Contact Equivalence Problem for Linear Hyperbolic Equations

Oleg I. Morozov
Department of Mathematics, Snezhinsk Physical and Technical Academy,
Snezhinsk, 456776, Russia
oim@foxcub.org

Abstract. We consider the local equivalence problem for the class of linear second order hyperbolic equations in two independent variables under an action of the pseudo-group of contact transformations. É. Cartan’s method is used for finding the Maurer – Cartan forms for symmetry groups of equations from the class and computing structure equations and complete sets of differential invariants for these groups. The solution of the equivalence problem is formulated in terms of these differential invariants.

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Introduction

In the present paper, we find necessary and sufficient conditions for two equations from the class of linear second order hyperbolic equations

$$u_{tx} = T(t, x) u_t + X(t, x) u_x + U(t, x) u$$

(1)

to be equivalent under an action of the contact transformation pseudo-group. We use Élie Cartan’s method of equivalence, [1] - [5], in its form developed by Fels and Olver, [6, 7], to compute the Maurer – Cartan forms, the structure equations, the basic invariants, and the invariant derivatives for symmetry groups of equations from the class. All differential invariants are functions of the basic invariants and their invariant derivatives. The differential invariants parametrize classifying manifolds associated with given equations. Cartan’s solution to the equivalence problem states that two equations are (locally) equivalent if and only if their classifying manifolds (locally) overlap.

The symmetry classification problem for classes of differential equations is closely related to the problem of local equivalence: symmetry groups of two equations are necessarily isomorphic if these equations are equivalent, while the converse statement is not true in general. The symmetry analysis of linear second order hyperbolic equations [11] is done by Lie, [16] Vol. 3, pp 492-523]. Two semi-invariants $H = -T_t + T X + U$ and $K = -X_x + T X + U$ were discovered by Laplace, [15]. These functions are unaltered under an action of the pseudo-groups of linear transformations $\mathbf{u} = c(t, x) \mathbf{u}$. In [22], Ovsiannikov found the invariants $P = K H^{-1}, Q = (\ln |H|)_{t,x} H^{-1}$ and used them to
classify equations \((1)\) with non-trivial symmetry groups. In [9, th 2.3], [10, § 10.4.2], it was claimed that the invariants \(P\) and \(Q\) form a basis of differential invariants for equations \((1)\), while all the other invariants are functions of \(P\) and \(Q\) and their invariant derivatives. In [14], a basis of five invariants and operators of invariant differentiation are found in the case \(P_x \neq 0\). In the case \(P_t \neq 0\) and \(P_x \neq 0\) two bases of four invariants are computed in [12].

In [18], the invariant version of Lie’s infinitesimal method was developed and applied to the symmetry classification of the class \((1)\).

The symmetry classification problem and invariants for the class of linear parabolic equations \(u_{xx} = T(t,x) u_t + X(t,x) u_x + U(t,x) u\) are studied in [16, 23, 11, 13] by Lie’s infinitesimal method. In [20, 21], Cartan’s method is applied to solve the contact equivalence problem for this class.

The paper is organized as follows. In Section 1, we begin with some notation, and use Cartan’s equivalence method to find the invariant 1-forms and the structure equations for the pseudo-group of contact transformations on the bundle of second-order jets. In Section 2, we briefly describe the approach to computing Maurer-Cartan forms and structure equations for symmetry groups of differential equations via the moving coframe method of Fels and Olver. In Section 3, the method is applied to the class of hyperbolic equations \((1)\). Finally, we make some concluding remarks.

1. Pseudo-group of contact transformations

In this paper, all considerations are of local nature, and all mappings are real analytic. Let \(E = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n\) be a trivial bundle with the local base coordinates \((x^1, \ldots, x^n)\) and the local fibre coordinate \(u\); then by \(J^2(E)\) denote the bundle of the second-order jets of sections of \(E\), with the local coordinates \((x^i, u, p_i, p_{ij})\), \(i, j \in \{1, \ldots, n\}, i \leq j\). For every local section \((x^i, f(x))\) of \(E\), the corresponding 2-jet \((x^i, f(x), \partial f(x)/\partial x^i, \partial^2 f(x)/\partial x^i \partial x^j)\) is denoted by \(j_2(f)\). A differential 1-form \(\vartheta\) on \(J^2(E)\) is called a contact form if it is annihilated by all 2-jets of local sections: \(j_2(f)^*\vartheta = 0\). In the local coordinates every contact 1-form is a linear combination of the forms \(\vartheta_0 = du - p_i dx^i, \vartheta_i = dp_i - p_{ij} dx^j, i, j \in \{1, \ldots, n\}, p_{ji} = p_{ij}\) (here and later we use the Einstein summation convention, so \(p_i dx^i = \sum_{i=1}^n p_i dx^i\), etc.) A local diffeomorphism

\[
\Delta : J^2(E) \to J^2(E), \quad \Delta : (x^i, u, p_i, p_{ij}) \mapsto (\overline{x}^i, \overline{u}, \overline{p}_i, \overline{p}_{ij}),
\]

\((2)\)

is called a contact transformation if for every contact 1-form \(\vartheta\) the form \(\Delta^* \vartheta\) is also contact. We use Cartan’s method of equivalence, [5, 24], to obtain a collection of invariant 1-forms for the pseudo-group of contact transformations on \(J^2(E)\). For this, take the coframe \(\{\vartheta_0, \vartheta_i, dx^i, dp_i | i, j \in \{1, \ldots, n\}, i \leq j\}\) on \(J^2(E)\). A contact transformation...
(2) acts on this coframe in the following manner:

$$\Delta^* \begin{pmatrix} \overline{v}_0 \\ \overline{v}_i \\ dx_i \\ dp_{ij} \end{pmatrix} = S \begin{pmatrix} \vartheta_0 \\ \vartheta_k \\ dx^k \\ dp_{kl} \end{pmatrix},$$

where $S : J^2(\mathcal{E}) \to \mathcal{G}$ is an analytic function, and $\mathcal{G}$ is the Lie group of non-degenerate block matrices of the form

$$\begin{pmatrix} a & \tilde{a}_k^k & 0 & 0 \\ \tilde{g}_i & h_i^k & 0 & 0 \\ \tilde{c}^i & \tilde{f}_{ik} & b_k^k & r_{ikl} \\ \tilde{s}_{ij} & \tilde{w}_{ij}^k & \tilde{z}_{ijk} & \tilde{q}_{ij}^{kl} \end{pmatrix}.$$ 

In these matrices, $i, j, k, l \in \{1, \ldots, n\}$, $r_{ikl}$ are defined for $k \leq l$, $\tilde{s}_{ij}$, $\tilde{w}_{ij}^k$, and $\tilde{z}_{ijk}$ are defined for $i \leq j$, and $\tilde{q}_{ij}^{kl}$ are defined for $i \leq j, k \leq l$.

Let us show that $\tilde{d}^2 = 0$. Indeed, the exterior (non-closed!) ideal $\mathcal{I} = \text{span}\{\vartheta_0, \vartheta_i\}$ has the derived ideal $\mathcal{I} = \{\omega \in \mathcal{I} | d\omega \in \mathcal{I}\} = \text{span}\{\vartheta_0\}$. Since $\Delta^* \mathcal{T} \subset \mathcal{I}$ implies $\Delta^* (\mathcal{I} \otimes \mathcal{T}) \subset \mathcal{I}$, we obtain $\Delta^* \vartheta_0 = a \vartheta_0$.

For convenience in the following computations, we denote by $(B_i^j)$ the inverse matrix for $(b_i^j)$, so $b_i^j B_j^k = \delta_i^k$, by $(H_i^j)$ denote the inverse matrix for $(h_i^j)$, so $h_i^j H_j^k = \delta_i^k$, change the variables on $\mathcal{Q}$ such that $g_i = \tilde{g}_i a^{-1}$, $f_{ij} = \tilde{f}_{ik} H_j^k$, $c_i = \tilde{c}_i a^{-1} - f_{ik} g_k$, $s_{ij} = \tilde{s}_{ij} a^{-1} - \tilde{w}_{ij}^k H_m^m g_m - \tilde{z}_{ijm} B_k^m c^k$, $w_{ij}^k = \tilde{w}_{ij}^k H_m^m - \tilde{z}_{ijm} B_k^m f_{ik}$, $z_{ijk} = \tilde{z}_{ijm} B_k^m$, $q_{ij}^{kl} = \tilde{q}_{ij}^{kl} - \tilde{z}_{ijm} B_m^m r_{mkl}$, and define $Q_{k'l'}^{kl}$ by $Q_{k'l'}^{kl} q_{ij}^{kl'} = \delta_i^k \delta_j^l$.

