The \((f,g)\)-inversion formula and its applications: the \((f,g)\)-summation formula

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Abstract

A complete characterization of two functions \(f(x,y)\) and \(g(x,y)\) in the \((f,g)\)-inversion is presented. As an application to the theory of hypergeometric series, a general bibasic summation formula determined by \(f(x,y)\) and \(g(x,y)\) as well as four arbitrary sequences is obtained which unifies Gasper and Rahman’s, Chu’s and Macdonald’s bibasic summation formula. Furthermore, an alternative proof of the \((f,g)\)-inversion derived from the \((f,g)\)-summation formula is presented. A bilateral \((f,g)\)-inversion containing Schlosser’s bilateral matrix inversion as a special case is also obtained.

\textit{Key words}: Bilateral formal power series, \((f,g)\)-inversion, \((f,g)\)-expansion formula, basic, elliptic, bibasic, bilateral hypergeometric series, transformation formula, \((f,g)\)-summation formula.

1 Introduction

As well-known, matrix inversions over the complex field \(\mathbb{C}\), play a very important role in the theory of basic hypergeometric series. Recall that a matrix inversion is define to be a pair of infinite lower triangular matrices \(F = (f_{n,k})_{n,k \in \mathbb{Z}}\) and \(G = (g_{n,k})_{n,k \in \mathbb{Z}}\) satisfying

\[
\sum_{i=k}^{n} f_{n,i}g_{i,k} = \delta_{n,k},
\]

where \(\delta\) denotes the Kronecker delta, \(\mathbb{Z}\) denotes the set of integers. If such a pair of matrices \(F\) and \(G\) are not lower triangular, we call it bilateral matrix inversion. In our previous paper [23], we have established the following matrix inversion, named the \((f,g)\)-inversion.

\textbf{Theorem 1} Let \(f(x,y)\) and \(g(x,y)\) be two arbitrary functions over the complex field \(\mathbb{C}\) in variables \(x, y\). Suppose that \(g(x,y)\) is antisymmetric, i.e., \(g(x,y) = \)
Let \( F = (f_{n,k})_{n,k \in \mathbb{Z}} \) and \( G = (g_{n,k})_{n,k \in \mathbb{Z}} \) be two matrices with entries given by

\[
f_{n,k} = \frac{\prod_{i=k}^{n-1} f(x_i, b_k)}{\prod_{i=k+1}^{n} g(b_i, b_k)} \quad \text{and} \quad g_{n,k} = \frac{f(x_k, b_k) \prod_{i=k+1}^{n} f(x_i, b_n)}{f(x_n, b_n) \prod_{i=k}^{n-1} g(b_i, b_n)},
\]

respectively. \((1.1)\) \((1.2)\)

Then \( F = (f_{n,k})_{n,k \in \mathbb{Z}} \) and \( G = (g_{n,k})_{n,k \in \mathbb{Z}} \) is a matrix inversion if and only if \( f(x,y) \in \text{Ker}\, L_3^{(g)} \) or \( f(x,y) \in \text{Ker}\, L_3 \), where \( \text{Ker}\, L_3^{(g)} \) and \( \text{Ker}\, L_3 \) denote the sets of functions \( f(x,y) \) such that for all \( a,b,c,x \in \mathbb{C} \)

\[
g(a,b) f(x,c) - g(a,c) f(x,b) + g(b,c) f(x,a) = 0 \quad \text{and} \quad f(a,b) f(x,c) - f(a,c) f(x,b) + f(b,c) f(x,a) = 0,
\]

respectively. \((1.3)\) \((1.4)\)

Our results in [23] state that the \((f,g)\)-inversion covers all known matrix inversions such as the Gould-Hsu formula, Krattenthaler’s inversion formula, as well as Warnaar’s elliptic matrix inversion. The aim of this paper, as the further developments in work of our paper [23], is twofold. First, we will give a complete characterization of \( f \) and \( g \) such that \( f \in \text{Ker}\, L_3^{(g)} \), which was left as an open problem in [23]. The answer is presented in Section 2. Second, we will set up a rather general summation formula via such a pair of functions \( f \) and \( g \). Later as we will see, this summation formula unifies all known bibasic summation formulas which are extensions of Jackson’s q-analogue of Whipple’s summation formula for a terminating very-well-poised balanced \( \phi_7[15,2.6.2] \) and are due to several authors (cf.[1,3,8,15,16,31,33]). For a good survey about this kind of summation formulae, we refer the reader to Gasper and Rahman’s book [15, Sections 3.6-3.7] and for their \( U(n+1)\)-extensions to [3,25]. In addition, from the \((f,g)\)-summation formula, we obtain a very short proof of the \((f,g)\)-inversion and a bilateral \((f,g)\)-inversion containing Schlosser’s bilateral matrix inversion as a special case, both are presented in Section 3.

Notations and conventions. Throughout this paper, for convenience, we will mainly work on the setting of formal bilateral series over the complex field \( \mathbb{C} \), and use the notation \( \mathbb{C}[x,y] \) to denote the ring of formal bilateral series in \( x,y \). As usual, we employ the following standard notations for the q-shifted factorial:
\[(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i)\]

\[(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n.\]

2 The complete characterizations of Ker\(L_3\) and Ker\(L_3^{(g)}\)

In this section, we will characterize two nonzero functions \(f(x, y)\) and \(g(x, y)\) required by the \((f, g)\)-inversion. For this, we first introduce some notation and preliminaries.

**Definition 1** Let \(f(x, y)\) and \(g(x, y)\) be two arbitrary nonzero functions over \(C\) in variables \(x, y\). Suppose that for all four numbers \(a, b, c, x\),

\[g(a, b)f(x, c) - g(a, c)f(x, b) + g(b, c)f(x, a) = 0,\]

then \(f(x, y)\) is called orthogonal to \(g(x, y)\). Further, \(f(x, y)\) is called self-orthogonal if \(f(x, y)\) is orthogonal to itself.

Write \(f(x, y) \perp g(x, y)\), or in short, \(f \perp g\) if \(f(x, y)\) is orthogonal to \(g(x, y)\). As it stands, \(f \perp g\) does not mean \(g \perp f\). This definition allows us to rewrite

\[\text{Ker}\(L_3^{(g)}\) = \{ f | f \perp g \} \}; \text{Ker}\(L_3\) = \{ f | f \perp f \}.\]

Since the set Ker\(L_3^{(g)}\) is of value to the \((f, g)\)-inversion, it seems necessary to find the explicit expressions of \(f\) and \(g\). The possibility for us to do this is within the ring of formal bilateral series. At first, we need

**Lemma 1** Let \(f(x, y) = \sum_{i,j=\infty}^{\infty} \lambda(i, j)x^iy^j \in C[x, y]\). Then \(f \perp f\) if and only if for any integers \(m, i, j, k \in Z\),

\[\lambda(m, i)\lambda(k, j) - \lambda(k, i)\lambda(m, j) + \lambda(k, m)\lambda(i, j) = 0.\] (2.1)

Based on (2.1), it is easy to show that \(f(x, y)\) is antisymmetric, i.e., \(f(x, y) = -f(y, x)\).

**Lemma 2** Let \(g(x, y) = \sum_{i,j=\infty}^{\infty} c(i, j)x^iy^j, f(x, y) = \sum_{i,j=\infty}^{\infty} \lambda(i, j)x^iy^j\). Then \(f \perp g\) if and only if for any integers \(m, i, j, k \in Z\),

\[c(m, i)\lambda(k, j) - c(m, j)\lambda(k, i) + c(i, j)\lambda(m, k) = 0.\] (2.2)
Before we give the explicit expressions of $f$ and $g$, it is better for us to work out whether there exists any connection between $g \perp g$ and $f \perp g$. This idea allows us to set up

**Lemma 3** Let $g(x, y) = \sum_{i,j=-\infty}^{\infty} c(i, j)x^i y^j \neq 0$. Then $g \perp g$ if and only if there are at least two integers $m_0, k_0$, such that $c(m_0, k_0) \neq 0$ and such that for arbitrary integers $i, j \in \mathbb{Z}$, there holds that

$$c(m_0, k_0)c(i, j) - c(m_0, i)c(k_0, j) + c(m_0, j)c(k_0, i) = 0. \quad (2.3)$$

**Proof.** Obviously, since $g \neq 0$, there is at least one nonzero coefficient $c(m_0, k_0)$. Let $g \perp g$. Then by Definition 1, we see that (2.3) holds. Now, assume that (2.3) holds for arbitrary integers $i, j$. Without any loss of generality, let $c(m_0, k_0) = 1$. Thus, for any quadruple of fixed $i, j, i_1, j_1 \in \mathbb{Z}$, there must hold that

$$\begin{align*}
  &c(i, j) = c(m_0, j)c(i, k_0) - c(k_0, j)c(i, m_0); \\
  &c(i_1, j_1) = c(m_0, j_1)c(i_1, k_0) - c(k_0, j_1)c(i_1, m_0); \\
  &c(i, i_1) = c(m_0, i_1)c(i, k_0) - c(k_0, i_1)c(i, m_0); \\
  &c(j, j_1) = c(m_0, j_1)c(j, k_0) - c(k_0, j_1)c(j, m_0); \\
  &c(i, j_1) = c(m_0, j_1)c(i, k_0) - c(k_0, j_1)c(i, m_0); \\
  &c(i_1, j) = c(m_0, j)c(i_1, k_0) - c(k_0, j)c(i_1, m_0).
\end{align*}$$

Set

$$\begin{align*}
  &c(m_0, i) = x_1, c(m_0, j) = x_2, c(m_0, i_1) = x_3, c(m_0, j_1) = x_4; \\\n  &c(k_0, i) = -y_1, c(k_0, j) = -y_2, c(k_0, i_1) = -y_3, c(k_0, j_1) = -y_4.
\end{align*}$$

Insert these into the preceding relations to arrive at

$$\begin{align*}
  &c(i, j) = x_2y_1 - x_1y_2, c(i, i_1) = x_3y_1 - x_1y_3, c(i, j_1) = x_4y_1 - x_1y_4; \\
  &c(i_1, j_1) = x_4y_3 - x_3y_4, c(j, j_1) = x_4y_2 - x_2y_4, c(i_1, j) = x_2y_3 - x_3y_2.
\end{align*}$$

And then by a bit of straightforward calculation, it is easily seen that

$$c(i_1, j_1)c(i, j) - c(i_1, j)c(i, j_1) + c(j_1, j)c(i, i_1) = 0,$$

which gives that $g \perp g$.

The next lemma gives the relationship between $g \perp g$ and $f \perp g$. 

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4
Lemma 4 Let $f(x, y) = \sum_{i,j=-\infty}^{\infty} \lambda(i, j)x^iy^j, g(x, y) = \sum_{i,j=-\infty}^{\infty} c(i, j)x^iy^j \in C[x,y]$ be two nonzero bilateral series, $g(x, y) = -g(y, x)$. Then $g \perp g$ provided that $f \perp g$.

