ON THE LAPLACIAN SPECTRUM OF $k$-SYMMETRIC GRAPHS

SUNYO MOON AND HYUNGKEE YOO

Abstract. For some positive integer $k$, if the finite cyclic group $\mathbb{Z}_k$ can act freely on a graph $G$, then we say that $G$ is $k$-symmetric. In 1985, Faria showed that the multiplicity of Laplacian eigenvalue 1 is greater than or equal to the difference between the number of pendant vertices and the number of quasi-pendant vertices. But if a graph has a pendant vertex, then it is at most 1-connected. In this paper, we investigate a class of 2-connected $k$-symmetric graphs with a Laplacian eigenvalue 1. We also identify a class of $k$-symmetric graphs in which all Laplacian eigenvalues are integers.

1. Introduction

A simple graph $G = (V, E)$ is a combinatorial object consisting of a finite set $V$ and a set $E$ of unordered pairs of different elements of $V$. The elements of $V$ and $E$ are called the vertices and the edges of the graph $G$, respectively. For a given graph $G$, the vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively.

Let $G$ be a graph with enumerated vertices. The Laplacian matrix $L(G)$ of $G$ is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix of $G$. Thus the Laplacian matrix is symmetric. Note that the Laplacian matrix can be considered a positive-semidefinite quadratic form on the Hilbert space generated by $V(G)$. Since the Laplacian matrix contains information on the structure of the graph, it has been studied importantly in various applied fields including artificial neural network research using graph shaped data [13, 14].

Let $G$ be a graph with $n$ vertices. For a square matrix $M$, we denote the characteristic polynomial of $M$ by $\mu(M, x)$. A root of the characteristic polynomial of Laplacian matrix $L(G)$ is called a Laplacian eigenvalue of $G$. Denote the all eigenvalues of $L(G)$ by $\lambda_n(G) \leq \lambda_{n-1}(G) \leq \cdots \leq \lambda_1(G)$. It is well-known that $\lambda_n(G) = 0$ and $\lambda_1(G) \leq n$. The multiset of Laplacian eigenvalues of $G$ is called the Laplacian spectrum of $G$. The Laplacian spectrum of the complement graph $\overline{G}$ of $G$ is satisfying

$$0 = \lambda_n(\overline{G}) \leq n - \lambda_1(G) \leq \cdots \leq n - \lambda_{n-1}(G).$$

The Laplacian spectrum shows us several properties of the graph. For instance, Kirchhoff [15] proved that the number of spanning tree of a connected

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graph $G$ with $n$ vertices is $\frac{1}{n}\lambda_1(G)\cdots\lambda_{n-1}(G)$. Let $m_G(\lambda)$ denote the multiplicity of $\lambda$ as a Laplacian eigenvalue of $G$. Note that the multiplicity of 0 is equal to the number of connected components of $G$.

The connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. A graph $G$ is said to be $t$-connected if $\kappa(G) \geq t$. If a graph is $t$-connected, then it is $(t-1)$-connected. Fiedler [9] proved that the second smallest Laplacian eigenvalue of $G$ is less than or equal to $\kappa(G)$.

A pendant vertex of $G$ is a vertex of degree 1. A quasi-pendant of $G$ is a vertex adjacent to a pendant. We denote the number of pendants of $G$ by $p(G)$, and the number of quasi-pendant vertices by $q(G)$. In [8], Faria showed that for any graph $G$,

$$m_G(1) \geq p(G) - q(G).$$

It implies that if $p(G)$ is greater than $q(G)$, then $G$ has a Laplacian eigenvalue 1. Also, such graph $G$ is at most 1-connected. In [1], Barik et al. found trees with a Laplacian eigenvalue 1 even though the right-hand side of the above inequality is 0. Since a tree has connectivity 1, we focus on 2-connected graph with a Laplacian eigenvalue 1.

The simplest way to obtain a 2-connected graph with a Laplacian eigenvalue 1 is the Cartesian product. The Cartesian product $G \square H$ of graphs $G$ and $H$ is the graph with the vertex set $V(G) \times V(H)$ such that two vertices $(v, v')$ and $(w, w')$ are adjacent if $v = w$ and $v'$ is adjacent to $w'$ in $H$, or if $v' = w'$ and $v$ is adjacent to $w$ in $G$. Fiedler [9] showed that the Laplacian eigenvalues of the Cartesian product $G \square H$ are all possible sums of Laplacian eigenvalues of $G$ and $H$. If either $G$ or $H$ has a Laplacian eigenvalue 1, then 1 is a Laplacian eigenvalue of $G \square H$. Spacapan [20] showed that the connectivity of $G \square H$ is

$$\kappa(G \square H) = \min\{\kappa(G)|H|, \kappa(H)|G|, \delta(G \square H)\},$$

where $\delta(G \square H)$ is the minimum degree of $G \square H$. Remark that if $G$ and $H$ are connected graphs, then $G \square H$ is 2-connected. Thus we concentrate a 2-connected graph that does not decompose nontrivial graphs under the Cartesian product. If a graph does not admit the nontrivial Cartesian product decomposition, then the graph is called prime with respect to the Cartesian product. In this paper, we prove the following theorem.

