Study on the vibration of functionally graded microcantilevers immersed in fluids under photothermal excitation

S Gu, Y Q Song* and Q Zheng
State Key Laboratory for Strength and Vibration of Mechanical Structures, School of Aerospace Engineering, Xi’an Jiaotong University, Xi’an 710049, People’s Republic of China

*Corresponding author: yqsong@mail.xjtu.edu.cn

Abstract. In this study, the dynamic response of FGM (Functionally graded materials) microcantilever immersed in fluids under high-frequency photothermal excitation was investigated theoretically. The temperature along the length of microcantilever can be obtained analytically by using Fourier heat conduction theory. The axial thermal stress varying along the thickness can be obtained by the temperature distribution. Using concept of physical neutral surface and thermal stress, photothermal driving force was obtained analytically by using thermoelastic theory. The hydrodynamic force was presented by means of Sader’s method. Based on the Euler-Bernoulli beam model, effective bending modulus and effective density of the FGM cantilever, dynamical deflection fields in fluids can be obtained analytically by using mode superposition method. Theoretical analysis showed that the influence of volume fraction in vacuum or air is more significant than in fluids, and the volume fraction has a less influence when the dimension of microcantilever get smaller. This study can be valuable to users and designers of FG microcantilever-based structures in MEMS/NEMS.

1. Introduction
Microcantilever-based structures are widely applied in Micro-Electro-Mechanical System and Nano-Electro-Mechanical System (MEMS and NEMS). In the practical application, the dynamical response of the beam is strongly decided by the properties of materials and surrounding medium. Many researches show experimentally illustrate theoretically that fluid environment have a crucial effect on the dynamic response of cantilever. Chu [1] implemented approximate hydrodynamic loading function in the derivation of the inviscid result for a cantilever with rectangular cross section. Sader [2] presented an explicit formulae and results by analysing the dynamic response of a cantilever immersed in a viscous fluid theoretically, then Van Eysden and Sader [3] derived explicit analytical formulas for both flexural and torsional resonant frequencies. Dufour et al. [4] presented an overview about hydrodynamic force for transverse bending, lateral bending and elongation modes. Heinisch M et al. [5] studied the frequency response of tuning forks immersed in a sample liquid and showed the rectangular cross-sectioned tuning fork was more sensitive to mass density than circular cross-sections. Schultzet al. [6] developed a new model to analyse the lateral-mode vibration of Timoshenko cantilever in viscous fluids and showed the Timoshenko effects was important for lateral-mode cantilevers of small slenderness ratios. Ghader Rezazadehet al. [7] investigated the effects on the response of the sensor immersed in fluids and utilized these effects to measure the properties of fluids. Dong and Song [8] showed the length of cantilever had
a significant effect on resonant response and quality factor for the cantilevers immersed in fluids. Song et al [9] showed the coating materials had a significant effect on the resonance frequency of microcantilever.

FGMs as novel composite materials have a high performance in the extreme mechanical and thermal operating condition [10]. The properties of FGMs vary continuously from one interface to the other, and this variation can be controlled by volume fraction of the materials which are composed of. The material properties are usually assumed to vary in the thickness direction following power law distribution [11], and a volume fraction index n is used to represent the rangeability of material properties across the FG microcantilever thickness.

In recent years, many researchers focused on vibration analysis of FGM beams. Sherman D [12] studied the mechanical behaviour of ceramic–metal laminate under thermal shock. Librescu et al. [13] studied thermoelastic modelling and behaviour of FGM thin walled rotating beams in a high temperature environment. Hichem Bellifa et al. [14] developed a new first-order shear deformation theory and concept of the neutral surface position to study dynamic behaviours of FGM plates. X Xu [15] studied thermoelastic damping (TED) of functionally graded material (FGM) microbeams and showed the influence of volume fraction to TED. L.Li et al. [16] showed the influence of high-order nonlinearity on the dynamical behaviours in the thermal environment should be included.