Proof. Since $g(x, y)$ is a nonzero bilateral series in $x, y$, there exists at least one nonzero coefficient $c(m_0, k_0)$. Under this condition, solving $f \perp g$, i.e.,

$$c(m, k)\lambda(i, j) - c(m, j)\lambda(i, k) + c(k, j)\lambda(i, m) = 0$$

immediately gives that for arbitrary $i, j$,

$$\lambda(i, j) = \frac{c(m_0, j)\lambda(i, k_0) + c(j, k_0)\lambda(i, m_0)}{c(m_0, k_0)}.$$  

(2.5)

Substitute each $\lambda(i, j)$ in (2.4) by (2.5) and then rearrange the corresponding result and find

$$\frac{\lambda(k, k_0)}{c(m_0, k_0)} [c(m, i)c(m_0, j) - c(m, j)c(m_0, i) + c(i, j)c(m_0, m)]$$

$$+ \frac{\lambda(k, m_0)}{c(m_0, k_0)} [c(m, i)c(j, k_0) - c(m, j)c(i, k_0) + c(i, j)c(m, k_0)] = 0.$$  

Note that $k, m, i, j$ are arbitrary and $f(x, y)$ is nonzero, there are at least one integer $k$, such that $\lambda(k, k_0) \neq 0$ or $\lambda(k, m_0) \neq 0$. Without any loss of generality, suppose $\lambda(k, k_0) \neq 0$. Now, set $j = k_0$. Since $c(k_0, k_0) = 0$, which follows from $g(x, y) = -g(y, x)$, it is easily seen that the second sum in the left-hand side of this identity vanishes. So the result is that for arbitrary $i, m \in \mathbb{Z},$

$$\frac{\lambda(k, k_0)}{c(m_0, k_0)} [c(m, i)c(m_0, k_0) - c(m, k_0)c(m_0, i) + c(i, k_0)c(m_0, m)] = 0,$$

which is equivalent to

$$c(m, i)c(m_0, k_0) - c(m, k_0)c(m_0, i) + c(i, k_0)c(m_0, m) = 0.$$  

By Lemma 3, it is easily seen that $g(x, y) \in \ker L_3$.

Lemma 4 provides a necessary condition for two functions $f$ and $g$ such that $f \perp g$. These lemmas in together makes it possible to find the explicit expressions for such $f$ or $g$. In what follows, it is always assumed that $g \perp g$. 

5
Theorem 2 Suppose that \( f(x, y) = \sum_{i,j=0}^{\pm \infty} \lambda(i, j)x^iy^j \in C[x, y] \). Then \( f \perp f \) if and only if there exist two complex sequences \( \{p_i\}_{i \in \mathbb{Z}} \) and \( \{q_i\}_{i \in \mathbb{Z}} \) such that \( p_m = 0, q_k = 0, \) and \( p_k = -q_m \neq 0 \),

\[
f(x, y) = P(x)Q(y) - P(y)Q(x), \tag{2.6}
\]

where \( P(x) = \sum_{i=-\infty}^{\pm \infty} p_i x^i, \) \( Q(x) = \frac{1}{q_m} \sum_{i=-\infty}^{\pm \infty} q_i x^i. \)

Proof. By Lemma 1, it is easily seen that \( f \perp f \) is equivalent to

\[
\lambda(i, j)\lambda(m, k) + \lambda(i, k)\lambda(j, m) - \lambda(i, m)\lambda(j, k) = 0. \tag{2.7}
\]

Assume that \( \lambda(m_0, k_0) \neq 0 \). In this case, it is easy to solve (2.7) and to get

\[
\lambda(i, j) = \frac{\lambda(i, m_0)\lambda(j, k_0) - \lambda(i, k_0)\lambda(j, m_0)}{\lambda(m_0, k_0)}. \tag{2.8}
\]

Starting with this identity, we proceed to show the theorem in question by proving that (2.7) is equivalent to (2.8) and then expressing the solution given by (2.8) in terms of generating function. At first, by the definition, it is easily seen that (2.7) implies (2.8). We need only to show is that (2.7) can be reduced from (2.8). To do this, replacing each \( \lambda(\bullet, \bullet) \) by (2.8) in the left-hand side of (2.7) to calculate

\[
\begin{align*}
\lambda(i, j)\lambda(m, k) + \lambda(i, k)\lambda(j, m) - \lambda(i, m)\lambda(j, k) &= \frac{\lambda(i, m_0)\lambda(j, k_0) - \lambda(i, k_0)\lambda(j, m_0)}{\lambda(m_0, k_0)} \cdot \frac{\lambda(m, m_0)\lambda(k, k_0) - \lambda(m, k_0)\lambda(k, m_0)}{\lambda(m_0, k_0)} \\
&\quad - \frac{\lambda(i, m_0)\lambda(k, k_0) - \lambda(i, k_0)\lambda(k, m_0)}{\lambda(m_0, k_0)} \cdot \frac{\lambda(j, m_0)\lambda(m, k_0) - \lambda(j, k_0)\lambda(m, m_0)}{\lambda(m_0, k_0)} \\
&\quad + \frac{\lambda(i, m_0)\lambda(m, k_0) - \lambda(i, k_0)\lambda(m, m_0)}{\lambda(m_0, k_0)} \cdot \frac{\lambda(j, m_0)\lambda(k, k_0) - \lambda(j, k_0)\lambda(k, m_0)}{\lambda(m_0, k_0)}.
\end{align*}
\]

Now, expand and rearrange the right-hand side of this identity to arrive at
\[
\lambda(i, j)\lambda(m, k) + \lambda(i, k)\lambda(j, m) - \lambda(i, m)\lambda(j, k) \\
= \frac{\lambda(i, m_0)\lambda(k, k_0)}{\lambda(m_0, k_0)^2} [\lambda(j, k_0)\lambda(m, m_0) + \lambda(j, m_0)\lambda(m, k_0) - \lambda(m, k_0)\lambda(j, m_0)] \\
- \frac{\lambda(i, m_0)\lambda(j, k_0)}{\lambda(m_0, k_0)^2} [\lambda(m, k_0)\lambda(k, m_0) + \lambda(k, m_0)\lambda(m, k_0) - \lambda(m, k_0)\lambda(k, m_0)] \\
- \frac{\lambda(i, k_0)\lambda(j, m_0)}{\lambda(m_0, k_0)^2} [\lambda(m, m_0)\lambda(k, k_0) + \lambda(k, m_0)\lambda(m, k_0) - \lambda(m, k_0)\lambda(k, k_0)] \\
+ \frac{\lambda(i, k_0)\lambda(k, m_0)}{\lambda(m_0, k_0)^2} [\lambda(j, m_0)\lambda(m, k_0) + \lambda(j, k_0)\lambda(m, m_0) - \lambda(m, m_0)\lambda(j, k_0)].
\]

Obviously, each term within the curly braces can be further simplified. The corresponding result is

\[
\lambda(i, j)\lambda(m, k) + \lambda(i, k)\lambda(j, m) - \lambda(i, m)\lambda(j, k) \\
= \frac{\lambda(i, m_0)\lambda(k, k_0)\lambda(j, k_0)}{\lambda(m_0, k_0)^2} - \frac{\lambda(i, m_0)\lambda(j, k_0)\lambda(k, k_0)}{\lambda(m_0, k_0)^2} - \frac{\lambda(i, k_0)\lambda(j, m_0)\lambda(k, k_0)}{\lambda(m_0, k_0)^2} + \frac{\lambda(i, k_0)\lambda(k, m_0)\lambda(j, m_0)}{\lambda(m_0, k_0)^2} \\
= 0.
\]

Finally, we turn to express \(\lambda(i, j)\) given by (2.8) in terms of generating function. To do this, suppose that \(\{p_i\}_{i \in \mathbb{Z}}\) and \(\{q_i\}_{i \in \mathbb{Z}}\) are arbitrary complex sequences such that \(p_{k_0} = -q_{m_0} \neq 0\). Define that

\[
\lambda(i, m_0) = p_i, \lambda(i, k_0) = q_i, i \in \mathbb{Z}.
\]

Note that \(p_{m_0} = 0, q_{k_0} = 0\), because \(f(x, y) = -f(y, x)\). Hence, \(\lambda(i, j)\) given by (2.8) may be rewritten as

\[
f(x, y) = P(x)Q(y) - P(y)Q(x), \quad (2.9)
\]

where \(P(x) = \sum_{i=-\infty}^{+\infty} p_i x^i, Q(x) = \sum_{i=-\infty}^{+\infty} q_i x^i\). This gives the complete proof of theorem.

**Corollary 1** The following functions \(f(x, y)\) are self orthogonal.

\[
f(x, y) = x - y, (y - x)(1 - \frac{xy}{d}), (x - y)(1 - \frac{b}{axy}).
\]

As an important case that both \(f(x, y) \in \text{Ker} L_3\) and \(f(x, y)\) is infinite bilateral series, we obtain an alternate proof of the addition formula of the theta function.
Corollary 2 Define \( \theta(x) = (x; q)_\infty (\frac{4}{x}; q)_\infty \) (\(|q| < 1\)), and \( f(x, y) = y\theta(xy)\theta(\frac{x}{y}) \). Then \( f(x, y) \) is self orthogonal.

Proof. In Theorem 2, set \((m_0, k_0) = (1, 0)\) and

\[
p_i = \begin{cases} 
q^{m^2-m}, & \text{if } i = 2m; \\
(q; q)_{\infty}^2, & \text{if } i = 2m + 1; \\
0, & \text{if } i = 2m + 1
\end{cases}
\]

and

\[
q_i = \begin{cases} 
-q^{m^2}, & \text{if } i = 2m + 1; \\
(q; q)_{\infty}^2, & \text{if } i = 2m.
\end{cases}
\]

A direct calculation gives the solution

\[
f(x, y) = \frac{1}{(q; q)_{\infty}^2} \sum_{i, j = -\infty}^{+\infty} q^{i^2+i+j^2} (x^{2i}y^{2j+1} - y^{2i}x^{2j+1}).
\]

By the definition of \( \theta(x) \) and Jacobi’s triple product identity

\[
\sum_{i = -\infty}^{+\infty} (-1)^i q(i)_q x^i = (x; q)_\infty (\frac{q}{x}; q)_\infty (q; q)_\infty,
\]

it follows that \( f(x, y) = y\theta(xy)\theta(\frac{x}{y}) \).

The next theorem gives the explicit expression of \( f(x, y) \in \text{Ker} \mathcal{L}^{(g)}_3 \).

Theorem 3 Let \( g(x, y) = \sum_{i,j = -\infty}^{\infty} c(i, j) x^i y^j \neq 0 \) such that \( g \perp g \). Then \( f \perp g \) if and only if there exists \( c(m_0, k_0) \neq 0 \), such that

\[
f(x, y) = \frac{1}{c(m_0, k_0)} \left( P(x)[x^{m_0}]g(x, y) - Q(x)[x^{k_0}]g(x, y) \right),
\]

where \( P(x) = \sum_{i = -\infty}^{+\infty} p_i x^i \), \( Q(x) = \sum_{i = -\infty}^{+\infty} q_i x^i \), \( p_i, q_i \) are arbitrary sequences, \( [x^{m_0}]g(x, y) \) stands the coefficient of \( x^{m_0} \) in \( g(x, y) \).

