Hessenberg Pairs of Linear Transformations

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Abstract

Let $K$ denote a field and $V$ denote a nonzero finite-dimensional vector space over $K$. We consider an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i)–(iii) below.

(i) Each of $A, A^*$ is diagonalizable on $V$.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that

$$A^*V_i \subseteq V_0 + V_1 + \ldots + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that

$$AV_i^* \subseteq V_0^* + V_1^* + \ldots + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

We call such a pair a \textit{Hessenberg pair} on $V$. In this paper we obtain some characterizations of Hessenberg pairs. We also explain how Hessenberg pairs are related to tridiagonal pairs.

Keywords: Leonard pair, tridiagonal pair, $q$-inverting pair, split decomposition.

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1 Introduction

In [1] Definition 1.1 Ito, Tanabe and Terwilliger introduced the notion of a \textit{tridiagonal pair} of linear transformations. Loosely speaking, this is a pair of diagonalizable linear transformations on a nonzero finite-dimensional vector space, each of which acts on the eigenspaces of the other in a certain restricted way. In [1] Theorem 4.6 Ito et. al. showed that a tridiagonal pair induces a certain direct sum decomposition of the underlying vector space, called the \textit{split decomposition} [1, Definition 4.1]. In order to clarify this result, in the present paper we introduce a generalization of a tridiagonal pair called a \textit{Hessenberg pair}. Our main results are summarized as follows. Let $V$ denote a nonzero finite-dimensional vector space, and let $(A, A^*)$ denote a pair of diagonalizable linear transformations on $V$. We show that if $(A, A^*)$ induces a split decomposition of $V$, then $(A, A^*)$ is a Hessenberg pair on $V$. Moreover the
converse holds provided that $V$ has no proper nonzero subspaces that are invariant under each of $A$, $A^*$. 

The rest of this section contains precise statements of our main definitions and results. We will use the following terms. Let $\mathbb{K}$ denote a field and $V$ denote a nonzero finite-dimensional vector space over $\mathbb{K}$. By a linear transformation on $V$, we mean a $\mathbb{K}$-linear map from $V$ to $V$. Let $A$ denote a linear transformation on $V$ and let $W$ denote a subspace of $V$. We call $W$ an eigenspace of $A$ whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that

$$W = \{v \in V \mid Av = \theta v\}.$$ 

In this case $\theta$ is called the eigenvalue of $A$ corresponding to $W$. We say $A$ is diagonalizable whenever $V$ is spanned by the eigenspaces of $A$.

**Definition 1.1.** By a Hessenberg pair on $V$, we mean an ordered pair $(A, A^*)$ of linear transformations on $V$ that satisfy (i)–(iii) below.

(i) Each of $A, A^*$ is diagonalizable on $V$.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that

$$A^*V_i \subseteq V_0 + V_1 + \ldots + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

(iii) There exists an ordering $\{V^*_i\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that

$$AV^*_i \subseteq V^*_0 + V^*_1 + \ldots + V^*_{i+1} \quad (0 \leq i \leq \delta),$$

where $V^*_{-1} = 0$, $V^*_{\delta+1} = 0$.

**Note 1.2.** It is a common notational convention to use $A^*$ to represent the conjugate-transpose of $A$. We are not using this convention. In a Hessenberg pair $(A, A^*)$ the linear transformations $A$ and $A^*$ are arbitrary subject to (i)–(iii) above.

**Note 1.3.** The term Hessenberg comes from matrix theory. A square matrix is called upper Hessenberg whenever each entry below the subdiagonal is zero [2, p. 28].

