WHIRLING INJECTIONS, SURJECTIONS, AND OTHER FUNCTIONS BETWEEN FINITE SETS

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Abstract. This paper analyzes a certain action called “whirling” that can be defined on any family of functions between two finite sets equipped with a linear (or cyclic) ordering. Many maps of interest in dynamical algebraic combinatorics, such as rowmotion of order ideals, can be represented as a composition of “toggling” involutions, each of which modifies its object only locally. Similarly whirling is made up of locally-acting whirling maps which directly generalize toggles, but cycle through more than two possible outputs. In this first paper on whirling, we consider it as a map on subfamilies of functions between finite sets.

For whirling acting on the set of injections or the set of surjections, we prove that within each whirling orbit, any two elements of the codomain appear as outputs of functions the same number of times. This result can be stated in terms of the homomesy phenomenon, which occurs when a statistic has the same average across every orbit. We further explore homomesy results and conjectures for whirling on restricted-growth words, which correspond to set partitions. These results extend the collection of combinatorial objects for which we have interesting dynamics and homomesy, and open the door to considering whirling in other contexts.

Keywords: dynamical algebraic combinatorics, homomesy, injection, noncrossing partition, RG-word, set partition, surjection, toggles, toggle group, whirling, whirls

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1. Introduction

In this paper, we explore the orbits associated with an action that can be defined on any family \(\mathcal{F}\) of functions \(f\) from one finite set \(S\) to another finite set \(T\), where \(S\) comes with a linear ordering and \(T\) comes with a cyclic ordering. Without loss of generality we may take \(S = [n] := \{1, 2, \ldots, n\}\) with its usual ordering and \(T = [k]\) with the mod \(k\) cyclic ordering. The action is generated by an operation \(w : \mathcal{F} \to \mathcal{F}\) (“the whirling map”) that is in turn defined as the composition of simpler maps \(w_i : \mathcal{F} \to \mathcal{F}\) (“whirling at \(i\)”, with \(i\) in \([n]\)) that repeatedly add 1 mod \(k\) to the value of \(f\) at \(i\) until we get a function in \(\mathcal{F}\). Our main results

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are examples of the *homomesy* phenomenon, introduced by the second and third authors [PR15] and defined as follows.

**Definition 1.1.** Suppose we have a set $S$, an invertible map $w : S \to S$ such that every $w$-orbit is finite, and a function ("statistic") $f : S \to \mathbb{R}$. Then we say the triple $(S, w, f)$ exhibits *homomesy* if there exists a constant $c \in \mathbb{R}$ such that for every $w$-orbit $O \subseteq S$,

$$\frac{1}{\#O} \sum_{x \in O} f(x) = c.$$  

In this case, we say that the function $f$ is *homomesic with average* $c$, or *c-mesic*, under the action of $w$ on $S$.

Although introduced relatively recently, the homomesy phenomenon has been discovered to be widespread throughout combinatorial dynamical systems. See [Rob16] for a survey article. Since it is a new area of research, there is still much to learn about it, including the best techniques for proving homomesy, what types of actions and statistics commonly yield homomesy, and what consequences can arise from homomesy.

Early on, almost all proven homomesy results were for actions whose order was straightforwardly computable in general and relatively small compared to the size of the ground set $S$. So it is rarer and notable to find homomesies for actions with unpredictable orbit sizes, as is the case with the maps in this paper, which also generally have unknown order. Other such examples can be found in [EFG+16, ELM+23]. In some cases, a statistic which is always an integer has a non-integer average across every orbit, which means all of the orbit sizes must be multiples of the constant average’s denominator. This consequence of homomesy appears in Corollary 2.22.

The whirl maps $w_i$ are a generalization of the *toggle* maps that are heavily studied in dynamical algebraic combinatorics. Toggles were first introduced as a way of writing the rowmotion map as a composition of simpler operations on order ideals [CF95]. Striker later generalized them to a much wider variety of settings [Str18]. There have been many interesting homomesy and periodicity phenomena found for actions defined in terms of toggles; see e.g., [PR15, Rob16, Had21, EFG+16, JR18, ELM+23]. In recent work, Plante and the third author consider a whirling action on the set of $k$-bounded $P$-partitions of a poset $P$. They prove this to be in equivariant bijection with rowmotion acting on order ideals of the poset $P \times [k]$ for any finite poset $P$. They then leverage this to obtain periodicity and homomesy results for rowmotion acting on the poset $V \times [k]$, which has surprisingly good dynamical properties [PR24]. A key point here is that beyond the intrinsic interest of whirling as a generalization of toggling, its study can facilitate our understanding of rowmotion questions that arise independently.

In Section 2 we consider the whirling action on injections and on surjections between two finite sets. In both cases, we prove that within each orbit, any two elements $1, 2, \ldots, k$ of the codomain appear as outputs of functions the same number of times. This was originally conjectured by the second author. We also prove this for a family of functions that naturally generalizes injections and conjecture it for a similar family that generalizes surjections. Then in Section 3, we explore the whirling action on various families of restricted growth words, which are functions that encode set partitions.

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1Greek for “same middle”
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2. Whirling on m-injections and m-surjections between finite sets

2.1. The whirling map on m-injections and m-surjections.

Definition 2.1. A function \( f : S \to T \) is m-injective if every element \( t \in T \) appears as an output of \( f \) at most \( m \) times, i.e., \( \#f^{-1}(t) \leq m \) for every \( t \in T \). An m-injective function is also called an m-injection. Injections are the same as 1-injections.

Definition 2.2. A function \( f : S \to T \) is m-surjective if every element \( t \in T \) appears as an output of \( f \) at least \( m \) times, i.e., \( \#f^{-1}(t) \geq m \) for every \( t \in T \). An m-surjective function is also called an m-surjection. Surjections are the same as 1-surjections.

We denote the set of all m-injective functions from \([n]\) to \([k]\) by Inj\(_m\)(\(n, k\)) and the set of all m-surjective functions from \([n]\) to \([k]\) by Sur\(_m\)(\(n, k\)). We write a function \( f : [n] \to [k] \) in one-line notation as \( f(1)f(2)\cdots f(n) \). For example, the function \( f(1) = 5, f(2) = 1, f(3) = 2, f(4) = 1 \) is written \( f = 5121 \). (All of our examples will have codomain \([k]\) with \( k \leq 9 \).) Note that m-injections are not injective in general. They are called “width-m restricted functions” in \cite{Wal}, which discusses their enumeration.

Definition 2.3. For \( f \in \mathcal{F} \subseteq [k]^{[n]} \) define a map \( w_i : \mathcal{F} \to \mathcal{F} \), called whirling at index \( i \), as follows: repeatedly add 1 \( (\mod k) \) to the value of \( f(i) \) until we get a function in \( \mathcal{F} \). The new function is \( w_i(f) \). (Here we represent our residues \( (\mod k) \) within \([k]\) rather than \( \{0,1,2,\ldots,k-1\} \).)

Example 2.4. Let \( \mathcal{F} = \text{Inj}_3(6,4) \) and \( f = 422343 \). To compute \( w_3(f) \), we first add 1 to \( f(3) = 2 \) and get the function 423343, which is not 2-injective since 3 appears as an output three times. So, we add 1 again to the third position and get 424343, which is also not 2-injective. Adding 1 \( (\mod 4) \) once more gives a 2-injective function 421343 = \( w_3(f) \).

Remark 2.5. The map \( w_i \) depends heavily on the family of functions \( \mathcal{F} \) on which it acts. For example,

- if \( \mathcal{F} = \text{Inj}_3(7,3) \), then \( w_1(1221332) = 3221332 \),
- if \( \mathcal{F} = \text{Sur}_1(7,3) \), then \( w_1(1221332) = 2221332 \),
- if \( \mathcal{F} = \text{Sur}_2(7,3) \), then \( w_1(1221332) = 1221332 \).

Thus, whenever we talk about whirling, we must first make clear what \( \mathcal{F} \) is. In this section \( \mathcal{F} \) either refers to Inj\(_m\)(\(n, k\)) or Sur\(_m\)(\(n, k\)) for some positive integers \( n, k, m \). In the next section, we will consider a different family of functions called restricted growth words, which are in bijection with set partitions.

Remark 2.6. In the case that \( \mathcal{F} = \text{Sur}_m(n, k) \), the only possible reason we could not add 1 \( \mod k \) to \( f(i) \) is because the value \( f(i) \) is only an output of the function \( m \) times. (Note that in the previous sentence, the first \( f(i) \) is a dynamic value, subject to change as we update \( f \), while the second is a value, associated with a particular \( f \); we trust that this dual usage will not create confusion.) Therefore, in this case, \( w_i \) either adds 1 \( \mod k \) to \( f(i) \), or it leaves \( f \) alone. This is not the case when \( \mathcal{F} = \text{Inj}_m(n, k) \).
Definition 2.7. For a family of functions \( \mathcal{F} \) from \([n]\) to \([k]\), we define the (left-to-right) whirling map, denoted \( w : \mathcal{F} \to \mathcal{F} \), to be the map that whirls at indices \( 1, 2, \ldots, n \) in that order. So \( w = w_n \circ \cdots \circ w_2 \circ w_1 \).

Example 2.8. Let \( \mathcal{F} = \text{Inj}_1(4, 7) \). Then
\[
2753 \xmapsto{w_1} 4753 \xmapsto{w_2} 4153 \xmapsto{w_3} 4163 \xmapsto{w_4} 4165
\]
so \( w(2753) = 4165 \).

