Expansive actions of countable amenable groups, homoclinic pairs, and the Myhill property

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This is joint work with Tullio Ceccherini-Silberstein.
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“...the Garden of Eden theorem can be generalized to a suitable class of hyperbolic actions ...”
Dynamical systems

A dynamical system is a pair \((X, G)\), where \(X\) is a compact metrizable topological space, \(G\) is a countable group acting continuously on \(X\). The space \(X\) is called the phase space. If \(f: X \to X\) is a homeomorphism, the d.s. \((X, Z)\), where \(nx := f^n(x)\) \(\forall n \in \mathbb{Z}, \forall x \in X\), is also denoted \((X, f)\).

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is also denoted \((X, f)\).
Examples of Dynamical systems

Example (Arnold's cat)
This is the d.s. \((T^2, f)\), where \(f\) is the homeomorphism of the 2-torus \(T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\) given by \(f: T^2 \to T^2((x_1, x_2) \mapsto (x_2, x_1 + x_2))\).

Example (Shifts and subshifts)
We take a discrete finite space \(A\), called the alphabet or the set of states, and a countable group \(G\). The associated shift is the d.s. \((A^G, G)\), where \(A^G = \{x: G \to A\}\) is equipped with the product topology and \(G\) acts on \(A^G\) by \((gx)(h) := x(g^{-1}h)\) \(\forall g, h \in G, \forall x \in A^G\).

An element of \(A^G\) is called a configuration. A subsystem of the shift (i.e., a pair \((X, G)\), were \(X \subset A^G\) is a closed \(G\)-invariant subspace) is called a subshift.
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The Ledrappier subshift is the subshift \((X,\mathbb{Z}_2)\) over the alphabet 
\(A := \{0, 1\} = \mathbb{Z}/2\mathbb{Z}\) consisting of all \(x : \mathbb{Z}_2 \to A\) such that 
\(x(g) = x(g + e_1) + x(g + e_2)\) \(\forall g \in \mathbb{Z}_2\),
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Homoclinicity

Let \((X, G)\) be a dynamical system. Let \(d\) be a metric on \(X\) that is compatible with the topology. Definition: Two points \(x, y \in X\) are called homoclinic if

\[
\lim_{g \to \infty} d(gx, gy) = 0,
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i.e., for every \(\varepsilon > 0\), there exists a finite subset \(F \subset G\) such that

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d(gx, gy) < \varepsilon \quad \forall g \in G \setminus F.
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Homoclinicity is an equivalence relation on \(X\). This relation is \(G\)-invariant and does not depend on the choice of \(d\).
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Example Consider Arnold's cat \((T^2, f)\).

Equip \(T^2 = \mathbb{R}^2 / \mathbb{Z}^2\) with its Euclidean structure.

The homoclinicity class of a point \(x \in T^2\) is \(D \cap D'\), where \(D\) is the line passing through \(x\) whose slope is the golden mean \(\phi := (1 + \sqrt{5})/2\) and \(D'\) is the line passing through \(x\) and orthogonal to \(D'\).

Each homoclinicity class is countably-infinite.

Example Consider the full shift \((A^G, G)\) over a finite alphabet \(A\) and a countable group \(G\).

Two configurations \(x, y \in A^G\) are homoclinic if and only if they coincide outside of a finite subset of \(G\).

Thus, each homoclinicity class is countably-infinite as soon as \(A\) has more than one element and \(G\) is infinite.

Example Consider the Ledrappier subshift \((X, \mathbb{Z}_2)\).

Observe that if two configurations \(x, y \in X\) coincide on the horizontal line \(\mathbb{Z} \times \{n\} \subset \mathbb{Z}^2\), then they coincide on \(\mathbb{Z} \times \{n+1\}\).

Therefore, the homoclinicity relation is trivial: the homoclinicity class of every configuration \(x \in X\) is reduced to \(x\).
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Pre-injective endomorphisms

Let \((X, G)\) be a dynamical system.

**Definition**

A continuous map \(\tau: X \to X\) is an endomorphism of the d.s. \((X, G)\) if it is \(G\)-equivariant, i.e.,

\[ \tau(gx) = g\tau(x) \quad \forall g \in G, x \in X. \]

**Remark**

An endomorphism of a shift (or subshift) is also called a cellular automaton.

**Definition**

An endomorphism \(\tau: X \to X\) of the d.s. \((X, G)\) is called pre-injective if its restriction to each homoclinicity class is injective.

