Global Existence and Uniqueness of Solution of Atangana–Baleanu Caputo Fractional Differential Equation with Nonlinear Term and Approximate Solutions

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In this paper, a class of fractional order differential equation expressed with Atangana–Baleanu Caputo derivative with nonlinear term is discussed. The existence and uniqueness of the solution of the general fractional differential equation are expressed. To present numerical results, we construct approximate scheme to be used for producing numerical solutions of the considered fractional differential equation. As an illustrative numerical example, we consider two Riccati fractional differential equations with different derivatives: Atangana–Baleanu Caputo and Caputo derivatives. Finally, the study of those examples verifies the theoretical results of global existence and uniqueness of solution. Moreover, numerical results underline the difference between solutions of both examples.

1. Introduction

Fractional calculus is considered to be a generalization of the integer-order calculus. This field has a long motivated history that started with Leibniz's letter to Hospital, where the meaning of half derivative was first considered. From this time, fractional order differential equations were used in different fields of mathematical modeling such as epidemiology [1–5], rheology [6, 7], and computer science. Owning to their ability of use in many applications, many active related theoretical areas of research have been developed with common focus: adapting existing results and methods of integer order calculus to fractional ones. Some of those various generalization that are specially developed for fractional order differential equations are existence and uniqueness of solutions [8–10], numerical methods of resolution [11–13], Lie symmetry analysis method [14–16], and stability of fractional system [17–19]. Additionally, research studies have proposed different approaches of fractional derivatives as Riemann–Liouville, Caputo, Caputo–Fabrizio, Caputo–Hadamard, Grunwald–Letnikov, and Atangana–Baleanu derivatives.

Many mathematicians have developed many problems of existence and uniqueness [8, 10] of fractional differential equations. Where some known methods related to the fixed point theory are often used such as Banach and Kranoselkii fixed point theorems. They have also established many fundamental theorems of local and global existence and uniqueness, and the conditions of each theorem could change by changing the derivative of the fractional differential problem. Then, many authors investigated the existence and uniqueness of general fractional differential equation with different forms.

Lin in [9] have proposed an organized analysis of local and global existence and uniqueness of a general problem expressed with Caputo fractional derivative $C_0^\alpha$: 
\[ C_{D_{0+}}^{\alpha}y(t) = f(t, y(t)), n - 1 < \alpha < n, y^{(k)}(t_0) = y_{(k)}^{(0)}, k = 0, \ldots, n - 1. \]

(1)

A similar study was established by Zhang in [10], where the same problem is proposed with a Riemann–Liouville fractional derivative.

Chidouh et al. in [20] have considered the existence and uniqueness of local positive solution in \( C[0, 1] \) of the next fractional differential equation using the Caputo fractional derivative:

\[ C_{D_{0+}}^{\alpha}y(t) + \omega(t) = \varepsilon(t, \sigma(t)), \quad 0 < t < T, \sigma(0) = \sigma_0 > 0, \]

(2)

where \( \varepsilon: [0, 1] \times [0, \infty) \rightarrow [0, \infty) \) is supposed to be continuous. The mathematical methods used in [20] for proving results of existence and uniqueness are different from ones used in [9]. Moreover, the choice of the last problem is argued by the wide use of this type of problem in the rheology field. This type of problem could characterize the behavior of viscoelastic materials, where \( \varepsilon \) is the stress and \( \sigma \) is the strain of the material. Recently, Jarad et al. in [8] and Syam in [21] investigated the local existence and uniqueness of solutions are related directly to the properties of the differential operator of the following general equation:

\[ ABC_{D_{0+}}^{\alpha}y(t) = f(t, y(t)), \quad 0 < \alpha < 1, y(0) = y_0, \]

(3)

where the authors proved the local existence using Banach fixed point theorem for solutions \( y \in H^1[0, 1] \) or \( y \in C[a, b] \). \( ABC_{D_{0+}}^{\alpha} \) is the Atangana–Baleanu Caputo fractional derivative.

