ANALYSIS OF POSITIVE SOLUTIONS FOR A CLASS OF SEMIPOSITONE $p$-LAPLACIAN PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We study positive solutions to (singular) boundary value problems of the form:

\[ \begin{cases} 
- (\varphi_p(u'))' = \lambda h(t) \frac{f(u)}{u^\alpha}, & t \in (0, 1), \\
 u'(1) + c(u(1))u(1) = 0, \\
 u(0) = 0,
\end{cases} \]

where $\varphi_p(u) := |u|^{p-2}u$ with $p > 1$ is the $p$-Laplacian operator of $u$, $\lambda > 0$, $0 < \alpha < 1$, $c : [0, \infty) \to (0, \infty)$ is continuous and integrable, and $h : (0, 1) \to (0, \infty)$ is continuous and integrable. We assume that $f \in C[0, \infty)$ is such that $f(0) < 0$, $\lim s \to \infty f(s) = \infty$ and $\frac{f(s)}{s^{p-1}}$ has a $p$-sublinear growth at infinity, namely, $\lim s \to \infty \frac{f(s)}{s^{p-1}} = 0$. We will discuss nonexistence results for $\lambda \approx 0$, and existence and uniqueness results for $\lambda \gg 1$. We establish the existence result by a method of sub-supersolutions and the uniqueness result by establishing growth estimates for solutions.

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1. Introduction. We study positive solutions to (singular) boundary value problems of the form:

\[
\begin{cases}
   - (\varphi_p(u'))' = \lambda h(t) \frac{f(u)}{u^\alpha}, & t \in (0, 1), \\
   u'(1) + c(u(1))u(1) = 0, \\
   u(0) = 0,
\end{cases}
\]

where \( \varphi_p(u) := |u|^{p-2}u \) with \( p > 1 \) is the \( p \)-Laplacian operator of \( u \), \( \lambda > 0 \), \( 0 < \alpha < 1 \), \( c : [0, \infty) \to (0, \infty) \) is continuous and \( h : (0, 1) \to (0, \infty) \) is continuous and integrable. We assume that \( f \in C[0, \infty) \) is such that \( f(0) < 0 \), \( \lim_{s \to \infty} f(s) = \infty \) and \( \frac{f(s)}{s^p} \) has a \( p \)-sublinear growth at infinity, namely, \( \lim_{s \to \infty} \frac{f(s)}{s^p} = 0 \).

In the case when \( f(0) < 0 \) and \( \alpha = 0 \), (1) is referred to as a semipositone problem. In the case when \( f(0) < 0 \) and \( \alpha \neq 0 \), (1) is referred to as an infinite semipositone problem. The study of positive solutions to these problems is very challenging since ranges of positive solutions must include regions where \( f \) is negative as well as where \( f \) is positive.

The boundary value problem (1) arises in the study for radially symmetric steady states of reaction diffusion equations of the form:

\[
\begin{cases}
   - \Delta_p u = \lambda K(|x|) \frac{f(u)}{u^\alpha}, & \text{in } \Omega, \\
   \frac{\partial u}{\partial \eta} + c(u)u = 0, & |x| = r_0, \\
   u(x) \to 0, & |x| \to \infty,
\end{cases}
\]

where \( \Delta_p := \text{div}(\nabla |\nabla u|^{p-2}) \), \( 1 < p < N \), \( \Omega = \{x \in \mathbb{R}^N \mid |x| > r_0 > 0\} \) and \( \frac{\partial u}{\partial \eta} \) is the outward normal derivative of \( u \) on \( |x| = r_0 \). Here for the case when \( c(u) \) is a positive constant (Robin boundary condition case) there is a rich history of results (see [1], [5] and [11]), while this is not the case when \( c(u) \) is not a constant. The case when \( c(u) \) is not a constant occurs naturally in various applications, see [6], [16] where they discuss models arising in chemical reaction theory, and see [3], [4] and [7] where they discuss models arising in population dynamics. In particular, in population dynamics, the case where \( c(u) \) is not a constant occurs when species exhibit strong density dependent behavior at habitat boundaries. Restricting the analysis to positive radial solutions, by a Kelvin type transformation, namely the change of variable \( r = |x| \) and \( t = \left( \frac{x}{r_0} \right)^{\frac{N-p}{p}} \), (2) reduces to analyzing the two point boundary value problem (1).

In [13], the author studied (1) when the solution satisfies Dirichlet boundary conditions both at \( t = 0 \) and at \( t = 1 \). Here we consider positive solutions \( u \) such that \( u \in \mathcal{C}[0, 1] \cap \mathcal{C}^1(0, 1) \) and \( u(t) > 0 \) for \( t \in (0, 1) \). However, for such a \( u \), \( u'(t) \) is strictly increasing for \( t \approx 0 \) since \( f(0) < 0 \). Thus \( u'(0) := \lim_{t \to 0^+} u'(t) \) is well-defined and finite, so \( u \in \mathcal{C}^1(0, 1) \). Hence, in this paper, we will study positive solutions \( u \) of (1) such that \( u \in \mathcal{C}^1[0, 1] \) and \( u(t) > 0 \) for \( t \in (0, 1) \).

We first establish the following nonexistence result:

**Theorem 1.1.** There exists no positive solution of (1) for \( \lambda \approx 0 \).
Finally, we assume:

\(H_4\) \(h\) is a strictly decreasing \(C^1\) function on \((0,1)\) and \(h := \inf_{t \in (0,1)} h(t) > 0\),
and establish the following existence result:

**Theorem 1.2.** Let \((H_1) - (H_4)\) hold. Then (1) has a positive solution \(u\) for \(\lambda \gg 1\) such that \(\|u\|_\infty \to \infty\) as \(\lambda \to \infty\). In fact, \(\inf_{t \in [a_0,1]} u(t) \to \infty\) as \(\lambda \to \infty\) for any given \(a_0 \in (0,1)\).

Finally, we assume:

\((H_5)\) \(f \in C^1(0,\infty)\) such that \(\limsup_{s \to 0+} s f'(s) < \infty\),
\((H_6)\) there exist \(q \in (0,p - 1)\) and \(b_0 > 0\) such that \(\frac{f(s)}{s^q}\) is nonincreasing for \(s \in [0,b_0,\infty)\),
\((H_7)\) there exists \(c_0 > 0\) such that \(c(s)\) is nondecreasing for \(s > c_0\),

and establish the following uniqueness result:

**Theorem 1.3.** Let \(\alpha = 0\) and \((H_1) - (H_7)\) hold. Then (1) has a unique positive solution for \(\lambda \gg 1\).

When \(p = 2\) and \(\alpha = 0\), the authors in [2] established the nonexistence of a positive solution for \(\lambda = 0\) and the existence result for \(\lambda \gg 1\). In [10], the authors extended these results to the case when \(p = 2\) and \(\alpha \neq 0\). Theorems 1.1 - 1.2 are extensions of these results in [2] and [10] to the case \(p > 1\). Further, in [9], the uniqueness result for \(\lambda \gg 1\) was established when \(p = 2\) and \(\alpha = 0\). Theorem 1.3 is the extension of this uniqueness result to the case when \(p > 1\) and \(\alpha = 0\). These extensions to the case \(p \neq 2\) are nontrivial and very challenging due to the presence of the nonlinear \(p\)-Laplacian operator. Further, we do not require the concavity assumption on \(f\) as in [9], instead, we use the weaker condition \((H_6)\).

We establish the existence result by introducing a method of sub-supersolutions for (1). In particular, we impose additional assumptions on subsolutions, namely, by a subsolution of (1), we mean a function \(\psi \in C^1[0,1]\) that satisfies \(\psi(t) \geq D d(t, \partial [0,1]) \kappa^*\) for some \(D > 0\) and \(\kappa^* > 0\) such that \(\alpha \kappa^* + \eta < 1\), and

\[
\begin{aligned}
- (\varphi_p(\psi'))' &\leq \lambda h(t) \frac{f(\psi)}{\psi^\alpha}, \ t \in (0,1), \\
\psi'(1) &+ c(\psi(1)) \psi(1) \leq 0, \\
\psi(0) &= 0.
\end{aligned}
\]

Here \(d(t, \partial S) := \min_{x \in \partial S} |t - x|\). By a supersolution of (1), we mean a function \(\phi \in C^1[0,1]\) that satisfies \(\phi(t) > 0\) for \(t \in (0,1)\) and

\[
\begin{aligned}
- (\varphi_p(\phi'))' &\geq \lambda h(t) \frac{f(\phi)}{\phi^\alpha}, \ t \in (0,1), \\
\phi'(1) &+ c(\phi(1)) \phi(1) \geq 0, \\
\phi(0) &= 0.
\end{aligned}
\]

We establish:

**Lemma 1.4.** Let \((H_3)\) hold. Assume that there exist a subsolution \(\psi\) and a supersolution \(\phi\) of (1) such that \(\psi \leq \phi\) on \([0,1]\). Then (1) has at least one solution \(u \in C^1[0,1]\) satisfying \(\psi \leq u \leq \phi\) on \([0,1]\).

