Reducing the detection of genuine entanglement of \( n \) qubits to two qubits

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Abstract
We propose a criterion for the detection of genuine entanglement of pure multiqubit states. To this aim, we define an operator called the losing one qubit operator, which is different from the reduced density operator. The states obtained from a multiqubit state by applying the losing one qubit operator are referred to as its projected states. We show that all of the projected states of a pure product \( n \)-qubit state are pure product states provided that it cannot be written as a product of a single qubit state and a genuinely entangled \((n - 1)\)-qubit state. We also show that a pure \( n \)-qubit state is genuinely entangled provided that the state has at least two genuinely entangled \((n - 1)\)-qubit projected states. By repeating the losing process, we reduce the detection of entanglement of pure \( n \)-qubit states to the one of pure two-qubit states. Also we write a LISP program for the reduction process.

Keywords  Product states · Genuinely entangled states · Maximally entangled states · Entanglement measure

1 Introduction
Quantum entanglement is considered as a unique quantum mechanical resource [1]. It is well known that entanglement takes a key rule in quantum information processing tasks, for e.g., quantum teleportation, quantum cryptography and quantum key distribution.

It is known that for GHZ of three qubits, tracing out of qubit i, the reduced density operator becomes completely unentangled, while for W of three qubits, tracing out of qubit i the reduced density operator remains entangled [2]. It is indicated that many
physical implementations of qubits, for example ion traps, optical lattices and linear optics, suffer from loss of qubits [3]. The entanglement resistant to particles loss via tracing out the particles was investigated [4].

Many efforts have been devoted to exploring criteria for detection of quantum entanglement [5–16]. A necessary condition for separability of a quantum system consisting of two subsystems is that a matrix, obtained by partial transposition of $\rho$, has only non-negative eigenvalues [5]. The criterion is referred to as PPT (Positive partial transpose). For $2 \times 2$ and $2 \times 3$ systems, the positivity of the partial transposition of a state is necessary and sufficient for its separability [6].

It was shown that for any separable state of a bipartite system, the sum of the singular values of the realigned matrix constructed from the density matrix is necessarily not greater than 1 [7]. The generalized reduction criterion for the separability of bipartite system in arbitrary dimensions was proposed [8]. The cross norm necessary criterion for the separability of density matrices for bipartite systems was studied [9].

Recently, Zwerger et al. showed that genuine entanglement of all multipartite pure states can be detected in a device-independent way via bipartite Bell inequalities [10]. The genuine $n$-qubit entanglement can be detected via the proportionality of two coefficient vectors [11].

In this paper, we propose the losing one qubit operator. Applying the losing one qubit operator to a state of $n$ qubits, we can obtain $n$ states of $(n-1)$ qubits which are referred to as the projected states. We demonstrate that the losing one qubit operator can reduce the detection of genuine entanglement of $n$ qubits to two qubits. In Sect. 2, we give necessary and sufficient conditions for two, three, and four qubits to be product states. In Sect. 3, we show that all of the projected states of an $n$-qubit product state are product states provided that the $n$-qubit product state cannot be written as a product of a single qubit state and a genuinely entangled $(n-1)$-qubit state. Thus, the losing one qubit operator can reduce the detection of the entanglement of $n$-qubit states to $(n-1)$-qubit ones and finally to two-qubit ones.

2 The necessary and sufficient conditions for product states of two, three, and four qubits

For given two vectors $v_1$ and $v_2$, if $v_1 = kv_2$, where $k$ is a complex number, then we said that $v_1$ is proportional to $v_2$. Specially, when $v_1 = 0$, then $k = 0$. For a set of vectors $v_1, v_2, \cdots, v_m$, if there is a vector $v_i \neq 0$ of the set of vectors such that any vector $v_\ell$ of the set of vectors is proportional to $v_i$, then we said that the set of vectors are proportional. Clearly, if at least two non-zero vectors of the set of the vectors are not proportional, then, the set of the vectors are not proportional.

