MICROPOLAR FLUID FLOWS WITH DELAY
ON 2D UNBOUNDED DOMAINS*

Wenlong Sun

Abstract In this paper, we investigate the incompressible micropolar fluid flows on 2D unbounded domains with external force containing some hereditary characteristics. Since Sobolev embeddings are not compact on unbounded domains, first, we investigate the existence and uniqueness of the stationary solution, and further verify its exponential stability under appropriate conditions – essentially the viscosity $\delta_1 := \min\{\nu, c_a + c_d\}$ is asked to be large enough. Then, we establish the global well-posedness of the weak solutions via the Galerkin method combined with the technique of truncation functions and decomposition of spatial domain.

Keywords Micropolar fluid flow, truncation function, well-posedness, exponential stability.

MSC(2010) 35B40, 35Q35, 35A01, 35A02.

1. Introduction

The micropolar fluid model is firstly derived by Eringen [12] in 1966, which is used to describe the fluids consisting of randomly oriented particles suspended in a viscous medium. The model can be described by the following equations:

$$\begin{cases}
\frac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r\text{rot}\omega + (u \cdot \nabla)u + \nabla p = f, \\
\nabla \cdot u = 0, \\
\frac{\partial \omega}{\partial t} - (c_a + c_d)\Delta \omega + 4\nu_r\omega + (u \cdot \nabla)\omega - (c_0 + c_d - c_a)\nabla \text{div}\omega - 2\nu_r\text{rot}u = \tilde{f},
\end{cases}$$

where $u = (u_1, u_2, u_3)$ is the velocity, $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity field of rotation of particles, $p$ represents the pressure, $f = (f_1, f_2, f_3)$ and $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ stand for the external force and moments, respectively. The positive parameters $\nu, \nu_r, c_0, c_a$ and $c_d$ are the viscosity coefficients. Actually, $\nu$ represents the usual Newtonian viscosity and $\nu_r$ is called the microrotation viscosity. Note that when the gyration is neglected, the micropolar fluid equations are reduced to the classical Navier-Stokes equations.

Micropolar fluid model takes an important role in the fields of applied and computational mathematics, there is a wide literature on the mathematical theory of

*The author was Partially supported by NSFC of China (No.11671134) and China Scholarship Council.
micropolar fluid model (1.1). Here we only illustrate some known results. First,
we must mention that Łukaszewicz [15] obtained fruitful results, including the exis-
tence and uniqueness of solutions for the stationary problems and the existence
of weak and strong solutions for the evolutionary problems, as well as the global
existence of solution for the heat-conducting flows and the applications of the mi-
cropolar fluids in lubrication theory and in porous media, etc. Also, there are some
other papers concentrating on the existence and uniqueness of solutions for the mi-
cropolar fluid flows. Galdi and Rionero [13] showed the existence and uniqueness
theorems, known in the theorem of the Navier-Stokes equations, are valid for the in-
compressible micropolar equations too. Yamaguchi [25] established the existence of
existence of solutions for the micropolar fluid model too. Galdi and Rionero [13]
showed the existence and uniqueness
of weak and strong solutions for the evolutionary problems, as well as the global
existence and uniqueness of solutions for the stationary problems and the existence
we must mention that Lukaszewicz [15] obtained fruitful results, including the ex-
istence and uniqueness of solutions in a bounded or unbounded
domain Ω ⊆ R^3 having a compact C^2-boundary. Zhang [28] investigated the global
existence and uniqueness of classical solutions to the 2D micropolar fluid flows with
fix partial dissipation and angular viscosity. Dong and Zhang [11] proved the global
existence and uniqueness of smooth solutions to the 2D micropolar fluid flows with
zero angular viscosity on unbounded domains. At the same time, the long time
behavior of solutions for the micropolar fluid model has been investigated from var-
ious aspects, see, e.g. [7–10,16,17,21,26,29]. However, to our knowledge, there are
very few articles about the micropolar fluid model with time delay. To date, we
have not found in the literature any work that considers the combination of delay
terms and unbounded domains.

In the real world, delay terms appear naturally, for instance as effects in wind
tunnel experiments (see [18]), in the equations describing the motions of the fluids.
The delay situations may also occur, for example, when we want to control the
system via applying a force which considers not only the present state but also
the history state of the system. There are some articles concerning the pullback
asymptotic behavior of solutions to the nonlinear evolution equations with delays
on bounded or unbounded domains, see, e.g. [3–6,14,20,23,24].

In this paper, we consider the situation that the velocity component in the x_3-
direction is zero and the axes of rotation of particles are parallel to the
3
axis. That is, \( u = (u_1, u_2, 0) \), \( \omega = (0, 0, \omega_3) \), \( f = (f_1, f_2, 0) \), \( \tilde{f} = (0, 0, \tilde{f}_3) \), \( g = (g_1, g_2, 0) \) and
\( \tilde{g} = (0, 0, \tilde{g}_3) \). Let \( \Omega \subseteq R^2 \) be an open set with boundary Γ that is not necessarily
bounded but satisfies the following Poincaré inequality:

\[
\text{There exists } \lambda_1 > 0 \text{ such that } \lambda_1 \| \varphi \|^2_{L^2(\Omega)} \leq \| \nabla \varphi \|^2_{L^2(\Omega)}, \forall \varphi \in H_0^1(\Omega). \quad (1.2)
\]

Then, we discuss the following equations of 2D non-autonomous incompressible
micropolar fluid flows:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - (\nu + \nu_r) \Delta u - 2\nu_r \nabla \times \omega + (u \cdot \nabla) u + \nabla p &= f + g(t, u_t), \quad \text{in } (0, T) \times \Omega, \\
\frac{\partial \omega}{\partial t} - \tilde{\alpha} \Delta \omega + 4\nu_r \omega - 2\nu_r \nabla \times u + (u \cdot \nabla) \omega &= \tilde{f} + \tilde{g}(t, \omega_t), \quad \text{in } (0, T) \times \Omega, \\
\nabla \cdot u &= 0, \quad \text{in } (0, T) \times \Omega, \\
u(0, \cdot) &= (u(0, \cdot), \omega(0, \cdot)) = (\phi_1(0, \cdot), \omega^0(\cdot)), \\
(u(t, \cdot), \omega(t, \cdot)) &= (\phi_1(t, \cdot), \phi_2(t, \cdot)), \quad \text{in } (0, T) \times \Omega.
\end{aligned}
\quad (1.3)
\]
where $T > 0$ is given, $\check{\alpha} := c_\alpha + c_\delta$, $x := (x_1, x_2) \in \Omega$, $(u^0, \omega^0)$ is the initial velocity filed, $g$ and $\hat{g}$ stand for the external force containing some hereditary characteristics $u_t$ and $\omega_t$, which are defined on $[-h, 0]$ as follows:

$$u_t = u_t(\cdot) := u(t + \cdot), \quad \omega_t = \omega_t(\cdot) := \omega(t + \cdot), \quad \forall t \geq 0.$$  \hspace{1cm} (1.4)

In addition, $(\phi_1, \phi_2)$ represents the initial datum in the interval of time $(-h, 0)$, where $h$ is a positive fixed number, and

$$\nabla \times u := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \nabla \times \omega := \left( \frac{\partial \omega_3}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1} \right).$$

For the sake of convenience, we introduce the following useful operators:

$$\begin{align*}
\langle Aw, \phi \rangle &:= \check{\nu} (\nabla u, \nabla \Phi) + \check{\alpha} (\nabla \omega, \nabla \phi_3), \quad \forall w = (u, \omega), \varphi = (\Phi, \phi_3) \in \hat{V}, \\
\langle B(u, w), \phi \rangle &:= ((u \cdot \nabla)w, \phi), \quad \forall u \in V, \ w = (u, \omega) \in \hat{V}, \ \forall \phi \in \hat{V}, \\
N(w) &:= (-2\nu_r \nabla \omega, -2\nu_r \nabla \times u + 4\nu_r \omega), \ \forall w = (u, \omega) \in \hat{V},
\end{align*}$$  \hspace{1cm} (1.5)

where $\check{\nu} = (\nu + \nu_r)$ and the notation $\hat{V}$ will be defined later. Then, we can formulate the weak version of equations (1.3) as follows:

$$\begin{align*}
\frac{\partial w}{\partial t} + Aw + B(u, w) + N(w) &= F(t, x) + G(t, w_t), \quad \text{in } (0, T) \times \Omega, \\
\nabla \cdot u &= 0, \quad \text{in } (0, T) \times \Omega, \\
w &= (u, \omega) = 0, \quad \text{on } (0, T) \times \Gamma, \\
w(0, x) &= w^0(x) = (u^0(x), \omega^0(x)), \ w(t, x) = \phi(t, x), \ t \in (-h, 0), \ x \in \Omega,
\end{align*}$$  \hspace{1cm} (1.6)

where $w := (u, \omega)$, $F(t) = F(t, x) := (f, \hat{f})$ and $G(t, w_t) := (g(t, u_t), \hat{g}(t, \omega_t))$.

There are two goals in writing this thesis.

The first goal is to prove the existence and uniqueness of the stationary solution and to verify its exponential stability, exactly, we reveal that when the viscosity $\delta_1 := \min\{\nu, c_\alpha + c_\delta\}$ is large enough, the weak solution of the evolutionary system (1.6) exponentially approaches the stationary solution as time increasing infinity. In this part, we need pay enough attention and give careful analysis for each term. In addition, the delay term will also increase the difficulty of the estimates.

The second goal is to establish the global well-posedness of the weak solution of system (1.6). Due to the lack of compact embedding in an unbounded domain, which will result in some obstacles in the process of using the classical Galerkin method to prove the existence of solutions. To overcome this difficulty, we utilize the Galerkin method combined with the technique of truncation function and the decomposition of spatial domain, and classical method to complete our purpose.

It is worth to mentioning that the existence and uniqueness of the weak solutions for the Navier-Stokes with delay on smooth bounded domains has been established by Caraballo and Real in [3]. Later, they investigated the asymptotic behaviour of the weak solutions in [4]. Afterwards, Garrido-Atienza and Marín-Rubio in [14] extended the results of [3] to unbounded domains. Moreover, they studied the existence and uniqueness of the stationary solution and its stability. We want to point out that the main idea of this paper originates from paper [14, 20, 27],

For the sake of convenience, we introduce the following useful operators:

$$\begin{align*}
\langle Aw, \phi \rangle &:= \check{\nu} (\nabla u, \nabla \Phi) + \check{\alpha} (\nabla \omega, \nabla \phi_3), \quad \forall w = (u, \omega), \varphi = (\Phi, \phi_3) \in \hat{V}, \\
\langle B(u, w), \phi \rangle &:= ((u \cdot \nabla)w, \phi), \quad \forall u \in V, \ w = (u, \omega) \in \hat{V}, \ \forall \phi \in \hat{V}, \\
N(w) &:= (-2\nu_r \nabla \omega, -2\nu_r \nabla \times u + 4\nu_r \omega), \ \forall w = (u, \omega) \in \hat{V},
\end{align*}$$  \hspace{1cm} (1.5)

where $\check{\nu} = (\nu + \nu_r)$ and the notation $\hat{V}$ will be defined later. Then, we can formulate the weak version of equations (1.3) as follows:

$$\begin{align*}
\frac{\partial w}{\partial t} + Aw + B(u, w) + N(w) &= F(t, x) + G(t, w_t), \quad \text{in } (0, T) \times \Omega, \\
\nabla \cdot u &= 0, \quad \text{in } (0, T) \times \Omega, \\
w &= (u, \omega) = 0, \quad \text{on } (0, T) \times \Gamma, \\
w(0, x) &= w^0(x) = (u^0(x), \omega^0(x)), \ w(t, x) = \phi(t, x), \ t \in (-h, 0), \ x \in \Omega,
\end{align*}$$  \hspace{1cm} (1.6)

where $w := (u, \omega)$, $F(t) = F(t, x) := (f, \hat{f})$ and $G(t, w_t) := (g(t, u_t), \hat{g}(t, \omega_t))$.

