A New Method for Computing Stationary Distribution and Steady-State Performance Measures of a Continuous-State Markov Chain with a Queuing Application

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Applications of stochastic models often involve the evaluation of steady-state performance, which requires solving a set of balance equations. In most cases of interest, the number of equations is infinite or even uncountable. As a result, numerical or analytical solutions are unavailable. This is true even when the system state is one-dimensional. This paper develops a general method for computing stationary distributions and steady-state performance measures of stochastic systems that can be described as continuous-state Markov chains supported on $\mathbb{R}$. The balance equations are numerically solved by properly constructing a proxy Markov chain with finite states. We show the consistency of the approximate solution and provide deterministic non-asymptotic error bounds under the supremum norm. Our finite approximation method is near-optimal among all approximation methods using discrete distributions, including the empirical distributions generated by a simulation approach. We apply the developed method to compute the stationary distribution of virtual waiting time in a $G/G/1+G$ queue and associated performance measures under certain mild but general differentiability and boundedness assumptions on the inter-arrival, service and patience time distributions. Numerical experiments validate the accuracy and efficiency of our method, and show it outperforms a standard Markov chain Monte Carlo method by several orders of magnitude. The developed method is also significantly more accurate than the available fluid approximations for this queue.

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1. Introduction. Many stochastic systems can be described by a Markov chain (MC). Computing steady-state performance measures for such systems involves solving a set of balance equations. As pointed out by [31], “in most cases of interest, the number of equations is infinite or too large and the equations cannot be solved analytically or numerically.” This is true even when the system state is one-dimensional. Such systems have numerous applications in operations [11, 2, 62, 58, 4, 7], finance and economics [55, 14, 47, 28].

This paper considers the steady-state performance evaluation problem of a (discrete time) MC with one-dimensional states. Such a problem can be formulated as:

$$\mathbb{E}[\phi(Z)] = \int_{\Omega} \phi(z) dp(z), \quad \text{(P1)}$$

where

$$p(x) = \int_{\Omega} \tau(x,u) dp(u), \quad \forall x \in \Omega. \quad \text{(P2)}$$

Here $\Omega \subseteq \mathbb{R}$ is a support, $z \in \Omega$ is the system state, $\phi: \Omega \to \mathbb{R}$ is a known performance function, $\tau: \Omega^2 \to \mathbb{R}_+$ is defined by the MC’s transition kernel, and $p$ is the stationary (probability) distribution that can be obtained by solving the balance equations (P2); see an example in Subsection 5.3. We
assume \( \Omega = [0,1] \) for the general results of this paper. However, our analysis can be adapted for other support \( \Omega' \subseteq \mathbb{R} \) using certain transformations (see Section 2).

We develop a deterministic finite approximation approach for (P1)-(P2). Specifically, we approximate the original MC with properly constructed and increasingly refined finite-state MCs, whose stationary distributions are computable. The idea of finite approximation is natural, whereas the development of a general approximation scheme with tractable error bounds is not known. Existing methods are often restricted to certain types of models, e.g., [6, 12, 26, 30], while others focus on countable-state systems, e.g., [63, 57, 41, 42, 43, 44, 38, 39]. The accuracy of existing methods is often validated by experiments only or asymptotic results, e.g., [49, 19, 21], but the computation of errors is not known, which limits their broader application. Thus, the development of a general finite approximation scheme with computable error bounds, as accomplished in this paper, fills an important fundamental gap in research on stochastic systems.

1.1. Contributions and Techniques.

A General and Computable Theory. We develop a general and consistent finite approximation scheme with tractable solutions and error bounds for computing the stationary distribution and associated steady-state performance measures of an MC supported on \( \mathbb{R} \) (Theorem 1, 2, 3). The MC’s support can be discrete, continuous or even mixed. Our approximate MC is constructed based on variation properties of the original MC’s transition kernel, and the error is obtained from a linear program with a relatively low computation cost.

Compared with simulation approaches using an empirical distribution as a proxy stationary distribution (e.g., Markov chain Monte Carlo (MCMC), discrete event simulation, etc.), our approach has better tractability because our approximate stationary distribution obtained from the constructed finite-state MC is explicit. The nature of error in these approaches is different. For simulation, the error of empirical distribution is usually quantified as probabilistic concentration bounds. In contrast, the finite approximation error is deterministic and guaranteed in the worst case.

An Illustration Example of G/G/1+G Queue. Our method can be used for any stochastic system that can be represented by an MC satisfying the settings in this paper. As an example, we consider a G/G/1+G queue, where a customer abandons the system once her deadline elapses before service is offered. The distributions of inter-arrival times, service times and patience times are all general, and a first-come-first-serve scheme is assumed. To the best of our knowledge, there is no known analytical or numerical solution for computing the virtual waiting time’s stationary distribution in such a queue. We compare our approach against benchmark results including an MCMC implementation, and a fluid approximation [59]. We find that our approach is significantly more accurate and efficient in obtaining the stationary distribution and evaluating a few illustrative example steady-state performance measures including abandonment probability, “no wait” probability, queue length, etc.

Efficient and Near-Optimal Approximation. Under a Lipschitz continuity assumption on the original MC’s transition kernel, our approach’s supremum-normed error in computing stationary distributions can be as small as \( O(N^{-1}) \), where \( N \) is the number of jumps (or states) used to construct the approximate MC. Moreover, our approach is near-optimal among all approximation methods using discrete distributions including the empirical distributions obtained by a simulation approach (Theorem 4). The empirical distributions can also be regarded as a discrete proxy stationary distribution for the original MC but converge as \( O(N^{-\frac{3}{2}}) \) according to the Dvoretzky–Kiefer–Wolfowitz (DKW) concentration inequality (Theorem 11.5 in [29]), where \( N \) is the number of jumps (or sample size) used in the empirical distribution.

Weak Assumptions. Our results are developed under the supremum norm, which has several advantages over the conventional total variation norm. The supremum norm requires weaker
assumptions on guaranteeing solution consistency. Moreover, it allows us to capture many consistency situations that are invalid under the total variation norm: when we apply finite approximation and use a finite-state MC to approximate another MC with a continuous stationary distribution, the total variation-normed error always equals 1. Thus, the total variation norm is not an adequate measure. In contrast, the supremum-normed error can approach 0 in a proper approximation.

Similar to [20], we do not require approximate MCs’ transition kernels converging under the operator norm, which is often seen in perturbation analysis but can be a strong assumption in practice. Our consistency results require compactness and stationary distribution uniqueness of the original MC’s transition kernel, while our error bound results require a weaker condition than the approximate finite-state MC’s irreducibility. Ergodicity can serve as a sufficient condition for computing error bounds (see Section 3).

A Novel Integral Equation Analysis Framework. Our results are based on a key observation that by adding a natural scaling constraint, many balance equations supported on $\mathbb{R}$ can be transformed into a canonical-form distributional integral equation. This class of distributional integral equations and their numerical solutions were analyzed in our companion paper [33], which provide a theoretical foundation for the current work.

1.2. Related Literature.
MC Finite Approximation. A significant literature has investigated the approximation of a countable-state MC by another (Table 1). However, these methods rely on the transition matrix structure, and are not extended for continuous-state MCs. [49] used finite-state MCs to approximate a continuous-state MC under a strong assumption of kernel continuity and provided a weak convergence result of stationary distributions. In practice, however, jumps and kernel discontinuities are common, and a weak convergence without tractable error bounds has limited applications. More recently, [20, 21, 22] used finite-rank sub-Markov kernels (which may not be a transition kernel of an MC) to approximate $V$-geometric ergodic MC kernels that could be continuous-state. However, $V$-geometric ergodicity can be a strong assumption [51]. In addition, the authors measured approximation errors by the total variation norm. As discussed in subsection 1.1, results using the total variation norm are weaker than that based on the supremum norm.

| Classification          | Model                  | Key Assumptions                                                                 | Main Results                              | References |
|-------------------------|------------------------|---------------------------------------------------------------------------------|-------------------------------------------|------------|
| Computational algorithms| Countable state        | Ergodic level dependent quasi-birth-and-death process.                         | Efficient algorithms, accuracy validated by experiments. | [5, 48, 13]|
| Perturbation analysis   | Countable state        | Irreducibility, positive recurrence, geometric ergodicity, uniform ergodicity, etc. | Explicit error bounds.                    | [56, 50, 1, 25, 36] |
| Truncation methods      | Countable state        | Irreducibility, positive recurrence, stochastic monotonicity, etc.               | Convergence in $L^1$ norm, $V$ norm, total variation norm, etc. | [52, 53, 16, 17, 24, 19, 37] |
| Error analysis          | Countable state        | Irreducibility, positive recurrence, upper Hessener structure, modulated drift condition, etc. | Errors in $L^1$ norm, supremum norm, V norm or total variation norm. | [63, 57, 35, 39, 38] |
| Block-based methods     | Countable state        | Block-structure, and block-monotonicity.                                        | Pointwise convergence, errors in supremum norm or total variation norm. | [32, 40, 41, 43] |
| Continuous state problem| Countable state        | Irreducibility, and positive recurrence.                                         | Pointwise convergence.                    | [60] |
| General methods         | Continuous state       | Transition kernel’s weak continuity and uniform weak convergence.               | Convergence in distribution.              | [49] |
| General state problem   | General state          | $V$-geometric ergodicity, weakened convergence, and uniform weak drift.          | Convergence and errors in total variation norm. | [20, 21, 22] |

Comparison with MCMC. Both MCMC and finite approximation evaluate the steady-state performance via generating a proxy stationary distribution. For the finite approximation,
the proxy stationary distribution is the approximate finite-state MC's stationary distribution, while for MCMC, the proxy stationary distribution is the empirical distribution of the sample. For MCMC, convergence rates and concentration bounds have been extensively studied, e.g., [15, 34, 61, 10, 9, 27, 7]. For a more systematic review, we refer the readers to [18, 54, 8].

1.3. Organization. We close this section with an outline of this paper. Section 2 introduces key notations and settings. Section 3 analyzes our finite approximation approach that uses an approximate finite-state MC to compute the original MC's stationary distribution. We provide error bounds for the approximate stationary distribution as well as the corresponding steady-state performance measures. Section 4 shows that under certain regularity conditions, the stationary distribution and steady-state performance measures computed by our approach are consistent under the supremum norm. Section 5 provides verifiable sufficient conditions ensuring the regularity conditions defined in Section 4. Section 6 illustrates the use of our approach with an example of G/G/1+G queue, and compares against benchmark results. Section 7 presents concluding remarks. All cited theorems from the literature are listed in Appendix A.

2. Preliminaries. We now present definitions and assumptions for analyzing our finite approximation. We assume support Ω = [0, 1] for the general results of this paper. Our analysis can be adapted for other support Ω′ ⊆ R, e.g., transforming an unbounded support R+ into (0, 1] via function e−x and then expanding support (0, 1] to [0, 1] via adding a (trivial) state 0. Our results (Theorem 1,2,3), conditions (Condition 1-4) and corresponding proofs can also be similarly developed for Ω′.

2.1. Notations and a Banach Space. We first construct a Banach space of right-continuous distribution functions. Let D be the collection of probability distribution functions of random variables defined on (Ω, B) with B being the Borel algebra for Ω. Define X := span(D) = \{ \sum_{k=1}^{n} a_k f_k \mid n \in \mathbb{N}_+, a_k \in \mathbb{R}, f_k \in D \}, which is also the linear space of distribution functions of all finite signed measures on (Ω, B). Let ||f||∞ be the supremum norm of f : R → R, and X be the closure of X with norm ||·||∞. Then, (X, ||·||∞) is a Banach space (Theorem 2.5 in [33]).

We next define operations on space X. We write “f_k converging to f on X” as “f_k ⇝ f” to emphasize uniform convergence. Let V(f; Γ) be the total variation of f : R → R on Γ ⊆ R:

\[ V(f; Γ) := \sup_{x_1 < x_2 < \ldots < x_n ; x_1, x_2, \ldots, x_n \in Γ ; n \in \mathbb{N}} \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})|, \]

where the supremum operation is over all partitions of Γ. For convenience, let \( V(f) := V(f; Ω) \). For function f(x, u) : R^n × R → R, we use \( V_u(f(x, u)) \) to denote the total variation of f(x, u) as a single-variable function of u with any fixed x ∈ R^n. Let I be the identity operator and ||·||O be the operator norm for an operator \( \mathcal{L} \) on X: ||\( \mathcal{L} \)||_O = sup_{f \in X, ||f||_∞ = 1} ||\( \mathcal{L} f \)||_∞.

We now define finite (transition) kernels \( T(Ω, B) \) in the distribution sense.

Definition 1 (Finite Kernel). Let T(Ω, B) be the collection of \( τ(x, u) : Ω^2 → R_+ \) such that (i) for all x ∈ Ω, \( τ(x, \cdot) \) is B-measurable, and (ii) for all u ∈ Ω, \( τ(\cdot, u) = 0 \) or there exists \( α > 0 \) such that \( ατ(\cdot, u) \in D \). Particularly, \( τ \in T(Ω, B) \) is a Markov kernel if \( τ(\cdot, u) \in D \) for all u ∈ Ω. Both finite and Markov kernels will be used to describe MCs’ transitions and balance equations. It is possible to formulate a kernel \( τ \in T(Ω, B) \) into a continuous linear operator on space X.

