Rigidification of pseudo–Riemannian Manifolds by an Elliptic Equation

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Abstract

We study the solvability of the equation for the smooth function \( \omega \), \( H^\omega = -\kappa \omega g \) on a geodesically complete pseudo–Riemannian manifold \((M, g)\), \( H^\omega \) being the covariant Hessian of \( \omega \). A similar equation was considered by Obata and Gallot in the Riemannian case for positive values of the constant \( \kappa \); the result was that the manifold must be the canonical sphere. In this generalized setting we obtain a range of possibilities, depending on the sign of \( \kappa \), the signature of the metric and the value of a certain first integral of the equation: the manifold is shown to be of constant sectional curvature or a warped product with suitable factors depending on the cases.

1 Introduction.

Given a Riemannian manifold \((M, g)\), we denote by \( H^\omega \) the Hessian of the smooth real function \( \omega \). It is a classical result that the existence of a solution of \( H^\omega = -\omega g \) constrains the curvature to be equal to 1 [Ob62]. Such theorem can be generalized by the equation

\[
H^\omega = -\kappa \omega g ,
\]

(1.1)

where now \( \kappa \in \mathbb{R} \) and \((M, g)\) is a pseudo-Riemannian manifold.

This equation was considered in the literature only in the Riemannian case and for positive values of the constant \( \kappa > 0 \) by Obata [Ob62] and Gallot [Ga79]. In particular Obata states the theorem (Obata’s Theorem): if \((M, g)\) is geodesically complete and \( \omega \) is a nontrivial solution to eq. (1.1) with \( \kappa > 0 \), then \((M, g)\) is isometric to the canonical sphere of radius \( R = \frac{1}{\sqrt{\kappa}} \).

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Unfortunately the proof of this nice theorem cannot be generalized to this context. The purpose of this paper is twofold; first we provide a proof that works independently of the sign of $\kappa$ in the Riemannian case (the sphere being replaced by a negatively curved hyperboloid for $\kappa < 0$). Secondly, the technique of the proof can be applied to the pseudo–Riemannian cases as well.

For historical reasons we name the equation (1-1) “Obata’s Equation” and a solution of it a “Obata’s function”.

A related equation was considered by Kerbrat in [Ke81]; in his case the equation is of third order in $\omega$, namely

$$\nabla_Y H^\omega(X, Z) = -\kappa (2d\omega(Y)(X, Z) + d\omega(X)(Y, Z) + (X, Y)d\omega(Z)),$$

which is presented and studied in the case $\kappa = -1$; the author then proves a certain rigidification assuming the existence of at least two linearly independent solutions or a full rigidification if the critical set of the solution is nonempty, a result that corresponds to our Thm. (3.1) in the present context.

The relation between these equations is elucidated by Gallot in [Ga79]. The author shows that (for $\kappa > 0$) the two equations are the first two of a sequence of equations $E_n$ of $n$-th order equations characteristic of the canonical sphere and he shows that their solutions are the harmonic homogeneous polynomials of degree $n - 1$ in the Euclidean space in which the sphere is embedded: the solutions of equation $E_n$ are proper eigenfunctions of the Laplacian.

The study of solvability of Obata’s equation depends also on the value of a first integral, namely

$$\|\Omega\|^2 + \kappa \omega^2 = h,$$

As outlined above, we generalize Obata’s theorem in two directions: first, we allow any real value of $\kappa$ while keeping the assumption on the Riemannian signature of the metric. The result is that the manifold is of constant sectional curvature only if $\kappa h > 0$ ($h$ being the first integral in Eq. (1-3), while in all other cases we have only the splitting of the metric in a “warped–product” metric (cf. [On83]), without any further constraint on the Riemannian curvature.

Secondly, we consider the same equation on a pseudo–Riemannian manifold. Once more we obtain that the sectional curvature must be a constant in the case $\kappa h > 0$ while for $\kappa h < 0$ we obtain only the splitting in a warped product metric. However we also obtain an “intermediate” rigidification in the case $\kappa \neq 0$, $h = 0$, with a necessary condition of asymptotic flatness on the level surfaces of $\omega$. This has no analog in the Riemannian case, where for $h = 0$ (and a fortiori $\kappa < 0$) we only have the warped–product structure. In order to construct a nontrivial example, in Prop. (iv) we define a geodesically complete manifold supporting such an Obata’s function by gluing two suitable incomplete manifolds along a lightlike smooth hypersurface. This manifold does not have constant curvature in general, showing that this is not a necessary condition, and is not globally a warped-product. Another case which is not present in the Riemannian setting is for $h = \kappa = 0$, because in Riemannian signature this would immediately imply that $\omega \equiv 0$ (using eq. (1-3) and (1-4)); on the contrary, in pseudo–Riemannian curvature the equation has nontrivial solutions corresponding to a null Killing vector field (Thm. (iv), part (iii)).

The paper is organized as follows: in Section (2) we set some notation recalling notion and properties of warped products and we establish some preliminary results on geodesic
completeness of pseudo–Riemannian warped products. In Section (3) we state and prove
the theorem of rigidification for Riemannian manifolds. Many statements do not rely on the
signature and hence can be rephrased without any change in the pseudo–Riemannian case.
In Section (4) we extend this theorem to the case of arbitrary signature: since the result and
the proof depend strongly on the relative signs of \( \kappa h \), we split the theorem in Thms. 4.1,
4.3 4.4 in order to avoid too many case distinctions. Finally Section (5) is devoted to
the study of a maximal set of Obata’s functions, where it is shown that if there exist more
than one solution, then the distribution spanned by their gradients is involutive and foliates
the manifold in submanifold of constant sectional curvature.

2 Preliminaries

We first recall the notion of warped product. Let \((B, g_B)\) and \((F, g_F)\) be two pseudo–Riemannian manifolds. Consider the smooth manifold
\(M := B \times F\) with the canonical projections
denoted by \(\pi : M \to B\) and \(\sigma : M \to F\). Given an arbitrary smooth map \(\alpha : B \to \mathbb{R}^+\) we define a (pseudo)–Riemannian metric \(g = g_\alpha\) on \(M\) (called warped metric \([On83]\)
\[g_\alpha := \pi^* g_B + (\alpha \circ \pi)^2 \sigma^* g_F.\] (2-4)
Let \(X,Y\) be sections of \(\Gamma(\pi^*TB)\) and \(U,V\) of \(\Gamma(\sigma^*TF)\) and let \(A\) denote the gradient of \(\alpha\).
The sectional curvature is given in terms of the sectional curvatures of \(B\) and \(F\) as follows
\([BG01]\):
\[K_{XY} = K_B^{XY}; \quad K_{XV} = -H_\alpha(X, X) / \alpha ||X||^2; \quad K_{UV} = K_{U^0V^0} - ||A||^2 / \alpha^2,\] (2-5)
where in these formulas the norms are the pseudo–lengths.

It is a well-known result that in the Riemannian case a warped product \(B \times_\alpha F\) is
geospectically complete iff both factors are and \(\alpha > 0\) \([B069]\). The pseudo–Riemannian case is
much less studied in general. If the base \(B\) is one dimensional we can establish the following
lemmas.

**Lemma 2.1** Let \(M = B \times_\alpha F\) be a pseudo–Riemannian warped product with \(B\) (anti)–Riemannian and both factors geodesically complete.
If \(\epsilon := \inf_B \alpha > 0\) then \(M\) is geodesically complete.

**Proof.** Consider the equation for a geodesic \(\gamma(s)\) of type space, time or light and set correspondingly \(C = +1, -1, 0\): decomposing the vector \(\dot{\gamma} = X + V = \pi^*(\dot{\gamma}) + \sigma^*(\dot{\gamma})\) we get (cf. \([BG01]\))
\[\nabla^B_X X = \frac{\|V\|^2}{\alpha} A, \quad \nabla^F_V V = -2 \frac{X(\alpha)}{\alpha} V.\] (2-6)
Computing the rate of change of the norms we get
\[- \frac{d}{ds} (\|V\|^2) = \frac{d}{ds} (\|X\|^2) = 2 \frac{X(\alpha)}{\alpha} (C - \|X\|^2).\] (2-7)
Integrating the equation once we obtain
\[\|X\|^2 = (C - \|V\|^2) = \frac{\alpha_0^2 (\|X_0\|^2 - C)}{\alpha^2 \circ \pi(\gamma(s))} + C \leq \frac{\alpha_0^2 (\|X_0\|^2 - C)}{\epsilon^2} + |C|\] (2-8)
This shows that the square norm of $X$ is bounded, hence the curve $\pi \circ \gamma : I \to B$ has finite length for any value of the parameter. This and geodesic completeness (which, for (anti–Riemannian manifolds is equivalent to completeness) proves that the curve $\pi \circ \gamma$ is defined for any $s \in \mathbb{R}$. Moreover the norm of $V$

$$\|V\|_F^2 = \frac{1}{\alpha^2} \|V\|^2 = \frac{C - \|X\|^2}{\alpha^2}\quad (2.9)$$

is bounded as well since $\|X\|^2$ is. Now $V$ is parallel translated because the projection of $\gamma$ on the second factor is a pregeodesic whose parameter is bounded by a suitable affine parameter, hence $\sigma \circ \gamma$ is defined for any $s \in \mathbb{R}$ as well.

