THE STABLE MONOMORPHISM CATEGORY OF A FROBENIUS CATEGORY

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Abstract. For a Frobenius abelian category \( A \), we show that the category \( \text{Mon}(A) \) of monomorphisms in \( A \) is a Frobenius exact category; the associated stable category \( \text{Mon}(A) \) modulo projective objects is called the stable monomorphism category of \( A \). We show that a tilting object in the stable category \( \text{A} \) of \( A \) modulo projective objects induces naturally a tilting object in \( \text{Mon}(A) \). We show that if \( A \) is the category of (graded) modules over a (graded) self-injective algebra \( A \), then the stable monomorphism category is triangle equivalent to the (graded) singularity category of the (graded) \( 2 \times 2 \) upper triangular matrix algebra \( T_2(A) \). As an application, we give two characterizations to the stable category of Ringel-Schmidmeier.

1. Introduction

Let \( A \) be an abelian category. Denote by \( \text{Mor}(A) \) the category of morphisms in \( A \) ([3, p.101]): the objects are morphisms in \( A \) and the morphisms are given by commutative squares in \( A \). It is an abelian category ([17, Proposition 1.1]). We are mainly concerned with the full subcategory \( \text{Mon}(A) \) of \( \text{Mor}(A) \) consisting of monomorphisms in \( A \), which is called the monomorphism category of \( A \). It is an additive subcategory of \( \text{Mor}(A) \) which is closed under extensions, thus it becomes an exact category in the sense of Quillen ([22, Appendix A]).

In the case that the abelian category \( A \) is the module category over a ring, the monomorphism category \( \text{Mon}(A) \) is known as the submodule category. Recently it is studied intensively by Ringel and Schmidmeier ([34, 35, 36]). If the ring is \( \mathbb{Z}/(q^p) \) with \( p \geq 2 \) and \( q \) a prime number, the study of the submodule category goes back to Birkhoff ([8]; see also [1]). The case that the ring is \( k[t]/(t^p) \) with \( k \) a field is studied by Simson ([37]) and also by Beligiannis ([7]). In this case, the study of indecomposable objects in \( \text{Mon}(A) \) shows an example of the typical trichotomy phenomenon “finite/tame/wild” in the representation theory of finite dimensional algebras, where the trichotomy depends on the parameter \( p \); see [36, Section 6]. Moreover, the case where the abelian category \( A \) is given by the graded module category over the graded algebra \( k[t]/(t^p) \) with \( \text{deg } t = 1 \) plays an important role in [36]; in this case, the

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monomorphism category $\text{Mon}(A)$ is denoted by $\mathcal{S}(\tilde{p})$. It is a Frobenius exact category ([27]; also see Lemma 2.1 and compare [22, Section 5]). Then by [18, Chapter I, Theorem 2.6] its stable category $\mathcal{S}(\tilde{p})$ modulo projective objects is triangulated. A very recent and remarkable result due to Kussin, Lenzing and Meltzer claims that the stable category $\mathcal{S}(\tilde{p})$ is triangle equivalent to the stable category of vector bundles on the weighted projective lines of type $(2,3,p)$; see [27]. Recall that a similar trichotomy phenomenon “domestic/tubular/wild” occurs in the classification of indecomposable vector bundles on the weighted projective lines of type $(2,3,p)$, while the trichotomy again depends on the parameter $p$; see [29, 26]. In this paper, we will call the triangulated category $\mathcal{S}(\tilde{p})$ the \textit{stable category of Ringel-Schmidmeier}.

The present paper studies the monomorphism category $\text{Mon}(A)$ of a Frobenius abelian category $A$, in particular, the stable category $A$ modulo projective objects is triangulated. We show that $\text{Mon}(A)$ is a Frobenius exact category and then the stable category $\text{Mon}(A)$ modulo projective objects is triangulated; it is called the \textit{stable monomorphism category of $A$}. Recently this category is also studied by Iyama, Kato and Miyachi ([21]). Observe that the triangulated categories above are algebraical in the sense of Keller. We have a well-behaved notion of \textit{tilting object} for an algebraical triangulated category ([24]). We prove that a tilting object in $A$ induces naturally a tilting object in $\text{Mon}(A)$; see Theorem 3.2. Moreover, if the category $A$ is the (graded) module category over a (graded) self-injective algebra $A$, we relate the category $\text{Mon}(A)$ to the category of (graded) Gorenstein-projective modules and then to the (graded) singularity category of the $2 \times 2$ upper triangular matrix algebra $T_2(A)$ of $A$ (for $T_2(A)$, see [17, p.115] and [3, Chapter III, Section 2]); see Theorem 4.1. We are inspired by a computational result by Li and Zhang on Gorenstein-projective modules ([30]; compare [7, 21]). Here, the Gorenstein-projective module is in the sense of Enochs and Jenda ([16, Chapter 10]), and the singularity category is in the sense of Orlov ([32, 33]; compare [10, 19]).

