STACK SEMANTICS AND THE COMPARISON OF MATERIAL AND STRUCTURAL SET THEORIES

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Abstract. We extend the usual internal logic of a (pre)topos to a more general interpretation, called the stack semantics, which allows for “unbounded” quantifiers ranging over the class of objects of the topos. Using well-founded relations inside the stack semantics, we can then recover a membership-based (or “material”) set theory from an arbitrary topos, including even set-theoretic axiom schemas such as collection and separation which involve unbounded quantifiers. This construction reproduces the models of Fourman-Hayashi and of algebraic set theory, when the latter apply. It turns out that the axioms of collection and replacement are always valid in the stack semantics of any topos, while the axiom of separation expressed in the stack semantics gives a new topos-theoretic axiom schema with the full strength of ZF. We call a topos satisfying this schema autological.

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1. Introduction

It is well-known that elementary topos theory is roughly equivalent to a weak form of set theory. The most common statement is that the theory of well-pointed topoi with natural numbers objects and satisfying the axiom of choice (a.k.a. ETCS; see [Law64]) is equiconsistent with “bounded Zermelo” set theory (BZC). This is originally due to Cole [Col73] and Mitchell [Mit72]; see also Osius [Osi74]. Moreover, the proof is very direct: the category of sets in any model of BZC is a model of

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ETCS, while from any model of ETCS one can construct a model of BZC out of well-founded relations. (The same ideas apply to stronger and weaker theories, but for clarity, in the introduction we will speak mostly about topoi.)

In fact, Lawvere and others have observed that ETCS can itself serve as a foundation for much mathematics, replacing traditional membership-based set theoretic foundations such as BZC or ZFC. Although the term “set theory” often refers specifically to membership-based theories, it is clear that ETCS is also a set theory, in the inclusive sense of “a theory about sets.” Pursuant to this philosophy, we will therefore use the term material set theory for theories such as BZC and ZFC which are based on a global membership predicate, and structural set theory for theories such as ETCS which take functions as more fundamental. (Some explanation of this terminology can be found in the appendix.)

Whatever names we use for them, the relationship between these two types of theory is like other bridges between fields of mathematics, in that it augments the toolkit of each side by the ideas and techniques of the other. Of central importance to such a programme is that under this equivalence, certain constructions in material set theory, such as forcing, can be identified with certain constructions in topos theory, such as categories of sheaves. However, there are surprising difficulties in making this precise, regarding the issue of well-pointedness.

It is only a well-pointed topos which can be regarded as a structural set theory and to which the Cole-Mitchell-Osius theory can be applied directly, since only in a well-pointed topos is an object, like a set, “determined by its elements” $1 \to X$. But the categorical constructions in question almost never preserve well-pointedness, so some trick is necessary to make the correspondence work.

Of course, the trick is well-known in essence: any topos has an “internal logic” (often called the Mitchell-Benabou language) which appears much more set-theoretic. In the internal logic, we can speak about “elements” of objects, and the semantics automatically interprets statements about these elements into more appropriate categorical terms. For instance, a morphism $f : X \to Y$ in an arbitrary topos is an epimorphism if and only if the statement “for all $y \in Y$, there exists an $x \in X$ with $f(x) = y$” is true in the internal logic.

This trick is quite powerful, and in fact it is fundamental to the whole field of categorical logic. However, the internal logic of a topos is not a set theory of any sort (material or structural), but rather a “higher-order type theory.” In particular, it does not include unbounded quantifiers, i.e. quantifiers ranging over all sets. This is problematic for the programme of bridging between material and structural set theories, since many of the most interesting axioms of set theory (especially separation, collection, and replacement) involve such unbounded quantifiers.

Secondary tricks which have been used to get around this difficulty. The first was proposed by Fourman [Fou80]: if $S$ is a complete topos (such as a Grothendieck topos), then we can mimic the construction of the von Neumann hierarchy inside $S$. We define inductively an object $V_\alpha$ for each (external) ordinal $\alpha$, setting $V_0 = 0$, $V_{\alpha+1} = PV_\alpha$, and taking colimits at limit stages (hence the need for $S$ to be complete). Now the internal logic can speak about “elements” of the objects $V_\alpha$, which inherit a “membership” structure from their construction and can be shown to model material set theories. Hayashi [Hay81] independently

\[1\text{Recall that a topos is well-pointed if 1 is a generator and there is no map } 1 \to 0. \text{ As we will see later, this notion requires some revision in an intuitionistic context.}\]
proposed a more general approach in which the $V_\alpha$ can be replaced by any class of objects satisfying suitable closure conditions.

While this construction works quite well, we find it somewhat unsatisfying for several reasons. Firstly, it appears to depend on the non-elementary hypothesis of completeness, which is not preserved by some constructions on toposes (such as realizability and filterquotients). Secondly, it appears to depend on power objects, which would make it inapplicable to compare pretoposes with “predicative” material set theories. Finally, the treatment of unbounded quantifiers is asymmetrical—they appear in the material set theory as if by magic, but there is no natural, category-theoretic, treatment of unbounded quantifiers on the topos side.

A more recent solution, begun by Joyal and Moerdijk in [JM95], is to consider not just a topos, but a category of classes in which it sits as the “small objects.” This is a category-theoretic version of material class-set theories such as Bernays-Gödel or Morse-Kelley. A category of classes contains a single object $V$, “the class of all sets,” which admits a membership relation which satisfies the axioms of material set theory in the internal logic of the category of classes. This approach, called algebraic set theory, avoids some of the problems with the Fourman-Hayashi approach. It is purely elementary at the level of the category of classes, and applies as well in the absence of powersets. And the internal logic of the category of classes provides a compelling category-theoretic treatment of “unbounded” quantifiers—they are simply bounded quantifiers over the class of all sets.

However, algebraic set theory also has its own disadvantages, foremost among which is the fact that instead of studying a topos by itself, one must now always somehow construct, and always carry along, a category of classes containing it. One would ideally wish for a canonical embedding of any topos into a category of classes, under which all desirable topos-theoretic structure corresponds equivalently to suitable structure on the category of classes. It is shown in [AF06, APW06] that every small topos can be embedded in some category of classes, via a construction which is in some ways canonical. However, the resulting category of classes will not, in general, satisfy axioms such as separation, even for toposes which might be expected to validate them (for instance, those which validate their material counterparts under the Fourman-Hayashi interpretation). A different construction is given by [ABSS07a, ABSS07b], which does in some cases produce categories of classes satisfying separation, but it is very non-category-theoretic in flavor (in the sense that it distinguishes between isomorphic objects) and certainly less canonical.

A second problem with algebraic set theory is that one has to worry about whether any given categorical construction can be extended from elementary toposes (where the results are usually known and easy) to any categories of classes that may contain them. In general, this turns out to be true, but the proofs often require a great deal of work (see, for instance, [AGLW09, vdBM07, vdBM08, vdBM09]).

I do not mean to denigrate the great advances in understanding produced by algebraic set theory, and the Fourman-Hayashi interpretation has also been very fruitful. The relationship between the two is also known: the “ideal semantics” of [ABSS07a, ABSS07b] generalizes the forcing semantics of Hayashi. However, it would undoubtedly also be useful to have a unified, canonical, elementary, and category-theoretic treatment of unbounded quantifiers over an arbitrary (pre)topos. The interpretations of Fourman-Hayashi and algebraic set theory would then, ideally, turn out to be special cases or generalizations of this unified framework.
The thesis of this paper is that such a thing is indeed possible; it is what we call the stack semantics. We can motivate this semantics in two ways. The first is to observe that higher-order type theory (that is, the internal logic of a topos) looks very much like a fragment of structural set theory (i.e., the theory of well-pointed topoi). Types correspond to objects of the category, and each term $x$ of type $A$ corresponds to a morphism $1 \to A$. In this way any assertion in type theory can be translated into an assertion in the first-order theory of well-pointed categories. However, the latter theory also includes quantifiers over objects of the category, so it is natural to seek a generalization of the internal logic which models the first-order theory of well-pointed categories, rather than higher-order type theory.

The stack semantics is exactly such a generalization. (When the logic is intuitionistic, as it is in the stack semantics, the notion of “well-pointed” has to be modified a bit; see §3. For clarity, we usually refer to this modified property constructive well-pointedness; thus what the stack semantics really models is the first-order theory of constructively well-pointed topoi.)

From this point of view, the stack semantics can be defined directly by generalizing the “Kripke-Joyal” description of the internal logic via a forcing relation. This description is thus closely related to Hayashi’s forcing interpretation of material set theory, and also to the forcing semantics defined in [ABSS07b, ABSS07a].

A second motivation for the stack semantics comes from trying to make the logic of a category of classes more canonical and category-theoretic. Specifically, if $S$ is a small topos, an obvious “canonical” candidate for a category of classes containing it is the category $\text{Sh}(S)$ of sheaves for the coherent topology on $S$. (The category of ideals defined in [AF06, AFW06] is a certain well-behaved subcategory of $\text{Sh}(S)$.) If we then want to interpret “unbounded” quantifiers over objects of $S$ as “bounded” quantifiers in the internal logic of $\text{Sh}(S)$, we need a suitable object to serve as “the class of sets.” Or, more category-theoretically, we want an internal category in $\text{Sh}(S)$ to serve as “the category of sets,” and ideally we would like this internal category to also be canonically determined by $S$.

Now $S$ of course has a canonical representation as a stack over itself, namely the “self-indexing” or “fundamental fibration” $X \mapsto S/X$. This is not an internal category in $\text{Sh}(S)$, since it is not a strict functor, only a pseudofunctor. But we can choose some strictification of it, which is therefore an internal category in $\text{Sh}(S)$, and let our “unbounded” quantifiers run over this. This is precisely the approach taken in [Awo97, Ch. V], where it was additionally observed that in the internal logic of $\text{Sh}(S)$, the self-indexing is (in our terminology) constructively well-pointed. However, now we take the additional step of isolating the particular fragment of the internal logic of $\text{Sh}(S)$ whose only “unbounded” quantifiers range over the self-indexing. Modulo one small caveat, this fragment is canonically determined by the topos $S$, and when expressed in the sheaf semantics of $\text{Sh}(S)$ via a forcing relation over the site $S$, it agrees with the stack semantics defined using the first approach.

The caveat is that there is one other thing this logic depends on: the chosen strictification of the self-indexing. However, this choice only affects the truth values of “non-category-theoretic” formulas, such as those which assert equalities between objects. In the stack semantics proper, we exclude such formulas by restricting to a dependently typed logic in which there simply is no equality predicate between

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2 Alternately, one can use a type theory allowing propositions which quantify over the universe Type. We will not do this, but it has certain advantages.
objects. (This is also precisely the necessary restriction so that the truth of such formulas be invariant under equivalence of categories.) This restricted logic can then be naturally identified with a fragment of the internal logic of the 2-category $\mathcal{S}t(S)$ of stacks on $S$. (The internal logic of a 2-category is not well-known, but it can be defined by close analogy with that of a 1-category; see [Shu].)

The existence of these multiple approaches to the stack semantics contributes to the feeling that it is a natural part of topos theory, though heretofore seemingly unappreciated. In fact, however, it is implicitly present throughout the literature. Topos theory abounds with definitions, arguments, and theorems that are said to be “internal” statements, but in many cases this cannot be directly understood as an assertion in the usual internal logic due to the presence of unbounded quantifiers. As a very simple example, the external topos-theoretic “axiom of choice” (AC) asserts that every epimorphism splits. This involves an unbounded quantification over all objects of the topos, so it is not directly internalizable. Instead, the “internal axiom of choice” (IAC) asserts (in one formulation) that for any object $A$, the functor $\Pi_A : S/A \to S$ preserves epimorphisms. To a beginner, it may not be obvious why this is the “internalization” of AC. But it turns out that IAC is equivalent to the validity in the stack semantics of the statement “every epimorphism splits.”

The proper way of internalization of such statements does soon become second nature to experts, and it has even started to appear more openly in the literature, such as in the “informal arguments” given in [MP02]. However, to my knowledge it has never before been given a formal, precise expression. This is what the stack semantics provides: a formal definition of the language whose statements we can expect to internalize, along with a procedure for performing that internalization and a theorem that the resulting semantics is sound. Thus, it has the potential to simplify and clarify many arguments in topos theory, by reducing them to internalizations of easier proofs in structural set theory.

(This effect is even more pronounced when the stack semantics is generalized to the full internal logic of $\mathcal{S}t(S)$. In [Shuc] we will show that almost any argument in the “naive” theory of large categories, such as Freyd’s adjoint functor theorem, or Giraud’s theorem characterizing Grothendieck topoi, can be interpreted in the generalized stack semantics to immediately yield the corresponding result for indexed categories over any base topos.)

However, in the present paper we are only concerned with the basic theory of the stack semantics, and its application to comparison of material and structural set theories. In particular, by generalizing the higher-order type theory of the internal logic to a semantics for structural set theory, the stack semantics enables us to reconstruct an interpretation of material set theory from any topos, by simply performing a version of the Cole-Mitchell-Osius construction inside the stack semantics. As suggested previously, this unifies, rather than replaces, the existing approaches of Fourman-Hayashi and of algebraic set theory. On the one hand, if the topos is complete, then the Fourman-Hayashi model can be identified with that derived from the stack semantics (in fact, one version of Hayashi’s approach essentially is the stack semantics model). On the other hand, if a topos $S$ is embedded inside a category of classes $C$ (and $S$ “generates” $C$ in a suitable sense), then the stack semantics of $S$ agrees with a fragment of the internal logic of $C$ (as summarized above in the case $C = \text{Sh}(S)$), and in good cases the resulting models of material set theory can also be identified.
In a sense, these equivalences are really only an expected “consistency check.” We believe that where the stack semantics really shows its worth is in suggesting topos-theoretic counterparts of additional set-theoretic axioms such as separation, collection, and replacement, which until now have been largely absent from the literature. With a little thought, it is not hard to write down axioms for structural set theory which correspond under the Cole-Mitchell-Osius equivalence to the material-set-theoretic axioms of separation, collection, and replacement. By then interpreting these axioms in the stack semantics, we can obtain true topos-theoretic versions of such axioms which are applicable to an arbitrary topos.

It turns out that under interpretation in the stack semantics, the axioms of collection and of separation behave quite differently. The axiom of collection (and hence that of replacement) is always valid in the stack semantics of any topos. (The fact that forcing semantics always validate collection has been observed elsewhere, e.g. [ABSS07a].) It follows that the models of material set theory constructed from the stack semantics always validate the material version of collection, providing a neat proof of the well-known fact that intuitionistically, collection without separation adds no proof-theoretic strength. In fact, collection emerges as precisely the dividing line separating the “internal” stack semantics from the “external” logic: a constructively well-pointed topos satisfies collection (in the “external” sense, not in its stack semantics) if and only if validity in its stack semantics is equivalent to “external” truth for all sentences.

On the other hand, the stack-semantics version of separation seems to be a new topos-theoretic property. We call a topos with this property autological. Autology is also closely connected to the stack semantics: a topos is autological if and only if its stack semantics is representable, in the same sense that the usual Kripke-Joyal semantics is always representable. (This motivates the name: a topos is autological if the logic of its stack semantics can be completely “internalized” by representing subobjects.) Autology is also quite common and well-behaved; for instance, we will show in §9 that all complete topoi are autological, and in in [Shul] we will study its preservation by other topos-theoretic constructions (including realizability).

Moreover, since autology is simply the axiom of separation in the stack semantics, the Cole-Mitchell-Osius model of material set theory constructed from an autologous topos must validate both collection and separation. Thus, autology provides a natural and fully topos-theoretic strengthening of the notion of elementary topos, which is equiconsistent with full ZF set theory, and which “explains” why complete and realizability topos are known to model (intuitionistic) ZF. These two cases have been addressed by algebraic set theory in [ABSS07a, ABSS07b], but as remarked above, the construction there is non-category-theoretic and non-elementary.

In fact, the relationship of autology to the axiom of separation for categories of classes is unidirectional: if a category of classes satisfies separation, then its topos of small objects is autological, but not conversely. The issue is that since autology is a first-order axiom schema, it can only speak about proper classes which are definable in some sense over the topos, but the objects of a category of classes need not be definable in any such way.

One might, however, ask whether every autological topos could be embedded in some category of classes satisfying separation, analogously to how every model

\footnote{I am indebted to Peter Freyd for this noun form, which is definitely easier on the tongue than “autological-ness”.
}
of ZFC can be embedded in some model of Bernays-Gödel class-set theory (namely, the model consisting of its definable proper classes). We will show in [Shua] that the answer is almost yes: any autological topos can be embedded in a 2-category of “definable stacks” which satisfies separation, and which satisfies 2-categorical versions of the axioms of algebraic set theory. However, it seems unlikely that this can be improved to a 1-category of classes, since this would require some sort of definable rigidification of the self-indexing. This suggests that perhaps we should study 2-categories of large categories instead of categories of classes. Such a modification is also category-theoretically natural, since in practice we generally only care about elements of a proper class insofar as they form some large category. See [Shua, Shuc] for further exploration of this idea.

The plan of the paper is as follows. We begin by defining carefully the material and structural set theories we plan to compare. Since material set theories are quite familiar, we list their axioms briefly in §2 while in §§3–5 we spend rather more time on the axioms of structural set theory. In both cases we include “predicative” theories as well as “impredicative” ones. Then in §6 we describe a version of the Cole-Mitchell-Osius comparison that is valid even predicatively, since the construction does not appear to be in the literature in this generality.

The core of the paper is §7, in which we introduce the stack semantics and prove its basic properties, including the interpretations of the axioms of separation and collection mentioned above. It is then almost trivial to derive the desired interpretations of material set theory; we say a few words about this in §8. Then in §9 and 10 we show that this interpretation agrees with both the Fourman-Hayashi model and with the models of algebraic set theory. Appendix A is devoted to some philosophical remarks comparing material and structural set theory, particularly as foundations for mathematics.

Our basic reference for topos-theoretic notions is [Joh02]. One important thing to note is that we will use intuitionistic logic in all contexts. This is in contrast to the common practice in topos theory of reasoning about the topos itself using classical logic, though the internal logic of the topos is intuitionistic. Our approach is necessary because we want all of our arguments to remain valid in the stack semantics, which, like the internal logic of a topos, is generally intuitionistic. We treat classical logic as an axiom schema (namely, \( \varphi \lor \neg\varphi \) for all formulas \( \varphi \)) which may or may not be satisfied by any given model.

All of our theories, material and structural, are of course properly expressed in a formal first-order language. However, to make the paper easier to read, we will usually write out logical formulas in English, trusting the reader to translate them into formal symbols as necessary. To make it clear when our mathematical English is to be interpreted as code for a logical formula, we will often use sans serif font and corner quotes. Thus, for example, in a structural set theory, “every surjection has a section” represents the axiom of choice.

If \( M \) models some theory, by a **formula in** \( M \) we mean the result of substituting elements of \( M \) for some of the free variables of a formula \( \varphi \) in a well-typed way. The substituted elements of \( M \) are called **parameters**. If parameters are substituted for all the free variables of \( \varphi \) (including the case when \( \varphi \) had no free variables), we say \( \varphi \) is a **sentence in** \( M \). There is an obvious notion of when a sentence \( \varphi \) in \( M \) is **true**, which we denote \( M \models \varphi \) (read \( M \) satisfies \( \varphi \)). Recall that our metatheory, in which we formulate this notion of truth, is intuitionistic.
Many of the axioms we state will be axiom schemas depending on some formula $\varphi$. In each such case, $\varphi$ may have arbitrary additional unmentioned parameters. Equivalently, such axiom schemas can be regarded as implicitly universally quantified over all the unmentioned variables of $\varphi$.

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2. Material set theories

Our material set theories will be theories in one-sorted first order logic, with a single binary relation symbol $\in$, whose variables are called sets. (We will briefly mention the possibility of “atoms” which are not sets at the end of this section.) In such a theory, a $\Delta^0_0$-quantifier is one of the form $(\exists x \in a)$ or $(\forall x \in a)$, and a $\Delta^0_0$-formula (also called restricted or bounded) is one with only $\Delta^0_0$-quantifiers. The prefix “$\Delta^0_0$” on any axiom schema indicates that the formulas appearing in it are restricted to be $\Delta^0_0$; we sometimes use the adjective “full” to indicate the lack of any such restriction.

Definitions 2.1.

- We write $\{x \mid \varphi(x)\}$ for a set $a$ such that $x \in a \iff \varphi(x)$, if such a set exists.
- A relation from $a$ to $b$ is a set $r$ of ordered pairs $(x,y)$ such that $x \in a$ and $y \in b$, where $(x,y) = \{\{x\},\{x,y\}\}$.
- A relation $r$ from $a$ to $b$ is entire (or a many-valued function) if for all $x \in a$ there exists $y \in b$ with $(x,y) \in r$. If each such $y$ is unique, $r$ is a function.
- A relation $r$ is bi-entire if both $r$ and $r^\prime = \{(y,x) \mid (x,y) \in r\}$ are entire.

The following set of axioms (in intuitionistic logic) we call the core axioms; they form a subsystem of basically all material set theories.

- Extensionality: "for any $x$ and $y$, $x = y$ if and only if for all $z$, $z \in x$ iff $z \in y$".
- Empty set: "the set $\emptyset = \{\}$ exists".
- Pairing: "for any $x$ and $y$, the set $\{x,y\}$ exists".
- Union: "for any $x$, the set $\{z \mid (\exists y \in x)(z \in y)\}$ exists".
- $\Delta^0_0$-Separation: For any $\Delta^0_0$-formula $\varphi(x)$, "for any $a$, $\{x \in a \mid \varphi(x)\}$ exists".
- Limited $\Delta^0_0$-replacement: For any $\Delta^0_0$-formula $\varphi(x,y)$, "for all $a$ and $b$, if for every $x \in a$ there exists a unique $y$ with $\varphi(x,y)$, and moreover $z \subseteq b$ for all $z \in y$, then the set $\{y \mid (\exists x \in a)\varphi(x,y)\}$ exists".

The first five of these are standard, while the last, though very weak, is unusual. We include it because we want our core axioms to imply a well-behaved category of sets, for which purpose the other five are not quite enough. It is satisfied in basically all material set theories, since it is implied both by ordinary replacement (even $\Delta^0_0$-replacement), and by the existence of power sets (using $\Delta^0_0$-separation).

We now list some additional axioms of material set theory, pointing out the most important implications between them. Of course, there are many possible set-theoretic axioms; we will consider only the most common and important ones.

One major axis along which set theories vary is “impredicativity,” expressed by the following sequence of axioms (of increasing strength):

- Exponentiation: "for any $a$ and $b$, the set $b^a$ of all functions from $a$ to $b$ exists".
- Fullness: "for any $a,b$ there exists a set $m$ of entire relations from $a$ to $b$ such that for any entire relation $f$ from $a$ to $b$, there is a $g \in m$ with $g \subseteq f$".
• **Power set:** If for any set \( x \), the set \( Px = \{ y \mid y \subseteq x \} \) exists\(^7\).

Each of these is strictly stronger than the preceding one, but in the presence of either of the following:

• **\( \Delta_0 \)-classical logic:** for any \( \Delta_0 \)-formula \( \varphi \), we have \( \varphi \lor \neg \varphi \).

• **Full classical logic:** for any formula \( \varphi \), we have \( \varphi \lor \neg \varphi \).

the hierarchy of predicativity collapses, since power sets can then be obtained (using limited \( \Delta_0 \)-replacement) from sets of functions into a 2-element set. Hence classical logic is also impredicative. Also considered impredicative is:

• **(Full) Separation:** For any formula \( \varphi(x) \), \( \forall x \in a \{ x \mid \varphi(x) \} \) exists\(^7\).

Full separation and \( \Delta_0 \)-classical logic together imply full classical logic, since we can form \( a = \{ \emptyset \mid \varphi \} \subseteq \{ \emptyset \} \), and then \( (\emptyset \in a) \lor (\emptyset \notin a) \) is an instance of \( \Delta_0 \)-classical logic that implies \( \varphi \lor \neg \varphi \).

We also have the collection and replacement axioms.

• **Collection:** for any formula \( \varphi(x,y) \), if for all \( x \in a \) there is a \( y \) with \( \varphi(x,y) \), then there is a \( b \) such that for all \( x \in a \), there is a \( y \in b \) with \( \varphi(x,y) \), and for all \( y \in b \), there is a \( x \in a \) with \( \varphi(x,y) \).

• **Replacement:** for any formula \( \varphi(x,y) \), if for every \( x \in a \) there exists a unique \( y \) with \( \varphi(x,y) \), then the set \( \{ y \mid (\exists x \in a)\varphi(x,y) \} \) exists\(^7\).

With an evident extension of 2.1 to formulas, we can state collection as “if \( \varphi \) is entire on \( a \), then there exists \( b \) such that \( \varphi \) is bi-entire from \( a \) to \( b \).” Collection also evidently implies replacement. Classical ZF set theory is usually defined using replacement rather than collection, but in the presence of the other axioms of ZF, collection can be proven. (Replacement and full classical logic together also imply full separation.) This is no longer true in intuitionistic logic, though, or in \( \Delta_0 \)-classical logic together imply full classical logic, since we can form \( a = \{ \emptyset \mid \varphi \} \subseteq \{ \emptyset \} \), and then \( (\emptyset \in a) \lor (\emptyset \notin a) \) is an instance of \( \Delta_0 \)-classical logic that implies \( \varphi \lor \neg \varphi \).

We state the axiom of infinity as the existence of the set \( \omega \) of finite von Neumann ordinals. Together with the core axioms, this implies the Peano axioms for \( \omega \), including induction for any property to which we can apply separation. For a stronger version of induction, we need a separate axiom.

• **Infinity:** If there exists a set \( \omega \) such that \( \emptyset \in \omega \), if \( x \in \omega \) then \( x \cup \{ x \} \in \omega \), and if \( z \) is any other set such that \( \emptyset \in z \) and if \( x \in z \) then \( x \cup \{ x \} \in z \), then \( \omega \subseteq z \).

• **Induction:** for any formula \( \varphi \), if \( \varphi(\emptyset) \) and \( \varphi(x) \Rightarrow \varphi(x \cup \{ x \}) \) for any \( x \in \omega \), then \( \varphi(x) \) for all \( x \in \omega \).

There is also:

• **Choice:** If for any set \( a \), if for all \( x \in a \) there exists a \( y \in x \), then there exists a function \( f \) from \( a \) to \( \bigcup a \) such that \( f(x) \in x \) for all \( x \in a \).

By Diaconescu’s argument [Dia75], choice and power set together imply \( \Delta_0 \)-classical logic. There are of course many variants of choice, such as countable, dependent, or “multiple” choice, or the “presentation axiom” (a.k.a. “EPSets” or “COSHET,” the Category Of Sets Has Enough Projectives), but we will confine our attention to the ordinary version.

Finally, we have the family of “foundation-type” axioms. Recall that a set \( x \) is called transitive if \( z \in y \in x \) implies \( z \in x \). A set \( X \) with a binary relation \( \prec \) is called
well-founded if whenever $Y \subseteq X$ is inductive, in the sense that $(\forall y \prec x)(y \in Y)$ implies $x \in Y$, then in fact $Y = X$. It is extensional if $(\forall z \in X)(z \prec x \iff z \prec y)$ implies $x = y$.

- **Set-induction**: for any formula $\varphi(x)$, "if for any set $y$, $\varphi(x)$ for all $x \in y$ implies $\varphi(y)$, then $\varphi(x)$ for any set $x$".
- **Foundation**, a.k.a. $\Delta_0$-Set-induction: like set-induction, but only for $\Delta_0$-formulas.
- **Transitive closures**: "every set is a subset of a smallest transitive set".
- **Mostowski’s principle**: "every well-founded extensional relation is isomorphic to a transitive set equipped with the relation $\in$".

The usual “classical” formulation of foundation is

- **Regularity**: "if $x$ is nonempty, there is some $y \in x$ such that $x \cap y = \emptyset$".

