On decidability of amenability in computable groups

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Received: 11 August 2020 / Accepted: 31 January 2022 / Published online: 3 March 2022
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Abstract
The main result of the paper states that there is a finitely presented group $G$ with decidable word problem where detection of finite subsets of $G$ which generate amenable subgroups is not decidable.

Keywords Computational groups · Amenability · Intrinsically computable relations

Mathematics Subject Classification 03D45 · 20F10

1 Introduction

In this paper we discuss the following problem. Given algebraic property of groups $\mathcal{P}$ is there a finitely presented group $G_\mathcal{P}$ with decidable word problem such that there is no algorithm to determine whether or not the subgroup generated by an arbitrary finite set of words in the given generators of $G_\mathcal{P}$ has the property $\mathcal{P}$? If one removes the requirement of decidability of the word problem, this task becomes easier. For example Baumslag, Boon and Neumann have proved in [5] a general theorem which gives a positive answer for a long list of properties (see also Section 3 in [21]). If we insist on decidability of the word problem the question becomes hard. Although the methods of contemporary algorithmic geometric group theory have become very advanced (for example see [25–27]) we do not know any example where they work for typical properties $\mathcal{P}$. Usually they are applied to properties of different kinds:
the conjugacy problem [26], the power problem [27] or the problem of solvability of exponential equations [7].

The main result of our paper is a theorem which for some natural properties \( P \) states existence of a finitely presented group \( G_P \) with decidable word problem where detection of finite subsets of \( G_P \) which generate subgroups satisfying \( P \) is not decidable. We emphasize the case when \( P \) is the property of amenability. The reason will be explained in the second part of the introduction.

The proof uses an idea which is completely new in geometric group theory. It consists of two steps. Firstly by application of the techniques of \textit{intrinsically computable relations} of Ash, Knight and Nerode from computable model theory we build a computable group with undecidable amenability problem for subgroups. In Sect. 3 using some kind of Higman’s embedding we transform it into a group which is also finitely presented. The construction of the first step is quite general and can be applied to other properties studied in group theory. Therefore we present it in a very general form.

For deeper discussion of the results we now introduce the main definitions of the paper.

A first-order structure \( M \) with domain \( N \) is called \textit{computable}, if all relations and operations of \( M \) are computable and this condition is satisfied uniformly. In group theory the terminology is slightly different. Let \( G = \langle X | R \rangle \) be a countable group generated by \( X \subseteq G \). The group \( G \) is called \textit{recursively presented} (see Section IV.3 in [20]) if \( X \) can be identified with \( N \) (or with some \( \{0, \ldots, n\} \)) so that the set of relators \( R \subseteq (X \cup X^{-1})^* \) is computably enumerable. When \( G \) is recursively presented and has decidable word problem, then there is an algorithm computing a 1–1 function \( f : N \to (X \cup X^{-1})^* \) such that each element of \( G \) is uniquely presented by a word from \( f(N) \) (we neglect the easy case when \( G \) is finite). The group operations of \( G \) induce a group with domain \( N \) which is isomorphic to \( G \). Since each computable group can be obviously viewed as a recursively presented one, we see that recursively presented groups with decidable word problem exactly correspond to computable groups.

Amenability is one of the basic concepts of mathematics. It was created by Banach, von Neumann and Tarski in the beginning of the twentieth century and nowadays it has become fundamental in dynamical systems, group theory, logic, measure theory, etc., see [29]. Very recently Cavaleri and Moriakov have begun investigations of algorithmic aspects of this topic, see \[ 8, 9\] and \[ 22\].

We now give two equivalent definitions of amenability. Let \( G \) be a discrete group and \( D \) be a finite subset of \( G \). Given \( n \in N \), we say that a finite \( F \subseteq G \) is an \( \frac{1}{n} \)-\textit{Følner set} with respect to \( D \) if

\[
\forall x \in D \quad \frac{|F \setminus xF|}{|F|} \leq \frac{1}{n}.
\]

The group \( G \) is called \textit{amenable} if for every \( n \in N \) and every finite \( D \subset G \) there is a finite \( F \subset G \) which is \( \frac{1}{n} \)-Følner with respect to \( D \) (this is so called Følner’s condition).

