ON THE RELATIVE $n$-TENSOR NILPOTENT DEGREE OF FINITE GROUPS

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Abstract. In this paper, we generalize the concepts of the relative commutativity degree $d(G, N)$ of a subgroup $N$ of a finite group $G$ and also the tensor degree of a finite group. We introduce the relative $n$-tensor nilpotent degree of a finite group $G$ with respect to a subgroup $H$ of $G$ and investigate some bounds on which.

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1 Introduction and Preliminaries

All groups considered in this paper are finite. Let $G$ be a group with a normal subgroup $N$. Then $(G, N)$ is said to be a pair of groups. Let $G$ and $N$ act on each other and on themselves by conjugation. The nonabelian tensor product $G \otimes N$ is the group generated by the symbols $g \otimes n$ subject to the relations

$$gg^' \otimes n = (g^' \otimes g)(g \otimes n)$$

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for all \( g, g' \) in \( G \) and \( n, n' \) in \( N \). For an element \( x \in G \), we consider the tensor centralizer of \( x \) as

\[
C_G^\otimes(x) = \{a \in G : a \otimes x = 1_{G \otimes G}\}
\]

which is a subgroup of \( G \). The intersection of all tensor centralizers of elements of \( G \) is called the tensor center of \( G \) and is denoted by \( Z^\otimes(G) \). The commutator map \( \kappa : G \otimes N \rightarrow [G, N] \) which is given by \( g \otimes n \mapsto [g, n] \) for all \( g \in G \) and \( n \in N \), is an epimorphism of groups and we denote \( \ker \kappa \) by \( J_2(G) \). We define the tensor upper central series of \( G \) as

\[
Z^\otimes_1(G) = Z^\otimes(G) \quad \text{and} \quad Z^\otimes_n(G) = \{a \in G : [a, x_1, \ldots, x_{n-1}] \otimes x_n = 1 \text{ for all } x_1, \ldots, x_n \in G\}
\]

for all \( n \geq 2 \). In fact \( Z^\otimes_n(G)/Z^\otimes_{n-1}(G) = Z^\otimes_{n-1}(G/Z^\otimes(G)) \) for all \( n \geq 1 \). Hence we have the ascending tensor central series as

\[
1 \leq Z^\otimes_1(G) \leq Z^\otimes_2(G) \leq Z^\otimes_3(G) \leq \cdots
\]

In [1], the concept of the tensor degree of a group is introduced as

\[
d^\otimes(G) = \frac{|\{(x, y) \in G \times G : x \otimes y = 1_{G \otimes G}\}|}{|G|^2}
\]

which may be considered as the distance of \( G \) from being equal to \( Z^\otimes(G) \), because \( d^\otimes(G) = 1 \) if and only if \( Z^\otimes(G) = G \). On the other hand, one may easily check that \( d^\otimes(G) = 1 \) if and only if \( G \) is abelian.

One of the most important concepts in probabilistic group theory is the commutativity degree \( d(G) \) of a finite group \( G \). It is defined in [4]. Erfanian et. al. in [3] generalized the notation of \( d(G) \) by defining the relative commutativity degree of a pair of groups \( (G, N) \). Let \( N \) be a subgroup of \( G \). The relative commutativity degree \( d(G, N) \) is the probability of commuting an element of \( N \) with an element of \( G \). It is obviously seen that \( d(G) = d(G, G) \) and \( d(G, N) = 1 \) if and only if \( N \) is contained in the center of \( G \). They also proved the following theorem:
Theorem 1.1. (See [3] Theorem 3.9) Let $H$ and $N$ be two subgroups of $G$ such that $N \triangleleft G$ and $N \subseteq H$. Then

$$d(H, G) \leq d(H/N, G/N)d(N),$$

equality holds if $N \cap [H, G] = 1$.

Theorem 1.2. [7] Let $G$ be a group and $p$ be the smallest prime divisor of the order of $G$. Then

$$d(G) \leq \frac{|Z\otimes(G)|}{|J_2(G)|} \left(1 - \frac{1}{|J_2(G)|}\right) \leq d(G) - \frac{(p-1)(|Z(G)| - |Z\otimes(G)|)}{|p(G)|}.$$ 

The special case when $Z\otimes(G) = 1$ is described by the next result and has analogies with Theorem 2.8 in [6]. There are analogous to the commutativity degree of groups in [1, 2, 7, 8].

