Higher-order and fractional discrete time crystals in clean long-range interacting systems

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Discrete time crystals are periodically driven systems characterized by a response with periodicity nT, with T the period of the drive and n > 1. Typically, n is an integer and bounded from above by the dimension of the local (or single particle) Hilbert space, the most prominent example being spin-1/2 systems with n restricted to 2. Here we show that a clean spin-1/2 system in the presence of long-range interactions and transverse field can sustain a huge variety of different ‘higher-order’ discrete time crystals with integer and, surprisingly, even fractional n > 2. We characterize these non-equilibrium phases of matter thoroughly using a combination of exact diagonalization, semiclassical methods, and spin-wave approximations, which enable us to establish their stability in the presence of competing long- and short-range interactions. Remarkably, these phases emerge in a model with continuous driving and time-independent interactions, convenient for experimental implementations with ultracold atoms or trapped ions.

Introduction.— The study of condensed matter systems out of equilibrium has attracted growing interest in recent years, accounting among others for the discovery of dynamical phase transitions [1, 2], quantum scars [3] and, particularly, discrete time crystals (DTCs) [4–10]. A n-DTC is a non-equilibrium phase of matter breaking the discrete time translational symmetry of a periodic (i.e., Floquet) drive. In the thermodynamic limit, the defining feature of a n-DTC is a subharmonic response at 1/n-th of the drive frequency (n > 1), which is robust to perturbations and which persists up to infinite time [10]. Following the first seminal proposals [4–7], DTCs have been widely investigated both theoretically and experimentally [10–20].

In these systems, heating to a featureless ‘infinite temperature’ state is typically avoided by introducing either disorder, which leads to a (Floquet) many-body-localized (MBL) phase [5, 7], or dissipation, which removes energy from the system compensating for absorption from the driving [21, 22]. Alternatively, in clean (i.e., non-disordered) non-dissipative systems, heating can be escaped with all-to-all interactions [10, 23, 24]. In presence of long-range (but not all-to-all) interactions, very recent work has shown that heating and transport can be significantly slowed down [25–27], a fact that has been exploited for the realization of prethermal DTCs, for which a subharmonic response is expected to persist for several time decades before the ultimate onset of thermalization [9, 12, 14, 28–30].

In this context, most work has focused on spin-1/2 systems, which have largely been shown to exhibit a 2-DTC where at every Floquet period the spins (approximately) oscillate between the states |↑⟩ and |↓⟩ leading to period doubling (i.e., n = 2). This fact naturally emerges from the dimension 2 of the local Hilbert space of the spins, and can be generalized to n-DTCs in models of n-dimensional clocks [23, 31]. Another well-studied setting is that of bosons in a gravitational field bouncing on an oscillating mirror [8], where the single-particle Hilbert space dimension is infinite (as the particle’s position is continuous) and where DTCs with arbitrary integer [32] and fractional [33] n have been shown.

Very recently, Ref. [29] has provided the theoretical framework to study Floquet, clean, long-range interacting systems, in which novel prethermal phases of matter are expected. While their framework allows for the possibility of n-DTCs with n larger than the size of the local (or single-particle) Hilbert space, their concrete examples are limited to n = 2. From our analysis below, we see that part of the difficulty in numerically observing what we call ‘higher-order’ DTCs may lie in their emergence at system sizes which are typically beyond the reach of exact diagonalization.

Here, we overcome this limitation by considering a system amenable to a set of complimentary methods, which enable us to discover an unusually rich dynamical phase diagramhosting a zoo of novel, exotic, non-equilibrium phases of matter. More specifically, we show that a clean spin-1/2 chain in the presence of long-range interactions (Fig. 1a) can sustain higher-order n-DTCs with integer and, remarkably, even fractional n > 2 (e.g., n = 3, 4, 8/3 and beyond). In each of these dynamical phases, the system responds with a characteristic frequency 1/n (in units of the drive frequency) that is robust to perturbations of the drive and that persists for several time decades. In the spectrum of the system’s magnetization, such a robustness manifests as plateaus of the spectral line at frequencies 1/n when varying the strength of an external magnetic field, resulting in a ‘fragmented’ phase diagram (Fig. 1b). The sequence of rigid plateaus hinges on the presence of interactions among the spins, and is intriguingly reminiscent of the plateau structure of the Fractional Quantum Hall Effect.

We thoroughly characterize these novel dynamical phases, and employ a spin-wave approximation to show their stability for sufficiently long-range interactions. Remarkably, in contrast to the commonly used switching protocols, we test our claims considering a continuous Floquet drive with constant-in-time interactions and a monochromatic transverse field. This choice, which in theoretical studies for spins is typically avoided in favor of binary or kicking drives, makes our model...
by summarizing the results, proposing possible experimental implementations and outlining directions for future research.

