ON THE CONSTRUCTION OF $K$-OPERATORS IN FIELD THEORIES AS SECTIONS ALONG LEGENDRE MAPS

Arturo Echeverría-Enríquez,
Departamento de Matemática Aplicada y Telemática
Campus Norte UPC, Módulo C-3
C/ Jordi Girona 1-3,
E-08034 Barcelona. Spain

Jesús Marín-Solano †
Departamento de Matemática Económica, Financiera y Actuarial, UB
Av. Diagonal 690. E-08034 Barcelona. Spain

Miguel C. Muñoz-Lecanda‡ Narciso Román-Roy‡
Departamento de Matemática Aplicada y Telemática
Campus Norte UPC, Módulo C-3
C/ Jordi Girona 1-3,
E-08034 Barcelona. Spain

Acta Applicandae Mathematicae 77(1) (2003) 1-40

Abstract

The “time-evolution $K$-operator” (or “relative Hamiltonian vector field”) in mechanics is a powerful tool which can be geometrically defined as a vector field along the Legendre map. It has been extensively used by several authors for studying the structure and properties of the dynamical systems (mainly the non-regular ones), such as the relation between the Lagrangian and Hamiltonian formalisms, constraints, and higher-order mechanics.

This paper is devoted to defining a generalization of this operator for field theories, in a covariant formulation. In order to do this, we use sections along maps, in particular multivector fields (skew-symmetric contravariant tensor fields of order greater than 1), jet fields and connection forms along the Legendre map. As a relevant result, we use these geometrical objects to obtain the solutions of the Lagrangian and Hamiltonian field equations, and the equivalence among them (specially for non-regular field theories).

Key words: Jet bundles, multivector fields, connections, jet fields, sections along maps, first order field theories, Lagrangian and Hamiltonian formalisms.

AMS s.c. (2000): 51P05, 53C05, 53C80, 55R10, 58A20, 58A30, 70S05.
PACS (1999): 02.40.Hw, 02.40.Vh, 11.10.Ef, 11.10.Kk
1 Introduction

The so-called time-evolution $K$-operator in mechanics (also known by some authors as the relative Hamiltonian vector field) is a tool which has mainly been developed in order to study the Lagrangian and Hamiltonian formalisms for singular mechanical systems and their equivalence. It was first introduced in a non-intrinsic way in [3] as an “evolution operator” to connect both formalisms, as a refinement of the technique used in [34]. This operator was later defined geometrically in two different but equivalent ways [9], [24] for autonomous dynamical systems, and in [8] for the non-autonomous case. In [24], a further different geometric construction is given, using a canonical map introduced by Tulczyjew [47].

The following is a summary of relevant results obtained using this operator:

- The equivalence between the Lagrangian and Hamiltonian formalisms is proved by means of this operator in the following way: there is a bijection between the sets of solutions of Euler-Lagrange equations and Hamilton equations, even though the dimensions of the final constraint submanifold in both formalisms are not the same, in general (see [3], [25]).

- The complete classification of constraints is achieved. All the Lagrangian constraints can be obtained from the Hamiltonian ones using the $K$-operator in the following way: at each level of the Lagrangian constraint algorithm, every Lagrangian constraint which is projectable by the Legendre map is associated with a Hamiltonian one of the preceding level of the Hamiltonian constraint algorithm, which is first-class with respect to the primary constraints, whereas the non-projectable Lagrangian constraints are associated with the second-class Hamiltonian ones (see [3]).

- Noether’s theorem is proved and the relation between the generators of gauge and “rigid” symmetries in the Lagrangian and Hamiltonian formalisms is studied. Thus, each Lagrangian Noether infinitesimal symmetry can be obtained from a Hamiltonian generator of symmetries, which is a conserved quantity (see [21], [22], [26], [28], [29], [32]).

- This operator has been applied to study Lagrangian systems whose Legendre map has generic singularities; that is, it degenerates on a hypersurface (see [43], [44]).

Most of these results have also been generalized for higher-order Lagrangian systems [10], [27], [30], [31], and for the case of more general types of singular differential equations on manifolds (implicit systems of equations) [25]. Finally, although a covariant description of this operator was not available, it has also been used to study several characteristics of some physical models in field theory, namely the bosonic string [1], [2], [26].

Our aim is to generalize the definition, properties and some of the applications of this operator for field theories (specifically, the non-regular ones) in order to describe the relationship between the Lagrangian and Hamiltonian formalisms. In particular, in this first work we will study how to obtain the solutions of Lagrangian and Hamiltonian field equations by means of this operator, and the relation between them, while the applications concerning constraints and symmetries will be the subject of further research.

We follow the procedure outlined in [24] and [8], which is based on the concept of section along a map [42], of which this operator is a particular case. The properties of these sections and their applications in mechanics have been extensively analyzed in different situations [6], [11], [12], [13], [14], [36], [41]. Our standpoint is the multisymplectic jet bundle formulation of Lagrangian
and Hamiltonian field theories. The essential geometric objects to be dealt with are transversal distributions in the corresponding bundles, which we study from three equivalent points of view: connections, jet fields and (classes of) transversal multivector fields. The first two are extensively used in the standard bibliography (see, for instance, [46]), while the third is introduced as a more convenient algebraic tool for certain applications. Furthermore, each formulation has particular characteristics which make easier to prove the results of the work.

The organization of the paper is as follows: In Section 2, we review first the definition and the main properties of the evolution operator $K$ for autonomous mechanics. Secondly, we state the main characteristics of multivector fields and their relation with jet fields and connections in jet bundles. Then we review the Lagrangian and Hamiltonian multisymplectic formalisms of field theories. Section 3 is devoted to a study of the concept and properties of multivector fields, jet fields and connections along maps, in the context of the jet bundle description of field theories. Next, the extended and restricted covariant field operators are defined in three equivalent ways, and their existence and non-uniqueness is proved. Finally, in Section 4, some properties of these operators are studied; namely, how they can be used to obtain the solutions of the Lagrangian and Hamiltonian field equations both for regular and singular theories (on the submanifolds where they exist), and hence the equivalence between the solutions of field equations in the Lagrangian and Hamiltonian formalisms are obtained in a straightforward way.

Throughout this paper $\pi: E \to M$ will be a fiber bundle ($\dim M = m$, $\dim E = N + m$), where $M$ is an oriented manifold with volume form $\omega \in \Omega^m(M)$. $\pi^1: J^1E \to E$ is the jet bundle of local sections of $\pi$, and $\bar{\pi}^1 = \pi \circ \pi^1: J^1E \to M$ gives another fiber bundle structure. $(x^\alpha, y^A, v_A^\alpha)$ will denote natural local systems of coordinates in $J^1E$, adapted to the bundle $E \to M$ ($\alpha = 1, \ldots, m$; $A = 1, \ldots, N$), and such that $\omega = dx^1 \wedge \ldots \wedge dx^m = d^m x$. Manifolds are real, paracompact, connected and $C^\infty$. Maps are $C^\infty$. Sum over crossed repeated indices is understood.

2 Preliminary considerations

2.1 The evolution operator $K$ in (autonomous) mechanics

(See [24] for details).

Let $\sigma: F \to B$ be a fiber bundle, and $\Phi: A \to B$ a differentiable map (we assume that $\Phi(A)$ is a submanifold of $B$). A section along $\Phi$ is a map $\mathcal{T}: A \to F$ such that $\sigma \circ \mathcal{T} = \Phi$. So we have

\[
\begin{array}{ccc}
\mathcal{T} & F \\
\Phi & \downarrow \sigma \\
A & B
\end{array}
\]

In particular, if $F$ is either $\Lambda^m TB$, $\Lambda^k T^* B$ or $\bigotimes^k (T^* B \otimes TB)$, the sections are called $m$-vector fields, $k$-forms, or $(k, m)$-tensor fields along $\Phi$, and the sets of these elements are denoted by $X^m(A, \Phi)$, $\Omega^k(A, \Phi)$, and $T^k_m(A, \Phi)$, respectively. Obviously, every section $s: B \to F$ of the projection $\sigma$ defines a section along $\Phi$ by $\mathcal{T} = s \circ \Phi$.

Contractions between tensor fields along maps are defined in a natural way. In this paper we will only use the following: if $\mathcal{T} \in T^k_m(A, \Phi)$ and $\mathcal{T}' \in T^l_r(A, \Phi)$, then

\[
[i(\mathcal{T})\mathcal{T}'](p) := i(\mathcal{T}(p))[\mathcal{T}'(p)], \quad \text{for every } p \in A
\]
so that, if \( n \geq m \) then \( i(T)T' \in T^{k+n-m}(A, \Phi) \), and if \( m = n, k = r = 0 \), then \( i(T)T' \in C^{\infty}(A) \). Of course, if \( n < m \), then \( i(T)T' = 0 \).

Let \((TQ, \Omega_L, E_L)\) be a Lagrangian system, \(\mathcal{FL}: TQ \rightarrow T^*Q\) the Legendre map, \(\Omega \in \Omega^2(T^*Q)\) the canonical form, and the canonical projections \(\pi_Q: T^*Q \rightarrow Q\), \(\tau_{TT^*Q}: TT^*Q \rightarrow T^*Q\).

The evolution operator \(K\) associated with the Lagrangian system \((TQ, \Omega_L, E_L)\) is a map \(K: TQ \rightarrow TT^*Q\) satisfying the following conditions:

1. \((\text{Structural condition})\): \(K\) is a vector field along \(\mathcal{FL}\),
   \[\tau_{T^*Q} \circ K = \mathcal{FL}\]

2. \((\text{Dynamical condition})\): \(\mathcal{FL}^*[i(K)(\Omega \circ \mathcal{FL})] = dE_L\).

3. \((\text{Second-order condition})\): \(T\pi_Q \circ K = \text{Id}_{TQ}\).

The existence and uniqueness of this operator can be proved, and its local expression (using natural coordinates in \(TQ\) and \(T^*Q\)) is

\[
K = v^A \left( \frac{\partial}{\partial q^A} \circ \mathcal{FL} \right) + \frac{\partial L}{\partial q^A} \left( \frac{\partial}{\partial p^A} \circ \mathcal{FL} \right)
\]

By definition, \(\varphi: \mathbb{R} \rightarrow TQ\) is an integral curve of \(K\) if

\[
\mathcal{FL} \circ \dot{\varphi} = K \circ \varphi
\]

so we have the diagram

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\varphi} & TQ \\
\downarrow{\varphi} & & \downarrow{\mathcal{FL}} \\
& TTQ & \xrightarrow{\mathcal{FL}} \ x \rightarrow T^*Q
\end{array}
\]

Moreover, \(\varphi = \dot{\phi}\), for \(\phi: \mathbb{R} \rightarrow Q\) (that is, \(\varphi\) is holonomic).

The main properties of this operator are the following:

- If there exists an Euler-Lagrange vector field \(X_L \in \mathfrak{X}(TQ)\) for \((TQ, \Omega_L, E_L)\) (that is, a holonomic vector field verifying that \(i(X_L)\Omega_L = dE_L\)), then \(\varphi: \mathbb{R} \rightarrow TQ\) is an integral curve of \(X_L\) if, and only if, it is an integral curve of \(K\); that is, relation (1) holds.

As a direct consequence of this fact, the relation between \(K\) and \(X_L\) is

\[
\mathcal{FL} \circ X_L = K
\]

In general, if the dynamical system is not regular, Euler-Lagrange vector fields exist only on a submanifold \(S \hookrightarrow TQ\).

- If there exists a Hamilton-Dirac vector field \(X_H \in \mathfrak{X}(T^*Q)\) associated with the Lagrangian system \((TQ, \Omega_L, E_L)\) (that is, a vector field solution of the Hamilton-Dirac equations in the Hamiltonian formalism), then \(\psi: \mathbb{R} \rightarrow T^*Q\) is an integral curve of \(X_H\) if, and only if,

\[
\dot{\psi} = K \circ T\pi_Q \circ \dot{\psi}
\]
So we have the diagram

\[
\begin{array}{ccc}
TQ & \xrightarrow{\mathcal{FL}} & T^*Q \\
\downarrow{\mathcal{T}\pi_Q} & & \downarrow{\mathcal{T}\pi_Q} \\
T\pi_Q & \xrightarrow{K} & T\mathcal{T}^*Q
\end{array}
\]

As a consequence, the relation between \( K \) and \( X_H \) is

\[ X_H \circ J = K \]  \hspace{1cm} (4)

In general, if the dynamical system is not regular, Hamilton-Dirac vector fields exist only on a submanifold \( P \hookrightarrow T^*Q \).

- If \( \xi \in C^\infty(T^*Q) \) is a Hamiltonian constraint, then \( i(K)(d\xi \circ J) \) is a Lagrangian constraint.

Relations (1), (2), (3) and (4) show how the Lagrangian and Hamiltonian descriptions can be unified by means of the evolution operator \( K \).

2.2 Multivector fields, jet fields and connections in jet bundles

(See [17] and [18] for the proofs and other details of the following assertions).

Let \( E \) be a \( n \)-dimensional differentiable manifold. Sections of \( \Lambda^m(TE) \) are called \textit{multivector fields} in \( E \), or more precisely, \textit{m-vector fields} in \( E \) (they are contravariant skew-symmetric tensors of order \( m \) in \( E \)). We will denote by \( \mathfrak{X}^m(E) \) the set of \( m \)-vector fields in \( E \). \( Y \in \mathfrak{X}^m(E) \) is said to be \textit{locally decomposable} if, for every \( p \in E \), there exists an open neighbourhood \( U_p \subset E \) and \( Y_1, \ldots, Y_m \in \mathfrak{X}(U_p) \) such that \( Y = Y_1 \wedge \ldots \wedge Y_m \). \textit{Contraction} of multivector fields and tensor fields in \( E \) is the usual one.

We can define the following equivalence relation: if \( Y, Y' \in \mathfrak{X}^m(E) \) are non-vanishing \( m \)-vector fields, and \( U \subseteq E \) is a connected open set, then \( Y \sim Y' \) if there exists a non-vanishing function \( f \in C^\infty(U) \) such that \( Y' = fY \). Equivalence classes will be denoted by \( \{ Y \}_U \). There is a one-to-one correspondence between the set of \( m \)-dimensional orientable distributions \( D \) in \( TE \) and the set of the equivalence classes \( \{ Y \}_E \) of non-vanishing, locally decomposable \( m \)-vector fields in \( E \). If \( Y \in \mathfrak{X}^m(E) \) is non-vanishing and locally decomposable, the distribution associated with the class \( \{ Y \}_U \) is denoted \( D_U(Y) \) (if \( U = E \) we write \( D(Y) \)). A non-vanishing, locally decomposable \( m \)-vector field \( Y \in \mathfrak{X}^m(E) \) is said to be \textit{integrable} (resp. \textit{involutive}) if its associated distribution \( D_U(Y) \) is integrable (resp. involutive). Of course, if \( Y \in \mathfrak{X}^m(E) \) is integrable (resp. involutive), then so is every \( m \)-vector field in its equivalence class \( \{ Y \} \), and all of them have the same integral manifolds. Moreover, the \textit{Frobenius’ theorem} allows us to say that a non-vanishing and locally decomposable \( m \)-vector field is integrable if, and only if, it is involutive.

Let us consider the following situation: if \( \pi: E \rightarrow M \) is a fiber bundle (\( \dim M = m \)), we are concerned with the case where the integral manifolds of integrable \( m \)-vector fields in \( E \) are sections of \( \pi \). Thus, \( Y \in \mathfrak{X}^m(E) \) is said to be \( \pi \)-transverse if, at every point \( y \in E \), \( (i(Y)(\pi^*\omega))_y \neq 0 \), where \( \omega \in \Omega^m(M) \) is the volume form in \( M \). Hence, \( D(Y) \) is a complementary of the vertical subbundle, and so there exists a unique connection whose horizontal subbundle is just \( D(Y) \). If \( Y \in \mathfrak{X}^m(E) \) is
integrable, it is π-transverse if, and only if, its integral manifolds are local sections of π: E → M. In this case, if φ: U ⊂ M → E is a local section with φ(x) = y and φ(U) is the integral manifold of Y through y, then T_y(Im φ) is D_y(Y). Integral sections φ of Y can be characterized by the commutativity of the diagram

\[
\begin{array}{ccc}
\Lambda^m TM & \xrightarrow{\Lambda^m T\phi} & \Lambda^m TE \\
\sigma_M & \downarrow & \downarrow fY \\
M & \xrightarrow{\phi} & E
\end{array}
\]

that is, by the condition

\[
\Lambda^m T\phi = fY \circ \phi \circ \sigma_M
\]  

(5)

where σ_M denotes the natural projection and f ∈ C^∞(E) is a non-vanishing function (observe that we are really characterizing the entire class \{Y\} of integrable m-vector fields).

From the above comments we conclude that classes of locally decomposable and π-transverse m-vector fields \{Y\} ⊂ Χ^m(E) are in one-to-one correspondence with orientable Ehresmann connection forms \(\nabla\) in \(\pi:E \to M\) (orientable in the sense that their associated horizontal distribution is orientable), and hence with orientable jet fields Ψ: E → \(J^1E\) [46]. Observe that this correspondence is characterized by the fact that the horizontal subbundle associated with Ψ (and \(\nabla\)) coincides with \(D(Y)\). Furthermore, the orientable jet field Ψ (and the connection form \(\nabla\)) is integrable if, and only if, so is \(Y\), for every \(Y \in \{Y\}\).