Proof. Suppose that \( f(x, y) = \sum_{i,j = -\infty}^{+\infty} \lambda(i, j) x^i y^j \). Observe that \( f \perp g \) is equivalent to, by Lemma 4, the relation

\[
\begin{align*}
& c(m, i)c(k, j) - c(k, i)c(m, j) + c(k, m)c(i, j) = 0; \\
& c(m, i)\lambda(k, j) - c(m, j)\lambda(k, i) + c(i, j)\lambda(k, m) = 0.
\end{align*}
\]

Let \( m_0 \) and \( k_0 \) be two integers such that \( c(m_0, k_0) \neq 0 \). Use this to solve (2.11) and find that
\[
\lambda(k, j) = \frac{c(m_0, j)\lambda(k, k_0) + c(j, k_0)\lambda(k, m_0)}{c(m_0, k_0)}.
\] (2.12)

Proceeding as before, we need only to show that (2.11) and (2.12) are equivalent to each other. At first, (2.11) implies (2.12), which is obvious. Conversely, (2.12) implies (2.11), too. To show this, replacing each \(\lambda(\bullet, \bullet)\) by (2.12) in the left-hand side of the second identity of (2.11) directly gives that

\[
c(m, i)\lambda(k, j) - c(m, j)\lambda(k, i) + c(i, j)\lambda(k, m)
= c(m, i) \frac{c(m_0, j)\lambda(k, k_0) + c(j, k_0)\lambda(k, m_0)}{c(m_0, k_0)}
- c(m, j) \frac{c(m_0, i)\lambda(k, k_0) + c(i, k_0)\lambda(k, m_0)}{c(m_0, k_0)}
+ c(i, j) \frac{c(m_0, m)\lambda(k, k_0) + c(m, k_0)\lambda(k, m_0)}{c(m_0, k_0)}
= \frac{\lambda(k, k_0)}{c(m_0, k_0)} [c(m, i)c(m_0, j) - c(m, j)c(m_0, i) + c(i, j)c(m_0, m)]
+ \frac{\lambda(k, m_0)}{c(m_0, k_0)} [c(m, i)c(j, k_0) - c(m, j)c(i, k_0) + c(i, j)c(m, k_0)].
\]

Observe that \(g(x, y) \in \operatorname{Ker}L_3\) gives

\[
\begin{cases}
    c(m, i)c(m_0, j) - c(m, j)c(m_0, i) + c(i, j)c(m_0, m) = 0; \\
    c(m, i)c(j, k_0) - c(m, j)c(i, k_0) + c(i, j)c(m, k_0) = 0.
\end{cases}
\]

Thus, the previous identity becomes

\[
c(m, i)\lambda(k, j) - c(m, j)\lambda(k, i) + c(i, j)\lambda(k, m) = 0.
\]

So (2.11) follows. Therefore, (2.12) gives the desired function. Next, let \(\{p_i\}\) and \(\{q_i\}\) be arbitrary complex sequences. Define

\[
P(x) = \sum_{i=-\infty}^{\infty} p_i x^i, Q(x) = \sum_{i=-\infty}^{\infty} q_i x^i,
\]

and \(\lambda(i, k_0) = p_i, \lambda(j, m_0) = q_j\). Then (2.12) can be rewritten as

\[
\lambda(i, j) = \frac{c(m_0, j)p_i + c(j, k_0)q_i}{c(m_0, k_0)}.
\]
which leads to
\[
    f(x, y) = \frac{1}{c(m_0, k_0)} \sum_{i,j=-\infty}^{\infty} (c(m_0, j)p_i + c(j, k_0)q_i)x^iy^j
\]
\[
    = \frac{1}{c(m_0, k_0)} \sum_{i=-\infty}^{\infty} p_ix^i \sum_{j=-\infty}^{\infty} c(m_0, j)y^j - \frac{1}{c(m_0, k_0)} \sum_{i=-\infty}^{\infty} q_ix^i \sum_{j=-\infty}^{\infty} c(k_0, j)y^j
\]
\[
    = \frac{1}{c(m_0, k_0)} \left( P(x)[x^{m_0}]g(x, y) - Q(x)[x^{k_0}]g(x, y) \right). 
\]
(2.13)

Note that the above results in together provides us with a constructive way to derive different \((f, g)\)-inversions and expansion formulae, when this is necessary.

**Corollary 3** The following function pairs \((f, g)\) satisfy that \(f \in \text{Ker} \mathcal{L}_3(g)\).

\[(f, g) = (P(x) + yQ(x), x - y), \left( (1 - axy)(1 - b\frac{x}{y}), (x - y)(1 - \frac{b}{axy}) \right), \]
\[
    \left( (x + y)(x + \frac{b}{ay}), (x - y)(1 - \frac{b}{axy}) \right). 
\]

**Proof.** By Corollary 1, each \(g \in \text{Ker} \mathcal{L}_3\). It only needs to verify \(f \perp g\) by Maple in a straightforward way. Left to the reader.

### 3 The \((f, g)\)-summation formula

As mentioned in Section 1, we start with a general summation formula related with \(f\) and \(g\) satisfying \(f \perp g\).

**Lemma 5** Let \(\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}\) be arbitrary sequences such that none of the denominators in (3.1) vanish. Then for any integer \(m \geq 0\),

\[
    \sum_{k=0}^{m} f(a_k, b_k)g(c_k, d_k) \frac{\prod_{j=1}^{k-1} f(a_j, c_j) \prod_{j=1}^{k-1} g(b_j, d_j)}{\prod_{j=1}^{k} f(a_j, d_j) \prod_{j=1}^{k} g(b_j, c_j)}
\]
\[
    = \frac{\prod_{j=1}^{m} f(a_j, c_j) \prod_{j=1}^{m} g(b_j, d_j)}{\prod_{j=1}^{m} f(a_j, d_j) \prod_{j=1}^{m} g(b_j, c_j)} + C 
\]
(3.1)

provided \(f \perp g\), i.e., \(f \in \text{Ker} \mathcal{L}_3^{(g)}\), where the constant \(C\) is uniquely decided by the initial conditions.
The idea of the proof relies on the difference equation: let \( \Delta S_n = S_{n+1} - S_n = t_n \). Then by telescoping, \( \sum_{k=0}^{n} t_k = S_n + C \). Conversely, if \( \sum_{k=0}^{n} t_k = S_n + C \) and \( t_k (k \neq n) \) is independent of \( n \), then \( t_n = \Delta S_n \). The constant \( C \) is given by the initial condition.

**Proof.** Define

\[
F(m) = \frac{\prod_{j=1}^{m} f(a_j, c_j) \prod_{i=1}^{m} g(b_j, d_j)}{\prod_{i=1}^{m} f(a_j, d_j) \prod_{j=1}^{m} g(b_j, c_j)}, \quad (3.2)
\]

and assume further that

\[
F(m) = \sum_{k=0}^{m} f(a_k, b_k) G(k) \frac{\prod_{j=0}^{k-1} g(b_j, d_j)}{\prod_{j=1}^{k} f(x_i, d_j)} + C. \quad (3.3)
\]

Thus, as mentioned above, we need only to show that \( G(n) \) is independent of \( m \). We proceed to do this in a straightforward calculation. Suppose

\[
\frac{\prod_{j=1}^{m} f(a_j, c_j) \prod_{i=1}^{m} g(b_j, d_j)}{\prod_{i=1}^{m} f(a_j, d_j) \prod_{j=1}^{m} g(b_j, c_j)} = \sum_{k=0}^{m} f(a_k, b_k) G(k) \frac{\prod_{j=0}^{k-1} g(b_j, d_j)}{\prod_{j=1}^{k} f(x_i, d_j)} + C. \quad (3.4)
\]

Calculation with the assumption gives that

\[
G(k) = \frac{F(k) - F(k-1) \prod_{j=1}^{k} f(a_j, d_j)}{f(a_k, b_k) \prod_{j=0}^{k-1} g(b_j, d_j)} = \frac{1}{f(a_k, b_k)} \left\{ \frac{F(k)}{F(k-1)} - 1 \right\} F(k-1) \frac{\prod_{j=1}^{k} f(a_j, d_j)}{\prod_{j=0}^{k-1} g(b_j, d_j)}. \quad (3.5)
\]

Observe that the term within the curly braces

\[
\frac{F(k)}{F(k-1)} - 1 = \frac{f(a_k, c_k) g(b_k, d_k)}{g(b_k, c_k) f(a_k, d_k)} - 1 = \frac{f(a_k, c_k) g(b_k, d_k) - g(b_k, c_k) f(a_k, d_k)}{g(b_k, c_k) f(a_k, d_k)}.
\]

Simplify the numerator in the latter expression further by means of the known condition that \( f \perp g \), namely say,

\[
f(a_k, c_k) g(b_k, d_k) - g(b_k, c_k) f(a_k, d_k) = f(a_k, b_k) g(c_k, d_k), \quad (3.6)
\]

to get

\[
\frac{F(k)}{F(k-1)} - 1 = \frac{f(a_k, b_k) g(c_k, d_k)}{g(b_k, c_k) f(a_k, d_k)}.
\]
The production after some calculation is

\[ F(k - 1) \prod_{j=1}^{k} f(a_j, d_j) \prod_{j=0}^{k-1} g(b_j, d_j) = \frac{\prod_{j=1}^{k-1} f(a_j, c_j) \prod_{j=1}^{k-1} g(b_j, d_j) \prod_{j=0}^{k-1} f(a_j, d_j) \prod_{j=0}^{k-1} g(b_j, d_j)}{g(b_0, d_0) \prod_{j=1}^{k-1} g(b_j, c_j)}. \]

Insert all these notes into (3.5). It gives

\[ G(k) = \frac{1}{f(a_k, b_k) g(c_k, d_k)} \frac{f(a_k, d_k)}{g(b_0, d_0)} \frac{\prod_{j=1}^{k-1} f(a_j, c_j)}{\prod_{j=1}^{k-1} g(b_j, c_j)}. \]

As expected, \( G(k) \) is indeed independent of \( m \). Thus, (3.4) holds. Inserting \( G(k) \) into (3.4) and making some simplifications, we get the desired result at once.

**Remark.** Note that the constant \( C \) in Lemma 5 is not unique. If we define

\[ \prod_{j=1}^{0} f(a_j, c_j) g(b_j, d_j) = 1, \prod_{j=1}^{-1} f(a_j, c_j) g(b_j, d_j) = 0, \]

then we find that \( C = -1 \). If we redefine

\[ \prod_{j=1}^{0} f(a_j, c_j) g(b_j, d_j) = 1, \prod_{j=1}^{-1} f(a_j, c_j) g(b_j, d_j) = \frac{1}{f(a_0, c_0) g(b_0, d_0)}, \]

then

\[ C = \frac{f(a_0, d_0) g(c_0, b_0)}{f(a_0, c_0) g(b_0, d_0)}. \]

To avoid this confusion, also with an effort to extend this result to the setting of bilateral summation, we employ the convention of defining (cf. [14, Eq.(3.6.12)])

\[ \prod_{j=k}^{m} A_j = \begin{cases} A_k A_{k+1} \cdots A_m, & m \geq k; \\ 1, & m = k - 1; \\ (A_{m+1} A_{m+2} \cdots A_{k-1})^{-1}, & m \leq k - 2 \end{cases} \]

over \( \mathbb{Z} \).
Now, we are in a position to show our main theorem which gives the bilateral form of Lemma 5.