This proves that $M$ is geodesically complete.

**Lemma 2.2** Let $M = \mathbb{R} \times \alpha \Sigma$ be a pseudo–Riemannian manifold of type $(r,p)$ with metric

$$ds^2 = -dt^2 + \alpha^2(t)g_{\Sigma},$$

Suppose $\alpha$ is monotonic out of a bounded set and $\inf_{t \in \mathbb{R}} \alpha(t) = 0$.

If $\alpha$ is integrable at either $+\infty$ or $-\infty$ then $M$ is not geodesically complete.

An inextensible incomplete geodesic reaches the spacelike boundary of $\Sigma$ for a finite value of the affine parameter. If the base $\mathbb{R}$ is (anti)Riemannian, then the geodesic reaches the (spacelike) timelike boundary of $\Sigma$.

**Proof.** The proof is based on the previous formulas. Suppose that the factor $\mathbb{R}$ is of Riemannian signature; if $s$ is the affine parameter of the geodesic, the equation for the geodesic coordinate $t$ of $\mathbb{R}$ can be recast into

$$\frac{\alpha(t)dt}{\sqrt{\alpha^2(t_0)(l_0^2 + C) - C\alpha^2(t)}} = ds,$$

where $C = +1, -1, 0$ according to the type space, time or light of the geodesic.

For timelike geodesics ($C = -1$) for which $|l_0| > 1$ we can have unbounded trajectories in the direction(s) where $\alpha$ goes to zero (namely one or both of $\pm \infty$). If the integral converges in a neighborhood of that point then the geodesic reaches the boundary of $\mathbb{R}$ in a finite value of the affine parameter and hence the manifold is incomplete: the projection on the fiber is timelike as follows from the computation of norms. Then it is easy to see that also this projection reaches the boundary at the same value of the affine parameter. Indeed let $\phi(s)$ be the projection of the geodesic $\gamma(s)$ on $\Sigma$ and let $\sigma(s)$ be the affine (timelike) length of $\phi(s)$ computed with the natural metric of $\Sigma$; then from $-\alpha^2(t(s)) \left( \frac{dt}{\alpha(t(s))} \right)^2 = \|V\|^2 = C - \|X\|^2$ we get

$$d\sigma = \frac{dt}{\alpha(t(s))\sqrt{1 + \frac{\alpha^2(t)C}{\alpha^2(t_0)(C+l_0^2)}}},$$

which shows that $\sigma$ diverges as $t$ tends towards the infinity where $\alpha$ vanishes (and this happens at a finite value of the affine parameter $s$).

Similar reasoning holds for the remaining two cases ($C = 1, 0$); moreover one realizes that there are no complete inextensible spacelike or lightlike geodesics.

If we change the signature of $\mathbb{R}$ we only have to swap the role of time and space in the previous derivation. \[\square\]
Example 2.1 In the case $\alpha(t) = \exp(t)$ (which we will encounter later) we see that $\alpha$ is integrable at $t = -\infty$. Let $\gamma(s)$ be an inextensible incomplete geodesic: if it is timelike or null then it has no “turning point” in $t$ namely $\dot{t} \neq 0$ and $t < 0$ if the geodesic reaches the boundary in the future. If it is spacelike then it has a “turning point” and it is incomplete in both directions.

3 Riemannian case

Throughout this section the manifold $(M, g)$ will be of Riemannian signature, although Lemmas 3.1, 3.2, 3.3 and much of the proof of Thm. 3.1 apply without modifications to the pseudo-Riemannian case as well. Given a smooth function $\omega : M \to \mathbb{R}$ we will denote the (contravariant) gradient of $\omega$ by $\Omega$. The Hessian $H^\omega$ will be the second covariant differential w.r.t. the Levi–Civita connection of $(M, g)$, namely

$$H^\omega(X, Y) = g(\nabla_X \Omega, Y). \quad (3-12)$$

Lemma 3.1 If $\omega$ is an Obata’s function then

$$\|\Omega\|^2 + \kappa \omega^2 = h \quad (3-13)$$

for some constant $h \in \mathbb{R}$.

Proof. Taking derivative along the vector $X$ of $\|\Omega\|^2 + \kappa \omega^2$ we obtain

$$X (\|\Omega\|^2 + \kappa \omega^2) = 2H^\omega(X, \Omega) + \kappa \omega g(X, \Omega) = 0 \quad (3-14)$$

This result does not rely on the signature and holds in the pseudo–Riemannian case as well.

Lemma 3.2 If $\omega$ is an Obata’s function then the curves generated by $\Omega$ are pregeodesics.

Proof. Obata’s equation $H^\omega(X, Y) = g(\nabla_X \Omega, Y) = -\kappa \omega g(X, Y)$ is equivalent to $\nabla_X \Omega = -\kappa \omega X$. In particular $\nabla_\Omega \Omega = -\kappa \omega \Omega$, and hence $\Omega$ is parallel. Once more this result does not rely on the signature and holds in the pseudo–Riemannian case as well.

Lemma 3.3 If $\omega$ is an Obata’s function on a pseudo–Riemannian manifold $(M, g)$ with first integral $h \neq 0$ and if $\Sigma_0 := \omega^{-1}(0)$ is not empty, then it is a smooth totally geodesic surface.

Proof. From the equation

$$\|\Omega\|_{\Sigma_0}^2 = h \neq 0 \quad (3-15)$$

follows that $\Sigma_0$ is smooth. Let $p \in \Sigma_0$ and $X \in T_p \Sigma_0$ and consider the geodesic starting at $p$ with initial tangent vector $X$, denoted by $\gamma(t)$, $t$ being the affine parameter. A simple computation gives

$$\frac{d^2}{dt^2} \omega(\gamma(t)) = \frac{d}{dt} g(\dot{\gamma}, \dot{\gamma}) = g(\nabla_{\dot{\gamma}} \Omega, \dot{\gamma}) = -\kappa \omega(\gamma(t)) ||\dot{\gamma}||^2. \quad (3-16)$$

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Therefore the function $\chi(t) := \omega(\gamma(t))$ satisfies a second order ODE $\chi'' = \pm \kappa \chi$ or $\chi'' = 0$ according to the type space, time or light of the geodesic. In our case $\chi(0) = \chi'(0) = 0$ and hence $\chi(t) \equiv 0$, so that the geodesic remains in the level surface $\Sigma_0 = \omega^{-1}(0)$. 

**Lemma 3.4** If $\omega$ is an Obata’s function with first integral $h = \|\Omega\|^2 + \kappa \omega^2$ on the geodesically complete Riemannian manifold $(M, g)$ then $J = \omega(M) \subset \mathbb{R}$ is the closure of the interval

1. $J = (-\sqrt{h/\kappa}, \sqrt{h/\kappa})$ if $\kappa, h > 0$;
2. $J = (\sqrt{|h|/\kappa}, \infty)$ (or $J = (-\infty, -\sqrt{|h|/\kappa})$) if $h < 0$, $\kappa > 0$;
3. $J = \mathbb{R}$ if $h > 0$, $\kappa \leq 0$.

**Proof.** The cases $h, \kappa < 0$ or $h = \kappa = 0$ cannot occur in the Riemannian case because of the positiveness of the metric and eq. (3-13).

