Extreme positions of regularly varying branching random walk in random environment

A. Bhattacharya · Z. Palmowski

Abstract. In this article, we consider a Branching Random Walk (BRW) on the real line where the underlying genealogical structure is given through a supercritical branching process in i.i.d. environment and satisfies Kesten-Stigum condition. The displacements coming from the same parent are assumed to have jointly regularly varying tails. Conditioned on the survival of the underlying genealogical tree, we prove that the appropriately normalized (depends on the expected size of the $n$-th generation given the environment) maximum among positions at the $n$-th generation converges weakly to a scale-mixture of Frechet random variable. Furthermore, we derive the weak limit of the extremal processes composed of appropriately scaled positions at the $n$-th generation and show that the limit point process is a member of the randomly scaled scale-decorated Poisson point processes (SScDPPP). Hence, an analog of the predictions by Brunet and Derrida \cite{BD19} holds.

Keywords. Regular variation · branching random walk · point process · extreme values · random environment · weak limit · maximum position

Mathematics Subject Classification (2010) Primary 60J70, 60G55 · Secondary 60J80

This work is partially supported by Polish National Science Centre Grant No. 2018/29/B/ST1/00756, 2019-2022.

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1 Introduction

The evolution of the discrete-time BRW with branching process in i.i.d. environment can be described as follows: Let $\Theta$ be a collection of probability measures on $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Then we consider a sequence $Y = (Y_i : i \geq 0)$ of $\Theta$-valued i.i.d. random elements. The process $Y$ will be referred to as the random environment or environment sequence. We assume that the process starts with one initial ancestor and refer to it as the zeroth generation. The position of the initial ancestor is assumed to be zero. Conditioned on the sequence $Y$ at the instant $n$ ($n \geq 1$), each of the particles in the $(n - 1)$-th generation dies after producing an independent copy of the following point process (with real atoms)

$$L_n := \sum_{i=1}^{Z_i^{(n)}} \delta_{X_i},$$

(1.1)

where the law of $Z_i^{(n)}$ is given by $Y_{n-1}$. The newborn particles form the $n$-th generation. The atom corresponding to a newborn particle is called its displacement. Position of a particle is defined to be its displacement translated by the position of its parent. If we run this procedure for a long enough time, the resultant process will be called the BRW with BPRE. In this article, we are mainly interested in the order of fluctuation and joint asymptotic behaviour of a few of the large positions at the $n$-th generation.

Assumptions on the genealogical structure: The size of the $i$-th generation will be denoted by $Z_i$ for $i \geq 0$. We have assumed that $Z_0 = 1$. A generic element of $\Theta$ will be denoted by $\theta$ and $E_\theta(\xi)$ will denote the expectation of the random variable $\xi$ distributed according to the law $\theta$. We will use $P_{Y}(\cdot)$ to denote the conditional probability $P(\cdot|Y)$. Define

$$S := \bigcap_{i \geq 1} \{Z_i > 0\}. \quad (1.2)$$

Let $q_e(Y) := P(S|Y)$ denote the probability of the survival conditioned on the environment and $P^*(\cdot) = P(\cdot|S)$ denote the probability of an event given the genealogical tree survives forever. The conditional expectations associated to the conditional probabilities $P_Y$ and $P^*$ will be denoted respectively by $E_Y$ and $E^*$. $\mathbb{I}(A)$ denote the indicator function of the event $A$. Define $\log^+(x) = \log(x \vee 1)$ and $\log^-(x) = \log(x^{-1} \vee 1)$. With these notations, we are now ready to write down the assumptions on the genealogical tree given by BPRE.

Assumptions 11. $(Y_i : i \geq 0)$ is a sequence of i.i.d. random variables which satisfy the following conditions:

$$E \left| \log P(\xi > 1|Y_0) \right| < \infty, \quad (1.3)$$

$$E \left( \log^+ E(\xi|Y_0) \right) < E \left( \log^+ E(\xi|Y_0) \right) < \infty, \quad (1.4)$$

$$\text{and } E \left( \frac{1}{E(\xi|Y_0)} \mathbb{E} \left[ \mathbb{I}(\xi \geq 2)\log \xi|Y_0 \right] \right) < \infty. \quad (1.5)$$
The implications of these assumptions are listed formally in the following lemma (see Theorem 1 in [7]).

**Lemma 1** (Growth of supercritical BP in i.i.d. environment) Consider the sequence \( \{W_n : n \geq 1\} \) where \( Z_0 = \pi_0 = 1 \) and

\[
\pi_n := E_Y(Z_n) = \prod_{i=0}^{n-1} E\{\xi_i\}
\]

for every \( n \geq 1 \). Then the random variables \( \{W_n : n \geq 1\} \) form a non-negative martingale sequence with respect to the filtration \( \{\sigma\{Z_0, Z_1, Z_2, \ldots, Z_n; Y\} : n \geq 0\} \) and hence there exists a random variable \( W \) such that \( \lim_{n \to \infty} W_n = W \) \( \mathbb{P}_Y \)-almost surely. If additionally the conditions (1.3), (1.4) and (1.5) hold, then

1. \( E_Y(W) = 1 \) almost surely;
2. \( 1 - q_x(Y) = \mathbb{P}_Y(S^c) = \mathbb{P}_Y(W > 0) \) almost surely, that is, the random variable \( W \) is positive \( \mathbb{P}_Y \)-almost surely conditioned on the survival of the genealogical tree.

**Assumptions on the displacements.** We assume that the displacements have jointly regularly varying tails if they are coming from the same parent. This framework allows to accommodate the structures like multivariate regular variation, asymptotic full-dependence and asymptotic tail-independence among the tails of \( X \) as particular subclasses. Referring to [32], we can use the \( M_0 \) convergence for the measures on the space \( (\mathbb{R}^N_0, \rho_0) \) where \( \mathbb{R}^N_0 = \mathbb{R}^N \setminus \{\theta_\infty\} \), \( \{\theta_\infty\} \) is the origin of \( \mathbb{R}^N \) and \( \rho_0(x, y) = \sum_{i \geq 1}(|x_i - y_i| \wedge 1)2^{-i} \) for \( x, y \in \mathbb{R}^N \).

Let \( B(\mathbb{R}^N_0) \) be the \( \sigma \)-field generated by the open balls in the relative topology and \( C^+(\mathbb{R}^N_0) \) be the space of all non-negative bounded and continuous functions which vanish in the neighbourhood of \( \theta_\infty \). Let \( M_0(\mathbb{R}^N_0) \) denote the space of all non-null measures \( \zeta \) such that \( \int f \, d\zeta < \infty \) for every \( f \in C^+(\mathbb{R}^N_0) \). We say \( \zeta_n \) converges in \( M_0 \) topology to the measure \( \zeta \) if \( \int f \, d\zeta_n \to \int f \, d\zeta \) as \( n \to \infty \) for every \( f \in C^+(\mathbb{R}^N_0) \) and denote it by \( \zeta_n \stackrel{M_0}{\to} \zeta \). We refer to Section 3 and 4 in [32] for a detailed discussion. For any positive scalar \( a > 0 \) we denote \( a \cdot x := (ax_i : i \geq 1) \) where \( x = (x_i : i \geq 1) \in \mathbb{R}^N \). We recall that an increasing sequence of positive numbers \( \{\gamma_n : n \geq 1\} \) is said to be regularly varying of index \( \beta \), if \( \gamma_{1/n} / \gamma_n \to a^\beta \) for every \( a > 0 \).

**Definition 12** (Regularly varying measure). A measure \( \kappa \in M_0(\mathbb{R}^N_0) \) is said to be regularly varying if there exists a measure \( \zeta \in M(\mathbb{R}^N_0) \) and an increasing regularly varying sequence of positive real numbers \( \{\gamma_n : n \geq 1\} \) such that

\[
\gamma_n \kappa(n \cdot) \xrightarrow{M_0} \zeta \quad \text{where} \quad \kappa(n \cdot A) = \kappa(\{n \cdot x : x \in A\}) \quad \text{for every} \quad A \in B(\mathbb{R}^N_0). \]

Then the limit measure \( \zeta \) exhibits a homogeneity property given by \( \zeta(a \cdot A) = a^{-\beta} \zeta(A) \) for some \( \beta \geq 0 \) and every \( A \in B(\mathbb{R}^N_0) \) and we write it as \( \kappa \in RV_\beta(\mathbb{R}^N_0, \zeta) \).

With these notations and definitions, we are now ready to state the assumptions on \( X \) and these assumptions will be used throughout the paper.
**Assumptions 13.** 1. The displacements $X = (X_i : i \geq 1)$ are independent of BPRE $Z = (Z_i : i \geq 0)$.

2. The displacements are identically distributed with regularly varying tails, that is,

$$P(|X_1| > t) = t^{-\alpha}L(t) \text{ and } \lim_{t \to \infty} \frac{P(X_1 > t)}{P(|X_1| > t)} = p \in [0, 1],$$

where $L$ is a slowly varying function and $\alpha > 0$.

3. We assume that the tails of displacements are jointly regularly varying when they are coming from the same parent, i.e.,

$$P(X \in \cdot | Y_0) \in RV_{\alpha}(\mathbb{R}_0^N, \nu_*),$$

where $\nu_* \in M_0(\mathbb{R}_0^N)$.

**Weak convergence of the extremal processes and maximum position.**

The genealogical structure is denoted by $T = (V, E)$ where $V$ and $E$ denote the set of all vertices and edges respectively. We use Ulam-Harris labelling for the vertices $v \in V$. The position and the displacement assigned to the particle/vertex $v$ is denoted by $S(v)$ and $X(v)$ respectively. For every vertex $v$, $|v|$ denotes the generation of the vertex $v$. Let $M_n^{(i)}$ denote the $i$-th largest position at the $n$-th generation for $i \geq 1$ and

$$M_n := M_n^{(1)} = \max_{|v|=n} S(v)$$

denote the maximum position at the $n$-th generation. In this article, we will address the order of the fluctuation of $(M_n^{(i)} : 1 \leq i \leq k)$ and also derive their joint asymptotic behaviour when they are divided by $B_n$.

It follows from SLLN that $n^{-1} \log \pi_n \stackrel{a.s.}{\longrightarrow} E(\log E(\xi|Y_0)) > 0$ and thus,

$$\pi_n \stackrel{a.s.}{\longrightarrow} \infty.$$ 

Conditioned on $Y$, define

$$B_n := \inf_{s \geq 1} \{s > 0 : P(|X_1| > s) < \pi_n^{-1}\}. \quad (1.9)$$

If $L(x) = \text{const}$, then $B_n = \pi_n^{1/\alpha}$ otherwise $B_n = \pi_n^{1/\alpha} \tilde{L}(\pi_n)$ where $\tilde{L}$ is another slowly varying function (see Proposition 0.8(v) in [42]). We now define the point process formed by the normalized positions in the $n$-th generation as

$$N_n := \sum_{|v|=n} \delta_{B_n^{-1}S(v)}. \quad (1.10)$$

Let $\mathcal{M}([0, \infty])$ denote the space of all point measures on the space $\mathbb{R}_0 = [\infty, \infty] \setminus \{0\}$ which does not put any mass at $\pm \infty$. Let $C^+_{K, N}(\mathbb{R}_0)$ be the space of all non-negative bounded continuous functions with compact support which is bounded away from 0. We say that a sequence $(\kappa_n : n \geq 1)$ of measures in
\( \mathcal{M}(\mathbb{R}_0) \) converges vaguely to a measure \( \kappa \) if \( \lim_{n \to \infty} \int f \, d\kappa_n = \int f \, d\kappa \) for all \( f \in C^+_K(\mathbb{R}_0) \). It is known that \( \mathcal{M}(\mathbb{R}_0) \) is a Polish space when equipped with the topology induced by vague convergence (see Proposition 3.17 in [42] and Section 2.3 in [32] for a discussion on the differences between \( M_0 \) and vague topology). The metric induced by the vague convergence will be called vague metric and will be denoted by \( d_v \). These observations have been used to develop the weak convergence theory for point processes based on Laplace functionals (see Proposition 3.19 in [42]). According to the physicists Brunet and Derrida [19], the strong and non-trivial dependence structure among the positions at the \( n \)-th generation forces the weak limit to exhibit some cluster phenomena and also some invariance property (related to the tail-behaviour of the displacements). For the displacements with regularly varying tails, it turns out that the limit must be member of randomly scaled scale-decorated Poisson point process (SScDPPP) (introduced in [11]). Here, we would like to introduce the scaling operator \( \mathcal{S} : \mathcal{M}(\mathbb{R}_0) \to \mathcal{M}(\mathbb{R}_0) \) such that \( \mathcal{S}_a(\sum_{l \geq 1} J_l \delta_{b_l}) = \sum_{l \geq 1} J_l \delta_{ab_l} \) for every \( a > 0 \) and \( \mathbb{R}_0 := \mathbb{R} \setminus \{0\} \). We are now ready to define SScDPPP.

**Definition 14 (SScDPPP).** A point process \( \mathcal{J} \) is called a scale-decorated Poisson point process (ScDPPP) with intensity measure \( \kappa \) and scale decoration \( \mathcal{D} \) (denoted by \( \mathcal{J} \sim \text{ScDPPP}(\kappa, \mathcal{D}) \)) if there exists a Poisson random measure \( \Lambda^\infty = \sum_{i \geq 1} \delta_{\lambda_i} \) on \( (0, \infty) \) with intensity measure \( \kappa \) and a point process \( \mathcal{D} \in \mathcal{M}(\mathbb{R}_0) \) such that

\[
\mathcal{J} = \sum_{i=1}^{\infty} \mathcal{A}_i \mathcal{D}_i, \tag{1.11}
\]

where \( (\mathcal{D}_i : i \geq 1) \) is a collection of independent copies of \( \mathcal{D} \). Consider a positive random variable \( U \) independent of the point process \( \mathcal{J} \), then the point process \( \mathcal{J}_* \equiv \mathcal{J}_U \mathcal{J} \) is called SScDPPP and denoted by \( \mathcal{J}_* \sim \text{SScDPPP}(\kappa, \mathcal{D}, U) \).

Proposition 3.2 in [12] allows us to verify whether a point process is a SScDPPP or not by looking at the invariance property of its Laplace functional. Furthermore, the maximum among the atoms of SScDPPP is distributed according to the law of a Frechét random variable multiplied by an independent positive random variable. The following theorem confirms that the weak limit of the extremal processes also belongs to the class SScDPPP and summarizes the contribution of this article.

**Theorem 15 (Weak limit of the sequence of extremal processes).** If Assumptions[11] and[13] hold, then there exists a SScDPPP \( \mathcal{N}_* \) such that \( \mathcal{N}_n \) converges weakly to \( \mathcal{N}_* \) conditioned on the event \( S \) in the space \( \mathcal{M}(\mathbb{R}_0) \) equipped with vague topology. Hence, for every \( x > 0 \), we have

\[
\lim_{n \to \infty} \mathbb{P}(M_n \leq B_n x | S) = \lim_{n \to \infty} \mathbb{P}(\mathcal{N}_n(x, \infty) = 0 | S) = \mathbb{E}^* (e^{-x^{-\alpha}Q}) \tag{1.12}
\]

for some positive random variable \( Q \).
Our main Theorems 21 and 22 that will be presented later provide a detailed picture of $N_\ast$ and the random variable $Q$, hence contain more information about the joint asymptotic behaviour of the extreme positions. To avoid the notional and other technical complexities, we present here the explicit description of $N_\ast$ only for one basic case.

Remark 1 (Weak limit of the extremal processes when displacements are i.i.d. and survival of the genealogical tree is certain) We need some notations for the explicit description. Recall that $W$ is the limit of the martingale sequence $(\pi_n^{-1}Z_n : n \geq 1)$ (see Lemma 1). If the tree $T$ does not have any leaf, then $W > 1$ almost surely and we do not need to condition on the survival of the genealogical tree. Consider the measure $\nu_\ast \in \mathcal{M}((0, \infty))$ defined as

$$\nu_\ast^+(dx) := \alpha x^{-\alpha-1} 1_x (x > 0) \, dx. \tag{1.13}$$

Let $\mathcal{P} = \sum_{l \geq 1} \delta_{\xi_l}$ be the Poisson process on the real line with mean measure $\nu_\ast^+$ and independent copies of $W$. Consider a collection $(\mathcal{E}_l : l \geq 1)$ is a sequence of independent copies of the random variable $\mathcal{E}$ such that $\mathbb{P}(\mathcal{E} = 1) = 1 - \mathbb{P}(\mathcal{E} = -1) = p$. The collection $(\mathcal{E}_l : l \geq 1)$ is assumed to be independent of $\mathcal{P}$ and $W$. Let $\mathcal{Y}'$ be an independent copy of the environment sequence $\mathcal{Y}$ and independent of the Poisson process $\mathcal{P}$. Let $\mathcal{Y}'_0 = (Y'_0, Y'_{-1}, Y'_{-2}, \ldots, Y'_1, Y'_0)$ be the segments of the environment sequence reversed in time for $i \geq 1$. It will be our convention as well that $Z_0 = 1$ conditioned on $\mathcal{Y}'_1 : 0$. Conditioned on $\mathcal{Y}'$, we define a random variable $\mathcal{R}$ with the probability mass function

$$\mathbb{P}(\mathcal{R} = r | \mathcal{Y}') = \frac{1}{C_0(\mathcal{Y}')} \sum_{i=0}^{\infty} \frac{\mathbb{P}(Z_i = r | \mathcal{Y}'_{i-1} : 0)}{\mathbb{E} \left( Z_i | \mathcal{Y}'_{i-1} : 0 \right)}, \tag{1.14}$$

where $C_0(\mathcal{Y}') = \sum_{i=0}^{\infty} \left[ \mathbb{E}(Z_i | \mathcal{Y}'_{i-1} : 0) \right]^{-1} < \infty$ for almost all environment (see Lemma 2). Conditioned on $\mathcal{Y}'$, $(\mathcal{R}_l : l \geq 1)$ is a collection of independent copies of the random variable $\mathcal{R}$ and assumed to be independent of $W$, $\mathcal{P}$ and $(\mathcal{E}_l : l \geq 1)$. We are now ready for the description for the weak limit.

