DUALITY BETWEEN $S^2P^4$ AND THE DOUBLE QUINTIC SYMMETROID

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Abstract. Let $\mathcal{X} = S^2P^4$ be the second symmetric product of $P^4$ and $\mathcal{Y}$ the double cover of the symmetric determinantal quintic hypersurface in $P^{14}$ considered in [33]. We study homological properties of $\mathcal{X}$ and $\mathcal{Y}$ which indicate the homological projective duality between (suitable noncommutative resolutions of) $\mathcal{X}$ and $\mathcal{Y}$. Among other things, we construct good desingularizations $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and also a dual Lefschetz collection in $D^b(\tilde{\mathcal{X}})$ and a Lefschetz collection in $D^b(\tilde{\mathcal{Y}})$. These are expected to give respective (dual) Lefschetz decompositions of suitable noncommutative resolutions of $D^b(\mathcal{X})$ and $D^b(\mathcal{Y})$.

1. Introduction

In the previous work [16], we have encountered an interesting new geometry of Reye congruences in dimension three through our study of mirror symmetry of Calabi-Yau manifolds. Reye congruences in dimension two have been attracting attention for long in relation to geometries of Enriques surfaces [6]. In dimension three, they define smooth Calabi-Yau threefolds. It was found in [16] that each of these Calabi-Yau threefolds is paired with another smooth Calabi-Yau threefold which arises naturally in the projective geometry of Reye congruences.

Let $\mathcal{X} = S^2P(V)$ be the symmetric product of the projective space $P(V) \cong P^n$. In terms of the so-called Chow form, we can embed $\mathcal{X}$ into the projective space $P(S^2V)$ of symmetric $(n+1) \times (n+1)$ matrices. Then $\mathcal{X}$ is identified with the Chow variety of 0-cycles of length two, and may also be identified with the rank $\leq 2$ locus of $P(S^2V)$ in the natural stratification by matrix rank. It is a well-known fact in classical projective geometry that this stratification is reversed to the corresponding one in the dual projective space $P(S^2V^*)$.

The Reye congruences are defined as general linear sections of $\mathcal{X}$ by $(n+1)$ linear forms on $P(S^2V)$. Since giving $(n+1)$ linear forms is equivalent to fixing a linear subspace $L \simeq C^{n+1} \subset S^2V^*$ in the vector space $S^2V^*$ dual to $S^2V$, the corresponding Reye congruence may be written by $X = \mathcal{X} \cap P(L^\perp)$. In this form, one may notice that the Reye congruence is in accord with the Mukai’s constructions [29] of Fano manifolds associated to homogeneous spaces. For example, in his classification of prime Fano threefolds, the Fano threefolds of genus 7, 8, 9, 10 are constructed by a similar linear sections of suitable homogeneous spaces. Furthermore, as an outcome of his construction, in the case of genus 7 for example, it was found that the intermediate Jacobian of the Fano threefold is isomorphic to the Jacobian of the curve which is obtained as the ‘orthogonal’ linear section of the projective dual of the homogeneous space.
Observing this similarity, we first considered in [16] the Hessian hypersurface in the dual projective space by \( H = \mathcal{H} \cap \mathbb{P}(L) \), where \( \mathcal{H} \) represents the rank \( \leq n \) locus in \( \mathbb{P}(S^2 V^*) \). Assume that \( n \) is even. While the Reye congruence \( X \) is a smooth Calabi-Yau manifold, \( H \) is a Calabi-Yau variety which is singular along a codimension two subvariety. The new geometry found in [loc.cit.] is a double covering \( Y \to H \) branched along the singular locus of \( H \). It was shown that \( Y \) is a smooth Calabi-Yau threefold when \( n = 4 \).

To clarify the relations to the previous construction, let us note that \( H \) is the determinantal hypersurface of degree \( n + 1 \). We call \( H \) symmetroid in this paper. When \( n \) is even, we will define the double covering \( Y \to H \) branched along the rank \( \leq (n - 1) \) locus of \( \mathcal{H} \) (Proposition 4.2.2). We call \( Y \) double symmetroid. \( Y \) is singular for \( n \geq 4 \) but still has nice properties in view of the minimal model program (Proposition 4.3.1). We show that \( Y = Y \cap \mathbb{P}(L) \) is a Calabi-Yau variety in general and is smooth when \( n = 4 \) as studied in the previous work.

The recent proposal in [25, 28], called homological projective duality, describes the Mukai’s construction in terms of the derived category of coherent sheaves and a suitable decomposition (Lefschetz decomposition) of it. More generally, for a singular variety, the proposal deals with the so-called noncommutative resolution which is a full subcategory of the derived category of a suitable resolution of the singularities. The classical examples of the Fano threefolds of genus 7, 8, 9, 10 due to Mukai, which are related to some nice homogeneous spaces, are described in this framework [24]. Also the homological projective dual of the Grassmann variety \( G(2, 7) \) was shown to be the noncommutative resolution of its projective dual variety \( \text{Pf}(7) \), the Pfaffian variety [28]. Under this duality, two Calabi-Yau threefolds are obtained [23, 28] as suitable linear sections in a similar way to the above. While we can observe many similarities between our case and the Grassmann-Pfaffian case, there are also many dis-similarities between the two. One important difference we should note is that \( X \) as well as \( Y \) are not homogeneous spaces although they admit natural quasi-homogeneous \( SL(n) \) actions.

In this article, we make a first step toward formulating the new geometry appeared in the previous work within the framework of the homological projective duality.

The homological projective duality, if applies to our case, provides a systematic way to describe the derived categories of the (noncommutative resolutions of) linear sections of \( X \) and \( Y \). Showing the homological projective duality in general consists of two major steps of finding suitable categorical/noncommutative resolutions and making suitable (dual) Lefschetz decompositions of them. The categorical/noncommutative resolutions of \( X \) and \( Y \), respectively, should be identified up to equivalences inside the derived categories \( D^b(X) \) and \( D^b(Y) \) as full subcategories by finding suitable resolutions \( \tilde{X} \to X \) and \( \tilde{Y} \to Y \).

A natural resolution of \( X \) is given by the Hilbert-Chow morphism \( \tilde{X} = \text{Hilb}^2 \mathbb{P}^n \to X \) (cf. Subsection 3.3). In this case, it should be rather easy to find the noncommutative resolution of \( X \) in the derived category \( D^b(X) \) based on the theory of [20] (see [19]). In contrast to this, the singularity of \( Y \) turns out to be more involved (see Subsection 5.6). Because of this complication, the theorem [20, Theorem 4.3] does not apply to this example, and having the noncommutative resolution...
seems to be a more difficult task although we will find a nice resolution \( \widetilde{Y} \to Y \) (Subsections 5.4, 5.6). In this paper, as a strong indication for the homological projective duality between \( Y \) and \( X \), we find a Lefschetz collection which generates a full subcategory of \( D^b(\widetilde{Y}) \) (Theorem 3.4.1) and also the corresponding dual Lefschetz collection in \( D^b(X) \) (Theorem 3.4.5). These two theorems are main results of this paper. We expect that these (dual) Lefschetz collections are actually the (dual) Lefschetz decompositions of the noncommutative resolutions of \( Y \) and \( X \).

The construction of this paper is as follows: In Section 3, we summarize some basic properties of the variety \( X \), and construct the Hilbert-Chow morphism \( \tilde{X} \to X \). Using these, we construct the dual Lefschetz collection in \( D^b(\tilde{X}) \). Also some basic properties of Calabi-Yau manifolds of Reye congruences are summarized. In Section 4, we introduce the determinantal hypersurface (symmetroid) \( H \), and using the geometry of singular quadric parametrized by \( H \), we define its double cover, i.e., the double symmetroid \( \tilde{Y} \). We define Calabi-Yau variety \( Y \) in \( Y \), and for \( n = 4 \), we determine topological invariants of \( Y \) from geometries of \( Y \) (Proposition 4.3.4). In Section 5, we study the birational geometry of \( Y \) by introducing a subvariety \( \overline{Y} \) in \( G(3, \wedge^3 V) \). Although we will not go into the details, we find that the birational geometry of \( Y \) has a close relation to the Hilbert scheme of conics on Grassmann \( G(3, V) \) (Proposition 5.2.1). As a resolution of the singularity of \( \overline{Y} \), we introduce the Grassmann bundle \( G(3, T(-1)^\wedge 2) \) over \( \mathbb{P}(V) \), and the so-called two ray game (Sarkisov link) of this Grassmann bundle is studied in detail to obtain our desingularization \( \widetilde{Y} \) of \( Y \) (see 5.3) and also Fig.4 in Section 6. The morphism \( \widetilde{Y} \to Y \) turns out to be a divisorial contraction which is negative with respect to the canonical divisor (Propositions 5.7.1 and 5.7.2). In Section 6, we define three locally free sheaves \( \tilde{S}_L^\ast \), \( \tilde{Q} \), and \( \tilde{T} \) on \( \tilde{Y} \) (Definitions 6.1.2 and 6.2.3), which will generate a Lefschetz collection in \( D^b(\tilde{Y}) \). In Section 7, we further study the exceptional divisor \( F_{\tilde{X}} \) of divisorial contraction \( \widetilde{Y} \to Y \). We describe some birational models of \( F_{\tilde{X}} \) in Proposition 7.3.2 and (7.4). This section is necessary for our cohomology calculations in the subsequent section. In Section 8, we find a Lefschetz collection in \( D^b(\tilde{Y}) \) and observe a certain duality between the quiver diagrams associated to the Lefschetz collection in \( D^b(\tilde{Y}) \) and the dual Lefschetz collection in \( D^b(X) \) respectively (3.9 and 8.1).

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Notation: Throughout the paper, we work over $\mathbb{C}$, the complex number field. We will use the following notation which simplifies lengthy formulas:

- $V$: a (fixed) $n + 1$ dimensional complex vector space. $\mathbb{P}^n := \mathbb{P}(V)$.
- $V_i$: an $i$-dimensional vector subspace of $V$.
- $\Omega(i) := \Omega_{\mathbb{P}(V)}(i)$ for $i \geq 2$.
- $T^{(-1)} := T_{\mathbb{P}(V)}(-1)$.
- $\mathcal{O}(i) := \mathcal{O}_{\mathbb{P}(V)}(i)$ for $i \in \mathbb{Z}$.

2. Preliminaries

For the computations of cohomology groups which appear in this paper, we use the Bott theorem about the cohomology groups of Grassmann bundles extensively.

For a locally free sheaf $\mathcal{E}$ of rank $r$ on a variety and a nonincreasing sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ of integers, we denote by $\Sigma^\beta \mathcal{E}$ the associated locally free sheaf with the Schur functor $\Sigma^\beta$.

**Theorem 2.0.1. (Bott theorem)** Let $\pi: G(r, \mathcal{A}) \to X$ be a Grassmann bundle for a locally free sheaf $\mathcal{A}$ on a variety $X$ of rank $n$ and $0 \to \mathcal{S} \to \mathcal{A} \to \mathcal{Q} \to 0$ the universal exact sequence. For $\beta := (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$, $\alpha_1 \geq \cdots \geq \alpha_r$, and $\gamma := (\alpha_{r+1}, \ldots, \alpha_n) \in \mathbb{Z}^{n-r}$, $\alpha_{r+1} \geq \cdots \geq \alpha_n$, we set $\alpha := (\beta, \gamma)$ and $\mathcal{V}(\alpha) := \Sigma^\beta \mathcal{S}^* \otimes \Sigma^\gamma \mathcal{Q}^*$. Finally, let $\rho := (n, n-1, \ldots, 1)$, and, for an element $\sigma$ of the $n$-th symmetric group $\mathfrak{S}_n$, we set $\sigma^\cdot(\alpha) := \sigma(\alpha + \rho) - \rho$.

1. If $\sigma(\alpha + \rho)$ contains two equal integers, then $R^i\pi_*\mathcal{V}(\alpha) = 0$ for any $i \geq 0$.
2. If there exists an element $\sigma \in \mathfrak{S}_n$ such that $\sigma(\alpha + \rho)$ is strictly decreasing, then $R^i\pi_*\mathcal{V}(\alpha) = 0$ for any $i \geq 0$ except $R^{l(\sigma)}\pi_*\mathcal{V}(\alpha) = \Sigma^{\sigma^\cdot(\alpha)}\mathcal{A}^*$, where $l(\sigma)$ represents the length of $\sigma \in \mathfrak{S}_n$.

**Proof.** See [4], [8], or [35, (4.19) Corollary].

In this paper, we adopt the following definition of Calabi-Yau variety and also Calabi-Yau manifold.

**Definition 2.0.2.** We say a normal projective variety $X$ a **Calabi-Yau variety** if $X$ has only Gorenstein canonical singularities, the canonical bundle of $X$ is trivial, and $h^i(\mathcal{O}_X) = 0$ for $0 < i < \dim X$. If $X$ is smooth, then $X$ is called a **Calabi-Yau manifold**. A smooth Calabi-Yau threefold is abbreviated as a **Calabi-Yau threefold**.
3. The Geometry of $S^2P(V)$

3.1. $S^2P(V)$ and quadrics in $P(V)$. Let $\mathcal{X} := S^2P(V)$ be the symmetric product of $P(V)$. Since we can identify $\mathcal{X}$ with the Chow variety of 0-cycles in $P(V)$ of length 2, $\mathcal{X}$ can be embedded in the projective space $P(S^2V)$ for the symmetric product $S^2V := \text{Sym}^2V$ in terms of the so-called Chow form:

$$(3.1) \quad w_{ij} = x_i y_j, \quad w_{ij} := x_i y_j + x_j y_i \quad (i \neq j),$$

where $w_{ij} = w_{ji}$ (1 ≤ $i, j$ ≤ $n + 1$) and $x_i, y_i$ (1 ≤ $i$ ≤ $n + 1$) are coordinates of $P(S^2V)$ and $P(V)$, respectively (cf. [13, Theorem 2.2]). If we view this as giving a morphism $P(V) \times P(V) \to P(S^2V)$, the isomorphism of $S^2P(V)$ to the Chow variety follows from the fact that $w_{ij}$ in (3.1) generates all the invariant polynomials under $x_i \leftrightarrow y_i$ [loc.cit.].

We may identify $P(S^2V)$ with the dual to the space of the symmetric (1, 1)-divisors on $P(V) \times P(V)$. $S^2P(V)$ in $P(S^2V)$ is defined by the zero set of all the $3 \times 3$ minors of $(n + 1) \times (n + 1)$ symmetric matrix $(w_{ij})$ representing the coordinate of $P(S^2V)$.

Quadrics in $P(V)$ are given by the equations $^tAxA = 0$ with $^tx = (x_1, \ldots, x_{n+1})$ and an $(n + 1) \times (n + 1)$ symmetric matrix $A$. We denote by $Q_A$ the quadric defined by a symmetric matrix $A$, and by $q_A(x)$ the quadratic form $^txAx$. The projectivization of the vector space of all the $(n + 1) \times (n + 1)$ symmetric matrices may be identified with $P(S^2V^*)$, which is dual to $P(S^2V)$. More explicitly, we write the dual pairing by

$$A \cdot w = \sum_{1 \leq i, j \leq n+1} a_{ij} w_{ij},$$

where $w = (w_{ij}) \in P(S^2V)$. Using (3.1), we have the equality

$$(3.2) \quad A \cdot w_{xy} = ^txAy,$$

where $w_{xy}$ is the image in $P(S^2V)$ of $x, y \in P(V) \times P(V)$.

3.2. Projective duals of $\text{Sec}^i \mathcal{J}_0$. When we restrict the morphism $P(V) \times P(V) \to P(S^2V)$ described above to the diagonal, we obtain the second Veronese morphism of $P(V)$. We denote by $\mathcal{J}_0 = v_{2}(P(V))$ its image. Then it is easy to see that $S^2P(V)$ is defined by all the $2 \times 2$ minors of the generic $(n + 1) \times (n + 1)$ symmetric matrix $(w_{ij})$, namely, $\mathcal{J}_0$ is the locus of symmetric matrices of rank one. By the characterization of $\mathcal{X}$ and $\mathcal{J}_0$ with rank condition as above, we see that $\mathcal{X}$ is the secant variety of $\mathcal{J}_0$, namely,

$$\mathcal{X} = \cup\{\langle p, q \rangle \mid p, q \in \mathcal{J}_0, p \neq q\},$$

where $\langle p, q \rangle$ is the line through $p$ and $q$.

The second Veronese variety $\mathcal{J}_0$ is one of the Scorza varieties classified by Zak (see [21, 5]). Associated to $\mathcal{J}_0$, we naturally have the tower of higher secant varieties. Recall that, for projective varieties $X, Y \subset P^N$, in general, their join is defined as

$$J(X, Y) := \cup\{\langle p, q \rangle \mid p \in X, q \in Y, p \neq q\}.$$
corresponds to a sum of \((i+1)\) matrices of rank one. In particular, it holds that
\(\text{Sec}^n\mathcal{X}_0 = \mathbb{P}(S^2V)\) and \(\text{Sec}^{n-1}\mathcal{X}_0\) is the hypersurface of degree \(n+1\) defined by the determinant of the generic symmetric matrix \((w_{ij})\). In summary, we have the tower of the secant varieties:
\[
\emptyset \subset \mathcal{X}_0 \subset \text{Sec}^1\mathcal{X}_0 \subset \text{Sec}^2\mathcal{X}_0 \subset \cdots \subset \text{Sec}^{n-1}\mathcal{X}_0 \subset \text{Sec}^n\mathcal{X}_0.
\]
(3.3)
\[
v_2(\mathbb{P}(V)) \rightarrow \mathbb{P}(S^2V)
\]
It is known that \(\dim \text{Sec}^i\mathcal{X}_0 = (i+1)n - \frac{i(i-1)}{2}\). This tower gives the orbit decomposition of the action of \(\text{SL}_{n+1}\) on \(\mathbb{P}(S^2V)\). Precisely, \(\text{Sec}^i\mathcal{X}_0 \setminus \text{Sec}^{i-1}\mathcal{X}_0\) is an \(\text{SL}_{n+1}\)-orbit for any \(i\) since any two symmetric matrices of the same rank are transformed to each other by \(\text{SL}_{n+1}\). It is known that \(\text{Sec}^{i+1}\mathcal{X}_0\) is the singular locus of \(\text{Sec}^i\mathcal{X}_0\). In particular, \(\mathcal{X}_0 = \text{Sing} \mathcal{X}\).

In the dual projective space \(\mathbb{P}(S^2V^*)\) to \(\mathbb{P}(S^2V)\), we consider the dual varieties \((\text{Sec}^i\mathcal{X}_0)^*\) of \(\text{Sec}^i\mathcal{X}_0\), namely, \((\text{Sec}^i\mathcal{X}_0)^*\) is the closure of the locus of the hyperplanes of \(\mathbb{P}(S^2V)\) tangent to \(\text{Sec}^i\mathcal{X}_0\) at smooth points. Verifying the tangent space of \(\text{Sec}^i\mathcal{X}_0\) at a general point, we see that \((\text{Sec}^i\mathcal{X}_0)^*\) is the locus of symmetric matrices \(A\) of rank \(\leq n-i\). In summary, we have the dual tower to (3.3):
\[
\mathbb{P}(S^2V^*) \supset (\mathcal{X}_0)^* \supset (\text{Sec}^1\mathcal{X}_0)^* \supset (\text{Sec}^2\mathcal{X}_0)^* \supset \cdots \supset (\text{Sec}^{n-1}\mathcal{X}_0)^* \supset \emptyset.
\]
(3.4)
\[
\mathcal{H} \supset (\mathcal{X}_0)^*.
\]
Throughout this paper, we set
\[
\mathcal{H} := (\mathcal{X}_0)^*.
\]
\(\mathcal{H}\) is the symmetric determinantal hypersurface in \(\mathbb{P}(S^2V^*)\), which is called the symmetroid.

3.3. \(\mathcal{F} = \text{Hilb}^2\mathbb{P}(V)\) and \(G(2,V)\). By the construction of the double cover \(\mathbb{P}(V) \times \mathbb{P}(V) \rightarrow \mathcal{X}\), we see that \(\mathcal{X}\) has quotient singularities of type \(\frac{1}{n}(1^n)\) along the singular locus \(\mathcal{X}_0 = v_2(\mathbb{P}(V))\). In particular, \(\mathcal{X}\) is Gorenstein if and only if \(n\) is even. Hereafter in this subsection, we assume that \(n\) is even. Now we set
\[
\mathcal{F} := \text{Hilb}^2\mathbb{P}(V),
\]
which is the Hilbert scheme of 0-dimensional subschemes of \(\mathbb{P}(V)\) of length 2 (we simply call it the Hilbert scheme of two points on \(\mathbb{P}(V)\)). Recall that the Hilbert-Chow morphism \(f: \mathcal{F} \rightarrow \mathcal{X}\) is the blow-up of \(\mathcal{X}\) along the singular locus \(\mathcal{X}_0\).

Since the 0-dimensional subscheme on \(\mathbb{P}(V)\) of length two determines a line on \(\mathbb{P}(V)\), we have a natural morphism \(g: \mathcal{F} \rightarrow G(2,V)\):
\[
\xymatrix{ \mathcal{F} \ar[r]^g & G(2,V). }
\]
Let \(\mathcal{F}\) be the universal subbundle of rank two on \(G(2,V)\). Then the Hilbert scheme \(\mathcal{F}\) of two points on \(\mathbb{P}(V)\) is isomorphic to \(\mathbb{P}(S^2\mathcal{F})\). This follows from the
fact that the fiber of \( g \) over a point \([l] \in G(2, V)\) is identified with \( \text{Hilb}^2 \mathbb{P}(l) = \mathbb{P}^2 \mathbb{P}(l) \simeq \mathbb{P}(S^2 l)\), where \( l \simeq \mathbb{C}^2 \) is the affine two plane representing \([l]\).

We set
\[
H_{\mathcal{X}} = f^* \mathcal{O}_{\mathcal{X}}(1) \quad \text{and} \quad L_{\mathcal{X}} = g^* \mathcal{O}_{G(2, V)}(1).
\]

By \( \mathcal{X} \simeq \mathbb{P}(S^2 \mathcal{F})\), we see that the morphism \( f \) follows from the natural morphism \( \mathbb{P}(S^2 \mathcal{F}) \to \mathbb{P}^2 \mathbb{P}(V)\). Hence \( H_{\mathcal{X}} \) is the tautological divisor for \( \mathbb{P}(S^2 \mathcal{F})\).

Let us denote by \( E_f \) the \( f \)-exceptional divisor. We see that \( E_f \simeq \mathbb{P}(\mathcal{F})\), namely, \( E_f \subset \text{Hilb}^2 \mathbb{P}(V) \) parameterizes pairs of points \( x \in \mathbb{P}(V) \) and lines \( l \) through \( x \).

Moreover, from the relative Euler sequence \( 0 \to \mathcal{O}_{\mathcal{X}} \to g^*(S^2 \mathcal{F}) \otimes \mathcal{O}_{\mathcal{X}}(1) \to T_{\mathcal{X}/G(2, V)} \to 0\), we have
\[
K_{\mathcal{X}} = -3H_{\mathcal{X}} - (n - 2)L_{\mathcal{X}}.
\]

On the other hand, since \( f : \mathcal{X} \to \mathcal{F} \) is the blow-up of \( \mathcal{X} \) along the singular locus \( \mathcal{X}_0\), we have
\[
K_{\mathcal{X}} = -(n + 1)H_{\mathcal{X}} + \frac{n - 2}{2}E_f.
\]

Therefore we have
\[
\frac{n - 2}{2}E_f \sim (n - 2)(H_{\mathcal{X}} - L_{\mathcal{X}}).
\]

### 3.4. Constructing a dual Lefschetz collection in \( D^b(\mathcal{X}) \).

We recall some basic definitions from the theory of triangulated categories (cf. [2, 3]).

**Definition 3.4.1.** An object \( \mathcal{E} \) in a triangulated category \( D \) is called an exceptional object if \( \text{Hom}(\mathcal{E}, \mathcal{E}) \simeq \mathbb{C} \) and \( \text{Hom}^*(\mathcal{E}, \mathcal{E}) = 0 \) for \( \bullet \neq 0 \).

**Definition 3.4.2.** A triangulated subcategory \( D' \) of \( D \) is called admissible if there are right and left adjoint functors for the inclusion functor \( i_* : D' \to D \).

**Definition 3.4.3.** A sequence \( D_1, \ldots, D_m \) of admissible triangulated subcategories in a triangulated category \( D \) is called a semiorthogonal collection if \( \text{Hom}_D(D_i, D_j) = 0 \) for any \( i > j \). Moreover, if \( D_1, \ldots, D_m \) generates \( D \), then it is called a semiorthogonal decomposition.

A semiorthogonal collection of exceptional objects \( \mathcal{E}_1, \ldots, \mathcal{E}_n \) is called an exceptional collection. Moreover, if \( \text{Hom}^*(\mathcal{E}_i, \mathcal{E}_j) = 0 \) holds for any \( i, j \) and any \( \bullet \neq 0 \), then it is called a strongly exceptional collection.

Hereafter, in this article, we restrict our attention to the cases of the derived categories of bounded complexes of coherent sheaves on a variety. In such cases, a special type of semiorthogonal collection plays an important role (cf. [25, 28]).

**Definition 3.4.4.** For a variety \( X \), a Lefschetz collection of \( D^b(X) \) is a semiorthogonal collection of the following form:
\[
D_0, D_1(1), \ldots, D_{m-1}(m - 1),
\]
where it holds that \( 0 \subset D_{m-1} \subset D_{m-2} \subset \cdots \subset D_0 \subset D^b(X) \) and \( (k) \) means the twist by \( L \otimes k \) with a fixed invertible sheaf \( L \). Moreover, if \( D_0, D_1, \ldots, D_{m-1}(m - 1) \) generate \( D^b(X) \), then it is called a Lefschetz decomposition.
Similarly, a dual Lefschetz collection of $\mathcal{D}^b(X)$ is a semiorthogonal collection of the following form:

$$
\mathcal{D}_{m-1}(-(m-1)), \mathcal{D}_{m-2}(-(m-2)), \ldots, \mathcal{D}_0,
$$

where it holds that $0 \subset \mathcal{D}_{m-1} \subset \mathcal{D}_{m-2} \subset \cdots \subset \mathcal{D}_0 \subset \mathcal{D}^b(X)$. Moreover, if $\mathcal{D}_{m-1}(-(m-1)), \mathcal{D}_{m-2}(-(m-2)), \ldots, \mathcal{D}_0$ generate $\mathcal{D}^b(X)$, then it is called a dual Lefschetz decomposition.

Now, based on the geometry of the projective bundle $\mathcal{X} = \mathbb{P}(S^2 \mathcal{F})$ over $G(2, V)$, we construct a dual Lefschetz collection in $\mathcal{D}^b(\mathcal{X})$ by restricting our attention to the case $n = 4$. Other cases of $n > 4$ should be done in a similar way, but we confine ourselves to this case to avoid possible complications.

We may naturally conceive the sheaves $\mathcal{O}_{\mathcal{X}}$ and $g^* \mathcal{F}^*$ as the objects in the (dual) Lefschetz collection. Recall the isomorphism $\mathcal{X} \simeq \mathbb{P}(S^2 \mathcal{F})$ and consider associated Euler sequence $0 \to \mathcal{O}_{\mathbb{P}(S^2 \mathcal{F})}(-1) \to g^* S^2 \mathcal{F} \to \mathcal{T}_{\mathbb{P}(S^2 \mathcal{F})/G(2, V)}(-1) \to 0$. Twisting this by $2L_{\mathcal{X}}$ we obtain an injection

$$
\varphi : \mathcal{O}_{\mathcal{X}}(-H_{\mathcal{X}} + 2L_{\mathcal{X}}) \to (g^* S^2 \mathcal{F})(2L_{\mathcal{X}}) \simeq g^* S^2 \mathcal{F}^*,
$$

where we use $\mathcal{F} \otimes \mathcal{O}_{G(2, V)}(1) = \Sigma^{(0, -1)} \mathcal{F}^* \otimes \Sigma^{(1, 1)} \mathcal{F}^* \simeq \mathcal{F}^*$. The cokernel of this injection, which is $\mathcal{T}_{\mathbb{P}(S^2 \mathcal{F})/G(2, V)}(-H_{\mathcal{X}} + 2L_{\mathcal{X}})$, plays a role in the following theorem:

**Theorem 3.4.5.**

1. Let

$$
(F_3, F_2, F_{1a}, F_{1b}) = (\mathcal{O}_{\mathcal{X}}, g^* \mathcal{F}^*, \text{Coker } \varphi, \mathcal{O}_{\mathcal{X}}(L_{\mathcal{X}}))
$$

be an ordered collection of sheaves on $\mathcal{X}$. Then $(\mathcal{K}_i)_{1 \leq i \leq 4} := (F_{1b}, F_{1a}, F_2, F_3)$ is a strongly exceptional collection of $\mathcal{D}^b(\mathcal{X})$, namely satisfies

$$
H^*(\mathcal{K}_i \otimes \mathcal{K}_j) = 0 \text{ for } 1 \leq i, j \leq 4 \text{ and } \bullet > 0
$$

and $H^0(\mathcal{K}_i \otimes \mathcal{K}_j) = 0$ ($i > j$), $H^0(\mathcal{K}_i \otimes \mathcal{K}_i) \simeq \mathbb{C}$ ($1 \leq i \leq 4$).

2. For $i < j$, $H^0(\mathcal{K}_i \otimes \mathcal{K}_j)$ are given by

$$
H^0(F_3 \otimes F_2) \simeq V^*, \quad H^0(F_3 \otimes F_{1a}) \simeq S^2 V^*, \quad H^0(F_3 \otimes F_{1b}) \simeq \wedge^2 V^*,
$$

$$
H^0(F_2 \otimes F_{1a}) \simeq V^*, \quad H^0(F_2 \otimes F_{1b}) \simeq V^*, \quad H^0(F_{1a} \otimes F_{1b}) = 0,
$$

which may be summarized in the following quiver diagram:

$$
\begin{align*}
\mathcal{F}_3 & \quad V^* \\
\wedge^2 V^* & \quad F_{1b} \\
\end{align*}
$$

3. Set $\mathcal{D}_{\mathcal{X}} := (F_{1b}, F_{1a}, F_2, F_3) \subset \mathcal{D}^b(\mathcal{X})$. Then

$$
\mathcal{D}_{\mathcal{X}}(-4), \ldots, \mathcal{D}_{\mathcal{X}}(-1), \mathcal{D}_{\mathcal{X}}
$$

is a dual Lefschetz collection, where $(-t)$ represents the twist by the sheaf $\mathcal{O}_{\mathcal{X}}(-tH_{\mathcal{X}})$. 
We prepare the following lemma for our proof of the theorem.

**Lemma 3.4.6.** Set $C_{ij} = K_i^\ast \otimes K_j$ (1 ≤ i, j ≤ 4). For $C = C_{ij}$, it holds that

$$H^\bullet(X, C(-t)) \simeq H^{8-\bullet}(X, C^\ast((t-5)))$$

for any t.

**Proof.** By the Serre duality and $K_{\bar{\mathscr{X}}} = -5H_{\bar{\mathscr{X}}} + E_f$, we have $H^\bullet(X, C(-t)) \simeq H^{8-\bullet}(X, C^\ast((t-5)H_{\bar{\mathscr{X}}} + E_f))$ for any t. By the exact sequence

$$0 \rightarrow C^\ast((t-5)) \rightarrow C^\ast((t-5)H_{\bar{\mathscr{X}}} + E_f) \rightarrow C^\ast((t-5)H_{\bar{\mathscr{X}}} + E_f)|_{E_f} \rightarrow 0,$$

we have only to show that $H^{8-\bullet}(E_f, C^\ast((t-5)H_{\bar{\mathscr{X}}} + E_f)|_{E_f}) = 0$. Since $E_f \rightarrow X_0 \simeq \mathbb{P}^4$ is a $\mathbb{P}^3$-bundle, it suffices to show the vanishing of cohomology groups of the restriction of $C^\ast((t-5)H_{\bar{\mathscr{X}}} + E_f)|_{E_f}$ to a fiber $E_f \rightarrow X_0$. Note that $\mathcal{O}_{\bar{\mathscr{X}}}(E_f)|_{E_f} \simeq \mathcal{O}_{\mathbb{P}^3}(-2)$ and $\mathcal{O}_{\bar{\mathscr{X}}}(H_{\bar{\mathscr{X}}})|_{E_f} \simeq \mathcal{O}_{\mathbb{P}^3}$. As we note in the end of Subsection 3.4.3, $E_f$ parameterizes pairs of a point $x \in \mathbb{P}(V)$ and a line $l$ through $x$. Therefore a fiber $E_f \simeq \mathbb{P}^3$ parameterizes lines through one fixed point. This implies that $g^\ast \mathcal{F}^\ast|_{E_f} \simeq \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$. Restricting the natural injection $\mathcal{O}_{\bar{\mathscr{X}}}(-H_{\bar{\mathscr{X}}}) \rightarrow g^\ast S^2\mathcal{F}$ to $E_f$, we have an injection

$$\mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2).$$

Therefore, by the definition of $\mathcal{F}_{1a}$, we have $\mathcal{F}_{1a}|_{E_f} \simeq \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$. Consequently, $C^\ast((t-5)H_{\bar{\mathscr{X}}} + E_f)|_{E_f}$ is a direct sum of $\mathcal{O}_{\mathbb{P}^3}(-1)$, $\mathcal{O}_{\mathbb{P}^3}(-2)$ and $\mathcal{O}_{\mathbb{P}^3}(-3)$ for any $C = C_{ij}$ and t, hence all of its cohomology groups vanish.

