Balance and Frustration in Signed Networks under Different Contexts

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Abstract
Frustration index is a key measure for analysing signed networks that has been underused due to its computational complexity. We use an optimisation-based method to analyse frustration as a global structural property of signed networks under different contexts. We provide exact numerical results on social and biological signed networks as well as networks of formal alliances and antagonisms between countries and financial portfolio networks. Molecular graphs of carbon and spin glass models are among other networks that we consider to discuss context-dependent interpretations of the frustration index. The findings unify the applications of a graph-theoretical measure in understanding signed networks.

Highlights:
Models involving signed networks in 6 different disciplines are analysed
An optimisation model is used for a wide range of computational experiments
Values of the frustration index in large graphs are computed exactly and efficiently
Applications of signed network balance and frustration are unified

Keywords:
Frustration index, Line index of balance, Signed graph, Integer programming, Optimisation, Balance theory
1 Introduction

The theory of structural balance [42] is an essential tool to understand the impact of local interactions on the global structure of signed networks. Cartwright and Harary identified cycles containing an odd number of negative edges [13] to be a source of tension that may influence the structure of signed networks in particular ways leading to a reduction in overall tension. Signed networks in which no such cycles are present satisfy the property of structural balance which is considered as a state with minimal tension [13]. For graphs that are not balanced, a distance from balance (a measure of partial balance) can be computed [7]. Among various measures [13, 26, 27, 62, 73] is the frustration index that indicates the minimum number of edges whose removal (or equivalently, negation) results in balance [1, 37, 76]. In this study, we focus on applications of the frustration index in different contexts beyond the structural balance of signed social networks.

The frustration index, also known as the line index of balance [37], is a key to frequently stated problems in many different fields of research [23, 24, 45, 46]. In biological networks, optimal decomposition of network into monotone subsystems—which is essential for understanding Drosophila segment polarity—is made possible by calculating the frustration index of the underlying signed graph [45]. In Physics, the frustration index provides the ground state of atomic magnet models [41, 46]. In international relations, the dynamics of alliances and enmities between countries can be investigated using the frustration index [23]. Frustration index can also be used as an indicator of network bi-polarisation in practical examples involving financial portfolios. For instance, some low risk portfolios have an underlying balanced graph containing negative edges [40]. Finally in chemistry, bipartite edge frustration can be used as a stability indicator of carbon allotropes known as fullerenes [24].

2 Computing the frustration index

From a computational viewpoint, calculating the frustration index of a signed graph is an NP-hard problem equivalent to the ground state calculation of a spin glass model without special structure [9, 59, 71]. Computation of the frustration index can also be reduced from the graph maximum cut (MAXCUT) problem which is known to be NP-hard [44].

The frustration index can be computed in polynomial time for planar graphs [44, 47] while in general graphs, the frustration is even NP-hard to approximate within any constant factor [44]. It is believed that the computational complexity of the problem has led to the lack of systematic investigations on computing the frustration index [5, 6]. There are some studies on approximating [2, 8, 14, 16] and computing [5, 6, 11, 12, 35] the frustration index in large signed graphs.

A closely related and more general problem is finding the minimum number of frustrated edges based on Davis’s definition of weak balance (generalised balance) [17] is referred to as the Correlation Clustering problem [31, 54] which is investigated more
comprehensively. The most recently developed method for computing the frustration index is based on an integer linear programming model and is capable of processing graphs with thousands of nodes and edges on an ordinary computer [5,6]. Facchetti, Iacono, and Altafini suggested a non-linear energy function minimisation model for finding the frustration index [27]. Their model was used as a basis of various non-exact optimisation techniques [25,45,54,55,74]. Using heuristic algorithms [45], estimations of the frustration index is provided for biological networks up to $1.5 \times 10^5$ nodes [45] and social networks with up to $10^5$ nodes [27,28]. Doreian and Mrvar have provided some upper bounds on the frustration index of signed international relation networks [6] using a polynomial-time algorithm [23,65]. We use their signed international relation dataset in Section 7 and analyse it using the exact values of the frustration index.

An analysis of the literature shows that there are only a few studies on exact methods for computing the frustration index in signed graphs of non-trivial size and order [5,5,12,44]. Among these methods, the integer linear programming models suggested by Aref et al. [5,6] are the only computational methods capable of processing the signed networks that we aim to analyse in this study.

**Our contribution**

The main objective served in this paper is analysing several problems from different disciplines that all can be looked at through the perspective of signed networks. The analysis is based on a graph-theoretic measure of structural balance. This involves computing the frustration index, a standard measure of balance overlooked for decades due to the inherent combinatorial complexity. We use an integer linear optimisation model to compute the frustration index.