In the presence of transitive closures, regularity is equivalent to the conjunction of $\Delta_0$-set-induction and $\Delta_0$-classical logic. Thus, in an intuitionistic theory, $\Delta_0$-set-induction is preferable. Of course, $\Delta_0$-set-induction follows from full set-induction, while the converse is true if we have full separation. Set-induction also implies ordinary induction, since $<$ and $\in$ are the same for von Neumann ordinals.

Transitive closures can be constructed either from set-induction and replacement (see [AR01]), or from infinity, ordinary (full) induction, and replacement. Mostowski’s original proof shows that his principle follows from replacement and full separation. Transitive closures and Mostowski’s principle are not usually considered “foundation” axioms, but we group them thusly for several reasons:

- The conjunction of transitive closures, Mostowski’s principle, and foundation is equivalent to the single statement “to give a set is the same as to give a well-founded extensional relation with a specified element.” One could argue that this sums up the objects of study of well-founded material set theory.
- As we will see in §6 it follows that the material set theories that can be constructed from structural ones are precisely those which satisfy these three axioms.
- Finally, the various axioms of anti-foundation are also naturally stated as “to give a set is the same as to give a (blank) relation with a specified element,” where (blank) denotes some stronger type of extensionality (there are several choices; see [Acz88]). Thus, these axioms naturally include analogues not just of foundation, but also of transitive closures and Mostowski’s principle.

Some particular material set theories with names include the following. Note that the prefix constructive is generally reserved for “predicative” theories (including, potentially, exponentiation and fullness, but never power sets or full separation) while intuitionistic refers to “impredicative” (but still non-classical) theories which may have power sets or full separation. None of these theories include our axiom of limited $\Delta_0$-replacement explicitly, but all include either power sets or a stronger version of replacement, both of which imply it.

- The standard **Zermelo-Fraenkel set theory with Choice** (ZFC) includes all the axioms mentioned above, although it suffices to assume the core axioms together with full classical logic, power sets, replacement, infinity, foundation, and choice.
- By **Zermelo set theory** (Z) one usually means the core axioms together with full classical logic, power sets, full separation, and infinity.
• The theory ZBQC of [ML86], called RZC in [MLM94], consists of the core axioms together with full classical logic, power sets, infinity, choice, and regularity.

• The Mostowski set theory (MOST) of [Mat01] consists of ZBQC together with transitive closures and Mostowski’s principle.

• By Kripke-Platek set theory (KP) is usually meant the core axioms together with full classical logic, a form of $\Delta_0$-collection (sufficient to imply our limited $\Delta_0$-replacement), and set-induction for some class of formulas.

• Aczel’s Constructive Zermelo-Fraenkel (CZF) consists of the core axioms together with fullness, collection, infinity, and set-induction; see [AR01]. His CZF$_0$ consists of the core axioms, replacement, infinity, and induction.

• The usual meaning of Intuitionistic Zermelo-Fraenkel (IZF) is the core axioms together with power sets, full separation, collection, infinity, and set-induction. This is the strongest theory that can be built from the above axioms without any form of classical logic or choice.

• The Basic Constructive Set Theory (BCST) of [AFW06] consists essentially of the core axioms together with replacement. Their Constructive Set Theory (CST) adds exponentials to this, and their Basic Intuitionistic Set Theory (BIST) adds power sets. (These theories also allow atoms; see below.)

• The Rudimentary Set Theory (RST) of [vdBM07] consists of the core axioms together with collection and set-induction.

So far we have considered only theories of pure sets, i.e. in which everything is a set. If we want to also allow “atoms” or “urelements” which are not sets, we can modify any of the above theories by adding a predicate $\Gamma x$ is a set$, whose negation is read as $\Gamma x$ is an atom$. We then add the following axioms:

• Decidability of sethood: $\Gamma$ every $x$ is either a set or an atom$.

• Only sets have elements: $\Gamma$ if $x \in y$, then $y$ is a set$.

• Set of atoms: $\Gamma$ the set $\{ x \mid x$ is an atom $\}$ exists$.

and modify the extensionality axiom so that it applies only to sets:

• Extensionality for sets: $\Gamma$ for any sets $x$ and $y$, $x = y$ if and only if for all $z$, $z \in x$ iff $z \in y$.

3. Structural set theory: well-pointed Heyting pretoposes

We now move on to structural set theories. Here we must be a little more careful with the ambient logic. The theory of categories can, of course, be formulated in ordinary first-order logic, but this is unsatisfactory because it allows us to discuss equality of objects. We need all our formulas to express “category-theoretic” facts, and thus we must avoid ever asserting that two objects are equal (rather than isomorphic). However, we cannot simply remove the equality predicate on objects from the ordinary first-order theory, because there are situations in which we do need to know that two objects are the same. For example, to compose two arrows $f$ and $g$, we need to know that the source of $g$ is the same as the target of $f$.

The solution to this problem, which has seemingly been rediscovered many times (see, for instance, [Bla79, Fre76, Mak95, Mak01]) is to use a language of dependent types. We give here a somewhat informal description in an attempt to avoid the complicated precise syntax of dependent type theory, but the reader familiar with DTT should easily be able to expand our description into a formal one.
Definition 3.1. The language of categories (with terminal object) consists of the following.

(i) There is a type of objects.
(ii) There is a specified object-term 1.
(iii) For object-terms $X$ and $Y$, there is a type $X \to Y$ of arrows from $X$ to $Y$.
(iv) For each object-term $X$, there is an arrow-term $1_X : X \to X$.
(v) For any object-terms $X$, $Y$, and $Z$ and each pair of arrow-terms $f : X \to Y$ and $g : Y \to Z$, there is an arrow-term $(g \circ f) : X \to Z$.
(vi) For any arrow-terms $f, g : X \to Y$, there is an atomic formula $(f = g)$.

Non-atomic formulas are built up from atomic ones in the usual way, using connectives $\top$, $\bot$, $\land$, $\lor$, $\Rightarrow$, $\neg$, and quantifiers $\exists$ and $\forall$. There are of course two types of quantifiers, for object-variables and for arrow-variables. Any category is a structure for this language, and we have the usual satisfaction relation $A \vDash \varphi$ for a category $A$ and a sentence $\varphi$ in $A$.

Of course, this theory uses very little of the full machinery of DTT, and in particular it may be said to live in the “first-order” fragment of DTT. In this way it is closely related to the logic FOLDS studied in [Mak95], although we are allowing some term constructors in addition to relations.

Our inclusion of a specified term 1, intended to denote a terminal object, is just for convenience; it will make it easier to single out the $\Delta_0$-formulas. The other main thing to note is that, as in FOLDS, the only atomic formulas are equalities between parallel arrows; the language does not allow us to even discuss whether two objects are equal. This implies that “truth is isomorphism- and equivalence-invariant,” in a sense we can make precise as follows.

Let $A$ be a category, let $\varphi$ be a sentence in $A$, and suppose we are given, for each object-parameter $A$ in $\varphi$, an isomorphism $A \xrightarrow{\cong} A'$ in $A$. Let $\varphi'$ denote the sentence in $A$ obtained from $\varphi$ by replacing each object-parameter $A$ by $A'$ and each arrow-parameter $A \xrightarrow{j} B$ by the composite

$$A' \xrightarrow{\cong} A \xrightarrow{j} B \xrightarrow{\cong} B'.$$

We call $\varphi'$ an isomorph of $\varphi$. Likewise, if $F : A \to B$ is any functor and $\varphi$ is a sentence in $A$, we write $F(\varphi)$ for the sentence in $B$ obtained by applying $F$ to all the parameters of $\varphi$. The following can then easily be proven by induction on the construction of formulas.

Lemma 3.2 (Isomorphism-invariance of truth). If $\varphi$ is a sentence in $A$ and $\varphi'$ an isomorph of $\varphi$, then $A \vDash \varphi$ if and only if $A \vDash \varphi'$.

Lemma 3.3 (Equivalence invariance of truth). If $F$ is fully faithful and essentially surjective, then $A \vDash \varphi$ if and only if $B \vDash F(\varphi)$.

In fact, the dependently typed theory of categories characterizes exactly those properties of categories which are invariant under equivalence in this sense; see [Bla79, Fre76, Mak95].

We now begin a listing of the axioms of the structural set theories we will consider. First of all, we must have a category, so we include:

- Identity: $1_Y \circ f = f = f \circ 1_X$.
- Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$. 

Many of the additional axioms of structural set theories assert well-known categorical properties. Recall that a **Heyting category** is a category $S$ satisfying the following axioms.

- **Finite limits:** $`S`$ has pullbacks, and $1$ is a terminal object$`$.
- **Regularity:** $`$every morphism factors as a regular epi followed by a monic, and regular epis are stable under pullback$`$.
- **Coherency:** $`$finite unions of subobjects exist and are stable under pullback$`$.
- **Dual images:** $`$for any $f : X \to Y$, the pullback functor $f^* : \text{Sub}(Y) \to \text{Sub}(X)$ has a right adjoint $\forall_f$.$`$

Here $\text{Sub}(X)$ denotes the poset of subobjects of $X$, i.e. of monomorphisms into $X$ modulo isomorphism. If $S$ also satisfies

- **Positivity/extensivity:** $`S`$ has disjoint and pullback-stable binary coproducts$`$.

it is called **positive** (or **extensive**). If it is positive and also satisfies:

- **Exactness:** $`$every equivalence relation is a kernel$`$.

it is called a **Heyting pretopos**. A Heyting pretopos which also satisfies:

- **Local cartesian closure:** $`$for every arrow $f : X \to Y$, the pullback functor $f^* : S/Y \to S/X$ has a right adjoint $\Pi_f$.$`$

(which implies dual images) is called a **$\Pi$-pretopos**. Finally, if it satisfies:

- **Power objects:** $`$every object $X$ has a power object $PX$.$`$

(which, together with finite limits, implies all the other axioms of a $\Pi$-pretopos) it is called an **(elementary) topos**. See, e.g., [Joh02, §A.1] for more details. In particular, we remind the reader that in a pretopos, every monic and every epic is regular, and thus it is balanced (every monic epic is an isomorphism). Also, the self-indexing of any regular category is a prestack for its regular topology (generated by regular epimorphisms), and it is a stack if the category is exact (such as a pretopos).

**Remark 3.4.** Axioms which assert the existence of adjoints (such as dual images and local cartesian closure) seemingly go beyond first-order logic, but they can easily be formulated in an elementary way, as simply asserting the existence of objects with a suitable universal property.

Three other important axioms are the following:

- **Booleanness:** $`$every subobject has a complement$`$.
- **Full classical logic:** for any formula $\varphi$, we have $\varphi \lor \neg \varphi$.
- **Infinity:** $`$there exists a natural number object (NNO)$`$.
- **Choice:** $`$every regular epimorphism splits$`$.

(If $S$ is not cartesian closed, then the definition of an NNO should be taken with arbitrary parameters; see [Joh02, A2.5.3].)

Each of the above axioms (beyond those of a Heyting pretopos) is, more or less obviously, a counterpart of some axiom of material set theory. The first easy observation is that each material axiom implies its corresponding structural axiom for the category of sets.

**Theorem 3.5.** If $V$ satisfies the core axioms of material set theory, then the sets and functions in $V$ form a Heyting pretopos $\text{Set}(V)$. Moreover:

(i) If $V$ satisfies exponentiation, then $\text{Set}(V)$ is locally cartesian closed.
(ii) If $V$ satisfies the power set axiom, then $\text{Set}(V)$ is a topos.
(iii) If $V$ satisfies $\Delta_0$-classical logic, then $\text{Set}(V)$ is Boolean.
(iv) If $\mathbf{V}$ satisfies full classical logic, so does $\mathbf{Set}(\mathbf{V})$.
(v) If $\mathbf{V}$ satisfies infinity and exponentials, then $\mathbf{Set}(\mathbf{V})$ has a NNO.
(vi) If $\mathbf{V}$ satisfies the axiom of choice, then so does $\mathbf{Set}(\mathbf{V})$.

Proof. We first show that $\mathbf{Set}(\mathbf{V})$ is a category. Using pairing and limited $\Delta_0$-replacement, we can form, for any set $a$ and any $x \in a$, the set $p_{a,x}$ of all pairs \( \{x, y\} \) for $y \in a$. Again using limited $\Delta_0$-replacement, we can form the set of all the $p_{a,x}$ for $x \in a$, and take its union, thereby obtaining the set of all pairs $\{x, y\}$ for $x \in a$. Applying this construction twice to $a \cup b = \bigcup \{a, b\}$, we can then use $\Delta_0$-separation to cut out the set $a \times b$ of ordered pairs $(x, y)$ where $x \in a$ and $y \in b$. With this in hand we can define function composition and identities using $\Delta_0$-separation.

Finite limits are straightforward: we already have products, $\{\emptyset\}$ is a terminal object, and $\Delta_0$-separation supplies equalizers. The construction of pullback-stable images, unions, and dual images is also easy from $\Delta_0$-separation. A stable and disjoint coproduct $a + b$ can as usual be obtained as a subset of $\{0, 1\} \times (a \cup b)$. And if $r \subseteq a \times a$ is an equivalence relation, $\Delta_0$-separation supplies the equivalence class of any $x \in a$, and limited $\Delta_0$-replacement then supplies the set of all such equivalence classes.

Exponentiation clearly implies cartesian closedness. For local cartesian closedness, suppose given $f: a \to b$ and $g: x \to a$. Then the fiber of $\Pi_f(g) \to b$ over $j \in b$ should be the set of all functions $s: f^{-1}(j) \to x$ such that $g \circ s = 1_{f^{-1}(j)}$; this can be cut out of $x^{f^{-1}(j)}$ by $\Delta_0$-separation. Finally, the entire set $\Pi_f(g)$ can be constructed from these and limited $\Delta_0$-replacement, since each element of each fiber is a subset of $a \times x$.

The relationship between power sets and power objects is analogous. Likewise, $\Delta_0$-classical logic is equivalent to saying that every subset has a complement, which certainly implies Booleanness of $\mathbf{Set}(\mathbf{V})$. The implication for full classical logic is also evident.

Now suppose that $\mathbf{V}$ satisfies infinity and exponentials, and let $\omega$ be as in the axiom of infinity; we define $0 \in \omega$ and $s: \omega \to \omega$ in the obvious way. Suppose given $A \xrightarrow{s} B \xleftarrow{t} B$; we must construct a unique function $f: A \times \omega \to B$ such that $f(1_A \times 0) = g \pi_1$ and $f(1_A \times s) = t f$. Let $R = \{(a, b) \in \omega \times \omega \mid a \in b\}$ with projection $\pi_2: R \to \omega$, and let $X$ be the exponential $(B \times \omega \to \omega)^{(A \times R + \omega)}$ in $\mathbf{Set}(\omega)$; thus an element of $X$ is equivalent to an $n \in \omega$ together with a function $f: A \times \{m \mid m \in n\} \to B$. Using $\Delta_0$-separation, we have the subset $Y \subseteq X$ of those $f \in X$ such that $f(a, 0) = g(a)$ and $f(a, s(m)) = t(f(a, m))$ whenever $s(m) \in n$. We then prove by induction that for all $n \in \omega$ there exists an $f \in Y$ with $n \in \text{dom}(f)$ and that for any two such $f, f'$ we have $f(n) = f'(n)$. The union of $Y$ is then the desired function.

Finally, if $\mathbf{V}$ satisfies choice, then from any surjection $p: e \to b$ we can construct, using limited $\Delta_0$-replacement, the set $\{p^{-1}(x) \mid x \in b\}$, and applying the material axiom of choice to this gives a section of $p$. \hfill \Box

However, not every Heyting pretopos deserves to be called a model of structural set theory. The distinguishing characteristic of a set, as opposed to an object of some more general category, is that a set is determined by its elements and nothing more. This is expressed by the following property.
Definition 3.6. A Heyting category $\mathbf{S}$ is **constructively well-pointed** if it satisfies the following.

(a) If $m: A \to X$ is monic and every $1 \xrightarrow{x} X$ factors through $m$, then $m$ is an isomorphism (**1 is a strong generator**).

(b) Every regular epimorphism $X \to 1$ splits (**1 is projective**).

(c) If 1 is expressed as the union of two subobjects $1 = A \cup B$, then either $A$ or $B$ must be isomorphic to 1 (**1 is indecomposable**).

(d) There does not exist a map $1 \to 0$ (**1 is nonempty**).

**Remark 3.7.** Of course, since regular epis are stable under pullback, 1 is projective if and only if for any regular epi $Y \xrightarrow{p} X$, every global element $1 \to X$ lifts to $Y$. Likewise, since unions are stable under pullback, 1 is indecomposable if and only if whenever $X = A \cup B$, every global element $1 \to X$ factors through either $A$ or $B$. In particular, if $\mathbf{S}$ is locally small in some external set theory, then $\mathbf{S}$ is constructively well-pointed if and only if $\mathbf{S}(1,-): \mathbf{S} \to \mathbf{Set}$ is a conservative coherent functor.

We immediately record the following.

**Proposition 3.8.** If $\mathbf{V}$ satisfies the core axioms of material set theory, then $\mathbf{Set}(\mathbf{V})$ is constructively well-pointed.

**Proof.** Functions $1 \to X$ in $\mathbf{Set}(\mathbf{V})$ are in canonical correspondence with elements of $X$, and a function is monic just when it is injective. Thus, if every map from 1 factors through a monic, it must be bijective, and hence an isomorphism, so 1 is a strong generator. Next, the epis in $\mathbf{Set}(\mathbf{V})$ are the surjections, and if $X \xrightarrow{p} 1$ is a surjection, then $X$ must be inhabited, hence $p$ splits. And if $1 = A \cup B$, then the unique element of 1 must be in either $A$ or $B$ by definition of unions, hence either $A$ or $B$ must be inhabited. Finally, 0 has no elements, so 1 is nonempty. □

Recall that classically, a topos is said to be **well-pointed** if 1 is a generator (i.e. for $f, g: X \to Y$, having $fx = gx$ for all $1 \xrightarrow{x} X$ implies $f = g$) and there is no map $1 \to 0$ (1 is nonempty). Thus, any constructively well-pointed topos is well-pointed in the classical sense. Conversely, the following is well-known (see, for instance, [MLM94] VI.1 and VI.10).

**Lemma 3.9.** Let $\mathbf{S}$ be a Heyting category satisfying full classical logic, and in which 1 is a nonempty strong generator. Then $\mathbf{S}$ is Boolean and constructively well-pointed.

**Proof.** For any object $X$, either $X$ is initial or it isn’t. If it isn’t, then there must be a global element $1 \to X$, since otherwise the monic $0 \to X$ would be an isomorphism (since 1 is a strong generator). Therefore, if $X$ is not initial, then $X \to 1$ is split epic. Now suppose that $X$ is any object such that $X \to 1$ is regular epic; since $0 \to 1$ is not regular epic, it follows that $X$ is not initial, and so $X \to 1$ must in fact be split epic; thus 1 is projective. Also, if $1 = A \cup B$, then $A$ and $B$ cannot both be initial, so one of them has a global element; thus 1 is indecomposable.

Now let $A \to X$ be a subobject and assume that $A \cup \neg A$ is not all of $X$. Then since 1 is a strong generator, there is a $1 \xrightarrow{x} X$ not factoring through $A \cup \neg A$. Let $V = x^*(A \cup \neg A)$. Then $V$ is a subobject of 1. If $V$ is not initial, then it has a global element and hence is all of 1, so $x$ factors through $A \cup \neg A$. But if $V$ is initial, then $x^*(A)$ must also be initial, which implies that $x$ factors through $\neg A$. This is a contradiction, so $A \cup \neg A = X$; hence $\mathbf{S}$ is Boolean. □
Corollary 3.10. Let $S$ be a topos satisfying full classical logic, which is well-pointed in the classical sense. Then $S$ is Boolean and constructively well-pointed.

Proof. In a topos, any generator is a strong generator. □

Thus, in the presence of full classical logic, our notion of constructive well-pointedness is equivalent to the usual notion of well-pointedness. However, Proposition 3.8 shows that a constructively well-pointed topos not satisfying full classical logic need not be Boolean. One can also construct examples showing that in the absence of full classical logic, a Boolean topos in which 1 is a nonempty generator need not be constructively well-pointed. It is true, however, even intuitionistically, that if 1 is projective, indecomposable, and nonempty in a Boolean topos, it must also be a generator; see [Awo97, V.3].

Other authors have also recognized the importance of explicitly assuming projectivity and indecomposability of 1 in an intuitionistic context. In [Awo97] a topos in which 1 is projective, indecomposable, and nonempty was called hyperlocal (but in [Joh92, Joh02] that word is used for a stronger, non-elementary, property). And in [Pal], indecomposability of 1 is assumed explicitly, while projectivity of 1 is deduced from a factorization axiom. We note that most classical properties of a well-pointed topos make use of projectivity and indecomposability of 1, and many of these are still true intuitionistically as long as the category is constructively well-pointed.

Lemma 3.11. Let $S$ be a constructively well-pointed Heyting category. Then:

(i) A morphism $p: Y \to X$ is regular epic if and only if every map $1 \to X$ factors through it.

(ii) Likewise, $f: Y \to X$ is monic if and only if every map $1 \to X$ factors through it in at most one way, and an isomorphism if and only if every map $1 \to X$ factors through it uniquely.

(iii) Given two subobjects $A \rightarrow X$ and $B \rightarrow X$, we have $A \leq B$ if and only if every map $1 \to X$ which factors through $A$ also factors through $B$.

(iv) $X$ is initial if and only if there does not exist any morphism $1 \to X$.

Proof. The “only if” part of (i) follows because 1 is projective. For the “if” part, note that a map $p: Y \to X$ in a regular category is regular epic iff its image is all of $X$, while if every $1 \to X$ factors through $p$ then it must factor through $\text{im}(p)$ as well; hence $\text{im}(p)$ is all of $X$ since 1 is a strong generator.

The “only if” parts of (ii) are obvious. If $f: Y \to X$ is injective on global elements, then the canonical monomorphism $Y \to Y \times_X Y$ is bijective on global elements, and hence an isomorphism; thus $Y$ is monic. And if $f$ is bijective on global elements, then by this and by (i) it must be both monic and regular epic, hence an isomorphism.

The “only if” part of (iii) is also obvious. If every $1 \to X$ which factors through $A$ also factors through $B$, then in the pullback

$$
\begin{array}{ccc}
A \cap B & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & X
\end{array}
$$

the map $A \cap B \to A$ must be an isomorphism, since 1 is a strong generator; hence its inverse provides a factorization of $A$ through $B$. 

Finally, the “only if” part of (iv) is the nonemptiness assumption, while if \( X \) has no global elements, then the map 0 \( \to X \) is bijective on global elements and hence an isomorphism. \( \square \)

**Remark 3.12.** In terms of the internal logic, indecomposability of 1 corresponds to the *disjunction property* (if \( \varphi \lor \psi \) is true, then either \( \varphi \) is true or \( \psi \) is true), while projectivity of 1 corresponds to the *existence property* (if \( (\exists x)\varphi(x) \) is true, then there is a particular \( a \) such that \( \varphi(a) \) is true). We might also call nonemptiness of 1 the *negation property* (\( \bot \) is not true).

**Remark 3.13.** (Constructive) well-pointedness has a very different flavor from all the other axioms of structural set theory we have encountered so far. In particular, it is not preserved by slicing, or by most other categorical constructions. This is to be expected, however, since “being the category of sets” is quite a fragile property.

We have now reached the point where we can define what we mean by a *structural set theory*: an extension of the axiomatic theory describing a constructively well-pointed Heyting pretopos. For example:

- Lawvere’s *Elementary Theory of the Category of Sets (ETCS)*, defined in [Law64], is the theory of a well-pointed topos with a NNO satisfying full classical logic and the axiom of choice.
- Palmgren’s *Constructive ETCS (CETCS)*, defined in [Pal], is the theory of a constructively well-pointed Π-pretopos with a NNO and enough projectives.
- We will refer to the theory of a constructively well-pointed topos with a NNO as **Intuitionistic ETCS (IETCS)**.

**Convention 3.14.** When working in a structural set theory, we usually write \( \text{Set} \) for the category in question. We call its objects *sets* and its arrows *functions*. We speak of morphisms \( 1 \to X \) as *elements* of \( X \), and we write \( x \in X \) synonymously for \( x: 1 \to X \). If \( f: X \to Y \) is a function and \( x \in X \), we write \( f(x) \) for \( f \circ x: 1 \to Y \). We speak of monomorphisms \( S \to X \) as *subsets* of \( X \) and write \( S \subseteq X \). For \( x \in X \) and \( S \subseteq X \), we write \( x \in S \) and say “\( x \) is contained in \( S \)” to mean that \( 1 \xrightarrow{x} X \) factors through \( S \to X \). Similarly, for \( S,T \subseteq X \) we write \( S \subseteq T \) to mean that \( S \to X \) factors through \( T \to X \). We also use the following terminology.

- An arrow-variable \( x: 1 \to X \) whose domain is 1 is a \( \Delta_0 \)-variable.
- An equality \((f = g)\) of arrow-terms \( f,g: 1 \to X \) with source 1 is a \( \Delta_0 \)-atomic formula.
- A quantifier over a \( \Delta_0 \)-variable is a \( \Delta_0 \)-quantifier.
- A formula whose only variables are \( \Delta_0 \)-variables, whose only atomic subformulas are \( \Delta_0 \)-atomic, and whose only quantifiers are \( \Delta_0 \)-quantifiers is a \( \Delta_0 \)-formula.

These conventions make doing mathematics in a structural set theory sound much more familiar. They also make it sound very much like the internal logic of a category \( S \). Recall that the latter is a type theory with types for all objects of \( S \), function symbols for all arrows in \( S \), and relation symbols for all monics in \( S \). The following lemma makes this correspondence precise.

**Lemma 3.15.** For any Heyting category \( S \), there is a canonical bijection (up to provable equivalence) between

(a) \( \Delta_0 \)-formulas in the language of categories with parameters in \( S \), and
(b) formulas in the internal first-order logic of \( S \).
Proof. Given a $\Delta_0$-formula as in [a], we construct a formula in the internal logic by following formally the informal description in [3.14]. Note that every arrow-term $1 \to X$ must be constructed as a composite of some number of parameters, possibly beginning with a single $\Delta_0$-variable. (The possibility of nontrivial terms of type $1 \to 1$ can be excluded, up to provable equivalence.) Thus, any such term such as $f \circ (g \circ x) : 1 \to X$ can be represented by a term $f(g(x)) : X$ in the internal logic. Similarly, $\Delta_0$-atomic formulas give atomic equalities between such terms, and connectives and $\Delta_0$-quantifiers translate in the obvious way.

In the other direction, we have to do a little more work because the internal logic usually includes function-symbols of higher arity, corresponding to arrows in $S$ whose domain is a cartesian product. Whenever a term such as $f(t_1, t_2) : Y$ occurs, for terms $t_1 : X_1$ and $t_2 : X_2$ and an arrow $f : X_1 \times X_2 \to Y$ in $S$, in the translation we introduce a new $\Delta_0$-variable $z : 1 \to X_1 \times X_2$ (in addition to those occurring in $t_1$ and $t_2$), represent $f(t_1, t_2)$ by $f \circ z$, and carry along an extra stipulation that $p_1 \circ z = t_1$ and $p_2 \circ z = t_2$, where $p_i : X_1 \times X_2 \to X_i$ are the product projections in $S$. This extra condition has then to be placed at an appropriate point in the resulting formula, but this process is familiar from the procedure of definitional extensions in first-order logic.