Of course \(\tau\) injective \(\Rightarrow\) \(\tau\) pre-injective but the converse implication is false in general.
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Examples of pre-injective but not injective endomorphisms

Example (Arnold’s cat)

The group endomorphism $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ given by $x \mapsto 2x$ is an endomorphism of Arnold’s cat $(\mathbb{T}^2, f)$. The kernel of $\tau$ is $\text{Ker}(\tau) = \{(0, 0), (1/2, 0), (0, 1/2), (1/2, 1/2)\}$. The endomorphism $\tau$ is surjective and pre-injective but not injective.

Example

The endomorphism $\tau$ of the full shift $(A^Z, Z)$ on the alphabet $A = \mathbb{Z}/2\mathbb{Z}$ defined by $\tau(x)(n) := x(n+1) + x(n) \forall x \in \{0, 1\}^Z, \forall n \in \mathbb{Z}$ is surjective and pre-injective but not injective.

Example (The Ledrappier subshift)

The constant map that sends each configuration $x \in X$ to the 0-configuration is an endomorphism of the Ledrappier subshift $(X, Z^2)$ that is pre-injective but neither injective nor surjective.
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The endomorphism \( \tau \) of the full shift \((A^\mathbb{Z}, \mathbb{Z})\) on the alphabet \( A = \mathbb{Z}/2\mathbb{Z} \) defined by

\[
\tau(x)(n) := x(n + 1) + x(n) \quad \forall x \in \{0, 1\}^\mathbb{Z}, \forall n \in \mathbb{Z}
\]

is surjective and pre-injective but not injective.

**Example (The Ledrappier subshift)**

The constant map that sends each configuration \( x \in X \) to the 0-configuration is an endomorphism of the Ledrappier subshift \((X, \mathbb{Z}^2)\) that is pre-injective but neither injective nor surjective.
Amenable groups

Let $G$ be a countable group. Definition

The group $G$ is called amenable if there exists a sequence $(F_n)_{n \geq 1}$ of non-empty finite subsets of $G$ such that

$$\lim_{n \to \infty} \frac{|F_n \setminus F_n^g|}{|F_n|} = 0 \quad \forall g \in G.$$ 

Such a sequence is called a Følner sequence for $G$.

- Every locally finite group is amenable.
- Every abelian group and, more generally, every solvable group is amenable.
- Every finitely generated group with subexponential growth is amenable.
- An example of a non-amenable group is provided by the free group on 2 generators. More generally, every group containing a non-abelian free subgroup is non-amenable.
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The Garden of Eden theorem

The following result is known as the Garden of Eden theorem:

Theorem (CMS-1999)

Let \( G \) be a countable amenable group and \( A \) a finite set. Then every endomorphism \( \tau \) of the shift \((A^G, G)\) satisfies

\[ \tau \text{ surjective} \iff \tau \text{ pre-injective}. \]

Moore [Moo-1963] proved = \( \Rightarrow \) for \( G = \mathbb{Z}^d \), Myhill [Myh-1963] proved \( \iff \) for \( G = \mathbb{Z}^d \), Ceccherini-Silberstein, Machtı and Scarabotti [CMS-1999] proved \( \iff \) in the general case.

The proof consists in showing that \( \tau \text{ surjective} \iff h_{\text{top}}(\tau(A^G), G) = h_{\text{top}}(A^G, G) \iff \tau \text{ pre-injective} \), where \( h_{\text{top}}(X, G) \) denotes the topological entropy of the d.s. \((X, G)\).
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Michel Coornaert (IRMA, University of Strasbourg) 
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Let $(X, G)$ be a dynamical system.

**Definition**
The d.s. $(X, G)$ has the Moore property if every surjective endomorphism of $(X, G)$ is pre-injective.

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The d.s. $(X, G)$ has the Myhill property if every pre-injective endomorphism of $(X, G)$ is surjective.

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A d.s. has the Moore-Myhill property if it has both the Moore and the Myhill property.
The Moore and the Myhill property

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The Moore and the Myhill property (continued)

Example
Arnold’s cat \((T^2, f)\) has the Moore-Myhill property. Indeed, it is easy to show that any endomorphism \(\tau\) of the cat is of the form \(\tau = m\text{Id} + nf\), for some \(m, n \in \mathbb{Z}\). Thus, with the exception of the 0-endomorphism, every endomorphism of the cat is both surjective and pre-injective.

Example
The GOE theorem says that the shift \(A\) has the Moore-Myhill property whenever \(A\) is finite and \(G\) is amenable. Bartholdi [Bar-2010] proved that if \(G\) is non-amenable then there is a finite set \(A\) such that \(A^G\) does not have the Moore property. It is known that if \(G\) contains a nonabelian free subgroup then there is a finite set \(A\) such that \(A^G\) does not have the Myhill property.

