The Causal approach for the electron-positron scattering in the Generalized Quantum Electrodynamics

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Abstract

In this paper we study the generalized electrodynamics contribution for the electron-positron scattering process, $e^- e^+ \rightarrow e^- e^+$, the Bhabha scattering. Within the framework of the standard model, for energies larger when compared to the electron mass, we calculate the cross section expression for the scattering process. This quantity is usually calculated in the framework of the Maxwell electrodynamics and, by phenomenological reasons, corrected by a cut-off parameter. On the other hand, by considering the generalized electrodynamics instead of Maxwell’s, we can show that the effects played by the Podolsky mass is actually a natural cut-off parameter for this scattering process. Furthermore, by means of experimental data of Bhabha scattering we will estimate its lower bound value. Nevertheless, in order to have a mathematically well defined description of our study we shall present our discussion in the framework of the Epstein-Glaser causal theory.
1 Introduction

Perturbative Quantum Electrodynamics (QED) is a gauge theory that presents a remarkable computational success. For instance, one may cite its impressive accuracy with the measurement of the anomalous magnetic moment of the electron and the muon [1]. However, it is a well-known fact that the standard model of particles is nothing more than an effective theory [2], due to its limited energy range. Because of that, there is room for different theories proposals in describing, for instance, the electromagnetic field; thus, if we subscribe to the standard lore of effective field theories, all possible terms allowed by the symmetries of the theory ought to be included. An interesting branch of such effective theories are the higher-order derivative (HD) Lagrangian functions [3]. They were initially proposed as an attempt to enhance and render a better ultraviolet behavior and renormalizability properties of physically relevant models. Moreover, in the electromagnetic field it is known that the Maxwell’s Lagrangian depends, at most, on first-order derivatives. However, one may add a second-order term in such a way that all original symmetries are preserved. In fact, it has been proved in the Ref. [4] that such term is unique when one requires that the theory’s linearity and Abelian $U(1)$ and Lorentz symmetry be preserved. As a result we have the generalized electrodynamics Lagrangian introduced by Bopp [5] and Podolsky and Schwed [6]. Moreover, an important feature of the generalized quantum electrodynamics (GQED) is that in the same way as the Lorenz condition is a natural gauge condition to the Maxwell electrodynamics, it has been showed that the generalized electrodynamics has its counter-part, the so-called the generalized Lorenz condition [7]:

$$\Omega[A] = (1 + \alpha^2 \Box) \partial^\mu A_\mu.$$  

A recent study via functional methods has showed that the electron self-energy and vertex part of the GQED are both ultraviolet finite at $\alpha$ order [8].
Moreover, despite the radiative functions previously evaluated, there are still some interesting scenarios where one may investigate for deviations of standard physics that would be rather important to be discussed, for instance, the scattering of standard model particles \cite{9}. Within these, we may cite the study of the Moeller scattering \cite{10}, $e^- e^- \rightarrow e^- e^-$, and Bhabha scattering \cite{11}, $e^- e^+ \rightarrow e^- e^+$, as offering some particularly interesting possibilities owing to their large cross section, which lead to a very good statistic. On the other hand, the modern linear electron collider allows experiments with highly precision measurements. Another important experimentation is the annihilation process $e^+ e^- \rightarrow \mu^- \mu^+$, and tau: $e^- e^+ \rightarrow \tau^- \tau^+$. We can cite studies in the literature analyzing deviations due to Lorentz-violating effects of the cross section in the electron-positron annihilation \cite{12}. Also, we may cite recurrent discussion in improving two-loop calculations for Bhabha’s scattering by contributions from QED \cite{13}.

The Bhabha scattering is one of the most fundamental reactions in QED processes, as well as in the phenomenological study in particle physics. Also, it is particularly important mainly because it is the process employed in determining the luminosity at $e^+ e^-$ collider. At collider operating at C.M. energies on $\mathcal{O}(100 \text{ GeV})$ the relevant kinematic region is the one in which the angle between the outgoing particles and the beam line is about of a few degrees only, in these regions the Bhabha scattering cross section is comparatively large and the QED contribution dominated. Since the luminosity value is measured with very high accuracy \cite{14}, it is necessary a precise theoretical calculation of the value for the Bhabha scattering cross section in order to keep the luminosity value error small.

Despite the incredible match between theoretical and experimental values in QED, there are some particular intriguing discrepancy between the QED results and the measurements that may give us, in principle, a window of possibility in proposing a modification to the QED-vertex and/or to the photon propagator; thus, we could, in principle, calculate a lower limit for the mass of a massive ”photon”. Nevertheless, since the generalized quantum electrodynamics is a good alternative to describe the interaction between fermions and photons, we shall consider this theory in order to calculate the $\alpha^2$ order correction to the usual QED differential cross section for the Bhabha scattering. For this purpose we shall consider the framework of the perturbative causal theory of Epstein-Glaser \cite{15}, specifically its momentum space form developed by Scharf et al \cite{16}.

Therefore, in this paper, we calculate within the framework of the Epstein-Glaser causal theory the contribution of the generalized electrodynamics for the cross section of the Bhabha scattering. This paper is organized as follows. In the Sect.2 by means of the theory of distributions, we introduce the analytic representation for the positive; moreover, we introduce an alternative gauge condition different to the generalized Lorentz condition. In the Sect.3 we obtain well defined (Feynman) electromagnetic propagator in the causal approach. Finally, in the Sect.4 we calculate the GQED correction for the Bhabha scattering, and by using the experimental data for this process we determine a lower bound value for the Podolsky’s mass. In the Sect. 5 we summarize the results, and present

\footnote{Simply called electromagnetic propagator, but since we introduce in this paper several propagators, thus we shall adopt the notation of \cite{17}.}
our final remarks and prospects.

2 Analytic Representation for Propagators

In order to develop the analytic representation for propagators we shall consider the Wightman’s formalism [18], this axiomatic approach guarantees that general physical principles are always obeyed. To formulate the analytic representation, we start by discussing the free scalar quantum field in this formalism.

The free scalar field, \( \phi \), is a general distributional solution of the Klein-Gordon-Fock equation \((\Box + m^2) \phi = 0\), and have the form

\[
\phi (x) = (2\pi)^{-3/2} \int d^4k \delta (k^2 - m^2) \tilde{a} (k) e^{-ikx},
\]

(2.1)

by considering \( \phi \) hermitian it follows that \( \tilde{a} (-k) = \tilde{a}^\dagger (k) \). The whole theory is formulated in terms of this field, understood as an operator-valued distribution, which generates the full Hilbert space from the invariant vacuum \(|\Omega\rangle\). Since \( \phi \) is a distribution this must be valued in a test function \( \mathcal{F} (\mathbb{R}^4) \), as a set of test functions. In this space we can also define the Fourier transformation of the scalar field, \( \hat{\phi} (k) \), as it follows

\[
\langle \hat{\phi} , \tilde{f} \rangle = \langle \phi , g \rangle = \int dk \hat{\phi} (k) \tilde{f} (k),
\]

(2.2)

where \( \tilde{f} \) is the inverse Fourier transformation of \( f \), and it is a well-behaved test function of \( \hat{\phi} \).

Formally, we can split the field into the positive and negative frequency components, defined in the distributional form as it follows

\[
\phi^{(\pm)} [f] |\Omega\rangle = \int d^4k \theta (k_0) \delta (k^2 - m^2) \tilde{a} (k) \tilde{f} (\pm k) |\Omega\rangle,
\]

(2.3)

where \( \phi^{(+)\rangle} \) is named the positive and \( \phi^{(-)} \) the negative part of the field. By the spectral condition, we obtain that we do not have components in \( k \in V^- \rightarrow -k \in V^+ \), then we see that the part associated to the \( \tilde{f} (-k) \) must be zero, while the part associated to \( \tilde{f} (k) \) must be nonzero, these conditions are satisfied only if

\[
\tilde{a} (k) |\Omega\rangle = 0, \quad \tilde{a}^\dagger (k) |\Omega\rangle \neq 0.
\]

(2.4)

From these conditions follow that the operators \( \tilde{a} (k) \) and \( \tilde{a}^\dagger (k) \) are interpreted as the operators of annihilation and creation, respectively.

In this formalism the central objects are the so-called Wightman’s functions. They are defined as the vacuum expected values (VEV) of a product of fields. For instance, the 2-points Wightman function for scalar fields is given by

\[
W_2 (x_1 , x_2) \equiv \langle \Omega | \phi (x_1) \phi (x_2) |\Omega\rangle = (2\pi)^{-2} \int d^4k W_2 (k) e^{-ik(x_1 - x_2)},
\]

(2.5)

\[\footnote{Actually, we will briefly review its development, since a detailed discussion can be found in the Ref. [19]}\]
where $\hat{W}_2(k)$ is the two-point Wightman function in the momentum space. Of course they are not functions in the strict sense, but tempered distributions. Moreover, we have that the Wightman function obeys the same equation as for the free field. Hence, as a consequence of the spectral condition, one can find that the two-point Wightman function in the momentum space is given by

$$\hat{W}_2(k) = \frac{1}{2\pi} \theta(k_0) \delta(k^2 - m^2).$$

(2.6)

Now with the necessary physical concepts and tools in hands, we shall now introduce the analytic representation for the propagators. In order to elucidate the content we shall discuss the case of scalar fields first.

