Stochastic Recursive Variance-Reduced Cubic Regularization Methods

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Abstract

Stochastic Variance-Reduced Cubic regularization (SVRC) algorithms have received increasing attention due to its improved gradient/Hessian complexities (i.e., number of queries to stochastic gradient/Hessian oracles) to find local minima for nonconvex finite-sum optimization. However, it is unclear whether existing SVRC algorithms can be further improved. Moreover, the semi-stochastic Hessian estimator adopted in existing SVRC algorithms prevents the use of Hessian-vector product-based fast cubic subproblem solvers, which makes SVRC algorithms computationally intractable for high-dimensional problems. In this paper, we first present a Stochastic Recursive Variance-Reduced Cubic regularization method (SRVRC) using a recursively updated semi-stochastic gradient and Hessian estimators. It enjoys improved gradient and Hessian complexities to find an \((\epsilon, \sqrt{\epsilon})\)-approximate local minimum, and outperforms the state-of-the-art SVRC algorithms. Built upon SRVRC, we further propose a Hessian-free SRVRC algorithm, namely SRVRC\(_{\text{free}}\), which only requires stochastic gradient and Hessian-vector product computations, and achieves \(\tilde{O}(dn\epsilon^{-2} \wedge d\epsilon^{-3})\) runtime complexity, where \(n\) is the number of component functions in the finite-sum structure, \(d\) is the problem dimension, and \(\epsilon\) is the optimization precision. This outperforms the best-known runtime complexity \(\tilde{O}(d\epsilon^{-3.5})\) achieved by stochastic cubic regularization algorithm proposed in Tripuraneni et al. (2018).

1 Introduction

Many machine learning problems can be formulated as empirical risk minimization, which is in the form of finite-sum optimization as follows:

\[
\min_{x \in \mathbb{R}^d} F(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),
\]

where each \(f_i: \mathbb{R}^d \to \mathbb{R}\) can be a convex or nonconvex function. In this paper, we are particularly interested in nonconvex finite-sum optimization, where each \(f_i\) is nonconvex. This is often the case for deep learning (LeCun et al., 2015). In principle, it is hard to find the global minimum of

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because of the NP-hardness of the problem (Hillar and Lim, 2013), thus it is reasonable to resort to finding local minima (a.k.a., second-order stationary points). It has been shown that local minima can be the global minima in certain machine learning problems, such as low-rank matrix factorization (Ge et al., 2016; Bhojanapalli et al., 2016; Zhang et al., 2018b) and training deep linear neural networks (Kawaguchi, 2016; Hardt and Ma, 2016). Therefore, developing algorithms to find local minima is important both in theory and in practice. More specifically, we define an \((\epsilon_g, \epsilon_H)\)-approximate local minimum \(x\) of \(F(x)\) as follows
\[
\|\nabla F(x)\|_2 \leq \epsilon_g, \quad \lambda_{\min}(\nabla^2 F(x)) \geq -\epsilon_H,
\] (1.2)
where \(\epsilon_g, \epsilon_H > 0\) are predefined precision parameters.

The most classic algorithm to find the approximate local minimum is cubic-regularized (CR) Newton method, which was originally proposed in a seminal paper by Nesterov and Polyak (2006). Generally speaking, in the \(k\)-th iteration, cubic regularization method solves a subproblem, which minimizes a cubic-regularized second-order Taylor expansion at the current iterate \(x_k\). The update rule can be written as follows:
\[
h_k = \arg\min_{h \in \mathbb{R}^d} \langle \nabla F(x_k), h \rangle + \frac{1}{2} \langle \nabla^2 F(x_k)h, h \rangle + \frac{M}{6} \|h\|^3_2,
\] (1.3)
\[
x_{k+1} = x_k + h_k,
\] (1.4)
where \(M > 0\) is a penalty parameter. Nesterov and Polyak (2006) proved that to find an \((\epsilon, \sqrt{\epsilon})\)-approximate local minimum of a nonconvex function \(F\), cubic regularization requires at most \(O(\epsilon^{-3/2})\) iterations. However, when applying cubic regularization to nonconvex finite-sum optimization in (1.1), a major bottleneck of cubic regularization is that it needs to compute \(n\) individual gradients \(\nabla f_i(x_k)\) and Hessian matrices \(\nabla^2 f_i(x_k)\) at each iteration, which leads to a total \(O(ne^{-3/2})\) gradient complexity (i.e., number of queries to the stochastic gradient oracle \(\nabla f_i(x)\) for some \(i\) and \(x\)) and \(O(ne^{-3/2})\) Hessian complexity (i.e., number of queries to the stochastic Hessian oracle \(\nabla^2 f_i(x)\) for some \(i\) and \(x\)). Such computational overhead will be extremely expensive when \(n\) is large as is in many large-scale machine learning applications.

To overcome the aforementioned computational burden of cubic regularization, Kohler and Lucchi (2017); Xu et al. (2017) used subsampled gradient and subsampled Hessian, which achieve \(\tilde{O}(ne^{-3/2} \wedge \epsilon^{-7/2})\) gradient complexity and \(\tilde{O}(ne^{-3/2} \wedge \epsilon^{-5/2})\) Hessian complexity. Zhou et al. (2018d) proposed a stochastic variance reduced cubic regularization method (SVRC), which uses novel semi-stochastic gradient and semi-stochastic Hessian estimators inspired by variance reduction for first-order finite-sum optimization (Johnson and Zhang, 2013; Reddi et al., 2016a; Allen-Zhu and Hazan, 2016), which attains \(O(n + n^{4/5}\epsilon^{-3/2})\) Second-order Oracle (SO) complexity. Zhou et al. (2018b); Wang et al. (2018); Zhang et al. (2018a) used a simpler semi-stochastic gradient compared with Zhou et al. (2018d), and semi-stochastic Hessian, which a better Hessian complexity, i.e., \(O(n + n^{2/3}\epsilon^{-3/2})\). However, it is unclear whether the gradient and Hessian complexities of the aforementioned SVRC algorithms can be further improved. Furthermore, all these algorithms need to use the semi-stochastic Hessian estimator, which is not compatible with Hessian-vector

\footnote{Second-order Oracle (SO) returns triple \([f_i(x), \nabla f_i(x), \nabla^2 f_i(x)]\) for some \(i\) and \(x\), hence the SO complexity can be seen as the maximum of gradient and Hessian complexities.}
product-based cubic subproblem solvers (Agarwal et al., 2017; Carmon and Duchi, 2016, 2018). Therefore, the cubic subproblem (1.4) in each iteration of existing SVRC algorithms has to be solved by computing the inverse of the Hessian matrix, whose computational complexity is at least $O(d^w)^2$. This makes existing SVRC algorithms not very practical for high-dimensional problems.

In this paper, we first show that the gradient and Hessian complexities of SVRC-type algorithms can be further improved. The core idea is to use a novel recursively updated semi-stochastic gradient and Hessian estimators, which are inspired by the recursive semi-stochastic gradient estimators used in Nguyen et al. (2017); Fang et al. (2018) for first-order finite-sum optimization. We show that such kind of estimators can also reduce the Hessian complexity, which has never been discovered before. In addition, in order to reduce the runtime complexity of existing SVRC algorithms, we further propose a Hessian-free SVRC method that can not only use the novel semi-stochastic gradient estimator, but also leverage the Hessian-vector product-based fast cubic subproblem solvers. Experiments on benchmark nonconvex finite-sum optimization problems illustrate the superiority of our newly proposed SVRC algorithms against the state-of-the-art.

In detail, our contributions are summarized as follows:

- We propose a new SVRC algorithm, namely SRVRC, which can find an $(\epsilon, \sqrt{\epsilon})$-approximate local minimum with $\tilde{O}(n\epsilon^{-3/2} + \epsilon^{-3})$ gradient complexity and $\tilde{O}(n + n^{1/2}\epsilon^{-3/2} + \epsilon^{-2})$ Hessian complexity. Compared with previous work, the gradient and Hessian complexity of SRVRC is strictly better than the algorithms in Zhou et al. (2018b); Wang et al. (2018); Zhang et al. (2018a), and better than that in Zhou et al. (2018d) in a wide regime.

- We further propose a new algorithm SRVRC<sub>free</sub>, which requires $\tilde{O}(d\epsilon^{-3} \land n\epsilon^{-2})$ runtime to find an $(\epsilon, \sqrt{\epsilon})$-approximate local minimum. The runtime of SRVRC<sub>free</sub> is strictly better than that of Agarwal et al. (2017); Carmon and Duchi (2016); Tripuraneni et al. (2018) when $n \gg 1$. The runtime complexity of SRVRC<sub>free</sub> is also better than that of SRVRC when $d$ is large.

For the ease of comparison, we list the gradient and Hessian complexity results of our algorithms as well as the baselines algorithms in Table 1 and the runtime complexity results in Table 2.

2 Other Related Work

In this section, we review additional related work, that is not discussed in the introduction section. Cubic Regularization and Trust-Region Methods Since cubic regularization was firstly proposed in Nesterov and Polyak (2006), there is a line of followup research. It was extended to adaptive regularized cubic methods (ARC) by Cartis et al. (2011a,b), which enjoy the same iteration complexity as standard cubic regularization while having better empirical performance. The first attempt to make cubic regularization a Hessian-free method was done by Carmon and Duchi (2016), which solves the cubic sub-problem by gradient descent, requiring total $\tilde{O}(dne^{-2})$ runtime. Agarwal

\[ w \] is the matrix multiplication constant, where $w = 2.37...$

\[ ^{3} \] The complexity for Natasha<sub>2</sub> to find $(\epsilon, \epsilon^{1/4})$-local minimum only requires $\tilde{O}(\epsilon^{-3.25})$. Here we adapt the complexity result for finding an $(\epsilon, \epsilon^{1/2})$-approximate local minimum.
Table 1: Comparisons of different methods to find an \((\epsilon, \sqrt{\rho \epsilon})\)-local minimum on the gradient complexity and the Hessian complexity.

| Algorithm          | Gradient Complexity | Hessian Complexity |
|--------------------|---------------------|--------------------|
| Cubic regularization (Nesterov and Polyak, 2006) | \(O\left(\frac{n}{\epsilon^{7/2}}\right)\) | \(O\left(\frac{n}{\epsilon^{7/2}}\right)\) |
| Subsampled cubic regularization (Kohler and Lucchi, 2017; Xu et al., 2017) | \(\tilde{O}\left(\frac{n}{\epsilon^{3/2}} \wedge \frac{1}{\epsilon^{7/2}}\right)\) | \(\tilde{O}\left(\frac{n}{\epsilon^{3/2}} \wedge \frac{1}{\epsilon^{7/2}}\right)\) |
| SVRC (Zhou et al., 2018d) | \(\tilde{O}\left(n + \frac{n^{4/5}}{\epsilon^{7/2}}\right)\) | \(\tilde{O}\left(n + \frac{n^{4/5}}{\epsilon^{7/2}}\right)\) |
| Lite-SVRC (Zhou et al., 2018b) | \(\tilde{O}\left(\frac{n}{\epsilon^{7/2}}\right)\) | \(\tilde{O}\left(n + \frac{n^{2/3}}{\epsilon^{5/2}}\right)\) |
| SVRC (Wang et al., 2018) | \(\tilde{O}\left(\frac{n}{\epsilon^{7/2}}\right)\) | \(\tilde{O}\left(n + \frac{n^{2/3}}{\epsilon^{5/2}}\right)\) |
| SVRC (Zhang et al., 2018a) | \(O\left(\frac{n}{\epsilon^{3/2}} \wedge \frac{n^{2/3}}{\epsilon^{5/2}}\right)\) | \(O\left(n + \frac{n^{2/3}}{\epsilon^{5/2}}\right)\) |
| SVRRC (Zhou et al., 2018b) | \(\tilde{O}\left(n \wedge \frac{1}{\epsilon^{7/2}} \wedge \frac{1}{\epsilon^{2}}\right)\) | \(\tilde{O}\left(n \wedge \frac{1}{\epsilon^{7/2}} + \frac{n^{3/2}}{\epsilon^{7/2}} \wedge \frac{1}{\epsilon^{2}}\right)\) |

et al. (2017) solved cubic sub-problem by fast matrix inversion based on accelerated gradient descent, which requires \(\tilde{O}(dn\epsilon^{-3/2} + dn^{3/4} \epsilon^{-7/4})\) runtime. In the pure stochastic optimization setting, Tripuraneni et al. (2018) proposed stochastic cubic regularization method, which uses subsampled gradient and Hessian-vector product-based cubic subproblem solver, and requires \(\tilde{O}(de^{-3.5})\) runtime. A closely related second-order method to cubic regularization methods are trust-region methods (Conn et al., 2000; Cartis et al., 2009, 2012, 2013). Recent studies (Blanchet et al., 2016; Curtis et al., 2017; Martínez and Raydan, 2017) proved that the trust-region method can achieve the same iteration complexity as the cubic regularization method. Xu et al. (2017) also extended trust-region method to subsampled trust-region method for nonconvex finite-sum optimization.