**Theorem 4** Let \( \{a_i\}, \{b_i\}, \{c_i\}, \{d_i\} \) be arbitrary sequences such that none of the denominators in (3.7) vanish. Then for any nonnegative integers \( m, n, \)

\[
\sum_{k=-n}^{m} f(a_k, b_k)g(c_k, d_k) \prod_{j=1}^{k-1} f(a_j, c_j) \prod_{j=1}^{k} g(b_j, d_j) \prod_{j=1}^{n-1} f(a_j, d_j) \prod_{j=1}^{n} g(b_j, c_j) = \prod_{j=1}^{m-1} f(a_j, c_j) \prod_{j=1}^{m} g(b_j, d_j) - \prod_{j=1}^{n-1} f(a_j, d_j) \prod_{j=1}^{n} g(b_j, c_j) \tag{3.7}
\]

provided \( f \perp g \).

**Proof.** Assume that

\[
\sum_{k=-n}^{m} t_k = \prod_{j=1}^{m} f(a_j, c_j) \prod_{j=1}^{m} g(b_j, d_j) - \prod_{j=1}^{n} f(a_j, d_j) \prod_{j=1}^{n} g(b_j, c_j).
\]

Apply the same argument as in Lemma 5 only to find

\[
t_k = f(a_k, b_k)g(c_k, d_k) \prod_{j=1}^{k-1} f(a_j, c_j) \prod_{j=1}^{k} g(b_j, d_j) \prod_{j=1}^{m} f(a_j, d_j) \prod_{j=1}^{m} g(b_j, c_j)
\]

for \( k \geq 0 \). However for \( k \leq -1 \),

\[
t_k = f(a_k, b_k)g(c_k, d_k) \prod_{j=1}^{k} f(a_j, d_j) \prod_{j=1}^{n} g(b_j, c_j) \prod_{j=k+1}^{m} f(a_j, c_j) \prod_{j=k+1}^{m} g(b_j, d_j) \prod_{j=k}^{n} f(a_j, d_j) \prod_{j=k}^{n} g(b_j, c_j).
\]

In this instance, by the above convention, \( t_k \) can be still rewrite as

\[
t_k = f(a_k, b_k)g(c_k, d_k) \prod_{j=1}^{k-1} f(a_j, c_j) \prod_{j=1}^{k} g(b_j, d_j) \prod_{j=1}^{m} f(a_j, d_j) \prod_{j=1}^{m} g(b_j, c_j)
\]

This yields

\[
\sum_{k=-n}^{m} f(a_k, b_k)g(c_k, d_k) \prod_{j=1}^{k-1} f(a_j, c_j) \prod_{j=1}^{k} g(b_j, d_j) \prod_{j=1}^{m} f(a_j, d_j) \prod_{j=1}^{m} g(b_j, c_j) = \prod_{j=1}^{m} f(a_j, c_j) \prod_{j=1}^{m} g(b_j, d_j) - \prod_{j=1}^{n} f(a_j, d_j) \prod_{j=1}^{n} g(b_j, c_j) + C,
\]

where \( C \) is independent of \( m, n \). When \( m = n = 0 \), then from \( f \perp g \) it follows \( C = 0 \). This gives the complete proof of (3.7).
From now on, we call (3.7) the \((f, g)\)-summation formula. As we will see later, the \((f, g)\)-summation formula indeed unifies and extends all known bibasic summation formulae. It should be pointed that the corresponding proofs of these summation formulas seem somewhat mysterious because they depend heavily on a tricky factorization of a difference of two four-term products into a four-term product (cf. [14, Eq.(3.6.10)]). In the author’s view, the \((f, g)\)-summation formula provides a natural interpretation about such “mysterious” phenomena. Before we turn to illustrate this, it is worth noting two particular cases of the \((f, g)\)-summation formula.

Set \(c_j = b_0, d_j = x, n = 0\) in (3.7). Then the \((f, g)\)-summation formula reduces to

**Corollary 4** With the assumption as in Theorem 4. Then for any integer \(m \geq 0\),

\[
\sum_{k=0}^{m} f(a_k, b_k) \prod_{j=0}^{k-1} f(a_j, b_0) \prod_{j=0}^{k-1} g(b_j, x) = \frac{\prod_{j=1}^{m} f(a_j, b_0) \prod_{j=1}^{m} g(b_j, x)}{\prod_{j=1}^{m} f(a_j, x)} \prod_{j=1}^{m} g(a_j, x) \quad (3.8)
\]

provided \(f \perp g\).

Another immediate consequence of Theorem 4 is Chu’s extension (cf. [8, Theorem A]) of Gasper’s bibasic summation formula, which turned out to be useful in deriving hypergeometric identities. To state Chu’s result, define

\[
\phi^*(x; m) = \prod_{i=m}^{\infty} (a_i + xb_i); \psi^*(y; m) = \prod_{i=m}^{\infty} (c_i + yd_i);
\]

\[
\phi(x; m) = \prod_{i=0}^{m-1} (a_i + xb_i) = \phi^*(x; 0)/\phi^*(x; m);
\]

\[
\psi(y; m) = \prod_{j=0}^{m-1} (c_j + yd_j) = \psi^*(y; 0)/\psi^*(y; m).
\]

It is easy to check that the definitions of \(\phi\) and \(\psi\) are confirmed with the previous convention. The next corollary is just Chu’s result.

**Corollary 5** Let \(\phi\) and \(\psi\) be given as above, \(m, n \in \mathbb{Z}\).

\[
(x - y) \sum_{k=m}^{n} (a_k d_k - b_k c_k) \frac{\phi(x; k)\psi(y; k)}{\phi(y; k + 1)\psi(x; k + 1)} = \frac{\phi(x; m)\psi(y; m)}{\phi(y; m)\psi(x; m)} - \frac{\phi(x; n + 1)\psi(y; n + 1)}{\phi(y; n + 1)\psi(x; n + 1)} \quad (3.9)
\]
Proof. Without any loss of generality, assume \(m,n \geq 0\). Take in (3.1) \(f(x,y) = g(x,y) = x - y\) and make the substitution

\[
a_i \rightarrow \frac{a_{i-1}}{b_{i-1}}, b_i \rightarrow \frac{c_{i-1}}{d_{i-1}};
\]

\[
c_i \rightarrow -x, d_i \rightarrow -y, \quad i = 0, 1, 2, \ldots
\]

So the products are

\[
\prod_{j=1}^{k} f(a_j, c_j) = \frac{\phi(x; k)}{b_0 b_1 \cdots b_{k-1}}, \quad \prod_{j=1}^{k} g(b_j, d_j) = \frac{\psi(y; k)}{d_0 d_1 \cdots d_{k-1}},
\]

and

\[
\frac{\prod_{j=1}^{k} f(a_j, c_j)}{\prod_{j=1}^{k} f(a_j, d_j)} = \frac{b_{k-1} \phi(x; k-1)}{\phi(y; k)}, \quad \frac{\prod_{j=1}^{k} g(b_j, d_j)}{\prod_{j=1}^{k} g(b_j, c_j)} = \frac{d_{k-1} \psi(y; k-1)}{\psi(x; k)}.
\]

Thus, (3.1) becomes

\[
(y - x) \sum_{k=0}^{n-1} \left( a_{k-1} d_{k-1} - b_{k-1} c_{k-1} \right) \frac{\phi(x; k-1) \psi(y; k-1)}{\phi(y; k) \psi(x; k)} = \frac{\phi(x; n) \psi(y; n)}{\phi(y; n) \psi(x; n)} + C.
\]

Substitute \(n\) by \(n + 1\) and \(k\) by \(k + 1\). This identity can be reformulated as

\[
(y - x) \sum_{k=0}^{n} \left( a_k d_k - b_k c_k \right) \frac{\phi(x; k) \psi(y; k)}{\phi(y; k+1) \psi(x; k+1)} = \frac{\phi(x; n + 1) \psi(y; n + 1)}{\phi(y; n + 1) \psi(x; n + 1)} + C.
\]

(3.10)

Replace \(n\) by \(m - 1\). So (3.10) becomes

\[
(y - x) \sum_{k=0}^{m-1} \left( a_k d_k - b_k c_k \right) \frac{\phi(x; k) \psi(y; k)}{\phi(y; k+1) \psi(x; k+1)} = \frac{\phi(x; m) \psi(y; m)}{\phi(y; m) \psi(x; m)} + C.
\]

(3.11)

Subtracting (3.11) from (3.10) leads to the desired result.

Next, we will exhibit some remarkable bibasic summation formulas from the \((f,g)\)-summation formula (3.7) by specializing \(f\) and \(g\) which are orthogonal to each other and related parameters. Among them are Subbarao and Verma’s summation formula [31], Gasper and Rahman’s bibasic summation formula [12], Chu’s formula [8] and Macdonald’ extensions [3]. These facts show convincingly that \(KerL_3^{(g)}\) provides indeed a rich source of bibasic summation formulae.
I. \((1, x - y)\)-summation formula.

**Corollary 6** With the assumption as in Theorem 4. Then for any integers \(m, n \geq 0\),

\[
\sum_{k=-n}^{m} (c_k - d_k) \frac{\prod_{j=1}^{k} (b_j - d_j)}{\prod_{j=1}^{k} (b_j - c_j)} = \frac{\prod_{j=1}^{m} (b_j - d_j)}{\prod_{j=1}^{m} (b_j - c_j)} - \frac{\prod_{j=-n}^{0} (b_j - c_j)}{\prod_{j=-n}^{0} (b_j - d_j)}.
\]

This identity follows from Theorem 4 by setting

\[
f(x, y) = 1, \ g(x, y) = x - y,
\]

and the simple fact that \(f \perp g\). It contains the following Subarao and Verma’s summation formula (cf. [31, Eq.(3.1)]) as a special case.

**Example 1**

\[
\sum_{k=-n}^{m} a z_k \left(1 - \frac{x_k}{az_k}\right) \left(1 - \frac{y_k}{az_k}\right) \frac{\prod_{j=1}^{k-1} (1 - x_j)(1 - y_j)}{\prod_{j=1}^{k} (1 - a_j)(1 - \frac{x_k y_k}{a_j})} = \frac{\prod_{j=1}^{m} (1 - x_j)(1 - y_j)}{\prod_{j=1}^{m} (1 - a_j)(1 - \frac{x_k y_k}{a_j})}.
\]

**Proof.** Actually, make the substitution at first in (3.12) \(b_j \mapsto t_j, c_j \mapsto t_j^{-1} - y_j, d_j \mapsto t_j^{-1} - u_j\), and then specificize parameters \(u_j \mapsto a z_j, t_j \mapsto x_j/u_j\). It is easy to check by direct calculation that

\[
\sum_{k=0}^{m} (c_k - d_k) \frac{\prod_{j=1}^{k-1} (b_j - d_j)}{\prod_{j=1}^{k} (b_j - c_j)} = \sum_{k=-n}^{m} \frac{(1 - t_k)(u_k - y_k)}{(1 - u_k)(1 - y_k)} \frac{1 - y_k}{1 - t_k y_k} \frac{k-1}{j=1} \frac{(b_j - d_j)}{(b_j - c_j)}
\]

\[
= \sum_{k=-n}^{m} \frac{(1 - t_k)(u_k - y_k)}{(1 - u_k)(1 - t_k y_k)} \frac{k-1}{j=1} \frac{(1 - x_j)(1 - y_j)}{(1 - a_j)(1 - \frac{x_k y_k}{a_j})}.
\]

Note that

\[
(1 - t_k)(u_k - y_k) = a z_k \left(1 - \frac{x_k}{az_k}\right) \left(1 - \frac{y_k}{az_k}\right);
\]

\[
(1 - u_k)(1 - t_k y_k) = (1 - a z_k)(1 - \frac{x_k y_k}{az_k}).
\]

Thus, the right-hand side of the previous identity can be simplified to

\[
\sum_{k=-n}^{m} a z_k \left(1 - \frac{x_k}{az_k}\right) \left(1 - \frac{y_k}{az_k}\right) \frac{\prod_{j=1}^{k-1} (1 - x_j)(1 - y_j)}{\prod_{j=1}^{k} (1 - b_j)(1 - \frac{x_k y_k}{a_j})}.
\]
This gives the desired result.

