Quantum nonequilibrium dynamics from Knizhnik-Zamolodchikov equations

Tigran A. Sedrakyan\textsuperscript{1} and Hrachya M. Babujian\textsuperscript{2}

\textsuperscript{1}Department of Physics, University of Massachusetts, Amherst, Massachusetts 01003, USA
\textsuperscript{2}A. Alikhanyan National Scientific Laboratory, Yerevan Physics Institute, Yerevan, Armenia

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We consider a set of quantum non-stationary models and show that their dynamics can be studied employing the links to Knizhnik-Zamolodchikov (KZ) equations for correlation functions in conformal field theories. We specifically consider the boundary Wess-Zumino-Novikov-Witten model where equations for correlators of primary fields are defined by modification of KZ equations and explore the links to dynamical systems. As an example of the workability of the proposed method, we provide exact solutions to dynamical systems that are specific multi-level generalizations of the two-level Landau-Zener system dubbed generalized Demkov-Osherov model. The method can be employed to study the nonequilibrium dynamics in multi-level systems from the solution of the corresponding KZ equations.

I. INTRODUCTION

There has been considerable interest in nonequilibrium dynamics of quantum systems (see e.g., Refs. \textsuperscript{11} over the last couple of decades that surged again recently in connection with quantum information problems\textsuperscript{12}. These include the dynamical discrete-state Bardeen-Cooper-Shriffer (BCS) pairing models\textsuperscript{13,14} where for example the interaction strength can be made time dependent, and various multi-level Landau-Zener tunneling models and their many body generalizations\textsuperscript{15-19}.

The cornerstone in the integrable properties of the discrete-state BCS hamiltonian is Richardson’s exact solution\textsuperscript{20,21} of such dynamical models leading to various exact solutions of such dynamical models with far-reaching applications in the theory of strongly correlated quantum matter. The exact solution is achieved by employing the connections to an exactly solvable Gaudin magnet\textsuperscript{22}. The connection is established upon rewriting the bilinears of spinful fermion operators in $s = 1/2$ operators using Schwinger fermion construction. As a result, the Richardson Hamiltonian is mapped to an exactly solvable spin-1/2 magnet with mutually commuting integrals of motion. Notably, the number of the integrals of motion is equal to the number of degrees of freedom in the system, a property that guarantees integrability. Moreover, these integrals of motion of the Richardson model are a linear combination of the corresponding integral of motion of the Gaudin magnet and $S^3$ component of the corresponding $s = 1/2$ operator.

Another profound connection is between the Gaudin magnet and a two-dimensional conformal field theory (CFT)\textsuperscript{23} associated with the $SU(2)$ group, namely the Wess-Zumino-Novikov-Witten (WZNW) theory. Generally, it is well established that theories, where current algebra is a fundamental symmetry, are governed by WZNW action, which has topological properties. The connection with magnet is through the correlation functions of the primary fields in CFT that satisfy a series of Knizhnik-Zamolodchikov (KZ)\textsuperscript{24} equations involving constants of motion of the Gaudin model. The correlation functions in $SU(2)$ WZNW have been obtained from the solution of KZ equations with constants of motion of the Gaudin model by one of us and a collaborator in Refs. \textsuperscript{49,50}. This approach was further developed in Ref. \textsuperscript{51} where a boundary $SU(2)$ WZNW model was identified and it was shown that the correlation functions of the primary fields there satisfy the series of modified KZ equations, where the Gaudin operators are replaced by constants of motion of the Richardson model. The approach was also used to obtain exact solutions to a variety of dynamical systems in Ref. \textsuperscript{52}. Other profound implications of KZ equations in gauge theories and quantum spin chains have recently been reported in Refs. \textsuperscript{53}.

Therefore, strikingly it turns out that the pairing Hamiltonian and a boundary WZNW CFT are related to one another in a nontrivial manner via the modification of KZ equations for primary fields in CFT that include integrals of motion of the pairing Hamiltonian. Consequently, it is expected that analytic properties of the correlation functions of conformal field theories and the spectral and dynamical properties of the pairing Hamiltonian can be explored on the same footing. One of the aims of the present work is to manifest this relation explicitly. To this end, we discuss the precise modification of the KZ equations and that these are satisfied by the correlation functions of primary fields in the boundary WZNW model. The main goal of the present work, however, is to study several dynamical systems that can be solved employing the connection through modified KZ equations, where time-dependence is explicit, but can be treated exactly upon solving the set of KZ equations.

As we see, through nontrivial correspondences, seemingly disconnected domains of theoretical physics are linked one with another. Following the same prescription above, several dynamical systems discussed below can be mapped on a variant of the pairing Hamiltonian, with the interaction parameter playing the role of a function of the time variable. Then, it can be shown that the Schrödinger equations for wave functions in these models lead to MKZ equations for correlation functions of primary fields in $SU(2)$ current algebra. Using the relation to the modified KZ equations, one can find the solutions of such dynamical models leading to various...
incarnations of the modified KZ equations in different situations. To this end, in the second part of the paper, we consider first the multi-level generalization of the two-level Landau-Zenner system, namely the Demkov-Osherov model, and its generalization. We show that the generalized Demkov-Osherov model and its partner generalized bow-tie models are linked to MKZ equations, which can be solved exactly. We present this analytical solution and discuss its implications.

The remainder of the paper is organized as follows. In section II, we construct the boundary WZNW model, which leads to MKZ. In section III, we consider the dynamical multi-level Landau-Zenner problem and its simplest realizations: the Demkov-Osherov model and its generalization. A class of exact solutions to these dynamical systems is presented. In the Appendix, we present the details of the construction of the boundary term in the WZNW model.