### 2.1 Analytic representation of the PF and NF propagators

Since the fundamental propagators are linear combinations of the positive (PF) and negative (NF) frequency parts of the propagator, it is rather natural to consider them here in our development. We define the PF propagator by the relation of the contraction between scalar fields:

$$\overline{\phi(x) \phi(y)} \equiv \left[ \phi^{(-)}(x), \phi^{(+)}(y) \right] = -iD^{(+)}_m(x-y).$$

(2.7)

Moreover, for a normalized vacuum, we have that the PF and NF propagators can be written as it follows

$$D^{(+)}_m(x-y) = i \left\langle \Omega \left| \left[ \phi^{(\mp)}(x), \phi^{(\pm)}(y) \right] \right| \Omega \right\rangle.$$

(2.8)

Now, by using the properties of the positive and negative parts of the field, Eq.(2.3), we may find the relation between the PF and NP propagators, as well as to the Wightman function

$$D^{(-)}_m(x-y) = -D^{(+)}_m(y-x) = -iW_2(y-x).$$

(2.9)

Hence, with the above results we can make use of the expression (2.6) to then obtain the PF and NF propagators written in the momentum space

$$\hat{D}^{(\pm)}_m(k) = \pm \frac{i}{2\pi} \theta(\pm k_0) \delta(k^2 - m^2) = \frac{i}{2\pi} \delta(k_0 \mp \omega_m) \delta(k^2 - m^2),$$

(2.10)

where $\omega_m = \sqrt{k^2 + m^2}$ is the frequency. In the first equality of Eq.(2.10) we can understood them as distributions in $k^2$, while in the second equality they can be understood equivalently either as distributions in $k_0$. Therefore, with the previous results we are able to determine the analytic representation of the propagator. For that we can make use the definition of the $\delta$-Dirac translated distribution (A.1). Hence, the propagators $\hat{D}_m^{(\pm)}$ can be defined by the following analytic representation [19]

$$\langle \hat{D}_m^{(\pm)} , \varphi \rangle = (2\pi)^{-2} \oint_{c_\pm} \frac{\varphi(k_0)}{k_0^2 - \omega_m^2} dk_0,$$

(2.11)

where $c_{\pm}$ is a counterclockwise closed path that contains only the positive (negative) poles of the Green’s function $\hat{g}(k) = \frac{1}{k^2 - \omega_m^2}$. 

5
We should emphasize that the above result Eq.(2.11) may be generalized for any free field. Moreover, since the PF and NF propagators are distributional solutions of the free field equations, then any linear combination of these is also a solution; for example, we may define the causal propagator distributional solution

\[
\hat{D}(k) = \hat{D}^+(k) + \hat{D}^-(k),
\]  

(2.12)

and its support follows from the Eq.(2.10):

\[
\text{Supp} \hat{D}(k) = \text{Supp} \hat{D}^+(k) \cup \text{Supp} \hat{D}^-(k) = \bar{V}^+ \cup \bar{V}^-.
\]  

(2.13)

In order to implement the generalization of the previous result (2.11), one can write the analytic representation for the causal propagator as it follows

\[
\langle \hat{D}, \phi \rangle = (2\pi)^{-2} \oint_{c_{all}} \hat{G}(k) \phi(k_0) dk_0,
\]  

(2.14)

where \(c_{all}\) are all counterclockwise closed paths that contain all individual poles, indicated by the variable \(k_0\), and \(\hat{G}(k)\) is the Green’s function associated to the free field equation. However, we may return to the PF and NF propagators from this quantity, for that purpose it is only necessary to split its support into the forward \(\bar{V}^+\) and backward \(\bar{V}^-\) cone, respectively,

\[
\langle \hat{D}^{(\pm)}, \phi \rangle = (2\pi)^{-2} \oint_{c_{\pm}} \hat{G}(k) \phi(k_0) dk_0.
\]  

(2.15)

where \(c_{+(-)}\) is a counterclockwise closed path that contains only the positive (negative) poles of the Green’s function \(\hat{G}(k)\).

### 2.2 The PF and NF electromagnetic propagators

The dynamics of the generalized electromagnetic theory is governed by the Lagrangian density, as it follows \[5, 6\]

\[
\mathcal{L}_\rho = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial_\mu F^{\mu\sigma} \partial^\nu F_{\nu\sigma},
\]  

(2.16)

where \(F_{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\) is the usual electromagnetic tensor field and \(a\) is the free Podolsky’s parameter with length dimension. This Lagrangian is invariant by \(U(1)\) gauge and Lorentz transformation. Usually the gauge-fixing procedure is performed by adding a Lagrange multiplier into the Lagrangian density, if we consider the Lorenz condition: \((\partial^\mu A_\mu)^2\). But, if instead we consider the generalized Lorenz condition \[7, 8\], we shall add the term: \([(1 + a^2 \Box) \partial^\mu A_\mu]^2\), this condition was proved to be the natural choice for the generalized electrodynamics; however, it increases the order of the field equation. Nevertheless, in order to preserve the order of the field equation, we may consider a third choice, for that we add the alternative gauge-fixing term into the Lagrangian:

\footnote{The regions \(\bar{V}^\pm\) are defined as: \(\bar{V}^\pm = \{x \mid x^2 \geq 0, \quad \pm x_0 \geq 0\}\).}
\[(\partial A) (1 + a^2 \Box) (\partial A) \] which is named as the *non-mixing gauge* and it is related to a pseudodifferential operator \[21\]. So, in this gauge condition, the total Lagrangian density is given by

\[
\mathcal{L}_P = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial_\mu F^{\mu\sigma} \partial_\nu F_{\nu\sigma} - \frac{1}{2\xi} (\partial A) (1 + a^2 \Box) (\partial A),
\]

where \(\xi\) is the gauge-fixing parameter. From this equation we can find the equation of motion,

\[
\varepsilon_{\mu\nu} (\partial) A^\nu \equiv (1 + a^2 \Box) \left[ (\Box g_{\mu\nu} - \partial_\mu \partial_\nu) + \frac{1}{\xi} \partial_\mu \partial_\nu \right] A^\nu = 0.
\]

We see that at the choice \(\xi = 1\), the equation of motion is reduced simply to:

\[
\Box A_\mu = 0,
\]

and a Proca sector

\[
(\Box + m_{a}^2) A_\mu = 0,
\]

where \(m_a = a^{-1}\) is the Podolsky’s mass. Moreover, in order to determine the analytic representation for the propagator we should first determine the Green’s function of the equation (2.18). In fact, it reads

\[
\hat{G}_{\mu\nu} (k) = \left[ g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{m_a^2} \right] \left( \frac{1}{k^2} - \frac{1}{k^2 - m_a^2} \right) - (1 - \xi) k_\mu k_\nu \frac{1}{(k^2)^2}.
\]

Thus, to find the PF and NF electromagnetic propagators via the analytic representation (2.15), it is convenient to calculate each one of the above terms separately:

1. Considering first the scalar case with Podolsky’s mass \(m_a\):

\[
\hat{G}_m (k) = \frac{1}{k^2 - \omega_{m_a}^2} = \frac{1}{(k_0 + \omega_{m_a})(k_0 - \omega_{m_a})},
\]

where \(\omega_{m_a} = \sqrt{k^2 + m_a^2}\). So, the poles are expressed in terms of the variable \(k_0\), thus from (2.15) the analytic representation for the PF and NF propagators is written as:

\[
\left\langle \hat{D}_{m_a}^{(\pm)}, \phi \right\rangle = (2\pi)^{-2} \int \frac{\varphi (p_0)}{(k_0 + \omega_{m_a})(k_0 - \omega_{m_a})} dk_0.
\]

Using the Cauchy’s integral theorem, also some distributional properties of the \(\delta\)-Dirac \[22\], see Appendix[A], we obtain the *PF and NF scalar propagators with mass \(m_a\) in the momentum space:

\[
\hat{D}_{m_a}^{(\pm)} (k) = \pm \frac{i}{2\pi} \theta (\pm k_0) \delta (k^2 - m_a^2).
\]

In particular, for the massless case, they have the form

\[
\hat{D}_0^{(\pm)} (k) = \pm \frac{i}{2\pi} \theta (\pm k_0) \delta (k^2).
\]
2. Now we shall consider the following Green’s function: 
\[ \hat{G}_0'(k) = \left( k_0^2 - \omega_0^2 \right)^{-2}, \]
which is associated to the dipolar massless scalar case [23]. Moreover, from the Eq. (2.15) the analytic representation for this term is given by
\[ \left\langle \hat{D}_0^{(\pm)} (k), \varphi \right\rangle = (2\pi)^{-2} \oint_{c_{\pm}} \frac{\varphi (k_0)}{(k_0^2 - \omega_0^2)^2} dk_0. \tag{2.26} \]

Using again the Cauchy’s integral theorem, we have that:
\[ \left\langle \hat{D}_0^{(\pm)} (k), \varphi \right\rangle = (2\pi)^{-2} \left\{ 2\pi i \frac{d}{dk_0} \left[ \frac{\varphi (k_0)}{(k_0 \pm \omega_0)^2} \right] \right\} _{k_0=\pm \omega_0} \]
\[ = \frac{i}{2\pi} \sum_{j=0}^{1} (-1)^j (1+j)! \frac{\varphi (1-j) (\pm \omega_0)}{(1-j)! (\pm 2\omega_0)^{2+j}}. \tag{2.27} \]

also, from the definition of the translated \( \delta \)-Dirac (A.2), it follows the expression
\[ \hat{D}_0^{(\pm)} (k) = -\frac{i}{2\pi} \sum_{j=0}^{1} (1+j)! \delta^{(1-j)} (k_0 \mp \omega_0) \tag{2.28} \]

Hence, using the distributional property (A.3) we obtain the \textit{PF and NF dipolar massless scalar propagators}
\[ \hat{D}_0^{(\pm)} (k) = \mp \frac{i}{2\pi} \theta (\pm k_0) \delta^{(1)} (k_0^2 - \omega_0^2). \tag{2.29} \]

Finally, we obtain that the expression for the \textit{PF and NF electromagnetic propagators} in the momentum space is given as follows
\[ \hat{D}_{\mu \nu}^{(\pm)} (k) = \left[ g_{\mu \nu} - (1 - \xi) \frac{k_\mu k_\nu}{m^2} \right] \left( \hat{D}_0^{(\pm)} (k) - \hat{D}_{m_a}^{(\pm)} (k) \right) - (1 - \xi) k_\mu k_\nu \hat{D}_0^{(\pm)} (k), \tag{2.30} \]
where the propagators \( \hat{D}_{m_a}^{(\pm)}, \hat{D}_0^{(\pm)} \) and \( \hat{D}_0^{(\pm)} \) are given by the Eqs. (2.24), (2.25) and (2.29), respectively.

