Non-Abelian Topological Order on the Surface of a 3D Topological Superconductor from an Exactly Solved Model

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Abstract

Three dimensional topological superconductors (TSCs) protected by time reversal ($T$) symmetry are characterized by gapless Majorana cones on their surface. Free fermion phases with this symmetry (class DIII) are indexed by an integer $\nu$, of which $\nu = 1$ is realized by the B-phase of superfluid $^3$He. Previously it was believed that the surface must be gapless unless time reversal symmetry is broken. Here we argue that a fully symmetric and gapped surface is possible in the presence of strong interactions, if a special type of topological order appears on the surface. The topological order realizes $T$ symmetry in an anomalous way, one that is impossible to achieve in purely two dimensions. For odd $\nu$ TSCs, the surface topological order must be non-Abelian. We propose the simplest non-Abelian topological order that contains electron like excitations, an even $\nu$ TSC surface, for which an explicit model is derived using a coupled layer construction. Both phases require electrons to transform as Kramers pairs, i.e. $T^2 = -1$ under time reversal. New topological phases of interacting fermions, based on bosonic topological states, are shown to introduce a $Z_2$ extension to the set of phases in this symmetry class.

I. INTRODUCTION

Recently it was pointed out that there exist exotic varieties of insulators and superconductors which form distinct phases of matter. This distinction is based on topological properties and hence falls outside the Ginzburg-Landau-Wilson symmetry based classification. On the other hand, these distinctions are often present only in the presence of certain symmetries, (e.g. time reversal $T$), leading to the terminology “symmetry protected topological phases” (SPTs). Many such phases can be realized at the level of non-interacting fermions \cite{Alicea} and several experimental realizations now exist. Their hallmark signature is the existence of gapless edge or surface modes. The best known example is the three dimensional topological insulator, which is protected by charge conservation and time reversal symmetry and has a single Dirac cone on its two dimensional surface. Another example is the three dimensional topological superconductor. Here, time reversal symmetry protects gapless Majorana cones at the surface. Within a free fermion (quadratic Hamiltonian) description, different topological phases are labeled by an integer $\nu$. The gaplessness of these surfaces is protected by the symmetry.

While a fairly complete picture exists of free fermion topological phases \cite{Alicea}, less is known about topological phases in the presence of interactions. Even if we restrict attention to SPTs (which may be adiabatically connected to a trivial gapped phase in the absence of symmetry), the qualitatively new phenomena that occur when strong interactions are present are only now beginning to be studied. Two advances in this general area include the result that the free fermion classification of SPT phases may be reduced in the presence of interactions, as shown for a class of 1D topological superconductors\cite{Hatsugai} and a class of 2D topological superconductors with $Z_2$ or reflection symmetry \cite{Kitaev}, and the discovery of SPT phases of bosons in different dimensions, that necessarily require interactions for their realization\cite{Kitaev} \cite{Jiang} \cite{Wang}.

An important lesson that emerged from studying interacting bosonic SPT phases is that the surface of a 3D phase can be gapped \textit{without} breaking the symmetry (either spontaneously or explicitly) if the surface develops topological order. In other words, the system must have deconfined anyons at the gapped, symmetric surface. Moreover, the anyons transform under the symmetries in a way that is disallowed in a strictly two dimensional system\cite{Wen}. Explicit examples of such symmetric topologically ordered surfaces have been constructed for 3D bosonic SPT phases \cite{Wang} \cite{Berg}. The simplest example is a bosonic version of the 3D topological superconductor protected by time reversal symmetry. In a conventional surface termination, time reversal symmetry is broken at the surface, leading to a gap. Domain walls between time reversed domains carry chiral edge modes with net chiral edge central charge $c_- = 8$, and so each domain is associated with half that chiral central charge. However it was realized that the surface can be gapped while retaining time reversal symmetry if it acquires the 3-fermion $Z_2$ gauge theory topological order\cite{Wen}. If this topological order were realized in 2D, it would always break $T$ symmetry, since it is associated with chiral edge states with $c_- = 4$ (exactly half the value associated with the surface domain walls). However, when realized on the boundary of a 3D system, no edge is present, and the theory can remain time reversal symmetric. Explicit constructions of such surface phases were given in \cite{Wen} \cite{Berg}.

While 3D SPT phases of bosons remain to be experimentally realized, it is interesting to consider the analogous topologically ordered surface states for the 3D fermionic topological insulators and superconductors, which we know are realized in Nature. The topologically ordered surface states provide a rare example of a...
qualitatively new phenomenon that lies beyond the free fermion description of these phases. Moreover, as we will see shortly, we will require non-Abelian topological order in some cases. This is a rare example where preserving symmetry requires not just topological order, but also particles with non-Abelian statistics, and may eventually help in realizing these exotic excitations.

In this paper we propose gapped, topologically ordered, $\mathcal{T}$-invariant terminations for 3D topological superconductors (topological insulators will be discussed in a separate publication). In terms of the free fermion classification these correspond to class DIII, where one has a $Z$ classification $[41,23]$; the $\nu = 1$ member of this class is the Balian-Werthamer (BW) $[24,25]$ state of the $B$ phase of liquid $^3$He, whose single Majorana cone describing the surface dispersion is protected by time reversal in the non-interacting setting. A principal feature of this surface is that a domain wall between regions of opposite $\mathcal{T}$ breaking is associated with a single chiral Majorana mode, which has chiral central charge $c_\sigma = \frac{1}{2}$. Hence each domain may be associated to $c_\sigma = \frac{1}{4}$, something that is impossible in a purely 2d fermionic system without topological order, where $c_\sigma$ is quantized in units of $\frac{1}{2}$. One strategy is then to look for patterns of fermionic 2d topological order (i.e. those that contain a fundamental fermion that has trivial braiding statistics with every other excitation and can be identified with the electron), and attempt to find a theory with both $c_\sigma = \frac{1}{4}$ mod $\frac{1}{2}$ and $\mathcal{T}$ symmetry. Fortunately this approach turns out to be fruitful, because 2d fermionic topological orders are extremely constrained.

First, we can rule out abelian theories, because these can only give $c_\sigma$ quantized in units of $\frac{1}{2}$, as can be seen from a $K$-matrix formulation. A well known non-Abelian example is the Moore-Read Pfaffian state $[26]$, but it is neither $\mathcal{T}$-invariant nor has the correct $c_\sigma$. However, the Pfaffian, with 12 quasiparticle types, is not the simplest non-Abelian fermionic theory. Indeed, from a classification standpoint, the smallest such theory contains only 4 quasiparticle types: in addition to the trivial particle and the electron, there is a self-segment $s$ and an its time reversed partner $\tilde{s}$. This is the integral spin sub-theory of $SU(2)_6$, equivalent to $SO(3)_6$ $[27,28]$. Serendipitously, $SU(2)_6$ has $c_\sigma = 2\frac{1}{4} = \frac{1}{2}$ mod $\frac{1}{2}$, and the braiding and fusion rules of its integral spin sub-theory are invariant under a $\mathcal{T}$ symmetry which exchanges $s$ and $\tilde{s}$. A strict 2d realization of this phase should break $\mathcal{T}$ as we argue below, based on the nontrivial edge chiral central charge $c_\sigma = 2\frac{1}{4}$ in one realization. However, it is conceivable that it may appear as a $\mathcal{T}$ invariant surface state of a 3D bulk, based on the statistics of the excitations. In that case one cannot interrogate the edge content, and hence a hidden $\mathcal{T}$ symmetry may exist. We substantiate this claim by explicitly constructing a 3D $\mathcal{T}$ symmetric model whose surface displays the $SO(3)_6$ topological order.