The advantage of formulating the problem as an optimisation model is not only exploring the details involved in a fundamental NP-hard problem, but also making use of powerful mathematical programming solvers like Gurobi to solve the NP-hard problem exactly and efficiently. This approach allows us to investigate several problems using a single method that applies to them all.

This paper begins by laying out the theoretical dimensions of the research in Section 3. The computational method is briefly discussed in Section 4 followed by a discussion on its efficiency. Numerical results on signed networks of six disciplines are provided in Sections 5–10. Section 11 provides a short conclusion and suggests future directions.

### 3 Preliminaries

#### 3.1 Notation

We consider undirected signed networks $G = (V, E, \sigma)$. The set of nodes is denoted by $V$, with $|V| = n$. The set of edges is represented by $E = \{e_1, e_2, \ldots, e_m\}$ that is partitioned into the set of positive edges $E^+$ and the set of negative edges $E^-$ with
\[ |E| = m, \quad |E^-| = m^-, \quad \text{and} \quad |E^+| = m^+ \quad \text{where} \quad m = m^- + m^+. \]

The sign function is denoted by \( \sigma : E \to \{-1, +1\}^m \).

We represent the \( m \) undirected edges in \( G \) as ordered pairs of vertices \( E = \{e_1, e_2, ..., e_m\} \subseteq \{(i, j) \mid i, j \in V, i < j\} \), where a single edge between nodes \( i \) and \( j \), \( i < j \), is denoted by \( (i, j) \), \( i < j \).

The entries of the symmetric adjacency matrix \( A \) are defined in Eq. (1).

\[
a_{ij} = \begin{cases} 
\sigma(i, j) & \text{if } (i, j) \in E \text{ or } (j, i) \in E \\
0 & \text{if } (i, j) \notin E
\end{cases}
\]  \quad (1)

We use \( G_r = (V, E, \sigma_r) \) to denote a reshuffled graph in which the sign function \( \sigma_r \) is a random mapping of \( E \) to \( \{-1, +1\} \) that preserves the number of negative edges.

A walk of length \( k \) in \( G \) is a sequence of nodes \( v_0, v_1, ..., v_k \) such that for each \( i = 1, 2, ..., k \) there is an edge from \( v_{i-1} \) to \( v_i \). If \( v_0 = v_k \), the sequence is a closed walk of length \( k \). If the nodes in a closed walk are distinct except for the endpoints, it is a cycle of length \( k \). The sign of a cycle is the product of the signs of its edges. Cycles with positive (negative) signs are balanced (unbalanced). A balanced graph is one with no unbalanced cycles.

### 3.2 Frustration count

Satisfied and frustrated edges are defined based on colourings of the nodes. Colouring the nodes with black and white, a frustrated (satisfied) edge \((i, j)\) is either a positive (negative) edge with different colours on the endpoints \(i, j\) or a negative (positive) edge with the same colours on the endpoints \(i, j\). Figure 1 (a) demonstrates an example signed graph in which positive and negative edges are represented by solid lines and dotted lines respectively. Figure 1 (b) illustrates two node colourings and their impacts on the frustrated edges represented by thick lines.

![Figure 1](image-url)

Figure 1: The impact of node colouring on the frustration of edges
To be more specific, for any signed graph $G = (V, E, \sigma)$, we can partition $V$ into two sets, denoted $X \subseteq V$ and $\bar{X} = V \setminus X$. We call $X$ the colouring set and we think of this partitioning as specifying a colouring of the nodes, where each node $i \in X$ is coloured black, and each node $i \in \bar{X}$ is coloured white. We let $x_i$ denote the colour of node $i \in V$ under $X$, where $x_i = 1$ if $i \in X$ and $x_i = 0$ otherwise.

We define the frustration count $f_G(X)$ as the number of frustrated edges of $G$ under $X$:

$$f_G(X) = \sum_{(i,j) \in E} f_{ij}(X)$$

where $f_{ij}(X)$ is the frustration state of edge $(i,j)$, given by

$$f_{ij}(X) = \begin{cases} 0, & \text{if } x_i = x_j \text{ and } (i,j) \in E^+ \\ 1, & \text{if } x_i = x_j \text{ and } (i,j) \in E^- \\ 0, & \text{if } x_i \neq x_j \text{ and } (i,j) \in E^- \\ 1, & \text{if } x_i \neq x_j \text{ and } (i,j) \in E^+ \end{cases}$$ (2)

The frustration index $L(G)$ of a graph $G$ can be found by finding a subset $X^* \subseteq V$ of $G$ that minimises the frustration count $f_G(X)$, i.e., solving Eq. 3. Note that both colourings in Figure 1 (b) are optimal.

$$L(G) = \min_{X \subseteq V} f_G(X)$$ (3)

Upper bounds can be readily provided for the frustration index such as $L(G) \leq m^-$ which states the obvious result that removing all negative edges gives a balanced graph. Akiyama proves that the largest frustration count of a graph is bounded by $m/2$ [3]. Note that $f_G(X)$ gives an upper bound on $L(G)$ for any $X \subseteq V$.

