HESSIAN ESTIMATES FOR NON-DIVERGENCE FORM ELLIPTIC EQUATIONS ARISING FROM COMPOSITE MATERIALS

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Abstract. In this paper, we prove that any \( W^{2,1} \) strong solution to second-order non-divergence form elliptic equations is locally \( W^{2,\infty} \) and piecewise \( C^2 \) when the leading coefficients and data are of piecewise Dini mean oscillation and the lower-order terms are bounded. Somewhat surprisingly here the interfacial boundaries are only required to be \( C^{1,\text{Dini}} \). We also derive global weak-type \((1,1)\) estimates with respect to \( A_1 \) Muckenhoupt weights. The corresponding results for the adjoint operator are established. Our estimates are independent of the distance between these surfaces of discontinuity of the coefficients.

1. Introduction and main results

Let \( \mathcal{D} \) be a bounded domain in \( \mathbb{R}^n \) that contains \( M \) disjoint sub-domains \( \mathcal{D}_1, \ldots, \mathcal{D}_M \) with \( C^{1,\text{Dini}} \) boundaries, that is, \( \mathcal{D} = (\bigcup_{j=1}^M \overline{\mathcal{D}_j}) \setminus \partial \mathcal{D} \). For more details about \( C^{1,\text{Dini}} \) boundaries, see Definition 2.2. We suppose that if the boundaries of two \( \mathcal{D}_j \) touch, then they touch on a whole component of such a boundary. We thus without loss of generality assume that \( \partial \mathcal{D} \subset \partial \mathcal{D}_M \).

We consider the following second-order elliptic equation in non-divergence form

\[
L \mathbf{u} := a^{ij} D_{ij} \mathbf{u} + b^i D_i \mathbf{u} + c \mathbf{u} = f \tag{1.1}
\]

in \( \mathcal{D} \), where the Einstein summation convention on repeated indices is used. Throughout this paper, the coefficients \( a^{ij}, b^i, \) and \( c \) are bounded by a positive constant \( \Lambda \). We assume that the principal coefficients matrices \( A = (a^{ij})^{n \times n}_{i,j=1} \) are defined on \( \mathbb{R}^n \) and uniformly elliptic with ellipticity constant \( \delta \in (0,1) \):

\[
\delta |\xi|^2 \leq a^{ij}(x)\xi^i \xi^j, \quad \forall \xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n.
\]

Without loss of generality, we may assume that \( A \) is symmetric, i.e., \( a^{ij} = a^{ji} \).

We are interested in the case when the coefficients and data are allowed to be discontinuous across the interfacial boundaries. Such problem, in particular for the corresponding divergence form equations, has been studied by many authors. See, for instance, [21, 20, 5, 23, 24, 1].

In this paper, we prove the piecewise \( C^2 \) regularity and local \( W^{2,\infty} \) estimate for \( W^{2,1} \) strong solutions of \((1.1)\) when the coefficients and \( f \) are piecewise Dini continuous in the \( L^1 \)-mean sense in each subdomains. Moreover, when the subdomains, \( \mathcal{D}_1, \ldots, \mathcal{D}_{M-1}, \) are away from the boundary \( \partial \mathcal{D} \), we prove global weak-type \((1,1)\) estimates with \( A_1 \) Muckenhoupt weights for any \( W^{2,1} \) strong solution of \((1.2)\) without imposing further conditions on the leading coefficients \( a^{ij} \) other than

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being piecewise Dini mean oscillation over an open set containing \( \overline{D} \); see Theorem 6.1.

Our argument is based on Campanato’s approach presented in [4, 18], the key point of which is to show that the mean oscillation of \( D^2u \) (or \( Du \), or \( u \), respectively) in balls vanishes in certain order as the radii of balls go to zero. The method was used recently for divergence and non-divergence form elliptic equations with coefficients satisfying certain conditions. For instance, in [24] the authors derived very general BMO, Dini, Hölder, and higher regularity estimates for weak solutions to the corresponding divergence form systems by using Campanato’s approach, where the estimates may depend on the distance between sub-domains. See also [5] in which both divergence form systems and non-divergence form equations were studied when the subdomains are laminar. In [10], the authors studied \( C^2 \) and weak-type \((1, 1)\) estimates of the solution to

\[
a^{ij}D_{ij}u = f. \tag{1.2}
\]

They showed that any \( W^{2, 2} \) strong solution to (1.2) is \( C^2 \) provided that the modulus of continuity of coefficients in the \( L^1 \)-mean sense satisfies the Dini condition. Later, the authors in [7] extended and improved the results in [10], by showing that any strong solution to elliptic equations in non-divergence form with zero Dirichlet boundary conditions is \( C^2 \) up to the boundary when the coefficients satisfy the same condition. The main obstacle in [4, 10] is that the usual argument based on \( L^p \) \((p > 1)\) estimates does not work because only the assumptions on the \( L^1 \)-mean oscillations of the coefficients and data are imposed. To overcome it, they used weak-type \((1, 1)\) estimates and adapted Campanato’s method in the \( L^p \) setting with \( p \in (0, 1) \). The above idea was also used in a recent paper [12], where the authors showed that \( W^{1, p} \), \( 1 \leq p < \infty \), weak solutions to divergence form elliptic systems are Lipschitz and piecewise \( C^1 \) under the same conditions on the coefficients and data as imposed before. Hence, this paper can be regarded as a companion paper of [12].

Similar to [12], an added difficulty is the lack of regularity of \( D^2u \) in one direction. For this, we adapt the scheme in [12] to our case. We point out that the coordinate system in our setting and [12] is chosen according to the geometry of the sub-domains and is different at each point. This is in contrast to [4, 7, 10], where the coordinate system is fixed. Therefore, our mean oscillation estimates depend on the balls under consideration, which makes the argument much more involved.

Denote by \( \mathcal{A} \) the set of piecewise constant functions in each \( D_j \), \( j = 1, \ldots, M \). We assume that \( A \) is piecewise Dini continuous in the \( L^1 \) sense in \( D \), that is,

\[
\omega_A(r) := \sup_{x_0 \in D} \inf_{\tilde{A} \in \mathcal{A}, \tilde{A}(x_0)} \int_{B_r(x_0)} |A(x) - \tilde{A}| \, dx \tag{1.3}
\]

satisfies the Dini condition, where \( B_r(x_0) \subset D \). For more details about the Dini condition, see Definition 2.1. For \( \varepsilon > 0 \) small, we set

\[
D_{\varepsilon} := \{ x \in D : \text{dist}(x, \partial D) > \varepsilon \}.
\]

Denote \( b := (b^1, \ldots, b^n) \).

Our first result reads that if the coefficients and \( f \) are piecewise Dini continuous in the \( L^1 \) sense, then any \( W^{2, p} \) strong solution to the above equation (1.1) is locally \( W^{2, \infty} \) and piecewise \( C^2 \).
Theorem 1.1. Let $\mathcal{D}$ be defined as above. Let $\varepsilon \in (0,1)$, $p \in (1, \infty)$, and $\gamma \in (0,1)$. Assume that $A, b, c, f$ are of piecewise Dini mean oscillation in $\mathcal{D}$ and $f \in L^\infty(\mathcal{D})$. If $u \in W^{2,p}(\mathcal{D})$ is a strong solution to (1.1) in $\mathcal{D}$, then $u \in C^2(\overline{\mathcal{D}} \cap \overline{\mathcal{D}}_s)$, $j = 1, \ldots, M$, and $Du$ is Lipschitz in $\mathcal{D}_s$. Moreover, for any fixed $x \in \mathcal{D}_s$, there exists a coordinate system associated with $x$, such that for all $y \in \mathcal{D}_s$, we have

$$|D_{x^j}u(x) - D_{x^j}u(y)| \leq C \int_0^{\varepsilon^2} \frac{\tilde{\omega}_\ell(t)}{t} dt \left( \|D^2u\|_{L^1(\mathcal{D})} + \int_0^1 \frac{\tilde{\omega}_f(t)}{t} dt + \|f\|_{L^\infty(\mathcal{D})} + \|u\|_{L^1(\mathcal{D})} \right)$$

$$+ C|x - y|^\gamma \left( \|D^2u\|_{L^1(\mathcal{D})} + \|f\|_{L^\infty(\mathcal{D})} + \|u\|_{L^1(\mathcal{D})} \right) + C \int_0^{\varepsilon^2} \frac{\tilde{\omega}_f(t)}{t} dt,$$  \hspace*{2cm} (1.4)

where $C$ depends on $n, M, p, \delta, \Lambda, \varepsilon, \omega, \gamma$, and the $C^{1,\text{Dini}}$ characteristics of $\partial \mathcal{D}_s$. $\tilde{\omega}_\ell(t)$ is a Dini function derived from $\omega_\ell(t)$; see (3.14).

We note that in the above theorem the interfacial boundaries are only required to be in $C^{1,\text{Dini}}$, which is the same condition as in [2]. This is in contrast to the usual Dirichlet boundary value problem in which case for the $C^2$ estimate the boundary of the domain is assumed to be in $C^{2,\text{Dini}}$. See, for example, [7].

Under the stronger condition that the coefficients and $f$ are piecewise Hölder continuous in $\mathcal{D}$, we further show that $D_{x^j}u$ is Hölder continuous.

Corollary 1.2. Let $\mathcal{D}$ be defined as above and each sub-domain has $C^{1,\mu}$ boundary with $\mu \in (0,1)$. Let $\varepsilon \in (0,1)$ and $p \in (1, \infty)$. Assume that $A, b, c, f \in C^{\alpha}(\overline{\mathcal{D}}_s)$ with $\alpha \in (0, \mu/(1 + \mu)]$. If $u \in W^{2,p}(\mathcal{D})$ is a strong solution to (1.1) in $\mathcal{D}$, then the assertions of Theorem 1.1 also hold true. Furthermore, (1.4) is replaced by

$$|D_{x^j}u(x) - D_{x^j}u(y)| \leq C|x - y|^\alpha \left( \sum_{j=1}^M |f_i|_{L^\infty(\mathcal{D})} + \|D^2u\|_{L^1(\mathcal{D})} + \|u\|_{L^1(\mathcal{D})} \right),$$

where $C$ depends on $n, M, \alpha, \mu, \delta, \Lambda, \varepsilon, p, |A|_{C^{\alpha}(\overline{\mathcal{D}}_s)}, |b|_{C^{\alpha}(\overline{\mathcal{D}}_s)}, |c|_{C^{\alpha}(\overline{\mathcal{D}}_s)}$, and the $C^{1,\mu}$ norms of $\partial \mathcal{D}_s$.

Remark 1.3. It follows from (1.3) that $D^2u \in C^{\alpha}(\overline{\mathcal{D}}_s \cap \overline{\mathcal{D}}_s)$. Indeed, from the proof of Theorem 1.1, we have $D^2u \in L^\infty_{\text{loc}}$ with

$$\|D^2u\|_{L^\infty(\mathcal{D}_s)} \leq C\|D^2u\|_{L^1(\mathcal{D})} + C \sum_{j=1}^M |f_i|_{L^\infty(\mathcal{D})} + C\|u\|_{L^1(\mathcal{D})}.$$

Then, $Du$ and $u$ are Lipschitz in $\mathcal{D}_s$. Since

$$D_{x^j}u = \frac{1}{a^{ij}} \left( f - b_i^{ij}D_iu - cu - \sum_{(i,j) \neq (i,u)} a^{ij}D_ju \right),$$

we also have $D_{x^j}u \in C^{\alpha}(\overline{\mathcal{D}}_s \cap \overline{\mathcal{D}}_s)$. Therefore, $u \in C^{2,\alpha}(\overline{\mathcal{D}}_s \cap \overline{\mathcal{D}}_s)$. By using interpolation inequalities, the term $\|D^2u\|_{L^1(\mathcal{D})}$ on the right-hand side of (1.4) and (1.5) can be dropped. We also point out that for $L^p$-viscosity solutions, a result similar to Corollary 1.2 was obtained in [23] when $\alpha \in \left(0, \mu/(n(1 + \mu]\right]$ by using a different argument.
To prove the above results, we need to consider the formal adjoint operator defined by

\[ L^* u = D_i (a^{ij} u) - D_j (b^i u) + c u, \]

and deal with the following boundary value problem

\[
\begin{aligned}
L^* u &= \text{div}^2 g & \text{in } & \mathcal{D}, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial g}{\partial \nu} & \text{on } & \partial \mathcal{D},
\end{aligned}
\]

where \( g = (g^{ij})_{i,j=1}^n \), \( \text{div}^2 g = D_i (g^{ij}) \) with \( g \in L^\infty (\mathcal{D}) \), and \( \nu \) is the unit outer normal vector on \( \partial \mathcal{D} \). For more details about the adjoint solution to (1.6), see Definition 2.4. By using a similar idea to that in the proof of Theorem 1.1, we also obtain the corresponding results for the adjoint problem (1.7).

**Theorem 1.4.** Let \( \mathcal{D} \) be defined as in Theorem 1.1. Let \( \varepsilon \in (0, 1) \), \( p \in (1, \infty) \), and \( \gamma \in (0, 1) \). Suppose that \( A, b, c, \) and \( g \) are of piecewise Dini mean oscillation in \( \mathcal{D} \), \( g \in L^\infty (\mathcal{D}) \). Let \( u \in L^p (\mathcal{D}) \) be a local adjoint solution of

\[ L^* u = \text{div}^2 g \quad \text{in } \mathcal{D}. \]

Then \( u \in L^\infty (\mathcal{D}_\varepsilon) \). Moreover, for any fixed \( x \in \mathcal{D}_\varepsilon \), there exists a coordinate system associated with \( x \), such that for any \( y \in \mathcal{D}_\varepsilon \), we have

\[
|\tilde{u}(x) - \tilde{u}(y)| \\
\leq C \int_0^{r - \| y - x \|} \frac{\tilde{\omega}_A(t)}{t} \, dt \cdot \left( \sum_{i=1}^{\infty} \frac{\tilde{\omega}_G(t)}{t} \, dt + \| g \|_{L^\infty(\mathcal{D})} + \| u \|_{L^p(\mathcal{D})} \right) \\
+ C |x - y|^\gamma \left( \| g \|_{L^\infty(\mathcal{D})} + \| u \|_{L^p(\mathcal{D})} \right) + C \int_0^{r - \| y - x \|} \frac{\tilde{\omega}_G(t)}{t} \, dt,
\]

where \( \tilde{u} = a^{mn} u - g^{mn} \) and \( C \) depends on \( n, M, \gamma, A, b, c, \) and the \( C^{\gamma, \text{Dini}} \) characteristics of \( \partial \mathcal{D} \).

**Corollary 1.5.** Let \( \mathcal{D} \) be defined as in Corollary 1.2. Let \( \varepsilon \in (0, 1) \) and \( p \in (1, \infty) \). Assume that \( A, b, c, \) and \( g \in C^{\gamma \alpha} (\mathcal{D}_\varepsilon) \) with \( \alpha \in (0, 1) \). If \( u \in L^p (\mathcal{D}) \) is a local adjoint solution of

\[ L^* u = \text{div}^2 g \quad \text{in } \mathcal{D}. \]

Then the assertions of Theorem 1.4 also hold true, and (1.7) is replaced with

\[
|\tilde{u}(x) - \tilde{u}(y)| \leq C |x - y|^\gamma \left( \sum_{i=1}^{M} |g|_{L^\infty(\mathcal{D})} + \| u \|_{L^p(\mathcal{D})} \right),
\]

where \( C \) depends on \( n, M, p, \alpha, \mu, \delta, \Lambda, \varepsilon, A, b, c, \) and the \( C^{1, \gamma, \text{Dini}} \) norms of \( \partial \mathcal{D} \).

**Remark 1.6.** Restricting to each \( \mathcal{D}_i \cap \mathcal{D}_e \), since \( u = (a^{mn})^{-1} (\tilde{u} + g^{mn}) \), \( a^{mn} \) and \( g^{mn} \) are in \( C^{\gamma \alpha} (\mathcal{D}_i) \), we conclude that \( u \in C^{\gamma \alpha} (\mathcal{D}_i \cap \mathcal{D}_e) \).

By using a duality argument and Theorem 1.4, we derive the following corollary.

**Corollary 1.7.** Let \( A, b, c, \) and \( f \) be as in Theorem 1.1. If \( u \in W^{2,1} (\mathcal{D}) \) is a strong solution to (1.1) in \( \mathcal{D} \), then we have \( u \in W^{2, p}_{\text{loc}} (\mathcal{D}) \) for some \( p \in (1, \infty) \), and the conclusion of Theorem 1.4 still holds.
Remark 1.8. Similar to Corollary 1.5, we can show that under the assumptions imposed in Theorem 1.4, if 

$$ u \in L^1(\mathcal{D}) $$

then 

$$ u \in L^p_{\text{loc}}(\mathcal{D}) $$

for some \( p \in (1, \infty) \), and the conclusion of Theorem 1.4 still holds true.

Throughout this paper, unless otherwise stated, \( C \) denotes a constant independent of the distance between sub-domains.

The rest of this paper is organized as follows. In Section 2, we fix our domain and the coordinate system. We also introduce some notation, definitions, and auxiliary results used in this paper. In Section 3, we provide the proofs of Theorem 1.1 and the coordinate system. We also introduce some notation, definitions, and auxiliary lemmas used in the paper.

2. Preliminaries

In this section, we fix our domain and list some notation, definitions, and auxiliary lemmas used in the paper.

2.1. Notation and definitions. We follow the notation and definitions from \([12]\). Write \( x = (x^1, \ldots, x^n) = (x', x^\circ) \), \( n \geq 2 \). Denote 

$$ B_r(x) := \{ y \in \mathbb{R}^n : |y - x| < r \}, \quad B'_r(x') := \{ y' \in \mathbb{R}^{n-1} : |y' - x'| < r \}, $$

and 

$$ B_r := B_r(0), \quad B'_r := B'_r(0'), \quad \mathcal{D},(x) := \mathcal{D} \cap B_r(x). $$

For a function \( f \) defined in \( \mathbb{R}^n \), we set 

$$ (f)_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} f(x) \, dx = \frac{1}{\mathcal{D}} \int_{\mathcal{D}} f(x) \, dx, $$

where \( |\mathcal{D}| \) is the \( n \)-dimensional Lebesgue measure of \( \mathcal{D} \). We shall use the notation 

$$ [u]_{\mathcal{D}} := \sup_{x \in \mathcal{D}} |D^k u(x)| \quad \text{and} \quad [u]_{k,\gamma;\mathcal{D}} := \sup_{x,y \in \mathcal{D}} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\gamma}}, $$

where \( k = 0, 1, \ldots, \gamma, \gamma \in (0,1) \). For \( k = 0 \), we denote \([u]_{\mathcal{D}} := [u]_{0,\gamma;\mathcal{D}}\) for abbreviation. We also define 

$$ [u]_{k,\mathcal{D}} := \sum_{j=0}^{k} [u]_{j,\mathcal{D}} \quad \text{and} \quad [u]_{k,\gamma;\mathcal{D}} := [u]_{k,\mathcal{D}} + [u]_{k,\gamma;\mathcal{D}}. $$

We denote \( C^{k,\gamma}(\mathcal{D}) \) to be the set of bounded measurable functions \( u \) that are \( k \)-times continuously differentiable in \( \mathcal{D} \) and \([u]_{k,\gamma;\mathcal{D}} < \infty \). Moreover, the following notation will be used:

$$ D_{x'} u = u_{x'}, \quad D^2_{x'} u = u_{xx'}, \quad D^2_{x'x'} u = u_{xxxx'}, \quad D^2_{x'} D_{x'} u = u_{xx'xx'}, \quad D^2_{x'} D^2_{x'} u = u_{xxxxx'}. $$

For a function \( f, k = 1, 2, \ldots, n, \text{and} h > 0 \), we define the finite difference quotient 

$$ \delta_h, k f(x) := \frac{f(x + he_k) - f(x)}{h}. $$

We refer the reader to \([15]\) for a related result.
Definition 2.1. We say that a continuous increasing function \( \omega : [0, 1] \rightarrow \mathbb{R} \) satisfies the Dini condition if \( \omega(0) = 0 \) and
\[
\int_0^t \frac{\omega(s)}{s} \, ds < +\infty, \quad \forall \ t \in (0, 1].
\]

Definition 2.2. Let \( k \geq 1 \) be an integer and \( D \subset \mathbb{R}^n \) be open and bounded. We say that \( \partial D \) is \( C^{k, \text{Dini}} \) if for each point \( x_0 \in \partial D \), there exists \( R_0 \in (0, 1/8) \) independent of \( x_0 \) and a \( C^k \) function (that is, \( C^k \) function whose \( k \)-th derivatives are Dini continuous) \( \varphi : B'_{R_0} \rightarrow \mathbb{R} \) such that (upon relabeling and reorienting the coordinates axes if necessary) in a new coordinate system \((x', x'')\), \( x_0 \) becomes the origin and
\[
D_{R_0}(0) = \{ x \in B_{R_0} : x'' > \varphi(x') \}, \quad \varphi(0') = 0,
\]
and \( D^k \varphi \) has a modulus of continuity \( \omega_0 \), a Dini function which is increasing, concave, and independent of \( x_0 \).