**Model and soluble limits.**— We consider a one-dimensional chain of \( N \) spins (\( N \to \infty \) in the thermodynamic limit) driven according to the following time-periodic Hamiltonian

\[
H(t) = \frac{J}{N^{\alpha}} \sum_{i,j=1}^{N} \frac{\sigma_i^z \sigma_j^z}{(r_{i,j})^\alpha} + \lambda \sum_{j=1}^{N} \sigma_j^x \sigma_j^{z+1} - \pi h[1 + \sin(2\pi t)] \sum_{j=1}^{N} \sigma_j^z,
\]

where \( \sigma_{j}^{x,y,z} \) denote the standard Pauli operators for the \( j \)-th spin, the Kac normalization \( N_{\alpha} = \sum_{j=2}^{N} \frac{1}{(r_{1,j})^\alpha} \) guarantees extensivity, periodic boundary conditions \( r_{i,j} = \min(|i-j|, L-|i-j|) \) are assumed, and both \( h \) and the drive frequency have been set to 1. \( J \) measures the strength of a power-law interaction with characteristic exponent \( \alpha \), \( \lambda \) is the strength of a nearest-neighbor interaction, and \( \pi h \) is the average over one drive period of the monochromatic transverse magnetic field.

The dynamics from an initially \( z \)-polarized state \( |\psi(0)\rangle = |\uparrow, \uparrow, \ldots, \uparrow\rangle \) is integrable in the non-interacting limit \( J = \lambda = 0 \), for which the magnetization \( m(t) = \langle \sigma_j^z(t) \rangle \) at stroboscopic times \( t = 0, 1, 2, \ldots \) reads \( m(t) = \cos(2\pi h t) \), that is \( h \) is the system’s characteristic frequency. The essential question to diagnose a \( n \)-DTC is whether, upon switching on the interactions, there exists a finite range of \( h \) for which the system’s characteristic frequency \( \nu \) remains instead locked to a constant value \( 1/n < 1 \). In the following we answer this question affirmatively not only for the well-known \( n = 2 \) case, but, if the interactions are sufficiently long-range, also for integer and even fractional \( n > 2 \).

**All-to-all interactions.**— For the sake of clarity, we first focus on the limit \( \alpha = \lambda = 0 \), i.e., the LMG model for all-to-all interactions, which allows for a transparent semiclassical interpretation of the various dynamical phases. The dynamics of the system can in this case be described by a semiclassical Gross-Pitaevskii equation (GPE) for the complex fields \( \psi_\uparrow \) and \( \psi_\downarrow \) (details in Supplementary Section I)

\[
\frac{d\psi_\uparrow}{dt} = \pi h[1 + \sin(2\pi t)]\psi_\downarrow - 4J|\psi_\uparrow|^2\psi_\uparrow,
\]

\[
\frac{d\psi_\downarrow}{dt} = \pi h[1 + \sin(2\pi t)]\psi_\uparrow - 4J|\psi_\downarrow|^2\psi_\downarrow,
\]

where we can identify \( |\psi_\uparrow|^2 - |\psi_\downarrow|^2 \to m = \langle \sigma_j^z \rangle \) and \( \psi_\uparrow = |\psi_\uparrow||\psi_\downarrow|e^{i\theta} \to \langle \sigma_j^z \rangle + i\langle \sigma_j^y \rangle \). The dynamics of the magnetization \( m \) is obtained integrating the GPE (2) from an initially \( z \)-polarized state \( |\psi_\uparrow(0)\rangle = 1, |\psi_\downarrow(0)\rangle = 0 \), and the corresponding Fourier transform \( \hat{m}(\nu) \) versus the magnetic field strength \( h \) is plotted in Fig. 1b. As it is well-known [10], the 2-DTC results in the system characteristic frequency \( \nu \) being locked to \( 1/2 \) for \( h \approx 1/2 \). Surprisingly, the same locking occurs at frequencies \( 1/n \) with integer and fractional \( n > 2 \) (e.g. \( n = 3, 4, 8/3 \)), giving rise to a fragmentation of the spectral line of \( \hat{m}(\nu) \) in plateaus of constant frequency for a finite
FIG. 2. Phase space structure of the dynamical phases. Poincaré maps of the semiclassical dynamics (2) for various magnetic field strengths $h$ and a fixed interaction $J = 0.5$. Red markers highlight the trajectory starting in the $\mathbf{z}$-polarized state ($m = 1, \theta = 0$, green asterisk). (a) Dynamical ferromagnet (F): the magnetization $m$ remains $\approx 1$ at all times; (b) Stroboscopic ferromagnet (sF): the magnetization $m$ changes sign during the micromotion and yet it remains positive at stroboscopic times; (c) 2-DTC: the system alternatively visits two islands of the phase space—one with $m \approx 1$ (numbered as 0) at even times, and the other with $m \approx -1$ (numbered as −1) at odd times; (d,e) Higher-order $n$-DTCs with integer $n = 4, 8$, respectively: the system visits cyclically $n$ islands of the phase space (accordingly numbered in red), with one tour of the islands corresponding to one complete revolution of the spins around the Bloch sphere. (f) Higher-order $n$-DTC with fractional $n = q/p = 8/3$: it takes $p$ revolutions of the spins for the system to tour $q$ islands of the phase space, resulting in a sharp magnetization oscillation frequency $\nu \approx p/q$. The insets on the right zoom on the island visited at times $t = 8k + 5$, $k = 0, 1, 2, \ldots$ for the 8-DTC (top) and the 8/3-DTC (bottom).

range of $h \approx 1/n$. Each of these plateaus signals a higher-order (possibly fractional) DTC, the width of the plateau being a signature of the DTC’s robustness.