The translation of atomic equalities between terms is obvious, but the internal logic includes relation symbols of arbitrary arity, with corresponding atomic formulas such as $R(x_1, x_2)$. However, such an atomic formula can be replaced by the $\Delta_0$-formula $(\exists z \in R)(p_1(z) = x_1 \land p_2(z) = x_2)$, where $p_i : R \to X_i$ are the jointly monic projections of the relation $R$ in $S$. Thus we can translate all atomic formulas, and the extension to connectives and quantifiers is straightforward. \hfill $\square$

Remark 3.16. Note that in the translation of function symbols of higher arity, the object $X_1 \times X_2$ is now a parameter of the resulting $\Delta_0$-formula, as are $p_1$ and $p_2$. Thus, we actually have to choose some particular cartesian product of $X_1$ and $X_2$ in $S$. If $S$ doesn’t come with “specified” products, then this requires some axiom of choice to define the entire bijective correspondence. But we will only need to apply the correspondence to particular formulas or finite sets of formulas, in which case there is no problem since we only need to make finitely many choices.

From now on we will identify the two types of formulas whose equivalence is shown in Lemma 3.15. The following observation shows that not only do the formulas themselves correspond, so do their “extensions,” in the senses appropriate to the internal logic and to structural set theory respectively.

Proposition 3.17. Let $S$ be a category with finite limits in which $1$ is a strong generator. Then the following are equivalent.

(i) $S$ is a constructively well-pointed Heyting category.

(ii) $S$ satisfies the schema of $\Delta_0$-separation: for any $\Delta_0$-formula $\varphi(x)$ with free variable $x : 1 \to X$, there exists a subobject $S \to X$ such that any map $x : 1 \to X$ factors through $S$ if and only if $\varphi(x)$ holds.

Proof. Assuming (i) and given a $\Delta_0$-formula $\varphi$ as in (ii), we construct $S$ as the usual “interpretation” in the internal logic of $S$ of the formula corresponding to $\varphi$ under Lemma 3.15. That is, we build $S$ by induction on the structure of $\varphi$, using intersections, unions, images, dual images, and so on in $S$. That this satisfies the required property also follows from an inductive argument. The cases of $T$
and ∧ are clear, while ⇒ and ∀ follow from the assumption that 1 is a strong generator. Finally, the cases of ∃, ∨, and ⊥ use the projectivity, indecomposability, and nonemptiness of 1, respectively.

Conversely, assume ∆₀-separation (ii) and that 1 is a strong generator. The argument of Lemma 3.11(ii) still applies to show that a morphism is monic iff it is injective on global elements. Let ℳ denote the class of monics and ℰ the class of morphisms that are surjective on global elements. Because 1 is a strong generator, any morphism in ℰ is extremal epic (i.e. factors through no proper subobject of its codomain). Moreover, ℰ is evidently stable under pullback.

For any map \( f: Y \to X \), apply ∆₀-separation to the formula \((∃ y \in Y)(f(y) = x)\) to obtain a monic \( m: S \to X \). The pullback of \( m \) along \( f \) is a monic which is bijective on global elements, hence an isomorphism; thus \( f \) factors through \( m \), and the factorization \( e: Y \to S \) is in ℰ by construction. Therefore, \((ℰ, ℳ)\) is a pullback-stable factorization system, from which it follows that \( S \) is a regular category and that \( ℰ \) is exactly the class of regular epics—and hence 1 is projective.

Now given monics \( m: A \to X \) and \( n: B \to X \), apply ∆₀-separation to the formula \((∃ a \in A)(m(a) = x) \lor (∃ b \in B)(n(b) = x)\) to obtain a monic \( S \to X \). Lemma 3.11(iii) still applies to show that \( A \subseteq S \) and \( B \subseteq S \) and that if \( A \subseteq T \) and \( B \subseteq T \) then \( S \subseteq T \); thus \( S = A \cup B \). The defining property of \( S \) (it contains precisely those \( 1 \to X \) factoring through either \( A \) or \( B \)) is evidently stable under pullback. Similarly, from the formula \( ϕ(x) = ⊥ \) we obtain a pullback-stable bottom element \( 0 \to X \), so \( S \) is a coherent category. Moreover, by the construction of unions and empty subobjects, it follows that 1 is indecomposable and nonempty. Finally, dual images can similarly be constructed by applying ∆₀-separation to a formula with a universal ∆₀-quantifier.

Thus, the definition of a constructively well-pointed Heyting category, though it may seem categorically technical, is equivalent to a very natural structural analogue of the ∆₀-separation axiom.

Remark 3.18. If \( S \) is a Π-pretopos or a topos, then the direction (b) → (a) of Lemma 3.15 can be extended to formulas in the appropriate higher-order logic involving dependent product types and/or power types, analogously to how we dealt with finite products, by choosing appropriate objects in \( S \) to introduce as extra parameters. Therefore, the direction (i) → (ii) of Proposition 3.17 can be applied to such formulas as well. We will need this generalization in what follows. Unfortunately, it seems difficult to precisely describe a class of formulas in the language of structural set theory which would satisfy higher-order versions of Lemma 3.15 and Proposition 3.17. This can be remedied by describing structural set theory as a type theory with quantification over types, as mentioned in footnote 2 on page 1.

4. Structural separation, fullness, and induction

There are several axioms from material set theory for which we have not yet considered structural versions. In this section, we consider separation, fullness, and induction; in the next we consider collection and replacement.

The structural separation axiom simply generalizes Proposition 3.17(ii) to unbounded quantifiers. For any formula \( ϕ(x) \), for any set \( X \), there exists a subobject \( S \to X \) such that for any \( x \in X \), we have \( x \in S \) if and only if \( ϕ(x) \).
Lemma 4.1. If $V$ satisfies the core axioms together with full separation, then $Set(V)$ satisfies separation.

Proof. Note that any formula $\varphi$ in $Set(V)$ in the language of categories may be translated into a formula $\hat{\varphi}$ in $V$ in the language of material set theory. Moreover, if $\varphi$ is $\Delta_0$, then so will $\hat{\varphi}$ be. Now, if $V$ satisfies material separation, it is easy to see that $S = \{ x \in X \mid \hat{\varphi}(x) \}$ has the desired property for structural separation. □

The structural axiom of fullness is also a direct translation of the material one.

• Fullness: For any sets $X, Y$ there exists a relation $R \to M \times X \times Y$ such that $R \to M \times X$ is regular epic, and for any relation $S \to X \times Y$ such that $S \to X$ is regular epic, there exists an $s \in M$ such that $(s, 1)^* R \subseteq S^\ast$.

Lemma 4.2. If $V$ satisfies the core axioms together with fullness, then $Set(V)$ satisfies structural fullness.

Proof. Just like the proofs for exponentials and power sets in Theorem 3.5. □

Remark 4.3. The axiom of fullness appears quite different from our statements of the structural axioms of exponentiation and powersets in §3. Specifically, the former refers explicitly to global elements and is thus only suitable in a well-pointed category, while the latter are phrased in a more “category-theoretic” way that makes sense in more generality.

In fact, however, in a well-pointed category, the existence of power objects is easily shown to be equivalent to the following more “set-like” version: for any $X$ there is an object $PX$ and a relation $RX \to X \times PX$ such that for any subset $S \to X$, there exists a unique $s \in PX$ such that $S \cong (1, s)^* R_X$. This is also true for exponentiation as long as we also assume the collection axioms from the next section. Conversely, from our axiom of fullness one can derive a more category-theoretic version by interpreting it in the stack semantics. So the difference is only an artifact of our chosen presentations.

The axiom of induction, of course, only makes sense in the presence of the axiom of infinity. The structural axiom of infinity (existence of an NNO) asserts that functions can be constructed by recursion, which implies Peano’s induction axiom in the sense that any subset $S \to \omega$ which contains $0: 1 \to \omega$ and is closed under $s: \omega \to \omega$ must be all of $\omega$. (The proof of Theorem 3.5.4 essentially shows that the converse holds in any Π-pretopos.) It follows from $\Delta_0$-separation that $\Delta_0$-formulas can be proven by induction; the axiom of full induction extends this to arbitrary formulas.

• Induction: For any formula $\varphi(x)$ with free variable $x \in \omega$, where $\omega$ is an NNO, if $\varphi(0)$ and $(\forall n \in \omega)(\varphi(x) \Rightarrow \varphi(sx))$, then $\varphi(x)$ for all $x \in \omega$.

Just as in material set theory, infinity and full separation together imply full induction. We also have:

Proposition 4.4. If $V$ satisfies the core axioms of material set theory, and also the axioms of infinity, exponentials, and induction, then $Set(V)$ has an NNO and satisfies induction.

Proof. Any formula $\varphi(x)$ in $Set(V)$ with $x \in \omega$ can be rewritten as a formula in $V$, to which the material axiom of induction can be applied. □
There is no particularly natural structural “axiom of foundation,” although in § we will mention a somewhat related property. We can, however, formulate a structural axiom which is closely related to the material axiom of set-induction.

- **Well-founded induction:** For any formula $\varphi(x)$ with free variable $x \in A$, if $A$ is well-founded under the relation $\prec$, and moreover if $\varphi(y)$ for all $y \prec x$ implies $\varphi(x)$, then in fact $\varphi(x)$ for all $x \in A^\gamma$.

This axiom is implied by separation, since then we can form $\{x \in A \mid \varphi(x)\}$ and apply the definition of well-foundedness. We can also say:

**Proposition 4.5.** If $V$ satisfies the core axioms of material set theory, and also the axioms of power set, set-induction, and Mostowski’s principle, then $\text{Set}(V)$ satisfies well-founded induction.

**Proof.** With power sets, any well-founded relation can be collapsed to a well-founded extensional one (an “extensional quotient” in the sense of §), which by Mostowski’s principle will be isomorphic to a transitive set. Thus, set-induction performed over the resulting transitive set can be used for inductive proofs over the original well-founded relation.

\[\square\]

5. **Structural collection and replacement**

We now turn to structural versions of the collection and replacement axioms. Various such axioms have been proposed in the context of ETCS (see [Col73, Osi74, Law03, McL04]), but none of these seem to be quite appropriate in an intuitionistic or predicative theory. Hence our axioms must be different from all previous proposals (though they are most similar to the replacement axiom of [McL04]).

The intuition behind structural collection is that since the elements of a set in a structural theory are not themselves sets, instead of “collecting” sets as elements of another set we must collect them as a family indexed over another set. Also, since the language of category theory is two-sorted, it is unsurprising that we have to assert collection for objects and morphisms separately. In fact, we find it conceptually helpful to formulate three axioms of collection, although the third one is automatically satisfied.

- **Collection of sets:** For any formula $\varphi(u, X)$, for any set $U$, if for every $u \in U$ there exists an $X$ with $\varphi(u, X)$, then there exists a regular epi $V \overset{p}{\longrightarrow} U$ and an $A \in \text{Set}/V$ such that for every $v \in V$ we have $\varphi(pv, v^*A)^\gamma$.

- **Collection of functions:** For any formula $\varphi(u, f)$, for any set $U$ and any $A, B \in \text{Set}/U$, if for all $u \in U$ there exists $u^*A \overset{f}{\longrightarrow} u^*B$ with $\varphi(u, f)$, then there exists a regular epi $V \overset{p}{\longrightarrow} U$ and a function $p^*A \overset{g}{\longrightarrow} p^*B$ in $\text{Set}/V$ such that for all $v \in V$, we have $\varphi(pv, v^*g)^\gamma$.

- **Collection of equalities:** For any set $U$, any $A, B \in \text{Set}/U$, and any functions $f, g: A \to B$ in $\text{Set}/U$, if $u^*f = u^*g$ for every $u \in U$, then there is a regular epi $V \overset{p}{\longrightarrow} U$ such that $p^*f = p^*g$.

We will say that $\text{Set}$ “satisfies collection” if it satisfies all three of these axioms. However, one, and sometimes two, of these are redundant.

**Proposition 5.1.** Let $\text{Set}$ be a constructively well-pointed Heyting category. Then:

(i) $\text{Set}$ satisfies collection of equalities.
(ii) If Set satisfies full separation and is a \( \Pi \)-pretopos, then it satisfies collection of functions.

**Proof.** In the situation of collection of equalities, let \( E \xrightarrow{f} A \) be the equalizer of \( f \) and \( g \) and let \( S = \forall y E \), where \( A \xrightarrow{\Delta} U \) is the structure map of \( A \in \text{Set}/U \). The assumption that \( u^*f = u^*g \) for every \( 1 \xrightarrow{u} U \) implies that every such \( u \) factors through \( S \). Since 1 is a strong generator, this implies \( S \cong U \), and therefore \( E \cong A \) and so \( f = g \); thus we can take \( V = U \) and \( p = 1_U \). This shows (i).

In the situation of (ii) let \( C = (B \to U)(A \to U) \) be the exponential in \( \text{Set}/U \), with projection \( C \xrightarrow{\Delta} U \). Then each \( c \in C \) corresponds to a map \( f_c : (gc)^*A \to (gc)^*B \). Using the axiom of separation, find a subobject \( V \xrightarrow{\rho} C \) such that \( c \in C \) is contained in \( V \) iff \( \varphi(f_c) \). By assumption, every \( u \in U \) lifts to some \( c \in V \), so since \( \text{Set} \) is well-pointed, the projection \( V \xrightarrow{gm} U \) is regular epi. Finally, there is an evident map \( h : (gm)^*A \to (gm)^*B \) such that \( \varphi(gmc,c^*h) \) for all \( c \in V \).

The appropriate structural formulation of the axiom of replacement is a bit more subtle than that of collection. The idea is that if we modify the hypotheses of collection by asserting unique existence, then passage to a cover \( V \to U \) should be unnecessary. As with collection, we may expect three versions for sets, functions and equalities, and of these the second two are easy and follow from collection.

**Proposition 5.2.** Let \( \text{Set} \) be a constructively well-pointed Heyting category.

(i) \( \text{Set} \) always satisfies replacement of equalities: \( \forall \) for any set \( U \), any \( A, B \in \text{Set}/U \), and any functions \( f, g : A \to B \) in \( \text{Set}/U \), if \( u^*f = u^*g \) for every \( u \in U \), then \( f = g \).

(ii) If \( \text{Set} \) satisfies collection of functions, then it satisfies replacement of functions: \( \forall \) for any set \( U \) and any \( A, B \in \text{Set}/U \), if for every \( 1 \xrightarrow{u} U \) there exists a unique \( u^*A \to u^*B \) such that \( \varphi(u,f) \), then there exists \( A \xrightarrow{g} B \) in \( \text{Set}/U \) such that for each \( 1 \xrightarrow{u} U \) we have \( \varphi(u,u^*f) \).

**Proof.** The proof of Proposition 5.1(i) already shows (i). For (ii) collection of functions gives us a regular epimorphism \( V \xrightarrow{p} U \) and a function \( h : p^*A \to p^*B \) in \( \text{Set}/V \) such that \( \varphi(pv,v^*h) \) for any \( v \in V \). If \( (r,s) : V \times_U V \to V \) is the kernel pair of \( p \), then for every \( z = (v_1,v_2) \in V \times_U V \) we have \( pv_1 = pv_2 = u \), say, and thus \( \varphi(u,v_1^*h) \) and \( \varphi(u,v_2^*h) \). By uniqueness, \( v_1^*h = v_2^*h \), and so

\[
z^*r^*h = v_1^*h = v_2^*h = z^*s^*h.
\]

By replacement of equalities, \( r^*h = s^*h \). So since the self-indexing of \( \text{Set} \) is a prestack, \( h \) descends to \( g : A \to B \) with \( p^*g = h \). Since \( p \) is regular epic, for any \( u \in U \) there exists a \( v \in V \) with \( pv = u \), so we have \( u^*h = v^*p^*h = v^*g \), whence \( \varphi(u,u^*h) \).

These two replacement axioms imply, in particular, that universal properties are reflected by global elements. Rather than give a precise statement of this, we present a paradigmatic example.

**Proposition 5.3.** Let \( \text{Set} \) be a constructively well-pointed Heyting category satisfying replacement of equalities and functions, and suppose we have \( A, B, E \in \text{Set}/X \) and a morphism \( v : E \times_X A \to B \) such that for all \( x \in X \), \( x^*v \) exhibits \( x^*E \) as an exponential \( (x^*B)(x^*A) \) in \( \text{Set} \). Then \( v \) exhibits \( E \) as an exponential \( B^A \) in \( \text{Set}/X \).
Proof. The assumption means that for any object \( C \) and any morphism \( f : C \times x^* A \to x^* B \), there exists a unique \( g : C \to x^* E \) such that \( f = x^* v \circ (g \times 1) \). In particular, given any \( D \in \text{Set} \) and morphism \( f : D \times X \to A \to B \), for every \( x \in X \) there is a unique \( g_x : x^* D \to x^* E \) such that \( x^* f = x^* v \circ (g_x \times 1) \). By replacement of functions, we have \( g : D \to E \) such that \( x^* f = x^* (v \circ (g \times 1)) \) for each \( x \in X \), or equivalently \( x^* f = x^* (v \circ (g \times 1)) \). By replacement of equalities, \( f = v \circ (g \times 1) \). Moreover, if \( f = v \circ (h \times 1) \) for some \( h : D \to E \), then for each \( x \in X \) we have \( x^* f = x^* v \circ (x^* h \times 1) \). By the universal property of the exponential \( x^* E \), we have \( x^* h = g_x = x^* g \), and hence \( h = g \) by replacement of equalities. \( \square \)

In future, we will invoke similar facts frequently, trusting the reader to supply analogous arguments as necessary. Note also that all universal properties reflected in this way are automatically pullback-stable.

We would also like an axiom of “replacement of sets,” which would ensure that the existence of an object satisfying a universal property is also reflected by global elements. Since no structural theory can determine an object of a category more uniquely than up to unique isomorphism, one natural such statement would be:

- **Replacement of sets:** If for every \( u \in U \) there is a set \( A \) with \( \varphi(u, A) \) which is unique up to unique isomorphism, then there is a \( B \in \text{Set}/U \) such that \( \varphi(u, u^* B) \) for all \( u \in U \).

If \( \text{Set} \) is additionally exact, then replacement of sets follows from collection, using the fact that in this case the self-indexing \( \text{Set}/(-) \) a stack for the regular topology. Moreover, just as in material set theory, we have:

**Proposition 5.4.** If \( S \) is a well-pointed Heyting category which satisfies full classical logic and replacement of sets, then it also satisfies separation.

Proof. For any formula \( \varphi \) with a free variable \( 1 \xrightarrow{x} X \), let \( \psi(x, A) \) be \( \varphi^x \) is terminal and \( \varphi(x) \), or \( A \) is initial and \( \neg\varphi(x) \). Since \( (\forall x)(\varphi(x) \lor \neg\varphi(x)) \) holds by classical logic, and initial and terminal objects are unique up to unique isomorphism, we can apply replacement of sets to obtain a \( B \in \text{Set}/X \) such that for any \( x \in X \), if \( \varphi(x) \) then \( x^* B \) is terminal, while if \( \neg\varphi(x) \) then \( x^* B \) is initial. It follows that \( B \to X \) is monic, and is the subobject classifying \( \varphi \) required by the separation axiom. \( \square \)

However, for most other purposes replacement of sets is basically useless, since hardly ever is a set determined up to absolutely unique isomorphism, only an isomorphism which is compatible with some additional functions whose existence is being asserted at the same time. For example, a cartesian product \( A \times B \) is only determined up to an isomorphism which is unique such that it respects the projections \( A \times B \to A \) and \( A \times B \to B \) with which the product comes equipped. In order to state a useful version of replacement, we invoke the notion of context from dependent type theory (which also simplifies greatly in our particular situation).

**Definition 5.5.** Let \( S \) be a category.

(i) A context \( \Gamma \) in \( S \) is a finite (ordered) list of typed variables, with the property that whenever an arrow-variable \( f : X \to Y \) occurs in \( \Gamma \), each of \( X \) and \( Y \) is either a parameter, or an object-variable occurring in \( \Gamma \) prior to \( f \).

(ii) A instantiation \( t \) of a context \( \Gamma \) in \( S \) is a well-typed assignation of objects and arrows in \( S \) to the variables of \( \Gamma \). We write \( t : \Gamma \) and use the notations \( X \mapsto X_A \) and \( (f : Y \to Z) \mapsto (f_A : Y_A \to Z_A) \), where if \( Y \) or \( Z \) is a parameter then \( Y_A \) or \( Z_A \) denotes simply that parameter.
(iii) A morphism of instantiations \( \vec{A} \to \vec{B} \) consists of
(a) For each object-variable \( X \) in \( \Gamma \), a morphism \( \alpha_X : X_A \to X_B \), such that
(b) For each arrow-variable \( f : X \to Y \) in \( \Gamma \), we have \( \alpha_Y \circ f_A = f_B \circ \alpha_X \).
Here if \( X \) or \( Y \) is a parameter, then \( \alpha_X \) or \( \alpha_Y \) denotes the identity arrow of that parameter.
(iv) If \( (\Gamma, \Delta) \) is a context (meaning the concatenation of \( \Gamma \) and \( \Delta \)), and \( \vec{A} \) is an instantiation of \( \Gamma \), then an extension of \( \vec{A} \) to \( \Delta \) is a \( \vec{B} \) such that \( (\vec{A}, \vec{B}) \) is an instantiation of \( \Delta \). Similarly we define extensions of morphisms.

Of course, instantiations of a given, fixed, context can be discussed (and quantified over) in the language of categories over \( S \). If \( \varphi \) is a formula whose free variables are those in the context \( \Gamma \), we write \( \varphi(\Gamma) \), and similarly \( \varphi(\vec{A}) \) for the instantiation of \( \varphi \) with these variables replaced by the parameters in \( \vec{A} \). The pullback of contexts and instantiations along a morphism \( p : V \to U \) is defined in the obvious way.

As a particularly important example, the context \( (X, x : 1 \to X) \) can be instantiated in \( \text{Set}/U \) by the pair \((U^*U, \Delta_U)\) consisting of the object \( U^*U = U \times U \) and the morphism \( \Delta_U : 1_U \to U^*U \). Moreover, this pair is the universal such instantiation over \( U \), in that for any \( p : 1 \to U \) we have an isomorphism of instantiations \( u^*(U^*U, \Delta_U) \cong (U, u) \). Similarly, for any \( V \xrightarrow{p} U \) we can consider \((V^*U, (1_V, p))\), which has the analogous property that \( v^*(V^*U, (1_V, p)) \cong (U, pv) \) for any \( v : 1 \to V \).

**Proposition 5.6.** Any constructively well-pointed Heyting category satisfying collection also satisfies collection of contexts: for any context \( (X, x : 1 \to X, \Gamma) \) and any formula \( \varphi(X, x, \Gamma) \), \( \forall \) for any set \( U \), if for every \( u \in U \) there exists \( \vec{A} : \Gamma \) extending \((U, u)\) such that \( \varphi(U, u, \vec{A}) \), then there exists a regular epi \( V \xrightarrow{p} U \) and a \( \vec{B} : \Gamma \) extending \((V^*U, (1_V, p))\) in \( \text{Set}/V \) such that for every \( v \in V \) we have \( \varphi(U, pv, v^*\vec{B}) \).

**Proof.** Simply apply collection of sets and functions repeatedly, using the fact that regular epimorphisms compose. \( \square \)

We can now improve “replacement of sets” to a more useful axiom.

- **Replacement of contexts:** for any context \( (X, x : 1 \to X, \Gamma) \) and any formula \( \varphi(X, x, \Gamma) \), \( \forall \) for any set \( U \), if for every \( u \in U \) there is an extension \( \vec{A} : \Gamma \) of \((U, u)\) such that \( \varphi(U, u, \vec{A}) \), and moreover \( \vec{A} \) is unique up to a unique isomorphism of instantiations extending \( 1_U \), then there is an extension \( \vec{B} : \Gamma \) of \((U^*U, \Delta_U)\) in \( \text{Set}/U \) such that \( \varphi(U, u, u^*\vec{B}) \) for all \( u \in U^\gamma \).

**Proposition 5.7.** Any constructively well-pointed Heyting pretopos satisfying collection also satisfies replacement of contexts.

**Proof.** By Proposition 5.6, there is a regular epi \( p : V \to U \) and a \( \vec{C} : \Gamma \) in \( \text{Set}/V \) such that for every \( v \in V \) we have \( \varphi(U, pv, v^*\vec{C}) \). Let \((r, s) : V \times_U V \to V \) be the kernel pair of \( f \), and consider \( r^*\vec{C} \) and \( s^*\vec{C} \) in \( \text{Set}/V \times_U V \). The assumption implies that when pulled back along any \( z : 1 \to V \times_U V \), these two instantiations of \( \Gamma \) become uniquely isomorphic. Thus, by replacement of functions and equalities applied some finite number of times, we actually have a unique isomorphism \( r^*\vec{C} \cong s^*\vec{C} \) in \( \text{Set}/V \times_U V \). Uniqueness implies that this isomorphism satisfies the cocycle condition over \( V \times_U V \times_U V \). Thus, since \( \text{Set} \) is a stack for its regular topology (applied some finite number of times), the entire instantiation \( \vec{C} : \Gamma \) descends to some \( \vec{B} : \Gamma \) in \( \text{Set}/U \). Since \( p \) is surjective, the desired property for \( \vec{B} \) follows. \( \square \)
In general, it seems that replacement of contexts does not imply collection, although we do not have a counterexample. As in material set theory, however, replacement suffices in the classical world.

**Proposition 5.8.** Over ETCS, the following are equivalent.

(i) Replacement of contexts.

(ii) Collection of sets.

(iii) Full separation and collection (of both sets and functions).

**Proof.** By Propositions 5.1(ii), 5.4, and 5.7, it suffices to prove that replacement of contexts implies collection of sets. Given a formula $\varphi(u, X)$, we can construct a context $\Gamma$ including both a set $X$ and a binary relation on it, and let $\psi(u, \Gamma)$ assert that $\varphi(u, X)$ holds and that the given relation is a well-ordering, which has the smallest possible order-type among well-ordered sets $X$ such that $\varphi(u, X)$. Since classically, any set admits a smallest well-ordering which is unique up to unique isomorphism, we can apply replacement of contexts to $\psi$ and thereby deduce collection of sets.

As promised, replacement of contexts implies that the existence of objects satisfying a universal property is reflected by global elements. Continuing the example of Proposition 5.3, we have the following.

**Proposition 5.9.** Let $\text{Set}$ be a constructively well-pointed Heyting pretopos satisfying collection. If $\text{Set}$ is cartesian closed, then it is locally cartesian closed.

**Proof.** Let $A, B \in \text{Set}/X$; we want to construct the exponential $(B \to X)^{(A \to X)}$. For $x \in X$, let $\varphi(x, E, e)$ assert that $e: E \times x^*A \to x^*B$ exhibits $E$ as the exponential $(x^*B)^{x^*A}$. This formula can be phrased as $\psi(X, x, \Gamma)$ for some $\Gamma$, although we note that $\Gamma$ involves not just $E$ and $e$ but also object-variables for $x^*A$, $x^*B$, and $E \times x^*A$, and arrow-variables giving the projections that exhibit these as two pullbacks and a cartesian product, respectively.

Now, since $\text{Set}$ is cartesian closed, for every $x \in X$ there exists $E$ and $e$ with $\varphi(x, E, e)$, and such are unique up to a unique isomorphism which respects all the structure. Therefore, replacement of contexts supplies an instantiation of $\Gamma$ extending $(X^*X, \Delta_X)$ in $\text{Set}/X$. This consists essentially of an object $C$ and a morphism $c: C \times X A \to B$. The conclusion of replacement of contexts implies that $C$ and $c$ satisfy the hypotheses of Proposition 5.3 so they must be an exponential $(B \to X)^{(A \to X)}$ as desired.

This is of course reminiscent of the way in which local cartesian closure comes “for free” in Theorem 3.5. We end this section by extending Theorem 3.5 to the axiom of collection.

**Theorem 5.10.** If $V$ satisfies the core axioms of material set theory along with collection, then $\text{Set}(V)$ also satisfies collection.