II. \((x - y, x - y)\)-summation formula.

**Corollary 7** With the assumption as in Theorem 4. Then for any integers \(m, n \geq 0\),

\[
\sum_{k=-n}^{m} (a_k - b_k)(c_k - d_k) \frac{\prod_{i=1}^{k-1}(a_j - c_j) \prod_{j=1}^{k-1}(b_j - d_j)}{\prod_{j=1}^{k}(a_j - d_j) \prod_{j=1}^{k}(b_j - c_j)} = \frac{\prod_{j=1}^{m}(a_j - c_j) \prod_{j=1}^{m}(b_j - d_j)}{\prod_{j=1}^{m}(a_j - d_j) \prod_{j=1}^{m}(b_j - c_j)} - \frac{\prod_{j=-n}^{0}(a_j - c_j) \prod_{j=-n}^{0}(b_j - d_j)}{\prod_{j=-n}^{0}(a_j - d_j) \prod_{j=-n}^{0}(b_j - c_j)}. \tag{3.14}
\]

This identity follows from Theorem 4 by setting

\[f(x, y) = x - y, g(x, y) = x - y.\]

By Corollary 1, we see that \(f \perp g\). It is worth mentioning that this result contains Subarao and Verma’s summation formula [31, Eq.(2.1)] with four independent bases. We restate it as follows:

**Example 2** Let \(\{u_i\}, \{v_i\}, \{w_i\}, \{z_i\}\) be arbitrary sequences such that none of the denominators in (3.15) vanish, \(m, n\) be nonnegative integers. Then the following holds,

\[
\sum_{k=-n}^{m} \frac{u_k v_k w_k z_k}{u_k v_k w_k} \left(1 - \frac{w_k z_k}{u_k v_k}\right) \left(1 - \frac{u_k z_k}{v_k w_k}\right) \left(1 - \frac{v_k z_k}{u_k w_k}\right) = \frac{\prod_{j=1}^{m}(1 - u_j^2)(1 - v_j^2)(1 - w_j^2)(1 - z_j^2)}{\prod_{j=1}^{m}(1 - \frac{u_j v_j w_j}{z_j})(1 - \frac{u_j v_j z_j}{w_j})(1 - \frac{w_j z_j u_j}{v_j})(1 - \frac{w_j z_j v_j}{u_j})} - \frac{\prod_{j=-n}^{0}(1 - u_j^2)(1 - v_j^2)(1 - w_j^2)(1 - z_j^2)}{\prod_{j=-n}^{0}(1 - \frac{u_j v_j w_j}{z_j})(1 - \frac{u_j v_j z_j}{w_j})(1 - \frac{w_j z_j u_j}{v_j})(1 - \frac{w_j z_j v_j}{u_j})}. \tag{3.15}
\]

**Proof.** Specify all parameters in (3.14) by

\[a_j = 1/A_{2,j}, b_j = 1/B_{2,j};\]
\[c_j = Y_j, d_j = X_j,\]

and then set parameters in the corresponding result by

\[A_{2,j}Y_j = A_{1,j}; B_{2,j}X_j = B_{1,j}.\]
Then, after some calculation, we arrive at

\[
\frac{\prod_{j=1}^{k-1}(1 - A_{1,j})}{\prod_{j=1}^{k}(1 - A_{2,j})} \frac{\prod_{j=1}^{k-1}(1 - B_{1,j})}{\prod_{j=1}^{k}(1 - B_{2,j})} \frac{A_{1,j}}{A_{2,j} B_{2,j}} = \sum_{k=-n}^{m} \frac{1}{A_{2,k}} - \frac{B_{1,k}}{B_{2,k}} A_{2,k} B_{2,k}
\]

where the term \( C_n \) is

\[
C_n = \prod_{j=-n}^{0} \frac{(1 - A_{2,j} B_{1,j}) (1 - B_{2,j} A_{1,j})}{(1 - A_{1,j})(1 - B_{1,j})}.
\]

In order to show Subarao and Verma’s result (3.15), we make the further substitution of parameters

\[
\begin{align*}
A_{1,j} &= \frac{u_j^2 + v_j^2}{1 + u_j^2 v_j}, \\
A_{2,j} &= \frac{u_j v_j}{1 + u_j^2 v_j}, \\
B_{1,j} &= \frac{w_j^2 + z_j^2}{1 + w_j^2 z_j}, \\
B_{2,j} &= \frac{w_j z_j}{1 + w_j^2 z_j},
\end{align*}
\]

which in turn gives

\[
\begin{align*}
A_{1,j}/A_{2,j} &= u_j/v_j + v_j/u_j, \\
B_{1,j}/B_{2,j} &= w_j/z_j + z_j/w_j.
\end{align*}
\]

Using these relations to calculate

\[
\frac{1 - A_{1,j}}{1 - A_{2,j} B_{1,j} B_{2,j}} = \frac{(1 - u_j^2)(1 - v_j^2)}{(1 - u_j v_j/z_j)(1 - w_j/v_j)}.
\]

Also, similar calculation gives that

\[
\frac{1 - B_{1,j}}{1 - B_{2,j} A_{1,j} A_{2,j}} = \frac{(1 - w_j^2)(1 - z_j^2)}{(1 - w_j z_j/v_j)(1 - w_j/v_j)};
\]

\[
A_{2,k} B_{2,k} = \frac{(1 - u_k v_k w_k z_k)(1 - w_k/z_k)(1 - w_k/v_k)(1 - u_k/v_k)}{(1 - u_k v_k w_k z_k)(1 - w_k/z_k)(1 - w_k/v_k)(1 - u_k/v_k)}.
\]

So, the production is
This also gives

\[ A_{2,k}B_{2,k} \prod_{j=1}^{k-1} \left( 1 - A_{1,j} \right) \prod_{j=1}^{k-1} \left( 1 - B_{1,j} \right) = u_k v_k w_k z_k \prod_{j=1}^{k} \left( 1 - u_j^2 \right) \left( 1 - v_j^2 \right) \left( 1 - w_j^2 \right) \left( 1 - z_j^2 \right) \]

Further calculation yields

\[
\begin{align*}
\frac{1}{A_{2,k}} - \frac{1}{B_{2,k}} &= \frac{1}{u_k v_k} \left( \frac{1 + u_k^2 v_k^2}{w_k z_k} - \frac{1 + w_k^2 z_k^2}{u_k} \right) = \frac{(u_k v_k w_k z_k - 1) (v_k u_k - z_k w_k)}{u_k v_k w_k z_k}; \\
\frac{A_{1,k}}{A_{2,k}} - \frac{B_{1,k}}{B_{2,k}} &= \frac{u_k}{v_k} + \frac{w_k}{u_k} - \frac{z_k}{w_k} = \frac{(v_k w_k - z_k u_k) (-u_k w_k + v_k z_k)}{u_k v_k w_k z_k}.
\end{align*}
\]

This also gives

\[
\frac{1}{A_{2,k}} - \frac{1}{B_{2,k}} \left( \frac{A_{1,k}}{A_{2,k}} - \frac{B_{1,k}}{B_{2,k}} \right) = \frac{(1 - u_k v_k w_k z_k) (u_k v_k - w_k z_k) (v_k w_k - u_k z_k) (u_k w_k - v_k z_k)}{(u_k v_k w_k z_k)^2}.
\]

Insert all these into (3.16) and rearrange the resulting identity. The result is (3.15).

Observe that by performing various substitutions we may yet deduce other summation formulas obtained by Subarao and Verma [31]. We leave them for the interested reader.

Further, making the substitution \( n \mapsto 0, b_j \mapsto b_0/b_j, c_j \mapsto 1, d_j \mapsto x \) in Corollary 7 gives Chu’s extension of Krattenthaler’s matrix inversion [8].

**Example 3** Let \( \{a_i\}, \{b_i\} \) be arbitrary sequences, \( x \) be indeterminate. Then for any integer \( m \geq 0 \),

\[
\sum_{k=0}^{m} \frac{b_0 - b_k a_k}{b_0 - b_0 a_0} \prod_{j=0}^{k-1} (1 - a_j) (b_0 - b_j x) \frac{1}{x^k} = \prod_{j=1}^{m} (1 - a_j) (b_0 - b_j x) \frac{1}{x^m}.
\]

**III.** \((1 - a x y)(1 - b x^2)_y, (x - y)(1 - \frac{b}{a x y})\)-summation formula.

Define that

\[
g(x, y) = (x - y)(1 - \frac{b}{a x y}), \quad f(x, y) = (1 - a x y)(1 - \frac{b}{x y}).
\]

Then by Corollary 3, \( f \perp g \). This reduces Theorem 4 to
Corollary 8  With the assumption as in Theorem 4. Then for any integer \( m, n \geq 0 \),

\[
\sum_{k=-n}^{m} (1 - a a_k b_k)(1 - b a_k b_k/c_k)(c_k - d_k)(1 - b/a_{k} d_k)
\]

\[
\prod_{j=1}^{m} (1 - a a_j c_j)(1 - b a_j c_j/c_j) \prod_{j=1}^{m} (b_j - d_j)(1 - b/a_{j} d_j)
\]

\[
\prod_{j=1}^{m} (1 - a a_j d_j)(1 - b a_j d_j/c_j) \prod_{j=1}^{m} (b_j - c_j)(1 - b/a_{j} c_j)
\]

\[
- \prod_{j=-n}^{m} (1 - a a_j d_j)(1 - b a_j d_j/c_j) \prod_{j=-n}^{m} (b_j - d_j)(1 - b/a_{j} d_j)
\]

\[
- \prod_{j=-n}^{m} (1 - a a_j d_j)(1 - b a_j d_j/c_j) \prod_{j=-n}^{m} (b_j - c_j)(1 - b/a_{j} c_j)
\]

\[ (3.17) \]

Identity (3.17) contains Gasper and Rahman’s indefinite bibasic summation formula [15, Eq.(3.6.3)] as a special case. In fact, set

\[ a_k = p^k, b_k = dq^k, c_k = 1, d_k = \frac{d}{x}. \]

It is easily seen that

\[
\sum_{k=-n}^{m} \frac{(1 - a d p^k q^k)(1 - b/d p^k q^{-k})}{(1 - a d)(1 - b/d)} \frac{(a, b; p)_k(x, a d^2/(bx); q)_k}{(d q, a d q/b; q)_k(a d p/x, b p x/d; p)_k} q^k
\]

\[
= \frac{(1 - a)(1 - b)(1 - x)(1 - a d^2/(bx))}{(1 - a d)(1 - b/d)(d - x)(1 - a d/(bx))} \left\{ \frac{(a p, b p; p)_m(x q, a d^2 q/(b x); q)_m}{(d q, a d q/b; q)_m(a d p/x, b p x/d; p)_m} \right. \\
\left. - \frac{(x/(a d), d/(b x); p)_{n+1}(1/d, b/(a d); q)_{n+1}}{(1/x, b x/(a d^2); q)_{n+1}(1/a, b/p)_{n+1}} \right\}.
\]

\[ (3.18) \]

Gasper and Rahman used (3.18) to set up a series of quadratic and cubic summation and transformation formulas of the basic hypergeometric series. See [14,15, Section 3.8] for more details.

