Snyder’s model—de Sitter special relativity duality and de Sitter gravity

Han-Ying Guo,1,2 Chao-Guang Huang,1 Yu Tian,3 Hong-Tu Wu,4 Zhan Xu5 and Bin Zhou6

1 Institute of High Energy Physics, Chinese Academy of Sciences, PO Box 918-4, Beijing 100049, People’s Republic of China
2 Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, People’s Republic of China
3 Department of Physics, Beijing Institute of Technology, Beijing 100081, People’s Republic of China
4 Department of Mathematics, Capital Normal University, Beijing 100037, People’s Republic of China
5 Physics Department, Tsinghua University, Beijing 100084, People’s Republic of China
6 Department of Physics, Beijing Normal University, Beijing 100875, People’s Republic of China

E-mail: hyguo@itp.ac.cn, huangcg@mail.ihep.ac.cn, ytian@bit.edu.cn, lobby_wu@yahoo.com.cn, zx-dmp@tsinghua.edu.cn and zhoub@bnu.edu.cn

Received 14 March 2007, in final form 20 June 2007
Published 31 July 2007
Online at stacks.iop.org/CQG/24/4009

Abstract

Between Snyder’s quantized space-time model in de Sitter space of momenta and the dS special relativity on dS-spacetime of radius $R$ with Beltrami coordinates, there is a one-to-one dual correspondence supported by a minimum uncertainty-like argument. Together with the Planck length $\ell_p, R \simeq (3/\Lambda_1)^{1/2}$ should be a fundamental constant. They lead to a dimensionless constant $g \sim \ell_p R^{-1} = (G \hbar c^{-3})^{1/2} \sim 10^{-61}$. These indicate that physics at these two scales should be dual to each other and there is in-between gravity of local dS-invariance characterized by $g$. A simple model of dS-gravity with a gauge-like action on umbilical manifolds may show these characteristics. It can pass the observation tests and support the duality.

PACS numbers: 04.90.+e, 04.50.+h, 03.30.+p, 02.40.Dr

1. Introduction

A long time ago, Snyder [1] proposed a quantized space-time model. In his model, Snyder started with a projective geometry approach to the de Sitter(dS)-space of momenta with a scale $\alpha$, which is proportional to $\hbar/a$ in [1], near or equal to the Planck scale. The energy and momentum of a particle were identified with the inhomogenous projective coordinates.
Then, the spacetime ‘coordinates’ were naturally defined as 4-translation generators $\hat{x}_\mu$ of dS-algebra $\mathfrak{so}(1, 4)$ in dS-space of momenta and became noncommutative. It is important that there is a new kind of uniform ‘velocity’ motion with constant ‘group velocities’ for some ‘wave packets’ in the model. In addition, it may indicate that correspondingly there is a new kind of uniform coordinate velocity motion for some particles in the dS-spacetime [2–13]. But this was not recognized.

Recently, in order to explain the Greisen–Zatsepin–Kuz'min effect, the doubly/deformed special relativity (DSR) was proposed [14]. In DSR, there is also a large energy–momentum scale $\kappa$ near the Planck energy, related to $a$ in Snyder’s model in addition to $c$. It was found that there is a close relation between Snyder’s model and DSR. In fact, DSR can be regarded as a generalization of Snyder’s model [15]. And most DSR models with $\kappa$-Poincaré algebra can be realized geometrically by means of particular coordinate systems on dS/Minkowski(Mink)/AdS-space of momenta [15] other than the inhomogenous projective coordinates used by Snyder’s model in the dS-space of momenta. Thus, there is a kind of coordinate transformation from Snyder’s model to some of DSR on the dS-space of momenta and vice versa.

In fact, the projective geometry approach to dS-space is basically equivalent to the Beltrami-like model (Beltrami model for short). Historically, de Sitter [16] first used the Beltrami coordinates [17, 18] for his solution of constant curvature, the dS-spacetime, in the course of debate with Einstein on ‘relative inertia’. Later, Pauli [19] mentioned this metric in the Euclidean signature.

In his first paper [20], Einstein assumed the rigid ruler at rest be Euclidean. For free space in large scale, it has less observational basis since it is not supported by the asymptotic behavior of our universe [22, 23]. Actually, once Einstein’s Euclid assumption is released, there should be three kinds of special relativity with ISO(1, 3), SO(1, 4), SO(2, 5)-invariance on Mink/dS/AdS-spacetime, respectively. This is in analogy with the remarkable historic issue on the Euclidean fifth axiom. Once the axiom is weakened, there are Euclid, Riemann and Lobachevski geometries of zero, positive and negative constant curvatures on an almost equal footing. In these four-dimensional geometries, say, there is a kind of special coordinate system. In these coordinates, the points, straight lines of linear form, metric and other geometric objects are invariant or transformed among themselves under linear transformations of ISO(4)-invariance or fractional linear transformations with a common denominator (FLTs) of SO(5), SO(1, 4)-invariance, respectively. For the Lobachevski plane, Beltrami [17] first introduced such coordinates. Then, changing the metric to a physical signature by an inverse Wick rotation [5], these spaces become Mink-, dS- and AdS-spacetime with invariance of ISO(1, 3), SO(1, 4) and SO(2, 3), respectively. And the geometric objects such as points, straight lines and Euclid or Beltrami metric become corresponding events, straight worldlines and Mink- or Beltrami metric with the signature of $(+, −, −, −)$, respectively.

For Euclid’s counterpart, there is Einstein’s special relativity in Mink-space based on the principle of relativity and the postulate on universal invariant of the speed of light $c$. What should correspond to the other two non-Euclidean counterparts? Those are just two other kinds of special relativity on dS/AdS-space based on the corresponding principle of relativity and the postulate on universal constants of the speed of light $c$ and the curvature radius $R$.

More concretely, say, for dS-space with radius $R$, Beltrami coordinates are in analogy with Mink-coordinates on Mink-space. It is precisely the Beltrami model of a dS-hyperboloid $H_R$ in a 5D Mink-space $dS \simeq H_R \subset M^{1,4}$. Via the Beltrami coordinate atlas, the dS-space can be covered patch by patch, in which particles and light signals move along the timelike or null geodesics being straight worldlines with constant coordinate velocities in each patch,
respectively. All these properties are invariant under FLTs of SO(1, 4) symmetry among Beltrami systems. In light of inertial motions in both Newtonian mechanics and Einstein’s special relativity, these particles and signals should be in free motion of inertia without gravity. Accordingly, the Beltrami coordinates and observers should also be of inertia and there should be the principle of relativity in dS-spacetime.

In 1970, Lu [2] first noticed these important properties and began to study dS/AdS-invariant special relativity on dS/AdS-space (dS/AdS special relativity for short) [3]. Recently, promoted by the observations of our dark universe, further studies are being made [4–13].

It is interesting to see that in terms of the Beltrami model of dS-space (denoted as BdS-space) [2–6, 16, 19], there is a dual one-to-one correspondence between Snyder’s quantized space-time model [1] and the dS special relativity [2–7]. Actually, the dS special relativity can be regarded and simply formulated as a spacetime counterpart of Snyder’s model for dS-space of momenta as long as the constant $\alpha$ in Snyder’s model as dS-radius of momenta near the Planck scale is replaced by $R$ as radius of dS-spacetime. Furthermore, via the constant $a$ in Snyder’s model (or $\kappa$ in DSR) or Planck length $\ell_P$ and the cosmological constant $\Lambda$, a dimensionless constant $g$ can be introduced:

$$\psi \equiv \kappa^2 / R^2 \rightarrow g^2 \sim G\ell_P^2 R^{-2} \simeq 10^{-122}.$$  \hspace{1cm} (1.1)

Since Newton’s gravitational constant appears, $g$ should describe gravity with local dS-invariance between these two scales. In fact, this dimensionless constant $g$ has appeared in a simple model of dS-gravity with dS-algebra as gauge algebra [24–27] to characterize self-interaction of gravity with local dS-invariance. In addition, an uncertainty-like argument can be given via a ‘tachyon’ dynamics in embedded space, which may support the dual one-to-one correspondence.

Based upon these important properties we may expect that there should be a duality between physics at the scale of Planck length $\ell_P = (G\ell_P^2 R^{-2})^{1/2}$ and dS-radius $R = (3/\Lambda)^{1/2}$ with the cosmological constant $\Lambda$ regarded as a fundamental one in the nature. That is, physics at two such fundamental scales are dual to each other in some ‘phase’ space, and in-between there is gravity of local dS-invariance characterized by the dimensionless constant $g$.

Thus, there no longer exists the puzzle on $\Lambda$ as the ordinary ‘vacuum energy’ in the viewpoint of dS special relativity and the duality above. This is due to the fact that the concept of ‘vacuum’ now is dS invariant and the so-called ‘vacuum energy’ calculated in the Mink-space becomes improper. In the viewpoint of dS special relativity, the question should be: what is the origin of the dimensionless constant $g$ and is it calculable?

From the point of view of general relativity, there is no room for special relativity on dS/AdS-space. In the point of view of dS/AdS special relativity, however, there is no gravity on dS/AdS-space. As previously explained, this ‘funny’ stuff in Einstein’s relativity is in analogy with the remarkable historic issue on the Euclid fifth axiom. Thus, we should explain how to describe gravity in the point of view of dS/AdS special relativity.

In parallel with the local Poincaré gauge theory of gravity [29–31], we suggest that gravity should be based on the localization of dS special relativity and described by a gauge-like dynamics characterized by the dimensionless constant $g$. We show how to localize the dS-hyperboloid $H_B \subset M^{1,4}$ at each event on a kind of umbilical manifold of local dS-invariance and also very briefly introduce a simple model with a Yang–Mills-type action [24–27] of such gravity with local dS-invariance. We show that this model supports the duality and also provides some hints on the above questions on the dimensionless constant $g$.

7 Recently, another version of dS special relativity has been proposed [21], but the principle of relativity, inertial frames and the transformations among them are not mentioned there.
This paper is arranged as follows. We first review the general properties of the Beltrami model of 4D Riemann sphere and the Beltrami model of dS-space via an inverse Wick rotation in section 2. Next, we recall some important issues in dS special relativity in terms of the Beltrami model of dS-spacetime and in Snyder’s model of quantum space-time (together with DSR) in terms of the Beltrami model of dS-space of momenta, respectively, in sections 3 and 4. Then, we show that there is a dual one-to-one correspondence between Snyder’s model and dS special relativity, which is supported by a minimum uncertainty-like argument indicated by a ‘tachyon’ dynamics and propose the duality for physics at and in-between the two scales in section 5. In section 6 we explain how to describe gravity based on localization of dS special relativity and briefly introduce a simple model of dS-gravity on a kind of umbilical manifold. Finally, we end with some concluding remarks.