In accordance with Cartan’s method of equivalence, we take the lifted coframe

$$\begin{pmatrix} \Theta_0 \\ \Theta_i \\ \Xi^i \\ \Sigma_{ij} \end{pmatrix} = S \begin{pmatrix} \vartheta_0 \\ \vartheta_k \\ dx^k \\ dp_{kl} \end{pmatrix} = \begin{pmatrix} a \vartheta_0 \\ g_i \Theta_0 + h_i^k \vartheta_k \\ c_i \Theta_0 + f_{ik} \Theta_k + b_i^k dx^k + r_{ikl} dp_{kl} \\ s_{ij} \Theta_0 + u_{ij}^k \Theta_k + z_{ijk} \Xi^k + q_{ij}^{kl} dp_{kl} \end{pmatrix},$$

on $J^2(\mathcal{E}) \times \mathcal{G}$. Expressing $du, dx^k, dp_k$, and $dp_{kl}$ from (3) and substituting them to $d\Theta_0$, we have

$$d\Theta_0 = da \wedge \vartheta_0 + a d\vartheta_0 = da a^{-1} \wedge \Theta_0 + a dx^i \wedge dp_i = da a^{-1} \wedge \Theta_0 + a dx^i \wedge \vartheta_i$$

$$= \Phi_0^0 \wedge \Theta_0 + a B_i^j H_i^m \Xi^k \wedge \Theta_m + a H_i^m R_{ijkl} \Sigma_{kl} \wedge \Theta_m$$

$$+ a H_i^m \left(B_i^j f_{kj} + R_{ijkl} w_{kl}^j\right) \Theta_j \wedge \Theta_m,$$

where

$$\Phi_0^0 = da a^{-1} + a H_i^m \left(B_i^j \left(c^k + R_{ijkl} s_{kl}\right) \Theta_m - g_m B_i^k \left(\Xi^k - c^k \Theta_0 - f_{kj} \Theta_j\right)\right) - g_m R_{ijkl} \left(\Sigma_{kl} - s_{kl} \Theta_0 - w_{kl}^m \Theta_m - z_{klm} \Xi^m\right)$$

and $R_{ijkl} = -r_{ikl'} B_i^j Q_{k'l'}^{kl}$.
The multipliers of $\Xi^k \wedge \Theta_m$, $\Sigma_{kl} \wedge \Theta_m$, and $\Theta_j \wedge \Theta_m$ in (4) are essential torsion coefficients. We normalize them by setting $a B^i_k H^m_1 = \delta^m_k$, $R^{ikl} = 0$, and $f^{kj} = f^{jk}$.

Therefore the first normalization is
\[ h^i_k = a B^i_k, \quad r^{ikl} = 0, \quad f^{kj} = f^{jk}. \tag{5} \]

Analysing $d\Theta_i$, $d\Xi^i$, and $d\Sigma_{ij}$ in the same way, we obtain the following normalizations:
\[ q^{kl}_{ij} = a B^k_i B^l_j, \quad s_{ij} = s_{ji}, \quad w^k_{ij} = w^k_{ji}, \quad z_{ijk} = z_{jik} = z_{ijk}. \tag{6} \]

After these reductions the structure equations for the lifted coframe have the form
\begin{align*}
d\Theta_0 &= \Phi^0_0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i, \\
d\Theta_i &= \Phi^i_0 \wedge \Theta_0 + \Phi^i_k \wedge \Theta_k + \Xi^k \wedge \Sigma_{ik}, \\
d\Xi^i &= \Phi^0_i \wedge \Xi^i - \Phi^i_k \wedge \Psi^0_0 \wedge \Theta_0 + \Psi^i_k \wedge \Theta_k, \\
d\Sigma_{ij} &= \Phi^k_{ij} \wedge \Sigma_{kj} - \Phi^0_{ij} \wedge \Sigma_{ij} + \Upsilon^0_{ij} \wedge \Theta_0 + \Upsilon^k_{ij} \wedge \Theta_k + \Lambda_{ijk} \wedge \Xi^k,
\end{align*}
where the forms $\Phi^0_0$, $\Phi^i_0$, $\Phi^k$, $\Psi^0_0$, $\Psi^i_k$, $\Upsilon^0_{ij}$, $\Upsilon^k_{ij}$, and $\Lambda_{ijk}$ are defined by the following equations:
\begin{align*}
\Phi^0_0 &= da a^{-1} - g_k \Xi^k + (c^k + f^{km} g_m) \Theta_k, \\
\Phi^i_0 &= dg_i + g_k d_{B^j_i} B^j_i - (g_i g_k + s_{ik} + c^j z_{ijk}) \Xi^k + c^k \Sigma_{ik} \\
&\quad + (g_i c^k + g_m f^{mk} - c^j w^k_{ij} + f^{mk} s_{im}) \Theta_k, \\
\Phi^k_i &= \delta^k_i da a^{-1} - d_{B^j_i} B^j_i + (g_i \delta^k_j - w^k_{ij} - f^{km} z_{jm}^i) \Xi^j + f^{km} \Sigma_{im} + f^{jm} w^k_{ij} \Theta_m, \\
\Psi^0_{ij} &= df^{ij} \Phi^0_j + c^k \Phi^k_i + (c^i f^{mj} g_m - c^k f^{mj} w^k_{ij}) \Theta_j - c^k f^{ij} \Sigma_{kj} \\
&\quad + c^k (f^{im} z^{jno} + w^k_{ij} - g_k \delta^k_j - g_j \delta^k_i) \Xi^j, \\
\Psi^{ij} &= df^{ij} + (f^{ik} \delta^j_m + f^{jk} \delta^i_m) \Phi^m_k + (c^i \delta^j_k + c^j \delta^i_k - f^{ij} g_j + f^{im} f^{jl} z_{klm}) \Xi^k \\
&\quad + f^{ij} (c^k + f^{km} g_m) \Theta_k - f^{ik} f^{jm} \Sigma_{km}, \\
\Upsilon^0_{ij} &= ds_{ij} - s_{ij} da a^{-1} + s_{kj} d_{B^m_i} B^m_i + s_{ik} d_{B^m_j} B^m_j + s_{ij} \Phi^0_0 + w^k_{ij} \Phi^0_k + z_{ijk} \Psi^0_0, \\
\Upsilon^k_{ij} &= dw^k_{ij} - w^k_{ij} da a^{-1} + (w^k_{ij} \delta^m_j + w^k_{jl} \delta^i_j) d_{B^m_i} B^m_j + (s_{ij} \delta^m_k + z_{ijkl} f^{mk} w^l_{m'lm'}) \Xi^m \\
&\quad + w^k_{ij} \Phi^m_k + f^{ik} (w^m_{il} \delta^m_j + w^m_{jl} \delta^m_i) \Sigma_{jm} + (c^k + f^{mk} g_m) \Sigma_{ij}, \\
\Lambda_{ijk} &= dz_{ijk} - 2 z_{ijk} da a^{-1} + (z_{ijkl} d_{B^m_k} B^m_k + z_{ikl} d_{B^m_j} B^m_j + z_{ijk} d_{B^m_l} B^m_l + z_{ijk} \Phi^0_0 \\
&\quad + z_{ijk} g_m \Xi^m + g_i \Sigma_{jk} + g_j \Sigma_{ik} + g_k \Sigma_{ij} - w^l_{ij} \Sigma_{lk} - w^l_{ik} \Sigma_{lj} - w^l_{jk} \Sigma_{li} \\
&\quad - f^{lm} (z_{imj} \Sigma_{kl} + z_{imk} \Sigma_{jl} + z_{jmk} \Sigma_{il}) \Sigma_{ij}.
\end{align*}