In this paper, the dynamic response of FG cantilever immersed in fluids is investigated. We first establish the simple modelling for the vibration of FG cantilever using classical Euler-Bernoulli beam theory. Then we analysis the vibration behaviours in fluids by using hydrodynamic force presented by Sader.

Results from our study shows the influence for volume index, beam thickness and fluid properties to dynamic response. This study can be valuable to users and designers of FG microcantilever-based structures in MEMS/NEMS.

2. Vibration of microcantilever under photothermal excitation
Consider a FG microcantilever with length $L$ ($0 \leq x \leq L$), thickness $d$ ($-d / 2 \leq d \leq d / 2$), and width $W$. The FG microcantilever is generally composed of two different materials at the top and the bottom surfaces (as shown in figure 1). The material properties of FG microcantilever are assumed to vary along the thickness direction according to power law distribution:

$$P(z) = (P_1 - P_2) \left( \frac{1}{2} + \frac{z}{d} \right)^n + P_2$$

(Figure 1. Geometry of a FG microcantilever)

Where $P_1$ and $P_2$ denote the values of the material properties (i.e. $\kappa$, $\rho$, $C$, $E$) of the top surface and bottom surface of the FG microcantilever respectively. Volume fraction index $n$ is a non-negative variable parameter represent the material distribution in the beam, different volume fractions of material can obtained by different values of $n$.

2.1 Temperature field

Due to the thermal diffusion length is much larger than the cantilever thickness and the laser spot diameter is comparable with the cantilever width, it is assumed that the temperature along the $z$ axis and $y$ axis is constant. Therefore, the time-dependent temperature distribution can be described by the one-dimensional heat diffusion equation[17].
Figure 2. Model of computing temperature field

Consider an element dx on the cantilever beam, and the heat flow increment through the element at x is expressed,

$$\Delta Q_x = WK_1 \frac{\partial T}{\partial t}$$

The heat transfer flow of convection between the boundary of element and the environment is expressed,

$$Q_c = 2h(W + d)dx(T - T_f)$$

The energy stored in element is,

$$E_{st} = W\zeta \frac{\partial T}{\partial x} dx$$

Where $K_1 = \int_{-d/2}^{d/2} \kappa(z)dz$ and $\zeta = \int_{-d/2}^{d/2} \rho(z)C(z)dz$. $\kappa(z)$ is the distribution of thermal conductivity along axis z, $\rho(z)$ is the density, $C(z)$ is the specific heat capacity, $h$ is the heat transfer coefficient. Substitute Eq. (2)—(4) to energy equilibrium equation $\Delta Q_x - Q_c = E_{st}$, and defining $\Delta T = T - T_f$, the governing equation of temperature field $\Delta T(x, t)$ can be obtained,

$$\frac{\partial \Delta T}{\partial t} = K \frac{\partial^2 \Delta T}{\partial x^2} - \beta \Delta T$$

Where $K = K_1/\zeta$ and $\beta = 2h(W + d)/(W\zeta)$

Taking the Fourier transform of Eq. (5) to obtain the temperature field in frequency domain,

$$\frac{\partial^2 \hat{T}(x, \omega)}{\partial x^2} - \frac{\beta + i\omega}{K} \hat{T}(x, \omega) = 0$$

Where

$$\Delta \hat{T}(x, \omega) = \int_{-\infty}^{\infty} \Delta T(x, t)e^{-i\omega t} dt$$

The boundary conditions for Eq. (6) are

1. Continuity condition of temperature on laser loading location is

$$\Delta \hat{T}(x_0^+) = \Delta \hat{T}(x_0^-)$$

2. Jump condition of heat flow on laser loading location is

$$\frac{\partial \Delta \hat{T}}{\partial x} \bigg|_{x=x_0^+} - \frac{\partial \Delta \hat{T}}{\partial x} \bigg|_{x=x_0^-} = \frac{\lambda P_0}{WK_1}$$

3. On clamped end

$$\Delta \hat{T} \bigg|_{x=0} = 0$$

4. heat convection on free end is
Where the heat conduction coefficient of FG beam.