Remark 2.9. On any family \( \mathcal{F} \), the map \( w_i \) (and thus \( w \) also) is invertible. Given \( f \in \mathcal{F} \), we get \( w_i^{-1}(f) \) by repeatedly subtracting 1 (mod \( k \)) from the value of \( f(i) \) until we get a function in \( \mathcal{F} \).

We now discuss orbits and homomesy for \( w \). Theorem 2.11 is the main result of this section.

Definition 2.10. For \( j \in [k] \), define \( \eta_j(f) = \# f^{-1}(\{j\}) \) to be the number of times \( j \) appears as an output of the function \( f \).

Theorem 2.11. Fix \( \mathcal{F} \) to be either \( \text{Inj}_m(n,k) \) or \( \text{Sur}_1(n,k) \) for positive integers \( n,k,m \). Then under the action of \( w \) on \( \mathcal{F} \), \( \eta_j \) is \( \frac{n}{k} \)-mesic for any \( j \in [k] \).

Note that \( \sum_j \eta_j(f) = n \) for all \( f \). It is clear that an equivalent statement to Theorem 2.11 is that \( \eta_i - \eta_j \) is \( 0 \)-mesic for any \( i,j \in [k] \). That is, \( i \) and \( j \) appear as outputs of functions the same number of times across any orbit. We also conjecture this result for \( m \)-surjections in general.

Conjecture 2.12. Let \( \mathcal{F} = \text{Sur}_m(n,k) \) for positive integers \( m,n,k \). Then under the action of \( w \) on \( \mathcal{F} \), \( \eta_j \) is \( \frac{m}{k} \)-mesic for any \( j \in [k] \).

Note that Conjecture 2.12 holds for \( k = 2 \) by Theorem 2.11 since \( \text{Sur}_m(n,2) = \text{Inj}_{m-n}(n,2) \). The conjecture has been verified by a computer for all triples \( (m,n,k) \) where \( n \leq 10 \) (for which there are finitely many relevant triples since \( \text{Sur}_m(n,k) = \emptyset \) if \( mk > n \)).

2.2. The proof of Theorem 2.11 for injections. In this subsection, we prove Theorem 2.11 when \( \mathcal{F} = \text{Inj}_1(n,k) \). Even though this is a special case of \( m \)-injections, it is easier to understand the proof in this simpler situation. We will utilize much of the same notation and terminology for the other cases.

Let \( \emptyset \) be an orbit under the action of \( w \) on \( \mathcal{F} \). We draw a board for the orbit \( \emptyset \) by placing some \( f \in \mathcal{F} \) on the top line. The function in row \( i+1 \) is \( w_i(f) \) for \( i \in [0, \ell(\emptyset) - 1] \), where \( \ell(\emptyset) \) is the length of \( \emptyset \). For example, in Figure 1 we show a board for an orbit on \( \mathcal{F} = \text{Inj}_1(3,6) \). From this board, we see that if we let \( f = 621 \), then \( w(f) = 342 \), \( w^2(f) = 563 \), and so on, and that \( w^{10}(f) = f \).

Notice that within an orbit board, if \( f \) is a given line, then the “partially whirled element” \((w_i \circ \cdots \circ w_1)(f)\) is given by the first \( i \) numbers on the line below \( f \) and the last \( n-i \) numbers of \( f \). (We say the line “below” the bottom line is the top line, as we consider the orbit board to be cylindrical.) For instance, in the first two lines of Figure 1 \( f = 621 \), \( w_1(f) = 321 \), and \((w_2 \circ w_1)(f) = 341 \).

We use the term reading the orbit board to refer to this action where we start at a certain position \( P \) of the board, and continue to the right until we reach the end of the line, and then go to the leftmost position of the line below and continue. This is because it is exactly
like reading a book (except that continuing past the bottom line means returning to the top line). When we refer to a certain position being $x$ positions “before” or “after” the position $P$, or say the “previous” or “next” position, we always mean in the reading order.

Definition 2.13. For a given position $P$ in an orbit board, let $(P, h)$ denote the position $h$ places after $P$ in the reading order (or $-h$ places before $P$ if $h < 0$). Let $(P, [a, b])$ denote the $(b - a + 1)$-tuple $((P, a), (P, a + 1), \ldots, (P, b - 1), (P, b))$.

Example 2.14. Consider the following orbit on Inj$_1(4, 5)$. Let $P$ be the position in the second row and second column, shown below surrounded by a black rectangle. Then $(P, [1, 4])$ consists of the four positions circled in red. Also $(P, [0, 4])$ is $(P, [1, 4])$ together with $P$, while $(P, [-1, 2])$ is the second row. As the orbit is cylindrical, the bottom right corner is both $(P, 14)$ and $(P, -6)$. Note that $P$ refers only to the position, not the value in that position. So $P \neq (P, 5)$ since they are different positions, even though both contain the value 3. Similarly, we will never write, e.g., $P = 3$.

Note that if $P$ is in row $i$ and column $j$, then $(P, [1, n])$ always consists of all positions to the right of $P$ in row $i$, followed with the leftmost $j$ positions of row $i + 1$. Also note that $P([-n, -1])$ consists of the rightmost $n - j + 1$ positions of row $i - 1$ followed by all $j - 1$ positions left of $P$ in row $i$.

Lemma 2.15. Suppose $i$ is in position $P$ of a board for the $w$-orbit $\emptyset$ on Inj$_1(n, k)$.

1. There is exactly one occurrence of $i + 1 \mod k$ within $(P, [1, n])$.
2. There is exactly one occurrence of $i - 1 \mod k$ within $(P, [-n, -1])$.

Proof. To prove (1), suppose position $P$ is in column $j$. So $f(j) = i$ for some $f \in \emptyset$. Then $(P, [1, n])$ contains the multiset of outputs of $(w_j \circ \cdots \circ w_1)(f)$ in some order. If $(w_{j-1} \circ \cdots \circ w_1)(f)$ does not have $i + 1 \mod k$ already as an output, then by definition $w_j$ changes the output corresponding to the input $j$ from $i$ to $i + 1 \mod k$. So there is an
occurrence of \(i + 1 \mod k\) within \((P, [1, n])\). Since \((w_j \circ \cdots \circ w_1)(f)\) cannot have any output more than once, this occurrence is unique.

The proof of (2) is analogous to (1) using the inverse of whirling instead.

![Figure 2](image-url)

Figure 2. We demonstrate, in six panels, how to place a \([6]\)-chunk on this \(w\)-orbit on \(F = \text{Inj}_1(3, 6)\). In panel \(i\) from left to right, the positions highlighted in green are the first \(i\) positions in the chunk. The circled entries (in the first five panels) are the next \(n = 3\) positions after the one containing the \(i\). We choose the unique \(i + 1\) among the circled entries to be in the chunk.

**Proof of Theorem 2.11** for \(F = \text{Inj}_1(n, k)\). The idea of the proof is to partition any given orbit into \([k]\)-chunks that contain every number \(1, 2, \ldots, k\) exactly once. Within any orbit \(O\), we will assume without loss of generality that \(1\) appears as an output at least as often as any other number \(2, \ldots, k\). This is because if \(i > 1\) appears as an output more times than \(1\), then we can renumber \(i, i + 1, \ldots, k, 1, \ldots, i - 1\) as \(1, 2, \ldots, k\), and the outputs remain the same relative to each other \((\mod k)\).

Choose a 1 in the orbit board and call this position \(P_1\). Then by Lemma 2.15(1), there exists a unique occurrence of 2 within \((P_1, [1, n])\); place this 2 in the same chunk as 1 and call its position \(P_2\). For every \(i\) in a chunk in position \(P_i\), choose the \(i + 1\) within \((P_i, [1, n])\) and call its position \(P_{i+1}\). Continue this until the chunk contains \(1, 2, \ldots, k\). Refer to Figure 2 for an example of this process. In each step shown, the circled entries are the next \(n = 3\) positions after the position placed in the chunk.

To start a new chunk, we choose a 1-entry that is not already part of a chunk, and continue the same process. If \(Q\) is the position of the \(i + 1\) in a chunk, the \(i\) in the same chunk clearly must be in \(Q([-n, -1])\). By Lemma 2.15(2), there is only one \(i\)-entry that can be in the same chunk as the given \((i + 1)\)-entry. Thus, our \([k]\)-chunks are disjoint.

Once every 1 in the orbit board is part of a completed \([k]\)-chunk, there are no more entries not already in a chunk, since we assumed 1 appears as an output at least as often as any other number. The chunking process shows that \(1, 2, \ldots, k\) appear as outputs the same number of times in any orbit.

**2.3. The proof of Theorem 2.11 for \(m\)-injections.** Now we prove Theorem 2.11 for the case \(F = \text{Inj}_m(n, k)\). We use a similar technique as for the \(\text{Inj}_1(n, k)\) case. We will again partition orbit boards into \([k]\)-chunks, where each chunk contains \(1, 2, \ldots, k\) and the instance of \(i + 1\) within a chunk is at most \(n\) positions after the instance of \(i\) (in the reading
Unlike in the $\text{Inj}_1(n,k)$ case, there is no longer a unique way of partitioning the orbit into chunks; see Figure 3 for two different ways to partition the orbit of $\text{Inj}_2(4,4)$ containing 1441 into $[4]$-chunks. However, all that matters to prove Theorem 2.11 is the existence of a partitioning into $[k]$-chunks. Unfortunately the proof becomes more complicated due to lack of uniqueness of the partition into chunks.