To achieve the uniqueness result, we first adapt ideas in [15] to derive useful growth estimations, derivative estimations and that \(u(t) \geq \lambda^{\frac{1}{p-1}} d(t, \partial [0,1])\) for \(\lambda \gg 1\).
Remark 1. A simple example of (1) satisfying our hypotheses is \( f(s) = s^{q^*} - 1, \) \( h(t) = \frac{1}{p^*} \) and \( c(s) = e^s \) where \( 0 < q^* < p - 1 \) and \( 0 < \alpha + \eta < 1. \)

In Section 2, we prove our method of sub-supersolutions (Lemma 1.4). In Section 3, we establish our nonexistence and existence results (Theorems 1.1 - 1.2). In Section 4, we establish the uniqueness result (Theorem 1.3).

2. The method of sub-supersolutions. For a subsolution \( \psi \) and a supersolution \( \phi \) such that \( \psi \leq \phi, \) we define the operator \( T : C[0, 1] \rightarrow C[0, 1] \) related to (1) by

\[
Tw(t) := \int_0^t \varphi_p^{-1} \left( \lambda \int_s^t h(r) \frac{f(\gamma(r, w))}{\gamma(r, w)^{\alpha}} dr - \varphi_p(c(w(1))) \right) ds, \tag{5}
\]

where \( \gamma : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) and \( \overline{c} : \mathbb{R} \rightarrow \mathbb{R} \) are defined by

\[
\gamma(t, s) := \begin{cases} 
\phi(t), & s > \phi(t), \\
\phi(t), & s \leq \phi(t), \quad \text{and} \\
\psi(t), & s < \psi(t),
\end{cases}
\]

and \( \overline{c}(t, s) := \begin{cases} 
c(\phi(1))\phi(1), & s > \phi(1), \\
c(s), & s \leq \phi(1), \quad \text{and} \\
c(\psi(1))\psi(1), & s < \psi(1),
\end{cases} \)

It follows that \( T \) satisfies the following properties:

Lemma 2.1. Let \((H_3)\) hold. Then \( T \) is completely continuous.

Proof. Let \( \{v_n\} \) be a bounded sequence in \( C[0, 1]. \) Let \( f^*(s) := \max_{0 \leq s \leq s} |f(r)| \) and \( c^*(s) := \max_{0 \leq s \leq s} c(r). \) Then we have

\[
|(Tv_n)'(t)| \leq \varphi_p^{-1} \left( \lambda \int_0^1 h(s) \frac{|f(\gamma(s, v_n))|}{\gamma(s, v_n)^{\alpha}} ds + \varphi_p(\overline{c}(v_n(1))) \right)
\]

\[
\leq \varphi_p^{-1} \left( \lambda f^*(\|\phi\|_{\infty}) \int_0^1 h(s) \frac{1}{\gamma^\alpha} ds + \varphi_p(c^*(\|\phi\|_{\infty})) \right)
\]

\[
\leq \varphi_p^{-1} \left( \lambda M_1 f^*(\|\phi\|_{\infty}) + \varphi_p(c^*(\|\phi\|_{\infty})) \right),
\]

where \( M_1 := \frac{1}{\|\phi\|_{\infty}} \left( \frac{1}{2} \int_0^1 h(s) \frac{1}{\gamma^\alpha} ds + \int_1^\infty \frac{h(s)}{1-\gamma^\alpha} ds \right) \) and \( \|\phi\|_{\infty} := \max_{t \in [0, 1]} |\phi(t)|. \) This implies that \( \|Tv_n\|_{\infty} \) is uniformly bounded, and hence \( \{Tv_n\} \) is uniformly bounded. By the Arzela-Ascoli Theorem, \( \{T(v_n)\} \) has a convergent subsequence in \( C[0, 1]. \)

Next we show that \( T \) is continuous. Let \( \{w_n\} \subset C[0, 1] \) be such that \( w_n \rightarrow w \) as \( n \rightarrow \infty \) for some \( w \in C[0, 1]. \) Since \( \overline{c} \) is continuous, \( \varphi_p(\overline{c}(w_n(1))) \) converges to \( \varphi_p(\overline{c}(w(1))) \) as \( n \rightarrow \infty. \) We also have

\[
\int_0^1 h(s) \left| \frac{f(\gamma(s, w_n))}{\gamma(s, w_n)^{\alpha}} - \frac{f(\gamma(s, w))}{\gamma(s, w)^{\alpha}} \right| ds
\]

\[
\leq \int_0^1 h(s) \left| \frac{f(\gamma(s, w_n)) - f(\gamma(s, w))}{\gamma(s, w_n)^{\alpha}} \right| ds + \int_0^1 h(s) \left| \frac{f(\gamma(s, w))}{\gamma(s, w_n)^{\alpha}} - \frac{f(\gamma(s, w))}{\gamma(s, w)^{\alpha}} \right| ds
\]

\[
\leq M_1 \|f(\gamma, w_n) - f(\gamma, w)\|_{\infty} + f^*(\|\phi\|_{\infty}) \int_0^1 h(s) \left| \frac{1}{\gamma(s, w_n)^{\alpha}} - \frac{1}{\gamma(s, w)^{\alpha}} \right| ds.
\]

Since the last term converges to 0 as \( n \rightarrow \infty \) by the Lebesgue Dominated Convergence Theorem, \( \int_0^1 h(s) \frac{f(\gamma(s, w_n))}{\gamma(s, w_n)^{\alpha}} ds \) converges uniformly to \( \int_0^1 h(s) \frac{f(\gamma(s, w))}{\gamma(s, w)^{\alpha}} ds \) as \( n \rightarrow \infty. \) For each \( n \) and \( t \in [0, 1], \) we have

\[
\varphi_p(\overline{c}(w_n(1))) \leq \varphi_p(c^*(\|\phi\|_{\infty})) \quad \text{and} \quad \int_0^1 h(s) \frac{f(\gamma(s, w_n))}{\gamma(s, w_n)^{\alpha}} ds \leq M_1 f^*(\|\phi\|_{\infty}).
\]
3. Proof of Lemma 1.4

3.1. Proof of Theorem 1.1. Let $\|\cdot\|_\infty$.

For $t \in [0, 1]$, we also have
\[
\varphi_p(c'(w(1))) \leq \varphi_p(c^*(\|\cdot\|_\infty)) \quad \text{and} \quad \int_t^1 h(s) \frac{f(\gamma(s, w))}{\gamma(s, w)^\alpha} \, ds \leq M_1 f^*(\|\phi\|_\infty).
\]

Let $M_2 := 2(\lambda M_1 f^*(\|\phi\|_\infty) + \varphi_p(c^*(\|\phi\|_\infty)))$. Since $\varphi_p^{-1}$ is uniformly continuous on $[-M_2, M_2]$, we obtain that $T w_n(t)$ converges uniformly to $T w(t)$ as $n \to \infty$. This implies that $\|T w_n - T w\|_\infty \to 0$ as $n \to \infty$, so $T$ is continuous. Hence Lemma 2.1 is proven.

**Proof of Lemma 1.4.** Note that $T$ defined by (5) is bounded on $C[0, 1]$ since
\[
|T w(t)| \leq \int_0^t \varphi_p^{-1} \left( \lambda \int_s^1 h(r) \frac{|f(\gamma(r, w))|}{\gamma(r, w)^\alpha} \, dr + \varphi_p(c(1)) \right) \, ds \leq \varphi_p^{-1}(M_2).
\]

This implies that there exists $M^* \gg 1$ such that $(I - T)(w) \neq 0$ for any $w \in C[0, 1]$ satisfying $\|w\|_\infty = M^*$. Then, by the Homotopy Invariance Theorem, we have
\[
\deg(I - T, B_{M^*}(0), 0) = \deg(I, B_{M^*}(0), 0) = 1,
\]
and there exists $w_0 \in C[0, 1]$ such that $T(w_0) = w_0$. It is easy to show $w_0 \in C^1[0, 1]$.

Now we claim that $w_0(t) \in [\psi(t), \phi(t)]$ for $t \in [0, 1]$. If our claim is true, then $\gamma(t, w_0(t)) = w_0(t)$ for $t \in [0, 1]$. This implies that $w_0$ is a positive solution of (1), and hence Lemma 1.4 is proven. To show $w_0(t) \geq \psi(t)$ for $t \in [0, 1]$, assume that there exists $t_0 \in (0, 1)$ such that $\psi(t_0) > w_0(t_0)$. Then two cases are followed:

(I) there exists $(a, b) \subset (0, 1)$ such that $\psi(t) > w_0(t)$ in $(a, b)$ and $\psi(a) = w_0(a)$ and $\psi(b) = w_0(b)$, or

(II) there exists $a \in (0, 1)$ such that $\psi(t) > w_0(t)$ in $(a, 1]$ and $\psi(a) = w_0(a)$.

