A regularity criterion for the solution of the nematic liquid crystal flows in terms of $\dot{B}^{-1}_{\infty,\infty}$-norm

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Abstract

In this paper, we investigate regularity criterion for the solution of the nematic liquid crystal flows in dimension three and two. We prove the solution $(u, d)$ is smooth up to time $T$ provided that there exists a positive constant $\varepsilon_0 > 0$ such that (i) for $n = 3$,

$$\|(u, \nabla d)\|_{L^\infty(0,T; \dot{B}^{-1}_{\infty,\infty})} \leq \varepsilon_0,$$

and (ii) for $n = 2$,

$$\|\nabla d\|_{L^\infty(0,T; \dot{B}^{-1}_{\infty,\infty})} \leq \varepsilon_0.$$

Keywords: Nematic liquid crystal flows; Navier–Stokes equations; heat flow; regularity criterion

2010 AMS Subject Classification: 76A15, 35B65, 35Q35

1 Introduction

Liquid crystal, which is a state of matter capable of flow, but its molecules may be oriented in a crystal-like way. Liquid crystals exhibit a phase of matter that has properties between those of a conventional liquid and those of a solid crystal, hence, it is commonly considered as the fourth state of matter, different from gases, liquid, and solid. There have been numerous attempts to formulate continuum theories describing the behaviour of liquid crystals flows, we refer to the seminal papers [7, 18]. To the present state of knowledge, three main types of liquid crystals are distinguished, nematic, termed smectic and cholesteric. The nematic phase appears to be the most common,

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where the molecules do not exhibit any positional order, but they have long-range orientational order.

In the present paper, we consider the following hydrodynamic model for the flow of the nematic liquid crystal material in \( n \)-dimensions \((n = 2 \text{ or } 3)\):

\[
\begin{align*}
\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla P &= -\lambda \nabla \cdot (\nabla d \odot \nabla d) \quad &\text{in } \mathbb{R}^n \times (0, +\infty), \\
\partial_t d + (u \cdot \nabla) d &= \gamma (\Delta d + |\nabla d|^2 d) \quad &\text{in } \mathbb{R}^n \times (0, +\infty), \\
\nabla \cdot u &= 0 \quad &\text{in } \mathbb{R}^n \times (0, +\infty), \\
(u, d)|_{t=0} &= (u_0, d_0) \quad &\text{in } \mathbb{R}^n,
\end{align*}
\]

where \( u(x,t) : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}^n \) is the unknown velocity field of the flow, \( d : \mathbb{R}^n \times (0, +\infty) \to \mathbb{S}^2 \), the unit sphere in \( \mathbb{R}^3 \), is the unknown (averaged) macroscopic/continuum molecule orientation of the nematic liquid crystal flow and \( P(x,t) : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R} \) is the scalar pressure, \( \nabla \cdot u = 0 \) represents the incompressible condition, \( u_0 \) is a given initial velocity with \( \nabla \cdot u_0 = 0 \) in distribution sense, \( d_0 : \mathbb{R}^n \to \mathbb{S}^2 \) is a given initial liquid crystal orientation field, and the constants \( \nu, \lambda, \gamma \) are positive constants that represent viscosity, the competition between kinetic energy and potential energy, microscopic elastic relaxation time for the molecular orientation field. The notation \( \nabla d \odot \nabla d \) denotes the \( n \times n \) matrix whose \((i,j)\)-th entry is given by \( \partial_i d \cdot \partial_j d \) \((1 \leq i, j \leq n)\), and there holds \( \nabla \cdot (\nabla d \odot \nabla d) = \Delta d \cdot \nabla d + \frac{1}{2} \nabla |\nabla d|^2 \). Since the concrete values of the constants \( \nu, \lambda \) and \( \gamma \) do not play a special role in our discussion, for simplicity, we assume that they all equal to one throughout this paper.

The above system \((1.1) - (1.4)\) is a simplified version of the Ericksen–Leslie model for the hydrodynamics of the nematic liquid crystals developed during the period of 1958 through 1968 (see \([7, 18]\)). It can be viewed as the incompressible Navier–Stokes equations (the case \( d = 1 \), see \([1, 9, 17, 19]\)) coupling the heat flow of a harmonic map (the case \((1.1) - (1.4)\), we refer the readers to \([13, 20, 26, 27, 29, 32]\) and the references therein.

Later, in \([24]\), they further proved that the one-dimensional spacetime Hausdorff measure of the singular set of the so-called suitable weak solutions is zero. For more researches about system \((1.1) - (1.4)\), we refer the readers to \([13, 20, 26, 27, 29, 32]\) and the references therein.
In this paper, we are interested in the local-in-time classical solution to system (1.1)–(1.4). Since the strong solutions of the heat flow of harmonic maps must be blowing up at finite time [5], we cannot expect that (1.1)–(1.4) has a global smooth solution with general initial data. By using standard methods, it is known that if the initial velocity \( u_0 \in H^s(\mathbb{R}^n, \mathbb{R}^n) \) with \( \nabla \cdot u_0 \) and \( d_0 \in H^{s+1}(\mathbb{R}^n, S^2) \) with \( s \geq n \), then there exists \( 0 < T_* < +\infty \) depending only on the initial value such that the system (1.1)–(1.4) has a unique local classical solution \((u, d)\) satisfying (see for example [32])

\[
\begin{align*}
  u &\in C([0, T_*); H^s(\mathbb{R}^n, \mathbb{R}^n)) \cap C^1([0, T_*); H^{s+1}(\mathbb{R}^n, \mathbb{R}^n)) \quad \text{and} \\
  d &\in C([0, T_*); H^{s+1}(\mathbb{R}^n, S^2)) \cap C^1([0, T_*); H^s(\mathbb{R}^n, S^2)).
\end{align*}
\] (1.5)

Here, we emphasize that such an existence theorem gives no indication as to whether solutions actually lose their regularity or the manner in which they may do so. Assume that \((0, T_*)\) is the interval for (1.5) holds, the purpose of this paper is to give some criterion to ensure the solution \((u, d)\) is smooth up to time \( T_* \).

