ASYMPTOTICS FOR THE RADON TRANSFORM 
ON HYPERBOLIC SPACES

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Abstract. Let $G/H$ be a hyperbolic space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let $K$ be a maximal compact subgroup of $G$. Let $D$ denote a certain explicit invariant differential operator, such that the non-cuspidal discrete series belong to the kernel of $D$. For any $L^2$-Schwartz function $f$ on $G/H$, we prove that the Abel transform $A(Df)$ of $Df$ is a Schwartz function. This is an extension of a result established in [2] for $K$-finite and $K \cap H$-invariant functions.

1. Introduction

The Radon transform $R$ on the hyperbolic spaces $G/H$,

$$Rf = \int_{N^*} f(nH) \, dn,$$

where $N^* \subset G$ is a certain unipotent subgroup, and the associated Abel transform $A$, were introduced and studied in [1] and [2]. Generalizing Harish-Chandra’s notion of cusp forms for real semisimple Lie groups, a discrete series is said to be cuspidal if it is annihilated by the Radon transform. In contrast with the Lie group case, however, non-cuspidal discrete series exist. For the projective hyperbolic spaces, these are precisely the spherical discrete series, but for some real non-projective hyperbolic spaces, there also exist non-spherical non-cuspidal discrete series.

Let $C^2(G/H)$ denote the space of $L^2$-Schwartz functions on $G/H$. Except for some boundary cases, $A$ maps $C^2(G/H)$ into Schwartz functions in the absence of non-cuspidal discrete series. On the other hand, $Af$ can be explicitly calculated for functions $f$ belonging to the non-cuspidal discrete series. To complete the picture, we prove below that $A$ essentially maps the orthocomplement in $C^2(G/H)$ of the non-cuspidal discrete series into Schwartz functions. To be more precise, let $\Delta_\rho = \Delta + \rho^2$, where $\Delta$ denotes the Laplace–Beltrami operator on $G/H$, and consider the $G$-invariant differential operator $D = \Delta_\rho \left( \Delta_\rho - \lambda_1^2 \right) \cdots \left( \Delta_\rho - \lambda_r^2 \right)$, where $\lambda_1, \ldots, \lambda_r$ are the parameters of the non-cuspidal discrete series. Then $A(Df)$ is a Schwartz function. This extends our previous result, [2, Theorem 6.1 (vi)], valid only for the dense $G$-invariant subspace of $C^2(G/H)$ generated by the $K$-irreducible $(K \cap H)$-invariant functions, to all Schwartz functions.

In [2] we also considered the exceptional case corresponding to the Cayley numbers $\mathbb{O}$. We expect our new result to hold for this case as well, but we have not been through the rather cumbersome details.

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2. The Radon transform

In this section, we define the Radon transform and the Abel transform for the projective hyperbolic spaces over the classical fields $F = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. We have tried to keep the presentation and notation to a minimum, see [1] and [2] for further details (including results and proofs).
Let $x \mapsto \overline{x}$ be the standard (anti-) involution of $\mathbb{F}$. Let $p \geq 0$, $q \geq 1$ be two integers, and consider the Hermitian form $[\cdot, \cdot]$ on $\mathbb{F}^{p+q+2}$ given by

$$[x, y] = x_1\overline{y}_1 + \cdots + x_{p+q+1}\overline{y}_{p+1} - x_{p+2}\overline{y}_{p+2} - \cdots - x_{p+q+1}\overline{y}_{p+q+1}, \quad (x, y \in \mathbb{F}^{p+q+2}).$$

Let $G = U(p+1, q+1; \mathbb{F})$ denote the group of $(p+q+2) \times (p+q+2)$ matrices over $\mathbb{F}$ preserving $[\cdot, \cdot]$. Thus $U(p+1, q+1; \mathbb{R}) = O(p+1, q+1)$, $U(p+1, q+1; \mathbb{C}) = U(p+1, q+1)$ and $U(p+1, q+1; \mathbb{H}) = Sp(p+1, q+1)$ in standard notation. Put $U(p; \mathbb{F}) = U(p, 0; \mathbb{F})$, and let $K = U(p+1; \mathbb{F}) \times U(q+1; \mathbb{F})$ be the maximal compact subgroup of $G$ fixed by the Cartan involution on $G$.