For the case (I), there exists $\hat{t} \in (a, b)$ such that $\psi'(\hat{t}) = w_0'(\hat{t})$ and $\psi'(t) > w_0'(t)$ for $t \in (a, \hat{t})$. However, for $t \in (a, \hat{t})$ we have
\[
\varphi_p(\psi'(t)) - \varphi_p(w_0'(t)) \leq \lambda \int_t^{\hat{t}} h(s) \left( \frac{f(\psi)}{\psi^{\alpha}} - \frac{f(\gamma(s, w_0))}{\gamma(s, w_0)^\alpha} \right) \, ds = 0.
\]

Since $\varphi_p$ is increasing, we obtain $\psi'(t) \leq w_0'(t)$ for $t \in (a, \hat{t})$. This is a contradiction. For the case (II), we have $\psi'(1) - w_0'(1) \leq -c(\psi(1))\psi'(1) + c(w_0(1)) = 0$ since $\psi(1) = w_0(1)$. Thus there exists $\hat{t} \in (a, 1]$ such that $\psi'(\hat{t}) = w_0'(\hat{t})$ and $\psi'(t) > w_0'(t)$ for $t \in (a, \hat{t})$. This is a contradiction by the same argument of the case (I). Hence $w_0(t) \geq \psi(t)$ for $t \in [0, 1]$. By a similar argument, we can show that $w_0(t) \leq \phi(t)$ for $t \in [0, 1]$. \qed

3. Proofs of Theorems 1.1 - 1.2.

3.1. Proof of Theorem 1.1. Since $f(0) < 0$ and $\lim_{s \to \infty} \frac{f(s)}{s^{q+1}} = 0$, there exists $K_1 > 0$ such that $\frac{f(s)}{s^\alpha} \leq K_1 s^{p-1}$ for $s \in (0, \infty)$. Assume that $u$ is a positive solution of (1). Then we have
\[
u(t) = \int_t^1 \varphi_p^{-1} \left( \lambda \int_s^1 h(r) \frac{f(u)}{u^{\alpha}} \, dr - \varphi_p(c(u(1))u(1)) \right) \, ds \leq \int_t^1 \varphi_p^{-1} \left( \lambda K_1 \int_s^1 h(r) u(r)^{p-1} \, dr \right) \, ds \leq \varphi_p^{-1} \left( \lambda K_1 H \|u\|_\infty^{-1} \right),
\]
where \( H := \int_1^h h(s)ds \). This implies that \( 1 \leq \varphi_p^{-1}(\lambda K_1 H) \). This is a contradiction for \( \lambda \approx 0 \). Hence there exists no positive solution of (1) for \( \lambda \approx 0 \).

3.2. Proof of Theorem 1.2. We first construct a subsolution of (1). We consider the eigenvalue problem:

\[
\begin{aligned}
-(\varphi_p(\xi'))' &= \lambda \varphi_p(\xi), \quad t \in (0, 1), \\
\xi(0) &= 0 = \xi(1).
\end{aligned}
\]  

(6)

It is well-known that (6) has a spectrum \( 0 < \lambda_1 < \lambda_2 < \lambda_3 \to \infty \) and the principal eigenvalue \( \lambda_1 \) is simple and isolated (see [12]). Let \( \xi \in C^1[0,1] \) be the corresponding eigenfunction satisfying \( ||\xi||_\infty = 1 \) and \( e > 0 \) on (0, 1). Then there exist \( K_2 > 0 \) and \( K_3 > 0 \) such that \( K_2 d(t, \partial[0,1]) \leq \xi(t) \leq K_3 d(t, \partial[0,1]) \). Let \( \kappa = \frac{\mu - \eta}{p - \mu} \). Note that \( \kappa > 1 \). Thus there exist \( m > 0 \), \( e > 0 \) and \( \mu > 0 \) such that \( \lambda_1|\xi(t)|^p - (\kappa - 1)(p - 1)|\xi(t)|^{p-1} \leq -m \) for \( t \in (0, \epsilon) \cup (1 - \epsilon, 1) \) and \( \xi(t) \geq \mu \) in \([\epsilon, 1-\epsilon] \). Choose \( A^* > 0 \) and \( l \in (0, p - 1 + \alpha) \) such that \( f(s) \geq A^* s^l \) for \( s > 1 \) from (H1). Let \( \psi := \lambda \varphi \alpha \) where \( \sigma \in (\frac{1}{p - 1 + \alpha}, \frac{1}{p - 1 + \alpha - 1}) \). Then we have

\[
-(\varphi_p(\psi'))' = \frac{\lambda^{\sigma(p-1)} K^{(p-1)}_p}{e^{\alpha + \eta}} \left( (\lambda e^{\alpha} \kappa^{p-1}) - (\kappa - 1)(p - 1)|e^{\alpha} \kappa^{p-1}|\psi' \right)
= \frac{\lambda^{\sigma(p-1)} K^{(p-1)}_p}{e^{\alpha + \eta}} \left( (\lambda e^{\alpha} \kappa^{p-1} - (\kappa - 1)(p - 1)|e^{\alpha} \kappa^{p-1}|\psi' \right).
\]

(7)

Since \( \sigma > \frac{1}{p - 1 + \alpha} \), we obtain that \( \frac{dK^\alpha }{m} \leq \lambda^{\sigma(p-1)+\alpha} \) for \( \lambda \gg 1 \) where \( f_s := \min_{s \in (0, 1)} f(s) \). For \( t \in (0, \epsilon) \) and \( \lambda \gg 1 \), we have

\[
-(\varphi_p(\psi'))' \leq \frac{-\lambda^{\sigma(p-1)} K^{(p-1)}_p}{e^{\alpha + \eta}} \leq \frac{-\lambda^{\sigma(p-1)} K^{(p-1)}_p}{(K_3 t)^{\eta} e^{\alpha}} \leq \frac{\lambda^{\sigma(p-1)} K^{(p-1)}_p}{t^{\eta} (\lambda e^{\alpha})^\alpha} \leq \lambda h(t) \frac{f_0}{\psi_\sigma}.
\]

By a similar argument for \( t \in (1 - \epsilon, 1) \) and \( \lambda \gg 1 \), we have

\[
-(\varphi_p(\psi'))' \leq \frac{-\lambda^{\sigma(p-1)} K^{(p-1)}_p}{e^{\alpha + \eta}} \leq \frac{-\lambda^{\sigma(p-1)} K^{(p-1)}_p}{(1-t)^{\eta} e^{\alpha}} \leq \frac{\lambda h}{1 - \epsilon} \frac{f_0}{\psi_\sigma}.
\]

Since \( \sigma < \frac{1}{p - 1 + \alpha} \), we have \( \lambda^{\sigma(p-1)} K^{(p-1)}_p \leq \lambda^{1-\sigma(p-1)+\alpha} \) for \( \lambda \gg 1 \). For \( \lambda \gg 1 \), we also have \( A^*(\lambda e^{\alpha} \kappa^{p-1}) \leq f(\kappa e^{\alpha} \kappa^{p-1}) \) for \( t \in [\epsilon, 1-\epsilon] \). Then for \( t \in [\epsilon, 1-\epsilon] \) and \( \lambda \gg 1 \) we have

\[
-(\varphi_p(\psi'))' \leq \frac{\lambda^{\sigma(p-1)} K^{(p-1)}_p}{e^{\alpha}} \leq \frac{\lambda^{\sigma(p-1)} K^{(p-1)}_p}{(1-t)^{\eta} e^{\alpha}} \leq \frac{\lambda h A^* (\lambda e^{\alpha})^\alpha}{1 - \epsilon} \leq \lambda h(t) \frac{f_0}{\psi_\sigma}.
\]

We also have \( f'(1) + c f(1) = 0 \) and \( \lambda > 1 \). Further, \( \psi(t) \geq \lambda^{\sigma(p-1)} K^{(p-1)}_p d(t, \partial[0,1]) \) for \( t \in [0, 1] \) and \( \lambda \) satisfies \( \alpha \kappa \eta < 1 \) since \( \kappa = \frac{\mu - \eta}{p - \mu} \). Thus \( \psi \) is a subsolution of (1) for \( \lambda \gg 1 \).

Next we construct a supersolution of (1). Let \( \tilde{z} \in C^1[0, 1] \) be the solution of the boundary value problem:

\[
\begin{aligned}
-(\varphi_p(\xi'))' &= \frac{h(t)}{p}, \quad t \in (0, 1), \\
\xi(0) &= 0 = \xi(1).
\end{aligned}
\]  

(7)

Then there exists \( K_4 > 0 \) such that \( \tilde{z}(t) \geq K_4 t \) for \( t \in [0, 1] \). Since \( f'(s) = \frac{f'_{s}(s)}{p - \alpha} \to 0 \) as \( s \to \infty \). Thus there
exists \( K_\lambda \gg 1 \) such that \( \psi(t) \leq K_\lambda \tilde{z}(t) \) for \( t \in [0,1] \) and \( f^*(K_\lambda \|\tilde{z}\|_\infty) \leq \frac{K_\lambda^2}{\|\tilde{z}\|_\infty^{p-1}} \).