4 Method and Material

In this section, we discuss a mathematical programming model in Eq. (4) that minimises the frustration count as the objective function over binary decision variables defined for nodes and edges of the graph. Note that, there are various ways to formulate the frustration count using variables defined over graph nodes and edges leading to various mathematical programming models that are investigated in details in [5].

4.1 Methodology

Computing the frustration index can be formulated as a binary linear model that counts the frustrated edges using binary variable $f_{ij} \ \forall (i,j) \in E$ to denote frustration of edge $(i,j)$ and $x_i \ \forall i \in V$ to denote the colour of node $i$. Constraints of the optimisation model are formulated according to the values of $f_{ij}$ in Eq. (2). Therefore, the minimum
frustration count under all node colourings \((x_1, x_2, \ldots, x_n) \subset \{0, 1\}^n\) is obtained by solving (4):

\[
\min_{x_i, f_{ij}} Z = \sum_{(i,j) \in E} f_{ij}
\]

s.t. \(f_{ij} \geq x_i - x_j \quad \forall (i, j) \in E^+
\)

\(f_{ij} \geq x_j - x_i \quad \forall (i, j) \in E^+
\)

\(f_{ij} \geq x_i + x_j - 1 \quad \forall (i, j) \in E^-
\)

\(f_{ij} \geq 1 - x_i - x_j \quad \forall (i, j) \in E^-
\)

\(x_i \in \{0, 1\} \quad \forall i \in V
\)

\(f_{ij} \in \{0, 1\} \quad \forall (i, j) \in E
\)

Several techniques are used to speed up the branch and bound algorithm for solving the binary programming model in Eq. (4). These techniques are discussed in [5, 6]. We implement the four speed-up techniques known as pre-processing data reduction, prioritised branching and fixing a colour, unbalanced triangle valid inequalities, and node degree valid inequalities [5, 6]. All details of developing the optimisation model can be found in [5, 6].

Aref et al. have tested their optimisation models [5, 6] on synthetic and real-world datasets using Gurobi version 7 on a desktop computer with an Intel Corei5 4670 @ 3.40 GHz and 8.00 GB of RAM running 64-bit Microsoft Windows 7 showing the efficiency of their model in computing the frustration index in comparison to other models in the literature [11, 12, 16, 23, 32, 35, 36, 39, 44, 45]. For detailed discussions on the efficiency of the binary linear programming model in Eq. (4), one may refer to [5].

For comparing the level of frustration among networks of different size and order, we use the normalised frustration index, \(F(G) = 1 - 2L(G)/m\). The normalised frustration index is a standard measure of partial balance suggested in a comparative study of structural balance measures [7] as the only measure satisfying axiomatic properties. Values of \(F(G)\) are within the range of \([0, 1]\) and greater values of \(F(G)\) represent closeness to a state of structural balance.

### 4.2 Material

We use a wide range of examples from different disciplines including six social signed networks ranging in size from 36 to 99917 edges in Section 5, four biological signed networks with 779-5215 edges in Section 6, one dynamic network of international relations with 51 time windows ranging in size from 362 to 1247 edges in Section 7, six financial portfolios with 10-55 edges over 9 years in Section 8, a wide range of molecular fullerene graphs with 360-9000 edges in Section 9, and finally Ising spin glass models with 32-79600 edges in Section 10. The data used in this study is made publicly available on Figshare research data sharing website.
The numerical results in this paper are obtained by solving the binary linear programming model (4) coupled with four speed-up techniques [5,6] using Gurobi Python interface. The hardware used for the computations is a virtual machine computer with 32 Intel Xeon CPU E5-2698 v3 @ 2.30 GHz processors and 32 GB of RAM running 64-bit Microsoft Windows Server 2012 R2 Standard unless stated otherwise.

5 Social networks

We use well-studied datasets of communities with positive and negative interactions and preferences as signed social network. This includes Read’s dataset for New Guinean highland tribes [68] and the last time frame of Sampson’s data on monastery interactions [70]. We also use graphs inferred from datasets of students’ choice and rejection [49,61]. A further explanation on the details of inferring signed graphs from the choice and rejection data can be found in [7]. The same data is used in some other studies of social networks [20,22].

