Article

New fractional Hadamard and Fejér-Hadamard inequalities associated with exponentially \((h, m)\)-convex functions

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Abstract: The aim of this paper is to establish some new fractional Hadamard and Fejér-Hadamard inequalities for exponentially \((h, m)\)-convex functions. These inequalities are produced by using the generalized fractional integral operators containing Mittag-Leffler function via a monotonically increasing function. The presented results hold for various kinds of convexities and well known fractional integral operators.

Keywords: Convex functions, exponentially \((h, m)\)-convex functions, Hadamard inequality, Fejér-Hadamard inequality, generalized fractional integral operators, Mittag-Leffler function.

1. Introduction and Preliminaries

Convex functions are very important in the field of mathematical inequalities. Nobody can deny the importance of convex functions. A large number of mathematical inequalities exist in literature due to convex functions. For more information related to convex functions and its properties (see, [1–3]).

Definition 1. A function \(\mu : I \rightarrow \mathbb{R}\) on an interval of real line is said to be convex, if for all \(\alpha, \beta \in I\) and \(\kappa \in [0, 1]\), the following inequality holds:

\[\mu(\kappa\alpha + (1-\kappa)\beta) \leq \kappa\mu(\alpha) + (1-\kappa)\mu(\beta).\]  

(1)

The function \(\mu\) is said to be concave if \(-\mu\) is convex.

A convex function is interpreted very nicely in the coordinate plane by the well known Hadamard inequality stated as follows:

Theorem 2. Let \(\mu : [\alpha, \beta] \rightarrow \mathbb{R}\) be a convex function such that \(\alpha < \beta\). The following inequalities holds:

\[\mu \left( \frac{\alpha + \beta}{2} \right) \leq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mu(\kappa) d\kappa \leq \frac{\mu(\alpha) + \mu(\beta)}{2}.\]

In [4], Fejér gave the generalization of Hadamard inequality known as the Fejér-Hadamard inequality stated as follows:

Theorem 3. Let \(\mu : [\alpha, \beta] \rightarrow \mathbb{R}\) be a convex function such that \(\alpha < \beta\). Also let \(v : [\alpha, \beta] \rightarrow \mathbb{R}\) be a positive, integrable and symmetric to \(\frac{\alpha + \beta}{2}\). The following inequalities hold:

\[\mu \left( \frac{\alpha + \beta}{2} \right) \int_{\alpha}^{\beta} v(\kappa) d\kappa \leq \int_{\alpha}^{\beta} \mu(\kappa) v(\kappa) d\kappa \leq \frac{\mu(\alpha) + \mu(\beta)}{2} \int_{\alpha}^{\beta} v(\kappa) d\kappa.\]  

(2)

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The Hadamard and the Fejér-Hadamard inequalities are further generalized in various ways by using different fractional integral operators such as Riemann-Liouville, Katugampola, conformable and generalized fractional integral operators containing Mittag-Leffler function etc. For more results and details (see, [5–21]).

Next we give the definition of exponentially convex functions.

**Definition 4.** [9,22] A function \( \mu : I \to \mathbb{R} \) on an interval of real line is said to be exponentially convex, if for all \( \alpha, \beta \in I \) and \( \kappa \in [0,1] \), the following inequality holds:

\[
e^{\mu(\alpha (1 - \kappa) + \beta)} \leq \kappa e^{\mu(\alpha)} + (1 - \kappa) e^{\mu(\beta)}.
\]

(3)

In [23], Rashid et al., gave the definition of exponentially \( s \)-convex functions.

**Definition 5.** Let \( s \in [0,1] \). A function \( \mu : I \to \mathbb{R} \) on an interval of real line is said to be exponentially \( s \)-convex, if for all \( \alpha, \beta \in I \) and \( \kappa \in [0,1] \), the following inequality holds:

\[
e^{\mu(\alpha (1 - \kappa) + \beta)} \leq \kappa e^{\mu(\alpha)} + (1 - \kappa) e^{\mu(\beta)}.
\]

(4)

In [24], Rashid et al., gave the definition of exponentially \( h \)-convex functions.

**Definition 6.** Let \( I \subseteq \mathbb{R} \) be an interval containing \((0,1)\) and let \( h : I \to \mathbb{R} \) be a non-negative function. Then a function \( \mu : I \to \mathbb{R} \) on an interval of real line is said to be exponentially \( h \)-convex, if for all \( \alpha, \beta \in I \) and \( \kappa \in [0,1] \), the following inequality holds:

\[
e^{\mu(\alpha (1 - \kappa) + \beta)} \leq h(\kappa)e^{\mu(\alpha)} + h(1 - \kappa)e^{\mu(\beta)}.
\]

(5)

In [25], Rashid et al., gave the definition of exponentially \( m \)-convex functions.

**Definition 7.** A function \( \mu : I \to \mathbb{R} \) on an interval of real line is said to be exponentially \( m \)-convex, if for all \( \alpha, \beta \in I \), \( m \in (0,1) \) and \( \kappa \in [0,1] \), the following inequality holds:

\[
e^{\mu(\alpha (1 - \kappa) + \beta)} \leq \kappa e^{\mu(\alpha)} + m(1 - \kappa)e^{\mu(\beta)}.
\]

(6)

In [26], Rashid et al., gave the definition of exponentially \((h,m)\)-convex functions.

