); however, in the applications, $a_{\varepsilon}$ (and $b_{\varepsilon}$) will be supported into a region of $\mathbb{R}^n_+ \times \mathbb{R}^n_-$ whose projection onto $\mathbb{R}^n_+$ is included into a coordinate chart of $X$. Thus, up to an invertible operator, we can consider (2.18) as an operator from $L^2(\mathbb{R}^n)$ to $L^2(X)$.

Following (2.16), we see that we have to study $H_{\varepsilon}J(\varphi_{\varepsilon}, a_{\varepsilon}) - J(\varphi_{\varepsilon}, a_{\varepsilon})P$. Since $P = p(hD)$ is a Fourier multiplier, we have obviously $J(\varphi_{\varepsilon}, a_{\varepsilon})P = J(\varphi_{\varepsilon}, a_{\varepsilon}P)$. On the other hand, we see easily that

$$H_{\varepsilon}(e^{i\varphi_{\varepsilon}}a_{\varepsilon}) = e^{i\varphi_{\varepsilon}}(p_{\varepsilon}(x,\partial_x\varphi_{\varepsilon})a_{\varepsilon} + hi^{-1}L_{\varepsilon}(x,\partial_x)a_{\varepsilon} + h^2P_{\varepsilon}(x,Dx)a_{\varepsilon}).$$  

(2.19)

Here, $L_{\varepsilon}(x,\partial_x)$ is a differential operator of order 1 defined as

$$L_{\varepsilon}(x,\partial_x) = w_{\varepsilon}(x,\xi)\partial_x + c_{\varepsilon}(x,\xi)$$
where we have set

\[ w_\epsilon(x, \xi) = (\partial_\xi p_\epsilon)(x, \partial_x \varphi_\epsilon), \quad c_\epsilon(x, \xi) = p_\epsilon(x, \partial_x) \varphi_\epsilon + ip_\epsilon^{(1)}(x, \partial_x \varphi_\epsilon). \]

All this shows that if we look for \( a_\epsilon = a_\epsilon^{(0)} + h a_\epsilon^{(1)} + \cdots + h^N a_\epsilon^{(N)} \), then

\[ H_\epsilon J(\varphi_\epsilon, a_\epsilon) - J(\varphi_\epsilon, a_\epsilon)P = J(\varphi_\epsilon, \tilde{a}_\epsilon) \tag{2.20} \]

with \( \tilde{a}_\epsilon = a_\epsilon^{(0)} + h a_\epsilon^{(1)} + \cdots + h^{(N+2)} a_\epsilon^{(N+2)} \). For \( 0 \leq j \leq N + 2 \), the functions \( \tilde{a}_\epsilon^{(j)} \) are given by

\[ \tilde{a}_\epsilon^{(j)} = (p_\epsilon(x, \partial_x \varphi_\epsilon) - p(\xi)) a_\epsilon^{(j)} - iL_\epsilon(x, \partial_x) a_\epsilon^{(j-1)} + P_\epsilon(x, D_x) a_\epsilon^{(j-2)}, \tag{2.21} \]

where we use the convention that \( a_\epsilon^{(k)} = 0 \) for \( k < 0 \) or \( k > N \).

Since we want to make \( \epsilon \) small, we look for \( \varphi_\epsilon(x, \xi) \) such that

\[ p_\epsilon(x, \partial_x \varphi_\epsilon) = p(\xi) \tag{2.22} \]

which is usually called the Hamilton-Jacobi equation. We also need to find \( a_\epsilon^{(k)} \) such that

\[
\begin{align*}
L_\epsilon(x, \partial_x) a_\epsilon^{(0)} &= 0, \tag{2.23} \\
L_\epsilon(x, \partial_x) a_\epsilon^{(k)} &= -iP_\epsilon(x, D_x) a_\epsilon^{(k-1)}, \quad k \geq 1, \tag{2.24}
\end{align*}
\]

which are the transport equations. The resolution of \( 2.22, 2.23 \) and \( 2.24 \) rely upon estimates on classical trajectories. The technical part leading to such estimates in the asymptotically hyperbolic case is the purpose of section \( \S \). Here we recall the general method. Assume that we can find \( S_\epsilon(t, x, \xi) \) defined on \( [0, \infty) \times \Gamma \), for some open set \( \Gamma \subset \mathbb{R}^2 \), such that

\[ \partial_t S_\epsilon = p_\epsilon(x, \partial_x S_\epsilon), \quad S_\epsilon(0, x, \xi) = x, \xi. \]

The existence of \( S_\epsilon \) will follow from suitable estimates on \( \phi_\epsilon^t \), the Hamilton flow of \( p_\epsilon \). In practice, \( \Gamma \) is such that

\[ \partial_t S_\epsilon(t, x, \xi) \to \infty, \quad \text{as } t \uparrow \infty. \tag{2.25} \]

Now, using the fact that \( S_\epsilon \) is a generating function of the flow, i.e.

\[ \phi_\epsilon^t(x, \partial_x S_\epsilon) = (\partial_t S_\epsilon, \xi), \tag{2.26} \]

the invariance of \( p_\epsilon \) by the flow and \( 2.24 \) imply that

\[ \lim_{t \uparrow \infty} p_\epsilon(x, \partial_x S_\epsilon) = p(\xi) \tag{2.27} \]

provided

\[ p(\xi) = \lim_{x \to \infty} p_\epsilon(x, \xi). \]

This shows that we have to built \( \varphi_\epsilon \) such that \( \partial_\xi \varphi_\epsilon = \lim_{t \uparrow \infty} \partial_t S_\epsilon \), or equivalently such that

\[ \partial_x \varphi_\epsilon(x, \xi) = \xi + \int_0^\infty \partial_t \partial_x S_\epsilon(t, x, \xi) \, dt. \]

Of course, if such a function \( \varphi_\epsilon \) exists it is not unique. A possible construction is the following

\[ \varphi_\epsilon(x, \xi) = x, \xi + \int_0^\infty \partial_t \left( S_\epsilon(t, x, \xi) - \tilde{S}_\epsilon(t, \xi) \right) \, dt. \tag{2.28} \]
Indeed, any solution $a(x, \xi)$ of $L \cdot a = 0$ satisfies

$$a(\hat{x}_c^t, \xi) = a(x, \xi) \exp \left( - \int_0^t c_c(\hat{x}_c^s, \xi) \, ds \right).$$

In the applications, we shall prove that $\hat{x}_c^t(x, \xi)$ is defined on $[0, \infty) \times \Gamma$ and satisfies (in a sense to be made precise)

$$\hat{x}_c^t \sim x + t \partial_x p(\xi), \quad t \uparrow \infty.$$  \hspace{1cm} (2.29)

This is actually well known in the Euclidean case (see for instance $[29, 21, 47, 15]$). Hence, if we look for $a_c^{(0)}$ such that $a_c^{(0)}(x, \xi) \to 1$ as $(x, \xi) \to \infty$ in $\Gamma$, we obtain

$$a_c^{(0)}(x, \xi) = \exp \left( \int_0^\infty c_c(\hat{x}_c^t, \xi) \, dt \right).$$ \hspace{1cm} (2.30)

Similarly, by the method of variation of constants, we see that any solution of $L \cdot a = \iota$ satisfies

$$a(\hat{x}_c^t, \xi) = \left( a(x, \xi) + \int_0^t \iota(\hat{x}_c^s, \xi) \exp \left( \int_s^t c_c(\hat{x}_c^u, \xi) \, du \right) \exp \left( - \int_0^u c_c(\hat{x}_c^v, \xi) \, dv \right) \right).$$

Thus if we look for a solution $a_c^{(k)}$ of $L \cdot a = \iota$ such that $a_c^{(k)}(x, \xi) \to 0$ as $(x, \xi) \to \infty$ in $\Gamma$, we get

$$a_c^{(k)}(x, \xi) = - \int_0^\infty \iota P_c(x, D_x) a_c^{(k-1)}(\hat{x}_c^t, \xi) \exp \left( \int_0^t c_c(\hat{x}_c^s, \xi) \, ds \right) \, dt.$$ \hspace{1cm} (2.31)

Here again, the convergence of the integrals is justified by the appropriate estimates on the functions $c_c$ and $P_c(x, D_x) a_c^{(k-1)}$ which need to be shown. Such estimates are well known in the Euclidean case and will follow from section 3 for the hyperbolic one.

The formulas (2.30) and (2.31) define $a_c^{(k)}$, for $k \geq 0$, in $\Gamma$. In section 5 we will explain how to define them globally, i.e. how to cut them off outside a suitable area.

Recall that we want to consider a factorization of the form $J(\varphi_c, a_c) J(\varphi, b_c)^*$ whose Schwartz kernel is

$$K_c(x, x') = (2\pi h)^{-n} \int e^{\frac{i}{h}(\varphi_c(x, \xi) - \varphi_c(x', \xi))} a_c(x, \xi, h) b_c(x', \xi, h) \, d\xi.$$  

This kernel is the one of a pseudo-differential operator since Kuranishi’s trick, namely

$$\varphi_c(x, \xi) - \varphi_c(x', \xi) = (x - x'). \vartheta_c(x, x', \xi)$$

which is of course obtained by Taylor’s formula, allows to write

$$K_c(x, x') = (2\pi h)^{-n} \int e^{\frac{i}{h}(x - x'). \vartheta_c(x, x', \theta, h)} \, d\theta.$$
provided the following map is a diffeomorphism for each \( x, x' \) belonging to the projections of the supports of \( a_\varepsilon \) and \( b_\varepsilon \):

\[
\xi \mapsto \vartheta_\varepsilon(x,x',\xi) = \int_0^1 \partial_x \varphi_\varepsilon(x' + t(x - x'),\xi) \, dt.
\]

Note that, in view of (2.28), we see that if \( \partial_x \partial_t S_\varepsilon(t,x,\xi) \) is small (which will indeed be the case), then the map \( \xi \mapsto \vartheta_\varepsilon \) is close to the identity and easily seen to be a diffeomorphism. In this case we have

\[
\vartheta_\varepsilon(x,x',\theta,h) = a_\varepsilon(x,\theta^{-1}(x,x',\theta),h) b_\varepsilon(x',\theta^{-1}(x,x',\theta),h) |\det \partial_\theta \theta^{-1}(x,x',\theta)|.
\]

Using a general elementary property of pseudo-differential operators, we have

\[
K_\varepsilon(x,x') \sim (2\pi h)^{-n} \sum_{j \geq 0} h^j \int e^{i(x-x') \cdot \theta} \sum_{|\alpha| = j} \frac{1}{\alpha!} \partial_{\theta}^\alpha D_{\theta}^\alpha \varphi_\varepsilon(x,x',\theta,h)|_{x'=x} \, d\theta.
\]

By identification of the powers of \( h \), this allows to find \( b_\varepsilon = b_\varepsilon^{(0)} + \cdots + h^N b_\varepsilon^{(N)} \) such that, modulo \( h^N \), the right hand side is the expansion of the Schwartz kernel of \( K_\varepsilon \), with the notation of (2.17). For instance, if the principal symbol of \( K_\varepsilon \) is \( \sigma(x,\theta) \), we must have

\[
a_\varepsilon^{(0)}(x,\theta^{-1}(x,\theta)) b_\varepsilon^{(0)}(x,\theta^{-1}(x,\theta)) |\det \partial_\theta \theta^{-1}(x,\theta)| = \sigma(x,\theta)
\]

which implies that

\[
b_\varepsilon^{(0)}(x,\xi) = \sigma(x,\vartheta_\varepsilon(x,x,\xi)) a_\varepsilon^{(0)}(x,\xi)^{-1} |\det \partial_\xi \vartheta_\varepsilon(x,x,\xi)|.
\]  \( (2.32) \)

More generally, one can get explicit expressions for \( b_\varepsilon^{(1)}, \cdots, b_\varepsilon^{(N)} \) and the important remark is that they are linear combinations of products of (derivatives of) \( a_\varepsilon^{(0)}(x,\xi)^{-1} \), \( a_\varepsilon^{(k)}(x,\xi) \) for \( k \geq 1 \), \( \vartheta_\varepsilon(x,x',\xi) \) (evaluated at \( x' = x \)) and the symbols of \( K_\varepsilon \) evaluated at \( (x,\vartheta_\varepsilon(x,x,\xi)) \). In practice, \( \vartheta_\varepsilon(x,x,\xi) - \xi \) will be small in the region that we will consider and \( \Gamma \) will be a neighborhood of \( \supp \sigma \), so that \( b_\varepsilon^{(0)}, b_\varepsilon^{(1)}, \cdots \) will only depend on \( \sigma \) and on the values of \( a_\varepsilon^{(0)}, a_\varepsilon^{(1)}, \cdots \) and \( \varphi_\varepsilon \) in \( \Gamma \). In particular (2.30) will ensure that \( a_\varepsilon^{(0)} \) doesn’t vanish.

3 Some estimates on the geodesics

In this technical section, we prove long time estimates on classical trajectories, in suitable areas of \( T^*X \), needed to justify Isozaki-Kitada’s method in the asymptotically hyperbolic case.

3.1 Results of this section

We consider the function defined for \( r \in \mathbb{R}, \rho \in \mathbb{R} \) and \( y, \eta \in \mathbb{R}^{n-1} \) by

\[
p_\varepsilon(r,y,\rho,\eta) = \rho^2 + e^{-2r}g_\varepsilon(r,y,\eta)
\]

where the function \( g_\varepsilon(r,y,\eta) \) defines a metric on \( \mathbb{R}^{n-1} \), i.e. \( g_\varepsilon \) is a smooth function which is an homogeneous polynomial of degree 2 w.r.t. \( \eta \) and such that for some \( C_0 > 0 \)

\[
C_0^{-1} |\eta|^2 \leq g_\varepsilon(r,y,\eta) \leq C_0 | \eta |^2,
\]  \( (3.1) \)
for all \( r \in \mathbb{R}, y, \eta \in \mathbb{R}^{n-1} \) and \( \epsilon \in [0,1] \). We assume furthermore that \( g_\epsilon \) is of the following form

\[
g_\epsilon(r, y, \eta) = g(y, \eta) + \epsilon e^{-r} \tilde{g}(e^{-r}, y, \eta)
\]  

(3.2)

where \( g \) and \( \tilde{g} \) satisfy the following estimates

\[
|\partial_x^k \partial_y^\alpha \partial_\eta^\beta \tilde{g}(x, y, \eta)| + |\partial_y^\alpha \partial_\eta^\beta g(y, \eta)| \leq C_{k, \alpha, \beta} |\eta|^{2-|\beta|}, \quad x \in [0,1], \ y, \eta \in \mathbb{R}^{n-1}.
\]  

(3.3)

Note that in particular, we have \( g_\epsilon(r, y, \eta) - g(y, \eta) = O(e^{-r})|\eta|^2 \). All these estimates are satisfied by the principal symbol of \( P_\epsilon \) in any chart at infinity \((R, \infty) \times \Omega \). Thus we can assume that this principal symbol is the restriction of some function \( p_\epsilon \) satisfying the above estimates. We are going to prove estimates on trajectories of some vector fields and these trajectories will turn out to lie inside \((R, \infty) \times \Omega \times \mathbb{R}^n \). This means that the results of the present section can be obviously considered as local results in \( T^\ast X \).

Our first goal is the study of the Hamiltonian flow \( \phi^t_\epsilon \) of the function \( p_\epsilon \), that is the solution of

\[
\dot{\phi}^t_\epsilon = \mathbf{H}_\epsilon(\phi^t_\epsilon)
\]

where the notation \( \dot{\cdot} \) stands for \( d/dt \) throughout the section and \( \mathbf{H}_\epsilon = \mathbf{H}_\epsilon(r, y, \rho, \eta) \) is the Hamiltonian vector field of \( p_\epsilon \), that is

\[
\mathbf{H}_\epsilon = \left( \begin{array}{c}
\partial_\rho p_\epsilon \\
\partial_\eta p_\epsilon \\
- \partial_r p_\epsilon \\
- \partial_\eta p_\epsilon
\end{array} \right) = \left( \begin{array}{c}
2\rho \\
e^{-2r} \partial_y g_\epsilon(r, y, \eta) \\
e^{-2r} \partial_\rho g_\epsilon(r, y, \eta)
\end{array} \right) 
\]  

(3.4)

We denote the components of the flow by \( (r^t_\epsilon, y^t_\epsilon, \rho^t_\epsilon, \eta^t_\epsilon) \). They are functions of the initial condition \((r, y, \rho, \eta)\). We remark that \( \phi^t_\epsilon \) is defined for \( t \in \mathbb{R} \) for if this was wrong, then \( |\phi^t_\epsilon| \) should blow up in finite time. This cannot happen since the conservation of energy, \( p_\epsilon = p_\epsilon \circ \phi^t_\epsilon \), implies easily that \( \dot{\phi}^t_\epsilon \) is bounded, hence \( \phi^t_\epsilon - \phi^0_\epsilon = O(t) \) can not blow up in finite time.

Let us introduce some notations. We consider energy intervals of width \( 2w \) defined by

\[
I(w) = (1 - w, 1 + w).
\]

In the end, \( w \) will be chosen small enough but for the time being we only assume that \( 0 < w < 1/2 \). We also introduce the outgoing (resp. incoming) parameters \( \sigma_+ > 0 \) and \( \delta_+ > 0 \) (resp. \( \sigma_- > 0 \) and \( \delta_- > 0 \)) defined by

\[
\sigma_\pm = (1 \mp w)^{1/2} - \delta_\pm.
\]

Of course, this makes sense provided \( 0 < \delta_\pm < (1 \mp w)^{1/2} \). The last two parameters we shall need are a positive real number \( R > 0 \) and an arbitrary open subset \( \Omega \subset \mathbb{R}^{n-1} \). We can now define the outgoing area \( \Upsilon^+(R, \sigma_+, w, \Omega) \) and the incoming area \( \Upsilon^-(R, \sigma_-, w, \Omega) \) by

\[
\Upsilon^\pm(R, \sigma_\pm, w, \Omega) := \{(r, y, \rho, \eta) \in \mathbb{R}^{2n} | p_0(r, y, \rho, \eta) \in I(w), \ \pm \rho > -\sigma_\pm, \ r > R, \ y \in \Omega \}.
\]  

(3.5)

The point of considering such areas is that we have a splitting

\[
\Upsilon^+(R, \sigma_+, w, \Omega) \cup \Upsilon^-(R, \sigma_-, w, \Omega) = \{(r, y, \rho, \eta) | p_0(r, y, \rho, \eta) \in I(w), \ r > R, \ y \in \Omega \}. 
\]  

(3.6)

The first result is the following.
Corollary 3.3. Let $0 < w < 1/2$, $\sigma_\pm > 0$ and $\Omega$ as above. There exists $R$ large enough and $C > 0$, depending only on $C_0$ and a finite number of constants $C_{k,\alpha,\beta}$ in (3.5), such that the following estimates hold

\begin{align}
  r_\epsilon^t &\geq r \pm t - C, \\
  \rho_\epsilon^t &\geq 0 \\
  |y_\epsilon^t - y| &\leq Ce^{-r}
\end{align}

for all $\epsilon \in [0,1]$, $(r,y,\rho,\eta) \in \Upsilon^\pm(R,\sigma_\pm, w, \Omega)$ and $t \geq 0$.

The meaning of these statements is that the properties are true on $\Upsilon^+$ (resp. $\Upsilon^-$) for $t \geq 0$ (resp. $t \leq 0$). This result says that the geodesics starting in outgoing (resp. incoming) areas stay in the neighborhood of infinity in $X$ for $t \geq 0$ (resp. $t \leq 0$). More precise estimates on the flow will be given in the theorem below. Note moreover that (3.9) shows that $y_\epsilon^t$ lie in any arbitrary small neighborhood of $\Omega$. It explains why our results can be localized in charts at infinity on $X$.

Remark 1. This proposition, or rather its proof, shows in particular that the flow of $p_\epsilon$ in $\Upsilon^\pm(R,\sigma_\pm, w, \Omega)$ depends only on the values of $p_\epsilon$ in the region $\{r \geq R - C\}$. This implies that if we replace $\tilde{g}$ by $\tilde{g}^{(\kappa)}$ depending on a parameter $\kappa \in [0,1]$, in (3.2), such that the constants $C_0$ and $C_{k,\alpha,\beta}$ can be chosen uniformly w.r.t. $\kappa$ for $r$ large enough, then the above proposition is true uniformly w.r.t. $\kappa$. In particular, this holds if one considers the principal symbol of (2.8) which involves

\begin{equation}
  \tilde{g}^{(\kappa)}(e^{-r}, y, \eta) = \Theta(\kappa r)^2 \tilde{g}(e^{-r}, y, \eta).
\end{equation}

Using this remark, we now claim that all the results listed in this subsection hold true uniformly w.r.t. $\kappa \in [0,1]$ if one considers (3.10).

Theorem 3.2. Let $w, \sigma_\pm$ and $\Omega$ be as in proposition 3.1. There exists $R$ large enough such that for all $\gamma$ defined by $\partial^\gamma = \partial_r^{\gamma_r} \partial_y^{\gamma_y} \partial_\rho^{\gamma_\rho} \partial_\eta^{\gamma_\eta}$, one can find $C_\gamma > 0$ satisfying

\begin{align}
  |\partial^\gamma (r_\epsilon^t - r - 2tr_\rho)| &\leq C_\gamma(t) e^{-(j+1)r} |\eta| \\
  |\partial^\gamma (y_\epsilon^t - y)| &\leq C_\gamma e^{-(j+1)r} |\eta| \\
  |\partial^\gamma (\rho_\epsilon^t - \rho)| &\leq C_\gamma e^{-(j+1)r} |\eta| \\
  |\partial^\gamma (\eta_\epsilon^t - \eta)| &\leq C_\gamma e^{-(j+1)r} |\eta|
\end{align}

for all $\epsilon \in [0,1]$ provided the following condition holds

\begin{equation}
  \pm t \geq 0 \quad \text{and} \quad (r,y,\rho,\eta) \in \Upsilon^\pm(R,\sigma_\pm, w, \Omega).
\end{equation}

Corollary 3.3. For $R$ large enough and for all $(r,y,\rho,\eta) \in \Upsilon^\pm(R,\sigma_\pm, w, \Omega)$ the following limits exist for all $\epsilon \in [0,1]$

\begin{align}
  y_\epsilon^\pm &= \lim_{t \to \pm \infty} y_\epsilon^t, \quad \eta_\epsilon^\pm = \lim_{t \to \pm \infty} \eta_\epsilon^t \quad \lim_{t \to \pm \infty} \rho_\epsilon^t = \pm p_\epsilon^{1/2}
\end{align}

where $p_\epsilon = p_\epsilon(r,y,\rho,\eta)$.

