Existence of Static Dyonic Black Holes in $4d\; N = 1$ Supergravity With Finite Energy

Fiki T. Akbar$^\sharp$ and Bobby E. Gunara$^{\flat,\sharp}$

$^\flat$Indonesian Center for Theoretical and Mathematical Physics (ICTMP)

and

$^\sharp$Theoretical Physics Laboratory

Theoretical High Energy Physics and Instrumentation Research Group,
Faculty of Mathematics and Natural Sciences,
Institut Teknologi Bandung
Jl. Ganesha no. 10 Bandung, Indonesia, 40132
email: ftakbar@fi.itb.ac.id, bobby@fi.itb.ac.id

Abstract

We prove the existence and the uniqueness of the static dyonic black holes in four dimensional $N = 1$ supergravity theory coupled vector and scalar multiplets. We set the near-horizon geometry to be a product of two Einstein surfaces, whereas the asymptotic geometry has to be a space of constant scalar curvature. Using these data, we show that there exist a unique solution for scalar fields which interpolates these regions.

1 Introduction

Four dimensional static spacetimes, which are closely related to the spherical symmetric black holes, have been intensely studied because it may be a simple model and useful to test an extension theory of Einstein’s general relativity. For example, a supersymmetric extension of Einstein’s general relativity called $N = 1$ supergravity is of interest to consider since in the rigid limit it may provide a unification theory.

Recently, some authors considered a class of extremal spherical symmetric black hole solutions of four dimensional $N = 1$ supergravity coupled to vector and chiral multiplets with the scalar potential turned on $[\Pi]$. These black holes are non-supersymmetric. The boundaries of the black holes are set to be spaces of constant scalar curvature. Near the horizon, the geometry becomes a product space formed by two Einstein surfaces, namely two-anti de Sitter surface (AdS$_2$) and two-sphere $S^2$. While, in the asymptotic region the geometry turns to a space of constant scalar curvature which is not Einstein.

At the boundaries the complex scalars are frozen and regular. If the value of the scalars in both regions coincides, then both asymptotic and near-horizon geometries may correspond to each other. This can be achieved if the Arnowitt-Deser-Misner (ADM)
mass is extremized which implies that the scalar charges vanish \[1\].

In this paper we prove locally and globally the existence of static dyonic black holes of \(N = 1\) local supersymmetry (supergravity) with general coupling of vector and scalar multiplets in four dimensions. These black holes are generally non-extremal and non-supersymmetric. We use the same setting for the geometry of the boundaries of the black holes as the previous case mentioned above. Our result here also covers the non-supersymmetric case of \(N = 1\) theory \[2\] and could be applied to the supersymmetric black holes in \(N = 2\) theory \[3\, 4\].

To prove the local existence and the uniqueness, we use a contraction mapping theorem to the complex scalar field equations of motions. By defining an integral operator, the existence and the uniqueness can be achieved by showing that the non-linear parts satisfy the local Lipshitz condition. This can be done by taking several assumptions in the following: First, both Kähler potential and Christoffel symbol are bounded above by \(U(n)\) symmetric Kähler potential and its Christoffel symbol, respectively. These estimates can be used to eliminate the quantity associated with Kähler metric in our analysis. Second, the potentials, namely the scalar potential and the black hole potential have to be at least \(C^2\)-functions and their derivative is locally Lipshitz function. Finally, the spacetime metric is at least a \(C^2\)-function.

The second step is to prove the global existence by showing that the energy functional is bounded above a positive constant between the boundaries. We also take that all functions namely the scalar fields, the potentials, and the spacetime metric have to be at least a \(C^2\)-function. After some computations, we find that the scalar potential should be vanished in the asymptotic region.

The organization of this paper can be mentioned as follows. In section 2 we review shortly \(N = 1\) supergravity coupled with vector and chiral multiplets in four dimensions. We describe some aspects of static spacetimes whose boundaries are spaces of constant scalar curvature in section 3. In section 4 we prove the local existence of radius dependent scalar fields which follows that such static black holes do exist in any compact interval. Finally, we employ an energy functional analysis to show the solutions could be globally defined in section 5.

