A new stochastic order based on discrete Laplace transform and some ordering results of the order statistics

Fatemeh Gharari and Masoud Ganji
Department of Statistics and Computer Science, University of Mohaghegh Ardabili, Ardabil, Iran

ABSTRACT
This paper aims to study a new stochastic order based upon discrete Laplace transforms. By this order, in a setup where the sample size is random, having discrete delta and nabla distributions, we obtain some ordering results involving ordinary and fractional order statistics. Some applications in frailty models and reliability are presented as well.

1. Introduction
Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) random variables from a distribution $F$. The $i$th order statistics (os) will be denoted by $X_{i:n}$, $i = 1, 2, \ldots, n$. For further details about os, we refer the reader to Arnold, Balakrishnan, and Nagaraja (1992).

In recent years, many researchers have studied stochastic comparisons of os for iid random variables when the sample size is fixed. Shaked and Wong (1997) have studied stochastic comparisons of os, $X_{1:n}$ and $X_{n:n}$. Some other results regarding stochastic comparisons of the os $X_{1:n}$ and $X_{n:n}$ can be obtained in Bartoszewicz (2001). Nanda et al. (2005) and Nanda and Shaked (2008) generalized the results proved in Shaked and Wong (1997) and Bartoszewicz (2001) to os other than the minimum and the maximum. They have studied the closure of the different partial orders under the formation of a $r$-out-of-$N$ system when the number of components $N$, forming the system, is a random variable with support $k, k+1, \ldots$, for some fixed positive integers $k$. An open problem in stochastic order theory is stochastic comparisons of fractional order statistics (fos) when the sample size is random.

The concept of fos was introduced by Stigler (1977) via a Dirichlet process definition. This type of os was developed as a technical device which may be viewed as defining a continuum of os for any sample size. There are two principle definitions for fos. The first definition presented by Jones (2002) as a random convex combination of $X_{i:n}$ and $X_{i+1:n}$, namely

$$X_{a:n} = X_{i+c:n} = (1-c)X_{i:n} + cX_{i+1:n},$$
where $1 < a < n$, $i = \lfloor a \rfloor$ denotes the largest integer less than or equal to $a$, $c = a - i$, and $X_{i:n}$ is the $i$th os from an independent and identically distributed sample size $n$ from distribution $F$. The second definition of fos is due to Stigler (1977). Let $X_{a:n} = F^{-1}(U_{a:n})$, where $U_{a:n}$ is fractional associated with the uniform (on $[0, 1]$) defined to be a quantity following the beta distribution (on $[0, 1]$) with parameters $a$ and $n - a + 1$, written Beta($a, n - a + 1$). This definition is appropriate for $1 < a < n + 1$. As an application of fos, non-parametric confidence intervals for quantiles can be calculated using fos (see Hutson 1999). Another application is to construct approximate prediction intervals of os based on fos (see Basiri, Ahmadi, and Raqab 2016). As noted by Hutson (1999), in general, fos cannot be calculated from the sample, since their indices do not need to take integer values. An open problem in order statistics theory is finding the distribution of a fos.

In this study, we try to find solutions for these two problems. To this end, first we present new definitions of os and fos by using discrete fractional calculus. On the other hand, we know that the existence of the delta and nabla discrete distributions were proved by using discrete fractional calculus (see Ganji and Gharari 2018a). We see that these two definition processes were generated on a construction equipment of delta and nabla calculus. Considering this important point, we can find solutions by looking for some relationships between os and fos with nabla and delta types of probability distributions. In this regard, some of important constructions will form the basis of our theory. First, we suppose that the sample size $N$ is a random variable having discrete nabla or delta distribution. Under this situation, we give two general expressions for the probability density functions (pdfs) of the $r$th os and fos in terms of the $r$th derivatives of the Laplace transform of $N$. For the next problem, we define a new stochastic order based on the ordinary and fractional derivatives of the Laplace transform of the sample size $N$ and compare os and fos in a setup where the sample size is random. This study has two main advantages in comparison with previous works regarding the comparison of os when the sample size is random. First, the new stochastic order appears the relation between the $i$th os and the $i$th derivative of the Laplace transform of the random sample size $N$, where $i$ and $N \in \mathbb{N}_1$. Therefore, we present a new method for the comparison of os other than the minimum and the maximum. On the other hand, we generalize the new stochastic order for the comparison of fos when the sample size is random. Similarly in this case, the new stochastic order reveals the relationship between $\gamma$th fos, $n - 1 < \gamma \leq n$, $n \in \mathbb{N}$ when the sample size $N$ is random, and the $\gamma$th fractional derivative of the Laplace transform of $N$, where $N \in \mathbb{N}_{x-1}$, $x - 1 > 0$.

The article is organized as follows. The second section contains some definitions and preliminary results that are needed for the main results to be established in the next sections. The third section is essential for starting our theory and contains a new definition of a fos by spacings. In the fourth section, we give another definition of a fos by fractional operators and there represent two general expressions for the pdfs of the $r$th os and fos when the sample size $N$ is random. Also, we introduce two new stochastic orderings based on both fractional and ordinary $r$th derivations of the Laplace transforms of $N$. At the next section, some stochastic orders based on discrete Laplace transforms are defined along with some results and properties. Some of their applications are presented in the final section.
2. Preliminaries

For a non-negative random variable $X$ with df $F$, its Laplace transform is defined as

$$L_X(s) = \int_0^\infty e^{-sx}dF(x).$$

Let $X$ and $Y$ be two non-negative random variables. $X$ is said to be smaller than $Y$ in the Laplace transform order (denoted by $X \leq_l Y$) if $L_Y(s) \leq L_X(s)$ for all $s > 0$. Shaked and Wong (1997) defined two stochastic orders based on ratios of the Laplace transforms: $X$ is said to be smaller than $Y$ in the Laplace transform ratio order (denoted by $X \leq_l Y$) if $L_Y(s)/L_X(s)$ is decreasing in $s > 0$; and $X$ is said to be smaller than $Y$ in the reversed Laplace transform ratio order (denoted by $X \leq_l Y$) if $[1 - L_Y(s)]/[1 - L_X(s)]$ is decreasing in $s > 0$. Crescenzo and Shaked (1996) applied a useful property of the Laplace transform ratio order to families of non-negative random variables that have the generalized semigroup property. Also, Li, Ling, and Li (2009) defined a stochastic order based on ratio of differentiated Laplace transforms: $X$ is said to be smaller than $Y$ in the differentiated Laplace transform ratio order (denoted by $X \leq_{d-l} Y$) if $L_Y'(s)/L_X'(s)$ is decreasing in $s > 0$. Now, let us introduce a new order which, as will be seen in Section 6, has some potential applications.

**Definition 2.1.** $X$ is said to be smaller than $Y$ in the $i-$ differentiated Laplace transform ratio order (denoted by $X \leq_{d-l} Y$) if $L_Y^{(i)}(s)/L_X^{(i)}(s)$ is decreasing in $s > 0$.

