Generalized Extended Momentum Operator

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Abstract
We study and generalize the momentum operator satisfying the extended uncertainty principle relation (EUP). This generalized extended momentum operator (GEMO) consists of an arbitrary auxiliary function of position operator, \( \mu(x) \), in such a combination that not only GEMO satisfies the EUP relation but also it is Hermitian. Next, we apply the GEMO to construct the generalized one-dimensional Schrödinger equation. Upon using the so called point canonical transformation (PCT), we transform the generalized Schrödinger equation from \( x \)-space to \( z \)-space where in terms of the transformed coordinate, \( z \), it is of the standard form of the Schrödinger equation. In continuation, we study two illustrative examples and solve the corresponding equations analytically to find the energy spectrum.

Keywords: Extended Uncertainty Principle, Generalized Momentum, Exact Solution

(Some figures may appear in colour only in the online journal)

1. Introduction

The idea of generalized momentum operator in quantum mechanics stems in the Heisenberg uncertainty principle (HUP) [1, 2]. Let’s recall that, any simultaneous measurements of incompatible physical quantities in quantum theory, such as position and momentum or time and energy, admit uncertainty. For the case of the position and momentum operators where the HUP relation states

\[
\Delta x \Delta p \geq \frac{1}{2} \left| \langle x, p \rangle \right| t
\]

the more precisely one measures the position of a particle the less information in momentum one gains. In order to modify the HUP, the minimum length is proposed in quantum gravity [3], string theory [4] and non-commutative spacetimes due to quantum field in quantized spacetimes [5, 6]. Correspondingly, the generalization of the uncertainty principle for minimum position and momentum has been presented in the form of the extended and generalized uncertainty principle (EGUP) in [1, 2] which is given by

\[
\Delta x \Delta p \geq \frac{1}{2} (1 + \alpha \Delta x^2 + \beta \Delta p^2). \tag{2}
\]

The implementation of quantum mechanics in gravity limits the measurement by a minimal length in which \( \beta \) is related to the Planck’s length [2] in accordance with the generalized uncertainty principle (GUP). For alternative GUP approaches we refer to the recent review papers [7] and the references therein. In the identical manner, the extended uncertainty principle (EUP) discusses the cutoff in the minimum momentum in the (anti) de sitter spacetime with the term \( \alpha \) representing the radius of the curved spacetime [8]. In [9, 10], upon using an extended momentum operator on the primordial perturbation spectrum during the inflation caused by quantum fluctuations is investigated. The extended form of the momentum operator is obtained on the (anti) de Sitter spacetime considering the deformation of the spacetime in [11]. The authors in the latter research, by taking the gravitational interactions into account and ignoring the curved spacetime, have also found the corrections to the temperature of black holes. In [12], the corrections of energy spectra for harmonic oscillator and Hydrogen atom is acquired inserting the extended form of the momentum operator in the de Sitter background. Again in [13], the quantum theory is investigated for the (Anti) de Sitter spacetime in which the special case of EUP is applied. Correspondingly, the deformed exponential and trigonometric functions are attained using the deformed-derivative and deformed-integral. Furthermore, in [14] the classical approach to the (Anti) de Sitter background, having deformed-mechanics, is comprehensively discussed in one
and $d-$dimensions. The study on the corrections to the Bekenstein-Hawking entropy in the Schwarzschild black holes has been done in [15] for the EUP and GUP cases with considerable outcomes. Interestingly, in [16] the exact solutions of the Klein–Gordon and Dirac equations in the deformed space for one-dimensional harmonic oscillator are obtained. Additionally, in [17] the impact of EUP on the thermodynamics of the system at high-temperature has been determined. In [17] in the context of EUP, the exact solutions of some $d-$dimensional potentials including an infinite box, a harmonic and pseudo-harmonic oscillators have been obtained. In this respect, the minimum momentum uncertainty imposed by the corresponding EUP, due to the obtained. In this respect, the minimum momentum uncertainty imposed by the corresponding EUP correction in send effect has been studied in the curved spacetime with de Sitter spacetime, has been found. The Ramsauer-Townsend effect has been studied in the curved spacetime with corresponding EUP correction in [18]. In the latter work authors considered the WKB approximation for one-dimensional Schrödinger equation upon utilizing various potentials. In [19], the effects of EUP and GUP on the entropy-area relation on the apparent horizon of the FRW have been studied. Costa Filho, et al in [20] showed that the translation operator relates the metric of spacetime to the generalized momentum. This, ultimately, gives the deformed Hamiltonian as a consequence of curved spacetime. Upon considering the EUP corrections very recently Hamil in [21] obtained the one-dimensional harmonic oscillator in AdS and dS background. Recently, Costa Filho et al in [22] have found connections between the quantum harmonic oscillator in a deformed space—due to its EUP corrections—and the Morse potential in the regular space.