**Local Minima Finding** Besides cubic regularization and trust-region type methods, there is another line of research for finding approximate local minima, which is based on first-order optimization. Ge et al. (2015); Jin et al. (2017a) proved that (stochastic) gradient methods with additive noise are able to escape from nondegenerate saddle points and find approximate local minima. Carmon et al. (2018); Royer and Wright (2017); Allen-Zhu (2017); Xu et al. (2018); Allen-Zhu and Li (2018); Jin et al. (2017b); Yu et al. (2017b,a); Zhou et al. (2018a); Fang et al. (2018) showed that by alternating first-order optimization and Hessian-vector product based negative curvature descent, one can find approximate local minima even more efficiently.

**Variance Reduction** Variance reduction techniques play an important role in our proposed algorithms. Variance reduction techniques were first proposed for convex finite-sum optimization, which use semi-stochastic gradient to reduce the variance of the stochastic gradient and improve the gradient complexity. Representative algorithms include Stochastic Average Gradient (SAG) (Roux et al., 2012), Stochastic Variance Reduced Gradient (SVRG) (Johnson and Zhang, 2013; Xiao and Zhang, 2014), SAGA (Defazio et al., 2014) and SARAH (Nguyen et al., 2017), to mention a
Table 2: Comparisons of different methods to find an $(\epsilon, \sqrt{\rho \epsilon})$-local minimum on the runtime.

| Algorithm                  | Runtime                                      |
|----------------------------|----------------------------------------------|
| Fast-Cubic \cite{Agarwal17} | $\tilde{O}(\frac{dn}{\epsilon^3} + \frac{dn^{3/4}}{\epsilon^{7/4}})$ |
| GradientCubic \cite{Carmon16} | $\tilde{O}(\frac{dn}{\epsilon^2})$            |
| Stochastic Fast Cubic \cite{Tripuraneni18} | $\tilde{O}(\frac{d}{\epsilon^{7/2}})$       |
| SRVRC\text{free} \cite{Thiswork} | $\tilde{O}(\frac{dn}{\epsilon^2} \land \frac{d}{\epsilon^3})$ |

d. few. For nonconvex finite-sum optimization problems, Garber and Hazan (2015); Shalev-Shwartz (2016) studies the case where each individual function is nonconvex, but their sum is still (strongly) convex. Reddi et al. (2016a); Allen-Zhu and Hazan (2016) extended SVRG to nonconvex finite-sum optimization, which is able to converge to first-order stationary point with better gradient complexity than vanilla gradient descent. Recently, Fang et al. (2018); Zhou et al. (2018c) further improve the gradient complexity for nonconvex finite-sum optimization to be (near) optimal.

### 3 Notation and Preliminaries

In this work, all index subsets are multiset. We use $\nabla f_I(x)$ to represent $1/|I| \cdot \sum_{i \in I} \nabla f_i(x)$ if $|I| < n$ and $\nabla F(x)$ otherwise. We use $\nabla^2 f_I(x)$ to represent $1/|I| \cdot \sum_{i \in I} \nabla^2 f_i(x)$ if $|I| < n$ and $\nabla^2 F(x)$ otherwise. For a vector $v$, we denote its $i$-th coordinate by $v_i$. We denote vector Euclidean norm by $\|v\|_2$. For any matrix $A$, we denote its $(i, j)$ entry by $A_{i,j}$, its Frobenius norm by $\|A\|_F$, and its spectral norm by $\|H\|_2$. For a symmetric matrix $H \in \mathbb{R}^{d \times d}$, we denote its minimum eigenvalue by $\lambda_{\text{min}}(H)$. For symmetric matrices $A, B \in \mathbb{R}^{d \times d}$, we say $A \succeq B$ if $\lambda_1(A - B) \geq 0$. We use $f_n = O(g_n)$ to denote that $f_n \leq Cg_n$ for some constant $C > 0$ and use $f_n = \tilde{O}(g_n)$ to hide the logarithmic factors of $g_n$. For $a, b \in \mathbb{R}$, $a \land b$ means $\min(a, b)$.

We begin with a few assumptions that are needed for later theoretical analyses of our algorithms.

The following assumption says that there is a bounded gap between the function value at the initial point $x_0$ and the minimal function value.

**Assumption 3.1.** For any function $F(x)$ and an initial point $x_0$, there exists a constant $0 < \Delta F < \infty$ such that

$$F(x_0) - \inf_{x \in \mathbb{R}^d} F(x) \leq \Delta F.$$  

We also need the following $L$-gradient Lipschitz and $\rho$-Hessian Lipschitz assumption.

**Assumption 3.2.** For each $i$, we assume that $f_i$ is $L$-gradient Lipschitz continuous and $\rho$-Hessian
Lipschitz continuous, where we have
\[ \|\nabla f_i(x) - \nabla f_i(y)\|_2 \leq L\|x - y\|_2, \forall x, y \in \mathbb{R}^d, \]
\[ \|\nabla^2 f_i(x) - \nabla^2 f_i(y)\|_2 \leq \rho\|x - y\|_2, \forall x, y \in \mathbb{R}^d. \]

Note that \( L \)-gradient Lipschitz is not required in the original cubic regularization algorithm (Nesterov and Polyak, 2006) and the SVRC algorithm in Zhou et al. (2018d). However, for most other SVRC algorithms (Zhou et al., 2018b; Wang et al., 2018; Zhang et al., 2018a), they need the \( L \)-gradient Lipschitz assumption.

In addition, we also need the difference between the stochastic gradient and the full gradient to be bounded.

**Assumption 3.3.** We assume that \( F \) has \( M \)-bounded stochastic gradient, where we have
\[ \|\nabla f_i(x) - \nabla F(x)\|_2 \leq M, \forall x \in \mathbb{R}^d, \forall i \in [n]. \]

It is worth noting that Assumption 3.3 is weaker than the assumption that each \( f_i \) is Lipschitz continuous, which has been made in Kohler and Lucchi (2017); Zhou et al. (2018b); Wang et al. (2018); Zhang et al. (2018a). We would also like to point out that we can make additional assumptions on the variances of the stochastic gradient and Hessian, such as the ones made in Tripuraneni et al. (2018). Nevertheless, making these additional assumptions does not improve the dependency of the gradient and Hessian complexities or the runtime complexity on \( \epsilon \) and \( n \). Therefore we chose not making these additional assumptions on the variances.

### 4 The Proposed SRVRC Algorithm

In this section, we present SRVRC, a novel algorithm which utilizes new semi-stochastic gradient and Hessian estimators compared with previous SVRC algorithms. We also provide a convergence analysis of the proposed algorithm.

#### 4.1 Algorithm Description

In order to reduce the computational complexity for calculating full gradient and full Hessian in (1.3), several ideas such as subsampled/stochastic gradient and Hessian (Kohler and Lucchi, 2017; Xu et al., 2017; Tripuraneni et al., 2018) and variance-reduced semi-stochastic gradient and Hessian (Zhou et al., 2018d; Wang et al., 2018; Zhang et al., 2018a) have been used in previous work. SRVRC follows this line of work. The key idea is to use a new construction of semi-stochastic gradient and Hessian estimators, which are recursively updated in each iteration, and reset periodically after certain number of iterations (i.e., an epoch). To be more specific, SRVRC takes different construction strategies for iteration \( t \) depending on whether \( \text{mod}(t, S) = 0 \) or not, where \( S \) is the epoch length. In the \( t \)-th iteration when \( \text{mod}(t, S) = 0 \), SRVRC will calculate a subsampled gradient \( \nabla f_{J_t}(x_t) \) and Hessian \( \nabla^2 f_{J_t}(x_t) \) at point \( x_t \) and set the semi-stochastic gradient \( v_t \) and Hessian \( U_t \).
Algorithm 1 Stochastic Recursive Variance-Reduced Cubic Regularization (SRVRC)

1: **Input:** Total iterations $T$, batch sizes $\{B_t^{(g)}\}_{t=1}^T$, $\{B_t^{(h)}\}_{t=1}^T$, cubic penalty parameter $\{M_t\}_{t=1}^T$, inner length $S$, initial point $x_0$, accuracy $\epsilon$ and Hessian Lipschitz constant $\rho$.

2: for $t = 0, \ldots, T - 1$ do
3:    if $\text{mod}(t, S) = 0$ then
4:        Sample index set $\mathcal{J}_t$, $|\mathcal{J}_t| = B_t^{(g)}$; $v_t \leftarrow \nabla f_{\mathcal{J}_t}(x_t)$
5:        Sample index set $\mathcal{I}_t$, $|\mathcal{I}_t| = B_t^{(h)}$; $U_t \leftarrow \nabla^2 f_{\mathcal{I}_t}(x_t)$
6:    else
7:        Sample index set $\mathcal{J}_t$, $|\mathcal{J}_t| = B_t^{(g)}$; $v_t \leftarrow \nabla f_{\mathcal{J}_t}(x_t) - \nabla f_{\mathcal{J}_t}(x_{t-1}) + v_{t-1}$
8:        Sample index set $\mathcal{I}_t$, $|\mathcal{I}_t| = B_t^{(h)}$; $U_t \leftarrow \nabla^2 f_{\mathcal{I}_t}(x_t) - \nabla^2 f_{\mathcal{I}_t}(x_{t-1}) + U_{t-1}$
9:    end if
10:   $h_t \leftarrow \arg\min_{h \in \mathbb{R}^d} m_t(h) := \langle v_t, h \rangle + \frac{1}{2} \langle U_t h, h \rangle + \frac{M_t}{6} \|h\|^3$
11:   $x_{t+1} \leftarrow x_t + h_t$
12:   if $\|h_t\|_2 \leq \sqrt{\epsilon/\rho}$ then
13:       return $x_{t+1}$
14:   end if
15: end for
16: (With high probability, this line will not be reached.)

As follows

$$v_t = \nabla f_{\mathcal{J}_t}(x_t), \quad U_t = \nabla^2 f_{\mathcal{I}_t}(x_t).$$

In the $t$-th iteration when $\text{mod}(t, S) \neq 0$, SRVRC constructs semi-stochastic gradient and Hessian $v_t$ and $U_t$ based on previous estimators $v_{t-1}$, $U_{t-1}$ recursively. More specifically, SRVRC generates index sets $\mathcal{J}_t, \mathcal{I}_t \subseteq [n]$, and calculates two subsampled gradients $\nabla f_{\mathcal{J}_t}(x_t), \nabla f_{\mathcal{J}_t}(x_{t-1})$, and two subsampled Hessians $\nabla^2 f_{\mathcal{I}_t}(x_t), \nabla^2 f_{\mathcal{I}_t}(x_{t-1})$. Then SRVRC sets $v_t$ and $U_t$ as

$$v_t = \nabla f_{\mathcal{J}_t}(x_t) - \nabla f_{\mathcal{J}_t}(x_{t-1}) + v_{t-1}, \quad (4.1)$$
$$U_t = \nabla^2 f_{\mathcal{I}_t}(x_t) - \nabla^2 f_{\mathcal{I}_t}(x_{t-1}) + U_{t-1}. \quad (4.2)$$

Note that this kind of $v_t$ has been used in first-order optimization algorithms before (Nguyen et al., 2017; Fang et al., 2018), while such $U_t$ is new and to our knowledge has never been used before. With semi-stochastic gradient $v_t$, semi-stochastic Hessian $U_t$ and $t$-th Cubic penalty parameter $M_t$, SRVRC constructs the $t$-th Cubic subproblem $m_t$ and solves for the solution to $m_t$ as $t$-th update direction, which is defined as

$$h_t = \arg\min_{h \in \mathbb{R}^d} m_t(h),$$

$$m_t(h) : = \langle v_t, h \rangle + \frac{1}{2} \langle U_t h, h \rangle + \frac{M_t}{6} \|h\|^3. \quad (4.3)$$

If $\|h_t\|_2$ is less than a given threshold which we set it as $\sqrt{\epsilon/\rho}$, SRVRC returns $x_{t+1} = x_t + h_t$ as its output. Otherwise, SRVRC updates $x_{t+1} = x_t + h_t$ and continues the loop.
1: **Input:** Total iterations $T$, batch sizes $\{B_t^{(g)}\}_{t=1}^T$, $\{B_t^{(h)}\}_{t=1}^T$, cubic penalty parameter $\{M_t\}_{t=1}^T$, inner length $S$, initial point $x_0$, accuracy $\epsilon$, Hessian Lipschitz constant $\rho$, gradient Lipschitz constant $L$ and failure probability $\xi$.

