Regional topology and approximative solutions of difference and differential equations

Janusz Migda
June 18, 2014

Abstract

We introduce a topology, which we call the regional topology, on the space of all real functions on a given locally compact metric space. Next we obtain a new versions of Schauder’s fixed point theorem and Ascoli’s theorem. We use these theorems and the properties of the iterated remainder operator to establish conditions under which there exist solutions, with prescribed asymptotic behavior, of some difference and differential equations.

Key words: regional topology, difference equation, differential equation, remainder operator, approximative solution, prescribed asymptotic behavior, Schauder’s theorem, Ascoli’s theorem.

AMS Subject Classification: 39A10

1 Introduction

Let \( \mathbb{N}, \mathbb{Z}, \mathbb{R} \) denote the set of positive integers, the set of all integers and the set of real numbers, respectively. Assume

\[ m \in \mathbb{N}, \quad t_0 \in [0, \infty), \quad I = [t_0, \infty). \]

In this paper we consider difference equations of the form

\[ \Delta^m x_n = a_n f(n, x_n) + b_n \tag{E1} \]

\[ n \in \mathbb{N}, \quad a_n, b_n \in \mathbb{R}, \quad f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \]

and differential equations of the form

\[ x^{(m)}(t) = a(t) f(t, x(t)) + b(t) \tag{E2} \]

\[ t \in I, \quad a(t), b(t) \in \mathbb{R}, \quad f : I \times \mathbb{R} \to \mathbb{R}. \]

We are interesting in the existence of solutions with prescribed asymptotic behavior. More precisely, we establish conditions under which, for a given \( \alpha \in (-\infty, 0] \) and solution \( y \) of the equation \( \Delta^m y = b \), there exists a solution \( x \) of \( \text{(E1)} \) such that \( x = y + o(n^\alpha) \).

Analogously, we establish conditions under which, for a given \( \alpha \in (-\infty, 0] \) and solution \( y \) of the equation \( y^{(m)} = b \) there exists a solution \( x \) of \( \text{(E2)} \) such that \( x = y + o(t^\alpha) \).
We obtain these results using a new version of the Schauder fixed point theorem and a new version of the Ascoli theorem. First, we introduce the notion of regional norm and regional topology. In the case of functional space, the regional topology is the topology of uniform convergence. This topology is nonlinear, but it is almost linear, see Remark 6.3. Using this topology we can state our version of the Schauder theorem.

Next, in functional spaces, we introduce the notion of homogeneous at infinity and stable at infinity family of functions. The first notion generalizes the notion of equiconvergence at infinity, see Remark 3.3 and Remark 6.5. We use the notion of homogeneity at infinity to state our version of the Ascoli theorem.

Our approach to the study of existence of solutions with prescribed asymptotic behavior is based on using the iterated remainder operator. In the discrete case properties of this operator were presented in [8], [11] and [13]. The basic properties of this operator in the continuous case we establish in the first part of Section 5.

If \( y \) is a sequence with known properties and \( x \) is a solution of (E1) such that \( x = y + o(1) \), then we say that \( x \) is a solution with prescribed asymptotic behavior. Writing the equality \( x = y + o(1) \) in the form \( y = x + o(1) \) we may say that \( y \) is an approximative solution of (E1).

In Section 4, using the iterated remainder operator we establish conditions under which a given solution \( y \) of the equation \( \Delta^n y = b \) is an approximative solution of (E1). Moreover, the technique of iterated remainder operator allow us to change \( o(1) \) by \( o(n^\alpha) \) for a given nonpositive real \( \alpha \). Hence we obtain an approximative solution which better approximates the solution \( x \).

In Section 5, we obtain an analogous result in the continuous case. In Section 6, we give some additional remarks on the regional topology.

This paper is a continuation of a cycle of papers [7]-[12]. Our studies, in continuous case, were inspired by papers [3, 6, 14, 15] (devoted to asymptotically linear solutions) and by papers [4, 5, 16, 17] (devoted to asymptotically polynomial solutions).

## 2 Notation and terminology

If \( p, k \in \mathbb{Z}, p \leq k \), then \( \mathbb{N}_p, \mathbb{N}(p, k) \) denote the sets defined by

\[
\mathbb{N}_p = \{p, p+1, \ldots \}, \quad \mathbb{N}(p, k) = \{p, p+1, \ldots, k\}.
\]

Let \( X \) be a set. We denote by

\[
F(X), \quad F_b(X)
\]

the space of all functions \( f : X \to \mathbb{R} \) and the space of all bounded functions \( f \in F(X) \) respectively. Let \( f, g \in F(X), F \subset F(X) \). Then

\[
|f|, \quad f + g, \quad f - g, \quad fg
\]

denotes the functions defined by a standard way. If \( Z \subset X \), then \( f|Z \) denotes the restriction \( f|Z : Z \to \mathbb{R} \). Moreover,

\[
F - F = \{f - g : f, g \in F\}, \quad F + g = \{f + g : f \in F\}.
\]

We say that \( F \) is pointwise bounded if for any \( t \in X \) the set \( F(t) = \{f(t) : f \in F\} \) is bounded. If \( X \) is a topological space, then \( C(X) \) denotes the space of all continuous
functions $f \in F(X)$.

Assume $X$ is a locally compact metric space. A function $f \in F(X)$ is called vanishing at infinity if for any $\varepsilon > 0$ there exists a compact subset $Z$ of $X$ such that $|f(t)| < \varepsilon$ for any $t \notin Z$. The space of all vanishing at infinity functions $f \in F(X)$ we denote by $F_0(X)$. Moreover

$$C_0(X) = C(X) \cap F_0(X).$$

Note that any continuous and vanishing at infinity function is bounded.

Assume $p \in \mathbb{N}$, $t_0 \in [0, \infty)$, $I = [t_0, \infty)$. Then

$$F_0(\mathbb{N}_p) = \{ x \in F(\mathbb{N}_p) : \lim_{n \to \infty} x_n = 0 \}, \quad F_0(I) = \{ f \in F(I) : \lim_{t \to \infty} f(t) = 0 \}.$$

Moreover, for $\alpha \in \mathbb{R}$ we use the following notation

$$o(n^\alpha) = \{ x \in F(\mathbb{N}_p) : \lim_{n \to \infty} x_n n^{-\alpha} = 0 \}, \quad o(t^\alpha) = \{ f \in C(I) : \lim_{t \to \infty} f(t) t^{-\alpha} = 0 \}.$$

Let $X$ be a metric space. For a subset $A$ of $X$ and $\varepsilon > 0$, we define an $\varepsilon$-framed interior of $A$ by

$$\text{Int}(A, \varepsilon) = \{ x \in X : B(x, \varepsilon) \subset A \},$$

where $B(x, \varepsilon)$ denotes an open ball of radius $\varepsilon$ about $x$. Moreover, we define an $\varepsilon$-ball about $A$ by

$$B(A, \varepsilon) = \bigcup_{x \in A} B(x, \varepsilon).$$

A subset $A$ of $X$ is called an $\varepsilon$-net for a subset $Z$ of $X$ if $Z \subset B(A, \varepsilon)$. A subset $Z$ of $X$ is said to be totally bounded if for any $\varepsilon > 0$ there exist a finite $\varepsilon$-net for $Z$.