**Proof of Theorem 3.4.5.** Note that the subcategory $D_{\mathscr{X}}$ is admissible if the property (1) holds [27 Theorem 3.2. a)]. Then, for the proof of (3), it suffices to verify

$$H^\bullet(K_i^\ast \otimes K_j(-t)) = 0 \ (1 \leq i, j \leq 4, 1 \leq t \leq 4)$$

for $\bullet > 0$, since $D_{\mathscr{X}}$ is generated by $K_i(1 \leq i \leq 4)$ (cf. [27 Lemma 2.2]). Therefore, the claims (1),(2),(3) follow from explicit evaluations of the cohomology groups. Since the computations are similar, we only explain some of them below. Note also that we may assume that $t = 0, 1, 2$ by Lemma 3.4.6 which simplifies the computations considerably.

As for $H^\bullet(X, C_{44}(-t)) = H^\bullet(X, \mathcal{O}_{\mathscr{X}}(-t))$ (0 ≤ t ≤ 2), these vanish except in the case where t = 0 since g is a $\mathbb{P}^2$-bundle and hence $Rg^\ast \mathcal{O}_{\mathscr{X}}(-t) = 0$ for $t = 1, 2$ and $i \geq 0$. $H^\bullet(X, \mathcal{O}_{\mathscr{X}})$ vanish except $H^0(X, \mathcal{O}_{\mathscr{X}})$ by the Kodaira vanishing theorem.

As for $H^\bullet(X, C_{34}(-t)) = H^\bullet(X, g^\ast \mathcal{F}^\ast(-tH_{\mathscr{X}}))$ (0 ≤ t ≤ 2), these vanish except in the case where $t = 0$ by a similar reason. We have

$$H^\bullet(X, g^\ast \mathcal{F}^\ast) \simeq H^\bullet(G(2, V), \mathcal{F}^\ast),$$

which vanish except $H^0(G(2, V), \mathcal{F}^\ast) \simeq \mathbb{V}^*$ by the Bott theorem 20.1.

As for $H^\bullet(X, C_{24}(-t)) = H^\bullet(X, \text{Coker } \varphi(-t))$ (0 ≤ t ≤ 2), consider the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathscr{X}}(-(t+1)H_{\mathscr{X}} + 2L_{\mathscr{X}}) \rightarrow S^2(g^\ast \mathcal{F}^\ast)(-t) \rightarrow \text{Coker } \varphi(-t) \rightarrow 0.$$

Since $g$ is a $\mathbb{P}^2$-bundle, $H^\bullet(X, \mathcal{O}_{\mathscr{X}}(-(t+1)H_{\mathscr{X}} + 2L_{\mathscr{X}}))$ vanish except possibly in the case where $t = 2$, and $H^\bullet(X, S^2(g^\ast \mathcal{F}^\ast)(-t))$ vanish except possibly in the case where $t = 0$. Since $K_{\mathscr{X}} = -3H_{\mathscr{X}} - 2L_{\mathscr{X}}$, each cohomology of
with a linear subspace \( L \) of \( \mathbb{P}^3 \) gives a Lefschetz decomposition of a categorical crepant resolution, if exists, Calabi-Yau manifold.

Since the singularity of the double symmetroid \( Y \) associated to the sheaf \( \mathcal{E} \), after making a nice resolution of \( X \), \( \mathcal{E} \) is equivalent to the noncommutative resolution \( D^b(\mathcal{X}_b) \) associated to the sheaf \( \mathcal{R} := f_*\text{End}(\mathcal{O}_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}}(-L_{\mathcal{X}})) \).

Detailed study will be presented in a future publication [19].

In the next section, we will introduce the double cover \( \mathcal{Y} \) of the symmetroid \( \mathcal{X} \). After making a nice resolution of \( \mathcal{Y} \) in Section [5] we find a Lefschetz collection in the derived category \( D^b(\mathcal{X}) \) in Theorem 8.1.1

\[
(D_{\mathcal{X}}(-4), \ldots, D_{\mathcal{X}}(-1), D_{\mathcal{X}}) \subset D^b(\mathcal{X}) 
\]

Since the singularity of the double symmetroid \( \mathcal{Y} \) is complicated, the theorem [20] Theorem 1] seems to be difficult to apply for the resolution \( \mathcal{Y} \) to obtain the categorical resolution \( \mathcal{D} \) of \( D^b(\mathcal{Y}) \). However, we expect that the Lefschetz collection (3.11) gives a Lefschetz decomposition of a categorical crepant resolution, if exists, of \( D^b(\mathcal{Y}) \).

3.5. Calabi-Yau manifold \( X \) of a Reye congruence. For generality of arguments below, consider \( \mathcal{X} \) for any \( n > 0 \). We choose a general linear subsystem \( P \in |\mathcal{O}_X(1)| \) of dimension \( n \). We regard \( P \) as a general \( n \)-plane in \( \mathbb{P}(\mathbb{S}^2V^*) \) associated with a linear subspace \( L \cong \mathbb{C}^{n+1} \subset \mathbb{S}^2V^* \), i.e., \( P = \mathbb{P}(L) \). Explicitly we assume the form \( P = |Q_{A_1}, Q_{A_2}, \ldots, Q_{A_{n+1}}| \) with the quadratic forms \( q_{A_i} (1 \leq i \leq n+1) \) on \( \mathbb{P}(V) \).

Let \( P^\perp = \mathbb{P}(L^\perp) \subset \mathbb{P}(\mathbb{S}^2V) \) be the orthogonal projective space with \( L^\perp \subset \mathbb{S}^2V \), and define \( X := \mathcal{X} \cap P^\perp \). \( X \) is naturally identified with the complete intersection...
in $\mathcal{X} = S^2\mathbb{P}(V)$,

$$X = \{ A_1 \cdot w_{xy} = \cdots = A_{n+1} \cdot w_{xy} = 0 \} \cap \mathcal{X}.$$ 

Since $P$ is general, the orthogonal space $P^\perp$ is disjoint from $\text{Sing } \mathcal{X}$ from dimensional reason, and hence $X$ is smooth by the Bertini theorem. Conversely several properties of $X$ follow from assuming only that $X$ is smooth; first of all, smoothness implies that $P^\perp$ is disjoint from $\text{Sing } \mathcal{X} = X_0$, hence we may consider $X$ is embedded in $\mathcal{X}$. Moreover, we see the following properties:

**Proposition 3.5.2.** The morphism $g : \mathcal{X} \to G(2, V)$ gives an isomorphism $X \cong g(X)$. The linear system $P$ of quadrics is free from the base points.

**Proof.** Both assertions follow from $X \cap \mathcal{X}_0 = \emptyset$ in $\mathcal{X}$.

We start with the first assertion. Since $X$ is fiberwise a linear section with respect to $g$, we have only to show that if a fiber $\ell$ of $g$ intersects $X$, then $X \cap \ell$ consists of only one point. Assume the contrary. Then $X \cap \ell$ is a linear subspace of $\ell$ of positive dimension. Since $E_f \cap \ell$ is a conic in $\ell$, we have $E_f \cap X \neq \emptyset$, a contradiction to that $X \cap \mathcal{X}_0 = \emptyset$ in $\mathcal{X}$.

As for the second assertion, we note that the base locus of $P$ is given by $\{ x | A \cdot w_{xx} = \langle x, A \rangle = 0 \}$ for any $[A] \in P$, where $A \cdot w_{xx} = \langle x, A \rangle$ follows from (3.2). This is empty if and only if $X \cap \mathcal{X}_0 = \emptyset$. □

Reye congruence is a line congruence in $G(2, V)$, which is nothing but the isomorphic image of $X$ under $g : \mathcal{X} \to G(2, V)$ (cf. [30]). In this paper we often identify $X$ with $g(X)$. Below, we present a characterization of $X \subset G(2, V)$ by the projective geometry of quadrics in $P$. For a point $[l] \in G(2, V)$, we denote by $P_l$ the subspace of $P$ which parameterizes quadrics in $\mathbb{P}(V)$ containing the line $l$, namely, if we write $l = \langle x, y \rangle$ with $x, y \in \mathbb{P}(V)$, then $P_l = \{ [A] \in P | \langle x, A \rangle = \langle y, A \rangle = 0 \}$.

**Proposition 3.5.2.** A point $[l] \in G(2, V)$ is contained in $X \subset G(2, V)$ if and only if $P_l$ is an $(n-2)$-plane.

**Proof.** First we show that, for any $[l] \in G(2, V)$, $\dim P_l \leq n-2$. Assume by contradiction that $\dim P_l \geq n-1$. Then all the quadrics $[Q_A] \in P$ contains $l$ or the restrictions of quadrics $[Q_A] \in P$ to $l$ reduce to a unique quadric on $l$. This is a contradiction to the second statement of Proposition 3.5.1.

Assume that $[l] \in X$. We have only to show $\dim P_l \geq n-2$. By Proposition 3.5.1, there exists a unique $w_{xy} \in X \subset \mathcal{X}$ such that $l = \langle x, y \rangle$. Since $X = \mathcal{X} \cap P^\perp$, we have $\langle x, A \rangle = \langle y, A \rangle = 0$ for any $[A] \in P$ by (3.2). Therefore $P_l = \{ [A] \in P | \langle x, A \rangle = \langle y, A \rangle = 0 \}$, and hence $\dim P_l \geq n-2$.

Conversely, assume that $\dim P_l = n-2$. We may choose a basis $A_1, \ldots, A_{n+1}$ of $P$ such that $A_1, \ldots, A_{n-1}$ form a basis of $P_l$. Then the restrictions of quadrics $[Q_A] \in P$ to $l$ form the pencil of quadrics on $l$ spanned by $Q_{A_1}|_l$ and $Q_{A_{n+1}}|_l$. The corresponding pencil of symmetric $(1, 1)$-divisors on $l \times l$ has two base points $(x, y)$ and $(y, x)$ with $x \neq y$ since the base points of this pencil are disjoint from the diagonal. Then note that $l = \langle x, y \rangle$. By the definition of $(x, y)$, we have $\langle x, A_n \rangle = \langle x, A_{n+1} \rangle = 0$. By the choice of $A_1, \ldots, A_{n-1}$, we have $\langle x, A_1 \rangle = \cdots = \langle x, A_{n-1} \rangle = 0$. Therefore $\langle x, A \rangle = 0$ for any $[A] \in P$. This implies that $w_{xy} \in X$ by (3.2). □
When $n$ is even, $X$ is a Calabi-Yau $(n - 1)$-fold and satisfies:

**Proposition 3.5.3.** $\pi_1(X) \simeq \mathbb{Z}_2$ and $\text{Pic } X \simeq \mathbb{Z} \oplus \mathbb{Z}_2$, where the free part of $\text{Pic } X$ is generated by the class $D$ of a hyperplane section of $X$ restricted to $X$.

**Proof.** Consider the complete intersection $\tilde{X}$ in $\mathbb{P}(V) \times \mathbb{P}(V)$ defined by the pull-back of $P$. Then we have the projection morphism $\pi_{\tilde{X}} : \tilde{X} \to X$.

By the Lefschetz theorem, $\pi_1(\tilde{X}) = \{1\}$. Since the map $\pi_{\tilde{X}}$ is an étale double cover, we have $\pi_1(X) \simeq \mathbb{Z}_2$.

Let $E$ be any effective divisor on $X$. Since $\pi_{\tilde{X}}^* E$ is $\mathbb{Z}_2$-invariant, it is of type $(m, m)$ with some non-negative integer $m$. We may choose a homogeneous equation $F_E$ of $\pi_{\tilde{X}}^* E$ as symmetric or skew-symmetric. If $F_E$ is symmetric, then $E \sim mD$.

Assume that $F_E$ is skew-symmetric. Let $\tilde{D}$ be a skew-symmetric $(1, 1)$-divisor. Then $\pi_{\tilde{X}}^* E - m\tilde{D} = \text{div } (\alpha)$, where $\alpha$ is a $\mathbb{Z}_2$-invariant rational function of $\tilde{X}$. Then $\alpha$ is the pull-back of a rational function $\beta$ of $X$, and we see that $E - m\pi_{\tilde{X}}^* \tilde{D} = \text{div } (\beta)$ by looking at the zero and pole of $\beta$. Therefore $\text{Pic } X$ is generated by the classes of $D$ and $\pi_{\tilde{X}}^* \tilde{D}$. Since $2D \sim 2\pi_{\tilde{X}}^* \tilde{D}$, we conclude that $\text{Pic } X \simeq \mathbb{Z} \oplus \mathbb{Z}_2$. $\square$

When $n = 4$, $X$ is a Calabi-Yau threefold with the following invariants [10]:

$$\deg(X) = 35, \ c_2, D = 50, \ h^{2,1}(X) = 26, \ h^{1,1}(X) = 1,$$

where $c_2$ is the second Chern class of $X$. 
4. The double symmetroid $\mathcal{V}$ and Calabi-Yau variety $Y$

Consider Calabi-Yau $(n-1)$-fold $X$ of a Reye congruence for arbitrary even $n$. Under the projective duality $\mathcal{V}$, we find that $X$ is naturally paired with another Calabi-Yau variety $H$ in the symmetroid $\mathcal{H}$ (see [10] for $n=4$). The geometry of the symmetroid $\mathcal{H}$ was studied in detail by Tjurin [22]. In particular, he defined a double cover $\mathcal{V} \rightarrow \mathcal{H}$, which we call double symmetroid. In this section, we elaborate Tjurin’s construction. By considering linear sections of this double cover, we obtain a Calabi-Yau variety $Y$ for arbitrary even $n$. For $n=4$, $Y$ turns out to be a smooth Calabi-Yau threefold. These Calabi-Yau threefolds $X$ and $Y$ arise naturally from the (dual) Lefschetz collections (3.10) and (3.11) assuming the projective homological duality [25] [28].

4.1. Resolution $\mathcal{U}$ of $\mathcal{H}$. Let us define the following projective bundle over $\mathbb{P}(V)$:

$$\mathcal{U} = \mathbb{P}(S^2(\Omega(1)))$$

where $\Omega(1)$ is the cotangent sheaf of $\mathbb{P}(V)$. Considering the (dual of the) Euler sequence $0 \rightarrow \mathcal{O}(-1) \rightarrow V \otimes \mathcal{O} \rightarrow T(-1) \rightarrow 0$, we see that there is a canonical injection $\Omega(1) \hookrightarrow V^* \otimes \mathcal{O}$, which entails the injection $S^2\Omega(1) \hookrightarrow S^2V^* \otimes \mathcal{O}$. With this injection, we have a morphism

$$i_u : \mathcal{U} = \mathbb{P}(S^2(\Omega(1))) \rightarrow \mathbb{P}(S^2V^*).$$

Proposition 4.1.1. (1) The image of the morphism $i_u$ coincides with $\mathcal{H}$. In particular, the morphism gives a resolution of $\mathcal{H}$.

(2) $\mathcal{U} \simeq \{(x, [A]) | Ax = 0\} \subset \mathbb{P}(V) \times \mathbb{P}(S^2V^*)$.

Proof. (1) Since the fiber of $\Omega(1)$ over a point $[V_1] \in \mathbb{P}(V)$ is $(V/V_1)^*$, the fiber of the projective bundle $\mathcal{U} \rightarrow \mathbb{P}(V)$ over $[V_1]$ is given by $\mathbb{P}(S^2(V/V_1)^*)$. The morphism $i_u$ sends $\mathbb{P}(S^2(V_1)^*)$ into $\mathbb{P}(S^2V^*)$. Then the image is identified with quadrics in $\mathbb{P}(V)$ which are singular at $[V_1]$, or equivalently, symmetric matrices whose kernels contain $[V_1]$. Hence the image is contained in the symmetroid $\mathcal{H}$. The surjectivity is clear since $\mathcal{H}$ consists of singular symmetric matrices. Finally, $\mathcal{U}$ is smooth since it is a projective bundle.

(2) Let us denote the r.h.s. by $\mathcal{U}'$. There is a natural projection from $\mathcal{U}'$ to $\mathbb{P}(V)$. Then the fiber $\mathcal{U}'_{[x]}$ over $[x] \in \mathbb{P}(V)$ is the projective space of singular symmetric matrices whose kernels contain $V_1 = \mathbb{C}x$. Namely, $\mathcal{U}'_{[x]}$ coincides with the isomorphic image of $\mathbb{P}(S^2(V/V_1)^*)$ above. Hence $\mathcal{U}' \simeq \mathcal{U}$. \hfill $\square$

We summarize the resolution in the diagram,

$$\begin{array}{ccc}
\mathbb{P}(V) & \xrightarrow{\pi} & \mathcal{U} = \mathbb{P}(S^2(\Omega(1))) \xrightarrow{i_u} \mathcal{H}.
\end{array}$$

4.2. The double covering $\mathcal{V}$ of $\mathcal{H}$. Here we construct the double cover $\mathcal{V}$ of $\mathcal{H}$ by considering $\frac{n+2}{2}$-planes contained in each singular quadric.

Let us first consider a variety $\mathcal{Z}$ which parameterizes the pairs of quadrics $Q$ and $\frac{n}{2}$-planes $\mathbb{P}(\Pi)$ such that $\mathbb{P}(\Pi) \subset Q$. To parametrize $\frac{n}{2}$-planes in $\mathbb{P}(V)$, consider the Grassmann $G(\frac{n+2}{2}, V)$. Let

$$\begin{array}{c}
0 \rightarrow \mathcal{W}^* \rightarrow V^* \otimes \mathcal{O}_{G(\frac{n+2}{2}, V)} \rightarrow \mathcal{U}^* \rightarrow 0
\end{array}$$
be the dual of the universal exact sequence on \( G(\frac{n+2}{2}, V) \), where \( W \) is the universal quotient bundle of rank \( \frac{n}{2} \) and \( \Pi \) is the universal subbundle of rank \( \frac{n+2}{2} \). For an \( \frac{n}{2} \)-plane \( \mathbb{P}(\Pi) \subset \mathbb{P}(V) \), there exists a natural surjection \( S^2V^* \to S^2H^0(\mathbb{P}(\Pi), O_{\mathbb{P}(\Pi)}(1)) \) such that the projectivization of the kernel consisting of the quadrics containing \( \mathbb{P}(\Pi) \). By relativizing this surjection over \( G(\frac{n+2}{2}, V) \), we obtain the following surjection: \( S^2V^* \otimes O_{G(\frac{n+2}{2}, V)} \to S^2\Pi^* \). Let \( E^* \) be the kernel of this surjection, and consider the following exact sequence:

\[
0 \to E^* \to S^2V^* \otimes O_{G(\frac{n+2}{2}, V)} \to S^2\Pi^* \to 0.
\]

Set \( \mathcal{Z} = \mathbb{P}(E^*) \) and denote by \( \rho_\mathcal{Z} \) the projection \( \mathcal{Z} \to G(\frac{n+2}{2}, V) \). By (4.3), \( \mathcal{Z} \) is contained in \( G(\frac{n+2}{2}, V) \times \mathbb{P}(S^2V^*) \). Since the fiber of \( E^* \) over \([\Pi]\) parameterizes quadrics in \( \mathbb{P}(V) \) containing \( \mathcal{Z} \), we obtain the following surjection:

\[
\mathcal{Z} = \{ ([\Pi], Q) \mid \mathbb{P}(\Pi) \subset Q \} \subset G(\frac{n+2}{2}, V) \times \mathbb{P}(S^2V^*).
\]

Note that \( Q \) in \([\Pi], Q\) \( \in \mathcal{Z} \) is a singular quadric since a smooth quadric does not contain \( \frac{n}{2} \)-planes. Hence the symmetroid \( \mathcal{H} \) is the image of the natural projection \( \mathcal{Z} \to \mathbb{P}(S^2V^*) \). Now we introduce

\[
\mathcal{Z} \xrightarrow{\pi_\mathcal{Z}} \mathcal{Y} \xrightarrow{\rho_\mathcal{Y}} \mathcal{H}
\]

by the Stein factorization of \( \mathcal{Z} \to \mathcal{H} \). By (4.3), the tautological divisor \( H_{\mathbb{P}(E^*)} \) of \( \mathbb{P}(E^*) \to G(\frac{n+2}{2}, V) \) is nothing but the pull-back of a hyperplane section of \( \mathcal{H} \). We set

\[
M_\mathcal{Z} := H_{\mathbb{P}(E^*)} = \pi_\mathcal{Z}^* \circ \rho_\mathcal{Y}^* O_{\mathcal{H}}(1).
\]

We denote by \( \mathcal{Z}[Q] \) the fiber of \( \mathcal{Z} \to \mathcal{H} \) over a point \([Q] \in \mathcal{H}\).

**Lemma 4.2.1.** For a quadric \( Q \) of rank \( n \), the fiber \( \mathcal{Z}[Q] \) is the orthogonal Grassmann \( OG(\frac{n}{2}, n) \) which consists of two connected components.

**Proof.** The quadric \( Q \) of rank \( n \) induces a non-degenerate symmetric bilinear form \( q \) on the quotient \( V/V_1 \), where \( V_1 \) is the 1-dimensional vector space such that \([V_1]\) is the vertex of \( Q \). Then \( \frac{n}{2} \)-planes on \( Q \) naturally correspond to the maximal isotropic subspaces in \( V/V_1 \) with respect to \( q \), which are parameterized by the orthogonal Grassmann \( OG(\frac{n}{2}, n) \). \( \square \)

**Proposition 4.2.2.** The morphism \( \mathcal{Y} \to \mathcal{H} \) is of degree two and is branched along the locus of quadrics of rank less than or equal to \( n-1 \).

**Proof.** By Lemma 4.2.1, the degree of \( \mathcal{Y} \to \mathcal{H} \) is two since \( \mathcal{Z}[Q] \) has two connected components for a quadric \( Q \) of rank \( n \). If a quadric \( Q \) has rank less than or equal to \( n-1 \), the family of \( \frac{n}{2} \)-planes in \( Q \) is connected. Hence we have the assertion. \( \square \)

By this proposition, we see that \( \mathcal{Y} \) parameterizes connected families of \( \frac{n}{2} \)-planes in singular quadrics in \( \mathbb{P}(V) \) (cf. Fig.1).

**Definition 4.2.3.** Related to the morphism \( \rho_\mathcal{Y} : \mathcal{Y} \to \mathcal{H} \), we define \( G_\mathcal{Y} \) to be the inverse image by \( \rho_\mathcal{Y} \) of the locus of quadrics of rank less than or equal to \( n-2 \).
Since $G_{\mathcal{Y}}$ is contained in the ramification locus of $\rho_{\mathcal{Y}}$, it is clearly isomorphic to the locus of quadrics of rank less than or equal to $n-2$.

$\mathcal{Y}$ has the following nice properties in view of the minimal model program.

**Proposition 4.2.4.** The Picard number of $\mathcal{Z}$ is two and $\pi_{\mathcal{Y}}: \mathcal{Z} \to \mathcal{Y}$ is a Mori fiber space. In particular, $\mathcal{Y}$ is a Q-factorial Gorenstein canonical Fano variety with Picard number one. Moreover, $\mathcal{Y}$ is smooth outside $G_{\mathcal{Y}}$ and the Fano index of $\mathcal{Y}$ is $\frac{n(n+1)}{2}$.

**Proof.** Since $\mathcal{Z}$ is a projective bundle over $G(\frac{n+2}{2}, V)$, its Picard number is two. Note that the relative Picard number of $\pi_{\mathcal{Y}}: \mathcal{Z} \to \mathcal{Y}$ is one, then we see that the Picard number of $\mathcal{Y}$ is one. Since a general fiber of $\pi_{\mathcal{Y}}$ is a Fano variety by Lemma 4.2.1, $\pi_{\mathcal{Y}}$ is a Mori fiber space. By [14, Lemma 5-1-5], $\mathcal{Y}$ is Q-factorial, and, by [9, Corollary 4.6], $\mathcal{Y}$ has only kawamata log terminal singularities. $\mathcal{Y}$ is a Gorenstein Fano variety since it is a double cover of a Gorenstein Fano variety $\mathcal{H}$ and the ramification locus has codimension greater than one by Proposition 4.2.2. Thus $\mathcal{Y}$ has only canonical singularities.

As we mentioned in Subsection 3.2, the singular locus of $\mathcal{H}$ is the locus of quadrics of rank less than or equal to $n-1$. Since $\rho_{\mathcal{Y}}: \mathcal{Y} \to \mathcal{H}$ is etale outside this locus, $\mathcal{Y}$ is smooth outside the inverse image of this locus. Moreover, by the proof of [10, Lemma 3.6], $\mathcal{H}$ has only ordinary double points along the locus of quadrics of rank $n-1$. Since $\rho_{\mathcal{Y}}$ is ramified along this locus, $\mathcal{Y}$ is smooth along the inverse image of this locus. Therefore $\mathcal{Y}$ is smooth outside $G_{\mathcal{Y}}$.

Since $K_{\mathcal{H}} = \mathcal{O}_{\mathcal{H}}(\frac{n(n+1)}{2})$, the Fano index of $\mathcal{Y}$ is $\frac{n(n+1)}{2}$. \qed

**Remark.** We will show that $\mathcal{Y}$ has only terminal singularities when $n = 4$ (see Proposition 5.2.2).

When $n = 4$, we have more detailed descriptions of the fibers of $\rho_{\mathcal{Y}}$.

**Proposition 4.2.5.** If $\text{rank} Q = 4$, then $\mathcal{Z}_{[Q]}$ is a disjoint union of two smooth rational curves. Each connected component is identified with a conic on $G(3, V)$. If $\text{rank} Q = 3$, then $\mathcal{Z}_{[Q]}$ is a smooth rational curve, which is also identified with a conic on $G(3, V)$. If $\text{rank} Q = 2$, then $\mathcal{Z}_{[Q]}$ is the union of two $\mathbb{P}^3$’s intersecting at one point. If $\text{rank} Q = 1$, then $\mathcal{Z}_{[Q]}$ is a (non-reduced) $\mathbb{P}^3$.

**Proof.** If $\text{rank} Q = 4$, the fiber $\mathcal{Z}_{[Q]}$ consists of two disconnected components, and is isomorphic to the orthogonal Grassmann $\text{OG}(2, 4)$ by Lemma 1.2.1. To be more explicit, let $[V_1] \in \mathbb{P}(V)$ be the vertex of $Q$. Then the quadric $Q$ is the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ with the vertex $[V_1]$. There are two distinct $\mathbb{P}^1$-families of lines on $\mathbb{P}^1 \times \mathbb{P}^1$. Each of the families can be understood as a corresponding conic on $G(2, V/V_1)$, which gives one of the connected components of $\text{OG}(2, 4)$. Under the natural map $G(2, V/V_1) \to G(3, V)$, we have a $\mathbb{P}^1$ family of 2-planes on $Q$ parameterized by a conic on $G(3, V)$.

If $\text{rank} Q = 3$, the vertex of the quadric $Q$ is a line $[V_2] \subset \mathbb{P}(V)$. The quadric $Q$ is the cone over a conic with the vertex $[V_2]$. The conic is contained in $\mathbb{P}(V/V_2) = G(1, V/V_2)$, and can be identified with a conic in $G(3, V)$ under the natural map $G(1, V/V_2) \to G(3, V)$.

If $\text{rank} Q = 2$, then the quadric $Q$ has a vertex $[V_3] \subset \mathbb{P}(V)$ and is the union of two 3-planes intersecting along the 2-plane $[V_3]$. Hence $\mathcal{Z}_{[Q]} \subset G(3, V)$ is given
by the union of the corresponding $\mathbb{P}^3$'s, i.e., $G(3, 4)$'s in $G(3, V)$, which intersect at one point $[V_3]$.

If rank $Q = 1$, then $Q$ is a double $3$-plane. Thus $\mathcal{Z}_Q$ is a (non-reduced) $\mathbb{P}^3 \cong G(3, 4)$. □

Fig.1. Quadrics $Q$ in $\mathbb{P}(V)$ and families of planes therein.
The singular loci of $Q$ are written by $[V_k]$ with $k = 5 - \text{rk} Q$. Also the parameter spaces of the planes in each $Q$ are shown. See also Fig.2 in Section 5.

We write by $G^1_{\mathcal{Y}}$ (resp. $G^2_{\mathcal{Y}}$) the inverse image under $\rho_{\mathcal{Y}}$ of the locus of quadrics of rank one (resp. two). We see that $G_{\mathcal{Y}} \cong \mathbb{S}^2 \mathbb{P}(V^*)$, $G^1_{\mathcal{Y}} \cong v_2(\mathbb{P}(V^*))$ and $G^2_{\mathcal{Y}} = G_{\mathcal{Y}} \setminus G^1_{\mathcal{Y}}$. Using these, we summarize our construction above for $n = 4$ in the following diagram:

$$
\begin{array}{c}
\mathcal{Y} \xrightarrow{\text{\scriptsize$\mathbb{P}^8$-bundle}} G(3, V) \\
\downarrow \pi_{\mathcal{Y}} \quad \quad \quad \downarrow \rho_{\mathcal{Y}}
\end{array}
$$

(4.4) \quad \quad \quad \quad G^1_{\mathcal{Y}} \subset G_{\mathcal{Y}} \subset \mathcal{Y}

where $\pi_{\mathcal{Y}}$ is a $\mathbb{P}^1$-fibration over $\mathcal{Y} \setminus G_{\mathcal{Y}}$ by Proposition 4.2.5. In Section 3 we will construct a nice desingularization $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$. There we will also study the geometry of $\tilde{\mathcal{Y}} \to \mathcal{Y}$ along the loci $G_{\mathcal{Y}}$ and $G^1_{\mathcal{Y}}$ in full details.

4.3. Calabi-Yau variety $Y$. Assume that a Reye congruence Calabi-Yau $(n-1)$-fold $X = \mathcal{Z} \cap P^\perp$ is given by a general $n$-plane $P$ in $\mathbb{P}(\mathbb{S}^2 V^*)$, i.e., $P = \mathbb{P}(L)$ with $L \cong \mathbb{C}^{n+1} \subset \mathbb{S}^2 V^*$. Define $H := \mathcal{H} \cap P$. Then $H$ is a determinantal hypersurface of degree $n + 1$ in $P \cong \mathbb{P}(L)$ and hence the canonical bundle of $H$ is trivial.

Let $Y$ be the pull-back of $H$ on $\mathcal{Y}$. According to [2], we say that $Y$ is orthogonal to $X$ and vice versa.

Proposition 4.3.1. $Y$ is a Calabi-Yau variety.

Proof. The canonical divisor $K_Y$ is trivial since $K_H \sim 0$ and the branch locus of $Y \to H$ has the codimension greater than or equal to two.
By Proposition $1.2.3$, $\mathcal{Y}$ has only Gorenstein canonical singularities and $h^i(\mathcal{O}_Y) = 0$ for $i > 0$. Then, by a version of the Bertini theorem, which says that general sections of Gorenstein canonical varieties are also Gorenstein and canonical, we see that $Y$ has only Gorenstein canonical singularities. It is standard to derive $h^i(\mathcal{O}_Y) = 0$ for $0 < i < \dim Y$ from $h^i(\mathcal{O}_Y) = 0$ for $i > 0$ by the Kodaira-Kawamata-Vieweg vanishing theorem.

In the rest of this paper, we restrict our attention to $n = 4$. In this case, $Y$ is smooth by Proposition $1.2.4$ since $Y \cap G_{\mathcal{Y}} = \emptyset$ by dimensional reason. This Calabi-Yau manifold $Y$ coincides with the double covering $Y$ defined in $[16]$, where $Y$ is called the (shifted) Mukai dual to $X$.

In the previous paper $[16]$ Prop.3.11 and Prop.3.12, we have determined invariants of the Calabi-Yau threefold $Y$. Here we reproduce these invariants using the construction summarized in $[14]$.

Let us first recall $[14]$ for the definition of $E^*$ and set $E := (E^*)^*$. Then we have

**Lemma 4.3.2.** $c_1(E) = c_1(\mathcal{O}_{G(3, V)}(4))$.

**Proof.** Note that $c_1(\mathcal{O}_{G(3, V)}(1))$ is given by the Schubert cycle $\sigma_1$, which is $c_1(U^*)$ in our notation. Since $\text{rk} U = 3$, we have $c_1(E) = c_1(S^2U^*) = 4c_1(U^*)$. Thus we have the assertion. □

Now let us note the relative Euler sequence associated with the projective bundle $\rho_{\mathcal{X}}: \mathcal{X} = \mathbb{P}(E^*) \to G(3, V)$:

(4.5) \[ 0 \to \mathcal{O}_{\mathcal{X}} \to \rho_{\mathcal{X}}^*E^*(M_{\mathcal{X}}) \to T_{\mathcal{X}} \to \rho_{\mathcal{X}}^*T_{G(3, V)} \to 0. \]

From this we obtain the following:

**Lemma 4.3.3.** Let $N_{\mathcal{X}} := \rho_{\mathcal{X}}^*\mathcal{O}_{G(3, V)}(1)$. $K_{\mathcal{X}} = -9M_{\mathcal{X}} - N_{\mathcal{X}}$ holds for the canonical divisor $K_{\mathcal{X}}$ on $\mathcal{X}$ and we have the following cohomologies for the sheaves on $\mathcal{X}$ and for $0 \leq k \leq 10$ ($-1 \leq k \leq 10$ for (2)):

(1) $H^*(\mathcal{O}_{\mathcal{X}}(-(k + 1)M_{\mathcal{X}} + N_{\mathcal{X}})) = 0$.