We will again use the notation $(P,h)$ and $(P,[a,b])$ as we did for the injections proof. We will always refer to the reading order.

**Figure 3.** Two different ways to partition the same $w$-orbit of $\text{Inj}_2(4,4)$ into $[4]$-chunks.

**Remark 2.16.** For any position $P$ in column $j$, $(P,[1,n])$ contains the multiset of outputs of $(w_j \circ \cdots \circ w_1)(f)$ for some $f \in \mathcal{O}$. Thus, any set of $n$ consecutive positions cannot contain more than $m$ equal values.

**Remark 2.17.** When describing the entire orbit board, it does not make sense to say that a particular position is “earlier” or “later” than another position, because the orbit board is cylindrical. However, when working within a tuple of $r$ consecutive positions, these notions are well-defined. Suppose we are in $(P,[a,b])$ for some position $P$ and $a, b \in \mathbb{Z}$. Then to say position $Q$ is earlier than position $R$ (equiv. $R$ is later than $Q$) within $(P,[a,b])$ means that $Q = (P,c)$ and $R = (P,d)$ for some $a \leq c < d \leq b$. Likewise, it makes sense to refer to the last (or earliest) occurrence of $3$ within $(P,[a,b])$, but it doesn’t make sense to refer to such within the entire orbit board.

**Lemma 2.18.** Suppose $i$ is the value in position $P$ of a board for the $w$-orbit $\mathcal{O}$ on $\text{Inj}_m(n,k)$.

1. There are at most $m$ occurrences of $i + 1 \mod k$ within $(P,[1,n])$.
2. If the position $(P,n)$ directly below $P$ does not contain $i + 1 \mod k$, then there are exactly $m$ occurrences of $i + 1 \mod k$ within $(P,[1,n-1])$.
3. There are at most $m$ occurrences of $i - 1 \mod k$ within $(P,[-n,-1])$. 
(4) If the position \((P, -n)\) directly above \(P\) does not contain \(i - 1 \mod k\) occurrences of \(i - 1 \mod k\) within \((P, -(n - 1), -1)\).

**Proof.** Suppose position \(P\) is in column \(j\). Then \(f(j) = i\) for some \(f \in \varnothing\). By Remark 2.16 there cannot be more than \(m\) occurrences of \(i + 1 \mod k\), proving (1).

If \((w_{j-1} \circ \cdots \circ w_1)(f)\) does not have \(i + 1 \mod k\) already \(m\) times as an output, then by definition \(w_j\) changes the output corresponding to the input \(j\) from \(i\) to \(i + 1 \mod k\). This proves (2).

The proofs of (3) and (4) are analogous to (1) and (2) using the inverse of whirling instead.

**Proof of Theorem 2.11** for \(\mathcal{F} = \text{Inj}_m(n, k)\). By the same relabeling argument from the injections proof, we will assume without loss of generality that within any given orbit \(\varnothing\), \(1\) appears as an output at least as often as any other number \(2, \ldots, k\).

Choose a 1 in the orbit board and call its position \(P_1\). Then by Lemma 2.18(2), there exists at least one occurrence of 2 in \((P_1, [1, n])\); pick a position \(P_2\) containing such a 2 and place it in the same chunk as \(P_1\). For every \(P_i\) containing the value \(i\), we select a position \(P_{i+1}\) in \((P, [1, n])\) containing the value \(i + 1\) and place it in the same chunk as \(P_i\). Continue this until the chunk contains 1, 2, \ldots, \(k\) (that is, until the \(k\) positions that comprise the chunk contain the respective values 1, 2, \ldots, \(k\)).

To start a new chunk, we choose a 1-entry that is not already part of a chunk, and wish to continue the same process. However, unlike in the case of injections, there is not necessarily a unique occurrence of \(i + 1\) within the next \(n\) positions after an \(i\), nor a unique occurrence of \(i - 1\) within the previous \(n\) positions before an \(i\). When placing an \((i + 1)\)-entry in the same chunk as \(i\) in position \(P\), we want to choose \(i + 1\) that is not already part of a chunk. When \((P, [1, n])\) contains such an \((i + 1)\)-entry, we choose one of them. However, such an \((i + 1)\)-entry may not exist depending on how we chose earlier chunks.

See the left side of Figure 4 for an example of this problem. Let \(P\) be the position of the 2 in the purple chunk. Then the only 3 in \((P, [1, n])\) is already part of the brown chunk. In this case, we reassign the part of the brown chunk starting with 3 to be in the purple chunk, and then continue where the purple chunk is complete and we now attempt to complete the brown chunk.

In general, suppose for a given \(i\) in position \(P\), all \((i + 1)\)-entries within \((P, [1, n])\) are already part of chunks. Then we claim that at least one of these \((i + 1)\)-entries is in the same chunk as an \(i\)-entry in \((P, [1, n - 1])\). To explain this we consider two cases.

**Case 1:** The entry directly below position \(P\) is \(i + 1\). Let \(Q = (P, n)\) be this position below \(P\). Then the \(i + 1\) in position \(Q\) is already in a chunk with an \(i\)-entry in \((Q, [-n, -1]) = (P, [0, n - 1])\). However, we know position \(P\) is not already in a chunk with an \((i + 1)\)-entry. So the \((i + 1)\)-entry in position \(Q\) must be in the same chunk as an \(i\)-entry in \((P, [1, n - 1])\).

**Case 2:** The entry directly below position \(P\) is not \(i + 1\). Then Lemma 2.18(2) implies that there are \(m\) occurrences of \(i + 1\) within \((P, [1, n - 1])\). Consider these \((i + 1)\)-entries. Each of them is in a chunk with an \(i\) that is at most \(n\) positions before it (and at least 1 position before). Thus, all of these corresponding \(i\)-entries are in \((P, [-(n-1), n-2])\). If they are all in \((P, [-(n-1), -1])\), then there would be \(m\) different \(i\)-entries in \((P, [-(n-1), -1])\). Adding in the \(i\) in position \(P\), this implies there are \(m + 1\) different \(i\)-entries in \((P, [-(n-1), 0])\), contradicting Remark 2.16. So at least one of the occurrences of \(i + 1\) within \((P, [1, n - 1])\) is in a chunk with an \(i\) in \((P, [1, n - 2])\).
Suppose that we get stuck in partitioning the orbit board into \( [k] \)-chunks. In this case, when creating a chunk \( C \) through \( \{1, \ldots, i\} \), we have \( i \) in position \( P \) and all \( (i + 1) \)-entries in \((P, [1, n])\) are already part of chunks. Then we will choose a chunk \( C' \) for which both the \( i \) and \( (i + 1) \)-entries are within \((P, [1, n])\); such a chunk exists by the above argument. We will take the positions of the \( i + 1, \ldots, k \)-entries of \( C' \) and make them part of the chunk \( C \) instead of \( C' \). Then it is just as if \( C \) was created previously, and we now must complete \( C' \). This can cause a chain reaction of chunk reassignment.

At any point, let \( Q \) be the position of \( i + 1 \) for which we reassign \( i + 1, \ldots, k \) to be in a different chunk. Then we are always choosing to reassign the \( i + 1, \ldots, k \)-entries with an \( i \) that is earlier within \((Q, [-n, -1])\) by the previous paragraph. This is well-defined by Remark 2.17. Therefore, the \( i + 1 \) in position \( Q \) can never later be reassigned to the same chunk as the \( i \)-entry it was previously with.

For example, suppose \( \mathcal{F} = \text{Inj}_3(6, 5) \). Suppose a position \( Q \) containing 4 is in the same chunk as the position \( (Q, -3) \) containing 3. Also suppose that the positions in \((Q, [-6, -1])\) with the value 3 are \((Q, -5), (Q, -3), \) and \((Q, -2)\). Then \( Q \) (and the corresponding 5-entry in the same chunk) might be reassigned to be in the same chunk as position \((Q, -5)\), in which case it then could not later be reassigned to the chunk with position \((Q, -3)\). Also, position \( Q \) cannot be reassigned to the chunk with position \((Q, -2)\), as it is later than \((Q, -3)\) in \((Q, [-6, -1])\).

Given \( i \in [k - 1] \), there are only finitely many positions with entry \( i \) that can be in the same chunk as any specific position with entry \((i + 1)\). If chunk reassignments continue to be necessary, then eventually for every pair \( i \in [k - 1] \) and position \( Q \) containing \( i + 1 \), \( Q \)
will be placed in the same chunk as the earliest position with \( i \) in \((Q, [-n, -1])\), in which case no more chunk reassignments are possible. So the process always terminates leading to a partition of the orbit board into \([k]\)-chunks. ■

2.4. **The proof of Theorem 2.11 for surjections.** In this section, we prove Theorem 2.11 in the case where \( F = \text{Sur}_1(n, k) \). We again use the notation \((P, h)\) and \((P, [a, b])\) as in the previous sections. Note that Remark 2.17 still applies here.

Throughout this section, \( i + 1, i + 2, i - 1, \) etc. (the entries in the orbit boards and the outputs of the functions) are all to be interpreted mod \( k \).

