A CHARACTERIZATION OF SATURATED FUSION SYSTEMS OVER ABELIAN 2-GROUPS

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Abstract. Given a saturated fusion system \( F \) over a 2-group \( S \), we prove that \( S \) is abelian provided any element of \( S \) is \( F \)-conjugate to an element of \( Z(S) \). As a consequence, if \( B \) is a 2-block of a finite group \( G \) such that all \( B \)-subsections are major, the defect groups of \( B \) are abelian. Our results were conjectured by Kühnshammer–Navarro–Sambale–Tiep [2] and generalize a Theorem of Camina–Herzog [3].

1. Introduction

This short note gives an example of how a conjecture in the modular representation theory of finite groups can be proved by showing its generalization to saturated fusion systems. We refer the reader to [1] for definitions and basic results regarding fusion systems and to [5] as a background reference on block theory. Here is our main theorem:

**Theorem 1.** Let \( F \) be a saturated fusion system on a 2-group \( S \) such that for any \( x \in S \), \( \text{Hom}_F(\langle x \rangle, Z(S)) \neq \emptyset \). Then \( S \) is abelian.

Since for any finite group \( G \) with Sylow 2-subgroup \( S \) the fusion system \( F_S(G) \) is saturated, the above theorem yields immediately the following corollary:

**Corollary 1** (Camina–Herzog). Let \( G \) be a finite group such that \( |G : C_G(x)| \) is odd for any 2-element \( x \) of \( G \). Then the Sylow 2-subgroups of \( G \) are abelian.

Corollary 1 was first proved by Camina–Herzog [3]. As they point out, it means that one can read from the character table of a finite group if its Sylow 2-subgroups are abelian. The proof of Camina–Herzog relies on the Theorem of Goldschmidt [4] about groups with strongly closed abelian 2-subgroups, whereas our approach is elementary and self-contained. More precisely, we show Theorem 1 by an induction argument which appears canonical in the context of fusion systems. Using the same idea, one can also give an elementary direct proof of Corollary 1 which does not use fusion systems; see Remark 2.1 for details.

We now turn attention to \( p \)-blocks of finite groups. Their Brauer categories, which we introduce in Definition 2.2, provide important examples of saturated fusion systems. Given a finite group \( G \) and a \( p \)-block \( B \) of \( G \), recall that a \( B \)-subsection of \( G \) is a pair \((x, b)\) such that \( x \) is a \( p \)-element and \( b \) is a block of \( C_G(x) \) with the property that the induced block \( b^G \) equals \( B \). A \( B \)-subsection \((x, b)\) is called major if the defect groups of \( b \) are also defect groups of \( B \). Theorem 1 applied to the Brauer category of a 2-block yields the following corollary, which we prove in detail at the end of this paper:

**Corollary 2.** Suppose \( B \) is a 2-block of a finite group \( G \) such that all \( B \)-subsections are major. Then the defect groups of \( B \) are abelian.

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2. Proofs

Proof of Theorem 1 Let $\mathcal{F}$ be a counterexample to Theorem 1 such that $|S|$ is minimal.

Step 1: We show that $N_S(x) = C_S(x)$ for any $x \in S$. By assumption, there exists $\varphi \in \text{Hom}_\mathcal{F}(x, S)$ such that $\varphi(x) \in Z(S)$. In particular, $\varphi(x)$ is fully normalized, so by [1 I.2.6(c)], there exists $\alpha \in \text{Hom}_\mathcal{F}(N_S(x), S)$ such that $\alpha(\langle x \rangle) = \varphi(\langle x \rangle)$. Then in particular, $\alpha(x) \in Z(S)$ and so $[\alpha(x), \alpha(N_S(\langle x \rangle))] = 1$. As $\alpha$ is injective, this implies $[x, N_S(\langle x \rangle)] = 1$.

Step 2: We prove that $U := \Omega_1(S)$ is elementary abelian. It is sufficient to show that any two involutions $u, v \in S$ commute. Note that $\langle u, v \rangle$ is a dihedral subgroup of $S$, so there exists $x \in S$ with $\langle u, v \rangle = \langle x, u \rangle$ and $x^u = x^{-1}$. Then by Step 1, $[x, u] = 1$, so $\langle u, v \rangle$ is a four-group, which implies $[u, v] = 1$.

Step 3: We now reach the final contradiction. By Step 2, $U$ is an elementary abelian subgroup of $S$, and since the image of an involution under an $\mathcal{F}$-morphism is again an involution, $U$ is strongly closed. Hence the factor system $\overline{\mathcal{F}} := \mathcal{F}/U$ as in [1 Section II.5] is well-defined and by [1 II.5.2, II.5.4] a saturated fusion system on $\overline{S} := S/U$. By construction of $\overline{\mathcal{F}}$ and by [1 I.4.7(a)] or [1 II.5.10], the morphisms of $\overline{\mathcal{F}}$ are precisely the group homomorphisms between subgroups of $\overline{S}$ which are induced by morphisms in $\mathcal{F}$. Hence, as $\text{Hom}_\mathcal{F}(\langle x \rangle, Z(S)) \neq \emptyset$ for any $x \in S$, we have also

$$\text{Hom}_{\overline{\mathcal{F}}}(\overline{\langle x \rangle}, \overline{Z(S)}) \neq \emptyset$$

Since $\overline{Z(S)} \leq Z(S)$, we have in particular that $\overline{\mathcal{F}}$ fulfills the assumptions of Theorem 1. So as $\mathcal{F}$ is a counterexample with $|S|$ minimal, $\overline{S}$ is abelian. Now by [1 Theorem 3.6], every morphism of $\overline{\mathcal{F}}$ extends to an element of $\text{Aut}_\overline{\mathcal{F}}(\overline{S})$. By construction of $\overline{\mathcal{F}}$, the elements of $\text{Aut}_\overline{\mathcal{F}}(\overline{S})$ are induced by elements of $\text{Aut}_\mathcal{F}(S)$ and whence leave $\overline{Z(S)}$ invariant. It follows now from (II) that $S = Z(S)U$, so $S$ is abelian as $U$ is abelian, contradicting $\mathcal{F}$ being a counterexample.