Definition 2 (Transition Operator). For kernel \( τ \in T(Ω, B) \), its corresponding transition operator \( \mathcal{L} \) on X is defined as follows: if \( f \in X \), then

\[ \mathcal{L} f(x) := \int_{Ω} τ(x, u) df(u), \quad \forall x ∈ Ω. \]
Otherwise, we select an arbitrary sequence \( \{f_k\}_{k \in \mathbb{N}} \subseteq X \) that converges to \( f \) on \( X \) and define
\[
\mathcal{L}f(x) := \lim_{k \to \infty} \mathcal{L}f_k(x), \quad \forall x \in \Omega.
\]
Then we can use \( f' = \mathcal{L}f \) to describe that an initial probability distribution \( f \) returns distribution \( f' \) after one transition in an MC with kernel \( \tau \). According to Theorem 3.1 of [33], transition operator \( \mathcal{L} \) in Definition 2 is a well-defined continuous linear operator iff the following holds.

**Condition 1 (Uniformly Bounded Variation).** For \( \tau \in T(\Omega, \mathcal{B}) \), \( \sup_{x \in \Omega} V_u(\tau(x,u)) < \infty \).

According to Condition 1, for all \( x \in \Omega \), \( \tau(x,u) \) as a single-variable function of \( u \) has a uniformly bounded total variation. In Definition 2, \( \mathcal{L}f \) is not defined uniformly, i.e., (1) vs. (2). This is because function \( f \in \bar{X}\setminus X \) as a distribution of measures is ill defined and the conventional definition (1) does not apply. Instead, we define (2) via convergence, which is natural because any continuous operator must have (2) as a property. With slight abuse of notation, we also write \( \mathcal{L}f(x) \) as \( \int_{\Omega} \tau(x,u)df(u) \) for all \( f \in \bar{X}\setminus X \).

### 2.2. A Finite Approximation Approach.

Our approximation approach is given below.

**Approximation 1.** Consider a discrete-time MC supported on \( \Omega \), denoted as the original MC. Its stationary distribution \( p \in D \) satisfies the following balance equations (by letting \( f = p \))

\[
f(x) = \int_{\Omega} \bar{\kappa}(x,u)df(u), \quad \forall x \in \Omega,
\]

where the Markov kernel \( \bar{\kappa} \in T(\Omega, \mathcal{B}) \). Also consider a sequence of discrete-time finite-state MCs supported on \( \Omega \), denoted as the approximate MCs and indexed by \( r \in \mathbb{N} \). For the \( r \)-th approximate MC, the states are \( 0 = c^{(r)}_1 < c^{(r)}_2 < ... < c^{(r)}_{J^{(r)}} = 1 \), the transition matrix is \( Q^{(r)} = \{q_{ij}^{(r)}\}_{i,j=1,2,...,J^{(r)}} \), and the stationary distribution is \( p^{(r)} \). We compute the original MC’s steady-state performance measure \( \int_{\Omega} \phi(x)dp(x) \) by using \( p \approx p^{(r)} \).

Section 4 will show how to properly construct the approximate MCs. We describe the \( r \)-th approximate MC’s balance equations in a distributional form, similar to (BE-1) for the original MC:

\[
f(x) = \int_{\Omega} \bar{\kappa}^{(r)}(x,u)df(u), \quad \forall x \in \Omega.
\]

Here transition kernel \( \bar{\kappa}^{(r)} \) is specified by the approximate finite-state MC’s transition matrix \( Q^{(r)} \):

\[
\bar{\kappa}^{(r)}(x,u) := \sum_{i=1}^{J^{(r)}} \bar{\omega}^{(r)}_i(x)1\{u \in (c^{(r)}_{i-1}, c^{(r)}_i]\}, \quad \forall x,u \in \Omega, r \in \mathbb{N},
\]

\[
\bar{\omega}^{(r)}_i(x) := \sum_{j=1}^{J^{(r)}} 1\{c^{(r)}_j \leq x\}q^{(r)}_{ij}, \quad x \in \Omega, i = 1, 2, ..., J^{(r)}, r \in \mathbb{N},
\]

and \( c^{(r)}_0 := -\infty \). Then \( p^{(r)} \) is a solution to (ABE-1) (by letting \( f = p^{(r)} \), \( r \in \mathbb{N} \)).

### 3. Errors of Finite Approximation.

Our main result of this section is the following bound.

**Theorem 1 (Errors in Stationary Distributions).** Consider an instance of approximate MC in Approximation 1. If the original MC’s kernel \( \kappa \) satisfies Condition 1 and the approximate
MC is irreducible, then the error \( \|p - p^{(r)}\|_\infty \leq e_1 \cdot e_2 \), where \( e_1 = \sup_{x, u \in \Omega} |\bar{\kappa}(x, u) - \bar{\kappa}^{(r)}(x, u)| \)
and \( e_2^{-1} = \min_{k=0,1,...,J^{(r)}} \{y_k^*\} \), and \( y_k^* \) is obtained from a linear program with \( a_0 \equiv 0 \) \( (k = 0, 1, ..., J^{(r)}) \):

\[
\min_{y_k \in \mathbb{R}, \{a_j\}_{j=1}^{J^{(r)}} \in [-1,1]^{J^{(r)}}} y_k \\
\text{s.t. } -y_k \leq a_j + \eta \delta_{jk}(1 - a_j) - \sum_{i=1}^{j} (1 + \sum_{s=1}^{j} q_{is}^{(r)}) \cdot (a_i - a_{i-1}) \leq y_k, \quad j = 0, 1, ..., J^{(r)}, \eta \in \{0, 1\}.
\]

Theorem 1 provides the following insight. The balance equations of both the original and approximate MCs are linear systems. The later can be regarded as a computable approximation of the former with a perturbation. Our results indicate that the approximation error, i.e., the solution error due to perturbation, is decided by a factor \( e_1 \) measuring the MC kernel difference (original vs. approximate), and factor \( e_2 \) measuring the approximate stationary distribution’s sensitivity.

Our proof is completed in three steps. First, we show that the original and approximate MCs’ balance equations can both be formulated into operator-form equations (Subsection 3.1). Next, we obtain an operator-form approximation error bound (Subsection 3.2). Lastly, we show the operator-form bound can be computed by \( e_1 \cdot e_2 \) (Subsection 3.3).

The error bounds for steady-state performance measures are provided in Subsection 3.4.

### 3.1. Transformation of Balance Equations.

We show that by adding a natural scaling constraint, balance equations of both the original and approximate MCs can be transformed into operator equations. Our transformation brings a key advantage: balance equations (BE-1) or (ABE-1) have at least one degree of freedom in its solution space, while by incorporating a natural scaling constraint, the transformed equations have a unique solution under appropriate assumptions. Our error bound and consistency analysis will both be based on this solution uniqueness.

**Lemma 1 (Balance Equation Reformulation I).** Given the settings and assumptions in Theorem 1, \( f = p \) and \( f = p^{(r)} \) are respectively solutions to

\[
f(x) = \xi(x) + \int_{\Omega} \kappa(x, u)df(u), \quad \forall x \in \Omega; \quad \text{(BE-2)}
\]
\[
f(x) = \xi(x) + \int_{\Omega} \kappa^{(r)}(x, u)df(u), \quad \forall x \in \Omega, \quad \text{(ABE-2)}
\]

where the inhomogeneous term \( \xi \in \bar{X} \) and the kernels \( \kappa, \kappa^{(r)} \in T(\Omega, \mathcal{B}) \) are specified as: \( \forall x, u \in \Omega, \)

\[
\xi(x) := -\mathbf{1}\{x \geq 0\}, \quad \kappa(x, u) := \mathbf{1}\{x \geq 0\} + \bar{\kappa}(x, u), \quad
\kappa^{(r)}(x, u) := \sum_{i=1}^{\eta} \mathbf{1}\{x \geq 0\} + \bar{\omega}_i^{(r)}(x) \cdot \mathbf{1}\{u \in (c_{i-1}^{(r)}, c_i^{(r)})\}.
\]

Equivalently (BE-2) and (ABE-2) can be stated as

\[
f = \xi + \mathcal{K}f, \quad \text{(BE-3)}
\]
\[
f = \xi + \mathcal{K}^{(r)}f, \quad \text{(ABE-3)}
\]

where operators \( \mathcal{K} \) and \( \mathcal{K}^{(r)} \) are respectively defined by Definition 2 with kernels \( \kappa \) and \( \kappa^{(r)} \).

**Proof:** Recall that \( p \) satisfies (BE-1) and an extra scaling constraint \( p(1) = 1 \) because \( p \) is a probability distribution supported on \( \Omega = [0, 1] \). Thus, \( f = p \) satisfies each of the following systems.

\[
f(1) = 1; \quad f(x) = \int_{\Omega} \bar{\kappa}(x, u)df(u), \quad \forall x \in \Omega.
\]
Thus, \( f = p \) is a solution to (BE-2). We next consider \( p^{(r)} \), which similarly satisfies (ABE-1) and an extra scaling constraint \( p^{(r)}(1) = 1 \) because \( p^{(r)} \) is a probability distribution supported on \( \Omega = [0, 1] \). Thus, \( f = p^{(r)} \) satisfies each of the following systems.

\[
\begin{align*}
\Rightarrow f(x) & = 1\{x \geq 0\} \cdot [f(1) - 1] + \int_{\Omega} \tilde{\kappa}(x, u)df(u), \forall x \in \Omega. \quad \text{(due to } 1\{x \geq 0\} \cdot [f(1) - 1] = 0) \\
\Rightarrow f(x) & = -1\{x \geq 0\} + \int_{\Omega} [1\{x \geq 0\} + \tilde{\kappa}(x, u)]df(u), \forall x \in \Omega. \quad \text{(due to } f(1) = \int_{\Omega} df(u) ) \\
\Rightarrow f(x) & = \xi(x) + \int_{\Omega} \kappa(x, u)df(u), \forall x \in \Omega. \quad \text{(due to definitions of } \xi \text{ and } \kappa) \\
\end{align*}
\]

(5)

Thus, \( f = p^{(r)} \) is a solution to (ABE-2). Lastly, we directly have the operator forms (BE-3) for (BE-2) and (ABE-3) for (ABE-2) via Definition 2. \( \square \)

### 3.2. Operator-Form Error Bound

Lemma 1 implies the following error bound.

**Lemma 2 (Operator-Form Errors in Stationary Distributions).** Given the settings and assumptions in Theorem 1, we have the following bound

\[
\|p - p^{(r)}\|_{\infty} \leq \|(I - K^{(r)})^{-1}\|_O \cdot \|Kp - K^{(r)}p\|_{\infty}. \tag{6}
\]

**Proof:** Suppose \( (I - K^{(r)})^{-1} \) exists. Because \( p = \xi + Kp \) and \( p^{(r)} = \xi + K^{(r)}p^{(r)} \) by Lemma 1,

\[
\begin{align*}
\|p - p^{(r)}\|_{\infty} & = \|(I - K^{(r)})^{-1} (p - K^{(r)}p)\|_{\infty} = \|(I - K^{(r)})^{-1} (p - \xi - K^{(r)}p)\|_{\infty} \\
& \leq \|(I - K^{(r)})^{-1} (Kp - K^{(r)}p)\|_{\infty} \leq \|(I - K^{(r)})^{-1}\|_O \cdot \|Kp - K^{(r)}p\|_{\infty},
\end{align*}
\]

which is the bound in (6). Next we show that \( (I - K^{(r)})^{-1} \) indeed exists. Define weight functions

\[
\omega_i^{(r)}(x) := 1\{x \geq 0\} + \tilde{\omega}_i^{(r)}(x) = 1\{x \geq 0\} + \sum_{j=1}^{j^{(r)}} 1\{c_j^{(r)} < x\}q_j^{(r)}, \quad x \in \Omega, i = 1, 2, ..., J^{(r)}, r \in \mathbb{N}. \tag{7}
\]

Recall the definitions of operator \( K^{(r)} \) and kernel \( \kappa^{(r)} \) in Lemma 1. We can write \( \kappa^{(r)} \) as:

\[
\kappa^{(r)}(x, u) = \sum_{i=1}^{j^{(r)}} [1\{x \geq 0\} + \tilde{\omega}_i^{(r)}(x) \cdot 1\{u \in (c_{i-1}^{(r)}, c_i^{(r)})\} = \sum_{i=1}^{j^{(r)}} \omega_i^{(r)}(x) \cdot 1\{u \in (c_{i-1}^{(r)}, c_i^{(r)})\},
\]

for all \( x, u \in \Omega, r \in \mathbb{N} \). Then we can also write operator \( K^{(r)} \) into the following equivalent form:

\[
K^{(r)} f(x) = \int_{\Omega} \kappa^{(r)}(x, u)df(u) = \sum_{i=1}^{j^{(r)}} \omega_i^{(r)}(x) \cdot [f(c_i^{(r)}) - f(c_{i-1}^{(r)})], \quad \forall x \in \Omega, r \in \mathbb{N}. \tag{8}
\]
According to Theorem 5.2 of [33], for a (finite-rank) operator \( K^{(r)} \) in the form of (8), \((I - K^{(r)})^{-1}\) exists if and only if the matrix \( H \) is non-singular, where \( H \) is defined by \( H_{ij} := \delta_{ij} - \omega_i^{(r)}(c_j^{(r)}) + \omega_{i+1}^{(r)}(c_j^{(r)}), \quad i, j = 1, 2, ..., J^{(r)}. \) Here \( \delta_{ij} := 1\{i = j\} \) \((i, j \in \mathbb{N})\) and \( \omega_j^{(r)}(x) := 0, \forall x \in \mathbb{R}. \) Then we only need to show \( H \) is non-singular. By the definitions of \( \{\omega_i^{(r)}\}_{i=1}^{J^{(r)}} \) in (7), we have
\[
H_{ij} = \begin{cases} 
\delta_{ij} - \sum_{k=1}^{J^{(r)}} q_{k,j}^{(r)} + \sum_{k=1}^{J^{(r)}} q_{k+1,j}^{(r)}, & i, j \in \{1, 2, ..., J^{(r)}\}; i \neq J^{(r)}, \\
\delta_{ij} - \sum_{k=1}^{J^{(r)}} q_{k,i}^{(r)} - 1, & i, j \in \{1, 2, ..., J^{(r)}\}; i = J^{(r)}.
\end{cases}
\]

