A note on centering in subsample selection for linear regression

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Centring is a commonly used technique in linear regression analysis. With centred data on both the responses and covariates, the ordinary least squares estimator of the slope parameter can be calculated from a model without the intercept. If a subsample is selected from a centred full data, the subsample is typically uncentred. In this case, is it still appropriate to fit a model without the intercept? The answer is yes, and we show that the least squares estimator on the slope parameter obtained from a model without the intercept is unbiased and it has a smaller variance covariance matrix in the Loewner order than that obtained from a model with the intercept. We further show that for noninformative weighted subsampling when a weighted least squares estimator is used, using the full data weighted means to relocate the subsample improves the estimation efficiency.

KEYWORDS
estimation efficiency, ordinary least squares, variance, weighted least squares

1 | INTRODUCTION

In recent years, due to the challenge of rapidly increasing volumes of data, one may have to select a small subsample from the full data so that available computational resources at hand can fully analyse the subsample and useful information can be drawn. For example, the data from the second strategic highway research program naturalistic driving study are over two million gigabytes (Dingus et al., 2015), and existing analyses are only done on a small proportion of it. As another example, the intelligent research in sight registry (Parke et al., 2017), as of 2022, has aggregated data on over four hundred million patient visits, and it is still growing rapidly every day. An analysis on the full data would not be possible for most practitioners due to the super large data volume and the relatively limited computing resources. Subsampling is also an important tool in a wide range of modern machine learning platforms. For instance, large-scale recommender systems such as YouTube and TikTok may receive over billions of training data points each day, creating tremendous difficulty to effectively train online models. Subsampling is often employed to reduce the data intensity so that online models can be updated in time (Wang et al., 2021).

Investigations on subsampling have been fruitful for linear regression. A popular technique is to use statistical leverage scores or their variants to construct subsampling probabilities; see Drineas et al. (2012), Ma et al. (2015), Yang et al. (2015), Nie et al. (2018), and the references therein. Wang et al. (2019) proposed the information-based optimal subdata selection (IBOSS) method, and the proposed deterministic selection algorithm has a high estimation efficiency and a linear computational time complexity. Pronzato and Wang (2021) developed an online subsample selection algorithm that achieves the optimal variance under general optimality criteria. Yu and Wang (2022) recommended using leverage scores to select subsamples deterministically.

The slope parameter is often the main focus, and the intercept may not be of interest when fitting a linear regression model. In this scenario, a widely used trick to simplify the calculation is to centre the data so that a linear model without the intercept can be used to calculate the slope estimator. If the intercept is needed, for example, for prediction, it can be calculated using the mean response and the means of the covariates together with the slope estimator. To be specific, consider the following linear regression model for the full data \( D_n = (X, y) \) of sample size \( n \),

\[
y = X\theta + \epsilon = \alpha 1_n + X\beta + \epsilon,
\]

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where \( \mathbf{y} = (y_1, ..., y_n)^\top \) is the response vector, \( \mathbf{Z} = (\mathbf{1}_n, \mathbf{X})\), \( \theta = (\alpha, \beta)\) with \( \alpha \) and \( \beta \) being the intercept and slope vector, respectively, \( \mathbf{e} = (e_1, ..., e_n)^\top \) is the model error satisfying \( \mathbb{E}(\mathbf{e}) = \mathbf{0} \) and \( \mathbb{V}(\mathbf{e}) = \sigma^2 \mathbf{I}_n \). \( \mathbf{1}_n \) is an \( n \times 1 \) vector of ones, and \( \mathbf{I}_n \) is the \( n \times n \) identity matrix.

To estimate \( \theta \), the ordinary least squares (OLS) estimator using the full data \( \mathcal{D}_n \) is

\[
\hat{\theta} = (\tilde{\alpha} \tilde{\beta})^{\top} = (\mathbf{Z}^\top \mathbf{Z})^{-1} \mathbf{Z}^\top \mathbf{y}.
\]

