Forward Discretely Self-Similar Solutions of the Navier-Stokes Equations

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Abstract. Extending the work of Jia and Šverák on self-similar solutions of the Navier-Stokes equations, we show the existence of large, forward, discretely self-similar solutions.

1 Introduction

Denote \( \mathbb{R}_+^4 = \mathbb{R}^3 \times (0, \infty) \). Consider the 3D incompressible Navier-Stokes equations for velocity \( u : \mathbb{R}_+^4 \to \mathbb{R}^3 \) and pressure \( p : \mathbb{R}_+^4 \to \mathbb{R} \),

\[
\frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \text{div} \, u = 0,
\]

(1.1)

in \( \mathbb{R}_+^4 \), coupled with the initial condition

\[
u_{|t=0} = u_0, \quad \text{div} \, u_0 = 0.
\]

(1.2)

The system (1.1) enjoys a scaling property: If \( u(x, t) \) is a solution, then so is

\[
u^{(\lambda)}(x, t) := \lambda u(\lambda x, \lambda^2 t)
\]

(1.3)

for any \( \lambda > 0 \). We say \( u(x, t) \) is self-similar (SS) if \( u = u^{(\lambda)} \) for every \( \lambda > 0 \). In that case, the value of \( u(x, t) \) is decided by its value at any time moment \( t = \frac{1}{2a} \) and

\[
u(x, t) = \lambda(t) U(\lambda(t)x), \quad \lambda(t) = \frac{1}{\sqrt{2at}},
\]

(1.4)

where \( U(x) = u(x, \frac{1}{2at}) \) and \( a > 0 \) is a parameter. On the other hand, if \( u = u^{(\lambda)} \) only for one particular \( \lambda > 1 \), we say \( u \) is discretely self-similar (DSS) with factor \( \lambda \), or \( \lambda \)-DSS. Its value in \( \mathbb{R}_+^4 \) is decided by its value in the strip \( x \in \mathbb{R}^3 \) and \( 1 \leq t < \lambda^2 \). We consider being SS as a special case of being DSS, and would say “strictly DSS” to exclude the former. We call them forward to indicate they are defined for \( t > 0 \). We can also consider (1.1) for

1. \( (x, t) \in \mathbb{R}^3 \times (-\infty, 0) \), or
2. \( x \in \mathbb{R}^3 \), \( u = u(x) \) is time independent.

For both cases the scaling law (1.3) still holds, and we define backward and stationary SS and DSS solutions in the same manner. In particular, a backward SS solution satisfies (1.4) with \( a < 0 \), a stationary SS solution satisfies

\[
u(x) = \lambda(x) U(\lambda(x)x), \quad \lambda(x) = \frac{1}{|x|},
\]

(1.5)
with \( U(x) = u(x) \), and the profile \( U(x) \) of the SS solution for all three cases satisfies Leray’s equations
\[
- \Delta U - aU - ax \cdot \nabla U + (U \cdot \nabla)U + \nabla p = 0, \quad \text{div} \, U = 0, \tag{1.6}
\]
with \( a > 0, a < 0 \) and \( a = 0 \) respectively. Note the stationary SS solutions are often called minus-one homogeneous solutions in the literature.

When \( u(x, t) \) is either SS or DSS, then so is \( u_0(x) \). Thus it is natural to assume
\[
|u_0(x)| \leq \frac{C_\ast}{|x|}, \quad 0 \neq x \in \mathbb{R}^3 \tag{1.7}
\]
for some constant \( C_\ast > 0 \) and look for solutions satisfying
\[
|u(x, t)| \leq \frac{C(C_\ast)}{|x|}, \quad \text{or} \quad \|u(\cdot, t)\|_{L^3, \infty} \leq C(C_\ast). \tag{1.8}
\]
Here by \( L^{q,r} \), \( 1 \leq q,r \leq \infty \), we denote the Lorentz spaces. In such classes, with sufficiently small \( C_\ast \), the unique existence of mild solutions – solutions of the integral equation version of (1.1) – has been obtained by Giga-Miyakawa \([6]\) and refined by Cannone-Meyer-Planchon \([3, 4]\). As a consequence, if \( u_0(x) \) is SS or DSS satisfying (1.7) with small \( C_\ast \) and \( u(x, t) \) is a corresponding solution satisfying (1.8) with small \( C(C_\ast) \), the uniqueness property ensures that \( u(x, t) \) is also SS or DSS, because \( u(\lambda) \) is another solution with same bound and same initial data \( u_0(\lambda) = u_0 \). For large \( C_\ast \), mild solutions still make sense but there is no existence theory since perturbative methods like the contraction mapping no longer work.

Alternatively, one may try to extend the concept of weak solutions (which requires \( u_0 \in L^2(\mathbb{R}^3) \)) to more general initial data. One such theory is local-Leray solutions, constructed by Lemarié-Rieusset \([12]\) (to be defined below). However, there is no uniqueness theorem for them and hence the existence of large SS or DSS solutions was unknown.

In a surprising recent preprint \([7]\), Jia and Šverák constructed SS solutions for every SS \( u_0 \) which is locally Hölder continuous. Their main tool is a local Hölder estimate of the solution near \( t = 0 \), assuming minimal control of the initial data in the large (see Theorem 3.2). This estimate enables them to prove a priori estimates of SS solutions, and then show their existence by applying the Leray-Schauder degree theorem. Note that this existence theorem does not assert uniqueness. In fact, non-uniqueness is conjectured in \([7]\).

In this note, as an attempt to understand \([7]\), we consider the existence of discretely self-similar solutions for DSS \( u_0 \) satisfying (1.7) with large \( C_\ast \).

We now recall the definition of local-Leray solutions, see \([12, 7]\). Suppose
\[
u_0 \in L^2_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3), \quad \|u_0\|_{L^2_{\text{loc}}} := \sup_{x_0 \in \mathbb{R}^3} \left( \int_{B_1(x_0)} |u_0|^2 \, dx \right)^{\frac{1}{2}} < \infty, \quad \text{div} \, u_0 = 0. \tag{1.9}
\]
A vector field \( u \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0, \infty)) \) is called a local-Leray solution of (1.1) if (i)
\[
\text{ess sup} \sup_{0 \leq t \leq R^2} \int_{B(x_0,R)} |u(x, t)|^2 \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B(x_0,R)} |\nabla u(x, t)|^2 \, dx \, dt < \infty, \tag{1.10}
\]
and
\[
\lim_{|x_0| \to \infty} \int_0^{R^2} \int_{B(x_0,R)} |u(x, t)|^2 \, dx \, dt = 0, \tag{1.11}
\]
for any \( R < \infty \), (ii) together with some distribution \( p \) in \( \mathbb{R}^4 \), they satisfy \( \Box u \) in \( \mathbb{R}^4 \) in the sense of distributions, (iii) \( \lim_{t \to 0^+} \| u(\cdot,t) - u_0 \|_{L^2(K)} = 0 \) for any compact set \( K \subset \mathbb{R}^3 \), and (iv) \( u \) is suitable, i.e., it satisfies the local energy inequality in the sense of Caffarelli, Kohn, and Nirenberg [2].

The class of local-Leray solutions contains both Leray-Hopf weak solutions (with \( u_0 \in L^2(\mathbb{R}^3) \)) and mild solutions (with \( u_0 \) in \( L^3(\mathbb{R}^3) \) or \( VMO^{-1} \)), and is strictly larger. It is useful for our purpose because it allows initial data of the size \(|u_0(x)| \sim \frac{C}{|x|} \), because the local energy inequality is valid, and because a priori local energy estimates are available (see Lemma 3.1).

