Uniform in $N$ Global Well-posedness of the Time-Dependent Hartree-Fock-Bogoliubov Equations in $\mathbb{R}^{1+1}$

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Abstract

In this article, we prove the global well-posedness of the time-dependent Hartree-Fock-Bogoliubov (TDHFB) equations in $\mathbb{R}^{1+1}$ with two-body interaction potentials of the form $N^{-1}v_N(x) = N^{\beta-1}v(N^\beta x)$ where $v$ is a sufficiently regular radial function $v \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$. In particular, using methods of dispersive PDEs similar to the ones used in [GM17], we are able to show for any scaling parameter $\beta > 0$ the TDHFB equations are globally well-posed in some Strichartz-type spaces independent of $N$, cf. [BBC16].

1 Introduction

Let us consider a closed system of $N$ spinless, identical, non-relativistic interacting Bosons in $\mathbb{R}^d$ for $d \leq 3$ with pairwise interaction potential $\lambda w$ where $\lambda$ is the coupling constant. The evolution of the system in the Bosonic space $\otimes_s^N L^2(\mathbb{R}^d)$ is governed by the linear Schrödinger equation

$$\frac{1}{i} \frac{\partial}{\partial t} \Psi_N(t, X_N) = \sum_{i=1}^{N} \Delta_{x_i} \Psi_N(t, X_N) - \lambda \sum_{1 \leq i < j \leq N} w(x_i - x_j) \Psi_N(t, X_N)$$

with $X_N = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}$. In this article, we are interested in the model where both the kinetic energy and the interaction potential energy are scaled in a similar fashion. In particular, since the Hamiltonian

$$H_N := \sum_{i=1}^{N} \Delta_{x_i} - \lambda \sum_{1 \leq i < j \leq N} w(x_i - x_j)$$

scales like $O(N) + \lambda O(N^2)$, then the energy of each particle is $O(1)$ provided the coupling constant $\lambda$ is of the order of $O(N^{-1})$, which we called the mean-field scaling of (2). With this scaling, we define the mean-field limit to be the singular limit of (2) as $N \to \infty$. Some of

\[1\] cf. [FL03]. A more rigorous definition of the mean-field limit refers to the factorization of the marginal density matrices, for a system of initially uncorrelated particles, into tensor products of mean fields in trace norm as $N \to \infty$. cf. Ch 1.11 of [Gol16].
the more recent works on the qualitative studies on the rate of convergence of mean-field limit toward Hartree dynamics can be found in [RS09, CLS11, CL11, Kuz15].

To physically motivate the mean-field model, let us consider $N$ particles inside a fixed box $\mathbb{R}^d$ with volume $V = \ell^d$ subjected to either Robin or Neumann boundary conditions. Furthermore, assume the particles interact through a two-body repulsive potential $w$ (with coupling constant $\lambda$ set to 1). Then the particles will uniformly spread themselves inside the box with an average separation distance of $N^{-1/d}\ell$ since the average volume occupied by a particle is $N^{-1/d}\ell^d$. In particular, we are interested in the dilute gas model, that is the case when $N^{-1/d}\ell \gg 1$. Following a scaling argument, one can show that the dynamics generated by the Hamiltonian (2) is equivalent to the dynamics generated by the rescaled Hamiltonian

$$\frac{1}{N^{3-d}} \sum_{i=1}^{N} \Delta y_i - \frac{1}{N} \sum_{1 \leq i<j \leq N} w_N(y_i - y_j)$$

where $w_N(y) = N^d w(Ny)$ provided we set the length scale of $y_i$ to order $1$. In the case $d = 3$, we see that (3) gives us a mean-field model for the particles in a unit box with interactions $v_N$. Finally, if we take the dilute limit, $N^{-1/d}\ell \rightarrow \infty$, in the box $\ell^3$, we essentially recover the mean-field limit of $N$ weakly interacting particles in the unit box. In particular, the 3D mean-field model in the unit box is equivalent to the strongly interacting dilute gas model in a box. We refer the interested reader to [Lew15, LSSY05, Gol16] for more in-depth discussions.

Motivated by the above discussion, we are lead to consider the mean-field Hamiltonian

$$H_{N, mf} = \sum_{i=1}^{N} \Delta x_i - \frac{1}{N} \sum_{1 \leq i<j \leq N} v_N(x_i - x_j)$$

where $v_N(x) = N^d \beta v(N^\beta x)$ for $d \leq 3$ and $v \in C^\infty \cap L^1(\mathbb{R})$ which is spherically symmetric. The reader should take note of the two scaling processes that are involved in the interactions of this mean-field model. Aside from the obvious mean-field scaling, we also have the short-range scaling of the interaction $v$ given by $v_N$ with a tuning parameter $\beta > 0$. Let us consider the dynamics generated by the mean-field Hamiltonian and let $\Psi_N$ be the solution to

$$\frac{1}{i} \frac{\partial}{\partial t} \Psi_N = H_{N, mf} \Psi_N$$

then by rescaling the solution, i.e. defining $\Phi(\tau, y) = \Psi_N(N^{-2\beta} \tau, N^{-\beta} y)$, we see the dynamics of the rescaled system is governed by the equation

$$\frac{1}{i} \frac{\partial}{\partial \tau} \Phi = \sum_{i=1}^{N} \Delta y_i \Phi - N^{(d-2)\beta-1} \sum_{1 \leq i<j \leq N} v(y_i - y_j) \Phi$$

The box model was used to simplify the exposition. Alternatively, we could have considered $N$ particles in $\mathbb{R}^d$ subjected to some harmonic trapping potential, i.e. $x$

$$H_N = \sum_{i=1}^{N} \{ \Delta x_i - v_{ext}(x_i) \} - \sum_{1 \leq i<j \leq N} w(x_i - x_j)$$

where $v_{ext}$ is small inside the box $[-L, L]$ and large otherwise.

To preserve the dynamics, we will need to rescale the time by a factor of $N^{-2}$.

Here we are assuming $x_i$ is on the length scale $\ell \sim N$.

It should be noted that the 1D and 2D mean-field model can only correspond to the weakly interacting dense gas model.
In the instance of \(d = 3\), we see, at least heuristically, the appearance of a critical scaling when \(\beta = 1\), which we called the Gross-Pitaveskii scaling. Some of the important works done for the case \(\beta = 1\) in illustrating the change in the effective dynamics and the emergence of the scattering length can be found in [ESY10, BdOS15, BCS17]. Moreover, it is heuristically clear that there is no critical scaling when \(d = 1, 2\). To be more specific, for \(d \leq 2\), the coupling constant for the interaction of the rescaled system is inversely proportional to the number of particles which means the mean-field scaling is more prominent than the short-range scaling effect. Thus, we do not expect to see any short scale correlation effects. One of the purposes of this article is to offer an initial step to a rigorous demonstration of the fact that there is no development of short scale correlation structure when \(d = 1\) for the effective description. The case \(d = 2\) for all \(\beta > 0\) is still open.

Another reason to consider the entire range of \(\beta\) in \(\mathbb{R}^{1+1}\) is inspired by the Lieb-Liniger model [LL63, Lie63] which is a 1D model for a system of ultracold Bose particles inside the torus endowed with a pairwise interaction given by the repulsive \(\delta\)-function, i.e. the Lieb-Liniger Hamiltonian for the \(N\)-particle Bose gas, in appropriate units, is

\[
H_N = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j) \tag{7}
\]

where \(c \geq 0\) denotes the repulsion strength. More specifically, one can view the Lieb-Liniger model on \(\mathbb{R}\) as a heuristic endpoint case of our analysis of the dynamics generated by (4) in the weak-coupling limit regime, \(c \to 0\).

Our interest in the model is twofold. From a physics point of view, the model has an important feature of being exactly solvable in the ground state with computable spectrum. Moreover, the recent advancement in the techniques of trapping and cooling atoms has opened up a variety of possible experimental studies for ultracold Bose gases that are effectively one-dimensional; for a comprehensive survey on the subject, we refer the reader to [BDZ08]. Hence a firm mathematical understanding of the dynamics generated by the Hamiltonian (7) is an indispensable theoretical tool to suggest further experimental investigation of certain 1D properties for ultracold Bose gases. In particular, an effective description of the dynamics generated by the Lieb-Liniger model would provide a simplified way to analyze the dynamics of these effectively one-dimensional Bose gases. From a mathematical perspective, the Lieb-Liniger model on \(\mathbb{R}\) is the simplest instance of a many-body quantum mechanical model with interaction given by the \(\delta\)-potential. Up to date, there is no rigorous results on the effective description of the evolution of any quantum system with \(\delta\)-interaction.