A theorem of A. Tarski states that the group \( G \) is not amenable if and only if \( G \) has a \textit{paradoxical decomposition}. The latter means that there exists a finite set \( K \subset G \)
and two families of subsets of $G$: $(A_k)_{k \in K}$ and $(B_k)_{k \in K}$ such that

$$G = \left( \bigsqcup_{k \in K} k \cdot A_k \right) \bigcup \left( \bigsqcup_{k \in K} k \cdot B_k \right) = \left( \bigsqcup_{k \in K} A_k \right) = \left( \bigsqcup_{k \in K} B_k \right).$$

Here we use a version of the definition given in [10], where some members $A_k$ or $B_k$ can be empty. The first author has shown in his master thesis (which is available on ArXiv: [16]) that given computable non-amenable $G$ there is an effective procedure which assigns to any $D$ without $\frac{1}{n}$-Følner sets a paradoxical decomposition consisting of computable pieces. Moreover the family of such $D$ is computable if and only if there is an algorithm which distinguishes all finite subsets of $G$ generating amenable subgroups. This observation explains our original motivation.

As we have already mentioned above standard methods of algorithmic group theory do not seem helpful for producing examples of the form $G_P$ as in the beginning of the section. It seems to us that Proposition 1 below describes basic obstacles in the field. It roughly says that when we build a computable group with undecidable amenability it should not resemble free groups. Note that standard finitely presented examples in group theory are usually amenable or “free-like”. For example they often satisfy the Tits alternative: it is virtually soluble (i.e. amenable) or contains a non-abelian free group. Thus the proposition suggests that in order to build non-amenable groups of the form $G_P$ we probably need new tools. In particular, computability theory techniques may become essential in this respect.

To present Proposition 1 we remind the reader that a group $G$ is called fully residually free if for any finite collection of nontrivial elements $g_1, \ldots, g_n \in G \setminus \{1\}$ there exists a homomorphism $\phi : G \to F$ onto a free group $F$ such that $\phi(g_1) \neq 1, \ldots, \phi(g_n) \neq 1, [18]$. The class of fully residually free groups as well as residually free groups has received a lot of attention mainly in connection with algorithmic and model-theoretic investigations in group theory, see for example [19] and [30].

**Proposition 1** If $G$ is a computable fully residually free group then the problem if a finite subset of $G$ generates an amenable subgroup is decidable.

**Proof** We use the well-known fact that abelian groups are amenable contrary to groups containing non-abelian free subgroups.

Let $G$ be a computable fully residually free group. It suffices to show that a finite $K \subset G$ generates a non-amenable subgroup if and only if there exist $x, y \in K$ such that $[x, y] \neq 1$.

$(\Rightarrow)$ Let us assume that $[x, y] = 1$ for all $x, y \in K$. Therefore subgroup $\langle K \rangle$ is a finitely generated abelian group, i.e. it is amenable.

$(\Leftarrow)$ Let us assume that there exist $x, y \in K$ with $[x, y] \neq 1$. Since $x, y, [x, y]$ are nontrivial elements of $G$ we have a homomorphism $\phi$ from $G$ into $F_2$ such that the set $\{\phi(x), \phi(y), \phi([x, y])\}$ consists of pairwise distinct elements. Clearly, $\langle \phi(x), \phi(y) \rangle$ is a free group of rank 2. Thus $\langle x, y \rangle$ is also a free subgroup of rank 2. In particular $\langle K \rangle$ is not amenable.

\[\square\]
1.1 Technical preliminaries

We do not assume that the reader has any special knowledge in recursion theory. We give all necessary definitions making the material available for group theorists. From now on we identify each finite set \( F \subseteq \mathbb{N} \) with its Gödel number. The following notion is convenient when one builds a computable copy of an abstract group.

**Definition 1** Let \( G \) be a group and \( \nu : \mathbb{N} \to G \) be a surjective function. We call the pair \( (G, \nu) \) a **numbered group**. The function \( \nu \) is called a **numbering** of \( G \). If \( g \in G \) and \( \nu(n) = g \), then \( n \) is called a number of \( g \).

Note that when the numbering \( \nu \) is a bijection and the set \( \text{Mult}_T := \{(i, j, k) : \nu(i)\nu(j) = \nu(k)\} \) is computable, the map \( \nu \) is an isomorphism of \( G \) with a computable group, and the latter is called a **computable copy** of \( G \). On the other hand if a numbering \( \nu \) of a group \( G \) is not necessary injective but the set \( \text{Mult}_T \) above is still computable, then

- \( G \) has a computable copy (possibly under another numbering). Indeed, in this case the set of the smallest numbers of the elements of \( G \) is computable. Enumerating this set by natural numbers we obtain a required 1–1 enumeration.
- In this case we also have that the set \( \{(n_1, n_2) : \nu(n_1) = \nu(n_2)\} \) is computable.