We now state our main theorems on the existence of forward discretely self-similar solutions. We first consider those with DSS factor close to one. We denote \( \lambda \) with no swirl, DSS with factor

\[
\lambda
\]

useful for our purpose because it allows initial data of the size

\[
\| u_0 \|_{C^1(B_1 \setminus B_{\lambda})} \leq C_* \text{, \ div } u_0 = 0, \text{ and } u_0 \text{ is DSS with factor } \lambda \in (1, \lambda_*]. \text{ Then there is a local-Leray solution } u \text{ of (1.1)} \text{ with initial data } u_0 \text{ that is DSS with factor } \lambda \text{ and, for } v(\cdot,t) := u(\cdot,t) - e^{t\Delta} u_0
\]

\[
\| u(x,t) \| \leq \frac{C}{|x| + \sqrt{t}} \quad \| v(x,t) \| \leq \frac{C\sqrt{t}}{|x|^2 + t}
\]

in \( \mathbb{R}^4_+ \) with \( C = (\gamma, C_*) \). It is also a mild solution in the class (1.12). If furthermore, \( \| u_0 \|_{C^{1,\beta}(\mathbb{R}^3 \setminus \{0\})} \leq C_* \) for some \( 0 < \beta < 1 \), then

\[
\| v(x,t) \| \leq \frac{C}{\sqrt{t}} \left( \frac{|x|}{\sqrt{t}} \right)^{-3} \log \left( \frac{|x|}{\sqrt{t}} \right), \quad |D_x v(x,t)| \leq \frac{C}{t} \left( \frac{|x|}{\sqrt{t}} \right)^{-3}
\]

in \( \mathbb{R}^4_+ \) with \( C = (\beta, C_*) \).

Note that \( \lambda - 1 > 0 \) has to be small enough.

A similar result is true for axisymmetric initial data with no swirl that is DSS with arbitrary factor. We recall that a vector field \( u \) in \( \mathbb{R}^3 \) is called \textit{axisymmetric} if in cylindrical coordinates \( r, \theta, z \) with \( (x_1, x_2, x_3) = (r \cos \theta, r \sin \theta, z) \) and \( r = \sqrt{x_1^2 + x_2^2} \), it is of the form

\[
u(x) = u^r(r, z)e_r + u^\theta(r, z)e_\theta + u^z(r, z)e_z.
\]

The components \( u^r, u^\theta, u^z \) do not depend upon \( \theta \) and the basis vectors \( e_r, e_\theta, e_z \) are

\[
  e_r = \left( \frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \quad e_\theta = \left( -\frac{x_2}{r}, \frac{x_1}{r}, 0 \right), \quad e_z = (0, 0, 1).
\]

It is called “no swirl” if \( u^\theta = 0 \). This class of vector fields is preserved under (1.1). If the initial data \( u_0 \in H^2(\mathbb{R}^3) \) is axisymmetric with no swirl, global in-time regularity of the solution was proved independently by Ukhouvskii-Yudovich [20] and Ladyzhenskaya [8]. See [11] for a refined proof. The case of general axisymmetric flow with \( u^\theta \neq 0 \) is open.

**Theorem 1.2** (Existence of axisymmetric DSS solutions with no swirl). For any \( 1 < \lambda < \infty, 0 < \gamma < 1, \) and \( C_* > 0 \), suppose \( u_0 \in C^\gamma_0(\mathbb{R}^3 \setminus \{0\}), \) is axisymmetric with no swirl, DSS with factor \( \lambda \), \( \text{div } u_0 = 0 \), and \( \| u_0 \|_{C^\gamma(B_1 \setminus B_1)} \leq C_* \). Then there is a
local-Leray solution \( u \) of (1.1) with initial data \( u_0 \) that is DSS with factor \( \lambda \), axisymmetric with no swirl, and satisfies (1.12) in \( \mathbb{R}^4 \) with \( C = C(\lambda, \gamma, C_*) \). It is also a mild solution in the class (1.12)_1.

If furthermore, \( \|u_0\|_{C^{1,\beta}(\overline{B_1})} \leq C_* \) for some \( 0 < \beta < 1 \), then it satisfies (1.13) in \( \mathbb{R}^4 \) with \( C = C(\lambda, \beta, C_*) \).

If \( C_* \) is small, the existence is known by [6, 3, 4]. Theorems 1.1 and 1.2 are concerned with large \( C_* \).

If one assumes higher regularity of \( u_0 \), say \( u_0 \in C^{3,\beta}_{loc}(\mathbb{R}^3 \setminus \{0\}) \) for some \( 0 < \beta < 1 \), the same proof of [7] Th 4.1 shows

\[
|v(x, t)| \leq \frac{Ct}{(|x| + \sqrt{t})^3}, \quad (|x|/\sqrt{t}) \geq C.
\]

This rate is optimal in view of the explicit spatial asymptotes for small SS solutions with smooth initial data in [1]. Eq. (1.13) is slightly worse than (1.16) by a log factor, but only assumes \( u_0 \in C^{1,\beta}_{loc} \). There is a gap between (1.12) and (1.13), especially when one takes \( 1 - \gamma = \beta \ll 1 \). It is probably because we require pointwise bound of the source term when we estimate the Stokes system. One may be able to narrow the gap by considering integral bounds of the source term.

Our approach follows that of [7], and relies heavily on the a priori estimates of the solutions, see Lemmas 3.3 and 3.4. One difference is that, instead of estimating a stationary solution of (1.6), we need to estimate a time dependent solution of (1.1). Another difference is the following: [7] first proves a priori estimates and constructs solutions for smooth initial data, and then gets solutions for \( C^\gamma \) data by approximation. In contrast, we prove a priori estimates and construct solutions for \( C^\gamma \) initial data directly. The reason for this change is that, at least for Theorem 1.1 we need the explicit dependence of \( \lambda_* \) on the local \( C^\gamma \)-norm of the data.

To extend these results, one may look for DSS solutions with DSS initial data of the form

\[
u_0 = u^1_0 + u^2_0
\]

where \( u^1_0 \) is SS and large, while \( u^2_0 \) is DSS with a large factor and is sufficiently small. When \( \lambda \) is large, a priori estimates seem unavailable, and one may try to study the linearized flow around \( u^1 \), a SS solution with initial data \( u^1_0 \). It turns out to be very challenging.

Another interesting problem is the non-uniqueness of local-Leray solutions conjectured by [7] and Sverák [18]: Considers SS solutions \( W_\sigma \) with SS initial data \( \sigma u_0 \), \( \sigma > 0 \). For \( \sigma \) small, \( W_\sigma \) is unique (see Lemma 3.4). However, when one increases \( \sigma \), one might get bifurcation. If the bifurcation is of saddle-node type, we get two SS solutions with the same initial data. If it is a Hopf bifurcation, the new solutions would be time periodic in the similarity variables \( (y = t^{-1/2}x \text{ and } s = \log t) \) and correspond to DSS solutions. These kind of DSS solutions \( u \) are different from those in Theorems 1.1 and 1.2 since their initial data \( u_0 \) are SS and only the difference \( v(\cdot, t) = u(\cdot, t) - e^{t\Delta}u_0 \) are strictly DSS. One may approximate \( u_0 \) by DSS data \( u^\varepsilon \) and take limits \( \varepsilon \to 0 \), and try to show that the strict DSS property of the corresponding solutions \( u^\varepsilon \) is somehow preserved in the limit. Of course this is purely speculation.

The existence question of discretely self-similar solutions also occur in two other instances for Navier-Stokes equations: (i) For singular backward solutions \( u(x, t) : \mathbb{R}^3 \times (-\infty, 0) \to \mathbb{R}^3 \) of (1.1), the nonexistence of SS solutions under some minimal integrability
assumptions was proved in \cite{14} and \cite{19}. The existence problem for DSS solutions under the same integrability assumptions is open; (ii) For stationary Navier-Stokes equations, the self-similar (minus-one homogeneous) solutions in $\mathbb{R}^3 \setminus \{0\}$ are shown by Šverák \cite{17} to be exactly those axisymmetric solutions found by Landau \cite{9} \cite{10}. The existence problem for strictly DSS solutions is open.

