Dilatonic AdS-Kerr Solution to AdS/CFT Correspondence

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Abstract

We consider the $AdS_5$ solution deformed by a non-constant dilaton interpolating between the standard AdS (UV region) and flat boundary background (IR region). We show that this dilatonic solution can be generalized to the case of a non-flat boundaries provided that the metric of the boundaries satisfies the vacuum Einstein field equations.

As an example, we describe the case when the four-dimensional boundaries represent the Kerr space-time.

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The recently discovered remarkable AdS/CFT correspondence between higher-dimensional classical supergravity and quantum gauge theory on the boundary (bulk/boundary correspondence) [1] gives a new insight to the understanding of strongly coupled gauge theories. In particular, the $AdS_5 \times S^5$ vacua of type-IIB supergravity correspond to the four-dimensional Super Yang Mills theory on the boundary. This stimulated attempts to get the description of running gauge coupling of Yang-Mills theory and QCD-confinement in the frames of type-0 superstring theory [2, 3, 4]. In another approach [5, 6, 7, 8, 9], the attempts to reproduce similar QCD-effects were based on the non-supersymmetric background solutions of type-IIB string theory which can be obtained due to the deformation of $AdS_5 \times S^5$ vacuum by a non-constant dilaton breaking the supersymmetry and conformal symmetry. The exact solution of a such sort was first given in [10] and we shall follow the notations of this work. Starting with ten-dimensional dilatonic gravity and the solution with topology of $AdS_5 \times S^5$, one can integrate out five coordinates of the sphere $S^5$ and obtain [10, 9] effective action for $AdS_5$ background

$$S = - \int d^5x \sqrt{-g} \left( R - \Lambda - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right),$$

and the equations for metric $g$ and dilaton field $\phi$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} - \frac{1}{2} \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \right) = 0,$$

$$\partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) = 0.$$  

Assuming that solutions for $g$ and $\phi$ depend only on coordinate $y \equiv x^5$, the following ansatz for metric was proposed

$$ds^2 = \sum_{\mu, \nu=0}^d g_{\mu\nu} dx^\mu dx^\nu = f(y) dy^2 + g(y) \sum_{i,j=0}^{d-1} \eta_{ij} dx^i dx^j,$$

where $\eta_{ij}$ is the metric of Minkowski background. As it was shown in [10, 9], the solution of these equations can be given in five dimensions by functions

$$g = y$$

and

$$f = \frac{3}{y^2 \left( \lambda^2 + \frac{c^2}{2y^4} \right)}.$$
where $\lambda^2 = -\Lambda$ is positive. The main peculiarity of this solution is the appearance of two boundaries corresponding to different limiting values of the dilaton. This solution interpolates between conformal AdS background (weak coupling regime, $y \to \infty$) and flat space at singular value of dilaton $\phi \to 6^{1/2} \ln y$ (strong coupling regime, $y \to 0$).

The aim of this note is to attract attention to the possibility of the generalization of this solution for non-flat four-dimensional boundaries. We shall show that any metric of the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f(y) dy^2 + g(y) \hat{g}_{ik} dx^i dx^k$$

represents the solution of dilatonic gravity equations (2) and (3) provided that the boundary metric $\hat{g}_{ik}$ satisfies the vacuum Einstein equations. This result is also valid for a $d$-dimensional boundary of $AdS_{d+1}$. As an example we consider in more details $d=4$ case when the four-dimensional boundary represents the Kerr geometry.

Let us consider $d+1$-dimensional metric (7) where $\hat{g}_{ik}$ is a $d$-dimensional boundary metric, $\mu, \nu, \ldots = 0, 1, \ldots d$, and $i, k, \ldots = 0, 1, \ldots d - 1$. In the matrix notations we have

$$g_{\mu\nu} = \begin{pmatrix} g(y) \hat{g}_{ik} & 0 \\ 0 & f \end{pmatrix},$$

while the contravariant form of this metric is given by

$$g^{\mu\nu} = \begin{pmatrix} g(y)^{-1} \hat{g}^{ik} & 0 \\ 0 & f^{-1} \end{pmatrix},$$

where $\hat{g}^{ik}$ is the contravariant form of the corresponding $d$-dimensional metric. The expressions for the connection coefficients $\Gamma_{\nu\lambda}^\mu$ are given in Appendix. For the Ricci curvature tensor we have

$$R_{dd} = -d \left( \frac{g''}{2g} + (\frac{g'}{2g})^2 + \frac{g'g''}{4fg} \right),$$

$$R_{id} = 0,$$

$$R_{ik} = \hat{R}_{ik} + g_{ik} \left[ \frac{g'^2}{4f^2} - \frac{g''}{2f} + (2 - d) \frac{(g')^2}{4fg} \right].$$

The important point is that $\hat{R}_{ik}$ represents Ricci curvature of $d$-dimensional boundary with the metric $\hat{g}_{ik}$ and the connections $\hat{\Gamma}_{ik}^l$. Similarly, the scalar curvature is

$$R = g^{-1} \hat{R} + \frac{d}{4f} \left[ 2 \frac{f'g'}{fg} - 4 \frac{g''}{g} + (3 - d) \left( \frac{g'}{g} \right)^2 \right].$$
Finally, the expressions for the Einstein tensor are given by \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) and are as follows:

\[
G_{dd} = \frac{d(d-1)}{8} (\frac{d'}{g})^2, \quad (14)
\]

\[
G_{id} = 0, \quad (15)
\]

\[
G_{ik} = \hat{G}_{ik} + g_{ik} \left[ \frac{(d-1)(d-4)(g')^2}{8fg} - \frac{(d-1)f'g'}{4f^2} + \frac{(d-1)g''}{2f} \right]. \quad (16)
\]

When the d-dimensional metric \( \hat{g}_{ik} \) satisfies the vacuum Einstein field equations \( \hat{G}_{ik} = 0 \), the components \( \hat{G}_{ik} \) drop out of the equations (2) and (3). As a result we come exactly to differential equations for functions \( f \) and \( g \) given in the paper [9] and to the expressions (5) and (6). Therefore, in the solutions (4), (5) and (6) the d-boundary metric \( \eta_{ik} \) can be replaced by any metric \( \hat{g}_{ik} \) satisfying the vacuum Einstein equations.