There are two goals in writing this thesis.

The first goal is to prove the existence and uniqueness of the stationary solution and to verify its exponential stability, exactly, we reveal that when the viscosity $\delta_1 := \min\{\nu, c_\alpha + c_\delta\}$ is large enough, the weak solution of the evolutionary system (1.6) exponentially approaches the stationary solution as time increasing infinity. In this part, we need pay enough attention and give careful analysis for each term. In addition, the delay term will also increase the difficulty of the estimates.

The second goal is to establish the global well-posedness of the weak solution of system (1.6). Due to the lack of compact embedding in an unbounded domain, which will result in some obstacles in the process of using the classical Galerkin method to prove the existence of solutions. To overcome this difficulty, we utilize the Galerkin method combined with the technique of truncation function and the decomposition of spatial domain, and classical method to complete our purpose.

It is worth to mentioning that the existence and uniqueness of the weak solutions for the Navier-Stokes with delay on smooth bounded domains has been established by Caraballo and Real in [3]. Later, they investigated the asymptotic behaviour of the weak solutions in [4]. Afterwards, Garrido-Atienza and Marín-Rubio in [14] extended the results of [3] to unbounded domains. Moreover, they studied the existence and uniqueness of the stationary solution and its stability. We want to point out that the main idea of this paper originates from paper [14, 20, 27],
Compared with the Navier-Stokes equations studied in [14], the angular velocity field $\omega$ of the micropolar microparticles in the micropolar fluid flows leads to a different nonlinear term $B(u, w)$ and an additional term $N(w)$ in the abstract equation (see (1.6)). For that reason, more delicate estimates and analysis are required in our studies.

Throughout this paper, we denote the usual Lebesgue space and Sobolev space (see [1]) by $L^p(\Omega)$ and $W^{m,p}(\Omega)$ endowed with norms $\| \cdot \|_p$ and $\| \cdot \|_{m,p}$, respectively.

$$\| \varphi \|_p := \left( \int_{\Omega} |\varphi|^p dx \right)^{1/p} \quad \text{and} \quad \| \varphi \|_{m,p} := \left( \sum_{|\beta| \leq m} \int_{\Omega} |D^\beta \varphi|^p dx \right)^{1/p}.$$ 

Especially, we denote $H^m(\Omega) := W^{m,2}(\Omega)$ and $H^1_0(\Omega)$ the closure of $\{ \varphi \in C_0^\infty(\Omega) \}$ with respect to $H^1(\Omega)$ norm.

\begin{align*}
V & := \mathcal{V}(\Omega) := \{ \varphi \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) \mid \varphi = (\varphi_1, \varphi_2), \nabla \cdot \varphi = 0 \}, \\
\hat{V} & := \hat{\mathcal{V}}(\Omega) := V \times C_0^\infty(\Omega), \\
H & := H(\Omega) := \text{closure of } V \text{ in } L^2(\Omega) \times L^2(\Omega), \text{ with norm } \| \cdot \|_H \text{ and dual space } H^*, \\
V & := V(\Omega) := \text{closure of } V \text{ in } H^1(\Omega) \times H^{1/2}(\Omega), \text{ with norm } \| \cdot \|_V \text{ and dual space } V^*, \\
\hat{H} & := \hat{H}(\Omega) := \text{closure of } \hat{V} \text{ in } (L^2(\Omega))^3, \text{ with norm } \| \cdot \|_{\hat{H}} \text{ and dual space } \hat{H}^*, \\
\hat{V} & := \hat{\mathcal{V}}(\Omega) := \text{closure of } \hat{V} \text{ in } (H^{1/2}(\Omega))^3, \text{ with norm } \| \cdot \|_{\hat{V}} \text{ and dual space } \hat{V}^*,
\end{align*}

where $\| \cdot \|_H$, $\| \cdot \|_V$, $\| \cdot \|_{\hat{H}}$ and $\| \cdot \|_{\hat{V}}$ are defined by

\begin{align*}
\|(u, v)\|_H & := \left( \|u\|^2_2 + \|v\|^2_2 \right)^{1/2}, \quad \|(u, v)\|_V := \left( \|u\|^2_{H^1} + \|v\|^2_{H^1} \right)^{1/2}, \\
\|(u, v, w)\|_{\hat{H}} & := \left( \|u\|^2_{\hat{H}} + \|v\|^2_{\hat{H}} + \|w\|^2_{\hat{H}} \right)^{1/2}, \quad \|(u, v, w)\|_{\hat{V}} := \left( \|(u, v)\|^2_{\hat{V}} + \|w\|^2_{\hat{V}^*} \right)^{1/2}.
\end{align*}

$(\cdot, \cdot)$— the inner product in $L^2(\Omega)$, $H$ or $\hat{H}$, $\langle \cdot, \cdot \rangle$— the dual pairing between $V$ and $V^*$ or between $\hat{V}$ and $\hat{V}^*$. Throughout this article, we simplify the notations $\| \cdot \|_2$, $\| \cdot \|_H$ and $\| \cdot \|_{\hat{H}}$ by the same notation $\| \cdot \|$ if there is no confusion. Furthermore, we denote

\begin{align*}
L^p(I; X) & := \text{space of strongly measurable functions on the closed interval } I, \\
\text{with values in the Banach space } X, \text{ endowed with norm} \\
\|\varphi\|_{L^p(I; X)} & := \left( \int_I \|\varphi\|^p_X dt \right)^{1/p}, \text{ for } 1 \leq p < \infty, \\
C(I; X) & := \text{space of continuous functions on the interval } I, \text{ with values in the Banach space } X, \text{ endowed with the usual norm}, \\
& \quad \hookrightarrow \quad \text{the compact embedding between spaces}.
\end{align*}

The rest of this paper is organized as follows. In section 2, we first make some preliminaries. Then, we concentrate on establishing the existence and uniqueness of the stationary, and further verifying its exponential stability, that is, under suitable conditions, the weak solution, the existence of which could be ensured by section 3, exponentially approaches the stationary solutions as time goes to $+\infty$. In section 3, we show the global well-posedness of the weak solutions.
2. Stability of stationary solutions

We divide this section into two subsections. In the first subsection, we make some necessary preliminaries. In the other subsection, we prove the stability of the stationary solutions.

2.1. Preliminaries

To begin with, let us give some useful properties and estimates about the operators defined in (1.5). That is,

**Lemma 2.1.**

1. The operator $A$ is linear continuous from $\hat{V}$ to $\hat{V}^*$ and from $D(A)$ to $\hat{H}$, and so is the operator $N(\cdot)$ from $\hat{V}$ to $\hat{H}$, where $D(A) := \hat{V} \cap (H^2(\Omega))^3$.

2. The operator $B(\cdot, \cdot)$ is continuous from $V \times \hat{V}$ to $\hat{V}^*$. Moreover, for any $u \in V$ and $w \in \hat{V}$, there holds

$$\langle B(u, \psi), \varphi \rangle = -(B(u, \varphi), \psi), \ \forall u \in V, \ \forall \psi \in \hat{V}, \ \forall \varphi \in \hat{V}. \quad (2.1)$$

**Proof.** (1) The continuity of the operators $A$ and $N(\cdot)$ can be deduced directly from their definition. The linearity of the operator $A$ is evident. So we only need check the linearity of the operator $N(\cdot)$. Indeed, for any $\phi = (\Phi, \phi_3) \in \hat{V}$ with $\Phi = (\phi_1, \phi_2)$ and $\psi = (\Psi, \psi_3) \in \hat{V}$ with $\Psi = (\psi_1, \psi_2)$, we have

$$N(\phi) - N(\psi)$$

$$= \langle -2\nu_r (\nabla \times \phi_3 - \nabla \times \psi_3), -2\nu_r (\nabla \times \Phi - \nabla \times \Psi) + 4\nu_r (\phi_3 - \psi_3) \rangle$$

$$= \langle -2\nu_r \left(\frac{\partial \phi_3}{\partial x_2} - \frac{\partial \psi_3}{\partial x_2} - \frac{\partial \phi_3}{\partial x_1} + \frac{\partial \psi_3}{\partial x_1}\right),$$

$$-2\nu_r \left(-\frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} + \frac{\partial \psi_1}{\partial x_2} + 4\nu_r (\phi_3 - \psi_3)\right)$$

$$= \langle -2\nu_r \left(\frac{\partial (\phi_3 - \psi_3)}{\partial x_2}, -\frac{\partial (\phi_3 - \psi_3)}{\partial x_1}\right),$$

$$-2\nu_r \left(-\frac{\partial (\phi_2 - \psi_2)}{\partial x_1} - \frac{\partial (\phi_1 - \psi_1)}{\partial x_2} + 4\nu_r (\phi_3 - \psi_3)\right)$$

$$= \langle -2\nu_r, \nabla^\perp (\phi_3 - \psi_3) - 2\nu_r \nabla \times (\Phi - \Psi) + 4\nu_r (\phi_3 - \psi_3) \rangle = N(\phi - \psi).$$

(2) The continuity of the operator $B(\cdot, \cdot)$ can also be obtained from its definition.

Next, we verify (2.1). In fact, for any $u \in V, w \in \hat{V}$, we have

$$\langle B(u, w), w \rangle = \langle (u \cdot \nabla) w, w \rangle$$

$$= \int_\Omega \left( u_1 \frac{\partial w_1}{\partial x_1} + u_2 \frac{\partial w_2}{\partial x_2} + u_3 \frac{\partial w_3}{\partial x_3} \right) w_1 w_2 w_3 (u_1, w_2, w_3) (w_1, w_2, w_3) dx = \sum_{j=1}^3 \sum_{i=1}^3 \int_\Omega u_i \frac{\partial w_j}{\partial x_i} w_j dx$$

$$= \sum_{j=1}^3 \sum_{i=1}^3 \frac{1}{2} \int_\Omega u_i \frac{\partial w_j^2}{\partial x_i} dx = \frac{1}{2} \sum_{j=1}^3 \sum_{i=1}^3 \left( u_i w_j^2 \right)_{\Omega} = \frac{1}{2} \sum_{j=1}^3 w_j^2 D_i u_i dx$$

$$= -\frac{1}{2} \sum_{j=1}^3 \int_\Omega \sum_{i=1}^3 w_j^2 D_i u_i dx = -\frac{1}{2} \sum_{j=1}^3 w_j (\nabla \cdot u) dx = 0. \quad (2.2)
Hence, (2.1) is valid as a consequence of (2.2). The proof is complete.