The rest of this note is structured as follows: In §2 we consider the Stokes system. In §3 we prove a priori estimates for DSS local-Leray solutions. In §4 we show their uniqueness for small initial data. Finally in §5 we prove their existence.

Notation. We denote $\langle z \rangle = (|z|^2 + 2)^{1/2}$ for $z \in \mathbb{R}^n$, $n \in \mathbb{N}$, and $A \lesssim B$ if there is a constant $C$, which may change from line to line, such that $A \leq CB$. We denote by $D_x^k u$ all $k$-th order partial derivatives of $u$ with respect to the variable $x$.

## 2 Stokes system

Consider the non-stationary Stokes system in $\mathbb{R}^3$ with a force tensor $f = (f_{ij})$

$$
\partial_t v - \Delta v + \nabla p = \nabla \cdot f, \quad \text{div} v = 0, \quad v|_{t=0} = 0.
$$

(2.1)

Here $(\nabla \cdot f)_j = \sum_k \partial_k f_{kj}$. If $f$ has sufficient decay, a solution is given by $v = \Phi f$, with

$$
(\Phi f)_i(x, t) = \int_0^t \int_{\mathbb{R}^3} \partial_{x_k} S_{ij}(x - y, t - s) f_{kj}(y, s) dy ds
$$

(2.2)

and $S = (S_{ij})$, the Oseen tensor, is the fundamental solution of the non-stationary Stokes system in $\mathbb{R}^3$ (see \cite{15} and \cite{16} page 27)

$$
S_{ij}(x, t) = \Gamma(x, t) \delta_{ij} + \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^3} \frac{\Gamma(y, t)}{|x - y|} dy, \quad Q_j(x, t) = \frac{\delta(t)}{4\pi} x_j, \quad |y|
$$

(2.3)

where $\Gamma(x, t) = (4\pi t)^{-n/2} e^{-x^2/4t}$ is the heat kernel. It is known in \cite{16} Theorem 1] that

$$
|D_x^\ell D_y^k S(t, x)| \leq C_{k, l} (|x| + \sqrt{t})^{-3-\ell-2k}, \quad (\ell, k \geq 0),
$$

(2.4)

where $D_x^\ell$ indicates $\ell$-th order derivatives with respect to the variable $x$.

When the bilinear operator $\Phi(u \otimes v) : X \times X \to X$ is well-defined on some Banach space $X$ of $\mathbb{R}^3$-valued fields on $\mathbb{R}^3_+$, we say $u(x, t)$ is a mild solution of (1.1) and (1.2) if

$$
u(x, t) = e^{t \Delta} u_0 - \Phi(u \otimes u)(., t) \quad \text{in} \quad X.
$$

(2.5)

We start with an integral estimate.

Lemma 2.1. Let $0 < a < 5, 0 < b < 5$ and $a + b > 3$. Then

$$
\phi(x, a, b) = \int_0^1 \int_{\mathbb{R}^3} (|x - y| + \sqrt{1 - t})^{-a} (|y| + \sqrt{t})^{-b} dy dt
$$

(2.6)

is well defined for $x \in \mathbb{R}^3$ and

$$
\phi(x, a, b) \lesssim R^{-a} + R^{-b} + R^{3-a-b}[1 + (1_{a=3} + 1_{b=3}) \log R]
$$

(2.7)

where $R = |x| + 2$. 

The cases $a, b \in \{3, 4\}$ are stated in [7, (4.12)]. We will also use $4 \leq a < 5$ and $2 \leq b < 3$.

**Proof.** Clearly $\phi \lesssim 1$ for $R \leq 8$. Consider now $R > 8$. Estimate the integral in 3 parts:

$$
\int_0^1 \int_{|y| > 2R} (\cdot) \lesssim \int_0^1 \int_{|y| > 2R} |y|^{-a-b} dy dt = CR^{3-a-b}, \quad (2.8)
$$

$$
\int_0^1 \int_{|y| < R/2} (\cdot) \lesssim \int_0^1 \int_{|y| < R/2} R^{-a} (|y| + \sqrt{t})^{-b} dy dt
$$

$$
\lesssim R^{-a} \left( \int_0^1 \int_{|y| < 1} + \int_0^1 \int_{1 < |y| < R/2} \right) (\cdot) \lesssim R^{-a} (1 + R^{3-b} (1 + 1_{b=3} \log R)), \quad (2.9)
$$

and similarly

$$
\int_0^1 \int_{R/2 < |y| < 2R} (\cdot) \lesssim \int_0^1 \int_{R/2 < |y| < 2R} (|x - y| + \sqrt{1 - t})^{-a} R^{-b} dy dt
$$

$$
\lesssim R^{-b} \int_0^1 \int_{|z| < 3R} (|z| + \sqrt{t})^{-a} dz dt \lesssim R^{-b} (1 + R^{3-a} (1 + 1_{a=3} \log R)). \quad (2.10)
$$

\[ \square \]

**Lemma 2.2.** Suppose $|f(x, t)| \leq \frac{1}{t} \left( \frac{\sqrt{t}}{|x| + \sqrt{t}} \right)^{2+m}$ in $\mathbb{R}^4_+$ for $0 \leq m < 1$. Then

$$
|\Phi f(x, t)| \lesssim \frac{1}{\sqrt{t}} \left( \frac{\sqrt{t}}{|x| + \sqrt{t}} \right)^{2+m}, \quad \forall (x, t) \in \mathbb{R}^4_+. \quad (2.11)
$$

**Proof.** By [24] and change of variables $x = \sqrt{t}\tilde{x}$, $y = \sqrt{t}\tilde{y}$, $s = ts$,

$$
|\Phi f(x, t)| \lesssim \int_0^t \int_{\mathbb{R}^3} (|x - y| + \sqrt{1 - s})^{-4} s^{m/2} (|y| + \sqrt{s})^{-2-m} dy ds \quad (2.12)
$$

$$
= t^{-1/2} \int_0^1 \int_{\mathbb{R}^3} (|\tilde{x} - \tilde{y}| + \sqrt{1 - s})^{-4} s^{m/2} (|\tilde{y}| + \sqrt{s})^{-2-m} dy ds. \quad (2.13)
$$

By $s^{m/2} \leq 1$ and Lemma 2.1 we get $|\Phi f(x, t)| \lesssim t^{-1/2} (|\tilde{x}|)^{-(2+m)}$, i.e., (2.11).

\[ \square \]

We now show Hölder estimates in space and time. Fix $0 < \theta < 1$. Denote a local parabolic Hölder estimate for $(x, t) \in \mathbb{R}^4_+$:

$$
[u]_\theta(x, t) := \sup_{\tilde{x}, \tilde{t}} \left\{ \frac{|u(x, t) - u(\tilde{x}, \tilde{t})|}{\delta^\theta} \right\} \quad \delta := |x - \tilde{x}| + \sqrt{|t - \tilde{t}|} \leq \frac{\sqrt{t}}{10}. \quad (2.14)
$$