(2) $H^*(\mathcal{O}_{\mathcal{X}}(-kM_{\mathcal{X}})) \simeq H^*(\rho_{\mathcal{X}}^*E^*(-(k-1)M_{\mathcal{X}})) \simeq \begin{cases} S^2V & (\bullet, k) = (0, -1) \\ \mathbb{C} & (\bullet, k) = (0, 0), (13, 10), \\ 0 & \text{(otherwise)} \end{cases}$

(3) $H^*(\rho_{\mathcal{X}}^*T_{G(3, V)}(-kM_{\mathcal{X}})) \simeq H^*(T_{\mathcal{X}}(-kM_{\mathcal{X}})) \simeq \begin{cases} \text{sl}(V) & (\bullet, k) = (0, 0) \\ \mathbb{C} & (\bullet, k) = (12, 10), \\ 0 & \text{(otherwise)} \end{cases}$

**Proof.** The claimed formula of $K_{\mathcal{X}}$ follows by taking the determinant of the Euler sequence $[14]$. We should note that rank $E = 9$ and $N_{\mathcal{X}} = \rho_{\mathcal{X}}^*\mathcal{O}_{G(3, V)}(1)$.

For the calculations of the cohomologies in (1), (2), (3), we use the Serre duality, the Kodaira vanishing theorem and also the Bott theorem $2.0.1$ as well as the defining exact sequence $[14]$ of $E^*$.

(1) By the Serre duality, we have $H^*(\mathcal{O}_{\mathcal{X}}(-(k + 1)M_{\mathcal{X}} + N_{\mathcal{X}}) \simeq H^{14-\bullet}(\mathcal{O}_{\mathcal{X}}((k-8)M_{\mathcal{X}} - 2N_{\mathcal{X}}))$. From this, the claimed vanishings follow for the range $0 \leq k \leq 7$ for all $\bullet$ since $\rho_{\mathcal{X}}: \mathcal{X} \to G(3, V)$ is a $\mathbb{P}^8$-bundle. When $k = 8$, the vanishing follows from $H^{14-\bullet}(\mathcal{O}_{\mathcal{X}}(-2N_{\mathcal{X}})) \simeq H^{14-\bullet}(G(3, V), \mathcal{O}_{G(3, V)}(-2)) = 0$. When $k =$
9, we need to evaluate \( H^{14-\bullet}(\mathcal{O}_\mathcal{X}(M_\mathcal{X} - 2N_\mathcal{X})) \approx H^{14-\bullet}(G(3, V), \mathcal{E}(-2)) \). By tensoring the dual of (4.3) by \( \mathcal{O}_{G(3,V)}(-2) \) and using the Bott theorem, it is easy to obtain the claimed vanishing. When \( k = 10 \), we have \( H^{14-\bullet}(\mathcal{O}_\mathcal{X}(2M_\mathcal{X} - 2N_\mathcal{X})) \approx H^{14-\bullet}(G(3, V), S^2\mathcal{E}(-2)) \). For the calculation of the cohomologies of \( S^2\mathcal{E}(-2) = S^2\mathcal{E} \otimes \mathcal{O}_{G(3,V)}(-2) \), we note the following exact sequence:

\[
0 \to \wedge^2(S^2\mathcal{I}) \to S^2V \otimes S^2\mathcal{I} \to S^2(S^2V) \otimes \mathcal{O}_{G(3,V)} \to S^2\mathcal{E} \to 0.
\]

Tensoring by \( \mathcal{O}_{G(3,V)}(-2) \) and using the Bott theorem, the claimed vanishing follows for all degree.

(2) Since the calculations of \( H^\bullet(\mathcal{O}_\mathcal{X}(-kM_\mathcal{X})) \) is easy, we omit them. For the cohomologies \( H^\bullet(\rho_2^*\mathcal{E}^*(-(k - 1)M_\mathcal{X})) \), when \( k = 0 \), we have to evaluate \( H^\bullet(\rho_2^*\mathcal{E}^*(M_\mathcal{X})) = H^\bullet(G(3, V), \mathcal{E}^* \otimes \mathcal{E}) \). This can be done by considering two short exact sequences: one from tensoring the defining exact sequence \( \mathcal{E} \) by \( \mathcal{E} \) and the other from tensoring the dual of \( \mathcal{E} \) by \( S^2\mathcal{I}^* \). The cases of other values of \( k \) are rather easy, so we omit their details.

(3) For the calculations of \( H^\bullet(\rho_2^*T_{G(3,V)}(-kM_\mathcal{X})) \), we use \( T_{G(3,V)} = \mathcal{I}^* \otimes \mathcal{W} \). For example, for \( k = 0 \), we evaluate \( H^\bullet(\rho_2^*T_{G(3,V)}) = H^\bullet(G(3, V), \mathcal{I}^* \otimes \mathcal{W}) \), which is non-vanishing only for \( \bullet = 0 \) with the result \( \Sigma^{1,0,0,0,-1}\mathcal{I}^* \simeq s_l(V) \simeq \mathbb{C}^{24} \). For \( k \geq 1 \), use the Serre duality and the defining exact sequence \( (4.3) \).

Finally the calculations of \( H^\bullet(T_\mathcal{X}(-kM_\mathcal{X})) \) are done with the relative Euler sequence \( (4.5) \) and also using the results obtained so far. Since they are straightforward, we omit them here.

Let \( M \) be the pull-back of \( O_H(1) \) to \( Y \). The following proposition refines the results in [16, Propositions 3.11 and 3.12]:

**Proposition 4.3.4.** The 3-fold \( Y \) is a simply connected Calabi-Yau 3-fold such that \( \text{Pic}Y = \mathbb{Z}[M], M^3 = 10, c_2(Y), M = 40 \) and \( e(Y) = -50 \). In particular, \( h^{1,1}(Y) = 1 \) and \( h^{1,2}(Y) = 26 \).

**Proof:** We have already shown that \( Y \) is a smooth Calabi-Yau threefold. Since \( Y \to H \) is a double cover, we have \( M^3 = 2c_1(\mathcal{O}_H(1))^3 = 10 \).

To calculate other invariants, we use the restriction of \( \pi_\mathcal{X} : \mathcal{X} \to \mathcal{Y} \) over \( Y \), which is a \( \mathbb{P}^1 \)-fibration. Set \( Z := \pi_\mathcal{X}^*(Y) \) and \( \pi_Z \) be the restriction of \( \pi_\mathcal{X} \) to \( Z \). We also set \( N_Z \) and \( M_Z \) be the restrictions of \( N_\mathcal{X} \) and \( M_\mathcal{X} \) to \( Z \), respectively.

Let us first note that we have \( K_Z = M_Z - N_Z \) for the canonical divisor by Lemma \[4.3.3\] since \( Z \) is a complete intersection of ten members of \( |M_\mathcal{X}| \). Also we note the following Koszul resolution of \( \mathcal{O}_Z \) as a \( \mathcal{O}_\mathcal{X} \)-module:

\[
0 \to \wedge^{10}(\mathcal{O}_\mathcal{X}(-M_\mathcal{X})^{\oplus 10}) \to \cdots \to \mathcal{O}_\mathcal{X}(-M_\mathcal{X})^{\oplus 10} \to \mathcal{O}_\mathcal{X} \to \mathcal{O}_Z \to 0.
\]

We observe the following isomorphisms:

\[
H^\bullet(Z, T_Z) \simeq H^\bullet(Z, \pi_Z^*T_Y) \simeq H^\bullet(Y, T_Y),
\]

where we note that \( H^\bullet(Y, T_Y) \) vanishes for \( \bullet = 0, 3 \) since \( Y \) is a Calabi-Yau threefold. The second isomorphism follows from the fact that \( Z \to Y \) is a \( \mathbb{P}^1 \)-fibration. To see the first isomorphism, let us note the exact sequence \( 0 \to T_{Z/Y} \to T_Z \to \pi_Z^*T_Y \to 0 \), from which we have \( T_{Z/Y} = \mathcal{O}_Z(-K_Z) \) since \( K_Y = 0 \) and \( T_{Z/Y} \) is an invertible sheaf. Then we have \( H^\bullet(Z, T_{Z/Y}) = H^\bullet(Z, \mathcal{O}_Z(-M_Z + N_Z)) \). Tensoring

\[\footnote{The authors would like to thank Prof. R.F. for suggesting this exact sequence.}\]
the resolution by $\mathcal{O}(-M_\mathcal{Y} + N_\mathcal{Y})$ and using (1) of Lemma we see the vanishing $H^\bullet(Z, T_\mathcal{Z}/Y) = 0$. This entails the first isomorphism.

Next let us consider the exact sequence

\begin{equation}
0 \to T_\mathcal{Z} \to T_\mathcal{Z}|_\mathcal{Z} \to \mathcal{O}_\mathcal{Z}(M_\mathcal{Z})^{\oplus 10} \to 0.
\end{equation}

Since $Z \to Y$ is a $\mathbb{P}^1$-fibration, we have $H^\bullet(Z, \mathcal{O}_\mathcal{Z}(M_\mathcal{Z})) \simeq H^\bullet(Y, \mathcal{O}_Y(M))$, where the r.h.s. vanish by the Kodaira vanishing theorem except $H^0(Y, \mathcal{O}_Y(M)) \simeq \mathbb{C}^5$. Therefore, by (4.7) and (4.8), we have

\begin{equation}
H^\bullet(T_\mathcal{Z}) \simeq H^\bullet(T_\mathcal{Z}|_\mathcal{Z}) \quad \text{for } \bullet \geq 2.
\end{equation}

We finally calculate the cohomology of the sheaf $T_\mathcal{Z}|_\mathcal{Z}$ as

\begin{equation}
H^0(T_\mathcal{Z}|_\mathcal{Z}) = \mathbb{C}^{\oplus 24}, \quad H^2(T_\mathcal{Z}|_\mathcal{Z}) = \mathbb{C}, \quad H^i(T_\mathcal{Z}|_\mathcal{Z}) = 0 (i \neq 0, 2).
\end{equation}

These results follow from tensoring the resolution by $T_\mathcal{Z}$ and using (3) of Lemma Now combining (4.10) with (4.9) and (4.7), we obtain $h^1(T_\mathcal{Z}) = 26$ and $h^2(T_\mathcal{Z}) = h^2(T_Y) = 1$.

Since $Y$ is a Calabi-Yau threefold, we have $e(Y) = 2(h^2(T_Y) - h^1(T_Y)) = -50$. Also by the Riemann-Roch formula, and the vanishings derived above, we have

\begin{equation}
\chi(Y, \mathcal{O}_Y(M)) = \frac{M^3}{6} + \frac{c_2(Y).M}{12} = \dim H^0(Y, \mathcal{O}_Y(M)) = 5.
\end{equation}

From this, we obtain $c_2(Y).M = 40$.

It remains to show the simply connectedness of $Y$. This will imply $\text{Pic} Y \simeq \mathbb{Z}$ since we have already shown $\rho(Y) = h^2(T_Y) = 1$. Let $W$ be a general 4-dimensional complete intersection of $\mathbb{Y}$ by members of $|M_\mathcal{Y}|$ containing $Y$. Then $W$ is a Fano 4-fold, and moreover smooth, since $\mathbb{Y}$ is smooth outside $G_\mathbb{Y}$ by Proposition and the codimension of $G_\mathbb{Y}$ in $\mathbb{Y}$ is 5. Hence we know that $W$ is simply connected. The simply connectedness of $Y$ follows from the Lefschetz theorem since $Y$ is an ample divisor on $W$. □

In a separate publication, we will show the derived equivalence $D^b(X) \simeq D^b(Y)$ using the properties of the (dual) Lefschetz collections (3.10) and (3.11).
5. The resolution $\overline{\mathcal{W}} \to \mathcal{W}$

We will restrict our attention to the case of $n = 4$ (dim $V = 5$).

5.1. Conics and planes in $G(3, V)$. Recall the Stein factorization $\mathcal{Z} \to \mathcal{W} \to \mathcal{H}$. As shown in Proposition 4.2.5, the fiber of $\mathcal{Z} \to \mathcal{W}$ over $y \in \mathcal{W} \setminus G_{\mathcal{W}}$ is a smooth conic in $G(3, V)$ which parametrizes planes contained in the corresponding quadric $\rho_{\mathcal{Z}}(y) = [Q_y] \in \mathcal{H}$. Writing this conic by $q_y$, we can represent $y \in \mathcal{W} \setminus G_{\mathcal{W}}$ by the pair $([Q_y], q_y)$. Let us note that each conic $q_y \subset G(3, V)$ determines a conic in $\mathbb{P}(\wedge^3 V)$ under the Plücker embedding, and in turn, determines a plane $\mathbb{P}_{q_y}^2$ in $\mathbb{P}(\wedge^3 V)$ which contains the conic.

If $\text{rank } Q_y = 4$, the plane $\mathbb{P}_{q_y}^2$ recovers the conic by

$$\mathbb{P}_{q_y}^2 \cap G(3, V) \subset \mathbb{P}(\wedge^3 V).$$

This fact can be seen as follows: First note that $Q_y$ determines a quadric $\overline{Q}_y$ in $\mathbb{P}(V/V_1)$ with $[V_1]$ being the vertex of $Q_y$. Note also that $q_y$ determines a conic $\overline{q}_y$ in $G(2, V/V_1)$ uniquely and the corresponding plane $\mathbb{P}_{\overline{q}_y}^2 \subset \mathbb{P}(\wedge^2 (V/V_1))$. Since $G(2, V/V_1)$ is a quadric in $\mathbb{P}(\wedge^2 (V/V_1))$ and $\mathbb{P}_{\overline{q}_y}^2 \not\subset G(2, V/V_1)$ holds for rank $Q_y = 4$, the intersection with $\mathbb{P}_{\overline{q}_y}^2$ recovers the conic $q_y$ and also $\overline{q}_y$.

If $\text{rank } Q_y = 3$, the plane $\mathbb{P}_{q_y}^2$ does not recover the conic $q_y$. To see this, we recall the fact that there are two types of planes contained in $G(3, V)$. The first one is a plane which is given by

$$P_{V_2} := \{ [\Pi] \in G(3, V) \mid V_2 \subset \Pi \} \cong \mathbb{P}^2$$

with some $V_2$. The second one is

$$P_{V_1, V_4} := \{ [\Pi] \in G(3, V) \mid V_1 \subset \Pi \subset V_4 \} \cong \mathbb{P}^2$$

with some $V_1$ and $V_4$ satisfying $V_1 \subset V_4$. We call $P_{V_2}$ a $\rho$-plane and $P_{V_1, V_4}$ a $\sigma$-plane. While these are (projective) planes in $G(3, V)$, we may (and will) consider them in $\mathbb{P}(\wedge^3 V)$ under the Plücker embedding $G(3, V) \subset \mathbb{P}(\wedge^3 V)$. When rank $Q_y = 3$, the conic $q_y$ parametrizes planes containing the vertex $\mathbb{P}(V_2)$ of $Q_y$, hence the corresponding plane $\mathbb{P}_{q_y}^2$ is a $\rho$-plane and the intersection becomes $\mathbb{P}_{q_y}^2 \cap G(3, V) = \mathbb{P}_{\overline{q}_y}^2$.

In [21 §3.1], the Hilbert scheme of conics in $G(3, V)$ has been studied in general. As an element of the Hilbert scheme, every conic $q$ in $G(3, V)$ determines the corresponding conic in $\mathbb{P}(\wedge^3 V)$ and then the corresponding plane $\mathbb{P}_{q}^2$. We follow [21] in the following definition:

**Definition 5.1.1.** A conic $q$ is called $\tau$-conic if $\mathbb{P}_{q}^2 \not\subset G(3, V)$ holds. $q$ is called $\rho$-conic and $\sigma$-conic, respectively, if the plane $\mathbb{P}_{q}^2$ is a $\rho$-plane and $\sigma$-plane.

Clearly the $\rho$-planes $P_{V_2}(V_2 \subset V)$ are parametrized by $G(2, V)$ while $\sigma$-planes $P_{V_1, V_4}(V_1 \subset V_4 \subset V)$ are parametrized by the flag variety $F(1, 4, V) \cong \mathbb{P}(\hat{D}(V)) \cong \mathbb{P}(\hat{D}(V^*))$. Since $G(2, V)$ and $F(1, 4, V)$ parametrize planes in $\mathbb{P}(\wedge^3 V)$, these define subvarieties in $G(3, \wedge^3 V)$, which we denote by $\overline{\mathcal{F}}_\rho$ and $\overline{\mathcal{F}}_\sigma$, respectively.
5.2. Smooth conics and conics of rank two. Denote by $\mathcal{U}$ the universal sub-bundle on $G(3, V)$, and regard $\mathbb{P}(\mathcal{U})$ as the universal family of the planes in $\mathbb{P}(V)$. Consider the natural projection

$$\varphi_{\mathcal{U}} : \mathbb{P}(\mathcal{U}) \to \mathbb{P}(V).$$

For a conic $q \subset G(3, V)$, we consider the restriction $\mathbb{P}(\mathcal{U}|_q)$ and its image under $\varphi_{\mathcal{U}}$.

**Proposition 5.2.1.** Smooth $\tau$- and $\rho$-conics correspond bijectively to points in $\mathfrak{Y} \setminus G_{\mathfrak{Y}}$.

**Proof.** We have seen in the beginning of the preceding subsection that each point $y \in \mathfrak{Y} \setminus G_{\mathfrak{Y}}$ determines a smooth conic $q_y$ in $G(3, V)$ (see also Proposition 1.2.3).

For the converse, suppose a smooth conic $q \subset G(3, V)$ is given. Note that the dual bundle $\mathcal{U}^*$ on $G(3, V)$ restricts as $\mathcal{U}^*|_q \simeq \mathcal{O}(1)_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$, or $\mathcal{O}(2) \oplus \mathcal{O}_{\mathbb{P}^2}$ since $\mathcal{U}^*$ is generated by its global sections and $\deg \mathcal{U}^*|_q = c_1(\wedge^3 \mathcal{U}^*|_q) = 2$. Let $Q$ be the image of $\mathbb{P}(\mathcal{U}|_q)$ under $\varphi_{\mathcal{U}}$. Then there are two possibilities; (i) the degree of $\mathbb{P}(\mathcal{U}|_q) \to Q$ is two, and $Q$ is a 3-plane, i.e., a quadric of rank 1, or (ii) the degree of $\mathbb{P}(\mathcal{U}|_q) \to Q$ is one and $Q$ is a quadric of rank 4 or 3 depending on the splitting type of $\mathcal{U}^*|_q$, i.e., $\mathcal{O}(1)_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ or $\mathcal{O}(2) \oplus \mathcal{O}_{\mathbb{P}^2}$, respectively (see Example 5.2.2 below). The case (i) is excluded by the condition $y = ([Q], q) \in \mathfrak{Y} \setminus G_{\mathfrak{Y}}$. Also if $Q$ is a 3-plane $\mathbb{P}(V_4)$, then $q \subset \{[U] \in G(3, V) \mid U \subset V_4\}$ and $\mathbb{P}^2_q$ must be a $\sigma$-plane, i.e., $q$ is a $\sigma$-conic by definition.

Thus we see that every smooth $\tau$- or $\rho$-conic determines a point $y = ([Q], q) \in \mathfrak{Y} \setminus G_{\mathfrak{Y}}$, and vice versa. \qed

We present some explicit examples of conics according to their ranks. It should be noted that the Hilbert scheme of conics admits the natural $SL(V)$-action, and hence the examples below describe the general properties of each orbit of the Hilbert scheme under the $SL(V)$-action.

**Example 5.2.2.** (Conics of rank 3) Taking a basis $e_1, \cdots, e_5$ of $V$, consider the subspaces, for example, to be $V_1 = \langle e_1 \rangle$, $V_4 = \langle e_1, e_2, e_3, e_4 \rangle$ and $V_2 = \langle e_4, e_5 \rangle$. Then typical examples of $\tau$-, $\rho$-, $\sigma$-conics, respectively, may be given explicitly in terms of the homogeneous coordinates of $G(3, V)$:

$$q_\tau = \left\{ \begin{array}{c}
\begin{array}{cccc}
s & t & 0 & 0 \\
0 & s & t & 0 \\
0 & 0 & 0 & 1
\end{array}
\end{array} \right\}, \quad q_\rho = \left\{ \begin{array}{c}
\begin{array}{cccc}
s^2 & st & t^2 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}
\end{array} \right\}, \quad q_\sigma = \left\{ \begin{array}{c}
\begin{array}{cccc}
s & t & 0 & 0 \\
0 & s & t & 0 \\
0 & 0 & 1 & 0
\end{array}
\end{array} \right\},$$

where $[s, t] \in \mathbb{P}^1$ parameterizes each conic $q$. The $\tau$-conic above is on a unique plane $P_{q_\tau} = \mathbb{P}((e_{135}, e_{245}, e_{235} + e_{145})) \subset \mathbb{P}(\wedge^3 V)$ and characterized by the conic $P_{q_\tau} \cap G(3, V) = \{p_{135}p_{245} - p_{145}^2 = 0\}$, where $p_{ijk}$ are the Plücker coordinates and we have introduced a notation $e_{ijk} := e_i \wedge e_j \wedge e_k$. The $\rho$-conic above is on a plane $P_{q_\rho} = \mathbb{P}((e_{145}, e_{245}, e_{345}))$ with its equation $p_{145}p_{345} - p_{245}^2 = 0$. Similarly, the $\sigma$-conic above is on the plane $P_{q_\sigma} = \mathbb{P}(\wedge^3 V)$ and the equation $p_{124}p_{234} - p_{134}^2 = 0$ for $q_\sigma$. \[ \]

Let $q$ be a $\tau$-conic of rank 2. Then $q$ is a pair of intersecting lines, say, $l_1$ and $l_2$. We may write $l_i = \{[\Pi] \mid V_i^{(i)} \subset \Pi \subset V_4^{(i)}\}$, where $\mathbb{P}(V_2^{(i)}) \subset \mathbb{P}(V)$ are lines and $\mathbb{P}(V_4^{(i)}) \subset \mathbb{P}(V)$ are 3-planes for $i = 1, 2$. Since $l_1 \cap l_2 = \emptyset$, it holds
dim\(V_2(1) \cap V_2(2)\) \geq 1. Since \(q\) is not a \(\rho\)-conic, \(V_2(1) \neq V_2(2)\). Therefore we see that \(dim(V_2(1) \cap V_2(2)) = 1\) and \(l_1 \cap l_2 = [V_2(1) + V_2(2)]\). If \(V_4(1) = V_4(2)\), then \(q\) is contained in \(P_{V_4(1),V_4(2)}\) and then \(q\) must be a \(\sigma\)-conic. Therefore \(V_4(1) \neq V_4(2)\).

In summary, \(q = l_1 \cup l_2\) is a \(\tau\)-conic of rank two iff
\[
V_4(1) \neq V_4(2) (d = 2, 4), V_4(1) \cap V_4(2) = V_2(1) + V_2(2).
\]

The other two types of conics of rank two may be described in a similar way.

**Example 5.2.3.** (Conics of rank 2) We present examples of \(\tau\)-, \(\rho\)-, and \(\sigma\)-conics of rank 2, respectively. (1) From the above description of \(\tau\)-conic of rank 2 may be given by \(q_\tau = l_1 \cup l_2\) with
\[
l_1 = \{[e_1, e_2, s e_3 + te_4] | [s, t] \in \mathbb{P}^1\}, \quad l_2 = \{[e_2, e_3, s e_1 + te_5] | [s, t] \in \mathbb{P}^1\}.
\]

It is easy to see that this conic is on a unique plane \(P_q^2 = \mathbb{P}(\langle e_{123}, e_{124}, e_{235} \rangle) \subset \mathbb{P}(\wedge^3 V)\). Then the equation of \(\xi_\tau\) can be read as \(P_{q_\tau} \cap G(3, V) = \{p_{124235} = 0\}\).

(2) Take \(V_1, V_2, V_4\) as in Example 5.2.2. Then \(P_{V_1, V_4} = \mathbb{P}(\langle e_{124}, e_{134}, e_{234} \rangle) \subset G(3, V)\) and \(P_{V_2} = \mathbb{P}(\langle e_{145}, e_{245}, e_{345} \rangle) \subset G(3, V)\). As a \(\rho\)-conic of rank 2 on the \(\sigma\)-plane \(P_{V_2}\), we may have \(q_\rho = l_3 \cup l_4\) with
\[
l_3 = \{[s e_1 + t e_2, e_4] | [s, t] \in \mathbb{P}^1\}, \quad l_4 = \{[s e_2 + t e_3, e_4] | [s, t] \in \mathbb{P}^1\}.
\]

The equation of this conic is \(p_{145p_{345}} = 0\). Similarly, as an example of \(\sigma\)-conic of rank 2 on the \(\sigma\)-plane \(P_{V_1, V_4}\), we may have \(q_\sigma = l_5 \cup l_6\) with
\[
l_5 = \{[s e_1 + t e_2, e_3, e_4] | [s, t] \in \mathbb{P}^1\}, \quad l_6 = \{[e_1, s e_2 + t e_3, e_4] | [s, t] \in \mathbb{P}^1\}.
\]

The equation of this conic is \(p_{124p_{235}} = 0\).

5.3. \(\mathcal{B}_3\) and \(\mathcal{F}\). Consider a smooth conic \(q = \{[\xi(s, t)] \in G(3, V) | [s, t] \in \mathbb{P}^1\}\), then the planes \(P(\xi(s, t)) \subset \mathbb{P}(V)\) contain (at least) a point \([V_1] \in \mathbb{P}(V)\) in common (see Proposition 5.2.1 and Example 5.2.2). Therefore we may assume the form \([\xi(s, t)] = [\xi^{(1)}(s, t)] = [\xi^{(2)}(s, t)] \subset \mathbb{P}(\wedge^3 V)\) with \(V_1 = \mathbb{C}v\). Corresponding plane \(P_q^2\) in \(\mathbb{P}(\wedge^3 V)\) is a point \([U] = [u_1, u_2, u_3] \in G(3, \wedge^3 V)\) determined by
\[
\xi^{(1)}(s, t) \wedge \xi^{(2)}(s, t) \wedge v = u_1 s^2 + u_2 s t + u_3 t^2 \quad (u_i \in \wedge^3 V).
\]

Note that \(u_i = \bar{u}_i \wedge v\) with some \(\bar{u}_i \in \wedge^2 V\), and \(\bar{u}_i\) may be considered in \(\wedge^2(V/V_1)\). Then the point \([U] \in G(3, \wedge^3 V)\) may be represented by \([\bar{U}] = [\bar{u}_1, \bar{u}_2, \bar{u}_3] \in G(3, \wedge^2(V/V_1))\). We write this by \([U] = [\bar{U} \wedge v] \subset \mathbb{P}(\wedge^2(V/V_1))\) which corresponds to \([\bar{U}]\).

For each \(v \in V\), we define a linear map \(e_v : \wedge^3 V \rightarrow \wedge^4 V\) by \(u \mapsto v \wedge u\). Consider the restriction \(e_v|_U : U \subset \wedge^3 V\), and set \(a_U = \{v \in V | e_v|_U = 0\}\). \(a_U\) is nothing but the annihilator of \(U\).

**Definition 5.3.1.** We define
\[
\mathcal{B}_3 = \{[[U], [V_1]] | V_1 \subset a_U\} \subset G(3, \wedge^3 V) \times \mathbb{P}(V),
\]
and \(\mathcal{F}\) to be the projection of \(\mathcal{B}_3\) to the first factor of \(G(3, \wedge^3 V) \times \mathbb{P}(V)\). We consider the reduced structure on \(\mathcal{F}\).
Since $e_v|_U = 0$ $(V_1 = \mathbb{C}v)$ implies that $U$ is the $\mathbb{C}$-span of non-vanishing vectors of the form $\bar{u}_i \wedge v$ $(i = 1, 2, 3)$ with $\bar{u}_i \in \wedge^2(V/V_1)$, the fiber of the natural projection $\mathcal{Y}_3 \to \mathbb{P}(V)$ over $[V_1] \in \mathbb{P}(V)$ can be identified with $G(3, \wedge^3(V/V_1))$. Hence we see that

$$\mathcal{Y}_3 = G(3, T(-1)^{\wedge^2}),$$

and in particular $\mathcal{Y}_3$ is smooth. In what follows, both $([U], [V_1]) \in \mathcal{Y}_3$ with $[U] \in G(3, \wedge^2(V/V_1))$ and $([U], [V_1])$ will be used to represent a point on $\mathcal{Y}_3$.

From the arguments above, it is clear that all planes $\mathbb{P}_q^2$ corresponding to smooth conics $q$ are contained in $\mathcal{Y}_3$. We also note that each point $([U], [V_1])$ of $\mathcal{Y}_3$ determines a conic $\bar{q}$ in $G(2, V/V_1)$ by the intersection $\mathbb{P}(\bar{U}) \cap G(2, V/V_1)$ in $\mathbb{P}(\wedge^2(V/V_1))$ if $\mathbb{P}(\bar{U}) \not\subset G(2, V/V_1)$. Then the conic $\bar{q} = \{[\xi^{(1)}(s, t), \xi^{(2)}(s, t)] \in G(2, V/V_1) \mid [s, t] \in \mathbb{P}^1\}$ determines a conic $q$ in $G(3, V)$ by $q = \{[\xi^{(1)}(s, t), \xi^{(2)}(s, t), v] \in G(3, V) \mid [s, t] \in \mathbb{P}^1\}$ with $V_1 = \mathbb{C}v$.

**Proposition 5.3.2.** (1) The image $\overline{\mathcal{Y}}$ of $\mathcal{Y}_3$ under the projection is described by

$$\overline{\mathcal{Y}} = \{[U] \in G(3, \wedge^3V) \mid \dim a_U \geq 1\}.$$

(2) $\overline{\mathcal{Y}}$ is singular along $\overline{\mathcal{Y}}_{\text{sing}}$ which is given by

$$\{[U] \in G(3, \wedge^3V) \mid \dim a_U = 2\} = \overline{\mathcal{Y}}_{\text{sing}}(\simeq G(2, V)).$$

(3) The morphism $\rho_{\mathcal{Y}_3} : \mathcal{Y}_3 \to \overline{\mathcal{Y}}$ is a small resolution with its fiber being $\mathbb{P}^1$ over each point of $\overline{\mathcal{Y}}_{\text{sing}}$ and is isomorphic over $\overline{\mathcal{Y}} \setminus \overline{\mathcal{Y}}_{\text{sing}}$.

**Proof.** The claim (1) is immediate from the definition.

(2) From the definition of the annihilator $a_U$, it is clear that the condition $\dim a_U \geq 1$ is satisfied only when $[U] \in G(3, \wedge^3V)$ takes the form

$$[U] = [\bar{u}_1 \wedge v, \bar{u}_2 \wedge v, \bar{u}_3 \wedge v]$$

with some non-zero vector $v \in V$ and $\bar{u}_i \in \wedge^2(V/V_1)$ with $V_1 = \mathbb{C}v$. It is easy to see that the possible values for $\dim a_U$ are 1 or 2, and the case $\dim a_U = 2$ occurs iff $[U]$ specializes further to

$$[U] = [a \wedge v_1 \wedge v_2, b \wedge v_1 \wedge v_2, c \wedge v_1 \wedge v_2].$$

Now let us note the duality $\wedge^3V \simeq \wedge^2V^*$, which follows from the non-degenerate pairing $\wedge^3V \times \wedge^2V \to \wedge^4V$. Using this, we consider the following sequence of maps:

$$S^2(\wedge^3V) \simeq S^2(\wedge^2V^*) \to (\wedge^2V^*) \wedge (\wedge^2V^*) \to \wedge^4V^* \simeq V.$$ 

We denote the composition of these maps by $\varphi$, and consider its restriction $\varphi_U := \varphi|_{S^2U} : S^2U \to V$. Using the form (5.1), we take a basis of $S^2U$ by $\{(\bar{u}_i \wedge v) \otimes_s (\bar{u}_j \wedge v)\}$, where $\otimes_s$ means the symmetric tensor product. Then the composite map above may be described by

$$(\bar{u}_i \wedge v) \otimes_s (\bar{u}_j \wedge v) \mapsto \bar{u}_i^* \otimes_s \bar{u}_j^* \mapsto \bar{u}_i^* \wedge \bar{u}_j^*,$$

where $\bar{u}_i^* \in \wedge^2(V/V_1)^*$ are determined by the non-degenerated pairing $\wedge^2(V/V_1) \times \wedge^2(V/V_1) \to \wedge^4(V/V_1)$ and are considered in $\wedge^2V^*$ via the natural inclusion $\wedge^2(V/V_1)^* \subset \wedge^2V^*$. We note that $\bar{u}_i^* \wedge \bar{u}_j^*$'s belong to the one dimensional subspace $\wedge^4(V/V_1)^* \subset \wedge^4V^*$. Hence if $[U]$ takes the form (5.1), then $\operatorname{rank} \varphi_U \leq 1$. It is also easy to see the converse. Therefore the condition $\dim a_U \geq 1$ is equivalent to the condition $\operatorname{rank} \varphi_U \leq 1$ on the rank, which we can express by the two-by-two minors of the matrix representing $\varphi_U$ (see Remark below). $\overline{\mathcal{Y}}$ is defined by taking the reduced
structure of the condition rank $\varphi_U \leq 1$. We will see in Lemma 5.4.1 below that the scheme structure coming from the rank condition contains embedded points along the singular locus of $\mathcal{W}$. Also we will show explicitly in 5.5 that the singularities of the variety $\mathcal{W}$ appear at the locus consisting of points $[U]$ of the form $[5.2]$ which determine the corresponding $\rho$-planes $P_{V_2}$ with $V_2 = \langle v_1, v_2 \rangle$ and vice versa. Hence the singular locus coincides with the set of $\rho$-planes $\mathcal{W}_\rho \simeq G(2, V)$.

