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SOME REMARKS ON THE NONLINEAR SCHRÖDINGER EQUATION WITH FRACTIONAL DISSIPATION.

MOHAMAD DARWICH AND LUC MOLINET.

Abstract. We consider the Cauchy problem for the $L^2$-critical focussing nonlinear Schrödinger equation with a fractional dissipation. According to the order of the fractional dissipation, we prove the global existence or the existence of finite time blowup dynamics with the log-log blow-up speed for $\|\nabla u(t)\|_{L^2}$.

1. Introduction

In this paper, we study the blowup and the global existence of solutions for the $L^2$-critical nonlinear Schrödinger equation (NLS) with a fractional dissipation term:

$$\begin{cases}
i u_t + \Delta u + |u|^\frac{4}{d} u + ia(-\Delta)^su = 0, (t,x) \in [0, \infty) \times \mathbb{R}^d, \quad d \in \mathbb{N}^*, \\ u(0) = u_0 \in H^r(\mathbb{R}^d) \end{cases} (1.1)$$

where $a > 0$ is the coefficient of friction, $s > 0$ and $r \in \mathbb{R}$.

The NLS equation ($a = 0$) arises in various areas of nonlinear optics, plasma physics and fluid mechanics to describe propagation phenomena in dispersive media. To take into account weak dissipation effects, one usually add a linear damping term as in the linear damped NLS equation (see for instance Fibich [12]):

$$iu_t + \Delta u + iau + |u|^pu = 0, \quad a > 0.$$  

However, in a wide range of situations a frequency-dependent attenuation has been observed (cf [7]). This motivates to rather complete the NLS equation with a laplacian term as in the following complex Ginzburg-Landau equation studied in Passota-Sulem-Sulem [32]:

$$iu_t + \Delta u - ia\Delta u + |u|^pu = 0, \quad a > 0,$$

Now, in many cases of practical importance the damping cannot be described by a local term even in the long-wavelength limit. In media with dispersion the weak dissipation is, in general, non local (see for instance Ott-Sudan [30]). It is thus quite natural to complete the NLS equation by a non local dissipative term in order to take into account some dissipation phenomena.

In this this paper we complete the $L^2$-critical NLS equation (1.2) with a fractional laplacian of order $2s$, $s > 0$, and study the influence of this term on the blow-up phenomena for this equation. The fractional laplacian is commonly used to model fractal (anomalous) diffusion related to the Lévy

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flights (see e.g. Stroock [35], Bardos and all [3], Hanyga [18]). It also appears in the physical literature to model attenuation phenomena of acoustic waves in irregular porous random media (cf. Blackstock [4], Gaul [17], Chen-Holm [7]).

Finally, note that the case of a nonlinear damping of the type \( i\alpha |u|^2u \), has been studied by Antonelli-Sparber and Antonelli-Carles-Sparber (cf. [1] and [2]). In this case the origin of the nonlinear damping term is multiphoton absorption.

Recall that the Cauchy problem for the \( L^2 \)-critical focusing nonlinear Schrödinger equation (\( a = 0 \)):

\[
\begin{cases}
iu_t + \Delta u + |u|^2u = 0 \\
u(0) = u_0 \in H^r(\mathbb{R}^d)
\end{cases}
\tag{1.2}
\]

has been studied by a lot of authors (see for instance [19], [8], [6]) and it is known that the problem is locally well-posed in \( H^r(\mathbb{R}^d) \) for \( r \geq 0 \). For any \( u_0 \in H^1(\mathbb{R}^d) \), a sharp criterion for global existence for (1.2) has been exhibited by Weinstein [37]: Let \( Q \) be the unique radial positive solution (see [5], [21]) to

\[
\Delta Q + Q|Q|^\frac{4}{d} = Q.
\tag{1.3}
\]

If \( \|u_0\|_{L^2} < \|Q\|_{L^2} \) then the solution of (1.2) is global in \( H^1(\mathbb{R}^d) \). This follows from the conservation of the energy and the \( L^2 \) norm and the sharp Gagliardo-Nirenberg inequality which ensures that

\[
\forall u \in H^1, \quad E(u) \geq \frac{1}{2} \left( \int |\nabla u|^2 \right) \left( 1 - \left( \frac{\int |u|^2}{\int |Q|^2} \right)^\frac{d}{2} \right).
\tag{1.4}
\]

Actually, it was recently proven that any solution of (1.2) emanating from an initial datum \( u_0 \in L^2(\mathbb{R}^d) \) with \( \|u_0\|_{L^2} < \|Q\|_{L^2} \) is global and scatters, i.e. \( u \) behaves like the linear evolution of an \( L^2 \) function for large time, \( u(t) \sim e^{it\Delta}u_+ \) (cf. [11]).

On the other hand, NLS has unique minimal mass blow-up solution in \( H^1 \) up to the symmetries of the equation (see [23]) that blow up at some time \( T > 0 \) with a \( H^1 \) norm that grows as \( \frac{1}{T} \).

In the series of papers [24], [25], [26], [27], Merle and Raphael studied the blowup for (1.2) with \( \|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \delta, \delta \) small and proved the existence of the blowup regime corresponding to the log-log law:

\[
\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \left( \frac{\log \log(T-t)}{T-t} \right)^\frac{1}{2}.
\tag{1.5}
\]
Recall that the evolution of (1.2) admits the following conservation laws in the energy space $H^1$:

$L^2$-norm: $m(u) = \|u\|_{L^2} = \left( \int |u(x)|^2 dx \right)^{1/2}$.

Energy: $E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{d}{4 + 2s} \|u\|_{L^{4+2}}^{4+2}$.

Kinetic momentum: $P(u) = \text{Im}(\int \nabla u(x) \bar{\Pi}(x) \, dx)$.

