Thermal QCD sum rules in the \( \rho^0 \) channel revisited

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Abstract. From the hypothesis that at zero temperature the square root of the spectral continuum threshold \( s_0 \) is linearly related to the QCD scale \( \Lambda \) we derive in the chiral limit for temperatures considerably smaller than \( \Lambda \) scaling relations for the vacuum parts of the Gibbs averaged scalar operators contributing to the thermal operator product expansion of the \( \rho^0 \) current-current correlator. The scaling with \( \lambda \equiv \sqrt{s_0(T)/s_0(0)} \), \( s_0 \) being the \( T \)-dependent perturbative QCD continuum threshold in the spectral integral, is simple for renormalization group invariant operators, and becomes nontrivial for a set of operators which mix and scale anomalously under a change of the renormalization point. In contrast to previous works on thermal QCD sum rules with this approach the gluon condensate exhibits a sizable \( T \)-dependence. The \( \rho \)-meson mass is found to rise slowly with temperature which coincides with the result found by means of a PCAC and current algebra analysis of the \( \rho^0 \) correlator.

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1 Introduction

QCD sum rules at finite temperature have been of intense interest since the pioneering work of Bochkarev and Shaposhnikov [1]. Since thermal field theory lacks asymptotically measurable states the empirical side, that is the spectral function of the QCD sum rule under consideration, must be parameterized. There are few guidelines for the parametrization of the spectral function by hadronic resonance and continuum contributions. However, as pointed out by several authors [2,3,4], the output of the sum rule at finite temperature or for nonvanishing chemical potential depends on the hadronic model leading to a specific spectral function.

Another complication due to finite temperature arises from the fact, that temperature is defined in a fixed reference frame which hence is singled out. Therefore, the Poincaré invariance of a field theory at \( T = 0 \) is partially broken for finite temperatures. Residual O(3) and translational invariance permit a wider set of operators to contribute to the operator product expansion (OPE) of the corresponding correlator. Using a background field method, Mallik [5] derived an OPE for the time-ordered thermal correlator of various quark field bilinears, where these new operators are included up to mass-dimension four. In contrast to the work of Hatsuda et. al. [2], which relies on the higher twist classification of operators originating from the analysis of Deep Inelastic Scattering (DIS) and which for nonvanishing spatial momentum components allows for contributions of O(3) non-invariant operators to the thermal OPE [4], a systematic O(3) invariant extension of the zero temperature OPE is obtained in Ref. [6]. Thereby the authors consider the mixing of the new operators under a change of the renormalization scale. However, using the background field method, the Wilson-coefficients for the radiative corrections of mass-dimension six could not be calculated, and hence have been omitted in Ref. [6]. These contributions are important, since they distinguish the vector from the axial vector channel and cancel a large part of the terms with mass-dimension four, yielding the experimental value of the \( \rho \)-meson mass at \( T = 0 \) with good stability [6].

In Refs. [2,3,4] the Gibbs average of the OPE is assumed to be saturated by the vacuum and the dilute pion gas contributions. Thereby, both the vacuum and the pion states are taken to be temperature independent. This leads to a \( T \)-independent gluon condensate [2], which is in contrast to lattice measurements [7] and to results obtained in effective meson models with scalar glueball fields [6]. The main point of this paper is to explore the consequences of a \( T \)-dependent vacuum for the \( T \) evolution of the gluon condensate and the \( \rho \)-meson mass in the chiral limit. To this end, we parametrize the \( T \)-dependence of the vacuum implicitly through the effective spectral variable \( s_0(T) \) which divides the hadronic part from the perturbative QCD domain of the spectral function. This ansatz

\footnote{At first sight this seems to violate rotational symmetry which is retained in the heat bath. Invariance with respect to O(3) transformations is, however, restored in Ref. [4] when performing the phase space integrals over pionic matrix elements introduced through the definition of the thermal average.}
can be justified for renormalization scales of the order of 1 GeV and for temperatures considerably less than the fundamental QCD scale $\Lambda$ (see Appendix A).