2: for $t = 0, \ldots, T - 1$ do

3: if $\text{mod}(t, S) = 0$ then

4: Sample index set $J_t$, $|J_t| = B_t^{(g)}$; $v_t \leftarrow \nabla f_{J_t}(x_t)$

5: else

6: Sample index set $J_t$, $|J_t| = B_t^{(g)}$; $v_t \leftarrow \nabla f_{J_t}(x_t) - \nabla f_{J_t}(x_{t-1}) + v_{t-1}$

7: end if

8: Sample index set $I_t$, $|I_t| = B_t^{(h)}$, set $U_t[\cdot] \leftarrow \nabla^2 f_{I_t}(x_t)[\cdot]$

9: $h_t \leftarrow \text{Cubic-Subsolver}(U_t[\cdot], v_t, M_t, 1/(16L), \sqrt{\epsilon/\rho}, 0.5, \xi/(3T))$ {See Algorithm 3 in Appendix G}

10: if $m_t(h_t) < -4\rho^{-1/2}\epsilon^{3/2}$ then

11: $x_{t+1} \leftarrow x_t + h_t$

12: else

13: $h_t \leftarrow \text{Cubic-Finalsolver}(U_t[\cdot], v_t, M_t, 1/(16L), \epsilon)$ {See Algorithm 4 in Appendix G}

14: return $x_{t+1} \leftarrow x_t + h_t$

15: end if

16: end for

17: (With high probability, this line will not be reached.)

The main difference between SRVRC and previous stochastic cubic regularization algorithms (Kohler and Lucchi, 2017; Xu et al., 2017; Zhou et al., 2018a,b; Wang et al., 2018; Zhang et al., 2018a) is that SRVRC adapts new semi-stochastic gradient and semi-stochastic Hessian estimators, which are defined recursively and have smaller asymptotic variance. The use of such semi-stochastic gradient has been proved to help reduce the gradient complexity in first-order nonconvex finite-sum optimization for finding stationary point (Fang et al., 2018). Our work takes one step further to apply it to Hessian, and we will later show that it helps reduce the gradient and Hessian complexities in second-order nonconvex finite-sum optimization for finding local minima (i.e., second-order stationary point).

### 4.2 Convergence Analysis

In this subsection, we present our theoretical results about SRVRC. While the idea of using variance reduction technique for cubic regularization is hardly new, the new semi-stochastic gradient and Hessian estimators in (4.1) and (4.2) bring new technical challenges in the convergence analysis.

To describe whether a point $x$ is a local minimum, we follow the original cubic regularization work (Nesterov and Polyak, 2006) to use the following criterion $\mu(x)$:
**Definition 4.1.** For any $x$, let $\mu(x)$ be

$$\mu(x) = \max \left\{ \|\nabla F(x)\|_2^{3/2}, -\frac{\lambda_{\min}(\nabla^2 F(x))}{\rho^{3/2}} \right\}.$$  \hfill (4.4)

It is easy to note that $\mu(x) \leq \epsilon^{3/2}$ if and only if $x$ is an $(\epsilon, \sqrt{\rho \epsilon})$-approximate local minimum. Thus, in order to find an $(\epsilon, \sqrt{\rho \epsilon})$-approximate local minimum, it suffices to find a point $x$ which satisfies $\mu(x) \leq \epsilon^{3/2}$.

The following theorem provides the convergence guarantee of SRVRC for finding an $O(\epsilon, \sqrt{\rho \epsilon})$-approximate local minimum.

**Theorem 4.2.** Under Assumptions 3.1, 3.2, 3.3, set the cubic penalty parameter $M_t = 100 \rho$ for any $t$ and the total iteration number $T \geq \Delta_F \rho^{1/2} \epsilon^{-3/2}/5$. For $t$ such that mod$(t, S) \neq 0$, set the gradient sample size $B^{(g)}_t$ and Hessian sample size $B^{(h)}_t$ as

$$B^{(g)}_t \geq \min \left\{ n, 144 M^2 \log^2 \left( \frac{2T}{\xi} \right) \frac{S \|x_t - x_{t-1}\|_2^2}{\epsilon^2} \right\},$$

$$B^{(h)}_t \geq \min \left\{ n, 36 \rho \log^2 \left( \frac{2T d}{\xi} \right) \frac{S \|x_t - x_{t-1}\|_2^2}{\rho \epsilon} \right\}.$$ \hfill (4.5-4.6)

For $t$ such that mod$(t, S) = 0$, set the gradient sample size $B^{(g)}_t$ and Hessian sample size $B^{(h)}_t$ as

$$B^{(g)}_t \geq \min \left\{ n, 144 L^2 \log^2 \left( \frac{2T}{\xi} \right) \frac{S \|x_t - x_{t-1}\|_2^2}{\epsilon^2} \right\},$$

$$B^{(h)}_t \geq \min \left\{ n, 36 L^2 \log^2 \left( \frac{2T d}{\xi} \right) \frac{S \|x_t - x_{t-1}\|_2^2}{\rho \epsilon} \right\}.$$ \hfill (4.7-4.8)

Then with probability at least $1 - \xi$, SRVRC outputs $x_{\text{out}}$ satisfying $\mu(x_{\text{out}}) \leq C \cdot \epsilon^{3/2}$, i.e., an $O(\epsilon, \sqrt{\rho \epsilon})$-approximate local minimum. $C$ is a universal constant.

Next corollary spells out the exact gradient complexity and Hessian complexity of SRVRC to find an $O(\epsilon, \sqrt{\rho \epsilon})$-approximate local minimum.

**Corollary 4.3.** Under the same conditions as Theorem 4.2, if set $S$ as

$$S = \sqrt{n \wedge \frac{L}{\rho \epsilon}},$$

and set $T$, $\{B^{(g)}_t\}$, $\{B^{(h)}_t\}$ as their lower bounds in (4.5)-(4.8), then with probability at least $1 - \xi$, SRVRC will output an $O(\epsilon, \sqrt{\rho \epsilon})$-approximate local minimum within

$$\tilde{O} \left( n \wedge \frac{L^2}{\rho \epsilon} + \frac{\sqrt{\lambda} \Delta_F}{\epsilon^{3/2}} \sqrt{n \wedge \frac{L^2}{\rho \epsilon}} \right).$$
stochastic Hessian evaluations and
\[ \tilde{O}\left( n^{3/2} + \frac{\sqrt{n}\Delta_F}{\epsilon^{3/2}} \left[ \frac{n}{\sqrt{n}} \left[ \frac{M^2\epsilon^{-2}}{\epsilon^{1/2}} + n^{1/2} \frac{\sqrt{n}L^2}{\rho\epsilon} + \frac{L^3}{\rho\epsilon^{3/2}} \right] \right] \right) \]

stochastic gradient evaluations.

**Remark 4.4.** For SRVRC, if we assume \( M, L, \rho, \Delta_F \) are constants, then its gradient complexity is
\[ \tilde{O}\left( n^{3/2} \wedge \frac{1}{\epsilon^3} \right), \]
and its Hessian complexity is
\[ \tilde{O}\left( n^{1/2} \wedge \frac{1}{\epsilon^3} + \frac{\sqrt{n}}{\epsilon^{3/2}} \wedge \frac{1}{\epsilon^2} \right). \]

**Remark 4.5.** Regarding Hessian complexity, suppose that \( \epsilon \ll 1 \), then the Hessian complexity of SRVRC can be simplified as \( \tilde{O}(n^{1/2} \wedge \epsilon^{-2}) \). Compared with existing SVRC algorithms (Zhou et al., 2018b; Zhang et al., 2018a; Wang et al., 2018), SRVRC outperforms the best-known Hessian sample complexity by a factor of \( n^{1/6} \wedge n^{2/3}\epsilon^{1/2} \). In terms of gradient complexity, SRVRC outperforms the algorithm in Zhang et al. (2018a) by a factor of \( n^{2/3}\epsilon^{1/2} \) when \( \epsilon \gg n^{-1/3} \), and by a factor of \( n\epsilon^{3/2} \) when \( n^{-2/3} \ll \epsilon \ll n^{-1/3} \). The gradient complexity of SRVRC also outperforms that of the algorithm in Zhou et al. (2018d) by a factor of \( n^{4/5}\epsilon^{3/2} \) when \( \epsilon \gg n^{-8/15} \).

## 5 Hessian-Free SRVRC

While SRVRC adapts novel semi-stochastic gradient and Hessian estimators to reduce both the gradient and Hessian complexities, it has three limitations for high-dimensional problems with \( d \gg 1 \): (1) it needs to compute and store the Hessian matrix, which needs \( O(d^2) \) computational time and storage space; (2) it needs to solve cubic subproblem \( m_t \) exactly, which requires \( O(d^w) \) computational time because it needs to compute the inverse of a Hessian matrix (Nesterov and Polyak, 2006); and (3) it cannot leverage the Hessian-vector product-based cubic subproblem solvers (Agarwal et al., 2017; Carmon and Duchi, 2016, 2018) because of the use of the semi-stochastic Hessian estimator.

### 5.1 Algorithm Description

We present a Hessian-free algorithm SRVRC\_free to address above limitations of SRVRC for high-dimensional problem, whose runtime complexity is linear in \( d \), and therefore works well in the high-dimension regime. SRVRC\_free uses the same semi-stochastic gradient \( \mathbf{v}_t \) as SRVRC. As opposed to SRVRC which has to construct semi-stochastic Hessian explicitly, SRVRC\_free only accesses to Hessian-vector product. In detail, at each iteration \( t \), SRVRC\_free subsamples index set \( \mathcal{I}_t \) and define a Hessian-vector product function \( U_t[\cdot] : \mathbb{R}^d \to \mathbb{R}^d \) as follows:

\[ U_t[\mathbf{v}] = \nabla^2 f_{\mathcal{I}_t}(x_t)[\mathbf{v}], \quad \forall \mathbf{v} \in \mathbb{R}^d. \]
Note that although the subproblem depends on $U_t$, SRVRC\textsubscript{free} never explicitly compute this matrix. Instead, it only provides the subproblem solver access to $U_t$ through Hessian-vector product function $U_t[\cdot]$. The subproblem solver performs gradient-based optimization to solve the subproblem $m_t(h)$ as $\nabla m_t(h)$ depends on $U_t$ only via $U_t[h]$. In detail, following Tripuraneni et al. (2018), SRVRC\textsubscript{free} uses Cubic-Subsolver (See in Algorithms 3 and 4 in Appendix G) and Cubic-Finalsolver from Carmon and Duchi (2016), to find approximate solution $h_t$ to the cubic subproblem in (4.3). Both Cubic-Subsolver and Cubic-Finalsolver only need to access gradient $v_t$ and Hessian-vector product function $U_t[\cdot]$ along with other problem-dependent parameters. With the output $h_t$ from Cubic-Subsolver, SRVRC\textsubscript{free} decides either to update $x_t$ as $x_t+1 ← x_t + h_t$ or to exit the loop. For the later case, SRVRC\textsubscript{free} will call Cubic-Finalsolver to output $h_t$, and takes $x_t+1 = x_t + h_t$ as its final output.