**Definition 8.** Let \( I \subseteq \mathbb{R} \) be an interval containing \((0,1)\) and let \( h : I \to \mathbb{R} \) be a non-negative function. Then a function \( \mu : I \to \mathbb{R} \) on an interval of real line is said to be exponentially \((h,m)\)-convex, if for all \( \alpha, \beta \in I \), \( m \in (0,1) \) and \( \kappa \in [0,1] \), the following inequality holds:

\[
e^{\mu(\alpha (1 - \kappa) + \beta)} \leq h(\kappa)e^{\mu(\alpha)} + mh(1 - \kappa)e^{\mu(\beta)}.
\]

(7)

**Remark 1.**

1. If we set \( h(\kappa) = \kappa \) and \( m = 1 \) in (7), then exponentially convex function (3) is obtained.
2. If we set \( h(\kappa) = \kappa^r \) and \( m = 1 \) in (7), then exponentially \( s \)-convex function (4) is obtained.
3. If we set \( m = 1 \) in (7), then exponentially \( h \)-convex function (5) is obtained.
4. If we set \( h(\kappa) = \kappa \) in (7), then exponentially \( m \)-convex function (6) is obtained.

Fractional integral operators also play important role in the subject of mathematical analysis. Recently in [27], Andrić et al., defined the generalized fractional integral operators containing generalized Mittag-Leffler function in their kernels as follows:

**Definition 9.** Let \( \psi,\sigma,\phi,I,c \in \mathbb{C}, \mathbb{R}(\sigma),\mathbb{R}(\phi),\mathbb{R}(l) > 0, \mathbb{R}(c) > \mathbb{R}(\zeta) > 0 \) with \( p \geq 0 \), \( r > 0 \) and \( 0 < q \leq r + \mathbb{R}(\sigma) \). Let \( \mu \in \mathcal{L}_1(\alpha, \beta) \) and \( u \in [\alpha, \beta] \). Then the generalized fractional integral operators \( \mathbb{Y}_{\psi,\phi,\sigma,l,c}^{\psi,\phi,\sigma,l,c} \) and \( \mathbb{Y}_{\psi,\phi,\sigma,l,c}^{\psi,\phi,\sigma,l,c} \mu \) are defined by:

\[
(\mathbb{Y}_{\psi,\phi,\sigma,l,c}^{\psi,\phi,\sigma,l,c} \mu)(u;p) = \int_{\mathbb{R}} (u - \kappa)^{\psi-1} \mathcal{E}_{\psi,\phi,\sigma,l,c}^{\psi,\phi,\sigma,l,c} (\psi(u - \kappa)^r; p) \mu(\kappa) d\kappa,
\]

(8)
\[ (Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu})(u; p) = \int_u^b (k - u)^{\phi - 1} E_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(\psi(k - u))^\prime; p)\mu(k)dk, \]  
where \( E_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(k; p) \) is the generalized Mittag-Leffler function defined as follows:

\[ E_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(k; p) = \sum_{n=0}^{\infty} \beta_p(\zeta + nq, c - \zeta) \frac{(c)_nq}{\Gamma(\sigma n + \phi)(l)_n}. \]

In [28], Farid defined the following unified integral operators:

**Definition 10.** Let \( \mu, \nu : [\alpha, \beta] \rightarrow \mathbb{R}, 0 < \alpha < \beta \) be the functions such that \( \mu \) be a positive and integrable and \( \nu \) be a differentiable and strictly increasing. Also, let \( \phi, \sigma, l, c, \zeta, \beta, \mu \in \mathbb{C}, \quad \Re(\phi), \Re(l) > 0, \Re(c) > \Re(\zeta) > 0 \) with \( p \geq 0, \sigma, r > 0 \) and \( 0 < q \leq r + \gamma \). Then for \( u \in [\alpha, \beta] \) the integral operators \( v Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu} \) are defined by:

\[ (v Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu})(u; p) = \int_u^b \frac{\gamma(v(u) - \nu(u))}{v(u) - \nu(u)} E_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(\psi(v(u) - \nu(u))^\prime; p)\mu(k)dk, \]  

and

\[ (v Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu})(u; p) = \int_u^b \frac{\gamma(v(k) - \nu(u))}{v(k) - \nu(u)} E_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(\psi(v(k) - \nu(u))^\prime; p)\mu(k)dk. \]

If we set \( \gamma(u) = u^\phi \) in (10) and (11), then we get the following generalized fractional integral operators containing Mittag-Leffler function:

**Definition 11.** Let \( \mu, \nu : [\alpha, \beta] \rightarrow \mathbb{R}, 0 < \alpha < \beta \) be the functions such that \( \mu \) be a positive and integrable and \( \nu \) be a differentiable and strictly increasing. Also, let \( \phi, \sigma, l, c, \zeta, \beta, \mu \in \mathbb{C}, \quad \Re(\phi), \Re(l) > 0, \Re(c) > \Re(\zeta) > 0 \) with \( p \geq 0, \sigma, r > 0 \) and \( 0 < q \leq r + \gamma \). Then for \( u \in [\alpha, \beta] \) the integral operators \( v Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu} \) are defined by:

\[ (v Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu})(u; p) = \int_u^b \frac{(v(u) - \nu(u))^\phi}{v(u) - \nu(u)} E_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(\psi(v(u) - \nu(u))^\prime; p)\mu(k)dk, \]  

and

\[ (v Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu})(u; p) = \int_u^b \frac{(v(k) - \nu(u))^\phi}{v(k) - \nu(u)} E_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(\psi(v(k) - \nu(u))^\prime; p)\mu(k)dk. \]

**Remark 2.** (12) and (13) are the generalization of the following fractional integral operators:

1. Setting \( v(u) = u \), the fractional integral operators (8) and (9), can be obtained.
2. Setting \( v(u) = u \) and \( p = 0 \), the fractional integral operators defined by Salim-Faraj in [29], can be obtained.
3. Setting \( v(u) = u \) and \( l = r = 1 \), the fractional integral operators defined by Rahman et al., in [30], can be obtained.
4. Setting \( v(u) = u \), \( p = 0 \) and \( l = r = 1 \), the fractional integral operators defined by Srivastava-Tomovski in [31], can be obtained.
5. Setting \( v(u) = u \), \( p = 0 \) and \( l = r = q = 1 \), the fractional integral operators defined by Prabhakar in [32], can be obtained.
6. Setting \( v(u) = u \) and \( \psi = p = 0 \), the Riemann-Liouville fractional integral operators can be obtained.