Our central tool is the Walker-Wang (WW) construction $[29,30]$, which, in essence, converts a given surface topological order it into a prescription for the bulk wave function (similar to the connection, one dimension lower, between quantum Hall wave functions and edge conformal field theories). The construction also provides an exactly soluble model to realize this wave function and surface topological order - so we know explicitly that it can be realized at the surface of an appropriate 3D system. Moreover in the case of fermionic topological phases it allows us to fix the transformation law for fermions under the protecting symmetry. Here we will see that the electrons must transform projectively, i.e. with $\mathcal{T}^2 = -1$ under time reversal symmetry. For constructing the exactly soluble 3D models, one is given the surface topological order, specified in terms of quasiparticle labels and fusion and braiding rules, collectively denoted $\mathcal{B}$ (mathematically, a premodular unitary category). When the only particle that braids trivially with everything in $\mathcal{B}$ is the identity - the so-called modular case - the WW models realize a confined 3+1 dimensional phase, with topological order $\mathcal{B}$ on the surface. Although the bulk is trivial, these models sometimes naturally allow for an incorporation of symmetry that can protect the surface topological order and hence result in a non-trivial 3D bosonic SPT $[21]$. This strategy was successfully applied to construct WW models that realize the 3 fermion surface topological order with time reversal symmetry, which realizes a 3D bosonic SPT phase $[21]$. In the present case, however, the input $\mathcal{B} = SO(3)_6$ contains the electron, which has trivial braiding statistics with all other excitations. We will see that this leads to a 3D WW model whose only bulk deconfined excitation is a $\mathcal{T}^2 = -1$ fermion - the electron - and whose surface realizes $SO(3)_6$. We note a minor caveat here. Since it is convenient to work entirely in terms of a bosonic WW model, rather than introducing fundamental fermions (electrons), the deconfined bulk fermions carry $Z_2$ gauge charge, and realize the $Z_2$ gauged 3D topological superconductor. Ungauging this theory by suppressing $Z_2$ flux loops is straightforward and yields the topological superconductor.

Finally, we discuss an Abelian topological order, the semion-fermion model, that is a candidate for the surface of an even $\nu$ topological superconductor. This theory also has three nontrivial excitations, a semion and its time-reversed partner, and a fermion that is identified with the electron. We argue that such a theory cannot be realized in 2D with $\mathcal{T}$ symmetry, but can appear on the surface of a 3D topological phase. In addition to a Walker-Wang model, we provide a coupled layer construction of this phase. The ingredients in each layer are topological orders that can be realized in 2D.

We also discuss the stability of bosonic SPT phases with just time reversal symmetry, in the presence of fundamental fermions. These phases can be thought of as being generated from bosons obtained by pairing fermions. While the classification of such phases Chen et al. $[14]$, Vishwanath and Senthil $[15]$ in the absence of fermions is $Z_2 \times Z_2$, we will see that this is reduced in the presence of fermions to a single $Z_2$. This gives rise
to a topological phase of interacting fermions that is not contained within the free fermion classification.

This paper is organized as follows. In Section II A the \( \text{SO}(3)_6 \) topological order is introduced and the action of \( \mathcal{T} \) on it is considered with an intuitive discussion of why it can be realized on the surface of a 3D system with \( \mathcal{T}^2 = -1 \) fermions in Section II B. The pertinent WW model is constructed in Section II C, while two caveats that entail minor modifications of the simplest WW construction are mentioned in IID. Next, in Section III the semion-fermion topological order is discussed, along with a coupled layer construction that allows for its realization on the surface of a 3D topological phase while retaining \( \mathcal{T} \). Finally, in Section IV we discuss the stability of bosonic SPT phases protected by \( \mathcal{T} \), in the presence of fundamental fermions. The Conclusions connect our results with the classification of fermionic TSCs and comment on future directions.

II. \( \text{SO}(3)_6 \) TOPOLOGICAL ORDER AND WALKER-WANG CONSTRUCTION OF THE 3D PHASE

A. \( \text{SO}(3)_6 \) Topological Order of Surface State

Before delving into the construction of the Walker-Wang model, let us describe the fusion and braiding properties of \( \text{SO}(3)_6 \). A useful viewpoint on this phase is to begin with the well known topological order \( \text{SU}(2)_k \) [21], which is a bosonic Read-Rezayi state [28, 32] with six quasiparticles labeled by spins \( j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\} \), and is described by the SU(2)\( _k \) Wess-Zumino-Witten chiral edge theory with chiral central charge \( c_- = 2\frac{1}{4} \). The topological spins of the quasiparticles are \( \theta_\beta = e^{\frac{2\pi i j_\beta}{9}} \). Thus \( j = 1 \) and \( j = 2 \) particles are self-ions, with topological spins \( \pm i \), while the \( j = 3 \) particle is a fermion. However, the fermion has mutual statistics with the half-integer spin particles. The latter can be eliminated if we introduce fundamental fermions (electrons) into the theory and condense the bound state of the \( j = 3 \) particle and the electron. This bound state can be condensed since it is a self-boson. However, since it has mutual statistics with the half-integer spin particles \( j = \frac{1}{2}, \frac{5}{2}, \frac{7}{2} \), they are confined, and one is left with just the integer spin particles. Of these, the fourth particle \( j = 3 \) has trivial braiding statistics with the remaining excitations, and is identified as the electron (note that this result in a non-modular theory, as in any topological order that contains fundamental fermions). Since only the integer representations of the spins remain in the final theory, we call it \( \text{SO}(3)_6 \) topological order [12]. The condensation process that converts the \( \text{SU}(2)_k \) to \( \text{SO}(3)_6 \) does not change the edge central charge and hence the final theory is also expected to have \( c_- = 2\frac{1}{4} \). We henceforth denote the two self-ions by \( s, \tilde{s} \), and the fermion by \( e: \{0, 1, 2, 3\} \rightarrow \{s, \tilde{s}, e\} \). Their fusion rules and topological spins are shown in Fig. I.

In our analysis we will need more information about the fusion and braiding, however: the data that enters the WW Hamiltonian requires the \( F \) and \( R \) symbols, which describe associativity of fusion (a quantum analogue of Clebsch-Gordon coefficients) and exchange of an arbitrary pair of quasiparticles, respectively, and uniquely determine \( \text{SO}(3)_6 \) as a premodular category [28]. For the definition of the \( F \) symbols, see Fig. 2, which also contains some of their symmetries. All of the non-trivial \( F \) symbols (i.e. those not equal to 1) can be obtained from those in Fig. 1 by using compositions of these symmetries, together with their invariance up to sign under \( s \leftrightarrow \tilde{s} \). A representative set of \( F \)-symbols which change sign under this exchange are also shown in Fig. 1.

B. Time Reversal Symmetry and \( \text{SO}(3)_6 \) Topological Order

We first elaborate on the question of time reversal symmetry and the \( \text{SO}(3)_6 \) topological order. As discussed in the overview, since this topological order is obtained by condensing particles in the \( \text{SU}(2)_6 \) theory, which is modular and has an edge central charge \( c_- = 2\frac{1}{4} \), we expect the same edge central charge in at least some realizations of this phase. Could one potentially realize the same \( \text{SO}(3)_6 \) topological order with a trivial edge? We now argue that it is impossible to realize this topological order in 2D without a chiral edge mode. This guarantees that it cannot be realized in a 2D system with \( \mathcal{T} \) symmetry, and thus, if realized on the surface of a 3D \( \mathcal{T} \) symmetric system, defines a 3D topological phase. For modular topological orders, where all non-trivial quasiparticles have non-trivial braiding statistics with at least one other particle, a powerful formula relates the topological properties to the chiral edge central charge \( c_- \mod 8 \):

\[
\frac{1}{D} \sum_n d_n^2 \theta_n = e^{i2\pi c_-/8}
\]  

