How do curved spheres intersect in 3-space?

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Abstract

The following problem was proposed in 2010 by S. Lando.

Let $M$ and $N$ be two unions of the same number of disjoint circles in a sphere. Do there always exist two spheres in 3-space such that their intersection is transversal and is a union of disjoint circles that is situated as $M$ in one sphere and as $N$ in the other? Union $M'$ of disjoint circles is situated in one sphere as union $M$ of disjoint circles in the other sphere if there is a homeomorphism between these two spheres which maps $M'$ to $M$.

We prove (by giving an explicit example) that the answer to this problem is “no”. We also prove a necessary and sufficient condition on $M$ and $N$ for existing of such intersecting spheres. This result can be restated in terms of graphs. Such restatement allows for a trivial brute-force algorithm checking the condition for any given $M$ and $N$. It is an open question if a faster algorithm exist.

The Lando Problem

We work entirely in the piecewise-linear (PL) category \footnote{A PL circle or circle is a closed broken line (polygon) without self-intersections in 3-space. A PL sphere or sphere is a polyhedron in 3-space (more precisely, 2-dimensional surface of the polyhedron), which is split into several parts by any circle lying on the polyhedron, i.e. is a polyhedron homeomorphic to $S^2$.}

Suppose $M$ and $M'$ are the unions of the same number of disjoint circles in spheres $S$ and $S'$. Then $M$ is situated in $S$ as $M'$ in $S'$ if there is a homeomorphism $f : S \to S'$ such that $f(M) = M'$.

The following problem suggested by S. Lando was one of the (unsolved) problems at the Moscow State University mathematical tournament for students and young professors 2010 (\cite{1}, problem MB-8).

Let $M$ and $N$ be two unions of the same number of disjoint circles in a sphere. Do there exist two spheres in 3-space whose intersection is transversal and is a union of disjoint circles that is situated as $M$ in one sphere and as $N$ in the other?

This problem appeared in the discussion of related papers \cite{3}, \cite{4}, \cite{5}.

In this paper we prove that the answer to Lando problem is “no” by giving an explicit example.
Figure 1: Two unions of $M$ (left) and $N$ (right) of 9 circles.

**Theorem 1** (an example). Let $M$ and $N$ be two unions of 9 disjoint circles in $S^2$ shown in Fig. 1. Then there are no two spheres in 3-space whose intersection is transversal and is a union of 9 disjoint circles that is situated as $M$ in one sphere and as $N$ in the other.

Figure 2: Bijection $h$ between two sets of three circles is realized by PL embeddings $f$, $g$.

In Theorem 2 (see below) we describe all the collections of circles which can be realized by two intersecting spheres. The precise meaning of the word “realized” is defined in the following paragraph.

Assume that $M$ and $N$ are two unions of disjoint circles in sphere $S^2$. Suppose there exists PL embeddings $f : S^2 \hookrightarrow \mathbb{R}^3$ and $g : S^2 \hookrightarrow \mathbb{R}^3$ such that intersection $f(S^2) \cap g(S^2)$ is transversal and $f(S^2) \cap g(S^2) = f(M) = g(N)$. These embeddings induce a bijection $h$ between sets of circles of $M$ and of $N$ (for circles $m \subset M$ and $n \subset N$ let $h(m) = n$ if $f(m) = g(n)$). Equivalently we may number circles of $M$ and of $N$ by $1, \ldots, k$ so that two circles corresponding to the same circle of $f(S^2) \cap g(S^2)$ have the same number. We say that $f$, $g$ realize $h$ (Fig. 2).

Theorem 2 (see below) gives a necessary and sufficient condition for the realizability of a bijection. In particular Theorem 2 can be used to prove Theorem 1. The following simple example shows that not every bijection is realizable.

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$\text{Map } f : A \to B$ is piecewise linear if $f$ is a simplicial map for some simplicial decompositions of $A$ and $B$.
Figure 3: Circles $A_0, A_1, A_2$, bijection $h$.

Example 1. Let $A_0, A_1, A_2$ be the circles situated in $S^2$ as shown in the Fig. 3. Let $M = N = A_0 \cup A_1 \cup A_2$. Let $h$ be a bijection defined by $h(A_0) = A_1$, $h(A_1) = A_0$ and $h(A_2) = A_2$. Then $h$ is not realizable.

The proof of Example 1 (see “Proofs”) demonstrates some of the ideas used in the proof of Theorem 2.

Let us introduce definitions necessary to state Theorem 2.

