We study the gauge covariance of the massive fermion propagator in three as well as four dimensional Quantum Electrodynamics (QED). Starting from its value at the lowest order in perturbation theory, we evaluate a non-perturbative expression for it by means of its Landau-Khalatnikov-Fradkin (LKF) transformation. We compare the perturbative expansion of our findings with the known one loop results and observe perfect agreement up to a gauge parameter independent term, a difference permitted by the structure of the LKF transformations.

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I. INTRODUCTION

In a gauge field theory, Green functions transform in a specific manner under a variation of gauge. In Quantum Electrodynamics (QED) these transformations carry the name Landau-Khalatnikov-Fradkin (LKF) transformations, \[1-3\]. Later, they were derived again by Johnson and Zumino through functional methods, \[4,5\]. These transformations are non-perturbative in nature and hence have the potential of playing an important role in addressing the problems of gauge invariance which plague the strong coupling studies of Schwinger-Dyson equations (SDE). In general, the rules governing these transformations are far from simple. The fact that they are written in coordinate space adds to their complexity. As a result, these transformations have played less significant and practical role in the study of SDE than desired. A consequence of gauge covariance are Ward-Green-Takahashi identities (WGTI), \[6-8\], which are simpler to use and, therefore, have been extensively implemented in the above-mentioned studies.

The LKF transformation for the three-point vertex is complicated and hampers direct extraction of analytical restrictions on its structure. Burden and Roberts, \[26\], carried out a numerical analysis to compare the self-consistency of various \textit{ansatze} for the vertex, \[10-12\], by means of its LKF transformation. In addition to these numerical constraints, indirect analytical insight can be obtained on the non-perturbative structure of the vertex by demanding correct gauge covariance properties of the fermion propagator. There are numerous works in literature, based upon this idea, \[12-17\]. However, all the work in this direction has been carried out in the context of massless QED3 and QED4. The masslessness of the fermions implies that the fermion propagator can be written only in terms of one function, the so called wavefunction renormalization, \( F(p) \). In order to apply the LKF transform, one needs to know a Green function at least in one particular gauge. This is a formidable task. However, one can rely on approximations based on perturbation theory. It is customary to take \( F(p) = 1 \) in the Landau gauge, an approximation justified by one loop calculation of the massless fermion propagator in arbitrary dimensions, see for example, \[19\]. The LKF transformation then implies a power law for \( F(p) \) in QED4 and a simple trigonometric function in QED3. To improve upon these results, one can take two paths: (i) incorporate the information contained in higher orders of perturbation theory and (ii) study the massive theory. As pointed out in \[16\], in QED4, the power law structure of the wavefunction renormalization remains intact by increasing order of approximation in perturbation theory although the exponent of course gets contribution from next to leading logarithms and so on \[7\]. In \[16\], constraint was obtained on the 3-point vertex by considering a power law where the exponent of this power law was not restricted only to the one loop fermion propagator. In QED3, the two loop fermion propagator was evaluated in \[18,21,22\], where it was explicitly shown that the the approximation \( F(p) = 1 \) is only valid upto one loop, thus violating the \textit{transversality condition} advocated in \[17\]. The result found there was used used in \[23\] to find the improved LKF transform.

In the present article, we calculate the LKF transformed fermion propagator in massive QED3 and QED4. We start with the simplest input which corresponds to the lowest order of perturbation theory, i.e., \( S(p) = 1/i\not{p} - m \) in the Landau gauge. On LKF transforming, we find the fermion propagator in an arbitrary covariant gauge.
case of QED3, we obtain the result in terms of basic functions of momenta. In QED4, the final expression is in the form of hypergeometric functions. Coupling $\alpha$ enters as parameter of this transcendental function. A comparison with perturbation theory needs the expansion of the hypergeometric function in terms of its parameters. We use the technique developed by Moch et al., for the said expansion. We compare our results with the one loop expansion of the fermion propagator in QED4 and QED3, and find perfect agreement up to terms independent of the gauge parameter at one loop, a difference permitted by the structure of the LKF transformations. We believe that the incorporation of LKF transformations, along with WGT identities, in the SDE can play a key role in addressing the problems of gauge invariance. For example, in the study of the SDE of the fermion propagator, only those assumptions should be permissible which keep intact the correct behaviour of the Green functions under the LKF transformations. We believe that of the fermion propagator in QED4 and QED3, and find perfect agreement up to terms independent of the