(1)

where \( d_n (D) \) are the individual (total) quantum dimensions of the quasiparticles. Unfortunately, we cannot directly apply this formula since we have a non-modular theory (the electron has trivial braiding with all quasiparticles). In fact we can easily see that the edge central charge can be changed by units of \( c_- = 2\frac{1}{4} \) without affecting the topological order, simply by putting the electrons in a \( p \pm ip \) topological superconductor phase. This suggests that the edge of \( \text{SO}(3)_6 \) is necessarily chiral. This result can be further argued as follows. Imagine that there was also a realization of the same topological order with a \( c_- = 0 \) edge, and consider it in conjunction with the existing \( c_- = 2\frac{1}{4} \) realization. Then, we could perform a reflection on the first phase and combine it with the second. In this way we will have realized a quantum double model, which can always be confined by condensing a bosonic quasiparticle. In this process the edge...
Fusion rules:
\[
\begin{align*}
    s \times s &= \tilde{s} \times \tilde{s} = 1 + s \\
    s \times \tilde{s} &= e + s + \tilde{s} \\
    s \times e &= \tilde{s} \quad \tilde{s} \times e = s \\
    e \times e &= 1
\end{align*}
\]

Quantum dimensions and topological spins:
\[
\begin{align*}
    d_1 &= d_e = 1, \\
    d_s &= d_{\tilde{s}} = 1 + \sqrt{2} \\
    \theta_1 &= 1, \\
    \theta_e &= -1, \\
    \theta_s &= i, \\
    \theta_{\tilde{s}} &= -i
\end{align*}
\]

F symbols:
\[
\begin{align*}
    &\left[ F_{s}^{\tilde{s},s} \right]_{s,s} = [ F_{s}^{s,s} ]_{\tilde{s},s} = -1 \\
    &\left[ F_{s}^{s,s} \right]_{1,s} = \left[ F_{s}^{s,s} \right]_{1,\tilde{s}} = \left[ F_{s}^{s,s} \right]_{1,s} = (1 + \sqrt{2})^{-\frac{1}{2}} \\
    &\left[ F_{s}^{s,s} \right]_{s,s} = \left[ F_{s}^{s,s} \right]_{s,\tilde{s}} = (2 + \sqrt{2})^{-1} \\
    &\left[ F_{s}^{\tilde{s},s} \right]_{s,s} = \left[ F_{s}^{\tilde{s},s} \right]_{\tilde{s},s} = -1 + \sqrt{2}
\end{align*}
\]

F symbols that change sign under \( s \leftrightarrow \tilde{s} \):
\[
\begin{align*}
    &\left[ F_{s}^{\tilde{s},s} \right]_{s,s} = \left[ F_{s}^{\tilde{s},s} \right]_{\tilde{s},s} = -1 \\
    &\left[ F_{s}^{s,s} \right]_{s,s} = \left[ F_{s}^{s,s} \right]_{\tilde{s},s} \\
    &\left[ F_{s}^{s,s} \right]_{s,\tilde{s}} = \left[ F_{s}^{s,s} \right]_{\tilde{s},s} \\
    &\left[ F_{s}^{\tilde{s},s} \right]_{s,\tilde{s}} = \left[ F_{s}^{\tilde{s},s} \right]_{\tilde{s},s} \\
    &\left[ F_{s}^{\tilde{s},s} \right]_{s,s} = \left[ F_{s}^{\tilde{s},s} \right]_{\tilde{s},s}
\end{align*}
\]

and all partners under \( s \leftrightarrow \tilde{s} \).

Central charge will not change and will remain \( c_- = 2 \frac{1}{2} \).

However, we now have eliminated all topological order.

A fermionic system with no topological order must have half integer quantized edge central charge, so we have arrived at a contradiction. Therefore there cannot be a 2D system with \( SO(3)_6 \) topological order and without a chiral edge state.

Let us now consider time reversal purely at the level of fusion and braiding rules. An examination of the topological spins in our theory (Fig. 1) suggests that it is time reversal invariant under the exchange \( s \leftrightarrow \tilde{s} \). Indeed, such an exchange together with complex conjugation gives a set of \( F \) and \( R \) symbols which must be gauge equivalent to the original ones, because \( SO(3)_6 \) is the unique theory with these topological spins and fusion rules. However, \( T \)-invariance of the WW Hamiltonian (constructed in the next section) requires more: we will need a time reversal transformation law which leaves the \( F \) and \( R \) symbols exactly invariant, not just invariant up to gauge transformation. Quick examination of \( F \) shows that there is no gauge in which such a transformation law takes the simple form \( s \leftrightarrow \tilde{s} \) followed by complex conjugation - this is because these two \( F \) symbols differ by a sign in our gauge and transform by complex conjugate phases under a gauge transform, so they cannot be complex conjugates in any gauge. However, the following more general \( T \) transformation law does work, and in fact forces \( T^2 = -1 \) on the electrons: we define \( T \) as the operation that exchanges \( s \leftrightarrow \tilde{s} \), complex conjugates, and multiplies by certain phase factors \( \alpha_{a,b}^{s} \) (see below) associated to the fusion spaces \( V_{a,b}^{s} \) in our anyon theory.

It is this \( T \) transformation which commutes with all of the \( F \) and \( R \) symbols; link invariants (i.e. braiding amplitudes) computed in the corresponding picture calculus are then invariant under it.

The phase factors \( \alpha_{a,b}^{s} \) are defined as follows. For vertices of type \( (s, \tilde{s}, e) \), \( \alpha_{a,b}^{s} \) is \( \pm i \) depending on the sign of the permutation that takes \( (s, \tilde{s}, e) \) to \( (a, b, c) \). Specifically,

\[
\alpha_{a,b}^{s} = \begin{cases} 
    i & \text{if } (s, \tilde{s}, e) \rightarrow (a, b, c) \\
    -i & \text{if } (s, \tilde{s}, e) \rightarrow (b, c, a) \\
    i & \text{if } (s, \tilde{s}, e) \rightarrow (c, a, b)
\end{cases}
\]

Figure 1: Braiding statistics and fusion data for \( SO(3)_6 \).

Figure 2: Graphical definition of \( F \) symbol, together with some identities satisfied by \( F \).
For vertices at which one or more of $a, b, c$ is the trivial anyon 1, we set $\alpha^{a,b}_{c} = 1$. The remaining vertices have all $a, b, c$ either $s$ or $\tilde{s}$. We take

$$\alpha^{s,s}_{s} = \alpha^{\tilde{s},\tilde{s}}_{\tilde{s}} = 1$$

and $\alpha^{a,b}_{c} = -i$ when $(a, b, c)$ consists of some permutation of two $s$’s and one $\tilde{s}$ or vice versa.

With this definition, $\mathcal{T}$ commutes with $F$. Since in our gauge the $F$ symbols are all real, this reads:

$$\alpha^{1,1}_{i} \alpha^{1,1}_{i} \langle F^{i,j,k} \rangle_{m,n} = \langle F^{\tilde{j},\tilde{j},k} \rangle_{m,n} \alpha^{1,1}_{i} \alpha^{1,1}_{i}$$

(2)

where $i \rightarrow \tilde{i}$ is just the permutation that fixes $i = 1, e$ and exchanges $i = s, \tilde{s}$. $\mathcal{T}$ also commutes with $R$:

$$\alpha^{j,i}_{k} (R^{i,j}_{k})^{*} = R^{\tilde{j},\tilde{i}}_{k} \alpha^{1,1}_{k}$$

(3)

These two relations are sufficient to show that Wilson lines computed in the corresponding picture calculus are invariant under $\mathcal{T}$. Note that $(\alpha^{j,i}_{k})(\alpha^{j,i}_{k})^{*} = 1$ for all vertex types except those where $(i, j, k)$ is a permutation of $(s, \tilde{s}, e)$, for which $(\alpha^{j,i}_{k})(\alpha^{j,i}_{k})^{*} = -1$. This means that $\mathcal{T}^{2}$ fixes all particle types, and gives minus signs to $(s, \tilde{s}, e)$ vertices. In any string net that respect the fusion rules there are an even number of such vertices, so $\mathcal{T}^{2}$ acts as the identity on them. In the exactly solved Walker-Wang model of the next section, this fact translates to the many body ground state being invariant under $\mathcal{T}$, with $\mathcal{T}^{2} = 1$. However, string-net configurations of the Walker-Wang model also include fusion-rule violating vertices such as $(1, 1, e)$; a consistent definition of time reversal that includes these will require them to have $\mathcal{T}^{2} = -1$, which will necessitate decorating the Walker-Wang model with additional spin 1/2’s (Kramers doublets).