Recalling Remark 2.6, \( w_i \) either adds 1 to the value of \( f(i) \) or it leaves \( f \) alone. Thus, given any \( i \in [n] \) and \( f \in \text{Sur}_1(n, k) \), \( w \) either adds 1 to \( f(i) \) or does not change the value \( f(i) \). Therefore, in an orbit board, such as the one in Figure 5, each entry in a column is either the same or one greater (mod \( k \)) as the entry directly above it.

**Definition 2.19.** Call a position \( P \) in an orbit board a **stoplight** if it has the same value as the position below it. This term comes from the analogy with traffic lights. As we look down a column, we always add 1 to the value, except we have to stop when a position is a stoplight. The stoplights are surrounded by circles in Figure 5.

**Lemma 2.20.** Let \( P \) be a position in an orbit board of \( \text{Sur}_1(n, k) \), and let \( i \) be the value in that position.

1. If \((P, [1, n - 1])\) does not contain \( i \) in any position, then the position \((P, n)\) directly below \( P \) contains the value \( i \).
2. If \((P, [1, n - 1])\) contains the value \( i \) in some position, then the position \((P, n)\) directly below \( P \) contains the value \( i + 1 \).

**Proof.** Suppose \( P \) is in column \( j \) and let \( f \) be the function on the row with \( P \). So \( f(j) = i \). Then \((P, [0, n - 1])\) consists of the multiset of entries of \((w_{j-1} \circ \cdots \circ w_1)(f)\).

If \((P, [1, n - 1])\) does not contain \( i \) in any position, then \((w_{j-1} \circ \cdots \circ w_1)(f)\) only contains \( i \) as an output once. So to maintain surjectivity, \(((w_j \circ \cdots \circ w_1)(f))(j) = i \), and hence \((P, n)\) contains the value \( i \).

On the other hand, if \((P, [1, n - 1])\) contains \( i \) in any position, then \((w_{j-1} \circ \cdots \circ w_1)(f)\) contains \( i \) as an output more than once. Thus applying \( w_j \) changes the output corresponding to \( j \) from \( i \) to \( i + 1 \), since we still get a surjective function. In this case, \((P, n)\) contains the value \( i + 1 \). ■

![Figure 5](image-url) An example \( w \)-orbit of \( \text{Sur}_1(8, 4) \) containing \( f = 31114424 \). The stoplights are colored in red and blue to distinguish the two cycles required to generate all of them via the method described in the proof.
Note that Lemma 2.20 implies that a position is a stoplight if and only if it does not contain the same value as any of the next \( n - 1 \) positions.

**Lemma 2.21.** Let \( P \) be a stoplight that contains \( i \). Let \( Q \) be the last position (in the reading order) in \( (P, [1, n - 1]) \) that contains \( i + 1 \). Then
\[
\begin{align*}
(1) \; & Q \text{ is a stoplight, and} \\
(2) \; & P \text{ is the last position containing } i \text{ within } (Q, [- (n - 1), -1]).
\end{align*}
\]

**Proof.** To prove (1), Lemma 2.20 assures us that it suffices to prove that \( (Q, [1, n - 1]) \) does not contain \( i + 1 \) in any position. Every position in \( (Q, [1, n - 1]) \) is either
- in \( (P, [1, n - 1]) \),
- the position below \( P \) (i.e., \( P, n \)), or
- the position directly below \( R \) for some \( R \) (strictly) between \( P \) and \( Q \).
Since \( Q \) is the last position in \( (P, [1, n - 1]) \) that contains \( i + 1 \), any position simultaneously in both \( (Q, [1, n - 1]) \) and \( (P, [1, n - 1]) \) cannot contain \( i + 1 \). The position below \( P \) contains \( i \) since \( P \) is a stoplight. Let \( R \) be a position between \( P \) and \( Q \). If the position below \( R \) contains \( i + 1 \), then \( R \) contains either \( i \) or \( i + 1 \). Since \( P \) is a stoplight, no position in \( (P, [1, n - 1]) \) contains \( i + 1 \); thus \( R \) cannot contain \( i \) (which proves (2)). If \( R \) contains \( i + 1 \), then \( R \) is not a stoplight because \( Q \) is a position in \( (R, [1, n - 1]) \) with the same value as \( R \). So the position below \( R \) cannot contain \( i + 1 \), meaning no positions in \( (Q, [1, n - 1]) \) contain \( i + 1 \).

\[\Box\]

**Proof of Theorem 2.11 for \( \mathcal{F} = \text{Sur}_1(n, k) \).** Within any column in an orbit board, if we skip over the stoplights, then every entry is \( 1 \mod k \) more than the entry above it. Since the positions in each column that are not stoplights are equidistributed between \( 1, 2, \ldots, k \), it suffices to show that the orbit’s stoplight entries are equidistributed between \( 1, 2, \ldots, k \).

If there are no stoplights in the orbit, then we are done. If there is a stoplight, pick one and call it \( P \). Let \( i \) be the entry in \( P \) and circle it. Then by Lemma 2.21(1), we can find another stoplight \( Q \) by taking the last \( (i + 1) \)-entry in \( (P, [1, n - 1]) \); circle \( Q \) as well. (For the same reason as \( m \)-injections (Remark 2.16), any \( n \) consecutive positions \( (P, [0, n - 1]) \) contains the multiset of values of some function in \( \text{Sur}_1(n, k) \) and thus contains a position with entry \( i + 1 \).) Then using Lemma 2.21 again, we see that the last \( (i + 2) \)-entry in \( (Q, [1, n - 1]) \) is a stoplight, which we circle as well. We can continue this and by the finiteness of positions in the orbit, we will eventually return to a stoplight we have already encountered. By Lemma 2.21(2), no two different stoplights can lead to the same stoplight during this process, so we return to \( P \).

This chain of stoplights clearly has the entries \( 1, 2, \ldots, k \) equidistributed by construction. In Figure 5, the cycle of stoplights determined by this method beginning with the ‘1’ in the fourth position of the top row are circled in red.

Now we look at the orbit again. If there are no stoplights not already circled, we are done. Otherwise, pick a stoplight, circle it, and begin the same process again. By Lemma 2.21(2), we will not circle a position circled in the previous chain.

We will continue this process until there are no more stoplights to circle. Thus the circled stoplights in the orbit will be equidistributed between \( 1, 2, \ldots, k \). In the example in Figure 5, two cycles are required; these are displayed with red and blue circles to distinguish them. \[\Box\]
2.5. **Consequences of the homomesy.** Let \( \mathcal{F} \) be either \( \text{Inj}_m(n, k) \) or \( \text{Sur}_1(n, k) \). Then given any \( j \in [k] \), \( \eta_j(f) \) is always an integer. Thus, Theorem 2.11 leads to a corollary about orbit sizes for the case when the average value of \( \eta_j \) across every orbit is not an integer.

**Corollary 2.22.** Suppose \( \mathcal{F} \) is either \( \text{Inj}_m(n, k) \) or \( \text{Sur}_1(n, k) \). The size of any \( w \)-orbit of \( \mathcal{F} \) is a multiple of \( \frac{k}{\gcd(n, k)} \).

**Proof.** Theorem 2.11 says that the average value of \( \eta_j \) across any \( w \)-orbit is \( \frac{n}{k} \). When reduced to lowest terms, the denominator of \( \frac{n}{k} \) is \( \frac{k}{\gcd(n, k)} \). Thus, the size of any orbit must be a multiple of \( \frac{k}{\gcd(n, k)} \) in order for the average value of \( \eta_j \) to be \( \frac{n}{k} \). \( \square \)

Corollary 2.22 gives an example where we can use the homomesy to prove a result that neither mentions homomesy nor the statistic that is homomesic. As the sizes of the orbits are generally unpredictable, this is the only known way to prove this. This is one of several instances where homomesy has been used to determine divisibility properties of a map, such as \( [\text{EFG}^+16] \) Corollary 4.8.

It is straightforward to see that Theorem 2.11 (and therefore Corollary 2.22 also) extend if we replace \( w \) by any product of the \( w_i \) maps, each used exactly once, in some order.

**Theorem 2.23.** Let \( \pi \) be a permutation on \([n]\) and \( w_{\pi} := w_{\pi(n)} \circ \cdots \circ w_{\pi(2)} \circ w_{\pi(1)} \). Fix \( \mathcal{F} \) to be either \( \text{Inj}_m(n, k) \) or \( \text{Sur}_1(n, k) \) for positive integers \( n, k, m \). Then under the action of \( w_{\pi} \) on \( \mathcal{F} \), \( \eta_j \) is \( \frac{n}{k} \)-mesic for any \( j \in [k] \).

**Proof.** Notice that for \( \mathcal{F} = \text{Inj}_m(n, k) \) or \( \mathcal{F} = \text{Sur}_m(n, k) \), we have

\[
f \in \mathcal{F} \iff f \circ \pi \in \mathcal{F}.
\]

Therefore whirling \( f \) at index \( \pi(i) \) is like whirling \( f \circ \pi \) at index \( i \), i.e., \( w_i(f \circ \pi) = w_{\pi_i}(f) \circ \pi \). So \( w(f \circ \pi) = w_{\pi}(f) \circ \pi \). Thus for any \( w_{\pi}-\)orbit \( \emptyset = (f_1, f_2, \ldots, f_l) \), there exists a corresponding \( w \)-orbit \( \emptyset' = (f'_1, f'_2, \ldots, f'_l) \) such that \( f'_i = f_i \circ \pi \) for all \( i \). Since \( \eta_j(f \circ \pi) = \eta_j(f) \), Theorem 2.11 implies \( \eta_j \) is also \( \frac{n}{k} \)-mesic on orbits of \( w_{\pi} \).

