Ramification groups of some finite Galois extensions of maximal nilpotency class over local fields of positive characteristic

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0 Introduction

Let $K$ be a complete discrete valuation field of equal characteristics $p > 0$. Assume that the residue field $k$ of $K$ is perfect. For a finite Galois extension of $K$, upper ramification groups are defined in [7], VI, §3. We are interested in the jumps of the upper ramification groups.

Let $L/K$ be an Artin-Schreier extension defined by $x^p - x = a$ for $a \in K$ such that the valuation $v(a)$ of $a$ is negative and prime to $p$. Then it is well known that the unique jump of the upper ramification groups is located at $-v(a)$. The number $-v(a)$ is the conductor of the extension $L/K$. Furthermore, the calculation of the conductor was generalized by Brylinski [4] to the cases where the extension is abelian. We give an analog in some non-abelian cases. Our result is related to that of Abrashkin [2], but our approach is totally different and the proofs are more elementary.

Fix an integer $2 \leq n \leq p$. Let $A$ be the nilpotent matrix of order $n$ defined by

$$A = (\delta_{i,j-1})_{ij} \in M(n)$$

where $\delta$ denotes the Kronecker delta and $M(n)$ denotes the group of matrices of order $n$. Let $R(t,x) = \sum_{i=0}^{p-1} \binom{x}{i} t^i \in \mathbb{F}_p[t,x]$ be the $(p - 1)$-st Maclaurin polynomial of $(1 + t)^x$ with respect to $t$. Define a morphism $A : \mathbb{G}_a \to GL(n)$ of algebraic groups by $A(x) = R(A,x) \in GL(n)$. Let $G \subset GL(n+1)$ be the unipotent algebraic group of dimension $n + 1$ over $\mathbb{F}_p$ defined by

$$G = \left\{ \begin{pmatrix} A(x) & y \\ 0 & 1 \end{pmatrix} \bigg| x \in \mathbb{G}_a, \ y \in \mathbb{G}_a^n \right\}.$$  \tag{0.2}

The order of the group $G(\mathbb{F}_p)$ is $p^{n+1}$ and $G$ is of nilpotency class $n$. When $n = 2$, $G$ is nothing but the algebraic group of upper triangular unipotent matrices of order
3. The group $G(\mathbb{F}_p)$ is one of the two non-abelian $p$-groups of the smallest order up to isomorphism.

We study a totally ramified Galois extension $M_n/K$ whose Galois group $G_{M_n/K}$ is isomorphic to $G(\mathbb{F}_p)$. Let $F : \mathbb{G}_a^n \to \mathbb{G}_a^n$ denote the component-wise Frobenius map. We show in Lemma 1.9 that there exist
\begin{equation}
a \in K, \alpha \in M_n, b \in K^n, \gamma = \begin{pmatrix} \gamma_n \\ \vdots \\ \gamma_1 \end{pmatrix} \in M_n^n, \quad (0.3)
\end{equation}
such that
\begin{equation}
\alpha^p - \alpha = a \in K, F(\gamma) - \gamma = A(-\alpha^p)b =: c = \begin{pmatrix} c_n \\ \vdots \\ c_1 \end{pmatrix} \in K(\alpha)^n. \quad (0.4)
\end{equation}
and $M_n = K(\alpha, \gamma_1, \ldots, \gamma_n)$. Let $r_n$ denote the largest rational number such that the $r_n$-th upper ramification group of $M_n/K$ is non-trivial. We may take $a, b$ satisfying the conditions of Lemma 3.2 without loss of generality. The aim of this paper is to express $r_n$ in terms of these $a, b$.

This extension is of maximum nilpotency class, i.e., the descending central series of its Galois group is the longest among the groups of the same order. Combined with the abelian cases, the calculation of the ramification groups for this case should give some insight when calculating the ramification groups of non-abelian Galois extensions in general. This is a motivation for us to consider the extension $M_n/K$.

This paper consists of three parts: Section 1, Section 2, and Section 3. In the first two sections, we give some preliminary results and settings. In the last section, we prove our main theorem, Theorem 3.3.

In Section 1 we give some basic results on Galois cohomology required to prove the main theorem. Using Galois cohomology, we give a combination of $a \in K, b \in K^n$ defining $M_n/K$. At the last of this section, we will give some calculations required in the last section.

In Section 2 we introduce filtrations $F_r K, F_r \Omega^1_K, F_r H^1(K)$ on $K, \Omega^1_K, H^1(K) := H^1(K, \mathbb{F}_p)$, and the graded modules $\text{Gr} K, \text{Gr} \Omega^1_K, \text{Gr} H^1(K)$ associated to these filtrations as in [1]. We investigate the relationship between the graded modules defined for $K$ and $L$, where $L/K$ is an Artin-Schreier extension. This is required since $M_n/K(\alpha)$ is a composition of $n$ Artin-Schreier extensions.

In Section 3 we give our main result, i.e., the calculation of $r_n$. We reduce the problem to the calculation of the conductor $m'_n$ of the Artin-Schreier extension of $L = K(\alpha)$ defined by $x^p - x = c_n$ under some conditions on $a, b$ as in Lemma 3.2. Take $c'_n, c''_n \in M_n$ such that
\begin{equation}
c_n - c''_n + c' = c'_n, -v_L(c'_n) = m'_n. \quad (0.5)
\end{equation}
Let
\begin{equation}
s_n = \max \left( -v_L \left( t \frac{dc_n}{dt} \right), -v_L \left( t \frac{dc''_n}{dt} \right) \right), \quad (0.6)
\end{equation}
where $t$ denotes a uniformizer of $L$. First, we express $s_n$ in terms of $a, b$ in Corollary 3.7(b). The main ingredient of the proof of Theorem 3.3 is to prove that the inequality in
\begin{equation}
m'_n = -v_L \left( t \frac{dc'_n}{dt} \right) \leq s_n \quad (0.7)
\end{equation}
is actually an equality, by calculating the image of $c_n$ in $\text{Gr}_{s_n} H^1(L)$ using the results from Section 2.

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1 Preliminaries

Let $B, C$ be groups, $\Gamma$ a profinite group acting continuously on $B$ and $C$, and $f, g : B \to C$ group homomorphisms preserving the actions of $\Gamma$. Assume that the map $h : B \to C$ defined by $y \mapsto f(y)g(y)^{-1}$ is a surjection. Let $A$ denote the inverse image of $\{1\}$ by $h$. Then $A$ is a subgroup of $B$ with a continuous action of $\Gamma$.

We consider the following sequence:

$$\{1\} \to A \hookrightarrow B \xrightarrow{h} C \to \{1\}. \quad (1.1)$$

This is “exact” in the sense that the images of the maps coincide with the inverse image of $\{1\}$ by the next map. Furthermore, the map $B \times A \to B \times_C B$ defined by $(y, x) \mapsto (y, yx)$ is a bijection. Nevertheless, this is not an exact sequence, because $h$ is not in general a group homomorphism. However, we can still consider the “long exact sequence of cohomology” for this “exact” sequence as in [7], VII, Annex.

**Proposition 1.1.** (a) For $y \in B$ such that $h(y) \in C^\Gamma$, the map $\zeta_y : \Gamma \to B$ defined by $\sigma \mapsto y^{-1}\sigma(y)$ is a 1-cocycle of $A$. Moreover, we can define a map $\delta : C^\Gamma \to H^1(\Gamma, A)$ as follows:

$$\delta(z) = \tilde{\zeta}_y, \quad (1.2)$$

where $h(y) = z \in C^\Gamma$, and $\tilde{\zeta}_y$ denotes the class of $\zeta_y$ as a 1-cocycle of $A$.

(b) The image of the map $\delta : C^\Gamma \to H^1(\Gamma, A)$ coincides with the inverse image of $\{1\}$ by $H^1(\Gamma, A) \to H^1(\Gamma, B)$.

(c) Take $z, z' \in C^\Gamma$. Let $\Gamma_z$ and $\Gamma_{z'}$ denote the intersections of the stabilizer subgroups of $\Gamma$ with respect to the elements in $h^{-1}(\{z\})$ and $h^{-1}(\{z'\})$ respectively. Then $\Gamma_z = \Gamma_{z'}$ if and only if there exists $y \in B^{\Gamma_z\Gamma_{z'}}$ such that $z' = f(y)z g(y)^{-1}$.

**Proof.** (a) For any $y \in B$ such that $h(y) \in C^\Gamma$, we have

$$\{yx \in B|x \in A\} = h^{-1}(\{h(y)\}) \supset \{\sigma(y)|\sigma \in G\}. \quad (1.3)$$

Thus, for any $\sigma \in \Gamma$, we have $y^{-1}\sigma(y) \in A$. Since for any $\sigma, \tau \in \Gamma$,

$$y^{-1}\sigma(y)\sigma(y^{-1}\tau(y)) = y^{-1}\sigma\tau(y), \quad (1.4)$$

$\zeta_y$ is a 1-cocycle of $A$. For any $x \in A$, $\zeta_{yx}$ is cohomologous to $\zeta_y$, since

$$\zeta_{yx}(\sigma) = x^{-1}\zeta_y(\sigma)x. \quad (1.5)$$

Thus $\delta$ is well-defined.

