Quasi-normal modes of a dielectric sphere
and some their implications

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Abstract

It is shown that the quasi-normal modes arise, in a natural way, when considering the oscillations in unbounded regions by imposing the radiation condition at spatial infinity with a complex wave vector $k$. Hence quasi-normal modes are not peculiarities of gravitation problems only (black holes and relativistic stars). It is proposed to consider the space form of the quasi-normal modes with allowance for their time dependence. As a result, the problem of their unbounded increase when $r \to \infty$ is not encountered more. The properties of quasi-normal modes of a compact dielectric sphere are discussed in detail. It is argued that the spatial form of these modes (especially so-called surface modes) should be taken into account, for example, when estimating the potential health hazards due to the use of portable telephones.

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I. INTRODUCTION

Quasi-normal modes (qnm) are widely used now in black hole physics and in relativistic theory of stellar structure (see, for example, Refs. [1, 2, 3]). The corresponding eigenfrequencies are complex numbers, however it is not due to the dissipative processes but it is a consequence of unbounded region occupied by the oscillating system. The latter naturally leads to the energy loss due to the wave emission (for example, gravity waves).

In this paper we would like to show that the quasi-normal modes are not the peculiarities of the gravitational problems only. Actually they appear, in a natural way, when considering the oscillating systems unbounded in space. The necessary condition for emergence of such modes is imposing the radiation condition at spatial infinity on the field functions. It is this condition that leads to the characteristic behaviour of the quasi-normal modes, namely, these solutions to the relevant equations exponentially decay in time when \( t \to \infty \) and simultaneously they exponentially rise at spatial infinity \( r \to \infty \). An interesting and physically motivated example is provided here by the oscillations of electromagnetic field connected with a compact dielectric sphere placed in unrestricted homogeneous media with a different refraction index or in vacuum. Taking here the formal limit \( \varepsilon \to \infty \) one passes to a perfectly conducting sphere (\( \varepsilon \) is the refraction index of the sphere material). In this case the quasi-normal modes describing the electromagnetic oscillations outside the sphere are tractable analytically. We propose to consider the spatial form of a quasi-normal modes with allowance for their time dependence. Doing in this way one can escape the exponential rise of quasi-normal modes at spatial infinity.

The eigenfrequencies of a dielectric sphere are complex \( \omega = \omega' - i\omega'' \), where \( \omega' \) is the free oscillation (radian) frequency and \( \omega'' \) is its relaxation time. These modes can be classified as the interior and exterior ones and, at the same time, as volume modes and surface modes.

In physical applications the surface modes turn out to be important, for example, when estimating the health hazards due to the use of portable telephones. The point is the eigenfrequencies of a dielectric sphere with physical characteristics close to those of a human head lay in the GSM 400 MHz frequency band which has been used in a first generation of mobile phone systems and now is considered for using again. In this situation one can assume that the surface modes excited by a cellular phone will lead to higher heat generation in the tissues close to a head surface as compared with the predictions of routine calculations.
in this field.

The layout of the paper is as follows. In Sec. II we show that the quasi-normal modes are the eigenfunctions of unbounded oscillating regions. As a simple example the quasi-normal modes of a perfectly conducting sphere are considered. The Sec. III is devoted to the consideration of the quasi-normal modes of a dielectric sphere. The main features of these modes are revealed and their classification is presented. The implication of these quasi-normal modes for estimation of the health hazards of portable telephones is considered in Sec. IV. In Conclusion (Sec. V) the main results are formulated and their relation to the general theory of open systems is discussed.

II. QUASI-NORMAL MODES AS THE EIGENFUNCTIONS OF UNBOUNDED OSCILLATING REGIONS

Here we show in the general case in what way complex frequencies and quasi-normal modes appear when considering harmonic oscillations in unbounded regions. Let a closed smooth surface $S$ divides the $d$-dimensional Euclidean space $\mathbb{R}^d$ into a compact internal region $D_{in}$ and noncompact external region $D_{ex}$. We consider here a simple scalar wave equation

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) u(t, x) = 0, \quad (2.1)$$

where $c$ is the velocity of oscillation propagation and $\Delta$ is the Laplace operator in $\mathbb{R}^d$. For harmonic oscillations

$$u(t, x) = e^{-i\omega t} u(x) \quad (2.2)$$

the wave equation (2.1) is reduced to the Helmholtz equation

$$(\Delta + k^2) u(x) = 0, \quad k = \omega / c. \quad (2.3)$$

The oscillations in the internal region $D_{in}$ are described by an infinite countable set of normal modes

$$u_n(t, x) = e^{-i\omega_n t} u_n(x), \quad n = 1, 2, \ldots \quad (2.4)$$

The spatial form of the normal modes (the functions $u_n(x)$) is determined by the boundary conditions which are imposed upon the function $u(x)$ on the internal side of the surface $S$. These conditions should fit the physical content of the problem under study. The set of
normal modes is a complete one. Hence any solution of (2.3) obeying relevant boundary conditions can be expanded in terms of the normal modes $u_n(x)$.

