Twisted Affine Integrable Hierarchies and Soliton Solutions

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Abstract
A systematic construction of a class of integrable hierarchy is discussed in terms of the twisted affine \( A_2^{(2)} \) Lie algebra. The zero curvature representation of the time evolution equations is shown to be classified according to its algebraic structure and according to its vacuum solutions. It is shown that a class of models admit both zero and constant (non-zero) vacuum solutions. Another consists essentially of integral non-local equations and can be classified into two sub-classes, one admitting only zero vacuum and another of constant strictly non-zero vacuum solutions. The two-dimensional gauge potentials in the vacuum play a crucial ingredient and are shown to be expanded in powers of the vacuum parameter \( v_0 \). Soliton solutions are constructed from vertex operators, which for the non-zero vacuum solutions correspond to deformations characterized by \( v_0 \).

Keywords Integrable hierarchies · Soliton solutions · Vertex operators · Kac-Moody algebras

1 Introduction
The connection between integrable hierarchies and the structure of affine Lie algebras has provided a series of important achievements such as the systematic construction and classification of time evolution of nonlinear equations, construction of soliton (multi) solutions, and conservation laws (see for instance [1–3]). More recently, the theory of affine Lie algebras has been systematically employed to derive and classify Bäcklund transformations (BT) [4–6]. These BT, in turn, were shown to naturally appear in describing integrable defects, in the sense that they connect two field configurations of the same equation of motion without breaking the integrability. Examples in connection to sine-Gordon, Tzitzeica-Bollough-Dood, and non relativistic models as well were considered, e.g., [7–9].

In particular, the Tzitzeica-Bollough-Dood (Tz-B-D) model corresponds to the relativistic equation based upon the twisted \( A_2^{(2)} \) affine algebra. The Tz-B-D for field \( \phi \) can be obtained from the untwisted \( A_2^{(1)} \) Toda model with fields \( \phi_1, \phi_2 \) identified together, i.e., \( \phi \equiv \phi_1 = \phi_2 \). A sequence of nonlinear time evolution equations can then be derived from the very same algebraic structure of the Tz-B-D model and zero curvature representation to constitute the \( A_2^{(2)} \) twisted affine hierarchy. In fact, such argument can be generalized mathematically to the affine \( A_2^{(2)} \) case by identifying symmetries of Dynkin diagram of \( A_2 \) under an automorphism [10].

It was shown in [8] and later in [5] that the Bäcklund transformation for the affine \( A_2^{(2)} \) hierarchy has a structure much more elaborated than its untwisted \( A_2^{(1)} \) counterpart involving an auxiliary external field. An elaborated study of Bäcklund transformation approach to integrable defects for affine twisted algebras was proposed in [11]. Such transformation was dubbed type II Bäcklund transformation [8] and is a general feature of twisted affine algebras that underlines an entire class of nonlinear equations.

In this paper, we extend the results of ref. [12] to the case of affine twisted algebras. We first consider the \( A_2^{(2)} \) affine algebra and construct two sub-hierarchies associated to some positive and negative grade, constant elements explained in section 3. The former is shown to admit both zero or constant (non-vanishing) vacuum soliton solutions. The negative sub-hierarchy, in turn, consists essentially of integral non-local
equations and can be classified into two sub-classes, one admitting zero vacuum and another of constant, non-zero vacuum solutions.

An interesting feature that naturally arises for non-zero vacuum solutions is that the vacuum structure for the two-dimensional gauge potentials involves both the grading of the algebraic structure of the affine algebra and powers of a parameter that characterizes the vacuum. The two concepts put together define a larger structure already encountered within the two loop Kac-Moody algebras [13, 14].

This paper is organized as follows. In section 2, we review the construction of integrable hierarchies in terms of decomposition of affine algebras and zero curvature representation. Next, in section 3, we discuss the construction of twisted $A_2^{(2)}$ affine algebra and classify the possible sub-hierarchies in terms of deformed vertex operators. Finally in section 5 we discuss the general construction for $A_2^{(2)}$ twisted affine algebra and derive a few simple examples. In section 6 we conclude and discuss further developments.

## 2 Construction of Integrable Hierarchies

Here, we review the construction of integrable hierarchies in terms of a graded affine Lie algebra (see for instance [1]). Consider an affine Lie algebra $G$ which can decompose according to a grading operator $Q$ as

$$G = \sum \delta \in \delta_a, [Q, \delta_a] \subset \delta_a, [\delta_a, \delta_b] \subset \delta_{a+b}, a, b \in Z.$$

(1)

Let $E \equiv E^{(1)} \in G$ be a semisimple grade 1 element, which decomposes the $G = \mathcal{K} \oplus \mathcal{M}$, where the kernel, $\mathcal{K}$ is defined to be

$$\mathcal{K} = \{ x \in \mathcal{K}, [x, E] = 0 \}$$

(2)

and $\mathcal{M}$ is its complement. The Lax operator is defined as

$$L = E + A_0$$

(3)

where $A_0 \in \mathcal{M} \cap \delta_0$.

The integrable hierarchy is constructed from the zero curvature representation

$$[\delta + E + A_0, \partial_{t^{(i)}} + D^{(N)} + \cdots + D^{(0)} + \cdots + D^{(−M)}] = 0$$

(4)

where $D^{(i)} \in \delta_0$, and due to the graded structure (1) the zero curvature Eq. (4) decomposes into

$$[E, D^{(N)}] = 0$$

(5)

[eq:6]

$$[E, D^{(N−1)}] + [A_0, D^{(N)}] + \partial_{t^{(i)}} D^{(N)} = 0$$

(6)

$$[E, D^{(−1)}] + [A_0, D^{(0)}] + \partial_{t^{(i)}} D^{(0)} = 0$$

(7)

$$[A_0, D^{(−M)}] + \partial_{t^{(i)}} D^{(−M)} = 0$$

(8)

Notice that the $D^{(i)}$, $i > 0$ can be solved recursively starting from (5) downwards. On the other hand, the $D^{(−1)}$, $j > 0$ are in turn solved starting from (8) upwards until we reach the zero grade component (7). This last Eq. (7) corresponds to the time evolution equations according to time $t_{NM}$ for fields parameterizing $A_0 \in \mathcal{M} \cap G_0$. Two sub-hierarchies are of particular interest, namely,

- The **positive sub-hierarchy** is obtained by setting $D^{−i} = 0, i > 0$. Equation (5) establishes the possible values for $N$ since it imposes the highest grade element $D^{(N)}$ to belong to the kernel of $E, D^{(N)} \in \mathcal{K}_E$.