The main differences between SRVRC and SRVRC\textsubscript{free} are two-fold. First, SRVRC\textsubscript{free} only needs to compute stochastic gradient and Hessian vector product, and both of these two actions only take $O(d)$ time (Rumelhart et al., 1986). Second, instead of solving cubic subproblem $m_t$ exactly, SRVRC\textsubscript{free} adopts approximate subproblem solver Cubic-Subsolver and Cubic-Finalsolver, both of which only need to access gradient and Hessian-vector product function, and again only take $O(d)$ time. Thus, SRVRC\textsubscript{free} is computational more efficient than SRVRC when $d \gg 1$.

5.2 Convergence Analysis

We now provide the convergence guarantee of SRVRC\textsubscript{free}, which ensures that SRVRC\textsubscript{free} will output an $(\epsilon, \sqrt{\rho}\epsilon)$-approximate local minimum.

**Theorem 5.1.** Under Assumptions 3.1, 3.2, 3.3, suppose $\sqrt{\tau} < L/(50\sqrt{\rho})$. Set the cubic penalty parameter $M_t = 100\rho$ for any $t$ and the total iteration number $T ≥ \Delta F^{1/2}\epsilon^{-3/2}$. Set the Hessian-vector product sample size $B_t^{(h)}$ as

$$B_t^{(h)} ≥ \min \left\{ n, 36 \frac{L^2 \log^2(3T\rho/\epsilon)}{\rho \epsilon} \right\}.$$  \hspace{1cm} (5.1)

For $t$ such that $\text{mod}(t, S) ≠ 0$, set the gradient sample size $B_t^{(g)}$ as

$$B_t^{(g)} ≥ \min \left\{ n, 144L^2 \log^2(3T/\xi) \frac{S\|x_t - x_{t-1}\|^2}{\epsilon^2} \right\}.$$ \hspace{1cm} (5.2)

For $t$ such that $\text{mod}(t, S) = 0$, set the gradient sample size $B_t^{(g)}$ as

$$B_t^{(g)} ≥ \min \left\{ n, 144 \frac{M^2 \log^2(3T/\xi)}{\epsilon^2} \right\}.$$ \hspace{1cm} (5.3)

Then with probability at least $1 - \xi$, SRVRC\textsubscript{free} outputs $x_{\text{out}}$ satisfying $\mu(x_{\text{out}}) ≤ C \cdot \epsilon^{3/2}$, i.e., an $O(\epsilon, \sqrt{\rho}\epsilon)$-approximate local minimum. $C$ is a universal constant.

The following corollary calculates the runtime complexity of SRVRC\textsubscript{free} to find an $O(\epsilon, \sqrt{\rho}\epsilon)$-approximate local minimum.
Corollary 5.2. Under the same conditions as Theorem 5.1, if set $S$ as

$$S = \sqrt{\frac{\rho e}{L}} \sqrt{n \wedge \frac{M^2}{\epsilon^2}},$$

and set $T, \{B_t^{(g)}\}, \{B_t^{(h)}\}$ as their lower bounds in (5.1)-(5.3), then with probability at least $1 - \xi$, SRVRC\textsubscript{free} will output an $O(\epsilon, \sqrt{\rho e})$-approximate local minimum within

$$\tilde{O}\left[d\left(n \wedge \frac{M^2}{\epsilon^2}\right) + \frac{d\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \left(n \wedge \frac{L\sqrt{n}}{\sqrt{\rho e}} \wedge \frac{LM}{\rho^{1/2} \epsilon^{3/2}}\right) + d\left(\frac{L \Delta_F e^2}{\epsilon^2} + \frac{L^{3/2}}{(\rho e)^{3/4}}\right) \left(n \wedge \frac{L^2}{\rho e}\right)\right] \tag{5.4}$$

er runtime.

Remark 5.3. For SRVRC\textsubscript{free}, if we assume $\rho, L, M, \Delta_F$ are constants, then its runtime complexity is

$$\tilde{O}\left(\frac{dn}{\epsilon^2} \wedge \frac{d}{\epsilon^3}\right). \tag{5.5}$$

Remark 5.4. For stochastic algorithms, the regime $n \to \infty$ is of most interest. In this regime, (5.5) becomes $\tilde{O}(de^{-3})$. Compared with other local minimum finding algorithms based on stochastic gradient and Hessian-vector product, SRVRC\textsubscript{free} outperforms the results achieved by Tripuraneni et al. (2018) and Allen-Zhu (2018) by a factor of $de^{-1/2}$. SRVRC\textsubscript{free} also matches the best-known result achieved by a recent first-order algorithm proposed in Fang et al. (2018).

![Figure 1: Plots of logarithmic function value gap with respect to runtime (in seconds) for nonconvex regularized binary logistic regression on (a) a9a (b) covtype, and for nonconvex regularized multiclass logistic regression on (c) MNIST.](image)

5.3 Discussions

We would like to further compare the runtime complexity between SRVRC and SRVRC\textsubscript{free}. In specific, SRVRC needs $O(d)$ time to construct semi-stochastic gradient and $O(d^2)$ time to construct semi-stochastic Hessian. SRVRC also needs $O(d^w)$ time to solve cubic subproblem $m_t$ for each iteration. Thus, with the fact that the total number of iterations is $T = O(\epsilon^{-3/2})$ by Corollary 4.3,
SRVRC needs
\[
\tilde{O}\left(d\left[\frac{n}{e^{3/2}} \wedge \frac{1}{e^3}\right] + d^2\left[n \wedge \frac{1}{e} + \sqrt{n} \wedge \frac{1}{e^2}\right] + \frac{d^w}{e^{3/2}}\right)
\]
runtime to find an \((\epsilon, \sqrt{\epsilon})\)-approximate local minimum if we regard \(M, L, \rho, \Delta_F\) as constants. Compared with (5.5), we conclude that \(\text{SRVRC}_{\text{free}}\) outperforms SRVRC when \(d\) is sufficiently large, which is in accordance with the fact that Hessian-free methods are superior for high dimension machine learning tasks. On the other hand, a careful calculation can show that the runtime of SRVRC can be less than that of \(\text{SRVRC}_{\text{free}}\) when \(d\) is moderately small. This is also reflected in our experiments.

6 Experiments

In this section, we present numerical experiments on different nonconvex Empirical Risk Minimization (ERM) problems and on different datasets to validate the advantage of our proposed SRVRC and \(\text{SRVRC}_{\text{free}}\) algorithms for finding approximate local minima. We use runtime as the performance measures.

**Baselines:** We compare our algorithms with the following algorithms: subsampled cubic regularization (Subsample Cubic) (Kohler and Lucchi, 2017), stochastic cubic regularization (Stochastic Cubic) (Tripuraneni et al., 2018), stochastic variance-reduced cubic regularization (SVRC) (Zhou et al., 2018d), sample efficient stochastic variance-reduced cubic regularization (Lite-SVRC) (Zhou et al., 2018b; Wang et al., 2018; Zhang et al., 2018a).

**Parameter Settings and Subproblem Solver** For each algorithm, we set the cubic penalty parameter \(M_t\) adaptively based on how well the model approximates the real objective as suggested in Cartis et al. (2011a,b); Kohler and Lucchi (2017). For SRVRC, we set gradient and Hessian batch sizes \(B_t^{(g)}\) and \(B_t^{(h)}\) as follows:

\[
B_t^{(g)} = B_t^{(h)} = B_t,
\]

\[
B_t^{(g)} = \lfloor B_t^{(g)}/S \rfloor, B_t^{(h)} = \lfloor B_t^{(h)}/S \rfloor, \quad \text{mod}(t, S) = 0,
\]

\[
B_t^{(g)} = \lfloor B_t^{(g)}/S \rfloor, B_t^{(h)} = \lfloor B_t^{(h)}/S \rfloor, \quad \text{mod}(t, S) \neq 0.
\]

For \(\text{SRVRC}_{\text{free}}\), we set gradient batch sizes \(B_t^{(g)}\) the same as SRVRC and Hessian batch sizes \(B_t^{(h)} = B_t^{(h)}\). We tune \(S\) over the grid \(\{5, 10, 20, 50\}\), \(B_t^{(g)}\) over the grid \(\{n, n/10, n/20, n/100\}\), and \(B_t^{(h)}\) over the grid \(\{50, 100, 500, 1000\}\) for the best performance. For Subsample Cubic, SVRC, Lite-SVRC and SRVRC, we solve the cubic subproblem using the cubic subproblem solver discussed in Nesterov and Polyak (2006). For Stochastic Cubic and \(\text{SRVRC}_{\text{free}}\), we use Cubic-Subsolver (Algorithm 3 in Appendix G) to approximately solve the cubic subproblem. All algorithms are carefully tuned for a fair comparison.

**Datasets and Optimization Problems** We use 3 datasets \(a9a\), \(covtype\) and \(MNIST\) from Chang and Lin (2011). For \(a9a\) and \(covtype\), we study binary logistic regression problem with a nonconvex regularizer \(\sum_{i=1}^d w_i^2/(1 + w_i^2)\) (Reddi et al., 2016b). For \(MNIST\), we study multi-class logistic regression with a nonconvex regularizer \(\sum_{i=1}^m \sum_{j=1}^d w_{i,j}^2/(1 + w_{i,j}^2)\), where \(m\) is number of classes.

We plot the logarithmic function value gap with respect to runtime in Figure 1. From Figure 1(a), 1(b) and 1(c), we can see that for the low dimension optimization task on \(a9a\) and \(covtype\),
our SRVRC outperforms all the other algorithms with respect to runtime. For high dimension optimization task MNIST, only Stochastic Cubic and SRVRC_free are able to make progress and SRVRC_free outperforms Stochastic Cubic. This is consistent with our discussions in Section 5.3.

7 Conclusions and Future Work

In this work we presented two faster SVRC algorithms namely SRVRC and SRVRC_free to find approximate local minima for nonconvex finite-sum optimization problems. SRVRC outperforms existing SVRC algorithms in terms of gradient and Hessian complexities, while SRVRC_free further outperforms the best-known runtime complexity for existing CR based algorithms. Whether our algorithms have achieved the optimal complexity under current assumptions is still an open problem, and we leave it as a future work.

A Proofs in Section 4

We define the filtration $\mathcal{F}_t = \sigma(x_0, \ldots, x_t)$ as the $\sigma$-algebra of $x_0$ to $x_t$. Without confusion, we assume $v_t$ and $U_t$ as the semi-stochastic gradient and Hessian, $h_t$ as the update parameter, $M_t$ as the cubic penalty parameter appearing in Algorithm 1 and Algorithm 2. We denote $m_t(h) := v_t^\top h + h^\top U_t h/2 + M_t \| h \|^3/6$ and $h^*_t = \text{argmin}_{h \in \mathbb{R}^d} m_t(h)$. In this section, we define $\delta = \xi/(2T)$ for the simplicity.

A.1 Proof of Theorem 4.2

To prove Theorem 4.2, we need the following lemmas from Zhou et al. (2018d) which characterize that $\mu(x_t + h)$ can be bounded by $\|h\|_2$ and the norm of difference between semi-stochastic gradient and Hessian.

Lemma A.1. (Zhou et al., 2018d) Suppose that $m_t(h) := v_t^\top h + h^\top U_t h/2 + M_t \| h \|^3/6$ and $h^*_t = \text{argmin}_{h \in \mathbb{R}^d} m_t(h)$. If $M_t/\rho \geq 100$, then for any $h \in \mathbb{R}^d$, we have

$$\mu(x_t + h) \leq 9 \left[ M^3 \rho^{-3/2} \| h \|^3 + M^3/2 \rho^{-3/2} \| \nabla F(x_t) - v_t \|^3/2 + \rho^{-3/2} \| \nabla^2 F(x_t) - U_t \|^3/2 + M^3 \rho^{-3/2} \| m_t(h) \|^3/2 + M^3 \rho^{-3/2} \| h \|^3 - \| h^*_t \|^3 \right].$$

Next lemma gives upper bounds on the inner product terms which will appear in our main proof.

