Betweenness centrality in corona product of Path and Star graphs

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Abstract. Betweenness centrality is a concept in graph theory which is currently widely used in network analysis. The network can be a computer network, a criminal network, social interaction network, etc. The concept of betweenness centrality studies the potential ability of a point to exert influence on a network. If there is a point in the shortest path between two points, then point has the potential to influence information passing through that path. This paper will investigate the concept of betweenness centrality in the corona operation between path and star graph. Corona operation is one of the operations on graph theory that works between two graphs. The corona operation will duplicate the second graph and connect each copy to the first graph.

1. Introduction
In a communication network, if there is a point in the communication path between point and , then has the potential to influence information between them. However, if does not fall in the communication channel, have no opportunity to influence the information. This case then led to the concept of betweenness centrality that introduced by Bavelas in 1948 [1]. This concept is defined mathematically by Freeman [2]. Freeman analogizes the Betweenness centrality of a vertex —we call ‘point’ as ‘vertex’ on the graph theory—as the probability of the existence of randomly in the shortest path between two other vertices. One of the betweenness centrality application is network analysis such as analyzing the street for store location [3] and analyzing eye-gaze data in autism spectrum disorder (ASD) [4].

In 2019, Sunnil and Kannan developed the concept of betweenness centrality in cartesian product graph [5]. Sunnil also investigated the properties of special graphs resulting from cartesian operations like hamming graph, hypercub graph, and product of cycle graph. In this paper, the development of the concept of betweenness centrality is combined with corona operations. This operation was introduced by Frucht [6]. One of the uses of corona surgery is Susilowati's research on the commutative characterization of corona product of graph with respect to metric dimension and local metric dimension [7,8], respectively. In 2017, Sharma introduced some of the structure and spectral properties of corona graphs [9]. The discussion in this paper just focused in the simple, finite and connected graph. Specifically, we will only discuss about betweenness centrality in corona product of path and star graph.
Let $G$ be a graph, the *order* of $G$ is the number of vertices of graph $G$, and the *size* of $G$ is the number of edges of graph $G$ [10]. The *distance* from a vertex $u$ to a vertex $v$ in $G$ denoted by $d_G(u, v)$ or simply $d(u, v)$ is the shortest path from $u$ to $v$, and of course $d(u, v) = d(v, u)$. The shortest path from $u$ to $v$ is called $u - v$ geodesic. The *degree* of $v \in G$ denoted by $\deg(v)$ is the number of vertices that adjacent to $v$ in $G$. A graph of order $n$ called as a Path Graph if it has $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and $E(P_n) = \{u_iu_{i+1} | i = 1, 2, \ldots, n - 1\}$. Let $G$ be a complete bipartite graph that can be partitioned into $C$ and $D$, if the cardinality of $C$ (or $D$) = 1, then $G$ is called *Star graph*. A vertex in $C$ is called the center of the star graph.

**Definition 1.1** ([5]) Let $G$ be a graph, for $x \in V(G)$, *betweenness centrality* of $x$, denoted by $B(x)$ is defined as:

$$B_G(x) = \sum_{\{u, v\} \subseteq V(G), \ u \neq v \neq x} \delta(u, v | x)$$

The pair-dependency $\delta(u, v | x)$ of the pair $\{u, v\}$ on $x$ is defined as $\delta(u, v | x) = \frac{\sigma(u, v | x)}{\sigma(u, v)}$ where $\sigma(u, v)$ be the number of $u - v$ geodesic and $\sigma(u, v | x)$ be the number of $u - v$ geodesic containing $x$. From there, $x$ is contained in the shortest $u - v$ path if only if $d(u, v) = d(u, x) + d(x, v)$. The number of $u - v$ geodesic that containing $x$ is $\sigma(u, v | x) = \sigma(u, x) \times \sigma(x, v)$.