Let $M$ and $N$ be two unions (not necessary nonempty) of disjoint circles in sphere $S^2$. Color connected components of $S^2 - N$ in black and white so that adjacent components have different colors. Union $M$ is \textit{on one side} (in this sphere) of $N$ if $M$ is contained in the union of same colored components of $S^2 - N$. Unions $M$ and $N$ are \textit{unlinked} (in this sphere) if $M$ is on one side of $N$ and $N$ is on one side of $M$. Equivalently unions $M$ and $N$ are \textit{unlinked} (in $S^2$) if $[M] = 0$ in $H_1(S^2 - N; \mathbb{Z}_2)$ and $[N] = 0$ in $H_1(S^2 - M; \mathbb{Z}_2)$.

Figure 4: (A) $M$ (solid) is on one side of $N$ (dashed) while $N$ is not on one side of $M$. (B) $M$ (solid) and $N$ (dashed), $N$ and $P$ (dotted) are unlinked, but $M$ and $P$ are not unlinked.

Unions $M$ and $N$ are always unlinked if $M$ or $N$ is empty. If $M$ is on one side of $N$ then $N$ is not necessary on one side of $M$ (Fig. 4 left). Unlinkedness is not transitive. That is, if $M$ and $N$, $N$ and $P$ are unlinked, then $M$ and $P$ are not necessarily unlinked (Fig. 4 right).

Let $M$ be a union of disjoint circles in sphere $S$. Suppose $A$ is a connected component of $S - M$. Denote by $\partial A$ the boundary of the closure of $A$.

**Theorem 2.** Let $M$ and $N$ be two unions of the same number of disjoint circles in $S^2$. Let $h$ be a bijection between sets of circles of $M$ and of $N$. Color connected components of $S^2 - M$ in two colors so that any two same colored components are not adjacent. Then $h$ is realizable if and only if $h(\partial A)$ and $h(\partial B)$ are unlinked for each two same-colored components $A$ and $B$ of $S^2 - M$.

We say that sphere with holes $P$ is \textit{properly embedded} in $D^3$ if $\partial P \subset \partial D^3$ and the interior of $P$ lies in the interior of $D^3$. Theorem 2 is proved using the following:
Embedding Extension Theorem. Let $M_1, \ldots, M_m$ be unions of disjoint circles in the sphere $S^2 = \partial D^3$. Then there exist properly embedded in $D^3$ disjoint spheres with holes $P_1, \ldots, P_m$ such that $\partial P_i = M_i$ for each $i = 1, \ldots, m$ if and only if $M_1, \ldots, M_m$ are pairwise unlinked.

Embedding Extension Theorem immediately implies the following:

Corollary 1. Let $M_1, \ldots, M_m$ be unions of disjoint circles in the sphere $S^2 = \partial D^3$. Suppose that for every $i, j$ there exist properly embedded in $D^3$ disjoint spheres with holes $P'_i, P'_j$ such that $\partial P'_i = M_i, \partial P'_j = M_j$. Then there exist properly embedded in $D^3$ disjoint spheres with holes $P_1, \ldots, P_m$ such that $\partial P_i = M_i$ for each $i = 1, \ldots, m$.

Note that analogous statement is false for all closed orientable 2-surfaces other than $S^2$. For instance:

Example 2. Let $M_1, M_2, M_3$ be unions of disjoint circles in the standard torus $T^2 \subset \mathbb{R}^3$. Let $M_1$ and $M_2$ be a single meridian each and let $M_3$ be a union of two meridians (Fig. 5). Then for every $i, j$ there exist disjoint spheres with holes $P'_i, P'_j$ whose interiors are inside $T^2$ and such that $\partial P'_i = M_i, \partial P'_j = M_j$. But there are no disjoint spheres with holes $P_1, P_2, P_3$ whose interiors are inside $T^2$ and such that $\partial P_1 = M_1, \partial P_2 = M_2, \partial P_3 = M_3$.

![Figure 5: Unions $M_1$ and $M_2$ consists of one meridian each and $M_3$ consists of two meridians.](image)

This example is similar to the famous Borromean rings example stated in the following way:

**Borromean rings.** Let $S^1_1, S^1_2, S^1_3$ be the Borromean rings in $S^3 = \partial D^4$. Then for every $i, j$ there exist properly embedded in $D^4$ disjoint disks $D^2_i, D^2_j$ such that $\partial D^2_i = S^1_i, \partial D^2_j = S^1_j$. But there are no properly embedded in $D^4$ disjoint disks $D^2_1, D^2_2, D^2_3$ such that $\partial D^2_1 = S^1_1, \partial D^2_2 = S^1_2, \partial D^2_3 = S^1_3$.