Definition 2.3. (1) We say \( w : \mathbb{R}^n \rightarrow [0, \infty) \) belongs to \( A_1 \) if there exists some constant \( C \) such that for all balls \( B \) in \( \mathbb{R}^n \),
\[
\int_B w(y) \, dy \leq C \inf_{x \in B} w(x).
\]
The \( A_1 \) constant \( [w]_{A_1} \) of \( w \) is defined as the infimum of all such \( C \)'s.
(2) We say \( w : \mathbb{R}^n \rightarrow [0, \infty) \) belongs to \( A_p \) for \( p \in (1, \infty) \) if
\[
\sup_B \frac{w(B)}{|B|} \left( \frac{w(B)}{|B|} \right)^{p-1} < \infty,
\]
where the supremum is taken over all balls in \( \mathbb{R}^n \). The value of the supremum is the \( A_p \) constant of \( w \), and will be denoted by \( [w]_{A_p} \).

The following definition is extracted from [15].

Definition 2.4. Let \( g \in L^p(B_1) \), \( 1 < p < \infty \) and \( 1/p + 1/p' = 1 \). We say that \( u \in L^p(B_1) \) is an adjoint solution of (1.6) if \( u \) satisfies
\[
\int_{B_1} uL'v = \int_{B_1} \text{tr}(gD^2v), \tag{2.1}
\]
for any \( v \in W^{2,p'}(B_1) \cap W^{1,p'}_0(B_1) \), where \( \text{tr}(gD^2v) = g^{ij}D_{ij}v \). By a local adjoint solution of
\[
L'u = \text{div}^2 g \quad \text{in } B_1,
\]
we mean a function \( u \in L^p_{\text{loc}}(B_1) \) that verifies (2.1) for any \( v \in W^{2,p'}_0(B_1) \).

2.2. Some properties of the domain, coefficients, and data. Below, we slightly abuse the notation. Let \( D \) be the unit ball \( B_1 \) and take \( x_0 \in B_{3/4} \). We now localize and fix our domain as follows. By suitable rotation and scaling, we may suppose that a finite number of sub-domains lie in \( B_1 \) and that they take the form
\[
x'' = h_j(x'), \quad \forall \ x' \in B_{1}', \ j = 1, \ldots, l \ (\leq M),
\]
with
\[-1 < h_1(x') < \cdots < h_l(x') < 1,
\]
and \( h_j(x') \in C^{1,\text{Dini}}(B_{1}') \). Set \( h_0(x') \equiv -1 \) and \( h_{l+1}(x') \equiv 1 \) so that we have \( l+1 \) regions:
\[
D_j := \{ x \in D : h_{j-1}(x') < x'' < h_j(x') \}, \quad 1 \leq j \leq l + 1.
\]
We may suppose that there exists some $D_{j_1}$ such that $x_0 \in B_{3/4} \cap D_{j_1}$ and the closest point on $\partial D_{j_1}$ to $x_0$ is $(x_0', h_{r}(x_0'))$, and $\nabla_s h_{r}(x_0') = 0'$ after a proper rotation. We introduce the $l + 1$ “strips”

$$\Omega_{j} := \{x \in D : h_{j-1}(x_0') < x'' < h_{j}(x_0')\}, \quad 1 \leq j \leq l + 1.$$  

Then we have the following result, which is [12, Lemma 2.3].

**Lemma 2.5.** There exists a constant $C_j$, depending on $n, l$ and the $C^{1, \text{Dini}}$ characteristics of $h_j, 1 \leq j \leq l$, such that

$$r^n |(D_j \Delta \Omega_j) \cap B_r(x_0)| \leq C_{\Omega_j}(r), \quad 1 \leq j \leq l + 1, \quad 0 < r < r_j := \frac{2}{3} \int_{0}^{R_0/2} \omega_0'(s) s \, ds,$$

where $D_j \Delta \Omega_j = (D_j \setminus \Omega_j) \cup (\Omega_j \setminus D_j)$, $R_0$ is defined in Definition 2.2, $\omega_0'$ denotes the left derivative of $\omega_0$, and $\omega_1(r) := \omega_0(2r + R)$ is a Dini function for some constant $R := R(r) > 2r$ satisfying

$$\int_{0}^{R} \omega_0'(2r + s) s \, ds = 3r/2.$$

Let $\hat{A}(\cdot) \in \mathcal{A}$ be a constant function in $D_{j_1}$ which corresponds to the definition of $\omega_{A_j}(r)$ in [12, 3]. We define piecewise constant (matrix-valued) functions

$$\bar{A}(x) = \hat{A}(\cdot), \quad x \in \Omega_{j_1}.$$  

Using $\hat{b}(\cdot)$ and $\hat{f}(\cdot)$, which are also constant functions in $D_{j_1}$, we similarly define piecewise constant functions $\bar{b}$ and $\bar{f}$. From Lemma 2.3 and the boundedness of $A$, we have

$$\int_{B_r(x_0)} |\bar{A} - \bar{A}| \, dx \leq C(n, \Lambda) r^n \sum_{j=1}^{l+1} |(D_j \Delta \Omega_j) \cap B_r(x_0)| \leq C\omega_1(r), \quad (2.2)$$

which is also true for $\hat{b}$ and $\hat{f}$.

### 2.3. Some auxiliary lemmas

We first recall the $W^{2,p}$-solvability for elliptic equations with leading coefficients which are variably partially VMO (vanishing mean oscillation) in the interior of $B_1$ and VMO near the boundary. We choose a cut-off function $\eta \in C_0^\infty(B_{7/8})$ with

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{3/4}, \quad |\nabla \eta| \leq 16.$$

Let $\bar{L}$ be the elliptic operator defined by

$$\bar{L}u = \bar{a}^{ij}D_{ij}u + b'D_{i}u + cu,$$

where $\bar{a}^{ij} = \eta a^{ij} + \delta(1 - \eta) \delta_{ij}$, $\delta$ is the ellipticity constant of $\bar{a}^{ij}$, and $\delta_{ij}$ is the Kronecker delta symbol. Consider

$$\begin{cases} Lu - \lambda u = f & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \quad (2.3)$$

where $\lambda \geq 0$ is a constant, $f \in L^p(B_1)$. Here $\bar{a}^{ij}$ satisfy the conditions of [3, Theorem 2.5] in the interior of the domain $B_1$, i.e., for a sufficiently small constant $\gamma_0 = \gamma_0(n, p, \delta) \in (0, 1)$ to be specified later, we can find a constant $r_0 \in (0, 1)$ such that the following holds: For any $x_0 \in B_1$ and $r \in (0, \min\{r_0, \text{dist}(x_0, \partial B_1)/2\})$ (so that
that the following hold. Assume that \( u \in W^{2,p}(B_1) \cap W^{1,p}_0(B_1) \),

\[
\|u\|_{L^p(B_1)} + \sqrt{\lambda} \|Du\|_{L^p(B_1)} + \|D^2u\|_{L^p(B_1)} \leq C \|Lu - \lambda u\|_{L^p(B_1)},
\]

provided that \( \lambda \geq \lambda_0 \), where \( C \) and \( \lambda_0 \) depend on \( n, p, \delta, \Lambda, \) and \( r_0 \).

(2) For any \( \lambda > \lambda_0 \) and \( f \in L^p(B_1) \), (2.3) admits a unique solution \( u \in W^{2,p}(B_1) \) \( \cap W^{1,p}_0(B_1) \). In the case when \( c \leq 0 \), we can take \( \lambda = 0 \).

In the Appendix, we shall prove a more general result in weighted Sobolev spaces. From [3, Theorem 2.5], the Sobolev embedding theorem, and a standard localization argument which is similar to that in the proof of [3, Lemma 4], we also have the following interior estimates.

**Lemma 2.7.** Let \( p \in (1, \infty) \). there exists a small constant \( \gamma_0 = \gamma_0(n, p, \delta) \in (0, 1) \) such that the following hold. Assume that \( u \in W^{2,p}_{loc}(B_1) \) satisfies \( a^{ij}D_{ij}u + b^i Du + cu = f \) in \( B_1 \), where \( f \in L^p(B_1) \). Then there exists a constant \( C = C(n, \delta, \Lambda, p, r_0) \) such that

\[
\|u\|_{W^{2,p}_{loc}(B_1)} \leq C\left(\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)}\right).
\]

In particular, if \( p > n \), it holds that

\[
|u|_{1,\gamma;B_1} \leq C\left(\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)}\right),
\]

where \( \gamma = 1 - n/p \) and \( C \) depends only on \( n, p, \delta, \Lambda, \) and \( r_0 \).

The adjoint operator corresponding to \( L \) is defined by

\[
L^*u := D_{ij}(\bar{a}^{ij}u) - D_{ij}(\bar{b}^i u) + cu.
\]

**Lemma 2.8.** Let \( q \in (1, \infty) \). Assume that \( g = (\hat{g}^{ij})_{i,j=1} \in L^q(B_1) \). Then, there is a \( \lambda_0 \geq 0 \) depending on \( n, q, \delta, \Lambda, \) and \( r_0 \), such that for any \( \lambda > \lambda_0 \),

\[
\begin{cases}
L^*u - \lambda u = \text{div}^2 \hat{g} & \text{in } B_1, \\
u = \frac{\hat{g}^{ij}}{\Lambda p} & \text{on } \partial B_1
\end{cases}
\]

admits a unique adjoint solution \( u \in L^q(B_1) \). Moreover, the following estimate holds,

\[
\|u\|_{L^q(B_1)} \leq C\|g\|_{L^q(B_1)},
\]

where \( C = C(n, q, \delta, \Lambda, r_0) \). In the case when \( c \leq 0 \), we can take \( \lambda = 0 \).
Proof. For \( f \in L^p(B_1) \) with \( 1/p + 1/q = 1 \), it follows from Lemma 2.6 that there exists a unique solution \( v \in W^{2,p}(B_1) \cap W^{1,q}_{0}(B_1) \) such that \( L \tilde{v} - \lambda \lambda v = f \) a.e. in \( B_1 \), provided that \( \lambda > \lambda_0 \). Moreover,

\[
\lambda \|v\|_{L^q(B_1)} + \|\lambda^{1/2} D_\xi v\|_{L^q(B_1)} + \|D^2 v\|_{L^q(B_1)} \leq C\|f\|_{L^p(B_1)}.
\]

(2.6)

Define the functional \( T : L^p(B_1) \to \mathbb{R} \) by

\[
T(f) := \int_{B_1} \text{tr}(g D^2 v) \, dx.
\]

(2.7)

Combining (2.6), (2.7), and Hölder’s inequality, we have

\[
|T(f)| \leq \|D^2 v\|_{L^q(B_1)} \|g\|_{L^p(B_1)} \leq C\|f\|_{L^p(B_1)} \|g\|_{L^p(B_1)}.
\]

Hence, \( T \) is a bounded functional on \( L^p(B_1) \). By the Riesz representation theorem, there exists a unique \( u \in L^q(B_1) \), such that

\[
T(f) = \int_{B_1} u f \, dx, \quad \forall f \in L^p(B_1).
\]

(2.8)

Moreover,

\[
\|u\|_{L^q(B_1)} \leq C\|g\|_{L^p(B_1)}.
\]

It follows from (2.7) and (2.8) that

\[
\int_{B_1} u(L \tilde{v} - \lambda v) \, dx = \int_{B_1} \text{tr}(g D^2 v) \, dx,
\]

that is, \( u \in L^q(\partial B_1) \) is the unique adjoint solution to (2.3).

Now we denote \( \tilde{L}_0u := \tilde{a}^{i}(x^n) D_{ij} u \), where \( \tilde{a}^{i}(x^n) \) satisfies the same ellipticity and boundedness conditions as \( a^{i}(x) \).

Lemma 2.9. Assume that \( u \in C^{1,1}_{loc}(B_1) \) satisfies \( \tilde{L}_0u = \tilde{f}(x^n) \) in \( B_1 \), where \( \tilde{f} \in L^\infty(\partial B_1) \). Then for any \( p \in (0, \infty) \), there exists a constant \( C = C(n, \delta, \Lambda, p) \) such that for any \( c \in \mathbb{R}^{(n-1) \times (n-1)} \),

\[
\|D^2 D_{x} u\|_{L^\infty(B_1)} \leq C\|D^2 D_{x} u - c\|_{L^p(B_1)}.
\]

(2.9)

Proof. By using the finite difference quotient technique and applying Lemma 2.7 with a slightly smaller domain, one can see that \( D^2 D_{x} u - c \in W^{2,q}_{loc}(B_1) \) satisfies

\[
\tilde{L}_0(D^2 D_{x} u - c) = 0 \quad \text{in } B_1
\]

and

\[
\|D^2 D_{x} u - c\|_{W^{2,q}(B_1)} \leq C\|D^2 D_{x} u - c\|_{L^2(B_1)}.
\]

(2.10)

If \( q > n \), we obtain

\[
\|D^2 D_{x} u - c\|_{W^{2,q}(B_1)} + \|D^2 D_{x} u\|_{L^\infty(B_1)} \leq C\|D^2 D_{x} u - c\|_{L^2(B_1)}.
\]

(2.10a)

Because \( \tilde{L}_0(D_x u) = 0 \) in \( B_1 \), one can see that

\[
D_{mn} D_{x} u = -\frac{1}{\tilde{a}^{mn}(x^n)} \sum_{(i,j) \neq (m,n)} \tilde{a}^{ij}(x^n) D_i D_j D_{x} u.
\]

Therefore, using (2.10) and \( \tilde{a}^{mn}(x^n) \geq \delta \), we have

\[
\|D_{mn} D_{x} u\|_{L^\infty(B_1)} \leq C\|D^2 D_{x} u - c\|_{L^2(B_1)}.
\]
which implies that

$$||D^2 D_{\nu} u||_{L^\infty(B_{1/2})} \leq C||D^2_{x} u - c||_{L^2(B_{1})},$$  \hfill (2.11)

For any $0 < p < 1 < \infty$, by using H"{o}lder's inequality, we get

$$||D^2_{x} u - c||_{L^p(B_{1/2})} \leq ||D^2_{x} u - c||_{L^q(B_{1/2})}^{1-q/p} ||D^2_{x} u - c||_{L^q(B_{1/2})}^{1/p},$$  \hfill (2.12)

Combining (2.10), (2.12), and H"{o}lder's inequality, we obtain

$$||D^2_{x} u - c||_{L^p(B_{1/2})} \leq C||D^2_{x} u - c||_{L^q(B_{1/2})}^{1-q/p} ||D^2_{x} u - c||_{L^q(B_{1/2})}^{1/p}$$

$$\leq \frac{1}{2} ||D^2_{x} u - c||_{L^\infty(B_{1/2})} + C||D^2_{x} u - c||_{L^q(B_{1/2})}.$$  \hfill (2.13)

By a well-known iteration argument (see, for instance, [8, Lemma 3.1 of Ch. V]), we get

$$||D^2_{x} u - c||_{L^p(B_{1/4})} \leq C||D^2_{x} u - c||_{L^p(B_{1})}, \forall p > 0.$$  \hfill (2.14)

Coming back to (2.11), we obtain (2.15). The lemma is thus proved.

In the proof of Theorem 1.4, we need to use the following

**Lemma 2.10.** Assume $u \in C^{0,1}_{loc}$ satisfies

$$L_0 u := D_i (\tilde{a}^{ij}(\chi^\nu) D_j u) = 0$$

in $B_1$. Then for any $p \in (0, \infty)$, there exists a constant $C = C(n, p, \bar{\delta}, \Lambda)$ such that for any constant $c \in \mathbb{R}$, we have

$$||Du||_{L^p(B_{1/2})} \leq C||u - c||_{L^p(B_{1})}.\hfill (2.16)$$

Proof. We first assume that $c = 0$. It directly follows from [13, Lemma 2.5] that for any $q \in (1, \infty), \hfill (2.17)

$$||u||_{W^{1,q}(B_{1/3})} \leq C||u||_{L^2(B_{1})}.$$  \hfill (2.14)

Then for $q > n$ by the Sobolev embedding theorem, we have

$$||u||_{L^\infty(B_{1/3})} \leq C||u||_{L^2(B_{1})}.\hfill (2.15)$$

For $0 < p < 1$, by using a similar argument used in deriving (2.13), we get

$$||u||_{L^p(B_{1/3})} \leq C||u||_{L^p(B_{1})}.\hfill (2.16)$$

For $k = 1, \ldots, n - 1$ and $h \in (0, 1/12)$, since $\tilde{a}^{ij}(\chi^\nu)$ is independent of $x'$, we have $L_0 (\delta_{ij} u) = 0$ in $B_{1/3}$. We thus use [12, Lemma 2.5] again and (2.16) to get

$$||\delta_{ij} u||_{W^{1,q}(B_{1/2})} \leq C||\delta_{ij} u||_{L^2(B_{1/3})} \leq C||D_{ij} u||_{L^2(B_{1/3})} \leq C||u||_{L^2(B_{1/3})} \leq C||u||_{L^p(B_{1})}, \forall p > 0.$$  \hfill (2.17)

Letting $h \to 0$ gives

$$||D_{ij} u||_{W^{1,q}(B_{1/2})} \leq C||u||_{L^p(B_{1})}, \forall p > 0. \hfill (2.18)$$

Moreover, notice that in $B_1$,

$$D_{ij} U = - \sum_{i=1}^{n-1} \sum_{j=1}^{n} \tilde{a}^{ij} D_{ij} u, \quad D_{ij} U = \sum_{i=1}^{n} \tilde{a}^{ij} D_{ij} u,$$

where $U := \tilde{a}^{ij}(\chi^\nu) D_{ij} u$. Then by using (2.15), (2.16), (2.17), and the boundedness of $\tilde{a}^{ij}(\chi^\nu)$, we obtain

$$||U||_{W^{1,q}(B_{1/2})} = ||U||_{L^q(B_{1/2})} + ||DU||_{L^q(B_{1/2})}$$
Then, we have
\[
\|\nabla u\|_{L^p(B_{1/2})} + C\left(\|D_x U\|_{L^p(B_{1/2})} + \|D_n U\|_{L^p(B_{1/2})}\right)
\]
\[
\leq C\|u\|_{W^{1,q}(B_{1/2})} + C\|DD_x u\|_{L^r(B_{1/2})}
\]
\[
\leq C\|u\|_{W^{1,q}(B_{1/3})} + C\|D_x u\|_{W^{1,q}(B_{1/2})} \leq C\|u\|_{L^p(B_1)}.
\]
Combining (2.17) and (2.18), by the Sobolev embedding theorem for \(q > n\), we have
\[
\|D_x u\|_{L^p(B_{1/2})} + \|U\|_{L^p(B_{1/2})} \leq C\|u\|_{L^p(B_1)}, \quad p > 0.
\]
Thus, by \(\bar{a}^{mn}(x^n) \geq \delta\), we get
\[
\|D_{u}\|_{L^p(B_{1/2})} \leq C\|u\|_{L^p(B_1)}, \quad p > 0.
\]
Now, replacing \(u\) with \(u - c\), we conclude (2.14). \(\square\)

We will also apply the following lemma, which is \([9, \text{Lemma 2.7}]\).