Now, for (1.1) with $a > 0$, there does not exist conserved quantities anymore. However, it is easy to prove that if $u$ is a smooth solution of (1.1) on $[0, T]$, then for all $t \in [0, T]$ it holds

$$\|u(t)\|_{L^2}^2 + a \int_0^t \|(-\Delta)^{s+1} u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 . \quad (1.6)$$

$$\frac{d}{dt} E(u(t)) = -a \int |(-\Delta)^{s+1} u(t)|^2 + a \text{Im} \int ((-\Delta)^s u(t)) u(t) |\nabla u(t)|^{4+2} \quad (1.7)$$

$$\frac{1}{2} \frac{d}{dt} P(u(t)) = a \text{Im} \int ((-\Delta)^s u(t)) \nabla u(t). \quad (1.8)$$

In [10], the first author studied the case $s = 0$. He proved the global existence in $H^1$ for $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, and showed that the log-log regime is stable by such perturbations (i.e. there exist solutions that blowup in finite time with the log-log law).

In [32], Passot, Sulem and Sulem proved that the solutions are global in $H^1(\mathbb{R}^2)$ for $s = 1$. However, their method does not seem to apply for any other values of $d$.

Our aim in this paper is to establish some results, for $s > 0$, on the global existence or the existence of finite time blowup dynamics with the log-log blow-up speed for $\|\nabla u\|_{L^2}$.

Let us now state our results:

**Theorem 1.1.** Let $d = 1, 2, 3, 4$ and $0 < s < 1$ then there exists $\delta_0 > 0$ such that $\forall a > 0$ and $\forall \delta \in [0, \delta_0]$, there exists $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$, such that the solution of (1.1) blows up in finite time in the log-log regime.

**Theorem 1.2.** Let $d \in \mathbb{N}^*$, $s \geq 1$ and $r \geq 0$. Then the Cauchy problem (1.1) is globally well-posed in $H^r(\mathbb{R}^d)$.

**Theorem 1.3.** Let $d \in \mathbb{N}^*$, $0 < s < 1$ and $a > 0$.

1. There exists a real number $0 < \gamma = \gamma(d) \leq \|Q\|_{L^2}$ such that for any initial datum $u_0 \in H^1(\mathbb{R}^d)$ with $\|u_0\|_{L^2} < \gamma$, the emanating solution $u$ is global in $H^1$ with an energy that is non increasing.

2. There does not exist any initial datum $u_0$, with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, such that the solution $u$ of (1.1) blows up at finite time $T^*$ and satisfies

$$\frac{1}{(T^* - t)^{\alpha}} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T^* - t)^{\beta}}, \quad \forall 0 < T - t \ll 1 ,$$

for some pair $(\alpha, \beta)$ satisfying $0 < \beta < \frac{1}{2s}$ and $\beta(1+s) - 1/2 < \alpha \leq \beta$. 

Remark 1.1. It is natural to expect that $\gamma = \|Q\|_{L^2}$. Indeed, for $u_0 \in H^1(\mathbb{R}^d)$ satisfying $\|u_0\|_{L^2} \leq \|Q\|_{L^2(\mathbb{R}^d)}$, (1.6) ensures that $\|u(t)\|_{L^2} < \|Q\|_{L^2(\mathbb{R}^d)}$ as soon as $t > 0$ and the solutions of the critical NLS equation with such initial data are global.

However, in contrary to the case $s = 0$, we do not succeed to prove this fact here since the constant $C_d$ appearing in the estimate on the energy (see subsection 2.3) seems not to be directly related to $Q$. Assertion (2) of Theorem 1.3 is a partial result in this direction since it ensures that we do not have any blowup in the log-log regime, for any $0 < s < 1$, and in the regime $1/4$, for any $0 < s < 1/2$, for initial data with critical or subcritical mass.

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2. Local and global existence results

In this section, we prove Theorem 1.2 and part (1) of Theorem 1.3. Theorem 1.2 will follow from an a priori estimate on the critical Strichartz norm whereas part (1) of Theorem 1.3 follows from a monotonicity of the energy.

2.1. Local existence result. Recall that the main tools to prove the local existence results for (1.2) are the Strichartz estimates for the associated linear propagator $e^{i\Delta t}$. These Strichartz estimates read

$$\|e^{i\Delta t} \phi\|_{L^q_t L^r_x(\mathbb{R}^d)} \lesssim \|\phi\|_{L^2(\mathbb{R}^d)}$$

for any pair $(q, r)$ satisfying $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $2 < q \leq \infty$. Such ordered pair is called an admissible pair.

For $a \geq 0$ and $s \geq 0$ we denote by $S_{a,s}$ the linear semi-group associated with (1.1), i.e. $S_{a,s}(t) = e^{i\Delta t - at(-\Delta)^s}$. It is worth noticing that $S_{a,s}$ is irreversible. The following lemma ensures that the linear semi-group $S_{a,s}$ enjoys the same Strichartz estimates as $e^{i\Delta t}$.

Lemma 2.1. Let $\phi \in L^2(\mathbb{R}^d)$ and $s > 0$. Then for every admissible pair $(q, r)$ it holds

$$\|S_{a,s}(\cdot) \phi\|_{L^q_t L^r_x(\mathbb{R}^d)} \lesssim \|\phi\|_{L^2(\mathbb{R}^d)}.$$

Proof. Setting, for any $t \geq 0$, $G_{a,s}(t, x) = \int e^{-ix\xi} e^{-at|\xi|^{2s}} d\xi$, it holds

$$S_{a,s}(t) \phi = G_{a,s}(t, \cdot) * e^{i\Delta t} \phi, \quad \forall t \geq 0.$$

Noticing that for $s > 0$, $\|G_{a,s}(t, \cdot)\|_{L^1} = \|G_{1,s}(1, \cdot)\|_{L^1}$ and that, according to Lemma 2.1 in [29], $G_{1,s}(1, \cdot) \in L^1(\mathbb{R}^d)$ for $s > 0$, we get

$$\|S_{a,s}(t) \phi\|_{L^q_t L^r_x(\mathbb{R}^d)} = \|G_{a,s}(t, \cdot) * e^{i\Delta t} \phi\|_{L^q_t L^r_x(\mathbb{R}^d)} \lesssim \|e^{i\Delta t} \phi\|_{L^q_t L^r_x(\mathbb{R}^d)} \lesssim \|\phi\|_{L^2}.$$
for (1.1) with \( a \geq 0 \) and \( s > 0 \). More precisely, we have the following statement:

**Proposition 2.1.** Let \( a \geq 0 \), \( s > 0 \), \( r \geq 0 \) and \( u_0 \in H^r(\mathbb{R}^d) \) with \( d \in \mathbb{N}^* \). There exists \( T > 0 \) and a unique solution \( u \in C([0,T];H^r) \cap L_T^{2+2} L_r^{4+2} \) to (1.1) emanating from \( u_0 \). In addition, there exists a neighborhood \( \mathcal{V}_{u_0} \) of \( u_0 \) in \( H^r(\mathbb{R}^d) \) such that the associated solution map is continuous from \( \mathcal{V}_{u_0} \) into \( C([0,T];H^r) \cap L_T^{2+2}([0,T] \times \mathbb{R}^d) \).