Next we are going to make explicit the above correspondence. Given the bundle \(\pi:E \to M\), denote by \{\(\Lambda^m TE\)\} the projective bundle associated with \(\Lambda^m TE\). Let \(\rho_E: \Lambda^m TE \to \{\Lambda^m TE\}\) and \{σ_E\}: \{\(\Lambda^m TE\)\} → E be the natural projections, and denote by \{\(\Lambda^m TE\)_y\} the fiber at \(y \in E\). We can define a map

\[\Upsilon_E: J^1E \to \{\Lambda^m TE\}\]

as follows: for every \(\bar{y} \in J^1E\) with \(\bar{y} \xrightarrow{\pi^1} y \xrightarrow{\overline{\xi}} x, \) if \(\phi: M \to E\) is a representative of \(\bar{y}\), then \(\Upsilon_E\) maps \(\bar{y} = j^1\phi(x)\) onto the projective class of m-vectors associated with the m-dimensional subspace \(\text{Im} T_x\phi\). An element of \{\(\Lambda^m TE\)_y\} belongs to \(\text{Im} \Upsilon_E\) if it is a class of π-transverse m-vectors at \(y \in E\), such that it has a representative which is decomposable. Denoting by \(D^m_y\) the set of those classes, we have that \(\text{Im} \Upsilon_E = \bigcup_{y \in E} D^m_y \equiv D^m TE\). The map \(\Upsilon_E\) is injective, and hence it is bijective onto its image \(D^m TE\). Then, there exists its inverse \(\Upsilon_E^{-1}\) (on \(D^m TE\)), which acts as follows: for every \((y, \{u_1 \wedge \ldots \wedge u_m\}) \in D^m TE\) with \(\pi(y) = x\), let \(\phi: M \to E\) be a local section such that \(T_x\phi(T_y\pi(u_i)) = u_i\), then \(\Upsilon_E^{-1}(y, \{u_1 \wedge \ldots \wedge u_m\}) = j^1\phi\).

Otherwise, if \(\{Y\} \in D^m TE\), there exists a unique connection form \(\nabla\) such that \(\{Y\}\) is the image in \(D^m TE\) of its horizontal distribution. Let

\[\Upsilon'_E: D^m TE \to \pi^*T^*M \otimes_E TE\]

be the map that associates to every element of \(D^m TE\) the corresponding connection form (observe that, given a horizontal distribution \(H \subset TE\), there is a unique map \(\nabla: E \to \pi^*T^*M \otimes_E TE\) such that \(\nabla(y)(T_{\pi(y)}M) = H_y\). Like \(\Upsilon_E\), the map \(\Upsilon'_E\) is injective, and hence it is bijective onto its image, which is just the set of Ehresmann connections in \(\pi:E \to M\).
So we have the diagram

\[
\begin{array}{c}
\pi^*T^*M \otimes_E TE \\
\downarrow \Upsilon'_E \\
\{\Lambda^m TE\} \supset D^m TE \\
\downarrow \rho_E \\
\Lambda^m TE \\
\end{array}
\quad \begin{array}{c}
\pi^*T^*M \otimes_{E^1} \pi T J^1 E \\
\downarrow \Upsilon'_j E \\
\{\Lambda^m T \pi_j^1\} \supset D^m T J^1 E \\
\downarrow \rho_{J^1 E} \\
\Lambda^m T \pi_j^1 \\
\end{array}
\]

Therefore, from a class \(\{Y\}: E \to D^m TE\) we obtain \(\Psi = \Upsilon_{E^1}^{-1} \circ \{Y\}\), and conversely, from \(\Psi\) we construct \(\{Y\} = \Upsilon_E \circ \Psi\). In the same way, from the class \(\{Y\}: E \to D^m TE\) we obtain \(\nabla = \Upsilon_{E^1} \circ \{Y\}\), and conversely, from \(\nabla\) we construct \(\{Y\} = \Upsilon_E^{-1} \circ \nabla\).

As an evident consequence of the existence of the bijections \(\Upsilon_E: J^1 E \to D^m TE\) and \(\Upsilon'_j E\) from \(D^m TE\) onto its image, \(D^m TE\) inherits the affine structure over \(\pi^*T^*M \otimes_E \nabla(\pi)\) common to \(J^1 E\) and the set of Ehresmann connections in \(\pi: E \to M\).

We want to characterize the integrable \(m\)-vector fields in \(J^1 E\) whose integral manifolds are canonical prolongations of sections of \(\pi\). Let \(\{X\}: J^1 E \to D^m T J^1 E \subset \{\Lambda^m T \pi_j^1\}\) be a class of non-vanishing, locally decomposable and \(\pi^1\)-transverse \(m\)-vector fields in \(J^1 E\), let \(\Psi: J^1 E \to J^1 J^1 E\) be its associated jet field, and \(\nabla: J^1 E \to \pi^1 T^*M \otimes_{J^1 E} T J^1 E\) its associated connection form. Now diagram (6) can be completed as follows

\[
\begin{array}{c}
\pi^*T^*M \otimes_E TE \\
\downarrow \Upsilon'_E \\
\{\Lambda^m TE\} \supset D^m TE \\
\downarrow \rho_E \\
\Lambda^m TE \\
\end{array}
\quad \begin{array}{c}
\pi^*T^*M \otimes_{E^1} \pi T J^1 E \\
\downarrow \Upsilon'_j E \\
\{\Lambda^m T \pi_j^1\} \supset D^m T J^1 E \\
\downarrow \rho_{J^1 E} \\
\Lambda^m T \pi_j^1 \\
\end{array}
\]

where the natural projection \(\tau \otimes T \pi_1: \pi^1 T^*M \otimes_{J^1 E} T J^1 E \to \pi^*T^*M \otimes_E TE\) acts in the following way: if \([\bar{y}, \zeta \otimes \bar{v}] \in \pi^1 T^*M \otimes_{J^1 E} T J^1 E\), with \(\bar{y} \in J^1 E\), \(\zeta \in T^{\pi_1(\bar{y})}M\), \(\bar{v} \in T_{\bar{y}}J^1 E\), then

\[
(\tau \otimes T \pi_1)[\bar{y}, \zeta \otimes \bar{v}] := [\pi^1(\bar{y}), \zeta \otimes T_{\bar{y}} \pi_1(\bar{v})]
\]

Then, bearing in mind the commutativity of this diagram, we have that \(\Psi = \Upsilon_{J^1 E}^{-1} \circ \{X\}\), and that \(\nabla = \Upsilon_{J^1 E} \circ \{X\}\), so

\[
\Upsilon_{J^1 E}^{-1} \circ \Upsilon_E^{-1} \circ \tau \otimes T \pi_1 \circ \nabla = \Upsilon_{E^1}^{-1} \circ \{\Lambda^m T \pi_1\} \circ \{X\} = j^1 \pi_1 \circ \Psi
\]

and, if \(X: J^1 E \to \Lambda^m T J^1 E\) is a representative of the class \(\{X\}\), denoting \(\varrho_E := \Upsilon_{E^1}^{-1} \circ \{\Lambda^m T \pi_1\} \circ \rho_{J^1 E}\), from the last equality we obtain that

\[
\Upsilon_{E^1}^{-1} \circ \Upsilon_E^{-1} \circ \tau \otimes T \pi_1 \circ \nabla = \varrho_E \circ X = j^1 \pi_1 \circ \Psi
\]
Definition 1 The jet field $\Psi$, its associated connection form $\nabla$ in $\tilde{\pi}^1: J^1E \to M$, and their associated class $\{X\}$ are said to be:

1. Semi-holonomic (or a Second Order Partial Differential Equation), if

$$\Upsilon^{-1}_E \circ \Upsilon'^{-1}_E \circ \tau \circ T\pi^1 \circ \nabla = \Upsilon^{-1}_E \circ \{\Lambda^mT\pi^1\} \circ \{X\} = j^1\pi^1 \circ \Psi = \text{Id}_{J^1E}$$

If $X \in \{X\}$ is a representative of this class, then it is a semi-holonomic $m$-vector field, and the above condition leads to

$$\varrho_E \circ X = \text{Id}_{J^1E}$$

2. Holonomic if they are integrable and their integral sections $\varphi: M \to J^1E$ are holonomic (that is, $\varphi = j^1\phi$, for some section $\varphi: M \to E$).

Then, it can be proved that the class $\{X\}$, and its associated jet field $\Psi$ and connection form $\nabla$ are holonomic if, and only if, they are integrable and semi-holonomic.

In a natural chart in $J^1E$, the local expressions of these elements are

$$X = \bigwedge_{\alpha=1}^m f \left( \frac{\partial}{\partial x^\alpha} + F^A_\alpha \frac{\partial}{\partial y^A} + G^A_{\alpha\nu} \frac{\partial}{\partial v^A_{\nu}} \right)$$

$$\Psi = (x^\alpha, y^A, v^A_{\alpha})$$

$$\nabla = dx^\alpha \otimes \left( \frac{\partial}{\partial x^\alpha} + F^A_\alpha \frac{\partial}{\partial y^A} + G^A_{\alpha\nu} \frac{\partial}{\partial v^A_{\nu}} \right)$$

(9)

where $f \in C^\infty(J^1E)$ is an arbitrary non-vanishing function. A representative of the class $\{X\}$ can be selected by the condition $i(X)(\tilde{\pi}^1_*\omega) = 1$, which leads to $f = 1$ in the above local expression. We will adopt this particular choice in the sequel.

Now, if these elements are integrable, and $\varphi(x) = (x^\alpha, y^A = \varphi^A(x), v^A_{\alpha} = \varphi^A_{\alpha}(x))$ is an integral section, then the components of $\varphi$ are solution of the system of partial differential equations

$$F^A_\alpha(x, \varphi^B(x), \varphi^B_\eta(x)) = \frac{\partial \varphi^A_\alpha}{\partial x^\alpha} ; \quad (A,B = 1,\ldots,N ; \eta,\alpha,\nu = 1,\ldots,m)$$

(10)

If these elements are semi-holonomic, their local expressions are the same as in (9) with $F^A_\alpha = v^A_{\alpha}$. In addition, if they are integrable, as their integral sections are holonomic, given a section $\phi(x) = (x^\alpha, y^A = \varphi^A(x))$, its jet prolongation $j^1\phi(x) = \left( x^\alpha, y^A = \varphi^A(x), v^A_{\alpha} = \frac{\partial \varphi^A_\alpha}{\partial x^\alpha}(x) \right)$ is an integral section iff the components of $\phi$ are solution of the second-order system of partial differential equations

$$G^A_{\alpha\nu}(x, \varphi^B(x), \varphi^B_\eta(x)) = \frac{\partial^2 \phi^A_{\alpha}}{\partial x^\nu \partial x^\alpha} ; \quad (A,B = 1,\ldots,N \quad \eta,\alpha,\nu = 1,\ldots,m)$$

(See [17] and [18] for details).

2.3 Lagrangian formalism for classical field theories

(See, for instance, [4], [17], [18], [23], [40], [45], [46], for details).
A classical field theory is described by its configuration fiber bundle $\pi: E \to M$; and a Lagrangian density which is a $\bar{\pi}^1$-semibasic $m$-form on $J^1E$. A Lagrangian density is usually written as $\mathcal{L} = \mathcal{L}\bar{\pi}^1 \omega$, where $\mathcal{L} \in C^\infty(J^1E)$ is the Lagrangian function associated with $\mathcal{L}$ and $\omega$. The Poincaré-Cartan $m$ and $(m+1)$-forms associated with the Lagrangian density $\mathcal{L}$ are defined using the vertical endomorphism $V$ of the bundle $J^1E$

$$\Theta_{\mathcal{L}} := i(\mathcal{V})\mathcal{L} + \mathcal{L} \in \Omega^m(J^1E) \ ; \ \Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}} \in \Omega^{m+1}(J^1E)$$

Then a Lagrangian system is a couple $(J^1E, \Omega_{\mathcal{L}})$. We will say that the Lagrangian system is regular if $\Omega_{\mathcal{L}}$ is 1-nondegenerate. In a natural chart in $J^1E$ we have

$$\Omega_{\mathcal{L}} = -\frac{\partial^2 \mathcal{L}}{\partial v^B_\nu \partial v^B_\alpha} dv^B_\nu \wedge dy^A \wedge d^{m-1}x^\alpha - \frac{\partial^2 \mathcal{L}}{\partial y^B \partial v^B_\alpha} dy^B \wedge dy^A \wedge d^{m-1}x^\alpha +$$

$$\frac{\partial^2 \mathcal{L}}{\partial v^B_\nu \partial v^A_\alpha} dv^B_\nu \wedge d^m x + \left( -\frac{\partial^2 \mathcal{L}}{\partial y^B \partial v^A_\alpha} - \frac{\partial \mathcal{L}}{\partial y^B} + \frac{\partial^2 \mathcal{L}}{\partial x^\alpha \partial v^B_\alpha} \right) dy^B \wedge d^m x$$

(where $d^{m-1}x^\alpha \equiv i \left( \frac{\partial}{\partial x^\alpha} \right) d^m x$); and the regularity condition is equivalent to $\det \left( \frac{\partial^2 \mathcal{L}}{\partial v^A_\alpha \partial v^B_\beta}(\bar{y}) \right) \neq 0$, for every $\bar{y} \in J^1E$. We must point out that, in field theories, the notion of regularity is not uniquely defined (for other approaches see, for instance, [5], [15], [16], [37], [38], [39]).

The Lagrangian problem associated with a Lagrangian system $(J^1E, \Omega_{\mathcal{L}})$ consists in finding sections $\phi \in \Gamma(M, E)$ ($\Gamma(M, E)$ denotes the set of sections of $\pi$), which are characterized by the condition

$$(j^1\phi)^* i(X)\Omega_{\mathcal{L}} = 0 \ , \ \text{for every } X \in \mathfrak{X}(J^1E)$$

In natural coordinates, if $\phi(x) = (x^\alpha, \phi^A(x))$, this condition is equivalent to demanding that $\phi$ satisfy the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial y^A} \bigg|_{j^1\phi} - \frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}}{\partial v^A_\alpha} \bigg|_{j^1\phi} = 0 \ ; \ \text{(for } A = 1, \ldots, N) \ (11)$$

The problem of finding these sections can be formulated equivalently as follows: finding a distribution $D$ of $T(J^1E)$ such that it is integrable (that is, involutive), $m$-dimensional, $\bar{\pi}^1$-transverse, and the integral manifolds of $D$ are the sections solution of the above equations. This is equivalent to stating that the sections solution of the Lagrangian problem are the integral sections of a class of holonomic $m$-vector fields $\{X_{\mathcal{L}}\} \subset \mathfrak{X}^m(J^1E)$, such that

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0 \ , \ \text{for every } X_{\mathcal{L}} \in \{X_{\mathcal{L}}\}$$

Taking into account the equivalence between classes of non-vanishing, locally decomposable and $\bar{\pi}^1$-transverse $m$-vector fields with orientable jet fields and connections, we can also state the above problem in two additional equivalent ways: finding a holonomic connection $\nabla_{\mathcal{L}}$ in $\bar{\pi}^1: J^1E \to M$ such that

$$i(\nabla_{\mathcal{L}})\Omega_{\mathcal{L}} = (m-1)\Omega_{\mathcal{L}}$$

or a holonomic jet field $\Psi_{\mathcal{L}}: J^1E \to J^1J^1E$, such that

$$i(\Psi_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$$

(where the contraction of jet fields with differential forms is defined in [17]). Semi-holonomic locally decomposable $m$-vector fields, jet fields and connections which are solution of these equations are called Euler-Lagrange $m$-vector fields, jet fields and connections for $(J^1E, \Omega_{\mathcal{L}})$. 
Their local expressions are (9) with \( F_A^\alpha = v_A^\alpha \), and where the coefficients \( G_{\alpha \nu}^A \) are related by the system of linear equations

\[
\frac{\partial^2 \mathcal{L}}{\partial v^\alpha \partial v^\nu} G_{\alpha \nu}^A = \frac{\partial \mathcal{L}}{\partial y^B} - \frac{\partial^2 \mathcal{L}}{\partial x^\nu \partial v^B} - \frac{\partial^2 \mathcal{L}}{\partial y^A \partial v^B} v^A \quad (A, B = 1, \ldots, N) \tag{12}
\]

Therefore, if \( j^1 \phi = (x^\mu, \phi^A, \frac{\partial \phi^A}{\partial x^\nu}) \) is an integral section of \( X_\mathcal{L} \), then \( v_\alpha^A = \frac{\partial \phi^A}{\partial x^\alpha} \), and hence the coefficients \( G_{\alpha \nu}^B \) must satisfy equations (10). As a consequence, the system (12) is equivalent to the Euler-Lagrange equations (11) for the section \( \phi \).