Our analysis also includes a signed network of US senators that is inferred by [60] through implementing a stochastic degree sequence model on Fowler’s Senate bill co-sponsorship data [33]. A larger social signed network we use is from the Wikipedia election dataset [51]. This dataset is based on all adminship elections before January 2008 in which Wikipedia users have voted for approval or disapproval of other users promotions to becoming administrators. We use an undirected version of the Wikipedia elections signed graph made publicly available in [52]. Four of the social signed networks are illustrated in Figure 2 where green and red edges represent positive and negative edges respectively. There are studies on the correlation clustering problem [54,55] and estimating frustration index [27] using Wikipedia elections network where many inconsistent directed edges are disregarded in order to “symmetrise” the data [27].

Our numerical results are shown in Table 1 where the average and standard deviation of the frustration index in 500 reshuffled graphs, denoted by \( L(G_r) \) and SD, are also provided for comparison.

| Graph               | n   | m  | m^- | \( L(G) \) | \( L(G_r) \) ± SD | Z score |
|---------------------|-----|----|-----|------------|-------------------|--------|
| Highland tribes     | 16  | 58 | 29  | 7          | 14.65 ± 1.38     | −5.54  |
| Monastery interactions | 18 | 49 | 12  | 5          | 9.71 ± 1.17      | −4.03  |
| Fraternity          | 17  | 40 | 17  | 4          | 7.53 ± 1.24      | −2.85  |
| Eastern college     | 17  | 36 | 16  | 6          | 6.48 ± 1.08      | −0.45  |
| US senate           | 100 | 2461 | 1047 | 331 | 965.6 ± 9.08 | −69.89 |
| Wikipedia elections | 7112 | 99917 | 21837 | 14532 | 21359.8 ± 241.3 | −28.29 |

As it is expected the six social signed networks are not totally balanced. However, the relatively small values of \( L(G) \) suggest low level of frustration in these networks.
Figure 2: Three signed networks inferred from the sociology and political science datasets and visualised using Gephi (colour version online)
In order to be more precise, we have implemented a very basic statistical analysis using Z scores $Z = \frac{L(G) - L(G_r)}{SD}$. These Z scores, provided in the right column of the Table 1, show how close the networks are to a state of balance. The results indicate that networks of Highland tribes, Monastery interactions, US senators, and Wikipedia users exhibit a level of frustration substantially lower than what is expected by chance.

Regarding performance of the optimisation model, a very basic binary linear formulation of the problem would solve the small social signed networks in a reasonable time on an ordinary computer. The binary linear programming model (4) solves such instances in split seconds, while for the senators network it takes a few seconds. For Wikipedia elections network, 10 hours of computation is required to find the optimal solution.

6 Biological networks

Some biological models are often used to describe interactions with dual nature between biological molecules in the field of systems biology. The interactions can be activation or inhibition and the biological molecules are enzymes, proteins or genes [56]. This explains the parallel between signed graphs and these types of biological networks. Interestingly, the concept of close-to-monotone [56] in systems biology is analogous to being close to a state of balance. Similar to negative cycles and how they lead to unbalance, existence of negative loops in biological networks indicates a system that does not display well-ordered behaviour [56].

We analyse large scale gene regulatory networks where nodes represent genes and positive and negative edges represent activating connections and inhibiting connections respectively. There are four signed biological networks analysed by [45]. They include two gene regulatory networks, related to two organisms: a eukaryote (the yeast Saccharomyces cerevisiae) [15] and a bacterium (Escherichia coli) [69]. Another signed network we use is based on the Epidermal Growth Factor Receptor (EGFR protein) pathway [64]. We also use a network based on the molecular interaction map of a white blood cell (macrophage) [63]. Figure 3 shows the four biological signed networks. The colour of edges correspond to the signs on the edges (green for $+1$ and red for $-1$). For more details on the four biological datasets, one may refer to [45].

The results for the four biological networks are shown in Table 2 where the average and standard deviation of the frustration index in 500 reshuffled graphs are also provided for comparison.

The results in Table 2 show that the level of frustration is very low for yeast and E.coli networks. In contrast for EGFR and macrophage, the level of frustration is very high, i.e., there are far more frustrated edges than we expect by chance. The Z score values in Table 2 show that the gene regulatory networks of yeast and E.coli are close to balance (close-to-monotone) confirming observations in systems biology [56]. Biological networks of the EGFR protein and that of the macrophage are different in nature and our results show that they are far from balanced.
The gene regulatory network of Saccharomyces cerevisiae \cite{15}

(b) The gene regulatory network of the Escherichia coli \cite{69}

(c) Epidermal growth factor receptor pathway \cite{64}

(d) Molecular interaction map of a macrophage \cite{63}

Figure 3: Four biological signed networks visualised using Gephi (colour version online)

The two smallest biological networks considered here (EGFR and macrophage) are the largest networks analysed in a recent study of balancing signed networks by negating minimal edges \cite{74} where the heuristic algorithm gives sub-optimal values of the frustration index \cite{74, Fig. 5}.