Referring to Definition 1.1, the orderings $\{V_i\}_{i=0}^d$ and $\{V^*_i\}_{i=0}^\delta$ are not unique in general. To facilitate our discussion of these orderings we introduce some terms. Let $(A, A^*)$ denote an ordered pair of diagonalizable linear transformations on $V$. Let $\{V_i\}_{i=0}^d$ (resp. $\{V^*_i\}_{i=0}^\delta$) denote any ordering of the eigenspaces of $A$ (resp. $A^*$). We say that the pair $(A, A^*)$ is Hessenberg with respect to $\{V_i\}_{i=0}^d$ (resp. $\{V^*_i\}_{i=0}^\delta$) whenever these orderings satisfy (1) and (2). Often it is convenient to focus on eigenvalues rather than eigenspaces. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^\delta$) denote the ordering of the eigenvalues of $A$ (resp. $A^*$) that corresponds to $\{V_i\}_{i=0}^d$ (resp. $\{V^*_i\}_{i=0}^\delta$). We say that the pair $(A, A^*)$ is Hessenberg with respect to $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^\delta$) whenever $(A, A^*)$ is Hessenberg with respect to $\{V_i\}_{i=0}^d$ (resp. $\{V^*_i\}_{i=0}^\delta$).
Definition 1.4. Let \((A, A^*)\) denote an ordered pair of linear transformations on \(V\). We say that the pair \((A, A^*)\) is irreducible whenever there is no subspace \(W\) of \(V\) such that \(AW \subseteq W\), \(A^*W \subseteq W\), \(W \neq 0\), \(W \neq V\).

We are primarily interested in the irreducible Hessenberg pairs. However for parts of our argument the irreducibility assumption is not needed.

As we will see in Proposition 2.4, for an irreducible Hessenberg pair the scalars \(d\) and \(\delta\) from Definition 1.1 are equal.

We now turn to the notion of a split decomposition. We will define this notion after a few preliminary comments. By a decomposition of \(V\) we mean a sequence \(\{U_i\}_{i=0}^{d}\) consisting of nonzero subspaces of \(V\) such that \(V = U_0 + U_1 + \cdots + U_d\) (direct sum).

For notational convenience we set \(U_{-1} = 0\), \(U_{d+1} = 0\). For an example of a decomposition, let \(A\) denote a diagonalizable linear transformation on \(V\). Then any ordering of the eigenspaces of \(A\) is a decomposition of \(V\).

Lemma 1.5. Let \(A\) denote a linear transformation on \(V\). Let \(\{U_i\}_{i=0}^{d}\) denote a decomposition of \(V\) and let \(\{\theta_i\}_{i=0}^{d}\) denote a sequence of mutually distinct elements of \(K\). Assume

\[(A - \theta_i I)U_i \subseteq U_{i+1}\] for \(0 \leq i \leq d\). (4)

Then \(A\) is diagonalizable and \(\{\theta_i\}_{i=0}^{d}\) are the eigenvalues of \(A\).

Proof: From (4) we see that, with respect to an appropriate basis for \(V\), \(A\) is represented by a lower triangular matrix which has diagonal entries \(\{\theta_i\}_{i=0}^{d}\), with \(\theta_i\) appearing \(\dim(U_i)\) times for \(0 \leq i \leq d\). Therefore \(\{\theta_i\}_{i=0}^{d}\) are the roots of the characteristic polynomial of \(A\). It remains to show that \(A\) is diagonalizable. From (4) we see that \(\prod_{i=0}^{d}(A - \theta_i I)\) vanishes on \(V\). By this and since \(\{\theta_i\}_{i=0}^{d}\) are distinct we see that the minimal polynomial of \(A\) has distinct roots. Therefore \(A\) is diagonalizable and the result follows. \(\square\)

Definition 1.6. Let \(d\) denote a nonnegative integer. Let \(A\) (resp. \(A^*\)) denote a diagonalizable linear transformation on \(V\) with eigenvalues \(\{\theta_i\}_{i=0}^{d}\) (resp. \(\{\theta_i^*\}_{i=0}^{d}\)). By an \((A, A^*)\)-split decomposition of \(V\) with respect to \(\{\theta_i\}_{i=0}^{d}\) (resp. \(\{\theta_i^*\}_{i=0}^{d}\)), we mean a decomposition \(\{U_i\}_{i=0}^{d}\) of \(V\) such that both

\[(A - \theta_{d-i} I)U_i \subseteq U_{i+1}\] for \(0 \leq i \leq d\). (5)

\[(A^* - \theta_i^* I)U_i \subseteq U_{i-1}\] (6)

for \(0 \leq i \leq d\).

