The 8V CSOS model and the $sl_2$ loop algebra symmetry of the six-vertex model at roots of unity

TETSUO DEGUCHI

Department of Physics, Ochanomizu University, 2-1-1 Ohtsuka
Bunkyo-ku, Tokyo 112-8610, Japan

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We review an algebraic method for constructing degenerate eigenvectors of the transfer matrix of the eight-vertex Cyclic Solid-on-Solid lattice model (8V CSOS model), where the degeneracy increases exponentially with respect to the system size. We consider the elliptic quantum group $E_{\tau,\eta}(sl_2)$ at the discrete coupling constants: $2N\eta = m_1 + im_2\tau$, where $N, m_1$ and $m_2$ are integers. Then we show that degenerate eigenvectors of the transfer matrix of the six-vertex model at roots of unity in the sector $S^Z \equiv 0 \pmod N$ are derived from those of the 8V CSOS model, through the trigonometric limit. They are associated with the complete $N$ strings. From the result we see that the dimension of a given degenerate eigenspace in the sector $S^Z \equiv 0 \pmod N$ of the six-vertex model at $N$th roots of unity is given by $2^{2S^Z_{\text{max}}/N}$, where $S^Z_{\text{max}}$ is the maximal value of the total spin operator $S^Z$ in the degenerate eigenspace.

1. Introduction

Recently, it has been explicitly discussed that the transfer matrix of the six-vertex model at roots of unity has the symmetry of the $sl_2$ loop algebra. Let us consider the XXZ spin chain under the periodic boundary conditions

$$H_{XXZ} = -J \sum_{j=1}^{L} \left( \sigma^X_j \sigma^X_{j+1} + \sigma^Y_j \sigma^Y_{j+1} + \Delta \sigma^Z_j \sigma^Z_{j+1} \right).$$

(1)

Here the parameter $\Delta$ is related to the $q$ variable of the quantum group $U_q(sl_2)$ as

$$\Delta = \frac{1}{2} (q + q^{-1})$$

(2)

When $q^{2N} = 1$, it was shown that the XXZ Hamiltonian commutes with the generators of the $sl_2$ loop algebra, which is an infinite dimensional algebra. Furthermore, it was shown by the Jordan-Wigner method for $N = 2$ and numerically for general $N$ that the dimensions of the degenerate eigenvectors are given by some powers of 2, which increase exponentially with respect to the system size $L$.

The exponential degeneracy of the $sl_2$ loop algebra should be important for the problem of the “completeness of the Bethe ansatz eigenvectors”. In fact, the $sl_2$
loop algebra symmetry has not been considered in the standard arguments of the string hypothesis. Thus, it seems that it is still open whether we can construct $2^L$ linearly independent eigenvectors of the XXZ spin chain at roots of unity for general $L$. The question should be related to so called singular Bethe ansatz solutions. In fact, it is numerically confirmed that the standard solutions of the Bethe ansatz equations determine only eigenvectors which have the highest weights of the $\mathfrak{sl}_2$ loop algebra. Furthermore, some important properties of complete $N$ strings have been discussed in association with the $\mathfrak{sl}_2$ loop algebra.

Interestingly, it was numerically suggested that the transfer matrix of the eight-vertex model at the discrete coupling parameters should have the degenerate eigenvectors corresponding to the degeneracy of the $\mathfrak{sl}_2$ loop algebra. Furthermore, it has been recently shown that some degenerate eigenspace of the eight-vertex model has dimension of $N^2 L/N$ if $L/N$ is an even integer.

Let us consider the XYZ Hamiltonian under the periodic boundary conditions

$$H_{XYZ} = -\sum_{j=1}^{L} (J_X \sigma_j^X \sigma_{j+1}^X + J_Y \sigma_j^Y \sigma_{j+1}^Y + J_Z \sigma_j^Z \sigma_{j+1}^Z)$$

(3)

where the coupling constants $J_X$, $J_Y$ and $J_Z$ are given by

$$J_X = J(1 + k \sin^2(2\eta)), \quad J_Y = J(1 - k \sin^2(2\eta)), \quad J_Z = J \csc(2\eta) \cot(2\eta)$$

(4)

Here $\sin(z)$, $\csc(z)$ and $\cot(z)$ denote the Jacobian elliptic functions with elliptic modulus $k$. We have called $2\eta$ the coupling parameter of the model. The number $N$ has been related to $2\eta$ by $2N\eta = 2m_1 K + im_2 K'$. The symbols $K$ and $K'$ denote the complete elliptic integrals of the first and second kinds, respectively.