In the next section we will see how to do this by binding Haldane chains to the $e$ links of the model. Ultimately, this ensures that the fundamental fermions in the theory (the electrons) must transform under time reversal as $\mathcal{T}^{2} = -1$. Since this is a important point, we elaborate on its origin in a more intuitive way below.

Emergence of $\mathcal{T}^{2} = -1$ for fermions in the Walker-Wang model: Let us briefly review the idea behind the Walker-Wang construction in order to make this point. The Hilbert space, defined on the links of the 3D lattice model, contains a state for each quasiparticle in the theory - in this case there are four states per link, labeled by the four particle types. The ground state wave function is a quantum superposition of loops labeled by these four indices (or four colors). At the vertices (we consider a trivalent lattice for simplicity - it is always possible to deform a lattice such as a cubic lattice into a trivalent one by splitting vertices), the loops are allowed to branch and combine according to the fusion rules. The amplitudes for these loop configurations are determined by imagining them as space-time Wilson loops of the 2+1D TQFT representing the surface topological order, and calculating the quantum amplitudes within that theory. Thus: $\Psi_{3D(C)} = \langle W(C) \rangle_{2+1TQFT}$. In practice, these amplitudes are implemented by writing down a parent Hamiltonian. Consider a part of the ground state wave function as shown in the top-left of Figure 3. The $s$ and $\tilde{s}$ particles of the theory, which have semion and “anti”-semion self statistics, fuse to give the electron $e$. Say this has an associated amplitude $\Psi$. Then, in order to preserve time reversal symmetry, one needs to not only complex conjugate this amplitude, but also multiply by a phase factor $\pm i$. Only then does the transformation commute with the $R$ moves, and produce a time reversal symmetric wave function. The choice of phase factor is fixed by the orientation of the vertex - eg, if we choose $i$ for the vertex where the $\{e, s, \tilde{s}\}$ appear on going around anti clockwise, we must choose $-i$ for the vertex of the opposite sense shown on the bottom left. Since time reversal exchanges $s$ and $\tilde{s}$, and hence these two vertices, we have the sequence $\Psi \xrightarrow{\mathcal{T}} i\Psi^{*} \xrightarrow{\mathcal{T}} -i(i\Psi^{*})^{*} = -\Psi$ and thus $\mathcal{T}^{2} = -1$. Since this vertex results in the creation of a fermion, we will ultimately associate this transformation law to the fermions in the theory. Note that in the Walker Wang models clockwise/counterclockwise orientations are fixed since the projection of the 3D lattice on the 2D plane is fixed a priori in order to define the model. Hence, one can imagine that a crystal axis is picked in defining the models, and the remaining sense along the axis, required to define a spinning particle, is provided
by the permutation of the three labels.

The time reversal symmetry described above assigns a phase factor that depends jointly on the state of three bonds that meet at a vertex. Such a non-on-site transformation, i.e. one that does not act independently on the physical variables that reside at links, is not a fully satisfactory implementation of symmetry. This is readily fixed by attaching additional link variables that are put into a ‘Haldane chain’ phase protected by time reversal symmetry, along the e strings. When these strings end (for example by splitting into an s, \( \tilde{s} \) pair, as in the figure [3]), the spin 1/2 excitation generated at the ends automatically provides the requisite phase factors to keep the state invariant under time reversal. Furthermore, the vertex term violating configurations mentioned previously are also made time reversal symmetric by this construction.

C. Review of Standard Walker-Wang construction

We now construct our exactly soluble model. First we review the standard Walker-Wang construction [29, 30], for the specific case of the premodular category \( \mathcal{B} = SO(3)_6 \). This model will have \( SO(3)_6 \) topological order on the surface and its only deconfined bulk excitation will be a fermion. Then we describe how to extend it by “gluing Haldane chains” [33] to the e-lines; the resulting model will be \( T \)-invariant under a natural on-site \( T \) symmetry, with the bulk deconfined fermion - the electron - carrying \( T^2 = -1 \).

Informally, the states in the WW model are string-nets obeying the fusion rules of \( \mathcal{B} \). The Hamiltonian is engineered in such a way that the ground state consists of a superposition of such string nets with amplitudes equal to their evaluation in the picture calculus of \( \mathcal{B} \). It is important to note that the WW models work with a particular planar projection of the 3D lattice, yielding a natural choice of framing. The only deconfined strings correspond to the so-called symmetric center \( \mathcal{Z}(\mathcal{B}) \): quasi-particles in \( \mathcal{B} \) which have trivial braiding with all other quasiparticles. In our case, these are just the electrons. Furthermore, the statistics of the bulk deconfined excitations are just those of \( \mathcal{Z}(\mathcal{B}) \), so our bulk deconfined quasiparticle is indeed a fermion [29, 44].

In order to explicitly describe the Walker-Wang model, we closely follow [29] and refer the reader there for further details. We start with a planar resolution of a trivalent lattice, as in Fig. 4. The links are labeled with the quasiparticle types of \( SO(3)_6 \). The Hamiltonian enforces fusion rules at the vertices, and contains some plaquette terms. For any quasiparticle \( s \) and plaquette (Fig. 5), there is a term in the Hamiltonian which acts on the links of that plaquette, labeled in Fig. 5. This term can change the labels of these links, and can also depend on the labels of adjoining links, which have primes on them (but cannot change these). Explicitly, the matrix element between a state with plaquette links \( (ab\bar{c}d)pqruvw \) and \( (a'b'c'd'p'q'r'u'v'w') \) is

\[
B_{p,q,r,u,v,w}^{a,b,c,d} = R_q^{q''}(R_c^{c''})^* R_p^{p''}(R_b^{b''})^* R_{r''}(F_w^{w''})^{a''} (F_r^{r''})^{a'} (F_v^{v''})^{a} (F_u^{u''})^{a} (F_p^{p''})^{a} (F_q^{q''})^{a} (F_r^{r''})^{a} (F_v^{v''})^{a} (F_u^{u''})^{a} (F_p^{p''})^{a} (F_q^{q''})^{a}.
\]

The intuition behind this complicated looking term is that it fuses in the loop \( s \) to the skeleton of the plaquette using multiple \( F \) moves, but in the process of doing so must use \( R \) symbols to temporarily displace certain links (\( c' \) and \( q' \) in Fig. 5). The Hamiltonian then contains a sum of all these plaquette terms, weighted by the quantum dimensions \( d_s \). It is possible to check that all of these terms commute, and the result is a model that satisfies the properties described above - again we refer the interested reader to [29] and [30] for more details.

We have thus constructed an exactly solved model which explicitly realizes the \( SO(3)_6 \) theory on its surface.
Figure 6: Sample spin configuration in the decorated Walker-Wang model. Black dots represent spin 1/2’s, and blue ellipses represent spin singlets. The leftmost vertex has a counterclockwise \((e, s, \bar{s})\) ordering of labels, and so prefers a down spin, whereas at the rightmost vertex the counterclockwise label ordering is \((e, \bar{s}, s)\) and an up spin is preferred (red).

D. Improved Walker Wang Model - Onsite \(T\) symmetry and ungauging the bulk topological order

1. Exactly soluble model with onsite \(T\)-symmetry

One way to make the model of Sec. [11C] time reversal invariant is by defining the \(T\) operator to act by \(s \leftrightarrow \bar{s}\) on link labels and the phase factors \(\alpha_{i,j}^{e,s,e}\) on vertices. As discussed in [11B], this commutes with the \(F\) and \(R\) symbols, and hence with the Walker-Wang Hamiltonian. However, in order to interpret this as the time reversal invariance of the \(\mathbb{Z}_2\)-gauged topological superconductor, we have to vary this definition slightly.