Let $d$ denotes a metric in $X$. A family $F \subset F(X)$ is called equicontinuous if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the condition $s, t \in X$, $d(s, t) < \delta$ implies $|f(s) - f(t)| < \varepsilon$ for any $f \in F$. For $m \in \mathbb{N}$ we use the rising factorial notation

$$n^m = n(n + 1) \ldots (n + m - 1) \quad \text{with} \quad n^0 = 1.$$

## 3 Regional topology

Let $X$ be a set. For a function $f \in F(X)$ we define a generalized norm $\|f\| \in [0, \infty]$ by

$$\|f\| = \sup\{|f(t)| : t \in X\}.$$

We say that a subset $F$ of $F(X)$ is ordinary if $\|f - g\| < \infty$ for any $f, g \in F$. We regard every ordinary subset $F$ of $F(X)$ as a metric space with metric defined by

$$d(f, g) = \|f - g\|. \quad (1)$$

Let $U \subset F(X)$. We say that $U$ is regionally open if $U \cap F$ is open in $F$ for any ordinary subset $F$ of $F(X)$. The family of all regionally open subsets is a topology on $F(X)$ which we call the regional topology. We regard any subset of $F(X)$ as a topological space with topology induced by the regional topology.

Note that a subset $U$ of $F(X)$ is a neighborhood of $f \in U$ if and only if there exists an
\( \varepsilon > 0 \) such that \( B(f, \varepsilon) \subset U \). Hence, a sequence \( (f_n) \) in \( F(X) \) is convergent to \( f \) if and only if for any the sequence \( \| f_n - f \| \) is convergent to zero. Therefore we can say that the regional topology in \( F(X) \) is the topology of uniform convergence.

More generally, let \( X \) be a real vector space. We say that a function \( \| \cdot \| : X \to [0, \infty] \) is regional norm if the condition \( \| x \| = 0 \) is equivalent to \( x = 0 \) and for any \( x, y \in X \) and \( \alpha \in \mathbb{R} \) we have

\[
\| \alpha x \| = |\alpha| \| x \|, \quad \| x + y \| \leq \| x \| + \| y \|.
\]

Hence, the notion of regional norm generalizes the notion of usual norm. If a regional norm on \( X \) is given, then we say that \( X \) is a regional normed space. If there exists a vector \( x \in X \) such that \( \| x \| = \infty \), then we say that \( X \) is extraordinary.

Assume \( X \) is a regional normed space. We say that a subset \( Z \) of \( X \) is ordinary if \( \| x - y \| < \infty \) for any \( x, y \in Z \). We regard every ordinary subset \( Z \) of \( X \) as a metric space with metric defined by \( (\| \cdot \|) \). Analogously as above we define a regional topology on \( X \). Let

\[
X_0 = \{ x \in X : \| x \| < \infty \}.
\]

Obviously \( X_0 \) is a linear subspace of \( X \). Moreover, the regional norm induces a usual norm on \( X_0 \). We say that \( X \) is a Banach regional space if \( X_0 \) is complete.

For \( p \in X \) let

\[
X_p = p + X_0.
\]

Then \( X_p \) is a maximal ordinary subset of \( X \) which contain \( p \). It is easy to see that \( X_p \) is the equivalence class of \( p \) under the relation defined by

\[
x \equiv y \Leftrightarrow \| x - y \| < \infty.
\]

We say that \( X_p \) is an ordinary component or a region of the space \( X \). It is easy to see that \( X_p \) is also a connected component of \( p \) in \( X \). From topological point of view, the space \( X \) is a disjoint union of all its regions. In particular, every region \( X_p \) is open and closed subset of \( X \). Moreover, for any \( p \in X \) the translation

\[
T_p : X_0 \to X_p, \quad T_p(x) = x + p
\]

is an isometry of \( X_0 \) onto \( X_p \). Hence a metric space \( X_p \) is metrically equivalent to the normed space \( X_0 \). Note also, that the translation \( T_p \) preserves convexity of subsets.

Now, we are ready to state and prove a generalization of the following theorem.

**Theorem (Schauder fixed point theorem)** Assume \( Q \) is a closed and convex subset of a Banach space \( X \), a map \( A : Q \to Q \) is continuous and the set \( AQ \) is totally bounded. Then there exists a point \( x \in Q \) such that \( Ax = x \).

**Theorem 3.1 (Generalized Schauder theorem)** Assume \( Q \) is a closed and convex subset of a regional Banach space \( X \), a map \( A : Q \to Q \) is continuous and the set \( AQ \) is ordinary and totally bounded. Then there exists a point \( x \in Q \) such that \( Ax = x \).

**Proof.** Choose \( p \in AQ \). Since \( AQ \) is ordinary, we have

\[
AQ \subset X_p. \tag{2}
\]

Let

\[
Q_p = Q \cap X_p, \quad Q_0 = Q_p - p.
\]
Then $Q_p$ is closed and convex. Moreover, by (2), $AQ_p \subset Q_p$. Let

$$A' : Q_p \to Q_p, \quad A'x = Ax, \quad U : Q_p \to Q_0, \quad Ux = x - p.$$ 

Then $Q_0 = T_p^{-1}Q_p$, and

$$T_p^{-1} : X_p \to X_0$$

is an isometry and preserves convexity. Hence $Q_0$ is a closed and convex subset of a Banach space $X_0$. Let

$$B : Q_0 \to Q_0, \quad B = U \circ A' \circ U^{-1}. $$

Then $B$ is continuous and

$$BQ_0 = (U \circ A' \circ U^{-1})Q_0 = (U \circ A')(U^{-1}Q_0) = (U \circ A')Q_p = U(A'Q_p). \quad (3)$$

Moreover,

$$A'Q_p = AQ_p \subset AQ.$$ 

Hence $A'Q_p$ is totally bounded and $U$ is an isometry. Therefore, by (3), $BQ_0$ is totally bounded. Thus, by the Schauder fixed point theorem, there exists a point $y \in Q_0$ such that $By = y$. Let $x = U^{-1}y$. Then

$$x = U^{-1}y = U^{-1}By = U^{-1}UA'U^{-1}y = A'U^{-1}y = A'y = Ax.$$ 

The proof is complete. \(\square\)

**Remark 3.1** In Example 6.3 we show, that the assumption of ordinarity of the set $AQ$, in the above theorem, is essential.

