Isospectrality and matrices with concentric circular higher rank numerical ranges

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Abstract
We characterize under what conditions \( n \times n \) Hermitian matrices \( A_1 \) and \( A_2 \) have the property that the spectrum of \( \cos tA_1 + \sin tA_2 \) is independent of \( t \) (thus, the trigonometric pencil \( \cos tA_1 + \sin tA_2 \) is isospectral). One of the characterizations requires the first \( \lceil \frac{n^2}{4} \rceil \) higher rank numerical ranges of the matrix \( A_1 + iA_2 \) to be circular disks with center 0. Finding the unitary similarity between \( \cos tA_1 + \sin tA_2 \) and, say, \( A_1 \) involves finding a solution to Lax’s equation.

Keywords: Isospectral, trigonometric pencil, higher rank numerical range, Lax pair.
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1 Introduction
Questions regarding rotational symmetry of the classical numerical range as well as the \( C^- \) numerical range have been studied in [1, 4, 6, 7, 8]; there is a natural connection with isospectral properties. In this paper we study the one parameter pencil \( \text{Re}(e^{-it}B) = \cos tA_1 + \sin tA_2 \), where \( A_1 = \text{Re}B = \frac{1}{2}(B + B^*) \) and \( A_2 = \frac{1}{2i}(B - B^*) \). We say that the pencil is isospectral when the spectrum \( \sigma(\text{Re}(e^{it}B)) \) of \( \text{Re}(e^{it}B) \) is independent of \( t \in [0, 2\pi) \); recall that the spectrum of a square matrix is the multiset of its eigenvalues, counting algebraic multiplicity. As our main result (Theorem 1.1) shows there is a natural connection between isospectrality and the rotational symmetry of the higher rank numerical ranges of \( B \).

Recall that the rank-\( k \) numerical range of a square matrix \( B \) is defined by
\[
\Lambda_k(B) = \{ \lambda \in \mathbb{C} : PBP = \lambda P \text{ for some rank } k \text{ orthogonal projection } P \}.
\]
This notion, which generalizes the classical numerical range when \( k = 1 \) and is motivated by the study of quantum error correction, was introduced in [2]. In [3, 10] it was shown that \( \Lambda_k(B) \) is convex. Subsequently, in [7] a different proof of convexity was given by showing the equivalence
\[
z \in \Lambda_k(B) \iff \text{Re}(e^{-it}z) \leq \lambda_k(\text{Re}(e^{-it}B)) \text{ for all } t \in [0, 2\pi).
\] (1)
Here \( \lambda_k(A) \) denotes the \( k \)th largest eigenvalue of a Hermitian matrix \( A \).

In order to state our main result, we consider words \( w \) in two letters. For instance, \( PPQ \), \( PQPQP \) are words in the letters \( P \) and \( Q \). The length of a word \( w \) is denoted by \( |w| \). When we write \( \text{na}(w, P) = l \) we mean that \( P \) appears \( l \) times in the word \( w \) (\( \text{na} = \text{number of appearances} \)). The trace of a square matrix \( A \) is denoted by \( \text{Tr} A \).

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Theorem 1.1. Let $B \in \mathbb{C}^{n \times n}$. The following are equivalent.

(i) The pencil $\text{Re}(e^{-it}B) = \cos t \text{Re}B + \sin t \text{Im}B$ is isospectral.

(ii) $\sum_{|w|=k, \text{na}(w, B^*)=l} \text{Tr} w(B, B^*) = 0$, $0 \leq l < \frac{k}{2}, 1 \leq k \leq n$.

(iii) For $1 \leq k \leq \lfloor n/2 \rfloor$ the rank-$k$ numerical range of $B$ is a circular disk with center 0, and $\text{rank}(\text{Re}(e^{-it}B))$ is independent of $t$.

(iv) $\text{Re}(e^{-it}B)$ is unitarily similar to $\text{Re}(B)$ for all $t \in [0, 2\pi)$.

Any of the conditions (i)-(iv) imply that $B$ is nilpotent.

Note that for a given matrix $B$ it is easy to check whether Theorem 1.1(ii) holds. For instance, when $n = 5$ one needs to check that $B$ is nilpotent (or, equivalently, $\text{Tr}B^k = 0$, $k = 1, \ldots, 5$) and satisfies

$$\text{Tr}B^2B^* = \text{Tr}B^3B^* = \text{Tr}B^4B^* = \text{Tr}B^5B^* + \text{Tr}B^2B^*BB^* = 0.$$ 

The paper is organized as follows. In Section 2 we prove our main result. In Section 3 we discuss the connection with Lax pairs.

## 2 Isospectral paths

We will use the following lemma.

