Phenomenology of Minimal Lepton Flavor Violation

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Abstract

We extend the effective theory of Minimal Lepton Flavor Violation (MLFV) by including four-lepton operators. We compute the rates for $\mu \to 3e$ and $\tau \to 3\ell$ decays and point out several new ways to test the hypothesis of MLFV. We also investigate to what extent it will be possible from (future) experimental information to pin down the contributions of different effective operators. In particular we look for experimental handles on quark-lepton operators of the type $\bar{\ell}_i \Gamma \ell_j \times \bar{q} \Gamma q$ by working out their contribution to hadronic processes such as $\tau \to \mu \pi^0$, $\pi^0 \to \mu \bar{e}$, $\Upsilon \to \tau \bar{\mu}$, as well as to purely leptonic decays such as $\mu \to 3e$ through loop effects.

1 Introduction

In a recent work \cite{1} we have extended the notion of Minimal Flavor Violation (MFV) \cite{2,3,4} to the lepton sector of beyond the Standard Model (SM) theories. The MFV hypothesis states that the irreducible sources of (lepton) flavor symmetry breaking are linked in a minimal way to the structures generating the observed pattern of fermion masses and mixing. While this idea has a straightforward and unique realization in the quark sector \cite{2,3,4} (the SM Yukawas are the only sources of quark-flavor symmetry breaking), the situation in the lepton sector is different, mainly due to the possibility/necessity to break the $U(1)$ symmetry associated with total lepton number.

The MFV framework has two particularly attractive features. On one hand, it implies a suppression of FCNC processes induced by new degrees of freedom at the TeV scale to a level which is consistent with present experimental constraints. Moreover, it provides a predictive framework that links the FCNC couplings to the fermion spectrum and mixing structure (up to an overall scale factor). For leptons it relates lepton-flavor mixing in the neutrino sector to lepton-flavor violation in the charged lepton sector.
In ref. [1] we found two ways to define the sources of flavor symmetry breaking in the lepton sector in a minimal and thus very predictive way: i) a scenario where the left-handed Majorana mass matrix is the only irreducible source of flavor symmetry breaking; and ii) a scenario with heavy right-handed neutrinos, where the Yukawa couplings define the irreducible sources of flavor symmetry breaking and the right-handed Majorana mass matrix has a trivial flavor structure.

For each scenario, according to the MLFV symmetry principle, we have constructed the basis of dimension six operators contributing to processes with only two charged leptons [1]. In this context we have investigated the sensitivity of processes such as $\ell_i \rightarrow \ell_j \gamma$ and $\mu \rightarrow e$ conversion to the scales of lepton flavor ($\Lambda_{LFV}$) and lepton number ($\Lambda_{LN}$) violation and we have pointed out distinctive predictions of MLFV as a function of $s_{13}$ and the CP violating phase $\delta$.

In this paper we extend previous work in several respects:

- we enlarge the MLFV effective theory to include operators contributing to four-lepton processes and we give predictions for decays such as $\mu \rightarrow 3e$ and $\tau \rightarrow 3\ell$.

- we investigate to what extent it will be possible to reconstruct from (future) experimental information the dynamics of a given MLFV model, which amounts to pin down the relative size of the couplings appearing in the effective lagrangian.

We tackle this second issue by working out a number of predictions for LFV decay rates. In particular, we study how LFV decays involving hadrons can be used to probe operators involving two leptons and two quarks (such as $\bar{\ell}_i \Gamma_{\ell_j} \times \bar{q} \Gamma_q$). We then show how, under suitable assumptions, the decay $\mu \rightarrow 3e$ can also be used to probe operators of the type $\bar{\ell}_i \Gamma_{\ell_j} \times \bar{q} \Gamma_q$ through loop effects.

The paper is organized as follows. In section 2 we recall the two realizations of MLFV introduced in Ref. [1] and give the complete basis of dimension six effective operators, including four-lepton operators. In section 3 we present results for the rates of various $\ell \rightarrow \ell'\ell''\ell'''$ decays at tree level in the effective field theory, explore their phenomenology, and point out new testable predictions of the MLFV scenario. In section 4 we discuss LFV decays involving hadrons, while in section 5 we calculate the loop-induced contribution to $\mu \rightarrow 3e$ from four-fermion operators involving two quark fields. We present our conclusions in section 6. Details concerning the phase space integration, the renormalization group analysis of section 5 and the matching to chiral perturbation theory are given respectively in appendices A,B, and C.

## 2 Extended Operator Basis

In this section we recall the two scenarios of MLFV identified in Ref. [1] and present the dimension six operator basis, extended to cover four-lepton processes. In order to formulate the minimal flavor violation hypothesis for leptons, one needs to identify the field content of the theory, the flavor symmetry group and the irreducible sources of symmetry breaking. Two possibilities arise (see ref. [1] for details).

**Minimal field content:** In this scenario the flavor symmetry group is $G_{LF} = SU(3)_L \times SU(3)_E$, acting on three left-handed lepton doublets $L^i_L$ and three right-handed charged...
lepton singlets $e_R^i$ (SM field content). The breaking of the $U(1)_{\text{LN}}$ is independent from the breaking of $G_{\text{LF}}$ and is associated to a very high scale $\Lambda_{\text{LN}}$. The irreducible sources of lepton-flavor symmetry breaking are $\lambda_{e}^i$ and $g_{\nu}^j$, defined by\footnote{Throughout this paper we use four-component spinor fields, and $\psi^c = -i \gamma^2 \psi^*$ denotes the charge conjugate of the field $\psi$. We also use $v = \langle H^0 \rangle \simeq 174$ GeV.}:

\[ \mathcal{L}_{\text{Sym.Br.}} = -\lambda_{e}^i \bar{e}_R^i (H^\dagger L_L^i) - \frac{1}{2 \Lambda_{\text{LN}}} g_{\nu}^j (\bar{L}_L^c \tau_2 H) (H^T \tau_2 L_L^j) + \text{h.c.} \]  

Extended field content: In this scenario the maximal flavor group is $G_{\text{LF}} \times SU(3)_{\nu_R}$, acting on three right-handed neutrinos, $\nu_R^j$, in addition to the SM fields. The right-handed neutrino mass term is flavor diagonal ($M^j_{\nu} = M_{\nu} \delta^j_\nu$, $|M_{\nu}| \gg v$) and its effect is to break $U(1)_{\text{LN}}$ as well as to break $SU(3)_{\nu_R}$ to $O(3)_{\nu_R}$. The remaining lepton-flavor symmetry is broken only by two irreducible sources, $\lambda_{e}^i$ and $\lambda_{\nu}^j$, defined by:

\[ \mathcal{L}_{\text{Sym.Br.}} = -\lambda_{e}^i \bar{e}_R^i (H^\dagger L_L^i) + i \lambda_{\nu}^j \bar{\nu}_R^j (H^T \tau_2 L_L^j) + \text{h.c.} \]  

In most SM extensions, at some scale $\Lambda_{\text{LFV}}$ above the electroweak scale and well below $\Lambda_{\text{LN}}$ (or $M_R$) there are new degrees of freedom carrying lepton flavor quantum numbers. As long as the underlying model respects MLFV, at scales below $\Lambda_{\text{LFV}}$ the physics of lepton flavor violation is described by the following effective lagrangian (up to higher dimensional operators):

\[ \mathcal{L} = \frac{1}{\Lambda_{\text{LFV}}} \sum_{i=1}^{5} \left( c_{LL}^{(i)} O_{LL}^{(i)} + c_{4L}^{(i)} O_{4L}^{(i)} \right) + \frac{1}{\Lambda_{\text{LFV}}^2} \left( \sum_{j=1}^{2} c_{RL}^{(j)} O_{RL}^{(j)} + \text{h.c.} \right) , \]  

with operators defined by:

\begin{align*}
O_{LL}^{(1)} &= \bar{L}_L^c \gamma^\mu \Delta L_L H^\dagger iD_\mu H \\
O_{LL}^{(2)} &= \bar{L}_L^c \gamma^\mu \tau^a \Delta L_L H^\dagger \tau^a iD_\mu H \\
O_{LL}^{(3)} &= \bar{L}_L^c \gamma^\mu \Delta L_L \tilde{Q}_L \gamma_\mu Q_L \\
O_{LL}^{(4d)} &= \bar{L}_L^c \gamma^\mu \Delta L_L \bar{d}_R \gamma_\mu d_R \\
O_{LL}^{(4u)} &= \bar{L}_L^c \gamma^\mu \Delta L_L \bar{u}_R \gamma_\mu u_R \\
O_{LL}^{(5)} &= \bar{L}_L^c \gamma^\mu \tau^a \Delta L_L \tilde{Q}_L \gamma_\mu \tau^a Q_L \\
O_{RL}^{(1)} &= g' H^\dagger \bar{e}_R \sigma^{\mu\nu} \lambda_e \Delta L_R B_{\mu\nu} \\
O_{RL}^{(2)} &= g H^\dagger \bar{e}_R \sigma^{\mu\nu} \tau^a \lambda_e \Delta L_R W_{\mu\nu}^a \\
O_{RL}^{(3)} &= (D_\mu H)^\dagger \bar{e}_R \lambda_e \Delta D_\mu L_L \\
O_{RL}^{(4)} &= \bar{e}_R \lambda_e \Delta L_L \tilde{Q}_L \lambda_d d_R \\
O_{RL}^{(5)} &= \bar{e}_R \sigma^{\mu\nu} \lambda_e \Delta L_L \tilde{Q}_L \lambda_{\mu\nu} \lambda d_R \\
O_{RL}^{(6)} &= \bar{e}_R \lambda_e \Delta L_L \bar{u}_R \lambda_U^a i \tau^2 Q_L \\
O_{RL}^{(7)} &= \bar{e}_R \sigma^{\mu\nu} \lambda_e \Delta L_L \bar{u}_R \sigma_{\mu\nu} \lambda_U^a i \tau^2 Q_L \\
\end{align*}

and

\begin{align*}
O_{4L}^{(1)} &= \bar{L}_L^c \gamma^\mu \Delta L_L \bar{L}_L^c \gamma_\mu L_L \\
O_{4L}^{(2)} &= \bar{L}_L^c \gamma^\mu \tau^a \Delta L_L \bar{L}_L^c \gamma_\mu \tau^a L_L \\
O_{4L}^{(3)} &= \bar{L}_L^c \gamma^\mu \Delta L_L \bar{e}_R \gamma_\mu e_R \\
O_{4L}^{(4)} &= \delta_{nj} \delta_{mi} \bar{L}_L^c \gamma^\mu L_L^i \bar{L}_L^m \gamma^\mu \gamma_\mu L_L^n \\
O_{4L}^{(5)} &= \delta_{nj} \delta_{mi} \bar{L}_L^c \gamma^\mu \tau^a L_L^i \bar{L}_L^m \gamma^\mu \tau^a L_L^n .
\end{align*}
The MLFV hypothesis fixes the couplings $\Delta$ and $\delta$ to be:

$$\Delta = \begin{cases} \frac{\Lambda_{LN}}{v^2} \hat{U} m_\nu^2 \hat{U}^\dagger & \text{minimal field content} \\ \frac{M_\nu}{v^2} \hat{U} m_\nu \hat{U}^\dagger & \text{extended field content, CP limit} \end{cases}$$

and

$$\delta = \delta^T = \begin{cases} \frac{\Lambda_{LN}}{v^2} \hat{U}^* m_\nu \hat{U}^\dagger & \text{minimal field content} \\ \frac{M_\nu}{v^2} \hat{U}^* m_\nu \hat{U}^\dagger & \text{extended field content} \end{cases}$$

in terms of the diagonal light neutrino mass matrix $m_\nu$ and the PMNS matrix $\hat{U}$. To arrive at the above results we have used that $\lambda_e \ll 1$ and assumed perturbative behavior in $\Delta$ and $\delta$, stopping at leading order in $(\Lambda_{LN} m_\nu)/v^2$ (and $(M_\nu m_\nu)/v^2$). Therefore, for consistency, in the extended field case the operators $O_{4L}^{(4),(5)}$ should be dropped.

The explicit form of the couplings $\Delta_{ij}$ in terms of neutrino mass splitting and mixing angles (using the parameterization of the PMNS matrix $\hat{U}$ of Ref. [5]) is reported in Ref. [11] for both minimal and extended field scenarios. Indicating by $s$ and $c$ the sine and cosine of the solar mixing angle and by $s_{13} \equiv \sin \theta_{13}$, the explicit form of $\delta_{ij}$ in the case of minimal field content (the only case we need) to first order in $s_{13}$ is,

$$\begin{align*}
\delta_{ee} &= \frac{\Lambda_{LN}}{v^2} \left[ c^2 e^{-i\alpha_1} m_{\nu_1} + s^2 e^{-i\alpha_2} m_{\nu_2} \right] \equiv \frac{\Lambda_{LN}}{v} d_{ee} \\
\delta_{e\mu} &= \frac{\Lambda_{LN}}{v^2} \left[ -s c e^{-i\alpha_1} m_{\nu_1} + s c e^{-i\alpha_2} m_{\nu_2} + s_{13} e^{i\delta} m_{\nu_3} \right] \equiv \frac{\Lambda_{LN}}{v} d_{e\mu} \\
\delta_{\mu\mu} &= \frac{\Lambda_{LN}}{v^2} \left[ s^2 e^{-i\alpha_1} m_{\nu_1} + c^2 e^{-i\alpha_2} m_{\nu_2} + m_{\nu_3} \right] \equiv \frac{\Lambda_{LN}}{v} d_{\mu\mu} \\
\delta_{e\tau} &= \frac{\Lambda_{LN}}{v^2} \left[ s c e^{-i\alpha_1} m_{\nu_1} - s c e^{-i\alpha_2} m_{\nu_2} + s_{13} e^{i\delta} m_{\nu_3} \right] \equiv \frac{\Lambda_{LN}}{v} d_{e\tau} \\
\delta_{\mu\tau} &= \frac{\Lambda_{LN}}{v^2} \left[ -s^2 e^{-i\alpha_1} m_{\nu_1} - c^2 e^{-i\alpha_2} m_{\nu_2} + m_{\nu_3} \right] \equiv \frac{\Lambda_{LN}}{v} d_{\mu\tau}.
\end{align*}$$

Throughout this paper we will assume maximal atmospheric mixing and the following central values for the remaining neutrino mixing parameters [5]: $\Delta m^2_{\text{sol}} = 8.0 \times 10^{-5} \text{eV}^2$, $\Delta m^2_{\text{atm}} = 2.5 \times 10^{-3} \text{eV}^2$, $\theta_{\text{sol}} = 33^\circ$.

## 3 $\ell \rightarrow \ell' \ell'' \ell'''$ at tree level in the effective theory

Within the effective theory described in the previous section we can now study decays of the type $\ell \rightarrow \ell' \ell'' \ell'''$. At tree level these decays receive contributions from the transition magnetic moment operators $O_{RL}^{(1),(2)}$ (photon exchange), the four-lepton operators $O_{4L}^{(i)}$, and from $O_{LL}^{(1),(2)}$ ($Z$ exchange).

### 3.1 Rates

The integrated decay rate for $\mu \rightarrow eee$ averaged over initial polarizations and summed over final polarizations is:

$$\Gamma_{\mu \rightarrow 3e} = \frac{n^4 |\Delta_{e\mu}|^2}{\Lambda_{\text{LFV}}^4} \left[ |a_+|^2 + 2|a_-|^2 - 8\text{Re}(a_0^a a_-) - 4\text{Re}(a_0^a a_+) + 6|a_0|^2 \right]$$

(9)
where $\Gamma_{\mu \to \nu \bar{\nu}} = m_\mu^5 / (v^4 1536 \pi^3)$, the $a_i$ coefficients are

$$
a_+ = \sin^2 \theta_w (c_{LL}^{(1)} + c_{LL}^{(2)}) + c_{4L}^{(3)}$$
$$a_- = (\sin^2 \theta_w - \frac{1}{2}) (c_{LL}^{(1)} + c_{LL}^{(2)}) + c_{4L}^{(1)} + c_{4L}^{(2)} + \frac{2\delta_{\mu \delta_e}^* \mu e}{\Delta_{\mu \mu}} (c_{4L}^{(4)} + c_{4L}^{(5)})$$
$$a_0 = 2e^2 (c_{RL}^{(1)} - c_{RL}^{(2)})$$

and $I$ is a phase space integral given by (see appendix A for details)

$$I \approx \frac{8}{3} \ln \left( \frac{m_\mu}{m_\mu} \right) - \frac{4}{3} \ln(2) - \frac{26}{9} + \mathcal{O} \left( \frac{m_e}{m_\mu} \right).$$

Note that $a_-$ depends not only on the Wilson coefficients but also on the ratio of effective FCNC couplings mediating $\mu \to e$ transitions, namely $\delta_{\mu \delta_e}^* \mu e$ and $\Delta_{\mu \mu}$. We shall study the consequences of this in Section 3.3. The rate for $\tau \to \mu \mu$ and for $\tau \to ee$ is given similarly, replacing $\Delta_{\mu \mu} \to \Delta_{\mu \tau}$, $\delta_{\mu \delta_e}^* \mu e \to \delta_{\mu \delta_e}^* \mu \mu$, and $\Delta_{\mu \mu} \to \Delta_{\mu \tau}$, $\delta_{\mu \delta_e}^* \mu e \to \delta_{\mu \delta_e}^* \mu \mu$, respectively, and $m_\mu \to m_\tau$.

The integrated rate for $\tau^- \to e^-\mu^- \mu^+$ is given by

$$\Gamma_{\tau \to e \mu \bar{\mu}} = \frac{v^4 |\Delta_{\mu \tau}|^2}{\Lambda_{\text{LFV}}^4} \left[ |a_+|^2 + |a_-|^2 - 4Re[a_0^* (a_+ + a_-)] + 12\bar{I}|a_0|^2 \right]$$

where $a_+, a_0$ are given in Eq. (10) while

$$\tilde{a}_- = (\sin^2 \theta_w - \frac{1}{2}) (c_{LL}^{(1)} + c_{LL}^{(2)}) + c_{4L}^{(1)} + c_{4L}^{(2)} + \frac{4\delta_{\mu \delta_e}^* \mu e}{\Delta_{\mu \tau}} (c_{4L}^{(4)} + c_{4L}^{(5)})$$

and $\tilde{I}$ is a phase space integral reported in appendix A. The rate for $\tau^- \to \mu^- e^- e^-$ is obtained by exchanging the labels $e \leftrightarrow \mu$ in $\Delta$ and $\delta$.

Finally, the rate for $\tau^- \to \mu^- \mu^- e^+$ is given by Eq. (11) replacing $\mu \to \tau$, but with

$$a_+ = a_0 = 0$$
$$a_- = \frac{2\delta_{\tau \delta_e} \mu \mu}{\Delta_{\mu \tau}} (c_{4L}^{(4)} + c_{4L}^{(5)})$$

(no photon and $Z$ exchange contribution) and similarly for $\tau^- \to e^- e^- \mu^+$, exchanging the labels $e \leftrightarrow \mu$ in $\delta$.

In order to explore the phenomenology of these results, it is convenient to express the rates in terms of dimensionless couplings with the large scale $\Lambda_{\text{LN}}$ (for minimal field content) or $M_{\nu}$ (for extended field content) factored out. Hence we introduce the quantities $a_{ij}, b_{ij}$ and $d_{ij}$ defined through

$$\Delta_{ij} = \begin{cases} \frac{\Lambda_{\text{LN}}}{v^2} a_{ij} & \text{minimal field content} \\ \frac{1}{M_{\nu}} b_{ij} & \text{extended field content} \end{cases}$$
and Eq. (8). These quantities depend only on low energy masses and mixing angles, and can be readily estimated. As an example we report the resulting expression for the $\mu \to 3e$ rate:

$$
\Gamma_{\mu \to 3e}/\Gamma_{\mu \to e\nu\bar{\nu}} = \left[|a_+|^2 + 2|a_-|^2 - 8\text{Re}(a_0^* a_-) - 4\text{Re}(a_0^* a_+) + 6|a_0|^2\right] \times
\left\{ \begin{array}{ll}
\left(\frac{\Lambda_{\text{LN}}}{\Lambda_{\text{LFV}}}\right)^4 |a_{e\mu}|^2 & \text{minimal field content} \\
\left(\frac{\nu M_e}{\Lambda_{\text{LFV}}}\right)^2 |b_{e\mu}|^2 & \text{extended field content}
\end{array} \right. 
$$  \quad (16)

### 3.2 Phenomenology: extended field content

Let us consider first the MLFV realization with extended field content, where the discussion is somewhat simpler. In this case the contributions of $O_4^{(4)}$ and $O_5^{(5)}$ to $\mu \to eee$ are negligible, since they involve the combination $\delta_{\mu e} \delta_{ee}$ which is suppressed by a small neutrino mass relative to the contributions to the amplitude that are proportional to $\Delta_{\mu e}$. As a consequence, the transition between families $i$ and $j$ is governed just by $b_{ij}$, the only allowed FCNC effective coupling. Moreover, for Wilson coefficients of $O(1)$, the expression in square brackets in Eq. (16) is $O(1)$ and therefore, for a given value of $(\nu M_\nu/\Lambda_{\text{LFV}}^2)$, the branching fraction for $\mu \to 3e$ is determined by $|b_{e\mu}|^2$.