As we will see in Corollary 3.4, the \((A, A^*)\)-split decomposition of \(V\) with respect to \(\{\theta_i\}_{i=0}^{d}\) (resp. \(\{\theta_i^*\}_{i=0}^{d}\)) is unique if it exists.

The main results of this paper are the following two theorems and subsequent corollary.
Theorem 1.7. Let $d$ denote a nonnegative integer. Let $A$ (resp. $A^*$) denote a diagonalizable linear transformation on $V$ with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$). Suppose that the pair $(A, A^*)$ is irreducible, and Hessenberg with respect to $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$. Then there exists an $(A, A^*)$-split decomposition of $V$ with respect to $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$.

Theorem 1.8. Let $d$ denote a nonnegative integer. Let $A$ (resp. $A^*$) denote a diagonalizable linear transformation on $V$ with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$). Suppose that there exists an $(A, A^*)$-split decomposition of $V$ with respect to $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$. Then the pair $(A, A^*)$ is Hessenberg with respect to $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$.

Combining Theorem 1.7 and Theorem 1.8 we obtain the following corollary.

Corollary 1.9. Let $d$ denote a nonnegative integer. Let $A$ (resp. $A^*$) denote a diagonalizable linear transformation on $V$ with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$). Assume the pair $(A, A^*)$ is irreducible. Then the following (i), (ii) hold.

(i) The pair $(A, A^*)$ is Hessenberg with respect to $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$.

(ii) There exists an $(A, A^*)$-split decomposition of $V$ with respect to $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$.

2 The Proof of Theorem 1.7

In this section we give a proof of Theorem 1.7. Along the way, we show that the scalars $d$ and $\delta$ from Definition 1.1 are equal. We will refer to the following setup.

Assumption 2.1. Let $A$ (resp. $A^*$) denote a diagonalizable linear transformation on $V$ with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^\delta$). Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^\delta$) denote the corresponding eigenspaces of $A$ (resp. $A^*$). We assume that the pair $(A, A^*)$ is irreducible and Hessenberg with respect to $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^\delta)$. For all integers $i$ and $j$ we set

$$V_{ij} = (V_0 + \cdots + V_i) \cap (V_0^* + \cdots + V_j^*). \quad (7)$$

We interpret the sum on the left in (7) to be 0 (resp. $V$) if $i < 0$ (resp. $i > d$). We interpret the sum on the right in (7) to be 0 (resp. $V$) if $j < 0$ (resp. $j > \delta$).

Lemma 2.2. With reference to Assumption 2.1, the following (i), (ii) hold for $0 \leq i \leq d$ and $0 \leq j \leq \delta$.

(i) $V_i^\delta = V_0 + \cdots + V_i$.

(ii) $V_{dj} = V_0^* + \cdots + V_j^*$.

Proof: (i) Set $j = \delta$ in (7) and use the fact that $V = V_0^* + \cdots + V_{\delta}^*$. (ii) Set $i = d$ in (7) and use the fact that $V = V_0 + \cdots + V_d$. □

Lemma 2.3. With reference to Assumption 2.1, the following (i), (ii) hold for $0 \leq i \leq d$ and $0 \leq j \leq \delta$.

(i) $(A - \theta_i I)V_{ij} \subseteq V_{i-1,j+1}$. 

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(ii) \((A^* - \theta^*_j I)V_{ij} \subseteq V_{i+1,j-1}\).

**Proof:** (i) Since \(V_i\) is the eigenspace of \(A\) corresponding to the eigenvalue \(\theta_i\), we have

\[
(A - \theta_i I) \sum_{h=0}^{i} V_h = \sum_{h=0}^{i-1} V_h.
\]

Using (2) we find

\[
(A - \theta_i I) \sum_{h=0}^{j} V_h^* \subseteq \sum_{h=0}^{j+1} V_h^*.
\]

Evaluating \((A - \theta_i I)V_{ij}\) using (7)–(9), we find it is contained in \(V_{i-1,j+1}\).