This corollary follows very easily from the motion equations, proposition 3.1 and theorem 3.2 since, in particular, $\dot{y}_\epsilon^t$, $\dot{\rho}_\epsilon^t$ and $\dot{\eta}_\epsilon^t$ are $O(e^{-2t})$. 

14
Another important consequence of the theorem is the following. Since $\nabla_{\rho, \eta} (\rho^l, \eta^l)$ is close to the identity matrix if $e^{-r} \eta$ is small enough, we can hope that $(\rho, \eta) \mapsto (\rho^l, \eta^l)$ is a diffeomorphism under suitable conditions. This is why we introduce

$$
\Gamma^\pm (R, \varepsilon, w, \Omega) = \{(r, y, \rho, \eta) \mid p_0(r, y, \rho, \eta) \in I(w), \ R > R, \pm \rho > 0, \ y \in \Omega, \ e^{-2r}g(y, \eta) < \varepsilon\}.
$$

Before studying the above diffeomorphism, let us note that any area $\Gamma^\pm$ can be reached in finite time from $\Upsilon^\pm$. Precisely we have

**Lemma 3.4.** Let $0 < w < w' < 1/2$ and $\sigma_\pm > 0$. Assume that $\Omega$ is bounded and that $\Omega \subset \Omega'$. Then, there exists $R > 0$ large enough such that for any $\varepsilon > 0$ and $R' > R$, there exists $T > 0$ such that

$$
\phi^l_\varepsilon \left( \Upsilon^\pm (R, \sigma_\pm, w, \Omega) \right) \subset \Gamma^\pm (R', \varepsilon, w', \Omega'), \quad \forall \pm t \geq T, \ \varepsilon \in [0, 1].
$$

This lemma follows easily from the previous results, the main tool being the fact that $e^{-r} \eta^l \rightarrow 0$ as $\pm t \rightarrow +\infty$. Now let us consider the map

$$
\Phi^l_\varepsilon : (r, y, \rho, \eta) \mapsto (r, y, \rho^l, \eta^l).
$$

Then we have the following result.

**Proposition 3.5.** Let $\Omega_0, \Omega_1$ be bounded, connected and such that $\Omega_1 \subset \Omega_0$. There exists $R_0, R_1 > 0$ large enough and $w_0, w_1, \varepsilon_0, \varepsilon_1 > 0$ small enough such that for all $\pm t \geq 0$

$$
\Gamma^\pm (R_1, \varepsilon_1, w_1, \Omega_1) \subset \Phi^l_\varepsilon (\Gamma^\pm (R_0, \varepsilon_0, w_0, \Omega_0)). \quad (3.16)
$$

Furthermore, $\Phi^l_\varepsilon$ is a diffeomorphism from $\Gamma^\pm (R, \varepsilon_0, w_0, \Omega_0)$ onto its range for all $\pm t \geq 0$. If we denote the inverse map by $(r, y, \rho, \eta) \mapsto (r, y, \rho^l, \eta^l)$ we have

$$
|\partial^\gamma (\rho^l - \rho)| + |\partial^\gamma (\eta^l - \eta)| \leq C_\gamma e^{-(j+1)r} |\gamma|, \quad \text{if} \quad \partial^\gamma = \partial^k_\rho \partial^l_\gamma \partial^k_\rho \partial^l_\gamma
$$

with $C_\gamma$ independent of $\varepsilon \in [0, 1], \pm t \geq 0$ and $(r, y, \rho, \eta) \in \Gamma^\pm (R_1, \varepsilon_1, w_1, \Omega_1)$.

The main application of this proposition is the resolution of the following eikonal equation

$$
\partial_t S^\pm_{\varepsilon} = p_{\varepsilon} (r, y, \partial_r S^\pm_{\varepsilon}, \partial_\gamma S^\pm_{\varepsilon}), \quad S^\pm_{\varepsilon} (0, r, y, \rho, \eta) = r \rho + y \eta \quad (3.18)
$$

with $S^\pm_{\varepsilon} = S^\pm_{\varepsilon} (t, r, y, \rho, \eta, \varepsilon)$ defined on $\Gamma^\pm (R_1, \varepsilon_1, w_1, \Omega_1)$ for any $\pm t \geq 0$. We can solve $\ref{3.18}$ since any $S$ which satisfy

$$
S(t, r, y, \rho^l, \eta^l, \varepsilon) = r \rho + y \eta + t \rho_{\varepsilon} - \int_0^t (r \partial_r \rho_{\varepsilon} + y \partial_\gamma \rho_{\varepsilon}) \circ \phi^l_{\varepsilon} \, ds \quad (3.19)
$$

solves $\ref{3.18}$. Thus the composition of the right hand side of $\ref{3.19}$ with the inverse of $\Phi^l_{\varepsilon}$ is a solution to $\ref{3.18}$. We shall show the following

**Proposition 3.6.** We have a solution $S^\pm_{\varepsilon}$ of $\ref{3.18}$ on $\Gamma^\pm (R_1, \varepsilon_1, w_1, \Omega_1)$ for $\pm t \geq 0$. Furthermore, if $R_1$ is large enough and $\varepsilon_1$ small enough, we have

$$
|\partial^\gamma (S^\pm_{\varepsilon} - r \rho - y \eta - t \rho_{\varepsilon})| \leq C_\gamma e^{-(j+1)r} |\gamma|, \quad \partial^\gamma = \partial^k_\rho \partial^l_\gamma \partial^k_\rho \partial^l_\gamma
$$

for $\varepsilon \in [0, 1], \pm t \geq 0$ and $(r, y, \rho, \eta) \in \Gamma^\pm (R_1, \varepsilon_1, w_1, \Omega_1)$. 

15
Lemma 3.8. Let us recall again that all the coming proofs in the next subsections work uniformly w.r.t. the parameter $\kappa \in [0, 1]$ since they rely on the estimates (3.1) and (3.3) which are uniform w.r.t. $\kappa$, when working with (3.10). We will omit this parameter not to burden the notations but we hope that the proofs are explicit enough to make this uniformity clear. Then the continuity w.r.t. $\kappa \in [0, 1]$, and actually the smoothness, is obvious since in the case of the flow (theorem 3.2) it follows from the standard result of smoothness of O.D.E. with respect to parameters and in the case of the diffeomorphism (propositions 3.5 and 3.6) it is due to the implicit functions theorem.

We now give the proofs of these results. We shall only consider the outgoing situation since the incoming one can be treated similarly.

3.2 Proof of proposition 3.1 and theorem 3.2

Most of the results that we are going to prove rely on the fact that

\[- \partial_r p_e = 2(p_e - \rho^2) + \mathcal{O}(e^{-r})(p_e - \rho^2).\]

(3.21)

The first step is the following.

Lemma 3.7. One can choose $\bar{R}$ large enough so that for all $(r, y, \rho, \eta) \in \Upsilon^+(\bar{R}, \sigma_+, w, \Omega)$ and all $\epsilon \in [0, 1]$ the following condition holds:

\[r_e^{t_1} \geq \bar{R} \quad \text{and} \quad \rho_e^{t_1} > 0 \quad \text{for some} \quad t_1 \geq 0 \Rightarrow r_e^t \geq \bar{R} \quad \text{and} \quad \rho_e^t \geq \rho_e^{t_1}, \quad \forall t \geq t_1.\]

Proof. First we remark that there exists $\bar{R} > 0$ and $c > 0$ such that

\[- \partial_r p_e \geq cg(y, e^{-r/\eta}) \quad r \geq \bar{R}, \quad y, \eta \in \mathbb{R}^{n-1}, \quad \epsilon \in [0, 1].\]

(3.22)

This follows easily from (3.1) and (3.3). It is the first example of application of (3.21). Then we consider the set

\[\mathcal{T} = \{t \geq t_1 \mid \rho_e^t \geq \rho_e^{t_1} \quad \text{and} \quad r_e^s \geq \bar{R}, \quad \forall s \in [t_1, t]\}.\]

It is clear that $t_1 \in \mathcal{T}$ thus $\mathcal{T}$ is a non empty interval. We shall prove that $T := \sup \mathcal{T}$ is $+\infty$. We argue by contradiction and assume that $T$ is finite. By continuity, it is clear that $T \in \mathcal{T}$, and that there exists $T' > T$ such that $\rho_e^{T'} > \rho_e^{t_1}/2$ for all $s \in [T_1, T']$. Now using the fact $r_e^{t_1} = 2\rho_e^{t_1}$ we get

\[r_e^s \geq r_e^{t_1} + (s - t_1)\rho_e^{t_1} \geq \bar{R}, \quad t_1 \leq s \leq T'.\]

By (3.22), this implies that $\rho_e^s \geq 0$ on $[t_1, T']$. In particular $\rho_e^s \geq \rho_e^{t_1}$ for $s$ in a larger interval than $[t_1, T]$ which is a contradiction.

In the next lemma, we show that $\rho_e^t$ reaches a positive value at some time $t_1$. More precisely we show that $\rho_e^{t_1} \geq 1/2$ for some positive time $t_1$ which is uniform with respect to the initial conditions in the outgoing area.

Lemma 3.8. There exists $\tilde{R}(w, \delta_+) > 0$ and $t_1(w, \delta_+) > 0$ such that for all initial condition $(r, y, \rho, \eta) \in \Upsilon^+(\bar{R}, \sigma_+, w, \Omega)$ and all $\epsilon \in [0, 1]$

\[r_e^{t_1} \geq \tilde{R} \quad \text{and} \quad \rho_e^{t_1} \geq \frac{1}{2}.\]
Proof. We first note that the result is clear if $r \geq \bar{R}$ and $\rho \geq 1/2$ by the previous lemma. Thus we assume that $\delta_+ - (1 - w)^{1/2} < \rho < 1/2$. We also choose $w < \tilde{w} < 1/2$ such that $\tilde{w} - w < \delta_+$. By possibly increasing $\bar{R}$ we may assume that (3.22) holds and that

$$p_0(r, y, \rho, \eta) \in I(w) \quad \text{and} \quad r \geq \bar{R} \Rightarrow p_e(r, y, \rho, \eta) \in I(\tilde{w}) \quad \forall \epsilon \in [0, 1]$$

(3.23) since $p_0 - p_e = \mathcal{O}(e^{-r})$ on $p_0^{-1}(I(w))$. We may assume moreover that $\bar{R}$ is large enough so that

$$\left| e^{-2r} \frac{\partial}{\partial r} e^{-r} \tilde{g} \left( e^{-r}, y, \eta \right) \right| \leq \frac{\delta_+}{2} \quad \text{if} \quad r \geq \bar{R} \quad \text{and} \quad p_e(r, y, \rho, \eta) \in I(\tilde{w}) \quad \forall \epsilon \in [0, 1].$$

(3.24)

Now we choose $\bar{R}$ such that

$$\bar{R} > \bar{R} + 4\frac{\sigma_+}{c_+ \delta_+}, \quad \text{with} \quad c_+ = 2^{1/2} - 1 - \frac{1}{2}.$$ 

We start by proving that for any $(r, y, \rho, \eta) \in T^+(\bar{R}, \sigma_+, w, \Omega)$ we have

$$r^1_\epsilon \geq \bar{R} \quad \text{and} \quad \rho^1_\epsilon \geq \rho \quad \forall \epsilon \in [0, 1].$$

(3.25)

To that end, we proceed as in the previous lemma. We consider the set

$$T = \{ t \geq 0 \mid r^*_s \geq \bar{R} \quad \text{and} \quad \rho^*_s \geq \rho, \quad \forall \epsilon \in [0, t] \}$$

which is non empty interval since $0 \in T$ and we show that $T := \sup T$ is $+\infty$. Assume that this is wrong, then $T$ belongs to $T$. Note that we may assume that $\rho^*_\epsilon \leq 0$ on $[0, t]$ since otherwise we can find $t_1 \in [0, T]$ such that $\rho^1_\epsilon > 0$ and we get a contradiction using lemma 3.7. Then (3.24) and the conservation of energy yields

$$\dot{\rho}^*_\epsilon \geq 2p^{1/2}_\epsilon (p^{1/2}_\epsilon + \rho^*_\epsilon) - \frac{\delta_+}{2} \geq c_+ \delta_+$$

(3.26)

where the second inequality follows from the fact that $p^{1/2}_\epsilon > (1/2)^{1/2}$ and

$$p^{1/2}_\epsilon + \rho \geq (1 - \tilde{w})^{1/2} - (1 - w)^{1/2} + \delta_+ \geq (1 - 1/2^{1/2}) \delta_+.$$ 

As a consequence we get $\rho^*_\epsilon - \rho \geq T c_+ \delta_+$ and this implies easily that

$$T < \frac{\sigma_+}{c_+ \delta_+}$$

since $\rho^*_\epsilon \leq 0$ and $\rho > \delta_+ - (1 - w)^{1/2}$. On the other hand, the first equation of motion yields

$$r^*_\epsilon \geq r - 2T E_+(w)^{1/2} > \bar{R} - 4T > \bar{R}.$$ 

Thus $\rho^*_\epsilon > \rho$ and $r^*_\epsilon > \bar{R}$. This shows that there exists $T' > T$ such that $\rho^*_\epsilon \geq 0$ and $r^*_\epsilon \geq \bar{R}$ on $[0, T']$ which is a contradiction and completes the proof of (3.24).

We shall now prove the existence of $t_1$ in the same spirit. Recall that we can assume that $r < 1/2$. Then there exists $t_0$ depending on $\epsilon$ and the initial conditions such that $\rho^{t_0}_\epsilon = 1/2$ otherwise $\rho^*_\epsilon < 1/2$ for all $t \geq t_0$ and

$$\dot{\rho}^*_\epsilon \geq 2(p^{1/2}_\epsilon + \rho^{1/2}_\epsilon)(p^{1/2}_\epsilon - \rho^{1/2}_\epsilon) - \frac{\delta_+}{2} \geq 2(2^{1/2} - 1/2)(1 - 2^{-1/2}) \delta_+ - \frac{\delta_+}{2} > \frac{\delta_+}{2}$$

(3.27)
since we can use (3.24) thanks to (3.25). This implies that $\rho^t_e \geq \rho + t\delta_+/2 \rightarrow +\infty$ as $t \rightarrow \infty$ which is forbidden by the energy conservation. This is a contradiction thus $\rho^t_e$ reaches the value $1/2$ at some positive time. Moreover (3.24) holds as long as $\rho^t_e$ is lower than $1/2$, thus if $t_0$ is the smallest positive time for which $\rho^t_e = 1/2$ we have $1/2 - \rho \geq t_0\delta_+/2$. This yields

$$t_0 < \frac{2}{\delta_+}(\sigma_+ + 1/2).$$

Using lemma (3.27) it is clear that $\rho^t_e \geq 1/2$ for $t \geq (2\sigma_+ + 1)/\delta_+$ and this completes the proof. □

**Proof of proposition 3.1.** We choose $R > \tilde{R}$ with $\tilde{R}$ as in lemma (3.8). Then for $t \geq t_1$, with $t_1$ as in lemma (3.8) we have

$$r^t_e \geq (t - t_1) + r^t_1.$$  \hspace{1cm} \text{(3.28)}

On the other hand, for $t \in [0, t_1]$ we have

$$r^t_e - r \geq -2c_1t$$

for any $c_1 > p_\epsilon(r, y, \rho, \eta)$ (note that such a $c_1$ can be chosen uniformly with respect to $\epsilon$ and the initial conditions in $\bar{T}^+(R, \sigma_+, w, \Omega)$). These two estimates yields (3.27). The proof of (3.8) follows directly from (3.21) since one knows that $e^{-r^t_e}$ is small uniformly for $t \geq 0$. Finally, the conservation of energy implies that $e^{-r^t_e}\eta^t_e$ is bounded and thus $\dot{\eta}^t_e = \mathcal{O}(e^{-r^t_e})$, by (3.7). This implies (3.9) and completes the proof. □

The rest of this subsection is now devoted to the proof of theorem 3.2. We start with the following lemma

**Lemma 3.9.** If $R$ is large enough, there exists $C > 0$ such that

$$|\eta^t_e - \eta| \leq Ce^{-r^t_e}|\eta|$$

for all $(r, y, \rho, \eta) \in \bar{T}^+(R, \sigma_+, w, \Omega)$, $\epsilon \in [0, 1]$ and $t \geq 0$.

**Proof.** Choosing $R$ large enough and using proposition 3.1 shows the existence of $C > 0$ such that

$$|\langle \partial_y g_\epsilon(r^t_e, y^t_e, \eta^t_e) \rangle| \leq cg_\epsilon(r^t_e, y^t_e, \eta^t_e)$$

since $r^t_e$ is large. The last motion equation and the conservation of energy then show that $\dot{\eta}^t_e$ is bounded and thus $\eta^t_e - \eta = \mathcal{O}(t)$. Putting this new estimate into the last motion equation and using the fact that $e^{-2r^t_e} = \mathcal{O}(e^{-2r^t_e})$ show that $\eta^t_e - \eta$ is bounded. Since $e^{-r^t_e}$ is bounded on $\bar{T}^+$, the lemma is proved provided $e^{-r^t_e}$ is away from any neighborhood of 0. Thus we assume now that $e^{-r^t_e}$ is small enough so that $\rho \geq 0$. Since $\rho^t_e \geq 0$ and $\rho \geq 0$ we have

$$\rho^2 \leq (\rho^t_e)^2 \leq p_\epsilon$$

and then, using the fact that $e^{-2r^t_e}g_\epsilon(r^t_e, y^t_e, \eta^t_e) = p_\epsilon - (\rho^t_e)^2$ we obtain

$$|\dot{\eta}^t_e| \leq c(p_\epsilon - \rho^2) \leq cC_0(e^{-r^t_e}|\eta|)^2.$$

This shows that $\eta^t_e - \eta = \mathcal{O}(te^{-r^t_e}|\eta|)$ and putting this estimate into the motion equation of $\eta^t_e$ as before we obtain $\eta^t_e - \eta = \mathcal{O}(e^{-r^t_e}|\eta|)$ thanks to the exponential decay in time of $e^{-r^t_e}$. □
We explain the strategy of the proof of theorem 3.2 with the case $\partial^r = \partial_t$. By standard results on ordinary differential equations we know that $\phi^t_\epsilon$ is smooth with respect to $\epsilon$. In particular, if we consider the matrix

$$M_\epsilon(r, y, p, \eta) = \begin{pmatrix} \frac{\partial^2}{\partial r^2}p_r & \frac{\partial^2}{\partial r p_y}p_r & \frac{\partial^2}{\partial r^2}p_\eta & \frac{\partial^2}{\partial r p_\eta}p_r \\ \frac{\partial^2}{\partial r^2}p_y & \frac{\partial^2}{\partial r p_\eta}p_y & \frac{\partial^2}{\partial r^2}p_r & \frac{\partial^2}{\partial r p_r}p_y \\ -\frac{\partial^2}{\partial y r^2}p_r & -\frac{\partial^2}{\partial y r}p_\eta & -\frac{\partial^2}{\partial y^2}p_r & -\frac{\partial^2}{\partial y p_r}p_\eta \\ -\frac{\partial^2}{\partial y^2}p_y & -\frac{\partial^2}{\partial y p_\eta}p_y & -\frac{\partial^2}{\partial y^2}p_r & -\frac{\partial^2}{\partial y p_r}p_y \end{pmatrix}$$  \hspace{1cm} (3.29)

applying $\partial_t$ to the motion equations yields

$$\dot{X}_\epsilon^t = M_\epsilon(t)X^t_\epsilon + Y^t_\epsilon$$

with the notations $X^t_\epsilon = \partial_t \phi^t_\epsilon$, $M_\epsilon(t) = M_\epsilon(\phi^t_\epsilon)$ and $Y^t_\epsilon = (\partial_t H_\epsilon)(\phi^t_\epsilon)$. By lemma 3.9 and proposition 3.1 imply easily the following key estimates

$$\tilde{M}_\epsilon(t) = M + O(\exp(-2r_\epsilon^t)g_\epsilon(r_\epsilon^t, y_\epsilon^t, \eta_\epsilon^t)),$$

$$\tilde{Y}_\epsilon^t = O\left(\exp(-3r_\epsilon^t)g_\epsilon(r_\epsilon^t, y_\epsilon^t, \eta_\epsilon^t)\right)$$  \hspace{1cm} (3.31)

where the matrix $M$ is given by

$$M = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, it is very easy to check that $M^2 = 0$ and this implies that

$$\exp(tM) = 1 + tM, \quad t \in \mathbb{R}.$$

We do a change of unknown function by considering $\tilde{X}_\epsilon^t = \exp(-tM)X^t_\epsilon$. It satisfies the equation

$$\tilde{X}_\epsilon^t = \tilde{M}_\epsilon(t)\tilde{X}_\epsilon^t + \tilde{Y}_\epsilon^t$$

where the matrix $\tilde{M}_\epsilon(t)$ and the vector $\tilde{Y}_\epsilon^t$ are given by

$$\tilde{M}_\epsilon(t) = \exp(-tM)(M_\epsilon(t) - M)\exp(tM),$$

$$\tilde{Y}_\epsilon^t = \exp(-tM)Y^t_\epsilon.$$  \hspace{1cm} (3.32)

Lemma 3.10 and proposition 3.1 imply easily the following key estimates

$$\tilde{M}_\epsilon(t) = O(\langle t \rangle^2 e^{-2t - r_\epsilon})$$

$$\tilde{Y}_\epsilon^t = O(\langle t \rangle \exp(-2t - 2r_\epsilon))$$  \hspace{1cm} (3.33)

We shall deduce the estimates on $\tilde{X}_\epsilon^t$ from (3.32), (3.33), (3.34) and the well known Gronwall’s lemma which we quote under the following form.