To see why, note that in the Walker Wang model, one has to make some choice about how it will act on vertices that violate the fusion rules. For example, consider a configuration of an electron string which terminates at a \((1, 1, e)\) fusion-rule violating vertex on one end, and at an \((s, \bar{s}, e)\) fusion-rule respecting vertex on the other. This is certainly not a low energy state with respect to the vertex or plaquette terms, but it is an allowed Hilbert space configuration. Unless we make the \((1, 1, e)\) vertex into a Kramers pair, which requires introducing extra spin\(-1/2\) degrees of freedom not present in the original Walker Wang model, we will have \(T^2 = -1\) on this configuration (from the one \((s, \bar{s}, e)\) vertex). But this cannot happen in a finite \(\mathbb{Z}_2\) gauged topological superconductor, where the odd charge sector has been projected out. Hence we are forced to introduce some additional spin\(-1/2\) degrees of freedom in order to have a \(T\)-symmetry consistent with that of a \(\mathbb{Z}_2\)-gauged topological superconductor - this is another argument for the introduction of additional spin \(1/2\) degrees of freedom.

Specifically, we proceed as follows: decorate each link with four extra states, to be thought of as two spin \(1/2\)’s, one associated with each of the two vertices adjacent to the link. Then add a term \(H_V\) to the Hamiltonian that projects these into a singlet, unless the link is labeled with an \(e\), in which case it forces the two spin \(1/2\)’s into singlets with spin \(1/2\)’s from other \(e\) lines adjacent to those vertices. If there are none (or if a total of \(3\ e\) lines meet at the vertex) there will be an unpaired spin \(1/2\) there. This is very similar to the construction in [33], and has the effect of binding Haldane chains to the \(e\) lines. At a \((s, \bar{s}, e)\) vertex we also add a term to \(H_V\) that energetically prefers either the up spin or the down spin depending on the sign of the permutation that takes \((s, \bar{s}, e)\) into a counter-clockwise labeling of the three links adjoining the vertex. Finally, we modify the plaquette terms in the original WW Hamiltonian in such a way that they move between low energy spin configurations. We will not explicitly write down the Hamiltonian, but simply note that the spin configuration is uniquely determined by a fusion rule respecting string net configuration - this gives us the wave function (see Fig. 6).

Although the above construction seems complicated, it is easy to check that the ground state satisfies all of the conditions enumerated above. Furthermore, we can even make all of the terms in the Hamiltonian commute, basically by making the plaquette terms act trivially on configurations which violate either the vertex terms or the terms which pin spin configurations to WW labelings. This is similar to the way that plaquette terms are made to all commute in Levin-Wen string net models Levin and Wen [34]. Upon doing so, the Hamiltonian commutes with a symmetry \(T\) which acts on the spin \(1/2\)’s in the usual way by complex conjugation followed by multiplication by \(\sigma_\delta\) (note the extra factor of \(i\) in our definition - it can be eliminated by a gauge transform if desired, it is only \(T^2 = -1\) that is important), and on the Walker-Wang labels by exchanging \(s \leftrightarrow \bar{s}\), and applying the phase factors \(\alpha_{a,b}^{e,s,e}\) on vertices at which all of the \(a, b, c\) are either \(s\) or \(\bar{s}\). Although this is not a true onsite action, it can be made so by a suitable gauge transformation [15]. In this system, all endpoints of \(e\) lines effectively contribute a minus sign to \(T^2\), and since the number of such endpoints is even in any configuration, the Hilbert space has \(T^2 = 1\).

2. Ungauging the bulk \(\mathbb{Z}_2\) topological order.

A second point has to do with the nature of the bulk phase in the WW model, when the surface topological order contains an electron, i.e. a fermion with trivial braiding statistics with all other excitations. In this case the fermionic excitation is deconfined in the bulk. However, note that all the degrees of freedom entering the microscopic WW model are bosonic, since they simply consist of local ‘qubits’ defined on the links of the lattice. Hence, in order to produce fermionic excitations in the bulk, one must have bulk topological order. This is readily seen as arising by ‘gauging’ the fermion parity. That is, the microscopic symmetry that is always present for
physical fermion - the conservation of their number modulo 2 - here is attributed to their $\mathbb{Z}_2$ gauge charge. Thus one has an emergent $\mathbb{Z}_2$ gauge theory in the bulk. Additionally, there are $\mathbb{Z}_2$ flux loops in the bulk that are gapped. Fermions circling these flux lines pick up a $\pi$ phase shift - indicating that the fermion parity has been gauged. However this bosonic theory is readily related to the free fermion topological phases by the following slave particle construction: say we decompose the physical boson destruction operator into a pair of fermionic partons, $b_i = f^\dagger_i f_i$. Time reversal is assumed to act projectively on the fermions $f^\dagger_i \rightarrow f_i^\dagger$, $f_i \rightarrow f_i^\dagger$. The fermions are then governed by a mean field Hamiltonian in a topological phase: $H = \sum_{ij} \left( t_{ij} f_i^\dagger f_j + \Delta_{ij} f_i^\dagger f_j + \text{h.c.} \right)$ (here $i$ refers to both space and spin index). To ‘ungauge’ the $\mathbb{Z}_2$ symmetry one simply introduces fundamental fermions $c_{r\sigma}$ in the model and condenses the pair amplitude $\langle \sum_{r} c^\dagger_{r\sigma} f_{r\sigma} \rangle$. Now, the $\mathbb{Z}_2$ flux loops are confined since they have nontrivial mutual statistics with the condensate. Hence the bulk topological order is removed, and one realizes the short range entangled topological phase in the bulk.

III. AN ABELIAN SURFACE TOPOLOGICAL ORDER AND COUPLED LAYER CONSTRUCTION

A. The Semion-Fermion State and Time reversal symmetry

So far, we have been discussing $SO(3)_6$, a non-abelian theory which we propose to be a candidate for the $T$-symmetric surface termination of a $\nu = 1$ (mod 2) topological superconductor. Now let us consider even $\nu$. In this section, we introduce an Abelian fermionic theory, which we call the semion-fermion theory, which turns out to also have an anomalous realization of $T$. The theory is the product of $U(1)_2$, which describes the universality class of bosonic fractional quantum Hall systems at 1/2 filling and has quasiparticle content $\{1,s\}$, with $s$ a semion ($\theta_s = i$), and the trivial fermionic theory, with quasiparticle content $\{1,f\}$, $f$ being a fermion. Letting $\bar{s} = sf$, we obtain the quasiparticles $\{1,s,\bar{s},f\}$ with topological spins $\{1,i,-i,-1\}$ (labeled $\mathbb{Z}_2^3 \times \mathbb{Z}_2$ in the notation of Ref. [28]). Potentially, one could imagine that this theory may be time reversal symmetric if $T$ exchanged the semion and anti-semion. However, this is not possible in a 2d system. This time the simple chiral central charge argument does not work. Instead, consider assigning fermion parity quantum numbers to the excitations (since the conservation of fermion parity of electrons is the other global symmetry present) . Since $s^2 = \bar{s}^2 = 1$, their fermion parity quantum numbers must be $\pm 1$. Also, since $s\bar{s} = f$, $s$ and $\bar{s}$ must have opposite fermion parities. The latter is not consistent with time reversal symmetry that exchanges them.