In this paper, we discuss an algebraic construction of degenerate eigenvectors of the eight-vertex cyclic Solid-on-Solid model (8V CSOS model), which is a variant of the eight-vertex Restricted Solid-on-Solid model (ABF model). Then, we show that through some limits, they give the degenerate eigenvectors of the six-vertex model in the sector $S^Z \equiv 0 \pmod{N}$ consisting of the complete $N$ strings.

2. The $\mathfrak{sl}_2$ loop algebra symmetry of the XXZ spin chain

Let us consider representations of the generators of $U_q(\mathfrak{sl}_2)$ on the $L$th tensor product of spin 1/2 representations.

$$q^{S^Z} = q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2}$$

(5)

$$S^\pm = \sum_{j=1}^{L} S_j^\pm = \sum_{j=1}^{L} q^{\sigma^+/2} \otimes \cdots q^{\sigma^+/2} \otimes \sigma_j^\pm \otimes q^{-\sigma^-/2} \otimes \cdots \otimes q^{-\sigma^-/2}$$

(6)

Let us introduce some symbols: $[n] = (q^n - q^{-n})/(q - q^{-1})$ for $n > 0$ and $[0] = 1$; $[n]! = \prod_{k=1}^{n} [k]$. Setting

$$S^{\pm(N)} = \lim_{q^{2N} \to 1} (S^\pm)^N/[N]!$$

(7)
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The operators $S^\pm(N)$ are non-vanishing and we have

$$S^\pm(N) = \sum_{1 \leq j_1 < \cdots < j_N \leq L} q^{\sigma_j^2} \otimes q^{\sigma_j^{(N-2)}} \otimes q^{\sigma_j^\pm} \otimes q^{\sigma_j^{(N-2)}} \otimes \cdots \otimes q^{\sigma_j^{(N-2)}} \otimes \cdots \otimes q^{\sigma_j^{(N-2)}}$$

(8)

The study of the symmetries of the XXZ Hamiltonian under periodic boundary conditions at roots of unity was initiated in Ref. [12]. $S^\pm(N)$ commute with the Hamiltonian $H$ when $S^z/N$ is an integer and $q^{2N} = 1$ holds. However, there exists a much larger symmetry algebra than that of $S^\pm(N)$. We remark that the XXZ Hamiltonian is associated with the affine quantum group $U_q(\hat{sl}_2)$. For instance, we may consider the following

$$T^\pm = \sum_{j=1}^{L} T_j^\pm = \sum_{j=1}^{L} q^{-\sigma^z / 2} \otimes \cdots \otimes q^{-\sigma^z / 2} \otimes \sigma_j^\pm \otimes q^{\sigma_j^z / 2} \otimes \cdots \otimes q^{\sigma_j^z / 2}$$

(9)

which is also obtained from $S^\pm$ by the replacement $q \to q^{-1}$. When $q^{2N} = 1$, we define $T^\pm(N)$ similarly as (9).

Let $T_{6V}(v)$ denotes the (inhomogeneous) transfer matrix of the six-vertex model. Then we can show the (anti) commutation relations when $S^z \equiv 0 \text{ (mod } N)$

$$S^\pm(N) T_{6V}(v) = q^N T_{6V}(v) S^\pm(N), \quad T^\pm(N) T_{6V}(v) = q^N T_{6V}(v) T^\pm(N)$$

(10)

and therefore in the sector $S^z \equiv 0 \text{ (mod } N)$ we have

$$[S^\pm(N), H] = [T^\pm(N), H] = 0.$$

(11)

Let us discuss the symmetry algebra. With the following identification

$$e_0 = S^{+}(N), \quad f_0 = S^{-}(N), \quad e_1 = T^{-}(N), \quad f_1 = T^{+}(N), \quad t_0 = -t_1 = -(-q)^NS^z / N,$$