We apply the theory of **intrinsically computable relations**, [2–4]. In fact we use the advanced version of it from [3]. According Sect. 2 of that paper the binary relation \( \bar{a} \leq_0 \bar{b} \) on tuples of the same length \( n \) from a computable structure \( M \) is defined to be the property that \( M \models \phi(\bar{a}) \) implies \( M \models \phi(\bar{b}) \) where \( \phi(x_1, \ldots, x_n) \) is atomic or the negation of an atomic formula with the Gödel number \( < n \) and variables among \( x_1, \ldots, x_n \).

The following statement is Theorem 2.1 from [3] under the additional assumption that the parameter \( \alpha \) which appears in the formulation given in [3] is equal to 1.

Let \( M \) be a computable structure and a relation \( R \subseteq M^s \) is computable on \( M \). Suppose that for any \( \bar{c} \) there is an \( s \)-tuple \( \bar{a} \notin R \) such that for any tuple \( \bar{a}_1 \) there exist \( \bar{a}' \in R \) and \( \bar{a}'_1 \) such that \( \bar{c}\bar{a}\bar{a}'_1 \leq_0 \bar{c}\bar{a}'\bar{a}_1 \) in \( M \). Then for every c.e. set \( C \) there is an isomorphism \( f \) from \( M \) onto a computable structure \( M' \) such that \( C \) and \( f(R) \) are of the same Turing degree.

**Remark 1** The formulation of Theorem 2.1 from [3] uses the notion “\( \alpha \)-friendliness” introduced in that paper. In our construction below we only consider the case when \( \alpha = 1 \). Thus it is worth noting here that 1-**friendliness** of a structure just means that the relation \( \leq_0 \) is computably enumerable on the set of tuples of arbitrary length. This condition is always satisfied in a computable structure. This is why we omit it.
2 A computable group with undecidable amenability

2.1 The construction

Let \((G_i, g_{i,1}, \ldots, g_{i,l})\), \(i \in 2\mathbb{N} + 1\) be a sequence of groups (indexed by odd numbers) such that the following conditions are satisfied.

- For every odd \(i\) the group \(G_i\) is generated by the tuple \(g_{i,1}, \ldots, g_{i,l}\).
- For every pair \(i \leq j\) the (directed and coloured) Cayley graph of \(G_j\) with respect to the generators \(g_{j,1}, \ldots, g_{j,l}\) has the same \(i\)-balls of 1 with the Cayley graph of \((G_i, g_{i,1}, \ldots, g_{i,l})\).

In terms of [11] these conditions exactly mean that the sequence of marked groups \((G_i, g_{i,1}, \ldots, g_{i,l})\) (i.e. a group with a distinguished tuple of generators), \(i \in 2\mathbb{N} + 1\), is convergent in the Grigorchuk’s topology. By compactness of the latter (see [11], Section 2.2) there is a marked group \((G_\infty, g_\infty,1, \ldots, g_\infty,l)\) which is the limit of the sequence, i.e. for every \(i \in 2\mathbb{N} + 1\) the Cayley graph of \(G_i\) with respect to the generators \(g_{i,1}, \ldots, g_{i,l}\) has the same \(i\)-balls of 1 with the Cayley graph of \((G_\infty, g_\infty,1, \ldots, g_\infty,l)\).

**Remark 2** A standard example of such a situation (which will be used below) is as follows. Let \(F_2\) be a 2-generated free group with the basis \(\{a, b\}\). Since \(F_2\) is residually finite, for every \(i \in 2\mathbb{N} + 1\) there is a finite group \(G_i\) and a surjective homomorphism \(\rho_i : F_2 \rightarrow G_i\) such that for the generators \(a_i = \rho_i(a)\) and \(b_i = \rho_i(b)\) the Cayley graph of \((G_i, a_i, b_i)\) has the same \(i\)-ball of 1 with the Cayley graph of \((F_2, a, b)\).

Let \(s\) be a positive natural number. We will assume below that for every \(i \in (2\mathbb{N} + 1) \cup \{\infty\}\)

- the group \(G_i\) is a computable group with domain \(\mathbb{N}\) or some \(\{1, \ldots, n\}\) (if the group is finite) and
- a computable \(s\)-ary relation \(R_i\) is fixed on \(G_i\) which is \(\text{Aut}(G_i)\)-invariant and contains the \(s\)-tuple of units \(\vec{1}\).