If \((J^1 E, \Omega_\mathcal{L})\) is a regular Lagrangian system, then the existence of classes of Euler-Lagrange \( m \)-vector fields for \( \mathcal{L} \) (or what is equivalent, Euler-Lagrange jet fields or connections) is assured, and in a local system of coordinates, these \( m \)-vector fields depend on \( N(m^2 - 1) \) arbitrary functions. For singular Lagrangian systems, the existence of Euler-Lagrange \( m \)-vector fields is not assured except perhaps on some submanifold \( S \hookrightarrow J^1 E \), and the number of arbitrary functions on which they depend is not the same as in the regular case, since it depends on the dimension of \( S \) and the rank of the Hessian matrix of \( \mathcal{L} \). Furthermore, locally decomposable and \( \tau^1 \)-transverse \( m \)-vector fields, solutions of the field equations can exist (in general, on some submanifold of \( J^1 E \), but none of them being semi-holonomic (at any point of this submanifold). As in the regular case, although Euler-Lagrange \( m \)-vector fields exist on some submanifold \( S \), their integrability is not assured except perhaps on another smaller submanifold \( I \hookrightarrow S \) such that the integral sections are contained in \( I \).

### 2.4 Hamiltonian formalism for classical field theories

(See, for instance, [7], [19], [20], [23], [33], [35], [40] for details).

For the Hamiltonian formalism of field theories, the choice of a multimomentum phase space or multimomentum bundle is not unique. In this work we take: \( J^1 \mathcal{L} \equiv \Lambda^m T^*E/\Lambda^0 m T^*E \), where \( \Lambda^m_1 T^*E \equiv \mathcal{M} \pi \) is the bundle of \( m \)-forms on \( E \) vanishing by the action of two \( \tau \)-vertical vector fields (sometimes it is called the extended multimomentum bundle), and \( \Lambda^0_0 T^*E \equiv \pi^* \Lambda^m T^*M \). We have the natural projections

\[
\tau^1: J^1 \mathcal{L} \rightarrow \mathcal{L} \quad \text{and} \quad \pi^1 = \pi \circ \tau^1: J^1 \mathcal{L} \rightarrow M
\]

Given a system of coordinates adapted to the bundle \( \pi: E \rightarrow M \), we can construct natural coordinates in \( J^1 \mathcal{L} \) and \( \mathcal{M} \pi \), which will be denoted as \((x^\alpha, y^A, p^\alpha_A)\) and \((x^\alpha, y^A, p^\alpha_A, p)\), respectively \((\alpha = 1, \ldots, m; A = 1, \ldots, N)\).

Now, if \((J^1 E, \Omega_\mathcal{L})\) is a Lagrangian system, we introduce the extended Legendre map associated with \( \mathcal{L} \), \( \tilde{\mathcal{L}}: J^1 E \rightarrow \mathcal{M} \pi \), in the following way:

\[
(\tilde{\mathcal{L}} \tilde{y})(Z_1, \ldots, Z_m) := (\Theta_\mathcal{L})_{\tilde{y}}(\tilde{Z}_1, \ldots, \tilde{Z}_m)
\]

where \( Z_1, \ldots, Z_m \in T_{\pi^1(z)}E \), and \( \tilde{Z}_1, \ldots, \tilde{Z}_m \in T_{\tilde{y}}J^1 E \) are such that \( T_{\tilde{y}}J^1 E \) are such that \( T_{\tilde{y}} \pi^1 \tilde{Z}_\alpha = Z_\alpha \). \( \tilde{\mathcal{L}} \) can also be defined as the “first order vertical Taylor approximation to \( \mathcal{L} \)” [7]). Hence, using the natural projection \( \mu: \mathcal{M} \pi = \Lambda^m_1 T^*E \rightarrow \Lambda^m_0 T^*E/\Lambda^0_0 T^*E = J^1 \mathcal{L} \), we define the restricted Legendre map associated with \( \mathcal{L} \) as \( \tilde{\mathcal{L}} := \mu \circ \tilde{\mathcal{L}} \). Their local expressions are

\[
\begin{align*}
\tilde{\mathcal{L}}^* x^\alpha &= x^\alpha, & \tilde{\mathcal{L}}^* y^A &= y^A, & \tilde{\mathcal{L}}^* p^\alpha_A &= \frac{\partial \mathcal{L}}{\partial v^\alpha} \\
\mathcal{L}^* x^\alpha &= x^\alpha, & \mathcal{L}^* y^A &= y^A, & \mathcal{L}^* p^\alpha_A &= \frac{\partial \mathcal{L}}{\partial v^\alpha}
\end{align*}
\]
Then, \((J^1E, \Omega_L)\) is a regular Lagrangian system if \(\mathcal{F}L\) is a local diffeomorphism (this definition is equivalent to that given above). Elsewhere \((J^1E, \Omega_L)\) is a singular Lagrangian system. As a particular case, \((J^1E, \Omega_L)\) is a hyper-regular Lagrangian system if \(\mathcal{F}L\) is a global diffeomorphism. A singular Lagrangian system \((J^1E, \Omega_L)\) is almost-regular if: \(\mathcal{P} := \mathcal{F}L(J^1E)\) is a closed submanifold of \(J^1E\) (we will denote the natural imbedding by \(\bar{j}_0; \mathcal{P} \hookrightarrow J^1E\), \(\mathcal{F}L\) is a submersion onto its image, and for every \(\tilde{y} \in J^1E\), the fibres \(\mathcal{F}L^{-1}(\mathcal{F}L(\tilde{y}))\) are connected submanifolds of \(J^1E\).

In order to construct a Hamiltonian system associated with \((J^1E, \Omega_L)\), first, recall that the multicotangent bundle \(\Lambda^mT^*E\) is endowed with canonical forms: \(\Theta \in \Omega^m(\Lambda^mT^*E)\) and the multisymplectic form \(\Omega := -d\Theta \in \Omega^{m+1}(\Lambda^mT^*E)\). But \(\mathcal{M}\pi \equiv \Lambda^mT^*E \subset \Lambda^mT^*E\) is a subbundle of \(\Lambda^mT^*E\). Then, if \(\lambda: \Lambda^mT^*E \hookrightarrow \Lambda^mT^*E\) is the natural imbedding, \(\Theta := \lambda^*\Theta\) and \(\Omega := -d\Theta = \lambda^*\Omega\) are canonical forms in \(\mathcal{M}\pi\), which are called the multimomentum Liouville \(m\) and \((m+1)\) forms. Their local expressions are

\[
\Theta = p_A^a dy^A \wedge dx^a + pd^m x, \quad \Omega = -dp_A^a \wedge dy^A \wedge dx^a - dp \wedge d^m x
\]

Observe that \(\mathcal{F}L^*\Theta = \Theta_L\) and \(\mathcal{F}L^*\Omega = \Omega_L\).

Now, if \((J^1E, \Omega_L)\) is a hyper-regular Lagrangian system, then \(\bar{\mathcal{P}} := \mathcal{F}L(J^1E)\) is a 1-codimensional imbedded submanifold of \(\mathcal{M}\pi\) (we will denote the natural imbedding by \(\bar{j}_0; \bar{\mathcal{P}} \hookrightarrow \mathcal{M}\pi\)), which is transverse to the projection \(\mu\), and is diffeomorphic to \(J^1\bar{E}\). This diffeomorphism is \(\mu^{-1}\), when \(\mu\) is restricted to \(\bar{\mathcal{P}}\), and also coincides with the map \(h := \mathcal{F}L \circ \mathcal{F}L^{-1}\), when it is restricted onto its image (which is just \(\bar{\mathcal{P}}\). This map \(h\) is called a Hamiltonian section, and can be used to construct the Hamilton-Cartan \(m\) and \((m+1)\) forms of \(J^1\bar{E}\) by making

\[
\Theta_h = h^*\Theta \in \Omega^m(J^1\bar{E}) \quad , \quad \Omega_h = h^*\Omega \in \Omega^{m+1}(J^1\bar{E})
\]

and the couple \((J^1\bar{E}, \Omega_h)\) is said to be the Hamiltonian system associated with the hyper-regular Lagrangian system \((J^1E, \Omega_L)\). Locally, the Hamiltonian section \(h\) is specified by the local Hamiltonian function \(H = p_A^a(\mathcal{F}L^{-1})^*v^A_{\alpha} - (\mathcal{F}L^{-1})^*L\), that is, \(h(x^\alpha, y^A, p_A^\alpha) = (x^\alpha, y^A, p_A^\alpha, -H)\). Then we have the local expressions

\[
\Theta_h = p_A^a dy^A \wedge dx^a - H d^m x, \quad \Omega_h = -dp_A^a \wedge dy^A \wedge dx^a + dH \wedge d^m x
\]

Of course \(\mathcal{F}L^*\Theta_h = \Theta_L\) and \(\mathcal{F}L^*\Omega_h = \Omega_L\).

The Hamiltonian problem associated with the Hamiltonian system \((J^1\bar{E}, \Omega_h)\) consists in finding sections \(\psi \in \Gamma(M, J^1\bar{E})\), which are characterized by the condition

\[
\psi^* i(X) \Omega_h = 0 \quad , \quad \text{for every } X \in \mathfrak{X}(J^1\bar{E})
\]

In natural coordinates, if \(\psi(x) = (x^\alpha, y^A(x), p_A^\alpha(x))\), this condition leads to the so-called Hamilton-De Donder-Weyl equations

\[
\frac{\partial y^A}{\partial x^\alpha}_{\psi} = \frac{\partial H}{\partial p_A^\alpha}_{\psi} \quad , \quad \frac{\partial p_A^\alpha}{\partial x^\alpha}_{\psi} = \frac{\partial H}{\partial y^A}_{\psi}
\]

The problem of finding these sections can be formulated equivalently as follows: finding a distribution \(D\) of \(T(J^1E)\) such that \(D\) is integrable (that is, involutive), \(m\)-dimensional, \(\tilde{\tau}^1\)-transverse, and its integral manifolds are the sections solution of the above equations. This is equivalent to stating that the sections solution of the Hamiltonian problem are the integral sections of a class of integrable and \(\tilde{\tau}^1\)-transverse \(m\)-vector fields \(\{X_H\} \subset \mathfrak{X}^m(J^1\bar{E})\) satisfying that

\[
i(X_H) \Omega_h = 0 \quad , \quad \text{for every } X_H \in \{X_H\}\]
As in the Lagrangian formalism, we can also state the above problem in two additional equivalent ways: finding an orientable connection $\nabla_{\mathcal{H}}$ in $\tilde{\tau}^1: J^{1*}E \to M$ such that

$$i(\nabla_{\mathcal{H}})\Omega_h = (m - 1)\Omega_h$$

or an orientable jet field $\Psi_{\mathcal{H}}: J^{1*}E \to J^1J^{1*}E$, such that

$$i(\Psi_{\mathcal{H}})\Omega_h = 0$$

$\tilde{\tau}^1$-transverse and locally decomposable $m$-vector fields, orientable jet fields and orientable connections which are solutions of these equations are called Hamilton-De Donder-Weyl (HDW) $m$-vector fields, jet fields and connections for $(J^{1*}E, \Omega_h)$.

Their local expressions in natural coordinates are

$$X_{\mathcal{H}} = \bigwedge_{\alpha=1}^m f \left( \frac{\partial}{\partial x^\alpha} + F^A_\alpha \frac{\partial}{\partial y^A} + G^\eta_{A\alpha} \frac{\partial}{\partial p^\eta_A} \right)$$

$$\Psi_{\mathcal{H}} = (x^\alpha, y^A, p^\alpha_A, F^A_\alpha, G^\eta_{A\alpha})$$

$$\nabla_{\mathcal{H}} = dx^\alpha \otimes \left( \frac{\partial}{\partial x^\alpha} + F^A_\alpha \frac{\partial}{\partial y^A} + G^\nu_{A\alpha} \frac{\partial}{\partial p^\nu_A} \right)$$

where $f \in C^\infty(J^{1*}E)$ is a non-vanishing function, and the coefficients $F^A_\alpha, G^\eta_{A\alpha}$ are related by the system of linear equations

$$F^A_\alpha = \frac{\partial H}{\partial p^A_\alpha}, \quad G^\nu_{A\alpha} = -\frac{\partial H}{\partial y^A}$$

Now, if $\psi = (x^\alpha, \psi^A(x), \psi^\alpha_A(x))$ is an integral section of $X_{\mathcal{H}}$ then

$$F^A_\alpha \circ \psi = \frac{\partial \psi^A}{\partial x^\alpha}; \quad G^\eta_{A\alpha} \circ \psi = \frac{\partial \psi^\alpha_A}{\partial x^\alpha}$$

which are the Hamilton-De Donder-Weyl equations for $\psi$. As above, a representative of the class $\{X\}$ can be selected by the condition $i(X)(\tilde{\tau}^1\omega) = 1$, which leads to $f = 1$.

The existence of classes of HDW $m$-vector fields, jet fields and connections is assured, and in a local system of coordinates they depend on $N(m^2 - 1)$ arbitrary functions.

In an analogous way, if $(J^1E, \Omega_{\mathcal{L}})$ is an almost-regular Lagrangian system, the submanifold $j_0: \mathcal{P} \to J^1E$, is a fiber bundle over $E$ (and $M$). The corresponding projections will be denoted by $\tau^1_0: \mathcal{P} \to E$ and $\tilde{\tau}^1_0: \mathcal{P} \to M$, satisfying that $\tau^1 \circ j_0 = \tau^1_0$ and $\tilde{\tau}^1 \circ j_0 = \tilde{\tau}^1_0$. In this case the $\mu$-transverse submanifold $\mathcal{P} \hookrightarrow M\pi$ is diffeomorphic to $\mathcal{P}$. This diffeomorphism is denoted $\tilde{\mu}: \mathcal{P} \to \mathcal{P}$, and it is just the restriction of the projection $\mu$ to $\mathcal{P}$. Then, taking $\tilde{h} := \tilde{\mu}^{-1} = \mathcal{F}\mathcal{L}_0 \circ \mathcal{F}\mathcal{L}_0^{-1}$, (where $\mathcal{F}\mathcal{L}_0$ and $\mathcal{F}\mathcal{L}_0$ are the restriction maps of $\mathcal{F}\mathcal{L}$ and $\mathcal{F}\mathcal{L}$ onto $\mathcal{P}$ and $\mathcal{P}$, respectively), we define the Hamilton-Cartan forms

$$\Theta^0_{\tilde{h}} = (j_0 \circ \tilde{h})^* \Theta; \quad \Omega^0_h = (j_0 \circ \tilde{h})^* \Omega$$

which verify that $\mathcal{F}\mathcal{L}_0^* \Theta^0_{\tilde{h}} = \Theta_{\mathcal{L}}$ and $\mathcal{F}\mathcal{L}_0^* \Omega^0_h = \Omega_{\mathcal{L}}$. Then $(J^{1*}E, \mathcal{P}, \Omega^0_h)$ is the Hamiltonian system associated with the almost-regular Lagrangian system $(J^1E, \Omega_{\mathcal{L}})$, and we have the following diagram

$$\begin{array}{c}
\mathcal{P} \\
\tilde{\tau}^1_0 \\
\mathcal{F}\mathcal{L}_0 \\
J^1E
\end{array} \quad \begin{array}{c}
\mathcal{F}\mathcal{L}_0 \\
\tilde{h} \\
\hat{\mu} \\
\mu
\end{array} \quad \begin{array}{c}
\mathcal{P} \\
\tilde{\tau}^1 \\
\mathcal{F}\mathcal{L}_0 \\
J^{1*}E
\end{array} \quad \begin{array}{c}
M \\
\tilde{\tau}^1_0 \\
\mathcal{F}\mathcal{L}_0 \\
J^{1*}E
\end{array}$$

(14)
The Hamiltonian problem associated with the Hamiltonian system \((J^1E, \mathcal{P}, \Omega^0_h)\) is stated as in the regular case, and the sections \(\psi_o \in \Gamma(M, \mathcal{P})\) solution of the Hamiltonian problem are the integral sections of a class of integrable and \(\bar{\tau}_0\)-transverse \(m\)-vector fields \(\{X_{\mathcal{H}_o}\} \subset \mathfrak{X}^m(\mathcal{P})\) satisfying that
\[
i(X_{\mathcal{H}_o})\Omega^0_h = 0, \quad \text{for every } X_{\mathcal{H}_o} \in \{X_{\mathcal{H}_o}\}
\]
As above, this is equivalent to finding an orientable connection \(\nabla_{\mathcal{H}_o}\) in \(\bar{\tau}_1: \mathcal{P} \to M\) such that
\[
i(\nabla_{\mathcal{H}_o})\Omega^0_h = (m - 1)\Omega^0_h
\]
or an orientable jet field \(\psi_{\mathcal{H}_o}: \mathcal{P} \to J^1\mathcal{P}\), such that
\[
i(\psi_{\mathcal{H}_o})\Omega^0_h = 0
\]
Now, not even the existence of these Hamilton-De Donder-Weyl \(m\)-vector fields, jet fields and connections for \((J^1E, \mathcal{P}, \Omega^0_h)\) is assured, and an algorithmic procedure in order to obtain a submanifold \(\mathcal{P}\) of \(\mathcal{P}\) where such \(m\)-vector fields, jet fields and connections exist, can be outlined. Of course, in general, the solution is not unique, but the number of arbitrary functions is not the same as above (it depends on the dimension of \(\mathcal{P}\)).