**Relation to graphs**

Suppose that $M$ is a union of disjoint circles in sphere $S^2$. Define ("dual to $M$") graph $G = G(S^2, M)$ as follows. The vertices are the connected components of $S^2 - M$. Two vertices are connected by an edge if the corresponding connected components are neighbors.
The definition of unlinked unions of circles can also be restated in terms of graphs. Let \( p \) and \( q \) be two sets of edges of a tree \( G \). Color connected components of the complement in \( G \) to the interiors of edges of \( q \) in black and white so that adjacent components have different colors. The set \( p \) is on the same side of \( q \) (in this tree \( G \)) if \( p \) is contained in the union of same-colored connected components of \( G - q \) (or, equivalently, if \( p \cap q = \emptyset \) and for each two vertices of edges of \( p \) there is a path in the tree connecting these two points, and containing an even number of edges of \( q \)). Sets \( p \) and \( q \) are unlinked (in this tree) if \( p \) is on the same side of \( q \) and \( q \) is on the same side of \( p \) (for example see Fig. 6).

Let \( G \) and \( H \) be two trees with the same number of edges. Color vertices of \( G \) in two colors so that any two same colored vertices are not adjacent. Bijection \( h \) between the sets of edges of \( G \) and \( H \) is called realizing if \( h(\delta A)^3 \) and \( h(\delta B) \) are unlinked (in \( H \)) for each two same-colored vertices \( A \) and \( B \) of \( G \).

Instead of a union of disjoint circles in a sphere let us consider its dual graph. Theorem 2 implies that a bijection between two sets of circles is realizable if and only if the corresponding bijection between the sets of edges of dual graphs is realizing.

Let \( G \) and \( H \) be two trees with \( k \) edges each. Given a bijection \( h \) between the sets of edges of \( G \) and \( H \) we can check algorithmically in at most \( O(k^2) \) time if \( h \) is realizing. So, there is a brute-force algorithm which finds a realizing bijection (if any) in \( O(k^2k!) \) time. We don’t know if the more efficient algorithm exists. More precisely there is the following open problem:

**Open problem 1.** Is there a “fast” algorithm, which takes as input two arbitrary trees \( G \) and \( H \) with \( k \) edges each and produces as output a realizing bijection (if any) between the sets of edges of \( G \) and \( H \)?

**Open problem 2.** Is there a tree \( G \) such that there is no realizing bijection between the sets of edges of \( G \) and \( H \), where \( H \) is the path graph with the same number of edges as \( G \)?

**Proofs**

*Proof of the Example* Assume to the contrary that there are PL embeddings \( f : S^2 \hookrightarrow \mathbb{R}^3 \) and \( g : S^2 \hookrightarrow \mathbb{R}^3 \) realizing \( h \).

\(^3\delta A \) is a set of all edges incident to \( A \)
Denote by $D$ the disk in $f(S^2) - g(S^2)$ bounded by $f(A_0)$ (Fig. 7). Denote by $C$ the cylinder in $f(S^2) - g(S^2)$ bounded by $f(A_1)$ and $f(A_2)$. Clearly $C$ and $D$ lie in 3-space on the same side of sphere $g(S^2)$. Circles $f(A_1) = g(h(A_1)) = g(A_0)$ and $f(A_2) = g(h(A_2)) = g(A_2)$ lie in sphere $g(S^2)$ on the different sides of the circle $f(A_0) = g(h(A_0)) = g(A_1)$. So $C$ intersects $D$. This contradicts to the assumption that $f$ is an embedding. \hfill $\Box$

**Proof of Theorem 1.** Assume to the contrary that there is a bijection $h$ between sets of circles of $M$ and of $N$ and PL embeddings $f : S^2 \leftrightarrow \mathbb{R}^3$ and $g : S^2 \leftrightarrow \mathbb{R}^3$ realizing $h$.

Denote the connected components of $f(S^2) - g(S^2)$ as shown in Fig. 1 (left). Consider disks $A_1, \ldots, A_4 \subset f(S^2)$. Without loss of generality we may assume that their interiors lie inside $g(S^2)$. Then the interior of component $C \subset f(S^2)$ lies inside $g(S^2)$ as well (since the intersection $f(S^2) \cap g(S^2)$ is transversal). Since $C, A_1, \ldots, A_4$ are disjoint, $C$ lies completely in one of the connected components of $\mathbb{R}^3 - g(S^2) \cup \bigsqcup A_i$. So all the 5 circles of $\partial C$ lie in the same connected component of $g(S^2) - \bigsqcup \partial A_i$ (this argument is generalized in the proof of Claim 1 below).