In the following, we generalize these stochastic orders for discrete Laplace transforms, which, as will be seen in the next sections, have some potential applications.

For real numbers $a$ and $b$, we denote $\mathbb{N}_a = \{a, a + 1, \ldots\}$ and $\mathbb{N}_b = \{b, b - 1, \ldots\}$. Throughout this paper, we apply discrete nabla and delta distributions having supports $\mathbb{N}_1$ and $\mathbb{N}_{x-1}$, $x > 0$, respectively. For further details about these distributions, we refer the reader to Ganji and Gharari (2018a). The delta and nabla discrete distributions include an extended class of discrete distributions. The geometric distribution (the number of independent trials required for first success), nabla discrete gamma distribution (Ganji and Gharari 2018a), nabla discrete Weibull distribution (Ganji and Gharari 2016), and nabla Mittag-Leffler distribution (Ganji and Gharari 2018b) are examples of nabla discrete distributions. Also, examples for delta discrete distribution are the geometric distribution (the number of failures for first success), delta discrete gamma distribution, delta discrete Weibull distribution, and delta Mittag-Leffler distribution.

Suppose that $X$ is a delta discrete random variable with values $x = \mathbb{N}_{x-1}$, $x > 0$. The delta discrete Laplace transform of probability mass function (pmf) of $X$ is defined as

$$L_X(t) = \sum_{x=x-1}^{\infty} \left( \frac{1}{1 + t} \right)^{\sigma(x)} f(x) = E[(1 + t)^{-\sigma(X)}] = E[e^{\sigma X}(\sigma(X), 0)] := M_{\sigma(X)}(-t),$$

where $t$ is the set of all regressive complex constants for which the series converges and $\sigma(x) = x + 1$. By applying the series expansion for $(1 + t)^{-x}$, it can be easily proved that $M_{\sigma(X)}(-t) = \sum_{k=0}^{\infty} E[-1]^k(\sigma(X))^t \frac{t^k}{k!}$, where
\[(\sigma(X))^k = \frac{\Gamma(\sigma(X) + k)}{\Gamma(\sigma(X))}.\]

This function generates the nabla moments of integer order of \(X\) as
\[E[(\sigma(X))^k] = (-1)^k \frac{d^k M_{\sigma(X)}(-t)}{dt^k} \bigg|_{t=0}.\]

**Definition 2.2.** Let \(X\) be a delta discrete random variable with pmf \(f\).

a. Its \(k\)th nabla moment is denoted by \(\mu_k^\nabla\) and is defined by \(\mu_k^\nabla := \sum_x (\sigma(X))^k f(x)\).

b. The nabla moment generating function (mgf) of \(X\) is given by \(M_{\sigma(X)}(t) = E[e^t(\sigma(X),0)]\), where \(e^t(\sigma(X),0)\) denotes the nabla exponential function defined on a time scale (see Bohner and Peterson (2003)).

c. reverse delta Laplace transform ratio order (written as \(\mathcal{L}_X(t)\)), if \(\mathcal{L}_X(t)/(1 - \mathcal{L}_Y(t))\) is increasing in \(s > 0\);

d. \(i\)-differentiated delta Laplace transform ratio order (I) (written as \(\mathcal{L}_Y^{(i)}(s)/\mathcal{L}_X^{(i)}(s)\)), if \(\mathcal{L}_Y^{(i)}(s)/\mathcal{L}_X^{(i)}(s)\) is decreasing in \(s > 0\).

Now, suppose that \(X\) is a nabla discrete random variable with values \(x = \mathbb{N}_1\). The nabla discrete Laplace transform of \(X\) is defined as
\[\mathcal{L}_X(t) = \sum_{x=1}^{\infty} (1-t)^{\rho(x)} f(x) = E[(1-t)^{\rho(X)}] = E[e^t(\rho(X),0)] := M_{\rho(X)}(-t),\]

where \(\rho(x) = x-1\) and \(t\) is the set of all regressive complex constants, for which the series converges. By using the series expansion for \((1-t)^\alpha\), it can be easily proved that
\[M_{\rho(X)}(-t) = \sum_{k=0}^{\infty} E[(-1)^k(\rho(X))^k] \frac{t^k}{k!},\]

where
\[(\rho(X))^k = \frac{\Gamma(\rho(X) + 1)}{\Gamma(\rho(X) + 1 - k)}.\]

This function generates the delta moments of integer order of \(X\) as
\[E[(\rho(X))^k] = (-1)^k \frac{d^k M_{\rho(X)}(-t)}{dt^k} \bigg|_{t=0}.\]

**Definition 2.4.** Let \(X\) be a nabla discrete random variable with pmf \(f\).

a. Its \(k\)th delta moment is denoted by \(\mu_k^\Delta\) and is defined by \(\mu_k^\Delta := \sum_x (\rho(X))^k f(x)\).
b. The delta mgf of $X$ is given by $M_{\rho(X)}(t) = E[e^t(\rho(X), 0)]$, where $e^t(\rho(X), 0)$ denotes the delta exponential function defined on a time scale (see Bohner and Peterson (2001, 2003)).

Also, $X \leq_{Lt} Y$, $X \leq_{Lt-r} Y$, $X \leq_{d^{(0)}-Lt-r} Y$ and $X \leq_{d^{(0)}-Lt} Y$ are defined for nabla random variables $X$ and $Y$ like delta random variables, but in this case there are two differences; these results are valid for the restriction $0 < s < 1$, and Definition 2.3 (d) is valid for $i \in \mathbb{N}$.

It would be interesting to see a unification of the mgfs and moments. For a given time scale $\mathbb{T}$, we present the construction of moments and the mgfs on time scales as

$$\hat{\mu}_k = E\left[\Gamma(k+1)\hat{h}_k(\eta(X))\right] \quad \text{and} \quad M_{\eta(X)}(-t) = E[\hat{e}_{\eta}(\eta(X), 0)], \quad x \in \mathbb{T}, \quad (2.1)$$

respectively. In order that the reader sees how ordinary moments, delta and nabla moments and also types of the mgfs follow from (2.1), it is necessary only in this point to know that

$$\hat{h}_k(x) = h_k(x) = \frac{x^k}{\Gamma(k+1)}, \quad \eta(x) = \sigma(x) = \rho(x) = x \quad \text{and} \quad \hat{e}_{\eta}(\eta(X), 0) = e^{-tx},$$

if $\mathbb{T} = \mathbb{R}^+$,

$$\hat{h}_k(x) = h_k(x) = \frac{x^k}{\Gamma(k+1)}, \quad \eta(x) = \sigma(x) \quad \text{and} \quad \hat{e}_{\eta}(\eta(X), 0) = (1 + t)^{-\sigma(x)},$$

if $\mathbb{T} = \mathbb{N}_{x-1}$, $x > 0$ and

$$\hat{h}_k(x) = h_k(x) = \frac{x^k}{\Gamma(k+1)}, \quad \eta(x) = \rho(x) \quad \text{and} \quad \hat{e}_{\eta}(\eta(X), 0) = (1 - t)^{\rho(x)},$$

if $\mathbb{T} = \mathbb{N}_1$.