Let $p \in M$ be any point where $\Omega_p \neq 0$. Consider the geodesic $\gamma(t)$ starting at $p$ and parallel to $\Omega$ (by Lemma 3.2), $t$ being the affine length parameter. The function $f(t) := \omega(\gamma(t))$ then satisfies (from eq. (3-13))

$$
(f'(t))^2 + \kappa(f(t))^2 = h. \tag{3-17}
$$

By virtue of the geodesic completeness, this is valid over the whole interval $t \in \mathbb{R}$. Integrating this simple ODE one obtains

$$
f(t) = \begin{cases} 
\sqrt{h/\kappa} \cos \left(\frac{\sqrt{h}t}{\kappa}\right) & \kappa > 0, \ (h > 0), \\
\sqrt{h/\kappa} \sinh \left(\frac{\sqrt{h}t}{\kappa}\right) & \kappa < 0, \ h > 0, \\
\sqrt{|h|/\kappa} \cosh \left(\frac{\sqrt{|h|}t}{\kappa}\right) & \kappa < 0, \ h < 0, \\
|\kappa|^{-\frac{1}{2}} \exp \left(\frac{\sqrt{|h|}t}{\kappa}\right) & \kappa < 0, \ h = 0, \\
\sqrt{h}t & \kappa = 0, \ (h > 0).
\end{cases} \tag{3-18}
$$

The intervals $J$ are just the ranges of the function $f(t)$ in the different cases. Note that the open intervals $J$ are constituted by all regular values of $\omega$. 

We can now state the theorem in the Riemannian case: the proof was given by the authors in [BG01]. For completeness we report the derivation of the result since part of the proof applies without changes to the pseudo–Riemannian case as well.

**Theorem 3.1** Let $(M, g)$ be any complete smooth Riemannian manifold of dimension greater than one (to avoid trivialities) such that there exists an Obata’s function $\omega$ with first integral $\omega^2 + ||\Omega||^2 = h$: denoting by $\Delta := \{x \in M : \langle \Omega_x, \Omega_x \rangle = 0\}$ the critical fibers of $\omega$, then

1. $(M \setminus \Delta, g)$ is isometric to a warped product $I \times_\alpha \Sigma_q$ where $I \subseteq \mathbb{R}$ is an open interval, $\Sigma_q := \omega^{-1}(q)$ for a regular value $q$, and $\alpha$ is a suitable function to be specified in the proof.
2. if $\Delta \neq \emptyset$ then $(M, g)$ is of constant curvature $K^{(M)} = \kappa$;
3. If $\kappa \leq 0 \leq h$ then the above holds globally (and $I = \mathbb{R}$).
More explicitly, the point $\psi$ being diffeomorphic by means of the diffeomorphism generated by the gradient $\Omega$ of $\{\}$ that is the metric on $p$ the coordinate on $J$, $X,Y$ injection. For all $\omega$ generated by $\Omega$ and starting at $q$ prove that the metric $\tilde{g}$ corresponds of regular values containing $q$: this is the interval defined in Lemma 3.4 according to the various cases.

The foliation induced on $\omega^{-1}(J) \subseteq M$ by the function $\omega$ is then regular, all level set of $\omega$ being diffeomorphic by means of the diffeomorphism generated by the gradient $\Omega$ of $\omega$

$$J \times \Sigma_q : \psi \rightarrow \omega^{-1}(J) \subseteq M.$$  \hfill (3-19)

More explicitly, the point $\psi(q, \sigma)$ is the (unique) point of $\Sigma_q := \omega^{-1}(q)$ lying on the geodesic generated by $\Omega$ and starting at $\sigma \in \Sigma_q$. Below we will denote by $\omega$ both the function and the coordinate on $J$. This definition implies (tautologically) that $\psi_* \partial_\omega = \frac{\Omega}{||\Omega||}$. We now prove that the metric $\tilde{g} := \psi^* g$ gives $J \times \Sigma_q$ the structure of warped product. Let $p_1$ and $p_2$ denote the projections onto the two factors of $J \times \Sigma_q$ and $i : \Sigma_q \to M$ be the natural injection. For all $X,Y$ in the tangent bundle of $\Sigma_q$ in $J \times \Sigma_q$ (i.e. in $\Gamma(p_2^* T\Sigma_q)$)

$$\tilde{g}(\partial_\omega, \partial_\omega) = \frac{1}{||\Omega||^2} = \frac{1}{h - \kappa \omega^2}; \quad \tilde{g}(\partial_\omega, X) = 0; \quad \tilde{g}(X, Y) = g(\psi_* X, \psi_* Y).$$

Let $X,Y \in \Gamma(p_2^* T\Sigma_q)$ such that $[\partial_\omega, X] = [\partial_\omega, Y] = 0$ and thus $[\Omega, \psi_* X] = [\Omega, \psi_* Y] = 0$: if we compute $\mathcal{L}_\partial g$ we get

$$\partial_\omega (\tilde{g}(X, Y)) = (\mathcal{L}_{\partial_\omega} \tilde{g})(X, Y) = \frac{1}{||\Omega||^2} \Omega \left( g(\psi_* X, \psi_* Y) \right) =$$

$$= \frac{1}{||\Omega||^2} \{ g(\nabla_\Omega \psi_* X, \psi_* Y) + g(\psi_* X, \nabla_\Omega \psi_* Y) \} =$$

$$= \frac{1}{||\Omega||^2} \{ g(\nabla_\psi X, \psi_* Y) + g(\psi_* X, \nabla_\psi Y) \} = \frac{1}{||\Omega||^2} 2H^2(X, Y) =$$

$$= \frac{-2\kappa \omega}{h - \kappa \omega^2} g(\psi_* X, \psi_* Y) = (\partial_\omega \ln |h - \kappa \omega^2|) \tilde{g}(X, Y), \quad (3-20)$$

That is the metric on $\{\omega\} \times \Sigma_q$ undergoes conformal rescaling under change of the base-point $\omega$. Hence

$$\hat{g} = \frac{1}{h - \kappa \omega^2} d\omega^2 + \frac{|h - \kappa \omega^2|}{h - \kappa q^2} i^* g$$

This proves the warped structure; introducing a geodesic coordinate $t$ in $J$ according to

$$dt^2 = \frac{1}{h - \kappa \omega^2} d\omega^2 \quad (3-21)$$

we obtain the metric $\hat{g}$ in the form

$$\hat{g} = dt^2 + \alpha^2(t) i^* g, \quad \alpha(t) := \sqrt{\frac{|h - \kappa \omega^2|}{h - \kappa q^2}} = \frac{1}{\sqrt{|h - \kappa q^2|}} |f'(t)|, \quad (3-22)$$
and $f(t)$ being given in eq. (3-18). According to the different cases the intervals $J$ of Lemma 3.4 expressed in the geodesic coordinate $t$ are

(1): $\varepsilon > 0$, $|h| > 0$, $J = (0, \pi/\sqrt{\varepsilon})$; 
(2): $\varepsilon < 0$, $|h| > 0$, $J = \mathbb{R}$; 
(3): $\varepsilon < 0$, $|h| < 0$, $J = (0, \infty)$; 
(4): $\varepsilon < 0$, $|h| = 0$, $J = \mathbb{R}$. 
(5): $\varepsilon = 0$, $|h| > 0$, $J = \mathbb{R}$.

We remark that the critical set $\Delta$ is not empty only in cases (1) and (3) above, since then the function $\omega$ takes on the critical values as seen in Lemma 3.4. Furthermore it follows from elementary Morse theory that since at those points the Hessian $H_{\omega} = \pm \sqrt{|h|/\varepsilon} g|_{\Delta}$ is nondegenerate and definite (positively or negatively), the critical points are isolated and are either maxima or minima of $\omega$. In case (1), i.e., $\varepsilon > 0$, $|h| > 0$, $\Delta$ is constituted by two isolated points, one maximum and one minimum $p_{\pm}$ with the critical values $\omega_{cr} = \pm \sqrt{|h|/\varepsilon}$; in case (3), i.e., $\varepsilon < 0 > |h|$, $\Delta$ is just one isolated point of minimum and $\omega(M) = \sqrt{|h|/\varepsilon} (\text{or maximum if } \omega(M) = (-\infty, -\sqrt{|h|/\varepsilon}))$ with critical value $\omega_{cr} = \sqrt{|h|/|\varepsilon|}$ ($\omega_{cr} = -\sqrt{|h|/|\varepsilon|}$).

Moreover, since the Hessian at the critical points is of definite signature (being proportional to the Riemannian metric), then the level surfaces of $\omega$ are topological spheres for values near to the critical values, as follows once more from elementary Morse theory. But since all regular level surfaces are diffeomorphic then all the level sets $\Sigma_q = \omega^{-1}(q)$ are topological spheres.