2.1 Two qubits

For two qubits, we can write any pure state of two qubits as $|\psi\rangle_{12} = \sum_{i=0}^{3} c_i |i\rangle$. 
Result 1 It is well known that $|\psi\rangle_{12}$ is a product state iff the following two vectors are proportional [11]

$$
( c_0 \ c_1 )^T, ( c_2 \ c_3 )^T.
$$

(1)

Note that the above two vectors are proportional iff the following equality holds.

$$
c_0c_3 = c_1c_2.
$$

(2)

Note also that $|c_0c_3 - c_1c_2|^2$ is just the concurrence of $|\psi\rangle_{12}$.

2.2 Three qubits

Result 2 Let $|\psi\rangle_{123} = \sum_{i=0}^{7} c_i |i\rangle$ be any pure state of three qubits. Then, $|\psi\rangle_{123}$ is a product state iff at least one of the following three pairs of vectors is proportional. Otherwise, it is genuinely entangled [11].

$$
( c_0 \ c_1 \ c_2 )^T, ( c_4 \ c_5 \ c_6 )^T, ( c_7 )^T.
$$

(3)

$$
( c_0 \ c_1 \ c_4 )^T, ( c_2 \ c_3 )^T, ( c_6 )^T.
$$

(4)

$$
( c_0 \ c_2 )^T, ( c_1 \ c_3 )^T, ( c_5 )^T.
$$

(5)

2.3 Four qubits

Result 3 Let $|\psi\rangle_{1234} = \sum_{i=0}^{15} c_i |i\rangle$. A pure product state $|\psi\rangle$ of four qubits can be written as (1). $|\varphi\rangle_1|\phi\rangle_{234}$, (2). $|\varphi\rangle_2|\phi\rangle_{134}$, (3). $|\varphi\rangle_3|\phi\rangle_{124}$, (4). $|\varphi\rangle_4|\phi\rangle_{123}$, (5). $|\varphi\rangle_{12}|\phi\rangle_{34}$, (6). $|\varphi\rangle_{13}|\phi\rangle_{24}$, or (7). $|\varphi\rangle_{14}|\phi\rangle_{23}$ iff the following corresponding set of vectors are proportional.

$$
(1). ( c_0 \ c_1 \ c_2 \ c_3 \ c_4 \ c_5 \ c_6 \ c_7 )^T, \\
( c_8 \ c_9 \ c_10 \ c_11 \ c_12 \ c_13 \ c_14 \ c_15 )^T;
$$

(6)

$$
(2). ( c_0 \ c_1 \ c_2 \ c_3 \ c_8 \ c_9 \ c_10 \ c_11 )^T, \\
( c_4 \ c_5 \ c_6 \ c_7 \ c_12 \ c_13 \ c_14 \ c_15 )^T;
$$

(7)

$$
(3). ( c_0 \ c_1 \ c_4 \ c_5 \ c_8 \ c_9 \ c_12 \ c_13 )^T, \\
( c_2 \ c_3 \ c_6 \ c_7 \ c_10 \ c_11 \ c_14 \ c_15 )^T;
$$

(8)

$$
(4). ( c_0 \ c_2 \ c_4 \ c_6 \ c_8 \ c_10 \ c_12 \ c_14 )^T, \\
( c_1 \ c_3 \ c_5 \ c_7 \ c_9 \ c_11 \ c_13 \ c_15 )^T;
$$

(9)

$$
(5). ( c_0 \ c_1 \ c_2 \ c_3 )^T, ( c_4 \ c_5 \ c_6 \ c_7 )^T, \\
( c_8 \ c_9 \ c_10 \ c_11 )^T, ( c_12 \ c_13 \ c_14 \ c_15 )^T;
$$