**Definition 2.5.** It is said that the random variable $X$ has a delta discrete gamma distribution with $(\alpha, \beta)$ parameters if its pmf is given by

$$Pr[X = x] = \frac{h_{\alpha-1}(x)\beta^x}{e_\beta(\sigma(x), 0)} = \frac{x^{\alpha-1}\beta^x}{\Gamma(x)(1 + \beta)^{\sigma(x)}}, \quad x \in \mathbb{N}_{x-1}, \quad (2.2)$$

where $\alpha > 0$, $\beta > 0$ and it is denoted by $\Gamma^\alpha(\alpha, \beta)$.

**Definition 2.6.** It is said that the random variable $X$ has a nabla discrete gamma distribution with $(\alpha, \beta)$ parameters if its pmf is given by

$$Pr[X = x] = \frac{h_{\alpha-1}(x)\beta^x}{e_\beta(\rho(x), 0)} = \frac{x^{\alpha-1}\beta^x(1 - \beta)^{\rho(x)}}{\Gamma(x)}, \quad x \in \mathbb{N}_1, \quad (2.3)$$

where $\alpha > 0$, $0 < \beta < 1$ and it is denoted by $\Gamma^\nabla(\alpha, \beta)$. 
3. A New definition of fos by spacings

Suppose that \( n - 1 \) points are dropped at random on the unit interval \((0, 1)\). We denote the ordered distances of these points from the origin by \( u_{i:n} \) (\( i = 1, 2, \ldots, n - 1 \)). We let \( V_i := \nabla u_{i:n} = u_{i:n} - u_{i-1:n} \) (\( u_{0:n} = 0 \)) and \( V_i' := \Delta u_{i:n} = u_{i+1:n} - u_{i:n} \) and call backward spacing and forward spacing, respectively. Similarly, we may have the same results for \( V_i' \) as \( W_i := \nabla V_{i:n} = V_{i:n} - V_{i-1:n} = u_{i:n} - 2u_{i-1:n} + u_{i-2:n} = \nabla^2 u_{i:n} \)

and

\[
W_i' := \Delta V_{i:n} = V_{i+1:n} - V_{i:n} = u_{i+1:n} - 2u_{i:n} + u_{i-1:n} = \Delta^2 u_{i:n}.
\]

By repeating \( m \) times, we obtain

\[
\nabla^m u_{i:n} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} u_{i-m-j:n} := Z_i
\]

and

\[
\Delta^m u_{i:n} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} u_{i+m-j:n} := Z_i',
\]

where \( \nabla^m \) and \( \Delta^m \) are \( m \)th order nabla and delta difference operators, respectively.

Now let \( \alpha > 0 \) with \( n' - 1 < \alpha < n' \). Using the definition of nabla and delta differences of order \( \alpha \) from discrete fractional calculus (see Goodrich and Peterson 2015), we have

\[
\nabla^\alpha u_{i:n} = \sum_{j=0}^{i-2} (-1)^j \binom{\alpha}{j} u_{i-j:n} := S_i, \quad i \in \mathbb{N}_{1+n'}
\]

and

\[
\Delta^\alpha u_{i:n} = \sum_{j=0}^{\alpha+i-1} (-1)^j \binom{\alpha}{j} u_{i+\alpha-j:n} := S_i', \quad i \in \mathbb{N}_{1+n'-\alpha}.
\]

Considering the indices of \( S_i' \) and \( S_i \), we proved new definitions of fos and os, respectively. Therefore, we could build a new definition of fos by using discrete fractional calculus (or nabla and delta fractional calculus). On the other hand, the existence of the delta and nabla discrete distributions were proved by using discrete fractional calculus (see Ganji and Gharari 2018a). It seems that there is a relationship between os and fos with these types of probability distributions. In the following, we are going to discover it with an interesting way.

4. Distribution of a fos by fractional operators

Stigler (1977) considered fos with both integer and non-integer sample sizes. Rohatgi and Saleh (1988) extended definition of df of the \( i \)th os, \( i \in \mathbb{N} \), with noninteger sample size. In this section, first we give two general expressions for the pdfs of the \( i \)th os and
fos when the sample size $N$ is a nabla or delta random variable. For doing this, we use both ordinary and fractional types of $i$th derivative of Laplace transform of $N$. Then, we introduce a new stochastic ordering based on the $i$th derivative of the Laplace transform $N$.

4.1. Distribution of an os when the sample size is random

Let $X_1, X_2, \ldots, X_N$ be a random sample size $N$ from a df $F$. Let $X_{1:N}, X_{2:N}, \ldots, X_{N:N}$ be the corresponding os when the sample size $N$ is a random variable itself.

Conditionally, given sample size $N = k$, let the df of the $i$th os $X_{i:k}$ be noted by $F_{i:k}(x|k)$ and the corresponding pdf by $f_{i:k}(x|k)$. We suppose that $N$ is a random variable with a nabla discrete distribution. In this case,

$$P_N(k) = P(N = k), \quad k \in \mathbb{N}_1.$$

The conditional df of $X_{i,\rho(N)}$, conditioned on the event $\{N \geq i\}$, is

$$F_{i,\rho(N)}(x) = \frac{1}{\pi_N} \sum_{k=i}^{\infty} F_{i,\rho(k)}(x|k)P_N(k), \quad (4.1)$$

where $\pi_N^i = P(N \geq i)$. It is well known that the df of the $i$th os, $i \in \mathbb{N}_1$, in a sample size $\rho(k)$ from a df $F$ is as

$$F_{i,\rho(k)}(x|k) = \sum_{j=0}^{\rho(k)} \binom{\rho(k)}{j} F^j(x)(1 - F(x))^{\rho(k) - j} \quad (4.2)$$

Obviously,

$$F_{i,\rho(k)}(x|k) = \int_0^{F(x)} \frac{\rho(k)!}{(i-1)! \rho(k)-i!} t^{i-1}(1 - t)^{\rho(k)-i} dt.$$

The $i$th derivative of $M_{\rho(N)}(t)$ exists and is obtained by formal differentiation as

$$(-1)^i M_{\rho(N)}^{(i)}(t) = \sum_{k=i}^{\infty} (\rho(k))^i(1 - t)^{\rho(k)-i} P_N(k), \quad i \in \mathbb{N}_1. \quad (4.3)$$

It follows from (4.3) and (4.1) that for $x \in \mathbb{R}$,

$$F_{i,\rho(N)}(x) = \frac{1}{\pi_N} \sum_{k=i}^{\infty} \int_0^{F(x)} \frac{\rho(k)!}{\Gamma(i)} t^{i-1}(1 - t)^{\rho(k)-i} P_N(k) dt$$

$$= \frac{1}{\pi_N^i(i - 1)!} \int_0^{F(x)} (-1)^i t^{i-1} M_{\rho(k)}^{(i)}(t) dt, \quad i \in \mathbb{N}_1.$$

In the special case, when $F$ has pdf $f$, we can immediately get the pdf of the $i$th os when the sample size is random as

$$f_{i,\rho(N)}(x) = \frac{(-1)^i F^{i-1}(x)f(x)}{\pi_N^i(i - 1)!} M_{\rho(N)}^{(i)}(F(x)), \quad x \in \mathbb{R}. \quad (4.4)$$
Note that this formula is valid for \( i = 1, 2, \ldots, \rho(N) \). Now, let us introduce a new order which, as will be seen in the next section, has some potential applications.