### 3 The field operators

#### 3.1 Sections along the Legendre maps in field theories

First remark that for multivector fields along maps the terminology introduced in Section 2.2 will be applied in a natural way. Thus, for instance:

- If \(\mathcal{X}\) is a \(m\)-vector field along \(\Phi\), it is locally decomposable if, for every \(p \in A\), there exists an open neighbourhood \(U_p \subset A\), and \(\mathcal{X}_1, \ldots, \mathcal{X}_m\), vector fields along \(\Phi\), such that \(\mathcal{X} = \mathcal{X}_1 \wedge \ldots \wedge \mathcal{X}_m\).

- If \(\mathcal{X}, \mathcal{X}'\) are non-vanishing multivector fields along \(\Phi\), and \(U \subset A\) is a connected open set, then \(\mathcal{X} \sim U \mathcal{X}'\) if there exists a non-vanishing function \(f \in C^\infty(U)\) such that \(\mathcal{X}' = f\mathcal{X}\).

Now, let \(\pi: E \to M\) be the configuration fiber bundle of a Lagrangian system \((J^1E, \Omega_E)\). The case we will consider consists in taking \(A = J^1E, B = \mathcal{M}\pi, \text{ and } \Phi = \bar{\mathcal{F}}L\).

**Definition 2**

1. A \(m\)-vector field along \(\bar{\mathcal{F}}L\) is a map \(\tilde{\mathcal{X}}: J^1E \to \Lambda^mT\mathcal{M}\pi\) such that
\[
\sigma_{\mathcal{M}\pi} \circ \tilde{\mathcal{X}} = \bar{\mathcal{F}}L
\]
where \(\sigma_{\mathcal{M}\pi}: \Lambda^mT\mathcal{M}\pi \to \mathcal{M}\pi\) is the natural projection.

2. A jet field along \(\bar{\mathcal{F}}L\) is a map \(\tilde{\mathcal{Y}}: J^1E \to J^1\mathcal{M}\pi\) such that
\[
\pi_{\mathcal{M}\pi}^1 \circ \tilde{\mathcal{Y}} = \bar{\mathcal{F}}L
\]
where \(\pi_{\mathcal{M}\pi}^1: J^1\mathcal{M}\pi \to \mathcal{M}\pi\) is the natural projection.
3. An Ehresmann connection form along \( \widetilde{\mathcal{L}} \) is a map \( \tilde{\nabla}: J^1E \to (\tilde{\tau}^1 \circ \mu)^*T^\ast M \otimes_{\mathcal{M}_\pi} T\mathcal{M}_\pi \) such that
\[
\kappa_{\mathcal{M}_\pi} \circ \tilde{\nabla} = \widetilde{\mathcal{L}}
\]
where \( \kappa_{\mathcal{M}_\pi}: (\tilde{\tau}^1 \circ \mu)^*T^\ast \mathcal{M}_\pi \otimes_M T\mathcal{M}_\pi \to \mathcal{M}_\pi \) is the natural projection, and satisfying that
\[
i(\tilde{\nabla})\tilde{\Xi} = \tilde{\Xi}
\]
for every \((\tilde{\tau}^1 \circ \mu)\)-semibasic 1-form \( \tilde{\Xi} \) along \( \widetilde{\mathcal{L}} \). (Observe that \( \tilde{\nabla} \) is a \((1,1)\)-tensor field along \( \widetilde{\mathcal{L}} \)).

Recall that a 1-form \( \Xi \) along \( \widetilde{\mathcal{L}} \) is \((\tilde{\tau}^1 \circ \mu)\)-semibasic if, for every \( Z \in \mathfrak{X}^{(\tilde{\tau}^1 \circ \mu)}(\mathcal{M}_\pi) \), and \( \check{y} \in J^1E \), we have that \( i(Z_{\widetilde{\mathcal{L}}(\check{y})})((\Xi(\check{y})) = 0 \). In particular, if \( \xi \in \Omega^1(\mathcal{M}_\pi) \) is a \((\tilde{\tau}^1 \circ \mu)\)-semibasic form, then \( \Xi := \xi \circ \widetilde{\mathcal{L}} \) is a \((\tilde{\tau}^1 \circ \mu)\)-semibasic 1-form along \( \widetilde{\mathcal{L}} \).

So we have the diagrams

In the same way as classes of locally decomposable and transverse \( m \)-vector fields in a fiber bundle are associated with orientable jet fields and connections [18], we have an analogous result in the current situation. In fact:

**Theorem 1** Classes of locally decomposable and \((\tilde{\tau}^1 \circ \mu)\)-transverse \( m \)-vector fields along \( \widetilde{\mathcal{L}} \) are in one-to-one correspondence with jet fields along \( \widetilde{\mathcal{L}} \), and hence with Ehresmann connection forms along \( \widetilde{\mathcal{L}} \).
(Proof) Bearing in mind (7), in the current situation we have the diagram

\[
\begin{array}{ccc}
\Lambda^m TE & \xrightarrow{\rho_E} & \Lambda^m T(\tau^1 \circ \mu) \\
& \downarrow & \\
\{\Lambda^m TJ^1 E\} \supset D^m TE & \xrightarrow{\{\Lambda^m T(\tau^1 \circ \mu)\}} & D^m TM\pi \subset \{\Lambda^m TM\pi\} \\
& \downarrow & \\
\gamma_E E & \xrightarrow{\gamma_E^{-1}} & \gamma'_{M\pi} \pi_{M\pi} \\
\pi^* T^* M \otimes E TE & \xrightarrow{\hat{\tau} \otimes T(\tau^1 \circ \mu)} & (\tau^1 \circ \mu)^* T^* M \otimes_{M\pi} TM\pi \\
\end{array}
\]

(where the natural projection \(\hat{\tau} \otimes T(\tau^1 \circ \mu)\) is defined in a similar way to \(\tau \otimes T\pi^1\) in (8)).

Now, if \(\{\tilde{X}\}: J^1 E \to D^m TM\pi \subset \{\Lambda^m TM\pi\}\) is a class of non-vanishing, locally decomposable and \((\tau^1 \circ \mu)\)-transverse \(m\)-vector fields along \(\tilde{F}\mathcal{L}\), then from this class \(\{\tilde{X}\}\) we obtain \(\tilde{\nabla}_X = \gamma_{M\pi}^{-1} \circ \{\tilde{X}\}\), and conversely, from a jet field \(\tilde{V}\) along \(\tilde{F}\mathcal{L}\) we construct \(\{\tilde{X}\} = \gamma_{M\pi} \circ \{\tilde{V}\}\). In the same way, from the class \(\{\tilde{X}\}\) we obtain \(\tilde{\nabla}_X = \gamma_{M\pi}^{-1} \circ \{\tilde{X}\}\), and conversely, from an Ehresmann connection form \(\tilde{\nabla}\) along \(\tilde{F}\mathcal{L}\) we construct \(\{\tilde{X}\} = \gamma_{M\pi}^{-1} \circ \tilde{\nabla}\).

The local expression of a representative of the class \(\{\tilde{X}\}\) of non-vanishing, locally decomposable and \((\tau^1 \circ \mu)\)-transverse \(m\)-vector fields along \(\tilde{F}\mathcal{L}\), and its associated jet field \(\tilde{\nabla}_X\) and connection form \(\tilde{\nabla}_X\) along \(\tilde{F}\mathcal{L}\) are

\[
\begin{align*}
\tilde{X} &= \bigwedge_{\alpha=1}^m F(x^\nu, y^B, v^B_\nu) \left[ \left( \frac{\partial}{\partial x^\alpha} \circ \tilde{F}\mathcal{L} \right) + f^A_\alpha(x^\nu, y^B, v^B_\nu) \left( \frac{\partial}{\partial y^A} \circ \tilde{F}\mathcal{L} \right) + g^{\eta}_{A\alpha}(x^\nu, y^B, v^B_\nu) \left( \frac{\partial}{\partial p^\eta_A} \circ \tilde{F}\mathcal{L} \right) + h_\alpha(x^\nu, y^B, v^B_\nu) \left( \frac{\partial}{\partial p} \circ \tilde{F}\mathcal{L} \right) \right] \\
\tilde{\nabla}_X &= \left( x^\alpha, y^A, \frac{\partial \mathcal{L}}{\partial x^\alpha}, \mathcal{L} - v^A_\alpha \frac{\partial \mathcal{L}}{\partial v^A_\alpha}; f^A_\alpha, g^{\eta}_{A\alpha}, h_\alpha \right) \\
\tilde{\nabla}_X &= (dx^\alpha \circ \tilde{F}\mathcal{L}) \otimes \left[ \left( \frac{\partial}{\partial x^\alpha} \circ \tilde{F}\mathcal{L} \right) + f^A_\alpha \left( \frac{\partial}{\partial y^A} \circ \tilde{F}\mathcal{L} \right) + g^{\eta}_{A\alpha} \left( \frac{\partial}{\partial p^\eta_A} \circ \tilde{F}\mathcal{L} \right) + h_\alpha \left( \frac{\partial}{\partial p} \circ \tilde{F}\mathcal{L} \right) \right]
\end{align*}
\]

Now, let \(\{\tilde{X}\}\) be a class of non-vanishing, locally decomposable and \((\tau^1 \circ \mu)\)-transverse \(m\)-vector fields along \(\tilde{F}\mathcal{L}\), let \(\tilde{\nabla}_X: J^1 E \to J^1 M\pi\) be its associated jet field along \(\tilde{F}\mathcal{L}\), and \(\tilde{\nabla}_X: J^1 E \to (\tau^1 \circ \mu)^* T^* M \otimes_{M\pi} TM\pi\) its associated Ehresmann connection form along \(\tilde{F}\mathcal{L}\). Then bearing in mind the commutativity of the diagram (15) (see also (18) below), and the relations among these elements, we have

\[
\begin{align*}
&(\tau^1 \circ \mu) \circ \tilde{\nabla}_X = \gamma_{E}^{-1} \circ \{\Lambda^m T(\tau^1 \circ \mu)\} \circ \{\tilde{X}\} = \gamma_{E}^{-1} \circ \gamma_{E}'^{-1} \circ \hat{\tau} \otimes T(\tau^1 \circ \mu) \circ \nabla
\end{align*}
\]
If $\bar{X}: J^1E \to \Lambda^mT\mathcal{M}\pi$ is a representative of the class $\{\bar{X}\}$, introducing
\[
\tilde{\theta}_E := \Upsilon^{-1}_E \circ \{\Lambda^mT(\tau^1 \circ \mu)\} \circ \rho_{\mathcal{M}\pi} = \Upsilon^{-1}_E \circ \rho_E \circ \Lambda^mT(\tau^1 \circ \mu)
\]
we have that
\[
\Upsilon^{-1}_E \circ \{\Lambda^mT(\tau^1 \circ \mu)\} \circ \{\bar{X}\} = \Upsilon^{-1}_E \circ \{\Lambda^mT(\tau^1 \circ \mu)\} \circ \rho_{\mathcal{M}\pi} \circ \bar{X} \equiv \tilde{\theta}_E \circ \bar{X}
\]
from the above equality we obtain
\[
j^1(\tau^1 \circ \mu) \circ \bar{\nabla}_{\bar{X}} = \tilde{\theta}_E \circ \bar{X} = \Upsilon^{-1}_E \circ \Upsilon'_E^{-1} \circ \tilde{\tau} \otimes T(\tau^1 \circ \mu) \circ \nabla
\]

**Definition 3** $\bar{\nabla}_{\bar{X}}$, its equivalent $\nabla_{\bar{X}}$, and their associated class $\{\bar{X}\}$ are said to be semi-holonomic if
\[
\Upsilon^{-1}_E \circ \Upsilon'_E^{-1} \circ \tilde{\tau} \otimes T(\tau^1 \circ \mu) \circ \nabla = \Upsilon^{-1}_E \circ \{\Lambda^mT(\tau^1 \circ \mu)\} \circ \{\bar{X}\} = j^1(\tau^1 \circ \mu) \circ \bar{\nabla}_{\bar{X}} = \text{Id}_{J^1E}
\]

If $\bar{X} \in \{\bar{X}\}$ is a representative of this class, then the above condition leads to
\[
\tilde{\theta}_E \circ \bar{X} = \text{Id}_{J^1E}
\]
and $\bar{X}$ is a semi-holonomic m-vector field along $\mathcal{FL}$.

In this case, we have completed the diagram (15) as follows

\[
\begin{array}{cccccc}
\pi^*T^*M \otimes_E TE & \xrightarrow{\tilde{\tau} \otimes T(\tau^1 \circ \mu)} & (\tau^1 \circ \mu)^*T^*M \otimes_{\mathcal{M}\pi} T\mathcal{M}\pi \\
\ \Upsilon'_E & \ | \ & \ | \\
\{\Lambda^mTJ^1E\} \supset D^mTE & \xrightarrow{\Upsilon^{-1}_E} & \{\Lambda^mT(\tau^1 \circ \mu)\} & \xrightarrow{\Upsilon'_E} & \{\Lambda^mT(\tau^1 \circ \mu)\} \\
\Upsilon_E & \ | \ & \ | \\
J^1E & \xrightarrow{\tilde{\theta}_E} & \{\bar{X}\} & \xrightarrow{j^1(\tau^1 \circ \mu)} & \{\tilde{\nabla}_{\bar{X}}\} & \xrightarrow{\Upsilon^{-1}_E} & \{\Lambda^mT(\tau^1 \circ \mu)\} \\
\rho_E & \ | \ & \ | \\
\Lambda^mTE & \xrightarrow{\Upsilon^{-1}_E} & \{\Lambda^mT(\tau^1 \circ \mu)\} & \xrightarrow{\rho_{\mathcal{M}\pi}} & \Lambda^mT\mathcal{M}\pi
\end{array}
\]

In order to obtain the corresponding local expressions, consider natural charts of adapted coordinates $(x^\alpha, y^A, p_A^\alpha, h_{\alpha})$ and $(x^\alpha, y^A, p_A^\alpha, p, f_\alpha^A, g_{A\alpha}^\eta, h_\alpha)$ in $J^1\mathcal{M}\pi$. Let $\bar{y} \in J^1\mathcal{M}\pi$ with $\bar{y} \xrightarrow{\tau^1} y \xrightarrow{\tau^1} x$. If $\tilde{\psi}: M \to \mathcal{M}\pi$ is a representative of $y$, then $\psi(x) = y$, and
\[
y \equiv (x^\alpha, y^A, p_A^\alpha, p) = (x^\alpha, \psi^A(x), \psi^\alpha_A(x), \psi(x)) \equiv \tilde{\psi}(x)
\]
\[
\bar{y} \equiv (x^\alpha, y^A, p_A^\alpha, f_\alpha^A, g_{A\alpha}^\eta, h_\alpha) = \left( x^\alpha, \psi^A(x), \psi^\alpha_A(x), \frac{\partial \psi^A}{\partial x^\alpha}(x), \frac{\partial \psi^\alpha_A}{\partial x^\alpha}(x), \frac{\partial \psi}{\partial x^\alpha}(x) \right) \equiv (j^1\tilde{\psi})(x)
\]
Now we can construct the section $\tau^1 \circ \tilde{\psi}: M \to E$, which is a representative of the point $\tilde{y} \in J^1 E$, such that $\tilde{y} \mapsto y \mapsto x$. Thus
\[
\tilde{y} \equiv (x^\alpha, y^A, v^A_\alpha) = \left( x^\alpha, \psi^A(x), \frac{\partial \psi^A}{\partial x^\alpha}(x) \right) \equiv [j^1(\tau^1 \circ \tilde{\psi})](x)
\]

On the other hand, recall that the map $j^1\tau^1$ is defined by $j^1\tau^1(\tilde{y}) := [j^1(\tau^1 \circ \tilde{\psi})](x)$, for every section $\tilde{\psi}$. Therefore, since $j^1\tau^1(\tilde{y}) = \tilde{y}$, we conclude that
\[
j^1\tau^1(x^\alpha, y^A, v^A_\alpha, f^A_\alpha) = (x^\alpha, y^A, f^A_\alpha)
\]
As a consequence, if $\tilde{Y}$ is a jet field along $\mathcal{FL}$, the condition of being semi-holonomic is locally equivalent to demanding that
\[
(x^\alpha, y^A, v^A_\alpha) \equiv \tilde{y} = (j^1\tau^1 \circ \tilde{\psi})(\tilde{y}) \equiv (j^1\tau^1 \circ \tilde{\psi})(x^\alpha, y^A, v^A_\alpha) = (j^1\tau^1 \circ \tilde{\psi}) \left( x^\alpha, \psi^A(x), \frac{\partial \psi^A}{\partial x^\alpha}(x) \right) = \left( x^\alpha, \psi^A(x), \frac{\partial \psi^A}{\partial x^\alpha}(x) \right) = (x^\alpha, y^A, f^A_\alpha)
\]
that is, $f^A_\alpha = v^A_\alpha$ (in (16)).