(3) Clearly, the fiber of $\mathfrak{B}_3 \to \mathcal{W}_{\text{sing}}$ over each point is $\mathbb{P}(V_2) \simeq \mathbb{P}^1$, and $\mathfrak{B}_3 \to \mathcal{W}$ is bijective over $\mathcal{W} \setminus \mathcal{W}_{\text{sing}}$.

Remark. The condition rank $\varphi_U \leq 1$ was suggested to the authors by the referee. If we write a point $[U]$ by the Plücker coordinates $p_{ijk}$ of $\wedge^3 V$ as $[U] = [p_{111}^{(1)}, p_{122}^{(2)}, p_{133}^{(3)}]$, then it is straightforward to obtain the $\dim V \times \dim S^2 U$ matrix which represents $\varphi_U$ in terms of these coordinates. For reader's convenience, we write the the matrix entries explicitly:

$$
C_{i(ab)} = \sum_{j,j',k,k'} \epsilon^{ijj'kk'} \left( \sum_{l,m,n} \epsilon^{klmn} p^{(a)}_{lmn} \right) \left( \sum_{l',m',n'} \epsilon^{k'l'm'n'} p^{(b)}_{l'm'n'} \right),
$$

where $\epsilon^{ijklm}$ is the signature function defined by $e_i^1 \wedge e_j^2 \wedge e_k^3 \wedge e_m^4 = \epsilon^{ijklm} e_1^1 \wedge e_2^2 \wedge e_3^3 \wedge e_4^4$ with the basis $e_i^1 (i = 1, \ldots, 5)$ of $V^*$.]

**Proposition 5.3.3.** The set of $\rho$-planes $\mathcal{W}_\rho = \mathcal{W}_{\text{sing}} \simeq G(2, V)$ and the set of $\sigma$-planes $\mathcal{W}_\sigma \simeq F(1, 4, V)$ in $G(3, \wedge^3 V)$ (see Subsection 5.1) are subvarieties in $\mathcal{W}$. They are given by

$$
\mathcal{W}_\rho = \left\{ \left[ (V/V_2) \wedge (\wedge^2 V_2) \right] \mid [V_2] \in G(2, V) \right\} \quad \text{and} \quad \mathcal{W}_\sigma = \left\{ \left[ \wedge^2 (V_4/V_1) \wedge V_1 \right] \mid [V_1 \subset V_4] \in F(1, 4, V) \right\}.
$$

**Proof.** Claims follow directly from the definitions.  

**5.4. The resolution $\mathcal{W} \to \mathcal{W}$.** We study the singularity of $\mathcal{W}$ along $\mathcal{W}_{\text{sing}}$, and find out to be related by the (anti-)flip with an exceptional set in $\mathfrak{B}_2$ of $\mathbb{P}^4 \times \mathbb{P}^5$ fibration over $\mathcal{W}_{\text{sing}}$ (see 5.9). For concreteness, we describe below the resolution in the local coordinates near a point $[U_0] = [e_1 \wedge e_4 \wedge e_5, e_2 \wedge e_4 \wedge e_5, e_3 \wedge e_4 \wedge e_5]$ in $\mathcal{W}_{\text{sing}}$, which can also be written by $[\left( (\wedge^2 V_2^0) \wedge (V/V_2^0) \right)]$ with $[V_2^0] = [e_4, e_5] \in G(2, V)$. Let us fix a basis of $\wedge^3 V$ by

$$
e_1 e_2 e_3 e_4 e_5:
e_1 e_2 e_3 e_4 e_5:
e_1 e_2 e_3 e_4 e_5:
e_1 e_2 e_3 e_4 e_5:$$

with $e_{ijk} := e_i \wedge e_j \wedge e_k$. We introduce an affine coordinate of $G(3, \wedge^3 V)$ centered at $[U_0]$ by the $3 \times 10$ matrix:

$$
[U] = \begin{bmatrix}
z_1 & y_{11} & y_{12} & y_{13} & x_{11} & x_{12} & x_{13} & 1 & 0 & 0 \\
z_2 & y_{21} & y_{22} & y_{23} & x_{21} & x_{22} & x_{23} & 0 & 1 & 0 \\
z_3 & y_{31} & y_{32} & y_{33} & x_{31} & x_{32} & x_{33} & 0 & 0 & 1
\end{bmatrix},
$$

where $z_i = y_{ij} = x_{ij} = 0$ represents $[U_0]$. With this coordinate, it is straightforward to evaluate (5.3). We denote by $\mathcal{I}_{\varphi_U}$ the ideal generated by all $2 \times 2$ minors of the matrix $(C_{i(ab)})$. Then, evaluating the primary decomposition of $\mathcal{I}_{\varphi_U}$ by Macaulay2 [14], we obtain
Lemma 5.4.1. The decomposition consists of two components: \( \mathcal{I}_{\Phi} = \mathcal{I}_{\Phi} \cap \mathcal{I}_{\Phi} \) with \( \mathcal{I}_{\Phi} \) and \( \mathcal{I}_{\Phi} \) representing the reduced part and embedded points, respectively. \( \mathcal{I}_{\Phi} \) is a radical ideal which is generated by
\[
\begin{vmatrix}
  y_{11} & y_{12} & y_{13} & y_{22} & y_{23} & y_{33} \\
  x_{11} & x_{12} & x_{13} & x_{22} & x_{23} & x_{33}
\end{vmatrix}
\]
and the \( 2 \times 2 \) minors of
\[
\begin{vmatrix}
  y_{11} & y_{12} & y_{13} & y_{22} & y_{23} & y_{33} \\
  x_{11} & x_{12} & x_{13} & x_{22} & x_{23} & x_{33}
\end{vmatrix}.
\]
The radical of the component \( \mathcal{I}_{\Phi} \) is generated by all the matrix entries of \( \begin{vmatrix}
  y_{11} & y_{12} & y_{13} & y_{22} & y_{23} & y_{33} \\
  x_{11} & x_{12} & x_{13} & x_{22} & x_{23} & x_{33}
\end{vmatrix} \), i.e., \( y_{ii}, x_{jj}, y_{ij} + y_{ji}, x_{ij} + x_{ji} \) \( (1 \leq i < j \leq 3) \), and
\[
\begin{vmatrix}
  y_{12} & y_{13} & y_{22} & y_{23} & y_{33} & y_{11} \\
  x_{12} & x_{13} & x_{22} & x_{23} & x_{33} & x_{11}
\end{vmatrix}.
\]

From the form of the ideal generated by the \( 2 \times 2 \) minors of \( \begin{vmatrix}
  y_{11} & y_{12} & y_{13} & y_{22} & y_{23} & y_{33} \\
  x_{11} & x_{12} & x_{13} & x_{22} & x_{23} & x_{33}
\end{vmatrix} \), we observe that the singular locus of the variety \( \mathcal{V}(\mathcal{I}_{\Phi}) \) is exactly along the variety \( \mathcal{V}(\mathcal{I}_{\Phi}) \) of the embedded points. Note that, due to the natural \( SL(V) \) action, these properties are valid for all other affine coordinates although we worked in a specific affine coordinate of \( G(3, \Lambda^3 V) \). \( \mathcal{V} \) is defined by the ideal \( \mathcal{I}_{\Phi} \) in each affine coordinate.

Now, let us note that among 21 variables \( (z_i, y_{ij}, x_{ij}) \in \mathbb{C}^{21} \), the parameters for the singular locus \( \mathcal{V}_{sing} = \mathcal{V}(\mathcal{I}_{\Phi}) \) are easily identified in the following form,
\[
[U] = \begin{pmatrix}
  a_2b_3 - a_3b_2 & 0 & b_3 - b_2 & 0 & -a_3 & a_2 & 0 & 0 & 0 \\
  a_1b_3 - a_3b_1 & 0 & b_1 & 0 & -a_3 & a_2 & 0 & 0 & 0 \\
  a_1b_2 - a_2b_1 & b_2 - b_1 & 0 & -a_2 & a_1 & 0 & 0 & 0 & 0 
\end{pmatrix}
\]
with \( [V] = [e_4 + a_1e_1 + a_2e_2 + a_3e_3, e_5 + b_1e_1 + b_2e_2 + b_3e_3] \). In this form, it is clear that \( \mathcal{V}_{sing} \cong G(2, V) \). Now, as a normal coordinate to \( \mathcal{V}_{sing} \) at \( [U_0] \), we introduce the following coordinates \( z_i \) and \( y_{ij}, x_{ij} \) \( (i \leq j) \) by
\[
[U] = \begin{pmatrix}
  z_1 & y_{11} & y_{12} & y_{13} & x_{11} & x_{12} & x_{13} & 1 & 0 & 0 \\
  z_2 & 0 & y_{22} & y_{23} & 0 & x_{22} & x_{23} & 0 & 1 & 0 \\
  z_3 & 0 & 0 & y_{33} & 0 & 0 & x_{33} & 0 & 0 & 1
\end{pmatrix}.
\]

Proposition 5.4.2. Using the normal coordinates to \( \mathcal{V}_{sing} \) at \( [U_0] \) above, \( \mathcal{V} \) is described by \( z_1 = z_2 = z_3 = 0 \) and the zeros of all \( 2 \times 2 \) minors of
\[
\begin{pmatrix}
  y_{11} & y_{12} & y_{13} & y_{22} & y_{23} & y_{33} \\
  x_{11} & x_{12} & x_{13} & x_{22} & x_{23} & x_{33}
\end{pmatrix},
\]
which describes the affine cone of the Segre embedding \( \mathbb{P}^1 \times \mathbb{P}^5 \). In particular, the variety \( \mathcal{V} \) is a normal variety.

Proof. The first claim is immediate from Lemma 5.4.1. Due to the \( SL(V) \)-action on \( \mathcal{V} \), all the singularities along \( \mathcal{V}_{sing} \) are isomorphic. Hence the second claim follows. \( \square \)

From the above proposition, we see that the singularity of \( \mathcal{V} \) can be resolved by introducing the ratio of the two lows or the six columns of \( \begin{pmatrix}
  y_{11} & y_{12} & y_{13} & y_{22} & y_{23} & y_{33} \\
  x_{11} & x_{12} & x_{13} & x_{22} & x_{23} & x_{33}
\end{pmatrix} \). These ratios introduce \( \mathbb{P}^1 \) or \( \mathbb{P}^5 \) along \( \mathcal{V}_{sing} \) as the exceptional sets. We can see that the former gives the resolution \( \mathcal{Y}_3 \rightarrow \mathcal{V} \). The latter resolution gives the (anti-)flip of \( \mathcal{Y}_3 \).

Definition 5.4.3. We define \( \mathcal{Y} \) to be the (small) resolution of \( \mathcal{V} \) with the exceptional set being a \( \mathbb{P}^5 \)-bundle over \( \mathcal{V}_{sing} \). We denote the exceptional set by \( G_p \). \( \square \)
By definition, the blow-up over the point $[U_0]$ is given by the ratio (representing a point $\mathbb{P}^5$)

\[(5.8) \quad y_{11} : y_{12} : \cdots : y_{33} = x_{11} : x_{12} : \cdots : x_{33} = S_{11} : S_{12} : S_{13} : S_{22} : S_{23} : S_{33}\]

for the transversal directions $(y_{ij}, x_{ij})_{i\leq j}$ introduced in \[5.6\] with $z_1 = z_2 = z_3 = 0$. Using this, one of the affine coordinate for the transversal directions may be written $(y_{11}, x_{11}, \frac{y_{12}}{y_{11}}, \frac{x_{12}}{x_{11}})_{i\leq j}$ and similarly for others.

If we use the both ratios, the exceptional set becomes $\mathbb{P}^1 \times \mathbb{P}^5$ fibered over $\mathcal{Y}_{\text{sing}}$ which is a divisor. We denote this blow-up by $\mathcal{Y}_2$ and the exceptional divisor by $F_p$. Then $\mathcal{Y}_3 \leftarrow \mathcal{Y}_2 \rightarrow \mathcal{Y}$ is the standard (anti-)flip. Now we summarize the resolutions in the following diagram (see also Fig.4):

\[(5.9)\]

\[
\begin{array}{c}
\mathcal{Y}_3 \quad \xleftarrow{\text{(anti-)flip}} \quad \mathcal{Y}_2 \\
\mathcal{Y} \quad \xleftarrow{\phi_{DS}} \quad G(3, V) \\
\end{array}
\]

5.5. Double spin decomposition and the construction of $\pi_{\hat{\phi}} : \mathcal{Y} \rightarrow \mathcal{H}$. Consider a point $[U] \in \mathcal{Y}$. By definition, we can express $[U] = [\hat{U} \wedge V_1]$ with some $V_1 = \mathcal{C} v$ and $[\hat{U}] \in G(3, \wedge^2(V/V_1))$. We describe $\wedge^3 \hat{U}$ in $\wedge^3(\wedge^2(V/V_1))$.

Let us note the following irreducible decomposition as $sl(V/V_1)$-modules (see \[12\] §19.1 for example):

\[(5.10) \quad \wedge^3(\wedge^2(V/V_1)) = S^2(V/V_1) \oplus S^2(V/V_1)^*.
\]

We will call this “double spin” decomposition since the components in the r.h.s. are identified with $V_{2\lambda}$ and $V_{\bar{\lambda}}$ as the $so(\wedge^2(V/V_1))$-modules, where $\lambda$ and $\bar{\lambda}$ represent the spinor and conjugate spinor weights, respectively (see \[loc. cit.\]). The projection to the second factor of \[5.10\] defines the following rational map

$$\pi^{DS}_2 : \mathbb{P}(\wedge^3(\wedge^2(V/V_1))) \dashrightarrow \mathbb{P}(S^2(V/V_1)^*).$$

We identify the projective space $\mathbb{P}(S^2(V/V_1)^*)$ with the space of quadrics in $\mathbb{P}(V/V_1)$. Also, by the natural inclusion $(V/V_1)^* \hookrightarrow V^*$, we consider the following composition

$$\pi^{DS}_2 : \mathbb{P}(\wedge^3(\wedge^2(V/V_1))) \dashrightarrow \mathbb{P}(S^2(V/V_1)^*) \hookrightarrow \mathbb{P}(S^2V^*),$$

and identify its image with the singular quadrics in $\mathbb{P}(V)$ which contain $V_1$ in the singular locus. We denote the restrictions of these rational maps to $G(3, \wedge^2(V/V_1)) \subset \mathbb{P}(\wedge^3(\wedge^2(V/V_1)))$ by

$$\varphi_{V_1} = \pi^{DS}_2|_{G(3, \wedge^2(V/V_1))}, \quad \varphi_{V_1} = \pi^{DS}_2|_{G(3, \wedge^2(V/V_1))}.$$

By definition, these rational maps have the same set of indeterminacy. Note also that the indeterminacy occurs when the projection to the second factor is zero in \[5.10\].
Lemma 5.5.1. For a point \([U] \in G(3, \Lambda^3 V)\), the following statements hold:

1. If \(a_U = V_1 (\dim a_U = 1)\), \([U]\) decomposes uniquely to \([U] = [\bar{U} \wedge V_1]\) and determines a point \(\varphi_{V_1}([\bar{U}])\).
2. If \(a_U = V_2 (\dim a_U = 2)\), \([U]\) decomposes into \([\bar{U} \wedge V_1] = [(V/V_2) \wedge (V_2/V_1) \wedge V_1]\) for each \(V_1 \subset V_2\). For any choice of \(V_1\), \([\Lambda^3 \bar{U}]\) is an indeterminacy point of the rational map \(\varphi_{V_1}\).

Proof. Both the claims follow from the explicit form (A.2) of the decomposition (6.10), where the Plücker coordinates of \(\Lambda^3 \bar{U}\) are related to symmetric matrices \([v_{ij}, w_{kl}]\) representing the decomposition. The indeterminacy of \(\varphi_{V_1}\) occurs exactly when \(w_{kl} = 0\). By the property (I.5) in Appendix, this implies \(\text{rank } v \leq 1\). Now we may assume \(v_{ij} = x_i x_j\) for some vector \(x \in V/V_1\). By the action of \(GL(V/V_1)\), we may further assume \(x_1 = 1\) and \(x_i = 0 (i = 2, 3, 4)\). Then we obtain the conditions \(p_{IJK} = 0\) except \(p_{124} = \frac{1}{2}\) for the Plücker coordinates of \(\Lambda^3 \bar{U}\), which determine \([\bar{U}]\) to be \([\bar{e}_1 \wedge \bar{e}_2, \bar{e}_1 \wedge \bar{e}_3, \bar{e}_1 \wedge \bar{e}_4]\). This implies \(\dim a_U = 2\). Hence, if \(\dim a_U = 1\), then \(\varphi_{V_1}([\bar{U}])\) is defined.

For the claim (2), we represent \([\bar{U}]\) as \([(V/V_2) \wedge (V_2/V_1)]\) by fixing \(V_1 \subset V_2\) arbitrarily. Using the \(GL(V/V_1)\) action, we may assume the form \([\bar{e}_2 \wedge \bar{e}_1, \bar{e}_3 \wedge \bar{e}_1, \bar{e}_4 \wedge \bar{e}_1]\) for \([\bar{U}]\). Then, from the relation (A.2), we obtain \(w_{kl} = 0\) and see that \([\Lambda^3 \bar{U}]\) is an indeterminacy point.

From the above lemma, and \(\dim a_U = 1\) or \(2\) for \([U] \in \mathcal{F}\), we have the following proposition (and definition):

**Proposition 5.5.2.** There is a rational map \(\varphi_{DS} : \mathcal{F} \dashrightarrow \mathcal{H} \subset \mathbb{P}(\mathcal{S}^2 \mathcal{V}^*)\) defined by \(\varphi_{DS}([U]) = \varphi_{V_1}([\bar{U}])\) through a decomposition \([U] = [\bar{U} \wedge V_1]\).

**Proposition 5.5.3.** The rational map \(\varphi_{DS} : \mathcal{F} \dashrightarrow \mathcal{H}\) extends to a morphism \(\bar{\varphi}_{DS} : \mathcal{F} \dashrightarrow \mathcal{H}\).

Proof. We show that the indeterminacy of the rational map \(\pi_{2DS}^{\mathcal{F}}\) is resolved by the blow-up \(\mathcal{Y} \to \mathcal{F}\). Since the indeterminacy occurs at the points \([U]\) with \(\dim a_U = 2\) (Lemma 5.5.1), we only need to analyze \(\varphi_{DS}\) near a point \([U] \in \mathcal{F}_{\text{sing}}\). Note that such a point can be written as \([U] = [(V/V_2) \wedge (\Lambda^2 V_2)]\) with \(V_2 = a_U\). By using the \(GL(V)\) action, we can further assume the form \([U_0]\) analyzed in Subsection 5.4. Let us write the affine coordinate (5.6) in terms of the ratio (6.8),

\[
[U] = \begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & t_{12} & t_{13} \\
0 & y_{11} & t_{22} & t_{23} \\
0 & 0 & 0 & t_{33}
\end{bmatrix}
\begin{bmatrix}
x_{11} & 1 & 1 & 0 \\
x_{12} & 0 & 1 & 0 \\
x_{13} & 0 & 0 & 1 \\
x_{14} & 0 & 0 & 1
\end{bmatrix},
\]

where the coordinate \((y_{ij}, x_{ij})_{i \leq j}\) for the normal direction to \(\mathcal{F}_{\text{sing}}\) are written by an affine coordinate \((y_{11}, x_{11}, t_i = \frac{S}{t_i})\) of the blow-up. Let us first consider the case \(x_{11} \neq 0\). In this case, we find a unique decomposition \([U] = [\bar{U} \wedge V_1]\) by \(V_1 = C_v\) with \(v = y_{11} e_4 + x_{11} e_5\). Writing by \(e_i\) the image of \(e_i\) in the quotient \(V_1\), we introduce the basis of \(\Lambda^2 V_1\) by \(e_{23}, e_{31}, e_{12}, e_{14}, e_{24}, e_{34}\) with \(e_{ij} := e_i \wedge e_j\). Then, writing \([\bar{U}] \in G(3, \Lambda^2 (V/V_1))\) explicitly with this basis, it is straightforward
to evaluate $\tilde{\varphi}_{V_1}([U])$ as

$$
\tilde{\varphi}_{V_1}([U]) = \begin{pmatrix}
2 & t_{12} & t_{13} & x_{11}t_{23} \\
t_{12} & 2t_{22} & t_{23} & x_{11}D \\
t_{13} & t_{23} & 2t_{33} & x_{11}t_{12}t_{33} \\
x_{11}t_{23} & x_{11}D & x_{11}t_{12}t_{33} & 2x_{11}^3t_{12}t_{23}t_{33}
\end{pmatrix},
$$

where we set $D = t_{12}t_{23} - t_{13}t_{22}$. Note that the $4 \times 4$ matrix above is written with respect to the basis $e_i(i = 1, 2, 3, 4)$ of the quotient $V/V_1$. For the case $y_{11} \neq 0$, we obtain the same expression but $x_{11}$ being replaced by $-y_{11}$ with the basis $e_i(i = 1, 2, 3, 5)$. These two cases are unified when we write the quadric $\varphi_{V_1}([U]) \in \mathcal{H} \subset \mathbb{P}(S^2V^*)$, by dividing the expressions by $x_{11}$ and $-y_{11}$, respectively, as

$$
\varphi_{V_1}([U]) = \begin{pmatrix}
2 & t_{12} & t_{13} & x_{11}t_{23} \\
t_{12} & 2t_{22} & t_{23} & x_{11}D \\
t_{13} & t_{23} & 2t_{33} & x_{11}t_{12}t_{33} \\
x_{11}t_{23} & x_{11}D & x_{11}t_{12}t_{33} & 2x_{11}^3t_{12}t_{23}t_{33}
\end{pmatrix}.
$$

The calculations are parallel for other affine coordinates of the blow-up, and we see that the rational map $\varphi_{DS}$ is extended to a morphism $\tilde{\varphi}_{DS}$ so that the point $[S_{ij}]$ of the exceptional set $\mathbb{P}^5$ is mapped to the following $3 \times 3$ matrix

$$
(5.11) \quad \tilde{\varphi}_{DS}([S_{ij}]) = \begin{pmatrix}
2S_{11} & S_{12} & S_{13} \\
S_{12} & 2S_{22} & S_{23} \\
S_{13} & S_{23} & 2S_{33}
\end{pmatrix} \in \mathbb{P}(S^2(V/V_2^*)).
$$

5.6. The morphism $\tilde{\varphi} : \tilde{\mathcal{Y}} \to \mathcal{Y}$. Here we prove the following proposition toward the end of this subsection:

**Proposition 5.6.1.** There exists a morphism $\rho_{\tilde{\varphi}} : \tilde{\mathcal{Y}} \to \mathcal{Y}$ which factors the morphism $\tilde{\varphi}_{DS} : \tilde{\mathcal{Y}} \to \mathcal{H}$ as $\tilde{\varphi}_{DS} = \rho_{\tilde{\varphi}} \circ \tilde{\varphi}$.

We recall that the morphism $\tilde{\varphi}_{DS}$ is defined by extending the rational map $\varphi_{DS} : \mathcal{Y} \to \mathcal{H}$ through the blow-up $\tilde{\mathcal{Y}} \to \mathcal{Y}$. Since the value of $\tilde{\varphi}_{DS}$ over the exceptional set is described by (5.11), studying the morphism $\varphi_{DS}$ over $\mathcal{Y} \setminus \mathcal{Y}_{sing}$ suffices to describe $\tilde{\varphi}_{DS}$. In particular, we have $\varphi_{DS}^{-1}(Q) = \varphi^{-1}_{DS}(Q)$ if $\varphi_{DS}^{-1}(Q) \neq \emptyset$, where the closure is taken in $\mathcal{Y}$.

By our definition of the rational map $\varphi_{DS}$, we note the following relation

$$
(5.12) \quad \varphi_{DS}^{-1} = \bigcup_{V_i \subset V} \varphi_{V_i}^{-1},
$$

where, although it is implicit, the inverse image $\varphi_{V_i}^{-1}$ should be considered with the wedge product $V_i \wedge : G(3, \wedge^3(V/V_i)) \hookrightarrow G(3, \wedge^3V)$. Also we note that $\varphi_{V_i}^{-1}$ above can be replaced by $\tilde{\varphi}_{V_i}^{-1}$ due to the canonical embedding $\mathbb{P}(S^2(V/V_1)^*) \hookrightarrow \mathbb{P}(S^2V^*)$.

We will study the fiber over each point $[Q] \in \mathcal{H}$ according to the rank of quadric $Q$. In the arguments below, we denote by $\text{Ker} Q$ the cone of the singular locus of $Q$, which is a linear subspace in $V$ of dimension $5 - \text{rank} Q$. 
Lemma 5.6.2. (1) For quadrics $Q$ of rank 4, $\tilde{\varphi}_{DS}^{-1}([Q])$ consists of two points.
(2) If rank $Q = 3$, then $\tilde{\varphi}_{DS}^{-1}([Q])$ is one point in the fiber of $\tilde{\varphi} \to \tilde{\varphi}$ over $[U] = [(V/V_2) \wedge (\wedge^2 V_2)]$ with $V_2 = \text{Ker } Q$.

Proof. (1) We use (5.12) to determine the inverse image. If rank $Q=4$, then $\tilde{\varphi}_{V_1}^{-1}([Q])$ is non-empty only for $V_1 = \text{Ker } Q$. It is useful to see $\tilde{\varphi}_{V_1}^{-1}([w])$ instead of $\varphi_{V_1}^{-1}([Q])$ by expressing the quadric $[Q] \in \mathbb{P}(S^2 V^*)$ with the corresponding 4×4 matrix $[w] \in \mathbb{P}(S^2(V/V_1)^*)$. The inverse image $\tilde{\varphi}_{V_1}^{-1}([w])$ can then be determined from the Plücker relation (A.3) for $G(3, \wedge^2(V/V_1))$ in terms of the “double spin coordinates” $[v, w]$. It is immediate from (I.2) in Appendix (and rank $w=4$) to see that $\tilde{\varphi}_{V_1}^{-1}([w])$ consists of two points $[v, w]$ satisfying $v.w = \pm \sqrt{\det wid_4}$. Then we have the claim for $\tilde{\varphi}_{DS}^{-1}([Q]) = \varphi_{DS}^{-1}([Q])$.

(2) As above, we consider $[w]$ which corresponds to a quadric $[Q]$ by choosing $V_1 \subset V_2$ ($V_2 = \text{Ker } Q$). Also, we may assume $V_2 = (e_4, e_5)$ using suitable $GL(V)$ actions. From (I.3) in Appendix, and the assumption rank $w = 3$, we see that $[w]$ cannot be the image of $\tilde{\varphi}_{V_1}$ for any choice of $V_1$. However, it is clear that there exists exactly one point in the exceptional set $\mathbb{P}^5$ which corresponds to the rank three matrix $w$ (see 5.11 and the equation above it).

Fig.2. The fibers of the morphism $\tilde{\varphi}_{DS} : \tilde{\varphi} \to \tilde{\varphi}$. $\mathcal{H}^i$ represent the loci of symmetric matrices of rank $i$. Note that $\mathcal{H}^k$ may be identified with $G^k_{\mathcal{X}}$ for $k = 1, 2$. Each fiber is interpreted in terms of the geometry of conics in Sections 5.1 and 5.2.

Let us now describe $\tilde{\varphi}_{DS}^{-1}([Q])$ for rank $Q = 2$. Set $V_3 = \text{Ker } Q$. Then the quadric considered in $\mathbb{P}(V/V_3)$ defines two points $[V^{(i)}_4/V_3] (i = 1, 2)$. Note that $V^{(1)}_4 \cap V^{(2)}_4 = V_3$ since the two points are distinct. With these data, we define a subset $\Gamma(V^{(1)}_4, V^{(2)}_4, V_3)$ in $\tilde{\varphi}$ by

$$
\Gamma(V^{(1)}_4, V^{(2)}_4, V_3) = \left\{ [V^{(1)}_2 V^{(2)}_2 \cup I_{V^{(2)}_2 V^{(2)}_2}] \ | \ V^{(1)}_2, V^{(2)}_2 \subset V_3, V^{(1)}_2 \neq V^{(2)}_2 \right\},
$$
where \(l_{V_i V_j}\) is a line in \(G(3, V)\) defined by \(l_{V_i V_j} = \{[\lambda^3 C^3] | V_2 \subset C^3 \subset V_4\}\) (cf. the lines \(l_i\) described in Subsection 3.2) and \([l \cup l']\) represents the projective plane containing \(l \cup l'\) with \(l \neq l'\) and \(l \cap l' \neq \emptyset\). By the condition \(V_2^{(1)} \neq V_2^{(2)}\), the intersection \(V_2^{(1)} \cap V_2^{(2)} = V_1\) determines a point in \(V_3\). Then we can decompose the set as \(\Gamma(V_4^{(1)}, V_4^{(2)}, V_3) = \bigcup_{V_1 \subset V_3} \Gamma(V_4^{(1)}, V_4^{(2)}, V_5)\) with the obvious notation.

**Lemma 5.6.3.** With the above definitions for quadrics of rank 2, we have
\[
\varphi^{-1}_{DS}([Q]) = \Gamma(V_4^{(1)}, V_4^{(2)}, V_3) \simeq \mathbb{P}(V_4^*) \times \mathbb{P}(V_3^*).
\]

**Proof.** Since the subspaces \(V_2^{(i)} \subset V_3\) are described by points in \(\mathbb{P}(V_3^*)\) for each, it is clear that the closure in \(\mathcal{Y}\) of \(\Gamma(V_4^{(1)}, V_4^{(2)}, V_3)\) is \(\mathbb{P}(V_3^*) \times \mathbb{P}(V_4^*)\). Similarly we see that the closure of \(\Gamma_{V_1}(V_4^{(1)}, V_4^{(2)}, V_3)\) is \(\mathbb{P}((V_3/V_1)^*) \times \mathbb{P}((V_3/V_1)^*) \simeq \mathbb{P}^1 \times \mathbb{P}^1\). We now show \(\varphi^{-1}_{V_1}([Q]) = \Gamma_{V_1}(V_4^{(1)}, V_4^{(2)}, V_3)\). The inclusion \(\varphi^{-1}_{V_1}([Q]) \supset \Gamma_{V_1}(V_4^{(1)}, V_4^{(2)}, V_3)\) can be verified by evaluating \(\varphi_{V_1}\) explicitly, for example, with \(V_1 = (e_1), V_2^{(1)} = (e_1, e_{i+1})\) and \(V_2^{(2)} = (e_1, e_2, e_3, e_{i+1})\). To show the equality, we consider as before the matrix \([w]\) which represents the quadric \(Q\) in \(\mathbb{P}(V/V_1)\) and describe \(\varphi^{-1}_{V_1}([w])\) using the Plücker relations (A.3) in terms of \([v, w]\). Changing the coordinate of \(V/V_1\) suitably, we may assume that \([w]\) is given in the form \(w_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\) with \(O_2\) being the \(2 \times 2\) zero matrix. Then by the properties (I.4) and (I.2), we obtain \(v = \begin{pmatrix} O_2 & O_2 \\ 1 & 2 \end{pmatrix} v_{12} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\). Now substituting \([v, w] = [v, tw_0](t \neq 0)\) into the equation in the first line of (A.3), we have
\[
v_{11}v_{22} - v_1^2 - t^2 = 0 \quad (t \neq 0).
\]
From this, we obtain \(\varphi^{-1}_{V_1}([w]) \simeq \mathbb{P}^1 \times \mathbb{P}^1\) for the closure. Since both sides of \(\varphi^{-1}_{V_1}([Q]) \supset \Gamma_{V_1}(V_4^{(1)}, V_4^{(2)}, V_3)\) have the same closure in \(\mathcal{Y}\), they must coincide. Now the claim follows since \(\varphi^{-1}_{DS}([Q]) = \bigcup_{V_1 \subset V_3} \varphi^{-1}_{V_1}([Q])\) and \(\varphi^{-1}_{DS}([Q]) = \varphi^{-1}_{DS}([Q])\).

Quadrics of rank 1 are determined by specifying \(V_4 = \text{Ker } Q\) or the corresponding points in \(\mathbb{P}(V^*)\). We now describe \(\varphi^{-1}_{DS}(V_4) := \varphi^{-1}_{DS}([Q])\). This inverse image is determined by careful analysis of the loci representing \(\rho\)-planes and \(\sigma\)-planes described in Proposition 5.3.3.

