On the Prime Radical of a Hypergroupoid

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Abstract: In this study, we give definitions of a prime ideal, a s-semiprime ideal and a w-semiprime ideal for a hypergroupoid K. For an ideal A of K we show that radical of A (R(A)) can be represented as the intersection of all prime ideals of K containing A and we define a strongly A-nilpotent element. For any ideal A of K, we prove that R(A)=∩(s-semiprime ideals of K containing A)= ∩(w-semiprime ideals of K containing A)={strongly A nilpotent elements}. For an ideal B of K put B^{0}=B and B^{n+1}=(B^{n})^{2}. If a hypergroupoid K satisfies the ascending chain condition for ideals then (R(A))^n⊂A for some n. For an ideal A of K we give a definition of right radical of A (R_r(A)). If K is associative then R(A)=R_r(A)=R_(n)(A).

Key words: Hypergroupoids, s-semiprime ideal, w-semiprime ideal, ascending chain

1. Hypergroupoids and Complete ℓ-Groupoids

Definition 1.1: A groupoid K is a system (K, ·), where K is a set and · is a binary operation on K.

Definition 1.2[1]: A complete ℓ-groupoid is a system (K, ·), where K is a complete lattice and · is a binary operation on K which satisfies the following conditions:

\[ a \cdot (Vb_1 \cdot t) = V (a \cdot b_1 \cdot t) \quad \text{for all} \quad a, b_1 \in K \]

for all a, b_1 ∈ K

Let K be a set and denote by 2^K the set of all its subsets.

Definition 1.3[2]: A multivariable binary operation on K is a map \( \vartheta : K \times K \rightarrow 2^K \). A hypergroupoid is a system (K, \( \vartheta \)), where K is a set and \( \vartheta \) is a multivariable operation on K.

From now on, we write a-b instead of \( \vartheta (a, b) \).

Let (K, ·) be a hypergroupoid. For A, B ∈ 2^K. A≠∅, B≠∅, put A-B={a-b | a∈A, b∈B} and A·B=∅ if a·b∈∅ for all \( a \in A \), \( b \in B \).

Let \( \vartheta \) be a ternary relation on K. Then (2^K, ·) is a complete ℓ-groupoid.

Conversely, If (2^K, ·) is a complete ℓ-groupoid then a restriction of the binary operation of 2^K to K is a multivariable operation on K and K is a hypergroupoid, with respect to this operation.

Let w be a ternary relation on K.

For (a, b)∈KxK, put a-b={xeK | (a, b, x)∈w}, then (K, ·) is a hypergroupoid.

Conversely, let (K, ·) be a hypergroupoid. Denote by w the set (a, b, c)∈KxKxK such that a·b≠∅ and c∈a·b. Then w is a ternary relation on K.

Hypergroupoids contain the following two classes of algebraic systems.
Definition 2.3: An element $h \in G$ is w-semiprime if $h \neq 1_G$ and $f(a)\neq h$, $G(\alpha \in \mathbb{A})$ implies that $a \neq h$.

Therefore every w-semiprime element is s-semiprime. For $a \in G$, $a \neq 1_G$, denote by $r^w_G(a)$ the intersection of all w-semiprime elements of $G$ containing $a$. Put $r^w_G(a)=1_G$ if there are not any element with this property. It is clear that $r^w_G(a)\leq r^s_G(a)\leq r^w_G(a)$ for all $a \in G$.

3. The Prime Radical of an Ideal

Definition 3.1: Let $K$ be a hypergroupoid. A right (left) ideal of $K$ is a subset $H$ such that $ha \subseteq H$ (respectively $ah \subseteq H$) for all $a \in K$, $h \in H$. An (two-side) ideal of $K$ is a subset $H$ such that $ha \subseteq H$ and $ah \subseteq H$ for all $a \in K$, $h \in H$.