(10)
We consider a product state of three qubits, for e.g.,
\[
\left( \begin{array}{c} c_0 c_1 c_4 c_5 \\ c_2 c_3 c_6 c_7 \end{array} \right)^T, \left( \begin{array}{c} c_8 c_9 c_{12} c_{13} \\ c_{10} c_{11} c_{14} c_{15} \end{array} \right)^T; \quad (11)
\]
\[
\left( \begin{array}{c} c_0 c_1 c_8 c_9 \\ c_2 c_3 c_{10} c_{11} \end{array} \right)^T, \left( \begin{array}{c} c_4 c_5 c_{12} c_{13} \\ c_6 c_7 c_{14} c_{15} \end{array} \right)^T. \quad (12)
\]

**Proof** We only prove Case (1): a pure product state \(|\psi\rangle = \sum_{i=0}^{15} c_i |i\rangle\) of four qubits can be written as \(|\varphi\rangle|\phi\rangle_{234}\) iff the the set of vectors in Eq. (6) are proportional.

\((\Longrightarrow). Let |\psi\rangle = |\varphi\rangle|\phi\rangle_{234}, where |\varphi\rangle = (\alpha|0\rangle_1 + \beta|1\rangle_1) and |\phi\rangle_{234} = \sum_{i=0}^{234} a_i |i\rangle_{234}. Clearly, |\psi\rangle = \alpha \sum_{i=0}^{234} a_i |0\rangle_1 |i\rangle_{234} + \beta \sum_{i=0}^{234} a_i |1\rangle_1 |i\rangle_{234}. Let v_1 be the first vector and v_2 be the second vector in Eq. (6). Then,

\[
\begin{align*}
v_1 &= (a a_0 \ldots a_7) \rangle \\
v_2 &= (\beta a_0 \ldots a_7) \rangle 
\end{align*}
\]

Thus, one can see that v_1 and v_2 are proportional.

\((\Longleftarrow). Conversely, if v_1 and v_2 are proportional, then we can write v_2 = kv_1 or v_1 = kv_2. Assume that v_2 = kv_1 and v_1 = \( (c_0 c_1 \ldots c_7) \rangle \). Then, via v_1 and v_2 we can write

\[
|\psi\rangle = \sum_{i=0}^{234} c_i |0\rangle_1 |i\rangle_{234} + k \sum_{i=0}^{234} c_i |1\rangle_1 |i\rangle_{234}
\]

\[
= |\varphi\rangle |\phi\rangle_{234},
\]

where |\varphi\rangle = (|0\rangle_1 + \beta |1\rangle_1) and |\phi\rangle_{234} = \sum_{i=0}^{234} c_i |i\rangle_{234}.

Thus, to determine that a state of four qubits is genuinely entangled, we need to determine that each of above seven sets of vectors are not proportional. We investigated the separability of four qubits by using permutations of qubits [11].

### 3 Losing one qubit operator

#### 3.1 Losing one qubit operator for three qubits

First we use the following example to demonstrate the losing one qubit operator. We consider a product state of three qubits, for e.g., \(|\psi\rangle_{123} = |\phi\rangle_2 |\varphi\rangle_3\), where

\[
|\phi\rangle_2 = (\alpha|0\rangle_2 + \beta |1\rangle_2 \quad \text{and} \quad |\varphi\rangle_3 = (a|00\rangle_3 + b|01\rangle_3 + c|10\rangle_3 + d|11\rangle_3).
\]

There are the following cases for losing qubits.

**Losing qubit 1:**

Let \(|\psi\rangle_{123} = \langle 0 |\psi\rangle_{123} + \langle 1 |\psi\rangle_{123}. Then, a calculation yields that
\[ |\psi^*\rangle_{123/1} = (\alpha|0\rangle_2 + \beta|1\rangle_2)((a + c)|0\rangle_3 + (b + d)|1\rangle_3). \]

Note that if the coefficients of \(|\psi^*\rangle_{123/1}\) vanish, then we also call \(|\psi^*\rangle_{123/1}\) a product state. Thus, clearly \(|\psi^*\rangle_{123/1}\) is a product state.