Let us define another matrix \( M \), obtained by replacing the first row of \( I - Q^{(r)^T} \) with \( 1^{T} \in \mathbb{R}^{J^{(r)}}: \)
\[
M_{ij} = \begin{cases} 
\delta_{ij} - q_{ij}, & i, j \in \{1, 2, ..., J^{(r)}\}; i \neq 1, \\
1, & i, j \in \{1, 2, ..., J^{(r)}\}; i = 1.
\end{cases}
\]

Because the approximate finite-state MC is irreducible, \( \text{rank}(M) = \text{rank}(I - Q^{(r)^T}) + 1 = J^{(r)}. \) It is easy to see that \( H^{T} \) can be transformed into \( M \) by matrices' elementary row and column operations. Thus, \( H \) is non-singular, \((I - K^{(r)})^{-1}\) exists and error bound (6) holds. \( \square \)

According to Lemma 1, error factor \( \|(I - K^{(r)})^{-1}\|_{O} \) in (6) measures the approximate stationary distribution's sensitivity (in the inhomogeneous term \( \xi \)), while error factor \( \|Kp - \bar{K}^{(r)}p\|_{\infty} \) measures the kernel difference between the original and approximate MCs.

### 3.3. Error Bound Computations

We complete the proof of Theorem 1 by showing that the operator-form error bound in Lemma 2 is further bounded by \( e_1 \cdot e_2 \), which is computable. The proof utilizes the following lemma.

**Lemma 3 (Theorem 5.3, [33]).** For an instance of \( K^{(r)} \) in Lemma 2, \( \|(I - K^{(r)})^{-1}\|_{O} = \min_{k=0,1,...,J^{(r)}} \{ y_k \} \), where \( y_k \) is obtained from a linear program with \( a_0 \equiv 0 \) \((k = 0, 1, ..., J^{(r)})\):

\[
\min_{y_k \in \mathbb{R}, \{ a_j \}_{j=1}^{J^{(r)}} \in [-1,1]^{J^{(r)}}} y_k \\
\text{s.t.} \quad -y_k \leq a_j + \eta \delta_{jk}(1-a_j) - \sum_{i=1}^{J^{(r)}} \omega_i^{(r)}(c_j^{(r)})[a_i - a_{i-1}] \leq y_k, \quad j = 0, 1, ..., J^{(r)}, \eta \in \{0, 1\}.
\]

**Proof of Theorem 1:** We first show that in the operator-form error bound (6), factor \( \|Kp - \bar{K}^{(r)}p\|_{\infty} \leq e_1 \). Because \( \|Kp(x) - \bar{K}^{(r)}p(x)\| = \left\| \int_{\Omega} \kappa(x,u)dp(u) - \int_{\Omega} \bar{K}^{(r)}(x,u)dp(u) \right\| \leq \int_{\Omega} \|\kappa(x,u) - \bar{K}^{(r)}(x,u)\| dp(u) \leq \sup_{x,u \in \Omega} \|\kappa(x,u) - \bar{K}^{(r)}(x,u)\|, \forall x \in \Omega, \) we have that
\[
\|Kp - \bar{K}^{(r)}p\|_{\infty} \leq \sup_{x,u \in \Omega} \|\kappa(x,u) - \bar{K}^{(r)}(x,u)\| \leq \sup_{x,u \in \Omega} \|\kappa(x,u) - \bar{K}^{(r)}(x,u)\| \leq e_1.
\]  

With regards to factor \( \|(I - K^{(r)})^{-1}\|_{O} \), it can be obtained by solving a series of linear programs as in Lemma 3. Because \( \omega_i^{(r)} = \bar{\omega}_i^{(r)} + 1 \) \((on \text{ domain } \Omega)\) for \( i = 1, 2, ..., J^{(r)} \) according to (7), we have that \( \|(I - K^{(r)})^{-1}\|_{O} = e_2. \) Therefore, \( \|p - \bar{p}\|_{\infty} \leq \|(I - K^{(r)})^{-1}\|_{O} \cdot \|Kp - \bar{K}^{(r)}p\|_{\infty} \leq e_1 \cdot e_2. \) \( \square \)

**Remark 1 (weaker conditions for Theorem 1).** The proof of Lemma 2 indicates that a necessary and sufficient condition for the error bound in (6) (or Theorem 1) is the non-singularity of the matrix \( M \) (rather than the approximate MC's irreducibility). \( M \) is obtained by replacing the first row of \( I - Q^{(r)^T} \) with \( 1^{T} \in \mathbb{R}^{J^{(r)}}. \) This further implies the following necessary and sufficient condition for Theorem 1: the approximate MC has an absorbing communicating class. For a brief proof, under this condition, \( \text{Ker}(I - Q^{(r)^T}) = \{\alpha \pi^{(r)}| \alpha \in \mathbb{R}\}, \) where \( \pi^{(r)} \) is the stationary probability vector. Thus, \( M \) is non-singular. On the other hand, if the approximate MC has two communicating classes not connected, then \( \text{rank}(M) \leq \text{rank}(I - Q^{(r)^T}) + 1 \leq (J^{(r)} - 2) + 1 < J^{(r)}, \) i.e., \( M \) is singular.
3.4. Errors in Steady-State Performance Measures. We now provide error bounds of performance measures computed by the approximate stationary distributions. Such an error can be expressed as \( \int_{\Omega} g(u) \, d\Delta p(u) = \int_{\Omega} g(u) \, dp - p'(u) \), where \( g \) is the performance function and \( p' \) is the approximate stationary distribution. Our error bound is based on the following observation: when we compute \( \int_{\Omega} g(u) \, d\Delta p(u) \), the integral is bounded by the total variation \( V(\Delta p) \) times the bound \( ||g||_{\infty} \). Using integral by parts, we switch the order of \( \Delta p \) and \( g \). In this way, the error is expected to be bounded by \( V(g) \) times \( ||\Delta p||_{\infty} \):

**Theorem 2 (Steady-State Performance Measure Errors).** For distributions \( p, p' \in D \), and a measurable function \( g(x) \) on \((\Omega, B)\) with a bounded total variation \( V(g) < \infty \), we have that

\[
\left| \int_{\Omega} g(x) dp(x) - \int_{\Omega} g(x) dp'(x) \right| \leq a V(g) \cdot ||p - p'||_{\infty},
\]

where \( a = 2 \). If \( g \) is continuous, \( a = 1 \).

**Proof:** Without loss of generality, we assume \( g \) is “centralized” such that \( g(1) = 0 \). We next extend \( g(x) \) such that \( g(x) = g(0), \forall x < 0 \) and \( g(x) = g(1), \forall x > 1 \). Note that integrations in \( (10) \) are also valid in the sense of Lebesgue–Stieltjes measure: let \( dh \) be the Lebesgue–Stieltjes measure associated with \( h: \mathbb{R} \rightarrow \mathbb{R} \). Then \( \int_{\Omega} g(x) dp(x) = \int_{\Omega} g(x) dp(x) \) and \( \int_{\Omega} g(x) dp'(x) = \int_{\Omega} g(x) dp'(x) \).

Define \( g_C(x) = \frac{g(x+1) + g(x+1)}{2} \), \( g_R(x) = g(x+1) \), \( p_C(x) = \frac{p(x+1) + p(x+1)}{2} \) and \( p'_C(x) = \frac{p'(x+1) + p'(x+1)}{2} \) for all \( x \in \Omega \). Then due to integral by parts (Theorem 21.67 in [23]), \( \int_{\Omega} g(x) dp(x) + \int_{\Omega} p'(x) dg_R(x) = g_C(1)p_C(1) - g_C(0)p_C(0) = 0 \). Similarly, \( \int_{\Omega} g_C(x) dp'(x) + \int_{\Omega} p_C'(x) dg_R(x) = 0 \). Thus, the steady-state performance measure error is bounded by

\[
\begin{align*}
\left| \int_{\Omega} g(x) dp(x) - \int_{\Omega} g(x) dp'(x) \right| & = \left| \int_{\Omega} [g(x) - g_C(x)] dp(x) - \int_{\Omega} p_C(x) dg_R(x) - \int_{\Omega} [g(x) - g_C(x)] dp'(x) - \int_{\Omega} p_C'(x) dg_R(x) \right| \\
& \leq \left| \int_{\Omega} [g(x) - g_C(x)] dp(x) \right| + \left| \int_{\Omega} [g(x) - g_C(x)] dp'(x) \right| + \left| \int_{\Omega} p_C(x) dg_R(x) \right| + \left| \int_{\Omega} p_C'(x) dg_R(x) \right|
\end{align*}
\]

The second term is bounded by \( V(g) \cdot ||p - p'||_{\infty} \). The first term is 0 if \( g \) is continuous. Otherwise,

\[
\begin{align*}
\left| \int_{\Omega} [g(x) - g_C(x)] dp(x) \right| &= \left| \int_{\Omega} \sum_{x \in J_1 \cup J_2} [g(x) - g_C(x)] \cdot [(p(x) - p(x-)) - (p'(x) - p'(x-))] \right| \\
& \leq \sum_{x \in J_1 \cup J_2} |g(x) - g_C(x)| \cdot ||p - p'||_{\infty} \leq \frac{V(g)}{2} \cdot ||p - p'||_{\infty} \leq V(g) \cdot ||p - p'||_{\infty}
\end{align*}
\]

where \( J_1 = \{ x \in \Omega \mid g(x-) \neq g(x+) \} \) and \( J_2 = \{ x \in \Omega \mid g(x-) = g(x+) \neq g(x) \} \). The transform from integral to summation in the second row is because \( V(g) < \infty \), and sets \( J_1 \) and \( J_2 \) are both countable. The inequality \( \sum_{x \in J_1 \cup J_2} |g(x) - g_C(x)| \leq \frac{V(g)}{2} \) is because every unit of difference in \( |g(x) - g_C(x)| \) yields at least two units of total variation in \( V(g) \). \( \square \)

4. Consistency of Finite Approximation. Theorem 1 has decomposed the finite approximation error into two factors: the kernel difference (original vs. approximate) and the approximate stationary distribution’s sensitivity. In this section, we show that under certain regularity conditions, the sensitivity measure is bounded while the kernel difference measure converges to 0. (Section 5 will provide verifiable conditions that ensuring these regularity conditions.)
THEOREM 3 (Consistency in Stationary Distributions). In Approximation 1, if the original MC’s kernel \( \bar{\kappa} \) satisfies Condition 1-3 given below, and the approximate MCs satisfy Condition 4 given below, then there exists \( r_0 \in \mathbb{N} \) such that for all \( r > r_0 \), stationary distribution \( p^{(r)} \) is unique, the approximation error bound in (6) holds and the bound converges to 0, i.e., \( p^{(r)} \rightarrow p \) as \( r \to \infty \).

The definitions and interpretations of Condition 1-4 are as follows.

Condition 1-2 for the original MC’s kernel \( \bar{\kappa} \) are designed such that it is possible to find a finite set of “representative” states to replace \( \Omega \) when computing stationary distributions. Condition 1 has been defined in Subsection 2.1. The intuition behind Condition 1 that to make finite approximation possible, the original MC’s transitions from the current state \( u \) should vary within limit as \( u \) varies from 0 to 1. Similarly, the original MC’s transitions to the next state \( x \) should vary within limit as \( x \) varies from 0 to 1: this motivates the following regularity condition for kernel \( \bar{\kappa} \).

CONDITION 2 (Càdlàg and Countable Jump Discontinuities). Consider kernel \( r \in T(\Omega, \mathcal{B}) \). For all \( \epsilon > 0 \), there exists a finite split of \( \Omega \) denoted by knots \( 0 = s_1 < s_2 < \ldots < s_N = 1 \) and intervals \( E_1 = [s_1, s_2), E_2 = [s_2, s_3), \ldots, E_N = [s_N, \infty) \) such that \( \forall x_1, x_2 \in E_i, i = 1, 2, \ldots, N \), we have \( \zeta_r(x_1, x_2) < \epsilon \) and \( v_r(x_1, x_2) < \epsilon \), where \( \zeta_r(\cdot, \cdot) \) and \( v_r(\cdot, \cdot) \) are distance functions respectively defined as \( v_r(x_1, x_2) := V_r(\tau(x_2, u) - \tau(x_1, u)) \) and \( \zeta_r(x_1, x_2) := |\tau(x_2, 1) - \tau(x_1, 1)|, \forall x_1, x_2 \in \Omega \).

