OPTIMAL MEAN-VARIANCE REINSURANCE IN A FINANCIAL MARKET WITH STOCHASTIC RATE OF RETURN

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Abstract. In this paper, we investigate the optimal investment and reinsurance strategies for a mean-variance insurer when the surplus process is represented by a Cramér-Lundberg model. It is assumed that the instantaneous rate of investment return is stochastic and follows an Ornstein-Uhlenbeck (OU) process, which could describe the features of bull and bear markets. To solve the mean-variance optimization problem, we adopt a backward stochastic differential equation (BSDE) approach and derive explicit expressions for both the efficient strategy and efficient frontier. Finally, numerical examples are presented to illustrate our results.

1. Introduction. In actuarial and financial theories, the problem of optimal investment in financial markets for an insurer has become more and more popular. One of the most classical work on optimal investment in the financial market can be seen in Browne [9], where the optimal investment strategy to maximize the expected exponential utility of terminal wealth is obtained, provided that the risk process is described by a Brownian motion with drift and the stock price follows a geometric Brownian motion. Afterwards, the parallel optimal investment problem is reconsidered in Yang and Zhang [32] and Wang et al. [31], where the risk processes are governed by a jump-diffusion process and an increasing pure jump process, respectively. However, in order to disperse the insurer’s risk and make the insurance company run more stable, new business (such as reinsurance) attracts an increasing attention. Among others, Schmidli [21] and Sun [25] investigated the optimal investment and/or reinsurance strategy for the problem of minimizing the ruin probability. Asmussen et al. [2] and Azcue and Muler [1] concentrated on constructing the optimal reinsurance and dividend strategy when the reserve process follows the diffusion approximation and Cramér-Lundberg model, respectively. The optimal problem of maximizing the expected utility of terminal wealth is also considered by many authors including Sun et al. [30] and Wang et al. [31]. Bäuerle

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first pointed out that the mean-variance criterion could also be of interest in insurance applications, and then studied the optimal reinsurance problem when the risk model is given by a classical compound Poisson process and the new business activity is governed by a nonnegative control, i.e., the retention level of new business constrained to be nonnegative. Subsequently, Bai and Zhang [4] considered the mean-variance problem under two constrained controls: reinsurance and investment, where the surplus process is modeled by a classical compound Poisson risk model and its diffusion approximation, respectively. Bi and Guo [6] studied the same mean-variance problem in a jump-diffusion financial market, which originally appeared in Merton [18], that is, the stock prices may jump to a new level and then follow a geometric Brownian motion. Aforementioned literature solving the underlying investment or reinsurance problems mainly concentrated on the stochastic control theory, and for more details readers may consult Fleming and Soner [11], Yong and Zhou [33] and references therein.

The mean-variance portfolio selection problem is to seek a best allocation of wealth in order to achieve the optimal trade-off between the expected investment return and its risk over a fixed time horizon. It is a significant criterion to measure the risk in financial theory, which was first proposed by Markowitz [17]. From then on, more and more researchers have devoted to this area. See, for example, Li et al. [13], Lim [16], Sun and Guo [26, 27], Zhou and Li [35] and so on. Recently, Bi et al. [7] derived the optimal reinsurance and investment strategies with and without bankruptcy prohibition under the mean-variance criterion. Sun et al. [28] studied the optimal mean-variance reinsurance and investment problem for a class of dependent risk model, where the claim number processes are correlated through a common shock. Sun et al. [29] considered the mean-variance asset-liability management problem by using the theory of BSDEs. Shen and Zeng [23] studied the optimal investment-reinsurance problem for a mean-variance optimizer, where the market price of risk depends on a Markovian, affine-form, square-root stochastic factor process.

In this paper, we proceed to study the optimal mean-variance investment and reinsurance problem. Suppose that the insurer’s risk process is governed by a compound Poisson process and the insurer faces a decision to invest his/her surplus into the financial market consisting of one risk-free asset and one risky asset. It is well known that the geometric Brownian motion model is a classical model to describe the stock dynamics and the instantaneous return rate is usually assumed to be a constant or deterministic. In reality, the price of assets such as stocks is influenced by supply and demand. For a surge in demand, the buyers will increase the price they are willing to pay, while the sellers will increase the price they wish to receive. For a surge in supply, the opposite happens. Supply and demand are created when investors shift allocation of investment between asset types. For example, at one time, investors may move money from government bonds to “tech” stocks; at another time, they may move money from “tech” stocks to government bonds. In each case, this will affect the price of both types of assets. In summary, bull and bear markets are always exist in real financial market (see, Fabozzi and Francis [10]). Intuitively, it is reasonable to suppose that the instantaneous return rate is stochastic (not deterministic), fluctuating around the mean return rate due to the changes of economic environment. Hence in our study of the optimal investment problem, the instantaneous return rate of the stock dynamics is assumed to be driven by an OU process, which is more reasonable compared with the classical model and more
suitable for the real financial market (see Liang et al. [14] and Rishel [20]). Besides, the insurer can also purchase new business (such as reinsurance) to spread his/her risk, with the retention level of new business acquired at time $t$ constrained to take nonnegative values. Our objective is to maximize the expected terminal wealth while minimizing the variance of the terminal wealth. Based on the framework of stochastic linear quadratic (LQ) control, the mean-variance problem is formulated as a bi-objective stochastic optimization problem and then we transform the bi-objective problem to a constrained variance-minimization problem. By the well-known Lagrangian duality method, the constrained variance-minimization problem is related to an equivalent min-max problem with a quadratic cost functional.

To our knowledge, when applying the dynamic programming principle (DPP) and the corresponding Hamilton-Jacobi-Bellman (HJB) equation approach to solve stochastic control problems, a verification theorem is needed to guarantee that the solution to HJB equation is indeed the value function. However, there are two main difficulties in studying our problem when applying DPP method. Firstly, the HJB equation has no longer a classical solution due to the constrained reinsurance strategy. Moreover, it is also hard to derive a viscosity solution by solving the partial integral-differential equation when incorporating stochastic factors into the model. Secondly, even if a viscosity solution can be derived, it is also difficult to provide the related verification theorem. Taking into account these difficulties, we investigate the underlying investment-reinsurance problem via a BSDE method. We first relate the solution of the mean-variance problem to a special type BSDE. By solving the BSDE, explicit expressions for the efficient strategy and efficient frontier of the mean-variance problem are derived. As a conclusion, the effects of OU parameters on the optimal strategies and the effects of parameters on the efficient frontier are illustrated by numerical analysis.

In summary, the main contributions of this paper are two-folds: (1) We incorporate OU process into the stock price model, which is much closer to the real financial market since the model could describe the features of bull and bear markets. (2) We apply a BSDE approach to solve this problem instead of the classical DPP method.

The paper is organized as follows. Section 2 establishes the market model and formulates the mean-variance problem. In Section 3, we concentrate on deriving the explicit solution of the BSDE and the optimal strategies for the variance-minimization problem. The efficient strategy and efficient frontier with closed-form expressions are derived in Section 4. Section 5 presents some numerical examples to analyze our results. Finally, Section 6 concludes the paper.

2. The model.

2.1. Some notations. Let $[0, T]$ be a finite time interval and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let \{\(W_1(t)\)\} and \{\(W(t)\)\} be two standard Brownian motions, \{\(N(t)\)\} be a Poisson process with intensity \(\lambda > 0\), and \{\(Y_i\)\} be a family of positive-valued i.i.d random variables. Throughout this paper, we assume that \(N(t)\) and \(Y_i\) are mutually independent and they are all independent of \(W_1(t)\) and \(W(t)\). We define \(\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}\) as a filtration generated by \(W_1(t), W(t), N(t)\) and \(Y_i\), satisfying the usual conditions. Denote by \(\mathbb{R}\) the set of real numbers. Then, we introduce the following notations:

- \(L^0_{\mathbb{F}}(0, T; \mathbb{R})\): the set of \(\mathbb{R}\)-valued, \(\mathbb{F}\)-adapted cadlag processes on \([0, T]\);
2.2. Problem formulation. Following the classical risk theory, we describe the claim process \( \{C(t)\}_{t \geq 0} \) of the insurer by

\[
C(t) = \sum_{i=1}^{N(t)} Y_i.
\]

It is clear that there exists a Poisson random measure \( N(dt, dy) \) with compensator \( \nu(dy)dt \), such that

\[
\sum_{i=1}^{N(t)} Y_i = \int_0^t \int_0^\infty yN(ds, dy),
\]

where \( \nu(dy) = \lambda F(dy) \), and \( F(y) \) is the distribution function of \( Y_i \). Here, we further assume that each claim size \( Y_i \) has finite moments up to order 4 (this condition guarantees that the stochastic integral with respect to the Poisson random measure in (32) is a local martingale). Then we obtain

\[
C(t) = \lambda \mu_1 t + \int_0^t \int_0^\infty y\tilde{N}(ds, dy),
\]

where \( \mu_1 = \mathbb{E}Y_i \), and \( \tilde{N}(dt, dy) = N(dt, dy) - \lambda dt F(dy) \) is compensated Poisson random measure. Then the risk process \( \{R(t)\}_{t \geq 0} \) of the insurer is modeled by

\[
dR(t) = c_0 dt - dC(t),
\]

where \( c_0 = (1 + \delta)\lambda \mu_1 \) is a constant representing the premium rate, and \( \delta > 0 \) is the safety loading of the insurance company.

We also assume that the insurer can purchase reinsurance/new business as described in Bäuerle [3]. Let \( q(t) \geq 0 \) denote the retention level of new business acquired at time \( t \). This implies that the insurer pays \( q(t)C(t) \) of a claim occurring at time \( t \) and the new business pays \( (1 - q(t))C(t) \). Then, the premium has to be paid at rate \( (1 + \eta)(1 - q(t))\lambda \mu_1 \), where \( \eta > \delta \) is the safety loading of the reinsurance company. Note that \( q(t) \in [0, 1] \) corresponds to a reinsurance cover, and \( q(t) \in (1, \infty) \) means that the company can take an extra insurance business from other companies, acting as a reinsurer for other cedents. Thus, the surplus process \( \{R^q(t)\}_{t \geq 0} \) with such a reinsurance strategy is given by

\[
dR^q(t) = (c_0 - (1 + \eta)(1 - q(t))\lambda \mu_1) dt - q(t)dC(t)
\]

\[
= [a\eta q(t) + c] dt - \int_0^\infty q(t)y\tilde{N}(dt, dy),
\]

where we denote \( a = \lambda \mu_1 \) and \( c = (\delta - \eta)a \).