The mean response is \( \mathbf{\bar{y}} = n^{-1} \mathbf{1}_n \mathbf{y} \), and the vector of column means for \( \mathbf{X} \) is \( \mathbf{r} = n^{-1} \mathbf{X}^\top \mathbf{1}_n \), so the centred data can be written as

\[
\mathbf{X}_c = \mathbf{X} - 1_n \mathbf{r}^\top = (\mathbf{1}_n - J_n) \mathbf{X},
\]

\[
\mathbf{y}_c = \mathbf{y} - 1_n \mathbf{y} = (\mathbf{1}_n - J_n) \mathbf{y},
\]

where \( J_n = n^{-1} 1_n 1_n^\top \) and the notation \( \circ \) means \( \mathbf{v} \circ \mathbf{w} = \mathbf{v}^\top \mathbf{w}^\top \) for a vector or matrix \( \mathbf{v} \). With centred data, it is well known that \( \hat{\beta} \) can be calculated as

\[
\hat{\beta} = (\mathbf{X}_c^\top \mathbf{X}_c)^{-1} \mathbf{X}_c \mathbf{y}_c.
\]

and if \( \alpha \) is of interest, \( \tilde{\alpha} = \mathbf{\bar{y}} - \mathbf{\bar{X}}^\top \hat{\beta} \).

An interesting question raises for centring in subsampling: If the full data are centred, do we have to centre the subsample to calculate the slope estimate if the model does not contain an intercept? We will show in this short note that it is better to not centre the subsample in this case. Since for a deterministically selected subsample, the OLS is applied (e.g. Pronzato & Wang, 2021; Wang et al., 2019), while for a randomly selected subsample with nonuniform probabilities, the weighted least squares (WLS) are often fitted (e.g., Ai et al., 2021; Yang et al., 2015; Zhang et al., 2021), we discuss these two types of estimators in Sections 2 and 3, respectively. Some numerical evaluations are provided in Section 4, and more technical details are given in Appendix A.

## 2 | DETERMINISTIC SELECTION WITH OLS

Let \( (\mathbf{X}^\ast, \mathbf{y}^\ast) \) denote the subsample of size \( r \) corresponding to the uncentred full data \( (\mathbf{X}, \mathbf{y}) \), and \( (\mathbf{X}_c^\ast, \mathbf{y}_c^\ast) \) be the subsample corresponding to the centred full data \( (\mathbf{X}_c, \mathbf{y}_c) \), that is, \( (\mathbf{X}_c^\ast) = \mathbf{X}^\ast - 1_n \mathbf{1}_c^\top \mathbf{y}_c^\ast = \mathbf{y}^\ast - 1_n \mathbf{y} \). In this section, we assume that the selection rule is non-random, and it may depend on \( \mathbf{X} \), but it does not depend on the response \( \mathbf{y} \). This type of subsampling methods includes the IBOSS that focuses on first-order linear regression models (Wang et al., 2019), the sequential online thinning that is designed for online streaming data (Pronzato & Wang, 2021), optimal design subsampling (Deldossi & Tommasi, 2021), and deterministic leverage score selection for model discrimination (Yu & Wang, 2022), among others. Subsampling methods in this category aim at estimating the true parameter, and they have a higher estimation efficiency in general, but they require strong assumptions on the correctness of the model so they should only be used when a linear regression model fits the data well. A subsample selected using this class of methods follows a linear regression model:

\[
\mathbf{y}^\ast = \mathbf{Z}^\ast \theta + \mathbf{e}^\ast = \alpha \mathbf{1}_r + \mathbf{X}^\ast \beta + \mathbf{e}^\ast,
\]

where \( \mathbf{Z}^\ast = (\mathbf{1}, \mathbf{X}^\ast) \), \( \mathbb{E}(\mathbf{e}^\ast) = \mathbf{0} \), \( \mathbb{V}(\mathbf{e}^\ast) = \sigma^2 \mathbf{I}_r \), \( \mathbf{1}_r \) is a \( r \times 1 \) vector of ones and \( \mathbf{I}_r \) is the \( r \times r \) identity matrix. The OLS based on the subsample is

\[
\hat{\theta} = (\tilde{\alpha} \tilde{\beta})^{\top} = (\mathbf{Z}^\ast \mathbf{Z}^\ast)^{-1} \mathbf{Z}^\ast \mathbf{y}^\ast.
\]

Clearly, \( \mathbf{X}_c^\ast \) and \( \mathbf{y}_c^\ast \) may not be centred; that is, their sample means are not zero. Can we simply use \( (\mathbf{X}_c^\ast, \mathbf{y}_c^\ast) \) to fit a model without the intercept to estimate \( \beta \), that is, use

\[
\hat{\beta}_c = (\mathbf{X}_c^\ast \mathbf{X}_c^\ast)^{-1} \mathbf{X}_c^\ast \mathbf{y}_c^\ast
\]

to estimate \( \beta \)? The answer is yes. Here, \( \hat{\beta}_c \) is not only unbiased but also has a smaller variance compared with \( \hat{\beta} \) in (7).