Now, the generalization of the integrability conditions (1) and (5) to the current situation leads to the following:

**Definition 4** Let $\tilde{\mathcal{X}}$ be a non-vanishing and locally decomposable $m$-vector field along $\mathcal{FL}$. A section $\varphi: M \to J^1 E$ is said to be an integral section of $\tilde{\mathcal{X}}$ if
\[
\Lambda^m T(\mathcal{FL} \circ \varphi) = f\tilde{\mathcal{X}} \circ \varphi \circ \sigma_M
\]
where $f \in C^\infty(J^1 E)$ is a non-vanishing function. $\tilde{\mathcal{X}}$ is said to be integrable if it admits integral sections. Thus, we have the diagram
\[
\begin{array}{ccc}
\Lambda^m TM & \xrightarrow{\Lambda^m T(\mathcal{FL} \circ \varphi)} & \Lambda^m TM \pi \\
\sigma_M \downarrow & & \sigma_{M \pi} \\
M & \xrightarrow{\varphi} & J^1 E \xrightarrow{\mathcal{FL}} M \pi
\end{array}
\]
$\tilde{\mathcal{X}}$ is said to be holonomic if its integral sections are holonomic, that is, $\varphi = j^1 \phi$ for some section $\phi: M \to E$.

Observe that we are characterizing the integrability of the entire class $\{\tilde{\mathcal{X}}\}$. Note also that this definition of integral section is equivalent to stating that the image of the section $\tilde{\varphi} = \mathcal{FL} \circ \varphi: M \to \Lambda^m TM \pi$ is an integral submanifold of the distribution $\mathcal{D}(\tilde{\mathcal{X}})$; that is, $T_{\tilde{\varphi}(x)}(\text{Im } \tilde{\varphi}) = [\mathcal{D}(\tilde{\mathcal{X}})]_{\tilde{\varphi}(x)}$, for every $x \in M$. Finally, it is important to remark that, if a $m$-vector field along $\mathcal{FL}$ is not integrable everywhere in $J^1 E$, it could be integrable on a submanifold $\mathcal{I} \hookrightarrow J^1 E$ (see the comment at the end of Section 2.3).
Remark: Of course, a class \( \{ \tilde{X} \} \) of \( m \)-vector fields along \( \tilde{\mathcal{FL}} \) is integrable (resp. holonomic) if, and only if, its associated jet field \( \tilde{\mathcal{Y}} \) and connection \( \tilde{\nabla} \) along \( \tilde{\mathcal{FL}} \) are also.

In addition, the class \( \{ \tilde{X} \} \) and its associated jet field \( \tilde{\mathcal{Y}} \) and connection \( \tilde{\nabla} \) are holonomic if, and only if, they are integrable and semi-holonomic (see the proof for multivector fields, jet fields and connections in jet bundles in [18]).

In a system of natural coordinates, if \( \varphi(x) = (x^\alpha, \varphi^A(x), \varphi^A_i(x)) \) is an integral section of \( \tilde{X} \) (see (16) for its local expression), then the following system of partial differential equations holds

\[
\begin{align*}
  f^A_\alpha(x^\beta, \varphi^C(x), \varphi^C_i(x)) &= \frac{\partial \varphi^A}{\partial x^\alpha}, \\
  g^n_{A\alpha}(x^\beta, \varphi^C(x), \varphi^C_i(x)) &= \frac{\partial^2 \mathcal{L}}{\partial x^\alpha \partial \psi^n_{\alpha}} + \frac{\partial \mathcal{L}}{\partial y^B \partial \psi^n_{\alpha}} \frac{\partial \varphi^B}{\partial x^\alpha} + \frac{\partial^2 \mathcal{L}}{\partial y^B \partial \psi^n_{\alpha}} \frac{\partial \varphi^B}{\partial x^\alpha} \\
  h_\alpha(x^\beta, \varphi^C(x), \varphi^C_i(x)) &= \frac{\partial \mathcal{L}}{\partial x^\alpha} + \frac{\partial \varphi^A}{\partial y^A} \frac{\partial}{\partial x^\alpha} - \varphi^n_{\alpha} \left( \frac{\partial^2 \mathcal{L}}{\partial x^\alpha \partial \psi^n_{\alpha}} + \frac{\partial \mathcal{L}}{\partial y^B \partial \psi^n_{\alpha}} \frac{\partial \varphi^B}{\partial x^\alpha} + \frac{\partial^2 \mathcal{L}}{\partial y^B \partial \psi^n_{\alpha}} \frac{\partial \varphi^B}{\partial x^\alpha} \right)
\end{align*}
\]

(20)

In particular, if \( \varphi \) is holonomic and \( \varphi = j^1 \phi \) with \( \phi(x) = (x^\alpha, \phi^A(x)) \), then \( \varphi^A \equiv \phi^A \), and \( \varphi^A_\alpha = \frac{\partial \phi^A}{\partial x^\alpha} \). Therefore the above system is second order.

As a final remark, we define the contraction of jet fields along \( \tilde{\mathcal{FL}} \) with differential forms along \( \tilde{\mathcal{FL}} \), as a natural extension of the same operation between jet fields and differential forms in a jet bundle. Thus, let \( \tilde{\mathcal{Y}} \) be a jet field along \( \tilde{\mathcal{FL}} \), then to every \( Z \in \tilde{\mathcal{X}}(M) \) we can associate a vector field \( \tilde{Z} \) along \( \tilde{\mathcal{FL}} \), which is given by

\[ \tilde{Z}(\tilde{y}) := (T_x \psi)(Z_x) \]

for every \( \tilde{y} \in J^1 E \), with \( \tilde{y} \overset{x^\alpha}{\mapsto} y \overset{x^\alpha}{\mapsto} x \), and \( \psi \in \tilde{\mathcal{Y}}(\tilde{y}) \). If \( Z = F^\alpha \frac{\partial}{\partial x^\alpha} \) the local expression of \( \tilde{Z} \) is

\[ \tilde{Z} = F^\alpha \left[ \left( \frac{\partial}{\partial x^\alpha} \circ \tilde{\mathcal{FL}} \right) \right] + f^A_\alpha \left( \frac{\partial}{\partial y^A} \circ \tilde{\mathcal{FL}} \right) + g^n_{A\alpha} \left( \frac{\partial}{\partial \psi^n_{\alpha}} \circ \tilde{\mathcal{FL}} \right) + h_\alpha \left( \frac{\partial}{\partial p} \circ \tilde{\mathcal{FL}} \right) \]

Definition 5 Let \( \Xi \) be a \( (m+p) \)-form along \( \tilde{\mathcal{FL}} \) (with \( p \geq 0 \)), then \( i(\tilde{\mathcal{Y}}) \Xi \) is an element of \( \Omega^m(M, \pi^1 \circ \mu) \otimes_M \Omega^p(M, \mathcal{FL}) \) defined as

\[ (i(\tilde{\mathcal{Y}}) \Xi)(\tilde{y}, Z_1_\ldots, Z_m, \tilde{X}_1_\ldots, \tilde{X}_p) := \Xi(\tilde{y}, Z_1_\ldots, Z_m, \tilde{X}_1_\ldots, \tilde{X}_p) \]

for \( Z_1_\ldots, Z_m \in \tilde{\mathcal{X}}(M) \), and \( \tilde{X}_1_\ldots, \tilde{X}_p \) vector fields in \( M \pi \) along \( \mathcal{FL} \).

This map is extended by zero to forms of degree \( k < m \), and it is a \( C^\infty(M) \)-linear and alternate on \( Z_1_\ldots, Z_m \) and \( \tilde{X}_1_\ldots, \tilde{X}_p \).

Remark: Observe that contracting a form \( \Xi \) along \( \mathcal{FL} \) with a jet field \( \tilde{\mathcal{Y}} \) along \( \mathcal{FL} \) is equivalent to contracting \( \Xi \) with a (suitable) representative of the class \( \{ \tilde{X} \} \) of \( m \)-vector fields along \( \mathcal{FL} \) associated with \( \tilde{\mathcal{Y}} \).

As is evident, all the definitions and results in this section can also be stated in a similar way for \( m \)-vector fields \( \mathcal{X} \) along the restricted Legendre map \( \mathcal{FL} \), and orientable jet fields \( \mathcal{Y} \) and connection forms \( \nabla \) along \( \mathcal{FL} \).
3.2 The extended field operators

Let \((J^1E, \Omega_L)\) be a Lagrangian system. Then:

**Definition 6**

1. An extended \(m\)-vector field operator \(\kappa\) associated with \((J^1E, \Omega_L)\) is a map \(\kappa: J^1E \to \Lambda^mT^*M \pi\) verifying the following conditions:
   
   (a) (Structural condition): \(\kappa\) is a non-vanishing, locally decomposable and \((\bar{\tau}^1 \circ \mu)\)-transverse \(m\)-vector field along \(\mathcal{F}\).
   
   (b) (Field equation condition): \(\mathcal{F}^*[(i(\kappa))((\Omega \circ \mathcal{F}))] = 0\).
   
   (c) (Semi-holonomy condition): \(\kappa\) is semi-holonomic.

2. An extended jet field operator \(\tilde{Y}_K\) associated with \((J^1E, \Omega_L)\) is a map \(\tilde{Y}_K: J^1E \to J^1M\pi\) verifying the following conditions:
   
   (a) (Structural condition): \(\tilde{Y}_K\) is an orientable jet field along \(\mathcal{F}\).
   
   (b) (Field equation condition): \(\mathcal{F}^*[(i(\tilde{Y}_K))(\Omega \circ \mathcal{F})] = 0\).
   
   (c) (Semi-holonomy condition): \(\tilde{Y}_K\) is semi-holonomic.

3. An extended connection operator \(\tilde{\nabla}_K\) associated with \((J^1E, \Omega_L)\) is a map \(\tilde{\nabla}_K: J^1E \to (\bar{\tau}^1 \circ \mu)^*T^*M \otimes_{M\pi} TM\pi\) verifying the following conditions:
   
   (a) (Structural condition): \(\tilde{\nabla}_K\) is an orientable Ehresmann connection form along \(\mathcal{F}\).
   
   (b) (Field equation condition): \(\mathcal{F}^*[(i(\tilde{\nabla}_K))(\Omega \circ \mathcal{F}) - (m - 1)(\Omega \circ \mathcal{F})] = 0\).
   
   (c) (Semi-holonomy condition): \(\tilde{\nabla}_K\) is semi-holonomic.

**Remark**: Note that the field equation condition of the first item in this definition defines not a single \(m\)-vector field along \(\mathcal{F}\), but classes of them. A representative can be selected by adding the following normalization condition

\[
i(\kappa)[((\bar{\tau}^1 \circ \mu)^*\omega \circ \mathcal{F})] = 1 \quad (21)
\]

which, in its turn, implies the \((\bar{\tau}^1 \circ \mu)\)-transversality condition.

**Theorem 2** A class of extended \(m\)-vector field operators \(\kappa\) is associated with an extended jet field operator \(\tilde{Y}_K\) and an extended connection operator \(\tilde{\nabla}_K\), and conversely.

(Proof) It follows from Theorem 1. The only point to be proved is the equivalence between the field equation conditions, which follows after a simple calculation in coordinates, using the local expressions (16).

**Theorem 3** (Existence and local multiplicity). There exist classes of extended \(m\)-vector field operators \(\kappa\) for \((J^1E, \Omega_L)\), and hence there exist also extended jet field operators \(\tilde{Y}_K\) and extended connection operators \(\tilde{\nabla}_K\). In a local system, they depend on \(N(m^2 - 1)\) arbitrary functions.
(Proof) First we analyze the local existence, and then their global extension.

Let \( \bar{y} \equiv (x^\alpha, y^A, v^A_\alpha) \in J^1E \), and \( \hat{y} = (y, X_y) \in \Lambda^mT\mathcal{M}\pi \) (where \( y \equiv (x^\alpha, y^A, p_A^A, p) \in T\mathcal{M}\pi \) and \( X_y \in \Lambda^mT_y\mathcal{M}\pi \)), such that \( \tilde{K}(\bar{y}) = \hat{y} \). First, the equation \( \sigma_{\mathcal{M}\pi} \circ \tilde{K} = \bar{F}L \) in condition 1 implies that \( y = \bar{F}L(\bar{y}) \), thus

\[
p^A_\alpha = \frac{\partial L}{\partial v^A_\alpha}, \quad p = L - v^A_\alpha \frac{\partial L}{\partial v^A_\alpha}
\]

On the other hand, from condition 1, \( X_y \) is a locally decomposable and \( (\tilde{r}^1 \circ \mu) \)-transverse \( m \)-vector at \( y \), hence

\[
X_y = \bigwedge_{\alpha=1}^m F(\bar{y}) \left( \frac{\partial}{\partial x^\alpha} \mid \bar{F}L(\bar{y}) \right) + f^A_\alpha(\bar{y}) \frac{\partial}{\partial y^A} \mid \bar{F}L(\bar{y}) + g^\eta_{A\alpha}(\bar{y}) \frac{\partial}{\partial p_A^\eta} \mid \bar{F}L(\bar{y}) + h_\alpha(\bar{y}) \frac{\partial}{\partial p} \mid \bar{F}L(\bar{y}) \right)
\]

In this way we can write

\[
\tilde{K} = \bigwedge_{\alpha=1}^m F \left[ \left( \frac{\partial}{\partial x^\alpha} \circ \bar{F}L \right) + f^A_\alpha \left( \frac{\partial}{\partial y^A} \circ \bar{F}L \right) + g^\eta_{A\alpha} \left( \frac{\partial}{\partial p_A^\eta} \circ \bar{F}L \right) + h_\alpha \left( \frac{\partial}{\partial p} \circ \bar{F}L \right) \right]
\]

Now, the semi-holonomy condition implies that \( f^A_\alpha = v^A_\alpha \).

Next, taking into account (13), we have that

\[
\bar{F}L^*[\tilde{K}(\Omega \circ \bar{F}L)] = (-1)^{m(m+1)/2} F \bar{F}L^* \left[ -v^A_\alpha (\partial p_A^\alpha \circ \bar{F}L) - (dp \circ \bar{F}L) + g^\eta_{A\alpha} (dy^A \circ \bar{F}L) \right] + \sum_{\eta \neq \alpha} (-1)^\eta (g^\eta_{A\eta} v^A_\alpha - g^\eta_{A\alpha} v^A_\eta) (dx^\alpha \circ \bar{F}L) + h_\alpha (dx^\alpha \circ \bar{F}L)
\]

\[
= (-1)^{m(m+1)/2} F \left[ -v^A_\alpha \left( \frac{\partial L}{\partial v^A_\alpha} \right) - d \left( L - v^A_\alpha \frac{\partial L}{\partial v^A_\alpha} \right) + g^\eta_{A\alpha} (dy^A \circ \bar{F}L) \right] + \sum_{\eta \neq \alpha} (-1)^\eta (g^\eta_{A\eta} v^A_\alpha - g^\eta_{A\alpha} v^A_\eta) \left( \frac{\partial L}{\partial x^\alpha} + \sum_{\eta \neq \alpha} (-1)^\eta (g^\eta_{A\eta} v^A_\alpha - g^\eta_{A\alpha} v^A_\eta) + h_\alpha \right) \right] \right]
\]

and then, from the field equation condition we obtain

\[
g^A_{\alpha\alpha} = \frac{\partial L}{\partial y^A}, \quad h_\alpha = \frac{\partial L}{\partial x^\alpha} - \sum_{\eta \neq \alpha} (-1)^\eta (g^\eta_{A\eta} v^A_\alpha - g^\eta_{A\alpha} v^A_\eta)
\]

Finally, if we apply the normalization condition (21), we can choose \( F = 1 \). In this way we have obtained, for this representative, the local expression

\[
\tilde{K} = \bigwedge_{\alpha=1}^m \left[ \left( \frac{\partial}{\partial x^\alpha} \circ \bar{F}L \right) + v^A_\alpha \left( \frac{\partial}{\partial y^A} \circ \bar{F}L \right) + g^\eta_{A\alpha} \left( \frac{\partial}{\partial p_A^\eta} \circ \bar{F}L \right) \right] + \left( \frac{\partial L}{\partial x^\alpha} - \sum_{\eta \neq \alpha} (-1)^\eta (g^\eta_{A\eta} v^A_\alpha - g^\eta_{A\alpha} v^A_\eta) \right) \left( \frac{\partial}{\partial p} \circ \bar{F}L \right)
\]

So, \( \tilde{K} \) is determined by the \( Nm^2 \) coefficients \( g^\eta_{A\alpha} \), which are related by the first group of \( N \) independent equations (22). Therefore, there are \( N(m^2 - 1) \) arbitrary functions.
These results allow us to assure the local existence of classes of extended \( m \)-vector field operators \( \tilde{K} \) satisfying the desired conditions. The corresponding global solutions are then obtained using a partition of unity subordinated to a covering of \( J^1E \) made of local natural charts \( \{ U_i \} \). Then, let \( \tilde{K}_i \) be the field operator in the corresponding open set \( U_i \). As every local class \( \{ \tilde{K}_i \} \) is associated with a local Ehresmann connection form \( \tilde{\nabla}_{\tilde{K}_i} \) along \( FL \) (by Theorem 2), and the convex combination of connection forms gives a connection form, \( \tilde{\nabla}_K = g^i \tilde{\nabla}_{\tilde{K}_i} \) is a (global) Ehresmann connection form along \( FL \) which is associated with the corresponding class. The class \( \{ \tilde{K} \} \) associated with \( \tilde{\nabla}_K \) is the global solution. In an analogous way, taking into account the affine structure of the fibers of \( J^1M \pi \), it is meaningful to construct convex combinations of sections \( \tilde{Y}_{K_i} \), so we can define a global jet field \( \tilde{Y}_K := g^i \tilde{Y}_{K_i} \) along \( FL \) associated with the class \( \{ \tilde{K} \} \).