In addition, suppose that the insurer is allowed to invest his/her surplus into a financial market consisting of one risk-free asset (bond or bank account) and one risky asset (stock or mutual fund). Specifically, the price of the risk-free asset is given by

\[
 dB(t) = rB(t)dt, \quad B(0) = 1, \quad r > 0.
\]
In this paper, we consider a model for a stock price which has features of bull and bear markets (see Rishel [20]), that is,

\[ dS(t) = S(t)(\mu(t)dt + \sigma dW_1(t)), \quad S(0) = s_0 > 0, \]

where \( \mu(t) = \mu + m(t) \), and \( m(t) \) is an OU process evolving as

\[ dm(t) = \alpha m(t)dt + \beta dW(t), \quad m(0) = m_0. \]

The quantities \( s_0, \mu > 0, \sigma, \alpha < 0, \beta \) are all known constants. For this model, the instantaneous growth rate \( \mu(t) \) of the stock price is random. If \( \mu \) is a mean growth rate for the stock price and there is a period for which \( \mu(t) \) is substantially larger than \( \mu \), this could be considered as a bull market. Conversely, when \( \mu(t) \) is substantially less than \( \mu \), this could be considered as a bear market. Here we will allow the two Brownian motions \( W(t) \) and \( W_1(t) \) to be correlated, and denote by \( \rho \) their correlation coefficient, i.e., \( \mathbb{E}[W(t)W_1(t)] = \rho t \). Now, we introduce a new standard Brownian motion \( W_2(t) \), which is independent of \( W_1(t) \). Then \( W(t) \) can be redefined by

\[ dW(t) := \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t). \]

**Remark 2.1.** It should be noted that the stock price model in our paper is different from the case in Zhou and Yin [36], where they apply regime-switching models to capture the features of bull and bear markets. The basic idea of regime-switching is to modulate the model with a continuous-time, finite-state Markov chain where each state represents a regime of the system or level of the economic indicator. In our paper, however, the instantaneous return rate of stock prices varies continuously according to a mean-reverting OU process.

Denote by \( \pi(t) \) the total money invested in the stock at time \( t \). In this paper, short-selling is allowed, i.e., \( \pi(t) \) is real-valued. Incorporating strategy \( \pi(\cdot) \) into (1) and denoting by \( X(\cdot) \) the wealth process of the insurer with \( X(0) = x \), we have

\[
dX(t) = \pi(t) \frac{dS(t)}{S(t)} + (X(t) - \pi(t)) \frac{dB(t)}{B(t)} + dR^q(t) \\
= [rX(t) + c + \theta(t)\pi(t) + aq(t)]dt \\
+ \sigma\pi(t)dW_1(t) - \int_0^\infty q(t)y\tilde{N}(dt, dy), \tag{2}
\]

where \( \theta(t) = \mu - r + m(t) \).

An admissible strategy \( \{u(t)\}_{t \in [0, T]} := \{(\pi(t), q(t))\}_{t \in [0, T]} \) is described as a two-dimensional stochastic process, which is defined as follows.

**Definition 2.2.** Let \( \mathbb{R}^+ = [0, +\infty) \) be the set of nonnegative real numbers. An investment-reinsurance strategy \( u(\cdot) \) is said to be admissible if \( (\pi(\cdot), q(\cdot)) \) are \( \mathbb{F} \)-predictable processes such that \( \pi(\cdot) \in L^2_\mathbb{F}(0, T; \mathbb{R}), q(\cdot) \in L^2_\mathbb{F}(0, T; \mathbb{R}^+) \cap L^{4,loc}_\mathbb{F}(0, T; \mathbb{R}^+), \) and in addition \( \mathbb{E}[\sup_{t \in [0, T]} |X(t)|^2] < \infty \). Denote by \( \mathcal{A} \) the set of all admissible investment-reinsurance strategies.

**Remark 2.3.** Different from the classical literature which only requires the square integrability of the reinsurance strategy, the condition \( q(\cdot) \in L^{4,loc}_\mathbb{F}(0, T; \mathbb{R}^+) \) is necessary in our paper to guarantee that the stochastic integral with respect to the Poisson random measure in (32) is a local martingale.
Remark 2.4. For any admissible strategy \( u(\cdot) = (\pi(\cdot), q(\cdot)) \), the stochastic differential equation (SDE) (2) admits a unique strong solution on \([0, T]\), i.e.,

\[
X(t) = xe^{rt} + \int_0^t e^{r(t-s)} [c + \theta(s)\pi(s) + a\eta q(s)]ds \\
+ \int_0^t e^{r(t-s)} \sigma \pi(s)dW_1(s) - \int_0^t \int_0^\infty e^{r(t-s)} q(s)g\tilde{N}(ds, dy). \tag{3}
\]

In fact, the conditions in Definition 2.2 guarantee that

\[
\mathbb{E} \left[ \int_0^t e^{r(t-s)} |\theta(s)\pi(s) + a\eta q(s)|ds \right] \\
\leq C \mathbb{E} \left[ \int_0^t |\theta(s)\pi(s)|ds + \int_0^t q(s)ds \right] \\
\leq C \mathbb{E} \left[ \left( \int_0^t \theta^2(s)ds \right)^{\frac{1}{2}} \left( \int_0^t \pi^2(s)ds \right)^{\frac{1}{2}} + \left( \int_0^t q^2(s)ds \right)^{\frac{1}{2}} \right] \\
\leq C \left[ \mathbb{E} \int_0^t \theta^2(s)ds \right]^\frac{1}{2} \left( \mathbb{E} \int_0^t \pi^2(s)ds \right)^\frac{1}{2} + \mathbb{E} \left( \int_0^t q^2(s)ds \right)^\frac{1}{2} \\
< \infty,
\]

and

\[
\mathbb{E} \left[ \int_0^t e^{2r(t-s)} \sigma^2 \pi^2(s)ds + \int_0^t \int_0^\infty e^{2r(t-s)} q^2(s)g^2 \lambda F(dy)ds \right] \\
\leq C \mathbb{E} \left[ \int_0^t (\pi^2(s) + q^2(s))ds \right] < \infty,
\]

where \( C \) is a generic constant. According to the above two inequalities, we can show that the solution (3) of the SDE (2) is well-defined.

Assumption 1. For a sufficiently large \( C \), \( \mathbb{E} \left[ e^{C \int_0^T \theta^2(t)dt} \right] < \infty \).

Remark 2.5. It should be noted that the exponential integrability condition in Assumption 1 is essential in Proposition 3.3 to assure that the Eq. (10) admits a unique solution, and meanwhile, enables us to give a complete proof of Proposition 3.5. Actually, the same assumption with respect to OU process has been appeared in Shen \[22\], and similar exponential integrability condition with respect to constant elasticity of variance (CEV) or Cox-Ingersoll-Ross (CIR) model has been used in the literature such as Shen \[23\], Shen et al. \[24\] and Zhang and Chen \[34\].

Now we formulate the insurer’s mean-variance optimal problem as the following definition.

Definition 2.6. The insurer’s mean-variance optimal problem is to maximize the expected terminal wealth \( \mathbb{E}[X(T)] \) and, in the meantime, to minimize the variance of the terminal wealth \( \text{Var}[X(T)] \) over \( u(\cdot) \in \mathcal{A} \), which can be formulated as the following bi-objective problem:

\[
\left\{ \begin{array}{l}
\min_{u(\cdot) \in \mathcal{A}} (J_1(u(\cdot)), J_2(u(\cdot))) = (\text{Var}[X(T)], -\mathbb{E}[X(T)]), \\
\text{subject to } (X(\cdot), u(\cdot)) \text{ satisfy (2)}.
\end{array} \right. \tag{4}
\]
Definition 2.7. In the bi-objective stochastic optimization problem (4), an admissible strategy \( u^*(\cdot) := (\pi^*(\cdot), q^*(\cdot)) \) is called an efficient strategy if no strategy \( u(\cdot) \in \mathcal{A} \) exists such that
\[
J_1(u(\cdot)) \leq J_1(u^*(\cdot)), \quad J_2(u(\cdot)) \leq J_2(u^*(\cdot)),
\]
and at least one of the above inequalities holds strictly. We call \((J_1(u^*(\cdot)), J_2(u^*(\cdot))) \in \mathbb{R}^2\) an efficient point. A collection of all efficient points is called an efficient frontier.

Remark 2.8. It should be noted that an efficient strategy is Pareto optimal. In other words, an efficient strategy is one for which there does not exist another admissible strategy that has higher mean and no higher variance, or has less variance and no less mean at the terminal time \( T \).

To solve the mean-variance problem (4), we first introduce a variance-minimization problem. That is, the problem of finding an admissible strategy such that the risk measured by the variance of the terminal wealth \( \text{Var}[X(T)] = \mathbb{E}[(X(T) - \xi)^2] \) is minimized, while the expected terminal wealth satisfies \( \mathbb{E}[X(T)] = \xi \) for a given \( \xi \in \mathbb{R} \). Therefore, the variance-minimization problem is the following constrained stochastic optimization problem parameterized by \( \xi \):
\[
\begin{align*}
\min_{u(\cdot) \in \mathcal{A}} & \quad J_{VM}(u(\cdot)) = \mathbb{E}[(X(T) - \xi)^2], \\
\text{subject to} & \quad \mathbb{E}[X(T)] = \xi, \\
& \quad (X(\cdot), u(\cdot)) \text{ satisfy (2)}. 
\end{align*}
\] (5)

We impose the following assumption throughout this paper.