II. MODIFIED KZ EQUATIONS AND BOUNDARY WZNW MODEL

Another connection is between the gapless generalization of the Heisenberg model to spin-\(\frac{1}{2}\) [50, 51]. Specifically, it was shown by Affleck and Haldane [52] that the connection reveals itself in identifying the quantum field theory corresponding to the critical point with the WZNW model with action

\[
S_{WZNW}(g) = \frac{k}{16\pi} \int_{S^2} d^2\zeta \, \text{tr} \left[ \partial_a g^i \partial^a g^i \right] - \frac{ik}{24\pi} \int_{S^2} d^2\zeta \epsilon^{\mu\nu\rho} \text{tr} \left[ g^1 \partial_\mu g^1 \partial_\nu g^1 \partial_\rho g^1 \right].
\]  

(1)

Here integration in the first term is over a two-dimensional manifold, \(S^2\), corresponding to a compactified complex plane parameterized by \((w, \bar{w})\) and group elements \(g(w, \bar{w}) \in SU(2)\). Integration in the second topological WZNW term is over a three-dimensional manifold \(\Sigma^3\), with \(x = (w, \bar{w}, \xi) \in \Sigma^3\), whose boundary at \(\xi = 0\) is the aforementioned sphere, \(S^2 = \partial \Sigma^3\) and the function \(g(w, \bar{w}; \xi = 0)\) is extended into the interior of the ball \(\xi \in [0, 1]\) in a non-unique way. The parameter \(k\) in Eq. (1) is an integer number, which is the level of the corresponding CFT and linked to spin of the chain model as \(k = 2s\). The WZNW action is invariant under conformal and non-Abelian current algebras.

In the seminal work Ref. 48, Knizhnik and Zamolodchikov have shown that \(N\)-point correlation functions, \(G(w_1, \ldots, w_N | \bar{w}_1, \ldots, \bar{w}_N) = \langle \phi_{s_{i1}}(w_1, \bar{w}_1) \cdots \phi_{s_{iN}}(w_N, \bar{w}_N) \rangle_{S_{WZNW}}\) of primary fields with spins \(0 \leq s_i \leq k/2\), \(i = 1, \ldots, N\) of the WZNW model Eq. (1), satisfy a system of first-order differential KZ equations. In the holomorphic sector these equations can be written as

\[
\left( (k + 2) \partial_{w_i} - \hat{H}_i^G \right) G(w_1) = 0,
\]  

(2)

where

\[
\hat{H}_i^G = \sum_{\ell \neq i} \frac{\hat{S}_i \cdot \hat{S}_\ell}{w_i - w_\ell},
\]  

(3)

are the integrals of motion of the Gaudin magnet model, with \(\hat{S}_i\) being a set of generators of \(SU(2)\) with \(i = 1, \ldots, N\). Gaudin model Hamiltonian \(\hat{H}_G\) is integrable and is linear combination of integrals of motion \(\hat{H}_i^G = 2 \sum_{l=1}^N w_l \hat{H}_l^G\). As we pointed in the introduction, a class of models including the celebrated Richardson’s pairing Hamiltonian [32, 33] whose conserved integrals of motion can be regarded as a generalization of those of the Gaudin magnet [24, 64]

\[
\hat{H}_i^R = \lambda \hat{S}_i^3 + \hat{H}_i^G, \quad \left[ \hat{H}_i^R, \hat{H}_j^R \right] = 0,
\]  

(4)

leading to modified KZ equations (MKZ), namely

\[
\left[ (k + 2) \partial_{w_i} - \hat{H}_i^R \right] G(w) = 0.
\]  

(5)

Below we will follow the idea put forward in Ref. [40] and show that MKZ equations, with \(l = 1, \ldots, N\), are satisfied by the primary fields of the boundary WZNW CFT.

A. Boundary WZNW term

Consider left and right boundary terms of the WZNW model based on left-flowing and right-flowing currents \(J^a(w) = \text{tr}[\hat{S}_a g^1(w) \partial_w g(w)]\), \(J^a(\bar{w}) = \text{tr}[\hat{S}_a g^1(\bar{w}) \partial_{\bar{w}} g(\bar{w})]\) \(a = 1, 2, 3\), with spin-\(s\) generators of \(SU(2)\) algebra, \(\hat{S}_a\), and

\[
S^L_{\text{bound}}(C) = \alpha \oint_C dw J^a(w),
\]

(6)

\[
S^R_{\text{bound}}(\bar{C}) = \alpha \oint_{\bar{C}} d\bar{w} \bar{J}^a(\bar{w}).
\]

Here the coefficient \(\alpha \in \mathbb{R}\) is real. Then, upon adding these currents to the action of the WZNW model, \(S_{WZNW}(g)\), we obtain the model dubbed boundary WZNW (BWZNW):

\[
S_{BWZNW}(g) = S_{WZNW}(g) + S^L_{\text{bound}}(C) + S^R_{\text{bound}}(\bar{C}).
\]  

(7)

We show below that the primary fields in this boundary theory satisfy MKZ equations Eq. (5). In Eq. (6) contours \(C\) and \(\bar{C}\) are largest contours encompassing complex numbers \(w_i\) and approaching \(\mathcal{C}_\infty\). Contour \(C\) have clockwise rotation. Both contours should contain all points \(w_i\) of the correlators, which are under consideration (see Fig. 1). Note also that due to conformal invariance, the left currents \(J^a(w), a = 1, 2, 3\), do not depend on \(\bar{w}\) and likewise the right currents, \(\bar{J}^a(\bar{w})\), do not depend on \(w\).