The paper is organized as follows: In section II we list the basic results for thermal dispersion relations and for the spectral function in the $\rho^0$-meson channel as already established in Refs. $[2]$ and $[3]$. Section III contains the thermal OPE for the invariant longitudinal amplitude of the $\rho^0$-meson correlator, and we briefly discuss the behavior of the nonscalar contributions of mass-dimension four under renormalization. The treatment of thermal operator averages is performed in section IV. With an implicit dependence on temperature, we derive a scaling relation with respect to $\lambda \equiv \sqrt{s_0(T)/s_0(0)}$ for $T$-dependent vacuum averages of scalar operators. This relation is easily implemented for renormalization group invariants (RGI), and becomes quite involved if one has to regard a set of operators which mix and scale anomalously under a change of the renormalization point. In section V we write out the Borel transformed sum rule and, by performing a logarithmic derivative with respect to the inverse squared Borel mass, obtain a sum rule for the $\rho^0$-meson mass which is the ratio of two moments. A numerical evaluation of the thermal sum rule is performed in section VI. Section VII summarizes and compares the results with those of previous approaches.

2 Thermal Dispersion relations, kinematic Invariants, and Spectral Function

We consider the thermal correlator of the time-ordered (T) product of two vector currents in the $\rho^0$-channel

$$T_{\mu\nu}(q, T) = Z^{-1-i} \int d^4x \ e^{iqx} e^{-\beta H} T j_{\mu}(x) j_{\nu}(0) , \quad (1)$$

where

$$j_{\mu}(x) = \frac{1}{2} \left[ \bar{u} \gamma_{\mu} u - \bar{d} \gamma_{\mu} d \right] \quad \text{and} \quad Z = e^{-\beta H} . \quad (2)$$

Here $H$ denotes the QCD Hamiltonian, and $\beta$ stands for the inverse of the temperature $T$. A sum rule for this correlator can be derived as $[3]$

$$T_{\mu\nu}(Q_0, |q|, T) = \frac{1}{\pi} \int_0^\infty dq_0^2 \ \frac{\Im T_{\mu\nu}(q_0^2, |q|)}{q_0^2 + Q_0^2} \ \tanh(\beta q_0^2/2) ,$$

$$Q_0^2 = -q_0^2 . \quad (3)$$

The frame of reference where temperature is defined moves with four-velocity $u_j$. With this additional covariant one can, for example, define the Lorentz scalars $\omega \equiv u_j q^j$ and $\bar{q} \equiv \sqrt{\bar{q}^2 - q^2}$. Imposing current conservation and symmetry under exchange of $\mu$ and $\nu$, the correlator of Eq. (3) can be decomposed as $[2]$

$$T_{\mu\nu}(q, T) = Q_{\mu\nu} T_1(q^2, \omega, T) + P_{\mu\nu} T_1(q^2, \omega, T) , \quad (4)$$

where $(g_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$, and the tensors $P_{\mu\nu}$, $Q_{\mu\nu}$ are given by

$$P_{\mu\nu} = -g_{\mu\nu} + (q_0 q_\nu - q^2 \bar{u}_\mu \bar{u}_\nu) q^2 \bar{q}^2 ,$$

$$Q_{\mu\nu} = \frac{q^4}{q^2} \bar{u}_\mu \bar{u}_\nu , \quad \bar{u}_\mu \equiv u_\mu - \omega q_\mu . \quad (5)$$

By evaluating $T_1 = u^\mu u^\nu T_{\mu\nu}$ and $T_2 = T_{\mu\nu}^2$ in the rest frame of the heat bath $(u_\mu = (1, 0, 0, 0))$, one can solve for the invariant amplitudes $T_1$ and $T_2$ as

$$T_1 = \frac{1}{q^2} T_2 , \quad T_2 = -\frac{1}{2} \left[ T_1 + \frac{q^2}{q^2} T_2 \right] , \quad (6)$$

and in the limit $q \to 0$ one obtains

$$T_1(q_0, q = 0) = q_0^2 T_1(q_0, q = 0) . \quad (7)$$

Since the sum rule of Eq. (3) holds for each component of $(T_{\mu\nu})$, it also holds for $T_1$ and $T_2$. With Eq. (4) one obtains sum rules for the invariant amplitudes $T_1$ and $T_2$. For example,

$$T_1(q_0^2, \bar{q}, T) = \int_0^\infty dq q_0^2 N_1(q_0^2, \bar{q}, T) \frac{q_0^2 + Q_0^2}{q_0^2 + Q_0^2} ,$$