When considering the oscillations in the external domain $D_{ex}$ one imposes, in addition to the conditions on the compact surface $S$, a special requirement concerning the behavior of the function $u(x)$ at large $r \equiv |x|$. In the classical mathematical physics [4] the radiation conditions, proposed by Sommerfeld [5, 6], are used here

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} u(r) = \text{const}, \quad \lim_{r \to \infty} r^{\frac{d-1}{2}} \left( \frac{\partial u}{\partial r} - iku \right) = 0.$$ (2.5)

For real values of the wave vector $k$ (for real frequencies $\omega$) the solution to Eq. (2.3), which obeys the radiation conditions (2.5) and reasonable boundary condition on a compact surface $S$, identically vanishes. In this case the Laplace operator entering the Helmholtz equation (2.3) has no eigenfunctions with real eigenvalues.

The physical content of the radiation conditions is very clear. They select only the oscillations with real frequencies driven by external sources which are situated in a compact spatial area. From the mathematical standpoint, these conditions ensure the uniqueness of the solutions to inhomogeneous wave or Helmholtz equations with external sources on the right-hand side, when these solutions are considered in the external region $D_{ex}$ or in the whole space $D_{in} + D_{ex}$ in the case of compound media.

Here the question arises, how to change minimally the conditions in the homogeneous problem at hand in order to get nonzero solutions, i.e., eigenfunctions in unbounded regions. The energy conservation law prompts a simple way to construct nonzero solutions to the homogeneous wave equation (2.1) or to the Helmholtz equation (2.3) describing the outgoing waves at spatial infinity, namely, one has to introduce complex frequencies $\omega = \omega' - i\omega''$, $\omega'' > 0$. We may hope that in this case the factor $e^{-\omega''t}$ will describe the decay of the initial solutions in time accounting the fact that outgoing waves take away the energy. In other words, we are dealing here with the radiation of the energy with the amplitude decaying in time.

Indeed, if we remove the requirement of reality of the wave vector $k$, then the homogeneous wave equation (2.1) and the Helmholtz equation (2.3) will have nonzero solutions with complex frequencies, these solutions obeying the radiation conditions (2.5) and a common boundary condition on a compact surface $S$ (for instance, Dirichlet or Neumann conditions). In quantum mechanics the radiation condition with a complex wave number $k$ is known
as the Gamov condition which singles out the resonance states in the spectrum of the Hamiltonian \[7, 8, 9\].

When introducing the radiation conditions and proving the respective uniqueness theorem in the textbooks [4] only the real wave vector \(k\) is considered. The possibility of existence of quasi-normal modes with complex frequencies satisfying the radiation conditions at spatial infinity with a complex \(k\) is not mentioned usually. We are aware only of a lone textbook where the eigenfunctions with complex frequencies are noted in this context. It is the article written by Sommerfeld in the book [6], where it is emphasized that the uniqueness in this problem is only up to the eigenfunctions with complex frequencies, i.e., up to the quasi-normal modes.

Thus imposing the radiation conditions with real \(k\) we remove the quasi-normal modes from our consideration only. However this cannot prevent the excitation of these modes in the real physical problem. Hence, when dealing with systems unbounded in space (open systems) one has always to investigate the consequences of qnm excitation. An example of such a problem will be considered in Sec. III.

As a very simple and physically motivated example of quasi-normal modes we consider here the oscillations of electromagnetic field outside a perfectly conducting sphere of radius \(a\). In this case the electric and magnetic fields are expressed in terms of two scalar functions \(f_{kl}^{\text{TE}}(r)\) and \(f_{kl}^{\text{TM}}(r)\) (Debye potentials [10]) which are the radial parts of the solutions to the scalar wave equation (2.1). Outside the perfectly conducting sphere placed in vacuum the solution to the Helmholtz equation (2.3) obeying the radiation conditions (2.5) has the form (\(d = 3\))

\[
f_{kl}(r) = C h_{l}^{(1)} \left( \frac{\omega}{c} r \right), \quad r > a, \quad (2.6)
\]

where \(h_{l}^{(1)}(z)\) is the spherical Hankel function of the first kind [11]. At the surface of perfectly conducting sphere the tangential component of the electric field should vanish. This leads to the following frequency equation for TE-modes

\[
h_{l}^{(1)} \left( \frac{\omega}{c} a \right) = 0, \quad l \geq 1 \quad (2.7)
\]

and for TM-modes

\[
\frac{d}{dr} \left( r h_{l}^{(1)} \left( \frac{\omega}{c} r \right) \right) = 0, \quad r = a, \quad l \geq 1. \quad (2.8)
\]