Let $\mathcal{H}$ be the subgroup of $\mathcal{G}$ defined by (5) and (6). We shall prove that the restriction of the lifted coframe (3) to $J^2(\mathcal{E}) \times \mathcal{H}$ satisfies Cartan’s test of involutivity. [24] def 11.7]. The structure equations remain unchanged under the following transformation
of the forms \( \Phi_0^0 \mapsto \tilde{\Phi}_0^0, \Phi_i^k \mapsto \tilde{\Phi}_i^k, \Phi_i^0 \mapsto \tilde{\Phi}_i^0, \Psi_{ij} \mapsto \tilde{\Psi}_{ij}, \Psi_{i0}^0 \mapsto \tilde{\Psi}_{i0}^0, \gamma_{ij}^0 \mapsto \tilde{\gamma}_{ij}^0, \gamma_{ij}^k \mapsto \tilde{\gamma}_{ij}^k, \lambda_{ij} \mapsto \tilde{\lambda}_{ij}, \), where

\[
\begin{align*}
\Phi_0^0 & = \Phi_0^0 + K \Theta_0, \\
\tilde{\Phi}_i^k & = \Phi_i^k + L_i^{kl} \Theta_l + M_i^k \Theta_0, \\
\tilde{\Phi}_i^0 & = \Phi_i^0 + M_i^k \Theta_k + N_i \Theta_0, \\
\tilde{\Psi}_{ij} & = \Psi_{ij} + P_{ij} \Theta_0 + S^{ij}_k \Theta_k - L_{ij}^{ij} \Xi^k, \\
\tilde{\Psi}_{i0}^0 & = \Psi_{i0}^0 + P_{i0}^j \Theta_j + T_i \Theta_0 + K \Xi^i - M_i^k \Xi^k, \\
\tilde{\gamma}_{ij}^0 & = \gamma_{ij}^0 + U_{ij} \Theta_0 + V_{ij} \Theta_k + W_{ij} \Xi^k + K \Sigma_{ij} + M_i^k \Sigma_{kj}, \\
\tilde{\gamma}_{ij}^k & = \gamma_{ij}^k + X_{ij}^{kl} \Theta_l + V_{ij} \Theta_0 + Y_{ij}^k \Xi^l + L_i \Sigma_{ij}, \\
\tilde{\lambda}_{ij} & = \lambda_{ij} + Z_{ijkl} \Xi^l + Y_{ijk} \Theta_l + W_{ijk} \Theta_0,
\end{align*}
\]

and \( K, L_i^{kl}, M_i^k, N_i, P_{ij}, S^{ijk}, T_i, U_{ij}, V_{ij}, W_{ijk}, X_{ij}^{kl}, Y_{ij}^k, \) and \( Z_{ijkl} \) are arbitrary constants satisfying the following symmetry conditions:

\[
\begin{align*}
L_i^{kl} & = L_i^{lk}, & P_{ij} & = P_{ji}, & S^{ijk} & = S^{jik} = S^{ikj}, & U_{ij} & = U_{ji}, & V_{ij}^k & = V_{ji}^k, \\
W_{ijk} & = W_{jik}, & X_{ij}^{kl} & = X_{ji}^{kl} = X_{ij}^{lk}, & Y_{ijkl} & = Y_{jkl} = Y_{ijl} = Y_{ijkl}, \\
Z_{ijkl} & = Z_{jikl} = Z_{ijlk} = Z_{ikjl}.
\end{align*}
\]

The number of such constants

\[
r^{(1)} = 1 + \frac{n^2 (n + 1)}{2} + n^2 + n + \frac{n (n + 1)}{2} + \frac{n (n + 1) (n + 2)}{6} + n + \frac{n (n + 1)}{2} + \frac{n^2 (n + 1)}{6} + \frac{n^2 (n + 1) (n + 2)}{4} + \frac{n^2 (n + 1) (n + 2)}{6} + \frac{n (n + 1) (n + 2)(n + 3)}{24} = \frac{1}{24} (n + 1)(n + 2)(11n^2 + 29n + 12)
\]

is the degree of indeterminacy of the lifted coframe, [24 def 11.2]. The reduced characters of this coframe, [24 def 11.4], are easily found: \( s_i' = \frac{1}{2} (n + 1)(n + 4) - i \) when \( i \in \{1, \ldots, n + 1\} \) and \( s_{n+1+j}' = \frac{1}{2} (n - 1 - j)(n + 2 - j) \) when \( j \in \{1, \ldots, n\} \). A simple calculation shows that \( r^{(1)} = s'_1 + 2 s'_2 + 3 s'_3 + \ldots + (2n + 1) s'_{2n+1} \). So the Cartan test is satisfied, and the lifted coframe is involutive.

It is easy to directly verify that a transformation \( \tilde{\Delta} : J^2(\mathcal{E}) \times \mathcal{H} \to J^2(\mathcal{E}) \times \mathcal{H} \) satisfies the conditions

\[
\tilde{\Delta}^* \Theta_0 = \Theta_0, \quad \tilde{\Delta}^* \Theta_i = \Theta_i, \quad \tilde{\Delta}^* \Xi^i = \Xi^i, \quad \tilde{\Delta}^* \Sigma_{ij} = \Sigma_{ij}
\]

if and only if it is projectable on \( J^2(\mathcal{E}) \), and its projection \( \Delta : J^2(\mathcal{E}) \to J^2(\mathcal{E}) \) is a contact transformation.
Since (10) imply \( \hat{\Delta}^* d\Theta_0 = d\Theta_0, \hat{\Delta}^* d\Theta_i = d\Theta_i, \hat{\Delta}^* d\Xi = d\Xi, \) and \( \hat{\Delta}^* d\Sigma_{ij} = d\Sigma_{ij}, \) we have
\[
\hat{\Delta}^* \left( \Phi_0^i \wedge \Theta_0 + \Xi^i \wedge \Theta_i \right) = \left( \hat{\Delta}^* \Phi_0^i \right) \wedge \Theta_0 + \Xi^i \wedge \Theta_i = \Phi_0^i \wedge \Theta_0 + \Xi^i \wedge \Theta_i,
\]
\[
\hat{\Delta}^* \left( \Phi_i^j \wedge \Theta_0 + \Xi^j \wedge \Theta_j + \Sigma_{ij} \right) = \hat{\Delta}^* \left( \Phi_i^j \right) \wedge \Theta_0 + \hat{\Delta}^* \left( \Phi_i^j \right) \wedge \Theta_j + \Xi^j \wedge \Sigma_{ij}
\]
\[
= \Phi_i^j \wedge \Theta_0 + \Phi_i^j \wedge \Theta_j + \Xi^j \wedge \Sigma_{ij},
\]
\[
\hat{\Delta}^* \left( \Phi_0^i \wedge \Xi^i - \Phi_i^j \wedge \Xi^j + \Psi^0 \wedge \Theta_0 + \Psi^k \wedge \Theta_k \right)
\]
\[
= \hat{\Delta}^* \left( \Phi_0^i \right) \wedge \Xi^i - \hat{\Delta}^* \left( \Phi_i^j \right) \wedge \Xi^j + \hat{\Delta}^* \left( \Psi^0 \right) \wedge \Theta_0 + \hat{\Delta}^* \left( \Psi^k \right) \wedge \Theta_k
\]
\[
= \Phi_i^j \wedge \Xi^i - \Phi_i^j \wedge \Xi^j + \Psi^0 \wedge \Theta_0 + \Psi^k \wedge \Theta_k.
\]
Therefore, we have the following transformation rules
\[
\hat{\Delta}^* \left( \Phi_0^i \right) = \Phi_0^i, \quad \hat{\Delta}^* \left( \Phi_i^j \right) = \Phi_i^j, \quad \hat{\Delta}^* \left( \Psi^0 \right) = \Psi^0,
\]
\[
\hat{\Delta}^* \left( \Psi^j \right) = \Psi^j, \quad \hat{\Delta}^* \left( \Phi_i^j \right) = \Phi_i^j, \quad \hat{\Delta}^* \left( \Psi^0 \right) = \Psi^0,
\]
\[
\hat{\Delta}^* \left( \Psi^k \right) = \Psi^k, \quad \hat{\Delta}^* \left( \Xi_{ijk} \right) = \Xi_{ijk}.
\]
where the constants \( K, ..., Z_{ijkl} \) in (9) are replaced by arbitrary functions on \( J^2(\mathcal{E}) \times H \) such that the same symmetry conditions (9) are satisfied.

2. Contact symmetries of differential equations

Suppose \( \mathcal{R} \) is a second-order differential equation in one dependent and \( n \) independent variables. We consider \( \mathcal{R} \) as a sub-bundle in \( J^2(\mathcal{E}) \). Let \( \text{Cont}(\mathcal{R}) \) be the group of contact symmetries for \( \mathcal{R} \). It consists of all the contact transformations on \( J^2(\mathcal{E}) \) mapping \( \mathcal{R} \) to itself. The moving coframe method, [3, 4], is applicable to find invariant 1-forms characterizing \( \text{Cont}(\mathcal{R}) \) is the same way, as the restriction of the lifted coframe (8) to \( J^2(\mathcal{E}) \times H \) characterizes \( \text{Cont}(J^2(\mathcal{E})) \). We briefly outline this approach.

