Pressure wave equation far from thermodynamics equilibrium

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Abstract. With the aim to model the propagation of sound pressure emitted by a source, based on the postulates of Far from equilibrium thermodynamics theory, a wave equation was constructed. The equation obtained is a classic generalized wave equation adding dissipative behavior as a function of the medium in which wave propagates. The new equation is of Maxwell-Cattaneo-Vernotte type. Also, an experimental activity was realized and the results were in accordance with theoretical solution of the new differential equation.

1. Introduction

Non equilibrium thermodynamics is a field theory whose finality is analyze phenomena that occurs in a thermodynamic system that is not in equilibrium.

Consider that thermodynamic variables that describe the non-equilibrium system are space and time functions. So, it is necessary that equations that describe variations of such variables are differential equations in partial derivatives respect space and time. The construction of such differential equations is made considering particular type of phenomena.

If the system is in a state out of equilibrium, but near of it, is valid apply the classic thermodynamic scheme which consider the existence of a State, Energy and Entropy equations as well as the laws of classic thermodynamics: the energy conservation law and increase of entropy principle. In this macroscopic theory we define thermodynamics fluxes like mass, heat, and linear momentum fluxes. And the thermodynamics forces, which are the gradients of temperature, concentration and pressure as the causes that induce non equilibrium phenomena.

Since thermodynamic variables are mathematical fields, the equations that describe their variations are the so-called conservation or balance equations. So, we have e.g. mass conservation, energy conservation, electric charge conservation, linear momentum conservation and entropy balance equation. All these in the Lineal Irreversible Thermodynamics (LIT).

When a thermodynamic system is far from equilibrium, to describe it, its necessary to use the Extended Irreversible Thermodynamics theory, in which the fluxes are also considered as thermodynamics variables. The variables are classified as conserved, those that fulfill a conservation equation; and fast, those that fulfill differential equations of other type, called relaxation equations.
1.1. Maxwell Cattaneo Vernotte equations

In different epochs and arguments several researchers have proposed that in a thermodynamic process, thermodynamic variables don’t change in a sudden way, and the speed of propagation of phenomena is not infinite. All go through a finite time different of zero and with a propagation speed that depends on the properties of the media.

Equations that describe the above were known since two centuries ago, Kohlrausch was the first to obtain this type of equation when examining the behavior of glass fibers; in 1867 Maxwell studying viscous bodies, concluded that the effort’s state disappears to a reason that depends on its value. Whereby the equations that assign a finite relaxation time or a reason of decay of a flux are known as Maxwell-Cattaneo-Vernotte type equation. These equations served as basis for a formulation of an undulatory version of Extended irreversible thermodynamic proposed by Gyarmati in 1977 [1].

1.1.1. The pressure wave equation

Based on non-equilibrium thermodynamics’ principles, it has been demonstrated [1] that the thermodynamic variable temperature fulfills the telegraphist equation

\[
\left( \tau_2 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - k_2 \nabla^2 T \right) = 0 ,
\]  

(1)

István Gyarmati, defined a set of variables \( \Gamma \) in terms of the basic thermodynamic variables

\[
\Gamma_1 \equiv T^{-1}, \Gamma_2 \equiv p T^{-1}, \Gamma_3 \equiv -(\mu_1 - \mu_k)T^{-1}, \ldots, \Gamma_f \equiv -(\mu_{k-1} - \mu_k)T^{-1}
\]  

(2)

where \( T \) is the temperature, \( p \) is the pressure, \( \mu_k \) is the k-esim chemical potential.