In this article, we are interested in studying the wellposedness of the time-dependent Hartree-Fock-Bogoliubov (TDHFB) equations which, in 3D, describes the quantum fluctuations of the Bose field around a Bose-Einstein condensate in the “absolute-zero temperature” model. These equations were first rigorously derived as Euler-Lagrange equations in [GM13], which was in turn is based on earlier works by the same authors with collaborator in [GMM10, GMM11]. Later, in [GM17], Grillakis and Machedon rederived the TDHBF equations as evolution equations for the Fock space marginal densities subjected to some reduced dynamics. The TDHFB equations are

\[
\frac{1}{i} \partial_t \varphi_t = \Delta \varphi_t - (v_N * \rho \Gamma_t) \varphi_t - \kappa (\Gamma_t \varphi_t) \varphi_t - \kappa (\Lambda_t \varphi_t) \bar{\varphi}_t \tag{8}
\]

\[\text{cf. [BBC}^+\text{16]}\]
\[
\frac{1}{i}\partial_t \Gamma_t = \Delta - v_N \rho_{\Gamma_t}, \Gamma_t - [\kappa(\Gamma_t^\varphi), \Gamma_t] - [\kappa(\varphi_t \otimes \varphi_t), \Gamma_t^\varphi]
\]

\[
\frac{1}{i}\partial_t \Lambda_t = \{\Delta - v_N \rho_{\Lambda_t}, \Lambda_t\} - \frac{1}{N}\kappa(\Lambda_t) - \{\kappa(\Lambda_t^\varphi), \Lambda_t\}
\]

\[
= [\Delta - v_N \rho_{\Gamma_t}, \Gamma_t] - [\kappa(\Gamma_t^\varphi), \Gamma_t^\varphi] - [\kappa(\varphi_t \otimes \varphi_t), \Gamma_t^\varphi] - \{\kappa(\Lambda_t^\varphi), \varphi_t \otimes \varphi_t\}
\]  

where \(\{A, B\} = AB^T + BA^T\), \(\Gamma_t^\varphi := \Gamma - \varphi \otimes \varphi\) and \(\Lambda_t^\varphi := \Lambda - \varphi \otimes \varphi\) and \(\kappa: \alpha \rightarrow \kappa(\alpha)\) has the integral kernel given by

\[
[k(\alpha)](x, y) = v_N(x - y)\alpha(x, y).
\]

A more explicit form of the equations in terms of the kernels can be found in Section 8.

Independently and in a different framework, Bach, Breteaux, Chen, Fröhlich, and Sigal derived equations closely related to the above equations in [BBC+16]. In particular, the two sets of equations are equivalent in the case of pure states. To be more precise, the triplet \((\phi_t, \gamma_t, \sigma_t)\), introduced in [BBC+16], corresponds to

\[
\phi_t = \sqrt{N}\varphi_t, \quad \gamma_t = N(\overline{\Gamma_t - \varphi_t \otimes \varphi_t}) = \frac{1}{2}(\text{ch}(2k_t) - 1), \quad \sigma_t = N(\overline{\Lambda_t - \varphi_t \otimes \varphi_t}) = \frac{1}{2}\text{sh}(2k_t)
\]

when written in the notations of [GM13, GM17]. See §2 for more details on the notation.

2 Notations and Main Statement

Let us indicate some of the notations adopted by the article.

**Remark 2.1.** We adopt the usual convention of identifying the collection of Hilbert-Schmidt operators on \(L^2(\mathbb{R}^d)\), denoted by \(L^2\), with their integral kernels in \(L^2(\mathbb{R}^d \times \mathbb{R}^d)\).

**Notations.** Following [GM17], we use the notations

\[
S_\pm := \frac{1}{i}\frac{\partial}{\partial t} - \Delta_x + \Delta_y \quad \text{and} \quad S := \frac{1}{i}\frac{\partial}{\partial t} - \Delta_x - \Delta_y
\]

to denote the two Schrödinger-type differential operators. Moreover, unless specified, \(x, y\) are real variables which means \(\Delta_x = \partial_{xx}\) and, similarly, \(\Delta_y = \partial_{yy}\). The two types of semilinear equations, corresponding to the above operators, considered are the inhomogeneous von-Neumann Schrödinger equation

\[
S_\pm \Gamma = F
\]

and the inhomogeneous Schrödinger equation

\[
\left( S + \frac{1}{N}v_N(x - y) \right) \Lambda = F
\]

where \(v_N(x) = N^3v(N^3x)\) and \(v \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})\).

Next, let us define the space for the initial data. For every \(s > 0\), we define the space

\[
\mathcal{X}^s = \{ (\varphi, \Gamma, \Lambda) \in H^s \times H^s_\text{Herm} \times H^s_\text{sym} \}.
\]
with $H^r$ being the Sobolev space $H^s(\mathbb{R})$, $H^s_{\text{Herm}}(\mathbb{R}^2)$ the Sobolev space $H^s_{\text{Herm}}(\mathbb{R}^2)$ restricted to functions $\Gamma$ such that $\Gamma(x,y) = \Gamma(y,x)$, and $H^s_{\text{sym}}$ the Sobolev space $H^s(\mathbb{R}^2)$ restricted to functions $\Lambda$ such that $\Lambda(x,y) = \Lambda(y,x)$. More specifically, $X^s$ is endowed with the norm

$$
\| (\varphi, \Gamma, \Lambda) \|_{X^s} := \| \langle \nabla_x \rangle^s \phi \|_{L^2(\mathbb{R})} + \| (\langle \nabla_x \rangle^2 \otimes 1 + 1 \otimes \langle \nabla_y \rangle^2)^{s/2} \Gamma \|_{L^2(\mathbb{R}^2)} + \| (\langle \nabla_x \rangle^2 \otimes 1 + 1 \otimes \langle \nabla_y \rangle^2)^{s/2} \Lambda \|_{L^2(\mathbb{R}^2)}.
$$

When the context is clear, we will use the symbol $\langle \nabla_{x,y} \rangle^s$ in place of $(\langle \nabla_x \rangle^2 \otimes 1 + 1 \otimes \langle \nabla_y \rangle^2)^{s/2}$.

The hyperbolic integral operators introduced in §1 are defined as follows

$$
\begin{align*}
\text{sh}(k) &:= k + \frac{1}{3!} k \circ k \circ k + \frac{1}{5!} k \circ k \circ k \circ k + \ldots \\
\text{ch}(k) &:= \delta + p(k) := \delta + \frac{1}{2!} k \circ k + \frac{1}{4!} k \circ k \circ k \circ k + \ldots
\end{align*}
$$

where $\circ$ indicates composition of operators. The symmetric kernel of $k$, $k(t,x,y) = k(t,y,x)$, is called the pair excitation function. The following are some useful trigonometric identities

$$
\begin{align*}
\text{sh}(2k) &= 2 \text{sh}(k) \circ \text{ch}(k), & \text{ch}(2k) &= \delta + 2 \text{sh}(k) \circ \text{sh}(k) \\
\text{ch}(k) \circ \text{ch}(k) - \text{sh}(k) \circ \text{sh}(k) &= \delta.
\end{align*}
$$

Lastly, we have adopted the usual conventional notation

$$
\rho_T(t,x) := \Gamma(t,x,x)
$$

to define the restriction of $\Gamma$ to the diagonal on the plane.

Let us state the main result of the article as follows

**Theorem 2.2** (Uniform in $N$ Local Wellposedness of the TDHFB in $\mathbb{R}^{1+1}$). Suppose $\beta > 0$ and $R > 0$. Then there exists $T(\beta,R) > 0$ and a function space $X_T$, both independent of $N$, such that given

$$(\varphi_0, \Gamma_0, \Lambda_0) \in \{(\varphi, \Gamma, \Lambda) \in X^s | \| (\varphi, \Gamma, \Lambda) \|_{X^s} < R\},$$

for some $s > 0$ to be determined, there exists a unique solution to the TDHFB equations with initial data $(\varphi_0, \Gamma_0, \Lambda_0)$ satisfying $(\varphi_t, \Gamma_t, \Lambda_t) \in C([0,T] \to X^s) \cap X_T$.

We refer the reader to §8 for the definition of $X_T$ and the corresponding $s$.

**Remark 2.3.** The local wellposedness of the TDHFB equations in Theorem 2.2 can be extended to global wellposedness. The idea behind the extension is to prove for $s$ sufficiently small the following estimates

$$
\begin{align*}
\| \langle \nabla_x \rangle^s \varphi(t,\cdot) \|_{L^2(dx)} &\lesssim 1 \\
\| \langle \nabla_{x,y} \rangle^s T(t,\cdot) \|_{L^2(dx dy)} &\lesssim 1 \\
\| \langle \nabla_{x,y} \rangle^s \Lambda(t,\cdot) \|_{L^2(dx dy)} &\lesssim 1
\end{align*}
$$

holds uniformly in $t$ and $N$, which is possible by using the conservation laws proved in [GM13], see §3.

**Remark 2.4.** Our result does not require the condition $V^2 \leq C(I - \Delta)$. More precisely, since we are working with $V(x) = N^{\beta - 1} v(N^\beta x)$, then we see that

$$
N^{\beta - 2} \int dx \ |v(x)|^2 |f(N^{-\beta} x)|^2 = \| Vf \|_{L^2(\mathbb{R})}^2 \lesssim \| f' \|_{L^2(\mathbb{R})}^2 + \| f \|_{L^2(\mathbb{R})}^2
$$

can only be true uniformly in $N$ provided $\beta < 2$. 


