Multidimensional Wavelets for Scalable Image Decomposition: Orbital Wavelets

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Abstract

Wavelets are closely related to the Schrödinger’s wave functions and the interpretation of Born. Similarly to the appearance of atomic orbital, it is proposed to combine anti-symmetric wavelets into orbital wavelets. The proposed approach allows the increase of the dimension of wavelets through this process. New orbital 2D-wavelets are introduced for the decomposition of still images, showing that it is possible to perform an analysis simultaneous in two distinct scales. An example of such an image analysis is shown.

Keywords

2D wavelets; anti-symmetric wavelets; orthogonal wavelets; image analysis.

1 Introduction

Wavelet transform methods are important tools in image processing due to their capabilities for multiresolution analysis and image decomposition [8,25]. Wavelet-based image processing find application in computer graphics [36], including radiosity, global illumination [39], and real volume data [28], and volume rendering [33]; being routinely considered as an approach for texture image decomposition, image coding, subband coding, fast image segmentation [1,3,20,23,44], and 3D signal processing [42]. In particular, wavelet transform coding [7,12,30,43] has emerged as a practical and fully-established tool which benefits image compressing methods, such as the JPEG 2000 standard [13,35,37,43], and multimedia schemes on Internet, such as data streaming over IP [4,31] and image querying [17]. Scalable coding for image, audio, and video largely adopts wavelet-based methods as demonstrated in the MPEG-4 codec [7,11,30,31,38].

Multiwavelets have been explored for the assessment of order/disorder in reconstructed biomedical images [45]. Further connections between information-theoretical entropy and wavelet analysis were explored in the context of geoscience [29]. Recent advances in the field of image decomposition include the proposition of special filters for spherical harmonics modeling [18] capable of multi-level decomposition suitable for 3D images with the introduction of the concept of spherical harmonics entropy [19].

In the same vein of exploring connections between different research fields, Ashmead reported [2] a link between quantum mechanics and wavelets. Despite the potentially deep mathematical meaning of such link, it has not been significantly explored in the context of wavelet decomposition and image analysis. Indeed, the wave nature of light can be deduced from the phenomenon of interference, the photoelectric effect, however, it seems to suggest a corpuscular nature of light. Theoretical physicists struggled to include observations like the photoelectric effect and the wave-particle duality into their formulations [41]. Erwin Schrödinger employed advanced mechanics to address such phenomena and developed an equation that relates the space-time in quantum mechanics. Because wavelets are localized in both time and frequency they offer significant advantages for the analysis of problems in quantum mechanics.

In this paper, we aim at proposing an alternative wavelet decomposition scheme. For such, we explore the above discussed link between wavelets and quantum mechanics and shed some light on some of these relations. Rather than

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seeking at wavelet features on particles or waves, we adapted some concepts of quantum mechanics to a novel wavelet decomposition for still images.

The paper is organized as follows. Section 2 details the main ideas behind the proposed wavelet system which is based on an interpretation from particle physics and quantum states [5, 14]. We describe the construction of the proposed orbital wavelets for the image decomposition. It is shown that the introduced derivation is naturally suitable for generating two-dimensional wavelets. In Section 3 we submit standard imagery to the proposed wavelet decomposition scheme and compare the results with standard wavelet decomposition. Section 4 concludes the paper.

2 Orbital Wavelet Decomposition

2.1 Particle Systems

The wave functions describing electronic orbitals can be combined generating atomic orbitals. Let us consider a two-particle non-interaction systems with particles located at position vectors \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), respectively, where each particle is equipped with wave functions \( \psi_\alpha(\mathbf{r}_1) \) and \( \psi_\beta(\mathbf{r}_2) \) at states \( \alpha \) and \( \beta \), respectively. The wave function that characterizes the orbital interaction of the two particles is furnished by a combination \( \psi_\alpha(\mathbf{r}_1) \) and \( \psi_\beta(\mathbf{r}_2) \) in two different configurations: symmetric (S) and anti-symmetric (A) combinations [13]. Such combinations are given by:

\[
\psi_S(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} \left[ \psi_\alpha(\mathbf{r}_1) \cdot \psi_\beta(\mathbf{r}_2) + \psi_\alpha(\mathbf{r}_2) \cdot \psi_\beta(\mathbf{r}_1) \right],
\]

\[
\psi_A(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} \left[ \psi_\alpha(\mathbf{r}_1)\psi_\beta(\mathbf{r}_2) - \psi_\alpha(\mathbf{r}_2)\psi_\beta(\mathbf{r}_1) \right],
\]

respectively. The anti-symmetric case can be conveniently written in matrix format as \( \psi_A(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} \det \begin{bmatrix} \psi_\alpha(\mathbf{r}_1) & \psi_\alpha(\mathbf{r}_2) \\ \psi_\beta(\mathbf{r}_1) & \psi_\beta(\mathbf{r}_2) \end{bmatrix} \).

A comparable concept in the scope of wavelets, also characterized by wave functions, would be the combination of different spatial wavelets [10].

2.2 Orbital Wavelets

2.2.1 Symmetric Case: Standard Wavelets

Usual wavelet image analysis combines one-dimensional (1D) wavelets to generate a two-dimensional (2D) wavelet [1, 44]. This can be done by considering a scaling function \( \varphi \) and a wavelet function \( \psi \), a version for each dimension, abscissa and ordinate [8]. Thus, we have a 2D scale function \( \varphi_{LL}(x, y) = \varphi(x) \cdot \varphi(y) \) and three 2D wavelet functions:

\[
\psi_{LH}(x, y) = \varphi(x) \cdot \psi(y),
\]

\[
\psi_{HL}(x, y) = \varphi(x) \cdot \psi(y),
\]

\[
\psi_{HH}(x, y) = \psi(x) \cdot \psi(y).
\]

The wavelets \( \psi_{LH}(x, y) \) and \( \psi_{HL}(x, y) \) naturally exhibit reflection symmetry with respect to the plane \( x = y \), i.e. \( \psi_{LH}(x, y) = \psi_{HL}(y, x) \) and \( \psi_{HL}(x, y) = \psi_{HL}(y, x) \). Such reflection symmetry is analogous to the symmetric wave function described in [4].

2.2.2 Anti-symmetric Case: Orbital Wavelets

Following this analogy, considering a single orthogonal wavelet system, we are compelled to pursue the definition of wavelets that could be regarded as the counterparts of the anti-symmetric wave function described in [2]. Thus the combination of \( \varphi(\cdot) \) and \( \psi(\cdot) \) should be arranged to provide anti-symmetry, i.e. the \( \hat{\psi}_{LH}(x, y) \) and \( \hat{\psi}_{HL}(x, y) \) wavelets.
A solution to the above requirement is to define an orbital-inspired combination (cf. (2)) of \(\varphi\) and \(\psi\) according to the following:

\[
\hat{\psi}_{LH}(x,y) \triangleq \frac{1}{\sqrt{2}} \left[ \varphi(x) \cdot \psi(y) - \psi(x) \cdot \varphi(y) \right].
\]  

(6)

In order to illustrate the effect of this definition we consider the case of the Meyer orthogonal wavelet [26]. Figure 1 shows the surface plots for the standard of the discussed functions.

Analogously, we define the wavelets related to the LL and HH decompositions according to the following expressions:

\[
\hat{\varphi}_{LL}(x,y) \triangleq \frac{1}{\sqrt{2}} \left[ \varphi^*(x) \cdot \varphi(y) + \varphi(y) \cdot \varphi(x) \right],
\]

(7)

\[
\hat{\psi}_{HH}(x,y) \triangleq \frac{1}{\sqrt{2}} \left[ \psi^*(x) \cdot \psi(y) - \psi(y) \cdot \psi(x) \right].
\]

(8)

The above definitions allows the analysis of images using continuous complex wavelets [21]. For real-valued wavelets, the above expressions collapse to the usual forms \(\varphi(x) \cdot \varphi(y)\) or \(\psi(x) \cdot \psi(y)\) present in standard wavelet analysis. Thus in the real case, although the proposed wavelet \(\hat{\varphi}_{LL}(x,y)\) in (7) is well-defined, the wavelet \(\hat{\psi}_{HH}(x,y)\) would collapse to zero.