Lemma A.2. (Zhou et al., 2018d) For any $h \in \mathbb{R}^d$, we have

$$\langle \nabla F(x_t) - v_t, h \rangle \leq \frac{M_t}{27} \| h \|^3 + \frac{2 \| \nabla F(x_t) - v_t \|^3/2}{M_t^{1/2}}, \quad (A.1)$$

$$\langle (\nabla^2 F(x_t) - U_t) h, h \rangle \leq \frac{2M_t}{27} \| h \|^3 + \frac{27}{M_t^{1/2}} \| \nabla^2 F(x_t) - U_t \|^3. \quad (A.2)$$

We also need the following two lemmas, which show that semi-stochastic gradient $v_t$ and Hessian $U_t$ are good approximations to true gradient and Hessian.
Lemma A.3. Suppose that \( \{B_k^{(g)}\} \) satisfies (4.5) and (4.7), then condition on \( \mathcal{F}_{[t/S],S} \), with probability at least \( 1 - \delta \cdot (t - [t/S] \cdot S) \), we have that for all \( [t/S] \cdot S \leq k \leq t \),
\[
\|\nabla F(x_k) - \nu_k\|_2^2 \leq 5\varepsilon^2.
\] (A.3)

Lemma A.4. Suppose that \( \{B_k^{(h)}\} \) satisfies (4.6) and (4.8), then condition on \( \mathcal{F}_{[t/S],S} \), with probability at least \( 1 - \delta \cdot (t - [t/S] \cdot S) \), we have that for all \( [t/S] \cdot S \leq k \leq t \),
\[
\|\nabla^2 F(x_k) - U_k\|_2^2 \leq 30\rho \varepsilon.
\] (A.4)

Given all the above lemmas, we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. Suppose that SRVRC breaks at iteration \( T^* - 1 \), then \( \|h_t\|_2 > \sqrt{\varepsilon/\rho} \) for all \( 0 \leq t \leq T^* - 1 \). We have
\[
F(x_{t+1}) \leq F(x_t) + \langle \nabla F(x_t), h_t \rangle + \frac{1}{2} \|h_t\|_2^2 + \frac{\rho}{6} \|h_t\|_2^3 + \frac{M_{t,S}}{12} \|h_t\|_2^3 + \|\nabla^2 F(x_t) - \nu_t\|_2^{3/2} + \frac{1}{2} \|h_t\|_2 \|\nabla F(x_t) - \nu_t\|_2 ,
\]
\[
\leq F(x_t) - 8\rho \|h_t\|_2^3 + \|\nabla^2 F(x_t) - \nu_t\|_2^{3/2} + \frac{1}{2} \|\nabla^2 F(x_t) - \nu_t\|_2^2 ,
\] (A.5)
where the second inequality holds due to the fact that \( m_t(h_t) \leq m_t(0) = 0 \) and \( \rho < M_t/2 \), last inequality holds because \( M_t = 100 \rho \). Then by Lemma A.4, with probability at least \( 1 - 2T\delta \), for all \( 0 \leq t \leq T - 1 \), we have that
\[
\|\nabla F(x_t) - \nu_t\|_2 \leq 10\varepsilon^{3/2} , \|\nabla^2 F(x_t) - \nu_t\|_2 \leq 200(\rho \varepsilon)^{3/2}
\] (A.6)
for all \( 0 \leq t \leq T - 1 \). Substituting (A.6) into (A.5), we have
\[
F(x_{t+1}) \leq F(x_t) - 8\rho \|h_t\|_2^3 + 3\rho^{-1/2} \varepsilon^{3/2} .
\] (A.7)
Taking summation for (A.7) from \( t = 0, ..., T^* - 1 \), we have
\[
\Delta_F \geq F(x_0) - F(x_{T^*}) \geq 8\rho \cdot T^* \cdot (\varepsilon/\rho)^{3/2} - 3\rho^{-1/2} \varepsilon^{3/2} \cdot T^* = 5\rho^{-1/2} \varepsilon^{3/2} \cdot T^* ,
\] (A.8)
which implies that \( T^* < T \). Thus, we have \( \|h_{T^*-1}\|_2 \leq \sqrt{\varepsilon/\rho} \). Denote \( \tilde{T} = T^* - 1 \), then we have
\[
\mu(x_{T^*+1}) = \mu(x_{T^*} + h_{T^*})
\]
\[
\leq 9 \left[ M_{T^*}^3 \rho^{-3/2} \|h_{T^*}\|_2^3 + M_{T^*}^3 \rho^{-3/2} \|\nabla F(x_{T^*}) - \nu_{T^*}\|_2^{3/2} + \rho^{-3/2} \|\nabla^2 F(x_{T^*}) - U_{T^*}\|_2 \right]
\]
\[
\leq C_1 \left[ \rho^{3/2} \|h_{T^*}\|_2^3 + \varepsilon^{3/2} \right]
\]
\[
\leq C_2 \varepsilon^{3/2} ,
\]
15
where $C_1 = 10^6$ and $C_2 = 10^7$. The first inequality holds due to Lemma A.1 where $\nabla m_{\tilde{T}}(h_{\tilde{T}}) = 0$ and $\|h_{\tilde{T}}\|_2 = \|h^*_T\|_2$. That implies our result.

A.2 Proof of Corollary 4.3

Proof of Corollary 4.3. Suppose that SRVRC breaks at $T^* - 1 \leq T - 1$ iteration. Taking summation for (A.7) from $t = 0$ to $T^* - 1$, we have

$$
\Delta_F \geq F(x_0) - F(x_{T^*}) \geq 8\rho \sum_{t=0}^{T^* - 1} \|h_t\|_2^3 - 3\rho^{-1/2} \epsilon^{3/2} \cdot T = 8\rho \sum_{t=0}^{T^* - 1} \|h_t\|_2^3 - 3/5 \cdot \Delta_F,
$$

which implies that $\sum_{t=0}^{T^* - 1} \|h_t\|_2^3 \leq \Delta_F/(5\rho)$. Thus,

$$
\sum_{t=0}^{T^* - 1} \|h_t\|_2^2 \leq (T^*)^{1/3} \left( \sum_{t=0}^{T^* - 1} \|h_t\|_2^3 \right)^{2/3} \leq \left( \frac{\Delta_F \rho^{1/2}}{5\epsilon^{3/2}} \right)^{1/3} \cdot \left( \frac{\Delta_F}{5\rho} \right)^{2/3} \leq \frac{\Delta_F}{\rho^{1/2} \epsilon^{1/2}}, \tag{A.9}
$$

where the first inequality holds due to Young’s inequality. We first consider the gradient complexity $\sum_{t=0}^{T^* - 1} B_t^{(g)}$, which can be bounded as

$$
\sum_{t=0}^{T^* - 1} B_t^{(g)} = \sum_{\text{mod}(t,S)=0} B_t^{(g)} + \sum_{\text{mod}(t,S)\neq 0} B_t^{(g)}
$$

$$
= \sum_{\text{mod}(t,S)=0} \min \left\{ n, 144 M^2 \log^2(d/\delta) e^2 \right\} + \sum_{\text{mod}(t,S)\neq 0} \min \left\{ n, 144 L^2 \log^2(d/\delta) S \|h_{t-1}\|_2^2 \right\}
$$

$$
\leq \tilde{O} \left[ n \wedge \frac{M^2 e^2}{\epsilon^2} + \frac{T^*}{S} \left( n \wedge \frac{M^2 e^2}{\epsilon^2} + \frac{L^2 S}{\epsilon^2} \sum_{t=0}^{T^* - 1} \|h_t\|_2^3 \right) \wedge nT^* \right]
$$

$$
\leq \tilde{O} \left[ n \wedge \frac{M^2 e^2}{\epsilon^2} + \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2} S} \left( n \wedge \frac{M^2 e^2}{\epsilon^2} + \frac{L^2 S}{\rho^{1/2} \epsilon^{3/2}} \right) \wedge n\Delta_F \rho^{1/2} \right]
$$

$$
= \tilde{O} \left( n \wedge \frac{M^2 e^2}{\epsilon^2} + \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \left[ \frac{n}{\sqrt{n} \wedge \|L(\rho e)^{-1/2}\|} + n \wedge \frac{\sqrt{n} L^2}{\rho e} \wedge \frac{L^3}{(\rho e)^{3/2}} \right] \right),
$$
where the second inequality holds due to (A.9). We then consider the Hessian complexity $\sum_{t=0}^{T^*-1} B_t^{(h)}$, which can be bounded as

\[
\sum_{t=0}^{T^*-1} B_t^{(h)} = \sum_{\text{mod}(t,S)=0} B_t^{(h)} + \sum_{\text{mod}(t,S)\neq0} B_t^{(h)}
\]

\[
= \sum_{\text{mod}(t,S)=0} \min \left\{ n, 36 \frac{L^2 \log^2(d/\delta)}{\rho \epsilon} \right\} + \sum_{\text{mod}(t,S)\neq0} \min \left\{ n, 36 \rho \log^2(d/\delta) \frac{S \|x_k - x_{k-1}\|^2}{\epsilon} \right\}
\]

\[
\leq \tilde{O} \left[ n \wedge \frac{L^2}{\rho \epsilon} + \frac{T^*}{S} \left( n \wedge \frac{L^2}{\rho \epsilon} \right) + \frac{n \rho S}{\epsilon} \sum_{t=0}^{T^*-1} \|h_t\|^2 \right]
\]

\[
\leq \tilde{O} \left[ n \wedge \frac{L^2}{\rho \epsilon} + \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2} S} \left( n \wedge \frac{L^2}{\rho \epsilon} \right) + \frac{\Delta_F \rho^{1/2} S}{\epsilon^{3/2}} \right]
\]

\[
= \tilde{O} \left[ n \wedge \frac{L^2}{\rho \epsilon} + \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \sqrt{n \wedge \frac{L^2}{\rho \epsilon}} \right],
\]

where the second inequality holds due to (A.9).

\[ \square \]

**B Proofs in Section 5**

In this section, we denote $\delta = \xi/(3T)$ for simplicity.

**B.1 Proof of Theorem 5.1**

We need the following two lemmas, which bound the variance of semi-stochastic gradient and Hessian estimators.

**Lemma B.1.** Suppose that $\{B_k^{(g)}\}$ satisfies (5.2) and (5.3), then condition on $\mathcal{F}_{[t/S] \cdot S}$, with probability at least $1 - \delta \cdot (t - [t/S] \cdot S)$, we have that for all $[t/S] \cdot S \leq k \leq t$,

\[
\|\nabla F(x_k) - v_k\|^2 \leq 5 \epsilon^2. \quad (B.1)
\]

**Proof of Lemma B.1.** The proof is very similar to that of Lemma A.3, hence we omit it. \[ \square \]

**Lemma B.2.** Suppose that $\{B_k^{(h)}\}$ satisfies (5.1), then condition on $\mathcal{F}_k$, with probability at least $1 - \delta$, we have that

\[
\|\nabla^2 F(x_k) - U_k\|^2 \leq 30 \rho \epsilon. \quad (B.2)
\]

**Proof of Lemma B.2.** The proof is very similar to that Proof of Lemma B.2, hence we omit it. \[ \square \]

We have the following lemma to guarantee that by Algorithm 3 Cubic-Subsolver, the output $h_t$ satisfies that sufficient decrease of function value will be made and the total number of iterations is bounded by $T'$.
Lemma B.3. Suppose that \( \| h_t^* \|_2 \geq \sqrt{\epsilon / \rho} \) or \( \| v_t \|_2 \geq \max\{ M_t \epsilon / (2 \rho), \sqrt{LM_t/2 (\epsilon / \rho)^{3/4}} \} \). We set \( \eta = 1/(16 L) \). Then for \( \epsilon < 16L^2 \rho / M_t^2 \), with probability at least \( 1 - \delta \), Solve\_Subproblem\( (U_t, v_t, M_t, \eta, \sqrt{\epsilon / \rho}, 0.5, \delta) \) will return \( h_t \) where \( m_t(h_t) \leq -M_t \rho^{-3/2} \epsilon^{3/2} / 24 \). Meanwhile, it takes total
\[
T' = \tilde{O} \left( \frac{L}{M_t \sqrt{\epsilon / \rho}} \right)
\] iterations.