**Lemma 3.10 (Gronwall’s lemma).** Let $u(t) \geq 0$ be a continuous function on $[t_0, +\infty)$ for some $t_0 \geq 0$. Assume that

$$u(t) \leq a \int_{t_0}^{t} u(s) \, ds + b, \quad t \geq t_0$$

for some $a \geq 0$ and $b \geq 0$. Then we have

$$u(t) \leq be^{a(t - t_0)}, \quad t \geq t_0.$$
Since $\tilde{X}_t^0 = X_t^0 = 0$, we can turn (3.32) into the following integral equation

$$
\tilde{X}_t^\epsilon = \int_0^t \tilde{M}_s(\epsilon) \tilde{X}_s^\epsilon \, ds + \int_0^t \tilde{Y}_s^\epsilon \, ds.
$$

Then Gronwall’s lemma combined with the estimates (3.34) shows that for any $T \geq 0$, there exists $C_T$ independent of the initial conditions in $\Sigma^+(R, \sigma^+, w, \Omega)$ and of $\epsilon$ such that

$$
|\tilde{X}_t^\epsilon| \leq C_T e^{-2r|\eta|}, \quad 0 \leq t \leq T. \tag{3.35}
$$

We can also write (3.32) as follows

$$
\tilde{X}_t^\epsilon = \int_t^T \tilde{A}_s(\epsilon) \tilde{X}_s^\epsilon \, ds + \int_0^t \tilde{Y}_s^\epsilon \, ds + \tilde{X}_T^\epsilon, \quad t \geq T.
$$

In particular, by choosing $T$ large enough we can show easily that

$$
|\tilde{X}_t^\epsilon| \leq \int_T^t |\tilde{X}_s^\epsilon| \, ds + C_T' e^{-2r|\eta|}, \quad t \geq T
$$

for some $C_T' > 0$. This is a simple consequence of the fact that $\tilde{M}_s(t) \to 0$ as $t \to +\infty$. Thus another application of Gronwall’s lemma shows that for some $C_T' \geq 0$

$$
|\tilde{X}_t^\epsilon| \leq C_T' e^{-2r|\eta|} e^{(t-T)}, \quad t \geq T. \tag{3.36}
$$

Putting (3.35) and (3.36) into (3.32) and using (3.33), one gets

$$
\dot{\tilde{X}}_t^\epsilon = \mathcal{O}\left(\langle t \rangle e^{-2r} e^{-2r|\eta|}\right), \quad t \geq 0.
$$

We can now come back to $X_t^\epsilon$ since one checks easily that $\dot{X}_t^\epsilon = M X_t^\epsilon + \exp(tM) \dot{X}_t^\epsilon$ which implies

$$
X_t^\epsilon = \exp(tM) \int_0^t \tilde{X}_s^\epsilon \, ds.
$$

Using the explicit form of $\exp(tM)$ we conclude that

$$
\begin{align*}
\partial^r r_t^\epsilon &= \mathcal{O}\left(\langle t \rangle e^{-2r} |\eta|\right), \\
|\partial^r y_t^\epsilon| + |\partial^r \rho_t^\epsilon| + |\partial^r \eta_t^\epsilon| &= \mathcal{O}\left(e^{-2r|\eta|}\right). \tag{3.37}
\end{align*}
$$

This completes the proof of theorem 3.2 if $\partial^\gamma = \partial_\gamma$. The other estimates for $|\gamma| = 0$ can be proved easily using lemma 3.4 and the motion equations. For $|\gamma| \neq 0$, we proceed by induction by applying $\partial^\gamma$ to the motion equations. \qed

Remark. Using the results of theorem 3.2 we can improve the estimates on $y_t^\epsilon$. Since

$$
y_t^\epsilon = y + \int_0^t e^{-2r_s^\epsilon} \partial_0 g_\epsilon(r_s^\epsilon, y_s^\epsilon, \eta_s^\epsilon) \, ds
$$

where $e^{-r_s^\epsilon} \partial_0 g_\epsilon(r_s^\epsilon, y_s^\epsilon, \eta_s^\epsilon)$ is bounded, we obtain easily

$$
|\partial^\gamma (y_t^\epsilon - y)| \leq C_\gamma e^{-(j_+ + 1)r} |\eta|, \quad j_+ = \max(j, 1). \tag{3.39}
$$
3.3 Proofs propositions 3.5 and 3.6

We start with a technical lemma.

**Lemma 3.11.** Assume that $\Omega$ is connected. Then $\Gamma^+(R, \varepsilon, w, \Omega)$ is connected.

**Proof.** Consider $(r_0, y_0, \rho_0, \eta_0)$ and $(r_1, y_1, \rho_1, \eta_1)$ in $\Gamma^+(R, \varepsilon, w, \Omega)$. We prove the result under the conditions $\eta_0 \neq 0$ and $\eta_1 \neq 0$ (the other cases are simpler). We first connect $(r_0, y_0, \rho_0, \eta_0)$ to $(r_1, y_1, \rho_0, \tau \eta_0)$ for some $\tau > 0$ by considering

$$((1 - s)r_0 + sr_1, y(s), \rho_0, \tau(s)\eta_0), \quad s \in [0, 1]$$

with $y(s)$ a path joining $y_0$ to $y_1$ in $\Omega$ and $\tau(s)$ the unique positive continuous function such that

$$\exp(-2((1 - s)r_0 + sr_1)) g(y(s), \tau(s)\eta_0) = \exp(-2r_0) g(y_0, \eta_0).$$

Then we connect $(r_1, y_1, \rho_0, \tau \eta_0)$ to $(r_1, y_1, \rho_1, \eta_1)$ using a path $(\rho(s), \eta(s))$ such that

$$\eta(0) = \tau \eta_0, \quad \eta(1) = \eta_1, \quad Q(\rho(s), \eta(s)) \in [E_0, E_1], \quad E_j = p_0(r_j, y_j, \rho_j, \eta_j), \quad j = 1, 2,$$

where $Q$ is the quadratic form on $\mathbb{R}^n$ defined by $Q(\rho, \eta) = \rho^2 + e^{-2r_1} g(y_1, \eta)$. This is possible since the fact that $n \geq 2$ implies that the region defined by $Q(\rho, \eta) \in [E_0, E_1]$ and $\rho > 0$ is connected in $\mathbb{R}^n$. □

We split the proof of proposition 3.5 into three lemmas.

**Lemma 3.12.** Let $0 < w < 1/2$ and $\Omega \subset \mathbb{R}^{n-1}$ be an open subset. There exists $\varepsilon > 0$ small enough and $R$ large enough such that $\Phi^\varepsilon$ is diffeomorphism from $\Gamma^+(R, \varepsilon, w, \Omega)$ onto its range, for all $\varepsilon \in [0, 1]$ and $t \geq 0$.

**Proof.** The estimates (3.13) and (3.14) show that $\nabla \Phi^\varepsilon$ is as close to the identity as we want by choosing $\varepsilon$ small enough. Under such a condition, $\Phi^\varepsilon$ is a local diffeomorphism onto its range, thus it is global if it is injective. If $\Phi^\varepsilon(r, y, \rho, \eta) = \Phi^\varepsilon(r', y', \rho', \eta')$ then $r = r'$ and $y = y'$. Moreover (3.13) and (3.14) implies that

$$(\rho', \eta') (r, y, \rho, \eta) - (\rho', \eta') (r, y, \rho', \eta') = (\rho - \rho', \eta - \eta') + O(\varepsilon) \quad (3.40)$$

thus if the left hand side vanishes we have $\rho - \rho' = O(\varepsilon)$ and $\eta - \eta' = O(\varepsilon)$. On the other hand, Taylor’s formula to the second order combined with (3.13) and (3.14) show that

$$(\rho', \eta') (r, y, \rho, \eta) - (\rho', \eta') (r, y, \rho', \eta') = (1 - O(\varepsilon)) (\rho - \rho', \eta - \eta') + O(\varepsilon) + O(\varepsilon) (\rho - \rho', \eta - \eta')^2$$

This implies that

$$| (\rho', \eta') (r, y, \rho, \eta) - (\rho', \eta') (r, y, \rho', \eta') | = (1 - O(\varepsilon)) |(\rho - \rho', \eta - \eta')|$$

if the left hand side of (3.40) vanishes and shows that $\rho = \rho'$ and $\eta = \eta'$, if $\varepsilon$ is small enough. □

**Lemma 3.13.** Let $\Omega_1, \Omega_2$ be connected open subsets of $\mathbb{R}^{n-1}$ such that $\Omega_1 \subseteq \Omega_2$. Then for all $\varepsilon_2$ small enough and $R_2$ large enough there exists $w_1, w_2, \varepsilon_1 > 0$ small enough and $R_1 > 0$ large enough so that $\Phi^\varepsilon$ holds for all $t \geq 0$ and all $\varepsilon \in [0, 1]$. 

21
Proof. The proof follows a rather standard scheme (see [45]). It is enough to find parameters such that

\[
\Gamma^+(R_1, \varepsilon_1, w_1, \Omega_1) \cap \Phi^j (\partial \Gamma^+(R_2, \varepsilon_2, w_2, \Omega_2)) = \emptyset \tag{3.41}
\]

\[
\Gamma^+(R_1, \varepsilon_1, w_1, \Omega_1) \cap \Phi^j (\Gamma^+(R_2, \varepsilon_2, w_2, \Omega_2)) \neq \emptyset. \tag{3.42}
\]

Actually, (3.42) is always satisfied since any element of the form \((r, y, 1, 0)\), with \(r\) large, is invariant by \(\Phi^j\). Thus we look for conditions on the parameters \(R_j, \varepsilon_j, w_j, \Omega_j\) ensuring the fact that (3.41) holds for all \(t \geq 0\) and \(\varepsilon \in [0, 1]\). Any element in the intersection of (3.41) can be written

\[
(r, y, \rho, \eta) = (r, y, \rho^j_0(r, y, \rho_0, \eta_0), \eta^j(r, y, \rho_0, \eta_0))
\]

with \((r, y, \rho_0, \eta_0) \in \partial \Gamma^+(R_2, \varepsilon_2, w_2, \Omega_2)\). Since \(\Omega_1 \in \Omega_2\), choosing \(R_1 > R_2\) implies that one of the following two conditions is satisfied

\[
\rho^2_0 + e^{-2r} g(y, \eta_0) = 1 \pm w_2, \tag{3.43}
\]

\[
eq 1 \pm w_2 \left(1 + O \left(\frac{\varepsilon_2}{R_2}, \eta_0\right)\right).
\]

Using (3.13) and (3.14) we see that

\[
\rho^2_0 + e^{-2r} g(y, \eta) = \rho^2_0 + e^{-2r} g(y, \eta_0) + O \left(\varepsilon_2^{1/2}\right) = 1 \pm w_2 \left(1 + O \left(\frac{\varepsilon_2}{w_2}\right)\right)
\]

Thus (3.43) can certainly not happen if the following conditions are satisfied simultaneously

\[
w_1 \ll w_2, \quad \varepsilon_2 \ll w_2^2. \tag{3.45}
\]

since \(\rho^2_0 + e^{-2r} g(y, \eta) = 1 + O \left(w_1\right)\). Similarly, we deduce easily from theorem 3.2 that

\[
eq e^{-2r} g(y, \eta) = e^{-2r} g(y, \eta_0) + O \left(e^{-r}\right) \approx \varepsilon_2 \left(1 + O \left(e^{-R_1}/\varepsilon_2\right)\right).
\]

Thus the conditions \(e^{-2r} g(y, \eta) < \varepsilon_1\) and (3.45) cannot hold simultaneously if

\[
\varepsilon_1 \ll \varepsilon_2 \quad \text{and} \quad e^{-R_1} \ll \varepsilon_2. \tag{3.46}
\]

The existence of parameters satisfying (3.43) and (3.46) is clear. We choose can choose for instance

\[
w_1 = w_2^2, \quad \varepsilon_2 = w_2^2, \quad \varepsilon_1 = w_2^4, \quad e^{-R_1} = w_2^4, \quad R_1 > R_2.
\]

Thus (3.41) and (3.42) hold. Since \(\Gamma^+(R_1, \varepsilon_1, w_1, \Omega_1)\) is connected and \(\Phi^j\) is an homeomorphism on a neighborhood of \(\Gamma^+(R_2, \varepsilon_2, w_2, \Omega_2)\) onto its range (by possibly decreasing \(w_2, \varepsilon_2, \Omega_2\) and increasing \(R_2\)), this implies (3.41).

\[\Box\]

**Lemma 3.14.** If \(\varepsilon_1 > 0\) is small enough, the estimates (3.17) hold on \(\Gamma^+(R_1, \varepsilon_1, w_1, \Omega_1)\) for \(t \geq 0\) and \(\varepsilon \in [0, 1]\).

**Proof.** By (3.10) we know that \((r, y, \rho^j_0, \eta^j_0) \in \Gamma^+(R_2, \varepsilon_2, w_2, \Omega_2)\) if \((r, y, \rho, \eta) \in \Gamma^+(R_1, \varepsilon_1, w_1, \Omega_1)\). Moreover if \(w_2\) is small enough, then \(\Gamma^+(R_2, \varepsilon_2, w_2, \Omega_2) \subset \Upsilon^+(R_2, -1/2, w_2, \Omega_2)\). Thus we can use the estimates (3.13) and (3.14) with the initial conditions \((r, y, \rho^j_0, \eta^j_0)\). This shows that

\[
|\eta^j_0 - \eta| \leq C e^{-r} |\eta^j_0|
\]

22
and by choosing \( r \) large enough so that \( Ce^{-r} < 1/2 \) we get
\[
|\tilde{\eta}_t - \eta| \leq 2Ce^{-r}|\eta|.
\]
Using this estimate and theorem 3.12, we obtain similarly
\[
|\tilde{\rho}_t^\epsilon - \rho| \leq Ce^{-r}|\tilde{\eta}_t^\epsilon| \leq Ce^{-r}|\eta|.
\]
This shows that (3.17) holds with \( \gamma = 0 \). For \( |\gamma| \geq 1 \) we proceed by induction, by differentiating the equality \( \Phi_t^\epsilon \circ (\Phi_t^\epsilon)^{-1} = \text{id} \). For instance, if \( \partial \gamma = \partial \epsilon \), we have
\[
(\nabla \Phi_t^\epsilon)|_{(\Phi_t^\epsilon)^{-1}}, \partial \epsilon (\Phi_t^\epsilon)^{-1} = -(\partial \Phi_t^\epsilon)|_{(\Phi_t^\epsilon)^{-1}},
\]
where the right hand side is \( \mathcal{O}(e^{-r}) \) and, on the left hand side, \( (\nabla \Phi_t^\epsilon) - 1 \) is small enough hence
\[
\partial \epsilon (\Phi_t^\epsilon)^{-1} = -(\nabla \Phi_t^\epsilon)^{-1}. \partial \epsilon (\Phi_t^\epsilon)^{-1} = \mathcal{O}(e^{-r}).
\]
We don’t go any further into details.

This lemma completes the proof of proposition 3.6.

**Proof of proposition 3.6.** The fact that \( S_t^+ \), defined as the composition of the right hand side of (3.19) with the inverse of \( \Phi_t^\epsilon \), solves (3.18) is a standard result. See for instance [19] or [41]. Thus we focus on the proof of (3.20). We first remark that (3.19) can be rewritten as follows
\[
r_t^\epsilon \rho_t^\epsilon + y_t^\epsilon, \eta_t^\epsilon = 2 \int_0^t (\rho_s^\epsilon)^2 \, ds + tp_\epsilon - \int_0^t e^{-r^*} \partial \eta g_t(r_s^\epsilon, y_s^\epsilon, \eta_s^\epsilon).e^{-r^*} \eta_s^\epsilon \, ds.
\]
This is easily obtained by integration by part using the motion equations. Furthermore
\[
\int_0^t (\rho_s^\epsilon)^2 \, ds = tp_\epsilon - \int_0^t e^{-2r^*}g_t(r_s^\epsilon, y_s^\epsilon, \eta_s^\epsilon) \, ds
\]
and this implies that
\[
(3.19) = r_t^\epsilon \rho_t^\epsilon + y_t^\epsilon, \eta_t^\epsilon - tp_\epsilon - \int_0^t e^{-r^*} \partial \eta g_t(r_s^\epsilon, y_s^\epsilon, \eta_s^\epsilon).e^{-r^*} \eta_s^\epsilon - 2e^{-2r^*}g_t(r_s^\epsilon, y_s^\epsilon, \eta_s^\epsilon) \, ds.
\]
where one must notice that the integral is convergent and \( \mathcal{O}(e^{-r}|\eta|) \). We rewrite (3.48) using the fact that \( p_\epsilon = p_\epsilon \circ \phi_t^\epsilon \). In particular, if we note \( \tilde{r}_t^\epsilon = r_s^\epsilon(r, y, \tilde{\rho}_t^\epsilon, \tilde{\eta}_t^\epsilon) \) and \( \tilde{y}_t^\epsilon = y_s^\epsilon(r, y, \tilde{\rho}_t^\epsilon, \tilde{\eta}_t^\epsilon) \) we obtain
\[
S_+(t, r, y, \rho, \eta, \epsilon) = \tilde{r}_t^\epsilon \rho + \tilde{y}_t^\epsilon, \eta - t \left( \rho^2 + e^{-2r^*}g_t(r, \tilde{y}_t^\epsilon, \eta) \right) + \mathcal{O} \left( e^{-r}|\eta| \right)
\]
\[
= \tilde{r}_t^\epsilon \rho + y, \eta - t \rho^2 + \mathcal{O} \left( e^{-r}|\eta| \right)
\]
using (3.39) to estimate \( \tilde{y}_t^\epsilon - y \) and the exponential decay w.r.t. \( t \) of \( e^{-2r^*}g_t(r_s^\epsilon, y_s^\epsilon, \eta_s^\epsilon) \). In order to estimate \( \tilde{r}_t^\epsilon \), we use the motion equations to write
\[
\tilde{r}_t^\epsilon = r + 2 \int_0^t \rho_s^\epsilon r_s^\epsilon, \tilde{\rho}_t^\epsilon, \tilde{\eta}_t^\epsilon \, ds, \quad \rho_s^\epsilon r_s^\epsilon, \tilde{\rho}_t^\epsilon, \tilde{\eta}_t^\epsilon = \rho + \int_s^t (\partial_r p_\epsilon)(r_s^\epsilon, \tilde{y}_u^\epsilon, \eta_u^\epsilon(r, y, \tilde{\rho}_t^\epsilon, \tilde{\eta}_t^\epsilon)) \, du.
\]
By (3.8) and theorem 3.2 we see that the second integral is \( \mathcal{O}(e^{-s} e^{-r}|\eta|) \) and this implies that
\[
\tilde{r}_t^\epsilon = r + 2t \rho + \mathcal{O}(e^{-r}|\eta|).
\]
and finally obtain
\[
S_+(t, r, y, \rho, \eta, \epsilon) = r\rho + y, \eta + t \rho^2 + \mathcal{O}(e^{-r}|\eta|).
\]
Note that the remainder \( \mathcal{O}(e^{-r}|\eta|) \) is explicit and theorem 3.2 combined with proposition 3.5 show that (3.20) holds. This completes the proof of proposition 3.6. \( \square \)
4 Pseudo-differential operators

4.1 Local theory

In this part, we consider pseudo-differential operators on \( \mathbb{R}^n \). We apply the results to operators on \( X \) in the next subsection.

**Definition 4.1.** For any \( m, m' \in \mathbb{R} \), the class \( S^{m,m'} \) is the set of smooth functions \( a \) such that, for all \( R > 0, \Omega \subseteq \mathbb{R}^{n-1} \) and \( k, l, \alpha, \beta \) there exists \( C \) such that
\[
|\partial^k_r \partial^l_s \partial^\alpha_{\rho} \partial^\beta_{\eta} a(r, y, \rho, \eta)| \leq C(\rho)^m \langle e^{-r \eta} \rangle^{m'},
\]
for all \( r \geq R, y \in \Omega, \rho \in \mathbb{R}, \eta \in \mathbb{R}^{n-1} \). We set \( S^{-\infty} = \cap_{m,m'} S^{m,m'} \).

As usual, the best constants \( C \) are semi-norms which define the topology of \( S^{m,m'} \). We also mention that we shall mainly consider cases where \( m, m' \in \mathbb{R}^{-} \).

We give two examples of special interest for us. If \( b \) belongs to \( S_{0,1}^{m,m'} \), that is if
\[
|\partial^k_r \partial^l_s \partial^\alpha_{\rho} \partial^\beta_{\eta} b(r, y, \rho, \eta)| \leq C(\rho)^m \langle \eta \rangle^{|\beta|}
\]
for \( r > 0, y \in \Omega, \rho \in \mathbb{R} \) and \( \eta \in \mathbb{R}^{n-1} \), then one checks easily that the function \( a \) defined by
\[
a(r, y, \rho, \eta) = b(r, y, \rho, e^{-r \eta})
\]
is an element of \( S^{m,m'} \). In particular, if \( f \in \mathcal{S} \) then \( f(\rho^2 + e^{-2r} g(y, \eta)) \in S^{-\infty} \). The second example is the following. If \( \Omega_0 \subseteq \Omega \) and \( w' > w \), there exists \( C > 0 \) such that for all \( R \) large enough and all \( a \in S^{-\infty} \) supported in \( \mathcal{Y}^+(R, \sigma_+, w, \Omega_0) \) (see (4.3)) we have
\[
a \circ \phi_\epsilon^{-t} \in S^{-\infty} \quad \text{and} \quad \text{supp } a \circ \phi_\epsilon^{-t} \subset \mathcal{Y}^+(R + t - C, \sigma_+, w', \Omega),
\]
for all \( t \geq 0 \) and \( \epsilon \in [0, 1] \). This follows from proposition 4.1 and theorem 3.2.

The pseudo-differential operators that we will use are of the form \( \operatorname{Op}_h(a) \), with
\[
\operatorname{Op}_h(a)u(r, y) = (2\pi)^{-n} \int e^{i r \rho - i y \eta} a(r, y, h\rho, h\eta) \hat{u}(\rho, \eta) \, d\rho d\eta, \quad h \in (0, 1],
\]
where \( \hat{u}(\rho, \eta) = \int e^{-i r \rho - i y \eta} u(r, \rho) \, dr dy \) is the Fourier transform of \( u \in \mathcal{S} \). They depend on the parameter \( h \in (0, 1] \) and it is natural to consider symbols depending on \( h \) as well. Following the standard definitions of \( \mathcal{H} \), we say that \( a = a(h) \) is an admissible symbol in \( S^{m,m'} \) and note
\[
a \sim a_0 + h a_1 + h^2 a_2 + \cdots ,
\]
to mean that for all \( N, a = a_0 + h a_1 + \cdots + h^{N-1} a_{N-1} + h^N r_N(h) \) with \( a_j \in S^{m,m'} \), independent of \( h \), and \( r_N(h) \) bounded family of \( S^{m,m'} \).