In [33], Mehmood et al., proved the following formulas for constant function:

\[ (v Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu})(u; p) = (v(u) - v(\alpha))^\phi E_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(\psi(v(u) - v(\alpha))^\prime; p) := v Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(u; p), \]

and

\[ (v Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu})(u; p) = (v(\beta) - v(u))^\phi E_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(\psi(v(\beta) - v(u))^\prime; p) := v Y_{\sigma,\phi}^{\nu,\lambda, c, \beta, \mu}(u; p). \]

The objective of this paper is to establish the Hadamard and the Fejér-Hadamard inequalities for generalized fractional integral operators (12) and (13) containing Mittag-Leffler function via a monotone function by using the exponentially \((h, m)\)-convex functions. These inequalities lead to produce the Hadamard
and the Fejér-Hadamard inequalities for various kinds of exponentially convexity and well known fractional integral operators given in Remark 1 and Remark 2. In Section 2, we prove the Hadamard inequalities for generalized fractional integral operators (12) and (13) via exponentially \((h,m)\)-convex functions. In Section 3, we prove the Fejér-Hadamard inequalities for these generalized fractional integral operators via exponentially \((h,m)\)-convex functions. Moreover, some of the results published in [26,33,34] have been obtained in particular.

2. Fractional Hadamard inequalities for exponentially \((h,m)\)-convex functions

In this section, we will give two versions of the generalized fractional Hadamard inequality. To establish these inequalities exponentially \((h,m)\)-convexity and generalized fractional integrals operators have been used.

**Theorem 12.** Let \(\mu, \nu : [\alpha, m\beta] \subset [0, \infty) \rightarrow \mathbb{R}, 0 < \alpha < m\beta\) be two functions such that \(\mu\) be integrable and \(\nu\) be differentiable. If \(\mu) be exponentially \((h,m)\)-convex, \(\nu\) be strictly increasing and \(h \in [0,1]\). Then for generalized fractional integral operators, the following inequalities hold:

\[
e^{\mu\left(\frac{\nu(a)+m(1-\nu(b))}{2}\right)} \nu_{\nu,\phi} \leq h\left(\frac{1}{2}\right) \left[ e^{\mu(\nu(a)+m(1-\nu(b)))} + me^{\mu((1-\nu(a))\nu + \nu(b))} \right].
\]

**Proof.** By the exponentially \((h,m)\)-convexity of \(\mu\), we have

\[
e^{\mu\left(\frac{\nu(a)+m(1-\nu(b))}{2}\right)} \leq h\left(\frac{1}{2}\right) \left[ e^{\mu(\nu(a)+m(1-\nu(b)))} + me^{\mu((1-\nu(a))\nu + \nu(b))} \right].
\]

Multiplying (17) with \(k^{\phi-1}E_{\nu,\phi,\delta}^{\alpha,\beta}(\psi\kappa; p)\) and integrating over \([0,1]\), we have

\[
e^{\mu\left(\frac{\nu(a)+m(1-\nu(b))}{2}\right)} \int_0^1 k^{\phi-1}E_{\nu,\phi,\delta}^{\alpha,\beta}(\psi\kappa; p) \, dk \leq h\left(\frac{1}{2}\right) \left[ \int_0^1 k^{\phi-1}E_{\nu,\phi,\delta}^{\alpha,\beta}(\psi\kappa; p) e^{\mu(\nu(a)+m(1-\nu(b)))} \, dk \right. + \left. m \int_0^1 k^{\phi-1}E_{\nu,\phi,\delta}^{\alpha,\beta}(\psi\kappa; p) e^{\mu((1-\nu(a))\nu + \nu(b))} \, dk \right].
\]

Setting \(\nu(u) = \nu(v\alpha + m(1-\nu(b))\) and \(\nu(v) = (1-\nu(a))\nu + \nu(b)\) in (18), then again from exponentially \((h,m)\)-convexity of \(\mu\), we have

\[
e^{\mu(\nu(a)+m(1-\nu(b)))} + me^{\mu((1-\nu(a))\nu + \nu(b))} \leq h(1) \left[ e^{\mu(\nu(a))} + me^{\mu((1-\nu(a))\nu + \nu(b))} \right] + mh(1) \left( e^{\mu(\nu(b))} + me^{\mu((1-\nu(a))\nu + \nu(b))} \right).
\]