Suppose that assumption 11 holds along with $\mathbb{P}(Z_1 \geq 1) = 1$. After some effort (see Subsection 4.5.1 in [32]), it can be shown that assumption 13 holds when the displacements are i.i.d. and the measure $\nu$ admits a product form (see 2111). As a consequence of our main result (Theorem 21), it follows that there exists a point process $\hat{\mathcal{N}}_\ast^{(\text{ind})}$ such that $N_\ast \Rightarrow \hat{\mathcal{N}}_\ast^{(\text{ind})}$ in the space $(\mathcal{M}(\mathcal{R}_0), d_\ast)$. Moreover,

$$\hat{\mathcal{N}}_\ast^{(\text{ind})} \overset{\text{d}}{=} \sum_{i=1}^{\infty} R^{(i)} \delta_{[C_0(\mathcal{Y}')(W)]_{1/\alpha} \xi_i} \overset{\text{d}}{=} \mathcal{J}_{[C_0(\mathcal{Y}')(W)]_{1/\alpha}} \sum_{i=1}^{\infty} \mathcal{J}_{\mathcal{E}_i} (R^{(i)} \delta_{\xi_i}) \sim \text{SScDPPP}(\nu_\ast^+, R\delta_{\mathcal{E}}, [C_0(\mathcal{Y}')(W)]^{1/\alpha}). \tag{1.15}$$

As a consequence, we have

$$\lim_{n \to \infty} \mathbb{P} \left( M_n \leq B_n x \right) = \mathbb{E} \left[ \exp \left\{ -W px^{-\alpha} C_0(\mathcal{Y}') \right\} \right]. \tag{1.16}$$
1.1 Literature review and the scope of the article

BRW turned out to be one of the most important model in the statistical physics due to its connections to the models like Gaussian multiplicative chaos, Gaussian free field, first passage percolation, last passage percolation, scale-free percolation (see [46]), randomized algorithms etc. Therefore, a vast literature on the extreme positions has been emerged (see [44] for a very nice review). Suppose that the displacements have exponentially decaying tail. The asymptotic analysis of the maximum position was initiated the fundamental works [29], [31] and [15]. The asymptotic order of the fluctuations and weak limit of the maximum position has mostly been studied in the last two decades; see [1], [8], [17]. As mentioned earlier, physicists Brunet and Derrida [19] predicted that the weak limit of the extremal processes exhibit certain invariance property and have clusters; see [45] for the mathematical description. This has been formalized in [34] for BRW relying on the work [35]. Recently, [38] and [30] studied asymptotic properties of the maximal position at the $n$-th generation in BRW with time-inhomogeneous environment.

The continuum analogue of the BRW with exponentially decaying tail is Branching Brownian Motion (BBM). It is known that the asymptotic behavior of the extreme positions in BBM is very similar to that of the BRW. The weak limit of the maximum position is known to satisfy the famous Fisher-Kolmogorov-Petrovski-Piskunov (FKPP) equation. For a survey in this field, see [9] and [16]. In the last two decades, the asymptotics of the extreme positions in BBM are investigated in [4], [5], [6] and [2] in detail. Recently, Hamel et. al. [28] studied the KPP equation when the underlying genealogical structure is in a periodic environment. Relying on this work, Lubetzky et. al. [33] studied the asymptotic behaviour of the maximum position in BBM in periodic environment. It has been proved that maximal position of BBM in periodic environment (branching rate of a particle depends on the state of the particle in a specific way), when centred by its median, converges weakly (in annealed sense) to a randomly shifted Gumbel random variable. It has been shown that the asymptotic order of median does not change much from [18] except the coefficient of log $t$. So it seems intuitively convincing that the asymptotic rate of growth of the extremes does not change much if the environment exhibits periodicity though the limit changes depending on environment. The asymptotic study of the point process is still open.

Asymptotic study of the extreme positions in BRW has been initiated in [22] and [23] when displacements have polynomially-decaying tails or regularly varying tails. The weak limit of the extremal process has been studied in detail in [11] and [12]; see also [36] and [41]. This article is inspired by the work of Lubetzky et. al. [33] though the set up is still quite different. We consider a random environment instead of the (deterministic) periodic environment and so, the order of fluctuation (the coefficient of log $t$ in the analysis [33]) is random (determined by the random environment). We have also been able to derive the weak limit of $N_n$ which provides more information of the joint asymptotic behavior of the extreme positions. Although the weak limit of $N_n$ belongs
to the class SScDPPP and $M_n$ is a scale-mixture of Frechét random variable which reminds \[12\], the laws of the clusters in $N_*$ and $Q$ in (1.12) are completely new. In particular, the description of the limit point process $N_*$ is more complicated and notational heavier than (1.15) when $T$ does not have any leaf and the displacements are not necessarily asymptotically tail-independent. A careful look at (1.15) show that the law of clusters/multiplicities $(R^{(l)})$ of a Poisson point depends on the underlying environment but, reversed in time. This type of picture is new in the context of extreme positions in BRW. This can be intuitively explained by two facts. We already know that a few displacements are large enough and hence, causes large positions in the $n$-th generation. The multiplicity of a Poisson point is actually the number of descendants in the $n$-th generation of a vertex with large displacement. So, we are looking at the most recent common ancestor of a bunch of large positions at the $n$-th generation and hence, the time reversal of the environment sequence appear. To derive the limit $N_*$, we have to go through a few approximation steps viz. one large displacement on a geodesic path, cutting the tree, pruning and regularization. In the first step, the asymptotic behaviour of the extreme positions at the $n$-th generation is connected to the extreme displacements upto the $n$-th generation. In the cutting step, we show that the large displacements can occur only at the last $o(n)$ generation and hence, the independence of $W$ and $Y'$ in the limit $N_*$. If we replace the regularly varying displacements by the displacements with exponentially-decaying tail, then the analogues of the above-mentioned are not yet known in the literature.

We believe that our analysis in this paper can also be adapted for the asymptotic study of the other heavy-tailed (for example, semiexponential or stretched-exponential) displacements which has been initiated in \[26\] and followed up recently in \[25\]. Although we considered i.i.d. environment in this article, it is believable that our steps can also be adapted to the environment which can be decomposed into i.i.d. blocks (regenerative environment). Recently, the large deviations of the extreme positions has attracted many researchers (see \[3\], \[10\], \[20\], \[21\], \[24\], \[27\]) which has not been explored in this article. We can also consider a multi-type BRW with branching process in i.i.d. environment (see \[14\], \[13\]).

Outline of the paper

The remainder of the paper is organized as follows. In the Section 2, we present the main result Theorem 21 with its consequences. A brief discussion on the steps to prove Theorem 21 is given in Section 3. The weak limit of the maximum position is derived in Section 5 from the weak limit of the sequence of extremal processes. The proofs of propositions used in Section 4 are based on some auxiliary lemmas which are proved in Appendix 6.
2 Main result and its consequences

In this section, we first state the main result Theorem 2.1 of the paper after introducing the necessary notations and auxiliary results. Then we come up with the consequences and implications of Theorem 2.1 with details. Some questions which we could not answer in this paper will be at the end of this section.

2.1 Main result

We begin with the observation that the supercritical branching process in i.i.d. environment grows geometrically with probability one.

Lemma 2 If the BPRE is supercritical (assumptions (1.3)–(1.4) hold) and the environment $Y$ is an i.i.d. sequence, then $P\left(\sum_{i=0}^{\infty} [E(Z_i|Y_0 : i-1)]^{-1} < \infty\right) = 1$.

The following result turns out to be crucial in deriving the Laplace functional of the weak limit $N_\ast$ of the sequence $N_n$ conditioned on the survival event $S$.

Lemma 3 If the assumptions in (1.3) hold along with the assumptions (1.3)–(1.4) on the underlying BPRE, then $\pi_n P(B_{n}^{-1}X \in \cdot | Y_0 : n-1) \overset{M_0}{\to} \nu(\cdot)$ on the space $M(\mathbb{R}^N)$ where

$$\nu(\cdot) := \frac{\nu_\ast(\cdot)}{\nu_\ast(\{|x_i:i \geq 1\} \in \mathbb{R}^N : |x_1| > 1)} \quad (2.1)$$

for the measure $\nu_\ast$ defined in (3).

Both above lemmas are proved in Appendix. We start from introducing the necessary notations. All introduced random sequences and variables are defined on the same probability space $(\Omega, \mathcal{F}, P)$.

(N1) Recall (see Lemma 1) that the random variable $W$ is the martingale sequence $(\pi_n^{-1}Z_n : n \geq 0)$ with respect to the filtration $(\sigma(Y; Z_0, Z_1, Z_2, \ldots, Z_{n-1}) : n \geq 1)$.

(N2) $Y' = (Y'_i : i \geq 0)$ is an independent copy of environment $Y$.

(N3) $[i : j]$ denotes the set $\{i, i+1, i+2, \ldots, j\}$ for $j > i \geq 0$ and the set $\{i, i-1, i-2, \ldots, j\}$ for $i > j \geq 0$. We define $Y'_i : 0 = (Y'_i, Y'_{i-1}, Y'_{i-2}, \ldots, Y'_0)$ for $i \geq 1$, $Y'_0 : 0 = Y_0$ and $Y'_{-1} : 0$ is empty. According to our convention $Z_0 \equiv 1$ conditioned on the filtration $Y''_{-1} : 0$ for the comfort of writing.

(N4) $Z_1^{(+)}$ denotes the random variable $Z_1$ conditioned to be positive that is, $P(Z_1^{(+)} = k) = P(Z_1 = k | Z_1 \geq 1)$.

(N5) The law of $Z_i$ conditioned on the environment segment $Y'_{n-1} : 0 = (Y'_{n-1}, Y'_{n-2}, \ldots, Y'_0)$ is the law of the number of descendants in the $n$-th generation of a particle in the $(n-i)$-th generation for every $i \in [1 : n-1]$. 
Given the environment \( Y \),

\[
\mathbb{P} ( \tilde{Z}_i^{(+)} = k | Y_{n-1} : 0 ) = \mathbb{P} ( \tilde{Z}_i^{(+)} = k | (Y_{i-1}', Y_{i-2}', \ldots, Y_i', Y_0') : \{ \tilde{Z}_i : 1 \})
\]

for all \( k \in \mathbb{N} \).

Conditioned on the environment \( Y'_{n-1} : 0 \), \( (\tilde{Z}_i^{(+),k} : k \geq 1) \) denotes a collection of independent copies of the random variable \( \tilde{Z}_i^{(+)} \) given \( (Y_{i-1}', Y_{i-2}', \ldots, Y_i') \) for every \( i \in [1 : n-1] \).

\( \text{Pow}(\mathcal{B}) := \{ C : C \subset \mathcal{B} \} \) is the power set of the set \( \mathcal{B} \).

Consider a Poisson random measure \( \mathfrak{P} \) on the space \( \mathbb{R}_0^\infty \) with the mean measure \( \nu \) (defined in (2.1)) such that

\[
\mathfrak{P} := \sum_{i \geq 1} \delta_{\xi^{(i)}} = \sum_{i=1}^{\infty} \delta_{\{\xi_1^{(i)}, \xi_2^{(i)}, \ldots\}}.
\]  

We assume that \( \mathfrak{P} \) is independent of the random variable \( W \) and environment sequence \( Y' \).

Let \( \{0_i\} \) be the origin of \( \mathbb{R}^i \). Conditioned on \( Y' \), consider a \( \prod_{i=1}^{\infty} \{\{i\} \times \{\mathbb{N}^i \setminus \{0_i\}\}\} \)-valued random variable \( (V, \mathbf{R}) = (V, (R_l : 1 \leq l \leq V)) \) with the following conditional probability mass function

\[
\mathbb{P}_{Y'}(V = v ; \mathbf{R} = r) = \frac{1}{C_1(Y')} \sum_{i=0}^{\infty} \frac{1}{E(Z_{i+1} | Y_i' : 0)} \mathbb{P} (Z_1 = v | Y_i') \prod_{m=1}^{v} \mathbb{P} (Z_i = r_m | Y_{i-1}' : 0)
\]  

for every \( v \geq 1 \) and \( r = (r_1, \ldots, r_v) \in \mathbb{N}^v \setminus \{0_v\} \) where \( C_1(Y') \) (thanks to Lemma 2) is the normalizing constant. The random vector \( (V, \mathbf{R}) \) is constructed independently of the random variable \( W \) and the Poisson process \( \mathfrak{P} \).

Given the environment \( Y' \), consider a collection \( ((V_i, \mathbf{R}^{(i)}) = (V_i, (R_k^{(i)} : 1 \leq k \leq V_i)) : l \geq 1 \) of independent copies of \( (V, \mathbf{R}) \) which are also independent of \( W \) and \( \mathfrak{P} \).

With these random variables and notations, we are now ready to present the main result of this article.

**Theorem 21.** If the set of assumptions in [11] and [13] holds, then there exists an SScDPPPP \( \mathbf{N}^\ast \) such that conditioned on the survival \( S \), \( \mathbf{N}^\ast \) converges weakly to an SScDPPPP \( \mathbf{N}_\ast \) in the space \( \mathcal{M}(\mathbb{R}_0^\infty) \) equipped with vague topology. Moreover, we have

\[
\mathbf{N}_\ast \overset{d}{=} \sum_{i=1}^{\infty} \sum_{k=1}^{V_i} R_k^{(i)} \delta_{(C_1(Y') W)^{1/\alpha k^{(i)}}}
\]  

and for \( f \in C_0^\infty(\mathbb{R}_0^\infty) \), the Laplace functional of \( \mathbf{N}_\ast \) can be written as follows

\[
\mathbb{E}^\ast(\exp\{-\mathbf{N}_\ast(f)\})
\]
\[ E^* \left[ \exp \left( -W \int_{\mathbb{R}^n \setminus (0^\infty)} \nu(dx) \sum_{i=0}^{\infty} \left( \mathbb{E}(Z_{i+1} | Y'_i : 0) \right)^{-1} \mathbb{P}(Z_1 \geq 1 | Y'_i) \right) \mathbb{E} \left[ \sum_{B \in \text{Pow}([1:Z^+(1)]) \setminus \{\emptyset\}} \left[ \mathbb{P}(Z_i \geq 1 | Y'_{i-1} : 0) \right]^{(|B| - |B|)} \left( 1 - \exp \left\{ - \sum_{k \in B} Z^{(+,k)}(x_k) \right\} \right) \left[ \mathbb{P}(Z_i = 0 | Y'_{i-1} : 0) \right] \right] \right]. \tag{2.5} \]

The proof of this theorem will be given in Section 3.

Remark 2 (Clusters of extremes in the limit and quenched law of the branching process.) It is clear that \( N^* \) is a Cox cluster process. The Poisson atoms \( (\xi_l : l \geq 1) \) appear because the collection \( (X_u : u \in T) \) are i.i.d. where \( X_u \) denote the displacement vector attached to the children of \( u \). One large displacement causes multiple large positions because of the ‘one large bunch of displacements’ phenomenon and the strong dependence structure among the positions. Hence the multiplicities \( \{(R^{(l)}_k : 1 \leq k \leq V_l) : l \geq 1)\) of the Poisson atoms and the law of the multiplicities is given by the quenched law of the generation-sizes in an independent environment. Thanks to the i.i.d. structure, the law of the environment sequence does not change even when reversed in time (this fact has been used in the proof of Proposition 34). The most interesting point here is that the law of the clusters involve the time-reversed environment. The main reason is that we are looking at the large positions at the \( n \)-th generation and then looking genealogically backward to their most recent common ancestor with the large displacement. We now would like to explain appearance of the random variable \( W \) and \( Y' \), and their independence in the limit \( N^* \). It follows from Lemma 1 and Lemma 2 that the sizes of the generations grow geometrically with high probability conditioned on the survival of the tree. If we combine this fact with the principle of a ‘single bunch of large displacements’, it follows that the large displacements can occur on in the last few \( o(n) \) with high probability) generations. So we cut the tree and ignore all the displacements except last few generations. Due to this cutting the genealogical tree very close to the generation \( n \), we create a forest containing i.i.d. trees with displacements and the size of the forest is comparable to the size of the \( n \)-th generation of the original genealogical tree. The random variable \( W \) appears from the normalized size of the forest. Thanks again to the i.i.d. structure of the environment sequence, the environment corresponding the first \( n - o(n) \) generations of the genealogical tree is independent of the environment of the forest.

Remark 3 (\( N^* \) as an SScDPPP) Due to lack of an explicit description of the measure \( \nu \), we can not derive an SScDPPP representation of \( N^* \). An SScDPPP representation has been derived in Corollary 2 when the distributions in \( \Theta \) have uniformly bounded support. In [12], to deal with the same challenge, the notion of the scaled Laplace functional has been proposed. We use that in the proof of Proposition 35.
2.2 Consequences

The consequences of Theorem 21 will be discussed now starting with the explicit description of weak limit of \((B_n^{-1}M_n : n \geq 1)\). Then the weak limit of \(N_n\) and \(B_n^{-1}M_n\) will be derived from Theorem 21 when displacements are asymptotically tail-independent and other choices of the dependence structures. We finally show that the main results in [12] and hence, [11] follow from Theorem 21.

2.2.1 Weak limit of the normalized maximum positions

Recall that \(M_n = \max |v| = S(v)\). The existence of the weak limit of \(B_n^{-1}M_n\) follows from the weak convergence of the extremal process \(N_n\). The following identity connects the maximum position to the extremal process

\[
\lim_{n \to \infty} \mathbb{P}^* (M_n \leq B_n x) = \lim_{n \to \infty} \mathbb{P}_n(x, \infty) = 0 = \mathbb{P}_* (N_* (x, \infty) = 0).
\]

Define \(M_*\) to be the weak limit of the sequence \(B_n^{-1}M_n\). As a consequence of \(N_*\) being an SScDPPP, it follows that \(M_*\) is a scale-mixture of Frechet random variables as stated in Theorem 15. But the explicit description of the random variable \(Q\) in (1.12) cannot be derived from the fact that \(N_*\) is SScDPPP. It is possible though due to the description of \(N_*\) in (2.4) which leads to an explicit description of the random variable \(Q\) in terms of the measure \(\nu\), the quenched description of the genealogical structure and \(W\). To state it, we need the following notation. Let us define

\[
G_0 := (-\infty, 1], \quad G_1 := (1, \infty] \quad \text{and} \quad H_{i_{1}, i_{2}, \ldots, i_{t}} := \prod_{j=1}^{t} G_{i_{j}} \times \prod_{j' = t+1}^{\infty} \mathbb{R},
\]

where \(i_{j} \in \{0,1\}\) for all \(j \in [1 : t]\).