For \( \bar{\kappa} \) satisfying Condition 2, the split knots divide support \( \Omega \) into intervals, on each of which the original MC’s transitions have a small variation as desired. Thus, for all \( \epsilon \), the corresponding split knots defined by Condition 2 can be used as a group of “representative” states. (Condition 2 is referred to as “càdlàg and countable jump discontinuities” because it describes the property of limit existence, right continuity and countable jumps for \( \tau(x, \cdot) \) respectively under the distance functions \( v_r(\cdot, \cdot) \) and \( \zeta_r(\cdot, \cdot) \) w.r.t. \( x \).)

We also require that the original MC has a unique stationary distribution.

CONDITION 3 (Unique Stationary Distribution). In Approximation 1, the original MC’s stationary distribution \( p \) is unique and the solution set of (BE-1) in \( \bar{\mathcal{X}} \) is \( \{\alpha p\}_{\alpha \in \mathbb{R}} \).

With regard to the approximate finite-state MC sequence, we assume that (a) their states are “representative” states as the split knots outlined in Condition 2, and (b) their transition matrices are defined by properly truncating the original MC’s kernel.

CONDITION 4 (Proper Truncation). In Approximation 1, we have that:

(i) (Representative states) There exist \( \{e^{(r)}\}_{r \in \mathbb{N}} \subseteq \mathbb{R}_+ \) such that \( \forall r \in \mathbb{N}, \forall x_1, x_2 \in [c_i^{(r)}, c_i^{(r+1)}] \cap \Omega, i = 1, 2, \ldots, J^{(r)} \), we have \( \zeta_r(x_1, x_2) < e^{(r)} \) and \( v_r(x_1, x_2) < e^{(r)} \). In other words, \( \{c_i^{(r)}\}_{i=1}^{J^{(r)}} (r \in \mathbb{N}) \) are defined by a sequence of finite splits for \( \bar{\kappa} \) in Condition 2 as \( \epsilon \to 0 \).

(ii) (Increasing partitions) \( \{c_i^{(r)}\}_{i=1}^{J^{(r)}} \subseteq \{c_i^{(r+1)}\}_{i=1}^{J^{(r+1)}}, \forall r_1 < r_2, r_1, r_2 \in \mathbb{N} \).

(iii) (Kernel truncation) \( q^{(r)}_{ij} = \bar{\kappa}(c_i^{(r)}, c_i^{(r)}), \forall i, j = 1, 2, \ldots, J^{(r)}, r \in \mathbb{N} \).

(iv) (Pointwise convergence) \( \lim_{r \to \infty} \bar{\kappa}^{(r)}(x, u) = \bar{\kappa}(x, u), \forall u, x \in \Omega \).

Theorem 2 and 3 imply the consistency of approximate steady-state performance measures.

COROLLARY 1 (Consistency in Steady-State Performance Measures). Given the settings and assumptions in Theorem 3 and a measurable performance function \( q(x) \) on \( (\Omega, \mathcal{B}) \) such that \( V(g) < \infty \), the steady-state performance measure associated with the approximate MC converges to that associated with the original MC. When we consider a set of performance functions \( \{g_i\}_{i \in \mathcal{S}} \) on \( (\Omega, \mathcal{B}) \) such that \( \sup_{i \in \mathcal{S}} V(g_i) < \infty \), the steady-state performance measures associated with the approximate MC uniformly converge to those associated with the original MC.
4.1. Proof Outline. Our proof is based on the following lemma and balance equations’ operator reformulation in Lemma 1, which indicates that \( f = p \) is a solution to \( f = \xi + Kf \) while \( f = p^{(r)} \) is a solution to \( f = \xi + K^{(r)}f \) \((r \in \mathbb{N})\).

**Lemma 4** (Lemma 2.3 of [33]). Consider function \( h \in X \) and linear operators \( \{K\} \cup \{K^{(r)}\}_{r \in \mathbb{N}} \) on \( X \). Suppose \( f = p \) and \( f = p^{(r)} \) \((r \in \mathbb{N})\) are respectively solutions to equations \( f = h + Kf \) and \( f = h + K^{(r)}f \) \((r \in \mathbb{N})\) on \( X \). If \( \{K\} \cup \{K^{(r)}\}_{r \in \mathbb{N}} \) further satisfy:

(A1) \( \{K^{(r)}\}_{r \in \mathbb{N}} \) are collectively compact, i.e., the set \( \{K^{(r)}f \mid r \in \mathbb{N}, f \in X, \|f\|_\infty \leq 1\} \) is relatively compact,

(A2) \( \{K^{(r)}\}_{r \in \mathbb{N}} \) are consistent operators to \( K \), i.e., \( K^{(r)}f \rightleftharpoons Kf, \forall f \in X \), and

(A3) \( (I - K)^{-1} \) exists,

then we have the following results:

(B1) there exists \( r_0 \in \mathbb{N} \) such that \( \forall r > r_0, p^{(r)} \) is unique and follows the error bound in (6),

(B2) consistency, i.e., \( p^{(r)} \rightleftharpoons p \), and

(B3) stability, i.e., \( \{\| (I - K^{(r)})^{-1} \|_\infty \mid r > r_0 \} \) is bounded.

Note (B1)-(B2) include all results of Theorem 3. Thus, we only need to show (A1)-(A3) to prove Theorem 3 according to Lemma 4. Our proof can be completed in two steps: in Subsection 4.2, we prove that Conditions 1, 2 and 4 imply collective compactness (A1) and operator consistency (A2). In Subsection 4.3, we prove that Condition 3 implies invertibility (A3).

Particularly, we show that under appropriate assumptions, our finite approximation approach is near-optimal in approximating the original MC’s stationary distribution among all approximation methods using discrete distributions (Subsection 4.4).

4.2. Collectively Compact and Consistent Operators. We use the (finite-rank) structure of \( \{K^{(r)}\}_{r \in \mathbb{N}} \) described in (8) and the following Lemma to show that Condition 1, 2 and 4 imply collective compactness (A1) and operator consistency (A2).

**Lemma 5** (Theorem 3.5 in [33]). For operators \( \{K\} \cup \{K^{(r)}\}_{r \in \mathbb{N}} \) as defined in Lemma 1, \( \{K^{(r)}\}_{r \in \mathbb{N}} \) are collectively compact and consistent operators iff

(i) (Pointwise convergence) \( \forall f \in X, \forall x \in \Omega, \lim_{k \to \infty} K^{(r)}f(x) = Kf(x) \).

(ii) (Uniform cadlag and countable jump discontinuities) for all \( \epsilon > 0 \), there exists a finite split of \( \Omega \) denoted by knots \( 0 = s_1 < s_2 < \ldots < s_N = 1 \) and intervals \( E_1 = [s_1, s_2], E_2 = [s_2, s_3], \ldots, E_{N-1} = \{s_N\} \) such that \( \forall r \in \mathbb{N} \), \( x, x_2 \in E_i, i = 1, 2, \ldots, N, \) we have \( \sum_{i=1}^{J(r)} |\Delta \omega_{i}^{(r)}(x_2) - \Delta \omega_{i}^{(r)}(x_1)| \leq \epsilon \). Here \( \Delta \omega_{i}^{(r)}(x) = \omega_{i+1}^{(r)}(x) - \omega_{i}^{(r)}(x), x \in \Omega, i = 1, 2, \ldots, J^{(r)}, r \in \mathbb{N} \); and particularly, \( \omega_{J^{(r)}+1}^{(r)}(x) \equiv 0, \forall r \in \mathbb{N} \).

**Proof of (A1)-(A2) in Lemma 4:** (Pointwise convergence.) We use dominated convergence. Recall definitions of \( \kappa = \kappa + 1 \) and \( \kappa^{(r)} = \kappa^{(r)} + 1 \) \((r \in \mathbb{N})\) on domain \( \Omega^{2} \) in Lemma 1. Then (iv) of Condition 4 implies \( \lim_{r \to \infty} \kappa^{(r)}(x, u) = \kappa(x, u), \forall x, u \in \Omega \). For all \( f \in X \), \( \kappa^{(r)}f \) can also be written as \( \kappa^{(r)}F(x) = \int_{\Omega} \kappa^{(r)}(x, u)df(u), \forall x \in \Omega, r \in \mathbb{N} \). Thus, for all \( f \in D \) and \( x \in \Omega \), by the dominated convergence, we have \( \lim_{r \to \infty} \kappa^{(r)}f(x) = \int_{\Omega} \lim_{r \to \infty} \kappa^{(r)}(x, u)df(u) = \int_{\Omega} \kappa(x, u)df(u) = Kf(x) \).

By the linearity, for all \( f \in X \), \( \lim_{r \to \infty} \kappa^{(r)}f(x) = Kf(x), \forall x \in \Omega \).

We next expand this result to \( f \in X \). Consider \( f = f^{'} + e, f \in X, f^{'} \in X \) and \( ||e||_{\infty} < \epsilon \). Here \( f^{'} \) can be regarded as an approximation to \( f \) in the subspace of \( X \) with a bounded error \( \epsilon \). For all \( x \in \Omega \), we have already proven \( \lim_{r \to \infty} \kappa^{(r)}f^{'}(x) = Kf^{'}(x) \). We only need to prove \( \lim_{r \to \infty} \limsup_{r \to \infty} |\kappa^{(r)}e(x) - \kappa e(x)| = 0 \). Assume there exists a constant \( C < \infty \), whose existence will be proven later, such that \( ||\kappa^{(r)}h - Kh||_{\infty} \leq C ||h||_{\infty} \) for all \( h \in X, r \in \mathbb{N} \). Then \( \lim_{r \to \infty} \limsup_{r \to \infty} |\kappa^{(r)}e(x) - \kappa e(x)| \leq \lim_{r \to \infty} C \epsilon = 0 \). Therefore, \( \lim_{r \to \infty} \kappa^{(r)}f(x) = Kf(x) \).

Lastly, we prove that there exists a constant \( C < \infty \) such that \( ||\kappa^{(r)}h - Kh||_{\infty} \leq C ||h||_{\infty} \) for all \( h \in X, r \in \mathbb{N} \). According to equation (SM2.8) in Theorem 3.1 of [33], for any kernel \( \tau \in T(\Omega, \mathcal{B}) \) satisfying Condition 1 and its associated operator \( \mathcal{L} \) in Definition 2, we have \( ||\mathcal{L}h||_{\infty} \leq ||\mathcal{L}||_{\infty} ||h||_{\infty} \leq ||h||_{\infty} \).
sup_{x \in \Omega} [2V_\nu(\tau(x,u) + \tau(x,1))] for all \( h \in \bar{X} \). Recall operators \( \mathcal{K} \) and \( \mathcal{K}^{(r)} (r \in \mathbb{N}) \) are respectively associated with kernels \( \kappa \) and \( \kappa^{(r)} (r \in \mathbb{N}) \). Then we have
\[
||\mathcal{K}h - \mathcal{K}^{(r)}h||_\infty \leq ||\mathcal{K}h||_\infty + ||\mathcal{K}^{(r)}h||_\infty \\
\leq ||h||_\infty \cdot \sup_{x \in \Omega} \left[ 2V_\nu(\kappa(x,u)) + \kappa(x,1) \right] + ||h||_\infty \cdot \sup_{x \in \Omega} \left[ 2V_\nu(\kappa^{(r)}(x,u)) + \kappa^{(r)}(x,1) \right] \\
\leq ||h||_\infty \cdot \sup_{x \in \Omega} \left[ 2V_\nu(\kappa(x,u)) + 2 \right] + ||h||_\infty \cdot \sup_{x \in \Omega} \left[ 2V_\nu(\kappa(x,u)) + 2 \right].
\]

Note here we use the facts \( \kappa(x,u) \leq \bar{\kappa}(x,u) + 1 \leq 2, \kappa^{(r)}(x,u) \leq \bar{\kappa}^{(r)}(x,u) + 1 \leq 2, \forall x, u \in \Omega \); and \( \sup_{x \in \Omega} V_\nu(\kappa^{(r)}(x,u)) = \sup_{x \in \Omega} V_\nu(\bar{\kappa}^{(r)}(x,u)) = \sup_{x \in \Omega} V_\nu(\kappa(x,u)) = \sup_{x \in \Omega} V_\nu(\bar{\kappa}(x,u)) \) due to (iii) in Condition 4. Thus, we can define \( C := 4 \sup_{x \in \Omega} V_\nu(\kappa(x,u)) + 4 = 4 \sup_{x \in \Omega} V_\nu(\bar{\kappa}(x,u)) + 4 \). Because \( \bar{\kappa} \) satisfies Condition 1, \( C < \infty \).