Fig. 1 shows the behavior of $|b_{e\mu}|^2$ as a function of (the poorly determined mixing angle) $s_{13}$ varying the lightest neutrino mass in the experimentally allowed range $0 \leq m_{\text{min}} \leq 0.2$ eV [5], in the case of normal ordering of the neutrino mass spectrum. The plots correspond to the two CP conserving values of the phase $\delta$ in the PMNS matrix: $\delta = 0$ (left panel) and $\delta = \pi$ (right panel). As can be seen, the present uncertainties in $s_{13}$ and the absolute scale of neutrino spectrum induce a variation of $|b_{e\mu}|^2$ over a couple of orders of magnitude. Similar results apply in the case of inverted spectrum. For $M_\nu = 6 \times 10^7 \Lambda_{\text{LFV}}^2/\nu$, which saturates the perturbative Yukawa coupling bound on $M_\nu$ when $\Lambda_{\text{LFV}} = 50$ TeV, one obtains $\mu \to 3e$ branching fractions of order $10^{-12}$, comparable to the 90% C.L. limit of $1.0 \times 10^{-12}$ [7]. The $\tau \to 3\mu$ branching ratio is governed by $|b_{\mu\tau}|^2$, which does not depend on $s_{13}$ and for given $m_{\text{min}}$ is typically two orders of magnitude larger than $|b_{e\mu}|^2$, leading to $B_{\tau \to 3\mu} \sim 10^{-10}$. This is below current experimental sensitivities [8, 9].

The inclusion of 4-lepton processes in the phenomenological analysis offers more ways to test the hypothesis of minimal lepton flavor violation. We report below two noteworthy features, which with appropriate changes (see next subsection) extend to the minimal field scenario as well.

- As pointed out in [1], testable predictions of MLFV involve ratios of FCNC transitions between two different families (e.g. $\mu \to e$ vs $\tau \to \mu$). In the case of three-lepton final states, several operators contribute and they pick a different phase space weight, in principle. So predictions are not as clean as in the case of lepton-photon final state. We find

$$
\frac{B_{\mu \to 3e}}{B_{\tau \to 3\mu}} = \frac{|b_{e\mu}|^2 I_{\mu \to 3e}}{|b_{\mu\tau}|^2 I_{\tau \to 3\mu}} \frac{1 + \frac{r_c}{I_{\mu \to 3e}}}{1 + \frac{r_c}{I_{\tau \to 3\mu}}}, 
$$  \quad (17)
Figure 1: $|b_{e\mu}|^2$ as a function of $s_{13}$ for normal neutrino mass hierarchy and the PMNS phase $\delta = 0$ (left) or $\delta = \pi$ (right). The shaded bands correspond to variations of the lightest neutrino mass in the range $0 \leq m_{\text{min}} \leq 0.2$ eV. The upper edge of the bands corresponds to $m_{\text{min}} = 0$.

where the phase space integrals are $I_{\mu\rightarrow 3e} = 9.886$, $I_{\tau\rightarrow 3\mu} = 3.26$ and $r_c$ is the following combination of Wilson coefficients (in the absence of contact operators $r_c = 0$):

$$r_c = \frac{1}{6|a_0|^2} \left( |a_+|^2 + 2|a_-|^2 - 8\text{Re}(a_0^* a_-) - 4\text{Re}(a_0^* a_+) \right). \quad (18)$$

Modulo the last ($r_c$ dependent) factor which is of $O(1)$, Eq. (17) has the same structure of the prediction for the ratio of radiative decays $B_{\mu\rightarrow e\gamma}/B_{\tau\rightarrow \mu\gamma} = |b_{e\mu}|^2/|b_{\mu\tau}|^2$, and will offer another way to test the MLFV pattern $B_{\tau\rightarrow \mu} \gg B_{\mu\rightarrow e} \sim B_{\tau\rightarrow e}$. A precise test requires knowledge of the ratio $r_c$, which can be determined if additional LFV modes are observed, as we now discuss.

- Ratios of FCNC transitions between the same two families (e.g. $\mu \rightarrow e\gamma$ vs $\mu \rightarrow 3e$ or $\tau \rightarrow \mu\gamma$ vs $\tau \rightarrow 3\mu$ and $\tau \rightarrow \mu e\bar{e}$) are determined by known phase space factors and ratios of various Wilson coefficients. If both $\mu \rightarrow e\gamma$ and $\mu \rightarrow eee$ processes are observed one can begin to disentangle the effects of photon exchange from those of contact interactions, through the ratio of rates

$$\frac{\Gamma_{\mu\rightarrow 3e}}{\Gamma_{\mu\rightarrow e\gamma}} = \frac{\alpha}{4\pi} \frac{I_{\mu\rightarrow 3e}}{I_{\mu\rightarrow e\gamma}} \times \left[ 1 + \frac{r_c}{I_{\mu\rightarrow 3e}} \right], \quad (19)$$

with $I_{\mu\rightarrow 3e}$ defined in Eq. (11) and $r_c$ in Eq. (18). The above ratio reduces to $\alpha/4\pi I_{\mu\rightarrow 3e}$ in the absence of contact interactions and offers a way to experimentally determine $r_c$. One obtains the same result for $\Gamma_{\tau\rightarrow 3\mu}/\Gamma_{\tau\rightarrow \mu\gamma}$ (with $I_{\mu\rightarrow 3e} \rightarrow I_{\tau\rightarrow 3\mu}$ and the same $r_c$) and a similar one for $\Gamma_{\tau\rightarrow \mu e\bar{e}}/\Gamma_{\tau\rightarrow \mu\gamma}$.

### 3.3 Phenomenology: minimal field content

In the case of minimal field content, the analysis of 4-lepton processes is complicated by the contributions of $O_{4L}^{(4)}$ and $O_{4L}^{(5)}$, implying that there are two FCNC effective couplings that mediate $\mu \rightarrow 3e$: $a_{\mu e}$ and $d_{\mu e}^s d_{e e}^s$ (see Eqs. (9) and (10)). This naturally leads us to consider two cases.
Figure 2: Range of $|a_{e\mu}|^2$ as a function of $s_{13}$. The shaded band corresponds to varying the PMNS phase $\delta$ between 0 and $2\pi$. The range of $|a_{e\mu}|^2$ is the same as this, but is anti-correlated: the maximum of $|a_{e\mu}|^2$, achieved at $\delta = 0$, corresponds to the minimum of $|a_{e\tau}|^2$.

(i) If the dynamics of the underlying MLFV model results in $c_{4L}^{(4)} + c_{4L}^{(5)} \ll c_{4L}^{(i)}$ for $i \neq 4, 5$, then the analysis of $\mu \rightarrow 3e$ processes parallels the one of the extended field content given in the previous section, with the changes $|b_{\mu e}|^2 \rightarrow |a_{\mu e}|^2$ and $(eM_e/\Lambda_{LFV})^2 \rightarrow (\Lambda_{LN}/\Lambda_{LFV})^2$. As before, for a given value of $\Lambda_{LN}/\Lambda_{LFV}$, $|a_{\mu e}|^2$ determines the $\mu \rightarrow 3e$ branching ratio. We show in Fig. 2 the range of allowed values of $|a_{e\mu}|^2$ as a function of $s_{13}$, varying the PMNS phase $\delta$ between 0 and $2\pi$. We see that, except for very small $s_{13}$, in the minimal field content scenario a ratio $\Lambda_{LN}/\Lambda_{LFV} = 10^{10}$ implies a branching fraction for $\mu \rightarrow eee$ as large as a few times $10^{-12}$, comparable the 90% C.L. limit of $1.0 \times 10^{-12}$.[4]. The range of $|a_{e\mu}|^2$ is the same as for $|a_{e\mu}|^2$, but is anti-correlated: the maximum of $|a_{e\mu}|^2$, achieved at $\delta = 0$, corresponds to the minimum of $|a_{e\tau}|^2$. The quantity $a_{\mu\tau}$ is independent of $s_{13}$, $|a_{\mu\tau}|^2 = 1.6 \times 10^{-51}$. In this scenario Eq. (17) remains valid with the substitution $|b_{ij}|^2 \rightarrow |a_{ij}|^2$, and $r_e$ (given in Eq. (18)) involves just ratios of Wilson coefficients that can be determined experimentally by considering the ratio $\Gamma_{\mu \rightarrow 3e}/\Gamma_{\mu \rightarrow e\gamma}$ as in Eq. (19).

(ii) If the underlying dynamics is such that all Wilson coefficients are of comparable magnitude, then it is important to address the relative size of the FCNC couplings $a_{\mu e}$ and $d_{\mu e}d_{ee}^*$, as there is a potential competition between them. In Fig. 3 we plot the ratio $r_{e\mu} \equiv |2d_{\mu e}d_{ee}^*/a_{\mu e}|$ as a function of $m_{\text{min}}$ for $0 \leq m_{\text{min}} \leq 0.2$ eV, in the case of normal hierarchy (left) and inverted hierarchy (right). The points correspond to a scanning in parameter space with $0 \leq s_{13} \leq 0.20$ and $0 \leq \alpha_{1,2}, \delta \leq 2\pi$ for the CP violating phases in the PMNS matrix. For a given value of $m_{\text{min}}$, the upper edge of the range corresponds to $s_{13} \rightarrow 0$. As can be seen, $r_{e\mu}$ displays a very strong dependence on the absolute mass scale of the neutrino spectrum. Typically we find $r_{e\mu} \gg 1$, except in the normal hierarchy case and for $m_{\text{min}} \rightarrow 0$. Moreover, for a given value of $m_{\text{min}}$ there are regions of parameter space where cancellations induced by CP violating phases drive $r_{e\mu}$ to zero.