**Proof.** If $V$ satisfies material collection, then given the setup of collection of sets for some formula $\varphi$, let $\psi(u, X)$ assert that $X$ is a Kuratowski ordered pair of the form $(u, X')$ with $\varphi(u, X')$. By material collection, let $V$ be a set such that for any $u \in U$, there is an $X \in V$ with $\psi(u, X)$, and for any $X \in V$, there is a $u \in U$ with $\psi(u, X)$. Then every element of $V$ is an ordered pair whose first element is in $U$,
so there is a projection \( V \to U \), which by assumption is surjective. We can then form 
\[
A = \{ (u, X'), a) \in V \times B \mid a \in X' \}.
\]
Then \( A \in \Set(V)/V \), and for any \( v = (u, X') \in V \), \( v^* A \) is isomorphic to \( X' \). Thus, by isomorphism-invariance, we have \( \varphi(u, v^* A) \), as required. Collection of functions is analogous.

The material axiom of replacement, however, does not seem to imply the structural one. This, we feel, is one of the reasons (though not the only one) that the material axiom of collection is necessary in practice.

6. Constructing material set theories

So far we have summarized the axioms of material set theory and structural set theory, and explained how most axioms of material set theory are reflected in structural properties of the resulting category of sets. We now turn to the opposite construction: how to recover a material set theory from a structural one. A “set” in material set theory, of course, contains much more information than a “set” in structural set theory, namely the membership relations between its elements, their elements, and so on. This gives rise to the idea of modeling a “material set” by a graph or tree with nodes depicting sets and edges depicting membership.

This basic idea was used by [Col73, Mit72, Osi74] in the first equiconsistency proofs of ETCS with BZC, and nearly identical constructions have been used for relative consistency proofs by others such as [Acz88] and [Mat01]. We are not aware of an exposition at our required level of generality, though. The constructions are the same as always, though, as long as we are careful about the definitions.

For the rest of this section, let \( \Set \) be a constructively well-pointed \( \Pi \)-pretopos with an \( \text{nno} \).

Definitions 6.1.

(i) A graph is a set \( X \), whose elements are called nodes, equipped with a binary relation \( \prec \).

(ii) If \( x \prec y \) we say that \( x \) is a child of \( y \).

(iii) A pointed graph is one equipped with a distinguished node \( \star \) called the root.

(iv) A pointed graph is accessible if for every node \( x \) there exists a path \( x = x_n \prec \cdots \prec x_0 = \star \) to the root.

(v) For any node \( x \) of a graph \( X \), we write \( X/x \) for the full subgraph of \( X \) consisting of those \( y \) admitting some path to \( x \). It is, of course, pointed by \( x \), and accessible.

“Accessible pointed graph” is abbreviated \( \text{APG} \).

Remark 6.2. The hypothesis on \( \Set \) is necessary to formalize “accessibility” and to define \( X/x \). Specifically, a pointed graph \( X_\prec \to X \times X \) with root \( \star : 1 \to X \) is accessible if for every \( x : 1 \to X \), there exists a nonzero finite cardinal \( [n] \) and a map \( [n] \to X_\prec \) realizing a path from \( x \) to \( \star \). (Recall that a finite cardinal is the pullback along a map \( 1 \to N \) of the second projection \( N_\prec \to N \) of the strict order relation on the \( \text{nno} \).) The definition of \( X/x \) is similar, using the extension of \( \Delta_0 \)-separation to functions described in [Remark 3.18].
We are using the terminology of \cite{Acz88}. The idea is that an arbitrary graph represents a collection of material-sets with $\prec$ representing the membership relation between them, a pointed graph represents a particular set (the root) together with all the data required to describe its hereditary membership relation, and an APG does this without any superfluous data (all the nodes bear some relation to the root). Thus, an arbitrary APG can be considered a picture of a possibly non-well-founded set: the root represents the set itself, its children represent the elements of the set, and so on.

We will henceforth restrict attention to graphs for which $\prec$ is well-founded, thereby ensuring that the models of material set theory we construct satisfy the axiom of foundation. It is also possible, by changing this requirement, to construct models satisfying the various axioms of anti-foundation (see \cite{Acz88}), but we will not do that here. Recall the definition:

**Definition 6.3.** A subset $S$ of a graph $X$ is **inductive** if for any node $x \in X$, if all children of $x$ are in $S$, then $x$ is also in $S$. A graph $X$ is **well-founded** if any inductive subset of $X$ is equal to all of $X$.

In the presence of classical logic, well-foundedness is equivalent to saying that every nonempty subset of $X$ has a $\prec$-least element, but in intuitionistic logic that version is both fairly useless and rarely satisfied.

With the above definition of well-foundedness, we can perform proofs by induction on a well-founded graph: by proving that a given subset is inductive, we conclude it is the whole graph. In order to prove a statement by well-founded induction, we must apply separation to the statement to turn it into a subset; thus either the statement must be $\Delta_0$, or we must have a stronger separation axiom. We can also use well-founded induction in more than one variable, since if $X$ and $Y$ are well-founded then so is $X \times Y$.

We record some observations about well-founded graphs.

**Lemma 6.4.** Any subset of a well-founded graph is well-founded with the induced relation.

**Proof.** For any $Z \subseteq X$, if $S \subseteq Z$ is inductive in $Z$, then the Heyting implication $(Z \Rightarrow S) \subseteq X$ is inductive in $X$. \hfill $\Box$

**Lemma 6.5.** If $X$ is a well-founded graph, then there does not exist any cyclic path $x = x_n \prec \cdots \prec x_0 = x$ in $X$.

**Proof.** If there were, then the Heyting complement of $\{x_0, \ldots, x_n\} \subseteq X$ would be inductive but not all of $X$. \hfill $\Box$

**Lemma 6.6.** If $X$ is a well-founded graph and $x \in X$, then $x: 1 \to X/x$ is a complemented subobject. In particular, it is decidable whether or not a node of a well-founded APG is the root.

**Proof.** By definition, every node $y \in X/x$ admits some path to $x$, of length $n \in N$ say. Since $0$ is a complemented subobject of $N$, either $n = 0$, in which case $y = x$, or $n > 0$. If $n > 0$, then if $x = y$ there would be a cyclic path from $x$ to itself; hence in this case $x \neq y$. Thus for all $y$, either $y = x$ or $y \neq x$, as desired. \hfill $\Box$

**Definition 6.7.** We will write $X/\#x$ for the complement of $x$ in $X/x$, i.e. the set of nodes admitting a path to $x$ of length $> 0$. In particular, $X/#\ast$ is the complement of the root.
We need one more requirement on our APGs, namely that they satisfy the axiom of extensionality.

**Definition 6.8.** An graph $X$ is **extensional** if whenever $x$ and $y$ are nodes such that $z \prec x \iff z \prec y$ for all $z$, then $x = y$.

For non-well-founded graphs, this definition would have to be strengthened in one of various possible ways.

**Remark 6.9.** An equivalent characterization of the universe of extensional well-founded APGs can be obtained by working with well-founded rigid trees instead. A tree is an APG in which every node $x$ admits a unique path to the root, and it is rigid if for any node $z$ and any children $x \prec z$ and $y \prec z$, if $X/x \cong X/y$, then $x = y$. Every extensional well-founded APG $X$ has an “unfolding” into a well-founded rigid tree $X^t$, whose nodes are the paths in $X$, and conversely every well-founded rigid tree is the unfolding of some extensional well-founded APG. Rigid trees are used by [Col73, Mit72] and [MLM94], while extensional relations are used by [Osi74] and [Joh77], but they result in essentially equivalent theories. (In the non-well-founded case, however, extensional relations seem more generally applicable than rigid trees.)

We now record some observations about extensionality and its interaction with well-foundedness.

**Definition 6.10.** An initial segment of a graph $X$ is a subset $Y \subseteq X$ such that $x \prec y \in Y$ implies $x \in Y$.

**Lemma 6.11.** Any initial segment of an extensional graph is extensional with the induced relation.

**Lemma 6.12.** If $X$ is a well-founded extensional graph and $X/x \cong X/y$, then $x = y$.

**Proof.** We prove this by well-founded induction. If $g : X/x \xrightarrow{\sim} X/y$, then for all $x' \prec x$ we have $X/x' \cong X/g(x')$, where $g(x') \prec y$; hence by induction $x' = g(x')$. Similarly, we have $y' = g^{-1}(y')$ for all $y' \prec y$. By extensionality, $x = y$. □

In the language of [Acz88, Lemma 6.12] says that well-founded extensional graphs are “Finsler-extensional.”

**Lemma 6.13.** Any automorphism of a well-founded extensional graph is the identity. Therefore, any two parallel isomorphisms of well-founded extensional graphs are equal.

**Proof.** Let $f : X \xrightarrow{\sim} X$ be an isomorphism; we prove by well-founded induction that $f(x) = x$ for all $x \in X$. But if $f(x') = x'$ for all $x' \prec x$, then extensionality immediately implies $f(x) = x$. □

**Definition 6.14.** We write $V(\text{Set})$ for the class of well-founded extensional APGs in $\text{Set}$.

Our goal is now to show that $V(\text{Set})$ is a model of a material set theory. We consider two APGs to be equal, i.e. to represent the same material-set, when they are isomorphic. And we model the membership relation $\in$ in the expected way:
Definition 6.15. If $X$ is an APG, the children of its root are called its members. We write $|X|$ for the set of members of $X$. If $X$ and $Y$ are APGs, we write $X \epsilon Y$ to mean that $X \cong Y/y$ for some member $y \in |Y|$.

Here is our omnibus theorem.

Theorem 6.16. Let $\textbf{Set}$ be a constructively well-pointed $\Pi$-pretopos with a NNO, and let $\varphi$ be a formula of material set theory with parameters in $\forall(\textbf{Set})$. Then $\forall(\textbf{Set}) \models \varphi$ whenever any of the following holds.

- $\varphi$ is the axiom of extensionality, empty set, pairing, union, exponentiation, infinity, foundation, transitive closures, Mostowski’s principle, or an instance of $\Delta_0$-separation.
- $\textbf{Set}$ satisfies structural fullness and $\varphi$ is material fullness.
- $\textbf{Set}$ is a topos and $\varphi$ is the power set axiom.
- $\textbf{Set}$ satisfies structural separation and $\varphi$ is an instance of material separation.
- $\textbf{Set}$ satisfies structural collection and $\varphi$ is an instance of material collection.
- $\textbf{Set}$ satisfies replacement of contexts and $\varphi$ is an instance of material replacement.
- $\textbf{Set}$ is Boolean and $\varphi$ is an instance of $\Delta_0$-classical logic.
- $\textbf{Set}$ satisfies full classical logic and $\varphi$ is an instance of full classical logic.
- $\textbf{Set}$ satisfies induction and $\varphi$ is an instance of full induction.
- $\textbf{Set}$ satisfies the axiom of choice and $\varphi$ is the material axiom of choice.
- $\textbf{Set}$ satisfies well-founded induction and $\varphi$ is an instance of set-induction.

We will prove this theorem with a series of lemmas, but first we need to introduce some auxiliary notions leading up to the construction of extensional quotients.

Definition 6.17. Let $X$ and $Y$ be graphs. A simulation from $X$ to $Y$ is a function $f : X \rightarrow Y$ such that

(i) if $x' \prec x$, then $f(x') \prec f(x)$, and

(ii) if $y \prec f(x)$, then there exists an $x' \prec x$ with $f(x') = y$.

A bisimulation from $X$ to $Y$ is a relation $R \rightarrow X \times Y$ such that both projections $R \rightarrow X$ and $R \rightarrow Y$ are simulations (where $R$ is considered as a full subgraph of $X \times Y$). A bisimulation is bi-entire if $R \rightarrow X$ and $R \rightarrow Y$ are surjective.

Note that if $f : X \rightarrow Y$ is a simulation, then the relation $(1, f) : X \rightarrow X \times Y$ is a bisimulation, which is bi-entire iff $f$ is surjective.

An obvious example of a simulation is the inclusion of an initial segment. The following lemma says that for well-founded extensional graphs, these are the only simulations.

Lemma 6.18. If $X$ is well-founded and extensional, then any simulation $f : X \rightarrow Y$ is injective, and isomorphic to the inclusion of an initial segment.

Proof. We show by well-founded induction that $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Suppose $f(x_1) = f(x_2)$; then for any $z_1 \prec x_1$, we have $f(z_1) \prec f(x_1) = f(x_2)$, so since $f$ is a simulation there is a $z_2 \prec x_2$ with $f(z_2) = f(z_1)$. Hence $z_1 = z_2$ by induction, so $z_1 \prec x_2$. Dually, $z_2 \prec x_1$ implies $z_2 \prec x_1$, so by extensionality $x_1 = x_2$. Finally, any injective simulation must be an initial segment. \(\Box\)

Lemma 6.19. If $R$ is a bi-entire bisimulation from $X$ to $Y$, then $X$ is well-founded if and only if $Y$ is so.
Proof. Suppose $X$ is well-founded and $S \subseteq Y$ is inductive. Let $T \subseteq X$ consist of those $x \in X$ such that $R(x, y)$ implies $y \in S$; we show that $T$ is inductive. Suppose $x \in X$ is such that $x' \prec x$ implies $x' \in T$, and suppose that $R(x, y)$. Then for any $y' \prec y$ there is an $x' \prec x$ with $R(x', y')$, whence $x' \in T$ and thus $y' \in S$. So since $S$ is inductive, $y$ must be in $S$. Thus $x \in T$, so $T$ is inductive. Since $X$ is well-founded, $T = X$, and then since $R$ is bi-entire, $S = Y$; thus $Y$ is well-founded.

Lemma 6.20. Any bi-entire bisimulation between extensional well-founded graphs must be an isomorphism.

Proof. Let $R \rightsquigarrow X \times Y$ be such. We show by well-founded induction that if $R(x, y_1)$ and $R(x, y_2)$, then $y_1 = y_2$. For if $z_1 \prec y_1$, then since $R$ is a bisimulation, there is a $w \prec x$ with $R(w, z_1)$, and again since $R$ is a bisimulation, there is a $z_2 \prec y_2$ with $R(w, z_2)$. By induction, $z_1 = z_2$, so that $z_1 \prec y_2$ for any $z_1 \prec y_1$. By symmetry, for any $z_2 \prec y_2$ we have $z_2 \prec y$, hence by extensionality $y_1 = y_2$.

By symmetry, if $R(x_1, y)$ and $R(x_2, y)$ then $x_1 = x_2$, so $R$ is functional in both directions. Since it is also bi-entire, it must be an isomorphism.

Lemma 6.21. If $X$ is well-founded, then it is extensional if and only if every bisimulation from $X$ to $X$ is contained in the identity.

Proof. Suppose first that $X$ is well-founded and extensional and that $R$ is a bisimulation with $R(x_1, x_2)$. Then $R$ is a bi-entire bisimulation from $X/x_1$ to $X/x_2$ (by ordinary induction on length of paths), so by Lemma 6.20 it must be an isomorphism $X/x_1 \cong X/x_2$. But $X$ is well-founded and extensional, so by Lemma 6.12 $x_1 = x_2$.

Now suppose that every bisimulation from $X$ to $X$ is contained in the identity, and also that $x, y \in X$ are such that $z \prec x \iff z \prec y$ for all $z$. Define $R(a, b)$ to hold if either $a = b$, or $a = x$ and $b = y$. Then $R$ is a bisimulation, and if it is contained in the identity, then $x = y$; hence $X$ is extensional.

In the language of [Acz88, Lemma 6.21] says that for well-founded graphs, extensionality is equivalent to “strong extensionality.”

Corollary 6.22. If $Y$ is well-founded and extensional, then any two simulations $f, g : X \Rightarrow Y$ are equal.

Proof. The image of $(f, g) : X \rightarrow Y \times Y$ is a bisimulation, hence contained in the identity.

Therefore, well-founded extensional graphs and simulations form a (large) preorder. In the language of material set theory, this preorder represents the partial order of transitive-sets and subset inclusions.

Lemma 6.23. If $X$ is a graph and $R$ is a bisimulation from $X$ to $X$ which is an equivalence relation, then its quotient $Y$ inherits a graph structure such that the quotient map $[-] : X \rightarrow Y$ is a simulation. Also, if $X$ is an APG, then so is $Y$.

Proof. Define $\prec$ on $Y$ to be minimal such that $[-]$ preserves $\prec$, i.e. $y_1 \prec y_2$ if there exist $x_1 \prec x_2$ in $X$ with $[x_1] = y_1$ and $[x_2] = y_2$. Now suppose that $y \prec [x]$. By definition this means that there exist $z_1 \prec z_2$ with $[z_1] = y$ and $[z_2] = [x]$, i.e. $R(z_2, x)$ holds. But $R$ is a bisimulation, so there exists $x' \prec x$ with $R(z_1, x')$, i.e. $[x'] = y$; hence $[\cdot]$ is a simulation. If $X$ is an APG, we define the root of $Y$ to be $[\ast]$; accessibility of $Y$ follows directly from that of $X$.
Of course, by Lemma 6.19, if $X$ is well-founded, then so is the quotient $Y$. If $R$ is the largest bisimulation on $X$, then its quotient is easily verified to be extensional. The largest bisimulation exists if $\text{Set}$ is a topos, or if it satisfies full separation, so in these situations every well-founded graph has an extensional quotient. In general, this seems not to be provable, but we can still construct extensional quotients in a useful amount of generality.

**Lemma 6.24.** Let $n$ be a fixed external natural number, let $X$ be a well-founded APG, and assume that $X/x$ is extensional whenever $x$ is a node that admits a path of length $n$ to the root. Then there is an extensional well-founded APG $\overline{X}$ and a surjective simulation $q$: $X \to \overline{X}$.

**Proof.** The proof is by external induction on $n$. (We will only need this lemma for $n \leq 3$, so the reader is encouraged not to worry too much about what this induction requires of the metatheory.) The base case is easy: since the root $\ast$ admits a path of length 0 to itself and $X \cong X/\ast$, we can take $\overline{X} = X$.

Now suppose the statement is true for some $n$, and let $X$ satisfy the hypothesis for $n + 1$. For any $k$, write $X_k$ for the set of nodes admitting a path of length $k$ to the root. Let the relation $R$ on $X$ be defined by $R(x, y)$ if there exists an isomorphism $X/x \cong Y/y$. (This can be constructed using $\Delta_\theta$-separation and local exponentials in $\text{Set}$.) Then $R$ is a bisimulation and an equivalence relation, so by Lemma 6.23 it has a quotient $Y$ which is again a well-founded APG.

We claim that $Y$ satisfies the hypothesis for $n$. Let $y \in Y_n$ and suppose that $y_1, y_2 \in Y/y$ satisfy $z \prec y_1 \Leftrightarrow z \prec y_2$: we must show $y_1 = y_2$. Applying the simulation property inductively, we have $y = [x]$ for $x \in X_n$, and $y_i = [x_i]$ with $x_i \in X/x$ for $i = 1, 2$. By Lemma 6.6 for each $i$ either $x_i = x$ or $x_i \in X/w_i$ for some $w_i \prec x$. If $x_1 = x$ and $x_2 = x$, then of course $y_1 = [x_1] = [x_2] = y_2$. If $x_1 = x$ and $x_2 \in X/w_2$ with $w_2 \prec x = x_2$, then there would be a cyclic path in $X$ from $x_2$ to itself, contradicting Lemma 6.5. Hence the only remaining case is when $x_i \in X/w_i$ with $w_i \prec x$ for both $i = 1, 2$.

Now this implies that $w_1 \in X_{n+1}$, so each $X/w_i$ is extensional, and thus so is each $X/x_i$ by Lemma 6.11. Therefore, by Lemma 6.13 the quotient map $[-]: X \to Y$ induces an isomorphism $X/x_i \cong Y/[x_i] = Y/y_i$. But $z \prec y_1 \Leftrightarrow z \prec y_2$ means that $Y/y_1 \cong Y/y_2$, and hence (by Lemma 6.6) also $Y/y_1 \cong Y/y_2$; thus we also have $X/x_1 \cong X/x_2$. Thus, by definition, $R(x_1, x_2)$, and so $y_1 = [x_1] = [x_2] = y_2$.

We have shown that $Y$ satisfies the hypothesis for $n$. Thus it has an extensional quotient $\overline{Y}$, and so the composite $X \to Y \to \overline{Y}$ is an extensional quotient of $X$. □

We can now start verifying the axioms of material set theory.

**Lemma 6.25.** The axiom of extensionality holds. That is, two well-founded extensional APGs $X$ and $Y$ are isomorphic iff $Z \in X \Leftrightarrow Z \in Y$ for all $Z$.

**Proof.** The “only if” direction is clear, so suppose that $Z \in X \Leftrightarrow Z \in Y$ for all $Z$. Then for every $x \in |X|$, we have $X/x \in Y$, hence $X/x \cong Y/y$ for some $y \in Y$. By Lemma 6.12 this $y$ must be unique, and conversely as well; hence we have a bijection $g$: $|X| \cong |Y|$ such that for any $x \in |X|$ there exists an isomorphism $h_x: X/x \cong Y/g(x)$ (which must be unique, by Lemma 6.13). Define a relation $R$ from $X$ to $Y$ such that $R(a, b)$ holds if:

(i) $a = \ast$ and $b = \ast$, or

(ii) there exists $x \in |X|$ such that $a \in X/x$, $b \in Y/g(x)$, and $h_x(a) = b$. 

Then $R$ is a bi-entire bisimulation, so by Lemma 6.20 it is an isomorphism. □

Lemma 6.26. The axioms of empty set, pairing, and union hold.

Proof. The empty set is represented by the APG with one node and no $\prec$ relations, which has no members.

If $X$ and $Y$ are extensional well-founded APGs, let $Z = X + Y + 1$ with $\prec$ induced from $X$ and $Y$ along with $\star_X \prec \star$ and $\star_Y \prec \star$, where $\star$ is the new point added. Since $X$ and $Y$ are extensional, $Z$ satisfies the hypothesis of Lemma 6.24 with $n = 1$, and its extensional quotient represents the pair $\{X,Y\}$.

Finally, if $X$ is an extensional well-founded APG, let $\|X\|$ denote the subset of those $x \in X$ such that $x \prec y \prec \star$ for some $y$, let $Y$ be the subset of $X$ consisting of those nodes admitting some path to a node in $\|X\|$, and define $Z = Y + 1$ with $\prec$ inherited from $Y$ and with $y \prec \star$ for each $y \in \|X\| \subseteq Y$, where $\star$ is the new element added. Then $Z$ satisfies the hypotheses of Lemma 6.24 with $n = 1$, so it has an extensional quotient, which is the desired union $\bigcup X$. □

Lemma 6.27. Cartesian products (using Kuratowski ordered pairs) exist in $\mathbb{V}(\text{Set})$.

Proof. Let $X$ and $Y$ be well-founded APGs, and consider the set

$$Z = (X/\star) + (Y/\star) + |X| + (|X| \times |Y|) + 1.$$

For $x \in |X|$ we write $x$ for its image in $(X/\star)$ and $x'$ for its image in the first copy of $|X|$. Similarly, we write $y$ for images in $Y/\star$, $(x,y)$ for images in the first copy of $|X| \times |Y|$, $(x,y)'$ for images in the second copy, and $\star$ for the final point. We define $\prec$ on $Z$ as follows:

- $\prec$ on $X/\star$ and $Y/\star$ is induced from $X$ and $Y$.
- $x \prec x'$ for all $x \in |X|$.
- $x \prec (x,y)$ and $y \prec (x,y)$ for all $y \in |Y|$.
- $x' \prec (x,y)'$ and $(x,y) \prec (x,y)'$ for all $x \in |X|$ and $y \in |Y|$.
- $(x,y)' \prec \star$ for all $x \in |X|$ and $y \in |Y|$.

It is straightforward to verify that $Z$ is then a well-founded APG. Since $X$ and $Y$ are extensional, $Z$ satisfies the hypothesis of Lemma 6.24 with $n = 3$. Its extensional quotient then represents the cartesian product of $X$ and $Y$. □

Lemma 6.28. $\mathbb{V}(\text{Set})$ satisfies the exponentiation axiom.

Proof. If $X$ and $Y$ are extensional well-founded APGs, let $Z$ be their material cartesian product as above, and define

$$W = (Z/\star) + |Y|^{|X|} + 1$$

with $\prec$ induced from $Z$ along with (using the notation of Lemma 6.27) $(x,y)' \prec f$ whenever $f(x) = y$, and $f \prec \star$ for any $f \in |Y|^{|X|}$. Then $W$ is a well-founded APG, and in fact is already extensional; we claim it represents the material function-set.

It is clear that if $F \in W$, then $F \in \mathbb{V}(\text{Set})$ is a function from $X$ to $Y$ in the sense of material set theory. Conversely, from any $F \in \mathbb{V}(\text{Set})$ which is a function from $X$ to $Y$, consider the subset of $|X| \times |Y|$ determined by those $(x,y)$ such that the Kuratowski ordered pair $\{\{X/x\}, \{X/x, Y/y\}\}$ is $\epsilon F$. This defines a function $|X| \to |Y|$ in $\text{Set}$, which therefore induces an $f \in |W|$ such that $F \cong W/f$. □
Observe that in particular, we have shown that for $X, Y \in \mathbb{V}(\text{Set})$, there is a 1-1 correspondence between functions $|X| \to |Y|$ in $\text{Set}$ and isomorphism classes of $\text{APG}$s $F \in \mathbb{V}(\text{Set})$ which represent functions from $X$ to $Y$ in the sense of material set theory. In fact, although $\mathbb{V}(\text{Set})$ need not satisfy limited $\Delta_0$-replacement, so that Theorem 3.5 does not apply to it directly, we still have:

**Lemma 6.29.** The sets and functions in $\mathbb{V}(\text{Set})$ form a category $\text{Set}(\mathbb{V}(\text{Set}))$, which can naturally be identified with a full subcategory of $\text{Set}$ closed under finite limits, subsets, quotients, and local exponentials. In particular, $\text{Set}(\mathbb{V}(\text{Set}))$ is a $\Pi$-pretopos.

**Proof.** We have observed closure under products and non-local exponentials. Closure under subsets is easy: if $U \subseteq |X|$, then the sub-graph $Y$ of $X$ consisting of the root and all nodes admitting a path to $U$ is a well-founded extensional $\text{APG}$ with $|Y| \cong U$. This then implies closure under finite limits.

For quotients, if $R$ is an equivalence relation on $|X|$, let $|X| \overset{q}{\to} Y$ be the quotient of $|X|$ by $R$ in $\text{Set}$; then the $\text{APG}$

$$Z = (X \sslash \ast) + Y + 1,$$

with $\prec$ inherited from $X$ along with $x \prec q(x)$ and $y \prec \ast$ for all $x \in |X|$ and $y \in Y$, is well-founded and extensional and has $|Z| \cong Y$.

Finally, given $f : |A| \to |B|$ and $g : |X| \to |A|$, we have the local exponential $h : \Pi f(g) \to |B|$ in $\text{Set}$, with counit $e : \Pi f(g) \times |B| |A| \to |X|$. Let $Z$ represent the material cartesian product of $A$ and $X$ as in Lemma 6.27 and consider the $\text{APG}$

$$W = (Z \sslash \ast) + \Pi f(g) + 1$$

with $\prec$ inherited from $Z$, along with $(x, y) \prec j$ whenever $f(x) = h(j)$ and $e(j, x) = y$, and $j \prec \ast$ for all $j \in \Pi f(g)$. Then $W$ is well-founded and extensional and $|W| \cong \Pi f(g)$; hence $\text{Set}(\mathbb{V}(\text{Set}))$ is closed under local exponentials. □

We return to verifying the axioms of material set theory.

**Lemma 6.30.** If $\text{Set}$ satisfies structural fullness, then $\mathbb{V}(\text{Set})$ satisfies material fullness.