**Theorem 1.1.** For any positive integer $n$, there is a 2-connected prime graph $G$ with respect to the Cartesian product,

$$m_G(1) \geq n.$$
copies of $n$-complete graph $K_n$ to the corresponding vertex of $\overline{K_n}$. Later, we examine that if $n \geq 2$, then $C(n, m)$ has the path $P_3$ as the induced subgraph, that is, it is not a cograph. In this paper, we also prove the following theorem.

**Theorem 1.2.** There are infinitely many pairs of positive integers $n$ and $m$, which make $C(n, m)$ a Laplacian integral graph.

This paper is organized as follows. In Section 2 We provide some linear algebra results needed for proof of main theorems. In Section 3 We define the $k$-symmetric graph by relaxing the condition of symmetric graph, and examine its properties. In Section 4 and Section 5 we prove the main theorems and related properties.

2. Preliminaries

In this section, we introduce some definitions and properties that will be used in this paper. The set of all $m \times n$ matrices over a field $\mathbb{F}$ is denoted by $M_{m \times n}(\mathbb{F})$. Denote $M_{n \times n}(\mathbb{F})$ by $M_n(\mathbb{F})$. We denote by $I_n$ and $J_n$ the $n \times n$ identity matrix and the $n \times n$ matrix whose entries are ones. Also, $1_n$ is the $n$-vector of all ones.

Let $A \in M_n(\mathbb{F})$ be a block matrix of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in M_m(\mathbb{F})$, $A_{12} \in M_{mx(n-m)}(\mathbb{F})$, $A_{21} \in M_{(n-m)xm}(\mathbb{F})$ and $A_{22} \in M_{n-m}(\mathbb{F})$. It is well known that if $A_{22}$ is invertible, then $\det A = \det A_{22} \det(A_{11} - A_{12}A_{22}^{-1}A_{21})$ (see [12, Chapter 0]).

For two matrices $A = (a_{ij}) \in M_{m \times n}(\mathbb{F})$ and $B \in M_{p \times q}(\mathbb{F})$, the Kronecker product of $A$ and $B$, denoted by $A \otimes B$, is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$ 

We state some basic properties of the Kronecker product (for more details, see [11, Chapter 4]):

(a) $A \otimes (B + C) = A \otimes B + A \otimes C$.
(b) $(B + C) \otimes A = B \otimes A + C \otimes A$.
(c) $(A \otimes B)(C \otimes D) = AC \otimes BD$.
(d) If $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
(e) $\det(A \otimes B) = (\det A)^n (\det B)^m$ for $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$.

A matrix $T \in M_n(\mathbb{F})$ of the form

$$T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \cdots & a_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}$$

is called a Toeplitz matrix. In [18], the authors gave a Toeplitz matrix inversion formula.
Theorem 2.1 (IS, Theorem 1). Let \( T = (a_{j-1})_{i,j=1}^{n} \) be a Toeplitz matrix and let \( f = (0, a_{n-1}, \ldots, a_{2} - a_{(n-2)}, a_{1} - a_{(n-1)})^{T} \) and \( e_1 = (1, 0, \ldots, 0)^{T} \). If each of the systems of equations \( Tx = f \), \( Ty = e_1 \) is solvable, \( x = (x_1, x_2, \ldots, x_n)^{T} \), \( y = (1, y_2, \ldots, y_n)^{T} \), then

(a) \( T \) is invertible;
(b) \( T^{-1} = T_1U_1 + T_2U_2 \), where

\[
T_1 = \begin{pmatrix}
y_1 & y_2 & \cdots & y_2 \\
y_2 & y_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
y_n & \cdots & y_2 & y_1
\end{pmatrix}, \quad U_1 = \begin{pmatrix}
1 & -x_n & \cdots & -x_2 \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & -x_n \\
0 & \cdots & 0 & 1
\end{pmatrix},
\]

\[
T_2 = \begin{pmatrix}
x_1 & x_2 & \cdots & x_2 \\
x_2 & x_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & x_n \\
x_n & \cdots & x_2 & x_1
\end{pmatrix}, \quad \text{and } U_2 = \begin{pmatrix}
0 & y_n & \cdots & y_2 \\
0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & y_n \\
0 & \cdots & 0 & 0
\end{pmatrix}.
\]

Corollary 2.2. Let \( aI_n + bJ_n \) be a matrix in \( M_n(\mathbb{F}) \). Then

(a) \( \det(aI_n + bJ_n) = a^{n-1}(a + nb) \).

(b) If \( aI_n + bJ_n \) is invertible, then its inverse matrix is

\[
\frac{1}{a(a + nb)}((a + nb)I_n - bJ_n).
\]

Proof. (a) It is easy to check that

\[
\det\begin{pmatrix}
a + b & b & \cdots & b \\
b & a + b & \cdots & b \\
\vdots & \vdots & \ddots & \vdots \\
b & b & \cdots & a + b
\end{pmatrix} = \det\begin{pmatrix}
a & 0 & 0 & \cdots & 0 \\
0 & a & 0 & \cdots & 0 \\
0 & 0 & a & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & a + nb
\end{pmatrix} = a^{n-1}(a + nb).
\]

Hence the determinant of \( aI_n + bJ_n \) is \( a^{n-1}(a + nb) \).