For the specific case \( \mathcal{F} = \text{Inj}_m(n, 2) \), we can restate our homomesy result in terms of toggle groups. Toggle groups were first introduced by Cameron and Fon-der-Flaass in the specific setting of order ideals of a poset \([\text{CF95}]\) and generalized more recently by Striker \([\text{Str18}]\) via the definition below.

**Definition 2.24 (Str18).** Let \( E \) be a set and \( \mathcal{L} \subseteq 2^E \) a set of “allowed” subsets of \( E \). Then to every \( e \in E \), we define its **toggle** \( t_e : \mathcal{L} \to \mathcal{L} \) as

\[
t_e(X) = \begin{cases} 
X \cup \{e\} & \text{if } e \notin X \text{ and } X \cup \{e\} \in \mathcal{L} \\
X \setminus \{e\} & \text{if } e \in X \text{ and } X \setminus \{e\} \in \mathcal{L} \\
X & \text{otherwise}
\end{cases}
\]

Each toggle \( t_e \) is a permutation on \( \mathcal{L} \). The **toggle group** is the subgroup of the symmetric group \( \mathcal{S}_\mathcal{L} \) on \( \mathcal{L} \) generated by \( \{t_e : e \in E\} \).

Let \( n > 0 \), \( 0 \leq r \leq n/2 \) be integers. In our case, the ground set \( E \) is \([n]\) and our set of allowed subsets is \( \mathcal{L}_r(n) := \{X \subseteq [n] | r \leq \#X \leq n - r\} \). Then for each \( e \in [n] \), the toggle \( t_e : \mathcal{L}_r(n) \to \mathcal{L}_r(n) \) is

\[
t_e(X) = \begin{cases} 
X \cup \{e\} & \text{if } e \notin X \text{ and } \#X \leq n - r - 1, \\
X \setminus \{e\} & \text{if } e \in X \text{ and } \#X \geq r + 1, \\
X & \text{otherwise}.
\end{cases}
\]
Corollary 2.25. Let \( \pi \) be a permutation on \([n]\) and \( T_\pi := t_{\pi(n)} \circ \cdots \circ t_{\pi(2)} \circ t_{\pi(1)} \). Then under the action of \( T_\pi \) on \( \mathcal{L}_r(n) \), the cardinality statistic is \( n/2 \)-mesic.

Proof. Let \( \mathcal{F} = \text{Inj}_{n-r}(n, 2) \). Any function \( f \in \mathcal{F} \) is clearly the indicator function of a subset \( S(f) \subseteq [n] \) with cardinality between \( r \) and \( n-r \), given by \( S(f) := \{ i \in [n] \mid f(i) = 1 \} \). This relation goes both ways so \( S \) is a bijection. It is straightforward to see that \( t_i \circ S = S \circ w_i \)
shown in the commutative diagram below.

\[
\begin{array}{ccc}
\text{Inj}_{n-r}(n, 2) & \xrightarrow{S} & \mathcal{L}_r(n) \\
\downarrow{w_i} & & \downarrow{t_i} \\
\text{Inj}_{n-r}(n, 2) & \xrightarrow{S} & \mathcal{L}_r(n)
\end{array}
\]

So \( T_\pi \circ S = S \circ w_\pi \), where \( w_\pi := w_{\pi(n)} \circ \cdots \circ w_{\pi(2)} \circ w_{\pi(1)} \) as in Theorem 2.23. Therefore any \( T_\pi \)-orbit on \( \mathcal{L}_r(n) \) can be written as \( (S(f_1), S(f_2), \ldots, S(f_\ell)) \) where \( (f_1, f_2, \ldots, f_\ell) \) is a \( w_\pi \)-orbit of \( \mathcal{F} \). Via Theorem 2.23 \( \eta_i \) has average \( n/2 \) on \( (f_1, f_2, \ldots, f_\ell) \). Since \( \eta_i(f) = \#S(f) \), the average cardinality in the \( T_\pi \)-orbit \( (S(f_1), S(f_2), \ldots, S(f_\ell)) \) is \( n/2 \).

Corollary 2.25 is one of many homomesic statistics discovered in toggle group actions defined as a product of every toggle each used exactly once. These results include toggles on order ideals and antichains of various posets, as well as on noncrossing partitions and independent sets of graphs [PR15, Rob16, Had21, EFG16, JRT13, ELM+23].

We also conjecture a generalization of Corollary 2.25 to the piecewise-linear realm. Every set \( S \in \mathcal{L}_r(n) \) is naturally associated to an indicator vector \( (v_1, v_2, \ldots, v_n) \in \{0, 1\}^n \) where \( v_i = 1 \) if \( i \in S \) and \( v_i = 0 \) if \( i \notin S \). For example the subset \( \{2, 5, 6\} \) of \([7]\) is associated with the vector \((0, 1, 0, 0, 1, 1, 0)\). Thus, we can think of \( \mathcal{L}_r(n) \) as the set \( \{ (v_1, v_2, \ldots, v_n) \in \{0, 1\}^n \mid r \leq v_1 + v_2 + \cdots + v_n \leq n-r \} \). To get the piecewise-linear version, we now allow the entries of the vector to be real numbers in the interval \([0, 1]\).

Definition 2.26. Let \( n > 0 \) be an integer and \( 0 \leq r \leq \frac{n}{2} \) be a real number. Define the polytope
\[
\mathcal{P}_r(n) = \{ (v_1, v_2, \ldots, v_n) \in [0, 1]^n \mid r \leq v_1 + v_2 + \cdots + v_n \leq n-r \}\.
\]

Definition 2.27. Let \( 1 \leq i \leq n \). We define the \textit{toggle} \( t_i : \mathcal{P}_r(n) \to \mathcal{P}_r(n) \) as follows. Given a vector \( v = (v_1, v_2, \ldots, v_n) \in \mathcal{P}_r(n) \), let
\[
L = \max\{0, r - v_1 - v_2 - \cdots - v_{i-1} - v_{i+1} - \cdots - v_n\},
\]
\[
R = \min\{1, n - r - v_1 - v_2 - \cdots - v_{i-1} - v_{i+1} - \cdots - v_n\}
\]
be the minimum and maximum values that we could change the value of \( v_i \) to and still obtain a vector in \( \mathcal{P}_r(n) \). Then
\[
t_i(v) = (v_1, v_2, \ldots, v_{i-1}, L + R - v_i, v_{i+1}, \ldots, v_n).
\]

Example 2.28. Let \( n = 7, r = 2 \), and consider the vector \((0.9, 0.6, 0.5, 0, 0, 1, 0.4) \in \mathcal{P}_2(7)\). We will apply the toggles \( T = t_2 \circ \cdots \circ t_2 \circ t_1 \) in a left-to-right order. To be in \( \mathcal{P}_2(7) \), the sum of the entries must be between \( 2 \) and \( 5 \). We first toggle \( t_1 \). The first entry is \( 0.9 \), and this entry can be anything from \( 0 \) to \( 1 \) and still yield of vector in \( \mathcal{P}_2(7) \). So \( t_1 \) changes the first entry to \( 0 + 1 - 0.9 = 0.1 \). Now we have \((0.1, 0.6, 0.5, 0, 0, 1, 0.4) \). To do the toggle \( t_2 \), the second entry can be anything from \( 0 \) to \( 1 \) and still be in \( \mathcal{P}_2(7) \), so we change it from \( 0.6 \) to
0 + 1 − 0.6 = 0.4, and we now have (0.1, 0.4, 0.5, 0, 0, 1, 0.4). We now do the toggle \( t_3 \). The third entry is 0.5, but the sum of the other entries is 1.9, so the third entry must lie in the interval \([0.1, 1]\). Thus \( t_3 \) changes the third entry to \((0.1 + 1 − 0.5 = 0.6) \). Applying \( t_4, t_5, t_6, t_7 \), we get that

\[
(0.9, 0.6, 0.5, 0, 0, 1, 0.4) \xrightarrow{T} (0.1, 0.4, 0.6, 1, 1, 0, 0.6).
\]

Applying \( T \) repeatedly, we get an orbit of length 1808, and the average sum of the entries is 7/2 for vectors in the orbit.

We have tested several random orbits and we conjecture that the homomesy in Corollary 2.25 extends to \( P_r(n) \). There is one caveat though. A vector with irrational entries probably does not always yield a finite orbit. Thus, we refer to the notion of asymptotic homomesy, discussed in [Vor18, Jos19, HJ22, Hop22], and which is clearly equivalent to the previous definition of homomesy over a finite orbit.

**Definition 2.29** ([Vor18, Definition 5.3.1]). Suppose we have a set \( S \), a map \( w : S \to S \), and a function ("statistic") \( f : S \to \mathbb{R} \). If there exists \( c \in \mathbb{R} \) such that for every \( x \in S \),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(w^i(x)) = c,
\]

then we say that \( f \) is homomesic with average \( c \) (or \( c \)-mesic) under the action of \( w \) on \( S \).