Losing qubit 2:

Let \(|\psi^*\rangle_{123/2} = 2\langle 0|\psi\rangle_{123} + 2\langle 1|\psi\rangle_{123}. \]

Then, a calculation yields that \(|\psi^*\rangle_{123/2} = (\alpha + \beta)|\psi\rangle_{13}. \]

Then, \(|\psi^*\rangle_{123/2}\) is an entangled state iff \(|\psi\rangle_{13}\) is an entangled state whenever \(\alpha + \beta \neq 0\).

Losing qubit 3:

Let \(|\psi^*\rangle_{123/3} = 3\langle 0|\psi\rangle_{123} + 3\langle 1|\psi\rangle_{123}. \]

Then, a calculation yields that \(|\psi^*\rangle_{123/3} = (a + b)(\alpha|00\rangle_12 + \beta|01\rangle_12) + (c + d)(\alpha|10\rangle_12 + \beta|11\rangle_12). \]

It is easy to see that the concurrence vanishes for \(|\psi^*\rangle_{123/3}\). Via Eq. (2), therefore \(|\psi^*\rangle_{123/3}\) is a product state.

So, for a product state of three qubits, there is at most one qubit \(i, i = 1, 2, \) or 3, such that \(|\psi^*\rangle_{123/i}\) is entangled.

A calculation yields the Table 1.

3.2 Reducing the detection of the entanglement of \(n\) qubits to \((n - 1)\) qubits

Definition: Let \(|\psi\rangle_{1...n}\) be a state of \(n(\geq 3)\) qubits and \(|\psi^*\rangle_{1...n/i} = i\langle 0|\psi\rangle_{1...n} + i\langle 1|\psi\rangle_{1...n}. \]

We call the \((n - 1)\) -qubit state \(|\psi^*\rangle_{1...n/i}\) a projected state of \(|\psi\rangle_{1...n}\) obtained by losing qubit \(i.\)

We can conclude the following theorem.

**Theorem 1** If \(|\psi\rangle_{1...n}\) is a product state of \(n\) qubits, then there exists at most one qubit \(i\) such that \(|\psi^*\rangle_{1...n/i}\) is genuinely entangled. That is, for \(j \neq i, |\psi^*\rangle_{1...n/j}\) are product states or zero.

**Proof** It is trivial for \(n = 2.\) For \(n \geq 3\) qubits, the proof is put in Appendix A. \(\square\)

**Corollary 1** Let \(|\psi\rangle_{1...n}\) be a state of \(n\) qubits. If there are at least two qubits \(i\) and \(j\) such that \((n - 1)\) -qubit projected states \(|\psi^*\rangle_{1...n/i}\) and \(|\psi^*\rangle_{1...n/j}\) are genuinely entangled, then the \(n\)-qubit state \(|\psi\rangle_{1...n}\) is genuinely entangled.

We give the following example to show that Corollary 1 is not necessary.

Let \(|\psi\rangle_{123} = |001\rangle + |010\rangle + |100\rangle + |111\rangle\) (belonging to W SLOCC class).

Then, \(|\psi^*\rangle_{123/1} = |\psi^*\rangle_{123/2} = |\psi^*\rangle_{123/3} = (|0\rangle + |1\rangle)(|0\rangle + |1\rangle). \]

Though \(|\psi\rangle_{123}\) is genuinely entangled, \(|\psi^*\rangle_{123/i}, i = 1, 2, 3,\) are product states.

**Corollary 2** If \(|\psi\rangle_{1...n}\) is an \(n\) -qubit product state but not as a product of a single qubit state and an \((n - 1)\) -qubit genuinely entangled state, then the projected state \(|\psi^*\rangle_{1...n/i}\) is a product state for any qubit \(i.\)
Proof} Ref. the proof of Theorem 1. □

From Corollary 2, it is easy to show that the following Corollary 3 is true.