(ii) Using (1) we find

\[
(A^* - \theta^*_j I) \sum_{h=0}^{i} V_h \subseteq \sum_{h=0}^{i+1} V_h.
\]

Since \(V_j^*\) is the eigenspace of \(A^*\) corresponding to the eigenvalue \(\theta^*_j\), we have

\[
(A^* - \theta^*_j I) \sum_{h=0}^{j} V_h^* = \sum_{h=0}^{j-1} V_h^*.
\]

Evaluating \((A^* - \theta^*_j I)V_{ij}\) using (7), (10), (11), we find it is contained in \(V_{i+1,j-1}\). □

**Proposition 2.4.** With reference to Assumption 2.1, the scalars \(d\) and \(\delta\) from Definition 2.1 are equal. Moreover,

\[
V_{ij} = 0 \quad \text{if} \quad i + j < d, \quad (0 \leq i, j \leq d).
\]

**Proof:** For all nonnegative integers \(r\) such that \(r \leq d\) and \(r \leq \delta\), we define

\[
W_r = V_{0r} + V_{1,r-1} + \cdots + V_{r0}.
\]

We have \(AW_r \subseteq W_r\) by Lemma 2.2(i) and \(A^*W_r \subseteq W_r\) by Lemma 2.3(ii). Now \(W_r = 0\) or \(W_r = V\) since the pair \((A, A^*)\) is irreducible. Suppose for the moment that \(r \leq d - 1\). Each term on the right in (13) is contained in \(V_0 + \cdots + V_r\) so \(W_r \subseteq V_0 + \cdots + V_r\). Thus \(W_r \neq V\) and hence \(W_r = 0\). Next suppose \(r = d\). Then \(V_{d0} \subseteq W_r\). Recall \(V_{d0} = V_0^*\) by Lemma 2.2(ii) and \(V_0^* \neq 0\) so \(V_{d0} \neq 0\). Now \(W_r \neq 0\) so \(W_r = V\). We have now shown that \(W_r = 0\) if \(r \leq d - 1\) and \(W_r = V\) if \(r = d\). Similarly \(W_r = 0\) if \(r \leq \delta - 1\) and \(W_r = V\) if \(r = \delta\). Now \(d = \delta\); otherwise we take \(r = \min(d, \delta)\) in our above comments and find \(W_r\) is both 0 and \(V\), for a contradiction. The result follows. □

**Lemma 2.5.** With reference to Assumption 2.1, the sequence \(\{V_{d-i,j}\}_{i=0}^{d}\) is an \((A, A^*)\)-split decomposition of \(V\) with respect to \((\{\theta_i\}_{i=0}^{d}; \{\theta^*_i\}_{i=0}^{d})\).
Proof: Observe that (5) follows from Lemma 2.3(i) and (6) follows from Lemma 2.3(ii). It remains to show that the sequence $\{V_{d-i,i}\}_{i=0}^d$ is a decomposition. We first show

$$V = \sum_{i=0}^d V_{d-i,i}. \quad (14)$$

Let $W$ denote the sum on the right in (14). We have $AW \subseteq W$ by Lemma 2.3(i) and $A^*W \subseteq W$ by Lemma 2.3(ii). Now $W = 0$ or $W = V$ by the irreducibility assumption. Observe that $W$ contains $V_{d0}$ and $V_{d0} = V^*_0$ is nonzero so $W \neq 0$. We conclude that $W = V$ and (14) follows. Next we show that the sum (14) is direct. To do this we show that

$$(V_{d0} + V_{d-1,1} + \cdots + V_{d-i+1,i-1}) \cap V_{d-i,i} \quad (15)$$

is zero for $1 \leq i \leq d$. Let $i$ be given. From the construction

$$V_{d-j,j} \subseteq V_{0}^* + V_1^* + \cdots + V_{i-1}^*$$

for $0 \leq j \leq i - 1$, and

$$V_{d-i,i} \subseteq V_0 + V_1 + \cdots + V_{d-i}.$$

Therefore (15) is contained in

$$(V_0 + V_1 + \cdots + V_{d-i}) \cap (V_0^* + V_1^* + \cdots + V_{i-1}^*). \quad (16)$$