Aref et al. compare the quality and solve time of their exact algorithm with that of recent heuristics and approximations implemented on the same datasets \cite{5, 6}. While data reduction schemes \cite{44} can take up to 1 day for these four biological networks and heuristic algorithms \cite{45} only provide bounds with up to 9% gap from optimality, the binary linear programming model \cite{4} equipped with the speed-up techniques solves the four biological networks to optimality in a few seconds on an ordinary computer \cite{5, 6}.
Table 2: The frustration index in various signed networks

| Graph   | \(n\) | \(m\) | \(m^-\) | \(L(G)\) | \(L(G_r) \pm SD\) | Z score |
|---------|-------|-------|---------|---------|------------------|--------|
| yeast   | 690   | 1080  | 220     | 41      | 124.3 ± 4.97     | -16.75 |
| E.coli  | 1461  | 3215  | 1336    | 371     | 653.4 ± 7.71     | -36.64 |
| EGFR    | 329   | 779   | 264     | 193     | 148.96 ± 5.33    | 8.26   |
| macrophage | 678 | 1425  | 478     | 332     | 255.65 ± 8.51    | 8.98   |

7 International relations

International relations between countries can be analysed using signed networks models and balance theory [21,48,50]. In earlier studies on balance of signed international relations, Harary used structural balance to explain the changes in international relations between 8 countries over 6 time frames [38].

In this section, we analyse the frustration index in a temporal political network of international relations. Doreian and Mrvar have used the Correlates of War datasets [65] to construct a signed network with 51 sliding time windows each having a length of 4 years [23]. This temporal network represents more than half a century of international relations among countries in the post second world war era starting with 1946-1949 time window and ending with 1996-1999 time window [23]. A dynamic visualisation of the network can be viewed using this [online link]. One may refer to [23, section 3.4] for a detailed explanation of using sliding time windows and other details involved in constructing the network.

In the first time window of the temporal network, network parameters are \(n = 64\), \(m = 362\) and \(m^- = 42\). In the last time window, these parameters are \(n = 151\), \(m = 1247\) and \(m^- = 147\). Figure 4 demonstrates the number of negative edges and the frustration index in the Correlates of War dataset. For this dataset, the binary linear programming model provides the exact values of the frustration index in less than 0.25 seconds on an ordinary computer with an Intel Corei5 4670 @ 3.40 GHz and 8.00 GB of RAM running 64-bit Microsoft Windows 7.

Doreian and Mrvar have also analysed the signed international network using the frustration index (under different name) [23] as a measure of balance. They used a polynomial time blockmodeling algorithm in Pajek for obtaining the frustration index which explains the sub-optimality of their numerical results for all 51 time windows of the network.

Bearing in mind that the size and order changes in each time window of the temporal network, we use the normalised frustration index, \(F(G) = 1 - 2L(G)/m\), in order to investigate the partial balance over time. Figure 5 demonstrates the normalised frustration index values indicating that the network has been close to a state of structural balance. We could have predicted this by the fact that \(F(G) \geq 1 - 2m^-/m\). The largest value of \(m^-/m\) in the Correlates of War dataset is 0.18 in the 1948-1951 time window. Therefore, \(F(G) \geq 0.82\) in all time windows of the Correlates of War dataset.

The data plotted in Figure 5 can also be used to statistically test the hypothesis
that signed networks become closer to balance over time \[4, 40\]. The Priestley-Subba Rao (PSR) test of non-stationarity \[66\] provides the means of a statistically rigorous hypothesis testing for stationarity of timeseries. We use an R implementation of the PSR test that is available in the fractal package in the CRAN repository. The p-value for variation of \( F(G) \) over time equals 0.04 indicating that there is strong evidence to reject the null hypothesis of stationarity.