- The **negative sub-hierarchy** obtained by taking $D^{(i)} = 0, i > 0$. An interesting particular example is the case where $N = 0, M = 1, i.e., t = t_{0,1}$ leading to

$$[A_0, D^{(−1)}] + \partial_{t^{(i)}} D^{(−1)} = 0$$

(9)

$$−\partial_{t^{(i)}} A_0 = 0$$

(10)

which can be solved in closed form for general Lie algebra by a change of variables $A_0 = B^{−1} \partial_{t^{(i)}} B$ and $D^{−1} = B^{−1} E^{−1} B$, $E^{−1} = E^t$ for some group element $B = e^{\delta_A}$. The Eq. (9) is automatically satisfied while (10) leads to the Leznov-Saveliev equation.

$$\partial_{t^{(i)}} (B^{−1} \partial_{t^{(i)}} B) − [E, B^{−1} E^{−1} B] = 0.$$

(11)

The above equation correspond to the relativistic Toda equations when the $(x, t_{0,1})$ variables are identified with the light cone coordinates, $(z, \bar{z})$. Explicit examples were constructed for $A_2^{(1)}$ and its Bäcklund transformations were discussed in [4, 16]. For $A_3^{(1)}$ and its generalization to $A_n^{(1)}$ we refer to [5, 6].

## 3 The Twisted $A_2^{(2)}$ Hierarchy

In this paper we shall discuss the construction of a class of integrable hierarchies connected to twisted affine algebras and discuss its classification in terms of their possible vacuum solutions. Let us start with the simplest case of the $A_2$ Lie algebra with positive roots
\[ a_1, a_2 \quad \text{and} \quad a_1 + a_2 \]  
\[ \sigma, \quad \text{an automorphism of order 2 such that} \quad \sigma(a_1) = a_2, \quad \sigma^2 = 1. \] 
Extending it to the Lie algebra,
\[ \sigma(a_1 \cdot H) = a_2 \cdot H, \quad \sigma(E_{a_1}) = E_{a_2}, \quad \sigma(E_{a_1+a_2}) = -E_{a_1+a_2} \] 
shows that it is consistent with \( E_{a_1+a_2} = [E_{a_1}, E_{a_2}]. \) The twisted affine \( A_2^{(2)} \) algebra is constructed by assigning integer affine indices to the even subalgebra under \( \sigma, \) i.e., \( T_a^{(m)} \) and \( T_a \) to grades, either \( E \) or \( H \) respectively. 
\[ \text{The structure of the kernel is such that the possible time evolution equations are derived from elements associated to grades, either} \quad N = 6n - 1 \quad \text{or} \quad N = 6n + 1. \] 
The simplest model is constructed by setting \( M = 0, N = 5, \) i.e., \( t = t_0 = 1 \) in the language of (4) (or \( N = 6n + 1 \text{ in (16)}. \)) Solving Eqs. (5)–(8) we find for the evolution Eq. (7)
\[ A_1 = E + \sqrt{h_1^0} + h_2^0, \]  
\[ A_5 = D^{(5)} + D^{(4)} + D^{(3)} + D^{(1)} + D^{(0)} \] 
\[ D^{(5)} = E^{(5)} = E_{-a_1} + E_{-a_2} + E_{(a_1+a_2)} \]  
\[ D^{(4)} = v(E_{a_1} - E_{a_2}) \]  
\[ D^{(3)} = \frac{1}{3}(v^2 + v_x)(h_{1}^0 - h_{2}^0) \]  
\[ D^{(2)} = -\frac{1}{3} \partial_x(v^2 + v)(E_{-a_1} - E_{-a_2}) \]  
\[ D^{(1)} = -\frac{1}{9} [(v^2 + v_x)^2 - 2\partial_x(v^2 + v_x)]E^{(1)} \]  
\[ D^{(0)} = -\frac{1}{9} [(v^5 - 5v^2v_x^2 - 5v^3v_x - 5v^2 - 20v^4v_x + 5v^5 + v^4)](h_{1}^0 + h_{2}^0) \] 
yielding the following equation of motion,
\[ 9v_x = -5v - 5v^3v_x + 5v^2v_x - 5v^2 - 20v^4v_x + 5v^5 + v^4. \] 

The second simplest model is constructed by setting \( M = 0, N = 7, \) i.e., \( t = t_{0.7} = 7 \) in the language of (4) (or \( N = 6n + 1 = 7 \text{ in (16)}. \)) Solving Eqs. (5)–(7) we find solution for \( D^{(0)}, i = 1, \ldots, 7 \) given in the appendix and find for the evolution Eq. (7),
\[ 81v_x = 28v^6 - 42v^3v^4 - 336v^5v^2v^3 + 28v^4 + 168v^2v^3 \] 
\[ + 21(2v^5v_x - 12v^3v_x + 2v^2v_x^2 + v_x^3)v^2 + 42v(4v_xv^2) \] 
\[ + 3v_x^3v_x^3 + 5v_xv_x^3 + 21v_x(11v_x^2v_x - v_x^3) \] 
\[ - 3(14v_x^2 + 21v_xv_x^4 + v_x^7) \] 
It is interesting to notice that the two equations (25) and (26) admit both, (i) zero vacuum, \( v = 0 \) or (ii) constant vacuum, \( v = v_0 \neq 0 \) soliton solutions. This is a general fact that can be extended to all models within the positive sub-hierarchy. 
We assume that for either zero or constant vacuum configurations, the two-dimensional gauge potentials \( A_x^{\text{vac}} \) and \( A_{v_x}^{\text{vac}} \) are constant affine Lie algebra elements. In general, the zero curvature representation for the vacuum configuration,
\[ [\partial_x + A_x^{\text{vac}}, \partial_x + A_x^{\text{vac}}] = [E + v_0(h_{1}^0 + h_{2}^0), D_{\text{vac}}^{(N)}] \] 
\[ + D_{\text{vac}}^{(N-1)} + \cdots + D_{\text{vac}}^{(0)} = 0 \] 
leads to the following explicit dependence of \( A_{i_x}^{\text{vac}} \) in terms of \( v_0 \)
\[ A_{0} = E(0+1) + v_0 E(0+1-1) + v_0^2 E(0+1-2) + \ldots + v_0^{6N+3} E(0), \]

\[ D_{0} = v_0^{N} E(0) \in G_k, \quad N = 6n + 1 \text{ and } D_{0}^{N} \text{ does not depend upon } v_0 \text{ Notice that (28) does not contain terms of single gradation, but several of different gradations.} \]

We now discuss in general terms the possible time evolutions for negative sub-hierarchies and its corresponding vacuum solutions. Two cases are to be considered:

- **Negative sub-hierarchy with zero vacuum solution,** \( v_0 = 0 \). The zero curvature representation (8) for \( v = 0 \) implies that \( E, D_{0}^{(-M)} = 0 \) since \( D_{0}^{(-M)} = 0 \) and hence \( D_{0}^{(-M)} \in \mathcal{K} \). It then follows from (16) and (17) that \( N = 0 \) and either \( M = 6n - 1 \) or \( M = 6n + 1 \).

