The structure of the curvature tensor of plane gravitational waves

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Abstract. Plane gravitational waves in the Riemann space of General Relativity is considered. The criterion of plane gravitational waves is used based on the analogy between plane gravitational and electromagnetic waves. The Theorem is proved that the action of the Lie derivative on the plane wave curvature 2-form in the direction of the vector generating the invariance group of this wave in the Riemann space is equal to zero. It is justified that the gravitational waves can be used to transmit information in the Riemann space.

1. Introduction
In previous papers, the structure of space-time formed by plane gravitational waves in the Riemann–Cartan space [1] and general affine-metric space [2] was studied. In these works, the criterion of plane gravitational waves was used, based on the analogy between plane gravitational and electromagnetic waves.

By analogy with a plane electromagnetic wave, the metric tensor of space-time must have a group of motions G₅, which does not change the isotropic hypersurface describing a plane wave front with constant amplitude.

In this paper, the structure of the curvature tensor is determined for the case of plane waves in the Riemann space of the General Relativity Theory.

2. Plane wave metrics in Riemann space
The generator (infinitesimal operator) of the group G₅ is the vector field

\[ \vec{X} = (a + b'x + c'y) \partial_x + b\partial_y + c\partial_y \], where \( b(u) \), \( c(u) \) are arbitrary functions, and \( b'(u) \), \( c'(u) \) are their derivatives \((a=const)\).

Vectors \( \vec{e}_\alpha (\alpha = 0, \hat{1}, \hat{2}, \hat{3}) : \vec{e}_0 = \partial_v , \vec{e}_1 = \partial_u , \vec{e}_2 = \partial_x , \vec{e}_3 = \partial_y \) are the basis vectors of a special coordinate system \( v, u, x, y \), where \( u \) is the phase of the wave (delayed time), and \( x \) and \( y \) are the coordinates on the wave surface \((u,v) = const\). The vector \( \vec{e}_v \) is covariant constant, isotropic and directed along the wave ray. The vector \( \vec{e}_u \) is also isotropic. Vectors \( \vec{e}_x \) and \( \vec{e}_y \)
are both space-like, commute and orthogonal to the vector $\mathbf{v}$. Basic vectors correspond to basic 1-forms $\theta^\alpha (\alpha = 0,1,2,3)$, $\theta^0 = dv$, $\theta^1 = du$, $\theta^2 = dx$, $\theta^3 = dy$, $d$ – an operator of external differentiation.

For a plane gravitational wave, the metric of the Riemann space $V_4$ satisfies the condition $L_X g_{\alpha\beta} = 0$, where $L_X$ is the Lie derivative in the direction of the vector $X$. This condition leads to a metric of Kundt type

$$\hat{g} = 2H(u,x,y)du^2 + 2дуdv - dx^2 - dy^2,$$

where

$$H(u,x,y) = \frac{1}{2}A(u)x^2 + B(u)xy + \frac{1}{2}C(u)y^2,$$

and the functions $A(u), B(u), C(u)$ satisfy the conditions

$$b^\alpha + bA(u) + cB(u) = 0, \quad c^\alpha + bB(u) + cC(u) = 0.$$

It is convenient to go over to a nonholonomic basis $\mathbf{e}_a (a=1,2,3,4)$:

$$\mathbf{e}_a = \mathbf{e}'_a, \quad \mathbf{e}_i = -H(u,x,y)\mathbf{e}_0 + \mathbf{e}_1, \quad \mathbf{e}_z = \mathbf{e}_2, \quad \mathbf{e}_3 = \mathbf{e}_4,$$

which corresponds to the following nonholonomic basic 1-forms:

$$\theta^0 = H(u,x,y)du + dv, \quad \theta^1 = du, \quad \theta^2 = dx, \quad \theta^3 = dy.$$

The transition from holonomic basic 1-forms to nonholonomic one is carried out by a linear transformation $\theta^a = h^a_\alpha \theta^\alpha$, where the matrix $h^a_\alpha$ is equal to

$$h^a_\alpha = \theta^a(\mathbf{e}_\alpha) = \begin{pmatrix} 1 & H & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

The components of the metric corresponding to the nonholonomic basis are equal:

$$g_{\alpha\beta} = \hat{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = h^a_\alpha h^b_\beta g_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\alpha\beta}.$$ 

The nonzero components of these metrics are equal, $g_{01} = g_{20} = 1, \quad g_{22} = g_{33} = -1$. The components of the metrics in the coordinate space are equal

$$g_{\alpha\beta} = \hat{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = h^a_\alpha h^b_\beta g_{ab}.$$ 

3. Plane waves of curvature in Riemann space

The curvature is determined by the tensor-valued curvature 2-form,

$$R^a_\beta = (2\Gamma^a_\beta)_\gamma^{\gamma} \wedge \theta^\gamma + (2\Gamma^a_{\gamma\mu})_\beta \theta^\gamma \wedge \theta^\mu,$$

where $\Gamma^a_{\beta\mu}$ is the 1-form of the Lorentzian connection, defined in terms of the Christoffel symbols and tetrad coefficients in the form, $\Gamma^a_{\beta\mu} = h^a_\rho h^\rho_\gamma \Gamma^\gamma_{\beta\mu} - h^a_\nu h^\nu_\rho \Gamma^\rho_{\beta\mu}$. The components of the non-zero Lorentzian connection 1-form are

$$\Gamma^0_{2\alpha} \theta^\alpha = \Gamma^2_{1\alpha} \theta^\alpha = du \partial_\alpha H, \quad \Gamma^0_{3\alpha} \theta^\alpha = \Gamma^3_{1\alpha} \theta^\alpha = du \partial_\alpha H.$$ 

The nonzero components of the curvature 2-form are equal,
The action of the Lie derivative on the curvature 2-form $R_a^b$ of a plane gravitational wave in the direction of the vector $\vec{X}$ generating the 5-parameter invariance group $G_5$ of this wave in the Riemann space is equal to zero, $L_{\vec{X}}R_a^b = 0$.

