Bridged Links and Tangle Presentations of Cobordism Categories

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Abstract: We develop a calculus of surgery data, called bridged links, which involves besides links also pairs of balls that describe one-handle attachments.

As opposed to the usual link calculi of Kirby and others this description uses only elementary, local moves (namely modifications and isolated cancellations), and it is valid also on non-simply connected and disconnected manifolds. In particular, it allows us to give a presentation of a 3-manifold by doing surgery on any other 3-manifold with the same boundary.

Bridged link presentations on unions of handlebodies are used to give a Cerf-theoretical derivation of presentations of 2+1-dimensional cobordisms categories in terms of planar ribbon tangles and their composition rules. As an application we give a different, more natural proof of the Matveev-Polyak presentations of the mapping class group, and, furthermore, find systematically surgery presentations of general mapping tori.

We discuss a natural extension of the Reshetikhin Turaev invariant to the calculus of bridged links. Invariance follows now - similar as for knot invariants - from simple identifications of the elementary moves with elementary categorial relations for invariances or cointegrals, respectively. Hence, we avoid the lengthy computations and the unnatural Fenn-Rourke reduction of the original proofs. Moreover, we are able to start from a much weaker “modularity”-condition, which implies the one of Turaev.

Generalizations of the presentation to cobordisms of surfaces with boundaries are outlined.
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0) Introduction and Survey of Results

The discovery of new algebraic structures in the form of quantum groups and braided tensor categories (BTC's) has led to new insights in several areas of mathematical physics, and, in particular, has stimulated renewed interest in low dimensional topology. This development was set off with the observation that the braid group representations obtained from quantum groups or BTC's can be used to construct knot and link invariants. The general strategy followed here is to associate to the “regular” singularities of a generic flat projection of a link, namely crossings, maxima, and minima, the defining braid- and rigidity-morphisms of a BTC. To the higher order singularities of projections of codimension one, (triple crossings, crossings at equal heights, saddles, tangential strands, etc.) which connected all generic projections to each other, one associates the respective relations that occur in the axioms of a BTC, i.e., the Artin relations (or hexagonal- and pentagonal equations), the rigidity constraints, inverse braid isomorphisms, etc.

Soon thereafter it was realized that the same structures can be used to define invariants of closed, compact three manifolds. The works of Turaev-Viro and of Reshetikhin-Turaev brought forward two types of invariants constructed from tensor categories, which are related to each other. In this article we shall focus on the second type described in [RT], since it is closer related to link invariants.

The starting point of the construction there are surgery presentations of a manifold by a framed link \( L \hookrightarrow S^3 \). In the description of [Wc] the components of the link are the curves along which two handles \( h^2 \) are attached to a four ball. The resulting three fold is then given as the boundary \( M = \partial(D^4 \cup h^2 \cup \ldots \cup h^2) \). From \( L \) the three fold \( M = M_L \) can also be described directly by a surgery along the components of \( L \), as in [Li].

The manifold invariant of [RT] is then defined as a weighted sum over the link invariants, described previously, evaluated on \( L \). The fact that still needs to be checked is that links presenting the same manifold yield the same invariant. In particular we need to know how two links \( L_1 \) and \( L_2 \) with \( M_{L_1} = M_{L_2} \) can be related.

An answer to this question is given by Kirby’s theorem, [Ki]. It states that \( L_2 \) can be obtained from \( L_1 \) by a series of two types of moves in the class of links in \( S^3 \). They are the two-handle slides (\( O_2 \)-moves) and the signature move (or \( O_1 \)-move), which corresponds to the replacement \( W \rightarrow W \#\mathbb{CP}^2 \) for the bounding four fold. This result is improved by the result in [FR], where it is shown that it suffices to consider only a special type of \( O_2 \)-moves, namely the \( \kappa \)-moves.

The main technical concern in both works is that the moves can be chosen such that one never leaves the class of presentations that use only two handles. For this purpose one also has to restrict the class of three folds on which we surger to connected and simply connected ones (like \( S^3 \)). If simply-connectedness is given up, Proposition 7 tells us that the Kirby formulation (but not that of [FR]) can be salvaged by including an additional move, which
is introduced as the \(\eta\)-move in Section 2.2.2). Clearly, for disconnected manifolds not even the results of [Wc] and [Li] hold, since surgeries along links do not change connectedness.

Consequently, the point of view chosen in the mentioned references involves a number of difficulties when dealing with the constructions of [RT]: First the algebraic verification of the \(\kappa\)-move is somewhat involved, since several pieces of the \(SL(2,\mathbb{Z})\) representation, that is associated to any modular BTC, have to be constructed by hand. Moreover, the verification is not like in the case of links a more or less obvious identification of elementary topological and algebraic constituents. Instead, it relies on the non trivial result in [FR], that presents all two handle slides as compositions of \(\kappa\)-moves. In the study of non semisimple invariants interpretations of two handle slides as the defining equation of an integral of a Hopf algebra have been found in [HKRL]. This, however, does not simplify the verification that the formulae given in [RT] are actually integrals.

In [RT] not only an invariant but more generally the construction of a TQFT, i.e., fiber functors on cobordism categories, is proposed. A rigorous construction of presentations of cobordisms, which generalizes the theory for closed manifolds, has so far been missing. For the special case of invertible cobordisms, which are given by elements of the mapping class group, tangle presentations have been constructed in [MP] using explicit presentations in terms of generators and relations, as in [Wj]. In order to generalize the surgery presentation as in [Wc], [Li], [Ki], and [FR], and for nice descriptions of the composition of cobordisms it is most convenient to start surgery on unions of handlebodies. These are neither connected nor simply connected and therefore do not fit in the conventional Kirby calculus.\footnote{After completion of the first version of this paper the author received the preprint [Sw]. The approach presented there is similar to the description of [RT], where the spines of embedded handlebodies are treated on the same footing as surgery ribbons. Presentations are thus reduced to the ordinary Kirby calculus. Unlike the description we derive here, the embeddings of the boundaries are variable, which makes the formulation of composition rules (let alone presentations of cobordism categories) more difficult.}

In order to shed more light on both of the outlined problems, it is most instructive, if we refrain from insisting on presentations, where only two handles are attached, but also include one handles into the surgery description. The manifolds we shall consider are thus of the form \(M = \partial(N^{(4)} \cup h^1 \cup \ldots \cup h^1 \cup h^2 \cup \ldots \cup h^2)\), and we make no connectedness assumptions on the boundary of \(N^{(4)}\). The attaching data is now encoded into, what we call a “bridged link”, which is embedded in the manifold \(\mathcal{L} \hookrightarrow M_0 = \partial N^{(4)}\), to be surgered on. The additional data that enters a bridged link are pairs of small spheres in \(M_0\), that indicate, where the one handles are to be attached. In analogy with presentations of groups this is like including more generators into the description and thereby providing more freedom in choosing a practical set of relations.

The one surgered manifold, \(\tilde{M} = \partial(N^{(4)} \cup h^1 \cup \ldots \cup h^1)\), is obtained by gluing the spheres of each pair together along an orientation reversing homeomorphism, after the interior three-balls have been removed. In particular, this type of surgery allows for a change of connectedness. To get from \(\tilde{M}\) to \(\tilde{M}'\) we attach the remaining two handles, i.e., we surger along an ordinary, framed link \(\hat{\mathcal{L}}\) in \(\tilde{M}\). The preimage of the components of \(\hat{\mathcal{L}}\) in \(M_0\) are
ribbons that may end in one surgery sphere and emerge at the respective spot on the partner sphere. A typical example of a bridged link is given in figure 2.3.

Manipulations with similar types of attaching data have appeared in the papers of, e.g., [FR]. A frequently used notation for one handle attachment due to Kirby is that of a “dotted circle”. However, the isoptopy classes of such attaching data are generally smaller, because associated surgery spheres cannot be moved independently.

With our definition, the analogue of the presentation theorems in [Wc] and [Li] is the following:

**Theorem 1** Suppose $M$ and $M_0$ are compact, oriented three folds, and $M$ is connected. Also assume there is a homeomorphism of boundaries:

$$\psi : \partial M_0 \cong \partial M.$$ 

Then there is a bridged link $\mathcal{L}$ in the interior of $M_0$ such that $\psi$ can be extended to a homeomorphism

$$\tilde{\psi} : (M_0)\mathcal{L} \cong M.$$ 

The “relations” between presentations, which we propose to choose, are given by the following five moves. See also, Proposition 6 in Section 2.3).

**Theorem 2** If for two bridged links $\mathcal{L}_1$ and $\mathcal{L}_2$ in the same compact manifold $M_0$ we have that $(M_0)\mathcal{L}_1 = (M_0)\mathcal{L}_2$, then they are related by a sequence of the following moves:

1. **Isotopies**: Regular isotopies of the bridged link $\mathcal{L} \hookrightarrow M_0$, where the ends of the ribbons stay attached to the spheres.

2. **Signature or $O_1$-move**: (as in [Ki])

3. **Isolated Cancellation**: If a component of the bridged link consists of a pair of spheres and a single ribbon which penetrates the spheres in only one pair of points, then this component can be discarded from the diagram. (This corresponds to figure 2.7, where the outer ribbons, a,b, and c, are omitted.)

4. **Modification (or Handle Trading)**: This corresponds to replacing a one handle of the four fold by a two handle. The operation on the bridged link is shown in figure 2.9 and explained in Section 2.2.2.5).

5. **One Slides and Isotopies over Components**: If a pair of surgery spheres (say $L_0$ and $L_0'$) lies in two different components of $M_0$, then we can push another surgery sphere ($H_i$) through as shown in figure 2.3. An isotopy of the link in a fixed $\tilde{M}$ gives rise isotopies of ribbons through spheres as indicated in figure 2.4.

(In the description of planar ribbon diagrams those are broken into opposite braid group actions on the the ribbons attached to the two surgery spheres, and pushing through loops.)
Since the original motivation of this work was to find cobordism presentations, the language and organization used in the derivation of these results sometimes suggests a specialization to the case where $M_0$ is a union of handle bodies $H$. As no references to the special properties of this choice are ever made, the proofs may be read literally in full generality.

All of the proposed moves in Theorem 2 are local. In particular, we have no two handle slides or $\eta$-moves in the list. Note also, that the fifth move can be omitted if $M_0$ is connected.

Diagrammatically, the modification-move is similar to the $\kappa$-move, but its topological derivation and its interpretation are far more elementary. Also, the algebraic computations involved in the invariance proof of [RT] become much easier, see Section 4.2. The proof of invariance we give here is thus closer in spirit to the proof of invariance for links as described in the beginning of the introduction. The relations between surgery and algebraic data are summarized in the table at the end of Section 4.2.1). In [KL] this correspondence is systematically put to use, in order to construct extended TQFT’s and the elementary cobordisms that belong to the algebraic generators are more explicitly identified.

Interestingly, we find - analogous to the correspondence in [HKRL] between 2-handles and integrals of Hopf algebras - an interpretation of 1-handles as cointegrals. Details and implications for non-semisimple TQFT’s will be discussed in separate papers.

As a byproduct of purely topological considerations, we will show that the modularity condition of [Tu] on the abelian BTC we start with can be equivalently replaced by the weaker condition:

\[ 1 \in \text{im}(S) \quad \text{(or, equivalently, } S^2(1) = 1, \text{ etc.)}, \]

where the $S$-matrix is as usual defined by the traces over the monodromies. If we think of $S$ as a generalized Fourier transform, this condition can be seen as an analogue of the point separation condition of the Stone-Weierstrass theorem. For the non-semisimple version of this condition see again [KL].

The second application of the bridged link calculus is the derivation of tangle presentations of connected cobordisms - or a central extension, $1 \to \Omega_4 \to \tilde{\text{Cob}}_3 \to \text{Cob}_3 \to 1$, thereof described in Chapter 1. We introduce so called “standard presentations” on unions of standard handle bodies with a fixed one-handle structure of the bounding four-fold. The moves under these restrictions are derived in Chapter 3. An additional move that has to be added to those for closed manifolds is the so called the $\sigma$-move, which also appeared in the combinatorial description of the mapping class group in [MP]. In fact, our presentation will entail a 3+1 dimensional, Cerf-theoretical proof of the results in [MP], which does not use the explicit presentations from [Wj]. The category $\mathcal{T}_g$ is described in the following theorem:

**Theorem 3** The category $\tilde{\text{Cob}}_3$ is naturally isomorphic to the category $\mathcal{T}_g$ of ordered sets of grouped, admissible, ribbon tangles in $\mathbb{R} \times [0,1]$, modulo relations.
The composition rules in $\mathcal{T}_g$ are as described in Sections 1.3) and 3.2.3).
The five equivalence relations are
Isotopies

The \( \tau \)-move (see figure 3.32)

The \( \sigma \)-move (see figure 3.27)

The \( \kappa \)-move (as a special \( O_2 \), see [FR])

The \( 0 \odot 0 \)-move (a special \( \eta \)-move, see Section 2.2.8))

The functor \( \widetilde{\text{Cob}}_3 \to \text{Cob}_3 \) is presented by a corresponding quotient \( \widetilde{T}_g \to T_g \) of tangle categories. In \( T_g \) we have in addition the \( O_1 \)-move, which allows us to omit the \( 0 \odot 0 \)-move.

In Section 4.1) tangle presentations are used to derive presentations of general mapping tori. We also discuss a diagram that is of interest in connection with the composition rules of the tangle categories. This tangle element gives rise to a canonical idempotent in a given quasitriangular Hopf algebra \( H \), whose image on a general representation of \( H \) is the maximal, self conjugate subrepresentation.

The proofs of Theorem 2 and related statements are based on the theory of stratifications of function spaces and their topology as developed by Cerf in [Ce]. The relevant facts are reviewed in Section 2.1). Another indication that the inclusion of one handle yields a more coherent picture is given by the observation that the space of codimension one Morse functions on a four fold with singularities of index one and two is connected, where as the corresponding space with only index two singularities is disconnected. (see Lemma 5).

In Theorem 2 the listed moves have specific meanings as to which part of the presentation they affect: The moves 1), 3) and 5) correspond to the elementary deformations of functions on a fixed bounding four fold. The \( O_1 \)-move is as usual the elementary move that changes the cobordism class of the four fold in \( \Omega_4 \), i.e., the class in the central extension \( \widetilde{\text{Cob}}_3 \), by connected summing with a \( \mathbb{C}P^2 \). Only the modification changes the four fold. The elementary operation it involves is the attachment of a five dimensional 2-handle to \( W \times [0,1] \) along a curve \( C \subset W \times \{1\} \). If \( C \) intersects the attaching data of a (four dimensional) 1-handle \( h^1 \subset W \) in exactly one point, then the surgered manifold \( (W)_C \) is given by trading \( h^1 \) for a two handle which is attached along an annulus surrounding \( C \).

In Section 4.3) we also describe the presentations for categories \( \widetilde{\text{Cob}}_3(N) \) of compact surfaces with \( N \) boundary components, which we work out completely in [KL]. The associated tangles contain \( N \) additional strands. A new non-trivial operation on the set \( \{ \widetilde{\text{Cob}}_3(N) \}_N \) is the glue tensor product \( \otimes_{\text{glue}} \), where we do not just take the disjoint unions of surfaces but also sew them along some of their boundary components. The main difficulty we encounter here is that the glue tensor of two standard surfaces is not canonically identified as a standard surfaces anymore. We briefly discuss the role of double categories, introduced in [KL] to describe the two gluing operations.
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1) Framed Cobordisms

1.1) Construction of Bounded Cobordisms

For the surgery presentation of closed three manifolds it is essential to find a bounding four manifold. The purpose of this section is find an analogous, practical notion of a bounded \(2 + 1\) cobordism category. We will define it as a special subcategory of \(3 + 1\) dimensional cobordisms. In order to organize the presentations according to their signature we introduce \(\bar{C}ob_3\), which is the subcategory of \(3 + 1\) cobordisms modulo five cobordisms.

The resulting category can equivalently be viewed as the one of surfaces and two-framed \(2+1\)-dim cobordisms, by a result of Atiyah.

1.1.1) \(2+1\) Cobordisms

We shall use the following conventions to describe a \(2+1\) dimensional cobordism category. An object is given by a sequence \(\bar{g} = (g_1, \ldots, g_K)\) of non-negative integers. (We admit \(K = 0\), i.e., \(\bar{g} = \emptyset\)). To any such sequence we associate a two dimensional surface \(\Sigma(\bar{g}) := \coprod K \Sigma_{g_j}\) with \(K\) components. Here the \(\Sigma_{g_j}\) are fixed, oriented coordinate surfaces, one for each genus.

A morphism from \(\bar{g}^-\) to \(\bar{g}^+\) is an (oriented) three dimensional cobordism. It consists of an oriented three manifold \(M\) with boundary \(\partial M = B_+ \amalg B_-\) together with an orientation reversing homeomorphisms \(\psi = \psi_+ \amalg \psi_-\) with \(\psi_\pm : B_\pm \rightleftharpoons \pm \Sigma(\bar{g}_\pm)\), which we shall call a chart.

The composition of (oriented) cobordisms is given by identifications along the boundaries, using the charts of the start and end boundary component respectively. We consider two cobordisms to be equivalent \((M, \psi) \sim (M', \psi')\), if there is a homeomorphism \(\chi : M \rightleftharpoons M'\) such that \(\psi' \circ \chi |_{\partial M} = \psi\).

The category also has an obvious tensor product, \(\otimes\), which is given by the disjoint union of surfaces and cobordisms.

**Definition 1** We denote by \(Cob_3\) the category of \(2+1\) dimensional cobordisms. The objects are sequences of non-negative integers \((\bar{g})\) and the morphisms \([M, \psi]\) are equivalence classes of cobording three manifolds. The composition structure is induced by the identification along boundaries.

1.1.2) Bounding \(3+1\) Cobordisms

The classical link presentation for closed manifolds is given on the standard manifold \(S^3\). Similarly, a presentation of manifolds with boundary \(\Sigma = \partial M\) should be given on a standard manifold with the same boundary \(\Sigma\). A nearby choice are unions of standard handlebodies \(H_g\) (one for every genus \(g\)). Since we also wish to describe compositions of presentations of cobordisms, we shall also consider the complementary handlebodies.

For a precise definition we fix for any non-negative integer \(g\) an unknotted embedding of a standard handlebody \(H_g^+ = H_g\) of genus \(g\) into \(S^3\) with a fixed orientation. We denote the
opposite handlebody \( H^g := S^3 - H_g \) and make the identification \( \Sigma_g := \partial H_g \), with induced orientation. Also, we write \( \mathcal{H}(\bar{g}^+, l, \bar{g}^-) \) for the (ordered) union of the handlebodies and \( l \) copies of \( S^3 \), giving rise to a standard cobordism from \( \bar{g}^- \) to \( \bar{g}^+ \).

Having this standard manifold for a given boundary we construct for every 2+1 cobordism \((M, \psi)\) a closed, oriented three manifold \( M^{cl} \) by

\[
M^{cl} = \left( \mathcal{H}(\bar{g}^+, l, \bar{g}^-) \right) \coprod \left( \{ \Sigma(\bar{g}^+) \coprod - \Sigma(\bar{g}^-) \} \times [0, 1] \right) \coprod M .
\]

Here \( \sim \) stands for the standard identification of the lower boundary of the cartesian product with the boundaries of the handlebodies. By a classical theorem of [Ro] we know that there always exists a compact four fold \( W \) such that

\[
M^{cl} \approx \partial W .
\]

This allows us to consider a subcategory of 3+1 dimensional cobordisms, which admits a full functor onto \( \text{Cob}_3 \). Its objects are of the form \( \Sigma \times [0, 1] \) for a closed, compact, and oriented two fold \( \Sigma \) and can thus be identified with the ones from \( \text{Cob}_3 \). Without referring to the bounded three fold as in \( (\mathcal{H}, \mathcal{H}) \), we define the morphisms as four folds \( W \) with special functions at the boundary. Specifically, we assume the existence of a height function \( g \) and a chart \( \psi \):

\[
g : U(\partial W) \rightarrow [0, 1] \text{ in a vicinity of the boundary}
\]

\[
\psi : \left( \{ \Sigma(\bar{g}^+) \coprod - \Sigma(\bar{g}^-) \} \times [0, 1] \right) \coprod \left( \mathcal{H}(\bar{g}^+, l, \bar{g}^-) \right) \hookrightarrow \partial W ,
\]

which satisfy the conditions:

1. \( g \) is smooth and has no critical points .
2. \( g^{-1}(0) = \psi\left( \mathcal{H}(\bar{g}^+, l, \bar{g}^-) \right) \)
3. \( g^{-1}(1) \subset \partial W \)
4. \( \text{im}(\psi) = \overline{g^{-1}([0, 1]) \cap \partial W} \) and \( g \circ \psi \) is the projection onto \( [0, 1] \) if restricted to \( \left( \Sigma(\bar{g}^+) \coprod - \Sigma(\bar{g}^-) \right) \times [0, 1] \)

Also, we shall always assume that the bounding four cobordism has components in one to one correspondence with the components of the three cobordism, i.e., we require

\[
\pi_0(g^{-1}(1)) \longrightarrow \pi_0(W)
\]

to be an isomorphisms.