(b) Note that the matrix \( aI_n + bJ_n \) is Toeplitz. Let

\[
x = (0, \ldots, 0)^{T} \quad \text{and} \quad y = \left(\frac{a + nb - b}{a(a + nb)}, \frac{-b}{a(a + nb)}, \ldots, \frac{-b}{a(a + nb)}\right)^{T}.
\]

Then \((aI_n + bJ_n)x = 0\) and \((aI_n + bJ_n)y = e_1\). By Theorem 2.1 the inverse of \( aI_n + bJ_n \) is

\[
\frac{1}{a(a + nb)}((a + nb)I_n - bJ_n).
\]

\[
\square
\]

3. k-Symmetric graphs

Symmetry is an important property of graphs. We deal with graphs that have symmetric property. Let \( G \) be a graph. An automorphism \( \varphi \) of a graph \( G \) is a permutation of \( V(G) \) such that \( \varphi(v) \) and \( \varphi(w) \) are adjacent if and only if \( v \) and \( w \) are adjacent where \( v \) and \( w \) are vertices of \( G \). The set of all automorphisms of \( G \) is called an automorphism group of \( G \) and denoted by \( \text{Aut}(G) \). A graph \( G \) is symmetric if \( \text{Aut}(G) \) acts transitively on both vertices of \( G \) and ordered pairs of adjacent vertices. This implies that \( G \)
is regular; that is, all vertices have the same degree. However, it is a very difficult problem to determine whether a given graph is a symmetric graph. Thus we concentrate the cyclic part of Aut(G). In this section, we define $k$-symmetric graphs and give some their properties. Also, we construct a $k$-symmetric graph from other $k$-symmetric graphs.

**Definition 3.1.** Let $k$ be a positive integer. A graph $G$ is $k$-symmetric if there is a subgroup $H$ of Aut($G$) such that $H$ is isomorphic to $\mathbb{Z}_k$ and $H$ freely act on vertices. A generator of $H$ is called a $k$-symmetric automorphism.

The above definition tells us that all graphs are 1-symmetric because the trivial group freely acts on any graph. If a graph $G$ with $n$ vertices is $n$-symmetric, then the automorphism group Aut($G$) has a cyclic subgroup $H$ which transitively acts on vertices. Thus $G$ is regular. However, the converse is not true even though $G$ is a symmetric graph. Before examining this, we check the following proposition.

**Proposition 3.2.** Let $G$ be a graph with $n$ vertices. If $G$ is $n$-symmetric, then either $G$ or its complement $\overline{G}$ have a Hamiltonian cycle.

**Proof.** Let $G$ be an $n$-symmetric graph with $n$ vertices, and let $\varphi$ be an $n$-symmetric automorphism of $G$. Choose a vertex $v$. If $v$ and $\varphi(v)$ are adjacent, then $\varphi^i(v)$ and $\varphi^{i+1}(v)$ are also adjacent for any integer $i$. Since $G$ is $n$-symmetric, the group generated by $\varphi$ acts freely and transitively on $V(G)$. Thus the sequence $v, \varphi(v), \ldots, \varphi^n(v)$ induces a Hamiltonian cycle of $G$. Suppose that $v$ and $\varphi(v)$ are not adjacent in $G$. Then $v$ and $\varphi(v)$ are adjacent in $\overline{G}$. Hence the sequence of vertices induces a Hamiltonian cycle of $\overline{G}$. \qed

For example, the Petersen graph in Figure 1 is 5-symmetric because the 5-fold rotation satisfies the 5-symmetric automorphism condition. The Petersen graph is a symmetric graph with 10 vertices. But since the Petersen graph is not Hamiltonian, it is not 10-symmetric. For any positive integer $k$, $k$-symmetric graphs are satisfying the following properties.

![Figure 1. The Petersen graphs with different bases for the 5-fold rotation](image)

**Proposition 3.3.** Let $G$ be a $k$-symmetric graph for some integer $k$ and let $d$ be a divisor of $k$. Then $G$ is a $d$-symmetric graph.

**Proof.** Let $\varphi$ be a $k$-symmetric automorphism of $G$ and let $k = k'd$ for some integer $k'$. Define an automorphism $\psi$ of $G$ by $\psi = \varphi^{k'}$. Then $\psi^d(v) = \varphi^{k'd}(v) = \varphi^k(v) = \text{id}_G(v)$ for all $v \in V(G)$. The subgroup $\langle \psi \rangle$ of Aut($G$) is isomorphic to $\mathbb{Z}_d$. Thus $G$ is a $d$-symmetric graph. \qed
Proposition 3.4. Let \( G_1 \) and \( G_2 \) be \( k \)-symmetric graphs for some integer \( k \). Then \( G_1 \cup G_2 \) is \( k \)-symmetric graph.