Let \( \phi := K_\lambda \tilde{z} \). Then we have

\[
-(\varphi_p(\phi')') = \frac{K_\lambda^{p-1} h(t)}{t^\alpha} \geq \lambda h(t) \frac{f^*(K_\lambda \|\tilde{z}\|_\infty)}{(K_\lambda \|\tilde{z}\|_\infty)^\alpha} \geq \lambda h(t) \frac{f(K_\lambda \tilde{z})}{(K_\lambda \|\tilde{z}\|_\infty)^\alpha} = \lambda h(t) \frac{f(\phi)}{\phi^\alpha}.
\]

We also have \( \phi'(1) + c(\phi(1))\phi(1) = 0 \) since \( \tilde{z}(1) = 0 = \tilde{z}'(1) \). Thus \( \phi \) is a supersolution of (1) such that \( \psi(t) \leq \phi(t) \) for \( t \in [0,1] \). By Lemma 1.4, there exists a positive solution \( u \) of (1) such that \( \psi(t) \leq u(t) \) for \( t \in [0,1] \). Further, \( \|u\|_\infty \to \infty \) as \( \lambda \to \infty \) since \( \|\psi\|_\infty \to \infty \) as \( \lambda \to \infty \).

Next we show that \( \inf_{t \in [a_0,1]} u(t) \to \infty \) as \( \lambda \to \infty \) for any constant \( a_0 \in (0,1) \). Let \( t_m \in (0,1) \) be the first point satisfying \( u(t) = 0 \). Let \( F_{\alpha}(s) := \int_0^s f(\varphi)^{\frac{1}{p}} \varphi \). By \( (H_1) - (H_2) \), there exist unique constants \( \beta \) and \( \theta \) such that \( 0 < \beta < \theta \), \( f(\beta) = 0 \) and \( F_{\alpha}(\beta) = 0 \). We first recall the results in Lemmas 2.1 - 2.3 from [15] which we restate as Lemmas 3.1 - 3.2 below:

**Lemma 3.1 ([15]).** Let \( (H_2) - (H_4) \) hold. If \( u \) is a positive solution of (1), then \( u \) has a unique interior maximum at \( t_m \) and \( u(t_m) > \theta \).

**Lemma 3.2 ([15]).** Let \( (H_2) - (H_4) \) hold. Let \( u \) be a positive solution of (1). If \( t_\beta \) and \( t_{\frac{1}{\alpha}+\theta} \) are the first points in \( (0,1) \) such that \( u(t_\beta) = \beta \) and \( u(t_{\frac{1}{\alpha}+\theta}) = \frac{\beta+\theta}{2} \), then \( t_\beta \leq O(\lambda^{-\frac{1}{p}}) \) and \( t_{\frac{1}{\alpha}+\theta} \leq O(\lambda^{-\frac{1}{p}}) \).

Next we establish the following property of \( u \) at \( t = 1 \) when \( \lambda \gg 1 \):

**Lemma 3.3.** Let \( (H_2)-(H_4) \) hold. If \( u \) is a positive solution of (1), then \( u(1) \to \infty \) as \( \lambda \to \infty \).

**Proof.** We first show that \( u(1) \geq \frac{\beta+\theta}{2} \) for \( \lambda \gg 1 \). Assume \( u(1) < \frac{\beta+\theta}{2} \). By Lemma 3.1, there exists \( \tilde{t}_\theta \in (t_m,1) \) such that \( u(\tilde{t}_\theta) = \theta \) and \( u'(\tilde{t}_\theta) < 0 \). Define \( E(t) := \lambda F_{\alpha}(u(t))h(t) + \frac{p-1}{p}|u'(t)|^p \). Then \( E(\tilde{t}_\theta) = \frac{p-1}{p}|u'(\tilde{t}_\theta)|^p > 0 \). Thus we obtain that \( E(1) > 0 \) since \( E(t) \) is strictly increasing for \( t \in (\tilde{t}_\theta,1) \). This implies that \( u(1) > 0 \). Since \( E(1) = \lambda F_{\alpha}(u(1))h(1) + \frac{p-1}{p}|u'(1)|^p \), we have

\[
c(u(1))u(1) = -u'(1) > \left(-\lambda \frac{p}{p-1} F_{\alpha}(u(1))h(1)\right)^{\frac{1}{p}}.
\]

This implies that \( u(1) \approx 0 \) for \( \lambda \gg 1 \) since \( u(1) < \frac{\beta+\theta}{2} \). Thus \( \frac{F_{\alpha}(u(1))}{u(1)} \approx -\infty \) for \( \lambda \gg 1 \), and hence \( \left(-\lambda \frac{p}{p-1} F_{\alpha}(u(1))h(1)\right)^{\frac{1}{p}} \gg 1 \) for \( \lambda \gg 1 \). Therefore \( c(u(1))u(1)^{\frac{p-1}{p}} \gg 1 \) for \( \lambda \gg 1 \). This is a contradiction since \( u(1) \approx 0 \) for \( \lambda \gg 1 \). Hence \( u(1) \geq \frac{\beta+\theta}{2} \) for \( \lambda \gg 1 \).
Next, let \( a_0 \in (0, 1) \) be any constant, that is independent of \( \lambda \). First we assume \( t_m > \frac{a_0 + 1}{2} \). By Lemma 3.2, for \( \lambda \gg 1 \) we have

\[
\begin{align*}
    u(a_0) &= \beta + \int_{t \beta}^{a_0} \varphi^{-1}_p \left( \lambda \int_{\frac{s}{a_0}}^{t_m} h(r) \frac{f(u)}{u^\alpha} \, dr \right) \, ds \\
    &\geq \int_{\frac{s}{a_0}}^{a_0} \varphi^{-1}_p \left( \lambda \int_{\frac{s}{a_0}}^{a_0} h(r) \frac{f(u)}{u^\alpha} \, dr \right) \, ds \\
    &\geq \int_{\frac{s}{a_0}}^{a_0} \varphi^{-1}_p \left( \lambda \frac{1 - a_0}{2} \frac{h}{\|u\|_{\infty}} f \left( \frac{\beta + \theta}{2} \right) \right) \, ds \\
    &\geq \frac{\lambda^{p-1}}{\|u\|_{\infty}^{\frac{p-1+a}{2}}} K_5,
\end{align*}
\] 

where \( K_5 := \frac{a_0}{2} \varphi^{-1}_p \left( \frac{1-a_0}{2} h f \left( \frac{\beta + \theta}{2} \right) \right) \). Thus \( \|u\|_{\infty}^{\frac{p-1+a}{2}} \geq \lambda^{p-1} K_5 \). By the Mean Value Theorem, we have

\[
u(t_m) - u(1) = -u'(\tilde{t})(1-t_m) \leq -u'(1) = c(u(1))u(1),
\]

where \( \tilde{t} \in (t_m, 1) \). Thus we have \((c(u(1)) + 1)u(1) \geq \|u\|_{\infty} \geq \lambda^{\frac{1}{p-1+a}} K_5^{\frac{p-1}{p-1+a}} \) if \( t_m > \frac{a_0 + 1}{2} \). Next we assume \( t_m \leq \frac{a_0 + 1}{2} \). Since \( u(1) \geq \frac{\beta + \theta}{2} \), we have

\[
c(u(1))u(1) = -u'(1) = \varphi^{-1}_p \left( \lambda \int_{t_m}^{1} h(s) \frac{f(u)}{u^\alpha} \, ds \right) \geq \lambda^{\frac{1}{p-1}} \frac{K_6}{\|u\|_{\infty}^{\frac{p-1+a}{2}}},
\]

where \( K_6 := \varphi^{-1}_p \left( \frac{1-a_0}{2} h f \left( \frac{\beta + \theta}{2} \right) \right) \). From (8), we have

\[
\lambda^{\frac{1}{p-1}} K_6 \leq c(u(1))u(1) \|u\|_{\infty}^{\frac{a}{2}} \leq (c(u(1)) + 1)u(1) \|u\|_{\infty}^{\frac{p-1+a}{2}}.
\]

Thus we obtain \((c(u(1)) + 1)u(1) \geq \lambda^{\frac{1}{p-1+a}} K_6^{\frac{p-1}{p-1+a}} \) if \( t_m \leq \frac{a_0 + 1}{2} \). Hence \((c(u(1)) + 1)u(1) \geq \lambda^{\frac{1}{p-1+a}} K^\frac{p-1}{p-1+a} \) where \( K^7 := \min\{K_5, K_6\} \). Thus \( u(1) \to \infty \) as \( \lambda \to \infty \).