Further, let \( n = 0, b = 0, d = 1 \) or \( n = 0, d = 1 \). In both cases, \( (1/d; q)_{n+1} = 0 \). After some simplification, this identity reduces to Gosper’s bibasic summation formula

\[
\sum_{k=0}^{m} \frac{1 - a p^k q^k (a; p)_k(1/x; q)_k}{1 - a} x^k = \frac{(a p; p)_m(q/x; q)_m}{(q; q)_m(a p x; p)_m} x^m
\]

\[ (3.19) \]
and Gasper’s bibasic summation formula

\[
\sum_{k=0}^{m} \frac{(1 - ap^k q^k)(1 - bp^k q^{-k})}{(1 - a)(1 - b)} (a, b; p)_k (x, a/(bx); q)_k q^k
\]

\[
= \frac{(ap, bp; p)_m (xq, aq/(bx); q)_m}{(q, aq/b; q)_m (ap/x, bpx; p)_m}.
\]

(3.20)

respectively.

**IV.** \((x + y)(x + \frac{b}{ax}); (x - y)(1 - \frac{b}{axy})\)-summation formula.

Define that

\[
g(x, y) = (x - y)(1 - \frac{b}{axy}),
\]

\[
f(x, y) = (x + y)(x + \frac{b}{ay}).
\]

Then \(f \perp g\). Therefore, Theorem 4 reduces to

**Corollary 9** With the assumption as in Theorem 4. Then for any integers \(m, n \geq 0\),

\[
\sum_{k=-n}^{m} (a_k + b_k)(a_k + \frac{b}{ab_k})(c_k - d_k)(1 - \frac{b}{ac_kd_k})
\]

\[
\times \prod_{j=1}^{k-1} (a_j - c_j)(a_j + \frac{b}{ac_j}) \prod_{j=1}^{k-1} (b_j - d_j)(1 - \frac{b}{ab_jd_j})
\]

\[
\prod_{j=1}^{k} (a_j + d_j)(a_j + \frac{b}{ad_j}) \prod_{j=1}^{k} (b_j - c_j)(1 - \frac{b}{ab_jc_j})
\]

\[
\prod_{j=1}^{m} (a_j + d_j)(a_j + \frac{b}{ad_j}) \prod_{j=1}^{m} (b_j - c_j)(1 - \frac{b}{ab_jc_j})
\]

\[
\prod_{j=-n}^{0} (a_j + d_j)(a_j + \frac{b}{ad_j}) \prod_{j=-n}^{0} (b_j - c_j)(1 - \frac{b}{ab_jc_j})
\]

\[
\prod_{j=-n}^{0} (a_j - c_j)(a_j + \frac{b}{ac_j}) \prod_{j=-n}^{0} (b_j - d_j)(1 - \frac{b}{ab_jd_j})
\]

(3.21)

**V.** \((y(1 - \frac{x}{y})(1 - \frac{xy}{d}), y(1 - \frac{x}{y})(1 - \frac{xy}{d}))\)-summation formula.

Define that

\[
g(x, y) = f(x, y) = y(1 - x/y)(1 - xy/d).
\]

Corollary 1 states that \(f \perp g\). In this case, Theorem 4 reduces to

**Corollary 10** With the assumption as in Theorem 4. Then the following
holds,
\[
\sum_{k=-n}^{m} (b_k - a_k)(1 - \frac{a_kb_k}{d})(d_k - c_k)(1 - \frac{c_kd_k}{d}) \times \prod_{j=1}^{k-1}(c_j - a_j)(1 - \frac{a_{c_j}}{d}) \prod_{j=1}^{k-1}(d_j - b_j)(1 - \frac{b_{d_j}}{d}) \prod_{j=1}^{m}(d_j - a_j)(1 - \frac{a_{d_j}}{d}) \prod_{j=1}^{m}(c_j - b_j)(1 - \frac{b_{c_j}}{d}) \prod_{j=1}^{0}(c_j - a_j)(1 - \frac{a_{c_j}}{d}) \prod_{j=1}^{0}(d_j - b_j)(1 - \frac{b_{d_j}}{d}) \prod_{j=1}^{0}(d_j - a_j)(1 - \frac{a_{d_j}}{d}) \prod_{j=1}^{0}(c_j - b_j)(1 - \frac{b_{c_j}}{d})
\]
(3.22)

An important case of this identity is Chu’s generalization of Gasper and Rahman’s formula, i.e., (3.18).

Example 4 (cf.[8]) With the assumption as Corollary 10. The following holds
\[
\sum_{k=0}^{m} (b_0 - a_kb_k)(b_k - a_kb_0/d) \frac{\prod_{j=1}^{k-1}(1 - a_j)(1 - a_j/d)(b_0 - b_j)(b_0 - b_j d/x)}{\prod_{j=1}^{k}(b_0 - b_j)(b_0 - b_j d)(1 - a_j/x)(1 - a_jx/d)} = \frac{x}{(d - x)(x - 1)} \prod_{j=1}^{m} (1 - a_j)(1 - a_j/d)(b_0 - b_j)(b_0 - b_j d)(1 - a_j/x)(1 - a_jx/d).
\]
(3.23)

Proof. Set in (3.22) \(n \mapsto 0, b_i \mapsto b_i d/b_0 = b_i', c_i \mapsto 1, d_i \mapsto x\). It is easy to check that \(b_0' = d\), and
\[
f(a_k, b_k') = (b_k d/b_0)(b_0 - a_kb_k)(b_k - a_kb_0/d)/(b_0 b_k) = d(b_0 - a_kb_k)(b_k - a_kb_0/d)/b_0^2,
\]
as well as
\[
\frac{\prod_{j=1}^{k-1} g(b_j', x)}{\prod_{j=1}^{k} f(a_j, x)} = \frac{1}{xb_0^{2k-2}} \frac{\prod_{j=1}^{k-1}(b_0 - b_j)(b_0 - b_j d/x)}{\prod_{j=1}^{k}(1 - a_j/x)(1 - a_jx/d)}.
\]

To simplify (3.22) by these calculation gives the desired result.

Apply the substitution \(b_i \mapsto d e b_i, c_i \mapsto 1\), and \(d_i \mapsto x/e\) to (3.22). Then we get
Example 5 (cf. [3, Eq.(4.32)]) With the assumption as Corollary 10. The following holds

\[ \sum_{k=0}^{m} (b_k - a_k/(de))(1 - c_k b_k) \]

\[ \times \frac{\prod_{j=1}^{k-1}(1 - a_j)(1 - a_j/d)}{\prod_{j=1}^{k}(1 - b_j de^2/x)(1 - xb_j)} \]

\[ \prod_{j=1}^{k-1}(1 - b_j e/x)(1 - \frac{a_j x}{de}) \prod_{j=1}^{k}(1 - db_j)(1 - eb_j) \]

\[ \frac{x}{(x-e)(x-d)} \left\{ \frac{(1 - a_0 e/x)(1 - \frac{a_0 x}{de})}{(1 - b_0 de^2/x)(1 - x b_0)(1 - a_0)(1 - a_0/d)} \right\} \cdot \frac{-\prod_{j=1}^{m}(1 - a_j)(1 - a_j/d)}{\prod_{j=1}^{m}(1 - b_j de^2/x)(1 - xb_j)} \]

\[ \frac{\prod_{j=1}^{m}(1 - b_j e/x)(1 - \frac{a_j x}{de})}{\prod_{j=1}^{m}(1 - db_j)(1 - eb_j)} \] \hspace{1cm} (3.24)

In his private communication with G. Bhatnagar, Macdonald extended this result further to the following [3, Theorems 2.27 and 2.29, p.200-201]

Example 6 Let \( \{a_i\}, \{b_i\}, \{c_i\}, \{d_i\} \) be arbitrary sequences, \( e \) be an indeterminate, \( m, n \) be nonnegative integers, such that none of the denominators in (3.25) vanish. Then

\[ \sum_{k=-n}^{m} e(1 - a_k b_k e)(b_k - a_k/(d_k e))(1 - c_k/e)(1 - d_k e/c_k) \]

\[ \times \frac{\prod_{j=1}^{k}(1 - a_j)(1 - a_j/d)}{\prod_{j=1}^{k}(1 - b_j e/x)(1 - \frac{a_j x}{de})} \prod_{j=1}^{k}(1 - db_j)(1 - eb_j) \]

\[ \prod_{j=1}^{k}(1 - b_j e/x)(1 - \frac{a_j x}{de}) \prod_{j=1}^{k}(1 - db_j)(1 - eb_j) \]

\[ \frac{\prod_{j=1}^{m}(1 - a_j)(1 - a_j/d)}{\prod_{j=1}^{m}(1 - b_j e/x)(1 - \frac{a_j x}{de})} \prod_{j=1}^{m}(1 - db_j)(1 - eb_j) \]

\[ \frac{\prod_{j=1}^{m}(1 - b_j e/x)(1 - \frac{a_j x}{de})}{\prod_{j=1}^{m}(1 - db_j)(1 - eb_j)} \] \hspace{1cm} (3.25)

Proof. As a key fact in Macdonald’s proof, Eq.(2.24) in [3] can be replaced by \( f \perp g \). In fact, for arbitrary sequences \( \{a_i\}, \{b_i\}, \{c_i\}, \{d_i\} \),

\[ f(a_k, c_k)g(b_k, d_k) - g(b_k, c_k)f(a_k, d_k) = f(a_k, b_k)g(c_k, d_k), \]

where \( f, g \) are given as Corollary 10. This identity remains valid under the substitution \( b_i \mapsto deb_i, c_i \mapsto 1 \), and \( d_i \mapsto c/e \). The result is

\[ f(a_k, 1)g(deb_k, c/e) - g(deb_k, 1)f(a_k, c/e) = f(a_k, deb_k)g(1, c/e). \]

Since \( c, d \) in this identity are arbitrary (if so, \( e \) is not free since each denominators in (3.25) should not equal zero), we set \( c \mapsto c_k, d \mapsto d_k \). As there should
be no confusion, we still write \( f(x, y) \) and \( g(x, y) \) for the resulting functions. Thus

\[
f(a_k, 1)g(d_k e b_k, c_k/e) - g(d_k e b_k, 1)f(a_k, c_k/e) = f(a_k, d_k e b_k)g(1, c_k/e).
\]

It can be reformulated, under the assumption that \( g(d_k e b_k, 1)f(a_k, c_k/e) \neq 0 \), as

\[
\frac{f(a_k, 1)g(d_k e b_k, c_k/e)}{g(d_k e b_k, 1)f(a_k, c_k/e)} - 1 = \frac{f(a_k, d_k e b_k)g(1, c_k/e)}{g(d_k e b_k, 1)f(a_k, c_k/e)}.
\]

Based on this identity and proceed as in Theorem 4. Assume again that

\[
\sum_{k=-n}^{m} \frac{\prod_{j=1}^{k-1} f(a_j, d_j e b_j)g(1, c_j/e)}{\prod_{j=1}^{k} g(d_j e b_j, 1)f(a_j, c_j/e)} = \prod_{j=1}^{m} \frac{f(a_j, 1)g(d_j e b_j, c_j/e)}{g(d_j e b_j, 1)f(a_j, c_j/e)} - C_n,
\]

where

\[
C_n = \frac{\prod_{j=-n}^{0} f(a_j, d_j) \prod_{j=-n}^{0} g(b_j, c_j)}{\prod_{j=-n}^{0} f(a_j, c_j) \prod_{j=-n}^{0} g(b_j, d_j)}.
\]

Solving this identity to get

\[
w_k = f(a_k, d_k e b_k)g(1, c_k/e) \frac{\prod_{j=1}^{k-1} f(a_j, 1)g(d_j e b_j, c_j/e)}{\prod_{j=1}^{k} f(a_j, d_j e b_j)g(1, c_j/e)}.
\]

It in turn reduces the previous identity to

\[
\sum_{k=-n}^{m} f(a_k, d_k e b_k)g(1, c_k/e) \frac{\prod_{j=1}^{k-1} f(a_j, 1)g(d_j e b_j, c_j/e)}{\prod_{j=1}^{k} g(d_j e b_j, 1)f(a_j, c_j/e)} = \prod_{j=1}^{m} f(a_j, 1)g(d_j e b_j, c_j/e) \frac{1}{\prod_{j=-n}^{0} g(d_j e b_j, 1)f(a_j, c_j/e)} - \prod_{j=-n}^{m} g(d_j e b_j, 1)f(a_j, c_j/e),
\]

which can be simplified to the desired result.