Following the general construction for the negative sub-hierarchy for \( N = 0, M = 1, \quad (t = t_{-1}) \) we find from (11) the Tzitzeica (Bullough-Dodd) model,

\[ \partial_t \partial_x \phi = e^{2\phi} - e^{-\phi} \tag{29} \]

where \( B = e^{\phi(a_y + a_z)} \), \( \phi = \int f_y v(y) dy \equiv d_{-1} \phi \). The Tzitzeica model (29) corresponds to the simplest model \( (N = 0, M = 1) \) within the negative sub-hierarchy and it is clear that it admits only zero vacuum solution, i.e., \( \phi_{vac} = 0 \). Other models within this subclass are associated to elements of the Kernel with either \( M = 6n - 1 \) or \( M = 6n + 1 \).

- **Negative sub-hierarchy with constant vacuum solution,** \( v_0 \neq 0 \). The zero curvature for the vacuum configuration in this case reads,

\[ [E + v_0 (h_{1}^{0} + h_{2}^{0}), D_{0}^{(-M)} + D_{0}^{(-M+1)} + \ldots + D_{0}^{(-1)}] = 0 \tag{30} \]

and implies that \( A_{0}^{vac} \) has the following \( v_0 \neq 0 \) dependence,

\[ A_{0}^{vac} = a_{0} h_{1}^{0} + a_{0} h_{2}^{0} D_{0}^{(-M+1)} + \ldots + a_{0}^{M+2} D_{0}^{(-2)} + \ldots + a_{0}^{M+1} D_{0}^{(-1)} \subseteq \mathcal{K}. \tag{31} \]

Starting from the lowest projection of (30), namely,

\[ [v_0 (h_{1}^{0} + h_{2}^{0}), D_{0}^{(-M)}] = 0 \tag{32} \]

implies that for \( v_0 \neq 0 \), Eq. (32) allows two possibilities, either

\[ D_{0}^{(-M)} = a_{0} (h_{1}^{0} + h_{2}^{0}) \quad \text{or} \quad D_{0}^{(-M)} = b (h_{1}^{0} - h_{2}^{0} + h_{1}^{0} - h_{2}^{0} + h_{1}^{0} - h_{2}^{0} + h_{1}^{0} - h_{2}^{0}), \]

and hence \( M = 6m \) or \( M = 6m - 3 \) respectively. The simplest model within the negative sub-hierarchy sector is obtained from (33) for \( M = 3 \), i.e., for \( t = t_{-3} \). Solving (8)–(7), we find

\[ D_{0}^{(-1)} = - 3 e^{-d^{(-1)} y} - 3 e^{-d^{(-1)} y} (E_{0}^{(0)} + E_{0}^{(0)}) + 6 e^{-2d^{(-1)} y} d_{-1}^{(-1)} (e^{-d^{(-1)} y} E_{0}^{(0)} + E_{0}^{(0)}) \]

\[ D_{0}^{(-2)} = 3 e^{-d^{(-2)} y} d_{-1}^{(-1)} (E_{0}^{(0)} - E_{0}^{(0)}) \]

\[ D_{0}^{(-3)} = h_{1}^{(-2)} - h_{2}^{(-2)}, \]

where \( d_{-1} f = \int f(y) dy \). It leads to the following non-local equation of motion,

\[ - \frac{1}{3} v_{t_{-3}} = e^{d_{-1} y} d_{-1} (e^{d_{-1} y} E_{0}^{(0)} + E_{0}^{(0)}) + 2 e^{d_{-1} y} d_{-1} (e^{d_{-1} y} E_{0}^{(0)} + E_{0}^{(0)}). \tag{34} \]

It is clear that Eq. (34) only admits non-zero vacuum solution \( v = v_0 \neq 0 \) as can easily be checked if we denote \( d_{-1} v = v_0 x \),

\[ 0 = e^{y_{0} x} d_{-1} (e^{-2y_{0} x} d_{-1} (e^{y_{0} x}) + 2 e^{-2y_{0} x} d_{-1} (e^{y_{0} x}) \]

\[ = e^{y_{0} x} d_{-1} (\frac{1}{v_{0} y} e^{y_{0} x} + 2 e^{-2y_{0} x} d_{-1} (\frac{1}{v_{0} y} e^{y_{0} x}) \]

\[ = e^{y_{0} x} (\frac{1}{v_{0} y} e^{y_{0} x} + 2 e^{-2y_{0} x} (\frac{1}{2v_{0} y} e^{y_{0} x}) = 0 \tag{35} \]

It can be also checked that \( v_0 = 0 \) does not satisfy (34) since \( d_{-1} v_{0} = 0 \) and henceforth is solution of (36). Also, for \( v_0 = 0 \) it is clear that \( W \neq 0 \) and it is not solution for (36).

### 4 Dressing and the Construction of Soliton Solutions

In this section we discuss the systematic construction of soliton solutions from the dressing formalism. The zero curvature representation implies the two-dimensional gauge potentials written in a pure gauge form, in particular for the vacuum configuration with \( v_0 \neq 0 \),

\[ \text{Springer} \]
\[ A_{vac}(v_0) = T_0^{-1} \partial T_0, \quad A_{vac}(v_0) = T_0^{-1} \partial T_0 \]  
\[ \text{(38)} \]

or \( T_0 = e^{(A_{vac} v_0)} \). Here \( T_0 \) denotes a key group element describing the different vacuum possibilities. Once the vacuum configuration is known, a nontrivial configuration, \( T = T_0 \Theta \) can be obtained by gauge transformation,

\[ A_\mu = \Theta^{-1} A_{vac} \Theta + \Theta^{-1} \partial \mu \Theta \]  
\[ \text{(39)} \]

The dressing method connects the vacuum to a nontrivial configuration by gauge transformation (39) [3]. In fact, there are two solutions for \( \Theta \),

\[ \Theta_+ = e^{(p(0))} e^{(1)} e^{(2)} \ldots \quad \text{and} \quad \Theta_- = e^{(p(-1))} e^{(-2)} \ldots \]  
\[ \text{(40)} \]