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II. FERMION PROPAGATOR AND THE LKF TRANSFORMATION

We start by expanding out the fermion propagator, in momentum and coordinate spaces respectively, in its most general form as follows:

$$S_F(p; \xi) = A(p; \xi) + \frac{B(p; \xi)}{p^2} = \frac{F(p; \xi)}{i\not{p} - \mathcal{M}(p; \xi)}, \quad (1)$$

$$S_F(x; \xi) = \not{\xi} X(x; \xi) + Y(x; \xi), \quad (2)$$

where $F(p; \xi)$ is generally referred to as the wavefunction renormalization and $\mathcal{M}(p; \xi)$ as the mass function. $\xi$ is the usual covariant gauge parameter. Motivated from the lowest order perturbation theory, we take

$$F(p; 0) = 1 \quad \text{and} \quad \mathcal{M}(p; 0) = m. \quad (3)$$

Perturbation theory also reveals that this result continues to hold true to one loop order for the wavefunction renormalization. Eqs. (1,2) are related to each other through the following Fourier transforms:

$$S_F(p; \xi) = \int d^4x e^{i\not{x}\cdot\not{p}} S_F(x; \xi) \quad (4)$$

$$S_F(x; \xi) = \int \frac{d^d p}{(2\pi)^d} e^{-i\not{x}\cdot\not{p}} S_F(p; \xi), \quad (5)$$

where $d$ is the dimension of space-time. The LKF transformation relating the coordinate space fermion propagator in Landau gauge to the one in an arbitrary covariant gauge reads:

$$S_F(x; \xi) = S_F(x; 0)e^{-i[\Delta(0) - \Delta(x)]}, \quad (6)$$

where

$$\Delta_d (x) = -i\xi e^2 \mu^{4-d} \int_0^\infty \frac{d^d p}{(2\pi)^d} \frac{e^{-i\not{p}\cdot x}}{p^2} \quad (7)$$

e$^2$ is the dimensionless electromagnetic coupling. Taking $\psi$ to be the angle between $x$ and $p$, we can write $d^d p = dpp^d-1 \sin^{d-2} \psi \sin \Omega_{d-2}$, where $\Omega_{d-2} = 2 \pi^{(d-1)/2} / \Gamma ((d - 1)/2)$. Hence

$$\Delta_d (x) = -i\xi e^2 \mu^{4-d} f(d) \int_0^\infty dpp^{d-5} \int_0^\pi d\psi \sin^{d-2} \psi e^{-ipx \cos \psi}, \quad (8)$$

where $f(d) = \Omega_{d-2}/(2\pi)^d$. Performing angular and radial integrations, we arrive at the following equation

$$\Delta_d (x) = -\frac{i\xi e^2}{16(\pi)^{d/2}} (\mu x)^{4-d} \Gamma \left(\frac{d}{2} - 2\right). \quad (9)$$

With these tools at hand, the procedure now is as follows:

- Start with the lowest order fermion propagator and Fourier transform it to coordinate space.
- Apply the LKF transformation law.
- Fourier transform the result back to momentum space.
III. THREE DIMENSIONAL CASE

Employing Eqs. (18, 19), the lowest order three dimensional fermion propagator in Landau gauge in the position space is given by

\[ X(x; 0) = -\frac{e^{-mx}(1 + mx)}{4\pi x^3}, \] (10)
\[ Y(x; 0) = -\frac{me^{-mx}}{4\pi x}. \]

Once in the coordinate space, we can apply the LKF transformation law using expression (3) explicitly in three dimensions:

\[ \Delta_3(x) = -\frac{i\alpha x}{2}, \] (11)

where \( \alpha = e^2/4\pi \). The fermion propagator in an arbitrary gauge is then

\[ S(x; \xi) = S_F(x; 0)e^{-(\alpha \xi/2)x}. \] (12)