Let $x_0 = (0, \ldots, 0, 1)^T$, where superscript $T$ indicates transpose. Let $H = U(p+1, q; \mathbb{F}) \times U(1; \mathbb{F})$ be the subgroup of $G$ stabilizing the line $\mathbb{F} \cdot x_0$ in $\mathbb{F}^{p+q+2}$. The reductive symmetric space $G/H$ can be identified with the projective hyperbolic space $\mathbb{X} = \mathbb{X}(p+1, q+1; \mathbb{F})$,

$$\mathbb{X} = \{ z \in \mathbb{F}^{p+q+2} : [z, z] = -1 \}/\sim,$$

where $\sim$ is the equivalence relation $z \sim zu$, $u \in \mathbb{F}^*$. Let $X_t$, for $t \in \mathbb{R}$, denote the element in the Lie algebra $\mathfrak{g}$ of $G$ with value $t$ in the $(1, p+q+2)$th and $(p+q+2, 1)$th matrix entries (the two opposite corners in the anti-diagonal), and 0 otherwise. Let $\mathfrak{a}_t$ denote the Abelian subalgebra given by $\{X_t \mid t \in \mathbb{R}\}$, let $a_t = \exp(X_t)$ denote the exponential of $X_t$, and also define $A_q = \exp(\mathfrak{a}_q)$.

Let (considered as row vectors)

$$u = (u_1, \ldots, u_p) \in \mathbb{F}^p \quad \text{and} \quad v = (v_q, \ldots, v_1) \in \mathbb{F}^q,$$

and let $w \in \text{Im} \mathbb{F}$ (i.e., $w = 0$ for $\mathbb{F} = \mathbb{R}$). Define $N_{u, v, w} \in \mathfrak{g}$ as the matrix given by

$$N_{u, v, w} = \begin{pmatrix}
-w & u & v & w \\
-\overline{u}^T & 0 & 0 & \overline{v}^T \\
-\overline{v}^T & 0 & 0 & -\overline{u}^T \\
-w & u & v & w
\end{pmatrix}.$$ 

Then $\exp(N_{u, v, w}) = I + N_{u, v, w} + 1/2N_{u, v, w}^2$, and a small calculation yields that

$$a_t \exp(N_{u, v, w}) \cdot x_0 = (\sinh t + 1/2e^t(|u|^2 - |v|^2) + e^tw, \overline{u}^T, -\overline{v}^T, \cosh t + 1/2e^t(|u|^2 - |v|^2) + e^tw)^T,$$

for any $t \in \mathbb{R}$.

Define the nilpotent subalgebra $\mathfrak{n}^*$ as follows, for $p \geq q$,

$$\mathfrak{n}^* = \{ N_{u, v, w} : u = (-\overline{v}^r, u'), v \in \mathbb{F}^q, u' \in \mathbb{F}^{p-q}\},$$

and, for $p < q$,

$$\mathfrak{n}^* = \{ N_{u, v, w} : v = (-\overline{u}^r, v'), u \in \mathbb{F}^p, v' \in \mathbb{F}^{q-p}\},$$

where $u^r, v^r$ means that the order of the indices is reversed. By abuse of notation, we leave out the superscript $r$ in what follows.

We finally also define the following $\rho$-factors. Let $d = \dim \mathbb{F}$, and let $\rho_q = \frac{1}{2}(dp + dq + 2(d-1)) \in \mathbb{R}$, and $\rho_1 = \frac{1}{2}((dp - dq) + 2(d-1)) \in \mathbb{R}$.

Let $N^* = \exp(\mathfrak{n}^*)$ denote the nilpotent subgroup generated by $\mathfrak{n}^*$. For functions $f$ on $G/H$, we define, assuming convergence,

$$Rf(g) = \int_{N^*} f(gn^*H) \, dn^* \quad (g \in G).$$

Let $f \in C^2(G/H)$, the space of $L^2$-Schwartz functions on $G/H$. From [1] and [2], we know that the Radon transform $Rf$ is a smooth function. Also, the integral defining $R$ converges uniformly on compact sets, and $R$ is $G$- and $\mathfrak{g}$-equivariant.

