The survival probability of the high-dimensional contact process with random vertex weights on the oriented lattice

Xiaofeng Xue *
Beijing Jiaotong University

Abstract: This paper is a further study of Reference [8]. We are concerned with the contact process with random vertex weights on the oriented lattice. Our main result gives the asymptotic behavior of the survival probability of the process conditioned on only one vertex is infected at \( t = 0 \) as the dimension grows to infinity. A SIR model and a branching process with random vertex weights are the main auxiliary tools for the proof of the main result.

Keywords: survival probability, contact process, oriented lattice.

1 Introduction

In this paper we are concerned with the contact process with random vertex weights on the oriented lattice \( \mathbb{Z}_d^+ \) for \( d \) sufficiently large, where \( \mathbb{Z}_+ = \{0, 1, 2, \ldots \} \). This paper is a further study of Reference [8], which deals with the critical value of the aforesaid process. First we introduce some notations and definitions. For \( x = (x_1, \ldots, x_d) \in \mathbb{Z}_d^+ \), we define

\[
\|x\| = \sum_{j=1}^{d} x_j
\]

as the \( l_1 \) norm of \( x \). For \( 1 \leq j \leq d \), we use \( e_j \) to denote the \( j \)th elementary unit vector of \( \mathbb{Z}_d^+ \), i.e.,

\[
e_j = (0, \ldots, 0, 1_{j \text{th}}, 0, \ldots, 0).
\]

We use \( O \) to denote the origin of \( \mathbb{Z}_d^+ \). For \( x, y \in \mathbb{Z}_d^+ \), we write \( x \to y \) when and only when

\[
y - x = e_j
\]

for some \( j \in \{1, 2, \ldots, d\} \).

*E-mail: xfxue@bjtu.edu.cn Address: School of Science, Beijing Jiaotong University, Beijing 100044, China.
For later use, we define an order $\prec$ on $\mathbb{Z}_d^+$. For $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{Z}_d^+$, $x \prec y$ when and only when there exists $j \in \{1, 2, \ldots, d\}$ such that $x_j < y_j$ while $x_i = y_i$ for any $i < j$.

Let $\rho$ be a random variable such that $P(\rho \in [0, M]) = 1$ for some $M \in (0, +\infty)$ and $P(\rho > 0) > 0$, then we assign an independent copy $\rho(x)$ of $\rho$ on each vertex $x \in \mathbb{Z}_d^+$. $\rho(x)$ is called the vertex weight of $x$. We assume all these vertex weights are independent. After the vertex weights are given, the contact process $\{C_t\}_{t \geq 0}$ on $\mathbb{Z}_d^+$ with vertex weights $\{\rho(x)\}_{x \in \mathbb{Z}_d^+}$ is a continuous time Markov process with state space

$$X = \{A : A \subseteq \mathbb{Z}_d^+\}$$

and transition rates function given by

$$C_t \rightarrow \begin{cases} C_t \setminus \{x\} & \text{at rate 1 if } x \in C_t, \\ C_t \cup \{x\} & \text{at rate } \lambda \sum_{y : y \to x} \rho(x) \rho(y) 1_{\{y \in C_t\}} \text{ if } x \notin C_t, \end{cases} \quad (1.1)$$

where $\lambda$ is a positive constant called the infection rate while $1_A$ is the indicator function of the event $A$.

Intuitively, the process describes the spread of an epidemic on $\mathbb{Z}_d^+$. Vertices in $C_t$ are infected while vertices out of $C_t$ are healthy. An infected vertex waits for an exponential time with rate one to become healthy while a healthy vertex $x$ may be infected by an infected vertex $y$ when and only when $y \to x$. The infection occurs at rate proportional to the product of the weights on these two vertices.

The (classic) contact process is introduced by Harris in [3], where $\rho \equiv 1$ and infection occurs between nearest (un-oriented) neighbors. For a detailed survey of the classic contact process, see Chapter 6 of [4] and Part 1 of [5].

The contact process with random vertex weights is first introduced in [7] on the complete graph $K_n$ by Peterson, where a phase transition consistent with the mean-field analysis is shown. In detail, infected vertices dies out in $O(\log n)$ units of time with high probability when $\lambda < \frac{1}{E(\rho)}$ or survives for $\exp\{O(n)\}$ units of time with high probability when $\lambda > \frac{1}{E(\rho^2)}$. In [5], Xue studies this process on the oriented lattice and gives the asymptotic behavior of the critical value of the process as the dimension $d$ grows to infinity. When $P(\rho = 1) = p = 1 - P(\rho = 0)$ for some $p \in (0, 1)$, the model reduces to the contact process on clusters of the site percolation, which is a special case of the model introduced in [1] with $n = 1$. In [1], Bertacchi, Lanchier and Zucca study the contact process on $G \times K_n$, where $G$ is the infinite open cluster of the site percolation while $K_n$ is the complete graph with $n$ vertices. Criteria judging whether the process survives is given.

If the i.i.d. weights are assigned on the edges instead of on the vertices, the model turns into the contact process with random edge weights, which is first introduced by Chen and Yao in [11], where a complete convergence theorem is shown.
2 Main results

In this section we give our main results. First we introduce some notations and definitions. We assume that \( \{\rho(x)\}_{x \in \mathbb{Z}^d} \) are defined under the probability space \( (\Omega_d, \mathcal{F}_d, \mu_d) \). The expectation with respect to \( \mu_d \) is denoted by \( E_{\mu_d} \). For \( \omega \in \Omega_d \), we denote by \( P_{\lambda, \omega} \) the probability measure of our model with vertex weights \( \{\rho(x, \omega)\}_{x \in \mathbb{Z}^d} \). \( P_{\lambda, \omega} \) is called the quenched measure. The expectation with respect to \( P_{\lambda, \omega} \) is denoted by \( E_{\lambda, \omega} \). We define \( P_{\lambda,d}(\cdot) = E_{\mu_d}[P_{\lambda,\omega}(\cdot)] = \int P_{\lambda,\omega}(\cdot) \mu_d(d\omega) \), which is called the annealed measure. The expectation with respect to \( P_{\lambda,d} \) is denoted by \( E_{\lambda,d} \).

For any \( A \subseteq \mathbb{Z}^d_+ \), we write \( C_t \) as \( C_t^A \) when \( C_0 = A \). If \( A = \{x\} \) for some \( x \in \mathbb{Z}^d_+ \), we write \( C_t^A \) as \( C_t^x \) instead of \( C_t^\{x\} \).

For \( \lambda > \frac{1}{E(\rho^2)} \), where \( E \) is the expectation with respect to \( \rho \), there is a unique solution \( \theta > 0 \) to the equation

\[
E\left(\frac{\lambda \rho^2}{1 + \lambda \rho \theta}\right) = 1.
\] (2.1)

Now we give the main result of this paper.

**Theorem 2.1.** For any \( \lambda > \frac{1}{E(\rho^2)} \) and \( \theta \) defined as in Equation (2.1),

\[
\lim_{d \to +\infty} P_{\lambda,d}(C_t^O \neq \emptyset, \forall t \geq 0) = E\left(\frac{\lambda \rho \theta}{1 + \lambda \rho \theta}\right).
\]

Theorem 2.1 gives the asymptotic behavior of the survival probability of the process conditioned on \( O \) is the unique initially infected vertex as the dimension \( d \) grows to infinity. The theorem only deals with the case where \( \lambda > \frac{1}{E(\rho^2)} \) because

\[
\lim_{d \to +\infty} P_{\lambda,d}(C_t^O \neq \emptyset, \forall t \geq 0) = 0
\]

for any \( \lambda < \frac{1}{E(\rho^2)} \) according to the main theorem given in [8], which shows that the critical value of the infection rate of the model converges to \( \frac{1}{E(\rho^2)} \) as \( d \to +\infty \).

When \( \rho \equiv 1 \), we have the following direct corollary.

**Corollary 2.2.** If \( \rho \equiv 1 \) and \( \lambda > 1 \), then

\[
\lim_{d \to +\infty} P_{\lambda,d}(C_t^O \neq \emptyset, \forall t \geq 0) = \frac{\lambda - 1}{\lambda}.
\]

The counterpart of Corollary 2.2 for the classic contact process on the lattice is given in [9].
The counterpart of Theorem 2.1 for the contact process with random edges weights on the (un-oriented) lattice is given in [10]. It is claimed in [10] that

$$\lim_{d \to +\infty} P_{\lambda,d}(C_t^O \neq \emptyset, \forall t \geq 0) = \frac{\lambda E \rho - 1}{\lambda E \rho}$$

for the process with edge weights which are independent copies of $\rho$ and infection rate $\lambda > \frac{1}{E \rho}$.

As a auxiliary tool for the proof of Theorem 2.1, we introduce a SIR (susceptible-infected-recovered) model with random vertex weights on $\mathbb{Z}_+^d$.

After the vertex weights $\{\rho(x)\}_{x \in \mathbb{Z}_+^d}$ are given, the SIR model $\{(S_t, I_t)\}_{t \geq 0}$ is a continuous-time Markov process with state spapce

$$X_2 = \{(S, I) : S, I \subseteq \mathbb{Z}_+^d, S \cap I = \emptyset\}$$

and transition rates function given by

$$(S_t, I_t) \to \begin{cases} (S_t, I_t \setminus \{x\}) & \text{at rate } 1 \text{ if } x \in I_t, \\ (S_t \setminus \{x\}, I_t \cup \{x\}) & \text{at rate } \frac{\lambda}{d} \sum_{y : y \to x} \rho(x) \rho(y) 1_{\{y \in I_t\}} \text{ if } x \in S_t. \end{cases} \quad (2.2)$$

For the SIR model, an infected vertex waits for an exponential time with rate one to become recovered while a recovered vertex can never be infected again.

We write $(S_t, I_t)$ as $(S_t^A, I_t^A)$ when $(S_0, I_0) = (\mathbb{Z}_+^d \setminus A, A)$, then it is easy to check that

$$P_{\lambda,d}(I_t^O \neq \emptyset, \forall t \geq 0) \leq P_{\lambda,d}(C_t^O \neq \emptyset, \forall t \geq 0).$$

One way to check this inequality is to utilize the basic coupling of Markov processes (see Section 3.1 of [3]), we omit the details. As a result, to prove Theorem 2.1 we only need to show that

$$\liminf_{d \to +\infty} P_{\lambda,d}(I_t^O \neq \emptyset, \forall t \geq 0) \geq E(\lambda \rho \theta 1 + \lambda \rho \theta) \quad (2.3)$$

and

$$\limsup_{d \to +\infty} P_{\lambda,d}(C_t^O \neq \emptyset, \forall t \geq 0) \leq E(\frac{\lambda \rho \theta}{1 + \lambda \rho \theta}) . \quad (2.4)$$

The proof of Theorem 2.1 is divided into three sections. In Section 3 we introduce a branching process $\{W_n\}_{n \geq 0}$ with random vertex weights on the oriented rooted tree $\mathbb{T}^d$. We will show that the probability that the process survives converges to $E(\frac{\lambda \rho \theta}{1 + \lambda \rho \theta})$ as $d \to +\infty$.

In Section 4 we give the proof of Equation (2.3). The proof relies on a coupling relationship between the branching process and the SIR model. A technique introduced in [8] is utilized.

In Section 5 we give the proof of Equation (2.4). The proof relies on a coupling relationship of the three aforesaid processes.
3 A branching process with vertex weights

In this section we introduce a branching process with random vertex weights on the oriented rooted tree. We denote by $\mathbb{T}^d$ the rooted tree that the root has $d$ neighbors while any other vertex on the tree has $d+1$ neighbors. We denote by $\Upsilon$ the root of the tree. There is a function $f : \mathbb{T}^d \to \{0, 1, 2, \ldots\}$ satisfies the following conditions.

1. $f(\Upsilon) = 0$.
2. $f(x) = 1$ for each neighbor $x$ of $\Upsilon$.
3. For any $y \neq \Upsilon$, there is one neighbor $u$ of $y$ that $f(u) = f(y) - 1$ while there are $d$ neighbors $v$ of $y$ that $f(v) = f(y) + 1$.

For $x, y \in \mathbb{T}^d$, we write $x \Rightarrow y$ when and only when $x$ and $y$ are neighbors and $f(y) = f(x) + 1$.

Intuitively, $\Upsilon$ is the ancestor of a family and has $d$ sons. Each other individual in this family has one father and $d$ sons. $x \Rightarrow y$ when and only when $y$ is a son of $x$.

We assume that $\{\rho(x)\}_{x \in \mathbb{T}^d}$ are i.i.d. copies of the random variable $\rho$, which is defined as in Equation (1). After the vertex weights are given, we assume that $Y(x)$ is an exponential time with rate one for each $x \in \mathbb{T}^d$ while $U(x, y)$ is an exponential time with rate $\frac{\lambda}{\rho(x)} \rho(y)$ for any $x, y \in \mathbb{T}^d$ that $x \Rightarrow y$. We assume that all these exponential times are independent under the given vertex weights. Then, the branching process $\{W_n\}_{n \geq 0}$ is defined as follows.

1. $W_0 = \Upsilon$.
2. For $n \geq 0$, $W_{n+1} = \{y : x \Rightarrow y \text{ and } U(x, y) < Y(x) \text{ for some } x \in W_n\}$.

$\{W_n\}_{n \geq 0}$ describes the spread of a SIR epidemic on $\mathbb{T}^d$. Initially, $\Upsilon$ is infected. A healthy vertex may only be infected by its father. If $x$ is infected, then $x$ waits for an exponential time with rate one to become recovered while waits for an exponential time with rate $\frac{\lambda}{\rho(x)} \rho(y)$ to infect the son $y$. The infection really occurs when and only when $y$ is infected before the moment $x$ is recovered, i.e., $U(x, y) < Y(x)$.

Similar with what we have done in Section 2, we denote by $\hat{P}_{\lambda, \omega}$ the quenched measure of the branching process with respect to the random environment $\omega$ in the space where $\{\rho(x)\}_{x \in \mathbb{T}^d}$ are defined. We denote by $\check{P}_{\lambda, d}$ the annealed measure. Note that according to our definition, for $x \Rightarrow y \Rightarrow z$, $U(x, y)$ and $U(y, z)$ are independent under $\hat{P}_{\lambda, \omega}$ while positively correlated under $\check{P}_{\lambda, d}$.

The branching process $\{W_n\}_{n \geq 0}$ with random vertex weights on the oriented tree $\mathbb{T}^d$ is first introduced in [6]. Some results obtained in [6] will be directly utilized in this section.

The following lemma is crucial for us to prove Theorem 2.1.

Lemma 3.1. For any $\lambda > \frac{1}{E(\rho^2)}$ and $\theta$ defined as in Equation (2.1),

$$\lim_{d \to +\infty} \hat{P}_{\lambda, d}(W_n \neq \emptyset, \forall \ n \geq 0) = E(\frac{\lambda \rho \theta}{1 + \lambda \rho \theta}).$$

The remainder of this section is devoted to the proof of Lemma 3.1. From now on we assume that $\lambda > \frac{1}{E(\rho^2)}$. For any $s \in [0, M]$, we define

$$F_d(s) = \hat{P}_{\lambda, d}\left(W_n = \emptyset \text{ for some } n \geq 0 \mid \rho(\Upsilon) = s \right).$$
then the following two lemmas are crucial for us to prove Lemma 3.1.