**Lemma 2.3.** Suppose $|f(x, t)| \leq (|x| + \sqrt{t})^{-2}$ in $\mathbb{R}^4_+$. Then $\Phi f$ is locally Hölder continuous in $x$ and $t$ with any exponent $0 < \theta < 1$ and for any $T \in (1, \infty)$

$$
[\Phi f]_\theta(x, t) \leq C_T (x)^{2}, \quad \forall x \in \mathbb{R}^3, \quad \forall 1 \leq t \leq T. \quad (2.15)
$$
\textbf{Proof.} We may assume }t = 1.\\We first show spatial Hölder estimate. For }h \in \mathbb{R}^{3} \text{ with } \delta = |h| < 0.1, \text{ we have}
\begin{equation}
|\Phi f(x + h, t) - \Phi f(x, t)| \leq I_1 + I_2 \tag{2.16}
\end{equation}
\begin{equation}
I_1 \leq \int_{0}^{1} \int_{|z| > 2\delta} |h| D_{x}^{2} S(x, s) |f(x - z, 1 - s)| \, dz \, ds \tag{2.18}
\end{equation}
\begin{equation}
I_2 \leq \int_{0}^{1} \int_{|z| < 2\delta} \delta^{\theta} (|z| + \sqrt{s})^{-4 - \theta} (|x - z| + \sqrt{1 - s})^{-2} \, dz \, ds \tag{2.19}
\end{equation}
By Lemma 2.1,
\begin{equation}
I_1 \lesssim \delta^{\theta} \langle x \rangle^{-2}. \tag{2.20}
\end{equation}
For }I_2, \text{ we have } |z + h| < 3\delta. \text{ If } |x| < 4\delta, \text{ splitting } 0 < t < 1 \text{ to } 0 < t < \frac{1}{2} \text{ and } \frac{1}{2} < t < 1 \text{ and using } (2.24), \text{ we have}
\begin{equation}
I_2 \lesssim \int_{0}^{1/2} \int_{|z| < 3\delta} (|z| + \sqrt{s})^{-4} \, dz \, ds + \int_{1/2}^{1} \int_{|z| < 7\delta} (|z| + \sqrt{1 - s})^{-2} \, dz \, ds \tag{2.21}
\end{equation}
Using
\begin{equation}
\int_{0}^{1} \int_{|z| < \delta} (|z| + \sqrt{s})^{-\alpha} \, dz \, ds \lesssim \begin{cases} \delta^{5 - \alpha}, & (2 < \alpha < 5), \\
\delta^{3} \log(1/\delta), & (\alpha = 2), \end{cases} \tag{2.22}
\end{equation}
(which can be shown by splitting \((0,1) = (0,\delta^{2}) \cup [\delta^{2},1)\)), \text{ we get}
\begin{equation}
I_2 \lesssim \delta + \delta^{3} \log(1/\delta) \lesssim \delta. \tag{2.23}
\end{equation}
If }4\delta < |x|, \text{ we have (using } (2.22) \text{ again)}
\begin{equation}
I_2 \lesssim \int_{0}^{1} \int_{|z| < 3\delta} (|z| + \sqrt{s})^{-4} (|x| + \sqrt{1 - s})^{-2} \, dy \, ds \tag{2.24}
\end{equation}
\begin{equation}
\lesssim \int_{0}^{1/2} \int_{|z| < 3\delta} (|z| + \sqrt{s})^{-4} \langle x \rangle^{-2} \, dz \, ds + \int_{1/2}^{1} \int_{|z| < 3\delta} (|x|^{2} + 1 - s)^{-1} \, dz \, ds \tag{2.25}
\end{equation}
\begin{equation}
\lesssim \delta \langle x \rangle^{-2} + \delta^{3} \log \frac{1 + |x|^{2}}{|x|^{2}} \lesssim \delta \langle x \rangle^{-2}, \tag{2.26}
\end{equation}
which is also bounded by the right side of } (2.20) \text{ (and much less if } \delta \ll 1).\\We next show temporal Hölder estimate. Take }\tau = \delta^{2} \text{ with } 0 < \delta < 0.1. \text{ (For } \tau < 0 \text{ we can reverse } t \text{ and } t + \tau). \text{ We have}
\begin{equation}
|\Phi f(x, 1 + \tau) - \Phi f(x, 1)| \leq I_1 + I_2 + I_3 \tag{2.27}
\end{equation}
\begin{equation}
I_3 := \int_{0}^{1 - \tau} \int |DS(z, 1 + \tau - s) - DS(z, 1 - s)| \cdot |f(x - z, s)| \, dz \, ds \tag{2.28}
\end{equation}
\begin{align}
+ \int_{1-\tau}^{1+\tau} \int |DS(z, 1+\tau-s)| \cdot |f(x-z, s)| \, dz \, ds \\
+ \int_{1-\tau}^{1} \int |\cdot DS(z, 1-s)| \cdot |f(x-z, s)| \, dz \, ds.
\end{align}

(2.29)

For \( I_1 \), by mean value theorem and (2.4),

\begin{align}
I_1 \lesssim \int_0^{1-\tau} \int \tau(z) (|z| + \sqrt{1-s})^{-\theta} (|x-z| + \sqrt{s})^{-2} \, dz \, ds
\end{align}

(2.31)

By Lemma 2.1

\begin{align}
I_1 \lesssim \delta^\theta \langle x \rangle^{-2}.
\end{align}

(2.33)

The two terms \( I_2 \) and \( I_3 \) are similar and it suffices to estimate \( I_3 \): By (2.4),

\begin{align}
I_3 \lesssim \int_{1-\tau}^{1} \int (|z| + \sqrt{1-s})^{-4} (|x-z| + 1)^{-2} \, dz \, ds.
\end{align}

(2.34)

Integrating in \( s \) first, we get

\begin{align}
I_3 \lesssim \int_{\mathbb{R}^3} \frac{\tau}{|z|^2 + \tau} |z|^{-2} \, dz.
\end{align}

(2.35)

If \( |x| \leq 2 \), then \( |x-z| + 1 \sim \langle z \rangle \) and

\begin{align}
I_3 \lesssim \int \frac{\delta^\theta}{|z|^\gamma} |z|^{-2} \langle z \rangle^{-2} \, dz \lesssim \delta^\theta.
\end{align}

(2.36)

If \( |x| > 2 \),

\begin{align}
I_3 \lesssim \int_{|z| < |x|/2} \frac{\tau}{|z|^2 + \tau} |z|^{-2} \, dz + \int_{|z| > |x|/2} \tau |z|^{-4} |x-z|^{-2} \, dz
\end{align}

(2.37)

\begin{align}
&= C\delta |x|^{-2} + C'\delta |x|^{-3} \lesssim \delta |x|^{-2}.
\end{align}

(2.38)

The equality here is obtained by rescaling.

Finally we give a Liouville lemma.

**Lemma 2.4.** If \( v(x,t): \mathbb{R}_+^4 \to \mathbb{R}^3 \) satisfies

\begin{align}
|v(x,t)| \leq Ct^{-1/2} \langle x/\sqrt{t} \rangle^{-1-\gamma} \quad \text{for some } 0 < \gamma < 1
\end{align}

and

\begin{align}
\partial_t v - \Delta v + \nabla p = 0, \quad \text{div} \, v = 0, \quad v|_{t=0} = 0,
\end{align}

(2.39)

for some distribution \( p \), then \( v \equiv 0 \).

It is similar to [7, Lemma 4.1 (i)], with exactly the same proof.
3 A priori estimates for DSS solutions

We first recall a couple estimates for local-Leray solutions from [7].

