Effects of Disorder on a 1-D Floquet Symmetry Protected Topological Phase

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Periodically driven systems, also known as Floquet systems, can realize symmetry protected topological (SPT) phases that cannot be found in equilibrium. Here, we seek to understand the effects of strong disorder on such SPT phases, working with a one-dimensional Floquet system belonging to Altland-Zirnbauer class BDI. Using the transfer matrix method, we find that the disordered system hosts an array of trivial, topological, and Floquet topological phases. We then explore the phase diagram in the case of uniformly distributed bonds. Our analytic results are confirmed by a numeric computation of the bulk topological invariants. Although all states are generically localized, near phase transitions the localization length of topological edge states diverges. We derive the critical exponents governing these divergences, finding that they take on equilibrium values, despite the fact that we are considering phases unique to non-equilibrium. We also discuss how our work sets the stage for studying interacting Floquet SPT phases.

I. INTRODUCTION AND MOTIVATION

Symmetry protected topological phases (SPT phases) have received significant attention over the past several years. More recently, the notion of SPT phases was generalized to periodically driven systems, also known as Floquet systems, and closely related quantum walk systems. This generalization includes phases that have no counterpart in static systems, so that these so-called Floquet SPT phases require entirely new theoretical classification. For instance, two-dimensional Floquet systems can have chiral edge modes even when all bands have zero Chern number, and one dimensional Floquet systems and quantum walks can have protected edge states despite having zero bulk winding number.

In the presence of interactions, a translationally invariant Floquet system continuously absorbs energy from the periodic drive and heats up to an infinite temperature, thus precluding the existence of an SPT phase or other types of quantum order. However, thermalization can be avoided if strong disorder is incorporated so that the system is many-body localized (MBL) and one dimensional Floquet systems and quantum walks can have protected edge states despite having zero bulk winding number.

In this section we briefly review Floquet theory. A Floquet system is a Hamiltonian \( H(t) \) that is periodic in time \( t \) with period \( T \). \( H(t) \) admits solutions \( |\psi(t)\rangle \) to the time-dependent Schrodinger equation that satisfy \( |\psi(t + T)\rangle = e^{-iET} |\psi(t)\rangle \). Here \( E \) is a real parameter defined modulo \( 2\pi/T \) called the quasienergy. The solutions \( |\psi(t)\rangle \) are called Floquet states. Equivalently, the Floquet states satisfy the Floquet eigenvalue equation

\[
U_{t+T,t}|\psi(t)\rangle = e^{-iET} |\psi(t)\rangle,
\]

where \( U_{t+T,t} \) is the operator that evolves states from time \( t \) to time \( t+T \). We will usually pick the initial time to be 0, writing \( U_{T,0} = U_T \). We can also define \( H_{\text{eff}} = \frac{1}{T} \log U_T \) so that

\[
H_{\text{eff}}|\psi(t)\rangle = E|\psi(t)\rangle.
\]

The interpretation of (2) is that if the system is examined stroboscopically with period \( T \), \( H(t) \) simulates a time independent Hamiltonian \( H_{\text{eff}} \) whose energies are defined modulo \( 2\pi/T \).

The notion of an SPT phase generalizes naturally to Floquet systems. A nontrivial Floquet SPT phase is characterized by Floquet edge states robust to perturbations that preserve the symmetries of the effective Hamiltonian \( H_{\text{eff}} \). The only subtle point is that we are interested in the symmetries of \( H_{\text{eff}} \), not those of the instantaneous Hamiltonian \( H(t) \).

II. FLOQUET THEORY

III. HAMILTONIAN AND ITS PHASES

We now introduce the model that we study in this work, and then discuss its symmetries and its result-
ing topological phases. The model consists of spinless fermions hopping on a one-dimensional lattice containing two sites $A$ and $B$ per unit cell. It is obtained by adding disorder to a special case of the periodically driven Su-Schrieffer-Heeger model introduced in Ref. [10].