Fractional differential equations studied in [8, 9] are expressed with the same form and different types of derivatives. However, results and conditions of their existence and uniqueness are different. In general, theorems of existence and uniqueness and stability of solutions are related directly to the properties of the differential operator of the studied problem. Here, we study the global existence and uniqueness of the Atangana–Baleanu Caputo fractional differential equation:

\[ ABC_{D_{0+}}^{\alpha}y(t) + N(y(t)) = g(t, y(t)), \quad 0 < \alpha < 1, y(0) = y_0, \]

(4)

with \( N(y(t)) \) is a nonlinear function. Our main results are about providing conditions of the global existence and uniqueness of solution of problem (4). Those results are developed using some other powerful methods like the Laplace transform and the completely monotonic property of the Mittag–Leffler function.

The paper is organized as follows. In Section 2, the local existence of solution of problem (4) is investigated. In Section 3, the global existence and uniqueness of solution of our problem is presented. Section 4 is devoted to proposing an approximate scheme to produce numerical solutions for fractional differential equation with Atangana–Baleanu Caputo derivative. In Section 5, we present some numerical results.

2. Local Existence of Solutions

We recall the definition of the fractional Atangana–Baleanu Caputo derivative [8, 21–23].

**Definition 1.** Let \( f \in H^1(0, a) \) be the fractional Atangana–Baleanu derivative of function \( f \) defined by the following integral:

\[ ABC_{D_{0+}}^{\alpha}f(t) = B(\alpha) \frac{1}{1 - \alpha} \int_0^t f'(s) E_{\alpha}[-\frac{(t - s)^{\alpha}}{1 - \alpha}] ds, \]

(5)

with \( 0 < \alpha < 1 \), \( B(\alpha) \) is normalizing positive function satisfying \( B(0) = 1 \) and \( B(1) = 1 \) and \( E_{\alpha} \) is the Mittag–Leffler function [24].

In the next theorem, we present the results of local existence of solutions of problem (4).

**Theorem 1.** Let \( f \in [0, b], \ B = \{ y \in \mathbb{R} / |y - y_0| \leq \delta \}, \ D = J \times B, \) and \( f(t, y) = g(t, y) - N(y) \). Assume that the functions \( f: D \rightarrow \mathbb{R}, \ g: D \rightarrow \mathbb{R} \) and \( N: B \rightarrow \mathbb{R} \) satisfy the following:

(i) \( \|f\|_{\infty} \leq (\delta B(\alpha)/(1 - \alpha)) \)

(ii) \( g(t, y) \) and \( N(y) \) are continuous functions

Then, let \( b^* = \min\{b; ((B(\alpha)\Gamma(\alpha + 1)/\alpha) - (\delta/\|f\|_{\infty} - (1 - \alpha/B(\alpha))))^{1/\alpha}\} \). There exists a function \( y: [0, b^*] \rightarrow \mathbb{R} \) solution of problem (4).

**Proof.** We introduce the set \( U = \{ y \in C[0, b^*]/ \|y - y_0\|_{\infty} \leq \delta \}, \) where \( U \) is a nonempty closed subset of the Banach space \( C[0, b] \), and let \( A \) defined on \( U \) be

\[ Ay(t) = y_0 + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha - 1} g(\xi, y(\xi))d\xi + \frac{1 - \alpha}{B(\alpha)} g(t, y(t)) - \frac{1 - \alpha}{B(\alpha)} N(y(t)) - \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha - 1} N(y(\xi))d\xi. \]

(6)
Using this operator, we may prove that $A$ has a fixed point.