Modularizing the Semion-Fermion Theory: A variant of this argument is the following. Consider the sixteen modular extensions of $\{1,s,\bar{s},f\}$, obtained by taking the product of $\{1,s\}$ with Kitaev’s sixteen modular extensions of $\{1,f\}$; these have chiral central charges $j/2$ (mod 8), $j = 0,\ldots,15$, and $j = 0$ could potentially be $T$-invariant. Since $U(1)_2$ has chiral charge 1, this $j = 0$ theory is the product of $U(1)_2$ and the $\nu = -2$ theory of $\mathbb{Z}_2^6$; the latter has quasiparticles $\{1, a, \bar{a}, f\}$ with $aa = \bar{a}a = f$, and $\theta_a = \theta_{\bar{a}} = e^{\pi i/4}$. Let us show that there is no way that $T$ could act and be consistent with the fusion rules. First, note that $T$ has to fix $1, f$ and exchange $s, sf$. On the remaining quasiparticles it must do one of two things: either $a \leftrightarrow sa$ and $\bar{a} \leftrightarrow s\bar{a}$, or $a \leftrightarrow sa$ and $\bar{a} \leftrightarrow s\bar{a}$. In the first case, $sf = a(sa)$ goes to $(sa)a = sf$ under $T$, which is inconsistent, and in the second case it also goes to $(s\bar{a})\bar{a} = sf$, again an inconsistency.

This argument shows that the original fermionic $\{1,s,\bar{s},f\}$ theory cannot exist in a 2D $T$-symmetric system, assuming that the above theories exhaust the modular extensions of $\{1,s,\bar{s},f\}$. The following argument establishes that this is so: given any realization of $\{1,s,\bar{s},f\}$, and forgetting $T$, we can continuously deform the Hamiltonian to any other one in this universality class. In particular, we can choose a special one, which is effectively a bosonic fractional quantum Hall layer in parallel with some $p + ip$ superconductor; we know that gauging this will yield one of our 16 modular extensions. Since the type of modular extension one obtains after gauging is a discrete object, it cannot change under continuous deformation. Hence, gauging any fermionic realization of $\{1,s,\bar{s},f\}$ - including the putative $T$-invariant one we started with - must yield one of the 16 above theories, none of which are $T$-invariant. Hence we have reached a contradiction.

However, it is possible to realize this theory in a $T$ symmetric way on the surface of a 3D topological phase. Just as for the case of $SO(3)_6$, we can define a $T$-symmetry that exchanges $s \leftrightarrow \bar{s}$ and acts with appropriate phase factors on the fusion vertices. We once again find that demanding $T$ to commute with $F$ and $R$ symbols requires $T^2 = -1$ on the $(s,\bar{s},f)$ vertices.

B. The Walker Wang Model

Once again, we can build a Walker-Wang model based on the semion-fermion theory. Because the theory is Abelian, we can actually use a simplified cubic lattice instead of the original trivalent one here [30]. The definition of $T$-symmetry also follows as in the $SO(3)_6$ case: Haldane chains are bound to the $f$ lines, so that $f$ becomes a deconfined bulk fermion with $T^2 = -1$: the electron. The Walker-Wang model once again describes the $\mathbb{Z}_2$-gauged version of the topological superconductor. For completeness, note that the $F$ and $R$ symbols of the semion-fermion theory are just products of those of the $\{1,f\}$ theory, in which the only non-trivial symbol
and 2j+1. Now consider the four bosonic operators associated with every unit cell (we suppress the coordinates that represent the spatial position in the plane):

\[
E_{j+} = e_{2j} S_{2j} m_{2j+1} S'_{2j+1} \\
M_{j+} = m_{2j} S_{2j} e_{2j+1} S_{2j+1} \\
E_{j-} = e_{2j} S_{2j} m_{2j-1} S'_{2j-1} \\
M_{j-} = m_{2j} S_{2j} e_{2j-1} S_{2j-1}
\]

It is readily seen that this set of four operators are bosonic and commute with one another. Hence they can be simultaneously ‘condensed’. The resulting state will confine any excitation that has nontrivial mutual statistics with these objects. In a system with periodic boundary conditions, the only excitation that has trivial mutual statistics with these four condensates is the bound state \( f_k = \psi_kb_k \) in each layer. This is the deconfined fermion in the bulk of the system. It is readily verified that all other anyonic excitations are confined in the bulk.

If however a surface is present, one can identify anyonic excitations that are deconfined within the surface. Consider a semi-infinite system with layers indexed by \( j = 0, 1, \ldots \infty \). Therefore, the condensates \( E_{j=0-}, M_{j=0-} \) are absent. Then, the surface excitations that are now liberated (Fig.7) are \( s_0 = e_0 S'_0 \), and \( \bar{s} = m_0 S_0 \). Their combination \( s_0 \bar{s}_0 = e_0 \bar{s}_0 m_0 s_0 = \psi_0 b_0 = f_0 \) is just the fermion in the theory. Also, note using Eqn. 4 that under time reversal symmetry:

\[
s_0 \xrightarrow{T} \bar{s}_0
\]

This is exactly the surface topological order that we sought. Moreover it transforms under time reversal in exactly the way we require.

We note a couple of important points that have been alluded to while discussing the alternate Walker-Wang construction. First, as discussed in the Appendix, the transformation law for fermions \( \psi \) and hence for \( f \) require that \( T^2 = -1 \) when acting on these objects. Second, we note that the models used here are entirely bosonic - i.e. they do not contain microscopic fermion excitations with trivial statistics with all other particles, that can be identified with electrons. Although there are fermionic excitations \( f \) that are deconfined in the bulk, these are emergent fermions, and hence have mutual statistics with a Z2 flux loop excitation. This leads to bulk Z2 topological order. To rectify this and produce a fermionic model free of bulk topological order, we introduce electron operators \( c_\sigma \) that transform as Kramers pairs under time reversal symmetry. Then, one can condense the combination \( e^f \psi \) which is a boson that transforms trivially under time reversal symmetry, since both components are Kramers pairs. As a consequence, flux excitations in the

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Figure 7: Coupled layer construction of semion-fermion topological order on the surface of a 3D topological superconductor (with even \( \nu \)). Each layer has a double-semion model and a \( \mathbb{Z}_2 \) gauge theory. The composite excitations enclosed by the brackets can be condensed simultaneously after which the only deconfined excitations in the bulk are electrons \( e = b\psi \) shown by the dashed green boxes. At the surface, two additional excitations emerge - shown by the dashed green lines - which are exchanged by \( T \). A key ingredient is that the two bosons (electric and magnetic charges) of the \( \mathbb{Z}_2 \) gauge theory are exchanged by \( T \) which requires \( T^2 = -1 \) acting on electrons.

\( \{1, s \} \) is defined by \( R^{1,s} = i \) and \( \{F^{s,s,s} \} = -1 \).

C. Coupled Layer construction of the Semion-Fermion Surface Topological Order

We provide a coupled layer construction of a 3D topological superconductor with a surface termination that realizes the semion-fermion Abelian topological order \( \{1, f \} \times U(1)^2 = \{1, s, \bar{s}, f \} \) with time reversal symmetry. The building blocks are (i) a \( \mathbb{Z}_2 \) toric code topological order with 4 quasiparticles, \( \{1, e, m, \psi \} \), in which time reversal exchanges the two bosonic particles. This is only possible if the fermion transforms projectively under time reversal symmetry as discussed in the Appendix, Sec.V (ii) a doubled semion topological order also with 4 quasiparticles, \( \{1, S, S', b \} \) where time reversal exchanges the semion and anti-semion. Collecting together the action of time reversal on these excitations:

\[
e \xrightarrow{T} m, \ S \xrightarrow{T} S', \ \psi \xrightarrow{T} -\psi
\]

Consider a layered structure where in each layer, a topological state of each of these two types is present (see Figure 7). We define a unit cell to consist of pairs of these layers, and unit cell \( j \) will be consisting of layers \( 2j \)}
bulk of the system are confined since they have mutual
statistics with this condensate, and there is no remain-
ing bulk topological order. Now, the \( f \) fermion may be
interpreted as the physical electron.