The solution satisfied boundary conditions to Eq. (6) is

\[
\begin{align*}
\Delta \hat{T}(x, \omega) &= C_1 e^{x} + C_2 e^{-\omega x}, (x < x_0) \\
\Delta \hat{T}(x, \omega) &= C_3 e^{x} + C_4 e^{-\omega x}, (x \geq x_0)
\end{align*}
\]  

Where,

\[
\begin{align*}
C_1 &= -C \left[ e^{2\omega} (r + H) + e^{-\omega_0 (r - H)} \right] \\
C_2 &= -C_1 \\
C_3 &= -C \left( e^{2\omega_0} - 1 \right)(r - H) \\
C_4 &= -C \left( e^{2\omega_0} - 1 \right)e^{2\omega} (r + H)
\end{align*}
\]

\[r = \frac{\beta + \sqrt{\beta^2 + \omega^2}}{2K} + i \frac{-\beta + \sqrt{\beta^2 + \omega^2}}{2K}, \quad C = -\frac{\lambda P_0}{WK_1} \times \frac{e^{-\omega_0 (r + H) + (r - H)}}{2r}
\]

2.2 Cantilever’s driving force

Due to the material properties of FG beam along the axis z is different, there is an axial force along the thickness of beam. The thermal stress can be expressed by thermoelastic theory,

\[
\sigma(z) = E(z) \alpha(z) \Delta \hat{T}(x, \omega)
\]

The bending moment distribution for a temperature profile along the cantilever is obtained

\[
M(x, \omega) = W \int_{d/2}^{d/2} E(z) \alpha(z) (z - z_0) \Delta \hat{T}(x, \omega) dx
\]

Where W is the cantilever width, z_0 is the position of the physical neutral axis with respect to geometric neutral surface,

\[z_0 = \frac{B_{22}}{B_{11}}
\]

Where,

\[
\{B_{11}, B_{22}, B_{33}\} = \int_{-d/2}^{d/2} E(z) \{1, \alpha, z_1\} dz
\]

Therefore, the effective force acting on the cantilever by the Euler-Bernoulli beam theory can be written as,

\[
F_{\text{drive}} = -\frac{\partial^2 M(x, \omega)}{\partial x^2} = -W \int_{-d/2}^{d/2} E(z) \alpha(z) (z - z_0) \frac{\partial^2 \Delta \hat{T}(x, \omega)}{\partial x^2} dx
\]

Moreover, the hydrodynamic force for a cantilever beam immersed in incompressible fluids oscillating was presented by Sader [2],

\[
F_{\text{hydro}}(x, \omega) = \frac{\pi}{4} \rho_f W \omega^2 \Gamma(\omega) Z(x, \omega)
\]

Where \(\rho_f\) is the fluid density, W is the beam width, \(\omega\) is the radial frequency of vibration, \(Z(x, \omega)\) is the response of vibration and \(\Gamma(\omega)\) is hydrodynamic function depends on the radial frequency.
2.3 Dynamic response

Upon the Euler-Bernoulli beam model, the governing equation for the vibration of FG microcantilever can be expressed as,

\[(EI)_{\text{eff}} \frac{\partial^4 Z(x,t)}{\partial x^4} + \rho_{\text{eff}} A \frac{\partial^2 Z(x,t)}{\partial t^2} = f(x,t)\]  \(20\)

Where

\[(EI)_{\text{eff}} = W \int_{-d/2}^{d/2} E(z)(z - z_0)^2 \, dz\]  \(21\)

\[\rho_{\text{eff}} = \int_{-d/2}^{d/2} \rho(z) \, dz\]

Where \(A\) is cross area of the beam, \((EI)_{\text{eff}}\) is effective bending stiffness derived by the bending modulus per unit length [18].