Finally, let \( T^* \) be the maximal time of existence of the solution \( u \) in \( H^r(\mathbb{R}^d) \), then

\[
T^* < \infty \implies \|u\|_{L_T^{2+2} L_r^{4+2}} = +\infty.
\] (2.1)

**2.2. Proof of Theorem 1.2.** Let \( u \in C([0,T];L^2(\mathbb{R}^d)) \) be the solution emanating from some initial datum \( u_0 \in L^2(\mathbb{R}^d) \). We have the following a priori estimates:

**Lemma 2.2.** Let \( u \in C([0,T];L^2(\mathbb{R}^d)) \) be the solution of (1.1) emanating from \( u_0 \in L^2(\mathbb{R}^d) \). Then

\[
\|u\|_{L^2_T L^2} \leq \|u_0\|_{L^2} \text{ and } \|(-\Delta)^{\frac{1}{2}} u\|_{L^2_T L^2} \leq \frac{1}{2a}\|u_0\|_{L^2}.
\] (2.2)

**Proof.** Assume first that \( u_0 \in H^\infty(\mathbb{R}^d) \). Then (1.6) ensures that the mass is decreasing as soon as \( u \) is not the null solution and (1.6) leads to

\[
\int_0^T \|(-\Delta)^{\frac{1}{2}} u(t)\|_{L^2}^2 dt = -\frac{1}{2a}(\|u(T)\|_{L^2}^2 - \|u_0\|_{L^2}^2) \leq \frac{1}{2a}\|u_0\|_{L^2}^2.
\]

This proves (2.2) for smooth solutions. The result for \( u_0 \in L^2(\mathbb{R}^d) \) follows by approximating \( u_0 \) in \( L^2 \) by a smooth sequence \((u_{0,n}) \subset H^\infty(\mathbb{R}^d)\). \(\Box\)

Note that the first estimate in (2.2) implies that \( \|u\|_{L^2_T L^2} \leq T^{1/2}\|u_0\|_{L^2} \) and thus by interpolation:

\[
\|\nabla u\|_{L^2_T L^2} \lesssim \|(-\Delta)^{\frac{1}{2}} u\|_{L^2_T L^2}^{\frac{1}{2}} \|u\|_{L^2_T L^2}^{\frac{1}{2}} \lesssim T^{\frac{1}{2} - \frac{1}{2}}
\] (2.3)

Interpolating now between (2.3) and the first estimate of (2.2) we get

\[
\|u\|_{L_T^{2+2} H^{\frac{2d}{2+2d}}} \lesssim T^{\frac{1}{2} - \frac{1}{2}}
\]

and the embedding \( H^{\frac{2d}{2+2d}}(\mathbb{R}^d) \hookrightarrow L^{2+2}(\mathbb{R}^d) \) ensures that

\[
\|u\|_{L_T^{2+2} L_r^{4+2}} \lesssim \|u\|_{L_T^{2+2} H^{\frac{2d}{2+2d}}} \lesssim T^{\frac{1}{2} - \frac{1}{2}}.
\]

Denoting by \( T^* \) the maximal time of existence of \( u \) in \( L^2(\mathbb{R}^d) \) and letting \( T \) tends to \( T^* \), this contradicts (2.1) whenever \( T^* \) is finite. This proves that the solutions are global in \( H^r(\mathbb{R}^d) \).

Moreover, for \( s = 1 \) we have \( \|u\|_{L_T^{2+2} L_r^{4+2}} \lesssim 1 \) for any \( T > 0 \) which ensures that

\[
\|u\|_{L_0^{2+2} H^{\frac{2d}{2+2d}}} \lesssim 1.
\]

\(\text{(1)}\) It is worth noticing that for \( r = 0 \), the neighborhood \( \mathcal{V}_{u_0} \) does not depend only on \( \|u_0\|_{H^r} \) but on the Fourier profile of \( u_0 \) (see for instance [20]).
2.3. Proof of Assertion 1 of Theorem 1.3. Note that the global existence for any \( u_0 \in L^2(\mathbb{R}^d) \) with \( \| u_0 \|_{L^2} \) small enough can be proven, as for the critical NLS equation, directly by a fixed point argument thanks to Lemma 2.1. This ensures the global existence in \( H^r(\mathbb{R}^d), \ r \geq 0, \) under the same smallness condition on \( \| u_0 \|_{L^2} \). We will not invoke this fact here and we will directly prove Assertion 1 of Theorem 1.3 by combining (1.4) and a monotony result on \( t \mapsto E(u(t)) \). To do this, we will work with smooth solutions and then get the result for \( H^1 \)-solutions by continuity with respect initial data.