**Corollary 3** If \(|\psi\rangle_{1...n}\), which is not as a product of a single qubit state and an \((n - 1)\)-qubit genuinely entangled state, has a genuinely entangled \((n - 1)\)-qubit projected state, then \(|\psi\rangle_{1...n}\) is genuinely entangled.

Via Corollary 1, for three qubits, we derive a simple sufficient condition for genuinely entangled states in Appendix B.

### 3.3 Reducing the detection of the entanglement of \(n\) qubits to two qubits

For any pure state \(|\psi\rangle_{1...n} = \sum_{i=0}^{2^n-1} a_i |i\rangle\), in light of Corollary 1, if there are at least two \((n - 1)\)-qubit projected states which are genuinely entangled, then the \(n\)-qubit state \(|\psi\rangle_{1...n}\) is genuinely entangled. For the \((n - 1)\)-qubit projected state \(|\psi^*\rangle_{1...n/i}\), in light of Corollary 1, if it has at least two \((n - 2)\)-qubit projected states which are genuinely entangled, then the \((n - 1)\)-qubit state \(|\psi^*\rangle_{1...n/i}\) is genuinely entangled. The process repeats until we get 4 (3, or 2)-qubit projected states. Therefore, detecting genuine entanglement of \(n\) qubits may reduce to four, three, or two qubits.

For four, three and two qubits, we have given the simple necessary and sufficient conditions for detecting genuine entanglement in Result 1, Result 2 and Result 3.

**Example 1** For the \(n\)-qubit W state, all the \((n - 1)\) qubit projected states are of the form \(\sqrt{\frac{n-1}{n}} W_{n-1} + \frac{1}{\sqrt{n}} |0\cdots0\rangle\), where \(W_{n-1}\) is the \((n - 1)\)-qubit W state. The \(k\)-qubit projected states are of the form \(\sqrt{\frac{k}{n}} W_k + \frac{n-k}{\sqrt{n}} |0\cdots0\rangle\). Specially, 2-qubit projected states have the form \(\sqrt{\frac{2}{n}} (|01\rangle + |10\rangle) + \frac{n-2}{\sqrt{n}} |00\rangle\). The latter state has nonzero concurrence and therefore is entangled. Consequently, via Corollary 1, the \(n\)-qubit W state is genuinely entangled.

**Example 2** Let the \(n\)-qubit state \(|\psi\rangle_{1...n} = \alpha |i_1 i_2 \cdots i_n\rangle + \beta |\bar{i}_1 \bar{i}_2 \cdots \bar{i}_n\rangle\), where \(\alpha \beta \neq 0\), \(i_1 = 0\) or 1 and \(\bar{i}_1 = 1 - i_1\). Clearly, after losing one qubit \(k\), we obtain the projected state \(|\psi^*\rangle_{1...n/k} = \alpha |i_1 \cdots i_{(k-1)} i_{(k+1)} \cdots i_n\rangle + \beta |\bar{i}_1 \cdots \bar{i}_{(k-1)} \bar{i}_{(k+1)} \cdots \bar{i}_n\rangle\). We can continue applying the losing one qubit operator to \(|\psi^*\rangle_{1...n/k}\). Finally, we obtain two-qubit projected states \(\alpha |z_1 z_2\rangle + \beta |\bar{z}_1 \bar{z}_2\rangle\). It is easy to know that \(\alpha |z_1 z_2\rangle + \beta |\bar{z}_1 \bar{z}_2\rangle\) is entangled. Therefore \(|\psi\rangle_{1...n}\) is genuinely entangled via Corollary 1. Specially, the \(n\)-qubit GHZ is genuinely entangled.