Generally, we are interested in calculating a correlation function of arbitrary spin \(0 \leq s_i \leq (k/2)\) primary fields at points \(w_i, i = 1, \ldots, N\) in the \(SU(2)\) boundary WZNW model (see Fig 1). Namely \(G(w_1, \ldots, w_N) = \langle \phi_{s_{i1}}(w_1, \bar{w}_1) \cdots \phi_{s_{iN}}(w_N, \bar{w}_N) \rangle_{S_{BWZNW}}\)
distances. It shrinks to the sum of contours $C_i$, $i = 1, \cdots, N$ encircling points $w_i$: $C = \bigsqcup_{i=1}^{N} C_i$.

\[ \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(z_N) \rangle_{SWZNW}, \]

where functional averaging is defined in a standard way $\langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_s = \int [D\phi] \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N)e^{-SWZNW-S_{\text{bound}}(C)}$. Due to this relation, one can evaluate any correlator in the boundary model via its relation to the correlation function of the same primary fields multiplied with $\Phi[C] = e^{-S_{\text{bound}}(C)}$ but with average in the bulk WZNW model. Namely, $G(z_1, \cdots, z_N) = \langle \Phi[C] \phi_{s_1}(z_1) \cdots \phi_{s_N}(z_N) \rangle_{SWZNW}$.

With this rearrangement and conceptually new averaging procedure, one can show that the primary fields satisfy the MKZ equation following the standard procedure of deriving the KZ equations in the ordinary WZNW model. We present a detailed discussion of the outlined procedure in the next subsection.

B. Modified Knizhnik-Zamolodchikov equations

The emergence of ordinary KZ equations in CFT is the consequence of the fact that by definition, primary fields $\phi_s(w)$ are null vectors in current algebra $SU(2)$. Namely the primary fields fulfill constraints

\[ [L_0 - \frac{\lambda}{k+2} J_0^a] \phi_s(w) = 0, \]

where operators $J_0^a$ and $J_{-1}^a$ are 0 and $-1$ coefficients of the Laurent series of the current $J^a(w)$, while $L_0 = \partial_w$ is the zero component of the Virasoro algebra. When they are acting on any field at $w \in C_1$, then one has

\[ J_0^a = \oint_{C_1} J^a(u) du, \quad J_{-1}^a = \oint_{C_1} \frac{J^a(u)}{u-w} du. \quad (8) \]

The null vector condition is the same also in the boundary model. Therefore its gives the following equation for correlation functions

\[ \langle [\partial_{w_1} - \frac{1}{k+2} J_0^a] \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{BWZNW} = \]

\[ (e^{\Phi(C)} [\partial_{w_1} - \frac{1}{k+2} J_0^a] \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{BWZNW} = 0. \quad (9) \]

Here we consider equation for the first primary field with argument $w_1$, therefore operator $J_0^a J_0^a$ is acting on $\phi_{s_1}(w_1)$. This means that in the integral representation of these operators, Eq. (8), contours should circle $w_1$. As one can see, in Eq. (9), there are two terms: The first term is the derivative, $\partial_{w_1}$. The second term is $J_0^a J_0^a$. By definition, the primary fields are those that are eigenstates of the current operator, $J^a(u)$. Namely, they fulfill the following operator algebra relation

\[ J^a(u)\phi_{s_1}(w_1) = \frac{S^a_i}{u-w_i}\phi_{s_1}(w_1) + \cdots. \quad (10) \]

Upon using this relation, the direct calculation of two terms in the null vector condition (9) for the first primary field at $w_1$ gives

\[ \langle e^{\alpha w_1} S^a_i \partial_{w_1} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{BWZNW} = \]

\[ e^{\alpha w_1} S^a_i \partial_{w_1} \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{BWZNW} \quad (11) \]

and

\[ \langle e^{\alpha w_j} S^a_j \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{BWZNW} = \]

\[ \sum_{j=1}^{N} \frac{S^a_j S^a_i}{w_i-w_j} e^{\alpha w_j} \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{BWZNW} + c_1 \alpha S^a_i e^{\alpha w_j} \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{BWZNW}, \quad (12) \]

where $c_1 = S^a_i S^a_i$ and the summation over repeating indices is assumed (see Appendix A for all pertinent calculations of the present subsection). Combining these two equations into the null vector condition, Eq. (9), for the correlation function

\[ G(w_1, \cdots, w_N) = \langle \phi_{s_1} \cdots \phi_{s_N} \rangle_{BWZNW} \]

\[ \langle e^{\alpha w_j} S^a_j \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{BWZNW}, \quad (13) \]

where we have used operator algebra (10) for primary field, we arrive at following MKZ equations

\[ [(k+2)\partial_{w_1} - \lambda S^a_i \sum_{j=2}^{N} \frac{S^a_j S^a_i}{w_i-w_j}] G(w_1, \cdots, w_N) = 0. \quad (14) \]

Here the coefficient $\lambda = \alpha (k+2 + c_1)$. Such an equation was established in Ref. 55 for the Maxwell-Bloch system. Below we consider a class of nonequilibrium systems whose dynamics can be studied from the exact solution of the corresponding MKZ equations.
III. MULTI-LEVEL LANDAU-ZENNER PROBLEM AND ITS DESCENDANTS

Integrable aspects of models with time-dependent Hamiltonians have recently been studied in Refs. 30,42. Here we present the solutions of the Demkov-Osherov [34] bow-tie (BT) model [35,36,37]. This is achieved from the exact solutions of the associated MKZ equations. We show that GBT contains a set of integrals of motion, which should be considered classical or, equivalently, large-spins $s \gg 1$. Using the off-shell Algebraic Bethe Ansatz (OSABA) technique developed in Refs. 50,51,52, we find the solution of this MKZ for correlation functions in GBT.