And he showed in his thermodynamics wave equations paper [1] that several combinations of thermodynamic variables fulfill a similar mathematical equation as (1)

\[
\dot{\tau} \frac{\partial^2 \Gamma}{\partial t^2} + \frac{\partial \Gamma}{\partial t} - k \nabla^2 \Gamma = 0.
\]  

(3)

Here we are only interested in \( \Gamma_2 \) since it contains the pressure that will be the vector field that describes the propagation of sound pressure, so that, must fulfill

\[
\tau_2 \frac{\partial^2 \Gamma_2}{\partial t^2} + \frac{\partial \Gamma_2}{\partial t} - k_2 \nabla^2 \Gamma_2 = 0 ,
\]  

(4)

which is equivalent to have

\[
\tau_2 \frac{\partial^2}{\partial t^2} \left( \frac{p}{T} \right) + \frac{\partial}{\partial t} \left( \frac{p}{T} \right) - k_2 \nabla^2 \left( \frac{p}{T} \right) = 0
\]  

(5)

If \( P = P (r^-, t) \) and \( T = T (r^-, t) \), the corresponding derivatives are

\[
\frac{\partial}{\partial t} \left( \frac{p}{T} \right) = \frac{1}{T} \frac{\partial p}{\partial t} - \frac{p}{T^2} \frac{\partial T}{\partial t} = \frac{1}{T} \left( \frac{\partial p}{\partial t} - \frac{p}{T} \frac{\partial T}{\partial t} \right),
\]  

(6)

\[
\frac{\partial^2}{\partial t^2} \left( \frac{p}{T} \right) = - \frac{1}{T^2} \frac{\partial T}{\partial t} \left( \frac{\partial p}{\partial t} - \frac{p}{T} \frac{\partial T}{\partial t} \right) + \frac{1}{T} \left[ \frac{\partial^2 p}{\partial t^2} - \left( \frac{p}{T} \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} \frac{\partial p}{\partial t} \right) \right]
\]  

(7)

knowing that \( \partial F / \partial t = F_t \) and \( (\partial ^2 F) / (\partial t ^2) = F_{tt} \) we can write equation (5) in the form

\[
\frac{\partial^2}{\partial t^2} \left( \frac{p}{T} \right) = \frac{1}{T^2} T_t \left( P_t - \frac{p T}{T} \right) + \frac{1}{T} \left( P_{tt} - \frac{p T}{T} T_{tt} + \frac{1}{T^2} T \left( P_t - \frac{p T}{T} T_t \right) \right)
\]  

(8)

simplifying equation (8) and writing equation (6) as equation (8) we get that
\[ \frac{\partial}{\partial t} \left( \frac{P}{T} \right) = \frac{1}{T} P_t - \frac{P}{T^2} T_t \, , \quad (9) \]

\[ \frac{\partial^2}{\partial t^2} \left( \frac{P}{T} \right) = \frac{1}{T} P_{tt} - \frac{P}{T^2} T_{tt} \, , \quad (10) \]

In the same way if

\[ \nabla^2 \left( \frac{P}{T} \right) = \frac{1}{T} \nabla^2 P - \frac{P}{T} \nabla^2 T \, , \quad (11) \]

So

\[ \nabla^2 \left( \frac{P}{T} \right) = \frac{1}{T} \nabla^2 P - \frac{P}{T} \nabla^2 T \, , \quad (12) \]

Substituting equations (9), (10) and (12) in equation (5) we obtain

\[ \tau_2 \left( \frac{1}{T} P_{tt} - \frac{P}{T^2} T_{tt} \right) + \frac{1}{T} P_t - \frac{P}{T^2} T_t \right) - k_2 \left( \frac{1}{T} \nabla^2 P - \frac{P}{T^2} \nabla^2 T \right) = 0 \, , \quad (13) \]

which can be written

\[ - \frac{P}{T^2} \left( \tau_2 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - k_2 \nabla^2 T \right) + \frac{1}{T} \left( \tau_2 \frac{\partial^2 P}{\partial t^2} + \frac{\partial P}{\partial t} - k_2 \nabla^2 P \right) = 0 \, , \quad (14) \]

But we know that all the left parenthesis is equal to zero, equation (1), so that

\[ \tau_2 \frac{\partial^2 P}{\partial t^2} + \frac{\partial P}{\partial t} - k_2 \nabla^2 P = 0 \, . \quad (15) \]

Then, we have a new partial differential equation that describes the pressure when is propagating in wave form

\[ \tau_p \frac{\partial^2 P(\vec{r}, t)}{\partial t^2} + \frac{\partial P(\vec{r}, t)}{\partial t} = D \nabla^2 P(\vec{r}, t) \quad (16) \]

where \( \tau_p \) and \( D \) are parameters that depends of the media in which sound propagates.