**Lemma 1.2** ([11]) Let $P_n$ be a path graph of order $n$ with $V(P_n) = \{u_i | i = 1, 2, \ldots, n\}$

$$B_{P_n}(u_i) = (i - 1)(n - i)$$

**Lemma 1.3** ([11]) Let $K_{1,n}$ be a star graph of order $n + 1$, for any $v \in V(K_{1,n})$

$$B_{K_{1,n}}(v) = \begin{cases} \binom{n}{2} & v \text{ is central vertex} \\ 0 & \text{other vertices} \end{cases}$$

**Definition 1.4** ([6]) Let $G$ be a graph of order $n$, *corona product* of graphs $G$ and $H$ denoted by $G \odot H$ is a graph obtained from $G$ and $H$ by taking one copy of $G$ and $n$ copies of $H$ and joining $i$-th vertex of $G$ by an edge to each vertex of $i$-th copy of $H$.

**Figure 1.** Figure of $P_5 \odot K_{1,5}$.

**Figure 2.** Figure of $K_{1,5}$.

**Figure 3.** Figure of $P_5$.

**Figure 1** show the corona product of $P_5$ and $K_{1,5}$, **Figure 2** show a star graph of ordo 6, and **Figure 3** show a path graph of ordo 5.
2. Methods
The method used in the preparation of this paper explained as follows.

i. Studying about graph theory, betweenness centrality, and corona operation in some scientific literatures.

ii. Studying the properties of path graph and star graph.

iii. Trying to find the betweenness centrality value of the path graph and the star graph of small order.

iv. Analyzing the results in step (iii) to determine the patterns and knowing the supporting properties of betweenness centrality such as distance and geodesic.

v. Conducting corona operations on small path and star graph.

vi. Analyzing the characteristics of the corona product to determine whether there is a relationship between the characteristics of the builder graph and the newly formed graph.

vii. Determine the formulation of the value of betweenness centrality in the corona product of path graph of size $n$ and star graph of size $1,m$.

viii. Proofing the results in step (vii).

3. Results and discussion
The main study of this paper will be presented by giving an explanation of the properties of the shortest path in corona product of graph, and betweenness centrality in corona product of path and star graph.

3.1. Some properties of shortest path in corona product of graph
In this section we discuss the property of the shortest path, the distance, and the pair dependency of two different vertices in corona product $G \odot H$ of the graphs $G$ of order $m$ and $H$ of order $n$. Let $V(G) = \{g_1, g_2, \ldots, g_m\}$ and $V(H) = \{h_1, h_2, \ldots, h_n\}$. For $1 \leq i \leq m$, the $i$-th copy of $H$ denoted by $H^i$, joined to vertex $g_i$ and let $V(H^i) = \{h_{i1}, h_{i2}, \ldots, h_{in}\}$.

**Lemma 3.1.1** Let $G \odot H$ be a corona product of graphs $G$ of order $m$ and $H$ of order $n$. For any $h_{ip}, h_{iq} \in V(H^i)$ with $1 \leq i \leq m$ and $1 \leq p, q \leq n$, then $d(h_{ip}, h_{iq}) \leq 2$.

**Proof.** For $h_k, h_m \in H$ where $d(h_k, h_h) = 1$, or $d(h_k, h_i) = 2$ is clear because $h_{ip}, h_{iq}$ are the copy of $h_p$ and $h_q$. If $d(h_{ip}, h_{iq}) > 2$, for $h_{ip}, h_{iq} \in V(H^i)$, there is $u_i \in V(G)$ where $d(h_{ip}, u_i) = 1$ and $d(u_i, h_{iq}) = 1$ then $d(h_{ip}, h_{iq}) = 2$. So, for any $h_{ip}, h_{iq} \in V(H^i)$, $d(h_{ip}, h_{iq}) \leq 2$.

**Lemma 3.1.2** For any $h_{ip}, h_{iq}$ vertices in graph $H$ where $d(h_{ip}, h_{iq}) = 2$ and $\sigma(h_{ip}, h_{iq}) = t$, then for $h_{ip}, h_{iq} \in V(H^i)$ in $G \odot H$, $\sigma(h_{ip}, h_{iq}) = t + 1$.