Outside of the critical locus $\Delta$ the level surfaces of $\omega$ are the same as those of $\alpha$ (spheres); as $\omega$ tends to a critical value $\alpha$ tends to zero (by its definition). To prove assertion (ii) we now compute the sectional curvature of $M$ on a plane spanned by $U, V \in \Gamma(T\Sigma_q)$. The calculation follows from the expression of the sectional curvature of a warped product:

$$K_{uv}^{(M)} = \frac{K_{\Sigma_q}^{(\alpha)} - (\alpha')^2}{\alpha^2}. \quad (3-23)$$

We shrink this topological sphere

$$\alpha \xrightarrow{t \to t_0} 0 \Leftrightarrow \omega \to \omega_{cr},$$

by parallel translating the two vertical vectors $U, V$ up to the critical point $p_{cr}$ along the flow generated by the gradient $\Omega$ (recall that $\Omega$ and the gradient of $\alpha$ generate pre-geodesics). For each such flow line $\gamma$, the projection on the fiber $\Sigma_q$ is constant, and the 2-plane spanned by $U, V$ does not change (each vector is just rescaled). At the end of this shrinking process we obtain two vectors in the tangent space $T_{p_{cr}} M$. Since we must obtain a well definite value of the sectional curvature of $M$ then we must have $K_{uv}^{(\Sigma_q)} = (\alpha'(t_0))^2$ independently of the “direction” of the geodesic, namely of the point on $\Sigma_q$, and of the two-plane. This proves that $\Sigma_q$ is a sphere because we have just proved that its sectional curvature is a constant.

Then, from Eq. (3-23) and from the explicit form of $\alpha$, it follows that the sectional curvature of the manifold $M \setminus \Delta$ is also constant $K^{(M)} = \varepsilon$ and hence $(M, g)$ is globally (by continuity) of constant sectional curvature, which proves part (ii). Notice that the fact that $\omega$ is a Morse function with spheres as level sets, implies here that $(M, g)$ is actually a round sphere or a non-quotiented hyperboloid (depending on the sign of $\varepsilon$).
If the critical locus $\Delta$ is empty (which corresponds to the remaining cases 2), 4), 5)) we have no constraint on the curvature of the leaf $\Sigma_q$, which can be any complete smooth Riemannian manifold.

4 The pseudo–Riemannian case

We now consider Obata’s equation (1-1) assuming that $g$ is pseudo–Riemannian with signature $(r,p)$ (both non-zero). We still have the same first integral as in Lemma (3.1), but now the square–norm of $\Omega$ can be of any sign and hence any combination of signs of $\kappa$ and $h$ is a priori allowed. From eq. 3-13 we have the following implications on the type of the vector $\Omega$ depending on the relative signs of these two constants.

| $\kappa$ | $h$ | Type of $\Omega$ |
|---------|-----|-----------------|
| $> 0$  | $> 0$ | depends          |
| $> 0$  | $< 0$ | timelike         |
| $> 0$  | $= 0$ | timelike or null |
| $< 0$  | $> 0$ | spacelike        |
| $< 0$  | $< 0$ | depends          |
| $< 0$  | $= 0$ | spacelike or null|
| $= 0$  | $> 0$ | spacelike        |
| $= 0$  | $< 0$ | timelike         |
| $= 0$  | $= 0$ | null             |

From Lemma 3.1 and Eq. (3-13) it follows that the critical values of $\omega$ can be only $\pm \sqrt{|h|/|\kappa|}$: we denote with $\Sigma_{\pm}$ the corresponding singular level–sets. These, contrarily to the Riemannian case, do not coincide with the set of stationary points because the latter are now saddle–points.

In order to extend Thm. (3.1) to the pseudo Riemannian case we will formulate Thms. (4.1, 4.2, 4.3, 4.4) rather than one single theorem in which we have many subcases. Only the case $\kappa \geq 0$ will be addressed because the case $\kappa < 0$ can be obtained easily from the case $\kappa > 0$ by exchanging the roles of “time” and “space”, that is by swapping $r$ and $p$ in the signature. Thus, for instance, the case $\kappa > 0$, $h > 0$ corresponds to the case $\kappa < 0$, $h < 0$ and so on.

We will work under the following

Common assumptions: $(M, g)$ is a geodesically complete, connected pseudo–Riemannian manifold; the signature of the metric is $(r, p)$ (with $r$ negative and $p$ positive eigenvalues, both nonzero) and there exists a nontrivial Obata’s function $\omega$ with first integral $\|\Omega\|^2 + \kappa \omega^2 = h$.

Theorem 4.1 Under the common assumptions, let $\kappa > 0$ and $h > 0$. Let $\Sigma_{\pm} = \omega^{-1} \left( \pm \sqrt{h/\kappa} \right)$ be the singular fibers. Then $\Sigma_{\pm}$ are both nonempty and $(M, g)$ has constant sectional curvature $K = \kappa$.

In each connected component of $M \setminus \Sigma_{\pm} \cup \Sigma_{\mp}$ the type of $\Omega$ is constant and $\omega$ takes value in one of these intervals

$$J_- = (\infty, -\sqrt{h/\kappa}) \ , \quad J_0 = (-\sqrt{h/\kappa}, \sqrt{h/\kappa}) \ , \quad J_+ = (\sqrt{h/\kappa}, +\infty) \ . \quad (4-24)$$
We denote with $M^+_{cc}$, $M^-_{cc}$, $M^0_{cc}$ the generic connected components of $\omega^{-1}(J_+)$, $\omega^{-1}(J_-)$, $\omega^{-1}(J_0)$.

The boundary of $M^\pm_{cc}$ is a light–cone with vertex in a critical point $p_\pm$ of $\omega$, with $\omega(p_\pm) = \pm \sqrt{h/\kappa}$. Moreover $M^\pm_{cc}$ is the set of points which are timelike related to $p_\pm$. The boundary of $M^0_{cc}$ is constituted by two light–cones with vertex at $p_\pm$ and the interior points are spacelike related to $p_\pm$.

Each connected component is isometric to an appropriate warped product $I \times_\alpha \Sigma$ as indicated below, where $d(\cdot, \cdot) : M \times M \rightarrow \mathbb{R}$ denotes the pseudo–distance (negative for timelike separation):

a) in $M^0_{cc}$

\[
\omega(p) = \sqrt{h/\kappa} \cos\left(\sqrt{\kappa} d(p, p_+)\right) ; \quad I = (0, \pi/\sqrt{\kappa}) \quad \alpha = \sqrt{h} \sin\left(\sqrt{\kappa} t\right) \quad (4.25)
\]

\[
ds^2 = dt^2 + \alpha^2(t)ds^2_\Sigma \quad (4.26)
\]

and $\Sigma$ is a hypersurface of type $(r, p - 1)$ and constant curvature $K^\Sigma = h$.

b) in $M^\pm_{cc}$ we have

\[
\omega(p) = \pm \sqrt{h/\kappa} \cosh\left(\sqrt{\kappa} d(p, p_\pm)\right) ; \quad I = (0, \infty) \quad \alpha = \sqrt{h} \sinh\left(\sqrt{\kappa} t\right) \quad (4.27)
\]

\[
ds^2 = -dt^2 + \alpha^2(t)ds^2_\Sigma \quad (4.28)
\]

and $\Sigma$ is a hypersurface of type $(r - 1, p)$ and constant curvature $K^\Sigma = -h$.

c) the singular fibers $\Sigma_{\pm}$ are union of light–cones with vertices in the stationary points of $\omega$.

**Proof.** We first prove that $\Sigma_{\pm}$ are both nonempty. Let $p$ be a regular point of $\omega$ where $\Omega$ is not null; such a point exists otherwise $\|\Omega\|^2 \equiv 0$ but then eq. (3-13) would imply that $\omega$ is a constant, in contradiction with eq. (1-1). Let $\gamma(s)$ be the geodesic generated by $\Omega$ starting at $p$. Then an argument similar to the Riemannian case shows that the value of $\omega$ along this geodesic will eventually reach one of the two values $\pm \sqrt{h/\kappa}$. More precisely, if $\Omega$ is timelike at $p$ it is spacelike, then $\omega(\gamma(s))$ is a trigonometric function which attains both values along the geodesic $\gamma$. If $\Omega$ is timelike at $p$, then $\omega(\gamma(s))$ is a hyperbolic cosine which reaches one of the two values along the geodesic.

Let $p_\pm$ be the first points on such a geodesic where $\omega(p_\pm) = \pm \sqrt{h/\kappa}$. Clearly $p_\pm \in \Sigma_{\pm}$ and hence they are not empty.

We now prove that $p_\pm$ are critical points. Suppose by contradiction that $\Omega(p_\pm) \neq 0$; then it should be a null vector from eq. (3-13), which is impossible because the geodesic $\gamma$ under consideration (generated by $\Omega$) is either timelike or spacelike. This argument shows also that any point $p$ in $M \setminus \Sigma_+ \cup \Sigma_-$ is geodesically connected with one or both type of the critical points $p_\pm$ (according the the cases). Therefore $p_\pm$ are isolated critical points of $\omega$ (the Hessian being non-degenerate). Moreover, they are saddle points and thus $\Omega$ is of every type in any neighborhood of the critical points since $\|\Omega\|^2 = h - \kappa \omega^2$ takes on positive and negative values in the neighborhood of the critical point.