We now give estimates on \( \operatorname{Op}_h(a) \) in Schatten classes.

**Proposition 4.2.** Let \( \chi \) be a bounded function supported in \( \mathbb{R}^+ \times \Omega \), with \( \Omega \subseteq \mathbb{R}^{n-1} \) and \( \nu > 0, q \geq 1 \) be positive real numbers such that \( \nu > (n-1)/q \). Then for any \( a \in S^{-\varepsilon-1/q, -\varepsilon-(n-1)/q} \), with \( \varepsilon > 0 \), the operator \( e^{-\nu r} \chi \operatorname{Op}_h(a) \) belongs to \( S_q \) and
\[
\| e^{-\nu r} \chi \operatorname{Op}_h(a) \|_q \leq C h^{-n/q}, \quad h \in (0, 1],
\]
where \( C \) depends on finitely many semi-norms of \( a \). In particular, if \( \nu > (n-1) \), then \( e^{-\nu r} \chi \operatorname{Op}_h(a) \) is trace class and
\[
\text{tr} \left( e^{-\nu r} \chi \operatorname{Op}_h(a) \right) = (2\pi h)^{-n} \int \cdots \int e^{-\nu r} \chi(r, y) a(r, y, \rho, \eta) d\rho d\eta dr dy.
\]
Proof. With no loss of generality we can assume that $\varepsilon$ is a small as we want and in particular that

$$\nu > 2\varepsilon + \frac{n-1}{q}.$$  \hfill (6.4)

We can also assume that $\chi(r, y) = \phi(r)\psi(y)$ with $\psi \in C_0^\infty(\Omega)$ and $\phi \equiv 1$ near infinity. The estimate \ref{4.4} will follow from the fact that we can write $e^{-\nu r} \chi \mathcal{O} b_h(a) = A_h B_h$ with

$$||A_h||_q \leq C h^{-n/q}, \quad ||B_h||_\infty \leq C, \quad h \in (0, 1].$$  \hfill (6.7)

In order to construct $A_h$, we pick $\tilde{\psi} \in C_0^\infty$ such that $\tilde{\psi}\psi = \psi$, and we set $A_h = A_h^1 \otimes A_h^2$ with

$$A_h^1 = e^{-\nu r} \phi(r) \langle h D_r \rangle^{-\varepsilon - 1/q}, \quad A_h^2 = \tilde{\psi}(y) \langle h D_y \rangle^{-\varepsilon -(n-1)/q}.$$ 

By standard estimates, we know that $A_h^1 = \mathcal{O}(h^{-1/q})$ in $S_q^0((L^2(\mathbb{R}, dr))$ and that $A_h^2 = \mathcal{O}(h^{-(n-1)/q})$ in $S_q^0((L^2(\mathbb{R}^{n-1}, dy)))$, thus the first estimate of \ref{6.7} holds. We now have to consider $B_h$ which we define, using $\tilde{\phi}$ such that $\tilde{\phi}\phi = \phi$, by

$$B_h = \langle h D_r \rangle^{\varepsilon + 1/q} \langle h D_y \rangle^{\varepsilon - (n-1)/q} e^{-\nu r} \tilde{\phi}(r) \psi(y) \mathcal{O} b_h(a).$$

In order to show the second estimate of \ref{4.4}, it is enough to show that $B_h = \mathcal{O}(b_h)$ with $b = b(h)$ bounded in $S^{0,0}$. We first consider $\langle h D_y \rangle^{\varepsilon + (n-1)/q} e^{-\nu r} \tilde{\phi}(r) \psi(y) \mathcal{O} b_h(a) = \mathcal{O}(b_1)$ with

$$b_1(r, y, \rho, \eta, h) = e^{-\nu r} \tilde{\phi}(r) (2\pi)^{1-n} \int e^{i y \xi} \langle h \eta + h \xi \rangle^{\varepsilon + (n-1)/q} \tilde{\phi}(\xi) \psi(y) \mathcal{O} b_2(r, \xi, h, \eta) d\xi$$  \hfill (6.9)

where $\tilde{\phi}$ is the Fourier transform of $\psi(y) \phi(r, y, \rho, \eta)$ with respect to $y$. By Peetre’s inequality we have $\langle h \eta + h \xi \rangle^{\varepsilon + (n-1)/q} \leq C \langle h \eta \rangle^{\varepsilon + (n-1)/q} \langle h \xi \rangle^{\varepsilon + (n-1)/q}$ and the integrand of \ref{6.9} is dominated, for any $M$, by

$$C \langle \xi \rangle^{-M} e^{-\nu r} \langle h \eta \rangle^{\varepsilon + (n-1)/q} \langle h \rho \rangle^{-\varepsilon - 1/q} \langle h \eta \rangle^{-\varepsilon -(n-1)/q} \leq C \langle \xi \rangle^{-M} \langle h \rho \rangle^{-\varepsilon - 1/q},$$

where we used \ref{6.5}. The same holds for the derivatives of $b_1$ and shows this $b_1 \in S^{-\varepsilon - 1/q, 0}$. Furthermore it depends continuously on $b$. Thus $\langle h D_y \rangle^{\varepsilon + 1/q} \mathcal{O} b_h(a) = \mathcal{O}(b_1)$ with $b$ bounded and the second estimate of \ref{6.7} holds. The proof of \ref{6.4} is well known. \hfill $\square$

This proposition is the key of theorem \ref{1.1}. For theorem \ref{1.2} we shall need the slightly stronger estimate \ref{6.10} below, in order to use \ref{6.13}. By standard results on pseudo-differential calculus we know that, for any $M \in \mathbb{R}$ and $b \in S^{0,0}$

$$\langle r \rangle^M \mathcal{O} b_h(a) \langle r \rangle^{-M} = \mathcal{O} b_h, \quad \bar{b} \in S^{0,0}.$$  \hfill (6.10)

Furthermore, for all $M$, $\langle r \rangle^M e^{-\nu r} \phi(r) \langle h D_r \rangle^{-\varepsilon - 1/q} \langle r \rangle^M$ still belongs to $S_q^0$ with norm $\mathcal{O}(h^{-n/q})$. Using this remark and the proof of proposition \ref{6.2} we see easily that

$$||e^{-\nu r} \langle r \rangle^M \mathcal{O} b_h(a) \langle r \rangle^M||_q \leq C h^{-n/q}, \quad h \in (0, 1],$$  \hfill (6.11)

for any $M \in \mathbb{R}$, $a \in S^{-\varepsilon - 1/q, -\varepsilon -(n-1)/q}$ and $\nu > (n-1)/q$. Furthermore, if $a^{(\kappa)}$ is a family of symbols which is bounded in $S^{-\varepsilon - 1/q, -\varepsilon -(n-1)/q}$ such that $a^{(\kappa)} \to a^{(0)}$ in $C^\infty(\mathbb{R}^{2n})$ (or $D'(\mathbb{R}^{2n})$), as $\kappa \downarrow 0$ and such that $\rho^2 + e^{-2r/\eta^2}$ is bounded, independently of $\kappa$, on their support then

$$e^{-\nu r} \langle r \rangle^M \mathcal{O} b_h(a^{(\kappa)}) \langle r \rangle^M \to e^{-\nu r} \langle r \rangle^M \mathcal{O} b_h(a^{(0)}) \langle r \rangle^M \quad \text{in } S_q^0, \quad \kappa \downarrow 0.$$  \hfill (6.12)

This follows simply from the fact that $e^{-\delta r} a^{(\kappa)} \to e^{-\delta r} a^{(0)}$ in $S^{-\varepsilon - 1/q, -\varepsilon -(n-1)/q}$ for any $\delta > 0$, and from the fact that we can choose $\delta$ small enough such that $\nu - \delta > (n-1)/q$.

Now we are going to study symbols and operator depending smoothly on $\varepsilon \in [0, 1]$. 

25
Definition 4.3. The class $S^{\nu, m, m'}_\epsilon$ is the set of functions $a_\epsilon(r, y, \rho, \eta)$ which are smooth w.r.t. $r, y, \rho, \eta$ and $\epsilon$ such that for all $R > 0$, $\Omega \subset \mathbb{R}^{n-1}$ and $j, k, l, \alpha, \beta$ there exists $C$ satisfying

$$\left| \partial_{\rho}^{j\alpha} \partial_{\rho}^{j\beta} a_\epsilon(r, y, \rho, \eta) \right| \leq Ce^{-(j+\nu)r} (\rho)^{m'} (\epsilon^{-r})^{m},$$

for all $r \geq R$, $y \in \Omega$, $\rho \in \mathbb{R}$, $\eta \in \mathbb{R}^{n-1}$. We set $S^{0, -\infty}_\epsilon = \bigcap_{m, m'} S^{\nu, m, m'}_\epsilon$.

The typical example of such symbols is given by $a_\epsilon(r, y, \rho, \eta) = f(\rho^2 + e^{-2r} g(r, y, \eta))$, which belongs to $a_\epsilon \in S^{0, -\infty}_\epsilon$ if $f \in S$. If we replace $a$ by $a_\epsilon \in S^{0, -\infty}_\epsilon$ then (4.12) holds with $S^{0, -\infty}_\epsilon$ instead of $S^{\infty}_\epsilon$. This class is particularly natural and convenient since proposition 4.2 shows that for all $a_\epsilon \in S^{\nu, m, m'}_\epsilon$

$$\left| \partial_{\rho}^{j\alpha} \partial_{\rho}^{j\beta} a_\epsilon(r, y, \rho, \eta) \right|_{l_j} \leq C \epsilon^{-n/j}, \quad \text{provided} \quad q > n - 1, \quad q \geq j \geq 1. \quad (4.12)$$

The main drawback of these classes is the following. If $a_\epsilon \in S^{\nu, m, m'}_\epsilon$ then we can't have in general $O_{\rho} a_\epsilon = O_{\rho} (\tilde{a}_\epsilon)$ with $\tilde{a}_\epsilon \in S^{\nu, m, m'}_\epsilon$, which is due to the fact that pseudo-differential operators do not preserve exponential decay. This problem can be overcome by considering properly supported operators. This is the purpose of what follows.

Recall that the Schwartz kernel of $O_{\rho} a_\epsilon$ is given by the following oscillatory integral

$$K_{\rho, \epsilon}(r, r', y, y') = (2\pi h)^{-n} \int e^{i(r-r')/\rho + i(y-y')/\eta} a_\epsilon(r, y, \rho, \eta) \, d\rho \, d\eta$$

which we can write, using $\theta \in C^\infty_0(\mathbb{R})$ and $\theta \equiv 1$ on $(-\delta, \delta)$, as $K_{\rho, \epsilon} = K^{\text{diag}}_{\rho, \epsilon} + K^{\text{off}}_{\rho, \epsilon}$ with

$$K^{\text{diag}}_{\rho, \epsilon}(r, r', y, y') = K_{\rho, \epsilon}(r, r', y, y') \theta(r - r') \theta(|y - y'|). \quad (4.13)$$

Definition 4.4. If $a_\epsilon \in S^{\nu, m, m'}_\epsilon$, we define $O_{\rho} a_\epsilon$ as the operator with Schwartz kernel $K^{\text{diag}}_{\rho, \epsilon}$.

The following proposition explains in which sense $O_{\rho} a_\epsilon - O_{\rho} (\tilde{a}_\epsilon)$ is small.

Proposition 4.5. Assume that $\nu \geq 0$, $m \geq 0$ and $m' \leq 0$. Let $\chi$ be a bounded function supported in $\mathbb{R}^+ \times \Omega$, with $\Omega \subset \mathbb{R}^{n-1}$. Let $K_{\rho, \epsilon}$ the operator with kernel $\chi(r, y)K^{\text{off}}_{\rho, \epsilon}(r, r', y, y')$. Then for all $s \in \mathbb{R}$, $j \geq 0$ and $N \geq 0$, there exists $C > 0$ such that

$$\left| \int e^{(j+r')/\rho + i(y-y')/\eta} a_\epsilon(r, y, \rho, \eta) \, d\rho \, d\eta \right| \leq C h^N, \quad h \in (0, 1], \quad \epsilon \in [0, 1]. \quad (4.14)$$

Proof. It is standard. On the support of $K^{\text{off}}_{\rho, \epsilon}$, we have either $|r - r'| \geq 1$ or $|y - y'| \geq 1$ and we can do as many integrations by parts as we want with $|r - r'|^{-1} h \partial_{\rho}$ or $|y' - y'|^{-2} h^2 \Delta_{\eta}$. This shows that $\partial_{\rho}^j K^{\text{off}}_{\rho, \epsilon}(r, r', y, y')$ is a linear combination of

$$h^N (2\pi h)^{-n} e^{-j} \int e^{i(r-r')/\rho + i(y-y')/\eta} b_{\eta, \epsilon}(r, r', y, y', h\rho, h\eta) \, d\rho \, d\eta$$

with $b_{\eta, \epsilon}$ a product of derivatives of $\partial_{\rho}^j a_\epsilon$, $\chi$ and $|r - r'|^{-1}$, $|y - y'|$. The result follows from (A.1) and (A.2). \qed

Note that the adjoint $K_{\rho, \epsilon}^\star$ satisfies (4.14) with $e^{(j+r')/r}$ on the right. This leads to the following definition.
Definition 4.6. A family of bounded operators \( A_{\epsilon,h} \) is said \( h^N \) negligible if for all \( j \geq 0 \) and \( M \in \mathbb{R} \), we can write \( \partial^j A_{\epsilon,h} = A_{\epsilon,h}^0 + A_{\epsilon,h}^1 + \cdots + A_{\epsilon,h}^j \) where, for \( 0 \leq k \leq j \),

\[
||e^{\nu_k r} r^{-M} A_{\epsilon,h}^k e^{\nu_k r} r^{-M}||_\infty \leq C_{N,M,j} h^N, \quad h \in (0,1], \quad \epsilon \in [0,1],
\]

for some pair \( \nu_k, \nu_{j-k} \geq 0 \) such that \( \nu_j + \nu_{j-k} = \nu + j \). It is called negligible if it is \( h^N \) negligible for all \( N \).

Proposition 4.6 shows that \( \chi Op_h^a(a_\epsilon) - \chi Op_h(a_\epsilon) \) is negligible, provided \( m \leq 0 \) and \( m' \leq 0 \). If \( A_{\epsilon,h} \) is negligible, then \( A_{\epsilon,h}^* \) is clearly negligible as well, hence all this shows that, by using standard methods for the calculus of the adjoint of a pseudo-differential, we obtain easily the following result.

Proposition 4.7. Let \( a_\epsilon \in S_{\nu,m,m'}^r \) be supported in \( (0,\infty) \times \Omega \) with \( \Omega \) bounded and \( m, m' \leq 0 \). Then \( Op_h(a_\epsilon)^* \) is the sum of \( Op_h(\tilde{a}_\epsilon) \) and of a negligible operator, with \( \tilde{a}_\epsilon \in S_{\nu,m,m'}^r \) such that

\[
\tilde{a}_{\epsilon,j} \sim \sum_j h^j \sum_{|\alpha|+k=j} \frac{1}{k!} \partial^k D^\alpha \partial^\alpha D_\eta a_\epsilon.
\]

We end this subsection with a simple remark. In practice, we will use the negligibility as follows: if \( A_{\epsilon,h} \) is \( h^N \) negligible, we can write for all \( j \geq 1 \), \( M \geq 0 \) and \( \epsilon > 0 \) as

\[
\partial^j A_{\epsilon,h} = h^N \sum_{k=0}^j e^{-\nu_k r} r^{-M} B^k_{\epsilon,h} r^{-M} e^{-\nu_{j-k}}
\]

where the operators \( B^k_{\epsilon,h} \) are bounded in operator norm and \( \nu_k + \nu_{j-k} = \nu+j-\epsilon \), with \( \nu_k, \nu_{j-k} \geq 0 \) for all \( k \).

### 4.2 Global theory and functional calculus

The pseudo-differential operators that we are going to consider on \( X \) will be of the form

\[
A = \sum_{k \in I} \tilde{\chi}_k Op_h(a^k) \chi_k, \quad I \text{ finite}
\]

For each \( k \), \( \chi_k \) and \( \tilde{\chi}_k \) are supported in the same chart which is either relatively compact or a chart at infinity, and \( a^k \) is a symbol expressed in the coordinates associated to the chart. If we work in a chart at infinity, we will always use the radial variable \( r \) and variables \( y_1, \ldots, y_{n-1} \) associated to the manifold at infinity \( Y \).

Let us recall some (standard) abuse of notations which are convenient. We use the same notation for \( \tilde{\chi}_k Op_h(a^k) \chi_k \) as an operator acting on \( L^2(\mathbb{R}^n) \) or on \( L^2(X) \). Furthermore if, for each \( k \in I \), \( \tilde{\chi}_k Op_h(a^k) \chi_k \) is bounded on \( L^2(\mathbb{R}^n) \) with the Lebesgue measure associated to the corresponding coordinates, then \( A \) is bounded on \( L^2(X) \) and its norm can be estimated by the sum of norms of \( \tilde{\chi}_k Op_h(a^k) \chi_k \) on \( L^2(\mathbb{R}^n) \). This is due to our choice of the density \( dvol \). The same remark holds for estimates in \( S_{\mathcal{Q}}(L^2(X)) \) and we will therefore use the notations \( ||.||_\infty \) and \( ||.||_\mathcal{Q} \), initially used on \( L^2(\mathbb{R}^n) \), for the respective norms of bounded operators and Schatten classes relative to \( L^2(X) \).

We now apply the results of the previous subsection to the analysis of functions of \( H_\epsilon \). We will only consider Schwartz functions \( f \) and use the Helffer-Sjöstrand formula,

\[
f(H_\epsilon) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \partial_s \bar{f}(s) e^{it} R_\epsilon(s) ds dt
\]
where \( \partial_z = \partial_x + i \partial_y \) and a \( \tilde{f} \) is a quasi-analytic extension of \( f \). i.e. a \( C^\infty \) function on \( \mathbb{C} \), supported in the strip \( |t| \leq 1 \) and such that

\[
\tilde{f}|_\mathbb{R} = f, \quad \left| \partial_z \tilde{f}(s + it) \right| \leq C_M |t|^M (s)^{-M}
\]

for all \( s, t \in \mathbb{R} \) and \( M \geq 0 \). We do not insist on the construction of \( \tilde{f} \) which can be chosen depending continuously on \( f \) and refer for instance to [10] for the details.

The formula (4.18) shows that we only have to study the pseudo-differential expansion of \( R_\varepsilon(z) \).

We recall that the pseudo-differential analysis of the resolvent is well known for operators on compact manifolds (or for elliptic operators on \( \mathbb{R}^n \)) and thus we will only focus on the calculations in charts at infinity. We look for a parametrix \( Q_\varepsilon(z) \) of \( R_\varepsilon(z) \) of the form (4.14) and more precisely

\[
Q_\varepsilon(z) = \sum_{k \in I} \chi_k Op_h(q^{k}_{\varepsilon}(z)) \chi_k
\]

where \( \sum_{k \in I} \chi_k = 1 \) is an admissible partition of unit, \( \chi_k \) is 1 near the support of \( \chi_k \), and \( q^k_\varepsilon(z) \) is an admissible symbol for each \( k \). We explain the construction of these symbols in a single chart at infinity and drop the index \( k \) for convenience. Since we want to get \( (H_\varepsilon - z)Q_\varepsilon(z) = 1 + O(h^\infty) \), we seek \( q_\varepsilon(z) \sim \sum_j h^j q_{j,\varepsilon}(z) \) satisfying, in the chart that we consider,

\[
\sum_{j,l,m} h^{j+l+m} (p^{(l)}_{j,l,m} \# q_{j,\varepsilon}) \sim 1.
\]

Here \( p^{(l)}_{j,l,m} \) is the symbols of \( H_\varepsilon - z \), i.e. \( p^{(l)}_{j,l,m} = p_{j,l,m} - z \) and \( p_{j,l,m} = p_{j,l,m}^{(l)} \) for \( l = 1, 2 \), and the notation \( (a \# b)_m \) stands for the \( m \)-th symbol of the (finite) expansion of the product \( Op_h(a)Op_h(b) \) if \( Op_h(a) \) is a differential operator. The condition (4.19) yields

\[
q_{0,\varepsilon}(z) = (\rho^2 + g_{\varepsilon}(r, y, e^{-r}\eta) - z)^{-1}
\]

(4.20)

\[
q_{j,\varepsilon}(z) = -q_{0,\varepsilon} \sum_{j_0 + j_1 + j_2 = j, \, j_1 < j} (p_{j_0}^{(j_0)} \# q_{j_1,\varepsilon})_{j_2}, \quad j \geq 1.
\]

(4.21)

This procedure is the standard one used by Seeley [50] and Helffer-Robert [27] but the point that we want to make here is the following: since the symbols of \( H_\varepsilon \) are of the form (4.11), it follows clearly that \( q_{j,\varepsilon}(z) \) defined as above is of the form (4.14) as well. By an easy induction, we get, for \( j \geq 1 \),

\[
q_{j,\varepsilon} = \sum_{l=1}^{2j-1} d_{l,j,\varepsilon}(p_z - z)^{-1-l},
\]

(4.22)

where \( d_{l,j,\varepsilon} \) is polynomial w.r.t. the variables \( \rho \) and \( e^{-r}\eta \) which is independent of \( z \) and a linear combination of elements of \( S^0_{\varepsilon,m,m'} \) with \( m + m' \leq 2l - j \). In particular we can write \( q_{j,\varepsilon}(r, y, \rho, \eta) = \tilde{q}_{j,\varepsilon}(r, y, \rho, e^{-r}\eta) \) for some \( \tilde{q}_{j,\varepsilon} \in S^0_{\varepsilon,1-j/2,-1-j/2} \) and we have

\[
q_{j,\varepsilon} \in S^0_{\varepsilon,m,m'}, \quad \text{for all } m, m' \leq 0 \text{ such that } m + m' = 2 - j.
\]

These remarks, combined with the fact that \( |p(p_z - z)^{-1}| \leq C(\Re z) / |\Im z| \), for \( p \in \mathbb{R} \) and \( z \notin \mathbb{R} \) show that the semi-norms of \( q_{j,\varepsilon} \) in \( S^0_{\varepsilon,m,m'} \) \( m + m' = -2 - j \) are dominated by

\[
C(\Re z)^M |\Im z|^M,
\]

(4.23)
for some $C, M$ depending on the semi-norm.

