0.1 Introduction. The Lusztig formula for characters of the simple finite-dimensional modules of the quantum group \( U_\epsilon(\mathfrak{sl}_k) \), where \( \epsilon = e^{2i\pi/n} \), gives the multiplicity of the Weyl module \( W_\mu \) of \( U_\epsilon(\mathfrak{sl}_k) \) with highest weight \( \mu \) in the simple \( U_\epsilon(\mathfrak{sl}_k) \)-module \( V_\lambda \) with highest weight \( \lambda \). Namely,

\[
[V_\lambda : W_\mu] = \sum_y (-1)^{(x,y)} P_{y,x}(1),
\]

where \( x \in \hat{S}_k \) is minimal such that \( \nu = \lambda.x^{-1} \) satisfies \( \nu_i < \nu_{i+1} \) for \( i = 1, 2 \ldots k - 1 \) and \( \nu_1 - \nu_k > 1 - k - n \), and \( \mu = \lambda.x^{-1}y \). This conjecture is proved by Kazhdan-Lusztig [KL] and Kashiwara-Tanisaki [KT]. The proof relies on an equivalence between the category of finite-dimensional \( U_\epsilon(\mathfrak{sl}_k) \)-modules and a category of negative-level representations of the affine algebra \( \hat{\mathfrak{sl}}_k \) which are integrable with respect to \( \mathfrak{sl}_k \). In [VV], Varagnolo and Vasserot propose a new approach to this conjecture, based on the geometric constructions of simple finite-dimensional \( U_q(\hat{\mathfrak{sl}}_k) \)-modules of [GV] and on the theory of canonical bases.

Let \( U_n^- \) be the generic Hall algebra of the cyclic quiver of type \( A^{(1)}_{n-1} \) (defined over the ring \( \mathbb{C}[q, q^{-1}] \)) and let \( B \) be the intersection cohomology basis of \( U_n^- \). Let \( \Lambda^\infty \) be the Fock space representation of \( U_n^- \) (see [KMS] and [VV]), and let \( B^\pm \) be the Leclerc-Thibon canonical bases of \( \Lambda^\infty \) (see [LT]). Varagnolo and Vasserot show that the Lusztig formula follows from the equality \( B|0\rangle = B^+ \), where \( |0\rangle \) is the vacuum vector of \( \Lambda^\infty \). In this paper, we give a direct proof of this equality, which can be thought of as a q-analogue of the Lusztig conjecture. This also yields a proof of the positivity conjecture for the basis \( B^+ \) (see [LLT], Conjecture 6.9 i)). Note that the equality \( B|0\rangle = B^+ \) does not follow from the general theory developed in [K] or [L] since the Fock space is not irreducible as a \( U_q(\mathfrak{sl}_n) \)-module. We also formulate and prove an analogue of the conjecture of Varagnolo and Vasserot for higher level Fock space representations \( \Lambda^\infty_s \) of \( U_n^- \) (see [JMMQ]). The corresponding canonical bases have been recently introduced by Uglov, [U].

Our proof relies on the construction of an isomorphism

\[
U_n^- \simeq U_q(\mathfrak{sl}_n) \otimes \mathbb{C}[z_1, z_2, \ldots].
\]

The central subalgebra \( \mathbb{C}[v, v^{-1}][z_1, z_2, \ldots] \) of \( U_n^- \) is linearly spanned by elements of the dual canonical basis of \( U_n^- \). By a quantum affine version of the Schur-Weyl duality, it also admits an interpretation in terms of the Bernstein center \( Z(\hat{H}_l) \) of the affine Hecke algebra \( \hat{H}_l \) of type \( A_l \) for \( l \leq n \). We consider
a basis \((a_\lambda)\) of the central subalgebra \(\mathbb{C}[v, v^{-1}][z_1, z_2, \ldots]\) of \(U_n\) corresponding to Schur polynomials and describe its action on \(\Lambda^\infty\). The equality \(B(0) = B^+\) follows from the characterization of \(B^+\) in terms of a lattice and the \(^{-}\)-involution of \(\Lambda^\infty\).

Finally, we note that a proof of the Lusztig conjecture (but not of its \(q\)-analogue) using the above ideas has been recently found by B. Leclerc (see [Le]).

### 0.2 Notations

Set \(S = \mathbb{C}[v], \ A = \mathbb{C}[v, v^{-1}]\) and \(K = \mathbb{C}(v)\). Throughout the paper we fix some integer \(n > 1\). Let \(q\) be a prime power and let \(F\) be a finite field with \(q^2\) elements. Let \((\epsilon_i), i \in \mathbb{Z}/n\mathbb{Z}\) be the canonical basis of \(\mathbb{N}\mathbb{Z}/n\mathbb{Z}\). For \(i \in \mathbb{Z}/n\mathbb{Z}\) and \(l \in \mathbb{N}^*, \) define the cyclic segment \([i; l]\) to be the image of the projection to \(\mathbb{Z}/n\mathbb{Z}\) of the segment \([i_0, i_0 + l - 1] \subset \mathbb{Z}\) for any \(i_0 \equiv i \pmod{n}\). A cyclic multisegment is a linear combination \(m = \sum_{i,l} a^l_i [i; l]\) of cyclic segments with coefficients \(a^l_i \in \mathbb{N}\). Let \(\mathcal{M}\) be the set of cyclic multisegments. For \(m \in \mathcal{M}\), we put \(|m| = \sum_{i,l} l a^l_i\) and dim \(m = \sum_{i,l} l (\epsilon_i + \ldots + \epsilon_{i+l-1}) \in \mathbb{N}\mathbb{Z}/n\mathbb{Z}\). Let \(\Pi\) denote the set of partitions and let \(\Pi_D \subset \Pi\) be the subset formed by partitions with at most \(D\) parts.

### 1 Hall algebra of the cyclic quiver

#### 1.1 Let \(Q\) be the quiver of type \(A_{n-1}^{(1)}\), i.e. the oriented graph with vertex set \(I = \mathbb{Z}/n\mathbb{Z}\) and edge set \(\Omega = \{(i, i+1): i \in I\}\). For any \(I\)-graded \(F\)-vector space \(V = \bigoplus_{i \in I} V_i\) let \(E_V \subset \bigoplus_{(i,j) \in \Omega} \text{Hom}(V_i, V_j)\) denote the space of nilpotent representations of \(Q\), i.e. collections of linear maps \((\sigma_i : V_i \to V_j, (i,j) \in \Omega)\) satisfying the following condition: for any \(i_0 \in I\) there exists \(r \in \mathbb{N}\) such that \(\sigma_{i_0} \ldots \sigma_{i_0} \sigma_{i_0} = 0\) for any \(i_2, \ldots, i_r\). The group \(G_V = \prod_{i \in I} GL(V_i)\) acts on \(E_V\) by conjugation. For each \(i \in I\) there exists a unique simple \(Q\)-module \(S_i\) of dimension \(\epsilon_i\), and for each pair \((i, l) \in I \times \mathbb{N}^*\) there exists a unique (up to isomorphism) indecomposable \(Q\)-module \(S_{i,l}\) of length \(l\) and tail \(S_i\). Furthermore, every nilpotent \(Q\)-module \(M\) admits an essentially unique decomposition

\[
M \simeq \bigoplus_{i,l} a^l_i S_{i,l}.
\] (1.1)

The classification of \(Q\)-modules is independent of the base field. We denote by \(\overline{m}\) the isomorphism class of \(Q\)-modules corresponding (by (1.1)) to the multisegment \(m = \sum_{i,l} a^l_i [i; l]\).

#### 1.2 We recall the Lusztig construction (see [L]). For any \(I\)-graded vector space \(V\), let \(C_G(E_V)\) be the set of \(G_V\)-invariant functions \(E_V \to \mathbb{C}\). For each \(d \in \mathbb{N}^I\), let us fix an \(I\)-graded vector space \(V_d\) of dimension \(d\). Given \(a, b \in \mathbb{N}^I\) such that \(a + b = d\), consider the diagram

\[
E_{V_a} \times E_{V_b} \overset{p_1}{\underset{p_3}{\rightleftharpoons}} E \overset{p_2}{\underset{p_3}{\rightarrow}} E_{V_d},
\]

where \(E\) is the set of triples \((x, \phi, \psi)\) such that \(x \in E_{V_d}\),

\[
0 \to V_a \overset{\phi}{\to} V_d \overset{\psi}{\to} V_b \to 0
\]
is an exact sequence of $I$-graded vector spaces and $\phi(V_n)$ is stable by $x$, and where $F$ is the set of pairs $(x, U)$ such that $U \subset V_d$ is an $I$-graded, $x$-stable subspace of dimension $a$.

Given $f \in C_G(EV_a)$ and $g \in C_G(EV_b)$, set

$$f \circ g = q^{-m(b,a)}(p_1); h \in C_G(V_d),$$

where $h \in C(F)$ is the unique function satisfying $p^* \circ (h) = p^* (fg)$ and $m(b,a) = \sum_{i \in I} a_i b_i + \sum_{(i,i_2) \in \Omega} a_{i2}$. Set $U_{q,n} = \bigoplus_d C_G(EV_d)$. Then $(U_{q,n}, \circ)$ is an associative algebra.

For $m \in M$ with $\dim m = d$, we let $O_m \subset EV_d$ be the $G_{V_d}$-orbit consisting of representations in the class $\overline{m}$, and we let $1_m \in C_G(V_d)$ be the characteristic function of $O_m$. Finally, we set $f_m = q^{-\dim O_m}1_m$, and if $d \in \mathbb{N}^I$, we let $f_d$ be the characteristic function of the trivial representation in $V_d$. Note that, by definition, $(f_m)_{m \in M}$ is a $C$-basis of $U_{q,n}$.

1.3 Let $a, b, d \in \mathbb{N}^I$ such that $a + b = d$. Fix a subspace $U$ of $V_d$ of dimension $a$ and a pair of graded vector space isomorphisms $U \simeq V_a$, $V_d/U \simeq V_b$. Consider the diagram

$$E_U \times E_{V/U} \xrightarrow{\rho} E \xrightarrow{i} EV_d,$$

where $E \subset EV_d$ is the subset of the representations preserving $U$. Set

$$\Delta_{a,b} : C_G(V_d) \to C_G(V_a) \otimes C_G(b), \quad f \mapsto q^{-n(b,a)p_i} f.$$

Here $n(b,a) = \sum_{(i,i_2) \in \Omega} b_i a_{i2} - \sum_{i \in I} a_i b_i$.