Theorem 1.3. [7] Let $G$ be a nonabelian group with $Z\otimes(G) = 1$ and $p$ be the smallest prime dividing $|G|$. Then $d\otimes(G) \leq \frac{1}{p}$.

2 Relative n-Tensor Nilpotent Degree of Groups

This section is devoted to define the concept of relative $n$-tensor nilpotent degree of a finite group $G$ and a subgroup $H$. Then we obtain some results on this concept.

Definition 2.1. Let $H$ be a subgroup of a finite group $G$. We define the relative $n$-tensor nilpotent degree of $H$ in $G$ as

$$d_n^\otimes(H, G) = \frac{|\{(h_1, \ldots, h_n, g) : [h_1, \ldots, h_n] \otimes g = 1_{H\otimes G}, h_i \in H, g \in G\}|}{|H|^n|G|}.$$ 

In the special case when $H = G$, it is called the $n$-tensor nilpotent degree of $G$ is denoted by $d_n^\otimes(G)$.

We begin with two following elementary results.

Lemma 2.2. Let $G$ be a group, $x \in G$ and $H \leq G$, then
(i) \([H : C^\otimes_G(x) \cap H] \leq [G : C^\otimes_G(x)]\);

(ii) Equality holds in (i), if \(G = HZ^\otimes(G)\). The converse is not true.

**Proof.** (i) Since \(C^\otimes_G(x) \leq G\), we have \(HC^\otimes_G(x) \subseteq G\) and hence
\[
|HC^\otimes_G(x)| = \frac{|H||C^\otimes_G(x)|}{|H \cap C^\otimes_G(x)|} \leq |G|.
\]
Therefore
\[
\frac{|H|}{|H \cap C^\otimes_G(x)|} \leq \frac{|G|}{|C^\otimes_G(x)|}.
\]

(ii) We know that \(Z^\otimes(G) = \cap_{x \in G} C^\otimes_G(x)\). So, if \(G = HZ^\otimes(G)\), then \(G = HC^\otimes_G(x)\), for all \(x \in G\). Thus,
\[
|HC^\otimes_G(x)| = \frac{|H||C^\otimes_G(x)|}{|H \cap C^\otimes_G(x)|} = |G|.
\]
Therefore
\[
[H : C^\otimes_G(x) \cap H] = [G : C^\otimes_G(x)], \quad (1)
\]
as required. For the converse, let equation (1) holds. Then obviously we have
\[
|HC^\otimes_G(x)| = |G|.
\]
This does not require to imply \(G = HZ^\otimes(G)\). For example, let \(G = Q_8 = \langle a, b | b^2 = a^4 = 1, b^{-1}ab = a^{-1} \rangle\) and \(H = \langle b \rangle\). By Lemma 4.2. of ([7]) we have \(Z^\otimes(G) = 1\) and hence \(G \neq HZ^\otimes(G)\). However, by proof of Theorem 4.3. of ([7]) we have \(C^\otimes_G(a^2) = \langle a \rangle\) and therefore \(G = HC^\otimes_G(a^2)\). \(\Box\)

**Theorem 2.3.** Let \(H \leq G\). Then \(d^\otimes_n(H, G) \leq [G : H]^{n+1} d^\otimes_n(G)\) for all \(n \geq 1\).
Proof. Using Lemma 2.2, we have

\[
\begin{align*}
\varnothing \otimes_n (H, G) &= \frac{1}{|H|^n |G|} \sum_{x_1 \in H} \cdots \sum_{x_n \in H} |C_G^\otimes([x_1, \ldots, x_n])| \\
&= \frac{1}{|H|^n} \sum_{x_1 \in H} \cdots \sum_{x_n \in H} \frac{|C_G^\otimes([x_1, \ldots, x_n])|}{|G|} \\
&\leq \frac{1}{|H|^n} \sum_{x_1 \in H} \cdots \sum_{x_n \in H} \frac{|C_G^\otimes([x_1, \ldots, x_n]) \cap H|}{|H|} \\
&\leq \frac{1}{|H|^n} \sum_{x_1 \in H} \cdots \sum_{x_n \in H} \frac{|C_G^\otimes([x_1, \ldots, x_n])|}{|H|} \\
&\leq \frac{|G|^{n+1}}{|H|^{n+1} |G|^{n+1}} \sum_{x_1 \in G} \cdots \sum_{x_n \in G} |C_G^\otimes([x_1, \ldots, x_n])| \\
&= [G : H]^{n+1} \varnothing_n (G)
\end{align*}
\]

\[
\square
\]