\( q(i) \in G_{\ldots}, \quad p(i) \in G, \) which can be determined by substituting (40) into (39). In particular \( e^{(p(0))} = B^{-1} e^{-x B} \). As a direct consequence \( \Theta_+\Theta_-^{-1} = T_0^{-1} g T_0 \) where \( g \) is a constant group element. It induces the general formula,

\[ < i | B e^{x} | j > = < i | T_0^{-1} g T_0 | j >, \quad i, j = 0, \ldots, \text{rank } G \]  
\[ \text{(41)} \]

where \( | i > \) denotes the highest weight state in the sense that \( \rho(i) | i > = 0 \), \( i = 1, 2, \ldots \) and \( \kappa \) is the central term of the affine Kac-Moody Lie algebra \( \hat{G} \) and \( g \) is a constant group element.\(^2\) Equation (41) relates physical fields parameterizing \( B \), obtained by integrating \( A_0 = B^{-1} \partial B \), explicitly in terms of space and time coordinates \((x,t)\) from \( T_0 \).

In order to obtain explicit soliton solutions, we choose

\[ g = e^{F(y)}, \quad [A_{mu}, F(y)] = \lambda_\mu(y) F(y). \]  
\[ \text{(42)} \]

\( F(y) \) is called vertex operator which, in general, has the property of being nilpotent, i.e., \( F(y)^{k+1} = 0 \), for some \( k \in Z \). It therefore follows from (42) that

\[ T_0^{-1} g T_0 = e^{(p(x,t,y) F(y))} = 1 + p(x,t,y) F(y) + \ldots + \frac{1}{k!} \rho(x,t,y) F(y)^k, \quad p(x,t,y) = e^{-\lambda_\mu(y) - \lambda_\mu(y) y_\mu}. \]  
\[ \text{(43)} \]

Soliton solutions are directly related to a particular choice of \( g \). In particular, for \( l \)-soliton solution, we consider

\[ g = e^{F(t_1) e^{F(t_2)}} e^{F(t_3)}, \quad T_0^{-1} g T_0 = e^{p_1 F(t_1) e^{p_2 F(t_2)} e^{p_3 F(t_3)}} \]  
\[ \text{(44)} \]

where \( p_1 = \rho(x,t;t'_1) \).

We should point out the different vacuum configurations induce different vertex operators. Consider for instance the constant vacuum configuration, \( A_{vac}(v_0), v_0 \neq 0 \) regarded as a deformation upon \( A_{vac}(v_0) = 0 \). Such deformation induces deformations to the vertex operators (parameterized in terms of \( v_0 \) and in turn to the soliton solutions (see for instance [12]). We now consider explicit solution for the equations derived in the previous section.

### 4.1 \( A_2^{(2)} \) Solitons

Consider the following auxiliary quantities\(^3\)

\[ \Omega_{(m+1)} = E_{a_1}^{(m)} + E_{a_2}^{(m)} + E_{a_1-a_2}^{(m+1/2)} + v_0 (h_1^{(m)} + h_2^{(m)}), \]  
\[ \text{(45)} \]

\[ \Gamma_{(m+5)} = E_{a_1}^{(m+1)} + E_{a_2}^{(m+1)} + E_{a_1+a_2}^{(m+1/2)} + v_0 (E_{a_1}^{(m+1/2)} - E_{a_2}^{(m+1/2)}) + \frac{v_0^2}{5} (h_1^{(m+1/2)} - h_2^{(m+1/2)}) \]  
\[ \text{(46)} \]

with the property that

\[ \Omega_{(m+1)} \Gamma_{(m+5)} = \frac{6l + 1}{2} \delta_{m+l+1,0} \]  
\[ \text{(47)} \]

The vacuum for the two-dimensional gauge potentials can be shown to be written entirely in terms of quantities (45) and (46), i.e.,

\[ A_{t_1}^{vac} = A_x^{vac} = \Omega_{(1)} \]  
\[ \text{(48)} \]

\[ A_{t_2}^{vac} = \Gamma_{(5)} - \frac{v_0^4}{32} \Omega_{(1)} \]  
\[ \text{(49)} \]

\[ A_{t_3}^{vac} = \Omega_{(7)} - \frac{v_0^2}{3} \Gamma_{(5)} + \frac{4v_0^6}{3^4} \Omega_{(1)} \]  
\[ \text{(50)} \]

\[ A_{t_4}^{vac} = \Gamma_{(11)} - \frac{v_0^4}{32} \Omega_{(7)} + \frac{5v_0^6}{3^2} \Gamma_{(5)} - \frac{2v_0^{10}}{3^6} \Omega_{(1)} \]  
\[ \text{(51)} \]

\[ A_{t_5}^{vac} = \Omega_{(13)} - \frac{v_0^2}{3} \Gamma_{(11)} + \frac{2v_0^4}{3^4} \Omega_{(7)} - \frac{7v_0^6}{3^5} \Gamma_{(5)} + \frac{35v_0^{12}}{3^8} \Omega_{(1)} \]  
\[ \text{(52)} \]

Notice that for the vacuum configuration \([A_{vac}, A_{vac}^{vac}] = 0 \).

\^2 Notice that, in order to introduce highest weight states \( | i > \) we have to extend the affine loop algebra to the central extended Kac-Moody algebra. As a consequence we have introduced an extra field \( \nu \) associated to the central term \( \kappa \).