2. Beltrami model of a Riemann sphere and de Sitter spacetime

2.1. Beltrami model of a Riemann sphere

A 4D Riemann sphere $S^4$ can be embedded in a 5D Euclid space $E^5$

$$S^4 : \delta_{AB}\xi^A\xi^B = l^2 > 0, \quad A, B = 0, \ldots, 4,$$

(2.1)

$$d s^2_E = \delta_{AB} d\xi^A d\xi^B = d\xi^I d\xi_\parallel.$$  

(2.2)

They are invariant under rotations of SO(5):

$$\xi \rightarrow \xi^S = S\xi, \quad S^T S = I, \quad \forall S \in \text{SO}(5).$$  

(2.3)

A Beltrami model $B_I$ of $S^4$ is the intrinsic geometry of $S^4$ with the Beltrami coordinate atlas. In a patch, say $U_{+4}$,

$$x^\mu := I^E_{\mu\xi}, \quad \xi^4 > 0, \quad \mu = 0, \ldots, 3.$$  

(2.4)

To cover $B_I \sim S^4$, one patch is not enough, but all properties of $S^4$ are well defined in the $B_I$ patch by path with

$$\sigma_E(x) := \sigma_E(x, x) = 1 + l^{-2} \delta_{\mu\nu} x^\mu x^\nu > 0,$$

(2.5)

$$d s^2_E = \left\{ \delta_{\mu\nu}\sigma_E^{-1}(x) - l^{-2}\sigma_E^{-2}(x)\delta_{\mu\lambda}x^\lambda \delta_{\nu\kappa}x^\kappa \right\} dx^\mu dx^\nu.$$  

(2.6)

It is clear that the inequality (2.5) and the Beltrami metric (2.6) are invariant under FLTs among Beltrami coordinates $x^\mu$, which can be written in a transitive form sending the point $A(a^\mu)$ to the origin $O(a^\mu = 0)$,

$$x^\mu \rightarrow \bar{x}^\mu = \pm \sigma_E^{-1/2}(a)\sigma_E^{-1}(a, x)(x^\nu - a^\nu)N^\mu_\nu,$$

$$N^\mu_\nu = O^\mu_\nu - l^{-2}\delta_{\nu\kappa}a^\kappa\left(\sigma_E(a) + \sigma_E^{1/2}(a)\right)^{-1}O^\mu_\kappa,$$

$$O := (O^\mu_\nu) \in \text{SO}(4).$$

(2.7)

There is an invariant for two points $A(a^\mu)$ and $B(b^\mu)$ on $B_I$

$$\Delta^2_E(a, b) = -l^2\left[\sigma_E^{-1}(a)\sigma_E^{-1}(b)\sigma_E^2(a, b) - 1\right].$$  

(2.8)

The arc length of the geodesic segment $AB$ connecting $A$ and $B$ is

$$L(A, B) = \int_A^B d s_E = l \arcsin(|\Delta_E(a, b)|/l).$$  

(2.9)
It may also be written as
\[ L(A, B) = \int_A^B ds_E = \int_{\alpha^0}^{\beta^0} dx^0 \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \] (2.10)
where \( \dot{x}^\mu := dx^\mu/dx^0 \) and \( g_{\mu\nu} \) is the Beltrami metric. From its variation, it follows a geodesic equation or alternatively
\[ \delta L(A, B) = \int_{\alpha^0}^{\beta^0} dx^0 \left\{ R^2 R_{\lambda\kappa\mu\nu} \frac{dx^\lambda}{ds} \frac{dx^\kappa}{ds} \frac{d}{dx^0} \left( \frac{dx^\mu}{dx^0} \right) \delta x^i \right. \\
+ \left. \frac{d}{dx^0} \left( \frac{dx^\mu}{dx^0} g^{\mu \nu} \delta x^\nu \right) \right\}, \quad i = 1, 2, 3, \] (2.11)
where \( R_{\lambda\kappa\mu\nu} \) is Riemann curvature tensor. \( \delta L(A, B) = 0 \) and \( \delta x^i = 0 \) on the initial and final hypersurfaces give rise to the equation of motion
\[ \frac{d}{ds} \frac{dx^i}{dx^0} = 0, \] (2.12)
which results in
\[ \frac{dx^i}{dx^0} = \text{consts.} \] (2.13)
Integrating it further gives
\[ x^i = \alpha^i x^0 + \beta^i; \quad \alpha^i, \beta^i = \text{consts.} \] (2.14)
Namely, the geodesics as shortest curves in the Beltrami model of the Riemann sphere are straight lines in linear form. In fact, equation (2.13) can be obtained directly from the first integral of the geodesic equation,
\[ q^\mu := \sigma^{-1}_E(x) \frac{dx^\mu}{ds} = \text{consts.} \] (2.15)
It is important that all these results are invariant under the FLT's (2.7) of \( \text{SO}(5) \) and globally true in the Beltrami atlas patch by patch.

In terminology of the projective geometry, Beltrami coordinates are inhomogenous projective ones. But, antipodal identification should not be taken here in order to preserve orientation. The great circles on the 4D Riemann-sphere (2.1) are mapped to straight lines (2.14) in \( B_1 \), and vice versa.

### 2.2. The Beltrami model of dS-spacetime via an inverse Wick rotation

From an inverse Wick rotation \([4, 5]\) of the 5D embedded space, which turns \( \xi^0 \) to be timelike, the Riemann sphere \( S^4 \subset E^5 \) (2.1) becomes the dS-hyperboloid \( H_t \subset M^{1,4} \):
\[ H_t : \eta_{AB} \xi^A \xi^B = \xi^i \mathcal{J} \xi = -l^2, \] (2.16)
\[ ds^2 = \eta_{AB} d\xi^A d\xi^B = d\xi^i \mathcal{J} d\xi, \] (2.17)
\[ \partial_P H_t : \eta_{AB} \xi^A \xi^B = \xi^i \mathcal{J} \xi = 0, \quad A, B = 0, \ldots, 4, \] (2.18)
where \( \mathcal{J} = (\eta_{AB}) = \text{diag}(1, -1, -1, -1, -1) \) and \( \partial_P \) denotes the projective boundary. They are invariant under dS-group \( \text{SO}(1, 4) \):
\[ \xi \to \xi^i = S\xi, \quad S^t \mathcal{J} S = \mathcal{J}, \quad \forall S \in \text{SO}(1, 4). \] (2.19)
The great circles on the Riemann sphere (1.1) now transfer to a kind of uniform ‘great circular’ motion of a particle with mass $m_l$ characterized by a conserved 5D angular momentum on $H_l \subset M^{1,4}$:

$$\frac{dL^{AB}}{ds} = 0, \quad L^{AB} := m_l \left( \xi_1 \frac{dx^B}{ds} - \xi_B \frac{dx^A}{ds} \right),$$

(2.20)

with an Einstein-like formula for mass $m_l$

$$- \frac{1}{2H^2} L^{AB} L_{AB} = m_l^2, \quad L_{AB} = \eta_{AC} \eta_{BD} L^{CD}. \quad (2.21)$$

Further, a simultaneous 3-hypersurface of $\xi^0 = \text{const}$ is an expanding $S^3$:

$$\delta_{ij} \xi^i \xi^j = l^2 + (\xi^0)^2, \quad I, J = 1, \ldots, 4,$$

(2.22)

d$d^2 = \delta_{ij} \xi^i \xi^j$. The generators of $so(5)$-algebra become the ones of dS-algebra $so(1, 4)$, which read ($\hbar = 1$)

$$\hat{L}_{AB} = \frac{1}{i} \left( \xi_A \frac{\partial}{\partial \xi^B} - \xi_B \frac{\partial}{\partial \xi^A} \right), \quad \xi_A = \eta_{AB} \xi^B,$$

(2.23)

or the Killing vector fields (without $i$) on dS-hyperboloid. They are globally defined on the dS-hyperboloid.

The first Casimir operator of the algebra is

$$\hat{C}_1 := -\frac{1}{2} l^{-2} \hat{L}_{AB} \hat{L}^{AB}, \quad \hat{L}^{AB} := \eta^{AC} \eta^{BD} \hat{L}_{CD}, \quad (2.24)$$

with eigenvalue $m_l^2$, which gives rise to the classification of the mass $m_l$.

It is clear that the BdS-space can be given either by the Beltrami model of the Riemann sphere via an inverse Wick rotation or by the generalized ‘gnomonic’ projection without antipodal identification of the dS-hyperboloid $H_l \subset M^{1,4}$.

Thus, there exists a Beltrami coordinate atlas covering dS-space patch by patch. On each patch, there are conditions and the Beltrami metric with $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1, -1)$:

$$\sigma(x) = \sigma(x, x) := 1 - l^{-2} \eta_{\mu\nu} x^\mu x^\nu > 0, \quad (2.25)$$

$$ds^2 = [\eta_{\mu\nu} \sigma^{-1}(x) + l^{-2} \eta_{\mu\nu} x^\mu x^\nu \sigma^{-2}(x)] dx^\mu dx^\nu. \quad (2.26)$$

Under FLTs of $SO(1, 4)$,

$$x^\mu = x^\mu (\sigma(a)) = \pm \sigma^{1/2}(a) \sigma^{-1}(a, x)(x^\nu - a^\nu) D^\mu_v, \quad D^\mu_v = L^\mu_v + l^{-2} \eta_{\mu\nu} a^\nu \sigma^{-1}(a + \sigma^{1/2}(a))^{-1} L^\mu_v,$$

(2.27)

$$L := (L^\mu_v) \in SO(1, 3),$$

which transform a point $A(a^\mu)$ with $\sigma(a^\mu) > 0$ to the origin, the system $S(x)$ transforms to $\bar{S}(\bar{x})$ and inequality (2.25) and equation (2.26) are invariant.

In such a BdS, the generators of FLTs or the Killing vectors read

$$\hat{\eta}_\mu = (\delta_{\mu\nu} - l^{-2} x_\mu x^\nu) \partial_\nu, \quad x_\mu := \eta_{\mu\nu} x^\nu,$$

(2.28)

$$\hat{L}_{\mu\nu} = x_\mu \hat{q}_\nu - x_\nu \hat{q}_\mu = x_\mu \partial_\nu - x_\nu \partial_\mu \in so(1, 3),$$

(2.29)

and form an $so(1, 4)$ algebra

$$[\hat{q}_\mu, \hat{q}_\nu] = l^{-2} \hat{L}_{\mu\nu}, [\hat{L}_{\mu\nu}, \hat{q}_\mu] = \eta_{\nu\kappa} \hat{L}_{\mu\kappa} - \eta_{\mu\kappa} \hat{L}_{\nu\kappa} - \eta_{\mu\nu} \hat{L}_{\kappa\kappa},$$

(2.30)

$$[\hat{L}_{\mu\nu}, \hat{L}_{\kappa\lambda}] = \eta_{\mu\kappa} \hat{L}_{\nu\lambda} - \eta_{\mu\lambda} \hat{L}_{\nu\kappa} + \eta_{\nu\kappa} \hat{L}_{\mu\lambda} - \eta_{\nu\lambda} \hat{L}_{\mu\kappa}.$$
Thus, for a set of ‘circular observers’, a set of observables that consist of a 5D angular momentum conserve for the uniform ‘great circular’ motions, satisfying an Einstein-like formula (2.21) corresponding to (2.24), etc.

It should be mentioned that for the dS-hyperboloid \( H_1 \subset M^{1,4} \), the above BdS-space with Beltrami coordinates defined by equation (2.4) is with respect to a timelike ‘gnomonic’ projection, i.e. \( x^0 \) is a temporal coordinate, and that there might be other kinds of Beltrami-like coordinate systems with respect to the null and spacelike ‘gnomonic’ projections, respectively (see appendix A). However, only the timelike ‘gnomonic’ projection discussed here should be taken, if we require that the transformations of \( \text{SO}(1, 3) \), as a subgroup of the dS group, acting on the coordinates, take the same form as that of the homogenous Lorentz group in the Minkowski case or that under \( R \to \infty \) all dS-transformations be back to the Poincaré ones.

It should also be noted that the 4D Riemann sphere \( S^4 \) may be regarded as an instanton with an Euler number \( e = 2 \) in the sense that it is a solution of the Euclidean version of gravitational field equations, which provides a tunneling scenario of BdS [5]. It will be shown that this is the case in a simple model of dS-gravity [24, 27] as in general relativity.

3. de Sitter special relativity

It has been shown that the dS special relativity can be formulated based on the principle of relativity [2–5] and postulate on universal invariants \( c \) and \( R [4–6] \). In fact, the most important properties in dS special relativity can be given in a BdS-model with \( l = R \) [4–6].

3.1. The law of inertia and the generalized Einstein’s formula

Why is there a law of inertia in the dS-spacetime? This can also be seen from another angle: the most general transformations are among inertial systems, in which a free particle moves of inertia with constant coordinate velocities.

In fact, Umow, Weyl and Fock studied what the most general transformations are between two inertial systems (see, e.g. [32]). If in an inertial system \( S(x) \), an inertial motion is described by

\[
\begin{align*}
  x^i &= x^i_0 + v^i (t - t_0), \\
  v^i &= \frac{dx^i}{dt} = \text{consts},
\end{align*}
\]

and in another inertial system \( S'(x') \), it may be described by

\[
\begin{align*}
  x'^i &= x'^i_0 + v'^i (t' - t'_0), \\
  v'^i &= \frac{dx'^i}{dt'} = \text{consts}.
\end{align*}
\]

Fock proved [32] that the most general transformations between two systems, \( x'^\mu = f^\mu(t, x^i) \), are fractional linear with the common denominator, which is the same as the FLTs in (2.27). In appendix B, we present a new proof for this theorem.

Thus, there is a law of inertia in Beltrami coordinates (2.4) of dS with the curvature radius \( l = R \): The free particles and light signals, without undergoing any unbalanced forces, should keep their uniform motions along straight worldlines in the linear forms in dS-space.

For such a free particle, there is a set of conserved observables along a timelike geodesic:

\[
\begin{align*}
  p^\mu &= \sigma^{-1}(x)m_R \frac{dx^\mu}{ds}, \\
  \frac{dp^\mu}{ds} &= 0, \\
  L^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu, \\
  \frac{dL^{\mu\nu}}{ds} &= 0.
\end{align*}
\]
These are just the inverse Wick rotation counterparts of (2.15) as the pseudo-4-momentum, pseudo-4-angular momentum of the particle and constitute a conserved 5D angular momentum (2.20).