The spherical Hankel function \(h_{l}^{(1)}(z)\) is \(e^{iz}\) multiplied by the polynomial in \(1/z\) of a finite order [11]. Hence frequency equations (2.7) and (2.8) have a finite number of roots which
are in the general case complex numbers. For \( l = 1 \) (the lowest oscillations) Eqs. (2.7) and (2.8) assume the form \((z = a \omega/c)\)

\[
h'^{(1)}(z) = \frac{1}{z} e^{iz} \left( 1 + \frac{i}{z} \right) = 0 \quad \text{(TE modes)},
\]

\[
\frac{d}{dz} \left( z h'^{(1)}(z) \right) = -\frac{i}{z^2} e^{iz} \left( z^2 + iz - 1 \right) = 0 \quad \text{(TM modes)}.
\]

Thus the lowest eigenfrequencies are

\[
\frac{\omega}{c} = -\frac{i}{a} \quad \text{(TE modes)},
\]

\[
\frac{\omega}{c} = -\frac{1}{2a} (i \pm \sqrt{3}) \quad \text{(TM modes)}.
\]

The complex eigenfrequencies lead to a specific time and spatial dependence of the respective natural modes and ultimately of the electromagnetic fields. So, with allowance of (2.11), we obtain

\[
e^{-i\omega t} f_{k1}^{\text{TE}}(r) = -iC \frac{a}{r} e^{(r-ct)/a} \left(1 - \frac{a}{r}\right), \quad r \geq a.
\]

Thus, the eigenfunctions are exponentially going down in time and exponentially going up when \( r \) increases. Such a time and spatial behaviour is a direct consequence of radiation conditions (2.5) and it is typical for eigenfunctions describing oscillations in external unbounded regions, the physical content and details of oscillation process being irrelevant. The eigenfunctions corresponding to complex eigenvalues are called quasi-normal modes keeping in mind their unusual properties [1]. The physical origin of such features is obvious, in fact we are dealing here with open systems in which the energy can be radiated to infinity. Therefore in open systems field cannot acquire a stationary state.

Quasi-normal modes do not obey the standard completeness condition and the notion of norm cannot be defined for them [1]. Therefore these eigenfunctions cannot be used for expansion of the classical field with the aim to quantize it and to introduce the relevant Fock operators. The treatment of these problems can be found, for example, in [12, 13, 14].

It is worthy to investigate the spatial form of the of quasi-normal modes with allowance for their time dependence. Indeed, these solutions have the character of propagating waves that are eventually going to spatial infinity. Let us take the point which is sufficiently far from the region with nontrivial dynamics in the system under study. Obviously, it has sense to say about the value of the quasi-normal mode at a given point only after arrival at this
point of the wave described by this mode. The maximal value of the quasi-normal mode is observed just at the moment of its arrival at this point. At the later moments the quasi-normal mode is dumping due to its characteristic time dependence. Indeed, taking into account all this we obtain for the maximal observed value of the quasi-normal mode (2.13) the following physically acceptable expression

\[
(e^{-i\omega t} f_{k1}^{TE}(r))_{\text{max-obs}} = -i C \frac{a}{r} \left(1 - \frac{a}{r}\right), \quad r \geq a.
\] (2.14)

Thus, in our consideration the problem of unbounded (exponential) rising of quasi-normal modes, when \( r \to \infty \), does not arise.

In order to associate with resonance phenomenon a single square integrable eigenfunction, rather sophisticated methods are used, for example, complex scaling [15] (known also as the complex-coordinate method or as the complex-rotational method).

The necessary condition for appearing the quasi-normal modes is imposing the radiation conditions at spatial infinity on the field functions. When other conditions are used at spatial infinity, the quasi-normal modes do not arise. For example, if we demand that the solution to the wave equation (2.1) becomes the sum of incoming and outgoing waves when \( r \to \infty \) then the spectrum of the variable \( k^2 \) in the Helmholtz equation (2.3) will be positive and continuous.

III. QUASI-NORMAL MODES OF A DIELECTRIC SPHERE

The same situation, with regard to quasi-normal modes, takes place when we consider the oscillations of compound unbounded media. In this case in both the regions \( D_{\text{in}} \) and \( D_{\text{ex}} \) the wave equations are defined

\[
\left(\Delta - \frac{1}{c_{\text{in}}^2} \frac{\partial^2}{\partial t^2}\right) u_{\text{in}}(t, x) = 0, \quad x \in D_{\text{in}},
\]

\[
\left(\Delta - \frac{1}{c_{\text{ex}}^2} \frac{\partial^2}{\partial t^2}\right) u_{\text{ex}}(t, x) = 0, \quad x \in D_{\text{ex}}
\] (3.1) (3.2)

with the matching conditions at the interface \( S \), for example, of the following kind

\[
u_{\text{in}}(t, x) = u_{\text{ex}}(t, x),
\]