Finally, we notice that when two copies of the semi-
fermion surface theory are brought together, their topo-
logical order can be removed without breaking time re-
versal symmetry. This is achieved by condensing the
\( s_1 \bar{s}_2 \) and \( s_2 \bar{s}_1 \) bound states in the two theories. \( s_1 \bar{s}_2 \)
and \( s_2 \bar{s}_1 \) have bosonic self and mutual statistics, there-
fore can be condensed together. Moreover, they map
into each other under time reversal, therefore the con-
densation process does not violate time reversal symme-
try. The resulting theory contains only a trivial fermion
and is hence not topologically ordered for a fermion sys-

em. We assume that the topological superconductor de-
scribed here is adiabatically connected to one of the free
fermion TSCs, and is not a new topological phase that
only appears in an interacting systems. Then, this is
saying that the surface state of a topological supercon-
derator can be made trivial without breaking symmetry
at least for some \( \nu \), implying the collapse of the \( Z \)
free fermion classification down to some finite \( Z_\nu \)
in interacting fermion systems. Kitaev has argued that in
the presence of interactions the integer classification of free
fermion superconductors in 3D is reduced to a \( Z_{16} \)
classification [46]. If so, together with our observation that
combining two copies of the semi-fermion model leads
to the identity, we conjecture that the semi-fermion model realizes the surface of the \( \nu = 8 \) phase.

IV. STABILITY OF 3D BOSONIC SPT PHASES
IN THE PRESENCE OF FERMIONS

Recently, symmetry protected topological phases of
bosons were classified - for example, with just time re-
versal \( T \) symmetry (the bosonic analog of the class DIII
systems discussed here), it was found that there were
three nontrivial phases composed together in a \( Z_2 \times Z_2 \)
structure. The nontrivial phases are generated by (i) a
group cohomology state and another state (ii) based on
Kitaev's \( E_8 \) state that lies outside the 'group coho-
mology' classification.

When discussing bosonic SPT phases one assumes that
the bosons are fundamental particles. However, we could
consider the possibility that the bosons are composites
(like spins or Cooper pairs) of fermions. In this case there
will also be gapped fermions in addition to the bosonic
degrees of freedom. Sometimes, the bosonic topological
phase can be unwound in the presence of fermions, or can
be related to one of the free fermion topological phases.
Several instances of this type were discussed in the con-
text of interacting fermionic SPT phases in 2D. Here we
will discuss the 3D systems, with special reference to the
topological superconductors.

A useful tool to discuss this question in the 3D case, is
the topologically ordered surface termination of the 3D
bosonic SPT phases. Based on this we will argue that
(a) the \( Z_2 \times Z_2 \) classification of bosonic SPTs is reduced
to a single \( Z_2 \) in the presence of fermions and (b) this is
a distinct state from the free fermion topological phases
(which themselves are believed to be reduced to a \( Z_{16} \)
classification). Assuming this result the minimal set of
topological phases of interacting fermions with only time
reversal symmetry is \( Z_{16} \times Z_{12} \).

Let us begin by discussing the surface topological or-
ders for the two Bosonic SPT states are (i) the three
fermion state, with quasiparticles \( \{1, f_1, f_2, f_3\} \) where
the three fermion have mutual semionic statistics and
transform with \( T^2 = -1 \) under time reversal, and (ii) the
\( Z_2 \) toric code topological order \( \{1, v_1, v_2, f\} \) where
the two bosonic excitations \( v_1, v_2 \) both transform as \( T^2 = -1 \)
Kramers doublets.

We argue that this classification collapses to a sin-
gle \( Z_2 \) in the presence of fermions \( \psi \) which transform as
\( T^2 = -1 \). To see this consider putting together this fund-
damental fermion \( \{1, \psi\} \) with these topological orders.
In each case we obtain:

\[
\begin{array}{lcc}
\text{Label} & 0 & 1 & 2 & 3 \\
B & 1 & \bar{v}_1 & v_2 & f \psi \\
F & \psi & v_1 \psi & v_2 \psi & f + \\
\end{array}
\]

Table I: The group cohomology state surface topological order
with \( v_{1,2} \) that transform as \( T^2 = -1 \) attached to electrons. The \( \pm \) value is the action of \( T^2 \) on that quasiparticle.

Table II: The beyond group cohomology state and related
two fermion surface topological order, attached to electrons.
The \( \pm \) value is the action of \( T^2 \) on that quasiparticle.

\[
\begin{array}{lcc}
\text{Label} & 0 & 1 & 2 & 3 \\
B & 1 & f_1 \psi & -f_2 \psi & -f_3 \psi \\
F & \psi & f_1 + & f_2 + & f_3 + \\
\end{array}
\]

Clearly the two tables above are identical - we can represent the particles as \( B_0 \ldots B_3, F_3 \ldots F_3 \) and they
have identical self and mutual statistics and transforma-
tion laws under time reversal. Therefore these topolog-
ical orders are identical in the presence of fundamental
fermions which transform under time reversal symmetry
as \( T^2 = -1 \). That is, some of the bosonic SPT phases
'unwind' in the presence of fundamental fermions [47].

Next, we need to argue that this topological phase is
not contained within the free fermion classification.
Given the \( Z_2 \) 'ness' of this phase, i.e. the fact that
two copies are equivalent to the identity, the only purely
fermionic phase we could compare it to the one with the
surface semion-fermion topological order. The two topo-
logical orders appear rather distinct but are they really
different, or could they be surface equivalent i.e. differ
by addition of a topological phase realizable in 2D? We

can argue that they indeed are different as follows. Con-
ider gauging the fermion parity to get a modular theory
(i.e. all excitations have nontrivial mutual statistics with one another). Then, the $Z_2$ topological order above becomes equivalent of a toric code with both bosonic particles transforming as $T^2 = -1$. This is not possible in 2D but possible on 3D surface. However, if we modularize the semion-fermion theory, we fail to get a time reversal invariant theory as discussed in Section IIIA. These different outcomes on modularizing the theory imply that these topological orders are distinct and not connected by adding phases that are realizable in 2D. We conclude that the interacting bosonic SPT phase is distinct from the free fermion topological phases. Thus we may conjecture that the set of topological phases of interacting fermions with only time reversal symmetry is at least $Z_{16} \times Z_2$.

V. CONCLUSION

We have provided two examples of time reversal symmetric topological orders, realized on the surface of 3D gapped fermionic systems, that are impossible to realize in a purely 2D system with time reversal symmetry. This immediately implies that these phases are 3D fermionic SPT phases protected by $T$. In particular, it is impossible to confine the surface states without breaking symmetry or closing an energy gap. If it were possible to do so, then one can eliminate the surface state on one face of a slab, and be left with the ‘impossible’ 2D topological order on the other. Since the slab can then be deformed into a 2D system, this leads to a contradiction. It is natural to identify these phases with the free fermion topological superconductors protected by $T$. However, we note that we have not proved this equivalence, e.g. by demonstrating that the free fermion Majorana cone edge states can annihilate one of our surface topological orders. If this could be shown, it would eliminate the logical possibility that the phases we are describing are some yet unknown topological superconductor of fermions which is only realized in the presence of interactions. Potentially, a classification of interacting SPT phases of fermions in 3D could resolve this question, but that is also currently unavailable (Ref. 15 attempts to study this question in some other symmetry classes). These are important directions for future work.