(Uniform càdlàg and countable jump discontinuities) Because \( \bar{\kappa} \) is \(+1\) on domain \( \Omega^2 \), the kernel \( \bar{\kappa} \) in (i) of Condition 4 can be replaced by \( \kappa \). Recall \( \epsilon^{(r)} (r \in \mathbb{N}) \) is associated with \( \{c_i^{(r)}\}_{i=1}^{J^{(r)}} \) in (i) of Condition 4. For all \( \epsilon > 0 \), we show that with \( r_1 = \inf \{ r \in \mathbb{N} | \epsilon^{(r)} < \frac{\epsilon}{2} \} \), knots \( \{c_i^{(r_1)}\}_{i=1}^{J^{(r_1)}} \) form a finite split satisfying (ii) of Lemma 5 with \( \epsilon \). Indeed, (a) for \( r \leq r_1 \), \( \forall x_1, x_2 \in [c_i^{(r_1)}, c_{i+1}^{(r_1)}) \cap \Omega, i = 1, 2, ..., J^{(r_1)} \), we have that \( x_1, x_2 \in [c_i^{(r)}, c_{i+1}^{(r)}) \cap \Omega \) for some \( i \in \{1, 2, ..., J^{(r)}\} \) due to the increasing partition property in (ii) of Condition 4 and \( \omega^{(r)}(x_2) = \omega^{(r)}(x_1) \) for \( i = 1, 2, ..., J^{(r_1)} + 1 \). Thus, \( \sum_{i=1}^{J^{(r)}} |\Delta \omega^{(r)}(x_2) - \Delta \omega^{(r)}(x_1)| = 0 \). (b) For \( r > r_1 \), we first define function \( c^{(r)}(x) := \sum_{i=1}^{J^{(r_1)}} \bar{c}_i^{(r_1)} \{x \in [c_i^{(r_1)}, c_{i+1}^{(r_1)}) \cap \Omega, i = 1, 2, ..., J^{(r_1)} \} \), we have that
\[
\sum_{i=1}^{J^{(r)}} |\Delta c^{(r)}(x_2) - \Delta c^{(r)}(x_1)| = V_\nu(\kappa^{(r)}(x_2, u) - \kappa^{(r)}(x_1, u)) + |\kappa^{(r)}(x_2, 1) - \kappa^{(r)}(x_1, 1)| \\
= \sup_{x \in \Omega} V_\nu(\kappa^{(r)}(c^{(r)}(x_2), u) - \kappa^{(r)}(c^{(r)}(x_1), u)) + |\kappa^{(r)}(c^{(r)}(x_2), 1) - \kappa^{(r)}(c^{(r)}(x_1), 1)| \\
\leq V_\nu(\kappa^{(r)}(c^{(r)}(x_2), u) - \kappa^{(r)}(c^{(r)}(x_1), u)) + |\kappa^{(r)}(c^{(r)}(x_2), 1) - \kappa^{(r)}(c^{(r)}(x_1), 1)| \\
\leq V_\nu(\kappa^{(r)}(c^{(r)}(x_2), c^{(r)}(x_1))) + \zeta(\kappa^{(r)}(c^{(r)}(x_2), c^{(r)}(x_1))) \leq 2c^{(r)} \leq \epsilon,
\]
where the last row follows \( c^{(r)}(x_2), c^{(r)}(x_1) \in [c_i^{(r_1)}, c_{i+1}^{(r_1)}) \cap \Omega \) for some \( i \in \{1, 2, ..., J^{(r)}\} \) due to the increasing partition property in (ii) of Condition 4. Recall \( \forall x_1, x_2 \in [c_i^{(r)}, c_{i+1}^{(r)}) \cap \Omega, i = 1, 2, ..., J^{(r)} \), we have \( \nu_{c^{(r)}}(x_1, x_2) \leq 2c^{(r)} \). Thus, \( \nu_{c^{(r)}}(c^{(r)}(x_2), c^{(r)}(x_1))) \leq 2c^{(r)} \leq \epsilon \). Lastly, due to (a) and (b), the finite split \( \{c_i^{(r_1)}\}_{i=1}^{J^{(r_1)}} \) satisfies (ii) of Lemma 5 with \( \epsilon \).

4.3. Invertibility. We use the following Lemma to show that Condition 3 implies (A3).

Lemma 6 (Lemma 2.4 and Theorem 3.4 in [33]). For kernel \( \kappa \) and its associated operator \( \mathcal{K} \) in Lemma 1, if \( \kappa \) satisfies Condition 1-2 and \( \mathcal{K} \) satisfies Ker(\( \mathcal{I} - \mathcal{K} \)) = \{0\}, then (\( \mathcal{I} - \mathcal{K} \))\(^{-1} \) exists.

This lemma is based on the Riesz-Schauder’s theorem: Condition 1-2 for \( \kappa \) imply the compactness of \( \mathcal{K} \), and Ker(\( \mathcal{I} - \mathcal{K} \)) = \{0\} implies the existence of (\( \mathcal{I} - \mathcal{K} \))\(^{-1} \) by the Riesz-Schauder’s theorem.

Proof of (A3) in Lemma 4: We only need to show (i) \( \kappa \) satisfies Condition 1-2 and (ii) Ker(\( \mathcal{I} - \mathcal{K} \)) = \{0\}. Verifying (i) is not hard since \( \kappa = \bar{\kappa} + 1 \) (on domain \( \Omega^2 \)) and \( \kappa \) satisfies Condition 1-2.

To prove (ii), we only need to show that (BE-3) has a unique solution in \( X \). To prove the solution uniqueness, we use the fact that (5) is reversible, i.e., for all \( f \in \bar{X} \), we have
\[
f(1) = 1; f(x) = \int_\Omega \bar{\kappa}(x,u)df(u), \forall x \in \Omega. \Leftrightarrow f(x) = \xi(x) + \int_\Omega \kappa(x,u)df(u), \forall x \in \Omega.
\]
(The proof will be presented shortly.) The system of equations on the right hand side (RHS) is exactly (BE-2), while the left hand side (LHS) has a unique solution \( p \in \bar{X} \) due to Condition 3. Thus, (BE-2) as well as its operator form (BE-3) has a unique solution \( p \in \bar{X} \).
Lastly, we show the equivalence in (11) to complete the proof.

(⇒) Consider any fixed \( f \in X \). The LHS of (11) implies \( f(x) = 1\{x \ge 0\} \cdot [f(1) - 1] + \int_{\Omega} \hat{\kappa}(x,u) df(u) = \xi(x) + f(1) + \int_{\Omega} \hat{\kappa}(x,u) df(u) \) for all \( x \in \Omega \). Because \( \kappa = \bar{\kappa} + 1 \) (on domain \( \Omega^2 \)) according to Lemma 1, we only need to prove \( f(1) = \int_{\Omega} df(u) \) to obtain the RHS. Indeed, if \( f \in X \), then \( f(1) = \int_{\Omega} df(u) \) because \( f \) defines a finite signed measure. If \( f \in X \setminus X \), then \( f(1) = \lim_{k \to \infty} f_k(1) = \lim_{k \to \infty} \int_{\Omega} df_k(u) \), where \( \{f_k\}_{k \in \mathbb{N}} \subseteq X \) is any sequence converging to \( f \) and the last equality is due to (2).

(⇐) Consider any fixed \( f \in X \). We have shown \( f(1) = \int_{\Omega} df(u) \) in (⇒). Thus, the RHS implies

\[
\int_{\Omega} df(u) = -1 + f(1) + \int_{\Omega} \tilde{\kappa}(x,u) df(u), \quad \forall x \in \Omega. \tag{12}
\]

Consider \( x = 1 \). We have \( f(1) = -1 + f(1) + \int_{\Omega} df(u) = -1 + 2f(1) \). Thus, \( f(1) = 1 \). Plugging this into (12) yields \( f(x) = \int_{\Omega} \tilde{\kappa}(x,u) df(u), \forall x \in \Omega \), which constitutes the LHS with \( f(1) = 1 \).

Now we complete the proof of Theorem 3 with results in Subsection 4.2.4.3.

**Proof of Theorem 3:** In Subsection 4.2.4.3, we have shown (A1)-(A3) in Lemma 4 hold. Thus, (B1)-(B2) in Lemma 4 also hold, i.e., we have the results of Theorem 3.

### 4.4. Near-Optimality of Finite Approximation.

We make the following assumption.

**Assumption 1.** In Approximation 1, we have (i) kernel \( \tilde{\kappa} \) is Lipschitz continuous on \( (x,u) \in \Omega^2 \), and (ii) there exists a small interval \( [x_1, x_2] \subseteq \Omega \) on which \( p \) is continuous and strictly increasing.

We let \( \delta := p(x_2) - p(x_1) > 0 \) denote the steady-state probability associated with the small interval in Assumption 1. Without loss of generality, we assume the approximate MC’s’ states \( \{c_i^{(r)} | i = 1, 2, ..., J^{(r)} \} \subseteq \mathbb{N} \) are density in \( \Omega \) and \( \max_{i=2,3,...,J^{(r)}} |c_i^{(r)} - c_{i-1}| = O(\frac{1}{J^{(r)}}) \). If the denseness or the convergence rate does not hold, we may replace the definition of \( \{c_i^{(r)} \}_{i=1}^{J^{(r)}} \) with \( \{c_i^{(r)} \}_{i=1}^{J^{(r)}} \cup \{\frac{1}{2^{\sigma(r)}} | i = 0, 1, ..., 2^{\sigma(r)} \} \), where \( \sigma(r) = \inf \{i \in \mathbb{N} | 2^i + 1 \ge J^{(r)}, i \ge r \} \) for \( r \in \mathbb{N} \).

The near-optimality of finite approximation is proven below:

**Theorem 4 (Near-optimality of Finite Approximation).** Consider the settings and assumptions in Theorem 3, the finite approximate distributions \( \{p^{(r)}\}_{r \in \mathbb{N}} \), which respectively have \( J^{(r)} \) jumps, and any other sequence of discrete distributions \( \{\hat{p}^{(r)}\}_{r \in \mathbb{N}} \subseteq \mathcal{D} \), which respectively have \( \hat{J}^{(r)} \) jumps. If Assumption 1 holds and \( \lim_{r \to \infty} \frac{J^{(r)}}{\hat{J}^{(r)}} = 1 \), i.e., the two sequences of discrete distributions have asymptotically the same number of jumps, then \( ||p^{(r)} - p||_\infty = O(||\hat{p}^{(r)} - p||_\infty) \), i.e., as approximators to \( p \), \( \{p^{(r)}\}_{r \in \mathbb{N}} \) are at least as accurate as \( \{\hat{p}^{(r)}\}_{r \in \mathbb{N}} \) by a constant multiplier.

**Proof:** We only need to show \( ||p^{(r)} - p||_\infty = O(\frac{1}{J^{(r)}}) \) while \( ||\hat{p}^{(r)} - p||_\infty \ge \frac{\delta}{2(J^{(r)} + 1)} \). Then

\[
||p^{(r)} - p||_\infty = O(||\hat{p}^{(r)} - p||_\infty) \quad \text{due to} \quad \lim_{r \to \infty} \frac{J^{(r)}}{\hat{J}^{(r)}} = 1.
\]

To bound \( ||\hat{p}^{(r)} - p||_\infty \), we only need to consider the small interval in Assumption 1, on which \( p \) is continuous and strictly increasing. For any discrete approximate distribution with \( J \) jumps, the approximation error on this interval is at least \( \frac{\delta}{2(J+1)} \) because \( p \) is continuous on this interval. Then the overall error is at least \( \frac{\delta}{2(J+1)} \). Thus, \( ||\hat{p}^{(r)} - p||_\infty \ge \frac{\delta}{2(J^{(r)} + 1)} \).

Lastly, we consider \( ||p^{(r)} - p||_\infty \). The approximation error bound is given in (6) of Lemma 2. Factor \( ||(I - K^{(r)})^{-1}||_\infty \) uniformly bounded according to (B3) of Lemma 4. Factor \( ||Kp - K^{(r)}p||_\infty \) is bounded by \( \sup_{x,u \in \Omega} |\tilde{\kappa}(x,u) - \tilde{\kappa}^{(r)}(x,u)| \) according to (9). Recall the definition of \( \tilde{\kappa}^{(r)} \) under Condition 4, which can be regarded as a step-wise function approximating \( \tilde{\kappa} \). Because \( \tilde{\kappa}(x,u) \) is Lipschitz continuous and \( \max_{i=2,3,...,J^{(r)}} |c_i^{(r)} - c_{i-1}| = O(\frac{1}{J^{(r)}}) \), we have that \( \sup_{x,u \in \Omega} |\tilde{\kappa}(x,u) - \tilde{\kappa}^{(r)}(x,u)| = O(\frac{1}{J^{(r)}}) \). In this way, the approximation error \( ||p^{(r)} - p||_\infty \) is also \( O(\frac{1}{J^{(r)}}) \).

According to Theorem 4, when we compute the original MC’s stationary distribution, the finite approximate distributions are at least as accurate as the empirical distributions generated by a simulation approach (e.g., discrete event simulation, MCMC, etc.) by a constant multiplier because the later are also discrete distributions.
5. Verifiable Sufficient Conditions.

5.1. Sufficient Conditions for Conditions 1, 2 and 4. Suppose \( \tau(x, u) \) is Lipschitz continuous w.r.t. \( u \), i.e., there exists \( L > 0 \) such that \(|\tau(x, u_2) - \tau(x, u_1)| \leq L|u_2 - u_1|, \forall x, u_1, u_2 \in \Omega \). Then Condition 1 holds because \( \forall x \in \Omega \), we have

\[
V_u(\tau(x, u)) = \sup_{0 \leq u_1 < u_2 < \ldots < u_n \leq 1; n \in \mathbb{N}} \sum_{i=2}^{n} |\tau(x, u_i) - \tau(x, u_{i-1})| \leq \sup_{0 \leq u_1 < u_2 < \ldots < u_n \leq 1; n \in \mathbb{N}} \sum_{i=2}^{n} L|u_i - u_{i-1}| \leq L.
\]

A stronger sufficient condition for Condition 1 is that the partial derivative \( \frac{\partial \tau(x, u)}{\partial x} \) is bounded or continuous for \((x, u) \in \Omega^2\), which yields \( V_u(\tau(x, u)) = \int_{\Omega} \left| \frac{\partial \tau(x, u)}{\partial x} \right| du \leq \sup_{x, u \in \Omega} \left| \frac{\partial \tau(x, u)}{\partial x} \right| < \infty \).