\[
\lambda_{\text{in}} \frac{\partial u_{\text{in}}(t, x)}{\partial n_{\text{in}}(x)} = \lambda_{\text{ex}} \frac{\partial u_{\text{ex}}(t, x)}{\partial n_{\text{ex}}(x)}, \quad x \in S,
\] (3.3) (3.4)
where \( n_{\text{in}}(x) \) and \( n_{\text{ex}}(x) \) are the normals to the surface \( S \) at the point \( x \) for the regions \( D_{\text{in}} \) and \( D_{\text{ex}} \), respectively. The parameters \( c_{\text{in}}, c_{\text{ex}}, \lambda_{\text{in}}, \) and \( \lambda_{\text{ex}} \) specify the material characteristics of the media. At the spatial infinity the solution \( u_{\text{ex}}(t, x) \) should satisfy the radiation conditions (2.5). For real \( k \) we again have only zero solution in this problem, both functions \( u_{\text{in}}(t, x) \) and \( u_{\text{ex}}(t, x) \) vanishing. However, the wave equations (3.1) and (3.2) have nonzero solutions with complex frequencies, i.e. quasi-normal modes, which satisfy the matching conditions (3.3) and (3.4) at the interface \( S \) and radiation conditions (2.5) at spatial infinity. It is important, that the frequencies of oscillations in internal (\( D_{\text{in}} \)) and external (\( D_{\text{ex}} \)) regions are the same. A typical example here is the complex eigenfrequencies of a dielectric sphere. This problem has been investigated by Debye in his PhD thesis concerned with the light pressure on a material particles [16].

Let us consider a sphere of radius \( a \), consisting of a material which is characterized by permittivity \( \varepsilon_1 \) and permeability \( \mu_1 \). The sphere is assumed to be placed in an infinite medium with permittivity \( \varepsilon_2 \) and permeability \( \mu_2 \). In the case of spherical symmetry the solutions to Maxwell equations are expressed in terms of two scalar Debye potentials \( \psi \) (see, for example, textbooks [10, 17]):

\[
\begin{align*}
E_{\text{TM}}^{\ell m} &= \nabla \times \nabla \times (r \psi_{\text{TM}}^{\ell m}), \\
H_{\text{TM}}^{\ell m} &= -i \omega \nabla \times (r \psi_{\text{TM}}^{\ell m}) \quad \text{(E-modes)}, \\
E_{\text{TE}}^{\ell m} &= i \omega \nabla \times (r \psi_{\text{TE}}^{\ell m}), \\
H_{\text{TE}}^{\ell m} &= \nabla \times \nabla \times (r \psi_{\text{TE}}^{\ell m}) \quad \text{(H-modes)}.
\end{align*}
\]

These potentials obey the Helmholtz equation and have the indicated angular dependence

\[
(\nabla^2 + k_i^2) \psi_{\ell m} = 0, \quad k_i^2 = \varepsilon_i \mu_i \frac{\omega^2}{c^2}, \quad i = 1, 2 \quad (r \neq a); \quad \psi_{\ell m}(r) = f_i(r)Y_{\ell m}(\Omega). \quad (3.6)
\]

Equations (3.6) should be supplemented by the boundary conditions at the origin, at the sphere surface, and at infinity. In order for the fields to be finite at \( r = 0 \) the Debye potentials should be regular here. At the spatial infinity we impose the radiation conditions with the goal to find the spectrum of eigenfunctions with complex frequencies (quasi-normal modes in the problem at hand). At the sphere surface the standard matching conditions for electric and magnetic fields should be satisfied [10].

In view of all this the Helmholtz equation (3.6) becomes now the spectral problem for the Laplace operator multiplied by the discontinuous factor \(-1/(\varepsilon(r) \mu(r))\)

\[
-\frac{1}{\varepsilon(r) \mu(r)} \Delta \psi_{\omega \ell m}(r) = \frac{\omega^2}{c^2} \psi_{\omega \ell m}(r), \quad r \neq a, \quad (3.7)
\]
where
\[ \varepsilon(r) \mu(r) = \begin{cases} 
\varepsilon_1 \mu_1, & r < a, \\
\varepsilon_2 \mu_2, & r > a. 
\end{cases} \]

In this problem the spectral parameter is \( \omega^2/c^2 \).

In order to obey the boundary conditions at the origin and at spatial infinity formulated above, the solution to the spectral problem (3.7) should have the form

\[ f_{\omega l}(r) = C_1 j_l(k_1 r), \quad r < a, \quad f_{\omega l}(r) = C_2 h_l^{(1)}(k_2 r), \quad r > a, \quad (3.8) \]

where \( j_l(z) \) is the spherical Bessel function and \( h_l^{(1)}(z) \) is the spherical Hankel function of the first kind [11], the latter obeys the radiation conditions (2.5).