Let \( \iota : \mathcal{R} \to J^2(\mathcal{E}) \) be an embedding. The invariant 1-forms of \( \text{Cont}(\mathcal{R}) \) are restrictions of the coframe (9)\( \iota \) to \( \mathcal{R} \): \( \theta_0 = \iota^* \Theta_0, \theta_i = \iota^* \Theta_i, \xi^i = \iota^* \Xi^i, \) and \( \sigma_{ij} = \iota^* \Sigma_{ij} \) (for brevity we identify the map \( \iota \times \text{id} : \mathcal{R} \times H \to J^2(\mathcal{E}) \times H \) with \( \iota : \mathcal{R} \to J^2(\mathcal{E}) \)). The forms \( \theta_0, \theta_i, \xi^i, \) and \( \sigma_{ij} \) have some linear dependencies, i.e., there exists a non-trivial set of functions \( E^0, E^i, F_i, \) and \( G^{ij} \) on \( \mathcal{R} \times H \) such that \( E^0 \theta_0 + E^i \theta_i + F_i \xi^i + G^{ij} \sigma_{ij} \equiv 0 \). These functions are lifted invariants of \( \text{Cont}(\mathcal{R}) \). Setting them equal to some constants
allows us to specify some parameters \( a, b^k, c_i, g_i, f^{ij}, s_{ij}, w^i_k \), and \( z_{ijk} \) of the group \( \mathcal{H} \) as functions of the coordinates on \( \mathcal{R} \) and the other group parameters.

After these normalizations, a part of the forms \( \phi^0_0 = \iota^* \Phi^0_0, \phi^k_i = \iota^* \Phi^k_i, \phi^0_i = \iota^* \Phi^0_i \), \( \psi^{ij} = \iota^* \Psi^{ij}, \psi^{0i} = \iota^* \Psi^{0i}, v^{ij}_{ij} = \iota^* \mathcal{Y}^{ij}_{ij}, \) and \( \lambda_{ijk} = \iota^* \Lambda_{ijk} \), or some their linear combinations, become semi-basic, i.e., they do not include the differentials of the parameters of \( \mathcal{H} \). From (11) and (3), we have the following statements: (i) if \( \phi^0_0 \) is semi-basic, then its coefficients at \( \theta_k, \xi^k \), and \( \sigma_{kl} \) are lifted invariants of Cont(\( \mathcal{R} \)); (ii) if \( \phi^0_i \) or \( \phi^k_i \) are semi-basic, then their coefficients at \( \xi^k \) and \( \sigma_{kl} \) are lifted invariants of Cont(\( \mathcal{R} \)); (iii) if \( \psi^{0i}, \psi^{ij} \), or \( \lambda_{ijk} \) are semi-basic, then their coefficients at \( \sigma_{kl} \) are lifted invariants of Cont(\( \mathcal{R} \)). Setting these invariants equal to some constants, we get specifications of some more parameters of \( \mathcal{H} \) as functions of the coordinates on \( \mathcal{R} \) and the other group parameters.

More lifted invariants can appear as essential torsion coefficients in the reduced structure equations

\[
\begin{align*}
  d\theta_0 &= \phi^0_0 \land \theta_0 + \xi^i \land \theta_i \\
  d\theta_i &= \phi^k_i \land \theta_0 + \phi^k_i \land \theta_k + \xi^k \land \sigma_{ik} \\
  d\xi^i &= \phi^0_i \land \xi^i - \phi^i_k \land \xi^k + \psi^{0i} \land \theta_0 + \psi^{jk} \land \theta_k \\
  d\sigma_{ij} &= \phi^k_i \land \sigma_{kj} - \phi^0_0 \land \sigma_{ij} + v^{0i}_i \land \theta_0 + v^{0j}_j \land \theta_k + \lambda_{ijk} \land \xi^k.
\end{align*}
\]

After normalizing these invariants and repeating the process, two outputs are possible. In the first case, the reduced lifted coframe appears to be involutive. Then this coframe is the desired set of defining forms for Cont(\( \mathcal{R} \)). In the second case, when the reduced lifted coframe does not satisfy Cartan’s test, we should use the procedure of prolongation. [24, ch 12].

3. Structure and invariants of symmetry groups for linear hyperbolic equations

We apply the method described in the previous section to the class of linear hyperbolic equations (11). Denote \( x^1 = t \), \( x^2 = x \), \( p_1 = u_t \), \( p_2 = u_x \), \( p_{11} = u_{tt} \), \( p_{12} = u_{tx} \), and \( p_{22} = u_{xx} \). The coordinates on \( \mathcal{R} \) are \( \{(t, x, u, u_t, u_x, u_{tt}, u_{xx})\} \), and the embedding \( \iota : \mathcal{R} \to J^2(\mathcal{E}) \) is defined by (11). At the first step, we analyse the linear dependence between the reduced forms \( \theta_0, \theta_i, \xi^i \), and \( \sigma_{ij} \). Without loss of generality we suppose that \( b^1_1 \neq 0 \) and \( b^2_2 \neq 0 \), then we find \( \sigma_{12} = E_1 \sigma_{11} + E_2 \sigma_{22} + E_3 \theta_0 + E_4 \theta_1 + E_5 \theta_2 + E_6 \xi^1 + E_7 \xi^2 \), where, for example, \( E_1 = -(b^1_1 b^2_2 + b^1_2 b^2_1)^{-1} b^1_1 b^1_2 \) and \( E_2 = -(b^1_1 b^2_2 + b^1_2 b^2_1)^{-1} b^1_1 b^2_2 \). Setting \( E_1, E_2, ..., E_7 \) equal to 0 sequentially, we have \( E_1 = 0 \Rightarrow b^1_1 = 0, E_2 = 0 \Rightarrow b^1_2 = 0, E_3 = 0 \Rightarrow s_{12} = -z_{112} c^1 - z_{122} c^2 + g_1 (b^2_2)^{-1} T + g_2 (b^1_1)^{-1} X - (b^1_1 b^2_2)^{-1} U, E_4 = 0 \Rightarrow u_{12} = -z_{112} f^{11} - z_{122} f^{12} - (b^2_1)^{-1} T, E_5 = 0 \Rightarrow w_{12}^2 = -z_{112} f^{11} - z_{122} f^{12} - (b^1_1)^{-1} X, E_6 = 0 \Rightarrow z_{112} = -a (b^1_1)^{-2} (b^2_1)^{-1} (T u_{tt} + (2 T X + 2 U - H) u_t + (X_1 + X^2) u_x + (U_1 + X U) u), \) and \( E_7 = 0 \Rightarrow z_{122} = -a (b^1_1)^{-2} (b^2_2)^{-2} (2 X u_{xx} + (T_x + T^2) u_t + (2 T X + 2 U - K) u_x + + \)

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\((U_x + T U) u\), where \(H = -T_t + T X + U\) and \(K = -X_x + T X + U\) are the Laplace invariants, \([14, 23, \S\, 9]\).

At the second step, we analyse the semi-basic forms \(\phi_j^i\) and \(\phi_j^0\). We have

\[ \phi_2^2 \equiv f^{12} \sigma_{11} + (g_1 + (b_1^1)^{-1} X) \xi^2 \quad (\text{mod } \theta_0, \theta_1, \theta_2, \xi^1), \]

therefore we take \(f^{12} = 0\), \(g_1 = -(b_1^1)^{-1} X\). This yields

\[ \phi_2^2 \equiv (-w_{11}^2 + a f^{22} (b_1^1)^{-2} (b_2^1)^{-1} (T u_{tt} + (2 T X + 2 U - H) u_t + (X_t + X^2) u_x + (U_t + X U) u)) \xi^1 \quad (\text{mod } \theta_0, \theta_1, \theta_2), \]

therefore we set \(w_{11}^2 = a f^{22} (b_1^1)^{-2} (b_2^1)^{-1} (T u_{tt} + (2 T X + 2 U - H) u_t + (X_t + X^2) u_x + (U_t + X U) u)\).

After that, we have

\[ \phi_2^1 \equiv (g_2 + (b_2^1)^{-1} T) \xi^1 + (-w_{22}^1 + a f^{11} (b_1^1)^{-1} (b_2^2)^{-2} (X u_{xx} + (T_x + T^2) u_t + (U_x + T U) u)) \xi^2 \quad (\text{mod } \theta_0, \theta_1, \theta_2), \]

so we set \(g_2 = -(b_2^1)^{-1} T\) and \(w_{22}^1 = a f^{11} (b_1^1)^{-1} (b_2^2)^{-2} (X u_{xx} + (T_x + T^2) u_t + (2 T X + 2 U - K) u_x + (U_x + T U) u)\).