So, let \( u \in C([0, T]; H^\infty(\mathbb{R}^d)) \) be a solution to (1.1) emanating from \( u_0 \in H^\infty(\mathbb{R}^d) \). Then it holds
\[
\frac{d}{dt} E(u(t)) = -a \int |(-\Delta)^{\frac{t}{4}} u(t)|^2 + a \text{Im} \int \left((-\Delta)^s u(t) \right) \left| u(t) \right|^3 u(t)
\]
and Hölder inequalities in physical space and in Fourier space lead to
\[
\int \left((-\Delta)^s u \right) \left| u \right|^4 \leq \left((-\Delta)^s u \right) \| u \|_{L^2}^4 \| u \|_{L^{\frac{4}{3}} + \frac{2}{1}}^3
\]
with
\[
\|(-\Delta)^s u \|_{L^2} \leq \left((-\Delta)^{\frac{s+1}{2}} u \right) \| u \|_{L^2}^{\frac{s+4}{s+2}}.
\]
Let us recall the following Gagliardo-Nirenberg inequality (2)
\[
\| u \|_{L^{\frac{4}{3} + \frac{2}{1}}} \leq C_{d}^{\frac{s+2}{d}} \| \nabla u \|_{L^2} \| u \|_{L^2}^{\frac{s+4}{s+2}}.
\]
This estimate together with Cauchy-Schwarz inequality (in Fourier space)
\[
\| \nabla u \|_{L^2}^2 \leq \left((-\Delta)^{\frac{s+1}{2}} u \right) \| u \|_{L^2}^2 \| u \|_{L^2}^{\frac{2}{s+4}}
\]
lead to
\[
\| u \|_{L^{\frac{4}{3} + \frac{2}{1}}} \leq C_{d}^{\frac{s+2}{d}} \|(-\Delta)^s u \|_{L^2}^{\frac{s+4}{s+2}} \| u \|_{L^2}^{\frac{4}{s+4}}.
\]
Combining the above estimates we eventually obtain
\[
\frac{d}{dt} E(u(t)) \leq a \|(-\Delta)^{\frac{s+1}{2}} u \|_{L^2}^2 \left(C_{d}^{\frac{s+1}{d}} \| u \|_{L^2}^{\frac{4}{d}} - 1 \right)
\]
which together with (2.2) implies that \( E(u(t)) \) is not increasing for \( \| u_0 \|_{L^2} \leq C_{d}^{1+\frac{4}{d}} \).

3. Proof of Assertion 2 of Theorem 1.3

Special solutions play a fundamental role for the description of the dynamics of (NLS). They are the solitary waves of the form \( u(t, x) = \exp(it)Q(x) \), where \( Q \) the unique positive radial solution to
\[
\Delta Q + Q|Q|^\frac{4}{d} = Q. \tag{3.1}
\]
The pseudo-conformal transformation applied to the “stationary” solution \( e^{it}Q(x) \) yields an explicit solution for (NLS)
\[
S(t, x) = \frac{1}{|t|^{\frac{d}{2}}} Q(\frac{x}{t}) e^{-\frac{|x|^2}{2} + \frac{t}{2}}
\]
\(^{(2)}\)It is proven in [37] that the constant \( C_d \) is related for \( d = 1, 2, 3 \) to the \( L^2 \)-norm of the ground state solution of
\[
2\Delta \psi - (\frac{t}{2} - 1)\psi + \psi^{\frac{d+1}{2}} = 0.
\]
which blows up at $T^* = 0$.

Note that
\[ \|S(t)\|_{L^2} = \|Q\|_{L^2} \text{ and } \|\nabla S(t)\|_{L^2} \sim \frac{1}{t} \]  
(3.2)

It turns out that $S(t)$ is the unique minimal mass blow-up solution in $H^1$ up to the symmetries of the equation (see [23]).

A known lower bound (see [8] and [6]) on the blow-up rate for (NLS) is
\[ \|\nabla u(t)\|_{L^2} \geq \frac{C(u_0)}{\sqrt{T-t}}. \]  
(3.3)

Note that this blow-up rate is strictly lower than the one of $S(t)$ given by (3.2) and of the log-log law given by (1.5).

To prove assertion 2 of Theorem 1.3, we will need the following result (see [16]):

**Theorem 3.1.** Let $(v_n)_n$ be a bounded family of $H^1(\mathbb{R}^d)$, such that:
\[ \limsup_{n \to +\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^d)} \leq M \quad \text{and} \quad \limsup_{n \to +\infty} \|v_n\|_{L^{4+2d} \mathbb{R}^d} \geq m. \]  
(3.4)

Then, there exists $(x_n)_n \subset \mathbb{R}^d$ such that:
\[ v_n(\cdot + x_n) \rightharpoonup V \text{ weakly}, \]
with
\[ \|V\|_{L^2(\mathbb{R}^d)} \geq \left( \frac{d}{d+1} \right)^{\frac{d}{2}} \frac{M^{d+1}}{M^2} \|Q\|_{L^2(\mathbb{R}^d)}. \]

Suppose that there exist an initial data $u_0$ with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, such that the corresponding solution $u(t)$ blows up at time $T > 0$ with the following behavior:
\[ \frac{1}{(T-t)\alpha} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T-t)^\beta}, \quad \forall t \in [0, T], \]  
(3.5)

where $\beta > 0$ and $\alpha \geq \beta(1+s) - 1/2$.

Recalling that
\[ E(u(t)) = E(u_0) - a \int_0^t K(u(\tau)) d\tau, \quad t \in [0, T], \]  
(3.6)

with $K(u(t)) = \int |(-\Delta)^{\frac{s-1}{2}} u|^2 - Im \int ((-\Delta)^s u) \overline{u}^\frac{4}{d} d\tau$, we obtain that
\[ E(u(t)) \lesssim E(u_0) + \int_0^t ((-\Delta)^s u) \overline{u}^\frac{4}{d} dx \lesssim E(u_0) + \int_0^t \|(-\Delta)^s u\|_{L^2} \|u\|_{L^{\frac{4}{2s}}}^{\frac{4}{d} + 1} \]
This last estimate together with
\[ \|u\|_{L^{\frac{4}{2s}} \mathbb{R}^d} \lesssim \|\nabla u\|_{L^2} \|u_0\|_{L^{2d} \mathbb{R}^d}^{\frac{4-d}{2d}} \quad \text{and} \quad \|(-\Delta)^s(u)\|_{L^2} \lesssim \|\nabla u\|_{L^2}^{2s} \|u\|_{L^{2s-2d} \mathbb{R}^d} \lesssim \|\nabla u\|_{L^2}^{2s} \]
yield
\[ E(u(t)) \lesssim E(u_0) + \int_0^t \|\nabla u\|_{L^2}^{2s} dx \]  
(3.7)
Note that assumption (3.5) ensures that
\[
0 \leq \int_0^T \frac{\|\nabla u(t)\|_{L^2(R^d)}^{2+2s} \ d\tau}{\|\nabla u(t)\|_{L^2(R^d)}^2} \leq (T - t)^{-2\beta(1+\alpha)+1+2\alpha} \to 0 \quad \text{as } t \nearrow T, \tag{3.8}
\]