Now we address the matching conditions at the sphere surface. By making use of Eqs. (3.5) we can write, in an explicit form, the radial (r) and tangential (t) components of electric and magnetic fields in the case of spherical symmetry. For TE-modes these equations read

\[ E_{klm,r}^{\text{TE}} = 0, \quad (3.9) \]
\[ E_{klm,t}^{\text{TE}} = a_{lm}(k) f_{kl}^{\text{TE}}(r) X_{lm}, \quad (3.10) \]
\[ H_{klm,r}^{\text{TE}} = \frac{1}{kr} \left( \frac{\varepsilon}{\mu} \right)^{1/2} \sqrt{l(l+1)} a_{lm}(k) f_{kl}^{\text{TE}}(r) Y_{lm}, \quad (3.11) \]
\[ H_{klm,t}^{\text{TE}} = \frac{i}{kr} \left( \frac{\varepsilon}{\mu} \right)^{1/2} a_{lm}(k) \frac{d}{dr} \left( r f_{kl}^{\text{TE}}(r) \right) X_{lm}^\perp, \quad (3.12) \]

and the same for the TM-modes

\[ E_{klm,r}^{\text{TM}} = -\frac{1}{kr} \left( \frac{\mu}{\varepsilon} \right)^{1/2} \sqrt{l(l+1)} b_{lm}(k) f_{kl}^{\text{TM}}(r) Y_{lm}, \quad (3.13) \]
\[ E_{klm,t}^{\text{TM}} = -\frac{i}{kr} \left( \frac{\mu}{\varepsilon} \right)^{1/2} b_{lm}(k) \frac{d}{dr} \left( r f_{kl}^{\text{TM}}(r) \right) X_{lm}^\perp, \quad (3.14) \]
\[ H_{klm,r}^{\text{TM}} = 0, \quad (3.15) \]
\[ H_{klm,t}^{\text{TM}} = b_{lm}(k) f_{kl}^{\text{TM}}(r) X_{lm}. \quad (3.16) \]

Here \( X_{lm} \) are the vector spherical harmonics [17]

\[ X_{lm}(\theta, \phi) = \frac{L Y_{lm}(\theta, \phi)}{\sqrt{l(l+1)}}, \quad l \geq 1, \quad (3.17) \]

where \( L \) is the angular momentum operator

\[ L = -i (r \times \nabla). \]
The vector spherical harmonic $\mathbf{X}_{lm}^\perp$ is obtained from $\mathbf{X}_{lm}$ after rotation by the angle $\pi/2$ around the normal $\mathbf{n} = \mathbf{r}/r$. From Eqs. (3.9) – (3.16) it follows, in particular, that the tangential components of electric field in TE- and TM-modes are orthogonal each other and the same holds for the magnetic field. It implies that the matching conditions on the sphere surface do not couple TE- and TM-modes.

At the sphere surface the tangential components of electric and magnetic fields are continuous (see Eqs. (3.9) – (3.16)). As a result, the eigenfrequencies of electromagnetic field for this configuration are determined [10] by the frequency equation for the TE-modes

$$\Delta_l^{\text{TE}}(a\omega) \equiv \sqrt{\varepsilon_1 \mu_2} j_l(k_1 a) h_l(k_2 a) - \sqrt{\varepsilon_2 \mu_1} j_l(k_1 a) h_l'(k_2 a) = 0$$ (3.18)

and by the analogous equation for the TM-modes

$$\Delta_l^{\text{TE}}(a\omega) \equiv \sqrt{\varepsilon_2 \mu_1} j_l'(k_1 a) h_l(k_2 a) - \sqrt{\varepsilon_1 \mu_2} j_l'(k_1 a) h_l'(k_2 a) = 0,$$ (3.19)

where $k_i = \sqrt{\varepsilon_i \mu_i} \omega/c$, $i = 1, 2$ are the wave numbers inside and outside the sphere, respectively, and $\hat{j}_l(z)$ and $\hat{h}_l(z)$ are the Riccati-Bessel functions [11]

$$\hat{j}_l(z) = z j_l(z) = \sqrt{\frac{\pi z}{2}} J_{l+1/2}(z), \quad \hat{h}_l(z) = z h_l^{(1)}(z) = \sqrt{\frac{\pi z}{2}} H_{l+1/2}^{(1)}(z).$$ (3.20)

In Eqs. (3.18) and (3.19) the orbital momentum $l$ assumes the values $1, 2, \ldots$, and prime stands for the differentiation with respect of the arguments $k_1 a$ and $k_2 a$ of the Riccati-Bessel functions.

