Homogenization of a Nonlocal Stochastic Schrödinger Equation with Oscillating Potential

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Abstract
We consider the homogenization of a nonlocal stochastic Schrödinger equation with a rapidly oscillating, periodically time-dependent potential. With help of a two-scale convergence technique, we establish a homogenization principle for this nonlocal stochastic partial differential equation. We explicitly derive the homogenized model. In particular, this homogenization principle holds when the nonlocal operator is the fractional Laplacian.

Keywords: Homogenization, nonlocal Laplace operator, Schrödinger Equation, effective dynamics, stochastic partial differential equation.

1 Introduction
The homogenization of stochastic partial differential equations has attracted a lot of attention recently [1, 2, 3], due to its importance in effective mathematical modeling and efficient simulation.

The Schrödinger equation is the fundamental equation in quantum physics for describing quantum mechanical behaviors. It quantifies the wave function of a physical system evolving over time. For the homogenization of deterministic Schrödinger equations, there are two different scalings. One is the semi-classical scaling [13, 14, 15], and the other one is the typical scaling of homogenization [16, 17].

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In the path integral approach\cite{18} to quantum physics, the integral over the Brownian trajectories leads to the usual (local) Schrödinger equation \cite{19}. Recent works on the path integrals over the Lévy paths (e.g., \cite{20}) lead to nonlocal Schrödinger equations. More physical investigations on fractional or nonlocal generalization of the Schrödinger equations may be found in, for example, \cite{21,22,23,24,25}.

As random disturbances may affect the qualitative behaviors drastically and result in new properties for this quantum model, stochastic Schrödinger equations have attracted attentions recently (e.g., \cite{26,27,28,29,30}).

In this paper, we will establish a homogenization principle for a nonlocal stochastic Schrödinger equation with a typical scaling and an oscillating potential. For stochastic homogenization problems, a two-scale convergence technique \cite{31,32,33,34} is available.

Specifically, we consider the homogenization for the following nonlocal stochastic Schrödinger equation (heterogeneous system) with a small positive scale parameter $\epsilon$:

$$
\begin{aligned}
\begin{cases}
\epsilon \partial_t u_{\epsilon} = \mathcal{A} u_{\epsilon} + \epsilon^{(1-\alpha)/2} \mathcal{V} u_{\epsilon} + g(u_{\epsilon}) dW_t + f, & x \in D, \\
 u_{\epsilon}(0, x) = h(x), & x \in D = (-1, 1), \\
 u_{\epsilon}(t, x) = 0, & x \in D^c = \mathbb{R} \setminus D,
\end{cases}
\end{aligned}
$$

where $u_{\epsilon} = u_{\epsilon}(t, x) : \mathbb{R}^+ \times D \rightarrow \mathbb{C}$ is an unknown wave function. The function $\mathcal{V}(t, x) = \mathcal{V}(\frac{x}{\epsilon}, \frac{x}{\epsilon})$ is potential, $g(v)$ is noise intensity and $W(t)$ is a Wiener process. Moreover, the nonlocal operator

$$
\mathcal{A} u = \mathcal{D}(\Theta \mathcal{D}^*) u,
$$

where $\Theta(x, y) = \Theta(\frac{x}{\epsilon}, \frac{y}{\epsilon})$ is of period 1, bounded and positive, and the linear operator $\mathcal{D}$ and its adjoint operator $\mathcal{D}^*$ are defined as follows.

Given functions $\beta(x, y)$ and $\gamma(x, y)$ with $\gamma$ antisymmetric ($\gamma(-x, -y) = -\gamma(x, y)$), the nonlocal divergence $\mathcal{D}$ on $\beta$ is defined as

$$
\mathcal{D}(\beta)(x) := \int_{\mathbb{R}} (\beta(x, y) + \beta(y, x)) \cdot \gamma(x, y) dy \quad \text{for } x \in \mathbb{R}.
$$

For a function $\phi(x)$, the adjoint operator $\mathcal{D}^*$ corresponding to $\mathcal{D}$ is the operator whose action on $\phi$ is given by

$$
\mathcal{D}^*(\phi)(x, y) = - (\phi(y) - \phi(x)) \gamma(x, y) \quad \text{for } x, y \in \mathbb{R}.
$$

Here we take $\gamma(x, y) = (y - x) \frac{1}{|y-x|^{1+\alpha}}$. As a special case, we set $\Theta$ to be 1, we have

$$
\frac{1}{2} \mathcal{D}^* \mathcal{D} = - (\Delta)^{\alpha/2}.
$$

The nonlocal Laplace operator $(-\Delta)^{\alpha/2}$ is defined as

$$
(-\Delta)^{\alpha/2} u(x) = \int_{\mathbb{R} \setminus \{0\}} \frac{u(y) - u(x)}{|y-x|^{1+\alpha}} dy,
$$

where the integral is in the sense of Cauchy principal value.
Remark 1. For a function $v(x, y)$, we define

$$(D^*_x v)(x, z, y) = -(v(z, y) - v(x, y))\gamma(x, z)$$

and

$$(D_x D^*_x v)(x, y) = 2\int_{\mathbb{R}} -(v(z, y) - v(x, y))\gamma^2(x, z)dz$$

$$= -(-\Delta)^{\alpha/2}v(x, y).$$

Our purpose is to examine the convergence of the solution $u^\epsilon$ of (1) in some probabilistic sense, as $\epsilon \to 0$, and to specify the limit $u_0$. We will see that the limit process $u_0$ satisfies the following nonlocal stochastic partial differential equation (homogenized system):

$$
\begin{cases}
idu_0 = -\Xi_1 (-\Delta)^{\alpha/2}u_0 - \Xi_2 (D\zeta)(x) - \Xi_3 \zeta(x) + g(u_0)dW_t + f \\
u_0(x, t) = 0, \quad (x, t) \in D^c \times (0, T), \\
u_0(0) = h(x), \quad x \in D,
\end{cases}
$$

(2)

where

$$
\Xi_1 = \int_{Y \times N} \Theta(y, \eta)dydn,
\Xi_2 = \int_{Y \times N \times Z} \Theta(y, \eta)D^*_y \chi dydn d\tau,
\Xi_3 = \int_{Y \times Z} \mathcal{V}(y, \tau) \chi(y, \tau)dyd\tau,
\zeta(x) = \frac{1}{|D|} \int_D (D^*_u_0)(x, z)dz.
$$

Structure of this paper. In order to motivate the theory, in Section 1, we present the heterogeneous system of nonlocal Schrödinger equation and the relationship between fractional Laplacian operator with the operator in the equation. In Section 2, we recall some function spaces and deal with the existence and uniqueness of the Schrödinger equation. In Section 3, we prove the homogenization theorem and derive the homogenized equation.