Denote by $\text{Id}(K)$ ($\text{Id}_{r}(K)$, $\text{Id}_{l}(K)$) the set of all ideals (respectively, right ideals, left ideals) of $K$. Put $\emptyset \in \text{Id}(K)$, $\emptyset \in \text{Id}_{r}(K)$, $\emptyset \in \text{Id}_{l}(K)$. Then $\text{Id}(K)$, $\text{Id}_{r}(K)$, $\text{Id}_{l}(K)$ are complete lattices with respect to the inclusion relation.

Proposition 3.2: Let $K$ be a hypergroupoid. Then:
1. $\bigcap_{t \in T} A_t \in \text{Id}(K)$ and $\bigcup_{t \in T} A_t \in \text{Id}(K)$ for any $A_t \in \text{Id}(K)$;
2. $\bigcap_{t \in T} B_t \in \text{Id}_{r}(K)$ and $\bigcup_{t \in T} B_t \in \text{Id}_{r}(K)$ for any $B_t \in \text{Id}_{r}(K)$;
3. $\bigcap_{t \in T} C_t \in \text{Id}_{l}(K)$ and $\bigcup_{t \in T} C_t \in \text{Id}_{l}(K)$ for any $C_t \in \text{Id}_{l}(K)$.

The proof is clear. We next consider the multiplication operation $A \cdot B$ on $2^K$.

Definition 3.3: Hypersemigroup is a hypergroupoid $K$ such that $(A \cdot B)C=A \cdot (B \cdot C)$ for any $A, B, C \subseteq 2^K$.

If $K$ is hypersemigroup then $A \cdot B \subseteq \text{Id}(K)$ for any $A, B \in \text{Id}(K)$. But there are a hypergroupoid $K$ and $A, B \in \text{Id}(K)$ such that $A \cdot B \notin \text{Id}(K)$. Therefore for any hypergroupoid $K$ we define a multiplication operation of ideals as follows:

For $A, B \in \text{Id}(K)$ denote by $A \cdot B$ the intersection of all ideals of $K$ containing the set $G=\{x \in a \cdot b, a \in A, b \in B\}$.

Multiplication operations on $\text{Id}_{r}(K)$ and $\text{Id}_{l}(K)$ are introduced similarly.

Proposition 3.4: For any hypergroupoid $K$, the lattices $\text{Id}(K)$, $\text{Id}_{r}(K)$, $\text{Id}_{l}(K)$ are complete $l$-groupoids with respect to above multiplication operations.

Proof: We give a proof for $\text{Id}(K)$ and the proofs for $\text{Id}_{r}(K)$ and $\text{Id}_{l}(K)$ are similar. Suppose $A, B \in \text{Id}(K)$, $t \in T$. It is clear that $A \cdot (\bigcup_{t \in T} B_t) \supseteq (\bigcup_{t \in T} A \cdot B_t)$.

Conversely the ideal $A \cdot (\bigcup_{t \in T} B_t)$ is the smallest ideal containing all elements $a \cdot b$, where $a \in A$, $b \in \bigcup_{t \in T} B_t$.

Let $a, b \in \text{Id}(K)$, $A \cdot (\bigcup_{t \in T} B_t)$. Since $b \in B_t$ for some $t \in T$ then $a \cdot b \in A \cdot B$. Therefore $A \cdot (\bigcup_{t \in T} B_t) \subseteq (A \cdot B_t)$.

Now, we apply the definitions and designations of the prime and semiprime elements of ordered groupoids to $2^K$, $\text{Id}(K)$, $\text{Id}_{r}(K)$, $\text{Id}_{l}(K)$. Put $R_0(A)=R(A)$, $r^w_G(A)=r^w_G(A)$, $r^w_G(A)=r^w_G(A)$ for $G=\text{Id}(K)$, $A \in \text{Id}(K)$.

Definition 3.5: An ideal $H$ is maximal if $H \neq K$ and $H \subseteq B \subseteq K$, $B \in \text{Id}(K)$ implies that $H=B$ or $B=K$.

For $a \in \text{Id}(K)$ denote by $[a]$ the intersection of all ideals of $K$ containing $a$.

Proposition 3.6: Let $K$ be a hypergroupoid. Then any maximal ideal of $K$ is prime if and only if $K=2^K$.