**Definition 4.1.** Let \( X \) and \( Y \) be nabla random variables with pdfs \( f \) and \( g \), dfs \( F \) and \( G \), and sfs \( \bar{F} \) and \( \bar{G} \), respectively. Then, \( X \) is said to be smaller than \( Y \) in the \( i \)-differentiated nabla Laplace transform ratio order (II) (written as \( X \leq_{d-LT-r} Y \)) if

\[
\frac{\pi_X^{(i)} L_Y^{(j)}(s)}{\pi_Y^{(i)} L_X^{(j)}(s)} \text{ is decreasing in } 0 < s < 1, \quad i \in \mathbb{N}_1.
\]

In the special case \( i = 1 \), we have \( P(X \geq 1) = P(Y \geq 1) = 1 \) and the order reduces to

\[
\frac{L_Y(s)}{L_X(s)} \quad (\text{written as } X \leq_{dLT-r} Y).
\]

### 4.2. Distribution of a fos when the sample size is fractional random

Let \( X_1, X_2, \ldots, X_N \) be a random sample size \( N \) from a df \( F \) with corresponding sf \( \bar{F} \). Suppose that \( N \) is a random variable with a delta discrete distribution. In this case,

\[
P_N(k) = P(N = k), \quad k \in \mathbb{N}_{\alpha-1}, \quad \alpha > 1,
\]

is the distribution of \( N \).

We let \( X_{\gamma;\sigma(N)+\gamma-n} \), \( n - 1 < \gamma \leq n \), \( n \in \mathbb{N} \) be the corresponding fos when \( N \) is a delta random variable itself. Note that, in the special cases \( \gamma = 1, 2, \ldots, n \), we get \( X_{1;\sigma(N)}, X_{2;\sigma(N)}, \ldots, X_{n;\sigma(N)} \) os, respectively. Also, in the more special case, if \( \alpha = 2 \) and the sample size is constant \( n \), for \( \gamma = 1, 2, \ldots, n \), we get \( X_{1;n}, X_{2;n}, \ldots, X_{n;n} \) os, respectively.

We are interested in obtaining the df of the \( \gamma \)th fos in the fractional sample size \( \sigma(k) + \gamma - n \), \( n - 1 < \gamma \leq n \), from a df \( F \) when \( N \) has a delta discrete distribution. Also, when \( F \) has pdf \( f \), an expression for the pdf \( f_{\gamma;\sigma(N)+\gamma-n} \) of the \( \gamma \)th fos is derived. Conditionally, given sample size with \( N = k \), the df of the \( \gamma \)th fos \( X_{\gamma;\sigma(k)+\gamma-n} \) is noted by \( F_{\gamma;\sigma(k)+\gamma-n}(x|k) \) and the corresponding pdf by \( f_{\gamma;\sigma(k)+\gamma-n}(x|k) \), where \( n - 1 < \gamma \leq n \).

We write the df of the \( \gamma \)th fos in a fractional sample size \( \sigma(k) + \gamma - n \) from a df \( F \) as

\[
F_{\gamma;\sigma(k)+\gamma-n}(x|k) = \sum_{j=\gamma}^{\infty} \binom{\sigma(k) + \gamma - n}{j} F^j(x)(1 - F(x))^j(\sigma(k) + \gamma - n - j). \quad (4.5)
\]

This is actually the same formula as before (4.2), since \( \binom{\sigma(k) + \gamma - n}{j} = 0 \) for \( \sigma(k) + \gamma - n < j \) when \( j \) is an integer or \( \gamma = n \). As an example, for \( \gamma = 1 \), we have the df of minimum os as

\[
F_{1;\sigma(k)}(x|k) = \sum_{j=1}^{\sigma(k)} \binom{\sigma(k)}{j} F^j(x)(1 - F(x))^j(\sigma(k) - j).
\]

Obviously,

\[
F_{\gamma;\sigma(k)+\gamma-n}(x|k) = \int_0^{F(x)} \frac{\Gamma(\sigma(k) + \gamma - n + 1)}{\Gamma(\gamma) \Gamma(\sigma(k) - n + 1)} t^{\gamma-1}(1 - t)^{\sigma(k) - n} dt.
\]
Now, we obtain fractional derivative of order $\gamma$ of $M_{\sigma(N)}(t)$. For further details on fractional operators, we refer the reader to Miller and Ross (1993). Because $f(t) = (1 + t)^{-\sigma(k)}$ is a function having a power series representation $\sum_{i=0}^{\infty} \frac{(\sigma(k))^{i}(-1)^{i}t^{i}}{\Gamma(i+1)}$ and also $D_{t}^{\gamma}t^{i} = \frac{\Gamma(i+1)}{\Gamma(i+1-\gamma)}t^{i-\gamma}$, where $D_{t}^{\gamma}$ denotes the $\gamma$th fractional derivative, we can write the $\gamma$th fractional derivative of this function as

$$D_{t}^{\gamma}(1 + t)^{-\sigma(k)} = \sum_{i=\gamma}^{\infty} \frac{(\sigma(k))^{i}(-1)^{i}t^{i-\gamma}}{\Gamma(i+1-\gamma)}.$$  

On the other hand, we have

$$\sum_{i=\gamma}^{\infty} \frac{\Gamma(k + 1 + i)(-1)^{i}t^{i-\gamma}}{\Gamma(k)\Gamma(i+1-\gamma)} = (-1)^{\gamma}(\sigma(k))^{\gamma}(1 + t)^{-\sigma(k) - \gamma}.$$  

So, we obtain

$$(-1)^{\gamma}D_{t}^{\gamma}M_{\sigma(N)}(t) = \sum_{k=\gamma}^{\infty} (\sigma(k))^{\gamma}(1 + t)^{-\sigma(k) - \gamma}P_{N}(k), \quad k \in \mathbb{N}_{\gamma-1}. \quad (4.6)$$  