Finally it is worth to mention that, very recently in some interesting works [23, 24], a different model—without modifying the Heisenberg algebra, preserving the canonical commutators—has been introduced to generate generalized uncertainty relations including GUP, EUP and extended generalized uncertainty principle (EGUP) based on the superposition of geometries (see also [25]). This model which is compatible with the equivalence principle, gives equivalent phenomenology at the level of the uncertainty relations however fundamentally different predictions when it is applied to simple systems, including free particles.

We aim, in the current research, to introduce a generalized extended momentum operator (GEMO), in the sense that: i) it satisfies the generalized EUP relation, as well as ii) it represents a physical quantity upon satisfying the standard Hermiticity condition. A limited form of such a generalization has been already considered in [26], where its non-Hermitian version has been discussed in [27].

This paper is organized as follows. In section 2 we introduce the generalized EUP together with the GEMO. In section 3 we apply the GEMO to the one-dimensional Schrödinger equation, and upon using PCT its transformed version is obtained. Two explicit examples are studied in this section and the paper is summarized in the Conclusion.

2. Generalized momentum operator

In the so called extended uncertainty principle (EUP) the commutation relation of position and momentum operators is generalized to be

$$[x, p] = i\hbar (1 + \mu(x))$$

in which $\mu(x)$ is a real well-defined function of position operator $x$. With the standard definition of the momentum operator, $p = -i\hbar \frac{dx}{dt}$ equation (3) becomes simply $[x, p] = i\hbar$ with $\mu(x) = 0$ while for the more interesting case where $\mu(x) = \alpha x^2$ it has been found that [9, 10, 14, 16–18, 21]

$$p = -i\hbar (1 + \alpha x^2) \frac{dx}{dt}$$

In finding (4), the only requirement was

$$[x, p] = i\hbar (1 + \alpha x^2)$$

however adding any function of the position operator i.e., $x$, to the obtained momentum operator mathematically does not change the commutation relation (5). Hence, a general momentum operator satisfying (5) may be written as

$$p = -i\hbar (1 + \alpha x^2) \frac{dx}{dt} + f(x)$$

in which $f(x)$ is a well-defined function of position operator $x$. To specify $f(x)$ we impose the Hermiticity condition on $p$ which is expected to be held by any physical quantity. This, in turn, results in (see appendix A)

$$f(x) = -i\alpha x$$

and consequently the Hermitian counterpart of the extended momentum operator (4) becomes

$$p = -i\hbar (1 + \alpha x^2) \frac{dx}{dt} - i\hbar \alpha x.$$

Next, we generalize the above result in terms of EUP corresponding to the commutation relation expressed in (3) by proposing the Hermitian GEMO defined by (see appendix A)

$$p = -i\hbar (1 + \alpha x) \frac{dx}{dt} - \frac{i\hbar d\mu(x)}{2}.$$ 

3. The generalized Schrödinger equation

Employing the generalized extended momentum operator (9), the corresponding generalized Schrödinger equation for a one-dimensional quantum particle becomes

$$i\hbar \frac{\partial}{\partial t}\psi(x, t) = \left(\frac{p^2}{2m} + V(x)\right)\psi(x, t),$$

which upon considering $V(x)$ a time-independent potential and $\psi(x, t) = e^{-i\phi/\hbar}(\tilde{\phi}(x))$ we find the time-independent
In this section we study a quantum particle whose generalized extended momentum operator is defined by equation (9) with \( \mu(x) = \alpha^2 x^2 \) in which \( \alpha \) is a real constant of dimension \( L^{-1} \).