Proof: Let $K=2^K$ and $M$ be a maximal ideal of $K$. Assume that $A \cdot B=\bigcup_{t \in T} A \cdot B_t=2^K$.

Since $a \in \text{Id}(K)$, $a \in \text{Id}(K)$.

Therefore $M=K$. This is a contradiction. Thus $M$ is prime.

Conversely, Let $K \neq 2^K$ and $a \in K \cdot 2^K$. We prove that $M=K \setminus \{a\}$ is a minimal ideal of $K$ and it is not prime. Let $b \in M \setminus \{a\}$. Then $a \cdot b \in M$ and $b \cdot a \in M$ for all $a, b \in K$.

Indeed, if there is $K$ such that $a \cdot b \in M$ then $a \cdot b \notin M$.

Hence $a \in K \cdot 2^K$. It is a contradiction. Thus $a \cdot b \notin M$ and $b \cdot a \notin M$ for any $a \cdot b \in K$. It is clear that $M$ is a maximal ideal. Prove that $M$ is not prime. By $a \in M$ we have $[a]=M$. But $[a]=\bigcup_{t \in T} A_t \subseteq M$. Therefore $M$ is not prime.

Remark: This proposition is known for semigroups\[5\].

Every sequence $\{x_0, x_1, \ldots, x_n, \ldots\}$, where $x_n=a$, $x_{n+1} \in [x_n]^2$, will be called an $s$-sequence of the element $a$.

Definition 3.7: Let $a \in \text{Id}(K)$. An element $a \in K$ is strongly $A$-nilpotent if every $s$-sequence of a meets $A$.
Remark: This definition is similar to the definition of the n-sequence\(^6\).

Denote by \(n(A)\) the set of all strongly A-nilpotent elements of K.

Theorem 3.8: Let K be a hypergroupoid. Then for any ideal A of K, we have \(n(A)=r^w(A)=r^s(A)=R(A)\).

Proof: From the definitions \(r^w(A)\), \(r^s(A)\), and \(R(A)\) we obtain \(r^s(A)\subseteq r^w(A)\subseteq R(A)\) for any \(A\in\text{Id}(K)\).

We prove that \(n(A)\subseteq r^s(A)\). If there is not an s-semiprime ideal of K containing A then \(r^s(A)=K\) and \(n(A)\subseteq r^s(A)\).

Assume that there exists an s-semiprime ideal of K containing A. Let \(a\in n(A)\) and \(b\in s\)-semiprime ideal of K containing A. We first prove that \(a\in S\) and \(b\in S\).

By continuing in this manner we obtain an s-sequence \(\{x_n, x_1, x_2, \ldots, x_{n+1}\}\) of the element a such that \(x_n\in S\) for all n. But this is a contradiction since every s-sequence of the element a meets A. Thus \(a\in S\) and \(b\in S\).

Now we prove that \(R(A)=n(A)\). If \(n(A)=K\) then \(n(A)=r^s(A)=r^w(A)=R(A)=K\). Let \(n(A)=K\). Hence there exists \(b\in K\) such that \(b\in n(A)\). Then there exists an s-sequence \(X=\{x_n, x_1, x_2, \ldots, x_{n+1}\}\) of the element b such that \(X\cap A=\emptyset\).

Conversely, if every ideal of K is radical then using the Theorem 3.8 we obtain that every \(s\)-semiprime ideal of K is radical. Let A be an ideal of K. Then \(A^2=\cup_{a\in A} [a]\) and \(A^3=\cup_{a\in A} [a^2]\) for all \(a\in A\).

Using the Proposition 3.4 we have \(A^{2n}\subseteq A, B_{2n}\subseteq A\), and \(B_{2n}\subseteq A\) for any \(a, b\in A\).

Remark: This corollary is an analog of the similar theorem for associative rings\(^6\).

Definition 3.12: Let \(A\in\text{Id}(K)\). An ideal B of K is \(A\)-nilpotent if \(B^{m}\subseteq A\) for some m.

Proposition 3.13: Let K be hypergroupoid and A, B be ideals of K. If C is B\(_n\)-nilpotent and B is A\(_n\)-nilpotent then C is A\(_n\)-nilpotent.