Then we have \(\phi_0^0 \equiv c^1 \sigma_{11} \quad (\text{mod } \theta_0, \theta_1, \theta_2, \xi^1, \xi^2), \phi_0^0 \equiv c^2 \sigma_{22} \quad (\text{mod } \theta_0, \theta_1, \theta_2, \xi^1, \xi^2), \)

so we set \(c^1 = 0\) and \(c^2 = 0\). Now we obtain

\[ \phi_0^0 \equiv K (b_1^1)^{-1} (b_2^1)^{-1} \xi^2 \quad (\text{mod } \theta_0, \theta_1, \theta_2), \]

\[ \phi_0^0 \equiv H (b_1^1)^{-1} (b_2^1)^{-1} \xi^1 \quad (\text{mod } \theta_0, \theta_1, \theta_2). \]

(12)

There are two possibilities now: \(H \equiv K \equiv 0\) or at least one of the Laplace invariants is not identically equal 0.

We denote by \(S_0\) the subclass of equations \([11]\) such that \(H \equiv K \equiv 0\). For an equation from \(S_0\) we use the procedures of absorption and prolongation, \([23]\), to compute the structure equations:

\[
\begin{align*}
d\theta_0 &= \eta_1 \land \theta_0 + \xi^1 \land \theta_1 + \xi^2 \land \theta_2, \\
d\theta_1 &= \eta_2 \land \theta_1 + \xi^1 \land \sigma_{11}, \\
d\theta_2 &= \eta_3 \land \theta_2 + \xi^2 \land \sigma_{22}, \\
d\xi^1 &= (\eta_1 - \eta_2) \land \xi^1 + \eta_4 \land \theta_1, \\
d\xi^2 &= (\eta_1 - \eta_3) \land \xi^2 + \eta_5 \land \theta_2, \\
d\sigma_{11} &= (2 \eta_2 - \eta_1) \land \sigma_{11} + \eta_6 \land \theta_1 + \eta_7 \land \xi^1, \\
d\sigma_{22} &= (2 \eta_3 - \eta_1) \land \sigma_{22} + \eta_8 \land \theta_2 + \eta_9 \land \xi^2, \\
d\eta_1 &= 0, \\
d\eta_2 &= \pi_1 \land \theta_1 + \eta_4 \land \sigma_{11} - \eta_6 \land \xi^1, \\
\end{align*}
\]
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In these equations, the forms $\eta_1$, ..., $\eta_9$ on $J^2(\mathcal{E}) \times \mathcal{H}$ depend on differentials of the parameters of $\mathcal{H}$, while the forms $\pi_1$, ..., $\pi_{10}$ depend on differentials of the prolongation variables. From the structure equations it follows that Cartan’s test for the lifted coframe \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, ..., \eta_9\} is satisfied, therefore the coframe is involutive.

The same calculations show that the symmetry group of the linear wave equation $u_{tx} = 0$ has the same structure equations, but with a different lifted coframe. All the essential torsion coefficients in the structure equations are constants. Thus, applying Theorem 15.12 of [24], we obtain the well-known result, [23, § 9]: every equation from $S_1$ is contact equivalent to the wave equation.

Now we return to the case of $H \neq 0$ or $K \neq 0$. Since we can replace $H$ and $K$ by renaming the independent variables $t \mapsto x$, $x \mapsto t$, we put $H \neq 0$ without loss of generality. Then we use (12) and take $b_2^2 = H(b_1^1)^{-1}$. After this, the form $\phi_1^1 + \phi_2^2 - 2 \phi_0^0$ becomes semi-basic. Since $\phi_1^1 + \phi_2^2 - 2 \phi_0^0 \equiv f^{11} \sigma_{11} + f^{22} \sigma_{22}$ (mod $\theta_1, \theta_2, \xi^1, \xi^2$), we take $f^{11} = 0$ and $f^{22} = 0$. Then we have $\phi_1^1 + \phi_2^2 - 2 \phi_0^0 \equiv -(w_{11}^1 + H^{-1}(b_1^1)^{-1}(H_t + 2XH)) \xi^1 - (w_{22}^1 + H^{-2}b_1^1 (H_x + 2TH)) \xi^2$ (mod $\theta_1, \theta_2$), so we take $w_{11}^1 = -H^{-1}(b_1^1)^{-1}(H_t + 2XH)$ and $w_{22}^1 = -H^{-2}b_1^1 (H_x + 2TH)$.

At the third step, we analyse the structure equations. After absorption of torsion they have the form

\begin{align*}
d\eta_3 &= \pi_2 \wedge \theta_2 + \eta_5 \wedge \sigma_{22} - \eta_8 \wedge \xi^2, \\
d\eta_4 &= -\pi_1 \wedge \xi^1 + \pi_3 \wedge \theta_1 + (\eta_1 - 2 \eta_2) \wedge \eta_4, \\
d\eta_5 &= -\pi_2 \wedge \xi^2 + \pi_4 \wedge \theta_2 + (\eta_1 - 2 \eta_3) \wedge \eta_5, \\
d\eta_6 &= 2 \pi_1 \wedge \sigma_{11} + \pi_5 \wedge \theta_1 + \pi_6 \wedge \xi^1 + (\eta_2 - \eta_1) \wedge \eta_6 - \eta_4 \wedge \eta_7, \\
d\eta_7 &= \pi_6 \wedge \theta_1 + \pi_7 \wedge \xi^1 - 3 \eta_6 \wedge \sigma_{11} + (3 \eta_2 - 2 \eta_1) \wedge \eta_7, \\
d\eta_8 &= 2 \pi_2 \wedge \sigma_{22} + \pi_8 \wedge \theta_2 + \pi_9 \wedge \xi^2 + (\eta_3 - \eta_1) \wedge \eta_8 - \eta_5 \wedge \eta_9, \\
d\eta_9 &= \pi_9 \wedge \theta_2 + \pi_{10} \wedge \xi^2 - 3 \eta_8 \wedge \sigma_{22} + (3 \eta_3 - 2 \eta_1) \wedge \eta_9.
\end{align*}

where the functions $P = KH^{-1}$ and $Q = (HH_{tx} - H_t H_x)H^{-3} = (\ln |H|)_{tx}H^{-1}$ are invariants of the symmetry group, and the 1-forms $\eta_1$, ..., $\eta_4$ depend on differentials of
parameters of the group $\mathcal{H}$ (these forms are not necessary the same as in the case of an equation from $S_1$).

We denote by $S_2$ the subclass of equations (11) such that $P_t \neq 0$. This subclass is not empty, since, for example, the equation $u_{tx} = t^2 x^2 u_t + u$ belongs to $S_2$. For an equation from $S_2$ we can normalize $P_t (b^1_t)^{-1}$, the only essential torsion coefficient in the structure equations (13), to 1 by setting $b^1_t = P_t$. Then, after prolongation, we have the involutive lifted coframe $\theta = \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, \eta_3\}$ with the structure equations

\begin{align*}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2, \\
d\theta_1 &= \eta_1 \wedge \theta_1 - P \theta_0 \wedge \xi^2 - J_2 \theta_1 \wedge \xi^1 - J_1 \theta_1 \wedge \xi^2 + \xi^1 \wedge \sigma_{11}, \\
d\theta_2 &= \eta_1 \wedge \theta_2 - \theta_0 \wedge \xi^1 + J_2 \theta_2 \wedge \xi^1 + J_1 \theta_2 \wedge \xi^2 + \xi^2 \wedge \sigma_{22}, \\
d\xi^1 &= J_1 \xi^1 \wedge \xi^2, \\
d\xi^2 &= J_2 \xi^1 \wedge \xi^2, \\
d\sigma_{11} &= \eta_1 \wedge \sigma_{11} + \eta_2 \wedge \xi^1 - \theta_0 \wedge \xi^2 + (Q + 1 - 2P) \theta_1 \wedge \xi^2 + 2 J_1 \xi^2 \wedge \sigma_{11}, \\
d\sigma_{22} &= \eta_1 \wedge \sigma_{22} + \eta_3 \wedge \xi^2 + (P - 2 + Q) \theta_2 \wedge \xi^1 - 2 J_2 \xi^1 \wedge \sigma_{22}, \\
d\eta_1 &= (P - 1) \xi^1 \wedge \xi^2, \\
d\eta_2 &= \pi_1 \wedge \xi^1 + \eta_1 \wedge \eta_2 - 3 J_1 \eta_2 \wedge \xi^2 + J_2 \theta_0 \wedge \xi^2 + (4P J_2 - 2Q J_2 - \mathbb{D}_1(Q) \\
&\quad - 2J_2 + 3) \theta_1 \wedge \xi^2 + (2 J_1 J_2 + 2 - 3P + 3Q - 2 \mathbb{D}_2(J_2)) \xi^2 \wedge \sigma_{11}, \\
d\eta_3 &= \pi_2 \wedge \xi^2 + \eta_1 \wedge \eta_3 + 3 J_2 \eta_3 \wedge \xi^1 + (2 J_1 (P + Q - 2) - \mathbb{D}_2(Q) - \mathbb{D}_2(P)) \theta_2 \wedge \xi^1 \\
&\quad + (2P - 3 - 2J_1 J_2 + 2 \mathbb{D}_2(J_2) + Q) \xi^1 \wedge \sigma_{22},
\end{align*}

