Constraints on coronal turbulence models from source sizes of noise storms at 327 MHz

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Abstract. We seek to reconcile observations of small source sizes in the solar corona at 327 MHz with predictions of scattering models that incorporate refractive index effects, inner scale effects and a spherically diverging wavefront. We use an empirical prescription for the turbulence amplitude $C_N^2(R)$ based on VLBI observations by Spangler and coworkers of compact radio sources against the solar wind for heliocentric distances $R \approx 10^{-50} R_\odot$. We use the Coles & Harmon model for the inner scale $l_i(R)$, that is presumed to arise from cyclotron damping. In view of the prevalent uncertainty in the power law index that characterizes solar wind turbulence at various heliocentric distances, we retain this index as a free parameter. We find that the inclusion of spherical divergence effects suppresses the predicted source size substantially. We also find that inner scale effects significantly reduce the predicted source size. An important general finding for solar sources is that the calculations substantially underpredict the observed source size. Three possible, non-exclusive, interpretations of this general result are proposed. First and simplest, future observations with better angular resolution will detect much smaller sources. Consistent with this, previous observations of small sources in the corona at metric wavelengths are limited by the instrument resolution. Second, the spatially-varying level of turbulence $C_N^2(R)$ is much larger in the inner corona than predicted by straightforward extrapolation Sunwards of the empirical prescription, which was based on observations between 10–50 $R_\odot$. Either the functional form or the constant of pro-
portionality could be different. Third, perhaps the inner scale is smaller than
the model, leading to increased scattering.
1. Introduction

Refractive scattering of radiation by density turbulence in the Sun’s corona and solar wind leads to angular broadening of embedded radio sources, and of cosmic sources observed through these media. The process is similar to the twinkling of stars and modified “seeing” caused by density turbulence in Earth’s atmosphere and ionosphere. This scattering process has been investigated for many years using geometrical optics [e.g., Steinberg et al. 1971] and the parabolic wave equation [e.g., Lee and Jokipii, 1975; Coles and Harmon 1989; Bastian 1994; Cairns 1998].

Scattering is thought to affect the observed properties of type II and III solar radio bursts in several ways: greatly increasing the angular sizes of the sources [e.g., Riddle 1974], causing the time profiles to have exponential decreases [e.g., Robinson and Cairns, 1998], and causing anomalously low brightness temperatures at decametric wavelengths [e.g., Thejappa & Kundu 1992].

The primary motivation of this paper is to investigate the constraints imposed on models of density turbulence in the solar corona by recent observations at 327 MHz [Mercier et al., 2006], made by combining visibilities from the Giant Metrewave Radio Telescope (GMRT) in Pune, India, and the Nancay Radioheliograph (NRH) in France. The maps of Mercier et al. [2006] show structures ranging from the smallest observed size of 49″ to that of the whole Sun, with dynamic ranges as high as a few hundred. These features make them the best meter wavelength snapshot maps of the solar corona to date. Mercier et al. [2006] found the smallest steady angular size of type I solar noise storms to be 49″ in their high dynamic range, full disk, 17 second snapshots. We therefore adopt the smallest observed
source size of 49″ in these maps to be a canonical number for comparison with our model predictions. The only other two-dimensional map showing small source sizes that we are aware of is that of Zlobec et al [1992], who observed a source as small as 30″ at 327 MHz. However, the dynamic range of their map was severely limited, and included only a very small range of size scales. It should be noted that the lower limit to the observed size is imposed by the resolution of the instrument if scattering by density turbulence is weak enough. The scattering calculations shown below imply that there is a possibility that smaller solar sources will be detected in the future by instruments with improved angular resolution.

In this paper we use a formalism based on the paraxial wave equation and the structure function, together with observationally based models for the density turbulence that scatter the radiation, to predict the size of sources in the solar corona at 327 MHz. Our results can be interpreted as the scatter-broadened image of an ideal point source in the solar corona. We have used an empirical model for the amplitude $C_N^2(R)$ of coronal turbulence that is based directly a fit to the scattering measure obtained from VLBI observations of cosmic sources broadened by scattering in the outer solar corona and inner solar wind. Here $R$ is the heliocentric distance. We have assumed that this model is valid throughout the corona, specifically at smaller $R$. We also consider the effects of spherical and plane wave propagation, variations of the inner scale $l_i(R)$ and power-law index $\alpha$ of the turbulence on the predicted source sizes. In most cases, we find that the models predict sizes that are at least an order of magnitude below the smallest observed size of 49″ at 327 MHz. Our formalism and analyses differ primarily from those of Bastian [1994] in the models for $C_N^2(R)$ and the electron density profile $n_e(R)$, while our applications are to
metric rather than centimetric and decimetric emissions. Since our predictions are much smaller and Bastian’s [1994] predictions much larger than 49'' at 327 MHz, the analyses demonstrate the importance of knowing $C_N^2(R)$, $n_e(R)$, and $l_i(R)$ much better for future observations and predictions of solar sources. These quantities are also relevant to the heating and outward flow of the coronal plasma, with activity localized to specific ranges of $R$ potentially leading to larger $C_N^2(R)$ and so enhanced scattering at, say, decimetric frequencies than expected at, say, metric frequencies.