Assumption 2. The value of the expected terminal wealth \( \xi \) satisfies
\[
\xi \geq \xi_{\text{min}} \triangleq x_0 e^{rT} + \frac{c}{r} (e^{rT} - 1). 
\]

Remark 2.9. In fact, this assumption is reasonable, since \( \xi_{\text{min}} \) could be considered as the future value of the insurer’s wealth at time \( T \) if the insurer allocates all the wealth to the risk-free asset and cedes all claims to the reinsurer.

Remark 2.10. It is well-known that the efficient strategy and efficient frontier are subsets of the variance-minimization strategy and variance-minimization frontier, respectively (see Bielecki et al. [8]). In addition, when \( \mathbb{E}[X(T)] \geq \xi_{\text{min}} \), the variance-minimization strategy and variance-minimization frontier are really the efficient strategy and efficient frontier, respectively. So we can get the solution to problem (4) directly from the solution to problem (5).

Before solving the problem (5), we firstly explore its feasibility, i.e., there exists an admissible strategy \( u(\cdot) \in \mathcal{A} \) such that the constraint \( \mathbb{E}[X(T)] = \xi \) is satisfied.

Theorem 2.11. Under Assumption 1, the problem (5) is feasible for every \( \xi \in \mathbb{R} \) if
\[
\mathbb{E}\left[ \int_0^T \theta^2(t) dt \right] > 0. 
\] (6)

Proof. It only remains to find an admissible strategy such that \( \mathbb{E}[X(T)] = \xi \). For any \( \alpha \in \mathbb{R} \), we consider the strategy \( u^\alpha(t) := (\pi^\alpha(t), q^\alpha(t)) = (\frac{\alpha}{\sigma} \theta(t), 1 - \frac{\alpha}{\sigma}) \), then
the associated wealth process, denoted by \( X^\alpha(t) \), corresponding to \( u^\alpha(t) \) follows

\[
X^\alpha(t) = xe^{rt} + \frac{\alpha}{\sigma^2} \int_0^t e^{r(t-s)} \theta(s) \, ds + \sigma W_1(s)
- \int_0^t \int_0^\infty e^{r(t-s)}(1 - \frac{\delta}{\eta}) y \tilde{N}(dt, dy).
\]

An application of Burkholder-Davis-Gundy inequality yields

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X^\alpha(t)|^2 \right] \leq C \mathbb{E} \left[ \left( \int_0^T \theta^2(s) \, ds \right)^2 + \int_0^T \theta^2(s) \, ds + \lambda T \int_0^\infty y^2 dF(y) \right] < \infty,
\]

where the last inequality follows from Assumption 1 and \( C \) is a constant. Moreover, since \( \mathbb{E} \left[ \int_0^T |\pi^\alpha(s)|^2 \, ds \right] \leq C \mathbb{E} \left[ \int_0^T \theta^2(s) \, ds \right] < \infty \), and \( q^\alpha(t) = 1 - \frac{\delta}{\eta} \) is a positive constant, it is not difficult to see that \( \pi^\alpha(t) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}) \) and \( q^\alpha(t) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^+) \cap L^{1,loc}_{\mathbb{F}}(0,T;\mathbb{R}^+) \). Therefore, \( u^\alpha(t) \) is indeed an admissible strategy for every \( \alpha \in \mathbb{R} \) due to Definition 2.2. In addition, (8) guarantees that the local martingales in (7) are uniformly integrable martingales. Taking expectation on both sides of (7) leads to

\[
\mathbb{E}[X^\alpha(T)] = xe^{rT} + \frac{\alpha}{\sigma^2} \mathbb{E} \left[ \int_0^T e^{r(T-s)} \theta^2(s) \, ds \right].
\]

In terms of (6), if we choose \( \alpha = \frac{\sigma^2(xe^{rT})}{\mathbb{E}[\int_0^T e^{r(T-s)} \theta^2(s) \, ds]} \), it holds that \( \mathbb{E}[X^\alpha(T)] = \xi \). \( \square \)

**Remark 2.12.** In fact, (6) is a mild and reasonable condition since the rate of return of the stock is larger than the risk-free interest rate in real financial market.

### 3. Solution to the variance-minimization problem: a BSDE approach.

In this section, we mainly focus on the variance-minimization problem (5) using a BSDE approach. Since (5) is a convex optimization problem, the equality constraint \( \mathbb{E}[X(T)] = \xi \) can be dealt with by introducing a Lagrange multiplier \( \zeta \in \mathbb{R} \). By the Lagrangian duality method, the optimal strategy of problem (5) can be derived via solving the following equivalent min-max problem:

\[
\begin{cases}
\max_{\zeta \in \mathbb{R}} \min_{u(.) \in A} J_{VM}(u(.), \zeta) = \mathbb{E}[(X(T) - (\xi - \zeta))^2] - \zeta^2, \\
\text{subject to } (X(.), u(.)) \text{ satisfy (2)}. 
\end{cases}
\]

In order to solve the min-max problem (9), it remains to first solve the following unconstrained stochastic optimization problem, namely the benchmark problem:

\[
\begin{cases}
\min_{u(.) \in A} J_{BM}(u(.)) = \mathbb{E}[(X(T) - l)^2], \\
\text{subject to } (X(.), u(.)) \text{ satisfy (2)}, 
\end{cases}
\]

where \( l = \xi - \zeta \), and then maximize \( \zeta \) over \( \mathbb{R} \) in the outer maximization problem of (9).
To this end, we first introduce the following BSDE:

\[ \begin{aligned}
    dP(t) &= \left[ \left( -2r + \frac{\sigma^2}{b^2} + \frac{\sigma^2(t)}{\sigma^2} \right)P(t) + \frac{\Lambda_1(t)}{P(t)} + \frac{2\theta(t)\Lambda_1(t)}{\sigma} \right]dt \\
    &\quad + \Lambda_1(t)dW_1(t) + \Lambda_2(t)dW_2(t), \\
    P(T) &= 1,
\end{aligned} \tag{10} \]

where \( b = \sqrt{\lambda \mu_2} \) and \( \mu_2 = EY^2 \). In fact, BSDE (10) is a backward stochastic Riccati equation (BSRE). Moreover, our paper is different from that in Lim [15] since the coefficient \( \theta(t) \) here is not uniformly bounded. The explicit solution \((P(t), \Lambda_1(t), \Lambda_2(t)) \) to (10) will be determined below.

**Theorem 3.1.** One solution \((P(t), \Lambda_1(t), \Lambda_2(t)) \) to BSRE (10) is given by

\[
    P(t) = \exp \left\{ \left( -2r + \frac{\sigma^2(t)^2}{b^2} \right)(t - T) + \frac{1}{2} K(t)\theta^2(t) + N(t)\theta(t) + M(t) \right\}, \tag{11}
\]

and

\[
    \begin{aligned}
    \Lambda_1(t) &= \beta \rho [K(t)\theta(t) + N(t)]P(t), \\
    \Lambda_2(t) &= \beta \sqrt{1 - \rho^2}[K(t)\theta(t) + N(t)]P(t),
\end{aligned} \tag{12}
\]

where \( K(t), N(t) \) and \( M(t) \) are solutions of the following ordinary differential equations (ODEs):

\[
\begin{align*}
    \dot{K}(t) &= (2\alpha - \frac{4\theta^2}{\sigma^2})K(t) - \beta^2(2\rho^2 - 1)K^2(t) - \frac{2}{\rho^2} \quad \text{with} \quad K(T) = 0; \\
    \dot{N}(t) &= (\alpha - \frac{2\theta^2}{\rho})N(t) - \beta^2(2\rho^2 - 1)K(t)N(t) + \alpha(r - \mu)K(t) \quad \text{with} \quad N(T) = 0; \\
    \dot{M}(t) &= \alpha(r - \mu)N(t) - \frac{1}{2} \beta^2(2\rho^2 - 1)N^2(t) + \frac{1}{2} \beta^2K(t) \quad \text{with} \quad M(T) = 0.
\end{align*} \tag{13}
\]

**Proof.** Differentiating on both sides of \( \theta(t) = \mu - r + m(t) \) with respect to \( t \) gives

\[
\begin{align*}
    d\theta(t) &= dm(t) = \alpha m(t)dt + \beta dW(t) \\
    &= \alpha(\theta(t) + r - \mu)dt + \beta \rho dW_1(t) + \beta \sqrt{1 - \rho^2}dW_2(t) \\
    &= [\alpha(r - \mu) + \alpha \theta(t)]dt + \beta \rho dW_1(t) + \beta \sqrt{1 - \rho^2}dW_2(t). \tag{14}
\end{align*}
\]

Applying Itô’s formula to \( P(t) \) in (11) and using (12)-(14) yield

\[
\begin{aligned}
    dP(t) &= \left\{ \frac{\partial P}{\partial t} + \frac{\partial P}{\partial \theta} \left[ \alpha(r - \mu) + \alpha \theta(t) \right] + \frac{1}{2} \beta^2 \frac{\partial^2 P}{\partial \theta^2} \right\}dt \\
    &\quad + \beta \rho \frac{\partial P}{\partial \theta}dW_1(t) + \beta \sqrt{1 - \rho^2} \frac{\partial P}{\partial \theta}dW_2(t) \\
    &= \left\{ \left( -2r + \frac{\sigma^2(t)^2}{b^2} \right)P(t) + \frac{1}{2} K(t)\theta^2(t)P(t) + \dot{N}(t)\theta(t)P(t) + \dot{M}(t)P(t) \\
    &\quad + [K(t)\theta(t) + N(t)][\alpha(r - \mu) + \alpha \theta(t)]P(t) \\
    &\quad + \frac{1}{2} \beta^2[K(t)\theta(t) + N(t)]^2P(t) + \frac{1}{2} \beta^2K(t)P(t) \right\}dt \\
    &\quad + \beta \rho [K(t)\theta(t) + N(t)]P(t)dW_1(t) \\
    &\quad + \beta \sqrt{1 - \rho^2}[K(t)\theta(t) + N(t)]P(t)dW_2(t) \\
    &= \left\{ \left( -2r + \frac{\sigma^2(t)^2}{b^2} + \frac{\sigma^2(t)}{\sigma^2} \right)P(t) + \frac{\Lambda_1^2(t)}{P(t)} + \frac{2\theta(t)\Lambda_1(t)}{\sigma} \right\}dt \\
    &\quad + \Lambda_1(t)dW_1(t) + \Lambda_2(t)dW_2(t),
\end{aligned}
\]
Remark 3.2. Theorem 3.1 provides us a solution \((P(t), \Lambda_1(t), \Lambda_2(t))\) to BSRE (10), once the ODEs (13) are solved. In fact, the ODEs (13) have explicit solutions, which are shown in Proposition 3.4. The next Proposition tells that the solution to BSRE (10) given by (11) and (12) must be unique.

