Oscillation of damped second order quasilinear wave equations with mixed arguments *

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Abstract: Following the previous work [1], we investigate the impact of damping on the oscillation of smooth solutions to some kind of quasilinear wave equations with Robin and Dirichlet boundary condition. By using generalized Riccati transformation and technical inequality method, we give some sufficient conditions to guarantee the oscillation of all smooth solutions. From the results, we conclude that positive damping can “hold back” oscillation. At last, some examples are presented to confirm our main results.

Keywords: Oscillation, quasilinear, wave equations, damped, mixed arguments.

Mathematics Subject Classification 2010: 34C10, 35L72, 81R20.

1 Introduction

Following the work [1], we continue to consider the oscillatory nature of quasilinear wave equation of the form

\[
\begin{align*}
    r(t)u^{\alpha-1}u_{tt} + p(x,t)u^{\alpha/2}u_t^2 + \hat{p}(x,t)u^{\alpha-1}u_t + f(u,x,m(t)) \\
    = a(t)\Delta u(x,t) + \sum_{k=1}^{k_1} a_k(t)\Delta u(x,\eta(t)), \quad (x,t) \in G,
\end{align*}
\]

where \( \alpha \) is the ratio of positive odd integers, \( G = \Omega \times (t_0, \infty), \ t_0 > 0, \ \Omega \) is a bounded domain in \( \mathbb{R}^N \) with a piecewise smooth boundary \( \partial \Omega \), and \( \Delta u(x,t) = \sum_{j=1}^{N} \frac{\partial^2 u(x,t)}{\partial x_j^2} \). Here the first order term \( \hat{p}(x,t)u^{\alpha-1}u_t \) is added to consider the damping effect on the equation. Assumptions on the coefficients \( (r, a, a_k) \) and functions \( (f, m \text{ and } \eta) \) are the same as in [1]. For convenient, we list them here:

(H1) \( r(t) \in C^1((t_0, \infty), \mathbb{R}_+) \); \( a(t), a_k(t) \in C((t_0, \infty), \mathbb{R}_+) \), \( k \in I_s = \{1, 2, \ldots, s\} \); \( p(x,t) \in C(G, R_+) \) and \( p(x,t) \geq (\alpha - 1)r(t) \) for \( t > t_0 \); \( \hat{p}(x,t) \in C(G, R) \);

(H2) \( m(t) \in C((t_0, \infty), R_+) \), \( \eta(t) \in C((t_0, \infty), R_+) \), \( m(t), \eta(t) \) is increasing, \( m(t) \geq t, \) and \( \lim_{t \to \infty} m(t) = \lim_{t \to \infty} \eta(t) = \infty \);

(H3) \( f \in C(R \times \Omega \times (t_0, \infty), R) \), \( f(u,x,t) > q(x,t)u^\alpha(x,t) \geq q(t)u^\alpha(x,t) \) for some positive function \( q(x,t) \in C(G, R_+) \) and \( q(t) = \min_{x \in \Omega} q(x,t) \).

When \( \hat{p}(x,t) \equiv 0, \) the authors in [1] give three oscillation criteria for quasilinear wave equation [1] under the Robin boundary condition:

\[
\frac{\partial u(x,t)}{\partial \gamma} + \psi(x,t)u(x,t) = 0, \quad (x,t) \in \partial \Omega \times R_+,
\]

and the Dirichlet boundary condition

\[
u(x,t) = 0, \quad (x,t) \in \partial \Omega \times R_+,
\]

\*This work is supported in part by the National Natural Science Foundation of China (Grant No. 11671237)
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where \( \gamma \) is the unit exterior normal vector to \( \partial \Omega \) and \( \psi(x,t) \) is a nonnegative continuous function on \( \partial \Omega \times R_+ \). In this paper, we are interested in the influence of the additional (damped) term \( \dot{p}(x,t)u^{\alpha-1}u_t \) on the oscillation of equation (1.1) (or (1.1) (1.3)).

As we all know, oscillation phenomena is very common in our life and describes a form of material movement. While damping is the characteristic that the oscillation amplitude decreases gradually due to external effects or the inherent causes of the system itself, which plays a very important role in control theory, machinery, aviation, mechanics and other fields. In 2015, some oscillation theorems of certain nonlinear fractional partial differential equation with damping were established in [2] by using differential inequality method as well as integral average method. However, there are no deviating arguments considered. In 2016, the authors [3] gave new characterizations on the oscillation of third-order nonlinear damped delay differential equations.