Combining all these together, we give two characterizations to the stable category $\mathcal{S}(\tilde{p})$ of Ringel-Schmidmeier. We characterize the stable category $\mathcal{S}(\tilde{p})$ as the bounded derived category of $T_2(kA_{p-1}) \cong kA_2 \otimes_k kA_{p-1}$; see Corollary 3.4. Here, for each $n \geq 1$, $A_n$ is the linear quiver with $n$ vertices and linear orientation, and $kA_n$ is the path algebra. We characterize the stable category $\mathcal{S}(\tilde{p})$ as the graded singularity category of $T_2(k)[t]/(tp)$, where the algebra $T_2(k)[t]/(tp)$ is graded such that $\deg T_2(k) = 0$ and $\deg t = 1$; see Corollary 4.7.

For the convention, throughout we fix a commutative artinian ring $R$. All artin algebras are artin $R$-algebras, and all categories and functors are $R$-linear. For an artin algebra $A$, denote by $\text{mod } A$ the category of finitely generated right $A$-modules and by $\text{proj } A$ the full subcategory consisting of projective modules. We denote by $A_A$ and $\underline{A}A$ the right and left regular modules of the artin algebra $A$, respectively. For triangulated categories and derived categories, we refer to [20, 18, 23, 24].

\section{Monomorphism Category}

Let $A$ be a Frobenius abelian category. Thus $A$ has enough projective objects and enough injective objects, and the class of projective objects coincides with the class of injective objects. Denote by $\mathcal{P}$ the full subcategory of $A$ consisting of projective
objects. Denote by \( \mathcal{A} \) the stable category of \( \mathcal{A} \) modulo \( \mathcal{P} \): the objects are the same as \( \mathcal{A} \), and the morphism spaces are factors of the morphism spaces in \( \mathcal{A} \) modulo those factoring through projective objects ([3, p.101]). The stable category \( \mathcal{A} \) is a triangulated category such that its shift functor is given by the quasi-inverse of the syzygy functor on \( \mathcal{A} \) and triangles are induced by short exact sequences in \( \mathcal{A} \); for details, see [18, Chapter I, Section 2].

Recall that \( \text{Mor}(\mathcal{A}) \) is the category of morphisms in \( \mathcal{A} \): the objects are morphisms \( \alpha: A \to B \) in \( \mathcal{A} \) and the morphisms are commutative squares in \( \mathcal{A} \), that is, of the form \((f,g): \alpha \to \alpha'\) where \( f: A \to A' \) and \( g: B \to B' \) are morphisms in \( \mathcal{A} \) such that \( \alpha' \circ f = g \circ \alpha \) (compare [3, p.101]). For an object \( \alpha: A \to B \) in \( \text{Mor}(\mathcal{A}) \), we write \( s(\alpha) = A \) and \( t(\alpha) = B \), which are called the source and target of \( \alpha \), respectively. Note that \( \text{Mor}(\mathcal{A}) \) is an abelian category such that a sequence \( \alpha' \to \alpha \to \alpha'' \) is exact if and only if the induced sequences of sources and targets are exact in \( \mathcal{A} \) ([17, Corollary 1.2]).

Recall that an exact category in the sense of Quillen is an additive category together with an exact structure, that is, a distinguished class of ker-coker sequences, which are called conflations, subject to certain axioms. Recall that a full additive subcategory of an abelian category which is closed under extensions has a natural exact structure such that conflations are just exact sequences with terms in the subcategory ([22, Appendix A] and [23, Section 4]). Moreover, there is a notion of Frobenius exact category and the associated stable category modulo projective objects is still triangulated; compare [18, p.10-11], [22, subsection 1.2 b)] and [23, Section 6].

Recall that our main concern is the monomorphism category \( \text{Mon}(\mathcal{A}) \), which is the full subcategory of \( \text{Mor}(\mathcal{A}) \) consisting of monomorphisms in \( \mathcal{A} \). We will consider the following two functors: the first functor \( i_1: \mathcal{A} \to \text{Mon}(\mathcal{A}) \) is defined such that \( i_1(A) = 0 \to A \) and \( i_1(f) = (0, f) \) where \( A \) is an object and \( f \) is a morphism in \( \mathcal{A} \); the second \( i_2: \mathcal{A} \to \text{Mon}(\mathcal{A}) \) is defined such that \( i_2(A) = A \xrightarrow{1_A} A \) and \( i_2(f) = (f, f) \). We observe that both functors are exact and fully faithful.

**Lemma 2.1.** Let \( \mathcal{A} \) be an abelian category. Then the monomorphism category \( \text{Mon}(\mathcal{A}) \) is an exact category such that conflations are given by sequences \( \alpha' \to \alpha \to \alpha'' \) with the induced sequences of sources and targets short exact in \( \mathcal{A} \).

Assume further that \( \mathcal{A} \) is Frobenius. Then the exact category \( \text{Mon}(\mathcal{A}) \) is Frobenius such that its projective objects are equal to direct summands of objects of the form \( i_1(P) \oplus i_2(P) \) where \( P \) is a projective object in \( \mathcal{A} \).

**Proof.** We observe that \( \text{Mon}(\mathcal{A}) \) is an additive subcategory of the abelian category \( \text{Mor}(\mathcal{A}) \) which is closed under extensions by Snake Lemma. Then it is an exact category with conflations induced by short exact sequences in \( \text{Mor}(\mathcal{A}) \); see Example 4.1 in [23].