In the general case (ii), Eq. (19) remains valid but now $r_e$ of Eq. (18) depends on ratios of Wilson coefficients and the ratio of FCNC effective couplings $r_{e\mu} \equiv |2d_{\mu e}d_{ee}^*/a_{\mu e}|$. 
Figure 3: Range of the ratio $|2d_{e\mu}d_{\mu e}/a_{e\mu}|$ as a function of $m_{\text{min}}(\text{eV})$ in the case of normal hierarchy (left) and inverted hierarchy (right). We use the range $0 \leq s_{13} \leq 0.20$ and we allow the CP violating phases in the PMNS matrix to vary between 0 and $2\pi$. For a given value of $m_{\text{min}}$, the upper edge of the range corresponds to $s_{13} \rightarrow 0$.

The analogue of Eq. (17) now reads

$$\frac{B_{\mu\rightarrow 3e}}{B_{\tau\rightarrow 3\mu}} = \frac{|a_{e\mu}|^2 I_{\mu\rightarrow 3e}}{|a_{\mu\tau}|^2 I_{\tau\rightarrow 3\mu}} \frac{1 + r_c(r_{e\mu})}{1 + r_c(r_{\mu\tau})}.$$  \hspace{1cm} (20)

If one could measure both $\Gamma_{\mu\rightarrow 3e}/\Gamma_{\mu\rightarrow e\gamma}$ and $\Gamma_{\tau\rightarrow 3\mu}/\Gamma_{\tau\rightarrow \mu\gamma}$ (which determine $r_c(r_{e\mu})$ and $r_c(r_{\mu\tau})$ respectively), then Eq. (20) provides another way to test the MLFV hypothesis, insensitive to specific model details.

4 Disentangling operators: hadronic processes

As data becomes available on different lepton flavor violating processes it will be possible to begin disentangling the contributions of different operators. The rates for $\mu \rightarrow e\gamma$ and $\mu \rightarrow 3e$ decays and for $\mu$-to-$e$ conversion in nuclei depend through different linear combinations on the Wilson coefficients in the effective lagrangian in Eq. (3). The ratio $\Gamma_{\mu\rightarrow 3e}/\Gamma_{\mu\rightarrow e\gamma}$ is sensitive to contact four-lepton operators already present in the operator basis ($O_{4L}^{(i)}$) or induced by $O_{LL}^{(1,2)}$ via $Z^0$ exchange (see Eq. (19)). Similarly, $\Gamma_{\mu\rightarrow e\text{ conversion}}/\Gamma_{\mu\rightarrow e\gamma}$ is sensitive to the contact terms $O_{LL}^{(i)}$ involving quarks either directly ($O_{LL}^{(3-5)}$) or via $Z^0$ exchange ($O_{LL}^{(1,2)}$). In absence of contact operators the latter ratio is,

$$\frac{\Gamma_{\mu\rightarrow e\gamma}}{\Gamma_{\mu\rightarrow e\gamma}} = \pi D_A^2$$  \hspace{1cm} (21)

where we use the notation of Ref. [10] for the dimensionless nucleus-dependent overlap integral $D_A$. Typical values for the overlap integral in the case of a light and heavy nucleus are [10]: $D_{Al} \simeq 0.036$ and $D_{Au} \simeq 0.19$. A deviation from the result of Eq. (21) could be attributed to the contribution of $O_{LL}^{(i)}$. 

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Figure 4: Ratio $\Gamma(\tau \rightarrow \pi\mu)/\Gamma(\tau \rightarrow \mu\nu\bar{\nu})$ in units of $10^{-27}(M_\nu v/\Lambda_{LFV}^2)^2$, in the extended field content scenario as a function of the lightest neutrino mass (in eV). The red (lower) curve is for the normal hierarchy of neutrino masses while the green (upper) curve is for the inverted hierarchy.

After the cancellation of the MECO experiment, which would have had a single event sensitivity to branching fraction for $\mu$-to-$e$ conversion of $2 \times 10^{-17}$, it seems that experimental searches of $\mu$-to-$e$ conversion are in the far future. This motivates us to explore the sensitivity of other processes to $O_{LL}^{(3-5)}$. In this section we study other decays where hadrons are involved and $O_{LL}^{(3-5)}$ contribute at tree level. Finally, in Sec. 5 we shall investigate whether $O_{LL}^{(3-5)}$ can give significant contributions to $\mu \rightarrow 3e$ through loop effects in models where the 4L operators are not generated at the scale $\Lambda_{LFV}$ (and therefore their coefficients can be discarded).

4.1 $\tau \rightarrow \pi^0\mu$

Consider $\tau \rightarrow \mu X$, where $X = \pi^0$, $\rho^0$, etc. With a non-vanishing left handed neutrino Majorana mass term, electroweak interactions will produce a non-zero amplitude at one loop. Roughly, this amplitude corresponds to an operator

$$O_{SM} = \left[ \frac{\alpha^2}{\sqrt{s} M_W^2 M_Z^2} \right] (\bar{U} m^2 \bar{U}^\dagger)_{\mu\tau} (\bar{\mu}_L \gamma^\sigma \tau_L) (\bar{q} \gamma_\sigma (\gamma_5) q).$$

This is much smaller than the contribution from the operators generated at the LFV scale. For example, for minimal content the factor in square brackets, of order $10^{-10}$ GeV$^{-4}$, is replaced by $1/v^4(\Lambda_{LN}/\Lambda_{LFV})^2 \sim 10^{-9}$ GeV$^{-4}(\Lambda_{LN}/\Lambda_{LFV})^2$, which for any reasonable value of the ratio $\Lambda_{LN}/\Lambda_{LFV}$ is orders of magnitude bigger.

A straightforward calculation gives

$$\Gamma(\tau \rightarrow \mu\pi^0) = \frac{1}{256\pi} \frac{m^3_{\tau} \Delta^2_{\mu\tau} f^2_\pi}{\Lambda_{LFV}^4} \left[ c_{LL}^{(1)} + c_{LL}^{(2)} - c_{LL}^{(4d)} + c_{LL}^{(4u)} + 2c_{LL}^{(5)} \right]^2,$$

where $f_\pi = 130$ MeV is the $\pi$ decay constant. Numerically we find, in the minimal model

$$\frac{\Gamma(\tau \rightarrow \mu\pi^0)}{\Gamma(\tau \rightarrow \mu\nu\bar{\nu})} = 5.2 \times 10^{-52} \left( \frac{\Lambda_{LN}}{\Lambda_{LFV}} \right)^4 \left[ c_{LL}^{(1)} + c_{LL}^{(2)} - c_{LL}^{(4d)} + c_{LL}^{(4u)} + 2c_{LL}^{(5)} \right]^2.$$
Using $\Lambda_{\text{LN}}/\Lambda_{\text{LFV}} = 10^9$ which corresponds to $\text{Br}(\mu \to e\gamma) \sim 10^{-13}$, and assuming the coefficients $c_{iLL}^{(i)}$ are of order unity, this yields $\text{Br}(\tau \to \mu\pi^0) \sim 10^{-15}$.

For the extended model we obtain

$$\frac{\Gamma(\tau \to \mu\pi^0)}{\Gamma(\tau \to \mu\nu\bar{\nu})} = 5.0 \times 10^{-27} \left(\frac{vM_{\mu}}{\Lambda_{\text{LFV}}^4}\right)^2 \left[c_{LL}^{(1)} + c_{LL}^{(2)} - c_{LL}^{(4d)} + c_{LL}^{(4u)} + 2c_{LL}^{(5)}\right]^2,$$

in the normal hierarchy case, assuming $m_{\text{min}} = m_{\nu_1} \ll m_{\nu_3} \approx \sqrt{\Delta m_{\text{atm}}^2}$. The result is plotted as a function of the lightest neutrino mass, $m_{\text{min}}$, in Fig. 4 both for normal and inverted hierarchies. If $M_{\nu} = 3 \times 10^5 \Lambda_{\text{LFV}}^2/v$, which again corresponds to $\text{Br}(\mu \to e\gamma) \sim 10^{-13}$, we obtain $\text{Br}(\tau \to \mu\pi^0) \sim 10^{-15}$.

### 4.2 $\pi^0 \to \mu^+e^-$

With almost no extra work we can compute

$$\Gamma(\pi^0 \to \mu^+e^-) = \frac{f^2 m_{\pi}^2 (m_{\mu}^2 - m_{\pi}^2)^2 |\Delta_{\mu\pi}|^2}{128 \pi m_{\pi}^2 \Lambda_{\text{LFV}}^4} \left[c_{LL}^{(1)} + c_{LL}^{(2)} - c_{LL}^{(4d)} + c_{LL}^{(4u)} + 2c_{LL}^{(5)}\right]^2.$$

Numerically, using $1/\tau(\pi^0) = 7.83$ eV,

$$\text{Br}(\pi^0 \to \mu^+e^-) = 1.4 \times 10^{-9} \left[c_{LL}^{(1)} + c_{LL}^{(2)} - c_{LL}^{(4d)} + c_{LL}^{(4u)} + 2c_{LL}^{(5)}\right]^2 \left\{\begin{array}{ll} (\Lambda_{\text{LN}}/\Lambda_{\text{LFV}})^4 |a_{e\mu}|^2 & \text{minimal} \\ (M_{\nu}v/\Lambda_{\text{LFV}}^2)^2 |b_{e\nu}|^2 & \text{extended} \end{array}\right\}.$$

Assuming $c_{LL}^{(i)} \sim O(1)$ and $\Lambda_{\text{LN}}/\Lambda_{\text{LFV}} = 10^9$ one expects $\text{Br}(\pi^0 \to \mu^+e^-) \sim 10^{-25}$.