For Fourier transforming back to momentum space, we use

\[ A(p; \xi) = -\frac{F(p; \xi)M(p; \xi)}{p^2 + M^2(p; \xi)} = \int d^3 x \ e^{ip \cdot x} Y(x; \xi) \] (13)
\[ iB(p; \xi) = -\frac{ip^2 F(p; \xi)}{p^2 + M^2(p; \xi)} = \int d^3 x \ p \cdot x \ e^{ip \cdot x} X(x; \xi). \]

Performing the angular integration, we get

\[ A(p; \xi) = -\frac{m}{p^2 + (2m + \alpha \xi)^2} \] (14)
\[ B(p; \xi) = \frac{1}{p} \int_0^\infty \frac{dx}{x^2} (1 + mx) \ [px \cos px - \sin px] e^{-(m + \alpha \xi/2)x}, \] (15)

and the radial integration then yields

\[ A(p; \xi) = -\frac{4m}{4p^2 + (2m + \alpha \xi)^2} \] (16)
\[ B(p; \xi) = \frac{4p^2 + \alpha \xi(2m + \alpha \xi)}{4p^2 + (2m + \alpha \xi)^2} + \frac{\alpha \xi}{2p} \arctan \left[ 2p/(2m + \alpha \xi) \right]. \] (17)

One can now arrive at the following expressions for the wavefunction renormalization and the mass function, respectively:

\[ F(p; \xi) = -\frac{\alpha \xi}{2p} \arctan \left[ 2p/(2m + \alpha \xi) \right] + \frac{2p(4p^2 + \alpha^2 \xi^2) - \alpha \xi(4p^2 + \alpha \xi(2m + \alpha \xi)) \arctan \left[ 2p/(2m + \alpha \xi) \right]}{2p(4p^2 + \alpha \xi(2m + \alpha \xi)) - \alpha \xi(4p^2 + (2m + \alpha \xi)^2) \arctan \left[ 2p/(2m + \alpha \xi) \right]}, \] (18)
\[ M(p; \xi) = \frac{8p^3m}{2p(4p^2 + \alpha \xi(2m + \alpha \xi)) - \alpha \xi(4p^2 + (2m + \alpha \xi)^2) \arctan \left[ 2p/(2m + \alpha \xi) \right]}. \] (19)

In the massless limit, one immediately recuperates the well-known results:

\[ F_{\text{massless}}(p; \xi) = 1 - \frac{\alpha \xi}{2p} \arctan \left[ \frac{2p}{\alpha \xi} \right], \] (20)
\[ M_{\text{massless}}(p; \xi) = 0. \]

In the weak coupling, we can expand out Eqs. (18, 19) in powers of \( \alpha \). To \( \mathcal{O}(\alpha) \), we find

\[ F(p; \xi) = 1 + \frac{\alpha \xi}{2p^3} \left[ (m^2 - p^2) \arctan \left[ p/m \right] - mp \right], \] (21)
\[ M(p; \xi) = m \left[ 1 + \frac{\alpha \xi}{2p^3} \left\{ (m^2 + p^2) \arctan \left[ p/m \right] - mp \right\} \right]. \] (22)

The expression for the wavefunction renormalization function fully matches with that obtained in [28], while the one for the mass function is also in agreement up to a term proportional to \( \alpha \xi^0 \), as allowed by the structure of the LKF transformations.
IV. FOUR DIMENSIONAL CASE

Employing Eqs. (1,2,3,5), the lowest order four dimensional fermion propagator in position space is given by

\[
X(x; 0) = -\frac{m^2}{4\pi^2 x^2} K_2(mx),
\]

\[
Y(x; 0) = -\frac{m^2}{4\pi^2 x} K_1(mx),
\]

where \( K_1 \) and \( K_2 \) are Bessel functions of the second kind. In order to apply the LKF transformation in four dimensions, we expand Eq. (9) around \( d = 4 - \epsilon \) and use the following identities

\[
\Gamma \left( -\frac{\epsilon}{2} \right) = -\frac{2}{\epsilon} - \frac{1}{\epsilon} \ln x + O(\epsilon),
\]

\[
x^\epsilon = 1 + \epsilon \ln x + O(\epsilon^2),
\]

to obtain

\[
\Delta_4(x) = i \frac{\xi e^2}{16\pi^2} \left[ \frac{2}{\epsilon} + \frac{2}{\epsilon} + 2 \ln \mu x + O(\epsilon) \right].
\]