2 Preliminaries

We now briefly discuss the well-posedness for the heterogeneous equation (1), and derive a few uniform estimates concerning the solution $u^\epsilon$. 

3
2.1 Function spaces

Let $\alpha \in (1,2)$ and $D = (-1,1)$, the classical fractional Sobolev space is

$$H^{\alpha/2}(D) = \{ u \in L^2(D) : \int_D \int_D \frac{|u(x) - u(y)|^2}{|x-y|^{1+\alpha}} dx dy < \infty \},$$

with the norm

$$||u||_{H^{\alpha/2}(D)}^2 = ||u||_{L^2(D)}^2 + \int_D \int_D \frac{|u(x) - u(y)|^2}{|x-y|^{1+\alpha}} dx dy.$$

For the fractional Laplacian operator, we have

$$(\mathcal{A}^\epsilon u, u)_{L^2(D)} = \frac{1}{2}(\Theta'(x,y)D^\epsilon u(x,y), D^\epsilon u(x,y))_{L^2(\mathbb{R} \times \mathbb{R})}$$

$$= \frac{1}{2} \int_D \int_{D^c} \Theta'(x,y) \frac{|u(x)|^2}{|y-x|^{1+\alpha}} dy dx + \frac{1}{2} \int_D \int_D \Theta'(x,y) \frac{|u(x) - u(y)|^2}{|x-y|^{1+\alpha}} dx dy.$$

Pose $\rho(x) := \int_{D^c} \frac{|x-y|^{-1-\alpha}}{|y-x|^{1+\alpha}} dy$. Since the fact $\Theta'(x,y)$ is positive and bounded, we then can define a weighted fractional Sobolev space without considering the function $\Theta'(x,y)$:

$$H^{\alpha/2}_\rho(D) := \{ u \in L^2(\mathbb{R}) : u|_{\mathbb{R}\setminus D} = 0, ||u||_{H^{\alpha/2}_{\rho}(D)} < \infty \},$$

equipped with the norm

$$||u||_{H^{\alpha/2}_\rho(D)} := \left( \frac{1}{2} \int_D \rho(x)|u(x)|^2 dx + \frac{1}{2} \int_D \int_D \frac{|u(x) - u(y)|^2}{|x-y|^{1+\alpha}} dx dy \right)^{\frac{1}{2}},$$

which immediately implies that $((\Delta)^{\alpha/2} u, u)_{L^2(D)} = ||u||_{H^{\alpha/2}_{\rho}(D)}^2$.

We consider $Y, N, Z$ as subset of $\mathbb{R}_y, \mathbb{R}_\eta, \mathbb{R}_\tau$ respectively (the space of variables $y, \eta$ and $\tau$ respectively).

Recall that $C_{per}(Y)$ is the subspace of $C(\mathbb{R})$ of $Y$-period functions. It is a Banach space under the supremum norm, whereas $L^p_{per}(Y)$ is a Banach space under the norm $||u||_{L^p_{per}(Y)} = (\int_Y |u(y)|^p dy)^{\frac{1}{p}}$.

The space $H^{\#}_{\#}(Y)$ is defined as

$$H^{\#}_{\#}(Y) = \{ u \in H_{per}(Y) | \int_Y u(y) dy = 0 \}.$$

The space $\mathcal{M}(D)$ is the space of functions in $C^\infty(D)$ with compact supports. We set

$$\mathcal{Y}((0,T) \times \Omega) = \{ v \in L^2((0,T) \times \Omega; H^{\alpha/2}_{\rho}(D)) : v' \in L^2((0,T) \times \Omega; H^{-\alpha/2}_{\rho}(D)) \}.$$

$\mathcal{Y}((0,T) \times \Omega)$ is provided with the norm

$$||v||_{\mathcal{Y}((0,T) \times \Omega)}^2 = ||v||_{L^2((0,T) \times \Omega; H^{\alpha/2}_{\rho}(D))}^2 + ||v'||_{L^2((0,T) \times \Omega; H^{-\alpha/2}_{\rho}(D))}^2$$

which makes it a Hilbert space.
2.2 Well-posedness

Let $B^\epsilon$ be the linear operator in $L^2(D)$ defined by

$$B^\epsilon u = -iA^\epsilon u \text{ for all } u \in D(B^\epsilon),$$

with domain

$$D(B^\epsilon) = \{ v \in H^\alpha_\rho(D) : A^\epsilon v \in L^2(D) \}.$$  

Then, $B^\epsilon$ is of skew-adjoint since $A^\epsilon$ is self-adjoint. Moreover, $B^\epsilon$ is the generator of a contraction semigroup $(G_t^\epsilon)_{t>0}$.

Now, let us check the existence and uniqueness for equation (1). The abstract problem for equation (1) is given by

$$\begin{aligned}
\begin{cases}
  u'_e = B_e u_e + F_e(u_e) + g(u_e) dW_t, \\
  u_e(0,x) = h(x),
\end{cases}
\end{aligned}$$  

where $F_e$ is defined in $L^2(0,T;L^2(D))$ by

$$F_e(v)(t) = -i\epsilon^{1-\alpha}V^e v - if(t).$$

Then $F_e(v)$ is locally Lipschitz. We can obtain the following lemma (Ichikawa [35]).

**Lemma 1.** Suppose $h \in D(B^\epsilon)$, $f \in C([0,T];L^2(D))$ and for all $\epsilon > 0$,

$$\epsilon^{(1-\alpha)/2}||V||_\infty \leq \beta,$$

where $\beta$ is a positive constant independent of $\epsilon$. We obtain the existence and uniqueness of solution $u_e(t) \in C([0,T];D(B^\epsilon)) \cap C^1([0,\tau(h)];L^2(D))$ for some $\tau = \tau(h) > 0$.