Multiplying (19) with \(h\left(\frac{1}{2}\right) k^{\phi-1}E_{\nu,\phi,\delta}^{\alpha,\beta}(\psi\kappa; p)\) and integrating over \([0,1]\), we have

\[
h\left(\frac{1}{2}\right) \left[ \int_0^1 k^{\phi-1}E_{\nu,\phi,\delta}^{\alpha,\beta}(\psi\kappa; p) e^{\mu(\nu(a)+m(1-\nu(b)))} \, dk + m \int_0^1 k^{\phi-1}E_{\nu,\phi,\delta}^{\alpha,\beta}(\psi\kappa; p) e^{\mu((1-\nu(a))\nu + \nu(b))} \, dk \right] \leq h\left(\frac{1}{2}\right) \left[ \left( e^{\mu(\nu(a))} + me^{\mu((1-\nu(a))\nu + \nu(b))} \right) \int_0^1 k^{\phi-1}E_{\nu,\phi,\delta}^{\alpha,\beta}(\psi\kappa; p) h(1) \, dk + m \left( e^{\mu(\nu(b))} + me^{\mu((1-\nu(a))\nu + \nu(b))} \right) \int_0^1 k^{\phi-1}E_{\nu,\phi,\delta}^{\alpha,\beta}(\psi\kappa; p) h(1-\nu(a)) \, dk \right].
\]
where the first inequality of (22) is obtained.

Corollary 1. Setting $m = 1$ in (16), the following inequalities for exponentially $h$-convex function can be obtained:

$$e^{\mu \left( \frac{v(\alpha) + v(\beta)}{2} \right)} (v(\alpha) + v(\beta); p) \leq h \left( \frac{1}{2} \right) \left[ \left( v \left( \frac{e^{\mu \left( \frac{v(\alpha) + v(\beta)}{2} \right)} - 1}{2} \right) \right) + \left( v \left( \frac{e^{\mu \left( \frac{v(\alpha) + v(\beta)}{2} \right)} + 1}{2} \right) \right) \right] (\nu(\alpha) + \nu(\beta); p)$$

where $\nu = \frac{\nu}{(v(\beta) - v(\alpha))^p}$.

Remark 3. 1. If we set $h(\kappa) = \kappa$ in (16), then [33, Theorem 8] is obtained.
2. If we set $h(\kappa) = \kappa$ and $m = 1$ in (16), then [33, Corollary 1] is obtained.
3. If we set $v(u) = u$ and $h(\kappa) = \kappa$ in (16), then [34, Theorem 2.1] is obtained.
4. If we set $v(u) = u$, $h(\kappa) = \kappa$ and $m = 1$ in (16), then [34, Corollary 2.2] is obtained.
5. If we set $v(u) = u$ in (16), then [26, Theorem 2.1] is obtained.

In the following we give another version of the Hadamard inequality for generalized fractional integral operators via exponentially $(h, m)$-convex functions.

Theorem 13. Let $\mu, \nu : [a, m\beta] \subset [0, \infty) \to \mathbb{R}$, $0 < \alpha < m\beta$ be two functions such that $\mu$ be integrable and $\nu$ be strictly increasing. Then for generalized fractional integral operators, the following inequalities hold:

$$e^{\mu \left( \frac{v(\alpha) + v(\beta)}{2} \right)} (v(\alpha) + v(\beta); p) \leq h \left( \frac{1}{2} \right) \left[ \left( v \left( \frac{e^{\mu \left( \frac{v(\alpha) + v(\beta)}{2} \right)} - 1}{2} \right) \right) + \left( v \left( \frac{e^{\mu \left( \frac{v(\alpha) + v(\beta)}{2} \right)} + 1}{2} \right) \right) \right] (\nu(\alpha) + \nu(\beta); p)$$

where $\nu = \frac{\nu}{(v(\beta) - v(\alpha))^p}$.

Proof. By the exponentially $(h, m)$-convexity of $\mu$, we have

$$e^{\mu \left( \frac{v(\alpha) + v(\beta)}{2} \right)} \leq h \left( \frac{1}{2} \right) \left[ \left( v \left( \frac{e^{\mu \left( \frac{v(\alpha) + v(\beta)}{2} \right)} - 1}{2} \right) \right) + \left( v \left( \frac{e^{\mu \left( \frac{v(\alpha) + v(\beta)}{2} \right)} + 1}{2} \right) \right) \right] (\nu(\alpha) + \nu(\beta); p).$$

Multiplying (23) with $\kappa^{-1} E_{\nu, \phi, J} (\kappa \nu; p)$ and integrating over $[0, 1]$, we have

$$e^{\mu \left( \frac{v(\alpha) + v(\beta)}{2} \right)} = \int_0^1 \kappa^{-1} E_{\nu, \phi, J} (\kappa \nu; p) d\kappa$$

where $\nu = \frac{\nu}{(v(\beta) - v(\alpha))^p}$.

Setting $v(u) = \frac{v(\alpha) + v(\beta)}{2}$ and $v(v) = \frac{v(\alpha) + v(\beta)}{2}$ in (24), then by using (12), (13) and (14), the first inequality of (22) is obtained.
Again from exponentially \((h, m)\)-convexity of \(\mu\), we have
\[
e^{\mu(\xi v(a)+m\frac{(2-\kappa)}{m}v(\beta))} + me^{\mu(\xi v(\beta)+m\frac{(2-\kappa)}{m}v(a))} \leq h \left( \frac{K}{2} \right) \left( e^{\mu(v(a))} + me^{\mu(v(\beta))} \right) + mh \left( \frac{2-K}{2} \right) \left( e^{\mu(v(\beta))} + me^{\mu(v(a))} \right).
\] (25)