**Lemma 3.2.** If \( \{d_l\}_{l \geq 1} \) is a subsequence of 1, 2, 3, \ldots such that
\[
\lim_{l \to +\infty} F_{d_l}(s) \exists := F(s)
\]
for any \( s \in [0, M] \), then
\[
F(s) = \frac{1}{1 + \lambda s \theta}.
\]

**Lemma 3.3.** For \( d \geq 1 \) and \( 0 \leq s < t \leq M \),
\[
|F_d(s) - F_d(t)| \leq \lambda(t-s)M.
\]

We first show how to utilize Lemmas 3.2 and 3.3 to prove Lemma 3.1. The proofs of Lemmas 3.2 and 3.3 are given at the end of this section.

**Proof of Lemma 3.1.** If Lemma 3.1 does not hold, then there are a constant \( \epsilon_0 > 0 \) and a subsequence \( \{a_l\}_{l \geq 1} \) of 1, 2, 3, \ldots such that
\[
|\hat{E}_{\lambda,a_l}(F_{a_l}(\rho)) - E\left(\frac{1}{1 + \lambda \rho \theta}\right)| > \epsilon,
\]
(3.1)
since
\[
\hat{P}_{\lambda,d}(W_n \neq \emptyset, \forall \ n \geq 0) = 1 - E_{\lambda,d}(F_d(\rho)).
\]
Since \( 0 \leq F_d(\cdot) \leq 1 \), according to a classic procedure of picking subsequences, there is a subsequence \( \{d_j\}_{j \geq 1} \) of \( \{a_l\}_{l \geq 1} \) such that
\[
\lim_{j \to +\infty} F_{d_j}(r) \exists := F_\Delta(r)
\]
for any \( r \in \mathbb{Q} \). It is obviously that \( F_d(s) \) is decreasing with \( s \) for each \( d \geq 1 \), then
\[
F_d(r_1) \geq F_d(s) \geq F_d(r_2) \quad \text{and} \quad F_\Delta(r_1) \geq F_\Delta(r_2)
\]
for any \( r_1 < s < r_2, r_1, r_2 \in \mathbb{Q} \). As a result, it is reasonable to define
\[
F_\Delta^-(s) = \lim_{r \downarrow s, r \in \mathbb{Q}} F_\Delta(r) \quad \text{and} \quad F_\Delta^+(s) = \lim_{r \uparrow s, r \in \mathbb{Q}} F_\Delta(r)
\]
for any \( s \not\in \mathbb{Q} \) and hence
\[
\limsup_{j \to +\infty} F_{d_j}(s) \leq F_\Delta^-(s) \quad \text{while} \quad \liminf_{j \to +\infty} F_{d_j}(s) \geq F_\Delta^-(s).
\]
By Lemma 3.3
\[
|F_\Delta(r_1) - F_\Delta(r_2)| \leq \lambda M(r_2 - r_1)
\]
for \( r_1 < s < r_2, r_1, r_2 \in \mathbb{Q} \). Therefore, let \( r_1 \uparrow s \) and \( r_2 \downarrow s \),
\[
F_\Delta^-(s) = F_\Delta^+(s) := F_\Delta(s)
\]
\[ \lim_{j \to +\infty} F_{d_j}(s) = F_\Delta(s) \]

for any \( s \notin \mathbb{Q} \). As a result,

\[ \lim_{j \to +\infty} F_{d_j}(s) := F_\Delta(s) \]

for any \( s \in [0,M] \). Then, by Lemma 3.2,

\[ F_\Delta(s) = \frac{1}{1 + \lambda s \theta} \]

for any \( s \in [0,M] \) and hence

\[ \lim_{j \to +\infty} \tilde{E}_{\lambda,d_j}(F_{d_j}(\rho)) = E\left( \frac{1}{1 + \lambda \rho \theta} \right). \]

However, this is contradictory with Equation (3.1) since \( \{d_j\}_{j \geq 1} \) is a subsequence of \( \{a_l\}_{l \geq 1} \). As a result, Lemma 3.1 holds and the proof is complete. \( \square \)

At last we give the proof of Lemmas 3.2 and 3.3.

**Proof of Lemma 3.2** For \( \Upsilon \Rightarrow y \), conditioned on \( Y(\Upsilon), \rho(\Upsilon), \rho(y) \), the probability that \( \Upsilon \) infects \( y \) is

\[ P_{\lambda,\omega}(U(\Upsilon, y) < Y(\Upsilon) \mid \rho(\Upsilon), Y(\Upsilon), \rho(y)) = 1 - e^{-\frac{\lambda}{2}\rho(\Upsilon)\rho(y)Y(\Upsilon)}. \]

If \( \{W_n\}_{n \geq 0} \) dies out, then for any \( y \) such that \( \Upsilon \) infects \( y \), the epidemic on the subtree consisted of \( y \) and its descendant must die out, the probability of which is \( F_d(\rho(y)) \). As a result,

\[ F_d(s) = \tilde{E}_{\lambda,d}\left[ H_d(Y(\Upsilon)) \right]^d = E\left[ H_d(Y_0) \right]^d, \] (3.2)
where
\[ H_d(t) = E\left(F_d(\rho) \left(1 - e^{-\frac{\Delta s t \rho}{\rho}} + e^{-\frac{\Delta s t \rho}{\rho}}\right)\right) \]
for any \( t \geq 0 \) and \( Y_0 \) is an exponential time with rate one defined under some space we do not care. According to our assumption, it is easy to check that
\[
\lim_{t \to +\infty} d(t) = -\lambda s t E\left(\rho(1 - F(\rho))\right).
\] (3.3)

According to the theory of calculus, if \( a_d \to 0, c_d \to +\infty \) and \( a_d c_d \to c \), then \((1 + a_d)^{c_d} \to e^c\). Therefore, by Equations (3.2) and (3.3),
\[
\lim_{t \to +\infty} F_d(s) = E e^{-\lambda s Y_0 \tilde{\theta}} = \frac{1}{1 + \lambda s \theta},
\] (3.4)

where \( \tilde{\theta} = E(\rho(1 - F(\rho))) \). As a result, \( F(s) = \frac{1}{1 + \lambda s \theta} \) for any \( s \) and we only need to show that \( \tilde{\theta} = \theta \). According to the definition of \( \tilde{\theta} \),
\[
\tilde{\theta} = E\left(\rho(1 - F(\rho))\right) = E\left(\rho(1 - \frac{1}{1 + \lambda s \theta})\right) = E\left(\frac{\rho^2 \tilde{\theta}}{1 + \lambda s \theta}\right).
\]

Therefore, to prove \( \theta = \tilde{\theta} \) we only need to show that \( \tilde{\theta} \neq 0 \). This fact follows directly from the conclusion that
\[
\lim_{d \to +\infty} \sup E(F_d(\rho)) < 1
\]
when \( \lambda > \frac{1}{E(\rho^2)} \), which is proved in [6].

\[ \Box \]

**Proof of Lemma 3.3.** We denote by \( \{W_n^s\}_{n \geq 0} \) the branching process conditioned on \( \rho(\Upsilon) = s \) and denote by \( \{W_n^t\}_{n \geq 0} \) the branching process conditioned on \( \rho(\Upsilon) = t \). We couple these two branching processes in a same probability space as follows. For any \( x \in \mathbb{T}^d \), we assume that these two processes utilize the same exponential time \( Y(x) \) with rate one. For any \( x \neq \Upsilon \) and \( x \Rightarrow z \), we assume that these two processes utilize the same exponential time \( U(x, z) \) with rate \( \lambda s \rho(x) \rho(z) \). For each \( y \) that \( \Upsilon \Rightarrow y \), we assume that \( \{W_n^s\}_{s \geq 0} \) utilizes an exponential time \( U_s(\Upsilon, y) \) with rate \( \lambda s \rho(y) \) while \( \{W_n^t\}_{t \geq 0} \) utilizes an exponential time
\[
U_t(\Upsilon, y) = \inf \{U_s(\Upsilon, y), U_{t-s}(\Upsilon, y)\},
\]

where \( U_{t-s}(\Upsilon, y) \) is an exponential time with rate \( \frac{\lambda}{d}(t - s) \rho(y) \) and is independent of \( U_s(\Upsilon, y), Y(\Upsilon) \) under the quenched measure. Therefore, \( U_t(\Upsilon, y) \) is an exponential
time with rate $\lambda t \rho(y)$. According to the coupling of $\{W^n_s\}_{n \geq 0}$ and $\{W^n_t\}_{n \geq 0}$,

$$|F_d(t) - F_d(s)| = P(\{W^n_t\}_{n \geq 0} \text{ survives while } \{W^n_s\}_{n \geq 0} \text{ dies out})$$

$$\leq P(W^n_t \neq W^n_s)$$

$$= P(U_{t-s}(\Upsilon, y) < Y(\Upsilon) < U_s(\Upsilon, y) \text{ for some } y)$$

$$\leq \sum_{y: \Upsilon \Rightarrow y} \tilde{E}_{\lambda, d} [e^{-\frac{1}{\lambda} \rho(y) Y(\Upsilon)} - e^{-\frac{1}{\lambda} \rho(y) Y(\Upsilon)}]$$

$$= \sum_{y: \Upsilon \Rightarrow y} \tilde{E}_{\lambda, d} \left[\frac{1}{1 + \frac{1}{\lambda} \rho(y)} - \frac{1}{1 + \frac{1}{\lambda} \rho(y)}\right]$$

$$= d E \left[\frac{\frac{1}{\lambda} (t - s) \rho}{(1 + \frac{1}{\lambda} \rho)(1 + \frac{1}{\lambda} \rho)}\right] \leq \frac{d \lambda}{d} (t - s) M = \lambda (t - s) M$$

and the proof is complete. 

4 Proof of Equation (2.3)

In this section we give the proof of Equation (2.3). For later use, we assume that there exists $\epsilon > 0$ that

$$P(\rho = 0 \text{ or } \rho \in [\epsilon, M]) = 1,$$  \hspace{1cm} (4.1)

where $M$ is defined as in Section 1.

This assumption is without loss of generality according to the following analysis. For $\rho$ not satisfying (4.1), we let $\rho_m = \rho 1_{\rho \geq 1/m}$, then $\rho \geq \rho_m$ and $\rho_m \to \rho$ as $m \to +\infty$. It is obviously that $\rho_m$ satisfies (4.1) while the process with weights respect to $\rho$ has larger probability to survive than that with weights respect to $\rho_m$.

As a result, if Equation (2.3) holds under assumption (4.1), then

$$\liminf_{d \to +\infty} P_{\lambda, d, \rho} (I_t^0 \neq \emptyset, \forall \ t \geq 0) \geq \liminf_{d \to +\infty} P_{\lambda, d, \rho_m} (I_t^0 \neq \emptyset, \forall \ t \geq 0) \geq E\left(\frac{\lambda \rho_m \theta_m}{1 + \lambda \rho_m \theta_m}\right)$$

for any sufficiently large $m$, where $P_{\lambda, d, \rho}$ is the annealed measure of the process with vertex weights which are i.i.d copies of $\rho$ while $\theta_m$ satisfies

$$E\left(\frac{\lambda \rho_m^2}{1 + \lambda \rho_m \theta_m}\right) = 1$$

and it is easy to check that $\lim_{m \to +\infty} \theta_m = \theta$. Let $m \to +\infty$, then Equation (2.3) holds for general $\rho$.

First we give a sketch of the proof, which is inspired by the approach introduced in [10]. We divide $\mathbb{Z}_+^d$ into two parts $\Gamma_1$ and $\Gamma_2$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset$. The first step is to show that with probability at least $E\left(\frac{\lambda \rho}{1 + \lambda \rho \theta}\right) + o(1)$ there exists a path
starting at $O$ on $\Gamma_1$ with length $O(\log d)$ that all the vertices on this path have ever been infected. The second step is to show that conditioned on the existence of the aforesaid path, the vertices on this path infects $K(d)$ vertices on $\Gamma_2$ through edges connecting $\Gamma_1$ and $\Gamma_2$ with high probability, where $K(d) \to +\infty$ as $d \to +\infty$. The third step is to show that conditioned on $K(d)$ vertices are initially infected on $\Gamma_2$, the SIR model on $\Gamma_2$ survives with high probability. To prove the first step, we construct a couple between the SIR on $Z^d_+$ and the branching process introduced in Section 3.

To give our proof, we introduce some definitions and notations. For sufficiently large $d$, we define $N(d) = \log(\log d)$ and

$$\Gamma_1 = \left\{ x = (x_1, x_2, \ldots, x_d) \in Z^d_+ : \sum_{i=d-\left\lfloor \frac{d}{N(d)} \right\rfloor +1}^{d} x_i = 0 \right\},$$

$$\Gamma_2 = \left\{ x \in Z^d_+ : \sum_{i=d-\left\lfloor \frac{d}{N(d)} \right\rfloor +1}^{d} x_i > 0 \right\},$$

$$\Gamma_3 = \left\{ x \in Z^d_+ : \sum_{i=d-\left\lfloor \frac{d}{N(d)} \right\rfloor +1}^{d} x_i = 1 \right\} \subseteq \Gamma_2.$$

For $n \geq 0$, we define

$$V_n = \left\{ x \in Z^d_+ : \|x\| = n \text{ and } x \in I_t^O \text{ for some } t \geq 0 \right\}$$

as the set of vertices which have ever been infected with $l_1$ norm $n$. Since in the SIR model, infection can not occur repeatedly between neighbors, $\{V_n\}_{n \geq 0}$ can be defined equivalently as the following way. For each $x \in Z^d_+$, let $\tilde{Y}(x)$ be an exponential time with rate one. For any $x, y$ that $x \to y$, let $\tilde{U}(x, y)$ be an exponential time with rate $\frac{1}{d} \rho(x) \rho(y)$. We assume that all these exponential times are independent under the quenched measure with respect to the given edge weights, then

1. $V_0 = \{O\}.$
2. For each $n \geq 0$,

$$V_{n+1} = \left\{ y \in Z^d_+ : x \to y \text{ and } \tilde{U}(x, y) < \tilde{Y}(x) \text{ for some } x \in V_n \right\}.$$

The intuitive explanation of the above definition is similar with that of the branching process introduced in Section 3. $\tilde{Y}(x)$ is time $x$ waits for to become recovered after $x$ is infected while $\tilde{U}(x, y)$ is the time $x$ waits for to infect $y$.

For later use, we define $\{\hat{V}_n\}_{n \geq 0}$ as the vertices which have ever been infected for the SIR model confined on $\Gamma_1$. In details,

1. $\hat{V}_0 = \{O\}.$
2. For each $n \geq 0$,

$$\hat{V}_{n+1} = \left\{ y \in \Gamma_1 : x \to y \text{ and } \tilde{U}(x, y) < \tilde{Y}(x) \text{ for some } x \in \hat{V}_n \right\}.$$
Let $\sigma > 0$ be an arbitrary constant that $\sigma \log(\lambda M^2) < \frac{1}{10}$, then we have the following lemma.