We further have

**Lemma 2.2** (see [16,19,26]).

1. There are two positive constants $c_1$ and $c_2$ such that
   \[ c_1 \langle A w, w \rangle \leq \| w \|_{\hat{V}}^2 \leq c_2 \langle A w, w \rangle, \quad \forall w \in \hat{V}. \]  
   (2.3)

2. There exists some positive constant $\lambda_0$ which depends only on $\Omega$, such that for any $(u, \psi, \phi) \in V \times \hat{V} \times \hat{V}$ there holds
   \[ |\langle B(u, \psi), \phi \rangle| \leq \begin{cases} 
   \lambda_0 \| u \|_1 \| \nabla u \|_1 \| \psi \|_1 \| \nabla \psi \|_1, \\
   \lambda_0 \| u \|_1 \| \nabla u \|_1 \| \psi \|_1 \| \nabla \psi \|_1. 
   \end{cases} \]  
   (2.4)

3. There exists a positive constant $c(\nu_r)$ such that
   \[ \| N(\psi) \| \leq c(\nu_r)\| \psi \|_{\hat{V}}, \quad \forall \psi \in \hat{V}. \]  
   (2.5)

In addition, \[ \delta_1 \| \psi \|_{\hat{V}}^2 \leq \langle A \psi, \psi \rangle + \langle N(\psi), \psi \rangle, \quad \forall \psi \in \hat{V}, \]  
   (2.6)

where $\delta_1 := \min \{ \nu, \alpha \}$.

Next, we recall a key lemma from [14] as follows.

**Lemma 2.3.** Let $I$ be a bounded open set of $\mathbb{R}^d$, and $X, E$ are two Banach spaces with $X \hookrightarrow E$. Consider $1 \leq r < q \leq \infty$. Suppose $F \subset L^r(I; E)$ satisfies

(i) $\forall \omega \subset I$, \( \sup_{f \in F} \| \Pi_h f - f \|_{L^q(\omega; E)} \to 0 \) as $h \to 0$,

where $\Pi_h$ is the translation function: $(\Pi_h f)(t) = f(t + h)$,

(ii) $F$ is bounded in $L^q(I; E) \cap L^1(I; X)$.

Then $F$ is precompact in $L^r(I; E)$.

Finally, we end this subsection with the definition of weak solution of (1.6).

**Definition 2.1.** For each $T > 0$, function $w$ is called a weak solution of (1.6) if, $w = (u, \omega) \in C^0([0, T]; \hat{H}) \cap L^2(-h, T; \hat{V})$ is such that for any $t \in (0, T)$ and any $\varphi \in \hat{V}$,

\[
\begin{align*}
\frac{d}{dt} \langle w(t), \varphi \rangle + \langle A w, \varphi \rangle + \langle B(u, w), \varphi \rangle + \langle N(w), \varphi \rangle &= \langle F(t, x), \varphi \rangle + \langle G(t, w_t), \varphi \rangle, \\
w(0) &= w^0, \quad w(t) = \phi(t), \quad t \in (-h, 0)
\end{align*}
\]

holds in the distribution sense of $D'(0, T)$.

### 2.2. Stability of stationary solutions

In this subsection, we prove the existence and uniqueness of stationary solutions to the micropolar fluid flows provided the viscosity is large enough, when the delay
term has a special form. Furthermore, in a little stronger conditions, we verify its exponential stability.

From now on, we suppose that $\rho(\cdot) \in C^1([0, T])$, $\rho(t) \geq 0$ for all $t \in [0, T]$, $h = \max_{t \in [0, T]} \rho(t) > 0$ and $\rho_* = \max_{t \in [0, T]} \rho'(t) < 1$, and the external force $F$ is independent of time, the delay term $G(t, w_t) = \hat{G}(w(t - \rho(t)))$ with $\hat{G} : \bar{H} \to \bar{H}$ satisfies

$$
\begin{align*}
&\begin{cases}
(i) \hat{G}(0) = 0,
(ii) \text{there exists } L_0 > 0 \text{ such that } \|\hat{G}(w) - \hat{G}(v)\| \leq L_0\|w - v\|, \forall \ w, \ v \in \bar{H}.
\end{cases}
\end{align*}
$$

In the following, we concentrate on establishing the existence and uniqueness of the stationary of (1.6). That is, to find a function $w^* = (u^*, \omega^*) \in \hat{V}$ such that

$$
(Aw^*, v) + \langle B(u^*, w^*), v \rangle + \langle N(w^*), v \rangle = (F, v) + \langle \hat{G}(w^*), v \rangle, \forall \ v \in \hat{V}. \quad (2.8)
$$

**Theorem 2.1.** Suppose that $\hat{G}$ satisfies (2.7) and $\delta_1 > \lambda_1^{-1} L_0$. Then, for any $F \in \hat{V}^*$,

1. there exists at least one solution to (2.8),

2. under the extra condition: $\lambda_1^2 (\delta_1 - \lambda_1^{-1} L_0)^2 > \lambda_0 \|F\|_{\hat{V}^*}$, there corresponds at most one solution to (2.8).

*Where the constant $\lambda_0$ comes from (2.4).*

**Proof.** (1) Existence of stationary solutions. Firstly, we take an orthonormal basis $\{v_j\}_{j=1}^\infty \subset \hat{V}$ of $\hat{H}$ such that the span $\{v_1, v_2, \ldots, v_n, \ldots\}$ is dense in $\hat{V}$. Denote $\hat{V}_m := \text{span}\{v_1, v_2, \ldots, v_m\}$ and consider the following problem:

$$
\begin{align*}
\begin{cases}
&\text{To find } w^m \in \hat{V}_m \text{ such that, for a fixed } q^m = (q^m_1, q^m_2, q^m_3) = (p^m, q^m_3) \in \hat{V}_m, \\
&\sigma(w^m, v^m) = (F, v^m) + \langle \hat{G}(q^m), v^m \rangle, \forall v^m \in \hat{V}_m,
\end{cases}
\end{align*}
$$

where $\sigma(w^m, v^m) := \langle Aw^m, v^m \rangle + \langle B(p^m, w^m), v^m \rangle + \langle N(w^m), v^m \rangle$. On one hand, it is not difficult to check that, for a fixed $q^m = (p^m, q^m_3) \in \hat{V}_m$, the function $\sigma(\cdot, \cdot)$ is bilinear, continuous and coercive in $\hat{V}_m \times \hat{V}_m$. On the other hand, the function $v^m \mapsto (F, v^m) + \langle \hat{G}(q^m), v^m \rangle$ is obviously linear and continuous. Therefore, by Lax-Milgram theorem, for each fixed $q^m = (p^m, q^m_3) \in \hat{V}_m$, there exists a unique solution to problem (2.9), which we denote as $w^m$. Then, consider the map $E_m(\cdot) : \hat{V}_m \mapsto \hat{V}_m$, which is defined by

$$
E_m(q^m) = E_m((p^m, q^m_3)) = (u^m, \omega^m) = w^m.
$$

Next, we are devoted to proving that, for each $m$, there exists at least one fixed point of the mapping $E_m(\cdot)$. This implies that there exists a $w^m \in \hat{V}_m$ satisfying

$$
\begin{align*}
&\langle Aw^m, \ v^m \rangle + \langle B(u^m, \ w^m), \ v^m \rangle + \langle N(w^m), \ v^m \rangle \\
&= (F, v^m) + \langle \hat{G}(w^m), \ v^m \rangle, \forall v^m \in \hat{V}_m. \quad (2.10)
\end{align*}
$$

In order to proceed, taking $v^m = w^m$ in (2.9), and using (1.2), (2.2), (2.6) and (2.7), we deduce

$$
\delta_1 \|w^m\|_{\hat{V}}^2 \leq (F, w^m) + \langle \hat{G}(q^m), w^m \rangle \leq \|F\|_{\hat{V}^*} \|w^m\|_{\hat{V}} + \|\hat{G}(q^m)\| \|w^m\|.
$$
which implies
\[ \delta_1 \| w^m \|_{\hat{V}} \leq \| F \|_{\hat{V}} + \lambda_1^{-1} L_0 \| q^m \|_{\hat{V}}. \]  
(2.11)

Since \( \delta_1 > \lambda_1^{-1} L_0 \), we can take \( k > 0 \) such that \( k(\delta_1 - \lambda_1^{-1} L_0) \geq \| F \|_{\hat{V}}, \) then,
\[ \delta_1 \| w^m \|_{\hat{V}} \leq k(\delta_1 - \lambda_1^{-1} L_0) + \lambda_1^{-1} L_0 \| q^m \|_{\hat{V}}. \]  
(2.12)

Now, define \( K_m = \{ q \in \hat{V}_m \mid \| q \|_{\hat{V}} \leq k \} \), which is a convex compact set of \( \hat{V} \).

Then, it follows from (2.12) that \( E_m(\cdot) \) maps \( K_m \) to \( K_m \). In the following, we are going to apply the Brouwer fixed point theorem to \( E_m(\cdot)|_{K_m} \). For this end, it only remains to show that \( E_m \) is continuous. Indeed, take \( q_i^m \in K_m \), \( i = 1, 2 \), and denote \( w_i^m = (u_i^m, \omega_i^m) = E_m(q_i^m) = E_m(p_i^m, q_i^m) \) the respective solutions of (2.9). Then, it holds that
\[
\langle A(w_i^m - w_2^m), v^m \rangle + \| B(p_i^m, w_i^m) - B(p_2^m, w_2^m), v^m \rangle + \langle N(w_i^m - w_2^m), v^m \rangle
= (\hat{G}(q_i^m) - \hat{G}(q_2^m), v^m), \quad \forall \ v^m \in \hat{V}_m.
\]

Particularly, choose \( v^m = w_1^m - w_2^m \), with the aid of (1.2), (2.2), (2.4), (2.6) and (2.7), the above inequality gives
\[
\begin{align*}
\delta_1 & \| w_1^m - w_2^m \|_{\hat{V}}^2 \\
\leq & \langle B(p_2^m, w_2^m) - B(p_1^m, w_1^m), v_1^m - v_2^m \rangle + \langle \hat{G}(q_1^m) - \hat{G}(q_2^m), w_1^m - w_2^m \rangle \\
\leq & \langle B(p_2^m - p_1^m, \omega_2^m), v_1^m - v_2^m \rangle + \| \hat{G}(q_1^m) - \hat{G}(q_2^m) \| \| w_1^m - w_2^m \| \\
\leq & \lambda_0 \| p_2^m - p_1^m \| \| \omega_2^m \|_{V_{1,2}} \| v_1^m - v_2^m \|_{\hat{V}} + \| \hat{G}(q_1^m) - \hat{G}(q_2^m) \| \| w_1^m - w_2^m \|_{\hat{V}} \\
+ & L_0 \| q_1^m - q_2^m \| \| w_1^m - w_2^m \|_{\hat{V}} \\
\leq & \lambda_1^{-\frac{1}{2}} \lambda_0 \| p_1^m - p_2^m \| \| \omega_1^m \|_{\hat{V}} \| w_1^m - w_2^m \|_{\hat{V}} + \lambda_1^{-1} L_0 \| q_1^m - q_2^m \| \| w_1^m - w_2^m \|_{\hat{V}} \\
\leq & (\lambda_1^{-\frac{1}{2}} \lambda_0 \| p_1^m - p_2^m \|_{\hat{V}} + \lambda_1^{-1} L_0) \| q_1^m - q_2^m \| \| w_1^m - w_2^m \|_{\hat{V}},
\end{align*}
\]
(2.13)
where we also used Hölder inequality and the facts \( q_i^m = (p_i^m, q_i^m), w_i^m = E_m(q_i^m) \in K_m \). The inequality (2.13) implies \( E_m \) is continuous. At this stage, we can conclude that, for each \( m \in \mathbb{N} \), there exists a \( w^m = (u^m, \omega^m) \in \hat{V}_m \) satisfying (2.10).