Theorem 2.4. Let \( H \leq G \). Then

\[
\varnothing_{n+1}(H, G) \leq \frac{1}{2}(1 + \varnothing_n(\frac{H}{H \cap Z^\otimes(G)})).
\]

Proof. Write \( \overline{H} \) for \( \frac{H}{H \cap Z^\otimes(G)} \) and for each \( x \in H \) let \( \overline{x} \) stands for \( x(H \cap Z^\otimes(G)) \) as an element of \( \overline{H} \). We know that

\[
\begin{align*}
\varnothing_{n+1}(H, G) &= \\
&= \frac{1}{|H|^{n+1} |G|} \{(x_1, \ldots, x_{n+1}, y) : [x_1, \ldots, x_{n+1}] \otimes y = 1_{H \otimes G}, x_i \in H, y \in G\}.
\end{align*}
\]
Therefore

\[ |H|^{n+1} |G| \leq d_{n+1}^\otimes (H, G) \]

\[ = |\{(x_1, \ldots, x_{n+1}, y) : [x_1, \ldots, x_{n+1}] \otimes y = 1, x_i \in H, y \in G\}| \]

\[ = \sum_{x_1 \in H} \cdots \sum_{x_{n+1} \in H} |C_G^\otimes ([x_1, \ldots, x_{n+1}])| \]

\[ = \sum_{x_1 \in H} \cdots \sum_{x_{n+1} \in H} \sum_{[x_1, \ldots, x_{n+1}] \in H \cap Z^\otimes (G)} |C_G^\otimes ([x_1, \ldots, x_{n+1}])| \]

\[ + \sum_{x_1 \in H} \cdots \sum_{x_{n+1} \in H} \sum_{[x_1, \ldots, x_{n+1}] \in H \cap Z^\otimes (G)} |C_G^\otimes ([x_1, \ldots, x_{n+1}])|. \]

On the other hand,

\[ d_n^\otimes (H) = \left( \frac{1}{|H|} \right)^n |\{(\overline{x_1, \ldots, x_{n+1}}) : [\overline{x_1, \ldots, x_n}] \otimes \overline{x_{n+1}} = 1, \overline{x_i} \in \overline{H}\}| = \left( \frac{|H \cap Z^\otimes (G)|}{|H|} \right)^n |\{(x_1, \ldots, x_{n+1}) : [x_1, \ldots, x_{n+1}] \in H \cap Z^\otimes (G), x_i \in H\}| \]

and we have

\[ \sum_{x_1 \in H} \cdots \sum_{x_{n+1} \in H} |C_G^\otimes ([x_1, \ldots, x_{n+1}])| = \]

\[ |H|^{n+1} d_n^\otimes (H) |G|. \]

Therefore

\[ |H|^{n+1} |G| d_{n+1}^\otimes (H, G) \leq \]

\[ |H|^{n+1} d_n^\otimes (\overline{G}) |G| + |H|^{n+1} d_{n+1}^\otimes (\overline{H}) |G| \]

and

\[ |H|^{n+1} |G| d_n^\otimes (\overline{H}) + \frac{|H|^{n+1} |G|}{2} = \frac{|H|^{n+1} |G|}{2} (1 + d_n^\otimes (\overline{H})). \]

Hence we have

\[ d_{n+1}^\otimes (H, G) \leq \frac{1}{2} (1 + d_n^\otimes (\overline{H})). \]
3 Tensor Nilpotent Groups

We are ready to define the concept of tensor nilpotency of a group:

**Definition 3.1.** Let $G$ be a group. Then $G$ is called tensor nilpotent if $Z^\otimes_n(G) = G$ for some $n \geq 0$. For a tensor nilpotent group $G$, the smallest $c \geq 0$ in which $Z^\otimes_c(G) = G$ is called the tensor nilpotency class or briefly the tensor class of $G$.