$$N_1(q_0^2, \bar{q}, T) = \pi^{-1} \Im T_1(tanh(\beta q_0^2/2)) . \quad (8)$$

As already indicated in the introduction, thermal field theory is handicapped by the lack of asymptotically measurable states, and therefore the spectral function (defined as the numerator of the integrand of the right-hand side of Eq. (5), that is $N_1(q_0^2, \bar{q}, T)$), must be modelled. In the following $s_0(T)$ denotes a temperature dependent threshold that divides the hadronic part of the spectrum from the perturbatively accessible QCD domain. It is suggested $[3]$, that in the hadronic region of integration, $0 \leq q_0^2 \leq s_0$, the correlator is saturated by the $\rho^0$-resonance and the two-pion continuum. Thereby, the pions are assumed to be noninteracting. The hadronic contributions $j_{\mu}^{\rho^0}$ and $j_{\mu}^{\pi}$ to the current $j_{\mu}$ of Eq. (3) are obtained by using the field-current-identity

$$j_{\mu}^{\rho^0} = m_\rho f_\rho \rho^0_{\mu} , \quad \text{where} \quad f_\rho(T = 0) = 153.5 \text{ MeV} , \quad (9)$$

and by appealing to the SU(2) flavor symmetry structure resulting in

$$j_{\mu}^{\pi} = \varepsilon_{abc} \rho^a \partial_\mu \pi^c . \quad (10)$$

Thereby, the pionic current is obtained by constructing the Noether current of the SU(2)$_V$ symmetry of a low-energy chiral Goldstone-field theory which to lowest order in the derivatives contains noninteracting fields. Since we only consider a dilute pion gas (one-pion states in Gibbs average) this free pion theory is relevant for our purposes.

Employing the noninteracting, finite temperature (real-time) $\rho^0$-meson and pion propagators $[1][2][3]$ for the calculation of the unitarity cuts for the tree diagram of the $\rho^0$ contribution and for the thermal pion loop, the
hadronic part of the spectral function has been calculated in Ref. [1]. In contrast to Ref. [2] and following Ref. [3] we approximate the imaginary part of the correlator above the threshold \( s_0(T) \) by perturbative QCD (pQCD). This perturbative piece thermal contributions are omitted since the fermionic distribution function \( n_F \) remains small for the relevant temperature and energy range [3]. In the limit \( q \to 0 \) and after a Borel transformation in \( Q_0 \) with Borel mass \( M \) [4] the spectral side of the sum rule of Eq. (8) reads

\[
T_l(M, T) = \frac{1}{\pi M^2} \int_0^\infty ds T_l(s, T) e^{-s/M^2} \tanh(\sqrt{s}/2T)
\]

\[
= \frac{1}{M^2} \int_0^\infty ds e^{-s/M^2} \rho(s, T)
\]

\[
= \frac{1}{M^2} \left( \int_0^{\infty} (f^2(T) e^{-m_T^2(T)/M^2} + J_0^{\pi \pi} + J_{T}^{\pi \pi} + J_0^{q \bar{q}}) \right)
\]

with \( s \equiv q^2 \),

and

\[
J_0^{\pi \pi} = \frac{1}{48\pi^2} \int_0^{s_0(T)} ds e^{-s/M^2} \nu_0^\pi, \quad J_0^{q \bar{q}} = \frac{1}{8\pi^2} \left( 1 + \frac{\alpha_s(\mu)}{\pi} \right)
\]

\[
J_T^{\pi \pi} = \int_0^{s_0(T)} ds e^{-s/M^2} \frac{1}{24\pi^2} \int_0^{s_0} ds \left( e^{-s/M^2} \nu_0^{\pi \pi} + \nu(3 - \nu^2)/2 \right) n_B(\sqrt{s}/2, T),
\]

where \( \mu = 1 \) GeV. Here the function \( v \) is defined as \( v(s, m_\pi) = \sqrt{1 - 4m_\pi^2/s} \), \( n_B \) denotes the Bose distribution, and \( J_0^{\pi \pi}(T), J_0^{q \bar{q}} \) are the vacuum (thermal) parts of the spectral integrals due to the \( \pi \pi \) and pQCD continua, respectively.

