Mathematical Modeling of Biofilm Development

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Abstract. We perform mathematical analysis of the biofilm development process. A model describing biomass growth is proposed: It arises from coupling three parabolic nonlinear equations: a biomass equation with degenerate and singular diffusion, a nutrient transport equation with a biomass-density dependent diffusion, and an equation of the Navier-Stokes type, describing the fluid flow in which the biofilm develops. This flow is subject to a biomass–density dependent obstacle. The model is treated as a system of three inclusions, or variational inequalities; the third one causes major difficulties for the system’s solvability. Our approach is based on the recent development of the theory on Navier-Stokes variational inequalities.

1 Introduction

It is quite important for our future to find clean and reproducible materials and energy resources. In this connection, biomass has been noticed for the last thirty years. Biomass growth is a process of aggregation of some living organisms transported in fluids (liquids or gaz), usually sticking to the walls of the fluid container, and thus influencing the flow itself. It also involves nutrient transport and consumption. It can occur in air, water, soil penetrated by any fluid, blood. Only little is known about mathematical models of this mechanism. In particular, the process occurs in fluids, but models coupled with hydrodynamics have been seldom analysed.

In [8], such a biomass growth model coupled with fluid dynamics has been proposed in the three dimensional space. However, as far as we know, no theoretical results appeared in this context. The model assumed a sharp interface between the (solid) biomass and the liquid. In the present paper we propose an analogous mathematical model of biomass growth dynamics in a fluid, postulating, in place of a sharp interface, a thicker layer, considered as a mixture of both phases — just as in the weak formulation of a solid–liquid phase transition.

For other formulations of biomass growth with taxis terms, see [7]. These formulations are not explicitly included in our formulation, but can be easily obtained by a modification.
Let us recall in more detail the mathematical full model proposed in [8]. Let \( \Omega \subset \mathbb{R}^3 \) be a container in which biomass growth takes place. The process is described in terms of three unknown functions \( \mathbf{v}(x,t), w(x,t) \) and \( u(x,t) \) which are respectively the velocity of the fluid, the nutrient concentration and the biomass density at a point \( x \in \Omega \) and time \( t \geq 0 \). They are governed by the following system:

\[
(H_0) \quad \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} = -\frac{1}{\rho} \nabla P, \quad \div \mathbf{v} = 0, \quad \text{in} \quad \{(x,t)|u(x,t) = 0\},
\]

where \( \rho \) is the constant density and \( P \) is the pressure in the fluid,

\[
(N_0) \quad w_t + \mathbf{v} \cdot \nabla w - \div (d(u)\nabla w) = -f(w)u \quad \text{in} \quad \Omega, \quad t > 0,
\]

where \( f(w) = \frac{k_1w}{k_2+w} \) for positive constants \( k_1, k_2 \),

\[
(B_0) \quad u_t - \Delta d_1(u) + bu = f(w)u \quad \text{in} \quad \Omega, \quad t > 0,
\]

subject to suitable initial and boundary conditions. This model is derived under the postulate that the fluid cannot penetrate into the solid biomass \( (u > 0) \), the nutrient is convective by \( \mathbf{v} \cdot \nabla w \) and diffusive with biomass-density dependent coefficient \( d(u) \), and the diffusion of biomass is very slow near the interface \( u = 0 \), but very fast near the maximum density \( u = u^* \). The function \( f \) is the nutrient consumption term and \( b \) is a positive constant.

In this paper, we propose some relaxations and modifications into the above model, postulating that:

(i) The biomass density \( u(x,t) \) is non-negative and it has the finite maximum value \( u^* \), i.e. \( 0 \leq u(x,t) \leq u^* \). For some \( \delta_0 \in (0,u^*) \), which is fixed, we postulate that the region of high density \( \delta_0 \leq u(x,t) \leq u^* \) is solid, and that of low density \( 0 < u(x,t) < \delta_0 \) is the interface layer between the solid biomass and the liquid. In such a layer, the behavior of \( u \) may correspond to the dynamics of planktonic biomass floating in the liquid, cf. e.g. [17].

This causes a biomass dependent constraint on the fluid’s velocity. The constraint is written as:

\[
|\mathbf{v}(x,t)| \leq p_0(u^\varepsilon(x,t)),
\]

where \( p_0(r) : (0,u^*) \to [0,\infty) \) is a \( C^1 \), non-negative and non-increasing function on \( (0,u^*) \) such that (see Fig.1(i)):

\[
\lim_{r \downarrow 0} p_0(r) = \infty, \quad p'_0(r) < 0, \quad \forall r \in (0,\delta_0), \quad p_0(r) = 0, \quad \forall r \in [\delta_0,u^*]; \tag{1.1}
\]

on the other hand \( u^\varepsilon := \rho_\varepsilon \ast u \) is the local spatial-average of \( u(x,t) \) by means of the usual mollifier \( \rho_\varepsilon(x) \) (see Section 2 for details).

(ii) The nutrient concentration \( w(x,t) \) is non-negative and has the threshold value 1, i.e. \( 0 \leq w(x,t) \leq 1 \). Also, we suppose that there is no nutrient supply from the exterior. The diffusion coefficient \( d(u) \) depends on the biomass density \( u \) and

\[
c_d \leq d(r) \leq c'_d, \quad |d(r_1) - d(r_2)| \leq L(d)|r_1 - r_2|, \quad \forall r_1, r_2 \in \mathbb{R}, \tag{1.2}
\]
Figure 1: Functions introduced in (i)–(iii): the obstacle function $p_0$, governing the flow velocity, the nutrient diffusivity $d$, the biomass diffusivity $d_1$.

where $c_d$, $c_d'$ and $L(d)$ are positive constants (see Fig.1(ii)). The function $f(w)u$, appearing in biomass density and nutrient transport equations, is called the nutrient consumption, and in our model we suppose that

$$f(w) \text{ is of } C^1 \text{ and Lipschitz in } w \in \mathbb{R}, \quad f(0) \leq 0 \text{ and } f(1) \geq 0. \quad (1.3)$$

(iii) Biomass is diffusive (slowly near $u = 0$, but fast near $u = u^*$), as well as convective by $v \cdot \nabla u$. The degenerate diffusion term $d_1(u)$ is strictly increasing in $u \in [0, u^*)$ and

$$d_1(0) = 0, \quad \lim_{r \downarrow 0} \frac{d_1(r)}{r} = 0, \quad \lim_{u \uparrow u^*} d_1(u) = \infty \quad (\text{see Fig.1(iii)}). \quad (1.4)$$

Note that we do not suppose $d_1$ to be continuous.

Now our relaxed/modified version for $\{(H_0), (N_0), (B_0)\}$ is described as a system of three evolution equations — one of them with a constraint — which is of the form:

$$(H)^\varepsilon \quad \nu_t + (v \cdot \nabla)v - \nu \Delta v = g - \frac{1}{\rho} \nabla P, \quad \text{div } v = 0, \quad |v| \leq p_0(u^\varepsilon), \quad \text{in } \Omega, \ t > 0,$$

$$(N) \quad w_t + v \cdot \nabla w - \text{div}(d(u)\nabla w) = -f(w)u \quad \text{in } \Omega, \ t > 0,$$

$$(B) \quad u_t + v \cdot \nabla u - \Delta d_1(u) + bu = f(w)u \quad \text{in } \Omega, \ t > 0.$$
(H₀) includes a constraint, meaning: no flow when u > 0, free flow when u = 0. This is a sharp interface model. The constraint in (H)ε, expressed in terms of p₀, is a blurred version of the previous one. The ‘blurring’ is governed by two parameters, ε and δ₀. As a matter of fact, we may reduce the number of parameters by taking δ₀ = δ₀(ε) with δ₀(ε) ↓ 0 as ε ↓ 0; still, as they are independent, we leave both. When ε ↓ 0 and δ₀ ↓ 0 in {(H)ε, (N), (B)}, we formally arrive at {(H₀), (N₀), (B₀)}. However, it seems quite difficult to carry out rigorously this limit procedure.

The main objective of this paper is to give an existence result for {(H)ε, (N), (B)}, fixing parameters ε > 0 and δ₀ > 0. The result is completely new and the model itself reasonable from the biological point of view, despite the approximating parameters.

From the mathematical point of view, (H)ε is going to be formulated in the solenoidal function space $H^1_{0,σ}(Ω)$, (N) and (B) in the dual space of $H^1(Ω)$. Each problem (H)ε, (N) and (B) is separately treated in the above-mentioned spaces (cf. [3, 5, 6, 9, 10]). However, the structure of our system {(H)ε, (N), (B)} is extremely complicated because of its quasi-variational structure (cf. [11, 15]). The main difficulty for the analysis arises from this complexity of the couplings, especially the one in (H)ε, which appears via the nonlinear and unknown–dependent constraint.

The organization of this paper is as follows. In section 2, we introduce the analytical framework. In sections 3, 4 and 5, we formulate each model apart: the biomass density evolution, the nutrient transport and the flow governed by a Navier-Stokes variational inequality, respectively. We also give a smooth approximation for each model and prove its convergence. Finally, in section 6, we formulate an approximate full system by coupling these three models, and prove existence of its solution by the Schauder fixed-point argument. Then, we construct a solution of our original problem {(H)ε, (N), (B)} as a limit of approximate solutions, making use of a recent important development on variational inequalities of the Navier-Stokes type, see [12]. Our main result is Theorem 6.2.

2 Functional framework

2.1 Functionals and their subdifferentials

For a general (real) Banach space X we denote by $X^*$ its dual. We denote by $|·|_X$ and $|·|_{X^*}$ the norms in X and $X^*$, and by $(·,·)_{X^*,X}$ the duality pairing between both spaces.

Now, let X be reflexive and consider a functional $ψ : X → R ∪ \{∞\}$. We say that:

ψ is proper, if $−∞ < ψ(z) ≤ ∞$ for all $z ∈ X$ and if it is not idetically $∞$;

ψ is lower semi-continuous (l.s.c.) on X, if $\liminf_{n → ∞} ψ(z_n) ≥ ψ(z)$ for any sequence {zₙ} converging to z in X;

ψ is convex on X, if $ψ(rz_1 + (1−r)z_2) ≤ rψ(z_1) + (1−r)ψ(z_2)$ for all $z_1, z_2 ∈ X$ and $r ∈ [0, 1]$.

For a proper, l.s.c. and convex function $ψ$ on X, the set

$$D(ψ) := \{z ∈ X \mid ψ(z) < ∞\}$$

is called the effective domain. For each $z ∈ D(ψ)$ we consider a subset of $X^*$

$$∂_{X^*,X}ψ(z) := \{z^* ∈ X^* \mid (z^*, v − z)_{X^*,X} ≤ ψ(v) − ψ(z), \forall v ∈ X\},$$
which is called the subdifferential of $\psi$ at $z$; we put $\partial_X^* X \psi (z) = \emptyset$ for $z \notin D(\psi)$. If $X$ is a Hilbert space and it is identified with its dual, the subdifferential of a proper, l.s.c. and convex function $\psi$ on $X$ is defined by using the inner product $(\cdot, \cdot)_X$ in place of the duality $\langle \cdot, \cdot \rangle_{X^*, X}$ and the subdifferential at $z \in X$ is denoted by $\partial_X \psi (z)$:

$$
\partial_X \psi (z) := \{ y \in X \mid (y, v - z)_X \leq \psi (v) - \psi (z), \forall v \in X \}.
$$

For fundamental concepts and basic properties of subdifferentials we refer to [1, 4, 14].

2.2 The domain

Throughout this paper, we fix:

$\Omega$, a bounded domain in $\mathbb{R}^3$ with smooth boundary $\Gamma := \partial \Omega$;

$\Gamma_0$, a compact subset of $\Gamma$, having positive surface measure;

$T$, which is an arbitrary positive real number, and we denote $Q = \Omega \times [0, T]$.

2.3 Function Spaces

We set up:

$$
H := L^2(\Omega), \quad V := H^1(\Omega).
$$

The norms $| \cdot |_H$ and $| \cdot |_V$ are defined as usual. Next, denote by $V_0$ the space

$$
V_0 := \{ z \in V \mid z = 0 \text{ a.e. on } \Gamma_0 \},
$$

with the norm $|z|_{V_0} := |\nabla z|_H$.

The condition $z = 0$ above is understood in the sense of trace. We assume always that the dual spaces $V^*$ and $V_0^*$ are equipped with the dual norms of $V$ and $V_0$, respectively.

By identifying $H$ with its dual space, we have

$$
V \subset H \subset V^*, \quad V_0 \subset H \subset V_0^*, \quad \text{with compact embeddings; (2.1)}
$$

throughout this paper, we fix a positive constant $c_0$ such that

$$
|z|_H \leq c_0 |z|_V \quad \forall z \in V, \quad |z|_H \leq c_0 |z|_{V_0} \quad \forall z \in V_0, \quad |z|_{V^*} \leq c_0 |z|_H, \quad |z|_{V_0^*} \leq c_0 |z|_H \quad \forall z \in H.
$$

(2.2)

For simplicity of notation, the inner product $(\cdot, \cdot)_H$ in $H$, the dualities $\langle \cdot, \cdot \rangle_{V^*, V}$ and $\langle \cdot, \cdot \rangle_{V_0^*, V_0}$ are denoted by $(\cdot, \cdot)$, $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_0$, respectively.

The duality mapping $F_0$ from $V_0$ onto $V_0^*$ is characterized by

$$
\langle F_0 z_1, z_2 \rangle_0 = \int_\Omega \nabla z_1 (x) \cdot \nabla z_2 (x) dx =: (F_0 z_1, F_0 z_2)_*, \quad \forall z_1, z_2 \in V_0,
$$

(2.3)

where the first equality defines $F_0$ and the second the induced inner product in $V_0^*$, denoted by $(\cdot, \cdot)_*$. From the definition of $F_0$ and $V_0$, it follows (cf. [14; §1]) that formally

$$
F_0 v = -\Delta v \text{ in } \Omega \text{ in the sense of distributions, } v = 0 \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma - \Gamma_0.
$$

(2.4)

Next, we consider solenoidal function spaces. Let

$$
\mathcal{D}_s := \{ z \in C_0^\infty (\Omega)^3 \mid \text{div} z = 0 \text{ in } \Omega \},
$$
\[ V_\sigma = \text{the closure of } D_\sigma \text{ in } H^1_0(\Omega)^3, \quad H_\sigma = \text{the closure of } D_\sigma \text{ in } L^2(\Omega)^3. \]

In these spaces the norms are given by

\[ |z|_{H_\sigma} := \left( \frac{2}{3} \int_\Omega |z^{(k)}(x)|^2 dx \right)^{\frac{1}{2}}, \quad \forall z = (z^{(1)}, z^{(2)}, z^{(3)}) \in H_\sigma, \]

and \[ |z|_{V_\sigma} := \left( \frac{2}{3} \int_\Omega |\nabla z^{(k)}(x)|^2 dx \right)^{\frac{1}{2}}, \quad \forall z = (z^{(1)}, z^{(2)}, z^{(3)}) \in V_\sigma. \]

Note that \( H_\sigma \) is a Hilbert space and by identifying it with its dual, we have

\[ V_\sigma \subset H_\sigma \subset V_\sigma^* \text{ with compact embeddings.} \tag{2.5} \]

We write \( \langle \cdot, \cdot \rangle_\sigma \) for the inner product in \( H_\sigma \) and \( \langle \cdot, \cdot \rangle_\sigma \) for duality between \( V_\sigma^* \) and \( V_\sigma \).