**Example 3** Let us check that \(|\psi\rangle_{1234} = |0000\rangle + |0111\rangle - |1111\rangle\) is genuinely entangled. A calculation produces that \(|\psi^*\rangle_{1234/1} = |000\rangle\), which is a product state, and \(|\psi^*\rangle_{1234/i} = |000\rangle + |011\rangle - |111\rangle\), \(i = 2, 3, 4\). Next, we show that \(|\psi^*\rangle_{1234/i}, i = 2, 3, 4\) are genuinely entangled. Let \(|\omega\rangle_{123} = |000\rangle + |011\rangle - |111\rangle\). A calculation yields that \(|\omega^*\rangle_{123/1} = |00\rangle\) and \(|\omega^*\rangle_{123/2} = |\omega^*\rangle_{123/3} = |00\rangle + |01\rangle - |11\rangle\), which is an entangled state of two qubits. So, via Corollary 1, the three-qubit states \(|\psi^*\rangle_{1234/i}, i = 2, 3, 4\) are genuinely entangled, and then \(|\psi\rangle_{1234}\) is genuinely entangled.
3.4 Maximally entangled states (MES)

MES can be defined via different ways such as entropy and LOCC. It is well known that the GHZ state can be regarded as the maximally entangled state of three qubits in several aspects.

Let $|\psi\rangle_{1\ldots n}$ be a state of $n$ qubits. If $k$ of all the $(n - 1)$-qubit projected states $|\psi^*\rangle_{1\ldots n/i}$ are genuinely entangled, then we say that $|\psi\rangle_{1\ldots n}$ has the entanglement measure of $k$. If all the $(n - 1)$-qubit projected states are genuinely entangled, then the state is called MES.

We next demonstrate the entanglement measure of some entangled states below.

1. After losing one qubit, the projected states of the $n$-qubit GHZ state are just the $(n - 1)$-qubit GHZ state. It means that losing one qubit operator preserves the GHZ-constructions. Clearly, GHZ is a MES.

2. For the W state of $n$ qubits, all the $(n - 1)$-qubit projected states are $|P\rangle = \frac{1}{\sqrt{n}}W_{n-1} + \frac{1}{\sqrt{n}}|0\ldots0\rangle$, which is not the W-states. It means that losing one qubit operator does not preserve the W-construction. Thus, the W-construction is fragile under losing one qubit operator. From Example 1, $|P\rangle$ is genuinely entangled. Therefore, W is a MES.

3. $|/\Phi_1\rangle_4 = |0001\rangle + |0010\rangle + |1100\rangle + |1111\rangle$ is genuinely entangled [16]. $|/\Phi_4\rangle_{1234/4} = |/\Phi_4\rangle_{1234/3} = |00\rangle + |11\rangle)(|0\rangle + |1\rangle)$ which are product states while $|/\Phi_4\rangle_{1234/2} = |/\Phi^*\rangle_{1234/1} = |001\rangle + |010\rangle + |100\rangle + |111\rangle$ which are genuinely entangled. Thus, $|/\Phi_4\rangle$ has the entanglement measure 2.

3.5 A program for losing one qubit operator

We write a LISP program for the formula for the projected states $|/\psi^*\rangle_{1\ldots n/i}$ and the procedure reducing detection of entanglement of $n$ qubits to two qubits in Appendix C. The program can detect the genuinely entangled states with the measure $\geq 2$. For example, the program can detect the following genuinely entangled states.

1. The states GHZ and W of three qubits
2. For four qubits, GHZ, W, Cluster state, Dicke state $|2, 4\rangle$, and Osterloh’s $|/\Phi_5\rangle$ and $|/\Phi_4\rangle$ states [16].
3. For five qubits, Osterloh’s $|/\Psi_2\rangle$, $|/\Psi_5\rangle$, and $|/\Psi_6\rangle$ states [16].
4. For six qubits, Osterloh’s $|/\Xi_2\rangle$, $|/\Xi_5\rangle$, $|/\Xi_6\rangle$, and $|/\Xi_7\rangle$ states [16].