**Proof.** For $h_{ip}, h_{iq} \in H^i$ there is $g_i \in V(G)$, where $d(h_{ip}, g_i) = 1$ and $d(g_i, h_{iq}) = 1$, then there is the shortest $h_{ip}, h_{iq}$ path containing $g_i$ with distance 2. So, there are $t + 1 h_{ip} - h_{iq}$ geodesic.

The consequence of Lemma 3.12 is $\delta(h_{ip}, h_{iq}|g_i) = \frac{1}{t+1}$, if $d(h_{ip}, h_{iq}) \geq 3$ then $\delta(h_{ip}, h_{iq}|g_i) = 1$ and $\delta(h_{ip}, h_{iq}|h_{ip}) = 0$ where $0 \leq r \leq m$ and $r \neq p, q$.

**Lemma 3.1.3** For any $g_i, g_j \in V(G)$, $h_{ip} \in V(H^i)$, and $h_{jq} \in V(H^j)$ in $G \odot H$, if $i \neq j$ then the $g_i - g_j$ geodesic was contained in the $h_{ip} - h_{jq}$ geodesic.

**Proof.** According to the Definition 1.4, if $i \neq j$ then the shortest path from $h_{ip} \in V(H^i)$ to $h_{jq} \in V(H^j)$ passing through $g_i$ and $g_j$, and its distance is $d(h_{ip}, h_{jq}) = d(h_{ip}, g_i) + d(g_i, g_j) + d(g_j, h_{jq}) \geq 3$. The consequence is $g_i - g_j$ geodesic was contained in $h_{ip} - h_{jq}$ geodesic.

**Corollary 3.1.4** For any $x \in V(G \odot H)$, $g_i, g_j \in V(G)$, $h_{ip} \in V(H^i)$, and $h_{jq} \in V(H^j)$, then $\delta(h_{ip}, h_{jq}|x) = \delta(h_{ip}, g_j|x) = \delta(g_i, h_{jq}|x) = \delta(g_i, g_j|x)$.
Proof. From Lemma 3.2.3 we know that \( h_{ip} - g_j \) geodesic and \( g_i, h_{jq} \) geodesic containing \( g_i - g_j \) geodesic, noted that \( h_{ip} \) and \( g_i \) (also \( h_{jq} \) and \( g_j \)) adjacent, then \( \sigma(h_{ip}, g_i) = \sigma(h_{jq}, g_j) = 1 \), so

\[
\sigma(h_{ip}, h_{jq}) = \sigma(h_{ip}, g_i) \times \sigma(g_i, g_j) \times \sigma(g_j, h_{jq}) = \sigma(g_i, g_j).
\]

\[
\sigma(h_{ip}, g_j) = \sigma(h_{ip}, g_i) \times \sigma(g_i, g_j) = \sigma(g_i, g_j).
\]

\[
\sigma(g_i, h_{jq}) = \sigma(g_i, g_j) \times \sigma(g_j, h_{jq}) = \sigma(g_i, g_j).
\]

The consequence is

\[
\delta(h_{ip}, h_{jq}|x) = \delta(h_{ip}, g_j|x) = \delta(g_i, h_{jq}|x) = \delta(g_i, g_j|x)
\]

3.2. Betweenness centrality in corona product of path and star graphs

According to the Definition 1.1, then for the corona product \( G \circ H \) of the graph \( G \) of order \( m \) and graph \( H \) of order \( n \), the betweenness centrality of \( x \in V(G \circ H) \) is:

\[
B_{G \circ H}(x) = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \delta(u_i, u_j|x) + \sum_{q=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \delta(u_i, v_{jq}|x)
+ \sum_{i=1}^{m} \sum_{j=1}^{m-1} \sum_{p=1}^{n} \delta(u_{ip}, v_{jq}|x)
+ \sum_{i=1}^{m} \sum_{j=1}^{m-1} \sum_{p=1}^{n} \sum_{q=1}^{n} \delta(v_{ip}, v_{jq}|x)
+ \sum_{i=1}^{m} \sum_{j=1}^{m-1} \sum_{p=1}^{n} \sum_{q=1}^{n} \delta(v_{ip}, v_{iq}|x)
\]

The corona product that discussed in this section are \( P_n \circ K_{1,m} \) and \( K_{1,m} \circ P_n \).