(b) Take a 1-cocycle $s : \Gamma \to A$ of $A$.  

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Assume that $H^1(\Gamma, A) \to H^1(\Gamma, B)$ sends the class $\bar{s}$ of $s$ to $1$. Since $A \subseteq B$, this implies that there exists $y \in B$ such that $s(\sigma) = y^{-1}\sigma(y)$. Since $s(\sigma) \in A$, we have

$$h(y) = f(y)g(y)^{-1} = f(y)h(s(\sigma))g(y)^{-1} = f(y)f(y^{-1}\sigma(y))g(y^{-1}\sigma(y))^{-1}g(y)^{-1}$$

$$= f(\sigma(y))g(\sigma(y))^{-1} = \sigma(f(y)g(y)^{-1}) = \sigma(h(y)).$$ (1.6)

Hence $h(y) \in C^\Gamma$ and $\delta(h(y)) = \bar{s}$.

Conversely, take $z \in C^\Gamma$. Let $y$ denote an element of $B$ satisfying $h(y) = z$ and let $s = \zeta_y$. Then we have $s(\sigma) = g^{-1}\sigma(y)$ for all $\sigma \in \Gamma$ and $\delta(z) = s$. Since this is a $B$-coboundary, we have that $H^1(\Gamma, A) \to H^1(\Gamma, B)$ sends $\bar{s}$ to $1$.

Assume $\Gamma_z = \Gamma_{z'}$. Then for all $\sigma \in \Gamma_z\Gamma_{z'} = \Gamma_z = \Gamma_{z'}$, $y_0 \in h^{-1}\{\{z\}\}$, and $y_0' \in h^{-1}\{\{z'\}\}$, we have $(y_0'y_0^{-1})\sigma(y_0'y_0^{-1}) = 1$ in $B$. Therefore we have $y_0'y_0^{-1} \in B^\Gamma\Gamma_{z'}$ for all $y_0 \in h^{-1}\{\{z\}\}$ and $y_0' \in h^{-1}\{\{z'\}\}$. Meanwhile, we have

$$f(y_0'y_0^{-1})g(y_0'y_0^{-1})^{-1} = f(y_0')f(y_0)^{-1}h(y_0')g(y_0'y_0^{-1})^{-1}$$

$$= f(y_0')g(y_0)^{-1} = h(y_0') = z'.$$ (1.7)

Therefore setting $y = y_0'y_0^{-1}$ for some $y_0 \in h^{-1}\{\{z\}\}$ and $y_0' \in h^{-1}\{\{z'\}\}$ yields $z' = f(y)g(y)^{-1}$.

Conversely, assume that there exists $y \in B^\Gamma\Gamma_{z'}$ such that $z' = f(y)g(y)^{-1}$. Take $y_0 \in h^{-1}\{\{z\}\}$ and let $y_0' = y_0$. Then $y_0' \in h^{-1}\{\{z'\}\}$, since $h(y_0') = f(y)h(y_0')g(y_0') = z'$. Since $y$ is fixed by $\Gamma_z\Gamma_{z'} \subseteq \Gamma_z\Gamma_{z'}$, we have $y_0'^{-1}\sigma(y_0') = y_0^{-1}\sigma(y_0) = 1$ for all $\sigma \in \Gamma_z\Gamma_{z'}$. Thus, $\Gamma_{z'} \supseteq \Gamma_z\Gamma_{z'}$. Hence, $\Gamma_{z'} \supseteq \Gamma_{z'}$. By symmetry, we also have $\Gamma_{z'} \supseteq \Gamma_z$. Thus, we have $\Gamma_{z'} \supseteq \Gamma_z$. □

**Proposition 1.2.** Let $G$ be a group. Assume that $G$ admits a descending normal series of subgroups $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_r = \{1\}$. For $0 \leq i < r$, let $\pi_i$ denote the canonical projection $G_i \to G_i/G_{i+1}$. Let $f, g : G \to G$ be group homomorphisms satisfying $f^{-1}(G_i) = g^{-1}(G_i) = G_i$ for all $i$. Define a map $h : G \to G$ by $y \mapsto f(y)g(y)^{-1}$, and for all $i$, let $h_i^{\prime} : G_i/G_{i+1} \to G_i/G_{i+1}$ be the morphism induced by $h$ and $\pi_i$. Then $h$ is surjective if $h_i^{\prime}$ is surjective for all $i$.

**Proof.** We prove this proposition by induction on the length $r$ of the descending normal series of $G$. If $r = 1$, then we have $G_1 = \{1\}$ and $\pi$ is the identity map. Hence $h$ is surjective if $h_0^{\prime}$ is surjective.

Suppose $n > 1$. Since $G_1$ admits a descending normal series of length $r - 1$, it suffices to show that $h$ is surjective if $h|_{G_1}$ and $h_0^{\prime}$ are surjective.

Suppose $h|_{G_1}$ and $h_0^{\prime}$ are surjective. Consider the following commutative diagram with exact rows.

$$\begin{array}{c}
\{1\} \xrightarrow{h|_{G_1}} G_1 \xrightarrow{h} G \xrightarrow{\pi} G/G_1 \xrightarrow{\{1\}} \\
\{1\} \xrightarrow{h_0^{\prime}} G_1^{\prime} \xrightarrow{h_0} G \xrightarrow{\pi} G/G_1 \xrightarrow{\{1\}}
\end{array}$$ (1.8)

We will prove that $h$ is surjective by a technique similar to that used to prove the five lemma. Note that we cannot simply apply the five lemma, because $h$ is not in general a homomorphism.

Take $c' \in G$. Since $h_0^{\prime}$ and $\pi$ are surjective, there exists $c \in G$ satisfying $h_0^{\prime}(\pi(c)) = \pi(c')$. By the commutativity of the diagram, $\pi(f(c)g(c)^{-1}) = \pi(c')$. Since $\pi$ is a homomorphism, $\pi(f(c)^{-1}c'g(c)) = 1$. Then by the exactness of the lower row, we have
\( f(c)^{-1}c'g(c) \in G_1 \). Since \( h|_{G_1} \) is surjective, there exists \( b \in G_1 \) satisfying \( h(b) = f(c)^{-1}c'g(c) \). We have

\[
h(cb) = f(c)h(b)g(c)^{-1} = f(c)f(c)^{-1}c'g(c)g(c)^{-1} = c'.
\]

(1.9)

Thus, \( h \) is surjective. \( \square \)

For a field \( K \) of characteristic \( p > 0 \), let \( K_s \) denote a separable closure of \( K \) respectively, and \( G_K = \text{Gal}(K_s/K) \) the absolute Galois group of \( K \). Let \( G \) be a unipotent group over \( K \), i.e., an algebraic subgroup of the group of unitriangular \( n \times n \) matrices over \( K \) for some \( n \).

**Definition 1.3.** We say that a unipotent group \( G \) over \( K \) is split if it admits a finite descending normal series of subgroups whose quotients are isomorphic to the additive group \( \mathbb{G}_a \) (cf. \([3]\), Definition 15.1).

**Remark 1.4.** By \([3]\), Theorem 15.5(ii), every connected unipotent group is split if \( K \) is perfect. However, this is not true if \( K \) is not perfect. According to \([3]\), V.3.4, if \( K \) is not perfect and \( t \) is an element of \( K - K^p \), then the algebraic subgroup \( \{ (x, y) \mid x^p - x - ty^p = 0 \} \) of \( \mathbb{G}_a \times \mathbb{G}_a \) is not split.

**Proposition 1.5.** (a) (Nguyễn) \([8]\) A connected unipotent group \( G \) over \( K \) is split if and only if \( H^1(G_L, G) = \{ 1 \} \) for every extension \( L/K \).

(b) Let \( G \) be a split unipotent group over \( F_p \). Let \( F: G \to G \) denote the morphism defined by the absolute Frobenius and let \( P: G \to G \) denote the map defined by \( y \mapsto F(y)y^{-1} \). Then \( P \) is surjective.

**Proof.** \([8]\) See \([8]\).

Since \( K_s \to K_s \) defined by \( x \mapsto x^p - x \) is surjective, by Proposition \( 1.2 \) \( P \) is surjective. \( \square \)

**Definition 1.6.** Fix an integer \( 2 \leq n \leq p \).

(a) Let \( A \) be the nilpotent matrix of order \( n \) defined by

\[
A = (\delta_{i,j-1})_{ij} \in M(n)
\]

where \( \delta \) denotes the Kronecker delta and \( M(n) \) denotes the group of the matrices of order \( n \). Let \( R(t, x) \) be the \((p - 1)\)-st Maclaurin polynomial of \((1 + t)^x\) with respect to \( t \), i.e.,

\[
R(t, x) = \sum_{i=0}^{p-1} \binom{x}{i} t^i \in F_p[t, x].
\]

(1.11)

Define a morphism \( A: \mathbb{G}_a \to GL(n) \) of algebraic groups by \( A(x) = R(A, x) \in GL(n) \).