For the well-known Navier–Stokes equations with dimension \( n \geq 3 \), there are many interesting sufficient conditions for regularity of solutions (see for example [2, 10, 16, 17]), and the Ladyzhenskaya–Prodi–Serrin condition (see [28, 19]) state that

\[
u \in L^\alpha(0, T_*; L^\beta(\mathbb{R}^n))\quad \text{for all } \frac{2}{\alpha} + \frac{n}{\beta} \leq 1, 2 \leq \alpha < \infty, n < \beta \leq \infty
\] (1.6)

ensure the smoothness of solution \( u \) up to time \( T_* \). The limiting case \( u \in L^\infty(0, T_*; L^n(\mathbb{R}^n)) \) in (1.6) has been proved by Escauriaza, Seregin and Šverák [8] by using the method of backward uniqueness of solution. Beale, Kato and Majda in [3] proved that the vorticity \( \omega = \nabla \times u \) does not belong to \( L^1(0, T_*; L^\infty(\mathbb{R}^n)) \) if \( T_* \) is the first finite singular time. On the other hand, as for the heat flow of harmonic maps into \( S^2 \), Wang [30] established that for \( n \geq 2 \), the condition \( \nabla d \in L^\infty(0, T; L^n(\mathbb{R}^n)) \) implies that the solution \( d \) is regular on \((0, T]\), i.e., \( d \in C^\infty((0, T] \times \mathbb{R}^n) \).

When \( n = 2 \), Lin, Lin and Wang obtained that the local smooth solution \((u, d)\) to (1.1)–(1.4) can be continued past any time \( T > 0 \) provided that there holds

\[
\int_0^T \|\nabla d(\cdot, t)\|_{L^4}^4 \, dt < \infty.
\]

Huang and Wang [14] established that if \( 0 < T_* < \infty \) is the first finite singular time of the smooth solutions \((u, d)\) to system (1.1)–(1.4), then

\[
\begin{align*}
  &\int_0^{T_*} (\|\omega\|_{L^\infty} + \|\nabla d\|_{L^2}^2) \, dx = \infty \quad \text{when dimension } n = 3; \\
  &\int_0^{T_*} \|\nabla d\|_{L^\infty}^2 \, dx = \infty \quad \text{when dimension } n = 2.
\end{align*}
\]

In the references cited above, we noticing that the scaling invariance property plays a particularly significant role. For system (1.1)–(1.4), it is clear that if \((u(x, t), d(x, t))\) is the solution of system (1.1)–(1.4), then

\[
(u_\lambda(x, t), d_\lambda(x, t)) := (\lambda u(\lambda x, \lambda^2 t), d(\lambda x, \lambda^2 t))
\] (1.7)
for any $\lambda > 0$ is also the solution of (1.1)–(1.3) with initial data $(u_0\lambda(x), d_0\lambda(x)) := (\lambda u_0(\lambda x), d_0(\lambda x))$. In fact, it is easy to verify that the space $L^n_\lambda(\mathbb{R}^n)$ and from a mathematical viewpoint, Besov space $\dot{B}^{-1}_\infty(\mathbb{R}^n)$ for any $\lambda > 0$ is also the solution of (1.1)–(1.3) with initial data $(u_0, d_0)$ in system (1.1)–(1.4), i.e., $L^n(\mathbb{R}^n)$ and $L^n(\mathbb{R}^n)$-norm of $(u(t), \nabla d(t))$ is invariant under the action of the scaling (1.7). Due to the facts that

$$L^n(\mathbb{R}^n) \subset \dot{B}^{-1}_\infty(\mathbb{R}^n) \text{ and } L^n(\mathbb{R}^n) \neq \dot{B}^{-1}_\infty(\mathbb{R}^n),$$

and from a mathematical viewpoint, Besov space $\dot{B}^{-1}_\infty(\mathbb{R}^n)$ is the largest scaling invariant space of the system (1.1)–(1.4), the purpose of this paper is to establish a regularity criterion for local-in-time smooth solutions of system (1.1)–(1.4) in term of the homogeneous Besov space $\dot{B}^{-1}_\infty$-norm.

Our main results are as follows:

**Theorem 1.1** For $n = 3$, $u_0 \in H^3(\mathbb{R}^3, \mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^4(\mathbb{R}^3, \mathbb{S}^2)$. Let $0 < T_* < \infty$ be the value such that the nematic liquid crystal flow (1.1)–(1.4) has a unique solution $(u, d)$ satisfying (1.5). If there exists a small positive constant $\varepsilon_0$ such that

$$\| (u, \nabla d) \|_{L^\infty(0, T_*; \dot{B}^{-1}_\infty)} \leq \varepsilon_0,$$

then $(u, d)$ is smooth up to time $t = T_*$.  

**Remark 1.2** In [8], Escauriaza, Seregin and Šverák used the fact that functions in $L^3(\mathbb{R}^3)$ has decay at infinity, which ensures that the solution of the 3D Navier–Stokes equations is smooth outside an big ball centered at origin so that the backward uniqueness theorem can be applied. We can not generalize the regularity criterion (1.8) as $(u, \nabla d) \in L^\infty(0, T_*; \dot{B}^{-1}_\infty(\mathbb{R}^3))$, since functions in $\dot{B}^{-1}_\infty(\mathbb{R}^3)$ is different from the functions in $L^3(\mathbb{R}^3)$, which has no decay at infinity.

As a byproduct of our proof of Theorem 1.1, we obtain the following corresponding criterion in dimension two. More precisely, we have

**Corollary 1.3** For $n = 2$, $u_0 \in H^2(\mathbb{R}^2, \mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^3(\mathbb{R}^2, \mathbb{S}^2)$. Let $0 < T_* < \infty$ be the value such that the nematic liquid crystal flow (1.1)–(1.4) has a unique solution $(u, d)$ satisfying (1.5). If there exists a small positive constant $\varepsilon_0$ such that

$$\| \nabla d \|_{L^\infty(0, T_*; \dot{B}^{-1}_\infty)} \leq \varepsilon_0,$$

then $(u, d)$ is smooth up to time $t = T_*$.  

The remaining parts of the paper is written as follows. Section 2, we recall the definition of Besov spaces and an useful inequality. Section 3 is devoted to proving Theorem 1.1 and Corollary 1.3. Throughout the paper, $C$ denotes the positive constant and its value may change from line to line; $\| \cdot \|_X$ denotes the norm of space $X(\mathbb{R}^3)$ or $X(\mathbb{R}^2)$.

\footnote{Here $W^{1,n}(\mathbb{R}^n)$ denotes the homogeneous Sobolev space on $\mathbb{R}^n$ (see e.g., [20]).}
2 Preliminaries and a key lemma

In this section, we will give the definition of the Besov spaces and an useful inequality. In order to define Besov spaces, we first introduce the Littlewood–Paley decomposition theory. Let $S(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions, for given $f \in S(\mathbb{R}^n)$, its Fourier transform $\mathcal{F}f = \hat{f}$ and its inverse Fourier transform $\mathcal{F}^{-1}f = \check{f}$ are, respectively, defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

and

$$\check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(x) d\xi.$$ 

More generally, the Fourier transform of any given $f \in S'(\mathbb{R}^n)$, the space of tempered distributions, is given by

$$<\hat{f}, g> = <f, \hat{g}>,$$

for any $g \in S(\mathbb{R}^n)$.