The composition of two such manifolds \( W_i, i = 1, 2 \), is given by identifications along the common boundary pieces \( \Sigma(\bar{g}) \times [0, 1] \) using the charts \( \psi \). The new functions, \( g \) and \( \psi \), are
the restriction of the old ones. If \( l_i \) is the number of \( S^3 \) components in the boundaries of \( W_i \cap g^{-1}(0) \) the respective number for the composition is

\[
l = l_1 + l_2 + \sum_{j=1}^{K} g_j.
\]

Since we wish to give presentations in only one \( S^3 \), we often redefine the composition by gluing in four balls, \( D^4 \), along the excessive \( S^3 \)'s, at the expense of introducing additional index 0 singularities for the composite of the Morse functions on the \( W_i \)'s. Typically, if \( M \) is connected, these will be cancelled with other index 1 singularities. A schematic picture of the composition is given in Figure 1.1.

![Figure 1.1: 3+1 Cobordism](image)

1.1.3) The Category \( \tilde{\text{Cob}}_3 \)

Suppose a four fold \( W_1 \) bounding \( M \) can be obtained from another four fold \( W_2 \) by surgeries in the interior. If we had in addition a Morse function \( f \) on \( W_1 \) we can easily find “modified” Morse functions on the surgered manifolds, which coincide with \( f \) outside the surgered pieces, and whose singularity structure differs from that of \( f \) in a specific way. As Morse functions on \( W \) define presentations of a three fold \( \subset \partial W \), surgeries on the four folds yield moves between presentations of the same \( M \) as boundary of either \( W_1 \) or \( W_2 \). The surgery, and thus the moves, only exist if \( W_1 \) and \( W_2 \) are cobordant with proper identifications of the boundaries. This leads us to consider a wider notion of equivalence of the special 3+1 cobordisms than just the homeomorphism type:

In order for two connected cobordisms \( (W_1, \psi_1) \) and \( (W_2, \psi_2) \) to be considered equivalent there shall be an orientation preserving homeomorphism \( \tilde{\chi} : \partial W_1 \to \partial W_2 \) which is compatible with the charts \( \psi_1 \) and \( \psi_2 \). As in three dimensions (see ([1.1])), we construct a closed four
The dimensional manifold $\hat{W}$ from the pieces $W_1, W_2$ and $[0, 1] \times \partial W_i$ using the identification $\hat{\chi}$. This manifold shall be the boundary of a five dimensional manifold $Q$. Since $\hat{\chi}$ was assumed to be orientation preserving an orientation on $\hat{W}$ is opposite to the orientation of exactly one constituent. Thus, by e.g. [J], the signature $\sigma(\hat{W})$ is the difference of signatures of $W_1$ and $W_2$ and it follows from [Wa2] that $Q$ exists if and only if $\sigma(W_1) = \sigma(W_2)$. If $(W_1, \psi_1)$ and $(W_2, \psi_2)$ are disconnected than we say that they are equivalent iff all of their connected components are equivalent.

**Definition 2** We denote $\tilde{\text{Cob}}_3$ the categories of $2+1$ cobordism bounding special $3+1$ cobordisms, modulo $4+1$ cobordisms. The objects are sequences of non-negative integers and the morphisms are the equivalence classes of four manifolds $[W, \psi]$, for which smooth boundary functions $g$ exist. The composition is given by identification of respective boundary pieces and shrinking of excessive $S^3$'s.

**1.1.4) Moves between Four Manifolds**

In the presentation of a three dimensional manifold we can change either the bounding four fold $W$ or the Morse function on it that describes a handle decomposition of $W$: The move that result from changing the Morse function and an elementary modification of $W$ are described in the next chapter. In this subsection we wish to give the possible changes in the handle decomposition of $W$, if we change representatives of $[W, \psi]$. Other than in [Ki] we need to account for the fact that $\pi_1(W)$ may be non trivial and surgeries on $W$ may thus be non local. Here we shall use also techniques and definitions that will be explained later in this paper.

If we give a surgery description of a connected cobordism $M$ starting from $M_0$ (e.g. $= \mathcal{H}(\tilde{g}^+, l, \tilde{g}^-)$) the four fold $W$ is given by $M_0 \times I$ with four dimensional $k$-handles attached to the upper boundary. It is clear, using cancellations and connectivity arguments, that we may omit all 0- and 4-handles. In fact we are interested in four folds, which admit a handle decomposition of the form

$$W \cong M_0 \times I \cup h^1 \cup \ldots \cup h^1 \cup h^2 \cup \ldots \cup h^2,$$

where the handles are attached to the boundary piece $\text{int}(M_0) \times \{1\}$.

Let us also introduce $\tilde{W} = M_0 \times I \cup h^1 \cup \ldots \cup h^1 \subset W$ and the three fold $\tilde{M}$ which is the upper boundary component such that $\partial \tilde{W} = M_0 \amalg_{\partial M_0} \tilde{M}$. One useful feature of $\tilde{M}$ is that it contains all homotopy, more precisely:

**Lemma 1** For $j = 0, 1$ the following are an isomorphism and an epimorphism:

$$\pi_j(\tilde{M}) \longrightarrow \pi_j(\tilde{W}) \longrightarrow \pi_j(W)$$

---

*I am indebted to Justin Roberts for remarking the lack of treatment of this point in an earlier version of this paper.*
Proof: In fact for $\pi_0$ both maps are isomorphisms. To show that the inclusion of $\tilde{M}$ into $\tilde{W}$ is an isomorphism it suffices to check that for any four fold $W$, $\pi_j(\partial(W \cup h^1)) \xrightarrow{\sim} \pi_j(W \cup h^1)$ is an isomorphism whenever $\pi_j(\partial W) \xrightarrow{\sim} \pi_j(W)$ is one (for every component). Since $(\partial W) \cup h^1$ is obtained from $\partial(W \cup h^1)$ by gluing in a $D^3 \times I$ along the 0-,1-connected piece $S^2 \times I$, it follows immediately that $\pi_j((\partial W) \cup h^1) \cong \pi_j(\partial(W \cup h^1))$ for $j = 0, 1$. In the case where $h^1$ is attached to two different connected components $\partial W^\alpha, \partial W^\beta$ of $\partial W$, with corresponding connected components $W^\alpha, W^\beta$ of $W$, the spaces $W \cup h^1$ and $\partial(W) \cup h^1$ are given - up to homotopy type - by replacing the respective components by $\partial W^\alpha \cup \partial W^\beta$ and $W^\alpha \cup W^\beta$. If $h^1$ is attached to the same component of $\partial W$ we end up with $\partial W \cup S^1$ and $W \cup S^1$. In both cases the assertion follows easily, since $\pi_1$ is freely generated by known parts. A simple Seifert van Kampen argument shows that $\pi(W \cup h^2) \cong \pi_1(W)/[C]$, where $C$ is the attaching curve of the two handle, completing the verification of Lemma 1.

For a four fold $W$ with a handle decomposition as in (1.4) we may attach a pair of two handles without changing the boundary. More precisely, assume we have attached an $h^2$ to $\tilde{W}$ along any curve $C \subset \tilde{W}$. Suppose $D$ is a small disk which intersects the attaching data of all other two handles exactly once in $C$. Then $\partial(\tilde{W} \cup h^2 \cup h^2) = \partial \tilde{W}$, where the second two handle is attached along $\partial D$ with framing induced by $D$. In the language of bridged links this corresponds to the “$\eta$-move”, and will be discussed in more detail in Section 2.2.2.8.

We may now describe the precise relation between four folds giving rise to the same classes in $\text{Cob}_3$.

Proposition 2 1. For any connected cobordism class $[W, \psi]$ there exists a representative $(W_0, \psi)$, such that $W_0$ is of the form (L4).

2. Suppose two connected four cobordisms $(W_j, \psi_j)$ with handle decompositions as in (L4) for $j = 1, 2$ give rise to the same class in $\text{Cob}_3$. Then there exists a representative $(W_3, \psi_3)$ which is of the form (L4) and can be obtained from either $W_1$ or $W_2$ by a sequence of $\eta$-moves.

Proof: As explained in the proof of Lemma 3 any $W$ with connected upper boundary $M$ is representable without 0- or 4-handles. To show the first assertion it thus remains to replace the 3-handles. Suppose the upper parts $M$ and $M'$ of the boundaries $\partial W$ and $\partial(W \cup h^3)$ are connected. Reading the cobordism piece from $M$ to $M'$ in different directions we have $W' := (M \times I) \cup h^3 \cong h^1 \cup (M' \times I)$. Using a modification as described in Section 2.2.2) we have another cobordism between the same manifolds $W'' = h^2 \cup (M' \times I) = (M \times I) \cup h^2$. We may connect $W''$ to a bunch of $\mathbb{C}P^2$'s or $\overline{\mathbb{C}P^2}$'s (which is a special type of two-handle attachent) such that $\sigma(W'') = \sigma(W')$. The closed manifold $W''' \sqcup_{\partial W} W'$ thus bounds some $Q^{(5)}$, so that we have $W \cup h^3 \sim W M W''' = W \cup h^2 \cup ... \cup h^2$.

By definition there is a connected five fold $Q$ cobording $W_1$ to $W_2$. As in [Ki] we may forget about 0- and 5-handles and make modifications on $Q$ that replace all 1- and 4-handles by 3- and 2- handles. We may push the remaining 2-handles close towards the boundary $W_1$ and the remaining 3-handles towards $W_2$. Hence $Q$ is given by the composite $Q = Q_1 \sqcup_{W_1} (-Q_2)$ of cobordisms, build up from handles of the same type. For both pieces
we have that $Q_j$ cobords $W_j$ to $W_3$, and that it is given by attaching five dimensional two handles $(W_j \times I) \cup h^2 \cup \ldots \cup h^2$. Now, a corresponding two surgery on a four fold $W$ is determined by an attaching curve $S^1 \hookrightarrow W$. A neighborhood $D^3 \times S^1$ is removed and an $S^2 \times D^2$ is glued in along the new boundary component $S^2 \times S^1$. It remains to be checked that this is the same as attaching the pair of (four dimensional) two handles described in the $\eta$-move above.

For a four fold $W$ of the form (1.4) we may apply Lemma 1 to find a (PL-) homotopy of the attaching curve to a path $C$ in $\tilde{M}$. By transversality we may choose $C \subset \tilde{M}$ to be without selfintersection and the homotopy in $W$ to be an ambient isotopy of curves. Thus we may assume that $S^2 \times D^2$ is glued in along a curve in $\tilde{M}$. Since all framings of the curve in four dimensions are homeomorphic, we may choose it such that the upper and lower hemisphere of the fiber $D^3 = D^3_+ \amalg D^3_-$ lie in an upper and a lower collar of $\tilde{M}$. I.e., we have $S^1 \times D^3_+ \hookrightarrow \tilde{M} \times [0, \pm 1] \subset \tilde{M} \times [-2, 2] \subset W$, and $S^1 \times D^3 \cap \tilde{M} \times \{0\} = S^1 \times D^2_\pm$. Correspondingly, we may decompose the sphere of the newly attached piece into two hemispheres $S^2 = S^2_+ \amalg S^2_\pm$. As a result we obtain the decomposition

\[
\begin{align*}
(M \times [-2, 2] - (S^1 \times D^3)) & \amalg S^1 \times S^2 \ D^2 \times S^2 \\
((M \times [-2, 0]) - (S^1 \times D^3)) & \amalg D^2 \times S^2 \ D^2 \times S^2 \ (M \times [0, 2]) - (S^1 \times D^3))
\end{align*}
\]

The first and fourth piece are clearly homeomorphic to $\tilde{M} \times I$ with $R_\pm = S^1 \times S^2 = \tilde{M}$ lying now in the upper (lower) boundary part and we may omit one of them (e.g. the last one) since the attachment is just thickening a boundary piece. The gluing of the second piece is nothing but a two handle attachment of $h^2 = D^2 \times S^2$ to $\tilde{M} \times I$ along the framed knot $R_\pm$. The same is true for the gluing of the third piece. Only now the attaching data $D^2 \times S^2 \hookrightarrow D^2 \times \partial(S^2) \subset \partial(\tilde{M} \times I \cup h^2)$ is inside of the surgered region. We may push a circle $\{p\} \times S^1 \subset D^2 \times \partial S^2$ outside of this region by moving $p \in D^2$ to a point $p' \in S^1$, so that it is a meridian $\{p'\} \times \partial D^2 \subset \partial R_\pm$ of the attaching knot of the first handle. Similarly, it follows that for a small interval $J \subset D^2$ the attaching ribbon $J \times \partial D^2_\pm$ can be pushed outside of the surgered region to an annulus as described in the $\eta$ move.

\[\square\]

1.2) A Central Extension by $\Omega_4$

By construction a morphism in $\tilde{Cob}_3$ bounds a morphism in the original category $Cob_3$. Explicitly, it is given by the restrictions

\[M = g^{-1}(1) \quad \text{and} \quad \psi|_{(1) \times \Sigma}, \quad (1.5)\]

which are clearly compatible with the compositions. In summary, we have a canonical functor,

\[\tilde{Cob}_3 \quad \tilde{\rightarrow} \quad Cob_3,\]

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which is full by Rohlins result stated in (1.2). In this section we investigate the structure of $\mathcal{D}$ in more detail. In particular, we interpret $\widetilde{\text{Cob}}_3$ as a central extension of $\text{Cob}_3$ by the cobordism group $\Omega_4$, which is related to different central extensions by [At], [Ko], and [RM].

### 1.2.1) The $\Omega_4$-Action and Anomalous TQFT’s

To a connected four fold $Y$ with boundary function $g$ we can canonically assign a natural transformation $\xi_Y$ of the identity functor on the subcategory of connected four cobordisms. We first apply the construction (1.1) to the identity $\text{id} = \Sigma(\bar{g}) \times [0,1]$ of $\text{Cob}_3$, so that $(\text{id})_{cl} \sim S^3$. From this we obtain $\xi_Y(\bar{g})$ by identification of $(\text{id})_{cl}$ with the boundary of $D^4 \# Y$, i.e., a four ball with $Y$ connected to it. Clearly, the result of composing a connected morphism $[W,\psi]$ in $\widetilde{\text{Cob}}_3$ with $\xi_Y$ on either side is $[W \# Y,\psi]$. In particular, we have $\xi_Y \circ \xi_Z = \xi_Y \# Z$.

The images of these transformations in $\widetilde{\text{Cob}}_3$ only depend on the class of $Y$ in the cobordism group $\Omega_4 \cong \mathbb{Z}$, and thus form a free abelian group generated by $y = [\xi_{\mathbb{CP}^2}]$. Therefore, we can view the functor $\mathcal{D}$ as a quotient map to the orbits of the morphisms under the free action of $\Omega_4 \subset \text{Nat}_{\widetilde{\text{Cob}}_3}(\text{id})$. In this picture we can define a projective (or anomalous) TQFT as a fiber functor on $\widetilde{\text{Cob}}_3$. (Examples are the TQFT’s constructed in [RT] and [Tu] using quantum groups, see also Section 5.)

Assuming that $\bar{g} = (0)$ is assigned to a one dimensional vectorspace, and the functor on $D^4 \in \text{End}(0)$ is non-zero, this yields a number $\theta \neq 0$ for $y$, which determines all of $\xi$. Clearly, the TQFT-functor factors through $\mathcal{D}$ into a fiber functor on $\text{Cob}_3$ (an ordinary or “anomalie free” TQFT) if and only if $\theta = 1$.

### 1.2.2) Another Two-Cocycle

The extension of $\text{Cob}_3$ is in fact non-split. The precise behavior can be given by a “two-cocycle” $\mu$ of $\text{Cob}_3$, which measures the non-additivity of the signature for the composite of two four manifolds $W_1$ and $W_2$. It turns out that $\mu$ only depends on the charts of $M_i = g_i^{-1}(1)$. We write

$$\psi_1 : \Sigma(\bar{g}) \to M_1 \amalg_2 \mathcal{H}^-(\bar{g}_1^-) \quad \psi_1 : \Sigma(\bar{g}) \to M_2 \amalg_2 \mathcal{H}^+(\bar{g}_2^+)$$

where $\bar{g} = \bar{g}_1^+ = \bar{g}_2^*$

with identifications $\sim$ along the common surfaces. We denoted by $\mathcal{H}^\pm(\bar{g})$ the respective unions of handlebodies. In the following discussion we shall disregard the extra $S^3$, which have no effect on the signature. To give explicit expressions we introduce the Lagrangian subspaces (in rational homology)

$$\Lambda_i = \ker(H_1(\psi_i)) \subset H_1(\Sigma(\bar{g}))$$

and

$$V^\pm = \ker(H_1(i_\pm)),$$

where $i_\pm$ is the inclusion of $\Sigma(\bar{g})$ into $\mathcal{H}^\pm(\bar{g})$ so that $H_1(\Sigma(\bar{g})) = V^+ \oplus V^-$.  

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If \( \omega \) is the skew form on \( H_1(\Sigma(\bar{g})) \), let us define a bilinear \( \phi' \) form on \( U' = \Lambda_1 + \Lambda_2 \) as follows. For \( \lambda \in U' \) denote \( \lambda = \lambda^+ + \lambda^- \) the components in the subspaces \( V^\pm \). Also, we choose a \( \lambda_i \in \Lambda_i \) with \( \lambda = \lambda_1 + \lambda_2 \). Then

\[
\phi(\lambda, \eta) = \omega(\lambda_2, \eta_1) - \omega(\lambda^+, \eta^-) \quad (1.6)
\]

It is easily checked that \( \phi \) is symmetric and does not depend on the choices of \( \lambda_i \) and \( \eta_i \). Also, it factors into a bilinear form \( \phi : U \times U \to \mathbb{R} \) on the quotient space

\[
U = \frac{\Lambda_1 + \Lambda_2}{\Lambda_1 \cap V^- + \Lambda_2 \cap V^+}.
\]

We denote by \( \mu(\psi_1, \psi_2) \) the signature of \( \phi \).

Applying a result from [Wa] we find the anomalie of the signatures:

**Proposition 3** We have

\[
\sigma(W_1 \circ W_2) = \sigma(W_1) + \sigma(W_2) + \mu(\psi_1, \psi_2) \quad (1.7)
\]

**Proof:** As in [Wa] we denote by \( Y_+ = W_2, Y_- = W_1, X_+ = M_2 \Pi \{\Sigma(\bar{g}_2^+) \times [0, 1]\} \Pi \mathcal{H}(\bar{g}_2^+) \Pi \mathcal{H}^-(\bar{g}) \) with orientation induced by \( Y_+ \), \( X_- = M_1 \Pi \{\Sigma(\bar{g}_1^-) \times [0, 1]\} \Pi \mathcal{H}^- (\bar{g}_1^-) \Pi \mathcal{H}(\bar{g}) \) with orientation opposite to \( Y_- \), \( X_0 = [0, 1] \times \Sigma(\bar{g}) \) with orientation from \( Y_- \), and \( Z = \Sigma(\bar{g}) \times 0 \Pi \Sigma(\bar{g}) \times 1 \) where the orientation on the 1 component is standard and on the 0 component opposite. (Consult again Figure 1.1)

With this orientation the bilinear form on the two component space \( H_1(Z) = V_0 \oplus V_1 \) is given by \( \tilde{\omega} = -\omega \oplus \omega \). From the inclusions of \( X_+, X_- \), and \( X_0 \) we have Lagrangian subspaces

\[
A = V_0^- \oplus \Lambda_1 \quad C = V_0^+ \oplus \Lambda_2
\]

and

\[
B = \left\{ (-x, x) : x \in H_1(\Sigma(\bar{g})) \right\},
\]

where we consider \( \Lambda_i \subset V_1 \). It is now straight forward to show that the form in [Wa] reduces to (1.6). Its signature is identified with the signature anomalie of the four folds. Also, the gluing of the four ball in the composition does not change the signature.