Note that, as in the functional case, a subset $U$ of $X$ is a neighborhood of a point $x$ if and only if there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. Hence, we have the following remark, which will be used in the proof of Theorem 3.3.

**Remark 3.2** If $T$ is a topological space, then a map $\varphi : X \to T$ is continuous at a point $x$ if and only if for any neighborhood $V$ of $\varphi(x)$ there exists an $\varepsilon > 0$ such that $\varphi(B(x, \varepsilon)) \subset V$.

Now, assume that $X$ is a locally compact metric space and $F \subset F(X)$. We say that $F$ is:

- vanishing at infinity if for any $\varepsilon > 0$ there exists a compact subset $Z$ of $X$ such that $|f(s)| < \varepsilon$ for every $s \notin Z$ and every $f \in F$.
- equicontinuous at infinity if for any $\varepsilon > 0$ there exists a compact subset $Z$ of $X$ such that $|f(s) - f(t)| < \varepsilon$ for every $s, t \notin Z$ and every $f \in F$.
- stable at infinity if for any $\varepsilon > 0$ there exists a compact subset $Z$ of $X$ such that $|f(s) - g(s)| < \varepsilon$ for any $s \notin Z$ and $f, g \in F$
- homogeneous at infinity if for any $\varepsilon > 0$ there exists a compact subset $Z$ of $X$ such that $|(f - g)(s) - (f - g)(t)| < \varepsilon$ for every $s, t \notin Z$ and every $f, g \in F$. 

5
Remark 3.3  Obviously, every family which is vanishing at infinity is also stable at infinity and equicontinuous at infinity. Moreover, every stable at infinity and every equicontinuous at infinity family is homogeneous at infinity. It is easy to see that a family $F \subset F(X)$ is stable at infinity if and only if $F - F$ is vanishing at infinity. Similarly $F$ is homogeneous at infinity if and only if $F - F$ is equicontinuous at infinity.

Example 3.1  Let

$$f, h : \mathbb{R} \to \mathbb{R}, \quad f(t) = \exp(-t^2), \quad h(t) = t^2,$$

$$F = \{f\}, \quad G = \{f, f+1\}, \quad H = \{h\}, \quad K = \{h, h+1\}.$$

Then $F$ is vanishing at infinity, $G$ is equicontinuous at infinity but not vanishing at infinity. $H$ is stable at infinity but not equicontinuous at infinity, $K$ is homogeneous at infinity but not stable at infinity and not equicontinuous at infinity.

In [1], Avramescu use the term evanescent solution for solution vanishing at infinity. If $X = \mathbb{N}$, then the notion of equicontinuity at infinity is equivalent to the notion of uniformly Cauchy family of sequences, see [2].

Now, we are ready to generalize the following theorem.

Theorem (Ascoli theorem)  If $X$ is a compact metric space, then every pointwise bounded and equicontinuous subset $F$ of $C(X)$ is totally bounded.

Theorem 3.2 (Generalized Ascoli theorem)  If $X$ is a locally compact metric space, then every equicontinuous, pointwise bounded and homogeneous at infinity subset $F$ of $C(X)$ is totally bounded.

Proof. Let $h \in F$ and

$$G = F - h = \{f - h : f \in F\}.$$  

Then $G$ is pointwise bounded, equicontinuous and equicontinuous at infinity. Let $\varepsilon > 0$. Choose a compact $Z \subset X$ such that

$$|g(s) - g(t)| < \varepsilon$$

for $g \in G$ and $s, t \notin Z$. Choose $s \in X \setminus Z$ and let

$$Y = Z \cup \{s\}.$$

Then $Y$ is a compact subset of the space $X$. By the Ascoli theorem, there exist

$$g_1, \ldots, g_n \in G$$

such that the family $g_1|Y, \ldots, g_n|Y$ is an $\varepsilon$-net for the set $\{g|Y : g \in G\}$. Let $g \in G$. Then there exists an index $i \in \mathbb{N}(1, n)$ such that $|g(y) - g_i(y)| \leq \varepsilon$ for any $y \in Y$. Let $t \in X \setminus Y$. Then

$$|g(t) - g_i(t)| \leq |g(t) - g(s)| + |g(s) - g_i(s)| + |g_i(s) - g_i(t)| \leq 3\varepsilon.$$

Hence $\{g_1, \ldots, g_n\}$ is a $3\varepsilon$-net for $G$. Therefore the set $G$ is totally bounded. Thus the set $F = G + h$ is also totally bounded. □.
Theorem 3.2 generalizes the Compactness criterion of C. Avramescu (see [16, page 1164] and Remark 6.5). In this paper, we do not need the Arzela theorem which states that if $X$ is a compact metric space, then every totally bounded subset of $C(X)$ is equicontinuous.

Now, we present the last ‘topological’ theorem which will be used in the proofs of our main theorems.

**Theorem 3.3** Let $X$ be a locally compact metric space, $h \in F(X)$, $\rho \in F_0(X)$ and 
\[
F = \{ f \in F(X) : |f - h| \leq |\rho| \}.
\]
Then $F$ is closed, convex, pointwise bounded and stable at infinity. Moreover

(a) if $\rho$ is continuous, then $F$ is ordinary,

(b) if $X$ is discrete, then $F$ is compact and ordinary.

**Proof.** Obviously, $F$ is convex, pointwise bounded and stable at infinity. Using Remark 3.2 it is easy to see that the map 
\[
F(X) \to F(X), \quad f \mapsto f - h
\]
is continuous. Similarly, using Remark 3.2 we can see that for any $x \in X$, the evaluation 
\[
e_x : F(X) \to \mathbb{R}, \quad e_x(f) = f(x)
\]
is continuous. Hence, for any nonnegative $\alpha$, the set 
\[
\{ f \in F(X) : |f(x) - h(x)| \leq \alpha \}
\]is closed in $F(X)$. Therefore the intersection 
\[
F = \bigcap_{x \in X} \{ f \in F(X) : |f(x) - h(x)| \leq |\rho(x)| \}
\]is closed. Assume $\rho$ is continuous, then $\|\rho\| < \infty$ and 
\[
\|f - g\| \leq \|f - h\| + \|g - h\| \leq 2\|\rho\|
\]
for any $f, g \in F$. Hence $F$ is ordinary.