Let

$$S_h := \{ \phi \in S(\mathbb{R}^n), \int_{\mathbb{R}^n} \phi(x) x^\gamma dx = 0, |\gamma| = 0, 1, 2, \cdots \}.$$ 

Then its dual is given by

$$S_h' = S'/S_h^\perp = S'/P,$$

where $P$ is the space of polynomial. Let us choose two nonnegative radial functions $\chi, \varphi \in S(\mathbb{R}^n)$ supported in $B = \{ \xi \in \mathbb{R}^n : |\xi| \leq \frac{4}{3} \}$ and $C = \{ \xi \in \mathbb{R}^n : \frac{4}{3} \leq |\xi| \leq \frac{8}{3} \}$ respectively, such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\},$$

and

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^n.$$ 

For $j \in \mathbb{Z}$, the homogeneous Littlewood–Paley projection operators $\hat{\Delta}_j$ and $\hat{\Delta}_j$ are, respectively, defined as

$$\hat{\Delta}_j f = \chi(2^{-j}D)f = 2^{nj} \int_{\mathbb{R}^n} \tilde{h}(2^j y) f(x-y) dy,$$

where $\tilde{h} = \mathcal{F}^{-1}\chi$, and

$$\hat{\Delta}_j f = \varphi(2^{-j}D)f = 2^{nj} \int_{\mathbb{R}^n} h(2^{-j} y) f(x-y) dy,$$

where $h = \mathcal{F}^{-1}\varphi$.

Informally, $\hat{\Delta}_j$ is a frequency projection to the annulus $\{ |\xi| \sim 2^j \}$, while $\hat{\Delta}_j$ is a frequency projection to the ball $\{ |\xi| \lesssim 2^j \}$. One can easily verify that $\hat{\Delta}_j \hat{\Delta}_k f = 0$ if $|j - k| \geq 2$. 

5
Let $s \in \mathbb{R}$, $p, q \in [1, +\infty]$, the homogeneous Besov space $\dot{B}_{p,q}^s(\mathbb{R}^n)$ is defined by those distributions $f$ in $\mathcal{S}'_k$ such that

$$\sum_{j \in \mathbb{Z}} (2^j \| \dot{\Delta}_j f \|_{L^p})^q < \infty,$$

with the norm

$$\| f \|_{\dot{B}_{p,q}^s} := \begin{cases} (\sum_{j \in \mathbb{Z}} 2^{jq} \| \dot{\Delta}_j f \|_{L^p}^q)^{\frac{1}{q}}, & 1 \leq q < +\infty, \\ \sup_{j \in \mathbb{Z}} \{ 2^j \| \dot{\Delta}_j f \|_{L^p} \}, & q = +\infty. \end{cases}$$

The following interpolation inequality will be used in the next section.

**Lemma 2.1** (see [1, 7]) Let $1 \leq q < p < \infty$ and $\alpha$ be a positive real number. A constant $C$ exists such that

$$\| f \|_{L^p} \leq C \| f \|_{\dot{B}_{p,q}^\alpha}^{1-\theta} \| f \|_{\dot{B}_{p,q}^\beta}^{\theta}$$

with $\beta = \alpha \left( \frac{p}{q} - 1 \right)$ and $\theta = \frac{q}{p}$, for all $f \in \dot{B}_{\infty,\infty}^{-\alpha}(\mathbb{R}^n) \cap \dot{B}_{q,q}^\beta(\mathbb{R}^n)$ with $n \geq 1$.

It is of interest to notice that the homogeneous Besov space $\dot{B}_{2,2}^s(\mathbb{R}^n)$ is equivalent to the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$. Hence, from Lemma 2.1 above, we have the following interpolation inequality:

$$\| f \|_{L^2} \leq C \| f \|_{\dot{B}_{2,2}^{s+\frac{n}{2}}}^{\frac{1}{2}} \| f \|_{\dot{B}_{\infty,\infty}^{-\frac{n}{2}}}^{1-\frac{1}{2}}$$

with $2 < q < \infty$ and $\alpha > 0$, (2.1)

for all $f \in \dot{H}^{\alpha(\frac{n}{2}-1)}(\mathbb{R}^n) \cap \dot{B}_{\infty,\infty}^{-\alpha}(\mathbb{R}^n)$ with $n \geq 1$.

### 3 The proofs of Theorem [1.1] and Corollary [1.3]

In this section, we shall give the proofs of Theorem [1.1] and Corollary [1.3]. We first need to prove the following lemma.

**Lemma 3.1** For $n = 2$ or $3$, $s \geq n$, $u_0 \in H^s(\mathbb{R}^n, \mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^{s+1}(\mathbb{R}^n, S^2)$, $0 < T_* < \infty$, let $(u, d)$ be a solution to system [1.1]–[1.4] satisfying [1.5], and there exists a small positive constant $\varepsilon_0$ such that

$$\|(u, \nabla d)\|_{L^\infty(0, T_*; \dot{B}_{\infty,\infty}^{-1})} \leq \varepsilon_0,$$

for $n = 3$; (3.1)

or

$$\| \nabla d \|_{L^\infty(0, T_*; \dot{B}_{\infty,\infty}^{-1})} \leq \varepsilon_0,$$

for $n = 2$. (3.2)

Then

$$\sup_{0 < t \leq T_*} \| \nabla u(\cdot, t) \|_{L^2}^2 + \| \Delta u(\cdot, t) \|_{L^2}^2 + \int_0^{T_*} \left( \| \nabla^2 u(\cdot, t) \|_{L^2}^2 + \| \nabla \Delta d(\cdot, t) \|_{L^2}^2 \right) dt \leq C_0,$$

(3.3)

where $C_0$ is a positive constant depending only on $u_0$, $d_0$ and $T_*$. 


Proof. We firstly notice that for all smooth solutions to system (1.1), one has the following basic energy law (see [22, 23]):

\[
\|u(t)\|_{L^2}^2 + \|\nabla d(t)\|_{L^2}^2 + \int_0^t (\|\nabla u(\tau)\|_{L^2}^2 + \|\Delta d + |\nabla d|^2(\tau)\|_{L^2}^2) d\tau \\
\leq \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2, \quad \text{for all } 0 < t < \infty,
\]
and when the space dimension \(n = 2\), the above energy inequality becomes energy equality.