3.6 Comparing reduced density operator with losing one qubit operator

1. Reduced density operator and losing one qubit operator are different. For the Bell state $|/\text{Bell}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the reduced density is $tr_2(|/\text{Bell}\rangle\langle/\text{Bell}|) = \frac{1}{2}I$. After losing one qubit, $|/\psi^*\rangle_{12/2} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Therefore, losing a qubit operator is different from the reducing density operator.

2. $|/\psi^*\rangle_{1\ldots n/i}$ is a pure state while the reducing density $tr_i(|/\psi\rangle_{1\ldots n}\langle/\psi|)$ may become a mixed state. For example, $tr_2(|/\text{Bell}\rangle\langle/\text{Bell}|) = \frac{1}{2}I$, which is a mixed state.
3. After losing one qubit, the \((n - 1)\)-qubit projected states of the \(n\) -qubit GHZ state are still the \((n - 1)\)-qubit GHZ state while for GHZ of three qubits, tracing out of a qubit, the reduced density operator becomes completely unentangled. It means that the entanglement property of the state GHZ is fragile under tracing out a qubit. Ref. Table 2.

4 Summary

In this paper, we show that all the projected states obtained via the losing one qubit operator are product states for a product state of \(n\) qubits but not as a product of a single qubit state and a genuinely entangled \((n - 1)\) -qubit state. Thus, if there are at least two \((n - 1)\)-qubit projected states which are genuinely entangled, then the state of \(n\) qubit is genuinely entangled. We can repeat the process until we get \(2(3,\text{or} \ 4)\)-qubit projected states for which we have the necessary and sufficient conditions to detect their separability. Thus, the losing one qubit operator can reduce the detection of entanglement for \(n\)-qubits to two qubits. We have written a LISP program for detection of entanglement for \(n\)-qubits.

Appendix A The proof of Theorem 1

The proof of Theorem is follows.

For \(n \geq 3\) qubits, there are three cases.

Case 1. \(|\psi\rangle_{1...n} = |\phi\rangle|\varphi\rangle|\omega\rangle\). Clearly, it is easy to see that \(|\psi^*\rangle_{12...n/i}\) is a product state for any \(i\).

Case 2. \(|\psi\rangle_{1...n} = |\phi\rangle_{q1...q_i}|\varphi\rangle_{qi+1...q_n}, \text{where} \; i \geq 2, \; n - i \geq 2.\)

Clearly, \(|\psi^*\rangle_{1...n/q_k} = |\phi^*\rangle_{q1...q_i/q_k}|\varphi\rangle_{qi+1...q_n}, \text{where} \; 1 \leq k \leq i.\) One can see that \(|\psi^*\rangle_{1...n/q_k}\) is a product state of \((n - 1)\) qubits or zero.

Case 3. \(|\psi\rangle_{1...n} = (\alpha|0\rangle_{q1} + \beta|1\rangle_{q1})|\varphi\rangle_{q2...q_n}, \text{where} \; |\varphi\rangle_{q2...q_n}\) is genuinely entangled. For example, \(|\psi\rangle_{123} = |0\rangle_{1}|\text{EPR}\rangle_{23}, |\psi^*\rangle_{123}/1 = |\text{EPR}\rangle_{23.}\)

Case 3.1. \(|\psi^*\rangle_{12...n/q_1} = (\alpha + \beta)|\varphi\rangle_{q2...q_n}, \text{which is genuinely entangled when} \; \alpha + \beta \neq 0 \text{ or zero when} \; \alpha + \beta = 0.\)

Case 3.2. \(|\varphi\rangle_{q2...q_n} = |0\rangle_{q_k}|\varphi^{(0)}\rangle_{q2...q_n/q_k} + |1\rangle_{q_k}|\varphi^{(1)}\rangle_{q2...q_n/q_k}, k \neq 1.\)

Thus, \(|\psi^*\rangle_{12...n/q_k} = (\alpha|0\rangle_{q1} + \beta|1\rangle_{q1})|\varphi^*\rangle_{q2...q_n/q_k}, \text{which is a product state.}\)