(b) Let \( G \subset GL(n+1) \) be the unipotent algebraic subgroup of dimension \( n + 1 \) over \( F_p \), defined by

\[
G = \left\{ \begin{pmatrix} A(x) & y \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{G}_a, \ y \in \mathbb{G}_a^n \right\}.
\]

(1.12)

Let

\[
G \supseteq Z_1G \supseteq \cdots \supseteq Z_nG = \{ 1 \}
\]

(1.13)

be the lower central series of \( G \).
Remark 1.7. We have for all $1 \leq j \leq n$,

$$Z_j G = \left\{ \left( \begin{array}{c} A(x) \\ 0 \\ 0 \\ 0 \\ y \\ 1 \\ 1 \\ \vdots \\ y_1 \\ 1 \\ \vdots \\ y_n \\ 1 \\ \vdots \\ y_j \\ 1 \\ \vdots \\ y_1 \\ 1 \\ \vdots \\ y_1 \\ \vdots \\ y_1 \\ 1 \end{array} \right) \in G ~|~ y = \left( \begin{array}{c} y_n \\ \vdots \\ y_1 \end{array} \right), \ x = y_1 = \cdots = y_j = 0 \right\}. \quad (1.14)$$

Moreover, $G$ is split since $F$ is perfect.

Definition 1.8. Fix an integer $2 \leq n \leq p$. Let $K$ be a field. Let $M_n/K$ denote a Galois extension whose Galois group is isomorphic to $G(\mathbb{F}_p)$. Let $K \subset M_1 \subset \cdots \subset M_n$ be the Galois subextensions of $M_n/K$ corresponding to the lower central series of $G(\mathbb{F}_p)$.

Assume that $K$ is of characteristic $p > 0$. We will apply Proposition 1.1 to $B = C = G(K_s)$, $\Gamma = G_K$, $f = F$, $g = \text{id}_{G(K_s)}$, where $G_K$ denotes the absolute Galois group over $K$, and $F : G(K_s) \to G(K_s)$ denotes the Frobenius map. By Proposition 1.5, the map $h : G(K_s) \to G(K_s)$ defined by $y \mapsto F(y)y^{-1}$ is a surjection. Thus we can apply Proposition 1.1.

We have $A = h^{-1}(\{1\}) = G(\mathbb{F}_p)$. Since $G_K$ acts on $G(\mathbb{F}_p)$ trivially, we can identify $H^1(G_K, G(\mathbb{F}_p))$ with the set of conjugacy classes of $\text{Hom}(G_K, G(\mathbb{F}_p))$ by $G(\mathbb{F}_p)$.

Lemma 1.9. (a) There exist $a \in K$, $b \in K^n$ such that the extension $M_n/K$ is defined by

$$\left( \begin{array}{c} A(x^p) \\ 0 \\ 1 \\ \vdots \\ y \\ 1 \\ \vdots \\ y_1 \\ 1 \\ \vdots \\ y_1 \\ 1 \end{array} \right) = \left( \begin{array}{c} A(a) \\ 0 \\ 1 \\ \vdots \\ y \\ 1 \\ \vdots \\ y_1 \\ 1 \end{array} \right), \quad (1.15)$$

where $F : \mathbb{G}_a^n \to \mathbb{G}_a^n$ denotes the component-wise Frobenius map.

(b) Take

$$\alpha \in K_s, \ \gamma = \left( \gamma_n \vdots \gamma_1 \right) \in K_s^n \quad (1.16)$$

such that

$$\alpha^p - \alpha = a, \ F(\gamma) - \gamma = A(-\alpha^p)b. \quad (1.17)$$

Then for all $1 \leq j \leq n$, we have $M_j = K(\alpha, \gamma_1, \ldots, \gamma_j)$.

Proof. (a) Let $\pi : G_K \to G_{M_n/K}$ denote the canonical projection. Let $\phi : G_{M_n/K} \to G(\mathbb{F}_p)$ be any isomorphism. Define $\delta : G(K) \to H^1(G_K, G(\mathbb{F}_p))$ as in Proposition 1.1. By Proposition 1.1 and Proposition 1.5, $\delta$ is a surjection. Thus, there exists

$$T = \left( \begin{array}{c} A(a) \\ 0 \\ 1 \end{array} \right) \in G(K) \quad (1.18)$$

such that $\delta(T)$ equals the conjugacy class of $\phi \circ \pi$. Then by definition of $\delta$ in Proposition 1.1, $M_n/K$ is defined by (1.15).

(b) By applying Proposition 1.1 to $K(\alpha)$, we may replace $T$ with

$$\left( \begin{array}{c} A(-\alpha^p) \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} A(a) \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} A(-\alpha) \\ 0 \\ 1 \end{array} \right)^{-1} = \left( \begin{array}{c} A(0) \\ 0 \\ 1 \end{array} \right) \left( A(-\alpha^p)b \right) \quad (1.19)$$

Thus $M_n = K(\alpha, \gamma_1, \ldots, \gamma_n)$. Note that $G_{M_n/M_j}$ acts trivially on $\alpha, \gamma_1, \ldots, \gamma_j$, but not trivially on $\gamma_{j+1}, \ldots, \gamma_n$. Therefore, $M_j = K(\alpha, \gamma_1, \ldots, \gamma_j)$. \qed
Lemma 1.10. Let \( l_1, \ldots, l_n \) and \( \lambda_1, \ldots, \lambda_n \) be sequences of integers and \( \lambda \) an integer. Assume that for all \( 2 \leq i \leq n \), they satisfy the following conditions:

(i) If \( l_i < -(p-1)\lambda + p\lambda_i \), then we have \( \lambda_{i-1} = -\lambda + \lambda_i \).

(ii) We have \( l_i \leq -(n-i)\lambda + \lambda_n \).

(iii) We have \( \lambda < \lambda_i \).

Then we have \( l_i < -(p-1)\lambda + p\lambda_i \), \( \lambda_i = (i-1)\lambda + \lambda_1 \) \( (1.20) \) and \( \lambda_i = (i-1)\lambda + \lambda_1 \) \( (1.21) \) for all \( 2 \leq i \leq n \).

Proof. We prove this lemma by induction on \( n \). It is clear that the lemma holds for \( n = 1 \).

Suppose \( n = j \). Assume that the lemma holds for \( n = j - 1 \).

By (ii) for \( i = j \), we have \( l_j \leq \lambda_j \). \( (1.22) \)

By (iii) for \( i = j \), we have \( \lambda_j < -(p-1)\lambda + p\lambda_j \). \( (1.23) \)

Thus we get \( (1.20) \) for \( i = j \). By (i) for \( i = j \), we have

\[
-(j - 1 - i)\lambda + \lambda_{j-1} = -(j - i)\lambda + \lambda_j.
\]

The integer \( \lambda \) and the sequences \( l_1, \ldots, l_{j-1} \) and \( \lambda_1, \ldots, \lambda_{j-1} \) clearly satisfy conditions (i) and (iii) of the lemma for \( n = j - 1 \). We will show that they also satisfy condition (iii).

By (1.21) and (ii) for \( 2 \leq i \leq j - 1 \), we have

\[
l_i \leq -(j - 1 - i)\lambda + \lambda_{j-1}
\]

for all \( 2 \leq j \leq i - 1 \). Therefore, by induction hypothesis, we have \( (1.20) \) and \( (1.21) \) for all \( 2 \leq j \leq i - 1 \). By \( (1.21) \) for \( i = j - 1 \) and \( (1.24) \), we have \( (1.21) \) for \( i = j \).

Hence, the lemma also holds for \( n = j \).

\[\square\]

Lemma 1.11. We have an equality

\[
\sum_{i=1}^{j} \frac{(-1)^{i-1}y}{(j-i)!(i-1)!(x+(i-1)y)} = \prod_{i=1}^{j} \frac{y}{x+(i-1)y}
\]

in \( \mathbb{Z}[1/(j-1)!], x, y, \frac{1}{\prod_{i=1}^{j} x+(i-1)y} \).