Equation (16) describes a non equilibrium wave phenomena. We can identify \( p \) as the sound pressure propagating like a wave into a media.

### 1.2. Solution of the new pressure wave equation

Since the sound propagates in all space and in the course of time, we propose a solution of equation (16), in rectangular coordinates, in the form:

\[ P(\vec{r}, t) = X(x) Y(y) Z(z) T(t) = R(x, y, z) T(t) \quad (17) \]

So, we have

\[ \frac{D \nabla^2 R(x, y, z)}{R(x, y, z)} - \left( \tau_p \frac{T''(t)}{T(t)} + \frac{T'(t)}{T(t)} \right) = 0 \quad (18) \]

\[ \left( \tau_p \frac{T''(t)}{T(t)} + \frac{T'(t)}{T(t)} \right) = -k \quad (19) \]

\[ \tau_p T''(t) + T'(t) + kT(t) = 0 \quad (20) \]
if we define

\[ 1 - 4\tau_p k = \mu, \]  

(21)

There are three possible cases to find \( T(t) \):

\[ T_{\mu>0}(t) = e^{-\frac{1}{2\tau_p}t} \left( A_1 e^{\frac{\sqrt{1+4\tau_p k}}{2\tau_p} t} + A_2 e^{\frac{\sqrt{1-4\tau_p k}}{2\tau_p} t} \right), \]  

(22)

\[ T_{\mu<0}(t) = e^{\frac{-1}{2\tau_p}t} \left[ A_1 \sin\left(\frac{\sqrt{1+4\tau_p k-1}}{2\tau_p} t\right) + A_2 \cos\left(\frac{\sqrt{1-4\tau_p k-1}}{2\tau_p} t\right) \right], \]  

(23)

\[ T_{\mu=0}(t) = e^{\frac{-1}{2\tau_p}t} (A_1 + A_2 t), \]  

(24)

We can write in general form:

\[ T_{\mu}(t) = e^{-\frac{1}{2\tau_p}t} Y_{\mu}(t), \]  

(25)

Now, from equations (18) and (19)

\[ \frac{D^2 R(x,y,z)}{R(x,y,z)} = -k, \]  

(26)

\[ \frac{\partial^2 R(x,y,z)}{\partial x^2} + \frac{\partial^2 R(x,y,z)}{\partial y^2} + \frac{\partial^2 R(x,y,z)}{\partial z^2} + \frac{k}{D} R(x,y,z) = 0, \]  

(27)

Knowing that:

\[ R(x, y, z) = X(x) Y(y) Z(z) \]  

(28)

and using the variables separation method proposed above, we obtain:

\[ \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\frac{k}{D}, \]  

(29)

If

\[ \frac{X''(x)}{X(x)} = -\xi_1^2, \quad \frac{Y''(y)}{Y(y)} = -\xi_2^2, \quad \frac{Z''(z)}{Z(z)} = -\xi_3^2, \]  

(30)

then

\[ \xi_1^2 + \xi_2^2 + \xi_3^2 = \frac{k}{D}, \]  

(31)

Renaming:

\[ X(x) = X_1(x_1), \quad Y(y) = X_2(x_2), \quad Z(z) = X_3(x_3) \]  

(32)

We have:

\[ X_n''(x_n) + \xi_n^2 X_n(x_n) = 0, \]  

\[ n = 1, 2, 3. \]  

(33)

whose solution is

\[ X_n(x_n) = A_n \cos(\xi_n x_n + \phi_n), \]  