Before introducing the formal causal method itself, it is rather interesting, by means of complementarity, to present some point remarks discussed previously for the scalar case which is suitable for the gauge field as well. With the PF and NF scalar propagators we may determine the \textit{causal propagator} (2.12):
\[ D_m (x) = D_m^{(+)} (x) + D_m^{(-)} (x). \tag{2.31} \]
The causal propagator can also be splitted into two important propagators: one in which indicates the propagation to the future and other to the past. These are the \textit{retarded} and \textit{advanced} propagators, which vanish for \( x_0 < 0 \) and \( x_0 > 0 \), respectively, in whatever referential, and are related to the causal propagator as follows
\[ D_m^R (x) = \theta (x_0) D_m (x), \quad D_m^A (x) = -\theta (-x_0) D_m (x). \tag{2.32} \]
From these very definitions, it follows another important distributional solution, the so-called Feynman propagator
\[ D^F_m(x) = \theta(x_0) D^{(+)}_m(x) - \theta(-x_0) D^{(-)}_m(x), \tag{2.33} \]
which is related to the vacuum expectation value of time-ordered products. Moreover, for the scalar case this distribution in the momentum space form can be written as it follows
\[ \hat{D}^F_m(k) = -\frac{1}{(2\pi)^2} \lim_{\varepsilon \to 0^+} \frac{1}{k^2 - m^2 + i\varepsilon}, \tag{2.34} \]
which differs from its Green’s function by the imaginary term \(i\varepsilon\), this addition process is named Feynman \(i\varepsilon\)-prescription \([17,24]\) and it is closely related with the Wick rotation technique. In general this prescription is given in order to handle the singularities in the propagators.

The causal approach takes into account only the general physical properties of the propagators into their deductions. For instance, when we write the scalar Feynman propagator using the definition from the retarded or advanced distribution (2.32) into (2.33), we obtain that
\[ D^F_m(x) = D^R_m(x) - D^{(-)}_m(x) = D^A_m(x) + D^{(+)}_m(x). \tag{2.35} \]
This is not a superfluous equivalence to the Wick rotation. Moreover, we may separate it into the positive and negative part one, and using the definition of the retarded and advanced propagators, Eq.(2.32), we may show that the Feynman propagator has the following causal property: Only positive-frequency solution can be propagating to the future and only negative-frequency solution can be propagating to the past. This general physical property and the general definition of the propagator in the distributional form are the starting point for the development of the Epstein-Glaser causal approach, in which no prescription is employed in dealing to the propagator’s poles.

In order to determine the electromagnetic propagator in the non-mixing gauge, we must first obtain the expression of the causal electromagnetic propagator as a sum of the PF and NF propagators (2.30) and using (2.31),
\[ \hat{D}_{\mu\nu}(k) = \left[g_{\mu\nu} - (1 - \xi) \frac{k_{\mu} k_{\nu}}{m^2_a} \right] \left[\hat{D}_0(k) - \hat{D}_{m_a}(k)\right] - (1 - \xi) k_{\mu} k_{\nu} \hat{D}'_0(k), \tag{2.36} \]
moreover, using the results (2.24), (2.25) and (2.29), respectively, the (causal) terms of (2.36) are written as it follows
\[ \hat{D}_0(k) = \frac{i}{2\pi} \text{sgn}(k_0) \delta(k^2), \quad \hat{D}_{m_a}(k) = \frac{i}{2\pi} \text{sgn}(k_0) \delta(k^2 - m^2_a), \quad \hat{D}'_0(k) = -\frac{i}{2\pi} \text{sgn}(k_0) \delta^{(1)}(k^2). \tag{2.37} \]
We can write this propagator in the configuration space,
\[ D_{\mu\nu}(x) = \left[g_{\mu\nu} + (1 - \xi) \frac{\partial_{\mu} \partial_{\nu}}{m^2_a} \right] \left[D_0(x) - D_{m_a}(x)\right] + (1 - \xi) \partial_{\mu} \partial_{\nu} D'_0(x), \tag{2.38} \]
where $D_0(x)$, $D_m(x)$ are the massless and massive Pauli-Jordan causal propagators, respectively. Moreover, their expressions are well known in the literature [23, 24],

$$D_0(x) = \frac{1}{2\pi} \text{sgn}(x_0) \delta(x^2),$$
$$D_m(x) = \frac{1}{2\pi} \text{sgn}(x_0) \left[ \delta(x^2) - \theta(x^2) \frac{m_m}{2\sqrt{x^2}} J_1 \left( \frac{m_m \sqrt{x^2}}{2} \right) \right],$$

(2.39)

whereas for the dipolar term we can show that it reads

$$D'_0(x) = -\frac{1}{8\pi} \text{sgn}(x_0) \theta(x^2) \left[ J_0 \left( M \sqrt{x^2} \right) \right]_{M=0} = -\frac{1}{8\pi} \text{sgn}(x_0) \theta(x^2),$$

(2.40)

where $J_0$ and $J_1$ are the zero-order and first-order Bessel’s functions, respectively. Since the support of the distribution $\delta(x^2)$ is contained in the surface of the backward and forward light-cone, and the support of $\theta(x^2)$ is contained in the closed forward light-cone, this assertion is valid for their derivatives as well. Thus, we have showed that the causal electromagnetic propagator $D_{\mu\nu}(x)$, Eq.(2.38), in the non-mixing gauge has causal support, i.e. $\text{Supp}(D_{\mu\nu}) \in \bar{V}^+ \cup \bar{V}^-$. 

3 The Epstein-Glaser causal method

In order to discuss scattering process in the framework of field theory we shall make use of the causal framework proposed by Epstein-Glaser [15, 16], a method that explicitly uses the causal structure as a powerful tool. One of the remarkable changes in this proposal is the introduction of a test function $g$, belonging to the Schwartz space, defined in the spacetime such that $g(x) \in [0, 1]$. The test function plays the role of switching the interaction in some region of the spacetime. Then, the scattering matrix, $S\text{-Matrix}$, is viewed necessarily as an operator-valued functional of $g$: $S = S[g]$. We shall now briefly review the main points of the Epstein-Glaser causal method.

Recalling that in the Epstein-Glaser approach the $S\text{-Matrix}$ can be written in the following formal perturbative series

$$S[g] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 dx_2 ... dx_n T_n(x_1, x_2, ..., x_n) g(x_1) g(x_2) ... g(x_n),$$

(3.1)

where we can identify the quantity $T_n$ as an operator-valued distribution, that is determined inductively term-by-term, and $g^{\otimes n}$ is its respective test function. Moreover, we have that the test function $g$ belongs to the Schwartz space $\mathcal{S}(M^4)$. It should be emphasized, however, that this formalism consider only free asymptotic fields acting on the Fock space to construct the $S\text{-Matrix} S[g]$. For instance, for GQED (as for QED), we have the free electromagnetic and fermionic fields: $A_\mu$, $\psi$ and $\bar{\psi}$. Then for GQED, $T_1$ takes the following form [8]:

$$T_1(x) = i e \cdot \bar{\psi}(x) \gamma^\mu \psi(x) : A_\mu(x),$$

(3.2)

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4 A detailed discussion can be found in the Refs. [25, 26].
where : : indicates the normal ordering and, in this approach, \( e \) is the normalized coupling constant.