### 4.3 $J/\psi \to \tau\mu$ and $\Upsilon \to \tau\mu$

The lepton flavor violating decays of heavier neutral mesons are interesting because decays into $\tau$’s become energetically allowed. By normalizing the decay rate for $V \to \ell\ell'$ (for $V = J/\psi, \Upsilon$) to the well measured $V \to ee$ we can make a prediction free of hadronic uncertainties:

$$\frac{\Gamma(V \to \ell_i\ell_j)}{\Gamma(V \to ee)} = \frac{1}{e^2} \left(\frac{m_V}{2v}\right)^4 \left|\tilde{C}_V\right|^2 \left\{\begin{array}{ll} (\Lambda_{\text{LN}}/\Lambda_{\text{LFV}})^4 |a_{ij}|^2 & \text{minimal} \\ (M_{\nu}v/\Lambda_{\text{LFV}}^2)^2 |b_{ij}|^2 & \text{extended} \end{array}\right\},$$

with

$$\tilde{C}_V = \left\{\begin{array}{ll} c_{LL}^{(1)} + c_{LL}^{(2)} \left(\frac{1}{2} - \frac{4}{3} \sin^2 \theta_w\right) + c_{LL}^{(3)} + c_{LL}^{(4d)} - c_{LL}^{(5)} & V = J/\psi \\ c_{LL}^{(1)} + c_{LL}^{(2)} \left(-\frac{1}{2} + \frac{2}{3} \sin^2 \theta_w\right) + c_{LL}^{(3)} + c_{LL}^{(4d)} + c_{LL}^{(5)} & V = \Upsilon \end{array}\right\}.$$

In particular, for $\Upsilon$ decays into $\tau\mu$ this gives

$$\text{Br}(\Upsilon \to \tau^+\mu^-) = \left[c_{LL}^{(1)} + c_{LL}^{(2)} \left(-\frac{1}{2} + \frac{2}{3} \sin^2 \theta_w\right) + c_{LL}^{(3)} + c_{LL}^{(4d)} + c_{LL}^{(5)}\right]^2 \times \left\{\begin{array}{ll} 3.2 \times 10^{-55} (\Lambda_{\text{LN}}/\Lambda_{\text{LFV}})^4 & \text{minimal} \\ 3.2 \times 10^{-30} (M_{\nu}v/\Lambda_{\text{LFV}}^2)^2 & \text{extended, normal h, } m_{\nu_{\text{min}}} = 0 \end{array}\right\} \times \left\{\begin{array}{ll} 4.3 \times 10^{-30} (M_{\nu}v/\Lambda_{\text{LFV}}^2)^2 & \text{extended, inverted h, } m_{\nu_{\text{min}}} = 0 \end{array}\right\}.$$
Assuming again $c_{LL}^{(i)} \sim O(1)$ and $\Lambda_{LN}/\Lambda_{LFV} = 10^9$ one expects $\text{Br}(\Upsilon \to \tau^+ \tau^-) \sim 10^{-20}$.

All the hadronic modes we have considered here, for acceptable values of the ratios $\Lambda_{LN}/\Lambda_{LFV}$ and $v M_\nu/\Lambda_{LFV}^2$ and of the Wilson coefficients, are predicted to be well below foreseeable experimental sensitivities [12]. On one hand this result can be considered as a generic prediction of minimal lepton flavor violation and can be used in the future to test this framework (e.g. observation of $\Upsilon \to \tau^+ \mu^-$ at B factories would strongly disfavor it). On the other hand we see that within minimal lepton flavor violation the most sensitive probes of $O_{LL}^{(3-5)}$ are: (i) $\mu \to e$ conversion in nuclei and (ii) purely leptonic processes such as $\mu \to 3e$ (to which $O_{LL}^{(3-5)}$ contribute via loops).

5 $\mu \to eee$ in the absence of 4L operators

The operators $O_{LL}^{(3-5)}$ can give rise to $\mu \to eee$ via one loop graphs. These contributions can be significant in models where the 4L operators are not generated at the scale $\Lambda_{LFV}$, or are generated with tiny Wilson coefficients. We will analyze now this dynamical scenario, which amounts to the assumption $c_{4L}(\Lambda_{LFV}) \ll c_{LL}(\Lambda_{LFV})$. In order to address the sensitivity of $\mu \to eee$ to $c_{LL}^{(3-5)}$, we organize our analysis as follows: first we use the renormalization group to evolve the effective lagrangian from $\mu \sim \Lambda_{LFV}$ down to $\mu \sim 1$ GeV. Then we take $\mu \to 3e$ matrix elements of the relevant operators, using perturbative QCD to deal with the heavy quarks and chiral perturbation theory to deal with the light quark loops.

5.1 RG analysis

We use the renormalization group equations (RGEs) to determine the low energy effective lagrangian. This is done in three steps: first, the RGE in the unbroken phase of the $SU(2) \times U(1)$ theory is used to compute the coefficients in the effective lagrangian in Eq. (3) down to a scale $\mu \sim M_Z$. In the second step the coefficients are matched to those of an effective lagrangian for the theory in the broken symmetry phase of $SU(2) \times U(1)$. And third, the coefficients of this effective lagrangian are computed at $\mu \sim 1$ GeV using the RGE for the theory with only $U(1)$ gauge group. The calculation is rather lengthy and technical, so the details are presented in Appendix B.

In the first step of the calculation we compute for $M_Z < \mu < \Lambda_{LFV}$ the mixing of coefficients $c_{LL}^{(i)}$ into the coefficients of the four-lepton contact interactions, $c_{4L}^{(i)}$, assuming $c_{4L}(\Lambda_{LFV}) \ll c_{LL}(\Lambda_{LFV})$. In the second step, an effective theory with $U(1)$ gauge field is constructed with effective lagrangian given by matching at $\mu \sim M_Z$ to the theory with
SU(2) × U(1) symmetry. The relevant operators in the new effective theory are

\[
\hat{O}_{LL}^{(3)} = \bar{e}_L \gamma^\mu \Delta e_L \left( \bar{u}_L \gamma_\mu u_L + \bar{d}_L \gamma_\mu d_L \right)
\]
\[
\hat{O}_{LL}^{(4d)} = \bar{e}_L \gamma^\mu \Delta e_L \bar{d}_R \gamma_\mu d_R
\]
\[
\hat{O}_{LL}^{(4u)} = \bar{e}_L \gamma^\mu \Delta e_L \bar{u}_R \gamma_\mu u_R
\]
\[
\hat{O}_{LL}^{(5)} = -\bar{e}_L \gamma^\mu \Delta e_L \left( \bar{u}_L \gamma_\mu u_L - \bar{d}_L \gamma_\mu d_L \right)
\]
\[
\hat{O}_{RL}^{(1)} = e\bar{v}_R \sigma^{\mu\nu} \lambda_e \Delta e_L F_{\mu\nu}
\]
\[
\hat{O}_{4L}^{(1)} = \bar{e}_L \gamma^\mu \Delta e_L \bar{e}_L \gamma_\mu e_L
\]
\[
\hat{O}_{4L}^{(3)} = \bar{e}_L \gamma^\mu \Delta e_L \bar{e}_R \gamma_\mu e_R
\]

where sums over quark generations are understood (the top quark has been integrated out), and the effective lagrangian is

\[
\mathcal{L} = \frac{1}{\Lambda_{\text{LFV}}^2} \sum_i \left( \hat{c}_{LL}^{(i)} \hat{O}_{LL}^{(i)} + \hat{c}_{4L}^{(i)} \hat{O}_{4L}^{(i)} \right) + \frac{1}{\Lambda_{\text{LFV}}^2} \left( \sum_j \hat{c}_{RL}^{(j)} \hat{O}_{RL}^{(j)} + \text{h.c.} \right)
\]

The coefficients \( \hat{c}_{LL,RL,4L} \), obtained by matching, are given in Eq. (55) of Appendix B.

Finally, in the third step we solve the RGE satisfied by \( \hat{c}_{4L}^{(1)} \) and \( \hat{c}_{4L}^{(3)} \). Ignoring the running of \( c_{LL/RL} \) (that is working to order \( g^2 \)), we finally obtain the following low scale (\( \mu < M_Z \)) Wilson coefficients:

\[
\hat{c}_{4L}^{(1)}(\mu) = -\frac{\alpha}{\pi} \left\{ \frac{1}{3} \ln \left( \frac{\mu}{M_Z} \right) \left[ c_{LL}^{(3)} + 4c_{LL}^{(4u)} - 3c_{LL}^{(4d)} - 7c_{LL}^{(5)} \right] + 2 \ln \left( \frac{\sqrt{2} m_t}{M_Z} \right) \left[ c_{LL}^{(3)} + c_{LL}^{(4u)} - c_{LL}^{(5)} \right] \right. \\
+ \ln \left( \frac{M_Z}{\Lambda_{\text{LFV}}} \right) \left[ \frac{1}{2 \cos^2 \theta_w} \left( c_{LL}^{(3)} + 2c_{LL}^{(4u)} + c_{LL}^{(4d)} \right) - \frac{3}{2 \sin^2 \theta_w} c_{LL}^{(5)} \right] \right\} \\
- \left( \frac{1}{2} - \sin^2 \theta_w \right) \left( c_{LL}^{(1)} + c_{LL}^{(1)} \right)
\]

\[
\hat{c}_{4L}^{(3)}(\mu) = -\frac{\alpha}{\pi} \left\{ \frac{1}{3} \ln \left( \frac{\mu}{M_Z} \right) \left[ c_{LL}^{(3)} + 4c_{LL}^{(4u)} - 3c_{LL}^{(4d)} - 7c_{LL}^{(5)} \right] + 2 \ln \left( \frac{\sqrt{2} m_t}{M_Z} \right) \left[ c_{LL}^{(3)} + c_{LL}^{(4u)} - c_{LL}^{(5)} \right] \right. \\
+ \ln \left( \frac{M_Z}{\Lambda_{\text{LFV}}} \right) \left[ \frac{1}{\cos^2 \theta_w} \left( c_{LL}^{(3)} + 2c_{LL}^{(4u)} + c_{LL}^{(4d)} \right) \right] \right\} + \sin^2 \theta_w \left( c_{LL}^{(1)} + c_{LL}^{(1)} \right).
\]

### 5.2 Matrix elements

At the order in \( g^2 \) we are working, the matrix elements of \( \hat{O}_{4L}^{(i,3)} \) have to be taken at tree level, while those of \( \hat{O}_{LL}^{(i)} \) have to be evaluated at one loop. Consider for definiteness the
Considering the graphs of Fig. 5 with heavy quark internal loops, we find

\[ \langle m \text{ in muon decays} \rangle = \frac{g^{\mu\nu} M_{\mu\nu}(p_1, p_2)}{\sqrt{m^2_{13} + m^2_{12}}}, \]

and the kinematic variables \( x = m^2_{13}/m^2_{12} = (p_{e_1} + p_e)^2/m^2_{12} \) and \( y = m^2_{23}/m^2_{12} = (p_{e_2} + p_e)^2/m^2_{12} \).