**Proof.** Analogously to exponentiation, if $R \overset{q}{\to} M \times X \times Y$ is a generic set of multi-valued functions from $|X|$ to $|Y|$, consider

$$W = (Z \sslash \ast) + M + 1$$

with $\prec$ induced from $Z$ along with $(x, y) \prec m$ if $R(m, x, y)$ and $m \prec \ast$ for all $m \in M$. Then $W$ is a well-founded $\text{APG}$ and satisfies the hypothesis of Lemma 6.27 with $n = 1$. Its extensional quotient represents a generic set of multi-valued functions, by a similar argument as for exponentiation. □

**Lemma 6.31.** If $\text{Set}$ is a topos, then $\mathbb{V}(\text{Set})$ satisfies the power set axiom, and $\text{Set}(\mathbb{V}(\text{Set})) \subseteq \text{Set}$ is a logical subtopos.

**Proof.** For an extensional well-founded $\text{APG}$ $X$, define

$$Y = (X \sslash \ast) + P|X| + \ast$$

with $\prec$ induced from $X$ along with $x \prec A$ whenever $x \in |X|$, $A \in P|X|$, and $x \in A$; and also of course $A \prec \ast$. This is an extensional well-founded $\text{APG}$ that represents the material power set of $X$. The second statement is immediate. □
Lemma 6.32. $\forall (\mathsf{Set})$ satisfies the axiom of infinity.

Proof. Let $\omega = N + 1$, with $\prec$ defined to be $<$ on $N$ together with $n \prec \star$ for all $n \in N$. This is an extensional well-founded APG which represents the von Neumann ordinal $\omega$. The infinity axiom follows from the universal property of the NNO.  

Lemma 6.33. If $\mathsf{Set}$ satisfies the axiom of choice, then so does $\forall (\mathsf{Set})$.

Proof. Let $X$ be a well-founded extensional APG such that $X \sslash x$ is inhabited for each $x \in |X|$. If $Y \subseteq X \times |X|$ consists of those $(y, x)$ such that $y \in X \sslash x$, then the projection $Y \to |X|$ is surjective, and hence has a section, say $s$. From $s$ we can easily construct a material choice function for $X$.  

Lemma 6.34. $\mathsf{Set}$ has transitive closures.

Proof. If $X$ is a well-founded extensional APG, let $T = (X \sslash \star) + 1$ with $\prec$ inherited from $X$ along with $x \prec \star$ for all nodes $x \in X$ (not just all members). Then $T$ is a well-founded extensional APG which represents the transitive closure of $X$.  

Lemma 6.35. $\mathsf{Set}$ satisfies Mostowski’s principle.

Proof. Since $\mathsf{Set}(\forall (\mathsf{Set}))$ is closed in $\mathsf{Set}$ under finite limits and subsets, any well-founded extensional graph $X$ constructed in the material set theory $\forall (\mathsf{Set})$ will induce such a graph in $\mathsf{Set}$. Therefore, $X + 1$, with $\prec$ induced from $X$ along with $x \prec \star$ for all nodes $x \in X$, is a well-founded extensional APG, i.e. an object of $\forall (\mathsf{Set})$. We then verify that it is transitive and isomorphic to $X$ in $\forall (\mathsf{Set})$.  

We now turn to the axiom schemata. For these, we need to be able to translate material formulas into structural ones. There is an obvious way to do this: if $\varphi$ is a formula in $\forall (\mathsf{Set})$, we define $\varphi_\epsilon$ in $\mathsf{Set}$ as follows:

- We replace the material “equality” symbol $=$ by isomorphism $\cong$ of APGs.
- We replace the material “membership” symbol $\in$ by the relation $\epsilon$.
- The connectives are unchanged.
- We replace quantifiers over material-sets by quantifiers over well-founded extensional APGs. For example, $(\exists x)\varphi(x)$ becomes “there exists a well-founded extensional APG $X$ such that $\varphi_\epsilon(X)$”.

It is easy to see that $\forall (\mathsf{Set}) \vDash \varphi$ if and only if $\mathsf{Set} \vDash \varphi_\epsilon$. This translation works quite well for the schemata involving arbitrary formulas.

Lemma 6.36. If $\mathsf{Set}$ satisfies separation, then so does $\forall (\mathsf{Set})$.

Proof. Let $\varphi(x)$ be a formula and $A \in \forall (\mathsf{Set})$. Using separation, let $U \subseteq |A|$ consist of precisely those $a \in A$ such that $\varphi_\epsilon(A/a)$. Define $B$ to consist of the root of $A$ together with all nodes admitting a path to some node in $U$. Then $B$ is a well-founded extensional APG, and for any $C \epsilon A$ we have $C \epsilon B$ iff $\varphi_\epsilon(C)$.  

Lemma 6.37. If $\mathsf{Set}$ satisfies full classical logic, then so does $\forall (\mathsf{Set})$.

Proof. Classical logic for $\mathsf{Set}$ implies $\varphi_\epsilon \lor \neg \varphi_\epsilon$ for any formula $\varphi$ in $\forall (\mathsf{Set})$.  

Lemma 6.38. If $\mathsf{Set}$ satisfies collection, then so does $\forall (\mathsf{Set})$.  


Proof. Suppose that \( A \in \mathbb{V}(\text{Set}) \) and that \( \varphi \) is a formula such that for any \( X \in A \), there exists a \( Y \in \mathbb{V}(\text{Set}) \) with \( \varphi(X, Y) \). This means that for any \( x \in |A| \), there exists a well-founded extensional APG \( Y \) such that \( \varphi(x, A, Y) \). By collection in \( \text{Set} \), there is a surjection \( V \xrightarrow{p} |A| \) and a pointed graph \( B \in \text{Set}/V \) (here “pointed” means we have a section \( s: V \to B \) over \( V \)) such that for each \( v \in V \), \( v^*B \) is a well-founded extensional APG and \( \varphi(x, A/p(v), v^*B) \). It is easy to show that \( B \), considered as a graph in \( \text{Set} \), is still well-founded. And since \( B \) is a graph in \( \text{Set}/V \), its relation \( \prec \) is fiberwise; thus for each \( v \in V \) we have \( B/s(v) \cong v^*B \), which is therefore extensional and accessible. It follows that \( B+1 \), with \( s(v) \prec \star \) for all \( v \in V \), satisfies the hypotheses of \cite[Lemma 6.24]{6} with \( n = 1 \). Its extensional quotient is then the set desired by the material collection axiom. \( \square \)

**Lemma 6.39.** If \( \text{Set} \) satisfies replacement of contexts, then \( \mathbb{V}(\text{Set}) \) satisfies material replacement.

Proof. Suppose that \( A \in \mathbb{V}(\text{Set}) \) and that \( \varphi \) is a formula such that for any \( X \in A \), there exists a unique \( Y \in \mathbb{V}(\text{Set}) \) with \( \varphi(X, Y) \). Thus, for any \( x \in |A| \), there exists a well-founded extensional APG \( Y \) such that \( \varphi(x, A, Y) \), and any two such \( Y \) are isomorphic. Since any such isomorphism is unique by \cite[Lemma 6.13]{6} replacement of contexts in \( \text{Set} \) supplies a pointed graph \( B \in \text{Set}/|A| \) such that for each \( x \in |A| \) we have \( \varphi(A/x, x^*B) \). The extensional quotient of \( B+1 \) then represents the set desired by material replacement. \( \square \)

**Lemma 6.40.** If \( \text{Set} \) satisfies full induction, then so does \( \mathbb{V}(\text{Set}). \)

Proof. If \( \varphi(x) \) is as in the statement of the material induction axiom, then \( \varphi_\star(X) \) is a statement about some \( X \in \omega \), i.e. \( X \in N \). But every \( X \in N \) is isomorphic to \( N/n \) for a unique \( n \in N \), so \( \varphi_\star(X) \) is equivalent to a statement \( \varphi'_\star(n) \) about some \( n \in N \), which can then be proven by the structural induction axiom. \( \square \)

**Lemma 6.41.** If \( \text{Set} \) satisfies well-founded induction, then \( \mathbb{V}(\text{Set}) \) satisfies set-induction.

Proof. Just like \cite[Lemma 6.40]{6}, using induction over the well-founded (extensional) relation in \( \text{Set} \) that underlies any object of \( \mathbb{V}(\text{Set}) \). \( \square \)

The schemata involving \( \Delta_0 \)-formulas require a little more work, since if \( \varphi \) is \( \Delta_0 \) then \( \varphi_\star \) need not be. Thus, we need to define a different translation for \( \Delta_0 \)-formulas. Suppose \( \varphi \) is a \( \Delta_0 \)-formula in \( \mathbb{V}(\text{Set}) \) with parameters \( A_1, \ldots, A_k \), each of which is a well-founded extensional APG. Then

\[
A_1^* + \cdots + A_k^* + 1,
\]

with \( \prec \) inherited from the \( A_i \) along with \( \star_i \prec \star \) for all \( 1 \leq i \leq k \), satisfies the hypothesis of \cite[Lemma 6.24]{6} with \( n = 1 \). Let \( T \) be its extensional quotient. We can now translate \( \varphi \) into a \( \Delta_0 \)-formula as follows, interpreting material-set variables not by APGs but by elements of \( T \).

- Each parameter \( A_i \) is replaced by \( |A_i| \in T \).
- The material “equality” symbol \( = \) is replaced by equality in \( T \).
- The material “membership” symbol \( \in \) is replaced by \( \prec \in T \).
- The connectives are unchanged.
- A \( \Delta_0 \)-quantifier of the form \( \exists x \in y \varphi \) is replaced by a \( \Delta_0 \)-quantifier of the form \( \exists x \in T)(x \prec y \wedge \varphi) \). Similarly, \( \forall x \in y \varphi \) is replaced by \( \forall x \in T)(x \prec y \Rightarrow \varphi) \).
We call the formula produced in this way $\varphi_{\Delta_0}$.

**Lemma 6.42.** $\varphi_\epsilon$ is equivalent to $\varphi_{\Delta_0}$.

*Proof.* [Lemma 6.12] implies that $T/x \cong T/y$ if and only if $x = y$. Similarly, if $T/x \in T/y$, then $T/x \cong (T/y)/y' = T/y'$ for some $y' \prec y$, whence $x = y'$ and so $x \prec y$. The converse is easy, so $T/x \in T/y$ if and only if $x \prec y$. Thus the atomic formulas correspond, and the connectives evidently do, so it remains to observe that $\Delta_0$-quantifiers are adequately represented by quantifiers over $T$, since by definition $X \in T/y$ if and only if $X \sim T/x$ for some $x \prec y$. □

**Lemma 6.43.** $\forall(\text{Set})$ satisfies $\Delta_0$-separation.

*Proof.* Just like [Lemma 6.36], but using $\varphi_{\Delta_0}$ instead of $\varphi_\epsilon$, and the $\Delta_0$-separation property of Proposition 3.17 instead of the separation axiom. □

**Lemma 6.44.** $\forall(\text{Set})$ satisfies foundation.

*Proof.* For any $\Delta_0$-formula $\varphi$ in $\forall(\text{Set})$ and any well-founded apg $X \in \forall(\text{Set})$, we can form $\{ x \in X \mid \varphi(X/x) \}$ using $\Delta_0$-separation, since $\varphi(X/x) = \varphi_{\Delta_0}(x)$. The hypothesis on $\varphi$ implies that this is an inductive subset of $X$, hence all of it. This implies the desired conclusion, since for every well-founded extensional apg $Y$ there is another one $X$ with $Y \in X$. □

**Lemma 6.45.** If $\text{Set}$ is Boolean, then $\forall(\text{Set})$ satisfies $\Delta_0$-classical logic.

*Proof.* For any $\Delta_0$-formula $\varphi$ in $\forall(\text{Set})$, $\Delta_0$-separation supplies a subset $\{ \emptyset \mid \varphi \} \subseteq \{ \emptyset \}$. If this is complemented in $\text{Set}$, then we must have $\varphi \lor \neg \varphi$. □

This completes the proof of [Theorem 6.16].

Of course, it is natural to ask to what extent the constructions $\forall(-)$ and $\text{Set}(-)$ are inverse. We have already seen the canonical inclusion $\text{Set}(\forall(\text{Set})) \hookrightarrow \text{Set}$, and the following is easy to verify.

**Lemma 6.46.** The inclusion $\text{Set}(\forall(\text{Set})) \rightarrow \text{Set}$ is an equivalence if and only if every object of $\text{Set}$ can be embedded into some well-founded extensional graph. □

I propose to call this property of $\text{Set}$ the **axiom of well-founded materialization**. (Other axioms of materialization would arise from using various kinds of non-well-founded graphs.)

Note that well-founded materialization follows from the axiom of choice, since then every object can be well-ordered and thus given the structure of an apg representing a von Neumann ordinal. We remark in passing that this implies that “all replacement schemata for ETCS are equivalent.”

**Proposition 6.47.** Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be axiom schemata for structural set theory such that for each $i = 1, 2$,

(i) if $\forall$ satisfies ZFC, then $\text{Set}(\forall)$ satisfies $\mathcal{T}_i$, and

(ii) if $\text{Set}$ satisfies ETCS+$\mathcal{T}_i$, then $\text{Set}(\forall)$ satisfies ZFC.

Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent over ETCS.

*Proof.* Since ETCS includes AC, for any model of ETCS we have $\text{Set} \simeq \text{Set}(\forall(\text{Set}))$. Thus, if $\text{Set} \models \mathcal{T}_1$, then by (i) we have $\text{Set}(\forall) \models \text{ZFC}$, hence by (i) we have $\text{Set} = \text{Set}(\forall(\text{Set})) \models \mathcal{T}_2$. The converse is the same. □
We have seen that our axioms of collection and replacement from \([\text{Mat01}]\) satisfy \([\text{Mat01}]\) and \([\text{Mat01}]\) so over ETCS they are equivalent to any other such schema. This includes the replacement axiom of McLarty \([\text{McL04}]\), the axiom CRS of Cole \([\text{Col73}]\), the axiom RepT of Osius \([\text{Osi74}]\), and the reflection axiom of Lawvere \([\text{Law05}]\). However, our axioms seem to be more appropriate in an intuitionistic context.

Returning to the material-structural comparison, on the other side we have:

**Proposition 6.48.** If \(V\) satisfies the core axioms of material set theory along with infinity, exponentials, foundation, and transitive closures, then there is a canonical embedding \(V \to \forall(\text{Set}(V))\). This map is an isomorphism if and only if \(V\) additionally satisfies Mostowski’s principle.

**Proof.** By assumption, any \(x \in V\) has a transitive closure \(TC(x)\), which is a well-founded extensional graph. If we define \(Y = TC(x) + 1\), with \(\prec\) induced by \(\in\) on \(TC(x)\) and with \(z \prec \ast\) for all \(z \in x\), then \(Y\) is a well-founded extensional APG, i.e. an object of \(\forall(\text{Set}(V))\). This construction gives a map \(V \to \forall(\text{Set}(V))\), and it is straightforward to verify that it preserves and reflects membership and equality.

Now if this embedding is an isomorphism, then clearly \(V\) must satisfy Mostowski’s principle, since \(\forall(\text{Set}(V))\) does so. Conversely, if \(V\) satisfies Mostowski’s principle, then every \(X \in \forall(\text{Set}(V))\) is isomorphic in \(V\) to a transitive set, and therefore equal in \(\forall(\text{Set}(V))\) to something in the image of the embedding. □

**Remark 6.49.** The theory \(M_0\) of \([\text{Mat01}]\) consists of the core axioms together with power sets and full classical logic. If \(V \models M_0\), then \(\forall(\text{Set}(V))\) is precisely the model \(W_1\) constructed in \([\text{Mat01}]\) \(\S 2\), which satisfies \(M_0\) plus regularity, transitive closures, and Mostowski’s principle, and inherits infinity and choice from \(V\). (In the presence of classical logic and power sets, the axiom of infinity is unnecessary for the construction \(\forall(-)\).) In particular, if \(V \models \text{ZBQC}\), then \(\forall(\text{Set}(V)) \models \text{MOST}\), while \(\forall(\text{Set}(V)) \cong V\) if we already had \(V \models \text{MOST}\).

So far we have concentrated on building models of pure sets only. However, we can also allow an arbitrary set of atoms: we fix some \(A \in \text{Set}\) and modify our definitions as follows. (The definitions not listed below need no modification.)

- An **A-graph** is a graph \(X\) together with a partial function \(\ell: X \to A\), such that \(\text{dom}(\ell)\) is a complemented subobject of \(X\), and if \(x \prec y\) then \(y \notin \text{dom}(\ell)\).
- Isomorphisms between A-graphs are, of course, required to preserve the labeling functions \(\ell\).
- An A-graph is **A-extensional** if
  1. \(\ell\) is injective on its domain, and
  2. for any \(x, y \notin \text{dom}(\ell)\), if \(z \prec x \iff z \prec y\) for all \(z \in X\) then \(x = y\).
- An A-simulation between A-graphs is a simulation \(f: X \to Y\) such that for any \(x \in X\), if either \(\ell(x)\) or \(\ell(f(x))\) is defined, then both are, and they are equal.

Note that if an A-graph \(X\) is an APG and \(\ast \in \text{dom}(\ell)\), then \(X\) can have no nodes other than the root, so \(X\) is simply an element of \(A\). Thus, accessible pointed A-graphs model atoms themselves in addition to sets that can contain atoms.

It is straightforward to verify that all the above lemmas, and the main theorem, still hold when extensional well-founded APGs are replaced by A-extensional well-founded accessible pointed A-graphs. The resulting material set theories, of course, satisfy the modified axioms for theories with atoms listed at the end of \([\text{Mat01}]\).
7. The stack semantics

We now introduce the stack semantics of a Heyting pretopos S, no longer assumed to be well-pointed. (In fact, we will work most of the time with a positive Heyting category, although a few facts require exactness.) In the introduction, we described two approaches to this semantics: first, as a direct generalization of the usual Kripke-Joyal semantics, and second, as a fragment of the internal logic of the category of sheaves or stacks. In this section we will take the first, more explicit, viewpoint; in \[\text{[III]}\] we will show how it agrees with the second.

Recall that our goal is to embed the usual internal type theory in a structural set theory, along the lines of \[\text{Lemma 3.15}\]. The interpretation of the internal type theory is usually defined by constructing, for any formula \(\varphi\) with (say) one free variable \(x\) of type \(A\) (an object of \(S\)), a subobject \([\varphi]\nrightarrow A\) which we think of as the subset \(\{x \in A \mid \varphi(x)\}\). This approach is very powerful and flexible—for instance, it generalizes to the internal logic of any fibered poset—but it seems inadequate to deal directly with unbounded quantifiers.

Thus, we start instead with the Kripke-Joyal semantics (see e.g. \[\text{MLM94, VI.6}\]), according to which \([\varphi]\nrightarrow A\) can be defined indirectly by characterizing the sub-presheaf \(S(-, [\varphi]) \rightarrow S(-, A)\), i.e. characterizing which maps \(U \rightarrow A\) factor through \([\varphi]\). The usual construction of \([\varphi]\) then becomes a proof that this subfunctor is representable. The stack semantics consists of defining a sub-presheaf \("S(-, [\varphi])") for any formula \(\varphi\) in a similar way, though it will not in general be representable. (This also makes the connection with \(\text{Sh}(S)\) clear, since this sub-presheaf is in fact a sub-sheaf and thus a genuine subobject in \(\text{Sh}(S)\).)

If \(U\) is an object of \(S\) and \(\varphi\) is a formula of category theory with parameters in \(S/U\), we say that \(\varphi\) is a formula over \(U\). Note that a formula over 1 is the same as a formula in \(S\) itself. We can think of a formula over \(U\) as an assertion taking place in each fiber. If moreover \(V\nrightarrow U\) is any map, its pullback \(p^*\varphi\) is a formula over \(V\) obtained by replacing each parameter of \(\varphi\) by its pullback along \(p\).

**Remark 7.1.** Since pullbacks are only defined up to isomorphism, the notation \(p^*\varphi\) is, strictly speaking, ambiguous. However, by isomorphism-invariance (Lemmas 3.2 and \[\text{[III]}\]), the particular choices made are irrelevant. Moreover, since only a finite number of choices are ever needed for any particular formula, making such choices requires no axiom of choice in the metatheory.

**Definition 7.2** (The stack semantics). Let \(S\) be a positive Heyting category, and let \(\varphi\) be a sentence over \(U\) in \(S\). The relation \(U \models \varphi\) is defined recursively as follows.

\[
\begin{align*}
(\models \text{a}) & \quad U \models (f = g) \text{ iff in fact } f = g. \\
(\models \top) & \quad U \models \top \text{ always.}
\end{align*}
\]

\[
(\models \bot) \quad U \models \bot \text{ iff } U \text{ is an initial object.}
\]

\[
(\models \land) \quad U \models (\varphi \land \psi) \text{ iff } U \models \varphi \text{ and } U \models \psi.
\]

\[
(\models \lor) \quad U \models (\varphi \lor \psi) \text{ iff } U = V \cup W, \text{ where } \overset{i}{\rightarrow} U \text{ and } W \overset{j}{\rightarrow} U \text{ are subobjects such that } V \models i^*\varphi \text{ and } W \models j^*\psi.
\]

\[
(\models \Rightarrow) \quad U \models (\varphi \Rightarrow \psi) \text{ iff for any } V\nrightarrow U \text{ such that } V \models p^*\varphi, \text{ also } V \models p^*\psi.
\]

\[
(\models \neg) \quad U \models \neg \varphi \text{ iff } U \models (\varphi \Rightarrow \bot).
\]

\[
(\models \exists_0) \quad U \models (\exists X)\varphi(X) \text{ iff there is a regular epimorphism } V \overset{p}{\rightarrow} U \text{ and an object } A \in S/V \text{ such that } V \models p^*\varphi(A).
\]


Similarly, $U \vdash (\exists f : A \to B) \varphi(f)$ iff there is a regular epimorphism $V \xrightarrow{p} U$ and an arrow $g : p^*A \to p^*B$ in $S/V$ such that $V \vdash p^*\varphi(g)$.

$(\models \forall V_0) U \models (\forall X) \varphi(X)$ iff for any $V \xrightarrow{p} U$ and any object $A \in S/V$, we have $V \models p^*\varphi(A)$.

$(\models \exists V_1) U \models (\exists f : A \to B) \varphi(f)$, where $A$ and $B$ are objects of $S/U$, iff for any $V \xrightarrow{p} U$ and any arrow $p^*A \xrightarrow{g} p^*B$ in $S/V$, we have $V \models p^*\varphi(j)$.

If $\varphi$ is a formula over 1 (i.e. a formula in $S$), we say $\varphi$ is valid if $1 \models \varphi$.

We occasionally write $U \models_{S} \varphi$ if we wish to emphasize the category $S$. We may also write $V \models \varphi$ instead of $V \models p^*\varphi$ if the map $p$ is obvious from context.

We now prove a couple of basic lemmas about the stack semantics.

**Lemma 7.3.** Let $\varphi$ be a sentence over $U$ in a positive Heyting category $S$.

1. If $\varphi'$ is an isomorph of $\varphi$, then $U \models \varphi'$ if and only if $U \models \varphi$.
2. If $U \models \varphi$, then for any $V \xrightarrow{p} U$ we have $V \models p^*\varphi$.
3. If $V \xrightarrow{p} U$ is a regular epimorphism and $V \models p^*\varphi$, then $U \models \varphi$.
4. If $U$ is initial, then $U \models \varphi$ for any $\varphi$.
5. If $U = V \sqcup W$ with $V \models \varphi$ and $W \models \varphi$, then $U \models \varphi$.

**Proof.** These are all inductions on formulas. (i) is easy, just like Lemma 3.2, and (ii) needs only the fact that regular epis and unions are stable under pullback and initial objects are strict (that is, any map $V \to 0$ is an isomorphism). Strictness of initial objects also immediately implies (iv).

For (iii), $\top$, $\bot$, and $\land$ are trivial. Atomic formulas, $\to$, and $\forall$ follow from pullback-stability of regular epimorphisms. For $\forall$, we take images, and for $\exists$, we just compose regular epis.

(v) is only slightly more involved. Once again, $\top$, $\bot$, and $\land$ are trivial, while atomic formulas, $\to$, and $\forall$ follow from pullback-stability of unions. $\lor$ is also easy since unions distribute over themselves. For $\exists_0$, if we have $U = V \sqcup W$ with regular epis $V' \to V$ and $W' \to W$, and objects $X \in S/V'$ and $Y \in S/W'$ such that $V' \models \varphi(X)$ and $W' \models \varphi(Y)$, then because coproducts are disjoint and stable, $X + Y \in S/(V' + W')$ pulls back to $X$ and $Y$ over $V$ and $W$, respectively; thus by the inductive hypothesis, $V' + W' \models \varphi(X + Y)$. And $V' + W' \to U$ is regular epic, so $U \models (\exists X) \varphi(X)$. The case of $\exists_1$ is analogous. $\Box$

We refer to Lemma 7.3(iii), (iv), and (v) as descent of forcing. Collectively, they say that “forcing descends along finite jointly effective-epimorphic families”.

**Lemma 7.4.** The usual rules of deduction for intuitionistic logic are sound for $\vdash$.

In other words, if $\varphi \models \psi$ is provable and $U \models \varphi$, then also $U \models \psi$.

**Proof.** Straightforward verification; see, for instance, Johnstone D1.3.1] for the axioms we have to prove. The only slightly nontrivial axioms are the rules for falsity, disjunction, and existential quantification. The axiom

$$\bot \Rightarrow \varphi$$

for falsity follows from Lemma 7.3(iv). For the axiom

$$(\varphi \Rightarrow \chi) \implies ((\psi \Rightarrow \chi) \implies ((\varphi \lor \psi) \Rightarrow \chi))$$

we must check that if $U \models (\varphi \Rightarrow \chi), U \models (\psi \Rightarrow \chi)$, and $U \models (\varphi \lor \psi)$, then $U \models \chi$.

But $U \models (\varphi \lor \psi)$ means $U = V \sqcup W$ with $V \models \varphi$ and $W \models \psi$, while the other
two hypotheses imply \( V \models \chi \) and \( W \models \chi \); so \( U \models \chi \) follows by descent of forcing \( \text{(Lemma 7.3(v))} \). Finally, for the axiom

\[
(\forall X)(\varphi(X) \Rightarrow \psi) \implies ((\exists X)\varphi(X) \Rightarrow \psi)
\]

we must check that if \( U \models (\forall X)(\varphi(X) \Rightarrow \psi) \) and \( U \models (\exists X)\varphi(X) \), then \( U \models \psi \). But \( U \models (\exists X)\varphi(X) \) means we have a regular epimorphism \( V \twoheadrightarrow U \) and \( A \in S/V \) with \( V \models p^*\varphi(A) \). The other assumption then implies that \( V \models p^*\psi \), so \( U \models \psi \) again follows by descent of forcing \( \text{(Lemma 7.3(iii))} \). Quantification over arrows is identical. \( \square \)

Now we consider the relationship of the stack semantics to the usual internal logic, by way of representing objects.

**Definition 7.5.** Let \( S \) be a positive Heyting category and \( \varphi \) a sentence in \( S \) over \( U \). We say that a subobject \( \llbracket \varphi \rrbracket \hookrightarrow U \) represents or classifies \( \varphi \) if for any \( V \to U \),

\[
V \models p^*\varphi \iff p \text{ factors through } \llbracket \varphi \rrbracket.
\]

Evidently \( \llbracket \varphi \rrbracket \) is unique up to isomorphism, if it exists. As suggested earlier, we can then prove that the usual constructions used in the internal logic do produce such classifying subobjects.