In its general formulation, the multilevel LZ problem is defined by a time-dependent Hamiltonian of the form

$$H_{LZ} = \hat{A} + \hat{B}t,$$

(15)

where $\hat{A}$ and $\hat{B}$ are $(n+1) \times (n+1)$ Hermitian matrices. In general, for arbitrary $\hat{A}$ and $\hat{B}$ matrices, corresponding time-dependent Schrödinger equation,

$$-i\partial_t \Psi(t) = H_{LZ} \Psi(t),$$

(16)

can be solved formally via utilizing the $T$-exponent representation of the wave function. The latter however is practically not very useful for computation of observables and other applications. Though, in some simple cases, a solution can be found. The simplest case is when matrix $\hat{B}$ has $n$ coinciding eigenvalues, and only one is different. Then the unitary matrix $U$, which diagonalizes $\hat{B}$, is degenerate on a subgroup of $n \times n$ unitary matrices. The latter can be used to diagonalize $n \times n$ minor of the matrix $\hat{A}$. After this procedure, the Hamiltonian $H_{LZ}$ acquires the form

$$H_{DO} = \begin{pmatrix}
  t + a_{00} & v_{01} & \cdots & v_{0n} \\
v_{01} & a_{01} & 0 & \cdots & 0 \\
v_{02} & 0 & a_{02} & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v_{0n-1} & 0 & \cdots & a_{0n-1} & 0 \\
v_{0n} & 0 & \cdots & 0 & a_{0n}
\end{pmatrix},$$

(17)

which was first defined in Ref. 34 and dubbed Demkov-Osherov (DO) model. BT model (parametrized by a set of constants \{r_i\}) is equivalent to the DO model and their Hamiltonians are linked by the following linear transformation:

$$a_{0i} = r_i t + t.$$

(18)

These models and their generalizations were discussed in Refs. 41,42.

A. The solution of Demkov-Osherov and bow-tie models

We look for the eigenvalues and wave functions of the Hamiltonian, $H_{DO} | x_m(t) = E_m^0(t) | x_m(t)$, in the form

$$| x_m(t) \rangle = \sum_{k=0}^{n} \gamma_k \frac{\gamma_k}{x_m(t) - \epsilon_k} | k \rangle,$$

(19)

where

$$| k \rangle = (0, \ldots, 1, \ldots, 0)^T, \quad k = 0, 1, \ldots n.$$  

(20)

Here in $| k \rangle$ the unity entry stands at the $k$'th place where the leftmost entry has number zero. Now, we assume that the energy eigenvalues corresponding to wave function, $| x_m(t) \rangle$, are of the form

$$E_m^0(t) = \frac{\gamma_0^2}{x_m(t) - \epsilon_0},$$

(21)

where the functions $x_m(t)$ and the connection between $\gamma_k$ and $\epsilon_k$ and parameters entering the DO Hamiltonian Eq. (17) are yet to be determined from self-consistency relations. This so far is an ansatz, which will simplify the final result considerably. In this basis, the DO Hamiltonian (17) acquires a simple form

$$H_{DO} = (t + a_{00}) | 0 \rangle \langle 0 | + \sum_{i=1}^{n} a_{0i} | i \rangle \langle i |$$

$$+ \sum_{i=1}^{n} v_{0i} (| 0 \rangle \langle i | + | i \rangle \langle 0 |).$$

(22)

We substitute now the expressions in Eq. (19) for eigenstate and Eq. (21) for energy into the Schrödinger equation. This yields

$$H_{DO} | x_m(t) = E_m^0(t) | x_m(t) \rangle.$$  

(23)

Consider the equation corresponding to basis element $| k = 0 \rangle$. The equation thus will acquire the following algebraic form:

$$(a_{00} + t) \frac{\gamma_0}{x_m(t) - \epsilon_0} + \sum_{i=1}^{n} v_{0i} \frac{\gamma_i}{x_m(t) - \epsilon_i}$$

$$= \frac{\gamma_0^2}{x_m(t) - \epsilon_0} \cdot \frac{\gamma_0}{x_m(t) - \epsilon_0}.$$  

(24)

It is straightforward to check that the eigenvalue equation will be fulfilled if the original parameters $v_{01}, \cdots v_{0n}$ and $a_{01}, \cdots a_{0n}$ are self-consistently connected with $\gamma_i$ and $\epsilon_i - \epsilon_0, i, \cdots n$ as

$$v_{0i} = \frac{\gamma_0 \gamma_i}{\epsilon_0 - \epsilon_i},$$

(25)

$$a_{0i} = \frac{\gamma_0^2}{\epsilon_i - \epsilon_0}, \quad i = 1, \cdots n.$$
From Eq. [24], the parameters \( a_{00} \) and the time variable, \( t \), are bound with relations

\[
a_{00} = \sum_{i=1}^{n} \frac{\gamma_i^2}{\epsilon_i - \epsilon_0}, \quad (26)
\]

\[
t = \sum_{i=0}^{n} \frac{\gamma_i^2}{x_m(t) - \epsilon_i}. \quad (27)
\]

Eq. (26) yields a condition under which the eigenvalue equation has a solution, while Eq. (27) for time gives polynomial equation of \( n + 1 \) order for \( x_m(t) \). That polynomial \( t \)-dependent equation defines \( m = 0, \cdots n \) solutions in the form of functions for \( x_m(t) \). Importanty, the relation Eq. (27) appears when we equate the time dependent terms in the left and right hand sides of Eq. [24].

Equations for \( |k\rangle \) with \( k \neq 0 \) are also fulfilled due to relations in Eq. (25). From the relations in Eq. (25), one can find

\[
\frac{v_{bi}}{a_{0i}} = -\frac{\gamma_k}{\gamma_0}, \quad i = 1, \cdots n, \quad (28)
\]

that directly relates parameters in the DO Hamiltonian Eq. (17), with the set of new parameters (\( \gamma_0; \{ \gamma_i \} \)).