A basic least-squares trendline fitted on Figure 5 data shows a positive slope. Therefore, our analysis confirms that the network has moved towards becoming more balanced over 1946-1999 period. The overlap of network time period with the cold war era may explain how international relations were getting closer to a global state of bi-polarity with countries clustered into two antagonist sides. This is contrary to the interpretation of Doreian and Mrvar of the their results on frustration index estimates without considering changes in network size and order over time \[23\].
8 Financial portfolios

There are studies investigating financial networks of securities modeled by signed graphs [40, 44]. Harary et al. originally suggested analyzing portfolios using structural balance theory [40]. They represented securities of a portfolio by nodes and the correlations between pairs of securities by signed edges [40]. They used $\pm 0.2$ as thresholds for considering signed edge between two securities of the portfolio. Simplifying a portfolio containing Dow Jones, London FTSE, German DAX, and Singapore STI to a signed graph with four nodes, they observed that the graph has remained in a state of balance from October 1995 to December 2000. Höffner, Betzler, and Niedermeier considered portfolios containing 60-480 stocks and thresholds of $\pm 0.325, \pm 0.35, \pm 0.375$ to evaluate the scalability of their algorithm for approximating the frustration index [44]. Their data is also analyzed in [30].

In this subsection, we consider well-known portfolios recommended by financial experts for having a low risk in most market conditions [10]. These portfolios are known as lazy portfolios and usually contain a small number of well-diversified securities [10]. We consider 6 lazy portfolios each consisting of 5-11 securities. Table 3 represents the six lazy portfolios and their securities.

The signed networks representing the lazy portfolios are generated by considering prespecified thresholds as in [40, 44]. We use the daily returns correlation data that can be found on the Portfolio Visualizer website [72] and thresholds of $\pm 0.2$ similar to [40]. Correlation coefficients with an absolute value greater than 0.2 are considered to draw signed edges between the securities with respect to the sign of correlation.

![A weighted complete graph representing correlation coefficients](image1)

![The portfolio signed graph produced by thresholding on $\pm 0.2$](image2)

Figure 6: Portfolio P3 illustrated as (a) weighted and (b) signed networks using Gephi (colour version online)

Figure 6 shows two networks of portfolio P3 based on February-May 2017 data.
Table 3: Six portfolios and their securities

| Portfolio | Mebane Faber | Larry Swedroe | FundAdvice Ultimate | David Swensen | David Swensen | Bill Schultheis |
|-----------|--------------|---------------|---------------------|---------------|---------------|-----------------|
| Portfolio | Ivy Simple Portfolio | | | | | |
| (P1) | | | | | | |
| (P2) | | | | | | |
| (P3) | | | | | | |
| (P4) | | | | | | |
| (P5) | | | | | | |
| (P6) | | | | | | |
| VEx | x | x | x | x | x | |
| VGSIX | x | x | x | x | x | |
| VIPSX | x | x | x | x | x | |
| VTMGX | x | x | x | x | x | |
| VIVAX | x | x | x | x | x | |
| NAESX | x | x | x | x | x | |
| EFV | x | x | x | x | x | |
| VFIX | x | x | x | x | x | |
| VFISX | x | x | x | x | x | |
| VISVX | x | x | x | x | x | |
| VTMX | x | x | x | x | x | |
| IJS | x | x | x | x | x | |
| TLT | x | x | x | x | x | |
| VFITX | x | x | x | x | x | |
| VBFX | x | x | x | x | x | |
| VGTSX | x | x | x | x | x | |
| GSG | x | x | x | x | x | |
| IEF | x | x | x | x | x | |
| VEU | x | x | x | x | x | |
| VNQ | x | x | x | x | x | |
| VTI | x | x | x | x | x | |

The nodes represent 11 securities of the portfolio and the colours of edges correspond to the correlation coefficient values between the securities (green for positive and red for negative correlation). Lighter colours in Figure 6 (a) represent smaller correlation coefficient values.

We analyse 108 monthly time frames for each of the six portfolios which correspond to the signed networks of each month within the 2008-2016 period. In a large number of time frames for each portfolio (74-79%), a totally balanced signed network with negative edges is observed. In a relatively small number of time frames (1-13%), the underlying network is unbalanced. Figure 7 illustrates the results which are consistent with the findings of Harary et al. in [40].

More detailed results on balance states and frustration index of six portfolios over time are provided in Figures 8 – 9. We observe that there are some months when several portfolios have an unbalanced underlying signed graph like 2013-11, 2015-
02 and 2016-11 in which four portfolios were unbalanced. One may think that this is because of the common securities in the portfolios. However, considering that P(1) does not have any security in common with other portfolios which suggests otherwise.

Regarding sensitivity of the results to the cut-off threshold, it is important to point out that other thresholds (like ±0.1 and ±0.3) also lead to balanced states being dominant. Using thresholds of ±0.1 leads to relatively more unbalanced states and less all-positive states, while thresholds of ±0.3 have the opposite effect.
9 Molecular graphs of fullerene

The bipartite edge frustration is a graph-theoretic measure that is closely related to the frustration index of signed graphs. Bipartite edge frustration, also referred to as network bipartivity, is the minimum number of edges that must be removed to make the graph bipartite [43]. Note that bipartite edge frustration of a graph is equal to the minimum frustration count of a signed graph whose edges are all negative.