Proof. Let \( \varphi_1 \) and \( \varphi_2 \) be \( k \)-symmetric automorphisms of \( G_1 \) and \( G_2 \), respectively. Then the automorphism \( \varphi_1 + \varphi_2 \) of \( G_1 \cup G_2 \) is defined by

\[
(\varphi_1 + \varphi_2)(v) = \begin{cases} 
\varphi_1(v), & \text{if } v \in V(G_1), \\
\varphi_2(v), & \text{if } v \in V(G_2).
\end{cases}
\]

Hence \( G_1 \cup G_2 \) is \( k \)-symmetric graph.

Let \( \varphi \) be a \( k \)-symmetric automorphism of a graph \( G \) and let \( \text{id}_H \) be the identity automorphism of a graph \( H \). Then the automorphism \( \varphi \times \text{id}_H \) of \( G \square H \) is \( k \)-symmetric. Thus we obtain the following proposition.

Proposition 3.5. Let \( G \) be a \( k \)-symmetric graphs for some integer \( k \). For any graph \( H \), the Cartesian product \( G \square H \) is \( k \)-symmetric graph.

Let \( G \) be a graph with a \( k \)-symmetric automorphism \( \varphi \). Then \( \mathbb{Z}_k \) acts on \( V(G) \) as follows. For any \( i \in \mathbb{Z}_k \) and \( v \in V(G) \), we define \( i \cdot v = \varphi^i(v) \). For any vertex \( v \), the orbit of \( v \) is denoted by \( \mathbb{Z}_k \cdot v \). Let \( B_\varphi \) be a minimal subset of \( V(G) \) such that

\[
\bigcup_{i=0}^{k-1} \varphi^i(B_\varphi) = V(G).
\]

Alternatively, \( B_\varphi \) is a minimal subset of \( V(G) \) such that

\[
\bigcup_{v \in B_\varphi} \mathbb{Z}_k \cdot v = V(G).
\]

The set \( B_\varphi \) is called a base of \( \varphi \). Since \( k \) choices are possible for each orbit, \( B_\varphi \) is not unique as drawn in Figure 3.1. Note that the size of the base \( B_\varphi \) is \( |V(G)| \).

Now we introduce how to construct a \( k \)-symmetric graph from other \( k \)-symmetric graphs for any positive integer \( k \). First we observe a graph join. Let \( H_1 \) and \( H_2 \) be graphs. The graph join \( H_1 \cup H_2 \) of \( H_1 \) and \( H_2 \) is a graph obtained by joining each vertex of \( H_1 \) to all vertices of \( H_2 \). Since every graph is 1-symmetric with respect to identity map, we can understand graph join \( H_1 \cup H_2 \) as a join of the bases \( V(H_1) \) and \( V(H_2) \) of \( \text{id}_{H_1} \) and \( \text{id}_{H_2} \). From this fact, we generalize graph join.

Definition 3.6. For \( i \in \{1, 2\} \), let \( G_i \) be a \( k \)-symmetric graph with a \( k \)-symmetric automorphism \( \varphi_i \), and let \( B_i \) be a chosen base of \( \varphi_i \). The \( k \)-symmetric join is a graph obtained by joining each vertex of \( \varphi_i^j(B_i) \) to all vertices of \( \varphi_j^i(B_2) \) for all \( j \in \mathbb{Z}_k \). The \( k \)-symmetric join is denoted by \( (G_1, \varphi_1, B_1) \cup_k (G_2, \varphi_2, B_2) \). If we choose arbitrary \( k \)-symmetric automorphisms and its bases of \( G_1 \) and \( G_2 \), then the \( k \)-symmetric join is simply denoted by \( G_1 \cup_k G_2 \).

The \( k \)-symmetric join preserves the \( k \)-symmetry. Because the \( k \)-symmetric automorphism of \( G_1 \cup_k G_2 \) is \( \varphi_1 + \varphi_2 \) and its base is \( B_1 \cup B_2 \). Definition 3.6 derives that the graph join is the 1-symmetric join. Note that, \( n \)-symmetric joins are not unique even if the base of each \( G_i \) is unique. For instance, the Cartesian product of 5-cycle \( C_5 \) with \( K_2 \) and the Petersen graph are both
5-symmetric joins of two 5-cycles, but they are not isomorphic as drawn in Figure 2.

Figure 2. The 5-symmetric join of two 5-cycles is $C_5 \square K_2$ if both automorphisms are $(1,2,3,4,5)$, and the Petersen graph if automorphisms are $(1,2,3,4,5)$ and $(1,3,5,2,4)$.

4. 2-CONNECTED $k$-SYMMETRIC GRAPHS WITH LAPLACIAN EIGENVALUE 1

In this section, we prove Theorem 1.1. First we consider the multiplicity of an integral Laplacian eigenvalue. Recall that for a given graph $G$, the partition $\pi = (V_1,V_2,\ldots,V_k)$ of $V(G)$ is an equitable partition if for all $i, j \in \{1,2,\ldots,k\}$ and for any $v \in V_i$ the number $d_{ij} = |N_G(v) \cap V_j|$ depends only on $i$ and $j$. The $k \times k$ matrix $L^\pi(G) = (b_{ij})$ defined by

$$b_{ij} = \begin{cases} -d_{ij}, & \text{if } i \neq j, \\ \sum_{s=1}^{k} d_{is} - d_{ij}, & \text{if } i = j \end{cases}$$

is called the divisor matrix of $G$ with respect to $\pi$.