(12)

we can show that they satisfy the defining relations of the sl$_2$ loop algebra:

$$[S^{+}(N), T^{+}(N)] = [S^{-}(N), T^{-}(N)] = 0,$$

(13)

$$[S^\pm(N), S^z] = \pm NS^\pm(N), \quad [T^\pm(N), S^z] = \pm NT^\pm(N),$$

(14)

$$S^{+(N)3} T^{-}(N) - 3S^{+(N)2} T^{-}(N) S^+(N) + 3S^{+(N)} T^{-}(N) S^+(N)^2 - T^{-}(N) S^{+(N)} = 0,$$

$$S^{-(N)3} T^{+}(N) - 3S^{-(N)2} T^{+}(N) S^-(N) + 3S^{-(N)} T^{+}(N) S^-(N)^2 - T^{+}(N) S^{-(N)} = 0,$$

$$T^{+(N)3} S^{-}(N) - 3T^{+(N)2} S^{-}(N) T^+(N) + 3T^{+(N)} S^{-}(N) T^+(N)^2 - S^{-}(N) T^{+(N)} = 0,$$

$$T^{-(N)3} S^{+}(N) - 3T^{-(N)2} S^{+}(N) T^-(N) + 3T^{-(N)} S^{+}(N) T^-(N)^2 - S^{+}(N) T^{-(N)} = 0,$$

(15)

and in the sector $S^z \equiv 0 \text{ (mod } N)$ we have

$$[S^{+}(N), S^{-}(N)] = [T^{+}(N), T^{-}(N)] = -(-q)^N\frac{2}{N} S^z.$$

(16)
3. The algebraic Bethe ansatz of the elliptic quantum group $E_{\tau, \eta}(sl_2)$

The elliptic algebra $E_{\tau, \eta}(sl_2)$ is an algebra generated by meromorphic functions of a variable $h$ and the matrix elements of a matrix $L(z, \lambda)$ with non-commutative entries, which satisfy the Yang-Baxter relation with a dynamical shift

$$R^{(12)}(z_1, \lambda - 2\eta h^{(3)})L^{(1)}(z_1, \lambda)L^{(2)}(z_2, \lambda - 2\eta h^{(1)}) = L^{(2)}(z_2, \lambda)L^{(2)}(z_1, \lambda - 2\eta h^{(2)})R^{(12)}(z_{12}, \lambda)$$

(17)

Here $h$ is a generator of the Cartan subalgebra $h$ of $sl_2$. Drinfeld’s quasi-Hopf algebra gives a natural framework for the dynamical Yang-Baxter relation, which can be derived from the standard quantum group $U_q(sl_2)$ through the twist.

The $R$-matrix of (17) is essentially that of the ABF model (the 8V RSOS model). Let $V$ be the two-dimensional complex vector space with the basis $e[1]$ and $e[-1]$. Here we denote $e[-1]$ also as $e[2]$, and let $E_{ij}$ denote the matrix satisfying $E_{ij}e[k] = \delta_{jk}e[i]$. Then, the $R$-matrix $R(z, \lambda) \in End(V)$ is given by

$$R(z, \lambda; \eta, \tau) = E_{11} \otimes E_{11} + E_{22} \otimes E_{22} + \alpha(z, \lambda)E_{11} \otimes E_{22} + \beta(z, \lambda)E_{22} \otimes E_{11}$$

$$+ \beta(z, \lambda)E_{21} \otimes E_{21} + \alpha(z, \lambda)E_{21} \otimes E_{21}$$

(18)

where $h = E_{11} - E_{22}$ and $\alpha(z, \lambda)$ and $\beta(z, \lambda)$ are defined by

$$\alpha(z, \lambda) = \frac{\theta(z)\theta(\lambda + 2\eta)}{\theta(z - 2\eta)\theta(\lambda)}$$

$$\beta(z, \lambda) = -\frac{\theta(z + \lambda)\theta(2\eta)}{\theta(z - 2\eta)\theta(\lambda)}$$

(19)

The theta function has been given by

$$\theta(z; \tau) = 2p^{1/4}\sin \pi z \prod_{n=1}^{\infty} (1 - p^{-2n})(1 - p^{2n}\exp(2\pi iz))(1 - p^{-2n}\exp(-2\pi iz))$$

(20)

where the nome $p$ is related to the parameter $\tau$ by $p = \exp(\pi i\tau)$ with $\text{Im } \tau > 0$.