Moreover, in the Epstein-Glaser method only mathematically well-defined distributional products are introduced, such as the following intermediate \( n \)-point distributions

\[
A'_{n} (x_1, \ldots, x_n) \equiv \sum_{P_2} \tilde{T}_{n_1} (X) T_{n-n_1} (Y, x_n), \quad (3.3)
\]

\[
R'_{n} (x_1, \ldots, x_n) \equiv \sum_{P_2} T_{n-n_1} (Y, x_n) \tilde{T}_{n_1} (X), \quad (3.4)
\]

where \( P_2 \) are all partitions of \( \{x_1, \ldots, x_{n-1}\} \) into the disjoint sets \( X, Y \) such that \( |X| = n_1 \geq 1 \) and \( |Y| \leq n - 2 \). Moreover, if the sums in (3.3) and (3.4) are extended over all partitions: \( P^0 \), including the empty set, important distributions may be obtained. These are namely the advanced and retarded distributions

\[
A_n (x_1, \ldots, x_n) \equiv \sum_{P^0_2} \tilde{T}_{n_1} (X) T_{n-n_1} (Y, x_n) = A'_{n} (x_1, \ldots, x_n) + T_{n} (x_1, \ldots, x_n), \quad (3.5)
\]

\[
R_n (x_1, \ldots, x_n) \equiv \sum_{P^0_2} T_{n-n_1} (Y, x_n) \tilde{T}_{n_1} (X) = R'_{n} (x_1, \ldots, x_n) + T_{n} (x_1, \ldots, x_n). \quad (3.6)
\]

By means of causal properties, one may then conclude that \( R_n \) and \( A_n \) have retarded and advanced support, respectively.

\[
\text{Supp } R_n (x_1, \ldots, x_n) \subseteq \Gamma^+_{n-1} (x_n), \quad \text{Supp } A_n (x_1, \ldots, x_n) \subseteq \Gamma^-_{n-1} (x_n), \quad (3.7)
\]

where \( \Gamma^\pm_{n-1} (x_n) = \{ (x_1, \ldots, x_n) \mid x_j \in \tilde{V}^\pm (x_n), \quad \forall \ j = 1, \ldots, n-1 \} \), and \( \tilde{V}^\pm (x_n) \) is the closed forward (backward) cone. These two distributions can not be determined by the induction assumption only, in fact, they are obtained by the splitting process [27] of the so-called causal distribution defined as it follows

\[
D_n (x_1, \ldots, x_n) \equiv R'_n (x_1, \ldots, x_n) - A'_n (x_1, \ldots, x_n) = R_n (x_1, \ldots, x_n) - A_n (x_1, \ldots, x_n). \quad (3.8)
\]

In the case of the GQED we can write \( D_n \) as it follows

\[
D_n (x_1, \ldots, x_n) = \sum_{k} d^k_n (x_1, \ldots, x_n) : \prod_j \psi (x_j) \prod_l \psi (x_l) \prod_m A (x_m) : , \quad (3.9)
\]

where \( d^k_n (x_1, \ldots, x_n) \) is the numerical part of the causal distribution \( D_n \). Moreover, by the translational invariance of \( d^k_n \) one may show that it depends only on the relative coordinates:

\[
d (x) \equiv d^k_n (x_1 - x_n, \ldots, x_{n-1} - x_n) \in \mathcal{F}' (\mathbb{R}^m), \quad m = 4 (n - 1). \quad (3.10)
\]

As it was emphasized above an important step in this inductive construction is the splitting process of the causal distribution, but its splitting at the origin \( \{x_n\} = \Gamma^+_{n-1} (x_n) \cap \Gamma^-_{n-1} (x_n) \) can be translated equivalently to its numerical part \( d \) to be splitted into the advanced and retarded distributions \( a \) and \( r \), respectively. Another important point to be analyzed is the convergence of the sequence \( \{\langle d, \phi_\alpha \rangle\} \), where \( \phi_\alpha \) has decreasing support when \( \alpha \to 0^+ \) and also belonging to the Schwartz space \( \mathcal{F} \).
From the aforementioned analysis we can find some natural distributional definitions. For instance, we may name \( d \) as being a distribution of singular order \( \omega \) if its Fourier transform \( \hat{d}(p) \) has a quasi-asymptotic \( \hat{d}_0(p) \neq 0 \) at \( p = \infty \) with regard to a positive continuous function \( \rho(\alpha) \), \( \alpha > 0 \), if the limit
\[
\lim_{\alpha \to 0^+} \rho(\alpha) \langle \hat{d}(\frac{p}{\alpha}), \phi(p) \rangle = \langle \hat{d}_0(p), \phi(p) \rangle \neq 0,
\] (3.11)
exists in \( \mathcal{D}'(\mathbb{R}^m) \). By the scaling transformation one may derive that the power-counting function \( \rho(\alpha) \) satisfies
\[
\lim_{\alpha \to 0} \frac{\rho(a\alpha)}{\rho(\alpha)} = a^\omega, \quad \forall \ a > 0,
\] (3.12)
with
\[
\rho(\alpha) \to \alpha^\omega L(\alpha), \text{ when } \alpha \to 0^+,
\] (3.13)
where \( L(\alpha) \) is a quasi-constant function at \( \alpha = 0 \). Of course, there is an equivalent definition of the above process in coordinate space, but, since the splitting process is more easily accomplished in the momentum space, this one suffices for our purposes. Moreover, we specify the splitting problem by requiring that the splitting procedure must preserve the singular order of the distributions. From these very definitions we have two distinct cases depending on the value of \( \omega \), these are \[26\]:

(i) Regular distributions - for \( \omega < 0 \). In this case the solution of the splitting problem is unique and the retarded distribution is defined by multiplying \( d \) by step functions, its form in the momentum space is given as it follows
\[
\hat{r}(p) = \frac{i}{2\pi} sgn(p^0) \int_{-\infty}^{+\infty} dt \frac{\hat{d}(tp)}{(1-t+sgn(p^0)i0^+)}.
\] (3.14)
identified as a dispersion relation without subtractions.

(ii) Singular distributions - for \( \omega \geq 0 \). Then the solution can not be obtained directly as in the regular case. But, after a careful mathematical treatment, it may be shown that the retarded distribution is given by the so-called central splitting solution
\[
\hat{r}(p) = \frac{i}{2\pi} sgn(p^0) \int_{-\infty}^{+\infty} dt \frac{\hat{d}(tp)}{t^{\omega+1}(1-t+sgn(p^0)i0^+)}.
\] (3.15)
identified as a dispersion relation with \( \omega + 1 \) subtractions.

4 Bhabha scattering

Now that we have obtained all the necessary tools and developed important ideas we can concentrate our attention in determining inductively the terms of the \( S \)-Matrix (3.1). In particular we are interested here in the term corresponding to the Bhabha scattering. Hence, in order to accomplish that, the perturbative program has its start when constructing the intermediate distributions:
\[
R'_2(x_1,x_2) = -T_1(x_2)T_1(x_1), \quad A'_2(x_1,x_2) = -T_1(x_1)T_1(x_2),
\] (4.1)
and subsequently construct the causal distribution $D_2$ as it follows

$$D_2 (x_1, x_2) = R'_2 (x_1, x_2) - A'_2 (x_1, x_2) = [T_1 (x_1), T_1 (x_2)].$$  

For GQED we have Eq.(3.2) as the first perturbative term: $T_1 (x) = i e : \psi (x) \gamma^\mu \psi (x) : A_\mu (x)$. Hence, after applying the Wick theorem for the normally ordering product, we obtain from all of these terms those associated with the Bhabha scattering contributions:

$$R'_2 (x_1, x_2) = e^2 : \psi (x_2) \gamma^\mu A_\mu (x_2) \psi (x_2) \psi (x_1) A_\nu (x_1) \gamma^\nu \psi (x_1) : ,$$  

$$A'_2 (x_1, x_2) = e^2 : \psi (x_1) \gamma^\mu A_\mu (x_1) \psi (x_1) \psi (x_2) A_\nu (x_2) \gamma^\nu \psi (x_2) : .$$  

We have that the electromagnetic contraction is defined as it follows:

$$\Gamma (x) A_\mu (x) A_\nu (y) \equiv \left[ A^{(-)}_\mu (x), A^{(+)}_\nu (y) \right] = i D^{(+)}_\mu \nu (x - y)$$

where $D^{(\pm)}_\mu \nu (x)$ are the PF and NF parts of the electromagnetic propagator, Eq.(2.30). After some calculation, we arrive at the following expression for the causal distribution (4.2):

$$D_2 (x_1, x_2) = -i e^2 : \psi (x_1) \gamma^\mu \psi (x_1) D_\mu \nu (x_1 - x_2) \psi (x_2) \gamma^\nu \psi (x_2) :$$

where $D_\mu \nu$ is the causal electromagnetic propagator, Eq.(2.38); moreover, we have showed above that this distribution has causal support: Supp $D_\mu \nu (x_1, x_2) \subseteq \Gamma_2^+ (x_2) \cup \Gamma_2^- (x_2)$. Nevertheless, in order to determine the singular order of this propagator, we shall follow the criterion in the momentum space, Eq.(3.11). Hence, from the expression (2.36) we write $\hat{D}_\mu \nu (k / \alpha)$ as it follows

$$\hat{D}_\mu \nu \left( k / \alpha \right) = -\frac{i}{2 \pi} sgn \left( k_0 / \alpha \right) g_{\mu \nu} \left[ \delta \left( \frac{k^2}{\alpha^2} \right) - \delta \left( \frac{k^2}{\alpha^2} - m_\alpha^2 \right) \right]$$

$$+ \frac{i}{2 \pi} sgn \left( k_0 / \alpha \right) \left( 1 - \frac{k_\mu k_\nu}{\alpha^2} \right) \delta \left( 1 \right) \left( \frac{k^2}{\alpha^2} \right)$$

$$+ \frac{i}{2 \pi} sgn \left( k_0 / \alpha \right) \left( 1 - \frac{k_\mu k_\nu}{\alpha^2} m_\alpha^2 \right) \delta \left( \frac{k^2}{\alpha^2} - m_\alpha^2 \right) - \delta \left( \frac{k^2}{\alpha^2} \right) .$$  

Now, we shall calculate this expression when $\alpha \to 0^+$, first we note that $sgn \left( k_0 / \alpha \right) = sgn (k_0)$ and, it should be emphasized that in order to obtain the correct singular order we must consider the whole distribution. Moreover, we may use the scale property (A.5) and the Taylor expansion (A.6) of the $\delta$-Dirac distribution to write down $\hat{D}_\mu \nu \left( k / \alpha \right)$ in the significant leading $\alpha$-order

$$\hat{D}_\mu \nu \left( k / \alpha \right) \approx \alpha^4 \left\{ \frac{i}{2 \pi} sgn (k_0) \left[ m_\alpha^2 \delta \left( 1 \right) \left( k^2 \right) \right] \right\} g_{\mu \nu} .$$  