Lemma 4.1 ([2,6]). Let $G$ be a graph and let $\pi = (V_1,\ldots,V_k)$ be an equitable partition of $G$ with divisor matrix $L^\pi(G)$. Then each eigenvalue of $L^\pi(G)$ is also an eigenvalue of $L(G)$.

In the following theorem, we obtain the multiplicity of $n$ of $k$-symmetric join of graphs where $n$ is the size of a base.

Theorem 4.2. Let $G_1,\ldots,G_l$ be $k$-symmetric graphs for some $k$ and let $G = G_1 \vee_k \cdots \vee_k G_l$ be the $k$-symmetric join of $G_1,\ldots,G_l$. Let $n = \frac{|V(G)|}{k}$. Then

$$m_G(n) \geq l - 1.$$ 

Proof. The partition $\pi = (V(G_1),\ldots,V(G_l))$ is an equitable partition of $G$. Then we have

$$L^\pi(G) = \begin{pmatrix} n - n_1 & -n_2 & \cdots & -n_l \\ -n_1 & n - n_2 & \cdots & -n_l \\ \vdots & \vdots & \ddots & \vdots \\ -n_1 & -n_2 & \cdots & n - n_l \end{pmatrix},$$

where $n_i = \frac{|V(G_i)|}{k}$ for $i = 1,\ldots,l$. Since the characteristic polynomial of $L^\pi(G)$ is $\mu(L^\pi(G),x) = x(x-n)^{l-1}$, by Lemma 4.1 we obtain

$$m_G(n) \geq l - 1.$$
If each $G_i$ in the above theorem is $n$-symmetric graph with $n$ vertices, then the size of a base of $G$ is $l$.

**Corollary 4.3.** Let $G_1, \ldots, G_l$ be $n$-symmetric graphs with $n$ vertices. Then for any their $n$-symmetric join $G$,

$$m_G(l) \geq l - 1.$$ 

Let $G$ be an $n$-symmetric graph with $n$ vertices. Take an $n$-symmetric automorphism $\phi$ of $G$. Let $G'$ be a graph that $n$-symmetric join of $m$ copies of $G$ along $\phi$. Then since each base of the copy of $G$ is a vertex, the base of $G'$ induces the $m$ complete graph $K_m$. Since $G'$ is constructed by same $n$-symmetric automorphism, $G'$ becomes the Cartesian product of $G$ and $K_m$.

**Corollary 4.4.** Let $G$ be $n$-symmetric graphs with $n$ vertices. Then for any positive integer $m$,

$$m_{K_m \circ G}(m) \geq m - 1.$$ 

By the Špacapan’s result [20] about the connectivity of the Cartesian product in Section 1, we realize that for any positive integer $m$, there is a $m$-connected graph $G$ with $m_G(m) \geq m - 1$.

Now consider a special case of $k$-symmetric join. For any $i \in \{1, \ldots, l\}$, let $G_i$ be a $k$-symmetric graph for some positive integer $k$ and let $\phi_i$ be an associated $k$-symmetric automorphism. Let $B^1_i$ be a base of $G_i$, and let $B^j_i = \phi^j_i(B^1_i)$. Recall that the union of $G_1, \ldots, G_l$ is also $k$-symmetric with the $k$-symmetric automorphism $\phi_1 + \cdots + \phi_l$ and the base $B^1_1 \cup \cdots \cup B^1_l$. Define a graph $G$ by $k$-symmetric joining $K_k$ and $G_1 \cup \cdots \cup G_l$. Then the subgraph induced by a base of $G$ has a cut-vertex as drawn in Figure 3 (a). From this fact, we can take an equitable partition $\pi = (V_0, V_1, \ldots, V_l)$ where $V_0 = V(K_k)$ and $V_i = V(G_i)$ for any $i \in \{1, \ldots, l\}$ as drawn in Figure 3 (b). Remark that for any distinct $i$ and $i'$, there is no edge connecting two subgraphs $G_i$ and $G_{i'}$ in $G$. To prove Theorem 1.1 we need the following two theorems.

![Figure 3. k-symmetric graph $G = K_k \vee_k (G_1 \cup \cdots \cup G_l)$](image)
Theorem 4.5. Let $G_1, \ldots, G_l$ be $k$-symmetric graphs for some positive integers $l$ and $k$, and let $G = \overline{K_k} \uplus_k (G_1 \cup \cdots \cup G_l)$. Then

$$m_G(1) \geq l - 1.$$ 

Proof. Suppose that $G_1, \ldots, G_l$ and $G$ are the graphs in the statement of the theorem. Let $V_0$ be the vertices set of $\overline{K_k}$ and let $V_i$ be the vertices set of $G_i$ for $i = 1, \ldots, l$. Our observation implies that the partition $\pi = (V_0, V_1, \ldots, V_l)$ is an equitable partition of $G$. Then the divisor matrix $L^\pi(G)$ is equal to

$$
\begin{pmatrix}
 n & -n_1 & -n_2 & \cdots & -n_l \\
 -1 & 1 & 0 & \cdots & 0 \\
 -1 & 0 & 1 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 -1 & 0 & 0 & \cdots & 1
\end{pmatrix},
$$

where $n_i = |V_i|$ for $i = 1, \ldots, l$ and $n = \sum_{i=1}^l n_i$. We can partition the matrix $xI - L^\pi(G)$ into four blocks as

$$
\begin{pmatrix}
 x-n & n_1 & n_2 & \cdots & n_l \\
 1 & x-1 & 0 & \cdots & 0 \\
 1 & 0 & x-1 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 1 & 0 & 0 & \cdots & x-1
\end{pmatrix}.
$$