Let us consider a triple \(V_1 \subset V_2 \subset V_4\), and one dimensional subspaces \((V/V_1) \wedge (\wedge^2 V_2)\) and \(\wedge^2 (V_4/V_2) \wedge V_1\) in the quotient space \(\wedge^3 V \mod V_4 \wedge (\wedge^2 V_2)\). We introduce an affine two plane in the quotient space by
\[
\mathbb{C}^2(V_1, V_2, V_4) = (V/V_4) \wedge (\wedge^2 V_2) \oplus \wedge^2 (V_4/V_2) \wedge V_1.
\]

Using this, we define the following subset in \(\mathcal{Y}\):
\[
\Gamma(V_4) = \left\{ [(V_4/V_2) \wedge (\wedge^2 V_2), \xi] \mid \xi \in \mathbb{C}^2(V_1, V_2, V_4), \mathbb{C} \xi \neq (V/V_4) \wedge (\wedge^2 V_2), V_1 \subset V_2 \subset V_4 \right\},
\]

where it should be noted that \([[(V_4/V_2) \wedge (\wedge^2 V_2), \xi]\) defines a (projective) plane in \(\mathbb{P}(\wedge^3 V)\) and also a point in \(\mathcal{Y}_{\text{sing}}\) if \(\mathbb{C} \xi = (V/V_4) \wedge (\wedge^2 V_2)\). We decompose this set into \(\Gamma(V_4) = \bigcup_{V_1 \subset V_4} \Gamma(V_1, V_4)\) with fixing \(V_1 \subset V_4\), and similarly \(\Gamma(V_4) = \bigcup_{V_2 \subset V_4} \Gamma(V_2, V_4)\) with fixing \(V_2 \subset V_4\).
Lemma 5.6.4. (1) $\tilde{\varphi}^{-1}_{DS}(\Gamma(V_4)) = \Gamma(V_4)$.  
(2) There is a birational morphism $\mathbb{P}(\mathcal{O}_{G(2,V_4)} \oplus \mathcal{U}'_{(2,V_4)}(1)) \to \Gamma(V_4)$ which contracts a divisor $E_\sigma = \mathbb{P}(\mathcal{U}'_{(2,V_4)}(1))$ to $\mathbb{P}(V_1)$, where $\mathcal{U}'_{(2,V_4)}$ is the universal subbundle of the Grassmann $G(2,V_4)$.

Proof. (1) As in the preceding lemma, it suffices to show the equality $\tilde{\varphi}^{-1}_{V_1}(\Gamma(V_1,V_4)) = \Gamma(V_1,V_4)$. The inclusion $\tilde{\varphi}^{-1}_{V_1}(\Gamma(V_1,V_4)) \supset \Gamma(V_1,V_4)$ is easily verified by taking $V_4 = \langle e_1,e_2,e_3,e_4 \rangle$ and $V_1 = \langle e_1 \rangle$, for example. To show the equality, we represent the quadric $Q$ by the corresponding matrix $[w] \in \mathbb{P}(S^2(V/V_1)^*)$ and determine $\tilde{\varphi}^{-1}_{V_1}(\Gamma(V_1,V_4))$. By assumption, we have rank $w = 1$ and hence $[w]$ can be written as $[a_k a_l]$ with some non-zero vector $a \in \mathbb{P}(V/V_1)^*$. Then from (I.5) in Appendix, we see that rank $w = 1$. Writing $v_{ij} = x_i x_j$ with $x \in V/V_1$ and also solving (A.3) we obtain $\tilde{\varphi}^{-1}_{V_1}(\Gamma(V_1,V_4)) = \{ [ x_i x_j, t a_k a_l ] \mid a x = 0, t \neq 0 \}$. From this, we see that the closure of $\tilde{\varphi}^{-1}_{V_1}(\Gamma(V_1,V_4)) = \Gamma(V_1,V_4)$ in $\tilde{\mathcal{Y}}$ is the cone over $v_2(\mathbb{P}^2) \cong \mathbb{P}^2$ from the vertex $[0,a_k a_l] \in \mathbb{P}(S^2(V/V_1) \oplus S^2(V/V_1)^*)$, which is isomorphic to $\mathbb{P}(1^3,2)$.

The same cone arises from the closure $\Gamma(V_1,V_4)$. To see this, let us write a point $[U] = [(V_4/V_2) \wedge (\wedge^2 V_2), \xi] \in \Gamma(V_1,V_4)$ with $\xi \in \mathbb{C}^2(V_1,V_2,V_4)$ and $[V_2] \in F(V_1,V_2)$ to $\mathbb{P}(V_2/V_1)$. Here we observe that, when $\mathbb{C}^2 = \wedge^2(V_4/V_2) \wedge V_1 \in \mathbb{C}^2(V_1,V_2,V_4)$, the corresponding point $[U] = [(V_4/V_2) \wedge (\wedge^2 V_2), \xi]$ is reduced to $[U_0] := [\wedge^2(V_4/V_1) \wedge V_1]$ which is constant for any choice of $[V_2]$. Otherwise $[U]$ varies when $[V_2]$ moves in $\mathbb{P}(V_2/V_1)$. This defines a cone over $\mathbb{P}^2 \cong \mathbb{P}(V_2/V_1)$ in $\Gamma(V_1,V_4)$ from the vertex $[U_0]$. Combined with the inclusion $\tilde{\varphi}^{-1}_{V_1}(\Gamma(V_1,V_4)) \supset \Gamma(V_1,V_4)$, we see that the equality $\tilde{\varphi}^{-1}_{V_1}(\Gamma(V_1,V_4)) = \Gamma(V_1,V_4)$ must hold.

(2) We note that the elements of $\Gamma(V_2,V_4)$ are parametrized by the lines $\mathbb{C} \xi$ contained in the union $\bigcup_{V_1 \subset V_2} \mathbb{C}^2(V_1,V_2,V_4)$ which simplifies to $(V_4/V_2) \wedge (\wedge^2 V_2) \wedge V_2$. Namely, the set $\Gamma(V_2,V_4)$ is parametrized by the projective plane $\mathbb{P}((V_4/V_2) \wedge (\wedge^2 V_2) \wedge V_2)$. Now moving $[V_2]$ in the Grassmann $G(2,V_4)$, we obtain the following projective bundle which parametrizes $\Gamma(V_4)$:

$$\mathbb{P}(\wedge^2 \mathcal{U}_{(2,V_4)} \oplus (\wedge^2 \mathcal{U}'_{2,V_4}) \oplus \mathcal{U}_{(2,V_4)}) \cong \mathbb{P}(\mathcal{O}_{G(2,V_4)} \oplus \mathcal{U}'_{G(2,V_4)}(1)), $$

where we use $\wedge^2 \mathcal{U}_{(2,V_4)} \cong \mathcal{O}_{G(2,V_4)}(1)$ and $\mathcal{U}_{(2,V_4)}(1) \cong \mathcal{U}'_{G(2,V_4)}$ for the universal bundle $\mathcal{U}'_{(2,V_4)}$. Since $\mathcal{U}'_{(2,V_4)}(V_2) \cong \wedge^2(V_4/V_2) \wedge V_1 = \{ \wedge^2(V_4/V_2) \wedge V_1 | V_1 \subset V_2 \}$ and $(V_4/V_2) \wedge (\wedge^2 V_2), \xi = [\wedge^2(V_4/V_1) \wedge V_1]$ holds for all $\mathbb{C} \xi \subset \mathcal{U}_{(2,V_4)}(V_2)$, we see that the divisor $\mathbb{P}(\mathcal{U}_{(2,V_4)}(1))$ collapses to $\{ [\wedge^2(V_4/V_1) \wedge V_1] | V_1 \subset V_4 \} \cong \mathbb{P}(V_4)$ in $\Gamma(V_4)$ as claimed.

Remark. $\tilde{\varphi}^{-1}_{DS}(\Gamma(V_4))$ is the weighted Grassmann $wG(2,5)$ as in [8 Example 2.5], which is defined in $\mathbb{P}(1^6,2^4)$ by the equations

$$\begin{pmatrix} 0 & x_{ij} \\ -x_{ij} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x_{12} x_{34} - x_{13} x_{24} + x_{14} x_{23} = 0,$$

where $[x_{ij}, y_k] = [x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, y_1, \ldots, y_4]$ is the (weighted) homogeneous coordinate of $\mathbb{P}(1^6,2^4)$. This $wG(2,5)$ has singularities along $x_{ij} = 0 \cong \mathbb{P}^3$ of type $\frac{1}{5}(1,1,1)$. Over the complement of this singular locus, we see $wG(2,5)$ has
a \mathbb{C}^2\text{-fibration over } G(2,4). In Fig.3, we have depicted schematically the fiber
\bar{\varphi}_{DS}^{-1}([V_4]) = \bar{\varphi}_{\bar{\mathcal{Y}}}^{-1}([V_4]) over [V_4] \in G_{\bar{\mathcal{Y}}}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{The fiber of $\bar{\varphi}_{DS}$ over $[V_4]$. Fibers are interpreted in terms of the geometry of conics in Sections 5.1 and 5.2. The contraction of $E_\sigma$ may be understood that the $\sigma$-conics (double lines) are reduced $\sigma$-planes on which they lie.}
\end{figure}

\textbf{Proof of Proposition 5.6.1.} It suffices to show that each fiber of $\bar{\varphi}_{DS}$ over quadrics of rank $\leq 3$ consists only one connected component, and two components over quadrics of rank 4 (see Proposition 4.2.2). We have seen these properties in the preceding three lemmas. \hfill \Box

\subsection{More properties of $\rho_{\bar{\mathcal{Y}}} : \bar{\mathcal{Y}} \to \mathcal{Y}$}

We show that the contraction $\rho_{\bar{\mathcal{Y}}} : \bar{\mathcal{Y}} \to \mathcal{Y}$ is a $K_{\bar{\mathcal{Y}}}$-negative extremal divisorial contraction.

\textbf{Proposition 5.7.1.} The exceptional locus of $\rho_{\bar{\mathcal{Y}}} : \bar{\mathcal{Y}} \to \mathcal{Y}$ is an SL($V$)-invariant prime divisor, which will be denoted by $F_{\color{red}\bar{\mathcal{Y}}}$.

\textbf{Proof.} By construction, $\rho_{\bar{\mathcal{Y}}}$ is SL($V$)-equivariant, and also $\bar{\mathcal{Y}}$ is smooth and $\rho(\bar{\mathcal{Y}}) = \mathcal{Y}$. By Proposition 4.2.3 \mathcal{Y} is a $\mathbb{Q}$-factorial Gorenstein Fano variety with Picard number one. Therefore $\rho_{\bar{\mathcal{Y}}}$ is neither a composite of non-trivial morphisms nor a small contraction. Hence we have the first assertion.

By Lemmas 5.6.2 and 5.6.3 (see also Fig.2), it is immediate to see that $\rho_{\bar{\mathcal{Y}}}(F_{\bar{\mathcal{Y}}}) = G_{\bar{\mathcal{Y}}}$.

\textbf{Proposition 5.7.2.} $K_{\bar{\mathcal{Y}}} = \rho_{\bar{\mathcal{Y}}}^* K_\mathcal{Y} + 2F_{\bar{\mathcal{Y}}}$. In particular, $\rho_{\bar{\mathcal{Y}}}$ is a $K_{\bar{\mathcal{Y}}}$-negative divisorial extremal contraction and $\mathcal{Y}$ has only terminal singularities with $\text{Sing} \mathcal{Y} = G_{\bar{\mathcal{Y}}}$.

\textbf{Proof.} $F_{\bar{\mathcal{Y}}}$ is generically a $\mathbb{P}^2 \times \mathbb{P}^2$-fibration over $G_{\bar{\mathcal{Y}}}$ (see Lemma 5.6.3). Let $r$ be a line in a ruling of the generic fiber $\Gamma \simeq \mathbb{P}^2 \times \mathbb{P}^2$ of $F_{\bar{\mathcal{Y}}}$. It is enough to show $K_{\bar{\mathcal{Y}}} \cdot r = -2$. Indeed, assume this, then by the adjunction formula

$$K_{\Gamma} = K_{F_{\bar{\mathcal{Y}}}}|_r = (K_{\bar{\mathcal{Y}}} + F_{\bar{\mathcal{Y}}})|_r,$$

it holds $(K_{\bar{\mathcal{Y}}} + F_{\bar{\mathcal{Y}}}) \cdot r = -3$, and hence $F_{\bar{\mathcal{Y}}} \cdot r = -1$. Set $K_{\bar{\mathcal{Y}}} = \rho_{\bar{\mathcal{Y}}}^* K_{\mathcal{Y}} + aF_{\bar{\mathcal{Y}}}$ with unknown $a$. Then it holds that $K_{\bar{\mathcal{Y}}} \cdot r = aF_{\bar{\mathcal{Y}}} \cdot r$ and we obtain $a = 2$ as claimed.
Now we show that \( K_{\tilde{Y}} \cdot r = -2 \). Let us choose \( r \) so that it intersects with the diagonal of \( \Gamma \). Let \( r' \) be the strict transform on \( \mathcal{Y}_2 \) of \( r \) and \( r'' \) the image of \( r' \) on \( \mathcal{Y}_3 \). Since \( \mathcal{Y}_2 \to \mathcal{Y} \) is the blow-up along \( G_\rho \), we have

\[
K_{\mathcal{Y}_2} \cdot r' = K_{\tilde{Y}} \cdot r + 1.
\]

Moreover, since \( K_{\mathcal{Y}_2} = \rho_{\mathcal{Y}_2}^* K_{\mathcal{Y}_3} + 5F_\rho \) (cf. (6.7)), we have

\[
K_{\mathcal{Y}_2} \cdot r' = K_{\mathcal{Y}_3} \cdot r'' + 5.
\]

Therefore it suffices to show \( K_{\mathcal{Y}_3} \cdot r'' = -6 \). The strict transform of \( \Gamma \) on \( \mathcal{Y}_2 \) has a natural \( \mathbb{P}^1 \times \mathbb{P}^1 \)-fibration over \( \mathbb{P}^2 \) since it is the blow-up of \( \Gamma \) along the diagonal. Its fiber is described in the proof of Lemma 5.6.3 and then \( r'' \) is a line in a fiber of \( \mathcal{Y}_3 \to \mathbb{P}(V) \). Therefore we have \( K_{\mathcal{Y}_3} \cdot r'' = -6 \).

Finally, \( \mathcal{Y} \) has only terminal singularity along \( G_\mathcal{Y} \) since the minimal discrepancy for a smooth point of \( \mathcal{Y} \) is equal to \( \dim \mathcal{Y} - 1 \). \( \square \)
6. Sheaves $\tilde{S}_L$, $\tilde{Q}$, $\tilde{T}$ on $\tilde{Y}$ and their properties

Here we introduce locally free sheaves $\tilde{S}_L$, $\tilde{Q}$, $\tilde{T}$ on $\tilde{Y}$, which will play central roles in our construction of the Lefschetz collection. For this, it will be helpful to have the following picture of the birational geometry of $\tilde{Y}$ (Fig.4).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{geometry.png}
\caption{Birational geometries of $\mathcal{Y}$. $F_{\tilde{Y}}$ in $\tilde{Y}$ represents the prime divisor parameterizing reducible conics on $G(3, V)$. Note that the diagram from $\mathbb{P}(V)$ to the endpoint $\mathcal{Y}$ defines a so-called Sarkisov link.}
\end{figure}

We will use the following convention without mentioning at each time:

$L_\Sigma$: the pull back on a variety $\Sigma$ of $\mathcal{O}(1)$ if there is a morphism $\Sigma \to \mathbb{P}(V).$

$M_\Sigma$: the pull back on a variety $\Sigma$ of $\mathcal{O}_\Sigma(1)$ if there is a morphism $\Sigma \to \mathcal{H}.$

6.1. Locally free sheaves $\tilde{S}_L$, $\tilde{Q}$ on $\tilde{Y}$.

Consider the following universal sequence of the Grassmann bundle $\mathcal{Y}_3 = G(3, \wedge^2 T(-1))$ over $\mathbb{P}(V)$ (cf. [11, p.434]):

$$0 \to S \to \pi_3^* (\mathcal{O}(1) \otimes 2) \to Q \to 0,$$

where $S$ is the relative universal subbundle of rank three and $Q$ is the relative universal quotient bundle of rank three. Similarly, we denote by $\tilde{S}$ the universal subbundle of rank three of the Grassmann $G(2, \wedge^3 V)$. Then

**Proposition 6.1.1.** $S^*(L_{\mathcal{Y}_3}) = \rho_{\mathcal{Y}_3}^* \tilde{S}^*.$

**Proof.** We may write a point of $\mathcal{Y}_3$ by $y = ([U], [V])$ (see the description right after Definition 5.3.1.), which is mapped to $[U] = [U \wedge V] \in \mathcal{Y}$. Note that $V_1 = \pi_3^* \mathcal{O}_\mathbb{P}(V)(-1)_{| y} = -L_{\mathcal{Y}_3}|_y$. Therefore $S(-L_{\mathcal{Y}_3}) = \rho_{\mathcal{Y}_3}^* \tilde{S}$ holds for the universal subbundles.

Now we have the following proposition (and definition):

**Proposition 6.1.2.** There exist locally free sheaves $\tilde{S}_L^*$ and $\tilde{Q}$ on $\tilde{Y}$ which satisfy

$$\rho_{\tilde{Y}_2}^* S^*(L_{\mathcal{Y}_2}) = \tilde{\rho}_{\tilde{Y}_2}^* \tilde{S}_L^* \text{ and } \rho_{\tilde{Y}_2}^* Q = \tilde{\rho}_{\tilde{Y}_2}^* \tilde{Q}.$$
Proof. We define $\tilde{S}^*_2$ to be the pullback of $\tilde{S}^*$ to $\tilde{\mathcal{F}}$, then the first claim is immediate by the commutativity of the morphisms in Fig.4. To see the existence of $\tilde{Q}$, consider the universal sequence (6.1) on $\mathcal{F}_3$. Let $[V_1, V_2]$ be a point on the exceptional locus $\mathcal{P}_\rho = F(1, 2, V) \to G(2, V)$ of the blow-up $\mathfrak{S}_3 \to \tilde{\mathcal{F}}$. Since $S_{[V_1, V_2]} = (V/V_1) \wedge (V_2/V_1)$ (see Proposition 6.3.3), we have $0 \to (V/V_2) \wedge (V_2/V_1) \to \wedge^2(V_1) \to Q_{[V_1, V_2]} \to 0$. Hence we have $Q_{[V_1, V_2]} \simeq \wedge^2(V_2)$, which implies $Q_{[\gamma]} \simeq \mathcal{O}^{\oplus 3}$ for a fiber $\gamma = \mathbb{P}^1$ of $F(1, 2, V) \to G(2, V)$. The last property ensures the existence of a locally free sheaf $\tilde{Q}$ on $\tilde{\mathcal{F}}$ such that $Q = \rho_{\mathcal{F}_3}^* \tilde{Q}$. From the commutativity of the diagram in Fig.4, the pull-back $\tilde{Q}$ of $Q$ to $\tilde{\mathcal{F}}$ has the claimed property. \hfill $\square$

6.2. Locally free sheaf $\tilde{F}$ on $\tilde{\mathcal{F}}$. Let us focus on the local geometry of the blow-up $\mathcal{F}_3 \to \tilde{\mathcal{F}}$ which is described by $\mathcal{P}_\rho = F(1, 2, V) \to G(2, V)$. We denote the universal sub-bundles of the partial flag variety $F(1, 2, V)$ by $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \mathcal{R}_V$, where we set $\mathcal{R}_V := V \otimes \mathcal{O}_{F(1, 2, V)}$ and rank $\mathcal{R}_k = k$. There is an exact sequence

$$0 \to \mathcal{R}_2/\mathcal{R}_1 \to \mathcal{R}_V/\mathcal{R}_1 \to \mathcal{R}_V/\mathcal{R}_2 \to 0. \tag{6.2}$$

It is sometimes useful to identify $F(1, 2, V)$ with the projective bundle $\mathbb{P}(T(-1))$ over $\mathbb{P}(V)$. Under this identification, the exact sequence above is nothing but the relative Euler sequence of the projective bundle. For example, we have $\mathcal{R}_2/\mathcal{R}_1 = \mathcal{O}_{\mathbb{P}(T(-1))}(-1)$ and $\mathcal{R}_V/\mathcal{R}_1 = \pi_F^* T(-1)$ with $\pi_F : \mathbb{P}(T(-1)) \to \mathbb{P}(V)$.

Proposition 6.2.1. Let $\pi_{\mathcal{F}_3} = \pi_{\mathcal{F}_2} \circ \rho_{\mathcal{F}_2}$ be the composition of $\rho_{\mathcal{F}_2} : \mathcal{F}_2 \to \mathcal{F}_3$ and $\pi_{\mathcal{F}_1} : \mathcal{F}_3 \to \mathbb{P}(V)$. Also denote by $\rho_{\mathcal{F}_2}|_{F_\rho}$ the restriction of the morphism $\rho_{\mathcal{F}_2}$ to the exceptional divisor $F_\rho \subset \mathcal{F}_2$. Then

1. There is a surjective morphism $\pi_{\mathcal{F}_3}^* \Omega(1) \to (\rho_{\mathcal{F}_2}|_{F_\rho})^* (\mathcal{R}_2/\mathcal{R}_1)^*$. 
2. The kernel of the surjective morphism

$$\mathcal{T}_2^* = \text{Ker} \left\{ \pi_{\mathcal{F}_3}^* \Omega(1) \to (\rho_{\mathcal{F}_2}|_{F_\rho})^* (\mathcal{R}_2/\mathcal{R}_1)^* \right\}$$

is a locally free sheaf on $\mathcal{F}_2$.

Proof. From the exact sequence (6.2), we have a surjection $(\mathcal{R}_V/\mathcal{R}_1)^* \to (\mathcal{R}_2/\mathcal{R}_1)^* \to 0$, and hence $(\rho_{\mathcal{F}_2}|_{F_\rho})^* (\mathcal{R}_V/\mathcal{R}_1)^* \to 0$. Here we note that $(\rho_{\mathcal{F}_2}|_{F_\rho})^* (\mathcal{R}_V/\mathcal{R}_1)^* = i^* \circ \pi_{\mathcal{F}_2}^* \Omega(1)$ with $i : F_\rho \to \mathcal{F}_2$. We now obtain the exact sequence

$$0 \to \mathcal{T}_2^* \to \pi_{\mathcal{F}_3}^* \Omega(1) \xrightarrow{\pi_{\mathcal{F}_3}^* \Omega(1)} (\rho_{\mathcal{F}_2}|_{F_\rho})^* (\mathcal{R}_2/\mathcal{R}_1)^* \to 0, \tag{6.3}$$

with the surjection claimed in (1) and $\mathcal{T}_2^*$ defined in (2). By taking $Ext(-, \mathcal{O}_{\mathcal{F}_2})$ of this sequence, we see that $\mathcal{T}_2^*$ is a locally free sheaf on $\mathcal{F}_2$ (see [13, III, Ex 6.6]). \hfill $\square$

Lemma 6.2.2. $\mathcal{T}_2^*|_{F_\delta} = \mathcal{O}_{\mathcal{F}_3}^{\oplus 4}$ for each fiber $\delta \simeq \mathbb{P}^1$ of $F_\rho \to G_\rho$.

Proof. Each fiber $\delta$ of $F_\rho \to G_\rho$ projects isomorphically to a fiber of $F(1, 2, V) \to G(2, V)$, and further to a line $\mathbb{P}^1$ in $\mathbb{P}(V)$. Therefore $\pi_{\mathcal{F}_3}^* \Omega(1)|_{\delta} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and also $(\rho_{\mathcal{F}_2}|_{F_\rho})^* (\mathcal{R}_2/\mathcal{R}_1)|_{\delta} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$. By restricting (6.3), we obtain

$$\mathcal{T}_2^*|_{\delta} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}(-1) \to 0,$$

which shows that there is a surjection $\mathcal{T}_2^*|_{\delta} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$ with its kernel being an invertible sheaf $\mathcal{L}$. Note that $\det \mathcal{T}_2^* \simeq \mathcal{O}_{\mathcal{F}_2}(-L_{\mathcal{F}_2} - F_\rho)$ from (6.3) and $(\mathcal{R}_2/\mathcal{R}_1)^* = \mathcal{O}_{\mathcal{F}_3}^{\oplus 4}$ for each fiber $\delta \simeq \mathbb{P}^1$ of $F_\rho \to G_\rho$. \hfill $\square$
$O_{\tilde{\mathbb{P}}(T(-1))}(1)$. Now, since $L_{\mathcal{O}_2}|_S = O_S(1)$ by definition and also $F_p \cdot \delta = -1$, we have $\det T_2^*|_S \simeq O_S$ and $L \simeq O_S$. Hence $T_2^*|_S \simeq O_S^{\oplus 4}$.

Define $T_2 := (T_2^*)^*$. From the above lemma, we have the following proposition (and definition):

**Proposition 6.2.3.** There exists a locally free sheaf $\tilde{T}$ on $\tilde{\mathbb{V}}$ such that

$$T_2 = \tilde{\rho}_{\mathcal{O}_2} \tilde{T}.$$ 

The following exact sequence will be used in our later calculations:

**Proposition 6.2.4.**

\begin{equation}
0 \to \pi_{\mathcal{O}_2} T(-1) \to T_2 \to (\rho_{\mathcal{O}_2}|_{F_p})^*(R_2/R_1)(F_p|_{F_p}) \to 0.
\end{equation}

**Proof.** By definition of $T_2^*$, we have $0 \to \pi_{\mathcal{O}_2} T \to \pi^* \Omega(1) \to (\rho_{\mathcal{O}_2}|_{F_p})^*(R_2/R_1)^* \to 0$. Now, by taking $\text{Hom}(-, O_{\mathcal{O}_2})$, we obtain:

$$0 \to \pi^* \Omega(1) \to T_2 \to \text{Ext}^1_{O_{\mathcal{O}_2}}((\rho_{\mathcal{O}_2}|_{F_p})^*(R_2/R_1)^*, O_{\mathcal{O}_2}) \to 0.$$ 

The claim follows from the isomorphism:

$$\text{Ext}^1_{O_{\mathcal{O}_2}}((\rho_{\mathcal{O}_2}|_{F_p})^*(R_2/R_1)^*, O_{\mathcal{O}_2}) \simeq (\rho_{\mathcal{O}_2}|_{F_p})^*(R_2/R_1)^*(F_p|_{F_p}),$$

which we derive by the spectral sequence

$$\text{Ext}^1_{O_{\mathcal{O}_2}}((\rho_{\mathcal{O}_2}|_{F_p})^*(R_2/R_1)^*, O_{\mathcal{O}_2}) \Rightarrow \text{Ext}^1_{O_{\mathcal{O}_2}}((\rho_{\mathcal{O}_2}|_{F_p})^*(R_2/R_1)^*, O_{\mathcal{O}_2}),$$

and also $\text{Ext}^1_{O_{\mathcal{O}_2}}(O_{\mathcal{O}_2}, O_{\mathcal{O}_2}) \simeq \omega_{\mathcal{O}_2} \otimes \omega_{\mathcal{O}_2}^{-1} \simeq O_{\mathcal{O}_2}(F_p|_{F_p})$, $\text{Ext}^1_{O_{\mathcal{O}_2}}(O_{\mathcal{O}_2}, O_{\mathcal{O}_2}) = 0$ if $j \neq 1$. 

6.3. Properties of $S^*, Q$ restricted on $\mathcal{P}_p$ and $\mathcal{P}$. As in the last subsection, we identify $\mathcal{P}_p = \mathbb{F}(1, 2, V)$ with the projective bundle $\mathbb{P}(T(-1))$ with $\pi_F : \mathbb{P}(T(-1)) \to \mathbb{P}(V)$. We introduce two divisors on $\mathcal{P}_p$:

$$H_{\mathcal{P}_p} = O_{\mathbb{P}(T(-1))}(1) \text{ and } L_{\mathcal{P}_p} := \pi_F^* O(1).$$

**Proposition 6.3.1.** $Q|_{\mathcal{P}_p} \simeq S^*(L_{\mathcal{O}_2})|_{\mathcal{P}_p}$ and $Q|_{\mathcal{P}_p} \simeq S^*(L_{\mathcal{O}_2})|_{\mathcal{P}_p}$.

**Proof.** Take a point $[V_1, V_2] \in \mathcal{P}_p$, then the universal sequence (6.1) restricts to

$$0 \to (V/V_2) \wedge (V_2/V_1) \to \wedge^2 (V/V_1) \to \wedge^2 (V/V_2) \to 0,$$

where $S|_{[V_1, V_2]} = (V/V_2) \wedge (V_2/V_1)$ and $Q|_{[V_1, V_2]} = \wedge^2 (V/V_2)$. By the wedge product, we have a natural map $S|_{[V_1, V_2]} \times Q|_{[V_1, V_2]} \to \wedge^4 (V/V_1)$. In fact, it is easy to see that this defines a non-degenerate form. Hence, noting $\wedge^4 (V/V_1) = L_{\mathcal{O}_2}|_{[V_1, V_4]}$, we obtain the claimed isomorphism $Q|_{\mathcal{P}_p} \simeq S^*(L_{\mathcal{O}_2})|_{\mathcal{P}_p}$.

The second isomorphism follows from a similar argument starting with the restrictions $S|_{[V_1, V_4]} = \wedge^2 (V_4/V_1)$ (see Proposition 6.3.3) and $Q|_{[V_1, V_4]} = (V/V_4) \wedge (V_4/V_1)$.

**Proposition 6.3.2.**

1. $Q|_{\mathcal{P}_p} \simeq \wedge^2 (R_2/R_2) \simeq (R_2/R_2)^*(H_{\mathcal{P}_p} + L_{\mathcal{P}_p})$.
2. $\det Q|_{\mathcal{P}_p} = 2(H_{\mathcal{P}_p} + L_{\mathcal{P}_p})$. 


Proof. (1) Since \((R_V/R_2)[V_1,V_2] = V/V_2\) and \(Q_{V_1,V_2} = \wedge^2(V/V_2)\), we have \(Q|_{\mathcal{P}} = \wedge^2(R_V/R_2).\) By taking the determinant of (6.2), we have
\[
\wedge^3(R_V/R_2) \simeq \wedge^4(R_V/R_1) \otimes (R_2/R_1)^* \simeq L_{\mathcal{P}} \otimes \mathcal{O}_P(T(-1))(1),
\]
where we use the relations \(R_V/R_1 = \pi_F^* T(-1)\) and \(R_2/R_1 = \mathcal{O}_P(T(-1))(-1)\) with \(\pi_F : \mathbb{P}(T(-1)) \to \mathbb{P}(V).\) Noting the non-degenerate pairing \(\mathcal{O}|_{\mathcal{P}} = \wedge^2(R_V/R_2) \simeq (R_V/R_2)^*(H_{\mathcal{P}} + L_{\mathcal{P}}).\) The claim (2) follows from \(\det Q|_{\mathcal{P}} = \wedge^3(\wedge^2(R_V/R_2)) \simeq (\wedge^3(R_V/R_2))^{\otimes 2}.\)

\section{Divisors on \(\mathcal{B}_2\) and \(\mathcal{B}_3\).}

Recall the universal sequence of the Grassmann bundle \(\mathcal{B}_3 = G(3,T(-1)^{\wedge 2});\)
\[
0 \to \mathcal{S} \to \pi_{\mathcal{P}_3}^* (T(-1)^{\wedge 2}) \to \mathcal{Q} \to 0.
\]
Taking the determinant and using \(\wedge^6(T(-1)^{\wedge 2}) = (\wedge^4(T(-1)))^{\otimes 3} = \mathcal{O}(3),\) we have
\[
(6.5) \quad \det \mathcal{Q} = \det \mathcal{S}^* + 3L_{\mathcal{P}_3} = \det \{S^*(L_{\mathcal{P}_3})\}.
\]
Also, since \(T_{\mathcal{P}_3/\mathbb{P}(V)} = S^* \otimes \mathcal{Q}\) (see [11, p.435]), we have
\[
(6.6) \quad K_{\mathcal{P}_3} = -\det(Q \otimes S^*) + 5L_{\mathcal{P}_3} = -3(\det Q + \det S^*) - 5L_{\mathcal{P}_3} = -6 \det Q + 4L_{\mathcal{P}_3},
\]
where we note \(\text{rank} \mathcal{S} = \text{rank} \mathcal{Q} = 3\) and we use (6.5) in the last equality.