Now we show that \( \inf_{\varepsilon \in [a_0, 1]} u(t) \to \infty \) as \( \lambda \to \infty \). Let \( \lambda \gg 1 \) be such that \( t_\beta < \frac{a_0}{2} \) and \( t_{a_0} < \frac{a_0}{2} \). If \( t_m > a_0 \), then we have

\[
\begin{align*}
    u \left( \frac{a_0}{2} \right) &= \beta + \int_{t \beta}^{a_0} \varphi^{-1}_p \left( \lambda \int_{\frac{s}{a_0}}^{t_m} h(r) \frac{f(u)}{u^\alpha} \, dr \right) \, ds \\
    &\geq \int_{\frac{s}{a_0}}^{a_0} \varphi^{-1}_p \left( \lambda \int_{\frac{s}{a_0}}^{a_0} h(r) \frac{f(u)}{u^\alpha} \, dr \right) \, ds \\
    &\geq \lambda^{\frac{1}{p-1}} K_8 \frac{K_8}{u(a_0)^{\frac{p-1+a}{2}}},
\end{align*}
\]

where \( K_8 := \frac{a_0}{2} \varphi^{-1}_p \left( \frac{a_0}{2} h f \left( \frac{\beta + \theta}{2} \right) \right) \). Since \( u \) is increasing on \([\frac{a_0}{2}, a_0]\), we have \( u(a_0) \geq u \left( \frac{a_0}{2} \right) \geq \lambda^{\frac{1}{p-1}} K_8 \frac{K_8}{u(a_0)^{\frac{p-1+a}{2}}} \). Thus \( u(a_0) \geq \lambda^{\frac{1}{p-1+a}} K_8^{\frac{p-1}{p-1+a}} \). If \( t_m \leq a_0 \), then \( u(a_0) \geq u(1) \) since \( u \) is decreasing on \([a_0, 1]\). Therefore \( u(a_0) \geq \min \{ \lambda^{\frac{1}{p-1+a}} K_8^{\frac{p-1+a}{p-1+a}}, u(1) \} \). Since \( u \) is concave on \([a_0, 1]\), we obtain that \( \inf_{\varepsilon \in [a_0, 1]} u(t) \geq \min \{ u(a_0), u(1) \} \geq \min \{ \lambda^{\frac{1}{p-1+a}} K_8^{\frac{p-1+a}{p-1+a}}, u(1) \} \). Hence \( \inf_{\varepsilon \in [a_0, 1]} u(t) \to \infty \) as \( \lambda \to \infty \). \(\)
4. Proof of Theorem 1.3. Here $\alpha = 0$. Without loss of generality, we assume $b_0 > \frac{4+6}{2}$ in $(H_0)$. Define $G : [b_0, \infty) \to \mathbb{R}$ by $G(s) := \frac{s^4}{(f(s))^{\frac{4}{p-1}}}$. Then $G$ is strictly increasing and $\lim_{s \to \infty} G(s) = \infty$ since $(H_0)$ and $\lim_{s \to \infty} \frac{f(s)}{s^{p-1}} = 0$. Further, $G^{-1}$ satisfies:

Lemma 4.1 ([8]). Let $(H_0)$ hold. For each $C > 0$, there exist positive constants $L_1$, $L_2$ (independent of $\lambda$) and $\tilde{\lambda} > 0$ such that

$$L_1 G^{-1}(\lambda^\frac{1}{p-1}) \leq G^{-1}(\lambda^\frac{1}{p-1} C) \leq L_2 G^{-1}(\lambda^\frac{1}{p-1})$$

for $\lambda > \tilde{\lambda}$ where $L_1 := \min \left\{ 1, C \right\}$, $L_2 := \max \left\{ 1, C \right\}$ and $\tilde{\lambda} := \left( \frac{G(b_0)}{\min(1, C)} \right)^{\frac{1}{p-1}}$.

Next we establish the following result by arguments similar to Lemma 2.8 in [15].

Lemma 4.2. Let $(H_1)$ – $(H_4)$ and $(H_6)$ hold. For $\lambda \gg 1$ there exist positive constants $C_1$ and $C_2$ (independent of $\lambda$) such that if $u$ is a positive solution of (1) then

$$C_1 G^{-1}(\lambda^\frac{1}{p-1}) d(t, \partial[0,1]) \leq u(t) \leq C_2 G^{-1}(\lambda^\frac{1}{p-1}) d(t, \partial[0,1])$$

Proof. First we show that for $\lambda \gg 1$ there exists $C_1 > 0$ such that

$$u(t) \geq C_1 G^{-1}(\lambda^\frac{1}{p-1}) d(t, \partial[0,1])$$

Consider the boundary value problem:

$$\begin{cases} - (\phi_p(z'(t)))' = h(t) (f(M) \chi_{[\frac{1}{4}, \frac{3}{4}]}(t) + f(0) \chi_{[\frac{3}{4}, 1)}(t)), & t \in (0, 1), \\ z(0) = 0 = z(1), \end{cases} \tag{10}$$

where $\chi_{S}(t) = 1$ for $t \in S$ and $\chi_{S}(t) = 0$ for $t \in [0,1] \setminus S$. Choosing $M_\lambda \gg 1$, (10) has a unique positive solution $z$ such that $z(t) \geq d(t, \partial[0,1])$ (see [15]). Since $\inf_{t \in [a, b]} u(t) \to \infty$ as $\lambda \to \infty$, for $\lambda \gg 1$ we have $- (\phi_p(u') - \lambda \phi_p(z'))' = \lambda h(t) f(u) - \lambda h(t) (f(M) \chi_{[\frac{1}{4}, \frac{3}{4}]}(t) + f(0) \chi_{[\frac{3}{4}, 1)}(t))(t)) \geq 0$. Then $u(t) \geq \lambda^\frac{1}{p-1} d(t, \partial[0,1])$ for $\lambda \gg 1$ by the comparison principle. Let $M_\lambda : (\geq 1)$ be the largest constant such that $u(t) \geq \lambda^\frac{1}{p-1} M_\lambda d(t, \partial[0,1])$. Then $u(t) \geq M_\lambda := \frac{\lambda^\frac{1}{p-1} M_\lambda}{4}$ on $[\frac{1}{4}, \frac{3}{4}]$. For $t \in [\frac{1}{4}, \frac{3}{4}]$ and $\lambda \gg 1$, we have

$$- f(M) (\phi_p(u'))' + \lambda f(M_\lambda) (\phi_p(z'))' = \lambda f(M) h(t) f(u) - \lambda f(M_\lambda) h(t) (f(M) \chi_{[\frac{1}{4}, \frac{3}{4}]} + f(0) \chi_{[\frac{3}{4}, 1)}(t)) \geq 0.$$ 

For $t \in [\frac{1}{4}, \frac{3}{4}]$ and $\lambda \gg 1$, we have

$$- f(M) (\phi_p(u'))' + \lambda f(M_\lambda) (\phi_p(z'))' = \lambda f(M) h(t) f(u) - \lambda f(M_\lambda) h(t) (f(M) \chi_{[\frac{1}{4}, \frac{3}{4}]}(t) + f(0) \chi_{[\frac{3}{4}, 1)}(t)) \geq 0.$$ 

Then $[f(M)]^\frac{1}{p-1} u(t) \geq \lambda^\frac{1}{p-1} [f(M_\lambda)]^\frac{1}{p-1} z(t)$ for $t \in [0,1]$ and $\lambda \gg 1$ by the comparison principle. Hence we have

$$u(t) \geq \lambda^\frac{1}{p-1} \frac{[f(M_\lambda)]^\frac{1}{p-1}}{[f(M)]^\frac{1}{p-1}} z(t).$$
This implies that $M_{\lambda} \geq \frac{[f(M)]^\frac{1}{p-1}}{[f(M)]^\frac{1}{p-1}}$ for $\lambda \gg 1$. Thus we obtain

$$
\frac{\lambda^\frac{1}{p-1} M_{\lambda}}{4} = G^{-1} \left( \frac{M_{\lambda}}{[f(M)]^\frac{1}{p-1}} \right) \geq G^{-1} \left( \frac{\lambda^\frac{1}{p-1}}{4[f(M)]^\frac{1}{p-1}} \right).
$$

By Lemma 4.1, for $\lambda \gg 1$ there exists $C_1 > 0$ such that

$$
4G^{-1} \left( \frac{\lambda^\frac{1}{p-1}}{4[f(M)]^\frac{1}{p-1}} \right) \geq C_1 G^{-1}(\lambda^\frac{1}{p-1}).
$$

Hence, for $\lambda \gg 1$ we have

$$
u \geq \lambda^\frac{1}{p-1} M_{\lambda} z \geq C_1 G^{-1}(\lambda^\frac{1}{p-1}) z \geq C_1 G^{-1}(\lambda^\frac{1}{p-1}) d(t, \partial[0, 1]).$$

Next we show that there exists $C_2 > 0$ such that $u(t) \leq C_2 G^{-1}(\lambda^\frac{1}{p-1}) t$ for $t \in [0, 1]$ and $\lambda \gg 1$. For $\lambda > \frac{2b_0}{\lambda^\frac{1}{p-1}}$, we have $\|u\|_\infty \geq b_0$. Hence $b_0 \leq \|u\|_1 \leq 2\lambda^\frac{1}{p-1} [f(\|u\|_1)]^\frac{1}{p-1} \varphi_p^{-1}(H)$. This implies that

$$
\|u\|_\infty \leq G^{-1} \left( \frac{\|u\|_1}{[f(\|u\|_1)]^\frac{1}{p-1}} \right) \leq G^{-1} \left( 2\lambda^\frac{1}{p-1} \varphi_p^{-1}(H) \right),
$$

where $\|u\|_1 := \|u\|_\infty + \|u\|_\infty$. By Lemma 4.1, for $\lambda \gg 1$ there exists $C_2 > 0$ such that

$$
G^{-1} \left( 2\lambda^\frac{1}{p-1} \varphi_p^{-1}(H) \right) \leq C_2 G^{-1}(\lambda^\frac{1}{p-1}).
$$

Thus we have $u(t) \leq \|u\|_\infty t \leq C_2 G^{-1}(\lambda^\frac{1}{p-1}) t$ for $\lambda \gg 1$. Hence the proof is complete.