The paper is organized as follows. In § 2 we summarize the scattering formalism and observations of the density turbulence. Coronal density models are described in § 3. The results are presented in § 4, including estimates of the predicted angular broadening, and the implications for coronal density turbulence. The conclusions are presented in § 5.

2. Angular broadening

We first consider the angular broadening predicted by the empirical formula of Erickson (1964):

$$\theta = 50 \left( \frac{\lambda}{D} \right)^2 \text{arcminutes}.$$  \hfill (1)

Here $\lambda$ is the observing (free-space) wavelength in meters and $D$ is the elongation in units of $R_\odot$. If we take $D = 1.056 R_\odot$, which is where 327 MHz emission would originate according to the hybrid density model described below, then $\theta \simeq 50'$ for 327 MHz, which is much larger (60 times) than the observed 49''. It points to a significant difference between the situation for observations of celestial background sources against the solar wind, for which Erickson’s (1964) formula is well accepted, and observations of solar radio events.
that originate in the solar corona. This difference will be specifically addressed in the Discussion section below.

2.1. Density turbulence

Density turbulence in the Sun’s corona and solar wind is modeled here by writing the three-dimensional isotropic spatial power spectrum \( S_n(k, R) \) of the fluctuating part \( \delta n \) of the electron density \( n_e \) as [cf., Lee & Jokipii 1975; Rickett 1977; Coles and Harmon, 1989; Bastian, 1994; Cairns 1998; Spangler, 2002]

\[
S_n(k, R) = \langle \left( \delta n \right)^2 \rangle (k, R) = C_N^2(R)k^{-\alpha}e^{-k^2/q_i^2}.
\] (2)

Here \( k \) and \( R \) are the (isotropic) wavenumber and radial distance (in units of \( R_\odot \)), respectively, \( C_N^2(R) \) models the level of turbulence, \( \alpha \) is the power-law index, and \( q_i \) is the wavenumber corresponding to the inner scale of the turbulence. While it is fairly well established that the turbulence spectrum largely follows the Kolmogorov scaling (with \( \alpha = 11/3 \)) at scales larger than about 100 km, there is some evidence that it flattens, with \( \alpha \) decreasing to values as low as 3, at scales between a few km and a few hundred km [Bastian, 1994]. There is also some evidence for variation of the turbulence power law spectrum with heliocentric distance [e.g., Efimov et al., 2008]. Furthermore, there is evidence for significant variation in the index between the slow and fast solar wind [Manoharan et al., 1994]. We therefore retain \( \alpha \) as a parameter. It may be noted that some authors use a power law index of \( 5/3 \) to describe the one-dimensional Kolmogorov spectrum; the index they refer to is equal to \( \alpha - 2 \).

The empirical model we use for \( C_N^2(R) \) was originally mooted by Armstrong & Woo [1980] and later refined, based on VLBI observations between 10–50 \( R_\odot \), by Spangler &
Sakurai [1995] and Spangler [2002] among others:

\[ C_N^2(R) = 1.8 \times 10^{10} \left( \frac{R}{10R_\odot} \right)^{-3.66}. \]  

(3)

The dimensions of \( C_N^2 \) depend on \( \alpha \), being \( m^{-\alpha-3} \). The normalizations for \( C_N^2 \) differ by about a factor of a few and the power-law index with \( R \) ranges from \(-3.66\) to \(-4\) in these works, presumably due to solar wind variability.

The inner scale \( l_i \) is modeled using Coles & Harmon’s [1989] model which agrees roughly with their observations,

\[ q_i(R) = \frac{\Omega_i(R)}{3V_A(R)} \equiv \frac{2\pi}{l_i(R)} = \frac{2\pi}{684 n_e(R)^{-1/2}} \text{ km}^{-1} \]  

(4)

where \( \Omega_i \) is the ion cyclotron frequency, \( V_A \) is the Alfvén speed and \( n_e \) is the electron density in cm\(^{-3}\). This model is interpreted conventionally in terms of cyclotron damping by MHD waves. We use this definition for the inner scale throughout this paper, except in two cases where we artificially set \( l_i \) equal to a very small value.

A popular alternative prescription for \( C_N^2 \) supposes that \( C_N^2 \) is \( \propto \) the square of the background electron density. Such a prescription has a constant of proportionality, which is often determined via observed values of the phase structure function (e.g., Bastian 1994). The magnitude of the phase structure function in turn, is very dependent on the elongation to which it is referenced. We discuss this issue further in § 4.

2.2. Plane vs spherical wave propagation

Scattering depends quantitatively on whether the wavefront is planar (1-D) or spherical (3-D). When a source is embedded in the scattering medium, it is often appropriate to adopt a formalism that includes the spherically diverging nature of the wavefront. The geometry for spherically diverging propagation is shown in Figure 1.
Similarly, when a plane wave illuminates the scattering medium, a 1-D planar formalism is standard. In this case an observer is typically sensitive only to scattering regions (eddies) with sizes of order the baseline length \( s \). In the spherically diverging situation, however, the observer is sensitive to a range of eddy sizes given by \( sa/b \), where \( a \) is the (continuously varying) distance of the scattering screen from the source and \( b \) is the distance of the observer from the source. In our situation, this is tantamount to saying that the effective baseline for spherical wave propagation is [Ishimaru, 1978]

\[
\text{s}_{\text{eff}} = sR/(R_1 - R_0).
\]  \hspace{1cm} (5)

This is the basic difference between Eqs (8) and (9) discussed below.