Proof. The components of the Lie derivative acting on an arbitrary vector in a nonholonomic basis $\vec{V} = V^a\partial_a$ are equal to

$$\left(L_{\vec{X}}\vec{V}\right)^a = X^b\partial_b V^a - V^c\partial_c X^a + X^b\partial^a\left[\vec{e}_b, \vec{e}_c\right]^c.$$ 

The components of the Lie derivative acting on an arbitrary 1-form $\sigma = \sigma_a\partial_a$ in a nonholonomic basis are equal to

$$\left(L_{\vec{X}}\sigma\right)_a = X^b\left(\vec{e}_b, \sigma_a\right) + \left(\vec{e}_b X^b\right)\sigma_b + X^b\left[\vec{e}_b, \vec{e}_c\right]^c\sigma_c.$$ 

Applying these formulas to the curvature 2-form, we find that the components of the Lie derivative acting on it have the form:

$$\left(L_{\vec{X}}R\right)^a_{\mu\nu} = X^d\left(\vec{e}_d R^a_{\mu\nu}\right) - \left(\vec{e}_d X^d\right)R^a_{\mu\nu} + \left(\vec{e}_d X^d\right)R^a_{\mu\nu} + \left(\vec{e}_d X^d\right)R^a_{\mu\nu}.$$

From here, for example, for the values $\mu = 1, \nu = 2$ we find,

$$\left(L_{\vec{X}}R\right)^0_{212} = \left(a + b'\left(u\right) x + c'\left(u\right) y\right)(\partial_a R^0_{212}) + b\left(\partial_a R^0_{212}\right) + c\left(\partial_a R^0_{212}\right).$$

Similarly, for all nonzero values of the Lie derivative, we obtain,

$$\left(L_{\vec{X}}R\right)^0_{212} = -b\partial_3^x H - c\partial_3^x H, \quad \left(L_{\vec{X}}R\right)^0_{213} = -c\partial_3^x H - b\partial_3^x H, \quad \left(L_{\vec{X}}R\right)^0_{312} = -b\partial_3^x H - c\partial_3^x H, \quad \left(L_{\vec{X}}R\right)^0_{313} = -c\partial_3^x H - b\partial_3^x H.$$

For a specific value of the function $H(u, x, y)$ of a plane gravitational wave, the values of all third-order derivatives with respect to the variables $x$ and $y$ are equal to zero. As a result, we obtain the proof of the theorem.

As a consequence of the expression for the function $H(u, x, y)$, we find that the components of the curvature tensor of the Riemann space depend on all three arbitrary functions that define the metrics,

$$R^0_{212} = -A(u), \quad R^0_{213} = -B(u), \quad R^0_{312} = -B(u), \quad R^0_{313} = -C(u),$$

$$R^2_{112} = -A(u), \quad R^2_{113} = -B(u), \quad R^3_{112} = -B(u), \quad R^3_{113} = -C(u).$$

In this case, the curvature scalar is equal to zero, and the Ricci tensor is equal to

$$R_{11} = A(u) + C(u).$$

The dependence of the curvature tensor on arbitrary functions allows us to assert that the curvature tensor of a plane gravitational wave of the Riemann space in General Relativity can transfer information encoded in the source of the gravitational wave using the arbitrary functions $A(u)$ and $C(u)$.

4. Conclusion

We have considered plane gravitational waves in the Riemann space of General Relativity using the criterion of plane gravitational waves based on the analogy between plane gravitational and electromagnetic waves. This criterion states that the metric tensor of the gravitational wave space-time has a group $G_5$ of motion, which leaves invariant the isotropic hypersurface of a plane wave front with constant amplitude.
The Theorem is proved that the action of the Lie derivative on the plane wave curvature 2-form in the Riemann space in the direction of the vector generating the invariance group $G_5$ of this wave is equal to zero.

In the Riemann space of General Relativity, the components of the curvature tensor depend on three arbitrary functions that determine the metrics. But Einstein's equations define only two functions, $A(u)$ and $C(u)$, because the Ricci tensor has the form, $R_{11} = A(u) + C(u)$, and the curvature scalar is equal to zero. The third function, $B(u)$, remains arbitrary. Therefore, the curvature tensor of plane gravitational waves can carry information encoded by these two functions in the wave source. This fact may be significant, if gravitational waves in Riemann space can be used in future to transmit information.

References
[1] Babourova O V, Frolov B N and Scherban’ V N 2013 *Gravit. Cosmol.* 19 144
[2] Babourova O V, Frolov B N, Khetseva M S and Markova N V 2018 *Class. Quantum Grav.* 35 175011