With these lemmas, we begin our proof of Theorem 5.1.

Proof of Theorem 5.1. Suppose that SRVRC\(_{\text{free}}\) breaks at iteration \( T^* - 1 \). Then \( T^* \leq T \), and we have that for all \( 0 \leq t < T^* \),
\[
F(x_{t+1}) \leq F(x_t) + \langle \nabla F(x_t), h_t \rangle + \frac{1}{2} (h_t, \nabla^2 F(x_t) h_t) + \frac{\rho}{6} \| h_t \|^3_2
\]
\[
= F(x_t) + m_t(h_t) + \frac{\rho - M_s t}{6} \| h_t \|^3_2 + \langle h_t, \nabla F(x_t) - v_t \rangle + \frac{1}{2} \left( h_t, (\nabla^2 F(x_t) - U_t) h_t \right)
\]
\[
\leq F(x_t) - \frac{M_s t}{12} \| h_t \|^3_2 + m_t(h_t) + \frac{\| \nabla F(x_t) - v_t \|^3_2}{5 \sqrt{\rho}} + \frac{1}{200 \rho^2} \| \nabla^2 F(x_t) - U_t \|^3_2
\]
\[
\leq F(x_t) + m_t(h_t) - 8 \rho \| h_t \|^3_2 + \frac{\| \nabla F(x_t) - v_t \|^3_2}{5 \sqrt{\rho}} + \frac{1}{200 \rho^2} \| \nabla^2 F(x_t) - U_t \|^3_2;
\]
(B.4)
where the second inequality holds due to the fact that \( \rho < M_t / 2 \), last inequality holds because \( M_t = 100 \rho \). By Lemma B.3 and union bound, we know that with probability at least \( 1 - T \delta \), we have
\[
m_t(h_t) \leq -M_t \rho^{-3/2} \epsilon^{3/2} / 24 \leq -4 \rho^{-1/2} \epsilon^{3/2},
\]
(B.5)
where we use the fact that \( M_t = 100 \rho \). By Lemma B.1 and Lemma B.2, we know that with probability at least \( 1 - 2T \delta \), for all \( 0 \leq t \leq T^* - 1 \), we have
\[
\| \nabla F(x_t) - v_t \|^3_2 / 2 \leq 10 \epsilon^{3/2}, \| \nabla^2 F(x_t) - U_t \|^3_2 \leq 200 (\rho \epsilon)^{3/2}.
\]
(B.6)
Substituting (B.5) and (B.6) into (B.4), we have
\[
F(x_{t+1}) - F(x_t) \leq -4 \rho^{-1/2} \epsilon^{3/2} - 8 \rho \| h_t \|^3_2 + 3 \rho^{-1/2} \epsilon^{3/2} \leq -8 \rho \| h_t \|^3_2 - 4 \rho^{-1/2} \epsilon^{3/2}.
\]
(B.7)
Taking summation for (B.7) from \( t = 0 \) to \( T - 1 \), we have
\[
\Delta_F \geq F(x_0) - F(x_T) \geq 8 \rho \sum_{t=0}^{T-1} \| h_t \|^3_2 + \rho^{-1/2} \epsilon^{3/2} \cdot T^*,
\]
(B.8)
From (B.8), we immediately know that \( T^* < T = 2 \Delta_F \rho^{1/2} \epsilon^{-3/2} / 2 \). Thus, with probability at least \( 1 - 3T \delta \), Cubic-Finalsolver is executed by SRVRC\(_{\text{free}}\) at \( T^* - 1 \) iteration. We have that \( \| v_t \|_2 < \max\{ M_t \epsilon / (2 \rho), \sqrt{LM_t/2 (\epsilon / \rho)^{3/4}} \} \) and \( \| h_{T^* - 1} \|_2 \leq \sqrt{\epsilon / \rho} \) by Lemma B.3.
The only thing left is to check that we indeed find a second-order stationary point, $x_{T^*}$, by Cubic-Finalsolver. We first need to check that the choice of $\eta = 1/(16L)$ satisfies that $1/\eta > 4(L + M_tR)$ by Lemma D.2, where $R$ is defined in Lemma D.2. We can check that with the assumption that $\|v_t\|_2 < \max\{M_t\epsilon/(2\rho), \sqrt{L}M_t/2(\epsilon/\rho)^{3/4}\}$, if $\epsilon < 4L^2\rho/M_t^2$, then $1/\eta > 4(L + M_tR)$ holds.

For simplicity, we denote $T = T^* - 1$. Then we have

$$\mu(x_{T} + \tilde{h}_T) \leq \left[ M_T^3 \rho^{-3/2} \|h_T\|^2_2 + M_T^3 \rho^{-3/2} \|\nabla F(x_{T}) - v_{T}\|^2_2 + \rho^{-3/2} \|\nabla^2 F(x_{T}) - U_{T}\|^2_2 \right]$$

$$+ M_T^3 \rho^{-3/2} \|\nabla m_{T}(h_{T})\|^2_2 + M_T^3 \rho^{-3/2} \|h_{T} - h_{T}^*\|^2_2$$

$$\leq 9 \left[ 2M_T^3 \rho^{-3/2} \|h_{T}^*\|^2_2 + M_T^3 \rho^{-3/2} \|\nabla F(x_{T}) - v_{T}\|^2_2 + \rho^{-3/2} \|\nabla^2 F(x_{T}) - U_{T}\|^2_2 \right]$$

$$+ M_T^3 \rho^{-3/2} \|\nabla m_{T}(h_{T})\|^2_2$$

$$\leq C_1 \epsilon^{3/2},$$

where the first inequality holds due to Lemma A.1, the second inequality holds due to the fact that $\|h_T\|_2 \leq \|h_T^*\|_2$ from D.2, the last inequality holds due to the facts that $\|\nabla m_{T}(h_{T})\|_2 \leq \epsilon$ from Cubic-Finalsolver and $\|h_{T}^*\|_2 \leq \sqrt{\epsilon/\rho}$ by Lemma B.3.

### B.2 Proof of Corollary 5.2

We have the following lemma to bound the total number of iterations $T''$ of Algorithm 4 Cubic-Finalsolver.

**Lemma B.4.** If $\epsilon < 4L^2\rho/M_t^2$, then Cubic-Finalsolver will break in iteration $T'' = O(L^{3/2}/(\rho \epsilon)^{3/4})$

**Proof of Corollary 5.2.** By (B.8), we have $\Delta F \geq 8\rho \sum_{t=0}^{T^* - 1} \|h_t\|^2_2$, which implies that

$$\sum_{t=0}^{T^* - 1} \|h_t\|^2_2 \leq (T^*)^{1/3} \left( \sum_{t=0}^{T^* - 1} \|h_t\|^2_2 \right)^{2/3} \leq \left( \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \right)^{1/3} \cdot \left( \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \right)^{2/3} = \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}}, \hspace{1cm} (B.9)$$

where the first inequality holds due to Young’s inequality. We first consider the total stochastic
gradient computations, \( \sum_{t=0}^{T^*-1} B_t^{(g)} \), which can be bounded as

\[
\sum_{t=0}^{T^*-1} B_t^{(g)} = \sum_{\text{mod}(t,S)=0} B_t^{(g)} + \sum_{\text{mod}(t,S)\neq 0} B_t^{(g)} \\
= \sum_{\text{mod}(t,S)=0} \min \left\{ n, 144 \frac{M^2 \log^2(d/\delta)}{\epsilon^2} \right\} + \sum_{\text{mod}(t,S)\neq 0} \min \left\{ n, 144L^2 \log^2(d/\delta) S \| h_{t-1} \|_2^2 \right\} \\
\leq \tilde{O} \left[ n \land \frac{M^2}{\epsilon^2} + \frac{T^*}{S} \left( n \land \frac{M^2}{\epsilon^2} \right) + \left( L^2 S \sum_{t=0}^{T^*-1} \| h_t \|_2^2 \right) \land n T^* \right] \\
\leq \tilde{O} \left[ n \land \frac{M^2}{\epsilon^2} + \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2} S} \left( n \land \frac{M^2}{\epsilon^2} \right) + \left( \frac{\Delta_F L^2 S}{\rho^{1/2} \epsilon^{3/2}} \right) \land n \right] \\
= \tilde{O} \left[ n \land \frac{M^2}{\epsilon^2} + \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \left( \frac{1}{S} \left( n \land \frac{M^2}{\epsilon^2} \right) + \left( \frac{L^2 S}{\rho \epsilon} \right) \land n \right) \right] \\
= \tilde{O} \left[ n \land \frac{M^2}{\epsilon^2} + \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \left( n \land \frac{L \sqrt{n}}{\sqrt{\rho \epsilon}} \land \frac{LM}{\rho \epsilon} \right) \right], \tag{B.10}
\]

where the last equality holds due to the selection of \( S \). We now consider the total amount of Hessian-vector computations \( \mathcal{T}_1 \), which includes \( \mathcal{T}_1 \) from Cubic-Subsolver and \( \mathcal{T}_2 \) from Cubic-Finalsolver. By Lemma B.3, we know that at \( k \)-th iteration of SRVRC\text{free}, Cubic-Subsolver has \( T^*_k = \tilde{O}(L/M_t \cdot \sqrt{\rho/\epsilon}) = \tilde{O}(L/\sqrt{\rho \epsilon}) \) iterations, which needs \( O(B^{(h)}_k) \) Hessian-vector computations. Thus, we have

\[
\mathcal{T}_1 = \sum_{k=0}^{T^*-1} \tilde{O}(T^*_k \cdot B^{(h)}_k) \leq \tilde{O} \left( T \cdot T^* \cdot \left[ n \land \frac{L^2}{\rho \epsilon} \right] \right) = \tilde{O} \left( T^* \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \left[ n \land \frac{L^2}{\rho \epsilon} \right] \right) = \tilde{O} \left( \frac{L \Delta_F}{\epsilon^2} \cdot \left[ n \land \frac{L^2}{\rho \epsilon} \right] \right). \tag{B.11}
\]

By Lemma B.4, we know that Cubic-Finalsolver will take \( T'' = O(L^{3/2}/(\rho \epsilon)^{3/4}) \) iterations. Since we need \( B^{(h)}_{T^*-1} \) Hessian-vector computations each iteration, thus we have

\[
\mathcal{T}_2 = O \left( B^{(h)}_{T^*-1} \cdot T'' \right) = \tilde{O} \left( T'' \left[ n \land \frac{L^2}{\rho \epsilon} \right] \right) = \tilde{O} \left( \frac{L^{3/2}}{(\rho \epsilon)^{3/4}} \cdot \left[ n \land \frac{L^2}{\rho \epsilon} \right] \right). \tag{B.12}
\]

Combining (B.10), (B.11) and (B.12), we know that the total stochastic gradient and Hessian-vector product computations are bounded as

\[
\tilde{O} \left[ n \land \frac{M^2}{\epsilon^2} + \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \left( n \land \frac{L \sqrt{n}}{\sqrt{\rho \epsilon}} \land \frac{LM}{\rho \epsilon^{3/2}} \right) + \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \left( n \land \frac{L^2}{\rho \epsilon} \right) T^* + \left( n \land \frac{L^2}{\rho \epsilon} \right) T'' \right] \\
= \tilde{O} \left[ n \land \frac{M^2}{\epsilon^2} + \frac{\Delta_F \rho^{1/2}}{\epsilon^{3/2}} \left( n \land \frac{L \sqrt{n}}{\sqrt{\rho \epsilon}} \land \frac{LM}{\rho \epsilon^{3/2}} \right) + \left( \frac{L \Delta_F}{\epsilon^2} + \frac{L^{3/2}}{(\rho \epsilon)^{3/4}} \right) \cdot \left( n \land \frac{L^2}{\rho \epsilon} \right) \right]. \tag{B.13}
\]

Finally, since each stochastic gradient computation and Hessian-vector product computation needs
O(d) runtime, then (5.4) holds with (B.13).