Proposition 3.3. Under Assumption 1, the solution \((P(t), \Lambda_1(t), \Lambda_2(t))\) given by (11) and (12) is the unique solution to BSRE (10).

Proof. Suppose that \((\tilde{P}(t), \tilde{\Lambda}_1(t), \tilde{\Lambda}_2(t))\) is another solution to BSRE (10), which may be different from \((P(t), \Lambda_1(t), \Lambda_2(t))\) obtained in (11) and (12). Set

\[
\log P(t) = (-2r + \frac{a^2\eta^2}{b^2})(t - T) + \frac{1}{2}K(t)\theta^2(t) + N(t)\theta(t) + M(t).
\]

Applying Itô formula to \log P(t), and changing the probability measure from \(P\) to \(\tilde{P}\) yields

\[
d\log P(t) = \left\{ -2r + \frac{a^2\eta^2}{b^2} + \frac{\theta^2(t)}{\sigma^2} + \frac{\Lambda_1^2(t)}{2P^2(t)} - \frac{\Lambda_2^2(t)}{2P^2(t)} + \frac{2\theta(t)\Lambda_1(t)}{\sigma P(t)} \right\} dt
\]

\[
+ \frac{\Lambda_1(t)}{P(t)} dW_1(t) + \frac{\Lambda_2(t)}{P(t)} dW_2(t)
\]

\[
= \left\{ -2r + \frac{a^2\eta^2}{b^2} + \frac{\theta^2(t)}{\sigma^2} + \frac{\Lambda_1^2(t)}{2P^2(t)} - \frac{\Lambda_2^2(t)}{2P^2(t)} \right\} dt
\]

\[
+ \frac{\Lambda_1(t)}{P(t)} d\tilde{W}_1(t) + \frac{\Lambda_2(t)}{P(t)} d\tilde{W}_2(t),
\]

where \(\tilde{P}\) is defined by

\[
\frac{d\tilde{P}}{dP} \bigg|_{\mathcal{F}_T} = \exp \left\{ - \int_0^T \frac{2\theta^2(t)}{\sigma^2} dt - \int_0^T \frac{2\theta(t)}{\sigma} dW_1(t) \right\} \triangleq Z(T). \tag{15}
\]

It is obvious that the stochastic exponential in (15) is an \((\mathcal{F}, P)\)-martingale due to Assumption 1. Define another probability measure \(\hat{P}\) by

\[
\frac{d\hat{P}}{dP} \bigg|_{\mathcal{F}_T} = \exp \left\{ - \int_0^T \frac{\Lambda_1^2(t)}{2P^2(t)} dt - \int_0^T \frac{\Lambda_1(t)}{P(t)} d\tilde{W}_1(t)
\]

\[
- \int_0^T \frac{\Lambda_2^2(t)}{2P^2(t)} dt + \int_0^T \frac{\Lambda_2(t)}{P(t)} d\tilde{W}_2(t) \right\}. \tag{16}
\]
Then the stochastic exponential in (16) is an \((\mathbb{F}, \hat{\mathbb{P}})\)-martingale. In fact,
\[
\hat{\mathbb{E}} \left[ \exp \left\{ \int_0^T \frac{\Lambda_1^2(t)}{2P^2(t)} dt + \int_0^T \frac{\Lambda_2^2(t)}{2P^2(t)} dt \right\} \right] \\
= \mathbb{E} \left[ Z(T) \exp \left\{ \int_0^T \frac{\Lambda_1^2(t)}{2P^2(t)} dt + \int_0^T \frac{\Lambda_2^2(t)}{2P^2(t)} dt \right\} \right] \\
\leq (EZ^2(T))^{\frac{1}{2}} \left( \mathbb{E} \exp \left\{ \int_0^T \frac{\Lambda_1^2(t)}{P^2(t)} dt + \int_0^T \frac{\Lambda_2^2(t)}{P^2(t)} dt \right\} \right)^{\frac{1}{2}} < \infty,
\]
which implies the Novikov condition is satisfied. According to Girsanov’s theorem, the processes
\[
\hat{W}_1(t) = \tilde{W}_1(t) + \int_0^t \frac{\Lambda_1(s)}{P(s)} ds
\]
and
\[
\hat{W}_2(t) = \tilde{W}_2(t) - \int_0^t \frac{\Lambda_2(s)}{P(s)} ds
\]
are both standard \((\mathbb{F}, \hat{\mathbb{P}})\)-Brownian motions. If we define
\[
\Delta \log P(t) := \log P(t) - \log \hat{P}(t),
\]
\[
\Delta \Lambda_i(t) := \frac{\Lambda_i(t)}{P(t)} - \frac{\Lambda_i(t)}{P(t)} = \frac{\Lambda_i(t)}{\tilde{P}(t)} \quad \text{for } i = 1, 2,
\]
then
\[
d\Delta \log P(t) = d[\log P(t) - \log \hat{P}(t)] \\
= \left\{ \left( \frac{\Lambda_1^2(t)}{2P^2(t)} - \frac{\tilde{\Lambda}_1^2(t)}{2\tilde{P}^2(t)} \right) - \left( \frac{\Lambda_2^2(t)}{2P^2(t)} - \frac{\tilde{\Lambda}_2^2(t)}{2\tilde{P}^2(t)} \right) \right\} dt \\
+ \left( \frac{\Lambda_1(t)}{P(t)} - \frac{\tilde{\Lambda}_1(t)}{\tilde{P}(t)} \right) d\tilde{W}_1(t) + \left( \frac{\Lambda_2(t)}{P(t)} - \frac{\tilde{\Lambda}_2(t)}{\tilde{P}(t)} \right) d\tilde{W}_2(t) \\
= - \left( \frac{1}{2} \Delta \Lambda_1^2(t) - \frac{1}{2} \Delta \tilde{\Lambda}_1^2(t) \right) dt + \Delta \Lambda_1(t) d\tilde{W}_1(t) + \Delta \Lambda_2(t) d\tilde{W}_2(t). (17)
\]
Under \(\hat{\mathbb{P}}\), (17) is a quadratic BSDE with the terminal value \(\Delta \log P(T) = 0\). Then (17) admits a unique solution \((\Delta \log P(\cdot), \Delta \Lambda_1(\cdot), \Delta \Lambda_2(\cdot)) \equiv (0, 0, 0)\) (see Kobylanski [12]). This means
\[
\log P(t) = \log \hat{P}(t), \quad \frac{\Lambda_i(t)}{\hat{P}(t)} = \frac{\Lambda_i(t)}{P(t)} = \frac{\tilde{\Lambda}_i(t)}{\tilde{P}(t)}, \quad \text{for } i = 1, 2,
\]
which immediately results in
\[
\hat{P}(t) = P(t), \quad \frac{\Lambda_i(t)}{\hat{P}(t)} = \frac{\Lambda_i(t)}{P(t)}, \quad \text{for } i = 1, 2.
\]
Hence, the solution \((P(t), \Lambda_1(t), \Lambda_2(t))\) given by (11) and (12) is the unique solution to BSRE (10), which completes the proof.

In the next Proposition, we derive closed-form solutions to the ODEs (13), which provides the explicit solution to BSRE (10).
Proposition 3.4. The explicit solutions of $K(t)$, $N(t)$ and $M(t)$ in (13) are given as follows:

\[
K(t) = \begin{cases} 
\frac{d_1 - d_2 e^{\sqrt{\Delta} (t - T)}}{1 - e^{\sqrt{\Delta} (t - T)}}, & \text{if } \Delta > 0, \\
\frac{(\alpha - 2\beta^2)^2 (t - T)}{\beta^2 (2\rho^2 - 1) [1 + (\alpha - 2\beta^2) (t - T)]}, & \text{if } \Delta = 0, \\
\frac{1}{2\beta^2 (2\rho^2 - 1)} \left\{ 2\alpha - \frac{4\beta \rho}{\sigma} + \sqrt{-\Delta} \tan \left[ \frac{\sqrt{-2\Delta}}{2} (t - T) + \arctan \frac{4\beta \rho}{\sqrt{-2\Delta}} \right] \right\}, & \text{if } \Delta < 0, 
\end{cases}
\]  

(18)

\[
N(t) = \alpha(r - \mu) \int_t^T K(s) e^{\int_s^t (\alpha - \frac{2\beta \rho}{\sigma} - \beta^2 (2\rho^2 - 1) \nu) \, ds} \, ds, 
\]

(19)

and

\[
M(t) = \int_t^T \left[ \alpha(r - \mu) N(s) + \frac{1}{2} \beta^2 K(s) - \frac{1}{2} \beta^2 (2\rho^2 - 1) N^2(s) \right] \, ds, 
\]

(20)

where

\[
\Delta = 4 \left( \alpha^2 - \frac{4\alpha \beta \rho}{\sigma} + \frac{2\beta^2}{\sigma^2} \right),
\]

\[
d_{1,2} = \frac{1}{2\beta^2 (2\rho^2 - 1)} \left( 2\alpha - \frac{4\beta \rho}{\sigma} \right) \pm \sqrt{\Delta}.
\]

Proof. Rewriting the first equality of (13) yields

\[
\frac{dK(t)}{dt} = \beta^2 (2\rho^2 - 1) K^2(t) + \left( \frac{4\beta \rho}{\sigma} - 2\alpha \right) K(t) + \frac{2}{\sigma^2}, \quad K(T) = 0.
\]