Applying the PCT (13) we obtain

\[
  z(x) = \frac{1}{\alpha} \arctan(\alpha x) \tag{15}
\]

or inversely

\[
  x(z) = \frac{1}{\alpha} \tan(\alpha z) \tag{16}
\]

where we set the integration constant \( z_0 \) to be zero. We note that, for \( x \in \mathbb{R} \), the transformed coordinate \( z \) is confined i.e., \( z \in \left(-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}\right) \) For a zero-potential configuration in \( x \)-space, i.e., \( V(x) = 0 \), one finds the corresponding particle in an infinite well in the \( z \)-space where

\[
  V(z) = \begin{cases} 
    0 & \text{if } |z| < \frac{\pi}{2\alpha} \\
    \infty & \text{elsewhere} 
  \end{cases} \tag{17}
\]

Let’s comment that while the particle is quasi-free in \( x \)-space with a modified momentum operator, it is in an infinite potential well with standard momentum operator in \( z \)-space. The quasi-free position dependent mass quantum particle, has been studied before in [28]. The eigenfunctions and eigenvalues of the reference equation (14) are given by

\[
  \chi_n(z) = \begin{cases} 
    \frac{2}{\sqrt{\pi}} \cos(n oz), & \text{odd } n \\
    \frac{2}{\sqrt{\pi}} \sin(n oz), & \text{even } n 
  \end{cases} \tag{18}
\]

and

\[
  E_n = \frac{n^2 \hbar^2 \alpha^2}{2m} \tag{19}
\]

respectively, where \( n = 1, 2, 3, \ldots \).

To clarify the validity of (17) and consequently (18) one may apply the proper boundary conditions as follow. The explicit form of the GEMO eigenfunctions in \( x \)-space are given in (B4) and (B5) and, from this, the \( z \)-space representations can be easily derived. From (B5) it is clear that the eigenfunctions \( \psi_p(x) \) do indeed tend to zero as \( x \rightarrow \pm \infty \). In \( z \)-space, this corresponds to the limit \( z \rightarrow \pm \pi/(2\alpha) \). We note that, these points are not formally part of the domain of \( \chi_n(z) \) (nor, for that matter, of \( \psi_p(z) \)). We may require \( \psi_p(z) \) to be arbitrarily close to zero as \( z \) approaches \( \pi/(2\alpha) \), which corresponds to \( x \) approaching \( \pm \infty \).

Moreover, here is the physical meaning of quasi-free, as opposed to genuinely free: The GEMO eigenfunctions are physical (normalizable) states, which vanish at spatial infinity, whereas the eigenfunctions of the canonical momentum operator are non-normalizable and do not vanish at spatial infinity. In fact, it is this condition that is paramount: since it is equivalent to imposing boundary conditions corresponding to an infinite square well in \( z \)-space, we know that \( V(x) = 0 \) transforms to \( V(z) \) given by equation (17) and not simply to \( V(z) = 0 \).

Finally, while the energy spectrum of the target Schrödinger equation, (11), is the same as (19) the normalized

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**Figure 1.** The plot of \( |\phi_n(x)|^2/1 \) in terms of \( ax \), equation (20), for \( n = 1, 2, 3 \) and 4. The quasi-free particle is confined within its own generalized momentum such that the particle is localized around \( x = 0 \) where its deviation from the standard momentum remains small.
eigenfunctions, after (12), are given by

\[ \phi_n(x) = \begin{cases} \sqrt{\frac{2^n}{\pi}} \frac{\cos(n \arctan(x/\alpha))}{\sqrt{1 + \alpha^2 x^2}}, & \text{odd } n \\ \sqrt{\frac{2^n}{\pi}} \frac{\sin(n \arctan(x/\alpha))}{\sqrt{1 + \alpha^2 x^2}}, & \text{even } n \end{cases} \]  

(20)

In figure 1 we plot the first four states corresponding to the quantum numbers \( n = 1, 2, 3 \) and \( 4 \). Unlike a free particle with standard momentum operator which is equally likely to be found everywhere on the position axis, here the extended momentum operator confines the particle to be localized around smaller \( \alpha \).