In the solar situation, radiation from an embedded coronal source is subject to scattering as it propagates to the observer. Since the radiation is generated near \( f_p \) and \( 2f_p \), scattering effects are expected to be largest in the source and in its vicinity, assuming that \( f_p(R) \) decreases monotonically with increasing \( R \). Spherical effects are expected to arise in two ways. Firstly, scattering will maximally distort an initially plane wavefront close to and in the source. Secondly, on a larger scale, the solar wind density is expected to be spherically symmetric, with radiation being refracted towards the radial direction. Accordingly, spherical divergence effects are expected to be vital. They are explicitly calculated below and shown to be quantitatively important. In contrast, the planar formalism is expected to be appropriate when the source, scattering region(s), and observer are all far apart, as assumed in calculations for pulsars and other celestial sources.
2.3. Structure function

The starting points for the expression we use for the scattering angle are equations (4)–(7) of Coles et al [1987] that specify the structure function and the mutual coherence function using the parabolic wave equation (PWE) formalism that includes small-angle refractive scattering and diffraction, but not reflection. For the sake of completeness, we reproduce them below. The asymptotic forms of the gradient of the phase structure function \( D(s, R) \) are

\[
\frac{\partial}{\partial R} D(s, R) = \frac{8\pi^2}{2^{\alpha-2} (\alpha - 2)} \frac{\Gamma(1 - (\alpha - 2)/2)}{\Gamma(1 + (\alpha - 2)/2)} C_N^2(R) r_e^2 \lambda^2 s^{\alpha-2}, \quad \text{for } s_{\text{eff}} \gg l_i(R), \tag{6}
\]

\[
\frac{\partial}{\partial R} D(s, R) = \frac{4\pi^2}{2^{\alpha-2}} \Gamma \left( 1 - \frac{(\alpha - 2)}{2} \right) C_N^2(R) r_e^2 \lambda^2 l_i(R)^{\alpha-4} s^2, \quad \text{for } s_{\text{eff}} \ll l_i(R), \tag{7}
\]

where \( r_e \) is the classical electron radius, \( \lambda \) is the observing wavelength and \( s_{\text{eff}} \) is the effective interferometer spacing. It is noted that this formalism is valid only for \( 2 < \alpha < 4 \); in particular, equations (6) and (7) diverge at \( \alpha = 4 \) owing to the behavior of the term \( \Gamma(1 - (\alpha - 2)/2) \). It may also be noted that the branches (6) and (7) do not meet at \( s_{\text{eff}} = l_i \); the ratio of (7) to (6) is equal to \( (1/2)(\alpha - 2)(l_i/s)^{\alpha-4} \Gamma(1 + (\alpha - 2)/2) \), and at \( s_{\text{eff}} = l_i \) this is equal to unity only for \( \alpha = 4 \).

The effective interferometer spacing \( s_{\text{eff}} \) is equal to \( s \) for the case of plane wave propagation, but is equal to \( sR/(R_1 - R_0) \) for spherical wave propagation, as discussed in § 2.2 and Figure 1. The phase structure function for the cases of plane wave and spherical wave propagation are

\[
D_p(s) = \int_{R_0}^{R_1} \frac{\partial}{\partial R} D(s, R) dR, \quad \text{for plane wave propagation}, \tag{8}
\]

\[
D_s(s) = \int_{R_0}^{R_1} \frac{\partial}{\partial R} D \left( \frac{sR}{R_1 - R_0}, R \right) dR, \quad \text{for spherical wave propagation}, \tag{9}
\]
where the lower limit of integration $R_0$ is the radial distance from which scattering is assumed to be effective (we take this to be equal to the fundamental emission level), and the upper limit $R_1$ corresponds to the observer (here at $R_1 = 1$ AU). All quantities are assumed to have spherical symmetry and the path is assumed to be radial.

Scattering depends sensitively on the ratio of the radiation frequency $f$ to the local electron plasma frequency $f_p(R)$ [Cairns, 1998]. Equations (16) and (22) of Cairns [1998] include the effects on refractive scattering that arise from $f_p(R)$ being non-zero and varying with position between the source and observer. By analogy with these equations we write

$$D_{pf}(s) = \int_{R_0}^{R_1} \frac{1}{1 - f_p(R)^2/f^2} \frac{\partial}{\partial R} D(s, R) dR$$

(10)

for plane wave propagation and

$$D_{sf}(s) = \int_{R_0}^{R_1} \frac{1}{1 - f_p(R)^2/f^2} \frac{\partial}{\partial R} D\left( \frac{sR}{R_1 - R_0}, R \right) dR$$

(11)

for spherical wave propagation, respectively.