It is suggested that the bipartite edge frustration is an indicator of chemical stability [29] in allotropes of carbon known as fullerenes. The graphs representing fullerene molecular structure are called fullerene graphs where nodes and edges correspond to atoms and bonds of a molecule respectively. Three fullerene graphs are visualised in Figure 10.

![Figure 10: Three fullerene graphs with $m \in \{360, 3240, 9000\}$ (colour version online)](image)

(a) C240 fullerene  
(b) C2160 fullerene  
(c) C6000 fullerene

Computing bipartite edge frustration of a graph in general is computationally intractable and heuristic methods are suggested for its approximation [43]. For fullerene
graphs that are planar and have a small density however, a polynomial time algorithm of complexity $O(n^3)$ exists for computation of bipartite edge frustration [24]. As the algorithm cannot process graphs as large as $n = 240$ [24], we use the binary linear model (4) to compute the bipartite edge frustration of giant fullerene molecules with up to 6000 carbon atoms. The results are provided in Table 4.

Table 4: The exact values of bipartite edge frustration for a range of fullerene graphs

| Fullerene graph | $m$ | $L(G)$ |
|-----------------|-----|--------|
| C240*           | 360 | 24     |
| C260            | 390 | 24     |
| C320            | 480 | 24     |
| C500            | 750 | 30     |
| C540*           | 810 | 36     |
| C720            | 1080| 36     |
| C960*           | 1440| 48     |
| C1500*          | 2250| 60     |
| C2160*          | 3240| 72     |
| C2940*          | 4410| 84     |
| C3840*          | 5760| 96     |
| C4860*          | 7290| 108    |
| C6000*          | 9000| 120    |

* Icosahedral fullerene

Among the fullerene graphs in Table 4 are the icosahedral fullerenes. These family of fullerenes have the structure of a truncated icosahedron. It is a conjecture that icosahedral fullerenes with $n$ vertices have a bipartite edge frustration of $L(G) = \sqrt{12n/5}$ [24, Conjecture 13]. The conjecture has not been tested due to the complexity involved in computing bipartite edge frustration [24]. Our results for icosahedral fullerene in Table 4 (marked with *) are in alignment with the conjecture.

Our computations for the fullerene graphs with 240–2940 atoms take from split second to a few minutes. The solve times for computing the bipartite edge frustration of C3840, C4860, and C6000 are 1790.98, 4088.11, and 5852.32 seconds respectively.

10 Ising spin glass models with $\pm 1$ interactions

Among the most notable application of the frustration index are the calculations of spin glass ground-state properties. The most simple and standard form of Ising spin glass models represent patterns of atomic magnets based on the interactions among spins and their nearest neighbours. A key objective in spin glass models with $\pm 1$ interactions is finding the spin configurations with the minimum energy. The standard nearest-neighbour Ising spin glass model with $\pm 1$ interactions and without external magnetic fields is explained in what follows.
Figure 11: Two and three dimensional spin glass models with 50% negative edges (colour version online)

Each spin is connected to its neighbours in a grid-shaped structure. Two connected spins may have matched or mismatched couplings. Each spin can either take an upward or a downward configuration. Frustration arises if and only if a matched (mismatched) coupling has different (same) spin configurations on the endpoints. The energy of a spin configuration is calculated based on the Hamiltonian function: 

$$H = - \sum_{i,j} J_{ij} s_is_j$$

in which the sum \(\sum_{i,j}\) is over the coupled spins. \(J_{ij}\) represent the couplings that are limited to \(\pm 1\) in the Ising model with the type of interactions relevant to this study. \(s_1, s_2, \ldots, s_n\) are the decision variables that take values \(+1\) or \(-1\) and represent upward/downward spin configurations. The Hamiltonian function of these spin glass models is very similar to the energy function in [27] that is also used in other studies [25, 45, 54, 55, 74]. Calculating the minimum value of \(H\) over all possible spin configurations is NP-hard for many structures [53].

Hartmann and collaborators have suggested efficient algorithms for computing the ground-state properties in 3-dimensional spin glass models with 1000 nodes [57] improving their previous contributions in 1-, 2-, and 3-dimensional [19, 41, 58] spin glass models. Recently, they have used a method for solving binary optimisation models to compute the ground state of 3-dimensional models containing up to \(268^3\) nodes [34].