In terms of $p^\mu$ and $L^{\mu\nu}$, the Einstein-like formula (2.21) becomes

$$-\frac{1}{2R^2} \mathcal{L}^{AB} \mathcal{L}_{AB} = E^2 - p^2c^2 - \frac{c^2}{2R^2} L^2 \epsilon_{(1,3)} = m_R^2 c^4,$$

and

$$E^2 = m_R^2 c^4 + p^2c^2 + \frac{c^2}{R^2} j^2 - \frac{c^4}{R^2} k^2,$$

with energy $E$, momentum $p^i$, $p_i = \delta_{ij} p^j$, ‘boost’ $k^j$, $k_i = \delta_{ij} k^j$ and 3-angular momentum $j^i$, $j_i = \delta_{ij} j^j$. It can be proved that they are Noether’s charges with respect to the Killing vectors (2.28).

It should be emphasized that since the generators in (2.23) are globally defined on the dS-hyperboloid, they should also be globally defined in the Beltrami atlas patch by patch. Thus, there is a set of globally defined ten Killing vectors in the Beltrami atlas, and correspondingly there is a set of ten Noether’s charges forming a 5D angular momentum $\mathcal{L}^{AB}$ in (2.20) globally in the Beltrami atlas, though the physical meaning of each Noether’s charge depends on the Beltrami coordinate patch used. This issue will be explored in detail elsewhere.

Note that $m_R^2$ is now the eigenvalue of the first Casimir operator of the dS-group, the same as that in (2.24) with $l = R$. And also note that we can introduce the Newton–Hooke constant $\nu$ [8] and link the curvature radius $R$ with the cosmological constant $\Lambda$:

$$\nu := \frac{c}{R} = \left(\frac{3c^2}{\Lambda}\right)^{1/2}, \quad \nu^2 \sim 10^{-35} s^{-2},$$

It is very tiny if $\Lambda$ is taken as the present value of the dark energy [8]. Thus, all local experiments, whose characteristic spatial and temporal scales are much smaller than $R$ and $\nu^{-1}$, respectively, cannot distinguish between Einstein’s special relativity and dS special relativity. Namely, all experiments that prove Einstein’s special relativity cannot exclude the dS special relativity.

The interval and thus the light cone can be well defined as the inverse Wick rotation counterparts of (2.8) and (2.9).

### 3.2. Two kinds of simultaneity and closed dS-cosmos

Different from Einstein’s special relativity, there are two kinds of simultaneity, and there is a relation between them reflecting the cosmological significance of dS special relativity.

The first is of the Beltrami time $t = x^0/c$ for the experiments of inertial observers with the principle of relativity. Similar to Einstein’s special relativity, one can define that two events $A$ and $B$ are simultaneous if and only if the Beltrami-time coordinate $x^0$ for the two events are same,

$$a^0 := x^0(A) = x^0(B) = b^0.$$

It is with respect to this simultaneity that free particles move along straight lines with uniform velocities. This simultaneity defines a 3+1 decomposition of the Beltrami metric in one patch [4].

---

8 The dark energy may have other (dynamical) contributions in addition to the cosmological constant. The assumption that the ‘observed’ dark energy be cosmological constant is just a working hypothesis.
The second simultaneity is for the proper time $\tau$ of clocks at rest in Beltrami coordinates with the spatial Beltrami coordinates $x^i = \text{const.}$ Namely, all events are simultaneous if and only if they correspond the same $\tau$.

The two time-scales are related by
\[ \tau = R \sinh^{-1}(R^{-1} \sigma^{-1/2}(x)ct). \] (3.8)

If $\tau$ is chosen as cosmic time by comoving observers with a cosmological principle, the Beltrami metric becomes its Robertson–Walker counterpart [4]
\[ ds^2 = d\tau^2 - d\ell^2 = d\tau^2 - \cosh^2(R^{-1}\tau) d\ell^2_0, \] (3.9)
with $d\ell^2_0$ being a three-dimensional Beltrami metric on an $S^3$ with radius $R$. It is an ‘empty’ accelerated expanding cosmological model with a slightly closed cosmos of order $R$.

Since the Beltrami coordinates are essential for the principle of relativity, the physical simultaneity corresponding to comoving observations should be related to the Beltrami coordinate systems in the most natural and simple way. The family of ‘static’ observers with spatial coordinates $x^i = \text{const.}$ in the Beltrami coordinates should be regarded as the comoving ones with respect to the corresponding proper-time simultaneity. In fact, the relevant kind of comoving coordinates of (3.9) are indeed the most natural and simplest in comparison with other kinds of comoving coordinates having flat or open three-dimensional cosmos, respectively.

If we take $R^2 = 3\Lambda^{-1}$, our universe is then asymptotic to the closed dS-cosmos (3.9). This is a different prediction from standard cosmological models in general relativity, in which there is an input parameter $k$ to characterize whether the universe is closed or not. This prediction does not conflict with the results of WMAP and SDSS [23]9 and can be further checked.

It is important that the dS-group as a maximum symmetry ensures that in dS-space there is the principle of relativity, the cosmological principle of dS-invariance and their relation as well. In dS-space there should be some types of inertial-comoving observers with a kind of two-time-scale clock, measuring the Beltrami time and the cosmic time. This reflects that there is an important link between the principle of relativity and the cosmological principle of dS-invariance. Thus, what should be done for those inertial-comoving observers is merely to switch their timers from cosmic time back to Beltrami time according to their relation and vice versa. Namely, when the observers carry out the experiments in their laboratories, they should take their timers switching Beltrami-time scale on and the cosmic-time scale off so as to act as inertial observers and all observations are of inertia, while when they take cosmic observations on the distant stars and the cosmic effects other than the cosmological constant as test objects they may just switch off the Beltrami-time scale and switch on the cosmic-time scale again, then they should hopefully act as comoving observers.

Similar issues also hold for the AdS-space.

3.3. On thermodynamical properties

In the ordinary approach to dS-space in general relativity, there should be the Hawking temperature and entropy at the horizon [33]. This leads to one of the dS-puzzles: why does dS-space of constant curvature look like a black hole and what is the statistical origin of the entropy?

From the viewpoint of dS special relativity, however, there is a different explanation [7]. From equation (3.8), it is easy to see that for the imaginary Beltrami time and the proper

9 The three year observations of WMAP (Table 12 of the last reference in [23]) show that the central values of $\Omega_k$ systematically lean to negative and the error bar for $1\sigma$ is entirely in the region $\Omega_k < 0$ for the data set WMAP+SDSS LRG.
time, there is no periodicity for the former since both the Beltrami-time axis and its imaginary counterpart are straight lines without coordinate acceleration, but there is such a period in the imaginary proper time that is inversely proportional to the Hawking temperature $\frac{\hbar}{(2\pi R k_B)}$ at the horizon. If the temperature Green’s function can still be applied here, this should indicate that for the horizon in Beltrami coordinates it is at zero temperature and there is no need to introduce entropy. In addition, in dS special relativity, the simultaneity in an inertial frame is defined by its Beltrami time. A ‘test’ mass moving along any worldline with the Beltrami coordinates $x^i = \text{const}$ is of inertia and has vanishing coordinate velocity. Its 4-proper acceleration, 3-(coordinate) acceleration, 4-(coordinate) acceleration and the relative (coordinate) acceleration between the two nearby observers are all zero. Thus, there is no force needed for an inertial observer to hold a ‘test’ mass in place when it tends to the event horizon. Namely, there is no ‘surface gravity’ on the event horizon in dS-spacetime in the viewpoint of dS special relativity [7].

On the other hand, it can be shown that the non-vanishing surface gravity on the dS-horizon given in general relativity is actually a kind of inertial force, which leads to the departure from an inertial motion. For example, the observer near the dS-horizon static with respect to the Killing time is of non-inertia. Therefore, although there are Hawking temperature and entropy $S = \frac{\pi R^2 c^3 k_B}{G \hbar}$ at the horizon in other coordinate-systems such as the static and the Robertson–Walker-like ones (3.9), they are not caused by gravity but by non-inertial motions. This is also an analogy with the relation between Einstein’s special relativity in Mink–space and the horizon in Rindler coordinates. The temperature at the Rindler horizon is caused by non-inertial motion rather than gravity [7].

4. Snyder’s quantized space-time and DSR
Snyder considered the homogenous quadratic form
$$-\eta^2 = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 := \eta^{AB} \eta_A \eta_B < 0,$$  (4.1)
partially inspired by Pauli. It is a model via homogenous (projective) coordinates of a 4D momentum space of constant curvature, a dS-space of momenta. Snyder’s model of (4.1) may be reviewed as a dS-hyperboloid (2.16) in 5D space of momenta, the inverse Wick rotation of (2.1), if we identify $\eta_A$ with $(\alpha/l)\xi_A$ with a common factor $(\alpha/l) \neq 0$ where $\alpha$ is near or equal to the Planck momentum, proportional to the inverse of the parameter $a$ in Snyder’s model:
$$H_a : \eta^{AB} \eta_A \eta_B = -\alpha^2,$$  (4.2)
$$d\sigma^2 = \eta^{AB} d\eta_A d\eta_B,$$  (4.3)
$$\partial_P H_a : \eta^{AB} \eta_A \eta_B = 0.$$  (4.4)
It should be pointed out that we may consider a massive ‘particle’ with a 5-momentum $P_A = m c \eta_A \eta_B$ moving on the dS-hyperboloid embedded in a five-dimensional Mink-space $H_l \subset M^{1,4}$ (2.16). Then, Snyder’s condition (4.1) or (4.2) of momenta indicates that it is satisfied by the particle’s 5-momentum $P_A$ if the particle could be a ‘tachyon’ with $m^2 < 0$:
$$\eta^{AB} P_A P_B = m^2 c^2 < 0.$$  (4.5)
In fact, this is a general issue not only for Snyder’s model, but also for other DSR models, which may be transformed from Snyder’s one (see, e.g. [15]).

Furthermore, a Beltrami model of dS-space of momenta may also be set up on a space of momenta:
$$p_0 := \alpha \frac{\eta_0}{\eta_4} = \alpha \frac{\xi^0}{\xi^4}, \quad p_i := \alpha \frac{\eta_i}{\eta_4} = \alpha \frac{\xi^i}{\xi^4}.$$  (4.6)
Quantum mechanically according to Snyder, in this ‘momentum picture’ the operators for the time coordinate $\hat{t}$ and the space coordinates $\hat{x}^i$ should be given by

$$
\hat{x}^i := i \left( \frac{\partial}{\partial p^i} + \alpha^{-2} p^j \frac{\partial}{\partial p^j} \right),
$$

$$
\hat{x}^0 := i \left( \frac{\partial}{\partial p^0} - \alpha^{-2} p^0 \frac{\partial}{\partial p^0} \right), \quad \hat{x}^0 = c\hat{t}, \quad p^\mu = \eta^{\mu\nu} p_\nu. \tag{4.7}
$$

They form an $\text{SO}(1, 4)$ algebra together with the ‘boost’ $\hat{M}^\mu = \hat{x}^i p^0 + \hat{x}^0 p^i$ and ‘3-angular momentum’ $\hat{L}^\mu = \frac{1}{2} \epsilon^{ijk} (\hat{x}^j p^k - \hat{x}^k p^j)$

$$
[\hat{x}^i, \hat{x}^j] = i \alpha^{-2} \epsilon^{ijk} \hat{L}^k, \quad [\hat{x}^i, \hat{x}^0] = i \alpha^{-2} \hat{M}^i, \quad [\hat{L}^i, \hat{L}^j] = \epsilon^{ijk} \hat{L}^k, \quad [\hat{M}^i, \hat{M}^j] = \epsilon^{ijk} \hat{M}^k, \quad \text{etc.} \tag{4.8}
$$

It should be noted that since $p^\mu$ as inhomogenous (projective) coordinates or Beltrami coordinates $q^\mu$ in (2.15) after an inverse Wick rotation, one coordinate patch is not enough to cover the $\text{dS}$-space of momenta in Snyder’s model [4]. Since the 4D projective space $\mathbb{P}^4$ is not orientable, in order to preserve the orientation the antipodal identification should not be taken. In Snyder’s model, the operators $\hat{x}^\mu$ are just four generators (2.28) of the $\text{dS}$-algebra and $\hat{L}^i, \hat{M}^i$ are just the rest six generators $\hat{L}_{\mu i}$ in (2.29) of Lorentz algebra $\text{so}(1, 3)$. Actually, the algebra (4.8) is the same as (2.30) in the space of momenta.