Proof. We may assume \( y = 1 \). Then it follows from the fact that the polynomial of degree \( j-1 \),

\[
f(x) = \sum_{i=1}^{j} \left( \frac{(-1)^{i-1}}{(j-i)!(i-1)!} \prod_{1 \leq i' \leq j, i' \neq i} (x+(i'-1)) \right)
\]

satisfies \( f(0) = f(-1) = \cdots = f(-(j-1)) = 1 \). \[\square\]
2 Filtrations

Let $K$ be a complete discrete valuation field of characteristic $p > 0$ with perfect residue field $k$ and $K_*$ a separable closure of $K$. Define filtrations $F_nK$ of $K$ as $F_nK = \mathfrak{m}_K^{\geq n}$.

Let $P : K_* \to K_*$ be the surjective map defined by $P(x) = x^p - x$. Then by Proposition 1.1 and Proposition 1.2, we can identify the cokernel of $P|_K : K \to K$ with $H^1(K) := H^1(G_K, \mathbb{F}_p) = \text{Hom}(G_K, \mathbb{F}_p)$ by the following isomorphism:

$$
\text{Coker } P|_K \to \text{Hom}(G_K, \mathbb{F}_p) \\
\begin{array}{c}
x \\
\mapsto (\sigma \mapsto \sigma(y) - y)
\end{array}
$$

(2.1)

where $y \in K_*$ satisfies $P(y) = x$.

Consider the map $K \to H^1(K)$ defined by the projection $K \to K/P(K) = \text{Coker } P|_K$. Let $F_nH^1(K)$ be the image of $F_n(K)$ by this map in $H^1(K) := H^1(K, \mathbb{F}_p)$.

Let $Gr_nK = F_nK/F_{n-1}K, Gr_nH^1(K) = F_nH^1(K)/F_{n-1}H^1(K)$ denote the graded quotients, and define the graded algebra $Gr K := \oplus_{n \in \mathbb{Z}} Gr_n K$. The graded algebra $Gr K$ is isomorphic to $k[t, t^{-1}]$.

The space $\Omega^1_K$ of Kähler differentials is a 1-dimensional $K$-vector space, and its submodule $\Omega^1_K$ is a free $\mathcal{O}_K$-module of rank 1. Let $d : K \to \Omega^1_K$ denote the canonical derivation. Let $F_n\Omega^1_K = \mathfrak{m}_K^{n-1}\Omega^1_K$ and define the graded quotient $Gr_n\Omega^1_K$ and the graded module $Gr \Omega^1_K$ as above. Note that this graded module is a $Gr K$ module. For $\chi \in \Omega^1_K$, let $v_K(\chi)$ denote the smallest integer $n$ such that $\chi \in F_{-n}\Omega^1_K$.

Lemma 2.1. (a) Let $t$ denote a uniformizer of $K$. The multiplication $K \to \Omega^1_K$ by $t^{-1}dt$ induces an isomorphism $\mu : Gr_nK \to Gr_n\Omega^1_K$. This isomorphism does not depend on the uniformizer $t$.

(b) The derivation $d$ induces a morphism $\partial : Gr_nK \to Gr_n\Omega^1_K$. We have $\partial = -n \cdot \mu$.

Proof. The multiplication by $t^{-1}dt$ induces an isomorphism $\mu$, since the multiplication is clearly an isomorphism and $F_n\Omega^1_K = t^{-1}dtF_nK$. For any uniformizer $t, t'$ of $K$, there exist $0 \neq a, b \in k$ and $c \in \mathcal{O}_K$ satisfying $t' = at + bt^2$. Hence we have

$$
(t')^{-1}dt' = t^{-1}(a + bt)^{-1}(adt + d(bt^2)).
$$

(2.2)

Since $(a + bt)^{-1} \in a^{-1} + \mathfrak{m}$ and $d(bt^2) \in \mathfrak{m}dt$, we have

$$
t^{-1}dt' \equiv t^{-1}dt \mod \mathcal{O}_Kdt.
$$

(2.3)

Therefore, $\mu$ does not depend on the uniformizer.

Let $t$ be a uniformizer of $K$. Since $d(t^{-n}) = -nt^{-n-1}dt$, $d$ induces $\partial$, and we have $\partial = -n \cdot \mu$.

Lemma 2.2. Let $L/K$ be a finite separable totally ramified extension of complete discrete valuation fields with residue field $k$, $\epsilon$ the ramification index of $L/K$, and $\delta$ the valuation of the different of $L/K$. For any integer $n$, let $n' = e(n + 1) - \delta - 1$.

The canonical morphisms $K \to L$ and $\Omega^1_K \to \Omega^1_L$ induce $F_nK \to F_{en}L, F_{n-1}K \to F_{en-1}L, F_n\Omega^1_K \to F_n\Omega^1_L$, and $F_{n-1}\Omega^1_K \to F_{n-1}\Omega^1_L$. There exists a unique non-zero
element $\theta \in \text{Gr}_{e-\delta-1} L \simeq k$ such that for all $n$, the diagram below is commutative, where $\mu$ is the morphism in Lemma 2.1(a)

$$\begin{array}{ccccc}
\text{Gr}_n K & \xrightarrow{\mu} & \text{Gr}_n L & \xrightarrow{\theta} & \text{Gr}_n' L \\
\downarrow & & \downarrow & & \\
\text{Gr}_n \Omega^1_K & \xrightarrow{\mu} & \text{Gr}_n' \Omega^1_L.
\end{array}$$

Moreover, for all $\chi \in \Omega^1_K$ such that $-v_K(\chi) = n$, we have $-v_L(\chi) = n' = c(n+1) - \delta - 1$.

Proof. Let $t_K$ and $t_L$ denote a uniformizer of $K$ and $L$ respectively. Let

$$\lambda = \frac{t_K^{-1} dt_K}{t_L^{-1} dt_L} \in F_{e-\delta-1} L.$$  

(2.5)

The following diagram is commutative:

$$\begin{array}{ccc}
K & \xrightarrow{\lambda} & L \\
\downarrow & & \downarrow \\
\Omega^1_K & \xrightarrow{\Omega^1_L},
\end{array}$$

(2.6)

where the lower horizontal arrow denotes the canonical morphism, and the left and right vertical arrows denote the multiplication by $t_K^{-1} dt_K$ and $t_L^{-1} dt_L$ respectively. Thus setting $\theta \in \text{Gr}_{e-\delta-1} L$ as the image of $\lambda$ makes the diagram (2.1) commutative. It follows from Lemma 2.1(a) that $\theta$ does not depend on the choices of $t_K$ and $t_L$. \hfill \Box

Lemma 2.3. Let $n$ be an integer. Consider $\nu : \text{Gr}_n K \to \text{Gr}_n H^1(K)$ induced by the canonical morphism $K \to H^1(K)$. If $n > 0$ and $p \nmid n$, the morphism $\nu$ is an isomorphism. If $n = 0$, the morphism $\nu$ is a surjection and $\text{Gr}_0 H^1(K)$ is isomorphic to $H^1(k)$.

Otherwise, the morphism $\nu$ is the zero-map.

Proof. Suppose $n > 0$ and $p \nmid n$. Since there is no $x \in K$ satisfying $v_K(P(x)) = -n$, we have $P(K) \cap (F_n K) \subset F_{n-1} K$, thus, the morphism $\nu$ is an isomorphism.

Suppose $n = 0$. The surjectivity follows from the definition of $\nu$. Since $F_{-1} K \subset P(K)$, we have

$$\text{Gr}_0 H^1(K) \simeq (F_0 K/F_{-1} K)/((F_0 K \cap P(K))/F_{-1} K)) \simeq k/(P(O_K)/m_K) = k/P(k) \simeq H^1(k).$$

(2.7)

Suppose $n > 0$ and $p \nmid n$. Take $n'$ such that $n = pn'$. For all $x \in F_n K$, there exists $y \in F_{n'} K$ such that $-v_K(x - P(y)) < n$. Thus, $\nu$ is the zero-map.

Suppose $n < 0$. Since $F_{-1} H^1(K) = 0$, $\nu$ is the zero-map. \hfill \Box

For a finite Galois extension $L/K$ such that $K$ is a complete discrete valuation field of characteristic $p > 0$, let $G_{L/K}$ denote its Galois group, and $L_{L/K}, U_{L/K}$ the sets of indices at jumps of the lower and upper ramification groups of $G_{L/K}$ respectively. For $i \geq -1$, let $G_{L/K,i}, G_{L/K}'$ denote the $i$-th lower and upper ramification group of $G_{L/K}$ respectively. Define the Herbrand function $\psi_{L/K}$ as in [2], IV, §3.

Lemma 2.4. Let $K$ be a complete discrete valuation field and $M/K$ a finite Galois extension. Let $L/K$ be a Galois subextension of $M/K$. Let $G = G_{M/K}$, $H = G_{M/L}$, and $\psi = \psi_{L/K}$. Then we have $G^i \cap H = H^{\psi(i)}$ for all $i \geq -1$. 