Let us record the explicit anomalies in two special cases of interest.

**Lemma 4**

1. The anomalie vanishes if one of the \( \Lambda_i \) coincides with a standard subspace \( V^\pm \).

2. For the torus \( \Sigma((1)) \) assume that \( e^\pm \in V^\pm \alpha_i \in \Lambda_i \) are non zero vectors in the one dimensional Lagrangian subspaces. Then the anomalie \( \mu(\psi_1, \psi_2) \) is the signature of the binary form \( \phi \), where

\[
\phi_{ij} = \omega(\alpha_i, e^-)\omega(e^-, e^+)\omega(e^+, \alpha_j) \quad \text{with} \quad i \leq j
\]
Remark: Our definition of a bounded cobordisms is different from the description in [Ko] or [Wk] and yields the actual signature of the linking diagram on a standard manifold. The cocycle gives also rise to a non-split central extension of mapping class groups, if we restrict to invertible morphisms of \( \widetilde{\text{Cob}}_3 \). It is related to well known central extensions, which are described in terms of canonical framings of \( TM \oplus TM \) in [At].

### 1.3) Ordering and Connectivity

We chose the objects of \( \widetilde{\text{Cob}}_3 \) and \( \text{Cob}_3 \) to be ordered sequences of surfaces. It is clear that reordering does not change the isomorphism class. In fact, we have canonical isomorphisms

\[
[\pi] : (g_1, \ldots, g_K) \mapsto (g_{\pi(1)}, \ldots, g_{\pi(K)})
\]

for every \( \pi \in S_K \), which are homeomorphic to unions of \( \Sigma_{g_j} \times [0,1] \). In \( \widetilde{\text{Cob}}_3 \) we bound the \([\pi]^{cl}\)’s by \( D^4 \)’s.

The fact that a cobordism \( M \) in \( \text{Cob}_3 \) is has connected components \( M_j \) is expressed by the formula

\[
M = [\pi](M_1 \otimes \ldots \otimes M_r).
\]

From this it follows easily that every cobordism \( M : \bar{g} \rightarrow \bar{g}' \) in \( \text{Cob}_3 \) can be written as a product of cobordisms of the form

\[
[\pi](N \otimes \text{id}_{\bar{g}'} \otimes \ldots \otimes \text{id}_{\bar{g}'_r}),
\]

where \( N \) is a connected cobordism and \([\pi]\) a permutation as in (1.8).

By the connectedness assumption in (1.3) a presentation will thus be a collection of presentations of the connected components. It is not efficient to try to evaluate the presentation of the composition of two cobordisms of the form (1.9) directly. We prefer to expand the product first by using the most elementary composition rules. These are to write \( M_1 \otimes \ldots \otimes M_r \) as the product of the commuting cobordisms \( id \otimes \ldots id \otimes M_j \otimes \ldots \otimes id \), the rule \([\alpha] \cdot [\pi] = [\alpha \circ \pi]\) for the permutations, and \([\pi]M_1 \otimes \ldots \otimes M_r = M_{\pi(1)} \otimes \ldots \otimes M_{\pi(r)}[\pi]'\), if \( \pi \) is the permutation of ordered subsets. It is not hard to see, that for the correct expansion and correct bracketing the only other compositions that will occur are of the form

\[
[\alpha] \cdot N \quad N \cdot [\alpha]
\]

and

\[
L := N \otimes id \cdot id \otimes M,
\]

where the cobordisms \( N, M \), and \( L \) are connected and \( \alpha \) is a permutation.
2) Enhanced Surgery Presentations

Assume we are given a compact $n$-manifold $X$ and a Morse Function $f : X \rightarrow [0, 1]$. It is an elementary fact (see [Mi]) that if $f$ has a singularity of index $j$ with non-degenerate critical value $a$, then the sublevel manifold $X^{\leq a+\epsilon} = f^{-1}([0, a+\epsilon])$ is homeomorphic to $X^{\leq a-\epsilon}$ with a $j$-handle $h_j$ attached at its boundary. This means, we have an embedding $S^{j-1} \times D^{n-j} \hookrightarrow f^{-1}(a-\epsilon) \subset \partial X^{\leq a-\epsilon}$, and we attach $h_j = D^j \times D^{n-j}$, along the respective piece in the boundary $\partial h_j = \partial D^j \times D^{n-j}$. The classical link presentations of [Wc] and [Li] of three folds result from attaching two handles to a four ball such that $M = \partial (D^4 \cup h^2 \cup \ldots \cup h^2)$. The theorem of Kirby (see [Ki]) states that two such presentations (with the same signature) can be related by sliding the handles across each other. Both results rely on the fact that the manifold $S^3 = \partial D^4$ we surger on is connected and simply connected. Neither is true for the manifolds $\mathcal{H}(\bar{g}^+, l, \bar{g}^-)$.

For these reasons we shall admit presentations that involve also surgeries with one handle. Most of this section is devoted to describe in detail the calculus of “bridged links”, which is the corresponding generalization of the link calculus for closed manifolds. Some elements of this description were also used in [FR]. We shall also discuss the calculus of “bridged ribbon graphs” analogous to [RT0] or [RT] for generic, planar projections of the link diagrams.

2.1) Some Cerf Theory

A presentation for a manifold using a Morse function $f$ as above is only defined if the critical points and critical values of $f$ are non-degenerate. Since the space of these functions is disconnected it is not always possible to deform the presentation of one function into the presentation given by another function. Nevertheless, if we admit also functions of codimension one we obtain again a connected space.

The content of Cerf’s theory [Ce] is to determine the connectivity of function spaces depending on their codimensions, and the behavior of the singularities once a path of function passes through a lower stratum. In this section we shall review the elements that are relevant for our purposes. We start by introducing the function spaces, that will describe the bridged link presentations and its moves.

2.1.1) Excellent and Codimension One Functions

For given $W$ and boundary function $g$ introduce a Riemannian structure such that the gradient flow of $g$ is parallel to the boundary piece $\psi([0, 1] \times \Sigma)$. As usual we introduce the space $\mathcal{F}$ of smooth functions on $W$ which coincide with $g$ near $\partial W$. Also, we denote by $\mathcal{F}^0$ the “excellent functions”, which are Morse functions with distinct critical values, and by $\mathcal{F}^1$ the codimension one functions. The latter set is the union of the set $\mathcal{F}^1_\alpha$ of functions with
distinct critical values and only one degenerate critical point, which is a birth (or death) point, and the set $\mathcal{F}_\beta^1$ of Morse functions for which exactly two critical values coincide.

We denote the stratification of codimension one:

$$\mathcal{F}_T = F^0 \cup F^1_\alpha \cup F^1_\beta$$

We also denote by

$$p\mathcal{F} \subset \mathcal{F} \quad p\mathcal{F}^0 \subset \mathcal{F}^0 \quad \text{etc.}$$

the subspaces of functions for which all singularities have index one or two, all index two singularities have higher values than the index one singularities, and all singularities are in general position.

Correspondingly, the codimension one strata in there are denoted

$$p\mathcal{F}^1_\alpha, \quad p\mathcal{F}^1_\beta, \quad p\mathcal{F}^0_i \quad i = 1, 2, 3.$$  

They are the functions with one 1-2 birth point, functions with exactly two critical values coinciding, excellent functions for which the ascending and descending manifolds of two index $i$ singularities intersect in a single trajectory, and functions where the descending manifold of a 2-singularity and the ascending manifold of a 1-singularity intersect with an intermediate level three fold in a curve and a sphere that are tangential to each other in exactly one point (and transversal in all other).

Their union with $p\mathcal{F}^0$ is denoted $p\mathcal{F}_T$.

It is occasionally useful to label the singularities of index one (three) as to which components of the surgered manifold are connected by the respective one handles.

**Definition 3**  
1. The label of a critical point $c$ of index one (three) is the pair $(\lambda, \mu)$ of connected components of $g^{-1}(0)$ which are intersected by the descending (ascending) manifold of $c$.

2. For a function $f \in p\mathcal{F}^0$ we denote by $\tilde{M}$ the level manifold $f^{-1}(y)$, where $y$ lies above all critical values of index one and below all critical values of index two.

It is clear that $\tilde{M}$ can be obtained from $\mathcal{H}$ by either connecting different component to each other or to copies of $S^1 \times S^2$. In fact the connection prescription may be entirely determined from the set of labels of the index one singularities.

**2.1.2) Paths and Deformations of Paths of Functions**

A basic result in [Ce] asserts that the space $\mathcal{F}_T$ is 0-connected and any path in $\mathcal{F}_T$ can be deformed to a path which is transversal to the lower dimensional strata $\mathcal{F}^1$. By the general theory of stable manifolds, a presentation changes along a path in $\mathcal{F}^0$ only by an isotopy. A complete set of “moves” for presentations on the same four fold are thus given by the (transversal) passages through $\mathcal{F}^1$.

A generic path of functions in $\mathcal{F}_T$ is conveniently illustrated by a graphic. This is a collection of paths $(f_t(c^i_t), t)$ where $c^i_t$ is a continuous family of critical points of $f_t$. The point where $f_t$ passes through $\mathcal{F}^1_\alpha$ is given by a beak joining an index $i$ and an index $i + 1$ critical point. Going through $\mathcal{F}^1_\beta$ corresponds to crossing of two components of the graphic.
Also, we draw a dashed line at points of $t$ where the ascending and the descending manifolds of two critical points of the same index intersect each other in a trajectory.

In particular, a path through the codimension one strata of $p\mathcal{F}_T$ is presented by the pieces of a graphic given in [2.2].

![Codimension-One Strata in Graphics](image)

Figure 2.2: Codimension-One Strata in Graphics

In the graphic for the intersection of the manifolds of index one critical points we also indicated the possible labels.

The next more general result in [Ce] concerns deformations of generic paths. They may always be chosen such that the paths pass transversally through singularities of codimensions two. For more general results on functions with framings and a useful and humorous summary of singularities of higher codimensions see [Ig]. The list of elementary deformations can be illustrated by graphics as in [Ce], [Ki], and [HW, pg26]:

1. **Independent Trajectories**: If the descending manifolds of a trajectory $c_t$ of singularities do not intersect the ascending manifolds of another trajectory $c'_t$ for any $t$ then the two trajectories can be moved independently from each other. (e.g., [HW, pg.65]). For the relevant examples see [Ki].

2. **Triangle Lemma**: For $i_1 + i_3 \leq 3$, $\inf(i_1, i_3) \leq i_2 - 1$ or $i_1 = i_2 = i_3 \leq 2$ we have the move:
3. **Beak Lemma**: If \( i > 0 \) we have the move

4. **Dovetail Lemma**:

The remaining moves are obtained from the above by reflections \( t \to 1 - t \) and \( f \to 1 - f \).

### 2.2) Bridged Link Presentations

We introduce an enhanced presentation of a three fold by adding both one and two
handles to another compact, oriented three fold, which does not have to be connected or
simply connected.

#### 2.2.1) Surgery Presentation from Bridged Links

From the structure of the singularities of a function \( f \in \mathcal{F}^0 \) we obtain a presentation of
the cobordism \( g^{-1}(1) \) by considering the intersections \((\simeq S^0, \simeq S^1)\) of the descending (or
unstable) manifolds of the critical points with \( g^{-1}(0) \). These yield together with framings
of their normal bundles a unique surgery prescription. To be more precise, let us introduce
the following standard handles:

1) A two handle \( h^2 = D^2_a \times D^2_b \) and two intervals \( I, J \cong [-1, 1] \) such that \( D^2_a = J \times I \).

2) A 2-sphere \( S^2 \subset \mathbb{R}^3 \) with meridian \( S^1 = \{(x, y, 0) \in S^2\} \) and involution \( \rho : S^2 \to S^2 : (x, y, z) \to (x, y, -z) \). Also denote by \( B^3 \) the oriented ball bounded by \( S^2 \) and by \( D^2_\pm \) the
upper and the lower hemisphere.

On \( \mathcal{H} = \mathcal{H}(\bar{m}^+, l, \bar{m}^-) \) for arbitrary \( l \) and a function \( f \in \mathcal{p}\mathcal{F}^0 \) a bridged link presentation
consists of the following data:
1) An orientation preserving embedding $\phi = (\phi_1, \phi_1', \ldots, \phi_m', \phi_m)$ of $2m$ copies of $B^3$.

The pairs of balls are neighborhoods of the points where the descending trajectories of the index one singularities intersect $H$. The three fold $\tilde{M}$ just above all index one singularities is then (for a suitable given Riemann structure) naturally homeomorphic to the manifold obtained by removing the interiors of the images of the three balls and identifying the pairs of boundaries using the isomorphisms

$$\bar{\phi}_j := \phi_j' \circ \rho \circ \phi_j^{-1}.$$ 

Occasionally, we shall prefer the picture of an actual handle attachment, where we glue in an additional $S^2 \times [0,1]$ in between the boundary components.

The surgery is described further by considering the intersections of the descending manifolds of the index two singularities with $\tilde{M}$. Close to the singularity, the normal (Morse) form of $f$ determines - up to homotopy - a trivialization of the normal bundle. This, in turn, is determined by a non vanishing section of the normal bundle in any level manifold, i.e., an embedding of the ribbon: $R := J \times \partial D^2_b \hookrightarrow \tilde{M}$, or more precisely of a link of ribbons. Undoing the index one surgery we obtain the next ingredient of the surgery data:

2) A map $\tau = (\tau_1, \ldots, \tau_s) : R \sqcup \ldots \sqcup R \hookrightarrow H$ of $s$ copies of the ribbon $R$ which is an embedding into the complement of the balls $\text{im}(\phi)$ except for a finite number of intervals $J \times \{p\} \subset R$. The right and left sided limits $\tau_j^\pm : J \times \{p\} \to H$ are embeddings into spheres $\phi_k(p)(S^2)$ and $\phi_k'(p)(S^2)$ such that

$$\bar{\phi}_k \tau_j^\pm = \tau_j^\mp.$$ 

Since we will be dealing with non-simply connected manifolds (with no canonical framings of tangent bundles over the one-skeletons) we preferred here the language of ribbons over that of framing numbers.

A typical picture of the “bridged link” is given in Figure 2.3. The action of the flip $\rho$ is indicated by the invariant meridians, drawn here as dashed lines.

It should be kept in mind that the two spheres may lie in different components of $H$ depending on the label of the critical point, and are thus not connected by any cancelling ribbon (as in move 4) of the next section).

The surgery on a component of the resulting ribbon link in $\tilde{M}$ is given as usual by extending the embedding to the tubes $D^2_a \times \partial D^2_b \supset J \times 0 \times \partial D^2_b$. Then we remove the images and glue in $\partial D^2_a \times D^2_b$ with reversed orientation.

2.2.2) Moves for Bridged Links

In this section we compile a list of “moves” of bridged link presentations that do not change the homeomorphy class of the presented three fold. Some of them come from deforming the Morse function through a codimension one stratum, some are obtained by modification of the bounding four fold.
1) Isotopies An isotopy is a superposition of isotopies of the three balls in $\mathcal{H}$ and of the ribbon link in $\hat{\mathcal{M}}$. More precisely, after identifying a pair of embedded spheres and their vicinity via the maps $\phi$ and $\phi'$ with the respective standard regions in $\mathbb{R}^3$, we may consider a pair of regions $F \subset S^2$ and $F' = \rho(F) \subset S^2'$. A collar $(U, F) \cong (F \times I, F \times 0)$ (opposite to the bounded balls) of one region, can be moved continuously to the other ending with the isomorphism:

$$F \times I \xrightarrow{\rho \times (1-t)} F' \times I.$$ 

Extending this isotopy (or its inverse) to the identity outside of $F$ and $F'$, we can move pieces of the link diagram through a gluing sphere as indicated in Figure 2.4.

We confined generic presentations to have to 1- and 2-singularities in general positions; yet, an isotopy over a surgery sphere also involves situations where the surgery spheres are not always transversal to the ribbons. At these points we can deform our path such that we pass only through the stratum $p_{F_{12}}$. This means only one ribbon is tangential to the sphere at a time, and this configuration results from pushing a small ribbon loop through the sphere.

2) One- (Handle) Slides They occur when the Morse function passes through $\mathcal{F}_{10}$ and the ascending manifold of an index one singularity $Lo$ is moved transversally through one of the descending trajectories of another index one singularity, $Hi$, with higher value. Considering a level manifold in between the two singularities it is clear that this corresponds to moving a ball with the attached ribbons through a gluing sphere, as was described above for links. A typical situation is suggested in Figure 2.5.

3) Two- (Handle) Slides This is the Kirby move, denoted $O_2$ in [Ki]. It occurs when the path of Morse functions passes through $\mathcal{F}_{20}$, and the intermediate manifolds of the two index two singularities are moved through each other transversally. At the point where the descending two fold of the singularity $Hi$ with higher value goes through the other critical point $Lo$ we first have the corresponding ribbons tangential along small pieces of their boundaries. To describe the result of the slide, the ribbon of $Lo$ is cut along its
middle and the closer part is connected to the ribbon $Hi$ as depicted in Figure 2.6. On an intermediate level manifold this is just an isotopy of the ribbon $Hi$ over the surgered piece along $Lo$.

4) **1-2- (Handle) Cancellation** The Cancellation Lemma of Smale (see, e.g. [HW] pg. 172) states that when for consecutive singularities $c_1$ and $c_2$ of index $i$ and $i + 1$ the intersection of the descending manifold of $c_2$ intersects the ascending manifold of $c_1$ in exactly one trajectory, then $c_1$ and $c_2$ can be canceled against each other. This is achieved along a path of codimension one functions passing through $F_1^1$, i.e., an $i - i + 1 -$ birth- or death-point.

Here we are only interested in the situation where $f$ passes through $pF_1^1$, and where the singularities are presented by a pair of balls $c_1$ and a ribbon $c_2$ intersecting the gluing sphere of $c_1$ in exactly one interval but no other spheres. In this situation we remove in addition to the balls a solid tube $D^2 \times I$ around $c_2$ and collapse the opposite hemispheres of the total removed region using $\rho$ and the framing of $c_2$, see Figure 2.7.

In order to explicitly show that this move preserves the surgered manifold we perform the one surgery, where we already removed the solid torus of the two surgery. This corresponds to identifying the remaining hemispheres, which leaves us with Figure 2.8, where the torus is empty. Completing the two surgery we obtain the right hand side of Figure 2.7.

5) **Modification (Handle Trading)** This move also changes the bounding four manifold, exchanging an index one singularity (or handle) for an index two singularity. Suppose we have a ribbon-component $R = S^1 \times [1, 2]$ which can be extended to an embedding of a disc $D = S^1 \times [2, 0]/S^1 \times 0$, (which may intersect other ribbons-components nicely). Then we duplicate $D$ and push the two copies away from each other. We complete each of the discs to a sphere, so that possible ribbons are attached to the outside, and define the maps $\phi$ and $\phi'$ such that $\bar{\phi}$ is the correct identification of the hemispheres $D$, see Figure 2.9.
The procedure of modification is described in [Wa]. To see that the identification of the hemispheres is the correct one we shall make the argument explicit: We parametrize a tubular neighborhood of $R$ by $J \times I \times \partial D_b^2$ and the thickened, bounded disc by $J \times D_c^2$, with identification $\partial D_b^2 = \partial D_c^2$. Instead of removing only the solid torus we also take out the thickened disc and glue it back later. The opposite torus $\partial D_a \times D_b$ consists of two parts $J \times 0 \times D_b^2$ and $Z \times D_b^2$ with $Z = (\partial D_a^2 - J \times 0)$, which are identified along two copies of $D_b$. If we glue the second part into the empty region along $Z \times \partial D_b^2$, we obtain th region in Figure 2.10 where the insides of the two spheres $D_b^2 \cup D_c^2$ are missing.