Let us put

$$a^\epsilon(u,v) = \int_D \int_D \Theta^\epsilon D^* u(x,z) \overline{D^* v(x,z)} dxdz.$$  

**Lemma 2.** Let $u^\epsilon$ be a solution of equation (1) with initial value $h \in L^2(D)$.

Suppose further that

$$f, f' \in L^2(0,T;L^2(D))$$

and

$$\frac{\partial V}{\partial \tau} \in L^\infty with \epsilon^{-1-\alpha/2}\frac{\partial V}{\partial \tau} \leq c_0,$$

$c_0$ being a constant independent of $\epsilon$. Then there exists a constant $c > 0$ independent of $\epsilon$ such that the solution $u_e$ of equation (1) verifies:

$$\sup_{0 \leq t \leq T} \epsilon E \sup_{0 \leq t \leq T} ||u'_e||^2_{L^2(D)} + \epsilon E ||u'_e||^2_{L^2(0,T;H^\alpha_\rho(D))} \leq c.$$
Proof. Applying Itô formula for $u_\epsilon(t)$, we have

$$
||u_\epsilon(t)||^2_{L^2} = ||u_\epsilon(0)||^2_{L^2} - \text{Re} \int_0^t 2i(A^\epsilon u_\epsilon(s) + e^{(1-\alpha)/2}V^\epsilon u_\epsilon(s), u_\epsilon(s))ds \\
- \text{Re} \int_0^t 2i(g(u_\epsilon(s)), u_\epsilon(s))dW_s - \text{Re} \int_0^t 2i(f, u_\epsilon(s))ds + \int_0^t ||g(u_\epsilon)||^2_{L^2}ds
$$

$$
= ||u_\epsilon(0)||^2_{L^2} + \text{Im} \int_0^t 2(g(u_\epsilon(s)), u_\epsilon(s))dW_s + \text{Im} \int_0^t 2(f, u_\epsilon(s))ds + \int_0^t ||g(u_\epsilon)||^2_{L^2}ds.
$$

By Burkholder-Davis-Gundy’s inequality, Hölder inequality and Young’s inequality, it refers that

$$
E \sup_{0 \leq t \leq T} 2\text{Im} \int_0^t 2(g(u_\epsilon(s)), u_\epsilon(s))dW_s
$$

$$
= E \sup_{0 \leq t \leq T} 2\text{Im} \int_0^t 2 \int_D g(u_\epsilon(s))\overline{u_\epsilon}(s)dW_s dx
$$

$$
\leq c_1E(\int_0^T ||\overline{u_\epsilon}(s)g(u_\epsilon(s))||^2_{L^2}ds)^{\frac{1}{2}}
$$

$$
\leq c_1E(\delta \sup_{0 \leq t \leq T} ||u_\epsilon(t)||^2_{L^2} + \frac{1}{3}(\int_0^T ||g(u_\epsilon(s))||^2_{L^2}ds)
$$

$$
\leq \frac{1}{3}E \sup_{0 \leq t \leq T} ||u_\epsilon(t)||^2_{L^2} + c_2E \int_0^T ||u_\epsilon(s)||^2_{L^2}ds + c_2.
$$

Then, we obtain

$$
\frac{2}{3}E \sup_{0 \leq t \leq T} ||u_\epsilon(t)||^2_{L^2} \leq ||u_\epsilon(0)||^2_{L^2} + c_3\int_0^T \sup_{0 \leq \tau \leq s} ||u_\epsilon(r)||^2_{L^2}dr + c_4.
$$

which implies from Gronwall inequality that

$$
E \sup_{0 \leq t \leq T} ||u_\epsilon(t)||^2_{L^2} \leq c_5,
$$

where the positive constant $c_5$ is independet of $\epsilon$.

Moreover, we also have

$$
E \sup_{0 \leq t \leq T} ||u_\epsilon(t)||^4_{L^2} \leq c_5.
$$

Next, taking the product in $L^2(D)$ of equation (1) with $u_\epsilon'$

$$
i||u_\epsilon'(t)||^2_{L^2} = (A^\epsilon u_\epsilon(t) + e^{(1-\alpha)/2}V^\epsilon u_\epsilon(t), u_\epsilon'(t)) + (g(u_\epsilon(t)), u_\epsilon'(t))dW_t + (f(t), u_\epsilon'(t)).
$$

By the preceding equality we have

$$
\text{Re}(A^\epsilon u_\epsilon(t) + e^{(1-\alpha)/2}V^\epsilon u_\epsilon(t), u_\epsilon'(t)) + \text{Re}(g(u_\epsilon(t)), u_\epsilon'(t))dW_t + \text{Re}(f(t), u_\epsilon'(t)) = 0.
$$

Since the fact that

$$
\frac{d}{dt}a^e(u_\epsilon(t), u_\epsilon(t)) = 2\text{Re}(A^\epsilon u_\epsilon(t), u_\epsilon'(t)),
$$


\[ e^{(1-\alpha)/2} \frac{d}{dt}(V^\alpha u_\varepsilon(t), u_\varepsilon(t)) = e^{(1-\alpha)/2}\left((\frac{\partial V}{\partial \tau})' u_\varepsilon(t), u_\varepsilon(t)\right) + 2\varepsilon^{(1-\alpha)/2} \text{Re}(V^\alpha u_\varepsilon(t), u_\varepsilon'(t)). \]

We have
\[ \frac{1}{2} \frac{d}{dt} a'(u_\varepsilon(t), u_\varepsilon(t)) + \frac{1}{2} e^{(1-\alpha)/2} \frac{d}{dt}(V^\alpha u_\varepsilon(t), u_\varepsilon(t)) - e^{(1-\alpha)/2}\left((\frac{\partial V}{\partial \tau})' u_\varepsilon(t), u_\varepsilon(t)\right) + \text{Re}(g(u_\varepsilon(t)), u_\varepsilon'(t))dW_t + \text{Re}\left(\frac{d}{dt}(f(t), u_\varepsilon(t)) - \text{Re}(f'(t), u_\varepsilon(t))\right) = 0. \]