Then the characteristic polynomial of $L^\pi(G)$ is

$$
\mu(L^\pi(G), x) = (x-1)^l \left( (x-n) - \frac{1}{x-1} \right) 
= (x-1)^{l-1} \left( (x-n)(x-1) - n \right) 
= (x-1)^{l-1} x (x - (n + 1)).
$$

By Lemma 4.1, we obtain $m_G(1) \geq l - 1$. \hfill \Box

Theorem 4.6. Let $G_1, \ldots, G_l$ be connected $k$-symmetric graphs for some integers $l, k \geq 2$. Then the graph $G = \overline{K_k} \uplus_k (G_1 \cup \cdots \cup G_l)$ is a 2-connected prime graph with respect to Cartesian product.

Proof. First we prove that the graph $G = \overline{K_k} \uplus_k (G_1 \cup \cdots \cup G_l)$ is 2-connected. Suppose that there is a cut-vertex $v$ of $G$ in $\overline{K_k}$. Since $k \geq 2$, there is another vertex $w$ in $\overline{K_k}$. Since $l \geq 2$, there are two independent paths $P_1$ and $P_2$ from $v$ to $w$ passing through $G_1$ and $G_2$, respectively. This implies that $v$ lies on the cycle $P_1 \cup P_2$, and hence $v$ is not a cut-vertex. Next we suppose that a cut-vertex of $G$ is not lying on $\overline{K_k}$. Without loss of generality, assume that a cut-vertex $v$ is in $G_1$. Since $l, k \geq 2$ and $G$ is connected, there is a cycle containing $v$ in $G$. It follows that $v$ is not a cut-vertex. Therefore, $G$ is 2-connected.

Now, we show that the graph $G$ is prime with respect to the Cartesian product. It is well-known that if two edges of a nontrivial Cartesian product are incident, then they are included in a subgraph $C_4$ of the Cartesian product. For any vertex $u$ of $G$ in $\overline{K_k}$, two incident edges $e$ and $f$ of $u$ such that the endpoints of $e$ and $f$ are contained in different graphs $G_i$ and $G_j$
for some integers $i$ and $j$. But there is no $C_4$ including $e$ and $f$. Thus we deduce that $G$ is prime.

Theorems [1, 5] and [14] imply Theorem [17]. Note that, if one of the graphs $G_1, \ldots, G_l$ is disconnected, then there is a counterexample. For example, if $G_1$ is $K_k$, then $G = K_k \cup (G_1 \cup \cdots \cup G_l)$ has a cut vertex.

5. LAPLACIAN INTEGRAL GRAPHS

In this section, we discuss $k$-symmetric graphs with integral Laplacian spectrum. Let $n$ and $m$ be positive integers. Since the $n$-complete graph $K_n$ is $n$-symmetric, the disjoint union of $m$ copies of $K_n$, denoted by $mK_n$, is also $n$-symmetric. We consider the $n$-symmetric join of $K_n$ and $mK_n$. Denote the graph $K_n \cup mK_n$ by $C(n, m)$. Now we observe that a graph $C(n, m)$ is not a cograph for $n \geq 2$. Let $v$ and $w$ be vertices in $K_n$. Then there are two vertices in $K_n$ which are adjacent to $v$ and $w$, respectively. Thus the graph $C(n, m)$ contains the path $P_4$ as an induced subgraph. We will show that a graph $C(n, m)$ is Laplacian integral for some positive integers $n$ and $m$. In the following theorem, we give the characteristic polynomial of $L(C(n, m))$.

**Theorem 5.1.** Let $n$ and $m$ be positive integers. Then the characteristic polynomial of $L(C(n, m))$ is

$$x(x - 1)^{n-1}(x - (n + 1))^{(m-1)(n-1)}(x - (m + 1))(x^2 - (m + n + 1)x + mn)^{n-1}.$$ 

**Proof.** The Laplacian matrix of $C(n, m)$ is

$$L(C(n, m)) = \begin{pmatrix} mI_n + L(K_n) & -I_n^T \otimes I_n \\ -I_n \otimes I_n & I_n \otimes (I_n + L(K_n)) \end{pmatrix}.$$ 