Now, let
\[
\rho(t) = \frac{\|\nabla Q\|_{L^2(R^d)}}{\|\nabla u(t)\|_{L^2(R^d)}} \quad \text{and} \quad v(t, x) = \rho^2 u(t, \rho x)
\]

and let \((t_k)\) be a sequence of positive times such that \(t_k \nearrow T\). We set \(\rho_k = \rho(t_k)\) and \(v_k = v(t_k, \cdot)\). The family \((v_k)\) satisfies
\[
\|v_k\|_{L^2(R^d)} = \|u(t_k, \cdot)\|_{L^2(R^d)} \leq \|u_0\|_{L^2(R^d)} \leq \|Q\|_{L^2(R^d)} \quad \text{and} \quad \|\nabla v_k\|_{L^2(R^d)} = \|\nabla Q\|_{L^2(R^d)}.
\]

The above estimate on \(v_k\) and (3.7) lead to
\[
0 \leq \frac{1}{2} \int \|\nabla v_k\|^2 \left(1 - \frac{\int \|v_k\|^2}{\|Q\|^2}\right) \leq E(v_k) = \rho_k^2 E(u(t)) \leq \rho_k^2 E(u_0) + \rho_k^2 \int_0^{t_k} \|\nabla u(\tau)\|_{L^2(R^d)}^{2+2s} \ d\tau
\]
which, together with (3.8), ensures that \(\lim_{k \to +\infty} E(v_k) = 0\). This forces
\[
\|v_k\|_{L^{3+2d}(R^d)} \to \frac{d+2}{d} \|\nabla Q\|_{L^2(R^d)} = \frac{d+2}{d} \|\nabla Q\|_{L^2(R^d)} \tag{3.9}
\]
and thus the family \((v_k)\) satisfies the hypotheses of Theorem 3.1 with
\[
m^{\frac{d}{3}+2} = \frac{d+2}{d} \|\nabla Q\|^2_{L^2(R^d)} \quad \text{and} \quad M = \|\nabla Q\|_{L^2(R^d)}.
\]

Hence, there exists a family \((x_k)\) \(\subset \mathbb{R}\) and a profile \(V \in H^1(\mathbb{R})\) with
\[
\|V\|_{L^2(R^d)} \geq \|Q\|_{L^2(R^d)}, \quad \text{such that,}
\]
\[
\rho_k^\frac{d}{2} u(t_k, \rho_k \cdot + x_k) \rightharpoonup V \in H^1 \quad \text{weakly.} \tag{3.10}
\]

Using (3.10), \(\forall A \geq 0\)
\[
\lim_{n \to +\infty} \int_{B(0,A)} \rho_n^d |u(t_n, \rho_n x + x_n)|^2 \ dx \geq \int_{B(0,A)} |V|^2 \ dx.
\]
But, since \(\lim_{n \to +\infty} \rho_n = 0\), \(\rho_n A < 1\) for \(n\) large enough and thus
\[
\liminf_{n \to +\infty} \sup_{y \in \mathbb{R} \atop |x-y| \leq 1} |u(t_n, x)|^2 \ dx \geq \liminf_{n \to +\infty} \int_{|x-x_n| \leq \rho_n A} |u(t_n, x)|^2 \ dx \geq \int_{|x| \leq A} |V|^2 \ dx.
\]
Since this it is true for all \(A > 0\) we obtain that
\[
\|u_0\|_{L^2}^2 \geq \liminf_{n \to +\infty} \sup_{y \in \mathbb{R} \atop |x-y| \leq 1} |u(t_n, x)|^2 \ dx \geq \|Q\|_{L^2}^2
\]
which contradicts the assumption \(\|u_0\|_{L^2} \leq \|Q\|_{L^2}\) and the desired result is proven.
4. Blow up solution.

In this section, we prove the existence of the explosive solutions in the case $0 < s < 1$.

**Theorem 4.1.** Let $0 < s < 1$ and $1 \leq d \leq 4$. There exist a set of initial data $\Omega$ in $H^1$, such that for any $0 < a < a_0$ with $a_0 = a_0(s)$ small enough, the emanating solution $u(t)$ to (1.1) blows up in finite time in the log-log regime.

The set of initial data $\Omega$ is the set described in [24], in order to initialize the log-log regime. It is open in $H^1$. Using the continuity with regard to the initial data and the parameters, we easily obtain the following corollary:

**Corollary 4.1.** Let $0 < s < 1$, $1 \leq d \leq 4$ and $u_0 \in H^1$ be an initial data such that the corresponding solution $u(t)$ of (1.2) blows up in the log-log regime. There exist $\beta_0 > 0$ and $a_0 > 0$ such that if $v_0 = u_0 + h_0$, $\|h_0\|_{H^1} \leq \beta_0$ and $a \leq a_0$, the solution $v(t)$ for (1.1) with the initial data $v_0$ blows up in finite time.

Now to prove Theorem 4.1, we look for a solution of (1.1) such that for $t$ close enough to blowup time, we shall have the following decomposition:

$$u(t,x) = \frac{1}{\lambda^2(t)}(Q_{b}(t) + e)(t, \frac{x - x(t)}{\lambda(t)})e^{\gamma(t)},$$

for some geometrical parameters $(b(t), \lambda(t), x(t), \gamma(t)) \in (0, \infty) \times (0, \infty) \times \mathbb{R}^d \times \mathbb{R}$, $\lambda(t) \sim \left\| \nabla u(0) \right\|_{L^2}^{-1}$, $b \sim -\frac{\lambda}{\chi(t)}$ where $\frac{d\chi}{dt} = \frac{1}{\lambda^2(t)}$. The profiles $Q_b$ are suitable deformations of $Q$ related to some extra degeneracy of the problem, in fact these profile $Q_b$ are a regularization of the exact self similar solutions to (1.2) which satisfy the nonlinear elliptic equation:

$$\Delta Q_b - Q_b + ib\frac{d}{2}Q_b + y.\nabla Q_b + Q_b|Q_b|^\frac{2}{s} = 0.$$\hspace{1cm}(4.2)