**Theorem 3.2.** For a finite group $G$, we have

$$d^\otimes_{n+1}(G) \leq \frac{1}{2^n}(2^n - 1 + d^\otimes(G/Z^\otimes_n(G)))$$

for all $n \geq 1$.

**Proof.** We know that

$$Z^\otimes_n(G)/Z^\otimes(G) = Z_{n-1}(G/Z^\otimes(G))$$

for all $n \geq 1$. We proceed by induction on $n$. For $n = 1$, by using Theorem 2.4, we have

$$d^\otimes_2(G) \leq \frac{1}{2}(1 + d(G/G \cap Z^\otimes(G))) = \frac{1}{2}(1 + d(G/Z^\otimes(G))).$$

Using Theorem 2.4 and the induction we have

$$d^\otimes_{n+1}(G) \leq \frac{1}{2}(1 + d^\otimes_n(G/G \cap Z^\otimes(G)))$$

$$\leq \frac{1}{2}(1 + \frac{1}{2^{n-1}}(2^{n-1} - 1 + d^\otimes(G/Z^\otimes_n(G))))$$

$$= \frac{1}{2}(1 + \frac{1}{2^{n-1}}(2^{n-1} - 1 + d^\otimes(G/Z^\otimes_n(G))))$$

$$= \frac{1}{2}(1 + \frac{1}{2^{n-1}}(2^{n-1} + 2^{n-1} - 1 + d^\otimes(G/Z^\otimes_n(G))))$$

$$= \frac{1}{2^n}(2^n - 1 + d^\otimes(G/Z^\otimes_n(G))),$$

as required. \(\square\)
**Theorem 3.3.** If $G$ is not a tensor nilpotent group of class at most $n$, then
\[ d_n^\otimes(G) \leq \frac{2^{n+2} - 3}{2^{n+2}}. \]

**Proof.** Since $G$ is not a tensor nilpotent group of class at most $n$, $Z_n^\otimes(G) \neq G$ and $G/Z_{n-1}^\otimes(G)$ is a nonabelian group. We know that $d^\otimes(G) \leq d(G)$, therefore using Theorem 2.2 in [3] implies that $d^\otimes(G/Z_{n-1}^\otimes(G)) \leq \frac{5}{8}$. So we have
\[
d_n^\otimes(G) \leq \frac{1}{2^{n-1}}(2^{n-1} - 1 + d^\otimes(G/Z_{n-1}^\otimes(G))) \\
\leq \frac{1}{2^{n-1}}(2^{n-1} - 1 + \frac{5}{8}) \\
= \frac{1}{2^{n-1}}(2^{n-1} - \frac{3}{2}) \\
= \frac{2^{n+2} - 3}{2^{n+2}},
\]
as required. \(\square\)

**Lemma 3.4.** If $G$ is tensor nilpotent of class at most $n$, then $G$ is nilpotent of class $n$.

**Proof.** We know that $Z_n^\otimes(G) \leq Z_n(G)$, so the result followes. \(\square\)

**Theorem 3.5.** If $G$ is a nontrivial group and $Z(G) = 1$, then
\[ d_n^\otimes(G) \leq \frac{2^n - 1}{2^n} \]

**Proof.** We proceed by induction on $n$. Let $n = 1$ since $Z(G) = 1$, $G$ is not nilpotent and Theorem 3 in [5] implies that $d(G) \leq \frac{1}{2}$. We know that $d^\otimes(G) \leq d(G) \leq \frac{1}{2}$. Therefore
\[
d_{n+1}^\otimes(G) \leq \frac{1}{2^n}(2^n - 1 + d^\otimes(G)) \\
\leq \frac{1}{2^n}(2^n - 1 + \frac{1}{2}) \\
= \frac{1}{2^n}(2^n - \frac{1}{2}) \\
= \frac{2^{n+1} - 1}{2^{n+1}}.
\]
Theorem 3.6. Let $H$ be a proper subgroup of $G$. Then for all $n \geq 1$, we have

(i) If $H \subseteq Z^\otimes_n(G)$, then $d_n^\otimes(H, G) = 1$.