**Theorem 22.** Under the assumptions stated in [11] and [13], for every \(x > 0\), we have \(\mathbb{P}^*(M_* \leq x) = \mathbb{E}^* (e^{-x^{-\alpha}Q})\) where

\[
Q := W \sum_{j=0}^{\infty} \left[ E(Z_{j+1}|Y'_j) \right]^{-1} \sum_{v=1}^{\infty} \mathbb{P}(Z_1 = v | Y'_j) \sum_{k=1}^{v} (1 - \mathbb{P}(Z_j = 0 | Y'_{j-1} : 0))^{k} \sum_{i_{1}, i_{2}, \ldots, i_{v} : i_{1} + i_{2} + \ldots + i_{v} = k} \nu(H_{i_{1}, i_{2}, \ldots, i_{v}}).
\]

The proof of this theorem will be given in Section 5.

**Remark 4** The distribution of the random variable \(Q\) depends on the law of the genealogical structure through the random variable \(W\) and the expected generation sizes given the environment \(Y'\). This type of the explicit description is missing in the BRW literature when displacements have exponentially decaying tail. To obtain the detailed description of \(Q\), the crucial role is played by the representation of \(N_*\) obtained in (2.4).
2.2.2 Genealogical tree does not have any leaf

Suppose that \( P(Z_1 \geq 1) = 1 \). Then \( P(S) = 1 \) and we do not need to condition on the survival of the genealogical tree. Let us denote the weak limit of \( N_n \) by \( \hat{N}_* \). In the expression for the Laplace functional of \( \hat{N}_* \), the genealogical random variables \((Z_i : i \geq 1)\) do not need to be conditioned to be positive. These observations yield the following result.

**Proposition 23.** In addition to the assumptions stated in Theorem 21, we assume that \( P(Z_1) = 1 \). Then there exists a point process \( \hat{N}_* \) such that \( N_n \) converges weakly to the point process \( \hat{N}_* \) in the space \( \mathcal{M}(\mathbb{R}_0) \) equipped with vague topology. Furthermore,

\[
\hat{N}_* \overset{d}{=} \sum_{l=1}^{\infty} \sum_{k=1}^{l} \hat{R}_l^{(k)} \delta_{C_2(Y^l)|W|^{1/\alpha}}(l),
\]

(2.9)

where

\[
P(\hat{V} = v; \hat{R} = r|Y') = \frac{1}{C_2(Y')} \sum_{i=0}^{\infty} P(Z_i = v|Y'_i) \prod_{m=1}^{n} P(Z_i = r_m|Y'_i),
\]

(2.10)

for every \( r \in \mathbb{N}^v \) and \( v \geq 1 \) and \( C_2(Y') \) being the normalizing constant. As a consequence, \( B_n^{-1}M_n \Rightarrow \hat{M}_* \) and for every \( x > 0 \), we have

\[
P(\hat{M}_* \leq x) = \mathbb{E} \left\{ \exp \left\{ -Wx^{-\alpha} \sum_{j=0}^{\infty} \left[ E(Z_{j+1}|Y'_j) \right]^{-1} \sum_{v=1}^{\infty} P(Z_1 = v|Y'_j) \right\} \sum_{i_1,i_2,\ldots,i_v \geq 1} \nu(\mathcal{H}_{i_1,i_2,\ldots,i_v}) \right\}.
\]

2.2.3 Tails of the displacements are asymptotically independent

We call the displacements \( X \) to be *asymptotically tail-independent* (see subsection 4.5.1 in [32] for a detailed discussion) if the limit measure \( \nu_* \) admits the following form

\[
\sum_{i=1}^{\infty} \otimes_{j=1}^{i} \delta_0 \otimes_{j'=i+1}^{\infty} \delta_0
\]

(2.11)

where

\[
\nu_\alpha(dx) = \alpha p x^{-\alpha-1} \mathbb{I}(x > 0) \, dx + \alpha (1-p) (-x)^{-\alpha-1} \mathbb{I}(x < 0) \, dx
\]

(2.12)

is an element in \( M_0(\mathbb{R}_0) \). Recall that \( P = \sum_{d \geq 1} \delta_d \) is the Poisson random measure on \((0, \infty)\) with mean measure \( \nu_\alpha^+ \) (see [113]) and assumed to be independent of the random variable \( W \) and environment sequence \( Y' \). Also
recall that \((\mathcal{E}_l : l \geq 1)\) is a collection of \(\{\pm 1\}\)-valued i.i.d. random variables such that \(\mathbb{P}(\mathcal{E}_1 = 1) = 1 - \mathbb{P}(\mathcal{E}_1 = -1) = p\) and independent of \(W\), environment sequence \(Y\) and Poisson random measure \(\mathcal{P}\). It follows immediately that \(\sum_{l \geq 1} \mathcal{F}_l \delta_{\mathcal{E}_l} = \sum_{l \geq 1} \delta_{\mathcal{E}_l} \) is a Poisson random measure on \(\mathbb{R}_0 = [\infty, \infty] \setminus \{0\}\) with mean measure \(\nu\). Conditioned on \(Y\), consider a collection \((R^{(l)} : l \geq 1)\) of independent copies of the random variable \(R\) such that

\[
\mathbb{P}(R = r \mid Y) = \frac{1}{C_3(Y)} \sum_{i=0}^{\infty} \frac{\mathbb{P}(Z_i = r \mid Y_{i-1} : 0)}{\mathbb{E}(Z_i \mid Y_{i-1} : 0)} \tag{2.13}
\]

for every \(r \geq 1\) where

\[
C_3(Y) = \sum_{i=0}^{\infty} \frac{\mathbb{P}(Z_i \geq 1 \mid Y_{i-1} : 0)}{\mathbb{E}(Z_i \mid Y_{i-1} : 0)} \tag{2.14}
\]

is a normalizing constant. Then the weak limit of \(N_n\) and \(M_n\) is characterized in the following theorem.

**Theorem 24.** Suppose that the assumptions \([14] \) and \([15] \) hold. Additionally, we assume the displacements to be asymptotically tail-independent. Then there exists a point process \(N^{(id)}_\ast\) such that \(N_n \Rightarrow N_\ast\) in the space \(\mathcal{M}(\mathbb{R}_0)\) equipped with the vague topology. Furthermore,

\[
N^{(id)}_\ast \overset{d}{=} \sum_{l=1}^{\infty} R^{(l)} \mathcal{F}_{[C_3(Y)W]^{1/\alpha}} \mathcal{E}_l = \mathcal{F}_{[C_3(Y)W]^{1/\alpha}} \sum_{l=1}^{\infty} \mathcal{F}_{R^{(l)}} \delta_{\mathcal{E}_l} \sim \text{SScDPP}[\nu^{\ast}, \mathcal{R}\delta_{\mathcal{E}}, [C_3(Y)W]^{1/\alpha}] \tag{2.15}
\]

with the Laplace functional given by

\[
\mathbb{E}^* \left[ \exp \left\{ -W \int_{\mathbb{R}_0} \nu^{\ast}(dx) \sum_{i=0}^{\infty} \frac{\mathbb{P}(Z_i \geq 1 \mid Y_{i-1} : 0)}{\mathbb{E}(Z_i \mid Y_{i-1} : 0)} \mathbb{E} \left( 1 - e^{-f(x)} Z_i^{(+) \ast} \mid Y_{i-1} : 0 \right) \right\} \right] \tag{2.16}
\]

for every \(f \in C^+_K(\mathbb{R}_0)\). As a consequence, for every \(x > 0\), we have

\[
\lim_{n \to \infty} \mathbb{P}^*(M_n \leq x) = \mathbb{P}^*(M^{(id)}_\ast \leq x) = \mathbb{E}^* \left[ \exp \left\{ -x \mathbb{E}^* \left[ \frac{C_3(Y)}{W} \right] \right\} \right]. \tag{2.17}
\]

**Remark 5** It follows from the form of \(\nu^{\ast}\) in \((2.11)\) that two displacements coming from the same parent can not be large simultaneously (principle of ‘single big jump’) if they are asymptotically tail-independent. So the sum over all possible subsets of \([1 : \infty] Z^{(+) \ast} \) vanishes and the expression \((2.10)\) follows after some algebraic treatments. As a consequence, the representation of \((2.15)\) also lacks an extra sum compared to that of \(N_\ast\). The same algebraic adjustments can be used to derive \((2.17)\) from Theorem 22 or \((2.17)\) can be approached directly from the explicit description of \(N^{(id)}_\ast\).
Remark 6 (Joint asymptotic distribution of the maximum and minimum position) Let us define
\[ \mathcal{I}_n := \min_{|v|=n} S(v). \]
As a consequence of Theorem 24, it follows that \( B_{n^{-1}}(\mathcal{I}_n, M_n) \) converges weakly to a random vector \((\mathcal{I}_*, M_*)\) conditioned on the survival of the underlying genealogical tree. Consider two positive real numbers \( x, y \). Then the joint distribution of the random vector \((\mathcal{I}_*, M_*)\) can be derived using the following identity
\[
\lim_{n \to \infty} \mathbb{P}^* \left( \mathcal{I}_n > -y; M_n \leq x \right) = \lim_{n \to \infty} \mathbb{P}^* \left( N_n(-\infty, -y) = 0; N_n(x, \infty) = 0 \right)
= \mathbb{P}^* \left( \mathcal{N}_*(-\infty, -y) = 0; \mathcal{N}_*(x, \infty) = 0 \right). \tag{2.18}
\]
Now we can ignore the clusters in (2.15) as they are positive almost surely and hence, have no role to play in computing the probability in (2.18). Thus we can write down the event in (2.18) in terms of \( \sum_{l \geq 1} \delta_{[C_3(Y')W]^{1/\alpha} \mathcal{E}_l} \) which is a Poisson random measure conditioned on \( W \) and \( Y' \). The following expression of (2.18) then follows after some algebra
\[
\mathbb{E}^* \left[ \exp \left\{ -WC_3(Y')(px^{-\alpha} + (1-p)y^{-\alpha}) \right\} \right]. \tag{2.19}
\]
Remark 7 (Joint asymptotic distribution of the first two maxima) It follows from Theorem 24 that \( B_{n^{-1}}(M_n^{(1)}, M_n^{(2)}) \) converges weakly conditioned on the event \( \mathcal{S} \) and let us denote the limit by \((M_*^{(1)}, M_*^{(2)})\). Recall that \( M_n^{(1)} = M_n \) and \( M_*^{(1)} = M_* \). Thanks to the description of \( N_*(\text{id}) \) in 2.15, we can derive an explicit description of the distribution of the random vector \((M_*^{(1)}, M_*^{(2)})\) using the following identity
\[
\lim_{n \to \infty} \mathbb{P}^* \left( M_n^{(1)} \leq B_n y; M_n^{(2)} \leq B_n x \right)
= \lim_{n \to \infty} \mathbb{P}^* \left( N_n(y, \infty) = 0; N_n(x, \infty) \leq 1 \right)
= \mathbb{P}^* \left( \mathcal{N}_*(y, \infty) = 0; \mathcal{N}_*(x, \infty) \leq 1 \right)
= \mathbb{P}^* \left( \mathcal{N}_*(x, \infty) = 0 \right) + \mathbb{P}^* \left( \mathcal{N}_*(y, \infty) = 0; \mathcal{N}_*(x, y] = 1 \right) \tag{2.20}
\]
for \( 0 < x < y \). Note that the cluster sizes \((R(l)) : l \geq 1\) are positive almost surely and hence, the expression in (2.20) equals
\[
\mathbb{P}^* \left( \sum_{l=1}^{\infty} \delta_{\mathcal{E}_l} \left( [C_3(Y')W]^{1/\alpha} x, \infty \right) = 0 \right)
+ \mathbb{P}^* \left( \sum_{l=1}^{\infty} \delta_{\mathcal{E}_l} \left( [C_3(Y')W]^{1/\alpha} y, \infty \right) = 0; \sum_{l=1}^{\infty} \delta_{[C_3(Y')W]^{1/\alpha} \mathcal{E}_l} (y, x) = 1 \right). \tag{2.21}
\]
Then some algebraic adjustments and properties of the Poisson random measure yield the following expression for joint distribution \((2.20)\)

\[
\begin{align*}
\mathbb{E}^* \left( \exp \left\{ -pWx^{-\alpha}C_3(Y') \right\} \right) \\
+ p \mathbb{E}^* \left[ W(x^{-\alpha} - y^{-\alpha})C_3(Y') \exp \left\{ -pWx^{-\alpha}C_3(Y') \right\} \right].
\end{align*}
\]

From the explicit expression of in \((2.22)\), one can derive an explicit description of the gap \(M_1^* - M_2^*\) between the first and the second largest position. In the context of statistical physics, the gap statistics \((M_k^* - M_{k+1}^* : k \geq 1)\) is very important (see [40], [37], [39] and references therein for example).

**Corollary 1** If \(P(Z_1 \geq 1) = 1\) holds in addition to the assumptions in Theorem \(24\), then the weak limit of \(N_n\) and \(M_n\) are given in Remark \(7\).

### 2.2.4 Tails of the displacements are asymptotically fully dependent

Suppose now that the displacements from the same parent are exactly same that is,

\[
\mathcal{X}^{(n)} = Z_1^{(n)} \delta_{X_1},
\]

where \(P(X_1 \in \cdot) \in \text{RV}_\alpha(R_0, \nu_\alpha)\) (see \((1.7)\) and \((2.12)\)). Then using page 196 in [43] observe that

\[
P(X_1 \in \cdot) \in \text{RV}_\alpha(R_0^N, \nu_f)\]

where \(\nu_f(\cdot) = \nu_\alpha(\{x \in \mathbb{R} : x \cdot 1_\infty \in \cdot\})\) (2.24)

and \(1_\infty \in \mathbb{R}^N\) with all its components equal to 1. Note that \((2.23)\) implies \((2.24)\) but, the converse is not necessary to hold. If \((2.24)\) holds for the displacement sequence \(X\), then we call the displacements \(X\) to be asymptotically full tail-dependent. Recall that \(P\) is a Poisson random measure on \(R_0\) with mean measure \(\nu_\alpha^+\) which is constructed independently of \(W\) and \(Y'\). Also recall that \((E_l : l \geq 1)\) is a collection of i.i.d. \(\{\pm 1\}\)-valued random variables independently of \(W\), \(Y'\) and \(P\). Conditioned on the environment sequence \(Y'\), \((R^{(l)}, V_l) : l \geq 1\) is a collection of independent copies of \((R, V)\) (specified in \((2.3)\)) and also constructed independently of \(W\), \(P\) and \((E_l : l \geq 1)\). The following result provides a description of the weak limit of \(N_n\) and \(M_n\) when the genealogical tree survives.

**Theorem 25.** Suppose that the assumptions \(11\) and \(13\) hold. In addition, we assume that \(P(X_1 \in \cdot) \in \text{RV}_\alpha(R_0^N, \nu_f)\) where \(\nu_f\) is given in \((2.24)\). Then there exists a point process \(N_{\text{full}}^{(n)}\) such that conditioned on the event \(S\), \(N_n \Rightarrow N_{\text{full}}\) in the space \(\mathcal{M}(R_0)\) equipped with the vague topology. Furthermore, we have

\[
N_{\text{full}}^{(n)} \overset{d}{=} \sum_{l=1}^{\infty} \left( \sum_{k=1}^{V_l} R^{(l)}_k \right) \delta_{C_1(Y')W^{|1/\alpha|}} \mathbb{E}^*_l
\]
and $N_{f}^{(full)}$ has the following Laplace functional

$$
\mathbb{E}^{*}\left[\exp\left\{-W\int_{R_{0}}^{\infty}\nu_{f}(dx)\sum_{i=0}^{\infty}\mathbb{P}(Z_{i} \geq 1|Y_{i}' : 0)\right\}\right]
$$

for every $f \in C_{\alpha}^{+}(\mathbb{R}_{0})$. As a consequence, for every $x > 0$, we have

$$
\mathbb{P}^{x}(M_{*}^{(full)} \leq x) = \lim_{n \to \infty} \mathbb{P}^{x}(M_{n} \leq x) = \mathbb{E}^{*}\left[\exp\left\{-Wx^{-a}C_{1}(Y')\right\}\right]
$$

(2.27)

Remark 8 The expression of the Laplace functional of $N_{*}^{(full)}$ in (2.26) follows from the expression in (2.25) substituting $\nu$ by $\nu_{f}$. The main reason to mention Theorem 25 is that the limit measure $N_{*}^{(full)}$ admits a representation as an SScDPPP. The form of $N_{*}^{(full)}$ easily follows from that of $N_{*}$ as either all the children coming from same parent are large or none of them. Suppose now that the genealogical tree does not have any leaf that is, $\mathbb{P}(Z_{1} \geq 1) = 1$. Then we do not need to condition on the survival of the genealogical tree and some simplifications are possible but we skip them here.

2.2.5 Multivariate regular variation and bounded progeny random variables

Let us assume that $\mathbb{P}(Z_{1} \leq K) = 1$ for some in integer $K$. Then we can use multivariate regular variation to model the tail-dependence among the displacements. In that case, we assume that the random vector $X_{[K]} = (X_{1}, X_{2}, \ldots, X_{K})$ is regularly varying on $\mathbb{R}^{K}$ that is, $\mathbb{P}(X_{[K]} \in \cdot) \in RV_{\alpha}(\mathbb{R}^{K} \setminus \{0_{K}\}, \nu^{(K)})$. The result in Theorem 24 holds with $\nu$ replaced by $\nu^{(K)}$ and we denote the limit by $N_{*}$. Note that the measure $\nu^{(K)}$ can be decomposed into radial measure $\nu^{(+)}_{f}$ on $(0, \infty)$ and angular measure $\zeta$ on $\mathbb{S}^{K-1} = \{x \in \mathbb{R}^{K} : ||x|| = 1\}$ where $||x|| = \sqrt{x_{1}^{2} + x_{2}^{2} + x_{3}^{2} + \ldots + x_{K}^{2}}$; see Theorem 6.1 in [33] for the decomposition. This decomposition leads to a precise description of the law of decoration of $N_{*}$ (similar to Corollary 5.3 in [12]).
Corollary 2 Suppose that assumptions 11 and 13 hold along with \( P(Z_1 \leq K) = 1 \). Then \( \tilde{N} \sim \text{SScDPP}(c_0, \nu^\alpha, \sum_{k=1}^{\mathbf{V}} R_k \delta_{A_k}, [C_1(Y^*)W]^{1/\alpha}) \) where \( A \) is a \( \mathbb{S}^{K-1} \)-valued random variable with the law \( c_0^{-1}\mathfrak{A} \) and \( c_0 = \varsigma(\mathbb{S}^{K-1}) \).