We now investigate several relations among basic divisors on \(\mathcal{B}_2.\)

Note that
\[
(6.7) \quad K_{\mathcal{P}_2} = \rho_{\mathcal{P}_2}^* K_{\mathcal{P}_3} + 5F_{\mathcal{P}}
\]
since \(\rho_{\mathcal{P}_2}\) is the blow-up along a smooth subvariety of codimension 6. By (6.6), we have
\[
(6.8) \quad K_{\mathcal{P}_2} = -6\rho_{\mathcal{P}_2}^* \det \mathcal{Q} + 4L_{\mathcal{P}_2} + 5F_{\mathcal{P}}.
\]

\textbf{Proposition 6.4.1.} The pull-back \(M_{\mathcal{P}_2}\) of \(\mathcal{O}_{\mathcal{P}_2}(1)\) is given by
\[
M_{\mathcal{P}_2} = \rho_{\mathcal{P}_2}^* (\det Q) - L_{\mathcal{P}_2} - F_{\mathcal{P}}.
\]

Proof. Recall the definition of \(\tilde{\varphi}_{DS}\) (Proposition 5.5.3) and the second relation of (A.1.1). We note that the Plücker coordinates of \(G(3,\wedge^2T(-1))\) are sections of \(\mathcal{S}^*\) and also \(\wedge^4T(-1) = \mathcal{O}(1)\) in (A.1.1) (cf. Lemma 6.4.2 below). \(\tilde{\varphi}_{DS}\) is defined by removing the zero in \(w_{kq}\) along the indeterminacy set \(\mathcal{P}_2\) of \(\varphi_{V_1}\), which is blown-up to the exceptional divisor \(F_{\mathcal{P}}\) under \(\rho_{\mathcal{P}_2}\). Hence, using commutativity of the diagram in Fig.4, we have \(\tilde{\varphi}_{DS} \circ \tilde{\varphi}_{DS}^* \mathcal{O}_{\mathcal{P}_2}(1) = \rho_{\mathcal{P}_2}^* (\det S^* + 2L_{\mathcal{P}_2}) - F_{\mathcal{P}}.\) The claim follows from (6.5) and the definition \(M_{\mathcal{P}_2} := \rho_{\mathcal{P}_2}^* \tilde{\varphi}_{DS}^* \mathcal{O}_{\mathcal{P}_2}(1).\)

\textbf{Lemma 6.4.2.}
\[
(6.9) \quad \wedge^3(T(-1)^{\wedge 2}) = (S^2(T(-1)) \otimes \mathcal{O}(1)) \oplus (S^2(T(-1))^* \otimes \mathcal{O}(2)).
\]

Proof. Globalizing the decomposition (6.10), we have \(\wedge^3(T(-1)^{\wedge 2}) = S^2(T(-1)) \otimes \mathcal{O}(a) \oplus S^2(T(-1))^* \otimes \mathcal{O}(b)\) with some integers \(a\) and \(b.\) To determine \(a\) and \(b,\) we restrict this equality to a line \(l \subset \mathbb{P}(V).\) Noting \(T(-1)|_l \simeq \mathcal{O}^{\otimes 3} \oplus \mathcal{O}(1),\) we see that \(a = 1\) and \(b = 2.\)
7. Birational models of $F_{\tilde{\mathcal{Y}}}$ and flattening of $F_{\tilde{\mathcal{Y}}} \to G_{\mathcal{Y}}$

In this section, we study the birational geometry of the divisor $F_{\tilde{\mathcal{Y}}} \subset \tilde{\mathcal{Y}}$ in details, and obtain its explicit description $F_{\tilde{\mathcal{Y}}} = \tilde{F}/\mathbb{Z}_2$ in Proposition 7.3.2. In particular, we obtain a natural flattening of the fibration $F_{\tilde{\mathcal{Y}}} \to G_{\mathcal{Y}}$ in Proposition 7.3.1 which is required for our cohomology calculations of the sheaves $\tilde{S}_L, \tilde{Q}, \tilde{F}$ on $\mathcal{Y}$ in Section 8. The properties summarized in Lemma 7.5.1 and Lemma 7.5.2 will be used in our proof of Lemma 8.1.3.

7.1. Birational models $F^{(k)}/\mathbb{Z}_2$ of $F_{\tilde{\mathcal{Y}}}$. Let us see that $F_{\tilde{\mathcal{Y}}}$ is birationally equivalent to the $\mathbb{Z}_2$-quotient of the following $\mathbb{Z}_2$-variety:

$$\tag{7.1} F^{(1)} := \{([V_2^{(1)}], [V_2^{(2)}]; [V_4^{(1)}], [V_4^{(2)}]) \mid V_2^{(1)} \cap V_4^{(1)} \cap V_4^{(2)} \geq 1\} \subset G(2, V) \times G(2, V) \times \mathbb{P}(V^*) \times \mathbb{P}(V^*),$$

where $\mathbb{Z}_2$ acts by the simultaneous exchanges $V_2^{(1)} \leftrightarrow V_2^{(2)}$ and $V_4^{(1)} \leftrightarrow V_4^{(2)}$. There is a natural morphism $F^{(1)}/\mathbb{Z}_2 \to \mathbb{S}^2\mathbb{P}(V^*)$, and its fiber over a point $([V_4^{(1)}], [V_4^{(2)}])$ with $V_4^{(1)} \neq V_4^{(2)}$ is isomorphic to $\mathbb{P}(V_4^*) \times \mathbb{P}(V_4^*)$ with $V_4 = V_4^{(1)} \cap V_4^{(2)}$. Therefore $F^{(1)}/\mathbb{Z}_2 \to \mathbb{S}^2\mathbb{P}(V^*)$ is birationally equivalent to $F_{\tilde{\mathcal{Y}}} \to G_{\mathcal{Y}}$ by Lemma 5.6.3.

It turns out that $F^{(1)}/\mathbb{Z}_2$ is not isomorphic to $F_{\tilde{\mathcal{Y}}}$ but we can reconstruct $F_{\tilde{\mathcal{Y}}}$ from $F^{(1)}/\mathbb{Z}_2$ explicitly. Our reconstruction of $F_{\tilde{\mathcal{Y}}}$ proceeds as follows; we first construct the (anti-)flip $F^{(2)} \dashrightarrow F^{(4)}$ of a resolution $F^{(2)} \to F^{(1)}$ (Lemma 7.3.2); then we describe the strict transform $D^{(4)}$ on $F^{(4)}$ of the inverse image $D^{(1)}$ in $F^{(1)}$ of the diagonal of $\tilde{G} = \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ (Lemma 7.3.1) and then consider a contraction of $D^{(4)} \subset F^{(4)}$ to obtain $F^{(4)} \to \tilde{F}$; finally we divide $\tilde{F} \to \tilde{G}$ by the naturally induced $\mathbb{Z}_2$-action to obtain $F_{\tilde{\mathcal{Y}}} \to G_{\mathcal{Y}}$ (Proposition 7.3.2). The entire picture is summarized in the diagram 7.3.

Now let us first construct a small resolution of $F^{(1)}$.

**Lemma 7.1.1.** Let

$$\tag{7.2} F^{(2)} := \{([V_2^{(1)}], [V_2^{(2)}]; [V_3]; [V_4^{(1)}], [V_4^{(2)}]) \mid V_2^{(1)} \cap V_2^{(2)} \subset V_3 \subset V_4^{(1)} \cap V_4^{(2)}\} \subset G(2, V) \times G(2, V) \times G(3, V) \times \mathbb{P}(V^*) \times \mathbb{P}(V^*)$$

and

$$\tag{7.3} \tilde{G'} := \{([V_3]; [V_4^{(1)}], [V_4^{(2)}]) \mid V_3 \subset V_4^{(1)} \cap V_4^{(2)}\} \subset G(3, V) \times \mathbb{P}(V^*) \times \mathbb{P}(V^*).$$

Then, 1) $\tilde{G'}$ is the blow-up of $\tilde{G} := \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ along the diagonal, and $F^{(2)}$ has a $\mathbb{P}^2 \times \mathbb{P}^2$-fibration structure over $\tilde{G'}$. In particular, $F^{(2)}$ is smooth. 2) Moreover, the natural morphism $F^{(2)} \to F^{(1)}$ is a small resolution.

**Proof.** The first part is almost obvious. We only show that $F^{(2)} \to F^{(1)}$ is a small resolution. Note that, since $V_3 = V_2^{(1)} \cap V_2^{(2)}$ holds for $F^{(2)}$ when $V_2^{(1)} \neq V_2^{(2)}$, or $V_3 = V_4^{(1)} \cap V_4^{(2)}$ holds for $F^{(2)}$ when $V_4^{(1)} \neq V_4^{(2)}$, the morphism $F^{(2)} \to F^{(1)}$ is isomorphic outside the diagonal set

$$\tag{7.4} \Delta_{F^{(1)}} := \{([V_2], [V_2]; [V_3], [V_4]) \mid V_2 \subset V_4\} \simeq F(2, 4, V) \subset F^{(1)}.$$
The fiber of $F(2) \to F(1)$ over a point $([V_2],[V_2];[V_4]) \in \Delta_{F(1)}$ is
\[
\{( [V_2],[V_2];[V_3];[V_4]) \mid [V_3] \in G(3,V), V_2 \subset V_3 \subset V_4 \} \simeq \mathbb{P}^3.
\]
Therefore the dimension of the exceptional set of $F(2) \to F(1)$ is equal to $\dim \Delta_{F(1)} + 1 = 9$, hence $F(2) \to F(1)$ is small.

Lemma 7.1.2. 1) $\mathcal{N}_{\gamma/F(2)} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 8}$ for non-trivial fibers $\gamma \simeq \mathbb{P}^1$ of $F(2) \to F(1)$. 2) There exists another small resolution $F(4) \to F(2)$ which is isomorphic outside $\Delta_{F(1)}$ and whose nontrivial fiber is isomorphic to $\mathbb{P}^2$. The variety $F(4)$ is constructed by taking the blow-up $F(3) \to F(2)$ along the exceptional locus of $F(2) \to F(1)$ and contracting the exceptional divisor of this blow-up in the other direction.

Remark. The small resolution $F(4) \to F(1)$ as in the above statement can be considered locally a family of the small resolution of a 4-dimensional singularity, which is studied in [22].

Proof. The main part of our proof is to determine the singularities of $F(1)$. To describe it, we set
\[
Z := \{( [V_2(1);[V_4(1)]) \mid V_2(1) \subset V_4(1) \} \subset G(2,V) \times \mathbb{P}(V^*)
\]
and consider the projection $F(1) \to Z$. Let $F^2_z(1)$ be the fiber of this projection over a point $z = ( [V_2(1);[V_4(1)]) \in Z$. To describe the fiber $F^2_z(1)$, we choose a basis $(e_1, \ldots, e_5)$ of $V$ such that $V_2(1) = \langle e_1, e_2 \rangle$ and $V_4(1) = \langle e_1, \ldots, e_4 \rangle = \langle 0, 0, 0, 0, 1 \rangle$. Then the conditions $V_2(1) \cap V_4(2) \subset V_4(1) \cap V_4(2)$ and $\dim (V_2(1) \cap V_4(2)) \geq 1$ for the points $([V_2(2);[V_4(2)]) \in G(2,V_4(1)) \times \mathbb{P}(V^*)$ on the fiber are easily analyzed to obtain
\[
F_z(1)^{(1)} = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}, [0,0,c_3,c_4,c_5] \mid \text{rank} \begin{pmatrix} a_3 & a_4 \\ b_3 & b_4 \\ c_4 & -c_3 \end{pmatrix} \leq 1 \right\}.
\]
From this, we see that $F_z(1)^{(1)}$ is singular only at the origin $o$ of the affine chart with $(b_1,b_2) \neq 0$ and $c_5 \neq 0$, i.e., $([V_2(2);[V_4(2)]) = ([V_2(1);[V_4(1)])$, and the singularity is isomorphic to the vertex of the cone over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2$.

We may consider $F(1) \to Z$ locally as an equisingular family of the cone over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2$. It is well-known that the cone $C$ over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2$ has exactly two small resolutions $p_1 : C_1 \to C$ and $p_2 : C_2 \to C$, where the exceptional locus $E_1$ of $p_1$ is a copy of $\mathbb{P}^2$ with $\mathcal{N}_{E_1/C_1} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$, and the exceptional locus $E_2$ of $p_2$ is a copy of $\mathbb{P}^1$ with $\mathcal{N}_{E_2/C_2} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$. We can conclude that $F(2) \to F(1)$ is locally a family of $p_2 : C_2 \to C$, and then we have the assertion (cf. [22]). $F(4) \to F(1)$ is nothing but locally a family of $p_1 : C_1 \to C$.

7.2. Divisors $D(k)$ in $F(k)$. Now, let $\Delta_{\mathbb{P}} = G(4,V)$ be the diagonal of $\widehat{G} = \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ and $D(1)$ the inverse image of $\Delta_{\mathbb{P}}$ by the natural morphism $F(1) \to \mathbb{P}(V^*) \times \mathbb{P}(V^*)$, namely,
\[
D(1) := \{( [V_2(1);[V_2(2);[V_4];[V_4]) \mid V_2(1) \cap V_2(2) \subset V_4, \dim (V_2(1) \cap V_2(2)) \geq 1 \} \subset F(1).
\]
Let
\[
D(2) := \{( [V_2(1);[V_2(2);[V_3];[V_4);[V_4]) \mid V_2(1), V_2(2) \subset V_3 \subset V_4 \} \subset F(2).
\]
The natural morphism $D^{(2)} \to D^{(1)}$ over $\Delta_{\mathbb{F}}$ is the restriction of the morphism $F^{(2)} \to F^{(1)}$ in Lemma 7.1.1. $D^{(2)}$ has a $\mathbb{P}^2 \times \mathbb{P}^2$-bundle structure over the flag variety $F(3, 4, V)$. In particular $D^{(2)}$ is a smooth variety. Hence $D^{(1)}$ is a prime divisor on $F^{(1)}$ and $D^{(2)}$ is its strict transform on $F^{(2)}$.

Lemma 7.2.1. Set

\begin{equation}
D^{(4)} := \{ ([V_1]; [V_2^{(1)}]; [V_2^{(2)}]; [V_4]) \mid V_2^{(1)}, V_2^{(2)} \subset V_4, \dim (V_2^{(1)} \cap V_2^{(2)}) \geq 1 \}
\end{equation}

Then $D^{(4)}$ is the strict transform on $F^{(4)}$ of $D^{(2)}$. Moreover, the restriction $D^{(2)} \to D^{(4)}$ of the (anti-)flip $F^{(2)} \to F^{(4)}$ is a family of Atiyah flops. Noting $D^{(2)}$ (resp. $D^{(4)}$) has a natural $\mathbb{P}^2 \times \mathbb{P}^2$-fibration structure over $F(3, 4, V)$ (resp. $F(1, 4, V)$), we obtain the following commutative diagram:

\begin{equation}
\begin{array}{ccc}
D^{(2)} & \xrightarrow{\text{Atiyah flop}} & D^{(4)} \\
\varphi^2 \times \varphi^2, \text{flip} & \searrow & \nearrow \varphi^2 \times \varphi^2, \text{flip} \\
F(3, 4, V) & \downarrow & D^{(1)} \downarrow F(1, 4, V) \\
& & \Delta_{\mathbb{F}}.
\end{array}
\end{equation}

Proof. Let $D^{(1)}_{[V_4]}$ be the fiber of $D^{(1)} \to \Delta_{\mathbb{F}}$ over the diagonal point $([V_4]; [V_4])$. Then, from the definition, we have

$$D^{(1)}_{[V_4]} = \{ ([V_2^{(1)}]; [V_2^{(2)}]) \mid V_2^{(1)}, V_2^{(2)} \subset V_4, \dim (V_2^{(1)} \cap V_2^{(2)}) \geq 1 \}$$

which is singular along $[V_2^{(1)}] = [V_2^{(2)}]$. As in the proof of Lemma 7.1.2, we fix $V_2^{(1)} = \langle e_1, e_2 \rangle$ and $V_4 = \langle e_1, e_2, e_3, e_4 \rangle$. Then the natural restriction $D^{(1)}_{[V_4]|_{[V_2^{(1)}]}}$ can be described by (7.5) as

$$D^{(1)}_{[V_4]|_{[V_2^{(1)}]}} = \left\{ \left( V_2^{(1)}, \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} \right) \mid \text{rank} \begin{bmatrix} a_3 & a_4 \\ b_3 & b_4 \end{bmatrix} \leq 1 \right\}.$$

In this form, it is clear that the two small resolutions $F^{(2)} \to F^{(1)}$ and $F^{(4)} \to F^{(1)}$ given in Lemma 7.1.2 restricts to $D^{(2)} \to D^{(1)}$ and $D^{(4)}' \to D^{(1)}$, respectively. We show the equality $D^{(4)'}$ to $D^{(4)}$.

Since the equality holds over the smooth locus of $D^{(1)}$, it suffices to see the correspondence between the exceptional set over the diagonal set $[V_2^{(1)}] = [V_2^{(2)}]$. To see this, we fix $[V_2^{(1)}] = \langle e_1, e_2 \rangle$ and consider $D^{(1)}_{[V_4]|_{[V_2^{(1)}]}}$ as above. Then the exceptional set of $D^{(4)'}$ over $D^{(4)}_{[V_4]|_{[V_2^{(1)}]}}$ consists of points $[V_1]$ such that $V_1 = V_2^{(1)} \cap \hat{V}_2^{(2)}(s, t) = \mathbb{C}(t, -s, 0, 0)$ with $[V_2^{(2)}(s, t)] = \begin{bmatrix} 1 & 0 & a_3 & a_4 \\ 0 & 1 & b_3 & b_4 \end{bmatrix}$ and

$$a_3 : b_3 = a_4 : b_4 = s : t \in \mathbb{P}^1.$$

This exactly describes $\mathbb{P}^1$ over the diagonal set of $D^{(4)}$.

Other statements follow directly from the definitions. \qed
Lemma 7.3.1. \[ (7.8) \]

**Lemma 7.3.1.** Let \( F(4) \) be as in Lemma 7.1.2. Then there exists a divisorial contraction \( F(4) \to \hat{F} \) which contracts the strict transform \( D(4) \) of \( D(1) \) to the locus isomorphic to the flag variety \( F(1, 4, V) \)(see (7.7)). The discrepancy of \( D(4) \) is two.

**Proof.** Let \( \Delta_p \) be the inverse image in \( \hat{G}' \) of \( \Delta_p \). Note that \( \Delta_p' \simeq F(3, 4, V) \). Let \( \Gamma \) be a fiber of \( D(2) \to \Delta'_p \), where we recall \( \Gamma \simeq \mathbb{P}^2 \times \mathbb{P}^2 \). Then, by the proof of Lemma 7.2.1, \( \Gamma \) intersects the flopping locus along the diagonal transversally. Take a line \( r \subset \mathbb{P}^2 \times \mathbb{P}^2 \) which is contained in a fiber of a projection \( \Gamma \to \mathbb{P}^2 \) and intersects the flopping locus. Then its strict transform \( r' \) on \( D(4) \) is contracted by the morphism \( D(4) \to F(1, 4, V) \). Since \( F(2) \to \hat{G}' \) is a \( \mathbb{P}^2 \times \mathbb{P}^2 \)-fibration and \( D(2) \) is the pull-back of \( \Delta'_p \), we see that \( K_{F(2)} \cdot r = -3 \) and \( D(2) \cdot r = 0 \). By the standard calculations of the changes of the intersection numbers by the flip, we have \( K_{F(4)} \cdot r' = -3 + 1 = -2 \) and \( D(4) \cdot r' = 0 - 1 = -1 \). These equalities of the intersection numbers still hold for any line in a ruling of a fiber of \( D(4) \to F(1, 4, V) \).

We show that \( -K_{F(4)} + 2D(4) \) is relatively nef over \( \hat{G} \). Let \( \gamma \) be a curve on \( F(4) \) which is contracted to a point \( t \) on \( \hat{G} \). If \( t \notin \Delta_p \), then \( -K_{F(4)} + 2D(4) \cdot \gamma > 0 \) since \( D(4) \cap \gamma = \emptyset \) and \( F(4) \to \hat{G} \) is a \( \mathbb{P}^2 \times \mathbb{P}^2 \) fibration outside \( \Delta_p \). If \( t \in \Delta_p \) and \( \gamma \) is an exceptional curve of \( F(4) \to F(1) \), then \( -K_{F(4)} + 2D(4) \cdot \gamma > 0 \) since \( -K_{F(4)} \cdot \gamma > 0 \) and \( D(4) \cdot \gamma > 0 \). In the remaining cases, \( t \in \Delta_p \) and \( \gamma \subset D(4) \). Therefore we have only to consider the relative nefness of \( -K_{F(4)} + 2D(4) \rvert_{D(4)} \) over \( \Delta_p \). Now we take for \( \gamma \) any line in a ruling of a fiber of \( D(4) \to F(1, 4, V) \). As we see above, \( -K_{F(4)} + 2D(4) \rvert_{D(4)} \cdot \gamma = 0 \). Therefore \( -K_{F(4)} + 2D(4) \rvert_{D(4)} \) is the pull-back of some divisor \( D_F \) on \( F(1, 4, V) \). It suffices to show \( D_F \) is relatively nef over \( \Delta_p \), which is true since an exceptional curve of \( D(4) \to D(1) \) is positive for \( -K_{F(4)} + 2D(4) \rvert_{D(4)} \) as above and is mapped to a curve on a fiber of \( F(1, 4, V) \to \Delta_p \). Therefore \( -K_{F(4)} + 2D(4) \) is relatively nef over \( \hat{G} \).

Finally, from the above argument, we see that \( -K_{F(4)} + 2D(4) 
\supset \cap \mathbb{N}(E(F(4)/\hat{G})) \) is generated by the numerical class of the curves on fibers of \( D(4) \to F(1, 4, V) \). In particular, \( -K_{F(4)} + 2D(4) 
\supset \cap \mathbb{N}(E(F(4)/\hat{G})) \subset (K_{F(4)})^{<0} \). Therefore, by Mori theory, there exists a contraction associated to this extremal face, which is nothing but the divisorial contraction contracting \( D(4) \) to \( F(1, 4, V) \).

By the equalities \( K_{F(4)} \cdot r' = -2 \) and \( D(4) \cdot r' = -1 \), we see that the discrepancy of \( D(4) \) is two. \[ \square \]

Recall the \( \mathbb{Z}_2 \)-action on \( F(1) \) described in (7.1). Since all the morphisms constructed to obtain \( \hat{F} \) from \( F(1) \) are \( \mathbb{Z}_2 \)-equivariant, the variety \( \hat{F} \) also has a naturally
induced $\mathbb{Z}_2$-action. We also note that
\[
G'_{\mathfrak{y}} := \tilde{G}'/\mathbb{Z}_2 \simeq \text{Hilb}^2 \mathbb{P}(V^*) .
\]

Now we have

**Proposition 7.3.2.** The $\rho_{\mathfrak{y}}$-exceptional divisor $F_{\mathfrak{y}}$ is isomorphic to $\tilde{F}/\mathbb{Z}_2$.

**Proof.** We compare the morphisms $a: F_{\mathfrak{y}} \to G_{\mathfrak{y}}$ and $b: \tilde{F}/\mathbb{Z}_2 \to G_{\mathfrak{y}}$. By Lemma 5.5 for example, it suffices to check the following properties hold for them:

- Both $F_{\mathfrak{y}}$ and $\tilde{F}/\mathbb{Z}_2$ are normal.
- the morphisms $a$ and $b$ are isomorphic to each other in codimension one.
- $-K_{F_{\mathfrak{y}}}$ and $-K_{\tilde{F}/\mathbb{Z}_2}$ are $\mathbb{Q}$-Cartier.
- $-K_{F_{\mathfrak{y}}}$ is $a$-ample and $-K_{\tilde{F}/\mathbb{Z}_2}$ is $b$-ample.

The variety $F_{\mathfrak{y}}$ is normal. Indeed, it satisfies the $S_2$ condition since it is a Cartier divisor on a smooth variety. Moreover, it satisfies the $R_1$ condition since it is a $\mathbb{P}^2 \times \mathbb{P}^2$-fibration outside the locus of codimension two by Lemmas 5.6.3 and 5.6.4.

We see also that the variety $\tilde{F}/\mathbb{Z}_2$ is normal by its explicit construction as above.

The morphisms $a$ and $b$ are isomorphic outside $G_1^{\mathfrak{y}}$ by Lemma 5.6.3 and the construction of $F^{(1)}/\mathbb{Z}_2$. Moreover, the inverse images of $G_1^{\mathfrak{y}}$ by the morphism $a$ has codimension two in $F_{\mathfrak{y}}$ by Lemma 6.6.1 (and Remark after it), and the inverse images of $G_1^{\mathfrak{y}}$ by the morphism $b$ has codimension two in $\tilde{F}/\mathbb{Z}_2$ by the construction of $\tilde{F}/\mathbb{Z}_2$. Therefore the morphisms $a$ and $b$ are isomorphic to each other in codimension one.

The divisor $-K_{F_{\mathfrak{y}}}$ is $\mathbb{Q}$-Cartier since $F_{\mathfrak{y}}$ is a divisor on the smooth variety $\tilde{\mathfrak{y}}$. Since the relative Picard number $\rho(\mathfrak{y}/\mathfrak{y})$ is one and $a$ is generically a $\mathbb{P}^2 \times \mathbb{P}^2$-fibration, we see that $-K_{F_{\mathfrak{y}}}$ is $a$-ample.

We see that similar facts hold for the morphism $b$. Also we see that $-K_{\tilde{F}/\mathbb{Z}_2}$ is $\mathbb{Q}$-Cartier. Indeed, by Lemma 7.3.4 the discrepancy of $D^{(4)}$ is two. Then, by the Kawamata-Shokurov base point free theorem, $-K_{F^{(4)}} - 2D^{(4)}$ is the pull-back of a Cartier divisor on $\tilde{F}$, which turns out to be the anti-canonical divisor $-K_{\tilde{F}}$. Thus $-K_{\tilde{F}/\mathbb{Z}_2}$ is $\mathbb{Q}$-Cartier.

To show that $-K_{\tilde{F}/\mathbb{Z}_2}$ is $b$-ample, it suffices to see the relative Picard number $\rho((\tilde{F}/\mathbb{Z}_2)/G_{\mathfrak{y}})$ is one because $b$ is generically a $\mathbb{P}^2 \times \mathbb{P}^2$-fibration. Let us note that the relative Picard number $\rho(F^{(2)}/\mathfrak{g}')$ is two since $F^{(2)} \to \hat{G}'$ is a $\mathbb{P}^2 \times \mathbb{P}^2$-fibration and it is easy to see that it is a composite of two $\mathbb{P}^2$-fibrations. Moreover we have $\rho^{22}(F^{(2)}/\mathfrak{g}') = 1$ since the two rulings of a fiber $\mathbb{P}^2 \times \mathbb{P}^2$ of $F^{(2)} \to \hat{G}'$ are exchanged by the $\mathbb{Z}_2$-action. Therefore $\rho^{22}(F^{(2)}) = 3$ since $\rho^{22}(\hat{G}') = 2$. It holds that $\rho^{22}(F^{(4)}) = 3$ since the flip preserves the Picard number and it is $\mathbb{Z}_2$-equivariant. Since a divisorial contraction decreases the Picard number at least by one, we have $\rho^{22}(\tilde{F}) \leq 2$. Now we see that $\rho((\tilde{F}/\mathbb{Z}_2)/G_{\mathfrak{y}})$ is one since $\rho(G_{\mathfrak{y}}) = 1$ and the morphism $\tilde{F}/\mathbb{Z}_2 \to G_{\mathfrak{y}}$ is non-trivial. Therefore we conclude that $-K_{\tilde{F}/\mathbb{Z}_2}$ is $b$-ample.
In summary, we have obtained the following diagram:

\[
\begin{array}{ccc}
F^{(3)} & \xrightarrow{(\text{anti-})\text{flip} \text{(Lem. 7.4.1)}} & F^{(4)} \\
\downarrow & & \downarrow \\
F^{(2)} & \xrightarrow{\text{div. cont.} \text{(Lem. 7.4.1)}} & F^{(3)} \\
\downarrow & & \downarrow \\
\hat{G}' & \xrightarrow{\text{div. cont.} \text{(Lem. 7.4.1)}} & \hat{G}' \\
\end{array}
\]

7.4. Flattening of \( F^{(3)} \to \hat{G}' \). Here we describe the fibers of \( F^{(3)} \to \hat{G}' \) in the diagram (7.9) and show the flatness of the morphism \( F^{(3)} \to \hat{G}' \).

**Proposition 7.4.1.**

1. The fiber of \( F^{(3)} \to \hat{G}' \) over a point \([V_3]; [V_4^{(1)}], [V_4^{(2)}]\) with \( V_4^{(1)} \neq V_4^{(2)} \) is \( \mathbb{P}(V_3^*) \times \mathbb{P}(V_4^*) \).

2. The fiber of \( F^{(3)} \to \hat{G}' \) over a point \([V_3]; [V_4], [V_4]\) is the union of the following two 4-dimensional varieties \( A \) and \( B \):
   - \( A \cong \mathbb{P}(\mathcal{O}_{V_3^*} \oplus T_{\mathbb{P}(V_3^*)}) \), which is isomorphic to the restriction of \( \mathbb{P}(\mathcal{O}_{\hat{G}(2, V_4)} \oplus U'_{\hat{G}(2, V_4)}(1)) \) over \( \mathbb{P}(V_3^*) = G(2, V_3) \subset G(2, V_4) \).
   - \( B \) is the blow-up of \( \mathbb{P}(V_3^*) \times \mathbb{P}(V_4^*) \) along the diagonal \( \Delta_{V_3^*} \). It is endowed with a morphism \( p_B: B \to \mathbb{P}(V_3) \) induced from the rational map
     \[
     \mathbb{P}(V_3^*) \times \mathbb{P}(V_4^*) \setminus \Delta_{V_3^*} \to \mathbb{P}(V_3)
     \]
     \[
     ([V_2^{(1)}], [V_2^{(2)}]) \mapsto [V_2^{(1)} \cap V_2^{(2)}],
     \]
     and is a \( \mathbb{P}^1 \times \mathbb{P}^1 \)-bundle over \( \mathbb{P}(V_3) \).

In particular the morphism \( F^{(3)} \to \hat{G}' \) is flat. Moreover, the intersection \( E_{AB} := A \cap B \) is \( \mathbb{P}(T_{\mathbb{P}(V_3^*)}) \) in \( A \), which is the restriction of \( E_\sigma = \mathbb{P}(U_{\hat{G}(2, V_4)}(1)) \) (cf. Lemma 5.3.4) over \( G(2, V_3) \), and also \( E_{AB} \) is the exceptional divisor of \( B \to \mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*) \) in \( B \).

**Proof.** Part (1) follows from the construction of \( F^{(2)} \to \hat{G}' \).

We show Part (2). The fiber of \( F^{(2)} \to \hat{G}' \) over a point \([V_3]; [V_4], [V_4]\) is \( \mathbb{P}(V_3^*) \times \mathbb{P}(V_4^*) \). The intersection of the fiber \( \mathbb{P}(V_3^*) \times \mathbb{P}(V_4^*) \) with the exceptional locus of \( F^{(2)} \to F^{(1)} \) is

\[
\{( [V_2]; [V_2]; [V_3]; [V_4], [V_4]) \mid V_2 \subset V_3 \} \cong \mathbb{P}^2,
\]

which is nothing but the diagonal of \( \mathbb{P}(V_3^*) \times \mathbb{P}(V_4^*) \). Therefore we have \( B \) as an irreducible component of the fiber of \( F^{(3)} \to \hat{G}' \) over the point \([V_3]; [V_4], [V_4]\).

Another component \( A \) is a \( \mathbb{P}^2 \)-bundle over the diagonal of \( \mathbb{P}(V_3^*) \times \mathbb{P}(V_4^*) \) since the exceptional divisor of \( F^{(3)} \to F^{(2)} \) is a \( \mathbb{P}^2 \)-bundle over the exceptional locus of \( F^{(2)} \to F^{(1)} \). As described in Lemma 7.1.2, it is introduced by the blow-up with respect to the three normal coordinates of the singular locus of \( F^{(1)} \). Hence \( A \) is given by the projective bundle \( \mathbb{P}(\mathcal{O}_\Delta_{V_3^*} \oplus \mathcal{N}_\Delta_{V_3^*}) \) over the diagonal. Now we
Lemma 7.5.1. Denote by $F$ the pull-backs of the divisor $D$ in $A$.

The pull-backs of $D$ in $A$ can be described by the restriction of $\mathcal{O}_{G(2,V_4)} \oplus \mathcal{O}_{V_4}(1)$ over $\mathbb{P}(2,V_4)$.

Remark. Recall the description (7.8) of the divisor $D$ in $F$. The component $B$ above is nothing but the restriction $D|_{([V_3],[V_4])}$ of $D$ over the point $([V_3],[V_4])$. Also the component $A$ can be identified in Lemma 5.6.4 (2) and also in Fig. 3.

By Proposition 7.3.1 and the results in the previous subsection summarized in (7.3), we obtain the flattening of $F_{\mathcal{Y}} \to G_{\mathcal{Y}}$.

\[
\begin{array}{ccc}
F^{(3)} & \to & F_{\mathcal{Y}} \\
\downarrow & & \downarrow \\
G' & \to & G_{\mathcal{Y}}.
\end{array}
\]

The flatness of the morphism $F^{(3)} \to G'$ is crucial for our proof of Lemma 8.1.3. By this property, we can reduce computations of cohomology groups on $F_{\mathcal{Y}}$ to those on $F^{(3)}$ and then those on special fibers of $F^{(3)} \to G'$.

7.5. The pull-backs of $S^2$, $\bar{Q}$ and $\bar{T}$ on $A$ and $B$. In this subsection, we consider the situation of Proposition 7.3.1 (2) fixing $V_3$ and $V_4$. We describe the pull-backs of the divisor $F_{\mathcal{Y}}$ and the sheaves $S^2$, $\bar{Q}$, $\bar{T}$ on $A$ and $B$ in the fiber of $F^{(3)} \to G'$.