Assume now that the abelian category \( \mathcal{A} \) is Frobenius. We will show first that objects of the form \( i_1(P) \) and \( i_2(P) \) are projective and injective. Recall that for an object \( \alpha \) in \( \text{Mon}(\mathcal{A}) \) we denote by \( s(\alpha) \) and \( t(\alpha) \) the source and target of \( \alpha \), respectively. We have the following natural isomorphisms

\[
\text{Hom}_{\text{Mon}(\mathcal{A})}(i_1(P), \alpha) \simeq \text{Hom}_{\mathcal{A}}(P, t(\alpha)) \quad \text{and} \quad \text{Hom}_{\text{Mon}(\mathcal{A})}(i_2(P), \alpha) \simeq \text{Hom}_{\mathcal{A}}(P, s(\alpha)).
\]
These isomorphisms show that the objects \( i_1(P) \) and \( i_2(P) \) are projective. Similarly, we have the following natural isomorphisms

\[
\text{Hom}_{\text{Mon}(\mathcal{A})}(\alpha, i_1(P)) \simeq \text{Hom}_{\mathcal{A}}(\text{Cok } \alpha, P)
\]

and

\[
\text{Hom}_{\text{Mon}(\mathcal{A})}(\alpha, i_2(P)) \simeq \text{Hom}_{\mathcal{A}}(t(\alpha), P).
\]

These isomorphisms show that the objects \( i_1(P) \) and \( i_2(P) \) are injective; here, we use that the functor \( \text{Cok} \) of taking the cokernels is exact on \( \text{Mon}(\mathcal{A}) \) by Snake Lemma.

Let \( \alpha \) be an object in \( \text{Mon}(\mathcal{A}) \). Take epimorphisms \( P \to s(\alpha) \) and \( P \to t(\alpha) \) with \( P \) projective in \( \mathcal{A} \). Then we have an epimorphism \( i_1(P) \oplus i_2(P) \to \alpha \) whose kernel lies in \( \text{Mon}(\mathcal{A}) \). This shows that the exact category \( \text{Mon}(\mathcal{A}) \) has enough projective objects. On the other hand, for the object \( \alpha \), take monomorphisms \( a: t(\alpha) \to P \) and \( b': \text{Cok } \alpha \to P \) with \( P \) projective in \( \mathcal{A} \). Denote by \( b \) the composite \( t(\alpha) \to \text{Cok } \alpha \overset{b'}{\to} P \) where the first morphism is the natural projection. Consider the following morphism in \( \text{Mor}(\mathcal{A}) \)

\[
\begin{pmatrix} a \circ \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}: \alpha \to i_2(P) \oplus i_1(P).
\]

It is a monomorphism and by a diagram-chasing its cokernel lies in \( \text{Mon}(\mathcal{A}) \). Then it becomes a conflation in \( \text{Mon}(\mathcal{A}) \). This shows that the exact category \( \text{Mon}(\mathcal{A}) \) has enough injective objects. From the argument above, it is direct to conclude that in the exact category \( \text{Mon}(\mathcal{A}) \) the class of projective objects coincides with the class of injective objects, and projective objects are direct summands of objects of the form \( i_1(P) \oplus i_2(P) \) where \( P \) is a projective object in \( \mathcal{A} \).

**Remark 2.2.** With a slightly modified proof as above, one can show that a similar result holds if the category \( \mathcal{A} \) is an exact category. In this case, one replaces \( \text{Mon}(\mathcal{A}) \) by the inflation category of \( \mathcal{A} \); compare [22, Section 5] and [21].

For a Frobenius abelian category \( \mathcal{A} \), we denote by \( \text{Mon}(\mathcal{A}) \) the stable category of \( \text{Mon}(\mathcal{A}) \) modulo projective objects; it is a triangulated category. We will call it the *stable monomorphism category* of \( \mathcal{A} \).

We observe that both the functors \( i_1 \) and \( i_2 \) are fully faithful and send projective objects to projective objects. Then they induce fully faithful triangle functors \( i_1: \mathcal{A} \to \text{Mon}(\mathcal{A}) \) and \( i_2: \mathcal{A} \to \text{Mon}(\mathcal{A}) \) ([18, p.23, Lemma 2.8]).

### 3. Tilting Objects in Stable Monomorphism Category

In this section, we will show that for a Frobenius abelian category \( \mathcal{A} \), a tilting object in the stable category \( \mathcal{A} \) induces naturally a tilting object in the stable monomorphism category \( \text{Mon}(\mathcal{A}) \). We characterize the stable category of Ringel-Schmidmeier as the bounded derived category of a finite dimensional algebra.

Following Keller, we recall that a triangulated category is *algebraical* provided that it is triangle equivalent to the stable category of a Frobenius exact category ([24,
subsection 8.7]). One has a well-behaved notion of tilting object in an algebraical triangulated category.