**Lemma 4.1.** For $\lambda > \frac{1}{E(\rho^2)}$,

$$
\liminf_{d \to +\infty} P_{\lambda,d}(\hat{V}_{\lfloor \sigma \log d \rfloor} \neq \emptyset) \geq E\left(\frac{\lambda \rho \theta}{1 + \lambda \rho \theta}\right).
$$

The proof of Lemma 4.1 is given in Subsection 4.2. As a preparation of this proof, we give a coupling of $\{W_n\}_{n \geq 0}$ and $\{V_n\}_{n \geq 0}$ in Subsection 4.1.

To execute the second step as we have introduced, we define

$$
D = \{ y \in \Gamma_3 : x \to y \text{ and } \hat{U}(x,y) < \hat{Y}(x) \text{ for some } x \in \bigcup_{n=0}^{\lfloor \sigma \log d \rfloor} \hat{V}_n \}
$$

as the set of vertices in $\Gamma_3$ which have been infected by vertices in $\bigcup_{n=0}^{\lfloor \sigma \log d \rfloor} \hat{V}_n$ through edges connecting $\Gamma_1$ and $\Gamma_2$. Let $K(d) = \lfloor \sqrt{\log N(d)} \rfloor$, then we have the following lemma.

**Lemma 4.2.** For $\lambda > \frac{1}{E(\rho^2)}$,

$$
\lim_{d \to +\infty} P_{\lambda,d}(|D| \geq K(d) \big| \hat{V}_{\lfloor \sigma \log d \rfloor} \neq \emptyset) = 1,
$$

where $|D|$ is the cardinality of $D$.

The proof of Lemma 4.2 is given in Subsection 4.3.

As we have introduced, the third step of the proof of Equation (2.3) is to show that the SIR model survives with high probability conditioned on $K(d)$ vertices are initially infected. In detail, we have the following lemma.

**Lemma 4.3.** If $\lambda > \frac{1}{E(\rho^2)}$, then there exists $m(d)$ for each $d \geq 1$ such that

$$
\lim_{d \to +\infty} m(d) = 1
$$

and

$$
P_{\lambda,d}(I^A_t \neq \emptyset, \forall t \geq 0) \geq m(d)
$$

for any $A \subseteq \Gamma_3$ with $|A| = K(d)$.

The proof of Lemma 4.3 is given in Subsection 4.4.

Now we show how to utilize Lemmas 4.1, 4.2 and 4.3 to prove Equation (2.3).

**Proof of Equation (2.3).** For $x, y \in \mathbb{Z}^d_+$, we write $x \Rightarrow y$ when there exists $x_1, x_2, \ldots, x_m$ for some integer $m \geq 1$ such that $x = x_0 \to x_1 \to x_2 \to \ldots \to x_m \to x_{m+1} = y$ and $\hat{U}(x_j, x_{j+1}) < \hat{Y}(x_j)$ for all $0 \leq j \leq m$. Then, according to the meanings of the exponential times $\hat{U}(\cdot, \cdot)$ and $\hat{Y}(\cdot)$,

$$
\bigcup_{t \geq 0} I^A_t = A \bigcup \{ y : x \Rightarrow y \text{ for some } x \in A \}.
$$

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Since each infected vertex becomes recovered in an exponential time with rate one, the infected vertices never die out when and only when there are infinite many vertices have ever been infected. Therefore,
\[\{I_t^A \neq \emptyset, \forall \ t \geq 0\} = \{|\{y : x \Rightarrow y \text{ for some } x \in A\}| = +\infty\} \] (4.2)
for any finite \(A\). According to the definition of \(D\), \(O \Rightarrow x\) for any \(x \in D\). As a result,
\[\{y : x \Rightarrow y \text{ for some } x \in D\} \subseteq \{y : O \Rightarrow y\} \subseteq \bigcup_{t \geq 0} I_t^O, \] (4.3)
since \(O \Rightarrow y\) when \(O \Rightarrow x\) and \(x \Rightarrow y\). By Equations (4.2) and (4.3),
\[P_{\lambda,d}(\{|\{y : x \Rightarrow y \text{ for some } x \in D\}| = +\infty\}) \leq P_{\lambda,d}(I_t^O \neq \emptyset, \forall \ t \geq 0). \] (4.4)

According to the conditional probability formula,
\[P_{\lambda,d}(\{|\{y : x \Rightarrow y \text{ for some } x \in D\}| = +\infty\}) \geq P_{\lambda,d}(\{|\{y : x \Rightarrow y \text{ for some } x \in D\}| = +\infty\} \bigg| |D| \geq K(d), \hat{V}_{[\sigma \log d]} \neq \emptyset) \times P_{\lambda,d}(\hat{V}_{[\sigma \log d]} \neq \emptyset). \] (4.5)

Note that \(D\) is a subset of \(\Gamma_3\) with cardinality at least \(K(d)\) while the event \(\{|\{y : x \Rightarrow y \text{ for some } x \in D\}| = +\infty\}\) is positively correlated with the event \(\{|D| \geq K(d), \hat{V}_{[\sigma \log d]} \neq \emptyset\}\). As a result, by Lemma 4.3 and Equation (4.5),
\[P_{\lambda,d}(\{|\{y : x \Rightarrow y \text{ for some } x \in D\}| = +\infty\}) \geq \inf \left\{ P_{\lambda,d}(\{|\{y : x \Rightarrow y \text{ for some } x \in A\}| = +\infty\} : \right. \]
\[\left. |A| = K(d) \text{ and } A \subseteq \Gamma_3 \right\} \]
\[= \inf \left\{ P_{\lambda,d}(I_t^A \neq \emptyset, \forall \ t \geq 0) : |A| = K(d) \text{ and } A \subseteq \Gamma_3 \right\} \geq m(d). \] (4.6)

By Equations (4.5), (4.6), Lemmas 4.1, 4.2 and 4.3,
\[\lim \inf_{d \to +\infty} P_{\lambda,d}(\{|\{y : x \Rightarrow y \text{ for some } x \in D\}| = +\infty\}) \geq E\left(\frac{\lambda \rho \theta}{1 + \lambda \rho \theta}\right) \lim_{d \to +\infty} m(d) = E\left(\frac{\lambda \rho \theta}{1 + \lambda \rho \theta}\right). \] (4.7)
Equation (2.3) follows directly from Equations (4.4) and (4.7).
4.1 The coupling between \( \{W_n\}_{n \geq 1} \) and \( \{V_n\}_{n \geq 1} \)

In this section, we give a coupling between the SIR model \( \{V_n\}_{n \geq 0} \) on \( \mathbb{Z}^d_+ \) and the branching process \( \{W_n\}_{n \geq 0} \) on \( \mathbb{T}^d \).

We let \( \{\rho(x)\}_{x \in \mathbb{Z}^d_+} \) be i.i.d copies of \( \rho \) as defined in Section 1. We let \( \{\tilde{Y}(x)\}_{x \in \mathbb{Z}^d_+} \) and \( \{\tilde{U}(x, y)\}_{x \rightarrow y} \) be exponential times with respect to \( \{\rho(x)\}_{x \in \mathbb{Z}^d_+} \) as defined at the beginning of this section. We let \( \{V_n\}_{n \geq 0} \) be the SIR model with respect to \( \tilde{Y}(\cdot) \) and \( \tilde{U}(\cdot, \cdot) \) as defined at the beginning of this section. Now we give the evolution of \( \{W_n\}_{n \geq 0} \) by induction.

We let \( W_0 = Y, \rho(Y) = \rho(O) \) and \( Y(Y) = \tilde{Y}(O) \), where \( O \) is the origin of \( \mathbb{Z}^d_+ \). For the \( d \) sons denoted by \( n_1, n_2, \ldots, n_d \) of \( Y \), we let \( \rho(n_i) = \rho(e_i), Y(n_i) = \tilde{Y}(e_i) \) and \( U(Y, n_i) = \tilde{U}(O, e_i) \) for each \( 1 \leq i \leq d \), where \( e_i \) is the elementary unit vector of \( \mathbb{Z}^d_+ \) as defined in Section 1. Then \( W_1 \) is defined according to the values of \( \{U(Y, n_i)\}_{1 \leq i \leq d} \) and \( Y(Y) \) as in Section 2.

For \( n \geq 1 \), if \( |V_n| = |W_n| \) and there is a bijection \( g_n : V_n \rightarrow W_n \) such that \( \rho(g_n(x)) = \rho(x) \) and \( Y(g_n(x)) = \tilde{Y}(x) \) for each \( x \in V_n \), then we say that our coupling is successful at step \( n \). It is obviously that our coupling is successful at step \( n = 1 \) since \( g_1 \) can be defined as \( g_1(e_i) = n_i \) for any \( e_i \in V_1 \).

If \( \{W_m\}_{m \leq n} \) is well defined and the coupling is successful at step \( m \) for all \( 1 \leq m \leq n \), then \( W_{n+1} \) is defined as follows. For any \( x \in V_n \), we define

\[
q(x) = \left\{ y : x \rightarrow y \text{ and } z \rightarrow y \text{ for some } z \in V_n \setminus \{x\} \right\},
\]

\[
\psi(x) = \left\{ y : x \rightarrow y \right\} \setminus q(x)
\]

and \( h(x) = d - |q(x)| \), then \( |\psi(x)| = h(x) \). For each \( x \in V_n \), we arbitrarily choose \( h(x) \) sons of \( g_n(x) \), which are denoted by \( w_1, w_2, \ldots, w_{h(x)} \). Giving the \( h(x) \) elements in \( \psi(x) \) an arbitrary order \( y_1, y_2, \ldots, y_{h(x)} \), then we let \( \rho(w_i) = \rho(y_i) \), \( Y(w_i) = \tilde{Y}(y_i) \) and \( U(g_n(x), w_i) = \tilde{U}(x, y_i) \) for each \( 1 \leq i \leq h(x) \). For any son \( u \) of \( g_n(x) \) which is not in \( \{w_1, w_2, \ldots, w_{h(x)}\} \), let \( Y(u) \) be an exponential time with rate one and \( \rho(u) \) be an independent copy of \( \rho \) that \( Y(u) \) and \( \rho(u) \) are independent of the aforesaid exponential times and vertex weights while let \( U(g_n(x), u) \) be an exponential time with rate \( \frac{\lambda \rho}{d} \rho(g_n(x)) \rho(u) \). Then, \( W_{n+1} \) is defined according to the values of \( \{Y(g_n(x))\}_{x \in V_n} \) and \( \{U(g_n(x), w)\}_{x \in V_n, g_n(x) \not\rightarrow w} \) as in Section 3.

If \( n \geq 2 \) is the first step that the coupling is not successful, then we let \( \{W_m\}_{m \geq n+1} \) evolves independently of \( \{V_m\}_{m \geq n+1} \).

From now on we assume that \( \{W_n\}_{n \geq 0} \) and \( \{V_n\}_{n \geq 0} \) are defined under the same probability space. The annealed measure is still denoted by \( P_{\lambda, d} \).

The remainder of this subsection is devoted to the proof of the following lemma.

**Lemma 4.4.** For \( \lambda > \frac{1}{E(\rho^2)} \) and \( \sigma \in (0, \frac{1}{10 \log(\lambda M^2)}) \),

\[
\liminf_{d \to +\infty} P_{\lambda, d}(V_{\lfloor \sigma \log d \rfloor} \neq \emptyset) \geq E\left( \frac{\lambda \rho \theta}{1 + \lambda \rho \theta} \right).
\]
The proof of Lemma 4.4 relies heavily on the following lemma.

**Lemma 4.5.** For given \( \sigma \in (0, \frac{1}{10 \log(\lambda M^2)}) \), we denote by \( B(d) \) the event that the coupling of \( \{V_n\}_{n \geq 0} \) on \( \mathbb{Z}^d_+ \) and \( \{W_n\}_{n \geq 0} \) on \( \mathbb{T}^d \) is successful at step \( m \) for all \( 1 \leq m \leq \lfloor \sigma \log d \rfloor \), then

\[
\lim_{d \to +\infty} P_{\lambda,d}(B(d)) = 1.
\]

The proof of Lemma 4.5 is given at the end of this subsection. Now we show how to utilize Lemma 4.5 to prove Lemma 4.4.

**Proof of Lemma 4.4.** On the event \( B(d) \), \(|W_m| = |V_m|\) for all \( 1 \leq m \leq \lfloor \sigma \log d \rfloor \). As a result,

\[
P_{\lambda,d}(V_{\lfloor \sigma \log d \rfloor} \neq \emptyset) \geq P_{\lambda,d}(W_{\lfloor \sigma \log d \rfloor} \neq \emptyset) - P_{\lambda,d}(B(d)^c),
\]

where \( B(d)^c \) is the complementary set of \( B(d) \). Therefore, by Lemmas 3.1 and 4.5

\[
\liminf_{d \to +\infty} P_{\lambda,d}(V_{\lfloor \sigma \log d \rfloor} \neq \emptyset) \geq \liminf_{d \to +\infty} P_{\lambda,d}(W_{\lfloor \sigma \log d \rfloor} \neq \emptyset) - \lim_{d \to +\infty} P_{\lambda,d}(B(d)^c) = \liminf_{d \to +\infty} P_{\lambda,d}(W_n \neq \emptyset, \forall n \geq 0) = E(\frac{\lambda \rho \theta}{1 + \lambda \rho \theta})
\]

and the proof is complete.

At the end of this subsection we give the proof of Lemma 4.5.

**Proof of Lemma 4.5.** First we claim that

\[
\lim_{d \to +\infty} P_{\lambda,d}\left( \sum_{m=0}^{\lfloor \sigma \log d \rfloor} |V_m| > d^{0.2} \right) = 0. \quad (4.8)
\]

Equation (4.8) follows from the following analysis. For a given oriented path \( \bar{l} : O = x_0 \to x_1 \to \ldots \to x_m \) on \( \mathbb{Z}^d_+ \),

\[
P_{\lambda,d}\left( \bar{U}(x_j, x_{j+1}) < \bar{Y}(x_j) \right) \text{ for all } 0 \leq j \leq m - 1 \leq \left( \frac{\lambda}{d M^2} \right)^m,
\]

since \( \bar{Y}(\cdot) \) is an exponential time with rate one while \( \bar{U}(\cdot, \cdot) \) is an exponential time with rate at most \( \frac{\lambda}{d M^2} \). The number of oriented paths starting at \( O \) with length \( m \) on \( \mathbb{Z}^d_+ \) is \( d^m \). As a result,

\[
E_{\lambda,d}|V_m| \leq \left( \frac{\lambda}{d M^2} \right)^m d^m = (\lambda M^2)^m,
\]

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since $x \in V_m$ when and only when there exists an oriented path $\tilde{I}: O = x_0 \to x_1 \to \ldots \to x_m = x$ that $\tilde{U}(x_j, x_{j+1}) < \tilde{Y}(x_j)$ for all $0 \leq j \leq m - 1$.