Furthermore, we assume that the multiplication in \(G_i\) and the characteristic function of \(R_i\) are given by a uniform algorithm on \(i\).

**Remark 3** Let \(G = \langle g_1, \ldots, g_l | \mathcal{R} \rangle\) be a recursively presented group with decidable word problem and let \((G, g_1, \ldots, g_l)\) be the limit of a sequence of finite marked groups. Then there is a sequence \((G_i, g_{i,1}, \ldots, g_{i,l}), i \in 2\mathbb{N} + 1\) which together with \((G, g_1, \ldots, g_l)\) (viewed as \((G_\infty, g_\infty,1, \ldots, g_\infty,l)\)) satisfies the assumptions given above. Indeed, fix a computable copy of \(G\) and view it as \(G\). Take any computable enumeration of multiplication tables of finite groups. For every \(i \in 2\mathbb{N} + 1\) determine the \(i\)-ball of the unit in \((G, g_1, \ldots, g_l)\). Then let \((G_i, g_{i,1}, \ldots, g_{i,l})\) be the finite group with the same \(i\)-ball such that its multiplication table has the minimal number.

We define the sequence \((G_i, \bar{g}_i), i \in \omega, \) of marked groups so that for even \(i\) the group \((G_i, \bar{g}_i)\) coincides with \((G_\infty, g_\infty,1, \ldots, g_\infty,l)\). For odd \(i\) we assume that \(G_i\) is as before. Let \(G = \sum_{i \in \omega} G_i\). Let

- \(\text{Odd}(x)\) to be the unary predicate on \(G\) for the subgroup \(\sum_{i \in 2\mathbb{N} + 1} G_i\);
– $Pr_i(x, y)$ be the relation from $G \times G$ that $y$ is the $G_i$-projection of $x$, $i \in \omega$;
– $R \subseteq G^s$ be the relation consisting of all $s$-tuples $\bar{g}$ such that the projection of $\bar{g}$ to every $G_i$ satisfies $R_i$, $i \in \omega$.

We define a numbering $v : \mathbb{N} \to G$ as follows. We fix a $1$–$1$ numbering of all finite subsets of $\mathbb{N} \times \mathbb{N}$ with pairwise distinct first coordinates. Given $k$ let $\{(k_1, l_1), \ldots, (k_s, l_s)\}$ be the subset with the number $k$ where $k_1 < \cdots < k_s$. Let $v(k)$ be the element of \( \sum_{i \in \omega} G_i \) represented by the sequence where the only non-unit elements are $l_j$ at the corresponding places of $\{k_1, \ldots, k_s\}$.

**Remark 4** If some $G_i$ are finite we modify $v$ so that when $\{(k_1, l_1), \ldots, (k_s, l_s)\}$ is a subset with the number $k$ then each $l_i$ is strictly less than $|G_i|$.

**Lemma 2** The numbered structure $(G, \cdot, 1, R, Odd, \{Pr_i\}_{i \in \omega}, v)$ is computable, i.e. the group operations of $G$ and the relations $R$, $Odd$ and $Pr_i$ are computable with respect to $v$.

**Proof** The structure $(G, \cdot, 1, R, Odd, \{Pr_i\}_{i \in \omega}, v)$ is computable by the definition of $v$ and the observation that an equation is satisfied in $G$ if and only if it is satisfied in all $G_i$. \(\square\)

Having this lemma we see that the structure $(G, \cdot, 1, R, \{Pr_i\}_{i \in \omega})$ has a computable copy (for example by an argument given in Sect. 1.1). From now on we identify $G$ with $\mathbb{N}$. The following theorem is the crucial technical statement.

**Theorem 3** Assume that $G^s_\infty \setminus R_\infty$ is non-trivial, but for all odd $i$ the relation $R_i$ coincides with $G^s_i$. Then the group $G$ has a computable copy so that $R$ is not computable.

**Proof** Let us verify the condition of Theorem 2.1 from [3] which is stated in Sect. 1.1. Let $\bar{c}$ be a tuple from $G$ and $t_1 < t_2 \cdots < t_r$ be the support of $\bar{c}$, i.e. the indices of those $G_i$ where elements from $\bar{c}$ have non-trivial projections. These numbers can be found algorithmically by Lemma 2.