The scattering angle is conventionally defined using a coherence scale $s_0$ in the following manner [e.g., Coles et al., 1987; Bastian, 1994]:

$$\theta_c = \left( \frac{2\pi s_0}{\lambda} \right)^{-1},$$

(12)

where

$$D_s(s_0) = 1$$

(13)

and $D_s(s)$ is either equal to $D_{pf}(s)$, defined by (10), or $D_{sf}(s)$, defined by (11), in appropriate limits. This scattering angle $\theta_c$ can be interpreted as the predicted size of an idealized point source.
2.4. Density Models

A model for $n_e(R)$ in the corona and solar wind is required to be able to predict the angular broadening. The density model is required for computing the inner scale, which is defined in the next subsection. Since there is no universally accepted model, we initially consider four representative density models. One is the four-fold Newkirk density model for the corona, based on eclipse observations [Newkirk, 1961]:

$$n_{4n}(R) = 4 \times 4.2 \times 10^4 \times 10^{1.32/R} \, \text{cm}^{-3},$$ \hspace{1cm} (14)

The second model is derived from the frequency drift rate of interplanetary type III bursts [Leblanc et al., 1998]:

$$n_{lb}(R) = 3.3 \times 10^5 R^{-2} + 4.1 \times 10^6 R^{-4} + 8 \times 10^7 R^{-6} \, \text{cm}^{-3}.$$ \hspace{1cm} (15)

The third model considered is due to Aschwanden et al. [1995]. It is based on the drift rates of type III bursts [Alvarez & Haddock, 1973] in the outer corona and solar wind ($f < 10$ MHz) and assumes an isothermal barometric atmosphere for the lower corona:

$$n_a(R) = \begin{cases} 
    n_1 \left( \frac{R - 1}{R_2} \right)^{-p}, & R > 1 + R_2 \\
    n_Q \exp \left( -\frac{R - 1}{\mu} \right), & R < 1 + R_2 
\end{cases} \hspace{1cm} (16)$$

where $p = 2.38$, $n_Q = 4.6 \times 10^8 \, \text{cm}^{-3}$, $n_1 = n_Q \exp(-p)$, $\mu = 0.1$ and $R_2 = p\mu$. The fourth model is a “hybrid”, using the Aschwanden & Benz [1995] model for the lower corona and the four-fold Newkirk model multiplied by a normalization factor (to ensure continuity) in the upper corona. In other words, the density $n_{hyb}(R)$ of the hybrid model is

$$n_{hyb}(R) = \begin{cases} 
    A n_{4n}(R), & R > 1 + R_2 \\
    n_a(R), & R < 1 + R_2 
\end{cases} \hspace{1cm} (17)$$

where $A = 0.324$ is the normalization factor that ensures continuity.
Figure 2 shows $f_p(R) = 8.97 n_e(R)^{1/2}$ kHz for all four density models, with $n_e$ in units of cm$^{-3}$.

The figure shows that the highest frequency predicted for $R > 1$ by the Leblanc et al. [1998] density model (Eq 15) is less than 100 MHz. Since our observing frequency is 327 MHz, this model is therefore unsuitable for our purposes. However, the four-fold Newkirk model (14), Aschwanden & Benz [1995] model (16), and hybrid model can account for $f_p = 327$ MHz for $R > 1$. However, since the Aschwanden & Benz [1995] model predicts unrealistically low densities (and consequently $f_p$) for $R > 1 + R_2 = 1.23$, only the hybrid model is considered further below. Fundamental emission at 327 MHz emanates from a heliocentric distance of 1.055 $R_\odot$ with this model. In order to avoid the singularity in the integrand in Eqs (10) and (11), we start the integration at $R_0 = 1.056 R_\odot$. In other words, we start the integration from a distance of approximately 700 km above the height at which 327 MHz fundamental emission originates. This distance is smaller than that corresponding to the frequency difference $\Delta f$ between the minimum frequency of fundamental emission at a given location and the local value of $f_p$, so that avoiding the singularity is correct. This positive frequency difference exists because conservation of energy and the wave dispersion relations force the standard nonlinear Langmuir wave processes for fundamental and harmonic emission to produce radiation of order several percent above $f_p$ and $2f_p$ [e.g., Cairns, 1987a,b], with the value of $\Delta f/f_p$ depending on the beam and plasma parameters.

2.5. Inner scale effects

We next discuss the need for including inner scale effects in our treatment. In general, inner scale effects are important if the baseline is smaller than the inner scale.
Figure 3 shows the inner scale (in km) given by Eq (4), as a function of heliocentric distance for some of the density models discussed in the preceding section. Clearly, the inner scale is quite dependent upon the density model. The inner scale for the Aschwanden & Benz [1995] model far exceeds those for the other models, since the density with this model for $R > 1.23$ is unrealistically low; we have therefore chosen not to depict the $l_i$ for this model in Figure 3. As explained in the preceding section, we choose to use only the hybrid model from now on, for it is by far the most realistic one.