Note that we cannot write a similar expression for \( \Delta_4(0) \) because of the presence of the term proportional to \( \ln x \). Therefore, we introduce a cut-off scale \( x_{min} \). Now

\[
\Delta_4(x_{min}) - \Delta_4(x) = -i \ln \left( \frac{x^2}{x_{min}^2} \right)^\nu,
\]

where \( \nu = \alpha \xi / 4\pi \). Hence

\[
S_F(x; \xi) = S_F(x; 0) \left( \frac{x^2}{x_{min}^2} \right)^{-\nu}.
\]

For Fourier transforming back to momentum space we use the following expressions

\[
A(p; \xi) = -\frac{F(p; \xi)M(p; \xi)}{p^2 + M^2(p; \xi)} = \int d^4 x \ e^{ip \cdot x} Y(x; \xi)
\]

\[
iB(p; \xi) = -\frac{ip^2 F(p; \xi)}{p^2 + M^2(p; \xi)} = \int d^4 x \ p \cdot x \ e^{ip \cdot x} X(x; \xi).
\]

On carrying out angular integration, we obtain:

\[
A(p; \xi) = -\frac{m^2}{p} \frac{2 \nu}{x_{min}^2 \int_0^\infty dx x^{-2\nu+1} K_1(mx) J_1(px)},
\]

\[
B(p; \xi) = -m^2 \int_0^\infty dx x^{-2\nu+1} K_2(mx) J_2(px).
\]

The radial integration then yields:

\[
A(p; \xi) = -\frac{1}{m} \left( \frac{m^2}{\Lambda^2} \right)^\nu \Gamma(1 - \nu) \Gamma(2 - \nu) \ _2F_1 \left( 1 - \nu, 2 - \nu; 2; -\frac{p^2}{m^2} \right),
\]

\[
B(p; \xi) = -\frac{p^2}{2m^2} \left( \frac{m^2}{\Lambda^2} \right)^\nu \Gamma(1 - \nu) \Gamma(3 - \nu) \ _2F_1 \left( 1 - \nu, 3 - \nu; 3; -\frac{p^2}{m^2} \right),
\]

where we have identified \( 2/x_{min} \rightarrow \Lambda \). The above equations imply
Let us switch off the coupling and put $\alpha = 0$ which implies $\nu = 0$. Now using the identity

$$2F_1(1, 2; 2; -p^2/m^2) = 2F_1(1, 3; 3; -p^2/m^2) = (1 + p^2/m^2)^{-1},$$

(35)

it is easy to see that

$$F(p; \xi) = 1$$

and

$$\mathcal{M}(p; \xi) = m,$$

(36)

which coincides with the lowest order perturbative result as expected.

### B. Case $m >> p$

In the limit $m >> p$, the hypergeometric functions in Eqs. (33,34) can be easily expanded in powers of $p^2/m^2$, using the identity

$$2F_1(\alpha, \beta; \gamma; -p^2/m^2) = 1 - \frac{\alpha \beta}{\gamma} \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^2}{m^2}\right)^2.$$

(37)

Retaining only $\mathcal{O}(p^2/m^2)$ terms, we arrive at:

$$F(p; \xi) = \frac{\Gamma(1 - \nu)\Gamma(2 - \nu)}{(1 - \nu/2)} \left[1 + \frac{2\nu}{3} \left(1 - \frac{5\nu}{8}\right) \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^2}{m^2}\right)^2\right] \left(\frac{m^2}{\Lambda^2}\right)^{\nu},$$

(38)

$$\mathcal{M}(p; \xi) = \frac{m}{(1 - \nu/2)} \left[1 + \frac{\nu}{6}(1 - \nu) \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^2}{m^2}\right)^2\right].$$

(39)