Let $t_{r+1}$ be the first even index greater than $t_r$. We define $\bar{a}$ to be a tuple of $G_{t_{r+1}}$, say $(a_{t_{r+1}}, \ldots, a_{t_{r+1}+s})$ which is not in $R_{t_{r+1}}$. We consider it as a tuple of elements of $G$ (assuming to be 1 for other coordinates).

Let $\bar{a}_1$ be any tuple from $G$, and let $n$ be the length of $\bar{c}\bar{a}_1$. We want to find $\bar{a}' \in R$ and $\bar{a}_1'$ as in the formulation (given in Sect. 1.1). In particular verifying $\bar{c}\bar{a}_1 \leq_0 \bar{c}\bar{a}_1'$ we only consider formulas of Gödel numbers $< n$. We may suppose that these formulas are as follows:

$$\{w_i(\bar{z}, \bar{x}, \bar{x}_1) = 1 : i \in I_1\} \cup \{v_i(\bar{z}, \bar{x}, \bar{x}_1) \neq 1 : i \in I_2\},$$

where $w_i$ and $v_i$ are group words. For a word $w(\bar{z}, \bar{x}, \bar{x}_1)$ let $w(\bar{c}, \bar{a}, \bar{a}_1)(t)$ be the word written in the generators $g_{t,1}, \ldots, g_{t,l}$ which is obtained by the substitution of the $G_i$-projections of elements from $\bar{c}\bar{a}_1$ into $w(\bar{z}, \bar{x}, \bar{x}_1)$ (before reductions). Let

$$n_0 = \max \bigcup \{|w_i(\bar{c}, \bar{a}, \bar{a}_1)(t)| : i \in I_1\} \cup \{|v_i(\bar{c}, \bar{a}, \bar{a}_1)(t)| : i \in I_2\} : t \in \text{supp}(\bar{c}\bar{a}_1)\} + 1.$$
Let $\hat{t}$ be the first odd index greater than $\max (\text{supp}(\overline{c\overline{a}_1}))$, such that the $n_0$-ball of 1 in the Cayley graph of $G_{t+1}$ is isomorphic to the $n_0$-ball of 1 in the Cayley graph of $G_{\hat{t}}$. Let us define $\overline{a'}\overline{a}_1$ as follows:

$$\overline{a'}\overline{a}_1(t) = \overline{a}\overline{a}_1(t) \text{ when } t \notin \{t_{r+1}, \hat{t}\},$$
$$\overline{a'}\overline{a}_1(t_{r+1}) = 1,$$
and the words of the sequence $\overline{a'}\overline{a}_1(\hat{t})$ coincide with ones of the sequence $\overline{a}\overline{a}_1(t_{r+1})$ under the correspondence $(g_{\hat{t}, 1}, \ldots, g_{l, 1}) \leftrightarrow (g_{t_{r+1}, 1}, \ldots, g_{t_{r+1}, l})$.

It is clear that $\overline{a'} \in R$. Since the sequences $\overline{c\overline{a}_1}$ and $\overline{c\overline{a'}\overline{a}_1}$ coincide on the sets of indices

$$\text{supp}(\overline{c\overline{a}_1}) \setminus \{t_{r+1}, \hat{t}\} = \text{supp}(\overline{c\overline{a'}\overline{a}_1}) \setminus \{t_{r+1}, \hat{t}\},$$

their realizations on the formulas of Gödel numbers $< n$ are equivalent on this part of the support. Note that $t_{r+1} \notin \text{supp}(\overline{c\overline{a'}\overline{a}_1})$ and $\hat{t} \notin \text{supp}(\overline{c\overline{a}_1})$. Thus to obtain the result it suffices to note that for any word $w$ appearing in the formula of the Gödel number $< n$ the equality $w(\overline{c}, \overline{a}, \overline{a}_1)(t_{r+1}) = 1$ is equivalent to $w(\overline{c}, \overline{a'}, \overline{a}_1)(\hat{t}) = 1$. The latter follows from the choice of $n_0$ and $\hat{t}$.

Since the conditions of Theorem 2.1 from [3] (see Sect. 1.1 above) are satisfied, applying it to a non-computable $C$ we have that there is an 1–1 numbering $\nu'$ of the group $G$ such that $(G, \nu')$ is a computable group where the relation $R$ is not computable with respect to $\nu'$.