We can now construct the global parametrix. We fix $N \geq 0$ and define it as

$$Q_{N, \epsilon}(z) = \sum_{k \in I} \tilde{\chi}_k \left( \sum_{j \leq N} \hat{h}^j \text{Op}_h (q_{j, \epsilon}(z)) \right) \chi_k,$$

with the $q_{j, \epsilon}(z)$ defined in each chart by the preceding procedure. Then we have

$$(H_\epsilon - z)Q_{N, \epsilon}(z) = \sum_{k \in I} \tilde{\chi}_k (H_\epsilon - z) \left( \sum_{j \leq N} \hat{h}^j \text{Op}_h (q_{j, \epsilon}(z)) \right) \chi_k + \sum_{k \in I} [H_\epsilon, \tilde{\chi}_k] \left( \sum_{j \leq N} \hat{h}^j \text{Op}_h (q_{j, \epsilon}(z)) \right) \chi_k. \quad (4.24)$$

**Lemma 4.8.** For each $k$ and $j$, the Schwartz kernel of $[H_\epsilon, \tilde{\chi}_k] \text{Op}_h (q_{j, \epsilon}(z)) \chi_k$ is $O(h^\infty)$ in the Schwartz space $S(\mathbb{R}^n)$. More precisely, if we note it $K(r, r', y, y', \epsilon, z, h)$ then we have

$$\left| \partial_{r, r', y, y'}^\gamma [H_\epsilon, \tilde{\chi}_k] \text{Op}_h (q_{j, \epsilon}(z)) \right| \leq C h^M e^{-M r} r^M (-M \langle y \rangle - M \langle y' \rangle)^{-M} \frac{\langle \text{Re } z \rangle^M}{\langle \text{Im } z \rangle^M}$$

for all multiindex $\gamma$, $M \geq 0$, $r, r' \geq 0$, $y, y' \in \mathbb{R}^{n-1}$, $\epsilon \in [0, 1]$, $h \in [0, 1]$ and $z \in \mathbb{R}$.

**Proof.** It follows easily by integrations by parts similar to those of lemma 4.9. The exponential decay is due to the following fact. We can choose $\tilde{\chi}_k = \tilde{\phi}_k(r) \tilde{\psi}_k(y)$, with $\tilde{\phi}_k \equiv 1$ near infinity. Then $[H_\epsilon, \tilde{\chi}_k]$ is either compactly supported w.r.t. r, or $|y - y'| \neq 0$ in which case we integrate by part using $\Delta_\epsilon$ and get as many powers of $e^{-r}$ as we want. We omit the other details. \qed

This lemma shows in particular that the second term of the right hand side of (4.24) is negligible, in the sense of definition 4.6. On the other hand, by construction, the first sum of the right hand side of (4.24) is

$$\sum_{k \in I} \tilde{\chi}_k \left( 1 + h^N \text{Op}_h (\hat{g}_{N, \epsilon}(z)) \right) \chi_k = 1 + h^N \sum_{k \in I} \tilde{\chi}_k \text{Op}_h (\hat{g}_{N, \epsilon}(z)) \chi_k$$

where $\hat{g}_{N, \epsilon}(z) = \sum_{j+i+m \geq 0} h^{j+i+m} |p_{i, \epsilon} \# q_{j, \epsilon}(z)|$. It is not hard to check that it belongs to $S^{0, -N/2, -N/2}_\epsilon$ (for instance) using the form of $q_{j, \epsilon}$. We can summarize our result as follows.

**Proposition 4.9.** The operators $H_\epsilon$ are essentially self-adjoint from $C_0^\infty(\mathbb{R})$ and the domain of their self-adjoint realizations is independent on $\epsilon \in [0, 1]$. We have for all $N$

$$R_\epsilon(z) = Q_{N, \epsilon}(z) + h^N R_{N, \epsilon}(z) \mathcal{R}_{N, \epsilon}(z). \quad (4.25)$$

Here $h^N R_{N, \epsilon}(z)$ is the sum of $\sum_{k \in I} h^N \tilde{\chi}_k \text{Op}_h (\hat{g}_{N, \epsilon}(z)) \chi_k$ and of the operators with kernel studied in lemma 4.8.

**Proof.** By the standard trick (see [13]), we see that $(H_\epsilon \pm i)^* \equiv -h^N R_{N, \epsilon}(z)^* = 1 + O(h^N)$ in operator norm on $L^2(X)$. This implies the existence of a unique
Lemma 4.10. Let $A$ be a differential operator on $X$ of order $m$ of the following form in any chart at infinity
\[ A = \sum_{l+|\alpha| \leq m} a_{l,\alpha}(r,y)(e^{-rD_y})^\alpha D_r^l \]
with $a_{l,\alpha}$ bounded. Then for all $\nu, q, k$ such that $\nu > (n-1)/q$ and $2k - m > n/q$, there exists $C$ such that
\[ \| e^{-\nu r h^m} A R_\epsilon^2(0)^{1+k} \|_q \leq C h^{-n/q}, \quad h \in (0, 1), \; \epsilon \in [0, 1]. \]
If $\nu = 0$, by convention the above norm is the operator norm $\| . \|_{\infty}$ and $h^{n/q} = 1$.

Note that this lemma implies the following estimate, for $\text{Im} z \neq 0$,
\[ \| e^{-\nu r h^m} A R_\epsilon^2(z)^{1+k} \|_q \leq C h^{-n/q} \frac{(\text{Re} z)^{1+k}}{|\text{Im} z|^{1+k}}, \quad h \in (0, 1), \; \epsilon \in [0, 1]. \]

**Proof.** We use the fact that $k! R_\epsilon(z)^{k+1} = \partial_z^k R_\epsilon(z)$ and Leibnitz rule into (4.10) to obtain
\[ R_\epsilon(i)^{k+1} = \frac{1}{k!} \left( \partial_z^k Q_{N,\epsilon}(i) \right)(i) \left( 1 - h^N \sum_{\mu \leq k+1} (H_\epsilon - i)^\mu \partial_z^\mu R_{N,\epsilon}(i) \frac{1}{\mu!(k-\mu)!} \right)^{-1} \tag{4.26} \]
for $h \leq h_0$ small enough. Note that $(H_\epsilon - i)^\mu \partial_z^\mu R_{N,\epsilon}(i)$ is bounded for all $\mu$. Using the form of the symbols of $Q_{N,\epsilon}$ given by (4.22) and by proposition 4.2 applied to $e^{-\nu r h^m} A \partial_z^k Q_{N,\epsilon}(i)$, we obtain the result for $h \leq h_0$. Note that we use the fact that $k! \partial_z^k q_{0,\epsilon}(z) = q_{0,\epsilon}(z)^{k+1}$ belongs to $S^{0-j-\epsilon-1/q} - |\alpha| - \epsilon - (n-1)/q$ for all $j, |\alpha|$ such that $j + |\alpha| = m$ and $\epsilon > 0$ small enough. The result for $h \leq 1$ follows by writing $R_\epsilon(i) = (\bar{h}^2 P_\epsilon + i)^{-1} (\bar{h}^2 P_\epsilon + i)_{R_\epsilon(i)}$ with $\bar{h} = \min(h_0, h)$. \qed

This proposition will be used in order to estimate $\partial_z^k (R_\epsilon(z) R_{N,\epsilon}(z))$ in Schatten classes. We shall also need estimates similar to (4.10). To that end, we first recall that for any $0 \leq M_0 \leq 1$
\[ \left[ (r)^{M_0}, R_\epsilon(z) \right] = -R_\epsilon(z) \left[ (r)^{M_0}, H_\epsilon \right] R_\epsilon(z) \]
where $\left[ (r)^{M_0}, H_\epsilon \right]$ is a differential operator of order 1 which is $H_\epsilon$ bounded since $[\partial_r, (r)^{M_0}]$ is bounded. This implies in particular that
\[ (r)^{M_0} R_\epsilon(z) (r)^{-M_0} = R_\epsilon(z) - R_\epsilon(z) \left[ (r)^{M_0}, H_\epsilon \right] R_\epsilon(z) (r)^{-M_0}. \]
Since any $M \geq 0$ can be written $1M_0$ with $l \in \mathbb{N}$, this formula can be iterated to show that
\[ \| (H_\epsilon + i)(r)^M R_\epsilon(z) (r)^{-M} \|_{\infty} \leq C \frac{(\text{Re} z)^M}{|\text{Im} z|^M}, \]
for some constant $C$. This completes the proof of Lemma 4.10.
for some $M' \geq 0$, uniformly w.r.t. $h \in (0, 1]$ and $\epsilon \in [0, 1]$. More generally, we can obtain rather easily for any $k \in \mathbb{N}$ and $M \geq 0$

$$\left\| (H_\epsilon + i)^{1+k} (r)^M R_\epsilon (z)^{1+k} (r)^{-M} \right\|_\infty \leq C \frac{(\operatorname{Re} z)^{M'}}{|\operatorname{Im} z|^{M'}}$$

(4.27)

for some $C$ and $M'$ independent of $\epsilon$ and $h$. On the other hand, $\partial_\epsilon^j R_\epsilon (z) = (-1)^j j! R_\epsilon (z) (V R_\epsilon (z))^j$ can be written for any $M \in \mathbb{R}$ as

$$\partial_\epsilon^j R_\epsilon (z) = (-1)^j j! \langle r \rangle^{-M} \left( \langle r \rangle^M R_\epsilon (z) \langle r \rangle^{-M} \right) \left( \tilde{V} \langle r \rangle^{-M} R_\epsilon (z) \langle r \rangle^M \right)^j \langle r \rangle^{-M}$$

where the differential operator $\tilde{V} = \langle r \rangle^M V (\langle r \rangle^M)$ is of the form $e^{-\nu r} h^2 A$ for any $\nu < 1$ with the notations of lemma 4.10. Using these remarks combined with lemma 4.10, we will obtain

**Lemma 4.11.** For any $q > n - 1$, $M \geq 0$ real and $j \geq 1$ integer, there exists $C, M'$ such that

$$\left\| \langle r \rangle^M (\partial_\epsilon^j R_\epsilon (z)) R_\epsilon (i)^{j-1} \langle r \rangle^M \right\|_{q/3} \leq C h^{-n/j/q} \frac{(\operatorname{Re} z)^{M'}}{|\operatorname{Im} z|^{M'}}$$

for all $h \in (0, 1]$ and $\epsilon \in [0, 1]$.

**Proof.** We proceed by induction on $j$. If $j = 1$, we write

$$\langle r \rangle^M \partial_\epsilon R_\epsilon (z) \langle r \rangle^M = (\langle r \rangle^M R_\epsilon (z) \langle r \rangle^{-M} e^{-\nu r}) (h^2 A \langle r \rangle^{-M} R_\epsilon (z) \langle r \rangle^M)$$

with $(n-1)/q < \nu < 1$ and $e^{-\nu r} h^2 A = \langle r \rangle^M V (\langle r \rangle^M)$. The first factor belongs to $S_q$ whereas the second one is bounded, by lemma 4.10. If $j \geq 2$, we have $\partial_\epsilon^j R_\epsilon (z) = -j (\partial_\epsilon^{j-1} R_\epsilon (z)) V R_\epsilon (z)$ and then the operator that we want to estimate can be written

$$-j (\langle r \rangle^M (\partial_\epsilon^{j-1} R_\epsilon (z)) R_\epsilon (i)^{j-2} \langle r \rangle^M) (\langle r \rangle^{-M} (H_\epsilon + i)^j V R_\epsilon (z) R_\epsilon (i) \langle r \rangle^M).$$

We use the the induction assumption for the first factor, which belongs to $S_{q/(j-1)}$. We estimate the second factor using lemma 4.10 and (4.27) and conclude using (4.10).

The main consequence of this lemma is the following.

**Proposition 4.12.** For each $N \geq N_0$ large enough and $M \geq 0$, $q > n - 1$ as above and $j \geq 1$ there exists $C$ and $M'$ such that

$$\left\| \langle r \rangle^M \partial_\epsilon^j (R_\epsilon (z) - Q_{N, \epsilon} (z)) \langle r \rangle^M \right\|_{q/3} \leq C h^{N-n/j/q} \frac{(\operatorname{Re} z)^{M'}}{|\operatorname{Im} z|^{M'}}$$

(4.28)

for all $\epsilon \in [0, 1]$ and $h \in (0, 1]$.

**Proof.** Since $h^N \partial_\epsilon^j (R_\epsilon (z) R_{N, \epsilon} (z))$ which is a linear combination of $h^N \partial_\epsilon^j (R_\epsilon (z) \partial_\epsilon^j R_{N, \epsilon} (z)$ with $j_1 + j_2 = j$, we have to estimates operators of the form

$$h^N (\langle r \rangle^M (\partial_\epsilon^{j+1} R_\epsilon (z)) R_\epsilon (i)^{j+1-1} \langle r \rangle^M) (\langle r \rangle^{-M} (H_\epsilon + i)^j R_\epsilon (i) \langle r \rangle^M).$$

The first factor can be estimated by proposition 4.11 and the second one by 4.10 and lemma 4.8.

Using formula (4.18), we get directly the following result.
Theorem 4.13. Let \( q > n - 1, f \in \mathcal{S}(\mathbb{R}) \) and \( N \geq 1 \). There exists a pseudo-differential operator \( Q_{N,\epsilon}^f = \sum_{l \leq N} h^l A_l \) with \( A_l \) of the form \( \langle \xi \rangle^l \), with symbols in \( S^{0,-\infty}_l \) such that for all \( M \geq 0 \)
\[
\left\| \partial_j^r (r^M (f(H_\epsilon) - Q_{N,\epsilon}^f)) (r^M) \right\|_{q/j} \leq C h^{N-nj/q}, \quad \epsilon \in [0,1].
\]
Here \( C \) depends on a finite number of semi-norms of \( f \). In each chart, the symbols of \( A_l \) are linear combinations of \( d_{\nu,p} f^{(\ell)}(p_\nu) \) and in particular the principal symbol is \( f(p_\nu) \).

4.3 Proofs of theorem 1.1 and lemma 2.1

Theorem 1.1 is a direct consequence of theorem 4.13. We obtain (1.10) by considering \( h = \epsilon^{1/2} \) and \( f \in \mathcal{S}(\mathbb{R}) \) such that \( f(\lambda) = e^{-\lambda} \) near the spectra of the operators. A priori, (1.10) involves all powers of the form \( \epsilon^{(k-n)/2} \), but a standard argument shows that the coefficients corresponding to odd \( k \) vanish, since they correspond to integrals of odd functions on the sphere.

For the proof of lemma 2.1 we have to show that \( \text{tr}\{ f(H_\epsilon^{(\kappa)})V^{(\kappa)} \}_q \to \text{tr}\{ f(H_\epsilon) V \}_q \) as \( \kappa \downarrow 0 \), for all \( f \in \mathcal{S}(\mathbb{R}) \). Using the explicit expressions of the symbols of \( Q_{N,\epsilon}^{(\kappa)} \) associated to \( H_\epsilon^{(\kappa)} \) (with the notation of theorem 4.13), it is easy to check that
\[
\left\{ V^{(\kappa)} Q_{N,\epsilon}^{(\kappa)} \right\}_q \to \left\{ V Q_{N,\epsilon} \right\}_q \quad \text{in the trace class}
\]
with a trace norm uniformly bounded by \( C(\text{Re } z)^M/|\text{Im } z|^M \) for some \( M \) and \( C \). Thus we are left to the study of the remainder which, via Helffer-Sjöstrand formula, reduces to the study of the remainder given by (4.26). By the resolvent identity, we have
\[
R_\epsilon^{(\kappa)}(z) - R_\epsilon(z) = -R_\epsilon^{(\kappa)}(z) \left( V^{(\kappa)} - V \right) R_\epsilon(z), \quad R_\epsilon^{(\kappa)}(z) = (H_\epsilon^{(\kappa)} - z)^{-1}
\]
with \( (V^{(\kappa)} - V)R_\epsilon(z) \to 0 \) in operator norm. This shows that \( R_\epsilon^{(\kappa)}(z) \to R_\epsilon(z) \) in operator norm. Using (4.26) with \( k = 0 \) for \( H_\epsilon^{(\kappa)} \) it is easy to check that \( H_\epsilon R_\epsilon^{(\kappa)}(i) \) is bounded in operator norm uniformly w.r.t. \( \kappa \). The resolvent identity then shows that \( H_\epsilon R_\epsilon^{(\kappa)}(i) \to H_\epsilon R_\epsilon(i) \) in operator norm and thus \( H_\epsilon R_\epsilon^{(\kappa)}(z) \) converges as well. By induction \( H_\epsilon^{k+1} R_\epsilon^{(\kappa)}(z)^{k+1} \to H_\epsilon^{k+1} R_\epsilon(z)^{k+1} \) in operator norm with a uniform bound of the form \( C(\text{Re } z)^M/|\text{Im } z|^M \) (apply \( \partial^\kappa_z \) to the resolvent identity for instance). Then, we see that \( \left\{ \partial^\kappa_z R_\epsilon^{(\kappa)}(z) \right\}_q \to R_\epsilon(z)^{j-1} \) converges in \( S_{\text{trace}} \) with the same kind of bound as in lemma 4.11 and we can repeat the proof of proposition 4.12. The expected convergence follows from Helffer-Sjöstrand formula and dominated convergence. \( \square \)

5 The proof of theorem 1.2

This section is entirely devoted to the proof of theorem 1.2 and more particularly to the proof of the asymptotic expansion. The continuity of \( \xi_q \) on the absolutely continuous spectrum is a consequence of the method. More precisely, we shall explicitly show that \( \xi_q(\mu, h) \) is continuous in a neighborhood of \( \mu = 1 \). The proof of the continuity of \( \xi_q(\lambda) \), which we omit, would follow from the same method using (1.13) for fixed \( h \).

5.1 Isozaki-Kitada’s method in the asymptotically hyperbolic case

Let us consider a covering of \( Y \) by a finite number of open sets \( U_{n-1} \), each one of them being a relatively compact subset of a coordinate patch \( \tilde{U}_{n-1} \) and consider their respective images on
\( \mathbb{R}^{n-1} \), i.e. \( \Omega \) and \( \tilde{\Omega} \), under the coordinates maps. For each such \( \Omega \), we choose \( \Omega_k \) for \( k = 1, \cdots, 5 \) such that

\[
\Omega = \Omega_5 \subset \Omega_4 \subset \Omega_3 \subset \Omega_2 \subset \Omega_1 \subset \tilde{\Omega}.
\]

Furthermore we can assume that \( \Omega_k \) is open and convex (this will be usefull only for proposition \( 5.3 \)) for all \( k \). Then we consider, the outgoing areas \( \Gamma_k^+ \supset \Gamma_{k+1}^+ \supset \cdots \supset \Gamma_5^+ \) defined by

\[
\Gamma_k^+ = \Gamma^+(R^k, \varepsilon^k, w^k, \Omega_k), \quad k = 1, \cdots, 5.
\]

We assume that \( R > 1 \) and \( \varepsilon, w \in (0, 1) \) thus it is clear that \( \Gamma_k^+ \subset \Gamma_{k-1}^+ \) for \( k = 2, \cdots, 5 \). More precisely one can always find a cutoff function supported in \( \Gamma_{k-1}^+ \) which is \( \equiv 1 \) on \( \Gamma_k^+ \). We can choose for instance

\[
\chi^k(r)\chi_{\Omega_k}(y)\chi_+(\rho)\chi_{\varepsilon^k} (e^{-2r} g(y, \eta)) \chi_{w^k}(\rho^2 + e^{-2r} g(y, \eta)) \quad \text{for } \rho \leq 1
\]

with smooth functions \( \chi^k, \chi_{\Omega_k}, \chi_{\varepsilon^k} \) and \( \chi_{w^k} \) such that

\[
\text{supp } \chi^k \subset (R^{k-1}, \infty) \quad \text{and } \quad \chi^k \equiv 1 \text{ on } (R^k, \infty),
\]

\[
\text{supp } \chi_{\Omega_k} \subset \Omega_{k-1} \quad \text{and } \quad \chi_{\Omega_k} \equiv 1 \text{ on } \Omega_k
\]

\[
\text{supp } \chi_{\varepsilon^k} \subset (-\infty, \varepsilon^{k-1}) \quad \text{and } \quad \chi_{\varepsilon^k} \equiv 1 \text{ on } (-\infty, \varepsilon^k),
\]

\[
\text{supp } \chi_{w^k} \subset (1 - w^{k-1}, 1 + w^{k-1}) \quad \text{and } \quad \chi_{w^k} \equiv 1 \text{ on } (1 - w^k, 1 + w^k).
\]

Since we will choose \( w \) and \( \varepsilon \) small, it is enough to consider \( \chi_+ \) supported into \((0, \infty)\) and such that \( \chi_+ \equiv 1 \text{ on } (1/2, \infty) \). Note that on \( \Gamma_k^+ \), we always have

\[
(1 - w^k - \varepsilon^k)^{1/2} \leq \rho \leq (1 + w^k)^{1/2}.
\]

We are now ready to construct the functions needed for Isozaki-Kitada’s method in the hyperbolic case. We start with the following proposition.

**Proposition 5.1.** For \( R \) large enough, and \( \varepsilon, w \) small enough, there exists a smooth function \( \varphi^+_\varepsilon(r, y, \rho, \eta) \) defined, for any \( \varepsilon \in [0, 1] \), on \( \Gamma_1^+ \) such that

\[
(\partial_r \varphi^+_\varepsilon)^2 + e^{-2r} g_0(r, y, \partial y \varphi^+_\varepsilon) = \rho^2.
\]

This function satisfies the following estimates for \( \varepsilon \in [0, 1] \) and \( (r, y, \rho, \eta) \in \Gamma_1^+ \)

\[
|\partial_r^j \partial y^k \partial^\alpha \partial y^\beta (\varphi^+_\varepsilon - r \rho - y \eta)| \leq C_{j,k,l,a,b} \varepsilon^{-(j+1)r} |\eta|.
\]

This proposition solves the equation \( 2.22 \) in the case \( p_\varepsilon(x, \xi) = \rho^2 + e^{-2r} g_0(r, y, \eta) \) and explains how differentiation w.r.t. \( \varepsilon \) provides exponential decay.