An integration on \([0, t]\) of the equality above yields,
\[ \frac{1}{2} a'(u_\varepsilon(t), u_\varepsilon(t)) + \frac{1}{2} e^{(1-\alpha)/2}(V^\alpha u_\varepsilon(t), u_\varepsilon(t)) - \frac{1}{2} e^{(1-\alpha)/2}(V^\alpha(0)h, h) \]
\[ = e^{(1-\alpha)/2} \left( \int_0^t ((\frac{\partial V}{\partial \tau})' u_\varepsilon(s), u_\varepsilon(s))ds - \int_0^t \text{Re}(g(u_\varepsilon(s)), u_\varepsilon'(s))dW_s \right) \]
\[ - \text{Re}(f(t), u_\varepsilon(t)) + \text{Re}(f(0), h) + \text{Re}\int_0^t (f'(s), u_\varepsilon(s))ds. \]

It follows that
\[ c_0||u_\varepsilon(t)||^2_{H_\alpha^\alpha/2} + 2\int_0^t \text{Re}(g(u_\varepsilon(s)), u_\varepsilon'(s))dW_s \]
\[ \leq \beta||u_\varepsilon(t)||^2_{L_2} + c_7||h||^2_{H_\alpha^\alpha/2} + \beta||h||^2_{L_2} \]
\[ + c_0||u_\varepsilon(t)||^2_{L^2(0,T;L_2^2(D))} + 2||f(t)||_{L_2}||u_\varepsilon(t)||_{L_2} \]
\[ + 2||f(0)||_{L_2}||h||_{L_2} + 2||f'||_{L_2(0,T;L_2^2(D))}||u_\varepsilon(t)||_{L^2(0,T;L_2^2(D))}. \]

We consider the expectation after integrating on \([0, T]\) the preceding inequality and using Burkholder-Davis-Gundy’s inequality, we have
\[ \mathbb{E}||u_\varepsilon||^2_{L_2^2(0,T;H_\alpha^\alpha/2(D))} \leq c_8, \]
where the positive constant \(c_8\) is independent of \(\varepsilon\). By equation (1), we have
\[ i \int_0^T < u'_\varepsilon(t), \bar{v}(t) > dt = \int_0^T a'(u_\varepsilon(t), v(t))dt + \int_0^T e^{(1-\alpha)/2}(V^\alpha u_\varepsilon(t), v(t))dt \]
\[ + \int_0^T (g(u_\varepsilon), v(t))dW_t + \int_0^T (f(t), v(t))dt \]
for all \(v \in L^2^2(0,T;H_\alpha^\alpha/2(D))\). Hence, we have
\[ \mathbb{E}||u_\varepsilon'||^2_{L_2^2(0,T;H_\alpha^{-\alpha/2}(D))} \leq c_9. \]

In summary, we deduce that
\[ \sup_\varepsilon \mathbb{E} \sup_{0 \leq t \leq T} ||u_\varepsilon'||^2_{L^2_2(D)} + \sup_\varepsilon \mathbb{E} ||u_\varepsilon||^2_{L_2^2(0,T;H_\alpha^{-\alpha/2}(D))} + \sup_\varepsilon \mathbb{E} ||u_\varepsilon'||^2_{L_2^2(0,T;H_\alpha^{-\alpha/2}(D))} \leq c. \]

From Lemma I and Lemma II, we obtain the global existence and uniqueness of equation (1) and (3).
### 3 Homogenization and Homogenized Equation

After proving several convergence results, we establish homogenization principle and derive the homogenized equation.

#### 3.1 Some convergence results

We now give some useful convergence results which is important for the final homogenization theorem. Let $Q = D \times (0, T)$ with $T \in \mathbb{R}_+^*$.

**Definition 1.** A sequence $(u_\epsilon) \in E \subset L^2(Q)$ is said to two-scale converge in $L^2(Q)$ to some $u_0 \in L^2(Q; L^2_{\text{per}}(Y \times N))$ if as $E \in \epsilon \to 0$,

$$\int_Q u_\epsilon(x,t)\psi(x,t)dxdt \to \int \int \int_{Q \times Y \times N} u_0(x,t,y,\tau)\psi(x,t,y,\tau)dxdydt$$

for all $\psi \in L^2(Q; C_{\text{per}}(Y \times N))$, where $\psi(x,t) = \psi(x,t,\frac{x}{\epsilon},\frac{t}{\epsilon}).$

**Lemma 3.** Let $E$ be a fundamental sequence. Then, a subsequence $E'$ can be extracted from $E$ such that, as $E' \ni \epsilon \to 0$,

$$u_\epsilon \to u_0 \quad \text{in} \quad L^2((0,T); H^{\alpha/2}_p(D)) \quad \text{weakly},$$

$$u_\epsilon \to u_0 \quad \text{in} \quad L^2(Q) \quad \text{two-scale}.$$  

Moreover, let

$$\psi_\epsilon = \psi_0 + \epsilon(1+\alpha/2)\psi_1, \quad \text{i.e.,} \quad \psi_\epsilon(x,t) = \psi_0(x,t) + \epsilon(1+\alpha/2)\psi_1(x,t,\frac{x}{\epsilon},\frac{t}{\epsilon}),$$

where

$$\psi_0 \in M(Q) \quad \text{and} \quad \psi_1 \in M(Q) \otimes [(C_{\text{per}}(Y)/\mathbb{C}) \otimes C_{\text{per}}(Z)].$$

For a further subsequence $\epsilon' \in E''$, we obtain

$$\int_0^T a^\epsilon(u^\epsilon,\psi^\epsilon)dt \to \int_{Q \times D \times Y \times N \times Z} \Theta(y,\eta)(D_x^*u_0+D_y^*u_1)(D_z^*\psi_0+D_y^*\psi_1)dxdzdyd\eta d\tau,$$

where $u_0 \in \mathcal{Y}(0,T), u_1 \in L^2(Q \times \Omega; L^2_{\text{per}}(Z; H^{\alpha/2}_p(Y))).$

**Proof.** Let $\phi_0(x,z,t) = (D_z^*\psi_0)(x,z,t), \phi_1(x,\frac{x}{\epsilon},\frac{t}{\epsilon},\frac{t}{\epsilon},\frac{t}{\epsilon}), \phi(x,z,t,\eta,\tau) = \phi_0(x,z,t) + \phi_1(x,y,t,\eta,\tau)$. For convenient, $\phi_0(x,z,t)$ and $\phi_1(x,\frac{x}{\epsilon},\frac{t}{\epsilon},\frac{t}{\epsilon},\frac{t}{\epsilon})$ are abbreviated as $\phi_0(x,z)$ and $\phi_1(x,\frac{x}{\epsilon},\frac{t}{\epsilon})$ respectively.