Note that our proof is very close to the case of $s = 0$ (see Darwich [10]). Actually, as noticed in [33], we only need to prove that in the log-log regime the $L^2$ norm does not grow, and the growth of the energy (resp the momentum) is below $\frac{1}{\lambda(t)}$ (resp $\frac{1}{\chi(t)}$). In this paper, we will prove that in the log-log regime, the growths of the energy and the momentum are bounded by:

$$E(u(t)) \lesssim \log(\lambda(t))\lambda(t)^{-2s}, \quad P(u(t)) \lesssim \log(\lambda(t))\lambda(t)^{\frac{2}{s+1}}.$$\hspace{1cm}(4.3)

Let us recall that a fonction $u : [0, T] \mapsto H^1$ follows the log-log regime if the following uniform controls on the decomposition (4.1) hold on $[0, T]$:

- Control of $b(t)$
  $$b(t) > 0, \quad b(t) < 10b(0).$$\hspace{1cm}(4.4)

- Control of $\lambda$:
  $$\lambda(t) \leq e^{-e^{-\frac{t}{100\chi(t)}}}$$\hspace{1cm}(4.5)

and the monotonicity of $\lambda$:

$$\lambda(t_2) \leq \frac{3}{2}\lambda(t_1), \quad \forall \ 0 \leq t_1 \leq t_2 \leq T.$$\hspace{1cm}(4.6)
Let $k_0 \leq k_+ \in [0,T]$ such that
\[
\frac{1}{2k_0} \leq \lambda(0) \leq \frac{1}{2k_0 - 1}, \quad \frac{1}{2k_+} \leq \lambda(T^+) \leq \frac{1}{2k_+ - 1}
\] (4.6)
and for $k_0 \leq k \leq k_+$, let $t_k$ be a time such that
\[
\frac{1}{2k_0} \leq \lambda(t_k) \leq \frac{1}{2k_0 - 1}, \quad \frac{1}{2k_+} \leq \lambda(t_{k+}) \leq \frac{1}{2k_+ - 1}
\] (4.7)
then we assume the control of the doubling time interval:
\[
t_{k+1} - t_k \leq k \lambda(t_k)^2.
\] (4.8)

The main point is to establish that (4.3)-(4.9) determine a trapping region for the flow. Actually, after the decomposition (4.1) of $u$, the log-log regime corresponds to the following asymptotic controls
\[
b_s \sim Ce^{-\frac{c}{b}}, \quad \frac{\lambda_s}{\lambda} \sim b
\] (4.10)
and
\[
\int_{\mathbb{R}^d} |\nabla \epsilon(t)|^2 \lesssim e^{-\frac{c}{b}},
\] (4.11)
where we have introduced the rescaled time
\[
\frac{ds}{dt} = \frac{1}{\lambda^2(t)}.
\]
In fact, (4.11) is partly a consequence of the preliminary estimate:
\[
\int_{\mathbb{R}^d} |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}} + \lambda^2(t) E(t).
\] (4.12)
One then observes that in the log-log regime, the integration of the laws (4.10) yields
\[
\lambda \sim e^{-\frac{c}{b}}, \quad b(t) \to 0, \quad t \to T.
\] (4.13)
Hence, the term involving the conserved Hamiltonian is asymptotically negligible with respect to the leading order term $e^{-\frac{c}{b}}$ which drives the decay (4.12) of $b$. This was a central observation made by Planchon and Raphael in [33]. In fact, any growth of the Hamiltonian algebraically below $\frac{1}{\lambda^2(t)}$ would be enough. In this paper, we will prove that in the log-log regime, the growth of the energy is estimated by:
\[
E(u(t)) \lesssim (\log (\lambda(t))) \lambda^{-2s}(t).
\] (4.14)
It then follows from (4.12) that:
\[
\int_{\mathbb{R}^d} |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}}.
\] (4.15)
An important feature of this estimate of $H^1$ flavor is that it relies on a flux computation in $L^2$. This allows one to recover the asymptotic laws for the geometrical parameters (4.10) and to close the bootstrap estimates of the log-log regime.

**Remark 4.1.** Actually, one also needs the bound on the momentum to control the geometrical parameters (see Lemma 7.2 in [10]).
4.1. Control of the energy and the kinetic momentum. Let us recall that we say that an ordered pair \((q, r)\) is admissible whenever \(\frac{2}{q} + \frac{d}{r} = d\frac{2}{2}\) and \(2 < q \leq \infty\). We define the Strichartz norm of functions \(u : [0, T] \times \mathbb{R}^d \mapsto \mathbb{C}\) by:

\[
\|u\|_{S^0([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{admissible}} \|u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)}
\]

and

\[
\|u\|_{S^1([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{admissible}} \|\nabla u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)}
\]

We will sometimes abbreviate \(S^i([0, T] \times \mathbb{R}^2)\) with \(S^i_T\) or \(S^i[0, T]\), \(i = 1, 2\). Now we will derive an estimate on the energy, to check that it remains small with respect to \(\lambda^{-2}\).

**Proposition 4.1.** Assuming that (4.4)-(4.9) hold, then the energy and kinetic momentum are controlled on \([0, T^+]\) by:

\[
|E(u(t))| \lesssim (\log \lambda(t))\lambda^{-2s}(t),
\]

\[
|P(u(t))| \lesssim (\log \lambda(t))\lambda^{-\frac{2s}{s+t}}(t).
\]

To prove Proposition 4.1, we will need the two following lemmas.