**Conjecture 2.30.** Let \( n > 0 \) be an integer and \( 0 \leq r \leq \frac{n}{2} \) be a real number. Let \( \pi \) be a permutation on \([n]\) and \( T_\pi := t_{\pi(1)} \circ \cdots \circ t_{\pi(n)} \). Then under the action of \( T_\pi \) on \( P_r(n) \), the cardinality statistic is homomesic with average \( n/2 \).

It is not uncommon for homomesies in the combinatorial realm to lift to the piecewise-linear realm; see [EP21, Jos19, HJ22, Hop22]. However, as far as we know, the only homomesies shown to lift to the piecewise-linear realm are on the order polytope or chain polytope of a poset. So Conjecture 2.30 is interesting as it would be an example outside of a poset.

### 3. Whirling Restricted Growth Words

A **partition** of the set \([n]\) is an unordered collection of disjoint subsets of \([n]\), called **blocks**, whose union is \([n]\). For example \(\{\{1, 2, 5\}, \{4, 7\}, \{3, 6\}\}\) is a partition of \([7]\). We often write a partition by writing each block without set braces or commas, and using a bar to separate blocks. For example, \(\{\{1, 2, 5\}, \{4, 7\}, \{3, 6\}\}\) would be written \(125|47|36\). Notice that \(125|47|36 = 125|36|47 = 63|152|47\) because the order of the blocks is unimportant as well as the order of the numbers within each block.

Another way of encoding a set partition is using its restricted growth word.

**Definition 3.1.** Let \( \pi \) be a partition of \([n]\) with \( k \) blocks. First order the blocks according to their least elements. The **restricted growth word** (or **RG-word**) according to \( \pi \) is the function \( f : [n] \to [k] \) where \( f(i) = j \) if \( i \) is in the \( j \)th block of \( \pi \).

**Example 3.2.** For the partition \(125|46|37\) of \([7]\), we reorder the blocks as \(125|37|46\). Then its RG-word is \(1123132\).

Let \( RG(n, k) \) denote the set of RG-words corresponding to partitions of \([n]\) with exactly \( k \) blocks and \( RG(n) \) the set of RG-words corresponding to all partitions of \([n]\) (without
specifying the number of blocks). That is
\[
\text{RG}(n) = \bigcup_{k=1}^{n} \text{RG}(n, k).
\]
It is clear that any RG-word corresponds to a unique set partition, and that an RG-word of length \(n\) is in \(\text{RG}(n, k)\) where \(k\) is the maximum entry.

Let \(\binom{n}{k}\) denote the number of partitions of \([n]\) with exactly \(k\) blocks and \(B_n\) denote the total number of partitions of \([n]\). These are called the Stirling numbers of the second kind and Bell numbers respectively; e.g., see [Sta11]. RG-words were first introduced by Hutchinson [Hut63] and have been studied recently by Cai and Readdy [CR17] for their connections to the \(q\) and “\(q-(1+q)\)” analogues for Stirling numbers of the second kind.

The following proposition is clear from the way RG-words are defined, since the blocks are ordered via least elements.

**Proposition 3.3.**

- A function \(f : [n] \to [k]\) is in \(\text{RG}(n, k)\) if and only if it is surjective and for all \(1 \leq j \leq k-1\), \(\min\{i \mid f(i) = j\} \leq \min\{i \mid f(i) = j+1\}\).
- A function \(f : [n] \to [n]\) is in \(\text{RG}(n)\) if and only if for any \(1 \leq j \leq n-1\),
  - if \(j+1\) is in the range of \(f\), then so is \(j\), and in this case,
  - \(\min\{i \mid f(i) = j\} \leq \min\{i \mid f(i) = j+1\}\).

The condition \(\min\{i \mid f(i) = j\} \leq \min\{i \mid f(i) = j+1\}\) means that in the one-line notation of \(f\), the first occurrence of \(j\) occurs before the first \(j+1\). This is where the term “restricted growth” comes from.

Note that we consider the codomain of functions in \(\text{RG}(n, k)\) to be \([k]\) and of \(\text{RG}(n)\) to be \([n]\). Therefore for \(\mathcal{F} = \text{RG}(n, k)\), \(w_i\) adds 1 mod \(k\) repeatedly to the value \(f(i)\) until we get a function in \(\mathcal{F}\), but for \(\mathcal{F} = \text{RG}(n)\), \(w_i\) adds 1 mod \(n\) repeatedly instead.

**Proposition 3.4.** Let \(\mathcal{F} = \text{RG}(n, k)\), \(f \in \mathcal{F}\), and \(i \in [n]\). Let \(f(i) = j\). Then \(w_i\) changes the output at \(i\) in the following way.

- If \(j \neq k\), then
  \[
  (w_i(f))(i) = \begin{cases} 
  j+1 & \text{if there exists } i' < i \text{ such that } f(i') = j, \\
  j & \text{if the only } i' < \min\{h \mid f(h) = j+1\} \text{ s.t. } f(i') = j \text{ is } i' = i, \\
  1 & \text{otherwise.}
  \end{cases}
  \]

- If \(j = k\), then
  \[
  (w_i(f))(i) = \begin{cases} 
  k & \text{if the only value } i' \text{ for which } f(i') = k \text{ is } i' = i, \\
  1 & \text{otherwise.}
  \end{cases}
  \]

Refer to Example 3.6 for an example of Proposition 3.4. This says that in the one-line notation of \(f\), if the value in position \(i\) is \(j \neq k\), then \(w_i\) adds 1 to it if it is not the first occurrence of \(j\). If it is the first occurrence of \(j\), then \(w_i\) leaves it alone if there is not another \(j\) to the left of the first \(j+1\), and otherwise changes the value to 1. In the case where the value in position \(i\) is \(k\), \(w_i\) leaves it alone if it is the only \(k\), and otherwise changes it to 1.

**Proof.** Case 1: \(f(i) \neq k\). We have two subcases to consider.

**Case 1a:** There exists \(i' < i\) such that \(f(i') = j\). Then changing the value of \(f(i)\) will not change the RG-word criterion \(\min\{i \mid f(i) = j\} \leq \min\{i \mid f(i) = j+1\}\) from
Proposition 3.3 So \((w_i(f))(i) = j + 1\).

**Case 1b:** There does not exist \(i' < i\) such that \(f(i') = j\). Then we cannot change the value of \(f(i)\) to anything larger than \(j\) and still have an RG-word. If the only \(i' < \min\{h \mid f(h) = j + 1\}\) such that \(f(i') = j\) is \(i' = i\), then changing the value of \(f(i)\) to something other than \(j\) will violate \(\min\{i \mid f(i) = j\} \leq \min\{i \mid f(i) = j + 1\}\) and not be an RG-word. Otherwise, changing the value of \(f(i)\) to 1 results in an RG-word.

**Case 2:** \(f(i) = k\). Since an RG-word must be surjective, if there does not exist \(i' \neq i\) satisfying \(f(i') = k\), then we cannot change the value of \(f(i)\) to something other than \(k\). So \((w_i(f))(i) = k\). Otherwise, changing the value of \(f(i)\) to 1 still results in an RG-word, and so \((w_i(f))(i) = 1\).

For the \(F = \text{RG}(n)\) case, we have the following slightly different characterization of \(w_i\). We omit the proof as it is straightforward and similar to the proof of Proposition 3.3.

**Proposition 3.5.** Let \(F = \text{RG}(n)\), \(f \in F\), and \(i \in [n]\). Let \(f(i) = j\). Then

\[
(w_i(f))(i) = \begin{cases} 
  j + 1 & \text{if there exists } i' < i \text{ such that } f(i') = j, \\
  j & \text{if the only } i' < \min\{h \mid f(h) = j + 1\} \text{ s.t. } f(i') = j \text{ is } i' = i, \\
  1 & \text{otherwise.}
\end{cases}
\]

If there is no \(h\) satisfying \(f(h) = j + 1\), then we are not in the second case.

**Example 3.6.** For \(F = \text{RG}(7, 4)\),

\[
1213341 \xrightarrow{w_1} 1213341 \xrightarrow{w_2} 1213341 \xrightarrow{w_3} 1223341 \xrightarrow{w_4} 1221341 \xrightarrow{w_5} 1221341 \xrightarrow{w_6} 1212341 \xrightarrow{w_7} 1213241 \xrightarrow{w_8}
\]

so \(w(1213341) = 1221342\).

On the other hand, for \(F = \text{RG}(7)\),

\[
1213341 \xrightarrow{w_1} 1213341 \xrightarrow{w_2} 1213341 \xrightarrow{w_3} 1223341 \xrightarrow{w_4} 1221341 \xrightarrow{w_5} 1221341 \xrightarrow{w_6} 1212341 \xrightarrow{w_7} 1213241 \xrightarrow{w_8} 1212341 \xrightarrow{w_9}
\]

so \(w(1213341) = 1221321\).

Propositions 3.4 and 3.5 show that for \(F = \text{RG}(n, k)\) or \(F = \text{RG}(n)\), whirling at index \(i\) either adds 1 to the value \(f(i)\) or leaves \(f(i)\) alone or changes the value \(f(i)\) to 1.