As an example let us now consider the partial case \( d = 4 \) and the Kerr boundary metric \( \hat{g}_{ik} \) in the Kerr-Schild form

\[
\hat{g}_{ik} = \eta_{ik} + 2hk_{i}k_{k}, \quad (17)
\]

where \( h \) is the harmonic scalar function

\[
h = mr/(r^2 + a^2 \cos^2 \theta), \quad (18)
\]

and \( m \) is the mass parameter. This is an important solution of the vacuum Einstein equations describing the field of rotating black holes and modelling the gravitational field of spinning particle [11, 12]. Besides, this form allows one to get a simple comparison with the Minkowskian case.

The vector field \( k_{i}(x) \) is tangent to the principal null congruence and it is determined by the 1-form

\[
k_{i}dx^{i} = \frac{\sqrt{2}}{1 + Y\bar{Y}}[du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv], \quad (19)
\]

where

\[
2^{\frac{1}{2}}\zeta = x + iy, \quad 2^{\frac{1}{2}}\bar{\zeta} = x - iy,
\]

\[
2^{\frac{1}{2}}u = z + t, \quad 2^{\frac{1}{2}}v = z - t, \quad (20)
\]
are the Cartesian null coordinates, and $Y(x) = e^{i\phi} \tan \frac{\theta}{2}$ is the projective angular coordinate. The field $k_i$ is null in respect to Minkowski metric, $k_i k_k \eta^{ik} = 0$, as well as regarding the metric $\hat{g}_{ik}$

$$k_i k_k \hat{g}^{ik} = 0. \quad (21)$$

One can build the five dimensional field $k^\mu = (k^i, 0)$ which will obviously be the null field with respect to the full five-dimensional metric as well

$$k^\mu k^\nu g_{\mu\nu} = 0. \quad (22)$$

Therefore, the five-dimensional Kerr-AdS metric takes the form

$$ds^2 = f(y) dy^2 + g(y) [\eta_{ik} + 2hk_i k_k] dx^i dx^k, \quad (23)$$

where functions $f(y)$, $g(y)$ and $h$ are given by (14), (15) and (18), and vector field $k_i$ is determined by expressions (19) and (20).

In the region of parameters corresponding to spinning particles the black hole horizons disappear and a region of the rotating disk-like source is opened. The structure of this source should possess some properties of OCD-confinement and represents an old problem. The predicted exotic properties of the matter of this source [12] do not allow to construct them in four dimensions from a known sort of classical matter. The conjectured AdS/CFT correspondence gives a new approach to this problem and stimulates consideration of new models, in particular, of a bag-like models resembling the cosmic bubble models. In this case the bag-like source based on the supersymmetric domain wall models [13] can contain the AdS-region of a false vacuum inside the bag separated from the true vacuum of the outer region by a thin domain wall. As a field model, the supersymmetric version of $U(1) \times U(\tilde{1})$ Witten model [14] seems the most appropriate, since it provides the long range electromagnetic field out of core. A hypothetical mechanism of the formation of the bag-like Kerr source can be connected with a phase transition governed by the value of dilaton near the core.

The AdS-BH solution of another sort was considered in [15].

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2It can be also expressed as $Y = (z - ia - \tilde{r})/(x - iy)$, where $\tilde{r} = r + ia \cos \theta$ is complex radial distance.
Appendix

The connection coefficients are (no summation over $d$)

\[
\Gamma_{dd}^d = \frac{f'}{2f}, \quad \Gamma_{id}^d = 0, \quad \Gamma_{di}^d = 0, \quad \Gamma_{ik}^d = -\frac{g'g_{ik}}{2f}, \quad \Gamma_{dk}^i = \Gamma_{kd}^i = -\frac{g'd}{2g}, \quad \Gamma_{dd}^i = 0, \quad \Gamma_{jk}^i = \tilde{\Gamma}_{jk}^i.
\]

Here $\tilde{\Gamma}_{jk}^i$ are connections to d-dimensional metric $\hat{g}_{ik}$.

We have also the relations

\[
\Gamma_{d\mu}^\mu = \Gamma_{\mu d}^\mu = \frac{f'}{2f} + d\frac{g'}{2g}, \quad \Gamma_{i\mu}^\mu = \Gamma_{\mu i}^\mu = \tilde{\Gamma}_{i\mu}^\mu, \quad \Gamma_{d\nu}^\mu \Gamma_{d\mu}^\nu = \left(\frac{f}{2f}\right)^2 + d\left(\frac{g'}{2g}\right)^2, \quad \Gamma_{iv}^\mu \Gamma_{k\mu}^v = \tilde{\Gamma}_{ij}^l \tilde{\Gamma}_{kl}^j - g_{ik}\left(\frac{g'}{2g}\right)^2, \quad \Gamma_{dd}^\mu \Gamma_{\nu\mu}^\nu = \frac{f}{2f} \left(\frac{f'}{2f} + d\frac{g'}{2g}\right), \quad \Gamma_{ik}^\mu \Gamma_{\nu\mu}^\nu = \tilde{\Gamma}_{ik}^l \tilde{\Gamma}_{lj}^i - g_{ik}\left(\frac{f'}{2f} + d\frac{g'}{2g}\right).
\]

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