For $t \in [0, b^*]$ and $y \in U$, $N(y)$ and $g(t, y)$ are both continuous on the compact $D$, and then they are uniformly continuous there, and for arbitrary $\varepsilon > 0$, we can find $\delta_1 > 0$ and $\delta_2 > 0$, such that

$$\|N(y) - N(z)\|_\infty < \frac{\varepsilon \|f\|_\infty}{2\delta}, \quad \text{whenever} \ |y - z| < \delta_1,$$

and

$$\|g(t, y) - g(t, z)\|_\infty < \frac{\varepsilon \|f\|_\infty}{2\delta}, \quad \text{whenever} \ |y - z| < \delta_2.$$ 

Let $y, \tilde{y} \in U$. Hence, for $t \in [0, b^*]$,

$$|(Ay)(t) - (A\tilde{y})(t)| \leq \frac{1 - \alpha}{B(\alpha)} \|f\|_\infty + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \|f\|_\infty t^\alpha \int_0^t (t - \xi)^{\alpha - 1} d\xi$$

$$\leq \frac{1 - \alpha}{B(\alpha)} \|f\|_\infty + \frac{\alpha}{B(\alpha) \Gamma(\alpha + 1)} \|f\|_\infty t^\alpha$$

$$\leq \frac{1 - \alpha}{B(\alpha)} \|f\|_\infty + \frac{\alpha}{B(\alpha) \Gamma(\alpha + 1)} \|f\|_\infty b^* t^\alpha$$

$$\leq \frac{1 - \alpha}{B(\alpha)} \|f\|_\infty + \frac{\alpha}{B(\alpha) \Gamma(\alpha + 1)} \|f\|_\infty \left( \frac{\delta}{\|f\|_\infty} - \frac{1 - \alpha}{B(\alpha)} \right)$$

$$\leq \delta,$$

then $|Ay(t) - y_0| \leq \delta$ for all $t \in [0, b^*]$, which implies that $\|Ay - y_0\|_\infty \leq \delta$, using this formula, $A$ maps the set $U$ to itself.

Let $y, \tilde{y} \in U$. Hence, for $t \in [0, b^*]$,

$$|(Ay)(t) - (A\tilde{y})(t)| \leq \frac{1 - \alpha}{B(\alpha)} \|g(t, y(t)) - g(t, \tilde{y}(t))\| + \frac{1 - \alpha}{B(\alpha)} \|N(y(t)) - N(\tilde{y}(t))\|$$

$$+ \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t |g(\xi, y(\xi)) - g(\xi, \tilde{y}(\xi))| (t - \xi)^{\alpha - 1} d\xi$$

$$+ \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t |N y(\xi) - N \tilde{y}(\xi)| (t - \xi)^{\alpha - 1} d\xi.$$ 

For $\delta^* = \min(\delta_1, \delta_2)$ and $y, \tilde{y} \in U$, such that $\|y - \tilde{y}\|_\infty < \delta^*$, we have

$$\|Ay - A\tilde{y}\|_\infty \leq \frac{1 - \alpha}{B(\alpha)} \|f\|_\infty \frac{\varepsilon}{\delta} + \frac{1}{B(\alpha) \Gamma(\alpha)} \|f\|_\infty \frac{\varepsilon}{\delta} t^\alpha$$

$$\leq \frac{1 - \alpha}{B(\alpha)} \|f\|_\infty + \frac{\alpha}{B(\alpha) \Gamma(\alpha + 1)} \left( \frac{\delta}{\|f\|_\infty} - \frac{1 - \alpha}{B(\alpha)} \right),$$

$$\leq \frac{1 - \alpha}{B(\alpha)} \|f\|_\infty + \varepsilon - \frac{1 - \alpha}{B(\alpha)} \|f\|_\infty \leq \varepsilon.$$
The last equation proves the continuity of the operator $A$. Also, it is clear that $A(U) = \{AY : Y \in U\}$ is bounded.

Moreover, we shall prove that $A(y)(t)$ is continuous and $g(t, y(t))$ is continuous; then, it is sufficient to prove the continuity of the following operator:

$$
(Hy)(t) = \frac{\alpha}{B(a)\Gamma(a)} \int_0^t g(\xi, y(\xi))(t - \xi)^{a-1}d\xi,
$$

for $0 \leq t_1 \leq t_2 \leq b'$,

Thus, if $|t_1 - t_2| < \delta$, $|Hy(t_1) - Hy(t_2)| \leq \left(2\|g\|_{\infty}/B(a)\Gamma(a)\right)$, then the set $A(U)$ is equicontinuous.