Due to Lemma 2 one has a subsequence $E$, such that

$$u_\epsilon \to u_0 \quad \text{in} \quad L^2((0,T); H^{\alpha/2}_p(D)) \quad \text{weakly}.$$  

Then for a further subsequence $E'' \ni \epsilon'$, we have

$$\begin{align*}
\begin{cases}
  \{u_\epsilon'\} \quad \text{two-scale converges to} \quad u \in L^2(Q), \\
  D^* u_\epsilon' \quad \text{two-scale converges to} \quad U \in L^2(Q \times D),
\end{cases}
\end{align*}$$

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and there exists a function $U \in L^2(Q \times D)$ such that

$$
\int_0^T a(\varepsilon u', \psi') dt \to \int_{Q \times D \times Y \times N \times Z} \Theta(y, \eta) U(x, z, t, y, \eta, \tau) \phi(x, z, t, y, \eta, \tau) dx dz dy d\eta d\tau. 
$$

(5)

By the definition of $D^*$ and $D$, it follows that

$$
\int_0^T a(\varepsilon u', \psi') dt
= \int_Q \int_D \Theta^*(x, z)(D^* u')(x, z) \frac{\sqrt{\mu}}{\sqrt{\epsilon}} \left( x, z \right) dxdz dt + o(\varepsilon')
= \int_Q \left( u_0(x) \right) \left[ \phi_0(x, z) \Theta^*(x, z) + \phi_1(x, z) \Theta^*(x, z) \right] dxdz dt + o(\varepsilon')
= \Lambda_1' + \Lambda_2' + o(\varepsilon').
$$

For the first part of the right side,

$$
\Lambda_1' = \int_Q \int_D \left( D^* u_0 \right)(x, z) \phi_0(x, z) \Theta^*(x, z) dxdz dt
= \int_Q \left[ \phi_0(x, z) \Theta^*(x, z) + \phi_0(z, x) \Theta^*(z, x) \right] \gamma(x, z) dxdz dt.
$$

Let $\varepsilon'$ goes to 0, we have

$$
\Lambda_1' \to \int_Q \int_{Y \times N \times Z} u(x, y) \Theta(y, \eta) \left[ \phi_0(x, z) + \phi_0(z, x) \right] \gamma(x, z) dx dz dy d\eta d\tau
= \int_Q \int_{Y \times N \times Z} u(x, y) \Theta(y, \eta) \left( D_z \phi_0 \right)(x) dx dz dy d\eta d\tau.
$$

On the other hand, from the fact that $D^* u_0$ two-scale converges to $U \in L^2(Q \times D)$, we have

$$
\Lambda_1' = \int_Q \int_D \Theta^*(x, z) \left( D^* u_0 \right)(x, z) \frac{\sqrt{\mu}}{\sqrt{\epsilon}} \left( x, z \right) dxdz dt
\to \int_{Q \times D \times Y \times N \times Z} \Theta(y, \eta) U(x, z, t, y, \eta, \tau) \phi_0(x, z) dx dz dt dy d\eta d\tau.
$$

Then we have

$$
\int_{Q \times D \times Y \times N \times Z} \Theta(y, \eta) \left( U(x, z, t, y, \eta, \tau) - D^* u(x, y) \right) \phi_0(x, z) dx dz dt dy d\eta d\tau. 
$$

(6)
For the second part,
\[
\Lambda_2' = \int_Q \int_D (D^* u_\epsilon')(x, z) \Theta'(x, z) \psi_1(x, \frac{x}{\epsilon'}, \frac{z}{\epsilon'}) dx dz dt
\]
\[
= \int_Q u_\epsilon'(x) \int_D [\Theta'(x, z) \psi_1(x, \frac{x}{\epsilon'}, \frac{z}{\epsilon'}) + \Theta'(z, x) \psi_1(z, \frac{z}{\epsilon'}, \frac{x}{\epsilon'})] \gamma(x, z) dx dz dt
\]
\[
= \int_Q u_\epsilon'(x) \int_D [\Theta'(x, z) \psi_1(x, \frac{x}{\epsilon'}, \frac{z}{\epsilon'}) + \Theta'(z, x) \psi_1(z, \frac{z}{\epsilon'}, \frac{x}{\epsilon'})] \gamma(x, z) dx dz dt
\]
\[
+ \int_Q u_\epsilon'(x) \int_D [\Theta'(z, x) \psi_1(z, \frac{z}{\epsilon'}, \frac{x}{\epsilon'}) - \Theta'(z, x) \psi_1(x, \frac{x}{\epsilon'}, \frac{z}{\epsilon'})] \gamma(x, z) dx dz dt
\]
\[
= \Lambda_3' + \Lambda_4',
\]
where
\[
\Lambda_3' = \epsilon'(1-\alpha)/2 \int_Q u_\epsilon'(x) [D_y (\Theta \phi_1)(x, x')]' dx,
\]
and
\[
\Lambda_4' = \epsilon'(1+\alpha)/2 \int_Q u_\epsilon'(x) \int_D \Theta'(z, x) [\psi_1(z, \frac{z}{\epsilon'}) - \psi_1(x, \frac{x}{\epsilon'})] + \psi_1(x, \frac{x}{\epsilon'}) - \psi_1(z, \frac{z}{\epsilon'}) \gamma^2(x, z) dz dx
\]
\[
\to 0.
\]
By the fact that \((D_x \phi_1)' \to 0\), as \(\epsilon' \to 0\), we have
\[
\lim_{\epsilon' \to 0} \epsilon'(u_\epsilon', \psi_\epsilon') = \int_Q \int_{Y \times N \times Z} u(x, y) \Theta(y, \eta)(D_x \phi)(x) dx dy dy d\eta
\]
\[
+ \lim_{\epsilon' \to 0} \epsilon'(1-\alpha)/2 \int_Q u_\epsilon'(x) [D_y (\Theta \phi_1)(x, x')]' dx.
\]
By the two-scale convergence of \(u_\epsilon\),
\[
\int_Q \int_{Y \times N \times Z} u(x, y) D_y (\Theta \phi_1)(x, t, y, \tau) dz dx dt dy d\eta d\tau = 0.
\]
This yields in particular for any \(\phi\)
\[
\int_{Q \times D} \int_{Y \times N \times Z} (D_y u)(x, y)(\Theta \phi_1)(x, t, y, \eta, \tau) dx dz dx dt dy d\eta d\tau = 0,
\]
hence,
\[
(D_y u)(x, y) = 0,
\]
which means that \(u\) does not depend on \(y\). Then \(u = u_0\). Now, we set \(D_y (\Theta \phi_1) = 0\), we get
\[
\lim_{\epsilon' \to 0} \epsilon'(u_\epsilon', \psi_\epsilon') = \int_Q \int_{Y \times N \times Z} u(x, y) \Theta(y, \eta)(D_x \phi)(x) dx dy dy d\eta
\]
\[
= \int_{Q \times D} \int_{Y \times N \times Z} \Theta(y, \eta) D_x u_0(x, t, z) \phi(x, t, z, y, \eta, \tau) dz dx dt dy d\eta d\tau.
\]
We get that
\[
\int_{Q \times D} \int_{Y \times N \times Z} (U(x, t, z, y, \tau, \eta) - D_y^* u_0(x, t, z)) \Theta(y, \eta) \phi(x, t, z, y, \eta, \tau) \, dz \, dx \, dt \, dy \, d\tau = 0.
\]