Let \( \mathcal{T} \) be an algebraical triangulated category. Denote by \([1]\) the shift functor and by \([n]\) its \(n\)-th power for each \(n \in \mathbb{Z}\). An object \( T \) in \( \mathcal{T} \) is a \textit{tilting object} if the following conditions are satisfied:

\begin{enumerate}
  \item[(T1)] \( \text{Hom}_\mathcal{T}(T, T[n]) = 0 \) for \( n \neq 0 \);
  \item[(T2)] the smallest \textit{thick} triangulated subcategory of \( \mathcal{T} \) containing \( T \) is \( \mathcal{T} \) itself;
  \item[(T3)] \( \text{End}_\mathcal{T}(T) \) is an artin algebra having finite global dimension.
\end{enumerate}

Here, we recall that a triangulated subcategory of \( \mathcal{T} \) is called \textit{thick} if it is closed under taking direct summands. We point out that the notion of tilting object presented here is slightly different from, however closely related to, the ones in \([18]\) and \([24]\).

Recall that an additive category is said to be \textit{idempotent-split} provided that each idempotent \( e : X \rightarrow X \) admits a factorization \( X \xrightarrow{u} Y \xrightarrow{v} X \) such that \( uv = \text{Id}_Y \) ([18, Chapter I, 3.2]). Recall that for an artin algebra \( A \) having finite global dimension, the bounded derived category \( \text{D}^b(\text{mod } A) \) is algebraical and idempotent-split (see the proof of [18, Chapter I, Corollary 4.9]), and it has \( A_A \) as its tilting object.

The following remarkable result due to Keller claims that the converse holds true (compare [9, Theorem 1]).

\textbf{Lemma 3.1.} (Keller) \textit{Let \( \mathcal{T} \) be an idempotent-split algebraical triangulated category with a tilting object \( T \). Then there is a triangle equivalence}
\[
\mathcal{T} \cong \text{D}^b(\text{mod } \text{End}_{\mathcal{T}}(T)).
\]

\textit{Proof.} Set \( A = \text{End}_{\mathcal{T}}(T) \). By \([24, \text{Theorem 8.51 a]}\) there is a triangle functor \( F' : \mathcal{T} \rightarrow \text{D}(A') \) sending \( T \) to \( A' \), where \( A' \) is a differential graded algebra with the only nonzero cohomology \( H^0(A') \simeq A \) and \( \text{D}(A') \) is the (unbounded) derived category of differential graded (right) modules on \( A' \). By \([24, \text{subsection 8.4]}\) there is a triangle equivalence \( \text{D}(A') \cong \text{D}(\text{Mod } A) \) identifying \( A' \) with \( A_A \), where \( \text{Mod } A \) is the category of (not necessarily finitely generated) right \( A \)-modules. Consequently, there is a triangle functor \( F : \mathcal{T} \rightarrow \text{D}(\text{Mod } A) \) sending \( T \) to \( A \). Using (T1) and (T2) and applying Beilinson Lemma ([18, p.72, Lemma 3.4]), the triangle functor \( F \) is fully faithful. Then we may view \( \mathcal{T} \) as a triangulated subcategory of \( \text{D}(\text{Mod } A) \); moreover, since \( \mathcal{T} \) is idempotent-split, it is necessarily a thick subcategory of \( \text{D}(\text{Mod } A) \). By (T3) the artin algebra \( A \) has finite global dimension, and then the smallest thick triangulated subcategory of \( \text{D}(\text{Mod } A) \) containing \( A_A \) is \( \text{D}^b(\text{mod } A) \). From this we conclude that the essential image of \( F \) is \( \text{D}^b(\text{mod } A) \). Therefore \( F \) induces the required equivalence. \( \Box \)

Our first observation states that a tilting object in the stable category \( \mathcal{A} \) induces naturally a tilting object in the stable monomorphism category \( \text{Mon}(\mathcal{A}) \). Recall that for an artin algebra \( A \), \( T_2(A) = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \) is the \( 2 \times 2 \) upper triangular matrix algebra ([3, Chapter III, Section 2]).
**Theorem 3.2.** Let $\mathcal{A}$ be a Frobenius abelian category such that $T$ is a tilting object in its stable category $\underline{\mathcal{A}}$. Then $T' = i_1(T) \oplus i_2(T)$ is a tilting object in $\underline{\text{Mon}}(\mathcal{A})$; moreover, we have an isomorphism $\text{End}_{\underline{\text{Mon}}(\mathcal{A})}(T') \simeq T_2(\text{End}_{\underline{\mathcal{A}}}(T))$ of algebras.

**Proof.** Recall that $i_1 : \mathcal{A} \to \underline{\text{Mon}}(\mathcal{A})$ and $i_2 : \mathcal{A} \to \underline{\text{Mon}}(\mathcal{A})$ are fully faithful triangle functors. Observe that for objects $A$ and $B$ in $\mathcal{A}$, $\text{Hom}_{\underline{\text{Mon}}(\mathcal{A})}(i_2(A), i_1(B)) = 0$. So to check the condition (T1) for $T'$, it suffices to show that $\text{Hom}_{\underline{\text{Mon}}(\mathcal{A})}(i_1(T), i_2(T)[n]) = 0$ for $n \neq 0$. For this end, note that since $i_2$ is a triangle functor, we have

\[ i_2(T)[n] \simeq i_2(T[n]) = T[n] \frac{\text{Id}_{T[n]}}{\text{Id}_{T[n]}} T[n]. \]

Thus a morphism in $\text{Hom}_{\underline{\text{Mon}}(\mathcal{A})}(i_1(T), i_2(T)[n])$ is of the form $(0, f)$, where $f : T \to T[n]$ is a morphism in $\mathcal{A}$. By the condition (T1) for $T$, $f$ factors through a projective object $P$ in $\mathcal{A}$. Therefore the morphism $(0, f)$ factors through $i_1(P)$, which is projective in $\underline{\text{Mon}}(\mathcal{A})$; see Lemma 2.1. Hence $(0, f) = 0$ in the stable monomorphism category $\underline{\text{Mon}}(\mathcal{A})$.