**Remark 2.1.** We mean by \( H \subset V_0^*, \ H \subset V^* \), and \( H_\sigma \subset V_\sigma^* \) in (2.1) and (2.5) that \( \langle u, z \rangle_0 = \langle u, z \rangle \) for all \( u \in H, \ z \in V_0 \) and \( \langle u, z \rangle = \langle u, z \rangle \) for all \( u \in H, \ z \in V \) as well as \( \langle u, z \rangle_\sigma = \langle u, z \rangle_\sigma \) for all \( u \in H_\sigma, \ z \in V_\sigma \).

**Remark 2.2.** If \( v \in V_\sigma \), then \( v = 0 \) on \( \partial \Omega \) and \( v \cdot \nabla z = \text{div}(zv) \) for all \( z \in V \).

### 2.4 Space averaging

Given \( \mu \in (0, 1] \), a function \( u \in H \) and any smooth function \( \gamma \) on \( \mathbb{R}^3 \), we denote by \( \rho_\mu \ast (\gamma u) \) the convolution of the usual mollifier

\[ \rho_\mu (x) := \begin{cases} \frac{1}{N_\mu} \exp \left( -\frac{1}{\mu^2 - |x|^2} \right) & \text{ if } |x| < \mu, \\ 0, & \text{ otherwise,} \end{cases} \quad N_\mu = \int_\Omega \exp \left( -\frac{1}{\mu^2 - |x|^2} \right) dx, \]

and function \( \gamma(x)u(x) \), namely

\[ [\rho_\mu \ast (\gamma u)](x) := \int_{\mathbb{R}^3} \rho_\mu (x - y) \gamma(y) \tilde{u}(y) dy, \quad \forall x \in \Omega, \]

where \( \tilde{u} \) denotes the extension of \( u \) onto \( \mathbb{R}^3 \) by 0. Noting here that

\[ [\rho_\mu \ast (\gamma u)](x) = \int_\Omega \rho_\mu (x - y) \gamma(y) u(y) dy = (u, \gamma \rho_\mu (x - \cdot)), \]

we see that, in the case when \( \gamma = 0 \) on \( \Gamma_0 \)

\[ |\rho_\mu \ast (\gamma u)|_{C(\overline{\Omega})} \leq \left( \sup_{x \in \Omega} |\gamma \rho_\mu (x - \cdot)|_{V_0} \right) |u|_{V_0} \leq c_0 \left( \sup_{x \in \Omega} |\gamma \rho_\mu (x - \cdot)|_{V_0} \right) |u|_{H}, \tag{2.6} \]

and in the case when \( \gamma \equiv 1 \)

\[ |\rho_\mu \ast u|_{C(\overline{\Omega})} \leq \left( \sup_{x \in \Omega} |\rho_\mu (x - \cdot)|_{V} \right) |u|_{V^*} \leq c_0 \left( \sup_{x \in \Omega} |\rho_\mu (x - \cdot)|_{V} \right) |u|_{H}, \tag{2.7} \]

see (2.2). Similarly, if \( u \in W^{1,2}(0, T; V_0^*) \) and \( \gamma = 0 \) on \( \Gamma_0 \), then

\[ |\rho_\mu \ast (\gamma u)|_{W^{1,2}(0, T; C(\overline{\Omega}))} \leq \left( \sup_{x \in \Omega} |\gamma \rho_\mu (x - \cdot)|_{V_0} \right) |u|_{W^{1,2}(0, T; V_0^*)}, \tag{2.8} \]

and if \( u \in W^{1,2}(0, T; V^*) \), then

\[ |\rho_\mu \ast u|_{W^{1,2}(0, T; C(\overline{\Omega}))} \leq \left( \sup_{x \in \Omega} |\rho_\mu (x - \cdot)|_{V} \right) |u|_{W^{1,2}(0, T; V^*)}. \tag{2.9} \]
3 Biomass growth inclusion and its approximation

In order to describe the degenerate and singular diffusion for biomass density we use a non-negative, proper, l.s.c. and convex function $\hat{\beta}(\cdot)$ on $\mathbb{R}$ given by:

$$\hat{\beta}(r) := \begin{cases} \int_0^r d_1(s)ds, & \text{for } r \in [0,u^*], \\ \infty, & \text{otherwise,} \end{cases}$$

where $d_1$ is the function introduced in (i) in the introduction, satisfying (1.4). Its subdifferential $\beta := \partial \hat{\beta}$ in $\mathbb{R}$ is equal to $d_1$ except on a countable set, where $d_1$ is not necessarily continuous. In these points of discontinuity, it is given by $[d_1^- (r), d_1^+(r)]$, where $d_1^-(r) := \lim_{s \uparrow r} d_1(s)$ and $d_1^+(r) := \lim_{s \downarrow r} d_1(s)$ for $r \in (0, u^*)$, if $r \in (0, u^*)$, also $\beta(0) = (-\infty, 0]$ and $\beta(r) = \emptyset$ for $r < 0$ or $r \geq u^*$. Clearly, $D(\beta) = [0, u^*)$, $d_1(r) \in \beta(r)$ for any $r \in [0, u^*)$, $R(\beta) = \mathbb{R}$ and $\beta$ is strictly monotone in $\mathbb{R}$ (see Fig.1(iii)).

Now, we define the function $\varphi$ on $V_0^*$ by

$$\varphi(z) := \begin{cases} \int_{\Omega} \hat{\beta}(z(x))dx, & \text{if } z \in H \text{ and } \hat{\beta}(z) \in L^1(\Omega), \\ \infty, & \text{otherwise.} \end{cases}$$

Clearly, $\varphi(\cdot)$ is non-negative, proper and convex on $V_0^*$ with $D(\varphi)$ included in the subset \{z $\in H$ $|$ 0 $\leq$ z $\leq$ u* a.e. on $\Omega$\}. It follows that $\varphi$ is l.s.c. on $V_0^*$. Hence any level set of $\varphi(\cdot)$ is compact in $V_0^*$. We denote by $\partial_* \varphi(\cdot)$ the subdifferential of $\varphi(\cdot)$ in $V_0^*$, namely

$$\partial_* \varphi(z) := \partial_{V_0^*} \varphi(z) = \{z^* \in V_0^* \mid (z^*, v - z)_* \leq \varphi(v) - \varphi(z), \forall v \in V_0^*\}. \forall z \in D(\varphi).$$

Then we know (cf. [5, 6]) that

$$\partial_* \varphi(v) = \{F_0 \tilde{v} \mid \tilde{v} \in V, \tilde{v} \in \beta(v) \text{ a.e. on } \Omega\}, \forall v \in D(\partial_* \varphi) (\subset H). \quad (3.1)$$

Let $g \in L^2(0,T; V_0^*)$ and $u_0 \in D(\varphi)$. We denote by $CP(\varphi; g, u_0)$ the Cauchy problem

$$u' + \partial_* \varphi(u(t)) \ni g(t) \text{ in } V_0^* \text{ for a.e. } t \in [0,T], \quad u(0) = u_0.$$ 

By the general theory of evolution equations (cf. Appendix I) this Cauchy problem admits one and only one solution $u$ such that $u \in W^{1,2}(0,T; V_0^*)$ and $t \mapsto \varphi(u(t))$ is absolutely continuous on $[0,T]$. The following convergence result will be used later on.

**Lemma 3.1.** Let $u_0 \in H$ with $u_0 \in D(\varphi)$ and \{g\} \in $L^2(0,T; V_0^*)$ such that $g_n \rightharpoonup g$ weakly in $L^2(0,T; V_0^*)$ as $n \rightarrow \infty$. Then, the solution $u_n$ of $CP(\varphi; g_n, u_0)$ converges to the solution $u$ of $CP(\varphi; g, u_0)$ in $C([0,T]; V_0^*) \cap L^2(Q)$ and weakly in $W^{1,2}(0,T; V_0^*)$.

**Proof.** The convergences $u_n \rightharpoonup u$ weakly in $W^{1,2}(0,T; V_0^*)$ and strongly in $C([0,T]; V_0^*)$ are obtained by Proposition II of the Appendix (note that $D(\varphi)$ is compact in $V_0^*$). We show below the convergence in $L^2(Q)$. Taking the difference of two inclusions for $u_n$ and $u$, we have by (3.1)

$$u'_n - u' + F_0(\bar{u}_n - \bar{u}) = g_n - g \text{ in } V_0^*.$$
where \( \tilde{u}_n \in L^2(0, T; V_0) \) with \( \tilde{u}_n \in \beta(u_n) \) a.e. on \( Q \) and \( \tilde{u} \in L^2(0, T; V_0) \) with \( \tilde{u} \in \beta(u) \) a.e. on \( Q \). Now, take the inner product between both sides of the above relation and \( u_n - u \) in \( V_0^\ast \) to obtain
\[
\frac{1}{2} \frac{d}{dt} |u_n(t) - u(t)|^2_{V_0^\ast} + \int_\Omega (\tilde{u}_n(t) - \tilde{u}(t))(u_n(t) - u(t)) \, dx \leq (g_n(t) - g(t), u_n(t) - u(t)),
\]
for a.e. \( t \in [0, T] \). Integrating this inequality in time over \([0, t]\) yields
\[
\frac{1}{2} |u_n(t) - u(t)|^2_{V_0^\ast} + \int_0^t \int_\Omega (\tilde{u}_n - \tilde{u})(u_n - u) \, dx \, d\tau \leq \int_0^t (g_n - g, u_n - u) \, d\tau
\]
for all \( t \in [0, T] \), whence, by monotonicity,
\[
\lim_{n \to \infty} \int_Q (\tilde{u}_n - \tilde{u})(u_n - u) \, dx \, dt = 0.
\]

We derive from this convergence that \( u_n \to u \) in \( L^2(\Omega) \). In fact, by the strict monotonicity of \( \beta \) and \( 0 \in \beta(0) \), for any small \( \delta > 0 \) there is a constant \( C_\delta \in (0, 1) \) such that
\[
\bar{r}_1 - \bar{r}_2 \geq C_\delta \text{ if } r_1 - r_2 \geq \delta, r_1, r_2 \in D(\beta), \bar{r}_1 \in \beta(r_1) \text{ and } \bar{r}_2 \in \beta(r_2).
\]
Hence, putting \( E_{n, \delta} := \{(x, t) \in Q \mid |u_n(x, t) - u(x, t)| \geq \delta\} \), we observe that
\[
C_\delta \int_Q |u_n - u| \, dx \, dt = C_\delta \int_{E_{n, \delta}} |u_n - u| \, dx \, dt + C_\delta \int_{Q - E_{n, \delta}} |u_n - u| \, dx \, dt \\
\leq \int_Q (\tilde{u}_n - \tilde{u})(u_n - u) \, dx \, dt + \delta C_\delta |\Omega|,
\]
where \( |\Omega| \) denotes the volume of \( \Omega \). Accordingly, \( \limsup_{n \to \infty} \int_Q |u_n - u| \, dx \, dt \leq \delta T |\Omega| \).
Since \( \delta > 0 \) is arbitrary and \( 0 \leq u_n \leq u^* \) a.e. on \( Q \), we have \( u_n \to u \) in \( L^2(Q) \). \( \square \)

With the operator \( \partial_n \varphi \), the biomass growth equation \((B)\) with formal boundary condition \( u = 0 \) on \( \Gamma_0 \times (0, T) \) and \( \partial_n u = 0 \) on \( (\Gamma - \Gamma_0) \times (0, T) \) (cf. \((2.4)\)), is reformulated as the Cauchy problem:
\[
(B; w, v; u_0) \quad \begin{cases}
 u'(t) + \partial_n \varphi(u(t)) + v(t) \cdot \nabla u(t) + bu(t) \ni f(w(t))u(t) & \text{in } V_0^\ast, \\
 u(0) = u_0, 
\end{cases}
\]
where \( w, v, u_0 \) are given. More precisely, we have the following definition of solution.

**Definition 3.1.** Let \( w \in L^2(0, T; V) \cap L^\infty(Q), v \in L^2(0, T; V_\sigma) \) and \( u_0 \in H \) with \( \beta(u_0) \in L^1(\Omega) \). Then, a function \( u : [0, T] \to V_0^\ast \) is called a solution to \((B; w, v; u_0)\) if \( u \in W^{1,2}(0, T; V_0^\ast), 0 \leq u \leq u^* \) a.e. on \( Q \), and for a.e. \( t \in (0, T) \), \((3.2)\) is satisfied.