Lemma 3.1 ([7] Lemma 3.1). There are constants $0 < C_1 < 1 < C_2$ such that the following holds. Suppose $\text{div} u_0 = 0$, $A = \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |u_0(x)|^2 dx < \infty$ for some $R > 0$ and $u$ is a local-Leray solution with initial data $u_0$. Then for $\lambda = C_1 \min (A^{-2}R^2, 1)$,

$$\text{ess sup} \sup_{0 < t < \lambda R^2} \int_{B_R(x_0)} |u(x,t)|^2 dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} |\nabla u(x,t)|^2 dx dt \leq C_2 A. \quad (3.1)$$

Because (1.1) holds for $u$, for some $p(t) = p_{x_0,R}(t)$,

$$\sup_{x_0 \in \mathbb{R}^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} |p(x,t) - p(t)|^{3/2} dx dt \leq C_2 A^{3/2} R^{3/2}. \quad (3.2)$$

Theorem 3.2 ([7] Th 3.2). Suppose $\text{div} u_0 = 0$, $\|u_0\|_{L^2_{uloc}}^2 \leq A < \infty$, and $\|u_0\|_{C^\gamma(B_0)} \leq M < \infty$ for some $\gamma \in (0,1)$. Then there exists $T = T(A, \gamma, M) > 0$ such that any local-Leray solution $u$ with initial data $u_0$ satisfies $u \in C_\text{par}(\bar{B}_{1/4} \times [0,T])$ and

$$\|u\|_{C_\text{par}^{\gamma}(\bar{B}_{1/4} \times [0,T])} \leq C(A, \gamma, M). \quad (3.3)$$

Note that in the proof of [7] Th 3.1, which is needed for [7] Th 3.2, that the pressure is in $L^{5/3}$ in time does not follow from the elliptic equation it satisfies. Rather, it follows from [7] (3.3) and that the velocity is in $L_t^{10/3}$.

We now consider DSS solutions with factor close to one.

Lemma 3.3 (A priori estimates for DSS solutions with factor close to one).

(i) For any $0 < \gamma < 1$ and $C_* > 0$, there is $\lambda_* = \lambda_*(\gamma, C_*) \in (1,2)$ such that the following hold. Suppose $1 < \lambda \leq \lambda_*$ and $u$ is a forward $\lambda$-DSS local-Leray solution of (1.1) with $\lambda$-DSS initial data $u_0 \in C^\gamma_{\text{loc}}(\mathbb{R}^3 \backslash \{0\})$ satisfying $\text{div} u_0 = 0$ and $\|u_0\|_{C^\gamma(B_2 \backslash B_1)} \leq C_*$. Then $v(x,t) := u(x,t) - (e^{t\Delta} u_0)(x)$ satisfies, for some $C = C(\gamma, C_*)$,

$$|u(x,t)| \leq \frac{C}{|x| + \sqrt{t}}, \quad |v(x,t)| \leq \frac{C}{\sqrt{t}} \langle \frac{x}{\sqrt{t}} \rangle^{-2}, \quad (x,t) \in \mathbb{R}^4_+. \quad (3.4)$$

Moreover, $u$ is a mild solution of (1.1) and (1.2).

(ii) If furthermore $\|u_0\|_{C^{1,\beta}(\bar{B}_{2} \backslash B_{1})} \leq C_*$ for some $0 < \beta < 1$, then

$$|v(x,t)| \leq \frac{C}{\sqrt{t}} \langle \frac{x}{\sqrt{t}} \rangle^{-3} \log \langle \frac{x}{\sqrt{t}} \rangle, \quad |D_x v(x,t)| \leq \frac{C}{t} \langle \frac{x}{\sqrt{t}} \rangle^{-3} \quad (3.5)$$

in $\mathbb{R}^4_+$ for some $C = C(\beta, C_*)$.

Proof. (i) We will first show a weaker estimate

$$|v(x,t)| \leq \frac{C(\gamma, C_*)}{\sqrt{t}} \langle \frac{x}{\sqrt{t}} \rangle^{1-\gamma}, \quad (x,t) \in \mathbb{R}^4_+. \quad (3.6)$$
We first consider the region below the paraboloid,

\[ t \leq \frac{|x|^2}{R_1^2}, \quad (x, t) \in \mathbb{R}^4_+, \tag{3.7} \]

for some \( R_1 = R_1(\gamma, C_*) > 0 \) sufficiently large to be decided later. By Theorem 3.2 there exists \( T_1 = T_1(\gamma, C_*) > 0 \) such that for any \( x_0 \in \mathbb{R}^3 \) with \( 1 \leq |x_0| \leq \lambda \)

\[ \|u, v\|_{\text{par}, B_{1/\theta}(x_0) \times [0,T_1]} \leq C(\gamma, C_*), \tag{3.8} \]

where we have used that \( e^{\Delta} u_0 \) satisfies the same H"older estimate. Since \( v(x, 0) = 0 \), we get

\[ |v(x, t)| \leq C(\gamma, C_*) t^{\gamma/2}, \quad \frac{8}{9} \leq |x| \leq \lambda + \frac{1}{9}, \quad 0 \leq t \leq T_1. \tag{3.9} \]

Since \( u^{(k)}, k \in \mathbb{Z} \), is another local-Leray solution with same initial data \( u_0 \), the above estimate remains valid with \( v \) replaced by \( v^{(k)} \). Scaling back,

\[ |v(x, t)| < \frac{C(\gamma, C_*) t^{\gamma/2}}{(\sqrt{t} + |x|)^{1+\gamma}}, \tag{3.10} \]

in cylinder \( C_k = \{ x : \frac{8}{9} \lambda^k \leq |x| \leq \lambda^{k+1} \} \times [0, \lambda^2 T_1] \) for every \( k \in \mathbb{Z} \). Since \( \frac{8}{9} < \lambda < 2 \), the union \( \bigcup_{k \in \mathbb{Z}} C_k \) contains a set of the form in (3.7) with \( R_1 = 2T_1^{-1/2} \).

For the complement, by rescaling it suffices to prove an upper bound in the region

\[ t > \frac{|x|^2}{R_1^2}, \quad 1 \leq t \leq \lambda^2. \tag{3.11} \]

This region satisfies \( |x| < \lambda R_1 \). Let \( T = 4 \). (We will take \( T = 4\lambda^2 \) for the proof of Lemma 3.4) By Lemma 3.1 \( u \in (L_t^\infty L_x^2 \cap L_t^{10/3} H_x^1) \subset L_{t,x}^{10/3} \) in \( B_{\lambda R_1} \times (0,T) \) with the norm bounded by \( C(\gamma, C_*) \). Together with the decay for \( e^{\Delta} u_0 \) and estimate (3.10) for \( v \) in the region (3.7), we get \( \|u\|_{L_{t,x}^{10/3}(\mathbb{R}^3 \times (0,T))} \leq C(\gamma, C_*) \). Since \( p = (-\Delta)^{-1} \partial_i \partial_j u_i u_j \), we get

\[ \|p\|_{L_{t,x}^{10/3}(\mathbb{R}^3 \times (0,T))} \leq C(\gamma, C_*). \tag{3.12} \]

Denote shrinking annuli \( A_k = \{ x \in \mathbb{R}^3 : 2\lambda R_1 + k < |x| < 2\lambda R_1 + 20 - k \} \) for \( k = 1, 2, \ldots \).

Note that \( u \in L^\infty(A_k \times [\frac{1}{2}, T]) \) since the region is contained in the region (3.7). By regularity theory for (1.1), \( D_x^\ell u \in L^\infty(A_{k+1} \times [1, T]) \) for \( \ell \leq 3 \). By \( -\Delta p = (\partial_i u_j)(\partial_j u_i) \) and (3.12), we get \( D^\ell p \in L^\infty(A_{k+1} \times [1, T]) \) for \( \ell \leq 2 \). By (1.1), \( D^\ell u_t \in L^\infty(A_{k+1} \times [1, T]) \) for \( \ell \leq 1 \).