We consider $xI_{n(m+1)} - L(C(n, m))$ as a matrix over the field of rational functions $\mathbb{C}(x)$. Then the characteristic polynomial of $L(C(n, m))$ is

$$\mu(L(C(n, m)), x) = \det(xI_{n(m+1)} - L(C(n, m)))$$

$$= \det((x - m)I_n - I_n^T \otimes I_n)$$

$$= \det(I_n \otimes ((x - 1)I_n - L(K_n))) \det((x - m)I_n)$$

$$- (1 \otimes I_n)(I_n \otimes ((x - 1)I_n - L(K_n)))^{-1}(1 \otimes I_n)$$

$$= \det((x - 1)I_n - L(K_n))^m \det((x - m)I_n)$$

$$- m((x - 1)I_n - L(K_n))^{-1}.$$ 

Since $\det((x - 1)I_n - L(K_n)) = \mu(L(K_n), x)$, we obtain

$$\det((x - 1)I_n - L(K_n))^m = \mu(L(K_n), x - 1)^m$$

$$= (x - 1)^m(x - (n + 1))^{m(n-1)}.$$ 

Now, we compute the determinant of $(x - m)I_n - m((x - 1)I_n - L(K_n))^{-1}$. By Corollary 2.2 (b), we have

$$((x - 1)I_n - L(K_n))^{-1} = ((x - (n + 1))I_n + J_n)^{-1}$$

$$= \frac{1}{(x - 1)(x - (n + 1))}((x - 1)I_n - J_n).$$
This implies that
\[
\det \left( (x - m)I_n - m((x - 1)I_n - L(K_n))^{-1} \right) \\
= \det \left( (x - m)(x - 1)(x - (n + 1))I_n - m((x - 1)I_n - J_n) \right) \\
= \det \left( (x^3 - (m + n + 2)x^2 + (mn + m + n + 1)x - mn)I_n + mJ_n \right)
\]
\[
= (x^3 - (m + n + 2)x^2 + (mn + m + n + 1)x - mn)I_n + mJ_n
\]
By Corollary 2.2 (a), we have
\[
\det \left( (x^3 - (m + n + 2)x^2 + (mn + m + n + 1)x - mn)I_n + mJ_n \right)
= x(x - (m + 1))(x - (n + 1))(x - 1)^{n-1}(x^2 - (m + n + 1)x + mn)^{n-1}
\]
Hence the determinant of \((x - m)I_n - m((x - 1)I_n - L(K_n))^{-1}\) is
\[
\frac{x(x - (m + 1))(x^2 - (m + n + 1)x + mn)^{n-1}}{(x - (n + 1))^{n-1}(x - 1)}
\]
Therefore the characteristic polynomial of \(L(C(n, m))\) is
\[
x(x - 1)^{n-1}(x - (n + 1))(x - (m + 1))(x^2 - (m + n + 1)x + mn)^{n-1}
\]
□

The following corollary induces Theorem 1.2.

**Corollary 5.2.** Let \(n, m, k,\) and \(l\) be positive integers with \(l \neq 1\). Then

(a) If \(C(n, m)\) is Laplacian integral, then \(C(m, n)\) is also Laplacian integral.

(b) A graph \(C kl, (k + 1)(l - 1)\) is Laplacian integral.

(c) A graph \(C k^2 + k, k^2 + k\) is regular Laplacian integral.

**Proof.**

(a) It is obvious by Theorem 5.1

(b) If the quadratic \(x^2 - (m + n + 1)x + mn\) has two integer roots, then \(C(m, n)\) is Laplacian integral, by Theorem 5.1. Let \(k, l, r\) and \(s\) be positive integers with \(n = kl\) and \(m = rs\). Suppose that \(kr\) and \(ls\) are roots of the quadratic. Then, by Vieta’s formulas, we have \(kr + ls = rs + kl + 1\), that is,
\[
(s - k)r - (s - k)l + 1 = 0.
\]
If \(s = k\) then it is a contradiction. If \(s - k \neq 0\), then \(r - l + \frac{1}{s - k} = 0\). Since \(r\) and \(l\) are integers, \(s - k\) must be 1. Plugging \(s = k + 1\) into the equation (1), we have \(r = l - 1\). Since \(m\) is a positive integer, \(l\) is not equal to 1. Thus \(C kl, (k + 1)(l - 1)\) is Laplacian integral for any positive integers \(k\) and \(l \neq 1\).

(c) If \(m = n\), then \(C(n, m)\) is regular. By (b), a graph \(C k^2 + k, k^2 + k\) is regular Laplacian integral graph.

□

Now, we consider the \(n\)-complete graph \(K_n\) as a \(k\)-symmetric graph for some divisor \(k\) of \(n\). Note that a base of \(K_n\) as a \(k\)-symmetric graph is not unique, but the \(k\)-symmetric join of \(\overline{K}_k\) and \(mK_n\) is unique up to isomorphism. We denote by \(C(n, k, m)\) the graph \(\overline{K}_k \vee_k mK_n\). In the similar
way to the proof of Theorem 5.1, we get the characteristic polynomial of \( L(C(n, k, m)) \).

