Multi-critical behaviour in a self-dual Josephson junction array – fixed dimension renormalization versus large-N technique

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Abstract. Self-dual Josephson junction arrays are modelled by a theory of N components complex fields coupled to gauge fields. The multi-critical behaviour is analysed through a one loop renormalization group investigation at fixed dimension, which reveals the existence of charged infrared-stable fixed point solutions for large N. Further analysis of the model in the framework of the 1/N expansion confirms that the decoupled fixed point that was stable in the massive regime within the approximate one-loop renormalization group is destabilized. This is ascribed to the special interaction mediated by the mixed Chern-Simons term.

1. Introduction
Two-dimensional Josephson junction arrays (JJA) continue to draw interest due to their rich underlying physics [1]. In these systems, Cooper pairs and vortices have conserved currents, which in 2+1 dimension can be embodied as the curls of fictitious gauge fields. The resultant gauge field formulation [2] has Maxwell terms that express kinetic terms for charges and vortices and a mixed Chern-Simons term that describes the Lorentz force exerted by the vortices on the charges and the Magnus force exerted by the charges on the vortices. Together with the emerging gauge fields, complex scalar fields are added to account for quantum disorder due to electric and magnetic excitations in these systems [3]. The ensuing low energy effective model is a Ginzburg-Landau theory with two charged disorder fields coupled to Maxwell-mixed-Chern-Simons gauge fields. These competing scalar fields along with the fluctuating gauge fields lead to multi-critical behavior, which we report on here.

2. The model
The Euclidean Lagrangian of interest [4] is

\[ L = \left( \partial_{\mu} - i a_{\mu} \right) \Psi \left( \partial^{\mu} - i b^{\mu} \right) \Phi \quad + \left( \partial_{\mu} - i b_{\mu} \right) \Psi \left( \partial^{\mu} - i a^{\mu} \right) \Phi \quad + r_{1} \left| \Psi \right|^{2} + r_{2} \left| \Phi \right|^{2} + \frac{u}{6} \left( \left| \Psi \right|^{4} + \left| \Phi \right|^{4} \right) + \frac{w}{3} \left| \Psi \right|^{2} \left| \Phi \right|^{2} \]

\[ + \frac{1}{4e_{a}^{2}} \epsilon_{\mu\nu} f_{\mu\nu}^{2} + \frac{1}{4e_{b}^{2}} g_{\mu\nu}^{2} + i \kappa b_{\mu} e^{a\nu} \partial_{\nu} a_{a} \]

(1)
The gauge fields $a_\mu, b_\mu$ are associated with the currents of Cooper pairs and vortices and implicate two different gauge couplings. These are defined by the parameters of the JJA system [3]: $e_a = \sqrt{8E_C}$ and $e_b = 2\pi \sqrt{E_J}$ with $E_J$ the Josephson coupling energy measuring the strength for the phase-coupling between two superconductors and $E_C$ the charging energy required to add extra charges to neutral islands. The mixed Chern-Simons term has a coefficient $\kappa = 1/\pi$. The fields $\Phi$ and $\Psi$ represent the disorder of electric and magnetic charges which control the superconducting and insulating phases of the model. The superconducting phase emerges when the charge disorder field condenses while the magnetic disorder field resides in a vacuum; the insulating phase appears as a condensate of magnetic charges and a vacuum for electric disorder fields [3]. In this self-dual model, both types of excitations (electric-like and magnetic-like) are treated on an equal footing, leading to the distinctive feature of the action in Eq. (1) with equal quartic coupling constants of the matter fields. The other feature of the action in Eq. (1) is the presence of a mixed Chern-Simons term, which mediates the interaction between the electric and magnetic excitations. This term describes both the Lorentz force exerted by the vortices on the charges and the Magnus force exerted by the charges on the vortices [3]. This term also incorporates the symmetry between the two species $a_\mu, b_\mu$: integration by parts changes the mixed Chern-Simons term into something of the same form with $a_\mu$ and $b_\mu$ interchanged. To facilitate a more general theoretical description, we introduce a large $N$ generalization of the model by allowing the fields $\Phi$ and $\Psi$ to have each $N$ complex components.