### 3 Thermal Operator Product Expansion

In this work we use a thermal OPE for the invariant amplitude \( T_l \) which combines results of Ref. [3] and Ref. [2]. In Ref. [3] the expansion is only carried out up to operators of mass-dimension four. The nonscalar O(3) invariant contributions are expressed in terms of diagonal combinations with respect to the anomalous mixing matrix, resulting in a renormalization group invariant (RGI) (that is the total energy density) and a renormalization group non-invariant (RGI) contribution. The drawback of the OPE of Ref. [2] lies in the fact, that mass-dimension six contributions are omitted. These terms are important since they cancel a large part of the nonperturbative correction of mass-dimension four yielding at \( T = 0 \) the experimentally measured \( \rho \)-meson mass. In addition, it is the mass-dimension six part of the OPE that distinguishes at \( T = 0 \) the vector from the axial vector channel [3]. Therefore, we use together with the standard \( T = 0 \) scalar operators the results of Ref. [3] for the nonscalar mass-dimension six contribution. In the chiral limit, for \( q \to 0 \), and for two light quark flavors (\( u \) and \( d \)) [5], we then obtain the following expansion for the invariant \( T_l \)

\[
T_l(q^2, q = 0, T) = -\frac{1}{8\pi^2} \ln \left( \frac{Q_0^2}{\mu^2} \right) \left( 1 + \frac{\alpha_s(\mu)}{\pi} \right) + \frac{1}{Q_0^4} \left( \frac{\alpha_s}{\pi} F_{\mu \nu}^a F^{\mu \nu} a \right) T(\mu) + \frac{2}{\pi} \left( \theta_{00}^T T(\mu) + \frac{\alpha_s(\mu)}{\alpha_s(Q_0^2)} \delta/16 \left( \frac{3}{\theta_{00}^T - \theta_{00}^T(\mu) \right) \right) - \frac{\pi}{2Q_0^2} \left( \frac{\alpha_s}{\alpha_s(Q_0^2)} \right) (\bar{u} \gamma_\mu \gamma_5 t a u - \bar{d} \gamma_\mu \gamma_5 t a d)^2) T(Q_0) - \frac{\pi}{9Q_0^2} \langle \bar{u} \gamma_\mu \gamma_5 t a u - \bar{d} \gamma_\mu \gamma_5 t a d)^2 \rangle T(Q_0) + \frac{8\pi}{3Q_0^2} \langle \bar{u} \gamma_\mu \gamma_5 t a u - \bar{d} \gamma_\mu \gamma_5 t a d)^2 \rangle T(Q_0) + \frac{2}{\delta(\mu)} \left( \theta_{00} \right) + \frac{1}{2} \left( \theta_{00}^T - \theta_{00}^T(\mu) \right)
\]

where the nonscalar operators are understood to be symmetrized and to be made traceless with respect to the Lorentz indices. The generators \( t^a \) of color SU(3) in the fundamental representation are normalized to \( Tr t^a t^b = 2 \delta^{ab} \). There are, in principle, also O(3) mixed quark-gluon operators of twist 4. However, there is no experimental clue about their matrix elements as pointed out in Ref. [2]. There it was also indicated that bag model estimates of the nucleon matrix elements of these operators yield very small values and that the treatment of pions in the bag model is rather questionable due to their collective nature. Following Ref. [2] we simply omit these operators to obtain Eq. (13).

The Gibbs average in Eq. (13) is approximated by the vacuum and the dilute pion gas contributions in Ref. [3]. As the authors point out the scalar part of the Gibbs averaged OPE of Eq. (13) then has the same structure as the result for the vector correlator obtained in Ref. [5]. Based on PCAC, current algebra, and the LSZ reduction formula the authors of Ref. [13] find to order \( T^2 \) a mixing of the correlator of the vector with that of the axial vector channel due to finite temperature.

To one loop and for two quark flavors the constants \( b \) and \( \delta \) are given by [16]

\[
\delta = -\frac{2}{3} (\frac{16}{3} + 2), \quad b = 11 - \frac{2}{3}.
\]

At \( \mu = 1 \) GeV we take \( \alpha_s(\mu^2) = 0.36 \) [3]. Note, that in the case of mass-dimension four the operators a priori renormalized at \( Q_0^2 \) are already expressed by operators evaluated at \( \mu = 1 \) GeV. Only for \( \langle \frac{\delta}{\delta T} \theta_{00}^T - \theta_{00}^T(\mu) \rangle \) does this process of rescaling generate a renormalization group logarithm due to the nonvanishing anomalous dimension \( \delta \) of this operator.