Proof: Since C is B\(_n\)-nilpotent then \(C^{m}\subseteq B\) for some m. Hence \(C^{m+n}\subseteq (C^{m})^{n}\subseteq B^{m+n}\subseteq A\).

Theorem 3.14: Let K be a hypergroupoid satisfying the ascending chain condition for ideals. Then for any ideals A of K, R(A) is A\(_n\)-nilpotent.

Proof: Let \(A\in\text{Id}(K)\). Denote by \(\Sigma\) the set of all A\(_n\)-nilpotent ideals H of K such that \(H\subseteq A\). \(\Sigma\) is not empty since \(A\in\Sigma\). There exists a maximal element P in \(\Sigma\). We prove that P is s-semiprime. Let \(B_{2n}\subseteq P\). Then \(B_{2n}\subseteq B_{2n}\subseteq BP_{2n}\subseteq P\). By Proposition 3.13 the
ideal B∪P is A₀-nilpotent. By the maximality of P we have B∪P=P. Hence B⊆P. This means that P is s-semiprime. Since P⊇A then R(A)⊂P by Theorem 3.8. But P_n⊂A⊂R(A) for some n. Since R(A) is s-semiprime then P⊂R(A). Thus P=R(A).

Remark: This theorem is similar to the proposition for associative rings[7].

Corollary 3.15: Let K be hypergroupoid satisfying the ascending chain condition for ideals. Then the following conditions are equivalent:
1. K is an intersection of finite prime ideals and a
2. K doesn’t have a prime ideal;
3. K doesn’t have a s-semiprime ideal.

A proof follows from Theorem 3.14 and the definition of Pr. rad(K). Denote by Id(K) the set of all radical ideals of K. Id(K) is a complete lattice with respect to the inclusion relation. Denote by ∨ and ∧ the lattice operations in Id(K).

Theorem 3.16: Let K be a hypergroupoid. Then the lattice Id(K) satisfies the infinite ∧-distributive condition: 
A∩(B₁∪B₂) =(A∩B₁)∪(A∩B₂) for any A, B₁,B₂ ∈ Id(K)

Proof: The proof follows from Theorem 1.3[7].

Theorem 3.17: Let K be a hypergroupoid satisfying the ascending chain condition for ideals. Then any radical ideal of K is an intersection of finite prime ideals and a such representation is unique.

Proof: First we prove the following lemma.

Lemma: H∈Id(K) is prime ideal if and only if H is an ∧-indecomposable element of the lattice Id(K).

Proof: Let A be a prime ideal of K and A=A₁∧A₂, A₁, A₂∈Id(K). Then[7] 
A₁A₂⊂A₁∩A₂⊂R(A₁∩A₂)=A₁∧A₂=A. Hence A₁⊂A or A₂⊂A. Then A=A₁ or A=A₂. Let A be an ∧-indecomposable element in Id(K) and BC⊂A, B, C∈Id(K). Then R(B-C)⊂A. By the lemma 1.6[7] we have R(B)∩R(C)⊂R(B-C)⊂A. By the distributivity Id(K) we obtain A=Av(R(B)∩R(C))=(A∩R(B))∧(A∩R(C)). Then A=Av(R(B) or A=Av(R(C) since A is ∧-indecomposable. This means that B⊂R(B)⊂A or C⊂R(C)⊂A.

Thus A is prime. The lemma is proved. By the lemma and the Corollary[11] we obtain that every radical ideal of K is an intersection of finite prime ideals and a such representation is unique.

4. The Right Prime Radical of an Ideal

Definition 4.1: A right ideal H of K is maximal if H≠K and H⊂B⊂K, B∈Id_r(K), implies that H=B or B=K.

Proposition 4.2: Let K be a hypergroupoid such that A⊂K, A for all A∈Id_r(K). Then any maximal right ideal of K is prime element of Id_r(K).

Proof: Let M be a maximal right ideal of K and A-B=M, A, B∈Id_r(K). If A,M then M∪A=K. By Proposition 3.4 we have B⊂K=B=(M∪A)-B=MB∪AB⊂M.