Then, according to the Chebyshev’s inequality and the fact that $\sigma \log(\lambda M^2) < \frac{1}{10}$,

$$P_{\lambda, d}\left(\sum_{m=0}^{\lfloor \sigma \log d\rfloor} |V_m| > d^{0.2}\right) \leq d^{-0.2} \sum_{m=0}^{\lfloor \sigma \log d\rfloor} (\lambda M^2)^m \leq \frac{d^{-0.1}}{\lambda M^2 - 1},$$

Equation (4.8) follows from which directly.

For $d \geq 2$, we denote by $J(d)$ the event that $\tilde{U}(x, y) > \tilde{Y}(x)$ for any $x \in \bigcup_{m=0}^{\lfloor \sigma \log d\rfloor} V_m$ and any $y \in q(x)$. It is easy to check that there exists a vertex $y$ satisfying $x \to y, z \to y$ for given $x, z \in V_m$ when and only when $x - z = e_i - e_j$ for some $1 \leq i, j \leq d$ and such $y$ is unique that $y = x + e_j = z + e_i$. Hence,

$$|q(x)| \leq |V_m| - 1 < |V_m|$$

for any $x \in V_m$. By Equation (4.9) and the fact that $\tilde{Y}(\cdot)$ is an exponential time with rate one while $\tilde{U}(\cdot, \cdot)$ is an exponential time with rate at most $\frac{2}{d} M^2$ while the event $\sum_{m=0}^{\lfloor \sigma \log d\rfloor} |V_m| \leq d^{0.2}$ is negative correlated with the event $\tilde{U}(x, y) < \tilde{Y}(x)$ for $x \in \bigcup_{m=0}^{\lfloor \sigma \log d\rfloor} V_m, y \in q(x)$, we have

$$P_{\lambda, d}\left(\sum_{m=0}^{\lfloor \sigma \log d\rfloor} |V_m| \leq d^{0.2}\right)$$

$$\leq \sum_{m=0}^{\lfloor \sigma \log d\rfloor} E_{\lambda, d}\left(\sum_{l=0}^{\lfloor \sigma \log d\rfloor} |V_m| \lambda^l M^{l2} \sum_{m=0}^{\lfloor \sigma \log d\rfloor} |V_m| \leq d^{0.2}\right)$$

$$\leq \lambda M^2 \sum_{m=0}^{\lfloor \sigma \log d\rfloor} \sum_{m=0}^{\lfloor \sigma \log d\rfloor} |V_m| \leq d^{0.2}$$

$$\leq \lambda M^2 \sum_{m=0}^{\lfloor \sigma \log d\rfloor} d^{0.4} = \lambda M^2 d^{-0.6}.$$

By Equations (4.8) and (4.10),

$$\lim_{d \to +\infty} P_{\lambda, d}\left(\sum_{m=0}^{\lfloor \sigma \log d\rfloor} |V_m| \leq d^{0.2}\right) = 1.$$  

(4.11)

For $k < \lfloor \sigma \log d\rfloor$, conditioned on the event

$$J(d) \cap \{\sum_{m=0}^{\lfloor \sigma \log d\rfloor} |V_m| \leq d^{0.2}\} \cap \{\text{the coupling is successful at step } k\},$$

the coupling will be successful at step $k + 1$ if $U(g_k(x), y) > Y(g_k(x))$ for any $x \in V_k$ and any $y$ that $g_k(x) \Rightarrow y$ while $y \not= w_1, w_2, \ldots, w_h(x)$. Then, according to a similar
analysis with which leads to Equation (4.10),

\[
P_{\lambda,d} \left( \text{the coupling is successful at step } k+1 \right) \geq 1 - \frac{\lambda M^2}{d} \left( |J(d)| \sum_{m=0}^{[\sigma \log d]} |V_m| \right) \leq d^{0.2}
\]

By Equation (4.12) and the conditional probability formula,

\[
P_{\lambda,d} \left( B(d) \mid J(d), \sum_{m=0}^{[\sigma \log d]} |V_m| \leq d^{0.2} \right) \geq \left( 1 - \lambda M^2 d^{-0.6} \right)^{[\sigma \log d]}. \tag{4.13}
\]

Lemma 4.5 follows from Equations (4.11) and (4.13) directly since

\[
\lim_{d \to +\infty} \left( 1 - \lambda M^2 d^{-0.6} \right)^{[\sigma \log d]} = 1.
\]

\[\square\]

4.2 Proof of Lemma 4.1

In this subsection we give the proof of Lemma 4.1.

Proof of Lemma 4.1. There is an isomorphism \( \Phi : \mathbb{Z}_+^{d-\lfloor \frac{d}{N(d)} \rfloor} \to \Gamma_1 \) that

\[
\Phi(x_1, x_2, \ldots, x_{d-\lfloor \frac{d}{N(d)} \rfloor}) = (x_1, x_2, \ldots, x_{d-\lfloor \frac{d}{N(d)} \rfloor}, 0, \ldots, 0)
\]

for each \( x = (x_1, x_2, \ldots, x_{d-\lfloor \frac{d}{N(d)} \rfloor}) \in \mathbb{Z}_+^{d-\lfloor \frac{d}{N(d)} \rfloor} \), hence \( \{\tilde{V}_n\}_{n \geq 0} \) can be identified as the SIR model \( \{V_n\}_{n \geq 0} \) on \( \mathbb{Z}_+^{d-\lfloor \frac{d}{N(d)} \rfloor} \) with infection rate \( \frac{\lambda(d-\lfloor \frac{d}{N(d)} \rfloor)}{d} \). For given \( \lambda \in \left( \frac{1}{E(\rho^2)}, \lambda_1 \right) \) and \( \sigma \in (\sigma_1, \frac{1}{10 \log(M^2 \lambda)}) \),

\[
\frac{\lambda(d-\lfloor \frac{d}{N(d)} \rfloor)}{d} \geq \tilde{\lambda} \quad \text{and} \quad \sigma \log d \leq \tilde{\sigma} \log(d-\lfloor \frac{d}{N(d)} \rfloor)
\]

for sufficiently large \( d \). Then,

\[
P_{\lambda,d} \left( \tilde{V}_{[\sigma \log d]} \neq \emptyset \right) \geq P_{\lambda,d-\lfloor \frac{d}{N(d)} \rfloor} \left( V_{[\sigma \log d-\lfloor \frac{d}{N(d)} \rfloor]} \neq \emptyset \right)
\]
for sufficiently large $d$ and hence
\[
\liminf_{d \to +\infty} P_{\lambda,d}(\hat{V}_{[\sigma \log d]} \neq \emptyset) \geq \liminf_{d \to +\infty} P_{\lambda,d-\lfloor \frac{d}{N(d)} \rfloor}(V_{[\sigma \log d-\lfloor \frac{d}{N(d)} \rfloor]} \neq \emptyset) \geq E \left( \frac{\tilde{\lambda} \rho \theta}{1 + \lambda \rho \theta} \right)
\]
according to Lemma 4.4, where $\tilde{\theta}$ satisfies
\[
E \left( \frac{\tilde{\lambda} \rho^2}{1 + \lambda \rho \theta} \right) = 1
\]
and it is easy to check that $\lim_{\lambda \to \lambda'} \tilde{\theta} = \theta$.

Let $\tilde{\lambda} \to \lambda$, then the proof is complete.

4.3 Proof of Lemma 4.2

In this section we give the proof of Lemma 4.2. First we introduce some notations and definitions.

We let $\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_{[\sigma \log d]}$ be exponential times with rate one while $\Lambda_1, \ldots, \Lambda_{[\sigma \log d]}$ be exponential times with rate $\lambda M^2$. For $1 \leq i \leq [\sigma \log d]$ and $1 \leq j \leq \lfloor \frac{d}{N(d)} \rfloor$, let $\rho_{ij}$ be an independent copy of $\rho$ while $\hat{U}_{ij}$ is an exponential time with rate $\frac{1}{2} \epsilon \rho_{ij}$, where $\epsilon$ is defined as in Equation (4.1). According to the basic technique of measure theory, we can assume that $\{\rho_{ij} : 1 \leq i \leq [\sigma \log d], 1 \leq j \leq \lfloor \frac{d}{N(d)} \rfloor\}$ and $\{\rho(x) : x \in \mathbb{Z}_d^+\}$ are defined under the same space and independent under the annealed measure $P_{\lambda,d}$ while assume that $\hat{U}(\cdot, \cdot), \hat{U}, \hat{Y}(\cdot, \cdot), \hat{Y}, \Lambda$ are defined under the same space and independent under the quenched measure $P_{\lambda,\omega}$.

Lemma 4.6. Let $\xi_i = \sum_{j=1}^{\lfloor \frac{d}{N(d)} \rfloor} 1_{\{\hat{U}_{ij} < Y_i\}}$ for $1 \leq i \leq [\sigma \log d]$, then
\[
P_{\lambda,d} \left( |D| \geq K(d) \left| \hat{V}_{[\sigma \log d]} \neq \emptyset \right. \right) 
\geq P_{\lambda,d} \left( \sum_{i=1}^{[\sigma \log d]} \xi_i \geq K(d) \left| \hat{Y}_i < \Lambda_i \text{ for all } i \leq [\sigma \log d] \right. \right),
\]
and
\[
\lim_{d \to +\infty} E_{\lambda,d} \left( e^{-s \frac{N(d)}{\log d}} \sum_{j=1}^{[\sigma \log d]} \xi_j \left| \hat{Y}_i < \Lambda_i \text{ for all } i \leq [\sigma \log d] \right. \right) \exists = \Theta(s)
\]
for any $s > 0$, where $\lim_{s \to +\infty} \Theta(s) = 0$.

We give the proof of Lemma 4.6 at the end of this subsection. Now we show that how to utilize Lemma 4.6 to prove Lemma 4.2.
Proof of Lemma 4.2. By Chebyshev’s inequality, for any $s > 0$,
\[ P_{\lambda,d}\left( \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \xi_i \leq K(d) \left| \hat{Y}_i < \Lambda_i \right. \text{ for all } i \leq \lfloor \sigma \log d \rfloor \right) \]
\leq e^{-s \frac{K(d)N(d)}{\log d}} E_{\lambda,d}\left( e^{-s \frac{N(d)}{\log d} \sum_{j=1}^{\lfloor \sigma \log d \rfloor} \xi_j} \left| \hat{Y}_i < \Lambda_i \right. \text{ for all } i \leq \lfloor \sigma \log d \rfloor \right). \]

Then, according to Equation (4.15),
\[ \limsup_{d \to +\infty} P_{\lambda,d}\left( \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \xi_i \leq K(d) \left| \hat{Y}_i < \Lambda_i \right. \text{ for all } i \leq \lfloor \sigma \log d \rfloor \right) \leq \Theta(s) \]
for any $s > 0$, since $\lim_{d \to +\infty} \frac{K(d)N(d)}{\log d} = 0$.

Let $s \to +\infty$, since $\lim_{s \to +\infty} \Theta(s) = 0$, we have
\[ \lim_{d \to +\infty} P_{\lambda,d}\left( \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \xi_i \leq K(d) \left| \hat{Y}_i < \Lambda_i \right. \text{ for all } i \leq \lfloor \sigma \log d \rfloor \right) = 0. \] (4.16)

Lemma 4.2 follows from Equations (4.14) and (4.16) directly. \qed

Now we give the proof of Lemma 4.6.

Proof of Lemma 4.6. Equation (4.14) follows from the following analysis. Conditioned on $\hat{V}_{[\sigma \log d]} \neq \emptyset$, there are $O \to X_1 \to \ldots \to X_{[\sigma \log d]}$ that $X_i \in \hat{V}_i$ for $1 \leq i \leq \lfloor \sigma \log d \rfloor$. We choose $X_1, \ldots, X_{[\sigma \log d]}$ as follows. We let $X_{[\sigma \log d]}$ be the smallest one of $\hat{V}_{[\sigma \log d]}$ under the partial $\prec$. Then, we let $X_{[\sigma \log d]-1}$ be the smallest one of $\{ x \in \hat{V}_{[\sigma \log d]-1} : x \to X_{[\sigma \log d]} \}$ and $\bar{U}(x, X_{[\sigma \log d]}) < \bar{Y}(x)$. By induction, if $X_j$ is well defined for $i+1 \leq j \leq \lfloor \sigma \log d \rfloor$, then we define $X_i$ as the smallest one of
\[ \{ x \in V_i : x \to X_{i+1} \text{ and } \bar{U}(x, X_{i+1}) < \bar{Y}(x) \}. \]

For $1 \leq i \leq \lfloor \sigma \log d \rfloor$, let
\[ \zeta_i = \sum_{j=1}^{\lfloor \frac{d}{N(d)} \rfloor} 1\{ \bar{U}(X_i, x_i + e_j d - \lfloor \frac{d}{N(d)} \rfloor) < \bar{Y}(X_i) \} \text{ and } \eta_i = \sum_{j=1}^{\lfloor \frac{d}{N(d)} \rfloor} 1\{ \bar{U}_i < \bar{Y}(X_i) \}, \]

then
\[ |D| \geq \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \zeta_i \] (4.17)
according to our definition of $D$.