The operator \( \frac{\delta}{\delta T} F_{\mu \nu}^a F^{\mu \nu} a \) is an RGI. In the chiral limit, the fermionic and gluonic parts of the 00-component of the traceless energy-momentum tensor \( (\theta_{\mu \nu}) \) of one quark flavor QCD [17] are

\[
\theta_{00}^T = i \bar{q} \gamma_0 D_0 q, \quad \theta_{00}^T = -F_{\mu \lambda}^a F^{\mu \lambda} a + \frac{1}{4} g_{00} F_{\mu \lambda}^a F^{\mu \lambda} a, \quad D_0 \equiv \partial_0 - ig A_0^a \frac{t^a}{2}.
\]
with $\theta_{00}$ given by
\[ \theta_{00} = \theta_{00}^f + \theta_{00}^g. \] (16)

Since $\theta_{\mu\nu}$ is a conserved quantity the thermal average of $\theta_{00}$ is also an RGI. The expression for the covariant derivative $D_0$ in Eq. (15) contains the zeroth component of the gauge potential $A_0^a$ and the color SU(3) generators $t_a$ in the fundamental representation normalized to $Tr t_a t_b = 2\delta_{ab}$.

In Eq. (13) both mass-dimension six scalar operators are RGNIs, and we will consider their anomalous scaling and mixing in the next section. As for the O(3), non-scalar mass-dimension six contribution to Eq. (13) we neglect the anomalous scaling of the corresponding operator. It is known to belong to a set of operators which mix under renormalization. This set also contains mixed quark-gluon contributions, and following Ref. [2] we will omit them.

4 Gibbs averages

4.1 Scalar operators

As was suggested by several authors [1, 2, 3] the Gibbs averages of the scalar operators of Eq. (13) can be saturated by the vacuum and the one pion contributions. Thereby, both the vacuum and the pion states were assumed to exhibit no temperature dependence [2]. In the chiral limit the only quantities entering the sum rule, which can potentially describe thermal pion properties, are the pion decay constant, the only quantities entering the sum rule, which can potentially describe thermal pion properties, are the pion decay constant, the pion matrix elements of twist two operators. As for the former the $T$-dependence of the twist two pion averages of the scalar operators of Eq. (13) can be saturated by the vacuum and the one pion contributions. Therefore, for the former the $T$ dependence has been obtained in the imaginary time formulation of thermal chiral perturbation theory in Ref. [14]. There, the result for $f_\pi(T)$ is a decreasing function of temperature for $T \geq f_\pi$. On the contrary, the lattice simulation of Ref. [14] obtains a nearly $T$ independent pion decay constant up to the critical temperature which is understood as an artefact due to the large pion mass ($\sim 400$ MeV) used. As for the $T$-dependence of the twist two pion averages no information is available so far, and we have to use the pion-in-vacuum parton distributions to estimate the corresponding matrix elements. In accord with previous sum rule investigations at finite temperature [12, 13] and for consistency we then have no choice but to assume a $T$-independent pion decay constant.

The motivation for a $T$-dependent vacuum part in the Gibbs average of scalar operators emerges from a comparison of lattice data for the gluon condensate [3] with the result of Ref. [2]. The former approach indicates a sizable decrease above a critical temperature of about $T_c = 140$ MeV, whereas in the latter case the gluon condensate exhibits practically no temperature dependence even when leaving the chiral limit (less than 0.5% decrease at $T = 200$ MeV [2]).

On a more phenomenological level, the $T$-dependence of the gluon condensate has been obtained from the effective potential of theories based on the nonlinear $\sigma$-model, where the mesons are coupled to a scalar glueball field to mimic the breaking of scale invariance by the QCD vacuum [2, 14]. Depending on the normalization conditions used for the Bag constant and the glueball mass, in these calculations the $T$-evolution of the gluon condensate exhibits a strong decrease above critical temperatures ranging from $T_c = 140 - 400$ MeV.

The scalar operators appearing in the OPE of the thermal current correlator read
\[ \text{dim 4 : } O_4^a = \frac{\alpha_s}{\pi} F_{\mu\nu}^a F_{\mu\nu}^a \]
\[ \text{dim 6 : } O_6^{a_1} = \pi \alpha_s (\bar{u} \gamma_\mu \gamma_5 t^a u - i \bar{d} \gamma_\mu \gamma_5 t^a d)^2 , \]
\[ O_6^{a_2} = \pi \alpha_s (\bar{u} \gamma_\mu t^a u + \bar{d} \gamma_\mu t^a d)^2 . \] (17)