2.2.6 BRW with genealogical structure given by a simple Galton-Watson process

In this section, we assume that \( \Theta \) is singleton that is, contains exactly one probability distribution on \( \mathbb{N} \). Then the law of branching process \( Z = (Z_i : i \geq 1) \) is described through a simple supercritical Galton-Watson process. Assumption 11 implies that the progeny distribution satisfies Kesten-Stigum condition. Under this simpler set-up, the analogue of Theorem 21 has been obtained in [11] (assuming the displacements to be i.i.d. and have regularly varying tails) and [12] (assuming the displacements satisfy Assumption 13). Thus the main results of the aforementioned articles follow from Theorem 21.

To keep the discussion brief, it is enough to derive (4.44) in [12] from (2.5). Note that \( \pi_n = \mu^n \) where \( \mu = \mathbb{E}(\xi|\Theta) = \mathbb{E}(Z_1) > 1 \) (Assumption 1.4). Then it follows that \( (B_n : n \geq 1) \) (see (1.9)) is a deterministic sequence which satisfy \( \lim_{n \to \infty} \mu^n \mathbb{P}(|X_1| > B_n) = 1 \) that is, \( B_n \sim \mu^{n/\alpha} L(\mu^n) \) where \( L \) is a slowly varying sequence related to \( L \). To see (4.44), we need to replace the conditional probabilities and expectations by the unconditional ones and \( Z_1^{(+)} \) by \( U_1 \) in the exponent of Laplace functional of \( N \) in (2.5).

3 Proof of Theorem 21

The proof of Theorem 21 can be divided into five main steps viz. ‘at the most one large displacement on a typical path’, ‘cutting the tree’, ‘pruning the forest’, ‘regularization of the forest’ and ‘construction of the limit \( N \)’. Similar steps have also been used in [12] which can be explained by the ‘principle of a bunch of large displacements’ phenomenon. Still, the proofs in our context are much more complicated and, they require quite different treatment and estimates.

Step 1. (At the most one large displacement can occur on a geodesic path with high probability). Let \( l(v) \) denote the geodesic path from the root \( r \) to the vertex \( v \) for every \( v \in \mathbb{T} \). Then it follows that

\[
S(v) := \sum_{u \in l(v)} X(u)
\]

where \( X(u) \) denotes the displacement attached to the vertex \( u \). Define

\[
\tilde{N}_n := \sum_{|v|=n} \sum_{u \in l(v)} \delta_{B_{n}^{-1}X(u)},
\]

(3.1)

Recall that \( d \) denotes the metric induced by the vague topology on \( \mathcal{M}(\mathbb{R}_0^+) \). The following proposition connects the fluctuations of the extreme positions to that of extreme displacements.
**Proposition 31.** Under the assumptions of Theorem \ref{thm:main}, for every $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \mathbb{P}^* \left( d_v (N_n, \tilde{N}_n) > \varepsilon \right) = 0.$$ (3.2)

**Step 2.** (Cutting the genealogical tree.) Let $D_i$ denote the collection of all vertices at the $i$-th generation of the genealogical tree. In this step, we cut the tree at the $(n - \varrho)$-th generation for some large integer $\varrho < n$. As a result, a forest has been created containing $|D_{n-\varrho}|$ subtrees where $|D_{n-\varrho}|$ denotes the cardinality of the set $D_{n-\varrho}$. Let $I_\varrho(v)$ denote the members of $I(v)$ such that $|u \mapsto v| \leq \varrho - 1$ where $|u \mapsto v|$ denotes the number of vertices on the geodesic path. If we ignore the displacements appearing the the first $(n - \varrho)$ generations of $T$ in $\tilde{N}_n$, then we have the following point process

$$\tilde{N}_{n,\varrho} := \sum_{v} \sum_{u \in I_\varrho(v)} \delta_{B_n^{-1} X(u)}.$$ (3.3)

In the next proposition, we prove that the displacements in the first $(n - \varrho)$ do not contain the large ones with high probability.

**Proposition 32.** If the assumptions in Theorem \ref{thm:main} hold, then for every $\varepsilon > 0$, we have

$$\lim_{\varrho \to \infty} \lim_{n \to \infty} \mathbb{P}^* \left( d_v (\tilde{N}_n, \tilde{N}_{n,\varrho}) > \varepsilon \right) = 0.$$ (3.4)

Before moving to the next step, we observe that the displacement attached to the vertex $u$ in $\tilde{N}_{n,\varrho}$ affects multiple positions which is classical for the branching random walk and also explains the clusters/multiplicities of atoms in $N_*$. Let the collection of the subtrees in the forest be denoted by $\tilde{T}^{(i)} : 1 \leq i \leq |D_{n-\varrho}|$. Let $r_i$ denote the root of the $i$-th subtree in the forest. To appreciate this phenomenon, we now provide an alternative representation

$$\tilde{N}_{n,\varrho} := \sum_{i=1}^{|D_{n-\varrho}|} \sum_{u \in \tilde{T}^{(i)} \setminus \{r_i\}} A(u) \delta_{B_n^{-1} X(u)},$$ (3.5)

where $A(u)$ denotes the number of descendants in the $\varrho$-th generation of the subtree $\tilde{T}^{(i)}$ if $u \in \tilde{T}^{(i)}$. Thanks to Proposition \ref{prop:displacement} we can ignore the displacements attached to the root $r_i$ of the $i$-th subtree in the forest.

**Step 3.** (Pruning the subtrees in the forest). In this step, we modify each subtree in the forest $\tilde{T}^{(i)} : 1 \leq i \leq |D_{n-\varrho}|$ such that each vertex has at the most $\vartheta$ children in the next generation by deleting the extra ones with their line of descendants. We choose $\vartheta$ large enough so that each of the pruned subtrees remain supercritical. We sometimes refer $\vartheta$ to be the pruning threshold. The
pruned version of \( \hat{T}^{(i)} \) is denoted by \( T_i \). The version of the point process \( \hat{N}_{n, \varrho} \) associated to the pruned forest is given by

\[
\hat{N}_{n, \varrho} := \sum_{i=1}^{|D_{n-\varrho}|} \sum_{u \in \hat{T}^{(i)} \setminus \{r_i\}} A^{(\varrho)}(u) \delta_{B^{-1}_n X(u)},
\]

where \( A^{(\varrho)}(u) \) denotes the number of descendants in the \( \varrho \)-th generation of the pruned subtree \( \hat{T}^{(i)} \) if \( u \in \hat{T}^{(i)} \). The following result shows that the pruned forest contains the large displacement with high probability.

**Proposition 33.** Under the assumptions of Theorem 21, for every \( \varepsilon > 0 \) and \( \varrho \geq 1 \), we have

\[
\lim_{\varrho \to \infty} \limsup_{n \to \infty} \mathbb{P}^* \left[ d_v(\hat{N}_{n, \varrho}, \hat{N}_{n, \varrho}) \geq \varepsilon \right] = 0.
\]

**Step 4.** (Regularization of the forest). In this step, we add new children to each of the vertex, so that it has exactly \( \varrho \) children in the next generation. The modified subtree is a \( \varrho \)-regular tree and denoted by \( \tilde{T}^{(i)} \) for every \( i \geq 1 \). If \( u \in \tilde{T}^{(i)} \setminus T^{(i)} \), then \( A^{(\varrho)}(u) := 0 \). For every \( u \in \tilde{T}^{(i)} \), we replace the displacements of its \( \varrho \) children by an independent copy of the random vector \((X_1, X_2, \ldots, X_\varrho)\). The displacement attached to the vertex \( u \) will be denoted by \( X'(u) \). It is clear that

\[
\tilde{N}_{n, \varrho} := \sum_{i=1}^{|D_{n-\varrho}|} \sum_{u \in \tilde{T}^{(i)} \setminus \{r_i\}} A^{(\varrho)}(u) \delta_{B^{-1}_n X'(u)},
\]

With a slight abuse on notation, we use \( \tilde{N}_{n, \varrho} \) to denote the point process in the right hand side of (3.8). In the following proposition we show that \( \tilde{N}_{n, \varrho} \) converges weakly to the desired limit.

**Proposition 34.** Under the assumptions of Theorem 15

\[
\lim_{\varrho \to \infty} \lim_{n \to \infty} \mathbb{E}^* \left[ \exp \left\{ -\int f \, d\hat{N}_{n, \varrho} \right\} \right] = \mathbb{E}^* \left[ \exp \left\{ -\int f \, dN^* \right\} \right]
\]

for every \( f \in C^+_K(\mathbb{R}_0) \).

The final step is to check that the Laplace functional of \( \sum_{l \geq 1} \sum_{k=1}^{V_l} R_{k}^{(l)} \delta_{\zeta^{(l)}} \) equals that of \( N_* \) as the Laplace functional characterizes the distribution of the point processes uniquely. We also verify that the point process \( N_* \) is an SScDPPP based on its scaled Laplace functional and Theorem 2.4 in [10].

**Proposition 35.** The characterization on \( N_* \) in (2.4) holds true and \( N_* \) is an SScDPPP.

The proof of Theorem 21 follows from Propositions 31 and 35.
4 Proofs of the auxiliary results

4.1 At most one large displacement on every geodesic path

**Proof of Proposition 3.1** Fix a function \( f \in C^1_{\text{loc}}(\mathbb{R}) \) such that its support is contained in \( \{ x : |x| > \delta \} \). To prove this proposition, we have to show that

\[
\limsup_{n \to \infty} \mathbb{P}^* \left( \sum_{|v| = n} f(B_n^{-1}S(v)) - \sum_{|v| = n} \sum_{u \in I(v)} f(B_n^{-1}X(u)) \right) > \epsilon = 0. \tag{4.1}
\]

For \( \eta > 0 \) define

\[
G_{n,\eta} := \left\{ \bigcup_{|v| = n} \left\{ \sum_{u \in I(v)} \delta_{B_n^{-1}|X(u)|}(\eta/n, \infty) \geq 2 \right\} \right\}^c, \tag{4.2}
\]

where \( B^c \) denotes the complement of the event \( B \). We will first show that

\[
\lim_{n \to \infty} \mathbb{P}^*(G_{n,\eta}^c) = 0 \tag{4.3}
\]

and then prove that

\[
\lim_{\eta \to 0} \limsup_{n \to \infty} \mathbb{P}^* \left( \sum_{|v| = n} \left| \sum_{u \in I(v)} f(B_n^{-1}X(u)) - f(B_n^{-1}S(v)) \right| > \epsilon; G_{n,\eta} \right) = 0. \tag{4.4}
\]

Note that (4.1) follows from (4.3) and (4.4) as \(| \sum_{v=n} (f(B_n^{-1}S(v)) - \sum_{u \in I(v)} f(B_n^{-1}X(u))) | < \sum_{|v| = n} |f(B_n^{-1}S(v)) - \sum_{u \in I(v)} f(B_n^{-1}X(u))| \) almost surely.

**Proof of (4.3):** Observe that \( \mathbb{P}^*(G_{n,\eta}^c) \leq \mathbb{P}(G_{n,\eta}^c) / \mathbb{P}(\mathcal{S}) \) where \( \mathbb{P}(\mathcal{S}) > 0 \) for \( \mathcal{S} \) given in (1.2). Thus we only have to check that \( \mathbb{P}(G_{n,\eta}^c) \to 0 \) as \( n \to \infty \).

Note that

\[
\mathbb{P}^*(G_{n,\eta}^c) = \mathbb{E}^* \left[ 1 \wedge \mathbb{E}_Y \left( \mathbb{E}_Y^* \left[ \mathbb{P}^*_{|X|} G_{n,\eta}|D_n \right] \right) \right]
\]

\[
\leq \mathbb{E}^* \left[ 1 \wedge \mathbb{E}_Y \left( \sum_{v \in D_n} \mathbb{P}_Y^* \left[ \sum_{u \in I(v)} \delta_{B_n^{-1}|X(u)|}(\eta/n, \infty) \geq 2 |D_n| \right] \right) \right] \tag{4.5}
\]

using union bound as \( |D_n| \to 0 \) \( \mathbb{P}^* \)-almost surely. Note that \( B_n \) is measurable with respect to \( \sigma(Y) \) and \( (X(u) : u \in I(v)) \) are i.i.d. and independent of \( D_n \). Also observe \( B_n \) is positive \( \mathbb{P}^* \)-almost surely and the displacements also do not depend on the event \( \mathcal{S} \). Let \( (X_1^{(i)} : i \geq 1) \) denote a collection of independent copies of the random variable \( X_1 \) which is independent of the BRW. These observations together yield, for every \( v \in D_n \),

\[
\mathbb{P}_Y^* \left( \sum_{u \in I(v)} \delta_{B_n^{-1}|X(u)|}(\eta/n, \infty) \geq 2 |D_n| \right) = \mathbb{P}_Y \left( \sum_{i=1}^n \delta_{B_n^{-1}|X_1^{(i)}|}(\eta/n, \infty) \geq 2 \right) \tag{4.6}
\]
for almost all environments $\mathbf{Y}$. Conditioned on $\mathbf{Y}$, $\sum_{i=1}^{\infty} \delta_{B_{\delta}^{-1}X_i^{(i)}(\eta/n, \infty)}$ is a Binomial random variable with success probability $\mathbb{P}_\mathbf{Y}(\lvert X_1 \rvert > B_n \eta/n)$ as the environment affects the sequence $B_n$ but not the law of the displacements. Then the probability in the right hand side of (4.10) can be bounded from above by

$$\text{const} \cdot \delta \mathbb{P}_\mathbf{Y}(\lvert X_1 \rvert > B_n \eta/n)^2$$

(4.7)

almost surely conditioned on the environment $\mathbf{Y}$. Therefore, the conditional expectation in (4.5) can be given following upper bound

$$\text{const} \cdot \left( \mathbb{E}_\mathbf{Y}(Z_n) \right) \mathbb{P}_\mathbf{Y}(\lvert X_1 \rvert > B_n \eta) \left( \mathbb{P}_\mathbf{Y}(\lvert X_1 \rvert > B_n \eta/n) \right)^2 \left( \mathbb{E}_\mathbf{Y}(Z_n) \right)^{-1} \delta$$

(4.8)

for every environment $\mathbf{Y}$. Recall that $\mathbb{E}_\mathbf{Y}(Z_n) = \pi_n(\mathbf{Y}) = \pi_n$. For fixed $\epsilon > 0$ we now introduce the event

$$J_{n, \eta}(\epsilon) := \left\{ \mathbb{P}(\lvert X_1 \rvert > B_n \eta | \mathbf{Y}_0 : n_0 - 1) - \eta^{-\alpha} \right\} < \epsilon \right\}$$

(4.9)

Note that the first term in (4.8) is bounded from above by $(\eta^{-\alpha} + \epsilon)^2$ for all $n$ on the event $J_{n, \eta}(\epsilon)$. We can ignore the expectation on the event $J_{n, \eta}(\epsilon)$ as $\mathbb{P}(J_{n, \eta}(\epsilon))$ is negligibly large enough $n$ due to Lemma 3. To analyze the second term in (4.8), we will use the Potter’s bound. We can not apply it directly as $B_n$ is random. To handle this case, we need to prove first that

$$n^{-1} B_n \xrightarrow{p} \infty$$

(4.10)

using the fact that $\pi_n$ grows exponentially in $n$ with high probability. Let us assume (4.10) for the moment and proceed with the proof. Fix $\delta_1 \in (0, 1)$. It follows from Proposition 0.8(ii) in 12 that there exists a large enough number $t_0$ such that

$$\frac{\mathbb{P}(\lvert X_1 \rvert > xt)}{\mathbb{E}(\lvert X_1 \rvert > t)} \geq (1 - \delta_1)t^{-\alpha - \delta_1}$$

(4.11)

for all $t \geq t_0$ and $x > 1$. Define $\tilde{K}_{n, t_0} := \{n^{-1}B_n > t_0\}$. Then

$$\frac{\mathbb{P}(\lvert X_1 \rvert > B_n \eta | \mathbf{Y}_0 : n_0 - 1) - \eta^{-\alpha - \delta_1}}{\mathbb{P}(\lvert X_1 \rvert > B_n \eta/n | \mathbf{Y}_0 : n_0 - 1)} \geq (1 - \delta_1)n^{-\alpha - \delta_1}$$

(4.12)

on the event $\tilde{K}_{n, t_0}$. Hence we have the following upper bound for the expectation (4.5)

$$\mathbb{E} \left[ \min \left( 1, (\eta^{-\alpha} + \epsilon)^2 (1 - \delta_1)^{-2} n^{2\alpha + 2\delta_1} \pi_n^{-1} \mathbb{1}(\tilde{K}_{n, t_0}) \right) \right]$$

$$+ \mathbb{E} \left[ \min \left( 1, (\eta^{-\alpha} + \epsilon)^2 \left( \frac{\mathbb{P}(\lvert X_1 \rvert > B_n \eta/n | \mathbf{Y}_0 : n_0 - 1)}{\mathbb{P}(\lvert X_1 \rvert > B_n \eta | \mathbf{Y}_0 : n_0 - 1)} \right)^2 \pi_n^{-1} \mathbb{1}(\tilde{K}_{n, t_0}) \right] \right]$$

$$:= T_n^{(1)} + T_n^{(2)}$$

(4.13)
for arbitrarily small \( \epsilon, \delta \) and large enough \( n \). Using dominated convergence theorem we can conclude that the first term \( T_n^{(1)} \) converges to zero because \( n^{2a + 25} \pi_n^{-1} \xrightarrow{a.s.} 0 \). Moreover, the expression inside expectation in the second term is bounded by \( \mathbb{1}(\tilde{K}_{n,t}) \) and converges to zero due to (4.10). This completes the proof of (4.3).

**Proof of (4.10).** From the definition of sequence \( B_n \) given in (1.9), \( B_n = \pi_n^{1/\alpha} \tilde{L}(\pi_n) \) where \( \tilde{L} \) is another slowly varying function (see [42, Prop. 0.8(v)]).