3 Estimates for the Homogeneous $\Gamma$ Equation

In this section we prove a few estimates regarding the von-Neumann Schrödinger equation

$$\frac{1}{i} \frac{\partial}{\partial t} \Gamma + [-\Delta, \Gamma] = 0. \quad (15)$$

First, we shall establish a collapsing estimate for $\Gamma$. The reader should be aware of our attempt to keep track of the fractional derivative values. Keeping a record of these values allows us to show that the mapping use when implementing the fixed-point argument is indeed a self map.

**Proposition 3.1** (Collapsing Estimate). Suppose $\Gamma$ is a solution to $S_{\pm} \Gamma = 0$, then

$$\left\| \nabla_{x}^{1/2} \rho \Gamma(t, x) \right\|_{L^2(dt, dx)} \lesssim \left\| \Gamma_0(x, y) \right\|_{L^2(dx, dy)}. \quad (16)$$

**Proof.** Taking the spacetime Fourier transform of $\Gamma$ yields

$$\tilde{\rho} \Gamma(t, x) = \int dt dx e^{i\tau t - i\xi x} \rho \Gamma(t, x) = \int dt dy e^{i\tau t - i\xi x} \delta(x - y) \Gamma(t, x, y)$$

$$= \int d\eta dt dx dy e^{-i(-\tau t + (\xi - \eta) \cdot x + \eta y)} \Gamma(t, x, y) = \int d\eta dt e^{-i\tau \Gamma(t, \xi - \eta, \eta)}$$

$$= \int d\eta \delta(\tau - |\xi - \eta|^2 + \eta^2) \tilde{\Gamma}_0(\xi - \eta, \eta).$$

Taking the $L^2_{\tau, \xi} (\mathbb{R} \times \mathbb{R})$ norm of $\tilde{\nabla}_{x}^{1/2} \rho \Gamma$ and applying Cauchy-Schwarz gives us the estimate

$$\int d\tau d\xi \left| \tilde{\nabla}_{x}^{1/2} \rho \Gamma(t, x)(\tau, \xi) \right|^2 \lesssim \left\| \Gamma_0(x, y) \right\|_{L^2(dx, dy)}^2$$

since

$$\sup_{\tau, |\xi|} \int d\eta \delta(\tau - |\xi - \eta|^2 + \eta^2)|\xi| \lesssim 1.$$

Utilizing the above collapsing estimate, we prove a couple perturbed version of the collapsing estimate which will be crucial for our article.

**Lemma 3.2.** Suppose $\Gamma$ is a solution to $S_{\pm} \Gamma = 0$, then for any $\epsilon > 0$ we have that

$$\left\| \nabla_{x}^{\epsilon} \rho \Gamma(t, x) \right\|_{L^\infty(dt)L^2(dx)} \lesssim \left\| \nabla_{x, y}^{1/2+\epsilon} \Gamma(t, x, y) \right\|_{L^\infty(dt)L^2(dx, dy)}. \quad (17)$$

**Proof.** For any fixed $t$, it follows from the sharp trace theorem that we have

$$\left\| \nabla_{x}^{\epsilon} \rho \Gamma(t, x) \right\|_{L^2(dx)} \lesssim \left\| \nabla_{x, y}^{1/2+\epsilon} \Gamma(t, x, y) \right\|_{L^2(dx, dy)}.$$  

Hence taking the supremum in time yields the desired result. \qed
Proposition 3.3. Suppose $\Gamma$ is a solution to $S_{\pm} \Gamma = 0$, then for any $0 < \epsilon < \frac{1}{2}$ we have that
\[ \| \nabla_{\frac{3}{2}}^{\frac{1}{2} - \epsilon'} \rho \Gamma(t, x) \|_{L^q(dt)L^2(dx)} \lesssim \| \nabla_{x,y}^0 \Gamma(t, x, y) \|_{L^\infty(dt)L^2(dx,dy)} \] (18)
for some $\epsilon < \frac{1}{2} - \epsilon' < \frac{1}{2}$ and $\alpha, q$ will be stated in the proof.

Proof. Let us interpolate the estimates (16) and (17) to get
\[ \| \nabla_{\frac{3}{2}}^{\frac{1}{2} - \epsilon} \rho \Gamma \|_{L^q(dt)L^2(dx)} \lesssim \| \nabla_{x,y}^0 \Gamma(t, x, y) \|_{L^\infty(dt)L^2(dx,dy)} \]
and $\alpha$ is given by
\[ \alpha = \left( \frac{\frac{3}{2} + \epsilon}{\frac{3}{2} - \epsilon} \right) \epsilon'. \]
Moreover, we have that
\[ q = \frac{1 - 2\epsilon}{\frac{3}{2} - \epsilon' - \epsilon} \geq 2 \]
since $\epsilon' < \frac{1}{2} - \epsilon$. \hfill \qed

Corollary 3.4. Suppose $\Gamma$ is a solution to $S_{\pm} \Gamma = 0$, then for any $0 < \epsilon < 1/2$ we have that
\[ \| \nabla_{\frac{3}{2}}^{\frac{1}{2} - \epsilon'} \rho \Gamma(t, x) \|_{L^q(dt)L^2(dx)} \lesssim \| \nabla_{x,y}^0 \Gamma(t, x, y) \|_{L^\infty(dt)L^2(dx,dy)} \] (19)

Proof. Let us use $\delta$ to denote the $\epsilon$ from the previous proposition. Fix $\epsilon$, choose $\delta$ such that
\[ \frac{\frac{3}{2} + \delta}{\frac{3}{2} - \delta} = \frac{3}{2} - \epsilon \quad \Rightarrow \quad \delta = \frac{2(5 - 2\epsilon)}{1 - 2\epsilon} \]
and choose $q$ to be
\[ q = \frac{2}{(2 - \epsilon)(\frac{3}{2} - \epsilon)} \]
then we have the desired inequality. \hfill \qed

Remark 3.5. Heuristically we want the estimate
\[ \| \rho \Gamma(t, x) \|_{L^2(dt)L^\infty(dx)} \lesssim \| \nabla_{\frac{3}{2}}^{1/2} \rho \Gamma(t, x) \|_{L^2(dt, dx)} \lesssim \| \Gamma_0(x, y) \|_{L^2(dx, dy)} \]
but the estimate is a false endpoint of the Gagliardo-Nirenberg estimate. However, by using the above corollary and the fact that we are working on a finite interval $[0, T]$, we get that
\[ \| \rho \Gamma(t, x) \|_{L^2(dt)L^p(dx)} \lesssim \| \nabla_{\frac{3}{2}}^{\frac{1}{2} - \epsilon} \rho \Gamma(t, x) \|_{L^2(dt, dx)} \lesssim T^{some \ power} \| \nabla_{\frac{3}{2}}^{\frac{1}{2} - \epsilon} \rho \Gamma(t, x) \|_{L^2(dt, dx)} \lesssim T^{some \ power} \| \nabla_{x,y}^{(\frac{3}{2} - \epsilon)} \Gamma_0(x, y) \|_{L^2(dx, dy)}. \]
We will elaborate more on this point in the next section.
Next, let us establish the homogeneous Strichartz estimate for the linear operator $S_\pm$.

**Proposition 3.6** (Non-Endpoint Strichartz). *Suppose $\Gamma$ is a solution to $S_\pm \Gamma = 0$ with initial condition $\Gamma_0$ and $(q,r)$ is an admissible pair, i.e.*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

*where $(q,r) \in (2, \infty] \times [2, \infty]$. Then it follows*

$$\| e^{it(\Delta_x - \Delta_y)} \Gamma_0 \|_{L^q(dt) L^r(dx) L^2(dy)} \lesssim \| \Gamma_0 \|_{L^2(dx dy)}.$$  

**Proof.** The proof is essentially the same as the standard non-endpoint Strichartz estimate using both the $TT^*$ principle and Christ-Kiselev lemma. See [Tao06].

$\square$

### 4 Estimates for the Inhomogeneous $\Gamma$ Equation

Let us now consider the inhomogeneous $\Gamma$ equation

$$S_\pm \Gamma = F. \quad (22)$$

Observe the solution to the inhomogeneous equation can be written as

$$\Gamma(t, x, y) = e^{it(\Delta_x - \Delta_y)} \Gamma_0(x, y) + i \int_0^t e^{i(t-s)(\Delta_x - \Delta_y)} F(s, x, y) \, ds \quad (23)$$

which then yields

$$\rho_\Gamma(t, x) = [e^{it(\Delta_x - \Delta_y)} \Gamma_0](x, x) + i \int_0^t [e^{i(t-s)(\Delta_x - \Delta_y)} F](s, x, x) \, ds \quad (24)$$

Then it follows from the estimate [16] that

$$\| \nabla_x^{1/2} \rho_\Gamma \|_{L^2(0,T; L^2(dx dy))} \lesssim \| \Gamma_0 (x,y) \|_{L^2(dx dy)} + \int_0^T \| \nabla_x^{1/2} [e^{i(t-s)(\Delta_x - \Delta_y)} F](s, x, x) \|_{L^2(dx)} \, ds$$

$$\lesssim \| \Gamma_0 (x,y) \|_{L^2(dx dy)} + \| F \|_{L^1(0,T; L^2(dx dy))}.$$ 

Hence we have obtained the following proposition

**Proposition 4.1.** *Suppose $\Gamma$ solves $S_\pm \Gamma = F$, then for every $0 < \epsilon < 1$ we have*

$$\| \nabla_x^{1/2} \rho_\Gamma \|_{L^2(0,T; L^2(dx dy))} \lesssim \| \Gamma_0 \|_{L^2(dx dy)} + \| F \|_{L^1(0,T; L^2(dx dy))}. \quad (25)$$