Integrating the above equation on both sides with respect to time $t$, we obtain

\[
\int \frac{1}{\beta^2 (2\rho^2 - 1) K^2(t) + \left( \frac{4\beta \rho}{\sigma} - 2\alpha \right) K(t) + \frac{2}{\sigma^2}} \, dK(t) = t + C, 
\]

(21)

where $C$ is a constant to be determined later. Set

\[
\Delta = \left( \frac{4\beta \rho}{\sigma} - 2\alpha \right)^2 - \frac{8\beta^2 (2\rho^2 - 1)}{\sigma^2} = 4 \left( \alpha^2 - \frac{4\alpha \beta \rho}{\sigma} + \frac{2\beta^2}{\sigma^2} \right).
\]

When $\Delta > 0$, i.e., the characteristic function

\[
\beta^2 (2\rho^2 - 1) d^2 + \left( \frac{4\beta \rho}{\sigma} - 2\alpha \right) d + \frac{2}{\sigma^2} = 0
\]

has two different real solutions, namely,

\[
d_{1,2} = \frac{2\alpha - \frac{4\beta \rho}{\sigma} \pm \sqrt{\Delta}}{2\beta^2 (2\rho^2 - 1)}.
\]

Then, we have

\[
\int \frac{1}{\beta^2 (2\rho^2 - 1) K^2(t) + \left( \frac{4\beta \rho}{\sigma} - 2\alpha \right) K(t) + \frac{2}{\sigma^2}} \, dK(t) = \frac{1}{\beta^2 (2\rho^2 - 1) (d_1 - d_2)} \int \left( \frac{1}{K(t) - d_1} - \frac{1}{K(t) - d_2} \right) \, dK(t)
\]

(23)

\[
= \frac{1}{\beta^2 (2\rho^2 - 1) (d_1 - d_2)} \ln \frac{K(t) - d_1}{K(t) - d_2}.
\]

Combining (23) with (21), and taking into account the boundary condition $K(T) = 0$, we get the first equation of (18).
When $\Delta = 0$, i.e., the characteristic function (22) has a unique real solution, namely,

$$d_0 = \frac{\alpha - 2\beta\rho}{\beta^2(2\rho^2 - 1)}.$$

Then, we have

$$\int \frac{1}{\beta^2(2\rho^2 - 1)K^2(t) + \left(\frac{4\beta\rho}{\sigma} - 2\alpha\right)K(t) + \frac{2}{\sigma^2}}dK(t)$$

$$= \frac{1}{\beta^2(2\rho^2 - 1)} \int \frac{1}{(K(t) - d_0)^2}dK(t)$$

$$= \frac{1}{\beta^2(2\rho^2 - 1)(d_0 - K(t))}.$$

Combining (24) with (21), and taking into account the boundary condition $K(T) = 0$, we get the second equality of (18).

When $\Delta < 0$, i.e., the characteristic function (22) has no real solution. Thus, it follows that

$$\int \frac{1}{\beta^2(2\rho^2 - 1)K^2(t) + \left(\frac{4\beta\rho}{\sigma} - 2\alpha\right)K(t) + \frac{2}{\sigma^2}}dK(t)$$

$$= \frac{1}{\beta^2(2\rho^2 - 1)} \int \frac{1}{(K(t) - d)^2 + \left(\frac{2\pm \Delta}{2\beta^2(2\rho^2 - 1)}\right)^2}dK(t)$$

$$= \frac{2}{\sqrt{-\Delta}} \arctan\left[\frac{2\beta^2(2\rho^2 - 1)}{\sqrt{-\Delta}}(K(t) - d)\right].$$

Combining (25) with (21), and taking into account the boundary condition $K(T) = 0$, we get the third equality of (18). Substituting $K(t)$ into the second ODE of (13) leads to $N(t)$, which is described by (19). Plugging $K(t)$ and $N(t)$ back into the third ODE of (13), $M(t)$ in (20) is obtained. This completes the proof.

**Proposition 3.5.** Under Assumption 1, the solution $(P(t), \Lambda_1(t), \Lambda_2(t))$ obtained in Theorem 3.1 lies in $L^2(0, T; \mathbb{R}^+) \times L^2(0, T; \mathbb{R}) \times L^2(0, T; \mathbb{R})$. Moreover, there exists a sufficiently small constant $\epsilon$ such that

$$\epsilon \leq P(t) < e^{2r(T-t)}, \forall t \in [0, T], \mathbb{P}\text{-a.s.}$$

**Proof.** We first prove that $P(t)$ is uniformly bounded on $[0, T]$. Consider the reciprocal process of $P(\cdot)$ and denote by $Q(\cdot) = \frac{1}{P(\cdot)}$. Applying Itô’s formula to $Q(\cdot)$ leads to the following linear BSDE:

$$\left\{\begin{array}{l}
dQ(t) = \left[(2r - \frac{a_2\sigma^2}{\sigma^2} - \frac{\theta^2(t)}{\sigma^2})Q(t) + \frac{2\theta(t)\Gamma_1(t)}{\sigma} - \frac{\Lambda_2(t)\Gamma_2(t)}{P(t)}\right]dt \\
+ \Gamma_1(t)dW_1(t) + \Gamma_2(t)dW_2(t), \\
Q(T) = 1,
\end{array}\right.$$

where $\Gamma_1(t) = -\frac{\Lambda_1(t)}{P(t)}$ and $\Gamma_2(t) = -\frac{\Lambda_2(t)}{P(t)}$. Similar to the proof of Proposition 3.3, it is clear that the stochastic exponential $\tilde{Z}(T)$, when setting

$$\frac{d\tilde{P}}{dP}\bigg|_{\mathcal{F}_T} = \exp \left\{ - \int_0^T \frac{2\theta^2(t)}{\sigma^2}dt - \int_0^T \frac{2\theta(t)}{\sigma}dW_1(t) \\
- \int_0^T \frac{\Lambda_2(t)}{2P^2(t)}dt + \int_0^T \frac{\Lambda_2(t)}{P(t)}dW_2(t) \right\} \triangleq \tilde{Z}(T),$$
is an \((\mathcal{F}, \mathbb{P})\)-martingale. Changing the probability measure from \(\mathbb{P}\) to \(\tilde{\mathbb{P}}\) yields

\[
\left\{
\begin{array}{l}
\dif Q(t) = \left(2r - \frac{a^2 \eta^2}{b^2} - \frac{\theta^2(t)}{\sigma^2}\right)Q(t) \dif t + \Gamma_1(t) \dif \tilde{W}_1(t) + \Gamma_2(t) \dif \tilde{W}_2(t), \\
Q(T) = 1.
\end{array}
\right.
\tag{26}
\]

It is easy to see that the driver of \(Q(t)\) satisfies the stochastic Lipschitz condition defined in Bender and Kohlmann [5] with \(\alpha_1(t) = \left|2r - \frac{a^2 \eta^2}{b^2} - \frac{\theta^2(t)}{\sigma^2}\right| + \epsilon\) as its Lipschitz coefficient for any \(\epsilon > 0\). In fact, it is obvious that the conditions (H1)-(H3) and (H5) in Definition 2 of Bender and Kohlmann [5] hold. Hence we only need to verify that (H4) is valid. Set

\[
\alpha^2(t) := \alpha_1(t) = \left|2r - \frac{a^2 \eta^2}{b^2} - \frac{\theta^2(t)}{\sigma^2}\right| + \epsilon.
\]

Then we have

\[
A(T) := \int_0^T \alpha^2(t) \dif t \leq K \int_0^T \frac{\theta^2(t)}{\sigma^2} \dif t,
\]

and

\[
\mathbb{E}[e^{\beta A(T)} \cdot 1] \leq K \mathbb{E}[e^{\beta \int_0^T \frac{\theta^2(t)}{\sigma^2} \dif t}]
\]

\[
\leq K \mathbb{E}[\tilde{Z}(T) \cdot e^{\beta \int_0^T \frac{\theta^2(t)}{\sigma^2} \dif t}]
\]

\[
\leq K \left(\mathbb{E}Z^2(T)\right)^{\frac{1}{2}} \cdot (e^{2\beta \int_0^T \frac{\theta^2(t)}{\sigma^2} \dif t})^{\frac{1}{2}} < \infty,
\]

which indicates that (H4) is valid. According to Theorem 3 in Bender and Kohlmann [5], the BSDE (26) has a unique solution on \((\Omega, \mathcal{F}, \tilde{\mathbb{P}})\) satisfying

\[
\tilde{\mathbb{E}} \left[\int_0^T \alpha^2(t) Q^2(t) e^{\beta A(T)} \dif t\right] + \tilde{\mathbb{E}} \left[\int_0^T |\Gamma_1(t)|^2 e^{\beta A(T)} \dif t\right]
\]

\[
+ \tilde{\mathbb{E}} \left[\int_0^T |\Gamma_2(t)|^2 e^{\beta A(T)} \dif t\right] < \infty.
\tag{27}
\]