where the functions $J_1 = -P_{tx} H^{-1}$ and $J_2 = (H_t P_t - H P_{tt}) H^{-1} (P_t)^{-2}$ are invariants of the symmetry group of an equation from $S_2$, the operators

$$
\mathbb{D}_1 = \frac{\partial}{\partial \xi^1} = (P_t)^{-1} D_t, \quad \mathbb{D}_2 = \frac{\partial}{\partial \xi^2} = P_t H^{-1} D_x
$$

are invariant differentiations associated with $\xi^1$ and $\xi^2$. These operators are defined by the identity $dF = \mathbb{D}_1(F) \xi^1 + \mathbb{D}_2(F) \xi^2$, where $F = F(t, x)$ is an arbitrary function. The commutator identity for the invariant differentiations has the form

$$
[\mathbb{D}_1, \mathbb{D}_2] = -J_1 \mathbb{D}_1 - J_2 \mathbb{D}_2.
$$

We have $\mathbb{D}_1(P) = 1$, and, applying (15) to $P$, we obtain the syzygy

$$
J_1 = -\mathbb{D}_1(\mathbb{D}_2(P)) - J_2 \mathbb{D}_2(P).
$$

If $\mathbb{D}_2(P) \mathbb{D}_1(Q) \neq \mathbb{D}_2(Q)$, i.e., if $P_t Q_x \neq P_x Q_t$, then, applying (15) to $Q$ and using (16), we have

$$
J_2 = ([\mathbb{D}_1, \mathbb{D}_2] (Q) - \mathbb{D}_1(Q) \mathbb{D}_1(\mathbb{D}_2(P))) (\mathbb{D}_2(P) \mathbb{D}_1(Q) - \mathbb{D}_2(Q))^{-1}.
$$
Therefore, in this case the functions $P$ and $Q$ are a basis of differential invariants of the symmetry group. But $P$ and $Q$ are not necessary a basis in the case of their functional dependence, cf. [9, th 2.3], [10, § 10.4.2]. To prove this statement, we consider the equation

$$u_{tx} = u_t + \frac{2(p(t) - 1)}{q(t)(t + x)} u_x + \frac{2}{q(t)(t + x)^2} (1 - (p(t) - 1)(t + x)) u$$

with arbitrary functions $p(t)$ and $q(t)$ such that $p'(t) \neq 0$ and $q'(t) \neq 0$. For this equation we have

$$P = p(t), \quad Q = q(t), \quad J_2 = -\frac{2}{q'(t)(t + x)} - \frac{p''(t)}{(p'(t))^2} - \frac{q'(t)}{p'(t)q(t)}.$$
Remark 2. In [14] th 1, the following basis of invariants for the symmetry group of equation (11) is found: \{P, Q, J_3^1, J_3^2, J_3^3\}, where
\[
J_3^1 = H^{-3} (K H_{tx} + H K_{tx} - H_t K_x - H_x K_t),
\]
\[
J_3^2 = H^{-9} (H K_x - K H_x)^2 \left( H K H_{tt} - H^2 K_{tt} - 3 K H_t^2 + 3 H H_t K_t \right),
\]
\[
J_3^3 = H^{-9} (H K_t - K H_t)^2 \left( H K H_{xx} - H^2 K_{xx} - 3 K H_x^2 + 3 H H_x K_x \right).
\]

Using (10), we have the following expressions for invariants \(J_3^1, J_3^2, J_3^3\) in terms of \(P, Q, J_2\), and their invariant derivatives:
\[
J_3^1 = 2 P Q + \mathbb{D}_1(\mathbb{D}_2(P)) + J_2 \mathbb{D}_2(P),
\]
\[
J_3^2 = J_2 (\mathbb{D}_2(P))^2,
\]
\[
J_3^3 = \mathbb{D}_2(P) (\mathbb{D}_1(\mathbb{D}_2(P)) + J_2 \mathbb{D}_2(P)) - \mathbb{D}_2(\mathbb{D}_2(P)).
\]

The following operators of invariant differentiation are found in [14]:
\[
\tilde{X}_1 = H^{-3} (H K_x - K H_x) D_t, \quad \tilde{X}_2 = H^2 (H K_x - K H_x)^{-1} D_x.
\]

We have
\[
\tilde{X}_1 = \mathbb{D}_2(P) \mathbb{D}_1 \text{ and } \tilde{X}_2 = \mathbb{D}_2(P)^{-1} \mathbb{D}_2.
\]

Then in the case \(\mathbb{D}_2(P) \equiv 0 \equiv P_x\) the operator \(\tilde{X}_1\) is trivial, \(J_3^1 = 2 P Q\), \(J_3^2 = 0\), and \(J_3^3 = 0\). Therefore, the functions \(P, Q, J_3^1, J_3^2, J_3^3\) are not a basis of invariants of symmetry group for equation (17).

Remark 3. In the theorem of [12], two sets of functions are stated to be bases for invariants of symmetry groups of equations (11): the first set consists of functions \(P, Q, I = P P_x H^{-1}, \tilde{Q} = (\ln |K|)_{\nu} K^{-1}\), and the second set consists of the functions \(P, Q, I, \) and \(-J_2\). The operators of invariant differentiation are taken in the form \(\mathbb{D}_1 = P^{-1}_t D_t\) and \(\mathbb{D}_2 = P^{-1}_x D_x\). We have \(I = \mathbb{D}_2(P)\), therefore the function \(I\) can be excluded from both sets. Also we have
\[
\tilde{Q} = Q P^{-1} + J_2 \mathbb{D}_2(P) P^{-2} + \mathbb{D}_1(\mathbb{D}_2(P)) P^{-2} - \mathbb{D}_2(P) P^{-3},
\]

\(\mathbb{D}_1 = \mathbb{D}_1\), and \(\mathbb{D}_2 = (\mathbb{D}_2(P))^{-1} \mathbb{D}_2\). Therefore, in the case \(P_x = 0 = \mathbb{D}_2(P)\) we have \(I = 0\) and \(\tilde{Q} = Q P^{-1}\), so the functions \(P, Q, I, \) and \(\tilde{Q}\) are not a basis of invariants for the symmetry group of equation (17).

The function \(J_2\) and the operator \(\mathbb{D}_1\) are not defined when \(P_t \equiv 0\) (for example of this case we take the Moutard equation \(u_{tx} = U(t, x) u\)). So the second set of functions is not a basis of invariants of symmetry groups for the whole class (1).

Now we return to the case \(P_t \equiv 0\). Then the torsion coefficients in the structure equations (13) are independent of the group parameters, while \(dP = P_x b_1^1 H^{-1} \xi^2\). We denote by \(\mathcal{S}_3\) the subclass of equations (11) such that \(P_t \equiv 0, P_x \neq 0\). This subclass is not empty, since, for example, the equation \(u_{tx} = x^2 u_x + u\) belongs to \(\mathcal{S}_3\). For an equation from \(\mathcal{S}_3\) we normalize \(b_1^1 = H P_x^{-1}\). After absorption of torsion and prolongation, we obtain the involutive lifted coframe \(\theta = \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, \eta_3\}\) with the structure equations
\[
d\theta_0 = \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,
\]
\[d\theta_1 = \eta_1 \wedge \theta_1 - P \theta_0 \wedge \xi^2 - L \theta_1 \wedge \xi^2 + \xi^1 \wedge \sigma_{11},\]
\[d\theta_2 = \eta_1 \wedge \theta_2 - \theta_0 \wedge \xi^1 + L \theta_2 \wedge \xi^2 + \xi^2 \wedge \sigma_{22},\]
\[d\xi^1 = L \xi^1 \wedge \xi^2,\]
\[d\xi^2 = 0,\]
\[d\sigma_{11} = \eta_1 \wedge \sigma_{11} + \eta_2 \wedge \xi^1 + (Q + 1 - 2 P) \theta_1 \wedge \xi^2 + 2 L \xi^2 \wedge \sigma_{11},\]
\[d\sigma_{22} = \eta_1 \wedge \sigma_{22} + \eta_3 \wedge \xi^2 + (P - 2 + Q) \theta_2 \wedge \xi^1,\]
\[d\eta_1 = (P - 1) \xi^1 \wedge \xi^2,\]
\[d\eta_2 = \pi_1 \wedge \xi^1 - \eta_1 \wedge \eta_2 - 3 L \eta_2 \wedge \xi^2 - D_1(Q) \theta_1 \wedge \xi^2 + (3 Q - 3 P + 2) \xi^2 \wedge \sigma_{11},\]
\[d\eta_3 = \pi_2 \wedge \xi^2 + \eta_1 \wedge \eta_3 - (4 L + 1 - 2 PL - 2 Q L + D_2(Q)) \theta_2 \wedge \xi^1 + (Q - 3 + 2 P) \xi^1 \wedge \sigma_{22},\]

where the function \(L = (HP_{xx} - H_x P_x)(P_x)^{-2} H^{-1}\) is an invariant of the symmetry group, and the operators of invariant differentiation are \(D_1 = P_x H^{-1} D_t\) and \(D_2 = (P_x)^{-1} D_x\). We have \(D_1(P) = 0, D_2(P) = 1,\) and
\[\left[D_1, D_2\right] = L D_1.\]  \hspace{1cm} (19)