1.4 It is known that the structure constants of $(U_{q,n}, \circ, (\Delta_{a,b}))$ in the basis $(f_m)$ are values at $q = v^{-1}$ of some universal Laurent polynomials in $A$. Thus there exists an algebra $U^{-}_n$ defined over $A$ of which $U_{q,n}$ is the specialisation at $v = q^{-1}$. This algebra is called the generic Hall algebra. The algebra $U^{-}_n$ is naturally $\mathbb{N}^I$-graded and we will denote by $U^{-}_n[a]$ the graded component of degree $a \in \mathbb{N}^I$. Let $U^{-}(s\mathfrak{sl}_n)$ (resp. $U^{-}((\mathfrak{sl}_n)$) denote the rational form (resp. the Lusztig integral form) of the quantum affine algebra of type $A^{(1)}_{n-1}$, and let $e_i, k_i, f_i, i \in I$ (resp. $e^{(l)}_i, k^{(l)}_i, f^{(l)}_i, i \in I, l \in \mathbb{N}$) be the standard Chevalley generators (resp. their quantized divided powers). Let $U^{-}((\mathfrak{sl}_n)$ be the subalgebra of $U^{-}(\mathfrak{sl}_n)$ generated by $f^{(l)}_i, i \in I, l \in \mathbb{N}^*$. It is known that the map $f^{(l)}_i \mapsto f_i$ extends to an embedding of the algebras $U^{-}((\mathfrak{sl}_n) \hookrightarrow U^{-}_n$. Set $U^{-}_{K,n} = U^{-}_n \otimes_A K$.

1.5 Let $U^0$ be the commutative $A$-algebra generated by elements $k_d, d \in \mathbb{Z}^I$, satisfying

$$k_0 = 1, \quad k_a k_b = k_{a+b}, \quad \forall a, b \in \mathbb{Z}^I.$$ 

Set $\widehat{U^{-}}_n = U^{-}_n \otimes A U^0$ and put

$$(f \otimes k_a) \circ (g \otimes k_b) = v^{a \cdot d}(f \circ g) \otimes (k_{a+b}), \quad \forall g \in U^{-}_n[d], \forall f \in U^{-}_n,$$

where $a \cdot d = n(a, d) + n(d, a)$. Finally, define

$$\hat{\Delta} : \widehat{U^{-}}_n \to \widehat{U^{-}}_n \otimes \widehat{U^{-}}_n, \quad f \otimes k_c \mapsto \sum_{d=a+b} \Delta_{a,b}(f)(k_{b+c} \otimes k_c) \quad \forall f \in U^{-}_n[d].$$

It is proved in [Gr] that $(\widehat{U^{-}}_n, \circ, \hat{\Delta})$ is a bialgebra.
1.6 Define the following symmetric bilinear form on $\tilde{U}^-_n$:
\[
\langle f_m \otimes k_a, f_{m'} \otimes k_b \rangle = v^{-(a+b) \cdot d} \cdot \dim \text{Aut}(m) \cdot \frac{(1 - v^2)^{\vert m \vert}}{\vert \text{Aut}(m) \vert} \delta_{m,m'},
\]
where $d = \dim m$ and $\text{Aut}(m)$ stands for the group of automorphism of any representation in the orbit $O_m$. For any $f, g, h \in \tilde{U}^-_n$ we have \cite{LTV}
\[
\langle fg, h \rangle = \langle f \otimes g, \tilde{\Delta}(h) \rangle.
\]
It is clear that the restriction of $\langle , \rangle$ to the subalgebra $U^-_n$ is nondegenerate.

1.7 For $i \in I$, let $e'_i : U^-_n \to U^-_n$ be the adjoint of the left multiplication by $f_i$. It is a homogeneous operator of degree $-\epsilon_i$. Let $\tilde{f}_i, \tilde{e}_i$ be the Kashiwara operators (see \cite{K}, Section 3). Recall that a crystal basis of $U^-_n$ is a pair $(\mathcal{L}, B)$ where $B \subset \mathcal{L}/v\mathcal{L}$ is a $\mathbb{C}$-basis satisfying the following conditions:

i) for any $i \in I$ we have $\tilde{e}_i \mathcal{L}, \tilde{f}_i \mathcal{L} \subset \mathcal{L}$, and $\tilde{e}_i(B), \tilde{f}_i(B) \subset B \cup \{0\}$,

ii) for any $i \in I, b, b' \in B$ we have $\tilde{e}_i(b) = b'$ if and only if $\tilde{f}_i(b') = b$.

Set $\mathcal{L} = \bigoplus_m \mathbb{S} f_m$. Let $b_m$ denote the class of $f_m$ in $\mathcal{L}/v\mathcal{L}$ and set $B = \{b_m, m \in M\}$. The following result is proved in \cite{LT}. Theorem 4.1:

**Theorem.** The couple $(\mathcal{L}, B)$ is a crystal basis of $U^-_n$.

The crystal graph $\mathcal{C}$ of $U^-_n$ has the vertex set $\mathcal{M}$ and edges $m \rightarrow m'$ whenever $\tilde{f}_i(b_m) = b_{m'}$. It is explicitly described in \cite{LT}. Call a multisegment $m = \sum_{i,j} a_i^j[i;i,l]$ completely periodic if $a_i^j = a_j^i$ for all $l \in \mathbb{N}^*$ and $i, j \in I$. Let $\mathcal{M}^{\text{per}}$ be the set of all completely periodic multisegments. Then $\tilde{e}_i(b_m) = 0$ for each $m \in \mathcal{M}^{\text{per}}$ and $i \in I$, and the connected component of $\mathcal{C}$ containing $b_m$ is isomorphic to the crystal graph of $U^- (s\mathfrak{sl}_n)$.

1.8 For $m \in \mathcal{M}$, set
\[
b_m = \sum_{i,n} v^{-i + \dim O_m + \dim O_n} \dim \mathcal{H}_{O_m}^i (IC_{O_m}) f_n,
\]
where $\mathcal{H}_{O_m}^i (IC_{O_m})$ is the stalk over a point of $O_m$ of the $i$th intersection cohomology sheaf of the closure $\overline{O}_m$ of $O_m$. Then $B = \{b_m\}$ is the canonical basis of $U^-_n$, introduced in \cite{VL}. There exists a unique semilinear ring involution $x \mapsto \overline{x}$ of $U^-_n$ satisfying $\overline{b_m} = b_m$ (see \cite{VL}, Proposition 7.5). The element $b_m$ is characterized by the following two properties:

i) $\overline{b_m} = b_m$, and

ii) $b_m \in f_m + v\mathcal{L}$.

Call a multisegment $m = \sum_{i,j} a_i^j$ aperiodic if for each $l \in \mathbb{N}^*$ there exists $i \in I$ such that $a_i^l = 0$, and let $\mathcal{M}^{\text{ap}}$ be the set of aperiodic multisegments. Lusztig proved in \cite{L2} that $\{b_m \mid m \in \mathcal{M}^{\text{ap}}\}$ is the global canonical basis of $U^- (s\mathfrak{sl}_n)$.
1.9 Fix $D \in \mathbb{N}$. Let $\mathcal{S}_D$ (resp. $\hat{\mathcal{S}}_D$, resp. $\hat{H}_D$) be the symmetric group (resp. affine symmetric group, resp. affine Hecke algebra) of type $GL_D$. The $A$-algebra $\hat{H}_D$ is generated by elements $T_i^\pm, X_j^\pm$, $i \in [1, D-1]$, $j \in [1, D]$ with relations

\[
T_i T_i^{-1} = T_i^{-1} T_i, \quad (T_i + 1)(T_i - v^{-2}) = 0, \\
T_i T_{i+1} T_i = T_{i+1} T_i T_i, \quad |i - j| > 1 \Rightarrow T_i T_j = T_j T_i, \\
X_i X_i^{-1} = 1 = X_i^{-1} X_i, \\
T_i T_i X_i = v^{-2} X_i T_i, \quad j \neq i, i + 1 \Rightarrow X_j T_i = T_i X_j.
\]

Let $\hat{H}_\infty$ be the $A$-algebra generated by $T_i^\pm, X_j^\pm$, $i, j \in \mathbb{N}^*$ with the same relations as above.

The center of $\hat{H}_D$ is $Z(\hat{H}_D) = A[X_D^\pm, \ldots, X_D^\pm]^{\mathcal{S}_D}$. Set

\[
Z_D^+ = A[X_1, \ldots, X_D]^{\mathcal{S}_D}, \quad Z_D^- = A[X_1^{-1}, \ldots, X_D^{-1}]^{\mathcal{S}_D}.
\]

1.10 Let $\hat{A}(Z)$ be the $A$-linear span of vectors $x_i$, $i \in \mathbb{Z}$. Following [VV], Section 8.1, let $\hat{U}_n^-$ act on $\hat{A}(Z)$ by

\[
f_m(x_i) = \sum_{j \geq 1} \delta_{m, [(j-1)x_j + 1]}, \quad \forall m \in \mathcal{M}, \tag{1.3}
\]

\[
k_\alpha(x_i) = v^{-n(\alpha, x_i)} x_i, \quad \forall \alpha \in \mathbb{N}^J. \tag{1.4}
\]

Set $\otimes^D = (\hat{A}(Z))^{\otimes D}$ and let $\hat{U}_n^-$ act on $\otimes^D$ via the coproduct $\hat{\Delta}$. For $i = (i_1, \ldots, i_D) \in \mathbb{Z}^D$ we set $\otimes x_i = x_{i_1} \otimes \ldots \otimes x_{i_D} \in \otimes^D$. Then $\hat{H}_D$ acts on $\otimes^D$ on the right in the following way

\[
(\otimes x_i) T_k = \begin{cases} 
  v^{-2} \otimes x_i & \text{if } i_k = i_{k+1} \\
  v^{-1} \otimes x(i) s_k & \text{if } -n < i_k < i_{k+1} \leq 0 \\
  v^{-1} \otimes x(i) s_k + (v^{-2} - 1) \otimes x_i & \text{if } -n < i_{k+1} \leq 0,
\end{cases} \tag{1.5}
\]

\[
(\otimes x_i) X_j = \otimes x_{i(n_{\epsilon_j})}, \tag{1.6}
\]

where $k \in [1, D-1]$, $j \in [1, D]$ and $s_k \in \mathcal{S}_D$ is the $k$th simple transposition. Moreover, the actions of $\hat{U}_n^-$ and $\hat{H}_D$ on $\otimes^D$ commute (see [VV], Section 8.2).

1.11 Let $\Omega^D = \sum_i \text{Im}(1 + T_i) \subset \otimes^D$. For any $i \in \mathbb{Z}^D$ let $\land x_i$ be the class of $\otimes x_i$ in the quotient $\otimes^D / \Omega^D$. Then

\[
\{ \land x_i \mid i_1 > i_2 \ldots > i_D \}
\]

is a basis of $\otimes^D / \Omega^D$ (see [KMS], Proposition 1.3). If $\lambda = (\lambda_1 \geq \ldots \geq \lambda_D) \in \Pi_D$ set $| \lambda \rangle = \land x_i$ where $i_k = k - 1 - k, k = 1, \ldots, D$. Let $\Lambda^D$ be the $A$-linear span of the vectors $| \lambda \rangle$, $\lambda \in \Pi_D$. The representation of $\hat{U}_n^-$ on $\otimes^D$ descends to $\otimes^D / \Omega^D$ and restricts to $\Lambda^D$ (see [VV], 9.2). Note that $Z(\hat{H}_D)$ acts on $\otimes^D / \Omega^D$ and that $Z_D^-$ acts on $\Lambda^D$.