**Matrix elements of \( \hat{O}_{LL}^{(i)} \): heavy quark contributions**

Considering the graphs of Fig. 5 with heavy quark internal loops, we find

\[
\langle 3e | \hat{c}_{LL}^{(i)} \hat{O}_{LL}^{(i)} | \mu \rangle = -\frac{\alpha}{\pi} \left[ (2 F_c(y) - F_b(y)) c_{LL}^{(3)} + 2 F_c(y) c_{LL}^{(4u)} \right. \\
- F_b(y) c_{LL}^{(4d)} - (2 F_c(y) + F_b(y)) c_{LL}^{(5)} \left] \cdot M(p_1, p_2) \right.
\]

where \( F_q(x) \equiv F(m_q, m_q/m_{\mu})^2/x \), with

\[
F(m, z) = \begin{cases} 
\frac{1}{6} \ln \frac{2m^2}{\mu^2} - \frac{5}{18} - \frac{2}{3} + \frac{1}{3}(1 + 2z)\sqrt{4z - 1} \arctan \left( \frac{1}{\sqrt{4z - 1}} \right) & \text{for } 4z > 1 \\
\frac{1}{6} \ln \frac{2m^2}{\mu^2} - \frac{5}{18} - \frac{2}{3} + \frac{1}{6}(1 + 2z)\sqrt{1 - 4z} \ln \left( \frac{1 + \sqrt{1 - 4z}}{1 - \sqrt{1 - 4z}} \right) + i\pi & \text{for } 4z < 1
\end{cases}
\]

In muon decays \( m^2_{13}, m^2_{12} \sim q^2 < m^2_{12} \), and by using

\[
F(m, m^2/q^2) = \frac{1}{6} \log \frac{2m^2}{\mu^2} - \frac{q^2}{30m^2} + \ldots
\]

the heavy quark contribution can be safely approximated to a constant (local) term. Using the \( q^2/m^2_q \rightarrow 0 \) limit and the notation \( F_q \equiv 1/3 \log \left[ (\sqrt{2} m_q)/\mu \right] \) we then obtain:

\[
\langle 3e | \sum_i \hat{c}_{LL}^{(i)} \hat{O}_{LL}^{(i)} | \mu \rangle = -\frac{\alpha}{\pi} \left[ (2 F_c - F_b)c_{LL}^{(3)} + 2 F_c c_{LL}^{(4u)} - F_b c_{LL}^{(4d)} - (2 F_c + F_b)c_{LL}^{(5)} \right] \\
\times \left[ \langle \hat{O}_{4L}^{(1)} \rangle + \langle \hat{O}_{4L}^{(3)} \rangle \right]
\]
It is easy to check that the scale dependence of the above matrix elements cancels the one induced by $b, c$ loops in the Wilson coefficients $\hat{c}_L^{(i)}, \hat{c}_L^{(3)}$ given in Eq. (31).

**Matrix elements of $\hat{O}_{LL}^{(i)}$: light quark contributions**

The operators $\hat{O}_{LL}^{(i)}$ have the structure

$$\hat{O}_{LL}^{(i)} = \bar{e}_L \gamma^\mu \Delta e_L \frac{1}{2} \left[ V_\mu^{(i)} \pm A_\mu^{(i)} \right],$$

(38)

where the flavor structure of the currents $V_\mu^{(i)}, A_\mu^{(i)}$ can be read off Eq. (32). Only the vector current contributes to the matrix elements in question and we find:

$$\langle 3e|\hat{O}_{LL}^{(i)}|\mu\rangle = -i \frac{e^2}{2} \mathcal{M}^{\mu\nu}(p_1, p_2) \frac{1}{(p_2 + p_e)^2} \times$$

$$\times \int d^4x \epsilon^{x(p_2 + p_e)} \langle 0| T \left( V_\mu^i(0) V_\nu^{EM}(x) \right) |0 \rangle - \{p_1 \leftrightarrow p_2\}$$

(39)

Defining then

$$\int d^4x \epsilon^{xq} \langle 0| T \left( V_\mu^i(0) V_\nu^{EM}(x) \right) |0 \rangle = \frac{i}{(4\pi)^2} \left( g_{\mu\nu}q^2 - q_\mu q_\nu \right) \cdot \Pi^{i,EM}(q^2),$$

(40)

one arrives at:

$$\langle 3e|\hat{O}_{LL}^{(i)}|\mu\rangle = \frac{e^2}{2(4\pi)^2} \mathcal{M}(p_1, p_2) \Pi^{i,EM}((p_2 + p_e)^2) - \{p_1 \leftrightarrow p_2\}$$

(41)

In order to evaluate the VV correlator at low momentum transfer we use $SU(3) \times SU(3)$ chiral perturbation theory [13] to order $p^4$. Evaluating the one-loop diagrams depicted in Fig. 6 (with vertices from the $O(p^2)$ chiral lagrangian) and adding the counterterm contributions from the local $O(p^4)$ effective lagrangian [13], we find:

$$\Pi^{4u,EM}(q^2) = \left[ \sum_{i=\pi,K} f \left( \frac{m_i^2}{q^2}, \frac{m_i^2}{\mu_X^2} \right) - (4\pi)^2 \frac{7}{3} \left( 2H_1^r(\mu_X, \mu) + L_{10}^r(\mu_X) \right) \right]$$

(42)

The effective couplings $L_{10}$ and $H_1$ are defined in Ref. [13], and they cancel the loop-induced dependence on the chiral renormalization scale $\mu_X$. The constant $H_1$, however, depends on the renormalization scheme adopted in the short distance theory (at the quark level), which is in this case the $\overline{\text{MS}}$ scheme. This dependence is such as to cancel the $\mu$ dependence of the amplitude induced by the Wilson coefficients (see appendix C for details). Similarly we find $\Pi^{3,EM}(q^2) = 0$, $\Pi^{4d,EM}(q^2) = -\Pi^{4u,EM}(q^2)$, and $\Pi^{5,EM}(q^2) = 2\Pi^{4u,EM}(q^2)$. The final result for the light-quark loop matrix elements is

$$\langle 3e| \sum_i \hat{c}_L^{(i)} \hat{O}_{LL}^{(i)}|\mu\rangle = \frac{\alpha}{4\pi} \frac{1}{2} \left[ 2\hat{c}_L^{(4u)} + 2\hat{c}_L^{(5)} \right] \times$$

$$\times \mathcal{M}(p_1, p_2) \Pi^{4u,EM}((p_2 + p_e)^2) - \{p_1 \leftrightarrow p_2\}.$$
Over the physical region for $\mu \rightarrow 3e$ decay the function $\Pi^{4u,EM}(q^2)$ varies by less than 1%\(^2\). Approximating it to a constant we find:

$$
\langle 3e \rangle \sum_i \hat{c}_{\mu}^{(i)} \hat{O}_{\mu}^{(i)} | \mu \rangle = \frac{\alpha}{4\pi} \frac{1}{2} \left[ \hat{c}_{\mu}^{(4a)} - \hat{c}_{\mu}^{(4d)} + 2\hat{c}_{\mu}^{(5)} \right] \Pi^{4u,EM} (m_\mu^2) \times \left[ \langle \hat{O}_{\mu}^{(1)} \rangle + \langle \hat{O}_{\mu}^{(3)} \rangle \right] . \tag{44}
$$

In summary, the matrix element calculation results in:

$$
\langle 3e \rangle \sum_i \hat{c}_{\mu}^{(i)} \hat{O}_{\mu}^{(i)} | \mu \rangle = (\kappa_{\text{heavy} - q} + \kappa_{\text{light} - q}) \times \left( \langle \hat{O}_{\mu}^{(1)} \rangle + \langle \hat{O}_{\mu}^{(3)} \rangle \right) , \tag{45}
$$

with

$$
\kappa_{\text{heavy} - q} = -\frac{\alpha}{\pi} \left[ (2F_c - F_b)\hat{c}_{\mu}^{(3)} + 2F_c\hat{c}_{\mu}^{(4a)} - F_b\hat{c}_{\mu}^{(4d)} - (2F_c + F_b)\hat{c}_{\mu}^{(5)} \right] \\
\kappa_{\text{light} - q} = \frac{\alpha}{8\pi} \left[ \hat{c}_{\mu}^{(4a)} - \hat{c}_{\mu}^{(4d)} + 2\hat{c}_{\mu}^{(5)} \right] \Pi^{4u,EM} (m_\mu^2) . \tag{46}
$$

### 5.3 Rate

The result in Eq. (45) implies that in the scenario considered in this section the $\mu \rightarrow e\bar{e}e$ rate is given in closed form by Eq. (9), replacing the coefficients $a_\pm$ of Eq. (10) by:

$$
a_+ = \hat{c}_{\mu}^{(3)} + \kappa_{\text{heavy} - q} + \kappa_{\text{light} - q} , \\
a_- = \hat{c}_{\mu}^{(1)} + \kappa_{\text{heavy} - q} + \kappa_{\text{light} - q} , \tag{47}
$$

with $\hat{c}_{\mu}^{(1,3)}$ given by Eq. (31). Taking into account the $\mu$ dependence of $H^c_1(\mu, \mu)$ it is straightforward to verify that $a_\pm$ do not depend on the renormalization scale $\mu$.

While the formulas given above are quite general, in order to illustrate at which level $\mu \rightarrow e\bar{e}e$ decays probe $O_{\mu}^{(3-5)}$, let us assume that only one operator dominates. Moreover, let us focus on the operator contributing with the largest numerical coefficient, which turns out to be $O_{\mu}^{(5)}$. We find then:

$$
\Gamma_{\mu \rightarrow 3e/\Gamma_{\mu \rightarrow e\nu\bar{\nu}}} = 6 \times 10^{-3} \times \left| \hat{c}_{\mu}^{(5)} (\Lambda_{\text{LFV}}) \right|^2 \times \begin{cases} \left( \frac{\Lambda_{\text{LFV}}}{\Lambda_{\text{LFV}}} \right)^4 |a_{e\mu}|^2 & \text{minimal field content} \\
\left( \frac{\Lambda_{\text{LFV}}}{\Lambda_{\text{LFV}}} \right)^2 |b_{e\mu}|^2 & \text{extended field content} \end{cases} . \tag{48}
$$

---

\(^{2}\)For the low energy constants we use $L_{\text{EM}}^c(\mu, \mu) = -5.5 \times 10^{-3}$ (experiment, see e.g. [14]) and $|H^c_1(\mu, \mu = m, \mu = 1\text{GeV})| \leq 1/(64\pi^2)$ (naive dimensional analysis [15] bound).
A comparison of Eqs. (48) and (16) shows that the rates of $\ell \to \ell'\ell''\ell'''$ decays in the scenario $c_{4L}^{(i)}(\Lambda_{\text{LFV}}) \ll c_{LL}^{(j)}(\Lambda_{\text{LFV}})$ are typically suppressed relative to their tree level counterparts by $\sim 10^{-3}$. This factor is larger than the naive estimate $(\alpha/\pi)^2$ because of the presence of sizeable logarithms. Similar logarithmic enhancements were first pointed out within a non-MLFV effective theory approach to lepton flavor violation in Ref. [16] (in that reference the main focus was on the one-loop contributions to $\mu$-to-$e$ conversion in nuclei). Finally, let us notice that apart from the overall suppression, in this dynamical scenario we still find the pattern $\Gamma(\tau \to \mu) \gg \Gamma(\mu \to e) \sim \Gamma(\tau \to e)$, which is dictated by the MLFV hypothesis.