Fix an \( \epsilon_1 \in (0, e^\mu - 1) \) and define \( H_{n,\epsilon_1} := \{ |\pi_n^{1/n} - e^\mu| < \epsilon_1 \} \). Then, we can see that \( \lim_{n \to \infty} \mathbb{P}(H_{n,\epsilon_1}) = 0 \). Therefore, it will be enough to show that \( \liminf_{n \to \infty} n^{-1} B_n = \infty \) on the event \( H_{n,\epsilon_1} \). On this event \( H_{n,\epsilon_1} \), we can see that \( \pi_n \geq (e^\mu - \epsilon_1)^n \to \infty \) and therefore, we can use Potter’s bound to obtain a lower bound for \( \tilde{L}(\pi_n) \). Fix an \( \epsilon_2 \in (0, 1 \wedge \alpha^{-1}) \) and then we can choose \( n \) large enough to have \( B_n = \pi_n^{1/\alpha} \tilde{L}(\pi_n) \geq (1 - \epsilon_2)\pi_n^{1/\alpha - \epsilon_2} \geq (1 - \epsilon_2)(e^\mu - \epsilon_1)^n(1/\alpha - \epsilon_2) \). Our claim now follows immediately.

**Proof of (4.4).** Let us define \( T(v) := X(u) \) if \( \max_{v \in I(v)} |X(u')| \leq |X(u)| \). Note that absolute value of only one displacement on each path can exceed the threshold \( B_n \eta \) on the event \( G_{n,\eta} \) defined in (4.2). Thus \( T(v) \) is the only random variable on the path \( I(v) \) to exceed \( B_n \eta / n \) for every \( v \in D_n \) and makes the position \( S(v) \) large. This observation gives \( |f(B_n^{-1} S(v)) - \sum_{u \in I(v)} f(B_n^{-1} X(u))| = |f(B_n^{-1} S(v)) - f(B_n^{-1} T(v))| \) almost surely if \( \eta \in (0, \delta/2) \) as \( f \) is supported on \( \{ x : |x| > \delta \} \). Moreover, \( B_n^{-1} |S(v) - T(v)| \leq (n - 1) \eta / n < \eta \) almost surely for every \( v \in D_n \) on the event \( G_{n,\eta} \). This observation yields

\[
\sum_{v \in D_n} |f(B_n^{-1} S(v)) - f(B_n^{-1} T(v))| \leq \|f\| \eta \sum_{v \in D_n} \delta_{B_n^{-1} T(v)}((-\infty, -\delta/2) \cup (\delta/2, \infty)),
\]

where \( \|f\| \) is modulus of continuity of the function \( f \). Therefore, we have the following upper bound for the probability appearing in (4.4)

\[
\mathbb{P}^* \left( \|f\| \eta \sum_{v \in D_n} \delta_{B_n^{-1} T(v)} \{ x : |x| > \delta/2 \} \geq \epsilon \right)
\]

\[
\leq \mathbb{P}^* \left( \sum_{v \in D_n} \sum_{u \in I(v)} \delta_{B_n^{-1} X(u)} \{ x : |x| > \delta/2 \} \geq \epsilon \|f\|^{-1} \eta^{-1} \right)
\]

\[
= \mathbb{P}^* \left( \tilde{N}_n \left( \{ x : |x| > \delta/2 \} \right) > \epsilon \|f\|^{-1} \eta^{-1} \right). \quad (4.14)
\]

We will show in the steps (Propositions [32, 33]) that \( \tilde{N}_n \) converges \( \mathbb{P}^* \)-weakly to a point process \( N_n \) in the space of all point measures on \( \mathcal{M}(\mathbb{R}_0) \). If we let \( n \to \infty \), then the probability in (4.14) converges to \( \mathbb{P}^* \left( N_n \left( \{ x : |x| > \delta/2 \} \right) > \epsilon \|f\|^{-1} \eta^{-1} \right) \) and this probability converges to zero if we let \( \eta \to 0 \).

4.2 Cutting the tree and creation of the forest

**Proof of Proposition 32.** Consider a function \( f \in C^1_\mathbb{R}(\overline{\mathbb{R}_0}) \) such that support of the function \( f \) is contained in \( \{ x : |x| > \delta \} \). Then our aim will be to
show that
\[
\lim_{n \to \infty} \mathbb{P}^* \left( |\tilde{N}_n(f) - \tilde{N}_{n, \eta}(f)| > \varepsilon \right) = 0. \tag{4.15}
\]
Define
\[
O_{n, \eta} := \left\{ \sum_{|u| \leq n - \varrho} \delta_i(X(u)) (B_{n, \eta}, \infty) \geq 1 \right\}, \tag{4.16}
\]
that is, \(O_{n, \eta}\) is the event when the first \((n - \varrho)\) generations contains at least one large displacement. We will show that
\[
\lim_{n \to \infty} \mathbb{P}^*(O_{n, \eta}) = 0. \tag{4.17}
\]
Then \(\mathbb{P} \left( |\tilde{N}_n(f) - \tilde{N}_{n, \eta}(f)| > \varepsilon; \ O_{n, \eta} \right) = 0\) and hence the claim in (4.15) follows.

To establish (4.17) we first observe that
\[
\mathbb{P}^*(O_{n, \eta})
= \mathbb{E}^* \left[ \mathbb{P}^* \left( O_{n, \eta} \mid (D_i : 1 \leq i \leq n - \varrho); \ Y_{0 : n - 1} \right) \right]
\leq \mathbb{E}^* \left[ \min \left\{ 1, \sum_{|u| \leq n} \mathbb{P} \left( |X(u)| \geq B_{n, \eta} \mid (D_i : 1 \leq i \leq n - \varrho); \ Y_{0 : n - 1} \right) \right\} | Y_{0 : n - 1} \right]
\leq \mathbb{E}^* \left[ \min \left\{ 1, \mathbb{P} \left( |X| \geq B_{n, \eta} \mid Y_{0 : n - 1} \right) \sum_{i=1}^{n-\varrho} Z_i | Y_{0 : n - 1} \right\} \right]. \tag{4.18}
\]

We have used the union bound to get the upper bound. Then we used the fact that \(\mathbb{P}(B_{n-1}^{-1} X(u) \in A | (D_i : 1 \leq i \leq n - \varrho); \ Y_{0 : n - 1}) = \mathbb{P}(B_{n-1}^{-1} X_1 \in A | Y_{0 : n - 1})\) as the displacements are identically distributed and the probability depends on the branching mechanism through the random environment \((B_n, \text{ depends only on } Y_{0 : n - 1})\) rather than on the generation sizes \((D_i : 1 \leq i \leq n - \varrho)\). We now have the following upper bound for the (4.18)
\[
\mathbb{E}^* \left[ \min \left\{ 1, \mathbb{E} \left[ \sum_{i=1}^{n-\varrho} Z_i | Y_{0 : n - 1} \right] \mathbb{P}(|X| > B_{n, \eta} \mid Y_{0 : n - 1}) \right\} \right]
= \mathbb{E}^* \left[ \min \left\{ 1, \left[ \pi_n^{-1} \sum_{i=1}^{n-\varrho} \pi_i \right] \mathbb{P}(|X| > B_{n, \eta} \mid Y_{0 : n - 1}) \right\} \right]. \tag{4.19}
\]
Recalling that \(\lim_{n \to +\infty} \mathbb{P}(J_{n, \eta}(\varepsilon)) = 0\) for the event \(J_{n, \eta}(\varepsilon)\) defined in (4.9), we can conclude that it is enough to compute the expectation in (4.19) on the event \(J_{n, \eta}(\varepsilon)\). Note that
\[
\mathbb{E}^* \left[ \min \left\{ 1, \left[ \pi_n^{-1} \sum_{i=1}^{n-\varrho} \pi_i \right] \mathbb{P}(|X| > B_{n, \eta} \mid Y_{0 : n - 1}) \right\} \mathbb{1}(J_{n, \eta}(\varepsilon)) \right]
\]
\[
E^* \left[ \min \left( 1, (\theta^{-\alpha} + \epsilon) \sigma^{-1} \sum_{i=1}^{n-g} \pi_i \right) \right] \\
\leq (\theta^{-\alpha} + \epsilon) E^* \left[ \min \left( \theta^{-\alpha} + \epsilon, \sigma^{-1} \sum_{i=1}^{n-g} \pi_i \right) \right].
\]

Observe that the random variables \( (Y_j)_{0 \leq j \leq n - 1} \) are exchangeable and thus
\[
\sum_{i=0}^{n-g} \pi_i / \pi_n = \sum_{i=0}^{n-g} \left( \sum_{j=i+1}^{n-1} E(\xi | Y_j) \right)^{-1} \delta \sum_{i=0}^{n-1} \left( \prod_{j=1}^{i} E(\xi | Y_j) \right)^{-1} = \sum_{i=0}^{n-1} \pi_i^{-1}.
\]

(4.21)

It follows from Lemma 2 that the right hand side of (4.21) converges \( \mathbb{P} \)-almost surely to zero if we let \( n \to \infty \) and then \( g \to \infty \). We can use dominated convergence theorem then to show that the expectation in (4.20) converges to 0 as \( n \to \infty \) and \( g \to \infty \). Hence, the proof follows.

4.3 Pruning the random forest

**Proof of Proposition 33.** Consider a Lipschitz continuous function \( f \in C^*_T(\mathbb{R}_0) \) such that \( \text{support}(f) \subset \{ x : |x| > \delta \} \). To prove the theorem, it is enough to show that
\[
\lim \limsup_{\ell \to \infty} \text{limsup}_{n \to \infty} \mathbb{P}^* \left[ |\tilde{N}_{n,0,\ell}(f) - \tilde{N}_{n,0}(f)| > \epsilon \right] = 0.
\]

(4.22)

After cutting the original genealogical tree \( T \), we are remained with the forest \( (\tilde{T}_{\ell,i} : 1 \leq i \leq |D_{n,0}|) \). In the pruning step, we have modified the number of descendants of each of the particle and therefore, both the collection \((\tilde{D}_i^{(j)} : 1 \leq i \leq g; 1 \leq j \leq |D_{n,0}|) \) and \((A(u) : u \in \tilde{D}_i^{(j)} : 1 \leq i \leq g; 1 \leq j \leq |D_{n,0}|) \) have been changed to \((\tilde{D}_i^{(j)} : 1 \leq i \leq g; 1 \leq j \leq |D_{n,0}|) \) and \((A(u) : u \in \tilde{D}_i^{(j)} : 1 \leq i \leq g; 1 \leq j \leq |D_{n,0}|) \). We will compare the point processes \( \tilde{N}_{n,0,\ell} \) and \( \tilde{N}_{n,0} \) with the help of another intermediate point process \( N_{n,0,\ell} \). The point process \( N_{n,0,\ell} \) will be constructed on a forest containing “marked subtrees” \((T_{\ell,0}^{(j)} : 1 \leq i \leq |D_{n,0}|) \). The marked subtrees are constructed such that \( D^{(i,0)} \) is same as \( \tilde{D}_i^{(j)} \) where \( D^{(i,0)} \) denotes the the collection of vertices in the \( i \)-th generation of \( T_{\ell,i}^{(j)} \).

**Algorithm for construction of marked subtrees as pre-pruned subtrees.** Fix an \( i \in [1 : |D_{n,0}|] \). \( T_{\ell,i}^{(j)} \) is a tree constructed from \( T_{\ell,0}^{(j)} \) by assigning a number of \( c \)'s to labels of all vertices of \( T_{\ell,i}^{(j)} \) according to the following scheme. We consider the root of \( T_{\ell,0}^{(j)} \) and add a \( c \) to its label. We modify the labels generation wise, that is, we modify the labels of the vertices in the \( (j-1) \)-th generation and then move to the \( j \)-th generation. Assume that we
have modified the \((j - 1)\)-th generation and pick a vertex in the \(j\)-th generation. We add \(l\) many \(\circ\)'s to its label if it is one of the first (according to lexicographic order of the Ulam-Harris labels) \(\vartheta\) children of its parent in the \((j - 1)\)-th generation which has been assigned \((l - 1)\) \(\circ\)'s. Otherwise, we add \((l - 1)\) \(\circ\)'s to the chosen vertex. The tree \(\tilde{T}_{\vartheta,i}\) with the modified labels will be denoted by \(\hat{T}_{\vartheta,i}\). Note that a vertex in the \(j\)-th generation should have at least one \(\circ\) and can have at the most \((j + 1)\) \(\circ\)'s added to its label. If we consider the vertices with the \((j + 1)\) diamonds at the \(j\)-th generation in \(\tilde{T}_{\vartheta,i}\) for all \(j \geq 1\), then we get \(\tilde{T}_{\vartheta,i}\).

We now construct the point process \(\hat{N}_{n,\vartheta,\vartheta}\) indexed by the forest \((\tilde{T}_{\vartheta,i} : 1 \leq i \leq |D_{n,\vartheta}|)\). Define \(A^\circ(u)\) to be the number of descendants in \(D_{j,\vartheta,\vartheta}\) of \(u \in D_{j,\vartheta,\vartheta}\) generation with \(\vartheta - j + l\) \(\circ\)'s if \(u\) has \(l\) \(\circ\)'s. Note that if the vertex \(u \in D_{j,\vartheta,\vartheta}\) has \((j + 1)\) \(\circ\)'s then \(u \in D_{j,\vartheta,\vartheta}\) and \(A^\circ(u) = A^\circ(u)\). Define

\[
N_{n,\vartheta,\vartheta}^\circ := \sum_{i=1}^{D_{n,\vartheta}} \sum_{u \in \tilde{T}_{\vartheta,i}(\{t_i\})} A^\circ(u)\delta_{\hat{B}^{-1}_\vartheta X(u)}. \tag{4.23}
\]

We compare the point processes \(\hat{N}_{n,\vartheta,\vartheta}\) and \(\hat{N}_{n,\vartheta,\vartheta}\) with \(N_{n,\vartheta,\vartheta}^\circ\).

**Proposition 41.** Under the assumptions stated in Theorem 21, for every \(\varepsilon > 0\), \(\vartheta \geq 1\) and \(f \in C^+_K(\mathbb{R}_0^\times),\) we have

\[
\lim_{\vartheta \to \infty} \limsup_{n \to \infty} \mathbb{P}^* \left( \left| \hat{N}_{n,\vartheta,\vartheta}(f) - N_{n,\vartheta,\vartheta}^\circ(f) \right| > \varepsilon \right) = 0 \tag{4.24}
\]

and

\[
\lim_{\vartheta \to \infty} \limsup_{n \to \infty} \mathbb{P}^* \left( \left| N_{n,\vartheta,\vartheta}^\circ(f) - \hat{N}_{n,\vartheta,\vartheta}(f) \right| > \varepsilon \right) = 0. \tag{4.25}
\]

The proof of above proposition is given in Appendix. The proof of Proposition 43 follows immediately from above proposition.

4.4 Convergence of the Laplace functional

**Proof of Proposition 44.** We start with the observation that \(\mathbb{P}^*(\mathcal{B}) = q_\varepsilon^{-1} \mathbb{E}(1(\mathcal{B} \cap \mathcal{S}))\) for any event \(\mathcal{B}\) where \(q_\varepsilon := \mathbb{P}(\mathcal{S}) = \mathbb{E}(q_\varepsilon(Y))\) for \(\mathcal{S}\) is given in (4.22). Define

\[\mathcal{S}_{n,\vartheta} := \{ \text{ at least one of the } D_{n,\vartheta} \text{ subtrees is an infinite tree} \}.\]

Then it is clear that \(1(\mathcal{S}) = 1(|D_{n,\vartheta}| > 0) \cap \mathcal{S}_{n,\vartheta}\). These observations yield

\[
\mathbb{E}^* \left( e^{-\hat{N}_{n,\vartheta,\vartheta}(f)} \right) = q_\varepsilon^{-1} \mathbb{E} \left( e^{-\hat{N}_{n,\vartheta,\vartheta}(f)} 1(|D_{n,\vartheta}| > 0) \right)
\]

\[
- q_\varepsilon^{-1} \mathbb{E} \left( e^{-\hat{N}_{n,\vartheta,\vartheta}(f)} 1(|D_{n,\vartheta}| > 0) 1(\mathcal{S}_{n,\vartheta}^\circ) \right). \tag{4.26}
\]
Note that the second expectation in (4.26) is bounded from above by
\[
E \left[ \mathbb{1}(|D_n| > 0) \mathbb{E} \left( I(S_{n-\theta}^c) \bigg| D_n \right) \right] = E \left[ \mathbb{1}(|D_n| > 0) q_n^{|D_n|} \right]. \tag{4.27}
\]
Moreover, \( \mathbb{1}(|D_n| > 0) \xrightarrow{a.s.} \mathbb{1}(S) \) and \( |D_n| \xrightarrow{a.s.} \infty \) as \( n \to \infty \). Thus the term inside expectation in (4.27) converges to zero almost surely as \( n \to \infty \). Now the dominated convergence theorem can be applied to conclude that the expectation (4.27) converges to zero.

We will analyze the first expectation in (4.26) in the rest of the proof. Our next observation is that the point process \( \tilde{N}_{n,\theta} \) can be written as the superposition of point processes that is,
\[
\tilde{N}_{n,\theta} = \sum_{i=1}^{|D_n|} \sum_{u \in T_{n,\theta}(i)} A(i(u)) \delta_{B_n^{-1} X(i(u))} := \sum_{i=1}^{|D_n|} \tilde{N}^{(i)}_{n,\theta} . \tag{4.28}
\]
Note that \( \{\tilde{N}^{(i)}_{n,\theta}, 1 \leq i \leq |D_n|\} \) is a collection of i.i.d. point processes conditioned on the environment \( Y_{0:n-1} \). We can use this observation and obtain the following expression for the first expectation in (4.26),
\[
g_{c^{-1}} E \left[ \mathbb{1}(|D_n| > 0) \mathbb{E} \left( \exp \left\{ - \tilde{N}^{(1)}_{n,\theta}(f) \right\} \bigg| Y_{0:n-1} \right) \bigg| D_n \right] \\
= g_{c^{-1}} E \left[ \mathbb{1}(|D_n| > 0) \left( \mathbb{E} \left[ \exp \left\{ - \tilde{N}^{(1)}_{n,\theta}(f) \right\} \bigg| Y_{0:n-1} \right) \bigg| D_n \right] \right] \tag{4.29}
\]
as \( |D_n| \) is independent of \( \tilde{N}^{(1)}_{n,\theta} \) conditioned on \( Y_{0:n-1} \). Our aim now will be to compute the limit of the expectation in (4.29) as \( n \to \infty \). To accomplish this task, we need the following notations.