Remark 2.1 (Proof of the Theorem of Camina–Herzog). The proof of Corollary [1] in [3] also starts with showing that $U := \Omega_1(S)$ is abelian. An elementary proof of Corollary [1] which does not rely on the theory of fusion systems, can be given by applying induction to $N_G(U)/U$ similarly as in Step 3 of our proof of Theorem 1. This requires to show that $N_G(U)$ controls fusion in $G$. In fact, by [1 I.4.7(a)], it is true in general that any abelian subgroup which is strongly closed in $S \in \text{Syl}_p(G)$, controls fusion in $G$. However, in the special case we are in, there is a shorter argument to show that $N_G(U)$ controls fusion, which we give here:

We show first that $U \leq Z(S)$. Suppose by contradiction, there exists $a \in S$ with $[U, a] \neq 1$. Then we may choose $a$ of minimal order, which implies that $a$ acts as an involution on $U$. So $[U, a, a] = 1$ and thus $U$ normalizes $W := C_U(\langle a \rangle)$ as $U$ is abelian. By assumption of the Theorem, there exists $g \in G$ with $a \in Z(S^g)$. As $W \leq C_G(a)$, by Sylow’s Theorem we may assume $W \leq S^g$. Then $C_U(a)^{g^{-1}} \leq U$ as $U$ is strongly closed in $S$, and thus $C_U(a) \leq U^g$. Hence, since $U^g$ is abelian and $a \in Z(S^g)$,

$$U^g \leq C_G(W) \leq N_G(W).$$

Note that for any $h \in G$, $U^h$ is strongly closed in $S^h$, so if $U \leq S^h$, then $U = U^h$ is strongly closed in $S^h$. In particular, $U$ is strongly closed in any 2-subgroup containing $U$. Hence, as $U \leq N_G(W)$, it follows from Sylow’s Theorem that $U$ is conjugate to $U^g$ by an element of $N_G(W)$ and thus $U \leq C_G(W) \leq C_G(a)$, a contradiction. Thus $U \leq Z(S)$.

Let now $P \leq S$ and $x \in G$ such that $P^x \leq S$. By what we have just shown, $U, U^x \leq C_G(P^x)$. Again, as $U$ is strongly closed in any 2-subgroup containing $U$, there exists $c \in C_G(P^x)$ with $U^{xc} = U$. This proves that $N_G(U)$ controls fusion in $G$ as required.
Definition 2.2 (The Brauer category of a $p$-block). Let $G$ be a finite group, $p$ a prime, and $B$ a $p$-block of $G$. We refer the reader to [5, Section 5.9.1] for the definitions of subpairs, $B$-subpairs, the relation $\leq$ and its transitive closure $\leq$, which is an ordering on subpairs. The defect groups of $B$ are precisely the $p$-subgroups $D$ of $G$ which occur as the first component of a maximal $B$-subpair of $G$. Fixing a maximal $B$-subpair $(D, b_D)$, for any subgroup $Q \leq D$, there exists a unique block $b_Q$ of $QC_G(Q)$ such that $(\langle x \rangle, b_x) \leq (D, b_D)$. The Brauer category $\mathcal{F}_{(D, b_D)}(G, B)$ is the category whose objects are all subgroups of $D$ and, for $P, Q \leq D$, the set of morphisms from $P$ to $Q$ is given by

$$\text{Hom}_{\mathcal{F}_{(D, b_D)}(G, B)}(P, Q) = \{c_g : g \in G \text{ such that } (P^g, b_P^g) \leq (Q, b_Q)\},$$

where $c_g : P \rightarrow Q$ is defined via $c_g(x) = x^g$. It follows from [1, Theorem IV.3.2, Proposition IV.3.14] that the category $\mathcal{F}_{(D, b_D)}(G, B)$ is a saturated fusion system on $D$.

Proof of Corollary 2. Fix a maximal $B$-subpair $(D, b_D)$ of $G$ and set $\mathcal{F} := \mathcal{F}_{(D, b_D)}(G, B)$. Let $x \in D$. Then by [5, Theorem 5.9.3], there exists a unique block $b_x$ of $C_G(x)$ such that $(\langle x \rangle, b_x) \leq (D, b_D)$. Then $(x, b_x)$ is a $B$-subsection of $G$, which by assumption is major. Hence, by [5, Theorem 5.9.6], there exists $g \in G$ with $x^g \in Z(D)$ and $b_x^g = b_{C_G(x)}^g$. Note now the following general fact that follows from [5, Lemma 5.3.4] and the definition of $\leq$: If $(P, b_P)$ is a $B$-subpair with $P \leq Z(D)$ and $b_P = b_{C_G(P)}^P$, then $(P, b_P) \leq (D, b_D)$. Hence, we have $(\langle x^g \rangle, b_x^g) \leq (D, b_D)$ and thus $c_g : \langle x \rangle \rightarrow D$ is a morphism in $\mathcal{F}$ by definition of the Brauer category. Therefore, $\mathcal{F}$ fulfills the assumption of Theorem 1, which implies that $D$ is abelian. □

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