Corollary 4 There is a countable group $G$ which has a computable copy $G_{nA}$ such that the following problems are undecidable in $G_{nA}$:

(i) when does a finite subset of $G_{nA}$ generate an amenable subgroup?
(ii) when does a finite subset of $G_{nA}$ generate a group without free non-abelian subgroups?

Furthermore, there is a computable copy $G_{nT} \cong G$ such that the following problems are undecidable in $G_{nT}$:

(iii) when does a finite subset of $G_{nT}$ generate a finite subgroup?
(iv) when does a finite subset of $G_{nT}$ generate a subgroup having property (T)?

The group $G$ has the Haagerup property.
Proof Let \((G_i, a_i, b_i), i \in 2\mathbb{N} + 1,\) be the sequence defined in Remark 2. We consider the free group \((F_2, a, b)\) as \((G_\infty, g_1, \ldots, g_\infty, i)\) and define \((G_i, a_i, b_i), i \in \omega,\) taking even members to be equal to it. Let \(G\) be the computable group as in the construction and let

\[ R_i = \{(g_1, g_2) | \langle g_1, g_2 \rangle \text{ is not a free non-abelian subgroup of } G_i \}. \]

For an even \(i\) we have

\[ (g_1, g_2) \in R_i \iff g_1g_2 = g_2g_1. \]

Thus \(R_i\) is computable. Moreover it is easy to see that in \(G\) a pair \((g_1, g_2)\) satisfies \(R\) if and only if the subgroup \(\langle g_1, g_2 \rangle\) is not isomorphic to \(F_2\) (i.e. its even projections are cyclic). The latter is equivalent to the property that \(\langle g_1, g_2 \rangle\) is amenable. Applying Theorem 3 we obtain \(G_{nA}\) satisfying statement (i) of the corollary.

It is worth noting that in the group \(G\) the condition to be amenable subgroup coincides with the condition to be without non-abelian free subgroup. Thus conditions (i) and (ii) are equivalent.

Now let us consider the unary relation \(Odd\) on \(G\). Keeping \(G_\infty = F_2\) we see that its odd projections are the corresponding \(G_i\), but its even projections are trivial. Applying Theorem 3 to \(G\) and \(Odd\) we have \(G_{nT}\) satisfying statement (iii) of the corollary.

It is known that any free group does not have property (T) but finite groups have this property. Moreover property (T) is preserved under homomorphisms. Now it is easy to see that a finite subset of \(G\) generates a subgroup with property (T) if and only if it generates a finite subgroup. Thus for singletons such subsets are represented by \(Odd\). This proves (iv).

The last statement of the corollary follows from the facts that free and finite groups have the Haagerup property and the latter is preserved under direct sums (according chapt. 6 of [12] it is preserved under finite sums and holds in the group when all finitely generated subgroups satisfy it). □

2.2 Comments

(1) Is it possible to realize undecidability of conditions (i), (iii), (iv) of Corollary 4 (even separately) in computable torsion groups (or of bounded exponent)? By the positive solution of the restricted Burnside problem [31] if the group is of finite exponent then the size of a finite subgroup is bounded by a function depending on the number of the generators. Thus question (iii) is decidable in computable groups of finite exponent.

If \(G\) is \(B(m, n)\), the free Burnside group, and \(n\) is odd and sufficiently large, then by a theorem of S.V. Ivanov [17] any infinite subgroup of \(G\) contains a copy of \(B(m, n)\). Since \(B(m, n)\) are not amenable for sufficiently large odd \(n\), [1], question (i) is decidable in such \(G\).

This means that when \(n\) is large and odd and \((B(m, n), a_1, \ldots, a_m)\) (marked by the free generators) is a limit of finite marked groups \((G_i, \bar{g}_i), i \in 2\mathbb{N} + 1,\) then Theorem 3 gives a computable torsion group with undecidable question (i) of Corollary 4. It is an
open problem of geometric group theory if such a situation can be realized. V. Pestov notes in [24] (after G. Arzhantseva) that this question has positive solution provided that all hyperbolic groups are residually finite. The latter is a famous open problem of geometric group theory.