Condition 2 can be verified similarly. If \( \left| \frac{\partial \tau(x, u)}{\partial u} \right| < L' \) for all \((x, u) \in \Omega^2\), then for all \( x_1, x_2 \in \Omega \),

\[
v_r(x_1, x_2) = V_u(\tau(x_2, u) - \tau(x_1, u)) = \int_{\Omega} \left| \frac{\partial[\tau(x_2, u) - \tau(x_1, u)]}{\partial u} \right| du = \int_{\Omega} L'|x_2 - x_1| du \leq L'|x_2 - x_1|.
\]

If \( \left| \frac{\partial \tau(x, u)}{\partial x} \right| \leq L'' \) for all \( x \in \Omega \), then \( \zeta_r(x_1, x_2) = |\tau(x_2, 1) - \tau(x_1, 1)| \leq L''|x_2 - x_1| \) for all \( x_1, x_2 \in \Omega \). Let \( L''' = \max\{L', L''\} \). Then Condition 2 is satisfied: for every \( \epsilon > 0 \), we may define the corresponding split knots as \( \{0, \frac{1}{2^m}, \frac{2}{2^m}, \ldots, 1\} \), where \( n = \inf\{i \in \mathbb{N} \mid \frac{1}{2^m} \leq \epsilon \} \).

Regarding Condition 4, (i) simply requires that the approximate MCs’ states are specified in accordance with the finite splits in Condition 2. For (ii), if \( \{c_i^{(r_1)}\}_{i=1}^{j(r_1)} \) is not a subset of \( \{c_i^{(r_2)}\}_{i=1}^{j(r_2)} \), we may replace \( \{c_i^{(r_2)}\}_{i=1}^{j(r_2)} \) with \( \{c_i^{(r_1)}\}_{i=1}^{j(r_1)} \cup \{c_i^{(r_2)}\}_{i=1}^{j(r_2)} \). Item (iii) requires that the transition matrices of the approximate MCs are truncated from the original MC’s transition kernel. The verification of the limit in (iv) is not hard by using properties of \( \bar{\kappa} \) (e.g., the continuity of \( \bar{\kappa} \) may imply (iv); see an example in Subsection 5.3).

5.2. Sufficient Condition for Condition 3. We first define a class of accessible singletons.

**Definition 3 (Independent Mass Singleton).** For an original MC in Approximation 1, a singleton \( \{x_0\} \) with state \( x_0 \in \Omega \) will be called an independent mass singleton if after replacing the definition of \( \bar{\kappa}(\cdot, x_0) \) with an arbitrary \( h \in \mathbf{D} \), the MC with the modified kernel satisfies:

(i) a stationary distribution exists, and

(ii) every stationary distribution includes a mass at \( x_0 \).

If \( \{x_0\} \) is an independent mass singleton, then for all \( h \in \mathbf{D} \), there exists \( p' \in \mathbf{D} \) such that \( \int_{\{x_0\}} dp'(u) > 0 \) and \( p'(x) = \int_{\Omega} \bar{\kappa}'(x, u) dp'(u), \forall x \in \Omega \), where the modified kernel

\[
\bar{\kappa}'(x, u) := \left\{ \begin{array}{ll}
\bar{\kappa}(x, u), & u \neq x_0, \\
h(x), & u = x_0.
\end{array} \right.
\]

**Theorem 5 (Solution Uniqueness by Independent Mass Singleton).** In Approximation 1, if (i) kernel \( \bar{\kappa} \) satisfies Condition 1-2 and (ii) there exists an \( x_0 \in \Omega \) that forms an independent mass singleton \( \{x_0\} \), then Condition 3 holds.

Theorem 5 can be interpreted as follows: the original MC satisfies Condition 3 if it has a steady-state mass at \( x_0 \in \Omega \) such that the existence of stationary distribution as well as the existence of mass at \( x_0 \) is independent of the transitions from state \( x_0 \). In the rest of this subsection, we prove Theorem 5 and then give examples of independent mass singletons.
5.2.1. Proof of Theorem 5. We use the following two lemmas in our proof for Theorem 5.

**Lemma 7 (Balance Equation Reformulation II).** Given the settings and assumptions in Theorem 5, if constant \( q := \int_{x_0} dp(u) > 0 \), then \( f = \frac{1}{q} \) is a solution to

\[
f(x) = \xi_0(x) + \int_\Omega \kappa_0(x, u) df(u), \quad \forall x \in \Omega,
\]

(\( BE-2' \))

where the inhomogeneous term \( \xi_0 \in \tilde{\mathbf{X}} \) and the kernel \( \kappa_0 \in T(\Omega, \mathcal{B}) \) are specified as

\[
\xi_0(x) := \bar{\kappa}(x, x_0), \; x \in \Omega; \quad \kappa_0(x, u) := 1\{u \neq x_0\} \cdot \bar{\kappa}(x, u), \; x, u \in \Omega.
\]

Equivalently, \( BE-2' \) can be stated as

\[
f = \xi_0 + \mathcal{K}_0 f,
\]

(\( BE-3' \))

where operator \( \mathcal{K}_0 \) is defined by Definition 2 with kernel \( \kappa_0 \). Moreover, the solution space of \( BE-1 \) in \( \mathbf{X} \) is \( \text{span}(S) \) if \( S \neq \emptyset \), where \( S \) is the solution set of \( BE-2' \) or equivalently \( BE-3' \) in \( \mathbf{X} \).

**Proof:** As a stationary distribution for the original MC, \( p \) satisfies

\[
p(x) = \int_\Omega \bar{\kappa}(x, u) dp(u) = \int_{\{x\}} \bar{\kappa}(x, u) dp(u) + \int_{\Omega \setminus \{x\}} \bar{\kappa}(x, u) dp(u) = q \xi_0(x) + \int_\Omega \kappa_0(x, u) dp(u),
\]

for all \( x \in \Omega \). Multiplying both sides by \( \frac{1}{q} \) yields that \( f = \frac{\xi}{q} \) is a solution to \( BE-2' \).

With Definition 2, we directly have the operator form of \( BE-2' \), i.e., \( BE-3' \).

Now we only need to prove that the solution space of \( BE-1 \) in \( \mathbf{X} \) is \( \text{span}(S) \) if \( S \neq \emptyset \), where \( S \) is the solution set of \( BE-2' \) or equivalently \( BE-3' \) in \( \mathbf{X} \).

Next we use contradictions to show \( \tilde{S} \subseteq \text{span}(S) \). If \( \tilde{S} \) is not a subset to \( \text{span}(S) \), there exists \( f_2 \in \mathbf{X} \) such that (i) \( f_2 \in \tilde{S} \), (ii) \( f_2 \notin \text{span}(S) \), \( f_2 \neq f_1 \), \( f_2 \neq 0 \), and (iii) \( f_2(x_0) - f_2(x_0) = 0 \). Here (i) and (ii) are straightforward. With regards to (iii), if \( f_2(x_0) - f_2(x_0) = 0 \), we can construct a new \( f_3 \) such that \( f_3 \notin \tilde{S} \), which will satisfy (i)-(iii). Then we have \( f_2(x) = \int_\Omega \bar{\kappa}(x, u) df_2(u) = \int_\Omega 1\{u = x_0\} \bar{\kappa}(x, u) df_2(u) = \xi_0(x) - f_2(x_0) - f_2(x_0) + \int_\Omega \kappa_0(x, u) df_2(u), \forall x \in \Omega \).

Multiplying both sides by \( f_2(x_0) - f_2(x_0) \) yields that \( f_3(x) = \xi_0(x) + \int_\Omega \kappa_0(x, u) df_3(u), \forall x \in \Omega \), where \( f_3 := \frac{f_3(x)}{f_2(x_0) - f_2(x_0)} \). Therefore, \( f_3 \) is also a solution to \( BE-2' \) and \( f_3 \in \tilde{S} \). Because \( f_2 = f_2 \cdot [f_2(x_0) - f_2(x_0)]\), we have that \( f_2 \notin \text{span}(S) \), which contradicts \( f_2 \notin \text{span}(S) \).

**Lemma 8 (Solution Uniqueness by Compactness).** Given the settings and assumptions in Theorem 5, \( BE-3' \) has a unique solution in \( \mathbf{X} \) if \( \forall h \in \mathbf{D}, f = h + \mathcal{K}_0 f \) has a solution \( f \in \mathbf{X} \).

**Proof:** By the definition of \( \kappa_0, \kappa_0 \) satisfies Condition 1-2 as a result of \( \bar{\kappa} \) satisfying Condition 1-2. Then operator \( \mathcal{K}_0 \) is a well-defined compact linear operator (Theorem 3.1 and 3.4 in [33]). Then according to the Riesz-Schauder’s theorem (Lemma 2.4 in [33]), \( \text{Ker}(\mathcal{I} - \mathcal{K}_0) = \{0\} \) if \( \text{Im}(\mathcal{I} - \mathcal{K}_0) = \mathbf{X} \). We use this fact to prove Lemma 8.

\((\Leftarrow)\) Because for all \( h \in \mathbf{D} \), there exists an \( f \in \tilde{\mathbf{X}} \) such that \( f = h + \mathcal{K}_0 f \), we have that \( \mathbf{D} \subseteq \text{Im}(\mathcal{I} - \mathcal{K}_0) \). Because \( \text{Im}(\mathcal{I} - \mathcal{K}_0) \) is a linear space, \( \mathbf{X} \subseteq \text{Im}(\mathcal{I} - \mathcal{K}_0) \). Because \( \mathcal{K}_0 \) is a compact linear operator, \( \text{Im}(\mathcal{I} - \mathcal{K}_0) \) is closed. Thus, \( \text{Im}(\mathcal{I} - \mathcal{K}_0) = \mathbf{X} \) and \( \text{Ker}(\mathcal{I} - \mathcal{K}_0) = \{0\} \). Thus, equation \( f = \xi_0 + \mathcal{K}_0 f \), i.e., \( BE-3' \), has a unique solution in \( \mathbf{X} \).

\((\Rightarrow)\) Because equation \( f = \xi_0 + \mathcal{K}_0 f \) has a unique solution, we have that \( \text{Ker}(\mathcal{I} - \mathcal{K}_0) = \{0\} \). Therefore, \( \text{Im}(\mathcal{I} - \mathcal{K}_0) = \mathbf{X} \). Then for all \( h \in \mathbf{D} \), there exists an \( f \in \mathbf{X} \) such that \( f = h + \mathcal{K}_0 f \). □
Proof of Theorem 5: We first apply Lemma 7. According to Definition 3 for independent mass singleton \( \{ x_0 \} \), when \( h = \kappa(x, x_0) \), the transition kernel remains \( \kappa \) and has a stationary distribution \( p \) with a mass at \( x_0 \). Then we obtain \( q = \int_{\{ x_0 \}} dp(u) > 0 \). With all preconditions satisfied in Lemma 7, we obtain equation (BE-3') with a solution \( f = \frac{q}{p} \).

Next we consider an arbitrary \( h \in D \) replacing \( \kappa(\cdot, x_0) \). For the new MC with the modified kernel \( \kappa' \) defined in (13), we have a stationary distribution \( p' \) and a constant \( q' = \int_{\{ x_0 \}} dp'(u) > 0 \) due to Definition 3. By applying Lemma 7 to the new MC with the modified kernel \( \kappa' \), we obtain equation \( f = h + \kappa_0 f \) with a solution \( \frac{f}{q'} \). Thus, \( f = h + \kappa_0 f \) has a solution \( f \in X \) for all \( h \in D \). This result exactly satisfies the requirement of Lemma 8. Then (BE-3') has a unique solution. We have just shown \( f = \frac{q}{p} \) is a solution and thus the unique solution. By lemma 7, the solution space of balance equation (BE-1) is \( \{ \alpha p | \alpha \in \mathbb{R} \} \), and the original MC has a unique stationary distribution. \( \square \)

5.2.2. Examples of independent mass singletons. We give examples of independent mass singletons based on accessibility and return time.

Lemma 9 (Independent Mass Singleton Induced by Accessibility). In Approximation 1, \( x_0 \in \Omega \) forms an independent mass singleton if \( \exists m \in \mathbb{N} \) such that the probability of returning to \( x_0 \) within \( m \) transitions from any \( u \in \Omega \setminus \{ x_0 \} \) is lower bounded by \( \delta \), i.e., \( \mathbb{P}_u[\tau(x_0) \leq m] \geq \delta, \forall u \in \Omega \setminus \{ x_0 \} \). Here \( \tau_{A} \) represents the first return time to \( A \in B \) and \( \mathbb{P}_u \) represents the probability associated with the original MC’s kernel \( \kappa \) and the initial state \( u \).

Proof: By Definition 3, we need to show that for all \( h_1 \in D \), if we replace the definition of \( \kappa(\cdot, x_0) \) with \( h_1 \), the modified MC has a stationary distribution and a steady-state mass at \( x_0 \).