Finally, we will pass to the limit in (2.10) to obtain the existence of solutions of (2.8). Similar to (2.11), taking \( v^m = w^m \) in (2.10), we obtain
\[ \delta_1 \| w^m \|_{\hat{V}} \leq \| F \|_{\hat{V}}, \]  
that is,
\[ \| w^m \|_{\hat{V}} \leq \frac{1}{\delta_1 - \lambda_1^{-1} L_0} \| F \|_{\hat{V}}, \]  
(2.14)

So, we may extract a subsequence (denoting by the same symbol) \( \{ w^m \} \) such that
\[ w^m \to w \quad \text{weakly in } \hat{V}. \]  
(2.15)

Moreover, for any regular bounded set \( Q \subset \Omega \), we have the same uniform bounds of \( w^m|_Q \), which means, using the compact injection, that
\[ w^m|_Q \to w|_Q \quad \text{strongly in } (L^2(Q))^3. \]  
(2.16)
Based on the above argument, for a fixed \( v_j \in \{ v_j \}_{j=1}^{\infty} \), denote by \( Q_j \) the support of \( v_j \) (which is compact) and take \( Q \subset \Omega \) a bounded open set with smooth boundary containing \( Q_j \), then we not only have the weak convergence \( w^m \to w \) in \( \hat{V}(Q_j) \) but the strong convergence \( w^m \to w \) in \((L^2(Q_j))^3\). Furthermore, it holds that
\[
|\langle \hat{G}(w^m), v_j \rangle - \langle \hat{G}(w), v_j \rangle| \leq \| \hat{G}(w^m) - \hat{G}(w) \|_{(L^2(Q_j))^3} \| v_j \| \leq L_0 \| w^m - w \|_{(L^2(Q_j))^3} \| v_j \| \to 0, \quad \text{as } m \to \infty.
\]
At last, combining with (2.15), we may pass to the limit with respect to \( m \) to (2.8) under the extra condition \( \lambda w \)

one function

pletes the proof.

The strong convergence

Theorem 2.2. Assume that \( F \in \hat{V}^* \), \( \hat{G} \) satisfies (2.7) and the delay term \( G(t, w_t) \)
in (1.6) is given by \( G(t, w_t) = \hat{G}(w(t - \rho(t))) \). Suppose also that
\[
\delta_1 \lambda_0 > L_0 \quad \text{and} \quad \delta_1 \lambda_1 > \frac{L_0}{\sqrt{1 - \rho^2}} + \frac{\lambda_0^2 \| F \|_{\hat{V}^*}}{\delta_1 - \lambda_1^2 L_0}.
\]

Then, for all \( w_0 \in \hat{H} \) and \( \phi \in L^2(-h, 0; \hat{V}) \), the solution \( w(t) \) of (1.6) with \( F(t) \equiv F \) exponentially approaches the solution \( w^* \) of (2.8) as \( t \) goes to \( +\infty \). To be exact, there exists two positive constants \( r_1 \) and \( r_2 \) such that
\[
\| w(t) - w^* \|^2 \leq r_1 e^{-r_2 t}(\| w_0 - w^* \|^2 + \| \phi - w^* \|_{L^2(-h, 0; \hat{V})}^2), \quad \forall t \geq 0.
\]
Proof. Set \( \bar{w}(t) = w(t) - w^* \), and observe that

\[
\frac{d}{dt} (\bar{w}(t), v) + (A\bar{w}(t), v) + (B(u(t), w(t)), v) - (B(u^*, w^*), v) + (N(\bar{w}(t)), v) \\
= (\hat{G}(w(t - \rho(t))), v) - (\hat{G}(w^*), v) \quad \text{for all } v \in \hat{V}.
\]

In particular, it holds that

\[
\frac{1}{2} \frac{d}{dt} \| \bar{w}(t) \|^2 + (A\bar{w}(t), \bar{w}(t)) + (N(\bar{w}(t)), \bar{w}(t)) \\
= (\hat{G}(w(t - \rho(t))), \bar{w}(t)) - (\hat{G}(w^*), \bar{w}(t)) - (B(u(t), w(t)), \bar{w}(t)) + (B(u^*, w^*), \bar{w}(t)).
\]

From (1.2), (2.2) and (2.4), we see that

\[
| - (B(u(t), w(t)), \bar{w}(t)) + (B(u^*, w^*), \bar{w}(t)) | = |(B(u(t) - u^*, \bar{w}(t)), w^*) | \\
\leq \lambda_0 \| u(t) - u^* \|^2 \| \bar{w}(t) \|_\tilde{V} \| w^* \|^2 \| \bar{w}(t) \|_\tilde{V}^{\frac{1}{2}} \\
\leq \lambda_1^{-\frac{1}{2}} \lambda_0 \| \bar{w}(t) \|^2 \| w^* \|_\tilde{V}.
\]

In addition, it follows from (1.2) and (2.7) that

\[
(\hat{G}(w(t - \rho(t))), \bar{w}(t)) - (\hat{G}(w^*), \bar{w}(t)) \leq \| \hat{G}(w(t - \rho(t))) - \hat{G}(w^*) \| \| \bar{w}(t) \| \\
\leq L_0 \| w(t - \rho(t)) - w^* \| \| \bar{w}(t) \| \leq \lambda_1^{-1} L_0 \| \bar{w}(t - \rho(t)) \|_\tilde{V} \| \bar{w}(t) \|_\tilde{V}.
\]

Taking (2.6), (2.20)-(2.22) and the estimate \( \| w^* \|_\tilde{V} \leq \frac{\| F \|_{\hat{V}^*}}{\delta_1 - \lambda_1^{-1} L_0} \) (which can be deduced from (2.14)) into account, we have

\[
\frac{d}{dt} \| \bar{w}(t) \|^2 + 2\delta_1 \| \bar{w}(t) \|^2 \leq 2\lambda_1^{-1} L_0 \| \bar{w}(t - \rho(t)) \|_\tilde{V} \| \bar{w}(t) \|_\tilde{V} + \frac{2\lambda_1^{-\frac{1}{2}} \lambda_0 \| F \|_{\hat{V}^*}}{\delta_1 - \lambda_1^{-1} L_0} \| \bar{w}(t) \|^2 \\
\leq \left( \frac{\delta L_0}{\lambda_1} + \frac{2\lambda_1^{-\frac{1}{2}} \lambda_0 \| F \|_{\hat{V}^*}}{\delta_1 - \lambda_1^{-1} L_0} \right) \| \bar{w}(t) \|^2 + \frac{L_0}{\delta \lambda_1} \| \bar{w}(t - \rho(t)) \|^2_\tilde{V},
\]

where \( \delta \) is a positive constant specified later. Obviously, the above inequality yields

\[
\frac{d}{dt} (e^{rt} \| \bar{w}(t) \|^2) \\
\leq r_2 e^{rt} \| \bar{w}(t) \|^2 - 2\delta_1 e^{rt} \| \bar{w}(t) \|^2 \leq e^{rt} \left( \frac{\delta L_0}{\lambda_1} + \frac{2\lambda_1^{-\frac{1}{2}} \lambda_0 \| F \|_{\hat{V}^*}}{\delta_1 - \lambda_1^{-1} L_0} \right) \| \bar{w}(t) \|^2 \\
+ \frac{L_0}{\delta \lambda_1} e^{rt} \| \bar{w}(t - \rho(t)) \|^2_\tilde{V} \\
\leq \lambda_1^{-1} r_2 e^{rt} \| \bar{w}(t) \|^2 + \frac{2\lambda_1^{-\frac{1}{2}} \lambda_0 \| F \|_{\hat{V}^*}}{\delta_1 - \lambda_1^{-1} L_0} \| \bar{w}(t) \|^2 + \frac{L_0}{\delta \lambda_1} e^{rt} \| \bar{w}(t - \rho(t)) \|^2_\tilde{V},
\]

where \( r_2 \) is a positive constant which will be specified later. Consequently, for any \( t \in [0, T] \),

\[
e^{rt} \| \bar{w}(t) \|^2 \leq \| \bar{w}(0) \| + \lambda_1^{-1} (r_2 - 2\delta_1 \lambda_1 + \delta L_0 + \frac{2\lambda_1^{-\frac{1}{2}} \lambda_0 \| F \|_{\hat{V}^*}}{\delta_1 - \lambda_1^{-1} L_0}) \int_0^t e^{r(t - \theta)} \| \bar{w}(\theta) \|^2_\tilde{V} d\theta
\]
there exists

Assume that

In this section, we concentrate on proving the global existence, uniqueness and

3. Global well-posedness of the weak solutions

Now, we give a further estimate for (2.23). Set \( \eta(t) := t - \rho(t) \), then \( \eta(t) \) is a strictly increasing function. Since \( \rho(t) \in [0, h] \), we may conclude that \( \eta^{-1}(s) \leq s + h \).

Observe that

Substituting the above inequality into (2.23), we obtain

\[
\begin{align*}
&\leq \|\bar{w}(0)\|^2 + \frac{L_0 e^{r_2 h}}{\delta \lambda_1 (1 - \rho^* )} \int_{-h}^{t} e^{r_2 s} \|\bar{w}(s)\|_V^2 \, ds \\
&+ \lambda_1^{-1} (r_2 - 2 \delta_1 \lambda_1 + \delta L_0 + \frac{2 \lambda_1^2 \lambda_0}{\delta_1 - \lambda_1^{-1} L_0} ) \int_0^t e^{r_2 \theta} \|\bar{w}(\theta)\|_V^2 \, d\theta \\
&\leq \|\bar{w}(0)\|^2 + \frac{L_0 e^{r_2 h}}{\delta \lambda_1 (1 - \rho^* )} \int_{-h}^{0} \|\bar{w}(s)\|_V^2 \, ds \\
&+ \lambda_1^{-1} (\frac{L_0 e^{r_2 h}}{\delta (1 - \rho^* )} ) + r_2 - 2 \delta_1 \lambda_1 + \delta L_0 + \frac{2 \lambda_1^2 \lambda_0}{\delta_1 - \lambda_1^{-1} L_0} \int_0^t e^{r_2 \theta} \|\bar{w}(\theta)\|_V^2 \, d\theta.
\end{align*}
\]

Thanks to (2.18), choosing an appropriate \( \delta \) such that \( \frac{L_0}{\delta (1 - \rho^* )} + \delta L_0 = \frac{2L_0}{\sqrt{1 - \rho^*}} \), then there exists \( r_2 > 0 \) small enough such that

\[
\frac{L_0}{\delta (1 - \rho^* )} e^{r_2 h} + r_2 - 2 \delta_1 \lambda_1 + \delta L_0 + \frac{2 \lambda_1^2 \lambda_0}{\delta_1 - \lambda_1^{-1} L_0} \leq 0.
\]

Consequently, we deduce that

\[
\|\bar{w}(t)\|^2 \leq e^{-r_2 t} \|\bar{w}(0)\|^2 + \frac{L_0 e^{r_2 h}}{\delta \lambda_1 (1 - \rho^* )} e^{-r_2 t} \int_{-h}^{0} \|\bar{w}(s)\|_V^2 \, ds.
\]

Therefore, (2.19) is satisfied with \( r_1 = \max \{1, \frac{L_0 e^{r_2 h}}{\delta \lambda_1 (1 - \rho^* )} \} \). This completes the proof.