Similar to Snyder’s model, a quantized space-time in AdS-space of momenta can also be constructed.

As mentioned in the introduction, most DSR models with $\kappa$-Poincaré algebra can be realized geometrically by means of other kinds of coordinate systems on $\text{dS}$/AdS-space of momenta [15] other than the inhomogenous projective coordinates used by Snyder’s model in $\text{dS}$-space of momenta. Thus, the relation between these Snyder-like models and DSR (with $\kappa$-Poincaré algebra) may be given based on the coordinate transformations on 4D $\text{dS}$/AdS-space of momenta.

It is important that in these models, after an inverse Wick rotation, the inverses of ratios in (2.13) become ‘group velocity’ components of some ‘wave-packets’:

$$
\frac{\partial E}{\partial p^\mu} = \text{consts.} \quad E = p^0 c. \tag{4.9}
$$

Thus, there is a kind of uniform motion with component ‘group velocity’ or a law of inertia-like hidden in these models. In particular, when the correspondences of $\beta^i$ in (2.14) vanish, the ‘group velocity’ of a ‘wave-packet’ coincides with its ‘phase velocity’. This is similar to the case for a light pulse propagating in vacuum Mink-spacetime.

On the other hand, the treatment parallel to the one on the $\text{dS}$-space in general relativity will lead to the conclusions that there are ‘temperature’ $\tilde{T}_p$ and ‘entropy’ $\tilde{S}_p$ at the horizon in $\text{dS}$-space of momenta. It is unclear what sense $\tilde{T}_p$ makes in DSR models other than Snyder’s. However, the treatment parallel to the one in the viewpoint of $\text{dS}$ special relativity gives rise to the zero-‘temperature’ $\tilde{T}_p$ without ‘entropy’.

5. Duality between Planck scale and cosmological constant

5.1. A one-to-one correspondence between Snyder’s model and $\text{dS}$ special relativity

It is important to see that there is an interesting and important dual one-to-one correspondence in $\text{BdS}$ between Snyder’s model and $\text{dS}$ special relativity as shown in table 1.

The dual one-to-one correspondence should not be thought to happen accidentally. Since the Planck length and the cosmological constant provide a UV or an IR scale, respectively,
Table 1. Dual correspondence of Snyder’s model and dS special relativity.

| Snyder’s model                  | dS special relativity          |
|---------------------------------|--------------------------------|
| Momentum ‘picture’              | Coordinate ‘picture’           |
| BdS-space of momenta           | BdS-spacetime                  |
| Radius \(α \sim \) Planck scale | Radius \(R \sim \) Cosmic radius |
| Constant ‘group velocity’      | Constant 3-velocity            |
| Quantized space-time            | ‘Quantized’ momenta            |
| \(\hat{\xi}^i, \hat{t}\)      | \(\hat{p}_i, \hat{E}\)        |
| \(\mathcal{T}_p = 0\) without \(\mathcal{S}_p\) | \(T = 0\) without \(S\) |

This correspondence is a kind of the UV–IR connection and should reflect some dual relation between the physics at these two scales.

5.2. A minimum uncertainty-like argument and the dual correspondence

In fact, there is a minimum uncertainty-like argument that may indicate why there should be a one-to-one correspondence between Snyder’s model and dS special relativity.

Quantum mechanically, the coordinates and momenta cannot be determined exactly at the same time if there is an uncertainty principle in the embedded space:

\[
\Delta \xi^I \Delta \eta_I \geq \frac{\hbar}{2},
\]

where \(I = 1, \ldots, 4\) and the sum over \(I\) is not taken. It should be mentioned that here we simply employ the same notation of some observable for the expectation value of its operator over wave function in quantum mechanics.

Limited on the hyperboloid in embedded space, i.e. \(\Delta \xi^I \leq R\), and suppose that the momentum \(\eta_I\) conjugate to \(\xi^I\) also takes values on a hyperboloid. Then, \(\Delta \eta_I \leq \alpha\) and the minimum uncertainty relation implies \( Ra \sim \hbar\). Note that in this subsection \(R\) and \(\alpha\) are free radius parameters for dS-spacetime and dS-space of momenta, respectively. When the size of hyperboloid in the space of coordinates is Planck length, namely,

\[
\eta^{AB} \xi_A \xi_B = -\xi^2_P = -G\hbar c^{-3},
\]

the hyperboloid in the space of momenta then has Planck scale,

\[
\eta^{AB} \eta_{A\eta B} = -E_P^2/c^2 = -\hbar c^3/G < 0,
\]

which is equivalent to Snyder’s relation (4.1). In contrast, when the scale of hyperboloid in the space of momenta is

\[
\eta^{AB} \eta_{A\eta B} = -\frac{\Delta^2}{3},
\]

then we have relation (2.16). Therefore, the minimum uncertainty-like argument indicates a kind of UV–IR connection and the one-to-one correspondence listed in table 1 should reflect some duality between the physics at these two scales.

This argument may be further supported by a ‘tachyon’ dynamics as follows.

Suppose there is a free particle with mass \(m_R\) moving along a timelike curve \(C(\xi)\) on dS-hyperboloid \(H_R \subset M^{1,4}\) with an action

\[
S = \frac{1}{c} \int_{C(\xi)} d\xi L = m_Rc \int_{C(\xi)} d\xi \sqrt{\eta^{AB} \xi_A \xi_B}, \quad \xi^A = \frac{d\xi^A}{d\xi},
\]

(5.5)
where \( \varsigma \), dimension of length, is an affine parameter for the curve \( C(\varsigma) \) and it may be taken as \( \varsigma = s \), and \( L \) the Lagrangian.

A (vertical) variation and the variational principle lead to the Euler–Lagrangian equation equivalent to (2.20) so long as the affine parameter \( \varsigma \) being taken as \( s \). Horizontal variations as Lie derivatives with respect to those Killing vectors show that the 5D angular momentum is Noether’s charges. We may also introduce an action with a Lagrangian multiplier for the embedding condition (2.16) and the results are the same.

In order to find its phase space, we may take a Legendre transformation to get a Hamiltonian and suppose the basic Poisson brackets as usual. Alternatively, we may also take a ‘vertical’ differential \( d_\varsigma \) of the Lagrangian and from \( d_\varsigma^2 L = 0 \), we may further read off the symplectic form and canonical variables (see, e.g., [34]). Here \( d_\varsigma \) is nilpotent and anti-commutative with \( d_\varsigma \): above or it may also be regarded as a nilpotent ‘external’ variation. Thus, we have

\[
d_\varsigma L = \mathcal{E} + \frac{d}{d\varsigma} \Theta,
\]

(5.6)

where

\[
\mathcal{E} = -\eta_{AB} m_R c (\dot{\xi}^A - R^{-2} \xi^A) d_\varsigma \xi^B
\]

(5.7)

called Euler–Lagrange 1-form [34],

\[
\Theta = m_R c \eta_{AB} \dot{\xi}^A d_\varsigma \xi^B
\]

(5.8)

the symplectic 1-form, and \( \xi \) the parameter in the action (5.5). Then, from the nilpotency of \( d_\varsigma \), it follows a necessary and sufficient condition for the preserving of a symplectic form

\[
0 = d_\varsigma^2 L = \frac{d}{d\varsigma} \omega + d_\varsigma \mathcal{E},
\]

(5.9)

where the symplectic structure may be given in the canonical form as

\[
\omega = d_\varsigma \Theta = d_\varsigma P_B \wedge d_\varsigma \xi^B,
\]

(5.10)

Here \( P_B \) are canonical momentum for the canonical formalism. This symplectic structure (5.10) shows that \( (\xi^A, P_B) \) should be a set of canonical variables on the phase space with basic Poisson brackets given by the symplectic structure:

\[
\{ \xi^A, P_B \} = \delta^A_B.
\]

(5.11)

Then, the conserved quantities \( L^{AB} \) for the particle form an \( \mathfrak{so}(1, 4) \) algebra in Poisson bracket:

\[
\{ L^{AB}, L^{CD} \} = \eta^{AC} L^{BD} - \eta^{AD} L^{BC} + \eta^{BD} L^{AC} - \eta^{BC} L^{AD}.
\]

(5.12)

It should be noted that the canonical momenta \( P_B \) for the particle automatically satisfy (4.5), if we would identify \( P_A \) with \( \eta_A \). This should imply that the particle be a ‘tachyon’ on \( H_R \subset M^{1,4} \).

Quantum mechanically, the canonical variables \( (\xi^A, P_B) \) with their Poisson brackets (5.11) become

\[
\hat{P}_B := -i\hbar \frac{\partial}{\partial \xi^B}, \quad [\xi^A, \hat{P}_B] = -i\hbar \delta^A_B,
\]

(5.13)

in a coordinate picture, and the conserved quantities \( L^{AB} \) become the operators \( \hat{L}_{AB} \) of (2.23), which lead to a dS-algebra in Lie bracket. We may also write them in a momentum picture. Furthermore, on a ‘simultaneous’ 3-hypersurface \( \xi^0 = \text{const} \), we may have an uncertainty relation like (5.1) for such a ‘tachyon’ quantum mechanically. Thus, such a ‘tachyon’ dynamics may support why there is a one-to-one dual correspondence of Snyder’s model and dS special relativity.

We may also study such a ‘tachyon’ dynamics directly in a BdS-model. This should lead to the operators \( \hat{t} \) and \( \hat{x}^i \) in (4.7) in a momentum picture given by Snyder in his model.
5.3. The duality between Planck scale and cosmological constant

It is clear that both Snyder’s model and dS special relativity may deal with the motion of relativistic particles. In dS special relativity, the momenta of a particle are quantized and noncommutative, while in Snyder’s model, the coordinates of a particle are quantized and noncommutative. These are listed in table 1 for the one-to-one correspondence between them. As mentioned at the beginning, however, the dimensionless constant $g \sim \ell_P / R$ in (1.1) contains Newton’s gravitational constant and thus should describe some gravity. Therefore, we may make such a conjecture: the physics at these two scales should be dual to each other in some ‘phase’ space and there is in-between gravity of local dS-invariance characterized by $g$.

It is interesting to note that $g^2$ is in the same order of difference between $\Lambda$ and the theoretical quantum ‘vacuum energy’, there is no longer the puzzle in the viewpoint of the dS special relativity and gravity with local dS-invariance.

However, since $\Lambda$ is a fundamental constant as $c$, $G$ and $\hbar$, a further question should be: what is the origin of the dimensionless constant $g$? Is it calculable?

This is just the first question of the ‘top ten’ [35]: ‘Are all the (measurable) dimensionless parameters that characterize the physical universe calculable in principle or are some merely determined by historical or quantum mechanical accident and therefore incalculable?’

It is important to note that there are some hints on the answer for the dimensionless constant $g$. First, among 4D Euclid, Riemann and Lobachevski spaces there is only the Riemann sphere with a non-vanishing 4D topological number. Thus, if there is a quantum tunneling scenario for the Riemann sphere $S^4$ as an instanton of gravity to the dS-space, this could explain why the cosmological constant should be positive, i.e. $\Lambda > 0$. We should show in the next section that in a simple dS-Lorentz model of dS-gravity this is just the case.

Further, if the action of the dS-gravity is of the Yang–Mills type, then its Euclidean version is of a non-Abelian type with local SO(5) symmetry. Thus, due to the asymptotic freedom mechanism, the dimensionless coupling constant, say $g$, should be running and approaching to zero as the momentum tends to infinity. However, for the case of gravity, the momentum could not tend to infinity but the Planck scale so that the Euclidean counterpart of the dimensionless coupling constant should be very tiny.

Needless to say, there is still a long way to go to completely explain this problem.