Now, applying \(\nabla\) to the equation (1.1), multiplying the resulting equation by \(\nabla u\), integrating with respect to \(x\) over \(\mathbb{R}^n\) with \(n = 2\) or \(3\), and using integration by parts, we get

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^n} (\nabla u \cdot \nabla u) u dx - \int_{\mathbb{R}^n} \nabla (\nabla \cdot (\nabla d \odot \nabla d)) u dx.
\] (3.5)

Similarly, applying \(\nabla^2\) to the equation (1.2), multiplying the resulting equation by \(\nabla^2 d\), integrating with respect to \(x\) over \(\mathbb{R}^n\) with \(n = 2\) or \(3\), and using integration by parts, we get

\[
\frac{1}{2} \frac{d}{dt} \|\nabla^2 d(t)\|_{L^2}^2 + \|\nabla \Delta d(t)\|_{L^2}^2 = -\int_{\mathbb{R}^n} \nabla^2 (u \cdot \nabla d) \nabla^2 d dx + \int_{\mathbb{R}^n} \nabla^2 (|\nabla d|^2) \nabla^2 d dx.
\] (3.6)

Combining (3.5) and (3.6) together, and using the fact \(\nabla \cdot u = 0\), we get

\[
\frac{1}{2} \frac{d}{dt} (\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2) + (\|\Delta u(t)\|_{L^2}^2 + \|\nabla \Delta d(t)\|_{L^2}^2) \\
= -\int_{\mathbb{R}^n} [\nabla (u \cdot \nabla u) u - u \cdot \nabla^2 d u] dx - \int_{\mathbb{R}^n} \nabla (\nabla \cdot (\nabla d \odot \nabla d)) u dx \\
- \int_{\mathbb{R}^n} [\nabla^2 (u \cdot \nabla d) - u \cdot \nabla^2 d ] \nabla^2 d dx + \int_{\mathbb{R}^n} \nabla^2 (|\nabla d|^2) \nabla^2 d dx
\]

\[
\triangleq I_1 + I_2 + I_3 + I_4 + I_5.
\] (3.7)

By using the Hölder’s inequality, the interpolation inequality and Lemma 2.1 we obtain

\[
I_1 = \int_{\mathbb{R}^n} u \cdot \nabla u \nabla^2 u dx
\]

\[
\leq \begin{cases}
C \|u\|_{L^4} \|\nabla u\|_{L^4} \|\nabla^2 u\|_{L^2} \leq C \|u\|_{H^2}^{\frac{3}{2}} \|u\|_{B_{\infty, \infty}^{\frac{3}{2}}} \|\nabla u\|_{B_{\infty, \infty}^{\frac{3}{2}}} \|\Delta u\|_{L^2} \\
\leq C \|u\|_{B_{\infty, \infty}^{\frac{3}{2}}} \|\Delta u\|_{L^2}^2, \quad \text{for } n = 3
\end{cases}
\]

\[
\leq \begin{cases}
C \|u\|_{L^4} \|\nabla u\|_{L^4} \|\nabla^2 u\|_{L^2} \leq C \|u\|^{\frac{3}{2}}_{L^2} \|\nabla u\|_{L^2}^{\frac{3}{2}} \|\Delta u\|_{L^2} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \\
\leq \frac{1}{16} \|\Delta u\|_{L^2}^2 + C \|u\|_{L^2} \|\nabla u\|_{L^2}^2 \quad \text{(by using energy inequality (3.3))} \\
\leq \frac{1}{16} \|\Delta u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2, \quad \text{for } n = 2.
\end{cases}
\] (3.8)

\[
I_2 = \int_{\mathbb{R}^n} \nabla \cdot (\nabla d \odot \nabla d) \nabla^2 u dx = \int_{\mathbb{R}^n} |\nabla d \Delta d + \nabla (|\nabla d|^2/2)| \nabla^2 u dx = \int_{\mathbb{R}^n} \nabla d \Delta d \nabla^2 u dx
\]
By using facts where we have used the following Gagliardo–Nirenberg inequality

\[
\begin{gathered}
C\|\nabla d\|_{L^6} \|\Delta d\|_{L^3} \|\nabla^2 u\|_{L^2} \leq C\|\nabla d\|_{H^\frac{1}{6}} \|\nabla d\|_{B^{\frac{7}{3}}_{\infty, \infty}} \left( \|\Delta d\|_{H^\frac{1}{6}} \|\Delta d\|_{B^{\frac{7}{3}}_{\infty, \infty}} \right) \|\Delta u\|_{L^2}
\end{gathered}
\]

\[
\leq C\|\nabla d\|_{B^{\frac{11}{12}}_{\infty, \infty}} \|\nabla \Delta d\|_{L^2} \|\Delta u\|_{L^2}
\]

\[
\leq C\|\nabla d\|_{B^{\frac{11}{12}}_{\infty, \infty}} (\|\nabla \Delta d\|_{L^2}^2 + \|\Delta u\|_{L^2}^2), \quad \text{for } n = 3;
\]

\[
C\|\nabla d\|_{L^2} \|\Delta d\|_{L^2} \|\nabla^2 u\|_{L^2} \leq C\|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2} (\|\nabla \Delta d\|_{L^2}^2 + \|\Delta u\|_{L^2}^2) \|\Delta u\|_{L^2}
\]

\[
\leq \frac{1}{16} (\|\Delta u\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) + C\|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2}^2 \quad \text{(by } \text{3.3})
\]

\[
\leq \frac{1}{16} (\|\Delta u\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) + C\|\nabla^2 d\|_{L^2}^2, \quad \text{for } n = 2.
\]

Similar as the estimates of \( I_2 \), we obtain

\[
I_3 \leq \begin{cases} 
C\|\nabla d\|_{L^6} \|\Delta d\|_{L^3} \|\nabla^2 u\|_{L^2} \leq C\|\nabla d\|_{B^{\frac{11}{12}}_{\infty, \infty}} (\|\nabla \Delta d\|_{L^2}^2 + \|\Delta u\|_{L^2}^2), \quad \text{for } n = 3; \\
C\|\nabla d\|_{L^2} \|\Delta d\|_{L^2} \|\nabla^2 u\|_{L^2} \leq \frac{1}{16} (\|\Delta u\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) + C\|\nabla^2 d\|_{L^2}^2, \quad \text{for } n = 2.
\end{cases}
\]