We define the associated Abel transform $\mathcal{A}$ by $\mathcal{A}f(a) = a^{\rho_q}Rf(a)$, for $a \in A_q$. We are mainly interested in the values of $Rf$ and $\mathcal{A}f$ on the elements $a_s$, and thus define $Rf(s) = Rf(a_s)$, and,
similarly, \( A_f(s) = A_f(a_s) \), for \( s \in \mathbb{R} \). Let \( \Delta \) denote the Laplace–Beltrami operator on \( G/H \). Then, for \( f \in \mathcal{C}^2(G/H) \),

\[
A(\Delta f) = \left( \frac{d^2}{ds^2} - \rho_q^2 \right) A f \quad (s \in \mathbb{R}).
\]

Finally, for \( R > 0 \), let \( \mathcal{C}^R_\mathbb{R}(G/H) \) denote the subspace of smooth functions on \( G/H \) with support in the \((K\text{-invariant}) ‘ball’ \( \{ ka \cdot x_0 \mid |a| \leq R \} \) of radius \( R \). Similarly, let \( \mathcal{C}^\infty_\mathbb{R}(\mathbb{R}) \) denote the subspace of smooth functions on \( \mathbb{R} \) with support in \([-R,R]\), and let \( \mathcal{S}(\mathbb{R}) \) denote the Schwartz space on \( \mathbb{R} \).

3. The discrete series and the Abel transform

Let \( q > 1 \), or \( d > 1 \). The discrete series for the projective hyperbolic spaces can then be parametrized as

\[
\{ T_\lambda \mid \lambda = \frac{1}{2}(d - dp) - 1 + \mu_\lambda > 0, \mu_\lambda \in 2\mathbb{Z} \},
\]

see [1] and [2]. The spherical discrete series are given by the parameters \( \lambda \) for which \( \mu_\lambda \leq 0 \), including the 'exceptional' discrete series corresponding to \( \lambda > 0 \) for which \( \mu_\lambda < 0 \).

For \( q = d = 1 \), the discrete series is parameterized by \( \lambda \in \mathbb{R}\setminus\{0\} \) such that \( |\lambda| + \rho_q \in 2\mathbb{Z} \), and there are no spherical discrete series.

The parameters \( \lambda \) are, via the formula \( \Delta f = (\lambda^2 - \rho_q^2)f \), related to the eigenvalues of \( \Delta \) acting on functions \( f \) in the corresponding representation space in \( L^2(G/H) \).

We have a complete classification of the cuspidal and non-cuspidal discrete series for the projective hyperbolic spaces, also including information about the asymptotics of the Radon and Abel transforms:

**Theorem 1.** Let \( G/H \) be a projective hyperbolic space over \( \mathbb{R}, \mathbb{C}, \mathbb{H} \), with \( p \geq 0, q \geq 1 \).

(i) If \( d(q-p) \leq 2 \), then all discrete series are cuspidal.

(ii) If \( d(q-p) > 2 \), then non-cuspidal discrete series exists, given by the parameters \( \lambda > 0 \) with \( \mu_\lambda \leq 0 \). More precisely, if \( 0 \neq f \in \mathcal{C}^2(G/H) \) belongs to \( T_\lambda \), then \( A f(s) = C e^{s \lambda} \), with \( C \neq 0 \).

(iii) \( T_\lambda \) is non-cuspidal if and only if \( T_\lambda \) is spherical.

(iv) If \( p \geq q \) and \( f \in \mathcal{C}^R_\mathbb{R}(G/H) \), for \( R > 0 \), then \( A f \in \mathcal{C}^R_\mathbb{R}(\mathbb{R}) \).

(v) If \( d(q-p) \leq 1 \), and \( f \in \mathcal{C}^2(G/H) \), then \( A f \in \mathcal{S}(\mathbb{R}) \).