(2) In order to investigate more advanced properties of the group \( G \) from Corollary 4 it is interesting to know for which \( \alpha \) the relation \( \bar{a} \leq_\alpha \bar{b} \) from [3] is decidable in the group \( G \). In the case \( \alpha = 1 \) this relation exactly means that every existential formula satisfied by \( \bar{b} \) is satisfied by \( \bar{a} \). The authors do not even know if this relation is computably enumerable (=decidable) in free groups. We only know that there is an algorithm which decides if \( \bar{a} \) and \( \bar{b} \) belong to the same orbit. By [23] the latter is equivalent to the equality of existential types. It is worth noting that even if \( \leq_1 \) is decidable in \( F_2 \) in order to apply it to \( G \) we need a stronger result: decidability of the corresponding problem for direct sums of free groups. This problem seems to be more difficult.

(3) A relation \( R \) on a computable structure \( M \) is called intrinsically computable if it is computable in any computable copy of \( M \). The following statement is Theorem 3.1 from [2] (originally proved in [4]). It is a slightly simplified version of Theorem 2.1 from [3], which we used above.

Let \((M, R)\) be a computable structure whose existential diagram (i.e. the set of existential formulas with parameters which hold in \((M, R)\)) is computable. Then \( R \) is intrinsically computable on \( M \) if and only if both \( R \) and its complement are formally computably enumerable on \( M \).

In this formulation formally c.e. means that \( R \) is equivalent to a c.e. disjunction (possibly infinite) \( \bigvee \phi_n(\bar{x}, \bar{c}) \) of existential formulas over a tuple \( \bar{c} \). The following proposition is very close to the proof of Theorem 3.

**Proposition 5** Let \( G \) and \( R \) be the group and the relation defined in Theorem 3. Then \( \neg R \) is not formally computably enumerable on \( G \).

**Proof** Let \( \bar{c} \in G \) and let \( \bigvee \phi_n(x_1, \ldots, x_s, \bar{c}) \) be a disjunction of existential formulas of the group theory language. If \( \neg R \) is defined by this disjunction then each \( \phi_n(x_1, \ldots, x_s, \bar{c}) \) implies \( \neg R(x_1, \ldots, x_s) \). Moreover if \( t \) is an even index outside the support of \( \bar{c} \) and \( G_t \models \neg R(a_{t,1}, \ldots, a_{t,s}) \), then the tuple \( a_1, \ldots, a_s, \bar{c} \) with

\[
a_j(t) = a_{t,j} \text{ for } 1 \leq j \leq s, \text{ and } a_j(i) = 1 \text{ for } 1 \leq j \leq s \text{ and } i \neq t,
\]

realizes some \( \phi_n(x_1, \ldots, x_s, \bar{y}) \). Fixing this \( n \) assume that

\[
\phi_n(x_1, \ldots, x_s, \bar{y}) = \exists \bar{z} \phi'(x_1, \ldots, x_s, \bar{y}, \bar{z}),
\]

where \( \phi' \) is quantifier free. Let \( m \) be a number which is greater than the sum of the lengths of the words appearing in \( \phi'(x_1, \ldots, x_s, \bar{y}, \bar{z}) \). Take \( \bar{d} \) realizing \( \phi'(a_1, \ldots, a_s, \bar{c}, \bar{z}) \) in \( G \). Let \( \bar{d} \) be the projection of \( \bar{d} \) to \( G_t \). By the choice of \( G_t \)
with odd \( i \) there is a sufficiently large odd index \( \hat{t} \) outside the support of \( \mathbf{c} \mathbf{d} \) and a tuple
\[ a_{t,1}', \ldots, a_{t,s}', \hat{1}, \hat{d}' \in G_{t} \]
such that the map
\[ w(a_{t,1}, \ldots, a_{t,s}, \hat{1}, \hat{d}) \rightarrow w(a_{t,1}', \ldots, a_{t,s}', \hat{1}, \hat{d}') \]
where \( |w(x_{1}, \ldots, x_{s}, \bar{y}, \bar{z})| \leq m \),
induces an isomorphism from the substructure of the Cayley graph of \( G_{t} \) with respect to the generators \( a_{t,1}, \ldots, a_{t,s}, \hat{d} \) to the corresponding substructure of \( (G_{t}', a_{t,1}', \ldots, a_{t,s}', \hat{d}') \). Since \( G_{t} \models \phi'(a_{t,1}, \ldots, a_{t,s}, \hat{1}, \hat{d}) \), we have \( G_{\hat{t}} \models \phi' (a_{t,1}', \ldots, a_{t,s}', \hat{1}, \hat{d}') \). Let us define
\[ a'_{j}(\hat{t}) = a'_{\hat{t},j} \text{ for } 1 \leq j \leq s, \text{ and } a'_{j}(i) = 1 \text{ for } 1 \leq j \leq s \text{ and } i \neq \hat{t}, \text{ and } \]
\[ \bar{d}'(\hat{t}) = \bar{d}', \bar{d}'(t) = \bar{1}, \bar{d}'(i) = \bar{d}(i) \text{ for } i \notin \{\hat{t}, t\}. \]