Let us restate the previous statement in terms of graph $G(S^2, N)$ (Fig. 8). Denote by $G(C)$ the union of 5 edges of $G(S^2, N)$ corresponding to the circles of $\partial C$. Then $G(C)$ lies completely in one of the connected components of the compliment of $G(S^2, N)$ to the 4 edges corresponding to the circles of $\bigsqcup \partial A_i$. Since $G(S^2, N)$ has only 9 edges this means that $G(C)$ is a subtree of $G(S^2, N)$. Denote by $G(B)$ the union of 5 edges of $G(S^2, N)$ corresponding to the circles of $\partial B$. Likewise, $G(B)$ is a subtree of $G(S^2, N)$. 

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**Figure 7:** Circles $f(A_0) = g(A_1)$, $f(A_1) = g(A_0)$, $f(A_2) = g(A_2)$.

**Figure 8:** Graph $G(S^2, N)$. 
Since \( G(B) \cup G(C) = G(S^2, N) \), at least two of the three edges \( a, b, c \) of \( G(S^2, N) \) (Fig. 3) belong to one of subtrees \( G(B) \) or \( G(C) \). Without loss of generality we may assume that \( a, b \in G(B) \). But any subtree of \( G(S^2, N) \) containing both \( a \) and \( b \) has at least 6 edges while \( G(B) \) has only 5 edges. This contradicts the initial assumption.

**Proof of the “only if” part of Theorem 2.** Let \( A \) and \( B \) be two same colored components of \( S^2 - M \). Then \( f(A) \) and \( f(B) \) lie on the same side of \( g(S^2) \). So by the “only if” part of Embedding Extension Theorem \( \partial f(A) \) and \( \partial f(B) \) are unlinked in \( g(S^2) \). Then \( g^{-1}(\partial f(A)) = h(\partial A) \) and \( g^{-1}(\partial f(B)) = h(\partial B) \) are unlinked in \( S^2 \).

**Proof of the “if” part of Theorem 2.** Let \( g : S^2 \hookrightarrow \mathbb{R}^3 \) be any PL embedding. We define a PL embedding \( f : S^2 \hookrightarrow \mathbb{R}^3 \) such that \( f, g \) realize \( h \) by defining \( f(A) \) for every connected component \( A \) of \( S^2 - M \).

Color connected components of \( S^2 - M \) in black and white so that any two same colored components are not adjacent.

Let \( P_1, \ldots, P_m \) be the white components of \( S^2 - M \). By the assumption of the Theorem \( h(\partial P_1), \ldots, h(\partial P_m) \) are pairwise unlinked in \( S^2 \). So \( g(h(\partial P_1)), \ldots, g(h(\partial P_m)) \) are pairwise unlinked in \( g(S^2) \). By the “if” part of Embedding Extension Theorem there exist disjoint spheres with holes \( P'_1, \ldots, P'_m \) whose interiors are inside \( g(S^2) \) and such that \( \partial P'_i = g(h(\partial P_i)) \) for each \( i = 1, \ldots, m \). Define \( f(P'_i) := P'_i \) for each \( i \).

Likewise, let \( Q_1, \ldots, Q_n \) be the black components of \( S^2 - M \). By the “only if” part of Embedding Extension Theorem there exist disjoint spheres with holes \( Q'_1, \ldots, Q'_n \) whose interiors are outside \( g(S^2) \) and such that \( \partial Q'_j = g(h(\partial Q_j)) \) for each \( j = 1, \ldots, n \). Define \( f(Q_j) := Q'_j \) for each \( j \).

Image of \( f \) is the sphere \( \bigcup P'_i \cup \bigcup Q'_j \). Clearly, \( f \) and \( g \) realize \( h \).

**Proof of the “only if” part of Embedding Extension Theorem.** Consider a properly embedded in \( D^3 \) sphere with holes \( P_1 \). Add a “cap” (homeomorphic to a disk) in \( \mathbb{R}^3 - D^3 \) to every circle of \( \partial P_i \) such that the union of \( P_i \) with these caps is a sphere \( \hat{P}_i \). (In the smooth category we may assume that \( S^2 \) is a round sphere and that bounding circles of \( \partial P_i \) are round circles, none of them being an equator. Then for each circle of \( \partial P_i \) take the round sphere passing through this circle and the center of \( S^2 \). Take parts of such spheres lying in \( \mathbb{R}^3 - D^3 \) as these “caps”. Analogous, albeit slightly more complicated, construction is possible in the PL category).