Since $X_i \in \hat{V}_i$, $\rho(X_i) > 0$ and hence $\rho(X_i) \geq \epsilon$ by assumption (4.1). As a result, $\bar{U}(X_i, X_i + e_j d - \lfloor \frac{d}{N(d)} \rfloor)$ is an exponential time with rate at least $\frac{A}{d} \epsilon \rho(x_i + e_j d - \lfloor \frac{d}{N(d)} \rfloor)$,
where $\rho(X_i + e_{j+d-\frac{d}{N(d)}})$ is an independent copy of $\rho$. Therefore, $U_i(X_i, X_i + e_{j+d-\frac{d}{N(d)}})$ is stochastic dominated from above by $U_i$ and $\eta_i$ is dominated from above by $\xi_i$. Hence,

$$P_{\lambda,d}\left(\sum_{i=1}^{\lfloor \sigma \log d \rfloor} \xi_i \geq K(d) \bigg| \hat{V}_{[\sigma \log d]} \neq \emptyset \right) \geq P_{\lambda,d}\left(\sum_{i=1}^{\lfloor \sigma \log d \rfloor} \eta_i \geq K(d) \bigg| \hat{V}_{[\sigma \log d]} \neq \emptyset \right).$$

(4.18)

For any oriented path $\vec{l}: O \to l_1 \to \ldots \to l_{[\sigma \log d]}$, we use $\gamma(\vec{l})$ to denote

$$P_{\lambda,d}\left(X_i = l_i \text{ for all } 1 \leq i \leq [\sigma \log d] \bigg| \hat{V}_{[\sigma \log d]} \neq \emptyset \right).$$

Then,

$$P_{\lambda,d}\left(\sum_{i=1}^{[\sigma \log d]} \eta_i \geq K(d) \bigg| \hat{V}_{[\sigma \log d]} \neq \emptyset \right) = \sum_{\vec{l}} \gamma(\vec{l}) P_{\lambda,d}\left(\sum_{i=1}^{[\sigma \log d]} \eta_i(\vec{l}) \geq K(d) \bigg| X_i = l_i \text{ for all } 1 \leq i \leq [\sigma \log d], \hat{V}_{[\sigma \log d]} \neq \emptyset \right),$$

where

$$\eta_i(\vec{l}) = \sum_{j=1}^{\frac{d}{N(d)}} 1_{\{U_{ij} < \gamma(l_i)\}}.$$

The condition $\{X_i = l_i \text{ for all } 1 \leq i \leq [\sigma \log d], \hat{V}_{[\sigma \log d]} \neq \emptyset\}$ in Equation (4.19) is concerned with the values of $\gamma(l_i)$ and $\{U(l_i, l_i + e_j) : 1 \leq j \leq d - \frac{d}{N(d)}\}$ for $1 \leq i \leq [\sigma \log d]$. A worse condition for $\sum_{i=1}^{[\sigma \log d]} \eta_i(\vec{l}) \geq K(d)$ to occur is that

$$\gamma(l_i) < \inf \{U(l_i, l_i + e_j) : 1 \leq j \leq d - \frac{d}{N(d)}\}$$

for all $1 \leq i \leq [\sigma \log d]$, i.e.,

$$P_{\lambda,d}\left(\sum_{i=1}^{[\sigma \log d]} \eta_i(\vec{l}) \geq K(d) \bigg| X_i = l_i \text{ for all } 1 \leq i \leq [\sigma \log d], \hat{V}_{[\sigma \log d]} \neq \emptyset \right) \geq$$

$$P_{\lambda,d}\left(\sum_{i=1}^{[\sigma \log d]} \eta_i(\vec{l}) \geq K(d) \bigg| \gamma(l_i) < \inf \{U(l_i, l_i + e_j) : 1 \leq j \leq d - \frac{d}{N(d)}\} \right)$$

for all $1 \leq i \leq [\sigma \log d]$. (4.20)

Note that $\gamma(l_i)$ is with the same distribution as that of $\gamma_i$ while $\eta_i(\vec{l})$ is with the same distribution as that of $\xi_i$. Further more, $\inf \{U(l_i, l_i + e_j) : 1 \leq j \leq d - \frac{d}{N(d)}\}$
is an exponential time with rate at most
\[(d - \lfloor \frac{d}{N(d)} \rfloor) \frac{\lambda}{d} M^2 \leq \lambda M^2,\]
which is the rate of $\Lambda_i$. Then,
\[
P_{\lambda,d} \left( \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \eta_i(\vec{l}) \geq K(d) \mid \vec{Y}(l_i) < \inf \{ \overline{U}(l_i, l_i + e_j) : 1 \leq j \leq d - \lfloor \frac{d}{N(d)} \rfloor \} \right)
\]
for all $1 \leq i \leq \lfloor \sigma \log d \rfloor$
\[
\geq P_{\lambda,d} \left( \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \xi_i \geq K(d) \mid \hat{Y}_i < \Lambda_i \text{ for all } i \leq \lfloor \sigma \log d \rfloor \right).
\]
Therefore, by Equation (4.20),
\[
P_{\lambda,d} \left( \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \eta_i(\vec{l}) \geq K(d) \mid X_i = l_i \text{ for all } 1 \leq i \leq \lfloor \sigma \log d \rfloor, \hat{V}_{\lfloor \sigma \log d \rfloor} \neq \emptyset \right)
\]
\[
\geq P_{\lambda,d} \left( \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \xi_i \geq K(d) \mid \hat{Y}_i < \Lambda_i \text{ for all } i \leq \lfloor \sigma \log d \rfloor \right).
\]
By Equations (4.19) and (4.21),
\[
P_{\lambda,d} \left( \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \eta_i(\vec{l}) \geq K(d) \right) \geq \sum_{\vec{l}} \gamma(\vec{l}) P_{\lambda,d} \left( \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \xi_i \geq K(d) \mid \hat{Y}_i < \Lambda_i \text{ for all } i \leq \lfloor \sigma \log d \rfloor \right)
\]
\[
= P_{\lambda,d} \left( \sum_{i=1}^{\lfloor \sigma \log d \rfloor} \xi_i \geq K(d) \mid \hat{Y}_i < \Lambda_i \text{ for all } i \leq \lfloor \sigma \log d \rfloor \right),
\]
since $\sum_{\vec{l}} \gamma(\vec{l}) = 1$. Equation (4.14) follows directly from Equations (4.17), (4.18) and (4.22).

Equation (4.15) follows from the following analysis. According to the assumption of independence of the exponential times,
\[
E_{\lambda,d} \left( e^{-s \frac{N(d)}{\log d} \sum_{j=1}^{\lfloor \sigma \log d \rfloor} \xi_j} \mid \hat{Y}_i < \Lambda_i \text{ for all } i \leq \lfloor \sigma \log d \rfloor \right)
\]
\[
= E_{\lambda,d} \left( e^{-s \frac{N(d)}{\log d} \lambda_1} \mid \hat{Y}_1 < \Lambda_1 \right)^{\lfloor \sigma \log d \rfloor}.
\]
Let \( Y_0 \) be an exponential time with rate one while \( \Lambda_0 \) be an exponential time with rate \( \lambda M^2 \) and independent of \( Y_0 \), then by direct calculation,

\[
E_{\lambda,d}(e^{-sN(d)/\log d} \mid \hat{Y}_1 < \Lambda_1) = E(\Xi(\hat{Y}_0, s, d) \mid \hat{Y}_0 < \Lambda_0),
\]

where

\[
\Xi(t, s, d) = \left[ E(e^{-sN(d)/(1 - e^{-\lambda s/d})} + e^{-\lambda s/d}) \right]^{\lfloor \sigma \log d \rfloor}.
\]

Then, by Equation (4.23),

\[
E_{\lambda,d}(e^{-sN(d)/\log d} \sum_{j=1}^{\lfloor \sigma \log d \rfloor} \xi_j \mid \hat{Y}_i < \Lambda_i \text{ for all } i \leq \lfloor \sigma \log d \rfloor) = E(\Xi(\hat{Y}_0, s, d) \mid \hat{Y}_0 < \Lambda_0) \quad (4.24)
\]

By direct calculation, it is not difficult to check that

\[
\lim_{d \to +\infty} |\sigma \log d| (\Xi(t, s, d) - 1) = -s \sigma \lambda t E \rho.
\]

for any \( t > 0 \). As a result,

\[
\lim_{d \to +\infty} \left[ E(\Xi(\hat{Y}_0, s, d) \mid \hat{Y}_0 < \Lambda_0) \right]^{\lfloor \sigma \log d \rfloor} = E(e^{-s\lambda \hat{Y}_0 E \rho} \mid \hat{Y}_0 < \Lambda_0) := \Theta(s). \quad (4.25)
\]

Note that here we still use the theorem of calculus that \((1 + a_d)^{c_d} \to e^c\) when \( a_d \to 0 \), \( c_d \to +\infty \) and \( a_d c_d \to c \). Equation (4.15) follows directly from Equations (4.24) and (4.25).

\[\square\]

### 4.4 Proof of Lemma 4.3

In this subsection we give the proof of Lemma 4.3. The proof is inspired a lot by the approach introduced in [8]. First we introduce some definitions and notations. We let \( \{\vartheta_n\}_{n \geq 0} \) be the oriented random walk on \( \mathbb{Z}_d^+ \) such that

\[
P(\vartheta_{n+1} - \vartheta_n = e_i) = \frac{1}{d}
\]

for each \( n \geq 0 \) and \( 1 \leq i \leq d \). We let \( \{\nu_n\}_{n \geq 0} \) be an independent copy of \( \{\vartheta_n\}_{n \geq 0} \). From now on, we denote by \( \mathbb{P} \) the probability measure of \( \{\vartheta_n\}_{n \geq 0} \) and \( \{\nu_n\}_{n \geq 0} \) while denote by \( \mathbb{E} \) the expectation with respect to \( \mathbb{P} \). When we need to point out the dimension \( d \) of the lattice, we write \( \mathbb{P}_d \) and \( \mathbb{E}_d \). We write \( \vartheta_n \) (resp.
\(\nu_n\) as \(\vartheta_n^\nu\) (resp. \(\nu_n^\nu\)) when \(\vartheta_0 = x\) (resp. \(\nu_0 = x\)). For \(x, y \in \mathbb{Z}_+^d\) that \(x \neq y\) and \(|x| \leq |y|\), we define
\[
\tau_{x,y} = \inf \left\{ k \geq |y| - |x| : \vartheta_k^x = \nu_{k-|y|+|x|}^y \right\}.
\]
That is to say, \(\tau_{x,y}\) is the first moment when \(\{\vartheta_n^x\}_{n\geq0}\) visits some vertex on the path of \(\{\nu_n^y\}_{n\geq0}\). Note that
\[
|\vartheta_n| = |\vartheta_0| + n \text{ and } |\nu_n| = |\nu_0| + n
\]
according to the definition of the oriented random walk, hence
\[
k = l + |y| - |x|
\]
when \(\vartheta_k^x = \nu_l^y\) for some \(k, l\). For \(x, y \in \mathbb{Z}_+^d\) that \(|x| \leq |y|\), we introduce the following random variables. We define
\[
\tau_{0,x}^{x,y} = \begin{cases} 0 & \text{if } x = y, \\ \tau_{x,y} & \text{if } x \neq y. \end{cases}
\]
We let
\[
\tau_{1,x}^{x,y} = \inf \left\{ n \geq \tau_{0,x}^{x,y} : \vartheta_n^x = \nu_{n-|y|+|x|}^y, \vartheta_{n+1}^x = \nu_{n-|y|+|x|}^y \right\},
\]
\[
\kappa_{1,x}^{x,y} = \inf \left\{ n > \tau_{1,x}^{x,y} : \vartheta_n^x = \nu_{n-|y|+|x|}^y, \vartheta_{n+1}^x \neq \nu_{n-|y|+|x|}^y \right\},
\]
\[
\tau_{2,x}^{x,y} = \inf \left\{ n > \kappa_{1,x}^{x,y} : \vartheta_n^x = \nu_{n-|y|+|x|}^y, \vartheta_{n+1}^x = \nu_{n-|y|+|x|}^y \right\},
\]
\[
\kappa_{2,x}^{x,y} = \inf \left\{ n > \tau_{2,x}^{x,y} : \vartheta_n^x = \nu_{n-|y|+|x|}^y, \vartheta_{n+1}^x \neq \nu_{n-|y|+|x|}^y \right\},
\]
\[
\ldots 
\]
\[
\tau_{l,x}^{x,y} = \inf \left\{ n > \kappa_{l-1,x}^{x,y} : \vartheta_n^x = \nu_{n-|y|+|x|}^y, \vartheta_{n+1}^x = \nu_{n-|y|+|x|}^y \right\},
\]
\[
\kappa_{l,x}^{x,y} = \inf \left\{ n > \tau_{l,x}^{x,y} : \vartheta_n^x = \nu_{n-|y|+|x|}^y, \vartheta_{n+1}^x \neq \nu_{n-|y|+|x|}^y \right\},
\]
\[
\ldots 
\]
That is to say, \(\tau_{1,x}^{x,y}\) is the first moment \(n\) such that \(\vartheta_n^x = \nu_{n-|y|+|x|}^y\) and \(\vartheta_{n+1}^x = \nu_{n-|y|+|x|}^y\). For \(l \geq 1\), \(\kappa_{l,x}^{x,y}\) is the first moment \(n\) after \(\tau_{l,x}^{x,y}\) such that \(\vartheta_n^x = \nu_{n-|y|+|x|}^y\) and \(\vartheta_{n+1}^x \neq \nu_{n-|y|+|x|}^y\) while \(\tau_{l+1,x}^{x,y}\) is the first moment \(n\) after \(\kappa_{l,x}^{x,y}\) that
\[
\vartheta_n^x = \nu_{n-|y|+|x|}^y \text{ and } \vartheta_{n+1}^x = \nu_{n+1-|y|+|x|}^y.
\]
We define
\[
T(x, y) = \begin{cases} 
\sup \{ l \geq 0 : \tau_{l,x}^{x,y} < +\infty \} & \text{if } \tau_{0,x}^{x,y} < +\infty, \\
0 & \text{if } \tau_{0,x}^{x,y} = +\infty.
\end{cases}
\]
In this subsection we assume that \(d \geq 4\) such that \(T(x, y) < +\infty\) with probability one according to the conclusion given in [2] about the collision times of two independent oriented random walks.
For $1 \leq l \leq T(x, y)$, we define
\[ h_i^{x,y} = \kappa_i^{x,y} - r_i^{x,y}. \]
We let
\[
\begin{align*}
  f_0^{x,y} &= \left\{ \tau_0^{x,y} \leq n < \tau_1^{x,y} : \vartheta_n^x = \nu_n^{y} - \|y\| + \|x\| \right\}, \\
  f_1^{x,y} &= \left\{ \kappa_1^{x,y} < n < \tau_2^{x,y} : \vartheta_n^x = \nu_n^{y} - \|y\| + \|x\| \right\}, \\
  \ldots & \\
  f_T^{x,y} &= \left\{ \kappa_T^{x,y} < n < \tau_{T(x,y)}^{x,y} : \vartheta_n^x = \nu_n^{y} - \|y\| + \|x\| \right\}, \\
\end{align*}
\]
where $|A|$ is the cardinality of the set $A$ as we have introduced. Then, for $x, y$ that $\|x\| \leq \|y\|$, we define
\[
R(x, y) = \frac{T(x, y) + \sum_{i=0}^{T(x, y) - 1} f_i^{x,y} \sum_{i=0}^{T(x, y)} h_i^{x,y} + 4 \sum_{i=0}^{T(x, y)} f_i^{x,y}}{2 \sum_{i=0}^{T(x, y)} f_i^{x,y}} \left( \frac{\lambda E(\rho^2)}{d} \right) \sum_{i=1}^{T(x, y)} h_i^{x,y} \left( \frac{E(\rho^2)}{d} \right) 3 \sum_{i=1}^{T(x, y)} f_i(x, y) \\
\] when $\|y\| - \|x\| > \|y\|$ while define
\[
R(x, y) = \frac{T(x, y) + \sum_{i=0}^{T(x, y) - 1} f_i^{x,y} \sum_{i=0}^{T(x, y)} h_i^{x,y} + 4 \sum_{i=0}^{T(x, y)} f_i^{x,y} - 1}{2 \sum_{i=0}^{T(x, y)} f_i^{x,y} - 1} \left( \frac{\lambda E(\rho^2)}{d} \right) \sum_{i=1}^{T(x, y)} h_i^{x,y} \left( \frac{E(\rho^2)}{d} \right) 3 \sum_{i=1}^{T(x, y)} f_i(x, y) - 1 \left( E\rho \right) \\
\]
when $\|y\| - \|x\| = \tau_0^{x,y} = \tau_1^{x,y}$ and define
\[
R(x, y) = \frac{T(x, y) + \sum_{i=0}^{T(x, y) - 1} f_i^{x,y} \sum_{i=0}^{T(x, y)} h_i^{x,y} + 4 \sum_{i=0}^{T(x, y)} f_i^{x,y} - 1}{2 \sum_{i=0}^{T(x, y)} f_i^{x,y} - 1} \left( \frac{\lambda E(\rho^2)}{d} \right) \sum_{i=1}^{T(x, y)} h_i^{x,y} \left( \frac{E(\rho^2)}{d} \right) 3 \sum_{i=1}^{T(x, y)} f_i(x, y) - 1 \left( E\rho \right) \\
\]
when $\|y\| - \|x\| = \tau_0^{x,y} < \tau_1^{x,y}$. For $x, y$ that $\|x\| > \|y\|$, we define
\[
\tau_i^{x,y}, \kappa_i^{x,y}, h_i^{x,y}, T(x, y), f_i^{x,y}, R(x, y) \text{ as } \tau_i^{y,x}, \kappa_i^{y,x}, h_i^{y,x}, T(y, x), f_i^{y,x}, R(y, x). \\
\]
The following three lemmas is crucial for us to prove Lemma 16.
Lemma 4.7. There exists \( c_1 > 0 \) which does not depend on \( d \) such that
\[
P_d(\tau_{x,y} < +\infty) \leq \frac{c_1}{\sqrt{d}}
\]
for any \( d \geq 4 \), \( x, y \in \mathbb{Z}_d^+ \), \( x \neq y \).