**Proposition 4.2.** *Suppose $\Gamma$ solves $S_\pm \Gamma = F$, then for every $0 < \epsilon < 4/5$ we have*

$$\| \nabla_x^{1/2 - \epsilon} \rho_\Gamma \|_{L^2(0,T; L^2(dx dy))} \lesssim T^{\text{some power}} \left( \| \nabla_{x,y}^{(2-\epsilon)/\epsilon} \Gamma(t, x, y) \|_{L^\infty(dt) L^2(dx dy)} \right.$$

$$\left. + \| \nabla_{x,y}^{(2-\epsilon)/\epsilon} F \|_{L^1(0,T; L^2(dx dy))} \right) \quad (26)$$
Proof. Applying corollary 3.4 to (24) yields

\[ \| \nabla^\frac{1}{2} - \rho \Gamma(t, x) \|_{L^2(dtdx)} \]
\[ \lesssim T^{\text{some power}} \left( \| \nabla^\frac{1}{2} x y \Gamma(t, x, y) \|_{L^\infty L^2_y} + \int_0^T ds \| \nabla^\frac{1}{2} x y [e^{i(t-s)(\Delta_x - \Delta_y)} F](s, x, x) \|_{L^r_y L^q_x} \right) \]
\[ \lesssim T^{\text{some power}} \left( \| \nabla^\frac{1}{2} x y \Gamma(t, x, y) \|_{L^\infty(dtdx)L^2(dy)} + \| \nabla^\frac{1}{2} x y F \|_{L^1[0,T]L^2(dx)} \right). \]

To conclude the section, let us state the inhomogeneous Strichartz.

**Proposition 4.3.** Suppose \( \Gamma \) is a solution to \( S_\pm \Gamma = F \) with initial condition \( \Gamma_0 \) and \((q,r)\) and \((\tilde{q}, \tilde{r})\) are admissible pairs. Then it follows

\[ \| \Gamma(t, x, y) \|_{L^\infty(dt)L^r(dx)L^2(dy)} \lesssim \| \Gamma_0 \|_{L^2(dx)} + \| F \|_{L^\infty(dt)L^r(dx)L^2(dy)} \] (27)

and

\[ \| \nabla^\frac{1}{2} x y \Gamma(t, x, y) \|_{L^q(dt)L^r(dx)L^2(dy)} \lesssim \| \nabla^\frac{1}{2} x y \Gamma_0 \|_{L^2(dx)} + \| \nabla^\frac{1}{2} x y F \|_{L^\infty(dt)L^r(dx)L^2(dy)} \] (28)

5 Application of the Inhomogeneous \( \Gamma \) Estimates

The purpose of this section is to develop estimates for \( S_\pm \) However, as an immediate application of the previous two sections, we are now also ready to consider the local well-posedness of the following Hartree equation

\[ \frac{1}{i} \partial_t \Gamma = [\Delta - v_N * \rho \Gamma, \Gamma] \] (29)

or equivalently

\[ S_\pm \Gamma(t, x, y) = [v_N * \rho \Gamma(t, x) - v_N * \rho \Gamma(t, y)] \Gamma(t, x, y) \] (30)

in some Strichartz-type space, \( X \), equipped with the norm

\[ \| \Gamma \|_X := \| \nabla^\frac{1}{2} x y \rho \Gamma(t, x) \|_{L^2([0,T] \times X)} + \| (\nabla^\frac{1}{2} x y \Gamma(t, x, y) ) \|_{L^\infty([0,T] \times X)} \]

\[ + \| (\nabla^\frac{1}{2} x y \Gamma(t, x, y) ) \|_{L^4([0,T] \times X)} \]

where \( \epsilon \) is sufficiently small, say \( \epsilon < \frac{1}{2} \).

It suffices to close the estimate for (29) in \( X \). Let us consider three estimates on the nonlinearity. For the first estimate, we shall consider the following

\[ \| F \|_{L^1[0,T]L^2(dx)} \lesssim \| v_N * \rho \Gamma(t, x) \Gamma(t, x, y) \|_{L^1[0,T]L^2(dx)dy} \]

\[ \lesssim \| v_N * \rho \Gamma(t, x) \|_{L^2(dt)L^p(dx)} \| \Gamma(t, x, y) \|_{L^2(dt)L^r(dx)L^2(dy)} \]

\[ \lesssim \| v_N * \nabla^\frac{1}{2} x y \rho \Gamma(t, x) \|_{L^2(dtdx)} \| \Gamma(t, x, y) \|_{L^2(dt)L^r(dx)L^2(dy)} \]
where \( \tilde{T} \) and \( T \) are some power \( \frac{1}{2} - \epsilon \). Therefore, there exists some power \( \frac{3}{2} - \epsilon \) which again means there exists some power \( \frac{3}{2} - \epsilon \). Thus, \( \tilde{T} = (\frac{1}{2} - \epsilon)\frac{2}{3} \) and \( r = 2(1 - 2\epsilon)^{-1} \). It’s also easy to show \( \| \nabla_{x,y}^{\frac{1}{2}} F \|_{L^1[0,T]L^2(dx,dy)} \) also closes. Next, observe

\[
\| \nabla_{x,y}^{(\frac{3}{2} - \epsilon)\epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^1[0,T]L^2(dx,dy)} \lesssim \| \nabla_{x,y}^{\frac{3}{2} - \epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^1[0,T]L^2(dx,dy)}
\]

\[
+ \| \nabla_{x,y}^{\frac{3}{2} - \epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^2(dx,dy)} \| \Gamma(t, x, y) \|_{L^2(dx,dy)}
\]

\[
+ \| w_N * \rho_T(t, x) \|_{L^2(dx,dy)} \| \nabla_{x,y}^{(\frac{3}{2} - \epsilon)\epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^2(dx,dy)}
\]

where \( \tilde{p} = 2(5 - 2\epsilon)^{-1} \), \( \tilde{r} = [\epsilon^2 - \frac{5}{2}\epsilon + \frac{1}{2}]^{-1} \) and \( \tilde{q} = 2[(\frac{5}{2} - \epsilon)^{-1} \). Hence it follows

\[
\| \nabla_{x,y}^{(\frac{3}{2} - \epsilon)\epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^1[0,T]L^2(dx,dy)} \lesssim \| \nabla_{x,y}^{\frac{3}{2} - \epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^2(dx,dy)} \| \Gamma(t, x, y) \|_{L^2(dx,dy)}
\]

\[
+ \| w_N * \rho_T(t, x) \|_{L^2(dx,dy)} \| \nabla_{x,y}^{\frac{3}{2} - \epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^2(dx,dy)}
\]

Lastly, observe

\[
\| \nabla_{x,y}^{(\frac{3}{2} - \epsilon)\epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^1[0,T]L^2(dx,dy)} \lesssim \| \nabla_{x,y}^{\frac{3}{2} - \epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^2(dx,dy)} \| \Gamma(t, x, y) \|_{L^2(dx,dy)}
\]

\[
+ \| v_N * \rho_T(t, x) \|_{L^2(dx,dy)} \| \nabla_{x,y}^{\frac{3}{2} - \epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^2(dx,dy)}
\]

\[
+ \| w_N * \rho_T(t, x) \|_{L^2(dx,dy)} \| \nabla_{x,y}^{\frac{3}{2} - \epsilon} [v_N * \rho_T(t, x) \Gamma(t, x, y)] \|_{L^2(dx,dy)}
\]

As a result of the above calculation, we obtain the following proposition

**Proposition 5.1.** Suppose \( \Gamma \) solves (29) with Schwartz initial condition \( \Gamma_0 \) and \( v \in L^1(\mathbb{R}) \). Then the following estimate holds

\[
\| \Gamma \|_{X} \lesssim \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \Gamma_0 \|_{L^2(dx,dy)} + T^{\text{some power}} \| \tilde{\Gamma} \|_{\tilde{X}}.
\]

Thus, there exists \( T_0 > 0 \) such that for all \( 0 < T \leq T_0 \)

\[
\| \Gamma \|_{X} \lesssim \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \Gamma_0 \|_{L^2(dx,dy)}.
\]

Similarly, we can show that

\[
\| \partial_t \Gamma \|_{X} \lesssim \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \partial_t \Gamma_0 \|_{L^2(dx,dy)} + T^{\text{some power}} \| \partial_t \tilde{\Gamma} \|_{\tilde{X}}
\]

which again means there exists \( T_0 > 0 \) such that

\[
\| \partial_t \Gamma \|_{X} \lesssim \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \partial_t \Gamma_0 \|_{L^2(dx,dy)}.
\]
6 Homogeneous $\Lambda$ Equation

In this section we shall prove some estimates for the linear Schrödinger equations

$$\frac{1}{i} \frac{\partial}{\partial t} \Lambda - \Delta_x \Lambda - \Delta_y \Lambda = 0$$

(31)

which we will need later. As mentioned in the introduction, one of the main difficulties in the analysis of equation (12) is that the $L^p$-norms of the potential $N^{-1}v_N(x-y)$ are not uniformly bounded in $N$ when $p > 1$ and $\beta$ arbitrarily large since $N^{-1} \| v_N(x-y) \|_p \sim N^{-1+\beta(1-\frac{1}{p})}$. More precisely, from Proposition 6.2, we see the natural space to put the nonlinearity of equation (40) is in $L^1([0,T] \times L^2(\mathbb{R}^2))$. In particular, when handling the term $N^{-1}v_N(x-y)\Lambda(t,x,y)$ from equation (12) in $L^1([0,T] \times L^2(\mathbb{R}^2))$, we see there is no way (at least no simple way) to put the term $N^{-1}v_N(x-y)$ in $L^1(d(x-y))$. Thus, the purposes of §6 and textsection 7 are to develop sufficient amount of tools to handle $N^{-1}v_N(x-y)\Lambda(t,x,y)$ and all the nonlinearity coming from the TDHBF equations.