**Figure 4.** Figure of \( P_n \circ K_{1,m} \)  
**Figure 5.** Figure of \( K_{1,m} \circ P_n \)

**Figure 4** show the corona product of \( P_n \) and \( K_{1,m} \) or we denoted as \( P_n \circ K_{1,m} \) and **Figure 5** show the corona product of \( K_{1,m} \) and \( P_n \) or we denoted as \( K_{1,m} \circ P_n \).

**Theorem 3.2.1** Let \( P_n \circ K_{1,m} \) be a corona product of graphs \( P_n \) and \( K_{1,m} \) where \( V(P_n) = \{u_1, u_2, \ldots, u_n\} \) and \( V(K_{1,m}) = \{v_1, v_2, \ldots, v_m\} \).

(i) For any \( u_k \in V(P_n) \) where \( 1 \leq k \leq n \),

\[
B_{P_n \circ K_{1,m}}(u_k) = (m + 2)^2(k - 1)(n - k) + ((m + 1)^2 + (m + 1))(n - 1) + \frac{1}{2} \binom{m}{2}
\]

(ii) For any \( v_{kr} \in V(K_{1,m}) \) where \( 0 \leq r \leq m \),
Proof. Let $u_i, u_j \in V(P_n)$, $v_{ip} \in V(K_{1,m}^i)$, and $v_{jq} \in V(K_{1,m}^j)$, where $1 \leq i < j \leq n$, and $0 \leq p, q \leq m$.

(i) For $u_k \in V(P_n)$, $i, j \neq k$

From Corollary 3.1.4, $\delta(v_{ip}, v_{jq}|u_k) = \delta(u_i, v_{jq}|u_k) = \delta(v_{ip}, u_j|u_k) = \delta(u_i, u_j|u_k)$ and $\delta(v_{ip}, v_{kq}|u_k) = \delta(u_i, v_{kq}|u_k) = 1$ because $u_i - v_{kq}$ geodesic containing $u_k$.

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta(u_i, u_j|u_k) = B_{n,m}(u_k) = (k-1)(n-k)$$

$$\sum_{q=0}^{m} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \delta(u_i, v_{jq}|u_k) = (m+1)(n+1)(n-k)$$

$$\sum_{q=0}^{m} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \delta(v_{ip}, v_{jq}|u_k) = (m+1)(n-k)$$

$$\sum_{q=0}^{m} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \delta(v_{ip}, v_{kq}|u_k) = (m+1)^2(n-k)$$

The $v_{ip} - v_{iq}$ geodesic is not containing $u_k$ then

$$\sum_{q=p+1}^{m} \delta(v_{ip}, v_{iq}|u_k) = 0$$

For $v_{kp}, v_{kq} \in V(K_{1,m}^k)$ if $v_{kp}$ is the vertex center of $K_{1,m}^k$ then $\delta(v_{kp}, v_{kq}|u_k) = 0$, if both $v_{kp}$ and $v_{kq}$ is not the vertex center of $K_{1,m}^k$ then the number of $v_{kp} - v_{kq}$ geodesic is 2 where $d(v_{kp}, v_{kq}) = 2$, the first is containing the vertex center and the second is containing $u_k$.