But (16) is equal to $V_{d-i,i-1}$ and this is zero by (12), so (15) is zero. We have shown that the sum (14) is direct. Next we show that $V_{d-i,i} \neq 0$ for $0 \leq i \leq d$. Suppose there exists an integer $i \ (0 \leq i \leq d)$ such that $V_{d-i,i} = 0$. Observe that $i \neq 0$ since $V_{d0} = V^*_0$ is nonzero and $i \neq d$ since $V_{0d} = V_0$ is nonzero. Set

$$W = V_{d0} + V_{d-1,1} + \cdots + V_{d-i+1,i-1}$$

and observe that $W \neq 0$ and $W \neq V$ by our above remarks. By Lemma 2.3(ii), we find $A^*W \subseteq W$. By Lemma 2.3(i) and since $V_{d-i,i} = 0$, we find $AW \subseteq W$. Now $W = 0$ or $W = V$ by our irreducibility assumption, which yields a contradiction. We conclude that $V_{d-i,i} \neq 0$ for $0 \leq i \leq d$. We have now shown that the sequence $\{V_{d-i,i}\}_{i=0}^d$ is a decomposition of $V$ and we are done.

Theorem 1.7 is immediate from Lemma 2.5.

## 3 The Proof of Theorem 1.8

In this section we give a proof of Theorem 1.8. Along the way, we show that the split decomposition from Definition 1.6 is unique if it exists. The following assumption sets the stage.

Assumption 3.1. Let $d$ denote a nonnegative integer. Let $A$ (resp. $A^*$) denote a diagonalizable linear transformation on $V$ with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$). Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote the corresponding eigenspaces of $A$ (resp. $A^*$). We assume that there exists a decomposition $\{U_i\}_{i=0}^d$ of $V$ that is $(A, A^*)$-split with respect to $\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d$. 

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Lemma 3.2. With reference to Assumption 3.1, for $0 \leq i \leq d$ both
\begin{align}
U_i + U_{i+1} + \cdots + U_d &= V_0 + V_1 + \cdots + V_{d-i}, \\
U_0 + U_1 + \cdots + U_i &= V_0^* + V_1^* + \cdots + V_i^*.
\end{align}

Proof: First consider (17). We abbreviate
\[ W = U_i + U_{i+1} + \cdots + U_d, \quad Z = V_0 + V_1 + \cdots + V_{d-i}. \]
We show $W = Z$. To obtain $Z \subseteq W$, set $X = \prod_{h=0}^{i-1} (A - \theta_d - h I)$, and observe $Z = XV$ by elementary linear algebra. Using (5), we find $X U_j \subseteq W$ for $0 \leq j \leq d$, so $X V \subseteq W$ in view of (3). We now have $Z \subseteq W$. To obtain $W \subseteq Z$, set $Y = \prod_{h=i}^{d} (A - \theta_d - h I)$, and observe $Z = \{ v \in V \mid Y v = 0 \}$.

Using (5), we find $Y U_j = 0$ for $i \leq j \leq d$, so $Y W = 0$. Combining this with (19), we find $W \subseteq Z$. We now have $Z = W$ and hence (17) holds. Line (18) is similarly obtained using (6).

Lemma 3.3. With reference to Assumption 3.1
\[ U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}) \quad (0 \leq i \leq d). \]

Proof: Since $\{U_i\}_{i=0}^{d}$ is a decomposition of $V$,
\[ U_i = (U_0 + U_1 + \cdots + U_i) \cap (U_i + U_{i+1} + \cdots + U_d) \quad (0 \leq i \leq d). \]
Evaluating (21) using (17), (18) we obtain (20).

Corollary 3.4. With reference to Assumption 3.1, the split decomposition $\{U_i\}_{i=0}^{d}$ is uniquely determined by the given orderings of the eigenvalues $\{\theta_i\}_{i=0}^{d}$ and $\{\theta_i^*\}_{i=0}^{d}$.

Proof: Immediate from Lemma 3.3.