The conditional df of $X_{\gamma;\sigma(N)+\gamma-n}$ on the event $\{N \geq \gamma\}$ is

$$F_{\gamma;\sigma(N)+\gamma-n}(x) = \frac{1}{\pi_{N}} \sum_{k=\gamma}^{\infty} F_{\gamma;\sigma(k)+\gamma-n}(x|k)P_{N}(k),$$  

where $\pi_{N} = P(N \geq \gamma)$. It follows from (4.6) and (4.7) that for $x \in \mathbb{R}$,

$$F_{\gamma;\sigma(N)+\gamma-n}(x) = \frac{1}{\pi_{N}} \sum_{k=\gamma}^{\infty} \left[ F(x)^{\gamma} \frac{\Gamma(\sigma(k) + \gamma - n + 1)}{\Gamma(\gamma)\Gamma(\sigma(k) - n + 1)} t^{\gamma-1}(1 + t)^{-\sigma(k) - n}P_{N}(k) \right]dt$$

$$= \frac{1}{\pi_{N} \Gamma(\gamma)} \left[ F(x)^{\gamma} \frac{\Gamma(\sigma(k) + \gamma - n + 1)}{\Gamma(\gamma)\Gamma(\sigma(k) - n + 1)} t^{\gamma-1}(1 + t)^{-\sigma(k) - n}P_{N}(k) \right]dt$$

$$= \frac{1}{\Gamma(\gamma)\pi_{N}} \left[ (-1)^{\gamma}t^{\gamma-1}D_{t}^{\gamma}M_{\sigma(\sigma(k)-n)}(t)dt, \quad \gamma \in \mathbb{N}_{\gamma-1}.\right.$$  

In the special case, when $F$ has pdf $f$, we can immediately get the pdf of the $\gamma$th fos as

$$f_{\gamma;\sigma(k)+\gamma-n}(x) = \frac{(-1)^{\gamma}f(x)F_{x}^{\gamma-1}(x)}{\pi_{N} \Gamma(\gamma)F_{x}^{\gamma+1}(x)} D_{t}^{\gamma}M_{\sigma(\sigma(k)-n)} \left( \frac{F(x)}{F(x)} \right), \quad x \in \mathbb{R}. \quad (4.8)$$

Note that in the special case $\gamma = 1/2$ we get $(-1)^{\gamma} = i$, where $i$ is unit imaginary number. In the following, we are interested in comparing fos when the sample size is a random variable. Toward this end, we define a new stochastic order based on the $\gamma$th fractional derivative of the Laplace transform of $N$.

**Definition 4.2**. Let $X$ and $Y$ be delta discrete random variables with pdfs $f$ and $g$, dfs $F$ and $G$, and sfs $\tilde{F}$ and $\tilde{G}$, respectively. Then, $X$ is said to be smaller than $Y$ in the $\gamma$—differentiated delta Laplace transform ratio order (II) (written as $X \leq_{D^{\gamma-Lr-\gamma}} Y$) if

$$\left( \frac{\tilde{F}(x)}{\tilde{G}(x)} \right)^{\gamma} \leq \left( \frac{\tilde{F}(y)}{\tilde{G}(y)} \right)^{\gamma} \quad \forall x, y \in \mathbb{R}.$$
\[
\frac{\pi_Y^\gamma L_{(s)}^\gamma(s)}{\pi_X^\gamma L_{(s)}^\gamma(s)} \text{ is decreasing in } s > 0, \quad \gamma \in \mathbb{N}_{x-1}, \quad \alpha > 1.
\]

In the special case \( \gamma = \alpha - 1 \), we have \( P(X \geq \alpha - 1) = P(Y \geq \alpha - 1) = 1 \) and the recent definition reduces to \( L_Y^{(\alpha-1)}(s) / L_X^{(\alpha-1)}(s) \).

For the rest of this paper, we denote the \( i \)th os with \( X_i \) when the sample size \( N \) is a nabla random variable, and the \( \gamma \)th fos with \( X_{\gamma:N} \) when \( N \) is a delta random variable in the fractional sample size \( \sigma(N) + \gamma - n \).

### 5. A New stochastic order based on discrete Laplace transform

If \( X \) is a delta random variable with values \( \mathbb{N}_{\alpha-1} \), we denote its delta Laplace transform with \( \mathcal{L}_X(s) \) and we have

\[
\Psi_X(s) = 1 - \mathcal{L}_X(s) = \int_{\alpha-1}^\infty (1 - e^{\sigma(t)(\alpha,0)}) f(t) \Delta t
\]

or

\[
\Psi_X(s) = \sum_{t=\alpha-1}^\infty \left( 1 - \left( \frac{1}{1 + s} \right)^{\sigma(t)} \right) f(t),
\]

which can be considered as a mixture distribution of the Burr distribution with mean \( 1/t \), \( t > -1 \) and mixing distribution \( F \). Thus, \( \Psi \) is a distribution with a density

\[
\psi_X(s) = \int_{\alpha-1}^\infty \sigma(t)(1 + s)^{\sigma(t)-1} f(t) \Delta t, \quad s > 0, \quad t > -1.
\]

The random variable with this distribution is denoted by \( \xi(X) \). Assume that \( X \) and \( Y \) are delta discrete random variables with density functions \( f \) and \( g \), respectively. Recall that \( X \) is smaller than \( Y \) in the likelihood ratio order (denoted by \( X \leq_{lr} Y \)) if \( g(x)/f(x) \) is increasing in \( x \). Note that \( -\mathcal{L}_X(s) \) and \( -\mathcal{L}_Y(s) \) are density functions of \( \xi(X) \) and \( \xi(Y) \), respectively. It is easy to see that

\[
X \leq_{d-Lt-r} Y \iff \xi(Y) \leq_{br} \xi(X)
\]

and

\[
X \leq_{Lt(Lt-r,r-Lt-r)} Y \iff \xi(Y) \leq_{st(hr, rh)} \xi(X).
\]

The definitions of the stochastic orders that are mentioned above can be found, for example, in Shaked and Shanthikumar (1994).

Consider the Laplace transform of \( F \) as

\[
\mathcal{L}_X^*(s) = \int_{\alpha-1}^\infty e^{\sigma(t)(\alpha,0)} F(t) \Delta t
\]

and define the Laplace transform of \( F \) as

\[
\mathcal{L}_X^{**}(s) = \int_{\alpha-1}^\infty e^{\sigma(t)(\alpha,0)} \bar{F}(t) \Delta t,
\]

for all \( s > 0 \). By applying Corollary 2.14 and Example 2.5 in Goodrich and Peterson (2015), it is easy to verify
\[ L_X^* = \frac{L_X}{s(1 + s)^{2-t}}, \quad L_X^{**} = \frac{1 - L_X}{s(1 + s)^{2-t}}. \] (5.3)

Also, \( L_Y, L_Y^* \) and \( L_Y^{**} \) are defined for \( Y \) like \( X \).