In this paper, we consider the oscillation of the damped equation (1.1) with advanced (or delayed) arguments. To gain our aim, we first construct two new space variable integral average methods. From the results in this paper, we are convinced that positive damping can “hold back” the oscillation in some sense. As far as we know, there is no any results on the oscillation of quasilinear differential equation with damping. Then some oscillation criteria are obtained for nonlinear advanced (or delayed) arguments. To gain our aim, we first construct two new space variable integral average methods. From the results in this paper, we are convinced that positive damping can “hold back” the oscillation in some sense. As far as we know, there is no any results on the oscillation of quasilinear damped wave equations with mixed arguments. We refer [4–13] for more related results.

Before demonstrate our main results, we first give the definition of oscillation:

**Definition 1.1.** The solution \( u(x,t) \) of problem (1.1), (1.2) (or (1.1), (1.3)) is said to be oscillatory in the domain \( \Omega \) if for any positive number \( \mu \), there exist \( t > \mu \), \( x_1, x_2 \in \Omega \) such that \( u(x_1, t) > 0 \), \( u(x_2, t) < 0 \). Otherwise the solution \( u(x,t) \) is called nonoscillatory. If all of smooth solutions are oscillatory, we say the equation is oscillatory.

### 2 Main results

Now, we present the main results of this paper.

**Theorem 2.1.** Assume that (H1)-(H3) hold, \( \dot{p}(x,t) \geq 0 \) for any \( (x,t) \in G \), \( Q(t) = \frac{\alpha q(m(t))}{r(t)} \), \( p_1(t) = \frac{\min_{x \in \Omega} \dot{p}(x,t)}{r(t)} \). If
\[
\limsup_{t \to \infty} \int_{t_0}^{t} e^{-\int_{s}^{t} p_1(\tau)d\tau}ds = \infty,
\]
\[
\limsup_{t \to \infty} \int_{t_0}^{t} sQ(s)ds = \infty,
\]
and for sufficiently large \( t \geq T \geq t_0 \),
\[
m^{-1}(t) \int_{T}^{t} (s-T)Q(s)ms(s)ds + (t-T) \int_{t}^{\infty} Q(s)ds > 1.
\]
Then every smooth solution \( u(x,t) \) of (1.1) (1.2) (or (1.1) (1.3)) is oscillatory in \( \Omega \).

**Proof.** By (1.1) and (H1) (H3), we have
\[
r(t)(u^{\alpha-1}u_t + (\alpha - 1)u^{\alpha-2}u_t^2) + \dot{p}(x,t)u^{\alpha-1}u_t + q(m(t))u^\alpha(x,m(t))
\]
\[
< a(t)\Delta u(x,t) + \sum_{k=1}^{s} a_k(t)\Delta u(x,\eta(t)).
\]
Similar to the proof in [1], we get
\[
u''(t) + p_1(t)v'(t) < -Q(t)v(m(t)),
\]
2.1. Besides (2.1) we assume that there exists a function \( b \) and a function \( \hat{b} \) such that \( v^b(t) > T \). Then every smooth solution \( u(t) \) satisfying \( \beta(t) \leq \frac{b}{v(t)} \) for any \( t \geq T \). If we assume that \( v'(t) < 0 \), then

\[
v(t) = v(T) + \int_T^t v'(s)ds \leq v(T) + v'(T)e^{\int_T^{\tau'(t)}p_1(\tau)d\tau} \int_T^t e^{-\int_T^{\tau'(t)}p_1(\tau)d\tau}ds,
\]

for any \( t > T \), imply \( v(t) \to -\infty \) as \( t \to \infty \), which is a contradiction to the positivity of \( v(t) \). Therefore, \( v'(t) > 0 \), for any \( t \geq T \). Then we have

\[
v''(t) < -p_1(t)v'(t) - Q(t)v(m(t)) \leq -Q(t)v(m(t)) < 0.
\]

The following proof is similar to Theorem 2.1 in [1]. We can prove the theorem.