**Lemma 2.11.** Let \(\omega\) be a nonnegative bounded function. Suppose there is \(c_1, c_2 > 0\) and \(0 < \kappa < 1\) such that for \(\kappa t \leq s \leq t\) and \(0 < t < r\),
\[
c_1 \omega(t) \leq \omega(s) \leq c_2 \omega(t).
\]
Then, we have
\[
\sum_{\ell=0}^{\infty} \omega(\kappa^\ell r) \leq C \int_{0}^{r} \frac{\omega(t)}{t} \, dt,
\]
where \(C = C(\kappa, c_1, c_2)\).

\([7, \text{Lemma 4.1}]\) can be extended as follows by replacing the exponent \(p = 2\) with a general \(p \in (1, \infty)\). See also \([7, \text{Lemma 6.3}]\).

**Lemma 2.12.** Let \(\mathcal{D}\) be a bounded domain in \(\mathbb{R}^n\) satisfying
\[
|\mathcal{D}_r(x)| \geq A_0 r^n \quad \text{for all } x \in \overline{\mathcal{D}} \text{ and } r \in (0, \text{diam } \mathcal{D}),
\]
where \(A_0 > 0\) is a constant. Let \(p \in (1, \infty)\) and \(T\) be a bounded linear operator on \(L^p(\mathcal{D})\).
Suppose that for any \(\tilde{y} \in \mathcal{D}\) and \(0 < r < \mu\) \(\text{diam } \mathcal{D}\), we have
\[
\int_{\mathcal{D}(\tilde{y}, \tilde{r})} |Tb| \leq C_0 \int_{\mathcal{D}(\tilde{y}, \tilde{r})} |b|
\]
whenever \(b \in L^p(\mathcal{D})\) is supported in \(\mathcal{D}_r(\tilde{y})\), \(\int_{\mathcal{D}_r} b = 0\), and \(c > 1\), \(C_0 > 0\), \(\mu \in (0, 1)\) are constants. Then for any \(g \in L^p(\mathcal{D})\) and any \(t > 0\), we have
\[
||x \in \mathcal{D} : |T g(x)| > t|| \leq \frac{C}{t} \int_{\mathcal{D}} |g|,
\]
where \(C = C(n, c, C_0, \mathcal{D}, A_0, \mu, ||T||_{L^p \to L^p})\) is a constant.

3. Proofs of Theorem 1.1 and Corollary 1.2

In this section, we shall first consider the solution \(u \in W^{2,p}(\mathcal{D})\) to the equation (1.1) without lower-order terms. Then for the general case, we move lower-order terms to the right-hand side and use the \(L^p\)-estimates in Lemma 2.7.
3.1. Proof of Theorem 1.1. We first assume that $b^i \equiv c \equiv 0$. The general case will be outlined at the end of the proof. We fix $x_0 \in B_{3/4} \cap D_{y^r}, 0 < r \leq 1/4$, and take a coordinate system associated with $x_0$ as in Section 2.2. We shall derive an a priori estimate of the modulus of continuity of $D_{xx}u$ by assuming that $u \in C^{1,1}(B_{3/4})$. Denote
\[ L_{\bar{c}}u := \bar{a}^{ij}(x_0^i, x_0^j)D_{ij}u. \]
As before, we modify the coefficients $\bar{a}^{ij}(x_0^i, x_0^j)$ to get the following elliptic operator defined by
\[ \bar{L}_{\bar{c}}u := \bar{a}^{ij}D_{ij}u, \]
where $\bar{a}^{ij} = \eta \bar{a}^{ij}(x_0^i, x_0^j) + \delta(1 - \eta)\delta_{ij}$ with $\eta \in C_0^\infty(B_r(x_0))$ satisfying
\[ 0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{2r/3}(x_0), \quad |\nabla \eta| \leq 5/r. \]
We present several lemmas (and their proofs) that will provide key estimates for the proof of Theorem 1.1.

**Lemma 3.1.** Let $p \in (1, \infty)$ and $v \in W^{2,p}(B_r(x_0)) \cap W_0^{1,p}(B_r(x_0))$ be a unique solution of
\[
\begin{cases}
\bar{L}_{\bar{c}}v = F_{\chi_{B_{r/2}(x_0)}} & \text{in } B_r(x_0), \\
v = 0 & \text{on } \partial B_r(x_0),
\end{cases}
\]
where $F \in L^p(B_r(x_0))$. Then for any $t > 0$, we have
\[ |\{x \in B_{r/2}(x_0) : |D^2v(x)| > t\}| \leq \frac{C}{t} |F|_{L^1(B_r(x_0))}, \]
where $C = C(n, p, \delta) > 0$.

**Proof.** For simplicity, we set $x_0 = 0$ and $r = 1$. We modify the proof of [11, Lemma 2.12]. By using Lemma 2.6, we can see that the map $T : F \mapsto D^2v$ is a bounded linear operator on $L^p(B_{1/2})$, it suffices to show that $T$ satisfies the hypothesis of Lemma 2.12. We introduce a new matrix-valued function $\hat{a}^{ij} = \bar{a}^{ij}/\bar{a}^{mm}$, so that $\hat{a}^{mm} = 1$. Clearly, $\hat{a}^{ij}$ satisfies the ellipticity and boundedness conditions with a new ellipticity constant determined by $\delta$. Therefore, $v \in W^{2,p}(B_1) \cap W_0^{1,p}(B_1)$ satisfies
\[
\begin{cases}
\hat{a}^{ij}D_{ij}v = \hat{F}_{\chi_{B_{1/2}}} & \text{in } B_1, \\
v = 0 & \text{on } \partial B_1,
\end{cases}
\]
where $\hat{F} = F/\bar{a}^{mm}$. It is sufficient to show
\[ |\{x \in B_{1/2} : |D^2v(x)| > t\}| \leq \frac{C}{t} |\hat{F}|_{L^1(B_{1/2})}. \]
We take $c = 24$ and fix $\tilde{y} \in B_{1/2}, r \in (0, 1/4)$. Let $b \in L^p(B_1)$ be supported in $B_r(y) \cap B_{1/2}$ with mean zero and $v_1 \in W^{2,p}(B_1) \cap W_0^{1,p}(B_1)$ be a solution of
\[
\begin{cases}
\hat{a}^{ij}D_{ij}v_1 = b & \text{in } B_1, \\
v_1 = 0 & \text{on } \partial B_1, \quad (3.1)
\end{cases}
\]
the solvability of which follows from Lemma 2.6.
For any \( R \in [1, 1] \) such that \( B_{1/2} \setminus B_R(\tilde{y}) \neq \emptyset \) and \( g = (g_{ij})_{i,j=1}^{n} \in C_0^\infty((B_{2R}(\tilde{y}) \setminus B_R(\tilde{y})) \cap B_{1/2}) \), let \( v_2 \in W_{0}^{1,p'}(B_{1}) \) be a weak solution of
\[
\begin{aligned}
D_i(\bar{a}^{ij}D_jv_2 + D_j\bar{a}^{ij}v_2) &= \text{div}^2 g \quad \text{in } B_1, \\
v_2 &= 0 \quad \text{on } \partial B_1,
\end{aligned}
\]
where \( 1/p + 1/p' = 1 \) and \( \bar{A} = (\bar{a}^{ij}) \) are defined as follows,
\[
\bar{a}^{ij} = 1; \quad \bar{a}^{ij} = \hat{a}^{ij} \text{ for } i, j \in \{1, \ldots, n-1\};
\]
\[
\bar{a}^{n\cdot} = 0 \quad \text{for } j \in \{1, \ldots, n-1\}.
\]
It is easy to check that \( \bar{A} \) satisfies the ellipticity and boundedness conditions with a new ellipticity constant determined by \( \delta \). Also, \( D_i\bar{a}^{ij}, j = 1, \ldots, n-1 \) is bounded and \( D_n\bar{a}^{in} = 0 \). Since \( g = 0 \), \( \bar{a}^{ij} = \hat{a}^{ij}(x') \), and \( \bar{a}^{in} = 1 \) in \( B_{R/12}(\tilde{y}) \subset B_{2/3} \), we get
\[
D_i(\hat{a}^{ij}(x')D_jv_2) = 0 \quad \text{in } B_{R/12}(\tilde{y}).
\]
By the De Giorgi-Nash-Moser estimate, we see that \( v_2 \) is Hölder continuous in \( B_r(\tilde{y}) \) and
\[
[v_2]_{B_r(\tilde{y})} \leq [v_2]_{\tilde{B}_{R/12}(\tilde{y})} \leq CR^{-\gamma - \bar{\gamma}} ||v_2||_{L^p(B_{R/12}(\tilde{y}))},
\]
where \( \gamma \in (0, 1) \) and \( C > 0 \) depending only on \( n, \delta \) and \( \Lambda \). On the other hand, one observe that
\[
\sum_{i,j=1}^{n} D_i(\hat{a}^{ij}D_jv_2 + D_j\bar{a}^{ij}v_2)
\]
\[
= \sum_{i,j=1}^{n-1} D_i(\hat{a}^{ij}D_jv_2 + D_j\bar{a}^{ij}v_2) + 2 \sum_{i,j=1}^{n-1} D_n(\hat{a}^{n\cdot}D_jv_2 + D_j\bar{a}^{n\cdot}v_2) + D_n(\bar{a}^{in}D_nv_2)
\]
\[
= \sum_{i,j=1}^{n-1} D_{ij}(\hat{a}^{ij}v_2) + 2 \sum_{i,j=1}^{n-1} D_{nj}(\hat{a}^{n\cdot}v_2) + D_{nn}v_2 = \sum_{i,j=1}^{n} D_{ij}(\hat{a}^{ij}v_2).
\]
Here, we used the fact that \( \hat{a}^{in} = 1 \). Therefore, we see that \( v_2 \) is also an adjoint solution of
\[
\begin{aligned}
D_{ij}(\hat{a}^{ij}v_2) &= \text{div}^2 g \quad \text{in } B_1, \\
v_2 &= 0 \quad \text{on } \partial B_1.
\end{aligned}
\]
Hence, by using Lemma 2.8, we get
\[
||v_2||_{L^p(B_{1})} \leq C||g||_{L^p(B_{1})} = C||g||_{L^p((B_{2R}(\tilde{y}) \cap B_{1/2})).}
\]
By (3.1), (3.3), and the hypothesis on \( b \), we have
\[
\int_{(B_{R/12}(\tilde{y}) \cap B_{1/2})} D_{ij}v_1g^{ij} = \int_{B_{R/12}(\tilde{y}) \cap B_{1/2}} bv_2 = \int_{B_{R/12}(\tilde{y}) \cap B_{1/2}} b(v_2 - v_2(\tilde{y})).
\]
Then by using (3.3) and (5.6), we bound the absolute value of the right-hand side above by
\[
\frac{R}{R} R^{-\bar{\gamma}} ||v_2||_{L^p((B_{R}(\tilde{y}) \cap B_{1/2}) \cap B_{R/12}(\tilde{y}))} \leq C R^{-\bar{\gamma}} \sum_{i,j=1}^{n} D_{ij}(\hat{a}^{ij}v_2)||v_2||_{L^p((B_{R}(\tilde{y}) \cap B_{1/2}) \cap B_{R/12}(\tilde{y}))}
\]
By duality, we have
\[ \|D^2 v_1\|_{L^1((B_{2^r}(\bar{y})) \cap B_{1/2})} \leq C \left( \frac{r}{R} \right)^{\gamma} R^{-\gamma} \|b\|_{L^1((B_{2^r}(\bar{y})) \cap B_{1/2})}. \]

Hence, by Hölder’s inequality, we get
\[ \|D^2 v_1\|_{L^1((B_{2^r}(\bar{y})) \cap B_{1/2})} \leq C \left( \frac{r}{R} \right)^{\gamma} \|b\|_{L^1((B_{2^r}(\bar{y})) \cap B_{1/2})}. \]

(3.7)

Let \( N \) be the smallest positive integer such that \( B_{1/2} \subset B_{2^{N-1}r}(\bar{y}) \). By taking \( R = cr, 2cr, \ldots, 2^{N-1}cr \) in (3.7) and summarizing, we have
\[ \int_{B_{1/2}(\bar{y})} |D^2 v_1| \, dx \leq C \sum_{k=1}^{N} 2^{-k} \|b\|_{L^1((B_{2^k}(\bar{y})) \cap B_{1/2})} \leq C \int_{B_{r}(\bar{y}) \cap B_{1/2}} |b| \, dx. \]

Therefore, \( T \) satisfies the hypothesis of Lemma 3.1 and the proof is finished. \( \square \)

Denote
\[ \phi(x_0, r) := \inf_{q \in \mathbb{R}^{n \times n}} \left( \int_{B_{r}(x_0)} |D x r u - q|^q \, dx \right)^{1/q}, \]
where \( q \in (0, 1) \) is some fixed exponent. First of all, by Hölder’s inequality, we have
\[ \phi(x_0, r) \leq \left( \int_{B_{r}(x_0)} |D x r u|^q \, dx \right)^{1/q} \leq Cr^{-\gamma} \|D x r u\|_{L^1(B_{r}(x_0))}, \]

(3.8)

where \( C = C(n) \).

**Lemma 3.2.** For any \( \gamma \in (0, 1) \) and \( 0 < \rho \leq r \leq 1/4 \), we have
\[ \phi(x_0, p) \leq C \left( \frac{p}{r} \right)^{\gamma} r^{-\gamma} \|D x r u\|_{L^1(B_{r}(x_0))} + C \alpha_A(p) \|D^2 u\|_{L^{\infty}(B_{r}(x_0))} + C \alpha_f(p), \]

where \( C = C(n, p, \delta, \gamma) \), and \( \alpha_A(t) \) is a Dini function derived from \( \alpha_A(t) \).

**Proof.** For any \( t > 0 \), by using Lemma 3.1 with \( F = f(x) - f(x_0, x^n) + (\partial_i f(x) - a_i(t)) D_i u \) and (3.2), we have
\[ |x \in B_{r/2}(x_0) : |D^2 v(x)| > t| \leq \frac{C}{t} \int_{B_{r/2}(x_0)} |F| \, dx \]
\[ \leq \frac{C}{t} \left( \int_{B_{r/2}(x_0)} |f(x) - f(x_0, x^n)| \, dx + \int_{B_{r/2}(x_0)} |(\partial_i^f(x) - a_i(t)) D_i u| \, dx \right) \]
\[ \leq \frac{C}{t} \left( r^\gamma \alpha_f(r) + r^\gamma \alpha_A(r) \|D^2 u\|_{L^{\infty}(B_{r}(x_0))} \right), \]

(3.10)

where \( \alpha_A(r) := \alpha_A(r) + \alpha_1(r) \). Therefore, for any given \( q \in (0, 1) \), we have
\[ \int_{B_{r/2}(x_0)} |D^2 v|^q \, dx = \int_{0}^{\infty} q t^{q-1} \|x \in B_{r/2}(x_0) : |D^2 v(x)| > t| \, dt \]
\[ = \left( \int_{0}^{\tau} + \int_{\tau}^{\infty} \right) q t^{q-1} |x \in B_{r/2}(x_0) : |D^2 v(x)| > t| \, dt \]
\[ \leq C \tau^q |B_{r}(x_0)| + \frac{Cq}{1 - q} \tau^q \left( r^\gamma \alpha_f(r) + r^\gamma \alpha_A(r) \|D^2 u\|_{L^{\infty}(B_{r}(x_0))} \right). \]
By choosing a suitable $\tau$, we have
\[
\left(\int_{B_r(x_0)} |D^2v|^q \, dx\right)^{1/q} \leq C(\bar{\omega}_A(r)||D^2u||_{L^\infty(B_r(x_0))} + \bar{\omega}_f(r)).
\] (3.11)

Let $w = u - v$, which satisfies $L_{\nu}^q w = f(x', x^n)$ in $B_{2r}(x_0)$. By Lemma 2.9 with a suitable scaling, we see that for any $q \in \mathbb{R}^{n(n-1)}$,
\[
||D^2D_xw||^q_{L^\infty(B_{2r}(x_0))} \leq Cr^{-(n+q)} \int_{B_{2r}(x_0)} |D_xw - q|^q \, dx.
\]
Hence, for any $\nu \in (0, 1/4)$, we have
\[
||D_{xx}w - (D_{xx}w)_{B_{2r}(x_0)}||^q_{L^\infty(B_{2r}(x_0))} \leq C(\nu r)^{\frac{n+q}{n}}||D^2D_xw||^q_{L^\infty(B_{2r}(x_0))}
\]
\[
\leq C\nu^{\frac{n+q}{n}} \int_{B_{2r}(x_0)} |D_{xx}w - q|^q \, dx.
\]
That is,
\[
\left(\int_{B_{2r}(x_0)} |D_{xx}w - (D_{xx}w)_{B_{2r}(x_0)}|^q \, dx\right)^{1/q} \leq C_0\nu \left(\int_{B_{2r}(x_0)} |D_{xx}w - q|^q \, dx\right)^{1/q},
\] (3.12)
where $C_0 > 0$ is a constant depending on $n, p, \delta$, and $\Lambda$. Recalling that $u = w + v$, by using (3.11) and (3.12), we obtain
\[
\left(\int_{B_{2r}(x_0)} |D_xu - (D_xw)_{B_{2r}(x_0)}|^q \, dx\right)^{1/q} \leq C_0\nu \left(\int_{B_{2r}(x_0)} |D_xu - q|^q \, dx\right)^{1/q} + C\nu^{\frac{q}{n}} \left(\bar{\omega}_A(r)||D^2u||_{L^\infty(B_{2r}(x_0))} + \bar{\omega}_f(r)\right).
\]
Since $q \in \mathbb{R}^{n(n-1)}$ is arbitrary, we obtain
\[
\phi(x_0, \nu r) \leq C_0\nu \phi(x_0, r) + C\nu^{\frac{q}{n}} \left(\bar{\omega}_A(r)||D^2u||_{L^\infty(B_{2r}(x_0))} + \bar{\omega}_f(r)\right).
\]
For any $\gamma \in (0, 1)$, fix a $\nu \in (0, 1/4)$ sufficiently small such that $C_0\nu \leq \nu^{\gamma'}$. Then
\[
\phi(x_0, \nu r) \leq \nu^{\gamma'} \phi(x_0, r) + C\nu^{\frac{q}{n}} \left(\bar{\omega}_A(r)||D^2u||_{L^\infty(B_{2r}(x_0))} + \bar{\omega}_f(r)\right).
\]
By iterating, for $j = 1, 2, \ldots$, we obtain
\[
\phi(x_0, \nu^j r) \leq \nu^{j\gamma'} \phi(x_0, r) + C\nu^{\frac{q}{n}} \left(\bar{\omega}_A(r)||D^2u||_{L^\infty(B_{2r}(x_0))} + \bar{\omega}_f(r)\right),
\] (3.13)
where
\[
\bar{\omega}_*(t) = \sum_{i=1}^\infty \nu^i \left(\bar{\omega}_*(\nu^{-i} t) \chi_{\nu^{-i} t \leq 1} + \bar{\omega}_*(1) \chi_{\nu^{-i} t > 1}\right),
\] (3.14)
which is a Dini function; see (3, Lemma 1), and satisfies (2.19).