Note that \( \beta(u_0) \in L^1(\Omega) \) implies \( 0 \leq u_0 \leq u^* \) a.e. on \( \Omega \). So as to be explicit for the sense of \((3.2)\), we note that on account of \((3.1)\) and Remarks 2.1, 2.2, the solution \( u \) of \((B; w, v, u_0)\) satisfies the following variational equality: there is \( \tilde{u} : [0, T] \to V_0 \) such that
\[
\tilde{u}(x, t) \in \beta(u(x, t)) \quad \text{a.e. on } Q,
\]
\[
\langle u'(t), z \rangle_0 + \int_\Omega \nabla \tilde{u}(t) \cdot \nabla z \, dx - \int_\Omega u(t)v(t) \cdot \nabla z \, dx + (bu(t), z) \\
= (f(w(t))u(t), z), \quad \forall z \in V_0, \quad \text{for a.e. } t \in (0, T).
\]
In order to solve \((B; w, \mathbf{v}; u_0)\), we approximate it by the following problem including a real positive parameter \(\mu \downarrow 0\):
\[
(B; w, \mathbf{v}, u_0)_\mu \quad \begin{cases}
u'(t) + \partial_x \varphi(u(t)) + \mathbf{v}(t) \cdot \nabla [\rho_\mu \ast (\gamma_\mu u(t))] + b(t) \\ \ni f(\rho_\mu \ast w(t))u(t) & \text{in } V_0^* \text{ for a.e. } t \in [0, T], \\ u(0) = u_0,
\end{cases}
\]
where \(\{\gamma_\mu (\cdot)\}_{\mu \in (0, 1]}\) is a family of smooth functions on \(\mathbb{R}^3\) such that
\[
\gamma_\mu (y) \begin{cases} = 0, & \text{if } \text{dist}(y, \Gamma_0) \leq \frac{1}{2} \mu, \\ \in [0, 1], & \text{if } \frac{1}{2} \mu < \text{dist}(y, \Gamma_0) < \mu, \\ = 1, & \text{if } \text{dist}(y, \Gamma_0) \geq \mu,
\end{cases}
\]
for all \(\mu \in (0, 1]\) and \(\gamma_\mu (\cdot)\) is continuous in \(C(\Omega)\) with respect to \(\mu \in (0, 1]\). We have \(0 \leq \gamma_\mu (y) \leq 1\) and \(\gamma_\mu (y) \to 1\) for any \(y \in \Omega\) as \(\mu \downarrow 0\).

**Remark 3.1.** When \(\mu = 0\), \((B; w, \mathbf{v}, u_0)_\mu = (B; w, \mathbf{v}, u_0)\).

**Proposition 3.1.** Assume that (1.4) holds and let \(\mu \in (0, 1]\). Let \(\mathbf{v}\) and \(w\) be given functions such that
\[
\mathbf{v} \in L^2(0, T; \mathbf{V}_\sigma) \cap L^\infty(0, T; \mathbf{H}_\sigma),
\]
\[
w \in W^{1, 2}(0, T; V^*) \cap L^2(0, T; V), \ 0 \leq w \leq 1 \text{ a.e. on } Q.
\]
Also, let \(u_0 \in H\) be such that \(\hat{\beta}(u_0) \in L^1(\Omega)\). Then, there exists one and only one solution \(u\) to \((B; w, \mathbf{v}, u_0)_\mu\). This solution is such that \(t \to \hat{\beta}(u(t))\) is absolutely continuous on \([0, T]\). Moreover, there is a non-negative, bounded and non-decreasing function \(B_0(\cdot)\) on \([0, \infty) \times [0, \infty)\), independent of the parameter \(\mu \in (0, 1]\), such that
\[
|u|^2_{W^{1, 2}(0, T; V_0^*)} + \sup_{t \in [0, T]} |\hat{\beta}(u(t))|_{L^1(\Omega)} \leq B_0 \left( |\mathbf{v}|_{L^2(0, T; \mathbf{H}_\sigma)}, |\hat{\beta}(u_0)|_{L^1(\Omega)} \right).
\]

For the proof of Proposition 3.1 we prepare two lemmas.

**Lemma 3.2.** Assuming (3.4) we have, for all \(v \in V_0\) and \(t \in [0, T]\),
\[
|f(\rho_\mu \ast w(t))v|_{V_0} \leq 3(c_0 + 1) \left\{ L(f)|\Omega|^{\frac{1}{2}} \sup_{\bar{x} \in \bar{\Omega}} |\rho_\mu(x - \cdot)|_{V} + \max_{0 \leq r \leq 1} f(r) \right\} |v|_{V_0}
\]
\[
= M^\mu_{V_0} |v|_{V_0},
\]
where \(L(f)\) is the Lipschitz constant of \(f\) and \(c_0\) is the constant from (2.2).

**Proof.** First we note that
\[
|\nabla [\rho_\mu \ast w](x, t)|^2 = \sum_{i=1}^{3} \left| (\rho_\mu \ast w)(x_i, t) \right|^2 = \sum_{i=1}^{3} \left| \int_{\Omega} \rho_\mu(x_i, x - y)w(y, t)dy \right|^2 \leq \sum_{i=1}^{3} \left( \int_{\Omega} |\rho_\mu(x_i, x - y)|dy \right)^2 \leq |\Omega||\rho_\mu(x - \cdot)|^2_{V}.
\]
By (3.6),
\[
|f(\rho \ast w)v|_{V_0}^2 = \int_{\Omega} |\nabla [f(\rho \ast w)v]|^2 dx
\leq \left( L(f)^2 |\nabla (\rho \ast w)|^2_{C(\Omega)} + \max_{0 \leq r \leq 1} f(r)^2 \right) \int_{\Omega} |\nabla v|^2 dx
\]
Thus the required inequality is obtained.

**Lemma 3.3.** Assuming (3.4) we have, for all \( z \in H \) and \( t \in [0, T] \):
\[
|v(t) \cdot \nabla [\rho \ast (\gamma \ast z)]|_{V_0} \leq M_2^\mu |z|_{V_0}^\mu |v(t)|_{H_\sigma}, \quad |f(\rho \ast w(t))z - bz|_{V_0} \leq M_3^\mu |z|_{V_0}^\mu,
\]
where \( M_2^\mu := \sup_{x \in \Omega} |\gamma \ast \rho \ast (x \ast \cdot)|_{V_0} \) and \( M_3^\mu := M_1^\mu + b \).

**Proof.** For any \( z \in H \) we have by (2.6) and Remarks 2.1, 2.2:
\[
|v(t) \cdot \nabla [\rho \ast (\gamma \ast z)]|_{V_0} = \sup_{v \in V_0, |v|_{V_0} \leq 1} \langle v(t) \cdot \nabla [\rho \ast (\gamma \ast z)], v \rangle_0 = \sup_{v \in V_0, |v|_{V_0} \leq 1} \int_{\Omega} \langle \rho \ast (\gamma \ast z)v(t) \rangle v dx
\]
\[
= \sup_{v \in V_0, |v|_{V_0} \leq 1} \int_{\Omega} \{-\rho \ast (\gamma \ast z)v(t) \cdot \nabla v dx \} \leq |\rho \ast (\gamma \ast z)|_{C(\Omega)} |v(t)|_{H_\sigma}
\]
\[
\leq \left( \sup_{x \in \Omega} |\gamma \ast \rho \ast (x \ast \cdot)|_{V_0} \right) |z|_{V_0}^\mu |v(t)|_{H_\sigma} = M_2^\mu |z|_{V_0}^\mu |v(t)|_{H_\sigma}.
\]
Next, we see from Lemma 3.2 that for any \( z \in H \)
\[
|f(\rho \ast w(t))z|_{V_0} = \sup_{v \in V_0, |v|_{V_0} \leq 1} \langle f(\rho \ast w(t))z, v \rangle_0 = \sup_{v \in V_0, |v|_{V_0} \leq 1} \langle z, f(\rho \ast w(t))v \rangle_0
\]
\[
\leq |z|_{V_0}^\mu \sup_{v \in V_0, |v|_{V_0} \leq 1} |f(\rho \ast w(t))v|_{V_0} \leq M_1^\mu |z|_{V_0}^\mu.
\]
Therefore,
\[
|f(\rho \ast w(t))z - bz|_{V_0} \leq |f(\rho \ast w(t))z|_{V_0} + b|z|_{V_0} \leq M_2^\mu |z|_{V_0}^\mu.
\]
Thus (3.7) is obtained.

**Proof of Proposition 3.1.** We shall prove the proposition in three steps.
(Step 1) Assume that \( v \in C([0, T]; V_\sigma) \). By virtue of Lemma 3.3, our perturbation term
\[
h(t, z) := f(\rho \ast w(t))z - bz - v(t) \cdot \nabla [\rho \ast (\gamma \ast z)]
\]
is Lipschitz continuous in \( z \in V_0^* \) and continuous in \( t \), so that it satisfies the condition (h4) in Appendix III. The other conditions (h1) -- (h3) are easily checked. Therefore, the existence-uniqueness of a (strong) solution \( u \) of \( (B; w, v; u_0)_\mu \) is a direct consequence of
and in the same way with the Remark 2.2 we obtain
\[ T \text{ space of all weakly continuous functions from } [0, T] \text{ a.e. on } Q, \text{ these regularities imply } u \in C_w([0, T]; H), \text{ where } C_w([0, T]; H) \text{ stands for the space of all weakly continuous functions from } [0, T] \text{ into } H. \]

Next, we show the uniform estimate (3.5). We observe that, by (2.2), and as \(|\rho_\mu| \leq 1,\)
\[
|f(\rho_\mu \ast w(t))u(t) - bu(t)|_{V_0^*} \leq c_0|f(\rho_\mu \ast w(t))u(t) - bu(t)|_H \leq c_0 \left\{ \max_{0 \leq r \leq 1} f(r)u^* + bu^* \right\} |\Omega|^{1/2},
\]
and in the same way with the Remark 2.2 we obtain
\[
|v(t) \cdot \nabla[\rho_\mu \ast (\gamma_\mu u(t))]|_{V_0^*} = |\text{div}[\rho_\mu \ast (\gamma_\mu u(t))v(t)]|_{V_0^*} = \sup_{v \in V_0, |v|_{V_0} \leq 1} \left\{ -\int_{\Omega} \rho_\mu \ast (\gamma_\mu u(t))v(t) \cdot \nabla v dx \right\} \leq u^*|v(t)|_{H_\sigma} \sup_{v \in V_0, |v|_{V_0} \leq 1} |v|_{V_0} = u^*|v(t)|_{H_\sigma}.
\]

These inequalities imply that the perturbation term \(h(t, u(t))\) satisfies
\[
|h(\cdot, u)|_{L^2(0,T;V_0^*)} \leq M_4(1 + |v|_{L^2(0,T;H_\sigma)})
\]
for a positive constant \(M_4\) independent of \(\mu \in (0, 1]\), \(v\) and \(u\). Accordingly, from Appendix I, Proposition I(3), it follows that (3.5) holds for a non-negative increasing function \(B_\rho(\cdot, \cdot, \cdot)\).

(Step 2) In the general case of \(v \in L^2(0, T; V_\sigma)\), we choose a sequence \(\{v_n\}\) in \(C([0, T]; V_\sigma)\) such that \(v_n \to v\) in \(L^2(0, T; V_\sigma)\) (as \(n \to \infty\)). According to the result of (Step 1), \((B; w, v_n; u_0)\mu\) admits a unique solution \(u_n\) which enjoys the uniform estimate (3.5). Therefore we can choose a subsequence \(\{u_{n_k}\}\) from \(\{u_n\}\) and a function \(u \in W^{1,2}(0, T; V_0^*)\) with \(\sup_{t \in [0, T]} |\hat{\beta}(u(t))|_{L^1(\Omega)} < \infty\) such that
\[
u_{n_k} \to u \text{ in } C([0, T]; V_0^*) \text{ and weakly in } W^{1,2}(0, T; V_0^*), \quad \sup_{k \geq 1, \ t \in [0, T]} |\hat{\beta}(u_{n_k}(t))|_{L^1(\Omega)} < \infty.
\]

Now it follows from Lemma 3.1 and Lemma 3.3 that
\[
f(\rho_\mu \ast w)u_{n_k} - bu_{n_k} - \nabla(\rho_\mu \ast (\gamma_\mu u_{n_k})) \to f(\rho_\mu \ast w)u - bu - \nabla(\rho_\mu \ast (\gamma_\mu u)) \text{ in } L^2(0, T : V_0^*).
\]

As a consequence, by Proposition II in the appendix, \(u_{n_k}\) converges in \(C([0, T]; V_0^*)\) to the solution of \((B; w, v, u_0)\mu\). Clearly this solution coincides with \(u\).

(Step 3) We now show uniqueness of solution. Let \(u\) and \(\bar{u}\) be two solutions of \((B; w, v; u_0)\mu\). Then it follows from the appendix, Proposition I, (2), and from Lemma 3.3, that
\[
\frac{1}{2}|u(t) - \bar{u}(t)|_{V_0^*}^2 \leq -\int_0^t \left( f(\rho_\mu \ast w)(u - \bar{u}) - b(u - \bar{u}) - \nabla(\rho_\mu \ast (\gamma_\mu u - \bar{u})) \right) u - \bar{u})_s d\tau \leq (M_2^\mu|v|_{L^{\infty}(0,T;H_\sigma)} + M_3^\mu) \int_0^t |u - \bar{u}|_{V_0^*}^2 d\tau.
\]

Therefore, by the Gronwall inequality, we have \(u = \bar{u}\) on \([0, T]\). \(\square\)
Proposition 3.2. Assume \([1.4]\) and let \(u_0 \in H\) be such that \(\hat{\beta}(u_0) \in L^1(\Omega)\). Take any \(\mu \in [0, 1]\) and let \(\{\mu_n\}\) be a non-increasing sequence in \((0, 1]\) such that \(\mu_n \downarrow \mu\) (as \(n \to \infty\)). Let \(\{v_n\}\) and \(\{w_n\}\) be sequences such that

\[
\begin{align*}
\{v_n\} & \text{ is bounded in } L^\infty(0, T; H_\sigma), \quad v_n \to v \text{ weakly in } L^2(0, T; V_\sigma), \\
0 & \leq w_n \leq 1 \text{ a.e. on } Q, \ w_n \to w \text{ in } L^2(Q) \text{ and weakly in } W^{1,2}(0, T; V^*),
\end{align*}
\]

(3.8)

(as \(n \to \infty\)). Then \(u_n\), the solution of \((B; w_n, v_n, u_0)_{\mu_n}\), converges to the solution \(u\) of \((B; w, v, u_0)_{\mu}\) in the sense that

\[u_n \to u \text{ in } C([0, T]; V_0^*) \cap L^2(Q) \text{ and weakly in } W^{1,2}(0, T; V_0^*),\]

(3.9)

and

\[\int_0^T \varphi(u_n(t))dt \to \int_0^T \varphi(u(t))dt.\]

(3.10)