Lemma 3.5. With reference to Assumption 3.1, for $0 \leq i \leq d$ both
\begin{align}
A^* V_i &\subseteq V_0 + V_1 + \cdots + V_{i+1}, \\
A V_i^* &\subseteq V_0^* + V_1^* + \cdots + V_{i+1}^*.
\end{align}
Moreover $(A, A^*)$ is a Hessenberg pair on $V$.

Proof: To obtain (22), observe
\[ A^* V_i \subseteq A^* \sum_{h=0}^{i} V_h = A^* \sum_{h=d-i}^{d} U_h \quad (\text{by (17)}) \]
\[ \subseteq \sum_{h=d-i}^{d} U_h \quad (\text{by (6)}) \]
\[ = \sum_{h=0}^{i+1} V_h \quad (\text{by (17)}). \]
To obtain (23), observe

\[ AV_i^* \subseteq A \sum_{h=0}^{i} V_h^* \]
\[ = A \sum_{h=0}^{i+1} U_h \quad \text{(by (18))} \]
\[ \subseteq \sum_{h=0}^{i+1} U_h \quad \text{(by (5))} \]
\[ = \sum_{h=0}^{i+1} V_h^* \quad \text{(by (18)).} \]

\[ \square \]

Theorem 1.8 is immediate from Lemma 3.5.

We finish this section with a comment.

Corollary 3.6. With reference to Assumption 3.1, for \( 0 \leq i \leq d \) the dimensions of \( V_{d-i} \), \( V_i^* \), \( U_i \) are the same.

Proof: Recall that \( \{V_i\}_{i=0}^d \) and \( \{U_i\}_{i=0}^d \) are decompositions of \( V \). By this and (17),

\[ \dim(U_i) + \dim(U_{i+1}) + \cdots + \dim(U_d) = \dim(V_0) + \dim(V_1) + \cdots + \dim(V_{d-i}) \]

for \( 0 \leq i \leq d \). Consequently, the dimensions of \( V_{d-i} \) and \( U_i \) are the same for \( 0 \leq i \leq d \). A similar argument using (18) shows that the dimensions of \( V_i^* \) and \( U_i \) are the same for \( 0 \leq i \leq d \). The result follows. \[ \square \]

4 Hessenberg pairs and tridiagonal pairs

In this section, we explain how Hessenberg pairs are related to tridiagonal pairs. Using this relationship we show that some results [1, Lemma 4.5], [1, Theorem 4.6] about tridiagonal pairs are direct consequences of our results on Hessenberg pairs. We start by recalling the definition of a tridiagonal pair.

Definition 4.1. [1, Definition 1.1] By a tridiagonal pair on \( V \), we mean an ordered pair \((A, A^*)\) of linear transformations on \( V \) that satisfy (i)–(iv) below.

(i) Each of \( A, A^* \) is diagonalizable on \( V \).

(ii) There exists an ordering \( \{V_i\}_{i=0}^d \) of the eigenspaces of \( A \) such that

\[ A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \quad (24) \]

where \( V_{-1} = 0, V_{d+1} = 0 \).
Let \( (A, A^*) \) denote an ordered pair of diagonalizable linear transformations on \( V \). Let \( \{V_i\}_{i=0}^d \) (resp. \( \{V_i^*\}_{i=0}^\delta \)) denote any ordering of the eigenspaces of \( A \) (resp. \( A^* \)). We say that the pair \( (A, A^*) \) is \textit{tridiagonal with respect to} \( \{V_i\}_{i=0}^d ; \{V_i^*\}_{i=0}^\delta \). We now use this relationship to obtain some results on tridiagonal pairs.

In Proposition 4.4 we showed how Hessenberg pairs are related to tridiagonal pairs. We now use this relationship to obtain some results on tridiagonal pairs.

**Proposition 4.4.** Let \( A \) (resp. \( A^* \)) denote a diagonalizable linear transformation on \( V \) with eigenspaces \( \{V_i\}_{i=0}^d \) (resp. \( \{V_i^*\}_{i=0}^\delta \)). Then the following (i)–(iv) are equivalent.