Let $\otimes^\infty$ be the $A$-linear span of semi-infinite monomials

\[
\land x_i = x_{i_1} \otimes x_{i_2} \otimes \ldots
\]
Hence, by (1.5), (1.6), let \( \lambda_k \) denote the class of \( \otimes X_i \) in the quotient \( \otimes X_i / \Omega \). If \( \lambda \in \Pi \) we set \( |\lambda| = \wedge X_i \) where \( i = k + 1 - k \).

Finally, set \( A = \bigoplus \mathbb{K}|\lambda \rangle \). It is shown in [KMS], Section 10.1 that \( U^-_n \) acts on \( \Lambda \). Its restriction to \( U^-_n \) is a graded subalgebra of \( U^-_n \).

### 2 A central subalgebra of \( U^-_n \).

#### 2.1 Set \( R = \bigcap_i \ker e'_i = \bigcap_i \ker e_i \subset U^-_n \) and put \( R \otimes \mathbb{K} = R \otimes \mathbb{K} \). For simplicity, we set \( s = (1, 1 \ldots 1) \in \mathbb{N}^I \). We first show

**Proposition.** The following properties hold.

i) \( R \) is a graded subalgebra of \( U^-_n \) satisfying \( R = R \) and

\[
\dim R = \begin{cases} 0 & \text{if } \alpha \notin \mathbb{N}s \\ p(k) & \text{if } \alpha = ks. \end{cases} \tag{2.1}
\]

where \( p(k) \) is the number of partitions of the integer \( k \).

ii) The subalgebras \( R \) and \( U^-_n \otimes \mathbb{K} \) commute and the multiplication map \( m \) induces an isomorphism

\[
m : U^-_n \otimes \mathbb{K} R \rightarrow U^-_n.
\]

iii) The pair \( (R, \Delta) \) is a bialgebra.

**Proof.** It is clear that \( R \) is graded. Moreover, it follows from [LT], Theorem 4.1, that

\[
\dim (R \otimes \mathbb{K}[\alpha] \cap \mathbb{L}/v(R \otimes \mathbb{K}[\alpha] \cap \mathbb{L})) = \begin{cases} 0 & \text{if } \alpha \notin \mathbb{N}s \\ p(k) & \text{if } \alpha = ks. \end{cases}
\]

Hence \( R \otimes \mathbb{K}[\alpha] \cap \mathbb{L} \) is a free \( S \)-module of the given dimension, and (2.1) follows. That \( R \) is a subalgebra is a consequence of the following equality

\[
e'_i(ab) = v^{-\omega t_i(\alpha)}ae'_i(b) + e'_i(a)b, \quad \forall a \in R[\alpha], b \in R[\beta] \tag{2.2}
\]

where we set \( \omega t_i(\gamma) = 2\gamma_i - \gamma_i - \gamma_i+1 \) for all \( \gamma \in \mathbb{N}^I \). To prove (2.2), observe that for any \( c \in U^-_n \), we have

\[
\langle e'_i(ab), c \rangle = \langle ab, f_i c \rangle = \langle a \otimes b, (f_i \otimes 1 + k e_i \otimes f_i) \Delta(c) \rangle = \sum \langle a, f_i c' \rangle \langle b, c'' \rangle + \langle a, k e_i \otimes f_i \rangle \langle b, f_i c'' \rangle = \langle c'_i(a)b + v^{-\omega t_i(\alpha)}ae'_i(b), c \rangle,
\]

where we use Sweedler’s notation \( \Delta(c) = \sum c' \otimes c'' \).

We now prove the last statement in i). For \( i \in I \), consider \( e''_i : U^-_n \rightarrow U^-_n \), \( x \mapsto e''_i(x) \). We claim that, for all \( i, j \in I \),

\[
e'_i e''_j = v^{\omega t_i(\gamma)}e''_j e'_i. \tag{2.3}
\]
To prove (2.3), set $S = e'_j e''_j - v^{wt_j} e''_j e'_j$. Then
\[ e'_j e''_j f_{ls} = e'_j (f_{ls} e''_j + v^{-1} f_{ls} e_j) = f_{ls} e'_j e''_j + v f_{ls} e_j e''_j + v^{wt_j} f_{ls} e'_j e_j + v^{\delta_{j}} - \delta_{j-1} f_{ls} e_j e_j. \]

Similarly,
\[ v^{wt_j} e_j e'_j f_{ls} = v^{wt_j} e_j e'_j f_{ls} e''_j + v^{wt_j} e_j e''_j f_{ls} e'_j + v f_{ls} e_j e''_j + v^{\delta_{j}} - \delta_{j-1} f_{ls} e_j e_j. \]

Hence $S f_{ls} = f_{ls} S$ for all $l \in \mathbb{N}^*$. In an analogous fashion, $S f_k = f_k S$ for all $k \in I$. Moreover, by [GP], the algebra $U^-_{\mathfrak{k},n}$ is generated by $f_k$, $k \in I$ and $f_{ls} l \in \mathbb{N}^*$. Hence $S = 0$ and (2.3) is proved.

We now turn to (ii). By (2.3) we have $e''_j (Ker e'_j) \subset Ker e'_j$. Hence $e''_j (R) \subset R$. But $e''_j$ is a homogeneous operator of degree $-e_j$, and by (2.1) the only non-zero graded components of $R$ are located in degrees $l$s, $l \in \mathbb{N}^*$. Thus $e''_j (R) = 0$ for all $j$, and $R \subset \bigcap_j Ker e''_j = \mathbb{R}$. Hence $\mathbb{R} = R$ as desired.

By (2.3) we have
\[ e'_j (f_j x) = \delta_{ij} x = e'_i (x f_j), \quad \forall x \in R, \ j \in I. \]

Hence $R$ is $ad f_j$-stable. But $ad f_j$ is homogeneous of degree $e_j$. This again implies that $\langle ad f_j \rangle_R = 0$. Thus the subalgebras $U^-_{\mathfrak{s}_n}$ and $R$ commute.

The operators $e'_j$ are locally nilpotent. For any $u \in U^-_{\mathfrak{s}_n}$ there exists a sequence $i_1, i_2, \ldots, i_r$ such that $e'_{i_1} \cdots e'_{i_r} u \in R$. Since the operators $e'_i$ are proportional, it follows that $e_{i_1} \cdots e_{i_r} u \in R$. Then $u = f_{i_1} \cdots f_{i_r} e_{i_1} \cdots e_{i_r} u$. Therefore the multiplication map $m : U^-_{\mathfrak{s}_n} \otimes_R R \to U^-_{\mathfrak{s}_n}$ is surjective. By [LTY], Theorem 4.1 we have and section 1.8 we have
\[ \dim R_K [\beta] = \# \{ m \in M^{\text{per}} \mid \dim m = \beta \}. \]

Moreover, it is well known (see Section 1.8) that
\[ \dim U^-_{\mathfrak{s}_n} [\alpha] = \# \{ m \in M^{\text{op}} \mid \dim m = \alpha \}, \]
\[ \dim U^-_{\mathfrak{s}_n} [\gamma] = \# \{ m \in M \mid \dim m = \gamma \}. \]

Thus,
\[ \sum_{\alpha + \beta = \gamma} \dim U^-_{\mathfrak{s}_n} [\alpha] + \dim R_K [\beta] = \dim U^-_{\mathfrak{s}_n} [\gamma]. \]

This implies that $m$ is injective, and (ii) follows.

To prove (iii), note that by definition, $R = (\sum_i f_i U^-_{\mathfrak{s}_n})^\perp$. Let $x \in R$, $y \in \sum_i f_i U^-_{\mathfrak{s}_n}$ and $u \in U^-_{\mathfrak{s}_n}$. Then $yu, uy \in \sum_i f_i U^-_{\mathfrak{s}_n}$ by (ii). Thus
\[ (\hat{\Delta}(x), y \otimes u) = (x, yu) = 0, \]
\[ (\hat{\Delta}(x), u \otimes y) = (x, uy) = 0. \]

Hence $\hat{\Delta}(x) \in (U^0 R)^{\otimes 2}$. Finally, it follows from (i) and section 1.5 that the map $U^0 R \to R$, $k_n u \mapsto u$ is an algebra homomorphism, which implies that $(R, \Delta)$ is a bialgebra. □
2.2 For \( D \in \mathbb{N} \) set \( \Gamma_D = \mathbb{A}[y_1, \ldots, y_D]^{\Sigma_D} \), and let \( \Gamma \) be the ring of symmetric functions in the variables \( y_i, i \in \mathbb{Z} \). For \( k \in \mathbb{Z}^* \), denote by \( p_k^D \), \( p_k \) the \( k \)-th power sum in \( \Gamma_D \) and \( \Gamma \). Recall that \( \Gamma = \mathbb{A}[p_1, p_2, \ldots] \) and that \( \Gamma \) is equipped with a canonical cocommutative bialgebra structure \( \Delta : \Gamma \rightarrow \Gamma \otimes \Gamma \), \( p_k \rightarrow p_k \otimes 1 + 1 \otimes p_k \) (see [KMS], I, 5). Let us denote by \( \rho_D : U_n^{-} \rightarrow \text{End} \Lambda^D \) (resp. \( \rho : U_n^{-} \rightarrow \text{End} \Lambda^\infty \)) the representations of the Hall algebra on \( \Lambda^D \) and \( \Lambda^\infty \) (see Sections 1.10 and 1.11). Identify \( \Gamma_D \) with \( Z^{-} \) via \( y_i \mapsto \chi_i^{-1} \) and let \( \sigma_D : \Gamma_D \rightarrow \text{End} \Lambda^\infty \) (resp. \( \sigma : \Gamma \rightarrow \text{End} \Lambda^\infty \)) be the representations of the center of the affine Hecke algebras (see [KMS], Section 1.1).

**Proposition.** There exists a graded bialgebra isomorphism \( \iota : (R, \Delta) \xrightarrow{\sim} (\Gamma, \Delta_0) \) such that \( \rho|_R = \sigma \circ \iota \).

**Proof.** The action of \( U_n^{-}(\hat{\mathfrak{sl}}_n) \) on \( \Lambda^\infty \) extends to a level 1 action of the whole quantum affine algebra \( U(\mathfrak{sl}_n) \), which commutes to the action of \( \Gamma \). Moreover, by [KMS], Proposition 2.3, the Fock space \( \Lambda^\infty \) decomposes as
\[
\Lambda^\infty = L(\Lambda_0) \otimes_k \Gamma
\]
where \( L(\Lambda_0) \) is the \( U(\mathfrak{sl}_n) \)-submodule generated by the highest weight vector \( v_0 \) in the irreducible representation of \( U(\mathfrak{sl}_n) \) with highest weight \( \Lambda_0 \).