**Proof.** We give the proof only in the case \(\mu = 0\) (see Remark 3.1), the others being similar. On account of the uniform estimate \([3.5]\), \(\{u_n\}\) is bounded in \(W^{1,2}(0, T; V_0^*)\) and \(0 \leq u_n \leq u^*\) a.e. on \(Q\). Therefore there is a subsequence of \(\{u_n\}\), that we still denote by \(\{u_n\}\), such that \(u_n \to u\) in \(C([0, T]; V_0^*)\) (as \(n \to \infty\)) for a certain function \(u\) satisfying the estimate \([3.5]\). Now, put \(g_n(t) := f(\rho_{\mu_n} * w_n(t)) u_n(t) - b u_n(t) - v_n(t) \cdot \nabla[\rho_{\mu_n} * (\gamma_{\mu_n} u_n(t))]
\]
and \(g(t) := f(w(t)) u(t) - b u(t) - v(t) \cdot \nabla u(t)\). Since \(\{g_n\}\) is bounded in \(L^2(0, T; V_0^*)\), it follows from Lemma 3.1 that \(\{u_n\}\) is relatively compact in \(L^2(Q)\), and hence converges to \(u\) in \(L^2(Q)\). This shows that \(\gamma_{\mu_n} u_n \to u\) in \(L^2(Q)\) as well as \(\rho_{\mu_n} * (\gamma_{\mu_n} u_n) \to u\) in \(L^2(Q)\). Besides, \(g_n \to g\) weakly in \(L^2(0, T; V_0^*)\), which is seen as follows. Observe that

\[g_n - g = (f(\rho_{\mu_n} * w_n) - f(w)) u_n + f(w) (u_n - u) - b (u_n - u) - v \cdot \nabla[\rho_{\mu_n} * (\gamma_{\mu_n} u_n - u)] - (v_n - v) \cdot \nabla[\rho_{\mu_n} * (\gamma_{\mu_n} u_n)].\]

From the assumption \([3.8]\) with \([2.6]\) and \([2.9]\), it follows that the first four terms at the right hand side converge to 0 in \(C([0, T]; V_0^*)\), and the last one converges weakly to 0 in \(L^2(0, T; V_0^*)\). Therefore, the limit \(u\) is a unique solution of \((B; w, v, u_0)\), and \([3.9]\) and \([3.10]\) hold by Proposition II in Appendix II. \(\square\)

4 Nutrient transport equation and its approximation

Given functions \(u \in C_w([0, T]; H)\) with \(0 \leq u \leq u^*\) a.e. on \(Q\) and \(v \in L^2(0, T; V_\sigma)\), our nutrient transport equation is treated in the form:

\[
(N; u, v, w_0) \begin{cases}
\displaystyle w'(t) + \partial_v \Phi'(u; w(t)) + v(t) \cdot \nabla w(t) = -f(w(t)) u(t) & \text{ in } V^* \text{ for a.e. } t \in [0, T], \\
w(0) = w_0,
\end{cases}
\]

where the initial datum \(w_0\) is prescribed in \(H\), satisfying \(0 \leq w_0 \leq 1\) a.e. on \(\Omega\), \(f(\cdot)\) satisfies \([1.3]\), and \(\Phi'(u; \cdot)\) is a non-negative, continuous and convex function on \(V\) defined by

\[\Phi'(u; w) := \frac{1}{2} \int_\Omega d(u(x, t)) |\nabla w(x)|^2 dx, \quad \forall w \in V,\]
with the function $d(\cdot)$ satisfying (1.2); $\partial_V \Phi^f(u; \cdot)$ is the subdifferential of $\Phi^f(u; \cdot)$ from $V = D(\partial_V \Phi^f(u; \cdot))$ into $V^*$. We see that $\partial_V \Phi^f(u; \cdot)$ is singlevalued, linear and maximal monotone from $V$ into $V^*$, satisfying
\[
\langle \partial_V \Phi^f(u; w), z \rangle = \int_\Omega d(u(x,t)) \nabla w(x) \cdot \nabla z(x) dx, \; \forall w, z \in V, \; \forall t \in [0, T].
\]

**Definition 4.1.** Let $u \in C_w([0, T]; H)$ with $0 \leq u \leq u^*$ a.e. on $Q$ and $v \in L^2(0, T; V_\sigma)$. Then, for any $w_0 \in H$ with $0 \leq w_0 \leq 1$ a.e. on $\Omega$, a function $w : [0, T] \to V$ is called a solution of $(N; u, v, w_0)$, if $w \in L^2(0, T; V) \cap L^\infty(Q), w' \in L^2(0, T; V^*), w(0) = w_0$ and
\[
w'(t) + \partial_V \Phi^f(u; w(t)) + v(t) \cdot \nabla w(t) = -f(w(t))u(t) \text{ in } V^* \text{ for a.e. } t \in [0, T]. \quad (4.1)
\]

**Remark 4.1.** We shall construct a solution $w$ such that $0 \leq w \leq 1$ a.e. on $Q$.

**Remark 4.2.** If $w \in L^\infty(Q)$ or $v \in L^2(0, T; H_\sigma) \cap L^\infty(Q)^3$, we have $\nabla w \cdot v = \text{div}(wv) \in L^2(0, T; V^*)$, cf. Remark 2.2. Indeed, assume $w \in L^\infty(\Omega)$, then, for all $z \in L^2(0, T; V),\int_0^T \langle \text{div}(wv), z \rangle dt = - \int_0^T \int_\Omega wv \cdot \nabla z dx dt \leq |w|_{L^\infty(Q)} |v|_{L^2(0, T; H_\sigma)} |z|_{L^2(0, T; V)}.
\]

The other case is analogous.

**Remark 4.3.** If $v \in L^2(0, T; H_\sigma) \cap L^\infty(Q)^3$, the linear operator $w \to v(t) \cdot \nabla w$ is continuous from $V$ into $V^*$ and maximal monotone. Indeed, by Remark 2.2,
\[
\int_\Omega (v(x, t) \cdot \nabla w(x)) w(x) dx = \frac{1}{2} \int_\Omega \text{div} (v(x, t)w(x)^2) dx = 0 \quad (4.2)
\]

for all $w \in V$ and $t \in [0, T]$. Therefore the sum $w \to \partial_V \Phi^f(u; w) + v(t) \cdot \nabla w$ is linear, continuous, maximal monotone and coercive from $V$ into $V^*$.

We recall the general theory on evolution equations with monotone operators in Banach spaces (cf. [3; Chapter 4]) for the solvability of $(N; u, v, w_0)$. On account of Remark 4.3, this gives the following lemma.

**Lemma 4.1.** Assume that $u \in C_w([0, T]; H)$ with $0 \leq u \leq u^*, v \in L^2(0, T; V_\sigma) \cap L^\infty(Q)$ and (1.2) is satisfied. Then we have:

1. For any $f^* \in L^2(0, T; V^*)$ and $w_0 \in H$ the Cauchy problem
\[
\begin{cases}
w'(t) + \partial_V \Phi^f(u; w(t)) + v(t) \cdot \nabla w(t) = f^*(t) \text{ in } V^* \text{ for a.e. } t \in [0, T], \\
w(0) = w_0,
\end{cases}
\]

admits one and only one solution $w$ such that $w \in L^2(0, T; V)$ and $w' \in L^2(0, T; V^*)$.

2. Let $w_i$ be the solution of (4.3) with $w_0 = w_{i0} \in H$ and $f^* = f_i^* \in L^2(0, T; V^*)$ for $i = 1, 2$. Then, for all $t \in [0, T]:$
\[
\frac{1}{2} |w_1(t) - w_2(t)|_H^2 + c_d \int_0^t |\nabla (w_1 - w_2)|^2 dx dt \leq \frac{1}{2} |w_{10} - w_{20}|_H^2 + \int_0^t \langle f_1^* - f_2^*, w_1 - w_2 \rangle dt, \quad (4.4)
\]

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We prove now the existence-uniqueness result for \((N; u, v, w_0)\).

**Proposition 4.1.** Assume that \(u \in C_w([0, T]; H)\) with \(0 \leq u \leq u^*\ a.e.\ on\ Q\), \([1.2]\), \([1.3]\) are satisfied, \(v \in L^2(0, T; V_\sigma)\), and \(w_0 \in H\) with \(0 \leq w_0 \leq 1\ a.e.\ on\ \Omega\). Then the problem \((N; u, v, w_0)\) admits one and only one solution \(w\). This solution satisfies
\[
0 \leq w \leq 1\ a.e.\ on\ Q,
\]
and
\[
|w(t)|_H^2 + 2c_d \int_0^t |\nabla w|^2 H d\tau \leq e^{2u^*L(f)T} |w_0|_H^2, \quad \forall t \in [0, T].
\]

**Proof.** We prove the proposition in three steps.

(Step 1) Assume first that \(v \in L^\infty(Q)^3\). We are going to construct the solution \(w\) of \((N; u, v, w_0)\) by the contraction mapping principle. Let \(T_1 \in (0, T]\) be a time such that \(2u^*L(f)T_1 < 1\) and, using Lemma 4.1, define a mapping \(\mathcal{N}: C([0, T_1]; H) \to C([0, T_1]; H)\), which assigns to each \(\bar{w} \in C([0, T_1]; H)\) the solution \(w\) of \((4.3)\) on \([0, T_1]\) with \(f^* = f(\bar{w})u\), namely \(w = \mathcal{N}\bar{w}\). Then, for any \(\bar{w}_i \in C([0, T_1]; H)\), \(i = 1, 2\), we observe from \((4.4)\) that
\[
\frac{1}{2} |w_1(t) - w_2(t)|_H^2 + c_d \int_0^t \int_\Omega |\nabla (w_1 - w_2)|^2 dx d\tau
\leq u^* \int_0^t |f(\bar{w}_1) - f(\bar{w}_2)||w_1 - w_2|_H d\tau
\leq u^* L(f) \int_0^t |\bar{w}_1 - \bar{w}_2|_H |w_1 - w_2|_H d\tau,
\]
for all \(t \in [0, T_1]\), so that
\[
|w_1 - w_2|_{C([0, T_1]; H)} \leq 2u^* L(f) T_1 |\bar{w}_1 - \bar{w}_2|_{C([0, T_1]; H)}.
\]
This shows that \(\mathcal{N}\) is strictly contractive in \(C([0, T_1]; H)\) and it has a unique fixed point \(w\) in \(C([0, T_1]; H)\), namely \(w = \mathcal{N}w\), which is a unique solution of \((4.3)\) on the time interval \([0, T_1]\). It is a routine work to construct a unique solution \(w\) of \((N; u, v; w_0)\) on the whole interval \([0, T]\) by a finite number of time-steps.

(Step 2) Still assume that \(v \in L^\infty(Q)^3\), and recall that \(0 \leq w_0 \leq 1\ a.e.\ on\ \Omega\). Then we show that the solution \(w\) of \((4.3)\) constructed in Step 1 satisfies \(0 \leq w \leq 1\ a.e.\ on\ Q\). To do so, multiply \((4.1)\) by \(-w^-\) (= the negative part of \(w\)) and integrate the both sides in time to get by \((1.3)\)
\[
\frac{1}{2} |w^-(t)|_H^2 + c_d \int_0^t \int_\Omega |\nabla w^-|^2 dx d\tau \leq u^* L(f) \int_0^t |w^-(\tau)|_H^2 d\tau, \quad \forall t \in [0, T].
\]
Applying the Gronwall’s lemma to this inequality, we obtain that \(|w^-(t)|_H = 0\) for all \(t \in [0, T]\), namely \(w \geq 0\ a.e.\ on\ Q\). Similarly, by multiplying \((4.1)\) by \((w - 1)^+\) (= the positive part of \(w - 1\)), and integrating the both sides in time, we conclude that \(|(w - 1)^+|_H = 0\), namely \(w \leq 1\ a.e.\ on\ Q\). Thus \(0 \leq w \leq 1\ a.e.\ on\ Q\, and \(w\) is the solution of \((N; u, v, w_0)\) in the sense of Definition 4.1.

(Step 3) For general \(v\), we approximate \(v \in L^2(0, T; V_\sigma)\) by a sequence \(\{v_n\}\) from \(L^2(0, T; V_\sigma) \cap L^\infty(Q)^3\) such that \(v_n \to v\) in \(L^2(0, T; V_\sigma)\) (as \(n \to \infty\)). By virtue of Steps 1 and 2, for each \(n\), the problem
\[
w''_n(t) + \partial_{\nu^*} \Phi^*(u; w_n(t)) + v_n(t) \cdot \nabla w_n(t) = -f(w_n)u(t)\quad \text{in}\ V^*,
\]
\[
w_n(0) = w_0,
\]
and
\[
w_n(t) \to w(t)\quad \text{as}\ n \to \infty
\]
has a unique solution $w_n$ such that $w_n \in L^2(0,T;V)$, $w_n' \in L^2(0,T;V^*)$ and $0 \leq w_n \leq 1$ a.e. on $Q$. Multiplying (4.6) by $w_n$, we obtain by (4.2) that

$$\frac{1}{2}|w_n(t)|_H^2 + c_d \int_0^t |\nabla w_n(\tau)|_H^2 d\tau \leq \frac{1}{2}|w_0|_H^2 + u^* L(f) \int_0^t |w_n|_H^2 d\tau, \quad \forall t \in [0,T],$$

(4.7)

which implies, with the Gronwall inequality, that $\{w_n\}$ is bounded in $L^2(0,T;V)$. We also infer from “0 $\leq w_n \leq 1$” and Remark 4.2 that $v_n \cdot \nabla w_n = \text{div}(w_n v_n)$ is bounded in $L^2(0,T;V^*)$. Consequently, by (4.6), $\{w_n\}$ is bounded in $L^2(0,T;V^*)$. Therefore there exist a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and a function $w \in L^2(0,T;V)$ with $0 \leq w \leq 1$ a.e. on $Q$, such that $w_{n_k} \to w$ weakly in $L^2(0,T;V)$. Furthermore, on account of the Aubin's compactness theorem [2], we have $w_{n_k} \to w$ in $L^2(Q)$. Now it is easy to see, by letting $k \to \infty$ in (4.6) with $n = n_k$, that the limit $w$ satisfies (4.1) and the same type of energy inequality as (4.7) holds for $w$. We easily get the estimate (4.5) from it. Uniqueness of solution and (4.5) are obtained by the Gronwall inequality. \□

**Proposition 4.2.** Assume that (1.2) and (1.3) hold, $w_0 \in H$ with $0 \leq w_0 \leq 1$ a.e. on $\Omega$. Let $\{v_n\}$ and $\{u_n\}$ be sequences such that $0 \leq u_n \leq u^*$ a.e. on $Q$ for all $n$, and

$$v_n \to v \text{ weakly in } L^2(0,T;V_\sigma), \quad u_n \to u \text{ in } C([0,T];V^*_0) \cap L^2(Q).$$

(4.8)

Then, the solution $w_n$ of $(N;u_n,v_n,w_0)$ converges to the solution $w$ of $(N;u,v,w_0)$ in the sense that

$$w_n \to w \text{ in } L^2(Q) \text{ and weakly in } L^2(0,T;V), \quad w_n' \to w' \text{ weakly in } L^2(0,T;V^*).$$