**Proof.** We follow the principle explained in subsection \( 2.5 \) using the function \( S^+_\varepsilon(t, r, y, \rho, \eta) \) given by proposition \( 3.6 \). Since \( S^+_\varepsilon \) solves \( 4.15 \), and is a generating function of the flow (see \( 2.26 \)) we have

\[
\partial_t S^+_\varepsilon = p_\varepsilon(r, y, \partial_t S^+_\varepsilon, \partial y S^+_\varepsilon) = p_\varepsilon(\partial_\rho S^+_\varepsilon, \partial_\eta S^+_\varepsilon, \rho, \eta)
\]

\[
= \rho^2 + e^{-2r} S^+_\varepsilon g_0(\partial_\rho S^+_\varepsilon, \partial_\eta S^+_\varepsilon, \eta)
\]

where one remark that the last term of the second line is \( O(e^{-r-2\rho}) \) by \( 5.20 \) hence is integrable. Thus we can use formula \( 2.28 \) with \( S^+_\varepsilon(t, \rho, \eta) = t \rho^2 \) to define \( \varphi^+_\varepsilon \), and then \( 5.3 \) is a direct consequence of proposition \( 3.6 \).

The next lemma is a preparation lemma for the resolution of the transport equations. We do not prove it since it can be obtained very similarly to the estimates on the geodesics of section \( 5 \).
Lemma 5.2. There exist $R$ large enough and $w, \varepsilon$ small enough, the following result holds: for all $(r, y, \rho, \eta) \in \Gamma^+_2$ and $\varepsilon \in [0, 1]$, the solution $(\vec{r}_e^t, \vec{y}_e^t)(r, y, \rho, \eta)$ of
\[
\frac{d}{dt} \left( \begin{array}{c} \vec{r}_e^t \\ \vec{y}_e^t \end{array} \right) = \left( e^{-2\vec{r}_e^t}(\partial_{\eta} g_e)(\vec{r}_e^t, \vec{y}_e^t, \partial_{\varphi} g_e^+ (\vec{r}_e^t, \vec{y}_e^t, \rho, \eta)) \right), \quad \left( \begin{array}{c} \vec{r}_e^0 \\ \vec{y}_e^0 \end{array} \right) = \left( \begin{array}{c} r \\ y \end{array} \right)
\]
is defined for all $t \geq 0$ and satisfies $(\vec{r}_e^t, \vec{y}_e^t, \rho, \eta) \in \Gamma^+_1$. Furthermore, we have the estimates
\[
|\partial^j_t \partial^k_x \partial^\rho \partial^\eta \partial^\alpha \partial^\beta (\vec{r}_e^t - r - 2\rho t)| + |\partial^j_t \partial^k_x \partial^\rho \partial^\eta \partial^\alpha \partial^\beta (\vec{y}_e^t - y)| \leq C_{j,k,l,a,b} e^{-(j+1)r} |\eta| \tag{5.4}
\]
for all $\varepsilon \in [0, 1]$ and $(r, y, \rho, \eta) \in \Gamma^+_2$.

This lemma, which gives in particular a precise sense to (2.28) in the current context, allows to define functions $a^{(0)}_e, \ldots, a^{(N)}_e$ of $r, y, \rho, \eta \in \Gamma^+_2$ according to the formulas (2.29), (2.31) where one has of course to replace $\vec{y}_l^t$ by $\vec{r}_l^t, \vec{y}_l^t$ and use the following explicit expression
\[
c_e = \partial^2 \varphi + e^{-2r} g_e(r, y, \rho, \eta) \varphi + e^{-r} \sum_{l+|\alpha|=1} \vec{v}_l^j (e^{-r}, y) (e^{-r} \partial^\rho \partial^\eta \partial^\alpha \partial^\beta),
\]
where $\vec{v}_l^j = v_0^l + \varepsilon (v^l - v_0^l)$, with the notations of (1.2). By (5.2) we see easily that $c_e(r, y, \rho, \eta) = O(e^{-ct} \langle \eta \rangle)$ instead of $|\eta|$ which is caused by the term $\partial^2 \varphi^+$ with $l = 1$ in the expression of $c_e$. This implies that the integral in (2.30) is convergent since (5.2) and (5.4) show that, for some $c > 0$,
\[
c_e(r, y, \rho, \eta) = O(c^{-ct} \langle \eta \rangle), \quad t \geq 0.
\]
Thus $a^{(0)}_e$ is well defined on $\Gamma^+_2$. By induction, one checks that, for $m \geq 1$, we have
\[
P_e(r, y, D_r, D_y) a^{(m-1)}_e(r, y, \rho, \eta) = O(c^{-ct} \langle \eta \rangle), \quad t \geq 0,
\]
on $\Gamma^+_2$, for all $\varepsilon \in [0, 1]$ and thus $a^{(m)}_e$ is well defined on $\Gamma^+_2$ for all $m$. More generally, by mean of proposition 5.1, lemma 5.2 and 5.4, we obtain easily the following result.

Proposition 5.3. The functions $a^{(0)}_e, \ldots, a^{(N)}_e$ defined in $\Gamma^+_2$ by (2.29) and (2.31) satisfy
\[
\left| \partial^j_t \partial^k_x \partial^\rho \partial^\eta \partial^\alpha \partial^\beta (a^{(0)}_e(r, y, \rho, \eta) - 1) \right| \leq C_{j,k,l,a,b} e^{-(j+1)r} \langle \eta \rangle,
\]
\[
\left| \partial^j_t \partial^k_x \partial^\rho \partial^\eta \partial^\alpha \partial^\beta a^{(m)}_e(r, y, \rho, \eta) \right| \leq C_{j,k,l,a,b} e^{-(j+1)r} \langle \eta \rangle, \quad 1 \leq m \leq N
\]
for all $\varepsilon \in [0, 1]$ and $(r, y, \rho, \eta) \in \Gamma^+_2$.

The details of the proof are left to the reader. They follow from the explicit expressions of the functions $a^{(m)}_e$.

In order to consider functions defined on $\mathbb{R}^{2n}$, we choose a cutoff $\chi_{2,3}(r, y, \rho, \eta)$ of the form (5.4) which is supported in $\Gamma^+_2$ and $\equiv 1$ on $\Gamma^+_3$. Then we multiply the functions $a^{(0)}_e, \ldots, a^{(N)}_e$ by $\chi_{2,3}$ and define
\[
a_e = \chi_{2,3} \left( a^{(0)}_e + h a^{(1)}_e + \cdots + h^N a^{(N)}_e \right).
\]
Then, following Isozaki-Kitada’s method as explained in subsection 2.3, we consider
\[
H_e J(\varphi^+ + a_e) - J(\varphi^+ + a_e) P, \quad \text{with} \quad P = h^2 D^2_e.
\]
Of course, the choice of $P$ follows from (2.27) since $\lim_{r \to \infty} p_\epsilon(r, y, \rho, \eta) = \rho^2 =: p(\rho)$. The operator $H_\epsilon J(\varphi_\epsilon^+, a_\epsilon) - J(\varphi_\epsilon^+, a_\epsilon)P$ has the following expression

$$J \left( \varphi_\epsilon^+, a_\epsilon^\prime \right) + h^{N+2} J \left( \varphi_\epsilon^+, \chi_{2,3} P_\epsilon(r, y, D_r, D_y) a_\epsilon^{(N)} \right)$$

(5.5)

where $a_\epsilon^\prime$ is a linear combination of products of derivatives (of order $\geq 1$) of $\chi_{2,3}$ and of derivatives of $a_\epsilon^{(0)}, \ldots, a_\epsilon^{(N)}$. The first term of (5.5) is produced by all the derivatives due to $H_\epsilon$, which may fall on $\chi_{2,3}$. The amplitude of the second term is nothing but $\chi_{2,3}(\tilde{a}_\epsilon^{(0)} + \cdots + h^{N+2} \tilde{a}_\epsilon^{(N+2)})$ with the notations of subsection 2.3. This follows from the construction of $\varphi_\epsilon^+$ and $a_\epsilon^{(0)}, \ldots, a_\epsilon^{(N)}$ which satisfy respectively Hamilton-Jacobi’s equation and the transport equations on the support of $\chi_{2,3}$. The negligibility of (5.5), or more precisely of (2.47), will be a consequence of the next lemma.

**Lemma 5.4.** For $R$ large enough, and $\epsilon, w$ small enough we have the following property: for any symbol $b(r, y, \rho, \eta)$ supported in $\Gamma_\epsilon^+$ and such that, for all $M$,

$$\left| \partial_r^k \partial_y^\alpha \partial_r^\beta \partial_y^\gamma h(r, y, \rho, \eta) \right| \leq C_{k,l,\alpha,\beta,M} (r)^{-M}$$

we have the following estimates for $s \geq 0$

$$\left| \langle r \rangle^M J \left( \varphi_\epsilon^+, a_\epsilon^\prime \right) U(s) J(\varphi_\epsilon^+, b)^* \langle r \rangle^M \right|_\infty \leq C_M h^{M} (s)^{-M},$$

(5.6)

$$\left| \langle r \rangle^M J \left( \varphi_\epsilon^+, \chi_{2,3} P_\epsilon(r, y, D_r, D_y) a_\epsilon^{(N)} \right) U(s) J(\varphi_\epsilon^+, b)^* \langle r \rangle^M \right|_\infty \leq C_M h^{-n_0} (s)^{-M}$$

(5.7)

for all $M$ and some universal constant $n_0$. Here $U(s) = e^{-i \pi P}$ is the propagator of $P$.

Note that the power $h^{-n_0}$ on the right hand side of (5.7) is harmless since we have a power $h^{N+2}$ in (5.5), with $N$ arbitrarily large. The proof of this lemma is not very hard and follows from suitable integrations by parts on the Schwartz kernels of the operators which are explicitly given by oscillatory integrals. However its proof is a bit long and we have postponed it to appendix A.

The last step of the construction is the factorization of pseudo-differential operators. This is the purpose of the following proposition.

**Proposition 5.5.** There exists $R$ large enough and $\epsilon, w$ small enough such that the following property holds: for any symbol $c_\epsilon \in S$, supported in $\Gamma_\epsilon^+$, one can find symbols $b_\epsilon^{(0)}, \ldots, b_\epsilon^{(N)} \in S$ supported in $\Gamma_\epsilon^+$ such that

$$J(\varphi_\epsilon^+, a_\epsilon) J(\varphi_\epsilon^+, b_\epsilon)^* - \text{Op}_h(c_\epsilon) \text{ is } h^N \text{ negligible}$$

with $b_\epsilon = b_\epsilon^{(0)} + h b_\epsilon^{(1)} + \cdots + h^N b_\epsilon^{(N)}$.

**Proof.** As explained in subsection 2.3 we need to study the map

$$(\rho, \eta) \mapsto \Phi_\epsilon^+(r, y, r', y', \rho, \eta) = \int_0^1 \partial_{r,y} \varphi_\epsilon^+(r' + t(r - r'), y' + t(y - y'), \rho, \eta) \, dt$$

(5.8)

It is then clear that if $(r, y, \rho, \eta)$ and $(r', y', \rho, \eta) \in \Gamma^+(R, \epsilon, w, \Omega)$ with $\Omega$ convex, then $(r' + t(r - r'), y' + t(y - y'), \rho, \eta) \in \Gamma^+(R, C_0 \epsilon, w', \Omega)$ for all $t \in [0, 1]$, and any $w' > w$ which can be chosen as close to $w$ as we want by decreasing $\epsilon$. Here $C_0$ is given by (3.4). In particular,

$$\left| \partial^\gamma \left( \Phi_\epsilon^+ - (\rho, \eta) \right) \right| \leq C_\epsilon e^{-((j+1) \min(r, r')) |\eta|}$$

35
with \( \partial r = \partial_t \partial_x^k \partial_{y^l} \partial_y^m \partial_x^n \partial_y \partial_y^j \partial_y \), for all \((r, y, \rho, \eta)\) and \((r', y', \rho, \eta)\) in \( \Gamma^+(R, \varepsilon, w, \Omega) \). Using the formula (5.9) (with \( \theta_\varepsilon = \Phi^\varepsilon \)), we can define \( b_\varepsilon^{(0)} \) and by iterations \( b_\varepsilon^{(1)}, \ldots \), from \( c_\varepsilon \). In particular, if \( c_\varepsilon \) is supported in \( \Gamma^+_\varepsilon \), \( b_\varepsilon^{(0)}, b_\varepsilon^{(1)}, \ldots \) will be clearly supported in \( \Gamma^+_\varepsilon \) if \( R \) is large enough and \( \varepsilon, w \) small enough. In order to check that the difference \( J(\varphi^+, a_\varepsilon)J(\varphi^+, b_\varepsilon)^* - \partial h_\varepsilon(\varepsilon) \) is \( h^N \) negligible, we look at its Schwartz kernel which we split into three terms using the partition of unit

\[
1 = \theta_0(r-r') + \theta_+(r-r') + \theta_+(r'-r)
\]

with \( \theta_0 = 1 \) close to 0 and \( \theta_+ \) supported in \([1, \infty)\). The term corresponding to \( \theta_0 \) behaves nicely by construction. The off diagonal part is \( O(h^\infty) \) and is the sum of two oscillatory integrals which are either \( O(e^{-(j+n)r}) \) or \( O(e^{-(j+n)r'}) \) after application \( \partial_\varepsilon \), according to the sign of \( r-r' \). We don’t go any further into details.

\[\square\]

### 5.2 The dependence on \( \kappa \) and the remainders

Our next task is explain how to deal with ‘remainders’. In subsection 5.1 or in section 4, we have seen why and how a lot of expansions of operators, in powers of \( h \), could be obtained. It is now necessary to show why the contributions of the remainders of such expansions can indeed be neglected in the expected expansion. We introduce the notation \( \equiv^n_N \), which we will use extensively in the next subsection. Its meaning is the following.

**Definition 5.6.** We write

\[
\text{tr } (T^{(k)}_\varepsilon(t,h)) \equiv^n_N 0
\]

if \( T^{(k)}_\varepsilon(t,h) \) is a family of trace class operators for \( \kappa \in (0,1] \), whose trace is \( C^{q-1} \) with respect to \( \varepsilon \in [0,1] \), measurable with respect to \( t \in \mathbb{R} \), and if there exists \( C_{N,n_1} \), independent of \( h \), such that

\[
\lim_{\kappa \downarrow 0} \left\{ \text{tr } (T^{(k)}_\varepsilon(t,h)) \right\}_q \text{ exists in } \mathcal{S}'(\mathbb{R}) \text{ and belongs to } L^1(\mathbb{R},dt),
\]

\[
\int_{-\infty}^{+\infty} \lim_{\kappa \downarrow 0} \left\{ \text{tr } (T^{(k)}_\varepsilon(t,h)) \right\}_q \text{ dt } \leq C_{N,n_1} h^{N-n_1}.
\]

If the same result holds when integrating on \([0,+,\infty)\), then we will write \( \equiv^n_{N,+} \) instead of \( \equiv^n_N \).

If \( S^{(k)}_\varepsilon(t,h) \) is another family of trace class operators for \( \kappa \in (0,1] \), then

\[
\text{tr } (T^{(k)}_\varepsilon(t,h)) \equiv^n_{N} \text{ tr } (S^{(k)}_\varepsilon(t,h)) \iff \text{tr } (T^{(k)}_\varepsilon(t,h) - S^{(k)}_\varepsilon(t,h)) \equiv^n_{N} 0,
\]

and similarly for \( \equiv^n_{N,+} \).

Operators (or rather traces) satisfying (5.9) and (5.10) are of interest since

\[
\lim_{\kappa \downarrow 0} \mathcal{F}^{-1}_h \left\{ \text{tr } (T^{(k)}_\varepsilon(t,h)) \right\}_q = \mathcal{F}^{-1}_h \lim_{\kappa \downarrow 0} \left\{ \text{tr } (T^{(k)}_\varepsilon(t,h)) \right\}_q = O(h^{N-n_1-1}) \text{ in } C^0(\mathbb{R}).
\]

In practice, it is enough to show, for instance, that \( \lim_{\kappa \rightarrow 0} \{ T^{(k)}_\varepsilon(t,h) \}_q =: \{ T^{(0)}_\varepsilon(t,h) \}_q \) exists, in the weak topology of bounded operator and that

\[
\left\| \{ T^{(0)}_\varepsilon(t,h) \}_q \right\|_1 \leq C_N h^{N-n_1} \psi(t), \quad t \in \mathbb{R}, \quad h \in (0,1],
\]

\[
\lim_{\kappa \rightarrow 0} \text{tr } \{ T^{(k)}_\varepsilon(t,h) \}_q = \text{tr } \{ T^{(0)}_\varepsilon(t,h) \}_q \quad t \in \mathbb{R}, \quad h \in (0,1],
\]

\[
\left\| \{ T^{(k)}_\varepsilon(t,h) \}_q \right\|_1 \leq C h^{-n_1} (t)^{n_2}, \quad t \in \mathbb{R}, \quad h \in (0,1], \quad \kappa \in [0,1]
\]
for some $L^1$ function $\psi$ in $(5.12)$, and some $n_2 \geq 0$ in $(5.14)$. This last condition could be weakened since we could clearly allow $C$ to depend arbitrarily on $h$. Usually, $(5.13)$ and $(5.14)$ are easy to check and the non trivial part of the job is to show $(5.12)$.

Before giving explicit examples, let us explain how we will use definition $(5.6)$. Recall that we are studying $(2.5)$ which is nothing but $T_{\epsilon,N}(t,h)$ such that

$$
\text{tr} \left( f(H_{\epsilon}^{(\kappa)})U_{\epsilon}^{(\kappa)}(t)V^{(\kappa)}\tilde{f}(H_{\epsilon}^{(\kappa)}) \right) = \frac{n_1}{N} \text{tr} \left( T_{\epsilon,N}^{(\kappa)}(t,h) \right)
$$

for some $n_1$ independent of $N$, and moreover such that $\{\text{tr}(T_{\epsilon,N}^{(\kappa)}(t,h))\}_{q}$ converges in $\mathcal{S}'(\mathbb{R})$ as $\kappa \downarrow 0$ with a limit in $L^1(\mathbb{R},dt)$ satisfying

$$
\frac{1}{2\pi h} \int_{-\infty}^{+\infty} e^{\pi \mu t} \lim_{\kappa \downarrow 0} \left\{ \text{tr} \left( T_{\epsilon,N}^{(\kappa)}(t,h) \right) \right\}_q dt = h^{-n} \sum_{k=0}^{N} h^{k} \alpha_k(\mu) + O(h^{N-n})
$$

with $\alpha_k \in C^0(I)$ and $O(h^{N-n})$ understood in the topology of $C^0(I)$, then we get the existence of the expansion $(1.11)$. This is due to $(5.11)$ since it implies that

$$
\xi_q(\mu,h) - \frac{1}{2\pi h} \int_{-\infty}^{+\infty} e^{\pi \mu t} \lim_{\kappa \downarrow 0} \left\{ \text{tr} \left( T_{\epsilon,N}^{(\kappa)}(t,h) \right) \right\}_q dt = O(h^{N-n_1+1}) \in C^0
$$

with $N-n_1+1$ arbitrarily large if $N$ is large.

We shall now give explicit and useful examples of operators satisfying $(5.9)$ and $(5.10)$. To that end, we will use extensively the fact that for all $f \in C^\infty_{0}(\mathbb{R})$, in the strong sense,

$$
U_{\epsilon}^{(\kappa)}(t)f(H_{\epsilon}^{(\kappa)}) \rightarrow U_{\epsilon}(t)f(H_{\epsilon}), \quad \kappa \downarrow 0
$$

with a norm bounded by $\sup |f|$. We omit the proof of this easy fact since it follows by section $6$ or more directly, by Helffer-Sjöstrand formula for instance. We will also use the following well known lemma, which is valid if $B^{(\kappa)}$ is a family of bounded operators and $T^{(\kappa)}$ a family of $\mathbf{S}_q$.

**Lemma 5.7.** If $B^{(\kappa)} \rightarrow B^{(0)}$ strongly and $T^{(\kappa)} \rightarrow T^{(0)}$ in $\mathbf{S}_q$ then $B^{(\kappa)}T^{(\kappa)} \rightarrow B^{(0)}T^{(0)}$ in $\mathbf{S}_q$.

Let $\mathcal{R}_{\epsilon,N}^{(\kappa)}(h) = V^{(\kappa)} \left( \tilde{f}(H_{\epsilon}^{(\kappa)}) - 
abla \tilde{f}(H_{\epsilon}^{(\kappa)}) \right)$, with the notations of theorem $(4.13)$ where we include the dependence on $\kappa$.