Now we recall Lemma 2.9 in [15] which we restate as Lemma 4.3 below. We also provide a proof since there was an error in the arguments in [15].

**Lemma 4.3 ([15]).** Let $(H_1)-(H_4)$ and $(H_6)$ hold. Let $\gamma_0 \leq \gamma < 1$ where $\gamma_0 := \frac{C_1}{C_2}$. For $\lambda \gg 1$ there exists $\delta > 0$ (independent of $\lambda$) such that if $u$ and $v$ are positive solutions of (1) then $\frac{C_1 \gamma_0 G^{-1}(\lambda^\frac{1}{p-1})}{2} \leq |sv'(t) + (1 - s)\gamma u'(t)| \leq C_2 G^{-1}(\lambda^\frac{1}{p-1})$ for $s \in [0, 1]$ and $t \in [0, \delta]$.

**Proof.** Let $s \in [0, 1]$ and $y(t) := sv'(t) + (1 - s)\gamma u'(t)$. By (11) and (12), we have $|y(t)| \leq s\|u\|_\infty + (1 - s)\|u\|_\infty \leq C_2 G^{-1}(\lambda^\frac{1}{p-1})$.

Next we show that $|y(t)| \geq \frac{C_1 \gamma_0 G^{-1}(\lambda^\frac{1}{p-1})}{2}$ for $t \in [0, \delta]$. Clearly, $|y(0)| \geq C_1 \gamma_0 G^{-1}(\lambda^\frac{1}{p-1})$ by Lemma 4.2. We note that $u'$ is differentiable on $(0, 1)$ for the case $1 < p \leq 2$. By the Mean Value Theorem, we have

$$
|u'(t_2) - u'(t_1)| = \left| \varphi_p^{-1} \left( \int_{t_1}^{t_2} h(s)f(u)ds \right) - \varphi_p^{-1} \left( \int_{t_1}^{t_2} h(s)f(u)ds \right) \right| \\
\leq \frac{1}{p-1} \lambda^\frac{1}{p-1} [f(\|u\|_\infty)]^\frac{1}{p-1} \left( \int_{t_1}^{t_2} h(s)ds \right)^{\frac{2-p}{p-1}} \left( \int_{t_1}^{t_2} h(s)ds \right) \quad (13)
$$

\begin{align*}
&\leq N_1 \lambda^\frac{1}{p-1} [f(\|u\|_\infty)]^\frac{1}{p-1} |t_2 - t_1|^{1-\eta} - |t_1 - \eta| \\
&\leq N_1 \lambda^\frac{1}{p-1} [f(\|u\|_\infty)]^\frac{1}{p-1} |t_2 - t_1|^{1-\eta}
\end{align*}
for any $t_1$ and $t_2 \in [0, 1]$ where $f^*(s) := \max_{0 \leq r \leq s} |f(r)|$ and $N_1 := \frac{d\eta}{(p-1)(1-\eta)}$. Hence $u \in C^2(0,1) \cap C^{1,\alpha^*}[0,1]$ where $\alpha^* = 1 - \eta$. For the case $p > 2$, noting that 
\[|\phi_p^{-1}(b) - \phi_p^{-1}(a)| \leq 2^{\frac{p-2}{p}}|\phi_p^{-1}(b-a)| \quad \text{for } a, b \in \mathbb{R}, \]
we have
\[|u'(t_2) - u'(t_1)| = |\phi_p^{-1}\left( \lambda \int_{t_1}^{t_2} h(s)f(u)ds \right) - \phi_p^{-1}\left( \lambda \int_{t_1}^{t_2} h(s)f(u)ds \right) | \leq 2^{\frac{p-2}{p}} \lambda \int_{t_1}^{t_2} h(s)f(u)ds \geq (14)
\]
for any $t_1$ and $t_2 \in [0, 1]$ where $N_2 := \left( \frac{d^{p-2}}{1-\eta} \right)^{\frac{1}{p}}$. This implies that $u \in C^{1,\alpha^*}[0,1]$ with $\alpha^* = \frac{1-\eta}{p-1}$. Hence for $p > 1$, any positive solution $u$ of (1) belongs to $C^{1,\alpha^*}[0,1]$ for some $\alpha^* \in (0,1)$. Indeed, noting that $f^*(s) = f(s)$ for $s \gg 1$, it follows from (13) and (14) that we have $\|u\|_{1,\alpha^*} \leq \lambda \frac{1}{\beta} \|f\|_{1,\alpha^*} M_p$ for some $M_p > 0$ when $\lambda \gg 1$ where $\|u\|_{1,\alpha^*} := \|u\|_1 + \|u'\|_{\alpha^*}$ and $\|u\|_{\alpha^*} := \sup_{a \neq b \in (0,1)} \frac{|u(b) - u(a)|}{|b-a|^{\alpha^*}}$. Thus we have
\[\|u\|_{1,\alpha^*} = G^{-1}\left( \frac{\|u\|_{1,\alpha^*}}{f(\|u\|_{1,\alpha^*})} \right) \leq G^{-1}\left( \lambda^{\frac{1}{\alpha^*}} M_p \right).
\]
Without loss of generality, we obtain $\|u\|_{1,\alpha^*} \leq C_2 G^{-1}(\lambda^{\frac{1}{\alpha^*}})$ by Lemma 4.1. Hence $y \in C^\alpha[0,1]$ and $\|y\|_{\alpha^*} \leq C_2 G^{-1}(\lambda^{\frac{1}{\alpha^*}})$. Let $\delta > 0$ be such that $\delta^{\alpha^*} \leq \frac{C_1\gamma_0}{2C_2}$. Since $y \in C^\alpha[0,1]$, we have
\[|y(t) - y(0)| \leq C_2 G^{-1}(\lambda^{\frac{1}{\alpha^*}}) \|t\|^{\alpha^*} \leq C_2 G^{-1}(\lambda^{\frac{1}{\alpha^*}}) \delta^{\alpha^*}
\]
for $t \in [0,\delta]$. This implies that
\[|y(t)| \geq |y(0)| - |y(t) - y(0)| \geq \frac{C_1\gamma_0}{2} G^{-1}(\lambda^{\frac{1}{\alpha^*}}).
\]
Hence the proof is complete. 

Next we recall Lemma 2.10 and Lemma 2.12 in [15] which we restate below as Lemmas 4.4 - 4.5. For the convenience to the reader, we also provide the proofs of these lemmas. To state these results, let $u$ and $v$ be positive solutions of (1). We define $a_\lambda: [0, \delta] \rightarrow \mathbb{R}$ by
\[a_\lambda(t) := (p-1) \int_0^1 |s\ddot{u}'(t) + (1-s)\gamma \ddot{u}'(t)|^{p-2} ds,
\]
where $\ddot{u} := \frac{u}{G^{-1}(\lambda^{\frac{1}{\alpha^*}})}$ and $\ddot{v} := \frac{v}{G^{-1}(\lambda^{\frac{1}{\alpha^*}})}$. By Lemma 4.3, $a_\lambda(t) \in [C_\ast, C^\ast^*]$ where $C_\ast := (p-1) \min \left\{ \left( \frac{C_1\gamma_0}{2} \right)^{p-2}, C_2^{-2} \right\}$ and $C^\ast := (p-1) \max \left\{ \left( \frac{C_1\gamma_0}{2} \right)^{p-2}, C_2^{-2} \right\}$.

**Lemma 4.4 ([15]).** Let $\kappa_0$ be the solution of the boundary value problem:
\[-(a_\lambda(t)\kappa_0')(t)' = \begin{cases} 0, & t \in (0, a_\lambda], \\ h(t), & t \in (a_\lambda, \delta), \end{cases}\]

\[\]
\[ \kappa_0(0) = 0 = \kappa_0(\delta), \]

where \( t_\lambda \to 0 \) as \( \lambda \to \infty \). Then there exists a positive constant \( D \) (independent of \( \lambda \)) such that \( \kappa_0(t) \geq Dd(t, \partial[0, \delta]) \) for \( \lambda \gg 1 \).