Let us now review the construction of the eigenvectors of the elliptic algebra $E_{\tau, \eta}(sl_2)$ at the discrete coupling parameter: $2N\eta = m_1 + m_2\tau$, where $N$, $m_1$ and $m_2$ are any given integers. Here we note that $2N\eta = m_1 + m_2\tau$ corresponds to $2N\eta = 2m_1K + im_2K$ in (1). Hereafter we assume $m_2 = 0$ for simplicity. Let $W = V(z_1) \otimes \cdots \otimes V(z_L)$ be the $L$th tensor product of the evaluation modules $V_{\lambda_j}(z_j)$’s with $\Lambda_j = 1$ for all $j$. The transfer matrix $T(z)$ of $E_{\tau, \eta}(sl_2)$ is given by the trace of the $L$-operator acting on the module $W$

$$L(z, \lambda) = R^{(01)}(z - z_1, \lambda - 2\eta \sum_{j=2}^{L} h^{(j)})R^{(02)}(z - z_2, \lambda - 2\eta \sum_{j=3}^{L} h^{(j)}) \cdots R^{(0L)}(z - z_L, \lambda)$$

(21)

Let us consider the $m$th product of the creation operators $b(t_j)$’s on the vacuum. Let us assume the number $m$ satisfies the following condition

$$2m = L - rN, \quad r \in Z$$

(22)
Hereafter we also assume that \( r m_1 \) is even. We introduce a function \( g_c(\lambda) \) by
\[
g_c(\lambda) = e^{\lambda} \prod_{j=1}^{L} \frac{\theta(\lambda - 2nj)}{\theta(2\eta)}.
\]
Vector \( v_c \) is defined by \( v_c = g_c(\lambda)v_0 \), where \( v_0 \) is the highest weight vector of \( W \): \( hv_0 = L v_0 \). Then, making use of the fundamental commutation relations\(^4\) associated with \( b(z_j)'s \), we can show that \( b(t_1) \cdots b(t_m)v_c \) is an eigenvector of the transfer matrix \( T(z) \) with the eigenvalue \( C_0(z) \)
\[
C_0(w) = e^{-2\eta c} \prod_{j=1}^{m} \frac{\theta(w - t_j + 2\eta)}{\theta(w - t_j)} + e^{2\eta c} \prod_{j=1}^{m} \frac{\theta(w - t_j - 2\eta)}{\theta(w - t_j)} \prod_{\alpha=1}^{L} \frac{\theta(w - z_\alpha)}{\theta(w - z_\alpha - 2\eta)},
\]
if rapidities \( t_1, t_2, \ldots, t_m \) satisfy the Bethe ansatz equations
\[
\prod_{k=1}^{L} \frac{\theta(t_j - p_k)}{\theta(t_j - q_k)} = e^{-4\eta c} \frac{\theta(t_j - t_k + 2\eta)}{\theta(t_j - t_k - 2\eta)} \quad \text{for} \quad j = 1, \ldots, m.
\]
The “matrix elements” of the vector \( b(t_1) \cdots b(t_m)v_c \) is explicitly given by \( \Box \)
\[
b(t_1) \cdots b(t_m)v_c = (-1)^m e^{c(\lambda + 2q_m)} \sum_{P \in S_m} \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq L} \frac{\prod_{\alpha=1}^{L} \prod_{\beta=1}^{L} \theta(t_{P\alpha} - t_{P\beta} - z_{j_\alpha} - 2\eta(rN - j_\alpha + \alpha))}{\theta(t_{P\alpha} - z_{j_\alpha} - 2\eta)} \sigma_{j_1}^{-} \cdots \sigma_{j_m}^{-} |0\rangle.
\]
Here \( \sigma_j^{-} \) denotes the Pauli matrix \( \sigma^- \) acting on the \( j \)th site, \( S \) the symmetric group, \( |0\rangle \) the vacuum vector and \( f_{jk} = \theta(t_j - t_k - 2\eta)/\theta(t_j - t_k) \).