2 \(N = 1\) Supergravity Coupled with Vector and Chiral Multiplets

In this section, we give a short description about four dimensional \(N = 1\) supergravity coupled with vector and chiral multiplets. Here, we only write some useful terms for our analysis in the paper. For an excellent review, interested reader can further consult, for example, \[5\, 6\].

The theory consists of a gravitational multiplet \((e^A_\mu, \psi_\mu)\), \(nC\) chiral multiplet \((\phi^i, \chi^i)\) and \(nV\) vector multiplets \((A^a_\mu, \lambda^a)\) where Latin alphabets \(a, b = 1, ..., n_v\), and \(i, j = 1, ..., n_c\) show the number of multiplets and the Greek alphabets \(\mu, \nu = 0, ..., 3\) and \(\Lambda, \Sigma = 0, ..., 3\) show the spacetime and tangent space index respectively. Here \(e^A_\mu\), \(A^a_\mu\), and \(\phi\) are a vierbein, a gauge field, and a complex scalar, respectively, while \(\psi_\mu\), \(\lambda^a\), and \(\chi^a\) are the fermion fields.

Furthermore, the complex scalar fields span a \(2n_c\)-dimensional Hodge-Kähler manifold \(\mathcal{M}^{2n_c}\) endowed with metric \(g_{ij} = \partial_i \partial_j K(\phi, \bar{\phi})\) and a \(U(1)\) connection is defined by \(Q \equiv \)
for spacetime metric, In this section we set a scenario as follows. First of all, we consider the following ansatz

\[ \frac{1}{\sqrt{-g}} \mathcal{L} = -\frac{1}{2} R + h_{ab} \mathcal{F}^a_{\mu
u} \mathcal{F}^{b\mu\nu} + k_{ab} \mathcal{F}^a_{\mu\nu} \tilde{F}^{b\mu\nu} + g_{ij} (\phi^i \partial_{\mu} \phi^j \partial^\mu \phi^j) - V(\phi, \bar{\phi}) \]  

where \( V_S \) is the real function called the scalar potential which given by,

\[ V(\phi, \bar{\phi}) = e^K \left( g^{ij} \nabla_i W \nabla_j W - 3W \bar{W} \right) \]  

The quantity \( R \) is the Ricci scalar of four dimensional spacetime, whereas \( \mathcal{F}^a_{\mu\nu} \) is an Abelian field strength of \( A^a_\mu \), and \( \tilde{F}^b_{\mu\nu} \) is a Hodge dual of \( \mathcal{F}^a_{\mu\nu} \). The function \( h_{ab} \) and \( k_{ab} \) are the real and imaginary part of \( f_{ab} \) respectively and \( \nabla_i W = \partial_i W + K_i W \) is a covariant derivative in internal manifold, \( \mathcal{M}^{2\text{nc}} \). It is worth to mention that Lagrangian (2.1) is invariant under a set of supersymmetry transformations of the fields discussed in [5, 6].

The bosonic equations of motions can be obtained by varying the Lagrangian (2.1) with respect to \( g_{\mu\nu}, A^a_\mu, \) and \( \phi^i \), and by setting all the fermions are vanish at the level of equation of motion. The first equation is namely the Einstein field equation,

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = g_{ij} \left( \partial_\mu \phi^i \partial_\nu \phi^j + \partial_\nu \phi^i \partial_\mu \phi^j \right) - g_{ij} g_{\mu\nu} \partial_\rho \phi^i \partial_\rho \phi^j + 4 h_{ab} \mathcal{F}^a_{\mu\rho} \mathcal{F}^b_{\nu\sigma} - g_{\mu\nu} h_{ab} \mathcal{F}^a_{\rho\sigma} \mathcal{F}^{b\rho\sigma} + g_{\mu\nu} V \]  