In order to ascertain the importance of inner scale effects, we compare $l_i$ with the longest baseline (that determines the smallest source size), assuming that the source is situated at the fundamental emission level for 327 MHz, and the observer is at 1 AU. In order to compute the longest baseline, we set $49'' = 1.22\lambda/s$, where $49''$ is the observed source size and $\lambda$ is the (free space) observing wavelength (1 meter). This yields an effective baseline of $s \sim 5$ km. In order to ascertain the relevance of the inner scale (i.e., whether we should be using Eq 6 or 7) we need to compare the effective longest baseline $s_{\text{eff}}$ with the inner scale. It may be noted that a typical interferometer measurement involves a range of baselines, and while the longest baseline we have computed above is the one that limits the smallest observable source size, baselines shorter than this one do contribute to the overall measurement. However, our approach is appropriate because the longest baseline for a given source size is the largest length scale at which there is appreciable power. Inner scale effects are relevant only for the branch for which the baseline is $\ll$ the inner scale, as in Eq 7. If the longest relevant baseline is smaller than the inner scale, then it follows that the rest of the baselines in the problem automatically satisfy this criterion. Our approach thus provides a useful estimate of the importance of inner scale effects.
As discussed above, $s_{\text{eff}} = s$ for plane wave propagation, while $s_{\text{eff}} = sR/(R_1 - R_0)$ for spherical wave propagation. We show the ratio of $s_{\text{eff}}$ to $l_i$ in Figure 4.

When considering plane wave propagation, $s_{\text{eff}} = s$, and the solid line in Figure 4 shows that $s_{\text{eff}}$ is mostly $> l_i$. For heliocentric distances greater than about $30 R_\odot$, $s_{\text{eff}}$ does become somewhat smaller than $l_i$, but most of the contribution to the scattering kernel arises from distances well inside $30 R_\odot$. We should therefore use Eqs (6) and (10) for plane wave propagation. On the other hand, when considering spherical wave propagation, $s_{\text{eff}} = sR/(R_1 - R_0)$, and the dotted line in Figure 4 shows that $s_{\text{eff}}$ is $< l_i$ for all $R$. The appropriate equations to use for spherical wave propagation are therefore Eqs (7) and (11).

3. Results

3.1. Plane wave propagation

We first consider plane wave propagation, which is more appropriate for waves emanating from a background object that is far from the scattering medium.

3.1.1. $s_{\text{eff}} \gg l_i$

Since Figure 4 shows that $s_{\text{eff}} > l_i$ for plane wave propagation for $R < 30R_\odot$, the appropriate branch to use is Eq (6). At heliocentric distances greater than about $30 R_\odot$, $s_{\text{eff}}$ becomes marginally less than the inner scale, but we have verified numerically that this is immaterial, since most of the contribution to $\theta_c$ takes place well within $30 R_\odot$.

Figure 5 uses Eqs (6), (10) and (13) to predict the scattering angle $\theta_c(1 \text{ AU})$ at the Earth, in arcseconds, as a function of the power law index $\alpha$ for plane wave propagation at $f = 327 \text{ MHz}$. The thin line is for fundamental emission and the thick line is for second harmonic emission. Removal of the refractive index effect in Eq (10), meaning the factor
of \((1 - f_p^2/f^2)^{-1}\), causes a negligible change in the result. The predicted source size is slightly smaller for second harmonic emission.

### 3.1.2. \(s_{\text{eff}} \ll l_i\)

Although Figure 4 demonstrates that \(s_{\text{eff}} > l_i\) for plane wave propagation, we nevertheless investigate the predicted scattering angle for plane wave propagation, while using branch (7). The results are shown in Figure 6.

Evidently, the source sizes predicted approach the observed size for relatively steep turbulence spectra; for spectra that are steeper than Kolmogorov (i.e., \(\alpha > 11/3\)), the predicted source sizes exceed the observed one. In order to investigate inner scale effects we compute the scattering angle for plane wave propagation with the inner scale set to an artificially low value of 1 m, instead of being computed self-consistently from Eq 4. The results are shown using the heavy lines in Figure 7. When inner scale effects are thus removed, it is clear that the predicted source size increases, especially for flatter spectra.

### 3.2. Spherical wave propagation

As discussed earlier, spherical wave propagation is appropriate when the source is embedded in the scattering medium, as is the case here.

#### 3.2.1. \(s_{\text{eff}} \ll l_i\)

Since Figure 4 shows that \(s_{\text{eff}} \ll l_i\) for spherical wave propagation, the appropriate branch to use is (7).

The solid line in Figure 7 predicts \(\theta_c(1 \text{ AU})\) for spherical wave propagation, using (7), (10) and (13). Clearly, the predicted scattering angle is at least 25 times smaller than the observed one. The dashed line, on the other hand, is computed by artificially setting \(l_i = 1\text{ m}\), while still using (7). This is tantamount to neglecting inner scale effects. We
observe that for flat spectra, inner scale effects substantially reduce (by over an order of magnitude) the predicted $\theta_c$, but that this difference is progressively reduced as $\alpha$ increases. There are negligible differences between the results for fundamental and second harmonic emission, and removal of the refractive index effect causes a negligible change too.