For the term \( I_4 \), we have

\[
I_4 \leq \begin{cases} 
C\|\nabla u\|_{L^3} \|\nabla^2 d\|_{L^2}^2 \leq C\|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \|\nabla^2 d\|_{B^{\frac{7}{3}}_{\infty, \infty}} \left( \|\Delta d\|_{H^\frac{1}{6}} \|\Delta d\|_{B^{\frac{7}{3}}_{\infty, \infty}} \right) \\
\leq C\|\Delta u\|_{L^2} \|\nabla \Delta d\|_{L^2} \|\nabla d\|_{B^{\frac{7}{3}}_{\infty, \infty}} \quad \text{by } \text{3.3.4} \\
\leq \frac{1}{16} (\|\Delta u\|_{L^2}^2 + C\|\nabla d\|_{B^{\frac{11}{12}}_{\infty, \infty}} \|\nabla \Delta d\|_{L^2}^2) + C, \quad \text{for } n = 3; \\
C\|\nabla u\|_{L^3} \|\nabla^2 d\|_{L^2}^2 \leq C\|\nabla u\|_{L^2} \|\Delta u\|_{L^2} (\|\nabla \Delta d\|_{L^2} \|\nabla^2 d\|_{B^{\frac{7}{3}}_{\infty, \infty}})^2 \\
\leq C\|\Delta u\|_{L^2} \|\nabla \Delta d\|_{L^2} \|\nabla d\|_{B^{\frac{7}{3}}_{\infty, \infty}} \quad \text{by } \text{3.3.4} \\
\leq \frac{1}{16} (\|\Delta u\|_{L^2}^2 + C\|\nabla d\|_{B^{\frac{11}{12}}_{\infty, \infty}} \|\nabla \Delta d\|_{L^2}^2) + C, \quad \text{for } n = 2,
\end{cases}
\]

where we have used the following Gagliardo–Nirenberg inequality

\[
\|\nabla u\|_{L^3} \leq C\|\nabla u\|_{L^2} \|\Delta u\|_{L^2}, \quad \text{when } n = 3; \\
\|\nabla u\|_{L^3} \leq C\|\nabla u\|_{L^2} \|\Delta u\|_{L^2}, \quad \text{when } n = 2.
\]

By using facts \(|d| = 1\) and \(\Delta d \cdot d = -|\nabla d|^2\), we see that

\[
I_5 = \int_{\mathbb{R}^n} \nabla^2 (|\nabla d|^2 d) \nabla^2 dd dx
\]

\[
= \int_{\mathbb{R}^n} \nabla (2\nabla^4 dd + |\nabla d|^2 d) \nabla^2 dd dx
\]

\[
= \int_{\mathbb{R}^n} (2\nabla^3 d \nabla dd + 2|\nabla^2 d|^2 d + 5\nabla^2 d |\nabla d|^2 d) \nabla^2 dd dx
\]

\[
= \int_{\mathbb{R}^n} (2\nabla^3 d \nabla dd + 2|\nabla^2 d|^2 d + 5\nabla^2 d \Delta dd) \nabla^2 dd dx
\]

\[
\leq C \int_{\mathbb{R}^n} (|\nabla^3 d| \|\nabla d\| + |\nabla^2 d|^3 + |\nabla d| \|\Delta d\|) dx
\]

\[
\leq C\|\nabla d\|_{L^3} \|\nabla^2 d\|_{L^3} \|\nabla \Delta d\|_{L^2} + C\|\nabla^2 d\|_{L^4}^2 \|\Delta d\|_{L^3}
\]

\[
\leq C\|\nabla \Delta d\|_{L^2} (\|\nabla d\|_{H^\frac{1}{6}} \|\nabla d\|_{B^{\frac{7}{3}}_{\infty, \infty}})^2 (\|\nabla^2 d\|_{H^\frac{1}{6}} \|\nabla^2 d\|_{B^{\frac{7}{3}}_{\infty, \infty}}) + C\|\nabla^2 d\|_{H^\frac{1}{6}} \|\nabla^2 d\|_{B^{\frac{7}{3}}_{\infty, \infty}}^3
\]

\[
\leq C\|\nabla d\|_{B^{\frac{1}{3}}_{\infty, \infty}} \|\nabla \Delta d\|_{L^2}, \quad \text{for } n = 2 \text{ or } 3.
\]
Inserting (3.8)–(3.12) into (3.7), one gets
\[
\frac{1}{2} \frac{d}{dt} \left( \| \Delta u(t) \|_{L^2}^2 + \| \nabla^2 d(t) \|_{L^2}^2 \right) + \| \Delta u(t) \|_{L^2}^2 + \| \nabla \Delta d(t) \|_{L^2}^2 \\
\leq \left\{ \begin{array}{ll}
\frac{1}{2} \| \Delta u \|_{L^2}^2 + C(\| \nabla d \|_{L^\infty} + \| \nabla \|_{L^\infty})(\| \Delta u \|_{L^2}^2 + \| \nabla \Delta d \|_{L^2}^2) + C \\
= \frac{1}{4} \| \Delta u \|_{L^2}^2 + C(\| \nabla d \|_{L^\infty})(\| \Delta u \|_{L^2}^2 + \| \nabla \Delta d \|_{L^2}^2) + C \\
\leq \frac{1}{2} \| \Delta u \|_{L^2}^2 + C \varepsilon_0(\| \Delta u \|_{L^2}^2 + \| \nabla \Delta d \|_{L^2}^2) + C, \quad \text{for } n = 3; \\
\frac{1}{4} \| \Delta u \|_{L^2}^2 + \| \nabla \Delta d \|_{L^2}^2 + C(\| \nabla u \|_{L^2}^2 + \| \nabla^2 \|_{L^2}^2) + C \\
\leq \frac{1}{4} \| \Delta u \|_{L^2}^2 + \| \nabla \Delta d \|_{L^2}^2 + C \varepsilon_0(\| \Delta u \|_{L^2}^2 + \| \nabla^2 \|_{L^2}^2) + C, \quad \text{for } n = 2.
\end{array} \right.
\]
By taking the $\varepsilon_0$ in (3.1) or (3.2) small enough, and noticing that there holds equality $\| \nabla^2 d \|_{L^2}^2 = \| \Delta d \|_{L^2}^2$, we get
\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla u(t) \|_{L^2}^2 + \| \Delta d(t) \|_{L^2}^2 \right) + \frac{1}{2} \left( \| \Delta u(t) \|_{L^2}^2 + \| \nabla \Delta d(t) \|_{L^2}^2 \right)
\leq \left\{ \begin{array}{ll}
C, & \text{for } n = 3; \\
C(\| \nabla u \|_{L^2}^2 + \| \Delta d \|_{L^2}^2) + C, & \text{for } n = 2.
\end{array} \right.
\]
(3.13)

Then, by integrating with respect to $t$ over $[0; T_*]$ for $n = 3$, or by using the Gronwall’s inequality for $n = 2$, it follows from (3.13) that estimate (3.3) is established. This completes the proof of Lemma 3.1.

**Proof of Theorem 4.1.** By using standard method, we only need to give the a priori estimates to control $\| u(t) \|_{H^3} + \| \nabla d(t) \|_{H^3}$ for any $0 \leq t \leq T_*$ in terms of $u_0$, $d_0$ and $\varepsilon_0$. To this end, we need to introduce the following commutator and product estimates (see [15] [11] [19]):
\[
\| \Lambda^\alpha(f \cdot g) - f \Lambda^\alpha g \|_{L^p} \leq C(\| \nabla f \|_{L^{p_1}} \| \Lambda^{\alpha - 1} g \|_{L^{q_1}} + \| \Lambda^\alpha f \|_{L^{p_2}} \| g \|_{L^{q_2}}); \\
\| \Lambda^\alpha(f g) \|_{L^p} \leq C(\| f \|_{L^{p_1}} \| \Lambda^\alpha g \|_{L^{q_1}} + \| \Lambda^\alpha f \|_{L^{p_2}} \| g \|_{L^{q_2}})
\]
(3.14) (3.15)

with $\alpha > 0$, $1 < p, p_1, p_2, q, q_1, q_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Here $\Lambda := (-\Delta)^{\frac{1}{4}}$.