C Proofs of Technical Lemmas in Section A

C.1 Proof of Lemma A.3

We need the following lemma:

**Lemma C.1.** Condition on $F_k$, with probability at least $1 - \delta$, we have

$$
\|\nabla f_{J_k}(x_k) - \nabla f_{J_k}(x_{k-1}) - \nabla F(x_{k-1}) + \nabla F(x_k)\|_2 \leq 6L \sqrt{\frac{\log(1/\delta)}{B_k^{(g)}}} \|x_k - x_{k-1}\|_2.
$$

We also have

$$
\|\nabla f_{J_k}(x_k) - \nabla F(x_k)\|_2 \leq 6M \sqrt{\frac{\log(1/\delta)}{B_k^{(g)}}}.
$$

**Proof of Lemma A.3.** First, we have $v_t - \nabla F(x_t) = \sum_{k=\lfloor t/S \rfloor \cdot S}^t u_k$, where

$$
u_k = \nabla f_{J_k}(x_k) - \nabla f_{J_k}(x_{k-1}) - \nabla F(x_{k-1}) + \nabla F(x_k), \quad k > \lfloor t/S \rfloor \cdot S,$n
$$

$$
u_k = \nabla f_{J_k}(x_k) - \nabla F(x_k), \quad k = \lfloor t/S \rfloor \cdot S.$n

Meanwhile, we have $\mathbb{E}[u_k|F_{k-1}] = 0$. Condition on $F_{k-1}$, for $\text{mod}(k, S) \neq 0$, from Lemma C.1, we have that with probability at least $1 - \delta$ the following inequality holds:

$$
\|u_k\| \leq 6L \sqrt{\frac{\log(1/\delta)}{B_k^{(g)}}} \|x_k - x_{k-1}\| \leq \sqrt{\frac{\epsilon^2}{4S \log(1/\delta)}},
$$

where the second inequality holds due to (4.5). For $\text{mod}(k, S) = 0$, with probability at least $1 - \delta$, we have

$$
\|u_k\| \leq 6M \sqrt{\frac{\log(1/\delta)}{B_k^{(g)}}} \leq \frac{\epsilon}{\sqrt{4 \log(1/\delta)}}.
$$

where the second inequality holds due to (4.7). Condition on $F_{\lfloor t/S \rfloor \cdot S}$, by union bound, with probability at least $1 - \delta \cdot (t - \lfloor t/S \rfloor \cdot S)$ (C.3) or (C.4) holds for all $\lfloor t/S \rfloor \cdot S \leq k \leq t$. Then for given $k$, by vector Azuma-Hoeffding inequality in Lemma F.1, condition on $F_k$, with probability at
least $1 - \delta$ we have
\[
\|v_k - \nabla F(x_k)\|_2^2 = \left\| \sum_{k=[t/S]:S}^{t} u_k \right\|_2^2 \\
\leq 9 \log(d/\delta) \left[ (t - \lfloor t/S \rfloor \cdot S) \cdot \frac{\epsilon^2}{4S \log(d/\delta)} + \frac{\epsilon^2}{4 \log(1/\delta)} \right] \\
\leq 9 \log(1/\delta) \cdot \frac{\epsilon^2}{2 \log(1/\delta)} \\
\leq 5 \epsilon^2.
\] (C.5)

Finally, by union bound, we have that with probability at least $1 - 2\delta \cdot (t - \lfloor t/S \rfloor \cdot S)$, for all $\lfloor t/S \rfloor \cdot S \leq k \leq t$, we have (C.5) holds.

\[\square\]

### C.2 Proof of Lemma A.4

We need the following lemma:

**Lemma C.2.** Condition on $F_k$, with probability at least $1 - \delta$, we have the following concentration inequality
\[
\left\| \nabla^2 f_{I_k}(x_k) - \nabla^2 f_{I_k}(x_{k-1}) - \nabla^2 F(x_k) + \nabla^2 F(x_{k-1}) \right\|_2 \leq 6\rho \sqrt{\frac{\log(d/\delta)}{B_k^{(h)}}} \|x_k - x_{k-1}\|_2.
\] (C.6)

We also have
\[
\left\| \nabla^2 f_{I_k}(x_k) - \nabla^2 F(x_k) \right\|_2 \leq 6L \sqrt{\frac{\log(d/\delta)}{B_k^{(h)}}}.
\] (C.7)

**Proof of Lemma A.4.** First, we have $U_t - \nabla^2 F(x_t) = \sum_{k=[t/S]:S}^{t} V_k$, where
\[
V_k = \nabla^2 f_{I_k}(x_k) - \nabla^2 f_{I_k}(x_{k-1}) - \nabla^2 F(x_k) + \nabla^2 F(x_{k-1}), \quad k > [t/S] \cdot S, \\
V_k = \nabla f_{I_k}(x_k) - \nabla F(x_k), \quad k = [t/S] \cdot S
\]

Meanwhile, we have $\mathbb{E}[V_k | \sigma(V_{k-1}, ..., V_0)] = 0$. Condition on $F_{k-1}$, for $\text{mod}(k, S) \neq 0$, from Lemma C.2, we have that with probability at least $1 - \delta$, the following inequality holds :
\[
\|V_k\|_2 \leq 6\rho \sqrt{\frac{\log(d/\delta)}{B_k^{(h)}}} \|x_k - x_{k-1}\|_2 \leq \sqrt{\frac{\rho \epsilon}{S \log(d/\delta)}},
\] (C.8)

where the second inequality holds due to (4.6). For $\text{mod}(k, S) = 0$, with probability at least $1 - \delta$, we have
\[
\|V_k\|_2 \leq 6L \sqrt{\frac{\log(d/\delta)}{B_k^{(h)}}} \leq \sqrt{\frac{\rho \epsilon}{\log(d/\delta)}},
\] (C.9)
where the second inequality holds due to (4.8). Condition on \(F_{\lceil t/S \rceil \cdot S}\), by union bound, with probability at least \(1 - \delta \cdot (t - \lceil t/S \rceil \cdot S)\) (C.8) or (C.9) holds for all \([t/S] \cdot S \leq k \leq t\). Then for given \(k\), by Matrix Azuma inequality Lemma F.2, condition on \(F_k\), with probability at least \(1 - \delta\)

\[
\|U_k - \nabla^2 F(x_k)\|_2^2 = \left\| \sum_{k=[t/S] \cdot S}^t V_k \right\|_2^2 \\
\leq 9 \log(d/\delta) \left[ (t - \lceil t/S \rceil \cdot S) \cdot \frac{\rho \epsilon}{S \log(d/\delta)} + \frac{\rho \epsilon}{\log(d/\delta)} \right] \\
\leq 9 \log(d/\delta) \cdot \frac{\rho \epsilon}{\log(d/\delta)} \\
\leq 30 \rho \epsilon. \tag{C.10}
\]

Finally, by union bound, we have that with probability at least \(1 - 2\delta \cdot (t - \lceil t/S \rceil \cdot S)\), for all \([\lceil t/S \rceil \cdot S \leq k \leq t]\), we have (C.10) holds.

\[\square\]

## D Proofs of Technical Lemmas in Section B

### D.1 Proof of Lemma B.3

We have the following lemma which guarantees the effectiveness of Cubic-Subsolver, Algorithm 3:

**Lemma D.1.** (Carmon and Duchi, 2016) Let \(A \in \mathbb{R}^{d \times d}\) and \(\|A\|_2 \leq \beta\), \(b \in \mathbb{R}^d\), \(\tau > 0\), \(\zeta > 0\), \(\epsilon' \in (0, 1)\), \(\delta' \in (0, 1)\) and \(\eta < 1/(8\beta + 2\tau \zeta)\). We denote that \(g(h) = b^\top h + h^\top Ah/2 + \tau/6 \cdot \|h\|_3^2\) and \(s = \arg\min_{h \in \mathbb{R}^d} g(h)\). Then with probability at least \(1 - \delta'\), if

\[
\|s\|_2 \geq \zeta \text{ or } \|b\|_2 \geq \max\{\sqrt{\beta \tau/2 \zeta^3/2}, \tau \zeta^2/2\}, \tag{D.1}
\]

then \(x = \text{Solve Subproblem}(A, b, \tau, \eta, \zeta, \epsilon', \delta')\) satisfies that \(g(x) \leq -(1 - \epsilon') \tau \zeta^3/12\).

**Proof of Lemma B.3.** We simply set \(A = U_t\), \(b = v_t\), \(\tau = M_t\), \(\eta = (16L)^{-1}\), \(\zeta = \sqrt{\epsilon'/\rho}\), \(\epsilon' = 0.5\) and \(\delta' = \delta\). We have \(\|U_t\|_2 \leq L\), then we set \(\beta = L\). With the choice of \(M_t\) where \(M_t = 100\rho\) and the assumption that \(\epsilon < 16L^2\rho/M_t^2\), we can check that \(\eta < 1/(8\beta + 2\tau \zeta)\). We also have that \(s = h_t^*\) and (D.1) holds. Thus, we have

\[
m_t(h_t) = g(x) \leq -(1 - \epsilon') \tau \zeta^3/12 \leq -M_t \rho^{-3/2} \epsilon^{3/2}/24.
\]

By the choice of \(T'\) in Cubic-Subsolver, we have

\[
T' = \frac{480}{\eta \tau \zeta \epsilon'} \left[ 6 \log \left(1 + \sqrt{\alpha/\delta'}\right) + 32 \log \left(\frac{12}{\eta \tau \zeta \epsilon'}\right) \right] = \tilde{O} \left(\frac{L}{M_t \sqrt{\epsilon'/\rho}}\right).
\]

\[\square\]
D.2 Proof of Lemma B.4

We have the following lemmas which provide the guarantee for the dynamic of gradient steps in Cubic-Finalsolver.

**Lemma D.2.** (Carmon and Duchi, 2016) For \( b, A, \tau \), suppose that \( \|A\|_2 \leq L \). We denote that \( g(h) = b^T h + h^T A h/2 + \tau/6 \cdot \|h\|_2^3 \), \( s = \arg \min_{h \in \mathbb{R}^d} g(h) \), and let \( R \) be

\[
R = \frac{L}{2\tau} + \sqrt{\left( \frac{L}{2\tau} \right)^2 + \frac{\|b\|_2^2}{\tau}} \tag{D.2}
\]

Then for Cubic-Finalsolver, suppose that \( \eta < (4(L+\tau R))^{-1} \), then each iterate \( \Delta \) in Cubic-Finalsolver satisfies that \( \|\Delta\|_2 \leq \|s\|_2 \), and \( g(h) \) is \( L + 2\tau R \)-smooth.

**Lemma D.3.** (Carmon and Duchi, 2016) We denote that \( g(h) = b^T h + h^T A h/2 + \tau/6 \cdot \|h\|_2^3 \), \( s = \arg \min_{h \in \mathbb{R}^d} g(h) \), then \( g(s) \geq \|b\|_2^2 \|s\|_2^2 / 2 - \tau \|s\|_2^3 / 6 \).