Lemma 4.8. For given \( \lambda > \frac{1}{E(\rho^2)} \), there exists \( d_0 \geq 4 \) and \( c_2 > 0 \) which does not depend on \( d \) such that
\[
E_d\left(R(x,y) \mid \tau(x,y) < +\infty\right) \leq c_2
\]
for any \( d \geq d_0 \), \( x, y \in \mathbb{Z}_d^+ \).

Lemma 4.9. For \( A \subseteq \mathbb{Z}_d^+ \),
\[
P_{\lambda,d}\left(I^A_t \neq \emptyset, \forall t \geq 0\right) \geq \frac{|A|^2}{|A|^2 - |A|} \left(1 + \frac{c_1c_2}{\sqrt{d}}\right) + |A|c_2
\]
for any \( d \geq d_0 \), \( A \subseteq \mathbb{Z}_d^+ \).

The proofs of Lemmas 4.7-4.9 will be given later. Now we give the proof of Lemma 4.3.

Proof of Lemma 4.3. For \( x \neq y \), according to the definition of \( R(x,y)\), \( R(x,y) = 1 \) when \( \tau_{x,y} = +\infty \). Therefore, by Lemmas 4.7 and 4.8,
\[
E_d(R(x,y)) = \mathbb{P}_d(\tau_{x,y} = +\infty) + E_d(R(x,y)1_{\{\tau_{x,y} < +\infty\}}) \leq 1 + \frac{c_1c_2}{\sqrt{d}}
\]
for any \( d \geq d_0 \), \( x, y \in \mathbb{Z}_d^+ \). By Lemma 4.8 and Equation (4.26),
\[
\sum_{x \in A} \sum_{y \in A} E_d(R(x,y)) \leq |A|c_2 + (|A|^2 - |A|)\left(1 + \frac{c_1c_2}{\sqrt{d}}\right).
\]
By Lemma 4.9 and Equation (4.27),
\[
P_{\lambda,d}\left(I^A_t \neq \emptyset, \forall t \geq 0\right) \geq \frac{|A|^2}{(|A|^2 - |A|)\left(1 + \frac{c_1c_2}{\sqrt{d}}\right) + |A|c_2}
\]
for any \( d \geq d_0 \), \( A \subseteq \mathbb{Z}_d^+ \).

Let \( m(d) = \frac{|K(d)|^2}{(|K(d)|^2 - |K(d)|)\left(1 + \frac{c_1c_2}{\sqrt{d}}\right) + |K(d)|c_2} \), then \( \lim_{d \to +\infty} m(d) = 1 \) while
\[
P_{\lambda,d}\left(I^A_t \neq \emptyset, \forall t \geq 0\right) \geq m(d)
\]
for any \( A \subseteq \mathbb{Z}_d^+ \), \( |A| = K(d) \) by Equation (4.28) and the proof is complete. \( \square \)

Now we give the proof of Lemma 4.7.
Proof of Lemma 4.7. Let
\[ \tau_{O,O} = \inf\{ n \geq 1 : \vartheta_n^O = \nu_n^O \}, \]
then by the conclusion given in [2], there exists \( c_3 > 0 \) which does not depend on \( d \) that
\[ P_d(\tau_{O,O} < +\infty) \leq \frac{1}{d} + \frac{c_3}{d^2} \] (4.29)
for all \( d \geq 4 \). Since \( P_d(\tau_{O,O} = 1) = \frac{1}{d} \), according to the spatial homogeneity of \( \mathbb{Z}_+^d \),
\[ (1 - \frac{1}{d}) P_d(\tau_{i,j} < +\infty) = P_d(2 \leq \tau_{O,O} < +\infty) \leq \frac{c_3}{d^2} \] (4.30)
for any \( d \geq 4, 1 \leq i < j \leq d \). For \( x, y \) that \( x \neq y \) and \( \|x\| = \|y\|, \|x - y\| \) is an even at least two. Let
\[ \check{\tau}_{x,y} = \inf\{ n \geq 0 : \|\vartheta_n^x - \nu_n^y\| = 2 \}, \]
then, according to the strong Markov property,
\[ P_d(\tau_{x,y} < +\infty) = P_d(\check{\tau}_{x,y} < +\infty) P_d(\tau_{e_1,e_2} < +\infty) \leq \frac{c_3}{d(d-1)}, \] (4.31)
since \( \|\vartheta_{n+1}^x - \nu_{n+1}^y\| - \|\vartheta_n^x - \nu_n^y\| \in \{0, -2, 2\} \) for each \( n \).
For \( x, y \) that \( \|x\| < \|y\| \),
\[ P_d(\tau_{x,y} < +\infty) = P_d(\vartheta_{\|y\| - \|x\|}^x = \|y\|) + P_d(\tau_{x,y} < +\infty | \vartheta_{\|y\| - \|x\|}^x \neq \|y\|) P_d(\vartheta_{\|y\| - \|x\|}^x \neq \|y\|) \]
\[ \leq P_d(\vartheta_{\|y\| - \|x\|}^x = \|y\|) + P_d(\tau_{x,y} < +\infty | \vartheta_{\|y\| - \|x\|}^x \neq \|y\|) \] (4.32)
Since \( \|\vartheta_{\|y\| - \|x\|}^x\| = \|y\| \), by Equation (4.31) and the strong Markov property,
\[ P_d(\tau_{x,y} < +\infty | \vartheta_{\|y\| - \|x\|}^x \neq \|y\|) \leq \frac{c_3}{d(d-1)} \] (4.33)
By Equation (4.29), there exists \( c_4 > 0 \) which does not depend on \( d \) that
\[ P_d(\vartheta_{\|y\| - \|x\|}^x = \|y\|)^2 = P_d(\vartheta_{\|y\| - \|x\|}^x = \|y\|) P_d(\nu_{\|y\| - \|x\|}^y = \|y\|) \]
\[ \leq P_d(\vartheta_{\|y\| - \|x\|}^x = \|y\|) P_d(\vartheta_{\|y\| - \|x\|}^O = \nu_{\|y\| - \|x\|}^O) \]
\[ \leq P_d(\tau_{O,O} < +\infty) \leq \frac{c_4}{d} \]
As a result,
\[ P_d(\vartheta_{\|y\| - \|x\|}^y = \|y\|) \leq \frac{\sqrt{c_4}}{\sqrt{d}} \] (4.34)
By Equations (4.32), (4.33) and (4.34),
\[ P_d(\tau_{x,y} < +\infty) \leq \frac{\sqrt{c_4}}{\sqrt{d}} + \frac{c_3}{d(d-1)} \] (4.35)
for any \( d \geq 4, x, y \in \mathbb{Z}_d^+ \) that \( \|x\| < \|y\| \). Lemma 4.7 follows directly from Equations (4.31) and (4.35).

Now we give the proof of Lemma 4.8.

**Proof of Lemma 4.8** According to the definition of \( R(\cdot, \cdot) \), for any \( x, y \in \mathbb{Z}_d^+ \),

\[
R(x, y) \leq \frac{c_5^2}{d^2} T(x, y) + \sum_{i=0}^{T(x, y)} f_i \rho_i \left( 1 + \frac{\lambda M^2}{d} \right) T(x, y) + 2 \sum_{i=1}^{T(x, y)} \rho_i \left( \sum_{i=0}^{T(x, y)} f_i + 4 \rho_i \right) M \left( 6T(x, y) + 4 \sum_{i=0}^{T(x, y)} f_i \right),
\]

where \( c_5 > 0 \) is a constant which depends on \( M, E(\rho^2), E\rho, \lambda \) and does not depend on \( d \). According to the strong Markov property, for any positive integers \( T, \{h_i\}_{i=1}, \{f_i\}_{i=0} \),

\[
P_d\left( T(x, y) = T, h_i = h_i, f_i = f_i \quad \text{for} \quad 0 \leq i \leq T \quad \text{and} \quad \tau_{x,y} < +\infty \right) \leq \sum_{i=0}^{T} f_i \sum_{i=0}^{T} h_i \sum_{i=1}^{T} \rho_i \sum_{i=0}^{T} f_i \sum_{i=0}^{T} h_i.
\]

By Equations (4.36) and (4.37),

\[
P_d\left( T(x, y) = T, h_i, f_i = f_i \quad \text{for} \quad 0 \leq i \leq T \quad \text{and} \quad \tau_{x,y} < +\infty \right) \leq \left( \frac{c_3}{d^2} \right)^{T+1} \left( \frac{1}{d} \right)^{T} \sum_{i=1}^{T} h_i.
\]

By Equations (4.39) and (4.38),

\[
E_d\left( R(x, y) \bigg| \tau_{x,y} < +\infty \right) \leq c_5 \sum_{T=0}^{+\infty} \sum_{f_0=1}^{+\infty} \sum_{f_1=1}^{+\infty} \cdots \sum_{f_T=1}^{+\infty} \sum_{h_T=1}^{+\infty} \left( \frac{c_3}{d^2} \right)^{T} \sum_{i=0}^{T} f_i + 2 \frac{\lambda M^2}{d} \sum_{f_i=0}^{T} h_i + 4 \frac{\lambda M^2}{d} \sum_{f_i=0}^{T} f_i \sum_{i=0}^{T} h_i
\]

\[
\times 2 \frac{T+2}{d^2} \sum_{i=0}^{T} f_i \left( 1 + \frac{\lambda M^2}{d} \right) T \sum_{i=0}^{T} f_i \sum_{i=0}^{T} h_i \sum_{i=0}^{T} \rho_i \sum_{i=0}^{T} f_i \sum_{i=0}^{T} h_i \sum_{i=0}^{T} \rho_i \sum_{i=0}^{T} f_i \sum_{i=0}^{T} h_i
\]

\[
= \tilde{c}_5 \sum_{T=0}^{+\infty} \left( \frac{c_7(d)}{d^2} \right)^{T} \sum_{f_0=1}^{+\infty} \sum_{f_1=1}^{+\infty} \cdots \sum_{f_T=1}^{+\infty} \left( \frac{c_8(d)}{d^2} \right)^{T} \sum_{i=1}^{+\infty} f_i
\]

\[
\sum_{h_1=1}^{+\infty} \cdots \sum_{h_T=1}^{+\infty} \left( c_9(d) \right)^{T} \sum_{i=1}^{+\infty} \rho_i \sum_{i=0}^{T} f_i \sum_{i=0}^{T} h_i \sum_{i=0}^{T} \rho_i \sum_{i=0}^{T} f_i \sum_{i=0}^{T} h_i \sum_{i=0}^{T} \rho_i \sum_{i=0}^{T} f_i \sum_{i=0}^{T} h_i,
\]

\[26\]
where \( \hat{c}_5 = \frac{2c_5(1 + \frac{\lambda M^2}{d})^4M^4}{(E(\rho^2))^2} \), \( c_7(d) = \frac{2c_3(1 + \frac{\lambda M^2}{d})^4M^6}{d^2(E(\rho^2))^3} \), \( c_8(d) = \frac{2c_3(1 + \frac{\lambda M^2}{d})^4M^4}{d(E(\rho^2))^3} \) and
\[
\hat{c}_5 < \frac{3M^4c_5}{(E(\rho^2))^2} \text{ and } c_9(d) \leq c_{10}
\]
for sufficiently large \( d \). For sufficiently large \( d \),
\[
c_8(d) \leq \frac{1}{2} \text{ and } \frac{c_7(d)c_8(d)}{1 - c_8(d)} \frac{c_{10}}{1 - c_{10}} \leq \frac{1}{10}
\]
since \( \lim_{d \to +\infty} c_7(d) = \lim_{d \to +\infty} c_8(d) = 0 \). As a result, by Equation (4.39),
\[
\mathbb{E}_d\left( R(x, y) \right| \tau_{x, y} < +\infty \right) \leq \frac{1}{1 - c_8(d)} \frac{3M^4c_5}{(E(\rho^2))^2} \sum_{T=0}^{+\infty} \left( c_7(d) \frac{c_8(d)}{1 - c_8(d)} \frac{c_{10}}{1 - c_{10}} \right)^T
\]
for sufficiently large \( d \). Let
\[
c_2 = \frac{20M^4c_5}{3(E(\rho^2))^2}
\]
and the proof is complete.

At the end of this subsection, we give the proof of Lemma 4.9.

Proof of Lemma 4.9. For each \( m \geq 1 \) and each \( x \in \mathbb{Z}^d_+ \), we define
\[
L_m(x) = \{ \bar{x} = (x_0, x_1, \ldots, x_m) : x_0 = x, x_i \to x_{i+1} \text{ for all } 0 \leq i \leq m - 1 \}
\]
as the set of oriented paths starting at \( x \) with length \( m \).