(vi) Assume \( d(q-p) > 1 \). Let \( D \) be the \( G \text{-invariant differential operator} \( \Delta_p(\Delta_p-\lambda_1^2) \ldots (\Delta_p-\lambda_r^2) \), where \( \lambda_1, \ldots, \lambda_r \) are the parameters of the non-cuspidal discrete series, and \( \Delta_p = \Delta + \rho_q^2 \).

Then \( A(Df) \in \mathcal{S}(\mathbb{R}) \), for \( f \in \mathcal{C}^2(G/H) \).

The above theorem is almost identical to [2] Theorem 6.1, except for item (vi), which was only proved for functions in the (dense) \( G \text{-invariant subspace} \ V \) of \( \mathcal{C}^2(G/H) \) generated by the \( K \)-irreducible \( (K \cap H) \)-invariant functions. Additionally, [2] Theorem 6.1] furthermore included the exceptional case corresponding to the Cayley numbers \( O \).

**Remark** (including the reformulation of (vi)) also holds for the real non-projective spaces \( SO(p+1,q+1,\mathbb{C})/SO(p+1,q,\mathbb{C}) \), except for item (iii), due to the existence of non-cuspidal non-spherical discrete series corresponding to negative and odd values of \( \mu_\lambda \) in the exceptional series, see [1] Section 5).

The conditions in (vi) essentially state that \( A f \) is a Schwartz function if \( f \) is perpendicular to all non-cuspidal discrete series. The factor \( \Delta_p \), however, seems to be necessary (except in the real case with \( q-p \) odd), even for the case \( d(q-p) = 2 \), where there are no non-cuspidal discrete series.

In the next section, we prove Theorem [1](vi).

4. Proof of Theorem [1](vi)

First we note, following [2] Section 10], that the Schwartz decay conditions are satisfied near \(-\infty\) for \( A(f) \), and thus also for \( A(Df) \). This leaves us to study the Abel transform near \(+\infty\).
Let $f \in C^2(G/H)$, and write $f[x] = f(gH)$, where $x = g \cdot x_0$. From $\Box$ and $\Box$, we get
\[
Rf(s) = \int_{N^*} f(a_s n^* H) du^* = \int_{\mathbb{R}^{dq-dp} \times \mathbb{R}^d \times \mathbb{R}^{d-1}} f([\sinh s - 1/2]v'[2] + e^s w, -u, -v', \cosh s - 1/2]v'[2] + e^s w)] du' dw.
\]
Let $v' = |v'|^2 \nu, v = -\sinh s + 1/2]v'[2]$, such that $|v'|^2 = 1 + 2e^{-s}v - e^{-2s}$, and $\nu = e^s w$. Then,
\[
Rf(s) = e^{-ds} \int_{\sinh s}^{\infty} \int_{\mathbb{R}^{dq-dp-1} \times \mathbb{R}^d \times \mathbb{R}^{d-1}} f([\nu - v, -u, -(1 + 2e^{-s}v - e^{-2s})^{1/2} \nu, e^{-s}v - \nu + \bar{\nu}]) \times (1 + 2e^{-s}v - e^{-2s})^{(d_q-d_p)/2-1} dv d\nu d\bar{\nu},
\]
where $S^r$ is the unit sphere in $\mathbb{R}^r$.

We will use the identification of $X = X(p + 1, q + 1; \mathbb{F})$ with
\[
X = \{z \in \mathbb{F}^{p+q+2} : [z, z] < 0\}/\sim.
\]
and identify a function $f$ on $X$ with a homogeneous function on $\{z \in \mathbb{F}^{p+q+2} : [z, z] < 0\}$ of degree zero.

We now identify $\mathbb{F}^{p+q+2}$ with $\mathbb{R}^{d(p+q+2)}$ such that the coordinates satisfy $\Re z_j = x_{dj}$, for $j = 1, \ldots, p + q + 2$. Consider the real hyperbolic space $\overline{X} = \{z \in \mathbb{F}^{p+q+2} : [z, z] = -1\}$. The group $G = O(d(p + 1), d(q + 1))$ acts transitively on $\overline{X}$. Let $K = O(d(p + 1)) \times O(d(q + 1))$ denote the standard maximal compact subgroup of $G$. Let $U(\mathfrak{t})$, respectively $U(\mathfrak{k})$, denote the universal enveloping algebra of the Lie algebra $\mathfrak{f}$ of $K$, respectively of the Lie algebra $\mathfrak{k}$ of $K$.