It is not difficult to show now that in the connected regions of $M \setminus \Sigma_+ \cup \Sigma_-$ (where $\Omega$ is spacelike or timelike)

\[
\omega_{\text{space}}(p) = \sqrt{h/\kappa} \cos\left(\sqrt{\kappa} d(p, p_+)\right) \quad (4.29)
\]

\[
\omega_{\text{time}}(p) = \pm \sqrt{h/\kappa} \cosh\left(\sqrt{\kappa} d(p, p_\pm)\right) \quad (4.30)
\]
In eq. (4-29) \( p \) is restricted to belong to the region of points which are spacelike w.r.t. \( p_+ \) while in eq. (4-30) \( p \) is timelike related to \( p_\pm \). The singular fibers of \( \omega \) are constituted by the light–cones through the points \( p_\pm \). Moreover, following any spacelike geodesic emerging from \( p_+ \) we reach a stationary point \( p_- \) and vice versa: of course in general we need not reach the same stationary point where we started from, as this depends on the global topology of the manifold which cannot be fixed in this context (see Remark 4.1).

In order to prove that the sectional curvature is constant we can adapt the argument used in part (ii) of the proof of Thm. 3.1 except that now we must approach the stationary points \( p_\pm \) from spacelike or timelike directions. In these two cases the geometry of the level surfaces of \( \omega \) is different, since they are (locally) modeled on hyperboloids of different signatures. Nonetheless this is sufficient to prove that the sectional curvature of the surfaces \( \omega^{-1}(q) \) are constants: this in turn forces the sectional curvature of \( M \) to be constant and equal to \( \kappa \) on the complement of the light–cone. Smoothness of \( (M,g) \) completes the proof.

An example for \( k = 1, h = 1 \) on the quadric \(-t^2 + x^2 + y^2 = 1\) with \( \omega = x \): this is the paradigmatic case up to a covering.

**Remark 4.1** In the case \( \kappa > 0, h > 0 \), (or \( \kappa < 0, h < 0 \)) we have proved that \((M,g)\) must be of constant sectional curvature and now we can describe \( \omega \) explicitly. If we realize \((M,g)\) as a suitable quadric (or a covering of it), then \( \omega \) is just any linear function in the ambient pseudo–Riemannian flat manifold restricted to the quadric, provided that its gradient is not null.

In this case we have more than two critical points only if the manifold is a covering of the above quadric. Consider for instance the quadric \(-Z^2 - Y^2 + \sum dX^2 = -1\) in a flat spacetime with metric \(-dZ^2 - dY^2 + \sum dX^2\) (called “Anti de Sitter spacetime”) and the function \( \omega = Z \), (this corresponds to the case \( \kappa = -1, h = -1 \)); since this space has a nontrivial fundamental group \( \pi_1 \simeq \mathbb{Z} \), we may pass to its universal covering (or other coverings), and the function \( \omega \) would have many critical points.
We now consider the case $\kappa > 0$ and $\eta < 0$. Contrarily to the previous one we do not obtain a complete rigidification and we are left with an arbitrariness in the metrics that may occur on the leaves of Obata’s function. This is similar to what happens in the Riemannian case with $\kappa < 0$, $\eta > 0$.

An example for $\kappa = 1$ and $\eta = -1$ on the quadric $-t^2 + x^2 + y^2 = 1$ with $\omega = t$: this is not the paradigmatic case as in general the manifold does not have constant curvature.

**Theorem 4.2** Under the common assumptions, let now $\kappa > 0$ and $\eta < 0$. Then $\Omega$ is everywhere timelike and $(M, g)$ is globally isometric to a warped product $\mathbb{R} \times_\alpha \Sigma$ with
\[
\omega(x) = \sqrt{|\eta|/\kappa} \sinh (\sqrt{\kappa} t) ; \quad \alpha(t) = \sqrt{|\eta|} \cosh (\sqrt{\kappa} t) ,
\]
ds^2 = -dt^2 + \alpha^2(t) ds^2_{\Sigma},
x := (t, \sigma) \in \mathbb{R} \times \Sigma
\]

where $\Sigma$ is any geodesically complete, connected pseudo-Riemannian submanifold of type $(r - 1, p)$.

**Proof.** In this case $\Omega$ is everywhere timelike because $|\Omega|^2 = -|\eta| - \kappa \omega^2 \leq -|\eta| < 0$; in particular this implies that the foliation induced by the level surfaces of $\omega$ is smooth and all leaves are diffeomorphic by means of the flow induced by the gradient $\Omega$. By integrating $\omega$ along the geodesic generated by its gradient $\Omega$ we find that it can be written, in term of a suitably shifted affine parameter, as
\[
\omega(\gamma(t)) = \sqrt{|\eta|/\kappa} \sinh (\sqrt{\kappa} t) .
\]

In particular $\Sigma_0 := \omega^{-1}(0)$ is non-empty and hence, by Lemma 3.3, it is a totally geodesic smooth hypersurface of type $(r - 1, p)$. This proves that $M$ is diffeomorphic to $\mathbb{R} \times \Sigma_0$, where we use the coordinate $t := \frac{1}{\sqrt{|\eta|}} \sinh^{-1} \left( \frac{\sqrt{\kappa}}{|\eta|} \omega \right)$ for the factor $\mathbb{R}$. The same computation as
Theorem 4.3 Under the common assumptions, let now \( \kappa > 0 \) and \( \mathfrak{h} = 0 \).

(i) Then \( \Omega \) is almost everywhere timelike.

(ii) The two subsets \( M_\pm = \omega^{-1}(\mathbb{R}^\pm) \subset M \) are isometric to a warped product \( \mathbb{R} \times_\alpha \Sigma_\pm \) with

\[
\omega(p_\pm) = \pm \sqrt{1/\mathfrak{h}} \exp \left( \sqrt{\kappa} t \right); \quad \alpha(t) = \exp \left( \sqrt{\kappa} t \right),
\]

\[
ds^2 = -dt^2 + \alpha^2(t)ds_{\Sigma_\pm}^2,
\]

\[
p_\pm = (t, \sigma_\pm) \in \mathbb{R} \times \Sigma_\pm,
\]

and \( \Sigma_\pm \) are geodesically complete pseudo–Riemannian manifolds of type \((r - 1, p)\).

(iii) Necessary condition for geodesic completeness is that the sectional curvatures of \( \Sigma_\pm \) vanish at least as \( O(\sigma^{-1}) \) along any spacelike geodesic, \( \sigma \) being the natural length in \( \Sigma_\pm \).

(iv) If the sectional curvature of \((M, g)\) is bounded, then \( \Sigma_\pm \) are both flat and hence the sectional curvature of \( M \) is actually constant \( K = \kappa \).

Proof. We assume that \( \omega \) is not identically zero, hence either one or the other of \( M_\pm := \omega^{-1}(\mathbb{R}^\pm \setminus \{0\}) \) is not empty: without loss of generality we assume \( M_+ \) not empty.

The same reasoning as in the Riemannian case shows that \( \omega \) has the form in the statement of this theorem, where for now \( \Sigma_+ \) is just a geodesically complete pseudo–Riemannian hypersurface of the appropriate type. Then \((M_+, g)\) is isometric to \( \mathbb{R} \times_\alpha \Sigma_+ \), with \( \alpha(t) = \exp(\sqrt{\kappa} t) \) and metric \( ds^2 = -dt^2 + \alpha^2(t)ds_{\Sigma_+}^2 \). Such a warping function falls into the class of functions considered in Lemma 2.2, hence \((M_+, g)\) is not geodesically complete and \( M_+ \) is a proper subset of \( M \).

We now prove that \( M_- \) is nonempty as well and that \( \Omega \) never vanishes so that each connected component of \( \Sigma_0 := \omega^{-1}(0) \) is a smooth null hypersurface.

Indeed, let \( p \) be any point of \( \Sigma_0 \) and consider any geodesic \( \gamma(s) \) starting from inside \( M_+ \) and arriving at \( p \) at \( s = s_0 \). Then it is possible to compute explicitly \( \omega(\gamma(s)) \) for \( s < s_0 \) using Eq. (2.10) and prove that

\[
d\omega(p)(\gamma(s_0)) = \lim_{s \to s_0} \frac{d}{ds} \omega(\gamma(s)) \neq 0.
\]

(4.34)

This implies that:

i) \( d\omega \) does not vanish at any \( p \in \Sigma_0 = \omega^{-1}(0) \) and thus

ii) \( M_- \) is nonempty.