For \(F = \text{RG}(n, k)\) or \(F = \text{RG}(n)\), \(w_1\) acts trivially since any RG-word \(f\) satisfies \(f(1) = 1\). Thus, \(w = w_1 \circ \cdots \circ w_3 \circ w_2\) on these families of functions. So when we consider generalized toggle orders we define \(w_\pi = w_\pi(n) \circ \cdots \circ w_\pi(3) \circ w_\pi(2)\) where \(\pi\) is a permutation of \(\{2, 3, \ldots, n\}\).

**Definition 3.7.** Let

\[
I_{i \to 1}(f) = \begin{cases} 
  1 & \text{if } f(i) = 1, \\
  0 & \text{if } f(i) \neq 1.
\end{cases}
\]

The main homomesy result is the following.

**Theorem 3.8.** Let \(n \geq 2\). Fix \(F\) to be either \(\text{RG}(n)\) or \(\text{RG}(n, k)\) for some \(1 \leq k \leq n\). Let \(\pi\) be a permutation of \(\{2, 3, \ldots, n\}\). Under the action of \(w_\pi\) on \(F\), \(I_{i \to 1} - I_{j \to 1}\) is \(0\)-mesic for any \(i, j \in \{2, 3, \ldots, n\}\).

See Figure 6 for an illustration of Theorem 3.8 for the orbits under the action of \(w\) (i.e., \(\pi\) is the identity) over \(F = \text{RG}(5, 3)\). This theorem is another instance where there is homomesy under an action that produces unpredictable orbit sizes, and for which the order of the map is unknown in general.

For RG-words, results for \(w\) need not necessarily extend to other \(w_\pi\) products like they did for \(\text{Inj}_m(n, k)\) and \(\text{Sur}_m(n, k)\), since a rearrangement of an RG-word is not always an
Let \( f \) ranging over \( F \) be either \( \text{RG}(n) \) or \( \text{RG}(n, k) \) for some \( 1 \leq k \leq n \). Let \( \pi, \sigma \) be permutations on \( \{2, 3, \ldots, n\} \) where \( \sigma(i) = \pi(i+1) \) for \( 2 \leq i \leq n-1 \) and \( \sigma(n) = \pi(2) \). Then for any \( w_\pi \)-orbit \( O = (f_1, f_2, \ldots, f_\ell) \), there is a \( w_\sigma \)-orbit \( O' = (f'_1, f'_2, \ldots, f'_\ell) \) for which \( f'_j = w_\pi(2)(f_j) \) for all \( j \in [\ell] \).

Also, given \( h \in [n] \), the multiset of values \( f(h) \) as \( f \) ranges over \( O \) is the same as that for \( f \) ranging over \( O' \).

**Proof.** Let \( O = (f_1, f_2, \ldots, f_\ell) \) be a \( w_\pi \)-orbit satisfying \( w_\pi(f_j) = f_{j+1} \) for all \( j \). Consider the subscripts to be mod \( \ell \), the length of the orbit, so \( f_{\ell+1} = f_1 \) for instance. Note that \( w_\pi = w_\pi(2) \circ \cdots \circ w_\pi(3) \circ w_\pi(1) \) and \( w_\sigma = w_\pi(2) \circ w_\pi(1) \circ \cdots \circ w_\pi(3) \). Therefore, if we let \( f'_j = w_\pi(2)(f_j) \) for all \( j \), then \( w_\sigma(f'_j) = f'_{j+1} \) for all \( j \). So \( O' = (f'_1, f'_2, \ldots, f'_\ell) \) is a \( w_\sigma \)-orbit.

For \( h \neq \pi(2) \), \( f_\pi(h) = f'_j(h) \). Also, \( f'_j(\pi(2)) = f_{i+1}(2) \). Thus for any \( h \in [n] \), the multiset of values \( f(h) \) is the same for \( f \) ranging over \( O \) is the same as that for \( O' \).

**Remark 3.10.** Suppose we have \( w_\pi = w_\pi(2) \circ \cdots \circ w_\pi(3) \circ w_\pi(1) \) and \( w_\sigma \) is some cyclic shift of this order of composition. Then by applying Lemma 3.9 repeatedly, we get that every

| \( f \) | \( w(f) \) | \( g \) | \( w(g) \) |
|---|---|---|---|
| \( f \) | 12123 | \( g \) | 12213 |
| \( w(f) \) | 11231 | \( w(g) \) | 11223 |
| \( w^2(f) \) | 12312 | \( w_2(g) \) | 12331 |
| \( w^3(f) \) | 12323 | \( w^3(g) \) | 12132 |
| \( w^4(f) \) | 12131 | \( w^4(g) \) | 12233 |
| \( w^5(f) \) | 12232 | \( w^5(g) \) | 11213 |
| \( w^6(f) \) | 11233 | \( w^6(g) \) | 12321 |
| \( w^7(f) \) | 12311 | \( w^7(g) \) | 12332 |
| \( w^8(f) \) | 12322 | \( w^8(g) \) | 12133 |
| \( w^9(f) \) | 12333 | \( w^9(g) \) | 12133 |
| \( w^{10}(f) \) | 12223 | \( w^{10}(g) \) | 12133 |
| \( w^{11}(f) \) | 11223 | \( w^{11}(g) \) | 12133 |
| \( w^{12}(f) \) | 12231 | \( w^{12}(g) \) | 12133 |
| \( w^{13}(f) \) | 11231 | \( w^{13}(g) \) | 12133 |
| \( w^{14}(f) \) | 12312 | \( w^{14}(g) \) | 12133 |
| \( w^{15}(f) \) | 12133 | \( w^{15}(g) \) | 12133 |
\( w_\pi \)-orbit \( \mathcal{O} \) corresponds uniquely with a \( w_\sigma \)-orbit \( \mathcal{O}' \) where the multisets of \( f(h) \) values are the same for \( f \) ranging over \( \mathcal{O} \) as for \( \mathcal{O}' \).

For example, on \( \mathcal{F} = \text{RG}(6) \), \( w_3 \circ w_5 \circ w_4 \circ w_6 \circ w_2 \) and \( w_4 \circ w_6 \circ w_2 \circ w_3 \circ w_5 \) satisfy the above orbit correspondence. However in general two orders for applying the whirling maps do not yield the same orbit structure. For example, \( w_3 \circ w_5 \circ w_4 \circ w_6 \circ w_2 \) does not have the same orbit structure as \( w_5 \circ w_6 \circ w_3 \circ w_2 \circ w_4 \) because these are not cyclic rotations of each other.

**Lemma 3.11.** Let \( \mathcal{F} \) be either \( \text{RG}(n) \) or \( \text{RG}(n,k) \) for some \( 1 \leq k \leq n \) and \( \pi \) be a permutation of \( \{2,3,\ldots,n\} \). Let \( i = \pi(2) \) and suppose \( 2 \leq i \leq n - 1 \). Then for \( f \in \mathcal{F} \), if \( f(i) = 1 \), then \( (w_\pi^{-1}(f))(i) \geq (w_\pi^{-1}(f))(i + 1) \).

**Proof.** Let \( f \in \mathcal{F} \) such that \( f(i) = 1 \). Let \( g = w_\pi^{-1}(f) \). Since \( w_i \) is the first map we apply when we apply \( w_\pi \) to \( g \), \( w_i \) changes the value of \( g(i) \) to 1. By Propositions 3.4 and 3.5, either \( i \) is the least \( i' \) satisfying \( g(i') = g(i) \), or \( g(i) = k \) (the latter option only being possible or making sense if \( \mathcal{F} = \text{RG}(n,k) \)).

Assume by way of contradiction that \( g(i) < g(i + 1) \). This could only be possible in the case where \( i \) is the least \( i' \) satisfying \( g(i') = g(i) \). From the restricted growth condition, \( g(i + 1) = g(i) + 1 \). Then Propositions 3.4 and 3.5 say that \( w_i \) would leave the value \( g(i) \) as is, not change it to 1, which is a contradiction. Thus, \( g(i) \geq g(i + 1) \).

**Lemma 3.12.** Let \( i \in \{2,3,\ldots,n\} \).

1. The number of \( f \in \text{RG}(n,k) \) satisfying \( f(i) = 1 \) is \( \binom{n-1}{k} \).

2. The number of \( f \in \text{RG}(n) \) satisfying \( f(i) = 1 \) is \( B_{n-1} \).

**Proof.** The condition \( f(i) = 1 \) means 1 and \( i \) are in the same block of the partition of \( [n] \) corresponding to \( f \). Such set partitions can be formed by first choosing a partition of \( \{2,3,\ldots,n-1\} \) and placing 1 in the same block as \( i \).

In particular, the counting formulas in Lemma 3.12 do not depend on \( i \), which is what is important for the proof of Theorem 3.8.

**Proof of Theorem 3.8** Let \( \mathcal{O} \) be a \( w_\pi \)-orbit. The goal is to prove that in \( \mathcal{O} \), there are the same number of functions \( f \in \mathcal{O} \) satisfying \( f(i) = 1 \) as those satisfying \( f(j) = 1 \), when \( 2 \leq i, j \leq n \). We will approach this by showing that \( \mathcal{O} \) has the same number of functions \( f \) satisfying \( f(i) = 1 \) as \( f(i + 1) = 1 \), for \( 2 \leq i \leq n - 1 \).

Let \( i \in \{2,3,\ldots,n-1\} \). Without loss of generality, assume \( i = \pi(2) \). This is because by Remark 3.10, we could cyclically change \( \pi \) to get this condition, without changing the property we wish to prove for orbits of \( w_\pi \).