In view of

\[
d\left[Q(t) \exp \left\{\int_0^t \left(-2r + \frac{a^2 \eta^2}{b^2} + \frac{\theta^2(t)}{\sigma^2}\right) \dif s\right\}\right]
\]

\[
= \exp \left\{\int_0^t \left(-2r + \frac{a^2 \eta^2}{b^2} + \frac{\theta^2(t)}{\sigma^2}\right) \dif s\right\} \times (\Gamma_1(t) \dif \tilde{W}_1(t) + \Gamma_2(t) \dif \tilde{W}_2(t)),
\tag{28}
\]

an application of Burkholder-Davis-Gundy inequality yields

\[
\tilde{\mathbb{E}} \left[\sup_{0 \leq t \leq T} \left|\int_0^t \exp \left\{\int_0^s \left(-2r + \frac{a^2 \eta^2}{b^2} + \frac{\theta^2(v)}{\sigma^2}\right) \dif v\right\} \Gamma_1(s) \dif \tilde{W}(s)\right|\right]
\]

\[
\leq K \tilde{\mathbb{E}} \left[\left(\int_0^T \exp \left\{2 \int_0^t \left(-2r + \frac{a^2 \eta^2}{b^2} + \frac{\theta^2(s)}{\sigma^2}\right) \dif s\right\} \Gamma_1^2(t) \dif t\right)^{\frac{1}{2}}\right]
\]

\[
\leq K \tilde{\mathbb{E}} \left[\left(\int_0^T \exp \left\{2 \int_0^t \alpha_1(t) \dif t\right\} \Gamma_1^2(t) \dif t\right)^{\frac{1}{2}}\right]
\]

\[
\leq K \tilde{\mathbb{E}} \left[\exp \left\{2 \int_0^t \alpha_1(t) \dif t\right\}\right] + K \tilde{\mathbb{E}} \left[\int_0^T \Gamma_1^2(t) \dif t\right] < \infty,
\]
where the last inequality follows from (27) and Assumption 1. Therefore, the stochastic integrals on the right-hand side of (28) are uniformly integrable martingales under $\bar{\mathbb{P}}$. Combining with the terminal value $Q(T) = 1$, we have

$$Q(t) \cdot \exp \left\{ \int_0^t \left( -2r + \frac{a^2 \eta^2}{b^2} + \frac{\theta^2(s)}{\sigma^2} \right) ds \right\}$$

$$= \mathbb{E} \left[ \exp \left\{ \int_0^T \left( -2r + \frac{a^2 \eta^2}{b^2} + \frac{\theta^2(s)}{\sigma^2} \right) ds \right\} | \mathcal{F}_t \right],$$

which leads to

$$Q(t) = \mathbb{E} \left[ \exp \left\{ \int_t^T \left( -2r + \frac{a^2 \eta^2}{b^2} + \frac{\theta^2(s)}{\sigma^2} \right) ds \right\} | \mathcal{F}_t \right]$$

$$= e^{-2r(T-t)} \mathbb{E} \left[ \exp \left\{ \int_t^T \left( \frac{a^2 \eta^2}{b^2} + \frac{\theta^2(s)}{\sigma^2} \right) ds \right\} | \mathcal{F}_t \right]$$

$$> e^{-2r(T-t)}.$$

Therefore, we have $0 < P(t) < e^{2r(T-t)}$, $\forall t \in [0, T]$, $\mathbb{P}$-a.s.

It is easy to show that $K(t)$ in (11) does not explode over $[0, T]$ (see Lemma 3.4 in Shen and Zeng [23]). Recalling $\Lambda_i(t)$ in (12) for $i = 1, 2$ and $0 < P(t) < e^{2r(T-t)}$, $\forall t \in [0, T]$, $\mathbb{P}$-a.s., we have

$$\mathbb{E} \left[ \int_0^T |\Lambda_i(t)| dt \right] \leq C \mathbb{E} \left[ \int_0^T P^2(t) dt \right] \leq C T \mathbb{E} \left[ \sup_{t \in [0, T]} P^2(t) \right] < +\infty,$$

for some constant $C > 0$. This implies $\Lambda_i(t) \in L_+^2(0, T; \mathbb{R})$, for $i = 1, 2$.

We next show that there is a sufficiently small constant $\epsilon > 0$ such that $P(t) \geq \epsilon$ for all $t \in [0, T]$, $\mathbb{P}$-a.s. Denote $E = \{ t \in [0, T) : K(t) < 0 \}$. In fact, according to Theorem 3.1, we obtain

$$P(t) = \exp \left\{ - 2r + \frac{a^2 \eta^2}{b^2} (t - T) + \frac{1}{2} K(t) \left( \theta(t) + \frac{N(t)}{K(t)} \right)^2 + M(t) - \frac{N^2(t)}{2K(t)} \right\}$$

$$\geq \exp \left\{ - 2r - \frac{a^2 \eta^2}{b^2} (t - T) + M(t) - \frac{N^2(t)}{2K(t)} \right\}$$

for any $t \notin E$. Let

$$\epsilon = \inf_{t \in [0, T]} \exp \left\{ - 2r - \frac{a^2 \eta^2}{b^2} (t - T) + M(t) - \frac{N^2(t)}{2K(t)} \right\}.$$

We have $P(t) \geq \epsilon$, $\forall t \notin E$, $\mathbb{P}$-a.s. Suppose that the statement is not true for the case of $t \in E$, then for each large enough positive integer $n$, there exists $t_n \in [0, T]$ and $\Omega_n \subset \Omega$ satisfying $\mathbb{P}(\Omega_n) > 0$ such that

$$P(t_n, \omega_n) < \frac{1}{n}, \quad \forall \omega_n \in \Omega,$$

which implies that for each large enough positive integer $N$, there exists $t_N \in [0, T]$ and $\Omega_N \subset \Omega$ satisfying $\mathbb{P}(\Omega_N) > 0$ such that

$$|\theta(t_N, \omega_N)| > N, \quad \forall \omega_N \in \Omega.$$

This contradicts with the fact that $\mathbb{E}[|\theta(t)|] < \infty$ for all $t \in [0, T]$. Therefore, there exists a positive constant $\epsilon$ such that $P(t) \geq \epsilon$, $\mathbb{P}$-a.s., which completes the proof. \(\square\)

**Remark 3.6.** Proposition 3.5 provides an accurate estimate for $P(t)$, which is necessary in the outer maximization problem of (9) in Theorem 3.7.
We now devote to deriving the optimal investment and reinsurance strategies for the variance-minimization problem (5), which are expressed in terms of the solutions to BSRE (10).

**Theorem 3.7.** Under Assumptions 1 and 2, the optimal strategy and the optimal cost functional of the variance-minimization problem (5) (or the equivalent min-max problem (9)) are respectively given by

\[
\begin{align*}
\pi^*(t) &= -\left(\frac{\theta(t)}{\sigma^2} + \frac{\Lambda_2(t)}{P(t)}\right)(X(t) - g^*(t)), \\
g^*(t) &= -\frac{a_\eta}{\sigma^2}(X(t) - g^*(t)),
\end{align*}
\]

and

\[
J_{VM}(\pi^*(\cdot), \xi^*) = \frac{P(0)}{e^{2\sigma T} - P(0)}(\xi - \xi_{min})^2,
\]

where

\[
g^*(t) = (\xi - \xi^*)e^{-r(T-t)} + \frac{c}{r}[e^{-r(T-t)} - 1],
\]

with

\[
\xi^* = -\frac{P(0)}{e^{2\sigma T} - P(0)}(\xi - \xi_{min}).
\]

**Proof.** We will prove this theorem in three steps. Firstly, we contributes to solving the inner minimization problem of (9). Define \( Y(t) = X(t) - g(t) \), where \( g(t) = (\xi - \xi + \frac{\xi}{T})e^{r(t-T)} - \frac{\xi}{r} \). Combining (2) with \( g(t) \) yields

\[
\begin{align*}
dY(t) &= [r Y(t) + \theta(t) \pi(t) + a\eta q(t)]dt + \sigma \pi(t)dW_1(t) - \int_0^\infty q(t)Y(t)\,d\tilde{N}(dt, dy), \\
Y(0) &= x - g(0).
\end{align*}
\]

Then applying Itô's formula to \( P(t)Y^2(t) \) leads to

\[
\begin{align*}
d[P(t)(X(t) - g(t))^2] &= d[P(t)Y^2(t)] \\
= Y^2(t) \left[ -2\sigma^2 + \frac{\sigma^2 \eta^2}{2} \right] P(t) + \frac{\theta^2(t)P(t)}{\sigma^2} + \frac{2\theta(t)\Lambda_1(t)}{\sigma} + \frac{\Lambda_2^2(t)}{P(t)} \right] dt \\
&\quad + P(t)[2\sigma^2 Y^2(t) + 20(t)\pi(t)Y(t)] + 2a\eta q(t)Y(t) + \sigma^2 \pi^2(t) + b^2 \eta^2(t) \right] dt \\
&\quad + 2\sigma \pi(t) Y(t)\Lambda_1(t)dt + [\Lambda_1(t)Y^2(t) + 2\sigma \pi(t)P(t)Y(t)]dW_1(t) + \Lambda_2(t)Y^2(t)dW_2(t) \\
&\quad + \int_0^\infty P(t)[q(t)Y(t) - 2q(t)g(t)Y(t)]\,d\tilde{N}(dt, dy) \\
= P(t) \left[ \frac{\sigma \pi(t)}{\sigma} + \left(\frac{\theta(t)}{\sigma} + \frac{\Lambda_1(t)}{P(t)}\right)Y(t) \right]^2 \\
&\quad + \left[ b\eta(t) + \frac{a\eta}{b} Y(t) \right]^2 \right] dt \\
&\quad + [\Lambda_1(t)Y^2(t) + 2\sigma \pi(t)P(t)Y(t)]dW_1(t) + \Lambda_2(t)Y^2(t)dW_2(t) \\
&\quad + \int_0^\infty P(t)[q(t)Y(t) - 2q(t)g(t)Y(t)]\,d\tilde{N}(dt, dy). \quad (32)
\end{align*}
\]