This means that the causal propagator $\hat{D}_\mu \nu$ is a regular distribution with singular order

$$\omega \left( \hat{D}_\mu \nu \right) = -4 < 0 ,$$
this result is more regular from the one obtained in the usual QED theory \[16\], where \( \omega^{\text{Max}} = -2 \). Therefore, as we have determined all the necessary conditions, we are now in position to evaluate the retarded distribution Eq.\,(3.6). For this purpose we use the regular splitting formula (3.14) to write

\[
\hat{R}_{\mu\nu}(k) = \left[ g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{m_a^2} \right] \frac{i}{2\pi} \text{sgn} (k_0) \int dt \frac{(\hat{D}_0 (tk) - \hat{D}_{ma} (tk))}{1 - t + \text{sgn} (k_0) i0^+} + \infty_{-}\infty \cdot \text{sgn} (k_0) i0^+.
\]

In order to evaluate the dispersion integrals one can make use of the explicit expression for the propagators: \( \hat{D}_0 (k), \hat{D}_{ma} (k) \) and \( \hat{D}'_0 (k) \), Eq.\,(2.37). Finally, we find the following expression for the electromagnetic retarded propagator

\[
\hat{R}_{\mu\nu}(k) = \left[ g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{m_a^2} \right] [\hat{R}_0 (k) - \hat{R}_{ma} (k)] - (1 - \xi) k_\mu k_\nu \hat{R}'_0 (k), \tag{4.10}
\]

where we have defined the quantities for \( k^2 > 0 \)

\[
\hat{R}_0 (k) = -(2\pi)^{-2} \frac{1}{k^2 + \text{sgn} (k_0) i0^+}, \quad \hat{R}_{ma} (k) = -(2\pi)^{-2} \frac{1}{k^2 - m_a^2 + \text{sgn} (k_0) i0^+}, \quad \hat{R}'_0 (k) = -(2\pi)^{-2} \left( \frac{1}{k^2 + \text{sgn} (k_0) i0^+} \right)^2. \tag{4.11}
\]

Although from the definition (4.2) it follows several terms we can consider only those terms of interest, i.e., those associated with the Bhabha scattering contribution: \( T_2 (x_1, x_2) \). This contribution is obtained from the relation:

\[
T_2 (x_1, x_2) = R_2 (x_1, x_2) - R'_2 (x_1, x_2), \tag{4.12}
\]

where \( R_2 \) is the retarded part of \( D_2 \). From the previous results we have that

\[
R_2 (x_1, x_2) = -ie^2 : \psi(x_1) \gamma^\mu \psi(x_1) R_{\mu\nu} (x_1 - x_2) \psi(x_2) \gamma^\nu \psi(x_2) :. \tag{4.13}
\]

Then it follows that the complete contribution \( T_2 \) can be written in the following form:

\[
T_2 (x_1, x_2) = -ie^2 : \psi(x_1) \gamma^\mu \psi(x_1) D^F_{\mu\nu} (x_1 - x_2) \psi(x_2) \gamma^\nu \psi(x_2) :, \tag{4.14}
\]

where \( D^F_{\mu\nu} \) is defined, in the momentum space, by the relation:

\[
\hat{D}^F_{\mu\nu}(k) = \hat{R}_{\mu\nu}(k) - \hat{R}'_{\mu\nu}(k). \tag{4.15}
\]

Therefore, by replacing the expressions for \( \hat{R}_{\mu\nu} \) and \( \hat{R}'_{\mu\nu} = \hat{D}^{(-)}_{\mu\nu} \), Eqs.\,(4.10) and (2.30), respectively, we obtain the following expression for the propagator \( \hat{D}^F_{\mu\nu} \):

\[
\hat{D}^F_{\mu\nu}(k) = \left[ g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{m_a^2} \right] \left( \hat{D}_0^F (k) - \hat{D}_{ma}^F (k) \right) - (1 - \xi) k_\mu k_\nu \hat{D}'_0^F (k) \tag{4.16}
\]
with the quantities $\hat{D}_0^F$, $\hat{D}_{m_a}^F$ and $\hat{D}_0^F$ given by\footnote{They are obtained from the Eqs.\,(2.24), (2.25), (2.29) and (4.11).}
\begin{equation}
\begin{aligned}
\hat{D}_0^F(k) &= -(2\pi)^{-2} \frac{1}{k^2 + i0^+}, \\
\hat{D}_{m_a}^F &= -(2\pi)^{-2} \frac{1}{k^2 - m_a^2 + i0^+}, \\
\hat{D}_0^F(k) &= -(2\pi)^{-2} \left( \frac{1}{k^2 + i0^+} \right)^2.
\end{aligned}
\end{equation}

This is the generalized photon propagator in the non-mixing gauge condition.\footnote{And it is clearly different from the expression obtained in the Ref.\,[8].} Moreover, it should be emphasized that all the poles are well-defined in the expression (4.16), where we have not used neither the Feynman $i\epsilon$-prescription nor the Wick rotation [17][24].

### 4.1 Bhabha’s Cross Section

By definition, the transition probability $\mathcal{P}_{fi}$ is given as it follows [16][17][24]
\begin{equation}
\mathcal{P}_{fi} \equiv \left( |\psi_f \rangle , |\psi_i \rangle \right)^2 = |S_{fi}|^2 = \left| \left\langle \psi_f \left| S_i^{(1)} \right| \psi_i \right\rangle \right|^2 + \cdots. \tag{4.18}
\end{equation}

But, this quantity has no meaning if it is not written as a function of wave-packages and also considering the fermionic states:
\begin{equation}
\begin{aligned}
d_{s_i}^\dagger (p_i) b_{\sigma_i}^\dagger (q_i) |\Omega\rangle \to |\psi_i\rangle &= \int d^3 p_i d^3 q_1 \psi_i (p_1, q_1) d_{s_i}^{\dagger} (p_1) b_{\sigma_i}^{\dagger} (q_1) |\Omega\rangle, \\
d_{s_f}^\dagger (p_f) b_{\sigma_f}^{\dagger} (q_f) |\Omega\rangle \to |\psi_f\rangle &= \int d^3 p_2 d^3 q_2 \psi_f (p_2, q_2) d_{s_f}^{\dagger} (p_2) b_{\sigma_f}^{\dagger} (q_2) |\Omega\rangle,
\end{aligned}
\end{equation}
where $(q_i, \sigma_i)$ and $(q_f, \sigma_f)$ are the momentum and spin of the ingoing and outgoing electron, respectively, whereas $(p_i, s_i)$ and $(p_f, s_f)$ are the momentum and spin of the ingoing and outgoing positron, respectively. Moreover, to obtain the simplest case expression we shall consider a set of orthogonal wave-packages, and we take into account that the initial wave-packages are concentrated in $p_i$, $q_i$ and with fixed spins. Furthermore, if we consider the ingoing electron (1) as a target and define $v$ as the relative velocity of the ingoing positron, then, for an average cylinder of radius $R$ parallel to $v$ we arrive at [9][16]
\begin{equation}
\sum_f \mathcal{P}_{fi} (R) = \frac{1}{\pi R^2} \left[ (2\pi)^2 \frac{1}{|v|} \int d^3 p_2 d^3 q_2 \left| \mathcal{M}_{s_i s_f \sigma_i \sigma_f} (p_i, q_i, p_2, q_2) \right|^2 \delta (p_2 + q_2 - p_i - q_i) \right], \tag{4.21}
\end{equation}
where $\mathcal{M}$ is a distributional quantity related to the $S$-Matrix as it follows
\begin{equation}
S_{fi} = \delta (p_f + q_f - p_i - q_i) \mathcal{M}. \tag{4.22}
\end{equation}
The scattering cross section in the laboratory frame is given by [9][16]
\begin{equation}
\sigma \equiv \lim_{R \to \infty} \pi R^2 \sum_f \mathcal{P}_{fi} (R), \tag{4.23}
\end{equation}
and then it follows

\[
\sigma = (2\pi)^2 \frac{1}{|v|} \sum_{s_f, \sigma_f} \int d^3 p d^3 q \left| \mathcal{M}_{s_i, s_f, \sigma_i} (p_i, q_i, p_f, q_f) \right|^2 \delta \left( p_2 + q_2 - p_i - q_i \right),
\]

(4.24)

which can also be written in a Lorentz invariant form as a function of the normalized electron mass, \(m\), and energies of the ingoing electrons \([9, 16]\)

\[
\sigma = (2\pi)^2 \frac{E (p_i) E (q_i)}{\sqrt{(p_i q_i)^2 - m^4 s_f \sigma_f}} \sum_{s_f, \sigma_f} \int d^3 p d^3 q \left| \mathcal{M}_{s_i, s_f, \sigma_i} (p_i, q_i, p_f, q_f) \right|^2 \delta \left( p_2 + q_2 - p_i - q_i \right). \tag{4.25}
\]

Moreover, from this expression it is still possible to obtain the differential cross section in the center-of-mass reference \([9, 16]\)

\[
\left( \frac{d\sigma}{d\Omega} \right)_{c.m} = (2\pi)^2 \frac{E^2}{4} |\mathcal{M}|^2. \tag{4.26}
\]