One of the crucial tools for our analysis is the $X^{s,b}$ spaces (sometimes called the Bourgain spaces or dispersive Sobolev spaces) which is defined to be the closure of the Schwartz class, $\mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{R})$ with respect to the norm

$$\| u \|_{X^{s,b}} = \| (1 + |\xi|^2 + |\eta|^2)^s (1 + |\tau + |\xi|^2 + |\eta|^2)^b \tilde{u}(\tau,\xi,\eta) \|_{L^2(dt)\times L^2(d\xi d\eta)}.$$

For this paper, $s$ is always zero and we are only interested in defining the $X^{s,b}$ spaces for the operator $S$. Hence we dropped both the $s$ and $S$ labels from the norm to simplify the notation. For instance, we have $\| u \|_{X^b} = \| u \|_{X^{0,b}}$. We refer the interested reader to $[Tao06]$ for an more complete introduction to these spaces.

Same as the von-Neumann Schrödinger equation, we first obtain a collapsing estimate for the above equation.

**Proposition 6.1.** If $SA = 0$ then

$$\| p(t,x)\Lambda(t,x,x) \|_{L^2(dt dx)} \lesssim \| \Lambda_0(x,y) \|_{L^2(dx dy)}.$$

(32)

where $\tilde{p}(\tau,\xi) = |\tau - |\xi|^2|^{1/4}$.

**Proof.** Let us begin by taking the spacetime Fourier transform of the trace of $\Lambda$ to get

$$\Lambda(t,x,x) = \int dt dx \ e^{i\tau t - i|\xi|^2 x} \Lambda(t,x,x) = \int dt dx dy \ e^{i\tau t - i|\xi|^2 x} \delta(x-y) \Lambda(t,x,y)$$

$$= \int d\eta dt dx dy \ e^{-i(\tau t + (\xi - \eta) \cdot x + \eta \cdot y)} \Lambda(t,x,y) = \int d\eta dt \ e^{i\tau t} \tilde{\Lambda}(t,\xi - \eta,\eta)$$

$$= \int d\eta \ \delta(t - |\xi - \eta|^2 - |\eta|^2) \tilde{\Lambda}_0(\xi - \eta,\eta)$$

$$= \int d\eta \ \delta(t - |\xi - \eta|^2 - |\eta|^2) \tilde{\Lambda}_0(\xi - \eta,\eta).$$

Applying Cauchy-Schwarz inequality yields the following estimate

$$\int d\tau d\xi \ |(\tau - |\xi|^2)^{1/4} \Lambda(t,x,x)(\tau,\xi)|^2 \leq \sup_{\tau,\xi} I(\tau,\xi) \| \Lambda_0(x,y) \|_{L^2(dx dy)}^2.$$
where
\[
I(\tau, \xi) := \sqrt{\tau - |\xi|^2} \int d\eta \, \delta(\tau - |\xi - \eta|^2 - |\xi + \eta|^2).
\]

Observe, we have the identity
\[
\int d\eta \, \delta(\tau - |\xi - \eta|^2 - |\xi + \eta|^2) = \int d\eta \, \delta(\tau - \sqrt{\tau - |\xi|^2}) + \delta(\tau + \sqrt{\tau - |\xi|^2}) \frac{1}{4\sqrt{\tau - |\xi|^2}}
\]
\[
= \frac{1}{2\sqrt{\tau - |\xi|^2}}.
\]

Thus, it follows
\[
\int d\tau d\xi \, ||\tau - |\xi|^2|^{1/4} \Lambda(t, x, x)(\tau, \xi)|^2 \lesssim \|\Lambda_0(x, y)\|_{L^2(\mathbb{R}^d)}^2.
\]

Unfortunately, the homogeneous derivative \(p(t, x)\) of the trace of \(\Lambda\) is not of any immediate use to our studies of the nonlinear coupled equations. Since the nonlinearity in the coupled equations involves trace of \(\Lambda\), we need estimates that will allow us to control the trace of \(\Lambda\) by the spacetime derivative \(p(t, x)\) of trace of \(\Lambda(t, x, x)\). One such estimate is given by the following proposition.

**Proposition 6.2.** Suppose \(S\Lambda = 0\), then we have
\[
\|\Lambda(t, x, x)\|_{L^4(dt)L^2(dx)} \lesssim \|p(t, x)\Lambda(t, x, x)\|_{L^2(dt \, dx)}
\]

**Proof.** We shall prove the above estimate using a \(TT^*\) argument. Consider \(T : L^2_{t,x} \to L^4_{t}L^2_{x}\) defined by
\[
TF = \left(\frac{\hat{F}}{|\tau - |\xi|^2|^{1/4}}\right)^\vee
\]
then we see that \(TT^* : L^{4/3}_{t}L^2_{x} \to L^4_{t}L^2_{x}\) is given by
\[
TT^*F = \left(\frac{\hat{F}}{|\tau - |\xi|^2|^{1/2}}\right)^\vee = F \ast \left(\frac{1}{|\tau - |\xi|^2|^{1/2}}\right)^\vee =: F \ast K.
\]

By triangle inequality and Plancherel, we obtain the estimate
\[
\|K \ast F\|_{L^2(dx)} \leq \int ds \, \|\hat{K}(t - s, \xi)\hat{F}(s, \xi)\|_{L^2(dx)} \lesssim \int ds \, \frac{1}{|t - s|^{1/2}} \|\hat{F}(s, \cdot)\|_{L^2(dx)}
\]
since we have
\[
|\hat{K}(t - s, \xi)| = \left|\int_{-\infty}^{\infty} e^{i\tau(t-s)} \frac{e^{i|\xi|^2(t-s)}}{|\tau|^{1/2}} \, d\tau\right| \lesssim \frac{1}{|t - s|^{1/2}}
\]
which is independent of $\xi$. Thus, it follows
\[ \| TT^* F \|_{L^4(dt)L^2(dx)} \lesssim \left\| \int_{-\infty}^{\infty} ds \frac{\| \hat{F}(s, \cdot) \|_{L^2(\mathbb{R})}}{|t-s|^{1/2}} \right\|_{L^4(dt)}. \]

Now, apply Hardy-Littlewood-Sobolev inequality $\frac{n}{p} = \frac{n}{q} - n + \alpha$, with $n = 1, p = 4/3$ and $q = 4$ we have that
\[ \left\| \int_{-\infty}^{\infty} ds \frac{\| \hat{F}(s, \cdot) \|_{L^2(\mathbb{R})}}{|t-s|^{1/2}} \right\|_{L^4(dt)} \approx \left\| F(t, x) \right\|_{L^{4/3}(dt)L^2(dx)} \]
which means $TT^*$ is a bounded operator. Hence it follows from the $TT^*$ principle that $T$ is also a bounded operator, i.e.
\[ \| TF \|_{L^4(dt)L^2(dx)} \lesssim \| F \|_{L^2(dt\, dx)} \]
or equivalently
\[ \| F(t, x) \|_{L^4(dt)L^2(dx)} \lesssim \| |\tau - |\xi|^2|^{1/4} \hat{F}(\tau, \xi) \|_{L^2(d\tau d\xi)}. \]

\[ \square \]

As an immediate corollary of Proposition 6.2, we have that

\[ \textbf{Corollary 6.3.} \text{ Suppose $\Lambda$ solves $S\Lambda = 0$, then for every } 0 < \epsilon < 1 \text{ we have} \]
\[ \| \nabla_x \Lambda(t, x, x) \|_{L^4(dt)L^2(dx)} \lesssim \| \nabla_x \Lambda_0(x, y) \|_{L^2(dx\, dy)}. \] (34)

**Proof.** If $S\Lambda = 0$, then $S\nabla_{x+y}\Lambda = 0$. Applying the previous estimate, we obtain the estimate
\[ \| (\nabla_{x+y}\Lambda)(t, x, x) \|_{L^4(dt)L^2(dx)} \lesssim \| p(t, x)(\nabla_{x+y}\Lambda)(t, x, x) \|_{L^2(dt\, dx)} \]
\[ \lesssim \| \nabla_{x+y}\Lambda_0(x, y) \|_{L^2(dx\, dy)}. \]