6. How to describe gravity characterized by $g$?

6.1. Gravity as the localization of relativity with a gauge-like dynamics

In order to explain how to describe gravity, we should first recall Einstein’s equivalence principle, which says roughly that the laws of physics in a freely falling (nonrotating) lift are the same as those in inertial frames in a flat spacetime. Some scholars even say that the spacetime for a freely falling observer will be that of Einstein’s special relativity (known as Mink-spacetime) [36]. It is well known that in a Mink-spacetime the full Poincaré symmetry should hold. However, in Einstein’s general relativity only the local Lorentz symmetry is preserved in a local frame [29, 30, 37]. One may establish the gauge theory of local Poincaré symmetry for gravity [29–31], which may be regarded as the localization of Einstein’s special relativity with full Poincaré symmetry. Similarly, one may establish the gauge theory of local dS/AdS-symmetry based on the localization of dS/AdS special relativity, respectively. In this sense, the gauge theory of gravity should be based on the localization of corresponding special relativity with full symmetry.
Simply speaking, the spacetimes with gravity of local dS-invariance may be described as a kind of (3 + 1)-dimensional umbilical manifolds $M_{3+1} \subset \mathcal{H}_{3+1}$ as sub-manifolds of $(4 + 1)$-dimensional Riemann–Cartan manifolds $M_{4+1}$. In surface theory [38], a surface is umbilical if the normal curvatures at its each point are a constant. A sphere of radius $R$, $S^2 \subset E^3$, is such an umbilical surface, on which

$$g_{\mu\nu} = Rb_{\mu\nu}, \quad (6.1)$$

holds at each point, where $g_{\mu\nu}, b_{\mu\nu}$ are the first and second fundamental form of the manifold, respectively. The localization or the fibration of $S^2 \subset E^3$ may lead to a two-dimensional umbilical manifold $S^2$ with all points being umbilical as a submanifold of a three-dimensional manifold $E^3$. It is in this way the localization of a dS-hyperboloid $H_R \subset M_{4+1}$ could be realized [25].

Let us now illustrate how to construct such an $M_{3+1} \subset \mathcal{H}_{3+1} \subset M_{4+1}$. Suppose $\mathcal{H}_{3+1}$ is also of Riemann–Cartan with signature $-2$. Thus, for any given point $\forall p \in \mathcal{H}_{3+1}$, there is a local Mink-space as the tangent space at the point, $T_p(\mathcal{H}_{3+1})$, and given a vector $N_p = Rn_p$, of norm $R$ at the point with an $n_p$ as the unit base of space $N_p$ normal to $T_p(\mathcal{H}_{3+1})$ with a metric of dS-signature in $M_{4+1}$. Then the space $T_p \times N_p \cong M_p$ is tangent to $M_{4+1}$ at the point. Thus, under local dS-transformations on $T_p \times N_p \cong M_p$ there is a local hyperboloid structure $H_R \subset M_p$ isomorphic to the dS-hyperboloid $H_R \subset M_{4+1}$ in (2.16) at the point $p$ as long as $Rn_p = -r_p$ is taken, where $r_p$ is the radius vector. In fact, all these points consist of the umbilical manifold $M_{3+1} \subset \mathcal{H}_{3+1} \subset M_{4+1}$.

This construction can also be given in an opposite manner. Suppose there is a local $H_R \subset M_{4+1}$ anywhere and anytime tangent to the $M_{4+1}$ such that at a point $p \in \mathcal{H}_{3+1}$, the radius vector $r_p$ with norm $R$ of the $H_R \subset M_{4+1}$ is oppositely normal to the tangent Mink-space of $\mathcal{H}_{3+1}$, i.e. $r_p = -N_p$, at the point. Since this local Mink-space is also tangent to the $H_R \subset M_{4+1}$ at the point so that the point is an umbilical point in $M_{4+1}$. Thus, $\mathcal{H}_{3+1}$ consists of all these umbilical points and is a sub-manifold of the $M_{4+1}$, i.e., $\mathcal{H}_{3+1} \subset M_{4+1}$. Such Riemann–Cartan manifolds $\mathcal{H}_{3+1}$ are called umbilical manifolds with an umbilical structure of $H_R \subset M_{4+1}$ anywhere and anytime.

Therefore, on the co-tangent space $T^*_p$ at the point $p \in \mathcal{H}_{3+1}$ there is a Lorentz frame 1-form:

$$\theta^b = e_b^\mu dx^\mu, \quad \theta^b(\partial_\mu) = e_b^\mu; \quad e_a^\mu e_b^\nu = \delta^\mu_b, \quad e_a^\mu e_a^\nu = \eta^{\mu\nu}, \quad (6.2)$$

with respect to a Lorentz inner product:

$$\langle \partial_\mu, \partial_\nu \rangle = g_{\mu\nu}, \quad \langle e_a, e_b \rangle = \eta_{ab}, \quad \eta_{ab} = \text{diag}(1, -1, -1, -1). \quad (6.3)$$

Here, $\partial_\mu$ is the coordinate base of the tangent space $T_p$. The line element on $\mathcal{H}_{3+1}$ can be expressed as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} \theta^a \theta^b, \quad g_{\mu\nu} = \eta_{ab} e_a^\mu e_b^\nu. \quad (6.4)$$
There is a Lorentz covariant derivative a la Cartan:
\[ \nabla e \theta_b = \partial^e \theta_b (e_b) e_e, \quad \theta_b = B^a b \mu \, dx^\mu, \quad \theta^a_b (\partial_a) = B^a b \mu. \] 
(6.5)

\[ B^{ab} \mu = \eta^{bc} B^c e_{\mu} \in \text{so}(1, 3) \] are connection coefficients of the Lorentz connection 1-form \( \theta^{ab} = \eta^{bc} \theta^a c. \) The torsion and curvature can be defined as
\[ \Omega^a = d \theta^a + \theta^a b \wedge \theta^b = \frac{1}{2} T^a b \mu d x^\mu \wedge d x^v, \]
\[ T^a b \mu = \partial_b e^e - \partial_e e^e + B^e c \mu e^e - B^e c v e^e; \]
(6.6)
\[ \Omega^a = d \theta^a + \theta^a b \wedge \theta^b = \frac{1}{2} F^a b \mu v d x^\mu \wedge d x^v, \]
\[ F^a b \mu v = \partial_b B^a b \mu - \partial_a B^a b \mu + B^a c \mu B^c b \nu - B^a c v B^c b \mu. \]
(6.7)
They satisfy corresponding Bianchi identities.

It is easy to get a metric compatible affine connection \( \Gamma^\lambda_{\mu\nu} \) from the requirement
\[ e^\mu_{\nu} = \partial_b e^b - \Gamma^\lambda_{\mu\nu} e^b \lambda + B^e c \nu e^e - \Gamma^\lambda_{c v \mu} e^e = 0, \]
(6.8)where ‘;’ denotes the covariant derivative with respect to the affine connection \( \Gamma^\lambda_{\mu\nu} \) for spacetime indices and the spin connection \( B^e c \nu e^e \) for Lorentz-frame indices.

As just mentioned, at the point \( p \in \mathcal{H}^{1,3} \), there are a space \( N_p \) and its dual \( N_p^\perp \) normal to \( \mathcal{H}^{1,3} \) with a normal vector \( n \) and its dual \( v \) in \( T_p (\mathcal{M}^{1,3}) \) and \( T_p^* (\mathcal{M}^{1,3}) \), respectively. Namely, both \( \{ \partial_\mu, n; dx^\mu, v \} \) and \( \{ e_\mu, n; \theta^\mu, v \} \) span \( M_p^{1,4} = T_p^{1,3} \times N_p^\perp \) and \( M_p^{1,4*} = T_p^{1,3*} \times N_p^{1*} \). Let these bases satisfy the following conditions in addition to (6.3):
\[ dx^\lambda (n) = \theta^\mu (n) = 0, \quad v (\partial_\mu) = v (e_\mu) = 0, \quad n (v) = 1, \]
(6.9)
\[ \langle e_\mu, n \rangle = \langle \partial_\mu, n \rangle = 0, \quad \langle n, n \rangle = -1. \]
(6.10)
Then, the dS-Lorentz base \( \{ \hat{E}_\lambda \} \) and their dual \( \{ \hat{\Theta}^B \} \) \( A, B = 0, \ldots, 4 \) can be defined as
\[ \{ \hat{E}_\lambda \} = \{ e_\mu, n \}, \quad \{ \hat{\Theta}^B \} = \{ \theta^\mu, v \}. \]
(6.11)
And (6.3) and (6.9) can be expressed as
\[ \hat{\Theta}^B (\hat{E}_\lambda) = \delta^B_\lambda, \quad \langle \hat{E}_A, \hat{E}_B \rangle = \eta_{AB} = \text{diag} (1, -1, -1, -1, -1). \]
(6.12)
Introduce a normal vector \( N = R n \) with norm \( R \):
\[ N = R n = \hat{\xi}^A \hat{E}_A, \quad \langle \hat{\xi}^A \rangle = (0, 0, 0, 0, R), \quad \langle N, N \rangle = -R^2. \]
(6.13)
For the dS-Lorentz base, there are
\[ g_{\mu \nu} = \eta_{AB} \hat{E}_A^\lambda \hat{E}_B^\mu, \quad \eta_{AB} \hat{\xi}^A \hat{\xi}^B = 0, \quad \eta_{AB} \hat{\xi}^A \hat{\xi}^B \hat{\xi}^B = -R^2, \]
where
\[ \hat{E}_A^\lambda = \hat{\Theta}^A (\partial_\mu), \quad \{ \hat{E}_A^\lambda \} = \{ e_\mu, 0 \}. \]
(6.15)
The transformations, which maps \( M_p^{1,4} \) to itself and preserves the inner product, are
\[ \hat{E}_A \rightarrow E_A = (S')^A_B \hat{E}_B, \quad \hat{\Theta}^A \rightarrow \Theta = ( (S')^{-1} )^A_B \hat{\Theta}^B, \quad S^* J S = J, \]
(6.16)
where \( J = (\eta_{AB}), S = (S^A^B) \in \text{SO}(1, 4) \), the superscript \( t \) denotes the transpose as in (2.19). The transformed base is defined as the dS-base and its dual \( E_A, \Theta^B \), respectively:
\[ \Theta^A (E_B) = \delta^A_B, \quad \Theta^A (\partial_\mu) = E^A_\mu, \quad \langle E_A, E_B \rangle = \eta_{AB}. \]
(6.17)
\[ g_{\mu \nu} = \eta_{AB} E_A^\lambda E_B^\mu, \quad \eta_{AB} \hat{\xi}^A E^\mu = 0, \quad \eta_{AB} \hat{\xi}^A \hat{\xi}^B = -R^2. \]
(6.18)
where $E^\mu_\nu$ are the dS-frame coefficients. Obviously, these formulae reflect the local dS-invariance on $\mathcal{H}^{1,3}$ and equations in (6.18) show that there is a local four-dimensional hyperboloid $H^{1,3}_p \subset M^{4,4}_p$ tangent to $\mathcal{H}^{1,3}$ at the point $p$. Thus, (6.18) may be called the local dS-hyperboloid condition.

Now the dS-covariant derivative a la Cartan can be introduced

$$\hat{\nabla}_{\partial^\mu} E_B = \Theta^{C}_{B}(E_A) E_C. \quad (6.19)$$

$$\Theta^{AB} = \eta^{BC} \Theta^{A}_C \in so(1,4)$$

is the dS-connection 1-form. In the local coordinate chart \{x^\mu\} on $\mathcal{H}^{1,3}$,

$$\hat{\nabla}_{\partial^\mu} E_B = \Theta^{C}_{B}(\partial^\mu) E_C = B^{C}_{B\mu} E_C, \quad \hat{\nabla}_n E_B = \Theta^{C}_{B}(n) E_C = B^{C}_{B4} E_C, \quad (6.20)$$

$B^{AB}_\mu = \eta^{BC} B^A_{C\mu}$ and $B^{AB}_4 = \eta^{BC} B^A_{C4}$ denote the dS-connection coefficients on $\mathcal{H}^{1,3}$. There are also the dS-torsion $\Omega^{AB}_\mu$, curvature 2-forms $\Omega^{A}_4$, and their Bianchi identities.

Importantly, in light of the Gauss formula and the Weingarten formula in the theory of surfaces \[38\], from the dS-covariant derivative of the dS-Lorentz base (6.11) with properties of $\theta^a$, $\theta^a_\mu$, it follows a generalization of Gauss formula and Weingarten formula

$$\hat{\nabla}_{\partial^\mu} e_a = \theta^b_\mu (\partial^\mu) e_b - b_{ab} \theta^b_4 (\partial^\mu) n, \quad \hat{\nabla}_n e_a = b^a_b \theta^b_4 (\partial^\mu) e_a. \quad (6.21)$$

Here, $b_{ab}$ denotes the second fundamental form of the hypersurface. Since $\mathcal{H}^{1,3}$ is supposed to be an umbilical manifold, where every point satisfies the umbilical condition (6.1), these formulae read on $\mathcal{H}^{1,3}$

$$\hat{\nabla}_{\partial^\mu} e_a = \theta^b_\mu (\partial^\mu) e_b - R^{-1} \theta^a_4 (\partial^\mu) e_a, \quad \hat{\nabla}_n e_a = R^{-1} \theta^a_4 (\partial^\mu) e_a. \quad (6.22)$$

On the other hand, for the dS-Lorentz base from (6.19) there are

$$\hat{\nabla}_{\partial^\mu} e_a = \check{\Theta}^b_\mu (\partial^\mu) e_b + \check{\Theta}^4_\mu (\partial^\mu) n, \quad \hat{\nabla}_n e_a = \check{\Theta}^a_4 (\partial^\mu) e_a. \quad (6.23)$$

where $\check{\Theta}$ denotes the dS-connection $\Theta$ in the dS-Lorentz frame.