**Lemma 2.** Let $U \in U(\mathfrak{t})$, then $U$ maps $C^2(G/H)$ into itself.

**Proof.** The lemma is obvious for $d = 1$. So assume $d > 1$. We note that any element $x \in \overline{X}$ can be written as $x = ka \cdot x_0$, where $k \in K$, and $a = a_s, s \geq 0$. Let $\hat{H} = O(d(p + 1), d(q + 1) - 1)$, and let $\hat{m}$ denote the commutator of $A_q$ in the Lie algebra of $\hat{K} \cap \hat{H}$. Then $\mathfrak{t} = \mathfrak{t} + \mathfrak{m}$.

Let $U_k = Ad(k)U$, for $k \in K$, then $Uf = (Ad(k^{-1})U_k)f$. By the Campbell–Baker– Hausdorff formula, there exists an element $U_k^0 \in U(\mathfrak{t})$, such that $U_k = U_k^0$ modulo the left ideal generated by $\mathfrak{m}$. This implies that $Uf[ka \cdot x_0] = (Ad(k^{-1})U_k^0)f[ka \cdot x_0]$. The map $k \mapsto Ad(k^{-1})U_k^0$ is continuous into a finite dimensional subspace of $U(\mathfrak{t})$, and we can write $Uf[ka \cdot x_0] = (Ad(k^{-1})U_k^0)f[ka \cdot x_0] = \sum_i u_i(k)U_i f[ka \cdot x_0]$, for a finite set of elements $U_i \in U(\mathfrak{t})$ and continuous coefficients $u_i(k)$. It follows that $Uf$ is in $C^2(G/H)$. \(\Box\)

Define for $t = (t_1, t_2, t_3) \in \mathbb{R}^3$, the auxiliary function
\[
G_f(t_1, t_2, t_3) = \int_{\mathbb{R}^{dq-dp-1} \times \mathbb{R}^d \times \mathbb{R}^{d-1}} f([\nu + t_1, w; -u, t_2 \nu, t_3 + \bar{\nu}]) d\nu d\bar{\nu},
\]
and, with the identification $z = e^{-s}$, define the function $F(z) = e^{ds} Rf(s)$. Then, since $\sinh s = -(z - z^{-1})/2$, we get
\[
F(z) = \int_{(z-z^{-1})/2}^{\infty} G_f(-v, -(1 + 2zv - z^2)^{1/2}, z - v)(1 + 2zv - z^2)^{(d_q-d_p)/2-1} dv.
\]

**Lemma 3.** The function $G_f$ is homogeneous of degree $dp + d - 1$, for $t \in \{t \mid t_1^2 - t_2^2 - t_3 < 0\}$, $G_f$ is even in $t_2$, and satisfies $G_f(-t_1, t_2, -t_3) = G_f(t_1, t_2, t_3)$.

Let $X$ be the differential operator on $\mathbb{R}^3$ given by $t_3 \partial / \partial t_2 - t_2 \partial / \partial t_3$. For all $f \in C^2(G/H)$, and all $k, n \in \mathbb{N}$, there exists a constant $C$, such that
\[
|X^k G_f(t)| \leq C(t_2^2 + t_3^2)^{-d(q-p)/4}(1 + \log(t_2^2 + t_3^2))^{-N},
\]
for all $t \in \{t \mid t_1^2 - t_2^2 - t_3 < 0\}$. \(\Box\)
Proof. The first statement follows from the homogeneity of $f$ and the definition of $G_f$.

As before we identify $\mathbb{R}^{p+q+2}$ with $\mathbb{R}^{d(p+q+2)}$. Define, for $i = d(1 + 2p) + 1, \ldots, d(1 + p + q)$, the differential operator $D_i f(x) = x_i (d(p+q+2)) \partial / \partial x_i f(x) - x_i \partial / \partial x_i (d(p+q+2)) f(x)$. This operator is defined by the left action of an element $T_i$ in $O(d(q + 1))$ (with value 1 in the last entry of the $i$th row, value $-1$ in the last entry of the $i$th column, and 0 otherwise), and Lemma 2 thus gives that $D_i$ maps $C^2(G/H)$ into itself.