### B. Generalized Demkov-Osherov model: The link to KZ equations

DO, and BT models can be generalized by redefining the properties of eigenvalues of the matrix \( \tilde{B} \). Let us assume that the matrix \( \tilde{B} \) has \( n - 1 \) identical eigenvalues, \( b_2 \), and two other eigenvalues, \( b_1 \). Then diagonalizing the matrix, \( U \), we will have degeneracy over subgroup of \( SU(n - 1) \) matrices, which will allow diagonalizing \( (n - 1) \times (n - 1) \) diagonal minor of \( \tilde{A} \). Then, subtracting from the Hamiltonian Eq. (17) the diagonal matrix, \( b_2 t \times id \), and after appropriate rescaling of the time by the factor \( b_1 - b_2 \), one arrives at the following generalization of Demkov-Osherov model (GDO) given in the form of a simple matrix Hamiltonian:

\[
H_{GDO} = \begin{pmatrix}
(t + v_{00}) & v_{01} & v_{02} & \cdots & v_{0n} \\
v_{01} & (t + v_{11}) & v_{12} & \cdots & v_{1n} \\
v_{02} & v_{12} & a_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
v_{ij} & v_{ij} & 0 & \cdots & a_j & 0 \\
v_{on} & v_{1n} & 0 & \cdots & 0 & a_n
\end{pmatrix} \quad (29)
\]

Since the Hamiltonian is Hermitian, all parameters, \( v_{ij} \) and \( a_j, j = 2, \cdots n \), that are present in it, are real. In the bra-ket notations, the Hamiltonian Eq. (29) reads

\[
H_{GDO} = \sum_{k=0,1} \left( t + v_{kk} \right) |k\rangle \langle k| + \sum_{i=2}^{n} a_i |i\rangle \langle i|
\]

\[+ v_{01} (|0\rangle \langle 1| + |1\rangle \langle 0|) + \sum_{k=0,1;i=2} v_{ki} (|k\rangle \langle i| + |i\rangle \langle k|), \quad (30)\]

Below we are going to link the time-dependent Schrödinger equation \(-i\partial_\omega \Psi = H_{GDO} \Psi \), with MKZ equations and show that the solution of the latter satisfies the former. To this end, after Fourier transformation, the Schrödinger equation acquires the form of \( \omega \Psi = H_{o} \Psi \). The latter in detailed form reads

\[
\omega \psi_0 = (-i\partial_\omega + v_{00})\psi_0 + v_{01}\psi_1 + \sum_{k=2}^{n} v_{0k}\psi_k
\]

\[
\omega \psi_1 = (-i\partial_\omega + v_{11})\psi_1 + v_{01}\psi_0 + \sum_{k=2}^{n} v_{1k}\psi_k \quad (31)
\]

\[
\omega \psi_k = a_k\psi_k + v_{0k}\psi_0 + v_{1k}\psi_1, \quad k = 2, \cdots n.
\]

The solution of the last set of equations for \( \psi_k \) with \( k = 2, \cdots n \) is straightforward, and yields

\[
\psi_k = \frac{v_{0k}\psi_0 + v_{1k}\psi_1}{\omega - a_k}, \quad k = 2, \cdots n \quad (32)
\]

Substituting these expressions for \( \psi_k \) into the first two equations in (31), we obtain two equations for \( \psi_{0,1} \). The latter can be simplified further by the following transformation of the wave functions \( \psi_{0,1} = e^{i\omega^2/2}\hat{\psi}_{0,1} \). After this transformation, the linear in \( \omega \) term can be eliminated from the equations. Finally, we see that these two equations reduce to the following \( 2 \times 2 \) matrix equation

\[
i\partial_\omega \Phi = \begin{bmatrix} b_1^\mu S_1^\mu + \sum_{k=2}^{n} \frac{b_1^k}{\omega - a_k} \end{bmatrix} \Phi, \quad \Phi = (\psi_0, \psi_1)^T \quad (33)
\]

where the matrix \( S_1^\mu = (1, \sigma_1, \sigma_2, \sigma_3) \), \( \mu = 0, 1, 2, 3 \) is given by the unity and the set of three Pauli matrices (the presence of the unity operator in the set of spin operators implied that we have algebra \( su(2) \) instead of \( su(2) \)). In (33) vectors \( b_k \) are found to be

\[
b_1^0 = \frac{(v_{00} + v_{11})}{2}, \quad b_1^3 = \frac{(v_{00} - v_{11})}{2}, \quad b_1^- = b_1^+ = v_{01}, \quad (34)
\]

\[
b_k^0 = \frac{(v_{0k} + v_{1k})}{2}, \quad b_k^3 = \frac{(v_{0k} - v_{1k})}{2}, \quad b_k^+ = b_k^- = v_{0k}v_{1k}.
\]

One can see that Eq. (34) is surprisingly similar to one of the MKZ equations, Eq. (14), but instead of the full set of \( n \) quantum spin generators, we have only one spin generator \( S_i^\mu \) associated with energy \( \omega \) as a spectral parameter. Other \( n - 1 \) spins, associated with spectral parameters \( a_k, k = 2, 3, \cdots n \), are classical vectors, \( b_k^\mu \). They can be treated quasi-classically as large quantum spin limits.
The term corresponds to the $\lambda S^3$ modification in MKZ equations (4).

Now it is natural to ask whether there is a sufficient number of integrals of motion for the Hamiltonian in the Schrödinger equation (33), namely

$$H_1 = b_1^\mu S_1^\nu + \sum_{k=2}^{n} S_k^\mu b_k^\nu, \quad k = 2, \cdots n$$

(35)

to be integrable? One can expect that the set of operators

$$H_k = \sum_{k' \neq k} \frac{b_k^\mu b_{k'}^\nu}{a_k - a_{k'}} + \frac{b_k^\mu S_k^\nu}{a_k - \omega}, \quad k = 2, \cdots n$$

(36)

are candidates of integrals of motion, since they are large spin quasiclassical limits of the operators $H_j$ in the set of MKZ equations (34). Indeed, it appears that their commutators are equal to

$$[H_1, H_j] = \frac{1}{(a_i - \omega)(a_j - \omega)} b_i^\mu b_j^\nu [S_i^\mu, S_j^\nu] \quad (a_i = \omega, a_j = \omega)$$

$$= \frac{b_i^\mu b_j^\nu \epsilon_{abc}}{(a_i - \omega)(a_j - \omega)}, \quad i, j = 1, 2, \cdots n. \quad (37)$$