Choose a smooth cutoff function \( \zeta(x) \geq 0 \) with \( \zeta(x) = 1 \) when \( |x| < 2\lambda R_1 + 5 \) and \( \zeta(x) = 0 \) when \( |x| > 2\lambda R_1 + 6 \). Let \( w(x, t) \) be a solution supported in \( x \in A_4 \) of

\[ \text{div } w(x, t) = u(x, t) \cdot \nabla \zeta(x), \quad \|w(\cdot, t)\|_{H^2} \lesssim \|u(\cdot, t)\|_{H^1(A_3)} \quad \forall 1 \leq t \leq T, \tag{3.13} \]

by a construction uniform in \( t \), see e.g. [5, §III.3]. In particular \( \text{div } \partial_t w = \partial_t u \cdot \nabla \zeta \) and \( \|\partial_t w(\cdot, t)\|_{H^1} \lesssim \|\partial_t u(\cdot, t)\|_{L^2(A_3)} \). Let

\[ \tilde{u} = \zeta u - w. \tag{3.14} \]

One can check that \( \tilde{u} \) is a suitable weak solution of

\[ \partial_t \tilde{u} - \Delta \tilde{u} + (\tilde{u} \cdot \nabla) \tilde{u} + \nabla \tilde{p} = f, \quad \text{div } \tilde{u} = 0, \tag{3.15} \]
satisfying

\[ \tilde{u}(x, t) = 0 \quad \text{if} \quad |x| > 2\lambda R_1 + 6; \quad f(x, t) = 0 \quad \text{if} \quad x \notin A_4, \]

\[ \|f\|_{L^\infty H^1_2(\mathbb{R}^3 \times (1, T))} \leq C(\gamma, C_\ast), \]

and by Lemma 3.3

\[ \operatorname{ess sup}_{0 < t < T} \int_{\mathbb{R}^3} |\tilde{u}(x, t)|^2 dx + \int_0^T \int_{\mathbb{R}^3} |
abla \tilde{u}|^2 dx dt < C(\gamma, C_\ast). \]

By (3.15), there exists a time \( t_1 \in [1, 3/2] \) such that \( \tilde{u}(\cdot, t_1) \in H^1(\mathbb{R}^3) \) with \( |\tilde{u}(\cdot, t_1)|_{H^1} < C(\gamma, C_\ast) \). Thus, there is \( T_2 = T_2(\gamma, C_\ast) > 0 \) and a strong solution \( \tilde{u}(x, t) : \mathbb{R}^3 \times [t_1, t_1 + T_2) \rightarrow \mathbb{R}^3 \) of (3.15) with initial condition \( \tilde{u}(x, t_1) = \bar{u}(x, t_1) \). We may assume \( T_2 < 1 \).

By weak-strong uniqueness, we have \( \bar{u}(x, t) = \tilde{u}(x, t) \) for \( (x, t) \in \mathbb{R}^3 \times (t_1, t_1 + T_2) \).

If we take \( t_2 = t_1 + T_2/4 < 2 \) and \( \lambda_2^2 = 1 + T_2/4 \), we have \( [t_2, t_2\lambda^2] \subseteq (t_1, t_1 + T_2) \) whenever \( t_1 \in [1, 3/2] \) and \( 1 < \lambda \leq \lambda_\ast \). All spatial derivatives of \( \bar{u} \) are a priori bounded in \( \mathbb{R}^3 \times [t_2, t_2\lambda^2] \). Because \( u \) is discretely self-similar and agrees with \( \bar{u} \) in the region (3.11), we get \( |u(x, t)| \leq C \) in the region (3.11).

We have shown the weaker estimate (3.6). We now show (3.4). Let \( f = -u \otimes u \). By \( |(e^{t\Delta}u_0)(x)| \lesssim \langle t^{-1/2} \rangle^{-1} \) and (3.6),

\[ |f(x, t)| \lesssim \frac{1}{x^2 + t}, \quad (x, t) \in \mathbb{R}^4_+. \]

By Lemma 2.2 (with \( m = 0 \)), we get \( |\Phi f(x, t)| \lesssim \langle t^{-1/2} \rangle^{-2} \), where \( \Phi \) is defined in (2.2). Note \( \tilde{v} = v - \Phi f \) satisfies the bound (3.6) and the linear Stokes system in \( \mathbb{R}^4 \) with zero initial data and zero source. By Lemma 2.3 \( \tilde{v} \equiv 0 \). Thus we have \( v = \Phi f \) and (3.4).

(ii) Suppose now \( u_0 \) is in \( C^{1, \beta}_{loc} \) and \( \|\nabla u_0\|_{C^{\beta}(\overline{B(x_0, 1/2)})} \leq C_\ast \) for any \( x_0 \in \mathbb{R}^3 \) with \( 1 < |x_0| \leq \lambda \). The vorticity \( \omega = \operatorname{curl} u \) satisfies

\[ \partial_t \omega - \Delta \omega = -\operatorname{curl} \nabla \cdot (u \otimes u), \]

and \( \omega_0 = \omega(\cdot, 0) \) satisfies \( \|\omega_0\|_{C^{\beta}(\overline{B(x_0, 1/2)})} \leq C_\ast \).

Denote \( r_k = \frac{1}{k} - \frac{k}{100} \) and \( Q_k = B(x_0, r_k) \times (0, T_1) \) for \( k = 0, 1, \ldots \). The previous step shows \( u \in C^\beta_{par}(Q_0) \).

Let \( \eta(x) = \eta_0(x - x_0) \) where \( \eta_0(x) \) is a fixed smooth cut-off function with \( \eta_0(x) = 1 \) for \( |x| < r_2 \) and \( \eta_0(x) = 0 \) for \( |x| > r_1 \). Decompose \( \omega = \omega_1 + \omega_2 + \omega_3 \) where

\[ \omega_1(\cdot, t) = -\int_0^t e^{\Delta(t-s)}(\operatorname{curl} \nabla \cdot (u \otimes u\eta))(\cdot, s) ds, \]

\[ \omega_2(\cdot, t) = e^{t\Delta}(\eta\omega(\cdot, 0)), \]

\[ \omega_3 = \omega - \omega_1 - \omega_2. \]

Note \( (\partial_t - \Delta)\omega_3 = 0 \) in \( Q_2 \) and \( \omega_3(x, 0) = 0 \) for \( x \in B_{r_2}(x_0) \). Thus, with possibly a smaller \( T_1 \), both \( \omega_2 \) and \( \omega_3 \) are bounded in \( C^\beta_{par}(Q_3) \) by \( C(\beta, C_\ast) \). By singular integral estimates for heat equation, we have

\[ \|\omega_1\|_{L^q(\mathbb{R}^3 \times [0, T_1])} \leq C \|u \otimes u\eta\|_{L^q(\mathbb{R}^3 \times [0, T_1])} \leq C(\beta, C_\ast, q) \]

for any \( 1 < q < \infty \). We take \( q = \frac{5}{1 - \beta/2} \). Thus \( \omega = \sum_{i=1}^3 \omega_i \in L^q(Q_3) \). By elliptic estimate,

\[ \|\nabla u\|_{L^q(Q_4)} \leq C \|\operatorname{curl} u\|_{L^q(Q_3)} + C \|\div u\|_{L^q(Q_3)} + C \|u\|_{L^q(Q_4)} \leq C(\beta, C_\ast). \]
We now do a similar decomposition of \( \omega \) with \( \eta_0(x) = 1 \) for \( |x| < r_6 \) and \( \eta_0(x) = 0 \) for \( |x| > r_5 \). Again \( \omega_2 \) and \( \omega_3 \) are in \( C^0_{par}(Q_T) \). Rewrite
\[
\nabla \cdot (u \otimes u \eta) = u \cdot \nabla (u \eta) \in L^q(\mathbb{R}^3 \times [0,T_1]).
\]
(3.24)
By heat potential estimate
\[
[w_{1}]_{C^0_{par}(\mathbb{R}^3 \times [0,T_1])} \leq C \| u \cdot \nabla (u \eta) \|_{L^q(\mathbb{R}^3 \times [0,T_1])} \leq C(\beta, C_*) \text{.}
\]
(3.25)
Thus \( \omega = \sum_{i=1}^3 \omega_i \in C^0_{par}(Q_T) \). By elliptic estimate,
\[
\| \nabla u \|_{L^\infty(0,T; C^3(B(x_0, r_0)))} \leq C \| \nabla u \|_{C^3(B(x_0, r_0))},
\]
(3.26)
where the middle norms are \( C^3_{par}(Q_T) \)-norms. In particular we have shown
\[
|\nabla u(x,t)| \leq C(\beta, C_*) \text{,} \quad (1 \leq |x| \leq \lambda, \quad 0 \leq t \leq T_1).
\]
(3.27)
By the same scaling argument for (3.10), we get
\[
|\nabla u(x,t)| \leq \frac{C(\beta, C_*)}{(\sqrt{t} + |x|)^2}
\]
(3.28)
in the sub-paraboloid region \( (3.7) \).