(4.9)

**Proof.** From the uniform estimate (4.5) we observe that $\{w_n\}$ is bounded in $L^2(0,T;V)$ with $0 \leq w_n \leq 1$ a.e. on $Q$, so that $w_n' = -\partial_{\nu^*} \Phi^t(u_n;w_n) - \text{div}(v_n w_n) - f(w_n)u_n$ is bounded in $L^2(0,T;V^*)$. It follows from the Aubin’s compactness theorem [2] that $\{w_n\}$ is relatively compact in $L^2(Q)$. Therefore, there are a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and a function $\bar{w}$ so that $w_{n_k} \to \bar{w}$ in $L^2(Q)$ and weakly in $L^2(0,T;V)$ as well as $w_{n_k}' \to \bar{w}'$ weakly in $L^2(0,T;V^*)$. By these convergences and (4.8) we see that $\partial_{\nu^*} \Phi^t(u_{n_k};w_{n_k}) \to \partial_{\nu^*} \Phi^t(u;\bar{w})$ weakly in $L^2(0,T;V^*)$ and $-\text{div}(w_{n_k} v_{n_k}) - f(w_{n_k})u_{n_k} \to -\text{div}(\bar{w} v) - f(\bar{w})u$ weakly in $L^2(0,T;V^*)$ (as $k \to \infty$). Hence, by Remark 4.2, the limit $\bar{w}$ is a solution of $(N;u,v,w_0)$. By uniqueness we have $\bar{w} = w$, which implies that convergences (4.9) hold without extracting any subsequence from $\{w_n\}$. \□

As a regular approximation for $(N;u,v,w_0)$, we employ problem $(N;\rho_\mu \ast u,v,w_0)$, which is denoted by $(N;u,v,w_0)_\mu$ for any small parameter $\mu \in (0,1)$, namely

$$(N;u,v,w_0)_\mu \begin{cases} w'(t) + \partial_{\nu^*} \Phi^t(\rho_\mu \ast u; w(t)) + v \cdot \nabla w(t) = -f(w(t))\rho_\mu \ast u & \text{in } V^* \text{ for a.e. } t \in [0,T], \\ w(0) = w_0. \end{cases}$$

It is clear that Propositions 4.1, 4.2 are valid for this approximate problem by replacing $u$ by $\rho_\mu \ast u$.

5 Variational inequality of the Navier-Stokes type and its approximation

As was mentioned in the introduction, the biomass formation mechanism, together with the nutrient transport and consumption takes place in a fluid. At the same time, the
forming biomass becomes an obstacle for the flow. We model it by making use of a variational inequality of Navier-Stokes type.

Let \( p_0 : (0, u^*) \rightarrow \mathbb{R} \) be the same function as in (i) in the introduction, satisfying (1.4) (see Fig.1), and let \( u \) be a given function in \( C_w([0, T]; H) \) with \( 0 \leq u \leq u^* \) a.e. on \( Q \). Then, with the function \( u^\varepsilon := \rho_\varepsilon * u \) for a fixed small positive parameter \( \varepsilon \in (0, 1) \), the strong formulation of our variational inequality of the Navier-Stokes type is of the following form:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
|v(x, t)| \leq p_0(u^\varepsilon(x, t)) \text{ for } (x, t) \in Q; \\
\langle \dot{v}(t), v(t) - z \rangle_\sigma + \nu \int_\Omega \nabla v(x, t) \cdot \nabla (v(x, t) - z(x)) dx \\
+ \int_\Omega (v(x, t) \cdot \nabla) v(x, t) \cdot (v(x, t) - z(x)) dx \leq (g(t), v(t) - z)_\sigma \\
\forall z \in V_\sigma \text{ with } |z(x)| \leq p_0(u^\varepsilon(x, t)) \text{ for } x \in \Omega, \ t \in [0, T], \\
v(0) = v_0(x) \text{ for } x \in \Omega,
\end{array} \right.
\end{aligned}
\]

where \( \nu \) is positive constant (viscosity), \( v_0 \) a prescribed initial datum for \( v \) and \( g \in L^2(0, T; H_\sigma) \) a prescribed external force.

The existence-uniqueness of a strong solution to \((H; u, v_0, g)^\varepsilon\) is of course an open question just as the usual 3D Navier-Stokes equations. Therefore, we shall construct a weak solution of \((H; u, v_0, g)^\varepsilon\) in the variational sense.

**Definition 5.1.** Let \( u \in C_w([0, T]; H) \) with \( 0 \leq u \leq u^* \) a.e. on \( Q \), \( u^\varepsilon := \rho_\varepsilon * u \) and \( K(u^\varepsilon) \) be the class of test functions defined by

\[
K(u^\varepsilon) := \left\{ \eta \in C^1([0, T]; W^{1,4}_{0,\sigma}(\Omega)) \mid \text{supp}(\eta) \subset \hat{Q}(u^\varepsilon < \delta_0), \quad |\eta| \leq p_0(u^\varepsilon) \text{ on } Q \right\}
\]

where \( W^{1,4}_{0,\sigma}(\Omega) \) is the closure of \( D_\sigma(\Omega) \) in \( W^{1,4}_0(\Omega)^3 \) and \( \hat{Q}(u^\varepsilon < \delta_0) := \{(x, t) \in \Omega \times [0, T] \mid u^\varepsilon(x, t) < \delta_0\} \). Then, for a given initial datum \( v_0 \) and \( g \in L^2(0, T; H_\sigma) \), a function \( v : [0, T] \rightarrow H_\sigma \) is called a weak solution of \((H; u, v_0, g)^\varepsilon\), if the following conditions hold:

1. \( v \in L^2(0, T; V_\sigma) \) and \( \sup_{t \in [0, T]} |v(t)|_{H_\sigma} < \infty \);
2. the function \( t \rightarrow (v(t), \eta(t))_\sigma \) is of bounded variation on \([0, T]\) for any \( \eta \in K(u^\varepsilon) \);
3. \( v \) satisfies:

\[
\int_0^t (\dot{\eta}(\tau), v(\tau) - \eta(\tau))_\sigma d\tau + \nu \int_0^t \int_\Omega \nabla v(x, \tau) \cdot \nabla (v(x, \tau) - \eta(x, \tau)) dx d\tau
\]

\[
+ \int_\Omega \int_0^t (v(x, \tau) \cdot \nabla) v(x, \tau) \cdot (v(x, \tau) - \eta(x, \tau)) dx d\tau + \frac{1}{2} |\eta(t) - v(t)|_{H_\sigma}^2
\]

\[
\leq \int_0^t (g(\tau), v(\tau) - \eta(\tau))_\sigma d\tau + \frac{1}{2} |\eta(0) - v_0|_{H_\sigma}^2,
\]

\( \forall t \in [0, T], \forall \eta \in K(u^\varepsilon) \).
In the rest of this section, we propose an approximate problem \((H; u, v_0, g)^\varepsilon\) for \((H; u, v_0, g)^\varepsilon\). We begin with the approximation \(p_\mu(r)\) of \(p_0(r)\) with a small positive parameter \(\mu\) (actually \(\mu \in (0, \delta_0) \cap (0, 1)\)) with \(\mu < p_0(\mu)\). See Fig.2:

\[
p_\mu(r) := \begin{cases} 
p_0(\mu), & \text{for } r \in [0, \mu], 
p_0(r), & \text{for } r \in (\mu, p_0^{-1}(\mu)], 
mu, & \text{for } r \in (p_0^{-1}(\mu), u^*].
\end{cases}
\]  

(5.2)

Next, we approximate the obstacle function \(p_0(u^\varepsilon)\) by \(p_\mu((\gamma_\mu u)^\varepsilon)\) with \((\gamma_\mu u)^\varepsilon := \rho_\varepsilon*(\gamma_\mu u)\), where \(\gamma_\mu\) is given by (3.3). We put finally

\[
K_\mu((\gamma_\mu u)^\varepsilon; t) := \{z \in V_\sigma \mid |z(x)| \leq p_\mu((\gamma_\mu u)^\varepsilon(x,t)) \text{ a.e. } x \in \Omega\}, \ \forall \ t \in [0, T].
\]

Now, consider the following approximate problem \((H; u, v_0, g)^\mu\) for \((H; u, v_0, g)^\varepsilon\):

\[
(H; u, v_0, g)^\varepsilon := \begin{cases} 
v(t) \in K_\mu((\gamma_\mu u)^\varepsilon; t), & \forall t \in [0, T]; 
\langle v'(t), v(t) - z \rangle_\sigma + \nu \int_{\Omega} \nabla v(t) \cdot \nabla (v(t) - z) dx 
+ \langle G(v(t), v(t)), v(t) - z \rangle_\sigma 
\leq \langle g(t), v(t) - z \rangle_\sigma 
& \forall z \in K_\mu((\gamma_\mu u)^\varepsilon; t), \ \text{a.e. } t \in [0, T],
 v(0) = v_0;
\end{cases}
\]

where \(G\) is a nonlinear operator from \(V_\sigma \times V_\sigma \to V_\sigma^*\) given by

\[
\langle G(v, w), z \rangle_\sigma := \sum_{k,j=1}^3 \int_{\Omega} v^{(k)} \frac{\partial w^{(j)}}{\partial x_k} z^{(j)} dx
\]

for all \(v := (v^{(1)}, v^{(2)}, v^{(3)}), w := (w^{(1)}, w^{(2)}, w^{(3)})\) and \(z := (z^{(1)}, z^{(2)}, z^{(3)})\) in \(V_\sigma \cap L^\infty(\Omega)^3\).

**Remark 5.1** (a) By divergence freeness of \(v \in V_\sigma\), we have \(\langle G(v, v), v \rangle_\sigma = 0\).

(b) Also, \(G(v, v) \in H_\sigma\) for \(v \in K_\mu((\gamma_\mu u)^\varepsilon; t)\).

In order to describe the above variational inequality as an evolution inclusion of the subdifferential type we introduce time-dependent convex functions, \(\psi_\mu^t((\gamma_\mu u)^\varepsilon; \cdot)\), on \(H_\sigma\), of the following form:

\[
\psi_\mu^t((\gamma_\mu u)^\varepsilon; z) := \begin{cases} 
\nu \frac{|z|^2}{2} \langle v, z \rangle_\sigma, & \text{if } z \in K_\mu((\gamma_\mu u)^\varepsilon; t),\\ 
\infty, & \text{otherwise}
\end{cases}
\]

Figure 2: Approximating the obstacle function \(p_0\) by \(p_\mu\)
and denote by \( \partial \psi^\mu_\mu((\gamma_\mu u)_\mu^\varepsilon; \cdot) = \partial H_\mu \psi^\mu_\mu((\gamma_\mu u)_\mu^\varepsilon; \cdot) \) their subdifferential in \( H_\sigma \). We see that \( v^* \in \partial \psi^\mu_\mu((\gamma_\mu u)_\mu^\varepsilon; v) \) if and only if \( v \in K_\mu((\gamma_\mu u)_\mu^\varepsilon; t) \), \( v^* \in H_\sigma \) and

\[
(v^*, z - v)_\sigma \leq \nu \int_\Omega \nabla v(x) \cdot \nabla (z(x) - v(x)) \, dx, \quad \forall z \in K_\mu((\gamma_\mu u)_\mu^\varepsilon; t).
\] (5.3)

Now, we take \( v_0 \in K_\mu((\gamma_\mu u)_\mu^\varepsilon; 0) \) and consider the evolution inclusion:

\[
\begin{aligned}
&v'(t) + \partial \psi^\mu_\mu((\gamma_\mu u)_\mu^\varepsilon; v(t)) + G(v(t), v(t)) \ni g(t) \quad \text{in } H_\sigma \quad \text{for a.e. } t \in [0, T], \\
&v(0) = v_0,
\end{aligned}
\]

(5.4)

By Remark 5.1(b), (5.4) makes sense as an inclusion in \( H_\sigma \). If \( v \in W^{1,2}(0, T; H_\sigma) \), then (5.4) is equivalent to \((H; u, v_0, g)^\varepsilon_\mu\) by (5.3). A function \( v : [0, T] \to H_\sigma \) is called a (strong) solution to \((H; u, v_0, g)^\varepsilon_\mu\), if \( v \in W^{1,2}(0, T; H_\sigma) \cap C([0, T]; V_\sigma) \) and (5.4) holds.

**Proposition 5.1.** Let \( \mu \) be any small positive number, and let \( u \) be any function in \( W^{1,2}(0, T; V_\sigma^0) \) with \( 0 \leq u \leq u^* \) a.e. on \( Q \) (hence \( u \in C_w([0, T]; H) \)), let \( g \in L^2(0, T; H_\sigma) \). Also, let \( v_0 \) be any function in \( K_\mu((\gamma_\mu u)_\mu^\varepsilon; 0) \). Then \((H; u, v_0, g)^\varepsilon_\mu\) has one and only one solution \( v \), satisfying

\[
|v(t)|^2_{H_\sigma} + \nu \int_0^t |v(\tau)|^2_{V_\sigma} \, d\tau \leq |v_0|^2_{H_\sigma} + \frac{L^2}{\nu} \int_0^t |g(\tau)|^2_{H_\sigma} \, d\tau,
\]

(5.5)

where \( L_p \) is the Poincaré constant, i.e. \( |z|_{H_\sigma} \leq L_p |z|_{V_\sigma} \) for all \( z \in V_\sigma \). Moreover, there is a non-negative, bounded and non-decreasing function \( R_\mu(\cdot) \) on \([0, \infty) \times [0, \infty)\), depending only on \( \mu > 0 \), such that

\[
|v|^2_{W^{1,2}(0, T; H_\sigma)} + \nu \sup_{\frac{2}{2} \in [0, T]} |v(t)|^2_{V_\sigma} \leq R_\mu(|v_0|_{H_\sigma}, |g|_{L^2(0, T; H_\sigma)}).
\]

(5.6)

For the solvability of \((H; u, v_0, g)^\varepsilon_\mu\) we apply the general theory from Appendix III. To this end, we recall the following lemma, which is derived from the assumption \( u \in W^{1,2}(0, T; V_\sigma^0) \) (hence \( (\gamma_\mu u)_\mu^\varepsilon \in W^{1,2}(0, T; C(\Omega)) \) by (2.8)).