Applying $\Lambda^3$ on (1.1), multiplying $\Lambda^3 u$, integrating with respect to $x$ over $\mathbb{R}^3$, and using integration by parts, one obtains
\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^3 u(\cdot, t) \|_{L^2}^2 + \| \Lambda^4 u(\cdot, t) \|_{L^2}^2 = -\int_{\mathbb{R}^3} \Lambda^3 (u \cdot \nabla u) \cdot \Lambda^3 u \, dx - \int_{\mathbb{R}^3} \Lambda^3 (\Delta d \cdot \nabla d) \cdot \Lambda^3 u \, dx := I_6 + I_7.
\]
(3.16)

Noticing that the fact that $\text{div} \, u = 0$ implies $\int_{\mathbb{R}^3} \Lambda^3 \nabla (\frac{\text{div} \, d}{2}) \cdot \Lambda^3 u \, dx = 0$, it follows that
\[
I_6 = \int_{\mathbb{R}^3} [\Lambda^3 (u \cdot \nabla u) - u \cdot \nabla \Lambda^3 u] \cdot \Lambda^3 u \, dx
\leq C(\| \Lambda^3 (u \cdot \nabla u) - u \cdot \nabla \Lambda^3 u \|_{L^2} \| \Lambda^3 u \|_{L^3})
\leq C(\| \nabla u \|_{L^\infty} \| \Lambda^3 u \|_{L^2} \| \Lambda^3 u \|_{L^2})
\leq \frac{1}{4} \| \Lambda^4 u \|_{L^2}^2 + C(\| \nabla u \|_{L^2}^4)
\leq \frac{1}{4} \| \Lambda^4 u \|_{L^2}^2 + C C_0^7.
\]
(3.17)
where $C_0$ is the bounded positive constant in (3.3). Here we have used the following Gagliardo–Nirenberg inequalities:

$$
\|\nabla u\|_{L^3} \leq C\|\nabla u\|_{L^2}^{\frac{5}{6}}\|\Lambda^4 u\|_{L^2}^{\frac{1}{6}} \quad \text{and} \quad \|\Lambda^3 u\|_{L^3} \leq C\|\nabla u\|_{L^2}^{\frac{1}{6}}\|\Lambda^4 u\|_{L^2}^{\frac{5}{6}}.
$$

For $I_7$, applying the Hölder’s inequality and the Leibniz’s rule, we have

$$
I_7 = \int_{\mathbb{R}^3} \Lambda^2(\Delta d \cdot \nabla d) \cdot \Lambda^4 u \, dx \\
\leq \frac{1}{4}\|\Lambda^4 u\|_{L^2}^2 + C\int_{\mathbb{R}^3} |\Lambda^2(\Delta d \cdot \nabla d)|^2 \, dx \\
\leq \frac{1}{4}\|\Lambda^4 u\|_{L^2}^2 + C\int_{\mathbb{R}^3} (|\Lambda^4 d|^2|\nabla d|^2 + |\Lambda^2 d|^2|\Lambda^3 d|^2) \, dx \\
\leq \frac{1}{4}\|\Lambda^4 u\|_{L^2}^2 + C(\|\nabla d\|_{L^6}^2\|\Lambda^4 u\|_{L^6}^2 + \|\Lambda^2 d\|_{L^6}^2\|\Lambda^3 d\|_{L^6}^2) \\
\leq \frac{1}{4}\|\Lambda^4 u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^5 d\|_{L^2}^2 + C(\|\Delta d\|_{L^2}^\frac{1}{4} + \|\Delta d\|_{L^2}^\frac{5}{4}) \\
\leq \frac{1}{4}\|\Lambda^4 u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^5 d\|_{L^2}^2 + C(C_0^7 + C_0^{\frac{19}{2}}). 
$$

(3.18)

Here we have used the following Gagliardo–Nirenberg inequalities:

$$
\|\Lambda^4 d\|_{L^3} \leq C\|\Delta d\|_{L^2}^{\frac{5}{6}}\|\Lambda^5 d\|_{L^2}^{\frac{1}{6}}; \\
\|\Lambda^2 d\|_{L^6} \leq C\|\Delta d\|_{L^2}^{\frac{5}{4}}\|\Lambda^5 d\|_{L^2}^{\frac{1}{4}}; \\
\|\Lambda^3 d\|_{L^6} \leq C\|\Delta d\|_{L^2}^{\frac{5}{4}}\|\Lambda^5 d\|_{L^2}^{\frac{1}{4}}.
$$

Inserting (3.17) and (3.18) into (3.16), one gets

$$
\frac{d}{dt}\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^4 u\|_{L^2}^2 \leq \frac{1}{2}\|\Lambda^5 d\|_{L^2}^2 + C(C_0^7 + C_0^{\frac{19}{2}}). 
$$

(3.19)

Taking $\Lambda^4$ on (1.2), multiplying $\Lambda^4 d$, integrating with respect to $x$ over $\mathbb{R}^3$, and using integration by parts, one obtains

$$
\frac{1}{2}\frac{d}{dt}\|\Lambda^4 d\|_{L^2}^2 + \|\Lambda^5 d\|_{L^2}^2 = -\int_{\mathbb{R}^3} \Lambda^4(u \cdot \nabla d) \cdot \Lambda^4 d \, dx + \int_{\mathbb{R}^3} \Lambda^4(|\nabla d|^2 d) \cdot \Lambda^4 d \, dx := I_8 + I_9. 
$$

(3.20)