Taking the Fourier transform of Eq. (20) to obtain

\[\frac{(EI)_{\text{eff}} }{\partial x^4} \frac{\partial^4 Z(x,\omega)}{\partial x^4} - \rho_{\text{eff}} A\omega^2 Z(x,\omega) = F_{\text{drive}}(x,\omega) + F_{\text{hydro}}(x,\omega)\]  \(22\)

\[Z(x,\omega) = \sum_{n=1}^{\infty} A_n(\omega) \phi_n(x)\]  \(23\)

\[A_n(\omega) = \frac{\int_0^L F_{\text{drive}}(x,\omega) \phi_n(x) \, dx}{(EI)_{\text{eff}} \left\{ \int_0^L \left[ \frac{d^2 \phi_n(x)}{dx^2} \right]^2 \, dx - B(\omega) \right\}}\]  \(24\)

Where,

\[B(\omega) = \frac{\rho A\omega^4}{(EI)_{\text{eff}}} \left\{ 1 + \frac{\pi W^2 \rho_f}{4 A \rho} \Gamma(\omega) \right\}\]  \(25\)

3. Results and discussion

In this paper, a FG microcantilever composed of Gold (Au) and Silicon (Si) is considered as an exemplary study to investigate dynamic response of functionally graded microbeams immersed in fluid. Parameters used in calculation were shown in Table 1. In Figure 3 and Figure 5, the dimension of microcantilever is taken as: length \(L = 2500\, \mu m\), width \(W = 400\, \mu m\), thickness \(d = 200\, \mu m\).

### Table 1. Parameters used in calculation

| Parameter                              | Symbol | Value              |
|----------------------------------------|--------|--------------------|
| Density for Au                         | \(\rho_1\) | \(19.3 \times 10^1\, \text{kg/m}^3\) |
| Density for Si                         | \(\rho_2\) | \(2.33 \times 10^1\, \text{kg/m}^3\) |
| Young’s modulus for Au                 | \(E_1\) | \(0.8 \times 10^{11}\, \text{N/m}^2\) |
| Young’s modulus for Si                 | \(E_2\) | \(1.3 \times 10^{11}\, \text{N/m}^2\) |
| Thermal conductivity for Au            | \(\kappa_1\) | \(346\, \text{W/(m} \cdot \text{K)}\) |
| Thermal conductivity for Si            | \(\kappa_2\) | \(150\, \text{W/(m} \cdot \text{K)}\) |
| Thermal capacity for Au                | \(c_{\rho_1}\) | \(135\, \text{J/(kg} \cdot \text{K)}\) |
| Thermal capacity for Si                | \(c_{\rho_2}\) | \(695\, \text{J/(kg} \cdot \text{K)}\) |
| Coefficient of thermal expansion for Au| \(a_1\) | \(14.2 \times 10^{-6}\, \text{K}^{-1}\) |
| Parameter                                      | Value                  |
|-----------------------------------------------|------------------------|
| Coefficient of thermal expansion for Au        | \( a_2 = 3 \times 10^6 \text{ K}^{-1} \) |
| Heat transfer coefficient between gas and solids | \( h_1 = 10 \text{ W/(m}^2 \cdot \text{K)} \) |
| Heat transfer coefficient between liquids and solids | \( h_1 = 890 \text{ W/(m}^2 \cdot \text{K)} \) |
| Optical absorption coefficient                  | \( \lambda = 0.3 \)    |

Figure 3(a) shows different responses of FG microcantilever vary with volume fraction index \( n \), the first resonant frequency is 38749Hz, which \( n=0 \) means the cantilever is consisted in silicon, and frequency 10521Hz is first resonant frequency which the cantilever is consisted in gold. As Figure 3(a) shown, when \( n=0.01 \) , the natural frequency of FG beam in vacuum tends to the natural frequency of silicon cantilever, and tends to the natural frequency of gold cantilever when \( n = 3 \), this tendency denotes the accuracy of computing in this paper. With the index \( n \) decreasing, it can be noticed that the amplitude of FG beam increases.

Figure 3(b) shows the phase of deflection has almost 180 degree shift near the resonance frequency when the vibration of cantilever is in vacuum \( (\rho_{\text{vac}} = \eta_{\text{vac}} = 0) \).