Multiplying (25) with \(h \left( \frac{1}{2} \right) \kappa^{\theta-1} E_{c,\phi,L}^{\psi\kappa;\epsilon}(\psi\kappa;\epsilon)\) and integrating over \([0, 1]\), we have
\[
h \left( \frac{1}{2} \right) \left[ \int_0^1 \kappa^{\theta-1} E_{c,\phi,L}^{\psi\kappa;\epsilon}(\psi\kappa;\epsilon) e^{\left( \xi v(a)+m\frac{(2-\kappa)}{m}v(\beta) \right)} \, dk + m \int_0^1 \kappa^{\theta-1} E_{c,\phi,L}^{\psi\kappa;\epsilon}(\psi\kappa;\epsilon) e^{\left( \xi v(\beta)+m\frac{(2-\kappa)}{m}v(a) \right)} \, dk \right]
\leq h \left( \frac{1}{2} \right) \left[ \left( e^{\mu(v(a))} + me^{\mu(v(\beta))} \right) \int_0^1 \kappa^{\theta-1} E_{c,\phi,L}^{\psi\kappa;\epsilon}(\psi\kappa;\epsilon) h \left( \frac{K}{2} \right) \, dk \right.
\]
\[+ m \left( e^{\mu(v(\beta))} + me^{\mu(v(a))} \right) \int_0^1 \kappa^{\theta-1} E_{c,\phi,L}^{\psi\kappa;\epsilon}(\psi\kappa;\epsilon) h \left( \frac{2-K}{2} \right) \, dk \right].
\] (26)

Putting \(v(u) = \xi v(a)+m\frac{(2-\kappa)}{m}v(\beta)\) and \(v(v) = \xi v(\beta)+m\frac{(2-\kappa)}{m}v(a)\) in (26), then by using (12) and (13), the second inequality of (22) is obtained. \(\square\)

**Corollary 2.** Setting \(m = 1\) in (22), the following inequalities for exponentially \(h\)-convex function can be obtained:
\[
2e^{h\left( \frac{v(a)+v(b)}{2} \right)} \psi_{2c,\xi}^{\phi} \left( v(\frac{v(a)+v(b))}{2} \right) \leq h \left( \frac{1}{2} \right) \left[ \left( e^{\mu(v(a))} + me^{\mu(v(\beta))} \right) \left( \xi v(a)+m\frac{(2-\kappa)}{m}v(\beta) \right) \right) \right] \leq h \left( \frac{1}{2} \right) \left[ \left( e^{\mu(v(a))} + me^{\mu(v(\beta))} \right) \left( \xi v(a)+m\frac{(2-\kappa)}{m}v(\beta) \right) \right)
\] (27)

where \(\psi\) is same as in (21).

**Remark 4.**
1. If we set \(h(\kappa) = \kappa\) in (22), then [33, Theorem 9] is obtained.
2. If we set \(h(\kappa) = \kappa\) and \(m = 1\) in (22), then [33, Corollary 2] is obtained.
3. If we set \(v(u) = u\) and \(h(\kappa) = \kappa\) in (22), then [34, Theorem 2.4] is obtained.
4. If we set \(v(u) = u\), \(h(\kappa) = \kappa\) and \(m = 1\) in (22), then [34, Corollary 2.5] is obtained.
5. If we set \(v(u) = u\) in (22), then [26, Theorem 2.2] is obtained.

3. Fractional Fejér-Hadamard Inequalities for exponentially \((h, m)\)-convex functions

In this section, we will give two versions of the generalized fractional Fejér-Hadamard inequality. To establish these inequalities exponentially \((h, m)\)-convexity and generalized fractional integrals operators have been used.

**Theorem 14.** Let \(\mu, v : [\alpha, m\beta] \subset [0, \infty) \to \mathbb{R}, 0 < \alpha < m\beta\) be two functions such that \(\mu\) be integrable and \(v\) be differentiable. If \(\mu\) be exponentially \((h, m)\)-convex and \(\mu(v(v)) = \mu(v(\alpha)+mv(\beta) - mv(v))\) and \(v\) be strictly increasing. Also, let \(\gamma : [\alpha, m\beta] \to \mathbb{R}\) be a function which is non-negative and integrable. Then for generalized fractional integral operators, the following inequalities hold:
\[
e^{\mu(v(a)) + m\frac{(2-\kappa)}{m}v(\beta)} \left( v_{\gamma}^{\psi\kappa;\epsilon}(\psi\kappa;\epsilon) e^{\gamma v} \right) \left( v-1 \left( \frac{v(a)}{m} \right) ; p \right) \leq h \left( \frac{1}{2} \right) (1+m) \left( v_{\gamma}^{\psi\kappa;\epsilon}(\psi\kappa;\epsilon) e^{\gamma v} \right) \left( v-1 \left( \frac{v(a)}{m} \right) ; p \right)
\] (28)

\[
\leq h \left( \frac{1}{2} \right) \left( \frac{mv(\beta) - v(a)}{m\phi} \right) \left( e^{\mu(v(a))} + me^{\mu(v(\beta))} \right) \int_0^1 \kappa^{\theta-1} E_{c,\phi,L}^{\psi\kappa;\epsilon}(\psi\kappa;\epsilon) h(\kappa) \, dk
\]
\[+ m \left( e^{\mu(v(\beta))} + me^{\mu(v(a))} \right) \int_0^1 \kappa^{\theta-1} E_{c,\phi,L}^{\psi\kappa;\epsilon}(\psi\kappa;\epsilon) e^{\gamma((1-\kappa)\frac{v(a)}{m} + \kappa v(\beta))} h(\kappa) \, dk \]
where \( \tilde{\Psi} \) is same as in (16).