Let $c$ denote the vector of site annihilation operators:

$$c = \begin{pmatrix}
c_{A1} \\
c_{B1} \\
\vdots \\
c_{AN} \\
c_{BN}
\end{pmatrix},$$

(3)

where $c_{Ai}$ and $c_{Bi}$ annihilate a fermion at site $A$ and $B$, respectively, in unit cell $i$. The Hamiltonian is then defined by

$$H(t) = c^\dagger \mathcal{H}(t) c$$

(4)

where the first-quantized matrix $\mathcal{H}(t)$ is given by

$$\mathcal{H}(t) = \begin{cases}
\mathcal{H}_1(\{J_i\}) & 0 \leq t < \frac{1}{2} T \\
\mathcal{H}_2(\{K_i\}) & \frac{1}{2} T \leq t < \frac{3}{2} T \\
\mathcal{H}_1(\{J_i\}) & \frac{3}{2} T \leq t \leq T
\end{cases}$$

(5)

with $\mathcal{H}_1$ and $\mathcal{H}_2$ Hermitian matrices satisfying

$$c^\dagger \mathcal{H}_1(\{J_i\}) c = \sum_{i=1}^{N} 2 J_i (c_{Ai}^\dagger c_{Bi} + c_{Bi}^\dagger c_{Ai})$$

(6a)

$$c^\dagger \mathcal{H}_2(\{K_i\}) c = \sum_{i=1}^{N-1} 2 K_i (c_{Ai+1}^\dagger c_{Bi} + c_{Bi}^\dagger c_{Ai+1}).$$

(6b)

Here $\{J_i\}$ and $\{K_i\}$ are two sets of independent, identically distributed random variables. The first-quantized time evolution operator is then given by

$$\mathcal{U}_T = T e^{-i \int_0^T \mathcal{H}(t) dt} = e^{-\frac{i}{2} \mathcal{H}_1(\{J_i\})} e^{-\frac{i}{2} \mathcal{H}_2(\{K_i\})} e^{-\frac{i}{2} T \mathcal{H}_1(\{J_i\})},$$

(7)

where $T$ is the time ordering symbol.

In the translationally invariant limit, variants of this model have been explored in the context of driven superconductors and 1-D quantum walks. The so-called “simple quantum walk”, a special case of the quantum walk analogue of (4), has been studied numerically called “simple quantum walk”, a special case of the quantum walk analogue of (4), has been studied numerically in the presence of disorder. The phase factor $i$ does not affect the results that we will cite.

### IV. TRANSFER MATRIX METHOD

Here we discuss the the transfer matrix of the system and how it leads to a simple formula for $\nu_0$ and $\nu_\pi$. In general, the transfer matrix method is useful for solving 1-D lattice models with finite range hopping when translation symmetry is not available. First we review its application to static systems. In this case the goal is to convert a large (possibly infinite) eigenvalue equation $\mathcal{H}\psi = E\psi$ to a small matrix problem of the form

$$\begin{pmatrix}
\psi_{i+t} \\
\psi_{i+t+1} \\
\vdots \\
\psi_{i+t+s}
\end{pmatrix} = T(E) \begin{pmatrix}
\psi_i \\
\psi_{i+1} \\
\vdots \\
\psi_{i+s}
\end{pmatrix},$$

(9)

where $1 \leq t \leq s + 1$. Numerically iterating the transfer matrix allows one to compute its Lyapunov exponents, the smallest in magnitude of which gives the inverse localization length of the system. When symmetries constrain the form of the transfer matrix it is possible to obtain analytic information about the eigenstates.
For driven systems, we might hope to find a transfer matrix representation of the Floquet eigenvalue equations $U_T \psi = e^{-iE T} \psi$ or $\mathcal{H}_{\text{eff}} \psi = E \psi$. In general this is a hopeless task, because $U_T$ and $\mathcal{H}_{\text{eff}}$ will have infinite (though exponentially decaying) range even if the instantaneous Hamiltonian has finite range. Hamiltonian (5) is special in this respect, because it is a finite sequence of block diagonal Hamiltonians, so $U_T$ has finite range. Indeed, following the steps of Appendix A, we find that the transfer matrix has the two dimensional structure

\[
T_i(E) = \begin{pmatrix}
\psi_{A(i+1)} & \psi_{B(i+1)} \\
\psi_{A(i)} & \psi_{B(i)}
\end{pmatrix}.
\]