**Lemma 4.1.** Let \(u \in C([0, T]; H^1(\mathbb{R}^d))\) be a solution of (1.1). Then we have the following estimation:

\[
\|\nabla u\|_{L_T^2 L_x^2} + \|(-\Delta)^{\frac{1}{4} \frac{2}{3}} u\|_{L_T^2 L_x^2} \lesssim \|\nabla u_0\|_{L_x^2} + \|u\|_{L_T^2} \|\nabla u\|_{L_T^2 L_x^2}
\]

**Proof.** Multiply Equation 1.1 by \(\nabla u\), integrate and take the imaginary part, to obtain:

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + a \int |(-\Delta)^{\frac{1}{4} \frac{2}{3}} u|^2 \leq \int \|\nabla u\|^2 u \Delta u = \int \nabla (|u|^{\frac{3}{2}} u) \nabla u
\]

By integrating in time, we get

\[
\frac{1}{2} \|\nabla u\|_{L_T^2 L_x^2}^2 + a \|(-\Delta)^{\frac{1}{4} \frac{2}{3}} u\|_{L_T^2 L_x^2}^2 \leq \frac{1}{2} \|\nabla u_0\|_{L_x^2}^2 + \|\nabla (|u|^{\frac{3}{2}} u)\|_{L_T^2 L_x^2} \|\nabla u\|_{L_T^2 L_x^2}
\]

Dividing by \(\sqrt{2} \|\nabla u\|_{L_T^2 L_x^2} + a \|(-\Delta)^{\frac{1}{4} \frac{2}{3}} u\|_{L_T^2 L_x^2}\) we obtain:

\[
\|\nabla u\|_{L_T^2 L_x^2} + \|(-\Delta)^{\frac{1}{4} \frac{2}{3}} u\|_{L_T^2 L_x^2} \lesssim \|\nabla u_0\|_{L_x^2} + \|u\|_{L_T^2} \|\nabla u\|_{L_T^2 L_x^2}
\]

\(\square\)

**Lemma 4.2.** There exists a real number \(0 < \alpha \ll 1\) such that the following holds: Let \(u \in C([0, T]; H^1(\mathbb{R}^d))\) be the solution of (1.1) emanating from \(u_0 \in H^1\). For a fixed \(t \in [0, T]\) we set \(\Delta t = \alpha \|u_0\|_{L_T^2}^\frac{2}{2}\|u(t)\|_{H^1}^2\). Then \(u \in C([t, t + \Delta t]; H^1(\mathbb{R}^d))\) and we have the following controls

\[
\|u\|_{S^0([t, t + \Delta t])} \leq 2 \|u_0\|_{L_x^2}
\]

and

\[
\|u\|_{S^1([t, t + \Delta t])} + \|(-\Delta)^{\frac{1}{4} \frac{2}{3}} u\|_{L^2([t, t + \Delta t])L_x^2} \leq 2 \|u(t)\|_{H^1(\mathbb{R}^d)}
\]
Proof. We first assume that $\Delta t > 0$ is such that $t + \Delta t < T$. Then, according to Lemma 2.1, it holds

$$\left\| \int^t_0 S_{a,s}(t - \tau)|u(\tau)|^{\frac{3}{2}} u(\tau) d\tau \right\|_{S^1(t, t + \Delta t)} \lesssim \left\| u \right\|^3_{L^1(t, t + \Delta t)}.$$

Using the Hölder inequality we obtain:

$$\left( \int |u|^3 |\nabla u|^2 \right)^{\frac{1}{2}} \leq \left( \int |u|^{\frac{2(4+d)}{d}} \right)^{\frac{d}{2(4+d)}} \left( \int |\nabla u|^{\frac{2(4+d)}{d}} \right)^{\frac{d}{2(4+d)}}.$$

Integrating in time and applying again Hölder inequality we get:

$$\left\| u \right\|^\frac{3}{2} \|\nabla u\|_{L^1(t, t + \Delta t)} \leq \left( \int \left( \int |u|^{\frac{2(4+d)}{d}} dx \right)^{\frac{d}{2(4+d)}} dt \right)^{\frac{d}{2(4+d)}} \times \left( \int \left( \int |\nabla u|^{\frac{2(4+d)}{d}} dx \right)^{\frac{d}{2(4+d)}} dt \right)^{\frac{d}{2(4+d)}}.$$

Thus:

$$\left\| u \right\|^\frac{3}{2} \|\nabla u\|_{L^1(t, t + \Delta t)} \leq \left\| u \right\|^{\frac{3}{2}}_{L^{\frac{4+d}{d}}((t, t + \Delta t))} \left\| \nabla u \right\|_{L^{\frac{4+d}{d}}((t, t + \Delta t))} \left\| u \right\|_{S^1(t, t + \Delta t)}.$$

By Sobolev inequalities we have:

$$\left\| u \right\|_{L^{\frac{4+d}{d}}((t, t + \Delta t))} \leq \left\| u \right\|_{H^{\frac{2d}{d}}((t, t + \Delta t))},$$

which, according to (1.6), leads to

$$\left\| u \right\|^\frac{3}{2} \|\nabla u\|_{L^1(t, t + \Delta t)} \leq \left\| u \right\|^\frac{3}{2} \left\| u \right\|_{H^{\frac{2d}{d}}((t, t + \Delta t))} \left\| u \right\|_{S^1(t, t + \Delta t)}.$$

Now by interpolation we obtain for $d = 1, 2, 3, 4$:

$$\left\| u \right\|_{L^{\frac{4+d}{d}}((t, t + \Delta t))} \leq \left\| \left( \Delta \right)^{\frac{d}{2d}} \right\|_{L^{\frac{4+d}{d}}} \left\| u \right\|_{L^{\frac{4+d}{d}}} \left\| u \right\|_{S^1(t, t + \Delta t)} \left\| u \right\|_{S^1(t, t + \Delta t)}.$$

By Lemma 4.1 it holds

$$\left\| u \right\|_{L^{\infty}(t, t + \Delta t)} \left\| u \right\|_{H^1} \left\| u \right\|_{L^2} \lesssim \left\| u \right\|_{H^1} \left\| u \right\|_{L^2} \left\| u \right\|_{S^1(t, t + \Delta t)} \left\| u \right\|_{S^1(t, t + \Delta t)};$$

we finally get

$$\left\| u \right\|_{S^1(t, t + \Delta t)} \left\| u \right\|_{L^2(t, t + \Delta t)} \lesssim \left\| u \right\|_{H^1} \left\| u \right\|_{L^2} \left\| u \right\|_{S^1(t, t + \Delta t)}.$$

In view of (2.1) and a continuity argument, it follows that $u \in C([t, t + \Delta t]; H^1(\mathbb{R}^d))$ for some $\Delta t \sim \left\| u \right\|_{L^2} \left\| u \right\|_{H^1} \left\| u \right\|_{S^1(t, t + \Delta t)}$.