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Proof. By definitions of the lower and upper ramification groups, we have $G^i = G_{\psi_{M/K(i)}}$ and $H^\psi(i) = H_{\psi_{M/L(i)}^\psi(i)}$. By [7], IV, §3, Proposition 15, we have $\psi_{M/K} = \psi_{M/L} \circ \psi$. By [7], IV, §1, Proposition 2, $G_{\psi_{M/K(i)}} \cap H$ coincides with $H_{\psi_{M/K(i)}}$.

**Proposition 2.5.** Let $L/K$ be a ramified Artin-Schreier extension defined by $P(x) = a$ ($a \in K - P(K)$). Let $m_a > 0$ be the smallest integer such that the image of $a$ is in $F_{m_a}H^1(K)$.

(a) The valuation of the different of $L/K$ is $(m_a + 1)(p - 1)$.

(b) We have $U_{L/K} = \{m_a\}$.

(c) We have

$$\psi_{L/K(i)} = \max(i, pi - (p - 1)m_a). \tag{2.8}$$

(d) For an integer $n$ such that $p \nmid n$, let $n' = pn - (p - 1)m_a$ and $n'' = \max(n, n') = \psi_{L/K(n)}$. The canonical morphism $H^1(K) \to H^1(L)$ induces $F_n H^1(K) \to F_{n''} H^1(L)$ and $Gr_n H^1(K) \to Gr_{n''} H^1(L)$. Define $\theta$ as in Lemma 2.2. Then the $F_p$-linear map

$$u : Gr_{pn} L \to Gr_{n''} L$$

$$x \mapsto \begin{cases} \sqrt[n]{x} & (n < m_a) \\ \sqrt[n]{x} + \theta x & (n = m_a) \\ \frac{n}{m_a} \theta x & (n > m_a) \end{cases} \tag{2.9}$$

makes the diagram below commutative, where $\nu$ is defined as in Lemma 2.3.

$$\begin{array}{ccc}
Gr_n K & \xrightarrow{\nu} & Gr_{pn} L \\
\downarrow & & \downarrow u \\
Gr_n H^1(K) & \xrightarrow{\nu} & Gr_{n''} H^1(L).
\end{array} \tag{2.10}$$

If $n \neq m_a$, then the map $u$ is an isomorphism. If $n = m_a$, then the kernel of $u$ is generated by the image of $a$.

Proof. (a) The claim follows from the beginning (p. 42) of Section b) of [5].

(b) By (a) and [7], IV, §1, Proposition 4, we have $U_{L/K} = \{m_a\}$.

(c) This follows from the definition of $\psi_{L/K}$.

(d) Define $\mu$ as in Lemma 2.1(a). Take $y \in K$ such that $v_K(y) = -n$. Let $\eta$ be an element of $K_\eta$ such that $P(\eta) = y$. When $y \notin P(L)$, applying Lemma 2.4 to $L(\eta)/K$ we get $y \in F_{n''} H^1(L)$ by (d). When $y \in P(L)$, we have $y \equiv 0 \in H^1(L)$. Thus $y \in F_{n''} H^1(L)$ also in this case. Thus the canonical morphism $H^1(K) \to H^1(L)$ induces $F_n H^1(K) \to F_{n''} H^1(L)$ and $Gr_n H^1(K) \to Gr_{n''} H^1(L)$.

Let $\bar{y}$ denote the image of $y$ in $Gr_n K$. By (a) and Lemma 2.2 we have $-v_L(dy) = pn - (p - 1)m_a = n'$. Since $k$ is perfect, there exists $s \in L$ such that $v_L(y - s^p) > -np$. We have $v_L(s) = -n$. Let $s' = y - s^p$. Since $dy \neq 0$ in $\Omega^1_L$, we may assume $p \nmid v_L(s')$. Then we have $v_L(s') = v_L(ds')$. Since $ds' = dy$, we have $v_L(s') = -n'$.

We have $s' + s = y - P(s) \equiv y \in H^1(L)$. We will now write the images of $s'$ and $s$ in $Gr_{n'} L$ and $Gr_n L$ respectively in terms of $\bar{y}$. By Lemma 2.1(b) and Lemma 2.2, the image of $ds' = dy$ in $Gr_{n'} L$ equals

$$-n' \mu(s') = \mu(-n \bar{y}). \tag{2.11}$$
where \( q > r \).

Lemma 3.1. by [7], IV, §3, Proposition 14. It suffices to show that the elements of \( E/K \) are totally ramified. Recall that there exists \( i \in \mathbb{F}_p^\times \) such that \( a \equiv i(-\theta)^{-\frac{1}{n+m}} \) in \( \text{Gr}_{p^{n+m}} L \), and the kernel of \( u \) is generated by the class of \( a \).

\[ \sqrt[1]{G} \equiv \sqrt[1]{M} = \sqrt[1]{p} \]

Proof. We prove this lemma by descending induction on \( j \).

Suppose that \( j = n \). Let \( n \) be the largest element of \( U_{M_n/K} \). Then we have (3.3) for \( q > r_n \).

Suppose \( 1 \leq j < n \). Assume that we have (3.3) for \( q > r_{j+1} \). Let \( l, r_j \) be the largest elements of \( L_{M_{j+1}/K}, U_{M_j/K} \) respectively. Then we have \( G_{M_j/K} \subset G_{M_n/M_j} \) for \( q > r_j \) by [7], IV, §3, Proposition 14. It suffices to show \( G_{M_{j+1}/K} = G_{M_{j+1}/K,l} \). Since \( Z(G_{M_{j+1}/K}) = G_{M_{j+1}/K}/G_{M_{j+1}/M_{j+1}} \), and \( Z \) is a perfect group, we have \( G_{M_{j+1}/K,l} \) is an isomorphism, since the extension \( E/K \) is perfect.

\[ G_{M_{j+1}/K} = G_{M_{j+1}/K,l} \]

3 Calculation of the Ramification Groups

Let \( 2 \leq n \leq p \). Recall the algebraic group \( G \subset GL(n + 1) \) over \( \mathbb{F}_p \), and its lower central series

\[ G \supseteq Z_1G \supseteq \cdots \supseteq Z_nG = \{1\} \]  

(3.1)

of Definition 1.6(b).

Let \( K \) be a complete discrete valuation field, and \( K_s \) a separable closure of \( K \). Define \( P : K_s \to K_s \) by \( x \mapsto x^p - x \). Assuming that the residue field \( k \) of \( K \) is perfect, the image of \( a \) in \( H^1(L) \) equals \( 0 \). Thus, there exists \( i \in \mathbb{F}_p^\times \) such that \( a \equiv i(-\theta)^{-\frac{1}{n+m}} \) in \( \text{Gr}_{p^{n+m}} L \), and the kernel of \( u \) is generated by the class of \( a \).

\[ G \supseteq Z_1G \supseteq \cdots \supseteq Z_nG = \{1\} \]  

(3.1)

The image of \( s' \), \( s \), and \( \sqrt[1]{s} \) equals \( \frac{n^j}{\sqrt[1]{s}} \) in \( \text{Gr}_{n'} L \). By definition, the image of \( s \) in \( \text{Gr}_n L \) is \( \sqrt[1]{s} \).

Thus the image of \( s + s' \) equals

\[ \begin{cases} \sqrt[1]{s} & (n < m_a) \\ \sqrt[1]{s} + \sqrt[1]{s'} & (n = m_a) \\ \frac{n^j}{\sqrt[1]{s}} & (n > m_a) \end{cases} \]  

(2.12)

in \( \text{Gr}_{n'} L \). Since \( n' \equiv m_a \mod p \), the diagram (2.10) is commutative.

If \( n < m_a \), then the map \( u \) is an isomorphism, since \( k \) is perfect.

If \( n > m_a \), then the map \( u \) is an isomorphism, since \( n \) is prime to \( p \).

If \( n = m_a \), then the kernel of \( u \) is generated by \( (-\theta)^{-\frac{1}{n+m}} \). On the other hand, the image of \( a \) in \( H^1(L) \) equals \( 0 \). Thus, there exists \( i \in \mathbb{F}_p^\times \) such that \( a \equiv i(-\theta)^{-\frac{1}{n+m}} \) in \( \text{Gr}_{p^{n+m}} L \), and the kernel of \( u \) is generated by the class of \( a \).
Recall $A \in M(n)$, $R(t, x) \in \mathbb{F}_p[t, x]$, $A : \mathbb{G}_a \to GL(n)$ of Definition 1.6(a).

Lemma 3.2. Assume that $K$ is of characteristic $p > 0$ and $M_n/K$ is totally ramified. There exist $a \in K$, $b \in K^n$ satisfying the conditions of Lemma 1.9(a) and conditions (i)–(iii) below. Let

$$b = \begin{pmatrix} b_n \\ \vdots \\ b_1 \end{pmatrix},$$

and let $m_a, m_j (1 \leq j \leq n)$ denote $-v_K(a), -v_K(b_j)$ respectively.

(i) $m_a, m_1$ are positive and prime to $p$.