3.2.2. $s_{\text{eff}} \gg l_i$

In keeping with the spirit of our treatment for plane wave propagation, we investigate the predicted scattering angle for spherical wave propagation while using branch (6), which assumes that $s_{\text{eff}} > l_i$. We do this despite Figure 4’s prediction that $s_{\text{eff}}$ is $< l_i$ for spherical wave propagation.

The difference between fundamental and second harmonic emission are negligible.

3.2.3. Spherical vs plane wave propagation

Although we have investigated several different cases, our attention has been focussed mainly on two issues: first, the difference between the source sizes predicted for plane wave and spherical wave propagation, and, second, the influence of the inner scale. We now compare the plane wave and spherical wave results directly in Figure 9, assuming fundamental emission, and employing an inner scale that is computed self-consistently using Eq (4).

It is clearly evident from Figure 9 that spherical divergence effects decrease the predicted scattering angle by around two orders of magnitude as compared to the plane wave case. This is best seen by comparing the same branch (say, $s_{\text{eff}} > l_i$) for the plane wave and spherical wave cases. Thus, spherical divergence effects should be quantitatively important for scattered solar radio emission and plane wave results should be used with great caution.
4. Discussion and Summary

The highest resolution meter wavelength observations of the solar corona reveal compact sources around 49" in size at 327 MHz. The main aim of this paper is to employ an observationally-motivated model for the turbulence amplitude $C_N^2$, and see what it implies for the predicted scattering angle for radio sources located in the solar corona.

We reference our calculations to the same frequency (viz. 327 MHz) at which the smallest source size is observed. We employ the parabolic wave equation, together with the standard asymptotic forms for the phase structure function, which are valid for situations where the effective baseline is either much larger or much smaller than the inner scale.

We define the predicted scattering angle $\theta_c$ via Eq. (12) as the angle where the phase structure function falls to $1/e$ times its peak value. Effectively, this means that the scattering angles predicted here should be interpreted as the scatter-broadened image of an ideal point source in the solar corona. The real source will have an intrinsic size (i.e., it will not be a point source) and the observable source will be the convolution of the intrinsic source profile with $\theta_c$, provided there are no instrumental limitations. The results in this paper should therefore be regarded as lower limits to the observable source size set by scattering. The general consensus now seems to be that there is not much about the intrinsic source size that can be gleaned from scatter-broadened images [Bougeret & Steinberg, 1977; Melrose, 1980; Bastian, 1994]. However, if the intrinsic source size and $\theta_c$ are similar in size, then the observed source size will be larger than $\theta_c$ by a factor near $\sqrt{2}$.

Note that this factor cannot account for the large discrepancies between the minimum source size observed at 327 MHz [Mercier et al., 2006] and those predicted here. If, on
the other hand, the intrinsic source size is much larger, then scatter broadening does not
play a significant role.

We have included refractive index effects that can be important when the radiation is
emitted near the fundamental plasma level, but found them to be relatively unimportant.
The inner scale is included via the Coles and Harmon [1986] model, interpreted in terms
of cyclotron damping of MHD waves, and so depends primarily on the ambient electron
density. We employ a hybrid model for the electron density that yields reasonable heights
for meter wavelength emission at the fundamental. In view of the uncertainty in its value
in the inner corona, the power law index $\alpha$ characterizing the turbulent spectrum is taken
to be a free parameter. We consider both plane wave and spherical wave propagation.
For the geometry we consider, where the source is embedded in the scattering medium,
the spherical wave description is arguably more appropriate.

We have thus explored a wide variety of effects. We observe that there is no significant
difference in the predicted scattering angle between fundamental and second harmonic
emission. We also find that the removal of the refractive index effect causes a negligible
change in the predicted scattering angle. We find that the spherical divergence effect
results in a significant lowering of the predicted scattering angle (by around 2 orders of
magnitude). We find that removing inner scale effects by artificially setting the inner
scale to be equal to a very small value (instead of determining it self-consistently from
Eq 4) results in a significant enhancement of the predicted source size. The enhancement
is greatest for flatter spectra, where it can be a factor of around 50, and it progressively
disappears for steeper spectra.
As mentioned earlier, the power law index of the turbulent spectrum is a free parameter. There is a formal divergence at $\alpha = 4$ in Eqs (6) and (7), and we therefore limit the computations to a maximum value of $\alpha = 3.97$. The maximum value of the predicted scattering angle thus occurs at $\alpha = 3.97$.

For plane wave propagation, the predicted source size for Kolmogorov turbulence is around $10''$ lower than the observed one. For spherical wave propagation, we find that the maximum value of the predicted scattering angle is at least 25 times smaller than the observed one. It is emphasized that plane wave propagation is relevant to the well-accepted empirical formula of Erickson [1964], which predicts large source sizes, since it pertains to observations of celestial sources through the solar wind. Even so, with current estimates of $l_i(R)$ implying that $s_{\text{eff}} > l_i$ (Figure 4), additional scattering is required to bring Erickson’s result into quantitative agreement with the calculations here for plane wave scattering. Alternatively, the inner scale $l_i(R)$ should be smaller than that predicted by the Coles and Harmon [1989] model assumed here, so that $s_{\text{eff}} < l_i$.