**Proposition 7.6.** Every \( \Delta_0 \)-sentence in a positive Heyting category is classified.

**Proof.** First note that a \( \Delta_0 \)-sentence over \( U \) can be translated into a \( \Delta_0 \)-sentence in \( S \) with one free variable \( u \in U \). We interpret parameters in \( S/U \) as parameters in \( S \) by ignoring their structure maps to \( U \), while for any \( (X \to U) \in S/U \) we modify the \( \Delta_0 \)-quantifiers \( (\forall x \in X)\varphi \) and \( (\exists x \in X)\varphi \) to read instead \( (\forall x \in X)((fx = u) \Rightarrow \varphi) \) and \( (\exists x \in X)((fx = u) \land \varphi) \). Now we further translate this formula along the equivalence of \( \text{(Lemma 3.15)} \) to obtain a formula \( \tilde{\varphi}(u) \) in the usual internal logic of \( S \), with one free variable \( u : U \). Comparing definitions shows that for any \( p : V \to U \), we have \( V \models p^*\varphi \) in the stack semantics exactly when \( V \models \tilde{\varphi}(p) \) in the usual Kripke-Joyal semantics. It then follows, by the usual characterization of Kripke-Joyal semantics, that the standard representing subobject of \( \tilde{\varphi} \) also classifies \( \varphi \) in the above sense. \( \square \)

In particular, the usual internal logic of a positive Heyting category is exactly the \( \Delta_0 \)-fragment of the stack semantics. More precisely:

**Corollary 7.7.** A formula in the internal logic of \( S \) is valid if and only if its translate under the correspondence of \( \text{(Lemma 3.15)} \) is valid in the stack semantics.

If \( S \) is a \( \Pi \)-pretopos, then classifiers for all arrow-quantifiers can be constructed from those for \( \Delta_0 \)-quantifiers, since morphisms \( 1 \to Y^X \) are the same as morphisms \( X \to Y \). Likewise, if \( S \) is a topos, we can classify quantifiers over subobjects. However, there is no hope to classify arbitrary object-quantifiers in this way, and in general not all sentences will be classified. This suggests the following definition.

**Definition 7.8.** A positive Heyting category is autological if all sentences over all objects of \( S \) are classified.

In \( \text{Theorem 7.22} \), we will show that \( S \) is autological precisely when its stack semantics validates the structural axiom of separation from \( \text{[4]} \). But first, we need to
investigate more generally how validity in the stack semantics relates to “external” properties of \( S \) itself.

Like the ordinary internal logic which it generalizes, validity in the stack semantics can be interpreted as “local” truth. In particular, for “statements which are their own localizations,” validity in the stack semantics is equivalent to external truth. Chief among “statements which are their own localizations” are assertions of pullback-stable universal properties. More generally, by interpreting universal properties in the stack semantics we transform them into their “localizations.” In order to make this precise, we will reuse the notions of context and instantiation from the (closely related) discussion of replacement axioms in §5.

**Theorem 7.9.** Let \( S \) be a positive Heyting category, let \( (\Gamma, \Delta) \) be a context in \( S \) such that \( \Delta \) consists only of arrow-variables, and let \( \varphi(\Gamma) \) and \( \psi(\Gamma, \Delta) \) be formulas. Then the following are equivalent.

(i) \( 1 \vdash \Gamma \) for any \( \vec{A} : \Gamma \) such that \( \varphi(\vec{A}) \), there exists a unique \( \vec{f} : \Delta \) extending \( \vec{A} \) such that \( \psi(\vec{A}, \vec{f}) \).

(ii) For any \( U \) and any \( \vec{A} : \Gamma \) in \( S/U \) such that \( U \vdash \varphi(\vec{A}) \), there exists a unique \( \vec{f} : \Delta \) in \( S/U \) extending \( \vec{A} \) such that \( U \vdash \psi(\vec{A}, \vec{f}) \).

**Proof.** By the forcing interpretations of \( \forall \) and \( \exists \), statement (i) is equivalent to saying that for any \( U \) and any \( \vec{A} : \Gamma \) in \( S/U \) such that \( U \vdash \varphi(\vec{A}) \), there is a regular epi \( V \xrightarrow{p} U \) and an \( \vec{f} : \Delta \) in \( S/V \) extending \( p^*\vec{A} \) such that \( V \vdash \varphi(p^*\vec{A}, \vec{f}) \), and moreover for any \( W \xrightarrow{q} V \) and \( \vec{h} : \Delta \) in \( S/W \) extending \( q^*p^*\vec{A} \) such that \( W \vdash \varphi(q^*p^*\vec{A}, \vec{h}) \), we have \( \vec{f} = \vec{h} \).

In particular, \( \vec{f} \) is unique in \( S/V \) such that \( V \vdash \varphi(p^*\vec{A}, \vec{f}) \), and therefore all the arrows that make it up will satisfy the descent conditions over the kernel pair of \( p \).

Since any Heyting category is a prestack for its coherent topology, \( \vec{f} \) must descend to an instantiation \( \vec{g} \) of \( \Delta \) extending \( \vec{A} \) over \( U \), and descent of forcing implies that \( U \vdash \varphi(\vec{A}, \vec{f}) \); this proves (ii). The converse is easy. \( \square \)

The following example will hopefully help clarify in what sense this theorem is about pullback-stable universal properties.

**Example 7.10.** Let \( \xymatrix{ R \ar[r]^r \ar@{=}[d]^s & P \ar[r]^q & Q } \) be a diagram in \( S \) such that \( qr = qs \). Let \( \Gamma \) be the context \((X, f : P \to X)\), let \( \varphi(X, f) \) assert that \( fr = fs \), let \( \Delta \) be the context \( q : Q \to X \) extending \( \Gamma \), and let \( \psi(X, f, g) \) assert that \( qg = f \). (Note that \( R, P, Q, r, s, \) and \( q \) are parameters of these contexts.) Then Theorem 7.9(ii) becomes \( 1 \vdash \Gamma \vdash \vec{q} \) is a coequalizer of \( r \) and \( s \), while (ii) becomes the assertion that \( q \) is a pullback-stable coequalizer of \( r \) and \( s \).

Generalizing from this example in the evident way, we can interpret Theorem 7.9 as saying that \( 1 \vdash \Gamma \vdash \vec{B} \) has some universal property if and only if \( U^*\vec{B} \) has the given universal property in \( S/U \) for any \( U \in S \). Similarly, for asserting the existence of objects with universal properties, we have the following. Note that here we require exactness, just as we did for Proposition 5.7.

**Theorem 7.11.** Let \( S \) be a Heyting pretopos, let \( (\Gamma, \Delta) \) be a context in \( S \), and let \( \varphi(\Gamma) \) and \( \psi(\Gamma, \Delta) \) be formulas. Then the following are equivalent.
(i) $1 \vdash \gamma$ for any $\vec{A} : \Gamma$ such that $\varphi(\vec{A})$, there exists a $\vec{B} : \Delta$ extending $\vec{A}$ such that $\psi(\vec{A}, \vec{B})$, and any two such $\vec{B}$ are isomorphic by a unique isomorphism extending $1_{\vec{A}}$.

(ii) For any $U$ and any $\vec{A} : \Gamma$ in $\mathcal{S}/U$ such that $U \vdash \varphi(\vec{A})$, there exists a $\vec{B} : \Delta$ in $\mathcal{S}/U$ extending $\vec{A}$ such that $U \vdash \psi(\vec{A}, \vec{B})$, and any two such $\vec{B}$ are isomorphic by a unique isomorphism extending $1_{\vec{A}}$.

Proof. Just like the proof of [Theorem 7.9] but using the facts that any pretopos is a stack for its coherent topology, and that uniqueness of isomorphisms implies that they satisfy the cocycle condition.

Following on from [Example 7.10] we have the following.

**Example 7.12.** Let $\Gamma$ be the context $(R, P, r : R \to P, s : R \to P)$, let $\varphi = \top$, let $\Delta$ be the context $(Q, q : P \to Q)$ extending $\Gamma$, and let $\psi$ be the formula $\gamma q$ is a coequalizer of $r$ and $s$ considered in [Example 7.10]. Then [Theorem 7.11](i) becomes $1 \vdash \gamma$ every parallel pair has a coequalizer, while (ii) asserts that every slice category of $\mathcal{S}$ has pullback-stable coequalizers (for which it suffices that $\mathcal{S}$ itself has them).

Thus, we can interpret [Theorem 7.11] as saying that $1 \vdash \gamma$ for all $\vec{A}$, there exists $\vec{B}$ with some universal property if and only if for any $U$ and any $\vec{A}$ in $\mathcal{S}/U$, there exists $\vec{B}$ in $\mathcal{S}/U$ such that $p^*(\vec{A}, \vec{B})$ has the specified universal property in $\mathcal{S}/V$ for any $p : V \to U$. In the future, we will apply these theorems to all sorts of universal properties without further comment.

For statements $\varphi$ that do not express simple universal properties, however, validity in the stack semantics (expressed by $1 \vdash \varphi$) and external truth (expressed by $\mathcal{S} \models \varphi$) can have quite different meanings. For example, if $\varphi$ is $\gamma$ every regular epi splits, then $\mathcal{S} \models \varphi$ iff $\mathcal{S}$ satisfies the external axiom of choice (AC), while $1 \vdash \varphi$ iff $\mathcal{S}$ satisfies the internal axiom of choice (IAC). The most important example of the divergence of $\vdash$ and $\models$, however, is the following.

**Lemma 7.13.** For any positive Heyting category $\mathcal{S}$, we have $1 \vdash \gamma \mathcal{S}$ is constructively well-pointed and satisfies collection.$^\dagger$

Proof. We first show that $1 \vdash \gamma \mathcal{S}$ is constructively well-pointed. If $X \xrightarrow{f} U$ is in $\mathcal{S}/U$, $m : A \to X$ is monic, and $U \vdash \gamma$ every $x \xrightarrow{1} X$ factors through $A$, then in particular the generic element $\Delta_X : X \to f^*X$ in $\mathcal{S}/X$ factors through $m$; hence $m$ is split epic and thus an isomorphism. Thus $1 \vdash \gamma 1$ is a strong generator.$^\dagger$ Now since the initial object is strict, the existence of a map $1 \to 0$ is the same as saying that $1$ is an initial object. Then if $U \vdash \gamma 1_U$ is initial,$^\dagger$ $U$ must be initial; hence $1 \vdash \gamma 1$ is not initial.$^\dagger$ Similarly, if $U \vdash \gamma V \to 1_U$ is regular epic,$^\dagger$ then $V \to U$ is in fact regular epic, and thus (since $V$ has a section over itself) $U \vdash (\exists v : 1 \to V)$. Finally, if $U \vdash \gamma U = V \cup W$, then $U = V \cup W$, and so since $V$ and $W$ have sections over themselves, we have $U \vdash (\exists v : 1 \to V) \lor (\exists w : 1 \to W)$.

We now show $1 \vdash \gamma \mathcal{S}$ satisfies collection.$^\dagger$ Suppose that $1 \vdash (\forall u \in U)(\exists X)\varphi(u, X)$. Then $U \vdash (\exists X)\varphi(\Delta_U, X)$, so we have a regular epi $V \xrightarrow{p} U$ and an $A \in \mathcal{S}/V$ with $V \vdash \varphi(p, A)$. Thus, for any $W$ and any $W \xrightarrow{w} V$, we have $W \vdash \varphi(pw, v^*A)$, which is the desired conclusion. Collection of functions is analogous.

This validates our assertion that the stack semantics of any Heyting pretopos models a structural set theory. Therefore, from now on we avail ourselves of [Convention 3.14](conv) when speaking in the stack semantics.
Remark 7.14. Under the second approach to stack semantics, Lemma 7.13 says that if $S$ is small, any strictification of its self-indexing is well-pointed and satisfies collection as an internal category in the topos $\text{Sh}(S)$ of sheaves for the coherent topology of $S$. Since $\text{Sh}(S)$ is a coherent topos, it has enough points, so this is equivalent to saying that each stalk has these properties. This version of Lemma 7.13 was proven in [Awo97], and we will prove the analogous fact for the 2-category of stacks on $S$ in [Shua]; see also §10.

Remark 7.15. Theorem 7.11 implies that $1 \vdash \lozenge \text{the self-indexing of } S \text{ is a stack}$ is true in any Heyting pretopos. Thus, by Propositions 5.2 and 5.7, replacement of contexts is also always valid in the stack semantics of a Heyting pretopos.

Moreover, any additional axiom of structural set theory that is expressible as a pullback-stable universal property will be faithfully represented in the stack semantics. This includes all the structure mentioned in Theorem 3.5 except for AC and full classical logic. In particular, we can say:

• If $S$ is a Π-pretopos with a NNO satisfying the “internal presentation axiom” (there are enough internal projectives), then its stack semantics models CETCS.

• If $S$ is any topos with a NNO, then its stack semantics models IETCS.

Of course, Lemma 7.13 also says that the stack semantics of a Heyting pretopos is not just any structural set theory, but also satisfies collection. In particular, from any constructively well-pointed Heyting pretopos, we can construct another one which satisfies collection—albeit in an exotic logic (namely, the stack semantics of the original category). This implies the well-known fact that intuitionistically, the addition of collection does not change the consistency strength of a theory.

Remark 7.16. Since by Proposition 5.4 adding collection to full classical logic does change the consistency strength, it follows that full classical logic is not in general preserved by passage to the stack semantics. It is not hard to show that if the stack semantics of $S$ satisfies full classical logic, then $S$ is necessarily autological.

In fact, well-pointedness and collection are “precisely” the characteristic properties of the stack semantics, in the following sense.

Theorem 7.17. Let $S$ be a positive Heyting category; the following are equivalent.

(i) $S$ is constructively well-pointed and satisfies collection.

(ii) For any $U \in S$ and any sentence $\varphi$ over $U$, we have $U \models \varphi$ if and only if $S \models \text{u}^*\varphi$ for all $1 \to U$.

Proof. First assume (ii). In particular, this means that for any sentence $\varphi$ in $S$ we have $1 \models \varphi$ if and only if $S \models \varphi$. Since the stack semantics is always constructively well-pointed and satisfies collection, it follows from (ii) that $S$ is so, proving (i).

Now assume (i). We prove (ii) by structural induction on $\varphi$. For an atomic formula $(f = g)$ where $f, g : X \to Y$ in $S/U$, we can verify that $(f = g)$ is equivalent to $(\forall x \in X)(fx = gx)$, which is a $\Delta_0$-formula and hence classified by Proposition 7.6. Thus, strong-generation of 1 implies the result for atomic formulas.

The cases of $\top$ and $\wedge$ are obvious, as usual. If $U \models \bot$, then $U$ is initial, and thus (since 1 is nonempty) it has no global elements $1 \to U$; thus $S \models \bot$ for every such global element. Conversely, if $U$ has no global elements, then the map $0 \to U$ is a bijection on global elements, hence an isomorphism since 1 is a strong generator.
If $U \vdash (\varphi \lor \psi)$, then $U = V \cup W$ with $V \vdash \varphi$ and $W \vdash \psi$; hence by the inductive hypotheses, $S \models u^*\varphi$ for all $1 \to V$ and $S \models w^*\psi$ for all $1 \to W$. But since 1 is indecomposable, every $1 \to U$ factors through either $V$ or $W$, hence either $S \models u^*\varphi$ or $S \models w^*\psi$, so $S \models u^*(\varphi \lor \psi)$. Conversely, suppose $S \models u^*(\varphi \lor \psi)$ for each $1 \to U$, and consider the objects $U$ and $U + U$ over $U$, which pull back to $1$ and $1 + 1$ along any $1 \to U$. Applying collection of functions to the statement "either $u^*\varphi$ and $1 \to 1 + 1$ is the first injection, or $u^*\psi$ and $1 \to 1 + 1$ is the second injection", we obtain a regular epimorphism $V \twoheadrightarrow U$ and a suitable map $V \rightarrow V + V$ over $V$. This map decomposes $V$ as a coproduct $W + X$, and by the inductive hypothesis we have $W \models p^*\varphi$ and $X \models p^*\psi$; hence $U \models (\varphi \lor \psi)$.

If $U \models (\varphi \Rightarrow \psi)$, then for any $1 \to U$, if $S \models u^*\varphi$, then by the inductive hypothesis $1 \models u^*\varphi$ and so $1 \models u^*\psi$, hence $S \models u^*\psi$ by the inductive hypothesis. Thus $S \models u^*(\varphi \Rightarrow \psi)$. Conversely, if $S \models u^*(\varphi \Rightarrow \psi)$ for all $1 \to U$, then for any $V \to U$ with $V \models p^*\varphi$, for any $1 \to V$ we have $1 \models (pv)^*\varphi$, hence (by the inductive hypothesis) $S \models (pv)^*\varphi$, so $S \models (pv)^*\psi$, and thus $1 \models (pv)^*\psi$. By the inductive hypothesis, we have $V \models p^*\psi$; hence $U \models (\varphi \Rightarrow \psi)$.

If $U \models (\exists X)\varphi(X)$, then we have a regular epimorphism $V \twoheadrightarrow U$ and an $A \in S/V$ with $V \models \varphi(A)$. But for any $1 \to U$, by projectivity of 1 there is a $1 \to V$ with $pv = u$, and $1 \models v^*p^*\varphi(u^*A)$, hence $S \models (pv)^*\varphi(v^*A)$, and thus $S \models (\exists X)(pv)^*\varphi(X)$. Conversely, if $S \models (\exists X)u^*\varphi(X)$ for each $1 \to U$, then by collection of sets, there exists a regular epimorphism $V \twoheadrightarrow U$ and an $A \in S/V$ such that $S \models (pv)^*\varphi(v^*A)$ for all $1 \to V$, whence $V \models p^*\varphi(A)$ by the inductive hypothesis, and so $U \models (\exists X)\varphi(X)$. Existential quantification of morphisms is analogous.

Finally, if $U \models (\forall X)\varphi(X)$, then for any $1 \to U$ and $A \in S$ we have $1 \models u^*\varphi(A)$, hence $S \models u^*\varphi(A)$ by the inductive hypothesis, and so $S \models (\forall X)u^*\varphi(X)$. Conversely, if $S \models (\forall X)u^*\varphi(X)$ for all $1 \to U$, then for any $V \to U$ and $B \in S/V$, we have $S \models (pv)^*\varphi(v^*B)$ for any $1 \to V$, hence $V \models p^*\varphi(B)$ by the inductive hypothesis; hence $U \models (\forall X)\varphi(X)$. \qed

**Corollary 7.18.** The theory of a constructively well-pointed Heyting pretopos satisfying collection is complete for stack semantics over Heyting pretoposes. More precisely, if a statement is valid in the stack semantics of any Heyting pretopos (in any intuitionistic metatheory), then it can be proved from the axioms of a constructively well-pointed Heyting pretopos satisfying collection.

**Proof.** By standard completeness theorems for intuitionistic logic, it suffices to show that if $1 \models_S \varphi$ for any Heyting pretopos $S$ (in any intuitionistic metatheory), then $\textbf{Set} \models \varphi$ for any constructively well-pointed Heyting pretopos $\textbf{Set}$ that satisfies collection. However, for any such $\textbf{Set}$, [[Theorem 7.17]] implies that $\textbf{Set} \models \varphi$ if and only if $1 \models_{\textbf{Set}} \varphi$, and the latter is true by assumption. \qed

We now turn to our promised characterization of the axiom of separation in the stack semantics. First we need to know that the stack semantics is “local” and “idempotent”.

**Lemma 7.19.** For any $V \twoheadrightarrow U$ and any sentence $\varphi$ over $V$ in $S$, we have $V \models_S \varphi$  

if and only if $(V \twoheadrightarrow U) \models_{S/U} \varphi$. 

Proof. Straightforward from the definition, since unions and regular epis in $S/U$ are created in $S$. □

Lemma 7.20. For any sentence $\varphi$ over $V$ and any $V \xrightarrow{p} U$, we have $U \models V \models \varphi^\perp$ if and only if $V \models \varphi$.

Proof. By Lemma 7.19 it suffices to assume that $U = 1$. And since the stack semantics of $S$ satisfies collection, by Theorem 7.17 we have

\[(7.21) \quad 1 \models V \models \varphi \text{ if and only if } v^* \varphi \text{ holds for all } 1 \xrightarrow{w} V^\perp\]

In the "only if" direction, $(7.21)$ says that if $W$ is such that for any $Z \xrightarrow{q} W$ and $Z \xrightarrow{v} V$ we have $Z \models v^* \varphi$, then $W \models W*V \models W^* \varphi^\perp$. In particular, for $W = 1$, this says that if for any $Z \xrightarrow{v} V$ we have $Z \models v^* \varphi$, then $1 \models V \models \varphi^\perp$. But this hypothesis is satisfied as soon as $V \models \varphi$, so $V \models \varphi$ implies $1 \models V \models \varphi^\perp$.

In the "if" direction, $(7.21)$ says that for any $W \xrightarrow{v} V$, if $W \models \gamma^* V^\perp \models W^* \varphi^\perp$, then $W \models v^* \varphi$. In particular, for $W = V$ and $v = 1_V$, it says that if $V \models \gamma^* V \models V^* \varphi^\perp$, then $V \models \varphi$. But if $1 \models V \models \varphi^\perp$, then also $V \models \gamma^* V \models V^* \varphi^\perp$ by pullback, so $1 \models V \models \varphi^\perp$ implies $V \models \varphi$ as desired. □

Recall that we define $S$ to be autological if all sentences over all objects of $S$ are classified in the sense of Definition 7.3.

Theorem 7.22. Let $S$ be a Heyting pretopos; the following are equivalent.

(i) $S$ is autological.
(ii) $1 \models \gamma S$ is autological$^\perp$.
(iii) $1 \models \gamma S$ satisfies the axiom of separation$^\perp$.
(iv) (If $S$ is constructively well-pointed) $S$ satisfies separation and collection.

Proof. Lemma 7.20 implies that autology is a pullback-stable universal property, so Theorem 7.11 implies that $(i) \Rightarrow (ii)$. And by Theorem 7.17 the stack semantics is always constructively well-pointed and satisfies collection, so if we show that $(i) \Rightarrow (iv)$ then internalizing the same argument will show $(ii) \Rightarrow (iii)$.

Suppose $S$ is constructively well-pointed and autological: we show it satisfies Remark 7.17(ii). Note that the inductive proof of Remark 7.17(ii) only used collection in two cases, $\forall$ and $\exists$, and only in the $(\models)$ or $(\models)$ direction. Thus it will suffice to prove Remark 7.17(ii) for formulas of the form $(\varphi \lor \psi)$, $(\exists x) \varphi(x)$, or $(\exists f : X \rightarrow Y) \varphi(f)$, in each case assuming inductively that Remark 7.17(ii) holds for the constituent formulas.

Note first that all of these statements are obvious if $U = 1$. In the case of $\forall$, if $S \models (\varphi \lor \psi)$, then $S \models \varphi$ or $S \models \psi$, whence by induction $1 \models \varphi$ or $1 \models \psi$, and thus $1 \models (\varphi \lor \psi)$ via one of the decompositions $1 = 1 \cup 0$ or $1 = 0 \cup 1$. Similarly, in the second case, if $S \models (\exists x) \varphi(x)$, then there is some $A \in S$ with $S \models \varphi(A)$, whence by induction $1 \models \varphi(A)$, and so $1 \models (\exists x) \varphi(x)$ via the cover $1 \rightarrow 1$. Quantification over arrows is identical.

Now if $\chi$ is one of the formulas in question over a general $U$, by autology we can form $\llbracket \chi \rrbracket \rightarrow U$. By hypothesis, for any $1 \xrightarrow{u} U$ we have $S \models u^* \chi$, so the above remark shows that $1 \models u^* \chi$, whence $u$ factors through $\llbracket \chi \rrbracket$. But 1 is a strong generator, so $\llbracket \chi \rrbracket$ is all of $U$ and thus $U \models \chi$, as desired.

Thus, by Theorem 7.17 $S$ satisfies collection, so it remains to prove separation. Suppose given $U$ and $\varphi(u)$, and let $\psi \equiv U^* \varphi(\Delta_U)$. Then $\psi$ is a sentence over $U$ such that $u^* \psi$ is equivalent to $\varphi(u)$ for any $1 \xrightarrow{u} U$. Now consider $\llbracket \psi \rrbracket$. Then for
any $1 \overset{u}{\to} U$, by [Theorem 7.17] $S \vdash \varphi(u)$ is equivalent to $1 \models \varphi(u)$, and thus to $1 \models u^*\psi$, which is true iff $u$ factors through $[\psi]$; thus $[\psi]$ satisfies the conclusion of the separation axiom. Thus we have $[1] = \{[\psi]\}$.  

Now assume $S$ is well-pointed and satisfies separation and collection. For any sentence $\varphi$ over $U$, we apply separation to find a subobject $S \hookrightarrow U$ such that $1 \overset{u}{\to} U$ factors through $S$ if and only if $S \models u^*\varphi$. By [Theorem 7.17] $S \models \varphi$. And if $V \overset{p}{\to} U$ is such that $V \models p^*\varphi$, then for any $1 \overset{v}{\to} V$, $pv$ factors through $S$, which (since $S$ is well-pointed) implies that $V$ factors through $S$. Thus, $S = [\varphi]$, so $S$ is autological.

In particular, the implication (iv)$\Rightarrow$(i) of [Theorem 7.22] shows that if $V$ satisfies IZF, then $\mathbf{Set}(V)$ is autological, giving us our first examples of autological topoi.

We now present a quick proof that any complete topos over IZF (including any Grothendieck topos) is autological. Fix a model $V$ of IZF. As usual in material set theory, by a large category we mean a category whose objects and arrows form proper classes in the sense of $V$, i.e. consist of the sets in $V$ satisfying some first-order formulas. Recall that a topos is well-powered if and only if it is locally small, and that if it is complete it is also cocomplete. Note that any cocomplete topos also has a NNO, namely the countable copower of 1. The examples of most interest are of course Grothendieck toposes, i.e. categories of sheaves on a small site.

**Lemma 7.23.** If $S$ is a complete and well-powered topos over a model of IZF and $U = \bigcup_{i \in I} V_i$ in $S$ with $V_i \models \varphi$ for all $i \in I$, then $U \models \varphi$.

**Proof.** This is, of course, an extension of [Lemma 7.3(V)] to the infinitary case, but the proof requires the use of collection in $V$. The cases of atomic formulas, $\top$, $\bot$, and $\wedge$ are again easy, while $\vee$ follows from distributivity of unions over themselves, and $\Rightarrow$ and $\forall$ follow from pullback-stability of unions.

For $\exists$, if we have $V_i \models (\exists X)\varphi(X)$ for all $i \in I$, then for each $i$ there is an epimorphism $W \to V_i$ and an $A \in S/W$ such that $W \models \varphi(A)$. By collection in $V$, we have a set $Q$ such that for every $i \in I$ there is a quadruple $(i,W,p,A) \in Q$ such that $p: W \to V_i$ is an epimorphism, $A \in S/W$, and $W \models \varphi(A)$. Let $B = \prod_{(i,W,p,A) \in Q} A$ and $Z = \prod_{(i,W,p,A) \in Q} W$, which exist since $S$ is cocomplete. We then have an induced epimorphism $q: Z \to U$, and since $Z = \bigcup_{(i,W,p,A) \in Q} W$ and $B$ pulls back over each $W$ to the corresponding $A$, the inductive hypothesis implies that $Z \models \varphi(B)$. Therefore, $U \models (\exists X)\varphi(X)$. Existential quantification over arrows is analogous. □

**Theorem 7.24.** Any complete and well-powered topos over a model of IZF is autological.

**Proof.** Let $S$ be such a topos and $\varphi$ a sentence over $U$ in $S$. Using separation in $V$ and well-poweredness of $S$, let $K$ be a set of subobjects of $U$ representing each isomorphism class $V \to U$ such that $V \models \varphi$. Since $S$ is complete, $K$ has a union; call it $S$. By [Lemma 7.23] $S \models \varphi$. And given any $W \overset{p}{\to} U$ such that $W \models \varphi$, if $R$ is the image of $p$, then $R \models \varphi$, so $R \cong V$ for some $V \in K$; hence $R \subseteq S$ and thus $p$ factors through $S$. Thus, $S$ classifies $\varphi$, so $S$ is autological. □

**Remark 7.25.** This implies that in a Grothendieck topos of sheaves on some site, the definition of the stack semantics can be rephrased in terms of a forcing relation.
defined only over the site, just as for the usual internal logic. We omit the proof, which is very much like those of Theorems 7.17 and 10.8.