(10)

$T_i(E)$ is derived in Appendix A for all quasienergies. Here we only quote the results for $E = 0$ and $\pi/T$, which will be used to study the topological phases of the system:

\[
T_i(0) = \begin{pmatrix}
\sin \frac{1}{2} J_i T \cot \frac{1}{2} K_i T & 0 \\
0 & -\cos \frac{1}{2} J_i T \tan \frac{1}{2} K_i T
\end{pmatrix}
\]

(11a)

\[
T_i(\pi/T) = \begin{pmatrix}
\sin \frac{1}{2} J_i T \tan \frac{1}{2} K_i T & 0 \\
0 & \cos \frac{1}{2} J_i T \cot \frac{1}{2} K_i T
\end{pmatrix}
\]

(11b)

whose magnitude decays at rate

\[
\gamma_0 = \lim_{N \to \infty} \frac{1}{N} \log \left| \prod_{i=1}^{N} \frac{\sin \frac{1}{2} J_i T \cot \frac{1}{2} K_i T}{\cos \frac{1}{2} J_{i+1} T} \right|,
\]

(13)

where $\gamma_0$ is the signed Lyapunov exponent, or inverse localization length, of the state with energy quasienergy 0.

An edge state exists if $\gamma_0 < 0$ so that the wavefunction is normalizable, with the transition between trivial and topological phases occurring at $\gamma_0 = 0$. Since the localization length is the inverse of the absolute value of the Lyapunov exponent, we see that a change in $\nu_0$ is accompanied by a divergence in the localization length of states with quasienergy 0. Likewise, a change in $\nu_\pi$ is accompanied by a divergence in the localization length of states with quasienergy $\pi/T$. We will discuss this divergence later on.

The product (13) can be split into a sum of three averages: an average of $\{\log |\sin \frac{1}{2} J_i T|\}$, $\{\log |\cos \frac{1}{2} J_{i+1} T|\}$, and $\{\log |\cot \frac{1}{2} K_i T|\}$. Since each average is over independent, identically distributed random variables it converges to its corresponding statistical mean as we send $N \to \infty$. Letting an $\langle \cdots \rangle$ denote an expectation value over the disorder distribution, we find that

\[
\nu_0 = \begin{cases}
1 & \text{if } \gamma_0 < 0 \\
0 & \text{if } \gamma_0 > 0
\end{cases}
\]

(14a)

\[
\gamma_0 = \langle \log |\tan \frac{1}{2} J_i T|\rangle - \langle \log |\tan \frac{1}{2} K_i T|\rangle.
\]

(14b)
An identical analysis of the case $E = \pi/T$ leads to
\begin{equation}
\nu_\pi = \begin{cases} -1 & \text{if } \gamma_\pi < 0 \\
0 & \text{if } \gamma_\pi > 0 \end{cases}
\end{equation}
\begin{equation}
\gamma_\pi = -(\log |\tan \frac{1}{2} J'T|) - (\log |\tan \frac{1}{2} K'T|).
\end{equation}
As a check on these equations, consider the zero period limit, where $H_{\text{eff}}$ reduces to the average of $H(t)$ over a period. This average is exactly the Su-Schrieffer-Heeger model, which supports zero modes whenever $|\langle J_i/K_i \rangle| < \frac{\pi}{2}$. Indeed, the $e^{i\pi}$ reduces to this condition at small $T$.

V. PHASE DIAGRAM

We now explore the phase diagram implied by (14) and (15). We will fix $T$ and vary the remaining energy scales.

In a clean system, $J_i$ and $K_i$ are distributed according to point distributions at $J$ and $K$, respectively. Then Equations (14) and (15) reduce to the known phase diagram for uniform bonds, shown in Fig. (1a). The phase boundaries are located at
\begin{equation}
JT \pm KT = 2n\pi \text{ for } \nu_0
\end{equation}
\begin{equation}
JT \pm KT = (2n + 1)\pi \text{ for } \nu_\pi.
\end{equation}

We now introduce disorder. For the sake of definiteness, we take $J_i = J + w_j r_i$ and $K_i = K + w_K q_i$, where $r_i$ and $q_i$ are random variables uniformly distributed in $[-\frac{1}{2}, \frac{1}{2}]$. It is convenient to introduce the function
\begin{equation}
G(x) = \int_0^x du \log |\tan(u/2)|,
\end{equation}
which has period $2\pi$ and is antisymmetric about 0 and symmetric about $\pi/2$. In terms of $G$, the boundary at which $\nu_0$ and $\nu_\pi$ continue to vanish along the lines defined by (16) even for nonzero $w$.