For each \( \bar{x} = (x, x_1, \ldots, x_m) \in L_m(x) \), we denote by \( \pi_{\bar{x}} \) the event that \( \bar{U}(x_i, x_{i+1}) < \bar{Y}(x_i) \) for all \( 0 \leq i \leq m - 1 \), then for each \( x \in \mathbb{Z}^d_+ \),
\[
P_{\lambda, d}(\pi_{\bar{x}}) = E_{\mu_d}\left( \prod_{i=0}^{m-1} P_{\lambda, \omega}(\bar{U}(x_i, x_{i+1}) < \bar{Y}(x_i)) \right)
\]
\[
= E_{\mu_d}\left( \prod_{i=0}^{m-1} \frac{\lambda \rho(x_i)\rho(x_{i+1})}{1 + \frac{\lambda}{\rho(x)}\rho(x_{i+1})} \right)
\]
\[
= E\left( \prod_{i=0}^{m-1} \frac{\lambda \rho_i\rho_{i+1}}{1 + \frac{\lambda}{\rho_i}\rho_{i+1}} \right),
\]
(4.40)
where $\rho_0, \ldots, \rho_m$ are independent copies of $\rho$. For $x, y$ that $\|x\| \leq \|y\|$, $m \geq \|y\| - \|x\|$ and $\tilde{x} = (x, x_1, \ldots, x_m) \in L_m(x), \tilde{y} = (y, y_1, \ldots, y_m) \in L_m(y),$

$$P_{\lambda,d}(\pi_{\tilde{x}} \cap \pi_{\tilde{y}}) = E_{\lambda,d} \left( \prod_{i=0}^{m-\|y\|+\|x\|} P_{\lambda,\omega}(\tilde{U}(x_i, x_{i+1}) < \tilde{Y}(x_i)) \right) \quad (4.41)$$

$$\times \prod_{t=1}^{m-\|y\|+\|x\|} \prod_{j=m-\|y\|+\|x\|}^{m-1} G(x_t-\|y\|+\|x\|-1, y_{t-1}; x_t-\|y\|+\|x\|, y_t) \quad (4.42)$$

where

$$G(x, y; u, v) = P_{\lambda,\omega}(\tilde{U}(x, u) < \tilde{Y}(x), \tilde{U}(y, v) < \tilde{Y}(y))$$

for $x \to u$ and $y \to v$.

By direct calculation, for $x, y$ that $\|x\| = \|y\|,$

$$G(x, y; u, v) = \left\{ \begin{array}{ll}
\frac{32}{\lambda^2} (1+\frac{2}{\lambda} \rho(x) \rho(u))^{\lambda^2} & \text{if } x = y \text{ and } u = v, \\
\frac{32}{\lambda^2} \rho(x) \rho(u) & \text{if } x = y \text{ and } u \neq v,
\end{array} \right. \quad (4.43)$$

According to the definition of the SIR model, for given $A \subseteq \mathbb{Z}^d_+$, $m \geq 1$ and $x \in A$, if $\pi_{\tilde{x}}$ occurs for some $\tilde{x} = (x, x_1, \ldots, x_m) \in L_m(x)$, then

$$x_m \in \bigcup_{t \geq 0} I_t^A.$$ 

As a result, on the event $\bigcap_{m=1}^{+\infty} \bigcup_{x \in A} \bigcup_{\tilde{x} \in L_m(x)} \pi_{\tilde{x}},$ there are infinite many vertices have ever been infected and hence

$$P_{\lambda,d} (I_t^A \neq \emptyset, \forall t \geq 0) \geq P_{\lambda,d} \left( \bigcap_{m=1}^{+\infty} \bigcup_{x \in A} \bigcup_{\tilde{x} \in L_m(x)} \pi_{\tilde{x}} \right). \quad (4.44)$$

We use $\chi_{\tilde{x}}$ to denote the indicator function of $\pi_{\tilde{x}},$ then by the Cauchy-Schwartz’s
inequality and the dominated convergence theorem,

\[
P_{\lambda,d} \left( \bigcap_{m=1}^{+\infty} \bigcup_{x \in A} \bigcup_{\bar{x} \in L_m(x)} \pi_{\bar{x}} \right) \\
\geq \lim_{m \to +\infty} P_{\lambda,d} \left( \bigcup_{x \in A} \bigcup_{\bar{x} \in L_m(x)} \pi_{\bar{x}} \right) \\
= \lim_{m \to +\infty} P_{\lambda,d} \left( \sum_{x \in A} \sum_{\bar{x} \in L_m(x)} \chi_{\bar{x}} > 0 \right) \\
\geq \limsup_{m \to +\infty} \frac{\left[ E_{\lambda,d} \left( \sum_{x \in A} \sum_{\bar{x} \in L_m(x)} \chi_{\bar{x}} \right) \right]^2}{\sum_{x \in A} \sum_{\bar{x} \in L_m(x)} \sum_{y \in L_m(y)} P_{\lambda,d} \left( \pi_{\bar{x}} \cap \pi_y \right)}.
\]

By Equation (4.40), for given \( m \geq 1 \) and \( \bar{x} \in L_m(x) \), \( P_{\lambda,d}(\pi_{\bar{x}}) \) does not depend on the choice of \( \bar{x} \) and \( x \). Therefore, according to the fact that \( |L_m(x)| = d^m \),

\[
P_{\lambda,d} \left( \bigcap_{m=1}^{+\infty} \bigcup_{x \in A} \bigcup_{\bar{x} \in L_m(x)} \pi_{\bar{x}} \right) \geq \liminf_{m \to +\infty} \frac{1}{|A|^2} \sum_{x \in A} \sum_{y \in A} \sum_{\bar{x} \in L_m(x)} \sum_{\bar{y} \in L_m(y)} \frac{1}{d^m} P_{\lambda,d} \left( \pi_{\bar{x}} \cap \pi_y \right) P_{\lambda,d} \left( \pi_{\bar{x}} \cap \pi_y \right).
\]

We use \( \vartheta_x^m \) to denote the random path \( (\vartheta_1^x, \ldots, \vartheta_m^x) \) while use \( \vartheta_y^m \) to denote the path \( (\nu_1^y, \ldots, \nu_m^y) \), then by Equation (4.44),

\[
P_{\lambda,d} \left( \bigcap_{m=1}^{+\infty} \bigcup_{x \in A} \bigcup_{\bar{x} \in L_m(x)} \pi_{\bar{x}} \right) \geq \liminf_{m \to +\infty} \frac{1}{|A|^2} \sum_{x \in A} \sum_{y \in A} \sum_{\bar{x} \in L_m(x)} \sum_{\bar{y} \in L_m(y)} \frac{1}{d^m} P_{\lambda,d}(\pi_{\vartheta_x^m} \cap \pi_{\vartheta_y^m}) P_{\lambda,d}(\pi_{\vartheta_x^m} \cap \pi_{\vartheta_y^m}).
\]

For \( m \) sufficiently large and \( x, y \) that \( \|x\| \leq \|y\| \), we bound \( P_{\lambda,d}(\pi_{\vartheta_x^m} \cap \pi_{\vartheta_y^m}) \) from above according to the following procedure. For the denominator

\[
P_{\lambda,d}(\pi_{\vartheta_x^m}) P_{\lambda,d}(\pi_{\vartheta_y^m}) = \mu_d \left( \prod_{i=0}^{m-1} \frac{\lambda \rho(\vartheta_i) \rho(\vartheta_{i+1})}{1 + \lambda \rho(\vartheta_i) \rho(\vartheta_{i+1})} \right) \mu_d \left( \prod_{i=0}^{m-1} \frac{\lambda \rho(\nu_i) \rho(\nu_{i+1})}{1 + \lambda \rho(\nu_i) \rho(\nu_{i+1})} \right),
\]

29
if \( l \geq ||y|| - ||x|| \) satisfies that \( \vartheta_t^T = \nu_{l-||y||+||x||}^y \), then

\[
\begin{align*}
&\frac{1}{1+\frac{M}{\nu}} \frac{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)} \geq \frac{1}{1+\frac{M}{\nu}} \frac{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}, \\
&\frac{1}{1+\frac{M}{\nu}} \frac{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)} \geq \frac{1}{1+\frac{M}{\nu}} \frac{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}, \\
&\frac{1}{1+\frac{M}{\nu}} \frac{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)} \geq \frac{1}{1+\frac{M}{\nu}} \frac{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}, \\
&\frac{1}{1+\frac{M}{\nu}} \frac{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)} \geq \frac{1}{1+\frac{M}{\nu}} \frac{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}{\rho(\vartheta_{l+1})\rho(\vartheta_l^T)}.
\end{align*}
\]

For the numerator \( P_{\lambda,d}(\pi_{\varphi_{m_1}} \cap \pi_{\varphi_{m_2}}) \) with expression given by Equation (4.41),

\[
\begin{align*}
&G(\vartheta_{l+1}^{\nu}, \nu_{l-||y||+|x|}^y; \vartheta_l^{\nu}, \nu_{l-||y||+|x|+1}^y) \leq \frac{1}{2} \rho(\vartheta_l^{\nu})\rho(\vartheta_{l+1}^{\nu}) \\
&\text{if } \vartheta_{l+1}^{\nu} \neq \nu_{l-||y||+|x|}^y \text{ and } \vartheta_{l+1}^{\nu} \neq \nu_{l-||y||+|x|+1}^y, \\
&G(\vartheta_{l+1}^{\nu}, \nu_{l-||y||+|x|}^y; \vartheta_l^{\nu}, \nu_{l-||y||+|x|+1}^y) \leq \frac{2}{\nu} \rho(\vartheta_l^{\nu})\rho(\vartheta_{l+1}^{\nu})\rho(\nu_{l-||y||+|x|+1}^{\nu}), \\
&\text{if } \vartheta_{l+1}^{\nu} \neq \nu_{l-||y||+|x|}^y \text{ and } \vartheta_{l+1}^{\nu} \neq \nu_{l-||y||+|x|+1}^y, \\
&G(\vartheta_{l+1}^{\nu}, \nu_{l-||y||+|x|}^y; \vartheta_l^{\nu}, \nu_{l-||y||+|x|+1}^y) \leq \frac{2}{\nu} \rho(\vartheta_l^{\nu})\rho(\nu_{l-||y||+|x|}^y)\rho(\nu_{l-||y||+|x|+1}^{\nu}), \\
&\text{if } \vartheta_{l+1}^{\nu} \neq \nu_{l-||y||+|x|}^y \text{ and } \vartheta_{l+1}^{\nu} \neq \nu_{l-||y||+|x|+1}^y.
\end{align*}
\]

According to the aforesaid inequalities, \( \frac{P_{\lambda,d}(\pi_{\varphi_{m_1}} \cap \pi_{\varphi_{m_2}})}{P_{\lambda,d}(\pi_{\varphi_{m_1}})P_{\lambda,d}(\pi_{\varphi_{m_2}})} \) is bounded from above by an upper bound \( R_m(x, y) \). According to our assumption of the independence between the exponential times, the expression of \( R_m(x, y) \) can be simplified by canceling common factors in the numerator and denominator. For example, if \( l < k \) that \( \vartheta_l^{\nu} = \nu_{l-||y||+|x|}^y \) and \( \vartheta_k^{\nu} = \nu_{k-||y||+|x|}^y \) while \( \vartheta_l^{\nu} \neq \nu_{j-||y||+|x|}^y \) for any \( l < j < k \), then both the numerator and denominator have the factor

\[
\left( \frac{E(\prod_{i=1}^{k-l-1} \nu_{l}^{2}(1+\rho_{l}\rho_{l+1}))}{\prod_{i=1}^{k-l-2}(1+\rho_{l}\rho_{l+1})} \right)^2
\]

that can be canceled, where \( \rho_1, \ldots, \rho_{k-l-1} \) are independent copies of \( \rho \). As a result, it is not difficult to check that

\[
\lim_{m \to +\infty} R_m(x, y) = R(x, y)
\]

and hence

\[
P_{\lambda,d}(\bigcap_{m=1}^{+\infty} \bigcup_{x \in A} \bigcup_{x \in L_m(x)} \pi x) \geq \lim_{m \to +\infty} \frac{1}{|A|^2} \sum_{x \in A} \sum_{y \in A} E_d(R_m(x, y))
\]

\[
= \frac{1}{|A|^2} \sum_{x \in A} \sum_{y \in A} E_d(R(x, y))
\]

according to Equation (4.45). Lemma 4.9 follows directly from Equations (4.43) and (4.46).
5 Proof of Equation (2.4)

In this section we give the proof of Equation (2.4). We still assume that the vertex weight $\rho$ satisfies (4.1). The assumption is without loss of generality according to the following analysis. For general $\rho$ not satisfying (4.1), we let

$$\hat{\rho}_m = \begin{cases} \rho & \text{if } \rho \geq \frac{1}{m}, \\ \frac{1}{m} & \text{if } \rho < \frac{1}{m}, \end{cases}$$

then $\hat{\rho}_m \geq \rho$ and $\lim_{m \to +\infty} \hat{\rho}_m = \rho$. Therefore,

$$P_{\lambda, d, \rho}(C^O_t \neq \emptyset, \forall t \geq 0) \leq P_{\lambda, d, \hat{\rho}_m}(C^O_t \neq \emptyset, \forall t \geq 0).$$

If Equation (2.4) holds under assumption (4.1), which $\hat{\rho}_m$ satisfies, then

$$\limsup_{d \to +\infty} P_{\lambda, d, \rho}(C^O_t \neq \emptyset, \forall t \geq 0) \leq \limsup_{d \to +\infty} P_{\lambda, d, \hat{\rho}_m}(C^O_t \neq \emptyset, \forall t \geq 0) \leq E\left(\frac{\lambda \hat{\rho}_m \theta_m}{1 + \lambda \hat{\rho}_m \theta_m}\right),$$

where $\hat{\theta}_m$ satisfies

$$E\left(\frac{\lambda \hat{\rho}_m^2}{1 + \lambda \hat{\rho}_m \theta_m}\right) = 1$$

and it is easy to check that $\lim_{m \to +\infty} \hat{\theta}_m = \theta$. Let $m \to +\infty$, then Equation (2.4) holds for general $\rho$.

For each $n \geq 0$, we define

$$\beta_n = \{x \in \mathbb{Z}^d_+: \|x\| = n \text{ and } x \in \bigcup_{t \geq 0} C^O_t\}$$

as the vertices that with $l_1$ norm $n$ and have ever been infected with respect to the contact process with $O$ as the unique initially infected vertex.

Since each infected vertex waits for an exponential with rate one to become healthy, the infected vertices never die out when and only when there are infinite many vertices that have ever been infected. Furthermore, since $x$ infects $y$ only if $x \to y$,

$$\{C^O_t \neq \emptyset, \forall t \geq 0\} = \{\beta_n \neq \emptyset \text{ for all } n \geq 0\}.$$  \hspace{1cm} (5.1)

The proof of Equation (2.4) relies heavily on Equation (5.1) and the following two lemmas.

Lemma 5.1. Let $\{W_n\}_{n \geq 0}$ be the branching process with random vertex weights defined as in Section 3 and $\sigma \in (0, \frac{1}{10 \log(\lambda M^2)})$, then

$$\liminf_{d \to +\infty} \widehat{P}_{\lambda, d}(W_{\lfloor \sigma \log d \rfloor} = \emptyset) \geq E\left(\frac{1}{1 + \lambda \rho \theta}\right),$$

where $\widehat{P}_{\lambda, d}$ is the annealed measure of the branching process defined as in Section 3.
Lemma 5.2. Let \( \{V_n\}_{n \geq 0} \) be defined as in Section 4 and \( \sigma \in (0, \frac{1}{10 \log(\lambda M^2)}) \), then

\[
\lim_{d \to +\infty} \left[ P_{\lambda,d}(\beta_{\lfloor \sigma \log d \rfloor} = \emptyset) - P_{\lambda,d}(V_{\lfloor \sigma \log d \rfloor} = \emptyset) \right] = 0.
\]

The proof of Lemma 5.1 is given in Subsection 5.1. The core idea of the proof is to show that the branching process survives with high probability conditioned on \( W_{\lfloor \sigma \log d \rfloor} \neq \emptyset \). The proof of Lemma 5.2 is given in Subsection 5.2. The core idea of the proof is to construct a coupling of \( \{\beta_n\}_{n \geq 0} \) and \( \{V_n\}_{n \geq 0} \) such that \( \beta_{\lfloor \sigma \log d \rfloor} = V_{\lfloor \sigma \log d \rfloor} \) with high probability. Now we show how to utilize Lemmas 5.1 and 5.2 to prove Equation (2.4).