(ii) For all $2 \leq j \leq n$, we have $p \nmid m_j$ if $m_j > 0$.

(iii) If $n \leq p - 1$ and $m_a = m_1$, then the images $\bar{a}, \bar{b}_1$ of $a, b_1$ in $Gr_{m_a} K = Gr_{m_1} K$ respectively are linearly independent over $\mathbb{F}_p$.

Proof. Take $a \in K$, $b \in K^n$ as in Lemma 1.9(a). By Proposition 1.1(c), we may replace

$$\begin{pmatrix} A(a) & b \\ 0 & 1 \end{pmatrix}$$

by

$$\begin{pmatrix} A(s^p) & F(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A(a) & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A(s) & t^{-1} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} A(a + P(s)) & A(s^p)b - A(a + P(s))t + F(t) \\ 0 & 1 \end{pmatrix}$$

(3.6)

for $s \in K$, $t \in M_n^n$, if $A(s^p)b - A(a + P(s))t + F(t) \in K^n$.

When $n \leq p - 1$, let

$$S(t, x) = \frac{R(t, x) - 1}{t} = \sum_{i=0}^{p-2} \binom{x}{i+1} t^i \in \mathbb{F}_p[t, x],$$

(3.7)

and define a morphism $\nu : \mathbb{G}_a \to \mathbb{G}_a^n$ of algebraic varieties by

$$\nu(x) = S(A, x) \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{G}_a^n.$$

(3.8)

Then for any $x, y \in \mathbb{G}_a$, the morphism $\nu$ satisfies

$$\nu(x + y) = \nu(x) + A(x)\nu(y).$$

(3.9)

In particular, we have

$$\nu(x^p) - A(P(x))\nu(x) = F(\nu(x)) - A(P(x))\nu(x) = \nu(P(x)).$$

(3.10)

We have the following fact.
**Fact (\textit{*})** For any

\[ t = \begin{pmatrix} t_n \\ \vdots \\ t_1 \end{pmatrix} \in K^n, \quad (3.11) \]

the first component from the bottom of \((1 - A(a))t\) is 0, and the \(j\)-th component from the bottom of \((1 - A(a))t\) depends only on \(t_1, \ldots, t_{j-1}\) for all \(2 \leq j \leq n\).

We may define operations (I), (II) and (III) as follows.

(I) Choose appropriate \(s \in K\), set \(t = 0\), and replace \((a, b)\) by \((a + P(s), A(s^p)b)\), so that we have \(p \nmid m_a\) if \(m_a > 0\). We know by Lemma 2.3 that such \(s\) exists.

(II) Choose appropriate \(t_j \in K\ (1 \leq j \leq n)\) successively, set \(s = 0\), and replace \((a, b)\) by \((a, b + (1 - A(a))t + \bar{F}(t) - t)\), so that we have \(p \nmid m_j\) if \(m_j > 0\) for all \(1 \leq j \leq n\). We know by Lemma 2.3 and Fact (\textit{*}) that such \(t_j\ (1 \leq j \leq n)\) exist.

(III) Assume \(n \leq p - 1\), \(m_a = m_1\), and \(i = a^{-1}b_1 \in \mathbb{F}_p\). Take \(\alpha \in M_n\) such that \(\alpha^p - \alpha = a\) as in Lemma 1.9(b). Set \(s = 0\), \(t = i\bar{v}(\alpha)\), and replace \((a, b)\) by \((a, b - iv(\alpha))\), so that we have \(m_a > m_1\). This is valid since we have

\[ -A(a)v(\alpha) + F(v(\alpha)) = v(a) \quad (3.12) \]

by \(3.10\).

Define \(\alpha, \gamma_j \in M_n\ (1 \leq j \leq n)\) as in Lemma 1.9(b). Since \(M_n\) is totally ramified, \(K(\alpha), K(\gamma_1)\) are ramified. By Proposition 2.5(b), we have \(m_a, m_1 > 0\).

Performing (I) and (II) successively, conditions (I) and (II) are satisfied.

Suppose \(n \leq p - 1\), \(m_a = m_1\) and \(a^{-1}b_1 \in \mathbb{F}_p\). Performing (III), condition (III) is satisfied. However, since (III) may change \(m_j\ (1 \leq j \leq n)\), we have to perform (I) again to ensure conditions (I) and (II) are satisfied. Since (II) does not make \(m_1\) larger, condition (III) remains satisfied.

Take \(a \in K\), \(b \in K^n\) satisfying the conditions of Lemma 3.2. Define \(r_j\ (1 \leq j \leq n)\) as in Lemma 3.1.

Let

\[ \omega = \begin{pmatrix} \omega_n \\ \vdots \\ \omega_1 \end{pmatrix} = A(-a)db \in (\Omega^1_K)^n, \quad (3.13) \]

where \(db\) denotes the component-wise derivation.

We will now state our main theorem.

**Theorem 3.3.** Let \(K\) be a complete discrete valuation field of equal characteristics \(p > 0\). Assume that the residue field \(k\) of \(K\) is perfect. Let \(v_K\) denote the valuation of \(K\) and \(\Omega^1_K\) defined at the beginning of Section 2. Take \(K \subset M_1 \subset \cdots \subset M_n\) as in Definition 7.8. Assume that \(M_n/K\) is totally ramified. Take \(a \in K\), \(b \in K^n\) satisfying the conditions of Lemma 3.2. Define \(b_1, \ldots, b_n \in K\) as in \(7.7\). Let \(m_a = -v_K(a)\), \(m_j = -v_K(b_j)\ (1 \leq j \leq n)\). Define the sequence \(m_a \leq r_1 \leq \cdots \leq r_n\) as in Lemma 3.4 and \(\omega_j\ (1 \leq j \leq n)\) as in \(3.13\). Then we have

\[ r_j = \max \left( \max_{1 \leq i \leq j} \left( \frac{j - i}{p} m_a - v_K(\omega_i) \right), \frac{(j + p - 2)m_a + m_1}{p} \right) \quad (3.14) \]

for all \(2 \leq j \leq n\).
We will prove this theorem at the last of this paper.

Define \( \alpha, \gamma_j \) \((1 \leq j \leq n)\) as in Lemma 1.9. Let

\[
c = \begin{pmatrix} c_n \\ \vdots \\ c_1 \end{pmatrix} = A(-\alpha^p)b, \tag{3.15}
\]

and let \( m'_j \) \((2 \leq j \leq n)\) denote the smallest integer such that \( c_j \in L \) defines an element of \( F_{m'_j}H^1(L) \). Then we have \( P(\gamma_j) = c_j \) and \( M_j = K(\alpha, \gamma_1, \ldots, \gamma_j) \) for all \( 1 \leq j \leq n \) by Lemma 3.4.

**Lemma 3.4.** (a) Let \( 2 \leq j \leq n \). We have

\[
-v_L(c_j) \leq p \max_{i \leq j}((j - i)m_a + m_i) \tag{3.16}
\]

and

\[
-v_L(dc_j) = \max_{i \leq j}((j - i)m_a - v_L(\omega_i)). \tag{3.17}
\]

Furthermore, there exists a unique \( 1 \leq i \leq j \) satisfying

\[
-v_L(dc_j) = (j - i)m_a - v_L(\omega_i). \tag{3.18}
\]

For this unique \( i \), the image of \( dc_j \) in \( \text{Gr}_{-v_L(dc_j)} \Omega^1_L \) equals that of \( \left( \frac{-\alpha}{j - i} \right) \omega_i \).

(b) We have \( p \nmid m'_j = \psi_{L/K}(r_j) \) for \( 2 \leq j \leq n \), and \( m_a < m'_2 < \cdots < m'_n \).

**Proof.** (a) Since \( d(\alpha + \alpha) = d(\alpha^p) = 0 \) in \( \Omega^1_L \), we have

\[
c = A(-a - \alpha)b, \quad dc = A(-a - \alpha)db = A(-\alpha)\omega. \tag{3.19}
\]

Thus we have

\[
c_j = \sum_{i=1}^{j} \left( \frac{-a - \alpha}{j - i} \right) b_i \tag{3.20}
\]

and

\[
dc_j = \sum_{i=1}^{j} \left( \frac{-\alpha}{j - i} \right) \omega_i. \tag{3.21}
\]

We get (3.16) from (3.20). By Lemma 2.2 and Proposition 2.3, we have

\[
-v_L \left( \left( \frac{-\alpha}{j - i} \right) \omega_i \right) = (j - i)m_a - v_L(\omega_i)
\]

\[
= (j - i)m_a - pv_K(\omega_i) - (p - 1)m_a \equiv (j - i + 1)m_a \mod p. \tag{3.22}
\]

Therefore, the valuations of the terms in the right-hand side of the equation (3.21) do not coincide with each other. Thus we have (3.17) and the rest of the claim.