Now, for any $\rho$ satisfying $0 < \rho \leq r \leq 1/4$, we take $j$ to be the integer satisfying $\nu^{j+1} < \rho / r \leq \nu^j$. Then, by (3.13) and (2.19), we have
\[
\phi(x_0, \rho) \leq C\left(\nu^{\gamma'} \phi(x_0, r) + C\nu^{\frac{q}{n}} \left(\bar{\omega}_A(r)||D^2u||_{L^\infty(B_{2r}(x_0))} + \bar{\omega}_f(r)\right)\right).
\] (3.15)
Hence, (3.9) follows from (3.8) and (3.15). □
Lemma 3.3. We have
\[ \|D^2 u\|_{L^\infty(B_{1/4})} \leq C\|D^2 u\|_{L^1(B_{3/4})} + C\left( \int_0^{1\omega_f(t)/t} dt + \|f\|_{L^\infty(B_1)} \right), \]  
where \( C > 0 \) is a constant depending only on \( n, p, \delta, \gamma, \) and \( \omega_A. \)

Proof. Let \( \kappa \in (0, 1/4) \) be the constant in the proof of Lemma 3.2. Let \( \{q_{x_0, k^j r}\}_{k=0}^{\infty} \in \mathbb{R}^{n \times (n-1)} \) be such that
\[ \phi(x_0, \kappa^k r) = \left( \int_{B_{\kappa^j r}(x_0)} |D_{x\nu} u - q_{x_0, k^j r}|^q \, dx \right)^{1/q}. \]

Since
\[ |q_{x_0, kr} - q_{x_0, j}|^q \leq |D_{x\nu} u - q_{x_0, r}|^q + |D_{x\nu} u - q_{x_0, kr}|^q, \]
by taking average over \( x \in B_{x\nu}(x_0) \) and taking the \( q \)-th root, we obtain
\[ |q_{x_0, kr} - q_{x_0, j}| \leq C(\phi(x_0, kr) + \phi(x_0, r)). \]

By iterating, we have
\[ |q_{x_0, k^j r} - q_{x_0, r}| \leq C \sum_{j=0}^{K} \phi(x_0, \kappa^j r). \]  

(3.17)

Notice that (3.13) implies
\[ \lim_{K \to \infty} \phi(x_0, \kappa^K r) = 0. \]

Thus, by using the assumption that \( Du \in C^{0,1}(B_{3/4}) \) and the Lebesgue differentiation theorem, we obtain for a.e. \( x_0 \in B_{3/4}, \)
\[ \lim_{K \to \infty} q_{x_0, k^j r} = D_{x\nu} u(x_0). \]

On the other hand, (3.14) implies that \( \tilde{\omega}_A \) and \( \tilde{\omega}_f \) satisfy (2.19). Therefore, by taking \( K \to \infty \) in (3.17), using (3.13) and Lemma 2.11, for a.e. \( x_0 \in B_{3/4}, \) we have
\[ |D_{x\nu} u(x_0) - q_{x_0, r}| \leq C \sum_{j=0}^{\infty} \phi(x_0, \kappa^j r) \]
\[ \leq C \left( \phi(x_0, r) + \|D^2 u\|_{L^\infty(B_{x\nu}(x_0))} \int_0^{r}\frac{\tilde{\omega}_A(t)}{t} \, dt + \int_0^{r}\frac{\tilde{\omega}_f(t)}{t} \, dt \right). \]  

(3.18)

By averaging the inequality
\[ |q_{x_0, r}|^q \leq |D_{x\nu} u - q_{x_0, r}|^q + |D_{x\nu} u|^q \]
over \( x \in B_{r}(x_0) \) and taking the \( q \)-th root, we have
\[ |q_{x_0, r}| \leq 2^{1/q-1}\phi(x_0, r) + 2^{1/q-1}\left( \int_{B_{r}(x_0)} |D_{x\nu} u|^q \, dx \right)^{1/q}. \]

Therefore, combining (3.13) and (3.8), we obtain for a.e. \( x_0 \in B_{3/4}, \)
\[ |D_{x\nu} u(x_0)| \leq Cr^{-n}\|D_{x\nu} u\|_{L^1(B_{x\nu}(x_0))} \]
\[ + C \left( \|D^2 u\|_{L^\infty(B_{x\nu}(x_0))} \int_0^{r}\frac{\tilde{\omega}_A(t)}{t} \, dt + \int_0^{r}\frac{\tilde{\omega}_f(t)}{t} \, dt \right). \]
For any $x_1 \in B_{1/4}$ and $0 < r < 1/4$, we take the supremum of the above inequality over $B_r(x_1)$ to get
\[
\|D_{xx}u\|_{L^\infty(B_r(x_1))} \leq Cr^{-n}\|D^2u\|_{L^1(B_r(x_1))}
+ C\left(\|D^2u\|_{L^\infty(B_r(x_1))} \int_0^r \frac{\tilde{\omega}_A(t)}{t} \, dt + \int_0^r \frac{\tilde{\omega}_f(t)}{t} \, dt\right).
\]
Recalling that $a^{ij}(x)D_{ij}u(x) = f(x)$, one can see that
\[
D_{nn}u = \frac{1}{a_{nn}} \left(f - \sum_{(i,j)\neq(n,n)} a^{ij}D_{ij}u\right).
\]
Therefore, we have
\[
\|D^2u\|_{L^\infty(B_r(x_1))} \leq Cr^{-n}\|D^2u\|_{L^1(B_r(x_1))} + C\left(\|D^2u\|_{L^\infty(B_r(x_1))} \int_0^r \frac{\tilde{\omega}_A(t)}{t} \, dt \right.
+ \left. \int_0^r \frac{\tilde{\omega}_f(t)}{t} \, dt + \|f\|_{L^\infty(B_1)}\right).
\]
We fix $r_0 < 1/4$ such that for any $0 < r \leq r_0$,
\[
C \int_0^r \frac{\tilde{\omega}_A(t)}{t} \, dt \leq \frac{1}{4^n}.
\]
Then, for any $x_1 \in B_{1/4}$ and $0 < r \leq r_0$, we get
\[
\|D^2u\|_{L^\infty(B_r(x_1))} \leq 4^{-n}\|D^2u\|_{L^\infty(B_{r_0}(x_1))} + Cr^{-n}\|D^2u\|_{L^1(B_{r_0}(x_1))}
+ C\left(\int_0^r \frac{\tilde{\omega}_f(t)}{t} \, dt + \|f\|_{L^\infty(B_1)}\right).
\]
For $k = 1, 2, \ldots$, denote $r_k = 3/4 - (1/2)^k$. For $x_1 \in B_{r_k}$ and $r = (1/2)^{k+2}$, we have $B_{2r}(x_1) \subset B_{r_{k+1}}$. We take $k_0 \geq 1$ sufficiently large such that $(1/2)^{k_0+2} \leq r_0$. It follows that for any $k \geq k_0$,
\[
\|D^2u\|_{L^\infty(B_{r_k})} \leq 4^{-n}\|D^2u\|_{L^\infty(B_{r_{k+1}})} + Cr^{k_0}\|D^2u\|_{L^1(B_{r_{k+1}})}
+ C\left(\int_0^r \frac{\tilde{\omega}_f(t)}{t} \, dt + \|f\|_{L^\infty(B_1)}\right).
\]
By multiplying the above by $4^{-kn}$ and summing over $k = k_0, k_0 + 1, \ldots$, we have
\[
\sum_{k=k_0}^\infty 4^{-kn}\|D^2u\|_{L^\infty(B_{r_k})}
\leq \sum_{k=k_0+1}^\infty 4^{-((k+1)n)}\|D^2u\|_{L^\infty(B_{r_{k+1}})} + C\|D^2u\|_{L^1(B_{r_{k+1}})}
+ C\left(\int_0^r \frac{\tilde{\omega}_f(t)}{t} \, dt + \|f\|_{L^\infty(B_1)}\right).
\]
Recalling the assumption that $u \in C^{1,1}(B_{3/4})$, the summations on both sides are convergent, and we finally obtain (5.16). The lemma is proved.

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** By (5.18), for $r \in (0, 1/8)$, we have
\[
\sup_{x_0 \in B_{1/8}} |D_{xx}u(x_0) - q_{x_0,r}| \leq C \sup_{x_1 \in B_{1/8}} \phi(x_0, r) + C\|D^2u\|_{L^\infty(B_{1/4})} \int_0^r \frac{\tilde{\omega}_A(t)}{t} \, dt + C \int_0^r \frac{\tilde{\omega}_f(t)}{t} \, dt =: C\psi(r). \tag{3.19}
\]
We recall that for each $x_0$, the coordinate system and thus $x'$ are chosen according to $x_0$. By Lemma 3.2, for any $r \in (0, 1/8)$, we obtain
\[
\sup_{x_0 \in B_{1/8}} \phi(x_0, r) \leq C \left( r^2 \|D^2 u\|_{L^1(B_{1/4})} + \omega_A(r) \|D^2 u\|_{L^\infty(B_{1/4})} + \omega_f(r) \right). \tag{3.20}
\]

Suppose that $y \in B_{1/8} \cap D_{j_1}$, $j_1 \in [1, l + 1]$. Clearly, if $|x_0 - y| \geq 1/32$, then
\[
|D_{x} u(x_0) - D_{x} u(y)| \leq 2 \|D^2 u\|_{L^\infty(B_{1/4})} \|D^2 u\|_{L^1(B_{1/4})}
\]
\[
\leq C|x_0 - y| \left( \|D^2 u\|_{L^1(B_{1/4})} + \int_0^1 \frac{\omega_f(t)}{t} dt + \|f\|_{L^\infty(B_{1/4})} \right), \tag{3.21}
\]
where we used (3.16) in the second inequality. Otherwise, if $|x_0 - y| < 1/32$, we set $r = |x_0 - y|$ and discuss it further according to the following two cases:

**Case 1.** If
\[
r \leq 1/16 \max \{\text{dist}(x_0, \partial D_{j_1}), \text{dist}(y, \partial D_{j_1})\},
\]
then $j_0 = j_1$. We define
\[
\phi(x_0, r) := \inf_{Q \in K^{R_n}} \left( \int_{B_{r(x_0)}} \|D^2 u - Q\| dx \right)^{1/q}.
\]
For any $q = (q_{ij}) \in \mathbb{R}^{n \times (n-1)}$, we define $\tilde{Q} := (\tilde{Q}_{ij}) \in \mathbb{R}^{n \times n}$ by
\[
\tilde{Q}_{ij} = q_{ij} \quad \text{for } i = 1, \ldots, n, j = 1, \ldots, n - 1; \quad \tilde{Q}_{ii} = q_{ii} \quad \text{for } i = 1, \ldots, n - 1;
\]
\[
\tilde{Q}_{mn} = \frac{1}{a^{mn}(x)} \left( f - \sum_{(i,j) \neq (n,n)} a^{ij} \tilde{Q}_{ij} \right), \tag{3.22}
\]
where $\tilde{a}^{ij}$ and $\tilde{f}$ are constant functions corresponding to $a^{ij}$ and $f$, respectively. Combining
\[
D_{mn} u(x) = \frac{1}{a^{mn}(x)} \left( f - \sum_{(i,j) \neq (n,n)} a^{ij} D_{ij} u \right),
\]
and (3.9), we reach the following estimate: for any $\gamma \in (0, 1)$ and $0 < \rho \leq r < 1/8$, we have
\[
\phi(x_0, \rho) \leq C \left( \phi(x_0, \rho) + \omega_f(\rho) + \omega_A(\rho) \|f\|_{L^\infty(B_{r(x_0)})} + \|D_{x} u\|_{L^\infty(B_{r(x_0)})} \right).
\]
By using the same argument that led to (3.19), we obtain
\[
\sup_{x_0 \in B_{1/8}} \|D^2 u(x_0) - Q_{x_0, \rho}\|^\rho \leq C \sup_{x_0 \in B_{1/8}} \phi(x_0, r) + C \left( \|D^2 u\|_{L^\infty(B_{1/4})} + \|f\|_{L^\infty(B_{1/4})} \right) \int_0^\gamma \frac{\omega_A(t)}{t} dt + C \int_0^\gamma \frac{\omega_f(t)}{t} dt,
\]
where $Q_{x_0, \rho} \in \mathbb{R}^{n \times n}$ satisfying
\[
\phi(x_0, r) = \left( \int_{B_{r(x_0)}} \|D^2 u - Q_{x_0, \rho}\| dx \right)^{1/q}.
\]
Then by the triangle inequality, we have
\[
\|D^2 u(x_0) - D^2 u(y)\|^\rho \leq \|D^2 u(x_0) - Q_{x_0, \rho}\|^\rho + \|D^2 u(z) - Q_{x_0, \rho}\|^\rho + \|D^2 u(z) - X^T Q_{x_0, \rho} X\|^\rho
\]
Therefore, similar to (3.23), we have
\[ \|D_y^2 u(y) - X^T Q_{y,y} X\|_p, \] (3.24)
where \( X = (X_{ij}) \) is an \( n \times n \) matrix, and \( X_{ij} = \frac{\partial^2 f}{\partial y_i \partial y_j} \) for \( i, j = 1, \ldots, n \). We use \( D_y \) to denote derivatives in the coordinate system associated with \( y \), so that
\[ D_y^2 u(y) = X^T D_y^2 u(y) X. \]

Therefore, similar to (3.25), we have
\[
\|D_y^2 u(y) - X^T Q_{y,y} X\| = |X^T (D_y^2 u(y) - Q_{y,y}) X| \leq C\|D_y^2 u(y) - Q_{y,y}\|
\leq C\phi(y, r) + C\left(\|D^2 u\|_{L^\infty(B_{1/4})} + \|f\|_{L^\infty(B_{1/4})}\right) \int_0^r \frac{\hat{\omega}_A(t)}{t} dt + C \int_0^r \frac{\hat{\omega}_I(t)}{t} dt.
\]
We take the average over \( z \in B_r(x_0) \cap B_r(y) \) in (3.24), and then take the \( q \)-th root to get
\[
\|D_y^2 u(x_0) - D_y^2 u(y)\| \leq C \left(\psi(r) + \|f\|_{L^\infty(B_{1/4})} \int_0^r \frac{\hat{\omega}_A(t)}{t} dt\right).
\]
By using (3.16), (3.19), and (3.21), we obtain
\[
\|D_y^2 u(x_0) - D_y^2 u(y)\|
\leq C \int_0^{\|\phi\| r} \frac{\hat{\omega}_A(t)}{t} dt
+ C \int_0^{\|\phi\| r} \frac{\hat{\omega}_I(t)}{t} dt \left(\|D^2 u\|_{L^\infty(B_{1/4})} + \int_0^1 \frac{\hat{\omega}_I(t)}{t} dt + \|f\|_{L^\infty(B_{1/4})}\right),
\]
where \( \gamma \in (0, 1) \) is arbitrary.

**Case 2.** If \( r > 1/16 \max \{\text{dist}(x_0, \partial D_{y_i}), \text{dist}(y, \partial D_{y_i})\} \), then by the triangle inequality, we have
\[
\|D_{x_i} u(x_0) - D_{x_i} u(y)\|
\leq \|D_{x_i} u(x_0) - q_{x_i,r}\|_p + \|q_{x_i,r} - q_{y,r}\|_p + \|D_{y,y} u(y) - q_{y,r}\|_p + \|D_{y,y} u(y) - D_{x_i} u(y)\|_p
\leq C\psi(r) + \|D_{x_i} u(z) - q_{x_i,r}\|_p + \|D_{y,y} u(z) - q_{y,r}\|_p + \|D_{y,y} u(z) - D_{x_i} u(z)\|_p
+ \|D_{y,y} u(y) - D_{x_i} u(y)\|_p, \quad \forall z \in B_r(x_0) \cap B_r(y).
\]

In order to estimate the last two terms in (3.26), we first notice that
\[
D_{y,y} u(y) - D_{x_i} u(y) = X^T D_y^2 u(y) X^{-1} I_1 - D_y^2 u(y) I_1
= (X^T - I) D_y^2 u(y) I_1 + X^T D_y^2 u(y) (X^{-1} - I) I_1, \quad (3.27)
\]
where \( I \) is the \( n \times n \) identity matrix and \( I_1 = (\delta_{ij}) \) is an \( n \times (n - 1) \) matrix. On the other hand, we suppose that the closest point on \( \partial D_{y_i} \) to \( y \) is \( (y', h_{y_i}(y')) \), and let
\[
\nu_2 = \frac{- \nabla_x h_{y_i}(y')}{\sqrt{1 + |\nabla_x h_{y_i}(y')|^2}}
\]
be the unit normal vector at \( (y', h_{y_i}(y')) \) on the surface \( \{(y', t) : t = h_{y_i}(y')\} \). The corresponding tangential vectors are
\[
\tau_{2,1} = (1, 0, \ldots, 0, D_{y_i} h_{y_i}(y'))^T, \ldots, \tau_{2,n-1} = (0, 0, \ldots, 1, D_{y^{n-1}_i} h_{y_i}(y'))^T.
\]
We define the projection operator by
\[ \text{proj}_b a = \frac{\langle a, b \rangle}{\langle a, a \rangle} a, \]
where \( \langle a, b \rangle \) denotes the inner product of the vectors \( a \) and \( b \). Then apply the Gram-Schmidt process as follows:
\begin{align*}
\hat{t}_{2,1} &= \tau_{2,1}, \\
\hat{t}_{2,1} &= \frac{\hat{t}_{2,1}}{\| \hat{t}_{2,1} \|}, \\
\hat{t}_{2,2} &= \tau_{2,2} - \text{proj}_{\hat{t}_{2,1}} \tau_{2,2}, \\
\hat{t}_{2,2} &= \frac{\hat{t}_{2,2}}{\| \hat{t}_{2,2} \|}, \\
\vdots & \\
\hat{t}_{2,n-1} &= \tau_{2,n-1} - \sum_{j=1}^{n-2} \text{proj}_{\hat{t}_{2,j}} \tau_{2,n-1}, \\
\hat{t}_{2,n-1} &= \frac{\hat{t}_{2,n-1}}{\| \hat{t}_{2,n-1} \|}.
\end{align*}
Similarly, we denote \( v_1 = (0', 1)^T \) to be the unit normal vector at \((x_0', h_{x_0'}(x_0'))\), and the corresponding tangential vectors are
\[ \tau_{1,1} = (1, 0, \ldots, 0, 0)^T, \ldots, \tau_{1,n-1} = (0, 0, \ldots, 1, 0)^T. \]
It follows from the proof of Lemma 2.5 that the upper bound of \( |\nabla_x h_j(y')| \) is \( C \omega_1(r) \), \( j = 1, \ldots, M \). Then we have
\[ |v_1 - v_2| = \left| (0', 1)^T - \frac{(- \nabla_x h_j(y'), 1)^T}{\sqrt{1 + \nabla_x h_j(y')^2}} \right| \leq C \omega_1(|x_0 - y|), \]
which is also true for \( |\tau_{1,i} - \hat{t}_{2,i}|, i = 1, \ldots, n - 1 \). Thus, coming back to (3.27), we obtain
\[ |D_{xx'} u(y) - D_{yy'} u(y)| \leq C \| D^2 u \|_{L^1(B_{1/4})} \omega_1(|x_0 - y|). \] (3.28)
The penultimate term of (3.26) is also bounded by the right-hand side of (3.28). Coming back to (3.26), we take the average over \( z \in B_r(x_0) \cap B_r(y) \) and take the \( q \)-th root to get
\begin{align*}
&|D_{xx'} u(x_0) - D_{xx'} u(y)| \\
&\leq C \left( \psi(r) + \phi(x_0, r) + \phi(y, r) + \| D^2 u \|_{L^1(B_{1/4})} \omega_1(|x_0 - y|) \right) \\
&\leq C \left( \psi(r) + \| D^2 u \|_{L^1(B_{1/4})} \omega_1(|x_0 - y|) \right).
\end{align*}
It follows from (3.16), (3.19), and (3.20) that
\begin{align*}
|D_{xx'} u(x_0) - D_{xx'} u(y)| \\
&\leq C |x_0 - y| \| D^2 u \|_{L^1(B_{1/4})} + C \int_0^{\|x_0 - y\|} \frac{\tilde{w}_f(t)}{t} \, dt \\
&\quad + C \int_0^{\|x_0 - y\|} \frac{\tilde{w}_f(t)}{t} \, dt \cdot \left( \| D^2 u \|_{L^1(B_{1/4})} + \int_0^1 \frac{\tilde{w}_f(t)}{t} \, dt + \| f \|_{L^1(B_{1/4})} \right). \quad (3.29)
\end{align*}
Thus, we finish the proof of Theorem 1.1 without lower-order terms under the assumption that \( u \in C^{1,1}(B_{3/4}) \).
Now we remove the assumption that \( u \in C^{1,1}(B_{3/4}) \). For this, it follows from the interior regularity in [7] for the non-divergence form elliptic equations that we only need to show that for any \( x_0 \in \partial D_m, m = 1, \ldots, M - 1 \), there is a neighborhood of \( x_0 \) in which \( Du \) is Lipschitz. In the case when \( \partial D_m \) is smooth, say \( C^{2,\alpha} \) with \( \alpha \in (0,1) \), we can use the technique of locally flattening the boundaries, and an approximation argument, which is similar to that in the proof of [12, Theorem 1.1].