**Proposition 5.8.** Let $I$ be an interval such that $(1.13)$ holds. Then all $f, \tilde{f} \in C^\infty_{0}(I)$, with $f \tilde{f} = f$, we have

$$
\text{tr} \left( f(H_{\epsilon}^{(\kappa)})U_{\epsilon}^{(\kappa)}(t)\mathcal{R}_{\epsilon,N}^{(\kappa)}(h) \right) \equiv \frac{n_1}{N} 0, \quad n_1 = n + q.
$$

**Proof.** We only show $(5.12)$. We first rewrite the trace as

$$
\text{tr} \left( (r)^{-M} f(H_{\epsilon}^{(\kappa)})U_{\epsilon}^{(\kappa)}(t) (r)^{-M} \langle r \rangle^{M} \mathcal{R}_{\epsilon,N}^{(\kappa)}(h) \langle r \rangle^{M} \right)
$$

37
and then apply \( \partial \gamma^{-1} \). By Leibnitz rule and lemma 2.2, we obtain an explicit expression of this derivative. We note that, for any \( 0 \leq k \leq q - 1 \),

\[
\langle r \rangle^M \partial^{M,k} R^{(k)}_{\epsilon,N}(h) \langle r \rangle^M = O(h^{N-(q-k)n/q}) \quad \text{in } S_{q/(q-k)}
\]

(5.16)

and that this estimate holds for \( \kappa \geq 0 \), continuously w.r.t \( \kappa \) and \( \epsilon \), which will allow to let \( \kappa \downarrow 0 \). This can be seen with the same argument as the one used for the proof of lemma 3.1 in subsection 4.3. On the other hand, using the notations of lemma 2.2, where we now take \( \kappa \) into account, we can write \( \langle r \rangle^{-M} \) \ref{2.10} \( \langle r \rangle^{-M} \) as the integral over \( F' \) of

\[
h^{-j} \left( V^{(\kappa)} \langle r \rangle^{-M} \partial^{\kappa}_{\epsilon} S^{(\kappa)}_{\epsilon} \langle r \rangle^{M} \right) \left( \langle r \rangle^{-M} U^{(\kappa)} \langle t_0 \rangle \langle r \rangle^{-M} \right) \left( \langle r \rangle^{-M} V^{(\kappa)} \partial^{\kappa}_{\epsilon} S^{(\kappa)}_{\epsilon} \langle r \rangle^{M} \right) \ldots
\]

\[
\ldots \left( \langle r \rangle^{-M} U^{(\kappa)} \langle r \rangle^{-M} \right) \left( \langle r \rangle^{-M} \partial^{\kappa+1}_{\epsilon} S^{(\kappa)}_{\epsilon} \langle r \rangle^{-M} \right). \quad \text{(5.17)}
\]

Note that we used the notation 2.8. Then, as before, it is not hard to check that

\[
\langle r \rangle^{M} (V^{(\kappa)})^{j} \partial^{\kappa}_{\epsilon} S^{(\kappa)}_{\epsilon} \langle r \rangle^{M} \rightarrow \langle r \rangle^{M} V_{\tau} \partial^{\kappa}_{\epsilon} S^{(\kappa)}_{\epsilon} \langle r \rangle^{M} \quad \text{in } S_{q/(q-1-\tau)}
\]

for \( \tau = 0 \) or 1. By lemma 5.7 we can let \( \kappa \downarrow 0 \) in \ref{2.17}, and using \ref{1.5} we get for \( \kappa = 0 \),

\[
\left\| \left\langle r \right\rangle^{M} \partial^{\kappa-1-k}_{\epsilon} R^{(k)}_{\epsilon,N}(h) \left\langle r \right\rangle^{M} \right\|_{1} \leq C h^{N-n-j} \prod_{l=0}^{j} \left\| \left\langle r \right\rangle^{-M} U^{(\kappa)}_{\epsilon} \left\langle t_{l} \right\rangle \left\langle r \right\rangle^{-M} \right\|_{1}
\]

(5.18)

Then the result follows from \ref{2.11}.

The same method can be applied to other kind of operators. We only explain which modifications to do. For instance, if \( \partial^{j}_{\epsilon} R^{(\kappa)}_{\epsilon,N}(h) \) can be written in any chart as \ref{2.10} \( \times e^{-\tau} \) then everything works almost as well. This typically will happen when one considers remainders of the pseudo-differential expansion of the adjoint of \( V^{(\kappa)} f(H^{(\kappa)}_{\epsilon}) \) or with negligible operator involved in proposition 5.6. Actually, in this case we don’t have \ref{5.10} anymore, but the exponential weights in \ref{5.10} can be put on both sides of \ref{5.17} which then becomes trace class and we can conclude similarly. This situation is also of interest when one has to consider the remainder involved in \ref{2.41}. For the latter, the exponential weights are provided by the differentiation w.r.t. \( \epsilon \) in view of the explicit form of the symbols in term of the flow and theorem 2.2.

We can also study the contribution of the right hand side of \ref{2.17} using lemma 5.4. Here there is one more integral over \( s \) but we clearly have \( L^1 \) estimates by lemma 5.4 and the convolution trick works again. Note finally that in this case we will choose \( n_1 = n + q + n_0 \).

5.3 The core of the proof

Below, we will use extensively the notation \( \equiv_{N}^{n_1} \) of the previous subsection with \( n_1 = n + q + n_0 \), which is independent of \( N \).

We start from the semi-classical Fourier transform of \ref{2.4} with \( f \) supported in \( \{1 - w^5/4, 1 + w^5/4\} \) and \( f = 1 \) close to 1. We furthermore assume that the coefficients of \( V \) (hence of \( V^{(\kappa)} \)) are all supported in \( \{r > R\} \). We shall indicate how to modify the proof in the general case, however recall that it is the case for the principal symbol.

Using theorem 4.13 we get a pseudo-differential expansion of \( V^{(\kappa)} f(H^{(\kappa)}_{\epsilon}) \) whose symbols can be split in two parts using a partition of unit associated to \ref{3.6} so that get, for any \( N \),

\[
\text{tr} \left( f(H^{(\kappa)}_{\epsilon}) U^{(\kappa)}_{\epsilon}(t) V^{(\kappa)} f(H^{(\kappa)}_{\epsilon}) \right) \equiv_{N}^{n_1} \sum_{k \leq N, \Omega} h^k \text{tr} \left( f(H^{(\kappa)}_{\epsilon}) U^{(\kappa)}_{\epsilon}(t) Op(h \gamma_{\epsilon,k}) \right) \quad \text{(5.18)}
\]
where $\gamma^{(\kappa)}_{c,\omega} \in S^{1,\infty}$ and either $\text{supp}\gamma^{(\kappa)}_{c,\omega} \subset \gamma^{+} (R, w^{5}/2, \sigma_{+}, \Omega)$ or $\text{supp}\gamma^{(\kappa)}_{c,\omega} \in \gamma^{-} (R, w^{5}/2, \sigma_{-}, \Omega)$ for some $\Omega$ as described in the beginning of subsection 5.1. Let us consider only one term of the right hand side of (5.13) corresponding to a symbol supported in $\gamma^{+} (R, w^{5}/2, \sigma_{+}, \Omega)$ (the case $-$ is similar). Then, using the trick 2.13, lemma 5.3 and Egorov’s theorem we see that

$$\text{tr} \left( f(H^{(\kappa)}_{c})U^{(\kappa)}_{c}(t)O_{\kappa}(\gamma^{(\kappa)}_{c,\omega}) \right) \equiv n_{1} \sum_{l \leq N} h^{l} \text{tr} \left( f(H^{(\kappa)}_{e})U^{(\kappa)}_{e}(t)O_{\kappa}(\gamma^{(\kappa)}_{e,\omega}) \right)$$

(5.19)

with $\gamma^{(\kappa)}_{c,\omega} \in S^{1,\infty}$ supported in $\Gamma^{+}_{c}$. Here again, we study only one term of the right hand side of 5.19. We split the integral corresponding to the inverse Fourier transform 5.10 into two parts and consider first

$$\frac{1}{2\pi} \int_{0}^{+\infty} e^{\pi h t} \text{tr} \left( f(H^{(\kappa)}_{e})U^{(\kappa)}_{e}(t)O_{\kappa}(\gamma^{(\kappa)}_{c,\omega}) \right) \ dt. \quad (5.20)$$

The interest of considering positive times and a symbol supported in $\Gamma^{+}_{c}$ is that we can use 2.14 with the results of subsection 5.1. Using the notations of this subsection, we have

$$\text{tr} \left( f(H^{(\kappa)}_{e})U^{(\kappa)}_{e}(t)O_{\kappa}(\gamma^{(\kappa)}_{c,\omega}) \right) \equiv n_{1} \sum_{N_{+}} h_{N_{+}} \text{tr} \left( f(H^{(\kappa)}_{e})J(\varphi^{+}_{e,\omega}, a^{+}_{e})U(t)J(\varphi^{+}_{e,\omega}, b^{+}_{e})^{*} \right)$$

(5.21)

and, by centrality, we can rewrite the right hand side as

$$\text{tr} \left( J(\varphi^{+}_{e,\omega}, b^{+}_{e})^{*} f(H^{(\kappa)}_{e})J(\varphi^{+}_{e,\omega}, a^{+}_{e})U(t) \right) \equiv n_{1} \sum_{m \leq N} h_{m} \text{tr} \left( O_{\kappa}(c^{(\kappa)}_{e,m})U(t) \right)$$

(5.22)

with $c^{(\kappa)}_{e,m} \in S^{1,\infty}_{c}$. Now the contribution of each term of the right hand side of 5.22 to the expected asymptotic is easy to obtain since

$$\lim_{\kappa \to 0} \frac{1}{2\pi} \int_{0}^{+\infty} e^{\pi \mu t} \left\{ \text{tr} \left( O_{\kappa}(c^{(\kappa)}_{e,m})U(t) \right) \right\} \ dt = \frac{1}{2\pi} \int_{0}^{+\infty} \int \int \int \int \int \int \int \int c_{m}(r, y, \rho, \eta) e^{\pi \mu - \rho \eta} dr d \eta d \rho d \int \ dt$$

thanks to the easy and crucial remark that

$$c_{m} = \lim_{\kappa \to 0} \left\{ c^{(\kappa)}_{e,m} \right\} \in S^{1,\infty}_{c}.$$

(5.23)

The stationary phase theorem (in the variables $t, \rho$) yields easily the asymptotic of the last integral in integer powers of $h$ with coefficients which are smooth functions of $\mu$ in the neighborhood of 1. Note that we use the fact that $\rho$ is close to 1 on the support of $c_{m}$ (see for instance 5.2) thus the Hessian of the phase $(\mu - \rho \eta)^{t}$ is non degenerate w.r.t. $t, \rho$ at the stationary point $t = 0, \rho = \mu^{1/2}$.

We still have to explain how to deal with the contribution of negative times. Here we use the following trick due to Robert

$$\frac{1}{2\pi} \int_{-\infty}^{0} e^{\pi \mu t} \text{tr} \left( f(H^{(\kappa)}_{e})U^{(\kappa)}_{e}(t)O_{\kappa}(\gamma^{(\kappa)}_{c,\omega}) \right) \ dt = \frac{1}{2\pi} \int_{0}^{+\infty} e^{-\pi \mu t} \text{tr} \left( \tilde{f}(H^{(\kappa)}_{e})U^{(\kappa)}_{e}(t)O_{\kappa}(\gamma^{(\kappa)}_{c,\omega})^{*} \right) \ dt$$

which is a simple consequence of the fact that $\text{tr}(A^{*}) = \overline{\text{tr}(A)}$ combined with the centrality of the trace. Then we can use the same method since

$$\text{tr} \left( \tilde{f}(H^{(\kappa)}_{e})U^{(\kappa)}_{e}(t)O_{\kappa}(\gamma^{(\kappa)}_{c,\omega})^{*} \right) \equiv n_{1} \sum_{m \leq N} h_{m} \text{tr} \left( \tilde{f}(H^{(\kappa)}_{e})U^{(\kappa)}_{e}(t)O_{\kappa}(\gamma^{(\kappa)}_{c,\omega})^{*} \right)$$

39
where the symbols \( \tilde{\zeta}_{k,m}^{(\kappa)} \) are derivatives of the complex conjugate of \( \tilde{\zeta}_{k,d}^{(\kappa)} \). Since they are supported in \( \Gamma_{r}^{k} \) as well, we can repeat the same method and the conclusion follows.

This completes the proof under the condition that the coefficients of \( V \) vanish outside \( \{ r > R \} \). Otherwise, we proceed as follows. We write \( V = V_0 + V_{\infty} \) with \( V_{\infty} \) supported in \( \{ r > R \} \) and \( V_0 \) compactly supported outside \( \{ r > R + 1 \} \). The contribution of \( V_{\infty} \) (or more precisely \( V_{\infty}^{(\kappa)} \)) is treated as before and thus we only have to consider

\[
\text{tr } \left( f(H_{\kappa}^{(\kappa)}(t))V_{0}\tilde{f}(H_{\kappa}^{(\kappa)}) \right).
\]

Let \( K \) be the compact subset of \( T^*X \) defined by the conditions \( r \leq R + 1 \) and \( |p_e - 1| \leq w_5/4 \). By the non trapping condition, there exists \( T > 0 \) and \( R' \) such that \( \Pi_{X}^{\kappa} = \Pi_{X}^{\kappa} \cap \{ R < r < R' \} \), if \( \Pi_{X} \) is the projection onto the base. Then we choose \( r_0 \) such that \( \Pi_{X}^{\kappa} \cap \{ r > r_0 \} \) for \( |t| \leq T \) and this ensure the fact that \( \phi_{I}^{\kappa} = \phi_{I}^{\kappa} \) for \( |t| \leq T \). Thus we can use the trick \( (2) \) with \( K_{r} = V_{0}\tilde{f}(H_{\kappa}^{(\kappa)}) \) combined with Egorov’s theorem, and then repeat the same method.

\[ \square \]

### A Proof of lemma 5.4

Let us first recall that for any smooth and bounded function \( a(x, x', \xi) \) defined on \( \mathbb{R}^{3n} \) the operator \( A_{h} \), with Schwartz kernel \( A_{h}(x, x') \) defined as

\[
A_{h}(x, x') = (2\pi h)^{-n} \int e^{i\frac{h}{2}(x-x')^{\xi}} a(x, x', \xi) \, d\xi
\]

is bounded on \( L^{2}(\mathbb{R}^{n}) \) and we have the following bound

\[ ||A_{h}||_{\infty} \leq C \max_{|\alpha+\alpha'|+|\beta| \leq n_{0}} \sup_{x,x',\xi} \left| \partial_{x}^{\alpha} \partial_{x'}^{\alpha'} \partial_{\xi}^{\beta} a(x, x', \xi) \right|, \quad h \in (0, 1] \quad (A.1) \]

for some constants \( C \) and \( n_{0} \) independent of \( a \) and \( h \). This is the usual Calderon-Vaillancourt’s theorem. This kind of operators preserve polynomial decay in the sense that, for any \( M \in \mathbb{R} \) we have

\[ ||\langle x \rangle^{M} A_{h} \langle x \rangle^{-M} \rangle||_{\infty} \leq C_{M} \max_{|\alpha+\alpha'|+|\beta| \leq n_{M}} \sup_{x,x',\xi} \left| \partial_{x}^{\alpha} \partial_{x'}^{\alpha'} \partial_{\xi}^{\beta} a(x, x', \xi) \right|, \quad h \in (0, 1] \quad (A.2) \]

for some \( C_{M} \) and \( n_{M} \) depending only on \( M \). This is an easy consequence of (A.1).

Below, we shall use (A.1) as follows. Assume that we have an operator \( K_{h}^{*} \), with Schwartz kernel \( K_{h}^{*} \) of the form

\[
K_{h}^{*}(r, y, r', y') = (2\pi h)^{-n} \int e^{i\frac{h}{2}(r-r')^{\rho} + \frac{h}{2}(y-y')^{\eta}} e^{i\psi^{\kappa}(r, y, r', r', \rho, \eta)} a_{h}^{*}(r, y, r', y', \rho, \eta) \, d\rho d\eta
\]

with a smooth function \( \psi^{*} \) real valued and such that, on the support of \( a_{h}^{*} \),

\[ \left| \partial^{\gamma} \psi^{*}(r, y, r', y', \rho, \eta) \right| \leq C_{\gamma} \langle \kappa \rangle, \]

for all \( \gamma \). If we know moreover that for some \( N \geq 0 \) and \( M \geq 0 \), we have

\[ \left| \partial^{\gamma} a_{h}^{*}(r, y, r', y', \rho, \eta) \right| \leq C_{\gamma} h^{N} \langle \kappa \rangle^{-M} (r)^{M} (r')^{-M} \]

40
then we obtain the estimate
\[ \left\| (r)^M K_h^s \langle r \rangle^M \right\|_\infty \leq C h^{N-n_0} \langle s \rangle^{-M-n_0}. \]  
(A.3)

This follows simply from (A.1) by considering \( a = e^{i\omega^* / h} a_h^s \).

For any \( a(r, y, \rho, \eta) \), the kernel of \( J(\varphi^+, a) U(s) J(\varphi^+, b) \) is
\[ (2\pi h)^{-n} \int e^{i \Phi^*_s(r, y, r', y', \rho, \eta)} a(r, y, \rho, \eta) b(r', y', \rho, \eta) \, d\rho d\eta \]  
(A.4)

where the phase function \( \Phi^*_s \) is real valued and given by
\[ \Phi^*_s(r, y, r', y', \rho, \eta) = \varphi^+_s(r, y, \rho, \eta) - s\rho^2 - \varphi^+_s(r', y', \rho, \eta). \]

We shall use extensively the fact that for any \( \gamma \), \( \partial^\gamma (\varphi_s(r, y, \rho, \eta) - s\rho - y\eta) = O(\varepsilon^k) \) on \( \Gamma^+_k \). This is a direct consequence of (5.2) and show that, if \( \supp r \subset \Gamma^+_k \), then
\[ \partial_r \Phi^*_s(r, y, r', y', \rho, \eta) = r - r' - 2s\rho + O(\varepsilon^2) \]  
(A.5)
\[ \partial_y \Phi^*_s(r, y, r', y', \rho, \eta) = y - y' + O(\varepsilon^2) \]  
(A.6)

Recall moreover that, in view of (5.2), there exists \( C_1, C_1 > 0 \) such that
\[ c_1 < \rho < C_1. \]  
(A.7)
on the support of \( a(r, y, \rho, \eta) b(r', y', \rho, \eta) \), if \( \supp r \subset \Gamma^+_k \) and \( \supp b \subset \Gamma^+_k \).

Let us start the proof of (5.6). Here we shall use the fact that \( a'_s \in \Gamma^+_k \). More precisely, we need to study symbols \( a(r, y, \rho, \eta) \) of the form
\[ (\partial_r^k \chi_{2,3}(r, y, \rho, \eta)) \partial^\alpha a(r, y, \rho, \eta), \quad (e^{-|\alpha|} \partial_y^s \chi_{2,3}(r, y, \rho, \eta)) \partial^\alpha a(r, y, \rho, \eta) \]

with \( k \geq 1, |\alpha| \geq 1 \). We first consider the symbols \( a \) involving \( \partial_r^k \chi_{2,3} \).

1— If \( \partial_r \) falls on \( \chi_{R^3} \).

In this case, we have \( R^2 \leq r \leq R^3 \), hence we get the fast decay w.r.t. to \( r \). Furthermore \( r' \geq R^4 \) on the support of \( b \), thus we have \( r - r' \leq -R^4 + R^3 \ll 0 \) for \( R \) large, and we obtain
\[ \partial_r \Phi^*_s(r, y, r', y', \rho, \eta) \leq -1 - 2s\rho \leq -1 - 2c_1 s. \]  
(A.8)

Thus we can integrate by part with \( h (\partial_r \Phi^*_s)^{-1} D_r \) in (A.3), and we get as many powers of \( h \langle s \rangle^{-1} \) as we want.

2— If \( \partial_r \) falls on \( \chi_{x^3} \).

Then, we have \( \varepsilon^3 \leq e^{-2r} g(y, \eta) \leq \varepsilon^2 \); thus we get
\[ e^{-2r'} g(y', \eta) \leq \varepsilon^4 \ll \varepsilon^3 \leq e^{-2r} g(y, \eta). \]

Since \( e^{-2r'} \langle y \rangle^2 \leq C_0 e^{-2r'} g(y', \eta) \) and \( e^{-2r} g(y, \eta) \leq C_0 e^{-2r} \), this implies easily that
\[ e^{2r''} - e^{-2r'} \leq C_0^2 \varepsilon. \]

Thus \( r - r' = O(\log \varepsilon) \ll 0 \). In particular \( r \leq r' \) and since we have as much powers of \( \langle r' \rangle^{-1} \) as we want, we use the simple fact that \( \langle r \rangle \langle r' \rangle^{-1} \) is bounded to get as many powers of \( \langle r \rangle^{-1} \) as we want. Furthermore, (A.3) still holds, for \( \varepsilon \) small enough, thus we can integrate by part as before
and get as many powers of $h(s)^{-1}$ as we wish.

3— If $\partial_r$ falls on $\chi_{w^3}$.

Here we have $|\rho^2 + e^{-2r} g(y, \eta) - 1| \geq w^3$. On the other hand we also have

$$\rho^2 + e^{-2r} g(y, \eta) = \rho^2 + e^{-2r'} g(y', \eta) + \left( e^{-2r} g(y, \eta) - e^{-2r'} g(y', \eta) \right)$$

(A.9)

with $|\rho^2 + e^{-2r'} g(y', \eta) - 1| \leq w^4$. Thus, on the support of $a(r, y, \rho, \eta)b(r', y', \rho, \eta)$, we obtain

$$|\rho^2 + e^{-2r} g(y, \eta) - 1| \geq w^3 \quad \text{and} \quad |\rho^2 + e^{-2r} g(y, \eta) - 1| \leq w^4 + \mathcal{O}(\varepsilon^2).$$

(A.10)

If we choose $\varepsilon \leq w^2$ small enough, then (A.10) can not hold and then $a(r, y, \rho, \eta)b(r', y', \rho, \eta) \equiv 0$.

We continue our analysis by considering symbols involving $e^{-|\alpha| r} \partial_y^\alpha \chi_{2,3}$.

4— If $e^{-r} \partial_y$ falls on $\chi_{w^3}$.

Then $y \in \Omega_2 \setminus \Omega_1$ and $y' \in \Omega_4 \in \Omega_3$, thus $|y - y'| \geq c > 0$. Using (A.6), with $\varepsilon$ small enough, we have $|\partial_y \Phi|^2 \geq c/2$ and we can integrate by part using $h^2[\partial_y \Phi]^2 - \Delta_y$. This provides as many powers of $h$ as we want. Furthermore the factor $e^{-r}$ yields the fast decay w.r.t. $r$. We still need to explain how to get fast decay w.r.t. $s$. To that end, we introduce the following partition of unit

$$1 = \theta_- \left( r - r' \over 1 + s \right) + \theta_0 \left( r - r' \over 1 + s \right) + \theta_+ \left( r - r' \over 1 + s \right)$$

(A.11)

Here $\theta_-$ is supported in $(-\infty, c_1)$, $\theta_+$ in $(3C_1, +\infty)$ and we can assume that, on the support of $\theta_0$, we have

$$c_1/2 \leq r - r' \over 1 + s \leq 4C_1.$$

On this support, we can write $1 = \langle r - r' \rangle^M \langle r - r' \rangle^{-M} = \langle r - r' \rangle^M \mathcal{O}(s)^{-M}$, for any $M$. Since we already have fast decay w.r.t. $r$ and $r'$, the fast decay w.r.t. $s$ follows. On the support of $\theta_-((r - r')/(1 + s))$ (resp. $\theta_+$), using (A.6) and (A.7), we see that,

$$\partial_y \Phi (r, y, r', y', \rho, \eta) \leq -c_1 s + c_1 + \mathcal{O}(\varepsilon^2) \quad \text{(resp.} \geq C_1 s + 3C_1 + \mathcal{O}(\varepsilon^2))$$

(A.12)

thus, for $s$ large enough, we can integrate by part as in 1— and get as many powers of $\langle s \rangle^{-1}$ as we want.