Comparing with (6.22), it follows

$$\check{\Theta}^{ab}_\mu (\partial^\mu) = \theta^{ab}_\mu = B^{ab}_\mu, \quad \check{\Theta}^{a4}_\mu (\partial^\mu) = R^{-1} \theta^a_4 (\partial^\mu) = R^{-1} e^a_\mu. \quad (6.24)$$

Namely, the dS-connection in the dS-Lorentz frame reads

$$B^{AB}_\mu = B^{ab}_\mu, \quad B^{4a}_\mu = R^{-1} e^a_\mu. \quad (6.25)$$

This is just the connection introduced in [24–28]. Here, it is recovered from the local dS-invariance with local dS-connection in the dS-Lorentz frame.

For the dS-connection (6.25), its curvature reads

$$F^{AB}_{\mu\nu} = \left( \begin{array}{cc} B^{ab}_{\mu\nu} & R^{-1} e^a_\mu \\ -R^{-1} e^b_\mu & 0 \end{array} \right) \in so(1,4). \quad (6.26)$$

where $e^a_{\mu\nu} = e^a_{\mu\nu} e^b_\nu - e^a_{\mu\nu} e^b_\nu = \eta_{ab} e^b_{\mu\nu}$, $F^{ab}_{\mu\nu}$ and $T^a_{\mu\nu}$ are curvature (6.7) and torsion (6.6) of the Lorentz connection.

6.3. A simple model of dS-gravity

Now we consider the simple model of such dS-gravitational fields with a gauge-like action. The same dS-connection with different dynamics has also been explored in [39–43].

The total action of the model with source may be taken as

$$S_T = S_{\text{GYM}} + S_M. \quad (6.27)$$
where $S_M$ is the action of the source with minimum coupling, and $S_{GYM}$ the gauge-like action in the Lorentz gauge of the model as follows [24, 26, 27]:

$$S_{GYM} = \bar{h} \frac{g^2}{4} \int_M d^4x \epsilon Tr (\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu})$$

$$= - \int_M d^4x \epsilon \left[ \frac{h}{4g^2} F_{\mu\nu}^{ab} F_{ab}^{\mu\nu} - \chi (F - 2A) - \frac{\chi}{2} T_a^{\mu\nu} T_a^{\mu\nu} \right].$$  \hspace{1cm} (6.28)

Here, $\epsilon = \det (e_a^\mu)$, a dimensionless constant $g$ should be introduced as usual in the gauge theory to describe the self-interaction of the gauge field, $\chi$ is a dimensional coupling constant related to $g$ and $R$, and $F = \frac{1}{2} F_{\mu\nu}^{ab} e_a^\mu e_b^\nu$ the scalar curvature of the Cartan connection, the same as the action in the Einstein–Cartan theory. In order to make sense in comparing with the Einstein–Cartan theory, we should take $R = (3/\Lambda)^{1/2}$, $\chi = c^3/(16\pi G)$ and $g^{-2} = 3\hbar^{-1} \Lambda^{-1}$. $g^2$ defined here is the same order as the one introduced in equation (1.1) in the sense of the duality. This is why we have used the same symbol in different cases.

The field equations can be given via the variational principle with respect to $e_a^\mu$, $B^{ab\mu}$, $T_{\mu\nu}^{a}$, and $T_{\mu\nu}^{G}$:

$$T_{\mu\nu}^{a} := - \frac{1}{4\epsilon} \frac{\delta S_{M}}{\delta e_{a}^{\mu}},$$  \hspace{1cm} (6.31)

$$T_{\mu\nu}^{G} := g^{-2} T_{\mu\nu}^{a} + 2\chi T_{\mu\nu}^{a},$$  \hspace{1cm} (6.32)

are the tetrad form of the stress–energy tensor for matter and gravity, respectively, where

$$T_{\mu\nu}^{F} := - \frac{1}{4\epsilon} \frac{\delta S_{M}}{\delta e_{a}^{\mu}} \int d^4x e Tr (F_{\nu\kappa} F^{\nu\kappa})$$

$$= e_{a}^{\mu} Tr (F^{\mu\lambda} F_{\lambda\kappa}) - 1/4 e_{a}^{\mu} Tr (F^{\nu\sigma} F_{\nu\sigma})$$  \hspace{1cm} (6.33)

is the tetrad form of the stress–energy tensor for curvature and

$$T_{\mu\nu}^{T} := - \frac{1}{4\epsilon} \frac{\delta S_{M}}{\delta e_{a}^{\mu}} \int d^4x e (e^a_b T_{\nu\kappa}^{b} T_{\mu\kappa}^{b})$$

$$= e_{a}^{\mu} T_{\kappa\lambda}^{b} T_{\mu\nu}^{a} - 1/4 e_{a}^{\mu} T_{\kappa\lambda}^{b} T_{\mu\nu}^{a}$$  \hspace{1cm} (6.34)

is the tetrad form of the stress–energy tensor for torsion;

$$S_{Ma}^{\mu} := \frac{1}{2\sqrt{-g}} \frac{\delta S_{M}}{\delta B_{ab\mu}^{a}}$$  \hspace{1cm} (6.35)

and $S_{Gab}^{\mu}$ are spin currents for matter and gravity, respectively. Especially, the spin current for gravity can be divided into two parts:

$$S_{Gab}^{\mu} = S_{Fab}^{\mu} + 2S_{Tab}^{\mu},$$  \hspace{1cm} (6.36)

10 In what follows, the unit of $c = 1$ is used.
where
\[ S_{\text{Fab}}^\mu := \frac{1}{2\sqrt{-g}} \frac{\delta}{\delta B_{\mu}^{ab}} \int d^4x \sqrt{-g} F, \]
\[ = -e_{ab}^{\mu\nu} Y_{\lambda,\nu} e_{ab}^{\lambda\mu} \]  
(6.37)

\[ S_{\text{Tab}}^\mu := \frac{1}{2\sqrt{-g}} \frac{\delta}{\delta B_{\mu}^{ab}} \frac{1}{4} \int d^4x \sqrt{-g} T e_{ab}^{\nu\lambda}, \]
\[ = T_{[a}^{\mu\lambda} e_{b]\lambda}. \]  
(6.38)

are the spin current for curvature \( F \) and torsion \( T \), respectively.

For the case of spinless matter and torsion-free gravity, the curvature \( F_{ab\mu\nu} \) becomes the torsion-free curvature \( R_{ab\mu\nu} \), the gravitational action (6.28) becomes
\[ S_{\text{GYM}} = \frac{1}{4 g^2} \int d^4x e \text{Tr}_{\mathcal{D}'5}(R_{\mu\nu} R^{\mu\nu}) \]
\[ = -\int d^4x e \left[ \frac{1}{4 g^2} R_{ab\mu\nu} R_{ab}^{\mu\nu} - \chi R - 2\Lambda \right]. \]  
(6.39)

and the field equations (6.29) and (6.30) become the Einstein–Yang equations [44] with a \( \Lambda \)-term
\[ R_{ab}^{\mu\nu} = \frac{1}{4} e_{ab}^{\mu\nu} R + \Lambda e_{ab}^{\mu\nu} = -8\pi G (T_{M a}^{\mu} + g_{\mu\nu} T_{M b}^{\nu}), \]  
(6.40)
\[ R_{ab}^{\mu\nu} \parallel_{\nu} = 0. \]  
(6.41)

Now, \( \parallel \) is the covariant derivative compatible with the Christoffel and Ricci rotation coefficients. \( T_{M a}^{\mu} = e_{a}^{\mu} T_{R a}^{\mu} \) the tetrad form of the stress–energy tensor of Riemann curvature \( R_{ab}^{\mu\nu} \in \mathfrak{so}(1, 3) \). It can be shown [45] that
\[ T_{R a}^{\mu} = R_{[ab]k}^{\mu} R_{\lambda k}^{ab} - \frac{1}{4} R_{[ab]k}^{\mu} R_{\lambda k}^{ab} \]
\[ = \frac{1}{2} (R_{\kappa\lambda\mu}^{ab} R_{[ab]k}^{\mu} R_{\lambda k}^{ab} + R_{\kappa\lambda\mu}^{ab} R_{[ab]k}^{\mu} R_{\lambda k}^{ab}) \]
\[ = 2C_{\kappa\lambda\mu\nu} R_{[ab]}^{\mu} + \frac{R_{[ab]}^{\mu}}{3} \left( R_{[ab]}^{\mu} - \frac{1}{4} R_{[ab]}^{\mu} \right). \]  
(6.42)

where \( R_{\kappa\lambda\mu\nu} \) is the Riemann curvature tensor, \( R_{\kappa\lambda\mu\nu}^{*} = \frac{1}{2} R_{\kappa\lambda\sigma\rho}^{*} e_{\sigma\rho}^{\mu\nu} \) is the right dual of the Riemann curvature tensor, \( C_{\kappa\lambda\nu\mu} \) is the Weyl tensor. In the last step in (6.42), the Géhéniau–Debever decomposition for the Riemann curvature,
\[ R_{\mu\nu\kappa\lambda} = C_{\mu\nu\kappa\lambda} + E_{\mu\nu\kappa\lambda} + G_{\mu\nu\kappa\lambda}, \]  
(6.43)

is used [46], where
\[ E_{\mu\nu\kappa\lambda} = \frac{1}{2} \left( g_{\mu\kappa} S_{\nu\lambda} + g_{\mu\lambda} S_{\nu\kappa} - g_{\mu\kappa} S_{\nu\lambda} - g_{\mu\lambda} S_{\nu\kappa} \right). \]  
(6.44)
\[ G_{\mu\nu\kappa\lambda} = \frac{R}{12} \left( g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\kappa} \right). \]  
(6.45)
\[ S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}. \]  
(6.46)

It is easy to see that for the dS-space the ‘energy–momentum’-like tensor in (6.42) vanishes and the dS-space satisfies Einstein–Yang equations (6.40) and (6.41). Therefore, the
4D Riemann sphere is just the gravitational instanton in this model. In fact, it can be proved that any solution of the ordinary vacuum Einstein equation (with $\Lambda$ term) has a vanishing ‘energy–momentum’-like tensor and does exactly satisfy Einstein–Yang equations (6.40) and (6.41). So, this simple model can pass the observation tests on a solar scale. It can also be shown [47, 48] that Einstein–Yang equations admit a ‘Big Bang’ solution. Different from general relativity, $T_{\mu \nu}$ could play a role as ‘dark stuff’.

Returning to the field equations (6.29) and (6.30), all the terms other than the Einstein tensor $R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu}$ that can be picked up from $F_{\mu \nu} - \frac{1}{2} F g_{\mu \nu}$ by means of the relation between the Lorentz connection $B^{ab} \mu$ and the Ricci rotation coefficient $\gamma^{ab} \mu$ should play an important role as some ‘dark stuff’ in the viewpoint of general relativity. Thus, the model may provide a more reasonable framework for the analysis of dark stuff in the ‘precise cosmology’. Furthermore, since the field equations are of the Yang–Mills type, there should be gravitational potential waves of both metric and Cartan connection, including the gravitational metric waves in general relativity.

In fact, this simple model can be viewed as a kind of dS-gravity in a ‘special gauge’ and the 4D pseudo-Riemann–Cartan manifolds should be 4D umbilical manifolds with local dS-space together with ‘gauged’ dS-algebra anywhere and anytime. In this model, there is the $\Lambda$ from the ‘gauge’ symmetry so that it is not just a ‘dummy’ constant at classical level as in general relativity. It is important that even in this simple model the dS-gravity is characterized by a dimensionless coupling constant $g$ with $g^2 \sim G h^{-3} R^{-2}$. This supports the Planck scale-$\Lambda$ duality.

Note that with the help of the connection (6.25) valued in $so(1, 4)$, different gravitational dynamics can be constructed. For example, in MacDowell and Mansouri’s approach [39], the dynamics is different from here. Recently, via a symmetry broken mechanics for the connection (6.25) from the dS-algebra down to the Lorentz algebra has been explored [43] following [40], and via a BF-type topological theory Einstein’s equation with cosmological constant in general relativity has been discussed (see, e.g. [42]).