4. The eigenvectors of the 8V CSOS model

Let us replace \( \lambda \) with \( \lambda + \lambda_0 \) in the \( L \)-operator \(^{24} \) on \( W \). Here \( \lambda_0 \) is independent of \( \lambda \). Then, the \( R \)-matrix \( R(z, \lambda + \lambda_0) \) is related to the Boltzmann weights \( w(a, b, c, d; z, \lambda_0) \) of the 8V CSOS model through the following relation
\[
R(z, -2\eta d + \lambda_0)e[c - d] \otimes e[b - c] = \sum_a w(a, b, c, d; z, \lambda_0)e[b - a] \otimes e[a - d].
\]
Here \( a, b, c, d \) denote the spin variables of the IRF (the Interaction Round a Face) model which take integer values. \( \Box \) The spin variables have the constraint that the difference between the values of two nearest-neighboring spins should be given by \( \pm 1 \). Furthermore, for the 8V CSOS model discussed in Refs. \(^{13,14} \), the spin variables take the restricted values such as 0, 1, \ldots, \( N - 1 \) where the values \( 0 \) and \( N - 1 \) can be assigned for adjacent spins.

Through the relation \(^{24} \), we can show that the transfer matrix \( T(z) \) of \( E_{2|N}(sl_2) \) acting on the “path basis” corresponds to that of the 8V CSOS model \(^{13,14} \). Here we note that a “path” is given by a sequence of spin values satisfying the constraints on adjacent spins. Explicitly we consider the following \( \Box \)
\[
|a_1, a_2, \ldots, a_L(\lambda) = \delta(\lambda + 2\eta a_1) e[a_1 - a_2] \otimes e[a_2 - a_3] \otimes \cdots \otimes e[a_L - a_1].\]
Here for the 8V CSOS model, we assume that $a_L - a_1 \equiv \pm 1 \pmod{N}$. Expressing the eigenvector $b(t_1) \cdots b(t_m)v_c$ of $T(z)$ in terms of the path basis, we obtain that of the transfer matrix of the 8V CSOS model.

5. The degenerate eigenvectors of the transfer matrix of the 8V CSOS model

Let us now assume that out of $m$ rapidities $t_1, \ldots, t_m$, the first $R$ rapidities $t_j$ for $j = 1, \ldots, R$ are of standard ones satisfying the Bethe ansatz equations (24) with $m$ replaced by $R$, while the remaining $NF$ rapidities are formal solutions given by

$$t_{(\alpha, j)} = t_{(\alpha)} + \eta(2j - N - 1) + \epsilon r_j^{(\alpha)}, \quad \text{for} \quad j = 1, \ldots, N.$$  

We call the set of $N$ rapidities $t_{(\alpha,1)}, \ldots, t_{(\alpha,N)}$, the complete $N$-string with center $t_{(\alpha)}$. Here the index $\alpha$ runs from 1 to $F$. Furthermore, we assume that the index $(\alpha, j)$ corresponds to the number $R + N(\alpha - 1) + j$ for $1 \leq \alpha \leq F$ and $1 \leq j \leq N$. We note that the complete strings were suggested in Ref. 24 in another context.

Using the fundamental commutation relations, we can show when $\epsilon \neq 0$

$$T(z)b(t_1) \cdots b(t_{R+NF})v_c = C_0(z)b(t_1) \cdots b(t_{R+NF})v_c$$

$$+ \left( \sum_{j=1}^{R} + \sum_{j=R+1}^{R+NF} \right) C_j b(t_1) \cdots b(t_{j-1})b(z)b(t_{j+1}) \cdots b(t_{R+NF})v_c.$$  

We divide eq. (29) by $\epsilon$, and send $\epsilon$ to zero. Then, we can show that each of the terms of eq. (29) indeed converges, by making use of the following formula

$$\prod_{1 \leq \alpha < \beta \leq m} f_{\alpha \beta} = \prod_{1 \leq \alpha < \beta \leq m} f_{\alpha \beta} \times \prod_{1 \leq j < k \leq m} \left( \frac{\theta(t_j - t_k + 2\eta)}{\theta(t_j - t_k - 2\eta)} \right)^{H(P^{-1}j-P^{-1}k)},$$  

(30)

for $P \in S_m$. Here $H(x)$ denotes the Heaviside step function: $H(x) = 1$ for $x > 0$, $H(x) = 0$ otherwise. The symbol $P \in S_m$ denotes an element $P$ of the symmetric group of $m$ elements, where $j$ is sent to $Pj \in \{1,2,\ldots,m\}$ for $j = 1,\ldots,m$. The formula (30) has been proven in Ref. 24.