(ii) If $H \not\subseteq Z^\otimes_n(G)$ and $H/H \cap Z^\otimes(G)$ is tensor nilpotent of class at most $n - 1$, then $d_n^\otimes(H, G) = 1$.

(iii) If $H \not\subseteq Z^\otimes_n(G)$ and $H/H \cap Z^\otimes(G)$ is not tensor nilpotent of class at most $n - 1$, then $d_n^\otimes(H, G) \leq \frac{2^{n+2} - 3}{2^{n+2}}$.

Proof. (i) If $H \subseteq Z^\otimes_n(G)$, then $[h_1, \ldots, h_n] \otimes x = 1$ for all $h_1, \ldots, h_n \in H$ and $x \in G$. So

$$d_n^\otimes(H, G) = \frac{1}{|H|^n|G|} \sum_{h_1 \in H} \cdots \sum_{h_n \in H} |C^\otimes_G([h_1, \ldots, h_n])| = 1.$$ 

(ii) Since $H/H \cap Z^\otimes(G)$ is a tensor nilpotent group of class at most $n - 1$, for all $\overline{h_1}, \ldots, \overline{h_n}$ in $H/H \cap Z^\otimes(G)$ where $\overline{h_i} = h_iH/H \cap Z^\otimes(G)$, $h_i \in H$, $i = 1, \ldots, n$ and $[\overline{h_1}, \ldots, \overline{h_{n-1}}] \otimes \overline{h_n} = 1$. We know that there exists homomorphism $H/H \cap Z^\otimes(G) \otimes H/H \cap Z^\otimes(G) \rightarrow (H/H \cap Z^\otimes(G))'$ given $[\overline{h_1}, \ldots, \overline{h_{n-1}}] \otimes \overline{h_n} \rightarrow [\overline{h_1}, \ldots, \overline{h_n}]$. Since $[\overline{h_1}, \ldots, \overline{h_{n-1}}] \otimes \overline{h_n} = 1$, then $[\overline{h_1}, \ldots, \overline{h_n}] = 1$. Therefore there exist $h_1, \ldots, h_n$ in $H$ such that $[h_1, \ldots, h_n]H \cap Z^\otimes(G) = H \cap Z^\otimes(G)$, so $[h_1, \ldots, h_n] \in H \cap Z^\otimes(G)$. Therefore for all $x$ in $G$, we have $[h_1, \ldots, h_n] \otimes x = 1$. Hence, $C^\otimes_G([h_1, \ldots, h_n]) = G$. Thus,

$$|H^n||G|d_n^\otimes(H, G) = \{|(h_1, \ldots, h_n, x) \in H^n \times G : [h_1, \ldots, h_n] \otimes x = 1\|$$

$$= \sum_{h_1 \in H} \cdots \sum_{h_n \in H} |C^\otimes_G([h_1, \ldots, h_n])| = |H|^n|G|,$$

and so, $d_n^\otimes(H, G) = 1$.

(iii) Since $H/H \cap Z^\otimes(G)$ is not tensor nilpotent of class at most $n - 1$, by Theorem 3.3 we have

$$d_{n-1}^\otimes(\frac{H}{H \cap Z^\otimes(G)}) \leq \frac{2^{n+1} - 3}{2^n}.$$
Hence
\[ d_n^\otimes(H, G) \leq \frac{1}{2}(1 + d_n^\otimes(\frac{H}{H \cap Z^\otimes(G)})) \]
\[ \leq \frac{1}{2}(1 + \frac{2n+1 - 3}{2n+1}) \]
\[ \leq \frac{1}{2}\left(\frac{2n+1 + 2n+1 - 3}{2n+1}\right) \]
\[ = \frac{2n+2 - 3}{2n+2}. \]

\[ \square \]

**Theorem 3.7.** Let \( G \) be finite group, \( H \) and \( N \) be subgroups of \( G \) such that \( N \trianglelefteq G \) and \( N \subseteq H \). Then
\[ d_n^\otimes(H, G) \leq d_n^\otimes(H/N, G/N). \]

