The Use of the Finite Element Method for the Calculation of Spatial Rod Systems in Consideration the Large Displacements

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Abstract. The paper presents a methodology for the analytical calculation of systems with unilateral constraints. This technique is based on the Finite Element Method in the form of a classical mixed method. The approach to the formulation of a system of resolving equations for spatial rod systems with large displacements of nodes is described in detail. The following conditions are used for this: static equilibrium equations for all nodes of the system, the kinematic conditions that describe the nodal displacements and the conditions of continuity. In the first step, the calculation is performed in a linear setting. In the transition to the solution of the problem in a non-linear formulation, the resulting solution on the non-deformable scheme is used as the initial approximation. Further calculation is carried out in the form of an iterative process based on the stepwise loading procedure and do the calculation in a linear setting at each step, taking into account the new geometry of the system achieved in the previous step.

Introduction

With the development of computer technologies in the middle of the twentieth century, the finite element method gained widespread distribution in building structures calculation. In general, in software complexes this method is implemented in the form of a displacement method, but in solving problems in a nonlinear formulation this form gives large discrepancies with exact solutions, this is shown in detail in the paper [1,2,7]. In such problems, more accurate solutions make it possible to obtain the finite element method in the form of a mixed method. For rod finite elements in the basic system of the mixed method, are regarded as both forces and displacements [3,4,9]. In view of the simplicity of the basic system, the elements of the response matrix are obtained directly from static and kinematic conditions and continuity conditions.

Main part

The application of the finite element method in mixed form will be considered using the example of a cable-structure calculation. We take the design scheme in the form of the upper absolutely rigid body supported on the column. The lower ring is suspended on the cables to the upper disk. For convenience of theoretical calculations, we represent the lower ring in the form of a polygonal polygon inscribed in a circle. In this problem, a hexagonal polygon.

To compose a system of resolving equations, consider a node of a symmetric construction with adjacent rods. A fragment of the main system of the mixed method is shown in Fig. 1.
In this figure, the point $O$ is the center of the lower ring with radius $R$, $i$ is the considered node of the lower ring, $i'$ is the node of the upper ring.

Such a system is kinematically and geometrically variable and can retain its geometric configuration only with a self-balancing load statically equivalent to zero.

The considered system must first be calculated as a system of inextensible elements in order to find its equilibrium configuration under a given nodal load, similar to the calculation of the thread with the allowance.

![Figure 1. Fragment of the main system of the mixed method](image)

Consider the formulation of the equilibrium equations in general form.

Static equilibrium conditions of the node $i$:

\[ \sum X = N_{i,j} \cdot \cos \alpha + N_{i,i+1} + N_{i,i-1} \cdot \cos \beta = 0, \]  
\[ \sum Y = N_{i,i-1} - N_{i,i+1} \cdot \sin \beta = 0, \]  
\[ \sum Z = -N_{i,j} \cdot \sin \alpha + P_i, \]  

From (3) we obtain that $N_{i,j} = \frac{P_i}{\sin \alpha}$.  

It follows from (2) that $N_{i,i-1} = N_{i,i+1}$.  

Then, substituting the obtained values of (4) and (5) into (1), we obtain

\[ N_{i,i-1} = N_{i,i+1} = -\frac{P_i}{2 \cdot \tan \alpha \cdot \cos \beta}. \]

This expression shows that the forces in the elements of the ring adjacent to the $i$-th node are equal to each other and depend only on the load applied at the node $P_i$. When we move to another node, i.e., $i+1$ or $i-1$, we get other efforts, which is a contradiction. This contradiction is eliminated under the condition $P_i = P = \text{const}$, then the ring is uniformly compressed and the forces in all the cables are also the same.

To obtain a solution to this problem for different values of the node load $P_i$, it is necessary to proceed to the solution of the problem in a nonlinear formulation. In this case, the solution obtained in the linear formulation from the non-deformable scheme must be used as the initial approximation in the iterative process.

As soon as additional unequal loads $\Delta P_i$ are applied to the nodes, the system will change the configuration and all its nodes will shift to the values $\Delta x_i, \Delta y_i, \Delta z_i$ [5,8]. These values will determine
the position of each rod in space. We find the quantities $\Delta x_i, \Delta y_i, \Delta z_i$ as a function of the node loads $\Delta P_i + \Delta P_i$ for a system of non-deformable rods.

The basic system of the mixed method is obtained by introducing into each free node $i$ ($i = 1$, ..., 6) three links that prevent linear displacements of the node in the direction of the local coordinate axis and the elimination of connections in the section of each of the rods of the system. Thus, in the basic system we have 30 unknowns: 12 power unknowns (by the number of rods) and 18 kinematic unknowns (three unknowns at each fixed node). In accordance with this, the system of resolving equations in the linear formulation of the problem will consist of 30 equations.