The frequency equations for a dielectric sphere of permittivity $\varepsilon$ placed in vacuum follow from Eqs. (3.18) and (3.19) after putting there

$$\varepsilon_1 = \varepsilon, \quad \varepsilon_2 = \mu_1 = \mu_2 = 1.$$ (3.21)

The roots of these equations have been studied in the Debye paper [16] by making use of an approximate method. As the starting solution the eigenfrequencies of a perfectly conducting sphere were used. These frequencies are different for electromagnetic oscillations inside and outside sphere. Namely, inside sphere they are given by the roots of the following equations ($l \geq 1$)

$$j_l\left(\frac{\omega}{c} a\right) = 0 \quad \text{(TE-modes)}$$ (3.22)

$$\frac{d}{dr}\left( r j_l\left(\frac{\omega}{c} a\right) \right) = 0, \quad r = a \quad \text{(TM-modes)}$$ (3.23)
while outside sphere they are determined by Eqs. (2.7) and (2.8). The frequency equations for perfectly conducting sphere (2.7), (2.8) and (3.22), (3.23) can be formally derived by substituting (3.21) into frequency equations (3.18) and (3.19) and taking there the limit $\varepsilon \to \infty$.

Approximate calculation of the eigenfrequencies of a dielectric sphere without using computer [16] didn’t allow one to reveal the characteristic features of the respective eigenfunctions (quasi-normal modes). The computer analysis of this spectral problem was accomplished in the work [18] where the experimental verification of the calculated frequencies was accomplished also by making use of radio engineering measurements.

These studies enable one to separate all the dielectric sphere modes into the interior and exterior modes and, at the same time, into the volume and surface modes. It is worth noting that all the eigenfrequencies are complex

$$\omega = \omega' - i\omega''.$$

Thus we are dealing with ”leaky modes”.

The classification of the modes as the interior and exterior ones relies on the investigation of the behaviour of a given eigenfrequency in the limit $\varepsilon \to \infty$. The modes are called ”interior” when the product $ka = \sqrt{\varepsilon} a/c$ remains finite in the limit $\varepsilon \to \infty$, provided the imaginary part of the frequency ($\omega''$) tends to zero. The modes are referred to as ”exterior” when the product $ka/\sqrt{\varepsilon} = \omega a/c$ remains finite with growing $\omega''$. In the first case the frequency equations for a dielectric sphere (3.18) and (3.19) tend to Eqs. (3.22) and (3.23) and in the second case they tend to Eqs. (2.7) and (2.8). The order of the root obtained will be denoted by the index $r$ for interior modes and by $r'$ for exterior modes. Thus $\text{TE}_{lr}$ and $\text{TM}_{lr}$ denote the interior TE- and TM-modes, respectively, while $\text{TE}_{lr'}$ and $\text{TM}_{lr'}$ stand for the exterior TE- and TM-modes.

For fixed $l$ the number of the modes of exterior type is limited because the frequency equations for exterior oscillations of a perfectly conducting sphere (2.7) and (2.8) have finite number of solutions (see the preceding Section). In view of this, the number of exterior TE- and TM-modes is given by the following rule. For even $l$ there are $l/2$ exterior TE-modes and $l/2$ exterior TM-modes, for odd $l$ the number of the modes $\text{TE}_{lr'}$ is $(l + 1)/2$ and the number of the modes $\text{TM}_{lr'}$ equals $(l - 1)/2$. 

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An important parameter is the $Q$ factor

$$Q_{\text{rad}} = \frac{\omega'}{2\omega''} = 2\pi \frac{\text{stored energy}}{\text{radiated energy per cycle}}.$$  \hspace{1cm} (3.25)

For exterior modes the value of $Q_{\text{rad}}$ is always less than 1, hence these modes can never be observed as sharp resonances. At the same time for $\varepsilon$ greater than 5, the $Q_{\text{rad}}$ for interior modes is greater than 10 and it can reach very high values when $\varepsilon \to \infty$.

For physical implications more important is the classification in terms of volume or surface modes according to whether $r > l$ or $l > r$. For volume modes the electromagnetic energy is distributed in the whole volume of the sphere while in the case of surface modes the energy is concentrated in the proximity of the sphere surface. The exterior modes are the first roots of the characteristic equations and it can be shown that they are always surface modes.

![Electric energy density](image)

**FIG. 1**: Electric energy density $r^2 E_t^2$ for the surface (A) and volume (B) TE-modes of a dielectric sphere with $\varepsilon = 40$ placed in vacuum.

Figure 1 shows a typical spatial behaviour of the surface and volume modes of a dielectric sphere.

Thus a substantial part of the sphere modes (about one half) belong to the interior surface modes. It is important that respective frequencies are the first roots of the characteristic equations.

In order to escape the confusion, it is worth noting here that the surface modes in the problem in question obey the same boundary conditions at the sphere surface and when $r \to \infty$ as the volume modes do. Hence, these surface modes cannot be classified as the evanescent surface waves propagating along the interface between two media (propagating waves along dielectric waveguides, surface plasmon waves on the interface between metal
bulk and adjacent vacuum \[19, 20\] and so on). When describing the evanescent waves one imposes the requirement of their exponential decaying away from interface between two media. In this respect the evanescent surface wave differ from the modes in the bulk.