In the case \(D_1(Q) \neq 0\) we apply (19) to \(Q\) and obtain \(L = [D_1, D_2](Q)(D_1(Q))^{-1}\). Therefore, in this case the functions \(P\) and \(Q\) are a basis for the set of differential invariants of the symmetry group. But if \(D_1(Q) = 0\), then the functions \(P\) and \(Q\) are not necessary a basis. For example, consider the equation
\[u_{tx} = -\frac{2(p(x) - 1)}{q(x)(t + x)} u_t + u_x + \frac{2}{q(x)(t + x)^2} (p(x) + (p(x) - 1)(t + x)) u\]

where \(p(x)\) and \(q(x)\) are arbitrary functions such that \(p'(x) \neq 0\) and \(q'(x) \neq 0\). This equation has the following invariants:
\[P = p(x), \quad Q = q(x), \quad L = \frac{2}{p'(x)(t + x)} + \frac{p''(x)}{(p'(x))^2} + \frac{q'(x)}{p'(x) q(x)}\].

We have \(D_1(Q) = 0, D_2(Q) = q'(x)(p'(x))^{-1}\), and by induction the only non-trivial higher order differential invariants \(D_2(Q)\) depend on \(x\). Since \(L_t \neq 0\), the function \(L\) is independent of \(P, Q,\) and all their invariant derivatives. Thus for the whole subclass \(S_3\) we should take the functions \(P, Q,\) and \(L\) as a basis for the set of differential invariants of symmetry group. The \(s\)-th order classifying manifold associated with the coframe \(\theta\) and an open subset \(U \in \mathbb{R}^2\) can be taken in the form
\[C^{(s)}(\theta, U) = \{(P(x), Q_{jk}(t, x), L_{jk}(t, x)) \mid 0 \leq j + k \leq s, \ (t, x) \in U\} \hspace{1cm} (20)\]

with \(Q_{jk} = D_1(D_2(Q))\) and \(L_{jk} = D_1(D_2(L))\). Then two equations from \(S_3\) are equivalent under a contact transformation if and only if their second order classifying manifolds (20) are (locally) overlap.
Now we consider the case $P \equiv \text{const.}$ Then we have $dQ = Q_t (b_1^t)^{-1} \xi_1 + Q_x b_1^x H^{-1} \xi_2$. We denote by $\mathcal{S}_4$ the subclass of equations \[\mathcal{S}_4\] such that $P \equiv \text{const.}, Q_t \neq 0$. This subclass is not empty, since, for example, the equation $u_{tx} = (t - x)^3 u_x + (t - x)^2 u$ belongs to $\mathcal{S}_4$. For an equation from $\mathcal{S}_4$ we normalize $b_1^x = Q_t$. Then after absorption of torsion and prolongation we have the involutive lifted coframe $\theta = \{\theta_0, \theta_1, \theta_2, \xi_1, \xi_2, \sigma_1, \sigma_2, \eta_1, \eta_2, \eta_3\}$ with the structure equations

\[
d\theta_0 = \eta_1 \land \eta_0 + \xi_1 \land \theta_1 + \xi_2 \land \theta_2,
\]

\[
d\theta_1 = \eta_1 \land \theta_1 - P \theta_0 \land \xi_1 - M_2 \theta_1 \land \xi_1 - M_1 \theta_1 \land \xi_2 + \xi_1 \land \sigma_1,
\]

\[
d\theta_2 = \eta_1 \land \theta_2 - \eta_0 \land \xi_1 + M_2 \theta_2 \land \xi_1 + M_1 \theta_2 \land \xi_2 + \xi_2 \land \sigma_2,
\]

\[
d\xi_1 = M_1 \xi_1 \land \xi_1,
\]

\[
d\xi_2 = M_2 \xi_1 \land \xi_2,
\]

\[
d\sigma_1 = \eta_1 \land \sigma_1 + \eta_2 \land \xi_1 + (Q + 1 - 2P) \theta_1 \land \xi_2 + 2M_1 \xi_2 \land \sigma_1,
\]

\[
d\sigma_2 = \eta_1 \land \sigma_2 + \eta_3 \land \xi_2 + (P - 2 + Q) \theta_2 \land \xi_1 - 2M_2 \xi_1 \land \sigma_2,
\]

\[
d\eta_1 = (P - 1) \xi_1 \land \xi_2,
\]

\[
d\eta_2 = \pi_1 \land \xi_1 + \eta_1 \land \eta_2 - 3M_1 \eta_2 \land \xi_2 - (1 + 2M_2 + 2Q M_2 - 4P M_2) \theta_1 \land \xi_2
\]

\[
\quad + (Q - 2M_1 M_2 - 3P - 2D_1(M_1) + 2) \xi_2 \land \sigma_1,
\]

\[
d\eta_3 = \pi_2 \land \xi_2 + \eta_1 \land \eta_3 + 3M_2 \eta_3 \land \xi_1 - (4M_1 - 2M_1 P - 2M_1 Q + D_2(Q)) \theta_2 \land \xi_1
\]

\[
\quad + (2M_1 M_2 + 2P - 3 + 2D_1(M_1) + 3Q) \xi_1 \land \sigma_2,
\]

where the functions $M_1 = -Q_{tx} H^{-1}$ and $M_2 = (H,Q_t - H Q_{tt}) H^{-1} Q_t^{-2}$ are invariants of the symmetry group, and the operators of invariant differentiation are $D_1 = Q_t^{-1} D_t$ and $D_2 = Q_t H^{-1} D_x$. We have $[D_1,D_2] = -M_1 D_1 - M_2 D_2$. Since $D_1(Q) = 1$, then, applying the commutator identity to $Q$, we have the syzygy $M_1 = -D_1(D_2(Q)) - M_2 D_2(Q)$. The functions $Q$ and $M_2$ are a basis for the set of all invariants of the symmetry group of an equation from $\mathcal{S}_4$. We take the $s$-th order classifying manifold associated with the coframe $\theta$ and an open subset $U \in \mathbb{R}^2$ in the form

\[
C^{(s)}(\theta,U) = \{(P,Q_{jk}(t,x),M_{2jk}(t,x)) \mid 0 \leq j + k \leq s, (t,x) \in U\}
\]

with $Q_{jk} = D_1^j(D_2^k(Q))$ and $M_{2jk} = D_1^j(D_2^k(M_2))$. Then two equations from $\mathcal{S}_4$ are equivalent under a contact transformation if and only if their second order classifying manifolds are (locally) overlap.