These operators will commute if all classical limits of spins, $b_k^\mu, k = 1, 2, \cdots n$, are parallel: $b_k^\mu |b_j^\nu$. A straightforward analysis of expressions in Eq. (34) for $b_k^\mu$ shows that the parallel property will be fulfilled if

$$v_{ij} = \gamma_i \gamma_j, \quad i, j = 0, 1, \cdots n. \quad (38)$$

Then we have a complete set of MKZ equations, where $\omega$ plays the role of the first spectral parameter while others are $a_k, k = 2, \cdots n$. Moreover, MKZ equations, Eq. (6), with this set of Hamiltonians have a solution because zero curvatures condition is fulfilled

$$[H_i, H_j] = 0, \quad i, j = 0, 2, \cdots n, \quad (39)$$

where $a_0 = \omega$. Hence, the solution of GDO model, with "parallel" conditions Eq. (38) on parameters, is defined by the solution of the MKZ equations with one quantum and $n-1$ classical spins. Wave function of GDO model, $\Phi(\omega, a_2, \cdots a_n)$, thus can be treated as correlation function $G(\omega, a_1)$ in MKZ equation.

C. Solution of the generalized Demkow-Osherov model

The solutions of KZ and MKZ equations based on OSABA were formulated in Refs. 50,51,65. The situation in the GDO model is simpler. Since all vectors, $b_i$ are parallel it follows that all matrices $b_k^\mu S_i^\nu$ are commuting. Therefore, one can work with them as with ordinary commuting numbers.

The straightforward inspection yields that due to commutativity

$$\partial_\omega (\omega - a_i) b_i^\mu S_i^\nu = \frac{b_i^\mu S_i^\nu}{\omega - a_i} (\omega - a_i) b_i^\mu S_i^\nu.$$

(40)

Therefore, one can derive the general solution of MKZ equations based on the Hamiltonian $H_1$ given in Eq. (35) and integrals of motion $H_i, i = 2, \cdots n$ given by Eq. (36). Namely, the solution of

$$\partial_\omega \Phi(\omega, \{a_i\}) = -i H_i \Phi(\omega, \{a_i\}),$$

$$\partial_{a_i} \Phi(\omega, \{a_i\}) = -i H_i \Phi(\omega, \{a_i\})$$

(41)

we construct in several steps. As the first step, one can look for $\Phi(\omega, \{a_i\})$ in the form

$$\Phi(\omega, \{a_i\}) = \prod_{j > i} (a_i - a_j)^{-ib_i^\mu b_j^\nu} \Phi(\omega, \{a_i\})$$

(42)

Here vectors $b_i^\mu$ are defined by expressions provided in Eqs. (34) and (38). For the new "wavefunction" $\Phi_0$ we obtain a new equation that is similar to Eq. (41), wherein the RHS of the second equation in the operator $H_1$ one retains only the second term. As the second step, in the same second equation one can separate the $\mu = 0$ component of vectors $b_i^\mu$ from the rest in terms that contain them. This gives

$$\partial_\omega \Phi_0(\omega, \{a_i\})$$

$$= -i \left( b_0^\mu + b_1 S + \sum_{k=2}^{n} \frac{b_k^0}{\omega - a_k} + b_k b_S \right) \Phi(\omega, \{a_i\})$$

$$= -i \left( \frac{b_1^0}{a_1 - \omega} + \frac{b_1 S}{a_1 - \omega} \right) \Phi_0(\omega, \{a_i\}).$$

(43)

Here we define the $S = (\sigma^1, \sigma^2, \sigma^3)$, where $\sigma^j, j = 1, 2, 3$, are Pauli matrices. For $\Phi_0$ one will then obtain

$$\Phi_0(\omega, \{a_i\}) = e^{-i b_0^\mu n} \prod_{k=2}^{n} (\omega - a_k)^{-ib_k^\mu} \Phi_s(\omega, \{a_i\}),$$

(44)

where $\Phi_s(\omega, \{a_i\})$ is still unknown. After this one arrives at the equations for the function $\Phi_s(\omega, a_i)$ of the form

$$\partial_\omega \Phi_s(\omega, \{a_i\}) = -i \left( b_1 S + \sum_{k=2}^{n} \frac{b_k S}{\omega - a_k} \right) \Phi_s(\omega, \{a_i\})$$

$$\partial_{a_i} \Phi_s = -i \frac{b_1 S}{a_i - \omega} \Phi_s(\omega, \{a_i\}).$$

(45)

Earlier we have observed that commutativity of the Hamiltonians $H_i$ for different $i$ leads to the condition of all vectors $b_i$ being parallel to each other, i.e.

$$b_i = \beta \hat{n},$$

(46)

where $\hat{n}$ is an arbitrary unit vector and $\beta_i$ are the norms of vectors $b_i$. Below all these norms will be linked to the parameters of the GDO Hamiltonian.

Let us now rewrite Eqs. (43) using the newly introduced notations:

$$\partial_\omega \Phi_s(\omega, \{a_i\}) = -i \left( \beta_1 \omega + \sum_{k=2}^{n} \frac{\beta_k \omega}{\omega - a_k} \right) \Phi_s(\omega, \{a_i\}),$$

$$\partial_{a_i} \Phi_s(\omega, \{a_i\}) = -i \left( \frac{\beta_1 \omega}{a_i - \omega} \right) \Phi_s(\omega, \{a_i\}).$$

(47)
The eigenvectors and eigenvalues of \( \mathbf{nS} \) here are defined as

\[
\mathbf{nS}\xi_m = m\xi_m, \quad (48)
\]

where \( \xi_m \) represents a two component spinor eigenvalue with \( m = \pm 1 \). One can directly compute \( \xi_m \) by solving eigenvalue equations directly in components. Using this fact we are now able to finally write the solution \( \Phi_s^m \) in the form

\[
\Phi_s^m = \exp(-i\omega\beta_1 m) \prod_{k=2}^n (\omega - a_k)^{-i\beta_k m}\xi_m. \quad (49)
\]

As the last step, we collect all parts of the solution and putting them together obtain

\[
\Phi^m = \exp[-i\omega(b_1^0 + \beta_1 m)] \prod_{j>i} (a_i - a_j)^{-i(b_0^{ij} + \beta_j \gamma_i)j}
\times \prod_{k=2}^n (\omega - a_k)^{-i(b_0^k + \beta_k m)}\xi_m. \quad (50)
\]

As our equations are linear, the general solution will be a linear combination of the two different solutions corresponding to two eigenvalues with \( m = \pm 1 \).