Furthermore, the plateau at $\nu = 0$ for $h \approx 0$ signals the tendency of the spins to remain aligned along $\mathbf{z}$ in a dynamical ferromagnetic phase (F). This corresponds to macroscopic quantum self-trapping of weakly driven bosons in a double well [34, 35], which can in fact be exactly mapped to the LMG model (details in Supplementary Section I). For $h \approx 1, 2, 3, \ldots$, the spins complete approximately $1, 2, 3, \ldots$ revolutions around the Bloch sphere at each drive period, respectively, and yet maintain a preferential alignment along $\mathbf{z}$ at stroboscopic times, in what may be called a stroboscopic-ferromagnetic phase (sF).

Our results are confirmed by exact diagonalization studies. Thanks to the all-to-all coupling of the LMG model, the dynamics is in fact confined to the symmetric sector, whose size grows only linearly with the number of spins $N$. This allows a scaling analysis extended up to large system sizes, showing a progressive emergence of the spectral line plateaus for an increasing number of spins $N$. For the standard 2-DTC, the plateau is clearly visible already for $N \gtrapprox 10$, whereas, crucially, for the 4-DTC it appears only for $N \gtrapprox 100$ (see details in Supplementary Section II). This observation strongly suggests that signatures of the higher-order $n$-DTCs arise for larger system sizes as compared to the standard 2-DTC, making them generally elusive to exact diagonalization techniques. This fact might explain the difficulties in observing higher-order DTCs in the past and motivates the choice of model (1) in the first place.

The stroboscopic dynamics generated by the GPE (2) can be conveniently described with Poincaré maps, popular tools in dynamical systems theory that here provide an immediate and transparent interpretation of the dynamical phases. In Fig. 2, the trajectory starting in the $\mathbf{z}$-polarized state (green asterisk) is highlighted with red markers. For a weak drive $h \approx 0$, the spins tend to remain aligned along $\mathbf{z}$ in a dynamical ferromagnetic phase (a), giving rise to a Poincaré map which closely resembles the phase portrait of undriven bosons in a double well [34, 35]. For $h \approx 1$, the micromotion consists of approximately an entire revolution of the spins around the Bloch sphere per period (not shown), with a preferential alignment restored at stroboscopic times despite the detuning in the magnetic field strength (b). For $h \approx 1/n$ and $n = 2, 3, 4$ in (c), (d) and (e), respectively, the $n$-DTC results in the presence of $n$ ‘islands’ in the phase space which the system visits sequentially jumping from one to the next at each drive period. In $n$ drive periods, the system visits all the $n$ islands once, and
The robustness of the higher-order time crystals is induced by the interactions, justifying their classification as non-equilibrium phases of matter. For concreteness, we show this for the 4-DTC in the LMG model. (a,b) Magnetization \( m(t) \) at stroboscopic times (left) and respective Fourier transform \( \langle \tilde{m}(\nu) \rangle \) (right) for a slightly detuned magnetic field strength \( h = 1/4 + \epsilon \), with \( \epsilon \ll 1 \), originates in envelopes (that is, beatings) with period \( \sim 1/\epsilon \), resulting in the Fourier transform \( \tilde{m} \) being peaked at \( \nu \approx h \) and in trivial dynamics (a). Crucially, stronger interactions can compensate the mistake in the flipping field (b): the envelopes in \( m(t) \) disappear, the peak in \( \tilde{m} \) is set back to the subharmonic frequency \( \nu = 1/n \), and the discrete time symmetry is broken. The subharmonic peak magnitude \( \langle \tilde{m}(1/4) \rangle \) can be used to trace out the 4-DTC phase in the \((J, h)\) plane (c). The 4-DTC phase opens up from the integrable point \( J = 0, h = 1/4 \) for increasing interactions, in analogy with the opening of the standard 2-DTC from \( J = 0, h = 1/2 \) [15]. This opening confirms that larger interactions \( J \) allow the higher-order DTCs to bear larger detunings in the field \( h \). However, at even larger \( J \gtrsim 0.8 \) semiclassical chaos sets in and the time crystalline order is broken irrespectively of \( h \).