The second equation is the equation for gauge fields,

\[ \partial_\nu \left( \varepsilon^{\mu\nu\rho\sigma} \sqrt{-g} \mathcal{G}_{a\rho\sigma} \right) = 0 \]  

where

\[ \mathcal{G}_{a\rho\sigma} = k_{ab} \mathcal{F}^b_{\rho\sigma} - h_{ab} \tilde{F}^b_{\rho\sigma} \]  

is an electric field strengths tensor. Finally, we have the scalar field equations,

\[ \frac{g_{ij}}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi^j \right) + \partial_\kappa g_{ij} \partial_\nu \phi^j \partial_\nu \phi^k = \partial_\rho h_{ab} \mathcal{F}^a_{\mu\nu} \mathcal{F}^{b\mu\nu} - \partial_\rho k_{ab} \mathcal{F}^a_{\mu\nu} \tilde{F}^{b\mu\nu} - \partial_\rho V \]  

with \( g \equiv \det(g_{\mu\nu}) \). In addition, we have a Bianchi identity

\[ \partial_\nu \left( \varepsilon^{\mu\nu\rho\sigma} \sqrt{-g} \mathcal{F}^a_{\rho\sigma} \right) = 0 \]  

comes from the definition of \( \mathcal{F}^a_{\rho\sigma} \).

### 3 General Setting: Static Spacetimes

In this section we set a scenario as follows. First of all, we consider the following ansatz for spacetime metric,

\[ ds^2 = -e^{A(r)} dt^2 + e^{B(r)} dr^2 + e^{C(r)} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]  

\[ 3 \]
which is static and has a spherical symmetry. The functions \( A(r) \), \( B(r) \), and \( C(r) \) are smooth real functions. Among the functions \( A(r) \), \( B(r) \), and \( C(r) \), only two of them are independent since one can redefine the radial coordinate \( r \) to absorb one of them.

The solution for the gauge field equations, (2.4), and the Bianchi identity, (2.7), can be obtained by simply taking the case where the only non-zero component of the field strength tensor are \( F^a_{01}(r) \) and \( F^a_{23}(\theta) \). The solutions are,

\[
F^a_{01} = \frac{1}{2} e^{\frac{1}{2}(A+B) - C} (h^{-1})^{ab} (k_{bc} g^{ce} - q_b)
\]

\[
F^a_{23} = -\frac{1}{2} g^a \sin \theta ,
\]

with \( q_a \) and \( g^a \) are electric and magnetic charges, respectively \([7]\). Then, by assuming that the scalar field is function of radial coordinate only, the scalar field equations, (2.6), can be written as

\[
g_{ij} \bar{\phi}_j'' + \bar{\partial}_k g_{ij} \bar{\phi}_j' \bar{\phi}_k' + \frac{1}{2} (A' - B' + 2C') g_{ij} \bar{\phi}_j' = e^B \left( e^{-2C} \bar{\partial}_k V_{BH} + \bar{\partial}_i V \right) ,
\]

with \( \nu' \equiv d\nu/dr \) denotes the derivative respect to radial coordinate. The function \( V_{BH} \) is called the black hole potential \([4, 8]\) and it has the form

\[
V_{BH} \equiv -\frac{1}{2} (g q) \mathcal{M} \left( \begin{array}{c} g \\ q \end{array} \right) ,
\]

where

\[
\mathcal{M} = \left( \begin{array}{cc} h + k h^{-1} & -k h^{-1} \\ -h^{-1} k & h^{-1} \end{array} \right) .
\]

Now we can construct a class of solutions of equations (2.3) and (3.3) for particular regions, namely near asymptotic and near horizon regions discussed in \([1]\). Around asymptotic region, the scalar fields are frozen, namely

\[
\lim_{r \to \infty} \phi_i'' = 0 ,
\]

\[
\lim_{r \to \infty} \phi_i' = \phi_0' ,
\]

which implies that both the black hole potential and the scalar potential become constant. The black hole solutions can be written as,

\[
ds^2 = -\Lambda(r) dt^2 + \Lambda^{-1}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,
\]

with

\[
\Lambda(r) \equiv 1 + \frac{2\eta}{r} + \frac{V^0_{BH}}{r^2} - \frac{1}{3} V_0 \nu^2 ,
\]

and

\[
V^0_{BH} \equiv V_{BH}(\phi_0, \bar{\phi}_0) , \quad V_0 \equiv V(\phi_0, \bar{\phi}_0) .
\]