Let \( X \) be a delta random variable with df \( F \) and sf \( \frac{1}{1 + s} \). Denote \( R_X^*(n) = (-1)^{n-1} s^n / (n - 1)! (d^{n-1} / ds^{n-1}) L_X^{**}(s), \quad n \geq 1, \quad 0 < s, \) and let \( R_X^*(0) = 1 \) for all \( 0 < s \). \( R_X^*(n) \) is the reliability function corresponding to a discrete random variable, say, \( N_Y \). Note that, by Theorem 1.58 (Integration by part) in Goodrich and Peterson (2015), \( R_X^*(n) \) can be written as

\[
R_X^*(n) = \int_{x=1}^{\infty} \sum_{i=n}^\infty (1 - \frac{s}{1 + s})^i \frac{\sigma(x)^i}{i!} (s/1 + s)^i \Delta F(x)
= \begin{cases} 
\int_{x=1}^{\infty} \Gamma^\nu(n, \sigma(x)) \bar{F}(x) \Delta x, & n = 1, 2, \ldots \\
1, & n = 0
\end{cases}
\]

where \( \eta = s/1 + s \).

The following result gives a delta Laplace transform characterization of the order \( \leq sl \).

**Theorem 5.1.** Let \( X \) and \( Y \) be two delta random variables, and let \( N_s(X) \) and \( N_s(Y) \) be as described above. Then,

\[ X \leq sl Y \Rightarrow N_s(X) \leq sl N_s(Y). \]

If \( X \) is a nabla random variable with values \( N_1 \), we denote its nabla Laplace transform with \( L_X(s) \) and we have

\[ \Psi_X(s) = 1 - L_X(s) = \int_{0}^{\infty} (1 - e_s^*(\rho(t), 0)) f(t) \nabla t \]

or

\[ \Psi_X(s) = \sum_{i=1}^{\infty} (1 - (1 - s)^{-1}) f(t), \]

which can be considered as a mixture distribution of the beta distribution with mean \( 1/t, (t > 1) \) and a mixing distribution \( F \). Thus, \( \Psi \) is a distribution with a density function

\[ \psi_X(s) = \int_{0}^{\infty} \rho(t)(1 - s)^{\rho(t)-1} f(t) \nabla t, \quad 0 < s < 1, \quad t > 1. \]

The random variable with this distribution is denoted by \( \xi(X) \). Obviously, \( -L_X(s) \) and \( -L_Y(s) \) are density functions of \( \xi(X) \) and \( \xi(Y) \), respectively. Also, the relations (5.2) and (5.1) are valid for random variables \( \xi(X) \) and \( \xi(Y) \). Consider the Laplace transform of \( F \) as

\[ L_X^*(s) = \int_{0}^{\infty} e_s(\rho(t), 0) F(t) \nabla t \]

and define the Laplace transform of \( \bar{F} \) as
\[
L_X^* (s) = \int_0^\infty e^{st} \rho(t) F(t) dt,
\]
for all \(0 < s < 1\). By using Theorem 3.82 and Example 3.36 in Goodrich and Peterson (2015), it is easy to verify

\[
L_X^* = \frac{1}{s} L_X, \quad L_X^{**} = \frac{1 - L_X}{s}.
\]

Also, \(L_Y, L_Y^*\) and \(L_Y^{**}\) are defined for \(Y\) like \(X\).

Let \(X\) be a nabla random variable with distribution function \(F\) and \(s\)
\[F(x) = 1 - \int_x^\infty F(t) dt,\]
for all \(0 < s < 1\):

Using (5.3) and (5.4), it is easy to verify the following result.

Theorem 5.2. Let \(X\) and \(Y\) be two nabla random variables, and let \(N_s(X)\) and \(N_s(Y)\) be as described above. Then,

\[X \leq_{st} Y \Rightarrow N_s(X) \leq_{st} N_s(Y).\]

Using (5.3) and (5.4), it is easy to verify the following result.

Theorem 5.3. Let \(X\) and \(Y\) be two delta (nabla) random variables with \(F\) and \(G\), respectively. Then, \(X \leq_{Lt} Y\) if and only if \(L_X^{**} \leq L_Y^{**}\).

If \(X \leq_{Lt} Y\), then

\[1 - E \left[ (1 + s)^\rho(X) \right] /s (1 + s)^{x-1} \leq 1 - E \left[ (1 + s)^\rho(Y) \right] /s (1 + s)^{x-1}\]

for all \(s > 0\) \((0 < s < 1)\). Letting \(s \to 0\), it is seen that

\[X \leq_{Lt} Y \Rightarrow E[\sigma(X)] \leq E[\sigma(Y)] \quad \text{and} \quad X \leq_{Lt} Y \Rightarrow E[\rho(X)] \leq E[\rho(Y)]\]

provided the expectations exist.

The next proposition characterizes the orders \(\leq_{Lt-r}\) and \(\leq_{r-Lt-r}\) by functions of the respective moments.
Theorem 5.4. Let $X$ and $Y$ be delta random variables that possess moments $\mu_k^\nabla$ and $\nu_k^\nabla$, $(k = 1, 2, \ldots)$, respectively. Then,

(a) $X \leq_{L_1} Y$ if and only if
\[
\sum_{k=0}^{\infty} \frac{(-s)^k}{k!} \nu_k^\nabla \text{ is decreasing in } s.
\]
(b) $X \leq_{r-L_1} Y$ if and only if
\[
\sum_{k=1}^{\infty} \frac{(-s)^k}{k!} \nu_k^\nabla \text{ is decreasing in } s.
\]

Proof. By using the series expansion $(1 - s)^{\rho(t)} = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} (\rho(t))^k$ and $\mu_0^\nabla = \nu_0^\nabla = 1$, the result follows easily from Definition 2.3.

Remark 5.5. If $X$ and $Y$ are nabla random variables, the same result holds by substituting $\mu_0^\nabla$ and $\nu_0^\nabla$ with their corresponding nabla types. What we need to prove is the series expansion $(1 - s)^{\sigma(t)} = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} (\sigma(t))^k$ and $\mu_0^\Delta = \nu_0^\Delta = 1$.

Theorem 5.6. Let $X$ and $Y$ be delta (nabla) random variables. If $X \leq_{L_1} Y$, or if $X \leq_{r-L_1} Y$, then $L_X(s) \geq L_Y(s)$ for all $s > 0$ ($0 < s < 1$), i.e., $X \leq_{L_1} Y$.

Proof. Denote $L_X(\infty) = \lim_{s \to \infty} L_X(s)$ ($L_X(1) = \lim_{s \to 1} L_X(s)$). Since $L_X(0) = 1$ and $L_X(\infty) = 0$ ($L_X(1) = 0$) we see that if $X \leq_{L_1} Y$, then
\[
\frac{L_Y(s)}{L_X(s)} \leq \frac{L_Y(0)}{L_X(0)} = 1,
\]
and if $X \leq_{r-L_1} Y$, then
\[
\frac{1 - L_Y(s)}{1 - L_X(s)} \geq \frac{1 - L_Y(\infty)}{1 - L_X(\infty)} = 1 \left( \frac{1 - L_Y(s)}{1 - L_X(s)} \geq \frac{1 - L_Y(1)}{1 - L_X(1)} = 1 \right).
\]

This proves the stated results.