To make the last statement quantitative, we introduce a decorrelator \( d^2(t) \) [24, 37]

\[
d^2(t) = (|\psi_\uparrow|^2 - |\psi_\downarrow|^2)^2 + (|\psi_\downarrow|^2 - |\psi_\uparrow|^2)^2,
\]

measuring the distance between two initially very close copies of the system evolving under Eq. (2). Specifically, we consider as perturbed initial condition \( 1 - \tilde{m} = \theta' = 10^{-6} \). The decorrelator time-average \( \langle d^2 \rangle_t \) can be used as a further diagnostic tool, with \( \langle d^2 \rangle_t \sim 1 \) corresponding to sensitivity to the initial conditions, that is, to classical chaos, which in turn signals quantum thermalization [36]. The complimentary information provided by \( \langle \tilde{m}(1/4) \rangle \) and \( \langle d^2 \rangle_t \) in (c) and (d), respectively, can therefore be used to distinguish the 4-DTC, the trivial, and the thermal phases.

**Power-law and nearest-neighbor interactions.**— As shown, the DTCs rely on the interactions being sufficiently (but not too) strong. Crucially, in contrast to the standard 2-DTC, higher-order DTCs also necessitate the interactions to be sufficiently long-range. We now explore the effects of non-all-to-all interaction on higher-order DTCs, particularly assessing their stability upon breaking the mean-field solvability of the dynamics with power-law \( (\alpha > 0) \) and nearest-neighbor \( (\lambda > 0) \) interactions.

In the LMG limit the dynamics remains restricted to the completely symmetric Hilbert subspace and the system can be described in terms of a collective spin. In contrast, we now have to account for the dynamic generation of spin-wave excitations, which can be treated within a spin-wave approximation. Following Refs. [38, 39], this approximation is built into a rotating frame \( \mathcal{R}' = (X, Y, Z) \) with \( Z \) tracking the macroscopic collective spin \( \frac{1}{N} \sum_{j=1}^{N} \langle \sigma_i \rangle \). With a Holstein-Primakoff transformation from spin degrees of freedom to bosonic degrees of freedom \( \sigma_i^X \rightarrow b_i + b_i^\dagger, \sigma_i^Y \rightarrow -i(b_i - b_i^\dagger) \) and a Fourier transform \( \tilde{b}_k = \frac{1}{N} \sum_{j=1}^{N} e^{-ikj} b_j \), we obtain the spin-wave dynamics (more details in Supplementary Section III). The central dynamical variable of this approximation is the density of spin-wave excitations \( \epsilon = \frac{\pi}{4} \sum_{k \neq 0} \langle \tilde{b}_k^\dagger \tilde{b}_k \rangle \).
Fig. 4. Stability and prethermalization with power-law and nearest-neighbor interactions. The higher-order and fractional DTCs survive, most likely in a prethermal fashion, when deviating from the LMG model. For concreteness, we focus on the 4-DTC at the LMG limit of all-to-all interactions, and consider the effects of power-law ($\alpha > 0$) and nearest-neighbor ($\lambda > 0$) interactions. (a) If the interactions are sufficiently long-range (that is $\alpha$ is small enough, here for a fixed $\lambda = 0.03$), the density of spin-wave excitations $\epsilon$ remains small throughout several time decades. Conversely, shorter-range interactions lead to the proliferation of spin-wave excitations that makes the system quickly thermalize destroying any time crystalline order [14].

(b) A sharp transition between these regimes is highlighted by the time-average $\langle \epsilon \rangle_t$ over $10^3$ periods versus $\alpha$ (at a fixed $\lambda = 0.03$). The critical $\alpha_c$ at which $\langle \epsilon \rangle_t$ crosses the threshold 0.1 (inset), grows and possibly saturates with the system size $N$, suggesting the stability of the 4-DTC in the thermodynamic limit $N \rightarrow \infty$. (c,d) The stability of the 4-DTC for a whole region of the parameter space surrounding the LMG point $\alpha = \lambda = 0$ is highlighted plotting the magnitude of the subharmonic peak $|\hat{n}_a(1/4)|$ and the average spin-wave density $\langle \epsilon \rangle_t$ in the $(\alpha, \lambda)$ plane.

which is treated at lowest non-trivial order. In the LMG limit ($\lambda = 0 = \alpha$), no spin-wave excitations are generated and $\epsilon = 0$ at all times. When departing from such a limit, two scenarios are possible (Fig. 4a): (i) $\epsilon$ rapidly reaches a plateau $\lesssim 0.1$ (up to some small fluctuations), for which we consider the spin-wave approximation consistent, or (ii) $\epsilon$ rapidly grows to values $\gtrsim 1$, for which the spin-wave approximation breaks down. Although the method is not exact and may fail to capture the very long-time physics, it suggests that (i) and (ii) correspond to prethermalization and thermalization, respectively [14, 38, 39].