Since \( \Lambda(r) \) is a positive definite function, the geometries of the black hole solution (3.7) has a constant scalar curvature which are neither Einstein nor symmetric space. Furthermore,
the scalar fields can be written down as,

$$
\phi^i = \phi^i_0 + \frac{\Sigma^i}{r} + \left( P(r) \left( g^{ij} \partial_j V_{BH} \right)_{\phi_0} + Q(r) \left( g^{ij} \partial_j V \right)_{\phi_0} \right) ,
$$

$$
\phi^{\prime i} = -\frac{\Sigma^i}{r^2} + \left( P'(r) \left( g^{ij} \partial_j V_{BH} \right)_{\phi_0} + Q'(r) \left( g^{ij} \partial_j V \right)_{\phi_0} \right) ,
$$

(3.10)

where $\Sigma^i$ are the scalar charges introduced by [9]. The functions $P(r)$ and $Q(r)$ are

$$
P(r) = \frac{1}{r} \int \left( \int \frac{A^{-1}}{r^3} \, dr \right) \, dr ,
$$

$$
Q(r) = \int \left( \int r \, A^{-1} \, dr \right) \, dr .
$$

(3.11)

In order to evade the singularity in (3.11), the frozen scalars should be critical points of both the black hole and scalar potentials, namely

$$
(\partial_i V_{BH})_{\phi_0} = 0 ,
$$

$$
(\partial_i V)_{\phi_0} = 0 .
$$

(3.12)

On the other hand, in the near-horizon limit the scalar fields are also frozen with respect to radial coordinates

$$
\lim_{r \to r_h} \phi^i \to \phi^i_h ,
$$

$$
\lim_{r \to r_h} \phi^{\prime i} \to 0 ,
$$

(3.13)

and in general $\phi^i_h \neq \phi^i_0$ which can further be viewed as the critical points of the so-called effective black hole potential $V_{\text{eff}}$ which is a function of both the black hole potential $V_{BH}$ and the scalar potential $V$

$$
V_{\text{eff}} \equiv 1 - \sqrt{1 - 4V_{BH}V} .
$$

(3.14)

Here, the black hole geometries are the product of a two dimensional surface $M^{1,1}$ and the two-sphere $S^2$ with radius $r_h \equiv r_h(g,q)$. It is worth mentioning that the near horizon solution of the scalar fields equation (3.3) has the form

$$
\bar{\phi}^{\prime i} = \frac{\ell^{-1}}{(r - r_h)^2} \left( g^{ij} \partial_{\phi^i} V_{\text{eff}}(p_h) \right) ,
$$

$$
\bar{\phi}^i = \bar{\phi}^i_h - \ell^{-1} \ln |r - r_h| \left( g^{ij} \partial_{\bar{\phi}^i} V_{\text{eff}}(p_h) \right) ,
$$

(3.15)

where $p_h \equiv (\phi_h, \bar{\phi}_h)$ and

$$
\ell^{-1} = \frac{V_{\text{eff}}^h}{\sqrt{1 - 4V_{BH}^h V_h}} ,
$$

(3.16)

is a constant which shows that $\phi^i_h$ are indeed the critical points of $V_{\text{eff}}$ in order to have a regular solution at the region. The Killing vector analysis at the horizon gives us an evidence that $M^{1,1}$ must be $\text{AdS}_2$ [10].
4 Local Existence

In this section we prove the local existence of non-trivial radius dependent solutions of the scalar equations of motions (3.3). This class of solutions interpolates between the two regions, namely the horizon and asymptotic regions.

Let $\mathcal{M}^{2n_c}$ be a $2n_c-$dimensional Kähler manifold spanned by scalar fields $(\phi^i, \bar{\phi}^i)$ with Kähler potential $K = K(\phi, \bar{\phi})$. In this paper, we consider the case where the Kähler potential of $\mathcal{M}^{2n_c}$ is bounded above by a $U(n_c)$ symmetric Kähler potential and satisfies several conditions,

$$K \leq \Phi(|\phi|), \quad (4.1)$$
$$|\Gamma| \leq |\tilde{\Gamma}|, \quad (4.2)$$

where $|\phi| = (\delta_{ij}\phi^i\phi^j)^{1/2}$ and $\tilde{\Gamma}$ is the Christoffel symbol of $\tilde{g}$. Then we have the following lemma [11].