Now carrying out an expansion in $\alpha$ and substituting $\nu = \alpha \xi/4\pi$, we get the following $\mathcal{O}(\alpha)$ expressions:

$$F(p; \xi) = 1 + \frac{\alpha \xi}{4\pi} \left[2\gamma - \frac{1}{2} + \frac{2p^2}{3m^2} + \ln \frac{m^2}{\Lambda^2}\right],$$

(40)

$$\mathcal{M}(p; \xi) = m \left\{1 + \frac{\alpha \xi}{8\pi} \left[1 + \frac{p^2}{3m^2}\right] \right\}. $$

(41)

Let us now compare these expressions against the one-loop perturbative evaluation of the massive fermion propagator, see e.g., [27]:

$$F_{\text{1-loop}}(p; \xi) = 1 - \frac{\alpha \xi}{4\pi} \left[C\mu^s + \left(1 - \frac{m^2}{p^2}\right)(1 - L)\right],$$

(42)

$$\mathcal{M}_{\text{1-loop}}(p; \xi) = m + \frac{\alpha m}{\pi} \left[\left(1 + \frac{\xi}{4}\right) + \frac{3}{4}(C\mu^s - L) + \frac{\xi m^2}{4 p^2}(1 - L)\right],$$

(43)
where

\[ L = \left( 1 + \frac{m^2}{p^2} \right) \ln \left( 1 + \frac{p^2}{m^2} \right), \]

\[ C = -\frac{2}{\varepsilon} - \gamma - \ln \pi - \ln \left( \frac{m^2}{\mu^2} \right). \]

Knowing the fermion propagator even in one particular gauge is a prohibitively difficult task. Therefore, Eqs. (3) have to be viewed only as an approximation. For the wavefunction renormalization \( F(p) \), this approximation is valid up to one loop order, whereas, for the mass function, it is true only to the lowest order. Therefore we cannot expect the LKF transform of Eqs. (3) to yield correctly each term in the perturbative expansion of the fermion propagator. However, it should correctly reproduce all those terms at every order of expansion which vanish in the Landau gauge at \( \mathcal{O}(\alpha) \) and beyond. Therefore, we expect Eq. (42) to be exactly reproduced and Eq. (43) to be reproduced up to the terms which vanish in the Landau gauge at \( \mathcal{O}(\alpha) \). After subtracting these terms, we call the resulting function as the \textit{subtracted} mass function:

\[ M_{1-\text{loop}}^{S}(p; \xi) = m + \alpha \xi \frac{m}{4\pi} \left[ 1 + \frac{m^2}{p^2} (1 - L) \right]. \quad (44) \]

In the limit \( m \to \infty \), the wavefunction renormalization acquires the form

\[ F(p; \xi)_{1-\text{loop}} = 1 + \frac{\alpha \xi}{4\pi} \left[ -C m^\varepsilon - \frac{1}{2} + \frac{2p^2}{3m^2} \right], \quad (45) \]

while the \textit{subtracted} mass function is

\[ M_{1-\text{loop}}^{S}(p; \xi) = m \left\{ 1 + \frac{\alpha \xi}{8\pi} \left[ 1 + \frac{p^2}{3m^2} \right] \right\}. \quad (46) \]

The last two expressions are in perfect agreement with Eqs. (40, 41) after we make the identification:

\[ -C m^\varepsilon \to 2\gamma + \ln \frac{m^2}{\Lambda^2}. \quad (47) \]

\section*{C. Case of weak coupling}

The case \( m \gg p \) is relatively easier to handle as we merely have to expand \( _2F_1(\beta, \gamma; \delta; x) \) in powers of \( x \) and retain only the leading terms. If we want to obtain a series in powers of the coupling alone, we need the expansion of the hypergeometric functions in terms of its parameters \( \beta \) and \( \gamma \). We follow the technique developed in \[24\]. One of the mathematical objects we shall use for such an expansion are the \textit{Z}-sums defined as:

\[ Z(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{n \geq i_1 > i_2 > \ldots > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}. \quad (48) \]