The other part, $J \times 0 \times D_b^2$ is first glued to the extra piece $J \times D_c^2$ along the cylinder $J \times \partial D^2$ giving $J \times S^2$ and the included ribbons are $J$ times an interval on $S^2$. Shrinking $J$ to a point, it is clear from the picture that we obtain exactly the desired one surgery.

To describe equivalences of cobordisms we need to include two more moves, which change the signature of $W$ and the number of components of $\mathcal{H}$:

6) **Signature** Add an unknotted, ribbon separated from the rest by a sphere with framing $\pm 1$. This is the Kirby move $\mathcal{O}_1$ and results from connecting a projective space $\mathbb{CP}^2\left(\overline{\mathbb{CP}^2}\right)$ to $W$. We sometimes denote such a link component by $\Phi$ ($\overline{\Phi}$).
7) **0-1-Cancellation** If for a bridged link presentation an $S^3$ component of $\mathcal{H}$ contains nothing but a single surgery sphere we may omit the $S^3$-component, and the respective pair of surgery spheres. (The content of the presentation in this situation is to connect the $S^3$ component to another component which is of course trivial.)

Finally, let us introduce a move which is a combination of the previous ones:

8) **$\eta$-move, $0 \circlearrowleft^0$-move** An $\eta$-move is the addition or subtraction of a pair of ribbons $0 \circlearrowleft$ where one ribbon extends to a disc which is intersected exactly once by the second ribbon $C$ but no other ribbon of the presentation. As described in the proof of Proposition 4 this corresponds to the two-surgery $W \to (W - S^1 \times D^3) \cup D^2 \times S^2$.

When the ribbon $C$ can also be contracted to an annulus, and the pair of ribbons can be isolated from the rest, we call the associated $\eta$-move an $0 \circlearrowleft^0$-move. It corresponds to the surgery $W \to W \# S^2 \times S^2$. Using two slides shows that $0 \circlearrowleft^0 = 0 \circlearrowright^{2n}$. Thus, we shall tacitly assume that the $0 \circlearrowright^0$-move also implies addition and removal of an isolated $0 \circlearrowright^1 = \Phi \cup \Phi$-piece. This corresponds to connected summing with the non-orientable $S^2$-bundle $S^2 \times S^2$.

**2.3) Equivalences of Bridged Links**

2.3.1) **Completeness of Moves for Bridged Links**

A surgery presentation of a three manifold (e.g., a morphism of $\widetilde{\text{Cob}}_3$), may be changed by either changing the bounding four fold $W$ within its $4+1$-cobordism class or by changing the handle decomposition of $W$. As indicated in the beginning of this chapter it is equivalent
to speak of excellent functions on $W$ instead of handles so that we may use the results of Cerf’s Theory from Section 2.1) and the correspondence between codimension one strata of $p\mathcal{F}_T$ and the handle slides cancellations given in Section 2.2.2). We know that any two excellent functions in $p\mathcal{F}_T$ can be connected by a path in $\mathcal{F}_T$. Yet, this space is too large as it contains singularities of all indices so that we have to deal with all types of handles. For moves between bridged links we are only interested in decompositions of the form (1.4), which is equivalent to considering only functions in $p\mathcal{F}_0 \subset p\mathcal{F}_T$. We need the following refinement of the connectivity result from [Ce].

**Lemma 5** The space $p\mathcal{F}_T$ is 0-connected.

**Proof:** The elimination of components of a graphic with index zero and index four can be literally taken from [Ki]. The trajectories of index three singularities start and end
in birth - and death - points, and, similarly as for index one trajectories in [Ki], can be rearranged as disjoint trajectories as indicated in first graphic of Figure 2.11, using the Beak Lemma. Introducing a double Dovetail we find the second graphic, and another application of the Beak Lemma and the Principle of Independent Trajectories yields the third picture. Removing the remaining Dovetails we find a graphic which contains only index one and two singularities. By the Principle of Independent Trajectories we can move all index one trajectories below the index two trajectories. We thus end up with a path in \( F_T \).

Put in a different language, Lemma 5 states that all handle decomposition of a four fold of the form \((1.4)\) can be related to each other by isotopies, 1- or 2-slides, and 1-2-cancellations. If we combine this result with Proposition 2 we arrive at the following completeness of moves:
Corollary 1 If two bridged links on $\mathcal{H}$ present the same manifold with boundary, then they are related by a sequence of the moves 1)- 8) from Section 2.2.2).

Note, that the modification has been added to the list, even though it was not used in the preceding arguments.
In fact it can be expressed as an $\eta$-move, where the long ribbon $C$ is chosen as a connecting strip between the two spheres, followed by a cancellation along $C$.

2.3.2) Reduction of Moves

It turns out that the complete set of relations between bridged links, as listed in Corollary 1, is highly redundant. It is not hard to imagine that there are many possibilities to select a minimal subset of moves from it, which generates all other moves and equivalences. Let us assume that the ribbons passing through a general surgery sphere penetrate at given intervals $I_1 \ldots I_k \ldots$. Then the set of moves that has been proposed in Theorem 4 of the introduction follows from the following reduction:

Proposition 6 Two bridged link presentations yield the same three manifold if they are related by the following moves:

1. Isotopies Aside from general isotopies of surgery spheres and ribbons, which are constant along at the intervals $I_j$, we have isotopies where ribbon pieces are pushed through surgery spheres of different components. These can be written as combinations of
   a) Opposite actions of the braid group of the two spheres.
   b) An untangled, unbraided loop being pushed through.

2. One Slides We may assume that the $Hi$ -sphere is pushed through Lo first, followed by the attached ribbons. More precisely, before the move we have ribbons $r_1 \ldots r_m$ going through $I_1 \ldots I_m$ of Lo and ribbons $r_{m+1} \ldots r_{n+m}$ going through $I_1 \ldots I_n$ of Hi. After the move we have $r_1 \ldots r_{n+m}$ going through $I_1 \ldots I_{n+m}$ of Lo. Also, we only consider one slides where the two Lo spheres are in different components.

3. void

4. Cancellation We only consider cancellation with no other ribbons but the cancelling one attached.

5. Modification As stated.

6. Signature As stated.

7. void

8. void
Proof: An isotopy over a surgery sphere can be deformed into an isotopy where the strands in a vicinity of the sphere are always radial, except for passages through the stratum \( p\mathcal{F}_{12} \). The latter give rise to the case 1.b). The ambiguity of pushing the strands into the prescribed positions \( I_j \) are given by elements of the braid group as described in 1.a) and 2\( \pi \)-twists of the strands, which can be reexpressed as the combination of two loops and one braid.

In order to verify 2) we deform a one slide such that first the \( Hi \)-sphere is pushed through \( Lo \), (the attached ribbons going through \( Lo \) will then be loops at \( Lo_0 \)) and then the ribbons at \( Hi \). Conjugating everything with isotopies at \( Lo \) we can confine ourselves to the situation where the ribbons are ordered as described.

If we have the general situation of a cancellation as in Figure 2.7 we may push the cancelling ribbon \( c_2 \) to the side and use modifications as in Figure 2.12 to reduce the cancellation to an isolated one. The reduction can also be found by using two slides. Specifically, we may separate one penetrating ribbon after the other from the surgery spheres by sliding it over the extra cancelling ribbon.

\[ \text{Isotopy} \quad \text{Modif.} \quad \text{Modif.} \quad \ldots \quad \text{Modif.} \quad \text{Isotopy} \quad \ldots \]

\[ \text{Figure 2.12: Specialization of Cancellation} \]

The two-slides can be expressed in terms of cancellations. This is shown in Figure 2.13 where we move the surgery sphere \( A \) along the \( Lo \) ribbon of the two-slide.

Move 6) is omitted if we are only interested in cobordisms of \( \tilde{\text{Cob}}_3 \). It is easy to see that the removal of \( 0 \mathcal{C} \ldots \) as in move 8) can also be given by a modification at the annulus followed by a 1-2-cancellation along the ribbon \( \mathcal{C} \). Finally, if we have a pair of surgery spheres in the same component we may move them close to each other and replace the moves 1a), 1b) and 2) by ordinary isotopies through an annulus, which are conjugated by a modification. \( \square \)

The following, alternative set of generating moves among ordinary links constitutes the proper generalization of Kirby’s original calculus to non-simply connected manifolds:

**Proposition 7** Suppose for two ordinary link presentations on a connected not necessarily simply connected manifold \( M_0 \) the surgered manifolds are homeomorphic. Then we can relate the presentations by a sequence of the following moves
If we consider only cobordisms in $\tilde{\text{Cob}}_3$ the $O_1$-move is again omitted.

If $M_0$ is also simply connected we may replace the $\eta$-move by the $0 \otimes 0$-move.

Proposition 7 can be directly derived from Proposition 6, thus yielding a more structured proof of Kirby’s original theorem. The strategy is to consider the formal spaces of bridged links ($BL$) and ordinary links ($L$) in $M_0$ and define the quotient space $(BL)$ and $(L)$, where the relations from the two previous Propositions have been imposed. We certainly have an inclusion $\iota: (L) \hookrightarrow (BL)$ and it is implied in the proof of Proposition 6 that this factors into a map $\bar{\iota}: (L) \rightarrow (BL)$. If we choose a path between associated surgery spheres for each bridged link we can define a map $p: (BL) \rightarrow (L)$ by moving the spheres close to each other along the paths and eliminating them with a modification. What needs to be established is that $p$ factors into a map $\bar{p}: (BL) \rightarrow (L)$, which is independent of the choice of the recombination paths. It is then obvious that $\bar{p}$ and $\bar{\iota}$ are inverses of each other so that the
two link-calculi are equivalent on connected manifolds. A detailed discussion will be given in a separate paper, [Ke3].

In this context the interpretation of the $\eta$-move is to compensate for the recombination ambiguity of the isolated cancellation diagram. For the same reason the $\eta$-move was introduced in [Ki], when trajectories of one-handles that started and ended in different birth and death points had to be eliminated. The graphic of the function is changed as indicated in Figure 2.14, where the vertical lines indicate modifications of the fourfold. If the recombination is, e.g., done along the trajectories ending at the birth point, the first modification is an addition of a $0^0\bigotimes^0$-element, and the second a subtraction of an $\eta$-configuration, where $\mathcal{C}$ runs along the trajectories of the surgery spheres.

![Figure 2.14: Elimination of One-Handles](image)

The fact that for simply connected manifolds $M_0$ a general $\eta$-move is the composite of $\mathcal{O}_2$-moves and a $\bigotimes^0$-move has been proven in [Ki]. A slightly more direct argument starts from a general PL-homotopy of the curve $\mathcal{C}$. This is made transversal, such that only at a finite number of times there may be single crossings of $\mathcal{C}$ with another segment $\mathcal{X}$, which can be a piece of $\mathcal{C}$ itself or another ribbon. If we move the annulus around $\mathcal{C}$ to this crossing it can be substituted by a two-slide of $\mathcal{X}$ over the annulus. For the change of framing of $\mathcal{C}$, but also the reexpression of the $0^0\bigotimes^0$-move, using auxiliary $\Phi$'s and $\mathcal{O}_1$-moves see [Ki].

Finally, note that the reduction of the $\eta$-move can also be taken from its five dimensional interpretation. In particular, it is clear that the attaching curve $\mathcal{C}$ for five dimensional 2-handle can be contracted inside of $M_0 \times I \cup h^2 \cup \ldots \cup h^2$.

2.4) Bridged Ribbon Graphs

The construction of the invariants of [RT] from a link presentation is preceded by a reduction of the presentation to spaces of so called ribbon graphs, which are generic projections of the links into $\mathbb{R}^2$.

In this paragraph we shall describe the analogous space of bridged ribbon graphs, which will contain not only ribbons but also special pairs of coupons. The relations we will impose on this space are the ones that appear in Proposition 3 and the usual relations that take care of isotopies and the ambiguity of choosing a projection.
We confine ourselves to presentation in a union of $K$ handlebodies $H_g$ or spheres $S^3$. They are given by the union of $K$ diagrams of bridged ribbon graphs in $\mathbb{R}^2$. Clearly, the interior of $H_g$ is homeomorphic to $\mathbb{R}^3$ with $g$ strands, extending to $\infty$, removed. Thus in a projection of a ribbon diagram into $\mathbb{R}^2$ we present the strands as infinite, parallel strips in vertical direction. The pieces of an ordinary directed ribbon graph as in [RT0] are then given by the elements depicted in Figure 2.15.

![Figure 2.15: Ribbon Segments](image)

Figure 2.15: Ribbon Segments

Here the black strands are the ones representing the holes in $H_g$. In a bridged ribbon graph we have in addition pairs of coupons, one with strands going up (labeled $(j,+)$) and one with the same number of strands going down. The two coupons, as in Figure 2.16, live in different $\mathbb{R}^2$’s of the K-component ribbon-graph, if they present surgery spheres in different handlebodies.

![Figure 2.16: Pairs of Coupons](image)

Figure 2.16: Pairs of Coupons

We have the usual relations for projections of ribbon graphs, e.g., $Rel_1 - Rel_{10}$ of [RT0]. In these relations we also include auxiliary black strands, whenever they make sense. Isotopies of the gluing sphere lead to the moves $Rel_{11} - Rel_{13}$.

The last relation also takes care of the ambiguity of identifying a surgery sphere with a $+$ or a $-$ coupon. It is understood that the upside down versions of these pictures are also included in the set of moves.

Next, we describe the $1a)$ and $1b)$ Isotopies, which are also used to resolve the ambiguity of moving general strands into the standard positions $I_1 \ldots I_n$. The moves are given by the pictures for $Rel_{14} - Rel_{15}$ and their mirror images.

The reduced one slide is given by $Rel_{16}$ and reflections. The modification and the reduced cancellation are given by $Rel_{17} - Rel_{18}$. We remind ourselves that $Rel_{17}$ does not imply the relations $Rel_{14}$ and $Rel_{15}$, since the coupons may be in different components.

Finally, we have the signature move $Rel_{19}$, which is the inclusion of an isolated, unknotted ribbon $\oplus$ with one twist.
Proposition 8

1. Two directed, bridged, ribbon graphs are projections of bridged link presentations of the same manifold if and only if they are related by relations \( \text{Rel}_1 - \text{Rel}_{19} \).

2. If the coupons \((j, \pm)\) lie in the same component of the presentation the moves \( \text{Rel}_{13}, \text{Rel}_{14}, \text{Rel}_{15}, \) and \( \text{Rel}_{16} \) follow from the other moves. In particular, if the presentation has only one component \( \text{Rel}_1 - \text{Rel}_{12} \) and \( \text{Rel}_{17} - \text{Rel}_{19} \) form a complete set of moves for presentations of cobordisms in \( \text{Cob}_3 \).

3. For presentations of connected morphisms in \( \text{Cob}_3 \) we can omit \( \text{Rel}_{19} \).
Figure 2.20: $Rel_{14}$

Figure 2.21: $Rel_{15}$

Figure 2.22: $Rel_{16}$

Figure 2.23: $Rel_{17}$  $Rel_{18}$
3) Standard and Tangle Presentations of Cobordisms

The results in Chapter 2 on bridged links are valid for presentations on any compact, oriented three manifold. In this chapter we wish to consider the special case of surgery presentation on the three fold $H$ discussed in Chapter 1. The boundaries and the charts of the presented cobordisms are always understood to be the canonical ones of $H$. We shall use the special form of $H$ to simplify the presentation even further.

To this end we first introduce “standard” bridged link presentations for which $\tilde{M}$ and the bridged link diagram in the handlebodies is normalized. The relevant presentation is thus in $S^3$ from where it can be reduced to tangles in the spirit of [RT].

A larger part of the discussion is devoted to the subtleties of the composition rules.

3.1) Standard Presentations on $S^3$

In this paragraph we introduce a standard presentation of a connected cobordism, and we show that every cobordism admits one such presentation. We begin with the surgery descriptions of $\tilde{M}$ on $H$ with only one extra $S^3$ and the standard link presentation in the $H^\pm_g$ components. The latter are indicated in Figures 3.24 and 3.25.

3.1.1) Standard Presentation of $\tilde{M}$ and the class $U$

The handle structure of a presentation can be normalized as prescribed in the following Lemma:

**Lemma 9** Suppose $g^{-1}(1)$, and $W$ (as in Chapter 1) are connected. Then after connecting a sufficient number of modifications on $W$ we can reduce the number of one handles in the decomposition of $W$ to the minimum $|\pi_0(g^{-1}(0))| - 1$.

If $g^{-1}(0) = \mathcal{H}(\tilde{g}^+, 1, \tilde{g}^-)$, and if the extra sphere $S^3$ is labeled by $\beta$, then we can assume that the index one singularities have labels $(\lambda, \beta)$ where $\lambda$ runs over all other components.

We denote by $U$ the isotopy classes of such presentations with the surgery spheres in certain preassigned positions.

**Proof**: It is always possible to find a Morse function in $\mathcal{F}^0$ such that all critical points are in general position. Using the Principle of Independent Trajectories we may deform $f$ such that the critical points of index $i$ have values in $]b_i, b_{i+1}[$, for given numbers $0 = b_0 < b_1 < ... < b_5 = 1$.

By the following process we may reduce the index one points to the minimal number. Let us consider the first critical value $c \in ]b_1, b_2[$ with label $(\alpha, \beta)$. Now, if $\alpha = \beta$ we may change the index of the singularity from one to two using a modification and push it above $b_2$. More generally, if $x \in ]b_1, b_2[$ is a critical value we consider the graph, whose vertices are the components of $\mathcal{H}$ and which are connected by an edge whenever there is a critical value...
in \([b_1,x]\) with the corresponding label. If the label of \(x\) introduces an edge at the component of the given graph we may replace it as above, since the descending trajectories end in the same component of \(f^{-1}(x - \epsilon)\).

We end up with a label graph with a minimal number of edges. That is, if we remove any edge from the graph the component it lies in will decomposes into two components.

The components of \(f^{-1}(b_i)\) consist of the \(D^4\)'s, that are created by the index zero points, and the components of \(g^{-1}(0)\). If a \(D^4\) appears in a label of a critical point, we know by minimality that the the other component of the label is different from this \(D^4\). Hence we may apply Smales Lemma and remove the \(D^4\)-component by a 0-1-cancellation.

We arrive at the situation where the labels contain only components of \(g^{-1}(0)\). Hence, \(\tilde{M}\) is the direct union of \(D^4\)'s and connected sums of components of \(g^{-1}(0)\).

Replacing \(f\) by \(1 - f\) we use the same argument to get rid of singularities of index three, since \(g^{-1}(1)\) is connected.

Now, the remaining singularities in \(f^{-1}([b_2, b_4])\) do not change the number of components of the level sets. Hence the singularities of index 0 and 4 come in pairs, which belong to additional components of \(W\). However we assumed \(W\) to be connected. This implies the absence of index 0 and 4 points.

A one-slide corresponds to replacing the labels of the singularities as indicated in the middle of Figure 2.2. It is now easy to see that the form of the one-singularities described in part 2) can be obtained by sliding the one handles across each other, i.e., the label graph can be moved into a star form with \(S^3\) in its center.

**Remark:** To obtain a standard presentation we could have also started with a connected manifold \(g^{-1}(0)\) right away. In fact, the cobordisms \((\tilde{W} = f^{-1}([0, b_2]), f)\) for \(f\)'s with a minimal number of index one singularities are all isomorphic. \(\tilde{M}\) is just the connected sum of the common \(S^3\) and each \(H_g^\pm\).

Also, there always exists a path of Morse functions connecting two such presentations, which have the singularity structure as in Lemma 2 on \(\tilde{W}\). Certainly, there are also paths of Morse functions for which the connecting one handles are slid across each other. However, the description of these will be rather complicated since the space of one handle configurations is not simply connected.