Similar as estimate of $I_6$, we have

$$
I_8 = -\int_{\mathbb{R}^3} [\Lambda^4(u \cdot \nabla d) - u \cdot \nabla \Lambda^4 d] \cdot \Lambda^4 d \, dx \\
\leq C\|\Lambda^4(u \cdot \nabla d) - u \cdot \nabla \Lambda^4 d\|_{L^2}^\frac{1}{2}\|\Lambda^4 d\|_{L^3} \\
\leq C\|\nabla d\|_{L^3}\|\Lambda^4 u\|_{L^2}\|\Lambda^4 d\|_{L^3} + C\|\nabla u\|_{L^2}\|\Lambda^4 d\|_{L^3} \\
\leq \frac{1}{4}\|\Lambda^4 u\|_{L^2}^2 + C(\|\Delta d\|_{L^2}^\frac{5}{4}\|\Lambda^4 d\|_{L^2}^\frac{1}{4} + \|\nabla u\|_{L^2}\|\Lambda^5 d\|_{L^2} + \|\Lambda^4 d\|_{L^3}) \\
\leq \frac{1}{4}\|\Lambda^4 u\|_{L^2}^2 + C(\|\Delta d\|_{L^2}^\frac{5}{4}\|\Lambda^5 d\|_{L^2}^\frac{1}{4} + \|\nabla u\|_{L^2}\|\Delta d\|_{L^2}^\frac{1}{2}\|\Lambda^5 d\|_{L^2} + \|\Lambda^4 d\|_{L^3})
$$

where $C_0$ is the bounded positive constant in (3.3). Here we have used the following Gagliardo–Nirenberg inequalities:
\[ \leq \frac{1}{4} \| \Lambda^4 u \|_{L^2}^2 + \frac{1}{4} \| \Lambda^5 d \|_{L^2}^2 + C(\| \Delta d \|_{L^2}^{14} + \| \nabla u \|_{L^2}^{24} + \| \Delta d \|_{L^2}^{24}) \]
\[ \leq \frac{1}{4} \| \Lambda^4 u \|_{L^2}^2 + \frac{1}{4} \| \Lambda^5 d \|_{L^2}^2 + C(C_0^7 + C_0^{12} + C_0^2), \]  
(3.21)

where we have used the Gagliardo–Nirenberg inequality:
\[ \| \Lambda^4 d \|_{L^2} \leq C \| \Delta d \|_{L^2}^{\frac{5}{2}} \| \Lambda^5 d \|_{L^2}^{\frac{7}{2}} \]
and \[ \| \Lambda^4 d \|_{L^3} \leq C \| \Delta d \|_{L^2}^{\frac{5}{2}} \| \Lambda^5 d \|_{L^2}^{\frac{7}{2}}. \]

To estimate \( I_9 \), by using the Leibniz’s rule, the fact \( |d| = 1 \), the Hölder’s inequality and the Young inequality, one obtains
\[
I_9 = \int_{\mathbb{R}^3} \Lambda^4(|\nabla d|^2 d) \Lambda^4 d d x = - \int_{\mathbb{R}^3} \Lambda^3(|\nabla d|^2 d) \Lambda^5 d d x
\]
\[ \leq C(\| \Lambda^5 d \|_{L^2}(\| \Delta d \|_{L^2} \| \Lambda^4 d \|_{L^3} + \| \Lambda^2 d \|_{L^4} \| \Lambda^3 d \|_{L^4} + \| \Delta d \|_{L^2} \| \Lambda^5 d \|_{L^6} + \| \Delta d \|_{L^2} \| \Lambda^5 d \|_{L^6}) \]
\[ \leq C(\| \Lambda^5 d \|_{L^2}(\| \Delta d \|_{L^2} \| \Lambda^4 d \|_{L^3} + \| \Delta d \|_{L^2} \| \Lambda^5 d \|_{L^6} \]) \]
\[ \leq \frac{1}{4} \| \Lambda^5 d \|_{L^2}^2 + C \| \Delta d \|_{L^2}^{14} \]
\[ \leq \frac{1}{4} \| \Lambda^5 d \|_{L^2}^2 + CC_0^7. \]  
(3.22)

Here we have used the following Gagliardo–Nirenberg inequalities:
\[ \| \Lambda^4 d \|_{L^3} \leq C \| \Delta d \|_{L^2}^{\frac{5}{2}} \| \Lambda^5 d \|_{L^2}^{\frac{7}{2}}; \]
\[ \| \Lambda^4 d \|_{L^4} \leq C \| \Delta d \|_{L^2}^{\frac{5}{2}} \| \Lambda^5 d \|_{L^2}^{\frac{7}{2}}; \]
\[ \| \Lambda^3 d \|_{L^6} \leq C \| \Delta d \|_{L^2}^{\frac{5}{2}} \| \Lambda^5 d \|_{L^2}^{\frac{7}{2}}; \]
\[ \| \Lambda^3 d \|_{L^6} \leq C \| \Delta d \|_{L^2}^{\frac{5}{2}} \| \Lambda^5 d \|_{L^2}^{\frac{7}{2}}. \]

Inserting (3.21) and (3.22) into (3.20), one gets
\[
\frac{d}{dt}(\| \Lambda^4 d \|_{L^2}^2 + \| \Lambda^5 d \|_{L^2}^2) \leq \frac{1}{2} \| \Lambda^4 u \|_{L^2}^2 + C(C_0^7 + C_0^{12} + C_0^2). \]  
(3.23)

Combining (3.19) and (3.23) together, and letting \( C_0 > 1 \), one obtains
\[
\frac{d}{dt}(\| \Lambda^3 u \|_{L^2}^2 + \| \Lambda^4 d(t) \|_{L^2}^2) \leq \frac{1}{2} (\| \Lambda^4 u \|_{L^2}^2 + \| \Lambda^5 d \|_{L^2}^2) \leq CC_0^{14}. \]

Hence integrating with respect to \( t \) over \([0, T_*] \), we have
\[
\sup_{0 < t \leq T_*} (\| \Lambda^3 u(t) \|_{L^2}^2 + \| \Lambda^4 d(t) \|_{L^2}^2) + \frac{1}{2} \int_0^{T_*} (\| \Lambda^4 u(\cdot, \tau) \|_{L^2}^2 + \| \Lambda^5 d(\cdot, \tau) \|_{L^2}^2) d\tau \leq C < \infty, \]
where \( C \) only depends on the initial data \((u_0, d_0)\), \( C_0 \) and \( T_* \). Therefore, we get
\[
\| u \|_{L^\infty(0,T_*;H^3)} + \| u \|_{L^2(0,T_*,H^5)} \leq C < \infty, \]
\[
\| d \|_{L^\infty(0,T_*;H^5)} + \| d \|_{L^2(0,T_*,H^3)} \leq C < \infty. \]

This completes the proof of Theorem 1.1. \( \square \)
Proof of Corollary 1.3. Similar as the proof of Theorem 1.1 we only give the a priori estimates to control \( \|u(t)\|_{H^2} + \|\nabla d(t)\|_{H^0} \) for any \( 0 \leq t \leq T_* \) in terms of \( u_0, d_0 \) and \( \varepsilon_0 \). To this end, let us firstly recall the following useful Gagliardo-Nirenberg inequalities in \( \mathbb{R}^2 \):

\[
\| \nabla u \|_{L^3} \leq C \| \nabla u \|_{L^2}^{\frac{3}{2}} \| \Lambda^3 u \|_{L^2}^{\frac{1}{2}}; \quad \| \Lambda^2 u \|_{L^3} \leq C \| \nabla u \|_{L^2}^{\frac{3}{2}} \| \Lambda^3 u \|_{L^2}^{\frac{1}{2}}; \quad \| \Lambda^2 d \|_{L^3} \leq C \| \Delta d \|_{L^2}^{\frac{3}{2}} \| \Lambda^4 d \|_{L^2}^{\frac{1}{2}}; \quad \| \Lambda^3 d \|_{L^3} \leq C \| \Delta d \|_{L^2}^{\frac{3}{2}} \| \Lambda^4 d \|_{L^2}^{\frac{1}{2}}.
\]