As a corollary of Theorem 5.6, we see that
\[
X \leq_{L_1} Y \rightarrow E[\sigma(X)] \leq E[\sigma(Y)], \quad X \leq_{r-L_1} Y \rightarrow E[\sigma(X)] \leq E[\sigma(Y)]
\]
\[
(X \leq_{L_1} Y \rightarrow E[\rho(X)] \leq E[\rho(Y)], \quad X \leq_{r-L_1} Y \rightarrow E[\rho(X)] \leq E[\rho(Y)]
\]
provided the expectations exist; see Equation (5.5). It is easily seen that
\[
-(1 - s)L_X(s)/E[\rho(X)], \quad \forall 0 < s < 1 \quad -(1 + s)L_X(s)/E[\sigma(X)], \quad \forall s > 0,
\]
is the nabla (delta) Laplace transform of the distribution
\[
\hat{F}(x) = \int_0^x \rho(u) \nabla F(u)/E[\rho(X)] = \int_{x-1}^{x} \sigma(u) \Delta F(u)/E[\sigma(X)],
\]
related to $F$, provided $0 < E[\rho(X)] < \infty (0 < E[\sigma(X)] < \infty)$. We denote the random variables corresponding to $\hat{F}$ and $\hat{G}$ by $\hat{X}$ and $\hat{Y}$, respectively. Then, we get $X \leq_{d-L_1} Y$ if and only if $\hat{X} \leq_{L_1} \hat{Y}$. This observation leads to this that $X \leq_{L_1} Y$ implies $X \leq_{d-L_1} Y$. 

Because \( X \leq_Y Y \) if and only if \( \bar{X} \leq_{\bar{Y}} Y \) and on the other hand \( \bar{X} \leq_{\bar{Y}} \bar{Y} \) implies \( \bar{X} \leq \bar{Y} \).

**Theorem 5.7.** Let \( n > 0 \) be a fixed integer. Let \( X_1, X_2, \ldots, X_n \) be a set of independent nabla random variables and let \( Y_1, Y_2, \ldots, Y_n \) be another set of independent nabla random variables. If \( X_j \leq_{Lt-r} Y_j, j = 1, 2, \ldots, n \), then \( X_1 + X_2 + \ldots + X_n \leq_{Lt-r} Y_1 + Y_2 + \ldots + Y_n \).

**Proof.** Since \( \mathcal{L}_{X_1+X_2+\ldots+X_n}(s) = \prod_{i=1}^{n} \mathcal{L}_{X_i}(s), \) we see that if \( \mathcal{L}_{Y_j}(s)/\mathcal{L}_{X_i}(s) \) is decreasing in \( s, j = 1, 2, \ldots, n \), then \( \mathcal{L}_{Y_1+Y_2+\ldots+Y_n}(s)/\mathcal{L}_{X_1+X_2+\ldots+X_n}(s) \) is also decreasing in \( s \).

Equations (4.4) and (4.8) give the following theorem.

**Theorem 5.8.** Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed non-negative random variables, and let \( N_1 \) and \( N_2 \) be nabla random variables which are independent of the \( X_i \). Then

\[
N_1 \leq_{Lt-r} N_2 \iff \sum_{i=1}^{n} X_i \leq_{Lt-r} \sum_{i=1}^{n} X_i.
\]

**Proof.** For \( j = 1, 2 \), we have

\[
\mathcal{L}_{X_1+X_2+\ldots+X_{N_j}}(s) = \sum_{k=1}^{\infty} P_{N_j}(k) \mathcal{L}_{X_1+X_2+\ldots+X_k}(s) = \sum_{k=1}^{\infty} P_{N_j}(k) \mathcal{L}_{X_1}^{k}(s) = \mathcal{L}_{X_1}(s) \mathcal{L}_{N_j}(1 - \mathcal{L}_{X_1}(s)).
\]

The stated results now follow from the assumptions.

**Theorem 5.9.** (a) Suppose \( N_1 \) and \( N_2 \) are nabla random variables, which are independent of \( X_i \)'s. \( N_1 \leq_{d-Lt-r} N_2 \) if and only if \( X_i;N_j \leq \bar{b}X_i;N_j, i = 1, 2, \ldots, \rho(k) \).

(b) Suppose \( N_1 \) and \( N_2 \) are delta random variables, which are independent of \( X_i \)'s. Then, we have \( N_1 \leq_{d-Lt-r} N_2 \) if and only if \( X_j;N_2 \leq \bar{b}X_j;N_1 \).

**Remark 5.10.** In the recent theorem, (a) in the special case \( i = 1 \), obviously \( P(N_1 \geq 1) = P(N_2 \geq 1) = 1 \) and so \( N_1 \leq_{d-Lt-r} N_2 \) if and only if \( X_1;N_2 \leq \bar{b}X_1;N_1 \);

(b) in the special case \( \gamma = \alpha - 1, \alpha \geq 1 \), obviously \( P(N_1 \geq \alpha - 1) = P(N_2 \geq \alpha - 1) = 1 \) and then we have \( N_1 \leq_{d(\alpha-1)-Lt-r} N_2 \) if and only if \( X_{\alpha-1};N_2 \leq \bar{b}X_{\alpha-1};N_1 \).

### 6. Applications

#### 6.1. Frailty models

In demography and survival analysis, in order to study the unobserved difference in the risk of death of individuals, frailty models were introduced to evaluate the effect of the variation on the observed hazards (Vaupel, Manton, and Stallard 1979). These models turn out to play an important role in providing insights and ideas to explain practical phenomena. The Laplace transform has an important role in the study of frailty models.
For instance, the derivatives of the Laplace transforms are used to obtain general results for the power variance function family. In frailty model, the joint survival function for a cluster of size \( n \) with covariate information \( X = (x'_1, \ldots, x'_n) \) is obtained from the joint conditional survival function by integrating out the frailty with respect to the frailty distribution as

\[
S_{X,f}(T_n) = \mathcal{L}(H_{X,c}(T_n)),
\]

where \( H_{X,c}(T_n) \) is the sum of the cumulative hazards of the \( n_i \) subjects in cluster \( i \), and \( T_n = (t_1, t_2, \ldots, t_n) \). For further details, we refer the reader to Duchateau and Janssen (2008). From the joint survival function, the joint density function for a cluster of size \( n \) with covariate information \( X = (x'_1, \ldots, x'_n) \) can be obtained as

\[
f_{X,f}(T_n) = \prod_{j=1}^{n} h_{X_i,c}(t_j)(-1)^n \mathcal{L}^{(n)}(H_{X,c}(T_n)).
\]

These expressions for the joint survival and density functions are useful in deriving the contribution of different clusters to the marginal likelihood. The likelihood contribution corresponds to the survival up to the time of censoring for censored subjects and the density at the event time for the subjects that experience the event (Duchateau and Janssen (2008)). Therefore, the conditional likelihood contribution of cluster \( i \) is given by

\[
\prod_{j=1}^{n_i} h_{x_{ij},c}(y_{ij})(-1)^{d_i} \mathcal{L}^{(d_i)} \left( \sum_{j=1}^{n_i} H_{y_{ij}}(y_{ij}) \right),
\]

where \( h_{x_{ij},c} \) is the conditional hazard function for observation \( j \), \( d_i = \sum_{j=1}^{n_i} \delta_{ij} \) is the number of events in cluster \( i \), and \( y_{ij} \) is the minimum of the event time and censoring time.