(b) By Lemma 2.4, we have \( G_{M_j/L}^{\psi_{L/K}}(r_j) = G_{M_j/K}^i \cap G_{M_j/L} \) for all \( i \geq -1 \). Thus by Lemma 3.1 \( U_{M_j/L} = \{ \psi_{L/K}(r_j) \} \cup U_{M_{j-1}/L} \). Note that \( \psi_{L/K}(r_j) \) is larger than all of the elements of \( U_{M_{j-1}/L} \) and that \( M_j = M_{j-1}(\gamma_j) \). Hence \( m'_j = \psi_{L/K}(r_j) \). We get \( r_1 \geq m_a \) from \( U_{M_j/K} \supset U_{L/K} = \{ m_a \} \) and Lemma 3.1. Since we have \( r_1 < r_2 < \cdots < r_n \) by Lemma 3.1, we have \( m_a < m'_2 < \cdots < m'_n \). Since \( c_j \notin P(L) \), by definition of \( c_j' \) (3.23), we have \( p \nmid m'_j > 0 \).
Take $c'_j, c''_j \in L (2 \leq j \leq n)$ such that

$$c_j - P(c'_j) = c'_j, -v_L(c'_j) = m'_j.$$  \hspace{1cm} (3.23)

Then we have $dc_j + dc''_j = dc'_j$, since $d(c''_j) = 0$.

**Lemma 3.5.** We have

$$-v_L(dc''_j) \leq \max_{1 \leq i \leq j} ((j - i)m_a + m_i).$$ \hspace{1cm} (3.24)

**Proof.** By definition of $c''_j$ (3.23), we have

$$-v_L(dc''_j) \leq -v_L(c''_j) \leq -\frac{1}{p}v_L(c_j).$$ \hspace{1cm} (3.25)

By (3.16), we have (3.24). \hfill \Box

Let $\bar{a}, \bar{b}_j (1 \leq j \leq n)$ denote the image of $a, b_j$ in $Gr_{ma} K, Gr_{m_j} K$ respectively, when $m_j \neq -\infty$.

**Proposition 3.6.** Suppose we have

$$-v_L(dc_n) \leq \max_{1 \leq i \leq n} ((n - i)m_a + m_i).$$ \hspace{1cm} (3.26)

(a) For all $2 \leq j \leq n$, We have

$$-v_K(\omega_j) < (j - 1)m_a + m_1$$ \hspace{1cm} (3.27)

(b) For all $1 \leq j \leq n$, we have

$$p \nmid m_j = (j - 1)m_a + m_1 > 0,$$ \hspace{1cm} (3.28)

and

$$\bar{b}_j = \frac{m_1}{(j - 1)!m_j} \bar{a}^{j-1}\bar{b}_1.$$ \hspace{1cm} (3.29)

(c) We have $n \leq p - 1$.

(d) For all $1 \leq j \leq n$, we have

$$-v_L(c_j) = pm_j = p(j - 1)m_a + pm_1,$$  \hspace{1cm} (3.30)

and the image of $c_j$ in $Gr_{pm_j} L$ equals $\frac{(-m_a\bar{a})^{j-1}\bar{b}_1}{\prod_{i=2}^{j} m_i} \neq 0$.

(e) For all $1 \leq j \leq n$, we have

$$-v_L(c''_j) = -v_L(dc''_j) = m_j = (j - 1)m_a + m_1$$ \hspace{1cm} (3.31)

and the image of $c''_j$ in $Gr_{m_j} L$ equals $\frac{(-m_a\sqrt{\bar{a}})^{j-1}\sqrt{\bar{b}_1}}{\prod_{i=2}^{j} m_i}$. 

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Proof. (a) By Lemma 2.2 and Proposition 2.5(a), we have 
\[- v_L(\chi) = -pv_K(\chi) - (p - 1)m_a\]
for all \(\chi \in \Omega_1^K\). Hence it suffices to show that for all \(2 \leq j \leq n\), we have
\[- v_L(\omega_j) < p((j - 1)m_a + m_1) - (p - 1)m_a. \tag{3.32}\]

We will apply Lemma 1.10 to 
\[\lambda = m_a, \quad \lambda_j = \max_{1 \leq i \leq j} ((j - i)m_a + m_i), \quad l_j = -v_L(\omega_j). \tag{3.33}\]

By definition of \(\omega \tag{3.13}\), we have
\[\omega_j = \sum_{i=1}^{j} \left(\frac{-a}{j - i}\right)db_j. \tag{3.34}\]

We have
\[- v_L \left(\left(\frac{-a}{j - i}\right)db_j\right) = p((j - i)m_a + m_i) - (p - 1)m_a \tag{3.35}\]
by Lemma 2.2 and Proposition 2.5(a). By (3.34) and the property of valuation, condition (i) of Lemma 1.10 is satisfied.

By (3.17) and (3.26), we get
\[(n - j)m_a - v_L(\omega_j) \leq \max_{1 \leq i \leq n} ((n - i)m_a + m_i). \tag{3.36}\]

Thus condition (ii) of Lemma 1.10 is satisfied.

Since \(m_a, m_1 > 0\), we have
\[m_a < (j - 1)m_a + m_1 \leq \max_{1 \leq i \leq j} ((j - i)m_a + m_i) \tag{3.37}\]
for all \(2 \leq j \leq n\). Thus condition (iii) of Lemma 1.10 is satisfied.

By applying Lemma 1.10, we get (3.32) for all \(2 \leq j \leq n\).

(b) Since \(db = A(a)\omega\), we have
\[- v_K(db_j) = m_j \leq \max_{1 \leq i \leq j} ((j - i)m_a - v_K(\omega_i)). \tag{3.38}\]

By (3.27) for \(2 \leq i \leq j\), we have
\[(j - i)m_a - v_K(\omega_i) < (j - 1)m_a + m_1 \tag{3.39}\]
for \(2 \leq i \leq j\). On the other hand, we have \(\omega_1 = db_1\). Thus, we have
\[(j - 1)m_a - v_K(\omega_1) = (j - 1)m_a + m_1. \tag{3.40}\]

Thus we have
\[- v_K(db_j) = m_j = \max_{1 \leq i \leq j} ((j - i)m_a - v_K(\omega_i)) = (j - 1)m_a + m_1 > 0 \tag{3.41}\]
by the property of valuation. By conditions (i) and (ii) of Lemma 3.2, we have (3.32) for all \(1 \leq j \leq n\). The image of \(db_j\) equals the image of \(\left(\frac{a}{j - 1}\right)db_1\) in \(Gr_{m_j} \Omega_1^K\). Hence we have (3.29).
By (3.28) for all $2 \leq j \leq n$. Since $(j - 1)m_a + m_1 \mod p$ $(1 \leq j \leq n)$ are different from each other and $m_1, \ldots, m_n$ are all prime to $p$ by (3.28), we have $n \leq p - 1$. By (3.16) and (3.28), we have $-v_L(c_j) \leq pm_j$. By (3.20), Lemma 1.11 and (3.29), the image of $c_j$ in $Gr_{pm_j}$, $L$ equals

$$\sum_{i=1}^{j} (-\bar{a})^{j-i} \frac{b_i}{(j-i)!} = \sum_{i=1}^{j} \frac{(-\bar{a})^{j-i}m_1}{(j-i)!(i-1)!m_i} \bar{a}^{i-1}\bar{b}_1 \neq 0.$$ (3.42)

Thus we have (3.30).

Since we have (3.30) and $m_j$ is prime to $p$ by (3.28), we have (3.31). Hence, the image of $c_j'$ in $Gr_{m_j}$, $L$ equals $\frac{(-m_a\sqrt{\bar{a}})^{j-1}\sqrt{\bar{b}_1}}{\prod_{i=2}^{j} m_i}$, since we have $\sqrt{x} = x$ for $x \in \mathbb{F}_p$. \hfill \Box

We will now express $\max(-v_L(dc_n), -v_L(de_n''))$ in terms of $a, b$.

**Corollary 3.7.** (a) The following conditions are equivalent:

(i) $-v_L(dc_n) < -v_L(de_n'')$.

(ii) $-v_L(dc_n) < \max_{1 \leq j \leq n}((n - j)m_a + m_j)$.

(iii) $-v_L(dc_n) < (n - 1)m_a + m_1$.

(b) We have

$$m_n' = -v_L(dc_n) \leq \max(-v_L(dc_n), -v_L(de_n'')) = \max_{1 \leq i \leq n} \left( (\frac{n - i - p + 1}{p}m_a - pv_K(\omega_i)), (n - 1)m_a + m_1 \right).$$ (3.43)

**Proof.** (i) (ii) = (iii): This follows from (3.24).

(iii) = (i): This clearly holds.

(iii) = (ii) = (iii): This follows from (3.31). By (i), (3.17), and (3.31), we have (3.43). \hfill \Box

We will now prove our main theorem.