For \( x_1 = \ldots = x_k = 1 \) the definition reduces to the Euler-Zagier sums, \[27,28\] :

\[ Z(n; m_1, \ldots, m_k; 1, \ldots, 1) = Z_{m_1, \ldots, m_k}(n). \quad (49) \]

Euler-Zagier sums can be used in the expansion of Gamma functions. For positive integers \( n \) we have \[24\]:

\[ \Gamma(n + \epsilon) = \Gamma(1 + \epsilon) \Gamma(n) \left[ 1 + \epsilon Z_1(n - 1) + \ldots + \epsilon^{n-1} Z_{1 \ldots 1}(n - 1) \right]. \quad (50) \]

The first sum \( Z_1(n - 1) \), e.g., is just the \( (n - 1) \)-th harmonic number, \( H_{n-1} \), of order 1:

\[ Z_1(n - 1) = \sum_{i=1}^{n-1} \frac{1}{i} \equiv H_{n-1}. \quad (51) \]
With these definitions in hand, we proceed to expand a hypergeometric function, \( {}_2F_1(1 + \varepsilon, 2 + \varepsilon; 2; x) \), as an example, assuming \(|x| < 1\):

\[
{}_2F_1(1 + \varepsilon, 2 + \varepsilon; 2; x) = 1 + \frac{\Gamma(2)}{\Gamma(1 + \varepsilon)\Gamma(2 + \varepsilon)} \sum_{n=1}^{\infty} \frac{\Gamma(1 + \varepsilon + n)\Gamma(2 + \varepsilon + n)}{\Gamma(2 + n)} \frac{x^n}{n!}.
\]

Employing Eq. (50), we can expand the last expression in powers of \(\varepsilon\) to any desired order of approximation. We shall be interested only in terms upto \(O(\alpha)\).

\[
{}_2F_1(1 + \varepsilon, 2 + \varepsilon; 2; x) = 1 + \sum_{n=1}^{\infty} x^n - \varepsilon \sum_{n=1}^{\infty} x^n + \varepsilon \sum_{n=1}^{\infty} \frac{2 + 3n}{n(n+1)} x^n + 2\varepsilon \sum_{n=1}^{\infty} H_{n-1} x^n.
\]

(52)

Performing the summations, we obtain

\[
{}_2F_1(1 + \varepsilon, 2 + \varepsilon; 2; x) = 1 + \frac{1}{1 - x} \left[ 1 - \varepsilon \left( 1 + \frac{1 + x}{x} \ln (1 - x) \right) \right].
\]

(53)

Similarly,

\[
{}_2F_1(1 + \varepsilon, 3 + \varepsilon; 3; x) = 1 + \frac{3}{1 - x} - \varepsilon \left( \frac{1}{x} + \frac{1}{2} \frac{1}{1 - x} + \left( \frac{1 + x}{x^2} + \frac{2}{1 - x} \right) \ln (1 - x) \right).
\]

(54)

Substituting back into Eqs. (33,34) and identifying \(\varepsilon = -\nu\), we obtain

\[
F(p; \xi) = 1 - \frac{\alpha \xi}{4\pi} \left[ -2\gamma - \ln \frac{m^2}{\Lambda^2} + \left( 1 - \frac{m^2}{p^2} \right) (1 - L) \right],
\]

\[
\mathcal{M}(p; \xi) = m + \frac{\alpha \xi m}{4\pi} \left[ 1 + \frac{m^2}{p^2} (1 - L) \right],
\]

(55)

which matches exactly onto the one loop result of Eqs. (12,14) after the same identification as before, i.e., (47). Therefore, we have seen that the LKF transformation of the bare propagator contains important information of higher orders in perturbation theory.

V. CONCLUSIONS

We have studied the gauge covariance of the massive fermion propagator in three as well as four dimensional QED through its LKF transformation, starting from its lowest order approximation. Eqs. (18,19,33,34) form the main result of this article. In the three dimensional case, the LKF transformation consists of basic functions of the momentum variable, whereas, in the four dimensional case, hypergeometric functions arise with electromagnetic coupling as parameter of these functions. Although our input is only the bare propagator, the corresponding LKF transformation, being non-perturbative in nature, contains useful information of higher orders in perturbation theory. For example, we have shown that a perturbative expansion of our results matches onto the known 1-loop results up to gauge independent terms at this order. This slight difference arises due to our approximated input and can be corrected systematically at the cost of increasing complexity of the integrals involved. We intend to carry out similar exercise for the 3-point fermion-boson vertex. LKF transformations of the propagator and the vertex impose useful constraints on the SDE and we believe that these transformations can be of immense help in addressing the problems of gauge invariance in the related studies.