**Proof.** We have
\[ |H|^n|G|d_n^\otimes(H, G) \]
\[ = |\{(h_1, \ldots, h_n, y) : [h_1, \ldots, h_n] \otimes y = 1, h_i \in H, y \in G\}| \]
\[ = \sum_{h_1 \in H} \cdots \sum_{h_n \in H} |C_G^\otimes([h_1, \ldots, h_n])| \]
\[ = \sum_{h_1 \in H} \cdots \sum_{h_n \in H} \frac{|C_G^\otimes([h_1, \ldots, h_n])N||C_N^\otimes([h_1, \ldots, h_n])|}{|N|} \]
\[ \leq \sum_{h_1 \in H} \cdots \sum_{h_n \in H} |C_{G/N}^\otimes([h_1N, \ldots, h_nN])||C_N^\otimes([h_1, \ldots, h_n])| \]
\[ = \sum_{t_1 \in H/N} \cdots \sum_{t_n \in H/N} \sum_{h_1 \in H} \cdots \sum_{h_n \in H} |C_{G/N}^\otimes([t_1, \ldots, t_n])||C_N^\otimes([h_1, \ldots, h_n])| \]
\[ \leq |N|^n \sum_{t_1 \in H/N} \cdots \sum_{t_n \in H/N} |C_{G/N}^\otimes([t_1, \ldots, t_n])| \]
\[ = |H/N|^n|G/N|d_n^\otimes(H/N, G/N)|N|^n+1 \]
\[ = |H|^n|G|d_n^\otimes(H/N, G/N). \]
Therefore

\[ d_n^\otimes(H, G) \leq d_n^\otimes(H/N, G/N). \]

\[ □ \]

**Corollary 3.8.** If \( N \leq G \), then \( d_n^\otimes(G) \leq d_n^\otimes(G/N) \)

**Proof.** Let \( H = G \). Then by using Theorem 3.7, the result follows.

\[ □ \]

### 4 Some Examples

In this section we compute the relative n-tensor nilpotent group for some groups.

**Example 4.1.** For the Symmetric group of degree 3, \( S_3 \), we have

\[ d_n^\otimes(S_3) \leq \frac{1}{2}(1 + d_{n-1}^\otimes(S_3)) \]

\[ \leq \frac{1}{2} \left( 1 + \frac{1}{2n-2}(2^{n-2} - 1 + d^\otimes(Z_n \otimes(S_3))) \right) \]

\[ \leq \frac{2^n - 1}{2^n}. \]

On the other hand, \( Z^\otimes(S_m) = 1 \). Therefore \( d_n^\otimes(S_m) \leq \frac{2^{n-1}}{2^n} \).

**Example 4.2.** Let \( G = C_4 \) and \( H = 2C_4 \). Then

\[ d_n^\otimes(2C_4, C_4) = \frac{1}{|2C_4|^2|C_4|} \sum_{h_1 \in 2C_4} \sum_{h_2 \in 2C_4} |G_{C_4}^\otimes([h_1, h_2])| \]

\[ = \frac{1}{2^2 \times 4} \times 16 = 1. \]

**Example 4.3.** Let \( G = D_8 = \langle a, b \mid a^4 = b^2 = 1, ba = a^{-1}b \rangle \) be the dihedral group of order 8 and \( H \) the subgroup generated by \( \{a^2, ab\} \).
Let $n \geq 3$. Since $G$ is a nilpotent group of class 2, we have $\gamma_n(G) = 1$. Hence
\begin{align*}
d^\otimes_n(H, D_8) &= \frac{1}{|H|^n |D_8|} \sum_{h_1 \in H} \cdots \sum_{h_n \in H} |C^\otimes_{D_8}([h_1, \ldots, h_n])| \\
&= \frac{1}{|H|^n \times 8} \times |H|^n = \frac{1}{8}.
\end{align*}

The same is true if we put $G = Q_8 = \langle a, b | a^4 = 1, a^2 = b^2, ba = a^{-1}b \rangle$, the quaternion group of order 8, and $H = \{a^2, ab\}$.

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