**Lemma 5.1 (cf. [11, Lemma 4.3] or [12, Lemma 2.2]).** There exists a positive constant \( C_\mu \), depending only on \( \mu \), which satisfies the following property: for each \( s, t \in [0, T] \) and \( z \in K_\mu((\gamma_\mu u)_\mu^\varepsilon; s) \) there is \( \tilde{z} \in K_\mu((\gamma_\mu u)_\mu^\varepsilon; t) \) such that

\[
|\tilde{z} - z|_{H_\sigma} \leq C_\mu |(\gamma_\mu u)_\mu^\varepsilon(t) - (\gamma_\mu u)_\mu^\varepsilon(s)|_{C(\Omega)}, \quad \psi^\mu_\mu((\gamma_\mu u)_\mu^\varepsilon); \tilde{z}) \leq \psi^\mu_\mu((\gamma_\mu u)_\mu^\varepsilon; z).
\]

Lemma 5.1 shows that problem (5.4) can be handled in the general framework of Appendix with the set-up:

\[
X := H_\sigma, \quad \{ \varphi^\mu(\cdot) := \{ \psi^\mu_\mu((\gamma_\mu u)_\mu^\varepsilon; \cdot) \} \in \Phi_c(M) \text{ with } M \geq |\varphi^\mu|_{W^{1,2}(0, T)}
\]

where

\[
a(t) := C_\mu \int_0^t \left. \frac{d}{d\tau} (\gamma_\mu u)_\mu^\varepsilon(\tau) \right|_{C(\Omega)} \, d\tau, \quad b(\cdot) \equiv 0, \quad h := \mathcal{G}.
\]
Proof of Proposition 5.1. We observe that, cf. Remark 5.1,

$$|G(v, w)|_{H^s} \leq p_0(\mu) |w|_{V^s}, \quad \forall v \in K_{\mu}((\gamma_{\mu}u)^{\varepsilon}; t), \forall w \in V_{\sigma}$$

and

$$|G(z_1, z_1) - G(z_2, z_2)|, z_1 - z_2)| \leq 9p_0(\mu)|z_1 - z_2|_{H^s}|z_1 - z_2|_{V_{\sigma}},$$

for all $z_i \in K_{\mu}((\gamma_{\mu}u)^{\varepsilon}; t), \ i = 1, 2$. This shows that the perturbation operator

$$h(t, z) := G(z, z), \quad \forall z \in K_{\mu}((\gamma_{\mu}u)^{\varepsilon}; t), \forall t \in [0, T],$$

fulfills condition ($h_4$) in Appendix III. Also, it is easy to see that this operator fulfills the other conditions ($h_1$)–($h_3$). Therefore, on account of Proposition III(1) in Appendix, the problem (5.4), namely ($H; u, v_0, g)^{\varepsilon}_{\mu}$, has one and only one solution $v \in W^{1,2}(0, T; H_{\sigma})$ such that $\tau \rightarrow \psi_{\mu}^{t}((\gamma_{\mu}u)^{\varepsilon}; v(t))$ is absolutely continuous on $[0, T]$. This implies that $v \in C([0, T]; V_{\sigma})$. By Proposition III(2), we obtain an estimate of the form (5.6).

Finally, we prove (5.5). Multiply the inclusion in (5.4) by $\nu$ and integrate in time over $[0, t]$ to get

$$\frac{1}{2} |v(t)|_{H^s}^2 + \int_0^t |v(\tau)|_{V^s}^2 d\tau + \int_0^t \langle G(v(\tau), v(\tau)), v(\tau) \rangle_{\sigma} d\tau \leq \frac{1}{2} |v_0|_{H_{\sigma}}^2 + \int_0^t (g(\tau), v(\tau))_{\sigma} d\tau.$$ (5.5)

By Remark 5.1(a), we immediately obtain (5.5) from the above inequality. \qed

Proposition 5.2. Let $\mu$ be any small positive number, and let $u_0 \in H$ with $0 \leq u_0 \leq u^*$ a.e. on $\Omega$. Let $\{u_n\}$ be a bounded sequence in $W^{1,2}(0, T; V^s_{\sigma})$ with $0 \leq u_n \leq u^*$ a.e. on $Q$ such that $u_n(0) = u_0$ and $u_n \rightharpoonup u$ in $C([0, T]; V^s_{\sigma})$. Then, for any $v_0 \in K_{\mu}((\gamma_{\mu}u)^{\varepsilon}; 0)$ and any $g \in L^2(0, T; H_{\sigma})$, the solution $v_n$ of ($H; u_n, v_0, g)^{\varepsilon}_{\mu}$ converges to the solution $v$ of ($H; u, v_0, g)^{\varepsilon}_{\mu}$ in the sense that

$v_n \rightharpoonup v$ in $C([0, T]; H_{\sigma}) \cap L^2(0, T; V_{\sigma})$ and weakly in $W^{1,2}(0, T; H_{\sigma}).$

Proof. Let us recall (cf. (2.8)) that $(\gamma_{\mu}u_n)^{\varepsilon} \rightarrow (\gamma_{\mu}u)^{\varepsilon}$ uniformly on $\overline{Q}$ (as $n \rightarrow \infty$). We show first that, for every $t \in [0, T]$, $\psi_{\mu}^{t}((\gamma_{\mu}u_n)^{\varepsilon}; \cdot) \rightarrow \psi_{\mu}^{t}((\gamma_{\mu}u)^{\varepsilon}; \cdot)$ on $H_{\sigma}$ in the sense of Mosco, as $n \rightarrow \infty$ (cf. Appendix II). To this end, assume that $\{z_n\}$ is any sequence in $H_{\sigma}$ with $\liminf_{n \rightarrow \infty} \psi_{\mu}^{t}((\gamma_{\mu}u_n)^{\varepsilon}; z_n) < \infty$ and $z_n \rightharpoonup z$ weakly in $H_{\sigma}$. It is enough to consider the case $z_n \in K_{\mu}((\gamma_{\mu}u_n)^{\varepsilon}; t)$ and $\{z_n\}$ is bounded in $V_{\sigma}$. In this case, $\|z_n(x)\| \leq p_\mu((\gamma_{\mu}u_n)^{\varepsilon}(x, t))$ for a.e. $x \in \Omega$ and $z_n \rightharpoonup z$ in $H_{\sigma}$ by the boundedness of $\{z_n\}$ in $V_{\sigma}$. This strong convergence yields $|z(x)| \leq p_\mu((\gamma_{\mu}u)^{\varepsilon}(x, t))$ for a.e. $x \in \Omega$, so that $z \in K_{\mu}((\gamma_{\mu}u)^{\varepsilon}; t)$, namely $\psi_{\mu}^{t}((\gamma_{\mu}u)^{\varepsilon}; z) < \infty$. As a consequence we have, as $z_n \rightharpoonup z$ weakly in $V_{\sigma}$, that

$$\liminf_{n \rightarrow \infty} \psi_{\mu}^{t}((\gamma_{\mu}u_n)^{\varepsilon}; z_n) \geq \psi_{\mu}^{t}((\gamma_{\mu}u)^{\varepsilon}; z).$$

Next, let $z$ be any function in $K_{\mu}((\gamma_{\mu}u)^{\varepsilon}; t)$. Then we construct the function $z_n$ by:

$$z_n(x) = \left(1 - \frac{1}{\mu} p_\mu((\gamma_{\mu}u_n)^{\varepsilon}(t)) - p_\mu((\gamma_{\mu}u)^{\varepsilon}(t))\right) z(x), \quad x \in \Omega.$$ (5.7)

Since $(\gamma_{\mu}u_n)^{\varepsilon} \rightarrow (\gamma_{\mu}u)^{\varepsilon}$ uniformly on $\overline{Q}$ as $n \rightarrow \infty$ and $p_\mu((\gamma_{\mu}u_n)^{\varepsilon}) \geq 1$ by (5.2), it follows (cf. [12, Lemma 2.2]) that $z_n \in K_{\mu}((\gamma_{\mu}u_n)^{\varepsilon}; t)$ for all large $n$ and $z_n \rightharpoonup z$ in $V_{\sigma}$ (hence
\[ \psi^{\mu}_n((\gamma_{\mu}u_n)^{\varepsilon}; z_n) \rightarrow \psi^{\mu}_n((\gamma_{\mu}u)^{\varepsilon}; z_n). \] Accordingly, \[ \psi^{\mu}_n((\gamma_{\mu}u_n)^{\varepsilon}; \cdot) \rightarrow \psi^{\mu}_n((\gamma_{\mu}u)^{\varepsilon}; \cdot) \] on \( H_{\sigma} \) in the sense of Mosco.

We are now in a position to apply Proposition II to the sequence of problems

\[ \nu'_n(t) + \partial \psi^{\mu}_n((\gamma_{\mu}u_n)^{\varepsilon}; \nu_n(t)) + G(\nu_n(t), \nu_n(t)) \geq g(t) \text{ in } H_{\sigma}, \quad \nu_n(0) = \nu_0. \tag{5.8} \]

We note that all the families \( \{\psi^{\mu}_n((\gamma_{\mu}u_n)^{\varepsilon}; \cdot)\} \), \( n = 1, 2, \ldots \), belong to the same class \( \Phi_c(M) \) for a large number \( M \), since, by assumption and \( (2.8) \), \( \{(\gamma_{\mu}u_n)^{\varepsilon}\} \) is uniformly bounded in \( W^{1,2}(0, T; C(\Omega)) \). Therefore, by virtue of Proposition 5.1, problem \( (5.8) \) has one and only one solution \( \nu_n \), and the uniform estimates \( (5.5) \) and \( (5.6) \) hold for each \( \nu_n \).

Hence, there is a subsequence \( \{\nu_{n_k}\} \) of \( \{\nu_n\} \) such that

\[ \nu_{n_k} \rightarrow \nu \text{ weakly in } W^{1,2}(0, T; H_{\sigma}) \] and weakly* in \( L^\infty(0, T; V_{\sigma}) \)

(5.9)

(as \( k \rightarrow \infty \)), which implies that

\[ \nu_{n_k} \rightarrow \nu \text{ in } C([0, T]; H_{\sigma}) \text{ and } G(\nu_{n_k}, \nu_{n_k}) \rightarrow G(\nu, \nu) \text{ weakly in } L^2(0, T; H_{\sigma}). \tag{5.10} \]

Therefore, by Proposition II, \( \nu \) solves \( (5.4) \). Furthermore, by uniqueness of solution to \( (5.4) \), we obtain \( (5.9) \) without extracting any subsequence from \( \{\nu_n\} \).

It remains to show the convergence \( \nu_n \rightarrow \nu \) in \( L^2(0, T; V_{\sigma}) \). We consider the function \( \tilde{\nu}_n \) given by

\[ \tilde{\nu}_n(x, t) := \left(1 - \frac{1}{\mu}\right先锋]_{C(\Omega)}(\nu(x, t), (x, t) \in Q. \]

Just as for \( (5.7) \) above, we observe from [12, Lemma 2.2] again that

\[ \tilde{\nu}_n(t) \in K_{\mu}((\gamma_{\mu}u_n)^{\varepsilon}; t) \text{ for all large } n \quad \text{and} \quad \tilde{\nu}_n \rightarrow \nu \text{ in } L^2(0, T; V_{\sigma}). \tag{5.11} \]

Since \( g - \nu'_n - G(\nu_n, \nu_n) \in \partial \psi^{\mu}_n((\gamma_{\mu}u_n)^{\varepsilon}; \nu_n) \), it follows from \( (5.3) \) that

\[ \int_0^T \left( \nu'_n(t) + G(\nu_n(t), \nu_n(t)) - g(t), \nu_n(t) - \tilde{\nu}_n(t) \right)_{\sigma} dt \]

\[ \leq \nu \int_0^T \int_{\Omega} \nabla \nu_n(x, t) \cdot \nabla (\tilde{\nu}_n(x, t) - \nu_n(x, t)) dx dt. \tag{5.12} \]

Here, the left hand side of \( (5.12) \) tends to 0 as \( n \rightarrow \infty \), since, from \( (5.10) \) and \( (5.11) \),

\[ \liminf_{n \rightarrow \infty} \int_0^T \left( \nu'_n(t) + G(\nu_n(t), \nu_n(t)) - g(t), \nu_n(t) - \tilde{\nu}_n(t) \right)_{\sigma} dt \]

\[ \geq \frac{1}{2} |\nu(T)|_{H_{\sigma}}^2 - \frac{1}{2} |\nu_0|_{H_{\sigma}}^2 - \int_0^T (\nu'(t), \nu(t))_{\sigma} dt = 0. \]

Therefore

\[ \limsup_{n \rightarrow \infty} \int_0^T \int_{\Omega} |\nabla \nu_n|^2 dx dt \leq \lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} \nabla \nu_n \cdot \nabla \tilde{\nu}_n dx dt = \int_0^T \int_{\Omega} |\nabla \nu|^2 dx dt. \]

This implies \( \nu_n \rightarrow \nu \) in \( L^2(0, T; V_{\sigma}) \). \( \square \)
6 Approximate full system and its convergence

Let $\varepsilon$ be a small positive parameter and fix it. For each small $\mu > 0$, consider the coupling $P^\varepsilon_\mu := \{(B; w, v, u_0)_\mu, (N; u, v, w_0)_\mu, (H; u, v_0, g)^\varepsilon_\mu\}$ as the approximation to our problem $P^\varepsilon := \{(B; w, v, u_0), (N; u, v, w_0), (H; u, v_0, g)^\varepsilon\}$.

More precisely, a triplet $\{u_\mu, w_\mu, v_\mu\}$ is called a solution of $P^\varepsilon_\mu$, if

(a) $u_\mu \in W^{1,2}(0, T; V^*_0)$, $t \to |\hat{\beta}(u_\mu(t))|_{L^1(\Omega)}$ is absolutely continuous on $[0, T]$, and $u_\mu$ is the solution of $(B; w_\mu, v_\mu, u_0)_\mu$;

(b) $w_\mu \in L^2(0, T; V)$, $w'_\mu \in L^2(0, T; V^*)$, $0 \leq w_\mu \leq 1$ a.e. on $Q$ and $w_\mu$ is the solution of $(N; u_\mu, v_\mu, w_0)_\mu$;

(c) $v_\mu \in W^{1,2}(0, T; H_\sigma) \cap C([0, T]; V_\sigma)$ and $v_\mu$ is the solution of $(H; u_\mu, v_\mu, g)^\varepsilon_\mu$.