Our results are exemplified in Fig. 4 for the 4-DTC. For $\lambda = 0.03$ in (a), the 4-DTC is stable (at least in a prethermal fashion) for $\alpha < \alpha_c \approx 1.4$, whereas thermalization quickly sets in for shorter-range interactions. The transition between these two dynamical phases is better assessed looking at the spin-wave density time average $\langle \epsilon \rangle_t$ in (b). The critical $\alpha_c$, at which $\langle \epsilon \rangle_t$ crosses 0.1 increases (and possibly saturates) with the number of spins $N$, giving evidence for the 4-DTC to be stable in the thermodynamic limit. The stability of the 4-DTC in the presence of competing power-law and nearest-neighbor interactions is investigated in the $(\alpha, \lambda)$ plane plotting the amplitude of the subharmonic peak $|\hat{n}_a(1/4)|$ in Fig. 4c and the time-averaged spin-wave density $\langle \epsilon \rangle_t$ in Fig. 4d. The 4-DTC is stable for a whole region of the parameter space surrounding the LMG model point ($\alpha = \lambda = 0$), that is, if the interactions are sufficiently long-range. Finally, note that the DTC is also robust to arbitrary perturbations to the initial state, as we have checked by injecting a small amount of spin-wave excitations at initial time.

Discussion and conclusion. — Higher-order DTCs in clean long-range interacting systems are qualitatively distinct from DTCs of MBL Floquet systems [29]. Indeed, the higher-order DTCs require the establishment of order along directions different from $\pm z$. For instance, in the 4-DTC the spins are approximately aligned along $-y$ and $+y$ at times $t = 1.5, \ldots$ and $t = 3.7, \ldots$, respectively. In an MBL system, a disordered magnetic field or short-range interaction along $z$ would immediately scramble the system when the spins are far from the $z$ axis, precluding the possibility of a higher-order DTCs. Thus, our work establishes that translationally-invariant systems with long-range interactions can circumvent these limitations [12, 29].

The choice of a continuous Floquet drive with constant-in-time interactions and monochromatic transverse magnetic field makes model (1) a prime candidate for experimental implementation. For instance, bosons in a double well [35] could be used to realize a truly all-to-all interacting, that is the LMG, model. In this case, the field pulses would be simply implemented lowering the barrier between the two wells to allow particle tunnelling, and the fact that no time modulation for the particle-particle interaction is necessary should provide a major simplification. Power-law interactions with tunable alpha $0 \leq \alpha \leq 3$ can instead be realized in trapped-ion experiments [16, 40–44].

In conclusion, we have discovered higher-order DTCs with a period that is not limited from above by the size of the local (or single-particle) Hilbert space. For a clean spin-1/2 chain with long-range interactions, the dynamical phase space fragments to host many higher-order $n$-DTCs with integer and even fractional $n > 2$ (e.g., $n = 3, 4, 5/3$), at least in a prethermal fashion. Furthermore, a ferromagnetic and a stroboscopic ferromagnetic dynamical phases were shown in which at stroboscopic times the spins display a preferential alignment. In the LMG limit of all-to-all interactions, the model (1) has a low-dimensional semiclassical limit, which links the $n$-DTCs to the multifrequency mode locking of some nonlinear discrete maps, which is ubiquitous in the natural sciences [45–52]. In this limit, these dynamical phases can be interpreted in terms of Poincaré maps. An Arnold’s tongue is associated to the higher-order DTCs, highlighting the role of the interactions in the rigidification of the DTCs against perturbations,
and thus the genuinely many-body nature of the DTCs. The fate of these dynamical phases in the presence of competing, mean-field breaking, long- and short-range interactions, has been investigated within a spin-wave approximation, which provides evidence for the stability of the higher-order DTCs in a whole region of the parameter space surrounding the LMG model, that is in a genuinely quantum setting with no semiclassical counterpart.

Future work should attempt to gain further analytical understanding regarding the role of long-range interactions in stabilizing the different higher-order DTCs. Most importantly, what are the allowed fractions $q/p$ that result in a $q/p$-DTC? Further study should assess in more detail the role of the Kac normalization, which was here adopted for numerical convenience and whose impact can be nonnegligible [53]. Finally, an intriguing question for future research regards the role of dimensionality on the fate of the dynamical phases of matter presented here.

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Supplementary Information for "Higher-order and fractional discrete time crystals in clean long-range interacting systems"

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The Supplementary Information are devoted to technical details of the derivations and complimentary results and is organized as follows. In Section I we show the mapping of the Lipkin-Meshkov-Glick (LMG) model for fully-connected spins to a model of bosons in a double well, and exploit it to obtain the Gross-Pitaevskii equation (GPE) in the limit of infinite number of spins \( N \). In Section II we present results from exact diagonalization, showing how the discrete time crystals (DTCs) emerge for an increasing number of spins \( N \). We show that higher-order \( n \)-DTCs emerge at much larger \( N \) as compared to the standard 2-DTC, which explains the difficulty to numerically observe them. In Section III we give an overview of the spin-wave approximation we employ, whereas in Section IV we show that a binary Floquet protocol also leads to results similar to the ones of the continuous driving discussed in the main text.

I) GROSS-PITAEVSKII EQUATION

In this section, we derive the semiclassical GPE of motion for the LMG model of \( N \) fully-connected spins (\( \lambda = \alpha = 0 \)). To this end, we first map the spin system into a model of bosons in a double well. We then derive Heisenberg equations for the bosonic operators, and we treat them semiclassically replacing the bosonic operators with complex numbers.