Note that \( \pi(\cdot) \in L^2_\mathbb{F}(0, T; \mathbb{R}) \) and Proposition 3.5, we have

\[
\begin{align*}
\int_0^T |\Lambda_1(t)Y^2(t) + 2\sigma \pi(t)P(t)Y(t)|^2 dt < +\infty, & \quad \mathbb{P}\text{-a.s.} \\
\int_0^T |\Lambda_2(t)Y^2(t)|^2 dt < +\infty, & \quad \mathbb{P}\text{-a.s.}
\end{align*}
\]
Moreover, since \( q(\cdot) \in L^2_T(0, T; \mathbb{R}^+) \cap L^{4,loc}_T(0, T; \mathbb{R}^+) \) and each claim size has finite moments up to order 4, we obtain
\[
\int_0^T \int_0^\infty |P(t)[q^2(t)y^2 - 2q(t)yY(t)]|^2 \lambda F(dy)dt \\
\leq K \left\{ \int_0^T q^4(t)dt \times \int_0^\infty y^4 dF(y) + \int_0^T q^2(t)dt \times \int_0^\infty y^2 dF(y) \right\} < \infty, \quad \mathbb{P}\text{-a.s.}
\]
where \( K \) is a positive constant. This implies that the stochastic integrals with respect to the two Brownian motions and the Poisson random measure in (32) are local martingales. Hence there exists a sequence of stopping times \( \{\tau_n\}_{n=1,2,...} \) such that \( \tau_n \uparrow \infty \) as \( n \to \infty \), and the local martingales are indeed martingales when stopped by \( \{\tau_n\}_{n=1,2,...} \). Integrating on both sides of (32) from 0 to \( T \wedge \tau_n \) and taking expectations result in
\[
\mathbb{E}[P(T \wedge \tau_n)Y^2(T \wedge \tau_n)] - P(0)(x - g(0))^2 \\
= \mathbb{E} \left[ \int_0^{T \wedge \tau_n} P(t) \left\{ \left[ \sigma \pi(t) + \left( \frac{\theta(t)}{\sigma} + \frac{\Lambda_1(t)}{P(t)} \right) Y(t) \right]^2 + \left[ bq(t) + \frac{\alpha_\eta}{b} Y(t) \right]^2 \right\} dt \right].
\]
In view of Definition 2.2, we have \( \mathbb{E}[\sup_{t \in [0,T]} |Y(t)|^2] = \mathbb{E}[\sup_{t \in [0,T]} |X(t) - g(t)|^2] < \infty \) for any admissible strategy \( u(\cdot) \). Furthermore, \( P(t) \) is uniformly bounded on \([0,T] \) due to Proposition 3.5. Applying the dominated convergence theorem to the left-hand side of (33) and the monotone convergence theorem to the right-hand side yield
\[
\mathbb{E}[X(T) - (\xi - \zeta)]^2 = \mathbb{E}[P(T)Y^2(T)] - P(0)(x - g(0))^2 \\
= \mathbb{E} \left[ \int_0^T P(t) \left\{ \left[ \sigma \pi(t) + \left( \frac{\theta(t)}{\sigma} + \frac{\Lambda_1(t)}{P(t)} \right) Y(t) \right]^2 + \left[ bq(t) + \frac{\alpha_\eta}{b} Y(t) \right]^2 \right\} dt \right].
\]
Therefore, the optimal strategy and the associated cost functional of the inner minimization problem of (9) are respectively given by
\[
\pi^*(t) = - \left( \frac{\theta(t)}{\sigma^2} + \frac{\Lambda_1(t)}{\sigma P(t)} \right) Y(t),
\]
\[
q^*(t) = - \frac{\alpha_\eta}{b^2} Y(t),
\]
and
\[
J_{BM}(u^*(\cdot)) = P(0)(x - g(0))^2.
\]
Secondly, we proceed to determine the optimal \( \zeta^* \) among all \( \zeta \in \mathbb{R} \). Now we consider the cost functional of the min-max problem (9):
\[
J_{VM}(u^*(\cdot), \zeta) = P(0)[x_0 - g(0)]^2 - \zeta^2 \\
= P(0)[x_0 - (\xi - \zeta)e^{-rT} + c \int_0^T e^{-rs}ds]^2 - \zeta^2. \quad (34)
\]
Since \( P(0)e^{-2rT} - 1 < 0 \) due to Proposition 3.5, we have
\[
\frac{\partial^2 J_{VM}(u^*(\cdot), \zeta)}{\partial \zeta^2} = 2(P(0)e^{-2rT} - 1) < 0.
\]
This indicates that (34) attains its maximum value
\[ J_{VM}(u^*(\cdot), \zeta^*) = \frac{P(0)}{e^{2rT} - P(0)} (\xi - \xi_{min})^2 \]
at \( \zeta^* \), which is given by (31).

Thirdly, we aim to verify that \( q^*(t) \) obtained in (29) is nonnegative. In fact, combining (2) with (30), we obtain
\[ d[X(t) - g^*(t)] = [X(t) - g^*(t)] \times \left\{ r - \theta(t) \left( \frac{\theta(t)}{\sigma^2} + \frac{\Lambda_1(t)}{\sigma P(t)} \right) - \frac{a^2 \eta^2}{b^2} \right\} dt \\
- \sigma \left( \frac{\theta(t)}{\sigma^2} + \frac{\Lambda_1(t)}{\sigma P(t)} \right) dW_1(t) + \frac{\eta}{b} \int_0^\infty y \tilde{N}(dt, dy) \}, \] (35)
with initial value
\[ X(0) - g^*(0) = x_0 - (\xi - \zeta^*) e^{-rT} + \frac{c}{r} \left[ 1 - e^{-rT} \right] \]
\[ = e^{-rT}[\xi_{min} - \xi + \zeta^*]. \]

In view of Assumption 2, Proposition 3.5 and (31), we conclude that \( X(0) - g^*(0) \leq 0 \). Note from Eq. (35) that whether \( X(t) - g^*(t) \) is positive or negative is completely determined by the initial value \( X(0) - g^*(0) \). Therefore, we have \( X(t) - g^*(t) \leq 0 \), which guarantees the nonnegativity of \( q^*(t) = -\frac{a\eta}{b^2} (X(t) - g^*(t)) \). Thus, we end this proof. \( \square \)

4. Efficient strategy and efficient frontier. In this section, we devote to investigating the efficient strategy and efficient frontier of the mean-variance problem (4). Before this, we first prove the admissibility of the optimal strategy obtained in Theorem 3.7.

**Theorem 4.1.** Under Assumptions 1 and 2, the optimal strategy given by (29) in Theorem 3.7 is admissible.

**Proof.** In terms of Theorem 3.7, \( \pi^*(\cdot) \) and \( q^*(\cdot) \geq 0 \) given by (29) are predictable.

In the following, we first show that SDE (2) corresponding to \( (\pi^*(\cdot), q^*(\cdot)) \) in (29) has a unique solution \( X(t) \in L^2_T(0, T; \mathbb{R}) \). In fact, suppose that
\[ dX_1(t) = X_1(t) \left\{ r - \theta(t) \left( \frac{\theta(t)}{\sigma^2} + \frac{\Lambda_1(t)}{\sigma P(t)} \right) - \frac{a^2 \eta^2}{b^2} \right\} dt - \sigma \left( \frac{\theta(t)}{\sigma^2} + \frac{\Lambda_1(t)}{\sigma P(t)} \right) dW(t) \}, \] (36)
d\[ dX_2(t) = \frac{\eta}{b} \int_0^\infty y X_2(t-) \tilde{N}(dt, dy) \],
with \( X_1(0) = x - g^*(0) \) and \( X_2(0) = 1 \). Then (35) is equivalent to \( X(t) - g^*(t) = X_1(t) \cdot X_2(t) \). Clearly, SDE (36) admits a unique solution \( X_2(\cdot) \) such that \( \mathbb{E}[\sup_{t \in [0, T]} |X_2(t)|^2] < \infty \). Moreover, since \( \Lambda_1(t) = \beta \rho K(t) \theta(t) + N(t) P(t) \), \( K(t) \) does not explode over \([0, T]\) and \( P(t) \) is uniformly bounded on \([0, T]\), all the conditions in Pardoux and Răşcanu [19, Theorem 3.26] are satisfied by the drift and diffusion terms of SDE (36). Thus, SDE (36) admits a unique solution \( X_1(\cdot) \) such that \( X_1(\cdot) \in L^2_T(0, T; \mathbb{R}) \). As a result, \( X(t) = X_1(t) \cdot X_2(t) + g^*(t) \in L^2_T(0, T; \mathbb{R}) \).

We next show that \( \mathbb{E}[\sup_{t \in [0, T]} |X(t)|^2] < \infty \). Substituting (29) into Eq. (33), we have
\[ \mathbb{E}[P(t \wedge \tau_n) (X(t \wedge \tau_n) - g^*(t \wedge \tau_n))^2] - P(0)(x - g^*(0))^2 = 0. \]
Since \( P(t) \geq \epsilon \) and \( \tau_n \uparrow \infty \), \( \mathbb{P} \text{-a.s.} \), as \( n \to \infty \), it follows from Fatou’s Lemma that
\[
P(0)(x - g^*(0))^2 = \lim_{n \to \infty} \mathbb{E}[P(t \wedge \tau_n)(X(t \wedge \tau_n) - g^*(t \wedge \tau_n))^2] \\
\geq \mathbb{E}[ \lim_{n \to \infty} P(t \wedge \tau_n)(X(t \wedge \tau_n) - g^*(t \wedge \tau_n))^2] \\
= \mathbb{E}[P(t)(X(t) - g^*(t))^2] \\
\geq c\mathbb{E}[(X(t) - g^*(t))^2].
\]

Applying the boundedness of \( g^*(\cdot) \) gives \( \mathbb{E}[\sup_{t \in [0,T]} |X(t)|^2] < \infty \), which yields \( q^*(\cdot) \in L^2_\mathbb{P}(0,T;\mathbb{R}^+) \). Combining (29) with (35) and using the Holder’s inequality, we have \( \mathbb{E}[\int_0^T \pi^*(t)\,dt] < \infty \) after some direct calculations due to Assumption 1, which means \( \pi^*(\cdot) \in L^2_\mathbb{P}(0,T;\mathbb{R}) \). Moreover, the almost-surely boundedness of \( X(\cdot) \) implies that \( q^*(\cdot) \in L^{4,loc}_\mathbb{P}(0,T;\mathbb{R}^+) \). Hence, it holds that \( q^*(\cdot) \in L^2_\mathbb{P}(0,T;\mathbb{R}^+) \)\( \cap L^{4,loc}_\mathbb{P}(0,T;\mathbb{R}^+) \). Therefore, \( \pi^*(\cdot) \) and \( q^*(\cdot) \) obtained in Theorem 3.7 satisfies all the conditions in Definition 2.2, which guarantee that \( \pi^*(\cdot) \) and \( q^*(\cdot) \) given by (29) are admissible.

Now, we provide the following straightforward theorem to summarize the main result, i.e., the efficient strategy and efficient frontier parameterized by \( \xi \) of the mean-variance problem (4).