Therefore \( M = M_+ \cup \Sigma_0 \cup M_- \), \( \Sigma_0 \) being a smooth light–like hypersurface; smoothness is
guaranteed by the nonvanishing of $d\omega$.

The other half $M_-$ is also isometric to a similar warped product $M_- \simeq \mathbb{R} \times_\alpha \Sigma_-$ with the same $\alpha(t)$: the boundary $\Sigma_0$ is at $t = -\infty$ for both. We now prove that $\Sigma_\pm$ must be asymptotically flat in the spacelike direction: this follows from the requirement that the sectional curvature of $M$ does not blow up as we follow a geodesic which crosses the boundary of $M_+$ and from the fact that any geodesic $\gamma(s)$ which crosses the boundary projects onto a spacelike pregeodesic $\phi(s)$ in $\Sigma_\pm$.

Indeed, from Lemma (2.2) we see that in the case $\alpha(x(s)) = \exp(\sqrt{\kappa} x(s)) = O(s - s_0)$ (when the geodesic is unbounded towards $t = -\infty$) which implies that the length of the pregeodesic $\phi(s)$ grows as $1/(s - s_0)^2$.

Considering the expression of the sectional curvature of $M_\pm = \mathbb{R} \times_\alpha \Sigma_\pm$

$$K_{U\bar{U}}^{M_\pm} = \frac{K_{U\bar{U}}^{\Sigma_\pm} + (\alpha'(t))^2}{\alpha^2(t)} = \frac{K_{U\bar{U}}^{\Sigma_\pm} + \kappa e^{2\sqrt{\kappa}t}}{e^{2\sqrt{\kappa}t}} = K_{U\bar{U}}^{\Sigma_\pm} O \left( \frac{1}{(s - s_0)^2} \right) + \kappa ,$$

we see that $K_{U\bar{U}}^{\Sigma_\pm}, (U, U' \in T_{\phi(s)} \Sigma_\pm)$, must be infinitesimal w.r.t. $(s - s_0)^2 = O(\sigma^{-1})$ (now $\sigma$ is the length parameter of the pregeodesic $\phi$ in $\Sigma_\pm$), namely

$$K_{U\bar{U}}^{\Sigma_\pm} = O \left( \frac{1}{\sigma} \right)$$

(4-36)

along the pregeodesic $\phi(s) \subset \Sigma_\pm$.

Finally, we prove that boundedness of the total sectional curvature $K$ implies that it is constant. Indeed, using again eq. (4-35) on the horizontal geodesic generated by $\Omega$ (which projects to a constant in $\Sigma_\pm$) and then sending $t \to -\infty$, we see that $K_{U\bar{U}}^{\Sigma_\pm}$ must identically vanish in $\Sigma_\pm$ and hence they are flat manifolds. In this case then, the sectional curvature of $(M, g)$ is constant and equal to $\kappa$.  

In the next proposition we seek to show that there exist examples in which the manifolds $\Sigma_\pm$ are not flat but only asymptotically flat in the spatial directions.

Notice the sign + in front of $\alpha'$ which comes from the negative signature of $-dt^2$.  

\[4\text{Notice the sign + in front of } \alpha' \text{ which comes from the negative signature of } -dt^2\]
An example for $k = 1, h = 0$ on the quadric $-t^2 + x^2 + y^2 = 1$ with $\omega = t - y$: this is not the paradigmatic case as in general the leaves $\Sigma_{\pm}$ must only be asymptotically flat.

**Proposition 4.1** Let $\Sigma_{\pm}$ be two geodesically connected, semi–Riemannian manifolds of type $(r - 1, p)$ diffeomorphic to $\mathbb{R}^r$ and such that along any spacelike geodesic the sectional curvature vanishes faster than any power of $\frac{1}{r}$, $\sigma$ being the geodesic distance. Set $\alpha(t) = \exp(\sqrt{\omega}t)$ and form the two warped products $(M_{\pm}, g_{\pm})$ as in
\[ ds^2_{\pm} := -dt^2 + \alpha^2(t)d\sigma_{\Sigma_{\pm}}^2. \quad (4-37) \]

Then we can smoothly glue them in a geodesically complete manifold $(M, g)$ along a null hypersurface $\Sigma_0$: this hypersurface is connected except in the case of signature $(r, 1)$, where it is constituted of two connected components.

On this manifold the function defined by $\omega(t, \sigma)|_{M_{\pm}} = \pm \frac{1}{\sqrt{\alpha(t)}}\alpha(t)$, $\omega|_{\Sigma_0} \equiv 0$ is smooth and satisfies $H^\omega = -\omega g$, $\|\Omega\|^2 = -\omega^2$.

**Proof.** We define the geodesic boundary of $M_{\pm}$ and extend there the metric by means of the exponential map. We work on $M_+$, the arguments being identical for $M_-$. Fix an arbitrary point $p_0 \equiv (t_0, \sigma_0) \in M_+$ and consider the exponential map $\exp$ at this point. We saw in Lemma 22 that there exist inextensible incomplete geodesics $\gamma(s)$ of any type in $M_+$ starting from $p_0 \equiv (t_0, \sigma_0)$ and that in all these cases the projection of $\gamma$ on $\Sigma_{\pm}$ is a spacelike pregeodesic. Therefore $\exp : D_{p_0} \subset T_{p_0}M_+ \to M_+$ is defined on a proper star-shaped subset $D_{p_0}$ of the tangent space. In view of the discussion of the properties of incomplete geodesics, the boundary $\partial D_{p_0}$ of $D_{p_0}$ is made of three pieces, according to the type of the incomplete geodesic; writing $T_{p_0}M_+ \ni X = \pi_+ X + \sigma_+ X = v + V$, we have the boundary corresponding to the incomplete spacelike geodesics
\[
\mathcal{G} := \{ \mathcal{X} = (v, V) \in T_{p_0}M_+ \simeq \mathbb{R} \oplus T_{\sigma_+}\Sigma_+ : \]
\[-v^2 + \|V\|^2 = d_+^2(X) > 0, v \neq 0 \}, \quad (4-38)
\]
to the incomplete timelike geodesics
\[
\mathcal{T} := \{ \mathcal{X} = (v, V) \in T_{p_0}M_+ \simeq \mathbb{R} \oplus T_{\sigma_+}\Sigma_+ : \|V\|^2 > 0, \]
\[-v^2 + \|V\|^2 = -d_-^2(X) < 0, v < 0 \}, \quad (4-39)
\]
and to the incomplete lightlike geodesics
\[
\mathcal{L} := \{ \mathcal{X} = (v, V) \in T_{p_0}M_+ \simeq \mathbb{R} \oplus T_{\sigma_+}\Sigma_+ ; \]
\[\|V\|^2 = d_0^2(X) > 0, v^2 = d_0^2(X) \} . \quad (4-40)
\]
In these formulas the three functions $d_+, d_-, d_0$ are the upper extrema of the maximal interval of definition of the geodesics and will explicitly computed in equations (4-41, 4-42, 4-43). By their definition $d_+$ and $d_-$ are homogeneous invariant functions of their arguments because they depend only on the “direction” $X/\sqrt{\|X\|^2}$. On the contrary $d_0$ is homogeneous of degree $-1$ because a null geodesic with initial tangent vector $\lambda X$, $\lambda \in \mathbb{R}_+$ and $\|X\|^2 = 0$ is defined on the interval $(0, \frac{1}{\lambda} d_0(X))$. 

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Consider the set–theoretical union \( \overline{M}_{+} := M_{+} \cup \partial D_{p_{0}} \); we presently define a smooth structure of manifold–with–boundary on it. Indeed let \( U \) be a generic open neighborhood of a point \( p \) belonging to \( \partial D_{p_{0}} \subset T_{p_{0}}M_{+} \); the corresponding open neighborhood of \( p \) in \( \overline{M}_{+} \) will be \((U \cap \partial D_{p_{0}}) \cup \exp(U \cap D_{p_{0}})\). Since \( \exp \) restricted to \( U \cap D_{p_{0}} \) is a local diffeomorphism, we have thus defined a smooth structure of manifold–with–boundary on the set \( M_{+} \cup \partial D_{p_{0}} \).