\^3 Notice that these quantities contain terms of different \( Q \) gradations. In order to have all terms with the same grade we may introduce a second loop, \( v_0 = w \) and define a generalized grading operator \( \hat{Q} = Q + w \frac{\partial}{\partial w} \) as proposed within the two-loop affine algebra, see [13, 14].
For the negative sub-hierarchy with constant vacuum, \( v_0 \neq 0 \),
\[
A_{\text{vac}}^{\text{inc}}_{t,\text{m=+5}} = 3v_0^{-2} \Gamma(-6m+5),
\]
and henceforth,
\[
A_{\text{vac}}^{\text{inc}}_{t,\text{m=+6+5}} = E^{(-6m+5)}, \quad m = 1, 2, \ldots
\]
A vertex operator satisfying (42) for \( v_0 \neq 0 \) can then be constructed. Consider
\[
F(\gamma) = \sum_j 3^j [(v_0 - \gamma)(v_0 + \gamma)]^{-3/2} [(2v_0 - \gamma)(2v_0 + \gamma)]^{-j} e^{\{h^{(0)}_1 + h^{(0)}_2 + a_1 E_{a_1}^{(0)} + E^{(j+1/2)}_{a_2} + \cdots\}},
\]
\[
\rho(t_{j-1}, x) = e^{-t_{j-1} \lambda_{j-1} t_{j-1}^5} = e^{-\gamma (v_0 - \gamma)^5 t_{j-1}^5},
\]
\[
\rho(t_{j-1}, x) = e^{-t_{j-1} \lambda_{j-1} t_{j-1}^5} = e^{-\gamma (v_0 - \gamma)^5 t_{j-1}^5},
\]
and similarly for higher values of \( j \). For the negative sub-hierarchy with non-zero vacuum, we find
\[
\lambda_{-3} = \frac{2^2 \gamma}{v_0 \gamma (v_0 - \gamma)^2},
\]
\[
\lambda_{-6} = \frac{2^3 \gamma}{v_0 \gamma (v_0 - \gamma)^3},
\]
For \( v_0 = 0 \), the vertex operator (57) becomes
\[
F(\gamma) = \sum_j 3^j \gamma^{2j} e^{\{h^{(0)}_1 + h^{(0)}_2 + \cdots\}},
\]
\[
\rho(t_{j-1}, x) = e^{-t_{j-1} \lambda_{j-1} t_{j-1}^5} = e^{-\gamma (v_0 - \gamma)^5 t_{j-1}^5},
\]
It coincides with the vertex constructed in [15] for the Tzitzeica model where \( \gamma = \sqrt{3}c \) and (74) satisfy
\[
\rho(t_{j-1}, x) = e^{-t_{j-1} \lambda_{j-1} t_{j-1}^5} = e^{-\gamma (v_0 - \gamma)^5 t_{j-1}^5},
\]
leading to
\[
\rho(t_{j-1}, x) = e^{-t_{j-1} \lambda_{j-1} t_{j-1}^5} = e^{-\gamma (v_0 - \gamma)^5 t_{j-1}^5},
\]
\[ \rho(t_{m+5}, x) = e^{-\frac{r x + \frac{1}{2} t}{\tau^2}} \rho(t_{m+5}, x) . \]  

Equations (68)–(69), (72)–(73) and (77)–(78) establishes the space-time dependence for each particular model.

### 4.2 One-Soliton Solution

Here we explicitly derive the one-soliton solution for the \( A_2^{(2)} \) hierarchy. Propose the following ansatz,

\[ v(x, t_{m+1}) = v_0 + \partial_x \ln \left( \tau_0 / \tau_1 \right) \]

where

\[ \tau_i = 1 + \langle i | F | i \rangle \rho + \langle i | FF | i \rangle \rho^2 , \]

\[ | i \rangle, \ i = 0, 1 \] labels the two fundamental weights of \( A_2^{(2)} \) and \( \rho(x, t_{m+1}) \) is constructed from Eqs. (75) and (76) for \( m = 0 \) and \( m = 1 \) respectively.

\[ \rho(x, t_{m+1}) = e^{-\frac{r x}{\tau^2}} \rho \]

The matrix elements can be evaluated in terms of representations of the affine \( A_2^{(2)} \) Kac-Moody algebra where (see for instance [15]).

\[ (h_1^{(0)} + h_2^{(0)}) | 0 \rangle = 0, \quad \kappa | 0 \rangle = 2 | 0 \rangle \]

\[ (h_1^{(0)} + h_2^{(0)}) | 1 \rangle = 1 | 1 \rangle, \quad \kappa | 1 \rangle = 2 | 1 \rangle \]

It therefore follows using vertex (77) we find

\[ c_0^{(1)} = \langle 0 | F | 0 \rangle = -\frac{2(\gamma + v_0)}{3 \gamma} e, \]

\[ c_0^{(2)} = \langle 0 | F^2 | 0 \rangle = \frac{(\gamma^2 - 4v_0^2)(\gamma + v_0)}{36 \gamma^2(\gamma - v_0)} e^2 , \]

\[ c_1^{(1)} = \langle 1 | F | 1 \rangle = \frac{\gamma - 2v_0}{3 \gamma} e, \]

\[ c_1^{(2)} = \langle 1 | F^2 | 1 \rangle = \frac{(\gamma - 2v_0)^2}{36 \gamma^2} e^2 , \]

Notice that the matrix elements (83) are independent of space-time and hence are the same for all models within the hierarchy. For other models of the \( A_2^{(2)} \) hierarchy with zero vacuum \( v_0 = 0 \) we find the same functional expression for the tau-functions with \( \rho(x, t_N) = e^{-\frac{r x + \frac{1}{2} t}{\tau^2}} \), \( N = -1, 3, \ldots \) in the limit \( v_0 \to 0 \). The matrix elements (83) evaluated with either vertices (77) or (74) yield solution for all models within the hierarchy, i.e.,

\[ v(x, t_k) = \partial_x \phi(x, t_k) = v_0 + \partial_x \ln \left( \frac{1 + c_0^{(1)} \rho(x, t_k) + c_0^{(2)} \rho^2(x, t_k)}{1 + c_1^{(1)} \rho(x, t_k) + c_1^{(2)} \rho^2(x, t_k)} \right) \]

In order to check consistency of solutions (84) with equations of motion let us consider for instance \( t_5 \) Eq. (25) and propose

\[ \tau_0(x, t) = 1 + \omega_1 \rho_1(x, t) + \gamma_1 \rho_1(x, t)^2 , \]

\[ \tau_1(x, t) = 1 + \omega_2 \rho_1(x, t) + \gamma_2 \rho_1(x, t)^2 \]

with

\[ \rho_1(x, t) = e^{t \sqrt{r} \rho} , \]

Substituting in (25), after taking the least common multiple we find by collecting powers of \( \rho \). The lowest power of \( \rho \) leads to

\[ -5f_1^3 g_0^2 + 5f_1^4 f_2^2 + 9g_1 = 0 . \]

If we set \( f_1 = \gamma \) we find agreement with (81), i.e.,

\[ g_1 = -\frac{1}{9} (5v_0^2 \gamma^3 - 5v_0^4 \gamma^3) . \]

Inserting in the coefficient of \( \rho^2 \) we find,

\[ 6\gamma(\gamma_2 - 1)(\gamma - v_0) = (\omega_1 - \omega_2)(\omega_1 + \omega_2) \]

\[ + \omega_1 (\gamma - v_0)(\gamma - 2v_0) . \]