For further calculations, we introduce the following notation and assumptions in accordance with [6,10, 11]. We take the numbering of the nodes of the lower ring by Arabic numerals from 1 to 6, and the numbering of the nodes of the upper ring from 1' to 6'. The rods connecting the upper and lower rings are numbered from 1-1' to 6-6'. Geometrical parameters of the system:

Emerging reactions $R_{1,1}, R_{1,2}, R_{1,3}$, breaks in cross-section $\Delta_{1,1}, \Delta_{1,2}, \Delta_{1,6}, \Delta_{3,6}$, unknown movements $q_{1,1} = q_1, q_{1,2} = q_2, q_{1,6} = q_6$, efforts $N_{1,1}, N_{1,2}, N_{1,6} = N_6$.

Let us write the resolving equations for the node 1 in the linear formulation:

For a system with deviations from a given geometry (Fig. 2), we will compose the equations of static equilibrium and geometric equations in local coordinate systems. Figure 2 shows the displacement of the inclined rod 1'-1 to a new position caused by the displacement of the node 1 by the values $\Delta x_i, \Delta y_i, \Delta z_i$. 

From the geometric relations it follows that:
\[ d \cdot \cos \alpha + \Delta x_1^2 + \Delta y_1^2 + d \cdot \sin \alpha + \Delta z_1^2 = d^2. \]

Hence, after formation, we obtain:
\[ 2d \cdot \Delta x_1 \cdot \cos \alpha + \Delta z_1 \cdot \sin \alpha + \Delta x_1^2 + \Delta y_1^2 + \Delta z_1^2 = 0. \]  
(7)

The values of the orientation nodes of the rod 1'-1 necessary for further calculations in the new deflected position are determined by the following expressions:

\[ \cos \alpha_{1-1'} = \frac{d + \Delta z_1}{d}; \quad tg \alpha_{1-1'}^x = \frac{\Delta y_1}{d \cdot \cos \alpha + \Delta x_1}; \quad tg \alpha_{1-1'}^y = \frac{d \cdot \cos \alpha + \Delta x_1}{\Delta y_1}. \]

(8)

Similarly, for rods 1-2 and 6-1
\[ d \cdot \cos 30^\circ + \Delta y_{1-2}^2 + d \cdot \cos 60^\circ + \Delta x_{1-2}^2 + \Delta z_{1-2}^2 = d^2. \]

(9)

where \( \Delta z_{1-2} = z_2 - z_1 \), \( \Delta y_{1-2} = y_2 \cdot \cos 60^\circ - y_1 \), \( \Delta x_{1-2} = x_2 \cdot \cos 60^\circ - x_1 \), \( x_1, y_1, z_1 \) - moving node 1 in its local coordinate system,
\( x_2, y_2, z_2 \) - moving node 2 in the local coordinate system.
\[ d \cdot \cos 30^\circ + \Delta y_{6-1}^2 + d \cdot \cos 60^\circ + \Delta x_{6-1}^2 + \Delta z_{6-1}^2 = d^2. \]

(10)

where \( \Delta z_{6-1} = z_6 - z_1 \), \( \Delta y_{6-1} = y_6 \cdot \cos 60^\circ - y_1 \), \( \Delta x_{6-1} = x_6 \cdot \cos 60^\circ - x_1 \),
\[ \cos \alpha_{6-1'} = \frac{\Delta z_{6-1'}}{d}; \quad tg \alpha_{6-1'}^x = \frac{d \cdot \cos 30^\circ + \Delta y_{6-1'}}{d \cdot \cos 60^\circ + \Delta x_{6-1'}}; \quad tg \alpha_{6-1'}^y = \frac{d \cdot \cos 30^\circ + \Delta y_{6-1'}}{d \cdot \cos 60^\circ + \Delta x_{6-1'}}. \]

When the node 1 is shifted to a new position under the influence of a load with displacements along the coordinate axis:
\[ \Delta x_1 = q_1^1, \quad \Delta y_1 = q_2^1, \quad \Delta z_1 = q_3^1. \]
The new length of the thread 1-1' will be determined by the expression
\[ \tilde{l}_{1-1'}^2 = l + \Delta x_1^2 + 0 + \Delta y_1^2 + 0 + \Delta z_1^2. \]