**Lemma 2.1.** The actions of \( R \) and \( \Gamma \) on \( \Lambda^\infty \) commute.

**Proof.** Let \( (a_D)_D \) be a family of operators such that \( a_D \in \text{End} \Lambda^D \). We say that \( (a_D)_D \) satisfies property (*) if there exists \( N > 0 \) such that
\[
\forall s, r \in \mathbb{N}, r \geq s + N, \forall i_1, \ldots, i_r,
\]
\[
i_{i+1} = i_l - 1 \text{ if } l \geq s \Rightarrow a_r(x_{i_1} \wedge \ldots \wedge x_{i_r}) = a_{s+N}(x_{i_1} \wedge \ldots \wedge x_{i_{s+N}}) \wedge x_{i_{s+N}+1} \wedge \ldots \wedge x_{i_r}.
\]

It is easy to check that \( (p_k^D)_D \) satisfies (*) for any \( k \in \mathbb{N}^* \) and that for any \( u \in U_n^{-} \) the family \( (\rho_D(u))_D \) satisfies (*). As a consequence, the family \( ([\rho_D(u), p_k^D])_D \) satisfies (*) with constant, say \( N \). Given \( x_{i_1} \wedge x_{i_2} \wedge \ldots \) with \( i_l = 1 - l \) for \( l \geq s \), we have
\[
[\rho(u), p_k](x_{i_l}) = [p_{s+N}(u), p_k^{s+N}](x_{i_1} \wedge \ldots \wedge x_{i_{s+N}}) \wedge x_{i_{s+N}+1} \wedge \ldots = 0
\]
since the actions of \( U_n^{-} \) and \( \Gamma_{s+N} \) on \( \Lambda^{s+N} \) commute (see Section 1.10).

The proof of the following lemma will be given in the appendix (Section 4.1.)

**Lemma 2.2.** The actions of \( R \) and \( U(\mathfrak{sl}_n) \) on \( \Lambda^\infty \) commute.

It follows from Lemma 2.2 that for any \( x \in R \) there exists \( \tilde{x} \in \Gamma \) such that
\[
x.(v_0 \otimes 1) = v_0 \otimes \tilde{x}. \text{ Let } i \text{ be the map } R \rightarrow \Gamma, x \mapsto \tilde{x}. \text{ Lemmas 2.1 and 2.2 imply that } p(x) = \sigma(i(x)) \in \text{End} \Lambda^\infty. \text{ Moreover, by } [VAV] \text{ Section 9.3 we have } U_n^{-}[0] = \Lambda^\infty. \text{ Hence } i \text{ is surjective. The map } i \text{ is a graded algebra homomorphism, where } \deg p_k = ks. \text{ Since } R \text{ and } \Gamma \text{ have the same graded dimensions, it follows that } i \text{ is an isomorphism.}
Finally, we prove that $i$ is compatible with the bialgebra structures. For $D_1, D_2 ∈\mathbb{N}$, consider the map

$$μ_{D_1, D_2} : Λ^{D_1} \otimes Λ^{D_2} → Λ^{D_1+D_2}, \quad ∧x_i ∧ x_j → ∧x_{ij}$$

where $ij$ is the sequence $i$ followed by the sequence $j$. Then, by Section 1.11, (1.4) and the inclusion $R ⊂ Σ_i U_i [s]$, and the geometric description of $U_i [s]$.

Proof. Let $k ∈\mathbb{N}$ and put $c_k = i^{-1}(p_k)$. Choose $N \gg 0$ such that property $(*)$ is satisfied by $(ρ_D(u))$ for any $u ∈ \bigoplus_{t ≤ k} R[t]$ with constant $N$. Thus,

$$∀ s, r, D ∈\mathbb{N}, r ≥ s + D, ∀ i ∈ Z^D \text{ such that } i_{t+1} = i_t - 1 \text{ if } l ≥ s,$$

$$c_k(x_i ∧ \ldots ∧ x_{i_D}) = p_k^{s+D}(x_{i_1} ∧ \ldots ∧ x_{i_s}) ∧ x_{i_{s+D}} ∧ \ldots ∧ x_{i_D}.$$

In particular, if $i ∈ Z^{D_1}$ and $j ∈ Z^{D_2}$ both satisfy the condition in $(*)$ then

$$μ_{D_1, D_2}(ρ_{D_1} ⊗ ρ_{D_2})Δ(c_k)(∧x_i ∧ x_j) = μ_{D_1, D_2}(1 ⊗ c_k + c_k ⊗ 1)(∧x_i ∧ x_j)$$

It is easy to see that this implies that $Δ(c_k) = 1 ⊗ c_k + c_k ⊗ 1$ as desired. ■

Propositions 2.1 and 2.2 together imply

**Theorem.** There exists a graded algebra isomorphism $U^+_n \simeq U^+(\hat{a}_n) ⊗_Λ R$ where $R ≃ Λ[z_1, z_2, \ldots]$ with $\deg z_i = is ∈ \mathbb{N}^+$. 

2.3 Let us denote by $ρ'_D$ and $σ'_D$ the representations of $U^+_n$ and $Γ_D$ on $⊗^D$. Then in fact:

**Proposition.** There exist graded algebra morphisms $i_D : R → Γ_D$, $D ∈ \mathbb{N}^+$ such that $i = \lim_{→} i_D$ and $ρ'_D = σ'_D ∘ i_D$.

**Proof.** It follows from (1.3) and from the fact that $R$ belongs to the center of $U^+_n$ that, for any $k ∈\mathbb{N}$ there exists $k' ∈\mathbb{N}$ such that

$$∀ i ∈ \mathbb{Z}, \quad c_k x_i = x_{i+nk'}.$$

Then, by Prop. 2.2, for all $D ∈\mathbb{N}$,

$$∀ i ∈ Z^D, \quad ρ(D)(c_k)(x_{i_1} ⊗ \ldots ⊗ x_{i_D}) = σ'_D(p_k^D)(x_{i_1} ⊗ \ldots ⊗ x_{i_D}).$$

In particular, $ρ(D)(c_k) = σ_D(p_k^D)$ for all $D ∈\mathbb{N}$. This implies $k = k'$ and proves the proposition. ■

2.4 Let $\{b_m^*\}_{m ∈\mathbb{M}}$ be the dual basis of $\{b_m\}_{m ∈\mathbb{M}}$ with respect to $(\ , \ )$.

**Proposition.** We have $R = Σ_{m ∈\mathbb{M}_{tor}} C[v, v^{-1}]b_m^*$.

**Proof.** By definition we have

$$R = \bigcap_i \text{Ker } e_i' = \left( \sum_i f_i U_{k,n}^- \right)^⊥.$$

From [4], Th.14.3.2 and from the geometric description of $U^-_n$ in terms of Frobenius traces of perverse sheaves on $E_{Q_d}$, $d ∈ \mathbb{N}^{2/2}$ (see [VV]) it follows that

$$\sum_i f_i U_{k,n}^- = \bigoplus_{i, m ∈f_i C} C[v, v^{-1}]b_m = \bigoplus_{m ∈\mathbb{M}_{tor}} C[v, v^{-1}]b_m.$$

This proves the Proposition. ■
3 Proof of the Varagnolo-Vasserot conjecture

3.1 Set \( L_\Lambda = \bigoplus_\Lambda S(\lambda) \subset \Lambda^\infty \). Leclerc and Thibon have defined a semilinear involution \( a \mapsto \pi a \) on \( \Lambda^\infty \) such that

i) \( |0\rangle = |0\rangle \),

ii) \( \pi u a = u \pi a \) for all \( u \in U_n^-, a \in \Lambda^\infty \),

iii) \( \pi k a = p_k a \) for all \( k \in \mathbb{N}^*, a \in \Lambda^\infty \),

(see \([LT]\), \([VV]\)).

For \( \lambda \in \Pi \), set \( m(\lambda) = \sum_i [1-i, \lambda_i - i] \in M \). To simplify notations, put \( f_\lambda = f_{m(\lambda)} \) and \( b_\lambda = b_{m(\lambda)} \). If \( \lambda \in \Pi \), let \( n(\lambda) \) be the partition \( ((\lambda_1)^n, (\lambda_2)^n, \ldots) \). Thus \( \mathcal{M}^{reg} = \{ m(n(\lambda)), \lambda \in \Pi \} \). Leclerc and Thibon introduced in \([LT]\) two canonical bases \( B^\pm = \{ b^\pm_\lambda, \lambda \in \Pi \} \) of \( \Lambda^\infty \) characterized by

\[
\quad b^\pm_\lambda = b^\pm_\lambda, \quad b_\lambda^+ |\lambda\rangle + v \bigoplus_{\mu < \lambda} S |\mu\rangle, \quad b_\lambda^- |\lambda\rangle + v^{-1} \bigoplus_{\mu < \lambda} |\mu\rangle.
\]

(3.1)

The following was conjectured in \([VV]\) and is the main result of this paper:

**Theorem.** For all \( \lambda \in \Pi \) we have \( b_\lambda |0\rangle = b_\lambda^+ \).

The rest of this section is devoted to the proof of this theorem.

3.2 Recall that a partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \) is called \( n \)-regular if \( \lambda_i > \lambda_{n+i} \) for all \( i \) such that \( \lambda_i \neq 0 \). Let \( \Pi^{reg} \) be the set of all \( n \)-regular partitions. We first show

**Lemma 3.1.** We have \( b_\lambda |0\rangle = b_\lambda^+ \) if \( \lambda \in \Pi^{reg} \).

**Proof.** Consider the scalar product \( (, ) \) on \( \Lambda^\infty \) for which \( \{ |\lambda\rangle \} \) is an orthonormal basis. Recall that \( \Lambda^\infty \) is isomorphic to \( L(\Lambda_0) \otimes_\Lambda \Gamma \) as a \( U^-_n \otimes_\Lambda \Gamma \)-module. It is shown in \([LT2]\) that the restriction of \( (, ) \) and of the involution \( a \mapsto \pi a \) to \( L(\Lambda_0) \) coincide with the Kashiwara scalar product and involution defined on any simple integrable \( U^-_n \)-module (\([K]\), Sections 2 and 6). Thus the lower crystal basis of \( L(\Lambda_0) \) is a subset of \( \{ \pm b_\lambda^+ \} \). Note that \( m(\lambda) \in \mathcal{M}^{reg} \) if and only if \( \lambda \) is \( n \)-regular. Therefore, by Section 1.8 and the general theory of canonical bases

\[
\lambda \in \Pi^{reg} \Rightarrow b_\lambda |0\rangle \subset \{ \pm b_\lambda^+ \}.
\]

(3.2)

Moreover, by \([VV]\), Section 9.2, for any \( \lambda \in \Pi \) and any orbit \( O \subset \bar{O}_\lambda \setminus O_\lambda \) we have

\[
f_\lambda |0\rangle \in |\lambda\rangle + \bigoplus_{\mu < \lambda} \mathbb{A}|\mu\rangle, \quad f_O |0\rangle \in \bigoplus_{\mu < \lambda} \mathbb{A}|\mu\rangle.
\]

(3.3)

It is now clear from (3.1) and (3.2), (3.3) that \( b_\lambda |0\rangle = b_\lambda^+ \) if \( \lambda \) is an \( n \)-regular partition. \( \blacksquare \)
3.3 Set $\mathcal{L}^{\text{reg}} = \bigoplus_{m \in \mathcal{M}^{\text{reg}}} \mathcal{S}b_m$. It is known that $\mathcal{L}^{\text{reg}}$ is the smallest $\mathbb{S}$-submodule of $U_{\mathbb{Z}}$ containing 1 and stable by the operators $f_i$, $i \in I$ (c.f. [K] and Section 1.8). Set $\mathcal{L}_R = \mathcal{L} \cap R$. If $V$ is any $\mathbb{S}$-module $V$, we let $\overline{\mathcal{V}} = V \otimes_{\mathbb{S}} \mathbb{C}[[v]]$ be its completion with respect to the $v$-adic topology.