The unbiasedness of \( \hat{\beta}_c \) has been noticed in Yu and Wang (2022). We show it here for completeness. Note that

\[
\mathbf{y}_c^\ast = \mathbf{y}^\ast - \mathbf{\bar{y}} \mathbf{1}_r = \alpha \mathbf{1}_r + \mathbf{X}^\ast \beta + \mathbf{e}^\ast - (\alpha + \mathbf{\bar{X}}^\top \hat{\beta} + \mathbf{\bar{e}}) \mathbf{1}_r = \mathbf{X}_c^\ast \beta + \mathbf{e}^\ast - \mathbf{\bar{e}} \mathbf{1}_r,
\]

where \( \mathbf{\bar{e}} \) is the average of \( e_1, ..., e_n \) and therefore \( \mathbf{\bar{e}} \sim \mathcal{N}(0, n^{-1} \sigma^2) \). We then know that

\[
\mathbb{E}(\mathbf{\bar{e}} \mathbf{1}_r) = 0 \quad \text{and} \quad \mathbb{V}(\mathbf{\bar{e}} \mathbf{1}_r) = n^{-1} \sigma^2 \mathbf{1}_r \mathbf{1}_r^\top.
\]
\[
\hat{\beta}_c = (X_c^*X_c^*)^{-1}X_c^*(\beta + e^* - r1) = \beta + (X_c^*X_c^*)^{-1}X_c^*(e^* - r1).
\]  

(10)

Thus, we have the unbiasedness \(E(\hat{\beta}) = \beta\) from the above representation.

The following proposition shows that \(\hat{\beta}_c\) has a smaller variance than \(\hat{\beta}\) in the Loewner order.

**Proposition 1.** Assume that \(Z^*\) is full rank. Let \(X^*\) be the vector of subsample covariate means, that is, \(X^* = r^{-1}X^T1_r\). The variances of \(\hat{\beta}_c\) and \(\hat{\beta}\) satisfy that

\[
\text{Var}(\hat{\beta}|X) - \text{Var}(\hat{\beta}_c|X) = \sigma^2 \left( \frac{r}{1 - d} + \frac{r^2}{n} \right) (X_c^*X_c^*)^{-1}(X^* - \bar{x})^2(X_c^*X_c^*)^{-1},
\]

where \(d = \frac{r(X^* - \bar{x})^T(X_c^*X_c^*)^{-1}(X^* - \bar{x})}{r - 1} < 1\).

**Remark 1.** The smaller variance of \(\hat{\beta}_c\) indicates that even if the full data are not centred, it would be better to shift the subsample by \((\bar{x},\bar{y})\) and then fit a model without the intercept than fitting a model with an intercept directly.

**Remark 2.** The matrix on the right hand side of (11) is of rank one, so we do not expect the difference between \(\text{Var}(\hat{\beta}|X)\) and \(\text{Var}(\hat{\beta}_c|X)\) to be large, especially when the dimension of \(\beta\) is high. Nevertheless, we recommend \(\hat{\beta}_c\) because its calculation is as easy as that of \(\hat{\beta}\).

If the intercept \(\alpha\) is of interest, it can be estimated by using the subsample means \(\bar{x}^*\) and \(\bar{y}^*\). However, using the full data means \(\bar{x}\) and \(\bar{y}\) is usually significantly more efficient. This is also observed in the numerical examples of Wang et al. (2019). Specifically, the estimator

\[
\hat{\alpha}_c = \bar{y} - X^T\hat{\beta}_c
\]

(12)

is typically much more efficient than \(\hat{\alpha}\) defined in (7). By direct calculations, we obtain that

\[
\text{Var}(\hat{\alpha}_c|X) = \sigma^2 \left\{ \frac{1}{n} + \bar{x}^T\text{Var}(\hat{\beta}_c|X)\bar{x} \right\}
\]

and

\[
\text{Var}(\hat{\alpha}|X) = \sigma^2 \left\{ \frac{1}{r} + \bar{x}^2\text{Var}(\hat{\beta}|X)\bar{x}^2 \right\}.
\]