Now, assume that $X$ is discrete. Then $\rho$ is continuous and, by (a), $F$ is ordinary. Let $G$ be defined by 
\[
G = \{ f \in F(X) : |f| \leq \rho \}.
\]
Choose an $\varepsilon > 0$. Every compact subset of $X$ is finite. Since $\rho$ is vanishing at infinity, there exists a finite subset $Z$ of $X$ such that $|\rho(x)| \leq \varepsilon$ for any $x \notin Z$. For any $z \in Z$ choose a finite $\varepsilon$-net $H_z$ for the interval $[-\rho(z), \rho(z)]$ and let 
\[
H = \{ g \in G : g(z) \in H_z \text{ for } z \in Z \text{ and } g(x) = 0 \text{ for } x \notin Z \}.
\]
Then $H$ is a finite $\varepsilon$-net for $G$. Hence $G$ is a complete and totally bounded metric space and so, $G$ is compact. Therefore $F = G + h$ is also compact. $\Box$. 

7
4 Approximative solutions of difference equations

In this section we establish fundamental properties of the iterated remainder operator $r^m$ in the discrete case. Next, in Theorem 4.1, we obtain our first main result.

**Lemma 4.1** Assume $m \in \mathbb{N}$, $x \in \text{SQ}$ and

$$\sum_{n=1}^{\infty} n^{m-1} |x_n| < \infty. \tag{4}$$

Then there exists exactly one sequence $z \in \text{SQ}$ such that

$$z_n = o(1) \quad \text{and} \quad \Delta^m z = x. \tag{5}$$

The sequence $z$ is defined by

$$z_n = (-1)^m \sum_{j=n}^{\infty} \frac{(j-n+1)^{m-1}}{(m-1)!} x_j. \tag{6}$$

Moreover, if $k \in \mathbb{N}(0, m-1)$, then

$$\Delta^k z_n = (-1)^{m-k} \sum_{j=n}^{\infty} \frac{(j-n+1)^{m-1-k}}{(m-1-k)!} x_j. \tag{7}$$

**Proof.** By Lemma 4 in [12], there exists exactly one sequence $z \in \text{SQ}$ such that (5) is satisfied. The sequence $z$ is defined by

$$z_n = (-1)^m \sum_{k=0}^{\infty} \frac{(k+1)(k+2) \cdots (k+m-1)}{(m-1)!} x_{n+k}. \tag{8}$$

Replacing $n+k$ by $j$ in (8) we obtain (6). Moreover, (7) is a consequence of the proof of Lemma 3 in [12]. □.

**Lemma 4.2** Assume $m \in \mathbb{N}$, $\alpha \in (-\infty, 0]$, $x \in \text{SQ}$,

$$\sum_{n=1}^{\infty} n^{m-1-\alpha} |x_n| < \infty$$

and $z \in \text{SQ}$ is defined by (6). Then $z_n = o(n^\alpha)$.

**Proof.** The assertion follows from Lemma 4.2 of [11]. □.

**Definition 4.1** Assume $m, p \in \mathbb{N}$, $x \in F(\mathbb{N}_p)$ and

$$\sum_{n=p}^{\infty} n^{m-1} |x_n| < \infty.$$ 

We define $r^m x \in F(\mathbb{N}_p)$ by

$$r^m x_n = \sum_{j=n}^{\infty} \frac{(j-n+1)^{m-1}}{(m-1)!} x_j. \tag{9}$$
Lemma 4.3 Assume \( m, p \in \mathbb{N}, k \in \mathbb{N}(0, m), \alpha \in (-\infty, 0], \)

\[
\sum_{n=p}^{\infty} n^{m-1-\alpha}|x_n| < \infty,
\]

and \( n \in \mathbb{N}_p. \) Then we have

\[ r^m|x|_n \leq \sum_{j=n}^{\infty} j^{m-1}|x_j|, \quad (10) \]

\[ \Delta^k r^m x = (-1)^k r^{m-k} x = o(n^{\alpha-k}), \quad (11) \]

\[ \Delta^m r^m x = (-1)^m x, \quad r^m x = o(n^{\alpha}). \quad (12) \]

**Proof.** Obviously, (10) is a consequence of (9). If \( k \in \mathbb{N}(0, m - 1), \) then, by (6) and (7), we have

\[ \Delta^k r^m x = (-1)^{-k} r^{m-k} x = (-1)^k r^{m-k} x. \]

Moreover, by (5), we get \( \Delta^m r^m x = (-1)^m x. \) Using Lemma 4.2 and the equality

\[ m - 1 - \alpha = (m - k) - 1 - (\alpha - k) \]

we have

\[ r^{m-k} x = o(n^{\alpha-k}). \]

Hence, we obtain (11). Taking \( k = m \) and \( k = 0 \) in (11) we obtain (12). \( \square \)

Theorem 4.1 Assume \( m, p \in \mathbb{N}, U \subseteq \mathbb{R}, M \geq 1, \mu > 0, \alpha \in (-\infty, 0], \)

\[ f : \mathbb{N}_p \times \mathbb{R} \to \mathbb{R}, \quad ||f|_{\mathbb{N}_p \times U}|| \leq M, \quad (13) \]

\( f|_{\mathbb{N}_p \times U} \) is continuous, \( a, b, y \in \text{F}(\mathbb{N}_p), \Delta^m y = b, \)

\[ M \sum_{n=p}^{\infty} n^{m-1-\alpha}|a_n| \leq \mu, \quad \text{and} \quad y(\mathbb{N}_p) \subseteq \text{Int}(U, \mu). \quad (14) \]

Then there exists a sequence \( x \in \text{F}(\mathbb{N}_p) \) such that \( x_n = y_n + o(n^{\alpha}) \) and

\[ \Delta^m x_n = a_n f(n, x_n) + b_n \]

for \( n \geq p. \)

**Proof.** Let \( \rho = r^m|a| \) and

\[ Q = \{ x \in \text{F}(\mathbb{N}_p) : |x - y| \leq M\rho \}. \quad (15) \]

Let \( x \in Q \) and \( n \in \mathbb{N}_p. \) Then, using (10) and (14), we have

\[ |x_n - y_n| \leq M\rho_n = Mr^m|a|_n \leq M \sum_{j=n}^{\infty} j^{m-1}|a_j| \leq \mu. \]