Solving for \( \gamma_1 \) and inserting in the coeff. of \( \rho^3 \),

\[ -36\gamma_2 \gamma^2 (\gamma - v_0) = (\omega_1 - \omega_2)(\gamma_2 - 2v_0) \]

\[ + \omega_1 (\gamma - v_0)(\gamma - 2v_0) . \]

which can be solved for \( \gamma_2 \). The coefficient of \( \rho^4 \) leads to

\[ 20\gamma^3 (\omega_1 - \omega_2)^3 (\gamma_2 - 2v_0)^2 (17\gamma^2 - 5v_0^2) \]

\[ (\omega_1 (\gamma - 2v_0) + 2\omega_2 (\gamma + v_0)) = 0 , \]

and henceforth gives the following non-trivial, \( (w_1 \neq w_2) \) solution,

\[ \omega_2 = \frac{\omega_1 (\gamma - 2v_0)}{2(\gamma + v_0)} , \quad \gamma_1 = \frac{\omega_1^2 (\gamma^2 - 4v_0^2)}{16(\gamma^2 - v_0^2)} , \quad \gamma_2 = \frac{\omega_1^2 (\gamma - 2v_0)^2}{16(\gamma + v_0)^2} \]

The coefficients of higher powers of \( \rho \) all vanish when (92) is taken into account. In order to fit with solution (84), we set \( f_1 = \gamma \), \( \alpha = \frac{c}{3\gamma} \) and \( \alpha_1 = 2(\gamma + v_0) \) leading to

\[ g_1 = \frac{1}{9} (-5v_0^3 \gamma^3 + 5v_0^4 \gamma^3) , \quad \alpha_1 = c_0^{(1)} \quad \alpha_2 = c_1^{(1)} \]

and

\[ \gamma_1 = c_0^{(2)} \quad \gamma_2 = c_1^{(2)} . \]
5 Twisted Affine $A^{(2)}_{2r}$ Algebra

We shall now discuss the construction of a class of integrable hierarchies based upon the twisted affine Lie algebra $A^{(2)}_{2r}$. In general, consider the following second-order automorphism for $A_{2r}$,

$$
\sigma(a_i) = a_{2r-i+1}, \quad \sigma(a_{2r}) = a_{2r-1}, \quad \sigma(a_{2r-1}) = a_{2r}, \quad \sigma(a_i) = a_i
$$

(95)
diagrammatically illustrated according to the Dynking diagram.

Extending to the Lie algebra we find,

$$
\sigma(\alpha_i \cdot H) = \alpha_{2r-i+1} \cdot H, \quad i = 1, \ldots, r
$$

$$
\sigma(E_m) = \zeta_a E_{\sigma(a)}, \quad \zeta_a = \pm 1.
$$

(96)

E.g.,

$$
\zeta_{a_i} = +1, \quad i = 1, \ldots, r,
$$

$$
\zeta_{a_i + a_{i+1}} = -1, \quad i = 1, \ldots, r - 1,
$$

$$
\vdots
$$

$$
\zeta_{a_1 + \cdots + a_i} = (-1)^{i-j}, \quad i = 1, \ldots, r, \quad j = i, i+1, \ldots, r.
$$

(97)

The twisted affine $A^{(2)}_{2r}$ algebra is constructed by assigning integer affine indices to the even sub-algebra under $\sigma$, i.e., $T^{(m)}_a$, $\sigma(T_m) = T_a$ and semi-integer indices to the odd part, $T^{(m+1/2)}_a$, $\sigma(T_m) = -T_a, m \in \mathbb{Z}, a = 1, \ldots, \dim A_{2r}$.

We now define the grading operator (principal gradation)

$$
Q = 2(2r + 1)d + \sum_{i=1}^{2r} \mu_i \cdot H
$$

(98)

where $d$ is the derivation operator$^4$ and $\mu_i, i = 1, \ldots, 2r$ are the fundamental weights of the $A_{2r}$ Lie algebra. It therefore follows that $Q$ induces a decomposition of $A^{(2)}_{2r}$ into graded subspaces,

$$
G_{2(2r+1)m+0} = \{(1 + \sigma)E^{(m)}_{a_i}, \quad i = 1, \ldots, r
$$

(99)

$$
G_{2(2r+1)m+1} = \{(1 + \sigma)E^{(m+1/2)}_{a_i}, \quad \sigma = 1, \ldots, r, (1 - \sigma)E^{(m+1/2)}_{-(a_1 + \cdots + a_2)} \}
$$

(100)

$$
G_{2(2r+1)m+1} = \{(1 - \sigma)E^{(m+1/2)}_{a_i}, \quad i = 1, \ldots, r \}
$$

(101)

$$
G_{2(2r+1)m+[2(2r+1)+1]} = \{(1 - \sigma)E^{(m+1/2)}_{a_i}, \quad i = 1, \ldots, r \}
$$

(102)

The Lax operator is constructed by specifying a constant grade 1 operator

$$
E \equiv E^{(1)} = \sum_{i=1}^{2r} E^{(0)}_{a_i} + E^{(1/2)}_{-(a_1 + \cdots + a_2)} \in G_1
$$

(103)

which, in turn, decomposes the affine algebra $\hat{G} = K \oplus M$ where $K$ denotes the Kernel of $E$ such that $x \in K, \{E, x\} = 0$ and $M$ is its complement. The Lax operator is then defined to be

$$
L \equiv E + A_0 = E + \sum_{i=1}^{r} v_i(x, \tau)(1 + \sigma)h^{(0)}_i.
$$

(104)

$$
L \equiv E + A_0 = E + \sum_{i=1}^{r} v_i(x, \tau)(1 + \sigma)h^{(0)}_i.
$$

(106)

Following the algebraic data developed in Appendix 2 we now illustrate with the first few models for the $A^{(2)}_{2r}$ hierarchy. Let us redefine $v_1 \equiv v$ and $v_2 \equiv w$ for simplicity. The simplest case is given by $N = 3$, yielding the system of equations of motion,

$$
5v_3 = -3v^2(v_x - 2w_x) + 3v(4vw_x - 2ww_x - v_{2s} + w_{2s})
$$

$$
-3w(v_{2s} - v_{2x}) - 3v_x^2 + 3w_x^2 - 3w^2v_x + 2v_{3s} - 3w_{3x}
$$

(105)
Clearly, constant vacuum \((\nu, w) = (v_0, w_0) \neq 0\), as well as zero vacuum \((\nu, w) = (0, 0)\), are both solutions of \((107)-(108)\).