Lemma 2.1. Let $M(t) \in \mathbb{C}^{n \times n}$ for $t$ ranging in some domain. Then the spectrum $\sigma(M(t))$ is independent of $t$ if and only if $\text{Tr}M(t)^k$, $k = 1, \ldots, n$, are independent of $t$.

Proof. The forward direction is trivial. For the other direction, use Newton’s identities to see that the first $n$ moments of the zeros of a degree $n$ monic polynomial uniquely determine the coefficients of the polynomial, and thus the zeros of the polynomial. This implies that $\text{Tr}M(t)^k$, $k = 1, \ldots, n$, uniquely determine the eigenvalues of the $n \times n$ matrix. Thus, if $\text{Tr}M(t)^k$, $k = 1, \ldots, n$, are independent of $t$, then the spectrum $\sigma(M(t))$ is independent of $t$.

Proof of Theorem 1.1. Consider the trigonometric polynomials $f_k(t) = 2^k \text{Tr}[\text{Re}(e^{-it}B)^k], k = 1, \ldots, n$. The coefficient of $e^{i(2l-k)t}$ in $f_k(t)$ is given by $\sum_{|w|=k, \text{na}(w, B^*)=l} \text{Tr} w(B, B^*)$. By Lemma 2.1 the spectrum of $\text{Re}(e^{-it}B)$ is independent of $t$ if and only for $k = 1, \ldots, n$ and $2l \neq k$ the coefficient of $e^{i(2l-k)t}$ in $f_k(t)$ is 0. Due to symmetry, when they are 0 for $2l < k$ they will be 0 for $2l > k$. This gives the equivalence of (i) and (ii).

In particular note that when $l = 0$, we find that $\text{Tr}B^k = 0$, $k = 1, \ldots, n$, and thus $B$ is nilpotent.

Next, let us prove the equivalence of (i) and (iii). Assuming (i) we have that $\text{Re}B$ and $-\text{Re}B$ have the same spectrum, so $\text{Re}B$ has $\lfloor n/2 \rfloor$ nonnegative eigenvalues. As the spectrum of $\text{Re}(e^{-it}B)$ is independent of $t$, we have that $\text{Re}(e^{-it}B)$ has $\lfloor n/2 \rfloor$ nonnegative eigenvalues for all $t$, guaranteeing the rank-$k$ numerical range is nonempty for $k \leq \lfloor n/2 \rfloor$. Next, since $\lambda_k(\text{Re}(e^{-it}B))$ is independent of $t$, it immediately follows from the characterization of the numerical range of $B$ that $\lambda_k(B), 1 \leq k \leq \lfloor n/2 \rfloor$, is a circular disk with center 0. Also, (i) clearly implies that rank$(e^{-it}B)$ is independent of $t$.

Conversely, let us assume (iii). If the rank-$k$-numerical range of $B$ is $\{z : |z| \leq r\}$ for some $r > 0$ then $\lambda_k(\text{Re}(e^{-it}B))$ is constant. This also yields that $\lambda_{n+1-k}(\text{Re}(e^{-it}B)) = -\lambda_k(-\text{Re}(e^{-it}B))$. When for $1 \leq k \leq \lfloor n/2 \rfloor$ we have that $\lambda_k(B)$ has a positive radius, we obtain that (i) holds. Next, let us suppose $\Lambda_l(B)$ has radius zero, and $l$ is the least integer with this property. Then, as before, we may conclude that $\lambda_k(\text{Re}(e^{-it}B))$ is a positive constant for $1 \leq k < l$. We also
have, for $\ell \leq k \leq \lceil n/2 \rceil$, that $\lambda_k(\Re(e^{-itB})) = 0$ for some $t$. As we require $\text{rank} \Re(e^{-itB})$ to be independent of $t$, we find that for $\ell \leq k \leq \lceil n/2 \rceil$, $\lambda_k(\Re(e^{-itB})) = 0$ for all $t$. Again using $\lambda_{n+1-k}(\Re(e^{-itB})) = -\lambda_k(-\Re(e^{-itB}))$, we arrive at (i).

The equivalence of (i) and (iv) is obvious. □

Remark. The condition that $\text{rank} \Re(e^{-itB})$ is independent of $t$ in Theorem 1.1(iii) is there to handle the case when $\Lambda_k(B)$ has a zero radius. Indeed, it can happen that $\Lambda_k(B) = \{0\}$ without $\lambda_k(\Re(e^{-itB}))$ being independent of $t$; one such example is a diagonal matrix with eigenvalues $1, 0, -1, i$. It is unclear whether this can happen for a matrix whose higher rank numerical ranges are disks centered at 0.