We now rewrite
\[
v_i(x,t) = \int_0^t \int_{\mathbb{R}^3} S_{ij}(x - y, t - s) g_j(y, s) \, dy \, ds
\]
(3.29)
with
\[
g = -u \cdot \nabla u, \quad |g(y, s)| \leq \frac{C(\beta, C_*)}{(\sqrt{s} + |y|)^3}.
\]
(3.30)
Thus for \( \ell = 0, 1 \), by (2.3) and change of variables \( x = \sqrt{t} \bar{x}, \ y = \sqrt{t} \bar{y}, \ s = t \bar{s} \),
\[
|D^\ell v(x,t)| \leq \int_0^t \int_{\mathbb{R}^3} (|x - y| + \sqrt{t} - s)^{-3-\ell} (\sqrt{s} + |y|)^{-3} \, dy \, ds
\]
(3.31)
\[
= t^{-1+\ell/2} \int_0^t \int_{\mathbb{R}^3} \left( \frac{|x - y|}{\sqrt{t}} + 1 - s \right)^{-3-\ell} (\sqrt{s} + |y|)^{-3} \, dy \, ds.
\]
(3.32)
By Lemma 2.1, we get (3.5).

We next consider axisymmetric DSS flow with no swirl.

**Lemma 3.4** (A priori estimates for axisymmetric DSS flow with no swirl),
\[(i) \text{ For any } 1 < \lambda < \infty, \ 0 < \gamma < 1, \text{ and } C_* > 0, \text{ suppose } u \text{ is a forward } \lambda \text{-DSS local-Leray solution of (1.1) with } \lambda \text{-DSS initial data } u_0 \in C^\infty_{loc}(\mathbb{R}^3 \setminus \{0\}) \text{ that is axisymmetric with no swirl, DSS with factor } \lambda, \text{ div } u_0 = 0, \text{ and } \|u_0\|_{C^\gamma(\overline{B}_1 \setminus B_1)} \leq C_* \text{. Then } u \text{ satisfies (3.1) with constant } C = C(\lambda, \gamma, C_*). \text{ Moreover, } u \text{ is a mild solution of (1.1) and (1.2).}
\]
(ii) If furthermore \( \|u_0\|_{C^{1,\beta}(\overline{B}_1 \setminus B_1)} \leq C_* \) for some \( 0 < \beta < 1 \), then (3.5) hold with constant \( C = C(\lambda, \beta, C_*) \).
\]
Proof. We use the same proof of Lemma 3.3 (i) until we get a time \( t_1 \in [1, 3/2] \), a \( T_2 = T_2(\gamma, C_s) \in (0, 1) \) and that \( \tilde{u} \) agrees with a strong solution in \([t_1, t_1 + T_2)\). For \( t_2 = t_1 + T_2/4 \), all spatial derivatives of \( \tilde{u}(x, t_2) \) are a priori bounded. Since \( \tilde{u} \) has compact spatial support, we get
\[
\|\tilde{u}(\cdot, t_2)\|_{H^2(\mathbb{R}^3)} \leq C(\gamma, C_s). \tag{3.33}
\]
By [11, Th 1],
\[
\|\tilde{u}(\cdot, t)\|_{H^1(\mathbb{R}^3)} \leq C(\gamma, C_s), \quad \|\tilde{u}(\cdot, t)\|_{H^2(\mathbb{R}^3)} \leq C(\gamma, C_s, t), \tag{3.34}
\]
for \( t_2 \leq t \leq T = 4 \lambda^2 \), which contains the interval \([t_2, t_2 \lambda^2] \). Since \( u \) is \( \lambda \)-DSS and \( u \) agrees with \( \tilde{u} \) in the region \( t \geq |x|^2/R_1^2 \) and \( 1 \leq t \leq T \), we have shown the boundedness of \( u \) in the same region. The rest of the proof of Lemma 3.3 then goes through. \( \square \)

4 Uniqueness of DSS solutions with small data

When the initial data \( u_0 \) is small in \( L^{3, \infty}(\mathbb{R}^3) \), the existence theorem of [6, 33, 4] says that there is a unique mild solution \( u_{\text{mild}}(x, t) \) in the class
\[
\|u_{\text{mild}}(t)\|_{BC_w([0, \infty); L^{3, \infty}(\mathbb{R}^3))] \leq C\|u_0\|_{L^{3, \infty}(\mathbb{R}^3)}.
\tag{4.1}
\]
Above \( BC_w \) means bounded weak-star continuous \( L^{3, \infty} \)-valued functions of time. It is also known that \( u_{\text{mild}} \) is a local-Leray solution, see [12]. However, for our application later, we will need uniqueness in a larger class of solutions.

Lemma 4.1. Let \( u_0 \) satisfy the assumptions of Lemma 3.3 or Lemma 3.4 with \( C_s \) sufficiently small. Then any \( \lambda \)-DSS local-Leray solution \( u(t) \) of (1.1) with initial data \( u_0 \) must agree with the mild solution \( u_{\text{mild}} \) constructed by [6, 33, 4].

In particular, \( u(t) \) is allowed to be large in \( BC_w([0, \infty); L^{3, \infty}(\mathbb{R}^3)) \). Such a statement that “large equals small” is not known for general solutions with small \( L^{3, \infty} \) data. The lemma is only for DSS solutions and relies on the estimates of Lemmas 3.3 and 3.4.

Proof. Denote \( W(x, t) = u_{\text{mild}}(x, t) \) and \( v(x, t) = u(x, t) - u_{\text{mild}}(x, t) \). They are both DSS with factor \( \lambda \) and satisfy (3.4). Thus
\[
|W(x, t)| < \frac{C}{(|x| + \sqrt{t})}, \quad |v(x, t)| < \frac{C\sqrt{t}}{(|x| + \sqrt{t})^2}, \tag{4.2}
\]
in \( \mathbb{R}^4_+ \). Note that \( v \) satisfies
\[
\partial_t v - \Delta v + (W + v) \cdot \nabla v + v \cdot \nabla W + \nabla p = 0, \quad \text{div } v = 0. \tag{4.3}
\]
We have \(-\Delta p = \sum_{i,j} \partial_i \partial_j ((W + v)_i v_j + v_i W_j) \) and hence \(|p(\cdot, t)|_{L^q(\mathbb{R}^3)} \lesssim \|(x)^{-2} \|_{L^q(\mathbb{R}^3)} < \infty \) for \( 1 \leq t \leq \lambda^2 \) and \( 1 < q < \infty \).