For the negative sub-hierarchy, the simplest model corresponds to \(M = -1\), \(t = t_{-1}\).

\[
A_\nu = E^{(1)} + B^{-1} \partial_w B, \quad A_e = B^{-1} E^{(-1)} B,
\]

where \(E^{(-1)} = E^1 \in \mathcal{K}\) and \(v = \partial_y \phi_1, \ w = \partial_x \phi_2\) leading to the equations of motion,

\[
\begin{align*}
\partial_y \phi_1 & = e^{2\nu_1} \phi_1 - e^{-2\phi_1}, \\
\partial_y \phi_2 & = e^{-\phi_1} \phi_2 - e^{-2\phi_1}.
\end{align*}
\]

It is clear that \([E, D^{\nu_1}] = 0\) and henceforth \((101)\) allows only zero vacuum solution, \(\phi^{\nu_1}_{\nu_0} = 0, \ i = 1, 2\). The next model is obtained for \(M = 5, \ t = t_{-5}\) and equation of motion

\[
\begin{align*}
v_i & = -e^{-d_{i}\nu_0} d_i^{-1} \left[ e^{-d_{i}\nu_0} \left( c_{3} - a_{3}\right) \right] \\
& + e^{-d_{i}\nu_2} d_i^{-1} \left[ e^{-d_{i}\nu_2} \left( c_{3} - b_{3}\right) \right] \\
& - 2e^{-d_{i}\nu_0} d_i^{-1} \left[ e^{-d_{i}\nu_0} \left( c_{3} - b_{3}\right) \right]
\end{align*}
\]

and

\[
\begin{align*}
w_i & = e^{-d_{i}\nu_0} d_i^{-1} \left[ e^{-d_{i}\nu_0} \left( c_{3} - a_{3}\right) \right] \\
& - 2e^{-d_{i}\nu_0} d_i^{-1} \left[ e^{-d_{i}\nu_0} \left( c_{3} - b_{3}\right) \right]
\end{align*}
\]

where

\[
\begin{align*}
a_{3} & = -e^{-d_{i}\nu_0} d_i^{-1} \left[ e^{-d_{i}\nu_0} \left( c_{3} - a_{3}\right) \right] \\
b_{3} & = 4e^{-d_{i}\nu_0} d_i^{-1} \left[ e^{-d_{i}\nu_0} \left( c_{3} - b_{3}\right) \right] \\
c_{3} & = e^{-d_{i}\nu_0} d_i^{-1} \left[ e^{-d_{i}\nu_0} \left( c_{3} - b_{3}\right) \right]
\end{align*}
\]

After a tedious but straightforward calculation, it can be verified that \(v = v_0 \in w = w_0\) are indeed solutions of the equations of motion as expected by extending the arguments derived for \(A_2^{(2)}\) case in arguments in \((32)\).

6 Comments and Further Developments

In this paper, we have studied the construction of integrable hierarchies associated to the \(A_2^{(2)}\) twisted affine algebra. We have discussed explicitly the \(A_2^{(2)}\) case as an example and discussed in detail the algebraic construction of the positive and negative sub-hierarchies. An important ingredient in classifying the models is the structure of the zero curvature in the vacuum configuration, i.e., \(A_0^{\nu_1}\). These algebraic quantities contain terms of different \(Q\) gradations. The sub-hierarchies were shown to be classified according to (i) a grading structure of algebraic nature (\(Q\)-gradation) and (ii) its zero or non-zero vacuum solution. We now argue that the two concepts can be put together in terms of the two-loop Kac-Moody algebra proposed many years ago in connection with conformal affine Toda models \([13, 14]\).

\[
\begin{align*}
[T_{m,r}^{a}, T_{n,s}^{b}] & = \frac{i}{\theta} T_{m+n,r+s}^{a} + \kappa g_{abs} r \delta_{r+s,0} \delta_{m+n,0} \\
& + \tilde{\kappa} g_{abs} m \delta_{m+n,0} \delta_{r+s,0}
\end{align*}
\]

where \(\kappa\) and \(\tilde{\kappa}\) denote two central terms, \(d\) and \(\tilde{d}\), the two derivation operators,

\[
[d, T_{m,r}^{a}] = m T_{m,r}^{a}, \quad [\tilde{d}, T_{m,r}^{a}] = r T_{m,r}^{a}.
\]

For the centerless version of \((114)\), the generators can be realized as

\[
T_{m,r}^{a}(z,w) = \sum_{m,r \in \mathbb{Z}} T_{m,r}^{a} z^{-m} w^{-r}, \quad d = z \frac{\partial}{\partial z}, \quad \tilde{d} = w \frac{\partial}{\partial w}.
\]

Considering \(Q = 6d + \left( \mu_1 + \mu_2 \right) \cdot H\) of section 2 and defining \(w = v_0\) to describe the second loop, under the new gradation \(\tilde{Q} = Q + \tilde{d}\) decomposes the affine algebra \(\tilde{G} = \bigoplus \tilde{G}_a\) and

\[
A_\nu = E^{(1)} + v_0 (h_0^{(0)} + h_2^{(0)}) \in \tilde{G}_1.
\]

Moreover, all terms in \(Q_{\nu+1}^{(a)} \in \tilde{G}_{\nu+1}\) in \((45)\) and \(1_{\nu+5} \in \tilde{G}_{\nu+5}\) in \((46)\) have the same grade according to the \(\tilde{Q}\) gradation and so does the terms in \(A_0^{\nu_1}\) in \((28)\) and \(A_1^{\nu_1}\) in \((31)\). The central extension \(\kappa\) plays a role to ensure highest weight states \(|i >| is responsible for introducing the field \(v\) in \((41)\). The role of the second central term \(\kappa\) is still not clear in such context and is a subject of interest in future developments.

As a generalization of ref. \([15]\), we have constructed soliton solutions for all models within the \(A_2^{(2)}\) hierarchy. Those with non-zero vacuum involves the construction of deformed vertex operators in terms of the vacuum parameter \(v_0\). A few simple examples within the \(A_2^{(1)}\) positive sub-hierarchy (mKdV) \([12]\) show that these deformed solutions satisfy Bäcklund transformation. It would be interesting to extend and understand the
structure of (twisted) deformed integrable defects in the lines of [11] and to consider algebras other then \( A_{2s}^{(2)} \).