6 Conclusions

The MLFV hypothesis provides a framework for discussing and analyzing the predictions of a large class of models where Lepton Flavor Violation arises solely from lepton mass matrices. The framework is attractive because it is both very general and fairly predictive. We have extended the results of Ref. [1], which introduced the MLFV hypothesis and analyzed $\mu$-to-$e$ conversion in Nuclei and radiative decays, e.g., $\mu \to e\gamma$, to decays involving hadrons, e.g., $\tau \to \mu\pi$, and to decays involving only charged leptons, e.g., $\mu \to 3e$. To this end we extended the operator basis for the effective lagrangian introduced in [1]. In particular, we found five new purely leptonic operators, given in Eq. (5), which were omitted in [1] because they do not contribute directly to the processes considered there.

In the event that several LFV processes are observed and their rates measured one could begin to test the MLFV hypothesis since it predicts definite patterns of relative magnitudes of rates. We have therefore computed the rates for the hadronic processes $\tau \to \ell\pi$ ($\ell = \mu, e$), $\pi^0 \to \mu^+e^-$ and $V \to \tau\mu$ ($V = J/\psi, \Upsilon$), and for purely charged lepton decays $\mu \to 3e$, $\tau \to 3\ell$ ($\ell = e, \mu$), $\tau \to \ell\ell\ell'$ and $\tau \to \ell\ell'\ell''$ (($\ell, \ell') = (e, \mu), (\mu, e)$). One definite prediction of the MLFV hypothesis is that the rates for decays involving hadrons are exceedingly small. For example, if the scales of LFV and LNV are such that $\text{Br}(\mu \to e\gamma) \sim 10^{-13}$, then $\text{Br}(\tau \to \mu\pi^0) \sim 10^{-15}$, $\text{Br}(\pi^0 \to \mu^+e^-) \sim 10^{-25}$ and $\text{Br}(\Upsilon \to \tau^+\mu^-) \sim 10^{-20}$, well below the sensitivity of foreseeable experiments.

We analyzed the decays involving only charged leptons first at tree level and then through loop processes, in order to investigate the contribution of the operators involving quarks (under the assumptions that the coefficients of the 4-lepton operators of Eq. (5) are negligibly small). Some general conclusions apply to both cases: (i) LFV $3\ell$ decays display the same pattern $B_{\tau \to \mu} \gg B_{\mu \to e} \sim B_{\tau \to e}$ as the radiative decays, and hence offer an alternative way to test the MLFV hypothesis; (ii) ratios of transitions between the same two families, as, for example $\Gamma(\mu \to 3e)/\Gamma(\mu \to e\gamma)$, are determined by ratios of various Wilson coefficients and therefore the combined measurement would be valuable in disentangling the contributions of different operators.

More specifically, the tree level rates for decays involving only charged leptons can be very significant. If the coefficients in the effective lagrangian are of order 1 and the ratio of scales of LNV and LFV is as large as allowed by the condition that the Yukawa couplings are perturbative, then in the case of minimal field content $\text{Br}(\mu \to 3e)$ could
easily exceed the present 90\% C.L. limit of $1.0 \times 10^{-12}$. In the extended field content case, under the same assumptions, this branching fraction is comparable to the current experimental limit, and the fractions for LFV $\tau$ decays are three orders of magnitude smaller than the current limits of $1 - 3 \times 10^{-7}$[8,9]. Note that while the rates for LFV $\tau$ decays are larger than for $\mu \to 3e$, the higher experimental sensitivity of the latter is on the verge of placing stringent bounds on the theory and therefore it would be highly desirable to pursue higher sensitivity in this mode.

If the new physics giving rise to the effective lagrangian at the LFV scale is tied to the quark sector in such a way that only operators involving quarks and leptons together were produced by this dynamics, then the decays to $3\ell$ would not proceed at tree level. To investigate this scenario we computed the rates for $3\ell$ decays through loops, assuming the coefficients of the 4-lepton operators of Eq. (5) are negligibly small. Alternatively, since the rates for processes involving hadrons are exceedingly small, a more sensitive probe of them may arise through their loop effect on $3\ell$ decays. Not surprisingly we found the rates are suppressed relative to their tree level counterparts by $\sim 10^{-3}$. The calculation is interesting in its own right. It involves the computation of a low energy effective lagrangian obtained by integrating out the heavy quarks and the $W^\pm$ and $Z^0$ vector bosons, and also requires the use of the chiral lagrangian to properly describe the low energy physics involving light quarks. As a side result a new sum rule for the Gasser-Leutwyler counter-term $H_1^r$ was derived in Appendix C.

A discrimination of various dynamical scenarios within MLFV will be possible only when more than one LFV decay mode is observed. This remains true even when analyzing data beyond the minimal flavor violation hypothesis [17]. We therefore emphasize that it is highly desirable to complement existing experimental efforts [18,8,9] and pursue experimental searches of all the LFV $\mu$ and $\tau$ decay modes.

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A  Differential distributions and phase space for $\ell \to \ell'\ell''\ell'''$

The differential rate for $\mu \to ee\bar{e}$ in the variables $x = m_{13}^2/m_{\mu}^2 = (p_{e_1} + p_e)^2/m_{\mu}^2$ and $y = m_{23}^2/m_{\mu}^2 = (p_{e_2} + p_{e})^2/m_{\mu}^2$ is:

$$
\frac{d^2\Gamma}{dxdy} = \frac{m_{\mu}^5}{2^9 \pi^3 A_{\text{HFV}}^4} \frac{1}{xy} \left[ 2 \left( 2x^3 - 2x^2 + x + 2y^3 - 2y^2 + y \right) |a_0|^2 
+ |a_-|^2 \left( -8yx^3 - 16y^2x^2 + 8yx^2 - 8y^3x + 8y^2x \right) + |a_+|^2 \left( -2yx^3 + 2yx^2 - 2y^3x + 2y^2x \right) 
+ a_0 \left( (8y^2 + 8y^2x - 8yx) a_+^* + (-2yx^2 - 2y^2x) a_+^* \right) 
+ (a_+ (-2yx^2 - 2y^2x) + a_- (8y^2 + 8y^2x - 8yx)) a_0^* \right] .
$$

The phase space integral $I$ appearing in Eq. (49) reads:

$$
I = \int_{4\tilde{m}^2}^{(1-\tilde{m})^2} dx \int_{y_{\text{min}}}^{y_{\text{max}}} dy \left\{ \frac{1}{xy} \left[ y(1-x) + x(1-y) - 2(1-x-y)(x^2+y^2) \right] + 2(1-x-y) \right\}
$$

in terms of $x = (p_{e_1} + p_e)^2/m_{\mu}^2$, $y = (p_{e_2} + p_{e})^2/m_{\mu}^2$, and $\tilde{m} \equiv m_{\ell}/m_{\mu}$. The integration limits are given by:

$$
y_{\text{min}} = \frac{1}{2} \left[ 1 - x + 3\tilde{m}^2 \mp \sqrt{(x - 4\tilde{m}^2)((1-x - \tilde{m})^2 - 4\tilde{m}^2x)} \right] .
$$

Numerically, $I = 9.886$ for $\mu \to ee\bar{e}$ $I = 17.4$ for $\tau \to ee\bar{e}$ and $I = 3.26$ for $\tau \to \mu\mu\bar{\mu}$. An exact analytic expression for $I$ cannot be readily obtained, but by expanding in powers of $\tilde{m}$ the integrand in (50) after performing the $y$-integral, keeping the full dependence on $\tilde{m}$ in the limits of $x$-integration gives the result in Eq. (51), which numerically gives 10.4 for $\mu \to ee\bar{e}$.

The differential rate for $\tau^- \to e^-\mu^-\mu^+$ in the variables $x = m_{12}^2/m_{\tau}^2 = (p_{e} + p_{\mu}^-)^2/m_{\tau}^2$ and $y = m_{23}^2/m_{\tau}^2 = (p_{\mu}^- + p_{\mu}^+)^2/m_{\tau}^2$ is

$$
\frac{d^2\Gamma}{dxdy} = \frac{m_{\tau}^5}{2^8 \pi^3 A_{\text{LFV}}^4} \frac{1}{xy} \left[ 2 \left( 2x^2 + 2yx - 2x - y + 1 \right) |a_0|^2 + (2xy - 2x^2y) |a_-|^2 
+ (2y^2 - 4xy + 2x^2y + 2xy) |a_+|^2 + a_0 \left( (2y^2 + 2xy - 2y) a_+^* - 2xya_+^* \right) 
+ (a_+ (2y^2 + 2xy - 2y) - 2a_-xy) a_0^* \right] .
$$

The integral over phase space $\tilde{I}$, appearing in Eq. (52), is given by ($\tilde{m} = m_{\mu}/m_{\tau}$)

$$
\tilde{I} = \int_{\tilde{m}^2}^{(1-\tilde{m})^2} dx \int_{y_{\text{min}}}^{y_{\text{max}}} dy \left\{ 1 - y - 2x(1-x-y) \right\} .
$$

The numerical value is 1.50 for $\tau^- \to e^-\mu^-\mu^+$ and 8.49 for the corresponding integral in the case of $\tau^- \to \mu^-e^-e^+$. Note that the limits of integration are different in these cases than in $\mu \to ee\bar{e}$. 

19
B Low Energy Effective Lagrangian

In this appendix we give the details of the calculation described in Sec. 5.1, that is, the use of the renormalization group equations to determine the low energy effective lagrangian. As explained there, this is done in three steps: first, the RGE in the unbroken phase of the $SU(2) \times U(1)$ theory is used to compute the coefficients in the effective lagrangian in Eq. (3) down to a scale $\mu \sim M_Z$. In the second step the coefficients are matched to those of an effective lagrangian for the theory in the broken symmetry phase of $SU(2) \times U(1)$. And third, the coefficients of this effective lagrangian are computed at $\mu \sim 1$ GeV using the RGE for the theory with only $U(1)$ gauge group.