Map to bosons in a double well

The Schwinger's oscillator model of angular momentum connects the algebra of angular momentum and the algebra of two bosonic modes [54]. Here, we show this connection explicitly for the collective spin of a system of \( N \) spin-1/2.

Given \( p = (p_1, p_2, \ldots, p_N) \) a permutation of the indexes 1, 2, \ldots, \( N \), we say \( P_p \) the permutation operator acting as

\[
P_p|s_1, s_2, \ldots, s_N\rangle = |s_{p_1}, s_{p_2}, \ldots, s_{p_N}\rangle,
\]

where \( s_i \in \{\uparrow, \downarrow\} \). The permutation operators are used to build the symmetrization operator \( S \), defined as

\[
S = \frac{1}{\sqrt{N!}} \sum_p P_p.
\]

We say \( |n_\uparrow, n_\downarrow\rangle \) the symmetrized state with \( n_\uparrow \) spins up and \( n_\downarrow \) spins down, that is

\[
|n_\uparrow, n_\downarrow\rangle = \frac{1}{\sqrt{n_\uparrow!n_\downarrow!}} S|\uparrow, \uparrow, \ldots, \uparrow, \downarrow, \ldots, \downarrow\rangle_{n_\uparrow, n_\downarrow}.
\]

Since \( N \) is fixed and \( n_\downarrow = N - n_\uparrow \), the notation \( |n_\uparrow, n_\downarrow\rangle \) is actually slightly redundant. In the following, we nevertheless prefer to stick with this notation for the sake of clarity. The states \( |n_\uparrow, n_\downarrow\rangle \) form a basis for the Hilbert subspace of completely symmetrized states. Given an operator \( O \) commuting with the symmetrization operator \( S \), the action of \( O \) on this subspace is therefore fully-characterized by its action on the states \( |n_\uparrow, n_\downarrow\rangle \).

In particular, we consider the 'collective' operators \( \left( \sum_{j=1}^{N} \sigma^\alpha_j \right) \) with \( \alpha = x, y, z \), which indeed all commute with \( S \).
Introducing standard bosonic operators $a_{↑}, a_{↓}, a_{↑}^\dagger, a_{↓}^\dagger$ for the two bosonic modes labeled by $↑$ and $↓$, we can thus write

$$\sum_{j=1}^{N} \sigma_j^x = a_{↓}^\dagger a_{↑} + a_{↑}^\dagger a_{↓}, \quad \sum_{j=1}^{N} \sigma_j^y = -i \left( a_{↑}^\dagger a_{↓} - a_{↓}^\dagger a_{↑} \right), \quad \sum_{j=1}^{N} \sigma_j^z = n_{↑} - n_{↓}$$

(11)

with $n_{↑} = a_{↑}^\dagger a_{↑}$ and $n_{↓} = a_{↓}^\dagger a_{↓}$.

The Hamiltonian (1) in the LMG limit ($\alpha = \lambda = 0$) reads

$$H = \frac{J}{N} \sum_{i,j=1}^{N} \sigma_i^x \sigma_j^x - \pi \hbar [1 + \sin(2\pi t)] \sum_{j=1}^{N} \sigma_j^x,$$

(12)

and is thus rewritten in terms of the bosonic operators as

$$H = \frac{J}{N} \langle n_{↑} - n_{↓} \rangle^2 - \pi \hbar [1 + \sin(2\pi t)] (a_{↑}^\dagger a_{↓} + a_{↓}^\dagger a_{↑}).$$

(13)

We elaborate on the first term of Eq. (13) using $m = \frac{n_{↑} - n_{↓}}{N}$ and noting that

$$\begin{pmatrix} \frac{n_{↑}}{N} \\ \frac{n_{↓}}{N} \end{pmatrix} = \begin{pmatrix} \frac{1+m}{2} \\ \frac{1-m}{2} \end{pmatrix} \Rightarrow \frac{n_{↑} + n_{↓}}{N^2} = \frac{1-m^2}{4},$$

(14)

from which

$$m^2 = \left( \frac{n_{↑} - n_{↓}}{N} \right)^2 = \frac{n_{↑}^2 + n_{↓}^2 - 2n_{↑}n_{↓}}{N^2} = \frac{n_{↑}^2 + n_{↓}^2}{N^2} + \frac{m^2}{2} - \frac{1}{2},$$

(15)

and, isolating $m^2$,

$$m^2 = 2 \frac{n_{↑}^2 + n_{↓}^2}{N^2} - 1 = 2 \frac{n_{↑}(n_{↑} - 1) + n_{↓}(n_{↓} - 1)}{N^2} - 1 + \frac{2}{N}.$$

(16)

Setting $U = 4J$ and $\tau(t) = \pi \hbar [1 + \sin(2\pi t)]$, up to irrelevant additional constant terms, the Hamiltonian thus reads

$$H(t) = -\tau(t) (a_{↑}^\dagger a_{↓} + a_{↓}^\dagger a_{↑}) + \frac{U}{2N} (n_{↑}(n_{↑} - 1) + n_{↓}(n_{↓} - 1)),$$