The crucial result of this paper is that the predicted source sizes are considerably smaller than the observed lower limit of $49''$ when spherical wave propagation and inner scale effects are included, as they should be for sources in the solar corona. This broad trend of the models substantially underpredicting the source size can be interpreted in three ways that are not exclusive and can occur in combination. First, it could imply that source sizes much smaller than those that have been observed so far actually exist in the solar corona, and can potentially be observed. All the instances of observations of small sources to date have been limited by the instrument resolution; it is therefore quite likely that smaller sources can be detected when instrument resolutions are improved. Second, this
broad trend can be taken to imply that our naive extrapolation of the empirical form for
the turbulence amplitude $C_N^2(R)$ to the inner corona is not justified. The results could
be taken to imply that $C_N^2(R)$ in the inner corona is far higher than suggested by the
empirical formula (3). This could be due to the functional form for $C_N^2(R)$ increasing
more rapidly with decreasing $R$ or due to a larger normalization factor or both effects.
Third, the model (4) may significantly overestimate $l_i(R)$, meaning that the turbulent
cascade extends to smaller length scales (larger $q$) and leads to more scattering. These
proposals all appear reasonable, and we regard all three as viable.

Finally, we discuss the connection between our work and that of Bastian [1994]. The
methodology is similar, and we investigate similar issues such as the effects of the inner
scale, turbulence index $\alpha$, and spherical versus planar wave propagation. Bastian’s [1994]
findings are contrary to ours: we find that our model predictions are substantially below
the minimum observed size of $49''$, while Bastian’s [1994] model predictions are substan-
tially above $49''$. Thus, a priori, both models need revision. A major difference is in the
choice of a model for $C_N^2$. Bastian [1994] uses a model for $C_N^2$ which is proportional to
the square of the background electron density and assumes the density model of Riddle
(1974), which also involves a constant of proportionality. These two constants of propor-
tionality are absorbed into one and fixed by normalizing the structure function $D_{20}(10\text{km})$
for a baseline of 10 km, an observing wavelength of 20 cm, and an elongation of $5 R_\odot$. In
contrast, as explained earlier, the $C_N^2$ model we use is determined by an empirical fit to
VLBI scattering observations between 10–50 $R_\odot$; this was motivated by the need to use
a $C_N^2$ model that is derived as directly as possible from observations. A minor matter is
that Bastian [1994] discusses the disk to limb variation in the predicted scattering angle,
whereas our treatment is valid only for sources that are reasonably close to disk center. In order to do so, we would need to use the general formalism used here, together with an integration path that incorporates the appropriate extra path length needed for sources that are displaced from the disk center.

An appropriate means of comparing the normalizations of the two treatments is thus to compare the normalizations of the structure function. Using (10) – (13), we write the structure functions as

\[
D_{sf}(s) = \frac{4 \pi^2 s^2 \theta_{c sf}^2}{\lambda^2}, \\
D_{pf}(s) = \left(\frac{2 \pi s}{\lambda}\right)^{\alpha-2} \theta_{c pf}^{\alpha-2},
\]

where \(\theta_{c sf}(s)\) is the value of \(\theta_c\) for spherical wave propagation using branch (7) and \(\theta_{c pf}(s)\) corresponds to plane wave propagation for branch (6). Then using our models for \(C_{N}^2(R), l_i(R),\) and \(n_{hyb}(R)\) we find that \(D_{sf}(s) = 2.8 \times 10^{-3} \text{rad}^2\) for \(s = 10\text{ km}, \lambda = 91\text{ cm}\) (corresponding to 327 MHz), \(\alpha = 11/3\) and a starting height corresponding to 327 MHz fundamental emission. The same prescription and parameters yield \(D_{pf}(s) = 7.7 \times 10^{-3} \text{ rad}^{5/3}\). Since the structure functions we derive are based on integrations over heliocentric distance, we cannot assign a specific elongation to them.

In comparison, Bastian normalizes \(C_{N}^2\) by assuming \(D_{20\text{cm}}(10\text{km}) = 4–12 \text{ rad}^2\), based on measurements of by Coles and Harmon [1989] and Armstrong et al. [1990] of cosmic sources (implying primarily planar wave effects) at an elongation of \(5 R_\odot\). In order to normalize Bastian’s [1994] values for the structure function to a wavelength of 91 cm, we concentrate on the structure function for spherical wave propagation. Inspection of Eqs [11] and [18] reveals that \(D_{sf}(s) \propto \lambda^2\). Therefore, \(D_{20\text{cm}}(10\text{km}) \equiv D_{91\text{cm}}(10\text{km}) \equiv 21\). Bastian’s [1994] range of values for \(D_{20\text{cm}}(10\text{km})\) thus corresponds to \(D_{91\text{cm}}(10\text{km}) = \ldots\).
The difference in $D_{91\text{cm}}(10\text{km})$ between the two prescriptions is thus a factor of $\approx (30 - 90) \times 10^3$, corresponding to a factor $\approx 170 - 300$ in $\theta_c$.