Let \( \zeta_R(x) = \zeta(x/R) \) where \( \zeta(x) \geq 0 \) is a smooth cut off function with \( \zeta(x) = 1 \) for \(|x| < 1 \) and \( \zeta(x) = 0 \) for \(|x| > 2 \). Multiplying (4.3) by \( v \zeta_R \) and integrating by parts over \( \mathbb{R}^3 \times [1, \lambda^2] \), we get
\[
\left[ \int_{\mathbb{R}^3} \frac{|v(x, t)|^2}{2} \zeta_R(x) \, dx \right]_{t=1}^{\lambda^2} + \int_1^{\lambda^2} \int_{\mathbb{R}^3} (|\nabla v|^2 \zeta_R - v \otimes W : (\nabla v) \zeta_R) \, dx dt = I_R \tag{4.4}
\]
\[ I_R := \int_1^{\lambda^2} \int_{\mathbb{R}^3} \frac{1}{2} |v|^2 \Delta \zeta_R + \left( \frac{|v|^2}{2} (W + v) + (W \cdot v + p)v \right) \cdot \nabla \zeta_R \, dx \, dt. \] (4.5)

By Lemma 3.3 and \( p \in L_1^\infty L_t^q \), \( I_R \) converges to 0 as \( R \to \infty \). The first term in (4.4) also converges and it converges to
\[ \left[ \int_{\mathbb{R}^3} \frac{|v(x,t)|^2}{2} \, dx \right]_{t=1}^{\lambda^2} = \frac{\lambda - 1}{2} \int |v(x,1)|^2 \, dx \geq 0. \] (4.6)

We have shown
\[ \int_1^{\lambda^2} \int_{\mathbb{R}^3} (|\nabla v|^2 \zeta_R - v \otimes W : (\nabla v) \zeta_R) \, dx \, dt \leq o(1) \] (4.7)
as \( R \to \infty \). Since
\[ \left| \int \int v \otimes W : (\nabla v) \zeta_R \, dx \, dt \right| \leq \frac{1}{2} \int \int |\nabla v|^2 \zeta_R + C \int \int |v W|^2 \zeta_R \] (4.8)
and the last term converges as \( R \to \infty \), we get \( \int \int |\nabla v|^2 \zeta_R \to \int \int |\nabla v|^2 < \infty \) as \( R \to \infty \) and by Hardy inequality
\[ \int \int |\nabla v|^2 \leq \int \int v \otimes W : \nabla v \lesssim C_* \left( \int \int \frac{|v|^2}{x^2} \right)^{\frac{1}{2}} \left( \int \int |\nabla v|^2 \right)^{\frac{1}{2}} \lesssim C_* \int \int |\nabla v|^2. \] (4.9)
Thus when \( C_* \) is sufficiently small we get \( \int_1^{\lambda^2} \int_{\mathbb{R}^3} |\nabla v|^2 \, dx \, dt = 0 \). Thus \( v \equiv 0 \).

5 Existence of large DSS solutions

In this subsection we prove Theorems 1.1 and 1.2. Let
\[ U(x,t) = (e^{i \Delta} u_0)(x). \] (5.1)

By the assumption on \( u_0 \), \( U \) is \( \lambda \)-DSS and
\[ |U(x,t)| \leq \frac{C C_*}{\sqrt{t}} \left( \frac{x}{\sqrt{t}} \right)^{-1} = \frac{C C_*}{\sqrt{x^2 + 2t}}. \] (5.2)

Introduce a parameter \( \sigma \in [0,1] \). We look for a solution \( u(x,t) \) of (1.1) of the form
\[ u(x,t) = \sigma U(x,t) + v(x,t), \quad u(x,0) = \sigma u_0(x). \] (5.3)
The difference \( v \) satisfies the nonhomogeneous Stokes system (2.1) with
\[ f = - (\sigma U + v) \otimes (\sigma U + v). \] (5.4)
We expect (which is true at least for small \( \sigma \)) that \( v \) is \( \lambda \)-DSS and
\[ f(x,t) = \lambda^3 f(\lambda x, \lambda^2 t), \quad \forall (x,t) \in \mathbb{R}^4_+. \] (5.5)
In view of Lemmas 3.3 and 3.4, we also expect
\[
|v(x,t)| \lesssim \frac{1}{\sqrt{t}} \left( \frac{x}{\sqrt{t}} \right)^{-1-\gamma}, \quad |f(x,t)| \lesssim \frac{1}{t} \left( \frac{x}{\sqrt{t}} \right)^{-2} = \frac{1}{x^2 + 2t}.
\] (5.6)
The decay rate of \( v \) can be improved but it is not needed.

We now set up the framework for the application of the Leray-Schauder theorem. Let
\[
Q = \mathbb{R}^3 \times [1, \lambda^2],
\] (5.7)
and the Banach space \( X = X(\lambda) \):
\[
X = \left\{ v \in C(Q; \mathbb{R}^3) : \quad \text{div} \ v = 0, \quad \|v\|_X < \infty, \quad v(x,1) = \lambda v(\lambda x, \lambda^2), \quad \forall x \in \mathbb{R}^3 \right\},
\] (5.8)
where
\[
\|v\|_X := \sup_{(x,t) \in Q} \langle x \rangle^{1+\gamma} |v(x,t)|.
\] (5.9)
For each \( v \in X \), we define its DSS extension by
\[
Ev(x,t) = \lambda^k v(\lambda^k x, \lambda^{2k} t), \quad \text{for } (x,t) \in \mathbb{R}^4_+, \quad (5.10)
\]
where \( k \) is the unique integer so that \( 1 \leq \lambda^{2k} t < \lambda^2 \).

We now define an operator \( K : X \times [0,1] \to X \) by
\[
K(v, \sigma) := -\Phi((\sigma U + Ev) \otimes (\sigma U + Ev))|_{Q}, \quad \forall v \in X, \quad \forall \sigma \in [0,1].
\] (5.11)
Above \( \Phi \) is defined by (2.2). Note that for \( v \in X \) with \( \|v\|_X < M \) and \( 0 \leq \sigma \leq 1 \), the force \( f = -(\sigma U + Ev) \otimes (\sigma U + Ev) \) satisfies (5.5) and (5.6), and \( \Phi(f) \) defined by (2.2) satisfies (2.1) and is \( \lambda \)-DSS. By Lemma 2.4, its restriction to \( Q , K(v, \sigma) = -\Phi(f)|_{Q} \), is inside \( X \) and \( \|K(v, \sigma)\|_X \leq C(C_* + M)^2 \). Thus \( K \) indeed maps bounded sets in \( X \times [0,1] \) into bounded sets in \( X \).

Furthermore, \( K \) is compact because its main term \( \Phi(\sigma U \otimes \sigma U)|_{Q} \) is one dimensional while the other terms of \( K \) have extra decay by Lemma 2.4 and are Hölder continuous in \( x \) and \( t \) by Lemma 2.3.

We have now a fixed point problem
\[
v = K(v, \sigma) \quad \text{in } X
\] (5.12)
that satisfies the following:

1. \( K(v, \sigma) : X \times [0,1] \to X \) is compact by the previous discussion,

2. it is uniquely solvable in \( X \) for small \( \sigma \) by (6.3) and Lemma 4.1 thus the Leray-Schauder degree is nonzero, and

3. we have a priori estimate in \( X \) for solutions \( v \) of (5.12) uniformly for all \( \sigma \in [0,1] \) by Lemma 3.3 or 3.4.

By Leray-Schauder degree theorem (see e.g. [13]), there is a solution \( v \in X \) of (5.12) with \( \sigma = 1 \). It follows that \( Ev \) satisfies (2.1) with \( f = -(U + Ev) \otimes (U + Ev) \), and hence \( u = U + Ev \) is a \( \lambda \)-DSS local-Leray solution of (1.1) with initial data \( u_0 \). \( \square \)
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