For simplicity, we shall not consider the polarizations of the ingoing and outgoing fermions; hence, we shall consider the sum over \(s_f\) and \(\sigma_f\), and the average over \(s_i\) and \(\sigma_i\). Finally, we can write the differential cross section in the following form:

\[
\left( \frac{d\sigma}{d\Omega} \right)_{c.m} = \frac{e^4}{32 (2\pi)^2 E^2} \mathcal{F} (s, t, u), \tag{4.27}
\]

with \(s, t, u\) being the Mandelstam variables \([9]\), defined as follows

\[
s = (p_i + q_i)^2 = (p_f + q_f)^2 = 2m^2 + 2(q_i, p_i) = 2m^2 + 2(p_f, q_f), \tag{4.28}
\]

\[
t = (p_i - p_f)^2 = (q_i - q_f)^2 = 2m^2 - 2(p_f, p_i) = 2m^2 - 2(q_i, q_f), \tag{4.29}
\]

\[
u = (p_f - q_i)^2 = (p_i - q_f)^2 = 2m^2 - 2(p_f, q_i) = 2m^2 - 2(p_i, q_f), \tag{4.30}
\]

and the function \(\mathcal{F} (s, t, u)\) is given by Eq.\((B.25)\)

\[
\mathcal{F} (s, t, u) = \frac{[s^2 + u^2 + 8m^2 t - 8m^4]}{t^2 \left(1 - \frac{t}{m_a^2}\right)^2} + \frac{[u^2 + t^2 + 8m^2 s - 8m^4]}{s^2 \left(1 - \frac{s}{m_a^2}\right)^2} + 2 \frac{[u^2 - 8m^2 u + 12m^4]}{st \left(1 - \frac{s}{m_a^2}\right) \left(1 - \frac{t}{m_a^2}\right)}. \tag{4.31}
\]

Moreover, we see that when \(m_a \to \infty\) this expression reduces to the QED one \([16]\), a fact that is in accordance with the relation from the Podolsky’s to Maxwell theory.

We shall now consider the different energy regime by taking into account as reference the electron mass \(m = 0.510\ MeV\) and that the Podolsky mass is larger than this value; for instance, in \([8]\) it was obtained that \(m_a \geq 37.59\ GeV\). Moreover, the differential cross section is conventionally evaluated at the center-of-mass frame, where we have the relations

\[
p_i = -q_i \equiv \mathbf{p}, \quad p_f = -q_f, \quad E (p_i) = E (q_i) \equiv E, \quad E (p_f) = E (q_f). \tag{4.32}
\]

Hence, from the energy-momentum conservation it follows that: \(E (p_i) = E (p_f) \rightarrow |\mathbf{p}_f| = |\mathbf{p}_i|\), and also by defining \(\theta\) as the center-of-mass scattering angle, the Mandelstam variables \((4.30)\) reads

\[
s = 4m^2 + 4\mathbf{p}^2 = 4E^2, \quad u = -4\mathbf{p}^2 \cos^2 \frac{\theta}{2}, \quad t = -4\mathbf{p}^2 \sin^2 \frac{\theta}{2}. \tag{4.33}
\]

These results allow we obtain the following outcomes for the differential cross section:

\footnote{A detailed calculation of this quantity can be found in the Appendix\([8]\).}
1. First, for the energy regime
\[
\frac{(s - 4m^2)}{m^2} \sim \frac{|t|}{m^2} \sim \frac{|u|}{m^2} \ll 1, \quad (4.34)
\]
we have that the dominant contribution reads
\[
\mathcal{F}(s, t, u) \approx 8m^4 \left( \frac{1}{t^2} \right). \quad (4.35)
\]
Thus, we arrive at the following differential cross section:
\[
\left( \frac{d\sigma}{d\Omega} \right)_{c.m} = \frac{\alpha^2 m^2}{16 (E^2 - m^2)^2} \left( \frac{1}{\sin^4 \frac{\theta}{2}} \right). \quad (4.36)
\]
This is the known non-relativistic limit of the Bhabha’s formula \[9\].

2. For the energy regime
\[
\frac{(s - 4m^2)}{m^2} \sim \frac{|t|}{m^2} \sim \frac{|u|}{m^2} \sim 1, \quad (4.37)
\]
we find that the dominant contribution is
\[
\mathcal{F}(s, t, u) \approx \frac{s^2 + u^2 + 8m^2t - 8m^4}{t^2} + \frac{u^2 + t^2 + 8m^2s - 8m^4}{s^2} + \frac{2u^2 - 8m^2u + 12m^4}{st}. \quad (4.38)
\]
Hence, we obtain the following expression for the differential cross section:
\[
\left( \frac{d\sigma}{d\Omega} \right)_{c.m} = \frac{\alpha^2}{16E^2} \left[ \frac{2 (E^2 - m^2)^2 (1 + \cos^4 \frac{\theta}{2}) + 4 (E^2 - m^2) m^2 \cos^2 \frac{\theta}{2} + m^4}{(E^2 - m^2)^2 \sin^4 \frac{\theta}{2}} 
+ \frac{(E^2 - m^2)^2 (1 + \cos^2 \theta) + 4 (E^2 - m^2) m^2 + 3m^4}{E^4} 
- \frac{4 (E^2 - m^2)^2 \cos^4 \frac{\theta}{2} + 8 (E^2 - m^2) m^2 \cos^2 \frac{\theta}{2} + 3m^4}{E^2 (E^2 - m^2) \sin^2 \frac{\theta}{2}} \right], \quad (4.39)
\]
this is the known Bhabha’s formula \[9\].

3. At last, for the energy regime
\[
m^2 \ll (s - 4m^2) \sim |t| \sim |u| < m_a^2, \quad (4.40)
\]
we have that the terms associated to the Podolsky’s mass must be considered in the dominant contribution. Thus, in the leading order term in \( \frac{E}{m_a} \)
\[
\mathcal{F}(s, t, u) \approx \left( \frac{s^2 + u^2 + t^2}{t^2} + \frac{u^2 + t^2}{s^2} + \frac{2u^2}{st} \right) + \frac{2}{m_a^2} \left( \frac{s^2 + 2u^2}{t} + \frac{2u^2 + t^2}{s} \right). \quad (4.41)
\]
Finally, after some manipulation, we get the following expression for the differential cross section
\[
\left( \frac{d\sigma}{d\Omega} \right)_{c.m} = \frac{\alpha^2}{256E^2} \left( \cos^2 \theta + 7 \right)^2 - \frac{\alpha^2}{32m_a^2} \frac{(45 \cos \theta + 6 \cos 2\theta + 3 \cos 3\theta + 42)}{\sin^2 \frac{\theta}{2}}, \quad (4.42)
\]
this expression shows an additional term to the usual ultra-relativistic limit Bhabha’s formula, which we shall name as the \textit{GQED correction to the ultra-relativistic limit Bhabha’s formula}.
Nevertheless, from the expressions (4.27) and (4.31) we can also look the differential cross section at the "high-energy" regime

\[
\left( \frac{d\sigma}{d\Omega} \right)_{c.m} = \frac{\alpha^2}{2s} \left[ \frac{s^2 + u^2}{t^2 \left(1 - \frac{t}{m_a^2}\right)^2} + \frac{u^2 + t^2}{s^2 \left(1 - \frac{s}{m_a^2}\right)^2} + 2 \frac{u^2}{st \left(1 - \frac{s}{m_a} \right) \left(1 - \frac{t}{m_a}\right)} \right],
\]

(4.43)

this relation is identical to the phenomenological formula of the Bhabha scattering. Hence, we can identify the free parameter \( m_a \) as related with the phenomenological cut-off parameter \( \Lambda_+ \) [28]. In [29] (TASSO collaboration) is presented measurements of the differential cross sections at 95% of confidence and at total central mass energy \( 12 \, \text{GeV} \leq \sqrt{s} \leq 46,8 \, \text{GeV} \), that results into the value \( m_a \geq 370 \, \text{GeV} \), and by other experiment measurements [30] we have obtained values of the same order of magnitude.

Moreover, we can identify the first term in Eq.(4.42) as the QED contribution and the second as the GQED correction. Thus, the GQED correction to the Bhabha’s scattering may be evaluated by the following formula

\[
\delta = \left( \frac{d\sigma}{d\Omega} \right)_{c.m}^{\text{GQED}} - \left( \frac{d\sigma}{d\Omega} \right)_{c.m}^{\text{QED}},
\]

(4.44)

thus, we can find the expression

\[
\delta = -8 \left( \frac{E}{m_a} \right)^2 \left( \sin^2 \frac{\theta}{2} \right) \left( 45 \cos \theta + 6 \cos 2\theta + 3 \cos 3\theta + 42 \right) \left( \cos 2\theta + 7 \right)^2.
\]

(4.45)

On the other hand, as aforementioned, the relevance in studying the Bhabha scattering is mainly because it is the process employed in determining the luminosity \( L \) at \( e^- e^+ \) collider; in fact, we have \( L = N_{Bha}/\sigma_{th} \), where \( N_{Bha} \) is the rate of Bhabha events and \( \sigma_{th} \) is the Bhabha scattering cross section obtained by theoretical calculation [31]. There are two kinematical regions of special interest for the luminosity measurements: one is the small-angle Bhabha (SABh) process, which is found at scattering angle below \( 6^\circ \), and it is mainly dominated by the \( t \)-channel photon-exchange; while the another one, the large-angle Bhabha (LABh) process, is found for scattering angle above \( 6^\circ \), it gets important contributions from various \( s \)-channel (annihilation) exchange. The SABh process \( e^- e^+ \rightarrow e^- e^+ \) is employed in determining the luminosity and, hence, the absolute normalization of the cross section expression for all other \( e^- e^+ \) collisions. Moreover, since luminosity is dominated by photon-exchange its contribution is calculable, in principle, by the perturbative QED with arbitrary precision. Thus, in the GQED approach we can calculate the lowest effect of this theory to the SABh process. Thus, expanding Eq.(4.45) for small angles, \( \theta \ll 1 \), we have that the correction reads

\[
\delta = -\left( \frac{\sqrt{s}}{m_a} \right)^2 \frac{3}{4} \theta^2.
\]

(4.46)
Besides, we can rewrite the expression (4.42) for the differential cross section at small-angles

\[
\left( \frac{d\sigma}{d\Omega} \right)_{c.m.} = \frac{4\alpha^2}{E^2 \theta^4} \left\{ 1 - \left[ \frac{1}{2} + 3 \left( \frac{E}{m_a} \right)^2 \right] \theta^2 + \cdots \right\}. \tag{4.47}
\]

Thus, we can see that the GQED deviation for the luminosity decreases at the second order term the usual QED contribution.