**Proof.** Multiplying (17) with \( \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\gamma((1-\kappa)\frac{v(a)}{m}+\kappa v(\beta))} \) and integrating over \([0, 1]\), we have

\[
\begin{align*}
e^\mu \left( \frac{v(a)+\kappa v(\beta)}{2} \right) & \int_0^1 \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\gamma((1-\kappa)\frac{v(a)}{m}+\kappa v(\beta))} \, d\kappa \\
& \leq h \left( \frac{1}{2} \right) \left[ \int_0^1 \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\mu((1-\kappa)\frac{v(a)}{m}+\kappa v(\beta))} \, d\kappa \\
& \quad + m \int_0^1 \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\mu((1-\kappa)\frac{v(a)}{m}+\kappa v(\beta))} \, d\kappa \right].
\end{align*}
\]

(29)

Setting \( v(\nu) = (1-\kappa)\frac{v(a)}{m} + \kappa v(\beta) \) in (29), then by using (13) and assumption \( \mu(v(\nu)) = \mu(v(\alpha) + m\kappa v(\beta) - m\kappa v(\nu)) \), the first inequality of (28) is obtained.

Now multiplying (19) with \( h \left( \frac{1}{2} \right) \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\gamma((1-\kappa)\frac{v(a)}{m}+\kappa v(\beta))} \) and integrating over \([0, 1]\), we have

\[
\begin{align*}
h \left( \frac{1}{2} \right) & \left[ \int_0^1 \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\mu((1-\kappa)\frac{v(a)}{m}+\kappa v(\beta))} \, d\kappa \\
& \quad + m \int_0^1 \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\mu((1-\kappa)\frac{v(a)}{m}+\kappa v(\beta))} \, d\kappa \right] \\
& \leq h \left( \frac{1}{2} \right) \left( \mu(v(\nu)) + me^{\mu(v(\beta))} \right) \int_0^1 \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\gamma((1-\kappa)\frac{v(a)}{m}+\kappa v(\beta))} \, d\kappa \\
& \quad + m \left( \mu(v(\beta)) + me^{\mu(v(\beta))} \right) \int_0^1 \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\gamma((1-\kappa)\frac{v(a)}{m}+\kappa v(\beta))} \, d\kappa \right].
\end{align*}
\]

(30)

Setting \( v(\nu) = (1-\kappa)\frac{v(a)}{m} + \kappa v(\beta) \) in (30), then by using (13) and assumption \( \mu(v(\nu)) = \mu(v(\alpha) + m\kappa v(\beta) - m\kappa v(\nu)) \), the second inequality of (28) is obtained.

**Corollary 3.** Setting \( m = 1 \) in (28), the following inequalities for exponentially \( h \)-convex function can be obtained:

\[
\begin{align*}
e^\mu \left( \frac{v(a)+\kappa v(\beta)}{2} \right) \left( vY_{\alpha,\phi,J}\tilde{\Psi}e^{\gamma\nu} \right) (\alpha; p) & \leq 2h \left( \frac{1}{2} \right) \left( vY_{\alpha,\phi,J}\tilde{\Psi}e^{\mu(\nu)}e^{\gamma\nu} \right) (\alpha; p) \\
& \leq h \left( \frac{1}{2} \right) \left( v(\nu) - v(\alpha) \right) \tilde{\Psi} \left[ \mu(v(\nu)) + \mu(v(\beta)) \right] \int_0^1 \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\gamma((1-\kappa)v(\nu)+\kappa v(\beta))} \, d\kappa \\
& \quad + \int_0^1 \kappa^{\theta-1}E_{\alpha,\phi,J}(\psi\kappa^{\alpha}; p)e^{\gamma((1-\kappa)v(\alpha)+\kappa v(\beta))} h(1-\kappa) \, d\kappa \right],
\end{align*}
\]

(31)

where \( \tilde{\Psi} \) is same as in (21).

**Remark 5.**
1. If we set \( h(\kappa) = \kappa \) in (28), then [33, Theorem 10] is obtained.
2. If we set \( h(\kappa) = \kappa \) and \( m = 1 \) in (28), then [33, Corollary 3] is obtained.
3. If we set \( v(\nu) = u \) and \( h(\kappa) = \kappa \) in (28), then [34, Theorem 2.7] is obtained.
4. If we set \( v(\nu) = u, h(\kappa) = \kappa \) and \( m = 1 \) in (28), then [34, Corollary 2.8] is obtained.
5. If we set \( v(\nu) = u \) in (28), then [26, Theorem 2.3] is obtained.

In the following we give another generalized fractional version of the Fejér-Hadamard inequality.

**Theorem 15.** Let \( \mu, \nu : [a, m\beta] \subset [0, \infty) \to \mathbb{R}, 0 < a < m\beta \) be two functions such that \( \mu \) be integrable and \( \nu \) be differentiable. If \( \mu \) be exponentially \((h, m)\)-convex and \( \mu(v(\nu)) = \mu(v(\nu) + m(v(\beta) - m\kappa v(\nu))) \) and \( \nu \) be strictly increasing. Also, let \( \gamma : [a, m\beta] \to \mathbb{R} \) be a function which is non-negative and integrable. Then for generalized fractional integral operators, the following inequalities hold:
\[ e^{\left(\frac{v(\alpha) + m\mu(\beta)}{2}\right)} \left( Y_{\phi(\alpha),\phi(\beta)}^{\psi,\phi}, \frac{v(\alpha) + m\mu(\beta)}{2} \right) e^{\gamma_{ov}} \left( v(\alpha) \right) ; p \) 
\leq h \left( \frac{1}{2} \right) \left( v(\alpha) - v(\beta) \right) \int_{0}^{1} k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) h \left( \frac{K}{2} \right) dK 
+ m \left[ e^{\mu(v(\beta))} + m\mu \left( \frac{v(\alpha)}{m} \right) \right] \int_{0}^{1} k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) h \left( \frac{2 - \kappa}{2} \right) dK , \] 
(32) 

where \( \phi \) is same as in (16).