The simplest case to consider is $w_J = w_K = w$. Starting from the clean system, we increase $w$ and ask what happens to the phase boundaries in the $(JT, KT)$ plane. Using (18) and the fact that $G(x + 2\pi) = G(x)$ and $G(x + \pi) = -G(x)$, we see that $\gamma_0$ and $\gamma_\pi$ continue to vanish along these lines defined by (16) even for nonzero $w$.

VI. REAL-SPACE WINDING NUMBER

There are infinitely many such disorder induced transitions, corresponding to $wT = 2n\pi$ for integer $n$. These occur because the time evolution operator $e^{i\lambda T}$ depends on both $J'/T$ and $K'/T$ and mod $2\pi$, and whenever $wT = 2n\pi$ the distribution for $J'/T$ and $K'/T$ returns to itself.

Next we can ask what happens if $w_J \neq w_K$. Consider $w_J T, w_K T < 2\pi$ and $w_K > w_J$. In this case the phase boundaries shift in the $(JT, KT)$ plane relative to their $w = 0$ positions. The qualitative distinction at moderate disorder is that the crossing of the $\nu_0$ phase boundaries at $(JT, KT) = (0,0), (\pi, \pi)$ and the $\nu_\pi$ phase boundaries at $(JT, KT) = (0,\pi), (\pi, 0)$ open up. For instance, at $(JT, KT) = (\pi, \pi)$
\begin{equation}
\gamma_0 = -\frac{2}{w_J T} G\left(\frac{1}{2} w_J T\right) + \frac{2}{w_K T} G\left(\frac{1}{2} w_K T\right)
\end{equation}
so the crossing opens horizontally. An explicit example is shown in Figures (b) and (c). If we continue to increase $w_K T$ up to $2\pi$, the phase boundaries flatten out into vertical lines. If we instead consider $w_K < w_J$, then the crossings open vertically rather than horizontally.

The general expression for $\nu_0$ and $\nu_\pi$ in Floquet systems exhibiting chiral symmetry was worked out by Refs. 32 and 33 have generalized the notion of a winding number to disordered systems by expressing (21) in real space. Following the procedure
developed in their work, we numerically compute \( \nu \) and \( \nu' \), thus obtaining \( \nu_0 \) and \( \nu_\pi \). The results are shown in Fig. 2 and are in full agreement with the analytic results derived above.

**VII. LOCALIZATION LENGTHS AND CRITICAL EXPONENTS**

One dimensional systems are expected to localize in the presence of disorder. \(^{23}\) To check that this holds for our Floquet system, we have used the full transfer matrix \(^{27}\) to numerically compute the localization length of the system as a function of quasienergy. Except for states with quasienergies 0 and \( \pi/T \) which delocalize when \( \nu_0 \) and \( \nu_\pi \) change, respectively, all states are indeed localized in the presence of disorder. Therefore the system is a promising candidate for being many-body localized in the presence of interactions.

We now examine the transition between distinct localized phases, deriving the critical exponents governing the divergence of the localization length of states with quasienergy 0 and \( \pi/T \) at phase transitions. We do this by finding the form of (18) near a phase transition. We present the result here, with a careful analysis in appendix B.