Lemma 7.5.1. Denote by $A_{\mathcal{Y}}$ the image of $A$ on $\mathcal{Y}$, by $A_{\mathcal{Y}_2}$ the strict transform on $\mathcal{Y}_2$ of $A_{\mathcal{Y}}$, and by $A_{\mathcal{Y}_3}$ the image on $\mathcal{Y}_3$ of $A_{\mathcal{Y}_2}$. Then,

1. $A \to A_{\mathcal{Y}}$ is the contraction of $E_{AB} \simeq \mathbb{P}(T_{\mathbb{P}(V_3)})$ to $\mathbb{P}(V_3)$. $\mathbb{P}(V_3)$ is given by the image of $B$ by the morphism $P_B$ in Proposition 7.3.1 (2), and is equal to the singular locus of $A_{\mathcal{Y}}$ (see Fig. 3).
2. $A_{\mathcal{Y}_2} = \Gamma(V_3,V_4)$ with $\Gamma(V_3,V_4) := \cup_{V_2 \subset V_3} \Gamma(V_2,V_4)$,
3. $A_{\mathcal{Y}_3} \to A_{\mathcal{Y}}$ is the blow-up along the image on $A_{\mathcal{Y}}$ of the section of $A_3$ of $A \to \mathbb{P}(V_3)$ associated to the injection $\mathcal{O}_{\mathbb{P}(V_3)} \to \mathcal{O}_{\mathbb{P}(V_3)} \oplus T_{\mathbb{P}(V_3)}$,
4. let $\hat{A} \to A$ be the blow-up of $A$ along the section $A_3$. Then there exists a natural morphism $\hat{A} \to A_{\mathcal{Y}_3}$, which is the blow-up of $A_{\mathcal{Y}_3}$ along its singular locus, and
5. $A_{\mathcal{Y}_3} \simeq A_{\mathcal{Y}_3}$. Moreover, $\rho_{\mathcal{Y}_3}|_{A_{\mathcal{Y}_3}}$ is isomorphic to $\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))$ and $\rho_{\mathcal{Y}_3}|_{A_{\mathcal{Y}_3}} : A_{\mathcal{Y}_3} \to \mathbb{P}(V_3)$ is a quadric cone fibration.
Proof. The claims (1)–(3) follow from Lemmas 5.6.4, 7.4.1. The claim (4) is almost obvious.

Now we prove (5). By the proof of Lemma 5.6.4, the fiber of \( \rho_{\mathcal{G}} : A_{\mathcal{G}} \to \mathbb{P}(V'_3) \) over \([V] \in \mathbb{P}(V'_3)\), which is defined similarly to \( \mathcal{T}(V_1, V_4) \). Noting that \( \mathcal{T}(V_1, V_4) \) is the cone over \( \mathcal{Q}_2 \mathcal{P}(V_4/V_1) \), we see that \( \mathcal{T}(V_1, V_4) \) is the cone over \( \mathcal{Q}_2 \mathcal{P}(V_4/V_1) \), namely, \( \mathcal{T}(V_1, V_4) \) is the quadric cone. Therefore, \( \rho_{\mathcal{G}} : A_{\mathcal{G}} \to \mathbb{P}(V'_3) \) is a quadric cone fibration. In particular, \( \rho(A_{\mathcal{G}}) = 2 \). On the other hand, we have \( \rho(A_{\mathcal{G}}) = 2 \) since \( \rho(A_{\mathcal{G}}) = 1 \) and \( A_{\mathcal{G}} \to A_{\tilde{\mathcal{G}}} \) is a simple blow-up. Thus \( A_{\mathcal{G}} \to A_{\tilde{\mathcal{G}}} \) must be an isomorphism since it is birational.

Finally we show \( \mathcal{P}_{\rho|A_{\mathcal{G}}} \simeq \mathbb{P}(T_{\mathbb{P}(V'_3)}(-1)) \). Note that \( \mathcal{P}_{\rho|A_{\mathcal{G}}} \) is isomorphic to the exceptional divisor \( G \) of \( \tilde{\mathcal{A}} \to \mathcal{A} \), which we determine from now on. Let \( I_{sA} \) be the ideal sheaf of the section \( s_A \) in \( A \). Note that \( \mathcal{O}_{\mathbb{P}(V'_3)\mathcal{P}(V_4)}(1)|s_A = \mathcal{O}_{sA} \).

Tensoring \( 0 \to I_{sA} \to \mathcal{O}_A \to \mathcal{O}_{sA} \to 0 \) with \( \mathcal{O}_{\mathbb{P}(V'_3)\mathcal{P}(V_4)}(1) \) and pushing forward to \( \mathbb{P}(V'_3) \), we see that \( I_{sA}/I_{sA}^2 \simeq \mathcal{O}_{sA} \). Therefore \( G \) is isomorphic to \( \mathbb{P}(T_{\mathbb{P}(V'_3)}) \). It is well-known that \( \mathcal{P}(T_{\mathbb{P}(V'_3)}) \) is isomorphic to the incident variety \( \{(V_1, V_2) | V_1 \subset V_2 \} \subset \mathbb{P}(V'_3) \times \mathbb{P}(V'_3) \), which is also isomorphic to \( \mathbb{P}(T_{\mathbb{P}(V'_3)}(-1)) \).

For a locally free sheaf \( \mathcal{E} \) on \( \tilde{\mathcal{A}} \), we denote by \( \mathcal{E}_A \) and \( \mathcal{E}_B \) its pull-backs on \( A \) and \( B \), respectively unless stated otherwise. Denote by \( H_A \) the pull back on \( A \) of \( \mathcal{O}_{\mathbb{P}(V'_3)}(1) \), and \( F_A \) and \( F_B \) the pull-backs of (the line bundle) \( F_{\mathcal{G}} \) to \( A \) and \( B \), respectively.

Lemma 7.5.2. (1) \( F_A \sim -(EA_B + 2H_A) \), \( (\tilde{\mathcal{S}}^*_L)_A \simeq \tilde{\mathcal{O}}_A \simeq \mathcal{O}_A \oplus \mathcal{V} \), and \( T_A \simeq \mathcal{O}_A \oplus \mathcal{V} \), where \( \mathcal{V} \) is a locally free sheaf obtained as a unique nonsplit extension

\[ 0 \to \mathcal{O}_A(H_A + E_{AB}) \to \mathcal{V} \to \mathcal{O}_A(H_A) \to 0. \]

(2) \( \mathcal{O}_B(F_B) \simeq p_B^*\mathcal{O}_{\mathbb{P}(V_3)}(-1) \), \( (\tilde{\mathcal{S}}^*_L)_B \simeq \tilde{\mathcal{O}}_B \simeq \mathcal{O}_B \oplus p_B^*T_{\mathbb{P}(V'_3)}(-1) \), and \( \tilde{T}_B \simeq \mathcal{O}_B^2 \oplus p_B^*T_{\mathbb{P}(V'_3)}(-1) \), where \( p_B : B \to \mathbb{P}(V'_3) \) is as in Proposition 7.4.1.(2).

Proof.

Step 1. \( \det(\tilde{\mathcal{S}}^*_L)_A = \det \tilde{\mathcal{O}}_A = E_{AB} + 2H_A \).

By \( \tilde{\mathcal{O}}_A \), we have only to determine \( \det \tilde{\mathcal{O}}_A \). Let \( L_\tilde{\mathcal{A}} \) be the pull-back of \( \mathcal{O}_{\mathbb{P}(V'_3)}(1) \), and \( G \) the exceptional divisor for \( \tilde{\mathcal{A}} \to \mathcal{A} \). Note that \( G \) is the pull-back of the exceptional divisor \( F_\mathcal{G} \to \mathcal{G} \). By Proposition 6.4.1 we have \( \det \tilde{\mathcal{O}}_A = G + L_\tilde{\mathcal{A}} \) since \( M_\mathcal{G} \) is trivial on a fiber of \( F_{\mathcal{G}} \to G_\mathcal{G} \). Therefore it suffices to show \( E_{AB} + 2H_\tilde{\mathcal{A}} - G = L_\tilde{\mathcal{A}} \), where \( H_\tilde{\mathcal{A}} \) and \( E_{AB} \) are the pull-backs on \( \tilde{\mathcal{A}} \) of \( H_A \) and \( E_{AB} \), respectively. Note that we can write \( aE_{AB} + bH_\tilde{\mathcal{A}} - cG = L_\tilde{\mathcal{A}} \) with some
Since $\widehat{E}_{AB} \cap g = \emptyset$, we have $a\widehat{E}_{AB} + bH_\mathcal{A}|_{\widehat{E}_{AB}} = L_\mathcal{A}$. Since $\widehat{E}_{AB} \simeq E_{AB} \simeq \mathbb{P}(T_{\mathbb{P}(V_1^*)})$, we may consider $\widehat{E}_{AB}$ as the incident variety $\{(V_1, V_2) | V_1 \subset V_2 \subset \mathbb{P}(V_1) \times \mathbb{P}(V_2^*) \}$. Also, since $E_{AB}$ is the tautological divisor with respect to $\mathcal{O}_{\mathbb{P}(V^*_1)} \oplus T_{\mathbb{P}(V_1^*)}$, $\widehat{E}_{AB}|_{E_{AB}} = E_{AB}|_{E_{AB}}$ is the tautological divisor with respect to $T_{\mathbb{P}(V_1^*)}$. Hence $\widehat{E}_{AB}|_{E_{AB}}$ is the restriction of a $(1, -2)$-divisor of $\mathbb{P}(V_1) \times \mathbb{P}(V_2^*)$. This is equivalent to that $a = 1$ and $b = 2$. Now we note the equality $a\widehat{E}_{AB}|_G + bH_\mathcal{A}|_G - cG|_G = bH_\mathcal{A}|_G - cG|_G = L_\mathcal{A}|_G$. Recall that the conormal bundle of $s_A$ in $A$ is $\Omega_{\mathbb{P}(V_1^*)}$ as in the proof of Lemma 7.5.1. Therefore $-G|_G$ is the tautological divisor with respect to $T_{\mathbb{P}(V_1^*)}$, which is the restriction of a $(1, -2)$-divisor of $\mathbb{P}(V_1) \times \mathbb{P}(V_2^*)$. This is equivalent to that $b = 2$ and $c = 1$.

**Step 2.** $F_A = -(E_{AB} + 2H_A)$. 

By Proposition 6.3.1, we have $L_\mathcal{A} = \det \widetilde{\mathcal{Q}} - M_{\mathcal{A}}$, where $L_\mathcal{A}$ is the image of $L_{\mathcal{A}}$ on $\mathcal{A}$. Therefore, by (6.3), we have $K_{\mathcal{A}} = -6 \det \widetilde{\mathcal{Q}} + 4L_{\mathcal{A}} = -2 \det \widetilde{\mathcal{Q}} - 4M_{\mathcal{A}}$. Further, by Proposition 5.7.2 (2), we have $-2 \det \widetilde{\mathcal{Q}} - 4M_{\mathcal{A}} = -10M_{\mathcal{A}} + 2F_{\mathcal{A}}$. Therefore, since the pull-back of $M_{\mathcal{A}}$ on $A$ is trivial, we have $F_A = - \det \mathcal{Q}_A$. Consequently, we obtain $F_A = -(E_{AB} + 2H_A)$ by the equality in Step 1.

**Step 3.** $(S^2_{\mathcal{A}})_A \simeq \mathcal{Q}_A \simeq \mathcal{O}_A \oplus \mathcal{V}$.

We investigate the restriction of the universal exact sequence (6.3) on $A_{\mathcal{A}}$. Let $S_{A_{\mathcal{A}}}$ and $\mathcal{Q}_{A_{\mathcal{A}}}$ be the restrictions of $S$ and $\mathcal{Q}$, respectively. Then we obtain

$$0 \to S_{A_{\mathcal{A}}} \to \pi_{A_{\mathcal{A}}}(T(-1)^{\wedge 2} \mathcal{O}(1)) \to \mathcal{Q}_{A_{\mathcal{A}}} \to 0. \tag{7.11}$$

Note the following isomorphisms,

$$\wedge^2(T(-1)^{\mathcal{O}(1)} \mathcal{O}(1)) \simeq \wedge^2(T_{\mathbb{P}(V_3)}(1) \oplus V/V_3 \mathcal{O}(1)) \simeq \mathcal{O}(V_3) \otimes T_{\mathbb{P}(V_3)}(-1) \otimes \wedge^2(V/V_3) \mathcal{O}(V_3). \tag{7.12}$$

Let $([\hat{U}], [V_1])$ be a point in $A_{\mathcal{A}} \subset \mathcal{A}$ with $[U] = [\hat{U}, V_1]$ (see Definition 5.3.1). Since the morphism $A_{\mathcal{A}} \simeq A_{\mathcal{A}} \to A_{\mathcal{A}}$ is given by $([\hat{U}], [V_1]) \mapsto [U]$ and $A_{\mathcal{A}} = \mathbb{P}(V_3, V_4)$, we can assume the following form of $[U]$ (see Lemma 5.6.4):

$$[U] = ([V_4/V_2] \wedge (V_2/V_1), a(V_4/V_4) \wedge (V_2/V_1) + b \wedge^2 (V_4/V_2) \wedge V_1),$$

with $V_1 \subset V_2 \subset V_3$. For simplicity, we write $[U] \in A_{\mathcal{A}}$ with $[V_1]$ being implicit.

By definition, we have $S_{A_{\mathcal{A}}}|_U = U$ for the fiber of $S_{A_{\mathcal{A}}}$ over $[U] \in A_{\mathcal{A}}$. Similarly, the fiber of the pull-back $L_{A_{\mathcal{A}}}|_U$ of $\mathcal{O}_{\mathbb{P}(V_3)}(1) \simeq \wedge^2 T_{\mathbb{P}(V_3)}(-1)$ is given by $L_{A_{\mathcal{A}}}|_U = \wedge^2(V_3/V_2) \wedge (V_2/V_1)$. Hence we see that $S_{A_{\mathcal{A}}}$ contains $L_{A_{\mathcal{A}}}$ as a direct summand. Let us write $S_{A_{\mathcal{A}}} = S_{\mathcal{A}_{\mathcal{A}}} + L_{A_{\mathcal{A}}}$ with a locally free sheaf $S_{\mathcal{A}_{\mathcal{A}}}$ of rank two on $A_{\mathcal{A}}$. We note that $S_{\mathcal{A}_{\mathcal{A}}}$ is contained in $L_{A_{\mathcal{A}}} \oplus (V/V_3 \otimes \pi_{\mathcal{A}_{\mathcal{A}}}(T_{\mathbb{P}(V_3)}(-1)))$ since this does not contain (the pull-back on $A_{\mathcal{A}}$ of) the factor $\wedge^2(V/V_3) \mathcal{O}(V_3)$ in (7.12). Therefore, we have the following exact sequence from (7.11):

$$0 \to S_{\mathcal{A}_{\mathcal{A}}} \to V/V_3 \otimes \pi_{\mathcal{A}_{\mathcal{A}}}(T_{\mathbb{P}(V_3)}(-1)) \to \mathcal{Q}_{\mathcal{A}_{\mathcal{A}}} \to 0,$$

where $\mathcal{Q}_{\mathcal{A}_{\mathcal{A}}} \simeq \mathcal{Q}_{\mathcal{A}_{\mathcal{A}}} \otimes \mathcal{O}_{A_{\mathcal{A}}}$.

Consider the pull-back $S'_{\mathcal{A}}$ on $\mathcal{A}$ of $S'_{\mathcal{A}_{\mathcal{A}}}$. This contains a subbundle of rank one whose fiber at a point over $[U]$ is isomorphic to $V_4/V_3 \otimes V_2/V_1$. Since $V_3$ and $V_4$ are fixed, the vector space $V_4/V_3$ is the fiber of the trivial bundle on $\mathcal{A}$. Also, since $V_1$ is the fiber of $-L_\mathcal{A}$ and $V_2$ is the fiber of the pull-back of $\Omega_{\mathbb{P}(V_3^*)}(1)$, we see...
that $V_2/V_1$ is the fiber of $O_A(-H_A + L_A)$ by taking the determinants. Therefore $S'_A(-L_A)$ is presented as an extension,
\[ 0 \to O(-H_A) \to S'_A(-L_A) \to O(-H_A - \hat{E}_{AB}) \to 0, \]
where the quotient is determined by taking determinants. Since $O(-H_A)$, $S'_A(-L_A)$, and $O(-H_A - \hat{E}_{AB})$ are the pull-backs of locally free sheaves on $A$, we have
\[ 0 \to O_A(-H_A) \to (S'_L)_A \to O_A(-H_A - E_{AB}) \to 0, \]
where $(S'_L)_A$ is the sheaf which represents $S'_A(-L_A)$ by the pull-back. The sequence does not split since $H_A$ is not a locally free sheaf on $\hat{A}$, while $(S'_L)_A$ is. Since
\[ \text{Ext}^1(O_A(-H_A - E_{AB}), O_A(-H_A)) \simeq H^1(A, O_A(E_{AB})) \simeq H^1(\mathbb{P}(V'_A), O_{\mathbb{P}(V'_A)} \oplus \Omega^1_{\mathbb{P}(V'_A)}) \simeq \mathbb{C} \]
we see that $(S'_L)_A \simeq V$, and hence $(S'_L)_A \simeq V \oplus O_A$, with a locally free sheaf $V$ as described in (1).

Let $Q'_A$ be the pull-back on $\hat{A}$ of $Q'_{A, g_2}$. Taking a basis of $V$ so that $V_1 = \langle e_3 \rangle, V_2 = \langle e_1, e_3 \rangle, V_3 = \langle e_1, e_2, e_3 \rangle$ and $V_4 = \langle e_1, e_2, e_3, e_5 \rangle$, we see that there is a surjective map from $Q'_A$ to the invertible sheaf whose fiber at a point over $[\hat{U}]$ is isomorphic to $V_3/V_2 \otimes V/V_4$. We identify this invertible sheaf with $O_{\hat{A}}(H_A)$. Therefore $Q'_A$ is presented as an extension:
\[ 0 \to O(H_A + \hat{E}_{AB}) \to Q'_A \to O(H_A) \to 0, \]
where the kernel is determined by taking determinants. Therefore we see that $Q'_A$ is also the pull-back of $\mathcal{V}$ and $\mathcal{Q}_A \simeq \mathcal{V} \oplus O_A$ as we have determined $(S'_L)_A$.

**Step 4.** $\mathcal{T}_A \simeq O^\mathcal{Q}_A \oplus \mathcal{V}$. By Lemma 7.11, $\mathcal{Q}|_{A, g_2} \simeq \mathbb{P}(T_{\mathbb{P}(V)_3}(-1))$ and this lifts to $\mathcal{Y}_2$ isomorphically. Therefore, restricting (6.3) to $A, g_2$, we obtain
\[ 0 \to T^*_{A, g_2} \to \pi^*_{A, g_2} \Omega^1_{\mathbb{P}(V'_3)}(1) \oplus O^\mathcal{Q}_{\mathbb{P}(V'_3)}(1) \to O_{\mathbb{P}(T_{\mathbb{P}(V)_3}(-1))}(1) \to 0, \]
where we set $T^*_{A, g_2} = T^*_{A, g_2, \nu, \pi_{A, g_2}, \pi_{A, g_2} \vert A, g_2}$ and used $R_2/R_1 = O_{\mathbb{P}(T(-1))}(-1)$. Since $H^0(O_{\mathbb{P}(T_{\mathbb{P}(V)_3}(-1))}(1)) = H^0(\Omega^1_{\mathbb{P}(V'_3)}(1)) = 0$, we have
\[ T^*_{A, g_2} \simeq O^\mathcal{Q}_{A, g_2} \oplus \mathcal{V}', \]
where $\mathcal{V}'$ is the kernel of the map
\[ \pi^*_{A, g_2} \Omega^1_{\mathbb{P}(V'_3)}(1) \to O_{\mathbb{P}(T_{\mathbb{P}(V)_3}(-1))}(1). \]
Let us consider a point $[\hat{U}] \in A, g_2$. We note that the fiber of $\pi^*_{A, g_2} \Omega^1_{\mathbb{P}(V'_3)}(1)$ at $[\hat{U}] \in A, g_2$ is isomorphic to $(V_3/V_1)^*$. In a similar way to the arguments after (7.12), the vector space $(V_3/V_2)^*$ can be considered to be the fiber of $O_A(-H_A)$ at a point over $[V_2] \in \mathbb{P}(V_3)$. Therefore we have a natural injection
\[ O_A(-H_A) \to \pi^*_{A} \Omega^1_{\mathbb{P}(V'_3)}(1), \]
where the cokernel $\mathcal{K}_1$ is an invertible sheaf and $\pi_{A, g_2}$ is the naturally induced map $\hat{A} \to \mathbb{P}(V_3).$. By taking the determinants, we see that $\mathcal{K}_1 = O_A(H_A - L_A)$. We show that the composite morphism $O_A(-H_A) \to O_{\mathbb{P}(T_{\mathbb{P}(V)_3}(-1))}(1)$ of the injection above with the pull-back of (7.14) is zero. Here we note that $\mathbb{P}(T_{\mathbb{P}(V)_3}(-1))$ lifts
isomorphically on $\hat{A}$. Indeed, $H_{\mathbb{P}(\Omega_{\mathbb{P}(V_3)}^1(1))}$ is nothing but the pull-back of $H_A$ by the map $\hat{A} \to A$. Since $H_{\mathbb{P}(\Omega_{\mathbb{P}(V_3)}^1(1))} = H_{\mathbb{P}(\Omega_{\mathbb{P}(V_3)}^1(1))} - L$, where $L$ is the pull-back of $\mathcal{O}_{\mathbb{P}(V_3)}(1)$ to $\mathbb{P}(\Omega_{\mathbb{P}(V_3)}^1(-1))$, we have only to show $H^0(2H_{\mathbb{P}(\Omega_{\mathbb{P}(V_3)}^1(1))} - L) = 0$, which follows from the Bott theorem 2.0.1 by noting $H^0(2H_{\mathbb{P}(\Omega_{\mathbb{P}(V_3)}^1(1))} - L) \simeq H^0(S^2(T_{\mathbb{P}(V_3)})(-1)) \otimes \mathcal{O}_{A_{\mathbb{P}(V_3)}^2}(-1))$. Therefore we have an injection $\mathcal{O}_{\hat{A}}(-H_{\hat{A}}) \hookrightarrow \mathcal{V}'_{\hat{A}}$, where $\mathcal{V}'_{\hat{A}}$ is the pull-back of $\mathcal{V}$ on $\hat{A}$ and the cokernel $K_2$ has the following expression as an extension:

$$0 \to K_2 \to \mathcal{O}_{\hat{A}}(H_{\hat{A}} - L_{\hat{A}}) \to H_{\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))} \to 0.$$ 

Taking $\mathcal{E}xt^\bullet(-, \mathcal{O}_{\hat{A}})$ of this exact sequence, we see that $K_2$ is also an invertible sheaf by [15 III, Ex 6.6]. By taking the determinants, we see that $K_2 = \mathcal{O}_{\hat{A}}(H_{\hat{A}} - L_{\hat{A}} - F_{\rho}|_{\hat{A}})$, where $F_{\rho}|_{\hat{A}}$ is the pull-back of $F_{\rho}$. Since $M_{\mathbb{P}(V_3)}(A_{\mathbb{P}(V_3)^2}) = 0$, we have $\det \mathcal{Q}_{\hat{A}} = L_{\hat{A}} + F_{\rho}|_{\hat{A}}$ by Proposition 6.4.1, where $\mathcal{Q}_{\hat{A}}$ is the pull-back of $\mathcal{Q}$. Therefore, $K_2 = \mathcal{O}_{\hat{A}}(H_{\hat{A}} - \det \mathcal{Q}_{\hat{A}}) = \mathcal{O}_{\hat{A}}(-H_{\hat{A}} - E_{AB})$, where the second equality follows from Step 1. Therefore $\mathcal{V}'_{\hat{A}}$ fits into the following expression as an extension:

$$0 \to \mathcal{O}_{\hat{A}}(-H_{\hat{A}}) \to \mathcal{V}'_{\hat{A}} \to \mathcal{O}_{\hat{A}}(-H_{\hat{A}} - E_{AB}) \to 0.$$ 

Consequently, we have $\overline{T}_A \simeq \mathcal{O}^B_{\mathbb{A}} \oplus \mathcal{V}$ as we have determined $(\overline{S}_L)_A$.

**Step 5.** $F_B$, $(\overline{S}_L)_B$, $\overline{Q}_B$ and $\overline{T}_B$.

By Lemma 5.3.1(1), the image of $B$ on $F_{\mathbb{P}}$ is the $\mathbb{P}(V_3)$ contained in $A_{\mathbb{P}}$. Therefore, $F_B$, $(\overline{S}_L)_B$, $\overline{Q}_B$ and $\overline{T}_B$, respectively, are the pull-backs of the restrictions of $F_{\mathbb{P}}$, $\overline{Q}$, and $\overline{T}$ to $\mathbb{P}(V_3)$. Since $F_{\mathbb{P}}|_{A_{\mathbb{P}}} \simeq -(E_{AB} + 2H_{A})|_{A_{\mathbb{P}}}$ by Step 2, and this is the pull-back of $\mathcal{O}_{\mathbb{P}(V_3)}(-1)$, we have $F_B = p_B^*\mathcal{O}_{\mathbb{P}(V_3)}(-1)$. Also, since $T_A \simeq \mathcal{O}_A \oplus (\overline{S}_L)_A \simeq \mathcal{O}_A \oplus \overline{Q}_A$ as above, we have $\overline{T}_B \simeq \mathcal{O}_B \oplus (\overline{S}_L)_B \simeq \mathcal{O}_B \oplus \overline{Q}_B$. Thus we have only to determine $\overline{T}_B$. Recall that $\mathbb{P}(V_3)$ in $A_{\mathbb{P}}$ consists of the points of the form $[U] = [\wedge^2(V_4/V_1) \wedge V_1] \in \mathscr{P}$ with $V_1 \subset V_3$. Therefore $\mathbb{P}(V_3)$ is disjoint from $G_{\rho}$ (Definition 4.4.2 Fig. 4), and then, by (6.6), we have $\overline{T}_B \simeq p_B^*(T(-1)|_{\mathbb{P}(V_3)}) \simeq \mathcal{O}_B^{\oplus 2} \oplus p_B^*(T_{\mathbb{P}(V_3)}(-1)).$ 

□
DUALITY BETWEEN $S^2\mathbb{P}^4$ AND THE DOUBLE QUINTIC SYMMETROID

8. The Lefschetz collection in $\mathcal{D}^b(\tilde{Y})$

Using the sheaves $\tilde{S}_L$, $\tilde{Q}$, $\tilde{T}$ and divisors introduced in Section 6, we describe a Lefschetz collection in $\mathcal{D}^b(\tilde{Y})$, which shows an interesting duality between the (dual) Lefschetz collection obtained in Theorem 3.4.5.

8.1. Results. Our results on the sheaves $\tilde{S}_L$, $\tilde{Q}$, $\tilde{T}$ and $\mathcal{O}_{\tilde{Y}}$ are summarized in the following theorem.

**Theorem 8.1.1.**

1. Let
   $$ (E_3, E_2, E_{1a}, E_{1b}) = ((\tilde{S}_L)^*, \tilde{T}, \mathcal{O}_{\tilde{Y}}, \tilde{Q}(\mathcal{M}_{\tilde{Y}})) $$
   be an ordered collection of sheaves on $\tilde{Y}$. Then $(\tilde{B}_i)_{1 \leq i \leq 4} := (E_3, E_2, E_{1a}, E_{1b})$ is a strongly exceptional collection, namely, it satisfies
   $$ H^i(\tilde{B}_i^* \otimes \tilde{B}_j) = 0 \text{ for } 1 \leq i, j \leq 4 \text{ and } \bullet > 0 $$
   and $H^0(\tilde{B}_i^* \otimes \tilde{B}_j) = 0$ ($i > j$), $H^0(\tilde{B}_i^* \otimes \tilde{B}_i) = \mathbb{C}$ ($1 \leq i \leq 4$).

2. For $i < j$, $H^0(\tilde{B}_i^* \otimes \tilde{B}_j)$ are given by
   $$ H^0(E_3^* \otimes E_2) \simeq V, \ H^0(E_3^* \otimes E_{1a}) \simeq \wedge^2 V, \ H^0(E_3^* \otimes E_{1b}) \simeq S^2 V, \ H^0(E_2^* \otimes E_{1a}) \simeq V, \ H^0(E_2^* \otimes E_{1b}) \simeq 0, $$
   and may be summarized in the following diagram:

3. Set
   $$ D_{\tilde{Y}} := \langle E_3, E_2, E_{1a}, E_{1b} \rangle \subset \mathcal{D}^b(\tilde{Y}). $$
   Then
   $$ D_{\tilde{Y}}, D_{\tilde{Y}}(1), \ldots, D_{\tilde{Y}}(9) $$
   is a Lefschetz collection, namely, for $1 \leq i, j \leq 4$ and $\bullet > 0$ it holds that
   $$ H^*(\tilde{B}_i^* \otimes \tilde{B}_j(-t)) = 0 \text{ (1 \leq t \leq 9)}, $$
   where $(-t)$ represents the twist by the sheaf $\mathcal{O}_{\tilde{Y}}(-t\mathcal{M}_{\tilde{Y}})$.

The rest of this section is devoted to our proof of Theorem 8.1.1 where we compute the cohomology groups $H^*(\tilde{B}_i^* \otimes \tilde{B}_j(-t))$ ($0 \leq t \leq 9$). Our strategy is to reduce the computations of cohomology groups on $\tilde{Y}$ to those on $\mathcal{Y}_3$ and use the Bott Theorem 2.0.1 for the $G(3,6)$-bundle $\mathcal{Y}_3 \rightarrow \mathbb{P}(V)$. This idea works for small values of $t$ as we formulate in Proposition 8.1.6 below. The computations will be completed in the next subsection.
Lemma 8.1.2. \( K_{\mathcal{\tilde{Y}}} = -10M_{\mathcal{\tilde{Y}}} + 2F_{\mathcal{\tilde{Y}}} \)

Proof. We have \( K_{\mathcal{\tilde{Y}}} = -10M_{\mathcal{\tilde{Y}}} \) by Proposition 4.2.1. Then from Proposition 5.7.2 (2), \( K_{\mathcal{\tilde{Y}}} = \rho_{\mathcal{\tilde{Y}}}^* K_{\mathcal{Y}} + 2F_{\mathcal{\tilde{Y}}} = -10M_{\mathcal{\tilde{Y}}} + 2F_{\mathcal{\tilde{Y}}} \).

Let us introduce

\[ \tilde{C}_{ij} = B_i^* \otimes B_j \ (1 \leq i, j \leq 4) \]

for the ordered collection \( (\tilde{B}_i)_{1 \leq i \leq 4} \).

Lemma 8.1.3. For \( \tilde{C} = \tilde{C}_{ij} \) as above, it holds

\[ H^\bullet(\mathcal{\tilde{Y}}, \tilde{C}(-t)) \simeq H^{13-\bullet}(\mathcal{\tilde{Y}}, \tilde{C}^*(t - 10)) \]

for any integer \( t \).

Proof. Note that each of the cohomology groups \( H^\bullet(\mathcal{\tilde{Y}}, \tilde{C}(-t)) \) is Serre dual to

\[ H^{13-\bullet}(\mathcal{\tilde{Y}}, \tilde{C}^*((t - 10)M_{\mathcal{\tilde{Y}}} + 2F_{\mathcal{\tilde{Y}}})) \]

Considering the exact sequence

\[ 0 \to \tilde{C}^*((t - 10)M_{\mathcal{\tilde{Y}}} + (i - 1)F_{\mathcal{\tilde{Y}}}) \to \tilde{C}^*((t - 10)M_{\mathcal{\tilde{Y}}} + iF_{\mathcal{\tilde{Y}}}) \to \tilde{C}^*((t - 10)M_{\mathcal{\tilde{Y}}} + iF_{\mathcal{\tilde{Y}}})|_{F_{\mathcal{\tilde{Y}}}} \to 0, \]

it suffices to show

\[ H^{13-\bullet}(F_{\mathcal{\tilde{Y}}}, A_i) = 0 \text{ for } i = 1, 2 \text{ and any } \bullet, \]

where we set

\[ A_i := \tilde{C}^*((t - 10)M_{\mathcal{\tilde{Y}}} + iF_{\mathcal{\tilde{Y}}})|_{F_{\mathcal{\tilde{Y}}}}. \]

Recall the diagrams (7.9) and (7.10). Since the morphism \( \tilde{F} \to F_{\mathcal{\tilde{Y}}} \) is finite, and \( \tilde{F} \) has only rational singularities by its construction, the vanishing follows from the vanishings of the cohomology groups of the pull-backs of \( A_i \) on \( F^{(3)} \). Since the morphism \( F^{(3)} \to \tilde{G} \) is flat by Proposition 7.4.1, we have only to show the vanishing along its fibers. By the upper semi-continuity of cohomology groups on fibers, it suffices to prove the vanishing on fibers over points of the diagonal subset of \( \tilde{G} \). By Proposition 7.4.1 such fibers are of the form \( A \cup B \). Note that the restriction of the pull-back of \( M_{\mathcal{\tilde{Y}}} \) to the fibers is trivial. Therefore, by Lemma 7.5.2 it suffices to show the vanishing of the following cohomology groups:

\[ H^\bullet(A \cup B, C_{A \cup B}^*(iF_{A \cup B}))(i = 1, 2), \]

where \( C_{A \cup B} \) and \( F_{A \cup B} \) are the pull-backs of \( \tilde{C} \) and \( F_{\mathcal{\tilde{Y}}} \) to \( A \cup B \), respectively. Tensoring \( C_{A \cup B}^*(iF_{A \cup B}) \) with the Mayor-Vietri sequence, we have

\[ 0 \to C_{A \cup B}^*(iF_{A \cup B}) \to C_A^*(iF_A) \otimes C_B^*(iF_B) \to C_{A \cap B}^*(iF_{A \cap B}) \to 0, \]

where \( C_A, C_B, \) and \( C_{A \cap B} \) are the restrictions of \( C_{A \cup B} \) to \( A, B \) and \( A \cap B \) respectively, and \( F_{A \cap B} \) is the restriction of \( F_{A \cup B} \) to \( A \cap B \). By Lemma 7.5.2 (1), we can easily show the vanishing of \( H^\bullet(A, C_A^*(iF_A)) \). Moreover, by Lemma 7.5.2 (2), we easily see that the restriction maps \( H^\bullet(B, C_B^*(iF_B)) \to H^\bullet(A \cap B, C_{A \cap B}^*(iF_{A \cap B})) \) are isomorphisms. Therefore we have the vanishing of \( H^\bullet(A \cup B, C_{A \cup B}^*(iF_{A \cup B})) \). \( \square \)

Remark. Form the above lemma, we conjecture that \( \mathcal{D}_{\mathcal{\tilde{Y}}}, \mathcal{D}_{\mathcal{\tilde{Y}}}(1), \ldots, \mathcal{D}_{\mathcal{\tilde{Y}}}(9) \) generates a strongly crepant categorical resolution (cf. [26]).
Let us introduce a sequence of sheaves on \( \mathcal{Y}_2 \)

\[(B_i)_{1\leq i \leq 4} = (\rho_{\mathcal{Y}_2}^* S(-L_{\mathcal{Y}_2}), T_{2}^*, O_{\mathcal{Y}_2}, \rho_{\mathcal{Y}_2}^* Q^*),\]

and define \( C_{ij} = B_i \otimes B_j \) for \( 1 \leq i, j \leq 4 \). Note that \( B_i \) have its corresponding form to \( \tilde{B}_i \) for \( i = 1, 2, 3 \), but \( B_4 \) is defined by removing the twist of \( M_{\mathcal{Y}_2} \) from \( \mathcal{B}_4 \).