**Lemma 1.** Let $\mathcal{M}^{2n_c}$ be a Kähler manifold with Kähler potential $K = K(\phi, \bar{\phi})$. If $\mathcal{M}^{2n_c}$ satisfies (4.1), (4.2) and

$$F(|\phi|) \equiv \frac{1}{4|\phi|^2} \left( \Phi'' - \frac{\Phi'}{|\phi|} \right)$$

with $\Phi' = \partial \Phi / \partial |\phi|$ and $\epsilon$ is a non negative constant, then we have the following estimates

$$|K| \leq \epsilon/6 |\phi|^6 + C_1 |\phi|^4 + C_2 |\phi|^2 + C_3, \quad (4.4)$$
$$|\Gamma| \leq 2\epsilon |\phi|^3 + C_1 |\phi|. \quad (4.5)$$

Let $\phi : I \subset \mathbb{R} \to \mathcal{M}^{2n_c}$ be a curve in $\mathcal{M}^{2n_c}$ satisfying differential equations,

$$\bar{\phi}^i'' + F(r)\bar{\phi}^i' = -\Gamma^i_{jk} \bar{\phi}^j \bar{\phi}^k + g^{i\bar{k}} (G(r)\partial_k V_{BH} + H(r)\partial_k V), \quad (4.6)$$

where $F(r) \equiv \frac{1}{2} (A' - B' - 2C')$, $G(r) \equiv e^{B-2C}$ and $H(r) \equiv e^{B}$. We assume that the functions $F(r), G(r)$ and $H(r)$ are at least $C^2$- real functions. Introducing the new fields,

$$\pi^i = \phi^i', \quad \bar{\pi}^i = \bar{\phi}^i', \quad (4.7)$$

then we can write equation (4.6) as a first order equations,

$$\bar{\phi}^i' = \pi^i, \quad \bar{\pi}^i = F(r)\bar{\pi}^i = -\Gamma^i_{jk} \bar{\pi}^j \pi^k + g^{i\bar{k}} (G(r)\partial_k V_{BH} + H(r)\partial_k V), \quad (4.8)$$

together with their complex conjugate.

Let $u = (z^i, \pi^i)$, then we can write equation (4.8) as

$$\frac{d\bar{u}}{dr} = \mathcal{J}(u, \bar{u}, r), \quad (4.9)$$

with

$$\mathcal{J}(u, \bar{u}, r) = \begin{bmatrix} -F(r)\bar{\pi}^i - \Gamma^i_{jk} \bar{\pi}^j \pi^k + g^{i\bar{k}} (G(r)\partial_k V_{BH} + H(r)\partial_k V) \end{bmatrix}. \quad (4.10)$$
The local existence and uniqueness of the equation (4.9) is established using contraction mapping theorem by showing the operator $\mathcal{J}$ is a locally Lipshitz. Let $J \equiv [r_h, r_h + \delta] \subset I$ be a real interval with $\delta$ is a small real constant and $U \subset T\mathcal{M}^{2\kappa_c}$ is an open set. We prove the following lemma,

**Lemma 2.** Let $\mathcal{J}$ be an operator defined by (4.10). If the potentials $V$ and $V_{BH}$ are at least a $C^2$ function and satisfying,

$$
|\partial_k V_{BH}(\tilde{\phi}) - \partial_k V_{BH}(\phi)| \leq C_4 |\tilde{\phi} - \phi|, \\
|\partial_k V(\tilde{\phi}) - \partial_k V(\phi)| \leq C_5 |\tilde{\phi} - \phi|,
$$

(4.11)

then the operator $\mathcal{J}$ is a locally Lipshitz with respect to $u$.

**Proof.** From definition of the operator $\mathcal{J}$ in equation (4.10), we have the following estimate,

$$
|\mathcal{J}(r, u(r))|_U \leq |\pi(r)| + |F(r)||\pi(r)| + \left(1 + |\Gamma(\phi(r), \tilde{\phi}(r))| + 1\right) |\pi(r)|^2 \\
+ |G(r)| |\partial_k V_{BH}| + |H(r)| |\partial_k V|.
$$

(4.12)

Since $\mathcal{M}^{2\kappa_c}$ satisfying lemma [1] then using equation (4.1), we can write (4.12) as,

$$
|\mathcal{J}(r, u(r))|_U \leq (|F(r)| + 1) |\pi(r)| + \left(2\epsilon |\phi(r)|^2 + C_1 |\phi(r)| + 1\right) |\pi(r)|^2 \\
+ |G(r)| |\partial_k V_{BH}(\tilde{\phi}(r)) - \partial_k V_{BH}(\phi(r))| \\
+ |H(r)||\partial_k V(\tilde{\phi}(r)) - \partial_k V(\phi(r))|.
$$

(4.13)

Since $F(r), G(r), H(r)$ are at least a $C^2$ real function, then their value are bounded on any closed interval. Hence, $|\mathcal{J}(r, u(r))|_U$ is bounded on $J$.