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APPENDIX

Most of the integrals involved in this paper are listed below for a quick reference \[29,30\] :

\[
\int_0^\pi d\psi \sin d-2 \psi \cos \psi e^{-ipx \cos \psi} = -i \frac{\pi}{2} \left( \frac{px}{2} \right)^{1-\frac{d}{2}} \Gamma \left( \frac{d-1}{2} \right) J_{\frac{d}{2}}(px),
\]
\[
\int_0^\infty x^{d/2-1} J_{d/2}(ax) = \frac{\Gamma(d/2)}{2^{a-d/2} a^{d/2}}.
\]

For the three dimensional case, the needed integrals are :

\[
\int_0^\pi d\theta \sin \theta e^{-ipx \cos \theta} = 2 \sin \frac{px}{a} J_1(p),
\]
\[
\int_0^\pi d\theta \cos \theta \sin \theta e^{-ipx \cos \theta} = 2i \left[ \frac{\cos px}{px} - \frac{\sin px}{(px)^2} \right],
\]
\[
\int_0^\infty dp \frac{p^3}{(p^2 + m^2)} \left[ \frac{\cos px}{px} - \frac{\sin px}{(px)^2} \right] = -\frac{\pi}{2} \frac{(1 + mx)}{x^2} e^{-mx},
\]
\[
\int_0^\infty dp \frac{p^2}{(p^2 + m^2)} = \frac{\pi}{2} e^{-mx},
\]
\[
\frac{1}{p} \int_0^\infty dx \frac{x e^{-ax}}{x^2} \left[ px \cos px - \sin px \right] = -1 + a \arctan \frac{p}{a},
\]
\[
\frac{1}{p} \int_0^\infty dx \frac{x e^{-ax}}{x} \left[ px \cos px - \sin px \right] = \frac{a}{a^2 + p^2} - \frac{1}{p} \arctan \frac{p}{a},
\]
\[
\int_0^\infty dx \frac{px e^{-(m+\alpha \xi/2)x}}{m + \alpha \xi/2 + \frac{p^2}{2}} = \frac{p}{(m + \alpha \xi/2 + \frac{p^2}{2}).
\]

For the four dimensional case, we used the following integrals in particular :

\[
\int_0^\pi d\theta \sin^2 \theta e^{-ipx \cos \theta} = \frac{\pi}{px} J_1(p),
\]
\[
\int_0^\infty dp \frac{p^{\mu+1}}{(p^2 + m^2)^{\mu+1}} = \frac{m^\mu \mu!}{2^\mu \Gamma(\mu+1)} K_{\mu-\mu}(mx),
\]
\[
\int_0^\infty dx x^{-\lambda} K_{\mu}(ax) J_{\nu}(bx) = \frac{a^{\nu-1} b^\nu}{2^\nu + 1 \Gamma(1 + \nu)} \Gamma \left( \frac{\nu + \lambda + \mu + 1}{2} \right) \Gamma \left( \frac{\nu - \lambda - \mu + 1}{2} \right)
\]
\[
\times \ 2F1 \left( \frac{\nu + \lambda + \mu + 1}{2}, \frac{\nu - \lambda - \mu + 1}{2}; \nu + 1; -\frac{b^2}{a^2} \right).
\]

Some of the series used in our calculation are as follows :

\[
\sum_{n=1}^{\infty} H_{n-1} x^n = -\frac{x \ln(1-x)}{1-x},
\]
\[
\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)} x^n = -\frac{2 + x}{4x} - \frac{(1 + x^2) \ln(1-x)}{2x^2},
\]
\[
\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} x^n = \frac{2 - x}{2x} + \frac{(1 - x) \ln(1-x)}{x^2}.
\]
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