These elements satisfy the conditions of Definition 6. In particular:

- The classes \( \{ \tilde{K} \} \) are made of non-vanishing, locally decomposable and \((\tau^1 \circ \mu)\)-transverse \( m \)-vector fields along \( FL \), since they are associated with orientable jet fields and connections along \( FL \).

- The field equation condition holds for every \( \tilde{K} \), because it holds for every \( \tilde{K}_i \), and \( \tilde{K} \) is a linear combination \( \tilde{K} = f^i \tilde{K}_i \). As a consequence, the equivalent field equation conditions hold for \( \tilde{\nabla}_K \) and \( \tilde{Y}_K \).

- The semi-holonomy of \( \tilde{K} \) is proved starting from the semi-holonomy of \( \tilde{K}_i \), and using that \( \{ K \} \) is associated with \( \tilde{Y}_K := g^i \tilde{Y}_{K_i} \), which is semi-holonomic because so are \( \tilde{Y}_{K_i} \).

Remarks:

- Observe that the existence of these extended field operators does not depend on the regularity of the Lagrangian system.

- The class \( \tilde{K} \) (and hence the associated \( \tilde{Y}_K \) and \( \tilde{\nabla}_K \)) is integrable if Definition 4 holds for it. Observe that, if \( \tilde{K} \), and hence the associated \( \tilde{Y}_K \) and \( \tilde{\nabla}_K \) are integrable, they are holonomic, since they are semi-holonomic.

Among the multiplicity of extended field operators, we will mainly be interested in those which are integrable, and hence holonomic. Thus, if \( \varphi \equiv j^1 \phi(x) = \left( x^\mu, \phi^A, \frac{\partial \phi^A}{\partial x^\nu} \right) \) is an integral section, then \( v^A_\alpha = \frac{\partial \phi^A}{\partial x^\alpha} \), and from (20) and (22) we obtain that \( \phi \) must be a solution of the following system

\[
\frac{\partial^2 L}{\partial x^\alpha \partial v^A_\alpha} + \frac{\partial^2 L}{\partial y^B \partial v^A_\alpha} \frac{\partial \phi^B}{\partial x^\alpha} + \frac{\partial^2 L}{\partial v^B_\alpha \partial v^A_\alpha} \frac{\partial^2 \phi^B}{\partial x^\alpha \partial x^\nu} = \frac{\partial L}{\partial y^A}
\]

which are just the Euler-Lagrange equations for \( \phi \) (see Theorem 5 for a precise statement of this comment).

In general, we know there is no way of assuring that this system is integrable in \( J^1E \) (even in the hyper-regular case). In the most favourable cases, a submanifold \( I \hookrightarrow J^1E \) could exist such that there are (classes of) integrable \( m \)-vector field operators \( \tilde{K} \) on \( I \) which are tangent to \( I \). As a consequence, the above system is integrable on \( I \) and the corresponding integral sections solution are in \( I \). (See also [18] and [19] for a discussion of the integrability of multivector fields).
3.3 The restricted field operators

In field theory there is another kind of field operator which can be defined, and which is justified because the multimomentum bundle where the Hamiltonian formalism of field theories takes place is really $J^{1*}E$, instead of $M\pi$, and hence, in this case, the relevant Legendre map is $FL$ instead of $\tilde{FL}$.

**Definition 7**  
1. Given an extended $m$-vector field operator $\tilde{K}$, the restricted $m$-vector field operator $K$ associated with $\tilde{K}$ is 
$$K := \Lambda^m T \mu \circ \tilde{K}$$

2. Given an extended jet field operator $\tilde{Y}_K$, the restricted jet field operator $Y_K$ associated with $\tilde{Y}_K$ is 
$$Y_K := j^1 \mu \circ \tilde{Y}_K$$

3. Given an extended connection operator $\tilde{\nabla}_K$, the restricted connection operator $\nabla_K$ associated with $\tilde{\nabla}_K$ is 
$$\nabla_K := (\tau_\mu \otimes T \mu) \circ \tilde{\nabla}_K$$

So, we have the diagrams

\begin{align*}
\begin{array}{c}
\begin{array}{c}
\tilde{K} \\
\downarrow \Lambda^m T \pi \\
J^1 E \\
\downarrow FL \\
J^{1*} E
\end{array} & \begin{array}{c}
K \\
\downarrow \Lambda^m T \mu \\
J^{1*} E \\
\downarrow \sigma J^{1*} E \\
J^1 E
\end{array} \\
\begin{array}{c}
\tilde{Y}_K \\
\downarrow j^1 \mu \\
J^1 E \\
\downarrow FL \\
J^{1*} E
\end{array} & \begin{array}{c}
Y_K \\
\downarrow \pi^1 J^{1*} E \\
J^1 E \\
\downarrow FL \\
J^{1*} E
\end{array} \\
\begin{array}{c}
\tilde{\nabla}_K \\
\downarrow (\tau^1 \circ \mu)^* T^* M \otimes_M \pi T \pi \\
J^{1*} E \\
\downarrow FL \\
J^{1*} E
\end{array} & \begin{array}{c}
\nabla_K \\
\downarrow \kappa J^{1*} E \\
J^{1*} E
\end{array}
\end{array}
\end{align*}

(where the natural projection $\tau_\mu \otimes T \mu$ is defined in a similar way to $\tau \otimes T \pi^1$ in (8)).

The restricted field operators, $K$, $Y_K$, and $\nabla_K$ are $m$-vector fields, jet fields and Ehresmann connection forms along the Legendre map $FL$, respectively. In particular, it is obvious that every restricted $m$-vector field operator $K$ is non-vanishing, locally decomposable and $\tilde{\tau}^1$-transverse.

**Remark:** In an analogous way to Theorems 1 and 2, we can prove that for every class of extended $m$-vector field operators $\{\tilde{K}\}$, and its associated $\tilde{Y}_K$ and $\tilde{\nabla}_K$, the class of restricted $m$-vector field operators $\{K\} := \{\Lambda^m T \mu\} \circ \{\tilde{K}\}$ is associated with $Y_K = j^1 \mu \circ \tilde{Y}_K$, and $\nabla_K = \tau_\mu \otimes T \mu \circ \tilde{\nabla}_K$.

**Proposition 1**  
1. $\{K\}$ and its associated $Y_K$ and $\nabla_K$ are semi-holonomic.

2. $K$ is integrable if, and only if, the corresponding $\tilde{K}$ is also. That is, $\varphi: M \rightarrow J^1 E$ is an integral section of $\{K\}$ if, and only if, it is an integral section of $\{\tilde{K}\}$ too.

3. $Y_K$ and $\nabla_K$ are integrable if, and only if, the corresponding $\tilde{Y}_K$ and $\tilde{\nabla}_K$ are also.

(Proof)
1. For every $\tilde{K} \in \{\tilde{K}\}$, and $K = \Lambda^m T \circ \tilde{K}$, we have that

$$\Upsilon^{-1} \circ \rho_E \circ \Lambda^m T \circ K = \Upsilon^{-1} \circ \rho_E \circ \Lambda^m T \circ \tilde{K} = \Upsilon^{-1} \circ \rho_E \circ \Lambda^m T(\tau \circ \mu) \circ \tilde{K} = \text{Id}_{J^1 E}$$

2. Let $\varphi : M \to J^1 E$ be a section. We have

$$\Lambda^m T FL \circ \Lambda^m T \circ \varphi = \Lambda^m T (\mu \circ \bar{FL}) \circ \Lambda^m T \circ \varphi = \Lambda^m T \mu \circ \Lambda^m T \bar{FL} \circ \Lambda^m T \varphi$$

and, on the other hand, for every $\tilde{K} \in \{\tilde{K}\}$, and $K = \Lambda^m T \mu \circ \tilde{K}$, we have:

$$\mathcal{K} \circ \varphi \circ \sigma_M = \Lambda^m T \mu \circ \tilde{K} \circ \varphi \circ \sigma_M$$

therefore, if $f \in C^\infty(J^1 E)$ is a non-vanishing function, then

$$\Lambda^m T \bar{FL} \circ \Lambda^m T \circ \varphi = f \circ \tilde{K} \circ \varphi \circ \sigma_M \Rightarrow \Lambda^m T FL \circ \Lambda^m T \circ \varphi = f \circ \mathcal{K} \circ \varphi \circ \sigma_M$$

Conversely, if the last relation holds, then the first one is true for some non-vanishing function $g \in C^\infty(J^1 E)$.

3. It is a straightforward consequence of all the above results.

The coordinate expressions of these elements are

$$\mathcal{K} = \bigwedge_{\alpha=1}^m F \left[ \left( \frac{\partial}{\partial x^\alpha} \circ FL \right) + v^A_\alpha \left( \frac{\partial}{\partial y^A} \circ FL \right) + g^n_{A\alpha} \left( \frac{\partial}{\partial p^A_n} \circ FL \right) \right]$$

$$\Upsilon_{\mathcal{K}} = \left( x^\alpha, y^A, \frac{\partial L}{\partial v^A_\alpha}, v^A_\alpha, g^n_{A\alpha} \right)$$

$$\nabla_{\mathcal{K}} = (dx^\alpha \circ FL) \otimes \left[ \left( \frac{\partial}{\partial x^\alpha} \circ FL \right) + v^A_\alpha \left( \frac{\partial}{\partial y^A} \circ FL \right) + g^n_{A\alpha} \left( \frac{\partial}{\partial p^A_n} \circ FL \right) \right]$$

with the same relation as above for the coefficients $g^n_{A\alpha}$. Of course, the normalization condition

$$i(\mathcal{K})(\bar{\tau}^{1*} \omega \circ FL) = 1$$

allows to take $F = 1$, and selects a representative on each class $\{\mathcal{K}\}$ of restricted field operators. This implies $\bar{\tau}^1$-transversality condition for $\mathcal{K}$.

**Remark:** In the particular case $M \equiv \mathbb{R}$, the expressions (22), (23) and (24) lead to the local expressions of the extended and restricted $K$-operators of time-dependent mechanical systems given in [8].

### 4 Properties of the field operators

#### 4.1 The Lagrangian equations

Next we study the properties of the field operators in relation to the Lagrangian equations. As we have stated three different but equivalent approaches to the concept of field operator (namely jet fields, Ehresmann connection forms or classes of $m$-vector fields along the Legendre maps), we use the most suitable in each case, in order to make the proofs easier.

The result (in mechanics) that we want to generalize is relation (2) (on a submanifold $S \hookrightarrow TQ$). Thus, we have the following relation between the field operators and the solutions of the Lagrangian field equations (Euler-Lagrange jet fields, multivector fields and connections):
Theorem 4 Let \((J^1 E, \Omega_L)\) be a Lagrangian system.

1. Let \(\widetilde{\mathcal{Y}}_K\) be an extended jet field operator associated with \((J^1 E, \Omega_L)\). If there exist a jet field \(\Psi_L: J^1 E \to J^1 J^1 E\), and a \(\pi^1\)-transverse submanifold \(\mathcal{S}: S \hookrightarrow J^1 E\), such that

\[
j^1 F_L \circ \Psi_L = \frac{\widetilde{\mathcal{Y}}_K}{S}
\]  

then \(\Psi_L\) is an Euler-Lagrange jet field for \((J^1 E, \Omega_L)\), on \(S\).

Conversely, given an Euler-Lagrange jet field \(\Psi_L\) for \((J^1 E, \Omega_L)\), then (25) defines an extended jet field operator \(\widetilde{\mathcal{Y}}_K\) for \((J^1 E, \Omega_L)\), on \(S\).

2. Let \(\{\tilde{K}\}\) be a class of extended \(m\)-vector field operators associated with \((J^1 E, \Omega_L)\). If there exist a class of \(m\)-vector fields \(\{X_L\} \subset \mathfrak{X}^m(J^1 E)\), and a \(\pi^1\)-transverse submanifold \(\mathcal{S}: S \hookrightarrow J^1 E\), such that, for every \(\tilde{K} \in \{\tilde{K}\}\),

\[
\Lambda^m T \tilde{F}_L \circ X_L = \frac{\tilde{K}}{S} \quad \text{for some } X_L \in \{X_L\}
\]  

then \(\{X_L\}\) is a class of Euler-Lagrange \(m\)-vector fields for \((J^1 E, \Omega_L)\), on \(S\).

Conversely, given a class of Euler-Lagrange \(m\)-vector fields \(\{X_L\}\) for \((J^1 E, \Omega_L)\), then (26) defines a class of extended \(m\)-vector field operators \(\{\tilde{K}\}\) for \((J^1 E, \Omega_L)\), on \(S\).

3. Let \(\tilde{\nabla}_K\) be an extended connection operator associated with \((J^1 E, \Omega_L)\). If there exist an Ehresmann connection form \(\nabla_L: J^1 E \to \pi^1 T^* M \otimes J^1 E \otimes J^1 E\), and a \(\pi^1\)-transverse submanifold \(\mathcal{S}: S \hookrightarrow J^1 E\), such that

\[
(\tilde{\varepsilon}_{T^* M} \otimes T \tilde{F}_L) \circ \nabla_L = \frac{\tilde{\nabla}_K}{S}
\]  

(where \(\tilde{\varepsilon}_{T^* M}: \pi^1 T^* M \to (\pi^1 \circ \mu)^* T^* M\) is the natural identification), then \(\nabla_L\) is an Euler-Lagrange connection for \((J^1 E, \Omega_L)\), on \(S\).

Conversely, given an Euler-Lagrange connection \(\nabla_L\) for \((J^1 E, \Omega_L)\), then (27) defines an extended jet field operator \(\tilde{\nabla}_K\) for \((J^1 E, \Omega_L)\), on \(S\).

(Proof) First, as a guideline for the proof, consider the following diagram (which, in general, is
not commutative unless restricted to the appropriate submanifolds):

1. We must prove that both the semi-holonomy condition, and the field equation condition hold for \( \tilde{\psi}_K \) if, and only if, they hold for \( \psi_L \). In this proof all the equalities hold on \( S \).

On the one hand, and in relation to the semi-holonomy, we have that

\[
j^1(\tau^1 \circ \mu) \circ \tilde{\psi}_K = j^1(\tau^1 \circ \mu) \circ j^1 \overline{F_L} \circ \psi_L = j^1(\tau^1 \circ \mu \circ \overline{F_L}) \circ \psi_L = j^1 \pi^1 \circ \psi_L
\]

which relates the semi-holonomy of \( \tilde{\psi}_K \) and \( \psi_L \).

On the other hand, for the field equation we obtain

\[
\overline{F_L}^* [i(\tilde{\psi}_K)(\Omega \circ \overline{F_L})] = \overline{F_L}^* [i(j^1 \overline{F_L} \circ \psi_L)(\Omega \circ \overline{F_L})] = i(\psi_L)(\overline{F_L}^*\Omega) = i(\psi_L)\Omega_L
\]

hence the field equation condition holds for \( \tilde{\psi}_K \) if, and only if, the Lagrangian field equation holds for \( \psi_L \).

2. Bearing in mind the commutativity of the diagram (28) (on the appropriate submanifolds), from (25) we obtain that

\[
j^1 \overline{F_L} \circ \psi_L = \tilde{\psi}_K \quad \Leftrightarrow \quad j^1 \overline{F_L} \circ \psi_L \circ \gamma_{J^1 E}^{-1} \circ \{X_L\} = \gamma_{M\pi}^{-1} \circ \{K\}
\]

\[
\Leftrightarrow \quad \psi_L \circ j^1 \overline{F_L} \circ \gamma_{J^1 E}^{-1} \circ \{X_L\} = \Lambda^m T \overline{F_L} \circ \{X_L\} = \{K\}
\]

which leads to the relation (26), for every representative \( X_L \in \{X_L\} \). As \( \{K\} \) is the class associated with \( \tilde{\psi}_K \), it is made of semi-holonomic \( m \)-vector fields along \( \overline{F_L} \). In addition, following the same reasoning as in the above item, it is proved that all of them verify the field equation condition.

3. Finally, following the same pattern as in the last item, (25) leads to (27), and as \( \overline{\nabla}_K \) is associated with \( \tilde{\psi}_K \), it is semi-holonomic. In addition, following the same reasoning as in item 1, it is proved that it verifies the field equation condition.
Comment: Observe that relations (25), (26) and (27) arise because the commutativity of diagram (28) demands it.

As a straightforward consequence of this Theorem, and the Remark after Definition (7), we obtain:

**Corollary 1** If relations (25), (26) and (27) hold for the extended field operators, then the following ones hold for their associated restricted field operators:

\[
\begin{align*}
\mathbf{j}^1 \mathcal{F} \circ \Psi \big|_S &= \mathcal{Y}_K \\
\Lambda^m T \mathcal{F} \circ X_L \big|_S &= K \quad \text{for every } K \in \{K\}, \text{ and for some } X_L \in \{X_L\} \\
(\varepsilon_{T^*M} \otimes T \mathcal{F} \big|_S) \circ \nabla_L &= \nabla_K \\
\text{(with } \varepsilon_{T^*M} : \tilde{\pi}_1^* \rightarrow \tilde{\tau}_1^*)
\end{align*}
\]

Then, assuming all these relations, we have:

**Theorem 5** \( \varphi : M \xrightarrow{\varphi_S} S \xrightarrow{j_S} J^1 E \) is an integral section of \( \{K\} \) if, and only if, it is an integral section of \( \{X_L\} \) too. Moreover, every integral section \( \varphi := j_S \circ \varphi_S \) is an holonomic section. That is, the class \( \{K\} \), and its associated \( \mathcal{Y}_K \) and \( \nabla_K \) are integrable if, and only if, the class \( \{X_L\} \), and its associated \( \mathcal{Y}_L \) and \( \nabla_L \) are integrable too.