**Theorem 2.2.** Suppose (H1)-(H3) hold, \( Q(t) \), \( p_1(t) \), and \( \hat{p}(x,t) \) are the same as in Theorem 2.1. Besides (2.7) we assume that there exist a constant \( \beta \) and a function \( \tau'(t) \in C((t_0, \infty), R_+) \) satisfying \( \tau(t) \leq t \), \( \tau'(t) \geq 0 \), \( \tau''(t) \leq 0 \), such that the first order inequation

\[
v'(t) - Q_1(t)v(m(\tau(t))) > 0
\]

has no positive solution, where \( Q_1(t) = \int_{t_0}^{t+\beta} Q_0(s)ds \) and \( Q_0(t) = \min\{Q(t), Q(\tau(t))(\tau'(t))^2\} \). Then every smooth solution \( u(x,t) \) of (1.1) (1.2) (or (1.1)(1.3)) is oscillatory in \( G \).

**Proof.** Based on the proof of Theorem 2.1, there exists \( T > t_0 \) such that \( v'(t) > 0 \) and \( v''(t) < -Q(t)v(m(t)) \) for any \( t \geq T \), then the subsequent proof of this theorem is similar to Theorem 2.3 in [1]. We omit the details here.

**Remark 2.1.** Compared the two theorems with Theorem 2.1 and Theorem 2.3 in [1], we are convinced that the additional assumption (2.7) make oscillation more difficult. Specifically, when the damping coefficient \( \hat{p}(x,t) \leq 0 \), the assumption (2.7) satisfied naturally, those two theorems are equivalent to the corresponding theorems in [1]. When \( \hat{p}(x,t) > 0 \), the assumption (2.7) really make things, which means positive damping can “hold back” oscillation in some sense.

From the proof of Theorem 2.1, we notice that the positivity of \( \hat{p}(x,t) \) make a very important role to get the inequality (2.6). Next, we give two other sufficient theorems in which the positivity condition of damping coefficient \( (\hat{p}(x,t) > 0) \) is not necessary.

**Theorem 2.3.** Suppose (H1)-(H3) hold, \( Q(t) \), \( p_1(t) \), and \( \hat{p}(x,t) \) are the same as in Theorem 2.1. Besides (2.7) we assume that there exists a function \( b(t) \in C^1((t_0, \infty), R_+) \) such that

\[
\limsup_{t \to \infty} \int_T^t \left( b(s)Q(s) - \frac{b'(s)}{b(s)} - p_1(s) \right)^2 \frac{b(s)}{4} ds = \infty, \text{ for some } T > t_0.
\]

Then every smooth solution \( u(x,t) \) of (1.1) (1.2) (or (1.1)(1.3)) is oscillatory in \( G \).

The proof of the above theorem is similar to Theorem 2.2 in [1]. We omit the details here.

**Theorem 2.4.** Assume that (H1)-(H3) hold, \( Q_*(t) = aq(m(t)) \), \( h(t) = \min_{x \in \Omega} \hat{p}(x,t) - r'(t) \).

(1) If

\[
\limsup_{t \to \infty} \int_{t_0}^t r^{-1}(s)e^{-\int_T^t \frac{b(s)}{4}d\tau} ds = \infty,
\]

where \( v = \frac{1}{C} \int_\Omega u^b(x,t)dx \) for the Robin boundary condition, and \( v = \frac{1}{C} \int_\Omega v(x)u^b(x,t)dx \) for the Dirichlet boundary condition. Multiply both sides of (2.3) by \( e^{\int_p p_1(s)ds} \), we get

\[
\left( v'(t)e^{\int_p p_1(s)ds} \right)' < -Q(t)v(m(t))e^{\int_p p_1(s)ds} < 0,
\]

which implies that \( v'(t)e^{\int_p p_1(s)ds} \) does not change sign eventually, i.e. \( v'(t) \) does not change sign eventually, then there exists a \( T > t_0 \) such that either \( v'(t) < 0 \) or \( v'(t) > 0 \) for any \( t \geq T \). If we assume that \( v'(t) < 0 \), then

\[
v(t) = v(T) + \int_T^t v'(s)ds \leq v(T) + v'(T)e^{\int_T^{\tau'(t)}p_1(\tau)d\tau} \int_T^t e^{-\int_T^{\tau'(t)}p_1(\tau)d\tau}ds,
\]

for any \( t > T \), imply \( v(t) \to -\infty \) as \( t \to \infty \), which is a contradiction to the positivity of \( v(t) \). Therefore, \( v'(t) > 0 \), for any \( t \geq T \). Then we have

\[
v''(t) < -p_1(t)v'(t) - Q(t)v(m(t)) \leq -Q(t)v(m(t)) < 0\tag{2.6}
\]