Lemma 3.2. The multiplication defines a graded isomorphism

$$\overline{\mathcal{L}}^{\text{reg}} \otimes_{\mathbb{C}[[v]]} \overline{\mathcal{L}}_R \xrightarrow{\sim} \overline{\mathcal{L}}.$$

Proof. Every multisegment $n$ decomposes in a unique way as $n = p + a$ where $p \in \mathcal{M}^{\text{per}}$ and $a \in \mathcal{M}^{\text{ap}}$. Since $b_n$ belongs to the connected component of the crystal graph $C$ containing 1, there exists a sequence $i_1, \ldots, i_r$ such that $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot 1 \equiv f_n \pmod{v \mathcal{L}}$. Moreover, by Section 1.7 there exists $x \in \mathcal{L}_R$ such that $x \equiv f_p \pmod{v \mathcal{L}}$. Since the left multiplication by $x$ commutes with the $\tilde{f}_i$ (see the proof of Proposition 2.1), we have

$$xf_{i_1} \cdots \tilde{f}_{i_r} \cdot 1 = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot x$$

$$\equiv \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot f_p \pmod{v \mathcal{L}}$$

$$\equiv f_n \pmod{v \mathcal{L}}$$

Hence the multiplication map induces an isomorphism modulo $v$, and hence an isomorphism over $\mathbb{C}[[v]]$. \hfill \blacksquare

3.4 For $\nu \in \Pi$ let $s_\nu \in \Gamma$ be the Schur polynomial and set $a_\nu = i^{-1}(s_\nu) \in R$. Then $\{a_\nu, \nu \in \Pi\}$ is an $\mathbb{A}$-basis of $R$.

Lemma 3.3. The following holds:

i) $v^{(n-1)|\lambda|}a_{\mu\lambda} \subseteq L_{\lambda}$,

ii) $a_{\mu}(0) \in (-v)^{(n-1)|\mu|}(n\mu + vL_{\lambda})$.

iii) More generally, let $\lambda \in \Pi$ and write $m(\lambda) = m(\lambda') + m(n\mu)$ where $\lambda' \in \Pi^{\text{reg}}$. Then

$$a_{\mu|\lambda'} \subseteq (-v)^{-(n-1)|\mu|}(\lambda + vL_{\lambda}).$$

Proof. Statement i) follows from [LT], Theorem 6.3. Statements ii) and iii) are proved as in [LT], Theorem 6.7. \hfill \blacksquare

Let us denote by $<$ the order on multisegments such that $m \leq n$ if $O_m \subset O_n$.

Proposition. There holds

i) $a_\lambda \subseteq (-v)^{-(n-1)|\lambda|}(f_{n\lambda} + vL_{\lambda})$,

ii) $L_{\mathbb{R}} \cdot L_{\lambda} \subseteq L_{\lambda}$ and $L[0] \subseteq L_{\lambda}$.

Proof. Let $i_j$ be statement i) restricted to all $\lambda \in \Pi$ with $|\lambda| \leq k$, and let $ii_j$ be statement ii) restricted to $\bigoplus_{k' \leq k} L_{\mathbb{R}}[k']$, and $L[d]$, $d = (d_1, \ldots, d_n)$, $d_i \leq k$.

We will prove $i_j$ and $ii_j$ by induction. The case $k = 1$ is a consequence of Lemma 3.2 and the following formula:

$$a_{(1)} = \sum_{|m| = a} (v - v^{-1})^{(n-1)|r| - \sum_{i=1}^{|r|} (i-1)f_m. \quad (3.4)$$
Indeed, let $x$ denote the r.h.s of (3.4). A direct computation using [LTV], Proposition 4.1, shows that $x \in R$. Since $\dim R[\tau] = 1$, we have $a_{(1)} = cx$ for some $c \in K$. Using [LTV] Theorem 6.3, we see that the coefficient of $|\langle n \rangle|$ in $a_{(1)}|0\rangle$ is equal to 1. On the other hand, by (3.3), we have

$$cx|0\rangle = c \sum_{\mu \in \mathbb{N}} (v - v^{-1})^{(n-1) - \sum_\nu (l_i - 1) f_{\nu}|0\rangle} \in c|\langle n \rangle\rangle + \bigoplus_{\mu < \langle n \rangle} A|\mu\rangle$$

Therefore $c = 1$ and (3.4) is proved.

For $\lambda, \mu \in \Pi$ and $x \in R$ let $\Delta_{\lambda, \mu}(x) \in A$ be the coefficient of $f_{\alpha} \otimes f_{\mu}$ in $\Delta(x)$, where $\Delta(x)$ is expressed in the basis $(f_{\alpha} \otimes f_{\mu})_{\alpha, \mu}$. For $\lambda, \mu, \nu \in \Pi$, $|\lambda| = |\mu| + |\nu|$ we let $c_{\lambda, \mu, \nu} \in \mathbb{N}$ be the Littlewood-Richardson multiplicity (see [M], Section 5).

**Lemma 3.4.** For all $\lambda, \mu \in \Pi$, $l \in \mathbb{N}^*$ and $m \in \mathcal{M} \setminus \mathcal{M}_{\text{per}}$ we have

$$\Delta_{(1)^l, \mu}(f_{n\lambda}) \in c_{(1)^l, \mu} + v\mathcal{S}, \quad \text{and} \quad \Delta_{(1)^l, \mu}(f_m) \in v\mathcal{S}.$$  

**Proof.** See the appendix.\[\]

Now let $k > 1$ and suppose that $i)_{k-1}$ and $ii)_{k-1}$ hold. Let $d_\nu \in \mathbb{Z}$, $\nu \in \Pi$, be such that $\{v^{d_\nu} a_\nu\}$ is a $S$-basis of $L_{R}$. It follows from the crystal graph $C$ of $U_n^-$ that $e_i^\prime(b_m^0) = 0$ for all $i \in I$ if and only if $m \in \mathcal{M}_{\text{per}}$. Hence

$$a_\nu = (-v)^{-d_\nu} \left( \sum_{\sigma \in \Pi} \alpha_\sigma f_{n\sigma} + \sum_{1} \beta_1 f_{l \in \mathcal{M}} \right)$$  

for some $\alpha_\sigma \in \mathbb{C}$ and $\beta_1 \in v\mathcal{S}$. Thus, by Lemma 3.4,

$$\Delta_{(1)^l, \mu}(a_\nu) \in (-v)^{-d_\nu} \left( \sum_{\sigma} \alpha_\sigma c_{(1)^l, \mu} + v\mathcal{S} \right).$$  

(3.6)

On the other hand, by [M], Section 5.3 and Proposition 2.3, we have

$$\Delta(a_\nu) = \sum_{|\lambda| + |\mu| = k} c_{\lambda, \mu} a_\lambda \otimes a_\mu.$$  

(3.7)

Using the induction hypothesis $i)_{k-1}$ we obtain, for $|\lambda|, |\mu| \neq 0$

$$\Delta_{\lambda, \mu}(a_\nu) \in (-v)^{-(n-1)|\nu|} (c_{\lambda, \mu} + v\mathcal{S}).$$

(3.8)

For any $\nu \in \Pi$ there exists some $l \in \mathbb{N}^*$ and $\mu \in \Pi$ such that $c_{(1)^l, \mu} = 1$ (see [M], (5.17)). Combining (3.6), (3.8), we see that $d_\nu \geq (n-1)|\nu|$. In particular, it follows from Lemma 3.3 $i)$ that $v^{d_\nu} a_\nu \mathcal{L}_\lambda \subset \mathcal{L}_\lambda$. Since $\{v^{d_\nu} a_\nu\}_{|\nu| = k}$ is an $S$-basis
of $\mathcal{L}_R[k\tau]$ we obtain the first statement of $ii)_k$. The second part of Statement $ii)_k$ follows from Lemmas 3.1 and 3.2. Then $ii)_k$ implies

$$a_\nu |0\rangle \in (-v)^{-d_\nu} \left( \sum_{\sigma} \alpha_{\sigma} f_{\nu \sigma} + v\mathcal{L}[k\tau] \right) |0\rangle$$

$$\in (-v)^{-d_\nu} \left( \sum_{\sigma} \alpha_{\sigma} f_{\nu \sigma} |0\rangle + v\mathcal{L}_\Lambda \right).$$

Using (3.3), $ii)_k$ and Lemma 3.3 $ii)$, we get

$$d_\nu = (n - 1)|\nu|, \quad \alpha_\nu = 1, \quad \alpha_\sigma \neq 0 \Rightarrow \sigma \leq \nu.$$  

Finally, combining (3.5) and (3.6) now yields

$$\forall l \in \mathbb{N}, \forall \mu \in \Pi, \quad \sum_{\sigma < \nu} c^\sigma_{(1)^l, \mu} \alpha_\sigma = 0.$$  

For any fixed $\nu$, this system is nondegenerated and admits the unique solution $(\alpha_\sigma)_{\sigma < \nu} = 0$. Indeed let $(\ , \ )$ be the nondegenerate symmetric bilinear form on $\Gamma$ for which $\{s_\lambda\}$ is an orthonormal basis, and set $X = \sum_\sigma \alpha_\sigma s_\sigma$. Then for any $\mu$ and $l$ we have

$$\langle s_{(1)^l} s_\mu, X \rangle = \left( \sum_{\sigma} c^\sigma_{(1)^l, \mu} s_\sigma, X \right) = \sum_{\sigma} c^\sigma_{(1)^l, \mu} \alpha_\sigma = 0.$$  

Moreover $\langle s_{(k)} X \rangle = 0$ since $X \in \bigoplus_{\sigma < \nu} \Lambda s_\sigma \subset \bigoplus_{\sigma < (k)} \Lambda s_\sigma$. But

$$\Gamma|k\rangle = \Lambda s_{(k)} \oplus \sum_{l=1}^{k-1} (1) s_{(1)^l} \Gamma|k-l\rangle.$$  

Hence $X = 0$. Statement $i)_k$ is proved and the induction is complete. \hfill \blacksquare