The same result holds for the extended field operators \( \{\tilde{K}\} \), \( \tilde{\mathcal{Y}}_K \) and \( \tilde{\nabla}_K \).

(Proof) If \( \varphi : M \xrightarrow{\varphi_S} S \xrightarrow{j_S} J^1 E \) is an integral section of \( K \in \{K\} \) (on \( S \)), we have

\[
\Lambda^m T (\mathcal{F} \circ \varphi) = \Lambda^m T \mathcal{F} \circ \Lambda^m T \varphi = f \mathcal{F} \circ \varphi \circ \sigma_M = f (\Lambda^m T \mathcal{F} \circ X_L) \circ \varphi \circ \sigma_M
\]

where \( f \in C^\infty(J^1 E) \) is a non-vanishing function. However, if we recall that \( \ker T_{\varphi(x)} \mathcal{F} \mathcal{L} \) are \( \pi^1 \)-vertical vectors, and then also \( \tilde{\pi}^1 \)-vertical, from the above equality we can conclude that

\[
\Lambda^m T \varphi = g X_L \circ \varphi \circ \sigma_M
\]

\( g \in C^\infty(J^1 E) \) being another non-vanishing function. So \( \varphi := j_S \circ \varphi_S \) is an integral section of \( X_L \in \{X_L\} \) (on \( S \)). The converse is proved by reversing this reasoning.

As \( \varphi \) are integral sections of semi-holonomic \( m \)-vector fields, they are holonomic sections necessarily.

Finally, the result for the extended field operators is a consequence of Proposition 1.

**Remarks:**

- If \( (J^1 E, \Omega_L) \) is hyper-regular (recall that, in this case, \( \overline{\mathcal{F} \mathcal{L}} \) is a diffeomorphism onto its image, and \( \mathcal{F} \mathcal{L} \) is a diffeomorphism), then \( S = J^1 E \), and the correspondence between field operators and the corresponding Euler-Lagrange solutions of the Lagrangian field equations is one-to-one.

  If \( (J^1 E, \Omega_L) \) is almost-regular, then not only one, but a family of Euler-Lagrange solutions of the Lagrangian field equations is associated with every field operator.

- In addition, if the integrability condition holds only in a submanifold \( I \hookrightarrow S \), Theorem 5 holds only on \( I \).

Observe also that this Theorem establishes the property analogous to the first one given in Section 2.1 for the evolution operator \( K \) in mechanics.
4.2 The Hamiltonian equations

Now we will study the properties of the field operators in relation to the Hamiltonian equations (see the Section 2.4 for the necessary background).

The result (in mechanics) that we want to generalize is relation (4) (on a submanifold \( S \hookrightarrow TQ \)). Thus, we have the following relation between the field operators and the solutions of the Lagrangian field equations (HDW jet fields, multivector fields and connections):

**Theorem 6** Let \((J^1E, \Omega_L)\) be an almost-regular Lagrangian system, and \((J^{1*}E, \mathcal{P}, \Omega^0_h)\) its associated Hamiltonian system.

1. Let \(\tilde{\mathcal{Y}}_K\) be an extended jet field operator associated with \((J^1E, \Omega_L)\). If there exist a jet field \(\Psi_{H_o}: \mathcal{P} \rightarrow J^1\mathcal{P}\), and a \(\tilde{\pi}^1\)-transverse submanifold \(\tilde{J}_S: S \hookrightarrow J^1E\), such that
   \[
   j^1\tilde{\mathcal{Y}}_0 \circ j^1\tilde{h} \circ \Psi_{H_o} \circ \mathcal{F}L_0 = \tilde{\mathcal{Y}}_K
   \]  
   then \(\Psi_{H_o}\) is a Hamilton-De Donder-Weyl jet field for \((J^{1*}E, \mathcal{P}, \Omega^0_h)\), on \(P = \mathcal{F}L(S)\).
   Conversely, given a Hamilton-De Donder-Weyl jet field \(\Psi_{H_o}\) for \((J^{1*}E, \mathcal{P}, \Omega^0_h)\), on a submanifold \(P \hookrightarrow \mathcal{P}\), then (29) defines a jet field \(\tilde{\mathcal{Y}}_K\) along \(\mathcal{F}L\), on every submanifold \(S \hookrightarrow J^1E\) such that \(\mathcal{F}L(S) = P\), which satisfy the structural and the field equation conditions of Definition 6, but not the semi-holonomy condition necessarily.

2. Let \(\tilde{K}\) be a class of extended field operators associated with \((J^1E, \Omega_L)\). If there exist a class of \(m\)-vector fields \(\{X_{H_o}\} \subset \mathcal{X}^m(\mathcal{P})\), and a \(\tilde{\pi}^1\)-transverse submanifold \(\tilde{J}_S: S \hookrightarrow J^1E\), such that, for every \(\tilde{K} \in \{\tilde{K}\}\),
   \[
   \Lambda^m T\tilde{\mathcal{Y}}_0 \circ \Lambda^m T\tilde{h} \circ X_{H_o} \circ \mathcal{F}L_0 = \tilde{K} \quad ; \text{for some } X_{H_o} \in \{X_{H_o}\},
   \]  
   then \(\{X_{H_o}\}\) is a class of Hamilton-De Donder-Weyl \(m\)-vector fields for \((J^{1*}E, \mathcal{P}, \Omega^0_h)\), on \(P = \mathcal{F}L(S)\).
   Conversely, if \(\{X_{H_o}\}\) is a class of Hamilton-De Donder-Weyl \(m\)-vector fields for \((J^{1*}E, \mathcal{P}, \Omega^0_h)\), on a submanifold \(P \hookrightarrow \mathcal{P}\), then the above relation defines a class of \(m\)-vector fields \(\{\tilde{K}\}\) along \(\mathcal{F}L\), on every submanifold \(S \hookrightarrow J^1E\) such that \(\mathcal{F}L(S) = P\), which satisfy the structural and the field equation conditions of Definition 6, but not the semi-holonomy condition necessarily.

3. Let \(\tilde{\nabla}_K\) be an extended connection operator associated with \((J^1E, \Omega_L)\). If there exist an Ehresmann connection form \(\nabla_{H_o}: \mathcal{P} \rightarrow \tilde{\pi}_0^{1*}\mathcal{T}^*M \otimes_\mathcal{P} \mathcal{T}\mathcal{P}\), and a \(\tilde{\pi}^1\)-transverse submanifold \(\tilde{J}_S: S \hookrightarrow J^1E\), such that
   \[
   (\tilde{\varepsilon}_0^{0*} \mathcal{T}\tilde{F}L_0) \circ \nabla_{H_o} \circ \mathcal{F}L_0 = \tilde{\nabla}_K
   \]  
   (where \(\tilde{\varepsilon}_0^{0*} : \tilde{\pi}_0^{1*}\mathcal{T}^*M \rightarrow (\tilde{\pi}_1 \circ \mu)^*\mathcal{T}^*M\) is the natural identification), then \(\nabla_{H_o}\) is a Hamilton-De Donder-Weyl connection for \((J^{1*}E, \mathcal{P}, \Omega^0_h)\), on \(P = \mathcal{F}L(S)\).
   Conversely, given a Hamilton-De Donder-Weyl connection \(\nabla_{H_o}\) for \((J^{1*}E, \mathcal{P}, \Omega^0_h)\), on a submanifold \(P \hookrightarrow \mathcal{P}\), then the above relation defines an Ehresmann connection form \(\tilde{\nabla}_K\) along \(\mathcal{F}L\), on every submanifold \(S \hookrightarrow J^1E\) such that \(\mathcal{F}L(S) = P\), which satisfy the structural and the field equation conditions of Definition 6, but not the semi-holonomy condition necessarily.

If \((J^1E, \Omega_L)\) is a hyper-regular Lagrangian system, and \((J^{1*}E, \Omega^0_h)\) its associated Hamiltonian system, then the same results hold (with \(S = J^1E\)). In the converse statements however, the jet field \(\tilde{\mathcal{Y}}_K\), Ehresmann connection form \(\tilde{\nabla}_K\), and classes of \(m\)-vector fields \(\{\tilde{K}\}\) along \(\mathcal{F}L\) also satisfy also the semiholonomy condition, and hence they are extended field operators for \((J^1E, \Omega_L)\).
(Proof) First we prove item 2. As a standpoint, consider the following diagram, (which, in general, is not commutative unless restricted to the appropriate submanifolds) (see also diagram (14)):

\[ \begin{array}{ccc}
\Lambda^m T M_\pi & \xrightarrow{\sigma M_\pi} & M_\pi \\
\Lambda^m T \tilde{j}_0 & \xrightarrow{\tilde{\sigma}} & \tilde{\mathcal{F}} L \\
\Lambda^m T \tilde{\mathcal{P}} & \xrightarrow{\tilde{\sigma} \mathcal{P}} & \Lambda^m T \tilde{h} \\
\end{array} \]

Then we have that
\[
\tilde{\mathcal{F}} L^* [i(\tilde{K})(\Omega \circ \tilde{\mathcal{F}} L)] = (\tilde{j}_0 \circ \tilde{h} \circ \mathcal{F} L_0)^* [i(\Lambda^m T \tilde{j}_0 \circ \Lambda^m T \tilde{h} \circ X_{H_o} \circ \mathcal{F} L_0)(\Omega \circ \tilde{\mathcal{F}} L)]
\]
\[
= \mathcal{F} L_0^* (i(\tilde{h} \circ \tilde{h}^*) [i(\Lambda^m T \tilde{j}_0 \circ \tilde{h} \circ X_{H_o} \circ \mathcal{F} L_0)(\Omega \circ \tilde{\mathcal{F}} L)])
\]
\[
= \mathcal{F} L_0^* [i(\tilde{h} \circ X_{H_o} \circ \mathcal{F} L_0)(\Omega \circ \tilde{\mathcal{F}} L)]
\]

where all the equalities hold on \( S \). But, as \( \mathcal{F} L_0 \) is a submersion, we obtain that
\[
\tilde{\mathcal{F}} L^* [i(\tilde{K})(\Omega \circ \tilde{\mathcal{F}} L)] = 0 \iff \tilde{h} \circ X_{H_o} \circ \mathcal{F} L_0 = 0
\]
hence the field equation condition holds for \( \tilde{K} \) on \( S \) if, and only if, the Hamiltonian field equation holds for \( X_{H_o} \) on \( P \).

The proof of items 1 and 3 follow the same pattern as the proof of items 2 and 3 of Theorem 4.

For hyper-regular systems, the proof of these properties is the same, but taking into account that now \( \mathcal{P} = J^{1*} E \), \( \mathcal{F} L_0 = \mathcal{F} L \), and \( h = \tilde{j}_0 \circ \tilde{h} \). In addition, the classes of HDW \( m \)-vector fields, HDW jet fields, and HDW connections are defined everywhere in \( J^{1*} E \). Thus, the only addendum is to prove that, if \( X_\mathcal{H} \) is a Hamilton-De Donder-Weyl \( m \)-vector field for \( (J^{1*} E, \Omega_h) \), then its associated \( m \)-vector field along \( \mathcal{F} L \), \( \tilde{K} \), is semi-holonomic. As \( X_\mathcal{H} \in \mathcal{X}^M(J^{1*} E) \), by definition \( \sigma_{J^{1*} E} \circ X_\mathcal{H} = \text{Id}_{J^{1*} E} \), then, recalling the definition of the map \( \bar{\mathcal{L}}_E : \Lambda^m T M_\pi \to J^1 E \) (see (17)), and taking into account that \( \mathcal{F} L \) is a diffeomorphism, we have that
\[
\bar{\mathcal{L}}_E \circ \tilde{K} = \bar{\mathcal{L}}_E \circ \Lambda^m T \tilde{\mathcal{P}} \circ X_\mathcal{H} \circ \mathcal{F} L = \mathcal{F} L^{-1} \circ \sigma_{J^{1*} E} \circ X_\mathcal{H} \circ \mathcal{F} L = \text{Id}_{J^{1*} E}
\]
which is the condition for \( \tilde{K} \) (and hence for \( \bar{J}_K \) and \( \bar{\mathcal{V}}_K \)) to be semi-holonomic. That is, we have the following diagram

\[ \begin{array}{ccc}
\Lambda^m T J^{1*} E & \xrightarrow{\sigma_{J^{1*} E}} & X_\mathcal{H} \\
\Lambda^m T M_\pi & \xrightarrow{\sigma_{M_\pi}} & M_\pi \\
\end{array} \]

\[ \begin{array}{ccc}
J^{1*} E & \xrightarrow{\mathcal{F} L} & J^1 E \\
& \xrightarrow{h} & M_\pi \\
\end{array} \]

\[ \begin{array}{ccc}
\Lambda^m T J^{1*} E & \xrightarrow{\tilde{\mathcal{L}}_E} & \Lambda^m T M_\pi \\
\end{array} \]
Comment: Observe that relations (29), (30) and (31) arise because the commutativity of diagram (32) demands it.

As a straightforward consequence of this Theorem, and the Remark after Definition (7), we obtain:

Corollary 2 If the relations (29), (30) and (31) hold for the extended field operators, then the following ones hold for their associated restricted field operators:

\[
\begin{align*}
& j^1_{\mathcal{j}_0} \circ \Psi_{\mathcal{H}_o} \circ \mathcal{F}\mathcal{L}_0 = s \mathcal{Y}_\mathcal{K}, \\
& \Lambda^m T_{\mathcal{j}_0} \circ X_{\mathcal{H}_o} \circ \mathcal{F}\mathcal{L}_0 = s \mathcal{K} \quad \text{for every } \mathcal{K} \in \{\mathcal{K}\}, \text{ and for some } X_{\mathcal{H}_o} \in \{X_{\mathcal{H}_o}\} \\
& (\varepsilon^0_{T^* M} \otimes T \mathcal{F}\mathcal{L}_0) \circ \nabla_{\mathcal{H}_o} \circ \mathcal{F}\mathcal{L}_0 = s \nabla_\mathcal{K}, \quad \text{with } \varepsilon^0_{T^* M} : \mathcal{\bar{\tau}}_1 \oplus T^* M \rightarrow \mathcal{\bar{\tau}}_1 \oplus T^* M
\end{align*}
\]

(with the same restrictions in relation to the semi-holonomy condition).

Then assuming all these relations, we have:

Theorem 7 The class \(\{\mathcal{K}\}\), and its associated \(\mathcal{Y}_\mathcal{K}\) and \(\nabla_\mathcal{K}\) are integrable if, and only if, the class \(\{X_{\mathcal{H}_o}\}\), and its associated \(\mathcal{Y}_{\mathcal{H}_o}\) and \(\nabla_{\mathcal{H}_o}\) are integrable too. In particular:

1. Let \(\mathcal{F}\mathcal{L}_S : S \rightarrow P\) be the restriction of \(\mathcal{F}\mathcal{L}_0\) to \(S\) (that is, \(\mathcal{j}_0 \circ \mathcal{F}\mathcal{L}_S = \mathcal{F}\mathcal{L}_0 \circ \mathcal{j}_S\)). If \(\varphi : M \xrightarrow{\varphi_S} S \xrightarrow{\mathcal{j}_S} J^1 E\) is an integral section of \(\{\mathcal{K}\}\) on \(S\), then \(\psi_P : M \xrightarrow{\psi_P} P \xrightarrow{\mathcal{j}_P} P\) is an integral section of \(\{X_{\mathcal{H}_o}\}\) on \(P\), where \(\psi_P := \mathcal{F}\mathcal{L}_S \circ \varphi_S\).

2. Conversely, if \(\psi_P : M \xrightarrow{\psi_P} P \xrightarrow{\mathcal{j}_P} P\) is an integral section of \(\{X_{\mathcal{H}_o}\}\) on \(P\), then the section \(\varphi : M \xrightarrow{\varphi_S} S \xrightarrow{\mathcal{j}_S} J^1 E\), is an integral section of \(\{\mathcal{K}\}\) on \(S\), for every \(\varphi_S : M \rightarrow S \subseteq J^1 E\) such that \(\psi_P = \mathcal{F}\mathcal{L}_S \circ \varphi_S\).

The section \(\varphi_S\), and hence \(\varphi := \mathcal{j}_S \circ \varphi_S\), are holonomic if, and only if, the class \(\{\mathcal{K}\}\) is semi-holonomic (and hence is a class of field operators).