Still, it is convenient to view the one handles surgeries to be given by a Morse function when we describe compositions of cobordisms.

**3.1.2) Standard Links in \(H_g^\pm\)**

We define standard forms of a bridged links inside of the \(H_g^\pm\). They are depicted in Figures 3.24 and 3.25.

We start by drawing Heegaard diagrams on \(\Sigma_g\) that yield splittings of \(S^3\). They consist of curves \(\{A_1, \ldots, A_g\}\) on \(\partial H_g\) where the \(A_i\) are contractible to the inside of \(H_g^+\), and a Heegaard diagram \(\{B_1, \ldots, B_g\}\) where each \(B_j\) is contractible to the outside \(H_g^-\), and intersects the other diagram only once in the curve \(A_j\).
Moreover, we draw on the standard sphere that gives the one-handle attachment of $H_g^\pm$ a line $G^g$ which is disjoint from the equator and contains $2g$ intervals which we call in the order they are aligned along $G^g$.

$$I_1^i, I_1^o, I_2^i, \ldots, I_g^o \subset G^g$$ \hspace{1cm} (3.13)

Figure 3.24: Standard Link in $H_g^+$

In the components $H_g^-$ we thicken the curves $A_j$ to ribbons $\alpha_j$ in the surface and push them slightly off into the interior of $H_g^-$. We depict the surgery ball of the one handle attachment as the complement of a ball in $S^3$, which contains $H_g^+$ so that the surgery sphere surrounds $\Sigma_g$. For each $\alpha_j$ we introduce another ribbon $\gamma_j$, which starts at the interval $I_j^i$ and ends at the interval $I_j^o$. It shall follow a radial direction away from $\Sigma_g$ and surround $\alpha_j$ close to $\Sigma_j$ as depicted in Figure 3.25. The identification of the $\phi^-$-spheres for the 1-surgery is as they appear in the picture, i.e., the content of the ball on the left is inserted in the ball on the right part of the figure.

In the $H_g^+$ components we push the ribbons $\beta_j$ along the $B_j$-curves to the inside. Each $\beta_j$ connected to the surgery sphere $\phi^+$ as indicated in Figure 3.24 such that it starts at the interval $I_j^i$ and ends at the interval $I_j^o$.

Let us denote by $S$ the isotopy classes of these standard presentations.

3.1.3) The $\sigma$-Move and the Lemma of Connecting Annuli

Clearly, in a standard presentation the cobordism is entirely given by a bridged link diagram in $S^3$. Yet, in order to ensure the existence of such presentations and in order to
describe a complete set of moves as in Proposition we need to include a special version of the \( \eta \)-move combined with two slides and isotopies. We will call this move a \( \sigma \)-move. In the context of a combinatorial tangle presentation of the mapping class group an analogous move was introduced in [MP], where it was called \( K_3 \). This move also turns out to be a special case of the Lemma of “Connecting Annuli”, which we will discuss in the end of this section. The exceptional cases of this lemma are points of caution for the composition rule of presentations of general cobordisms.

The \( \sigma \)-move at the \( j \)-th handle in \( H_g^- \) is described as follows:

We introduce a disc \( \hat{\mathcal{B}}_j \) in \( H_g^- \) that is bounded by the curve \( B_j \). We may assume that all ribbons pass through the disc \( \hat{\mathcal{B}}_j \) transversally. The next step is to undo a cancellation along the disc. We then move the pair of surgery spheres around the handle of \( H_g \) with meridian \( A_j \), stretching the cancelling ribbon such that it coincides with the ribbon \( \alpha_j \). The spheres are recombined by a modification as indicated in Figures 3.26.

One half of the annulus of the modification is pushed through \( \phi^- \) such that the other half is identical with the \( \gamma_j \)-ribbon of the standard presentation. Furthermore, we push the strands that were intersecting \( \hat{\mathcal{B}}_j \) into the \( \mathbb{S}^3 \)-component, such that the disc \( \hat{\gamma}_j \) bounded by \( \gamma_j \) and the interval \( I_j \) on \( G^2 \) between \( I_j^1 \) and \( I_j^0 \) is only intersected by \( \alpha_j \).

If the \( \sigma \)-move is applied to a standard presentation it may be described as a move at
the surgery sphere \( \phi^- \subset S^3 \) as indicated in Figure 3.27. The annulus is what has been the \( \alpha_j \)-band before the move, and the lower loop is the second half of the newly created \( \gamma_j \)-ribbon.

We define the \( \sigma \)-move for \( H_g^+ \) in the same manner. Here the un-cancellation is done at the discs \( \hat{A}_j \) which are bounded by the \( A_j \)’s. Similarly, we push the modification annulus to the outside so that the annulus \( \hat{\beta}_j \) bounded by \( B_j \) and \( \beta_j \cup I_j \) is not intersected by any other ribbons.

The move in \( S^3 \) when applied to standard presentation is given in exactly the same way as for \( H_g^- \).

The \( \sigma \)-move (e.g. in \( H_g^- \)) can also be described as an \( \eta \)-move, where we introduce an \( \alpha_j \)-ribbon linked to the annulus, which will be extended to the \( \gamma_j \)-ribbon, and two slides of each ribbon passing through \( \hat{B}_j \) over \( \alpha_j \).

It is also a special case of the following Lemma for “Connecting Annuli”. It gives rules for replacing a piece of a link as in Figure 2.9 with only two strands passing through \( R \), such that we are left with a link with fewer components.

**Lemma 10** Suppose a disc bounded by a surgery ribbon is penetrated by two pieces of a ribbon diagram as indicated on the left of Figure 3.28. We assume that the ribbons are oriented
and that the orientation is defined by which face is upward in the plat graph. Moreover, we distinguish the cases where, if we follow the ribbon connected to $A^+$ in the diagram, we will return to the annulus at the point $A^-$, $B^-$, or $B^+$.

![Diagram](image)

**Figure 3.28: Connecting Annulus**

- $A^+...A^-$ The diagram may be substituted by two straight ribbons joining $A^\pm$ to $B^\pm$ respectively, as indicated in the middle of Figure 3.28.

- $A^+...B^-$ The diagram may be substituted by one straight ribbon joining $A^+$ to $B^+$ and one tangle joining $A^+$ to $B^-$, as indicated in the right of Figure 3.28.

- $A^+...B^+$ The ribbon piece running from $A^+$ to $B^+$ can be replaced by a ribbon for which the corresponding closed ribbons (as in the middle picture of 3.28) are isotopic.

**Proof:** In the case $A^+...A^-$ we can slide the ribbon with labels $A$ over the ribbon with labels $B$. Concluding with an $\eta$-move this gives us the move labeled $i$) of Figure 3.28.

For the case $A^+...B^-$ we start by adding to the diagram the handle in which the $A^+...B^-$ lives and do a modification. The surgery sphere connected to $A^+$ and $B^+$ is then pushed through the handle as described in Figure 3.29, so that it arrives in the correct position at the other sphere. Undoing the modification we obtain the desired move $ii$) in Figure 3.29.

For the last case we remark that after a modification we obtain two components of a bridged link, to which we can apply isotopies independently. They are different from those with fixed end points $A^\pm, B^\pm$ if $\pi_1(M)$ is non abelian.

3.1.4) **Existence of Standard Presentations and a Projection on $S$**

It is clear from Lemma 9 that we have an inclusion

$$\iota : S \hookrightarrow U.$$  \hspace{1cm} (3.14)

In this section we wish to show that $\iota$ is onto, i.e., induces an isomorphism, if we mod out the moves for link diagrams. More specifically, we shall construct a standard representative of the inverse of $\iota$ on link diagrams.
Proposition 11 Every cobordism admits a standard presentation. More precisely:

There is a canonical map

\[ \Psi : \mathcal{U} \to \mathcal{S} \]

that sends a general bridged link presentation in \( \mathcal{H} \) to a standard presentation of the same cobordism in \( \tilde{\text{Cob}}_3 \).

The composition

\[ \Psi \circ \iota : \mathcal{S} \to \mathcal{S} \]

is the composition of \( \sigma \)-moves at the surgery spheres in \( S^3 \), one for each handle.

In place of a proof we give the definition of \( \Psi \):

The first piece of \( \Psi \) is one \( \sigma \)-move at each handle of each \( H^\pm \) component.

As a result of this operation we obtain a bridged link which looks inside of a handlebody \( H^\pm \) as in Figures 3.24 or 3.25 with additional components \( \mathcal{L}'' \). By prescription of the \( \sigma \)-moves \( \mathcal{L}'' \) is disjoint from the discs (or annuli) \( \hat{\gamma}_j \), \( \hat{\beta}_j \), \( \hat{A}_j \), and \( \hat{B}_j \) that have been defined in Section 3.1.1). In fact, for sufficiently small, closed collars \( V_g^\pm \subset H^\pm \) of the surfaces \( \Sigma_g \cup \hat{A}_1 \cup \ldots \cup \hat{A}_g \cup \hat{\beta}_1 \cup \ldots \cup \hat{\beta}_g \) in \( H^+_g \) and \( \Sigma_g \cup \hat{B}_1 \cup \ldots \cup \hat{B}_g \cup \hat{\gamma}_1 \cup \ldots \cup \hat{\gamma}_g \) (containing all the \( \alpha_j \)) we may assume that \( \mathcal{L}'' \subset H^\pm_g - V^\pm_g \cong (S^2 - \Pi_{j=1}^g I_j) \times [0,1] \).

Moreover, we may choose a collar \( S^2 \times [1,1+\epsilon] \) of \( \phi^\pm \) in the \( S^3 \)-component such that the links different from the \( \beta_j \) and \( \gamma_j \) are all in \( K^\pm \cong (S^2 - \Pi_{j=1}^g I_j) \times [1,1+\epsilon] \). From a homeomorphism \( [0,1+\epsilon] \cong [1,1+\epsilon] \) we obtain a homeomorphism \( \delta : (H^\pm_g - V^\pm_g) \cup K^\pm_g \cong K^\pm_g \). It is clear that two links for which the parts \( \mathcal{L}'' \) are replaced by \( \delta(\mathcal{L}'') \) are isotopically equivalent.

Clearly, the result has the form of a standard presentation and we may give the definition of \( \Psi \) as the composition of the \( \sigma \)-moves and the maps \( \delta \).

\[ \square \]
3.1.5) Moves in a Standard Presentation

What remains to describe the class of connected cobordisms by bridged link diagrams in $S^3$ is a set of moves. The assertion of the next proposition is that the $\sigma$-move is the only additional move for standard presentations besides the proper specializations of the Kirby moves to the $S^3$ component:

**Proposition 12** Suppose we have two standard presentations on $\mathcal{H}$ of the same connected cobordism in $\text{Cob}_3$. Then they are related by a sequence of the five moves

1. Isotopies in $S^3$ (as in 1) of Section 2.2.2)) with fixed surgery spheres.
2. Two slides (as in 3) of Section 2.2.2)) where the $HI$-ribbon may be attached to a surgery sphere.
3. The signature move as in 6) of 2.2.2).
4. The $\sigma$-move for each handle.

In the second move we may confine ourselves to $\kappa$-moves. We may replace the third move by the $0 \bigcirc 0$-move, if we only wish to preserve the class of the cobordism in $\text{Cob}_3$.

**Proof**: Our task is to show that the set of moves listed in Proposition 11 applied to presentations in $\mathcal{S}$ can be expressed by the above moves. The basic idea is to conjugate other sequences of moves continuously with the projection $\Psi$.

An isotopy of a presentation in $\mathcal{U}$ can be decomposed into isotopies where a singularity of the ribbon diagram is moved through a specific disc, $\hat{B}_j$ (or $\hat{A}_j$), and isotopies that are constant across all $\hat{B}_j$. If we conjugate the first type of move with the map $\Psi$ we may push the entire isotopy into $S^3$, where it is given by a passage of a singularity through the respective loop (or annulus) from Figure 3.27.

In the same way we can express a two slide and an $\eta$-move by a two slide and an $\eta$-move which are constant close to the discs $\hat{B}_j$ conjugated by $\Psi$. As in the proof of Proposition 11 we may also push the rest of a move to the outside of the handlebody. Thus, a complete set of moves of $\mathcal{U}$ is given by $\Psi \pm 1$ and the types of moves from Proposition 11 which are constant on the handlebodies.

Since $S^3 - \bigcup \phi$ is simply connected we may use the arguments in the proof of Proposition 11 and replace the $\eta$-move by two slides and $0 \bigcirc 0$-moves. This yields the presentation of $\text{Cob}_3$.

By the same Proposition, we can replace the $0 \bigcirc 0$-moves by $O_1$-moves if we consider presentations of $\text{Cob}_3$. The reduction of two handle slides to $\kappa$-moves follows similarly from [FR].

Since we started from presentations in $\mathcal{S}$, and since $\Psi$ is in this case a combination of the $\sigma$-move in $S^3$, we conclude that the list given in Proposition 12 is complete.

\[\square\]
3.2) Tangle Presentation of Cobordisms

As in [RT] or Section 2.4) we seek to describe in this section a planar presentation of cobordisms. We shall use the reduction to standard presentations in $S^3$ to derive via a suitable projection a planar presentation of cobordisms in terms of ribbon graphs in $\mathbb{R} \times [0, 1]$, where the ribbons are allowed to end in the boundary of the strip. Thus the cobordisms are presented by admissible tangles, i.e., configurations of such ribbons, which fulfill a certain orientability condition.

The formulation of a complete set of moves is easily given using results from Section 2.4) and 3.1.5)

The composition rules contain a few subtleties: We shall give a Cerf theoretic derivation of a decoration rule for boundaries with many components. The rule has also been stated in a more rudimentary form in [Tu]. In our construction of presentations we also need to include a second rule related to the $\sigma$-move, which applies also to connected boundaries. In particular, we discuss the obstructions given by the Lemma of Connecting Annuli to naïve compositions of tangle diagrams.

3.2.1) From Standard Presentations to Admissible Tangles

The choice of the projection of a standard bridged link in $S^3$ to an admissible tangle depends on a few more conventions regarding the positions of the surgery spheres and the links in a fixed $S^3$.

To start with we fix two spheres $S^2_\pm \subset S^3$, which separate $S^3$ into three pieces homeomorphic to $[-1, 1] \times S^2$ and $D^3_\pm$. Inside the standard $S^2$ we fix a point $\infty$ and a homeomorphism $S^2 - \infty \cong \mathbb{R}^2$. For any pair $g_\pm$ we fix a standard alignment of the $K_+ + K_-$ surgery spheres of a standard presentation along the respective copy of $\mathbb{R}^2$. More precisely, we shall fix a sequence of intervals $G_j \subset \mathbb{R}$ for $j = 1, \ldots, K_+$ on the $x$-axis of $\mathbb{R}^2$. The standard position of the $j$-th surgery sphere $\phi_j$ is then specified by the property that $\phi_j^+ \cap S^2_\pm$, such that the special line on $\phi_j^+$, containing the intervals $I_s$, $s = 1, \ldots, g_j$, coincides with $G_j$. Also we require that the order in which the intervals appear on the $x$-axis is the same as in (3.13), see Figure 3.30.

Similarly, we define positions for the opposite spheres on $S^2$.

As in the remark of Section 3.1.1) we shall consider only Morse function $f|_W$ of the surgery presentation, such that the surgery-spheres are always in the defined positions, i.e., we will consider presentations on the manifold where the handlebodies are already connected to the $S^3$ by one-surgeries.

Next we introduce interior points $c_\pm \in D^3_\pm$ and (unbraided) lines $L_j^\pm$ in $D^3_\pm$ connecting $c_\pm$ to the surgery spheres $\phi_j^\pm$. The complement, $Q$, of the surgery balls, the points $c_\pm$, and the lines $L_j^\pm$ is clearly $S^2 \times [0, 1]$.

Moreover, we can introduce a homotopy of embeddings $f_t: Q \hookrightarrow Q$, such that $f_0$ is the identity along $\phi_j^+ \cap S^2_\pm$, $f_0 = id$, and $f_1$ maps $Q$ onto the $[-1, 1] \times S^2$-piece of $S^3$. Thus, for any ribbon diagram of a standard presentation and isotopy thereof with all ribbons in $Q - [-1, 1] \times \infty$, we have a canonical deformation to equivalent diagrams and isotopies in the $[-1, 1] \times \mathbb{R}^2$.
The diagram in $[-1, 1] \times \mathbb{R}^2$ is then projected into a strip $[-1, 1] \times X$ where $X \cong \mathbb{R} \subset \mathbb{R}^2$ is the $x$-axis, so that the $G_j$ are arranged along the lines $X_\pm := \{\pm1\} \times X$. If the diagram is in a general position the projection yields a ribbon graph (in the sense of [RT0]). It has the properties that to any of the $2 \sum j g_j^\pm$ intervals of $X_\pm$ a ribbon is attached, and, furthermore, the closed ribbons that result by inserting the strips $b_j^\pm$, as indicated in Figure 3.30, are all orientable (i.e., $\cong I \times S^1$). We call a ribbon graph in $[-1, 1] \times X$ with these properties an admissible tangle. An example with $\bar{g}^+ = (1, 2, 0)$ and $\bar{g}^- = (2, 3)$ is shown in Figure 3.31. Inserting the bands $b_j^\pm$ yields four closed, orientable ribbons.

We can always deform a given standard presentation into a position where all ribbons lie in $Q$. Yet, a general isotopy of standard diagrams can be merely chosen transversal to the complement of $Q$. I.e., the possible singularities occur when an individual ribbon passes transversally though a line $L_j$ or through $[-1, 1] \times \infty$. For a line $L_j$ this gives rise to the additional $\tau$-move at the group $G_j$ in the set of admissible tangles, which is given by Figure 3.32 and its reflections. Here, a strand is moved through the ribbons emerging from a group $G_j$ of $2g_j$ intervals on $X$. If we move a ribbon through $]-1, 1[\times\infty$ the corresponding move of admissible tangles is to push a ribbon through the strands of all groups at once, and can thus be written as a combination of $\tau$-moves.

**3.2.2) Moves for Admissible Tangles**

The possible moves for standard presentations are given in Proposition 12. It is straightforward to derive from this an equivalent set of moves for the admissible tangles. A general isotopy of a ribbon diagram can always be written as a composition of the moves in [RT0].

![Figure 3.30: Alignment of Surgery Spheres in $S^3$](image)
and the $\tau$-move. As usual for presentations of connected morphisms in $\tilde{\text{Cob}}_3$ we may forget the signature move 6). Since we give our presentation on a connected and simply connected manifold we can also use [FR] to replace the two slides by the $\kappa$-move, or by the remarks of Section 2.4) to use $\text{Rel}_{17}$ and $\text{Rel}_{18}$ for an enhanced bridged ribbon presentation. Given the projection rule of Figure 3.30, the form of the $\sigma$-move in the tangle presentation is obvious from the one given in Figure 3.27. Combining these observations with the results from Proposition 8 and Proposition 12, we may now formulate the presentation of cobordisms in terms of admissible tangles:

**Proposition 13** There is an isomorphism between morphisms in $\text{Cob}_3$ and the set of admissible, planar, ribbon tangles, $\tilde{T}_g$, divided by the following relations

1. Isotopies: $\text{Rel}_1 - \text{Rel}_{10}$, and $\tau$-move,

2. $\sigma$-move,

and, alternatively, for links in $S^3$

3. $\kappa$-move,

4. $\circ \cdot \circ \circ \circ \circ$-move,

or, for bridged links in $S^3$,
3. Isotopies Rel_{11} and Rel_{12},

4. Modification Rel_{17},

5. Cancellation Rel_{18}.

3.2.3) Compositions of Admissible Tangles

The composition of two cobordisms can be presented by a tangle that is build up from the original tangle diagrams. In a more fancy language this means that we can endow the set of admissible tangles with a composition structure of a category and extend the presentation to a functor.

As outlined in the end of Section 1.3) it is sufficient to give the rules for the compositions of tangles corresponding to products of cobordisms as in equations (1.11) and (1.12). We start with the first type, which is easier.