3. Global well-posedness of the weak solutions

In this section, we concentrate on proving the global existence, uniqueness and stability of the weak solution to system (1.6).

In order to establish the global well-posedness of the weak solutions, the following assumption is required.

(A) Assume that \( G : [0, T] \times L^2(-h, 0; \tilde{H}) \rightarrow (L^2(\Omega))^3 \) satisfies:

(i) For any \( \xi \in L^2(-h, 0; \tilde{H}) \), the mapping \( [0, T] \ni t \mapsto G(t, \xi) \in (L^2(\Omega))^3 \) is measurable.
(ii) \( G(\cdot, 0) = (0, 0, 0) \).

(iii) There exists a constant \( L_G > 0 \) such that for any \( t \in [0, T] \) and any \( \xi, \eta \in L^2(-h, 0; \hat{H}) \),

\[
\|G(t, \xi) - G(t, \eta)\| \leq L_G \|\xi - \eta\|_{L^2(-h, 0; \hat{H})}.
\]

(iv) There exists \( C_G \in (0, \delta_1) \) such that, for any \( t \in [0, T] \) and any \( w, v \in L^2(-h, T; \hat{H}) \),

\[
\int_0^t \|G(\theta, w) - G(\theta, v)\|^2 d\theta \leq C_G^2 \int_{-h}^t \|w(\theta) - v(\theta)\|^2 d\theta.
\]

Moreover, for any \( t \in [0, T] \), there exists a \( \gamma \in (0, 2\delta_1 - 2C_G) \) such that

\[
\int_0^t e^{\gamma \theta} \|G(\theta, w)\|^2 d\theta \leq C_G^2 \int_{-h}^t e^{\gamma \theta} \|w(\theta)\|^2 d\theta, \quad \forall w \in L^2(-h, T; \hat{H}).
\]

(v) If \( w^m \) converges to \( w \) weakly in \( L^2(-h, T; \hat{V}) \), weakly star in \( L^\infty(0, T; \hat{V}^*) \) and strongly in \( L^2(-h, T; (L^2(\Omega))^3) \) for a bounded open set \( \Omega \subset \Omega \) with smooth boundary, then \( G(\cdot, w^m) \) converges weakly to \( G(\cdot, w) \) in \( L^2(0, T; \hat{H}(\Omega)) \).

Now, we show the existence of the weak solutions in the following theorem.

**Theorem 3.1** (Existence). Assume that \( F(t, x) \in L^2(0, T; \hat{V}^*) \) and \( G(t, w_t) \) satisfies (A), then, for any given initial data \( w^0 \in \hat{H}, \phi \in L^2(-h, 0; \hat{V}) \), there corresponds at least one weak solution to system (1.6).

**Proof.** We will divide the proof into three steps.

**Step One: Local existence and uniqueness of the Galerkin approximate solutions.**

Consider an orthonormal basis \( \{v_j\}_{j=1}^\infty \subset \hat{V} \) of \( \hat{H} \) such that

\[
\text{span} \{v_1, v_2, \ldots, v_n, \ldots\} \text{ is dense in } \hat{V}.
\]

Denote \( \hat{V}_m := \text{span}\{v_1, v_2, \ldots, v_m\} \) and consider the projector

\[
P_m w := \sum_{j=1}^m (w, v_j) v_j, \quad w \in \hat{H} \text{ or } \hat{V}.
\]

For each \( T > 0 \), define \( w^m(t) := \sum_{j=1}^m \beta_{m,j}(t) v_j \), where the coefficients \( \beta_{m,j}(t) \) are desired to satisfy the following Cauchy problem of ordinary differential equations:

\[
\begin{aligned}
\frac{d}{dt}(w^m(t), v_j) + \langle Aw^m(t), v_j \rangle &+ \langle B(w^m(t), w^m(t)), v_j \rangle + \langle N(w^m(t)), v_j \rangle \\
&= \langle F(t, x), v_j \rangle + \langle G(t, w^m_t), v_j \rangle, \quad 1 \leq j \leq m, \ t \in [0, T],
\end{aligned}
\]

(3.1)

\[
w^m(0) = P_m w^0, \quad w^m(t) = P_m \phi(t), \quad t \in (-h, 0).
\]

Based on Theorem 2.1 in [14], the existence and uniqueness of the Galerkin approximate solution follows.

**Step Two: A priori estimates of the Galerkin approximate solutions.**
We now deduce a priori estimates to obtain the global existence of the Galerkin approximate solutions. Multiplying (3.1) by $\beta_{m,j}(t)$, summing up for $j$ from 1 to $m$ and using (2.2) and (2.6), we have
\[
\frac{1}{2} \frac{d}{dt} \| w^m(t) \|^2 + \delta_1 \| w^m(t) \|^2 < 0
\]
Multiplying (3.2) by $e^{\gamma t}$, we obtain
\[
\frac{d}{dt} (e^{\gamma t} \| w^m(t) \|^2) - \gamma e^{\gamma t} \| w(t) \|^2 + 2\delta_1 e^{\gamma t} \| w^m(t) \|^2 \leq 2e^{\gamma t} (F(t), w^m(t)) + 2e^{\gamma t} (G(t, w^m_j), w^m(t)).
\]
Let $0 \leq \theta \leq t \leq T$. Replacing the time variable $t$ with $\theta$ in the above inequality, then integrating it for $\theta$ over $[0, t]$ gives
\[
e^{\gamma t} \| w^m(t) \|^2 + (2\delta_1 - \gamma) \int_0^t e^{\gamma \theta} \| w(\theta) \|^2 d\theta \leq \| w^m(0) \|^2 + 2 \int_0^t e^{\gamma \theta} (F(\theta), w^m(\theta)) d\theta + 2 \int_0^t e^{\gamma \theta} (G(\theta, w^m_j), w^m(\theta)) d\theta.
\]
By Young’s inequality and assumption (A), we see that
\[
2 \int_0^t e^{\gamma \theta} (G(\theta, w^m_j), w^m(\theta)) d\theta \leq 2 \int_0^t e^{\gamma \theta} \| G(\theta, w_j^m) \| \| w^m(\theta) \| d\theta
\]
\[
\leq 2 (\int_0^t e^{\gamma \theta} \| G(\theta, w_j^m) \|^2 d\theta)^{1/2} (\int_0^t e^{\gamma \theta} \| w^m(\theta) \|^2 d\theta)^{1/2}
\]
\[
\leq C_G \int_{-h}^0 e^{\gamma \theta} \| w^m(\theta) \|^2 d\theta + 2C_G \int_0^t e^{\gamma \theta} \| w^m(\theta) \|^2 d\theta,
\]
and
\[
2 \int_0^t e^{\gamma \theta} (F(\theta), w^m(\theta)) d\theta \leq 2 \int_0^t e^{\gamma \theta} \| F(\theta) \| \| w^m(\theta) \| \| \hat{v} \| \| \hat{v} \| d\theta
\]
\[
\leq \alpha^{-1} \int_0^t e^{\gamma \theta} \| F(\theta) \|^2 \| \hat{v} \| \| \hat{v} \| d\theta + \alpha \int_0^t e^{\gamma \theta} \| w^m(\theta) \|^2 \| \hat{v} \| \| \hat{v} \| d\theta,
\]
where $\alpha \in (0, 2\delta_1 - \gamma - 2C_G)$. Substituting (4.4) and (4.5) into (4.3), we have
\[
e^{\gamma t} \| w^m(t) \|^2 + \beta \int_0^t e^{\gamma \theta} \| w(\theta) \|^2 d\theta \leq \| w^m(0) \|^2 + C_G \| P_m \|_{L^2([-h, 0; \Omega])} + \alpha^{-1} \int_0^t e^{\gamma \theta} \| F(\theta) \|^2 \| \hat{v} \| \| \hat{v} \| d\theta,
\]
where $\beta := 2\delta_1 - \gamma - 2C_G - \alpha > 0$. It is obtained easily from the above inequality that there exist two constants $k_1$ and $k_2$ (depending on $w^0, \phi, \delta_1, F, G, h, T$), but not on $m$ nor $t_*$ such that
\[
\sup_{t \in [0, t_*]} \| w^m(t) \|^2 \leq k_1, \quad \int_0^{t_*} \| w^m(\theta) \|^2 \| \hat{v} \| \| \hat{v} \| d\theta \leq k_2.
\]
Moreover, observe that \( w^m = P_m \phi \) in \((-h, 0)\) converges to \( \phi \) in \( L^2(-h, 0; \hat{V}) \). Thus, we can take \( t_\ast = T \) and obtain that

\[
\{w^m\} \text{ is bounded in } L^2(-h, T; \hat{V}) \cap L^\infty(0, T; \hat{H}), \tag{3.7}
\]

which together with the local existence obtained in step one gives to the global existence of the Galerkin approximate solution for all time \( t \in [0, T] \).

**Step Three: Existence of the global weak solutions.**

We will prove that the limit function of the Galerkin approximate solutions is a weak solution of (1.6). Using the diagonal procedure, we deduce from (3.7) that there exists a subsequence (which is still denoted by) \( \{w^m\} \), an element \( w \in L^\infty(0, T; \hat{H}) \cap L^2(0, T; \hat{V}) \) such that

\[
\begin{aligned}
&w^m \rightharpoonup^* \text{ weakly star in } L^\infty(0, T; \hat{H}) \quad \text{as } m \rightarrow \infty, \\
&w^m \rightharpoonup \text{ weakly in } L^2(0, T; \hat{V}) \quad \text{as } m \rightarrow \infty. \tag{3.8}
\end{aligned}
\]

Based on the above argument, we claim that, for any bounded open set \( \Omega \subset \Omega \), there exists a subsequence (depending on \( \Omega \) which we relabel) satisfying

\[
 w^m(t) \rightarrow w(t) \quad \text{strongly in } L^2(0, T; \hat{H}(\Omega)) \quad \text{as } m \rightarrow \infty. \tag{3.9}
\]

For the sake of clarity, we give the proof of (3.9) in the back of the present theorem.