It is remarkable to observe that, due to the zero potential i.e., \( V(x) = 0 \), the generalized momentum operator \( p \) and the Hamiltonian, \( H = \frac{p^2}{2m} \), commute. This implies that \( \phi_n(x) \) are the simultaneous eigenstates of the Hamiltonian and the momentum operator with eigenvalues \( E_n \) and \( \pm \sqrt{2mE_n} \), respectively. The extended form of the momentum operator, i.e.,

\[ p = -i\hbar \left( 1 + \alpha^2 x^2 \right) \frac{d}{dx} - i\hbar \alpha^2 x \]  

(21)

leads to a modified EUP where we expect a minimum uncertainty for the momentum operator to be observed. This can be seen from

\[ \delta x \delta p \geq \frac{1}{2} \left| \langle [x, p] \rangle \right| \]  

(22)

where \( [x, p] = i\hbar \left( 1 + \alpha^2 x^2 \right) \). The latter implies

\[ \delta x \delta p \geq \frac{\hbar}{2} (1 + \alpha^2 (x^2)) \]  

(23)

which after knowing \( (\delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \), one finds

\[ \delta x \delta p \geq \frac{\hbar}{2} \left( 1 + \alpha^2 ((\delta x)^2 - \langle x \rangle^2) \right) \]  

\[ \geq \frac{\hbar}{2} (1 + \alpha^2 ((\delta x)^2)). \]  

(24)

Finally, the momentum uncertainty is found to satisfy

\[ \delta p \geq \frac{\hbar}{2} \left( 1 + \alpha^2 ((\delta x)^2) \right) \]  

(25)

which obviously admits a minimum value for \( \delta p \) at \( \delta x = \frac{1}{\alpha} \) given by

\[ (\delta p)_{\text{min}} = \hbar \alpha. \]  

(26)

Using the eigenfunctions of the system, discussed above, equation (20), here we shall find \( \delta x \) and \( \delta p \) explicitly in order to investigate the minimum value of the momentum uncertainty (26). From the definition,

\[ (\delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 \]  

(27)

where

\[ \langle p^2 \rangle = \langle \phi_n | p^2 | \phi_n \rangle = 2mE_n \]  

(28)

and

\[ \langle p \rangle = \langle \phi_n | p | \phi_n \rangle = 0 \]  

(29)

which amount to \( \delta p = n/h\alpha \). Recall that, \( n = 1, 2, 3, \ldots \), the minimum uncertainty for the momentum operator occurs in the ground state where \( n = 1 \) and \( (\delta p)_{\text{min}} = \hbar \alpha \), in agreement with (26). On the other hand, since \( \langle x \rangle = 0 \) and \( \langle x^2 \rangle = \frac{2n - 1}{\alpha^2} \), we obtain \( \delta x = \frac{\sqrt{2n - 1}}{\alpha} \). Finally, one gets

\[ \delta x \delta p \geq n \sqrt{2n - 1} \hbar \]  

(30)

which clearly satisfies (23).

3.2. Illustrative example 2: \( \mu(x) = e^{-\gamma x} - 1 \)

In this final section we study another important generalized extended momentum operator with

\[ \mu(x) = e^{-\gamma x} - 1 \]  

(31)

in which \( \gamma \) is a positive real constant with dimension \( L^{-1} \). Similar to the previous example we apply the PCT (13) which implies

\[ z = \frac{1}{\gamma} (e^{\gamma x} - 1) \]  

(32)

and

\[ \phi(x) = e^{\gamma x/2} \chi(z) \]  

(33)

where we set the integration constant to be \( -\frac{1}{\gamma} \). Furthermore, we assume a modified-half-Morse type potential in the \( x \)-space, i.e.,

\[ V(x) = \begin{cases} \infty & x \leq 0 \\ V_0 \gamma^2 (1 - e^{\gamma x})^2 & x > 0 \end{cases} \]  

(34)

in which \( V_0 > 0 \) is a positive constant with dimension \( ML^2T^{-2} \). In \( z \)-space the potential becomes

\[ V(z) = \begin{cases} \infty & z \leq 0 \\ \frac{1}{\gamma} m \omega^2 z^2 & z > 0 \end{cases} \]  

(35)

where \( \omega^2 = \frac{2V_0 \hbar \gamma}{m} \). The standard solution of the quantum-half SHO is available in any textbook given by

\[ \chi_n(z) = \frac{1}{\sqrt{2^n n! \pi^2 \hbar}} \left( \frac{m \omega}{\sqrt{\hbar}} \right)^{1/4} e^{-m \omega z^2 / 2 \hbar} H_n \left( \frac{m \omega}{\sqrt{\hbar}} z \right), \]  

(36)

with Hermite polynomials defined by

\[ H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} (e^{-t^2}). \]  

(37)

Herein, the quantum numbers \( n = 0, 1, 2, 3, 4, \ldots \). Finally the eigenvalues and eigenfunctions of the original system in \( x \)-space are given by

\[ E_n = \hbar \omega \left( 2n + \frac{3}{2} \right) \]  