To prove the existence of a stationary distribution, the modified MC is \( \kappa \)-irreducible and recurrent. Moreover, \( \mathbb{E}_{x_0}[\tau(x_0)] \), the mean first return time from \( x_0 \) to \( x_0 \) is finite. Then a stationary distribution exists by Theorem 10.0.1 in [45].

(\( \kappa \)-irreducible.) Consider the modified MC. With any initial state \( u \in \Omega \setminus \{ x_0 \} \), we have that \( \tau(x_0) \), i.e., the first return time to \( x_0 \), is independent of \( h_1 \), i.e., the transition from \( x_0 \) to \( \Omega \). Therefore, we still have \( \mathbb{P}_u[\tau(x_0) \leq m] \geq \delta, \forall u \in \Omega \setminus \{ x_0 \} \) under the modified MC. This implies \( x_0 \) is accessible for all \( x \in \Omega \setminus \{ x_0 \} \), and thus accessible for all \( x \in \Omega \). Define measure \( \varphi: \varphi(\{ x_0 \}) = 1, \varphi(\Omega \setminus \{ x_0 \}) = 0 \). Then the modified MC is \( \psi \)-irreducible and thus \( \psi \)-irreducible.

(Recurrent.) Because \( \mathbb{P}_u[\tau(x_0) \leq m] \geq \delta, \forall u \in \Omega \setminus \{ x_0 \} \), the probability of returning to \( x_0 \) within \( m+1 \) transitions from any state in \( \Omega \) is lower bounded by \( \delta \), i.e., \( \mathbb{P}_u[\tau(x_0) \leq m+1] \geq \delta, \forall u \in \Omega \). By the strong Markov property, the modified MC with an initial state \( x_0 \) returns to \( \{ x_0 \} \) with probability 1, which implies the MC’s recurrence by Theorem 8.3.6 in [45].

(Finite mean first return time.) The above argument also implies that the modified MC’s mean first return time from \( x_0 \) to \( x_0 \) is upper bounded by \( \frac{m+1}{\delta} \). Since \( \{ x_0 \} \) as a singleton is a petite set, the modified MC has a stationary distribution due to the above three properties (\( \psi \)-irreducibility, recurrence and finite mean first return time) by Theorem 10.0.1 in [45].

Lastly, we prove the existence of mass at \( x_0 \). Let \( \Phi(t)(t \in \mathbb{N}) \) be the modified MC and \( p' \) be a stationary distribution. Because \( \mathbb{P}_u[\tau(x_0) \leq m+1] \geq \delta, \forall u \in \Omega \), we have that \( \max_{i=1,2,...,m+1} \mathbb{P}_u[\Phi(i) = x_0] \geq \frac{\delta}{m+1} \) for all \( u \in \Omega \). Thus, \( \Omega = \cup_{i=1,2,...,m+1} C_i \), where \( C_i := \{ u \in \Omega | \mathbb{P}_u[\Phi(i) = x_0] \geq \frac{\delta}{m+1} \} \). There must exist \( C_j \) where \( j = 1,2,...,m+1 \) such that \( \int_{C_j} dp'(u) > 0 \). Because after \( j \) transitions, the probability from any \( u \in C_j \) to \( x_0 \) is lower bounded by \( \frac{\delta}{m+1} \), we have that the steady-state probability of \( \Phi(t) = x_0 \) is greater than \( \frac{\delta}{m+1} \cdot \int_{C_j} dp'(u) > 0 \). \( \square \)

5.3. G/G/1+G Queue Example. Consider a waiting list with a single server who follows a first-come-first-serve rule. Arriving customers must wait in a buffer if the server is busy. A customer waiting in the buffer abandons after a random amount of patience time if her server has not been commenced, but never leaves during service. Once a customer being served completes
service, the just-freed server immediately turns to the next customer in buffer as long as the buffer is not empty. Customers’ inter-arrival times, patience times and service times form three independent and identically distributed sequences that are independent of each other. We aim to compute the following steady-state performance measures: the (customers’) probability of no wait, the probability of abandonment, and queue length.

For every customer, we define her virtual waiting time (VWT) as the amount of time she needs to wait before her service is commenced. Let \( t_n, s_n, y_n \) and \( w_n \) respectively be the inter-arrival time, service time, patience time and VWT of customer \( n(n \in \mathbb{N}) \). Let \( A(x) \) be the distribution of customers’ inter-arrival times, \( B(x) \) be the distribution of service times, and \( G(x) \) be the distribution of patience times. We make the following general assumptions:

**Assumption 2.** The distributions associated with the \( G/G/1+G \) queue satisfy:

(i) (bounded support) \( \exists \tilde{y}, \tilde{s} \in \mathbb{R}_+ \) such that \( G(\tilde{y}) = B(\tilde{s}) = 1 \) and \( \tilde{y} + \tilde{s} \leq 1 \).

(ii) (smoothness) \( \exists g_1, a_1, a_2 \in \mathbb{R}_+ \) such that \( G'(x) \leq g_1, A'(x) \leq a_1, |A''(x)| \leq a_2, \forall x \in \Omega \).

(iii) (ergodicity) \( \mathbb{P}[s_n - \tau_{n+1} < 0] > 0 \), i.e., the minimal service time is strictly smaller than the maximal inter-arrival time.

The assumption of \( \tilde{y} + \tilde{s} \leq 1 \) is without loss of generality and can be achieved by proper scaling. Item (ii) assumes smoothness of the inter-arrival time distribution in terms of the first- and second-order derivatives. Example distributions include exponential distributions, Erlang distributions, Gamma distributions (with a shape parameter no smaller than 2), distributions that can be expressed as a polynomial, etc. Similarly, item (ii) also assumes the smoothness of the patience time distribution in terms of the first-order derivative. Item (iii) is exactly the same with the ergodicity condition for \( G/G/1+G \) queue provided by [2].

Customers’ VWTs are described using an MC: for all \( n \in \mathbb{N} \), we have

\[
w_{n+1} = \begin{cases} w_n + s_n - t_{n+1} +, & \text{if } y_n > w_n, \\ w_n - t_{n+1} +, & \text{otherwise.} \end{cases}
\]

(14)

Due to (i) of Assumption 2, the MC is supported on \( \Omega = [0,1] \). Let \( p(x) \in \mathbb{D} \) be the stationary distribution of \( w_n \). The steady-state performance measures we aim to compute can be written as

\[
\mathbb{E}\phi_i(X) = \int_\Omega \phi_i(x)dp(x), \quad i \in \{1,2,3\},
\]

(15)

where \( \phi_1(x) = 1\{x = 0\} \) computes the probability of no wait, \( \phi_2(x) = G(x) \) computes the probability of abandonment, and \( \phi_3(x) = \lambda h(x) \) computes the queue length with \( h(x) := \int_{\mathbb{R}_+} \min\{y, x\} dG(y) \) and \( \lambda := 1/\int_{\mathbb{R}_+} x dA(x) \). The balance equation can be written as: \( \forall x \in \Omega \),

\[
p(x) = \int_\Omega \tilde{k}(x,u)dp(u), \quad \text{where } \tilde{k}(x,u) = \tilde{G}(u)D(x-u) + G(u)A^*(u-x).
\]

(16)

Here \( D(x) \) is the distribution of \( s_n - \tau_{n+1} \) and \( A^*(x) := 1 - A(x-) \). Thus, the MC in (14) defined by the VWTs is in accordance with the definition of original MC in Approximation 1.

We apply Approximation 1 and generate a sequence of finite-state MCs to approximate the original MC described in (14). The \( r \)-th MC has \( J^{(r)} = 2^r + 1 \) states. Its states \( \{c_i^{(r)}\}_{i=1}^{J^{(r)}} \) and transition matrix \( Q^{(r)} = \{q_{ij}^{(r)}\}_{i,j=1,2,\ldots,J^{(r)}} \) are respectively defined as

\[
c_i^{(r)} := \frac{(i-1)}{2^r}, \quad i = 1, 2, \ldots, J^{(r)}; \quad q_{ij}^{(r)} := \tilde{k}(c_i^{(r)}, c_j^{(r)}) - \tilde{k}(c_{i-1}^{(r)}, c_i^{(r)}), \quad i, j = 1, 2, \ldots, J^{(r)}.
\]

(17)

Particularly, \( c_0^{(r)} \equiv -\infty \). Let \( p^{(r)}(x) \) be the discrete stationary distribution of the \( r \)-th approximate MC. We verify Condition 1-4 so that Theorem 3 is applicable.
THEOREM 6. Consider a G/G/1+G queue with VWTs described as an MC in (14). The finite approximation outlined in (17) satisfies Condition 1–4 under Assumption 2.

Proof: (Verification of Condition 1.) By definition (16), the kernel \( \bar{\kappa}(x,u) \) can be written as

\[
\bar{\kappa}(x,u) = \int_{\Omega} [\bar{G}(u)A^*(u-x+s) + G(u)A^*(u-x)]dB(s), \quad \forall x,u \in \Omega.
\]

Then \( |\frac{\partial \bar{\kappa}(x,u)}{\partial u}| \leq \int_{\Omega} |\frac{\partial \bar{G}(u)}{\partial u}A^*(u-x+s) + \frac{\partial A^*(u-x+s)}{\partial u}\bar{G}(u) + \frac{\partial G(u)}{\partial u}A^*(u-x) + \frac{\partial A^*(u-x)}{\partial u}G(u)|dB(s) \leq \int_{\Omega} |g_1A^*(u-x+s) - A^*(u-x)| + a_1\bar{G}(u) + a_1G(u)|dB(s) \leq g_1 + a_1. \)

As discussed in Subsection 5.1, \( \bar{\kappa} \) satisfies Condition 1. (Verification of Condition 2.) By (18), \( |\frac{\partial^2 \bar{\kappa}(x,u)}{\partial x \partial u}| \leq \int_{\Omega} |\frac{\partial^2 \bar{G}(u)}{\partial x \partial u}A^*(u-x+s) + \frac{\partial A^*(u-x+s)}{\partial x}\bar{G}(u) + \frac{\partial^2 A^*(u-x)}{\partial x \partial u}G(u)|dB(s) \leq \int_{\Omega} |g_1a_1 + a_2\bar{G}(u) + a_1g_1 + a_2G(u)|dB(s) \leq 2g_1a_1 + a_2. \)

In addition, \( |\frac{\partial \bar{\kappa}(x,1)}{\partial x}| = |\frac{\partial A^*(1-x)}{\partial x}| \leq a_1. \) Then as discussed in Subsection 5.1, \( \bar{\kappa} \) satisfies Condition 2. (Verification of Condition 3.) By Theorem 5, we only need to show \( \{0\} \) is an independent mass singleton. Due to (iii) of Assumption 2, there exist \( l, \delta > 0 \) such that \( P[s_n - \tau_{n+1} \leq -l] \geq \delta. \) Then by (14), we have \( P[w_n \leq \max\{w_n - l, 0\}] \geq P[s_n - \tau_{n+1} \leq -l] \geq \delta. \) Thus, \( P_u[\tau_0 \geq \frac{l}{4}] \geq \delta \) for all \( u \in [0,1]\} \{0\}. \) By Lemma 9, \( \{0\} \) is an independent mass singleton. (Verification of Condition 4.) As discussed in Subsection 5.1, the approximate MCs’ states are in accordance with the finite splits in Condition 2. Thus, (i) of Condition 4 is satisfied. Item (ii) and (iii) of Condition 4 are directly satisfied by definition (17). Item (iv) of Condition 4 is guaranteed by the smoothness of \( \bar{\kappa} \) with \( \frac{\partial \bar{\kappa}(x,u)}{\partial u} \) and \( \frac{\partial \bar{\kappa}(x,u)}{\partial x} \) both bounded for all \( (x,u) \in \Omega^2. \) Here the bound of \( \frac{\partial \kappa(x,u)}{\partial u} \) has been provided when verifying Condition 2. With regards to the bound of \( \frac{\partial \kappa(x,u)}{\partial x} \), we have \( |\frac{\partial \kappa(x,u)}{\partial x}| \leq \int_{\Omega} |\frac{\partial A^*(u-x+s)}{\partial x}\bar{G}(u) + \frac{\partial A^*(u-s)}{\partial x}G(u)|dB(s) \leq \int_{\Omega} |a_1\bar{G}(u) + a_1G(u)|dB(s) \leq a_1. \)

6. Numerical Experiments.

6.1. Settings. We consider steady-state performance evaluation of an overloaded G/G/1+G queue. The steady-state performance measures we compute are given in (15). The inter-arrival times are assumed to follow a scaled Erlang distribution: \( A(x) = A_0(\lambda x), A_0(x) \sim \text{Er}(2, 2). \) The service times follow a scaled Beta distribution: \( B(x) = B_0\left(\frac{\mu x}{2}\right), B_0(x) \sim \text{Beta}(2, 2). \) The patience times also follow a scaled Beta distribution: \( G(x) = G_0(2x), G_0(x) \sim \text{Beta}(3, 4). \) Here \( \lambda \) and \( \mu \) are respectively the arrival intensity and service rate. We respectively compute for the models (a) \( (\lambda, \mu) = (4.1, 4) \) and (b) \( (\lambda, \mu) = (5, 4). \)

We compare our finite approximation approach against a standard MCMC implementation and a fluid approximation provided by [59]. Our finite approximation approach is as outlined in (17). In MCMC, we use Gibbs sampler with a burn-in period of \( 10^6 \) steps, and then collect one sample for every \( 10^3 \) steps. In fluid approximation, the stationary distribution of VWT is considered to be a one-point distribution concentrated at \( w_{\text{Fluid}} = G^{-1}(1 - \frac{\mu}{\lambda}). \) We refer to the approximate MC’s stationary distribution in finite approximation, the empirical distribution in MCMC, and the one-point distribution in fluid approximation as proxy stationary distributions of the VWT because for all these three approaches, we are essentially using the proxy stationary distributions to compute steady-state performance measures. Moreover, the accuracy of the proxy stationary distribution decides that of steady-state performance evaluation. We compute and compare:

- Balance equation residuals. Because the true stationary distribution is unavailable, we are not able to directly compute errors of proxy stationary distributions. Instead, we use the \( L^\infty \) residues in balance equations (16) to measure the proxy stationary distributions’ accuracy. We compute residues by discretizing support \([0, 1]\) and enumerating \( \{x \in [0, 1]|10^7 x \in \mathbb{N}\}. \)
- Computation time.
Approximate/estimated steady-state performance measures. Note that finite and fluid approximations yield deterministic approximators, while MCMC yields probabilistic estimators.