Hence

$$\sum_{p=0}^{m-1} \sum_{q=p+1}^{m} \delta(v_{kp}, v_{kq}|u_k) = 0 + \frac{1}{2} \binom{m}{2}$$

Then, the betweenness centrality of $u_k$ is

$$B_{n\circ K_{1,m}}(u_k) = (m+2)^2(k-1)(n-k) + ((m+1)^2 + (m+1))(n-k) + \frac{1}{2} \binom{m}{2}$$

(ii) For $v_{kr} \in V(K_{1,m}^k)$, $i, j \neq k$, $0 \leq r \leq m$

According to Corollary 3.1.4 and because of the $u_i - u_j$ geodesic is not containing $v_{kr}$ then $\delta(v_{ip}, v_{jq}|v_{kr}) = \delta(u_i, v_{jq}|v_{kr}) = \delta(v_{ip}, u_j|v_{kr}) = \delta(u_i, u_j|v_{kr}) = 0$. The $u_i - v_{kq}$ geodesic and $v_{ip} - v_{iq}$ geodesic is also not containing $v_{kr}$, hence $\delta(v_{ip}, v_{kq}|v_{kr}) = \delta(u_i, v_{kq}|v_{kr}) = 0$ and $\delta(v_{ip}, v_{iq}|v_{kr}) = 0$. If $v_{kr}$ is the vertex center of $K_{1,m}^k$ then $\delta(v_{kp}, v_{kq}|v_{kr}) = \frac{1}{2}$ because the $v_{kp} - v_{kq}$ geodesic is 2 the first is containing the vertex center and the second is containing $u_k$. If $v_{kr}$ is not the vertex center of $K_{1,m}^k$ then $\delta(v_{kp}, v_{kq}|v_{kr}) = 0$. Then, the betweenness centrality of $v_{kr}$ is

$$B_{n\circ K_{1,m}}(v_{kr}) = \frac{1}{2} \binom{m}{2}, \quad v_{kr} \text{ central vertex of } K_{1,m}^k$$

$$0, \quad \text{otherwise}$$
Theorem 3.2.2 Let \( K_{l,m} \odot P_n \) be a corona product of both graphs \( K_{l,m} \) and \( P_n \) where \( V(K_{l,m}) = \{v_1, v_2, \ldots, v_m\} \) and \( V(P_n) = \{u_1, u_2, \ldots, u_n\} \).

(i) For any \( v_k \in V(K_{l,m}) \) where \( 0 \leq k \leq m, \)
\[
B_{K_{l,m} \odot P_n}(v_k) = \begin{cases} 
\left( \frac{n}{2} \right)^2, & v_k \text{ is central vertex of } K_{l,m} \\
0, & \text{others}
\end{cases}
\]

(ii) For any \( u_{kr} \in \{P_n^k\} \) where \( 1 \leq r \leq n, \)
\[
B_{K_{l,m} \odot P_n}(u_{kr}) = \begin{cases} 
0, & r = 1 \vee r = n \\
\frac{1}{2}, & \text{others}
\end{cases}
\]

Proof. Let \( v_i, v_j \in V(K_{l,m}), u_{ip} \in V(P_n^i), \) and \( u_{jq} \in V(P_n^j), \) where \( 0 \leq i < j \leq m, \) and \( 1 \leq p, q \leq n. \)

(i) For \( v_k \in V(P_n^k), i, j \neq k \)

From Corollary 3.1.4, \( \delta(u_{kp}, u_{kq}|v_k) = \delta(v_i, u_{jq}|v_k) = \delta(u_{ip}, v_j|v_k) = \delta(u_i, u_j|v_k) \)
and \( \delta(u_{ip}, u_{kq}|u_k) = \delta(v_i, u_{kq}|u_k) = 1 \) because \( v_i - u_{kq} \) geodesic containing \( v_k. \)

\[
\sum_{l=0}^{m-1} \sum_{j=0}^{m-1} \delta(v_i, v_j|v_k) = B_{K_{l,m}}(v_k) = \begin{cases} 
\left( \frac{n}{2} \right)^2, & v_k \text{ is central vertex} \\
0, & \text{others}
\end{cases}
\]