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The Ledrappier subshift \((X, Z^2)\) has the Moore property (since every endomorphism is pre-injective) but does not have the Myhill property (since the 0-endomorphism is pre-injective but not surjective).
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Subshifts of finite type and strongly irreducible subshifts

Definition

A subshift \( X \subset A^G \) is said to be of finite type if there exist a finite subset \( \Omega \subset G \) and a subset \( P \subset A^\Omega \) such that

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X = \{ x \in A^G : (gx)_{\Omega} \in P \text{ for all } g \in G \}. 
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Definition

A subshift \( X \subset A^G \) is said to be strongly irreducible if there exists a finite subset \( \Delta \subset G \) with the following property:

if \( \Omega_1 \) and \( \Omega_2 \) are finite subsets of \( G \) such that there is no element \( g \in \Delta \) such that the set \( \Omega_1 g \) meets \( \Omega_2 \) (i.e., \( \Omega_1 \Delta \cap \Omega_2 = \emptyset \)) then, given any two configurations \( x_1, x_2 \in X \), there exists a configuration \( x \in X \) such that \( x|_{\Omega_1} = x_1|_{\Omega_1} \) and \( x|_{\Omega_2} = x_2|_{\Omega_2} \).
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The Moore-Myhill property for strongly irreducible subshifts of finite type

Fiorenzi extended the Garden of Eden theorem in the following way:

Theorem (Fio-2003)

Let $G$ be a countable amenable group and $A$ a finite set. Then every strongly irreducible subshift of finite type $X \subset A^G$ has the Moore-Myhill property.

Example

The hard sphere model is the subshift $X \subset \{0,1\}^{\mathbb{Z}^d}$ consisting of all $x: \mathbb{Z}^d \to \{0,1\}$ with no two 1s appearing at Euclidean distance 1 on $\mathbb{Z}^d$.

The hard sphere model is strongly irreducible and of finite type. Thus, it has the Moore-Myhill property.

Remark

For $d = 1$, the hard sphere model is also called the golden mean subshift because its topological entropy is equal to the golden mean.
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Let $G$ be a countable amenable group and $A$ a finite set. Then every strongly irreducible subshift $X \subset A^G$ has the Myhill property.

Example

The even subshift is the subshift $X \subset \{0, 1\}^\mathbb{Z}$ consisting of all bi-infinite sequences $x: \mathbb{Z} \to \{0, 1\}$ such that the number of 1s between any two 0s is even. The even subshift is strongly irreducible. Therefore the even subshift has the Myhill property. Note that the even subshift is not of finite type. Actually, Fiorenzi [Fio-2000] proved that the even subshift does not have the Moore property: it admits endomorphisms that are surjective but not pre-injective.
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Let $(X, G)$ be a dynamical system and let $d$ be a metric on $X$ that is compatible with the topology. Definition: The d.s. $(X, G)$ is expansive if there is a constant $\varepsilon > 0$ such that, for all distinct points $x, y \in X$, there exists $g \in G$ such that $d(gx, gy) \geq \varepsilon$. This definition does not depend on the choice of $d$. Example: Arnold’s cat is expansive. Example: All shifts and subshifts are expansive.
Expansive dynamical systems

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Let \((X, G)\) be a dynamical system and let \(d\) be a metric on \(X\) that is compatible with the topology.

**Definition**

The d.s. \((X, G)\) is **expansive** if there is a constant \(\varepsilon > 0\) such that, for all distinct points \(x, y \in X\), there exists \(g \in G\) such that

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The Myhill property for a class of expansive dynamical systems

Theorem (CC-2015b)

Let $X$ be a compact metrizable space equipped with a continuous action of a countable amenable group $G$. Suppose that the d.s. $(X, G)$ is expansive and that there exist a finite set $A$, a strongly irreducible subshift $\Sigma \subset A^G$, and a continuous, surjective, $G$-equivariant and uniformly finite-to-one map $\pi: \Sigma \to X$.

Then the dynamical system $(X, G)$ has the Myhill property.
The Myhill property for a class of expansive dynamical systems

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Smale’s Axiom A diffeomorphisms

Let $f : M \to M$ be a diffeomorphism of a smooth compact manifold $M$. A closed $f$-invariant subset $\Lambda \subset M$ is hyperbolic if the restriction to $\Lambda$ of the tangent bundle of $M$ splits as a direct sum of two invariant subbundles $E^s$ and $E^u$ such that, with respect to some (or equivalently any) Riemannian metric on $M$, the differential $df$ is uniformly contracting on $E^s$ and uniformly expanding on $E^u$.

A point $x \in M$ is called non-wandering if for every neighborhood $U$ of $x$, there is an integer $n \geq 1$ such that $f^n(U)$ meets $U$. The set $\Omega(f)$ consisting of all non-wandering points of $f$ is a closed invariant subset of $M$.