From the above three cases, we can conclude if \(|\psi\rangle_{1...n}\) is a product state of \(n\) qubits, then there exists at most one qubit \(q_i\) such that \(|\psi^*\rangle_{12...n/q_i}\) is genuinely entangled. That is, for other \(q_j, \; |\psi^*\rangle_{12...n/q_j}\) are product states.
Appendix B A simple sufficient condition for genuinely entangled states of three qubits

Via Corollary 1, for three qubits, we derive a simple sufficient condition for genuinely entangled states by losing one qubit operator. Let $|\psi\rangle_{123} = \sum_{i=0}^{7} c_i |i\rangle$ be any pure state of three qubits.

A calculation yields that

$$|\psi^*\rangle_{123/1} = (c_0 + c_4)|00\rangle + (c_1 + c_5)|01\rangle + (c_2 + c_6)|10\rangle + (c_3 + c_7)|11\rangle.$$ 

Via Eq. (2), clearly, $|\psi^*\rangle_{123/1}$ is entangled iff

$$(c_0 + c_4)(c_3 + c_7) \neq (c_1 + c_5)(c_2 + c_6). \tag{16}$$

A calculation yields that

$$|\psi^*\rangle_{123/2} = (c_0 + c_2)|00\rangle + (c_1 + c_3)|01\rangle + (c_4 + c_6)|10\rangle + (c_5 + c_7)|11\rangle.$$ 

Via Eq. (2), $|\psi^*\rangle_{123/2}$ is entangled iff

$$(c_0 + c_2)(c_5 + c_7) \neq (c_1 + c_3)(c_4 + c_6). \tag{17}$$

A calculation yields that

$$|\psi^*\rangle_{123/3} = (c_0 + c_1)|00\rangle + (c_2 + c_3)|01\rangle + (c_4 + c_5)|10\rangle + (c_6 + c_7)|11\rangle.$$ 

Via Eq. (2), $|\psi^*\rangle_{123/3}$ is entangled iff

$$(c_0 + c_1)(c_5 + c_7) \neq (c_2 + c_3)(c_4 + c_5). \tag{18}$$

Via Corollary 1, $|\psi\rangle_{123} = \sum_{i=0}^{7} c_i |i\rangle$ is genuinely entangled if at least two of Eqs. (16, 17, 18) hold.

Appendix C A formula for the projected states

Let $|\psi\rangle_{1...n} = \sum_{i=0}^{2^n-1} a_i |i\rangle$ be any pure state of n qubits. Then, we can write $|\psi\rangle_{1...n} = |0\rangle_k |\psi^{(0)}\rangle_{1...n/k} + |1\rangle_k |\psi^{(1)}\rangle_{1...n/k}$. Therefore, $|\psi^*\rangle_{1...n/k} = |\psi^{(0)}\rangle_{1...n/k} +$
\[|\psi^{(1)}_{1\cdots n/k}\rangle. \text{Denoting } \ell = 2^{n-k}, \text{then }|\psi^*\rangle_{1\cdots n/k}\]

\[= \sum_{i=0}^\ell (a_0 \times \ell \cdots i + a_1 \times \ell \cdots i) |0\cdots 0\rangle_{k-1} \]

\[\quad + \sum_{i=0}^\ell (a_2 \times \ell \cdots i + a_3 \times \ell \cdots i) |0\cdots 01\rangle_{k-1} \cdots \]

\[\quad + \sum_{i=0}^\ell (a_2(2^{k-4}) \times \ell \cdots i + a_2(2^{k-3}) \times \ell \cdots i) |1\cdots 10\rangle_{k-1} \]

\[\quad + \sum_{i=0}^\ell (a_2(2^{k-2}) \times \ell \cdots i + a_2(2^{k-1}) \times \ell \cdots i) |1\cdots 10\rangle_{k-1}.\]

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