If we consider a standard presentation with surgery spheres $\phi_j^+$ in positions $G_1, \ldots, G_K$ we may use an isotopy to bring the spheres into positions $G_{\pi(1)}, \ldots, G_{\pi(K)}$ for a given permutation $\pi \in S_K$, so that they still describe the same cobordism. The effect of the composition in (1.11) is to permute the handlebodies $H_g$, so that the bridged link with the $\phi_j$’s moved into new positions is in fact a standard presentation of the composite. On the level of tangle diagrams the isotopy will be given by a braid $b \in B_K$ of groups of strands, such that its class in $S_K$ is $\pi$. The example $\pi = (1,3) \in S_3$ is depicted in the left of Figure 3.33. To see that the definition does not depend on the choice of the braid element $b$ we observe that a standard generator of the pure braid group (see right of Figure 3.33) can be eliminated using $\tau$-moves.

The composition of two cobordisms is given by gluing corresponding components of the standard surfaces together. The bridged link presentation of the composite three fold close to a gluing surface $\Sigma_g$ is obtained from the original presentations by taking the union of the bridged links from $H_g^+$ and $H_g^-$ yielding a bridged link in an $S^3$-component. In a standard presentation the ribbon diagram in this $S^3_j$-component for the $j$-th surface is depicted in $b$).
Figure 3.33: Permutation of Components, Pure Braid

It is connected to the standard $S^3$ of either factor of the cobordism product by the surgery spheres $\phi^\pm$. Their partners in the standard $S^3_{N/M}$-components of the two cobordisms are sketched in a) and c) of the same figure.

The presentation is immediately simplified by carrying out the one-surgery along $\phi^-$ explicitly, which can also be thought of as a $0-1$-cancellation. In terms of the elements of Figure 3.34 this means that we insert the content of the ball bounded by $\phi^- \subset \bar{g}^3_j$ into the region bounded by $\phi^- \subset \bar{g}^3_M$. As a result the $S^3_j$-component disappears and the vicinity of the $\phi^-$-sphere in the $S^3_M$-components is replaced by the link diagram around $\phi^+ \subset S^3_N$.

We are left with a bridged link presentation on the union of the standard $S^3$'s, that are part of the presentation of either cobordism, and the handle bodies $H^\pm_g$, which are not glued, and on which the presentation at the boundaries is of standard form. The presentation of the composite itself is however not standard since we have two instead (of one) $S^3$-components, and the pairs of surgery spheres $\phi_j$, that survives the simplification described above, connect them to each other.

To obtain a standard presentation we need to be more specific about the ordering of the surfaces and the connectivity of the cobordisms. We assume that we are in the situation of equation (1.12).

We denote the morphisms $M : \bar{g}^1 \to \bar{g}^2 \otimes \bar{g}^3$ and $N : \bar{g}^1 \otimes \bar{g}^2 \to \bar{q}$, with $\bar{g}^2 \neq \emptyset$, so that $L : \bar{g}^1 \otimes \bar{p} \to \bar{q} \otimes \bar{g}^3$. On the standard sphere $S^2$ we mark the groups $G^1_j, G^2_j, \text{and } G^3_j$ in the given order. We move the surgery spheres in the corresponding positions of the spheres $S_M := S^2_{+,M}$ and $S_N := S^2_{-,N}$ in the copies of $S^3$, which define the tangle presentations of $M$ and $N$. Thus, the surgery spheres in $S_N$ which belong to the first group and the spheres in $S_M$, which belong to the third group connect to corresponding spheres in the $H^\pm_g$'s. The spheres in the second groups of $S_N$ and $S_M$ connect to each other as described above.

We modify the presentation by introducing a 1-2-birth-point in $M$ given by a pair of surgery spheres $\phi$ and $\phi'$ connected by a ribbon $R$. We choose one of the spheres $\phi_{j_0}$ in the second group $G^2$ and push the sphere $\phi'$ through it into the $S^3$ component of the other
cobordism. We may arrange it that $\phi$ sits inside the $D^3_-$ of $N$ and $\phi'$ inside $D^3_+$ of $M$. We then expand the spheres $\phi$ and $\phi'$ until they coincide with $S_N$ and $S_M$ respectively. Since $\pi_0(Diff(S^2)^{\pm})$ is trivial we may deform the isomorphisms $\phi \to S_N$ and $\phi' \to S_M$ such that they are compatible with the respective identifications with standard spheres (and the arrangement of the other surgery spheres).

We now carry out the index one surgery explicitly by gluing the $D^3_+ \cup S^2 \times I$-piece of the $N$-presentation to the $S^2 \times I \cup D^3_-$ piece of the $M$ presentation along $S_N$ and $S_M$. The result is a bridged link diagram in a single $S^3$, with a natural decomposition $D^3_+ \cup S^2 \times I \cup S^2 \times I \cup D^3_-$. Along the sphere in the middle the presentation has the form as in Figure 3.35.

The respective surgery spheres of the second group are in the correct position for the modification move, or, in the case of the sphere $\phi_j$ through which we chose to move $\phi'$, a cancellation along the ribbon $R$. The spheres of the first and third group are pushed into corresponding positions on $S^2_{-M}$ and $S^2_{+N}$ respectively.

Resizing the $S^2 \times I \sqcup_S S^2 \times I$ part we obtain a standard presentation, and for a generic projection a presentation of the composite in terms of admissible tangles. It is now clear how the composition rule of tangles should look like in order to give a functorial presentation of the cobordism category $\tilde{Cob}_3$. Its definition is summarized in Figure 3.36.

The boxes $M$ and $N$ stand for the tangle presentations of the respective cobordisms. The content of the boxes $\Phi_{g_j}$ is given in Figure 3.37. Note that $\Phi_{g_{j_0}}$ is the only group that is not decorated with an annulus.

Figure 3.34: Composition of $H^\pm_g$'s
3.2.4) Naïve Compositions, Connecting Annuli, and Closed Tangles

In situations, where the ribbons of a presentation that start from an interval \( I \) at the lower end of the presentation do not return there in an interval different from the partner interval of \( I \) we may apply the Lemma of Connecting Annuli and replace the boxes \( \Phi_u \) (or the respective pieces in Figure 3.37) by vertical strands.

Let us give a simple example with one boundary component, where this cancellation is not possible:

We choose cobordisms \( \Sigma_2 \rightarrow \emptyset \) and \( \emptyset \rightarrow \Sigma_2 \) given by the tangle presentations on the left side of Figure 3.38.

The naïve composition of these tangles yields the \( 0 \odot 0 \) link that can be removed all together, i.e., we obtain \( S^3 \). The correct composition leaves us with the Whitehead link \( W \) on the right side of Figure 3.38. The three manifold \( M_W \) presented by it is however nontrivial. In particular, we have \( \pi_1(M_W) = \mathbb{Z}(x) \oplus \mathbb{Z}/2(y) \). For a description of \( M_W \), when the components of the Whitehead link have framings \( \pm 1 \) consult [Rf].

The composition rules can be simplified at the expense of starting with a more specialized class of tangles. Let us define the space of closed tangles, \( \mathcal{T}_g \), by the property that the ribbon \( R_j^\pm \) starting at an interval \( I_j^\pm \subset X_\pm \) ends in the partner interval \( I_j^o \subset X_\pm \). An even smaller space is given by the special, closed tangles, \( \mathcal{S}_g \subset \mathcal{T}_g \), for which each ribbon \( R_j^\pm \) with the additional segment \( b_j^\pm \) inserted bounds a disks \( D_j \). We require that \( D_j \) is penetrated by only one ribbon, which has to be different from the \( R_j^\pm \)’s.

It is not hard to show that every closed tangle is equivalent to a special closed tangle if we admit the moves of Section 3.2.2). The category of (specialized) closed tangles is

\begin{align*}
\pi_1(M_W) &= \mathbb{Z}(x) \oplus \mathbb{Z}/2(y) \\
&= \mathbb{Z}(x, y, z) / (x^{-1}zx = zyx^{-1} = yzy^{-1})
\end{align*}

Dividing by these relations gives the asserted group.

\footnote{The link \( W \) can be presented as the closure of the braid \( \sigma_1^{-2} \sigma_2^{-2} \sigma_1^2 \sigma_2 \). With this we may compute, as in [Bi] or [Rf], \( \pi_1(S^3 - L) \). It is the free group in \( x, y, \) and \( z \) with relations \( x^{-1}zx = zyx^{-1} = yzy^{-1} \).

The loops along the ribbons are \( zy^{-1} \) and \( xyx^{-1}y^{-1}x^{-1}yxy \). Dividing by these relations gives the asserted group.}
endowed with the *naïve composition rule*. This means we omit the insertions $\Phi_g$, but keep the decoration rule.

There is a full functor $\ell : \tilde{T} \rightarrow s\tilde{T} : t \mapsto \ell(t)$ from the total space of admissible tangles to the space of special, closed tangles. In order to define it we observe that the tangle in Figure 3.37 can be written as a (naïve) composition of its upper and its lower half

$$\Phi_g = (S^+)^g \circ (S^-)^g.$$

For $\check{g}^\pm = \sum_j g_j^\pm$ we then set

$$\ell(t) := (S^-)^{\check{g}^+} \circ t \circ (S^+)^{\check{g}^-}. \quad (3.15)$$

The composition $\circ$ used here is just placing the tangles on top of each other.
Mainly by using $\sigma$-moves it is easy to see that $\mathcal{C}$ and the inclusion factor into isomorphisms, once we impose the relations on the tangle categories. Summarily, we have

$$\tilde{\text{Cob}}_3 \rightarrow \tilde{T}_g \rightarrow \tilde{sT}_g \rightarrow \tilde{cT}_g$$

(3.16)
4) Applications and Implications

The applications we shall be concerned in the first part of this chapter are of purely topological nature. They regard special classes of manifolds for which we derive presentations using a tangle presentations of the mapping class group, derived from the results of the previous chapter.

In the second part we extend the Reshetikhin-Turaev invariant to presentations with bridged links and give a simple proof of invariance. We shall discuss the algebraic implications of the Lemma \[\text{[L]}\] for Connecting Annuli. From this we will find a canonical natural transformation for a BTC and a canonical, central element of a quasitriangular Hopf algebra, which projects onto selfconjugate objects.

We conclude with some remarks on how the presentations of cobordisms of two folds with boundary are obtained. In particular we explain how the operation of a “glue tensor product” acts on the presentations.

4.1) Invertible Cobordisms and Presentations of Mapping Tori

An interesting family of morphisms in \(\widetilde{\text{Cob}}_3\) are the invertible cobordisms \(\text{Aut}(\Sigma)\) for a connected surface \(\Sigma\). They are given by \(M = \Sigma \times I\), equipped with possibly non canonical charts, as for \(\psi = \psi' \Pi id\) with \(\psi' \in \text{Diff}(\Sigma)^+\). For example it is not hard to find the standard presentations for different Dehn twists, and construct a presentation of the mapping class group of \(\Sigma\) in terms of tangles. This presentation is identical to the one constructed by \([MP]\).

However, in \([MP]\) the use of Cerf theory in 3+1 dimensions was avoided by referring to the explicit presentation of the mapping class group of \([Wj]\) in terms of generators and relations.

From the tangle form of \(\pi_0(\text{Diff}(\Sigma)^+)\) we derive surgery presentations of Heegaard splittings and general mapping tori over \(S^1\), with fiber \(\Sigma\). In a few examples we compare those to known presentation. Namely, lens spaces and “planar presentations” of trivial bundles of the form \(\partial(N^{(d-j)} \times D^j)\).

4.1.1) Tangle Presentation of the Mapping Class Groups

We start this section with a derivation of the presentation of the mapping class group in the category of admissible tangles. To begin with, we remark that the trivial tangle, \(I : (g) \to (g)\), given by \(2g\) vertical ribbons, represents the identity cobordism in \(\widetilde{\text{Cob}}_3\). This follows directly from Theorem \[\text{[R]}\] and identity the \(I \cdot t = t \cdot I = t\), which is by the composition rules holds obviously for any tangle \(t\). As an instructive exercise let us derive this explicitly from the topological situation:

In the corresponding standard presentation we obtain two concentric surgery spheres \(\phi^+\) and \(\phi^-\) in \(S^3\) with \(2g\) straight, radial lines joining them. As in the derivation of the
composition rule in Section 3.2.3) we carry out the one-surgery for the $\phi^-$-sphere explicitly, i.e., we do a 0-1-cancellation.

The resulting link-presentation of the cobordisms consists of the left side of Figure 3.25 and the left side of Figure 3.24 where we consider the two spheres as partners.

We perform the one-surgery along these spheres, too, so that we have a presentation on the one-component manifold $H^+_g \# H^-_g$, without one-surgery data. The detailed result is given in Figure 4.39.

Now, we can use an isotopy which slides the one handles with meridians $A_j$ of the inner handlebody over the $\beta_j$-ribbons so that the meridians of the inner and the outer handlebody are aligned. Using $\eta$-moves we may then remove all surgery ribbons. The resulting cobordism are just two nested copies if the same handle body, i.e., $\Sigma \times I$ with canonical charts at the boundaries. This is the identity cobordism.

![Identity on $\Sigma_g$](image)

The mapping class group of a surface $\Sigma$ is generated by Dehn twists $\delta_C$ along a sufficient number of Jordan curves $C$ on $\Sigma$. The corresponding cobordism $< C > := (\Sigma \times I, id, \delta_C)$ is equivalent to one of the form $< C > = (\widetilde{\Sigma} \times I, id, \mathbb{I})$. Here $\widetilde{\Sigma} \times I$ is the same manifold with canonical boundary maps, but a surgery done inside. The surgery presentation is given by pushing the curve $C$ inside the thickened surface $\Sigma \times I$ and inserting a ribbon with framing number 1 with respect the canonical framing of $T(S^3)_1$. In Figure 4.40 the cylindrical neighborhood of $C$ is shown; the equality follows from an ordinary isotopy.

The symbol $\Phi$ is short hand for $|n|$ isolated ribbons with framing number $\text{sgn}(n) = \pm 1$. (It may be omitted if we are only interested in the cobordism classes in $\text{Cob}_3$). The admissible tangles giving the presentation of the Dehn twists can be obtained by composing the $< C >$ with the standard presentation of the identity. We obtain an additional link component in $H^+_g$, which can be pushed into the $S^3$ component with the map $\Psi$ (or only $\sigma$-moves for the affected handles).
Specifically, we obtain Figure [4.41] by inserting a 1-framed ribbon along \( A_j \) in \( H^+_g \), and moving it along \( \beta_j \) and through \( \phi^+ \) into the \( S^3 \) component of the presentation. A Dehn twist along the curve \( B_j \) is given by placing a small one-framed ribbon around \( \alpha_j \) in \( H^-_g \). A \( \sigma \)-move at the \( j \)-th handle leaves us with Figure [4.42].

The curve \( C_j \) is the one intersecting \( B_j \) and \( B_{j+1} \) exactly once and no other \( A \) or \( B \) curve. For the tangle presentation in Figure [4.43] we insert the respective ribbon in \( H^+_g \) move it towards \( \phi^+ \) along the pieces of \( \beta_j \) and \( \beta_{j+1} \) that emerge ‘radially’ from the intervals \( I_j^o \) and \( I_{j+1}^i \).

Finally, the curve \( D_j \) is the one opposite to \( A_j \), i.e., it intersects only \( B_j \). If the respective ribbon is inserted in \( H^-_g \) ot can be pushed onto a curve \( \mathcal{E} \) on the sphere \( \phi^+ \). Now, \( \mathcal{E} \) separates \( \phi^+ \) into two hemispheres one containing the intervals \( I_1^o, I_1^i, ..., I_j^i \) and the other containing the intervals \( I_j^o, I_{j+1}^i, ..., I_g^i \). Thus in the \( S^3 \) the respective ribbon can be moved around the surgery sphere into the \( S^2 \times I \)-piece of the presentation in two ways. The results are depicted in Figure [4.43]. Clearly the two possibility differ by exactly one \( \tau \)-move since in one instance \( \mathcal{E} \) was moved through the special line \( L \) from Figure [3.30] in another it was not.

The equalities in all of the pictures follow from \( \kappa \)-moves.

Let us consider the subring, \( \mathcal{T} \), of cobordisms that are generated by the \( A_j \)'s, the \( B_j \)'s, and tangles presenting generators of the pure braid group \( P_{2g} \), as depicted in Figure [3.33]. For compositions it follows by induction that a strand of a tangle in \( \mathcal{T} \) starting at an interval in \( X_- \) will end either at its partner interval on \( X_- \) or at its own copy in \( X_+ \). Thus, by
Lemma \[\text{Lemma 10}\] we may use the naïve composition rule for all elements in \(\mathcal{T}\), so that \(\mathcal{T}\) is in fact a group. Clearly, all tangles of the mapping class group presentation lie in \(\mathcal{T}\). In fact the converse is also true.

**Proposition 14** The group \(\mathcal{T}\) is isomorphic to \(\pi_0(\text{Diff}(\Sigma)^+)\) via the presentation in terms of admissible tangles.

This remark follows immediately if we use that all invertible cobordisms are of the form \((\Sigma \times I, id, \psi)\). However, the generators of \(P_{2g}\) can also be produced directly using special Dehn twists. For example, a Dehn twist along the curve \(A_j \ast B_j \ast A_j^{-1} \ast B_j^{-1}\) yields a full twist of the strands at \(I^i_j\) and \(I^o_j\). (Here \(\ast\) is the composition of paths as in \(\pi_1\).) Similarly, we obtain from a twist along a curve along \(D_i \ast D_j^{-1}\) (not intersecting the A’s and B’s anywhere else) the full twist of the strands \(I^o_i, \ldots, I^o_j\).

It is often more useful to replace the generators \(A_j\) by the generators

\[
S^+_j = B_jA_jB_j, \quad (S^-_j := (S^+_j)^{-1}) \tag{4.17}
\]
which is identical to the tangle $S^+$ introduced in \((3.13)\).

Let us remark as a word of warning that, e.g., for $g = 2$ the simple braid of the strands $I^o_j$ and $I^o_i$ presents a non-invertible cobordism $F$. This follows immediately from $F \circ A_1 = F \circ A_2$. In particular the naïve composition is not applicable. Note also, that the simple braid on a pair, $I^o_j, I^o_i$, is equivalent to $S^+_j \circ S^+_j$.

The results in [MP], specifically Propositions 5.2 and 6.1, follow directly from Proposition [14]. When comparing our presentation to the one in [MP] we have to keep in mind that we have to include the $\tau - move$, since we consider surfaces without punctures.

A more detailed analysis of the presentation of $\text{Diff}(\Sigma)$ as a product of the pure braid group $P_{2g}$, the group $\mathbb{Z}^g$, generated by $S^\pm_j$, and the group $\mathbb{Z}^g$ generated by $B_j$, should also give an alternative proof of the results in [Wj].

4.1.2) Presentations of Manifolds from $T = \pi_0(\text{Diff}(\Sigma))$

From the presentation of cobordisms we may derive link presentations in $S^3$ of closed manifolds. We start with the easier example.

**Heegaard-Splittings:**

Any three fold $M$ can be presented by a Heegaard-splitting, i.e., we glue $H^+_g$ to $H^-_g$, where the boundary identification is given by an element in $\psi \in \text{Diff}(\Sigma)^+$, with associated cobordism $< \psi >$. Now, the manifolds $H^\pm_g$ may also be considered as cobordisms with tangle presentation as in Figure [1.43]. Hence $M$ may be written as the composite of the three cobordisms $H^- \circ < \psi > \circ H^+$. 

Figure 4.45: Presentations of $H^\pm_g : \emptyset \leftrightarrow \Sigma$
For the composite \( < \psi > \circ H_g^+ \) we simply close the tangle presenting \(< \psi > \) at the bottom with the ribbons \( b_j \). It is clear from the description of \( T \) that the strand emerging from \( I_j^i \) at \( X_+ \) ends at \( I_j^o \). We may apply \( \eta \)-moves to these \( g \) different strands after composing with \( H_g^- \). Thus the link presentation is obtained from the presentation of \(< \psi > \circ H_g^+ \) by omitting the strands ending in \( X_+ \), (and adding annuli if the closure of a strand goes through several intervals).