To be specific, from the assumption that \( u \in C^{1,1}(B_{3/4}) \), we can find a small \( r_0 > 0 \) and a \( C^{2,\alpha} \) diffeomorphism of flattening the boundary \( \partial D_m \cap B_{r_0}(x_0) \): \( y = \Phi(x) = (\Phi^1(x), \ldots, \Phi^n(x)) \), which satisfies \( \Phi(x_0) = 0, \det D\Phi = 1 \), and

\[
\Phi(\partial D_m \cap B_{r_0}(x_0)) = \Phi(B_{r_0}(x_0)) \cap \{ y^n = 0 \}.
\]

Let \( \tilde{u}(y) := u(x) \), which satisfies

\[
\partial^i D_{ij} \tilde{u} = \bar{h},
\]

where \( \partial^i D^j D_i \tilde{u} \), \( \tilde{h}(y) = \tilde{f}(y) - \partial^i D_i \partial^j D^i \tilde{u} \), and \( \tilde{f}(y) = f(x) \), which are also of piecewise Dini mean oscillation in \( \Phi(B_{r_0}(x_0)) \). Then, it suffices to show that \( \tilde{D} \tilde{u} \) is Lipschitz near 0. We take the standard mollification of the coefficients \( \partial^i D^j \tilde{u} \) and data \( \tilde{h} \) in the \( y' \) direction with a parameter \( \varepsilon > 0 \). Then we get a uniform Lipschitz estimate independent of \( \varepsilon \) by using [3, Theorem 3] and the a priori \( W^{2,\infty} \) estimate in Lemma 5.16. Finally, we take the limit as \( \varepsilon \to 0 \) by following the proof of [3, Theorem 3].

For \( D_m \) with \( C^{1,\text{Dini}} \) boundary, we shall approximate \( D_m \) by a sequence of increasing smooth domains \( \{ D_m^{k} \}_{k=1}^\infty \), which can be constructed via the regularized distance function \( \rho(x) \) such that \( \rho(x) \sim \text{dist}(x, \partial D_m) \) for any \( x \in D_m \) close to \( \partial D_m \), is of class \( C^{1,\text{loc}}(D_m) \), and has uniform \( C^{1,\text{Dini}} \)-characteristics. For the existence and properties of \( \rho \), we refer the reader to [22, Theorem 2.1] and [11, Lemma 5.1]. We set

\[
D_{m}^{k} := \{ x \in D_m : \rho(x) > 1/k \}, \quad k = 1, 2, \ldots,
\]

which have uniform \( C^{1,\text{Dini}} \)-characteristics. Note that \( \partial D_m \cap B_{2r}(x_0) \) can be represented by

\[
x^n = \varphi(x'),
\]

where \( \varphi \) is a \( C^{1,\text{Dini}} \) function. Similarly, for any fixed \( k = 1, 2, \ldots, \partial D_m^{k} \cap B_{2r}(x_0) \) can be represented by

\[
x^n = \varphi_k(x'),
\]

where \( \varphi_k, k = 1, 2, \ldots, \) are \( C^{1,\text{Dini}} \) functions with uniform \( C^{1,\text{Dini}} \)-characteristics. Next we approximate \( a^{ij} \) and \( f \) by

\[
a^{ij}_k(x', x^n) = a^{ij}(x', x^n + \varphi(x') - \varphi_k(x')), \quad f_k(x', x^n) = f(x', x^n + \varphi(x') - \varphi_k(x')),
\]

which are of piecewise Dini mean oscillation in subdomains \( B_{2r_0}(x_0) \cap D_m^{k} \) and \( B_{2r_0}(x_0) \setminus D_m^{k} \) with uniform moduli of continuity, and as \( k \to \infty \),

\[
a^{ij}_k \to a^{ij} \quad a.e., \quad f_k \to f \quad \text{in } L^p(B_{r_0}(x_0)).
\]

After that, we can find a sequence of solutions \( u_k \in W^{2,p}(B_{r/2}(x_0)) \) that converges to \( u \) almost everywhere with a uniform \( C^{1,1} \) norm in \( B_{r_0}(x_0) \), by modifying the coefficients \( a^{ij}_k \) which is similar to the argument in Section 2.3. Finally, passing to the limit finishes the proof of Theorem 5.14 under the assumption that \( b^i \equiv c \equiv 0 \).
For the general case, we rewrite (1.1) as
\[ a^j D_{ij} u = f - b^i D_i u - cu =: f_0. \]
Then we have
\[
\alpha \omega (r) \leq \alpha \omega (r) + \|Du\|_{L^\infty(B_{\gamma}(x_0))} \omega (r) + \gamma [Du]_j; B_{\gamma}(x_0) \|\gamma (B_{\gamma}(x_0)) \\
+ \|u\|_{L^\infty(B_{\gamma}(x_0))} \omega (r) + \gamma [u]_j; B_{\gamma}(x_0) \|\gamma (B_{\gamma}(x_0)),
\]
where \( \gamma \in (0, 1) \) and \( B_{\gamma}(x_0) \subseteq B_{3/4} \). By using Lemma 2.7,
\[
\|u\|_{L^\infty(B_{\gamma}(x_0))} + \|Du\|_{L^\infty(B_{\gamma}(x_0))} + [u]_j; B_{\gamma}(x_0) + [Du]_j; B_{\gamma}(x_0)
\leq C \left( \|u\|_{L^1(B_j)} + \|f\|_{L^1(B_1)} \right).
\]
Therefore, applying (3.31) to (3.30), and using (3.29) (or (3.28), (3.24)), we have
\[
|D_{xx'} u(x_0) - D_{xx'} u(y)|
\leq C \int_0^{\gamma - y} \frac{\tilde{\omega}_A(t)}{t} dt \cdot \left( \|D^2 u\|_{L^1(B_{\gamma}(x_0))} + \int_0^{\gamma - y} \frac{\tilde{\omega}_A(t)}{t} dt + \|f_0\|_{L^1(B_{\gamma}(x_0))} \right)
+ C|\gamma - y| \|D^2 u\|_{L^1(B_{\gamma}(x_0))} + C \int_0^{\gamma - y} \frac{\tilde{\omega}_A(t)}{t} dt
\leq C \int_0^{\gamma - y} \frac{\tilde{\omega}_A(t)}{t} dt \cdot \left( \|D^2 u\|_{L^1(B_{\gamma}(x_0))} + \int_0^{\gamma - y} \frac{\tilde{\omega}_A(t)}{t} dt + \|f\|_{L^\infty(B_1)} + \|u\|_{L^1(B_1)} \right)
+ C|\gamma - y| \left( \|D^2 u\|_{L^1(B_{\gamma}(x_0))} + \|f\|_{L^\infty(B_1)} + \|u\|_{L^1(B_1)} \right) + C \int_0^{\gamma - y} \frac{\tilde{\omega}_A(t)}{t} dt.
\]

3.2. Proof of Corollary 1.2 Similar to the proof of Theorem 1.1, we take \( x_0 \in B_{3/4} \cap \mathcal{D}_{\mu} \). Let \( A^{(i)} \in C^0(\mathcal{D}), 1 \leq i \leq l + 1 \), be matrix-valued functions, \( b^{(i)} \) and \( f^{(i)} \) be in \( C^1(\mathcal{D}) \). Define the piecewise constant (matrix-valued) functions
\[
\tilde{A}(x) = A^{(i)}(x_0, h(x_0)), \quad x \in \Omega_i.
\]
From \( b^{(i)} \) and \( f^{(i)} \), we similarly define piecewise constant functions \( \tilde{b} \) and \( \tilde{f} \). By Lemma 2.5 in this case we have \( \omega_1 (r) \sim r^{\mu/(1 + \rho)} \). Therefore, we get the following result, which is similar to [21] Lemma 5.2.

Lemma 3.4. Let \( A, \tilde{A}, b, \tilde{b}, f, \) and \( \tilde{f} \) be defined as above. There exists a constant \( C > 0 \), depending only on \( \max_{1 \leq j \leq l+1} \|A\|_{C^0(\mathcal{D})}, \max_{1 \leq j \leq l+1} ||b||_{C^0(\mathcal{D})}, \max_{1 \leq j \leq l+1} ||f||_{C^0(\mathcal{D})}, \) and \( n, l, \mu, \alpha, \delta, \Lambda \), such that for any \( x_0 \in B_{3/4} \) and \( r \in (0, 1] \),
\[
\int_{B_{\gamma}(x_0)} |A - \tilde{A}| \ dx + \int_{B_{\gamma}(x_0)} |b - \tilde{b}| \ dx + \int_{B_{\gamma}(x_0)} |f - \tilde{f}| \ dx \leq C r^\mu.
\]
Corollary 1.2 directly follows from (3.21), (3.25), and (3.29) by taking \( \gamma \in (\alpha, 1) \).
4. Proofs of Theorem 1.4 and Corollary 1.5

The proof of Theorem 1.4 is similar to that of Theorem 1.1. Again, we first assume that \( b^i \equiv c \equiv 0 \). The adjoint operator corresponding to \( L_{x_0}^* \) is defined by

\[
\tilde{L}_{x_0}^* u := D_{ij}(\bar{a}^i(x) - a^i(x))u.
\]

Similarly, we define the modified operator

\[
\tilde{L}_{x_0}^* u := D_{ij}(\bar{a}^i u).
\]

Then, we have

\[
\tilde{L}_{x_0}^* u = D_{ij}(\bar{a}^i(x) - a^i(x))u + \text{div}^2 g.
\]

To prove Theorem 1.4, we first present a lemma that is an adjoint version of Lemma 2.1:

**Lemma 4.1.** Let \( p \in (1, \infty) \) and \( v \in L^p(B_r(x_0)) \) be a unique solution to the adjoint problem

\[
\begin{cases}
\tilde{L}_{x_0}^* v = \text{div}^2 (G_{x_0} B_{1/2}(x_0)) & \text{in } B_r(x_0), \\
v = 0 & \text{on } \partial B_r(x_0),
\end{cases}
\]

where \( G \in L^p(B_{r/2}(x_0)) \). Then for any \( t > 0 \), we have

\[
|x \in B_{r/2}(x_0) : |v(x)| > t| \leq \frac{C}{t} \|G\|_{L^p(B_{r/2}(x_0))},
\]

where \( C = C(n, p, \delta) > 0 \).

**Proof.** For simplicity, we set \( x_0 = 0 \) and \( r = 1 \). By Lemma 2.8, the map \( T : G \mapsto v \) is a bounded linear operator on \( L^p(B_{1/2}) \). As before, we take \( c = 24 \). For \( \bar{y} \in B_{1/2} \) and \( r \in (0, 1/4) \), let \( \tilde{g} = (B^{ij})_{i,j=1}^{n} \in L^p(B_1) \) be a matrix-valued function supported in \( B_r(\bar{y}) \cap B_{1/2} \) with mean zero, and

\[
b^i = B^{ij} + \bar{a}^{ij}B^{nn}, \quad (i, j) \neq (n, n), \quad b^{nn} = B^{nn}.
\]

By Lemma 2.8, there exists an adjoint solution \( v_1 \in L^p(B_1) \) of the problem

\[
\begin{cases}
\tilde{L}_{x_0}^* v_1 = \text{div}^2 b & \text{in } B_1, \\
v_1 = 0 & \text{on } \partial B_1.
\end{cases}
\]

For any \( R \in [c \delta, 1] \) such that \( B_{1/2} \setminus B_R(\bar{y}) \neq \emptyset \) and \( f \in C_0^\infty((B_{2R}(\bar{y}) \setminus B_R(\bar{y})) \cap B_{1/2}) \), let \( v_2 \in W_{0}^{2,p'}(B_1) \) be a strong solution of

\[
\begin{cases}
\tilde{L}_{x_0}^* v_2 = f & \text{in } B_1, \\
v_2 = 0 & \text{on } \partial B_1.
\end{cases}
\]

By using Definition 2.4,

\[
D_{nn}v_2 = \frac{1}{a^{nn}} \left(f - \sum_{(i,j)\neq(n,n)} \bar{a}^{ij}D_{ij}v_2\right),
\]

the matrix \( B \) is supported in \( B_r(\bar{y}) \cap B_{1/2} \) with mean zero, and \( f = 0 \) in \( B_{2r}(\bar{y}) \), we have

\[
\int_{(B_{2r}(\bar{y}) \setminus B_r(\bar{y})) \cap B_{1/2}} v_1 f = \int_{B_1} D_{ij} v_2 \bar{a}^{ij} = \int_{B_r(\bar{y}) \cap B_{1/2}} \sum_{(i,j)\neq(n,n)} D_{ij} v_2 B^{ij}.
\]
Coming back to (4.1), we use (4.2) to get
\[= \int_{B_{1/2}(\bar{y})} \sum_{(i,j) \neq (n,n)} (D_{ij}v_2 - D_{ij}v_2(\bar{y}))B^{ij}. \quad (4.1)\]

Recalling that in \(B_{r}(\bar{y}) \subset B_{R/24}(\bar{y}) \subset B_{2/3r}, \bar{a}^{ij}(x) = \bar{\alpha}^{ij}(x_0', x^n), \) we see that \(v_2\) satisfies
\[
\bar{\alpha}^{ij}(x_0', x^n)D_{ij}v_2 = 0 \quad \text{in} \ B_{R/24}(\bar{y}).
\]

By using Lemma 3.9 with a suitable scaling, we have
\[
\|D^2D_{x'}^2v_2\|_{L^\infty(B_{1/2}(\bar{y}))} \leq CR^{-1}\|D^2v_2\|_{L^p(B_{R/24}(\bar{y}))}
\]
\[
\leq CR^{-1}\|D^2v_2\|_{L^p(B_{1/2}(\bar{y}))} \leq CR^{-1}\|f\|_{L^p((B_{24}(\bar{y})), B_{4/3}(\bar{y}))}. \quad (4.2)
\]

Coming back to (4.1), we use (4.2) to get
\[
\int_{(B_{24}(\bar{y}) \setminus B_{1/2}(\bar{y})) \cap B_{1/2}} v_1f \leq CrR^{-1}\|B\|_{L^1(B_{1/2}(\bar{y}) \cap B_{1/2})}\|f\|_{L^p((B_{24}(\bar{y})), B_{4/3}(\bar{y}))}. \]

The rest of the proof is identical to that of Lemma 3.1 and thus omitted. \(\square\)

The following lemma is an analogy of Lemma 3.2. Set
\[
\phi(x_0, r) := \inf_{q \in \mathbb{R}} \left( \int_{B_{r}(x_0)} |\tilde{u} - q_0|^q \, dx \right)^{1/q},
\]
where \(\tilde{u}(x) = a^{mn}(x)u(x) - g^{mn}(x).\) We recall that the coordinate system is chosen according to each \(x_0.\)

**Lemma 4.2.** For any \(\gamma \in (0, 1)\) and \(0 < \rho \leq r \leq 1/4,\) we have
\[
\phi(x_0, r) \leq C\left( \frac{\rho}{r} \right)^{\gamma} r^{-\gamma} ||u||_{L^\infty(B_r(x_0))} + C\bar{\omega}_A(\rho)\left( ||\tilde{u}||_{L^\infty(B_r(x_0))} + ||g||_{L^\infty(B_r(x_0))} \right) + C\bar{\omega}_x(\rho), \quad (4.3)
\]
where \(C = C(n, p, \delta, \gamma) > 0,\) and \(\bar{\omega}_*(t)\) is a Dini function derived from \(\omega_*(t).\)

**Proof.** By Lemma 3.1 with \(G = (A(x) - A(x_0))u + g(x) - g(x_0', x^n)\) and the argument that led to (3.11), we have
\[
||x \in B_{r/2}(x_0) : |v(x)| > t || \leq C \left( \frac{r^n}{t} |\tilde{u}_A(r)| + r^n |\tilde{\omega}_A(r)||u||_{L^\infty(B_r(x_0))} \right).
\]

Therefore, for any given \(q \in (0, 1),\) we have
\[
\left( \int_{B_{r/2}(x_0)} |v|^q \, dx \right)^{1/q} \leq C(\tilde{\omega}_A(r) + \tilde{\omega}_x(r)||u||_{L^\infty(B_r(x_0))}). \quad (4.4)
\]

Let \(w = u - v \in L^p(B_r(x_0)),\) which satisfies \(\tilde{L}_w^* w = \text{div}^2 \tilde{g}(x_0', x^n)\) in \(B_{r/2}(x_0).\) Denote
\[
\tilde{a}^{ij}(x_0', x^n) := \frac{\bar{a}^{ij}(x_0', x^n)}{\bar{a}^{mn}(x_0', x^n)}, \quad \tilde{w} := \tilde{a}^{mn}(x_0', x^n)w - g^{mn}(x_0', x^n).\]

Then \(\tilde{a}^{mn}(x_0', x^n) = 1,\) and
\[
D_{ij}(\tilde{a}^{ij}(x_0', x^n)\tilde{w}) = D_{ij}\left(\tilde{a}^{ij}(x_0', x^n)w - \tilde{a}^{ij}(x_0', x^n)\tilde{g}^{mn}(x_0', x^n)\right)
\]
\[
= D_{ij}(\tilde{a}^{ij}(x_0', x^n)w) - \text{div}^2 \tilde{g}(x_0', x^n) = 0 \quad \text{in} \ B_{r/2}(x_0).\]

Now we prove that \(D\tilde{w} \in L^p_{\text{loc}}(B_{r/2}(x_0)).\) We choose \(\eta \in C^\infty_0(B_{r/4}(x_0))\) with
\[
0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{r/5}(x_0), \quad |D\eta| \leq 40/r.
Then
\[ D_{ij}(\bar{a}^{ij}(x_0', x^n)\bar{\omega} \eta)) = 2D_i(\bar{a}^{ij}(x_0', x^n)\bar{\omega} D_{j\eta}) - \bar{a}^{ij}(x_0', x^n)\bar{\omega} D_{ij\eta}. \] (4.5)

Next we show that \( \bar{\omega} \in W^{1,p}(B_{r/5}(x_0)). \) For \( k = 1, \ldots, n-1 \) and \( 0 < |h| < r/12, \) we take the finite difference quotient on both sides of (4.5) to get
\[ D_{ij}(\bar{a}^{ij}(x_0', x^n)\delta_{h,k}(\bar{\omega} \eta)) = 2\delta_{h,k}D_i(\bar{a}^{ij}(x_0', x^n)\bar{\omega} D_{j\eta}) - \bar{a}^{ij}(x_0', x^n)\delta_{h,k}(\bar{\omega} D_{ij\eta}) \] (4.6)
for any \( x \in B_{r/3}(x_0). \) For the first term of right-hand side in (4.6), we have
\[ \bar{a}^{ij}(x_0', x^n)\bar{\omega} D_{ij\eta} \in L^p(B_{r/2}(x_0)). \] For the second term of right-hand side in (4.6), we consider
\[
\begin{cases}
\Delta V = -\bar{a}^{ij}(x_0', x^n)\delta_{h,k}(\bar{\omega} D_{ij\eta}) & \text{in } B_{r/3}(x_0), \\
V = 0 & \text{on } \partial B_{r/3}(x_0).
\end{cases}
\]

We temporarily suppose that \( \bar{a}^{ij}(x_0', x^n)\bar{\omega} D_{ij\eta} \) is smooth. Then for each \( x \in B_{r/3}(x_0), \)
\( k = 1, \ldots, n-1 \) and \( 0 < |h| < r/12, \) we have
\[
\bar{a}^{ij}(x_0', x^n)\delta_{h,k}(\bar{\omega} D_{ij\eta}) = \delta_{h,k}(\bar{a}^{ij}(x_0', x^n)\bar{\omega} D_{ij\eta})
\]
\[
= \int_0^1 D_k(\bar{a}^{ij}(x_0', x^n)\bar{\omega}(x + the_k)\bar{\omega} D_{ij\eta}(x + the_k)) dt \cdot e_k
\]
\[
= D_k \left( \int_0^1 \bar{a}^{ij}(x_0', x^n)\bar{\omega}(x + the_k)\bar{\omega} D_{ij\eta}(x + the_k) dt \cdot e_k \right).
\]