Noting the identity
\[ (\nabla_{x+y}\Lambda)(t, x, x) = \frac{1}{2} \nabla_x (\Lambda(t, x, x)), \] (35)
we get the estimate
\[ \| \nabla_x \Lambda(t, x, x) \|_{L^4(dt)L^2(dx)} \lesssim \| \nabla_{x+y}\Lambda_0(x, x) \|_{L^2(dx\, dy)}. \]

Interpolating above estimate with the estimate
\[ \| \Lambda(t, x, x) \|_{L^4(dt)L^2(dx)} \lesssim \| \Lambda_0(x, x) \|_{L^2(dx\, dy)} \]
yields the desired result. \[ \square \]

Let us also record the following Strichartz estimate for the homogeneous $\Lambda$ equation

\[ \textbf{Proposition 6.4 (Non-endpoint Strichartz). Suppose $\Lambda$ is a solution to $S\Lambda = 0$ with initial condition $\Lambda_0$ and $(q, r)$ is an admissible pair as defined in Proposition 3.6. Then it follows} \]
\[ \| e^{it(\Delta_x+\Delta_g)}\Lambda_0 \|_{L^q(dt)L^r(dx)L^2(dy)} \lesssim \| \Lambda_0 \|_{L^2(dx\, dy)}. \] (36)
**Proposition 6.5.** For any number $1^+ > 1$ and arbitrarily close to 1 there exists $\delta > 0$ such that the following estimate holds

$$\| F \|_{X^{-1/2+\delta}} \lesssim T^{\text{some power}} \| F \|_{L^2(0,T) L^1+ (dx) L^2(dy)}. \quad (37)$$

**Proof.** By Proposition 6.4 and Lemma 2.9 in [Tao06], we have the estimate

$$\| F \|_{L^4([0,T] \ L^\infty(dx) L^2(dy)} \lesssim \| F \|_{X^{1/2+\delta}} \quad \text{for all } \delta > 0. \quad (38)$$

Moreover, from (38) we also get the dual estimate

$$\| F \|_{X^{-1/2-\delta}} \lesssim \| F \|_{L^{4/3}([0,T] L^1(dx) L^2(dy)} \lesssim T^{1/4} \| F \|_{L^2([0,T] L^1(dx) L^2(dy)). \quad (39)$$

By linearly interpolating (39) with

$$\| F \|_{X^{-1/2+1/2}} = \| F \|_{L^2([0,T] L^2(dx) L^2(dy)}$$

yields

$$\| F \|_{X^{-1/2+\lambda}} \lesssim T^{\text{some power}} \| F \|_{L^2([0,T] L^{1+} (dx) L^2(dy)} \quad \text{for } -\delta < \lambda < \frac{1}{2} \text{ and some number } 1^+ \text{ depending on } \lambda. \text{ In particular, for any number } 1^+ \text{ arbitrarily close to 1 we can choose } \delta \text{ sufficiently small such that (37) holds.} \quad \square$$

**7 Inhomogeneous $\Lambda$ Equation**

Let us apologize to the reader for the fact that the $\epsilon$ used in this section is equivalent to $(\frac{3}{2} - \epsilon)\epsilon$ used in the other sections.

Consider the inhomogeneous equation

$$S\Lambda = F \quad (40)$$

then it follows from the $X^{a,b}$ energy estimate\(^7\) and Proposition 6.5 we have

$$\| \Lambda(t, x, x) \|_{L^4(dt) L^2(dx)} \lesssim \| \Lambda_0 \|_{L^2(dx dy)} + \| F \|_{X^{-1/2+\delta}} \lesssim \| \Lambda_0 \|_{L^2(dx dy)} + T^{\text{some power}} \| F \|_{L^2(dt) L^{1+} (d(x-y)) L^2(d(x+y))}.$$

Summarizing the above result we obtain the following proposition

**Proposition 7.1.** Suppose $\Lambda$ solves $S\Lambda = F$, then we have

$$\| \Lambda(t, x, x) \|_{L^4(dt) L^2(dx)} \lesssim \| \Lambda_0 \|_{L^2(dx dy)} + T^{\text{some power}} \| F \|_{L^2(dt) L^{1+} (d(x-y)) L^2(d(x+y))}. \quad (41)$$

Using the above proposition, we establish the following proposition

**Proposition 7.2.** Suppose $\Lambda$ solves (12) with initial condition $\Lambda_0$. Then we have

$$\| \Lambda(t, x, x) \|_{L^4(dt) L^2(dx)} \lesssim \| \Lambda_0(x, y) \|_{L^2(dx dy)} + \| F \|_{X^{-1/2+\delta}}. \quad (42)$$

\(^7\text{cf. Tao06 section 2.6}\)
Proof. Since by Proposition 6.2 we have
\[ \| \Lambda(t, x, x) \|_{L^2(dx)} \lesssim \| \Lambda_0(x, y) \|_{L^2(dy)}, \]
then it follows from Lemma 2.9 in [Tao06],
\[ \| F \|_{L^4(dt)L^2(dx)} \lesssim \| F \|_{X^{1/2+\delta}} \]
for any \( \delta > 0 \). In particular, applying the \( X^{s,b} \) energy estimate we get that
\[ \| \chi(t) \Lambda \|_{X^{1/2+\delta}} \lesssim \frac{1}{N} v_N \| \chi(t) \Lambda \|_{L^2(dx)} \lesssim \| F \|_{X^{1/2+\delta}} + \| \Lambda_0(x, y) \|_{L^2(dy)}. \]
Applying Proposition 6.5, we see that
\[ \| \chi(t) \Lambda \|_{X^{1/2+\delta}} \lesssim \frac{1}{N} T^{\text{some power}} \| v_N \chi(t) \Lambda \|_{L^2(dt)L^1+((d(x-y))L^2(dx+y))} \]
\[ \lesssim \frac{1}{N} T^{\text{some power}} \| v_N \|_{L^1+((d(x-y))L^2(dx+y))} \| \chi(t) \Lambda \|_{L^2(dt)L^2((d(x-y))L^2(dx+y))} \]
\[ \lesssim \frac{1}{N^{1-\beta/2}(1+)} T^{\text{some power}} \| \chi(t) \Lambda \|_{L^4(dt)L^2((d(x-y))L^2(dx+y))} \]
\[ \lesssim \frac{1}{N^{1-\beta/2}(1+)} T^{\text{some power}} \| \chi(t) \Lambda \|_{X^{1/2+\delta}} \]
Hence for \( 1+ \) sufficiently close to 1 we are in the perturbative regime. This allows us to absorb the contribution from the potential term \( \frac{1}{N} v_N (x-y) \Lambda \) when \( N \) is sufficiently large. \( \Box \)

Using the above proposition we could show that

**Corollary 7.3.** Suppose \( \Lambda \) solves (12) with initial condition \( \Lambda_0 \). Then for every \( 0 < \epsilon < 1 \) we have
\[ \| \nabla_x \Lambda(t, x, x) \|_{L^4(dt)L^2(dx)} \lesssim \| \nabla_{x+y} \Lambda_0(x, y) \|_{L^2(dx+dy)} + \| \nabla_{x+y} F \|_{X^{1/2+\delta}}. \]

**Proof.** Taking the spatial derivative \( \nabla_{x+y} \) of (12) yields
\[ \left( S + \frac{1}{N} v_N (x-y) \right) \nabla_{x+y} \Lambda = \nabla_{x+y} F \]
since \( [\nabla_{x+y}, N^{-1} v_N (x-y)] = 0 \). Hence by Proposition 7.2 we obtain the estimate
\[ \| (\nabla_{x+y} \Lambda)(t, x, x) \|_{L^4(dt)L^2(dx)} \lesssim \| \nabla_{x+y} \Lambda_0(x, y) \|_{L^2(dx+dy)} + \| \nabla_{x+y} F \|_{X^{1/2+\delta}}. \]
Again, noting the identity (35), we obtain the estimate
\[ \| \nabla_x \Lambda(t, x, x) \|_{L^4(dt)L^2(dx)} \lesssim \| \nabla_{x+y} \Lambda_0(x, y) \|_{L^2(dx+dy)} + \| \nabla_{x+y} F \|_{X^{1/2+\delta}}. \]
Interpolating (12) with (45) yields the desired result. \( \Box \)

Now, let us record some Strichartz estimates.
Proposition 7.4. Suppose $\Lambda$ is a solution to $S\Lambda = F$ with initial condition $\Lambda_0$ and $(q, r)$ is an admissible pair. Then it follows

$$
\| \Lambda(t, x, y) \|_{L^q(dt)L^r(dx)L^2(dy)} \lesssim \| \Lambda_0 \|_{L^2(dx,dy)} + \| F \|_{L^q(dt)L^r(dx)L^2(dy)}.
$$

(46)

In particular, it follows

$$
\| \nabla_{x,y}^\varepsilon \Lambda(t, x, y) \|_{L^q(dt)L^r(dx)L^2(dy)} \lesssim \| \nabla_{x,y}^\varepsilon \Lambda_0 \|_{L^2(dx,dy)} + \| \nabla_{x,y}^\varepsilon F \|_{L^q(dt)L^r(dx)L^2(dy)}.
$$

(47)

Remark 7.5. Let us note that Proposition 7.4 also holds for solution to (12) when $N$ is sufficiently large. To be more precise, by interpolation, we can show

$$
\frac{1}{N} \| \nabla_x^\varepsilon [v_N(x - y)]\Lambda \|_{L^{4/3}[0,T]L^1(dy)\Lambda L^2(dx)} \lesssim \frac{T^{1/2}}{N^{1-\varepsilon\beta}} \| \Lambda \|_{L^1[0,T]L^\infty(dy)\Lambda L^2(dx)}.
$$

(48)

Thus, for any $\beta > 0$, there exists $\epsilon = \epsilon(\beta)$ such that $1 - \epsilon\beta > 0$.