**Proof of Lemma B.4.** In Cubic-Finalsolver we are focusing on minimizing \( m_{T^*-1}(h) \). We have that \( \|v_t\|_2 \leq \max\{M_t \varepsilon / (2\rho), \sqrt{L M_t / 2 (\varepsilon / \rho)^{3/4}} \} \) and \( \|h_{T^*-1}^*\|_2 \leq \sqrt{\varepsilon / \rho} \) by Lemma B.3. We can check that \( \eta = (16L)^{-1} \) satisfies that \( \eta < (4(L+\tau R))^{-1} \), where \( R \) is defined in Lemma D.2, when \( \varepsilon < 4L^2 \rho / M_t^2 \). From Lemma D.2 we also know that \( m_{T^*-1} \) is \( L + 2M_{T^*-1} R \)-smooth, which satisfies that \( 1/\eta > 2(L + 2M_{T^*-1} R) \). Thus, by standard gradient descent analysis, to get a point \( \Delta \) where \( \|\nabla m_{T^*-1}(\Delta)\|_2 \leq \epsilon \), Cubic-Finalsolver needs to run

\[
T'' = O\left( \frac{m_{T^*-1}(\Delta_0) - m_{T^*-1}(h_{T^*-1}^*)}{\eta^2} \right) = O\left( \frac{L m_{T^*-1}(\Delta_0) - m_{T^*-1}(h_{T^*-1}^*)}{\epsilon^2} \right)
\tag{D.3}
\]

iterations, where we denote \( \Delta_0 \) is the starting point of Cubic-Finalsolver. By directly computing, we have \( m_{T^*-1}(\Delta_0) \leq 0 \). We also have

\[
-m_{T^*-1}(h_{T^*-1}^*) \leq \|v_{T^*-1}\|_2 \|h_{T^*-1}^*\|_2^2 / 2 + M_t \|h_{T^*-1}^*\|^3_2 / 6
\]

\[
= O\left( (\varepsilon + \varepsilon^{3/4} \sqrt{L / \rho^{1/4}}) \cdot \sqrt{\varepsilon / \rho} + \rho (\varepsilon / \rho)^{3/2} \right)
\]

\[
= O\left( \varepsilon^{3/2} / \sqrt{\rho} + \varepsilon^{5/4} L^{1/2} / \rho^{3/4} \right).
\]

Thus, (D.3) can be further bounded as

\[
T'' = O\left( \frac{L}{\sqrt{\rho \varepsilon}} + \frac{L^{3/2}}{(\rho \varepsilon)^{3/4}} \right) = O\left( \frac{L^{3/2}}{(\rho \varepsilon)^{3/4}} \right) = O\left( \frac{L \Delta_F}{\epsilon^2} \right)
\tag{D.4}
\]

when \( \varepsilon < 4L^2 \rho / M_t^2 \). \( \square \)
E Proofs of Additional Lemmas in Appendix C

E.1 Proof of Lemma C.1

Proof of Lemma C.1. We only need to consider the case where \( B_k^{(g)} = |\mathcal{J}_k| < n \). For each \( i \in \mathcal{J}_k \), let

\[
a_i = \nabla f_i(x_k) - \nabla f_i(x_{k-1}) - \nabla F(x_k) + \nabla F(x_{k-1}),
\]

(E.1)

then we have \( \mathbb{E}_i a_i = 0 \), \( a_i \) i.i.d., and

\[
\|a_i\|_2 \leq \|\nabla f_i(x_k) - \nabla f_i(x_{k-1})\|_2 + \|\nabla F(x_k) - \nabla F(x_{k-1})\|_2 \leq 2L\|x_k - x_{k-1}\|_2,
\]

where the second inequality holds due to the \( L \)-smoothness of \( f_i \) and \( F \). Thus by vector Azuma-Hoeffding inequality in Lemma F.1, we have that with probability at least \( 1 - \delta \),

\[
\left\| \nabla f_{\mathcal{J}_k}(x_k) - \nabla f_{\mathcal{J}_k}(x_{k-1}) - \nabla F(x_k) + \nabla F(x_{k-1}) \right\|_2 = \frac{1}{B_k^{(g)}} \left\| \sum_{i \in \mathcal{J}_k} \left[ \nabla f_i(x_k) - \nabla f_i(x_{k-1}) - \nabla F(x_k) + \nabla F(x_{k-1}) \right] \right\|_2 \\
\leq 6L \sqrt{\log(d/\delta) B_k^{(g)}} \|x_k - x_{k-1}\|_2.
\]

(E.2)

For each \( i \in \mathcal{J}_k \), let

\[
b_i = \nabla f_i(x_k) - \nabla F(x_k),
\]

then we have \( \mathbb{E}_i b_i = 0 \) and \( \|b_i\|_2 \leq M \). Thus by vector Azuma-Hoeffding inequality in Lemma F.1, we have that with probability at least \( 1 - \delta \),

\[
\left\| \nabla f_{\mathcal{J}_k}(x_k) - \nabla F(x_k) \right\|_2 = \frac{1}{B_k^{(g)}} \left\| \sum_{i \in \mathcal{J}_k} \left[ \nabla f_i(x_k) - \nabla f_i(x_k) - \nabla F(x_k) \right] \right\|_2 \leq 6M \sqrt{\log(d/\delta) B_k^{(g)}}.
\]

(E.3)

□

E.2 Proof of Lemma C.2

Proof of Lemma C.2. We only need to consider the case where \( B_k^{(h)} = |\mathcal{I}_k| < n \). For each \( i \in \mathcal{I}_k \), let

\[
A_i = \nabla^2 f_i(x_k) - \nabla^2 f_i(x_{k-1}) - \nabla^2 F(x_k) + \nabla^2 F(x_{k-1}),
\]

then we have \( \mathbb{E}_i A_i = 0, A_i^\top = A_i, A_i \) i.i.d. and

\[
\|A_i\|_2 \leq \left\| \nabla^2 f_i(x_k) - \nabla^2 f_i(x_{k-1}) \right\|_2 + \left\| \nabla^2 F(x_k) - \nabla^2 F(x_{k-1}) \right\|_2 \leq 2\rho \|x_k - x_{k-1}\|_2,
\]
where the second inequality holds due to $\rho$-Hessian Lipschitz continuous of $f_i$ and $F$. Then by Matrix Azuma inequality Lemma F.2, we have that with probability at least $1 - \delta$,

$$
\|\nabla^2 f_{I_k}(x_k) - \nabla^2 f_{I_k}(x_{k-1}) - \nabla^2 F(x_k) + \nabla^2 F(x_{k-1})\|_2
\leq \frac{1}{B_k^{(b)}} \|\sum_{i \in I_k} \left[ \nabla^2 f_i(x_k) - \nabla^2 f_i(x_{k-1}) - \nabla^2 F(x_k) + \nabla^2 F(x_{k-1}) \right]\|_2
\leq 6\rho \sqrt{\frac{\log(d/\delta)}{B_k^{(b)}}} \|x_k - x_{k-1}\|_2.
$$

For each $i \in I_k$, let

$$B_i = \nabla^2 f_i(x_k) - \nabla^2 F(x_k),$$

then we have $E_i B_i = 0$, $B_i^\top = B_i$, and $\|B_i\|_2 \leq 2L$. Then by Matrix Azuma inequality in Lemma F.2, we have that with probability at least $1 - \delta$,

$$
\|\nabla^2 f_{J_k}(x_k) - \nabla^2 F(x_k)\|_2 = \frac{1}{B_k^{(b)}} \|\sum_{i \in I_k} \left[ \nabla^2 f_i(x_k) - \nabla^2 F(x_k) \right]\|_2 \leq 6L \sqrt{\frac{\log(d/\delta)}{B_k^{(b)}}},
$$

which implies the result.

\[\square\]

F Auxialiry Lemmas

We have the following vector Azuma-Hoeffding inequality:

**Lemma F.1.** (Pinelis, 1994) Consider $\{v_k\}$ be a vector-valued martingale difference, where $E[v_k|\sigma(v_1, \ldots, v_{k-1})] = 0$ and $\|v_k\|_2 \leq A_k$, then we have that with probability at least $1 - \delta$,

$$
\left\| \sum_k v_k \right\|_2 \leq 3 \sqrt{\log(1/\delta) \sum_k A_k^2} \quad (F.1)
$$

We have the following Matrix Azuma inequality:

**Lemma F.2.** (Tropp, 2012) Consider a finite adapted sequence $\{X_k\}$ of self-adjoint matrices in dimension $d$, and a fixed sequence $\{A_k\}$ of self-adjoint matrices that satisfy

$$
E[X_k|\sigma(X_{k-1}, \ldots, X_1)] = 0 \text{ and } X_k^2 \preceq A_k^2 \text{ almost surely.}
$$

Then we have that with probability at least $1 - \delta$,

$$
\left\| \sum_k X_k \right\|_2 \leq 3 \sqrt{\log(d/\delta) \sum_k \|A_k\|_2^2}. \quad (F.2)
$$

26
G  Additional Algorithms and Functions

Due to space limit, we include the approximate solvers (Carmon and Duchi, 2016) for the cubic subproblem in this section for the purpose of self-containedness.

Algorithm 3 Cubic-Subsolver($A[:,]$, $b$, $\tau$, $\eta$, $\zeta$, $\epsilon'$, $\delta'$)

1: $x = \text{CauchyPoint}(A[:,], b, \tau)$
2: if $\text{CubicFunction}(A[:,], b, \tau, x) \leq -(1 - \epsilon')\tau\zeta^3/12$ then
3: return $x$
4: end if
5: Set
6: 
7: $T' = \frac{480}{\eta\tau\zeta\epsilon'} \left[ 6 \log \left( 1 + \sqrt{d/\delta'} \right) + 32 \log \left( \frac{12}{\eta\tau\zeta\epsilon'} \right) \right]$ 
8: Draw $q$ uniformly from the unit sphere, set $\tilde{b} = b + \sigma q$ where $\sigma = \tau^2\zeta^3\epsilon'/(\beta + \tau\zeta)/576$
9: $x = \text{CauchyPoint}(A[:,], b, \tau)$
10: for $t = 1, \ldots, T - 1$ do
11: $x \leftarrow x - \eta \cdot \text{CubicGradient}(A[:,], \tilde{b}, \tau, x)$
12: if $\text{CubicFunction}(A[:,], \tilde{b}, \tau, x) \leq -(1 - \epsilon')\tau\zeta^3/12$ then
13: return $x$
14: end if
15: end for
16: return $x$

Algorithm 4 Cubic-Finalsolver($A[:,]$, $b$, $\tau$, $\eta$, $\epsilon_g$)

1: $\Delta \leftarrow \text{CauchyPoint}(A[:,], b, \tau)$
2: while $\|\text{CubicGradient}(A[:,], b, \tau, \Delta)\|_2 > \epsilon_g$ do
3: $\Delta \leftarrow \Delta - \eta \cdot \text{CubicGradient}(A[:,], b, \tau, \Delta)$
4: end while
5: return $\Delta$

H  Additional Experimental Setups

For binary logistic regression problem with a nonconvex regularizer on $a9a$ and $c0untype$, we are given training data $\{x_i, y_i\}_{i=1}^n$, where $x_i \in \mathbb{R}^d$ and $y_i \in \{0, 1\}$ are feature vector and output label corresponding to the $i$-th training example. The nonconvex penalized binary logistic regression is formulated as follows

$$
\min_{w \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n y_i \log \phi(x_i^\top w) + (1 - y_i) \log[1 - \phi(x_i^\top w)] + \lambda \sum_{i=1}^d \frac{w_i^2}{1 + w_i^2},
$$

27
1. **Function:** CauchyPoint($A[·], b, \tau$)
2. **return** $-R_c b / \|b\|_2$, where

$$R_c = \frac{-b^\top A[b]}{\tau \|b\|_2^2} + \sqrt{\left(\frac{-b^\top A[b]}{\tau \|b\|_2^2}\right)^2 + \frac{2 \|b\|_2^2}{\tau}}$$

3. **Function:** CubicFunction($A[·], b, \tau, x$)
4. **return** $b^\top x + x^\top A[x]/2 + \tau \|x\|_2^3/6$

5. **Function:** CubicGradient($A[·], b, \tau, x$)
6. **return** $b^\top + A[x] + \tau \|x\|_2 x/2$

where $\phi(x)$ is the sigmoid function and $\lambda = 10^{-3}$.

For multiclass logistic regression problem with a nonconvex regularizer on MNIST, we are given training data $\{x_i, y_i\}_{i=1}^n$, where $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}^m$ are feature vectors and multilabels corresponding to the $i$-th data points. The nonconvex penalized multiclass logistic regression is formulated as follows

$$\min_{W \in \mathbb{R}^{m \times d}} -\frac{1}{n} \sum_{i=1}^n \langle y_i, \log[\text{softmax}(Wx_i)] \rangle + \lambda \sum_{i=1}^m \sum_{j=1}^d \frac{w_{i,j}^2}{1 + w_{i,j}^2},$$

where $\text{softmax}(a) = \exp(a) / \sum_{i=1}^d \exp(a_i)$ is the softmax function and $\lambda = 10^{-3}$.

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