If $\text{Per}(f)$ denotes the set of periodic points of $f$, one always has $\text{Per}(f) \subset \Omega(f)$.

Definition

One says that $f$ is Axiom A if $\Omega(f)$ is hyperbolic, and $\text{Per}(f)$ is dense in $\Omega(f)$.

If $f$ is Axiom A, then $\Omega(f)$ can be uniquely written as a disjoint union of closed invariant subsets $\Omega(f) = X_1 \cup \cdots \cup X_k$, such that the restriction of $f$ to each $X_i$ is topologically transitive (spectral decomposition theorem).

These subsets $X_i$ are called the basic sets of $(M, f)$.}

Michel Coornaert (IRMA, University of Strasbourg)

Expansive actions of countable amenable groups

December 8, 2015 20 / 23
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A dynamical system \((X,G)\) is topologically mixing if, given any two non-empty open subsets \(U, V \subset X\), one has \(U \cap gV \neq \emptyset\) for all but finitely many \(g \in G\).

Corollary (CC-2015a) Let \(f\) be an Axiom A diffeomorphism of a smooth compact manifold \(M\). Suppose that \(X\) is a topologically mixing basic set of \((M,f)\). Then the dynamical system \((X,f|_X)\) has the Myhill property.

Proof. The fact that the dynamical system \((X,f|_X)\) satisfies the hypotheses of the theorem follows from results obtained by Rufus Bowen in the 1970s.
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Let $f : M \to M$ be a diffeomorphism of a smooth compact manifold $M$. One says that $f$ is Anosov if the whole manifold $M$ is hyperbolic for $f$.

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Example (Hyperbolic toral automorphisms) Consider a matrix $A \in \text{GL}_n(\mathbb{Z})$ with no eigenvalue of modulus 1. Then $A$ induces a topologically mixing Anosov diffeomorphism $f_A$ of the $n$-torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$. One says that $f_A$ is a hyperbolic toral automorphism.

Arnold's cat is the hyperbolic toral automorphism associated with the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Every Anosov diffeomorphism of $T^n$ is topologically conjugate to a hyperbolic toral automorphism. In particular, every Anosov diffeomorphism of $T^n$ is topologically mixing.

Theorem (CC-2015a) Let $f$ be an Anosov diffeomorphism of the $n$-torus $T^n$. Then the d.s. $(T^n, f)$ has the Moore-Myhill property.
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Let \( f: M \rightarrow M \) be a diffeomorphism of a smooth compact manifold \( M \).

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References

[Bar-2010] L. Bartholdi, Gardens of Eden and amenability on cellular automata, J. Eur. Math. Soc. 12 (2010), no. 1, 241–248.

[CC-2012] T. Ceccherini-Silberstein, M. Coornaert, The Myhill property for strongly irreducible subshifts over amenable groups, Monatsh. Math. 165 (2012), 155–172.

[CC-2015a] T. Ceccherini-Silberstein, M. Coornaert, A Garden of Eden theorem for Anosov diffeomorphisms on tori, arXiv:1506.06945.

[CC-2015b] T. Ceccherini-Silberstein, M. Coornaert, Expansive actions of countable amenable groups, homoclinic pairs, and the Myhill property, arXiv:1508.07553.

[CMS-1999] T. Ceccherini-Silberstein, A. Mach, F. Scarabotti, Amenable groups and cellular automata, Ann. Inst. Fourier 49 (1999), 673–685.

[Fio-2000] F. Fiorenzi, The Garden of Eden theorem for sofic shifts, Pure Math. Appl. 11 (2000), no. 3, 471–484.

[Fio-2003] F. Fiorenzi, Cellular automata and strongly irreducible shifts of finite type, Theoret. Comput. Sci. 299 (2003), 477–493.

[Gro-1999] M. Gromov, Endomorphisms of symbolic algebraic varieties, J. Eur. Math. Soc. (JEMS) 1 (1999), 109–197.

[Moo-1963] E. F. Moore, Machine models of self-reproduction, Proc. Sympos. Appl. Math., Vol. 14, pp. 17-34, Amer. Math. Soc., Providence, R. I., 1963.

[Myh-1963] J. Myhill, The converse of Moore's Garden-of-Eden theorem, Proc. Amer. Math. Soc. 14 (1963), 685–686.
References

[Bar-2010] L. Bartholdi, *Gardens of Eden and amenability on cellular automata*, J. Eur. Math. Soc. **12** (2010), no. 1, 241–248.

[CC-2012] T. Ceccherini-Silberstein, M. Coornaert, *The Myhill property for strongly irreducible subshifts over amenable groups*, Monatsh. Math. **165** (2012), 155–172.

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