An approach to address such degeneracy is to consider daughter wavelets at different scales. Therefore let us consider the 1D orthogonal [22,24] wavelet mother \(\psi(x)\) equipped equipped with her daughter wavelets \(\{\psi_{a,b}(x)\}_{a \neq 0, b \in \mathbb{R}}\).
The formalism shown in (8) can be extended by considering the inclusion of two wavelets \( \psi_{a_1, b}(\cdot) \) and \( \psi_{a_2, b}(\cdot) \) resulting in the following definition.

**Definition 1** The function 2D-orbital at the scales \( \{a_1, a_2\} \) is defined by:

\[
\hat{\psi}_{HH}(x, y) \triangleq \frac{1}{\sqrt{2}} \det \begin{bmatrix}
\psi_{a_1, b}(x) & \psi_{a_1, b}(y) \\
\psi_{a_2, b}(x) & \psi_{a_2, b}(y)
\end{bmatrix},
\]

which can be rewritten as:

\[
\hat{\psi}_{HH}(x, y) = \frac{1}{\sqrt{2|a_1||a_2|}} \psi^* \left( \frac{x - b}{a_1} \right) \cdot \psi \left( \frac{y - b}{a_2} \right) - \frac{1}{\sqrt{2|a_1||a_2|}} \psi \left( \frac{x - b}{a_2} \right) \cdot \psi^* \left( \frac{y - b}{a_1} \right).
\]

The condition \( a_1 \neq a_2 \) eliminates the degeneration \( \hat{\psi}_{HH}(x, y) = 0 \). This is to some extent in connection to the Pauli Exclusion Principle [13], which states that with a single-valued many-particle wave function is equivalent to requiring the wave function to be antisymmetric. An antisymmetric two-particle state is represented as a sum of states in which one particle is in state \( \alpha \) and the other in state \( \beta \). Besides, the relationship \( \hat{\psi}_{HH}(y, x) = -\hat{\psi}_{HH}(x, y) \) ensures the desired asymmetry. Here, we use the same wavelet-mother, but at different scales. The 2D decomposition stated in Definition 1 results in a strict 2D wavelet.

### 2.3 Mathematical Properties

**Proposition 1** The previously defined 2D-orbital function has oscillatory behavior satisfying the following properties:

(i) \( \int_{-\infty}^{\infty} \hat{\psi}_{HH}(x, y) \, dx = 0 \),

(ii) \( \int_{-\infty}^{\infty} \hat{\psi}_{HH}(x, y) \, dy = 0 \),

(iii) \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\psi}_{HH}(x, y) \, dx \, dy = 0 \).

**Proof:** It follows that

\[
\int_{-\infty}^{\infty} \hat{\psi}_{HH}(x, y) \, dx = \frac{1}{\sqrt{2}} \psi_{a_2, b}(y) \cdot \psi^*_{a_1, b}(x) - \frac{1}{\sqrt{2}} \psi_{a_2, b}(x) \cdot \psi^*_{a_1, b}(y),
\]

where

\[
\overline{\psi}_{a,b}(x) \triangleq \int_{-\infty}^{\infty} \psi_{a,b}(x) \, dx.
\]

Therefore, the property (i) derives from the fact that \( \psi_{a,b}(x), a = \{a_1, a_2\} \) be individual wavelets. The demonstration for property (ii) is similar, considering that

\[
\int_{-\infty}^{\infty} \hat{\psi}_{HH}(x, y) \, dy = \frac{1}{\sqrt{2}} \psi_{a_2, b}(y) \cdot \psi^*_{a_1, b}(x) - \frac{1}{\sqrt{2}} \psi_{a_2, b}(x) \cdot \psi^*_{a_1, b}(y).
\]