Hereby, $O_4^a$ is an RGI, and $O_6^{a_1}, O_6^{a_2}$ can be expanded into a basis of scalar four-quark operators $P_1, P_2, ..., P_6$ [14] with
\[ P_1 = \bar{\psi}_L \gamma_\mu \psi_L \bar{\psi}_R \gamma_\mu \psi_R \]
\[ P_2 = \bar{\psi}_L \gamma_\mu t^a \psi_L \bar{\psi}_R \gamma_\mu t^a \psi_R \]
\[ P_3 = \bar{\psi}_L \gamma_\mu \psi_L \bar{\psi}_R \gamma_\mu t^a \psi_L + (L \rightarrow R) \]
\[ P_4 = \bar{\psi}_L \gamma_\mu t^a \psi_L \bar{\psi}_R \gamma_\mu t^a \psi_L + (L \rightarrow R) \]
\[ P_5 = \bar{\psi}_L \gamma_\mu \lambda^b \psi_L \bar{\psi}_R \gamma_\mu \lambda^b \psi_R \]
\[ P_6 = \bar{\psi}_L \gamma_\mu \lambda^b \psi_L \bar{\psi}_R \gamma_\mu \lambda^b \psi_R , \]
\[ \psi_{L(R)} = (u_{L(R)}, d_{L(R)})^T . \] (18)

where $t^a$ and $\lambda^b$ are the color SU(3) Gell-Mann and flavor SU(2) Pauli matrices, respectively. They are normalized to $Tr t^a t^b = 2\delta^{ab}$. The left-(right-) handed spinors are given by
\[ \psi_{L(R)} = \frac{1}{2} (1 \pm \gamma_5) \psi . \] (19)

Using the relation
\[ \tau^c \tau^c_{mn} = 2(\delta_{in} \delta_{jm} - \frac{1}{N} \delta_{ij} \delta_{mn}) \] (20)

for the SU(N) generator matrices $\tau^c (Tr \tau^a \tau^b = 2\delta^{ab})$ and applying Fierz transformations, one obtains the following decomposition of the flavor singlet part [4] (for a derivation see the Appendix B)
\[ O_6^{a_1} = \pi \alpha_s \left( \frac{5}{3} P_4 + \frac{32}{9} P_3 - 2 P_5 \right) \]
\[ O_6^{a_2} = \pi \alpha_s (P_4 + 2 P_2) . \] (21)

Note that the contributions which must be subtracted for a direct comparison. However, at finite temperature the perturbative part, which contributes to the partition function for momenta larger than, say, 1.5 GeV, is for $T < 200$ MeV severely Boltzmann suppressed and hence hardly influences the $T$ dependence of the thermal average.

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Note that the contributions which must be subtracted for a direct comparison. However, at finite temperature the perturbative part, which contributes to the partition function for momenta larger than, say, 1.5 GeV, is for $T < 200$ MeV severely Boltzmann suppressed and hence hardly influences the $T$ dependence of the thermal average.
The sets of operators $P_1, \ldots, P_4$ and $P_5, P_6$ mix independently under renormalization with the respective one-loop mixing matrices $\delta$ and $\tilde{\delta}$.

$$\delta = \begin{pmatrix}
0 & 3/2 & 0 & 0 \\
16/3 & 17/3 & 0 & -2/3 \\
0 & -2/3 & 0 & -11/6 \\
0 & -20/9 & -16/3 & 8/9
\end{pmatrix}, \quad \tilde{\delta} = \begin{pmatrix}
7/3 & 16/3 \\
3/2 & 0
\end{pmatrix}. \quad (22)

The eigenvalues (proportional to the anomalous dimensions of the diagonal combinations) read

$$\delta_1 = 7.043, \quad \delta_2 = 3.501, \quad \delta_3 = -2.891, \quad \delta_4 = -1.097, \quad \delta_5 = 8, \quad \text{and} \quad \delta_6 = -1. \quad (23)$$