Wang et al. (2019) have shown that \(\text{Var}(\hat{\beta}|X)\) converges to zero faster than \(r^{-1}\) if the support of the covariate distribution is not bounded. For this scenario, the dominating term in \(\text{Var}(\hat{\alpha}_c|X)\) is often \(\bar{x}^2\text{Var}(\hat{\beta}_c|X)\bar{x}^2\), which converges to zero faster than \(\sigma^2 r^{-1}\), while the dominating term in \(\text{Var}(\hat{\alpha}|X)\) is \(\sigma^2 r^{-1}\).

### 3 | NONUNIFORM RANDOM SUBSAMPLING WITH WLS

A large class of subsample selection methods are through nonuniform random sampling such as the leverage sampling and its variants (Ma et al., 2015; Yang et al., 2015), robust active sampling (Nie et al., 2018), and optimal sampling (Zhang et al., 2021). In this scenario, the inverse probability WLS approach is typically applied on the subsample, and the subsample estimator is often proposed as an “estimator” of the full data estimator. Subsampling methods in this categorical may not be as efficient as the deterministic selection methods in the previous section if the assumed linear model is the data generating model, but they require weaker assumptions on the correctness of the model. For this type of approaches, the exact variance of the resulting estimator may not be defined, so our discussions are on the asymptotic variance, which we use \(\text{Var}_a\) to denote. Properties of random subsampling estimators are more complicated, and we focus on the scenario that \(r = o(n)\) so that the contribution of the randomness from the full data to the asymptotic variance is negligible.

Assume that a subsample of size \(r\) is randomly selected according to nonuniform probabilities \(x_1, ..., x_n\), where \(x_i\) is the probability that the \(i\)th observation is selected in each sample draw. For leveraging sampling, \(x_i\)’s are proportional to statistical leverage scores; for robust active sampling and optimal sampling, \(x_i\)’s are derived to minimize the asymptotic mean squared error under mis-specified and correctly specified models, respectively. Here, we abuse the notations and use \(X^*, y^*, e^*\), and \(Z^*\) again to denote subsample quantities. We need to point out that (6) does not hold for a randomly selected subsample.

Let \(w = (w_1, ..., w_n)^T\) be the vector of weights where \(w_i\)’s are proportional to \(x_i^{-1}\). To ease the discussion, we assume that \(\|w\| = 1\). Let \(W\) be the corresponding \(n \times n\) diagonal weighting matrix, that is, \(w = W1_n\), and let \(w^*\) and \(W^*\) be the weighting vector and matrix, respectively, for the selected subsample. The WLS estimator is
\[ \hat{\theta}_w = (\bar{x}_w \hat{\beta}_w)^T = \left( Z^T W^* Z^* \right)^{-1} Z^T W^* y^*. \] (13)

It has been shown that \( \hat{\theta}_w \) is asymptotically unbiased towards the full data OLS \( \hat{\theta} \), and its asymptotic variance has been derived in the literature (see Ma et al., 2015; Wang et al., 2022; Yu et al., 2022, etc.). In our notation, the asymptotic variance of \( \hat{\theta}_w \) given \( D_n \) is

\[ \text{Var}_a(\hat{\theta}_w|D_n) = \frac{C}{r} \left( Z^T Z \right)^{-1} Z^T W E^2 Z \left( Z^T Z \right)^{-1}, \] (14)

where \( C = \sum_{i=1}^n w_i^{-1} \) and \( E = \text{diag}(e_1, \ldots, e_n) \) with \( e_i \) being the residuals from the full data OLS estimator. Note that \( Z^T W E^2 Z = \sigma^2 \sum_{i=1}^n w_i e_i^2 z_i^T \), so if the weights \( w_i \) do not involve \( e_i \)’s, then \( Z^T W E^2 Z = \sigma^2 \sum_{i=1}^n w_i z_i^T \left( 1 + o_P(1) \right) \) under reasonable conditions. Thus, the asymptotic variance can be written as

\[ \text{Var}_a(\hat{\theta}_w|D_n) = \frac{C r^2}{r} \left( Z^T Z \right)^{-1} Z^T W Z \left( Z^T Z \right)^{-1}, \] (15)

from which we obtain that

\[ \text{Var}_a(\hat{\beta}_w|D_n) = \frac{C r^2}{r} \left( X^T X_n \right)^{-1} X^T W X_n \left( X^T X_n \right)^{-1}. \] (16)