Clearly, all same colored connected components of \( S^2 - M_i = S^2 - \partial P_i \) lie on the same side of \( \hat{P}_i \). And since \( P_i \) and \( P_j \) are disjoint, \( S^2 \cap \hat{P}_j = M_j \) lie on one side of \( \hat{P}_i \), i.e. in the union of same colored components of \( S^2 - M_i \).

So \( M_j \) lie on one side of \( M_i \) by definition. Likewise, \( M_i \) lie on one side of \( M_j \). Therefore \( M_i \) and \( M_j \) are unlinked.

To prove the “if” part we require the following claim. Proof of the claim is postponed.

**Claim 1.** Let \( P_1, \ldots, P_n \) be properly embedded in \( D^3 \) pairwise disjoint spheres with holes. Let \( M \) be a union of disjoint circles in \( S^2 = \partial D^3 \) such that \( M \) and \( \partial P_i \) are unlinked for every \( i \). Then \( M \) lies in one connected component of \( D^3 - (P_1 \cup \cdots \cup P_n) \).

**Proof of the “if” part of Embedding Extension Theorem.** This proof was suggested by A. Novikov. It is simpler than our original proof.

Use induction on number of circles in \( M_1 \cup \cdots \cup M_m \).

Let \( p \) be a circle of \( M_1 \cup \cdots \cup M_m \) bounding an open disk \( D \) in \( S^2 \) disjoint with \( M_1 \cup \cdots \cup M_m \) (\( p \) corresponds to an edge of \( G(S^2, M_1 \cup \cdots \cup M_m) \) issuing out of a leaf vertex). We may assume that \( p \subset M_1 \). Denote by \( M'_1 \) the union of circles \( M_1 - p \) (note that \( M'_1 \) may be empty).
Unions $M_1', M_2', \ldots, M_m'$ are pairwise unlinked. By the inductive hypothesis there are properly embedded in $D^3$ disjoint spheres with holes $P_1', P_2', \ldots, P_m'$ such that $\partial P_i = M_i$ for each $i = 2, \ldots, m$ and $\partial P_1' = M'_1$. Let $D'$ be a disk obtained from the closure of $D$ by a slight deformation so that the interior of $D'$ is in the interior of $D^3$ and $\partial D' = p$. By Claim 1 each two points of $M_1 = M_1' \cup p$ can be connected by a path in the interior of $D^3$ disjoint with $P_2', \ldots, P_m'$. So we can connect $D'$ with $P_1'$ by a tube in the interior of $D^3$ disjoint with $P_2', \ldots, P_m'$. Then we obtain a sphere with holes. Denote it by $P_1$. We have $\partial P_1 = p \sqcup \partial P_1' = M_1$, $P_1$ is properly embedded in $D^3$ and $P_1$ is disjoint with $P_2', \ldots, P_m'$. The inductive step is proved.

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Figure 9: Paths $l$, $l''$.

**Proof of Claim 1**. Take any two points $A, B \in M$. Denote by $l$ a path in $D^3$ connecting $A$ and $B$ such that $\bar{l} := \#(l \cap \bigcup_{i=1}^n P_i)$ is minimal (minimal by $l$, objects $A, B, M, D^3, P_1, \ldots, P_n$ are fixed). Assume to the contrary that $l$ is not as required, i.e. $\bar{l} > 0$. Since $M$ is on one side of $\partial P_i$, number $\#(l \cap P_i)$ is even for each $i$. (If $m = 2$, we may even obtain that $\#(l \cap P_1) = 0$ and stop here.) Then $\#(l \cap P_i) \geq 2$ for some $i$. Denote by $Q$ and $R$ two consecutive points of $l \cap P_i$. Denote by $Q'$ the point of $l$ slightly before $Q$ and by $R'$ the point of $l$ slightly after $R$ (Fig. 9). Since $P_i$ is connected, $Q$ and $R$ can be connected by a path in $P_i$. So $Q'$ and $R'$ can be connected by a path $l'$ very close to $P_i$ but not intersecting $P_i$. Path $l'$ does not intersect any of $P_1, \ldots, P_n$ because it is very close to $P_i$ and $P_1, \ldots, P_n$ are pairwise disjoint. Substitute the part of $l$ between $Q'$ and $S'$ by $l'$. Denote the obtained path by $l''$. Then $\overline{l''} = \bar{l} - 2$. This contradicts to the minimality of $\bar{l}$. Thus $l$ is as required.

\[\square\]

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