**Proof.** Let \( t_m \in [0, \delta] \) be such that \( \|\kappa_0\|_\infty = \kappa_0(t_m) \). Then \( t_m \in (t_\lambda, \delta) \) and \( \kappa_0 \) can be written as

\[
\kappa_0(t) = \begin{cases} 
\int_0^t \frac{1}{a_\lambda(s)} \int_s^{t_m} h(r) \chi_{[t_\lambda, t_m]} dr ds, & t \in [0, t_m], \\
\int_t^\delta \frac{1}{a_\lambda(s)} \int_{t_m}^s h(r) dr ds, & t \in (t_m, \delta].
\end{cases}
\tag{15}
\]

By upper and lower estimates of \( \kappa_0(t_m) \), we obtain \( \frac{b}{\kappa_0} (\delta - t_m)^2 \leq \frac{b}{\kappa_0} t_m \) and \( \frac{b}{\kappa_0} s^2 \leq \frac{b}{\kappa_0} (\delta - t_m) \). This implies that there exist \( N_3 \in (0, \frac{\delta}{2}) \) and \( N_4 \in (0, \frac{\delta}{2}) \) (independent of \( \lambda \)) such that \( N_3 \leq t_m \leq \delta - N_4 \). Then for \( t \in (0, \frac{N_3}{2}) \) and \( \lambda \gg 1 \) we obtain

\[
\kappa_0^*(t) = \frac{1}{a_\lambda(t)} \int_t^{t_m} h(s) \chi_{[t_\lambda, t_m]} ds \geq \frac{h_N_3}{2C^*},
\]

and

\[
\kappa_0 \left( \delta - \frac{N_4}{2} \right) \geq \int_{\delta - N_4}^\delta \frac{1}{a_\lambda(s)} \int_{\delta - N_4}^{s - \frac{N_4}{2}} h(r) dr ds \geq \frac{hN_4^2}{4C^*}.
\]

Thus \( \kappa_0(t) \geq D \) for \( t \in \left[ \frac{\delta}{2}, \delta \right] \) and \( \lambda \gg 1 \) where \( D := \frac{h}{4C^*} \min\{N_3^2, N_4^2\} \). Similarly, for \( t \in (\delta - \frac{N_4}{2}, \delta) \) and \( \lambda \gg 1 \) we deduce

\[
\kappa_0^*(t) = -\frac{1}{a_\lambda(t)} \int_{t_m}^{t} h(s) ds \leq -\frac{hN_4}{2C^*},
\]

and

\[
\kappa_0 \left( \frac{N_3}{2} \right) \geq \int_0^{\frac{N_3}{2}} \frac{1}{a_\lambda(s)} \int_{\frac{N_3}{2}}^s h(r) dr ds \geq \frac{hN_3^2}{4C^*}.
\]

Thus \( \kappa_0(t) \geq D (\delta - t) \) for \( t \in \left[ \frac{\delta}{2}, \delta \right] \) and \( \lambda \gg 1 \). Hence we obtain \( \kappa_0(t) \geq Dd(t, \partial[0, \delta]) \) for \( t \in [0, \delta] \) and \( \lambda \gg 1 \).

**Lemma 4.5.** Let \( \kappa_0^* \) be the solution of the boundary value problem:

\[
-\left( a_\lambda(t) \kappa_0^{*'}(t) \right)' = \begin{cases} 
h(t), & t \in (0, t_\lambda], \\
0, & t \in (t_\lambda, \delta),
\end{cases}
\]

\[
\kappa_0^*(0) = 0 = \kappa_0^*(\delta).
\]

Then there exists a positive constant \( D^* \) (independent of \( \lambda \)) such that \( \kappa_0^*(t) \leq D^*t^1-\eta d(t, \partial[0, \delta]) \) for \( \lambda \gg 1 \).

**Proof.** Let \( t_m \in [0, \delta] \) be such that \( \|\kappa_0^*\|_\infty = \kappa_0^*(t_m) \). Then \( t_m \in (0, t_\lambda) \) and \( \kappa^*_0 \) can be written as

\[
\kappa_0^*(t) = \begin{cases} 
\int_0^t \frac{1}{a_\lambda(s)} \int_s^{t_m} h(r) dr ds, & t \in [0, t_m], \\
\int_t^\delta \frac{1}{a_\lambda(s)} \int_{t_m}^s h(r) dr ds, & t \in (t_m, \delta].
\end{cases}
\tag{16}
\]

For \( t \in [0, t_m] \) we have

\[
\kappa_0^*(t) = \int_0^t \frac{1}{a_\lambda(s)} \int_s^{t_m} h(r) dr ds \leq \frac{1}{C^*} \int_0^t \int_s^{t_m} \frac{d}{r^\eta} dr ds \leq \frac{dt^1-\eta t}{C^*(1-\eta)}.
\]
Since $\kappa_0^*$ has a maximum at $t_m$ we have

$$\kappa_0^*(t) \leq D^* t \gamma^{-\eta} for t \in \left[0, \frac{\delta}{2}\right],$$

where $D^* := \frac{d}{C^* (1 - \eta)}$. For $t \in [t_m, \delta]$ we have

$$\kappa_0^*(t) = \int_t^\delta \frac{1}{a_0(s)} \int_{t_m}^s h(r) \chi(t, t_m) dr ds \leq \frac{1}{C^*} \int_t^\delta \int_{t_m}^s \frac{d}{r^n} dr ds \leq \frac{dt^{1-\eta}(\delta - t)}{C^*(1 - \eta)}.$$

This implies

$$\kappa_0^*(t) \leq D^* t^{1-\eta}(\delta - t) for t \in \left[\frac{\delta}{2}, \delta\right].$$

Hence $\kappa_0^*(t) \leq D^* t^{1-\eta}d(t, \partial[0, \delta])$ for $\lambda \gg 1$. \qed

Now we establish the proof of Theorem 1.3. Let $\lambda \gg 1$ satisfying Theorem 1.2, Lemmas 4.1 - 4.3 and $\inf_{t \in (\frac{1}{2}, 1]} u(t) \geq \max\{c_0, k_0\}$ for any positive solution of (1). Let $u$ and $v$ be positive solutions of (1). By Lemma 4.2, we obtain that $v \geq \gamma_0 u$ on $[0, \frac{1}{2}]$ where $\gamma_0 := \frac{c_1}{c_2}$. Let $\gamma = (\gamma_0 \lambda)$ be the largest constant such that $v \geq \gamma u$ on $[0, 1]$.

First we show that $\gamma \geq \gamma_0$ when $\lambda \gg 1$. Assume that $\gamma < \gamma_0$ for $\lambda \gg 1$. Then $v(t) > \gamma u(t)$ for $t \in (0, \frac{1}{2}]$, and there exists $t_3 \in (\frac{1}{2}, 1]$ such that $v(t_3) - \gamma u(t_3) = 0$ since $\gamma$ is the largest constant such that $v \geq \gamma u$ on $[0, 1]$. Let $t_4 \in (\frac{1}{2}, 1]$ be the first point such that $v(t_4) - \gamma u(t_4) = 0$. If $t_4 = 1$, then $v'(1) - \gamma u'(1) = -c(v(1))v(1) + c(u(1))u(1) \geq 0$ by $\gamma (H_2)$. This implies that $v'(1) - \gamma u'(1) = 0$. Let $t_5 \in (0, 1)$ be the largest point such that $v(t_5) - \gamma u(t_5) > 0$ and $v'(t_5) - \gamma u'(t_5) = 0$. We can also choose $t_{b_0} \in (0, 1)$ such that $\gamma u(t_{b_0}) = b_0$ since $v(1) = \gamma u(1)$ and $v(1) > 1$ for $\lambda \gg 1$. Let $t_6 := \max\{t_5, t_{b_0}\}$. By $(H_6)$, we have

$$0 \geq -\varphi_p(v'(1)) + \varphi_p(\gamma u'(1)) - (-\varphi_p(v'(t_6)) + \varphi_p(\gamma u'(t_6)))$$

$$= \lambda \int_{t_6}^1 h(s) \left(f(v) - \gamma^{-1} f(u)\right) ds$$

$$\geq \lambda \int_{t_6}^1 h(s) \left(f(\gamma u) - \gamma^{-1} f(u)\right) ds$$

$$\geq \lambda (\gamma^{-\eta} - \gamma^{-1}) \int_{t_6}^1 h(s) f(u) ds.$$

This is a contradiction since $\int_{t_6}^1 h(s) f(u) ds > 0$. Hence $t_4 \in (\frac{1}{2}, 1)$. Then $v(t_4) - \gamma u(t_4) = 0$ and $v'(t_4) - \gamma u'(t_4) = 0$. By the above argument, we again get a contradiction. Hence $\gamma \geq \gamma_0$.