Furthermore, for $u, \tilde{u} \in U$, we have the following estimate,

$$
|\mathcal{J}(r, \tilde{u}(r)) - \mathcal{J}(r, u(r))|_U \leq (|F(r)| + 1) |\tilde{\phi}(r) - \phi(r)| + |\pi(r)|^2|\Gamma - \Gamma| \\
+ (|\Gamma'|)|\tilde{\pi}(r) + \pi(r)| + 1|\tilde{\pi}(r) - \pi(r)| \\
+ |G(r)||\partial_k V_{BH}(\tilde{\phi}(r)) - \partial_k V_{BH}(\phi(r))| \\
+ |H(r)||\partial_k V(\tilde{\phi}(r)) - \partial_k V(\phi(r))|.
$$

(4.14)

Using equation (4.1) and (4.11), we can write the equation (4.14) as

$$
|\mathcal{J}(r, \tilde{u}(r)) - \mathcal{J}(r, u(r))|_U \leq \left\{ |F(r)| + C_4 |G(r)| + C_5 |H(r)| + |\pi(r)|^2 \\
\left(2\epsilon |\tilde{\phi}(r)| |\phi(r) + \phi(r)| + |\phi(r)|^2 + C_1\right) |\tilde{\phi}(r) - \phi(r)| \\
+ \left\{ 2\epsilon |\tilde{\phi}(r)|^3 + C_1 |\phi(r)|\right\} |\tilde{\pi}(r) + \pi(r)| + 1\right\} |\tilde{\pi}(r) - \pi(r)|.
$$

(4.15)

Hence, we have

$$
|\mathcal{J}(r, \tilde{u}) - \mathcal{J}(r, u)|_U \leq C(|\tilde{u}|, |u|) |\tilde{u} - u|.
$$

(4.16)

Equation (4.16) proves that $\mathcal{J}$ is a locally Lipshitz with respect to $u$. \hspace{1cm} \Box
Write equation (4.9) in form of integral equation,
\[ \mathbf{u}(r) = \mathbf{u}(r_h) + \int_{r_h}^{r} J(s, \mathbf{u}(s), s) \, ds. \] (4.17)

Let
\[ X = \{ u \in C(J, T^N_M) : \mathbf{u}(r_h) = \mathbf{u}_0, \sup_{r \in J} |u(r)| \leq M \}, \] (4.18)
equipped with the norm,
\[ \| u \|_X = \sup_{r \in J} |u(r)|. \] (4.19)

Introducing an operator \( K \) which defined as follow,
\[ K(\mathbf{u}(r)) = \mathbf{u}_0 + \int_{r_h}^{r} ds J(s, \mathbf{u}(s)). \] (4.20)

Based on equation (4.17), if \( \mathbf{u}(r) \) is a solution of the differential equations (4.9), then \( \mathbf{u}(r) \) is a fixed point of operator \( K \). The existence and the uniqueness of the fixed point of the operator is guaranteed by the contraction mapping principle.

In the following lemma, we prove that the operator \( K \) is a mapping from \( X \) to itself and it is a contraction mapping.

**Lemma 3.** Let \( K \) be an operator defined by equation (4.20). There is a positive constant \( \delta \) such that \( K \) is a mapping from \( X \) to itself and \( K \) is a contraction mapping on \( J = [r_h, r_h + \delta] \).