We will now identify \( u \), the \( j \)-th vertex in the \( i \)-th generation by the pair \((i,j)\) for every \( 1 \leq j \leq \theta^i \) and \( 1 \leq i \leq q \). Define
\[
\tilde{S}_{\theta, \theta} := \{ 0 : \theta^0, \theta^1, \ldots, \theta^\theta \}, \quad \tilde{R}_0 = \mathbb{R}^{\theta^0 + \theta^1 + \cdots + \theta^\theta} \setminus \{ 0 \}, \quad 0 \text{ is the origin of } \mathbb{R}^{\theta^0 + \theta^1 + \cdots + \theta^\theta}
\]
and
\[
\tilde{A} := (A_{i,j} : 1 \leq j \leq \theta^i; \ 1 \leq i \leq q), \quad \tilde{X} := (X_{i,j} : 1 \leq j \leq \theta^i; \ 1 \leq i \leq q).
\]
Using these notations, the conditional expectation in (4.29) equals
\[
\left[ 1 - \pi_{n,-\theta} \sum_{\tilde{a} \in \tilde{S}_{\theta, \theta}} \mathbb{P} (\tilde{A} = \tilde{a} \big| Y_{n-\theta} : n-1) U_{n}(\tilde{a}) \right] \bigg|_{D_n} , \tag{4.30}
\]
amost surely, where
\[
U_{n}(\tilde{a}) := \left( \int_{\tilde{R}_0} \pi_n \mathbb{P} (B_n^{-1} \tilde{X} \in d \tilde{x} \big| Y_{0:n-1}) \right)
\]
\[
1 - \exp \left\{ -\sum_{i=1}^{\varrho} \sum_{j=1}^{\varrho'} a_{i,j} f(x_{i,j}) \right\}.
\] 
(4.31)

To compute its limit, we will prove some facts on the tail behaviour of the random vector \( \bar{X} \).

Let 
\[
\nu^{(\vartheta)} := \nu \circ \text{PROJ}_{\vartheta}^{-1},
\]
where \( \text{PROJ}_{\vartheta}(x) := (x_1, x_2, \ldots, x_\vartheta) \) when \( x = (x_i : i \geq 1) \in \mathbb{R}^N \). It follows from the Lemma [3] combined with Theorem 4.1 in [32], via the continuous mapping theorem that
\[
\pi_n \mathbb{P} \left( B_{n^{-1}} \bar{X} \in d \tilde{x} | Y_{0:n-1} \right) \xrightarrow{\text{a. s.}} \tau(d \tilde{x}) = \sum_{i=1}^{\varrho} \sum_{j \in F_i} \tau_{i,j}(d \tilde{x})
\]  
(4.32)

almost surely in the space \( \mathcal{M}_0(\tilde{\mathbb{R}}) \) where
\[
F_i := \{1, \vartheta + 1, 2\vartheta + 1, \ldots, \varrho^{i-1} + 1\} = \{l \vartheta + 1 : 0 \leq l \leq \varrho^{i-1} - 1\} 
\]  
(4.33)

for every \( i \geq 1 \) and
\[
\tau_{i,j} := \delta_0 \otimes \delta_0 \otimes \ldots \otimes \delta_0 \otimes \nu^{(\vartheta)} \otimes \delta_0 \otimes \delta_0 \otimes \ldots \otimes \delta_0 
\]  
(4.34)

for all \( j \in F_i \) and \( j \leq i \leq \varrho \). Let \( g \in C^+_K(\tilde{\mathbb{R}}) \). Then the convergence stated in (4.32) implies that
\[
\int_{\tilde{\mathbb{R}}} g(\tilde{x}) \pi_n \mathbb{P}(B_{n^{-1}} \bar{X} \in d \tilde{x} | Y_{0:n-1}) \xrightarrow{\text{a. s.}} \int_{\tilde{\mathbb{R}}} g(\tilde{x}) \tau(d \tilde{x}).
\]  
(4.35)

Note that
\[
\left( 1 - \exp \left\{ -\sum_{i=1}^{\varrho} \sum_{j=1}^{\varrho'} a_{i,j} f(x_{i,j}) \right\} \right) \in C^+_K(\tilde{\mathbb{R}})
\]  
(4.36)

for \( f \in C^+_K(\mathbb{R}) \) and for any fixed \( \tilde{\mathbb{a}} \in \tilde{S}_{\vartheta, \varrho} \). Therefore, for each fixed \( \tilde{\mathbb{a}} \in \tilde{S}_{\vartheta, \varrho} \) we have
\[
U_n(\tilde{\mathbb{a}})
\]
\[
\xrightarrow{\text{a. s.}} \sum_{i=1}^{\varrho} \sum_{j \in F_i} \int_{\mathbb{R}} \tau_{i,j}(d \tilde{x}) \left( 1 - \exp \left\{ -\sum_{i'=1}^{\varrho} \sum_{j'=1}^{\varrho'} a_{i',j'} f(x_{i',j'}) \right\} \right)
\]
\[
= \sum_{i=1}^{\varrho} \sum_{j \in F_i} \int_{\mathbb{R}^\vartheta \setminus \{0\}} \nu^{(\vartheta)}(d(x_1, x_2, \ldots, x_\vartheta)) \left( 1 - \exp \left\{ -\sum_{k=1}^{\varrho} a_{i,j+k} f(x_k) \right\} \right)
\]
\[
:= \overline{U}(\tilde{\mathbb{a}})
\]  
(4.37)

using the description of \( \tau_{i,j} \) given in (4.34). We will use this observation to compute the limit of the expectation in (4.29) given in the following lemma.
Lemma 4 Recall that $Y'$ is an independent copy of the environment sequence $Y$. If the assumptions stated in Theorem \[21\] hold, then the expression obtained in (4.38) converges to

$$E^* \left[ \exp \left\{ -W \left( \prod_{j=0}^{\varrho-1} E(\xi|Y_j') \right)^{-1} \sum_{\tilde{A} \in \mathbb{S}_{\varrho, \vartheta}} P (\tilde{A} = \tilde{a} | Y_{0: \varrho-1}) U(\tilde{a}) \right\} \right]. \quad (4.38)$$

as $n \to \infty$.

The proof of this lemma is given in Appendix. The most important point to note here is that $W$ appears in the exponent as a limit of $|D_n - \varrho|/n_{\varrho - \vartheta}$ which depends on $Y_{0: n-\varrho-1}$. The law of $\tilde{A}$ does depend on the the segment $Y_{n-\varrho: n-1}$ which is independent of $Y_{0: n-\varrho-1}$. $U_n(\tilde{a})$ involves the segment of the environment $Y_{0: n-1}$ but its limit $U(\tilde{a})$ does not depend on the environment. This explains the appearance of an independent copy $Y'$ in the exponent and independence of $Y'$ with $W$. We continue now the proof of Proposition \[34\]. Our next step will be computing the limit of the expression obtained in (4.38) as $\vartheta \to \infty$. For that, we need a more explicit expression for the sum in the exponent as (4.38) as the law of $\tilde{A}$ depends on $\vartheta$. We will rewrite the sum in the exponent explicitly.

Lemma 5 If the assumptions stated in Theorem \[15\] hold, then we have

$$\sum_{\tilde{A} \in \mathbb{S}_{\varrho, \vartheta}} P (\tilde{A} = \tilde{a} | Y_{0: \varrho-1}) U(\tilde{a})$$

$$= \sum_{i=1}^{\varrho} \int_{\mathbb{R}^n \setminus \{0_i\}} \nu^{(\varrho)}(x) \ E (Z_{i-1}^{(\varrho)} | Y_{0: i-2}) \ P (Z_1^{(\varrho)} \geq 1 | Y_{i-1}) E \left[ \sum_{Q \in \text{Pow}(\{\{1:Z_i^{(\varrho, +)}\}\}) \setminus \{\emptyset\}} \left[ P (Z_{i-1}^{(\varrho)} = 0 | Y_{i: \varrho-1}) \right]^{Z_i^{(\varrho, +)} - |Q|} \right]$$

$$1 - \exp \left\{ - \sum_{k \in Q} \tilde{Z}_{i}^{(\varrho, +, k) f(x_k)} \right\} \right] |Y_{i-1: \varrho-1}|. \quad (4.39)$$

The proof of this lemma is also given in Appendix. Combining Lemma \[5\] and Proposition \[1\] we obtain the following expression for (4.38)

$$E^* \left[ \exp \left\{ -W \left( \prod_{j=0}^{\varrho-1} E(\xi|Y_j') \right)^{-1} \sum_{i=1}^{\varrho} \int_{\mathbb{R}^n \setminus \{0_i\}} \nu^{(\varrho)}(d \tilde{x}) \ E (Z_i^{(\varrho)} | Y_{0: i-2}) \ P (Z_1^{(\varrho)} \geq 1 | Y_{i-1}) E \left[ \sum_{Q \in \text{Pow}(\{\{1:Z_i^{(\varrho, +)}\}\}) \setminus \{\emptyset\}} \left[ P (Z_{i-1}^{(\varrho)} = 0 | Y_{i: \varrho-1}) \right]^{Z_i^{(\varrho, +)} - |Q|} \right] \right\} \right]. \quad (4.40)$$
Observe that, if we let \( \vartheta \to \infty \), then \( Z_i^{(\vartheta,+)} \xrightarrow{a.s.} Z_i^{(+)} \), \( \tilde{Z}_{g-i}^{(\vartheta)} \xrightarrow{a.s.} \tilde{Z}_{g-i}^{(+)} \), \( \mathbb{E} (Z_i^{(\vartheta)}|Y_{0:i-2}) \xrightarrow{a.s.} \mathbb{E}(Z_{g-i}|Y_{0:i-2}) \) and \( \tilde{Z}_{g-i}^{(\vartheta,+)} \xrightarrow{a.s.} \tilde{Z}_{g-i}^{(+)} \) for every \( i \in [1: g] \) conditioned on the environment \( Y_{0:g-1} \). Using the dominated convergence theorem we can conclude that the Laplace functional in (4.40) converges to

\[
\mathbb{E}^* \left[ \exp \left\{ -W \left( \prod_{j=0}^{\vartheta-1} \mathbb{E}(\xi|Y_{j}^{'}) \right) \right\} \sum_{i=1}^{\vartheta} \int_{\mathbb{R}^n \setminus \{0, \infty \}} \nu(dx) \mathbb{E} [Z_{i-1}|Y_{0:i-2}] \right]
\]

\[
\mathbb{P}(Z_1 > 0|Y_{i-1}^{'}) \mathbb{E} \left[ \sum_{Q \in \text{Pow}([1:Z_i^{(+)}) \setminus \{0\}} \mathbb{P}(Z_{g-i} \geq 1|Y_{i}^{'}: \vartheta-1)|Q| \left( 1 - \exp \left\{ -\sum_{k \in Q} \tilde{Z}_{g-i}^{(+,k)} f(x_k) \right\} \right) \right].
\]

(4.41)

The next step will be to take \( \vartheta \to \infty \). We will first use the i.i.d. structure of the environment to derive some distributional identity. Note that

\[
\left( \prod_{j=0}^{\vartheta-1} \mathbb{E}(\xi_j)(\xi) \right)^{-1} \mathbb{E} (Z_{i-1}|Y_{0:i-2}) = \left( \mathbb{E} \left[ Z_{g-i+1}|Y_{i-1}^{'}: \vartheta-1 \right] \right)^{-1}
\]

(4.42)

almost surely conditioned on \( Y^{'} \). Moreover, \( Y_{0}^{'}: \vartheta-1 \) is exchangeable being i.i.d. that is, \( Y_{0}^{'}: \vartheta-1 \overset{d}{=} (Y_{0}^{'}: \vartheta-1, Y_{0}^{'}: \vartheta-2, Y_{0}^{'}: \vartheta-3, \ldots, Y_{0}^{'}: \vartheta-1) \). BPRE with environment \( (Y_{0}^{'}: \vartheta-1, 0) \) means that the progeny distribution for each particle in the \( i \)-th generation is given by \( Y_{i-1}^{'}: \vartheta-1 \) for all \( i \in [0: g-1] \). Then the sum in the exponent in (4.42) equals

\[
\sum_{i=1}^{\vartheta} \left( \mathbb{E}(Z_{g-i+1}|Y_{i-1}^{'}: \vartheta-1) \right)^{-1} \mathbb{P}(Z_1 \geq 1|Y_{i-1}^{'}: \vartheta-1) \mathbb{E} \left[ \sum_{Q \in \text{Pow}([1:Z_i^{(+)}) \setminus \{0\}} \mathbb{P}(Z_{g-i} \geq 1|Y_{i}^{'}: \vartheta-1)|Q| \left( 1 - \exp \left\{ -\sum_{k \in Q} \tilde{Z}_{g-i}^{(+,k)} f(x_k) \right\} \right) \right]
\]

\[
\overset{d}{=} \sum_{i=1}^{\vartheta} \left( \mathbb{E} \left( Z_{g-i+1} \right| Y_{i-1}^{'}: \vartheta-1 \right) \right)^{-1} \mathbb{P}(Z_1 \geq 1|Y_{i}^{'}: \vartheta-1) \mathbb{E} \left[ \sum_{Q \in \text{Pow}([1:Z_i^{(+)}) \setminus \{0\}} \mathbb{P}(Z_{g-i} \geq 1|Y_{i}^{'}: \vartheta-1)|Q| \left( 1 - \exp \left\{ -\sum_{k \in Q} \tilde{Z}_{i}^{(+,k)} f(x_k) \right\} \right) \right]
\]
\[
\text{Extreme positions of BRWRE 31}
\]
\[
= \sum_{i=0}^{\varrho-1} \left( E(Z_{i+1} | Y'_i : 0) \right)^{-1} P(Z_1 \geq 1 | Y'_i) E \left[ \sum_{Q \in \text{Pow}(\{1 : Z_1^{(+)}\} \setminus \emptyset)} [P \left( \sum_{k \in Q} Z_i^{(+)} f(x_k) \right) Y'_i : 0] \right]
\]
\[
\left( 1 - \exp \left\{ - \sum_{k \in Q} Z_i^{(+)} f(x_k) \right\} \right) Y'_i : 0 \right]
\]
\[
(4.43)
\]
almost surely where rearrangement \((i \mapsto \varrho - i)\) of sum has been used to obtain the third equality. Note that the random sum derived in \((4.43)\) has the following upper bound
\[
\sum_{i=0}^{\varrho-1} \left[ E(Z_{i+1} | Y'_i : 0) \right]^{-1} = \sum_{i=0}^{\varrho-1} \left[ \pi_i(Y'_i) \right]^{-1} \xrightarrow{a.s.} \sum_{i=0}^{\infty} \left[ \pi_i(Y'_i) \right]^{-1} (4.44)
\]
as \(\varrho \to \infty\). It follows from Lemma 2 that the series in \((4.44)\) is finite \(P\)-almost surely. Therefore, we conclude that \((4.43)\) converges almost surely to
\[
\sum_{i=0}^{\infty} \left[ E(Z_{i+1} | Y'_i : 0) \right]^{-1} P(Z_1 \geq 1 | Y'_i) E \left[ \sum_{Q \in \text{Pow}(\{1 : Z_1^{(+)}\} \setminus \emptyset)} [P \left( \sum_{k \in Q} Z_i^{(+)} f(x_k) \right) Y'_i : 0] \right]
\]
\[
\left( 1 - \exp \left\{ - \sum_{k \in Q} Z_i^{(+)} f(x_k) \right\} \right) Y'_i : 0 \right]
\]
\[
(4.45)
\]
We can now use dominated convergence theorem to see that Laplace functional obtained in \((4.41)\) converges to the desired limit given in \((2.5)\).