(ii) For \( u_{kr} \in \{P_n^k\} \) where \( 1 \leq r \leq n, \)

\[
\sum_{l=0}^{m-1} \sum_{j=0}^{m-1} \delta(u_{ip}, u_{jq}|v_k) = \begin{cases} 
\left( \frac{n}{2} \right)^2, & v_k \text{ is central vertex} \\
0, & \text{others}
\end{cases}
\]

The \( u_{ip} - u_{iq} \) geodesic is not containing \( v_k \) then
\[
\sum_{l=0}^{m-1} \sum_{j=0}^{m-1} \delta(u_{ip}, u_{jq}|v_k) = 0
\]

There is \( \left( \frac{n}{2} \right) \) pairs of vertices in \( P_n^k \) where \( (n - 1) \) pairs of distance 1, \( (n - 2) \) pairs of distance 2, and \( \left( \frac{n}{2} \right) - (2n + 3) \) pairs of distance more than 3. Based on the consequence of Lemma 3.1.2,
\[
\sum_{p=1}^{n-1} \sum_{q=p+1}^{n} \delta(u_{kp}, u_{kq}|v_k) = \frac{n - 2}{2} + \left( \frac{n}{2} \right) - (2n + 3) = \frac{(n - 2)^2}{2}
\]
Then, the betweenness centrality of $v_k$ is

$$B_{K_{1,m} \circ P_n}(v_k) = \begin{cases} (n+1)^2 \left( \frac{m}{2} \right) + (n^2+n)m + \frac{(n-2)^2}{2}, & v_k \text{ is central vertex of } K_{1,m} \\ (n^2+n)m + \frac{(n-2)^2}{2}, & \text{others} \end{cases}$$

(ii) For $u_{kr} \in V(P^k_n)$, $i, j \neq k, 1 \leq r \leq n$

According to Corollary 3.1.4 and because of the $v_i - v_j$ geodesic is not containing $u_{kr}$ then $\delta(u_{ip}, u_{jq} | u_{kr}) = \delta(v_i, u_{jq} | u_{kr}) = \delta(u_{ip}, v_j | u_{kr}) = \delta(v_i, v_j | u_{kr}) = 0$. The $u_i - v_{kq}$ geodesic and The $v_{ip} - v_{iq}$ geodesic is also not containing $u_{kr}$, hence $\delta(u_{ip}, u_{kq} | u_{kr}) = \delta(v_i, u_{kq} | u_{kr}) = 0$ and $\delta(u_{ip}, u_{iq} | u_{kr}) = 0$. For $u_{kp}, u_{kq} \in V(P^k_n)$, if $r < p, q$ or $r > p, q$ then $\delta(u_{kp}, u_{kq} | u_{kr}) = 0$. Hence, if $r = 1$ or $r = n$ then $\delta(u_{kp}, u_{kq} | u_{kr}) = 0$. If $q = p + 1$ then $\delta(u_{kp}, u_{kq} | u_{kr}) = 0$. If $q = p + 2$ and $p < r < q$ then $\delta(u_{kp}, u_{kq} | u_{kr}) = 1/2$. If $q \geq p + 3$, according to the consequence of Lemma 3.1.2, then $\delta(u_{kp}, u_{kq} | u_{kr}) = 0$.

Then, the betweenness centrality of $v_{kr}$ is

$$B_{K_{1,m} \circ P_n}(u_{kr}) = \begin{cases} 0, & r = 1 \lor r = n \\ 1, & \text{others} \end{cases}$$

4. Conclusion

The betweenness centrality of a vertex in corona product $P_n \circ K_{1,m}$ has a relationship to the betweenness centrality of it in $P_n$. The betweenness centrality of a vertex in corona product $K_{1,m} \circ P_n$ is also has a relationship with in $K_{1,m}$. The properties of the pair dependency in Corollary 3.1.4 show the cause of this relationship. For general corona product $G \circ H$, the pair dependency of the vertex in one copy of $H$ and a vertex in other copies has a value as the pair dependency of two vertices in $G$ that joined to the both copies of $H$.

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