**Proof.** From definition of the operator \( K \) in equation (4.20), we have the following estimate,
\[ \| K(\mathbf{u}) \|_X \leq \| \mathbf{u}_0 \|_X + \sup_{r \in J} \int_{r_h}^{r} ds |J(s, \mathbf{u}(s))| \]
\[ \leq \| \mathbf{u}_0 \|_X + \sup_{r \in J} (|J(r_h)| + CM |\mathbf{u}|) (r - r_h) \]
\[ \leq \| \mathbf{u}_0 \|_X + \delta (|J(r_h)| + MC_M). \] (4.21)

If we choose,
\[ \delta \leq \frac{1}{CM} \frac{1}{CM_M + \| J(r_h) \|}, \] (4.22)
then \( K(\mathbf{u}) \) is a mapping from \( X \) to itself.

Furthermore, for \( \mathbf{u}, \mathbf{u} \in X \) we have,
\[ \| K(\mathbf{u}) - K(\mathbf{u}) \|_X \leq \sup_{r \in J} \int_{r_h}^{r} ds |J(s, \mathbf{u}(s)) - J(s, \mathbf{u}(s))| \]
\[ \leq \sup_{r \in J} \sup_{0 \leq s \leq r} r |J(s, \mathbf{u}(s)) - J(s, \mathbf{u}(s))| \]
\[ \leq C_M \delta \| \mathbf{u} - \mathbf{u} \|_X. \] (4.23)

Since \( \delta \) satisfies the inequality (4.22), then the operator \( K \) is a contraction mapping.

Then, by contraction mapping theorem, the operator \( K \) admits a unique fixed point. Hence, for each initial value, there exist a unique local solution of the differential equation (4.9).
5 Finite Energy Conditions

In order to have global solutions in \( J_\infty \equiv [r_h, +\infty) \), we need to consider finite energy conditions which ensure the existence of such solutions. This aspect is discussed in this section as follows.

Let us first turn to the component of the tensor energy momentum

\[
T_{00} = e^{A-B} g^{ij} \phi'^i \phi'^j + e^{A-2C} V_{BH} + e^A V ,
\]

whose energy functional is given by

\[
E = \int \sqrt{-g} T_{00} \, d^3 x = 4\pi \int_{r_h}^{+\infty} e^{B+A+C} \left( e^{-B} g^{ij} \phi'^i \phi'^j + e^{-2C} V_{BH} + V \right) dr .
\]

Defining \( J_L \equiv [r_h, L] \) and \( J_A \equiv [L, +\infty) \) for finite and large \( L > r_h \). Suppose all functions \( A(r), B(r), \) and \( C(r) \) and all scalar fields \( \phi(r) \) together with potentials \( (V, V_{BH}) \) are at least \( C^2 \)-function. On \( J_A \) we have \( A(r) = -B(r) = \ln\Lambda(r) \), and \( C(r) = 2 \ln r \) where \( \Lambda(r) \) is given in (3.8). Moreover, the scalar fields \( \phi(r) \) tend to have the form (3.10) and the potentials \( (V, V_{BH}) \) become fixed, namely \( (V_0, V_{BH}^0) \). It turns out that in order to have finite energy, on \( J_A \) the scalar potential \( V \) should be vanished. So, the energy (5.2) has to be

\[
E \leq 4\pi \sup_{r \in J_L} \int_{r_h}^L e^{B+A+C} \left( e^{-B} g^{ij} \phi'^i \phi'^j + e^{-2C} V_{BH} + V \right) dr + C_0 (L, \Sigma, \phi_0, \tilde{\Sigma}, \tilde{\phi}_0) .
\]

for \( C_0 > 0 \) and \( \Sigma^i \) are the scalar charges. The first term in the right hand side of (5.3) is also bounded due to the \( C^2 \)-smoothness of all functions, fields, and the potentials. This shows that \( E \) is bounded above a positive constant.

Therefore, we have proven,

**Theorem 1.** Let \( M^{2nc} \) be a Kähler manifold spanned by scalar fields with potential Kähler \( K \). Let \( V \) and \( V_{BH} \) are scalar and black hole potentials defined by equations (2.2) and (3.4), respectively. If \( M^{2nc} \) satisfies lemma (7) and \( V \) and \( V_{BH} \) satisfy condition (4.11), then for each initial value \( u_0 \), there is positive constant \( \delta \) such that the differential equation (4.9) admits a unique local solution on \( [r_h, r_h + \delta] \). In particular, this local solution interpolates between two region, namely horizon and asymptotic regions, if the scalar potential \( V \) vanishes in the asymptotic region.

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