From the equation (6), we deduce that
\[
\int_{Q \times D} \int_{Y \times N \times Z} (U(x, t, z, y, \tau, \eta) - D_y^* u_0(x, t, z)) \Theta(y, \eta) \phi_1(x, t, y, \eta, \tau) \, dz \, dx \, dt \, dy \, d\tau = 0.
\]

Since the fact that \(D_y(\Theta \phi_1) = 0\), we can deduce that there exists a unique function \(u_1 \in L^2(\Omega; L^2_{\text{per}}(Z; H^{\alpha/2}(Y)))\) such that
\[
U(x, t, z, y, \tau, \eta) - (D_y^* u_0)(x, t, z) = (D_y^* u_1)(x, t, y, \tau, \eta).
\]

This ends the proof of Lemma 3.

**Remark 2.** This setting for the two scale convergence method has a very unique feature in that, the limit of the sequence depends on additional variable which does not appear in the weak limit.

**Remark 3.** If the limit in Lemma 3 can be shown to be unique then convergence of the whole sequence occurs.

Let us introduce the form
\[
\hat{a}(w, v) = \int_{Y \times N \times Z} \Theta(y, \eta)(D_y^* w \cdot \overline{D_y^* v}) \, dy \, d\tau \, d\eta
\]
for all \(w, v \in L^2_{\text{per}}(Z; H^{\alpha/2}_\#(Y))\).

Further, let \(\xi\) be the unique function defined by
\[
\hat{a}(\xi, v) = \int_{Y \times N} \overline{V_d} \, dy \, d\tau,
\]
for all \(v \in L^2_{\text{per}}(Z; H^{\alpha/2}_\#(Y))\).

**Lemma 4.** For all \(\psi_0 \in \mathcal{M}(Q)\), for the subsequence \(E''\) in Lemma 3, we have
\[
\int_Q e^{(1-\alpha)/2} u_\varepsilon \mathcal{V}(x, t) \psi_0 \, dx \, dt \to \int_Q \int_{Y \times N} u_1(x, t, y, \tau) \psi(y, \tau) \, dx \, dt \, dy \, d\tau.
\]

**Proof.** From the equation (7), we can conclude
\[
\int_Q e^{(1-\alpha)/2} u_\varepsilon \mathcal{V}(x, t) \psi_0 \, dx \, dt = e^{(1-\alpha)/2} \int_Q u_\varepsilon \psi_0 \mathcal{D}(D^* \xi) \, dx \, dt.
\]
Since the fact that, for every function $\Phi \in \mathcal{E}$, we have $D\Phi = \epsilon^{1/\alpha}(Dy\Phi)^\ast$. Let $\Phi = \Theta D^\ast (y_x)\xi$, we have

$$\int_Q \epsilon^{(1-\alpha)/2} \int_{t \in (0, T)} u(x, t) \psi_0 dt = \int_{t \in (0, T)} \int_{R^2} u(x, t) \psi_0 D(\Theta D^\ast (y_x)\xi) \psi_0 dt$$

and

$$\int_Q \Theta \int_{t \in (0, T)} \int_{R^2} \Theta \int_{t \in (0, T)} \int_{R^2} \Theta \int_{t \in (0, T)} \int_{R^2} \Theta (y(x, t, z)) \psi_0 u_0 \psi_0 dxdydzdt$$

Hence the conclusion in this lemma follows.

\[\square\]

### 3.2 Homogenization Theorem

In this section, we will verify the main result that gives the homogenization principle of equation (1) and homogenized equation.