3. Renormalization analysis at fixed dimension

3.1. Massless regime

To study the critical properties of the model, I develop a renormalization analysis at fixed dimension. This approach is suitable here because of the Chern-Simons term, which is an inherently three dimensional object. The calculation is done when both $r_1$ and $r_2$ are tuned to their critical values (i.e. $r_1 = 0, r_2 = 0$) by appropriately choosing the renormalization conditions for the vertex functions. Infrared divergences resulting from massless scalar propagators are handled by defining the renormalized coupling constants at finite values of the external momenta, and the renormalization beta functions are obtained by taking derivatives of the couplings with respect to the logarithm of the external momentum.

Up to one-loop order, the renormalized dimensionless coupling constants of the model satisfy the following flow equations [4]:

\begin{align}
\beta(\hat{\mu}) &= (2\eta_\mu - 1)\hat{\mu} + \frac{N + 4}{24}\hat{\mu}^2 + \frac{N}{24}\hat{\mu}^2 + \hat{\mu}_s^2 f(\hat{m}) \\
\beta(\hat{\nu}) &= (\eta_\nu + \eta_\phi - 1)\hat{\nu} + \frac{N + 1}{12}\hat{\nu}\hat{\mu} + \frac{\hat{\nu}^2}{12} - \hat{\nu}_s^2 \hat{\nu}_s^2 g(\hat{m}) \\
\beta(\hat{\nu}_s) &= -\hat{\nu}_s^2 \left(1 - \frac{N}{16}\hat{\nu}_s^2\right), \quad \beta(\hat{\phi}_s) = -\hat{\phi}_s^2 \left(1 - \frac{N}{16}\hat{\phi}_s^2\right)
\end{align}

where $\eta_\mu = d \ln(Z_\mu)/d \ln(p)$ and $\eta_\phi = d \ln(Z_\phi)/d \ln(p)$ are the anomalous dimensions of the scalar fields. $\eta_\mu, \eta_\phi, f(\hat{m}), g(\hat{m})$ are function of $\hat{m} = \hat{\nu}_s^2 / \pi$ and have well defined limits.
The fixed points of the model are obtained by finding the zeros of the beta functions. Since the beta functions (4) associated with the gauge couplings have nontrivial solutions when $\hat{e}^2 = \hat{e}^2 = 16 / N$, these lead to the fully charged fixed points (FP). One important observation about these charged FP is that the decoupled fixed point at $w = 0$ which was possible in the neutral case is no longer attainable. The result of the flow equations (2) and (3) is summarized in Fig. 1 for two different values of $N$. As Fig. 1 illustrates, the structure of the flow diagram in the critical plane depends critically on the number of components $N$. When $N<12$, there are two fixed points, one with two repulsive eigenvalues and one with one attractive and one repulsive eigenvalue. When $N>12$, there are four fixed points one of which has all attractive eigenvalues and should be identified with the critical point of the model.

![Flow diagram](image)

**FIG. 1** Flow diagrams in the $w-u$ plane in the massless regime with initial condition $\hat{e}^2 = \hat{e}^2 = 16 / N$. The arrows show the direction of the flow as $p \rightarrow 0$. (a) $N < 12$. (b) $N > 12$.

3.2. Massive regime

It is also useful to examine the properties of this field theory from the point of view of perturbative RG in the massive regime. This will be compared with $1/N$ expansion predictions. Construction of the beta functions of the couplings is done in the standard fashion [5], and the study of the stability of the fixed points reveals that for $N < 2$ the stable fixed point is the one with the enlarged $U(N)$ symmetry (Fig. 2a). For $2 < N < 4$, on the other hand, the mixed fixed point becomes the stable one (Fig 2b). Finally, for $N > 4$ the decoupled fixed point is stable (Fig. 2c).
4. Large N analysis

The fixed-dimension renormalization approach presented in the previous section has the drawback that it lacks a small parameter to control the calculation. To rectify this, I investigate the renormalized zero-momentum four-point couplings of the model by using large-N technique. The advantage of this analysis is that $1/N$ provides an expansion parameter which allows the summation of infinite classes of Feynman graphs. It turns out that this model is renormalizable in the $1/N$ expansion, and its critical exponents can be obtained in a systematic way [5].

The construction proceeds in a standard fashion, making the assumption that $u$ and $w$ are both of order $1/N$. The next-to-leading $1/N$ corrections to the renormalized couplings are found as a function of the bare couplings and the renormalized mass. These are used to find the $1/N$ correction to the $\beta$-functions and the fixed point values of the couplings.