Let \( g \) represent one of \( JT, KT, wJT, \) or \( wKT \) and \( g_c \) the corresponding critical value. Let \( l_0 \) and \( l_\pi \) denote the localization lengths of states with quasienergy 0 and \( \pi/T \), respectively. Then generally

\[
l_0 \sim |g - g_c|^{-1} \text{ when } \nu_0 \text{ changes } \quad (22a)
\]
\[
l_\pi \sim |g - g_c|^{-1} \text{ when } \nu_\pi \text{ changes. } \quad (22b)
\]

Anomalous scaling of the form

\[
l_0 \sim \left(\frac{g - g_c}{g} \log |g - g_c| \right)^{-1} \text{ when } \nu_0 \text{ changes } \quad (23a)
\]
\[
l_\pi \sim \left(\frac{g - g_c}{g} \log |g - g_c| \right)^{-1} \text{ when } \nu_\pi \text{ changes. } \quad (23b)
\]

appears upon tuning \( JT \) or \( wJT \) when \( (JT)_c = \pm \frac{1}{2}(wJT)_c + n\pi \), and upon tuning \( KT \) or \( wKT \) when \( (KT)_c = \pm \frac{1}{2}(wKT)_c + n\pi \). The region of anomalous scaling has codimension 2 and is therefore not a generic transition point. Both the standard and anomalous scaling match results for the time independent system.\(^{25,22}\) This is notable because the phase transition where \( \nu_\pi \) changes is unique to Floquet systems.

**VIII. DISCUSSION**

To summarize, we have found that disorder shifts the topological phase boundaries relative to the clean system.
according to Equations (14) and (15). It is also noteworthy that the domain of application of the real-space winding number has been extended to include Floquet systems. In addition we found that the exponent for the divergence of the localization length of topological edge states at phase transitions takes on equilibrium values, despite the fact that phases with nonzero $\nu_\pi$ are unique to equilibrium. We confirmed that the system is localized away from these transitions. Hence, the overall picture is very similar to that for static systems belonging to class BDI.

We also mention that our work is closely connected to recent work by V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi[13], in which the authors consider a driven Ising model and show that disorder protects Floquet spin-glass order against interactions. They discuss how the model that they can be mapped to a Floquet superconductor that exhibits Majorana edge modes with energy 0 and $\pi/T$. In the noninteracting limit, the BdG Hamiltonian describing this superconductor system maps onto the Hamiltonian that we consider in this work, up to an intercell rotation by $\frac{1}{\sqrt{2}}(1+\sigma_z)\otimes 1$. They add weak interactions to a point corresponding to $(JT, KT) = (3.06, 1.885)$, $(w_jT, w_KT) = (0.078, 2.198)$ in our phase diagram, finding that the so called $\pi$ spin-glass corresponding to $\pi/T$ Majoranas survives.

The natural question for future work is whether the analogous result holds for our formulation of the problem; that is, whether the many-body system described by [4] supports edge modes when weak interactions are included.

We would like to acknowledge that while finalizing this manuscript we became aware of a previous work by T. Rakovszky and J. K. Asboth[19], in which the authors derive equivalent results for the quantum walk analogue of the Floquet system studied here.

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Appendix A: Details of Transfer Matrix Computation

In this Appendix we outline the steps leading from the Floquet eigenvalue equation $\mathcal{U}_T \psi = e^{-iET} \psi$ to the transfer matrix [10]. It is useful to instead consider the operator $\mathcal{U}_T' = e^{-\frac{i}{2}T\mathcal{H}_2((K_i))} e^{-\frac{i}{2}T\mathcal{H}_1((J_i))}$ which satisfies $\mathcal{U}_T' \psi' = e^{-iET} \psi'$, where $\psi' = \Lambda^\dagger \psi$ and $\Lambda = e^{-\frac{i}{2}T\mathcal{H}_1((J_i))}$. This transformation minimizes the range of the matrix involved. Physically, it is equivalent to a shift in starting time of the periodic drive.

Letting site index $Ai$ correspond to matrix index $2i - 1$ and site index $Bi$ to matrix index $2i$,

$$
\begin{pmatrix}
F_1 & f_1 & 0 & 0 & 0 & \cdots \\
F_1 G_1 & F_1 G_1 & f_2 g_1 & 0 & 0 & \cdots \\
f_1 g_1 & F_1 g_1 & F_2 G_1 & f_2 g_1 & 0 & \cdots \\
0 & 0 & f_2 G_2 & F_2 G_2 & f_3 g_2 & \cdots \\
0 & 0 & f_2 g_2 & F_2 g_2 & F_3 G_2 & f_3 g_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
\psi_{A1} \\
\psi_{B1} \\
\psi_{A2} \\
\psi_{B2} \\
\psi_{A3} \\
\vdots \\
\end{pmatrix}
= e^{-iET}
\begin{pmatrix}
\psi_{A1}' \\
\psi_{B1}' \\
\psi_{A2}' \\
\psi_{B2}' \\
\psi_{A3}' \\
\vdots \\
\end{pmatrix}
$$