Furthermore, from Eq.(4.45) we can see that the maximum value of the correction is obtained for a scattering angle: \( \theta = 90^0 \). Hence, for this angle we see that the differential cross section (4.42) takes the form

\[
\left( \frac{d\sigma}{d\Omega} \right)_{c.m.} = \frac{9\alpha^2}{16E^2} \left( 1 - \frac{E^2}{m_a^2} \right). \tag{4.48}
\]

From this equation, we may state that pure electromagnetic contribution is present at the regime \( E < 0.5 m_a \).

As an additional remark, we may cite that in the advanced electron-positron collider, the International Linear Collider (ILC), the accelerated beams will collide at the energy in the center of mass: \( \sqrt{s} = 500 \text{ GeV} \) and, as small angle to the order of \( \theta = 1.41 \times 10^{-2} \text{ rad} \); hence, using the previous result for the Podolsky mass \( m_a \geq 370 \text{ GeV} \), we can estimate the GQED correction at the order of \( \delta = -0.027\% \), value within expectation of high precision measurements of the luminosity.

5 Concluding remarks

In this paper we have discussed the Bhabha scattering in the framework of the generalized quantum electrodynamics. The theory was quantized in the framework of the causal method of Epstein and Glaser, where this perturbative program give us consistent results with general physical requirements, such as causality, and also mathematically well-defined quantities, because it is embedded into the realm of the distribution theory.

In this approach are taken into account asymptotically free conditions, thus for the fermionic particles it was considered the Dirac equation and for the photon the Podolsky free field equation; moreover, in the latter we considered as the gauge condition the so-called non-mixing gauge condition. With this gauge condition we were able to find a suitable expression for the free electromagnetic propagator, where its physical content becomes more clear. Also, since the physical result of the transition amplitude of a given scattering process is not affected by a particular value of the gauge-fixing parameter \( \xi \), we have considered \( \xi = 1 \) in our calculations.

In our analysis of the Bhabha scattering we found that our "high-energy" formula have the same form as the phenomenological formula which consider the cut-off parameter or Feynman regulator \( \Lambda_+ \). Thus by identifying the Podolsky mass as this cut-off parameter, we were able to find a bound value for the free parameter \( m_a \). Hence, from the electron-positron scattering with \( 12 \text{ GeV} \leq \sqrt{s} \leq 46.8 \text{ GeV} \), it follows that: \( m_a > 370 \text{ GeV} \). Moreover, from this result, we can estimate the GQED correction to the luminosity for SABh process: \(-0.027\%\), according with the high precision measurements expectations of future advanced electron-positron collider (ILC). When we compare our
result with the value of the Podolsky’s mass obtained in the Ref. [8], $m_a \geq 37.59 \text{ GeV}$, we see that even this previous result was obtained considering experimental data of electron’s anomalous magnetic moment, this previous result is one lower order. This support the idea that a electron-positron collider is an excellent experiment to study new particle physics.

We believe that the generalized quantum electrodynamics stands as a reasonable leptonic-photon interacting theory and, moreover, it can cope with many “high energy” deviation experimental results, by introducing the Podolsky’s mass. Once we have developed all principal ideas of the causal inductive program, given in the Secs. 2 and 3 we are in position to go further and proceed in some analyses: (i) A first step in this direction, may be the explicit computation and discussion of the other second-order terms; in particular, the GQED one-loop radiactive corrections, such as the vacuum polarization, the fermion self-energy, and the three-point vertex function. For this purpose, we will make use of the full strength of the Epstein-Glaser causal approach, which will lead naturally to well defined and regularized quantities. (ii) On the other hand, we may study some other physical properties which were not considered as part of the constructed axioms, like the discrete symmetries: parity, time-reversal and charge conjugation. In particular, we want to examine whether or not the GQED is a normalizable model and if the gauge invariance is present, i.e., showing that the Ward-Takahashi-Fradkin identities are satisfied perturbatively order-by-order in the perturbative causal approach. These issues and others will be further elaborated, investigated and reported elsewhere.

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A Dirac’s delta distribution properties

In this appendix we summarize some handful properties the $\delta$-Dirac distribution [22]. Let $\varphi$ be a test function, we have the definition for the $\delta$-Dirac translated distribution

$$\delta (\pm \omega_m) = \langle \delta (k_0 \mp \omega_m), \varphi (k_0) \rangle . \] (A.1)$$

Moreover, from this definition we may also obtain its derivative:

$$\delta^{(1-j)} (\pm \omega_0) = (-1)^{(1-j)} \left\langle \delta^{(1-j)} (k_0 \mp \omega_0), \varphi (k_0) \right\rangle \] (A.2)$$

Another important relation involves

$$\theta (\pm \alpha) \delta^{(1)} (\alpha^2 - \beta^2) = \pm \sum_{j=0}^{1} \frac{(1+j)!}{(1-j)!} \left[ \frac{\delta^{(1-j)} (\alpha \mp \beta)}{(\pm 2 \beta)^{2+j}} \right] , \quad \beta > 0 , \] (A.3)$$

in particular, we have:

$$\theta (\pm k_0) \delta (k_0^2 - \omega_m^2) = \delta (k_0 \mp \omega_m) \frac{2 \omega_m}{2 \omega_m} . \] (A.4)$$
Two important identities from the $\delta$-Dirac distribution are its scale property and its Taylor expansion, respectively,

$$\delta^{(n)}\left(\frac{x}{\beta}\right) = |\beta|^{n}\delta^{(n)}(x), \quad n = 0, 1, \ldots \tag{A.5}$$

$$\delta(x - \beta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta^{(n)}(x) \beta^n. \tag{A.6}$$

**B  Transition amplitude for Podolsky’s photon exchange**

To calculate the matrix amplitude for the Bhabha scattering, we shall consider the following initial state $|\psi_i\rangle$ and the final state $|\psi_f\rangle$

$$|\psi_i\rangle = d_{s_i}^\dagger(p_i) b_{\sigma_i}^\dagger(q_i) |\Omega\rangle, \tag{B.1}$$

$$|\psi_f\rangle = d_{s_f}^\dagger(p_f) b_{\sigma_f}^\dagger(q_f) |\Omega\rangle, \tag{B.2}$$

where $(q_i, \sigma_i)$ and $(q_f, \sigma_f)$ are the momentum and spin of the ingoing and outgoing electron, respectively, and, $(p_i, s_i)$ and $(p_f, s_f)$ are the momentum and spin of the ingoing and outgoing positron, respectively. Thus, the transition amplitude in the second coupling constant order, in the causal approach, is given by the expression:

$$S_{fi} = \langle \psi_f | S_2 | \psi_i \rangle = \frac{1}{2!} \left\langle \psi_f \left| \int d^4x_1 d^4x_2 T_2(x_1, x_2) \right| \psi_i \right\rangle. \tag{B.3}$$

Substituting the expression $T_2$ from (4.11) into the above result we have that

$$S_{fi} = -\frac{ie^2}{2!} \int d^4x_1 d^4x_2 \left\langle \psi_f \left| \psi(x_1) \gamma^\mu \psi(x_1) D_{\mu\nu}^F(x_1 - x_2) \psi(x_2) \gamma^\nu \psi(x_2) \right| \psi_i \right\rangle. \tag{B.4}$$

We have that the Dirac field free solutions are given by [17]

$$\psi(x) = (2\pi)^{-3/2} \int d^3p \left[ b_{s_1}(p_1) u_{s_1}(p_1) e^{-ip_1x} + d_{s_1}^\dagger(p_1) v_{s_1}(p_1) e^{ip_1x} \right], \tag{B.5}$$

$$\bar{\psi}(x) = (2\pi)^{-3/2} \int d^3p \left[ b_{s_2}^\dagger(p_2) \bar{u}_{s_2}(p_2) e^{ip_2x} + d_{s_2}(p_2) \bar{v}_{s_2}(p_2) e^{-ip_2x} \right], \tag{B.6}$$

where the pair of operators $b_{s_1}, b_{s_1}^\dagger$ and $d_{s_1}, d_{s_1}^\dagger$ satisfy the anticommutation relations: \{ $b_{s_1}(p_1), b_{s_1}^\dagger(p)\} = \delta_{s_1 s_2}(p_1 - p) = \{ d_{s_1}(p_1), d_{s_1}^\dagger(p)\}$, while any other anticommutation relations vanish. Hence, we can obtain

$$\psi(x) b_{s}^\dagger(p) |\Omega\rangle = |\Omega\rangle (2\pi)^{-3/2} u_{s}(p) e^{-ipx}, \tag{B.7}$$

$$\bar{\psi}(x) d_{s}^\dagger(p) |\Omega\rangle = |\Omega\rangle (2\pi)^{-3/2} \bar{v}_{s}(p) e^{ipx}, \tag{B.8}$$

$$\langle \Omega | b_{s}(p) \psi(x) = (2\pi)^{-3/2} \bar{u}_{s}(p) e^{ipx} |\Omega\rangle, \tag{B.9}$$

$$\langle \Omega | d_{s}(p) \psi(x) = (2\pi)^{-3/2} v_{s}(p) e^{ipx} |\Omega\rangle. \tag{B.10}$$
Thus, after some algebraic manipulation, the transition amplitude, Eq. (B.4), can be written as:

\[ S_{fi} = -i e^2 (2\pi)^{-6} \int d^4x_1 d^4x_2 D_{\mu\nu}^F (x_1 - x_2) \]

\[ \times \left[ \bar{\psi}_{s_f} (p_f) \gamma^\mu v_{s_f} (p_f) u_{\sigma_f} (q_f) \gamma^\nu u_{\sigma_i} (q_i) e^{i(p_f - p_i) x_1 + i(q_f - q_i) x_2} \]

\[ - \bar{\psi}_{s_i} (p_i) \gamma^\mu u_{\sigma_i} (q_i) \bar{u}_{\sigma_f} (q_f) \gamma^\nu v_{s_f} (p_f) e^{i(-p_i - q_i) x_1 + i(q_f + p_f) x_2} \right]. \]  \hspace{1cm} (B.11)