The condition \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\psi}_{HH}(x, y) \, dx \, dy = 0 \) follows from Fubini’s theorem [33], regardless of the order of integration. \( \square \)
Hereafter we assume an orthogonal wavelet system with unitary energy. In other words, the following conditions hold true:

(i) the inner product \( \langle \psi_{a_1,b}, \psi_{a_2,b} \rangle = 0 \), i.e., the following integrals cancel out \( \forall a_1 \neq a_2 \):

\[
\int_{-\infty}^{\infty} \psi_{a_1,b}(x) \cdot \psi_{a_2,b}^*(x) \, dx = \int_{-\infty}^{\infty} \psi_{a_1,b}^*(x) \cdot \psi_{a_2,b}(x) \, dx = 0,
\]

and

\[
\int_{-\infty}^{\infty} |\psi_{a_1,b}(x)|^2 \, dx = 1.
\]

It is also noteworthy that

\[
\langle \psi_{a_1,b}, \psi_{a_2,b} \rangle^* = \langle \psi_{a_2,b}, \psi_{a_1,b} \rangle.
\]

Proposition 2 The 2D-orbital functions have normalized energy.

Proof: Expanding the expression \( |\hat{\psi}_{HH}(x,y)|^2 = \hat{\psi}_{HH}(x,y) \cdot \hat{\psi}_{HH}^*(x,y) \) yields cross-product terms in the following form

\[
cross(x,y) \triangleq -\psi_{a_1,b}(x) \cdot \psi_{a_2,b}^*(y) \cdot \psi_{a_1,b}^*(y) \cdot \psi_{a_2,b}(y)
\]

and its complex conjugate \( \text{cross}^*(x,y) \). Performing the integration with respect to \( x \) and \( y \) yields:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{cross}(x,y) \, dx \, dy = -2 \cdot \psi_{a_2,b}^*(y) \psi_{a_1,b}(y) \cdot \psi_{a_1,b}^*(x) \psi_{a_2,b}(x) \, dx \, dy,
\]

and

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{cross}(x,y) \, dx \, dy = -2 \cdot \psi_{a_2,b}(x) \psi_{a_1,b}(x) \cdot \psi_{a_1,b}^*(y) \psi_{a_2,b}(y) \, dx \, dy.
\]

Invoking the orthogonality condition we obtain that all cross terms are void. The remaining possibly nonnull terms are:

\[
|\hat{\psi}_{HH}(x,y)|^2 = \frac{1}{2} |\psi_{a_1,b}(x)|^2 \cdot |\psi_{a_2,b}(y)|^2 \cdot |\psi_{a_1,b}(y)|^2 \cdot |\psi_{a_2,b}(x)|^2
\]

and therefore, because of the normalized energy condition, we have:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{\psi}_{HH}(x,y)|^2 \, dx \, dy = 1.
\]

\( \square \)

It is possible (more easily) to combine orthogonal 1D-wavelets and use them to build a new 2D-wavelet.

Proposition 3 The 2D-orbital function is a 2D wavelet.

Proof: Let \( \Psi(\omega) \) and \( \Psi_{a,b}(\omega) \) be the Fourier transforms of the wavelet \( \psi(t) \) and the daughter wavelets \( \psi_{a,b}(t) \), respectively. If the admissibility condition holds \( [6, 8] \),

\[
\int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} \, d\omega < \infty
\]
then \( \int_{-\infty}^{\infty} \frac{\Psi_{a,b}(\omega)}{|\omega|} \, d\omega < \infty \), since \( \Psi_{a,b}(\omega) = \sqrt{|a|} \Psi(a\omega)e^{-jwb} \) \cite{25}. Let us now evaluate the condition for the 2D case. If the Fourier transform pair \( \hat{\psi}_{HH}(x,y) \leftrightarrow \hat{\Psi}_{HH}(u,v) \) does exist, then the 2D spectrum of \( \hat{\psi}_{HH} \) can be computed in terms of the Fourier spectrum of \( \hat{\psi} \):

\[
\hat{\Psi}_{HH}(u,v) = \sqrt{|a_1a_2|} \left[ \hat{\Psi}(a_1u)\hat{\Psi}^*(a_2v) - \hat{\Psi}(a_2u)\hat{\Psi}^*(a_1v) \right].
\]  