The proof of the above theorem is similar to Theorem 2.2 in [1]. We omit the details here.
and there exist $T_1 > t_0$, $b \in C^1((t_0, \infty), \mathbb{R}_+)$ such that

$$
\limsup_{t \to \infty} \int_{T_1}^t \left[ b(s)Q_+(s) - \frac{1}{4} r(s)b(s) \left( \frac{b'(s)}{b(s)} - \frac{h(s)}{r(s)} \right)^2 \right] ds = \infty \quad (2.10)
$$

(2) If

$$
\limsup_{t \to \infty} \int_{t_0}^t r^{-1}(s)e^{-\int_{r_0}^s \frac{h(t)}{r(t)} ds} ds < \infty,
$$

besides (2.10) we further assume that there exist $T_*, T_2 \geq t_0$, and $T_3 \geq T_*$ such that

$$
\limsup_{t \to \infty} \int_{r_0}^t Q_+(s)\theta(m(s))e^{\int_{r_0}^s \frac{h(t)}{r(t)} dt} ds = \infty,
$$

where $\theta(t) = \int_t^\infty \frac{1}{r(s)} e^{-\int_r^s \frac{h(t)}{r(t)} dt} ds$. Then every smooth solution $u(x, t)$ of (1.1) (or (1.1)(1.2)) is oscillatory in $G$.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x, t)$ of problem (1.1) (or (1.1)(1.2)). Without loss of generality, we may assume that $u(x, t) \geq 0$, $u(x, m(t)) \geq 0$, $u(x, \eta(t)) \geq 0$ in $\Omega \times (t_1, \infty)$. Considering the left hand of (1.1), we have

$$
[r(t)u^{\alpha-1}ut + r(t)(\alpha-1)u^{\alpha-2}u_t^2 + r'(t)u^{\alpha-1}u_t] + [p(x, t) - r'(t)]u^{\alpha-1}u_t
$$

$$
+ q(m(t))u^\alpha(x, m(t)) < a(t)\Delta u(x, t) + \sum_{k=1}^s a_k(t)\Delta u(x, \eta(t)),
$$

(2.13)

where we have used the facts that $p(x, t) \geq (\alpha-1)r(t)$ in $(H_1)$ and $f(u, x, m(t)) > q(m(t))u^\alpha(x, m(t))$ in $(H_3)$.

Similar the calculations in [1], we set $v_1 = \frac{1}{a} \int_\Omega u^\alpha(x, t)dx$ for the Robin boundary condition, and $v_2 = \frac{1}{a} \int_\Omega \varphi(x)u^\alpha(x, t)dx$ for the Dirichlet boundary condition. There yields $v_1, v_2$ satisfy the following inequality

$$
(r(t)v'(t))' + h(t)v'(t) + Q_+v(m(t)) < 0,
$$

(2.14)

for any $t > t_1 > t_0$.

Next, we claim that there exists $t_2 \geq t_1$ such that $v(t)$ is monotonic on $[t_2, +\infty)$. In fact, from (2.14), we have

$$
(r(t)v'(t))' + h(t)v'(t) < -Q_+v(m(t)) \leq 0.
$$

By setting $y(t) = r(t)v'(t)$, we get

$$
y'(t) + \frac{h(t)}{r(t)}y(t) < 0.
$$

(2.15)

Multiply both sides of (2.15) by $e^{\int_{r_0}^t \frac{h(s)}{r(s)} ds}$, we get

$$
\left( y(t)e^{\int_{r_0}^t \frac{h(s)}{r(s)} ds} \right)' < 0,
$$

(2.16)

which implies that $y(t)e^{\int_{r_0}^t \frac{h(s)}{r(s)} ds}$ is decreasing and $y(t)$ is eventually of one sign. Then $v'(t)$ has a fixed sign for all sufficiently large $t$ and we have one of the following:

$$
\begin{cases}
\text{Case (1)} : \; v'(t) > 0; \\
\text{Case (2)} : \; v'(t) < 0.
\end{cases}
$$