**Theorem 6.1.** Let $\mu \in (0, \delta_0) \cap (0, 1)$ with $\mu < p_0(\mu)$. Assume that $u_0 \in H$ is such that $\hat{\beta}(u_0) \in L^1(\Omega)$, $w_0 \in H$ with $0 \leq w_0 \leq 1$ a.e. on $\Omega$ and $v_0 \in V_\sigma \cap C(\overline{\Omega})^3$ with $|v_0| < p_0(\varepsilon_0)$ on $\overline{\Omega}$, where $u_0^2(x) = \int_0^1 \rho(x - y)u_0(y)dy$ for all $x \in \Omega$. Let $g \in L^2(0, T; H_\sigma)$. Then, for all small positive number $\mu$ the approximate system $P^\varepsilon_\mu$ has at least one solution $\{u_\mu, w_\mu, v_\mu\}$.

**Proof.** We put

$$
X(u_0) := \left\{ u \left| \begin{array}{c}
|u|^2_{W^{1,2}(0, T; V^*_0)} + \sup_{t \in [0, T]} |\hat{\beta}(u(t))|_{L^1(\Omega)} \\
\leq B_0 \left( T^2 \|v_0\|_{H_\sigma}, |\hat{\beta}(u_0)|_{L^1(\Omega)} \right), \\
u = u_0
\end{array} \right\},
$$

where $B_0(\cdot)$ is the same function as in (3.5) of Proposition 3.1. Note that $X(u_0)$ is non-empty, compact and convex in $C([0, T]; V^*_0)$. By assumption, for each $u \in X(u_0)$ we see that $|v_0| \leq p_\mu((\gamma_\mu u)^\varepsilon(-,0))$ on $\Omega$ for all small $\mu > 0$, since $(\gamma_\mu u)^\varepsilon \to u^\varepsilon$ in $C(\overline{Q})$ by $\gamma_\mu v \to u_0$ in $H$ as $\mu \downarrow 0$. This implies that $v_0 \in K_\mu((\gamma_\mu u)^\varepsilon; 0)$ for all small $\mu > 0$, so that $(H; u, v_\mu, g)^\varepsilon_\mu$ is uniquely solved. Now, denote the solution by $S_1u =: v$. Then, according to Proposition 5.1, $v \in W^{1,2}(0, T; H_\sigma) \cap C([0, T]; V_\sigma)$ and there is a positive constant $R_\mu(|v_0|_{V_\sigma}, |g|_{L^2(0, T; H_\sigma)}) =: R_\mu$, depending on the parameter $\mu$, $|v_0|_{V_\sigma}$ and $|g|_{L^2(0, T; H_\sigma)}$, such that (cf. (5.6))

$$
|v|^2_{W^{1,2}(0, T; H_\sigma)} + \frac{\nu}{2} \sup_{t \in [0, T]} |v(t)|^2_{V_\sigma} \leq R_\mu.
$$

Put

$$
Y(v_0) := \left\{ v \left| \begin{array}{c}
|v|^2_{W^{1,2}(0, T; H_\sigma)} + \frac{\nu}{2} \sup_{t \in [0, T]} |v(t)|^2_{V_\sigma} \leq R_\mu, \\
v(0) = v_0
\end{array} \right\}.
$$

Then, $S_1$ is a mapping from $X(u_0)$ into $Y(v_0)$.

Next, for each pair of $u \in X(u_0)$ and $v$ of $Y(v_0)$ we solve $(N; u, v, w_0)_\mu$ and denote its solution by $S_2(u, v) =: w$. By Proposition 4.1, estimate (4.5) holds for $w$. Put

$$
Z(w_0) := \{ w \in L^2(0, T; V) \cap L^\infty(0, T; H) \ | \ (4.5) \text{ holds}, \ w(0) = w_0 \}.
$$

Then $S_2$ is a mapping from $X(u_0) \times Y(v_0)$ into $Z(w_0)$.
Furthermore, for each \( w \in Z(w_0) \) and \( v \in Y(v_0) \) we solve \( (B; w, v, u_0)_\mu \) and denote the solution by \( S_3(w, v) := \bar{u} \). It follows from Proposition 3.1 with (3.5) that \( \bar{u} \in X(u_0) \), so that \( S_3 \) can be considered as a mapping from \( Z(w_0) \times Y(v_0) \) into \( X(u_0) \). Finally we define a mapping \( S \) from \( X(u_0) \) into itself by

\[
S u := S_3(S_2(u, S_1 u), S_1(u)), \quad \forall u \in X(u_0).
\]

In order to apply the fixed-point theorem for compact mappings we show continuity of \( S \). Assume that \( u_n \in X(u_0) \) and \( u_n \to u \) in \( C([0, T]; V_0^*) \). Then, by the definition of \( X(u_0) \), \( u_n \to u \) weakly in \( W^{1,2}(0, T; V_0^*) \). Moreover, by lower semicontinuity of \( \bar{\beta} \cdot \nabla \) \( u \in X(u_0) \). Therefore, \((\gamma \mu u_n)^\varepsilon := \rho \varepsilon \ast (\gamma \mu u_n) \in W^{1,2}(0, T; C(\Omega)) \) by (2.8) and

\[
|((\gamma \mu u_n)^\varepsilon|_{W^{1,2}(0,T; C(\Omega))} \leq R_{\mu}', \quad |(\gamma \mu u)^\varepsilon|_{W^{1,2}(0,T; C(\Omega))} \leq R_{\mu}',
\]

for some positive constant \( R_{\mu}' \) and \((\gamma \mu u_n)^\varepsilon \to (\gamma \mu u)^\varepsilon \) uniformly on \( \bar{Q} \) (as \( n \to \infty \)). As we have seen in Propositions 5.1 and 5.2, the problem \((H; u_n, v_0, g)_\mu \) has a unique solution \( v_n \) in \( W^{1,2}(0, T; H_\sigma) \cap C([0, T]; V_\sigma) \). Moreover, \( v_n \) converges in \( C([0, T]; H_\sigma) \cap L^2(0, T; V_\sigma) \) and weakly in \( W^{1,2}(0, T; H_\sigma) \) to the solution \( v \) of \((H; u, v_0, g)_\mu \). This fact implies that

\[
S_1 u_n \to S_1 u \text{ in } C([0, T]; H_\sigma) \cap L^2(0, T; V_\sigma) \text{ and weakly in } W^{1,2}(0, T; H_\sigma).
\]

Next, as for the sequence \( \{S_2(u_n, v_n)\} \) with \( v_n := S_1 u_n \), we obtain from Proposition 4.1 that \((N; u_n, v_n, w_0)_\mu \) has a unique solution \( w_n := S_2(u_n, v_n) \) in \( L^2(0, T; V) \) with \( w_n' \in L^2(0, T; V^*) \) and \( 0 \leq w_n \leq 1 \) a.e. \( Q \), satisfying the uniform estimate

\[
|w_n(t)|_H^2 + 2c_d \int_0^t |\nabla w_n|^2_H d\tau \leq e^{2u^* L(f)T} |w_0|_H^2, \quad \forall t \in [0, T]. \tag{6.1}
\]

This estimate shows that \( w_n' = -\partial_x \Phi(\rho \mu * u_n; w_n) - \nabla w_n - f(w_n) \rho \mu * u_n \) is bounded in \( L^2(0, T; V^*) \) (as \( n \to \infty \)), so that it follows from the Aubin’s compactness theorem [2] that \( \{w_n\} \) is relatively compact in \( L^2(Q) \). Now, we choose a subsequence \( \{w_{n_k}\} \) of \( \{w_n\} \) so as to satisfy \( w_{n_k} \to w \) in \( L^2(Q) \) (as \( k \to \infty \)) for some function \( w \). Then, by (6.1), \( w_{n_k} \to w \) weakly in \( L^2(0, T; V) \), \( w_{n_k}' \to w' \) weakly in \( L^2(0, T; V^*) \) and \( 0 \leq w \leq 1 \) a.e. on \( Q \). Besides, since \( \rho \mu * u_{n_k} \to \rho \mu * u \) in \( L^2(Q) \) (cf. (2.9)), we have

\[
f_{n_k} := \nabla w_{n_k} + f(w_{n_k}) \rho \mu * u_{n_k} \to f := \nabla w + f(w) \rho \mu * u \text{ in } L^2(0, T; V^*).
\]

As a consequence, letting \( k \to \infty \), we see that \( w \) is the solution of \((N; u, v, w_0)_\mu \). Since the solution of \((N; u, v, w_0)_\mu \) is unique, the above convergences hold without extracting any subsequence from \( \{w_n\} \), that is,

\[
S_2(u_n, v_n) = w_n \to w := S_2(u, v) \text{ in } L^2(Q) \text{ and weakly in } L^2(0, T; V),
\]

and \( w_n' \to w' \) weakly in \( L^2(0, T; V^*) \).

Moreover, by the convergences of \( \{v_n\} \) and \( \{w_n\} \) obtained above, Proposition 3.2 implies that the solution of \((B; w_n, v, u_0)_\mu \) converges to that of \((B; w, v, u_0)_\mu \) in \( C([0, T]; V_0^*) \). Namely,

\[
S u_n = S_3(w_n, v_n) \to S_3(w, v) = S u \text{ in } C([0, T]; V_0^*).
\]
Thus, $S$ is continuous in $C([0,T];V_0^\varepsilon)$. Accordingly, it follows from the Schauder fixed-point theorem that $S$ admits at least one fixed-point, $u_\mu = Su_\mu$. It is easy to see that this fixed-point $u_\mu$ with the solutions $v_\mu$ of $(H;u_\mu,v_0,g)^\varepsilon$ and $w_\mu$ of $(N;u_\mu,v_\mu,w_0)_\mu$ gives a set of solutions of our problem $P^\varepsilon_\mu$. \hfill \square

Now, we summarize the uniform estimate on approximate solutions $\{u_\mu, w_\mu, v_\mu\}$; we have automatically
\begin{equation}
0 \leq u_\mu \leq u^*, \quad 0 \leq w_\mu \leq 1 \quad \text{a.e. in } Q.
\end{equation}

Furthermore, by our construction of approximate solutions, there is a positive constant $A_0$ depending only on the data $u_0$, $w_0$, $v_0$, $g$, $\beta$, $f$ and $d_0$ such that
\begin{equation}
|u_\mu|_{W^{1,2}(0,T;V_0^\varepsilon)} + \max_{t \in [0,T]} |\beta(t, \cdot)|_{L^1(\Omega)} + |w_\mu|_{L^2(0,T;V)} + |v_\mu|_{L^2(0,T;V^*)} + |v_\mu|_{L^\infty((0,T;H_\sigma))} \leq A_0
\end{equation}
for all small $\mu > 0$. On account of the uniform estimates (6.2), (6.3), Lemma 3.1 and Proposition 4.2, there is a sequence $\{\mu_n\}$ with $\mu_n \downarrow 0$ (as $n \to \infty$) and a triplet $\{u, w, v\}$ of functions such that
\begin{equation}
u^n := u_{\mu_n} \to u \text{ in } L^2(Q) \text{ and weakly in } W^{1,2}(0,T;V_0^\varepsilon)
\end{equation}
\begin{equation}
u^n := w_{\mu_n} \to w \text{ in } L^2(Q), \text{ weakly in } L^2(0,T;V), \text{ weakly } w_n' \to w' \text{ weakly in } L^2(0,T;V^*),
\end{equation}
\begin{equation}
u^n := v_{\mu_n} \to v \text{ weakly in } L^2(0,T;V_\sigma) \text{ and weakly } v_n^\sigma \to v^\sigma \text{ in } L^\infty(0,T;H_\sigma).
\end{equation}

In the rest of this section we shall show that $\{u, w, v\}$ is a solution of the limit problem $P^\varepsilon$. To this end, we make use of recent important results about the convergence of $\{v_n\}$, which was obtained in the authors' work [12].

**Theorem 6.2.** Let $\varepsilon$ be a small positive number and fix it. Assume that $u_0 \in H$ with $\beta(u_0) \in L^1(\Omega)$, $w_0 \in H$ with $0 \leq w_0 \leq 1$ a.e. on $\Omega$, $g \in L^2(0,T;H_\sigma)$, and
\begin{equation}
v_0 \in W^{1,4}_{0,\sigma}(\Omega), \text{ supp}(v_0) \subset \{x \in \Omega \mid p_0(u_0^\varepsilon(x)) > 0\}, \text{ } |v_0| < p_0(u_0^\varepsilon) \text{ on } \overline{\Omega},
\end{equation}
where $u_0^\varepsilon = \rho_\varepsilon * u_0$. Then there exists at least one set of functions $\{u, w, v\}$ such that
(i) $u$ is a solution of $(B;w,v,u_0)$ in the sense of Definition 3.1.
(ii) $w$ is a solution of $(N;u,v,w_0)$ in the sense of Definition 4.1.
(iii) $v$ is a weak solution of $(H;u,v_0,g)^\varepsilon$ in the sense of Definition 5.1.

**Proof.** Fix $\varepsilon \in (0,1]$. Let $\{u_n, w_n, v_n\}$ be the same sequence of approximate solutions as in (6.4)-(6.7) with the limit $\{u, w, v\}$. As for the convergences of $(B;w_n,v_n,u_0)_{\mu_n}$ and $(N;u_n,v_n,w_0)_{\mu_n}$, by (6.4) and (6.5), we see that
\begin{equation}\rho_{\mu_n} * (\gamma_{\mu_n} u_n) \to u, \quad \rho_{\mu_n} * u_n \to u \text{ in } L^2(Q),
\end{equation}
and
\begin{equation}\rho_{\mu_n} * w_n \to w \text{ in } L^2(Q) \text{ and weakly in } L^2(0,T;V).
\end{equation}
Therefore, by Propositions 3.2 and 4.2 the limits \( u \) and \( w \) are solutions of \((B; w, v, u_0)\) and \((N; u, v, w_0)\), respectively. Thus (i) and (ii) hold. In order to complete the proof of Theorem 6.2 it remains to prove (iii).