Proof of Equation (2.4). We couple \( \{W_n\}_{n \geq 0} \) and \( \{V_n\}_{n \geq 0} \) under the same probability space as what we have done in Subsection 4.1. Recalling that we define \( B(d) \) as the event that the coupling is successful at step \( m \) for all \( m \leq \lfloor \sigma \log d \rfloor \), then

\( V_{\lfloor \sigma \log d \rfloor} = W_{\lfloor \sigma \log d \rfloor} \)

on the event \( B(d) \). Therefore, by Lemma 4.5,

\[
\left| \hat{P}_{\lambda,d}(W_{\lfloor \sigma \log d \rfloor} = \emptyset) - P_{\lambda,d}(V_{\lfloor \sigma \log d \rfloor} = \emptyset) \right| \leq 2P_{\lambda,d}(B(d)^c) \to 0
\]

as \( d \to +\infty \) and hence

\[
\liminf_{d \to +\infty} P_{\lambda,d}(V_{\lfloor \sigma \log d \rfloor} = \emptyset) \geq E\left(\frac{1}{1 + \lambda \rho \theta}\right)
\]

according to Lemma 5.1. Then, by Lemma 5.2,

\[
\liminf_{d \to +\infty} P_{\lambda,d}(\beta_{\lfloor \sigma \log d \rfloor} = \emptyset) \geq E\left(\frac{1}{1 + \lambda \rho \theta}\right)
\]

and hence

\[
\liminf_{d \to +\infty} P_{\lambda,d}(\beta_n = \emptyset \text{ for some } n \geq 0) \geq E\left(\frac{1}{1 + \lambda \rho \theta}\right). \tag{5.2}
\]

By Equation (5.2),

\[
\limsup_{d \to +\infty} P_{\lambda,d}(\beta_n \neq \emptyset \text{ for all } n \geq 0) \leq E\left(\frac{\lambda \rho \theta}{1 + \lambda \rho \theta}\right). \tag{5.3}
\]

Equation (2.4) follows from Equations (5.1) and (5.3) directly.
5.1 Proof of Lemma 5.1

In this subsection, we give the proof of Lemma 5.1. First we introduce some notations and definitions. For sufficiently large $d$, let $N(d) = \log(\log d)$ defined as in Section 4. For each $x \in T^d$, we give the $d$ sons of $x$ an order $x(1), x(2), \ldots, x(d)$. Then, we define

(1) $\hat{W}_0 = \mathcal{Y}$.
(2) For each $n \geq 0$, $\hat{W}_{n+1} = \{y : \text{there exists } x \in \hat{W}_n \text{ that } y = x(i) \text{ for some } i \leq d - \left\lfloor \frac{d}{N(d)} \right\rfloor \text{ and } U(x,y) < Y(x)\}.$

According to the aforesaid definition, $\{\hat{W}_n\}_{n \geq 0}$ is a branching process with random vertex weights on a subtree of $T^d$ while this subtree is isomorphic to $T^{d - \left\lfloor \frac{d}{N(d)} \right\rfloor}$. For each $n \geq 0$, $\hat{W}_n \subseteq W_n$. The following lemma is crucial for us to prove Lemma 5.1.

**Lemma 5.3.** For $\sigma \in (0, \frac{1}{10 \log(\lambda M^2)})$,

$$
\limsup_{d \to +\infty} \hat{P}_{\lambda,d}(\hat{W}_{\lfloor \sigma \log d \rfloor} \neq \emptyset) \leq E\left(\frac{\lambda \rho \theta}{1 + \lambda \rho \theta}\right) \quad (5.4)
$$

and hence

$$
\liminf_{d \to +\infty} \hat{P}_{\lambda,d}(\hat{W}_{\lfloor \sigma \log d \rfloor} = \emptyset) \geq E\left(\frac{1}{1 + \lambda \rho \theta}\right)
$$

for any $\sigma \in (0, \frac{1}{10 \log(\lambda M^2)})$.

For given $\lambda > \frac{1}{E(\rho \theta)}$ and $\sigma \in (0, \frac{1}{10 \log(\lambda M^2)})$, we choose arbitrary $\hat{\lambda} \in (\lambda, +\infty)$ and $\hat{\sigma} \in (0, \sigma)$. For sufficiently large $d$, we define

$$
\hat{d} = \inf\{k : k \lfloor \frac{k}{N(k)} \rfloor \geq d\},
$$
then it is easy to check that \( \lim_{d \to +\infty} \frac{\hat{\lambda}}{d} = 1 \) and hence

\[
\frac{\hat{\lambda}}{d} \geq \frac{\lambda}{d} \quad \text{while} \quad \hat{\sigma} \log d \leq \sigma \log d
\]  
(5.5)

for sufficiently large \( d \). As we have introduced, \( \{\hat{W}_n\}_{n \geq 0} \) on \( \mathbb{T}^d \) can be identified with \( \{W_n\}_{n \geq 0} \) on \( \mathbb{T}^d - \lfloor \frac{d}{N(d)} \rfloor \) with a scaling of the infection rate \( \lambda \). As a result, by Equation (5.5),

\[
\hat{P}_{\lambda, \hat{d}}(\hat{W}_{\lfloor \hat{\sigma} \log \hat{d} \rfloor} = \emptyset) \leq \hat{P}_{\lambda, d}(W_{\lfloor \sigma \log d \rfloor} = \emptyset)
\]  
(5.6)

for sufficiently large \( d \). By Equations (5.4) and (5.6),

\[
\liminf_{d \to +\infty} \hat{P}_{\lambda, d}(W_{\lfloor \sigma \log d \rfloor} = \emptyset) \geq \liminf_{d \to +\infty} \hat{P}_{\lambda, \hat{d}}(\hat{W}_{\lfloor \hat{\sigma} \log \hat{d} \rfloor} = \emptyset) \geq E\left(\frac{1}{1 + \lambda \rho \hat{\theta}}\right),
\]  
(5.7)

where \( \hat{\theta} \) satisfies

\[
E\left(\frac{\lambda \rho^2}{1 + \lambda \rho \hat{\theta}}\right) = 1
\]

and it is easy to check that \( \lim_{\lambda \to \hat{\lambda}} \hat{\theta} = \theta \).

Let \( \hat{\lambda} \to \lambda \), then Lemma 5.1 follows directly from Equation (5.7).

At the end of this subsection we give the proof of Lemma 5.3.

Proof of Lemma 5.3 We define

\[
\hat{D} = \{y : \text{there exists } x \in \bigcup_{m=0}^{\lfloor \sigma \log d \rfloor} \hat{W}_m \\
\text{that } y = x(i) \text{ for some } i \geq d - \lfloor \frac{d}{N(d)} \rfloor + 1 \text{ and } U(x, y) < Y(x)\}.
\]

It is easy to check that \( u \in \bigcup_{m \geq 0} W_m \) and \( \rho(u) > 0 \) for any \( u \in \hat{D} \). According to nearly the same analysis as that in the proof of Lemma 4.2,

\[
\lim_{d \to +\infty} \hat{P}_{\lambda, d}(|\hat{D}| \geq K(d)|\hat{W}_{\lfloor \sigma \log d \rfloor} \neq \emptyset) = 1,
\]  
(5.8)

where \( K(d) = \lfloor \frac{\sqrt{\log d}}{N(d)} \rfloor \) defined as in Section 4.

For any \( u \in \hat{D} \), we denote by \( T_u \) the subtree of \( \mathbb{T}^d \) rooted at \( u \) and consisted of \( u \) and its descendants. For \( u \in D \), \( \rho(u) > 0 \) and hence \( \rho(u) \geq \epsilon \) by assumption \( 4.1 \).

Then, the SIR model confined on \( T_u \) with \( u \) as the unique initially infected vertex has survival probability at least

\[
1 - F_d(\epsilon),
\]  
(5.9)
where $F_d(\epsilon)$ is defined as in Section 4. It is easy to check that $T_u \cap T_v = \emptyset$ for any $u, v \in D, u \neq v$, then by Equation (5.9),
\begin{equation}
P_{\lambda,d}\left(W_n \neq \emptyset \text{ for all } n \geq 0 \mid |\hat{D}| \geq K(d)\right) \geq 1 - \left[1 - (1 - F_d(\epsilon))\right]^{K(d)} = 1 - F_d(\epsilon)^{K(d)},
\end{equation}
since all the vertices in $\hat{D}$ have ever been infected.

By Lemmas 3.1 and 3.2, we have
\[
\lim_{d \to +\infty} F_d(\epsilon) = \frac{1}{1 + \lambda \epsilon}
\]
and hence $\lim_{d \to +\infty} 1 - F_d(\epsilon)^{K(d)} = 1$. Then by Equation (5.10),
\begin{equation}
\lim_{d \to +\infty} P_{\lambda,d}\left(W_n \neq \emptyset \text{ for all } n \geq 0 \mid |\hat{D}| \geq K(d)\right) = 1. \tag{5.11}
\end{equation}

Lemma 5.3 follows from Equations (5.8) and (5.11) directly.

\section*{5.2 Proof of Lemma 5.2}

In this subsection we give the proof of Lemma 5.2. First we couple $\{\beta_n\}_{n \geq 0}$ and $\{V_n\}_{n \geq 0}$ under the same probability space. Let $\{\hat{Y}(x)\}_{x \in \mathbb{Z}^d_+}$ and $\{\hat{U}(x,y)\}_{x \in \mathbb{Z}^d_+, x \to y}$ be defined as in Section 4, then $\{V_n\}_{n \geq 0}$ is defined as in Section 4 according to the values of $\{\hat{Y}(x)\}_{x \in \mathbb{Z}^d_+}$ and $\{\hat{U}(x,y)\}_{x \in \mathbb{Z}^d_+, x \to y}$. For any $x, y \in \mathbb{Z}^d_+, x \to y$, let $\tilde{U}_2(x,y)$ be an independent copy of $\hat{U}(x,y)$ under the quenched measure. We assume that all these exponential times are independent under the quenched measure. For the contact process, we let $\hat{Y}(x)$ be the time $x$ waits for to become healthy after the first moment when $x$ is infected. We let $\hat{U}(x,y)$ be the time $x$ waits for to infect $y$ after the first moment when $x$ is infected. If $\hat{U}(x,y) < \hat{Y}(x)$, then after the first infection from $x$ to $y$, $x$ waits for $\tilde{U}_2(x,y)$ units of time to infect $y$ again, i.e., $x$ infects $y$ at least twice before becoming healthy when $\hat{U}(x,y) + \tilde{U}_2(x,y) < \hat{Y}(x)$. Following the above definitions, $\{V_n\}_{n \geq 0}$ and $\{\beta_n\}_{n \geq 0}$ are coupled under the same probability space and it is obviously that $V_n \subseteq \beta_n$ for each $n \geq 0$.

Let $J(d)$ be defined as in Section 4 i.e., the event that $\hat{U}(x,y) > \hat{Y}(x)$ and $\hat{U}(z,y) > \hat{Y}(z)$ for any $x, y, z$ that $x, z \in \bigcup_{m=0}^{\lfloor x \log d \rfloor} V_m$ and $x, z \to y$. On the event $J(d)$, if $V_{\lfloor x \log d \rfloor} \neq \beta_{\lfloor x \log d \rfloor}$, then there must exist repeated infection from some $x$ to $y$ that $x \to y$ for the contact process, i.e.,
\[
\hat{U}(x,y) + \tilde{U}_2(x,y) < \hat{Y}(x).
\]

For each $m \geq 1$, $L_m(O)$ is the set of oriented paths on $\mathbb{Z}^d_+$ starting at $O$ with length $m$ defined as in Section 4. For each $\tilde{I} : O = l_0 \to l_1 \to l_2 \to \ldots \to l_m$ in $L_m(O)$, we denote by $\tilde{A}_{\tilde{I}}$ the event that $\tilde{U}(l_i, l_{i+1}) < \hat{Y}(l_i)$ for all $0 \leq i \leq m - 2$ and
\[
\tilde{U}(l_m - 1, l_m) + \tilde{U}_2(l_m - 1, l_m) < \hat{Y}(l_m - 1),
\]

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then according to the aforesaid analysis,

\[ P_{\lambda,d}(V_{[\sigma \log d]} \neq \beta_{[\sigma \log d]}, J(d)) \leq \sum_{m=0}^{[\sigma \log d]} \sum_{\tilde{l} \in L_m} P_{\lambda,d}(\tilde{A}_{l}). \]  

(5.12)

Now we give the proof of Lemma 5.2.

**Proof of Lemma 5.2.** For each \( \tilde{l} \in L_m \), since \( \tilde{U}(\cdot, \cdot), \tilde{U}_2(\cdot, \cdot) \) are exponential times with rate at most \( \frac{\lambda M^2}{d} \) while \( \tilde{Y}(\cdot) \) is an exponential time with rate one, it is easy to check that

\[ P_{\lambda,d}(\tilde{A}_{l}) \leq \left( \frac{\lambda M^2}{d} \right)^{m-1} \left( \frac{\lambda M^2}{d} \right)^2 = \frac{\lambda^{m+1}M^{2m+2}}{d^{m+1}}. \]

Since \( |L_m| = d^m \) and \( \sigma < \frac{1}{10 \log(\lambda M^2)} \), by Equation (5.12),

\[ P_{\lambda,d}(V_{[\sigma \log d]} \neq \beta_{[\sigma \log d]}, J(d)) \leq \sum_{m=0}^{[\sigma \log d]} d^m \frac{\lambda^{m+1}M^{2m+2}}{d^{m+1}} \leq \frac{\lambda^2 M^4 d^{-0.9}}{\lambda M^2 - 1}. \]  

(5.13)

By Equation (5.13),

\[ |P_{\lambda,d}(\beta_{[\sigma \log d]} = \emptyset) - P_{\lambda,d}(V_{[\sigma \log d]} = \emptyset)| \leq 2P_{\lambda,d}(\beta_{[\sigma \log d]} \neq V_{[\sigma \log d]}) \leq 2P_{\lambda,d}(\beta_{[\sigma \log d]} \neq V_{[\sigma \log d]}, J(d)) + 2P_{\lambda,d}(J(d)^c) \leq \frac{2\lambda^2 M^4 d^{-0.9}}{\lambda M^2 - 1} + 2P_{\lambda,d}(J(d)^c). \]

By Equation (4.11),

\[ \lim_{d \to +\infty} P_{\lambda,d}(J(d)^c) = 0. \]  

(5.15)

Lemma 5.2 follows directly from Equations (5.14) and (5.15).

\[ \square \]

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