**Theorem 4.2.** Under Assumptions 1 and 2, the efficient strategy \( \{\pi^*\xi(t), q^*\xi(t)\} : \xi \geq \xi_{\text{min}} \) of the mean-variance problem (4) is given by
\[
\begin{align*}
\pi^*\xi(t) &= -\left(\frac{\theta(t)}{\sigma^2} + \frac{\lambda(\xi)}{\sigma P(t)}\right)(X(t) - g^*(t)), \\
q^*\xi(t) &= -\frac{\alpha}{\beta}(X(t) - g^*(t)),
\end{align*}
\]
and the efficient frontier parameterized by \( \xi \) (where \( \xi = \mathbb{E}[X(T)] \)) is represented as
\[
\text{Var}[X(T)] = \frac{P(0)}{e^{2\sigma T} - P(0)}(\xi - \xi_{\text{min}})^2,
\]
where (with a slight abuse of notation) \( g^*(t) \) is given by (30) and \( P(t) \) follows from Theorem 3.1.

**Proof.** The proof is adapted from Theorem 3.7 and the relation between the mean-variance problem (4) and the variance-minimization problem (5), thereby is omitted.

From Theorem 4.2, we get easily the following corollary which corresponds to Theorem 2.3 of Bai and Zhang [4].

**Corollary 4.3.** If \( \alpha = \beta = 0 \), the efficient strategy \( \{\pi^*\xi(t), q^*\xi(t)\} : \xi \geq \xi_{\text{min}} \) of the mean-variance problem (4) is reduced to
\[
\begin{align*}
\pi^*\xi(t) &= -\frac{\mu - r \xi_{\text{min}}}{\sigma^2}(X(t) - g^*(t)), \\
q^*\xi(t) &= -\frac{\alpha}{\beta}(X(t) - g^*(t)),
\end{align*}
\]
and the efficient frontier parameterized by \( \xi \) (where \( \xi = \mathbb{E}[X(T)] \)) is represented as
\[
\text{Var}[X(T)] = \frac{1}{e^{(\frac{\mu T}{\sigma^2} + \frac{\mu - r \xi_{\text{min}}^2}{\sigma^2})T} - 1}(\xi - \xi_{\text{min}})^2,
\]
with \( g^*(t) \) given by (30).
Remark 4.4. Corollary 4.3 indicates that our result can be seen as an extension of Bai and Zhang [4] to the OU framework, which captures the features of bull and bear markets.

5. Sensitivity analysis. In this section, we will first demonstrate numerically how the settings of OU process affects the insurer’s investment and reinsurance strategy, and then discuss the effects of parameters on the efficient frontier obtained in Section 4. Throughout this section, unless otherwise stated, the parameters take the following values: \(x_0 = 4, T = 10, r = 0.04, \mu = 0.05, m_0 = 0.02, \sigma = \rho = 0.05, \alpha = -0.04, \beta = 0.03, \lambda = 1, \mu_1 = 0.6, \mu_2 = 0.4, \delta = 0.2, \eta = 0.3\).

Figure 1 shows us the path of OU process \(m(t)\) with parameters \(\alpha = -0.04\) and \(\beta = 0.03\). From Figure 1, we see that the value of \(m(i\Delta)\) decreases from the initial value \(m_0 = 0.02\) to 0, and then stays under the level 0 for the first half of the time period, which reflects the bear market. Due to the mean-reverting property of OU process, the value of \(m(i\Delta)\) increases from negative to positive, and then stays above the level 0 for the second half of the time period, which means the bear market is turning to the bull market.

Figure 2 shows us the two optimal investment strategies with parameters \(\alpha = -0.04, \beta = 0.03\) and \(\alpha = \beta = 0\), respectively. We can conclude from Figure 2 that, the optimal investment strategy is more stable over time in the financial market when there are no features of bull and bear markets (i.e., \(\alpha = \beta = 0\)). In other words, the optimal investment strategy is more likely to fluctuate over time in the financial market when incorporating the features of bull and bear markets (i.e., \(\alpha = -0.04, \beta = 0.03\)). This is because the insurer would like to invest less in the stock when it is a bear market and this investment portion increases when the bear market is about to turn bull market. On the other hand, the insurer would like to invest more in the stock when it is a bull market and this investment...
portion decreases when the bull market is about to turn bear market. That is, the optimal investment strategy has a mean-reverting characteristic when incorporating the features of bull and bear markets into the stock price model.

Figure 2. the optimal investment strategy with $\alpha = -0.04, \beta = 0.03$ and $\alpha = \beta = 0$.

Figure 3 contributes to the comparison of the two optimal reinsurance strategies with parameters $\alpha = -0.04, \beta = 0.03$ and $\alpha = \beta = 0$, respectively. From Figure 3, we see that the retention level of reinsurance in Bai and Zhang [4] is bigger than that in the current paper for the first half of the time period $[0, T]$, and decreases quickly with the passage of time. This implies that the incorporation of OU process (features of bull and bear markets) does affect the insurance portfolio of the insurer. One possible explanation for this phenomenon is that when it is a bull market, the expected return of the insurer increases and, at the same time, the exposure to the financial risk is reduced. Under such circumstances, the insurer has the power to undertake more insurance risks, and hence the retention level of reinsurance becomes larger. Conversely, when it is a bear market, the financial risk of the insurer becomes larger and in this case, the insurer prefers to purchase more reinsurance to hedge the forthcoming insurance risk.

Figure 4 illustrates the effect of the parameter $\alpha$ on the efficient frontier of the mean-variance problem in this paper. Assume that the value of $\alpha$ takes -0.04, -0.10 and -0.16, respectively, and the other parameters are fixed. Figure 4 shows that, under the same $\text{Var}[X(T)]$, $\mathbb{E}[X(T)]$ increases with $\alpha$. This is obvious since $-\alpha$ represents the mean-reverting rate of the instantaneous growth rate. That is, the smaller $-\alpha$, i.e., the bigger $\alpha$, the more slowly stock price moves downward in the bull market. Therefore, under the same risk, the insurer would like to invest more into the risky asset to reach a higher expected terminal wealth. On the other hand, the smaller $-\alpha$, the more slowly stock price moves upward in the bear market.
the i−th discrete time point in \([0,T]\) with \(\Delta = T/100\)

optimal reinsurance strategy \(q^*(i\Delta)\)

\(\alpha = -0.04, \beta = 0.03\)

\(\alpha = \beta = 0\)

\(E\[X(T)\]

\(\text{Var}[X(T)]\)

\(\alpha = -0.04\)

\(\alpha = -0.10\)

\(\alpha = -0.16\)

Therefore, under the same expected terminal wealth, the insurer would like to invest more into the risk-free asset and less to the risky asset to reduce the risk.

In Figure 5, we analyze the effect of the parameter \(\lambda\) on the efficient frontier of the mean-variance problem. Assume that \(\lambda\) takes 1.0, 1.2 and 1.4, respectively, and the other parameters are fixed. We can conclude from Figure 5 that, for \(E[X(T)]\) large
Figure 5. the effect of $\lambda$ on the efficient frontier.

enough, $\text{Var}[X(T)]$ decreases with $\lambda$ under the same $\mathbb{E}[X(T)]$. It is clear that the insurer receives more premium as $\lambda$ increases. Thus, to obtain the same expected terminal wealth, the insurer would like to invest more into the risk-free asset and less into the risky asset to hedge the more frequently claims, which results in the decreasing of variance of the terminal wealth. However, for $\mathbb{E}[X(T)]$ small enough, the frequency of claims is the main risk the insurer undertakes. Thus, the bigger $\lambda$ the more risks under the same expected terminal wealth.

Figure 6 investigates the impact of the parameter $\eta$ on the efficient frontier of the mean-variance problem. Assume that $\eta$ takes 0.20, 0.25 and 0.30, respectively, and the other parameters are fixed. In Figure 6 we find that for $\mathbb{E}[X(T)]$ large enough, $\text{Var}[X(T)]$ decreases with $\eta$ under the same $\mathbb{E}[X(T)]$. That is, as the safety loading of the reinsurer increases, the insurer would like to undertake more claims to reduce the premium ceded to the reinsurer. Thus, to obtain the same expected terminal wealth, the insurer would like to invest more into the risk-free asset and less into the risky asset to hedge the forthcoming claims, which results in the decreasing of variance of the terminal wealth. However, for $\mathbb{E}[X(T)]$ small enough, claims are the main risk for the insurer and the insurer have to rely on the reinsurer to hedge the risk associated with claims. Thus, under the same risks, the insurer will cede more premium to the reinsurer and obtain less expected terminal wealth as $\eta$ increases. Specially, if the insurer invest all the wealth into the risk-free asset and cedes all the claims to the reinsurer, the bigger $\eta$, the more premiums given to reinsurer, which leads to the less expected terminal wealth.

Figure 7 investigates the impact of the parameter $r$ on the efficient frontier of the mean-variance problem. Assume that $r$ takes 0.02, 0.03 and 0.04, respectively, and the other parameters are fixed. Figure 7 shows that, under the same $\text{Var}[X(T)]$, $\mathbb{E}[X(T)]$ increases with $r$. Obviously, as the interest rate $r$ increases, the expected terminal return of the insurer increases accordingly without investing more into the
5.5 6 6.5 7 7.5 8 8.5 9
0
5
10
15
E[X(T)]
Var[X(T)]

η =0.20
η =0.25
η =0.30

Figure 6. the effect of η on the efficient frontier.

4 5 6 7 8 9 10
0
2
4
6
8
10
12
14
16
18
20
E[X(T)]
Var[X(T)]

r=0.02
r=0.03
r=0.04

Figure 7. the effect of r on the efficient frontier.

risky asset. In other words, under the same risk \( \text{Var}[X(T)] \), the bigger \( r \) the bigger \( \text{E}[X(T)] \).

6. Conclusion. In this paper, we apply a BSDE method to deal with the optimal mean-variance problem when the risk model is described by a classical compound Poisson process. The price of the risky asset is driven by an OU process, which
makes our model more realistic since it can characterize the bull and bear markets. Instead of solving the HJB equation and presenting the verification theorem, we transform our problem to solving the corresponding BSDE, and then derive the explicit expressions for the efficient strategy and efficient frontier by applying the results of the BSDE.

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