Let us first introduce some functions spaces. We consider the space $F_0 = L^2((0, T); H^\alpha/2(D)) \times L^2(Q; L^2_{\text{per}}(Z; H^\alpha/2(Y)))$, provided with the norm

$$\|u\|_{F_0}^2 = \|u_0\|_{L^2((0, T); H^\alpha/2(D))}^2 + \|u_1\|_{L^2(Q; \mathcal{M}(Y))}^2$$

which makes it Hilbert space. We consider also the space $F_\infty = \mathcal{M}(Q) \times [\mathcal{M}(Q) \otimes \mathcal{C}_{\text{per}}(Y) / \mathcal{C} \otimes \mathcal{C}_{\text{per}}(Z)]$, which is a dense subspace of $F_0$. For $u = (u_0, u_1)$ and $v = (v_0, v_1) \in H^\alpha/2 \times L^2(Q; L^2_{\text{per}}(Z; H^\alpha/2(Y)))$, we set
\[
a(u,v) = \int_{D \times D \times Y \times N \times Z} \Theta(y, \eta)(D^*_zu_0 + D^*_yu_1)(D^*_zv_0 + D^*_yv_1)dxdzdyd\eta d\tau.
\]

From the assumption of function \(g, f, \Theta\) and \(V\), we have the following lemma.

**Lemma 5.** Suppose \(f \in L^2(0, T; L^2(D))\), the variational problem

\[
\begin{cases}
  u = (u_0, u_1) \in F_0^1 \text{ with } u_0(0) = h \\
  \int_0^T <u'_0(t), \nu_0(t)> dt = \frac{1}{2} \int_0^T a(u(t), v(t))dt \\
  + \int_0^T (g(u_0), v_0)dW_t + \int_0^T (f(t), v_0(t))dt \\
  + \int_0^T (\int_{Y \times Z} u_1V(y, \tau)dyd\tau, v_0(x))_{L^2(D)}dt \\
  + \int_0^T \int_D \int_{Y \times N} \rho(x)\Theta(y, \eta)u_0(x)\nu_0(x)dxdtdyd\eta
\end{cases}
\]

for all \(v = (v_0, v_1) \in F_0^1\) admits at most one solution.

Next, we will show \(u = (u_0, u_1)\), where \(u_0, u_1\) is defined in Lemma 3.

**Theorem 1.** (Homogenization Theorem) Suppose the hypotheses of Lemma 1 and Lemma 3 are satisfied. For fixed \(\epsilon > 0\), let \(u_\epsilon\) be the solution of equation (1). Then as \(\epsilon \to 0\), we have

\[
u_\epsilon \to u_0 \quad \text{in} \quad L^2((0, T); H^{\alpha/2} \rho(D)) - \text{weakly},
\]

\[
u_\epsilon \to u_0 \quad \text{in} \quad L^2(Q) - \text{strongly}.
\]

Furthermore, \(u = (u_0, u_1) \in F_0^1\) is the unique solution of equation (8).

**Proof.** Thanks to the Lemma 3, there are some subsequence \(E'\) extracted from \(E\) and some vector function \(u = (u_0, u_1) \in F_0^1\) such that the convergence is satisfied when \(E' \ni \epsilon \to 0\).

Thus, according to Lemma 5, the theorem is certainly proved if we can show that \(u\) verifies equation (8).

Indeed, we begin by verifying that \(u_0(0) = h\). Let \(v \in H^{\alpha/2} \rho\) and \(\varphi \in C^1([0, T])\) with \(\varphi(T) = 0\). By integration by parts, we have

\[
\int_0^T <u'_\epsilon(t), v > \varphi(t)dt + \int_0^T <u_\epsilon(t), v > \varphi'(t)dt = - <h, v > \varphi(0),
\]

we pass to the limit in the preceding equality as \(\epsilon \to 0\). we obtain

\[
\int_0^T <u'_\epsilon(t), v > \varphi(t)dt + \int_0^T <u_\epsilon(t), v > \varphi'(t)dt = - <h, v > \varphi(0).
\]
Since $\varphi$ and $\nu$ are arbitrary, we see that $u_0(0) = h$. Finally, let us prove the variational equality of (8). We let $\psi^\varepsilon \in L^2(Q; C_{per}(Y \times Z))$, then there are two functions $\psi_0 \in M(Q)$ and $\psi_1 \in M(Q) \otimes [C_{per}(Y)/C_{per}(Z)]$, such that

$$\psi^\varepsilon = \psi_0 + \varepsilon^{(1+\alpha)/2} \psi_1,$$

i.e., $\psi^\varepsilon(x,t) = \psi_0(x,t) + \varepsilon^{(1+\alpha)/2} \psi_1(x,t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon})$

By equation (1), one has

$$\int_0^T <u^{\varepsilon}_t(t), \bar{\psi}^\varepsilon(t)> dt = \frac{1}{2} \int_0^T a^\varepsilon(u^\varepsilon(t), \bar{\psi}^\varepsilon(t)) dt + \int_0^T \epsilon^{(1-\alpha)/2} \bar{\psi}^\varepsilon u^\varepsilon(t), \psi^\varepsilon(t) dt$$

$$+ \int_0^T (g(u^\varepsilon), \psi^\varepsilon(t)) dW_t + \int_0^T (f(t), \psi^\varepsilon(t)) dt$$

$$+ \int_0^T \int D \Theta^\varepsilon(x, z) \rho(x) u^\varepsilon(t) \psi^\varepsilon(t) dx dz dt.$$  (12)

The aim is to pass to the limit in the above equation as $\varepsilon$ goes to 0. First, we have

$$\int_0^T <u^{\varepsilon}_t(t), \bar{\psi}^\varepsilon(t)> dt \rightarrow - \int_Q u^\varepsilon \frac{\partial \bar{\psi}^\varepsilon}{\partial t} dx dt.$$  (11)

Thus, we have

$$\int_0^T <u^{\varepsilon}_t(t), \bar{\psi}^\varepsilon(t)> dt \rightarrow - \int_Q u^\varepsilon \frac{\partial \bar{\psi}^\varepsilon}{\partial t} dx dt = \int_0^T <u^\varepsilon_0(t), \bar{\psi}^\varepsilon_0(t)> dt,$$

as $\varepsilon \rightarrow 0$. Next, we have

$$\int_0^T a^\varepsilon(u^\varepsilon(t), \bar{\psi}^\varepsilon(t)) dt \rightarrow \int_0^T a(u(t), \phi(t)) dt,$$

where $\phi = (\psi_0, \psi_1)$. In fact, from Lemma 3, we obtain

$$\int_0^T a^\varepsilon(u^\varepsilon(t), \bar{\psi}^\varepsilon(t)) dt \rightarrow \int_0^T a(u(t), \phi(t)) dt,$$

as $\varepsilon$ goes to 0. On the other hand,

$$\int_0^T \epsilon^{(1-\alpha)/2} (\nabla^\varepsilon u^\varepsilon(t), \bar{\psi}^\varepsilon(t)) dt = \epsilon^{(1-\alpha)/2} \int_Q \nabla^\varepsilon u^\varepsilon \bar{\psi}^\varepsilon_0 dx dt + \epsilon \int_Q \nabla^\varepsilon u^\varepsilon \bar{\psi}^\varepsilon_1 dx dt$$