By using the \( W^{1,p} \) estimate for the Poisson equation, we have
\[ ||V||_{W^{1,p}(B_{r/3}(x_0))} \leq C||\bar{\omega}||_{L^p(B_{r/3}(x_0))}.
\]

Coming back to (4.6), we use the local estimate for the adjoint operator to get
\[ ||\delta_{h,k}(\bar{\omega} \eta)||_{L^p(B_{r/3}(x_0))} \leq C||\bar{\omega}||_{L^p(B_{r/3}(x_0))}.
\]

This estimate holds if \( \bar{a}^{ij}(x_0', x^n)\bar{\omega} D_{ij\eta} \) is smooth, and thus is valid by approximation for \( \bar{a}^{ij}(x_0', x^n)\bar{\omega} D_{ij\eta} \in L^p(B_{r/3}(x_0)). \) Therefore, we let \( h \to 0 \) to obtain
\[ ||D_{ij}\bar{\omega}||_{L^p(B_{r/3}(x_0))} \leq C||\bar{\omega}||_{L^p(B_{r/3}(x_0))}.
\]

Similarly, we have for any \( k \geq 1, \) \( D_k^j\bar{\omega} \in L^p(B_{r/3}(x_0)) \). By using the fact that \( \bar{a}^{ij}(x_0', x^n) = 0 \) for \( i = 1, \ldots, n-1, \) and \( \bar{a}^{ij}(x_0', x^n) = 1 \) (cf. (3.2)), we have
\[ D_m\bar{\omega} = \sum_{i,j=1}^{n-1} D_i(\bar{a}^{ij}(x_0', x^n)D_j\bar{\omega}) - \sum_{j=1}^{n-1} D_j(\bar{a}^{ij}(x_0', x^n)D_j\bar{\omega}) \] in \( B_{r/3}(x_0). \)

We now apply [8 Corollary 4.4] to conclude that \( D_m\bar{\omega} \in L^p(B_{r/3}(x_0)) \) and \( \bar{\omega} \in W^{1,\delta}(B_{r/3}(x_0)). \) Therefore, by repeating the same line that led to (4.6), we have
\[ D_i(\bar{a}^{ij}(x_0', x^n)D_j\bar{\omega}) = D_j(\bar{a}^{ij}(x_0', x^n)\bar{\omega}) = 0 \] in \( B_{r/3}(x_0). \)

For any \( q_0 \in \mathbb{R}, \) by Lemma 2.10 with a suitable scaling, we have
\[ ||D\bar{\omega}||_{L^q(B_{r/3}(x_0))} \leq C r^{-(n+q)} \int_{B_{r/3}(x_0)} |\bar{\omega} - q_0|^q \ dx.
\]

Thus, similar to (3.12), we obtain
\[ \left( \int_{B_{r/3}(x_0)} |\bar{\omega} - (\bar{\omega})_{B_{r/3}(x_0)}|^q \ dx \right)^{1/q} \leq C_0 k \left( \int_{B_{r/3}(x_0)} |\bar{\omega} - q_0|^q \ dx \right)^{1/q}. \]
Then by using (4.4), we get
\[
\left( \int_{B_r(x_0)} |\bar{u} - (\bar{w})_{B_r(x_0)}|^q \, dx \right)^{1/q}
\leq 2^{1/q-1} \left( \int_{B_r(x_0)} |w - (\bar{w})_{B_r(x_0)}|^q \, dx \right)^{1/q} + C \left( \int_{B_r(x_0)} |(a^{mn}(x) - \bar{a}^{mn}(x'))u + g^{mn}(x') - \bar{g}^{mn}(x')v|^q \, dx \right)^{1/q}
\]
\[
\leq C_0 \kappa \left( \int_{B_r(x_0)} |\bar{u} - q_0|^q \, dx \right)^{1/q} + C \kappa^{-n/q} \left( |\bar{u}|_{L^\infty(B_r(x_0))} + |\bar{w}|_{L^\infty(B_r(x_0))} \right).
\]
Therefore, similar to the argument that led to (3.19), we have
\[
\phi(x_0, x') \leq \kappa^{r'} \phi(x_0, r) + C |u|_{L^\infty(B(x_0))} \bar{\omega}_A(r) + C \bar{\omega}_g(r)
\]
\[
\leq \kappa^{r'} \phi(x_0, r) + C \left( |\bar{u}|_{L^\infty(B(x_0))} + |g|_{L^\infty(B(x_0))} \right) \bar{\omega}_A(r) + C \bar{\omega}_g(r),
\]
which implies (4.3). The lemma is proved.

Finally, we give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** We use the same argument that led to proof of Theorem 1.1 and list the main differences. By Lemma 1.2, for any \( r \in (0, 1/8) \), we have
\[
\sup_{x_0 \in B_{1/8}} \phi(x_0, r) \leq C \left( \bar{r}^{\prime} |\bar{u}|_{L^1(B_{1/4})} + \bar{\omega}_A(r) \right) \left( |\bar{u}|_{L^\infty(B_{1/4})} + |g|_{L^\infty(B_{1/4})} \right) + \bar{\omega}_g(r).
\]

Similar to (3.19), we have
\[
\sup_{x_0 \in B_{1/8}} |\bar{u}(x_0) - q_{x_0,r}|
\leq C \sup_{x_0 \in B_{1/8}} \phi(x_0, r) + C \left( |\bar{u}|_{L^\infty(B_{1/4})} + |g|_{L^\infty(B_{1/4})} \right) \int_0^r \frac{\bar{\omega}_A(t)}{t} \, dt + C \int_0^r \frac{\bar{\omega}_g(t)}{t} \, dt,
\]
where \( q_{x_0,r} \in \mathbb{R} \) satisfying
\[
\phi(x_0, r) = \left( \int_{B_r(x_0)} |\bar{u} - q_{x_0,r}|^q \, dx \right)^{1/q}.
\]

By repeating the same line of the proof of (3.16), we have
\[
|\bar{u}|_{L^\infty(B_{1/4})} \leq C |\bar{u}|_{L^1(B_{1/4})} + C \left( \int_0^{r_1} \frac{\bar{\omega}_A(t)}{t} \, dt + |g|_{L^\infty(B_{1/4})} \right).
\]

Using a similar argument as in the proof of Theorem 1.1, for any \( y_0 \in B_{1/8} \cap D_{j_1}, j_1 \in [1, l + 1], \) we have the following two cases:

**Case 1.** If \( |x_0 - y_0| \leq 1/32 \), we set \( r = |x_0 - y_0| \). Recalling the definition of \( \bar{u} \), we see that \( a^{mn} \) and \( g^{mn} \) depend on the coordinate system. Under the coordinate system associated with \( y_0 \), we use the notation \( \bar{u} \). Then, similar to (3.28), we get
\[
|\bar{u}(y_0) - \bar{u}(y_0)| \leq C \left( |u|_{L^\infty(B_{1/4})} + |g|_{L^\infty(B_{1/4})} \right) \omega_1(|x_0 - y_0|).
\]
Thus, we obtain
\[ |\bar{u}(x_0) - \bar{u}(y_0)| \leq C \int_0^{x_0 - y_0} \frac{\tilde{\alpha}_A(t)}{t} dt + \left( \|u\|_{L^1(B_1)} + \int_0^1 \frac{\tilde{\alpha}_G(t)}{t} dt + \|g\|_{L^\infty(B_1)} \right) \]
\[ + C|x_0 - y_0|^{\nu} \left( \|u\|_{L^1(B_1)} + \int_0^1 \frac{\tilde{\alpha}_G(t)}{t} dt + \|g\|_{L^\infty(B_1)} \right). \]  \hspace{1cm} (4.8)

Case 2. If \(|x_0 - y_0| \geq 1/32\), then
\[ |\bar{u}(x_0) - \bar{u}(y_0)| \leq C|x_0 - y_0|^{\nu} \left( \|u\|_{L^1(B_1)} + \int_0^1 \frac{\tilde{\alpha}_G(t)}{t} dt + \|g\|_{L^\infty(B_1)} \right). \] \hspace{1cm} (4.9)

The theorem is proved when \(b^i \equiv c \equiv 0\).

For the general case, we rewrite the equation as
\[ D_{ij}(a^{ij}u) = \text{div} \ g + D_i(b^j u) - cu. \]

Consider
\[
\begin{cases}
\Delta w = D_i(b^j u) - cu & \text{in } B_1, \\
w = 0 & \text{on } \partial B_1.
\end{cases}
\]

Then, by the \(W^{1,p}\) estimate, we have
\[ \|w\|_{W^{1,p}(B_1)} \leq C\|u\|_{L^p(B_1)}. \] \hspace{1cm} (4.10)

Hence, we get
\[ D_{ij}(a^{ij}u) = \text{div}^2 g + w. \]

Then by using the local estimate for the adjoint operator and (4.10), we have
\[ \|u\|_{L^{p^*}(B_{1/2})} \leq C \left( \|g + w\|_{L^{p^*}(B_1)} + \|u\|_{L^p(B_1)} \right) \leq C \left( \|g\|_{L^\infty(B_1)} + \|u\|_{L^p(B_1)} \right), \]
where \(1/p^* = 1/p - 1/n\) if \(p < n\) and \(p^* \in (p, \infty)\) is arbitrary if \(p \geq n\). By a bootstrap argument, for any \(q \in (1, \infty)\) we have \(w \in W^{1,q}_{loc}(B_1)\) and
\[ \|w\|_{W^{1,q}_{loc}(B_{1/2})} \leq C \left( \|g\|_{L^\infty(B_1)} + \|u\|_{L^p(B_1)} \right). \]

By Morrey’s inequality, we can take a sufficiently large \(q > n\) such that \(w \in C^\beta_{loc}(B_1)\) with \(\beta = 1 - n/q > \max(\gamma, \mu/(1 + \mu))\), and
\[ \|w\|_{C^{\beta}(B_{1/2})} \leq C \left( \|g\|_{L^\infty(B_1)} + \|u\|_{L^p(B_1)} \right). \]

Denote \(g^* := g + w\), and we have
\[ \omega_{\tilde{\alpha}}(r) \leq \omega_{\tilde{\alpha}}(r) + r^\beta |w|_{C^{\beta}(B_{1/2})}. \]

We can replace \(g\) with \(g^*\) in (4.8) and (4.9), respectively, to get
\[ |\bar{u}(x_0) - \bar{u}(y_0)| \]
\[ \leq C \int_0^{x_0 - y_0} \frac{\tilde{\alpha}_A(t)}{t} dt + \left( \|u\|_{L^1(B_1)} + \int_0^1 \frac{\tilde{\alpha}_G(t)}{t} dt + \|g\|_{L^\infty(B_1)} \right) \]
\[ + C|x_0 - y_0|^{\nu} \left( \|g\|_{L^\infty(B_1)} + \|u\|_{L^p(B_1)} + \int_0^1 \frac{\tilde{\alpha}_G(t)}{t} dt \right) + C \int_0^{x_0 - y_0} \frac{\tilde{\alpha}_G(t)}{t} dt, \]
and
\[ |\bar{u}(x_0) - \bar{u}(y_0)| \leq C|x_0 - y_0|^\gamma \left( \int_0^1 \frac{\tilde{w}(t)}{t} dt + \|g\|_{L^\infty(B_1)} + \|u\|_{L^p(B_1)} \right). \]

Theorem 1.4 is proved. \hfill \Box

Corollary 1.5 follows from (1.7) by using Lemma 3.4 and taking \( \gamma \in (\alpha, 1) \).

5. Proof of Corollary 1.7

In this section, we will use the idea in [3, 15] to prove that if \( u \in W^{2,1}(\mathcal{D}) \) verifies \( Lu = f \) a.e. in \( \mathcal{D} \) with \( f \in L^p(\mathcal{D}) \) for some \( p \in (1, \infty) \), then \( u \in W^{2,p}_{loc}(\mathcal{D}) \).

Proof of Corollary 1.7. We rewrite the equation (1.1) as
\[ Lu - \lambda_0 u = f - \lambda_0 u =: f_0, \]
where \( \lambda_0 \) is a large fixed constant and \( f_0 \in L^p(\mathcal{D}) \). Without loss of generality, we may assume that \( 1 < p < n/(n - 1) \). Let \( \zeta \in C_c^\infty(\mathcal{D}_0) \) with \( \zeta \equiv 1 \) in \( \mathcal{D}' \subset \subset \mathcal{D}_0 \), and \( 0 \leq \zeta \leq 1 \). Let \( \varphi \in C_c^\infty(\mathcal{D}_0) \). We shall show that
\[ \left| \int_{\mathcal{D}_0} D_{ij}(u\zeta)\varphi^{ij} dx \right| \leq C\|\varphi\|_{L^p(\mathcal{D})} \left( \|f\|_{L^p(\mathcal{D})} + \|u\|_{W^{2,1}(\mathcal{D})} \right), \]
where \( 1/p + 1/p' = 1 \). Let \( u_\alpha \in C_c^\infty(\mathcal{D}_0) \) be a sequence of functions converging to \( u \) in \( W^{2,1}_{loc}(\mathcal{D}_0) \) as \( \alpha \to 0 \). Then for any \( \varphi \in C_c^\infty(\mathcal{D}_0) \), we have
\[ \int_{\mathcal{D}_0} D_{ij}(u\zeta)\varphi^{ij} dx = \lim_{\alpha \to 0} \int_{\mathcal{D}_0} D_{ij}(u_\alpha\zeta)\varphi^{ij} dx. \tag{5.1} \]
By using the same idea that led to Lemma 2.8, we modify the coefficients \( \alpha^{ij} \) to get
\[ \tilde{a}^{ij}(x) = \eta a^{ij}(x) + \delta(1 - \eta)\delta_{ij}, \]
where \( \eta \in C_c^\infty(\mathcal{D}) \) is a cut-off function satisfying
\[ 0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } \mathcal{D}_0, \quad |\nabla \eta| \leq C(n, \varepsilon). \]
Then by using Lemma 2.8 there exists an adjoint solution \( v \in L^{p'}(\mathcal{D}) \) to
\[ \begin{cases}
D_{ij}(\tilde{a}^{ij}v) - D_i(b^{i}v) + (c - \lambda_0)v = D_{ij}\varphi^{ij} & \text{in } \mathcal{D}, \\
v = 0 & \text{on } \partial \mathcal{D},
\end{cases} \]
and
\[ \|v\|_{L^{p'}(\mathcal{D})} \leq C\|\varphi\|_{L^p(\mathcal{D})}. \tag{5.2} \]
Therefore, for any \( w \in W^{2,p}(\mathcal{D}) \cap W^{1,p}_0(\mathcal{D}) \), we have
\[ \int_{\mathcal{D}} v(\tilde{a}^{ij}D_{ij}w + b^{i}D_{i}w + (c - \lambda_0)w) dx = \int_{\mathcal{D}} \varphi^{ij}D_{ij}w dx. \]
It is easy to see that \( u_\alpha\zeta \in W^{2,p}(\mathcal{D}) \cap W^{1,p}_0(\mathcal{D}) \) for any \( \alpha > 0 \). Then,
\[ \int_{\mathcal{D}} v(\tilde{a}^{ij}D_{ij}(u_\alpha\zeta) + b^{i}D_{i}(u_\alpha\zeta) + (c - \lambda_0)(u_\alpha\zeta)) dx = \int_{\mathcal{D}} \varphi^{ij}D_{ij}(u_\alpha\zeta) dx. \tag{5.3} \]
It follows from Theorem 1.4 that \( v \in L^\infty(D_1) \). Since \( u_\sigma \to u \) in \( W^{2,1}_{\text{loc}}(D_\sigma) \) as \( \sigma \to 0 \), we thus use (5.1) and (5.3) to get

\[
\int_{D_1} q^{ij} D_{ij}(u \zeta) \, dx = \int_{D_1} v(a^{ij} D_{ij}(u \zeta) + b^i D_i(u \zeta) + (c - \lambda_0)u \zeta) \, dx
\]

\[
= \int_{D_1} v(a^{ij} D_{ij} u + b^i D_i u + (c - \lambda_0)u) \zeta \, dx
\]

\[
+ \int_{D_1} (v a^i D_i \zeta u + u v b^i D_i \zeta) \, dx + 2 \int_{D_1} v a^i D_i u D_i \zeta \, dx.
\]

By using (5.2), we obtain

\[
\left| \int_{D_1} v(a^{ij} D_{ij} u + b^i D_i u + (c - \lambda_0)u) \zeta \, dx \right|
\leq C \|v\|_{L^\infty(D_1)} \|f\|_{L^\infty(D_1)} \leq C \|\varphi\|_{L^\infty(D_1)} \left( \|f\|_{L^\infty(D_1)} + \|u\|_{W^{1,1}(D_1)} \right),
\]

\[
\left| \int_{D_1} (v a^i D_i \zeta u + u v b^i D_i \zeta) \, dx \right| \leq C \|v\|_{L^\infty(D_1)} \|u\|_{L^\infty(D_1)} \leq C \|\varphi\|_{L^\infty(D_1)} \|u\|_{W^{1,1}(D_1)},
\]

and

\[
\left| \int_{D_1} v a^i D_i u D_i \zeta \, dx \right| \leq C \|v\|_{L^\infty(D_1)} \|Du\|_{L^\infty(D_1)} \leq C \|\varphi\|_{L^\infty(D_1)} \|u\|_{W^{1,1}(D_1)}.
\]

Therefore, we get

\[
\left| \int_{D_1} q^{ij} D_{ij}(u \zeta) \, dx \right| \leq C \|\varphi\|_{L^\infty(D_1)} \left( \|u\|_{W^{1,1}(D_1)} + \|f\|_{L^\infty(D_1)} \right).
\]

We thus have \( u \in W^{2,p}(D') \), and

\[
\|u\|_{W^{2,p}(D')} \leq C \left( \|u\|_{W^{1,1}(D)} + \|f\|_{L^\infty(D)} \right).
\]

Corollary 1.7 is thus proved. \( \square \)

6. **Weak-type \((1,1)\) estimates**

In this section, we consider the case when the sub-domains \( D_1, \ldots, D_{M-1} \) are away from \( \partial D \). In this case, we denote \( \delta_0 = \min_{1 \leq i \leq M-1} \text{dist}(\partial D_i, \partial D) \). We derive global weak-type \((1,1)\) estimates with respect to an \( A_1 \) Muckenhoupt weight \( w \) for solutions to the non-divergence form equation without lower-order terms and the corresponding adjoint problem. Denote

\[
w(D) := \int_D w(x) \, dx, \quad \|f\|_{L^p_w(D)} := \left( \int_D |f|^p w \, dx \right)^{1/p}, \quad p \in [1, \infty),
\]

and

\[
W^{2,p}_w(D) := \{ u : u, Du, D^2 u \in L^p_w(D) \}.
\]

**Theorem 6.1.** Let \( p \in (1, \infty) \), \( D \) have a \( C^{1,1} \) boundary, and \( w \) be an \( A_1 \) Muckenhoupt weight. Suppose that the coefficients \( A = (a^{ij})_{i,j=1} \) are of piecewise Dini mean oscillation over an open set containing \( \overline{D} \). For \( f \in L^p_w(D) \), let \( u \in W^{2,p}_w(D) \) be a strong solution to

\[
\begin{cases}
  a^{ij} D_{ij} u = f & \text{in } D, \\
  u = 0 & \text{on } \partial D.
\end{cases}
\]
Then for any $t > 0$, we have

$$w\left(\{x \in \mathcal{D} : |D^2 u(x)| > t\}\right) \leq \frac{C}{t} ||f||_{L^1_\mu(\mathcal{D})},$$

where $C$ depends on $n, M, p, \partial, \Lambda, \delta_0, [w]_{A_1}$, the $C^{1,1}$ norm of $\partial \mathcal{D}$, and the $C^{1,1}$ characteristics of $\partial \mathcal{D}$, $j = 1, \ldots, M - 1$. Moreover, the linear operator $T : f \mapsto D^2 u$ can be extended to a bounded operator from $L^1_\mu(\mathcal{D})$ to weak-$L^1_\mu(\mathcal{D})$.

**Remark 6.2.** From the proof below we can see that the result in Theorem 6.4 still holds for equations with lower-order terms provided $L^1 \leq 0$ so that the weighted $W^{2,p}$ solvability is available; see Theorem 7.1.