In the region $\mu \gtrsim M_Z$ (neglecting $SU(2) \times U(1)$ symmetry breaking) a one loop computation gives

$$
\begin{align*}
\mu \frac{d c_{4L}^{(1)}}{d \mu} &= -\frac{1}{18} N_c N_g g_1^2 \frac{g_2^2}{4\pi^2} [c_{LL}^{(3)} + 2c_{LL}^{(4u)} - c_{LL}^{(4d)}] \\
\mu \frac{d c_{4L}^{(2)}}{d \mu} &= \frac{1}{6} N_c N_g g_2^2 \frac{g_2^2}{4\pi^2} c_{LL}^{(5)} \\
\mu \frac{d c_{4L}^{(3)}}{d \mu} &= -\frac{1}{9} N_c N_g g_1^2 \frac{g_2^2}{4\pi^2} [c_{LL}^{(3)} + 2c_{LL}^{(4u)} - c_{LL}^{(4d)}],
\end{align*}
$$

(54)

where we have denoted by $N_c$ the number of colors and by $N_g$ the number of quark generations. On the right hand side of these equations we have neglected terms proportional to $c_{4L}^{(i)}$ because we are considering a situation with $c_{4L}(\Lambda_{\text{LFV}}) \ll c_{LL}(\Lambda_{\text{LFV}})$. A full solution of the RGE requires equations for $c_{LL}^{(i)}(\mu)$ too; however, since $g_{1,2}$ are small, we retain only up to order $g_{1,2}^2$ in the amplitudes, and these terms drop out.

In the second step, an effective theory with $U(1)$ gauge symmetry is constructed with effective lagrangian given by matching at $\mu \sim M_Z$ to the theory with $SU(2) \times U(1)$. In this process $W^\pm, Z^0,$ and top quark are integrated out. The relevant operators are given
in Eq. (32), and the effective lagrangian in Eq. (33). Matching coefficients gives
\[ c_{LL}^{(3)}(M_Z) = c_{LL}^{(3)}(M_Z) - \frac{1}{6} \sin^2 \theta_w [c_{LL}^{(1)}(M_Z) + c_{LL}^{(2)}(M_Z)] + \frac{1}{6} \alpha Q_t F_i N_c [c_{LL}^{(3)}(M_Z) + c_{LL}^{(4u)}(M_Z) - c_{LL}^{(5)}(M_Z)] \]
\[ c_{LL}^{(4u)}(M_Z) = c_{LL}^{(4u)}(M_Z) - \frac{2}{3} \sin^2 \theta_w [c_{LL}^{(1)}(M_Z) + c_{LL}^{(2)}(M_Z)] + \frac{2}{3} \alpha Q_t F_i N_c [c_{LL}^{(3)}(M_Z) + c_{LL}^{(4u)}(M_Z) - c_{LL}^{(5)}(M_Z)] \]
\[ c_{LL}^{(4d)}(M_Z) = c_{LL}^{(4d)}(M_Z) + \frac{1}{3} \sin^2 \theta_w [c_{LL}^{(1)}(M_Z) + c_{LL}^{(2)}(M_Z)] - \frac{1}{3} \alpha Q_t F_i N_c [c_{LL}^{(3)}(M_Z) + c_{LL}^{(4u)}(M_Z) - c_{LL}^{(5)}(M_Z)] \]
\[ c_{LL}^{(5)}(M_Z) = c_{LL}^{(5)}(M_Z) - (\frac{1}{2} - \frac{1}{2} \sin^2 \theta_w) [c_{LL}^{(1)}(M_Z) + c_{LL}^{(2)}(M_Z)] - \frac{1}{2} \alpha Q_t F_i N_c [c_{LL}^{(3)}(M_Z) + c_{LL}^{(4u)}(M_Z) - c_{LL}^{(5)}(M_Z)] \]
\[ c_{4l}^{(1)}(M_Z) = c_{4l}^{(1)}(M_Z) + c_{4l}^{(2)}(M_Z) - (\frac{1}{2} - \sin^2 \theta_w) [c_{LL}^{(1)}(M_Z) + c_{LL}^{(2)}(M_Z)] - \frac{\alpha}{\pi} Q_t F_i N_c [c_{LL}^{(3)}(M_Z) + c_{LL}^{(4u)}(M_Z) - c_{LL}^{(5)}(M_Z)] \]
\[ c_{4l}^{(3)}(M_Z) = c_{4l}^{(3)}(M_Z) + \sin^2 \theta_w [c_{LL}^{(1)}(M_Z) + c_{LL}^{(2)}(M_Z)] - \frac{\alpha}{\pi} Q_t F_i N_c [c_{LL}^{(3)}(M_Z) + c_{LL}^{(4u)}(M_Z) - c_{LL}^{(5)}(M_Z)] \]
\[ c_{RL}^{(1)}(M_Z) = c_{RL}^{(1)}(M_Z) - c_{RL}^{(2)}(M_Z) \]

Finally, for the third step we need the RGE satisfied by \( \hat{c}_{4l}^{(1)} \) and \( \hat{c}_{4l}^{(3)} \):
\[ \mu \frac{d\hat{c}_{4l}^{(1)}}{d\mu} = \mu \frac{d\hat{c}_{4l}^{(3)}}{d\mu} = -\frac{e^2 N_c}{9 \cdot 4 \pi^2} \left[ (2n_u - n_d)\hat{c}_{LL}^{(3)} + 2n_u \hat{c}_{LL}^{(4u)} - n_d \hat{c}_{LL}^{(4d)} - (2n_u + n_d)\hat{c}_{LL}^{(5)} \right] \]  
\[ (56) \]
Here \( n_u \) and \( n_d \) again stand for the number of up-type and down-type quarks, respectively. It is easy to check that if one adds to the one loop amplitude from \( \hat{O}_{LL}^{(i)} \), \( i = 3, \ldots, 5 \), the tree level amplitude from \( \hat{c}_{4l}^{(1,3)} \), and one does not re-sum leading logs (i.e., one works to order \( \epsilon^2 \) only), then the \( \mu \) dependence cancels in the amplitude (the \( \mu \) dependence in \( \hat{c}_{4l}^{(1,3)} \) cancels that of the loop amplitude).

The solution to these equation is given in Sec. 5.1 in Eq. (34).

C A sum rule for \( H_1^r(\mu_{\chi}, \mu) \)

Let us return to the issue of matching the quark-level description with the chiral effective theory calculation. In particular, in this appendix we will show the explicit dependence of the chiral coupling \( H_1 \) on the MS renormalization scale \( \mu \).

The light quark contribution to the matrix element \( \langle 3e|\hat{O}_{LL}^{(i)}|\mu \rangle \) in eq. (11) is UV divergent, being proportional to \( \Pi^{EM}(q^2) \). In dimensional regularization the divergent
correlator reads (work with $i = 4u$ for definiteness)

$$
\Pi^{4u,\text{EM}}(q^2) = \frac{N_c Q_u}{4(4\pi)^2} \left[ \frac{4}{3} \left( \frac{2}{4-d} - \gamma + \log 4\pi \right) - 8F(m_u, m_u^2/q^2) \right], \tag{57}
$$

with $F(m_u, m_u^2/q^2)$ defined in Eq. (36). The $\overline{\text{MS}}$ prescription amounts to subtracting the term proportional to $\left( \frac{2}{4-d} - \gamma + \log 4\pi \right)$. Writing $\Pi(q^2) = \Pi(q^2_0) + \Pi(q^2) - \Pi(q^2_0)$, with $q^2_0$ in the regime of validity of perturbation theory, the $\overline{\text{MS}}$ subtracted correlator can be written as:

$$
\Pi^{4u,\text{EM}}_{\overline{\text{MS}}}(q^2) = -\frac{4 N_c Q_u}{3 (4\pi)^2} \left[ \log \frac{2q^2_0}{\mu^2} - \frac{5}{3} \right] + \Pi^{4u,\text{EM}}(q^2) - \Pi^{4u,\text{EM}}(q^2_0) \tag{58}
$$

where we have used the notation $\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$. The first term has the correct short distance $\mu$ dependence. The remaining terms can be expressed through a (once subtracted) dispersion relation as

$$
\Pi^{4u,\text{EM}}(q^2) - \Pi^{4u,\text{EM}}(0) = \frac{q^2}{\pi} \int_{s_{\text{th}}}^{\infty} ds \frac{\text{Im}\Pi^{4u,\text{EM}}(s)}{s(s - q^2 - i\epsilon)} \tag{60}
$$

If $q^2 \ll \Lambda_{\chi_{SB}}^2$, we can saturate the spectral function with the $\pi\pi$ and $KK$ threshold contributions, up to higher order terms in $q^2/\Lambda_{\chi_{SB}}^2$. Doing so and comparing with the chiral perturbation theory result in eq. (42) leads to a matching condition involving the low energy constant $L_{10}$ and the contact term $H_1$:

$$
\frac{8}{3} (2H_1^r(\mu, \mu) + L_{10}^r(\mu)) + \frac{2}{3(4\pi)^2} \log \left( \frac{m_K m_\pi}{\mu^2} \right) = \frac{4 N_c Q_u}{3 (4\pi)^2} \left[ \log \frac{2q^2_0}{\mu^2} - \frac{5}{3} \right] + \hat{\Pi}^{4u,\text{EM}}(q^2_0) \tag{61}
$$

A few comments are in order:

- The LHS does not depend on $\mu_\chi$ (due to chiral RG for $H_1$ and $L_{10}$).
- The RHS does not depend on $q^2_0$. This can be seen by separating the dispersion integral for $\hat{\Pi}^{4u,\text{EM}}(q^2_0)$ in an IR and UV region, and then using the free quark spectral function in the UV regime. This gives back a logarithmic dependence on $q^2_0$ that cancels the one in the other term.
- Eq. (61) explicitly shows how $H_1$ depends on the short distance renormalization conventions and in particular on the scale $\mu$. It also displays a purely non-perturbative contribution to the matching, namely $\hat{\Pi}^{4u,\text{EM}}(q^2_0)$. In absence of reliable estimates of $\hat{\Pi}^{4u,\text{EM}}(q^2_0)$, in our numerics we use the experimental value for $L_{10}^r$ and a bound for $H_1(\mu_\chi = m_\rho, \mu = 1 \text{GeV})$ based on naive dimensional analysis.
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