(A1)

where $F_i = \cos J_i T, f_i = -i \sin J_i T, G_i = \cos K_i T$, and $g_i = -i \sin K_i T$. Examining rows $2i$ and $2i + 1$ of (A1), we find

$$M_i \begin{pmatrix} \psi_{Ai} \\ \psi_{Bi} \end{pmatrix} = N_i \begin{pmatrix} \psi_{Ai+1} \\ \psi_{Bi+1} \end{pmatrix}$$

(A2)

where

$$M_i = \begin{pmatrix} f_i G_i & F_i G_i - e^{-iET} \\ f_i g_i & F_i g_i \end{pmatrix}$$

(A3a)

$$N_i = \begin{pmatrix} -f_{i+1} g_i & f_{i+1} g_i \\ e^{-iET} - F_{i+1} G_i & -f_{i+1} G_i \end{pmatrix}.$$  

(A3b)

In order to switch back to a transfer matrix equation for $\mathcal{U}_T$, note that $\Lambda$ is block diagonal with blocks separated by unit cells. Letting $\Lambda_i$ denote the $i$th block of this matrix,

$$\Lambda_i = \begin{pmatrix} \cos \frac{1}{2} J_i T & -i \sin \frac{1}{2} J_i T \\ -i \sin \frac{1}{2} J_i T & \cos \frac{1}{2} J_i T \end{pmatrix},$$

(A4)

this implies

$$M_i \Lambda_i \dagger \begin{pmatrix} \psi_{Ai} \\ \psi_{Bi} \end{pmatrix} = N_i \Lambda_i+1 \begin{pmatrix} \psi_{Ai+1} \\ \psi_{Bi+1} \end{pmatrix},$$

(A5)

so that

$$T_i(E) = \Lambda_i+1 N_i^{-1} M_i \Lambda_i \dagger.$$  

(A6)

Plugging in $F_i, G_i, f_i, g_i$ and performing the matrix multiplication we find
\[ T_i(E) = \frac{1}{\sin K_i T} \left( -\sin \frac{1}{2} J_i T \left( \cos ET + \cos K_i T \right) \right) \left( \frac{\cos \frac{1}{2} J_i T}{\sin \frac{1}{2} J_i T} \sin ET \right) - \left( \frac{\cos \frac{1}{2} J_i T}{\sin \frac{1}{2} J_i T} \left( \cos ET - \cos K_i T \right) \right) \right). \quad (A7) \]

The boundary conditions quoted in section [IV] are derived by combining the very first row of (A1) for \( E = 0, \pi/T \) with the equations for \( \psi_{A1} \) and \( \psi_{B1} \), in terms of \( \psi_{A1} \) and \( \psi_{B1} \).

**Appendix B: Localization Length**

In this appendix we derive Equations (22) and (23) for the divergence of localization lengths, using an analysis similar to Ref. [32]. For definiteness, we examine the localization length \( l_0 \) of states with quasienergy 0, and we consider tuning one of \( JT \), \( w_J T \) with the remaining parameters fixed. The analysis is the same if we consider \( l_\pi \) instead, or if we tune one of \( KT \), \( w_K T \).

It is useful to introduce an additional function

\[ F(x, a) = G(x + a) - G(x - a) \quad (B1) \]

where \( G(x) \) is defined in (17). Setting \( (x, a) = (JT, \frac{1}{2} w_J T) \) and using (B1),

\[ \gamma_0 = \pm \frac{1}{2a} F(x, a) - \frac{1}{w_K T} F(KT, \frac{1}{2} w_K T). \quad (B2) \]

Let \((x_c, a_c) = ((JT)_c, \frac{1}{2} (w_J T)_c)\) correspond to a critical point where \( \nu_0 \) changes value, and let \((\delta x, \delta a) = (x - x_c, a - a_c)\). At the transition, \( \gamma_0 = 0 \), so \( \gamma_0 \) is given by the non-constant piece of the expansion of \( \frac{1}{2a} F(x, a) \) in terms of \( \delta x \) and \( \delta a \).