The boundary of \( M_{+} \) as appears via the exponential map in \( T_{p_{0}}M_{+} \) here depicted for the special value \( \kappa = 1 \): in abscissa appears the square root of the projection \( V = \sigma_{*}X \), which is spacelike for all incomplete geodesics. The curve is obtained solving the implicit equations \( \|X\|^2 = -d_{t}^{2}(X) \) and \( \|X\|^2 = d_{s}^{2}(X) \), where \( d_{s,t} \) are given in the text.

The definition of the boundary seems to rely on the choice of the base–point \( p_{0} \); we now prove that this is not the case. Indeed we only need to show that the three functions \( d_{t}, d_{l}, d_{s} \) do not depend on the point: but this is obvious because, integrating eq. (2-10) with \( \alpha(t) = \exp(\sqrt{\kappa}t) \), we obtain

\[
\begin{align*}
  d_{l}(X) &= \frac{1}{\alpha(t_0)v} \int_{-\infty}^{t_0} \alpha(t)dt = \frac{1}{\sqrt{\kappa}v} \\
  d_{t}(X) &= \frac{\alpha(t)dt}{\sqrt{\alpha^2(t_0)(t_0^2 - 1) + \alpha^2(t)}} = \frac{1}{\sqrt{\kappa}} \ln \left( \frac{t_0 + 1}{t_0 - 1} \right) \\
  d_{s}(X) &= \frac{\alpha(t)dt}{\sqrt{\alpha^2(t_0)(t_0^2 + 1) - \alpha^2(t)}} = \frac{1}{\sqrt{\kappa}} \arcsin \left( \frac{1}{\sqrt{t_0^2 + 1}} \right),
\end{align*}
\]

where \( t_0 = \sqrt{\frac{v^2}{1 - v^2}} \). We can thus remove the subscript \( p_{0} \) and set \( \overline{M}_{+} := \partial D \cup M_{+} =: \Sigma_{0} \cup M_{+} \), where \( \Sigma_{0} \) is \( \partial D \) when thought in \( \overline{M}_{+} \).

As for the topology of the boundary, if the set \( ||V||^2 > 0 \) is disconnected, then so is \( \Sigma_{0} \); this happens only if the signature of the metric is \((r,1)\) in which case there are two connected
components.
We perform a similar construction in $M_-$ and identify the two copies of the boundary. We therefore obtain a smooth manifold $M = M_- \cup \Sigma_0 \cup M_+$. We must now define the metric on $TM|_{\Sigma_0}$: to this purpose we consider the canonical realization of the space of constant curvature $\kappa$ as a quadric in a suitable semi-Riemannian flat manifold. Then we can identify $\Sigma_0$ with the intersection of this quadric with a null hyperplane, and define the metric on $\Sigma_0$ as the pull-back of the metric on the quadric. Since the sectional curvature on $M$ (which is insofar defined only on the complement of $\Sigma_0$) tends to a constant on $\Sigma_0$ with all its derivatives tending to zero (this follows from the assumptions on the asymptotic flatness of $\Sigma_{\pm}$), we have that the metric is smoothly defined also on $\Sigma_0$. To conclude the proof we only have to show that $\omega$ satisfies Obata’s equation: but clearly it satisfies the equation on $M_{\pm}$ and then by continuity on the whole $M$.

We conclude with

**Theorem 4.4** Under the common assumptions, let now $\kappa = 0$ then

i) if $h > 0$ then $\Omega$ is everywhere spacelike and $(M, g)$ is globally isometric to a direct product $\mathbb{R} \times \Sigma$ with

$$\omega(x) = \sqrt{h} t; \quad ds^2 = dt^2 + h ds_\Sigma^2 \quad x = (t, \sigma) \in \mathbb{R} \times \Sigma$$

(4-44)

and $\Sigma$ is any geodesically complete pseudo–Riemannian manifold of type $(r, p - 1)$.

ii) if $h < 0$ then $\Omega$ is everywhere timelike and $(M, g)$ is globally isometric to a direct product $\mathbb{R} \times \Sigma$ with

$$\omega(x) = \sqrt{|h|} t; \quad ds^2 = -dt^2 + |h| ds_\Sigma^2 \quad x = (t, \sigma) \in \mathbb{R} \times \Sigma$$

(4-45)

and $\Sigma$ is any geodesically complete pseudo–Riemannian manifold of type $(r - 1, p)$.

iii) if $h = 0$ then, apart from the constant solution, $\Omega$ is a null Killing vector of $(M, g)$ and the level surfaces of $\omega$ are all totally geodesic.

**Proof.**

**Cases i), ii).** Each level surface is non–singular: $(M, g)$ is globally isometric to $\mathbb{R} \times \Sigma$ (with the appropriate types) as a direct product ($\alpha$ is now a constant). Such a manifold is clearly geodesically complete iff $\Sigma$ is complete: no further requirement is needed on $\Sigma_0$ coming from geodesic completeness or smoothness.

**Case iii).** There is the obvious solution $\omega = \text{const}$ which is trivial and implies no requirements whatsoever on the pseudo–Riemannian structure of $(M, g)$.

Let us consider a non-constant solution. The equation $H_\omega \equiv 0 \leftrightarrow \nabla \Omega = 0$ proves that $\Omega$ is parallel and hence it is a never vanishing Killing vector.

Moreover each level surface is totally geodesic: indeed, if $\varphi(s)$ is a geodesic starting at a point of $\Sigma_\varphi$ with initial velocity tangent to $\Sigma_\varphi$ then we have

$$\frac{d^2}{ds^2} \omega(\varphi(s)) = \frac{d}{ds} < \Omega_{\varphi(s)}, \dot{\varphi}(s) > = < \nabla_{\dot{\varphi}} \Omega, \dot{\varphi} > + < \Omega, \nabla_{\dot{\varphi}} \dot{\varphi} > = 0 .$$

(4-46)
Hence $\omega(\varphi(s)) = \omega(\varphi(0)) + A s$, but $A$ must vanish because $A = <\Omega, \dot{\varphi}(0)> = 0$.

A typical example of Case (iii) of Thm. 4.4 is the following: take $\mathbb{R}^{n+2}$ with coordinates $\omega, \xi, \eta_1, \ldots, \eta_n$ and consider the metric

$$ds^2 = A d\omega^2 + d\omega d\xi + d\omega \sum_{j \geq 1} B_j d\eta_j + \sum_{i, j \geq 1} C_{ij} d\eta_i d\eta_j,$$

(4-47)

where $A, B_i, C_{ij}$ are arbitrary functions independent of $\xi$ (with the only requirement that the metric be nondegenerate). Then $\Omega = \frac{\partial}{\partial \xi}$ and it is a null vector; the condition that $\Omega$ is a Killing vector is ensured by the fact that the coefficients do not depend on $\xi$.

Thus there are not stringent rigidification in this case as well.

5 Foliation associated to Obata’s equation

We now study the implications of the existence of more than one solution: the metric will be pseudo-Riemannian unless otherwise stated.

Note that for any Obata’s function $\omega$ we have $K_{\Omega, X} = \kappa$. Moreover there cannot exist two Obata’s functions corresponding to different values of the constant $\kappa$ as we show in the next simple lemma.

Lemma 5.1 If $\omega_1, \omega_2$ satisfy $H\omega_i = -\kappa_i \omega_i g$, $i = 1, 2$ for some constants $\kappa_i$, then $k_1 = k_2$.

Proof. This follows from $K_{\Omega_1, \Omega_2} = \kappa_1 = \kappa_2$.

We introduce the following natural definition.

Definition 5.1 A maximal system of Obata’s functions is a $m$-tuple \{ $\omega_1, \ldots, \omega_m$ \} of solutions such that their gradients are almost everywhere linearly independent.

The integer $m$ is a pseudo–Riemannian invariant of the manifold $(M, g)$. As we saw in the proof of Thms. 4.1, 4.2, 4.3, 4.4 if a Obata’s function has any critical point, then the manifold must be of constant sectional curvature: this happens whenever $\kappa_i > 0$ and under hypotheses of geodesic completeness of the manifold. Then we can show that there exist $n + 1$ nontrivial solutions of the equation which are obtained by realizing the manifold in the canonical way as a suitable quadric in a flat semi-Riemannian manifold $\mathbb{R}^{n+1}$ and hence using the linear coordinates of $\mathbb{R}^{n+1}$: one of these solution is functionally but not linearly dependent on the remaining, which then constitute a maximal system in the sense above.

In view of this remark we have the

Theorem 5.1 The manifold $(M, g)$ has constant sectional curvature if and only if there exists a maximal system of solutions with $m = \dim(M)$.