Let \( f \in \mathcal{O} \) be a function satisfying \( f(i) = 1 \) and \( f(i + 1) \neq 1 \). By Lemma 3.11, \( (w_\pi^{-1}(f))(i) \geq (w_\pi^{-1}(f))(i + 1) \). Now let \( r \) be the least positive integer satisfying \( (w_\pi^{-1}(f))(i) \geq (w_\pi^{-1}(f))(i + 1) \). Such an \( r \) must exist because when repeatedly applying \( w_\pi \) to \( f \), we must eventually cycle back around to \( w_\pi^{-1}(f) \).

Let \( g = w_\pi^{r-1} \). Then

\[
(1) \quad g(i) < g(i + 1) \quad \text{and} \quad (w_\pi(g)(i))(i) \geq (w_\pi(g))(i + 1).
\]

By the way whirling at an index works, either \( (w_\pi(g))(i) = g(i) + 1 \) or \( (w_\pi(g))(i + 1) = 1 \). Suppose \( (w_\pi(g))(i + 1) \neq 1 \). Then \( (w_\pi(g))(i + 1) = g(i + 1) \) or \( (w_\pi(g))(i + 1) = g(i) + 1 \), the latter of which violates Equation (1) because \( (w_\pi(g))(i) = g(i) + 1 \). So \( (w_\pi(g))(i + 1) = g(i + 1) \). Then from Equation (1), we have \( (w_\pi(g))(i) = g(i) + 1 = g(i + 1) \). However, we
apply \(w_{i+1}\) after \(w_i\), so \(w_{i+1}\) would have to add 1 to the value \(g(i + 1)\) by Propositions 3.4 and 3.5. This contradicts \(w_\pi(g)(i + 1) = g(i + 1)\). Thus,
\[
(w_\pi^w(f))(i + 1) = (w_\pi(g))(i + 1) = 1.
\]

By Lemma 3.11 \((w_\pi^w(f))(i) \neq 1\) for all \(1 \leq m \leq r\). Therefore, given any function \(f \in \mathcal{O}\) satisfying \(f(i) = 1\) and \(f(i + 1) \neq 1\), as we repeatedly apply \(w_\pi\) to \(f\), we obtain a function that sends \(i + 1\) to 1 before we get another function that sends \(i\) to 1. This implies
\[
\# \{ f \in \mathcal{O} \mid f(i) = 1 \} \leq \# \{ f \in \mathcal{O} \mid f(i + 1) = 1 \}.
\]

By Lemma 3.12 \# \{ \substack{f \in \mathcal{F} \mid f(i) = 1 \} \} = \# \{ \substack{f \in \mathcal{F} \mid f(i + 1) = 1 \} \}. Therefore, since \# \{ \substack{f \in \mathcal{O} \mid f(i) = 1 \} \} \leq \# \{ \substack{f \in \mathcal{O} \mid f(i + 1) = 1 \} \} for every \(w_\pi\)-orbit \(\mathcal{O}\) over \(\mathcal{F}\), we must have \# \{ \substack{f \in \mathcal{O} \mid f(i) = 1 \} \} = \# \{ \substack{f \in \mathcal{O} \mid f(i + 1) = 1 \} \} for every orbit \(\mathcal{O}\).

Theorem 3.8 implies another homomesic statistic only for the action of \(w_\pi\) only on \(\text{RG}(n, k)\).

**Corollary 3.13.** Let \(n \geq 2\) and \(\pi\) a permutation of \(\{2, 3, \ldots, n\}\). Under the action of \(w_\pi\) on \(\text{RG}(n, k)\), the statistic
\[
f \mapsto \binom{k}{2}f(2) - f(n)
\]
is homomesic with average \(k(k - 2)\).

**Proof.** For \(f \in \text{RG}(n, k)\), \(f(2)\) is always 1 or 2. Consider a \(w_\pi\)-orbit \(\mathcal{O}\) with length \(|\mathcal{O}|\). If \(f(2) = 2\) and \(f(n) = k\) for every \(f \in \mathcal{O}\), then the total value of \(f \mapsto \binom{k}{2}f(2) - f(n)\) across \(\mathcal{O}\) is \(\frac{k(k-1)}{2}2\ell(\mathcal{O}) - k\ell(\mathcal{O}) = k(k-2)\ell(\mathcal{O})\).

Suppose instead that there are \(c\) functions \(f \in \mathcal{O}\) that satisfy \(f(2) = 1\). Then this decreases the total value of \(\binom{k}{2}f(2) - f(n)\) across \(\mathcal{O}\) by \(\binom{k}{2}c\).

From Theorem 3.8 there are also \(c\) functions \(f \in \mathcal{O}\) satisfying \(f(n) = 1\). If \(f(n) \neq k\), then \(n\) cannot be the least \(i\) for which \(f(i) = f(n)\). Thus, when \(f(n) \neq k\), \(w_\pi\) adds 1 to the value of \(f(n)\). When \(f(n) = k\), either \(w_\pi(f)(n) = 1\) or \(w_\pi(f)(n) = k\) via Proposition 3.4. So there are also \(c\) functions satisfying \(f(n) = j\) for any \(j \in [k - 1]\). In relation to the original case where \(f(n) = k\) for all \(f\), this decreases the total value of \(f(n)\) across \(\mathcal{O}\) by \(c(1 + 2 + 3 + \cdots + (k - 1)) = \binom{k}{2}c\).

Therefore, the total value of \(f \mapsto \binom{k}{2}f(2) - f(n)\) across \(\mathcal{O}\) is
\[
k(k - 2)\ell(\mathcal{O}) - \binom{k}{2}c + \binom{k}{2}c = k(k - 2)\ell(\mathcal{O}).
\]
Hence this statistic has average \(k(k - 2)\) on \(\mathcal{O}\).

We finish the paper with some conjectural homomesies for whirling on the set of RG-words corresponding to noncrossing partitions.

**Definition 3.14.** A partition of \([n]\) is said to be **noncrossing** if whenever \(i < j < k < \ell\), it is not the case that \(i\) and \(k\) belong to one block of \(\pi\) with \(j\) and \(\ell\) belonging to another block. A **noncrossing RG-word** is an RG-word whose corresponding partition is noncrossing. Let \(\text{RG}_{nc}(n, k)\) and \(\text{RG}_{nc}(n)\) denote the sets of all noncrossing RG-words in \(\text{RG}(n, k)\) and \(\text{RG}(n)\) respectively.

The proof of the following proposition is obvious.
Proposition 3.15. An RG-word $f$ is noncrossing if and only if in the one-line notation, we do not have $abab$ in order (not necessarily consecutively) for some $a \neq b$.

Example 3.16. Let $f = 122133143$. Then $f \in \text{RG}(9, 4)$ but $f \not\in \text{RG}_{nc}(9, 4)$ because the red values are 1313.

Due to the noncrossing condition, there is not such a simple description (like Proposition 3.4 or 3.5) of how $w_i$ acts for $F = \text{RG}_{nc}(n, k)$ or $F = \text{RG}_{nc}(n)$. In the following example, we see $w_i$ can decrease $f(i)$ to something other than 1, or increase $f(i)$ by more than one.

Example 3.17. For $F = \text{RG}_{nc}(6, 4)$,

$$123442 \xrightarrow{w_1} 123442 \xrightarrow{w_2} 123442 \xrightarrow{w_3} 123242 \xrightarrow{w_4} 123242 \xrightarrow{w_5} 123244$$

so $w(123442) = 123244$.

For $F = \text{RG}(6)$,

$$123442 \xrightarrow{w_1} 123442 \xrightarrow{w_2} 123442 \xrightarrow{w_3} 123442 \xrightarrow{w_4} 123242 \xrightarrow{w_5} 123242 \xrightarrow{w_6} 123224$$

so $w(123442) = 123224$.

We conjecture homomesy for whirling noncrossing RG-words. Again $w_1$ acts trivially for $F = \text{RG}_{nc}(n)$ or $F = \text{RG}_{nc}(n, k)$, so for simplicity we ignore $w_1$.

Conjecture 3.18. Let $n \geq 2$. Fix $F$ to be either $\text{RG}_{nc}(n)$ or $\text{RG}_{nc}(n, k)$ for some $1 \leq k \leq n$. Let $\pi$ be a permutation of $\{2, 3, \ldots, n\}$. Under the action of $w_\pi$ on $F$, $I_{2\mapsto 1} - I_{n\mapsto 1}$ is 0-mesic.

The proof of Theorem 3.8 relied on the fact from Lemma 3.12 that for $F = \text{RG}(n, k)$ or $F = \text{RG}(n)$, $\# \{ f \in F \mid f(i) = 1 \} = \# \{ f \in F \mid f(j) = 1 \}$ given any $2 \leq i, j \leq n$. For $F = \text{RG}_{nc}(n, k)$ or $F = \text{RG}_{nc}(n)$, we only conjecture this when $i = 2, j = n$. For example, over the set of noncrossing partitions on $[4]$, there are five partitions containing 1 and 2 in the same block, four partitions containing 1 and 3 in the same block, and five partitions containing 1 and 4 in the same block.

Under the whirling left-to-right order ($\pi$ is the identity) Conjecture 3.18 has been confirmed for all $n \leq 9$ and relevant $k$ values ($k \in [n]$). It has also been tested and confirmed for many random whirling orders (with $n \leq 9$).

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