Let \( \mathcal{D} \) be a locally free sheaf on \( \mathcal{Y} \) and \( D := \tilde{\rho}_{\mathcal{Y}_2}^* \mathcal{D} \). Since \( \tilde{\rho}_{\mathcal{Y}_2} : \mathcal{Y}_2 \to \mathcal{Y} \) is a blow-up of a smooth variety, it holds that

\[(8.4) \quad H^\bullet(\mathcal{Y}, \mathcal{D}(-t)) \simeq H^\bullet(\mathcal{Y}_2, D(-t)),\]

where \( (-t) \) on the right hand side represents the twist by \( O_{\mathcal{Y}_2}(-tM_{\mathcal{Y}_2}) \). By Propositions \(6.1.2, 6.2.3\) and Lemma \(8.1.3\) for our cohomology calculations, it suffices to know

\[(8.5) \quad H^\bullet(\mathcal{Y}, \tilde{C}_{4i}(-t)) = H^\bullet(\mathcal{Y}_2, C_{4i}(-t+1)) \quad (t = 0, 1, \ldots, 6)\]

\[(8.6) \quad H^\bullet(\mathcal{Y}, \tilde{C}_{4j}(-t)) = H^\bullet(\mathcal{Y}_2, C_{4j}(-t-1)) \quad (t = 0, 1, \ldots, 4)\]

for \( 1 \leq i, j \leq 3 \) and

\[(8.7) \quad H^\bullet(\mathcal{Y}, \tilde{C}_{13}(-t)) = H^\bullet(\mathcal{Y}_2, C_{13}(-t)) \quad (t = 0, 1, \ldots, 5)\]

for \( 1 \leq i, j \leq 3 \) or \( i = j = 4 \).

**Lemma 8.1.4.** For the computations of \((8.5)\) and \((8.6)\), we may replace \( T_2 \) by \( \rho_{\mathcal{Y}_2}^* T(-1) \) except in one case \( C_{24}(1) = T_2 \otimes \rho_{\mathcal{Y}_2}^* Q^*(M_{\mathcal{Y}_2}) \). From now on we may use the following relations:

\[C_{42}(-t - 1) = (\rho_{\mathcal{Y}_2}^* Q \otimes \pi_{\mathcal{Y}_2}^* \Omega(1))(-t - 1),\]

\[C_{42}(-t + 1) = (\pi_{\mathcal{Y}_2}^* T(-1) \otimes \rho_{\mathcal{Y}_2}^* Q^*)(t - 1) \quad (t \neq 0),\]

\[C_{12}(-t) = (\mathcal{B}_i^* \otimes \pi_{\mathcal{Y}_2}^* \Omega(1))(-t) \quad (i = 1, 3),\]

\[C_{2j}(-t) = (\pi_{\mathcal{Y}_2}^* T(-1) \otimes \mathcal{B}_j)(-t) \quad (j = 1, 3),\]

\[C_{22}(-t) = (\pi_{\mathcal{Y}_2}^* T(-1) \otimes \pi_{\mathcal{Y}_2}^* \Omega(1))(-t).\]

**Proof.** First we consider \( C_{12}(-t) \). By the exact sequence \((6.3)\), we have

\[0 \to T_{2}^* \otimes D(-t) \to \pi_{\mathcal{Y}_2}^* \Omega(1) \otimes D(-t) \to (\rho_{\mathcal{Y}_2}^* |_{F_\rho})^*(R_{2}/R_{1})^* \otimes D(-t)|_{F_\rho} \to 0\]

for a locally free sheaf \( D \) on \( \mathcal{Y}_2 \). For \( D = O_{\mathcal{Y}_2}, \rho_{\mathcal{Y}_2}^* Q, \rho_{\mathcal{Y}_2}^* S^*(L_{\mathcal{Y}_2}) \) or \( T_2 \), it holds that

\[H^\bullet(F_{\rho}, (\rho_{\mathcal{Y}_2}^* |_{F_\rho})^*(R_{2}/R_{1})^* \otimes D(-t)|_{F_\rho}) = 0 \quad \text{for any } t\]

by the Leray spectral sequence for \( \tilde{\rho}_{\mathcal{Y}_2} |_{F_\rho} : F_\rho \to G_{\rho} \) since \( \tilde{\rho}_{\mathcal{Y}_2} |_{F_\rho} \) is a \( \mathbb{P}^1 \)-bundle and the restriction of \((\rho_{\mathcal{Y}_2}^* |_{F_\rho})^*(R_{2}/R_{1})^* \otimes D(-t)|_{F_\rho} \) to its fiber is a direct sum of \( O_{\mathcal{Y}_2}, (-1) \) (cf. the proof of Lemma \(6.2.2\)). Therefore we have

\[H^\bullet(\mathcal{Y}_2, T_{2}^* \otimes D(-t)) \simeq H^\bullet(\mathcal{Y}_2, \pi_{\mathcal{Y}_2}^* \Omega(1) \otimes D(-t))\]

for \( D = O_{\mathcal{Y}_2}, \rho_{\mathcal{Y}_2}^* Q, \rho_{\mathcal{Y}_2}^* S^*(L_{\mathcal{Y}_2}) \) or \( T_2 \) and for any \( t \).

Next we consider \( C_{2j}(-t) \) except \( C_{24}(1) \). By the exact sequence \((6.4)\), we have

\[0 \to \pi_{\mathcal{Y}_2}^* T(-1) \otimes D(-t) \to T_2 \otimes D(-t) \to (\rho_{\mathcal{Y}_2}^* |_{F_\rho})^*(R_{2}/R_{1})^* \otimes D(-tM_{\mathcal{Y}_2} + F_\rho)|_{F_\rho} \to 0\]

for a locally free sheaf \( D \) on \( \mathcal{Y}_2 \). Set \( D = O_{\mathcal{Y}_2}, \rho_{\mathcal{Y}_2}^* Q^*, \rho_{\mathcal{Y}_2}^* S^*(L_{\mathcal{Y}_2}), \) or \( \pi_{\mathcal{Y}_2}^* \Omega(1) \). We show the vanishing of

\[(8.7) \quad H^\bullet(F_{\rho}, (\rho_{\mathcal{Y}_2}^* |_{F_\rho})^*(R_{2}/R_{1}) \otimes D(-tM_{\mathcal{Y}_2} + F_\rho)|_{F_\rho}).\]
By Proposition 6.3.1 we have only to consider the case where \( D = \mathcal{O}_{\mathcal{X}_2}, \rho_{\mathcal{X}_2} Q^* \) or \( \pi_{\mathcal{X}_2}^* \Omega(1) \). For \( 0 \leq t \leq 4 \), the vanishing of (8.7) follows from the Leray spectral sequence for \( \rho_{\mathcal{X}_2} |_{F_t} : F_t \to \mathcal{P}_t \) since \( \rho_{\mathcal{X}_2} |_{F_t} \) is a \( \mathbb{P}^5 \)-bundle and the restriction of \((\rho_{\mathcal{X}_2} |_{F_t})^*(\mathcal{R}_2^*/\mathcal{R}_1) \otimes D(-tM_{\mathcal{X}_2} + F_t)|_{F_t}\) to its fiber is a direct sum of \( \mathcal{O}_{\mathcal{X}_2}(-t+1) \) by Proposition 6.4.1. Therefore we may assume that \( t = 5 \) from now on. Each of the cohomology groups (8.7) is Serre dual to
\[
H^{12-\bullet}((\mathcal{R}_2/\mathcal{R}_1)^* \otimes \overline{D}(-\det Q - L_{\mathcal{X}_2})|_{\mathcal{P}_t})
\]
by 6.8 and Proposition 6.4.1. Since \( \rho_{\mathcal{X}_2} \) is the blow-up of a smooth variety and \( D \) is the pull-back of a locally free sheaf \( \overline{D} \) on \( \mathcal{X}_2 \), each of the cohomology groups (8.8) is isomorphic to
\[
H^{12-\bullet}((\mathcal{R}_2/\mathcal{R}_1)^* \otimes \overline{D}(-\det Q - L_{\mathcal{X}_2})|_{\mathcal{P}_t}).
\]
Using Proposition 6.3.2 (1) and (2), we can write
\[
(\mathcal{R}_2/\mathcal{R}_1)^* \otimes \overline{D}(-\det Q - L_{\mathcal{X}_2})|_{\mathcal{P}_t} = \begin{cases} \mathcal{O}_{\mathcal{P}_t}(-H_{\mathcal{P}_t} - 3L_{\mathcal{P}_t}) & \text{for } \overline{D} = \mathcal{O}_{\mathcal{X}_2} \\ (\mathcal{R}_2/\mathcal{R}_1)^* \otimes \overline{D}(-2L_{\mathcal{P}_t}) & \text{for } \overline{D} = Q^* \\ (\mathcal{R}_2/\mathcal{R}_1)(-H_{\mathcal{P}_t} - 3L_{\mathcal{P}_t}) & \text{for } \overline{D} = \pi_{\mathcal{X}_2}^* \Omega(1), \end{cases}
\]
where we use \((\mathcal{R}_2/\mathcal{R}_1)^* = \mathcal{O}_{\mathbb{P}^n(T(\pi(-1))_1) = H_{\mathcal{P}_t}}\). All the cohomology groups of the restrictions of these sheaves to a fiber of \( \pi_{\mathcal{X}_2} |_{\mathcal{P}_t} : \mathcal{P}_t = F(1, 2, V) \to \mathbb{P}(V) \) vanish. Hence, by the Leray spectral sequence for \( \pi_{\mathcal{X}_2} |_{\mathcal{P}_t} \), the cohomology groups (8.9) vanish, too.

By the following simple lemma, we can reduce most of the computations of cohomology groups to those on \( \mathcal{X}_3 \):

**Lemma 8.1.5.**

1. \( R^q \rho_{\mathcal{X}_2}^* \mathcal{O}_{\mathcal{X}_2}(tF_t) = 0 \) for any \( t \leq 5 \) and \( q > 0 \).
2. \( \rho_{\mathcal{X}_2}^* \mathcal{O}_{\mathcal{X}_2}(tF_t) = \mathcal{O}_{\mathcal{X}_2} \) for \( t > 0 \).

**Proof.** Part (1) follows from the relative Kodaira vanishing theorem since \( tF_t - K_{\mathcal{X}_2} \equiv_{\mathcal{X}_2} (t - 5)F_t \) is \( \rho_{\mathcal{X}_2} \)-nef and \( \rho_{\mathcal{X}_2} \)-big if \( t \leq 5 \).

Part (2) is well-known. \( \square \)

Now define an ordered collection of sheaves on \( \mathcal{X}_3 \),
\[
(\overline{B}_i)_{1 \leq i \leq 4} = (S(-L_{\mathcal{X}_2}), \pi_{\mathcal{X}_2}^* \Omega(1), \mathcal{O}_{\mathcal{X}_2}, Q^*),
\]
and set \( \overline{C}_{ij} = \overline{B}_j \otimes \overline{B}_j \).

**Proposition 8.1.6.** The cohomology groups on \( \mathcal{X}_2 \) in the r.h.s. of (8.5) and (8.6) can be evaluated by using
\[
H^\bullet(\mathcal{X}_3, C_{ij}(-t)) \simeq H^\bullet(\mathcal{X}_3, \overline{C}_{ij}(-t \det Q + tL_{\mathcal{X}_2})) \quad (t = 0, 1, ..., 5)
\]
for \( 1 \leq i, j \leq 4 \) except the cases of \( C_{14}(1) \) (\( 1 \leq i \leq 3 \)). (We read \( t = -1 \) and \( t = 1 \) in the r.h.s. of (8.5) as \( t \) here.)

**Proof.** Note that we may assume that the relation \( C_{ij} = \rho_{\mathcal{X}_2}^* \overline{C}_{ij} \) holds for \( C_{i4}(t) \) except \( C_{24}(1) \) by Lemma 8.1.4. Then by the Leray spectral sequence for \( \rho_{\mathcal{X}_2} \), and Proposition 6.4.1 and Lemma 8.1.5 we have the claimed isomorphisms for the range of \( t \) in (8.5) and (8.6) except the cases \( C_{14}(1) \) (\( 1 \leq i \leq 3 \)), for which \( t = -1 \). \( \square \)
8.2. Calculations on \( \mathcal{Y}_3 \). Here first we calculate the r.h.s. of (8.10) postponing the cases \( C_{4i}(1) (1 \leq i \leq 3) \) to the latter half of this subsection. We use the Leray spectral sequence associated with \( \pi_{\mathcal{Y}_3} : \mathcal{Y}_3 \rightarrow \mathbb{P}(V) \).

Let \( G \approx \mathbb{G}(3,6) \) be a fiber of \( \pi_{\mathcal{Y}_3} \). Since \( L_{\mathcal{Y}_3} \), \( \pi_{\mathcal{Y}_3}^* T(-1) \) and \( \pi_{\mathcal{Y}_3}^* \Omega(1) \) are pullbacks of sheaves on \( \mathbb{P}(V) \), and also \( \det Q|_G = \mathcal{O}_G(1) \) holds, we can write the restrictions of \( \mathcal{C}_{ij}(1-t \det Q + tL_{\mathcal{Y}_3}) \) to \( G \) by using the following sheaves:

\[
\begin{align*}
S^*|_G \otimes S|_G(-t), & \quad S^*|_G(-t), & \quad S^*|_G \otimes Q^*|_G(-t), & \quad (0 \leq t \leq 5) \\
S|_G(-t), & \quad \mathcal{O}_G(-t), & \quad Q^*|_G(-t), & \quad (0 \leq t \leq 5) \\
Q|_G \otimes S|_G(-t), & \quad \mathcal{O}_G(-t) & \quad (1 \leq t \leq 5)
\end{align*}
\]

(8.11)

where \(-t\) represents the twist by \( \mathcal{O}_G(-t) \).

**Lemma 8.2.1.** All the cohomology groups of the sheaves in (8.11) vanish except

\[
\begin{align*}
H^0(S^*|_G \otimes S|_G), & \quad H^0(S^*|_G), & \quad H^0(\mathcal{O}_G), & \quad \text{and } H^0(Q|_G \otimes Q^*|_G).
\end{align*}
\]

**Proof.** We can verify these properties by using the Bott theorem [2.0.1]. \( \square \)

**Proposition 8.2.2.** Consider the cohomologies \( H^*(\mathcal{Y}_3, C_{ij}(-t)) \) except the cases \( C_{4i}(1) (1 \leq i \leq 3) \) as in Proposition 8.1.6. The r.h.s. of (8.10) vanishes except possibly

\[
H^*(\mathcal{Y}_3, C_{ij}) \simeq H^*(\mathbb{P}(V), \pi_{\mathcal{Y}_3}^* \mathcal{C}_{ij}) \quad \text{with } 1 \leq i, j \leq 3 \text{ or } i = j = 4.
\]

**Proof.** This follows from the isomorphisms (8.10) and Lemma 8.2.1. \( \square \)

For the evaluations of the r.h.s. of (8.12), we can use the Bott theorem for \( \pi_{\mathcal{Y}_3} : \mathcal{Y}_3 \rightarrow \mathbb{P}(V) \). All non-vanishing sheaves turns out to be

\[
\begin{align*}
\pi_{\mathcal{Y}_3}^* \mathcal{C}_{11}, & \quad \pi_{\mathcal{Y}_3}^* \mathcal{C}_{12}, & \quad \pi_{\mathcal{Y}_3}^* \mathcal{C}_{13} \quad \mathcal{O}, & \quad T(-1)^\wedge 2 \otimes \Omega(1), & \quad T(-1)^\wedge 2 \\
\pi_{\mathcal{Y}_3}^* \mathcal{C}_{22}, & \quad \pi_{\mathcal{Y}_3}^* \mathcal{C}_{23} \quad \simeq & \quad T(-1) \otimes \Omega(1), & \quad T(-1) \\
\pi_{\mathcal{Y}_3}^* \mathcal{C}_{33} \quad & \quad \mathcal{O}
\end{align*}
\]

and \( \pi_{\mathcal{Y}_3}^* \mathcal{C}_{44} \simeq \mathcal{O} \).

We can compute the cohomology groups on \( \mathbb{P}(V) \) by applying the Bott theorem [2.0.1] again. For the calculations, we first evaluate irreducible decompositions, for example,

\[
T(-1)^\wedge 2 \otimes \Omega(1) \simeq \Sigma^{(0,0,0,-1)} \Omega(1) \oplus \Sigma^{(1,0,-1,-1)} \Omega(1)
\]

by the Littlewood-Richardson rule. In this way, we finally obtain the following non-vanishing cohomology groups:

\[
\begin{align*}
H^0(\mathcal{Y}_3, C_{11}), & \quad H^0(\mathcal{Y}_3, C_{12}), & \quad H^0(\mathcal{Y}_3, C_{13}) \quad \mathbb{C}, & \quad V, & \quad \wedge^2 V \\
H^0(\mathcal{Y}_3, C_{22}), & \quad H^0(\mathcal{Y}_3, C_{23}) \quad \simeq & \quad \mathbb{C}, & \quad V \\
H^0(\mathcal{Y}_3, C_{33}) \quad & \quad \mathbb{C}
\end{align*}
\]

and \( H^0(\mathcal{Y}_3, C_{44}) \simeq \mathbb{C} \).

Now we turn our attention to the cases \( H^*(\mathcal{Y}_3, C_{4i}(1)) (1 \leq i \leq 3) \) for which the isomorphism (8.10) does not apply.
(1) \( H^\bullet(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{C}}_{14}) \simeq H^\bullet(\mathcal{Y}_2, \mathcal{C}_{14}(1)) \simeq H^0(\mathcal{Y}_2, \mathcal{C}_{14}(1)) \simeq S^2V. \)

For \( C_{14}(1) = \rho_{\mathcal{Y}_2}^* \mathcal{T}_{14}(1), \) we have
\[
C_{14}(1) = \rho_{\mathcal{Y}_2}^* (S^*(L_{\mathcal{Y}_2}) \otimes Q^*) (M_{\mathcal{Y}_2}) \simeq \rho_{\mathcal{Y}_2}^* (S^* \otimes Q^*(\det \mathcal{Q})) (-F_\rho)
\]
by Proposition 6.41. Consider the exact sequence
\[
0 \rightarrow \rho_{\mathcal{Y}_2}^* (S^* \otimes Q^*(\det \mathcal{Q})) (-F_\rho) \rightarrow \rho_{\mathcal{Y}_2}^* (S^* \otimes Q^*(\det \mathcal{Q})) \rightarrow (\rho_{\mathcal{Y}_2} | F_\rho)^* (S^* \otimes Q^*(\det \mathcal{Q})|_{\mathcal{P}_\rho}) \rightarrow 0.
\]
We evaluate the cohomology of the middle term by \( H^\bullet(\mathcal{Y}_3, S^* \otimes Q^*(\det \mathcal{Q})) \simeq \bigoplus_i H^{\bullet-i}(\mathbb{P}(V), R^i_{\pi_{\mathcal{Y}_3}*}(S^* \otimes Q^*(\det \mathcal{Q}))). \) By the Bott theorem for \( \pi_{\mathcal{Y}_3} : \mathcal{Y}_3 \rightarrow \mathbb{P}(V), \) it is easy to see that the only non-trivial term comes from \( \pi_{\mathcal{Y}_3*}(S^* \otimes Q^*(\det \mathcal{Q})) \simeq \Sigma^{(1,1,0,0,0)} T(-1)^{\wedge 2}. \) To use the Bott theorem again for the cohomology over \( \mathbb{P}(V), \) we note the following plethysm of the Schur functors:
\[
\Sigma^{(1,1,0,0,0)} T(-1)^{\wedge 2} \simeq \Sigma^{(2,0,0,0)} T(-1) \oplus \Sigma^{(1,1,1,1)} T(-1) \oplus \Sigma^{(2,1,0,0)} T(-1).
\]
Then the only non-vanishing result comes from the first summand, and we finally obtain \( H^\bullet(\mathcal{Y}_3, S^* \otimes Q^*(\det \mathcal{Q})) = H^0(\mathbb{P}(V), \Sigma^{(2,0,0,0)} T(-1)) \simeq S^2V. \)

Now, let us note that \( H^\bullet(F_{\rho}, (\rho_{\mathcal{Y}_2} | F_\rho)^* (S^* \otimes Q^*(\det \mathcal{Q})|_{\mathcal{P}_\rho})) \simeq H^\bullet(\mathcal{P}_\rho, S^* \otimes Q^*(\det \mathcal{Q})|_{\mathcal{P}_\rho}). \) By Propositions 6.5.1 and 6.3.2 we have \( S^* \otimes Q^*(\det \mathcal{Q})|_{\mathcal{P}_\rho} \simeq (R_V / R_2) \otimes (R_V / R_2)^*(2H_{\mathcal{P}_\rho} + L_{\mathcal{P}_\rho}). \) In a similar way to the above calculations, by considering the fibration \( \mathcal{P}_\rho \rightarrow \mathbb{P}(V), \) we see that
\[
H^\bullet(\mathcal{P}_\rho, (R_V / R_2) \otimes (R_V / R_2)^*(2H_{\mathcal{P}_\rho} + L_{\mathcal{P}_\rho})) \simeq H^\bullet(\mathcal{P}_\rho, \mathcal{O}_{\mathcal{P}_\rho}, (2H_{\mathcal{P}_\rho} + L_{\mathcal{P}_\rho})).
\]
The r.h.s. vanishes for \( \bullet \neq 0. \) For \( \bullet = 0, \) we note that \( H^0(\mathcal{P}_\rho, \mathcal{O}_{\mathcal{P}_\rho}, (2H_{\mathcal{P}_\rho} + L_{\mathcal{P}_\rho})) \) is isomorphic to \( H^0(\mathbb{P}(V), S^2 T(-1) \otimes \mathcal{O}(-1)). \) Then this vanishes by the Bott theorem 2.0.1 on \( \mathbb{P}(V). \)

This completes our calculation of \( H^\bullet(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{C}}_{14}). \)

(2) \( H^\bullet(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{C}}_{34}) \simeq H^\bullet(\mathcal{Y}_2, \mathcal{C}_{34}(1)) \simeq 0. \)

For \( C_{34}(1) = \rho_{\mathcal{Y}_2}^* \mathcal{T}_{34}(1), \) we have
\[
C_{34}(1) = \rho_{\mathcal{Y}_2}^* (Q^*(\det \mathcal{Q} - L_{\mathcal{Y}_2}))(F_\rho).
\]
by Proposition 6.41. Since the following calculations proceed exactly in the same ways as above, we only sketch them.

First, we have \( H^\bullet(\mathcal{Y}_2, \rho_{\mathcal{Y}_2}^* (Q^*(\det \mathcal{Q} - L_{\mathcal{Y}_2}))) \simeq H^\bullet(\mathcal{Y}_3, Q^*(\det \mathcal{Q} - L_{\mathcal{Y}_2})), \) and then evaluate this by the Bott theorem to be \( H^\bullet(\mathcal{P}(V), \Sigma^{(1,1,0,0,0)} T(-1)^{\wedge 2} \otimes \mathcal{O}(-1)). \) We use the plethysm \( \Sigma^{(1,1,0,0,0)} T(-1)^{\wedge 2} \simeq \Sigma^{(2,1,0,0)} T(-1). \) Then we evaluate \( H^\bullet(\mathcal{P}(V), \Sigma^{(1,1,0,0,0)} T(-1)^{\wedge 2} \otimes \mathcal{O}(-1)) \simeq H^\bullet(\mathcal{P}(V), \Sigma^{(1,0,0,0,1)} \mathcal{O}(1)). \) which vanish for all \( \bullet. \)

This completes our calculation of \( H^\bullet(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{C}}_{34}). \)
(3) $H^\bullet(\tilde{\mathcal{Y}}, \tilde{\mathcal{C}}_{24}) \simeq H^\bullet(\mathcal{Y}_2, \mathcal{C}_{24}(1)) \simeq H^0(\mathcal{Y}_2, \mathcal{C}_{24}(1)) \simeq V$.

Finally, for $\mathcal{C}_{24}(1) = T_2 \otimes \rho_2^* Q^*(M_{\mathcal{Y}_2})$, we consider the following exact sequence which we derive from (6.4):

$$0 \to \pi_{2}\otimes \rho_2^* Q^*(M_{\mathcal{Y}_2}) \to T_2 \otimes \rho_2^* Q^*(M_{\mathcal{Y}_2}) \to (\rho_{\mathcal{Y}_2} |_{F_\rho})^* (R_2/R_1)^* \otimes \rho_2^* Q^*(M_{\mathcal{Y}_2} + F_\rho) |_{F_\rho} \to 0.$$

Then for our purpose it suffices to compute $H^\bullet(\mathcal{Y}_2, \pi_{2}\otimes \rho_2^* Q^*(M_{\mathcal{Y}_2}))$ and also $H^\bullet(F_\rho, (\rho_{\mathcal{Y}_2} |_{F_\rho})^* (R_2/R_1)^* \otimes \rho_2^* Q^*(M_{\mathcal{Y}_2} + F_\rho) |_{F_\rho})$. We can compute the former in a similar way to the above two cases, and we see that they all vanish. Using (1) and (2) of Propositions 6.3.2, 6.4.1, we see that the latter cohomologies are isomorphic to $H^\bullet(\mathcal{P}_V, (R_V/R_2))$. Now, from the defining exact sequence (6.2) of $R_V/R_2$, we obtain $H^\bullet(\mathcal{P}_V, (R_V/R_2)) \simeq H^\bullet(\mathcal{P}(V), T(-1))$, which vanish except $H^0(\mathcal{P}(V), T(-1)) \simeq V$.

This completes our calculation of $H^\bullet(\tilde{\mathcal{Y}}, \tilde{\mathcal{C}}_{24})$.

Now our calculations of the cohomology groups (8.10) and the cases (1)-(3) above complete our proof of Theorem 8.1.1.
APPENDIX A. THE “DOUBLE SPIN” COORDINATES OF G(3, 6)

In this appendix, we set \( V_4 = \mathbb{C}^4 \) with the standard basis. We can write the irreducible decomposition \( 5.10 \) as

\[
\wedge^3 (\wedge^2 V_4) = \Sigma^{(3,1,1,1)} V_4 \oplus \Sigma^{(2,2,2,0)} V_4 \simeq S^2 V_4 \oplus S^2 V_4^*,
\]

where \( \Sigma^3 \) is the Schur functor. We define the projective space \( \mathbb{P}(\wedge^3 (\wedge^2 V_4)) = \mathbb{P}(S^2 V_4 \oplus S^2 V_4^*) \). The homogeneous coordinate of \( \mathbb{P}(S^2 V_4 \oplus S^2 V_4^*) \) is naturally introduced by \([v_{ij}, w_{kl}]\), where \( v_{ij} \) and \( w_{kl} \) are entries of \( 4 \times 4 \) symmetric matrices. Let \( I = \{\{i, j\} \mid 1 \leq i < j \leq 4\} \) the index set to write the standard basis of \( \wedge^2 V_4 \), then the homogeneous coordinate of \( \mathbb{P}(\wedge^3 (\wedge^2 V_4)) \) is naturally given by the \([p_{IJK}]\) where \( p_{IJK} \) is totally anti-symmetric for the indices \( I, J, K \in I \). These two coordinates are related by the above irreducible decomposition. Focusing the different symmetry properties of the Schur functors, it is rather straightforward to decompose \( p_{IJK} \) into the two components. When we use the signature function defined by \( e_1 \land e_2 \land e_3 \land e_4 \) by \( e^{i_1 i_2 i_3 i_4} e_1 \land e_2 \land e_3 \land e_4 \) with a basis \( e_1, \ldots, e_4 \) of \( V_4 \), they are given by

\[
(A.1) \quad v_{ij} = \frac{1}{6} \sum_{k,l,m,n} e^{klnm} p_{ijkl}[mn], \quad w_{kl} = \frac{1}{6} \sum_{a,b,c,m,n,q} e^{abc} e^{klmn} p_{ijkl}[am][bn][cq],
\]

where the square brackets in \( p_{ijkl}[mn] \) represents the anti-symmetric extensions of the indices, i.e., \( p_{ijkl}[j][k] = p_{ijkl}[j][k] \) for \( i < j \) while \( p_{ijkl}[j][k] = -p_{ijkl}[j][k] \) for \( i \geq j \). For convenience, we write them in the following (symmetric) matrices:

\[
(A.2) \quad v = (v_{ij}) = \begin{pmatrix}
2p_{124} & p_{134} + p_{125} & p_{234} + p_{126} & p_{146} - p_{245} \\
2p_{135} & p_{235} + p_{136} & p_{156} - p_{345} & 2p_{236} \\
2p_{236} & p_{256} - p_{346} & 2p_{456}
\end{pmatrix},
\]

\[
(A.2) \quad w = (w_{kl}) = \begin{pmatrix}
2p_{356} & -p_{346} - p_{256} & p_{345} + p_{156} & p_{235} - p_{136} \\
2p_{246} & -p_{245} - p_{146} & p_{126} - p_{234} & 2p_{145} \\
2p_{145} & p_{134} - p_{125} & 2p_{123}
\end{pmatrix},
\]

where we ordered the index set \( I \) as \( \{1, 2, \ldots, 6\} = \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}\} \). Inverting the relations \( A.2 \), we can write the Plücker relations among \( p_{IJK} \) in terms of the entries of \( v \) and \( w \). After some algebra, we find:

**Proposition A.1** The Plücker ideal \( I_G \) of \( G(3, 6) \subset \mathbb{P}(\wedge^3 (\wedge^2 V_4)) \) is generated by

\[
(A.3) \quad |v_{IJ} - \epsilon_I \epsilon_J |w_{IJ}| \quad (I, J \in I),
\]

\( (v.w)_{ij}, \quad (v.w)_{ij} - (v.w)_{ji} \quad (i \neq j, 1 \leq i, j \leq 4), \)

where \( \check{I} \) represents the complement of \( I \), i.e., \( x \in \check{I} \) such that \( x \cup I = \{1, 2, 3, 4\} \) and similarly for \( \check{J} \). \( |v_{IJ}| \) and \( |w_{IJ}| \) represent the \( 2 \times 2 \) minors of \( v \) and \( w \), respectively, with the rows and columns specified by \( I \) and \( J \). \( \epsilon_{IJ} \) is the signature of the permutation of the ‘ordered’ union \( I \cup \check{I} \). \( (v.w)_{ij} \) is the \( ij \)-entry of the matrix multiplication \( v.w \).

For all \( [v, w] \in V(I_G) \simeq G(3, 6) \), we show the following relations \( (I.1)-(I.5) \):

\[
(I.1) \quad \det v = \det w.
\]
By the Laplace expansion of the determinant of a $4 \times 4$ matrix $v$, we have $\det v = \sum_{J \in I} \epsilon_J |v_{IJ}||v_{I\bar{J}}|$. Then, using the first relations of (A.3), we obtain the equality.

(I.2) $v.w = \pm \sqrt{\det w} id_4$, where $id_4$ is the $4 \times 4$ identity matrix.

Note that the second line of (A.3) may be written in a matrix form $v.w = a id_4$, where $a$ is a scalar. Then, by (I.1), we have $v.w = (v.w)^2 = d^4$ and $\det v \cdot w = (\det w)^2 = d^4$, where $v, w, d$ are scalars.

(I.3) $rk w \neq 3$ and also $rk v \neq 3$.

Assume $rk w = 3$, then from (I.2) we have $v.w = 0$, which implies $rk v \leq 1$. However, this contradicts the first relations of (A.3). Hence $rk w \neq 3$. By symmetry, we also have $rk v \neq 3$.

(I.4) $rk w = 2 \Leftrightarrow rk v = 2$.

When $rk w = 2$, we see $rk v \geq 2$ by the first relations of (A.3). From (I.1) and (I.3), we must have $rk v = 2$. The converse follows in the same way.

(I.5) $rk w \leq 1 \Leftrightarrow rk v \leq 1$.

This is immediate from the first relations of (A.3).

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