If the centred data are sampled and used to construct an estimator of \( \beta \) directly

\[ \hat{\beta}_{w,c} = \left( X^c W^* X^c \right)^{-1} X^c W^* y^c, \] (17)

then since the full data means \( \bar{x} \) and \( \bar{y} \) are nonrandom functions of the full data \( D_n \), existing results (e.g., Ai et al., 2021; Yu et al., 2022; Wang et al., 2022) are applicable to \( \hat{\beta}_{w,c} \). This tells us that \( \hat{\beta}_{w,c} \) is asymptotically unbiased towards \( \beta \), and its asymptotic variance is the same as that of \( \hat{\beta}_w \) shown in (16). Thus, for noninformative random subsampling with WLS, if the original full data are centred, we can ignore the intercept as well.

Interestingly, if we use weighted means of the full data to relocate the subsample, we have an improved estimator of \( \beta \). Denote \( \bar{y}_w = W^* \bar{y} \) and \( \bar{x}_w = X^T W^* \bar{x} \) as the weighted mean response and the weighed mean covariate vector, respectively. Let \( y_{w,c} = y - \bar{y}_w 1_n \) and \( X_{w,c} = X - 1_n \bar{x}_w \) be the centred response vector and design matrix using the weighted means, respectively, and let \( y^c_{w,c} \) and \( X^c_{w,c} \) be the corresponding selected subsample quantities. A better subsample estimator for \( \beta \) is

\[ \hat{\beta}_{w,c} = \left( X^c_{w,c} W^* X^c_{w,c} \right)^{-1} X^c_{w,c} W^* y^c_{w,c}. \] (18)

Again, since \( \bar{y}_w \) and \( \bar{x}_w \) are nonrandom functions of the full data \( D_n \), existing results show that \( \hat{\beta}_{w,c} \) is asymptotically unbiased with asymptotic variance

\[ \text{Var}_a(\hat{\beta}_{w,c}|D_n) = \frac{C r^2}{r} \left( X^c_{w,c} X^c_{w,c} \right)^{-1} X^c_{w,c} W X_{w,c} \left( X^c_{w,c} X^c_{w,c} \right)^{-1}. \] (19)

The following result shows that \( \hat{\beta}_{w,c} \) has a smaller asymptotic variance than \( \hat{\beta}_w \).

**Proposition 2.** The asymptotic variances in (16) and (19) satisfy that \( \text{Var}_a(\hat{\beta}_{w,c}|D_n) \leq \text{Var}_a(\hat{\beta}_w|D_n) \), and the equality holds if \( x_{w,c} = x \).

**Remark 3.** The above result relies on the asymptotic representation in (15) that requires that the weights do not involve the residuals. Since the weights are inversely proportional to the sampling probabilities, this means that the sampling probabilities are noninformative; that is, they do not depend on the responses. For informative subsampling such as the A- or L- optimal subsampling (Ai et al., 2023; Wang et al., 2022), there is no definite ordering between \( \text{Var}_a(\hat{\beta}_{w,c}|D_n) \) and \( \text{Var}_a(\hat{\beta}_w|D_n) \).

Similar to the case of deterministic selection, if the intercept is of interest, it can be estimated by
Empirical MSEs of subsample estimators for the intercept and slope

|                | Uniform | IBOSS | Leverage |
|----------------|---------|-------|----------|
|                | WI      | WOI   | WI       | WOI     | WI     | WOI     |
| Case 1         |         |       |          |         |        |         |
| $\alpha$       | 88.422  | 76.342| 27.094   | 26.608  | 83.463 | 73.260  |
| $\beta$        | 18.745  | 18.482| 12.874   | 12.740  | 17.834 | 17.641  |
| Case 2         |         |       |          |         |        |         |
| $\alpha$       | 81.065  | 79.155| 12.693   | 12.579  | 152.350| 22.085  |
| $\beta$        | 0.683   | 0.668 | 0.064    | 0.062   | 0.400  | 0.396   |
| Case 3         |         |       |          |         |        |         |
| $\alpha$       | 54.336  | 45.025| 1.505    | 0.655   | 67.008 | 15.929  |
| $\beta$        | 13.418  | 13.259| 0.564    | 0.558   | 10.520 | 10.500  |