IV. IMPLICATION OF QNM OF A DIELECTRIC SPHERE FOR ESTIMATION OF THE HEALTH HAZARD OF PORTABLE TELEPHONES

Here we shall argue that the features of the quasi-normal modes of a dielectric sphere (namely, existence of surface and volume modes) should be taken into account, in particular, when estimating the potential health hazards due to the use of the cellular phones. The safety guidelines in this field \[21\] are based on the findings from animal experiments that the biological hazards due to radio waves result mainly from the temperature rise in tissues\(^1\) and a whole-body-averaged specific absorption rate (SAR) below 0.4 W/kg is not hazardous to human health. This corresponds to a limits on the output from the cellular phones (0.6 W at 900 MHz frequency band and 0.27 W at 1.5 GHz frequency band). Obviously, the local absorption rate should be also considered especially in a human head \[23\].

In such studies the following point should be taken into account. The parts of human body (for example, head) pose the eigenfrequencies of electromagnetic oscillations like any compact body. In particular, one can anticipate that the eigenfrequencies of human head are close to those of a dielectric sphere with radius \(a \approx 8\) cm and permittivity \(\varepsilon \approx 40\) (for human brain \(\varepsilon = 44.1\) for 900 MHz and \(\varepsilon = 42.8\) for 1.5 GHz \[23\]). Certainly, our model is very rough, however for the evaluation of the order of the effect anticipated (see below) it is sufficient. By making use of the results of calculations conducted in the work \[18\] one can easily obtain the eigenfrequencies of a dialectic sphere with the parameters mentioned above. For TE\(_{11}\) modes with \(l = 1, 2, 3\) we have, respectively, the following frequencies: 280 MHz, 420 MHz, and 545 MHz. For TM\(_{11}\) modes with \(l = 1, 2, 3\) the resonance frequencies are 425 MHz, 540 MHz, and 665 MHz. The imaginary parts of these eigenfrequencies are very small so the \(Q\) factor in Eq. \((3.25)\) responsible for radiation is greater than 100.

These eigenfrequencies belong to a new GSM 400 MHz frequency band which is now being standardized by the European Telecommunications Standards Institute. This band

\[^1\text{In principle, non-ionizing radiation can lead also to other effects in biological tissues}\]
was primarily used in Nordic countries, Eastern Europe, and Russia in a first generation of mobile phone system prior to the introduction of GSM.

Due to the Ohmic losses the resonances of a dielectric sphere in question are in fact broad, overlapping and, as the result, they cannot be manifested separately. Indeed, the electric conductance $\sigma$ of the human brain is rather substantial. According to the data presented in Ref. $[23]$ $\sigma \approx 1.0 \, \text{S/m}$. The eigenfrequencies of a dielectric dissipative sphere with allowance for a finite conductance $\sigma$ can be found in the following way. As known $[24]$ the effects of $\sigma$ on electromagnetic processes in a media possessing a common real dielectric constant $\varepsilon$ are described by a complex dielectric constant $\varepsilon_{\text{diss}}$ depending on frequency

$$\varepsilon_{\text{diss}} = \varepsilon + i \frac{4\pi \sigma}{\omega}. \quad (4.1)$$

The eigenfrequencies $\omega$, calculated for a real $\varepsilon$, are related to eigenfrequencies $\omega_{\text{diss}}$ for $\varepsilon_{\text{diss}}$ by the formula $[24]$

$$\omega_{\text{diss}} = \frac{\omega}{\sqrt{\varepsilon_{\text{diss}}}} \approx \omega - 2\pi i \frac{\sigma}{\varepsilon}. \quad (4.2)$$

The corresponding factor $Q_{\text{diss}}$ is

$$Q_{\text{diss}} = \frac{\omega_{\text{diss}}'}{2\omega_{\text{diss}}'} \approx \frac{\varepsilon \omega}{4\pi \sigma}. \quad (4.3)$$

Substituting in this equation the values $\omega/2\pi = 0.5 \cdot 10^9 \, \text{Hz}$, $\varepsilon = 40$, $\sigma = 1 \, \text{S/m} = 9 \cdot 10^9 \, \text{s}^{-1}$ one finds

$$Q_{\text{diss}} \approx \frac{20}{18} \approx 1. \quad (4.4)$$

Thus the real spectrum of electromagnetic oscillations in the problem under study is practically a continuous band around the frequency 400 MHz. The radiation of the cellular telephone with frequency laying in this band, will excite (practically with the same amplitudes) all the neighbouring modes of a dissipative dielectric sphere. In order to get the upper bound for the anticipated effect (see below) we assume that the number of excited modes is sufficiently large so that the half of these modes are surface modes.$^2$ Thus one can expect that the resulting spatial configuration of electric and magnetic fields inside a dielectric sphere with Ohm losses will follow, to some extent, the spatial behaviour of the relevant natural modes of the sphere, volume and surface ones. It is obvious that due to excitation of

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$^2$ As it was shown in preceding Section this relation between the number of the surface and volume modes holds only for the spectra as a whole.
surface modes the maximum values of electric and magnetic fields inside dissipative sphere will be shifted to its outer part $r > a/2$.