(3.24)

Now, applying \( \Lambda^2 \) on (1.1), multiplying \( \Lambda^2 u \) and integrating with respect to \( x \) over \( \mathbb{R}^3 \), and using (3.14), the Hörmander’s inequality, (3.14) and the Young inequality, one obtains

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \Lambda^2 u(\cdot, t) \|_{L^2}^2 + \| \Lambda^3 u(\cdot, t) \|_{L^2}^2 &= - \int_{\mathbb{R}^2} \Lambda^2 (u \cdot \nabla u) \cdot \Delta^2 u dx - \int_{\mathbb{R}^2} \Lambda^2 (\Delta d \cdot \nabla d) \cdot \Lambda^2 u dx \\
&\leq C [\| \Lambda^2 (u \cdot \nabla u) \|_{L^2} \| \Lambda^2 u \|_{L^3} + \| \nabla d \|_{H^0} \| \Lambda^3 d \|_{L^3} \| \Lambda^3 u \|_{L^2} + \| \Lambda^3 u \|_{L^2} \| \Lambda^2 d \|_{L^2}] \\
&\leq C [\| \Lambda^2 u \|_{L^2}^2 \| \nabla u \|_{L^3} + \| \nabla d \|_{H^0} \| \Lambda^3 d \|_{L^3} \| \Lambda^3 u \|_{L^2} + \| \Lambda^3 u \|_{L^2} \| \Lambda^2 d \|_{L^2}] \\
&\leq C [\| \nabla u \|_{L^3}^2 + \| \Lambda^3 u \|_{L^2}^2 \| \nabla d \|_{L^2} + \| \nabla d \|_{L^2}^2 \| \Lambda^3 u \|_{L^2} + \| \Lambda^3 u \|_{L^2} \| \nabla d \|_{L^2}] \\
&\leq C [\| \nabla u \|_{L^3}^2 + \| \Lambda^3 u \|_{L^2}^2 + \| \nabla d \|_{L^2}^2 \| \Lambda^3 u \|_{L^2} + \| \Lambda^3 u \|_{L^2} \| \nabla d \|_{L^2}] \\
&\leq C [\| \nabla u \|_{L^3}^2 + \| \Lambda^3 u \|_{L^2}^2 + \| \nabla d \|_{L^2}^2 \| \Lambda^3 u \|_{L^2} + \| \Lambda^3 u \|_{L^2} \| \nabla d \|_{L^2}] \\
&\leq C [1 + \| \nabla u \|_{L^2}^2 + \| \nabla d \|_{L^2}^2] \\
&\leq C_0,
\end{align*}
\]

where we have used the energy equality (3.4), and \( C_0 \) is the positive constant defined in Lemma 3.4.

Taking \( \Lambda^3 \) on (1.2), multiplying \( \Lambda^3 d \), integrating with respect to \( x \) over \( \mathbb{R}^2 \), and using (3.14), the Hörmander’s inequality, (3.14) and the Young inequality, one obtains

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \Lambda^3 d(\cdot, t) \|_{L^2}^2 + \| \Lambda^4 d(\cdot, t) \|_{L^2}^2 &= - \int_{\mathbb{R}^2} \Lambda^3 (u \cdot \nabla d) \cdot \Lambda^3 d dx + \int_{\mathbb{R}^2} \Lambda^3 (\nabla d^2 d) \cdot \Lambda^3 d dx \\
&\leq C [\| \Lambda^3 (u \cdot \nabla d) \|_{L^2} \| \Lambda^3 d \|_{L^2} + \| \nabla d \|_{H^0} \| \Lambda^3 d \|_{L^3} \| \Lambda^3 d \|_{L^2} + \| \Lambda^3 d \|_{L^2} \| \nabla d \|_{L^2}] \\
&\leq C [\| \Lambda^3 (u \cdot \nabla d) \|_{L^2}^2 + \| \nabla d \|_{L^2}^2 \| \Lambda^3 d \|_{L^2} + \| \nabla d \|_{L^2} \| \Lambda^3 d \|_{L^2}] \\
&\leq C [\| \nabla u \|_{L^3}^2 + \| \Lambda^3 u \|_{L^2}^2 + \| \nabla d \|_{L^2}^2 \| \Lambda^3 u \|_{L^2} + \| \Lambda^3 u \|_{L^2} \| \nabla d \|_{L^2}] \\
&\leq C [\| \nabla u \|_{L^3}^2 + \| \Lambda^3 u \|_{L^2}^2 + \| \nabla d \|_{L^2}^2 \| \Lambda^3 u \|_{L^2} + \| \Lambda^3 u \|_{L^2} \| \nabla d \|_{L^2}] \\
&\leq C_0.
\end{align*}
\]

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Hence integrating with respect to $t$ and (3.26) together, we obtain
\[\frac{d}{dt} \left( \|\Delta u\|_{L^2}^2 + \|\Lambda^3 d\|_{L^2}^2 \right) + (\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^4 d\|_{L^2}^2) \leq C(1 + C_0^6), \]
where we have used the energy equality (3.4), and $C_0$ defined in Lemma 3.1. Combining (3.25) and (3.26) together, we obtain
\[\frac{d}{dt} \left( \|\Lambda^2 u\|_{L^2}^2 + \|\Lambda^3 d\|_{L^2}^2 \right) \leq C(1 + C_0^6). \]
Hence integrating with respect to $t$ over $[0, T_\ast]$, we have
\[\sup_{0 < t < T_\ast} \left( \|\Lambda^2 u(t)\|_{L^2}^2 + \|\Lambda^3 d(t)\|_{L^2}^2 \right) + \int_0^{T_\ast} \left( \|\Lambda^3 u(\tau)\|_{L^2}^2 + \|\Lambda^4 d(\tau)\|_{L^2}^2 \right) d\tau \leq C < \infty, \]
where $C$ only depends on the initial data $(u_0, d_0)$, $C_0$ and $T_\ast$. Therefore, we get
\[\|u\|_{L^\infty(0, T; H^2)} + \|u\|_{L^2(0, T; H^3)} \leq C < \infty, \]
\[\|d\|_{L^\infty(0, T; H^2)} + \|d\|_{L^2(0, T; H^3)} \leq C < \infty. \]
This completes the proof of Corollary 1.3. \qed

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