(17)

where $n_{↑} = a_{↑}^\dagger a_{↑}$ and $n_{↓} = a_{↓}^\dagger a_{↓}$. That is, the LMG model in the symmetric subspace is mapped to a model for $N$ bosons in a double well (the two wells being labeled by $↑$ and $↓$).
Equations of motion

The bosonic representation of Eq. (S17) is particularly convenient to obtain dynamical equations. The Heisenberg equations for the bosonic operators read ($\hbar = 1$)

$$\frac{d\hat{a}_\uparrow}{d(it)} = [H(t), a_\uparrow] = \tau(t)a_\downarrow - \frac{U}{N}n_\uparrow a_\uparrow,$$

$$\frac{d\hat{a}_\downarrow}{d(it)} = [H(t), a_\downarrow] = \tau(t)a_\uparrow - \frac{U}{N}n_\downarrow a_\downarrow.$$  \hfill (S18)

In the limit $N \to \infty$, upon replacing $a_\uparrow \to \sqrt{N}\psi_\uparrow$ and $a_\downarrow \to \sqrt{N}\psi_\downarrow$, with $\psi_\downarrow$ complex fields, we finally derive the following Gross-Pitaevskii equation

$$\frac{d\psi_\uparrow}{d(it)} = \tau(t)\psi_\downarrow - U|\psi_\uparrow|^2\psi_\uparrow,$$

$$\frac{d\psi_\downarrow}{d(it)} = \tau(t)\psi_\uparrow - U|\psi_\downarrow|^2\psi_\downarrow.$$  \hfill (S19)

For an operator $\hat{O} = f(a_\uparrow, a_\downarrow, a_\uparrow^+, a_\downarrow^+)$ written as a function $f$ of the bosonic operators, the beyond-mean-field dynamics of the expectation value $O(t) = \langle \hat{O} \rangle(t)$ can be generally computed within a Truncated Wigner approximation (TWA) as

$$O(t) \approx \langle f(\psi_\uparrow(t), \psi_\downarrow(t), \psi_\uparrow^*(t), \psi_\downarrow^*(t))\rangle_{\psi_\uparrow(0), \psi_\downarrow(0)}$$  \hfill (S20)

where $\langle \ldots \rangle_{\psi_\uparrow(0), \psi_\downarrow(0)}$ denotes the average over an ensemble of stochastic semiclassical initial conditions $\psi_\uparrow(0)$ and $\psi_\downarrow(0)$ that are drawn according to the quantum initial condition, and then evolve in time with the GPE (S19).

In particular, let us consider as initial condition the symmetrized state $|\psi(0)\rangle$ with magnetization $m' = 1 - \delta$, with $0 < \delta \ll 1$ and $m'N$ integer (which, since we assume $N \to \infty$, does not restrict the possible values for $m'$)

$$|\psi(0)\rangle = |n_\uparrow', n_\downarrow'\rangle,$$

$$n_\uparrow' = m'N, \quad n_\downarrow' = (1 - m')N,$$  \hfill (S21)

for which the TWA is performed considering the following ensemble of initial conditions

$$\psi_\uparrow'(0) = \sqrt{1 - \delta} e^{i\theta_0(0)}, \quad \psi_\downarrow'(0) = \sqrt{\delta} e^{i\theta_1(0)},$$  \hfill (S22)

with $\theta_0(0)$ and $\theta_1(0)$ independent uniform random numbers between 0 and 2$\pi$. Thanks to a gauge transformation, we can always change the initial conditions (S22) into

$$\psi_\uparrow'(0) = \sqrt{1 - \delta}, \quad \psi_\downarrow'(0) = \sqrt{\delta} e^{i\theta_0},$$  \hfill (S23)

where $\theta_0$ is also a random number between 0 and 2$\pi$. Consider now the limit of $\delta \to 0$, that is of $|\psi'(0)\rangle \to |\psi(0)\rangle = |\uparrow, \uparrow, \ldots, \uparrow\rangle$. In this limit, the ensemble of stochastic initial conditions (S23) shrinks in the phase space of complex coordinates $\psi_\uparrow'(0)$ and $\psi_\downarrow'(0)$ towards the point $\psi_\uparrow(0) = 1 - \psi_\downarrow(0) = 1$. If the GPE is nonchaotic, the points of the shrunk ensemble follow close trajectories, so that the TWA average in Eq. (S20) can be actually replaced by the evaluation of $f$ for a single trajectory of the ensemble, say the one starting in $\psi_\uparrow(0) = 1 - \psi_\downarrow(0) = 1$. In contrast, if the GPE is chaotic, because of sensitivity to initial conditions, the ensemble quickly spreads, scrambling across the classical phase space. In this case the ensemble trajectories at long-times (in S20) interfere destructively, washing out any time oscillation of $O$: the TWA results in thermalization.