Next we denote by $\mathcal{S}_5$ the subclass of equations \[\mathcal{S}_5\] such that $P \equiv \text{const.}, Q_t \equiv 0, Q_x \neq 0$. This subclass is not empty, since, for example, the equation

\[
u_{tx} = -\frac{2(\lambda - 1)}{q(x)(t + x)} u_t + u_x + \frac{2(\lambda + (\lambda - 1)(t + x))}{q(x)(t + x)^2} u
\]
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has the invariants $P = \lambda \equiv \text{const.}$, $Q = q(x)$, and belongs to $S_5$. For an equation from $S_5$ we normalize $b_1^i = H Q_x^{-1}$. Then after absorption of torsion and prolongation we have the involutive lifted coframe $\Theta = \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, \eta_3\}$ with the structure equations

\[
d\theta_0 = \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,
\]

\[
d\theta_1 = \eta_1 \wedge \theta_1 - P \theta_0 \wedge \xi^2 - N \theta_1 \wedge \xi^2 + \xi^1 \wedge \sigma_{11},
\]

\[
d\theta_2 = \eta_1 \wedge \theta_2 - \theta_0 \wedge \xi^1 + N \theta_2 \wedge \xi^2 + \xi^2 \wedge \sigma_{22},
\]

\[
d\xi^1 = N \xi^1 \wedge \xi^2, \quad d\xi^2 = 0,
\]

\[
d\sigma_{11} = \eta_1 \wedge \sigma_{11} + \eta_2 \wedge \xi^1 + (Q + 1 - 2 P) \theta_1 \wedge \xi^2 + 2 N \xi^2 \wedge \sigma_{11},
\]

\[
d\sigma_{22} = \eta_1 \wedge \sigma_{22} + \eta_3 \wedge \xi^2 + (P - 2 + Q) \theta_2 \wedge \xi^1,
\]

\[
d\eta_1 = (P - 1) \xi^1 \wedge \xi^2, \quad d\eta_2 = \pi_1 \wedge \xi^1 + \eta_1 \wedge \eta_2 - 3 N \eta_2 \wedge \xi^2 + (2 - 3 P + 3 Q) \xi^2 \wedge \sigma_{11},
\]

\[
d\eta_3 = \pi_2 \wedge \xi^2 + \eta_1 \wedge \eta_3 + (2 N (P + Q - 2) - 1) \theta_2 \wedge \xi^1 + (2 P + Q - 3) \xi^1 \wedge \sigma_{22},
\]

where the function $N = (H Q_{xx} - H_x Q_x) H^{-1} Q_x^{-2}$ is an invariant of the symmetry group, and the operators of invariant differentiation are $\mathbb{D}_1 = Q_x H^{-1} D_t$ and $\mathbb{D}_2 = Q_x^{-1} D_x$. We have $[\mathbb{D}_1, \mathbb{D}_2] = -N \mathbb{D}_1$, $\mathbb{D}_1(Q) = 0$, and $\mathbb{D}_2(Q) = 1$. The functions $Q$ and $N$ are a basis for the set of all invariants of the symmetry group of an equation from $S_5$. We take the $s$-th order classifying manifold associated with the coframe $\Theta$ and an open subset $U \subseteq \mathbb{R}^2$ in the form

\[
C^s(\Theta, U) = \{(P, Q(x), \mathbb{D}_1^j(\mathbb{D}_2^k(N))(t, x)) \mid 0 \leq j + k \leq s, \ (t, x) \in U\}.
\]

(22)

Then two equations from $S_5$ are equivalent under a contact transformation if and only if their second order classifying manifolds (22) are (locally) overlap.

Finally, we denote by $S_6$ the subclass of equations \([\text{II}] \) such that $P \equiv \text{const.}$, $Q \equiv \text{const}$. This subclass is not empty, since, for example, the equation

\[
u_{tx} = -t u_t - \lambda x u_x - \lambda t x u
\]

(23)

has the invariants $P = \lambda$ and $Q = 0$, while the Euler - Poisson equation

\[
u_{tx} = 2 \mu^{-1} (t + x)^{-1} u_t + 2 \lambda \mu^{-1} (t + x)^{-1} u_x - 4 \lambda \mu^{-2} (t + x)^{-2} u
\]

(24)

has the invariants $P = \lambda$ and $Q = \mu$, \([\text{23}] \ $ 9.2 \). For an equation from $S_6$ after absorption of torsion and prolongation we have the involutive lifted coframe $\Theta = \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, \eta_3, \eta_4\}$ with the structure equations

\[
d\theta_0 = \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,
\]

\[
d\theta_1 = \eta_2 \wedge \theta_1 - P \theta_0 \wedge \xi^2 + \xi^1 \wedge \sigma_{11},
\]
\[ d\theta_2 = (2\eta_1 - \eta_2) \wedge \theta_2 - \theta_0 \wedge \xi^1 + \xi^2 \wedge \sigma_{22}, \]
\[ d\xi^1 = (\eta_1 - \eta_2) \wedge \xi^1, \]
\[ d\xi^2 = (\eta_2 - \eta_1) \wedge \xi^2, \]
\[ d\sigma_{11} = (2\eta_2 - \eta_1) \wedge \sigma_{11} + \eta_3 \wedge \xi^1 + (Q + 1 - 2P) \theta_1 \wedge \xi^2, \]
\[ d\sigma_{22} = (3\eta_1 - 2\eta_2) \wedge \sigma_{22} + \eta_4 \wedge \xi^2 + (P - 2 + Q) \theta_2 \wedge \xi^1, \]
\[ d\eta_1 = (P - 1) \xi^1 \wedge \xi^2, \]
\[ d\eta_2 = (P - Q - 1) \xi^1 \wedge \xi^2, \]
\[ d\eta_3 = \pi_1 \wedge \xi^1 - (2\eta_1 - 3\eta_2) \wedge \eta_3 + (3(Q - P) + 2) \xi^2 \wedge \sigma_{11}, \]
\[ d\eta_4 = \pi_2 \wedge \xi^2 + (4\eta_1 - 3\eta_2) \wedge \eta_4 + (3(Q - 1) + 2P) \xi^1 \wedge \sigma_{22}. \]

All the invariants of the symmetry group for an equation from \( S_6 \) are constants, and the classifying manifold is a point. Thus an equation from \( S_6 \) is equivalent to one of equations \([23]\) or \([24]\) with the same values of \( P \) and \( Q \), \([23]\ § 9.2\].

The results of the above calculations are summarized in the following statement:

**Theorem.** The class of linear hyperbolic equations \([1]\) is divided into the six subclasses \( S_1, S_2, \ldots, S_6 \) invariant under an action of the pseudo-group of contact transformations:

- \( S_1 \) consists of all equations \([1]\) such that \( H \equiv 0 \) and \( K \equiv 0 \);
- \( S_2 \) consists of all equations \([1]\) such that \( P_t \not\equiv 0 \);
- \( S_3 \) consists of all equations \([1]\) such that \( P_t \equiv 0 \) and \( P_x \not\equiv 0 \);
- \( S_4 \) consists of all equations \([1]\) such that \( P \equiv \text{const.} \) and \( Q_t \not\equiv 0 \);
- \( S_5 \) consists of all equations \([1]\) such that \( P \equiv \text{const.} \), \( Q_t \equiv 0 \), and \( Q_x \not\equiv 0 \);
- \( S_6 \) consists of all equations \([1]\) such that \( P \equiv \text{const.} \) and \( Q \equiv \text{const.} \).

Every equation from the subclass \( S_1 \) is locally equivalent to the linear wave equation \( u_{tx} = 0 \).

Every equation from the subclass \( S_6 \) is locally equivalent to either equation \([23]\) when \( Q = 0 \) or to the equation \([24]\) when \( Q \not\equiv 0 \).

For the subclass \( S_2 \), the basic invariants are \( P, Q \), and \( J_2 \), the operators of invariant differentiation are \( \mathbb{D}_1 = P_t^{-1} D_t \) and \( \mathbb{D}_2 = P_t H^{-1} D_x \).

For the subclass \( S_3 \), the basic invariants are \( P, Q \), and \( L \), the operators of invariant differentiation are \( \mathbb{D}_1 = P_x H^{-1} D_t \) and \( \mathbb{D}_2 = P_x^{-1} D_x \).

For the subclass \( S_4 \), the basic invariants are \( Q, M_1 \), and \( M_2 \), the operators of invariant differentiation are \( \mathbb{D}_1 = Q_t^{-1} D_t \) and \( \mathbb{D}_2 = Q_t H^{-1} D_x \).

For the subclass \( S_5 \), the basic invariants are \( Q \) and \( N \), the operators of invariant differentiation are \( \mathbb{D}_1 = Q_x H^{-1} D_t \) and \( \mathbb{D}_2 = Q_x^{-1} D_x \).

Two equations from one of the subclasses \( S_2, S_3, S_4, \) or \( S_5 \) are locally equivalent to each other if and only if the classifying manifolds \([18], [21], [21], \) or \([22]\) for these equations locally overlap.
Conclusion

In this paper, the moving coframe method of [6] is applied to the local equivalence problem for the class of linear second-order hyperbolic equations in two independent variables under an action of the pseudo-group of contact transformations. The class is divided into the six invariant subclasses. For all the subclasses, the Maurer-Cartan forms for symmetry groups, the bases of differential invariants and the invariant differentiation operators are found. This allowed to solve the equivalence problem for the whole class of linear hyperbolic equations. It is shown that the moving coframe method is applicable to structurally intransitive symmetry groups. The method uses linear algebra and differentiation operations only and does not require analysing overdetermined systems of partial differential equation or using procedures of integration.

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