Next we show that $\gamma \geq 1$. Assume that $\gamma < 1$. Let $\bar{u} = \frac{u}{G^{-1}(\lambda \frac{p}{p - 1})}$ and $\bar{v} = \frac{v}{G^{-1}(\lambda \frac{p}{p - 1})}$. Since $G^{-1}(\lambda \frac{p}{p - 1}) = \lambda \frac{p}{p - 1} f(\lambda \frac{p}{p - 1}) (G^{-1}(\lambda \frac{p}{p - 1}))$, we have

$$-(\varphi_p(\bar{v}))' = h(t) \frac{f(u)}{f(G^{-1}(\lambda \frac{p}{p - 1}))} \text{ and } -(\varphi_p(\bar{u}))' = h(t) \frac{f(v)}{f(G^{-1}(\lambda \frac{p}{p - 1}))}.$$
Thus we have

Without loss of generality, we assume $b_0 > \frac{\beta + \theta}{\beta}$ and $f\left(\frac{b_0}{\gamma_0}\right) \geq |f(0)|$ hold. Let $I := \{t \in (0, \delta) \mid u(t) \geq \frac{b_0}{\gamma_0}\}$ and $J := (0, \delta) \setminus I$. Then $I = \{t_1, \delta\}$ and $J = (0, t_1)$ where $t_1 := \min\{t \in (0, \delta) \mid u(t) \geq \frac{b_0}{\gamma_0}\}$. On $I$, it follows from (H$_0$) that

$$f(\gamma u) - \gamma^{p-1}f(u) \geq (\gamma^q - \gamma^{p-1})f(u) \geq m_1(1 - \gamma),$$

where $m_1 := (p - 1 - q)\gamma_0 f\left(\frac{b_0}{\gamma_0}\right) \min\{1, \gamma_0^{p-2-q}\}$. On $J$, we have

$$|f(\gamma u) - \gamma^{p-1}f(u)| \leq |f(\gamma u) - f(u)| + (1 - \gamma^{p-1})|f(u)| \leq (1 - \gamma)|u'| + (1 - \gamma)(p - 1)\max\{1, \gamma_0^{p-2}\}f\left(\frac{b_0}{\gamma_0}\right) \leq \frac{1 - \gamma}{\gamma_0}f'(\zeta) + (1 - \gamma)(p - 1)\max\{1, \gamma_0^{p-2}\}f\left(\frac{b_0}{\gamma_0}\right) \leq m_2(1 - \gamma),$$

where $\zeta \in (\gamma u, u)$ and $m_2 := \frac{1}{\gamma_0}\sup_{s \in (0, \gamma_0)} sf'(s) + (p - 1)\max\{1, \gamma_0^{p-2}\}f\left(\frac{b_0}{\gamma_0}\right)$. Thus we have

$$-\left(\frac{f(G^{-1}\frac{1}{\lambda}t)}{1 - \gamma}a_\lambda(t)(\tilde{v}' - \gamma\tilde{u}')\right)' \geq \begin{cases} m_1h(t), & t \in I, \\ -m_2h(t), & t \in J. \end{cases}$$

We note that for $\tilde{h}(t) = m_1h(t)\chi_I(t)$, the function $U : [0, \delta) \rightarrow \mathbb{R}$ defined by $U(t) := \int_0^t \frac{1}{a_\lambda(s)} \int_s^\delta \tilde{h}(r)drds - \int_0^t \frac{1}{a_\lambda(s)} \int_0^s \tilde{h}(r)drds$ is continuous on $[0, \delta]$ and satisfies $U(0) < 0 < U(\delta)$. Thus there exists $t^* \in I$ such that $U(t^*) = 0$. Then

$$\kappa(t) = \begin{cases} \int_0^t \frac{1}{a_\lambda(s)} \int_s^\delta \tilde{h}(r)drds, & t \in (0, t^*), \\ \int_t^\delta \frac{1}{a_\lambda(s)} \int_0^s \tilde{h}(r)drds, & t \in (t^*, \delta), \end{cases}$$

is the solution of the boundary value problem:

$$\begin{cases} - (a_\lambda(t)\kappa'(t))' = \tilde{h}(t), & t \in (0, \delta), \\ \kappa(0) = 0 = \kappa(\delta). \end{cases} \tag{17}$$

By a similar argument, we obtain the solution $\tilde{\kappa}$ of (17) when $\tilde{h}(t) = m_2h(t)\chi_J(t)$. Using the linearity of the equation (17), we conclude that (17) has the unique solution $\pi = \kappa - \tilde{\kappa}$ for $\tilde{h}(t) = m_1h(t)\chi_I(t) - m_2h(t)\chi_J(t)$. By the comparison principle, we have

$$\tilde{v}(t) - \gamma\tilde{u}(t) \geq \frac{1 - \gamma}{f(G^{-1}\frac{1}{\lambda}t)}\pi(t)$$

for $t \in (0, \delta)$. We also note that $t_1 \to 0$ and $|J| \to 0$ as $\lambda \to \infty$ by Lemma 4.2 where $|J|$ is the length of $J$. Then by Lemmas 4.4 - 4.5, we obtain that $\kappa(t) \geq Dd(t, \partial\{0, \delta\})$ and $|\kappa(t) - \pi(t)| \leq D^*|J|^{1-\eta}d(t, \partial\{0, \delta\})$ for $\lambda \gg 1$. Thus we have

$$\pi(t) \geq \kappa(t) - |\kappa(t) - \pi(t)| \geq (D - D^*|J|^{1-\eta})d(t, \partial\{0, \delta\}) \geq \frac{D}{2}d(t, \partial\{0, \delta\})$$

for $t \in [0, \delta]$ and $\lambda \gg 1$. Since $d(t, \partial\{0, \delta\}) = t$ for $t \in [0, \frac{\delta}{2}]$, we have

$$\pi(t) \geq \frac{D}{2}d(t, \partial\{0, 1\})$$
for $t \in [0, \frac{4}{2}]$. This implies that

$$v(t) - \gamma u(t) \leq \frac{1 - \gamma}{f(G^{-1}(\lambda \overline{r}))} \gamma u(t) \leq \frac{\overline{D}(1 - \gamma)}{2f(G^{-1}(\lambda \overline{r}))} t$$

for $t \in [0, \frac{4}{2}]$. Then we obtain

$$v(t) \geq (\gamma + \epsilon) u(t)$$

for $t \in [0, \frac{4}{2}]$ by Lemma 4.2 where $\epsilon_\lambda := \frac{\overline{D}(1 - \gamma)}{2C_2 f(G^{-1}(\lambda \overline{r}))}$. Then we have $v \left(\frac{4}{2}\right) \geq \gamma u \left(\frac{4}{2}\right) + \hat{\epsilon}_\lambda$ where $\hat{\epsilon}_\lambda := \epsilon_\lambda \lambda_1 G^{-1}(\lambda \overline{r}) \frac{\delta}{2}$. Now we claim that $v(1) > \gamma u(1)$. If not, $v(1) = \gamma u(1)$. By $(H_\lambda)$, we have $v' = -\gamma u + \epsilon u + \epsilon(u) u + c v(1) v \geq 0$. This implies that $v'(1) = \gamma u(1)$ since $v \geq \gamma u$. Let $t^* \in (0, 1)$ be the largest critical point such that $v'(t^*) - \gamma u'(t^*) = 0$, $v(t^*) - \gamma u(t^*) > 0$ and $v'(t) - \gamma u'(t) \leq 0$ for $t \in [t^*, 1]$. If $t^* \geq t_1$, then by $(H_\delta)$ we have

$$0 = -\varphi_p(v'(1)) + \varphi_p(\gamma u'(1))$$

$$= \lambda \int_{t^*}^1 h(s) \left( f(v) - \gamma^{p-1} f(u) \right) ds$$

$$\geq \lambda \int_{t^*}^1 h(s) \left( f(\gamma u) - \gamma^{p-1} f(u) \right) ds$$

$$\geq \lambda (\gamma^{p-1} - \gamma^{p-1}) \int_{t^*}^1 h(s) f(u) ds.$$

This implies that $\int_{t^*}^1 h(s) f(u) ds = 0$. This is a contradiction. Thus $t^* < t_1$. Then we have

$$0 \leq -\varphi_p(v'(t_1)) + \varphi_p(\gamma u'(t_1)) = \lambda \int_{t_1}^{t_1} h(s) \left( f(v) - \gamma^{p-1} f(u) \right) ds.$$

Thus we obtain

$$0 = -\varphi_p(v'(1)) + \varphi_p(\gamma u'(1))$$

$$= \lambda \int_{t}^{t_1} h(s) \left( f(v) - \gamma^{p-1} f(u) \right) ds + \lambda \int_{t_1}^1 h(s) \left( f(v) - \gamma^{p-1} f(u) \right) ds$$

$$\geq \lambda (\gamma^{p-1} - \gamma^{p-1}) \int_{t_1}^1 h(s) f(u) ds.$$

This implies that $\int_{t_1}^1 h(s) f(u) ds = 0$. This is a contradiction. Hence $v(1) > \gamma u(1)$. Let $\hat{\epsilon}_\lambda := v(1) - \gamma u(1)$. On $(\frac{4}{2}, 1]$, by $(H_\delta)$ we have

$$-(\varphi_p(v'))' = \lambda h(t) f(v) \geq \lambda h(t) f(\gamma u) \geq \lambda \gamma^{p-1} h(t) f(u).$$

But

$$-(\varphi_p((\gamma u + \epsilon_\lambda'))') = \lambda \gamma^{p-1} h(t) f(u),$$

where $\epsilon_\lambda := \min\{\hat{\epsilon}_\lambda, \hat{\epsilon}_\lambda\}$. Thus $v \geq \gamma u + \epsilon_\lambda$ on $(\frac{4}{2}, 1]$ by the comparison principle. By Lemma 4.2, we have

$$v \geq \gamma u + \epsilon_\lambda \geq \left( \gamma + \epsilon_\lambda \frac{1}{C_2 G^{-1}(\lambda \overline{r})} \right) u.$$
Thus \( v \geq (\gamma + \epsilon^\#) u \) on \([0, 1]\) where \( \epsilon^\# := \min \{ \epsilon_\lambda, \frac{\epsilon_0}{C_G G^{-1}(\lambda+1)} \} \). This is a contradiction for the maximality of \( \gamma \). Hence \( \gamma \geq 1 \). This implies that \( v \equiv u \) on \([0, 1]\).

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