(34)
Therefore, the general solution of the pressure wave equation is:

\[ P(\vec{r}, t) = e^{-\frac{1}{2\tau_p} t} \cdot Y_\mu(t) \cdot \prod_{n=1}^{3} A_n \cos(\xi_n x_n + \phi_n), \quad (35) \]

We know that our phenomenon to study must behave, in time, as a harmonized underdamped movement, so that \( \mu < 0 \) and \( Y_\mu(t) \) corresponds to equation (8), that we can write it as:

\[ T_{\mu<0}(t) = A_M e^{-\frac{1}{2} t} \cos(\omega t + \varphi), \quad (36) \]

with:

\[ \tau = 2\tau_p, \quad \omega = \sqrt{\frac{4\tau_p k-1}{4 \tau_p^2}}, \quad (37) \]

And:

\[ \prod_{n=1}^{3} A_n \cos(\xi_n x_n + \phi_n) = \mathbb{R} \{ e^{\vec{\xi} \cdot \vec{r}} + \phi \} = \mathbb{R} \{ e^{\vec{\xi} \cdot \vec{r}} - \phi \} = \cos(\vec{\xi} \cdot \vec{r} - \omega t + \psi) \quad (38) \]

Then we can write (35) as:

\[ P(\vec{r}, t) = A e^{-\frac{1}{2} t} \cos(\vec{\xi} \cdot \vec{r} - \omega t + \psi) \quad (40) \]

2. Experimental activity

Using the LabVIEW software and data acquisition card NI6008 of National Instruments, an experiment was carried out to quantify the propagation of sound in the air, i.e. the emission from the sound source of the acoustic pressure. A fingerboard was used as a sound source, a microphone as a sound sensor connected to a preamplifier and this to the interface. The distance between the fingerboard and the microphone, 10 cm., remained fixed.

The pressure of the sound emitted by the fingerboard when struck, causes the air to compress and expand, dissipating energy as the phenomenon is happening.

The graphic shown in Figure 1 was built with the experimental data, which shows the behavior of the amplitude of sound pressure as a function of time, and clearly, we see how the amplitude decays exponentially.

It was observed that the propagation of sound is a process that happens out of equilibrium and that the sound is attenuated over time.

![Figure 1. Graphics shows time evolution of sound pressure amplitude.](image)
The equation of the adjusted curve has been obtained with the experimental data

\[ P(t) = 0.485 e^{-\frac{t}{0.9075}} \cos(1231.99t) \] (41)

from (41) we can recognize that:

time decay \( \tau = 0.9075 \) seg. and angular frequency \( \omega = 1231.99 \) rad/s

The wave number \( \xi \) can be obtained from the equation \( \omega = v \xi \), with \( v = 334 \) m/s, the sound speed in the air, \( \xi = \omega /v = 1231.99/334 = 3.69 \) m\(^{-1}\)

So, the complete expression for \( P(x, t) \) is

\[ P(x, t) = 0.485 e^{-\frac{t}{0.9075}} \cos(3.69x + 1231.99t) \] (42)

This expression is consistent with the mathematical form of the solution (40) of the new wave differential equation.

3. Results

A new differential equation was obtained that describes the evolution in time and space of the Sound Pressure, which behaves as a wave. The solution of the differential equation was obtained in Cartesian coordinates. An experimental single-dimensional detection and quantification of the sound pressure was carried out, showing that sound wave decays due to media conditions in which it propagates.

The result of the experiment was in total agreement with the mathematical solution of the new wave equation.

4. Conclusions

Based on far from equilibrium thermodynamics theory, for non equilibrium phenomena, it was possible to deduce a new differential equation that describes the propagation behaviour of pressure sound in the air as a function of space and time.

This differential equation has the mathematical form of the so-called telegraphist equation, and it’s a generalization of the classic wave equation, but now includes a new term which represents the physical damping situation or exponential decay of amplitude due to the propagation in the media.

Theory and experiment show that sound pressure behaves as an undulatory descrescent phenomena.

5. References

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