Length increment:
\[ \Delta l_i = \Delta l_{i-1} = \bar{L}_{i-1} - l_{i-1}^q = l_i \sqrt{\left(1 + \frac{\Delta x_i}{l_i}\right)^2 + \left(\frac{\Delta y_i}{l_i}\right)^2 + \left(\frac{\Delta z_i}{l_i}\right)^2} - l_i \]

or

\[ \Delta l_{i-1} = l_i \sqrt{\left(1 + \frac{q_i}{l_i}\right)^2 + \frac{q_{i-1}^2}{l_{i-1}^2} + \frac{q_i^2}{l_i^2} - 1} \]

The length increment is expressed in terms of the force in the string,

\[ \Delta l - \Delta l_{i-1} = \frac{q_i \cdot l_i}{EF} \]

The angle of inclination of the deformed string to the horizontal plane \( xOy \):

\[
\cos \alpha_i = \sqrt{1 + \frac{\Delta x_i^2 + \Delta y_i^2}{l_i^2}} = \sqrt{1 - \frac{\left(\frac{\Delta z_i}{l_i}\right)^2}{\left(1 + \frac{\Delta x_i}{l_i}\right)^2 + \left(\frac{\Delta y_i}{l_i}\right)^2 + \left(\frac{\Delta z_i}{l_i}\right)^2}}, \tag{11}
\]

\[
\sin \alpha_i = \frac{\Delta z_i}{l_i} \sqrt{1 - \frac{\left(\frac{\Delta z_i}{l_i}\right)^2}{\left(1 + \frac{\Delta x_i}{l_i}\right)^2 + \left(\frac{\Delta y_i}{l_i}\right)^2 + \left(\frac{\Delta z_i}{l_i}\right)^2}}. \tag{12}
\]

In subsequent calculations, the use of expressions like (11) and (12) to compose solving equations and their subsequent solution by an integration method is inconvenient. Since the calculations can be limited in the first approximation only to first-order quantities of smallness, one can formally expand the function \( f(x) = \sqrt{f(x)} \) in a Taylor series with the retention in it of only the quantities of the first order of smallness.

For a thread between nodes 1 and 2 in the coordinate system adopted for the node 1 of the rod 1'-11, the movements of its ends in the \( xOy \) coordinate system are found using dependecies:

\[
\Delta x_{1-2} = \left(\Delta x^q - \Delta x^q\right) \cdot \cos 60^\circ - \left(\Delta y^q + \Delta y^q\right) \cdot \cos 30^\circ, \tag{13}
\]

\[
\Delta y_{1-2} = \left(\Delta x^q + \Delta x^q\right) \cdot \cos 30^\circ + \left(\Delta y^q - \Delta y^q\right) \cdot \cos 60^\circ, \tag{13}
\]

\[
\Delta z_{1-2} = \Delta z^2 - \Delta z^1. \tag{13}
\]

In the deflected position, the equilibrium conditions (1) of node 1 take the following form:

\[
R^1_1 = R^1_1 = -\tilde{q}_4^1 \cdot \cos \alpha^1_1 + q^6_6 \cdot \cos 60^\circ + \beta^6_6 + q^5_5 \cdot \cos 60^\circ - \beta^5_5 = 0, \tag{13}
\]

\[
R^2_2 = R^2_1 = -\tilde{q}_4^2 \cdot \cos \alpha^2_1 \cdot \sin \beta^1_1 + q^6_6 \cdot \sin 60^\circ + \beta^6_6 - \tilde{q}_5^1 \cdot \sin 60^\circ - \beta^5_5 = 0, \tag{13}
\]

\[
R^1_3 = R^1_3 = \bar{P} - \tilde{q}_4^1 \cdot \sin \alpha^1_1 + q^6_6 \cdot \sin \alpha^1_1 + q^5_5 \cdot \sin \alpha^1_4 + \tilde{q}_5^1 \cdot \sin \alpha^1_4 = 0. \tag{13}
\]

Equations of consistency (continuity) of deformation:

\[
\Delta l_{1-1} = \frac{\tilde{q}_4^1 \cdot l_{1-1}}{EF} - \Delta l_{1-1} = 0, \tag{13}
\]

\[
\Delta l_{1-2} = \frac{\tilde{q}_5^1 \cdot l_{1-2}}{EF} - \Delta l_{1-2} = 0. \tag{13}
\]

Equations for the remaining nodes of the polygonal ring are obtained cyclically.
Conclusions
The proposed method allows us to reduce the solution of the nonlinear system of equations (13) to linear calculation with a stepwise loading of the construction with iterative refinement of the linear solution at each step of the addition. The refinement of the solution at all stages of the calculation.

The correctness of the solution obtained is verified by using a kinematic action instead of force action in the form of a controlled displacement of the node. This method of calculating by the finite element method in the form of a classical mixed method makes it possible to do this at any step of loading by changing the form of the loading parameter.

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