One can express final answer in terms of parameters \( \{\gamma_i\}, i = 1, \ldots, n \) by use of the expressions in Eqs. (34) and (38) ensuring parallelism of \( b \) vectors. In terms of parameters \( \gamma \), the \( b \) vectors have the following form

\[
b_1^0 = \frac{\gamma_0^2 + \gamma_1^2}{2}, \quad b_3^0 = \frac{\gamma_2^2 - \gamma_1^2}{2}, \quad b_1^\pm = \gamma_0\gamma_1, \quad (51)
\]

\[
b_k^0 = \frac{\gamma_0^2 + \gamma_1^2}{2}\gamma_k^2, \quad b_k^0 = \frac{\gamma_2^2 - \gamma_1^2}{2}\gamma_k^2, \quad b_k^\pm = \gamma_0\gamma_1\gamma_k^2,
\]

from where it follows that

\[
\beta_1 = b_1^0 = \frac{\gamma_0^2 + \gamma_1^2}{2}, \quad (52)
\]

\[
\beta_k = b_k^0 = \frac{\gamma_0^2 + \gamma_1^2}{2}\gamma_k^2, \quad k = 2, 3, \ldots, N.
\]

Finally, by use of Eqs. (51) and (52), one can simplify the expression in Eq. (30). Taking into account the factor \( e^{i\omega t/2} \) see below Eq. (32), the exact solution of GDO model acquires the form

\[
\Phi(\omega, \{a_i\}) = e^{i\omega t/2} \prod_{j>i}(a_i - a_j)^{-2\beta_j \beta_i}
\times \prod_{k=2}^n (\omega - a_k)^{-i\beta_k(1+m)} e^{-i\beta_k(1+m)}\xi_m, \quad (53)
\]

with \( m = \pm 1 \).

Fourier transform of Eq. (53) for \( m = \pm 1 \) defines the real-time evolution of the wave function

\[
\Phi(t, a_2, \ldots, a_n) = \int_{-\infty}^{\infty} d\omega \Phi(\omega, a_2, \ldots, a_n) e^{i\omega t}. \quad (54)
\]

Appropriately normalized wave function gives the system’s evolution starting from the initial state at \( t = -\infty \). Namely

\[
\Phi(t, a_2, \ldots, a_n) = G(t, a_2, \ldots, a_n)\Phi(-\infty, a_2, \ldots, a_n), \quad (55)
\]

where \( G(t, a_2, \ldots, a_n) = T \exp\{i \int_{-\infty}^{t} dt' H_{\text{GDO}}\} \) is the time evolution operator where \( T \) stands for time ordering. It defines the amplitudes of transmission probabilities between diabatic states at \( t = -\infty \) and \( t = \infty \). We can see from here that the state with \( m = -1 \) has trivial \( \omega \) dependence, \( \Phi(\omega) \sim e^{i\omega t/2} \), which translates into a similar oscillatory time-dependence of the wave-function \( \Phi(t) \sim e^{-i\omega t/2} \), that gives zero transmission probability between diabatic states.

The analytical form of Fourier transformation of the wave function for arbitrary \( n \) is cumbersome. However, for the simple case of \( n = 2 \), it is straightforward, yielding the extract transmission probability \( P \) of the Landau-Zener problem

\[
P = e^{-2\pi(\gamma_0^2 + \gamma_1^2)\gamma_2^2}. \quad (56)
\]

This is in full accordance with Landau-Zenner’s result.

IV. CONCLUSIONS

We have presented a set of quantum dynamical systems, which are linked to modified Knizhnik-Zamolodchikov equations and can be solved exactly upon employing this connection. Those are the BCS paring model (based on Richardson’s exact solution), multi-level Landau-Zener tunneling models, and their realizations as a generalization of Demkov-Osherov and bow-tie models. In these systems, interaction couplings can be considered to be time-dependent linearly, leading to the extension of KZ equations. This link is surprising and based on the fact that all these models contain integrals of motion of Gaudin magnets. This inherent property of integrability leads to MKZ equations. Using the link and integrable properties, we solve GDO and GBT models exactly. We believe this link of dynamical systems to MKZ equations is not limited and can be extended to other systems with time-dependent Hamiltonians.

Moreover, it is known that KZ equations are written for correlation functions of WZWN models. Here we have (re)addressed the question posed in Ref. 30, namely what is the CFT for which MKZ defines correlation functions? Using a different method, we have shown that the corresponding model is the WZWN model with a boundary term discussed first in Ref. 30.

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V. APPENDIX A

A. First term with derivative in the null-vector condition (5)

From the operator algebra Eq. (10), where the current $J^a(u)$ is acting on primary field at the point $w_j$, we have

$$J_0 \phi_{s_i}(w_i) = \oint_{C_i} du J^a(u) \phi_{s_i}(w_i)$$

$$= \oint_{C_i} \frac{S^a_j}{w-w_i} \phi_{s_i}(w_i) = S^a_i \phi_{s_i}(w_i).$$

(57)

Here we have used the fact that since $J^0(u)$ is acting on the primary field at $w_i$, the contour $C_i$ is circling the position $w_i$, after which Cauchy integration is applied.