Hence, using the inclusion \( y(\mathbb{N}_p) \subseteq \text{Int}(U, \mu), \) we obtain \( x(\mathbb{N}_p) \subseteq U. \) Therefore, by (13),

\[ |f(n, x_n)| \leq M \quad (16) \]
for any $x \in Q$ and $n \in \mathbb{N}_p$. For $n \geq p$ let
\[ \bar{x}_n = a_n f(n, x_n). \]
Then $|\bar{x}| \leq M|a|$. Thus $r^m|\bar{x}| \leq Mr^m|a| = M\rho$. Now, we define a sequence $Ax$ by
\[ Ax = y + (-1)^m r^m \bar{x}. \]
Then
\[ |Ax - y| = |r^m \bar{x}| \leq r^m |\bar{x}| \leq M\rho. \]
Hence, by (15), $Ax \in Q$. Thus
\[ AQ \subset Q. \]
Let $\varepsilon > 0$. There exist $q \in \mathbb{N}_p$ and $\alpha > 0$ such that
\[ M \sum_{n=q}^{\infty} n^{m-1} |a_n| < \varepsilon \quad \text{and} \quad \alpha \sum_{n=p}^{q} n^{m-1} |a_n| < \varepsilon. \]
Let
\[ W = \{(n, t) : n \in \mathbb{N}(p, q), \quad |t - y_n| \leq \mu\} \]
Then $W$ is compact and, by (14), $W \subset \mathbb{N}_p \times U$. Hence $f|W$ is uniformly continuous.
Therefore there exists $\delta > 0$ such that if $((n, t_1), (n, t_2)) \in W$ and $|t_1 - t_2| < \delta$, then
\[ |f(n, t_1) - f(n, t_2)| < \alpha. \]
Choose $z \in Q$ such that $\|x - \bar{z}\| < \delta$. Let $\bar{z} = a_n f(n, z_n)$. Then
\[ \|Ax - Az\| = \sup_{n \geq p} |r^m \bar{x}_n - r^m \bar{z}_n| = \sup_{n \geq p} |r^m (\bar{x} - \bar{z})_n| \leq r^m |\bar{x} - \bar{z}|_p \]
\[ \leq \sum_{n=p}^{\infty} n^{m-1} |\bar{x}_n - \bar{z}_n| \leq \sum_{n=p}^{q} n^{m-1} |\bar{x}_n - \bar{z}_n| + \sum_{n=q}^{\infty} n^{m-1} |\bar{x}_n - \bar{z}_n| \]
\[ \leq \alpha \sum_{n=p}^{q} n^{m-1} |a_n| + 2M \sum_{n=q}^{\infty} n^{m-1} |a_n| < 3\varepsilon. \]
Hence, the mapping $A : Q \to Q$ is continuous. By Theorem 3.3, $Q$ is a compact subset of the metric space
\[ F(\mathbb{N}_p)_y = y + F_b(\mathbb{N}_p). \]
Hence $AQ \subset Q$ is totally bounded. Therefore, by Theorem 3.1, there exists $x \in Q$ such that $Ax = x$. Then
\[ x = y + (-1)^m r^m \bar{x} \]
and, by (12),
\[ \Delta^m x = \Delta^m y + \Delta^m((-1)^m r^m \bar{x}) = b + \bar{x}. \]
Therefore
\[ \Delta^m x_n = a_n f(n, x_n) + b_n \]
for $n \geq p$. Moreover, using (12), we have
\[ x = y + (-1)^m r^m \bar{x} = y + o(n^\alpha). \]
The proof is complete. $\Box$. 

10
5 Approximative solutions of differential equations

In this section we establish fundamental properties of the iterated remainder operator \( r^m \) in the continuous case. Next, in Theorem 5.1, we obtain our second main result.

**Lemma 5.1** Let \( a \in \mathbb{R}, \ b \in (a, \infty) \), \( f \in C[a, \infty) \), \( m \in \mathbb{N}(0) \). Then
\[
\int_a^b \int_t^b \frac{(s-t)^m}{m!} f(s) ds dt = \int_a^b \frac{(s-a)^{m+1}}{(m+1)!} f(s) ds.
\]

**Proof.** Let \( H: [a, b] \times [a, b] \to \mathbb{R}, \)
\[
H(t, s) = \frac{(s-t)^m}{m!} f(s).
\]
Then \( H \) is continuous and
\[
\int_a^b \int_t^b \frac{(s-t)^m}{m!} f(s) ds dt = \int_a^b \int_t^b H(t, s) ds dt = \int_a^b f(s) \int_t^s \frac{(s-t)^m}{m!} dt ds
\]
\[
= \int_a^b f(s) \left[ -\frac{(s-t)^{m+1}}{(m+1)!} t \right]_a^s ds = \int_a^b \frac{(s-a)^{m+1}}{(m+1)!} f(s) ds.
\]
\[ \square \]

**Lemma 5.2** Assume \( m \in \mathbb{N}, \ t_0 \in [0, \infty) \), \( f \in C[t_0, \infty) \) and
\[
\int_{t_0}^{\infty} s^{m-1} |f(s)| ds < \infty. \quad (17)
\]
Then there exists exactly one function \( F: [t_0, \infty) \to \mathbb{R} \) such that
\[
F = o(1) \quad \text{and} \quad F^{(m)} = f. \quad (18)
\]
The function \( F \) is defined by
\[
F(t) = (-1)^m \int_t^\infty \frac{(s-t)^{m-1}}{(m-1)!} f(s) ds. \quad (19)
\]
Moreover, if \( k \in \mathbb{N}(0, m-1) \), then
\[
F^{(k)}(t) = (-1)^{m-k} \int_t^\infty \frac{(s-t)^{m-1-k}}{(m-1-k)!} f(s) ds. \quad (20)
\]
**Proof.** Let \( \varphi_0 = \psi_0 = f \). For \( k \in \mathbb{N}(1, m) \) let \( \varphi_k, \psi_k : [t_0, \infty) \to \mathbb{R}, \)
\[
\varphi_k(t) = \int_t^\infty \varphi_{k-1}(s) ds, \quad \psi_k(t) = \int_t^\infty \frac{(s-t)^{k-1}}{(k-1)!} f(s) ds.
\]
By (17), the integrals $\psi_k$ are convergent. Using Lemma 5.1 it is easy to see that $\phi_k = \psi_k$ for $k \in \mathbb{N}(0, m)$. Let $F = (-1)^m \psi_m$. Obviously $\varphi'_k = -\varphi_{k-1}$ for $k \in \mathbb{N}(1, m)$. Hence

$$F^{(k)} = (-1)^{m-k} \psi_{m-k}$$

for $k \in \mathbb{N}(1, m)$. By (17), we have

$$\int_{t_0}^{\infty} |\psi_{m-1}(s)|ds < \infty.$$  

Hence

$$\psi_m(t) = \int_{t}^{\infty} \psi_{m-1}(s)ds = o(1).$$

Therefore $F = o(1)$. Taking $k = m$ in (21) we obtain (18). Moreover, (20) is a consequence of (21). Now assume

$$G : [t_0, \infty) \to \mathbb{R}, \quad G^{(m)} = f \quad \text{and} \quad G = o(1).$$

Then $(G - F)^{(m)} = 0$ and so $G - F \in \text{Pol}(m - 1)$. Moreover $G - F = o(1)$. Since $\text{Pol}(m - 1) \cap o(1) = 0$, we obtain $G - F = 0$. The proof is complete. $\Box$.