(23)

From the generalized Parseval-Plancherel energy theorem \cite{8,34}, the cross-terms vanish due to the orthogonality. Thus, we have

\[
|\hat{\Psi}_{HH}(u,v)|^2 = \frac{|a_1a_2|}{2} |\hat{\Psi}(a_1u)|^2 \cdot |\hat{\Psi}(a_2v)|^2 + \frac{|a_1a_2|}{2} |\hat{\Psi}(a_2u)|^2 \cdot |\hat{\Psi}(a_1v)|^2.
\]  

(24)

Then, from the marginal admission conditions of the 1D daughter wavelets, we have that

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\hat{\Psi}_{HH}(u,v)|^2}{|u| \cdot |v|} \, du \, dv < \infty.
\]  

(25)

\[\square\]

### 3 Orbital Wavelet-based Image Decomposition

The standard 2D wavelet decomposition effects coefficient matrices representing vertical, horizontal, and diagonal structures denoted by sub-images \( \{L_1L_1, L_1H_1, H_1L_1, H_1H_1\} \) and \( \{L_2L_2, L_2H_2, H_2L_2, H_2H_2\} \) for the first and second level decompositions \cite{40}. Figure 2 illustrate the structures. Noted that the terms \( L_1L_1, L_2L_2, H_1H_1 \), and \( H_2H_2 \) on the main diagonal correspond to a part of the standard wavelet decomposition of the image into two levels. The proposed wavelet analysis results in a similar structure with sub-images \( \{\hat{L}_1L_1, \hat{L}_1H_1, \hat{H}_1L_1, \hat{H}_1H_1\} \) and \( \{\hat{L}_2L_2, \hat{L}_2H_2, \hat{H}_2L_2, \hat{H}_2H_2\} \) in a two-level decomposition. Despite the similarity, the sub-images are computed according to the orbital wavelets. The resulting scheme is depicted in Figure 3.

Considered the symlet wavelet of order 4 (\texttt{symlet4}) \cite{9}, we applied the proposed decomposition to the standard image \texttt{woman} \cite{27}, the resulting sub-images are shown in Figure 4. For a qualitative comparison, we included the sub-images obtained from the usual wavelet decomposition. The computation was performed in the Matlab environment \cite{27}. It is worth noting that the subtraction of the images resulting from \( \psi_{LH}(x,y) \) and \( \psi_{HL}(x,y) \) results in the image obtained by the wave function in (6).
Figure 3: Image decomposition scheme according to the 2D orbital wavelet decomposition. A two-level wavelet-orbital decomposition.

| $\hat{L}_1 \hat{L}_1$ | $\hat{H}_1 \hat{H}_1$ | $\hat{L}_2 \hat{L}_2$ | $\hat{L}_2 \hat{H}_2$ | $\hat{L}_1 \hat{H}_1$ |
|----------------------|----------------------|----------------------|----------------------|----------------------|
| $\hat{H}_1 \hat{L}_1$ | $\hat{H}_1 \hat{H}_1$ | $\hat{H}_2 \hat{L}_2$ | $\hat{H}_2 \hat{H}_2$ | $\hat{L}_1 \hat{H}_1$ |

Figure 4: First-level decomposition woman image using symlet4 wavelet as defined in Matlab according to (a) the standard wavelet decomposition and (b) the proposed decomposition.
4 Concluding Remarks and Future Work

This paper offers an alternative approach for image decomposition engendered by asymmetric orthogonal wavelets whose definition is inspired by the wave function theory from particle physics. Despite the focus being essentially on still image analysis, the proposed approach allows a fully scalable multimedia decomposition. It remains to be investigated the potential of this methodology for image compressing [12], in 3D processing and scalable coding for multimedia schemes [30]. Applications in other scenarios such as wavelet-based watermarking [16] or steganography [32] also deserve an investigation. To the best of our knowledge, this is the first work to link the exchanging formalism from particle wave functions to wavelets analysis.

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