In the same way

$$\left\| u \right\|_{S^0(t, t + \Delta t)} \lesssim \left\| u \right\|_{L^2} \left\| u \right\|_{H^1} \left\| u \right\|_{S^0} \left\| u \right\|_{S^0(t, t + \Delta t)}.$$
which ensures that

\[ \|u\|_{S^0[t, t + \Delta t]} \leq 2 \|u_0\|_{L^2(\mathbb{R}^d)}. \]

\[ \square \]

**Proof of Proposition 4.1**: According to (4.8), each interval \([t_k, t_{k+1}]\), can be divided into \(k\) intervals, \([\tau_k^j, \tau_k^{j+1}]\) of length less than \(\lambda(t_k)\). From (1.7), we have

\[ |E(u(\tau_k^{j+1})) - E(u(\tau_k^j))| \lesssim \int_{\tau_k^j}^{\tau_k^{j+1}} \int_{\mathbb{R}^d} (-(\Delta)^s u) \|u\|^\frac{4}{d+2} dx dt \leq \|\nabla u\|^\frac{2s}{d-2s} \|u\|^\frac{2s}{d-2s} \|u\|^\frac{2s}{d+2s}. \]

For notation convenience we set \(\Theta = [\tau_k^j, \tau_k^{j+1}] \times \mathbb{R}^d\). By Plancherel formula and Hölder inequality, it holds

\[ \|-(\Delta)^\frac{s}{2} u\|^2_{L^2(\mathbb{R}^d; \Theta)} \lesssim \|\nabla u\|^\frac{2s}{d-2s} \|u\|^\frac{2s}{d-2s} \|u\|^\frac{2s}{d+2s}. \]

Noticing that the fractional Leibniz rule (see [20]) leads to

\[ \|-(\Delta)^\frac{s}{2} \|u\|^2_{L^2(\mathbb{R}^d; \Theta)} \lesssim \|\nabla u\|^\frac{2s}{d-2s} \|u\|^\frac{2s}{d-2s} \|u\|^\frac{2s}{d+2s}. \]

Finally, we obtain

\[ |E(u(\tau_k^{j+1})) - E(u(\tau_k^j))| \lesssim \|u\|^\frac{2s}{d+2s} \|\nabla u\|^\frac{2s}{d-2s} \|u\|^\frac{2s}{d+2s}. \]

Since \((\frac{4}{d} + 2, \frac{4}{d} + 2)\) is an admissible pair, Lemma 4.2 yields

\[ |E(u(\tau_k^{j+1})) - E(u(\tau_k^j))| \lesssim \lambda(t_k)^{-2s}. \]

and summing over \(j\) we get

\[ |E(u(t_{k+1})) - E(u(t_k))| \lesssim k\lambda(t_k)^{-2s}. \]

Finally, taking \(T^+ = T\) and summing from \(k_0\) to \(k^+\), we obtain:

\[ |E(u(T^+)) - E(u(0))| \lesssim k^+ \lambda^{-2s} T^+ \lesssim \log(\lambda(T)) \lambda^{-2s}(T). \]

Note that the growth of the energy is small with respect to \(\frac{1}{T^+}\), because \(s < 1\).

Let us now proceed with the momentum. According to (1.8) we have:

\[ |P(u(\tau_k^{j+1})) - P(u(\tau_k^j))| \lesssim \int_{\tau_k^j}^{\tau_k^{j+1}} \int_{\mathbb{R}^d} \nabla u \nobreak\,(\Delta)^s u \bigg| dx dt. \]

But

\[ \int_{\mathbb{R}^d} \nabla u \nobreak\,(\Delta)^s u \bigg| = \int_{\mathbb{R}^d} ((\Delta)^\frac{s}{2} \Delta u) \ nobreak\,(\Delta)^\frac{s}{2} \Delta u \leq \|\nabla u\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^2(\mathbb{R}^d)} \]}

with

\[ \|\nabla u\|_{L^2(\mathbb{R}^d)} = \|\nabla u\|_{L^2(\mathbb{R}^d)} \]
and, by interpolation,
\[
\|(-\Delta)^{\frac{s}{2} + \frac{1}{2}} u\|_{L^2(\mathbb{R}^d)} \leq \|(-\Delta)^{\frac{s}{2} + \frac{1}{2}} u\|_{L^2(\mathbb{R}^d)}^{2\theta} \|u\|_{L^2(\mathbb{R}^d)}^{2-2\theta},
\]
where \(0 < \theta = \frac{2s+1}{2s+2} < 1\). Therefore we get
\[
|P(u(\tau_k^{j+1})) - P(u(\tau_k^j))| \lesssim (\tau_{k+1} - \tau_k)^{1-\theta}\|u_0\|_{L^2(\mathbb{R}^d)}^{2-2\theta}\|(-\Delta)^{\frac{s}{2} + \frac{1}{2}} u\|_{L^2(\Theta)}^{2\theta},
\]
and Lemma 4.2 ensures that
\[
|P(u(\tau_k^{j+1})) - P(u(\tau_k^j))| \lesssim \lambda^{2-2\theta}\lambda^{-2\theta} = \lambda^{2-4\theta} = \lambda^{\frac{2s}{s+1}}.
\]
Summing over \(j\) we obtain that:
\[
|P(u(\tau_k) - P(u(\tau_0)))| \lesssim k\lambda^{\frac{2s}{s+1}}
\]
and summing from \(k_0\) to \(k^+\), we finally get
\[
|P(u(T^+)) - P(u(0))| \lesssim \log(\lambda(T))\lambda(T)^{\frac{2s}{s+1}}.
\]
Note that the growth of the momentum is small with respect \(\lambda^{\frac{1}{s}}\) since \(1 - \frac{2s}{s+1} > 0\).

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