Lemma 5.3 Assume $m \in \mathbb{N}$, $t_0 \in [0, \infty)$, $\alpha \in (-\infty, 0]$, $f \in C[t_0, \infty)$,

$$\int_{t_0}^{\infty} s^{m-1-\alpha} |f(s)|ds < \infty$$

and $F : [t_0, \infty) \to \mathbb{R}$ is defined by (19). Then $F(t) = o(t^\alpha)$.

Proof. Let $g, G : [t_0, \infty) \to \mathbb{R}$,

$$g(s) = s^{-\alpha} f(s), \quad G(t) = \int_{t}^{\infty} \frac{(s-t)^{m-1}}{(m-1)!} t^{-\alpha} f(s)ds.$$

Then, by Lemma 5.2 $G = o(1)$. Moreover,

$$|t^{-\alpha} F(t)| = t^{-\alpha} \left| \int_{t}^{\infty} \frac{(s-t)^{m-1}}{(m-1)!} f(s)ds \right| = \left| \int_{t}^{\infty} \frac{(s-t)^{m-1}}{(m-1)!} t^{-\alpha} f(s)ds \right|$$

$$\leq \int_{t}^{\infty} \frac{(s-t)^{m-1}}{(m-1)!} t^{-\alpha} |f(s)|ds \leq \int_{t}^{\infty} \frac{(s-t)^{m-1}}{(m-1)!} s^{-\alpha} |f(s)|ds = G(t) = o(1).$$

Hence $F(t) = o(t^\alpha)$. $\Box$.

Definition 5.1 Assume $m \in \mathbb{N}$, $f \in C[t_0, \infty)$ and

$$\int_{t_0}^{\infty} s^{m-1} |f(s)|ds < \infty.$$  

We define $r^m f : [t_0, \infty) \to \mathbb{R}$ by

$$(r^m f)(t) = \int_{t}^{\infty} \frac{(s-t)^{m-1}}{(m-1)!} f(s)ds.$$ (22)
Lemma 5.4 Assume \( m \in \mathbb{N}, k \in \mathbb{N}(0, m), \alpha \in (-\infty, 0], f \in C[t_0, \infty) \) and

\[
\int_{t_0}^{\infty} s^{m-1-\alpha} |f(s)| \, ds < \infty.
\]

Then for \( t \geq t_0 \) we have

\[
r^m |f|(t) \leq \int_t^{\infty} s^{m-1} |f(s)| \, ds
\]  \hspace{1cm} (23)

Moreover,

\[
(r^m f)^{(k)} = (-1)^k r^{m-k} f = o(t^{\alpha-k}),
\]  \hspace{1cm} (24)

\[
(r^m f)^{(m)} = (-1)^m f, \quad r^m f = o(t^{\alpha}).
\]  \hspace{1cm} (25)

Proof. Inequality (23) is an easy consequence of (22). By (22), (19) and (20), we have

\[
(r^m f)^{(k)} = (-1)^k r^{m-k} f.
\]

Using Lemma 5.3 we obtain the right equality in (24). Taking \( k = m \) and \( k = 0 \) in (24) we obtain (25). \( \square \).

Theorem 5.1 Assume \( m \in \mathbb{N}, U \subset \mathbb{R}, M \geq 1, t_0 \in [0, \infty), \mu > 0, \alpha \in (-\infty, 0], \)

\[
I = [t_0, \infty), \quad f : I \times \mathbb{R} \to \mathbb{R}, \quad \|f|I \times U\| \leq M,
\]  \hspace{1cm} (26)

\( f|I \times U \) is continuous, \( a, b \in C(I), y \in C^m(I), y^{(m)} = b, \)

\[
M \int_{t_0}^{\infty} s^{m-1-\alpha} |a(s)| \, ds \leq \mu, \quad \text{and} \quad y(I) \subset \text{Int}(U, \mu).
\]  \hspace{1cm} (27)

Then there exists a function \( x \in C^m(I) \) such that \( x(t) = y(t) + o(t^{\alpha}) \) and

\[
x^{(m)}(t) = a(t) f(t, x(t)) + b(t)
\]

for \( t > t_0 \).

Proof. Define a function \( \rho \) and a subset \( Q \) of \( C(I) \) by

\[
\rho = r^m |a|, \quad Q = \{ x \in C(I) : |x - y| \leq M \rho \}.
\]

By (22) and (27) we have \( M \rho \leq \mu \). Let \( x \in Q \). Then \( |x - y| \leq \mu \). Hence, using the inclusion \( y(I) \subset \text{Int}(U, \mu) \), we get \( x(I) \subset U \). Therefore, by (26), we have

\[
|f(t, x(t))| \leq M
\]  \hspace{1cm} (28)

for any \( x \in Q \) and \( t \in I \). For \( t \geq t_0 \) let

\[
\bar{x}(t) = a(t) f(t, x(t)).
\]  \hspace{1cm} (29)

Then \( |\bar{x}| \leq M |a| \). Thus \( r^m |\bar{x}| \leq Mr^m |a| = M \rho \). Now, we define a function \( Ax \) by

\[
Ax = y + (-1)^m r^m \bar{x}.
\]
Then

\[ |Ax - y| = |r^m \bar{x}| \leq r^m |\bar{x}| \leq M \rho. \]

Hence \( Ax \in Q \). Thus

\[ AQ \subset Q. \]

Let \( \varepsilon > 0 \). By (27), there exist \( k \in I \) and \( \alpha > 0 \) such that

\[ M \int_k^\infty s^{m-1}|a(s)|ds < \varepsilon \quad \text{and} \quad \alpha \int_{t_0}^k s^{m-1}|a(s)|ds < \varepsilon. \]

Let

\[ W = \{(t, s) : t \in [t_0, k], \ |s - y(t)| \leq \mu\} \]

Then \( W \) is compact and \( f|W \) is uniformly continuous. Hence there exists \( \delta > 0 \) such that if \( (t, s_1), (t, s_2) \in W \) and \( |s_1 - s_2| < \delta \), then

\[ |f(t, s_1) - f(t, s_2)| < \alpha. \]

Choose \( z \in Q \) such that \( \|x - z\| < \delta \). Let \( \bar{z} = a(t) f(t, z(t)) \). Then, using (23) and (28), we obtain

\[
\|Ax - Az\| = \sup_{t \geq t_0} |r^m \bar{x}(t) - r^m \bar{z}(t)| = \sup_{t \geq t_0} |r^m(\bar{x} - \bar{z})(t)| \leq r^m|\bar{x} - \bar{z}|(t_0)
\]

\[
\leq \int_{t_0}^\infty s^{m-1}|\bar{x} - \bar{z}|(s)ds \leq \int_{t_0}^k s^{m-1}|\bar{x} - \bar{z}|(s)ds + \int_k^\infty s^{m-1}|\bar{x} - \bar{z}|(s)ds
\]

\[
\leq \alpha \int_{t_0}^k s^{m-1}|a(s)|ds + 2M \int_k^\infty s^{m-1}|a(s)|ds < 3\varepsilon.
\]