To state the corresponding results for the adjoint operator, we need to impose additional conditions for the coefficient $A$ and the Dini function introduced in Definition 2.2.

**Assumption 6.3.** 1) $A$ is of piecewise Dini mean oscillation in $\mathcal{D}$, and satisfies the following condition: there exists some constant $c_0 > 0$ such that for any $r \in (0, 1/2)$, $\omega_A(r) \leq c_0 (\ln r)^{-2}$.

2) For some constant $c_1, c_2 > 0$, $\omega_A'(R_0) \geq c_1$ and for any $R \in (0, R_0/2)$, $\omega_A(R) \leq c_2 (\ln R)^{-2}$.

**Theorem 6.4.** Let $p \in (1, \infty)$, $\mathcal{D}$ have a $C^{2,1}$ boundary, and $w$ be an $A_1$ Muckenhoupt weight. Under Assumption 6.3, the following hold. For $f = (f^i)_{i,j=1}^n \in L^p_\mu(\mathcal{D})$, let $u \in L^1_\mu(\mathcal{D})$ be a solution to the adjoint problem

$$\begin{align*}
D_j^*(a^{ij} u) &= \text{div} \ f & \text{in} \ \mathcal{D}, \\
u = \frac{f^{i,i}}{w} & \text{on} \ \partial \mathcal{D}.
\end{align*}$$

Then for any $t > 0$, we have

$$w\left(\{x \in \mathcal{D} : |u(x)| > t\}\right) \leq \frac{C}{t} ||f||_{L^1_\mu(\mathcal{D})},$$

where $C$ depends on $n, M, p, \partial, \Lambda, \delta_0, [w]_{A_1}$, the $C^{2,1}$ characteristics of $\partial \mathcal{D}$ and the $C^{1,1}$ characteristics of $\partial \mathcal{D}$, $j = 1, \ldots, M - 1$. Moreover, the bounded linear operator $T : f \mapsto u$ can be extended to a bounded operator from $L^1_\mu(\mathcal{D})$ to weak-$L^1_\mu(\mathcal{D})$.

**Remark 6.5.** The result in Theorem 6.4 still holds for the problem

$$\begin{align*}
\text{div} \ f &= \text{div} \ f & \text{in} \ \mathcal{D}, \\
u &= \frac{f^{i,i}}{w} & \text{on} \ \partial \mathcal{D},
\end{align*}$$

when $L^1 \leq 0$ by using Corollary 7.3 and the same argument as in the proof of Theorem 6.4.

Instead of Lemma 2.12 above, which was used in [7, 10, 11], in the proofs of Theorems 6.1 and 6.4, we apply a generalized version of it stated below since our argument and estimates depend on the coordinate system associated with a given point, as mentioned before.

**Lemma 6.6.** [12, Lemma 6.3] Let $w$ be a doubling measure and $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ satisfying (2.26). Let $p \in (1, \infty)$ and $T$ be a bounded linear operator on $L^p_w(\mathcal{D})$. [Proof follows here.]
Suppose that if for some \( f \in L^p_w(\mathcal{D}) \), \( t > 0 \), and some cube \( Q^k \) we have
\[
\int_{Q^k} |f| w \, dx \leq C_t t,
\]
where \( C_t \geq 1 \) and \( \{Q^k\}_k \) is a collection of “cubes” defined in [12, Appendix], then \( f \) admits a decomposition \( f = g + b \) in \( Q^k \), where \( g \) and \( b \) satisfy
\[
\int_{Q^k} |g|^p w \, dx \leq C_1 t^{p} w(Q^k), \quad \int_{D \setminus B_{\epsilon}(x_0)} |T(bX_{Q^k})| w \, dx \leq C_t t w(Q^k)
\]
with \( x_0 \in Q^k \) and \( r = \text{diam } Q^k \). Then for any \( f \in L^p_w(\mathcal{D}) \) and \( t > 0 \), we have
\[
w\{x \in \mathcal{D} : |Tf(x)| > t\} \leq \frac{C}{t} \int_{\mathcal{D}} |f| w \, dx,
\]
where \( C = C(n,c,\mathcal{D},C_1,|T|_{L^p_w \rightarrow L^1_w}) \) is a constant. Moreover, \( T \) can be extended to a bounded operator from \( L^1_w(\mathcal{D}) \) to weak-\( L^1_w(\mathcal{D}) \).

**Proof of Theorem 6.1.** By Theorem 7.1, we can see that the map \( T : f \mapsto D^2u \) is a bounded linear operator on \( L^p_w(\mathcal{D}) \). It suffices to prove that \( T \) satisfies the hypothesis of Lemma 7.4. For simplicity, we may assume that \( \mathcal{D} \) is contained in \( B_3 \) and \( A \) has piecewise Dini mean oscillation on \( B_{10} \). Let \( \{Q^k\}_k \) be a collection of dyadic “cubes” introduced in the proof of [7, Lemma 4.1]. Notice that the assumptions in Lemma 7.4 satisfies automatically for large cubes (i.e., small \( k \)) by taking a sufficiently large \( c \). We thus can assume that each \( Q^k \) is small enough so that they do not intersect with \( \bigcup_{j=1}^{M-1} \mathcal{Q}_j \) and \( \partial \mathcal{D} \) at the same time. The following proof proceeds in the same way as in [7, Theorem 1.10] except that in our case, we consider the decomposition of \( f \) with respect to the \( A_1 \) Muckenhoupt weight \( w \); that is, for some \( Q^k \) and \( t > 0 \), suppose
\[
t < \frac{1}{w(Q^k)} \int_{Q^k} |f| w \, dx \leq C_1 t,
\]
where \( C_1 \geq 1 \). For a fixed \( x_k \in Q^k \), we associate \( Q^k \) with a Euclidean ball \( B_{r_k}(x_k) \) such that \( Q^k \subset B_{r_k}(x_k) \), where \( r_k := \text{diam } Q^k \leq \delta_0/2 \). Let \( W \) be the nonnegative adjoint solution to
\[
D_{ij}(a_{ij} W) = 0 \quad \text{in } B_{10}, \quad W = 1 \quad \text{on } \partial B_{10}.
\]
Then \( W \) is in the reverse Hölder class, with constants which depend only on \( n, \delta, \) and \( \Lambda \):
\[
(W)_{B_{2r_k}(x_k)} \leq C(W)_{B_{r_k}(x_k)} \left( \frac{1}{B_{r_k}(x_k)} \int_{B_{r_k}(x_k)} W^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \leq C \int_{B_{r_k}(x_k)} W \, dx,
\]
whenever \( B_{2r_k}(x_k) \subset B_{10} \). Also, \( (W)_{B_{10}} \approx 1 \); see [1, 3, 10, 12]. Then we have the following global estimate: For any \( x_k \in \mathcal{D} \) and \( 0 < r_k \leq \delta_0/2 \), by using (4.7) when \( B_k \cap \partial \mathcal{D} = \emptyset \) and [2, Lemma 2.26] when \( B_k \cap \partial \mathcal{D} = \emptyset \), we get
\[
|W|_{L^\infty(D_{2r_k}(x_k))} \leq C^{-\frac{n}{n-1}} |W|_{L^1(\mathcal{D}_{r_k}(x_k))},
\]
where \( C \) depends on \( n,M,\delta,\Lambda,\delta_0,\omega_A \), the \( C^{1,1} \) norm of \( \partial \mathcal{D} \), and the \( C^{1,\text{Dini}} \) characteristics of \( \partial \mathcal{D}_j \), \( j = 1, \ldots, M - 1 \). We now decompose \( f = g + b \) in a given \( Q^k \) such
We thus get
\[ g := \frac{1}{W(Q^k_d)} \int_{Q^k_d} f W, \quad b = f - g. \]

Then
\[ \int_{Q^k_d} b W \, dx = 0, \] (6.3)

and by using (6.2), the definition of \( A_1 \) weights, and (6.3), we have
\[ |g| \leq \frac{||W||_{L^\infty(Q^k_d)}}{W(Q^k_d)} \int_{Q^k_d} |f| \, dx \leq \frac{C}{||W||_{L^\infty(Q^k_d)}} \int_{Q^k_d} |f| w \, dx \leq \frac{C}{w(Q^k_d)} \int_{Q^k_d} |f| w \, dx \leq Ct. \]

We thus get
\[ \int_{Q^k_d} |g|^p w \, dx \leq Ct^p w(Q^k_d). \]

Let \( u_1 \in W^{2,p} \) be the unique solution to
\[
\begin{cases}
\frac{\partial}{\partial x_i} D_{ij} u_1 = b \chi_{Q^k_d} & \text{in } D, \\
u_1 = 0 & \text{on } \partial D.
\end{cases}
\]

Set \( R_0 = \text{diam } D \) and \( c = 4R_0/\delta_0 \). For any \( R \in [c R_0, R_0] \) such that \( D \setminus B_R(x_k) \neq \emptyset \) and \( h \in C^\infty(D_2R(x_k) \setminus B_R(x_k)) \), in view of Corollary 7.5 below, there exists a unique adjoint solution \( u_2 \in L^{p',(p-1)} \) of
\[
\begin{cases}
D_{ij}(\alpha_i u_2) = \text{div}^2 h & \text{in } D, \\
u_2 = 0 & \text{on } \partial D.
\end{cases}
\]

Let \( \tilde{u}_2 := u_2/W \). Then by duality and (5.3), we have
\[
\int_D D_{ij} u_1 h^{ij} = \int_{Q^k_d} \tilde{u}_2 W b = \int_{Q^k_d} (\tilde{u}_2 - \tilde{u}_2(x_k)) W b.
\]

Similar to \([2, (6.6)-(3.8)]\) (see also \([2, 13, 20]\)), we have
\[
\|\tilde{u}_2 - \tilde{u}_2(x_k)\|_{L^\infty(Q^k_d)} \leq C \left( \frac{R_0}{R} \right)^{\alpha} \|\tilde{u}_2\|_{L^\infty(D_0 R/(2R_0)(x_k))} \
\leq C \left( \frac{R_0}{R} \right)^{\alpha} \frac{1}{W(B_0 R/(2R_0)(x_k))} \int_{D_0 R/(2R_0)(x_k)} |\tilde{u}_2| W \, dx 
= C \left( \frac{R_0}{R} \right)^{\alpha} \frac{1}{W(B_0 R/(2R_0)(x_k))} \int_{D_0 R/(2R_0)(x_k)} |u_2|,
\]

where \( \alpha \in (0, 1) \). Hence, we obtain
\[
\left| \int_{D_2R(x_k) \setminus B_R(x_k)} D_{ij} u_1 h^{ij} \right| \leq C \left( \frac{R_0}{R} \right)^{\alpha} \frac{||W||_{L^\infty(B_0 R/(2R_0)(x_k))}}{W(B_0 R/(2R_0)(x_k))} \|u_2\|_{L^1(D_0 R/(2R_0)(x_k))} \|b\|_{L^1(Q^k_d)} \
\leq C \left( \frac{R_0}{R} \right)^{\alpha} R^{-n} \|u_2\|_{L^1(D_0 R/(2R_0)(x_k))} \|b\|_{L^1(Q^k_d)} \
\leq C \left( \frac{R_0}{R} \right)^{\alpha} R^{-n} \left( \int_D |u_2|^{p'} w^{-\frac{n}{p'}} dx \right)^{\frac{1}{p'}} \left( \int_{D_2R(x_k)} w \, dx \right)^{\frac{1}{p'}} \frac{1}{\inf_{Q^k_d} w} \int_{Q^k_d} |b| w \, dx \\leq C \left( \frac{R_0}{R} \right)^{\alpha} \left( \int_{D_2R(x_k) \setminus B_R(x_k)} |b|^{p'} w^{-\frac{n}{p'}} dx \right)^{\frac{1}{p'}} \left( \int_{D_2R(x_k)} w \, dx \right)^{\frac{1}{p'}} \int_{Q^k_d} |b| w \, dx,
\]
where we used (6.2), Hölder’s inequality, the definition of $A_1$ Muckenhoupt weights, and (7.13). By duality, we have
\[
\|D^2u_1\|_{L^p(R_{2r}(x_1),B_1(x_1))} \leq C\left(\frac{P}{R}\right)^\alpha \left(\int_{B_1(x_1)} w \, dx\right)^{r-1} \|b\|_{L^q(S^*)}.
\]

Therefore, by Hölder’s inequality, we obtain
\[
\|D^2u_1\|_{L^p(R_{2r}(x_1),B_1(x_1))} \leq C\left(\frac{P}{R}\right)^\alpha \|b\|_{L^q(S^*)}.
\]

The rest of the proof is identical to that of Lemma 3.4 and thus omitted. Hence, we get the desired result by using Lemma 6.6. □

We will also use the following lemma.

Lemma 6.7 (Lemma 3.4 of [10]). Let $\omega$ be a nonnegative increasing function such that $\omega(t) \leq (\ln t)^{-2}$ for $0 < t \leq 1$, and $\tilde{\omega}$ be given as in (3.14) with $\omega$ in place of $\tilde{\omega}$. Then for any $r \in (0,1)$, we have
\[
\int_0^r \frac{\tilde{\omega}(t)}{t} \, dt \leq C \left(\ln \frac{4}{r}\right)^{-1},
\]
where $C > 0$ is some positive constant.

Proof of Theorem 6.4. By Corollary 7.5, one can see that the map $T : f \mapsto u$ is a bounded linear operator on $L^p_w(D)$. We follow the argument in the proof of Theorem 5.2 with minor modifications. Under the same conditions that $f \in L^p_w(D)$ and $Q^k$ as mentioned in the proof of Theorem 6.1, we decompose $f$ in a given cube $Q^k$ according to the following two cases.

(i) If $\text{dist}(x_i, \partial D) \leq \delta_0/2$, then $B_{R_i}(x_i) \cap D_j = \emptyset$, $j = 1, \ldots, M-1$. In this case, we take $y_k \in \partial D$ such that $|x_k - y_k| = \text{dist}(x_k, \partial D)$. Let
\[
g := \int_{Q^k} f \, dx, \quad b = f - g \quad \text{in } Q^k.
\]

Then $b|_{Q^k} = 0$ and
\[
|g| \leq \int_{Q^k} f \, dx \leq \frac{1}{|Q^k|} \inf_{Q^k} w \int_{Q^k} |f| \, dx \leq \frac{C}{w(Q^k)} \int_{Q^k} |f| \, dx \leq Ct,
\]
where we used the definition of $A_1$ weights and (6.1). Hence,
\[
\int_{Q^k} |g|^p \, dx \leq Cr^p w(Q^k).
\]

We now check the hypothesis regarding $b$. Let $v_1 \in L^p_w(D)$ be an adjoint solution of the problem
\[
\begin{align*}
D_{ij}(a_{ij}v_1) &= \text{div}^2(b_{ij}Q^k) \quad &\text{in } D, \\
v_1 &= b_{ij}Q^k \cdot \nu / (A \cdot \nu) &\text{on } \partial D,
\end{align*}
\]
the solvability of which follows from Corollary 7.5. Set $R_0 = \text{diam } D$ and $c = 4R_0/\delta_0$. Then for any $R \in \{r_k, R_0\}$ such that $D \setminus B_R(x_k) \neq \emptyset$ and $h \in C_0^\infty(D_{2R}(x_k) \setminus B_R(x_k))$, let $v_2 \in W^{2,p-1}(D)$ be a strong solution of
\[
\begin{align*}
&\text{div} a_{ij}v_2 = h &\text{in } D, \\
&v_2 = 0 &\text{on } \partial D.
\end{align*}
\]
By using Definition 2.4, the matrix $b$ is supported in $Q^k_n$ with mean zero, and $h \in C_0^\infty(\mathcal{D}_{2R}(x_i) \setminus B_R(x_i))$, we have
\[
\int_{\mathcal{D}_{2R}(x_i) \setminus B_R(x_i)} v_1 h = \int_{Q^k_n} D_{ij} v_2 b^{ij} = \int_{Q^k_n} (D_{ij} v_2 - D_{ij} v(x_i)) b^{ij}. \tag{6.4}
\]
Since $R \leq R_0$, $B_{\frac{3R_0}{2}R}(x_i)$ does not intersect with sub-domains. Also, $a^{ij} D_{ij} v_2 = 0$ in $\mathcal{D}_R(x_i)$. Then by flattening the boundary and using a similar argument that led to an a priori estimate of the modulus of continuity of $D^2 v_2$ in the proof of [2, Theorem 1.5], we have
\[
|D^2 v_2(x) - D^2 v_2(x_i)| \leq C \left( \left( \frac{|x - x_i|}{R} \right)^\gamma + \omega_A^\alpha(|x - x_i|) \right) R^{-\eta} \|D^2 v_2\|_{L^1(\mathcal{D}_{2R_0}(x_i))} \tag{6.5}
\]
for any $x \in Q^k_n \subset \mathcal{D}_{2R_0}(x_i)$, where $\gamma \in (0, 1)$ is a constant and for $0 < t \leq 1$,
\[
\omega_A^\alpha(t) := \tilde{\omega}_A(t) + \int_0^t \frac{\omega_A(s)}{s} ds + \omega_A(4t) + \int_0^t \frac{\omega_A(4s)}{s} ds,
\]
\[
\omega(t) := \tilde{\omega}_A(t) + \omega_A(4t) + \omega_A(4t), \quad \text{and} \quad \omega(t) := \sup_{s \in [t, 1]} \omega_A(s).
\]
Then, coming back to (6.4), we obtain
\[
\left| \int_{\mathcal{D}_{2R}(x_i) \setminus B_R(x_i)} v_1 h \right| \leq \frac{1}{\inf_{Q^k_n} w} \|b\|_{L^1(Q^k_n)} \|D^2 v_2 - D^2 v_2(x_i)\|_{L^\infty(Q^k_n)}
\]
\[
\leq \frac{CR^{-\eta}}{\inf_{Q^k_n} w} \|b\|_{L^1(Q^k_n)} \|D^2 v_2\|_{L^1(\mathcal{D}_{2R_0}(x_i))} \left( r_k^{R^{-\gamma}} + \left( \ln \frac{4}{r_k} \right)^{-1} \right)
\]
\[
\leq C \left( \int_{\mathcal{D}_{2R}(x_i)} w dx \right)^{1/2-1} \|b\|_{L^1(Q^k_n)} \left( \int_{\mathcal{D}} |D^2 v_2| w^{-\frac{1}{1-\gamma}} dx \right)^{1/2} \left( r_k^{R^{-\gamma}} + \left( \ln \frac{4}{r_k} \right)^{-1} \right)
\]
\[
\leq C \left( \int_{\mathcal{D}_{2R}(x_i)} w dx \right)^{1/2-1} \|b\|_{L^1(Q^k_n)} \left( \int_{\mathcal{D}_{2R}(x_i) \setminus B_R(x_i)} \|h\| w^{-\frac{1}{1-\gamma}} dx \right)^{1/2} \left( r_k^{R^{-\gamma}} + \left( \ln \frac{4}{r_k} \right)^{-1} \right),
\]
where we used (5.5), Lemma [5.7], the definition of $A_1$ Muckenhoupt weights, Hölder’s inequality, and the estimate
\[
\left( \int_{\mathcal{D}} |D^2 v_2| w^{-\frac{1}{1-\gamma}} dx \right)^{1/2-1} \leq C \left( \int_{\mathcal{D}} |h| w^{-\frac{1}{1-\gamma}} dx \right)^{1/2} = C \left( \int_{\mathcal{D}_{2R}(x_i) \setminus B_R(x_i)} \|h\| w^{-\frac{1}{1-\gamma}} dx \right)^{1/2}.
\]
The rest of proof is identical to that of Theorem 5.1. We thus obtain
\[
\int_{\mathcal{D}(B_{R_0}(x_i))} |v_1| w dx \leq C \int_{Q^k_n} |b| w dx \leq C \int_{Q^k_n} |f| w dx + C \int_{Q^k_n} |g| w dx \leq C tw(Q^k_n).
\]
That is,
\[
\int_{\mathcal{D}(B_{R_0}(x_i))} |Tb| w dx \leq C tw(Q^k_n).
\]
(ii) If $\text{dist}(x_i, \partial \mathcal{D}) \geq \delta_0/2$, then $B_{\delta_0}(x_i) \cap \partial \mathcal{D} = \emptyset$. In this case, we choose the coordinate system according to $x_i$. In $Q^k_n$, we set
\[
g^{ij} = \int_{Q^k_n} \left( f^{ij} - \frac{a_{ij}}{a^{m n}} f^{m n} \right) dx + \frac{a^{ij}}{a^{m n}} \int_{Q^k_n} f^{m n} dx, \quad (i, j) \neq (n, n), \quad g^{m n} = \int_{Q^k_n} f^{m n} dx,
\]
and \( b = f - g \).