The renormalization is performed according to the following prescription for the two- and four-point functions of $\Psi$ and $\Phi$:

\[
\Gamma_{11}^{(2)}(p) = \Gamma_{22}^{(2)}(p) = Z^{-1}[m_r^2 + p^2]
\]

\[
\Gamma_{11}^{(4)}(0) = Z^{-2} \frac{3u}{N} m_r
\]

\[
\Gamma_{12}^{(4)}(0) = Z^{-2} \frac{3w}{N} m_r
\]

(5)

The renormalized mass $m_r$ and the two-point function are related to the self-energy $\Sigma(p,m)$, $\Gamma^{(2)}(p) = p^2 + m_r^2 - \Sigma(p,m)$, and the dimensionless renormalized couplings $\hat{u}$ and $\hat{w}$ are expressed as

\[
\hat{u} = \frac{N}{3m_r} \left[ 1 + 2 \frac{\partial\Sigma(p,m)}{\partial p^2} \right]_{p=0} \Gamma_{1,1}^{(4)}(0,0,0,0)
\]

\[
\hat{w} = \frac{N}{3m_r} \left[ 1 + 2 \frac{\partial\Sigma(p,m)}{\partial p^2} \right]_{p=0} \Gamma_{1,2}^{(4)}(0,0,0,0)
\]

(6)

The large-N limit of the $\beta$-functions are found to be
As Fig. 3 shows, there are four fixed points: the Gaussian fixed point \((u, w) = (0, 0)\), which is unstable in both directions in the \(u\)-\(w\) plane; the decoupled fixed point, \((u, w) = (16\pi, 0)\) which is stable; the enlarged U(N) symmetric fixed point \((u, w) = (8\pi, 8\pi)\), which is unstable, and \((u, w) = (8\pi, -8\pi)\), which is also unstable. In the next-to-leading order, the \(\beta\)-functions can be expanded in a systematic way, and a numerical analysis reveals that the fixed point values of the couplings at the next order are shifted to \(\hat{\nu} = 16\pi (1 - 3.55/N)\) and \(\hat{\psi} = 34/N\). This shows that the decoupled fixed point which was stable in the massive regime within the approximate one-loop renormalization group is destabilized in the framework of the 1/N expansion. This is attributed to the interaction mediated by the mixed Chern-Simons term which generates a sequence of Feynman diagrams as shown in Fig. 9 and Fig 10 of reference [5], and which efficiently couples the fields \(\Phi\) and \(\Psi\).

5. Conclusion

In summary, I have formulated a field-theoretic approach that investigates the multi-critical behaviour of the low energy electric and magnetic excitations near the quantum critical point of a Josephson junction array system. In the massless regime, a one loop renormalization group analysis at fixed dimension of the beta functions reveals the existence of a rich phase diagram consisting of two sets of infrared-stable charged fixed points: partially charged solutions with respect to the gauge fields; and fully charged solutions above some critical value of N. Furthermore in the massless regime, the approximate one loop analysis shows that the decoupled fixed point which is stable in the neutral case (no gauge contribution) is no longer attainable in the presence of fluctuating gauge fields. When we examined the model in the massive regime, both the one-loop renormalization group at large N and the leading order of the 1/N expansion lead to a compatible result, which is the existence of a stable decoupled fixed point as in the neutral gauge. At this fixed point, the scalar fields \(\Phi\) and \(\Psi\) are effectively decoupled \((w = 0)\). However, analysis of the next-to-leading contribution of the 1/N expansion in the same regime reveals that the decoupled fixed point \((w = 0)\) that was stable within the approximate one-loop renormalization group in the massive regime (Fig. 2) is destabilized. This is attributed to the interaction mediated by the mixed Chern-Simons term which generates a sequence of
Feynman diagrams which efficiently couples the fields \( \Phi \) and \( \psi \) at the next-to-leading order of the \( 1/N \) expansion. Another way to view this result is to note that in the large \( N \) generalization of the model there are \( N \) components of each of the scalar fields and only one component of the gauge fields. Accordingly the effect of the mixed Chern-Simons term, which favors coupling the scalar fields, is underestimated at the leading order of the \( 1/N \) expansion and starts becoming noticeable at the next-to-leading order contribution.

References
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