**Proof.** Multiplying (23) with \( k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) \) and integrating over \([0, 1]\), we have

\[ e^{\left(\frac{v(\alpha) + m\mu(\beta)}{2}\right)} \int_{0}^{1} k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) dK 
\leq h \left( \frac{1}{2} \right) \left[ e^{\mu(v(\alpha))} + m\mu(v(\beta)) \right] \int_{0}^{1} k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) dK 
+ m \left[ e^{\mu(v(\beta))} + m\mu \left( \frac{v(\alpha)}{m} \right) \right] \int_{0}^{1} k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) h \left( \frac{2 - \kappa}{2} \right) dK . \] 
(33)

Setting \( v(\alpha) = \frac{v(\alpha)}{2} + \frac{v(\alpha)}{2} \) in (33), then by using (13) and assumption \( \mu(v(\alpha)) = \mu(v(\alpha) + m\mu(\beta) - m\mu(v(\alpha))) \), the first inequality of (32) is obtained.

Now multiply (25) with \( h \left( \frac{1}{2} \right) k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) \) and integrating over \([0, 1]\), we have

\[ h \left( \frac{1}{2} \right) \left[ e^{\mu(v(\alpha))} + m\mu(v(\beta)) \right] \int_{0}^{1} k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) dK 
+ m \left[ e^{\mu(v(\beta))} + m\mu \left( \frac{v(\alpha)}{m} \right) \right] \int_{0}^{1} k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) h \left( \frac{2 - \kappa}{2} \right) dK . \] 
(34)

Setting \( v(\alpha) = \frac{v(\alpha)}{2} + \frac{v(\alpha)}{2} \) in (34), then by using (13) and assumption \( \mu(v(\alpha)) = \mu(v(\alpha) + m\mu(\beta) - m\mu(v(\alpha))) \), the second inequality of (32) is obtained. \( \square \)

**Corollary 4.** Setting \( m = 1 \) in (32), the following inequalities for exponentially h-convex function can be obtained:

\[ e^{\left(\frac{v(\alpha) + v(\beta)}{2}\right)} \left( Y_{\phi(\alpha),\phi(\beta)}^{\psi,\phi}, \frac{v(\alpha) + v(\beta)}{2} \right) e^{\gamma_{ov}} \left( v(\alpha) \right) (a; p) \leq 2h \left( \frac{1}{2} \right) \left( Y_{\phi(\alpha),\phi(\beta)}^{\psi,\phi}, \frac{v(\alpha) + v(\beta)}{2} \right) e^{\gamma_{ov}} \left( v(\alpha) \right) (a; p) 
\leq h \left( \frac{1}{2} \right) \left( v(\beta) - v(\alpha) \right) \left[ e^{\mu(v(\alpha))} + m\mu(v(\beta)) \right] \int_{0}^{1} k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) h \left( \frac{2 - \kappa}{2} \right) dK 
+ \int_{0}^{1} k^{\phi-1} E^{\psi,\alpha,\beta} (\psi \sigma_{\phi}, \lambda) e^{\gamma} \left( \frac{x(v(\beta) + m\mu(\alpha)\nu)}{2} \right) h \left( \frac{2 - \kappa}{2} \right) dK , \] 
(35)

where \( \phi \) is same as in (21).

**Remark.** 1. If we set \( h(\kappa) = \kappa \) in (32), then [33, Theorem 11] is obtained.

2. If we set \( h(\kappa) = \kappa \) and \( m = 1 \) in (32), then [33, Corollary 4] is obtained.
Remark 7. By setting $h(\kappa) = \kappa^s$ and $m = 1$ in Theorems 12, 13, 14 and 15, the Hadamard and the Fejér-Hadamard inequalities for exponentially $s$-convex functions can be obtained. We leave it for interested reader.

4. Concluding remarks

In this article, we established the Hadamard and the Fejér-Hadamard inequalities. To established these inequalities generalized fractional integral operators and exponentially $(h, m)$-convexity have been used. The presented results hold for various kind of exponentially convexity and well known fractional integral operators given in Remarks 1 and 2. Moreover, the established results have connection with already published results.