\[
\int_{Q_n^k} |g|^p w \, dx \leq C t^p w(Q_n^k).
\]

Let

\[
\tilde{b}^{ij} = b^{ij} - \frac{a^{ij}}{a^{nn}} b^{nn}, \quad (i, j) \neq (n, n), \quad \tilde{b}^{nn} = b^{nn}.
\]

Then \( \tilde{b}_{Q_n^k} = 0 \). It then follow from the argument as in the first case that

\[
\int_{D_{2x}(x_k) \setminus B_r(x_k)} v_1 h = \int_{Q_n^k} D_{ij} v_2 \tilde{b}^{ij} = \int_{Q_n^k} \sum_{(i, j) \neq (n, n)} \sum_{(i, j) \neq (n, n)} (D_{ij} v_2 - D_{ij} v_2(x_k)) \tilde{b}^{ij}, \quad (6.6)
\]

where we used

\[
D_{nn} v_2 = - \sum_{(i, j) \neq (n, n)} \frac{a^{ij}}{a^{nn}} D_{ij} v_2 \quad \text{in} \quad B_r(x_k).
\]

Recalling that \( \alpha \leq R \leq R_0 \) so that \( B_{\alpha R/(4R_0)}(x_k) \cap \partial D = \emptyset \). Then by using a similar argument that led to (3.29) (or (3.23), (3.21)), we obtain

\[
|D_{xx} v_2(x) - D_{xx} v_2(x_k)| \leq CR^{-n} \left( \frac{|x - x_k|}{R} \right) + \int_0^{t(x-k)} \frac{\tilde{\omega}_A(t)}{t} dt \|D_{xx} v_2\|_{L^1(B_{\alpha R/(2R_0)}(x_k))}
\]

for any \( x \in Q_n^k \subset B_{\alpha R/(4R_0)}(x_k) \). Coming back to (5.6) and using a similar argument as in the case (i), we obtain

\[
\int_{D_{\frac{1}{2}, B_{\alpha} (x_k)}} |v_1| w \, dx \leq C \int_{Q_n^k} |\tilde{b}| w \, dx \leq C \int_{Q_n^k} |f| w \, dx + C \int_{Q_n^k} |g| w \, dx \leq Ct w(Q_n^k).
\]

By Lemma 6.6, the theorem is proved. \( \square \)

7. Appendix

In the appendix, we give the \( W^{2,p}_w \)-estimate and solvability for the non-divergence form elliptic equation in \( C^{1,1} \) domains with the zero Dirichlet boundary condition. Consider

\[
\begin{cases}
\lambda u - Lu = f & \text{in} \quad D, \\
u = 0 & \text{on} \quad \partial D,
\end{cases}
\]

where \( \lambda \geq 0, D \in C^{1,1} \), and \( Lu := a^{ij} D_{ij} u + b^{i} D_{i} u + cu \). Let \( p \in (1, \infty) \) and \( w \) be an \( A_p \) weight. Denote

\[
\tilde{W}^{2,p}_w(D) := \{ u \in W^{2,p}_w(D) : u = 0 \text{ on } \partial D \}.
\]

Now we impose the regularity assumptions on \( a^{ij} \). Let \( \gamma_0 = \gamma_0(n, p, \delta, [w]_{A_p}) \in (0, 1) \) be a sufficiently small constant to be specified. There exists a constant \( r_0 \in (0, 1) \)
such that $a^{ij}$ satisfy (2.4) in the interior of $D$ and are VMO near the boundary: for any $x_0 \in \partial D$ and $r \in (0, r_0)$, we have
\[
\int_{B_r(x_0)} |a^{ij}(x) - (a^{ij})_{B_r(x_0)}| \, dx \leq \gamma_0.
\]
In addition, $b^i$ and $c$ are bounded by a constant $\Lambda$. Then we have the following

**Theorem 7.1.** Let $p \in (1, \infty)$, $w \in A_p$, and $L \leq 0$. There exists a sufficient small constant $\gamma_0 = \gamma_0(n, p, \delta, \Lambda, [w]_{A_p}) \in (0, 1)$ such that under the above conditions, for any $\lambda \geq 0$ and $f \in L^p_w(D)$, there exists a unique $u \in W^{2, p}_w(D)$ satisfying (7.1).

Furthermore, there exists a constant $C = C(n, p, \delta, \Lambda, D, [w]_{A_p}, r_0)$ such that
\[
||u||_{W^{2, p}_w(D)} \leq C||f||_{L^p_w(D)}.
\]

First we note that when $\lambda \geq \lambda_1(n, p, \delta, \Lambda, [w]_{A_p}, r_0)$, the theorem follows from the proofs of [19, Theorems 6.3 and 6.4] combined with the argument in [19, Theorem 2.5] and [19, Sections 8.5]. To deal with the case when $\lambda \in [0, \lambda_1)$, we need the following lemmas. The first one is a local regularity of solutions in weighted Sobolev spaces.

**Lemma 7.2.** Let $1 < p \leq q < \infty$, $z \in \overline{D}$. Denote $D_r := D \cap B_r(z)$. Then if
\[
\xi u \in \dot{W}^{2, p}_w(D_{2R}) \quad \forall \xi \in C^\infty_0(B_{2R}(z)), \quad Lu \in L^q_w(D_{2R}),
\]
we have
\[
\xi u \in \dot{W}^{2, p}_w(D_{2R}) \quad \forall \xi \in C^\infty_0(B_{2R}(z)).
\]
Furthermore, there exists a constant $C = C(R, p, q, n, \delta, \Lambda, [w]_{A_p}, r_0)$ such that if (7.2) holds, then
\[
||u||_{W^{2, p}_w(D_{2R})} \leq C\left(||Lu||_{L^p_w(D_{2R})} + ||u||_{L^p_w(D_{2R})}\right).
\]

**Proof.** We follow the proof of [19, Theorem 11.2.3] when $w = 1$. For $q = p$, (7.3) is obvious and (7.4) is obtained by using the method in the proof of [19, Theorem 9.4.1]. For $q > p$, we define
\[
\alpha = \frac{n}{n-1} \quad \text{for } n \geq 2; \quad p(j) = \alpha/p, \quad j = 0, 1, \ldots, k-1, \quad p(k) = q,
\]
where $k-1$ is the last $j$ such that $p(j) < q$. Take $\Lambda$ sufficiently large that $\lambda - L$, as an operator acting from $\dot{W}^{2, p(j)}_w(D)$ onto $L^{p(j)}_w(D)$ for $j = 0, 1, \ldots, k$, is invertible. Denote
\[
f = Lu, \quad g = (L - \lambda)(\xi u) = \xi f + 2a^{ij}DuD_j\xi + u(L - \sigma - \lambda)\xi.
\]
By weighted Sobolev embedding theorem, see [14, Theorem 1.3], we have
\[
\xi u \in W^{1, p(j)}_w(D)
\]
for any $\xi \in C^\infty_0(B_{2R}(z))$. Hence, $g \in L^{p(j)}_w(D)$. By the choice of $\lambda$, the equation
\[
(L - \lambda)v = g
\]
has a solution in $\dot{W}^{2, p(j)}_w(D) \subset \dot{W}^{2, p}_w(D)$ which is unique in $\dot{W}^{2, p}_w(D)$. We thus obtain that for $j = 1$,
\[
v = \xi u \in \dot{W}^{2, p(j)}_w(D), \quad \forall \xi \in C^\infty_0(B_{2R}(z)).
\]
If $p(1) < q$, then by repeating this argument with $p(1)$ in place of $p$, we get (7.3) for $j = 2$. In this way we conclude (7.3).
Next we prove (7.4). By the choice of \( \lambda \), for \( j \geq 1 \) and any \( \xi, \eta \in C_c(\mathbb{B}_{2R}(z)) \) such that \( \eta = 1 \) on the support of \( \xi \), we have
\[
\|\xi u\|_{W^{2,\infty}_p(D)} \leq C\|\xi f + 2\alpha/j DuD_j \xi + u(L - c - \lambda)\xi\|_{L^\infty_p(D)}
\leq C\|f\|_{L^\infty_p(D_R)} + \|\eta u\|_{W^{2,\infty}_p(D)}
\leq C\|f\|_{L^\infty_p(D_R)} + \|\eta u\|_{W^{2,\infty}_p(D)}.
\]
where we used the weighted Sobolev embedding theorem in the last inequality. By iterating the above inequality, we obtain that for any \( \xi \in C_c(\mathbb{B}_{3R/2}(z)) \), there is an \( \eta \in C_c(\mathbb{B}_{3R/4}(z)) \) such that
\[
\|\xi u\|_{W^{2,\infty}_p(D)} \leq C\|f\|_{L^\infty_p(D_R)} + \|\eta u\|_{W^{2,\infty}_p(D)}.
\]
Finally, recalling the conclusion for the case when \( p = q \), we have
\[
\|\eta u\|_{W^{2,\infty}_p(D)} \leq C\|u\|_{W^{2,\infty}_p(D_{R/4})} \leq C\|f\|_{L^\infty_p(D_{R/2})} + \|u\|_{L^\infty_p(D_{R/2})}.
\]
This yields (7.4) and the lemma is proved. \( \square \)

Next we recall the resolvent operator of \( L - \lambda I \) by
\[
\mathcal{R}_\lambda : L^p_w(D) \to \dot{W}^{2,p}_w(D).
\]
Then \( \mathcal{R}_\lambda \) is a bounded operator for \( \lambda \geq \lambda_1 \). The following properties of \( \mathcal{R}_\lambda \) for \( \lambda \) large play an important role in proving Lemma 7.4 below.

**Lemma 7.3.** Let the coefficients of \( L \) be infinitely differentiable, \( L^1 \leq 0 \), and \( \lambda \geq \lambda_1 \). Then

1. For any bounded \( f \) and any \( \gamma \in (0, 1) \), we have \( \mathcal{R}_\lambda f \in C^{1+\gamma} \), \( \mathcal{R}_\lambda f = 0 \) on \( \partial D \), and in \( D \),
\[
|\mathcal{R}_\lambda f(x)| \leq \mathcal{R}_\lambda |f|(x) \leq \lambda^{-1} \sup_{x \in \partial D} |f(x)|. \tag{7.6}
\]

2. There exists an integer \( m_0 = m_0(n, p, \delta, \Lambda, [\mathcal{D}_1]_{A_{\gamma}}, r_0) \), such that for any \( f \in L^p_w(D) \), we have
\[
\sup_{x \in \partial D} |\mathcal{R}^m_{\lambda_1} f(x)| \leq C|f|_{L^p_w(D)}, \tag{7.7}
\]
where \( C = C(n, p, \delta, \Lambda, \mathcal{D}_1[w]_{A_{\gamma}}, r_0) \).

**Proof.** For \( f \in L^\infty(D) \), (7.4) is proved in [19, Theorem 11.2.1(3)]. To prove (7.7), we set \( \alpha = n/(n - 1), p(j) = \alpha^j p, \) and
\[
u_0 = f, \quad u_j = \mathcal{R}^j_{\lambda_1} f, \quad j \geq 1.
\]
Notice that for \( j \geq 1 \), we have
\[
\lambda_1 u_{j+1} - Lu_{j+1} = u_j.
\]
Therefore, by using the solvability and estimates for \( \lambda \) large, we have \( u_{j+1} \in \dot{W}^{2,\infty}_w(D) \) and
\[
\|u_{j+1}\|_{W^{2,\infty}_w(D)} \leq C\|u_j\|_{L^\infty_p(D)} \leq C\|u_j\|_{W^{0,\infty}_w(D)}.
\]
By using Lemma 7.2, we get
\[
\|u_{j+1}\|_{W^{2,\infty}_w(D)} \leq C\|u_j\|_{L^\infty_p(D)} + \|u_{j+1}\|_{L^\infty_p(D)}.
\]
Hence,
\[
\|u_{j+1}\|_{W^{2,\infty}_w(D)} \leq C\|u_j\|_{L^\infty_p(D)}. \tag{7.8}
\]
By the weighted embedding theorem, we have
\[ \|u_{j+1}\|_{W^{m,p}_w(D)} \leq C\|u_j\|_{L^p_w(D)}. \]

Iterating the above inequality yields that for \( j \geq 0 \), we obtain
\[ \|u_j\|_{L^p_w(D)} \leq C\|f\|_{L^p_w(D)}. \quad (7.9) \]

It follows from Hölder’s inequality and the definition of \( A_p \) weights that \( u_{j+1} \in W^{2p(j)/p}(D) \). Then we fix a \( j = j(n,p) \) by choosing \( p(j) > np/2 \). For such \( j \), we conclude from (7.8) and (7.9) that
\[ \sup_{x \in D} |u_{j+1}(x)| \leq C\|u_{j+1}\|_{W^{2p(j)/p}(D)} \leq C\|u_{j+1}\|_{W^{2p(j)/p}(D)} \leq C\|f\|_{L^p_w(D)}, \]

which shows that (7.7) holds with \( m_0 = j + 1 \). The lemma is proved. \( \square \)

Next we show the solvability when \( \lambda \geq \varepsilon_0 \) for a positive constant \( \varepsilon_0 > 0 \).

**Lemma 7.4.** Let \( p \in (1, \infty), w \in A_p, \varepsilon_0 > 0, \) and \( L1 \leq 0 \). Under the above conditions, for any \( \lambda \geq \varepsilon_0 \) and \( u \in W^{2p}_w(D) \), we have
\[ \|u\|_{W^{2p}_w(D)} \leq C\|\lambda u - Lu\|_{L^p_w(D)}, \quad (7.10) \]

where \( C \) depends on \( n, p, \delta, \Lambda, D, \varepsilon_0, [w]_{A_p}, \) and \( r_0 \).

**Proof.** As noted after Theorem 7.1, it suffices to prove the case when \( \varepsilon_0 \leq \lambda < \lambda_1 \). We follow the idea in [19, Section 11.3]. Here we list the main differences. By approximations, we may assume that the coefficients are smooth. Similar to the proof of [19, Theorem 8.5.6], we have
\[ \|u\|_{W^{2p}_w(D)} \leq C\|\lambda u - Lu\|_{L^p_w(D)} + \|u\|_{L^p_w(D)}. \]

Therefore, it suffices to prove for \( \varepsilon_0 \leq \lambda < \lambda_1 \), we have
\[ \|u\|_{L^p_w(D)} \leq C\|\lambda u - Lu\|_{L^p_w(D)}. \]

For this, we define
\[ f := \lambda u - Lu. \]

Then
\[ \lambda_1 u - Lu = (\lambda_1 - \lambda)u + f, \quad u = (\lambda_1 - \lambda)R_{\lambda_1}u + R_{\lambda_1}f. \]

By induction on \( m \), we have
\[ u = (\lambda_1 - \lambda)R_{\lambda_1}^m u + \sum_{i=0}^{m-1} (\lambda_1 - \lambda)R_{\lambda_1}^i R_{\lambda_1} f, \quad (7.11) \]

where \( m \geq 1 \) is any integer. We next introduce constants \( C_1 \) and \( M_m \) such that
\[ \|R_{\lambda_1} g\|_{L^p_w(D)} \leq C_1\|g\|_{L^p_w(D)}, \quad \forall g \in L^p_w(D), \quad M_m = \sum_{i=0}^{m-1} (\lambda_1 - \varepsilon_0)^i C_1^{i+1}. \quad (7.12) \]

By using (7.11), \( 0 < \lambda_1 - \lambda \leq \lambda_1 - \varepsilon_0 \), (7.12), (7.13), and (7.14), we obtain for \( m > m_0 \),
\[ \|u\|_{L^p_w(D)} \leq w(D)^{1/p}(\lambda_1 - \varepsilon_0)^m \sup_{\lambda \in D} \|R_{\lambda_1}^{m-m_0} R_{\lambda_1}^{m_0} u(x)\| + M_m\|f\|_{L^p_w(D)} \]
\[ \leq w(D)^{1/p}\lambda_1^{m_0} (1 - \varepsilon_0/\lambda_1)^m \sup_{\lambda \in D} \|R_{\lambda_1}^{m_0} u(x)\| + M_m\|f\|_{L^p_w(D)} \]
\[ \leq C_2 w(D)^{1/p}\lambda_1^{m_0} (1 - \varepsilon_0/\lambda_1)^m\|u\|_{L^p_w(D)} + M_m\|f\|_{L^p_w(D)}. \]
Fixing $m > m_0$ such that
\[ C_2 w(\Omega)^{1/p} \lambda_1^{m_0} (1 - \epsilon_0 / \lambda_1)^m \leq 1/2, \]
we get (7.10). The lemma is proved. \hfill \Box

We now give

**Proof of Theorem 7.1.** By using the method of continuity, we only need to prove that for any $u \in \tilde{W}^{2,p}(\Omega)$ and $\lambda \geq 0$, we have (7.10). To this end, without loss of generality we may assume that $\Omega \subset B_{2R_0}$, where $R_0 = \text{diam } \Omega$. We take the global barrier $v_0$ from [4] Lemma 11.1.2:

\[ v_0(x) = \cosh(4\epsilon_0 R_0) - \cosh(c_0|x|) \]

satisfies $Lv_0 \leq -1$, and $v_0 > 0$ in $B_{4R_0}$ and $v_0 = 0$ on $\partial B_{4R_0}$, where $c_0 > 0$ is a constant to be chosen. Next we introduce a new operator $L'$ by

\[ L'u = v_0^{-1}Lv_0 u. \]

Notice that in $\Omega \subset B_{2R_0}$, according to the construction of $v_0$, we have $L'1 \leq -\delta^*$ for a constant $\delta^* > 0$ depending on $n, \delta, \Lambda$, and $R_0$, provided that $c_0 = c_0(\delta, \delta, \Lambda)$ is sufficiently large. By using Lemma 7.4 applied to $L'' := L' + \delta^*$, we get

\[ \|u\|_{\tilde{W}^{2,p}(\Omega)} \leq C\|v_0^{-1}\|_{\tilde{W}^{2,p}(\Omega)} \]

\[ \leq C\|v_0^{-1}\|_{L^{\infty}(\partial \Omega)} \]

\[ \leq C\|v_0^{-1}\|_{L^{\infty}(\partial \Omega)} \]

\[ \leq C\|v_0^{-1}\|_{L^{\infty}(\partial \Omega)} \leq C\|\lambda - L\|_{L^{\infty}(\partial \Omega)}. \]

Hence, we finish the proof of the theorem. \hfill \Box

Finally we give the solvability of the adjoint operator of $L$ defined by

\[ L' u := D_i (a^{ij} u) - D_i (b^j u) + cu. \]

By using a similar argument in the proof of Lemma 2.8, from Theorem 7.1, we have

**Corollary 7.5.** Let $p \in (1, \infty)$, $w \in A_p$, and $L1 \leq 0$. Assume that $g = (g^{ij})_{i,j=1}^n \in L^p_w(\Omega)$. The coefficients $a^{ij}$, $b^j$, and $c$ satisfy the same conditions as imposed in Theorem 7.1. Then for any $\lambda \geq 0$,

\[ \begin{cases} 
L' u - \lambda u = \text{div}^2 g & \text{in } \Omega, \\
\lambda u = -\frac{\text{grad}^2 g}{\Lambda} & \text{on } \partial \Omega 
\end{cases} \]

admits a unique adjoint solution $u \in L^p_w(\Omega)$. Moreover, the following estimate holds

\[ \|u\|_{L^p_w(\Omega)} \leq C\|g\|_{L^p_w(\Omega)}, \]

where $C = C(n, p, \delta, \Lambda, \Omega, r_0, [w], \lambda_1)$. \hfill \Box

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