Then, the Arzela–Ascoli theorem [25] asserts that $A(U)$ is relatively compact. Furthermore, $U$ is closed convex. Then, Schauder’s fixed point theorem [25] claims that $A$ has a fixed point, which is a solution of our initial value problem (4).

3. Global Existence and Uniqueness of Solution

In this section, we present results of global existence and uniqueness of solution of the fractional differential equation (4).

**Theorem 2.** Let $a = (B(a)/(1 - a))$ and $b = (a/(1 - a))$, then the solution of the fractional differential equation (4) is given by

$$
y(t) = \frac{1}{a} f(t, y(t)) + y(0)E_{a,1}(-bt^a) + \int_0^t y(s)E_{a,1}(-bt(t - s)^a)ds,
$$

with $f(t, y) = g(t, y) - N(y)$.

**Proof.** The previous formula of theorem is obtained by applying the fundamental theorem of Laplace transform of equation (4). Then,

$$
ay(s)s^{a-1} + \frac{s^{a-1}y(0)}{s^a + b} = f(s, y(s)) + \frac{s^{a-1}y(0)}{s^a + b}a.
$$

In the last formula, we have used the following property of Laplace transform of Atangana–Baleanu Caputo derivative of a function $f$ [21].

$$
\mathfrak{L}^{ABC}_{\theta,f}(t)(s) = \frac{B(a)}{1 - a} f(s)\frac{s^{-a - 1}f(0)}{s^a + (a/(1 - a))}.
$$

From equation (14), we deduce that

$$
\frac{1}{b}y(s)s^{a - 1} + \frac{s^{a - 1}y(0)}{(1/b)s^a + 1} = f(s, y(s)) + \frac{1}{b} \frac{s^{a - 1}y(0)}{(1/b)s^a + 1} + \frac{1}{b} \frac{s^{a - 1}y(0)}{(1/b)s^a + 1} t.
$$

and then

$$
\mathfrak{L}^{-1}\left[\frac{1}{b}y(s)s^{a - 1} + \frac{s^{a - 1}y(0)}{(1/b)s^a + 1} + \frac{1}{b} \frac{s^{a - 1}y(0)}{(1/b)s^a + 1} t\right] = \mathfrak{L}^{-1}\left[f(s, y(s)) + \frac{1}{b} \frac{s^{a - 1}y(0)}{(1/b)s^a + 1} t\right].
$$
In addition, we have the relation of Laplace transform:

\[ \mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda} \]  

(18)

which implies that

\[ ay(t) = f(t, y(t)) + ay(0)E_{\alpha,1}(-bt^\alpha) - \int_0^t a y(s)E_{\alpha,1}'(-b(t-s)^\alpha)ds. \]  

(20)

Then, one can obtain the integral equation (13). \( \Box \)

The main theorem of global existence and uniqueness is based on the completely monotonic of the Mittag–Leffler function. Before stating this theorem, we recall the next useful lemma [26].

**Lemma 1.** The Mittag–Leffler function \( E_{\alpha,\beta}(-x) \) is completely monotonic when \( 0 < \alpha \leq 1 \) and \( \beta \geq 0 \). It yields that

\[ (-1)^n \frac{d^n}{dx^n}E_{\alpha,\beta}(-x) \geq 0, \quad \text{for all } n \in \mathbb{N}, \]  

(21)

and then \( 0 \leq E_{\alpha,\beta}(-x) \leq 1/\Gamma(\beta) \) where \( x > 0 \) and \( 0 < \alpha \leq 1 \).

**Theorem 3.** Let \( J = [0, b], \) \( B = \{ y \in \mathbb{R} | |y - y_0| \leq \delta \}, \) \( D = J \times B, \) \( f(t, y) = g(t, y) - N(y), \) and \( a = (B(a)/(1-\alpha)) \). Assume that the functions \( g(t, y) \) and \( N(y) \) satisfy the following:

(i) There exists \( M_1 > 0 \), such that \( |g(t, y) - g(t, \bar{y})| \leq M_1 |y - \bar{y}| \) for all \( t \in J \) and all \( y, \bar{y} \in B \)

(ii) There exists \( M_2 > 0 \), such that \( |N(y) - N(\bar{y})| \leq M_2 |y - \bar{y}| \) for all \( y, \bar{y} \in B \)

(iii) \( M_1 + M_2 < a \)

Then, there exists a unique solution \( y: [0, b] \rightarrow \mathbb{R} \) of problem (4).