However, there are several pieces of evidence that link the surface topological orders we mention with the free fermion topological superconductors. First, our topological orders are only possible if the electrons transform as $T^2 = -1$. This is also a requirement to realize the class DIII free fermion topological phases. Next, it is well known that odd $\nu$ topological superconductors have an odd number of chiral Majorana modes bound to surface domain walls between domains of opposite $T$ breaking. Indeed this is what we would find for the $SO(3)_6$ surface if the topological order is removed while breaking $T$. Imagine placing a 2D realization of the same topological order on the surface to obtain a quantum double model that can be confined to completely destroy topological order. Of course, in this process one breaks $T$ symmetry by the choice of the 2D topological order. If this is done in opposite ways, at the domain wall one expects an odd number of Majorana modes since the edge central charge of $SO(3)_6$ is $c_\nu = \pm 2\frac{1}{2}$. The difference between these two opposite values is consistent with the odd number of Majorana modes at the surface domain wall. Given the simple nature of the topological order in this case, it is tempting to attribute this to $\nu = 1$, although a proof of this requires a way to bridge these different descriptions. Similarly, if the semion-fermion topological order describes a 3D free fermion topological superconductor, it must correspond to even $\nu$, since it is compatible with a gapped edge when realized in 2D. We conjecture that this phase realizes $\nu = 8$ based on arguments presented above.

In principle, we can generate topologically ordered surfaces for all nontrivial $\nu$ by combining copies of the $SO(3)_6$ theory. However, given the non-Abelian nature of these theories it is difficult to simplify these tensor product topological orders and is left for future work.

The surface topological order provides a non-perturbative definition of the bulk topological phase and may be conceptually useful in classifying interacting fermion SPT states. They were also used here to discuss the stability of bosonic SPT phases with time reversal symmetry, in the presence of fundamental fermions. We found that the $Z_2 \times Z_2$ classification of the purely bosonic phases is ‘unwound’ down to $Z_2$ once fermions are introduced. This phase appears to be beyond the free fermion topological phases, the set of interacting topological superconductors protected by $T$ in 3D we argued to be given at least by $Z_{16} \times Z_2$.

In the future we hope to discuss analogous surface terminations for topological insulators, and attempt to connect this definition with other non-perturbative definitions of the surface of fermionic SPT phases. Finally, we wish to emphasize the remarkable fact that in some cases fermionic SPT phase provides a guarantee of non-Abelian topological order - if the surface of a $\nu = 1$ topological superconductor is found to be gapped and $T$ symmetric, it must contain non-Abelian excitations.

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Appendix: $\mathbb{Z}_2$ topological order in which time reversal exchanges electric and magnetic charges.

Consider the toric code model with the $\mathbb{Z}_2$ topological order, leading to 4 particles $\{1, e, m, \psi\}$, where the first particle is the identity, the next two are bosons (electric and magnetic charges) and the last is a fermion. The last three particles all have mutual semionic statistics with one another. Consider the action of time reversal symmetry $T$. The quasiparticles can transform projectively, and we can choose a pair of them to transform as $T^2 = -1$. Let one of these particles be the fermion $\psi$. The $\psi$ fermions are gapped in this topological phase. Consider modifying their dispersion so that they undergo a quantum phase transition into a topological superconductor (class DIII) of which there is a $\mathbb{Z}_2$ classification in 2D. Now, we can investigate how the excitations transform under time reversal symmetry. The fermions $\psi$ still transform as $T^2 = -1$. The electric and magnetic particles can both be interpreted as $\pi$ flux for the fermions. If we choose the electric particle, we can label the object that is obtained by attaching it to the fermion as the magnetic particle i.e. we use the fusion rule $\psi \times e = m$. In the language of the free fermion topological superconductor, if we call the $\pi$ flux the electric particle, then the magnetic particle has the opposite fermion parity. Now, it is readily seen that under time reversal symmetry: $e \xrightarrow{T} m$. This is related to the fact that fermion parity and time reversal anti commute when acting on a $\pi$ flux in this phase $[T (-1)^{N_T} = (-1)^{N_T} T]$. This can be seen by considering the class DIII topological superconductor as a tensor product of up spin and down spin electrons in a $p_x + ip_y$ and a $p_x - ip_y$ state respectively. Since time reversal exchanges these two systems they remain invariant. But a $\pi$ flux inserted through both superconductors will trap a pair of Majorana zero modes $\gamma_{\uparrow, \downarrow}$. Under time reversal we have: $\gamma_{\uparrow} \xrightarrow{T} \gamma_{\downarrow}$ and $\gamma_{\downarrow} \xrightarrow{T} -\gamma_{\uparrow}$. This ensures that the fermion parity $i\gamma_{\uparrow} \gamma_{\downarrow}$ for the vortex changes sign under time reversal symmetry.

We note that although this symmetry transformation can be achieved in 2D, it cannot be described by the K-matrix formulation of Abelian topological orders, despite the underlying topological order being an abelian one, specifically that of the $\mathbb{Z}_2$ toric code. Indeed, in a recent classification of Symmetry Enhanced Topological orders, using the K-matrix technique, this state was not produced. Here, time reversal symmetry exchanges the electric and magnetic particles, which have mutual statistics. In general it can be shown that when time reversal symmetry is implemented within the K-matrix, it cannot exchange anyons with nontrivial mutual statistics.

The proof is as follows. Consider the K matrix of a time reversal invariant state. This is an even dimensional $(2N \times 2N)$ symmetric matrix which can be diagonalized and brought into the form:

$$K = \sum_{\alpha=1}^{N} \lambda_{\alpha} [L_{\alpha} L^{T}_{\alpha} - R_{\alpha} R^{T}_{\alpha}] \quad (10)$$

Since this is a symmetric matrix, the eigenvalues and eigenvectors are real. The pairing of eigenvalues results from the fact that the sign of the eigenvalues $\lambda$ refer to the chirality of edge modes which should be coupled into time reversal symmetric nonchiral pairs. Hence for every left mover there must be a right mover which is related by time reversal $L_{\alpha} \xrightarrow{T} R_{\alpha}$. Now consider a quasiparticle represented by the integer vector $l$ and its time reversed partner $\tilde{l}$. Now expanding them in terms of the eigenstates:

$$l = \sum_{\alpha=1}^{N} [a_{\alpha} L_{\alpha} + b_{\alpha} R_{\alpha}]$$

where the coefficients are real numbers. Using the transformation of eigenvectors under time reversal symmetry, we can write the time reversed partner as:

$$\tilde{l} = \sum_{\alpha=1}^{N} [b_{\alpha} L_{\alpha} + a_{\alpha} R_{\alpha}]$$

Now, it is readily verified that the two quasiparticles have trivial mutual statistics since:

$$\tilde{l}^{T} K^{-1} \tilde{l} = 0$$

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[41] Although a related theory, where one reverses the direction of the Majorana mode, can be made T invariant. Since this may have application to 3D topological insulators, it will be discussed in a separate publication.

[42] Sometimes, only theories with k = 0 mod 4 are labeled SO(3)k, since otherwise they are non-modular. Since we are specifically interested in theories which contain the electron, which are necessarily non-modular, we will not make this distinction.

[43] This structure is reminiscent of the notion of G action in a braided G-crossed category [38, 39].

[44] It is actually known that the symmetric center Z(C) of any braided fusion category C comes in two types [40]: either it consists entirely of bosons and is isomorphic to the set of representations of some finite group G, in which case the bulk forms a (possibly twisted) G-gauge theory, or a supersymmetric version of this, where G contains some odd elements and the corresponding representations have even and odd sectors, corresponding to bosons and
fermions respectively.

[45] Indeed, if we take a gauge transformation to act by phase factor $\beta_{a,b}^{c}$ on the fusion space $V_{a,b}^{c}$, then any choice which satisfies $\beta_{s,s}^{a,b}\beta_{\bar{s},\bar{s}}^{a,b} = i$, $\beta_{s,s}^{a,b}\beta_{\bar{s},\bar{s}}^{a,b} = -i$, $\beta_{s,s}^{a,b}\beta_{\bar{s},\bar{s}}^{a,b} = -i$, and sets the other $\beta_{a,b}^{c}$ to be trivial, does the trick. The action of $\mathcal{T}$ is then truly onsite.

[46] A. Kitaev, (private communication).

[47] Related results have been obtained by C. Wang, A. Potter and T. Senthil (to appear) in the context of topological insulators.