In the limit $|\psi'(0)\rangle \to |\psi(0)\rangle = |\uparrow, \uparrow, \ldots, \uparrow\rangle$, it is therefore convenient to consider the following 'single-shot GPE' [24], to be run just once

$$O(t) \to \langle f(\psi_\uparrow(t), \psi_\downarrow(t), \psi_\uparrow^*(t), \psi_\downarrow^*(t))\rangle_{\psi_\uparrow(0) = 1, \psi_\downarrow(0) = 0},$$  \hfill (S24)

which is then expected to be accurate when nonchaotic, and to signal quantum thermalization when chaotic, which motivates the use of the symbol " $\to "$, rather than "$ = "$.

In particular, considering the observable $S = \frac{1}{N} \sum_{j=1}^{N} \vec{σ}_j$, from Eq. (S25) we thus write

$$\frac{\langle S^z \rangle + i\langle S^y \rangle}{2} \to \psi_\uparrow^*\psi_\downarrow, \quad \langle S^z \rangle \to |\psi_\uparrow|^2 - |\psi_\downarrow|^2.$$  \hfill (S25)
In a DTC evolving from an initially $z$-polarized state $|\psi(0)\rangle = |\uparrow, \uparrow, \ldots, \uparrow\rangle$, the spins are mostly aligned at all times. Imperfections in the alignment can be described within a Holstein-Primakoff transformation in terms of bosonic spin-wave quasiparticles $\hat{b}_k$. Crucially, the collective spin $\vec{S} = \frac{1}{N} \sum_{j=1}^{N} \hat{\vec{S}}_j$ rotates as a function of time in the lab frame. Therefore, the Holstein-Primakoff transformation has to be performed in a rotating frame $(X,Y,Z)$ such that the, say, $Z$ axis tracks the orientation of the collective spin at all times. This tracking is encoded in the condition $\langle S^X \rangle = \langle S^Y \rangle = 0$, from which the dynamics of the rotating frame is obtained self-consistently. On top of this, an approximation is made in that the Hamiltonian is expanded to lowest non-trivial order in the density of spin-wave excitations $\epsilon = \frac{2}{N} \sum_{k \neq 0} \langle \hat{b}_k^\dagger \hat{b}_k \rangle$, which should remain $\ll 1$ for the approximation to be consistent. This procedure results in a set of $2N$ ordinary differential equations similar to Eqs. (26) and (29) in the Supplemental Material of Ref. [38], describing the rotation of the new reference frame and the dynamics of the spin waves at the various momenta.

Finally, we remark the main differences between the implementation of the spin-wave approximation in our work and in Ref. [38]. First, Ref. [38] considers a constant-in-time Hamiltonian, whereas the parameters of the Hamiltonian in the present work are time-dependent. As a consequence, the parameters in the system of ordinary differential equations become time-dependent. Second, Ref. [38] considers a nearest-neighbor interaction on top of a fully-connected one, whereas we consider the
FIG. S2. **Phase diagram fragmentation for a binary Floquet driving.** This figure is in complete analogy with Fig. 1b of the main paper, but considers the binary Floquet protocol (S26). We plot the Fourier transform \( \tilde{m}(\nu) \) of the magnetization \( m = \langle \sigma_z \rangle \) in the plane of the transverse field strength \( h \) and of the frequency \( \nu \), for a fixed interaction strength \( J = 0.5 \) and computed over 500 periods. We observe that the spectral lines fragment in plateaus with constant frequency, signaling the \( n \)-DTCs, (dynamic) ferromagnet and stroboscopic-ferromagnet. In blue, we indicate the index \( n \) of some of the resolved DTCs.

more general case of nearest-neighbor interaction on top of a power-law one. The \( \cos k \) that appears in the equations of Ref. [38] is therefore substituted by a more generic \( \tilde{J}_k = \sum_{j=1}^N J_{r_{ij}} e^{-i \nu r_{ij} k} \) in ours, where \( J_{r_{ij}} \) contains both the nearest-neighbor and the power-law interactions.

**IV) BINARY DRIVING**

In this section, we compliment the results from the main paper showing that the fragmentation of the phase diagram to host a multitude of higher-order DTCs also occur for a binary Floquet protocol. In particular, we consider a periodic Hamiltonian with period 1 alternating fully-connected interaction and transverse field

\[
H(t) = \begin{cases} 
+2 \frac{J}{N} \sum_{i,j=1}^{N} \sigma_i^z \sigma_j^z & 0 < t \leq 0.5 \\
-2\pi h \sum_{j=1}^{N} \sigma_j^x & 0.5 < t \leq 1.
\end{cases}
\] (S26)

From the Hamiltonian (S26), in the limit \( N \to \infty \), we can derive a GPE in complete analogy with Eq. (S19). Solving it for the initially \( z \)-polarized state \( |\psi(0)\rangle = |\uparrow, \uparrow, \ldots, \uparrow\rangle \), we obtain the spectrum \( \tilde{m}(\nu) \) of Fig. S2. Also for this model, it is possible to check within a spin-wave approximation that the dynamical phases persist when deviating from the LMG limit of fully-connected spins, as long as the interaction range is large enough, in complete analogy with the results of Fig. 4 of the main paper.