1. The pair \( (A, A^*) \) is tridiagonal with respect to \( \{V_i\}_{i=0}^d ; \{V_i^*\}_{i=0}^\delta \).
2. The pair \( (A, A^*) \) is irreducible, and Hessenberg with respect to each of \( \{V_i\}_{i=0}^d ; \{V_i^*\}_{i=0}^\delta \), \( \{V_d-i\}_{i=0}^d ; \{V_d-i^*\}_{i=0}^\delta \), \( \{V_d^\delta-i\}_{i=0}^d ; \{V_d^\delta-i^*\}_{i=0}^\delta \).
3. The pair \( (A, A^*) \) is irreducible, and Hessenberg with respect to each of \( \{V_i\}_{i=0}^d ; \{V_i^*\}_{i=0}^\delta \), \( \{V_d-i\}_{i=0}^d ; \{V_d^\delta-i\}_{i=0}^\delta \).
4. The pair \( (A, A^*) \) is irreducible, and Hessenberg with respect to each of \( \{V_d-i\}_{i=0}^d ; \{V_d-i^*\}_{i=0}^\delta \).

**Proof:** Observe that \( \{V_i\}_{i=0}^d \) satisfies (24) if and only if both \( \{V_i\}_{i=0}^d \) and \( \{V_d-i\}_{i=0}^d \) satisfy (21). Similarly \( \{V_i^*\}_{i=0}^\delta \) satisfies (25) if and only if both \( \{V_i^*\}_{i=0}^\delta \) and \( \{V_d^\delta-i\}_{i=0}^\delta \) satisfy (2). The result follows. \( \square \)

In Proposition 4.4 we showed how Hessenberg pairs are related to tridiagonal pairs. We now use this relationship to obtain some results on tridiagonal pairs.

**Theorem 4.5.** [11 Lemma 4.5] Let \( (A, A^*) \) denote a tridiagonal pair as in Definition 4.1. Then the scalars \( d \) and \( \delta \) from that definition are equal.

**Proof:** Combine Proposition 2.4 and Proposition 4.4. \( \square \)

**Definition 4.6.** Let \( (A, A^*) \) denote an ordered pair of diagonalizable linear transformations on \( V \). Let \( \{\theta_i\}_{i=0}^d \) (resp. \( \{\theta_i^*\}_{i=0}^\delta \)) denote any ordering of the eigenvalues of \( A \) (resp. \( A^* \)). Let \( \{V_i\}_{i=0}^d \) (resp. \( \{V_i^*\}_{i=0}^\delta \)) denote the corresponding ordering of the eigenspaces of \( A \) (resp. \( A^* \)). We say that the pair \( (A, A^*) \) is \textit{tridiagonal with respect to} \( \{\theta_i\}_{i=0}^d ; \{\theta_i^*\}_{i=0}^\delta \) whenever \( (A, A^*) \) is tridiagonal with respect to \( \{V_i\}_{i=0}^d ; \{V_i^*\}_{i=0}^\delta \).
Theorem 4.7. [1, Theorem 4.6] Let $d$ denote a nonnegative integer. Let $A$ (resp. $A^*$) denote a diagonalizable linear transformation on $V$ with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$). Then the following (i)–(iv) are equivalent.

(i) The pair $(A, A^*)$ is tridiagonal with respect to $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$.

(ii) The pair $(A, A^*)$ is irreducible, and there exist $(A, A^*)$-split decompositions of $V$ with respect to each of $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$, $(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d)$, $(\{\theta_{i-d}\}_{i=0}^d; \{\theta^*_{i-d}\}_{i=0}^d)$, $(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d)$.

(iii) The pair $(A, A^*)$ is irreducible, and there exist $(A, A^*)$-split decompositions of $V$ with respect to each of $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$, $(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d)$.

(iv) The pair $(A, A^*)$ is irreducible, and there exist $(A, A^*)$-split decompositions of $V$ with respect to each of $(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_{i-d}\}_{i=0}^d)$, $(\{\theta_i\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d)$.

Proof: Combine Corollary 1.9 and Proposition 1.4.

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