We will see another way of proving autology in §10 the category of small objects in any category of classes which satisfies separation is autological. In [Shul] we will prove directly that autology is preserved by many other topos-theoretic constructions.

Remark 7.26. We have not mentioned the interpretation of fullness, induction, or well-founded induction in the stack semantics. Since these do not express simple universal properties, their stack-semantics versions are not generally equivalent to their “external” versions. One can, of course, write out explicitly in terms of $S$ what each of these axioms means in the stack semantics, and the result will be the appropriate version of that axiom for a non-well-pointed category. For example, the assertion that $1 \models "S \text{ satisfies fullness}"$ is equivalent to the categorical version of fullness given in [vdBDM07, vdBM07] (minus their ‘smallness’ conditions).

8. Material set theories in the stack semantics

We can now combine §§6 and 7 in the expected way: from any Heyting pretopos $S$, we obtain an interpretation of material set theory by interpreting the theory of $S$ in the stack semantics of $S$. In particular, we can construct interpretations of material set theory by starting with a model $V$, passing to the category $\mathbf{Set}(V)$, performing some category-theoretic construction on $\mathbf{Set}(V)$ (which generally destroys well-pointedness), then constructing the model of $6$ in the stack semantics of the resulting category.

There are several important questions to ask about this construction. The first is, which of the relevant category-theoretic properties of $\mathbf{Set}(V)$ are preserved by the constructions in question, and which others can be “forced” to hold or fail by a clever choice of such a construction? To study this in generality is not our aim in this paper, but we remark briefly on what is known.

(i) The most-studied case is that of elementary topoi, which are well-known to be preserved by all sorts of constructions, such as sheaves (on internal sites), coalgebras for left-exact comonads, Artin gluing, realizability, and filterquotients. Natural numbers objects are also usually preserved.

(ii) Realizability and sheaf constructions on $\Pi$-pretoposes (often also satisfying some additional axioms, such as fullness) are studied in [MP00, MP02, vdB05a, AGLW09, vdB05b, vdBM07, vdBM08, vdBM09], among other places (see also [AST]).

(iii) Of course, Booleanness and the axiom of choice are not usually preserved by any of these constructions.

(iv) On the other hand, we have seen that collection is always satisfied in the stack semantics.

(v) We will prove in [Shul] that the axiom of separation, in its stack-semantics form (i.e. autology as defined in §10) is also preserved by the constructions of sheaves, coalgebras, gluing, realizability, and filterquotients.

(vi) We are not aware of any results regarding the preservation of full induction or well-founded induction.

The second important question to ask is, how does this approach compare to existing category-theoretic constructions of interpretations of material set theory?
As mentioned in the introduction, such interpretations fall into two groups: the older approach of Fourman [Fou80] and Hayashi [Hay81], and the newer approach of algebraic set theory begun by Joyal and Moerdijk [JM95]. The next two sections are devoted to comparing our models with these; under general hypotheses they turn out to be equivalent.

The third important question to ask is, how does this approach compare with well-known constructions of models that remain entirely within the world of material set theory? We will not study this question here, but a partial answer to it can be extracted from our answers to the second question.

For instance, in [BS89, BS92] it is shown, using a construction of Freyd [Fre87], that the Fourman-Hayashi interpretation in any Boolean Grothendieck topos (defined over a model of ZFC) can be identified with a certain symmetric submodel of a Boolean-valued model of ZF. (Fourman [Fou80] already observed this in particular cases.) It then follows that the stack-semantics model in any such Grothendieck topos can also be so identified. Hayashi [Hay81] also asserted that his interpretation in a topos of sheaves on a complete Heyting algebra agrees with the corresponding Heyting-valued model of IZF, as in [Gra79]. Since Freyd’s construction also applies to non-Boolean topoi, it seems likely that the Fourman-Hayashi interpretation in an arbitrary Grothendieck topos can similarly be compared to symmetric submodels of Heyting-valued models as studied in [TT81], but to my knowledge this has not been written down. Likewise, in [KGrO05, vdB08] it is shown that the model of material set theory constructed via algebraic set theory from a realizability topos agrees with McCarty’s realizability model of IZF [Mcc84], and thus so does the stack-semantics model in a realizability topos. In principle, it should be possible to give a direct comparison between stack-semantics models and material-set-theoretic models, but we will not do so here.

9. The cumulative hierarchy in a complete topos

In this section we will show that in good cases, our interpretation of material set theory in the stack semantics is equivalent to the interpretations of Fourman [Fou80] and Hayashi [Hay81]. In fact, Hayashi defines his interpretation by way of a Kripke-Joyal semantics, which is thus quite evidently the same as the stack-semantics approach. Fourman uses instead representing subobjects, which he can only construct in a complete and well-powered topos. Additionally, both focus on a version of the “von Neumann hierarchy” constructed internally to the topos, which our approach shows to be unnecessary.

Let us begin by reformulating the stack-semantics interpretation of material set theory in terms of elements rather than slice categories. Let $S$ be a topos with a NNO, let $A$ be a well-founded extensional graph in $S$, and let $x: U \to A$ be a morphism. Let $\mathfrak{r}$ denote the subobject of $U^*A$ such that in the internal logic of $S/U$, $\mathfrak{r}$ contains precisely those $y \in A$ which admit a path to $x$.

Lemma 9.1. Let $x$ and $\mathfrak{r}$ be as above, and similarly $y: U \to A$ and $\overline{y}$.

(i) $U \models \mathfrak{r}$ is a well-founded extensional APG$^\approx$.
(ii) $U \models (\mathfrak{r} \equiv \overline{y})$ if and only if $x = y$.
(iii) $U \models (\mathfrak{r} \equiv \overline{y})$ if and only if $x \prec y$.
(iv) If $f: A \to B$ is a simulation, then $U \models (\mathfrak{r} \equiv \overline{f x})$. 

Furthermore, every well-founded extensional APG in $S/U$ is isomorphic to one of the form $\mathcal{T}$, for some well-founded extensional graph $A$ in $S$ and some $x: U \to A$.

Proof. Since $U^*: S \to S/U$ is a logical functor, and well-foundedness and extensionality are $\Delta_0$-properties in a topos, $U \models \forall U^* A$ is a well-founded extensional graph. The proof of (i) is then obvious by arguing internally, while (iv) follows from Lemma 6.18. The “if” directions of (ii) and (iii) are easy, while the “only if” directions can be shown by well-founded induction.

Finally, suppose $X$ is a well-founded extensional APG in $S/U$. Arguing in the stack semantics of $S$, consider $X$ as a graph in $S$ itself and form $X + 1$, with $\prec$ inherited from $X$ and with $\ast_u \prec \ast$, where for each $u \in U$, $\ast_u$ denotes the root of the fiber $u^* X$. Then $X + 1$ satisfies the hypotheses of Lemma 6.24 with $n = 1$; let $A$ be its extensional quotient. The section $U \to X$ (which equips $X$ with its root, as an APG in $S/U$) induces a map $x: U \to A$ and it is easy to verify that $X \cong \mathcal{T}$. □

Remark 9.2. Since isomorphisms between well-founded extensional graphs are unique when they exist, $U \models (\mathcal{T} \cong \mathcal{F})$ if and only if in fact $\mathcal{T} \cong \mathcal{F}$ in $S/U$, and similarly $U \models (\mathcal{F} \in \mathcal{G})$ if and only if there exists a $z: U \to \mathcal{G}$ such that $\mathcal{T} \cong \mathcal{G}/z$ in $S/U$.

Remark 9.3. Combining Lemma 9.1(ii) with (iv), we see that for $x: U \to A$ and $y: U \to B$, we have $\mathcal{T} \cong \mathcal{F}$ if and only if there is an epimorphism $C \to A$ and $g: B \to C$, and similarly for membership. Note that for any such pair $A, B$ there exists such a $C$ with $f$ and $g$, such as the extensional quotient of $A + B + 1$.

Theorem 9.4. The stack semantics interpretation of material set theory in a topos can equivalently be described as follows. Instead of well-founded extensional APGs in $S/U$, the parameters at stage $U$ are morphisms $x: U \to A$, where $A$ is any well-founded extensional graph in $S$. The forcing semantics of formulas $\varphi$ is defined inductively by:

- For $x: U \to A$ and $y: U \to B$, we say $U \models (x = y)$ if $f x = g y$ for some (hence any) simulations $f: A \to C$ and $g: B \to C$.
- Likewise, $U \models (x \prec y)$ if $f x \prec g y$ for $f$ and $g$ as above.
- The connectives are interpreted in the usual Kripke-Joyal way.
- We say $U \models (\exists x) \varphi(x)$ if there is an epimorphism $V \to U$ and an $x: V \to A$, for some well-founded extensional graph $A$, such that $V \models \varphi(x)$.
- Likewise, $U \models (\forall x) \varphi(x)$ if $V \models \varphi(x)$ for any $V \to U$, any well-founded extensional graph $A$, and any map $x: V \to A$.

Proof. Straightforward induction on formulas, using Lemma 9.1. □

This is exactly the interpretation of Hayashi [Hay81], except that he considers only morphisms $x: U \to A$ where $A$ belongs to some specified class $\mathcal{W}$ of well-founded extensional graphs. Of course, some closure conditions on $\mathcal{W}$ are then necessary to ensure that the axioms of material set theory are satisfied. Hayashi assumes that:

- For any $A, B \in \mathcal{W}$ there is a $C \in \mathcal{W}$ with simulations $A \to C$ and $B \to C$, and
- For any $A \in \mathcal{W}$ there is a $B \in \mathcal{W}$ with a simulation $PA \to B$.

Here $PA$ is the power-object of $A$, equipped with the well-founded extensional relation $\prec$ defined internally as follows: for subsets $H, K \subseteq A$, we have $H \prec K$ iff there exists $x \in K$ such that $H = \{ y \mid y \prec x \}$. Hayashi calls a $\mathcal{W}$ satisfying
these conditions a pre-universe, and proves that the above semantics valued in any pre-universe validates the core axioms together with power sets, foundation, and transitive closures, plus infinity if there is a NNO, and $\Delta_0$-classical logic if the topos is Boolean. We thus record:

**Theorem 9.5.** Let $S$ be a topos and let $\mathcal{U}$ be the maximal pre-universe consisting of all well-founded extensional graphs. Then the interpretation of material set theory in the stack semantics of $S$ is identical to Hayashi’s interpretation valued in $\mathcal{U}$. □

Hayashi also defines a universe to be a pre-universe such that

- For any (small) set $S \subseteq \mathcal{U}$, there is a $B \in \mathcal{U}$ such that for all $A \in S$ there exists a simulation $A \to B$.

Of course, this is no longer an elementary property; it only makes sense when working in some ambient set theory. Note that the class of all well-founded extensional graphs need not, in general, be a universe. It is a universe, however, if $S$ is cocomplete relative to the external set theory, since then $B$ can be obtained as the extensional quotient of $\coprod_{A \in S} A + 1$.

Hayashi proves that the axiom of collection is satisfied if $S$ is well-powered and $\mathcal{U}$ is a universe. Our approach via the stack semantics shows that in fact, these hypotheses are unnecessary: the axiom of collection is always satisfied, at least for the maximal pre-universe. Hayashi also claims that the axiom of separation is valid as soon as the subobject lattices of $S$ are complete Heyting algebras, even if $\mathcal{U}$ is only a pre-universe, but I don’t think this can be correct, since it would imply the relative consistency of collection assuming separation.

The one remaining difference between Hayashi’s approach and ours is that rather than the “maximal” pre-universe consisting of all well-founded extensional graphs, he is more interested in the “minimal” ones obtained by transfinitely iterating the power set operation, mimicking the construction of the von Neumann hierarchy in material set theory. This is also the approach taken by Fourman.

In order to make sense of this transfinite iteration, we of course need an external set theory containing ordinals along which we can perform induction. For convenience and familiarity, we take this to be a material set theory. Thus, for the rest of this section, let $V$ be a model of IZF, with $\text{Set} = \text{Set}(V)$ its category of sets. (Fourman and Hayashi assumed the metatheory to be ZFC, but this is unnecessary.)

Let $S$ be a complete and well-powered topos over $V$. We will describe the cumulative hierarchy in $S$, as constructed by Fourman and Hayashi. Recall that a (von Neumann) ordinal in $V$ is a transitive set on which $\in$ is a transitive relation (it is automatically well-founded, by the axiom of foundation). Since set-induction is valid in IZF, we can construct objects of $S$ by transfinite recursion on ordinals. Thus we can inductively define a well-founded extensional graph $V_\alpha$ for each ordinal $\alpha$ of $V$ such that $V_\alpha$ is the extensional quotient of $\coprod_{\beta < \alpha} PV_\beta$. (The well-founded relation on $PV_\beta$ is defined as above, by setting $H \prec K$ if there exists an $x \in K$ such that $H = \{ y \mid y \prec x \}$.)

**Remark 9.6.** Fourman and Hayashi defined the hierarchy $V_\alpha$ in a more traditional way, by conditioning on whether $\alpha$ is zero, a successor, or a limit:

$$V_0 = 0 \quad V_{\alpha+1} = PV_\alpha \quad V_\alpha = \text{colim}_{\beta < \alpha} V_\beta \quad (\alpha \text{ a limit})$$
Intuitionistically, this classification of ordinals is no longer valid. We could still try to define $V_\alpha = \text{colim}_{\beta < \alpha} PV_\beta$, but since intuitionistic ordinals are not necessarily linearly ordered, in general this colimit need not be extensional. However, in the classical case, all three definitions are equivalent.

Evidently, the class $\{V_\alpha\}$ is a universe in $S$, in Hayashi’s sense. Thus, it induces a class of well-founded extensional APGs in all slices of $S$, namely those of the form $\tau$ for some $\tau: U \to V_\alpha$. It is natural to ask whether this includes all well-founded extensional APGs, and thus whether Hayashi’s interpretation valued in $U = \{V_\alpha\}$ is the same as the stack-semantics interpretation. The answer is yes.

**Lemma 9.7.** In any complete and well-powered topos $S$, every (internally) well-founded extensional APG in $S/U$ is of the form $\tau$ for some ordinal $\alpha$ and some $x: U \to V_\alpha$.

**Proof.** Since every slice topos of a complete and well-powered topos is also such, and pullback functors $U^*$ preserve the construction of the $V_\alpha$ hierarchy, we may assume that $U = 1$. Thus, let $X$ be a well-founded extensional APG in $S$; the proof is a modified version of the standard recursion theorem for well-founded relations.

For a fixed ordinal $\alpha$, define an $\alpha$-attempt to be a partial function $X \to V_\alpha$ which is a simulation and whose domain is an initial segment of $X$. Using extensionality of $V_\alpha$, we can prove by induction on $X$ in the internal logic of $S$ that any two $\alpha$-attempts agree on their common domain, and that the union of any family of $\alpha$-attempts is an $\alpha$-attempt. Therefore, there is a unique largest $\alpha$-attempt $f_\alpha: X \to V_\alpha$. Furthermore, for any $\beta \geq \alpha$ there is a unique simulation $i_\alpha^\beta: V_\alpha \hookrightarrow V_\beta$, and we must have $i_\alpha^\beta f_\alpha = f_\beta$.

We now show that there exists an $\alpha$ such that $f_\alpha$ is defined on all of $X$. Consider the set $D$ of subobjects $S \hookrightarrow X$ such that $S$ is the domain of $f_\alpha$ for some $\alpha$; this exists since $S$ is well-powered and $V$ satisfies separation. By collection in $V$, there is a set $A$ of ordinals such that for any $S \in D$, $S = \text{dom}(f_\alpha)$ for some $\alpha \in A$.

Let $\beta = \bigcup_{\alpha \in A} \alpha$ and $\gamma = \beta^+$; we claim that $f_\gamma$ is a total function. Let $S_\gamma$ be its domain. First, note that by the construction of $A$, there exists an $\alpha \in A$ such that $S_\gamma$ is the domain of $f_\alpha$. Since the domains of $f_\alpha$ are increasing in $\alpha$, it follows that $S_\gamma$ is the domain of $f_\beta$ as well.

We will prove that $S_\gamma$ is an inductive subset of $X$, and hence must be all of $X$. Applying the Kripke-Joyal semantics, suppose given some $U$ and some $q: U \to X$ such that for any $p: U' \to U$ and any $q': U' \to X$ such that $q' \preceq q$, $q'$ factors through $S_\gamma$; we must show that $q$ factors through $S_\gamma$.

Let $T \subseteq U^* X$ be the subset defined in the internal logic by $\{ x \in X \mid x \prec q \}$. Then by assumption, $T$ factors through $U^* S_\gamma$, and therefore the domain of $U^* f_\beta$ contains $T$. Then $T$ is classified by a map $t: U \to PX$. Now the partial function $f_\beta$ determines a map $P f_\beta: PX \to PV_\beta$ by taking images, so by composition we have a map $P f_\beta \circ t: U \to PV_\beta$. But $PV_\beta \cong V_\gamma$, so we have a map $r: U \to V_\gamma$.

Let $\overline{T} \subseteq X$ be the image of $q: U \to X$. We can prove in the internal logic that if $q(u) = q(u')$ for some $u, u' \in U$, then $r(u) = r(u')$; hence $r$ descends to $\tau: \overline{T} \to V_\gamma$. We can also show, again internally, that if $x \in S_\gamma$ and $x = q(u)$, then $\tau(x) = f_\gamma(x)$, since both are equal to $\{ f_\beta(y) \mid y \prec x \}$. Therefore, if we let $\overline{S} = S_\gamma \cup \overline{T}$, then $f_\gamma$ and $\tau$ induce a map $q: \overline{S} \to V_\gamma$. Finally, we can show that $g$ is a simulation, hence its domain is a subset of $S_\gamma$; thus $\overline{T} \subseteq S_\gamma$, i.e. $q$ factors through $S_\gamma$. Hence $S_\gamma$ is inductive, and thus all of $X$, so $f_\alpha$ is a total function. $\square$
Therefore, we have:

**Theorem 9.8.** Let $S$ be a complete and well-powered topos, and let $\mathcal{U}$ be the above von Neumann universe in $S$. Then the interpretation of material set theory in the stack semantics of $S$ is identical to Hayashi’s interpretation valued in $\mathcal{U}$. □

Finally, we relate this to Fourman’s approach \cite{Fou80}. Instead of a forcing semantics, he works with representing subobjects. Thus, we still consider the parameters to be morphisms $x: U \rightarrow A$ for some well-founded extensional graph $A$ in a pre-universe $\mathcal{U}$, but now instead of a forcing relation we define, for each formula $\varphi$ at stage $U$, a subobject $[\varphi] \rightarrow U$, as follows.

- If $x: U \rightarrow A$ and $y: U \rightarrow B$, then $[x = y]$ is the equalizer of $fx$ and $gy$ where $f: A \rightarrow C$ and $g: B \rightarrow C$ are a pair of simulations.
- Similarly, $[x \in y]$ is the pullback of $C \times$ along $(fx, gy)$.
- The connectives $\bot, \top, \land, \lor, \Rightarrow$, and $\neg$ are interpreted using the Heyting algebra connectives in $\text{Sub}(U)$, as usual.
- Given a formula $\varphi(x)$ at stage $U$ with one free variable $x$, for each well-founded extensional graph $A$ consider the image of
  $$[\varphi(\pi_2)] \rightarrow U \times A \xrightarrow{\pi_1} U.$$  

  This is a subobject of $U$, call it $[(\exists x \in A)\varphi(x)]$. Since $S$ is well-powered, we can apply separation in $V$ to obtain the set of all subobjects of $U$ which are of the form $[(\exists x \in A)\varphi(x)]$ for some $A \in \mathcal{U}$. Since $S$ is cocomplete, we can then take the union of this set; call it $[(\exists x)\varphi(x)]$.

- Universal quantification is similar, using the dual image $\forall_x[(\varphi(\pi_2)]$ instead of the image $\exists_x[(\varphi(\pi_2)]$, and an intersection instead of a union.

These subobjects are essentially the same as those defined by Fourman, although he also considered only the von Neumann universe. Thus it remains only to check that they do, in fact, represent the forcing relation defined by Hayashi.

**Lemma 9.9.** Let $S$ be a complete and well-powered topos and $\mathcal{U}$ a pre-universe. Then for any formula $\varphi$ at stage $U$ and any map $W \xrightarrow{p} U$, we have $W \Vdash \varphi$ in Hayashi’s sense if and only if $p$ factors through $[\varphi]$ in Fourman’s sense.

**Proof.** Since $S$ is autological by \cite{Theorem 7.24}, the forcing relation of Hayashi is representable. Thus, it remains only to check that the subobjects defined by Fourman are the same as those produced by autology. This is evident for atomic formulas and connectives, so we consider only the quantifiers.

In Fourman’s construction of $[(\exists x)\varphi(x)]$, it is clear that $[(\exists x \in A)\varphi(x)] \Vdash (\exists x)\varphi(x)$ for each $A$, so by \cite{Lemma 7.23} we also have $[(\exists x)\varphi(x)] \Vdash (\exists x)\varphi(x)$. On the other hand, given $W \rightarrow U$ such that $W \Vdash (\exists x)\varphi(x)$, we have an epimorphism $Z \rightarrow W$ and a map $x: Z \rightarrow A$, for some $A \in \mathcal{U}$, such that $Z \Vdash \varphi(x)$. Thus, $Z \rightarrow U \times A$ factors through $[\varphi(\pi_2)]$, so $W \rightarrow U$ factors through $[(\exists x \in A)\varphi(x)]$, and hence also through $[(\exists x)\varphi(x)]$. Thus, Fourman’s subobject $[(\exists x)\varphi(x)]$ is in fact a representing object for $(\exists x)\varphi(x)$.

Universal quantification is even easier. If $W \Vdash (\forall x)\varphi(x)$, then for any $A \in \mathcal{U}$, $p: W \rightarrow U$ must factor through $[(\forall x \in A)\varphi(x)]$, whence it factors through their intersection $[(\forall x)\varphi(x)]$. Conversely, it suffices to show that $[(\forall x)\varphi(x)] \Vdash (\forall x)\varphi(x)$, but this follows since by definition, we have $[(\forall x)\varphi(x)] \Vdash (\forall x \in A)\varphi(x)$ for any $A \in \mathcal{U}$. □
Theorem 9.10. Let $S$ be a complete and well-powered topos, and let $\mathcal{U}$ be the above von Neumann universe. Then the interpretation of material set theory in the stack semantics of $S$ is identical to Fourman’s interpretation valued in $\mathcal{U}$. □

As remarked previously, it follows from known facts about the Fourman-Hayashi interpretation that the stack-semantics interpretation in Grothendieck toposes, such as sheaves on a locale or continuous group actions, can also be identified with the logic of standard material-set-theoretic constructions, such as Boolean- and Heyting-valued, permutation, and symmetric models. See [Fou80, BS89, BS92].

10. Categories of classes

We now turn to a comparison between the stack semantics and the methods of algebraic set theory, a field which was begun by [IM95] and has been carried forward by many others; see [AST] for a number of references. The idea of algebraic set theory is to consider, rather than a category whose objects represent sets, a larger category whose objects represent classes, equipped with a “notion of smallness” specifying which objects (and more generally which families) should be regarded as sets. In particular, we require that there is a “class of all sets.” We can then interpret “unbounded” quantifiers over all sets as bounded quantifiers in the internal logic of the category of classes.

There is not yet a universally accepted set of axioms for a category of classes, so we will at first restrict ourselves to a small core set of axioms.

Definition 10.1. A (representable) notion of smallness on a positive Heyting category $C$ consists of a distinguished map $E \xrightarrow{\pi} S$. Given $\pi$, a morphism $X \to U$ in $C$ is called small if there exist two pullback squares

$$
\begin{array}{ccc}
X & \xleftarrow{\theta_X} & E \\
\downarrow & & \downarrow \pi \\
U & \xleftarrow{p} & U'
\end{array}
$$

in which $U' \to U$ is a regular epimorphism. An object $X$ is called small if $X \to 1$ is a small map.

Observe that a morphism $X \to U$ is small if and only if $U \vdash \forall x \in S \text{ such that } X \cong x^*E$ in the stack semantics of $C$. We thus abbreviate this statement as $\forall X \text{ is small}$. Note also that a map $f: X \to Y$ in $C/U$ is small if and only if $U \vdash \forall y \in Y, \text{ the fiber } y^*X \text{ is small}$, which is precisely the intended intuition. Thus we abbreviate this statement as $\forall f \text{ is small}$. The possibility of this sensible “logical” interpretation depends, of course, on allowing passage to a cover in the definition of small maps, since we cannot expect classifying maps of small objects to be uniquely defined.

In the literature, it is more common to consider $C$ as equipped with a class of maps called “small” and then assert the existence of a representing map $\pi$ as an axiom. As observed in [MP02], however, defining the small maps from $\pi$ as above has the advantage that it immediately implies a few of the other common axioms for small maps, including:

- Small maps are closed under pullback.
• Small maps descend along regular epis, i.e. if the pullback of \( f : X \to U \) along a regular epi \( V \to U \) is small, so is \( f \).

• If \( X \to U \) and \( Y \to V \) are small maps, then so is \( X + Y \to U + V \).

Let \( S \) be the full subcategory of \( C \) determined by the small objects; we intend to define a translation from the stack semantics of \( S \) to the internal logic of \( C \). However, for \( S \) to have a well-behaved stack semantics, it must be itself a positive Heyting category. The following definition is closely related to those commonly assumed in the literature, but weaker (and thus more general) than most.

**Definition 10.2.** A category of classes is a positive Heyting category equipped with a notion of smallness which additionally satisfies the following axioms.

(i) \( 1 \models_C \text{"if } X \text{ is small, then a map } f : Y \to X \text{ is small if and only if } Y \text{ is small"} \).

(ii) \( 1 \models_C \text{"the full subcategory of small objects is closed under finite limits, finite coproducts, images, and dual images (and is therefore itself a positive Heyting category)"} \).

(iii) Every small map is exponentiable.

We have stated the first two conditions in terms of the stack semantics, since we believe it makes their real meaning clearer. Of course, if we work out what they mean explicitly in terms of \( C \), we see that they reduce to some well-known axioms for categories of classes. Specifically, the first is equivalent to:

- if \( g \) is small, then \( gf \) is small if and only if \( f \) is small.

Assuming this (and the representable definition of small maps as above), the second is equivalent to:

- For any \( U \), the full subcategory of \( C/U \) determined by the small maps is a positive Heyting category, and the inclusion functor preserves all the structure.

More explicitly, this is equivalent to the following axioms.

- All isomorphisms are small.
- \( 0 \) and \( 1 + 1 \) are small objects.
- If \( g : B \to C \) and \( gf : A \to C \) are small, then so is the composite \( \text{im}(f) \to B \to C \). (In particular, this follows if small maps are closed under quotients.)
- If \( m : A \to C \) is monic and small, and \( f : C \to D \) is small, then \( \forall_f(A) \to D \) is small.

Finally, if \( C \) is exact, then the third condition (exponentiability of small maps) is equivalent to

- \( 1 \models \text{"every small object is exponentiable"} \).

but if \( C \) is not exact, then [Theorem 7.11] does not apply, and so this latter property does not imply the actual existence of exponentials.

**Remark 10.3.** [Definition 10.2(i)] is sometimes called the “axiom of replacement,” but it actually has no connection to our axiom of replacement from §5. We prefer to think of it as saying that the notions of “small family” in \( S \) and \( C \) agree.