Now, fix an element \( v_j \) and let \( \varphi \in C^1([0, T]) \) with \( \varphi(T) = 0 \). Then, it follows from (3.1) that

\[
\begin{align*}
- \int_0^T (w^m(t), v_j \varphi'(t)) dt &+ \int_0^T (A w^m(t), v_j \varphi(t)) dt \\
&\quad + \int_0^T (B(w^m(t), w^m(t)), v_j \varphi(t)) dt + \int_0^T (N(w^m(t)), v_j \varphi(t)) dt \\
= &\langle w^m(0), v_j \rangle \varphi(0) + \int_0^T \langle F(t), v_j \varphi(t) \rangle dt + \int_0^T \langle G(t, w^m_t), v_j \varphi(t) \rangle dt. \tag{3.10}
\end{align*}
\]

In the following, we are committed to passing to the limit in (3.10) to obtain a weak solution. Choose a subsequence (denoted again by \( w^m \)) by using diagonal procedure that satisfies (3.9) for a sequence of regular bounded open sets \( \Omega_j \subset \Omega \) containing all supports of functions \( v_j \) of the basis. Observe that for every \( v \in \hat{V} \), by the density, there exists a sequence \( \{v_j\} \subset \hat{V} \) such that \( v_j \rightarrow v \) in \( \hat{V} \) as \( j \rightarrow \infty \). Namely, for any \( \epsilon > 0 \), there exists a \( n_\epsilon > 0 \) such that

\[
\|v_j - v\|_{\hat{V}} < \epsilon \quad \text{for all } j \geq n_\epsilon. \tag{3.11}
\]

Thus, from (3.8), (3.9) and assumption (A), it is not difficult to see that for the above \( \epsilon \), there exists a \( m_\epsilon \geq n_\epsilon \) such that, for all \( m \geq m_\epsilon \),

\[
\begin{align*}
&\quad \left| \int_0^T \langle G(t, w^m_t), v_m \varphi(t) \rangle dt - \int_0^T \langle G(t, w_t), v \varphi(t) \rangle dt \right| \\
= &\quad \left| \int_0^T \int_{\Omega} (G(t, w^m_t) - G(t, w_t)) \cdot v_m \varphi(t) dx dt + \int_0^T \int_{\Omega} G(t, w_t) \cdot (v_{m_t} - v) \varphi(t) dx dt \right| \\
= &\quad \left| \int_0^T \int_{Q_{n_\epsilon}} (G(t, w^m_t) - G(t, w_t)) \cdot v_{m_t} \varphi(t) dx dt + \int_0^T \int_{\Omega} G(t, w_t) \cdot (v_{m_t} - v) \varphi dx dt \right|
\end{align*}
\]
\[ \leq \|G(t, w_t^m) - G(t, w_t)\|_{L^2(0, T; \tilde{H}(\mathbb{Q}_{n}))} \|\varphi(t)\|_{L^2(0, T)} \|v_n\| \]
\[ + \|v_{n_*} - v\| \int_0^T \|G(t, w_t)\| \cdot |\varphi(t)| dt \]
\[ \leq C_1 \varepsilon. \]

Similarly, it holds that
\[ |\int_0^T (w^m(t), v_{n_*}, \varphi'(t)) dt - \int_0^T (w(t), v\varphi'(t)) dt| \leq C_2 \varepsilon, \]
\[ |\int_0^T (F(t), v_{n_*}, \varphi(t)) dt - \int_0^T (F(t), v\varphi(t)) dt| \leq C_3 \varepsilon \]
and
\[ |(w^m(0), v_{n_*})\varphi(0) - (w(0), v)\varphi(0)| \leq C_4 \varepsilon. \]

In addition, by (1.5), (2.4), (2.5), (3.7)-(3.9), (3.11) and Lemma 2.1, we have
\[ |\int_0^T (Aw^m(t), v_{n_*}, \varphi(t)) dt - \int_0^T (Aw(t), v\varphi(t)) dt| \]
\[ \leq \max\{\nu + \nu_t, \tilde{\alpha}\} \left( \int_0^T \int_{\Omega} \nabla (w^m - w) \cdot \nabla (v_{n_*} \cdot \varphi(t)) dx dt \right) \]
\[ + \int_0^T \|\nabla w(t)\| \|\nabla (v_{n_*} - v)\| \|\varphi(t)\| dt \]
\[ \leq \max\{\nu + \nu_t, \tilde{\alpha}\} \left( \int_0^T \int_{\Omega} \nabla (w^m - w) \cdot \nabla (v_{n_*} \cdot \varphi(t)) dx dt \right) \]
\[ + \|v_{n_*} - v\| \int_0^T \|\nabla w(t)\| \|\varphi(t)\| dt \leq C_5 \varepsilon, \]
\[ |\int_0^T (B(w^m, w_m), v_{n_*}, \varphi(t)) dt - \int_0^T (B(u, w), v\varphi(t)) dt| \]
\[ = |\int_0^T (B(u^m - u, w^m), v_{n_*}, \varphi(t)) dt + (B(u, w^m - w), v_{n_*}, \varphi(t)) \]
\[ + (B(u, w), (v_{n_*} - v)\varphi(t))) dt| \]
\[ \leq \lambda_0 \sup_{t \in [0, T]} |\varphi(t)| \left\{ \int_0^T \|u^m - u\|_H^{\frac{1}{2}} \|\nabla (u^m - u)\| \|\nabla w^m\| \|\nabla v_{n_*}\| \right. \]
\[ + \int_0^T \|u\|^{\frac{3}{2}} \|\nabla u\|^{\frac{3}{2}} \|\nabla w^m - w\|_H^{\frac{3}{2}} \|\nabla v_{n_*}\| \left. \right\} \]
\[ \leq \lambda_0 \sup_{t \in [0, T]} |\varphi(t)| \]
\[ \cdot \left\{ \|u^m - u\|_L^2(0, T; H(\mathbb{Q}_{n})) \|u^m - u\|_L^1(0, T; V) \|w^m\|_L^2(0, T; \tilde{H}) \|v_{n_*}\|_L^2(0, T; \tilde{H}) \|v_{n_*}\|_V \right. \]
\[ + \|u\|_L^1(0, T; H) \|u\|_L^2(0, T; V) \|w^m - w\|_L^2(0, T; H(\mathbb{Q}_{n})) \|w^m - w\|_L^2(0, T; \tilde{H}) \|v_{n_*}\|_L^2(0, T; \tilde{H}) \|v_{n_*}\|_V \]
which is bounded and open, such that
\[ X = (0,1) \times \mathbb{R}^2, \] with \( r \) and \( \text{Micropolar fluid flows with delay on 2D unbounded domains 371} \)
a truncation argument.
for lack of regularity on the boundary \( \Gamma \), but it does in
\[ \epsilon \]
\[ \text{Since } (w^m(t), v^m) \in H^1(\Omega), \]
\[ \text{we can pass to the limit in } (3.10) \text{ and obtain} \]
\[ -\int_0^T (w(t), v\varphi'(t))dt + \int_0^T (Aw(t), v\varphi(t))dt \]
\[ + \int_0^T (B(u, w), v\varphi(t))dt + \int_0^T (N(w(t)), v\varphi(t))dt \]
\[ = (w^0, \varphi(0)) + \int_0^T (F(t), v\varphi(t))dt + \int_0^T (G(t, w), v\varphi(t))dt, \forall v \in \hat{V}. \] (3.12)
Since \((w(t), v)\varphi(t) \in H^1(0, T)\), we also can obtain an analogous expression to (3.12) with \((w(0), v)\) instead of \((w^0, v)\). This implies \((w(0) - w^0, v) = 0\) for all \(v \in \hat{V}\), hence \(w(0) = w^0\). It makes sense at time \(t = 0\). Writing (3.12) for \(\varphi \in D(0, T), w\) satisfies (1.6) in the distribution sense. This completes the proof.

**Proof of (3.9).** For this purpose, we will check the situation fits to Lemma 2.3 with \(r = 2, q = +\infty, I = (0, T)\).
(1) if \(Q \subset \subset \Omega\), then there exists a finite covering of balls, denoted by \(\hat{Q} \subset \Omega\), which is bounded and open, such that \(X = (H^1(\hat{Q}))^3 \hookrightarrow \hookrightarrow E = (L^2(\hat{Q}))^3\).
(2) For a general \(Q \subset \Omega\) the above comment may be not true since \(Q \) and \(\Omega\) can share part of their boundaries. The compact injection from \(H^1\) may not hold for lack of regularity on the boundary \(\Gamma\), but it does in \(H^1_0\). Therefore, we consider a truncation argument.
Taking a function \(\chi(\cdot) \in C^1(\mathbb{R}^2), \chi(x) \in [0, 1], \forall x \in \mathbb{R}^2\) such that
\[ \chi(x) = \begin{cases} 
1, & |x| \leq 1, \\
0, & |x| \geq 4.
\end{cases} \]
Let $l > 0$ be large enough such that $Q \subset B(0, l)$ and denote
$$
\tilde{Q} := \Omega \cap B(0, 2l), \quad w^{m,l}(x) := w^m(x)\chi(|x|^2/\tilde{l}^2).
$$

Then $X = (H_0^1(\tilde{Q}))^3 \hookrightarrow E = (L^2(\tilde{Q}))^3$. We conserve the original functions $w^m(\cdot)$ on $\Omega \cap B(0, l)$ and, for clarity, continue the proof directly with $w^{m,l}(\cdot)$ instead of $w^m(\cdot)$.