For sizes 2, 3, and 4, the conditions in Theorem 1.1 are equivalent to $B$ being nilpotent and the numerical range of $B$ being rotationally symmetric.

Corollary 2.2. Let $B \in \mathbb{C}^{n \times n}$, $n \leq 4$. Then the spectrum of $\Re(e^{-itB}) = \cos t \Re B + \sin t \Im B$ is independent of $t$ if and only if $B$ is nilpotent and the numerical range is a disk centered at 0.

Proof. When $n = 2$, condition (ii) in Theorem 1.1 comes down to $\text{Tr}B = \text{Tr}B^2 = 0$. When $n = 3$ we get the added conditions that $\text{Tr}B^3 = \text{Tr}B^2B^* = 0$. When $n = 4$, we also need to add the conditions $\text{Tr}B^4 = \text{Tr}B^3B^* = 0$. The condition that $\text{Tr}B^k = 0, 1 \leq k \leq n$, is equivalent to $B$ being nilpotent. The corollary now easily follows by invoking Remarks 1-3 in [6]. □

To show that Corollary 2.2 does not hold for $n \geq 5$, note that the following example from [6],

$$B = \begin{pmatrix}
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

is nilpotent, has the unit disk as its numerical range, but $\text{Tr}B^2B^* = 1 \neq 0$.

3 Connection with Lax pairs

A Lax pair is a pair $L(t), P(t)$ of Hilbert space operator valued functions satisfying Lax’s equation:

$$\frac{dL}{dt} = [P, L],$$

where $[X, Y] = XY - YX$. The notion of Lax pairs goes back to [5]. If we start with $P(t)$, and one solves the initial value differential equation

$$\frac{d}{dt}U(t) = P(t)U(t), \quad U(0) = I, \quad (2)$$

then $L(t) := U(t)L(0)U(t)^{-1}$ is a solution to Lax’s equation. Indeed,

$$L'(t) = \frac{d}{dt}[U(t)L(0)U(t)^{-1}] =$$

$$P(t)U(t)L(0)U(t)^{-1} - U(t)L(0)U(t)^{-1}P(t)U(t)U(t)^{-1} = P(t)L(t) - L(t)P(t).$$

This now yields that $L(t)$ is isospectral. When $P(t)$ is skew-adjoint, then $U(t)$ is unitary.
In our case we have that $L(t) = \text{Re}(e^{-it}B)$, and our $U(t)$ will be unitary. This corresponds to $P(t)$ being skew-adjoint. When we are interested in the case when $P(t) \equiv K$ is constant, we have that $U(t) = e^{itK}$. Thus, we are interested in finding $K$ so that $e^{-itK}L(t)e^{itK} = L(0)$, where $L(t) = A_1 \cos t + A_2 \sin t$. If we now differentiate both sides, we find

$$-e^{-itK}KL(t)e^{itK} + e^{-itK}L'(t)e^{itK} + e^{-itK}L(t)e^{itK} = 0.$$ 

Multiplying on the left by $e^{itK}$ and on the right by $e^{-itK}$, we obtain

$$-A_1 \sin t + A_2 \cos t = L'(t) = [K, L(t)] = [K, A_1 \cos t + A_2 \sin t].$$

This corresponds to $[K, A_1] = A_2$ and $[K, A_2] = -A_1$, which is equivalent to $[K, B] = -iB$. We address this case in the following result, which is partially due to [8].

**Theorem 3.1.** Let $B \in \mathbb{C}^{n \times n}$. The following are equivalent.

(i) $e^{it}B$ is unitarily similar to $B$ for all $t \in [0, 2\pi)$.

(ii) $\text{Tr} \ w(B, B^*) = 0$ for all words $w$ with $\text{na}(w, B) \neq \text{na}(w, B^*)$.

(iii) There exists a skew-adjoint matrix $K$ satisfying $[K, B] = -iB$.

(iv) There exists a unitary matrix $U$ such that $UBU^* = B_1 \oplus \cdots \oplus B_r$ is block diagonal and each submatrix $B_j$ is a partitioned matrix (with square matrices on the block diagonal) whose only nonzero blocks are on the block superdiagonal.

Recall that Specht’s theorem [9] says that $A$ is unitarily similar to $B$ if and only if $\text{Tr} \ w(A, A^*) = \text{Tr} \ w(B, B^*)$ for all words $w$.

**Proof.** By Specht’s theorem $e^{it}B$ is unitarily similar to $B$ for all $t$ if and only if $\text{Tr} \ w(e^{it}B, e^{-it}B^*) = \text{Tr} \ w(B, B^*)$ for all $t$ and all words. When $\text{na}(w, B) \neq \text{na}(w, B^*)$ this can only happen when $\text{Tr} \ w(B, B^*) = 0$. When $\text{na}(w, B) = \text{na}(w, B^*)$, we have that $\text{Tr} \ w(e^{it}B, e^{-it}B^*)$ is automatically independent of $t$. This proves the equivalence of (i) and (ii).