From equations (17) and (18), we have

\[ Ay(t) = \frac{1}{a} f(t, y(t)) + y(0)E_{\alpha,1}(-bt^\alpha) - \int_0^t y(s)E_{\alpha,1}'(-b(t-s)^\alpha)ds, \]  

(22)

and then

\[ aAy(t) = f(t, y(t)) + 2ay(0)E_{\alpha,1}(-bt^\alpha) - ay(t) + a \int_0^t y'(s)E_{\alpha,1}(-b(t-s)^\alpha)ds. \]  

(23)

From the last equation, we have

\[ 2aaAy(t) = f(t, y(t)) + 2ay(0)E_{\alpha,1}(-bt^\alpha) + a \int_0^t y'(s)E_{\alpha,1}(-b(t-s)^\alpha)ds, \]  

(24)

and then

\[ Ay(t) = \frac{1}{2a} f(t, y(t)) + y(0)E_{\alpha,1}(-bt^\alpha) + \frac{1}{2} \int_0^t y'(s)E_{\alpha,1}(-b(t-s)^\alpha)ds. \]  

(25)

Let \( y, \bar{y} \in U \), we prove that the operator \( A \) is a contraction map, where we use equation (25) and the completely monotonic of the Mittag–Leffler function \( E_{\alpha,1}(-bt^\alpha) \). We have

\[ \|Ay(t) - A\bar{y}(t)\| \leq \frac{1}{2a} \|f(t, y) - f(t, \bar{y})\| + \frac{1}{2} \|E_{\alpha,1}(-b(t-s)^\alpha)\| \int_0^t \|y'(s) - \bar{y}'(s)\| ds \leq \left( \frac{M_1 + M_2}{2a} + \frac{1}{2} \right) \|y(t) - \bar{y}(t)\|. \]  

(26)
Then, if \( M_1 + M_2 < a \), operator \( A \) is contraction and admits a fixed point as a unique solution of problem (4). \( \square \)

**Remark 1.** The previous theorem provides conditions of existence and uniqueness of solution of problem (4). With this theorem, we can demonstrate that, for all \( b \) and \( \delta \) positives, the solution exists on \( U = \{ y \in C \mid [0,b]/|y - y_0| \leq \delta \}. \) Then, it exists on \( U' = C[0, +\infty[. \) As a result, we state the following corollary.

**Corollary 1.** Let functions \( g(t,y) \) and \( N(y) \) satisfy conditions of Theorem 3. Then, there exists a unique global solution \( y: [0, +\infty[ \rightarrow \mathbb{R}. \)

### 4. Illustrative Example

In this section, we consider the following Riccati fractional differential equation:

\[
\begin{aligned}
\text{ABC}_0^\alpha y(t) + y^2(t) &= 0.01 y + t, \\
y(0) &= 0,
\end{aligned}
\]

with \( \alpha = 0.8, 0.85, 0.9, \) and \( 0.95. \)

Under the notation of Theorem 3 and Corollary 1, we have the following:

(i) \( g(t,y) = 0.01 y + t \) and \( N(y) = y^2 \) are both continuous

(ii) \( g(t,.) \) and \( N(.) \) are locally Lipshitz functions

(iii) \( M_1 = 0.01 \) and \( M_2 = 2 \) are, respectively, Lipshitz constants of \( g(t,.) \) and \( N(.) \)

(iv) \( 0.01 + 2 < a, \) where \( a = (B(\alpha)/1 - \alpha) \)

Then, the Riccati fractional differential equation (27), expressed with Atangana–Baleanu derivative with orders \( \alpha = 0.8, 0.85, 0.9, \) or \( 0.95, \) has a unique global solution \( y \in C[0, +\infty[. \)