Now the exponentiability of the representing map \( \pi : E \to S \) implies that it determines an internal full subcategory of \( C \), i.e. an internal category whose object of objects is \( S_0 = S \) and whose object of morphisms is the exponential \( S_1 = (p_1^*E)p_2^*E \) in \( S_0 \times S_0 \). (See [Lol02, B2.3.5] and [JM95, §15].) Regarding this as “the category of sets,” we can then reason about it in the ordinary internal logic of \( C \). That is, for any \( U \in S \), if \( \varphi \) is a formula in the language of categories with object-parameters
given by maps $U \to S_0$ and arrow-parameters given by maps $U \to S_1$, then $\phi$ has a meaning in the internal first-order logic of $C$ and is classified by a subobject $[\phi] \hookrightarrow U$. Of course, in general $[\phi]$ may not be small.

Now suppose that $\phi$ is a formula over $U$ in $S$ in the sense of §7. Since each object-parameter of $\phi$ is a small map $X \to U$, there is a regular epi $U' \to U$ over which $X$ is a pullback of $\pi$, and similarly for arrow-parameters. Since $\phi$ has only finitely many parameters, we can find a single cover $U' \to U$ over which $X$ is a pullback of $\pi$, and thereby translate $\phi$ into a formula $\hat{\phi}$ at stage $U'$ of the internal language of $C$ about the internal category $S_1 \rightrightarrows S_0$.

However, without further axioms, there is little we can say about the relationship of $\phi$ to $\hat{\phi}$. Basically, the problem is that $S$ only knows about the small objects over $1$, but $\hat{\phi}$ knows about small maps over all objects of $C$. So we need to assume that $C$ is “generated by small objects” in a suitable sense.

**Definition 10.4.** A category of classes $C$ is well-generated by small objects if the following hold.

(i) The small objects are a strong generator: if $A \xrightarrow{f} B$ is a monomorphism in $C$ and every map $U \to B$ factors through $A$ when $U$ is small, then $f$ is an isomorphism.

(ii) The small objects are relatively projective: if $U$ is small and $A \xrightarrow{p} U$ is regular epi, there is a small $V$ and a regular epi $V \xrightarrow{q} U$ which factors through $p$.

(iii) The small objects are relatively indecomposable: if $U$ is small and $U = A \cup B$, there are small objects $V$ and $W$ such that $U = V \cup W$, $V \subseteq A$, and $W \subseteq B$.

(iv) The small objects are collectively nonempty: if $X$ is small and there is a map $X \to 0$, then $X \cong 0$.

**Remark 10.5.** Two of these conditions are well-known: (i) is called the axiom of small generators and (ii) is called the axiom of small covers (see [ABSS07b]). Note that (i) is always true if interpreted in the stack semantics, since all identities are small. By contrast, when (ii) is interpreted in the stack semantics of $C$, it becomes precisely the usual “collection axiom” of algebraic set theory (see [J M95]).

Condition (iii) is rarely stated explicitly, since it follows from (ii) if all complemented monics are small, which in turn follows from the common assumption that all diagonals are small. And of course, (iv) is always true. However, we have chosen to state all four axioms explicitly in order to complete the analogy with (constructive) well-pointedness.

**Remark 10.6.** If $S$ is small in some external set theory, then $C$ has a restricted Yoneda embedding $C \hookrightarrow \text{Sh}(S)$ into the category of sheaves for the coherent topology on $S$. In this case, $C$ is well-generated by small objects if and only if this functor is coherent and conservative. This should be compared with Remark 3.7, and likewise the proof of Theorem 10.8 should be compared with that of Theorem 7.17.

**Remark 10.6** suggests that $\text{Sh}(S)$ itself should be a “canonical” choice of a category of classes containing $S$, and in fact this is the case.

**Proposition 10.7.** For any small Heyting pretopos $S$, the category $\text{Sh}(S)$ of sheaves for the coherent topology on $S$ is a category of classes in which $S$ is a well-generating category of small objects.
**Sketch of proof.** The self-indexing of $S$ is a stack for its coherent topology, so if we rectify it to a strict functor $S^{pp} \to \mathbf{Cat}$, it will become an internal category in $\mathbf{Sh}(S)$ by the results of [Aw90] [V.1]. With $S$ the object of objects of this internal category and a suitable definition of $E$, this defines a representable notion of smallness. We then show that a map $Y \to X$ is small iff for any $U \in S$ and any map $y(U) \to X$, where $y: S \to \mathbf{Sh}(S)$ is the Yoneda embedding, the pullback $y(U) \times_Y Y$ is representable. In particular, $S$ is the category of small objects. It is then straightforward to verify the axioms. 

With this example in mind, the following theorem realizes our claim in the introduction that when $S$ is small, the stack semantics of $S$ can equivalently be defined as a fragment of the internal logic of $\mathbf{Sh}(S)$, which is “canonical” modulo the chosen strictification of the self-indexing. Recall the definition of a formula $\hat{\varphi}$ in the internal logic of $C$ from any formula $\varphi$ in $S$ in the language of a category.

**Theorem 10.8.** Let $C$ be a category of classes which is well-generated by small objects, and $S$ the full subcategory of small objects. Then for any $U \in S$, any formula $\varphi$ over $U$ in $S$, and any $p: V \to U$ in $S$, we have

$$V \models_S \varphi \iff V' \models_C \hat{\varphi}.$$  

Here $V' = V \times_U U'$, where $U' \to U$ is a cover chosen as in the definition of $\hat{\varphi}$.

Note that on the left, $\models$ refers to the stack semantics notion of forcing, while on the right it refers to the Kripke-Joyal semantics for the usual internal logic of $C$ (which is, of course, the same as the $\Delta_0$-fragment of the stack semantics of $C$, so there is no real ambiguity in the notation).

**Proof.** The proof is by induction on $\varphi$. The cases of atomic formulas, $\top$, $\bot$, and $\wedge$ are evident, so it remains to deal with $\lor$, $\Rightarrow$, $\exists$, and $\forall$. We adopt the convention that adding a prime symbol denotes passing to a regular-epi cover over which all parameters are represented by morphisms to $S_0$ or $S_1$.

**Disjunction:** if $V \models_S (\varphi \lor \psi)$, then we have $V = W_1 \cup W_2$ in $S$ with $W_1 \models_S \varphi$ and $W_2 \models_S \psi$. By the inductive hypothesis, $W'_1 \models_C \hat{\varphi}$ and $W'_2 \models_C \hat{\psi}$; thus $V' \models_C (\hat{\varphi} \lor \hat{\psi})$.

Conversely, if $V' \models_C (\hat{\varphi} \lor \hat{\psi})$, then $V' = A'_1 \cup A'_2$ with $A'_1 \models_C \hat{\varphi}$ and $A'_2 \models_C \hat{\psi}$. Let $A_i$ be the image of $A'_i$ in $V$, for $i = 1, 2$, so that $V = A_1 \cup A_2$. Since small objects are relatively indecomposable, we have $V = W_1 \cup W_2$ in $S$ with $W_1 \subseteq A_1$ and $W_2 \subseteq A_2$. Therefore, $W'_1 \models_C \hat{\varphi}$ and $W'_2 \models_C \hat{\psi}$, and so by the inductive hypothesis, $W_1 \models_S \varphi$ and $W_2 \models_S \psi$; hence $V \models_S (\varphi \lor \psi)$.

**Implication:** suppose first $V \models_S (\varphi \Rightarrow \psi)$ and let $A' \to V'$ be a map in $C$ such that $A' \models_C \hat{\varphi}$. Form the classifying subobject $[\hat{\psi}] \to A'$ in $C$. Then for any $W \to A'$ where $W \in S$, we have $W \models_S (\varphi \Rightarrow \psi)$ and $W' \models_C \hat{\psi}$, hence by the inductive hypothesis $W \models_S \varphi$; thus $W \models_S \psi$ and so (by the inductive hypothesis again) $W' \models_C \hat{\psi}$. Therefore, $W'$ factors through $[\hat{\psi}]$, and so $W$ factors through $[\hat{\psi}]$. Since the small objects are a strong generator, this implies $[\hat{\psi}] \cong A'$, and thus $A' \models_C \hat{\psi}$; hence $V' \models_C (\hat{\varphi} \Rightarrow \hat{\psi})$.

Conversely, suppose $V' \models_C (\hat{\varphi} \Rightarrow \hat{\psi})$ and let $W \to V$ be such that $W \models_S \varphi$. Then, by the inductive hypothesis, $W' \models_C \hat{\varphi}$, and so $W' \models_C \hat{\psi}$, hence $W \models_S \psi$. Thus, $V \models_S (\varphi \Rightarrow \psi)$. 


Existential quantification: suppose first that $V \models_S (\exists X)\varphi(X)$. Then we have $W \in S$, a regular epi $W \rightarrow V$, and an object $Y \in S/W$ such that $W \models_S \varphi(Y)$. Passing to a cover $W' \rightarrow W$ which represents $Y$ (by a map $y: W' \rightarrow S_0$, say) as well as all other parameters of $\varphi$, the inductive hypothesis implies that $W' \models_C \hat{\varphi}(y)$. Since $W' \rightarrow V'$ is regular epi, this implies $V' \models_C (\exists y)\hat{\varphi}(y)$.

Conversely, suppose $V' \models_C (\exists y)\hat{\varphi}(y)$. Then we have a cover $A' \rightarrow V'$ and a map $y: A' \rightarrow S_0$ with $A' \models_C \hat{\varphi}(y)$. Since small objects are relatively projective, there is a regular epi $W \rightarrow V$ in $S$ which factors through $A'$. Hence $W \models_C \hat{\varphi}(y)$, so by the inductive hypothesis, $W \models_S \varphi(Y)$ where $Y = y^*E$. Hence, $V \models_S (\exists X)\varphi(X)$.

Existential quantification over arrows is analogous.

Universal quantification: suppose first that $V' \models_C (\forall X)\hat{\varphi}(X)$. Then for any $W \rightarrow V$ in $S$ and $Y \in S/W$, we can pass to a further cover $W'' \rightarrow W' = W \times_V V'$ over which $Y$ is represented by some $y: W'' \rightarrow S_0$. Since $W'' \models_C (\forall Y)\hat{\varphi}(Y)$, we have $W'' \models_C \hat{\varphi}(y)$, and hence by the inductive hypothesis $W \models_S \varphi(Y)$. Thus, $V \models_S (\forall X)\varphi(X)$.

Conversely, suppose $V \models_S (\forall X)\varphi(X)$, let $A' \rightarrow V'$ be a map, take any $y: A' \rightarrow S_0$, and form the classifying subobject $[\hat{\varphi}(y)] \rightarrow A'$. Then for any map $W \rightarrow A'$ with $W \in S$, we have $W \models_S (\forall X)\varphi(X)$, and thus by the inductive hypothesis, $W \models_C \hat{\varphi}(y)$. Hence $W$ factors through $[\hat{\varphi}(y)]$, so since the small objects are a strong generator, we have $[\hat{\varphi}(y)] \cong A'$, and so $A' \models_C \hat{\varphi}(y)$. Therefore, $V' \models_C (\forall X)\hat{\varphi}(X)$.

Universal quantification over arrows is analogous. □

In particular, we have another way of producing autological (pre)topoi.

Definition 10.9. A category of classes satisfies the separation axiom if every monomorphism is small.

Corollary 10.10. If $C$ is a category of classes which is well-generated by small objects and which satisfies the separation axiom, then its category $S$ of small objects is autological.

Proof. By Theorem 10.8 any sentence $\varphi$ over $U \in S$ induces a sentence $\hat{\varphi}$ in the internal logic of $C$ at stage $U'$ for some regular epi $U' \rightarrow U$. Since small objects are relatively projective, we may as well assume that $U'$ is small. By the separation axiom, it follows that the usual classifying object $[\hat{\varphi}]$ in the internal logic of $C$ is small, and so by closure under images, the image of the composite $[\hat{\varphi}] \rightarrow U' \rightarrow U$ is a small subobject of $U$. The correspondence of Theorem 10.8 then directly implies that this image classifies $\varphi$ in the stack semantics of $S$, so $S$ is autological. □

There remains the question of how a given category $S$ can be embedded as the category of small objects in some category of classes $C$. We have seen that there is a “canonical” answer, namely $Sh(S)$, but this category has some problems. In particular, it fails to satisfy some desirable axioms of algebraic set theory, such as that any quotient of a small object is small, or that all diagonals are small. We didn’t need these axioms to prove Theorem 10.8 but we will need some of them below to construct and compare models of material set theory. This problem can be solved by restricting to the subcategory of sheaves whose diagonals are small; these are the ideals of $AF[06] AFW[06]$.

Another problem, however, which afflicts both sheaves and ideals, is that not all axioms satisfied by $S$ extend to corresponding properties of these categories of classes. Chief among the problematic axioms is separation; i.e. we have no converse
One solution to this problem is proposed in [ABSS07a, ABSS07b]: if $S$ can be equipped with an extra non-elementary structure called a superdirected structural system of inclusions (sdssi), then it can be embedded in a category of superideals which does satisfy separation. (The word “structural” in the term sdssi is backwards from our usage—an sdssi is a very non-structural notion, in that it distinguishes between isomorphic objects.) This includes both complete toposes and realizability toposes, providing a unified reason why such toposes can model IZF.

We prefer to view autology as the property characterizing those toposes which can model IZF, since unlike an sdssi, it is an elementary first-order property and can be phrased in a purely structural (i.e. category-theoretic) way. (By Corollary 10.10, any topos admitting an sdssi must be autological.) However, this would be most satisfying if any autological topos could be embedded in a category of classes satisfying the axiom of separation. One wants to consider the category of “first-order definable classes” over $S$, but it is nearly impossible to give this meaning in a purely structural way. However, we will show in [Shua] that there is a canonical answer if we generalize to a 2-category of classes—namely, we can consider the 2-category of stacks for the coherent coverage on $S$ which are “definable” from the stack semantics in a precise sense. This modification is quite reasonable from the point of view of structural set theory, since there a “proper class” is something quite different from a set: at best it is a groupoid, having only a notion of isomorphism between its objects, rather than a notion of equality.

We now turn to a comparison between models of material set theory in the stack semantics and in a category of classes. Since Theorem 10.8 shows that the underlying logics agree, it remains to compare the “material sets” in the two models. This is quite similar to the situation in §9: the main difference is that in a category of classes, the “material sets” are represented by morphisms into a single object $V$ (which is not, of course, small), rather than by morphisms into a family of objects making up a “pre-universe.”

(In fact, there is a formal relationship: a pre-universe in a topos $S$, in Hayashi’s sense, naturally forms a single object of the category of ideals, in which we can apply the methods of this section. This is the jumping-off point of [ABSS07a, ABSS07b], which not only constructs a category of ideals, but a corresponding forcing semantics that generalizes Hayashi’s.)

As remarked above, in order to define the object $V$ and perform this comparison, we need a little more structure on our category of classes. Since [Theorem 10.8] shows that the X-indexed family of small subobjects of $A$ we mean a monic $R \to A \times X$ such that the projection $R \to X$ is a small map. A powerclass object of $A$ is an object $P_sA$ equipped with a universal $P_sA$-indexed family of small subobjects of $A$. Note that in general, $P_sA$ need not be small even if $A$ is. It is shown in [JM95] that powerclass objects can be constructed when $C$ is exact. Authors who do not assume that $C$ is exact often assume directly that powerclass objects exist.

The following ad hoc definition summarizes what we need for our comparison.

**Definition 10.11.** A category of classes $C$ has a good material model if it satisfies the following.
(i) $\mathcal{C}$ has a small NNO.
(ii) If $f$ and $g$ are small, then so is $\Pi fg$.
(iii) $\mathcal{C}$ has powerclass objects.
(iv) If $p$ is regular epi and $fp$ is small, then $f$ is small (equivalently, $1 \vdash_C \lnot \text{``if } Y \to X \text{ is regular epi and } Y \text{ is small, then } X \text{ is small''} \).$
(v) The endofunctor $P_s$ has an indexed initial algebra $\mathcal{V}$, i.e. a $P_s$-algebra $\mathcal{V}$ such that $U^*\mathcal{V}$ is an initial algebra for all $U \in \mathcal{C}$.

Assumptions (i) and (ii) ensure that $\mathcal{S}$ is a $\Pi$-pretopos with a NNO, so that the theory of $\mathcal{S}$ can be applied in its stack semantics. The other three axioms are about describing the universe-object $\mathcal{V}$. Note that (iv) is necessary in order to make $P_s$ into a covariant functor in the usual way. In fact, $P_s$ can be made into a monad, using (iv) again to obtain the multiplication—but we emphasize that the relevant notion of "algebra" refers only to an algebra for $P_s$ as an endofunctor, not as a monad.) The indexed initial $P_s$-algebra $\mathcal{V}$ is constructed in [JM95] under the assumption of exactness and a subobject classifier, and in [vdBM07] under the assumption of $W$-types and "bounded exactness;" we will simply assume it to exist.

As for any initial algebra, the structure map $P_s\mathcal{V} \xrightarrow{v} \mathcal{V}$ is an isomorphism. Thus, by the universal property of $P_s$, the inverse of $v$ classifies a binary relation $\prec$ on $\mathcal{V}$ which is "set-based," i.e. the second projection $\mathcal{V} \xrightarrow{\prec} \mathcal{V}$ is small. It is not hard to show that $\prec$ is well-founded and extensional. An interpretation of material set theory in $\mathcal{C}$ is then obtained by considering $\mathcal{V}$ with $\prec$ as a model for material sets with $\in$ in the internal logic of $\mathcal{C}$. It satisfies many of the axioms automatically, while others are inherited from the corresponding categorical axioms for $\mathcal{C}$; see [JM95, MP02, vdBM07] for more details.

Remark 10.12. There is a slightly different thread in algebraic set theory, starting with [Sim99] and continuing with [AF06, AFW06], in which material set theory is not modeled in the initial $P_s$-algebra but in an arbitrary object $\mathcal{U}$ equipped with a monomorphism $P_s\mathcal{U} \to \mathcal{U}$. The elements of $\mathcal{U}$ not in the image of $P_s\mathcal{U}$ are then regarded as "atoms," and the axiom of foundation is not necessarily satisfied. We will focus on the situation described above, which seems better suited to a comparison with the stack semantics.

Now, from any morphism $x: U \to \mathcal{V}$ we can construct a well-founded extensional APG $\mathfrak{P}$ over $U$ as follows. First note that all our axioms are stable under pullback, so we may as well assume that $U = 1$. Now let $\hat{x}: N \to P_s\mathcal{V}$ be the unique map such that

$\begin{array}{c}
1 \xrightarrow{o} N \\
\eta_x \downarrow \quad \downarrow s \\
\eta \quad \mathcal{V} \xrightarrow{\hat{x}} \quad \mathcal{V} \\
\downarrow \quad \downarrow c \\
P_s\mathcal{V} \xrightarrow{\mathcal{V}} P_s\mathcal{V}
\end{array}$

commutes. Here $\eta: \mathcal{V} \to P_s\mathcal{V}$ is the unit of the monad $P_s$, while $c$ is the composite:

$P_s\mathcal{V} \xrightarrow{P_s(v^{-1})} P_sP_s\mathcal{V} \xrightarrow{\mu} P_s\mathcal{V}$

(where $\mu$ is the multiplication of the monad $P_s$). Now $\hat{x}$ classifies an $N$-indexed family of small subobjects of $\mathcal{V}$, i.e. a map $\mathfrak{P} \to N \times \mathcal{V}$ such that $\mathfrak{P} \to N$ is small. Since $N$ is also small, $\mathfrak{P}$ is a small object.
Working in the internal logic of \( \mathbf{C} \), well-foundedness of \( \mathcal{V} \) now implies that the projection \( \pi \to \mathcal{V} \) is monic. Since it is also a simulation, it follows that \( \pi \) is well-founded and extensional. It is rooted by \( x \) and accessible by definition, so it is a well-founded extensional APG, as desired. And just as in \([0]\) we can verify that

\[ (i) \quad U \models (\pi \cong \gamma) \text{ if and only if } x = y, \text{ and} \]
\[ (ii) \quad U \models (\pi \in \gamma) \text{ if and only if } x \prec y. \]

Thus it remains only to verify the analogue of \([9.7]\).

**Lemma 10.13.** Let \( U \in \mathbf{S} \), and let \( X \) be a well-founded extensional APG in \( \mathbf{S}/U \) which remains well-founded in \( \mathbf{C}/U \). Then there exists a unique \( x : U \to \mathcal{V} \) such that \( X \cong \pi \).

**Proof.** This is essentially the standard recursion theorem for well-founded relations. Since \( \mathcal{V} \) is an indexed initial algebra, it suffices to assume that \( U = 1 \). Define an attempt to be a partial function \( X \rightharpoonup \mathcal{V} \) which is a simulation and whose domain is a small initial segment of \( X \). We can then use the internal logic of \( \mathbf{C} \) to define “the set of all attempts” as a subobject of \( P_{\mathcal{V}}(X \times \mathcal{V}) \). Since \( X \) is well-founded, we can then prove by induction, as usual, that any two attempts agree on their common domain, that the union of attempts is an attempt, and that the union of all attempts has domain all of \( X \). The resulting simulation \( X \xrightarrow{f} \mathcal{V} \) then exhibits \( X \) as isomorphic to \( \pi \), where \( x \) is the composite \( 1 \xrightarrow{\pi} X \xrightarrow{f} \mathcal{V} \), and it is unique since \( \mathcal{V} \) is extensional. \( \square \)

However, without further axioms, there is no reason why every well-founded extensional APG in \( \mathbf{S}/U \) should also be well-founded in \( \mathbf{C}/U \). This will be true if \( \mathbf{C} \) satisfies the axiom of separation, but in general there might be new subobjects of \( X \) in \( \mathbf{C}/U \) that are not in \( \mathbf{S}/U \), so well-foundedness in \( \mathbf{C} \) is a stronger statement than well-foundedness in \( \mathbf{S} \). If we do have this stronger property, though, then we can derive the desired equivalence.

**Theorem 10.14.** Let \( \mathbf{C} \) be a category of classes which is well-generated by small objects and has a good material model, with \( \mathbf{S} \) its full subcategory of small objects, and assume that for all \( U \in \mathbf{S} \), every well-founded extensional graph in \( \mathbf{S}/U \) is also well-founded in \( \mathbf{C} \). Then the interpretation of material set theory in the stack semantics of \( \mathbf{S} \) is identical to its interpretation relative to \( \mathcal{V} \) in the internal logic of \( \mathbf{C} \).

**Proof.** We have seen that under the given hypotheses, isomorphism classes of well-founded extensional APGs in \( \mathbf{S}/U \) can be identified with maps \( U \to \mathcal{V} \), in a way which preserves the atomic formulas. The same inductive argument from \([10.8]\) then implies that the interpretations agree on all formulas. \( \square \)

**Corollary 10.15.** If \( \mathbf{C} \) is a category of classes which is well-generated by small objects and has a good material model, and moreover satisfies the separation axiom, then the interpretation of material set theory in the stack semantics of its category of small objects is identical to its interpretation relative to \( \mathcal{V} \) in the internal logic of \( \mathbf{C} \). \( \square \)

Of course, asking that well-foundedness in \( \mathbf{S} \) extends to \( \mathbf{C} \) is a version of the axiom of “full well-founded induction” relative to the containing category of classes \( \mathbf{C} \). But even if \( \mathbf{S} \) satisfies well-founded induction (in its stack semantics), this
property will generally fail to be inherited by categories of sheaves or ideals. This is the same problem we observed after Corollary 10.10, and it has the same solution. Namely, we will show in [Shua] that the 2-category of definable stacks on $S$ does inherit full well-founded induction from $S$, and that the stack semantics of $S$ can be identified with the internal language of this 2-category. Thus, if we are willing to generalize from categories of classes to “2-categories of classes,” stack semantics and algebraic set theory can be seen as two faces of the same coin.

**Appendix A. Material and structural foundations**

As mentioned in the introduction, we prefer to view the correspondence between set theories and (pre)toposes as a relationship between two different kinds of set theory, which we call *material* and *structural*. In material set theories such as ZFC, the elements of a set $X$ have an independent reality and identity, apart from their being collected together as the elements of $X$; frequently they are also sets themselves. The name “material” is a suggestion of Steve Awodey [Awo96]; they are also called “membership-based” set theories. By contrast, in structural set theories such as ETCS, the elements of a set $X$ have no existence or identity independent of $X$, and in particular are not sets themselves; they are merely abstract “elements” with which we build mathematical structures. We call these theories “structural” because they are closely aligned with the mathematical philosophy of “structuralism” (see e.g. [McL93, Awo04]); they are also called “categorical” set theories.

**Remark A.1.** In structural set theory, it is generally not meaningful to ask whether two elements of two different sets are equal, only whether two elements of a given ambient set are equal. In this way, structural set theory is very similar to type theory, and a case can be made that they are really different names for the same thing. There are undeniable differences in presentation and emphasis, but there is certainly a large space of theories which it is difficult to classify as one and not the other. In particular, as we remarked in footnote 2 on page 4, the stack semantics could equally well be phrased using a type theory with quantification over types.

Material set theory, of course, has a long history as a foundation for mathematics, while structural set theory has not gained as broad a following, although Lawvere first wrote down the axioms of ETCS nearly 50 years ago. According to its proponents, structural set theory provides a foundation for mathematics which is just as adequate as the foundation provided by material set theory, and which is moreover closer to mathematical practice and free of superfluous data.

In support of this thesis, we observe that in most of mathematics, sets serve merely as carriers of structure: group structure, ring structure, topological structure, etc. This view of mathematics was formally enshrined in Bourbaki’s “general theory of structure” [Bou68]. Moreover, these structures, and hence the sets from which they are built, are generally only ever considered up to isomorphism. It never matters to the working mathematician whether the natural numbers are defined in von Neumann’s way:

$$0 = \emptyset, \ 1 = \{0\}, \ 2 = \{0, 1\}, \ 3 = \{0, 1, 2\}, \ldots$$

or in some other way, such as:

$$0 = \emptyset, \ 1 = \{0\}, \ 2 = \{1\}, \ 3 = \{2\}, \ldots$$
since the two resulting versions of $\mathbb{N}$ are canonically isomorphic. In particular, the intricate global membership structure of material set theory is irrelevant outside of set theory proper. Indeed, Bourbaki had to carefully specify exactly how this superfluous data is to be forgotten, in order that all of their structures would be invariant under isomorphism. In structural set theory this process of careful forgetting is unnecessary, since there is no global $\in$ in the first place. (The complexity of the Cole-Mitchell-Osius construction summarized in \cite{ColeMitchellOsius} testifies to the amount of data that must be forgotten.)

From this point of view, the value of an equivalence between material and structural set theory is that it implies that in principle, either could serve equally well as a foundation for mathematics. Of course, what really matters is the naturalness of the encoding—structural set theory would not be a very attractive foundation if mathematics could only be encoded in it by way of material set theory! However, the observations above about mathematical structure provide good empirical evidence that in fact, the situation is reversed: material set theory provides a foundation for most mathematics only by way of structural set theory.

On the other hand, the material point of view is not without advantages, especially in set theory proper. Although the entire study of ZF could be reinterpreted structurally in terms of well-founded extensional relations, the result would be unnecessarily complicated. In particular, some of the powerful methods of material set theory seem difficult to duplicate structurally without significant circumlocution. This seems to be essentially the point expressed by Mathias \cite{Mathias}:

\begin{quote}
If one wants to do geometry, why bother with von Neumann ordinals? What do they do that is not achieved by arbitrary well-orderings? Perhaps nothing... But if, sated with geometry, one wants to do transfinite recursion theory, why make life hard by avoiding von Neumann ordinals?
\end{quote}

Thus, we believe that there should be no real conflict between the two kinds of set theory. Rather, as frequently happens in mathematics, we simply have two different viewpoints on the same underlying reality. And as always in such a situation, recognition of this means that we can switch between the two theories freely, using whichever is most convenient for any purpose. From this perspective, our goal in this paper has been to extend the language and tools of structural set theory so as to deal with unbounded quantifiers directly, facilitating more direct comparisons with the corresponding language and tools of material set theory. The extension we have arrived at (the stack semantics) also of course has potential applications in topos theory by itself, which we hope to explore in later work.

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