(2.17)

Case (1). Define a generalized Riccati transformation by

$$
w(t) = \frac{b(t)r(t)v'(t)}{v(t)} > 0.
$$
Since (2.14) and \( m(t) \geq t \), we get

\[
w'(t) = \left[ \frac{b(t)v'(t)}{v(t)} \right]' = r(t)v'(t) \left( \frac{b(t)}{v(t)} \right)' + (r(t)v'(t)) \frac{b(t)}{v(t)}
\]

\[
< r(t)v'(t) \left( \frac{b(t)}{v(t)} \right)' - [h(t)v'(t) + Q_*(t)v(m(t))] \frac{b(t)}{v(t)}
\]

\[
\leq \left( \frac{b'(t)}{b(t)} - \frac{h(t)}{r(t)} \right) w(t) - \frac{1}{r(t)b(t)} w^2(t) - b(t)Q_*(t)
\]

\[
= - \left[ \frac{1}{\sqrt{r(t)b(t)}} w(t) - \frac{\sqrt{r(t)b(t)}}{2} \left( \frac{b'(t)}{b(t)} - \frac{h(t)}{r(t)} \right) \right]^2 + \frac{r(t)b(t)}{4} \left( \frac{b'(t)}{b(t)} - \frac{h(t)}{r(t)} \right)^2 - b(t)Q_*(t)
\]

\[
\leq \frac{1}{4} r(t)b(t) \left( \frac{b'(t)}{b(t)} - \frac{h(t)}{r(t)} \right)^2 - b(t)Q_*(t).
\] (2.18)

Integration (2.18) from \( T_1 > t_0 \) to \( t \), yields

\[
w(T_1) > \int_{T_1}^{t} \left[ b(s)Q_*(s) - \frac{1}{4} r(s)b(s) \left( \frac{b'(s)}{b(s)} - \frac{h(s)}{r(s)} \right)^2 \right] ds,
\]

which contradicts (2.10).

**Case (2).** By (2.16), for \( \tau \geq T_* \geq t_0 \), we have

\[
y(T_*)e^{\int_{T_*}^{\tau} \frac{h(s)}{r(s)} ds} \geq y(\tau)e^{\int_{\tau}^{T_*} \frac{h(s)}{r(s)} ds},
\]

then

\[
v'(\tau) \leq \frac{r(T_*)}{r(\tau)} v'(T_*) e^{-\int_{T_*}^{\tau} \frac{h(s)}{r(s)} ds}.
\] (2.19)

Integrating (2.19) from \( t \) to \( \infty \),

\[
v(t) \geq -r(T_*)v'(T_*) \int_{t}^{\infty} \frac{1}{r(\tau)} e^{-\int_{\tau}^{T_*} \frac{h(s)}{r(s)} ds} d\tau := A\theta(t),
\] (2.20)

where \( A = -r(T_*)v'(T_*) > 0, \theta(t) = \int_{t}^{\infty} \frac{1}{r(\tau)} e^{-\int_{\tau}^{T_*} \frac{h(s)}{r(s)} ds} d\tau \). Using (2.20) in (2.14), we find

\[
- (r(t)v'(t))' > h(t)v'(t) + Q_*(t)v(m(t)) \geq h(t)v'(t) + Q_*(t)A\theta(m(t)).
\] (2.21)

Define the function \( V(t) = r(t)v'(t) < 0 \), and (2.21) yields

\[
V'(t) < -\frac{h(t)}{r(t)} V(t) - AQ_*(t)\theta(m(t)),
\] (2.22)

for any \( t > T_* \). Multiply both sides of (2.22) by \( e^{\int_{t}^{T_2} \frac{h(s)}{r(s)} ds} \), we get

\[
\left( V(t)e^{\int_{t}^{T_2} \frac{h(s)}{r(s)} ds} \right)' < -AQ_*(t)\theta(m(t))e^{\int_{t}^{T_2} \frac{h(s)}{r(s)} ds}.
\] (2.23)