4.5 Characterization of the point process \(N_\ast\)

**Proof of Proposition 35.** Our aim will now be to compute the Laplace functional of the point process given in \((2.4)\) and show that it equals \((2.5)\). We compute it with the help of another marked Cox process associated to it. Define
\[
\tilde{N}_\ast := \sum_{i=1}^{\infty} \delta_{(V_i, R^0 \cup \{C_i(Y') W^{1/\varpi^0}\})}.
\]
\[
(4.46)
\]
Note that \(\tilde{N}_\ast\) is an \(\bigcup_{i=1}^{\infty} \{v\} \times \{N_v^0 \setminus \{0_v\}\} \times \{\mathbb{R}^N \setminus \{0_\infty}\}\)-valued point process. The state space can be embedded into a much larger space \(\mathbb{N} \times N_v^0 \times \{\mathbb{R}^N \setminus \{0_\infty\}\}\) which is a metric space equipped with the metric
\[
d_{\ast}(w, z) := |u - v| + \sum_{i=1}^{\infty} 2^{-i} (1 \wedge |s_i - t_i|) + \sum_{i=1}^{\infty} 2^{-i} (1 \wedge |a_i - b_i|),
\]
\[
(4.47)
\]
where \( w = (u, (s_i : i \geq 1), (a_i : i \geq 1)) \) and \( z = (v, (t_i : i \geq 1), (b_i : i \geq 1)) \). We can use this metric \( d_w \) for the space \( \bigcup_{i=1}^{\infty} \{ \{ v \} \times N_0^i \} \times \{ \mathbb{R}^N \setminus \{ 0_\infty \} \} \) adding appropriate number of 0’s to its elements. Consider a bounded continuous function \( h : \bigcup_{i=1}^{\infty} \{ \{ v \} \times N_0^i \} \times \{ \mathbb{R}^N \setminus \{ 0_\infty \} \} \to [0, \infty) \) which vanishes in the \( d_w \)-neighborhood of the set \( \bigcup_{i=1}^{\infty} \{ \{ v \} \times N_0^i \} \times \{ 0_\infty \} \) where \( d_w(A, B) = \inf \{ d_w(x, y) : x \in A \text{ and } y \in B \} \) for any two subsets \( A \) and \( B \). Conditioned on \( Y' \) and \( W \), \( N_w \) is a marked (marks are i.i.d.) Poisson point process with mean measure

\[
\mathbb{E} \left[ N_w(\{ v \} \times K \times H) \right] = \nu([C_1(Y')W]^{-1/\alpha} \cdot H) \otimes \mathbb{P} \left[ (V, R) \in \{ v \} \times K | Y' \right],
\]

where \( v \in \mathbb{N}, K \subset \mathbb{N}_0^0 \setminus \{ 0_\infty \}, 0_\infty \) is zero of \( \mathbb{R}^v \) and \( H \subset \mathbb{R}_0^N \). Thus we can use Proposition 3.8 in [42] to obtain following expressions for \( \mathbb{E}^* \left[ e^{-N_w(h)} \right] \)

\[
\mathbb{E}^* \left[ \mathbb{E} \left( \exp \left\{ -\sum_{i=1}^{\infty} h(V_i, R^{(i)}, (C_1(Y')W)^{1/\alpha} \zeta^{(i)}) \right\} | W; Y' \right) \right] \\
= \mathbb{E}^* \left[ \exp \left\{ -\int_{\mathbb{R}^N \setminus \{ 0_\infty \}} \nu(d_x) \mathbb{E} \left( 1 - \exp \left\{ -h(V, R, (C_1(Y')W)^{1/\alpha} x) \right\} | W; Y' \right) \right\} \right] \\
= \mathbb{E}^* \left[ \exp \left\{ -C_1(Y')W \int_{\mathbb{R}^N \setminus \{ 0_\infty \}} \nu(d_x) \mathbb{E} \left( 1 - \exp \left\{ -h(V, R, x) \right\} | W; Y' \right) \right\} \right],
\]

(4.49)

using homogeneity property of the measure \( \nu \). We will use this expression to find the Laplace functional of the process given [24]. Consider \( f \in C_K^* (\mathbb{R}_0^0) \) and define

\[
h_0(w) := \sum_{i=1}^{n} s_if(x_i)
\]

(4.50)

where \( w = (u, (s_i : 1 \leq i \leq u), (a_i : i \geq 1)) \). Then it is clear that \( h_0 \) is a positive bounded continuous function which vanishes in the \( d_w \)-neighbourhood of \( \bigcup_{i=1}^{\infty} \{ \varnothing \} \times N_0^i \times \{ 0_\infty \} \). Note also that \( \mathbb{E}^* \left[ e^{-N_w(h_0)} \right] \) equals

\[
\mathbb{E}^* \left[ \exp \left\{ -\sum_{i=1}^{\infty} h_0(V_i, R^{(i)}, (C_1(Y')W)^{1/\alpha} \zeta^{(i)}) \right\} \right] \\
= \mathbb{E}^* \left[ \exp \left\{ -\sum_{i=1}^{\infty} \sum_{l=1}^{V_i} R^{(i)}_l f((C_1(Y')W)^{1/\alpha} \zeta^{(i)}) \right\} \right]
\]

(4.51)
which is indeed the Laplace functional of the process given (2.4). To show that it equals (2.5) we observe that it can transform to

$$
E \left[ \exp \left\{ - C_1(Y')W \int_{\mathbb{R}^n \setminus \{0\}} \nu(dx) \mathbb{E} \left( 1 - \exp \left\{ - \sum_{i=1}^{V} R_i f(x_i) \right\} \left| Y' \right. \right) \right\} \right]
$$

(4.52)
as \((V, R)\) conditioned on \(Y'\) does not depend on the random variable \(W\). The expression of the Laplace functional in (2.5) can be derived from (4.52) using the explicit form of of the random vector \(((V, R))\) conditioned on \(Y'\) in (2.3).

\(N_\ast\) is an SScDPPP. To prove the claim, we shall compute the scaled Laplace functional

$$
E^\ast \left( \exp \left\{ - \int S_y^{-1} f d N_\ast \right\} \right)
$$

for some \(y > 0\) and \(f \in C^+_K(\mathbb{R}_0)\) where \(S_y f(\cdot) = f(y \cdot)\) for every measurable function \(f : \mathbb{R}_0 \to [0, \infty)\). It follows from the expression in (2.5) that

$$
E^\ast \left( \exp \left\{ - \int S_y^{-1} f d N_\ast \right\} \right) = E^\ast \left[ \exp \left\{ - y^{-\alpha} c_f \Delta \right\} \right],
$$

(4.53)

where \(c_f > 0\) is some constant which depends on the function \(f \in C^+_K(\mathbb{R}_0)\) and \(\Delta\) is a positive random variable. Hence, we can use the condition (Prop2) in Proposition 3.2 in [12] to conclude that \(N_\ast\) is an SScDPPP.

5 Proof of Theorem 22

Observe that

$$
\lim_{n \to \infty} P^\ast (M_n \leq B_n x) = \lim_{n \to \infty} P^\ast (N_\ast (x, \infty) = 0)
$$

$$
= E^\ast \left[ P^\ast \left( \sum_{l=1}^{V_l} \sum_{k=1}^{R_k^{(l)}} \delta_{C_i(Y')W}^{1/\alpha c_k^{(l)}} (x, \infty) = 0 \mid Y'; W \right) \right].
$$

(5.1)

We decompose the event inside conditional probability appearing above into some disjoint events. Recall that in (2.7) we introduced \(H_{i_1, i_2, \ldots, i_t}^{(l)} = \prod_{j=1}^{t} G_{ij} \times \prod_{j=t+1}^{\infty} R\) where \(i_j \in \{0, 1\}\) for all \(j \in [1 : t]\). We also define

$$
O_{i_1, i_2, \ldots, i_t}^{(l)} := \{ u_{\infty} \in \mathbb{N}_0^N : \sum_{j \in R_{i_1, i_2, \ldots, i_t}^{(l)}} u_j \geq 1 \},
$$

(5.2)

where

$$
R_{i_1, i_2, \ldots, i_t}^{(l)} := \{ j \in [1 : t] : i_j = 1 \}.
$$

(5.3)

Then the conditional probability inside the expectation in (5.1) can be rewritten as follows

$$
P^\ast \left( \sum_{l=1}^{V_l} \sum_{k=1}^{R_k^{(l)}} \delta_{C_i(Y')W}^{1/\alpha c_k^{(l)}} \left( \{ v \times (x, \infty) \} = 0 \mid W, Y' \right) \right).
$$
\[\mathbb{P}^x \left( \sum_{v=1}^{\infty} \sum_{i_1, i_2, \ldots, i_v} \sum_{i_1 + i_2 + \ldots + i_v \geq 1} \delta(v, R^{(i)}, [C_1(Y^v)W]^{1/\alpha} x^{-1} \zeta^{(i)}) \right) = 0 \mid Y'; W)\]

\[= \mathbb{P}^x \left( \bigcap_{v=1}^{\infty} \bigcup_{i_1, i_2, \ldots, i_v} \bigcup_{i_1 + i_2 + \ldots + i_v \geq 1} \delta(v, R, [C_1(Y^v)W]^{1/\alpha} x^{-1} \zeta^{(i)}) \right) = 0 \mid W; Y' \right). \tag{5.4}\]

To obtain the last equality, we have used that \((\mathcal{H}^{(i)}_{i_1, i_2, \ldots, i_v} : i_j \in \{0, 1\}; j \in [1 : l])\) is a disjoint family of sets. Observe now that, conditioned on \(W\) and \(Y'\),

\[\sum_{l=1}^{\infty} \delta(v, R^{(i)}, [C_1(Y^v)W]^{1/\alpha} \zeta^{(i)})\]

is a marked Poisson random measure with mean measure \(\mathbb{P}((V, R) \in \cdot \mid Y') \otimes \nu(\cdot)\) and marks are conditionally i.i.d. Using this fact we obtain the following expression for the conditional probability in (5.4)

\[\exp \left\{ -C_1(Y')W x^{-\alpha} \sum_{v=1}^{\infty} \sum_{i_1, i_2, \ldots, i_v} \nu(\mathcal{H}^{(i)}_{i_1, i_2, \ldots, i_v}) \right\} \mathbb{P} (V = v; R \in \mathcal{O}^{(i)}_{i_1, i_2, \ldots, i_v} | Y') \right\} = \exp \left\{ -C_1(Y')W x^{-\alpha} \sum_{v=1}^{\infty} \sum_{k=1}^{v} \nu(\mathcal{H}^{(i)}_{i_1, i_2, \ldots, i_v}) \mathbb{P} (V = v; R \in \mathcal{O}^{(i)}_{i_1, i_2, \ldots, i_v} | Y') \right\}. \tag{5.5}\]

In next steps of the proof we derive a more compact form of the exponent obtained in (5.5) and show that the exponent is finite. Note that the components of the random vector \(R\) is exchangeable conditioned on \(W\) and \(Y\). Therefore, the conditional probability inside the sum in (5.5) equals

\[\mathbb{P}(V = v; R_1 + R_2 + \ldots + R_k \geq 1 | Y') = \frac{1}{C_1(Y')} \sum_{i=0}^{\infty} \left[ \mathbb{E}(Z_{i+1} | Y'_i, 0) \right]^{-1} \mathbb{P}(Z_1 = v | Y'_i)(1 - [\mathbb{P}(Z_i = 0 | Y'_{i-1}, 0)]^k). \tag{5.6}\]
Combining these together, we obtain following expression for the right hand side of \((5.1)\)

\[
\mathbb{E}^* \left( \exp \left\{ -W x^{-\alpha} \sum_{j=0}^{\infty} \left[ \mathbb{E}(Z_{j+1}|Y'_j:0) \right] -1 \sum_{v=1}^{\infty} \mathbb{P}(Z_1 = v|Y'_j) \right\} \sum_{k=1}^{v} (1 - [\mathbb{P}(Z_j = 0|Y_{j-1}:0)]^k) \sum_{i_1, i_2, \ldots, i_v} \nu(H_{i_1, i_2, \ldots, i_v}) \right),
\]

which completes the proof of \((2.8)\).

To show that the exponent in \((2.8)\) is finite almost surely note that first

\[
\sum_{j=0}^{\infty} \left[ \mathbb{E}(Z_{j+1}|Y'_j:0) \right] -1 \sum_{v=1}^{\infty} \mathbb{P}(Z_1 = v|Y'_j) \sum_{k=1}^{v} (1 - [\mathbb{P}(Z_j = 0|Y_{j-1}:0)]^k) \sum_{i_1, i_2, \ldots, i_v} \nu(H_{i_1, i_2, \ldots, i_v}) \leq \sum_{j=0}^{\infty} \left[ \mathbb{E}(Z_{j+1}|Y'_j:0) \right] -1 \sum_{v=1}^{\infty} \mathbb{P}(Z_1 = v) \sum_{i_1, i_2, \ldots, i_v} \nu(H_{i_1, i_2, \ldots, i_v}) \quad (5.7)
\]

\(\mathbb{P}\)-almost surely. Further,

\[
\sum_{i_1, i_2, \ldots, i_v} \nu(H_{i_1, i_2, \ldots, i_v}) = \sum_{k=1}^{v} \sum_{i_1, i_2, \ldots, i_v} \nu(H_{i_1, i_2, \ldots, i_v}) = \sum_{k=1}^{v} \nu \left( \bigcup_{i_1, i_2, \ldots, i_v} H_{i_1, i_2, \ldots, i_v} \right) \quad (5.8)
\]

as \((H_{i_1, i_2, \ldots, i_v} : i_j \in \{0, 1\}; j \in [1 : t])\) is a disjoint family of sets. Now note that

\[
\bigcup_{i_1, i_2, \ldots, i_v} H_{i_1, i_2, \ldots, i_v} \subset \prod_{j=1}^{k-1} \mathbb{R} \times (1, \infty) \times \prod_{j=k+1}^{\infty} \mathbb{R}
\]

and that \(\nu(\prod_{j=1}^{k-1} \mathbb{R} \times (1, \infty) \times \prod_{j=k+1}^{\infty} \mathbb{R})\) do not depend on \(k \geq 1\) as the displacements are identically distributed. We denote them by \(\nu^{(1)}(1, \infty)\). These observations lead to the following upper bound for \((5.7)\)

\[
\nu^{(1)}(1, \infty) \sum_{j=0}^{\infty} \left[ \mathbb{E}(Z_{j+1}|Y'_j:0) \right] -1 \sum_{v=1}^{\infty} v \mathbb{P}(Z_1 = v|Y'_j) = \nu^{(1)}(1, \infty) \sum_{j=0}^{\infty} \left[ \mathbb{E}(Z_j|Y'_{j-1}:0) \right]^{-1}
\]
which is finite \( P \)-almost surely as \( \sum_{i \geq 1} \pi^{-1}_i \) is finite \( P \)-almost surely by Lemma \[2\]. This completes the proof.

6 Appendix

Proof of Lemma \[2\]. Recall that \( \mu = \mathbb{E}(\log Z_1) \). From SLLN it follows that \( n^{-1} \log \pi_n \to \mu \) \( P \)-almost surely. Let \( \epsilon \in (0, \mu) \). This means that

\[
P(\pi_n \geq e^{n(\mu - \epsilon)} \text{ eventually}) = P(\pi_n^{-1} \leq e^{-n(\mu - \epsilon)} \text{ eventually}) = 1.
\]

Moreover, on the event \( \{\pi_n^{-1} \leq e^{-n(\mu - \epsilon)} \text{ eventually} \} \), we have that

\[
\sum_{n \geq 1} \pi_n^{-1} < \infty.
\]

Proof of Lemma \[3\]. Let \( g : \mathbb{R}^N \to [0, \infty) \) be a non-negative, bounded, continuous function which vanishes in the neighborhood of \( 0 \). For each \( \epsilon > 0 \), it is enough to prove

\[
P(\{\pi_n \int_{\mathbb{R}^N} P(B)^{-1}_n X \in d x | Y_0 : n^{-1} \} g(x) - \int_{\mathbb{R}^N} g(x) \nu(d x) \geq \epsilon \text{ i.o.} \}) = 0.
\]

(6.1)

Define \( b(t) := F^{\epsilon}_{\pi(t)}(1 - t^{-1}) = \inf \{s \geq 1 : P(\{|X_1| > s\} < t^{-1}) \} \). Then it follows from Definition (1.7) that

\[
\lim_{t \to \infty} t P(\{|X_1| > b(t)\}) = 1
\]

and

\[
\lim_{t \to \infty} P \left( \frac{[b(t)]^{-1} X \in \cdot}{P[|X| > b(t)]} \right) = \nu(\cdot).
\]

(6.2)

Therefore, there exists a large number \( t_0 \) such that

\[
|t \int_{\mathbb{R}^N} P(\{b(t)|^{-1} X \in d x | Y_0 : n^{-1}\} g(x) - \int_{\mathbb{R}^N} g(x) \nu(d x) | \leq \epsilon \text{ i.o.} \}) < \epsilon.
\]

(6.3)

for all \( t \geq t_0 \). We have the following upper bound for the probability in the left hand side of (6.1)

\[
P(\left\{ \pi_n \int_{\mathbb{R}^N} P(B)^{-1}_n X \in d x | Y_0 : n^{-1} \} g(x) - \int_{\mathbb{R}^N} g(x) \nu(d x) \geq \epsilon \text{ i.o.} \} \right.) + P(\pi_n \leq t_0 \text{ i.o.}.)
\]

(6.4)

Note that the first term equals zero using (6.3) and the second term also equals zero using \( \pi_n \xrightarrow{a.s.} \infty \). Hence, the proof is complete.

Proof of Proposition \[4\]. Proof of (4.24): Note that

\[
|\tilde{N}_{n,\epsilon}(f) - \tilde{N}_{n,0,\epsilon}^{\circ}(f)| \leq \|f\| \sum_{i=1}^{[D_{n-\epsilon}]} \sum_{u \in T_{e,i}} \left[A(u) - A^{(\epsilon)}(u)\right] \mathbb{1}(\{|X(u)| > B_{n,\delta}\})
\]

(6.5)
almost surely. Recalling the definition of $J_{n,\delta}(\epsilon)$ given in (6.1) we can derive the following upper bound for the probability in (6.2):

\[
\mathbb{E}^* \left[ \mathbb{P}^* \left( \left\| f \right\| \sum_{i=1}^{[D_{n_\epsilon}]} \sum_{u \in \tilde{T}_{1,},} \left[ A(u) - A^\circ (u) \right] \mathbb{I} \left( |X| > B_n \delta \right) \right) \frac{\mathbb{E}^* \left( |X(u)| > B_n \delta \right)}{B_n \delta} \right] + o(1)
\]

(6.6)

using Markov inequality. We now treat the conditional expectation in (6.6). The first observation is that $D_{n_\epsilon}$ and the displacements indexed by the forest $(\tilde{T}_{i,} : 1 \leq i \leq [D_{n_\epsilon}])$ are independent conditioned on $Y_{0: n-1}$. This observation leads to the following expression for the conditional expectation

\[
\mathbb{E}^* \left( \mathbb{I} \left( D_{n_\epsilon} \right) \right) \mathbb{E}^* \left( \mathbb{I} \left( X(u) > B_n \delta \right) \right)
\]

(6.7)

almost surely. We now note that the branching mechanism and $(\mathbb{I} (|X(u)| > B_n \delta) : u \in \tilde{T}_{i,})$ are independent distributed conditioned on the environment $Y_{0: n-1}$. Using the fact that $(\mathbb{I} (|X(u)| > B_n \delta) : u \in \tilde{T}_{i,})$ are identically distributed we can obtain the following upper bound

\[
\mathbb{P}(|X| > B_n \delta \mid Y_{0: n-1}) \sum_{i=1}^{\theta} \mathbb{E}^*(\sum_{u \in \tilde{D}_{i,}} A(u) - A^\circ (u) \mathbb{I}(|X(u)| > B_n \delta) \mid Y_{0: n-1})
\]

(6.8)

almost surely for the conditional expectation inside the sum in (6.7). We now note that conditioned on $Y_{0: n-1}$, $\tilde{D}_{i,} \overset{d}{=} Z_i$ are independent of $A(u)$ and $A^\circ (u)$ where $A(u) \overset{d}{=} \tilde{Z}_{0,-i}$ and $A^\circ (u) = \tilde{Z}_i^{(\circ)}$ if $u \in \tilde{D}_{i,}$, where $\tilde{Z}_i^{(\circ)}$ is the number of descendants in the $n$-th generation of a particle in the $(n-i)$-th generation conditioned on the environment $Y_{0: n-1}$ with progeny random variable $Z_i^{(\circ)} = Z_i \wedge \vartheta$. These observations together yield following expression

\[
\sum_{i=1}^{\theta} \mathbb{E}^*(\tilde{Z}_{0,-i} - \tilde{Z}_i^{(\circ)} | Y_{0: n-1})
\]

(6.9)

for the conditional expectation in (6.8). Combining the expressions obtained in (6.7) - (6.9) and using definition (1.9) of the set $J_{n,\delta}(\epsilon)$, we have the the following upper bound for the conditional expectation in (6.6)

\[
\mathbb{I}(J_{n,\delta}(\epsilon)) \left[ \pi_n \mathbb{P}(|X| > B_n \delta | Y_{0: n-1}) \right] \frac{\pi_{n-\epsilon}}{\pi_n} \sum_{i=1}^{\theta} \left( \prod_{j=0}^{i-1} \mathbb{E}(\xi | Y_{n-\epsilon+j}) \right)
\]
\[ \leq (\delta^{-\alpha} + \epsilon) \sum_{i=1}^{\rho} \left( \prod_{i=n-\rho}^{n-1} \mathbb{E}(\xi|Y_i) \right)^{-1} \mathbb{E}^* \left( \tilde{Z}_{n-i} - \tilde{Z}_{n-i}^{(\rho)} | Y_{0:n-1} \right) \]

\[ = (\delta^{-\alpha} + \epsilon) \sum_{i=1}^{\rho} \left( \prod_{j=1}^{n-\rho} \mathbb{E}(\xi|Y_{n-\rho+j}) \right)^{-1} \left[ \left( \prod_{j'=i}^{\rho-1} \mathbb{E}(\xi|Y_{n-\rho+j'}) \right) - \left( \prod_{j'=i}^{\rho-1} \mathbb{E}(\xi \wedge \vartheta|Y_{n-\rho+j'}) \right) \right] \]

\[ = (\delta^{-\alpha} + \epsilon) \sum_{i=1}^{\rho} \left[ 1 - \prod_{j=i}^{\rho-1} \mathbb{E}(\xi \wedge \vartheta|Y_{n-\rho+j}) \right] \]  

almost surely. Plugging this upper bound back in (6.5), we obtain

\[ \varepsilon^{-1} \|f\|(\delta^{-\alpha} + \epsilon) \sum_{i=1}^{\rho} \left( 1 - \mathbb{E} \left[ \prod_{j=0}^{\rho-i-1} \frac{\mathbb{E}(\xi \wedge \vartheta|Y_j)}{\mathbb{E}(\xi|Y_j)} \right] \right) \]  

(6.11)

using \((Y_{n-\rho+i+j} : 0 \leq j \leq \rho - i - 1) \overset{d}{=} (Y_j : 0 \leq j \leq \rho - i)\) for every \(1 \leq i \leq \rho\) due to stationarity of the environment. The upper bound then is independent of \(n\). Note that the sum and the product in (6.11) is finite for fixed \(\rho \geq 1\). Also note that the product is bounded by one almost surely and so we can apply the dominated convergence theorem as \(\vartheta \nearrow \infty\). Hence the upper bound in (6.11) converges to zero as \(\vartheta \nearrow \infty\).