Moreover, by means of the distributional property:

\[ \int d^4x_1 d^4x_2 D_{\mu\nu}^F (x_1 - x_2) e^{i p x_1 + i q x_2} = (2\pi)^6 \delta (p + q) \hat{D}_{\mu\nu}^F (p), \]  \hspace{1cm} (B.12)

we finally can write the transition amplitude in the following short form:

\[ S_{fi} = \delta (p_f + q_f - p_i - q_i) \mathcal{M}, \]  \hspace{1cm} (B.13)

where \( \mathcal{M} \) is the matrix amplitude for the Podolsky’s photon exchange, and it is explicitly defined as it follows:

\[ \mathcal{M} = -i e^2 \left[ \hat{D}_{\mu\nu}^F (p_f - p_i) \bar{\psi}_{s_i} (p_i) \gamma^\mu v_{s_f} (p_f) \bar{u}_{\sigma_f} (q_f) \gamma^\nu u_{\sigma_i} (q_i) \right. \]

\[ - \hat{D}_{\mu\nu}^F (-p_i - q_i) \bar{\psi}_{s_i} (p_i) \gamma^\mu u_{\sigma_i} (q_i) \bar{u}_{\sigma_f} (q_f) \gamma^\nu v_{s_f} (p_f) \right]. \] \hspace{1cm} (B.14)

As it is easily seen, the transition and matrix amplitude are both distributions, then they only have meaning when it is considered wave-package states.

In the calculation of the Bhabha scattering cross section, at Sect. 4, we have considered the sum over the final spins and averaged over the initial spins, thus we had make the substitution: \( |\mathcal{M}|^2 \rightarrow \frac{\pi}{4} \sum_{s_i,\sigma_i,s_f,\sigma_f} |\mathcal{M}|^2 \). Hence, from the expression (B.14), we can evaluate the quantity:

\[ |\mathcal{M}|^2 = e^4 \left[ \hat{D}_{\mu\nu}^F (p_f - p_i) \bar{u}_{\sigma_i} (q_i) \gamma^\nu u_{\sigma_f} (q_f) \bar{v}_{s_f} (p_f) \gamma^\mu v_{s_i} (p_i) \right. \]

\[ - \hat{D}_{\mu\nu}^F (-p_i - q_i) \bar{v}_{s_f} (p_f) \gamma^\nu u_{\sigma_f} (q_f) \bar{u}_{\sigma_i} (q_i) \gamma^\mu v_{s_i} (p_i) \]

\[ \times \left[ \hat{D}_{\alpha\beta}^F (p_f - p_i) \bar{u}_{s_i} (p_i) \gamma^\beta u_{\sigma_i} (q_i) \bar{u}_{\sigma_f} (q_f) \gamma^\alpha v_{s_f} (p_f) \right. \]

\[ - \hat{D}_{\alpha\beta}^F (-p_i - q_i) \bar{u}_{s_i} (p_i) \gamma^\beta u_{\sigma_i} (q_i) \bar{u}_{\sigma_f} (q_f) \gamma^\alpha v_{s_f} (p_f) \right]. \] \hspace{1cm} (B.15)

Nevertheless, in order to calculate \( \frac{\pi}{4} \sum_{s_i,\sigma_i,s_f,\sigma_f} |M|^2 \) we can use the completeness relations for the four-spinor [17]

\[ \sum_s u_s (p) \bar{u}_s (p) = \frac{\gamma p + m}{2E}, \quad \sum_s v_s (p) \bar{v}_s (p) = \frac{\gamma p - m}{2E}, \] \hspace{1cm} (B.16)
and after some calculation we obtain that
\[
\frac{1}{4} \sum_{s_i, \sigma_i, s_f, \sigma_f} |\mathcal{M}|^2 = \frac{e^4}{64E(p_i)E(q_i)E(p_f)E(q_f)} \times \left\{ \hat{D}_{\mu\nu}^E(p_f - p_i) \hat{D}_{\alpha\beta}^E(p_f - p_i) \Xi^{\nu\alpha}(q_i, q_f, m) \Xi^{\mu\alpha}(p_i, p_f, -m) + (p_f \rightleftharpoons -q_i) \\
- \hat{D}_{\mu\nu}^E(p_f - p_i) \hat{D}_{\alpha\beta}^E(-p_i - q_i) \Xi^{\nu\mu\alpha}(q_f, p_f, p_i, q_i) - (p_f \rightleftharpoons -q_i) \right\},
\]  
where \( E \) indicates the energy of the corresponding fermion, and we have defined the quantities
\[
\Xi^{\nu\alpha}(q_i, q_f, m) = tr \left[ \gamma^\nu (\gamma.p + m) \gamma^\alpha (\gamma.q + m) \right],
\]
\[
\Xi^{\nu\mu\alpha}(q_f, p_f, p_i, q_i) = tr \left[ \gamma^\nu (\gamma.q_f + m) \gamma^\mu (\gamma.p_f - m) \gamma^\alpha (\gamma.q_i + m) \right].
\]
Since the longitudinal part of the photon propagator \( D_{\mu\nu}^F \), Eq.(4.16), does not contributes to the transition amplitude, we can then choose, without loss of generality, the gauge-fixing parameter to \( \xi = 1 \). Also, using the \( \gamma \)-matrices properties we can evaluate:
\[
g_{\mu\nu} g_{\alpha\beta} \Xi^{\nu\alpha}(q_i, q_f, m) \Xi^{\mu\alpha}(p_i, p_f, -m) = 8 \left[ s^2 + u^2 + 8m^2 t - 8m^4 \right],
\]
\[
g_{\mu\nu} g_{\alpha\beta} \Xi^{\nu\mu\alpha}(q_f, p_f, p_i, q_i) = 8 \left[ -u^2 + 8m^2 u - 12m^4 \right],
\]
where \( s, t, u \) are the Mandelstam variables \([9]\). Finally, with the above results, we can find that
\[
\frac{1}{4} \sum_{s_i, \sigma_i, s_f, \sigma_f} |\mathcal{M}|^2 = \frac{e^4}{8E(p_i)E(q_i)E(p_f)E(q_f)} \times \left\{ \left[ s^2 + u^2 + 8m^2 t - 8m^4 \right] \left[ \hat{D}_0^E - \hat{D}_{ma}^E \right]^\dagger (\sqrt{t}) \left[ \hat{D}_0^E - \hat{D}_{ma}^E \right] (\sqrt{t}) + (s \rightleftharpoons t) \\
+ \left[ u^2 - 8m^2 u + 12m^4 \right] \left[ \hat{D}_0^E - \hat{D}_{ma}^E \right]^\dagger (\sqrt{s}) \left[ \hat{D}_0^E - \hat{D}_{ma}^E \right] (\sqrt{s}) + (s \rightleftharpoons t) \right\}.
\]
Nevertheless, from the definition of the massless and massive propagators, Eq.(4.17), we obtain the relations:
\[
\left[ \hat{D}_0^E - \hat{D}_{ma}^E \right]^\dagger (k) \left[ \hat{D}_0^E - \hat{D}_{ma}^E \right] (k) = (2\pi)^{-4} \frac{2}{k^4 \left( 1 - k^2/m_a^2 \right)^2},
\]
\[
\left[ \hat{D}_0^E - \hat{D}_{ma}^E \right]^\dagger (k) \left[ \hat{D}_0^E - \hat{D}_{ma}^E \right] (q) + (q \rightleftharpoons k) = (2\pi)^{-4} \frac{1}{k^2 q^2 \left( 1 - k^2/m_a^2 \right) \left( 1 - q^2/m_a^2 \right)}.
\]
Finally, substituting the above relations in (B.22) and after some simplifications, we obtain that:
\[
\mathcal{F}(s,t,u) = \frac{(2\pi)^4 s^2}{2e^4} \left[ \frac{1}{4} \sum_{s_i, \sigma_i, s_f, \sigma_f} |\mathcal{M}|^2 \right]
= \left[ s^2 + u^2 + 8m^2 t - 8m^4 \right] \frac{1}{s^2 \left( 1 - \frac{t}{m_a} \right)^2} + \left[ u^2 + t^2 + 8m^2 s - 8m^4 \right] \frac{1}{t^2 \left( 1 - \frac{s}{m_a} \right)^2} + 2 \left[ u^2 - 8m^2 u + 12m^4 \right] \frac{1}{st \left( 1 - \frac{s}{m_a} \right) \left( 1 - \frac{t}{m_a} \right)}.
\]
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