Let us consider the following function of variable $z$

$$G(z) = \sum_{a=1}^{N} e^{-4N\eta a} \prod_{j=a+1}^{N} \prod_{k=1}^{R} \theta(z-t_k + \eta(2j-N+1)) \prod_{j=a+1}^{N} \prod_{k=1}^{R} \theta(z-t_k + \eta(2j-N-3)) \prod_{\beta=1}^{L} \theta(z-z_\beta + \eta(2j-N-1))$$  

(31)

Hereafter we assume $\exp(4N\eta c) = 1$. Then, the centers $t_{(\alpha)}$'s are determined by

$$G(z = t_{(\alpha)}) = 0, \quad \text{for} \quad \alpha = 1, \ldots, F.$$  

(32)

We can show that the zeros of (32) also form complete $N$ strings, and also that the number of zeros of (32) is given by $L - 2R$, by using the Bethe ansatz equations (24). Thus, the number of independent solutions to (32) is given by $(L - 2R)/N,$
which leads to the dimension $2^{(L-2R)/N}$ through the binomial expansion. Thus, for the transfer matrix of the 8V CSOS model, any standard Bethe ansatz eigenvector with $R$ rapidities has the degeneracy of $2^{(L-2R)/N}$.

Let us now consider the connection of the CSOS model to the six-vertex model. Taking the trigonometric limit: $\tau \rightarrow i\infty$ and sending $\lambda_0$ to infinity with some gauge transformations, the $L$-operator of the 8V CSOS model becomes that of the six-vertex model. We may assume that the trigonometric limits of the $R$ rapidities of the Bethe ansatz equations (24) with $\exp(4\eta c) = 1$ satisfy the trigonometric Bethe ansatz equations of the six-vertex model. Then, the degenerate eigenvectors with $F$ complete $N$ strings for the 8V CSOS model become those of the six-vertex model with $F$ complete $N$ strings. Thus, we have shown that the corresponding degenerate eigenspace is spanned by the eigenvectors having complete $N$ strings, and also that the dimension is given by $2^{(L-2R)/N} = 2^{2S^Z_{\text{max}}/N}$ since the highest weight $S^Z_{\text{max}}$ is given by $L/2 - R$. The result should be consistent with the previous studies.

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References

1. T. Deguchi, K. Fabricius and B.M. McCoy, J. Stat. Phys. 102, 701 (2001).
2. K. Fabricius and B.M. McCoy, J. Stat. Phys. 103, 647 (2001).
3. K. Fabricius and B.M. McCoy, J. Stat. Phys. 104, 575 (2001).
4. K. Fabricius and B.M. McCoy, cond-mat/0108057.
5. T. Deguchi, cond-mat/0109078.
6. M. Takahashi and M. Suzuki, Prog. of Theor. Phys. 46, 2187 (1972).
7. A.N. Kirillov and N.A. Liskova, J. Phys. A 30, 1209 (1997).
8. J.D. Noh, D.-S. Lee and D. Kim, Physica A 287, 167 (2000).
9. Baxter, Ann. Phys. 70, 193 (1972).
10. R. Baxter, Ann. Phys. 76, 1 (1973); 76, 25 (1973); 76, 48 (1973).
11. N.A. Seaton, Phys. Rev. Lett. 60, 1347 (1988).
12. K. Jimbo, T. Yajima, J. Phys. Soc. Jpn. 52, 829 (1987).
13. Y. Akutsu, T. Deguchi and M. Wadati, J. Phys. Soc. Jpn. 57, 1173 (1988).
14. G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35, 193 (1984).
15. V. Pasquier and H. Saleur, Nucl. Phys. B330, 523 (1990).
16. C. Korff and B.M. McCoy, hep-th/0104124.
17. G. Felder and A. Varchenko, Commun. Math. Phys. 181 (1996) 741 .
18. G. Felder and A. Varchenko, Nucl. Phys. B 480 (1996) 485.
19. O. Babelon, D. Bernard, E. Billey, Phys. Lett. B 375, 89 (1996).
20. M. Jimbo, H. Konno, S. Odake, J. Shiraishi, hep-th/9712029.
21. L. Takhtajan and L. Faddeev, Russ. Math. Survey 34(3), 11 (1979).
22. T. Deguchi, cond-mat/0107260, to appear in J. Phys. A.