**Note:** $\alpha$ and $\beta$ are estimated from a model with an intercept; “WOI”: $\beta$ is estimated from a model without an intercept, and $\alpha$ is estimated using the full data means.

\[
\tilde{\alpha}_{wci} = \gamma_w - \mathbf{x}_w^T \hat{\beta}_{wci} \quad \text{or} \quad \tilde{\alpha}_{waci} = \gamma - \mathbf{x}^T \hat{\beta}_{waci}.
\]  

4 | NUMERICAL COMPARISONS

We provide some numerical simulations that compare the performance of the estimators discussed in previous sections. We generated data from model (1) with $n = 10^5$, $\alpha = 1$, $\beta = 1$, and $\epsilon \sim \mathcal{N}(0, \sigma^2 I_n)$ with $\sigma^2 = 9$. To generate rows of $\mathbf{X}$, we considered the following three distributions. Case 1: multivariate normal distribution $\mathcal{N}(\mathbf{0}, \Sigma)$, Case 2: multivariate log normal distribution $\exp \left( \mathcal{N}(\mathbf{0}, \Sigma) \right)$, and Case 3: multivariate $t$ distribution with degrees of freedom five $\mathcal{T}(0, \Sigma, 5)$. Here, the $(ij)$th element of $\Sigma$ is $0.5^{ij}$ in all cases. These cases were used because the skewness and tail heaviness of the covariate distribution have a significant impact on the performance of different subsampling methods. Case 1 is a light-tailed distribution, and Case 3 is a heavy tailed distribution. Existing asymptotic results on the subsampling methods mentioned in Section 3 require the forth moment of the covariate distribution to be finite, and Case 3 gives the heaviest tail within available asymptotic frameworks. Cases 1 and 3 are symmetric distributions while Case 2 is an asymmetric distribution. We implemented three subsampling methods: uniform sampling, IBOSS (Wang et al., 2019), and leverage sampling (Ma et al., 2015). We run the simulation for 1000 times to calculate the empirical mean squared errors (MSE) reported in Table 1.

We see that for the slope parameter, although not very significant, an estimator based on centred full data (WOI, uncentred subsample) has a smaller MSE than the counterpart based on uncentred full data (WI, centred subsample). For the intercept, the estimator based on the full data means (WOI) is better than the counterpart based on the subsample only (WI), and the improvement is quite significant. Furthermore, by comparing results for Case 1 and Case 3, we see that the improvement is more significant if the covariate distribution has a heavier tail. One reason is that the variation of the slope estimator also contributes to the MSE of the intercept estimator, and the slope estimator has a smaller variance if the covariate distribution has a heavier tail. The covariate distribution is asymmetric in Case 2, so the leverage sampling gives higher preferences for data points in one tail of the full data and the means of the selected subsample do not perform well. Using the full data means to replace the subsample means resulted in the most significant improvement for estimating the intercept in this case.

5 | SUMMARY

For a subsample selected from centred full data, although the subsample is uncentred, it is better to fit a model without an intercept to estimate the slope parameter if the subsampling rule does not depend on the response variable. If the full data are uncentred, it would be better to shift the location of the data by the full data (weighted) means for the OLS (WLS) and then fit a model without an intercept.

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DATA AVAILABILITY STATEMENT

The code that generated the results in the paper is available upon request to the author.

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APPENDIX A: TECHNICAL DETAILS

Proof of Proposition 1. Let $S$ be the $r \times n$ selection matrix consisting of zeros and ones that maps the full data to the subsample, that is, $y^* = Sy$ and $X^* = SX$. We know that

$$X^*_c = X^* - 1_nX^* - n^{-1}1_n1_n^TX = S(I_n - J_n)X,$$

(A1)

$$y^*_c = y^* - y^* 1_r = Sy - n^{-1}1_n1_n^T y = S(I_n - J_n)y.$$

(A2)

Thus,

$$\mathbb{V}(y^*_c | X) = \mathbb{V}(S(I_n - J_n)x) = \sigma^2(I_n - m^{-1}J_n).$$