The first term in Eq. (10) reads

$$\langle e^a \oint_{C_i} dw J^a(w) \partial_{w_i} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle$$

$$= \sum_k \frac{\alpha_k}{k!} \left( \oint_{C_i} dw J^a(w) \right)^k \partial_{w_i} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N)$$

(58)

Using Eq. (10), it is easy to calculate the linear in the current term in these series. Since, according to CFT, singularities in this operator product expansion may appear only at the positions $w_i$ of the primary field, we can shrink boundary contour $C$ into the sum of circles $C_i$ around those points: $C = \bigcup_{i=1}^{N} C_i$ (see Fig[1]). Then each term defined by the contour $C_i$ gives the action of the current on primary field $\phi_{s_i}(w_i)$. Using Eq. (10) for the action of current on the primary field, we obtain

$$\sum_{i=1}^{N} \oint_{C_i} dw J^a(w) \partial_{w_i} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N)$$

$$= \sum_{i=1}^{N} \oint_{C_i} dw \partial_{w_i} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N)$$

$$= \sum_{i=1}^{N} w_i S^a_i \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N).$$

(59)

Higher-order terms of current $J^a(w)$ will produce series, where $\sum_{i=1}^{N} \oint_{C_i} dw J^a(w)$ is replaced by $\sum_{i=1}^{N} w_i S^a_i$ and, therefore, one will get

$$\langle e^a \oint_{C_i} dw J^a(w) \partial_{w_i} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle$$

$$= e^a \sum_{i=1}^{N} w_i S^a_i \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N).$$

(60)

Eq. (60) reproduces Eq. (11) of the main text.

B. Second term in the null-vector condition (9)

After expanding the exponent and using expressions for the currents, the second term in (9) reads

$$I = \langle \Phi(C)(J^a_0,J^0_J) \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle$$

$$= \sum_{k=1}^{N} \frac{\alpha_k}{k!} \left( \oint_{C} dw J^a(w) \right)^k \oint_{C} du J^a(u) S^a_i \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N).$$

(61)

Here we have used the relation Eq. (51) for the action of $J^0_J$ on primary fields and expression Eq. (8) for $J^a_0$. Eq. (61) shows that besides primary fields at $w_i$, there are also currents $J^a(u)$ with which $J^a(w)$ in the exponent will have contractions according to $SU(2)$ current algebra relations

$$J^3(w)J^\pm(u) = \pm \frac{J^\pm(u)}{w-u}.$$

(62)

Therefore contour $C$ will include singularities not only at $w_i, i = 1 \cdots N$, but also in $u$ (see Fig[2]). Hence boundary contour again will shrink into the sum $C = \bigcup_{j=1}^{N} C_j$. As in Eq. (60), the contribution of the contours $C_j, j = 2, 1 \cdots N$ is equivalent to the replacement of $\sum_{i=1}^{N} \oint_{C_i} dw J^a(w)$ by $\sum_{i=1}^{N} w_i S^a_i$. As a result, from Eq. (61) one obtains

$$I = \langle \sum_{k=1}^{N} \frac{\alpha_k}{k!} \left( \oint_{C} dw J^a(w) \right)^k \oint_{C} du J^a(u) S^a_i \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle$$

(63)

The situation with $C_1$ is different. According to the CFT rules, the order of currents $J^a(w)$ and $J^a(u)$ defines the size of the contours in their integrals, namely, $C_1 \supset C_1$, see Fig[2].

Now, let us first analyze the linear term in the series of exponential in Eq. (63). After using the current algebra relation Eq. (64) and Cauchy integration over $w$, one obtains

$$\oint_{C_1} dw \oint_{C_1} du J^a(w) J^\pm(u) \frac{J^a(u)}{u-w}$$

$$= \oint_{C_1} dw \oint_{C_1} du J^\pm(u) \frac{J^a(u)}{u-w}$$

$$= \oint_{C_1} dw \oint_{C_1} du J^\pm(u)$$

(64)

Hence, for the exponent in Eq. (64), one will have

$$\langle e^a \oint_{C_1} dw J^a(w) \oint_{C_1} du J^\pm(u) \frac{J^a(u)}{u-w} \rangle$$

$$= \oint_{C_1} \oint_{C_1} du J^\pm(u) \frac{J^a(u)}{u-w}$$

and the entire second term Eq. (64) becomes

$$I = \langle \oint_{C_1} du e^a J^+u J^a(u) S^a_i \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle$$

(66)
FIG. 2: Boundary contour C shrinks to the sum of contours $C_i$, $i = 1, \cdots, N$. Here $C_1$ and $C_i$, $i = 2, \cdots, N$ are the contours in the integral expression of the operators $J^a_{s1}$, which they act on the primary fields, $\phi_{a_1}(w_i)$, at points $w_i$, $i = 1, \cdots, N$. We have that $C_1 \supset C_1$.

Using the extension of the relation Eq. [10] for primary fields to their product

$$J^a(u)\phi_{a_1}(w_1) \cdots \phi_{a_1}(w_N) = \sum_{i=1}^{N} S^a \phi_{a_1}(w_i) \cdots \phi_{a_1}(w_N).$$

and after Cauchy integration over $u$, one obtains

$$I = \left[ \sum_{i=2}^{N} e^{\alpha w_i} S^a_i S^a_{i-1} + e^{-\alpha w_i} S^a_i S^a_{i-1} \right] \frac{1}{w_i - w_i} \int_{C_1} e^{\alpha w_i} S^a_i S^a_{i-1} \left( \phi_{a_1}(w_1) \cdots \phi_{a_1}(w_N) \right) \, du.$$

Now, after using the identity

$$S^a_i e^{\mp \alpha w_i} = S^a_i e^{\pm \alpha w_i},$$

and performing Cauchy integration, one arrives at

$$I = \left[ \sum_{i=2}^{N} S^a_i S^a_{i-1} + \alpha S^a_i S^a_{i-1} \right] \times e^{\alpha \sum_{i=2}^{N} w_i S^a_i} \left( \phi_{a_1}(w_1) \cdots \phi_{a_1}(w_N) \right).$$

After defining $c_1 = S^a_i S^a_{i-1}$, we see that Eq. 70 coincides with the relation Eq. (12) of the main text.

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