**Proof of (4.25).** We start with the observation

\[ D^{(s,i)}_j \subset D^{(i)}_j \text{ and } A^\rho(u) = A^{(\rho)}(u) \text{ for all } u \in D^{(s,i)}_j \]  

for all \(i \geq 1\) and \(j \geq 1\). We then observe that

\[ N^\rho_{n,\rho,\vartheta}(f) - \tilde{N}^n_{n,\rho,\vartheta}(f) \]

\[ = \sum_{i=1}^{\lfloor D_{n,\rho} \rfloor} \sum_{\rho} \sum_{\vartheta \in D_j^{(i)} \setminus D_j^{(s,i)}} A^\rho(u) f(B_{n-1}^u X(u)) \]

\[ \leq \|f\| \sum_{i=1}^{\lfloor D_{n,\rho} \rfloor} \sum_{\rho} \sum_{\vartheta \in D_j^{(i)} \setminus D_j^{(s,i)}} A^\rho(u) \mathbb{I}(\|X(u)\| > B_n) \]  

(6.12)

almost surely. Then we can follow the similar steps like in the proof of (4.24) and derive the following upper bound for the probability in (4.25)

\[ \|f\| \varepsilon^{-1} \mathbb{E}^* \left[ \mathbb{I}(J_{n,\rho}(\epsilon)) \mathbb{E}^* \left( \sum_{i=1}^{\lfloor D_{n,\rho} \rfloor} \sum_{\rho} \sum_{\vartheta \in D_j^{(i)} \setminus D_j^{(s,i)}} A^\rho(u) \right) \right] \]

\[ \mathbb{I}(\|X(u)\| > B_n) | Y_{0:n-1} \right) + \mathbb{P}^* (J^\rho_{n,\rho}(\epsilon)), \]  

(6.13)

where \(J_{n,\rho}(\epsilon)\) is defined in (4.9) and \(\mathbb{P}^* (J^\rho_{n,\rho}(\epsilon))\) is negligible for large enough \(n\) due to Lemma 3. Using similar arguments like in the proof of (4.3) and
stationarity of the environment we can derive the following upper bound

\[ \epsilon^{-1}\|f\|((\delta - \alpha) + \epsilon) \mathbb{E}^* \left[ \left( \prod_{j=n-\varrho}^{n-1} \mathbb{E}(\xi|Y_j) \right)^{-1} \sum_{\vartheta \in \mathbb{D}^{(j-1)}} \sum_{\varphi \in \mathbb{D}^{(\vartheta)}} \lambda_e(u) \right] \]

\[ = \epsilon^{-1}\|f\|((\delta - \alpha) + \epsilon) \mathbb{E}^* \left[ \left( \prod_{j=n-\varrho}^{n-1} \mathbb{E}(\xi|Y_j) \right)^{-1} \sum_{j=1}^{\vartheta} \mathbb{E} \left( Z_j - Z_j^{(\vartheta)}|Y_{n-\varrho}; \cdot \right) \mathbb{E}(\tilde{Z}_{e-j}^{(\vartheta)}|Y_{n-\varrho}; \cdot) \right] \]

\[ \leq \epsilon^{-1}\|f\|((\delta - \alpha) + \epsilon) \mathbb{E}^* \left[ \left( \prod_{j=n-\varrho}^{n-1} \mathbb{E}(\xi|Y_j) \right)^{-1} \sum_{j=1}^{\vartheta} \mathbb{E} \left( Z_j - Z_j^{(\vartheta)}|Y_{0}; \cdot \right) \right] \]

\[ = \epsilon^{-1}\|f\|((\delta - \alpha) + \epsilon) \sum_{i=1}^{\vartheta} \left[ 1 - \mathbb{E} \left( \prod_{j=0}^{i-1} \frac{\mathbb{E}(\xi \wedge \vartheta|Y_{n-\varrho+j})}{\mathbb{E}(\xi|Y_{n-\varrho+j})} \right) \right] \]

\[ = \epsilon^{-1}\|f\|((\delta - \alpha) + \epsilon) \sum_{i=1}^{\vartheta} \left[ 1 - \mathbb{E} \left( \prod_{j=0}^{i-1} \frac{\mathbb{E}(\xi \wedge \vartheta|Y_{j})}{\mathbb{E}(\xi|Y_{j})} \right) \right]. \]

The upper bound obtained in (6.14) is independent of \( n \). The sum is finite as \( \varrho \) is finite. We can let \( \vartheta \to \infty \) and apply dominated convergence theorem to conclude that each term inside the sum converges to zero. Hence the proof is complete. \( \square \)

**Proof of Lemma 4.** Fix \( \epsilon > 0 \) and define

\[ H_{n,\epsilon} := \{ \max_{\tilde{\alpha} \in \mathbb{S}_{\varrho,\vartheta}} |U_n(\tilde{\alpha}) - U(\tilde{\alpha})| < \epsilon \}. \]  

It follows from (6.17) that

\[ \lim_{n \to \infty} \mathbb{P}(H_{n,\epsilon}^c) = 0 \]  

as \( |\mathbb{S}_{\varrho,\vartheta}| < \infty \). Therefore, it is enough to compute the limit of Laplace functional on the event \( H_{n,\epsilon} \)

\[ \lim_{n \to \infty} \mathbb{E} \left[ I(H_{n,\epsilon}) I(|D_{n-\varrho}| > 0) \left( \mathbb{E} \left[ \exp \left\{ - N_{n,\varrho,\vartheta} f(j) \right\} \right]|Y_{0}; n-1 \right)^{\mid D_{n-\varrho} \mid} \right] \]  

when \( \epsilon \) is arbitrarily small. By (6.18), the term inside the expectation equals

\[ I(H_{n,\epsilon}) \left[ 1 - \pi_n^{-1} \sum_{\tilde{\alpha} \in \mathbb{S}_{\varrho,\vartheta}} \mathbb{P}(\tilde{\alpha} = \tilde{\alpha}|Y_{0}; n-1) U_n(\tilde{\alpha}) \right]^{\mid D_{n-\varrho} \mid} I(|D_{n-\varrho}| > 0) \]

\[ \leq \left[ 1 - \pi_n^{-1} \sum_{\tilde{\alpha} \in \mathbb{S}_{\varrho,\vartheta}} \mathbb{P}(\tilde{\alpha} = \tilde{\alpha}|Y_{n-\varrho}; n-1) (U(\tilde{\alpha}) - \epsilon) \right]^{\mid D_{n-\varrho} \mid} I(|D_{n-\varrho}| > 0) \]

\[ = \left[ 1 - \pi_n^{-1} \left( \frac{n-\varrho}{\pi_n} \right) \sum_{\tilde{\alpha} \in \mathbb{S}_{\varrho,\vartheta}} \mathbb{P}(\tilde{\alpha} = \tilde{\alpha}|Y_{n-\varrho}; n-1) (U(\tilde{\alpha}) - \epsilon) \right]^{\mid D_{n-\varrho} \mid} \]
\[ = \left[ 1 - \pi_{n-\theta} \left( \prod_{j=n-\theta}^{n-1} \mathbb{E} Y_j(\xi) \right)^{-1} \sum_{\tilde{a} \in \tilde{S}_{\theta, \theta}} \mathbb{P} \left( \tilde{A} = \tilde{a} | Y_{n-\theta} : n-1 \right) \times \left( U(\tilde{a}) - \epsilon \right) \right]^{\pi_{n-\theta}(|D_{n-\theta}|)/\pi_{n-\theta}} \] (6.18)

almost surely. We first note that \( \pi_{n-\theta} \) and \( D_{n-\theta} \) are independent of \( Y_{n-\theta} : n-1 \) as the environment random variables are independently distributed. To use the i.i.d. structure of the environment, consider an independent copy \( Y' = (Y'_i : i \geq 0) \) of \( Y \) and then we can see that the expression in (6.18) equals in distribution to the following expression

\[ \left[ 1 - \pi_{n-\theta} \left( \prod_{j=n-\theta}^{n-1} \mathbb{E} Y'_j(\xi) \right)^{-1} \sum_{\tilde{a} \in \tilde{S}_{\theta, \theta}} \mathbb{P} \left( \tilde{A} = \tilde{a} | Y'_0 : \theta-1 \right) \times \left( U(\tilde{a}) - \epsilon \right) \right]^{\pi_{n-\theta}(|D_{n-\theta}|)/\pi_{n-\theta}} \mathbb{I}(|D_{n-\theta}| > 0). \] (6.19)

We know that \( \mathbb{I}(|D_{n-\theta}| > 0) \xrightarrow{a.s.} \mathbb{I}(\mathcal{S}) \) as \( n \to \infty \) and \( \pi_{n-\theta} \xrightarrow{a.s.} \infty \). Therefore, on the event \( \mathcal{S} \), the expression obtained in (6.19) converges almost surely to

\[ \exp \left\{ -W \left( \prod_{j=0}^{\theta-1} \mathbb{E} Y'_j(\xi) \right)^{-1} \sum_{\tilde{a} \in \tilde{S}_{\theta, \theta}} \mathbb{P} \left( \tilde{A} = \tilde{a} | Y'_0 : \theta-1 \right) \left( U(\tilde{a}) - \epsilon \right) \right\}. \] (6.20)

Then the dominated convergence theorem implies that

\[ \lim_{n \to \infty} \mathbb{E} \left[ \mathbb{I}(H_{n, \epsilon}) \mathbb{I}(|D_{n-\theta}| > 0) \left( \mathbb{E} \left[ e^{-\tilde{N}^{(1)}_{n-\theta, \theta}(f)} | Y_{0, n-1} \right] \right)^{|D_{n-\theta}|} \right] \leq \mathbb{E} \left[ \mathbb{I}(\mathcal{S}) \exp \left\{ -W \left( \prod_{j=0}^{\theta-1} \mathbb{E}(\xi|Y'_j) \right)^{-1} \sum_{\tilde{a} \in \tilde{S}_{\theta, \theta}} \mathbb{P} \left( \tilde{A} = \tilde{a} | Y'_0 : \theta-1 \right) \left( U(\tilde{a}) - \epsilon \right) \right\} \right]. \] (6.21)

We can follow a similar path starting from the derivation in (6.15) that

\[ \lim_{n \to \infty} \mathbb{E} \left[ \mathbb{I}(H_{n, \epsilon}) \mathbb{I}(|D_{n-\theta}| > 0) \left( \mathbb{E} \left[ e^{-\tilde{N}^{(1)}_{n-\theta, \theta}(f)} | Y_{0, n-1} \right] \right)^{|D_{n-\theta}|} \right] \geq \mathbb{E} \left[ \mathbb{I}(\mathcal{S}) \exp \left\{ -W \left( \prod_{j=0}^{\theta-1} \mathbb{E}(\xi|Y'_j) \right)^{-1} \sum_{\tilde{a} \in \tilde{S}_{\theta, \theta}} \mathbb{P} \left( \tilde{A} = \tilde{a} | Y'_0 : \theta-1 \right) \left( U(\tilde{a}) + \epsilon \right) \right\} \right]. \] (6.22)

The proof of this lemma follows from (6.21) and (6.22) by letting \( \epsilon \to 0 \). \( \square \)

**Proof of Lemma 5.** Plugging in the value of \( U(\tilde{a}) \), we get the following expression for the sum in the exponent

\[ \sum_{\tilde{a} \in \tilde{S}_{\theta, \theta}} \mathbb{P} \left( \tilde{A} = \tilde{a} | Y'_0 : \theta-1 \right) U(\tilde{a}) \]
Ulam-Harris labeling. Then (6.23) can be rewritten as follows

\[
\sum_{\tilde{A}} \mathbb{P} \left( \tilde{A} = \tilde{a} | Y_{0}^{\prime} \right) \left( 1 - \exp \left\{ - \sum_{k=1}^{\varrho} a_{i,j+k} f(x_k) \right\} \right)
\]

almost surely. The sums and integral are interchanges as they are finite. We want to take into account only those vertices \((i,j) \in \tilde{T}_{\varrho,1}\) which contribute to the sum in exponent of (6.23) that is, \(\sum_{k=1}^{\varrho} a_{i,j+k} > 0\). For that, we need to first ensure that \((i,j)\) do not belong to the class of vertices which are added at the regularization step that is, it is one of the vertex in the pruned subtree \(T_{\varrho,1}\). We also need to exclude the possibility of the vertex \((i,j)\) being a leaf of the pruned subtree \(T_{\varrho,1}\). We need a few new notation for mathematical formalization of this discounting procedure.

Fix \(i \in [1: \varrho]\). Our first aim will be to rewrite the set \(F_i\) defined in (6.23) considering those vertices only in the \(i\)-th generation which satisfies \(\sum_{k=1}^{\varrho} a_{i,j+k} > 0\). Note that \(\nu^{(\varrho)}\) takes into account all the large displacements with the vertices in the \(i\)-th generation which have same parent in the \((i-1)\)-th generation and the new vertices added during the regularization do not contribute the sum. Thus we can consider the vertices in \(\tilde{D}_{i}^{(1)}\) and divide them into \(\tilde{D}_{i-1}\) groups such that all the members in the same group have same parents and the displacements from different groups must have different parents. A generic vertex in the \((i-1)\)-th generation will be denoted by \(w\), the number of its descendants in \(\tilde{D}_i\) will be denoted by \(Z_w^{(\varrho)}\) and the collection of its descendants in \(\tilde{D}_i\) will be denoted by \((w,k): 1 \leq k \leq Z_w^{(\varrho)}\) honoring the Ulam-Harris labeling. Then (6.23) can be rewritten as follows.

\[
\sum_{\mathcal{A} \in \tilde{S}_{i,\varrho}} \mathbb{P} \left( \tilde{A} = \tilde{a} | Y_{0}^{\prime} \right) \sum_{j \in F_i} \left( 1 - \exp \left\{ - \sum_{k=1}^{\varrho} a_{i,j+k} f(x_k) \right\} \right)
\]

almost surely as \((Z_w^{(\varrho)}; (A^{(\varrho)}(w,k): 1 \leq k \leq Z_w^{(\varrho)}))_{w \in \tilde{D}_i}\) are i.i.d. conditioned on the environment \(Y_{0}^{\prime} \in \varrho - 1\). We just discounted all the vertices added during regularization but, still remained to discount the leaves in the pruned subtree. We will accomplish this task in the next step.
Note that the second expectation in (6.24) equals

\[
\mathbb{E} \left[ I \left( Z_1^{(d)} > 0 \right) \left( 1 - \exp \left\{ - \sum_{k=1}^{Z_1^{(d)}} Z_{\theta - i}^{(d,k)} f(x_k) \right\} \right) \right] \left| Y_{i-1} :_\theta \right._{-1} \\
= \mathbb{P} \left( Z_1^{(d)} \geq 1 \left| Y_{i-1} \right._{-1} \right) \mathbb{E} \left[ \left( 1 - \exp \left\{ - \sum_{k=1}^{Z_1^{(d)}} Z_{\theta - i}^{(d,k)} f(x_k) \right\} \right) \left| Y_{i-1} :_\theta \right._{-1} \right]
\]

almost surely. The term inside the sum equals

\[
\left( \mathbb{P}(Z_{\theta - i}^{(d)} \geq 0 \left| Y_{i} :_\theta \right._{-1}) \right)^{-1} \mathbb{E} \left[ \left( 1 - \exp \left\{ - \sum_{k=1}^{Z_{\theta - i}^{(d)}} Z_{\theta - i}^{(d,k)} f(x_k) \right\} \right) \left| Z_{\theta - i}^{(d)}, Y_{i-1} :_\theta \right._{-1} \right]
\]

Hence the proof of this lemma follows from (6.24) and (6.26).

\[\square\]

**Acknowledgements** The authors would like to thank Gennady Samorodnitsky for his comments.
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