Obviously, the condition (ii) in Lemma 2.3 follows from (3.7). In the following, we concentrate on verifying the condition (i). In fact, we will prove that, for the whole domain $\Omega$,
$$
\sup_{m \in \mathbb{N}} \|\Pi_h w^m - w^m\|_{L^2(0, T; (\Omega))} \to 0 \quad \text{as} \quad h \to 0. \tag{3.13}
$$

Consider $h > 0$ arbitrarily small. From (3.1), we see that, for $(t, t + h) \subset (0, T),
$$
\langle w^m(t + h) - w^m(t), v_j \rangle + \int_t^{t+h} \langle Aw^m(\theta), v_j \rangle d\theta + \int_t^{t+h} \langle B(w^m(\theta), w^m(\theta)), v_j \rangle d\theta \\
+ \int_t^{t+h} \langle N(w^m(\theta)), v_j \rangle d\theta
$$

= \int_t^{t+h} \langle F(\theta), v_j \rangle d\theta + \int_t^{t+h} \langle G(\theta, w^m_0), v_j \rangle d\theta.
$$

Multiplying the above inequality by $\beta_{m,j}(t + h) - \beta_{m,j}(t)$ and summing in $j$, we obtain
$$
\|w^m(t + h) - w^m(t)\|^2
$$
$$
= - \int_t^{t+h} \langle Aw^m(\theta), w^m(t + h) - w^m(t) \rangle d\theta \\
- \int_t^{t+h} \langle B(w^m(\theta), w^m(\theta)), w^m(t + h) - w^m(t) \rangle d\theta \\
- \int_t^{t+h} \langle N(w^m(\theta)), w^m(t + h) - w^m(t) \rangle d\theta + \int_t^{t+h} \langle F(\theta), w^m(t + h) - w^m(t) \rangle d\theta \\
+ \int_t^{t+h} \langle G(\theta, w^m_0), w^m(t + h) - w^m(t) \rangle d\theta. \tag{3.14}
$$

In the following, we estimate the terms on the right-hand side of (3.14) one by one. First, according to the definitions of the operators in (1.5), Lemma 2.1, Lemma 2.2, and using Hölder inequality, we have
$$
- \int_t^{t+h} \langle Aw^m(\theta), w^m(t + h) - w^m(t) \rangle d\theta
$$
$$\leq c_1^{-1} \|w^m(t + h) - w^m(t)\|_2 \int_t^{t+h} \|w^m(\theta)\|_2 d\theta, \tag{3.15}
$$
$$
- \int_t^{t+h} \langle B(w^m(\theta), w^m(\theta)), w^m(t + h) - w^m(t) \rangle d\theta
$$
$$\leq \lambda_0 \|w^m(t + h) - w^m(t)\|_2 \int_t^{t+h} \|w^m(\theta)\|^2 \|\nabla w^m(\theta)\|^{1/2} \|w^m(\theta)\|^{1/2} \|\nabla w^m(\theta)\|^{1/2} d\theta.$$

Then, since $0 \leq \lambda_0 \leq \frac{1}{2}$ and $\lambda_0 \leq \frac{1}{2}$, the integral inequality from Schwartz’s inequality that

\begin{equation}
\int_t^{t+h} \{ N(w^m(\theta)), w^m(t+h) - w^m(t) \} d\theta \\
\leq \| w^m(t+h) - w^m(t) \| \int_t^{t+h} \{ N(w^m(\theta)) \} d\theta \\
\leq c(\nu_r) \| w^m(t+h) - w^m(t) \| \int_t^{t+h} \| N(w^m(\theta)) \| \varphi d\theta.
\end{equation}

(3.17)

Next, it is easy to know from Schwartz’s inequality that

\begin{equation}
\int_t^{t+h} \{ F(\theta), w^m(t+h) - w^m(t) \} d\theta \\
\leq \| w^m(t+h) - w^m(t) \| \int_t^{t+h} \| F(\theta) \| \varphi \cdot d\theta.
\end{equation}

(3.18)

Finally, using (1.2) and Schwartz’s inequality, we obtain

\begin{equation}
\int_t^{t+h} \{ G(\theta, w^m_\theta), w^m(t+h) - w^m(t) \} d\theta \\
\leq \| w^m(t+h) - w^m(t) \| \int_t^{t+h} \| G(\theta, w^m_\theta) \| d\theta \\
\leq \lambda_1^{-\frac{1}{2}} \| w^m(t+h) - w^m(t) \| \int_t^{t+h} \| G(\theta, w^m_\theta) \| d\theta.
\end{equation}

(3.19)

Now, substituting (3.15)-(3.19) into (3.14) and integrating the resultant inequality from $0$ to $T-h$, we have

\begin{equation}
\| \Pi_h \| w^m(t) - w^m(t) \|_{L^2(0,T-h;L^2(\Omega)^3)}^2 = \int_0^{T-h} \int_\Omega | \Pi_h w^m(t) - w^m(t) |^2 dx dt \\
\leq \int_0^{T-h} \| w^m(t+h) - w^m(t) \| \varphi \int_t^{t+h} H_m(\theta) d\theta dt,
\end{equation}

where

\begin{equation}
H_m := (c^{-1} + c(\nu_r)) \| w^m(\theta) \| _\varphi + \lambda_0 \| w^m(\theta) \| \| w^m(\theta) \| \varphi + \| F(\theta) \| _\varphi + \lambda_1^{-\frac{1}{2}} \| G(\theta, w^m_\theta) \|.
\end{equation}

Set

\begin{equation}
\hat{\theta} = \begin{cases} 0, & \text{if } \theta \leq 0, \\
\theta, & \text{if } 0 < \theta < T - h, \\
T - h, & \text{if } \theta > T - h.
\end{cases}
\end{equation}

Then, since $0 \leq \hat{\theta} - \theta - h \leq h$, with the aid of (3.6) and the Fubini theorem, we obtain

\begin{equation}
\int_0^{T-h} \| w^m(t+h) - w^m(t) \| \varphi \int_t^{t+h} H_m(\theta) d\theta dt \\
\leq \int_0^{T} H_m(\theta) \int_{\theta-h}^{\theta} \| w^m(t+h) - w^m(t) \| \varphi dt d\theta
\end{equation}
\[
\leq \int_0^T H_m(\theta)(\int_{\mathbb{R}^n} \|w^m(t+h) - w^m(t)\|^2 dt)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} dt\right)^{\frac{1}{2}} d\theta \\
\leq \int_0^T H_m(\theta) \cdot 2h^{\frac{1}{2}} \left(\int_0^{T-h} \|w^m\|^2 dt\right)^{\frac{1}{2}} d\theta \leq 2(hk_2)^{\frac{1}{2}} \int_0^T H_m(\theta) d\theta. \quad (3.20)
\]

From (3.7) and the assumption (A), it is not difficult to conclude that \(\int_0^T H_m(\theta) d\theta\) is bounded. Therefore, (3.13) is valid, that is, the condition (i) in Lemma 2.3 holds. Consequently, with the help of Lemma 2.3, (3.9) follows.

**Remark 3.1.** On the assumption (A)(v). When condition (A)(v) is satisfied in all “good” \(Q \subset \Omega\) (actually, it is enough for all \(Q_j \supset supp(v_j)\)), then the convergence holds in \(L^2(0, T; \hat{H})\). Indeed, consider \(\varphi \in \hat{H}\). We check that \(\lim_{m \to \infty} (G(\cdot, w\langle m \rangle), \varphi) = (G(\cdot, w\langle \cdot \rangle), \varphi)\). Take a sequence \(\varphi_n \in \hat{V}\) such that \(\varphi_n \to \varphi\) in \(\hat{H}\), and fix \(\epsilon > 0\). Consider \(n_0\) such that \(\|G(\cdot, w\langle m \rangle)\|\|\varphi_n - \varphi\| \leq \frac{\epsilon}{2}\) and \(\|G(\cdot, w\langle \cdot \rangle)\|\|\varphi_n - \varphi\| \leq \frac{\epsilon}{2}\) for all \(n \geq n_0\). Observe that \(G(\cdot, w\langle m \rangle) - G(\cdot, w\langle \cdot \rangle, \varphi_n) \to 0\) so it is possible to choose \(m_\epsilon\) to conclude that the claim is true.

In the following, we investigate the uniqueness of the weak solution.

**Theorem 3.2** (Uniqueness). Under the conditions of Theorem 3.1, for any \(T > 0\) and initial data \(w^0 \in \hat{H}, \phi \in L^2(-h, 0; \hat{V})\), there corresponds at most one weak solution to system (1.6).

**Proof.** Let \(w^{(1)} = (u^{(1)}, \omega^{(1)})\) and \(w^{(2)} = (u^{(2)}, \omega^{(2)})\) be two solutions of (1.6) in the interval \([-h, T]\) with the same initial data. Denote \(w = (u, \omega) = w^{(1)} - w^{(2)}\), then \(w\) satisfies

\[
\frac{\partial w}{\partial t} + Aw + B(1, w^{(1)}) - B(u^{(2)}, w^{(2)}) + N(w) = G(t, w^{(1)}_t) - G(t, w^{(2)}_t), \quad t \in [0, T]. \quad (3.21)
\]

Testing (3.21) by \(w(t)\), we have

\[
\frac{1}{2} \frac{d}{dt}\|w(t)\|^2 + \langle Aw(t), w(t)\rangle + \langle B(1, w^{(1)}), w^{(1)}_t - w^{(2)}_t, w(t)\rangle + \langle N(w(t)), w(t)\rangle \\
= \langle G(t, w^{(1)}_t) - G(t, w^{(2)}_t), w(t)\rangle, \quad t \in [0, T]. \quad (3.22)
\]

By (2.2), (2.4), Young’s inequality and the facts

\[
\|u(t)\| \leq \|w(t)\| \quad \text{and} \quad \|\nabla u(t)\| \leq \|\nabla w(t)\|,
\]

we deduce that

\[
|\langle B(1, w^{(1)}), w^{(1)}_t - w^{(2)}_t, w(t)\rangle| \leq \lambda_0 \|u(t)\|^\frac{1}{2} \|\nabla u(t)\|^\frac{1}{2} \|\nabla w(t)\|^\frac{1}{2} \|\nabla w^{(1)}(t)\|
\leq \lambda_0 \|w(t)\| \|w(t)\| \|w^{(1)}(t)\| \|\varphi\| \leq \frac{\lambda_0^2}{4\delta_1} \|w^{(1)}(t)\|^2 \|w(t)\|^2 + \delta_1 \|w(t)\|^2. \quad (23.23)
\]

In addition, it is easy to see that

\[
(G(t, w^{(1)}_t) - G(t, w^{(2)}_t), w(t)) \leq \|G(t, w^{(1)}_t) - G(t, w^{(2)}_t)\| \|w(t)\|.
\]
where the positive constants $a_1$ and $a_2$ satisfy $a_1a_2 \geq \frac{1}{4}$. Taking (2.6) and (3.22)-(3.24) into account, we have

$$\frac{d}{dt}\|w(t)\|^2 + 2\delta_1 \|w(t)\|^2 \leq \frac{\lambda_0^2}{2\delta_1} \|w^{(1)}(t)\|^2 \|w(t)\|^2 + 2\delta_1 \|w(t)\|^2 + 2a_1 \|w(t)\|^2 + 2a_2 \|G(t, w^{(1)}_t) - G(t, w^{(2)}_t)\|^2. \quad (3.25)$$

Integrating the above inequality over $[0, t]$, we obtain from assumption (A) that

$$\|w(t)\|^2 \leq \frac{\lambda_0^2}{2\delta_1} \int_0^t \|w^{(1)}(\theta)\|^2 \|w(\theta)\|^2 d\theta + 2a_1 \int_0^t \|w(\theta)\|^2 d\theta + 2a_2 \int_0^t \|G(\theta, w^{(1)}_\theta) - G(\theta, w^{(2)}_\theta)\|^2 d\theta \leq \int_0^t \left( \frac{\lambda_0^2}{2\delta_1} \|w^{(1)}(\theta)\|^2 + 2a_1 + 2a_2 C^2_G \right) \|w(\theta)\|^2 d\theta. \quad (3.26)$$

Using the Gronwall inequality to (3.26) yields

$$\|w(t)\| = 0, \quad \forall t \in [0, T].$$

This completes the proof. □

Finally, we verify the stability of the weak solutions with respect to the initial data.