To check (T2) for $T'$, recall that each object $\alpha$ fits into a conflation

\[ i_2(s(\alpha)) \longrightarrow \alpha \longrightarrow i_1(\text{Cok } \alpha) \]

and thus into a triangle

\[ i_2(s(\alpha)) \longrightarrow \alpha \longrightarrow i_1(\text{Cok } \alpha) \longrightarrow i_2(s(\alpha))[1]. \]

Here as in Section 2, $s(\alpha)$ denotes the source of $\alpha$. Hence the smallest triangulated subcategory of $\underline{\text{Mon}}(\mathcal{A})$ containing $i_1(\mathcal{A})$ and $i_2(\mathcal{A})$ is $\underline{\text{Mon}}(\mathcal{A})$ itself. Now applying the condition (T2) of $T$, we infer that (T2) holds for $T'$.

Finally to see the condition (T3) for $T'$, it is direct to check that $\text{End}_{\underline{\text{Mon}}(\mathcal{A})}(T') \simeq T_2(\text{End}_{\underline{\mathcal{A}}}(T))$. Recall that the algebra $\text{End}_{\underline{\mathcal{A}}}(T)$ has finite global dimension. Then by [3, Chapter III, Proposition 2.6] we infer that $\text{End}_{\underline{\text{Mon}}(\mathcal{A})}(T')$ has finite global dimension. \hfill $\square$

We will give an application of Theorem 3.2. Let $A = \oplus_{n \geq 0} A_n$ be a positively graded artin algebra. Denote by $c$ the maximal integer such that $A_c \neq 0$. Consider the following upper triangular matrix algebra

\[
\begin{pmatrix}
A_0 & A_1 & \cdots & A_{c-2} & A_{c-1} \\
0 & A_0 & \cdots & A_{c-3} & A_{c-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_0 & A_1 \\
0 & 0 & \cdots & 0 & A_0
\end{pmatrix}
\]

Here the multiplication of $b(A)$ is induced from the one of $A$. This algebra is called the Beilinson algebra of $A$ in [12].

Denote by $\text{mod}^Z A$ the category of finitely generated $\mathbb{Z}$-graded $A$-modules with homomorphisms preserving degrees. We say that $A$ is graded self-injective provided that $\text{mod}^Z A$ is a Frobenius category. In fact, this is equivalent to that as a ungraded algebra $A$ is self-injective ([15, 12]). In this case, we denote by $\text{mod}^Z A$ the stable category of $\text{mod}^Z A$ modulo projective modules; it is a triangulated category.
We say that a graded algebra $A$ is right well-graded, provided that $A_c$, as a right $A_0$-module, is sincere in the sense of [3, p.317]. In fact, for a graded self-injective algebra $A$, it is right well-graded if and only if it is left well-graded; see [12, Lemma 2.2]. In this case we will simply say that the graded algebra $A$ is well-graded.

**Corollary 3.3.** Let $A = \oplus_{n \geq 0} A_n$ be a positively graded self-injective artin algebra which is well-graded. Suppose that $A_0$ has finite global dimension. Then there is a triangle equivalence

$$\text{Mon}(\text{mod}^Z A) \simeq \text{D}^b(\text{mod} T_2(b(A))).$$

**Proof.** By [3, Chapter III, Proposition 2.6], the Beilinson algebra $b(A)$ and then $T_2(b(A))$ has finite global dimension. By [12, Corollary 1.2] there is a triangle equivalence \( \text{mod}^Z A \simeq \text{D}^b(\text{mod} b(A)) \). In particular, there is a tilting object $T$ in \( \text{mod}^Z A \) with endomorphism algebra $b(A)$. We apply Theorem 3.2 to get a tilting object $T'$ in \( \text{Mon}(\text{mod}^Z A) \) whose endomorphism algebra is isomorphic to $T_2(b(A))$. Note that the stable monomorphism category $\text{Mon}(\text{mod}^Z A)$ is idempotent-split; in fact, it is even a Krull-Schmidt category. Then the result follows immediately from Lemma 3.1.

In what follows, we will apply the obtained results to the stable category of Ringel-Schmidmeier.

Let $k$ be a field and let $p \geq 2$ be an integer. Consider the truncated polynomial algebra $A = k[t]/(t^p)$ with $t$ an indeterminant; it is positively graded such that $\deg t = 1$. Observe that $A$ is graded self-injective and moreover it is well-graded. In particular, the category $\text{mod}^Z A$ of finitely generated graded $A$-modules is Frobenius. Following [36, subsection 0.4], we denote by $S(\bar{p})$ the category of pairs $(V, U)$, where $V$ is a graded module over $A$ and $U \subseteq V$ is a graded submodule, and the morphisms in this category are given by morphisms in the graded module category which respect the inclusion. There is a natural identification $S(\bar{p}) = \text{Mon}(\text{mod}^Z A)$ and then by Lemma 2.1 it is a Frobenius exact category. Hence its stable category $\bar{S}(\bar{p})$ modulo projective objects is triangulated. This triangulated category will be called the **stable category of Ringel-Schmidmeier**.