### 5. Numerical Results

#### 5.1. Numerical Euler Method for Fractional Caputo Differential Equation

Let us recall the fractional Euler method for the following fractional Caputo differential equation [11]:

\[
\begin{aligned}
\text{D}_0^\alpha y(t) &= f(t, y(t)), \\
y(0) &= 0,
\end{aligned}
\]

where \( \text{D}_0^\alpha y(t) \) is the Caputo fractional derivative, and by applying the fractional integral \( \mathbb{D}_0^\alpha \) on both sides of equation (28), we have

\[
y(t) = y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s))ds
\]

and the quantity \( |[\text{D}_0^\alpha f(t, y(t))]_{t=t_{n+1}}| \) is approximated by

\[
\sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j).
\]

Then,

\[
y_{n+1} = y_0 + h^n \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j),
\]

with

\[
y(0) = y_0, \quad b_{j,n+1} = \frac{1}{\Gamma(\alpha + 1)} [(n - j + 1)^\alpha - (n - j)^\alpha].
\]

5.2. Approximate Scheme for Solution of Fractional Atangana–Baleanu Caputo Differential Equation. Let us consider the following fractional Atangana–Baleanu Caputo differential equation:

\[
\begin{aligned}
\text{ABC}_0^\alpha y(t) &= f(t, y(t)), \\
y(0) &= 0.
\end{aligned}
\]

By applying the fractional integral \( \text{ABC}_0^\alpha \) on both sides [21], one can drive the next equation:

\[
y(t) - y(0) = \frac{1 - \alpha}{B(\alpha)} f(t, y(t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s))ds,
\]

and then

\[
y(t) = y(0) + \frac{1 - \alpha}{B(\alpha)} f(t, y(t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s))ds,
\]

and at \( t_{n+1}, \) we approximate \( y(t_{n+1}) \) by

\[
y_{n+1} = y_0 + \frac{1 - \alpha}{B(\alpha)} f(t_n, y_n) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha - 1} f(s, y(s))ds.
\]

Using the forward Euler approximation in [11], we obtain the approximate scheme:

\[
y'_{n+1} = y_0 + \frac{1 - \alpha}{B(\alpha)} f(t_n, y_n) + \frac{h^n\alpha}{B(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j),
\]

where

\[
y(0) = y_0, \quad b_{j,n+1} = \frac{1}{\Gamma(\alpha + 1)} [(n - j + 1)^\alpha - (n - j)^\alpha].
\]
**Theorem 4.** Assume that \( y(t) \) is the solution of (33), \( f(t, y) \) satisfies the Lipschitz condition with respect to \( y \) with a Lipschitz constant \( L \), and \( f(t, y(t)), y(t) \in C^1[0, T] \), and \( y_j (1 \leq j \leq N) \) are the solutions of the fractional Atangana-Baleanu equation using the approximate scheme (37).

Then, there exists \( C \) a positive constant independent of \( h \) and \( n \), where

\[
|e_{n+1}| \leq C(|e_n| + h + o(h)),
\]

if \( 0 < C < 1 \), we have

\[
|e_n| \leq \frac{C}{1-C} (h + o(h)).
\]

In this case, the approximate scheme (37) is convergent.

**Proof.** Denoting by \( e_{n+1} = y_{n+1} - y(t_{n+1}) \), where \( y_{n+1} \) is given by relation (37) and

\[
y(t_{n+1}) = y(0) + \frac{1 - \alpha}{B(\alpha)} f(t_{n+1}, y(t_{n+1})) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s)) ds,
\]

\[
|y_{n+1} - y(t_{n+1})| \leq \frac{1 - \alpha}{B(\alpha)} |f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, y(t_{n+1}))| + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s)) ds |
\]

\[
\leq \frac{1 - \alpha}{B(\alpha)} \left( (t_{n+1} - t_n)^\alpha + |f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, y(t_{n+1}))| \right) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s)) ds |
\]

\[
+ \frac{h^\alpha}{B(\alpha)} \sum_{j=0}^{n} b_j y_j - f(t_j, y_j)
\]

\[
+ \frac{h^\alpha}{B(\alpha)} \sum_{j=0}^{n} b_j y_j
\]

\[
+ \frac{h^\alpha}{B(\alpha)} \sum_{j=0}^{n} b_j y_j - f(t_j, y_j)
\]

\[
|f(t_n, y(t_n)) - f(t_{n+1}, y(t_{n+1}))| = |f(t_n, y(t_n)) - f(t_n + h, y(t_n + h))|
\]

\[
= h f'(t_n, y(t_n)) + o(h).
\]