(38)
and

\[
\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{\gamma \sqrt{2mV_0}}{\pi \hbar} \right)^{1/4} \exp \left[ -\frac{1}{2} \frac{\sqrt{2mV_0}}{\hbar} (e^{\gamma x} - 1)^2 - \gamma x \right] \times H_n \left( \frac{\sqrt{2mV_0}}{\hbar} (e^{\gamma x} - 1) \right) \tag{39}
\]

respectively. In figure 2 we plot the probability density \(|\psi_n(x)|^2/\gamma| in terms of \(\gamma x\) for the first three states with the dimensionless parameter \(\sqrt{2mV_0}/\hbar = 1\). It is observed that for a given \(\gamma\), in the higher level states, the probability density moves away from the infinite wall.

4. Conclusion

We introduced the GEMO in terms of an auxiliary function of position operator, \(\mu(x)\). The generalization refers to the fact that this modified momentum operator is Hermitian as well as satisfying the EUP commutation relation. Using this momentum operator we, then, constructed the one-dimensional Schrödinger equation in terms of \(\mu(x)\). Applying the PCT, we transformed the explicit form of the resulting Schrödinger equation, from \(x\)-space to \(z\)-space where it became simpler and familiar in terms of its similarity to the standard one-dimensional Schrödinger equation. Following that we solved the Schrödinger equation for two different settings of \(\mu(x)\). In the first case we considered \(\mu(x) = \alpha^2 x^2\) with a zero potential \(V(x) = 0\) and in the second case we set \(\mu(x) = e^{-\gamma x} - 1\) and \(V(x)\) given in (34). In both illustrative examples we found exact solutions for the energy eigenstates and eigenvalues. Moreover, in the first case we have shown that the momentum admits a minimum uncertainty which was expected. This was shown from the EUP commutation relation as well as explicitly from the energy eigenstates. Finally we add that, a two or three dimensional generalization could be naturally considered which we leave it open for further work.

Appendix A. Derivation of \(f(x)\)

The GEMO given in equation (9) is Hermitian, here is the proof. Let’s start with the general form of the GEMO given by

\[
p = (1 + \mu(x))p_0 + f(x) \tag{A1}
\]

in which \(p_0 = -i\hbar \frac{\partial}{\partial x}\) and \(x\) are the canonical momentum and position operators which are both Hermitian. Next, we impose the Hermiticity condition i.e., \(p = p^\dagger\) to find \(f(x)\). In this line, we write

\[
p^\dagger = ((1 + \mu(x))p_0 + f(x))^\dagger \tag{A2}
\]

which upon applying the identities \(\hat{A}\hat{B}^\dagger = \hat{B}^\dagger \hat{A}^\dagger\) and \((f(\hat{A}))^\dagger = f^\ast(\hat{A}^\dagger)\) one obtains

\[
p^\dagger = p_0(1 + \mu(x)) + f^\ast(x) \tag{A3}
\]

in which the * stands for the complex conjugate. We note that although \(f(x)\) is an unknown complex function \(\mu(x)\) is assumed to be a real function. Furthermore, the commutating relation of the canonical momentum operator and any function of the position operator is given by \([p_0, g(x)] = -i\hbar g(x)\) which implies

\[
p^\dagger = (1 + \mu(x))p_0 - i\hbar \mu'(x) + f^\ast(x). \tag{A4}
\]

Herein, a prime stands for the derivative with respect to \(x\). Equating (A1) and (A4) we obtain

\[
f(x) - f^\ast(x) = -i\hbar \mu'(x), \tag{A5}
\]

and consequently

\[
2 \text{Im}(f(x)) = -\hbar \mu'(x). \tag{A6}
\]

This is remarkable to observe that the Hermiticity condition i.e., \(p = p^\dagger\) only identifies the imaginary part of the gauge function \(f(x)\). However, to recover the canonical momentum when \(\mu(x) = 0\), we set the real part of \(f(x)\) to be zero and consequently

\[
f(x) = i \text{Im}(f(x)) = -\frac{1}{2}i\hbar \mu'(x). \tag{A7}
\]

Finally, the Hermitian GEMO can by written as equation (9).
Appendix B. Eigenvalues and eigenfunctions of the GEMO