Let now $\Psi = (v_{d(1+2p)+1}, \ldots, v_{d(1+p+q)}) \in \mathbb{R}^{d(q-p)-1}$. The operator

$$Y_\Psi = \sum_{i=2+2p}^{p+q} v_i D_i,$$

also maps $C^2(G/H)$ into itself, and

$$|Y_\Psi f(x)| \leq d(q - p) \max_{i} (|D_i f(x)|).$$

Applying the operator $X$ to the integrand in the definition of $G_f$, we get

$$X f(t_1, u; -u, t_2 \Psi, t_3) = t_3 (\sum_{i=2+2p}^{p+q} v_i D_i f(x)) (\sum_{i=2+2p}^{p+q} v_i D_i f(x)) (\sum_{i=2+2p}^{p+q} v_i D_i f(x)),$$

The inequality for $X^k f$ follows from repeated use of this formula and from the asymptotic estimates of functions in $C^2(G/H)$.

In particular, it follows that the function $v \mapsto X^k G_f(v, -v, -v)$ has the same parity as $k$.

**Lemma 4.** Let $k_0$ be the largest integer such that $k_0 < (dq - dp)/2$, and let $\epsilon = (dq - dp)/2 - k_0$. Define $t = t(z, v) = (v, -(1 + 2zv - z^2)^{1/2}, z - v)$. Then

(i) For $k \leq k_0$, the function $v \mapsto \partial^k / \partial z^k (G_f(t(z, v))) (1 + 2zv - z^2)^{(dq - dp)/2 - 1}$ is uniformly integrable over $\mathbb{R}$ for $z < 1$.

(ii) For $k \leq k_0$ odd, the function $v \mapsto \partial^k / \partial z^k (G_f(t(z, v))) (1 + 2zv - z^2)^{(dq - dp)/2 - 1}$ is an odd function of $v$ for $z = 0$.

Proof. Notice that $t^2_1 - t^2_2 - t^2_3 = -1$ and $t^2_2 + t^2_3 = 1 + v^2$, for $t = t(z, v)$, and that the integral is uniformly convergent for $0 \leq z \leq K < \infty$. The same holds with $G_f$ replaced by $X^k G_f$.

Repeated use of the formula $\partial / \partial z (G_f(t(z, v))) (1 + 2zv - z^2)^{(dq - dp)/2 - 1}$ yields (i), and together with the parity properties of $X^k G_f$ also gives (ii).

We notice that $\epsilon = 1$ if $d(q - p)$ is even, and $\epsilon = 1/2$ if $d(q - p)$ is odd, i.e., if $d = 1$ and $q - p$ is odd.

For $k < k_0$, the derivatives $\partial^k / \partial z^k$ of $G_f(t(z, v))(1 + 2zv - z^2)^{(dq - dp)/2 - 1}$ are zero at $v = -\sin \pi s = \frac{1}{2}(z - z^{-1})$, whence the integrand is at least $k_0$ times differentiable near $z = 0$, and we can compute the derivatives $d^k / dz^k F(z)$ by differentiating under the integral sign in (i).

If $k_0 > 0$, we can use Taylor's formula to express $F(z)$ as a polynomial of degree $k_0 - 1$, plus a remainder term involving $d^k_0 / dz^k_0 F(\xi)$, for some $0 < \xi(z) < z$,

$$F(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_{k_0 - 1} z^{k_0 - 1} + R_{k_0}(\xi) z^{k_0},$$

where $0 < \xi < z$, and

$$c_j = \frac{1}{j!} \int_0^\infty \int_{|z| = 0} \frac{dz}{dz^j} \left. \left( G_f(t(z, v))(1 + 2zv - z^2)^{(dq - dp)/2 - 1} \right) dv, $$

for $j \in \{0, \ldots, k_0 - 1\}$. The remainder term is given by:

$$R_{k_0}(\xi) = \frac{1}{k_0!} \int_{(\xi - \xi^{-1})/2}^{\xi} \left. \left( G_f(t(z, v))(1 + 2zv - z^2)^{(dq - dp)/2 - 1} \right) dv. $$
Consider $Af(s) = e^{\rho_z} Rf(s) = z^{-(\rho_1 - d)} F(z)$, which is equal to
\[
\epsilon_{0} z^{-(\rho_1 - d)} + \epsilon_{1} z^{-(\rho_1 - d - 1)} + \epsilon_{2} z^{-(\rho_1 - d - 2)} + \ldots + \epsilon_{k_0 - 1} z^{-\epsilon} + z^{-(\epsilon + 1)} R_{k_0} (\xi).
\]
Here we have used that $\rho_1 - d = d(q - p)/2 - 1$. For $j$ even, the exponents $-(d(q - p)/2 - 1 - j)$, for $j \in \{0, \ldots, k_0 - 1\}$, correspond to the parameters $\lambda_1, \ldots, \lambda_r$ for the non-cuspidal discrete series, and $c_j = 0$ for $j$ odd, since the integrand is an odd function.

For the real non-projective hyperbolic spaces the condition concerning the parity $j$ does not hold, but in that case all the exponents $-d(q - p)/2 - 1 - j$, for $j \in \{0, \ldots, k_0 - 1\}$, correspond to parameters $\lambda_1, \ldots, \lambda_r$ for the non-cuspidal discrete series, see [1] Section 3.

From the definition of the differential operator $D$ and (3), we see that $A(Df)$ at most has a contribution from the remainder term, and further that $A(Df)$ does not have a constant term at $\infty$, due to the term $d^2/ds^2$. If $\epsilon = 1/2$, the remainder term $e^{-1/2s}R_{k_0}(\xi s)$ is clearly rapidly decreasing, and we are thus left to consider the case $\epsilon = 1$, in which case $k_0 = d(q - p)/2 - 1$.

Consider the constant term $C_{R_{k_0}} = \lim_{s \to \infty} R_{k_0}(e^{-s})$, which could be non-zero. We want to show that $R_{k_0}(\xi) - C_{R_{k_0}}$ is rapidly decreasing at $+\infty$, where $\xi = \xi(s)$, with $0 < \xi < e^{-s}$. We also include the case $k_0 = 0$, where we put $\xi = e^{-s}$.

Define $H(z, v) = \frac{d}{dz} \int_{0}^{z} (G(t(z, v))(1 + zv - z^2)^{k_0})$. Then, for $\xi < z < 1$,
\[
R_{k_0}(\xi) - C_{R_{k_0}} = \int_{\xi - \xi^{-1}/2}^{\infty} (H(\xi, v) - H(0, v)) \, dv + \int_{-\infty}^{\xi^{-1}/2} H(0, v) \, dv = I_1(\xi) + I_2(\xi).
\]
For $I_1(\xi)$, there exists $\xi_1 = \xi_1(\xi, v) < \xi$, such that $H(\xi, v) - H(0, v) = \xi d/dz|_{z=\xi_1} H(z, v)$, and we get:
\[
I_1(\xi) < z \int_{-\infty}^{\infty} \left| \frac{d}{dz} \right|_{z=\xi_1} H(z, v) \, dv.
\]
By Lemma [1] the integrand is uniformly integrable for $z < 1$, and we conclude that $I_1(\xi)$ is bounded by $Ce^{-s}$.

For $s$ large, the function $H(0, v)$ is for every $N \in \mathbb{N}$ bounded by
\[
|H(0, v)| \leq C(1 + v^2)^{-(d(q - p)/4)v^{k_0} \log(1 + v^2)^{-N}},
\]
for some positive constant $C$. Using this, we find that
\[
I_2(z) < C \int_{-\sinh s}^{\sinh s} v^{-1}(\log(v))^{-N} \, dv = C(N - 1)^{-1}(\log(\sinh s))^{-N + 1} \leq Cs^{-N + 1}.
\]
It follows that $R_{k_0}(\xi) - C_{R_{k_0}}$ is rapidly decreasing at $+\infty$, whence $A(Df)$ is rapidly decreasing at $+\infty$, which finishes the proof of Theorem [1].

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