The equivalence of (i) and (iv) is proven in [8, Theorem 2.1]. We will finish the proof by proving (iv) $\rightarrow$ (iii) $\rightarrow$ (i).

Assuming (iv), let $K_j$ be a block diagonal matrix partitioned in the same manner as $B_j$ and whose $m$th diagonal block equals $imI$. Then $[K_j, B_j] = -iB_j$. Let $K = U^*(K_1 \oplus \cdots \oplus K_r)U$. Then $[K, B] = -iB$, proving (iii).

When (iii) holds, let $U(t) = e^{-Kt}$. Denote $\text{ad}_X Y = [X, Y]$. Then $e^XYe^{-X} = \sum_{m=0}^{\infty} \frac{1}{m!}\text{ad}^m_X Y$, and (iii) yields that

$$U(t)BU(t)^* = e^{-itK}Be^{itK} = \sum_{m=0}^{\infty} \frac{1}{m!}\text{ad}^m_{-itK}B = \sum_{m=0}^{\infty} \frac{(it)^m}{m!}B = e^{it}B,$$

yielding (i).

It is clear that if $B$ satisfies Theorem 3.1(i) it certainly satisfies Theorem 3.1(i). In general the converse will not be true, and the size of such a counterexample must be at least 4; indeed, if $B$ is a strictly upper triangular $3 \times 3$ matrix with $\text{Tr}B^2B^* = 0$ at least one of the entries above the
diagonal is zero, making $B$ satisfy Theorem 3.1(iv). An example that satisfies the conditions of Theorem 3.1 but does not satisfy those of Theorem 3.1 is

$$B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Indeed, it is easy to check that $\text{Tr}B^2B^* = \text{Tr}B^3B^* = 0$, but $\text{Tr}B^3B^*BB^* = -1 \neq 0$. A $5 \times 5$ example satisfying the conditions of Theorem 1.1 but not those of Theorem 3.1 is

$$\begin{pmatrix} 0 & 1 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

When $B$ satisfies the conditions of Theorem 3.1, the $K$ from Theorem 3.1(iii) will yield the unitary similarity $\text{Re}(e^{-it}B) = e^{-it}(\text{Re}B)e^{it}$. It is easy to find $K = -K^*$ satisfying $[K, B] = -iB$ as it amounts to solving a system of linear equations (with the unknowns the entries in the lower triangular part of $K$).

When $B$ satisfies the conditions of Theorem 1.1 but not those of Theorem 3.1, finding a unitary similarity $U(t)$ so that $\text{Re}(e^{-it}B) = U(t)(\text{Re}B)U(t)^*$ becomes much more involved. To go about this one could first find a solution $P(t)$ to Lax’s equation

$$-A_1 \sin t + A_2 \cos t = L'(t) = [P(t), L(t)] = [P(t), A_1 \cos t + A_2 \sin t],$$

which now will not be constant. Next, one would solve the initial value ordinary differential matrix equation (2).

To illustrate what a solution $P(t), U(t)$ may look like, we used Matlab to produce the following solution when $A_1 = \text{Re} B$ and $A_2 = \text{Im} B$ (and thus $L(t) = \text{Re}(e^{-it}B)$) with $B$ as in (3):

$$P(t) = \begin{pmatrix} -\frac{i}{2} & 0 & 0 & \frac{ie^{-2it}}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ \frac{ie^{2it}}{2} & 0 & 0 & \frac{3i}{2} \end{pmatrix},$$

$$V(t) = \begin{pmatrix} 1 - e^{-it} & -1 - e^{-it} & 1 - e^{-it} & 1 + e^{-it} \\ 2 & 1 & -1 & 2 \\ -2e^{it} & e^{it} & e^{it} & 2e^{it} \\ e^{2it} + e^{it} & -e^{2it} + e^{it} & e^{2it} + e^{it} & e^{2it} - e^{it} \end{pmatrix}, U(t) = V(t)V(0)^{-1}.$$ 

Note that the columns of $V(t)$ are the eigenvectors of $L(t)$; indeed, we have

$$L(t) = V(t)\text{diag}(-1, -\frac{1}{2}, \frac{1}{2}, 1)\text{diag}(-1, -\frac{1}{2}, \frac{1}{2}, 1)\text{diag}(-1, -\frac{1}{2}, \frac{1}{2}, 1)^{-1}.$$ 

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