In order to make a connection between spin glass models and signed graphs, we represent spins with nodes and spin configurations with node colours. Matched and mismatched couplings between spins are modeled as positive and negative edges respectively. If \(X^*\) represents the optimal colouring leading to \(L(G)\) for a given signed graph, the minimum value of the corresponding Hamiltonian function can be calculated by 

$$H(X^*) = - \sum_{i,j} a_{ij} (2x_i - 1)(2x_j - 1).$$
We use the binary linear model in Eq. (4) to compute the frustration index in spin glass models of various grid sizes and dimensions. We consider both 2D and 3D grid structures of different size as well as hypercubes of different dimensions. Figure 11 demonstrates 2- and 3-dimensional spin glass models with 50% negative edges. Hypercubes of 4–8 dimensions with 50% negative edges are illustrated in Figure 12.

For each spin glass model, we generate 10 grids and randomly assign $\pm 1$ to the edges to achieve the pre-defined proportion of negative edges based on our experiment settings. Table 5 represents the spin glass models of specific dimension (Dim.) and grid size (Gri.) in each row where the number of nodes and edges are also provided. For each spin glass model, there are three experimental settings with $m^-/m \in \{25\%, 50\%, 75\%\}$. The mean and standard deviation of the frustration index values for each experimental setting is provided in Table 5.

The results in Table 5 show that greater values of frustration is achieved in each spin glass model when $m^-/m = 50\%$ which confirms the intuition based on all cycles having an even length. While there are specialised computational models for each type of spin glass model [19, 34, 41, 57, 58], the binary linear programming model in Eq. (4) can also be used as a general purpose computational method for a wide range of spin.
glass models with ±1 interactions.

Table 5: Frustration index values of several spin glass models

| Dim. | Gri. | n    | m    | \(m^-/m = 25\%\)   | \(m^-/m = 50\%\)   | \(m^-/m = 75\%\)   |
|------|------|------|------|---------------------|---------------------|---------------------|
|      |      |      |      | \(L(G)\) mean±SD   | \(L(G)\) mean±SD   | \(L(G)\) mean±SD   |
| 2    | 50   | 2500 | 4900 | 691.1 ± 12          | 720.9 ± 9.2         | 687.7 ± 10.5        |
| 2    | 100  | 10000| 19800| 2814.1 ± 16         | 2938.2 ± 22.3       | 2802.5 ± 24.7       |
| 2    | 150  | 22500| 44700| 6416.1 ± 25.9       | 6698.5 ± 55.3       | 6396.3 ± 41.7       |
| 2    | 200  | 40000| 79600| 11449 ± 47          | 11930.3 ± 58.9      | 11411.8 ± 46        |
| 3    | 5    | 125  | 300  | 51.4 ± 1.7          | 52.4 ± 2.5          | 51 ± 3.2            |
| 3    | 10   | 1000 | 2700 | 491.5 ± 7.5         | 509.1 ± 4           | 491.6 ± 7           |
| 3    | 15   | 3375 | 9450 | 1762.1 ± 16.1       | 1839.1 ± 10.4       | 1761.1 ± 14.2       |
| 4    | 2    | 16   | 32   | 5.6 ± 0.8           | 4.8 ± 1             | 5.6 ± 0.8           |
| 5    | 2    | 32   | 80   | 14.5 ± 1.1          | 15 ± 1.2            | 15 ± 1.2            |
| 6    | 2    | 64   | 192  | 38.8 ± 2            | 41 ± 1.6            | 38 ± 2.2            |
| 7    | 2    | 128  | 448  | 94.6 ± 3.1          | 99.6 ± 3.2          | 96 ± 2.4            |
| 8    | 2    | 256  | 1024 | 232.4 ± 3.7         | 245.8 ± 3.6         | 231 ± 4.7           |

11 Conclusion

In this study, the frustration index is used for analysing a wide range of signed networks from sociology and political science [5], biology [6], international relations [7], finance [8], chemistry [9] and physics [10]. Our results contribute additional evidence that suggests many signed networks exhibit a relatively low level of frustration which indicates that they are relatively close to the state of structural balance. The numerical results also show the capabilities of the optimisation-based model in making new computations possible for large scale signed networks with up to \(10^5\) edges. In particular, the superiority of the model is evident in the numerical results with guaranteed solution quality provided for large scale social, biological, and political signed networks, molecular graphs of fullerenes, and spin glass models of large size and dimension.

While this study is an overview of the vast range of applications to which the structural balance theory can be applied, it is by no means an exhaustive survey of the frustration index applications. This research has a number of important implications to topics of interest in signed networks including sign change simulation models [18] and models of temporal signed networks [67]. As another future research direction, one may consider formulating frustration-based measures of network stability according to other theories of signed networks [75].
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