3.5 Proof of Theorem 3.1. Let $\lambda \in \Pi$. The multisegment $m(\lambda)$ decomposes in a unique way as $m(\lambda) = m(\lambda') + m(n\mu)$ for some partitions $\lambda' \in \Pi^{\text{reg}}$ and $\mu \in \Pi$. Moreover, from Section 1.7 the element $b_\lambda$ is in the connected component of the crystal graph $\mathcal{C}$ containing $b_{n\mu}$. Hence, by Proposition 3.4 $i)$ and the proof of Lemma 3.2, we have

$$b_\lambda \equiv f_\lambda \equiv (-v)^{(n-1)|\mu|} a_\mu f_{\lambda'} \pmod{v\mathcal{L}}.$$  

Then, by Proposition 3.4 $ii)$, Lemma 3.3 $i)$ and the fact that $b_{\lambda'} - f_{\lambda'} \in v\mathcal{L}$, we get

$$(-v)^{(n-1)|\mu|} a_\mu f_{\lambda'} |0\rangle \equiv (-v)^{(n-1)|\mu|} a_\mu b_{\lambda'} |0\rangle \pmod{v\mathcal{L}_\Lambda}.$$  

Finally, by Lemma 3.1, and Lemma 3.3 $i)$ and $iii)$, we have

$$(-v)^{(n-1)|\mu|} a_\mu b_{\lambda'} |0\rangle \equiv (-v)^{(n-1)|\mu|} a_\mu b_{\lambda'}^+ |0\rangle \pmod{v\mathcal{L}_\Lambda}$$

$$\equiv (-v)^{(n-1)|\mu|} a_\mu |\lambda'\rangle \pmod{v\mathcal{L}_\Lambda}$$

$$\equiv |\lambda\rangle \pmod{v\mathcal{L}_\Lambda}.$$  

Thus $b_\lambda |0\rangle \in |\lambda\rangle + v \bigoplus_{\mu < \lambda} S|\mu\rangle$. Finally, $\overline{b_\lambda |0\rangle} = \overline{b_{\lambda'} |0\rangle} = b_\lambda |0\rangle$ by Section 3.1. Hence $b_\lambda |0\rangle = b_\lambda^+$ as desired. \hfill \blacksquare
4 An analogue of the Varagnolo-Vasserot conjecture for higher-level Fock spaces

In this section we sketch the generalization of Theorem 3.1 to the case of the higher-level Fock spaces. We use the definitions and notations of [1].

5.1 Let \( l > 1 \) and \( s \in \mathbb{Z} \). Let \( \Lambda^{s+\infty} \) be the semi-infinite wedge product of levels \( l \) and \( n \) and charge \( s \) (see [1], Section 4.1). Let

\[
\mathcal{Z}^l(s) = \{ s_l = (s_1, \ldots, s_l) \in \mathbb{Z}^l \mid \sum_i s_i = s \}. 
\]

Recall that \( \Lambda^{s+\infty} \) is equipped with a distinguished \( \Lambda \)-basis \( \{ |\lambda_i, s_i\rangle \} \) where \( \lambda_i = (\lambda_1, \ldots, \lambda_l) \in \Pi^l \) and \( s_l \in \mathcal{Z}^l(s) \). It is endowed with three commuting left actions: \( \rho_{l,s} : U(\hat{sl}_n) \to \text{End} (\Lambda^{s+\infty}) \), \( \rho'_{l,s} : U'(\hat{sl}_l) \to \text{End} (\Lambda^{s+\infty}) \) where \( U'(\hat{sl}_l) \) denotes the Lusztig integral form of the quantum affine algebra of type \( A_{l-1}^{(1)} \) with quantum parameter \( v^{-1} \) and the action of a Heisenberg algebra \( \mathcal{H} \) generated by operators \( B_m, m \in \mathbb{Z}^+ \) (see [1], Sections 4.2 and 4.3). Moreover, \( \Lambda^{s+\infty} \) is an integrable module for \( U(\hat{sl}_n) \) and \( U'(\hat{sl}_l) \). We denote by \( \mathcal{H}^+ \) (resp. \( \mathcal{H}^- \)) the subalgebra of \( \mathcal{H} \) generated by \( B_{-m}, m \in \mathbb{N} \) (resp. \( B_m, m \in \mathbb{N} \)).

Set \( \Lambda^{s+\infty}_K = \Lambda^{s+\infty} \otimes_{\Lambda} K \). The Fock space \( \Lambda^{s+\infty}_K \) decomposes under these actions as follows (see [1], Theorem 4.10). Set

\[
\mathcal{A}_l^s = \{ s_l \in \mathcal{Z}^l(s) \mid s_1 \geq s_2 \geq \ldots \geq s_l, s_l - s_i \leq n \}. 
\]

Then for \( s_l \in \mathcal{A}_l^s \) the vector \( |0, s_l\rangle \) is singular for \( U(\hat{sl}_n) \), \( U'(\hat{sl}_l) \) and \( \mathcal{H} \), i.e we have

\[
U(\hat{sl}_n)^+|0, s_l\rangle = U'(\hat{sl}_l)^+|0, s_l\rangle = \mathcal{H}^+|0, s_l\rangle = 0, 
\]

and

\[
\Lambda^{s+\infty}_K = \bigoplus_{s_l \in \mathcal{A}_l^s} U(\hat{sl}_n) \cdot \mathcal{H} \cdot U'(\hat{sl}_l) |0, s_l\rangle. 
\]

Moreover, for any \( s_l \in \mathcal{Z}^l(s) \), \( \Lambda_{s_l} = \bigoplus_{\lambda_i \in \Pi^l} \Lambda_{s_l}(\lambda_i) \) is a \( U(\hat{sl}_n) \)-module, and \( |0, s_l\rangle \) generates an irreducible integrable \( U(\hat{sl}_n) \)-module \( V_{s_l} \) of highest weight \( \Lambda = \Lambda_{s_1} + \Lambda_{s_2} + \ldots + \Lambda_{s_l} \). Here \( \Lambda_i, 0 \leq i \leq n - 1 \) is the \( i \)th fundamental weight of \( \hat{sl}_n \) and we set \( \Lambda_i = \Lambda_j \) if \( i \equiv j \) (mod \( n \)).

Set \( \mathcal{L}_{\Lambda^{s+\infty}} = \bigoplus_{\lambda_i, s_i} \mathcal{L}_{\lambda_i, s_i}(\lambda_i) \) and \( \mathcal{L}_{s_l} = \bigoplus_{\lambda_i} \mathcal{L}_{\lambda_i, s_l}(\lambda_i) \) for any \( s_l \in \mathcal{Z}^l(s) \). Let \( b_{\lambda_i, s_l} \) denote the image of \( |\lambda_i, s_l\rangle \) in \( \mathcal{L}_{\Lambda^{s+\infty}} / o \mathcal{L}_{\Lambda^{s+\infty}} \). Set \( B_{\Lambda^{s+\infty}} = \{ b_{\lambda_i, s_l} \mid \lambda_i \in \Pi^l, s_l \in \mathcal{Z}^l(s) \} \) and for any \( s_l \in \mathcal{Z}^l(s) \) set \( B_{s_l} = \{ b_{\lambda_i, s_l} \mid \lambda_i \in \Pi^l \} \). The following is proved in [JMMO].

**Proposition.** The couple \( (\mathcal{L}_{\Lambda^{s+\infty}}, B_{\Lambda^{s+\infty}}) \) is a crystal basis of the \( U(\hat{sl}_n) \)-module \( \Lambda^{s+\infty}_K \).

The crystal graph structure of \( B_{\Lambda^{s+\infty}} \) is explicitly described in [JMMO].
In [4], Uglov has defined a semilinear involution \( a \mapsto \tau \) on \( \Lambda^{s+\infty} \) satisfying

i) \( |0, s_i \rangle = |0, s_i \rangle \) for any \( s_i \in \mathbb{Z}^l(s) \),

ii) \( \tau^m = \pi \tau \) for all \( u \in U(\hat{s}_n), U'(\hat{s}_l), a \in \Lambda^{s+\infty} \),

iii) \( B_m a = B_m \tau a \) for all \( m \in \mathbb{N}^*, a \in \Lambda^{s+\infty} \).

Uglov also introduced two canonical bases \( \{ \mathbf{b}_{\Lambda, s_i}^\pm \}_{\lambda_i \in \Pi^l, s_i \in \mathcal{Z}(s)} \) characterized by the following properties

\[
\mathbf{b}_{\Lambda, s_i}^\pm = \mathbf{b}_{\Lambda, s_i}^\pm,
\]

\[
\mathbf{b}_{\Lambda, s_i}^+) \in |\lambda_i, s_i \rangle + v \bigoplus_{\mu_i, t_i} S|\mu_i, t_i \rangle, \quad \mathbf{b}_{\Lambda, s_i}^-) \in |\lambda_i, s_i \rangle - v^{-1} \bigoplus_{\mu_i, t_i} S|\mu_i, t_i \rangle.
\]

The set \( \mathbf{B}_{s_i}^+ = \{ \mathbf{b}_{\lambda, s_i}^+ | \lambda_i \in \Pi^l \} \) is a basis of \( \Lambda_n \) which contains the lower canonical basis of the irreducible \( U(\hat{s}_n) \)-module \( V_{s_i} \).

5.2 By Theorem 2.2, we can extend the action of \( U(\hat{s}_n) \) on \( \Lambda^{s+\infty} \) to an action of \( U^- \) by setting \( \mu_i, s_i(i^{-1}(p_k)) = B_{-k} \). Recall that \( \mathcal{L} = \bigoplus_m \mathfrak{S} \mathfrak{M}_m \) and \( \mathcal{L}_R = \mathcal{L} \cap \mathbb{R} \) denote the integral lattices in \( U^- \) and \( \mathbb{R} \), respectively, and that \( a_{\mu} = i^{-1}(p_{\mu}) \in \mathbb{R} \) defines the Schur polynomial associated to \( \mu \in \Pi \) (see Section 3.4).

**Proposition.** We have, for any \( \mu \in \Pi \) and \( s_i \in \mathbb{Z}^l(s) \),

i) \( (-v)^{l(1)}|\mu\|_{L^{\Lambda^{s+\infty}}} \subset L_{\Lambda^{s+\infty}} \) and \( (-v)^{l(1)}|\mu\|_{L_{\Lambda^{s+\infty}}} \subset B_{s_i} \),

ii) \( \mathcal{L}(0, s_i) \subset L_{\Lambda^{s+\infty}} \).