In view of Lemma 4, we pass to the limit in (12). This yields,

$$\int_0^T \epsilon^{(1-\alpha)/2} (\nabla^\varepsilon u^\varepsilon(t), \bar{\psi}^\varepsilon(t)) dt \rightarrow \int_{Q \times Y \times N} u^\varepsilon \nabla \bar{\psi}^\varepsilon V dx dt dy d\tau,$$
as $\epsilon \to 0$.

Hence, passing to the limit in (11) leads to

$$i \int_{0}^{T} <u_0'(t), \bar{\psi}_0'(t)> dt = \frac{1}{2} \int_{0}^{T} a(u(t), \phi(t)) + \int_{Q \times Y \times N} u_1 \bar{\psi}_0 \nu dxdtdydt$$

$$+ \int_{0}^{T} (g(u_0), \psi_0(t))dW_t + \int_{0}^{T} (f(t), \psi_0(t))dt,$$

$$+ \int_{0}^{T} \int_{D} \int_{Y \times N} \rho(x) \Theta(x, y) u_0(x) \bar{\psi}_0(x) dxdtdydz$$

(13)

for all $\phi = (\psi_0, \psi_1) \in \mathcal{F}_0^{\infty}$. Moreover, since $\mathcal{F}_0^{\infty}$ is a dense subspace of $\mathcal{F}_1^0$, by [13] we see that $u = (u_0, u_1)$ verifies [3]. Thanks to the uniqueness of the solution for [3] and let the fact that the sequence $E$ is arbitrary, the theorem is proved.

For further needs we wish to give a simple representation of the function $u_1$.

Next, we consider the variational problem:

$$\begin{align*}
\hat{a}(\chi, v) &= \int_{Y \times N \times Z} \Theta(y, \eta) D_y^* v dxdydz, \\
\chi &\in L^2_{\text{per}}(Z; H^{\alpha/2}_{#}(Y)),
\end{align*}$$

for all $v \in L^2_{\text{per}}(Z; H^{\alpha/2}_{#}(Y))$. It determines $\chi$ in a unique manner.

**Lemma 6.** Under the assumption of Lemma 3, we have

$$u_1(x, t, y, \tau) = -\frac{1}{|D|} \int_{D} (D_z^* u_0)(x, t, z) dz \cdot \chi(y, \tau),$$

for almost all $(x, t, y, \tau) \in Q \times Y \times Z$.

**Proof.** In [3] choose the particular test function $v = (v_0, v_1) \in \mathcal{F}_0^1$ with $v_0 = 0$ and $v_1 = \phi \times v$, where $\phi \in \mathcal{M}(Q)$ and $v \in L^2_{\text{per}}(Z; H^{\alpha/2}_{#}(Y))$. This yields

$$0 = |D| \int_{D} \hat{a}(u_1, v) dx + \int_{D} \int_{D} (D_z^* u_0)(x, t, z) dz$$

$$\times \int_{Y \times N \times Z} (D_y^* v)(x, t, y, \tau) \Theta(y, \eta) dxdydz,$$

(14)

almost everywhere in $(x, t) \in Q$ and for all $v \in L^2_{\text{per}}(Z; H^{\alpha/2}_{#}(Y))$. By the fact that $u_1$ is the sole function in $L^2_{\text{per}}(Z; H^{\alpha/2}_{#}(Y))$ solving equation (14). Hence the lemma follows immediately.
3.3 Homogenized Equation

In this section, we will show that the limit process $u_0$ satisfies the following nonlocal stochastic Schrödinger equation (homogenized system):

\[
\begin{cases}
    idu_0 = -\Xi_1(-\Delta)^{\alpha/2}u_0 - \Xi_2(D\zeta)(x) - \Xi_3\zeta(x) + g(u_0) dW_t + f, \\
    u_0(x, t) = 0, \quad (x, t) \in D^c \times (0, T), \\
    u_0(0) = h(x), \quad x \in D,
\end{cases}
\]

where

\[
\Xi_1 = \int_{Y \times N} \Theta(y, \eta) dy dn,
\]
\[
\Xi_2 = \int_{Y \times N \times Z} \Theta(y, \eta) D^*\chi dy dn d\tau,
\]
\[
\Xi_3 = \int_{Y \times Z} V(y, \tau) \chi(y, \tau) dy d\tau,
\]
\[
\zeta(x) = \frac{1}{|D|} \int_D (D^*u_0)(x, z) dz.
\]

**Lemma 7.** Suppose the hypotheses of Lemma 1 and 2 are satisfied. Then equation (15) has at most one weak solution $u_0$.

**Proof.** We can see that if $u_0$ verifies equation (15) then $u = (u_0, u_1)$ satisfies equation (8). \(\square\)

**Theorem 2.** (Homogenized equation) Suppose the hypotheses of Lemma 1 and 2 are satisfied. Let $u_\epsilon$ be defined by equation (1). Then, as $\epsilon$ goes to 0, we have $u_\epsilon \to u_0$ in $\mathcal{Y}(0, T)$-weakly, where $u_0$ is the unique weak solution of equation (15) in $\mathcal{Y}(0, T)$.

**Proof.** Since the fact that, from any fundamental sequence $\epsilon' \in E$ one can extract a subsequence $\epsilon'$ such that as $\epsilon$ goes to 0, we have (9)-(10), and (13) holds for all $\phi = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty$, where $u = (u_0, u_1) \in \mathcal{F}_0^1$. Now, substituting $u_1$ in Lemma 6 to (13), a simple computation yields equation (15). \(\square\)

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