**Theorem 3.3 (Stability).** Assume that the conditions of Theorem 3.1 hold, and let $w^{(i)}(\cdot)$ with $i = 1, 2$ be two solutions of (1.6) in the interval $[-h, T]$ with initial data $w^{(i)}(0) = w^{(0)}(i)$ and $w^{(i)}(s, x) = \phi^{(i)}(s, x), s \in (-h, 0)$, respectively. Then

$$\|w^{(1)}(t) - w^{(2)}(t)\|^2 \leq \left( \|w^{(1)}(0) - w^{(2)}(0)\|^2 + 2a_2 C^2_G \|\phi^{(1)} - \phi^{(2)}\|^2 \right) \times \exp \left\{ \int_0^t \left( \frac{\lambda_0^2}{\delta_1} \|w^{(1)}(\theta)\|^2 + 2a_1 + 2a_2 C^2_G \right) d\theta \right\}, \quad (3.27)$$

$$\int_0^t \|w^{(1)}(\theta) - w^{(2)}(\theta)\|^2 d\theta \leq \delta_1^{-1} \|w^{(1)}(0) - w^{(2)}(0)\|^2 + \frac{\delta_1}{2a_2} C^2_G \|\phi^{(1)} - \phi^{(2)}\|^2 \times \left[ 1 + \int_0^t \left( \frac{\lambda_0^2}{\delta_1} \|w^{(1)}(\theta)\|^2 + 2a_1 + 2a_2 C^2_G \right) \times \exp \left\{ \int_0^\theta \left( \frac{\lambda_0^2}{\delta_1} \|w^{(1)}(s)\|^2 + 2a_1 + 2a_2 C^2_G \right) ds \right\} d\theta \right], \quad (3.28)$$

where the positive constants $a_1$ and $a_2$ satisfy $a_1a_2 \geq \frac{1}{4}$.

**Proof.** Set $u(\cdot) = u^{(1)}(\cdot) - u^{(2)}(\cdot), w(\cdot) = \omega^{(1)}(\cdot) - \omega^{(2)}(\cdot), w(\cdot) = (u(\cdot), \omega(\cdot)) = w^{(1)}(\cdot) - w^{(2)}(\cdot)$, then $w(t)$ is a solution of the following system:

$$\begin{cases}
\frac{\partial w(t)}{\partial t} + Aw + B(u^{(1)}_t, w^{(1)}) - B(u^{(2)}_t, w^{(2)}) + N(w) = G(t, w^{(1)}_t) - G(t, w^{(2)}_t), \\
w(0) = w^{(1)}(0) - w^{(2)}(0), \quad w(t) = \phi^{(1)} - \phi^{(2)}, \quad t \in (-h, 0).
\end{cases} \quad (3.29)$$
Testing (3.29) by \( w(t) \), it holds that
\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|^2 + \langle Aw(t), w(t) \rangle + \langle N(w(t)), w(t) \rangle = (G(t, w_t^{(1)}) - G(t, w_t^{(2)}), w(t)) - (B(u^{(1)}(t), w^{(1)}(t)) + B(u^{(2)}(t), w^{(2)}(t)), w(t)).
\]
(3.30)

Similar to (3.25), we can deduce from (3.30) that
\[
\frac{d}{dt} \|w(t)\|^2 + \delta_1 \|w(t)\|_V^2 \leq \frac{\lambda_2^2}{\delta_1} \|w^{(1)}(t)\|_V^2 \|w(t)\|^2 + 2a_1 \|w(t)\|^2 + 2a_2 \|G(t, u_t^{(1)}) - G(t, u_t^{(2)})\|^2,
\]
where \( a_1, a_2 \) comes from (3.25). Integrating the above inequality over \([0, t] \) and using assumption (A), we obtain
\[
\|w(t)\|^2 + \delta_1 \int_0^t \|w(\theta)\|_V^2 d\theta \leq \|w(0)\|^2 + \frac{\lambda_2^2}{\delta_1} \int_0^t \|w^{(1)}(\theta)\|_V^2 \|w(\theta)\|^2 d\theta + 2a_1 \int_0^t \|w(\theta)\|^2 d\theta + 2a_2 \int_0^t \|G(\theta, w_\theta^{(1)}) - G(\theta, w_\theta^{(2)})\|^2 d\theta
\]
\[
\leq \|w^0\|^2 + \int_0^t \left( \frac{\lambda_2^2}{\delta_1} \|w^{(1)}(\theta)\|_V^2 + 2a_1 + 2a_2 C_G^2 \right) \|w(\theta)\|^2 d\theta + 2a_2 C_G^2 \int_{-h}^0 \|\phi^{(1)} - \phi^{(2)}\|^2 d\theta.
\]
(3.31)

Hence,
\[
\|w(t)\|^2 \leq \|w^0\|^2 + 2a_2 C_G^2 \|\phi^{(1)} - \phi^{(2)}\|^2_{L^2(-h, 0, R)} + \int_0^t \left( \frac{\lambda_2^2}{\delta_1} \|w^{(1)}(\theta)\|_V^2 + 2a_1 + 2a_2 C_G^2 \right) \|w(\theta)\|^2 d\theta.
\]

Using the Gronwall’s inequality to the above inequality yields (3.27). Furthermore, (3.28) follows from (3.27) and (3.31). This completes the proof. \( \square \)

**Remark 3.2.** The existence and uniqueness of the stationary solution of (2.8) and the existence of the weak solutions to system (1.6) can be obtained in the same way in 3D unbounded domains. The key is the nonlinear term \( B(u, w) \), the estimates (which can be deduced by Hölder and Gagliardo-Nirenberg inequality) for this term is different in the cases for 2D and 3D domains. For more detail, one can refer to [22, §2, §3 and §10].

**Remark 3.3.** Existence, uniqueness and stability of the solution have been established under different conditions—essentially the viscosity \( \delta_1 := \min\{\nu, \bar{\alpha}\} \) is asked to be large enough. We also want to point out that there still much work to be done concerning the micropolar fluid flows with delay on unbounded domains. For example, we could study the attractors, further, investigate the regularity, boundedness and tempered behavior of the pullback attractors. These issues will be the topics of some other papers.
Acknowledgements

The authors thank warmly the anonymous reviewers for their pertinent comments and suggestions, which greatly improves the earlier manuscript. The authors also sincerely thank Professor Yeping Li of East China University of Science and Technology and Dr. Guowei Liu of Shanghai Jiao Tong University for their many useful suggestions and elaborate guidance.

References

[1] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] J. L. Boldrini, M. Duran and M. A. Rojas-Medar, Existence and uniqueness of strong solution for the incompressible micropolar fluid equations in domains of $\mathbb{R}^3$, Annali Della Università Di Ferrara Sezione Scienze Matematiche, 2010, 56, 37–51.
[3] T. Caraballo and J. Real, Navier-Stokes equations with delays, Proc. R. Soc. Lond., 2001, 457, 2441–2453.
[4] T. Caraballo and J. Real, Asymptotic behaviour of 2D-Navier-Stokes equations with delays, Proc. R. Soc. Lond., 2003, 459, 3181–3194.
[5] T. Caraballo and J. Real, Attractors for 2D-Navier-Stokes models with delays, J. Differential Equations, 2004, 205, 271–297.
[6] T. Caraballo, P. Marín-Rubio and J. Valero, Autonomous and nonautonomous attractors for differential equations with delays, J. Differential Equations, 2005, 208, 9–41.
[7] J. Chen, Z. Chen and B. Dong, Existence of $H^2$-global attractors of two-dimensional micropolar fluid flows, J. Math. Anal. Appl., 2006, 322, 512–522.
[8] J. Chen, B. Dong and Z. Chen, Uniform attractors of non-homogeneous micropolar fluid flows in non-smooth domains, Nonlinearity, 2007, 20, 1619–1635.
[9] G. Chen, Pullback attractor for non-homogeneous micropolar fluid flows in non-smooth domains, Nonlinear Anal., 2009, 10, 3018–3027.
[10] B. Dong and Z. Chen, Global attractors of two-dimensional micropolar fluid flows in some unbounded domains, Appl. Math. Comp., 2006, 182, 610–620.
[11] B. Dong and Z. Zhang, Global regularity of the 2D micropolar fluid flows with zero angular viscosity, J. Differential Equations, 2010, 249, 200–213.
[12] A. C. Eringen, Theory of micropolar fluids, J. Math. Mech., 1966, 16, 1–18.
[13] G. P. Galdi and S. Rionero, A note on the existence and uniqueness of solutions of the micropolar fluid equations, International Journal of Engineering Science, 1977, 15, 105–108.
[14] M. J. Garrido-Atienza and P. Marín-Rubio, Navier-Stokes equations with delays on unbounded domains, Nonlinear Analysis, 2006, 64, 1100–1118.
[15] G. Lukaszewicz, Micropolar fluids: Theory and Applications, Birkhäuser, Boston, 1999.
[16] G. Lukaszewicz, Long time behavior of 2D micropolar fluid flows, Math. Comput. Modelling, 2001, 34, 487–509.
[17] G. Łukaszewicz and A. Tarasińska, *On $H^1$-pullback attractors for nonautonomous micropolar fluid equations in a bounded domain*, Nonlinear Anal., 2009, 71, 782–788.

[18] A. Z. Manitius, *Feedback controllers for a wind tunnel model involving a delay: analytical design and numerical simulation*, IEEE Trans. Automat Control, 1984, 29, 1058–1068.

[19] P. Marín-Rubio and J. Real, *Attractors for 2D-Navier-Stokes equations with delays on some unbounded domains*, Nonlinear Anal., 2007, 67, 2784–2799.

[20] P. Marín-Rubio, J. Real and J. Valero, *Pullback attractors for a two-dimensional Navier-Stokes model in an infinite delay case*, Nonlinear Anal., 2011, 74, 2012–2030.

[21] B. Nowakowski, *Long-time behavior of micropolar fluid equations in cylindrical domains*, Nonlinear Anal.-RWA, 2013, 14, 2166–2179.

[22] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis, Second Edition*, Society for Industrial and Applied Mathematics, 1995.

[23] T. Taniguchi, *The exponential behavior of Navier-Stokes equations with time delay external force*, Discrete Contin. Dyn. Syst.-A, 2005, 12, 997–1018.

[24] Y. Wang and P. E. Kloeden, *Pullback attractors of a multi-valued process generated by parabolic differential equations with unbounded delays*, Nonlinear Anal., 2013, 90, 86–95.

[25] N. Yamaguchi, *Existence of global strong solution to the micropolar fluid system in a bounded domain*, Mathematical Methods in the Applied Sciences, 2005, 28, 1507–1526.

[26] C. Zhao, S. Zhou and X. Lian, *$H^1$-uniform attractor and asymptotic smoothing effect of solutions for a nonautonomous micropolar fluid flow in 2D unbounded domains*, Nonlinear Anal.-RWA, 2008, 9, 608–627.

[27] C. Zhao, *Pullback asymptotic behavior of solutions for a non-autonomous non-Newtonian fluid on two-dimensional unbounded domains*, J. Math. Phys., 2012, 53, 122702–1–23.

[28] Z. Zhang, *Global Regularity for the 2D Micropolar Fluid Flows with Mixed Partial Dissipation and Angular Viscosity*, Abstract and Applied Analysis, 2014, 1–6.

[29] C. Zhao, W. Sun and C. Hsu, *Pullback dynamical behaviors of the non-autonomous micropolar fluid flows*, Dynamics of PDE, 2015, 12, 265–288.