Definition 4.3: An element 1∈K is called identity of K if 1=a-1=a for all a∈K.

Remark: The conditions of Proposition 4.2 are satisfied for groupoids with 1. Thus there exists a right ideal in such groupoids.

For an element a∈K denote by [a], the intersection of all right ideals containing a. Every sequence {xₙ,..., xₙ,...} where xₙ=a, xₙₐ∈[xₙ]₂⁺, is called an s⁺-sequence of the element a.

Definition 4.4: Let A∈Id_r(K). An element a∈K is strongly A₀-nilpotent if every its s⁺-sequence meets a.

Denote by n₀(A) the set of all strongly A₀-nilpotent elements of K.

Proposition 4.5: Let K be a hypergroupoid. For any right ideal A of K are satisfied the following inequalities:
R(A)⊂n₀(A)⊂r⁺ (A)⊂r⁺ (A)⊂R_r(A).

Proof: A proof of n₀(A)⊂r⁺ (A) is similar to the proof of n(A)⊂r⁺ (A) as in the Theorem 3.8. The inequality R(A)=n₀(A) and definitions of n(A) and n₀(A).

Theorem 4.6: Let K be a hypergroupoid satisfying the following conditions:
(K,A)-B=K-(A,B), (A,K)-B=A-(K,B) for all A, B∈Id_r(K). Then
R(A)=n₀(A)=r⁺ (A)=r⁺ (A)=R_r(A) for all A∈Id_r(K).

Proof: By Proposition 4.5 it is enough to prove that R_r(A)⊂R(A).

Denote by P(K) the set of all prime ideals of K and P_r(K) the set of all right prime ideals of K. We prove that P(K)⊂P_r(K). Let Q∈P(K) and B⊂Q. B∈Id_r(K). Then, (B∪K-B)=(B∪K-C) = (B∪K-C)⊂(B∪K-C)⊂Q.

Note that B∪KB and C∪KC are ideals of K. Indeed K×(B∪KB) =K-B∪(K-B)∈B∪KB.

From (B∪KB) (C∪KC) Q we obtain B⊂B∪KB⊂Q or C⊂C∪KC⊂Q since Q is prime. This means Q∈P_r(K).
Thus $P(K) \subseteq P_s(K)$. Therefore we have $R_s(A) \subseteq R(A)$.

**Remark:** The conditions of this theorem are satisfied for hypersemigroup. Therefore the same theorem is given for nonasociative hypergroupoid $K$ and $A \in \text{Id}(K)$ such that $R(A) = R_s(A)$ and $R(A) \neq R_{\cdot}(A)$. Let $A \in \text{Id}_s(K)$. For $b \in K$ put $b^{(0)} = b$, $b^{(n+1)} = (b^{(n)})^2$.

**Definition 4.7:** An element $b \in K$ is $A_s$-nilpotent if $b^{(n)} \subseteq A$ for some $n$. An element $b \in K$ is $A_w$-nilpotent if $f(b) \subseteq A$ for some $f(b) \subseteq \langle b \rangle$.

Denote by $n^S_0(A)$ ($n^w_0(A)$) the set of all $A_s$-nilpotent (respectively, $A_w$-nilpotent) elements of $K$.

**Proposition 4.8:** For any ideal $A$ of $K$ are hold the following inequalities:

\[ R(A) \subseteq n_s(A) \subseteq n^S_0(A) \subseteq n^S_0(A) \subseteq R_o(A) \]
\[ R(A) \subseteq n_s(A) \subseteq n^w_0(A) \subseteq n^w_0(A) \subseteq R_w(A) \]

The proof is similar to the proof of Proposition 4.5.

**Theorem 4.9:** Let $K$ be a hypersemigroup satisfying the condition $K \cdot a = a \cdot K$ for all $a \in K$. Then $R(A) = n_s(A) = r_s(A) = R_o(A)$ for all $A \in \text{Id}(K)$.

The proof is similar to the proof of Theorem 4.6.

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