5— If $e^{-r} \partial_y$ falls on $\chi_{w^3}$. We proceed as in 2—

6— If $e^{-r} \partial_y$ falls on $\chi_{w^3}$. We proceed as in 3—

This completes the proof of (5.6) using (A.6).

We now turn to the proof of (5.7). Here we just need to get as many powers of $\langle r \rangle^{-1} \langle s \rangle^{-1}$ as we want. The estimates will rely upon the fact that

$$\left| \partial_{x} \partial_{\rho} \partial_{y} \partial_{\eta}^{\alpha} P_{r}(r, y, D_{r}, D_{y}) \alpha^{(N)} \right| \leq C_{k,l,\alpha,\beta} e^{-r} \langle \eta \rangle$$

(A.13)

on $\Gamma_{y}^{+}$, for all $k, l, \alpha, \beta$, by proposition (A.8) We proceed as follows. We introduce the partition of unit (A.11) and consider first what happens on the support of $\theta_0((r - r')/(1 + s))$. Here we remark that

$$e^{-r} \langle \eta \rangle \leq e^{-c_1 s/2-r' \langle \eta \rangle}$$

42
where $e^{-r'|\eta|}$ is bounded on the support of $b$. This yields an exponential decay in $s$. Furthermore, we have $r \leq r' + 4C_1s$, thus we can write for any $M$

$$1 = \langle r \rangle^{-M} (r')^M = \langle (r') \rangle^M O \left( \langle (r') \rangle^M \right).$$

(A.14)

The positive powers of $(r')$ and $\langle s \rangle$ are respectively controlled by the fast decay w.r.t. $r'$ of $b$ and the exponential decay w.r.t. $s$, we obtain thus the expected decay.

On the support of $\theta_-( (r-r')/(1+s) )$, we have similarly $r-r' \leq c_1 (1+s)$ and we can use (A.14) again. This yields $(r')^{-M}$ for any $M$ but we have to control $\langle s \rangle^M$. By the same method as in 4−−, using the upper bound given by (A.12), we get as many negative powers of $\langle s \rangle$ as we want.

The last step is to study what happens on the support of $\theta_+ ((r-r')/(1+s))$. We have $r \geq r' + 3C_1(s+1)$ and we get exponential decay in time, since

$$e^{-r(\eta)} \leq e^{-3C_1s-r' \langle \eta \rangle}.$$ 

Furthermore, by choosing $\varepsilon$ small enough, the lower bound in (A.12) shows that

$$\partial_\rho \Phi^*_\varepsilon (r,y,r',y',\rho,\eta) \geq C_1s + 2C_1 > 0.$$ 

Thus we can integrate by part and get as many negative powers of $\partial_\rho \Phi^*_\varepsilon$ as we wish. Then we note that

$$|\partial_\rho \Phi^*_\varepsilon (r,y,r',y',\rho,\eta)|^{-M} \leq C \langle r-r' \rangle^{-M} \langle s \rangle^M \leq C' \langle r-r' \rangle^{-M} \langle s \rangle^M$$

and the fast decay with respect to $r$ follows as before. This completes the proof. 

\section{Propagation estimates}

In this appendix, we give sufficient conditions leading to (1.13).

\begin{proposition}
Assume that there are positive numbers $w, h_0, C$ such that

$$\sup_{\delta > 0} \left\| \langle r \rangle^{-M} R(e \pm i\delta) \langle r \rangle^{-M} \right\|_{\infty} \leq C h^{-1},$$

(B.1)

for all $\mu \in I$, $h \in (0, h_0]$ and $\varepsilon \in [0, 1]$. Then (1.13) holds.

\end{proposition}

\begin{proposition}
Assume that the manifold $(X, G)$ is non trapping, and that the principal symbols of $P_0$ and $P_1$ coincide outside $\{ r > r_0 \}$. Let $I$ be a neighborhood of 1. Then there exists $r_0$ large enough, $h_0$ small enough and $C > 0$ such that (1.13) holds with $M = 1$.

The first proposition follows from Kato’s theory of smooth perturbations. We recall its simple proof for the sake of completeness and to emphasize the uniformity with respect to the parameters. The proof of the second one uses Mourre theory \cite{37} and more particularly a combination of the ideas of \cite{18} and \cite{20, 19}.

\begin{proof}[Proof of proposition B.1]
Since we can always write $f = f_1 f_2$ with $f_1, f_2 \in C^\infty_0 (I)$ and

$$\| \langle r \rangle^{-M} f(H) U_e(t) \langle r \rangle^{-M} \|_{\infty} \leq \| \langle r \rangle^{-M} f_1(H) U_e(t) \|_{\infty} \| \langle r \rangle^{-M} f_2(H) U_e(-t) \|_{\infty}$$

it is enough to estimate the norm of $\| \langle r \rangle^{-M} f(H) U_e(t) \|_{\infty}$ in $L^2 (\mathbb{R}, dt)$. Let $A_e = \langle r \rangle^{-M} f(H_e)$, then by Parseval’s identity (see \cite{31, 14}), we have, for all $u \in L^2(X)$,

$$2\pi h^{-1} \int_{\mathbb{R}} e^{-2\delta t/h} \| A_e U_e(t) u \|^2 \, dt = \int_{\mathbb{R}} \| A_e (R_e (\mu - i\delta) - R_e (\mu + i\delta)) u \|^2 \, d\mu.$$ 

(B.2)

43
On the other hand, \((i/2)(R_\epsilon(z) - R_z(\bar{z})) = (\text{Im } z) R_\epsilon(z)R_z(\bar{z})\) defines a nonnegative operator for \(\text{Im } z > 0\) and we denote its positive square root by \(K_\epsilon(z)\). We have trivially \(\|K_\epsilon(z)A^*_\epsilon u\|^2 = (2i\pi)^{-1}(A^*_\epsilon u, (R_\epsilon(z) - R_z(\bar{z}))A^*_\epsilon u)\) and this yields

\[
4 \int_{\mathbb{R}} \|AK_\epsilon(\mu + i\delta)u\|^2 \, d\mu \leq \tilde{C}h^{-1} \int_{\mathbb{R}} \|K_\epsilon(\mu + i\delta)u\|^2 \, d\mu = \tilde{C}h^{-1}\pi\|u\|^2 \tag{B.3}
\]

since \(\|A, K_\epsilon(\mu + i\delta)^2 u\| \leq \|K_\epsilon(\mu + i\delta)A^*_\epsilon\|\|K_\epsilon(\mu + i\delta)u\|\). Here \(\tilde{C}\) depends only on \(C\) in (B.1) and \(f\), and is uniform w.r.t. \(\epsilon, h, \delta\). The left hand side of (B.3) is nothing but the right hand side of (B.2) and the result follows easily. \(\square\)

**Proof of proposition B.2.** The estimate (B.1) follows directly from the method of [37] provided the following Mourre estimate holds, with \(E_\epsilon(I)\) the spectral projector of \(H_\tau\) on \(I\),

\[
E_\epsilon(I)\delta[H_\tau, A_\delta] E_\epsilon(I) \geq c\delta h E_\epsilon(I) \tag{B.4}
\]

for some constant \(c > 0\) independent of \(h \in (0, h_0]\) and \(\epsilon \in [0, 1]\). If \(\delta h E_\epsilon(I)\) holds then, the theory of Mourre shows that \(\|A_\delta + i^{-1}R_\epsilon(\mu + i\delta)A_\delta + i^{-1}\|_\infty = O((\delta h)^{-1})\) thus (B.1) holds, provided

\[
\|\langle r \rangle^{-1}A_\delta\|_\infty \leq C_1, \quad h \in (0, h_0]. \tag{B.5}
\]

This reduces the proof to the construction of \(A_\delta\). We now sketch its construction and refer to [18, 20, 45] for the details. The idea is to construct \(A_\delta = A_0^\delta + A_\infty^\delta\) with \(A_\infty^\delta\) supported near infinity and \(A_0^\delta\) compactly supported. Following [18], we define \(A_\infty^\delta\) as

\[
A_\infty^\delta = \frac{1}{2} \hat{f}(H_1) \left( \theta^2 \omega^2 u h D_r + h D_r \theta^2 \omega^2 u \right) \hat{f}(H_1) \tag{B.6}
\]

with \(\hat{f} \in C^\infty, \hat{f} = 1\) close to 1, and \(\theta = \theta(r/R)\) smooth, bounded and supported near infinity, say in \(r \geq R/2\). We also have

\[
\omega_S = \omega \left( \frac{2r - \log(h^2\Delta_Y + 1)}{S} \right), \quad u = 2r + S - \log(h^2\Delta_Y + 1), \quad S > 0,
\]

where the function \(\omega \in C^\infty(\mathbb{R})\) is supported in \([-1, \infty)\) and such that \(\omega \equiv 1\) on \([-1/2, \infty)\). Then using the calculations of [18] and pseudo-differential calculus, we see that

\[
f(H_1)\gamma[H_\tau, A_\delta]\gamma f(H_1) \geq h\theta\omega_S f^2(H_\tau)\omega_S\theta + i(R, S)\mathcal{O}(h) + \mathcal{O}(h^2) \tag{B.7}
\]

if \(f\) is supported where \(\hat{f} = 1\). Here \(\iota(R, S) \downarrow 0\) as \(R, S \to \infty\) and the notation \(\mathcal{O}(h^k)\) holds in operator norm, uniformly w.r.t. \(\epsilon\). Note that the spectral cutoff \(\hat{f}(H_1)\) in (B.6) doesn’t commute with \(H_\tau\) and we use the fact, among other ones, that \((f(H_1) - f(H_\epsilon))\theta h D_r = \mathcal{O}(R^{-\infty})\).

If there was no term \(\omega_S\) in the right hand side of (B.7), we would have done half of our program. How to neglect \(\theta(1 - \omega_S) f(H_\epsilon)\)? We can give a pseudo-differential expansion of \(\omega_S\) similar to the one given in section 4 and thus, up to an operator which is \(\mathcal{O}(h)\), we can replace \(1 - \omega_S\) by a pseudo-differential operator with principal symbol \(1 - \omega((2r - \log(g(y, \eta) + 1))/S)\). On its support, we have \(e^{-2r}g(y, \eta) + e^{-2r} \geq e^{S/2}\). On the other hand, on the support of the principal symbol of \(f(H_\epsilon)\) we have \(e^{-2r}g_y(r, y, \eta) < 3/2\) (if \(f\) is supported close to 1) and thus \(e^{-2r}g(y, \eta) < 3/2 + \mathcal{O}(e^{-r})\). All this shows that the principal symbol of \(\theta(1 - \omega_S)f(H_\epsilon)\) is identically 0 for all \(R\) and \(S\) large enough, which implies that

\[
\theta(1 - \omega_S)\hat{f}(h^2P_r) = \mathcal{O}(h) \tag{B.8}
\]
where one should notice that $\mathcal{O}$ depends on $R$ and $S$. Then we get

$$f(H_\epsilon)i[H_\epsilon, A_\epsilon]f(H_\epsilon) \geq \hbar \theta f^2(H_\epsilon)\theta + i(R, S)\mathcal{O}(h) + \mathcal{O}(h^2)$$  \tag{B.9}$$

where $\mathcal{O}(h^2)$ depends on $R, S$, but not $\mathcal{O}(h)$.

We will now construct $A_0^0 h$ by the method of [20, 45], such that

$$f(H_\epsilon)i[H_\epsilon, A_0^0 h]f(H_\epsilon) = \hbar \tilde{\theta}f(H_\epsilon)\tilde{\theta} + i(R, K)\mathcal{O}(h) + \mathcal{O}(h^2),$$  \tag{B.10}$$

with $\tilde{\theta} \in C_0^\infty$ such that $\theta^2 + \tilde{\theta}^2 = 1$, and $i(R, K) \downarrow 0$ as $R, K \uparrow \infty$. Here $K$ is another large parameter introduced below. If this holds, we easily get (B.4) by summing (B.9) and (B.10), choosing first $R$, $K$ and $S$ large enough, then $\hbar$ small enough and then by multiplying both side by $E_\epsilon(I)$. Note that we also use the fact that $[f(H_\epsilon), \theta] = \mathcal{O}(h)$.

The idea is to define $A_0^0 h$ as a (bounded) pseudo-differential operator whose principal symbol is the following function $a$, which is invariantly defined on $T^*X$

$$a = \tilde{\chi} \int_0^\infty \tilde{\theta}^2 \circ \phi_1^t dt \circ p_1$$

where $p_1$ and $\phi_1^t$ are the principal symbol of $H_1$ and the associated flow, and $\tilde{\chi} = \tilde{\chi}(r/K)$ is a $C_0^\infty$ function such that $\tilde{\chi} \tilde{\theta}^2 = \tilde{\theta}^2$. Note that the integral is convergent thanks to the non trapping condition. Then the crucial remark is that the Poisson bracket is

$$\{p_\epsilon, a\} = \tilde{\theta} f(p_1)\tilde{\theta}^2 + \mathcal{O}(K^{-1}) + \mathcal{O}(R^{-\infty})$$

with $\mathcal{O}(K^{-1})$ uniform w.r.t $\epsilon$ (but not w.r.t $R$) in the topology of smooth and bounded functions. This follows first from the fact that

$$\{p_1, \tilde{\theta}^2 \circ \phi_1^t\} = \{p_1, \tilde{\theta}^2\} \circ \phi_1^t = \frac{d}{dt} (\tilde{\theta}^2 \circ \phi_1^t)$$

and from the fact that $p_\epsilon = p_1$ for $r \leq r_0$ with $r_0$ which we can choose $\geq R$. Then (B.10) follows from semi-classical pseudodifferential calculus. (see [46]).

References

[1] S.M. Belov, A.V. Rybkin, Higher order trace formulas of the Buslaev-Faddeev-type for the half-line Schrödinger operator with long-range potentials, J. Math. Phys. 44, no. 7, 2748-2761 (2003).

[2] M.S. Birman, M.G. Krein, On the theory of wave operators and scattering operators, Dokl. Akad. Nauk SSSR, 144, 475-478 (1962).

[3] M.S. Birman, M.Z. Solomyak Remarks on the spectral shift function, Zap. Nauchn. Sem. Leningrad Math. Inst. Steklov (LOMI) 27, 33-46 (1972).

[4] N.V. Borisov, W. Miller, R. Schrader, Relative index theorems and supersymmetric scattering theory, Comm. Math. Phys. 114, no. 3, 475-513 (1988).

[5] D. Borthwick, C. Judge, P.A. Perry, Determinants of Laplacians and isopolar metrics on surfaces of infinite area, Duke Math. J. 118, no. 1, 61-102 (2003).
[6] J.M. Bouclet, *Traces formulae for relatively Hilbert-Schmidt perturbations*, Asympt. Anal. 32, no. 3-4, 257-291 (2002).

[7] ________, *Spectral distributions for long range perturbations*, J. Funct. Anal. (in press).

[8] V. Bruneau, V. Petkov, *Meromorphic continuation of the spectral shift function*, Duke Math. J. 116, no. 3, 389-430 (2003).

[9] G. Carron, *Déterminant relatif et la fonction Xi*, Amer. J. Math. 124, no. 2, 307-352, (2002).

[10] T. Christiansen, *Weyl asymptotics for the Laplacian on asymptotically Euclidean spaces*, Amer. J. Math. 121, no. 1, 1-22 (1999).

[11] ________, *Weyl asymptotics for the Laplacian on manifolds with asymptotically cusp ends*, J. Funct. Anal. 187, No. 1, 211-226 (2001).

[12] Y. Colin de Verdière, *Une formule de traces pour l’opérateur de Schrödinger dans ℝ³*, Ann. Sci. cole Norm. Sup. (4) 14, no. 1, 27-39, (1981).

[13] C. Cuevas, G. Vodev *Sharp bounds on the number of resonances for conformally compact manifolds with constant negative curvature near infinity*, Comm. in Partial Diff. Eq. 28, no. 9-10, 1685-1704 (2003).

[14] S. De Bièvre, P.D. Hislop, I.M. Sigal, *Scattering theory for the wave equation on non-compact manifolds*, Reviews in Math. Phys., Vol. 4, No. 4, 575-618 (1992).

[15] I. Derezinski, C. Gérard, *Scattering theory of classical and quantum N-particles systems*, Texts and Monographs in Physics, Springer-Verlag, (1997).

[16] M. Dimassi, J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series, 268. Cambridge University Press, (1999).

[17] R.G. Froese, *Upper bounds for the resonance counting function of Schrödinger operators in odd dimensions*, Canadian J. Math. 50, 538-546 (1998) and *Correction*, Canadian J. Math. 53, 756-757 (2001).

[18] R.G. Froese, P.D. Hislop, *Spectral analysis of second-order elliptic operators on noncompact manifolds*, Duke Math. J. Vol. 58, No. 1, 103-129 (1989).

[19] ________, *On the distribution of resonances for some asymptotically hyperbolic manifolds*, Journées EDP Nantes, exp. VII (2000).

[20] C. Gérard, A. Martinez, *Principe d’absorption limite pour des opérateurs de Schrödinger à longue portée*, C.R. acad. sci. Paris sér. I 306, 121-123, (1988).

[21] ________, *Prolongement méromorphe de la matrice de scattering pour des problèmes à deux corps à longue portée*, Ann. Inst. H. Poincaré Phys. Théor. 51, No.1, 81-110, (1989).

[22] C. Gérard, A. Martinez, D. Robert, *Breit-Wigner formulas for the scattering phase and the total scattering cross-section in the semi-classical limit*, Comm. Math. Phys. 121, no. 2, 323-336 (1989).

[23] I.C. Gohberg, M.G. Krein, *Introduction to the theory of linear non-selfadjoint operators*, Translations of Mathematical Monographs, Vol. 18. AMS. Providence, R.I. (1969).
C. Graham, M. Zworski, *Scattering matrix in conformal geometry*, Invent. Math. 152, no. 1, 89-118 (2003).

L. Guillopé, *Une formule de trace pour l’opérateur de Schrödinger dans $\mathbb{R}^n$*, Thèse de troisième cycle, Université de Grenoble (1981).

L. Guillopé, M. Zworski, *Scattering asymptotics for Riemann surfaces*, Ann. of Math. 145, 597-660 (1997).

B. Helffer, D. Robert, *Calcul fonctionnel par la transformation de Mellin et opérateurs admissibles*, J. Funct. Anal. 53, 246-268 (1983).

M. Hitrik, I. Polterovich, *Regularized traces and Taylor expansions for the heat semigroup*, J. London Math. Soc. (2) 68, no. 2, 402-418 (2003).

H. Isozaki, H. Kitada, *Modified wave operators with time independent modifiers*, J. Fac. Sci., Univ. Tokyo, Sect. I A 32, 77-104 (1985).

M.S. Joshi, A. Sá Barreto, *Inverse scattering on asymptotically hyperbolic manifolds*, Acta Math. 184, 41-86 (2000).

T. Kato, *Perturbation theory for linear operators*, Springer-Verlag (1980).

L.S. Koplienko, *Trace formula for non trace-class perturbations*, Sib. Math. J. 25, 735-743 (1984).

E. Korotyaev, A. Pushnitski *Trace formulae and high energy asymptotics for the Stark operator*, Comm. in Partial Diff. Eq. 28, no. 3-4, 817-842 (2003).

A. Majda, J. Ralston, *An analogue of Weyl’s theorem for unbounded domains I, II, III*, Duke Math. J. vol. 45 183-196 (1978), vol. 45 513-536 (1978), vol. 46 725-731 (1979).

R. Mazzeo, R.B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. 75, No. 2, 260-310 (1987).

R.B. Melrose, *Geometric scattering theory*, Stanford lecture, Cambridge Univ. Press (1995).

E. Mourre, *Absence of singular continuous spectrum for certain self-adjoint operators*, Comm. Math. Phys. 78, 391-408 (1981).

W. Müller, *Relative zeta functions, relative determinants and scattering theory*, Comm. Math. Phys. 192, no. 2, 309-347, (1998).

S.J. Patterson, P.A. Perry, *The divisor of Selberg’s zeta function for Kleinian groups*, Duke Math. J. 106, no. 2, 321-390 (2001).

P.A. Perry, *A Poisson summation formula and lower bounds for resonances in hyperbolic manifolds*, Int. Math. Res. Not. no. 34, 1837-1851 (2003).

V. Petkov, G. Popov, *Asymptotic behaviour of the scattering phase for non-trapping obstacles*, Ann. Inst. Fourier 32, 111-149, (1982).

V. Petkov, M. Zworski, *Breit-Wigner approximation and the distribution of resonances*, Comm. Math. Phys. 204, no. 2, 329-351 (1999), and *Erratum*, Comm. Math. Phys. 214, no. 3, 733-735 (2000).
[43] ________ Semi-classical estimates on the scattering determinant, Ann. Inst. H. Poincaré 2, no. 4, 675-711 (2001).

[44] M. Reed, B. Simon Methods of modern mathematical physics, IV. Analysis of operators, Academic Press (1975).

[45] D. Robert, Autour de l’approximation semi-classique, Progress in mathematics, 68, Birkhäuser (1987).

[46] ________ Asymptotique de la phase de diffusion à haute énergie pour des perturbations du second ordre du Laplacien, Ann. scient. Éc. Norm. Sup. 4e série, t. 25, 107-134 (1992).

[47] ________, Relative time delay for perturbations of elliptic operators and semi-classical asymptotics, J. Funct. Anal. 126, No. 1, 36-82 (1994).

[48] D. Robert, H. Tamura, Semi-classical estimates for resolvents and asymptotics for total scattering cross-sections, Ann. Inst. H. Poincaré Phys. Théor. 46, no. 4, 415-442 (1987).

[49] J.T. Schwartz, Non linear functional analysis, Gordon & Breach (1969).

[50] R.T. Seeley, Complex powers of an elliptic operator, Singular integrals (Proc. Sympos. Pure Math. Chicago, III 1966) A.M.S. R.I., 288-307 (1967).

[51] B. Simon, Resonances in one dimension and Fredholm determinants, J. Funct. Anal. 178, 396-420 (2000).

[52] A. Vasy, X.P. Wang, Smoothness and high energy asymptotics of the spectral shift function in many-body scattering, Comm. in Partial Diff. Eq. 27, no. 11-12, 2139-2186 (2002).

[53] D. Yafaev, Mathematical scattering theory. General theory, vol. 105, A.M.S. Rhode Island (1992).