The corresponding presentation of a lens space \( L(\psi) \) with \( \psi = \prod_j S^+_1 T^{n_j}_1 \) is, for example, easily identified with the familiar chain of unknots with framing numbers \( \{ n_j \} \).

**Mapping Tori:**

A more interesting case is provided by bundles over \( S^1 \) with a connected surface \( \Sigma \) as fiber. They are classified by conjugacy classes in \( \pi_0(\text{Diff}(\Sigma)) \) and can be given as the mapping torus of a representative. The main ingredient for the surgery description is to fix a pair of “rigidity morphisms” of \( \tilde{\text{Cob}}_3 \):

For a connected surface of genus \( g \) let us introduce cobordisms \( \theta : \Sigma \amalg \Sigma \to \emptyset \) and \( \theta' : \emptyset \to \Sigma \amalg \Sigma \) as indicated in Figure 4.46.

\[
\theta:
\begin{array}{c}
\includegraphics{diagram1.png}
\end{array}
\quad \theta':
\begin{array}{c}
\includegraphics{diagram2.png}
\end{array}
\]

Figure 4.46: Pairings \( \Sigma \times I : \Sigma \amalg -\Sigma \leftrightarrow \emptyset \)

It is easy to see that

\[
(\theta \otimes \text{id}) \circ (\text{id} \otimes \theta') = (\text{id} \otimes \theta) \circ (\theta' \otimes \text{id}) = \text{id} ,
\]

using the composition rules for admissible tangles. In general, a cobordism \( \Sigma \amalg \Sigma \to \emptyset \) may be constructed from \( \Sigma \times I \) with canonical boundary charts by composing one of the components of the chart with an orientation reversing map \( \psi \in \text{Diff}(\Sigma)^- \). Using the relations (4.18) and the general form of an invertible cobordism, it is easy to show that \( \theta \) is also of this form. For the closed composition we have the following identity:

**Lemma 15** The composition \( \theta \circ \theta' \) is homeomorphic to \( \Sigma \times S^1 \).

**Proof:** The direct proof is easy using the previous remarks on the general structure of cobordisms \( \emptyset \to \Sigma \amalg \Sigma \). In fact, it follows that \( \theta \) and \( \theta' \) are of the form \((\pm \Sigma \times I, \text{id}, \psi)\) from which the assertion follows immediately. Nevertheless, we wish to give a more complicated
proof which reveals another surgery presentation of $\Sigma \times S^1$ starting directly from the link diagram of $\theta \circ \theta'$.

For the composite we may use the naïve composition rule for one component and insert $\Phi_g$ and the extra ribbon $R$ for the other component. The elements in $\Phi_g$ may not be replaced by the identity (this would yield a presentation of the connected sum of $g - 1$ copies of $S^1 \times S^2$). However, we may apply an (un-) modification, which introduces $g$ pairs of surgery spheres. We do the same with $R$, so that we end up with $g + 1$ index one surgeries and $g$ index two surgeries. The resulting surgery presentation is now planar and is shown on the left of Figure 4.47. Following the first ribbon we pass through the pieces $1, \phi_0, \phi_1, \phi_1', 3, \phi_0, \phi_0', 4, \phi_1', \phi_1, 1$.

![Figure 4.47: Presentation of $\Sigma \times S^1$](image)

It is a general principle that if a surgery graph is planar the surgered four manifold $W$ can be given as a product $N \times I$ and the planar graph can be used to give a surgery description of the three fold $N$. More specifically, write $D^4 = D^3 \times I$, so that we have a standard piece $S^2 \times I = \partial(D^3) \times I \subset S^3 = \partial(D^4)$. Now, for a planar diagram the attaching curves for $j$-handles $S^j \times D^{3-j} \hookrightarrow S^3$ can be brought into the form $i \times id_I : S^j \times D^{2-j} \times I \hookrightarrow S^2 \times I$. It is clear that instead of attaching a four dimensional $j$-handle $h^j = D^{j+1} \times (D^{2-j} \times I)$ to $D^4 = D^3 \times I$ we may as well attach a three dimensional $j$-handle $Q^j = D^{j+1} \times D^{2-j}$ to $D^3$ and form the product with $I$.

In our example, attaching a one handle to $D^3$ along the discs $\phi_0, \phi_0' \subset S^2$ yields the solid torus $H_1$ on the right of Figure 4.47, where the ribbon pieces 1 and 2 (or 3 and 4) are glued to loops in $\partial H_1$. If we attach another $Q^1$ along discs $\phi_j, \phi_j'$ we obtain the depicted handlebody $H_2$ of genus 2, and the corresponding attaching (Jordan) curve for $Q^2$. It is now clear from the picture that the manifold $H_2 \cup Q^2$ is homeomorphic to the solid torus $H_1$ with another torus removed from the inside. It can also be described as $S^1 \times A_1$, where $A_1$ is an annulus. The total surgery will result in the manifold $S^1 \times A_g$, where $A_g$ is the disc $D^2$ with $g$ small discs removed from the inside.

The surgered four-manifold is therefore $W = S^1 \times A_g \times I$. But with $A_g \times I = H_g$ we find $M = \partial W = S^1 \times \Sigma$. \qed

**Remark:** Clearly, the class of manifolds that have a planar bridged link presentation and as above allows a reduction of dimensions is much larger then the class of manifolds with planar link diagrams (which are just connected sums of $S^1 \times S^2$'s).
Remark: It is in fact possible to reduce the presentation of $\Sigma \times S^1$ by two dimensions using $W = \Sigma \times D^2$. A surgery presentation of $\Sigma$ is given by attaching to $D^2$ 2g one handles in the way indicated in the left of Figure 4.48 and closing the manifold with a two-handle. Here the attaching curves are pairs of points in the boundary $S^1$ joined by a dashed line. The attaching curve for the two handle passes through every piece of $S^1$ exactly once. The corresponding link diagram for $\Sigma \times S^1$ is indicated on the right of Figure 4.48. For $g = 1$ this presentation is identical to the one above; for $g > 1$ the equivalence of the link presentations is left as an exercise to the reader.

Figure 4.48: Other Presentation of $\Sigma \times S^1$

It is clear now that the mapping torus of $\psi \in \text{Diff}(\Sigma)^+$ is given by the composition, $\theta' \circ (< \psi > \otimes \text{id}) \circ \theta$. We give a bloc diagram of the corresponding link presentation in Figure 4.49. In the box $t$ the tangle corresponding to $\psi$ is inserted. This concludes our discussion of surgery presentations of mapping tori.

4.2) On the Reshetikhin Turaev Invariant

The construction of the invariant of closed three dimensional manifolds as in [RT] is based on close relations between tangle categories and general, abelian BTC’s. The new elements of bridged links and admissible tangles we have encountered so far also have natural counterparts in semisimple BTC’s and finite-dimensional nicely-quasitriangular Hopf algebras.

The two aspects we wish to address here are an extension of the [RT] - invariant for bridged ribbon presentations as described in Section 2.4) and a discussion of the Lemma 10 for Connecting Annuli; in particular the structure of the transformation associated to the tangle in Figure 3.28.

4.2.1) Invariants of Three Manifolds from Bridged Ribbons

Throughout this section we use the same notation as in [Tu]. Starting point of the construction is an abelian, strict, semisimple, balanced BTC, with only a finite set $I$ of

---

This presentation has been communicated to me by Robion Kirby.
isomorphism classes of irreducible objects, each of which contains only one element. We denote the braid element by \( c_{i,j} : i \otimes j \rightarrow j \otimes i \) and the balancing \( \theta_j \in \text{End}(j) = C \), so that \( \theta_{i \otimes j} = c_{j,i} c_{i,j} \theta_i \otimes \theta_j \). Rigidity provides us with a pair of morphisms \( 1 \rightarrow X \otimes X^\vee \) and \( X^\vee \otimes X \rightarrow 1 \). We define the maps

\[
\rho_X : (\theta_X \otimes 1) c_{X^\vee,X} : \text{Hom}(1, X^\vee \otimes X) \xrightarrow{\sim} \text{Hom}(1, X \otimes X^\vee),
\]

with \( \rho_{X^\vee} \rho_X = \text{id} \) and the corresponding ones on \( \text{Hom}(X^\vee \otimes X, 1) \). Applying these to the rigidity morphisms we produce corresponding morphisms for the opposite product, \( \otimes' \).

These morphisms are associated to maxima and minima in a directed, colored ribbon graph. The morphisms \( 1 \rightarrow 1 \) associated to an annulus, with a morphism \( f : X \rightarrow X \) inserted, defines a canonical, generally cyclic, and \( \otimes \)-factorizable trace

\[
tr_X : \text{End}(X) \rightarrow C.
\]

As usual we define the \( S \)-matrix and the \( q \)-dimensions:

\[
S_{i,j} = tr_{i \otimes j}(c_{j,i} c_{i,j}) \quad \text{dim}(j) = S_{1,j} = tr_j(1)
\]

In the construction of a three manifold invariant it is usually required that the \( S \)-matrix is invertible. (This is sometimes called the “modularity” axiom.) In the bridged link formalism it suffices to start for a seemingly weaker condition. All we require is that there is a vector \( \hat{d} \) with \( S \hat{d} = 1 \), i.e.,

\[
\sum_j \hat{d}(j) S_{j,i} = \delta_{i,1}.
\]

It will turn out that (1.21) also implies \( \hat{d}(1) \neq 0 \), but we shall add this property here to the list of assumptions.
Let us now give a definition of a functional \( BR \to \{ BR \} \) on the bridged links in unions \( U = \coprod_{r=1}^{\omega} S^3 \), from which we wish to define an invariant of the three manifolds they are presenting. The prescription to compute \( \{ \} \) is as follows:

For each object \( X \) of the semisimple category we introduce bases
\[
\{ f^X_\alpha \}_{\alpha \in \Lambda_X} \subset Hom(X, 1) \quad \text{and} \quad \{ e^X_\alpha \}_{\alpha \in \Lambda_X} \subset Hom(1, X)
\]
with \( f^X_\alpha \circ e^X_\beta = \delta_{\alpha\beta} 1_1 \).

A coloration is now not only a labelling of the directed ribbons with elements \( l \in \mathcal{I} \) but in addition a labelling of a pair of coupons \((j, \pm)\) with an (the same) element of \( \Lambda_X \). Here, \( X = l_1 \otimes \ldots \otimes l_\omega \), where \( l_k \) are the labels of the ribbons entering the coupon \((j, +)\).

In the next step we associate to a plat bridged ribbon graph \( BR \) with coloration \( C \) a composition of elementary morphisms. As in [RT0] we insert braid and rigidity morphisms in place of the tangle elements depicted in Figure 2.12. Furthermore, we assign \( f^X_\alpha \) to the coupon \((j, +)\) with color \( \alpha \), and \( e^X_\alpha \) to \((j, -)\) respectively. For a closed, bridged ribbon we obtain a morphism \( 1 \to 1 \) in every component, \( S^3_\omega \), and thus a number \( F(BR, C, \omega) \). From this we define the number
\[
\{ BR \} := \sum_C \prod_{\omega=1}^r F(BR, C, \omega) \prod_L d(C(L)) , \quad (4.22)
\]
where
\[
d(j) := \frac{d(j)}{d(1)} .
\]

The product runs over all components \( L \) of the link diagram in \( \mathcal{M} \) and \( C(L) \in \mathcal{I} \) is the coloration associated to \( L \) by \( C \).

Let us also define two classical invariants of the bounding four fold \( W_\mathcal{L} \). The homology of \( W_\mathcal{L} \) only depends on the bridged link \( \mathcal{L} \). It is given by the cellular complex \( (C, \partial) \), where \( C_j \) has as a basis the handles, so \( dim(C_2) \) is the number of ribbons in the presentation, \( dim(C_1) \) is the number of pairs of coupons, and \( dim(C_0) \) is the number \( r \) of components of \( U \). The boundary operation \( \partial : C_2 \to C_1 \) is given by counting the number of times a ribbon passes through a surgery sphere including signs for directions. \( \partial : C_1 \to C_0 \) assigns to a basis element the difference of the components of \( U \) in which the two surgery spheres lie.

In particular we have for the Euler number of \( W_\mathcal{L} \) the formula
\[
\chi(\mathcal{L}) = dim(C_2) - dim(C_1) + dim(C_0) .
\]
The signature \( \sigma(\mathcal{L}) \) of \( W_\mathcal{L} \) is given by the signature of the linking matrix of \( \mathcal{L} \) restricted to the kernel of \( \partial \).

It is clear that the functionals \( \{ . \} \), \( \sigma \), and \( \chi \) are isotopy-invariants of the bridged link. The fact that \( \{ . \} \) is independent of the choice of the projection follows as in [RT0] from \( Rel_1 - Rel_{11} \). Naturality of the balancing and \( \theta_1 = 1 \) implies \( Rel_{12} \). Since all functionals
are obviously invariant under 0-1-cancellations, we may confine ourselves to the situation where \( r = 1 \), i.e., the presentation is on only one \( S^3 \). In order to determine how they change under the remaining moves 2), 3), and 4) of Theorem 2 let us explain some notations for bridged links:

For a given link \( L \) let \( L \cup \Box \Box \) be the link with an additional, isolated cancellation diagram as in Rel\(_{18}\) of Figure 2.23. Similarly we define \( L \cup \Diamond \) as the diagram where an isolated, -1-framed unknot is added. If a pair of coupons is in a position as on the right in Rel\(_{17}\) of Figure 2.23 we denote by \( L \cup \eta \) the link where the respective piece is replaced by the diagram on the left hand side of Rel\(_{17}\). We have the following “transformation rules”:

**Lemma 16**

\[
\begin{align*}
\{L \cup \Box \Box\} &= \{L\} \\
\{L \cup \eta\} &= \frac{1}{d(1)}\{L\} \\
\{L \cup \Diamond\} &= \{L\}\{\Diamond\} \\
\chi(L \cup \Box \Box) &= \chi(L) \\
\sigma(L \cup \Box \Box) &= \sigma(L) \\
\chi(L \cup \eta) &= \chi(L) + 2 \\
\sigma(L \cup \eta) &= \sigma(L) \\
\chi(L \cup \Diamond) &= \chi(L) + 1 \\
\sigma(L \cup \Diamond) &= \sigma(L) - 1
\end{align*}
\]

**Proof:** The relations for the \( O_1 \)- and cancellation moves follow from the multiplicativity of \( \{ \cdot \} \) and the additivity of \( \sigma \) and \( \chi \). E.g., we have \( \{L_1 \cup L_2\} \) if \( L_1 \) and \( L_2 \) are two disjoint links that can be separated by a 2-sphere in \( S^3 \). For the verification of \( \{\Box \Box\} = 1 \) we remark that \( 1 \in \mathbb{I} \) is the only coloration of the cancelling ribbon that contributes to \( F(\Box \Box, C) \) and \( d(1)e^1 \circ f^1 = 1 \) by construction.

In order to check the modification move we decompose \( id \in End(l_1 \otimes \ldots \otimes l_n) \) into sums of composites of projections onto and injections of irreducibles \( R \in \mathcal{I} \). They are presented by pairs of coupons as in the cancellation configuration with an additional ribbon \( R \) of color \( C(R) = k \) joining them. We compose this presentation of \( id \) with the modification configuration on the left side of Figure 2.23, and slide the modification annulus \( A \) between the coupons over \( R \). If we sum now over the colorations \( C(A) \) we easily find from equation (4.21) that we are left with only \( C(R) = 1 \) as a possible channel and an extra factor \( \frac{1}{d(1)} \).

It is obvious from Proposition 8 and Lemma 16 that there is only one way to construct an invariant from the given data that is multiplicative with respect to connected summing and for which \( \tau(S^3) = 1 \):

**Corollary 2** If \( D^2 = \hat{d}(1)^{-1} \) then the expression

\[
\tau(M) = D^{-\chi(L) - \sigma(L)}\{\Diamond\}^{\sigma(L)}\{L\}
\]

only depends on the three manifold that is presented by the bridged link \( L \).

Note that the Corollary implies the identity \( \{\Diamond\}\{\Diamond\} = D^{-2} \), which allows us to substitute one of the coefficient with an expression in \( \{\Diamond\} \). As outlined in the end of this section it is not hard to extend \( \tau \) to a Topological Field Theory for any BTC satisfying (4.21), using the same arguments as above. In other words, we can construct a \( \otimes \)-functor \( \tau : \hat{\text{fig}} \rightarrow \text{Vect}(C) \), which specializes to the invariant for cobordisms between empty.
surfaces. The vector spaces that is associated, e.g., to the torus $T = S^1 \times S^1$ is canonically identified with $\tau(T) = \text{Hom}(F, 1) \cong \mathbb{C}^T$, where $F := \bigoplus_{j \in I} j^\vee \otimes j$. For the cobordism from $T$ to itself, represented by the tangle $S^+$ as defined in (3.15), we can give the map $\tau(S^+)$ in terms of the matrix $S_{i,j}$ from equation (4.20). Clearly, $S^+$ is invertible in $\mathbb{C}^g$ with inverse $S^-$ and $\tau(S^-) = (S_{i,j^\vee})$ for a suitably scaled basis. This implies a purely algebraic statement, namely that a category which satisfies (4.21) also has the ‘stronger’ modularity property. This implication can also be proven formally, without reference to the topological situation:

**Lemma 17** For a semisimple BTC the following are equivalent:

1. 
   
   \[ 1 \in \text{im}(S) \]  
   \[ (4.24) \]

2. 
   
   \[ \sum_{j \in I} \dim(j)S_{j,i} = D^2\delta_{i,1} \]  
   \[ (4.25) \]

3. 
   
   $S$ is invertible.

**Proof**: We prove this lemma by showing that 1) implies 3), and that 3) implies 2).

1) $\Rightarrow$ 3): Let $\hat{d}(j)$ be as in (4.21). The “Verlinde-formula”

\[
\dim(k)^{-1}S_{i,k}S_{j,k} = \frac{S_{i\otimes j,k}}{S_{1,k}} = \sum_p N_{i,j,p}S_{p,k},
\]

is a simple consequence of the general cyclicity of $tr_X$. ($\dim(k) \neq 0$ is part of the semisimplicity condition). We multiply this with $\hat{d}(1)$ and sum over $k$. Using the symmetry of $S$ we arrive at the equation:

\[
SYS = C,
\]

where $C$ is the conjugation matrix and $Y$ is the diagonal matrix, whose entries are the numbers $\hat{d}(k)\dim(k)^{-1}$. Thus $S$ is invertible. Note also that invertibility of $Y$ implies $\hat{d}(1) \neq 0$.

3) $\Rightarrow$ 2): We have 1 and can use again equation (4.26) and $\dim(j) = \dim(j^\vee)$ to show:

\[
\sum_j \dim(j)S_{j,i} = (SCS)_{1,i} = (Y^{-1})_{1,i} = \delta_{1,i}\hat{d}(1)^{-1}.
\]

$\square$

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The fact that $S$ is invertible and that the vector $d(j)$ as defined in (4.22) is also a solution of (3) implies
\[ d(j) \mathcal{D}^2 d(j) = \dim(j) \] (4.27)

Moreover, if we specialize (2) to $i = 1$ we obtain the expression
\[ \mathcal{D}^2 = \sum_{j \in I} \dim(j)^2 \] (4.28)
as it was originally defined in [Tu]. It also follows immediately that in the notation of [Tu],
\[ \Delta = \{\otimes\} = \sum_{j \in I} \theta_j^{-1} \dim(j)^2 . \] (4.29)

Thus the invariant defined in (4.23) coincides with the construction from [Tu], where the input data have been the quantum dimensions $\dim(j)$ and the statistical phases $\theta_j$.

In conclusion, let us note that our proof of invariance avoided the use of the Kirby $\kappa$-move, and the result from [FR] that it generates all two-handle slides of links. Other than in [RT] the verification of invariance is straightforward and does not involve any algebraic computation, if we start from the quantities $\widehat{d}(j)$ and $\{\otimes\}$. The computations needed to compare it to the original definition are also much simpler than in [RT] where most parts of the $SL(2,\mathbb{Z})$-representation intrinsic to a BTC have to be constructed by algebraic computation rather than topological arguments. Thus the bridged link presentations and the condition (4.21) are a much more natural and convenient starting point for the construction of three manifold invariants than the Kirby-Fenn-Rourke-Calculus and the modularity condition of Turaev.