7. Concluding remarks

With plenty of dS-puzzles, the dark universe as an accelerated expanding one, asymptotic to a de Sitter space with a tiny cosmological constant $\Lambda$ [22, 23] greatly challenges Einstein’s theory of relativity as the foundation of physics in large scale. It is well known that symmetry and its localization play extremely important roles in modern physics. This should also be the case in large scale.

It is important that there should be three kinds of special relativity [2–12] and their contractions [8], and correspondingly there should also be three kinds of theories of gravity as localization of the relevant special relativity with some gauge-like dynamics, respectively. While nature may pick out one of them.

We have shown that there is a one-to-one correspondence between Snyder’s quantized space-time model and dS special relativity. In addition, a minimum uncertainty-like argument indicated by a ‘tachyon’ dynamics. Based on this correspondence and the argument, we have conjectured that there should be a duality in physics at the Planck scale and the cosmological constant, and there is in-between the gravity characterized by a dimensionless constant $g$.

Gravity in-between these two scales should be based on the localization of dS special relativity with a gauge-like dynamics of full localized symmetry characterized by the dimensionless coupling constant. A simple model of dS-gravity in the Lorentz gauge on umbilical manifold of local dS-invariance may support this point of view.
As the asymptotic behavior of our universe is no longer flat, rather quite possibly a Robertson–Walker-like dS-spacetime, our universe may already indicate that dS special relativity and dS-gravity with local dS-invariance should be the foundation of physics in the large scale.

Finally, it should be stressed that the cosmological constant $\Lambda = 3R^{-2}$ is regarded as a fundamental constant and chosen as the dark energy in cosmological observation. The latter is just a working hypothesis. In fact, it is unnecessary to assume that the entire dark energy is composed of the cosmological constant because our universe should be described based on the gravity with local dS-invariance rather than dS special relativity itself.

Acknowledgments

We would like to thank Professors Z Chang, Q K Lu, X C Song, S K Wang, K Wu and M L Yan, and Dr X N Wu for valuable discussions. This work is partly supported by NSFC under grant nos 90403023, 90503002, 10505004, 10547002, 10605005.

Appendix A. Three kinds of Beltrami coordinate systems

First let us take the 2D sphere as an example to illustrate the intuitive geometric picture of what the Beltrami coordinates mean. We often use a method called gnomonic projection to draw a map, i.e., the projection of the earth’s surface from the earth’s center to any plane not passing through the center. Under this kind of projection, the great circles on the earth turn into straight lines on the plane. Now if we consider the pseudo-sphere $H_l \subset M^{1,4}_{11}$ (2.16), the hyperplane of projection has three cases: timelike, spacelike and null. (With a slight abuse of terminology we also call the corresponding (pseudo-)gnomonic projection to be timelike, spacelike and null, respectively.) It can be proved, anyway, that the geodesics on the dS spacetime $H_l$ turn into straight (world) lines on the hyperplane under this kind of (pseudo-)gnomonic projection. These three kinds of hyperplanes of projection correspond to three kinds of Beltrami coordinate systems, which we will describe below.

In the case of a timelike hyperplane of projection, since any timelike hyperplane can be transformed to the hyperplane $\xi^4 = \text{const}$ under an $SO(1, 4)$ transformation, we can only consider the ‘gnomonic’ projection to the hyperplane $\xi^4 = l$, without loss of generality. If the antipodal identification has not been made, the ‘gnomonic’ projection is, basically, two to one: the $\xi^4 > 0$ part and $\xi^4 < 0$ part in the dS spacetime $H_l$ have the same image on the hyperplane $\xi^4 = l$. In order to make a coordinate patch, we first consider the $\xi^4 > 0$ part (called the $U^+_l$ patch), and then increase the number of patches to cover the rest portion. Let $P^+_l$ denote the hyperplane $\xi^4 = l$. Using mathematical language, every point $\xi \in H_l$ with $\xi^4 > 0$ has a one-to-one corresponding point $x \in P^+_l$, whose coordinates are

$$x^\mu \equiv \xi^\mu = l \frac{\xi^\mu}{\xi^4}, \quad \xi^4 = l.$$

(A.1)

These coordinates $x^\mu$ are just the inverse Wick rotated version of the Beltrami coordinates (2.4). From the definition (2.16) of $H_l$, it is easy to show that $x^\mu$ must satisfy the condition (2.25) and the metric (2.26) follows. On the hyperboloid $\sigma(x) = 0$ of two sheets, the metric

$^{11}$ Hereafter the terms timelike, spacelike and null for the hyperplane mean that the signature of the hyperplane is non-definite, negative definite and degenerate, respectively.
Figure 1. The 2D illustration of the Beltrami coordinates from the timelike gnomonic projection, where the hyperbola $\sigma(x) = 0$ is nearly one half of the projective boundary of the dS spacetime, and the interior $\sigma(x) > 0$ contains the points in the dS spacetime.

becomes singular. In fact, $\sigma(x) = 0$ is nearly one half of the projective boundary of $H_l$ [49] (see figure 1). The inverse of the metric (2.26) is

$$g^{\mu\nu} = \sigma(x)(\eta^{\mu\nu} - l^{-2}x^\mu x^\nu).$$ (A.2)

Now we consider the $\xi^4 < 0$ part (called the $U^-_4$ patch). It is obvious that this part can be gnomonically projected to the hyperplane $\xi^4 = -l$ (called $P^-_4$), with the corresponding Beltrami coordinates

$$x^\mu \equiv -l \frac{\xi^\mu}{\xi^4}.$$ (A.3)

It is not difficult to imagine that there should be at least eight such patches $\{U^\pm_\alpha, \alpha = 1, \ldots, 4\}$ to cover the whole $H_l$, with the Beltrami coordinates for $\alpha = 1, 2, 3$ defined as

$$x^\nu \equiv \pm l \frac{\xi^\nu}{\xi^\alpha}, \quad \nu = 0, \ldots, \hat{\alpha}, \ldots, 4, \quad \xi^\alpha \gtrless 0.$$ (A.4)

The hyperplanes of projection corresponding to $U^\pm_\alpha (\alpha = 1, \ldots, 4)$ are timelike. Evidently, these patches can be related by $SO(1, 4)$ transformations.

The Beltrami coordinate systems have an intimate relation with projective geometry. In fact, the Beltrami metric (2.26) can be obtained purely through methods in projective geometry, where the projective boundary (the hyperboloid $\sigma(x) = 0$ of two sheets for the case of timelike ‘gnomonic’ projection above) is just the so-called absolute [49]. The methods in projective geometry clearly show that, under the Beltrami coordinates, timelike, null or spacelike geodesics in the dS spacetime are straight lines crossing, tangent to or apart from the absolute, respectively.

This conclusion holds not only for the case of timelike ‘gnomonic’ projection, but also for the case of null and spacelike ‘gnomonic’ projections that will be discussed below, the only difference being the distinct absolute in each case. The Beltrami coordinates from the timelike ‘gnomonic’ projection has the advantage over those from the null and spacelike ‘gnomonic’ projections that all the relevant expressions tend to their Mink-counterparts under the flat limit $l \to \infty$.

12 The other half of the projective boundary of $H_l$ can be nearly described by $\sigma(x) = 0$ with the corresponding Beltrami coordinates on the $U^-_4$ patch (see below).
Figure 2. The 2D illustration of the Beltrami coordinates from the null gnomonic projection, where the parabola $\varsigma(x) = 0$ is nearly one sheet of the projective boundary of the dS spacetime, and the lower region $\varsigma(x) < 0$ contains the points in the dS spacetime.

In the case of a null hyperplane of projection, it is convenient to define the following ‘light-cone’ coordinates:

$$\xi^+ \equiv \frac{\xi^0 + \xi^4}{\sqrt{2}}, \quad \xi^- \equiv \frac{\xi^0 - \xi^4}{\sqrt{2}}.$$  \hspace{1cm} (A.5)

Also, without loss of generality, we can only consider the ‘gnomonic’ projection to the hyperplane $\xi^- = l$. The corresponding Beltrami coordinates on the $\xi^- > 0$ patch are

$$x^\mu \equiv l \frac{\xi^\mu}{\xi^-}, \quad \mu = +, 1, 2, 3,$$  \hspace{1cm} (A.6)

where $x^\mu$ satisfy

$$\varsigma(x) \equiv 2l x^+ - x^2 < 0.$$  \hspace{1cm} (A.7)

Under this kind of Beltrami coordinates, the metric reads

$$ds^2 = l^2 \varsigma^{-1}(x) dx_+^2 + l^2 \varsigma^{-2}(x)(l dx^+ - x \cdot dx_-)^2.$$  \hspace{1cm} (A.8)

On the paraboloid $\varsigma(x) = 0$, which is nearly one sheet of the projective boundary of the dS spacetime (see figure 2), the metric becomes singular. The corresponding inverse metric is

$$(g^{\mu\nu})^\prime = l^{-3} \varsigma(x) \begin{pmatrix} 2x_+ & x_i \\ x_i & l^2 \delta_{ij} \end{pmatrix}. $$  \hspace{1cm} (A.9)

In the case of a spacelike hyperplane of projection, again we can only consider the ‘gnomonic’ projection to the hyperplane $\xi^0 = l$, with the corresponding Beltrami coordinates on the $\xi^0 > 0$ patch:

$$x^\alpha \equiv l \frac{\xi^\alpha}{\xi^0}, \quad \alpha = 1, 2, 3, 4,$$  \hspace{1cm} (A.10)

where $x^\alpha$ satisfy

$$\hat{\sigma}(x) \equiv 1 - l^{-2} \delta_{\alpha\beta} x^\alpha x^\beta < 0.$$  \hspace{1cm} (A.11)

13 In this paper, we use bold italic letters $x, y$ etc to stand for 3D vectors.
14 The projective boundary of the dS spacetime has topology $S^3 \times \mathbb{Z}_2$. 
Under this kind of Beltrami coordinates, the metric is
\[ ds^2 = [\delta_{\alpha\beta} \hat{\sigma}^{-1}(x) + l^{-2} \delta_{\gamma\delta} x^\gamma x^\delta \hat{\sigma}^{-2}(x)] dx^\alpha dx^\beta. \] (A.12)

On the 3-sphere \( \hat{\sigma}(x) = 0 \) the metric becomes singular, which is one sheet of the projective boundary of the dS spacetime (see figure 3). The corresponding inverse metric is
\[ g^{\alpha\beta} = \hat{\sigma}(x)(\delta^{\alpha\beta} - l^{-2} x^\alpha x^\beta). \] (A.13)

**Appendix B. On the general transformations among inertial motions**

**Theorem** (Fock’s theorem). If, under a coordinate transformation \( (x^i) \rightarrow (x'^i) \) on an \( n \)-dimensional region, a straight line \( x^i(\lambda) = x^i_0 + \lambda v^i \) (where \( v^i \)’s are constants) is always transformed to be another straight line, then there must be constants \( A^i_j, B^i, C_i \) and \( C \) such that
\[ x'^i = A^i_j x^j + B^i, \] (B.1)
and
\[
\begin{vmatrix}
A_1^1 & A_1^2 & \ldots & A_1^n & B^1 \\
A_2^1 & A_2^2 & \ldots & A_2^n & B^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_n^1 & A_n^2 & \ldots & A_n^n & B^n \\
C_1 & C_2 & \ldots & C_n & C
\end{vmatrix} \neq 0. \] (B.2)

**Proof.** As assumed, there should be constants \( v^i \) and \( \lambda' = \lambda'(\lambda) \) such that the straight lines \( x^i(\lambda) = x^i_0 + \lambda v^i \) are transformed to be a straight line \( x'^i(\lambda') = x'^i_0 + \lambda' v^i, \) namely,
\[ x'^i(x_0 + \lambda v) = x'^i_0 + \lambda' v^i. \]

Differentiating it twice with respect to \( \lambda, \) we obtain
\[ v^i v^j \frac{\partial x'^i}{\partial x^j}(x_0 + \lambda v) = v^i \frac{d\lambda'}{d\lambda}. \]

\[ v^i v^j \frac{\partial^2 x'^i}{\partial x^j \partial x^k}(x_0 + \lambda v) = v^i \frac{d^2\lambda'}{d\lambda^2}. \]
Since we can always assume $d\lambda'/d\lambda > 0$, the above equations give

$$v^j v^k \frac{\partial^2 x^q}{\partial x^j \partial x^k} (x_0 + \lambda v) = v^j \frac{\partial x^q}{\partial x^j} \frac{d\lambda' / d\lambda}{d\lambda}. $$