**Appendix 1**

Here we display the solution for \( t \) time evolution for the \( A_{2s}^{(2)} \) model.

\[
D^{(7)} = E_{a_1}^{(1)} + E_{a_2}^{(1)} + E_{-a_1+a_2}^{(2)} \\
D^{(6)} = v(h_{1}^{2} + h_{2}^{2}) \\
D^{(5)} = -\frac{1}{3}(\nu^2 + v_x)(E_{-a_1}^{(1)} + E_{a_2}^{(1)}) - \frac{1}{3}(\nu^2 - 2v_x)E_{(a_1+a_2)}^{(1)} \\
D^{(4)} = -\frac{1}{3}(v^2 - v_x v - v_{x2})(E_{a_1}^{(1)} - E_{-a_2}^{(1)}) \\
D^{(3)} = -\frac{1}{9}(\nu^2 + 2v_x v^2 - 2v_{x2} v - v_{x3})(h_{1}^{2} - h_{2}^{2}) \\
D^{(2)} = \frac{1}{9}(4v_{x1} v^3 + 2v_{x2} v^2 + 4v_x v^2 - 2v_{x3} v - 4v_{x1} v_{x2}) \\
\]

\[
D^{(1)} = \frac{1}{81}(4\nu^6 - 24\nu v^4 - 18\nu v^2 + 12\nu^2 v^2 + 18\nu v^3_3 + 72\nu v_{x2} + 3v v_{x4} v + 4v_x^3 - 24\nu v_{x3} - 9v_{x2}) \\
- 3v_{x3}(E_{a_1}^{(0)} + E_{a_2}^{(0)}) + \frac{1}{81}(4v^6 + 12v_x v^4) \\
- 60v_x v^2 v - 36v^3 v_{x2} - 72\nu v_{x3} v_{x4} v \\
- 32v_{x3}^3 + 27v_{x2} v^3 + 30v_{x3} v_x - 18v_{x3} v_{x2} + 6v_{x3}(E_{a_1}^{(2)}) \\
\]

\[
D^{(0)} = \frac{1}{81}(4\nu^6 - 42\nu v^4 - 84\nu^2 v^2 + 21(2v_{x2} v^2 + v_{x4} v^2) \\
+ 7(4v^4 + 12v_{x1} v^3 v + 9v_{x2} v_{x3}) v + 4v_{x2} v_{x3} v - 21v_{x1} v_{x3} + 3v_{x3} (h_{1}^{0} + h_{2}^{0}) \\
A_{\nu^2} = E_{a_1}^{(1)} + E_{a_2}^{(1)} + E_{-a_1+a_2}^{(2)} + v_0(h_{1}^0 - h_{2}^0) \\
- \frac{v_0^2}{9} [3(E_{-a_1}^{(1)} + E_{-a_2}^{(1)}) - E_{(a_1+a_2)}^{(2)}] \\
+ 3v_0(E_{a_1}^{(2)} - E_{-a_2}^{(2)}) h_{1}^{0} + h_{1}^{2} \\
+ \frac{4v_0^6}{81} [E_{a_1}^{(0)} + E_{a_2}^{(0)} + E_{-a_1+a_2}^{(2)} + v_0(h_{1}^0 + h_{2}^{0})] \\
\]

**Appendix 2**

For the \( A_{4}^{(2)} \) case, we find

\[
\mathbf{g}_{10}^{(m)} = \{ h_{1}^{(m)} + h_{4}^{(m)}, h_{2}^{(m)} + h_{3}^{(m)} \} \\
\mathbf{g}_{10}^{(m+1)} = \{ E_{a_1}^{(m)} + E_{a_2}^{(m)} + E_{a_3}^{(m)} + E_{-a_1-a_2-a_3}^{(m+1)} \} \\
\mathbf{g}_{10}^{(m+2)} = \{ E_{a_1}^{(m)} - E_{a_1+a_2+a_3}^{(m+1)} - E_{-a_1-a_2-a_3}^{(m+1)} \} \\
\mathbf{g}_{10}^{(m+3)} = \{ E_{a_1+a_2-a_3}^{(m+1)} + E_{a_1+a_2+a_3}^{(m+1)} - E_{a_1-a_2-a_3}^{(m+1)} \} \\
\mathbf{g}_{10}^{(m+4)} = \{ E_{a_1}^{(m+1)} - E_{a_2}^{(m+1)} - E_{-a_1}^{(m+1)} \} \\
\mathbf{g}_{10}^{(m+5)} = \{ h_{1}^{(m+1)} - h_{4}^{(m+1)}, h_{2}^{(m+1)} - h_{3}^{(m+1)} \} \\
\mathbf{g}_{10}^{(m+6)} = \{ E_{a_1}^{(m+1)} - E_{a_2}^{(m+1)} - E_{a_3}^{(m+1)} \} \\
\mathbf{g}_{10}^{(m+7)} = \{ E_{-a_1+a_2-a_3}^{(m+1)} - E_{-a_1-a_2+a_3}^{(m+1)} - E_{a_1-a_2-a_3}^{(m+1)} \} \\
\mathbf{g}_{10}^{(m+8)} = \{ E_{a_1+a_2-a_3}^{(m+1)} - E_{a_1+a_2+a_3}^{(m+1)} + E_{a_1-a_2+a_3}^{(m+1)} \} \\
\mathbf{g}_{10}^{(m+9)} = \{ E_{a_1}^{(m+1)} + E_{a_2}^{(m+1)} + E_{a_3}^{(m+1)} - E_{-a_1-a_2-a_3}^{(m+1)} \}
\]

Let

\[
A_0 = v(h_{1} + h_{4}) + w(h_{2} + h_{3}), \\
E^{(1)} = E_{a_1}^{(0)} + E_{a_2}^{(0)} + E_{a_3}^{(0)} + E_{-(a_1+a_2+a_3)}^{(1)} \\
\]

and the Kernel of \( E \) is given by

\[
E^{(10 m+1)} = E_{a_1}^{(m)} + E_{a_2}^{(m)} + E_{a_3}^{(m)} + E_{-(a_1+a_2+a_3)}^{(m+1)} \\
E^{(10 m+3)} = E_{(a_1+a_2+a_3)}^{(m)} + E_{(a_1+a_2+a_3)}^{(m+1)} + E_{(a_1+a_2+a_3)}^{(m+1)} + E_{-(a_1+a_2+a_3)}^{(m+1)} \\
E^{(10 m+5)} = E_{-a_1+a_2-a_3}^{(m+1)} + E_{-a_1+a_2+a_3}^{(m+1)} + E_{-a_1+a_2+a_3}^{(m+1)} + E_{-a_1+a_2+a_3}^{(m+1)} \\
E^{(10 m+9)} = E_{a_1}^{(m+1)} + E_{a_2}^{(m+1)} + E_{a_3}^{(m+1)} + E_{-(a_1+a_2+a_3)}^{(m+1)}
\]
\(m \in \mathbb{Z}\). Following the general pattern developed for the \(A_2^{(2)}\) case, we expect to obtain positive sub-hierarchies associated to each element of the Kernel in (131)–(134).

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Declarations

Conflict of Interest The authors declare no competing interests.

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