Hence, the mapping \( A : Q \to Q \) is continuous. Assume \( x \in Q \). If \( t_1, t_2 \in I \), then

\[
|A(x)(t_1) - A(x)(t_2)| = |r^m \bar{x}(t_1) - r^m \bar{x}(t_2)|
\]

\[
= \left| \int_{t_1}^{t_2} r^m s^{m-1} \bar{x}(s)ds \right| \leq \int_{t_1}^{t_2} r^{m-1} \bar{x}(s)ds \leq \int_{t_1}^{t_2} r^{m-1} |\bar{x}|(s)ds.
\]

Moreover, if \( t \in I \), then

\[
r^{m-1} |\bar{x}|(t) \leq \int_t^\infty s^{m-2} |\bar{x}(s)|ds \leq \int_{t_0}^\infty s^{m-2} |a(s)f(s, x(s))|ds
\]

\[
\leq M \int_{t_0}^\infty s^{m-1}|a(s)|ds \leq \mu.
\]

Hence

\[
|A(x)(t_1) - A(x)(t_2)| \leq \mu \int_{t_1}^{t_2} ds = |t_2 - t_1| \mu.
\]

Therefore the family \( AQ \) is equicontinuous. By Theorem 3.3, \( Q \) is closed, convex, pointwise bounded, and stable at infinity. Hence \( AQ \) is pointwise bounded, equicontinuous and stable at infinity. By Theorem 3.2, \( AQ \) is totally bounded. By Theorem 3.1, there exists \( x \in Q \) such that \( Ax = x \). Then

\[
x = y + (-1)^m r^m \bar{x}.
\]
Using (25), we obtain
\[ x^{(m)} = y^{(m)} + ((-1)^m r^m \bar{x})^{(m)} = b + \bar{x}. \]

Therefore, by (29), we get
\[ x^{(m)}(t) = a(t) f(t, x(t)) + b(t) \]
for \( t > t_0 \). Moreover, using (30) and (25), we have
\[ x = y + ((-1)^m r^m \bar{x}) = y + o(t^\alpha). \]

The proof is complete. \( \square \).

6 Remarks

In this section, we give some additional remarks on the regional topology. In particular, in Remark 6.3 we show that if \( X \) is an extraordinary regional space, then the regional topology is almost linear but not linear. In Example 6.3 we show, that in Theorem 3.1 the assumption of ordinarity of the set \( AQ \) is essential. At the end of the section we show that if \( X \) is a locally compact but noncompact metric space, then the notion of equicontinuity at infinity of a family \( F \subset F(X) \) is equivalent to the known notion of equiconvergence at infinity.

Let \( X \) be a regional normed space. The following remark is a consequence of the fact, that \( X \) is a topological disjoint union of all its regions.

**Remark 6.1** A subset \( Y \) of \( X \) is closed in \( X \) if and only if, \( Y \cap X_p \) is closed in \( X_p \) for any region \( X_p \). If \( Z \) is a topological space, then a map \( \varphi : X \to Z \) is continuous if and only if \( \varphi|X_p \) is continuous for any region \( X_p \). If \( Y \subset X \) and \( p \in X \), then the set \( Y \cap X_p \) is closed and open in \( Y \).

Let \( Z \) be a linear subspace of \( X \) such that
\[ Z \cap X_0 = \{0\}, \]
\( q \in X \), and \( Y = q + Z \). If \( p \in Y \), then \( Y \cap X_p = \{p\} \). Hence, by Remark 6.1 we obtain:

**Remark 6.2** If \( Z \) is a linear subspace of \( X \) such that \( Z \cap X_0 = \{0\}, q \in X \), then the topology on \( q + Z \) induced from \( X \) is discrete.

**Example 6.1** A formula
\[ \| (x, y) \| = \begin{cases} |y| & \text{if } x = 0 \\ \infty & \text{if } x \neq 0 \end{cases} \]
define a regional norm on the space \( X = \mathbb{R}^2 \). A subset \( L \) of \( X \) is a region in \( X \) if and only if \( L \) is a vertical line. If \( L \) is a vertical line, then the topology induced on \( L \) from \( X \) is the usual Euclidean topology. On the other hand, the topology induced from \( X \) on any nonvertical line \( L \) is discrete. Moreover, if \( \varphi : \mathbb{R} \to \mathbb{R} \) is an arbitrary function, then the topology induced on the graph
\[ G(\varphi) = \{(x, \varphi(x)) : x \in \mathbb{R}\} \]
is discrete.
Example 6.2 Let \( x, y \in X \) and let \( I \) denote the line segment
\[
I = [x, y] = x + [0, y - x] = x + [0, 1](y - x) = \{x + \lambda(y - x) : \lambda \in [0, 1]\}.
\]
If \( \|y - x\| < \infty \), then
\[
[0, y - x] = [0, 1](y - x) \subset X_0.
\]
Hence, in this case, \( I \) is topologically equivalent to standard interval \([0, 1]\). Moreover, \( I \) is a closed subset of \( X \) and, by Remark 6.1, \( I \) is closed in \( X \).
Assume \( \|y - x\| = \infty \). Then
\[
X_0 \cap \mathbb{R}(y - x) = \{0\}
\]
and, by Remark 6.2, the topology induced on \( x + \mathbb{R}(y - x) \) from \( X \) is discrete. Moreover
\[
I \subset x + \mathbb{R}(y - x).
\]
Hence, in this case, the topology induced on \( I \) is discrete. If \( p \in X \), then \( I \cap X_p \) is one point or empty subset of \( X_p \) and, by Remark 6.1, \( I \) is closed in \( X \).
Using Remark 6.1 it is not difficult to see that the addition \( X \times X \to X \) is continuous. Moreover, if \( p \in X \), and \( \lambda \in \mathbb{R} \), then the translation and \( x \mapsto p + x \) and the homothety \( x \mapsto \lambda x \) are continuous maps from \( X \) to \( X \). Let \( \mu : \mathbb{R} \times X \to X \) denote the multiplication by scalars. If there exists an \( x \in X \) such that \( \|x\| = \infty \), then \( \mu(\mathbb{R} \times \{x\}) = \mathbb{R}x \) and, by Remark 6.2 the topology induced on \( \mathbb{R}x \) is discrete. Hence, the restriction \( \mu|\mathbb{R} \times \{x_0\} \) is discontinuous. This implies the discontinuity of \( \mu \). Hence we have

Remark 6.3 The addition
\[
X \times X \to X, \quad (x, y) \mapsto x + y
\]
is continuous. If \( p \in X \), then the translation
\[
X \to X, \quad x \mapsto p + x
\]
is a homeomorphism. If \( \lambda \) is a nonzero scalar, the the homothety
\[
X \to X, \quad x \mapsto \lambda x
\]
is a homeomorphism. If \( X \) is extraordinary, then the multiplication by scalars
\[
\mathbb{R} \times X \to X, \quad (\lambda, x) \mapsto \lambda x
\]
is discontinuous.