We now study the intermediate cases $0 < m < \dim(M)$; if we assume geodesic completeness then we are addressing only the cases $\kappa_i \leq 0$, $\forall i = 1..m$, otherwise the analysis will only be local.

We promptly have

Proposition 5.1 The manifold $(M, g)$ is foliated by submanifolds $S^m$ of dimension $m$ with constant sectional curvature $\kappa$: these foliations are totally geodesic.
Proof. We use Frobenius’s Theorem, showing that $\Omega_i$ form an involutive distribution. Indeed
\[
[\Omega_i, \Omega_j] = \nabla_{\Omega_i} \Omega_j - \nabla_{\Omega_j} \Omega_i = -\kappa \omega_j \Omega_i + \kappa \omega_i \Omega_j .
\] (5-48)
If we have a geodesic with initial vector $H(p_0) = \sum_{i=1}^m h^i \Omega_i(p_0)$ then clearly the vector field $H = \sum_{i=1}^m h^i \Omega_i$ generates this geodesic and hence the distribution is also totally geodesic: this proves that the intrinsic sectional curvature $K^S_m$ equals the sectional curvature of $M$ and hence
\[
K^S_m = K^M = \kappa ,
\] (5-49)
which ends the proof. ■

Before studying the foliation we anticipate the

Lemma 5.2 If the dimension $m$ of a maximal system is strictly less than the dimension of $M$, then any other solution is a linear combination of the basis $\omega_1, ..., \omega_m$.

Proof. Let $\omega$ be another solution, then from the assumption of maximality $\Omega = a^i \Omega_i$ for some functions $a^i$: we are to prove that $a^i$ must be constants. Indeed
\[
-\kappa \omega X = \nabla_X \omega = \langle X, A^i \rangle \Omega_i + a^i \nabla_X \Omega_i = \langle X, A^i \rangle \Omega_i - \kappa a^i \omega_i X .
\] (5-50)
This implies that
\[
\kappa(a^i \omega_i - \omega) X = \langle X, A^i \rangle \Omega_i .
\] (5-51)
Taking an arbitrary vector field $X$ not belonging to the span of the $\Omega_i$’s we must have $\omega = a^i \omega_i$, so that now $\langle X, A^i \rangle \Omega_i \equiv 0$ and hence, being $\Omega_i$ independent, $A^i \equiv 0$, namely $a^i$ are constants. ■

This lemma shows that if the maximal system is not total (i.e., $m < n$) then we lose one solution which should be functionally but not linearly dependent on the other.

We remark that

Proposition 5.2 The following formula holds
\[
\langle \Omega_i, \Omega_j \rangle = -\kappa \omega_i \omega_j + c_{ij} ,
\] (5-52)
for some constants $c_{ij}$.

Proof. We have
\[
X < \Omega_i, \Omega_j >= \langle \nabla_X \Omega_i, \Omega_j \rangle + \langle \nabla_X \Omega_j, \Omega_i \rangle =
-\kappa \omega_j < X, \Omega_j > -\kappa \omega_i < X, \Omega_i > = -kX(\omega_i \omega_j) ,
\] (5-53)

hence $\langle \Omega_i, \Omega_j \rangle + \kappa \omega_i \omega_j = c_{ij}$ is constant.

Moreover we can assume that the matrix $c_{ij}$ is diagonal up to a linear change of basis $\tilde{\omega}_i$. ■

We finally introduce the complementary foliation $F$ to the distribution spanned by $\{\Omega_i\}_{i=1,m}$. Clearly the fibers $\Sigma_{p_0}$ of $F$ are the joint level sets of $\{\omega_1, ..., \omega_m\}$, namely $\Sigma_{p_0} = \bigcap_{i=1}^m \omega_i^{-1}(\omega(p_0)) = \bigcap_{i=1}^m \Sigma_i^{p_0}$ whose second fundamental form is given by
\[
\alpha(X,Y) = -\sum_{i=1}^m \frac{H^{\omega_i}(X,Y)}{||\Omega_i||^2} \Omega_i = -\kappa < X, Y > \sum_{i=1}^m \left(\frac{\omega_i}{||\Omega_i||^2} \Omega_i\right) ,
\] (5-54)
for any $X, Y \in T\Sigma_{p_0}$. Eq. (5-54) that $\Sigma_{p_0}$ is totally umbilical. 

We now restrict to the Riemannian case and, referring to the classical definitions in [Mo88], we have

**Theorem 5.2** $\mathcal{F}$ is a Riemannian foliation and $g$ is bundle-like.

**Proof.** Consider the connection $\tilde{\nabla}$ of the normal bundle $\Gamma(T\Sigma^\perp)$ defined by

$$\begin{align*}
\tilde{\nabla}_X Z &= \left\{ \begin{array}{ll}
\pi[X,Z], & X \in \Gamma(T\Sigma) \\
\pi(\nabla_X Z), & X \in \Gamma(T\Sigma^\perp),
\end{array} \right.
\end{align*}
$$

where $Z \in \Gamma(T\Sigma^\perp)$ is a normal section and $\pi : TM \to T\Sigma^\perp$ is the natural projection. A classical result of [To] ensures that $\tilde{\nabla}$ is metric if and only if $\mathcal{F}$ is Riemannian and $g$ is bundle-like.

Writing $Z = a^i\Omega_i$ and letting $A^i = \text{grad}(a^i)$, we have for $X \in \Gamma(T\Sigma)$

$$\begin{align*}
[X, Z] &= \langle X, A^i \rangle \Omega_i + a^i[X, \Omega_i] = \\
&= \langle X, A^i \rangle \Omega_i + a^i(\nabla_X \Omega_i - \nabla_{\Omega_i} X) = \\
&= \langle X, A^i \rangle \Omega_i - a^i(\kappa \omega_i X + \nabla_{\Omega_i} X). 
\end{align*}
$$

On the other hand

$$\begin{align*}
\langle \nabla_{\Omega_i} X, \Omega_j \rangle &= -\langle X, \nabla_{\Omega_i} \Omega_j \rangle = \kappa \omega_j \langle X, \Omega_i \rangle = 0,
\end{align*}
$$

from which we obtain

$$\begin{align*}
\nabla_X Z &= \pi[X,Z] = \langle X, A^i \rangle \Omega_i.
\end{align*}
$$

We not take $X \in \Gamma(T\Sigma^\perp)$ and compute

$$\begin{align*}
\nabla_X Z &= \langle X, A^i \rangle \Omega_i + a^i\nabla_X \Omega_i = \langle X, A^i \rangle \Omega_i - \kappa a^i \omega_i X,
\end{align*}
$$

from which we obtain

$$\begin{align*}
\tilde{\nabla}_X Z &= \pi(\nabla_X Z) = \nabla_X Z.
\end{align*}
$$

Summarizing the expression for the connection we have

$$\begin{align*}
\tilde{\nabla}_X Z &= \left\{ \begin{array}{ll}
\langle X, A^i \rangle \Omega_i, & X \in T\Sigma \\
\nabla_X Z, & X \in T\Sigma^\perp
\end{array} \right.
\end{align*}
$$

The proof that $\tilde{\nabla}$ is a metric connection is a straightforward computation of $X\|Z\|^2$ and $2\langle \nabla_X Z, Z \rangle$ and subsequent comparison.

We conclude with a description of the infinitesimal automorphisms of $\mathcal{F}$ and $\mathcal{F}^\perp \equiv \mathcal{S}$ (the distribution spanned by $\Omega_1, \ldots, \Omega_m$).

For $X \in \Gamma(\mathcal{F})$ and $Z \in \Gamma(\mathcal{S})$ we obtain

$$[Z, X] = a^i\nabla_{\Omega_i} X - \langle X, A^i \rangle \Omega_i + \kappa a^i \omega_i X.
$$

This shows that $Z$ is an infinitesimal automorphism of $\mathcal{F}$ (namely $[Z, X] \in \mathcal{F}$) iff $\langle X, A^i \rangle = 0$. Thus

**Proposition 5.3** The infinitesimal automorphisms of $\mathcal{F}$ are the sections of $\Gamma(\mathcal{F})$ and combinations $a^i\Omega_i$ whose coefficients are constant on the leaves of $\mathcal{F}$.

Finally, $X$ is an infinitesimal automorphism of $\mathcal{F}^\perp = \mathcal{S}$ iff $\nabla_{\Omega_i} X + \kappa \omega_j X = 0$, namely iff $[X, \Omega_i] = 0$.

**Proposition 5.4** The infinitesimal automorphisms of $\mathcal{S}$ are the sections of $\Gamma(\mathcal{S})$ and all sections of $\mathcal{F}$ which commute with every $\Omega_i$. 

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