We note that the Beilinson algebra $b(A)$ of the graded algebra $A$ is isomorphic to the path algebra $k \mathbb{A}_{p-1}$ of the linear quiver $\mathbb{A}_{p-1}$ with $p - 1$ vertices and linear orientation (compare [33, Example 2.9]). Then the $2 \times 2$ upper triangular matrix algebra $T_2(b(A))$ is given by the following quiver with $2p - 2$ vertices subject to the commutativity relation

\[
\bullet \rightarrow \bullet \quad \cdots \quad \bullet \rightarrow \bullet \\
\bullet \longrightarrow \bullet \quad \cdots \quad \bullet \rightarrow \bullet
\]

We observe that $T_2(b(A)) \simeq k \mathbb{A}_2 \otimes_k k \mathbb{A}_{p-1}$. Let us mention that these diagrams and algebras are studied in [28].

Then we have the following immediate consequence of Corollary 3.3.
Corollary 3.4. Use the notation above. Then there is a triangle equivalence
\[ \mathcal{S}(\tilde{p}) \simeq D^b(\text{mod } kA_2 \otimes_k kA_{p-1}) . \]

Remark 3.5. Let us remark that taking into account of the results obtained in [26] and [28, Corollary 1.2], one may find a close relation between Corollary 3.4 and some results in [27].

Recall that \( T = \bigoplus_{i=0}^{p-2} (A/(t^i - 1)) \) is a tilting object in \( \text{mod}^A A \), where \( (i) \) denote the degree-shift functors ([31] and [15]). This assertion can be obtained from the proof of [33, Corollary 2.8] or [12, Corollary 1.2]. We apply Theorem 3.2 to deduce that \( T' = i_1(T) \oplus i_2(T) \) is a tilting object in \( \mathcal{S}(\tilde{p}) \), which yields the triangle equivalence in Corollary 3.4. We point out that this explicit tilting object is also obtained in [27, Lemma 4.7] via a different method.

4. Stable Monomorphism Category as Singularity Category

In this section, we will relate the stable monomorphism category of the (graded) module category of a (graded) self-injective algebra to the (graded) singularity category of the associated (graded) \( 2 \times 2 \) upper triangular matrix algebra. We characterize the stable category of Ringel-Schmidmeier as the graded singularity category of a finite dimensional graded algebra.

Let \( A \) be an artin algebra. Recall that the bounded homotopy category \( K^b(\text{proj } A) \) of projective modules is viewed naturally as a triangulated subcategory of \( D^b(\text{mod } A) \). Following [32, 33], we call the Verdier quotient triangulated category \( D^b(\text{mod } A) / K^b(\text{proj } A) \) the singularity category of \( A \); compare [10] and [19].

Recall that for an artin algebra \( A \), \( T_2(A) \) is the \( 2 \times 2 \) upper triangular matrix algebra of \( A \). We consider the following composite functor
\[ G_A : \text{Mon}(\text{mod } A) \hookrightarrow \text{mod } T_2(A) \longrightarrow D^b(\text{mod } T_2(A)) \longrightarrow D_{sg}(T_2(A)) . \]

Here, the first inclusion is obtained by regarding morphisms in \( \text{mod } A \) as (right) \( T_2(A) \)-modules ([3, Chapter III, Proposition 2.2]), the middle functor identifies modules with stalk complexes concentrated at degree zero ([20, p.40, Proposition 4.3]), and the last functor is the quotient functor.

Our second observation is as follows.

Theorem 4.1. Let \( A \) be a self-injective algebra. Then the functor \( G_A \) induces a triangle equivalence
\[ \text{Mon}(\text{mod } A) \simeq D_{sg}(T_2(A)) . \]

Before giving the proof, we recall several notions. Let \( A \) be an artin algebra. Following [5, p.400], an acyclic complex \( P^\bullet \) of projective \( A \)-modules is called totally acyclic if the Hom complex \( \text{Hom}_A(P^\bullet, A) \) is still acyclic (also see [25, Section 7]). An \( A \)-module \( M \) is said to be Gorenstein-projective if there is a totally acyclic complex \( P^\bullet \) such that its zeroth cocycle \( Z^0(P^\bullet) \) is isomorphic to \( M \) ([16, Chapter 10]). Recall that a module
$M$ is Gorenstein-projective if and only if $\text{Ext}^i_A(M, A) = 0$, $\text{Ext}^i_{A^{\text{op}}}(\text{Hom}_A(M, A), A) = 0$ for $i \geq 1$ and the natural map $M \to \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(M, A), A)$ is an isomorphism (compare [14, Definition (1.1.2)]).

We denote by $\text{Gproj}_A$ the full subcategory of $\text{mod}
A$ consisting of Gorenstein-projective modules. Observe that projective modules are Gorenstein-projective and thus $\text{proj}
A \subseteq \text{Gproj}_A$. Moreover, by [2, Proposition 5.1] the subcategory $\text{Gproj}_A$ is closed under extensions and taking direct summands (also see [16]), and then it is direct to see that $\text{Gproj}_A$ is a Frobenius exact category such that its projective objects are equal to projective $A$-modules ([6, Proposition 3.8(i)] and [13, Proposition 3.1(1)]). Denote by $\text{Gproj}_A$ its stable category modulo projective $A$-modules; it is a triangulated category.