The RT-construction of invariants can be generalized to non-semisimple BTC’s, as for instance the representation categories $mod - H$ of a quantum group $H$ at roots of unity. For finite dimensional, quasitriangular Hopf algebras $H$ the first definition was given by M. Hennings, and, independently, the generalization to arbitrary BTC’s with enough limits by V. Lyubashenko. Their properties and rules of computation have been studied by L. Kauffman, T. Ohtsuki, and D. Radford, see [HKLR]. Let us briefly outline the algorithm by which the invariant for $mod - H$ can be computed:

Along a component of a link elements of $H$ are inserted and moved as follows: If $\mathcal{R} \in H^{\otimes 2}$ is the $R$-matrix we insert the elements $\mathcal{R}_j^{(1)}$ in one component of an overcrossing and the elements $\mathcal{R}_j^{(2)}$ in the other, and replace the overcrossing by a singular crossing which does not distinguish over and under. The elements can be moved through extrema using the antipode and successive elements along a component can be multiplied. It is clear that we reduce a plat link to a link with only singular crossings and only one element (which may depend on a summation index) at a prescribed spot for each component. The invariant is obtained by evaluating the integral $\mu \in H^*$ on each of these elements and sum over all products.

An integral of a Hopf-algebra is (up to normalization) uniquely defined by the property $(\mu \otimes 1) \Delta(y) = 1 \mu(y)$. The topological translation of this condition is the $O_2$-move, since
doubling a piece of a link component corresponds in the above algorithm to taking the
coproduct of all of the elements along this piece. In the bridged link formalism we have a
priori no requirement on $\mu \in H^*$, since the two handle slides are not among the selected
set of generating moves. Still, we need to extend the algorithm to 1-handle attachments,
such that the computed invariant is consistent with the modification move. This dictates
the following prescription: For each pair of coupons we connect the $n$ incoming strands to
the ones going out at the other coupon along any path (with singular crossings) and insert
the elements $\Lambda_j^{(k)}$ in the $k$-the component, where $\Delta^{n-1}(\Lambda) = \sum_j \Lambda_j^{(1)} \otimes \ldots \otimes \Lambda_j^{(n)}$ and

$$\Lambda = (\mu \otimes 1)(\mathcal{R}^t \mathcal{R}) \in H . \quad (4.30)$$

Now, the move $Rel_{12}$ from Figure 2.18 imposes the cointegral constraint $y\Lambda = \Lambda y = \epsilon(y)\Lambda$, where $y$ can be any element in $H$, if $H$ is nicely quasitriangular, as, e.g., for doubles. Also, for doubles it is easy to infer from (4.30) that $\mu$ has to be a right integral whenever $\Lambda$ is a
cointegral. The cancellation move imposes the normalization $\mu(\Lambda) = 1$. In fact, it is a well
known result from Hopf algebra theory that $\mu(\Lambda) \neq 0$ no matter if $H$ is semisimple or not.
Finally it is a fact that for doubles $S(\Lambda) = \Lambda$, as required by move $Rel_{13}$ of Figure 2.19.

For a semisimple BTC the categorial integral of the braided Hopf algebra $H := F^\vee$ is $\mu = \sum_j \text{dim}(j) F(j \cup \cap j) \in \text{Hom}(1, F)$ and the cointegral $\Lambda$ is the projection onto
invariance.

The fact that $\Lambda^2 = 0$ for non-semisimple BTC’s (or equivalently that $\tau(S^1 \times S^2) = 0$) makes it impossible to extend the invariants to TQFT’s that observe the $\otimes$-rule.

We conclude this section with a table of notions and conditions that we have found to be related:

| SURGERY                      | BTC / HOPF ALGEBRA                  |
|------------------------------|-------------------------------------|
| One Handle                   | Projection on Invariance / Cointegral|
| Two Handle                   | Canonical Trace / Integral           |
| Framing                      | Balancing / Ribbon Graph Element    |
| Isotopies                    | Braid and Rigidity Relations / Quasitriangular |
| 0-1-Cancellation             | $\chi(\mathcal{L})$-Normalization with $\mathcal{D}$ or “Quantum-rank” |
| 1-2-Cancellation             | Non-degenerate Pairing of Invariance and Coinvariance ($\Leftrightarrow$ Semisimplicity) or Contraction of Integral and Cointegral |
| $\mathcal{O}_1$ or $\#\mathbb{CP}^2$-move | $\sigma(\mathcal{L})$-Normalization with $\Delta$ or Moduli |
| Modification-, $\eta$-, or $\cup h_2^{(5)}$-move | $1 \in \text{im}(S)$ / Double |

4.2.2) The Connecting Annuli and Selfconjugate Objects

In the case of topological surgery presentations we have to be careful about replacing a
piece of a link as in Figure 3.28 by straight strands as in the middle picture. However, for
the evaluation of the RT-invariant this modification can be made, if we properly modify the summation \( \sum \prod_{\text{dim}(C(L))} \) in the definition \( (\ref{eq:tau}) \) of \( \tau \).

In the simplest, non-trivial case of Lemma \( \ref{lem:link1} \) with a link \( \mathcal{L} \), where \( A^+ \) is connected to \( B^+ \), we have only one link component \( L_o \) going through the annulus. If we applied the naïve move as for \( A^+...A^- \) we would get a link \( \mathcal{L}' \), where \( L_o \) is replaced by two components \( L^+ \) and \( L^- \). The total number of components stays the same. For a given coloration with \( C(L_o) = j \) we may replace the annulus by a pair of coupons, for which we insert dual bases of the (co-)invariance of \( j \otimes j^\vee \). A possible choice is, such that the canonical morphisms (associated to the maxima and minima of a ribbon graph) are \( f : 1 \to j \otimes j^\vee \) and \( \dim(j) e : j \otimes j^\vee \to 1 \).

Thus we can compute the invariant from the link \( \mathcal{L}' \) by confining the summation to colorations with \( C(L_o) = C(L_o) \) in the relative orientation induced by \( L_o \), and omitting the \( \dim(j)^2 \)-contribution to the product \( \prod_{\text{dim}(C(L))} \) coming from \( L^\pm \).

The more interesting case is given by the situation \( A^+...B^- \). Here, we have again only one link component \( L_o \) passing through the annulus, but this time with linking number 2 instead of 0. This entails that the diagram is zero for colorations for which \( j = C(L_o) \) is not selfconjugate.

For a more precise statement we observe that for a selfconjugate, irreducible object \( k \) the map \( \rho_k \) as defined in \( (\ref{eq:rho}) \) is an involution on \( \text{Hom}(1, k \otimes k) \cong \mathbb{C} \), i.e., \( \rho_k = \pm 1 \).

The invariant may now be computed from \( \mathcal{L}' \) by substituting the \( \dim(j) \) contribution in the product \( \prod_{\text{dim}(C(L))} \) by \( D^{-1} \rho_j \), and confining the summation to colorations with \( C(L_o) = C(L_o)^\vee \).

In fact the tangle in Figure \( \ref{fig:cube} \) defines a natural transformation of the identity of the BTC, with endomorphisms \( \xi(X) : X \to X \) uniquely determined by its values on the simple objects:

\[
\xi(j) = \begin{cases} 
D^{-1} \dim(j)^{-1} \rho_j & \text{if } j = j^\vee \\
0 & \text{elsewise}
\end{cases}.
\]

Hence we may also look at the evaluation of \( \mathcal{L}' \) with the morphism \( \xi \) inserted along \( L_o \).

The natural transformation \( \xi \) in the representation category of a quantum double \( H \) is given by the remarkable, central element,

\[
\rho = \sum_j \Lambda' f_j \hat{u} \Lambda'' e_j,
\]

which projects onto the selfconjugate subrepresentation when it is applied to a general representation of \( H \).

In \( (\ref{eq:rho}) \) \( \Lambda \) is the cointegral of the double as discussed in the previous subsection. For the form of \( p \) and our conventions for coproduct, dual bases \( \{e_j\} \) and \( \{f_j\} \), and \( \hat{u} \) see, e.g., [Ke1].

The constructions of Sections 4.2.1) and 4.2.2) are not confined to closed manifolds but apply to cobordisms as well. Let us conclude this section with a summary of the construction of an anomalous TQFT from the tangle presentation developed in Chapter 3:

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To a (connected) cobordism $M$ in $\overline{\text{Cob}}_3$ we associate a corresponding admissible tangle, $t$, and to this the special, closed tangle $t_M := \text{cl}(t)$ as in (3.13). For a given coloration we obtain as in [RT0] a morphism:

$$F(t_M, C) : j_1^\vee \otimes j_1 \otimes \ldots \otimes j_{\tilde{g}}^- \to k_1^\vee \otimes k_1 \otimes \ldots \otimes k_{\tilde{g}}^+$$

Summing over colorations of the internal, closed ribbons we obtain, after application of $\text{Hom}(-, 1)$, the linear map:

$$\{t, j, k\} : \text{Hom}(j_1^\vee \otimes j_1 \otimes \ldots \otimes j_{\tilde{g}}^-, 1) \to \text{Hom}(k_1^\vee \otimes k_1 \otimes \ldots \otimes k_{\tilde{g}}^+, 1)$$

with

$$\{t, j, k\} = \sum_{\text{internal}} \prod_{\text{internal}} \dim(l) F(t_M, C).$$

Here $j$ and $k$ are short hand for the $\tilde{g}^\pm$ colors at the boundaries $X^\pm$. We introduce the canonical injections and projections between the total invariance and the product of the invariances of the groups $G_s$:

$$\text{Hom}(j_1^\vee \otimes j_1 \otimes \ldots \otimes j_{\tilde{g}}, 1) \xrightarrow{p} \bigotimes_{s=1}^K \text{Hom}(j_{s,1}^\vee \otimes j_{s,1} \otimes \ldots \otimes j_{s,g_s}, 1),$$

if we have $K$ components of genera $\{g_s\}$. Also, we introduce the “Euler number” of the tangle:

$$\chi^+(t) = (\# \text{ of internal components of } \text{cl}(t)) + \tilde{g}^+ - K^+ + 2,$$

where $\tilde{g}^+$ and $K^+$ are the total number of pairs of ribbons and the number of components at $X^+$, respectively.

For the object $F := \bigoplus_{j \in \mathcal{J}} j^\vee \otimes j$ we have a canonical decomposition:

$$\tau((g)) := \text{Hom}(F^\otimes g, 1) = \bigoplus_i \text{Hom}(j_1^\vee \otimes j_1 \otimes \ldots \otimes j_{g}, 1)$$

On this we define the linear map associated to the cobordism by the TQFT

$$\bar{\tau}(M) : \tau(g^-) = \bigotimes_{s=1}^{K^-} \tau((g_s^-)) \longrightarrow \tau(g^+) = \bigotimes_{s=1}^{K^+} \tau((g_s^+))$$

by the sum of the block entries:

$$\bar{\tau}(M) = D^{-\chi^+(t)} \bigoplus_{\{j, k\}} \bigoplus_{j \in \mathcal{J}} \prod \text{dim}(j) p_+ \circ \{t, j, k\} \circ i_-$$

Note, that we consider only the ribbons at $X^+$ in the additional product of $q$-dimensions. Also, the signature, $\sigma(\mathcal{L})$, does not appear in the formula, since we only want to have an anomalous functor $\bar{\tau}$ defined on $\overline{\text{Cob}}_3$. Compatibility of $\bar{\tau}$ with the composition rules of Section 3.2.3) follow from (4.25). Since we projected on the invariance of the individual groups the $\tau$-move is given by a pure braid with one 1-colored strand and therefore does not change the morphism. Invariance of the functor under all other moves follows easily from the results for closed manifolds.
4.3) Punctured Cobordisms and Glue-$\otimes$

The definition of $\text{Cob}_3$ has a natural generalization, which includes punctured surfaces. To be more precise we define for any $N \in \mathbb{Z}^+$ a cobordism category $\text{Cob}_3(N)$ as follows:

The objects are compact, oriented two folds $\Sigma$ with orientation preserving homeomorphisms

$$\zeta : S(N) := \coprod_{i=1}^N S^1 \to \partial \Sigma,$$

parametrizing the boundaries by a oriented, standard manifold $S(N)$. A morphism from a surface $\Sigma_1$ to $\Sigma_2$ is given by a three fold $M$, and coordinate maps from its boundary to the composed surface:

$$\psi : \partial M \to \Sigma_1 \coprod_{\zeta_1} \{ S(N) \times [0, 1] \} \coprod_{\zeta_2} \Sigma_2.$$

We also impose a similar notion of equivalence as for the closed case. The composition of morphisms is then given as usual by gluing the two three folds together along a common boundary piece $\Sigma_2$. We obtain a three fold with boundary

$$-\Sigma_1 \coprod_{\zeta_1} \{ S(N) \times [0, 1] \} \coprod_{\zeta_2^{-1} \zeta_2} \{ S(N) \times [0, 1] \} \coprod_{\zeta_3} \Sigma_3$$

For a suitable redefinition of the $\zeta$-coordinate maps we may replace the two middle parts by one $S(N) \times [0, 1]$.

The outlined axioms entail representations

$$\pi_0(\text{Diff}(\Sigma, \partial \Sigma)^+) \cong \text{Aut}_{\text{Cob}_3(N)}(\Sigma)$$

and

$$\pi_1(\partial \Sigma) \hookrightarrow \text{Nat}_{\text{Cob}_3(N)}(id, id).$$

Also, $\text{Cob}_3(1)$ has a natural structure of a braided tensor category and contains a canonical, braided Hopf algebra (see, e.g., [Ke1] and [Ke2]).

In [KL] we develop the analogous tangle presentation of $\text{Cob}_3(N)$. It is obtained from the one of $\text{Cob}_3 = \text{Cob}_3(0)$ by considering a surgery presentation of the corresponding cobordism in $\text{Cob}_3$ where the cylinders $S^1 \times [0, 1]$, are filled with tubes $D^2 \times [0, 1]$. The latter are tubular neighborhoods of $N$ strands joining the opposite surfaces of the cobordism, and we may assume that they are disjoint from the other surgery ribbons. A standard presentation of a connected element of $\text{Cob}_3(N)$ is thus given by a standard bridged link diagram in $\mathcal{H}$ with $N$ additional ribbons that start and end in opposite boundary components of the $\text{Cob}_3$-cobordism, and have a prescribed standard form inside the handlebodies $H^+_g$. The moves of the presentation are obtained by treating the additional ribbons like singularities with highest values, i.e., they may go through surgery spheres and can be slid over other two handles.

A far more interesting aspect of this family of cobordism categories is another type of "glue operation", which may be understood as a second independent composition among
relative cobordisms. We start in the definition with the choice of some orientation reversing
involution \( \rho' : S^1 \to S^1 \). From the manifolds \( S(N) \) and \( S(M) \) we select \( K \) components and
construct a functor

\[
Cob_3(N) \times Cob_3(M) \longrightarrow Cob_3(N + M - 2K)
\]
as follows:

For two surfaces \( \Sigma \in \text{Ob}(Cob_3(N)) \) and \( \Sigma' \in \text{Ob}(Cob_3(M)) \) the product is defined as the
sewed surface,

\[
\Sigma \bowtie \Sigma'
\]
where we glue respective components \( C \subset \Sigma \) and \( C' \subset \Sigma' \) of the boundaries together using
the identifications

\[
\zeta' \circ \rho \circ \zeta^{-1} : C \to C'.
\]

Moreover, we define the product of cobordisms \( M \) and \( M' \) in these categories as a quotient
space:

\[
M \bowtie M',
\]
where we use the identification along the cylindrical boundary pieces \( S^1 \times [0, 1] \cong T \subset \partial M \)
given by

\[
\psi' \circ (\rho \times \text{id}) \circ \psi^{-1} : T \to T'.
\]

Clearly, the definitions for objects and morphisms are compatible.

In [KL] we describe a very natural way of organizing these two type gluings over surfaces in terms of double categories,
whose ingredients are always two types of compositions. Specifically, the pastings over surfaces are viewed as vertical compositions,
and the gluings over the cylindrical pieces as horizontal compositions.

To describe the (horizontal) glue product in terms of standard presentations, we observe
that \( \Sigma \otimes_{\text{glue}} \Sigma' \) is homeomorphic to a standard surface \( \Sigma'' \) of genus \( g'' = g + g' + K - 1 \),
with \( N + M - 2K \) holes. This allows us to define a cobordism \( \mathfrak{N} : \Sigma'' \times [0, 1] \cong T \subset \partial M \)
given by

\[
\mathfrak{N} : \Sigma \bowtie \Sigma' \twoheadrightarrow \Sigma''.
\]

We present \( \mathfrak{N} \) in a way analogous to the generalized standard presentation, which we outlined
above. In this picture we start from a cobordism

\[
\hat{\mathfrak{N}} : \hat{\Sigma} \bowtie \hat{\Sigma}' \to \hat{\Sigma}'',
\]
where \( \hat{\Sigma}^* \) denotes the corresponding, closed manifold. Starting and ending at the original
punctures we include ribbons: \( N - K \) of them going from \( \hat{\Sigma} \) to \( \hat{\Sigma}'' \), \( M - K \) going from \( \hat{\Sigma}' \)
to \( \hat{\Sigma}'' \), and \( K \) going from \( \hat{\Sigma} \) to \( \hat{\Sigma}' \). An example for \( g = 2, g' = 1, N = 3, M = 4, K = 2 \) is
given in Figure 4.50.
Figure 4.50: Non-canonical: \( \mathfrak{R} : \Sigma \otimes_{\text{glue}} \Sigma' \to \Sigma'' \)

For two cobordisms \( M : \Sigma_1 \to \Sigma_2 \) and \( M' : \Sigma'_1 \to \Sigma'_2 \) let us consider the composite:

\[
M \hat{\otimes} M' := \mathfrak{R}_2 \circ (M \otimes M') \circ \mathfrak{R}_1^{-1} : \Sigma''_1 \to \Sigma''_2.
\]

Let us also briefly describe the basic topological transformation used to extract from the glue operation a compatible composition law for the tangles:

A pair \((T, T')\) of the \( K \) cylindrical pieces in the boundaries of \( M \) and \( M' \) are now joined by the ribbons in \( \mathfrak{R}_i \)'s. Thus \( M \hat{\otimes} M' \) is a morphism in \( \text{Cob}_3(N + M - 2K) \), with \( K \) solid tori removed from the inside. A vicinity of such a torus is depicted on the left of Figure 4.51.

Using an ‘\( S \)-transformation’, we find it to be homeomorphic to the region on the right side of Figure 4.51 times a circle. If we fill in a \( D^2_{aux} \times S^1 \), we obtain the glued tensor \( M \otimes_{\text{glue}} M' \) with an “irrelevant” \( D^2_{aux} \times S^1 \cong [0, 1] \times T \) inserted between \( T \) and \( T' \).
Thus if we simply *reinterpret* the $K$ closed puncture ribbons as surgery ribbons we obtain a presentation of $M \otimes_{\text{glue}} M'$. Using similar moves as for the case of closed surfaces this can be brought again into a standard form.

The fact that the family of $\aleph$’s for all pairs of standard surfaces is by no means canonical, which is the source of some problems:

For example one has to verify that the $\aleph$’s are chosen, such that composites for different orders of sewing the same surface together yield equivalent cobordisms between the different pieces and the glued standard surface.

More importantly, in the construction of extended TQFT’s for surfaces with boundaries, we can at most expect to be able to construct *pseudo functors* from the double category of relative cobordisms into a corresponding algebraic double category. This functor will respect the $\otimes_{\text{glue}}$-product only up to an equivalence depending on the choice of $\aleph$’s.

Specifically, in [KL] we consider the algebraic double category, in which the 0-Objects are $n$-fold tensor products, $C \otimes \ldots \otimes C$, of an abelian category $C$, the horizontal and vertical 1-arrows are functors, and the 2-arrows are natural transformations between functors. We can, however, obtain an honest functor between the two double categories, by admitting not one but a finite, combinatorial set of surfaces as 1-arrows for each homeomorphism class (characterized by number of components, genera, and holes).
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