Using Theorem 2.4 in [27], there exists a positive constant \( C \), such that

\[
\frac{h^\alpha}{B(\alpha)} \sum_{j=0}^{n} b_j f(t_j, y_j) - \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s)) ds \leq \frac{\alpha}{B(\alpha)} Ch.
\]

Using relations (43) and (44), along with the Lipshitz condition of the function \( f \), we have
\[ |y_{n+1} - y(t_{n+1})| \leq \frac{1 - \alpha}{B(\alpha)} L |e_n| + \frac{1 - \alpha}{B(\alpha)} |h f'(t_n, y(t_n)) + o(h)| + \frac{h^\alpha a L}{B(\alpha)} \sum_{j=0}^{n} h_j |y_j - y(t_j)| + \frac{\alpha}{B(\alpha)} C h, \]

(45)

Then, using Lemma 3.3.1 of [11], one can find a positive constant \( C \) independent of \( h \) and \( n \), where

(47)

\[ |e_{n+1}| \leq C \left( |e_n| + h + o(h) \right). \]

Moreover, if \( 0 < C < 1 \), one can complete the proof by induction, and we have

(48)

\[ |e_n| \leq \frac{C}{1-C} \left( h + o(h) \right), \quad n = 0, 1, \ldots, N. \]

Then, using Lemma 3.3.1 of [11], one can find a positive constant \( C \) independent of \( h \) and \( n \), where

(47)

\[ |e_{n+1}| \leq C \left( |e_n| + h + o(h) \right). \]

Moreover, if \( 0 < C < 1 \), one can complete the proof by induction, and we have

(48)

\[ |e_n| \leq \frac{C}{1-C} \left( h + o(h) \right), \quad n = 0, 1, \ldots, N. \]

5.3. Numerical Results of Riccati Fractional Differential Equation. In Figures 1 and 2, we present the approximate solution for the following Riccati fractional problem, using the fractional Caputo derivative and the fractional Atangana–Baleanu Caputo derivative:

\[ \begin{cases} \mathcal{D}_{0+}^\alpha y(t) + y^2(t) = 0.01 y + t, \\ y(0) = 0, \end{cases} \]

(49)

where \( \mathcal{D}_{0+}^\alpha \) could be the fractional Caputo derivative or the fractional Atangana–Baleanu Caputo derivative. In other words, we consider two fractional problems: one with the fractional Caputo derivative, and the second is expressed with the fractional Atangana–Baleanu Caputo derivative. We underline that the solution exists and it is unique of both problems. In Figure 1, the approximate solutions of Riccati fractional differential equation (49) are derived using Euler method for fractional Caputo derivative (31) and approximate scheme (37) for the fractional Atangana–Baleanu derivative, where orders of derivatives are \( \alpha = 0.95 \). In Figure 2, the numerical approximate solutions of the same Riccati fractional differential equation are presented with respect to different values of derivative order: \( \alpha = 0.95, 0.9, 0.86, \) and \( 0.85 \).

Remark 2. From Figures 1 and 2, one can remark that approximate solution of the fractional problem (49) is directly related to the type of the fractional derivative \( \mathcal{D}_{0+}^\alpha \). Consequently, there is a remarkable difference between exact solutions of Riccati equation (49) expressed with Caputo derivative and with Caputo Atangana–Baleanu derivative. In a matter of fact, this difference is explained in the following paragraph. This could be concluded by comparing numerical solutions computed using Euler scheme (31) and (37) where the step of interval subdivision is extremely small.