8 The TDHFB Equations

In this section we prove the local well-posedness of our system of nonlinear equations addressed in the introduction. First, let us write down the kernel form of the TDHFB equations

$$
\begin{align*}
\left\{ \frac{1}{i} \frac{\partial}{\partial t} - \Delta_{x_1} \right\} \varphi_t(x_1) &= - \int dy \left\{ v_N(x_1 - y) \rho_t(t, y) \right\} \cdot \varphi_t(x_1) \\
&\quad - \int dy \left\{ v_N(x_1 - y) (\Gamma_t(y, x_1) - \varphi_t(y) \varphi_t(x_1)) \varphi_t(y) \right\} \\
&\quad - \int dy \left\{ v_N(x_1 - y) (\Lambda_t(x_1, y) - \varphi_t(y) \varphi_t(x_1)) \varphi_t(y) \right\} \\
\left\{ \frac{1}{i} \frac{\partial}{\partial t} - \Delta_{x_1} - \Delta_{x_2} \right\} \Gamma_t(x_1, x_2) &= - \int dy \left\{ (v_N(x_1 - y) - v_N(x_2 - y)) \Lambda_t(x_1, y) \Lambda_t(y, x_2) \right\} \\
&\quad - \int dy \left\{ (v_N(x_1 - y) - v_N(x_2 - y)) \Gamma_t(x_1, y) \Gamma_t(y, x_2) \right\} \\
&\quad - \int dy \left\{ (v_N(x_1 - y) - v_N(x_2 - y)) \rho_t(t, y) \Gamma_t(x_1, x_2) \right\} \\
&\quad + 2 \int dy \left\{ (v_N(x_1 - y) - v_N(x_2 - y)) |\varphi_t(y)|^2 \varphi_t(x_1) \varphi_t(x_2) \right\}
\end{align*}
$$

(49)

$$
\begin{align*}
\left\{ \frac{1}{i} \frac{\partial}{\partial t} - \Delta_{x_1} - \Delta_{x_2} + \frac{1}{N} v_N(x_1 - x_2) \right\} \Lambda_t(x_1, x_2) &= - \int dy \left\{ (v_N(x_1 - y) + v_N(x_2 - y)) \rho_t(t, y) \Lambda_t(x_1, x_2) \right\} \\
&\quad - \int dy \left\{ (v_N(x_1 - y) + v_N(x_2 - y)) \Lambda_t(x_1, x_2) \Gamma_t(y, x_2) \right\} \\
&\quad - \int dy \left\{ (v_N(x_1 - y) + v_N(x_2 - y)) \Gamma_t(x_1, y) \Lambda_t(y, x_2) \right\}
\end{align*}
$$

(51)
\[ + 2 \int dy \{ (v_N(x_1 - y) + v_N(x_2 - y))|\varphi_t(y)|^2 \varphi_t(x_1)\varphi_t(x_2) \] 

The space \( X_T \) is some type of Strichartz spaces equipped with the following norms

\[
N_T(\varphi) := \| \langle \nabla_x \rangle^{\frac{3}{2} - \epsilon} \varphi(t, x) \|_{L^4[0,T]L^\infty(dx)} + \| \langle \nabla_x \rangle^{\frac{3}{2} - \epsilon} \varphi(t, x) \|_{L^\infty[0,T]L^2(dx)}
\]

\[
N_T(\Gamma) := \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \Gamma(t, x, y) \|_{L^4[0,T]L^\infty(dx)} + \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \Gamma(t, x, y) \|_{L^\infty[0,T]L^2(dy)}
\]

\[
N_T(\Lambda) := \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \Lambda(t, x, y) \|_{L^4[0,T]L^\infty(dx)} + \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \Lambda(t, x, y) \|_{L^\infty[0,T]L^2(dy)}
\]

\[
N_T(X) := N_T(\varphi)^2 + N_T(\Gamma) + N_T(\Lambda) \lesssim 1.
\]

Similarly, the following estimates hold for the time derivative of \( \varphi, \Gamma, \Lambda, \) i.e.

\[
N_T(\partial_t \varphi) \lesssim \| \langle \nabla_x \rangle^{\frac{3}{2} - \epsilon} \partial_t \varphi_0 \|_{L^2(dx)} + T^{some \ power} (N_T(\varphi)^2 + N_T(\Gamma) + N_T(\Lambda)) N_T(\varphi)
\]

\[
N_T(\partial_t \Gamma) \lesssim \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \Gamma_0 \|_{L^2(dx)} + T^{some \ power} (N_T(\Gamma)^2 + N_T(\Lambda)^2 + N_T(\varphi)^4)
\]

To present the main result of the article

**Theorem 8.1.** Suppose \( \varphi, \Gamma, \) and \( \Lambda \) solve (43), (50) and (51) respectively with Schwartz initial condition \( \varphi_0 \) and \( k_0 \). Then there exists \( N_0 \) such that for all \( N \geq N_0 \) we have that

\[
N_T(\varphi) \lesssim \| \langle \nabla_x \rangle^{\frac{3}{2} - \epsilon} \varphi_0 \|_{L^2(dx)} + T^{some \ power} (N_T(\varphi)^2 + N_T(\Gamma) + N_T(\Lambda)) N_T(\varphi)
\]

\[
N_T(\Gamma) \lesssim \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \Gamma_0 \|_{L^2(dx)} + T^{some \ power} (N_T(\Gamma)^2 + N_T(\Lambda)^2 + N_T(\varphi)^4)
\]

\[
N_T(\Lambda) \lesssim \| \langle \nabla_{x,y} \rangle^{\frac{3}{2} - \epsilon} \Lambda_0 \|_{L^2(dx)} + T^{some \ power} (N_T(\Gamma)N_T(\Lambda) + N_T(\varphi)^4).
\]

In particular, there exists \( T_0 \) such that for all \( T \leq T_0 \) we have that

\[
N_T(X) := N_T(\varphi)^2 + N_T(\Gamma) + N_T(\Lambda) \lesssim 1.
\]

We split the presentation of the proof of the theorem into two subsections.
8.1 Proof Theorem [8.1] for the \( \Gamma \) and \( \Lambda \) Equations

Since the term \((v_N + \rho_T) \cdot \Gamma\) has already been handled in Section 3, it suffices to consider only the terms \((v_N \Lambda) \circ \Lambda\) and \((v_N \Gamma) \circ \Gamma\). In particular, it suffices to consider just the derivative of the terms since any computation for the derivatives will encompass the non-derivative terms. Let us begin by observing

\[
\| \nabla_x^{(3 \epsilon - \epsilon)} \left[ (v_N \Gamma) \circ \Gamma \right] \|_{L^1[0,T]L^2(dx,dy)} \\
\leq \int dz \ |v_N(z)| \| \nabla_x^{(3 \epsilon - \epsilon)} \left[ \Gamma(x, x - z) \right] \|_{L^1[0,T]L^2(dx,dy)} \\
\leq \int dz \ |v_N(z)| \| \nabla_x^{(3 \epsilon - \epsilon)} \left[ \Gamma(x, x - z) \right] \|_{L^2[0,T]L^2(dx,dy)} \| \Gamma \|_{L^2[0,T]L^2(dx,dy)} \\
+ \int dz \ |v_N(z)| \| \Gamma \|_{L^2[0,T]L^2(dx,dy)} \| \nabla_x^{(3 \epsilon - \epsilon)} \|_{L^1[0,T]L^2(dx,dy)} \\
\lesssim T^{\text{some power}} \int dz \ |v_N(z)| \| \nabla_x^{(3 \epsilon - \epsilon)} \|_{L^1[0,T]L^2(dx,dy)} \| \Gamma \|_{L^2[0,T]L^2(dx,dy)} \\
+ T^{\text{some power}} \int dz \ |v_N(z)| \| \nabla_x^{(3 \epsilon - \epsilon)} \|_{L^1[0,T]L^2(dx,dy)} \| \Gamma \|_{L^2[0,T]L^2(dx,dy)}
\]

and

\[
\| \nabla_x^{(3 \epsilon - \epsilon)} \left[ (v_N \Lambda) \circ \Lambda \right] \|_{L^1[0,T]L^2(dx,dy)} \\
\leq \int dz \ |v_N(z)| \| \nabla_x^{(3 \epsilon - \epsilon)} \left[ \Lambda(x, x - z) \right] \|_{L^1[0,T]L^2(dx,dy)} \\
\leq \int dz \ |v_N(z)| \| \nabla_x^{(3 \epsilon - \epsilon)} \left[ \Lambda(x, x - z) \right] \|_{L^1[0,T]L^2(dx,dy)} \| \Lambda \|_{L^1[0,T]L^2(dx,dy)} \\
+ \int dz \ |v_N(z)| \| \Lambda \|_{L^1[0,T]L^2(dx,dy)} \| \nabla_x^{(3 \epsilon - \epsilon)} \|_{L^1[0,T]L^2(dx,dy)} \\
\lesssim T^{1/2} N_T(\Lambda)^2
\]