The eigenvalue equation of the GEMO is given by

\[ p\psi_p(x) = \psi_p(x) \quad \text{(B1)} \]

in which \( p \) and \( \psi_p(x) \) are the eigenvalue and corresponding eigenfunction of the GEMO \( p \), respectively. With \( p \) given by (9) we obtain

\[
-i\hbar (1 + \mu(x)) \frac{d\psi_p(x)}{dx} - i\hbar \frac{d\mu(x)}{2dx} \psi_p(x) = p\psi_p(x),
\]

equation (B2) can be simplified further in the form

\[
\frac{d\psi_p(x)}{\psi_p(x)} = \left( -\frac{\mu'(x)}{2} + \frac{p}{\hbar} \right) dx
\]

which after integration admits

\[
\psi_p(x) = \frac{C}{\sqrt{1 + \mu(x)}} \exp \left( \frac{ip}{\hbar} \int \frac{dx}{1 + \mu(x)} \right),
\]

where \( C \) is an integration constant playing the role of the normalization constant. Here we observe that the eigenvalue of the GEMO, \( p \) is real and continuous while the eigenfunction corresponding to any eigenvalue \( p \) is generally given by (B4). For example, in the first case where \( \mu(x) = \alpha^2 x^2 \) one obtains the normalized eigenfunctions to be

\[
\psi_p(x) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{1 + \alpha^2 x^2}} \exp \left( \frac{ip}{\alpha \hbar} \arctan \left( \frac{\alpha x}{1} \right) \right). \quad \text{(B5)}
\]

References

[1] Heisenberg W A 1927 Phys. Rev. D 43 172
[2] Kempf A 1994 J. Math. Phys. 35 4483
[3] Kempf A 1997 J. Math. Phys. 38 2093
[4] Konishi K, Paffuti G and Provero P 1990 Phys. Lett. B 234 276
[5] Kempf A, Mangan G and Mann R B 1995 Phys. Rev. D 52 1108
[6] Maggiore M 1993 Phys. Lett. B 304 65
[7] Amati D, Veneziano G and Ciafaloni M 1988 Phys. Lett. B 216 41
[8] Snyder H S 1947 Phys. Rev. 71 38
[9] Snyder H S 1947 Phys. Rev. 72 68
[10] Westkamp A N and Diab A M 2014 Int. J. Mod. Phys. D 23 1430025
[11] Westkamp A N and Diab A M 2015 Rep. Prog. Phys. 78 126001
[12] Bambi C and Urban F R 2008 Class. Quantum Grav. 25 095006
[13] Skara F and Perivolaropoulos L 2019 Phys. Rev. D 100 123527
[14] Perivolaropoulos L 2017 Phys. Rev. D 95 103523
[15] Mignemi S 2010 Mod. Phys. Lett. A 25 1697
[16] Ghosh S and Mignemi S 2011 Int. J. Theor. Phys. 50 1803
[17] Chung W S and Hassanabadi H 2017 Mod. Phys. Lett. A 32 175148
[18] Chung W S and Hassanabadi H 2018 J. Korean Phys. Soc. 71 13
[19] Wang F J, Gui Y X and Zhang Y 2009 Gen. Relativ. Gravit. 41 2381
[20] Hamil B and Merad M 2018 Eur. Phys. J. Plus 133 174
[21] Hamil B, Merad M and Birkandan T 2019 Eur. Phys. J. Plus 134 287
[22] Chung W S and Hassanabadi H 2018 Mod. Phys. Lett. A 33 1850150
[23] Zhu T, Ren J R and Li M F 2009 Phys. Lett. B 674 204
[24] Costa Filho R N, Braga J P, Lira J H and Andrade J S Jr 2016 Phys. Lett. B 755 367
[25] Hamil B 2019 Indian J. Phys. 93 1319
[26] Costa Filho R N, Alencar G, Skagerstam B S and Andrade J S Jr 2013 EPJ. 101 10099
[27] Lake M J 2019 Ukrainian J. Phys. 64 1036
[28] Lake M J, Miller M, Ganardi R F, Liu Z, Liang S D and Paterek T 2019 Class. Quantum Grav. 36 155012
[29] Lake M J, Miller M and Liang S-D 2020 Universe 6 56
[30] Mazharimousavi S H 2012 Phys. Rev. D 85 034102
[31] Costa Filho R N, Almeida M P, Farias G A Jr. and Andrade J S 2011 Phys. Rev. A 84 050102
[32] Mustafa O and Mazharimousavi S H 2006 Phys. Lett. A 358 259

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