Then the tuple \( a_{1}', \ldots, a_{s}', \bar{c}, \bar{d}' \) satisfies \( \phi'(x_{1}, \ldots, x_{s}, \bar{y}, \bar{z}) \) and the tuple \( a_{1}', \ldots, a_{s}', \bar{c} \) satisfies \( \phi_{n}(x_{1}, \ldots, x_{s}, \bar{y}) \) in \( G \). Since \( a_{1}', \ldots, a_{s}' \) satisfies \( R \), we obtain a contradiction. 

This proposition suggests that the group \( G \) and the relation \( R \) from the proof of Corollary 4(i) also satisfy the conditions of Theorem 3.1 from [2] (they are stronger than ones used in the proof of Corollary 4). Furthermore, according [4] to get the conclusion of Theorem 3.1 from [2] one can weaken decidability of the the existential diagram of the structure \((G, \cdot, 1, R)\) (this is the only remaining task) by decidability of the set of existential formulas of the diagram which contain only one \( R \)-literal. However the authors were not able to prove either of these conditions. The attempts which were made lead us to the following problem.

Is there a family of finite two-generated groups
\[ G = \{G_{l} = \langle a_{l}, b_{l} \rangle : l \in \omega \} \]
such that the universal (or elementary) theory of \( G \) is decidable and \((F_{2}, a, b)\) is a limit group of this family in the Grigorchuk’s topology?

It is worth noting that the elementary theory of \((F_{2}, a, b)\) is decidable [19].

### 3 A finitely presented group with undecidable amenability

The following theorem is based on Corollary 4. We will also use Theorem 1 of [14] (see also [15]).

**Theorem 6** There is a finitely presented group \( H_{nA} \) with decidable word problem such that detection of finite subsets of \( H_{nA} \) which generate amenable subgroups is not decidable. Detection of finite subsets of \( H_{nA} \) which generate subgroups which do not embed free non-abelian groups is also not decidable. Moreover, the corresponding finitely presented example \( H_{nT} \) can be found for properties (iii) and (iv) of Corollary 4.
Proof} Let $G_{nA}$ be a computable group given by Corollary 4. Our construction starts with the following statement of A. Darbinyan.

- For any computable group $G$ there is a two-generated group $G_1 = \langle c, s \rangle$ with decidable word problem and a computable embedding $\phi_1 : G \rightarrow G_1$.

This is a part of the statement of Theorem 1 of [14] and the definition of $\phi_1$ at line 19 on p.4927 of that paper. We only add that applying this theorem

- we consider our $G$ with respect to the generators given by any enumeration of $G = \{a^{(1)}, a^{(2)}, \ldots\}$,
- $\phi_1$ is denoted by $\phi$ in [14], and
- the definition of $\phi$ in [14] should be corrected by $a^{(i)} \rightarrow [c, c^{2^i-1}]$.

We apply this theorem to $G = G_{nA}$. Then it is clear that detection of finite subsets of $G_1$ which generate amenable subgroups is not decidable. Theorem 6 of [13] states that

- a finitely generated group $G_1$ with decidable word problem can be embedded into a finitely presented group with decidable word problem.

Applying this theorem we obtain the group $H_{nA}$ as in the formulation. Indeed the embedding $\phi_2 : G_1 \rightarrow H_{nA}$ is obviously defined by the restriction of $\phi_2$ to the finite set of generators of $G_1$. Thus $\phi_2$ is computable. Having this we see that the problem if a finite subset of $H_{nA}$ generates an amenable subgroup is not decidable.

The same argument can be applied to properties (iii) - (iv) of Corollary 4. \qed

Acknowledgements The authors are grateful to M. Cavaleri, T. Ceccherini-Silberstein and L. Kołodziejczyk for reading the paper and helpful remarks. We also thank M. Sapir for right advice concerning Higman’s embedding. The authors are grateful to the referee for remarks which substantially improved the exposition.

Funding This research was partially supported by (Polish) Narodowe Centrum Nauki, Grant UMO–2018/30/M/ST1/00668.

Availability of data and materials Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Code Availability Not applicable.

Declarations

Conflict of interest Not applicable.

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