Actually, we are going to show that (iii) is a direct consequence of [12], putting
\[
p(x, t) := p_0(u^\varepsilon(x, t)), \quad p_n(x, t) := p_{\mu_n}(\gamma_{\mu_n}u_n)^\varepsilon(x, t), \quad \forall (x, t) \in \overline{Q}, \quad \forall n \in \mathbb{N}.
\]
From this definition of \( p, p_n \) and the fact \((\gamma_{\mu_n}u_n)^\varepsilon \to u^\varepsilon \) in \( C(\overline{Q}) \) it is easy to see that
\[
\begin{cases}
0 < p_n < \infty \text{ on } \overline{Q}, & \forall n \in \mathbb{N}, \\
\forall \kappa \in (0, \infty), & p_n \to p \text{ uniformly on } \{(x, t) \in \overline{Q} \mid p(x, t) \leq \kappa\}
\end{cases}
\tag{6.8}
\]
and
\[
\begin{cases}
\forall M > 0 \text{ sufficiently large }, & \exists n_M \in \mathbb{N} \text{ such that} \\
\quad & p_n > M \text{ on } \{(x, t) \in \overline{Q} \mid p(x, t) > M\}, \quad \forall n \geq n_M.
\end{cases}
\tag{6.9}
\]
Under (6.8) and (6.9) it is proved in [12; Lemma 4.1, Theorem 1.1] that the sequence of solutions \( v_n \) of variational inequalities \((H; u_n, v_0, g)^\varepsilon_{\mu_n} \) of the Navier-Stokes type converges to \( v \), and, in addition to (6.6):

(1) For every \( t \in [0, T] \), \( v_n(t) \to v(t) \) weakly in \( H_\sigma \) and \(|v(x, t)| \leq p_0(u^\varepsilon(x, t))\) for a.e. \( x \in \Omega \).

(2) \( v_n \to v \) (strongly) in \( L^2(0, T; H_\sigma) \).

Moreover:

(3) For any test function \( \eta \in \mathcal{K}(u^\varepsilon) \), given in Definition 5.1, the real-valued function \( t \to (v(t), \eta(t))_\sigma \) is of bounded variation on \([0, T]\).

(4) The limit \( v \) satisfies (5.1).

In fact, by virtue of (1) and (2), the nonlinear term \((v_n \cdot \nabla)v_n \) converges to \((v \cdot \nabla)v\) in \( L^\frac{2}{3}(0, T; W^{-1, \frac{2}{3}}(\Omega)^3) \) (the dual space of \( L^4(0, T; W^{1, 4}_0(\Omega)^3) \)). Hence we can arrive at the variational inequality (5.1) by integrating by parts in time and letting \( n \to \infty \) in the variational inequality equivalent to \((H; u_{\mu_n}, v_0, g)^\varepsilon_{\mu_n} \). For the detailed proof, see [12]. \( \square \)

**Remark 6.1.** In Theorem 6.2, we do not know whether \( v(t) \) is continuous in time or not. However the initial condition \( v(0) = v_0 \) makes sense, because \( v(t) \) is defined for every \( t \in [0, T] \) and the real-valued function \( t \to (v(t), \eta(t))_\sigma \) is of bounded variation on \([0, T]\) for any test function \( \eta \). In particular, if \( \text{supp}(\eta) \) is included in the liquid region (namely the interior of \( \{(x, t) \in \overline{Q} \mid u^\varepsilon(x, t) = 0\} \)), \( (v(t), \eta(t))_\sigma \) is absolutely continuous in \( t \) on \([0, T]\) and \((v(0), \eta(0))_\sigma = (v_0, \eta(0))_\sigma \). For this result, see [12; Corollary 3.2, Remark 4.1].

**Remark 6.2.** A number of open questions concerning the mathematical modeling of biomass development remain. For instance, the limit problem as \( \delta_0 \to 0 \) is the sharp interface model mentioned in the introduction. It is expected that this question will be affirmatively solved. Another question is to characterize the limit procedure of \( \varepsilon \to 0 \); when the convolution parameter \( \varepsilon \) tends to 0, in which class of evolution inclusions the limit problem can be handled. This seems a very difficult question.
Appendix

Let $X$ be a real Hilbert space with inner product $(\cdot, \cdot)_X$ and norm $|\cdot|_X$. For a fixed (large) positive number $M$ we denote by $\Phi(M)$ the set of all families $\{\varphi^t(\cdot)\}_{t \in [0,T]}$ of non-negative proper, l.s.c. and convex functions $\varphi^t(\cdot)$ on $X$ satisfying the following conditions (\Phi1) and (\Phi2):

(\Phi1) $\min_{z \in X} \{ |z|_X^2 + \varphi^t(z) \} \leq M$ for all $t \in [0,T]$.

(\Phi2) There are non-negative real-valued functions $a(\cdot) \in W^{1,2}(0,T)$ and $b(\cdot) \in W^{1,1}(0,T)$ satisfying

$$|a|^2_{W^{1,2}(0,T)} + |b|_{W^{1,1}(0,T)} \leq M$$

and the following property that for each $s$, $t \in [0,T]$ and $z \in D(\varphi^s)$ there is an element $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_X \leq |a(t) - a(s)|(1 + \varphi^s(z)^{\frac{1}{2}}),$$

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |b(t) - b(s)|(1 + \varphi^s(z)).$$

We recall the fundamental results (cf. [4, 13, 18]) on the Cauchy problem

$$CP(\varphi^t; f, u_0) \quad \left\{ \begin{array}{l} u'(t) + \partial_X \varphi^t(u(t)) \ni f(t) \quad \text{in} \ X, \ t \in [0,T], \\ u(0) = u_0, \end{array} \right.$$ 

where $f \in L^2(0,T; X)$ and $u_0 \in D(\varphi^0)$ are prescribed as data. It is said that $u : [0,T] \to X$ is a (strong) solution of $CP(\varphi^t; f, u_0)$, if $u \in W^{1,2}(0,T; X)$, $u(0) = u_0$ and $f(t) - u'(t) \in \partial_X \varphi^t(u(t))$ for a.e. $t \in (0,T)$, where $u'(t) := \frac{du(t)}{dt}$.

We denote by $\Phi_c(M)$ the subclass of all families $\{\varphi^t\}$ in $\Phi(M)$ that satisfy the condition of level set compactness:

$$\{ z \in X \mid \varphi^t(z) \leq r \} \text{ is compact in } X, \ \forall r > 0, \ \forall t \in [0,T].$$

[I] Existence and uniqueness

First of all, we recall the results on the existence, uniqueness and uniform estimates of solutions upon data for $CP(\varphi^t; f, u_0)$.

Proposition I. (cf. [13; Chapter 1]). Let $\{\varphi^t\} \in \Phi(M)$. Then we have:

(1) For each $f \in L^2(0,T; X)$ and $u_0 \in D(\varphi^0)$ the Cauchy problem $CP(\varphi^t; f, u_0)$ admits one and only one solution $u$ such that $u \in W^{1,2}(0,T; X)$ and the function $t \to \varphi^t(u(t))$ is absolutely continuous on $[0,T]$.

(2) Let $\{f_i, u_{i0}\} \in L^2(0,T; X) \times D(\varphi^0)$, $i = 1, 2$, be two sets of data and denote by $u_i$ the solution of $CP(\varphi^t; f_i, u_{i0})$. Then we have, for all $s, t \in [0,T]$ with $s \leq t$:

$$\frac{1}{2} |u_1(t) - u_2(t)|^2_X \leq \frac{1}{2} |u_1(s) - u_2(s)|^2_X + \int_s^t (f_1(\tau) - f_2(\tau), u_1(\tau) - u_2(\tau))_X d\tau.$$
(3) There is a non-negative and non-decreasing function \(A_1 := A_1(M; n_1, n_2, n_3) : \mathbb{R}_+^4 \to \mathbb{R}_+\) such that
\[
|u|^2_{W^{1,2}(0,T;X)} + \sup_{0 \leq t \leq T} \varphi'(u(t)) \leq A_1(M; n_1, n_2, n_3),
\]
as long as \(u\) is the solution of \(CP(\varphi; f, u_0)\) with \(|f|_{L^2(0,T;X)} \leq n_1, |u_0|_X \leq n_2\) and \(\varphi^0(u_0) \leq n_3\).

[II] Convergence results

Next, we recall the concept of Mosco convergence (cf. [16]). Let \(\{\varphi_n\}\) a sequence of non-negative proper, l.s.c. and convex function on \(X\). Then it is said that \(\{\varphi_n\}\) converges to a non-negative, proper l.s.c. and convex function \(\varphi\) on \(X\) (as \(n \to \infty\)) in the sense of Mosco, if the following two conditions (\(m_1\)) and (\(m_2\)) are satisfied:

\((m_1)\) If \(z_n \to z\) weakly in \(X\), then \(\liminf_{n \to \infty} \varphi_n(z_n) \geq \varphi(z)\).

\((m_2)\) For every \(z \in D(\varphi)\) there is a sequence \(\{z_n\}\) in \(X\) such that
\[
z_n \to z \text{ in } X, \quad \varphi_n(z_n) \to \varphi(z).
\]

For other characterizations of the Mosco convergence see e.g. [1; Chapter 3], [14; section 8].

Proposition II. (cf. [13; Theorem 2.7.1]) Let \(\{\varphi_n^t\}\) be a sequence of families in \(\Phi(M)\) and \(\{\varphi^t\}\) \(\in \Phi(M)\) such that \(\varphi_n^t\) converges to \(\varphi^t\) in the sense of Mosco on \(X\) for every \(t \in [0,T]\). Also, let \(\{f_n\}\) be a sequence in \(L^2(0,T;X)\) such that \(f_n \to f\) in \(L^2(0,T;X)\), and \(\{u_{n0}\}\) be a sequence in \(X\) such that \(u_{n0} \in D(\varphi_0^t)\), \(\sup_{n \in \mathbb{N}} \varphi_n^0(u_{n0}) < \infty\) and \(u_{n0} \to u_0\) in \(X\). Then the solution \(u_n\) of \(CP(\varphi_n^t; f_n, u_{n0})\) converges to the solution \(u\) of \(CP(\varphi^t; f, u_0)\) in the sense that
\[
u_n \to u \text{ in } C([0,T];X), \quad \left| \int_0^T \varphi_n^t(u_n(t))dt \right| \to \int_0^T \varphi^t(u(t))dt
\]
and
\[
u_n \to u \text{ weakly in } W^{1,2}(0,T;X).
\]

In particular, if \(\{\varphi_n^t\} \in \Phi_c(M)\) and \(\{\varphi^t\} \in \Phi_c(M)\), then the condition “\(f_n \to f\) in \(L^2(0,T;X)\)” is replaced by “\(f_n \to f\) weakly in \(L^2(0,T;X)\)”

[III] A perturbation result

Finally, we present a perturbation result. Let \(\{\varphi^t\} \in \Phi_c(M)\) and let \(h(t, \cdot)\) be a single-valued mapping from \(D(\varphi^t)\) into \(X\) for each \(t \in [0,T]\) such that

\((h1)\) if \(v \in L^2(0,T;X)\) with \(v(t) \in D(\varphi^t)\) for a.e. \(t \in [0,T]\), then \(h(\cdot, v(\cdot))\) is strongly measurable on \([0,T]\),

\((h2)\) there are positive constants \(\alpha_1, \alpha_2, \alpha_3\) such that
\[
|h(t, z)|^2_X \leq \alpha_1 \varphi^t(z) + \alpha_1 |z|^2_X + \alpha_3, \quad \forall t \in [0,T], \forall z \in D(\varphi^t),
\]

Similarly, there are positive constants \(\beta_1, \beta_2, \beta_3\) such that
\[
|h(t, z)|^2_X \leq \beta_1 \varphi^t(z) + \beta_1 |z|^2_X + \beta_3, \quad \forall t \in [0,T], \forall z \in D(\varphi^t),
\]

and
\[
|h(t, z)|^2_X \leq \gamma_1 \varphi^t(z) + \gamma_1 |z|^2_X + \gamma_3, \quad \forall t \in [0,T], \forall z \in D(\varphi^t),
\]

with \(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3\) depending on \(M, n_1, n_2, n_3\).
Proposition III. [18; Theorem 2.1] Let $t_n \in [0, T], z_n \in X, \{\varphi^n(z_n)\}$ is bounded, $z_n \to z$ in $X$ and $t_n \to t$ (as $n \to \infty$), then $h(t_n, z_n) \to h(t, z)$ weakly in $X$.

(h4) for each $\delta > 0$ there exists a positive constant $C_\delta$ such that

$$|(h(t, z_1) - h(t, z_2), z_1 - z_2)_X| \leq \delta(z_1^* - z_2^*, z_1 - z_2)_X + C_\delta|z_1 - z_2|^2_X,$$

$\forall t \in [0, T]$, $\forall z_i \in D(\partial_X\varphi^t)$, $\forall z_i^* \in \partial_X\varphi^t(z_i)$, $i = 1, 2$.

Now, given $f \in L^2(0, T; X)$ and $u_0 \in D(\varphi^0)$, we consider the following perturbation problem, denoted by $CP(\varphi^t, h; f, u_0)$,

$$CP(\varphi^t, h; f, u_0) \begin{cases}
  u'(t) + \partial_X\varphi^t(u(t)) + h(t, u(t)) \ni f(t) & \text{in } X, \ t \in [0, T], \\
  u(0) = u_0.
\end{cases}$$

It is said that $u$ is a solution of $CP(\varphi^t, h; f, u_0)$, if it is a solution of $CP(\varphi^t; f - h(\cdot, u(\cdot)), u_0)$, namely if $u \in W^{1, 2}(0, T; X)$, $u(0) = u_0$ and $f(t) - u'(t) - h(t, u(t)) \in \partial_X\varphi^t(u(t))$ for a.e. $t \in (0, T)$. As to this perturbation problem we have similar results to Proposition I.

**Proposition III.** [18; Theorem 2.1] Let $\{\varphi^t\} \in \Phi_c(M)$ and $h(\cdot, \cdot)$ be a single-valued mapping satisfying (h1) – (h4). Then we have:

1. For each $f \in L^2(0, T; X)$ and $u_0 \in D(\varphi^0)$ problem $CP(\varphi, h; f, u_0)$ admits one and only one solution $u$ such that $u \in W^{1, 2}(0, T; X)$ and the function $t \to \varphi^t(u(t))$ is absolutely continuous on $[0, T]$.

2. There is a non-negative and non-decreasing function $A_2 := A_2(M, h; n_1, n_2, n_3) : R^+_3 \to R_+$, depending only on the class $\Phi_c(M)$, $h$ and three given positive constants $n_1, n_2, n_3$, such that

$$|u|^2_{W^{1, 2}(0, T; X)} + \sup_{0 \leq t \leq T} \varphi^t(u(t)) \leq A_2(M, h; n_1, n_2, n_3),$$

as long as $u$ is the solution of $CP(\varphi^t, h; f, u_0)$ with $|f|_{L^2(0, T; X)} \leq n_1$ and $u_0 \in D(\varphi^0)$ satisfying $|u_0|_X \leq n_2$ and $\varphi^0(u_0) \leq n_3$.

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