**Theorem 5.3.** Let \( n \) and \( m \) be positive integers. Let \( k \) be a divisor of \( n \) and let \( d = n/k \). Then the characteristic polynomial of \( L(C(n, k, m)) \) is

\[
x(x-1)^{m-1}(x-(n+1))^{m(n-1)-k+1}(x-(md+1))(x^2-(md+n+1)x+mdn)^{k-1}.
\]

**Proof.** The Laplacian matrix of \( C(n, k, m) \) is

\[
L(C(n, k, m)) = \begin{pmatrix}
mdI_k + L(K_k) & -1_m \otimes (I_k \otimes 1_d) \\
-1_m \otimes (I_k \otimes 1_d) & I_m \otimes (I_n + L(K_n))
\end{pmatrix}.
\]

Then the characteristic polynomial of \( L(C(n, k, m)) \) is

\[
\mu(L(C(n, k, m)), x) = \det(xI_{n(m+1)} - L(C(n, k, m))).
\]

Consider \( xI_{n(m+1)} - L(C(n, k, m)) \) as a matrix over the field of rational functions \( \mathbb{C}(x) \). Then

\[
\det(xI_{n(m+1)} - L(C(n, k, m))) = \det((x-1)I_n - L(K_n))^m = (x-1)^m(x-(n+1))^{m(n-1)}.
\]

Now, we compute \( \det((x-md)I_k - m(I_k \otimes 1_d)((x-1)I_n - L(K_n))^{-1}(I_k \otimes 1_d)) \). By Corollary 2.2 (b), we have

\[
((x-1)I_n - L(K_n))^{-1} = \frac{1}{(x-1)(x-(n+1))}((x-1)I_n - J_n).
\]

Note that the matrix \((x-1)I_n - J_n\) can be written in the Kronecker product form \((x-1)I_k \otimes I_d - J_k \otimes J_d\). It follows that

\[
(I_k \otimes 1_d)((x-1)I_n - L(K_n))^{-1}(I_k \otimes 1_d)
= (x-1)^{-1}(x-(n+1))^{-1}(I_k \otimes 1_d)((x-1)I_k \otimes I_d - J_k \otimes J_d)(I_k \otimes 1_d)
= (x-1)^{-1}(x-(n+1))^{-1}(d(x-1)I_k - d^2J_k).
\]

Then we have

\[
\det((x-md)I_k - m(I_k \otimes 1_d)((x-1)I_n + L(K_n))^{-1}(I_k \otimes 1_d))) = \frac{\det((x-md)(x-1)(x-(n+1))I_k - md(x-1)I_k + md^2J_k)}{(x-1)^k(x-(n+1))^k}.
\]
By Corollary 2.2 (a), we obtain

\[
\det \left( (x - md)(x - 1)(x - (n + 1))I_k - md(x - 1)I_k + md^2J_k \right)
= \det \left( (x^3 - (md + n + 2)x^2 + (md + 1)(n + 1)x - mdn)I_k + md^2J_k \right)
= x(x - (md + 1))(x - (n + 1))(x - 1)^k - 1(x^2 - (md + n + 1)x + mdn)^k - 1.
\]

Hence the determinant of $(x-md)I_k-m(I_k\otimes J_d)((x-1)I_n-L(K_n))^{-1}(I_k\otimes 1_d)$ is

\[
\frac{x(x - (md + 1))(x^2 - (md + n + 1)x + mdn)^k - 1}{(x - 1)(x - (n + 1))^k - 1}.
\]

Thus the characteristic polynomial of $L(C(n,k,m))$ is

\[
x(x - 1)^m - 1(x - (n + 1))^m - 1(x^2 - (md + n + 1)x + mdn)^k - 1.
\]

\[\square\]

The next two corollaries tell us about the relation between Laplacian integral graphs $C(n,n,m)$ and $C(n,k,m')$ for some positive integers $n$, $m$, $m'$ and $k$ with $k | n$.

**Corollary 5.4.** Suppose that $C(n,n,m)$ is Laplacian integral for some positive integers $n$ and $m$. Let $d$ be a divisor of $n$. If $m$ is divisible by $d$, then $C(n,\frac{n}{d},m')$ is Laplacian integral.

**Proof.** Suppose that $C(n,n,m)$ is Laplacian integral for some positive integers $n$ and $m$. Then the polynomial $x^2 - (m + n + 1)x + mn$ in the characteristic polynomial of $C(n,n,m)$ can be factored over the integers. Let $d$ be a divisor of $n$. By Theorem 5.3 it is enough to show that the quadratic in the characteristic polynomial of $C(n,\frac{n}{d},m')$ has integral roots. Since the quadratic is

\[
x^2 - \left( \frac{m}{d}dn + n + 1 \right)x + \frac{m}{d}dn = x^2 - (m + n + 1)x + mn,
\]

the graph $C(n,\frac{n}{d},m')$ is Laplacian integral. \[\square\]

**Corollary 5.5.** Suppose that $C(n,k,m)$ is Laplacian integral for some positive integers $n$, $m$ and $k$ with $k | n$. Let $d = n/k$. Then $C(n,n,md)$ is Laplacian integral.

**Proof.** The proof is similar that of Theorem 5.4. Since the quadratic in the characteristic polynomial of $C(n,n,md)$ is

\[
x^2 - (md + n + 1)x + mdn,
\]

it is easy to see that $C(n,n,md)$ is Laplacian integral. \[\square\]

**Declaration of Competing Interest**

There is no competing interest.
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Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

Email address: symoon89@hanyang.ac.kr

Institute of Mathematical Sciences, Ewha Womans University, Seoul 03760, Korea

Email address: hyungkee@ewha.ac.kr