The same result holds for the extended field operators \(\{\tilde{\mathcal{K}}\}\), \(\tilde{\mathcal{Y}}_\mathcal{K}\) and \(\tilde{\nabla}_\mathcal{K}\).
(Proof) If the system is almost-regular, consider the diagram

\[
\begin{array}{ccccccc}
\Lambda^m T M & \xrightarrow{\Lambda^m T (F \mathcal{L} \circ \varphi)} & \Lambda^m T \mathcal{P} & \xrightarrow{\Lambda^m T j_0} & \Lambda^m T J^{1*} E \\
\sigma_M & \xrightarrow{\psi_0} & X_{H_o} & \xrightarrow{K} & \sigma_{J^{1*} E} \\
M & \xrightarrow{\varphi} & J^1 E & \xrightarrow{F \mathcal{L}_0} & P & \xrightarrow{j_0} & J^{1*} E \\
\varphi_S & \xrightarrow{j_S} & \mathcal{F} \mathcal{L}_S & \xrightarrow{j_P} & \mathcal{F} \mathcal{L} \\
S & \xrightarrow{\mathcal{F} \mathcal{L}_S} & P
\end{array}
\]

(where $X_{H_o}$ denotes any extension of the HDW $m$-vector field solution on $P$ to $\mathcal{P}$).

1. From (19), $\varphi_S: M \to J^1 E$ is an integral section of $K \in \{K\}$ on $S$ if, and only if,

\[
\Lambda^m T (j_0 \circ F \mathcal{L}_0 \circ \varphi) = \Lambda^m T (F \mathcal{L} \circ j_S \circ \varphi_S) = fK \circ j_S \circ \varphi_S \circ \sigma_M
\]  

Then on the one hand we have that

\[
\Lambda^m T (F \mathcal{L} \circ j_S \circ \varphi_S) = \Lambda^m T j_0 \circ \Lambda^m T (F \mathcal{L}_0 \circ j_S \circ \varphi_S) = \Lambda^m T j_0 \circ \Lambda^m T (j_P \circ F \mathcal{L}_S \circ \varphi_S)
\]

and on the other hand,

\[
fK \circ j_S \circ \varphi_S \circ \sigma_M = f\Lambda^m T j_0 \circ X_{H_o} \circ F \mathcal{L}_0 \circ j_S \circ \varphi_S \circ \sigma_M = f\Lambda^m T j_0 \circ X_{H_o} \circ j_P \circ F \mathcal{L}_S \circ \varphi_S \circ \sigma_M = f\Lambda^m T j_0 \circ X_{H_o} \circ j_P \circ \psi_P \circ \sigma_M
\]

where $f \in C^\infty (J^1 E)$ is a non-vanishing function. Hence, as $j_0$ is an imbedding, we obtain that (34) is equivalent to

\[
\Lambda^m T \psi_0 = \Lambda^m T (j_P \circ \psi_P) = fX_{H_o} \circ j_P \circ \psi_P \circ \sigma_M
\]  

which is the condition for $\psi_P$ to be an integral section of $X_{H_o}$ on $P$.

2. The converse is proved by reversing the above reasoning. In addition, the sections $\varphi_S$ and $\varphi := j_S \circ \varphi_S$ are holonomic if, and only if, they are integral sections of semi-holonomic $m$-vector fields along the Legendre map.

If the system is hyper-regular the proof is analogous, but taking $\mathcal{P} = J^{1*} E$ and $F \mathcal{L}_0 = F \mathcal{L}$.

Finally, the result for the extended field operators is a consequence of Proposition 1. □

And as an immediate corollary of this Theorem, we obtain the following characterization for the Hamiltonian sections:
Theorem 8 \( \psi_0 \) is a section solution of the Hamiltonian problem if, and only if, the following relation holds for \( \psi := j_0 \circ \psi_0 = j_0 \circ j_P \circ \psi_P \):

\[
\Lambda^m T \psi = f K \circ j^1 (\tau^1 \circ \psi) \circ \sigma_M
\]

or, what is equivalent, if \( K = \Lambda^m T \mu \circ \hat{K} \), for \( \tilde{\psi} := j_0 \circ \tilde{h} \circ \psi_0 \) we have

\[
\Lambda^m T \tilde{\psi} = f \hat{K} \circ j^1 (\tau^1 \circ \mu \circ \tilde{\psi}) \circ \sigma_M
\]

(Proof) Bearing in mind the commutativity of diagram (33), and taking into account that (35) is the n.s.c. for \( \psi_0 \) to be an integral section of \( X_{H_0} \), we have that

\[
\Lambda^m T \psi = \Lambda^m T (j_0 \circ \psi_0) = f \Lambda^m T j_0 X_{H_0} \circ \psi_0 \circ \sigma_M = f \Lambda^m T j_0 \circ X_{H_0} \circ FL_0 \circ \varphi \circ \sigma_M
\]

\[
= f K \circ \varphi \circ \sigma_M = f K \circ j^1 \phi \circ \sigma_M = f K \circ j^1 (\tau^1 \circ \psi) \circ \sigma_M
\]

since \( \varphi \) is holonomic and, by construction, \( \phi = \tau_0^1 \circ \psi_0 = \tau^1 \circ \psi \), as the following diagram shows

\[
\begin{array}{ccc}
J^1 E & \xrightarrow{FL_0} & P \\
\downarrow j^1 \phi & & \downarrow \tau^1 \\
E & \xrightarrow{\pi^1} & J^1* E
\end{array}
\]

\[
\begin{array}{ccc}
P & \xrightarrow{j_0} & J^1* E \\
\downarrow \varphi & & \downarrow \psi \\
M & \xrightarrow{\tau^1} & \psi_0
\end{array}
\]

The relation involving \( \hat{K} \) is immediate.

Remarks:

- In both the almost-regular and hyper-regular cases, the correspondence between HDW solutions of the Hamiltonian equations and the corresponding type of extended field operator is one-to-one.

- In addition, if the integrability condition holds only in a submanifold \( I \hookrightarrow S \), then Theorems 7 and 8 only holds on \( I \) and \( FL(I) \) (which is assumed to be a submanifold of \( P \)).

- Observe also that Theorem 7, together with Theorem 5, establishes the equivalence between the Lagrangian and Hamiltonian formalisms.

In its turn, Theorem 8 establishes the analogous property to (3) for the evolution operator \( K \) in mechanics.

In the light of these results, the existence of a multiplicity of extended field operators is hardly surprising, since in Lagrangian and Hamiltonian field theories, solutions of the field equations are not unique.
5 Conclusions and outlook

The generalization of the so-called evolution operator of the autonomous mechanical systems to field theories is achieved. Our geometric framework is the multisymplectic jet bundle description of these theories.

First, the geometric characteristics of multivector fields, jet fields and connection forms along maps are stated (in the jet bundle context which is of interest to us) as a generalization of the corresponding ones for multivector fields, jet fields and connection forms in jet bundles. In particular, the existence of one-to-one correspondences between (a) the sets of equivalence classes of non-vanishing, locally decomposable and transverse $m$-vector fields along the (extended) Legendre map, and (b) the sets of orientable jet fields and orientable connection forms along this map, is proved and used for establishing some geometrical characteristics of these objects.

In this way, the extended field operators are defined as sections along the extended Legendre map in three equivalent ways: as non-vanishing, locally-decomposable and transverse $m$-vector fields; as orientable jet fields; and as orientable connection forms along this map; all of them satisfying the conditions of being both semi-holonomic and a solution of a suitable field equation, which involves the multisymplectic canonical structure of the extended multimomentum bundle of the theory. The existence of these field operators is proved and, as a relevant difference to mechanics, we also see that they are not uniquely determined (and these results do not depend on the regularity of the system). This is not a surprising fact, since solutions of field equations are not unique, even for regular field theories.

Furthermore, the so-called restricted field operators are also defined starting from the extended ones. So they are non-vanishing, locally-decomposable and transverse $m$-vector fields, orientable jet fields, or orientable connection forms along the restricted Legendre map $\mathcal{F}L$.

As the first properties of the field operators, we show how solutions of the Euler-Lagrange and Hamiltonian field equations (jet fields, multivector fields and connections) can be generated from these field operators; and conversely, starting from these solutions the field operators can be recovered. In particular, we prove that the integral sections of the field operators are the section solutions of the Euler-Lagrange equations, whereas their images by the Legendre map are the integral sections of the Hamilton-De Donder-Weyl equations. Furthermore, it is showed that these integral sections of the Hamilton-De Donder-Weyl equations can be characterized using only the field operators. Of course, all these relations hold on the submanifolds where solutions of field equations exist. All these results establish the relationship between the solutions of field equations in the Lagrangian and Hamiltonian formalisms for (singular) field theories.

In this way, the field operators are very efficient tools to unify the Lagrangian and Hamiltonian formalisms.

It is interesting to point out that our field operators are covariant objects, and hence they are not “evolution operators” in any sense. In order to define these evolution operators a previous space-time decomposition must be carried out on the base manifold $M$ of the theory.

In further research works these field operators will be used for carrying out a deeper analysis of properties of these theories, in the same way as the evolution operator is used in mechanics. For instance, either to set the complete relation between the Lagrangian and Hamiltonian constraint algorithms arising for almost-regular field theories, or to study the existence and characterization of symmetries.
Acknowledgments

We acknowledge the financial support of the CICYT BFM2002-03493. We wish to thank Mr. Jeff Palmer for his assistance in preparing the English version of the manuscript. We are also very grateful to Prof. X. Gràcia for his enlightening comments and suggestions. Thanks to the referees for their constructive comments.

References

[1] C. Batlle, J. Gomis, X. Gràcia, J.M. Pons, “Noether’s theorem and gauge transformations: applications to the bosonic string and CP^n−1-model”, J. Math. Phys. 30(6) (1989) 1345-1350.
[2] C. Batlle, J. Gomis, J.M. Pons, “Hamiltonian and Lagrangian constraints of the bosonic string”, Phys. rev. D 34(8) (1986) 2430-2432.
[3] C. Batlle, J. Gomis, J.M. Pons, N. Román-Roy, “Equivalence between the Lagrangian and Hamiltonian formalism for constrained systems”, J. Math. Phys. 27 (1986) 2953-2962.
[4] E. Binz, J. Sniatycki, H. Fisher, The Geometry of Classical fields, North Holland, Amsterdam, 1988.
[5] D.E. Betounes, “Extension of the classical Cartan-form”, Phys. Rev. D (3) 29(4) (1984) 599-606.
[6] J.F. Cariñena, “Section along maps in geometry and physics”, Rend. Sem. Mat. Univ. Pol. Torino 54(3) (1996) 245-256.
[7] J.F. Cariñena, M. Crampin, L.A. Ibort, “On the multisymplectic formalism for first order field theories”, Diff. Geom. Appl. 1 (1991) 345-374.
[8] J.F. Cariñena, J. Fernández-Núñez, E. Martínez, “Time-dependent K-operator for singular Lagrangians”, (unpublished) (1995).
[9] J.F. Cariñena, C. López, “The time evolution operator for singular Lagrangians”, Lett. Math. Phys. 14 (1987) 203-210.
[10] J.F. Cariñena, C. López, “The time evolution operator for higher-order singular Lagrangians”, J. Mod. Phys. 7 (1992) 2447-2468.
[11] J.F. Cariñena, C. López, E. Martínez, “A new approach to the converse of Noether’s Theorem”, J. Phys. A: Math. Gen. 22 (1989) 4777-4787.
[12] J.F. Cariñena, C. López, E. Martínez, “Sections Along a Map Applied to Higher-order Lagrangian Mechanics. Noether’s Theorem”, Acta Appl. Mathematicae 29 (1991) 127-151.
[13] J.F. Cariñena, E. Martínez, W. Sarlet, “Derivations of differential forms along the tangent bundle projection”, Diff. Geom. and Appl. 2(1) (1992) 17-43.
[14] J.F. Cariñena, E. Martínez, W. Sarlet, “Derivations of differential forms along the tangent bundle projection II”, Diff. Geom. and Appl. 3(1) (1993) 1-29.
[15] P. Dedecker, On the generalization of symplectic geometry to multiple integrals in the calculus of variations, Lecture Notes in Mathematics 570, Springer, Berlin (1977).
[16] P. Dedecker, “Problèmes variationnels dégénérés”, C.R. Acad. Sci. Paris Sér. A-B 286(12) (1978) A547-A550.
A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, “Geometry of Lagrangian first-order classical field theories”. *Forts. Phys.* **44** (1996) 235-280.

A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, “Multivector Fields and Connections. Setting Lagrangian Equations in Field Theories”. *J. Math. Phys.* **39**(9) (1998) 4578-4603.

A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, “Multivector Field Formulation of Hamiltonian Field Theories: Equations and Symmetries”, *J. Phys. A: Math. Gen.* **32** (1999) 8461-8484.

A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, “Geometry of Multisymplectic Hamiltonian First-order Field Theories”, *J. Math. Phys.* **41**(11) (2000) 7402-7444.

C. Ferrario, A. Passerini, “Symmetries and constants of motion for constrained Lagrangian systems: a presymplectic version of the Noether theorem”, *J. Phys. A: Math. Gen.* **23** (1990) 5061-5081.

J.A. García, J.M. Pons, “Rigid and gauge Noether symmetries for constrained systems”, *Int. J. Modern Phys. A* **15**(29) (2000) 4681-4721.

G. Giachetta, L. Mangiarotti, G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory*, World Scientific Pub. Co., Singapore (1997).

X. Gràcia, J.M. Pons, “On an evolution operator connecting Lagrangian and Hamiltonian formalisms”, *Lett. Math. Phys.* **17** (1989) 175-180.

X. Gràcia, J.M. Pons, “A generalized geometric framework for constrained systems”, *Diff. Geom. Appl.* **2** (1992) 223-247.

X. Gràcia, J.M. Pons, “A Hamiltonian approach to Lagrangian Noether transformations”, *J. Phys. A: Math. Gen.* **25** (1992) 6357-6369.

X. Gràcia, J.M. Pons, “Gauge transformations for higher-order Lagrangians”, *J. Phys. A: Math. Gen.* **28** (1995) 7181-7196.

X. Gràcia, J.M. Pons, “Canonical Noether symmetries and commutativity properties for gauge systems”, *J. Math. Phys.* **41**(11) (2000) 7333-7351.

X. Gràcia, J.M. Pons, “Singular Lagrangians: some geometric structures along the Legendre map”, *J. Phys. A: Math. Gen.* **34** (2001) 3047-3070.

X. Gràcia, J.M. Pons, N. Román-Roy, “Higher order Lagrangian systems: geometric structures, dynamics and constraints”, *J. Math. Phys.* **32** (1991) 2744-2763.

X. Gràcia, J.M. Pons, N. Román-Roy, “Higher order conditions for singular Lagrangian dynamics”, *J. Phys. A: Math. Gen.* **25** (1992) 1989-2004.

X. Gràcia, J. Roca, “Covariant and non-covariant gauge transformations for the conformal particle”, *Mod. Phys. Lett. A* **8**(19) (1993) 1747-1761.

F. Hélein, J. Kouneiher, “Finite dimensional Hamiltonian formalism for gauge and quantum field theories”, *J. Math. Phys.* **43**(5) (2002) 2306-2347.

K. Kamimura, “Singular Lagrangians and constrained Hamiltonian systems, generalized canonical formalism”, *Nuovo Cim. B* **69** (1982) 33-54.

I.V. Kanatchikov, “Canonical structure of Classical Field Theory in the polymomentum phase space”, *Rep. Math. Phys.* **41**(1) (1998) 49-90.

Y. Kosmann-Schwarzbach, *Vector fields and Generalized Vector Fields on Fibered Manifolds*, Lecture Notes in Mathematics **792**, Springer, New York (1980) 307.
[37] D. Krupka, “Regular Lagrangians and Lepagean forms”, Proc. on Diff. Geom. Appls. (Brno 1986), Math Ap. (East European Ser. 27), Reidel, Dordrecht (1987) 111-148.

[38] O. Krupkova, D. Smetanova, “On regularization of variational problems in first-order field theory”. Proc. on 20th Winter School on Geom. and Phys., Rend. Circ. Mat. Palermo (2) Suppl. 66 (2001) 133-140.

[39] O. Krupkova, D. Smetanova, “Legendre transformation for regularizable Lagrangians in field theory”, Lett. Math. Phys. 58(3) (2002) 189-204.

[40] M. de León, J. Marín-Solano, J.C. Marrero, “A Geometrical approach to Classical Field Theories: A constraint algorithm for singular theories”, Proc. on New Developments in Differential Geometry, L. Tamassi-J. Szenthe eds., Kluwer Acad. Press, (1996) 291-312.

[41] G. Pidello, W.M. Tulczyjew, “Derivations of differential forms in jet bundles”, Math. Pura et Aplicata 147 (1987) 249-265.

[42] W.A. Poor, Differential Geometric Structures, McGraw-Hill, New York 1981.

[43] F. Pugliese, A.M. Vinogradov, “On the geometry of singular Lagrangians”, J. Geom. Phys. 35 (2000) 35-55,

[44] F. Pugliese, A.M. Vinogradov, “Discontinuous trajectories of Lagrangian systems with singular hypersurface”, J. Math. Phys. 42(1) (2001) 309-329.

[45] G. Sardanashvily, Generalized Hamiltonian Formalism for Field Theory. Constraint Systems, World Scientific, Singapore (1995).

[46] D.J. Saunders, The Geometry of Jet Bundles, London Math. Soc. Lect. Notes Ser. 142, Cambridge, Univ. Press, 1989.

[47] W.M. Tulczyjew, “Les sous-varietés lagrangiennes et la dinamique hamiltonienne”, C.R. Acad. Sc. Paris t 283A (1976) 15-18.