**VI. \((f, f)-\text{elliptic summation formula.}\)**

As an immediate consequence of (3.8) being combined with the addition formula of the elliptic function (cf.[33]), i.e., Corollary 2, we obtain the following identity of the elliptic hypergeometric series, which is equivalent to after relabeling, as pointed out by the referee, (3.2) of [33] and (3.6) of [16].

**Corollary 11** Let \( \theta(x) \) be given as in Corollary 2, \( \{a_i\}, \{b_i\}, \{c_i\}, \{d_i\} \) be arbitrary sequences such that none of the denominators in (3.26) vanish, \( m, n \)
be nonnegative integers. Then the following holds,

\[
\sum_{k=-n}^{m} \frac{b_k}{c_k} \theta(a_k b_k) \theta(c_k d_k) \theta \left( \frac{a_k}{b_k} \right) \theta \left( \frac{c_k}{d_k} \right) \frac{\prod_{j=1}^{k-1} \theta(a_j c_j) \theta(a_j / c_j) \prod_{j=1}^{k-1} \theta(b_j d_j) \theta(b_j / d_j)}{\prod_{j=1}^{k} \theta(b_j c_j) \theta(b_j / c_j) \prod_{j=1}^{k} \theta(a_j d_j) \theta(a_j / d_j)} = \prod_{j=1}^{m} \theta(b_j c_j) \theta(b_j / c_j) \prod_{j=1}^{m} \theta(a_j d_j) \theta(a_j / d_j) \frac{\prod_{j=-n}^{0} \theta(b_j c_j) \theta(b_j / c_j) \prod_{j=-n}^{0} \theta(a_j d_j) \theta(a_j / d_j)}{\prod_{j=-n}^{0} \theta(a_j c_j) \theta(a_j / c_j) \prod_{j=-n}^{0} \theta(b_j d_j) \theta(b_j / d_j)}.
\]

(3.26)

Proof. By Corollary 2, we see that \( f \perp f \). Then the desired result is an immediate consequence of Theorem 4.

4 The \((f, g)\)-inversions from the \((f, g)\)-summation formula

In [23], the author introduced an operator in order to set up the \((f, g)\)-inversion. Some possibly approaches to show it in a simple style were suggested by the anonymous referee(s) and tried by the author. So far, we have found it can be proved by Krattenthaler’s operator method [20] (only for the case \( f = g \)) and induction [33]. Surprisingly, the \((f, g)\)-summation formula leads to an alternative proof of this matrix inversion, which is considerably simpler than one given in [23]. It is worth noting that the similar argument appeared in Bhatnagar’s proof of Krattenthaler’s matrix inversion [2].

The proof of Theorem 1. Set \( c_j = b_0, d_j = x \) in (3.7). Then the \((f, g)\)-summation formula reduces to

\[
\sum_{k=0}^{n} \frac{f(a_k, b_k)}{f(a_0, b_0)} \frac{\prod_{j=0}^{k-1} f(a_j, b_0)}{\prod_{j=1}^{k} g(b_j, b_0)} \frac{\prod_{j=0}^{k-1} g(b_j, x)}{\prod_{j=1}^{k} f(a_j, x)} = \prod_{j=1}^{n} f(a_j, b_0) \prod_{j=1}^{n} g(b_j, b_0) \frac{\prod_{j=1}^{n} f(a_j, x)}{\prod_{j=1}^{n} g(b_j, b_0)},
\]

(4.1)

Actually, this identity can also be reformulated as

\[
\sum_{k=0}^{n} \frac{f(a_k, b_k) \prod_{j=k+1}^{n} f(a_j, x)}{\prod_{j=k}^{n} g(b_j, x)} \left\{ \frac{g(b_0, x) \prod_{j=0}^{k-1} f(a_j, b_0)}{\prod_{j=1}^{k} g(b_j, b_0)} \right\} = \prod_{j=1}^{n} f(a_j, b_0) \prod_{j=1}^{n} g(b_j, b_0).
\]

(4.2)

Replace \( n \) by \( n - 1 \) and set \( x \mapsto b_n \). Then

\[
\sum_{k=0}^{n-1} \frac{f(a_k, b_k) \prod_{j=k+1}^{n-1} f(a_j, b_n)}{\prod_{j=k}^{n-1} g(b_j, b_n)} \left\{ \frac{\prod_{j=0}^{k-1} f(a_j, b_0)}{\prod_{j=1}^{k} g(b_j, b_0)} \right\} = \prod_{j=1}^{n-1} f(a_j, b_0) \prod_{j=1}^{n-1} g(b_j, b_0).
\]
which is
\[ \sum_{k=0}^{n} f(a_k, b_k) \frac{\prod_{j=k+1}^{n-1} f(a_j, b_n)}{\prod_{j=k+1}^{n} g(b_j, b_n)} \left\{ \frac{\prod_{i=0}^{k-1} f(a_j, b_0)}{\prod_{j=1}^{k} g(b_j, b_0)} \right\} = \delta_{0,n} \]

On the other hand, it is easily seen that
\[ \sum_{k=0}^{n} \frac{\prod_{j=k+1}^{n-1} f(a_j, b_k)}{\prod_{j=k+1}^{n} g(b_j, b_k)} \delta_{0,k} = \left\{ \frac{\prod_{i=0}^{n-1} f(a_j, b_0)}{\prod_{j=1}^{n} g(b_j, b_0)} \right\}. \]

The result is that two matrices \( F = (f_{n,k}) \) and \( G = (g_{n,k}) \), whose entries given by
\[ f_{n,k} = \frac{\prod_{i=0}^{n-1} f(a_i, b_k)}{\prod_{i=k+1}^{n} g(b_i, b_k)} \quad \text{and} \quad g_{n,k} = \frac{f(a_k, b_k) \prod_{i=k}^{n-1} f(a_i, b_n)}{f(a_n, b_n) \prod_{i=k}^{n-1} g(b_i, b_n)}, \]
are inverses of each other. As desired.

In a similar technique, we can obtain the following identity

**Theorem 5** *Preserve the convention as before. Then for \( m \geq 0, n \geq 1,\)
\[ \sum_{k=-n}^{m} f(a_k, b_k) \frac{\prod_{j=m}^{k-1} g(b_j, b_m)}{\prod_{j=m}^{k} f(a_j, b_m)} \frac{\prod_{j=1}^{k-1} f(a_j, b_n)}{\prod_{j=1}^{k} g(b_j, b_n)} = 0. \]

**Proof.** In (3.7), set \( c_j \mapsto b_n, d_j \mapsto x. \) Since \( g(b_n, b_n) = 0, \) it gives
\[ \sum_{k=-n}^{m} f(a_k, b_k)g(b_n, x) \frac{\prod_{j=m}^{k-1} f(a_j, b_n)}{\prod_{j=m}^{k} g(b_j, b_n)} \frac{\prod_{j=1}^{k-1} f(a_j, b_n)}{\prod_{j=1}^{k} g(b_j, b_n)} = \frac{\prod_{j=1}^{m} f(a_j, b_n)}{\prod_{j=1}^{m} g(b_j, b_n)} \frac{\prod_{j=1}^{m} f(a_j, b_n)}{\prod_{j=1}^{m} g(b_j, b_n)}. \] (4.3)

Set \( x \mapsto b_{m+1} \) again in this identity. The result is
\[ \sum_{k=-n}^{m} f(a_k, b_k)g(b_n, b_{m+1}) \frac{\prod_{j=m}^{k-1} f(a_j, b_n)}{\prod_{j=m}^{k} g(b_j, b_{m+1})} \frac{\prod_{j=1}^{k-1} f(a_j, b_{m+1})}{\prod_{j=1}^{k} g(b_j, b_{m+1})} = \frac{\prod_{j=1}^{m} f(a_j, b_n)}{\prod_{j=1}^{m} g(b_j, b_n)} \frac{\prod_{j=1}^{m} f(a_j, b_n)}{\prod_{j=1}^{m} g(b_j, b_n)}. \]

Reformulate it by the convention. It leads to
\[ \sum_{k=-n}^{m} f(a_k, b_k) \frac{\prod_{j=1}^{k-1} f(a_j, b_n)}{\prod_{j=1}^{k} g(b_j, b_{m+1})} \frac{\prod_{j=m+1}^{k-1} g(b_j, b_{m+1})}{\prod_{j=m+1}^{k} f(a_j, b_{m+1})} = \frac{\prod_{j=1}^{m} f(a_j, b_n)}{\prod_{j=1}^{m} g(b_j, b_{n})}. \]

26
Replace $m$ by $m - 1$. It gives
\[
\sum_{k=-n}^{m-1} f(a_k, b_k) \frac{\prod_{j=1}^{k-1} f(a_j, b_{-n}) \prod_{j=m}^{k} g(b_j, b_m)}{\prod_{j=1}^{k} f(a_j, b_{-n}) \prod_{j=1}^{k} g(b_j, b_m)} = -\frac{\prod_{j=1}^{m-1} f(a_j, b_{-n})}{\prod_{j=1}^{m} g(b_j, b_{-n})},
\]
which is equivalent to
\[
\sum_{k=-n}^{m} f(a_k, b_k) \frac{\prod_{j=1}^{k-1} g(b_j, b_m) \prod_{j=1}^{k} f(a_j, b_{-n})}{\prod_{j=1}^{k} f(a_j, b_{-n}) \prod_{j=1}^{k} g(b_j, b_{-n})} = 0.
\]
As desired.