(A3)

and therefore,

$$\mathbb{V}(\hat{y}_c | X) = (X^*_c X^*_c)^{-1}X^*_c \mathbb{V}(y^*_c | X) X^*_c (X^*_c X^*_c)^{-1}.$$

(A4)
\[ = \sigma^2 (X_c^T X_c)^{-1} X_c^T (I_n - m^{-1} J_d) X_c (X_c^T X_c)^{-1} \]  
\[ = \sigma^2 (X_c^T X_c)^{-1} - \sigma^2 m^{-1} (X_c^T X_c)^{-1} X_c^T J_d X_c (X_c^T X_c)^{-1}. \]  

Let \( X_{c_i} \) be the subsample design matrix centered by the subsample means, that is, \( X_{c_i} = X^* - 1_r \bar{x}^T \). From the facts that \( I_1^T X_{c_i} = 0^T \) and

\[ X_{c_i} = X^* - 1_r \bar{x}^T = X_{c_i} - 1_r (\bar{x} - \bar{x}^*)^T, \]

we know

\[ X_{c_i}^T J_d X_{c_i} = r^{-1} X_{c_i}^T 1_r 1_r^T X_{c_i} = r(\bar{x} - \bar{x}^*)^{\otimes 2}. \]

Thus, (A6) and (A8) give

\[ \mathbb{V}(\hat{\beta}|X) = \sigma^2 (X_{c_i}^T X_{c_i})^{-1} - \frac{\sigma^2 r n^{-1} (X_{c_i}^T X_{c_i})^{-1} (\bar{x} - \bar{x}^*)^{\otimes 2} (X_{c_i}^T X_{c_i})^{-1}}{1 - d}. \]

From (7), the variance of \( \beta \) is

\[ \mathbb{V}(\hat{\beta}|X) = \sigma^2 (X_{c_i}^T X_{c_i})^{-1}. \]

Note that (A7) implies

\[ X_{c_i}^T X_{c_i} = X^*^T X^* - r (\bar{x}^* - \bar{x})^{\otimes 2}. \]

Thus, we obtain

\[ (X_{c_i}^T X_{c_i})^{-1} = (X_{c_i}^T X_{c_i})^{-1} + \frac{r (X_{c_i}^T X_{c_i})^{-1} (\bar{x}^* - \bar{x})^{\otimes 2} (X_{c_i}^T X_{c_i})^{-1}}{1 - d}, \]

where \( d = r (\bar{x}^* - \bar{x})^T (X_{c_i}^T X_{c_i})^{-1} (\bar{x}^* - \bar{x})^T \). Here, \( 1 - d \) must be positive because (A11) implies that \( (X_{c_i}^T X_{c_i})^{-1} \geq (X_{c_i}^T X_{c_i})^{-1} \) and they are both positive-definite.

Combining (A9), (A10), and (A12) finishes the proof. \qed

Proof of Proposition 2. For positive definite matrices \( A_1, A_2, B_1, \) and \( B_2 \), if \( A_1 \preceq A_2 \) and \( B_2 \succeq B_1 \), then

\[ B_2^{1/2} A_1^{1/2} B_2^{1/2} \preceq B_2^{1/2} A_2^{1/2} B_2^{1/2} \Rightarrow (B_2^{1/2} A_1^{1/2} B_2^{1/2})^2 \preceq (B_2^{1/2} A_2^{1/2} B_2^{1/2})^2 \Rightarrow A_1^{1/2} B_2 A_1^{1/2} \succeq A_2^{1/2} B_2 A_2^{1/2}. \]

so \( A_1^{1/2} B_1 A_1^{1/2} \succeq A_1^{1/2} B_2 A_1^{1/2} \) and \( A_2^{1/2} B_2 A_2^{1/2} \). Thus, we only need to prove that

\[ X_{c_i}^T X_{nc} \preceq X_{nc}^T X_{nc}, \]

and the equality in both hold if \( X_{nc} = \bar{x} \). The proof finishes from the fact that

\[ X_{c_i}^T W X_{c} - X_{nc}^T W X_{nc} = X^T (w - n^{-1} 1_n) \otimes 2 X = (X_{nc} - \bar{x})^{\otimes 2} \succeq 0, \]

which can be verified by inserting \( X_{c} = (I_n - n^{-1} 1_n 1_n^T) X \) and \( X_{nc} = (I_n - 1_n w^T) X \). \qed