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References

[1] Niculescu, C., & Persson, L. E. (2006). Convex functions and their applications. A contemporary approach, CMS Books in Mathematics, vol. 23, Springer Verlag, New York.
[2] Pečarić, J., Proschan, F., & Tong, Y. L. (1973). Convex Functions, Partial Orderings and Statistical Applications. Academics Press, New York.
[3] Roberts, A. W., & Varberg, D. E. (1973). Convex Functions, Academics Press, New York, USA.
[4] Fejér, L. (1906). Überdie Fourierreihen II, Math Naturwiss Anz Ungar Akad Wiss, 24, 369-390.
[5] Abbas, G., & Farid, G. (2017). Hadamard and Fejér-Hadamard type inequalities for harmonically convex functions via generalized fractional integrals. The Journal of Analysis, 25(1), 107-119.
[6] Awan, M. U., Noor, M. A., & Noor, K. I. (2018). Hermite–Hadamard inequalities for exponentially convex functions. Applied Mathematics & Information Sciences, 12(2), 405-409.
[7] Chen, F. (2014). On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity. Chinese Journal of Mathematics, 2014, 7 pp.
[8] Chen, H., & Katugampola, U. N. (2017). Hermite–Hadamard-Fejér type inequalities for generalized fractional integrals. Journal of Mathematical Analysis and Applications, 446, 1274-1291.
[9] Dragomir, S. S., & Gomm, I. (2015). Some Hermite-Hadamard type inequalities for functions whose exponentials are convex. Studia Universitatis Babeş-Bolyai Mathematica, 60(4), 527-534.
[10] Farid, G. (2016). Hadamard and Fejér-Hadamard inequalities for generalized fractional integrals involving special functions. Konuralp Journal of Mathematics, 4(1), 108-113.
[11] Farid, G. (2018). A treatment of the Hadamard inequality due to m-convexity via generalized fractional integrals. Journal of Fractional Calculus and Applications, 9(1), 8-14.
[12] Farid, G., & Abbas, G. (2018). Generalizations of some fractional integral inequalities for m–convex functions via generalized Mittag–Leffler function. Studia Universitatis Babeş–Bolyai Mathematica, 63(1), 23-35.
[13] Farid, G., Rehman, A. U., & Tariq, B. (2017). On Hadamard–type inequalities for m–convex functions via Riemann–Liouville fractional integrals. Studia Universitatis Babeş–Bolyai Mathematica, 62(2), 141-150.
[14] Farid, G., Rehman, A. U., & Mehmood, S. (2018). Hadamard and Fejér-Hadamard type integral inequalities for harmonically convex functions via an extended generalized Mittag–Leffler function. Journal of Mathematics and Computer Science, 8(5), 630-643.
[15] Farid, G., Khan, K. A., Latif, N., Rehman, A. U., & Mehmood, S. (2018). General fractional integral inequalities for convex and m–convex functions via an extended generalized Mittag–Leffler function. Journal of inequalities and applications, 2018, 243 pp.
[16] İşcan, I. (2015). Hermite Hadamard Fejér type inequalities for convex functions via fractional integrals. Studia Universitatis Babeş–Bolyai Mathematica, 60(3), 355-366.
[17] Kang, S. M., Farid, G., Nazeer, W., & Mehmood, S. (2019). $(h, m)$-convex functions and associated fractional Hadamard and Fejér-Hadamard inequalities via an extended generalized Mittag-Leffler function. Journal of inequalities and applications, 2019, 78 pp.
[18] Kang, S. M., Farid, G., Nazeer, W., &Tariq, B. (2018). Hadamard and Fejér-Hadamard inequalities for extended generalized fractional integrals involving special functions. Journal of inequalities and applications, 2018, 119 pp.
[19] Mehreen, N., & Anwar, M. (2019). Hermite-Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications. *Journal of inequalities and applications, 2019*, 92 pp.

[20] Sarikaya, M. Z., Set, E., Yaldız, H., & Basak, N. (2013). Hermite–Hadamard’s inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling, 57*(9-10), 2403-2407.

[21] Sarikaya, M. Z., & Yıldırım, H. (2016). On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Mathematical Notes, 17*(2), 1049-1059.

[22] Antczak, T. (2001). (p, r)-invex sets and functions. *Journal of Mathematical Analysis and Applications, 263*, 355-379.

[23] Rashid, S., Noor, M. A., Noor, K. I., & Akdemir, A. O. (2019). Some new generalizations for exponentially s–convex functions and inequalities via fractional operators. *Fractal and Fractional, 3*(2), 24.

[24] Rashid, S., Noor, M. A., & Noor, K. I. (2019). Some generalize Riemann–Liouville fractional estimates involving functions having exponentially convexity property. *Punjab University Journal of Mathematics, 51*, 1-15.

[25] Rashid, S., Noor, M. A., & Noor, K. I. (2019). Fractional exponentially m-convex functions and inequalities. *International Journal of Analysis and Applications, 17*(3), 464-478.

[26] Rashid, S., Noor, M. A., & Noor, K. I. (2019). Some new estimates for exponentially (h, m)-convex functions via extended generalized fractional integral operators. *The Korean Journal of Mathematics, 27*(4), 843-860.

[27] Andrić, M., Farid, G., & Pečarić, J. (2018). A further extension of Mittag-Leffler function. *Fractional Calculus and Applied Analysis, 21*(5), 1377-1395.

[28] Farid, G. (2020). A unified integral operator and further its consequences. *Open Journal of Mathematical Analysis, 4*(1) (2020), 1-7.

[29] Salim, T. O., & Faraj, A. W. (2012). A generalization of Mittag–Leffler function and integral operator associated with fractional calculus. *Journal of Fractional Calculus and Applications, 3*(5), 1-13.

[30] Rahman, G., Baleanu, D., Quraishi, M. A., Purohit, S. D., Mubeen, S., & Arshad, M. (2017). The extended Mittag-Leffler function via fractional calculus. *Journal of Nonlinear Sciences and Applications, 10*(8), 4244-4253.

[31] Srivastava, H. M., & Tomovski, Ž. (2009). Fractional calculus with an integral operator containing a generalized Mittag–Leffler function in the kernel. *Applied Mathematics and Computation, 211*(1), 198-210.

[32] Prabhakar, T. R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Mathematical Journal, 19*, 7-15.

[33] Mehmood, S., Farid, G., Khan, K. A., & Yussouf, M. New Hadamard and Fejér-Hadamard fractional inequalities for exponentially m-convex function. *Engineering and Applied Scienee Letters, 3*(1), 45-55.

[34] Mehmood, S.,& Farid, G. (2020). Fractional Hadamard and Fejér-Hadamard inequalities for exponentially m-convex function. *Studia Universitatis Babeș–Bolyai Mathematica*, to appear.

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