Denoting by \( y^{ABC}(t) \) the exact solution of Atangana–Baleanu Riccati equation (49) and \( y^{C}(t) \) the exact solution of the fractional Caputo–Riccati equation (49). In the presented numerical results, we use an extremely small step of subdivision \( h \) of the interval \( [0,1] \) so that approximate solutions \( y_n^{ABC}(t) \) and \( y_n^{ABC}(t) \) converge, respectively, to \( y^{ABC}(t) \) and \( y^{ABC}(t) \). From the presented numerical results, we underline the following conclusions:

(i) \( \forall t \in [0,1], y_n^{ABC}(t) - y_n^{C}(t) \geq 0 \)

(ii) \( \forall t \in [0,1], \lim_{n \to \infty} y_n^{ABC}(t) - y_n^{C}(t) = y^{ABC}(t) - y^{C}(t) \geq 0 \)

(iii) From Figures 1 and 2, the solution \( y^{ABC}(t) \) is more advanced in value at each point of time than \( y^{C}(t) \)

(iv) From Figure 2, when the order \( \alpha \) of the derivative decreases, the gap between the two solutions \( y^{ABC}(t) \) and \( y^{C}(t) \) increases

(v) The curve of solution \( y^{ABC}(t) \) of equation of solution of problem (49) could present a predictive model of growth of population (with no saturation level) [28]
Figure 1: Numerical solutions of Riccati fractional differential equation, with $\alpha = 0.95$.

Figure 2: Numerical solutions of Riccati fractional differential equation, with $\alpha = 0.95, 0.9, 0.86,$ and $0.85$. 
(vi) The dynamic of solution $y^{ABC}(t)$ of the corresponding model of growth, is more augmented than the dynamic presenting by $y^C(t)$

(vii) In addition to the fractional order of derivative $\alpha$, the judicious choice of the type of the derivative $D_0^\alpha$, of the model (49) could make the solution $y(t)$ more suitable to describe real phenomena data of a studied mathematical model

6. Conclusion

In this paper, we are interested by a fractional differential equation involving Atangana–Baleanu Caputo and a nonlinear term $N(y)$. Using some tools of the fixed point theory and completely monotonic property, we successfully demonstrated the local existence and global existence and uniqueness of the solution of the considered problem. Moreover, we preferred to construct an approximate scheme suitable to derive numerical solution of our problem with Atangana–Baleanu Caputo derivative. The convergence of this scheme has been studied. We used this method along with Euler method to present numerical results of Riccati fractional differential equation expressed with Atangana–Baleanu Caputo or Caputo fractional derivatives. From those results, we remarked that the solution of the Riccati problem is directly related to the type of derivative used in its formulation: Caputo derivative $C \mathcal{D}_0^\alpha$ or Atangana–Baleanu Caputo derivative $ABC \mathcal{D}_0^\alpha$. Both derivatives are expressed by a Boltzmann integral of the general form $C(\alpha) \int_0^t f'(s)K_\alpha(t-s)ds$, with $C(\alpha)$ is a constant and $K_\alpha$ is a real function. The last function is called the memory kernel of the derivative which is related to the type its type. For Caputo derivative, the memory Kernel is $K_\alpha(t) = t^{-\alpha}$ which differs from the Kernel of Atangana–Baleanu Caputo derivative that is given by the Mittag–Leffler function $K_\alpha(t) = E_\alpha[-\alpha(t^\alpha)/(1-\alpha)]$. Kernel functions provide to each type of derivatives an important character called by the memory effect or history effect. In general, the solution of a fractional differential equation depends on all its past states and behaviors. The type of the memory Kernel of the derivative $D_0^\alpha$ describes how all past states of the solution in $[0,t]$ contributed to the determination of the solution at the present state $t$. The difference between Kernel functions of the Caputo and Atangana–Baleanu Caputo derivatives is the main reason of the remarkable difference between solutions of Riccati equations expressed, respectively, with Caputo and Atangana–Baleanu Caputo derivatives. This example shows the efficiency of our analysis. Finally, it is interesting to study the stability of this equation.

Data Availability

No data where used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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