Recall that an artin algebra $A$ is said to be Gorenstein if the regular modules $A_A$ and $A_A$ have finite injective dimensions ([19]). In this case the two dimensions are equal and the common value is denoted by $G\text{dim}
A$. We say that the Gorenstein algebra $A$ is 1-Gorenstein provided that $G\text{dim}
A \leq 1$
.

For an artin algebra $A$, denote by $\text{sub}
A$ the full subcategory of $\text{mod}
A$ consisting of submodules of projective modules; these modules are called torsionless modules. We remark that homological properties of torsionless modules are studied in [4].

The following result is well known.

**Lemma 4.2.** Let $A$ be a 1-Gorenstein algebra. Then we have $\text{Gproj}
A = \text{sub}
A$.

**Proof.** The inclusion $\text{Gproj}
A \subseteq \text{sub}
A$ is easy. On the other hand, assume that $M$ is a torsionless module. Consider a short exact sequence $0 \to M \to P \to M' \to 0$ with $P$ projective. Since the regular module $A_A$ has injective dimension at most one, using dimension shift, we infer that $\text{Ext}^i(M, A) = 0$ for $i \geq 1$. Then by [16, Corollary 11.5.3] (see also [13, Lemma 3.7] and [25, Proposition 7.13]), $M$ is Gorenstein-projective.

The next observation is essentially due to Li and Zhang ([30, Theorem 1.1]; also see [7, Example 4.17] and [21, Proposition 3.6]). Recall that for an artin algebra $A$, a morphism of (right) $A$-modules is identified with a (right) module over $T_2(A)$; in fact, this yields an equivalence $\text{Mor}(\text{mod}
A) \simeq \text{mod}
T_2(A)$ of categories; see [3, Chapter III, Proposition 2.2].

**Lemma 4.3.** Let $A$ be a self-injective algebra. Then we have an equivalence of categories

$$\text{Mon}(\text{mod}
A) \simeq \text{sub}
T_2(A).$$

**Proof.** Recall the equivalence $\text{Mor}(\text{mod}
A) \simeq \text{mod}
T_2(A)$. Observe that the regular module $T_2(A)_{T_2(A)}$ corresponds to the monomorphism $(0_{1d_A}) : A \to A \oplus A$. From this one infers that torsionless $T_2(A)$-modules correspond to monomorphisms in $\text{mod}
A$. On the other hand, the third paragraph of the proof of Lemma 2.1 already shows that for a monomorphism $\alpha$, there is a short exact sequence $0 \to \alpha \to (1_{1d_P}) \to \alpha' \to 0$ in $\text{Mor}(\text{mod}
A)$ such that $P$ is a projective $A$-module. Observe that the monomorphism $(0_{1d_P})$ corresponds to a projective $T_2(A)$-module. Therefore the monomorphism $\alpha$ corresponds to a torsionless $T_2(A)$-module. This completes the proof. □
We will recall the last ingredient in our proof. Let $A$ be an artin algebra. Consider the following composite of functors

$$F_A : \text{Gproj } A \hookrightarrow \text{mod } A \longrightarrow \mathbb{D}^b(\text{mod } A) \longrightarrow \mathbb{D}_{sg}(A)$$

where from the left side, the first functor is the inclusion, the second identifies modules with stalk complexes concentrated in degree zero ([20, p.40, Proposition 4.3]) and the last is the quotient functor. Observe that the additive functor $F_A$ vanishes on projective modules and then induces uniquely an additive functor $\text{Gproj } A \rightarrow \mathbb{D}_{sg}(A)$, which is still denoted by $F_A$.

The following important result is due to Buchweitz ([10, Theorem 4.4.1]) and independently due to Happel ([19, Theorem 4.6]); also see [13, Proposition 3.5 and Theorem 3.8].

**Lemma 4.4.** (Buchweitz-Happel) Let $A$ be an artin algebra. Then the functor $F_A : \text{Gproj } A \rightarrow \mathbb{D}_{sg}(A)$ is a fully faithful triangle functor. Moreover, if $A$ is Gorenstein, then the functor $F_A$ is dense and thus a triangle equivalence.