Figure 4 shows phase of FG microcantilever immersed in air and in water. In comparison to the phase in vacuum, the phase in air has almost 180 degree shift is similar to the phase in vacuum, however, the phase in water is delaying near resonance.
Table 2 shows the resonant frequency, frequency shifts and quality factor for different volume fraction index $n$ of the microcantilever immersed in air and water. It is obvious that there is a negligible shift to lower frequencies when the cantilever immersed in air ($\rho_{\text{air}} = 1.205 \text{kg/m}^3, \eta_{\text{air}} = 1.81 \times 10^{-5} \text{Pa} \cdot \text{s}$), however, when the cantilevers immersed in water ($\rho_{\text{water}} = 998 \text{kg/m}^3, \eta_{\text{water}} = 1.01 \times 10^{-3} \text{Pa} \cdot \text{s}$), there is a bigger shift to lower frequencies than in air, and the frequency shift decrease with the fraction index increasing. For the same fraction index, the quality factor have a huge decrement in water compared to in vacuum and air. And the quality factor increase obviously with the fraction index increasing. The quality factors of vibration for the cantilever in vacuum approach to infinite.

**Table 2** Frequency shifts and quality factors of vibration for the cantilever No.1 with fraction index $n$ and immersed in different fluids

| Fluids   | Resonant frequency/Hz | Frequency shift/% |
|----------|-----------------------|-------------------|
| vacuum   | 29364                 | 0                 |
|          | 19257                 | 0.5               |
|          | 15987                 | 1                 |
|          | 13866                 | 2                 |
|          | 13049                 | 3                 |
|          | 0                     | 0.1               |
|          | 0                     | 0.5               |
|          | 0                     | 1                 |
|          | 0                     | 2                 |
|          | 0                     | 3                 |
| air      | 29356                 | 0.027             |
|          | 19255                 | 0.01              |
|          | 15986                 | 0.006             |
|          | 13866                 | 0                 |
|          | 13049                 | 0                 |
|          | 0                     | 0                 |
| water    | 24589                 | 6.3               |
|          | 17519                 | 9.0               |
|          | 14879                 | 6               |
|          | 13085                 | 9.6              |
|          | 12377                 | 5.1              |

| Quality factor |
|----------------|
| Fluids   | $n$ | 0.1 | 0.5 | 1   | 2   | 3   |
| vacuum   |     |     |     |     |     |     |
| air($\times 10^9$) | 1.5 | 2.5 | 3.1 | 3.6 | 3.9 |
| water    | 79  | 118 | 133 | 166 | 176 |

Figure 5 investigate the frequency shifts and quality factor of vibration for cantilever immersed in water, and the variety with thickness of the beam increasing. As Figure 6(a) shown, for the same index $n$, the frequency shifts decrease slowly, it denotes decrease in the thickness play a major role in the frequency shift. Figure 6(b) shows the quality factor increase with the thickness increasing, and it is obviously that, the growth rate of quality factor increase with the index $n$ reducing. This phenomenon denotes slenderness parameter of cantilevers are larger, the fluid effect is more obvious.

**Figure 5.** Frequency shift (a) ($\omega_{\text{water}} - \omega_{\text{vacuum}} / \omega_{\text{vacuum}}$) and quality factor (b) of cantilevers with different thickness ($d=20\text{mm}, 50\text{mm}, 100\text{mm}, 150\text{mm}, 200\text{mm}$) immersed in water, and the length and width is constant length $L = 2500\mu\text{m}$, width $W = 400\mu\text{m}$.