**Theorem 6.1.** Let \( X_1 \) and \( X_2 \) be two population random variables, and let \( U_1 \) and \( U_2 \) be frailty random variables. Then,

\[
U_1 \leq d^{(n)} - \text{Lt}_{-r} U_2 \iff X_2 \leq h_{r} X_1,
\]

\[
U_1 \leq d^{(n)} - \text{Lt}_{-r} U_2 \iff X_2 \leq h_{r} X_1.
\]

**6.2. Reliability**

The hazard rate ordering is stronger than the usual stochastic order for random variables, which compares lifetimes with respect to their hazard rate functions. This ordering is useful in reliability theory and survival analysis owing to the importance of the hazard rate function in these areas. The following theorem presents a property of the hazard rate ordering of random variable \( N_s(X) \).

**Theorem 6.2.** Let \( X \) and \( Y \) be two delta (nabla) random variables, and let \( N_s(X) \) and \( N_s(Y) \) be as described before, associated with nabla (delta) random variables. Then,

\[
\bar{F}_Y \leq d^{(n)} - \text{Lt}_{-r} \bar{F}_X \iff N_s(Y) \leq h_{r} N_s(X).
\]

**Proof.** Using the forms of \( R_s^X(n) \) and \( \mathcal{L}_X^s (s) \), it is easy to verify the result.
6.3. Random extremes

For a sequence of independent and identical random variables $X_1$, $X_2$, ..., and a discrete nabla random variable $N$, which is independent of $X_i$, random extremes $X_{1:N} \equiv \min_{1 \leq i \leq N}$, and $X_{N:N} \equiv \max_{1 \leq i \leq N}$ are of potential applications in reliability, transportation, economics, etc. The next theorem presents a property of some stochastic orders of random extremes.

**Theorem 6.3.** Let $X_1$, $X_2$, ..., be a sequence of non-negative independent and identically distributed random variables. Let $N_1$ and $N_2$ be two nabla random variables, which are independent of the $X_i$.

(a) $N_1 \leq_{\text{Lt-Dt}} N_2$ if and only if $X_{1:N_1} \leq_{\text{hr}} X_{1:N_2}$, $X_{N_1:N_1} \leq_{\text{hr}} X_{N_2:N_2}$;

(b) $N_1 \leq_{\text{Lt-Dt}} N_2$ if and only if $\sum_{i=1}^{\rho(N_1)} X_i \leq_{\text{Lt-Dt}} \sum_{i=1}^{\rho(N_2)} X_i$;

(c) If $N_1 \leq_{\text{Lt}} N_2$, then $X_{1:N_1} \geq_{hr} X_{1:N_2}$, $X_{N_1:N_1} \leq_{hr} X_{N_2:N_2}$;

(d) If $N_1 \leq_{r_{\text{Lt}-r_{\text{Dt}}}} N_2$, then $X_{1:N_1} \geq_{hr} X_{1:N_2}$, $X_{N_1:N_1} \leq_{hr} X_{N_2:N_2}$.

**Proof.** (a) By using Equation (4.4), for $j = 1, 2$ we can write

$$f_{X_{1:N_j}}(x) = -f(x) \sum_{k=1}^{\infty} \rho(k) (1 - F(x))^{\rho(k)-1} P_{N_j}(k)$$

$$= \frac{d}{dx} \mathcal{L}_{N_j}(F).$$

Since $f_{X_{1:N_2}}(x)/f_{X_{1:N_1}}(x) = (d/dx) \mathcal{L}_{N_2}(F)/(d/dx) \mathcal{L}_{N_1}(F)$ has the common monotone as $\mathcal{L}_{N_2}(x)/\mathcal{L}_{N_1}(x)$, the desired result follows immediately. Note that Equation (4.4) is applicable only for $i = 1, 2, ..., \rho(k)$, so for the case $i = k$ we get

$$f_{X_{1:N_j}}(x) = \frac{d}{dx} \left\{ F(x) \sum_{k=1}^{\infty} (1 - F(x))^{\rho(k)} P_{N_j}(k) \right\}$$

$$= \frac{d}{dx} \left\{ F(x) \mathcal{L}_{N_j}(F) \right\}$$

and the proof runs a similar manner and hence was omitted. In order to prove (b), we can write

$$\frac{d}{ds} \mathcal{L}_{X_{1} + \ldots + X_{\rho(N_j)}}(s) = \sum_{k=1}^{\infty} \rho(k) P_{N_j}(k) \mathcal{L}_{X_{1}}^{\rho(k)-1}(s) \mathcal{L}_{X_{1}}^{-1}(s)$$

$$= \mathcal{L}_{X_{1}}^{-1}(s) \sum_{k=1}^{\infty} P_{N_j}(k) (1 - (1 - \mathcal{L}_{X_{1}}^{-1}(s)))^{\rho(k)-1}$$

$$= \frac{d}{ds} \mathcal{L}_{N_j}(1 - \mathcal{L}_{X_{1}}^{-1}(s)),$$

then, for all $s \geq 0$,

$$\frac{d}{ds} \mathcal{L}_{X_{1} + \ldots + X_{\rho(N_j)}}(s) = \frac{d}{ds} \mathcal{L}_{N_j}(1 - \mathcal{L}_{X_{1}}^{-1}(s)) \frac{d}{ds} \mathcal{L}_{N_j}(1 - \mathcal{L}_{X_{1}}^{-1}(s))^{-1}.$$
To prove (c), note that

$$\bar{F}_{X_1,N_j} = \sum_{k=1}^{\infty} (1 - F(x))^q(k) P_N(k) = \mathcal{L}_{N_j}(F(x)), \quad j = 1, 2.$$  

In order to prove $X_{1:N_1} \geq_{hr} X_{1:N_2}$, we need to show that $\bar{F}_{X_{1:N_2}}(x)/\bar{F}_{X_{1:N_1}}(x)$ is decreasing in $x \geq 0$, but this follows from the fact that

$$\frac{\bar{F}_{X_{1:N_2}}(x)}{\bar{F}_{X_{1:N_1}}(x)} = \frac{\mathcal{L}_{N_2}(F(x))}{\mathcal{L}_{N_1}(F(x))},$$

and from $N_1 \leq_{lt} r N_2$. Similarly, we have $\bar{F}_{X_{N_2}} = \bar{F}(x) \mathcal{L}_{N_j}(F)$. The proof of (d) is similar.

\[\square\]

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