When assuming the total number of the surface modes to be the same as those for the volume modes and consequently it is equal to a half of all the dielectric sphere modes, then one can anticipate that the temperature rise in the head tissues close to head surface may be by a factor 1.5 higher in comparison with the standard calculations using the numerical methods without special allowance for the spatial behaviour of the relevant natural modes.

However the numerical methods used for estimation of the temperature rise in human tissues due to the radio frequency irradiation do not take into account this effect. Indeed, such calculations (see, for example, paper [23]) are carried out in two steps. First the electric and magnetic fields inside the human body are calculated by solving the Maxwell equations with a given source (antenna of a portable telephone). The electric field gives rise to conduction currents with the energy dissipation rate $\sigma E^2/2$, where $\sigma$ is a conduction constant. In turn it leads to the temperature rising. The second step is the solution of the respective heat conduction equation (or more precisely, bioheat equation [23]) with found local heat sources $\sigma E^2/2$ and with allowance for all the possible heat currents. Hence, for this method the distribution of electric field inside the head is of primary importance. The spatial behaviour of the eigenfunctions characterize the system as a whole, and these properties cannot be taken into account by local methods for calculating the solution to partial differential equations (in our case, to the Maxwell equations).

V. CONCLUSION

We have shown that such different, at first glance, notions as quasi-normal modes in black hole physics, Gamov states in quantum mechanics, and quasi-bound states in the theory of open electromagnetic resonators have the same origin, namely, all these are the eigenmodes of oscillating unbounded domains. By making use of a simple but physically motivated example of electromagnetic oscillations outside a perfectly conducting sphere, which admits analytical treatment, we have easily shown the main features of such oscillations, namely, their exponential decaying in time and, simultaneously, their exponential grows at spatial infinity. These properties are a direct consequence of the radiation condition which is met by qnm at spatial infinity. It is shown also that the exponential rising of qnm at infinity
is not observable because the time dependence of qnm should be taken into account here. In the considered example the qnm are the outgoing spherical waves for $r > a$. This point is disregarded in all the attempts to treat qnm mathematically, in particular, to formulate the completeness condition and the relevant expansions in terms of qnm. We have considered the qnm modes in the problem without inhomogeneous potential (instead of the potential the boundary conditions at the surface of the sphere are introduced). It enables us to infer the conclusion stated above clear and easy.

It is argued also that imposing the standard radiation conditions with a real wave vector $k = \omega/c$ does not prevent us from the necessity to investigate the eigenmodes with complex $\omega'$s, i.e., qnm in a given problem.

The importance of this is demonstrated by investigating the role of the qnm in the problem of estimating the potential health hazards due to the use of portable telephones. The general analysis of the qnm spectra of a dielectric sphere with allowance for the dissipative processes enables us to estimate quantitatively the expected effect, namely, a possible temperature rise in the tissues laying in the outer part of the human head may be 1.5 times greater as compared with the inner part of the brain.

The predicted effect is, in some sense, analogous to the usual (but weakly manifested here) skin-effect. It is not surprising because the substantial conductivity of the sphere material plays a principal part in our consideration. Due to this conductivity the individual resonances of a sphere become very overlapping and, being not observed separately, many of them (volume and surface ones) are excited by the cellular telephone irradiation with the frequency in this band. In view of different spatial behaviour, the surface and volume modes will lead to different spatial distributions of the net heat inside the sphere (and also inside a human head).

When we have the quasi-normal modes instead the usual normal modes it implies that we are dealing with the open systems. Open systems admit a dual description: on the one hand, they can be considered from the “inside” point of view, treating the coupling to the environment as a – not necessarily small – perturbation. From this point of view, one can study the (discrete) eigenvalues of the system, the width of resonances and the resulting decay properties. On the other hand, open systems allow to take the “outside” point of view, considering the system as a perturbation of the environment. The typical quantity to be investigated from this point of view is the scattering matrix ($S$-matrix), i.e.
the amplitude for passing from a given incoming field configuration to a certain outgoing configuration as a function of energy \[25\].

The registration of the quasi-normal modes of a black hole is considered now as a possible way to detect this object \[1, 3\]. In this connection it is surpassing that till now the quasi-normal modes have not been used in an analogous acoustic problem, i.e., for description of sound generation by ringing body \[27, 28, 29\]. At the same time one can find in literature the statement (without proving) that we hear ringing quasi-normal modes of a bell when we hear the bell sound \[30\].

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