The next theorem gives the bilateral form of the $(f, g)$-inversion. For convenience, write
\[
\frac{\prod_{j=1}^{m} f(a_j, c_j) \prod_{j=1}^{m} g(b_j, d_j)}{\prod_{j=1}^{m} f(a_j, d_j) \prod_{j=1}^{m} g(b_j, c_j)} \quad \text{for} \quad \lim_{m \to +\infty} \frac{\prod_{j=1}^{m} f(a_j, c_j) \prod_{j=1}^{m} g(b_j, d_j)}{\prod_{j=1}^{m} f(a_j, d_j) \prod_{j=1}^{m} g(b_j, c_j)}.
\]

**Theorem 6** Let $f \perp g$, $M, N \in Z$, and $\{A_M\}$ be an arbitrary sequence such that for all $A_M, A_N$,
\[
\frac{\prod_{j=1}^{\infty} f(a_j, A_M) \prod_{j=1}^{\infty} g(b_j, A_N)}{\prod_{j=1}^{\infty} f(a_j, A_N) \prod_{j=1}^{\infty} g(b_j, A_M)} = \frac{\prod_{j=-\infty}^{0} f(a_j, A_N) \prod_{j=-\infty}^{0} g(b_j, A_M)}{\prod_{j=-\infty}^{0} f(a_j, A_M) \prod_{j=-\infty}^{0} g(b_j, A_N)}
\]
and the limit
\[
h(M) = \lim_{N \to M} \frac{\prod_{j=1}^{\infty} f(a_j, A_M) \prod_{j=1}^{\infty} g(b_j, A_N)}{\prod_{j=1}^{\infty} f(a_j, A_N) \prod_{j=1}^{\infty} g(b_j, A_M)} - \frac{\prod_{j=-\infty}^{0} f(a_j, A_N) \prod_{j=-\infty}^{0} g(b_j, A_M)}{\prod_{j=-\infty}^{0} f(a_j, A_M) \prod_{j=-\infty}^{0} g(b_j, A_N)} \neq 0.
\]
Then a pair of matrices $F = (F_{n,k})_{n,k \in Z}$ and $G = (G_{n,k})_{n,k \in Z}$ with entries given by
\[
F_{n,k} = \frac{f(a_k, b_k) \prod_{j=1}^{k-1} f(a_j, A_n)}{h(n) \prod_{j=1}^{k} g(b_j, A_n)} \quad \text{and} \quad G_{n,k} = \frac{\prod_{j=1}^{n} g(b_j, A_k)}{\prod_{j=1}^{n} f(a_j, A_k)}
\]
is a matrix inversion.

**Proof.** In (3.7), set $m, n \to \infty$. Then
\[
\sum_{k=-\infty}^{\infty} f(a_k, b_k) g(c_k, d_k) \frac{\prod_{j=1}^{k-1} f(a_j, c_j) \prod_{j=1}^{k-1} g(b_j, d_j)}{\prod_{j=1}^{k} f(a_j, d_j) \prod_{j=1}^{k} g(b_j, c_j)} = \frac{\prod_{j=1}^{\infty} f(a_j, c_j) \prod_{j=1}^{\infty} g(b_j, d_j)}{\prod_{j=1}^{\infty} f(a_j, d_j) \prod_{j=1}^{\infty} g(b_j, c_j)} - \frac{\prod_{j=-\infty}^{0} f(a_j, d_j) \prod_{j=-\infty}^{0} g(b_j, c_j)}{\prod_{j=-\infty}^{0} f(a_j, c_j) \prod_{j=-\infty}^{0} g(b_j, d_j)}.
\]

27
Given two integers \( M, N \), set further \( c_j \mapsto A_M, d_j \mapsto A_N \). Then

\[
\sum_{k=-\infty}^{\infty} f(a_k, b_k)g(A_M, A_N) \frac{\prod_{j=1}^{k-1} f(a_j, A_M) \prod_{j=1}^{k-1} g(b_j, A_N)}{\prod_{j=1}^{k} f(a_j, A_N) \prod_{j=1}^{k} g(b_j, A_M)} = \prod_{j=1}^{\infty} f(a_j, A_M) \prod_{j=1}^{\infty} g(b_j, A_M) - \prod_{j=1}^{0} f(a_j, A_N) \prod_{j=1}^{0} g(b_j, A_M).
\]

Define

\[
h_{M,N} = \frac{1}{g(A_M, A_N)} \left\{ \prod_{j=1}^{\infty} f(a_j, A_M) \prod_{j=1}^{\infty} g(b_j, A_N) \right. \left. \prod_{j=1}^{0} f(a_j, A_N) \prod_{j=1}^{0} g(b_j, A_M) \right\}.
\]

Reformulate (4.4) with this note to get

\[
\sum_{k=-\infty}^{\infty} f(a_k, b_k) \frac{\prod_{j=1}^{k-1} f(a_j, A_M) \prod_{j=1}^{k-1} g(b_j, A_N)}{\prod_{j=1}^{k} g(b_j, A_M) \prod_{j=1}^{k} f(a_j, A_N)} = h_{M,N}.
\]

The relation suggests that

\[
PG = H,
\]

where \( P = (P_{M,k}) \), \( G = (G_{k,N}) \), and \( H = (h_{M,N}) \) are three matrices with entries given by

\[
P_{M,k} = f(a_k, b_k) \frac{\prod_{j=1}^{k-1} f(a_j, A_M)}{\prod_{j=1}^{k} g(b_j, A_M)} \quad \text{and} \quad G_{k,N} = \frac{\prod_{j=1}^{k-1} g(b_j, A_N)}{\prod_{j=1}^{k} f(a_j, A_N)},
\]

respectively. It should be possible that \( H \) is diagonal. Indeed, such \( H \) exists under the known conditions, whose entries are given by

\[
h_{M,N} = \delta_{M,N} h(M),
\]

where

\[
h(M) = \lim_{N \to M} \frac{\prod_{j=1}^{\infty} f(a_j, A_M) \prod_{j=1}^{\infty} g(b_j, A_N) - \prod_{j=1}^{0} f(a_j, A_N) \prod_{j=1}^{0} g(b_j, A_M)}{g(A_M, A_N) \prod_{j=1}^{\infty} f(a_j, A_M) \prod_{j=1}^{\infty} g(b_j, A_N)} \neq 0.
\]

Thus, we get

\[
H^{-1} = \left( \frac{1}{h(M)} \delta_{M,N} \right).
\]
Finally, it follows from (4.6) that

\[ G^{-1} = H^{-1}P. \]

Equate the \((n,k)\)-entries on both sides of this matrix identity. The result is proved.

As an important case is the following bilateral matrix inversion due to Schlosser [28]. He obtained this result from an instance of Bailey’s very-well-poised \(6\phi_6\) summation theorem and used it successfully to derive a lot of summation formulas of bilateral hypergeometric series.

**Example 7** Let \(a, b\) and \(c\) be indeterminates. Then two infinite matrices \(A = (A_{n,k})\) and \(B = (B_{n,k})\) are inverses of each other where

\[
A_{n,k} = \left( \frac{aq/b, bq/a, aq/c, cq/a, bq, q/b, cq, q/c; q}{(q, q, aq/a, aq/bc, bcq/a, cq/b, bq/c; q)} \right)_\infty \\
\times \left( 1 - bcq^{2n}/a \right) \frac{(b; q)_{n+k}(a/c; q)_{k-n}}{(1 - bc/a) (cq; q)_{n+k}(aq/b; q)_{k-n}}
\]

(4.7)

and

\[
B_{n,k} = \frac{1 - aq^{2n}}{(1 - a)} \frac{(c; q)_{n+k}(a/b; q)_{n-k}}{(bq; q)_{n+k}(aq/c; q)_{n-k}} q^{n-k}.
\]

(4.8)

**Proof.** It suffices to set in Theorem 6

\[ f(x, y) = g(x, y) = (y - x)(1 - \frac{a}{bc}xy) \]

and make the substitution

\[ A_n \mapsto q^{-n}, a_j \mapsto bq^j, b_j \mapsto cq^j. \]

Some direct calculation gives

\[
\prod_{j=-\infty}^{\infty} \frac{f(a_j, A_M)}{g(b_j, A_M)} = \frac{(a/c, cq/a, b, q/b; q)}{(a/b, bq/a, c, q/c; q)}_\infty,
\]

\[
\prod_{j=-\infty}^{\infty} \frac{f(b_j, A_N)}{g(a_j, A_N)} = \frac{(a/b, bq/a, c, q/c; q)}{(a/c, cq/a, b, q/b; q)}_\infty,
\]

from which it follows that

\[
\frac{\prod_{j=1}^{\infty} f(a_j, A_M) \prod_{j=1}^{\infty} g(b_j, A_N)}{\prod_{j=1}^{\infty} f(a_j, A_N) \prod_{j=1}^{\infty} g(b_j, A_M)} = \frac{\prod_{j=-\infty}^{0} f(a_j, A_N) \prod_{j=-\infty}^{0} g(b_j, A_M)}{\prod_{j=-\infty}^{0} f(a_j, A_M) \prod_{j=-\infty}^{0} g(b_j, A_N)}
\]

and
\[ h(M) = (q, q, aq, q/a, aq/bc, bcq/a, cq/b, bq/c; q)_{\infty} \]
\[ q^{3M}(1 - bc/a)(1 - a/(cq^M))(1 - a/(bq^M)) \]
\[ (1 - b)(1 - a)(1 - a/b)(1 - a/c)(1 - bcq^{2M}/a). \]

Further calculation leads to

\[ P_{n,k} = \frac{(c - b)(1 - aq^{2k})(cq;q)_{n}(a/(cq);q^{-1})_{n-1}}{(b;q)_{n+1}(a/b; q^{-1})_{n}} (b; q)_{n+k}(a/c; q)_{k-n} q^{n+k}, \]
\[ G_{n,k} = \frac{q^k(a/(bq); q^{-1})_{k-1}(bq; q)_{k}}{(a/c; q^{-1})_{k}(c;q)_{k+1}} (c; q)_{n+k}(a/b; q)_{n-k} \]

Observe that there exist the following relations

\[ P = LAT, \quad G = T^{-1}BS, \]

where \( L, T, \) and \( S \) are three diagonal matrices with the corresponding diagonal entries given by

\[ L_{k,k} = \frac{q^k(c - b)(cq;q)_{k}(a/(cq);q^{-1})_{k-1}}{(b;q)_{k+1}(a/b; q^{-1})_{k}} (1 - bc/a) \]
\[ (1 - bcq^{2k}/a) \]
\[ (aq/b, bq/a, aq/c, cq/a, bq/b, cq/c; q)_{\infty}, \]
\[ (aq/b, bq/a, aq/c, cq/a, bq/b, cq/c; q)_{\infty}; \]
\[ T_{k,k} = q^k(1 - aq^{2k}); \]
\[ S_{k,k} = \frac{q^{2k}(a/(bq); q^{-1})_{k-1}(bq; q)_{k}}{(1 - a)(a/c; q^{-1})_{k}(c;q)_{k+1}}, \]

respectively.

Since that \( PG = H \) gives

\[ LABS = H, \]

direct calculation states that \( H = LS, \) so \( AB = (\delta_{n,k}). \) The desired result follows.

Note that the calculation for \( h(M) \) provides an alternative proof of the famous transformation (cf. [15, Ex. 5.21])

\[ (aq/b, bq/a, aq/c, cq/a, bq/b, cq/c; q)_{\infty}(1/b, 1/c, 1/d, bcd/a^2; q)_{\infty} - (aq/b, aq/c, aq/d, bcdq/a; q)_{\infty}(b/a, c/a, d/a, a/bcd; q)_{\infty} = (aq, q/a, aq/bc, bcq/a, aq/bd, bdq/a, cdq/a, aq/cd; q)_{\infty}. \]
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