Combining (17) and (21),

\[ \frac{\partial}{\partial x} F(x, a_c) = \log |\sin x + \sin a_c| - |\sin x - \sin a_c| \quad (B3a) \]
\[ \frac{\partial}{\partial a} F(x, a) = \log |\cos x - \cos x_c| - |\cos x + \cos x_c| \quad (B3b) \]

As long as \( x_c \neq \pm a_c + n\pi \), both functions above are analytic at \((x_c, a_c)\) and can be expanded in a taylor series. Integrating then gives \( \frac{1}{a_c} F(x, a_c) \) in terms of \( \delta x \)

\[ \frac{1}{a_c} F(x, a_c) = \int F(x, a_c) + C_1 \delta x + \frac{1}{2} C_2 \delta x^2 + O(\delta x^3), \quad (B4) \]

and \( \frac{1}{2} F(x, a) \) in terms of \( \delta a \)

\[ \frac{1}{a} F(x, a) = \int F(x, a) + \left( D_1 - \frac{F(x, a)}{a_c} \right) \delta a + \left( D_2 + \frac{F(x, a)}{a_c} - \frac{D_a}{a_c} \right) \delta a^2 + O(\delta a^3), \quad (B5) \]

where

\[ C_1 = \log |\sin x_c + \sin a_c| - |\sin x_c - \sin a_c| \quad (B6a) \]
\[ D_1 = \log |\cos x_c - \cos x_c| - |\cos x_c + \cos x_c| \quad (B6b) \]
\[ C_2 = 2 \frac{\cos x_c \sin a_c}{\sin^2 a_c - \sin^2 x_c} \quad (B6c) \]
\[ D_2 = 2 \frac{\cos x_c \sin a_c}{\cos^2 x_c - \cos^2 a_c}. \quad (B6d) \]

If the linear coefficient in (B4) vanishes, then either the quadratic coefficient is nonzero or \( F(x, a_c) \) is constant. Both scenarios correspond to a trajectory in parameter space that is tangent to the phase boundaries, which is not a phase transition. The same holds for (B5). Hence, at a true transition

\[ \gamma_0 \sim JT - (JT)_c \text{ on tuning } JT \quad (B7a) \]
\[ \gamma_0 \sim w_J T - (w_J T)_c \text{ on tuning } w_J T. \quad (B7b) \]

Since \( l_0 = |\gamma_0|^{-1} \), this gives

\[ l_0 \sim |JT - (JT)_c|^{-1} \text{ on tuning } JT \quad (B8a) \]
\[ l_0 \sim |w_J T - (w_J T)_c|^{-1} \text{ on tuning } w_J T. \quad (B8b) \]

Now consider \( x_c = \pm a_c + n\pi \), which corresponds to \( (JT)_c = \pm \frac{1}{2} (w_J T)_c + n\pi \). Then

\[ \frac{\partial}{\partial x} F(x, a_c) = \tilde{C} \log |\delta x| + \text{powers of } \delta x \quad (B9a) \]
\[ \frac{\partial}{\partial a} F(x, a) = \tilde{D} \log |\delta a| + \text{powers of } \delta a \quad (B9b) \]

for nonzero \( \tilde{C} \) and \( \tilde{D} \). Integrating this we find that \( \frac{1}{a_c} F(x, a_c) \) and \( \frac{1}{2} F(x, a) \) have leading non-constant terms proportional to \( \delta x \log |\delta x| \) and \( \delta a \log |\delta a| \), respectively. Thus

\[ l_0 \sim |(JT - (JT)_c) \log |JT - (JT)_c||^{-1} \text{ on tuning } JT \quad (B10a) \]
\[ l_0 \sim |(w_J T - (w_J T)_c) \log |w_J T - (w_J T)_c||^{-1} \text{ on tuning } w_J T. \quad (B10b) \]

This is the anomalous scaling in (23).