**Proof.** It will be convenient to use the dual indexation of elements of the basis \( \{|\lambda_i, s_i \rangle\} \) by pairs \((\lambda_n, s_n)\) where \( \lambda_n \in \Pi^n \) and

\[
s_n \in \mathbb{Z}^n(s) = \{(s_1, \ldots, s_n) \in \mathbb{Z}^n | \sum_i s_i = s \}
\]
as explained in [3], Section 4.1. In particular we set \( \Lambda_{s_n} = \bigoplus_{\mu_n} A|\mu_n, s_n \rangle \) and \( L_{\Lambda_{s_n}} = \bigoplus_{\mu_n} \mathfrak{S}|\mu_n, s_n \rangle \). Now let \( \mu_n \in \Pi^n \), \( s_n \in \mathbb{Z}^n(s) \). It follows from [3], Corollary 5.6, \( i' \) that \( (-v)^{l(1)}|\lambda_{\mu} \|_{\Lambda_{s_n}} \in L_{\Lambda_{s_n}} \) if \( (\mu_n, s_n) \) is \( l|\lambda| \)-dominant (see [3], Section 5.1). Let \( e_i, f_i, i = 0, \ldots, n-1 \) be the Kashiwara operators corresponding to the \( U(\hat{s}_n) \)-action on \( \Lambda^{s+\infty} \). By [3], Corollary 4.9, there exists a sequence \( j_1, \ldots, j_r \) and operators \( \hat{x}_1, \ldots, \hat{x}_r \) with \( \hat{x}_i \in \{e_j, f_j\} \) such that \( \hat{x}_1 \ldots \hat{x}_r |\mu_n, s_n \rangle \) is a sum of \( l|\lambda| \)-dominant vectors. Then

\[
\hat{x}_1 \ldots \hat{x}_r(-v)^{l(1)}|\lambda_{\mu} \|_{\mu_n, s_n} = (-v)^{l(1)}|\lambda_{\mu} \|_{\mu_n, s_n} \in L_{\Lambda_{s_n}}.
\]

But \( \hat{x}_1 \ldots \hat{x}_r \) defines an isomorphism \( \Lambda_{s_n} \to \hat{x}_1 \ldots \hat{x}_r \Lambda_{s_n} \), which restricts to an isomorphism

\[
L_{\Lambda_{s_n}} \otimes_{\Lambda} \mathbb{C}[v] \to \hat{x}_1 \ldots \hat{x}_r L_{\Lambda_{s_n}} \otimes_{\Lambda} \mathbb{C}[v] = (L_{\Lambda^{s+\infty}} \cap \hat{x}_1 \ldots \hat{x}_r L_{\Lambda_{s_n}}) \otimes_{\Lambda} \mathbb{C}[v].
\]

It follows that \( (-v)^{l(1)}|\lambda_{\mu} \|_{\mu_n, s_n} \in L_{\Lambda_{s_n}} \). Hence \( (-v)^{l(1)}|\lambda_{\mu} \|_{\lambda^{s+\infty}} \subset L_{\Lambda^{s+\infty}} \) and the first statement of \( i \) is proved. The second statement of \( i \) is proved in the same way using Lemma 3.3 and [3] Corollary 5.6 \( i' \).

By the general theory of canonical bases, \( \mathbf{b}_{m}^+(0, s_i) \in \mathbf{B}_{s_i}^+ \cap V_{s_i} \) for any \( m \in \mathcal{M}^{\mathbb{R}} \) (see [3], Section 4.4). Hence \( \mathcal{L}(0, s_i) \subset L_{\Lambda_{s_i}} \). Statement \( ii \) is now a consequence of Lemma 3.2. \blacksquare
Theorem. For any \( s_i \in \mathbb{Z}^l(s) \) we have \( B|0, s_i \rangle \subseteq B^+_i \cup \{0\} \).

Proof. By \([U]\), Section 4.4, the basis \( B^+_i \) contains the upper canonical basis of the irreducible \( U(sl_n) \)-module \( V_{s_i} \). Thus, by Section 1.11

\[
\mathbf{b}_m|0, s_i \rangle \in B^+_i \cup \{0\}, \quad \forall \; m \in M^{ap}.
\]

Now any \( m \in M \) decomposes as \( m = m(\lambda') + m(\mu) \) for some \( \lambda' \in \Pi^{\text{reg}} \) and \( \mu \in \Pi \). By Lemma 3.3 and Proposition 3.4,

\[
\mathbf{b}_m \equiv (-v)^{(n-1)|\mu|}a_{\mu}f_{\lambda'} \pmod{v\mathcal{L}}.
\]

Thus, by Proposition 5.2 ii) and the fact \( \mathbf{b}_{\lambda'} - f_{\lambda'} \in v\mathcal{L} \), we have

\[
\mathbf{b}_m|0, s_i \rangle \equiv (-v)^{(n-1)|\mu|}a_{\mu}f_{\lambda'}|0, s_i \rangle \equiv (-v)^{(n-1)|\mu|} \mathbf{b}_{\lambda'}|0, s_i \rangle \pmod{\mathcal{L}_{\lambda_i}}
\]

Now by Proposition 5.2 i),

\[
(-v)^{(n-1)|\mu|} \mathbf{b}_{\lambda'}|0, s_i \rangle \equiv \nu|s_i \rangle
\]

for some \( \nu \in \Pi^l \) if \( \mathbf{b}_{\lambda'}|0, s_i \rangle \in B^+_i \) and \( (-v)^{(n-1)|\mu|} \mathbf{b}_{\lambda'}|0, s_i \rangle = 0 \) if \( \mathbf{b}_{\lambda'}|0, s_i \rangle = 0 \). Moreover, by \([U]\), Proposition 4.12, \( \mathbf{b}_m|0, s_i \rangle = \mathbf{b}_m|0, s_i \rangle = \mathbf{b}_m|0, s_i \rangle \). Hence \( \mathbf{b}_m|0, s_i \rangle = \mathbf{b}_{\lambda_i, s_i} \) in the first case and \( \mathbf{b}_m|0, s_i \rangle = 0 \) in the second case. \( \blacksquare \)

Let \( C^0_{s_i} \) be the subgraph of the crystal graph of \( \Lambda_{s_i} \) corresponding to \( U^-\mathcal{L}|0, s_i \rangle \). The above theorem implies the following special case of the positivity conjecture of Uglov (see \( [U] \), Section 4):

Corollary. For any \( s_i \in \mathbb{Z}^l(s) \) and any \( \lambda_i \in C^0_{s_i} \) we have

\[
\mathbf{b}_{\lambda_i, s_i} \in \bigoplus_{\mu|v|} \mathbb{N}[v]|\mu|, s_i \rangle.
\]

5 Appendix

6.1 Proof of Lemma 2.2. Let us prove that for any \( u \in U^-_n \) we have :

\[
[\rho(e_i), \rho(u)] = \rho\left(\frac{k_i e''_i(u) - k_i^{-1} e'_i(u)}{v - v^{-1}}\right).
\]

This result is well-known for \( U^- (\hat{sl}_n) \) (see \( [K] \), Lemma 3.4.1). Let us prove it for \( \mathfrak{f}_{s_n}, l \in \mathbb{N}^* \). We use the presentation of \( \Lambda^\infty \) in terms of Young diagrams and the description of the representation \( \rho \) in terms of the Hall algebra \( U^\infty \) of the infinite quiver and the quantum enveloping algebra \( U(sl^\infty) \), as given in \( [VV] \). We keep the notations of \( [VV] \), Section 6. In particular, let \( \gamma_d : U^\infty \to U^\infty \) be the map defined in \( [VV] \), Section 6; For any \( d = (d_1, d_2, \ldots) \in \mathbb{N}^{\mathbb{Z}} \) and \( k \in \mathbb{N} \) we put

\[
h(d) = \sum_{i<j, i \equiv j} d_i (d_{j+1} - d_j), \quad d' = \sum_{j<n; j \equiv m} d_j \varepsilon_m, \quad k'' = \sum_{r<k; r \equiv k} \varepsilon_r.
\]
Let $\pi : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{I}$ be the reduction modulo $n$. To avoid confusion, we will denote by $e_i, f_i$ the elements of $U(\hat{\mathfrak{s}}_n)$ and $U_n$, and by $e_i, f_i$ the elements of $U(\hat{\mathfrak{s}}_{\infty})$. Finally, to simplify notations, we will write $\sum_{\alpha < \beta}$ for $\sum_{\alpha < \beta, \alpha \equiv \beta}$. We have, in $\text{End}(\Lambda^\infty)$, 

$$f_{\kappa} = \sum_{d \in \pi^{-1}(\kappa)} v^h(d) f_d k_d, \quad \text{and} \quad e_i = \sum_{j \equiv i} e_j k_j^{-1}.$$ 

By [VV], Section 5.2, the element $f_d|\lambda$ is zero for all $\lambda$ if there exists $j$ such that $d_j \notin \{0, 1\}$. Hence,

$$[e_i, f_{\kappa}] = \sum_{d \equiv i} v^h(d) [e_j k_j^{-1}, f_d k_d']$$

$$= \sum_{j \equiv i} v^h(d) \left( \sum_{l < j} w_l(d) e_j f_d - v^{-\sum_{l < m} d_l w_l(-e_j) f_d e_j} \right) k_j^{-1} k_d.'$$

$$= \sum_{j \equiv i} v^h(d) + \sum_{l < j} 2d_l - d_{l-1} - d_{l+1} [e_j, f_d] k_j^{-1} k_d.'$$

Recall that $f_d$ is equal to the ordered product $f_d = \ldots f_{i-1} f_i f_{i+1} \ldots$. Then

$$[e_j, f_d] = \begin{cases} 
\left( \prod_{r=-\infty}^{j-1} f_r \right) \frac{k_j - k_j^{-1}}{v - v^{-1}} \left( \prod_{s=j+1}^{\infty} f_s \right) k_j^{-1} k_d' & \text{if } d_j = 1 \\
0 & \text{if } d_j = 0 
\end{cases}.$$ 

Thus,

$$[e_i, f_{\kappa}] = \sum_{d \equiv i} v^h(d) + \sum_{l < j} 2d_l - d_{l-1} - d_{l+1} \left( \prod_{r=-\infty}^{j-1} f_r \right) \frac{k_j - k_j^{-1}}{v - v^{-1}} \left( \prod_{s=j+1}^{\infty} f_s \right) k_j^{-1} k_d'.$$

We have

$$h(d) + \sum_{l < j} (2d_l - d_{l-1} - d_{l+1}) = h(d - e_j) + \sum_{s > j} (d_{s+1} - d_s) + \sum_{l < j} (d_l - d_{l+1})$$

and

$$k_j^{-1} k_{d'} = k_{(d-e_j)'} \prod_{l < j} k_l^{-1} \prod_{r > j} k_r.$$ 

Therefore

$$[e_i, f_{\kappa}] = \sum_{d \in \pi^{-1}(\kappa - e_i)} v^h(d) f_d k_d' \sum_{j \equiv i} \sum_{d_j = 0} v^{\sum_{s > j}(d_{s+1} - d_s) + \sum_{l < j}(d_l - d_{l+1})} \frac{v^{d_j+1} k_j - v^{-d_j+1} k_j^{-1}}{v - v^{-1}} \prod_{l < j} k_l^{-1} \prod_{r > j} k_r.$$
Moreover, \( f_\lambda | \lambda \neq 0 \) if and only if \( k_t | \lambda = \nu | \lambda \) for all \( t \) such that \( \tilde{d}_t = 1 \). It follows that, in \( \text{End}(\Lambda) \),

\[
[e_i, f_{ks}] = \sum_{\tilde{d} \in \pi^{-1}(ks - \epsilon_i)} v^{h(\tilde{d})} f_{\tilde{d}q} \sum_{j \in I, d_j = 0} \nu^{l \in J, d_i = 0} \left( \frac{v^{d_i + 1} k_j - v^{-d_i + 1} k_j^{-1}}{v - v^{-1}} \right) \prod_{t < j, \tilde{d}_t = 0} k_t^{-1} \prod_{r > j, \tilde{d}_r = 0} k_r^{-1}
\]

\[
= \sum_{\tilde{d} \in \pi^{-1}(ks - \epsilon_i)} v^{l(\tilde{d})} f_{\tilde{d}q} \left( \prod_{t \in I} k_t - \prod_{r \in I} k_r^{-1} \right)
\]

Note that both infinite products make sense as elements of \( \text{End}(\Lambda) \) since for any \( \lambda \in \Pi \), \( k_t | \lambda = | \lambda \) for all but a finite number of values of \( t \). Hence

\[
[e_i, f_{ks}] = \frac{v^{k_j - v^{-1} k_i^{-1}}}{v - v^{-1}}
\]

\[
= \frac{v^{-1} k_j - v k_i}{v - v^{-1}} f_{ks - \epsilon_i}
\]

as desired.