Integrating (2.23) from \( T_2 \geq T_* \) to \( t \),

\[
V(t)e^{\int_{t}^{T_2} \frac{h(s)}{r(s)} ds} < V(T_2)e^{\int_{T_2}^{T_2} \frac{h(s)}{r(s)} ds} - A \int_{T_2}^{t} Q_*(s)\theta(m(s))e^{\int_{s}^{T_2} \frac{h(s)}{r(s)} ds} ds
\]

\[
\leq -A \int_{T_2}^{t} Q_*(s)\theta(m(s))e^{\int_{s}^{T_2} \frac{h(s)}{r(s)} ds} ds,
\] (2.24)

then

\[
v'(t) < -\frac{A}{r(t)e^{\int_{t}^{T_2} \frac{h(s)}{r(s)} ds}} \int_{T_2}^{t} Q_*(s)\theta(m(s))e^{\int_{s}^{T_2} \frac{h(s)}{r(s)} ds} ds.
\] (2.25)
Integrating \(2.25\) from \(T_3 \geq T_*\) to \(\infty\), we get
\[
v(T_3) > A \int_{T_3}^{\infty} \frac{1}{r(\tau)} e^{\int_{T_2}^{\tau} \frac{h(s)}{r(s)} ds} \int_{T_2}^{\tau} Q_*(s) \theta(m(s)) e^{\int_s^\tau \frac{b(\xi)}{h(\xi)} d\xi} ds d\tau,
\]
which is a contradiction to \(2.12\).

At last, following \(1.17\), we claim that the assumption \(\int_T^{t_1} r^{-1}(s) e^{-\int_s^T \frac{h(\xi)}{r(\xi)} d\xi} ds = \infty\) can odd out the Case (2) (i.e. \(v'(t) < 0\)). In fact, if \(v'(t) < 0\), then
\[
v(t) = v(T) + \int_T^t (v'(s) ds < v(T) + r(T)v'(T) e^{\int_T^t \frac{h(s)}{r(s)} ds} \int_T^t r^{-1}(s) e^{-\int_s^T \frac{h(\xi)}{r(\xi)} d\xi} ds,
\]
for any \(t > T > t_0\), imply \(v(t) \to -\infty\) as \(t \to \infty\), which is a contradiction to the positivity of \(v(t)\). Now, we finish the proof.

3 Examples

Example 3.1 As an illustrative example, we consider the following equation
\[
tu^4 u_{tt} + u^3 u_t^2 + u^4 u_t + 2u^5(x,t) = \Delta u(x,t) + \sum_{k=1}^{5} (3 + \cos kt) \delta u(x, t^\frac{t}{2}),
\]
with the boundary conditions: \(\frac{\partial u(x,t)}{\partial n} + tu(x,t) = 0, \ (x,t) \in \partial \Omega \times R_+.
\]
Here \(\alpha = 5, r(t) = t, p(x,t) = 1, \tilde{p}(x,t) = 1, q(t) = 1, m(t) = 2t, a(t) = 1, a_k(t) = 3 + \cos kt,\)
\(\eta(t) = \frac{1}{2}, f(u,x,t) = 2u^3(x,t), \ b(t) = 1,\) then \(Q_*(t) = 5\). It is easy to check that all hypotheses in case (1) of Theorem 2.4 are satisfied, so we conclude the equation \(3.1\) with Robin boundary condition is oscillatory.

Example 3.2 We consider the following equation
\[
t^2 u^2 u_{tt} + uu_t^2 + 2tu^2 u_t + 2t^4 u^3(x,t+1) = \Delta u(x,t) + \sum_{k=1}^{5} (1 + kt) \delta u(x, t+2),
\]
with the boundary conditions: \(u(x,t) = 0, \ (x,t) \in \partial \Omega \times R_+.
\]
Here \(\alpha = 3, r(t) = t^2, p(x,t) = 1, \tilde{p}(x,t) = 2t, q(t) = t^4, m(t) = t+1, a(t) = 1, a_k(t) = 1 + kt,\)
\(\eta(t) = t + 2, f(u,x,t) = 2t^4 u^3(x,t), \ b(t) = 1.\) It is easy to check that all hypotheses in case (2) of Theorem 2.4 are satisfied, so the equation \(3.2\) with Dirichlet boundary condition is oscillatory.

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