**Proof of Theorem 4.1.** We observe that by [11, Remark 3.5] (also see [17, 19]) the algebra $T_2(A)$ is $1$-Gorenstein and then we can apply Lemma 4.2. Then Lemma 4.3 yields an equivalence of categories $\text{Mon}(\text{mod } A) \simeq \text{Gproj } T_2(A)$. We observe that this equivalence preserves the exact structures, that is, the equivalence and its quasi-inverse preserve short exact sequences in $\text{Mon}(\text{mod } A)$ and $\text{Gproj } T_2(A)$. Therefore, this equivalence is an equivalence of Frobenius exact categories. Consequently, we have an induced equivalence of triangulated categories

$$\text{Mon}(\text{mod } A) \simeq \text{Gproj } T_2(A).$$

Then the result follows directly from Lemma 4.4. \qed

We will need a graded version of Theorem 4.1. Let $A = \oplus_{n \geq 0} A_n$ be a positively graded artin algebra. Denote by $\text{proj}^Z A$ the full subcategory of $\text{mod}^Z A$ consisting of projective objects. Following [33], one has the graded singularity category of $A$ defined by

$$\mathbb{D}^Z_{sg}(A) = \mathbb{D}^b(\text{mod}^Z A) / \mathbb{K}^b(\text{proj}^Z A).$$

For a graded module $M = \oplus_{i \in \mathbb{Z}} M_i$ and an integer $d \in \mathbb{Z}$, its shifted module $M(d)$ has the same module structure as $M$ while it is graded such that $M(d)_i = M_{d+i}$ for all $i \in \mathbb{Z}$. This defines automorphisms $(d) : \text{mod}^Z A \rightarrow \text{mod}^Z A$, which are called degree-shift functors. For graded modules $M, N$, we write $\text{HOM}_A(M, N) = \oplus_{i \in \mathbb{Z}} \text{Hom}_{\text{mod}^Z A}(M, N(i))$ and set $\text{EXT}_n^A(\cdot, \cdot)$ to be the $n$-th right derived functors ([31] and [15]).

An acyclic complex $P^\bullet$ in $\text{proj}^Z A$ is *totally acyclic* if the complex $\text{HOM}_A(P^\bullet, A)$ in $\text{proj}^Z A^{op}$ is acyclic. A graded $A$-module is called graded Gorenstein-projective provided that it is the zeroth cocycle of a totally acyclic complex. Thus we have a full subcategory $\text{Gproj}^Z A$ of $\text{mod}^Z A$ consisting of graded Gorenstein-projective modules and evidently $\text{proj}^Z A \subseteq \text{Gproj}^Z A$. As in the ungraded case, the category $\text{Gproj}^Z A$ is a Frobenius exact category with its projective objects equal to graded projective $A$-modules.
Recall that a graded artin algebra $A$ is said to be graded Gorenstein if the graded regular modules $A A$ and $A A$ have finite injective dimensions in $\text{mod}^Z A$ and $\text{mod}^Z A$, respectively. In this case the two dimensions are the same, which will be denoted by $G\dim^Z A$.

We observe the following fact, which guarantees in principle that most results in Gorenstein homological algebra hold true in the graded situation.

**Lemma 4.5.** Let $A$ be a positively graded artin algebra, and let $M$ be a graded $A$-module. Then we have

1. the module $M$ is graded Gorenstein-projective if and only if it is Gorenstein-projective as a ungraded module;
2. the algebra $A$ is graded Gorenstein if and only if it is Gorenstein as a ungraded algebra; in this case, we have $G\dim^Z A = G\dim A$.

**Proof.** For (1), it suffices to recall that a graded module $M$ is graded Gorenstein-projective if and only if $\text{EXT}^i_A(M, A) = 0, \text{EXT}^i_{A^op}(\text{HOM}_A(M, A), A) = 0$ for $i \geq 1$ and the natural map $M \to \text{HOM}_{A^op}(\text{HOM}_A(M, A), A)$ is an isomorphism of graded modules; moreover, for graded modules $M$ and $N$ we have for each $i$ a natural identification $\text{EXT}^i_A(M, N) = \text{Ext}^i_A(M, N)$ ([31, Corollary 2.4.7]). For (2), we observe that a graded module $M$ has finite injective dimension in $\text{mod}^Z A$ if and only if it has finite injective dimension as a ungraded module; moreover, the two dimensions are the same ([31, Theorem 2.8.7]).

One can show the graded analogues of Lemmas 4.2, 4.3 and 4.4. Using these, we have the following graded analogue of Theorem 4.1.

**Proposition 4.6.** Let $A = \oplus_{n \geq 0} A_n$ be a positively graded self-injective artin algebra. Denote by $T_2(A)$ the $2 \times 2$ upper triangular matrix algebra of $A$ which is graded such that $T_2(A)_n = T_2(A_n)$ for $n \geq 0$. Then we have a triangle equivalence

$$\text{Mon}(\text{mod}^Z A) \simeq D_{sg}(T_2(A)).$$

We apply Proposition 4.6 to the stable category of Ringel-Schmidmeier.

Let $k$ be a field and $p \geq 2$ be an integer. Recall from Section 3 that $A = k[t]/(t^p)$ with $\deg t = 1$, which is graded self-injective. We observe that $T_2(A)$ is isomorphic, as a graded algebra, to $T_2(k)[t]/(t^p)$, while the latter is graded such that $\deg T_2(k) = 0$ and $\deg t = 1$.

Recall that the category $\mathcal{S}(\tilde{p})$ is identified with $\text{Mon}(\text{mod}^Z A)$, and then the stable category $\mathcal{S}(\tilde{p})$ of Ringel-Schmidmeier is identified with $\text{Mon}(\text{mod}^Z A)$. Then the following is an immediate consequence of Proposition 4.6.

**Corollary 4.7.** Use the notation above. Then there is a triangle equivalence

$$\mathcal{S}(\tilde{p}) \simeq D_{sg}(T_2(k)[t]/(t^p)).$$
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