**Proof of Theorem 7.3.** It suffices to show the case where $j = n$, since the case where $j < n - 1$ can be reduced to the case $j = n$ by replacing $n$ by $j$.

Let

$$s_n = \psi_{L/K}(\max_{1 \leq i \leq n} \left( (\frac{n - i}{p}m_a - v_K(\omega_i)), (\frac{n + p - 2}{p}m_a + m_1) \right)).$$ (3.44)

It suffices to show $m_n' = \psi_{L/K}(r_n) = s_n$ to complete the proof, since $\psi_{L/K}$ is injective by Proposition 2.5(c).

Since $\frac{(n + p - 2)}{p}m_a + m_1 > m_a$, we have

$$\max_{1 \leq i \leq n} \left( (\frac{n - i}{p}m_a - v_K(\omega_i)), (\frac{n + p - 2}{p}m_a + m_1) \right) > m_a.$$ (3.45)

By Proposition 2.5(c), we have $\psi_{L/K}(x) = px - (p - 1)m_a$ if $x \geq m_a$. Then by Corollary 3.7(b), we get

$$s_n = \max(-v_L(dc_n), -v_L(de_n'')) \geq m_n'.$$ (3.46)
We have $m'_n = s_n$ when $-v_L(d_{cn}) \neq -v_L(d_{cn}''')$. It suffices to show that we have $m'_n = s_n$ also when $-v_L(d_{cn}) = -v_L(d_{cn}''')$.

Assume $-v_L(d_{cn}) = -v_L(d_{cn}''')$. Then by (3.24), the hypothesis (3.26) of Proposition 3.6 is satisfied. By Proposition 3.6, we have $n \leq p - 1$. By Lemma 2.2 and Proposition 2.5(a), we have $-v_L(\omega_i) = -pv_K(\omega_i) - (p - 1)m_a \equiv m_a \mod p$ for all $1 \leq i \leq n$.

Thus, by Lemma 3.4(a), there exists a unique integer $1 \leq j \leq n$ satisfying $-v_L(d_{cn}) = (n - j)m_a - v_L(\omega_j) \equiv (n - j + 1)m_a \mod p$. Combining with (3.31), we get

$$v_L(d_{cn}) - v_L(d_{cn}'''') = 0 \equiv (j - 2)m_a + m_1 \mod p.$$  

(3.47)

Since $m_1 = (i-1)m_a + m_1$ is prime to $p$ for all $1 \leq i \leq n$ by (3.28), we have $j \neq 2, \ldots, n+1$. Therefore, since $j$ is an integer satisfying $1 \leq j \leq n$, we have $j = 1$. Hence, we have $m_n = (n - 1)m_a + m_1 = v_L(d_{cn}''') = -v_L(d_{cn}) = (n - 1)m_a - v_L(\omega_1)$.  

(3.48)

Therefore, we have $m_1 = v_L(\omega_1)$. By definition, we have $\omega_1 = db_1$. Thus we have $m_1 = pm_1 - (p-1)m_a$ by Lemma 2.2 and Proposition 2.5(b). Hence we have $m_n = m_1$. By (3.28) we get $m_j = jm_a$ for all $1 \leq j \leq n$. Hence we have $s_n = -v_L(d_{cn}) = -v_L(d_{cn}''') = m_n$.

We have only to check that the image of $d_{cn} + d_{cn}''' = d_{cn}'$ in $\text{Gr}_{m_n} \Omega^1_L$ does not vanish. Define the $k$-linear isomorphism $\mu : \text{Gr} L \rightarrow \text{Gr} \Omega^1_L$ as in Lemma 2.2 and $\theta \in \text{Gr}_{-(p-1)m_a} L$ as in Lemma 2.2. Since we have (3.48), the image of $d_{cn}$ in $\text{Gr}_{m_n} \Omega^1_L$ equals that of $\begin{pmatrix} -\alpha \\ n - 1 \end{pmatrix}$ by Lemma 3.4(a). Thus, by Lemma 2.1(b), the image of $d_{cn}$ in $\text{Gr}_{m_n} \Omega^1_L$ equals $-m_1 \frac{(-\sqrt[n-1]{a})}{(n-1)!} \mu(\bar{b}_1)$.

Meanwhile, by Proposition 3.6, Lemma 2.1(b), and $m_j = jm_a$ for all $1 \leq j \leq n$, the image of $d_{cn}''$ in $\text{Gr}_{m_n} \Omega^1_L$ equals $-m_1 \frac{(-\sqrt[n-1]{a})}{(n-1)!} \mu(\sqrt[n-1]{b_1})$.

Define $u : \text{Gr}_{pm_n} L \rightarrow \text{Gr}_{m_n} L$ as in Proposition 2.5(a). Then the image of $d_{cn} + d_{cn}'' = d_{cn}'$ in $\text{Gr}_{m_n} \Omega^1_L$ equals $-m_1 \frac{(-\sqrt[n-1]{a})}{(n-1)!}$ times

$$\mu(\bar{b}_1 + \sqrt[n-1]{b_1}) = \mu(u(\bar{b}_1))$$  

(3.49)

by Proposition 2.5(a). Since $n \leq p - 1$, by condition (iii) of Lemma 3.2 and Proposition 2.5(a), we have $u(\bar{b}_1) \neq 0$. Thus the image of $d_{cn}'$ in $\text{Gr}_{m_n} L$ does not vanish.

\begin{proof}

Example 3.8. 1. We give an example for $n = 2$, the simplest case where Theorem 3.3 can be applied.

Let $n = 2$. Then $G$ is the algebraic group of upper triangular unipotent matrices of order 3 over $\mathbb{F}_p$. Then $M_2/K$ is defined by

$$\begin{pmatrix} 1 & x^p & y_2^p \\ 0 & 1 & y_1^p \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & b_2 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y_2 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix}$$  

(3.50)

for some $a, b_1, b_2 \in K$ satisfying the conditions of Lemma 3.2. We have

$$\omega_1 = db_1, \quad \omega_2 = db_2 - adb_1.$$  

(3.51)

\end{proof}
By Theorem 3.3, we have
\[ U_{M_2/K} = U_{M_1/K} \cup \{ r_2 \} \]  \hspace{1cm} (3.52)

where
\[ r_2 = \max \left( -v_K(db_2 - adb_1), \frac{m_a}{p} + m_1, m_a + \frac{m_1}{p} \right). \]  \hspace{1cm} (3.53)

We can calculate \( U_{M_1/K} \) by Proposition 2.5(b), since \( M_1/K \) is an abelian extension.

2. We give an example where the maximum of the right-hand side of (3.14) is achieved by the first term or the second term, depending on the parameters \( \eta, \eta' \).

Let \( p > 2, n = p - 1, \eta, \eta' \in \mathbb{Z}_{\geq 0}. \) For \( 1 \leq j \leq p - 1 \), define \( f_j(x) \in \mathbb{Z}_{\left[ \frac{1}{j}, x^{-1} \right]} \) as a polynomial of \( x^{-1} \) satisfying
\[ \frac{df_j}{dx} = -\left( x^{-\eta'p^{-2}} \right). \]  \hspace{1cm} (3.54)

Let \( M_n/K \) be the Galois extension defined by
\[ \left( \begin{array}{cc} A(x^p) & F(y) \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} A(a) & b \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} A(x) & y \\ 0 & 1 \end{array} \right), \]  \hspace{1cm} (3.55)

with \( a = t^{-\eta p^{-1}}, \ b_j = \epsilon f_j(t) \) (\( 1 \leq j \leq n \)), where \( t \) is a uniformizer of \( K \) and \( \epsilon \in k - \mathbb{F}_p \). These \( a, b \) satisfy the conditions of Lemma 3.2. Then by Proposition 2.5(b), we have \( U_{M_1/K} = \{ \eta p + 1, \eta'p + 1 \} \).

We have \( db = -\epsilon A(t^{-\eta p^{-1}})v(1)t^{-\eta'p^{-2}dt}. \) Thus \( \omega = -\epsilon v(1)t^{-2dt}. \) Thus we have \( -v_K(\omega_1) = \eta'p + 1 \) and \( -v_K(\omega_j) = -\infty \) for \( 2 \leq j \leq p - 1. \) Hence, by Theorem 3.3, we have
\[ r_j = \max \left( \frac{(j-1)(\eta p + 1) + p(\eta'p + 1)}{p}, \frac{(j + p - 2)(\eta p + 1) + \eta'p + 1}{p} \right) \]
\[ = (j-1) \left( \eta + \frac{1}{p} \right) + \eta' + 1 + (p - 1) \max(\eta, \eta'). \]  \hspace{1cm} (3.56)

Thus, if we have \( \eta \geq \eta' \) (resp. \( \eta \leq \eta' \)), then the maximum of the right-hand side of (3.14) is achieved by the first (resp. second) term.

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