This large difference $\approx (30 - 90) \times 10^3$ in the normalization of the structure function is primarily indicative of a corresponding difference in the normalization of $C_N^2$ between the two treatments; this is because neglect of inner scale effects increases $\theta_c$ by less than a factor of 10 for Kolmogorov turbulence in Figure 7. In this connection, we note that Bastian’s [1994] normalization for $D_{20\text{cm}}(10\text{km})$ is based on values of $D(s)$ measured at an elongation of 5 $R_\odot$. We also note (e.g., Fig 1 of Coles & Harmon [1989]) that values of $D(s)$ measured at larger elongations can be considerably lower (by as much as a few orders or magnitude, depending upon the elongation). This is significant, since the model for $C_N^2$ that we use in this paper is based on observations between 10 and 50 $R_\odot$.

In summary, the foregoing results demonstrate conclusively that spherical wave propagation effects are vital for solar sources, with plane wave predictions several orders of magnitude larger than the spherical predictions. Similarly, inner scale effects are quantitatively important, while fundamental versus harmonic radiation effects are relatively small. The results and discussion above demonstrate the importance of accurate models for $C_N^2(R)$ and to a lesser extent models of $l_i(R)$ and $\alpha(R)$. This paper’s prescription for $C_N^2$ (Eq [3]) is empirical and directly based on observations (but extrapolated to smaller $R$), does not have any normalization constants that need to be determined, and leads to scattered sizes for a point source that are smaller than the minimum source size observed to date ($49''$ by Mercier et al. [2006]). Thus smaller source sizes than $49''$ may be observable. In contrast, another well-known prescription [Bastian, 1994] predicts much stronger scattering with source sizes always larger than $49''$; while this is inconsistent with the
minimum source size observed to date at 327 MHz, it may provide the extra scattering
required to account for Erickson’s empirical angular broadening result for cosmic sources
viewed through the solar wind.

While this paper’s results extend and confirm previous theoretical results pertaining to
spherical vs. plane wave effects and provide the first explanation of the small source sizes
recently observed, it is also clear that more observational and theoretical work is required
on $C_N^2(R)$ especially, but also on $l_i(R)$ and $\alpha(k, R)$. This includes temporal variations over
the solar cycle but also spatial variations between radio source regions and other regions
of the corona. Increases in $C_N^2(R)$ and decreases in $l_i(R)$ would lead to more scattering.
Work on both $n_e(R)$ and $\delta n(R)/n_e(R)$ may be useful [e.g., Efimov et al., 2008; Cairns et
al., 2009]. It is quite possible that scattering observations and theory will provide useful
constraints on these five quantities and therefore on the processes heating the solar corona
and accelerating the solar wind.

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Figure 1. Geometry for spherically diverging wavefront, which is appropriate to a situation where the source is embedded in the scattering medium.
Figure 2. The plasma frequencies predicted by the four density models in the text are plotted against the height $h = R - R_\odot$ above the photosphere (in units of $R_\odot$). The solid line uses the Leblanc et al model (15), the dotted line uses the Aschwanden & Benz model (16), the dashed line uses the 4*Newkirk model (14) and the dash-dot line uses the “hybrid” model (17).
Figure 3. The inner scales (in units of km) predicted by some density models in the text are plotted against the height $h = R - R_\odot$ above the photosphere (in units of $R_\odot$). The solid line uses the Leblanc et al model (15), the dashed line uses the 4*Newkirk model (14), and the dash-dot line uses the hybrid model (17).
Figure 4. The ratio $s_{\text{eff}}/l_i$ is plotted against the height $h = R - R_\odot$ above the photosphere (in units of $R_\odot$). The solid line shows the ratio for plane wave propagation (i.e., with $s_{\text{eff}} = s$), while the dotted line shows the ratio for spherical wave propagation (i.e., with $s_{\text{eff}} = sR/(R_1 - R_0)$). The hybrid density profile is used for determining the density and the inner scale.
Figure 5. Predicted $\theta_c$ (1 AU) in $''$ at 327 MHz for plane wave propagation, as a function of $\alpha$, using (6), (10) and (13). The observed source size is 49$''$. The thin line is for fundamental emission and the thick line is for second harmonic emission.
Figure 6. Predicted $\theta_c$ (1 AU) in $''$ at 327 MHz for plane wave propagation as a function of $\alpha$, using Eqs (7), (10) and (13). The solid line is for fundamental emission and the dotted line is for second harmonic emission. The heavy lines are computed with $l_i$ set at an artificially low value of 1 m, while still using branch (7).
Figure 7. Predicted $\theta_c$ (1 AU) in $''$ at 327 MHz for spherical wave propagation as a function of $\alpha$, using (7), (11) and (13). The observed source size at 327 MHz is 49$''$. There is negligible difference between fundamental and second harmonic emission. The solid lines are computed with $l_i$ from prescription (4), while the dotted lines are computed with $l_i$ set to an artificially low value of 1 m, while still using branch (7).
Figure 8. Predicted $\theta_c$ (1 AU) in $''$ at 327 MHz for spherical wave propagation as a function of $\alpha$, using (6), (11) and (13). The difference between the predictions for fundamental and second harmonic emission is negligible.
Figure 9. Direct comparison of $\theta_c$ (1 AU) for spherical (solid lines) and plane wave (dotted lines) propagation. Fundamental emission is assumed.