For any point $(x')$, there is always a straight line passing through it, with the tangent vector $v^j \frac{\partial}{\partial x^j}$ at $(x')$. Obviously, $\frac{d\lambda' / d\lambda}{d\lambda}$ at $(x')$ depends not only on $(x')$ but also on $v^j$. Therefore, there will be a function $f(x, v)$ such that

$$v^j v^k \frac{\partial^2 x^q}{\partial x^j \partial x^k} = v^j \frac{\partial x^q}{\partial x^j} f(x, v).$$

Observe the above equation. We see that $f(x, v)$ must be linear for $v^j$. Thus, there are functions $f_i(x) = \frac{\partial x^q}{\partial x^i}$ such that

$$v^j v^k \frac{\partial^2 x^q}{\partial x^j \partial x^k} = v^j \frac{\partial x^q}{\partial x^j} f_i(x).$$  (B.3)

The Jacobian $J(x) = \det (\frac{\partial x^q}{\partial x^i})$ is nonzero everywhere. Its partial derivatives can be obtained in the standard way as

$$\frac{\partial J}{\partial x^i} = \frac{\partial^2 x^q}{\partial x^i \partial x^j} \Delta_j = \frac{\partial^2 x^q}{\partial x^i \partial x^j} \frac{\partial x^j}{\partial x^i} J,$$

where $\Delta_j = \frac{\partial x^q}{\partial x^i} J$ is the algebraic complement of $\frac{\partial x^q}{\partial x^i}$ in $J$. Applying equation (B.3) to the above yields

$$f_k = \frac{1}{n + 1} \partial \ln |J|. \quad \text{(B.4)}$$

From equation (B.3) we obtain

$$\frac{\partial^3 x^q}{\partial x^i \partial x^j \partial x^l} = \frac{\partial x^q}{\partial x^i} \frac{\partial f_k}{\partial x^l} + \frac{\partial x^q}{\partial x^j} \frac{\partial f_l}{\partial x^i} + \frac{\partial x^q}{\partial x^l} \frac{\partial f_i}{\partial x^j} + \frac{\partial x^q}{\partial x^l} \frac{\partial f_j}{\partial x^i} + 2 \frac{\partial x^q}{\partial x^j} \frac{\partial f_i}{\partial x^l} \frac{\partial f_j}{\partial x^k}.$$

Cycle the indices $j, k$ and $l$ in the above and add them together. We thus have an expression of

$$\frac{\partial^3 x^q}{\partial x^i \partial x^j \partial x^l} = \frac{1}{3} \left[ \frac{\partial x^q}{\partial x^j} \left( \frac{\partial f_i}{\partial x^k} + \frac{\partial f_k}{\partial x^i} \right) + \frac{\partial x^q}{\partial x^l} \left( \frac{\partial f_j}{\partial x^i} + \frac{\partial f_i}{\partial x^j} \right) + \frac{\partial x^q}{\partial x^i} \left( \frac{\partial f_k}{\partial x^j} + \frac{\partial f_j}{\partial x^k} \right) \right] + 4 \frac{\partial x^q}{\partial x^j} \frac{\partial f_i}{\partial x^k} f_i + 4 \frac{\partial x^q}{\partial x^k} \frac{\partial f_j}{\partial x^l} f_j + 4 \frac{\partial x^q}{\partial x^l} \frac{\partial f_i}{\partial x^j} f_i f_k. \quad \text{(B.5)}$$

Using the above two equations we obtain

$$\frac{\partial f_k}{\partial x^j} + \frac{\partial f_j}{\partial x^k} = 2 f_j f_k.$$  

Since the two terms on the left-hand side are equal (see, equation (B.4)), we have

$$\frac{\partial f_k}{\partial x^j} = f_j f_k. \quad \text{(B.6)}$$

Using the above equation and equation (B.4), we can verify that the partial derivatives of $f_i |J|^{1/(n+1)}$ are zero. Thus there are constants $C_i$ such that

$$f_i = -C_i |J|^{1/(n+1)}. \quad \text{(B.7)}$$

Substituting it into equation (B.4) we can solve

$$|J|^{1/(n+1)} = C_i x^i + C \quad \text{(B.8)}$$
with $C$ being an integral constant. And, from equations (B.3), (B.4) and (B.6) we can check that
\[ \frac{\partial^2}{\partial x^j \partial x^k} (x^i | J |)^{-1/(n+1)} = 0. \] (B.9)

Therefore, equation (B.1) is satisfied.

It can be verified that, if the coordinate transformation equation (B.1) is satisfied, then a straight line is always transformed to be a straight line. □

The Jacobian of the transformation (B.1) is, in fact,
\[ J(x) = \begin{vmatrix} \frac{A^i}{C^k}x^j + C^i \end{vmatrix}^{n+1}. \] (B.10)

In the proof, we have obtained that $| J | = 1/(C^i x^j + C^i)^{n+1}$. This implies that $D = \pm 1$ in the proof. In fact, we can always require that
\[ D = 1 \] (B.11)
in the transformation (B.1).

References

[1] Snyder H S 1947 Phys. Rev. 71 38
[2] Look K H (Q-K Lu) 1970 Why the Minkowski metric must be used? unpublished
[3] Look K H, Tsou C L (Z.L. Zou) and Kuo H Y (H-Y Guo) 1974 Acta Phys. Sin. 23 225
Look K H, Tsou C L (Z.L. Zou) and Kuo H Y (H-Y Guo) Nature (Shanghai, Suppl.)
Guo H-Y 1977 Kexue Tongbao (Chin. Sci. Bull.) 22 487 (in Chinese)
Guo H-Y 1982 Proc. 2nd Marcel Grossmann Meet. on GR ed R Ruffini (Amsterdam: North-Holland) p 801
Guo H-Y 1989 Nucl. Phys. B (Proc. Suppl.) 6 381
[4] Guo H-Y, Huang C-G, Xu Z and Zhou B 2004 Mod. Phys. Lett. A 19 1701
Guo H-Y, Huang C-G, Xu Z and Zhou B 2004 Phys. Lett. A 331 1
[5] Guo H-Y, Huang C-G, Xu Z and Zhou B 2005 Chin. Phys. Lett. 22 2477
Guo H-Y, Huang C-G, Tian Y, Xu Z and Zhou B 2005 Acta Phys. Sin. 54 2494 (in Chinese)
[7] Guo H-Y, Huang C-G and Zhou B 2005 Europhys. Lett. 72 1045
[8] Huang C-G, Guo H-Y, Tian Y, Xu Z and Zhou B 2004 Int. J. Mod. Phys. A 22 2535 (Preprint hep-th/0403013)
Tian Y, Guo H-Y, Huang C-G, Xu Z and Zhou B 2005 Phys. Rev. D 71 044030
[9] Lu Q K 2005 Commun. Theor. Phys. 44 389 Dirac’s conformal spaces and de Sitter spaces, in memory of the 100th anniversary of Einstein special relativity and the 70th anniversary of Dirac’s de Sitter spaces and their boundaries MCM-Workshop Series vol 1
[10] Guo H-Y, Huang C-G, Tian Y, Xu Z and Zhou B 2006 Snyders quantized space-time and de Sitter invariant relativity Preprint hep-th/0607016 (Front Phys. China at press)
Guo H-Y 2006 Invited talk given at the International Workshop on Noncommutative Geometry and Physics (Beijing, 7–10 November 2005) (Preprint hep-th/0607017)
[11] Guo H-Y 2006 On principle of inertia in closed universe Preprint hep-th/0611341 (Phys. Lett. B at press)
[12] Zhou B and Guo H-Y 2006 Differential Geometry and Physics (Proc. of 22rd ICDGMP, Tianjin, Aug. 20–25, 2005) ed M L Ge and W Zhang (Singapore: World Scientific) pp 503–12 (Preprint hep-th/0512235)
Guo H-Y, Zhou B, Tian Y and Xu Z 2007 Phys. Rev. D 75 026006
[13] Yan M L et al 2007 Comm. Theor. Phys. 48 27 (Preprint hep-th/0512319)
[14] Amelino-Camelia G 2002 Int. J. Mod. Phys. D 11 35
Amelino-Camelia G 2001 Phys. Lett. B 510 255
[15] Kowalski-Glikman J 2002 Phys. Lett. B 547 291
Kowalski-Glikman J and Nowak S 2003 Class. Quantum Grav. 20 4799
[16] de Sitter W 1917 Mon. Not. R. Astron. Soc. 78 3
[17] Beltrametti E 1868 Opere Mat. I 374
[18] See, e.g. Rosenfeld B A 1988 A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space (New York: Springer)
Aldrovandi R, Beltrán Almeida J P and Pereira J G 2006 de Sitter special relativity Class. Quantum Grav. 24 1385 (Preprint gr-qc/0606122)

Aldrovandi R, Beltrán Almeida J P and Pereira J G 2007 Some implications of the cosmological constant to fundamental physics Preprint gr-qc/0702065

Riess A G et al 1998 Astron. J. 116 1009

Perlmutter S et al 1999 Astrophys. J. 517 565

Spergel D N et al 2003 Astrophys. J. (Suppl.) 148 1

Tegmark M et al 2004 Phys. Rev. D 69 103501

Spergel D N et al 2006 Preprint astro-ph/0603449

Wu Y S, Li G D and Guo H-Y 1974 Kexue Tongbao (Chin. Sci. Bull.) 19 509

An I, Chen S, Zou Z L and Guo H-Y Kexue Tongbao (Chin. Sci. Bull.) 19 31

Guo H-Y 1976 Kexue Tongbao (Chin. Sci. Bull.) 21 31

Townsend P 1977 Phys. Rev. D 15 2795

Tseytlin A A 1982 Phys. Rev. D 26 3327

Yan M L, Zhao B H and Guo H-Y 1979 Kexue Tongbao (Chin. Sci. Bull.) 24 587

Yan M L, Zhao B H and Guo H-Y 1984 Acta Phys. Sin. 33 1377, 1386 (in Chinese)

Wise D K 2006 MacDowell–Mansouri gravity and Cartan geometry Preprint gr-qc/0611154

Kibble T W B 1961 J. Math. Phys. 2 212

Held F W, von der Heyde P, Kerlick G D and Nester J M 1976 Rev. Mod. Phys. 48 393 and references therein

Zou Z L et al 1979 Sci. Sin. 22 628 (in Chinese)

Kuo H Y 1982 Proc. 2nd M. Grossmann Meet. on GR (1979) ed R Ruffini (Amsterdam: North-Holland) p 475

Fock V 1964 The Theory of Space-Time and Gravitation (Oxford: Pergamon) and references therein

See, e.g. Gibbons G W and Hawking S W 1977 Phys. Rev. D 15 2738

Guo H-Y and Wu K 2003 J. Math. Phys. 44 5978

Gross D 2000 Millennium Madness: Physics Problems for the Next Millennium, Strings (University of Michigan, 10–15 July) http://feynman.physics.lsa.uchicago.edu/strings2000

Peacock J A 1998 Cosmological Physics (Cambridge: Cambridge University Press)

Cartan E 1922 C. R. Acad. Sci., Paris 174 437, 593, 734, 857, 1104

Spivak M 1999 A Comprehensive Introduction to Differential Geometry vol 3 3rd edn (Boston, MA: Publish or Perish)

MacDowell S W and Mansouri F 1977 Phys. Rev. Lett. 38 739

MacDowell S W and Mansouri F 1977 Phys. Rev. Lett. 38 1376 (erratum)

Stelle K S and West P C 1980 Phys. Rev. D 21 1466

Wilczek F 1998 Phys. Rev. Lett. 80 4051

Freidel L and Starodubtsev A 2005 Quantum gravity in terms of topological observables Preprint hep-th/0501191

Leclerc M 2006 Ann. Phys., NY 321 708

Guo H-Y, Wu Y S and Zhang Y Z 1973 Kexue Tongbao (Chin. Sci. Bull.) 18 72 (in Chinese)

Wu Y S, Zou Z L and Chen S 1973 Kexue Tongbao (Chin. Sci. Bull.) 18 119 (in Chinese)

Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 1980 Exact Solutions of Einstein’s Field Equations (Cambridge: Cambridge University Press)

Huang P and Guo H Y 1974 Kexue Tongbao (Chin. Sci. Bull.) 19 512

Huang P 1976 Kexue Tongbao (Chin. Sci. Bull.) 21 69 (in Chinese)

Han J-C 1981 Acta Astrophys. Sin. 1 131 (in Chinese)

Han J-C 1981 Chin. Astron. Astrophys. 5 357 S

Guo H-Y, Huang C-G, Tian Y, Xu Z and Zhou B in preparation