There are essentially three terms we need to deal with namely \((v_N \ast \rho_T) \Lambda\), \((v_N \Gamma) \circ \Gamma\) and \((v_N \Lambda) \circ \Gamma\). Similar to the handling of the nonlinear terms for the \( \Gamma \) equation, it suffices to look at just the derivatives of the nonlinear terms. For the first term observe

\[
\| \nabla_x^{(3 \epsilon - \epsilon)} \left[ (v_N \ast \rho_T) \Lambda \right] \|_{L^2[0,T]L^1+(d(x-y)+L^2)} \\
\lesssim \| v_N \ast \rho_T \|_{L^2(dt)L^2+(dx)} \| \nabla_x^{(3 \epsilon - \epsilon)} \Lambda \|_{L^1(dt)L^2(d(x-y)+L^2)} \\
+ \| v_N \ast \rho_T \|_{L^2(dt)L^2+(dx)} \| \Lambda \|_{L^1(dt)L^2(d(x-y)+L^2)} \\
\lesssim \| \nabla_x^{(3 \epsilon - \epsilon)} \rho_T \|_{L^2(dtL^2+(dx))} \| \nabla_x^{(3 \epsilon - \epsilon)} \Lambda \|_{L^1(dt)L^2(d(x-y)+L^2)} \\
+ \| \nabla_x^{(3 \epsilon - \epsilon)} \rho_T \|_{L^2(dtL^2+(dx))} \| \Lambda \|_{L^1(dt)L^2(d(x-y)+L^2)} \\
\lesssim N_T(\Gamma) N_T(\Lambda)
\]

The terms \( F = (v_N \Lambda) \circ \Gamma \) and \( \Lambda \circ (v_N \Gamma) \) are handled similarly.
8.2 Proof Theorem 8.1 for the \( \phi \) Equation

Lastly, let us deal with the \( \phi \) equation. Let us begin by stating the following Strichartz estimate.

**Proposition 8.2.** Suppose \( \phi \) is a solution to \( S\phi = F \) with initial condition \( \phi_0 \) and let \((q,r)\) be an admissible pair. Then it follows for all \( \alpha > 0 \) that

\[
\| \nabla_x^\alpha \phi \|_{L^q(dt) L^r(dx)} \lesssim \| \nabla_x^\alpha \phi_0 \|_{L^2(dx)} + \| \nabla_x^\alpha F \|_{L^{3/2}(dt) L^1(dx)}.
\]

(61)

We will deal with terms \( (v_N * \rho_T) \cdot \phi \) and \((v_N \Lambda) \circ \phi \) since the method will work equally well with the other two terms. For the first nonlinearity, we obtain the estimate

\[
\| \nabla_x^{(2\varepsilon - \epsilon)} [(v_N * \rho_T) \cdot \phi] \|_{L^{3/2}(dt) L^2(dx)} \lesssim \| v_N * \nabla_x^{(2\varepsilon - \epsilon)} \rho_T \|_{L^{3/2}(dt) L^2(dx)} \| \phi \|_{L^\infty(dt) L^2(dx)}
\]

\[
+ \| v_N * \rho_T \|_{L^{3/2}(dt) L^2(dx)} \| \nabla_x^{(2\varepsilon - \epsilon)} \phi \|_{L^\infty(dt) L^2(dx)}
\]

\[
\lesssim T^{\text{some power}} N_T(\Gamma) N_T(\phi).
\]

For the second nonlinear term, we have

\[
\| \nabla_x^{(2\varepsilon - \epsilon)} [(v_N \Lambda) \circ \phi] \|_{L^{3/2}(dt) L^2(dx)}
\]

\[
\lesssim \int dz \| v_N(z) \| \| \nabla_x^{(2\varepsilon - \epsilon)} \Lambda(t,x,x,z) \|_{L^{3/2}(dt) L^2(dx)} \| \phi(t,x,z) \|_{L^\infty(dt) L^2(dx)}
\]

\[
+ \int dz \| v_N(z) \| \| \Lambda(t,x,x,z) \|_{L^{3/2}(dt) L^2(dx)} \| \nabla_x^{(2\varepsilon - \epsilon)} \phi(t,x,z) \|_{L^\infty(dt) L^2(dx)}
\]

\[
\lesssim T^{\text{some power}} N_T(\Lambda) N_T(\phi).
\]

8.3 Global Wellposedness of the TDHFB Equations

In this subsection, we prove the global wellposedness of the TDHFB equations. Let us begin by recalling the number and energy conservation laws derived in §9 of \([GM13]^{8}\). Recall the total particle number is given by

\[
N := N \int dx \, \rho_T(t,x) = N \int dx \left\{ |\phi_t(x)|^2 + \frac{1}{N} \gamma_t(x,x) \right\}
\]

and the energy is defined by

\[
\mathcal{E} := N \left\{ \int dx \, |\nabla \phi_t(x)|^2 + \frac{1}{2} \int dxdy \, v_N(x-y)|\Lambda_t(x,y)|^2
\]

\[
+ \frac{1}{2N} \int dxdydz \, v_N(x-y) \phi_t(x) \, \text{sh}(k) \phi_t(y,z) + \phi_t(y) \, \text{sh}(k) \phi_t(x,z)|^2
\]

\[
+ \frac{1}{2N} \int dxdy \, |\nabla_x y \text{sh}(k)\phi_t(x,y)|^2
\]

\[
+ \frac{1}{4N^2} \int dxdy \, v_N(x-y) \left\{ |\gamma_t(x,y)|^2 + \gamma_t(x,x) \gamma_t(y,y) \right\}
\]

**Theorem 8.3** (Conservation Laws). Suppose \((\phi_t, \Gamma_t, \Lambda_t)\) solves the TDHFB equations and \(v \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})\). Then the total particle number and energy is conserved.

\(^8\)cf. Corollary 2.7. and Theorem 2.8 in \([BBC+16]^{8}\)
As an immediate corollary of Theorem 8.3, we have

**Corollary 8.4.** Let \((\varphi_t, \Gamma_t, \Lambda_t)\) be a solution to the TDHFB equations. Then there exists a constant \(C > 0\) such that for any \(T > 0\) and \(0 < s < 1\) we have that

\[
\sup_{t \in [0, T]} \| (\varphi_t, \Gamma_t, \Lambda_t) \|_{X^s} \leq C,
\]

where \(X^s\) is independent of \(N\).

**Proof.** The estimate for \(\varphi_t\) follows immediately by interpolating between the conservation of total particle number and conservation of energy. Next, applying Cauchy-Schwarz and the conservation of total particle number, we obtain the estimate

\[
\| \Gamma(t, \cdot) \|_{L^2(dx)} \leq \| \varphi(t, \cdot) \|_{L^2(dx)} + \frac{1}{N} \| \text{sh}(k) \|_{L^2(dx)} \lesssim 1. \tag{65}
\]

Similarly, using Cauchy-Schwarz and the conservation of energy, we obtain

\[
\begin{align*}
\| \nabla_x \Gamma(t, \cdot) \|_{L^2(dx)} &\leq \| \varphi_t \|_{L^2(dx)} \| \nabla \varphi_t \|_{L^2(dx)} + \frac{1}{N} \| \text{sh}(k) \|_{L^2(dx)} \| \nabla \text{sh}(k) \|_{L^2(dx)} \lesssim 1. \tag{66}
\end{align*}
\]

Interpolating (65) and (66) yields a desired bound for \(\Gamma_t\).

To uniformly bound \(\Lambda_t\), we shall use the trig identity (13) to get the estimate

\[
\| \Lambda(t, \cdot) \|_{L^2(dx)} \leq \| \varphi_t \|_{L^2(dx)} + \frac{1}{N} \| \text{sh}(k) \|_{L^2(dx)} + \frac{1}{N} \| \text{sh}(k) \|_{L^2(dx)} \| \text{ph}(k) \|_{L^2(dx)} \lesssim 1. \tag{67}
\]

By identity (14), we see that \(p \circ p + 2p = \text{sh} \circ \text{sh} \) which means

\[
\| p(k) \|_{L^2(dx)} \leq \| p \circ p + 2p \|_{\text{Tr}} = \| \text{sh}(k) \|_{L^2(dx)} \]

since \(p(k)(x, x) \geq 0\). Hence by the conservation of total particle number we have that

\[
\| \Lambda(t, \cdot) \|_{L^2(dx)} \lesssim 1.
\]

Similarly, we can show that \(\| \nabla_x \Lambda(t, \cdot) \|_{L^2(dx)} \lesssim 1\). \(\square\)

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