Error bounds/confidence intervals. Note that finite approximation yields strict deterministic bounds for steady-state performance measures, while MCMC yields confidence intervals. All confidence intervals are constructed via central limit theorem at the level of 99.9%. The fluid approximation approach does not have available computable error bounds.

For both model (a) and (b), we run a series of experiments using finite approximation and MCMC. In the r-th experiment, finite approximation uses an approximate MC with \( N = 2^r + 1 \) states, while MCMC collects \( N = 2^r + 1 \) samples. In this way, the proxy stationary distributions obtained from finite approximation and MCMC are both discrete distributions with \( N \) jumps, and \( N \) can be regarded as a measure of distribution granularity. Fluid approximation is computed once.

6.2. Results.

Finite Approximation vs. MCMC. We compare proxy stationary distributions’ \( L^\infty \) residues in balance equations (16) (Figure 1): finite approximation has smaller residues and converges faster than MCMC. Although the residues are not necessarily linear in solution errors for balance equations (16), we find the finite approximation residues converging as \( O\left(\frac{1}{N}\right) \) approximately, which is in accordance with the convergence rate provided in Subsection 4.4. With regards to MCMC, we find the residues converging as \( O\left(\frac{1}{\sqrt{N}}\right) \) approximately, which is in accordance with the convergence rate provided by the DKW inequality [29]. We also find that if we change the measure into \( L^1 \) residues, finite approximation still outperforms MCMC with similar advantages (results available from the authors).

Figure 1. A comparison of finite approximation vs. MCMC with respect to \( L^\infty \) residues in balance equations (16). (\( N = \) distribution granularity, i.e., the number of jumps in the proxy stationary distributions.)

Table 2 presents balance equation residues and corresponding computation times of different approaches. In comparison with MCMC, finite approximation needs significantly less time to achieve the same level of accuracy. For example, MCMC needs as many as \( 2^{18} + 1 \approx 260,000 \) samples and 42-minute computation time to reduce residues to the level of \( 1 \times 10^{-3} \) in both Model (a) and (b). Finite approximation only needs to construct an MC of \( 2^8 + 1 = 513 \) states to achieve the same level of residues, and the computation time is about 0.04 second. This shows the dramatic improvement our approach provides over the use of MCMC. As \( N \) increases, a sample of MCMC takes \( O(N) \) time as expected. For finite approximation, our computation based on iterations takes \( O(zN^2) \) time, where \( z \) is the number of iterations. This computation cost can be further reduced if the approximate finite-state MC has a factorization structure [3].

Lastly, we compare the accuracy and efficiency in performance evaluation (Figure 2-3).

Accuracy: finite approximators have significantly smaller errors than MCMC estimators. For example, when we compute \( \mathbb{E}\phi_1(X) \), or the probability of no wait, MCMC still has an error at the level 0.1% with as many as 260,000 samples and 42-minute computation time in both Models. The finite approximator achieves the same level of accuracy by constructing an MC of \( 2^8 + 1 = 257 \)
Table 2. A comparison of finite approximation vs. MCMC with respect to $L^\infty$ residues in balance equations (16), and computation time. ($N =$ distribution granularity, i.e., the number of jumps in the proxy stationary distributions; $CT =$ computation time/second.)

| $\log_2(N-1)$ | Finite approximation | $L^\infty$ residue | CT | MCMC | $L^\infty$ residue | CT |
|---------------|---------------------|--------------------|----|------|--------------------|----|
| 5             | 2.53E-01            | 1.36E-01           | 9.5E+00 | 3.04E+00 |
| 6             | 1.10E-02            | 5.70E-03           | 1.06E-01 | 1.15E-01 |
| 9             | 3.88E-03            | 4.03E-02           | 3.93E-02 | 4.17E-02 |
| 12            | 4.88E-04            | 7.66E-01           | 1.70E-02 | 4.95E+00 |
| 15            | 5.94E-05            | 3.25E+01           | 5.74E+03 | 3.31E+02 |
| 18            | 6.37E-06            | 2.31E+03           | 1.40E+03 | 2.53E+03 |

| $\log_2(N-1)$ | Finite approximation | $L^\infty$ residue | CT | MCMC | $L^\infty$ residue | CT |
|---------------|---------------------|--------------------|----|------|--------------------|----|
| 5             | 2.64E-01            | 1.69E-03           | 5.24E-04 | 8.11E-01 |
| 6             | 3.40E-02            | 6.04E-01           | 3.57E-02 | 4.12E+01 |
| 9             | 4.25E-03            | 4.10E-02           | 5.75E-02 | 4.85E+01 |
| 12            | 5.24E-04            | 8.11E-01           | 7.29E-03 | 3.27E+02 |
| 15            | 6.48E-05            | 3.69E+01           | 2.80E-03 | 3.27E+02 |
| 18            | 7.06E-06            | 2.49E+03           | 1.40E-03 | 2.53E+03 |

Figure 2. A comparison of finite approximation vs. MCMC with respect to steady-state performance evaluation, results from Model (a). ($N =$ distribution granularity, i.e., the number of jumps included in the proxy stationary distributions. Note that finite approximation provides approximate steady-state performance measures with bounds, while MCMC provides estimated steady-state performance measures with 99.9% confidence intervals.)

Evaluate $E \phi_1(X) = E1(X = 0)$ in Model (a).

Evaluate $E \phi_2(X) = ER(X)$ in Model (a).

Evaluate $E \phi_3(X) = E \lambda(X)$ in Model (a).

states and the computation time is around 0.02 second. Here relative errors are computed by using the average results of finite approximation ($N = 18$) and MCMC ($N = 21$) as reference values.

- Efficiency: The performance measure bounds provided by finite approximation are converging faster than the confidence intervals provided by MCMC. Figures 2-3 indicate that the finite approximation bound length is converging as $O\left(\frac{1}{N}\right)$, which is in accordance with Theorem 2 and the convergence rate provided in Subsection 4.4. By comparison, the MCMC confidence interval length is converging as $O\left(\frac{1}{\sqrt{N}}\right)$, which is in accordance with the central limit theorem. When $N = 2^{18} + 1 \approx 260,000$, the finite approximation bound length has reduced to $10^{-3}$, while the MCMC confidence interval length is still at the level of $10^{-2}$.

Finite Approximation vs. Fluid Approximation. Because the large market assumption does not hold in our experiments, the fluid approximation has huge errors (Table 3). The results show the dramatic advantage of our approach in analyzing limited-scale stochastic systems.

7. Conclusion. This paper develops a general and consistent finite approximation scheme with tractable solutions and error bounds for computing the stationary distribution and associated
steady-state performance measures of an MC supported on $\mathbb{R}$. The approximate MCs are constructed based on variation properties of the original MC's transition kernel. Our approach is also applicable for countable-state MCs, and those mixing countable states with continuous states. We have also shown that our approach is near-optimal among all approximation methods using discrete distributions, given a Lipschitz continuity assumption on the original MC's transition kernel. In terms of its rate of convergence, our approach is theoretically more efficient than using MCMC. Another possible reason for our approach’s advantage over MCMC is that in MCMC’s proxy stationary distribution (i.e., the empirical distribution), states are random and the jump sizes are uniform and not weighted. In contrast, finite approximation selects states and weighs jump sizes via constructing an appropriate transition matrix.

A series of numerical experiments show that the method developed in this paper outperforms benchmark results when compared with MCMC and fluid approximation in its efficiency and accuracy by several orders of magnitude. These results also show a major advantage of the approach developed in this paper in computing performance measures of limited-scale stochastic systems.
Appendix A: List of Cited Theorems and Lemmas from the Literature.

**Theorem 7 (Theorem 2.5 in [33]).** Let \( \bar{X} \) be the closure of \( X \) under the norm \( ||\cdot||_\infty \). Then \( (\bar{X}, ||\cdot||_\infty) \) is a Banach space.

**Theorem 8 (Theorem 3.1 in [33]).** For \( \tau \in T(\Omega,B) \), operator \( L \) in Definition 2 is a well defined continuous linear operator on \( \bar{X} \) if Condition 1 (uniformly bounded variation) holds. Conversely, if a continuous linear operator \( L \) on \( \bar{X} \) has the form (1), then Condition 1 must hold. Particularly,

\[
||L||_O \leq \sup_{x \in \Omega} [2V_o(\tau(x,u)) + \tau(x,1)].
\]

**Theorem 9 (Theorem 5.2 in [33]).** Consider an operator \( \mathcal{L} \) on \( \bar{X} \) in the form of \( \mathcal{L}f(x) := \sum_{i=1}^{J} \zeta_i(x)[f(c_i) - f(c_{i-1})], \forall x \in \mathbb{R}, f \in \mathcal{F} \). Then \( (I - \mathcal{L})^{-1} \) exists iff matrix \( H \) is non-singular, where \( H \) is defined by \( H_{ij} := \delta_{ij} - \zeta_i(c_j) + \zeta_{i+1}(c_j), i,j = 1,2,...,J \), and \( \zeta_j(x) \equiv 0, \forall x \in \mathbb{R} \).

**Theorem 10 (Theorem 21.67 in [23]).** Let \( \alpha \) and \( \beta \) be any two real-valued functions on \( \mathbb{R} \) of finite variation, and let \( \lambda_\alpha \) and \( \lambda_\beta \) be their corresponding Lebesgue-Stieltjes measures. Then \( a < b \) in \( \mathbb{R} \) implies

\[
\int_{[a,b]} \frac{\beta(x) + \beta(x)}{2} d\lambda_\alpha(x) + \int_{[a,b]} \frac{\alpha(x) + \alpha(x)}{2} d\lambda_\beta(x) = \alpha(b+)\beta(b+) - \alpha(a-)\beta(a-).
\]

**Theorem 11 (Theorem 3.4 in [33]).** For \( \tau \in T(\mathbb{R},B) \) satisfying Condition 1 (uniformly bounded variation), operator \( \mathcal{L} \) in Definition 2 is compact on \( \bar{X} \) iff Condition 2 (càdlàg and countable jump discontinuities) holds.

**Lemma 10 (Lemma 2.4 in [33]).** Consider a Banach space \( (Y, ||\cdot||) \) and a compact linear operator \( \mathcal{L} \). The following statements are equivalent:

(i) \( \text{Ker}(I - \mathcal{L}) = \{0\} \), i.e., \( \forall \xi \in Y, f = \xi + \mathcal{L}f \) has at most one solution \( f \in Y \);

(ii) \( \text{Im}(I - \mathcal{L}) = Y \), i.e., \( \forall \xi \in Y, f = \xi + \mathcal{L}f \) has at least one solution \( f \in Y \); and

(iii) \( (I - \mathcal{L})^{-1} \) exists, i.e., \( \forall \xi \in Y, f = \xi + \mathcal{L}f \) has a unique solution \( f \in Y \).

A corollary is: Consider a Banach space \( (Y, ||\cdot||) \), linear operators \( \mathcal{L} \) and \( \{\mathcal{L}^{(r)}\}_{r \in \mathbb{N}} \) on \( Y \). Suppose \( \{\mathcal{L}^{(r)}\}_{r \in \mathbb{N}} \) are collectively compact and consistent operators to \( \mathcal{L} \). The following statements are equivalent:

(i) \( \text{Ker}(I - \mathcal{L}) = \{0\} \), i.e., \( \forall \xi \in Y, f = \xi + \mathcal{L}f \) has at most one solution \( f \in Y \);

(ii) \( \text{Im}(I - \mathcal{L}) = Y \), i.e., \( \forall \xi \in Y, f = \xi + \mathcal{L}f \) has at least one solution \( f \in Y \); and

(iii) \( (I - \mathcal{L})^{-1} \) exists, i.e., \( \forall \xi \in Y, f = \xi + \mathcal{L}f \) has a unique solution \( f \in Y \).

**Theorem 12 (Theorem 10.0.1 in [45]).** If the MC is recurrent then it admits a unique (up to constant multiples) invariant measure. The invariant measure is finite (rather than merely σ-finite) if there exists a petite set \( C \) such that \( \sup_{x \in C} E_x[\tau_C] < \infty \).

**Theorem 13 (Theorem 8.3.6 in [45]).** Suppose an MC is \( \psi \)-irreducible. Then it is recurrent if there exists some petite set \( C \) such that with any initial state \( x \in C \), the MC returns to \( C \) with probability 1.

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