Figure 6 analysis the relation between the thicknesses of microcantilever and the ratio of frequency difference between $n=0.1$ and $n=3$ which the cantilever immersed in different fluids to frequency difference between $n=0.1$ and $n=3$.
difference between \( n=0.1 \) and \( n=3 \) which the cantilever in vacuum. For the same fluid, this ratio increase when the thickness of cantilever increase, it denotes the adjustment capability of volume fraction index reduce with the thickness decrease, in other words, the fluid effect play a more significant role in the frequency change than volume fraction index when the slenderness parameter of cantilevers are larger. For the index \( n \) is same, the properties of fluids have an influence on the adjustment capability of volume fraction index, it means the influence due to properties of fluids changing can’t be negligible when control the resonant frequency by adjusting the index \( n \).

\[
\frac{\omega_{v=0.1}^{\text{vacuum}} - \omega_{n=0.3}^{\text{fluid}}}{\omega_{v=0.1}^{\text{vacuum}} - \omega_{n=0.3}^{\text{vacuum}}}.
\]

**Figure 6.** Ratio of frequency difference is

Fluids include water, gasoline \((\rho_{\text{gasoline}} = 678\text{kg/m}^3, \eta_{\text{gasoline}} = 2.9 \times 10^{-4}\text{Pa} \cdot \text{s})\)
and blood \((\rho_{\text{blood}} = 1050\text{kg/m}^3, \eta_{\text{blood}} = 4 \times 10^{-3}\text{Pa} \cdot \text{s})\)

4. Conclusions

In this study, theoretical analysis showed that the influence of volume fraction in vacuum or air is more significant than in fluids, with volume fraction increasing, the frequency shifts become small and quality factor become big. And the volume fraction has a less influence when the dimension of microcantilever immersed in liquid get smaller. The fluid effect have a significant influence on dynamic response for small scale cantilever with the ratio of length, width and thickness fixed.

Acknowledgements

The work was supported by National Natural Science Foundation of China (Grants No. 11672224) and the Project Supported by Natural Science Basic Research Plan in Shaanxi Province of China (Program No.2018JM1023) and Y.Q.Song was grateful for the financial support from CSC scholarship.

Reference

[1] W.-H. Chu, Tech. Rep. No. 2, DTMB, Contract NObs-86396(X), Southwest Research Institute, San Antonio, Texas (1963).
[2] Sader J E 1998 *J. Appl. Phys.* **84**(1), 64-76.
[3] Van Eysden C A, Sader J E 2006 *J. Appl. Phys* **100**(11):114916.
[4] Dufour I, Lemaire E, Caillard B, Debéda, H, Lucat C, Heinrich S M, et al. 2014 *Sensors and Actuators B: Chemical*, 192, 664-672.
[5] Heinisch M, Voglhuber-Brunnmaier T, Reichel E K, et al. 2015 *Sensors and Actuators A: Physical*. **226**:163-174.
[6] Schultz J A, Heinrich S M, Josse F, et al. 2015 *J Microelectromech S.* **24**(4):848-860.
[7] Ghanbari M, Hossainpour S, Rezazadeh G 2015 Appl. Phys. A. 121(2):651-663.
[8] Dong T, Song Y. 2014 Chin J. Theor. Appl. Mech. 46(5), 703-709.
[9] Song Y, Dong T, Bai J, et al. 2016 Appl. Math. Model. S0307904X1630484X.
[10] Shanmugavel P, Bhaskar G B, Chandrasekaran M, Mani P S, Srinivasan S P 2012 Eur. J. Sci. Res. 68 (3) 412–439.
[11] Shyang H C, Chung Y L 2006 Int. J. Solids Struct. 43:3657–3674.
[12] Sherman D, Schlumm D 1999 J. Mater. Res. 14(09):3544-3551.
[13] Librescu L, Oh S Y, Song O 2005 J. Therm. Stresses 28(6-7):649-712.
[14] Bellifa H, Benrahou K H, Hadji L, et al. 2016 J Braz. Soc. Mech. Sci. Eng. 38:265–275
[15] X Xu, S R Li 2017 Chin J Theor Appl Mech, 49(2):308-316
[16] Li L, Liao W H, Zhang D, et al. 2018 Compos. Struct. 208:244-260
[17] Ramos D, Tamayo J, Mertens J, Calleja M 2006 J. Appl. Phys. 99(12), 902.
[18] Li X, Shih W Y, Aksay I A, Shih W 1999 J Am. Ceram. Soc. 82(7), 1733-1740.