Every region \( X_p \) is topologically equivalent to the normed space \( X_0 \). Hence \( X_p \) is connected. Moreover, \( X_p \) is open and closed subset of \( X \). Hence \( X_p \) is a maximal connected subset of \( X \). Therefore any connected subset of \( X \) is contained in certain region.
Obviously, \( X \) is a Hausdorff space and so, any compact subset of \( X \) is closed. Moreover, if \( C \subset X \) is compact and \( p \in X \), then \( C_p = C \cap X_p \) is closed and open in \( C \). Hence \( C_p \) is compact and
\[
C \subset C_{p_1} \cup \cdot \cdot \cdot \cup C_{p_n}
\]
for some $p_1, \ldots, p_n \in X$.

Now, assume that $C \subset X$ is compact and convex. Let $x, y \in C$ and let $I = [x, y]$. If $\|y - x\| = \infty$, then, by Example 6.2, $I$ is closed and discrete subset of $C$. Hence $I$ is compact and discrete. It is impossible. This means that $C$ is ordinary. Obviously, any convex and ordinary subset of $X$ is connected.

Therefore we obtain:

**Remark 6.4** Any connected subset of $X$ is ordinary. Any compact subset of $X$ is a finite sum of ordinary compact subsets. Any compact and convex subset of $X$ is connected and ordinary.

**Example 6.3** Let $X$ be a regional normed space, $x, y \in X$, $\|y - x\| = \infty$,

$$Q = [x, y], \quad A : Q \to Q, \quad Az = \begin{cases} x & \text{for } z \in (x, y] \\ y & \text{for } z = x \end{cases}.$$ Then $Q$ is closed and convex, $A$ is continuous, $AQ = \{x, y\}$ is totally bounded and $Az \neq z$ for any $z \in Q$.

Assume that $X$ is a locally compact, noncompact metric space.

We say that a real number $p$ is a limit at infinity of a function $f : X \to \mathbb{R}$ if for any $\varepsilon > 0$ there exists a compact subset $Z$ of $X$ such that $|f(t) - p| < \varepsilon$ for any $t \notin Z$. Then we write

$$p = \lim_{t \to \infty} f(t)$$

and say that $f$ is convergent at infinity. We say that a family $F \subset F(X)$ is equiconvergent at infinity if all functions $f \in F$ are convergent at infinity and for any $\varepsilon > 0$ there exists a compact $Z \subset X$ such that

$$|f(s) - \lim_{t \to \infty} f(t)| < \varepsilon$$

for any $f \in F$ and $s \notin Z$.

In the next remark we show that a family $F \subset F(X)$ is equiconvergent at infinity if and only if, it is equi-continuous at infinity.

**Remark 6.5** Obviously every equiconvergent at infinity family $F \subset F(X)$ is equi-continuous at infinity. If a family $F \subset F(X)$ is equi-continuous at infinity, then for any natural $n$ there exists a compact $Z_n \subset X$ such that $|f(s) - f(t)| < 1/n$ for every $f \in F$ and $s, t \notin Z_n$. For $n \in \mathbb{N}$ let

$$K_n = Z_1 \cup \cdots \cup Z_n.$$ Then $K_n$ is compact. Choose a sequence $(t_n)$ in $X$ such that $t_n \notin K_n$ for any $n$. If $f \in F$, then $(f(t_n))$ is a Cauchy sequence and there exists a limit $p$ of this sequence. It is easy to see that

$$p = \lim_{t \to \infty} f(t)$$

and $F$ is equiconvergent at infinity.

Note that if $X$ is a compact space, then every family $F \subset F(X)$ is equi-continuous at infinity but, in this case, equiconvergence at infinity is not defined.
References

[1] C. Avramescu, *Evanescent solutions for linear ordinary differential equations*, Elect. Jour. Qual. Th. Diff. Eqs., No. 9 (2002), 1-12.

[2] S. S. Cheng, W. T. Patula, *An existence theorem for a nonlinear difference equation*, Nonlinear Anal. 20 (1993), no. 3, 193-203.

[3] M. Ehrnstrom, *Linear asymptotic behaviour of second order ordinary differential equations*, Glasgow Math. J. 49 (2007) 105-120.

[4] T. G. Hallam, *Asymptotic behavior of the solutions of an nth order nonhomogeneous ordinary differential equation*, Trans. Amer. Math. Soc. 122 (1966), 177-194.

[5] Q. Kong, *Asymptotic behavior of a class of nonlinear differential equations of nth order*, Proc. Amer. Math. Soc., 103 (1988), 831-838.

[6] O. Lipovan, *On the asymptotic behavior of the solutions to a class of second order nonlinear differential equations*, Glasgow Math. J. 45 (2003), 179-187.

[7] M. Migda, J. Migda, *On the asymptotic behavior of solutions of higher order nonlinear difference equations*, Nonlinear Anal. 47 (2001), 4687-4695.

[8] J. Migda, *Asymptotic Properties of Solutions of Nonautonomous Difference Equations*, Arch. Math. (Brno) 46 (2010), 1-11.

[9] J. Migda, *Asymptotic Properties of Solutions of Higher Order Difference Equations*, Math. Bohem. 135 (2010), No 1, 29-39.

[10] J. Migda, *Asymptotically polynomial solutions of difference equations*, Adv. Difference Equ. (2013), 2013:92, 1-16.

[11] J. Migda, *Approximative solutions of difference equations*, Electron. J. Qual. Theory Differ. Equ. (2014), No. 13, 1–26.

[12] J. Migda, *Approximative full solutions of difference equations*, Int. J. Difference Equ. (2014), 9, 111–121.

[13] J. Migda, *Iterated remainder operator, tests for multiple convergence of series and solutions of difference equations*, submitted.

[14] A.B. Mingarelli, K. Sadarangani, *Asymptotic solutions of forced nonlinear second order differential equations and their extensions*, Electr. J. Differ. Eqs. 2007 (2007) 1-40.

[15] O.G. Mustafa, Y.V. Rogovchenko, *Asymptotic integration of a class of nonlinear differential equations*, Appl. Math. Lett. 19 (2006), 849-853.

[16] Ch, G. Philos, I. K. Purnaras, P. Ch. Tsamatos, *Asymptotic to polynomials solutions for nonlinear differential equations*, Nonlinear Anal. 59 (2004), 1157-1179.

[17] W. F. Trench, *Asymptotic behavior of solutions of an n-th order differential equation*, Rocky Mountain J. Math. 14 (1984), no. 2, 441-450.