Now suppose that Lemma 2.2 holds for \( u \in U_1[\alpha] \) and \( w \in U_1[\beta] \). Then

\[
\rho(e_i)\rho(uw) = \rho(u)\rho(e_i)\rho(w) + \rho \left( \frac{k'_i e''_i(w) - k_i^{-1} e'_i(u)}{v - v^{-1}} w \right)
\]

\[
= \rho(uw)\rho(e_i) + \rho \left( \frac{v k_i e''_i(w) - k_i^{-1} e'_i(w)}{v - v^{-1}} \right) + \rho \left( \frac{k_i e''_i(u) - k_i^{-1} e'_i(u)}{v - v^{-1}} w \right).
\]

Therefore

\[
[\rho(e_i), \rho(uw)] = \rho \left( \frac{k_i (v^{w_1(\alpha)} e''_i(w) + e''_i(u)w) - k_i^{-1} (v^{-w_1(\alpha)} e'_i(w) + e'_i(u)w)}{v - v^{-1}} \right)
\]

\[
= \rho \left( \frac{k_i (e''_i(uw) - k_i^{-1} e'_i(uw))}{v - v^{-1}} \right),
\]

i.e. Lemma 2.2 is true for the product \( uw \). This proves Lemma 2.2 since, by [GH] Theorem 3.1, \( U_{\kappa, n} \) is generated by \( U^- (\tilde{a}_n) \) and the elements \( f_{ks}, l \in \mathbb{N}^* \).

6.2 Proof of Lemma 3.4. Fix integers \( k, l \) with \( k > l \), and let \( \mu \in \Pi \), \( |\mu| = k - l \). Fix a \( L \)-graded vector space \( V \) of dimension \( ks \), a subspace \( V' \subset V \) of dimension \( (k - l)s \) and choose an element \( y \in O_{m(\mu)} \subset E_{V'} \). Finally, let \( m \in M \) with \( \dim m = ks \). By definition,

\[
\Delta_{(1)^{m}, \mu}(1_{m}, f_{\lambda} \otimes f_{\mu}) = \# I_{q = \nu - \epsilon} 1_{\lambda} \otimes 1_{\nu}.
\]

where \( I_{m} = \{ x \in O_{m} \subset E_{V'} \mid x|_{V'} = y \} \). Recall that \( \# I_{m} \) is a polynomial in \( q \), and that by the Lang-Weil theorem ([LW]), we have \( \# I_{m} = q^{\dim I_{m}} + O(q^{\dim I_{m}-1}) \). Moreover it follows from [Y4], (6.17), that \( \lambda^{(1)}_{\mu} \in \{ 0, 1 \} \).
and that $I^m = \emptyset$ if $m = m(n\lambda)$ with $c_{(1)^{j'},d'} = 0$. Hence Lemma 3.4 is equivalent
to the following dimension inequalities:

$$\dim O_m - \dim O_{n\mu} \geq 2 \dim I^m,$$

$$\dim O_m - \dim O_{n\mu} = 2 \dim I^m \iff m = m(n\lambda) \text{ for some } \lambda \in \Pi \text{ with } c_{(1)^{j'},d'} = 1. \quad (5.1)$$

For $i \in \mathbb{N}^*$, $j \in I$ and for any $x \in O_m$, let us set

\[ d_j^i = \dim \ker y_{[V_j]}^i - \dim \ker y_{[V_j]}^{i-1}, \]
\[ \tilde{d}_j^i = \dim \ker x_{[V_j]}^i - \dim \ker x_{[V_j]}^{i-1}, \]
\[ \theta_j^i = d_j^i - \tilde{d}_j^i. \]

Then $I^m \neq \emptyset$ if and only if $\theta_j^i \in \mathbb{N}$ and $\sum_i \theta_j^i = l$ for all $j \in I$. Note that $d_j^i = d_j^i$ for any $k, j, j'$ since $m(n\mu) \in \mathcal{M}^{\text{der}}$. A direct computation gives

$$\dim O_m = \sum_j \left( \sum_{k>i}(d_j^k + \tilde{d}_j^{k-1}) \right) = \sum_j \left( \sum_{k>i}(d_j^k + d_j^{k-1} + \theta_j^k + \theta_j^{k-1})(d_j^i + \theta_j^i) \right).$$

Similarly,

$$\dim O_m(n\mu) = \sum_j \left( \sum_{k>i}(d_j^k + d_j^{k-1}) \right).$$

Thus,

$$\dim O_m - \dim O_m(n\mu) = \sum_j \left( \sum_{k>i} \left( 2 \theta_j^k d_j^i + \sum_{k>i} (\theta_j^k + \theta_j^{k-1}) \theta_j^i \right) \right). \quad (5.2)$$

Now we compute $\dim I^m$. Fix a complementary subspace $U$ of $V'$ in $V$. An element $x \in E_V$ satisfying $\text{Im} x \subset V'$ and $x_{[V]} = y$ is uniquely determined by the collection of maps $x : U_j \rightarrow V_{j+1}$. Moreover,

$$\dim (\ker x_{[V_j]}^k) = \dim (\ker y_{[V_j]}^k) + \dim \ker x_{[U_j]}^k + \dim (x^k(U_j) \cap y^k(V'_j))$$
$$= \dim (\ker y_{[V_j]}^k) + \dim \ker x_{[U_j]}^k + \dim (x(U_j) \cap (y^{k-1})^{-1}(y^k(V'_j))).$$

Set $Y_k^j = \{0\}$ and $Y_k^j = (y^{k-1})^{-1}(y^k(V'_j))$ if $k > 1$. Then, for $k > 1$

$$\dim Y_k^j = \sum_{i<k} d_{j+1}^i + \sum_{i>k} d_j^i = \sum_{i \neq k} d_j^i. \quad (5.3)$$

Now,

$$\dim (\ker x_{[V_j]}^k) - \dim (\ker x_{[V_j]}^{k-1}) = d_j^k + \dim (x(U_j) \cap Y_j^k) - \dim (x(U_j) \cap Y_j^{k-1}).$$
Hence \( x \in O_m \) if and only if for all \( j \in I \)
\[
\dim \ker x_{U_j} = \theta^1_k, \\
\forall k > 1, \quad \dim(x(U_j) \cap Y^k_j) - \dim(x(U_j) \cap Y^{k-1}_j) = \theta^k_j.
\] (5.4)

The variety \( X_j \) of subspaces \( W_j \subset V'_{j+1} \) satisfying (5.4) is of dimension
\[
\dim X_j = \sum_{l \geq 2} (\dim Y'_j - \sum_{i=2}^l \theta^l_j) \theta^l_j.
\]

A direct computation using (5.3) now gives
\[
\dim I^m = \sum_j \left( \sum_{k \neq i} \theta^k_j d^i_j + \sum_{k > j} \theta^k_j \theta^l_j \right)
\] (5.5)

Thus the dimension inequalities (5.1) are consequences of (5.2), (5.3) and the following result:

Claim. For all collections of positive integers \( (\theta^j_i)_{i,j}, \ i = 1, \ldots, h, \ j \in \mathbb{Z}/n\mathbb{Z} \) satisfying \( \sum_i \theta^j_i = \ell \) there holds
\[
\sum_j \left( \sum_{k \neq i} \theta^k_j \theta^l_j \right) \geq \sum_j \left( \sum_{k > j} \theta^k_j \theta^l_j \right),
\] (5.6)

with equality if and only if \( \theta^k_j = \theta^k_{j'} \) for all \( k, j, j' \).

We argue by induction on \( h \) to prove (5.6). The claim is trivial if \( h = 1 \). Suppose that (5.6) is proved for all \( h' < h \). We first note that
\[
\sum_j \sum_{k > i} (\theta^k_{j-1} \theta^i_j - \theta^k_j \theta^i_j) = \sum_j \left( \sum_{k > i} \theta^k_{j-1} \theta^i_j - \theta^k_j \theta^i_j + \sum_{i < h} (l - \sum_{k < h} \theta^k_j) (\theta^k_{j-1} - \theta^k_j) \right)
\]
\[
= \sum_j \sum_{k > i} \theta^k_j (\theta^i_j - \theta^i_{j-1}).
\] (5.7)

Let us freeze variables \( \theta^k_j, k > 1 \) and consider \( G((\theta^1_j)) = \sum_j \sum_{k > 1} \theta^k_j (\theta^1_j - \theta^k_{j-1}) \).

Then a direct computation shows that \( G((\theta^1_j)) \) reaches its global minimum when for all \( j \)
\[
(\theta^1_j - \theta^1_{j-1}) - (\theta^1_{j+1} - \theta^1_j) = \sum_{k=2}^{h-1} (\theta^k_{j+1} + \theta^k_{j-1} - 2 \theta^k_j),
\]
i.e. when for all \( j \)
\[
\sum_{k=1}^{h-1} (\theta^k_{j+1} - \theta^k_j) = \sum_{k=1}^{h-1} (\theta^k_j - \theta^k_{j-1}).
\]

Since \( \sum_j \sum_{k=1}^{h-1} (\theta^k_{j+1} - \theta^k_j) = 0 \), this implies that \( \sum_{k=1}^{h-1} \theta^k_{j+1} = \sum_{k=1}^{h-1} \theta^k_j \) for all \( j \). But then \( \theta^k_j = l - \sum_{k=1}^{h-1} \theta^k_j = \theta^1_j + 1 \), and in this case
\[
\sum_j \sum_{k > i} (\theta^k_{j-1} - \theta^k_j) \theta^i_j = \sum_j \sum_{k > i} (\theta^k_j - \theta^k_{j-1}) \theta^i_j.
\]
The result then follows from the induction hypothesis.

Acknowledgments

I would like to thank my advisor Eric Vasserot for suggesting this problem to me and for his patience and guidance. I am indebted to B. Leclerc for many enlightening discussions and comments on this paper and to A. Braverman and D. Gaitsgory for interesting discussions.

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Olivier Schiffmann, ENS Paris, 45 rue d’Ulm, 75005 PARIS; schiffma@clipper.ens.fr