In Appendix A we argue that for temperatures considerably smaller than the fundamental QCD scale $A (A \approx 200 \text{ MeV})$, in the chiral limit, and for a renormalization scale $Q_0$ of the order of 1 GeV it should be possible to describe the $T$ dependence of the thermal average of a scalar operator implicitly through that of the $T$-dependent spectral continuum threshold $s_0(T)$. Meeting the above premises, we can derive a scaling relation with respect to $\lambda = \sqrt{s_0(T)/s_0(0)}$ for the vacuum average of a given scalar and diagonal operator $O$ with mass-dimension $d$ and anomalous dimension one-loop coefficient $\delta_0$. In the spirit of a random phase approximation, where correlations between particle-hole excitations in the ground state are included, we set up the QCD vacuum state as an expansion in terms of $k$-particle (off-shell) quark-antiquark and gluon fluctuations with vacuum quantum numbers, that is

$$|0\rangle = \sum_k |k\rangle, \quad \text{with}$$

$$|k\rangle = \sum_{i_1, i_2, \ldots, i_k} \int d^4p_1 \int d^4p_2 \cdots \int d^4p_k$$

$$C_{i_1, i_2, \ldots, i_k} (p_1, p_2, \ldots, p_k) |p_1; i_1\rangle \cdots |p_k; i_k\rangle. \quad (24)$$

Hereby, $i_j$ is a collective index labelling the particle species (fermion or boson), spatial quantum numbers, flavor (if fermionic), and color. The $C_{i_1, i_2, \ldots, i_k}$ denote the corresponding expansion coefficients. Then the vacuum average of $O$ has a representation of the following form

$$\langle 0|O|0\rangle(Q_0) = \sum_k \sum_{i_1, i_2, \ldots, i_k} \int d^4p_1 \int d^4p_2 \cdots \int d^4p_k$$

$$f_{i_1, i_2, \ldots, i_k} (p_1, p_2, \ldots, p_k; Q_0), \quad (25)$$

where the functions $f_{i_1, i_2, \ldots, i_k}$ have mass-dimension $d-4k$, and $Q_0$ denotes the scale at which the operator $O$ is renormalized. By asymptotic freedom, the integrand of Eq. (25) will be strongly suppressed, if the momenta $p_1, \ldots, p_k$ are hard. The finite value of $\langle 0|O|0\rangle(Q_0)$ has its origin in the strong dynamics of soft fluctuations in the vacuum [4]. A criterion distinguishing soft from hard fluctuations should roughly be given by the scale $s_0$. For sizable contributions to the integral of Eq. (25) we assume that the time- and space-like virtuality $p_j^2$ of each of the momenta $p_1, \ldots, p_k$ be less than $s_0$ and, in addition, that $(p_j^0)^2$ be less than $s_0$ (recall that through the presence of the heat bath $(p_j^0)^2$ is formally raised to a Lorentz scalar). With the above premises and the results of Appendix A we may assume that temperature effectively acts on the vacuum only implicitly through $s_0$ and that there is a linear relation between the $T$ dependent QCD scale $\Lambda_T$ and $\sqrt{s_0(T)}$. Thus there are only two independent scales to be considered: $s_0(T)$ and $Q_0$. Performing the integrations over the spatial components of the momenta in Eq. (25) we obtain

$$\langle 0|O|0\rangle(Q_0) =$$

$$\sum_k \sum_{i_1, i_2, \ldots, i_k} \int \sqrt{s_0(0)} d^4p_1 \cdots \int \sqrt{s_0(0)} d^4p_k$$

$$h_{i_1, i_2, \ldots, i_k} (p_1^0, p_2^0, \ldots, p_k^0) \times$$

$$\left\{ \begin{array}{l}
p_1^0, \ldots, p_{k-1}^0 \\
p_k^0 \end{array} \right\} \left\{ \begin{array}{l}
p_1^0 \quad \text{with} \\
\frac{p_k^0}{Q_0} \quad \sqrt{s_0(0)} \end{array} \right\}, \quad \cdots,$$

$$\left\{ \begin{array}{l}
p_k^0 \quad \text{with} \\
\frac{p_k^0}{Q_0} \quad \sqrt{s_0(0)} \end{array} \right\} = \frac{Q_0}{\sqrt{s_0(0)}}. \quad (26)$$

In Eq. (26) the functions $h_{i_1, i_2, \ldots, i_k}$ are homogeneous functions of $p_j^0$ ($j = 1, \ldots, k$) with mass-dimension $d-k$, and $g_{i_1, i_2, \ldots, i_k}$ are dimensionless functions of their dimensionless arguments. The $T$-dependent vacuum average (not to be confused with the Gibbs average) is then given as

$$\langle 0|O|0\rangle_T(Q_0) =$$

$$\sum_k \sum_{i_1, i_2, \ldots, i_k} \int \sqrt{s_0(T)} d^4p_1 \cdots \int \sqrt{s_0(T)} d^4p_k$$

$$h_{i_1, i_2, \ldots, i_k} (p_1^0, p_2^0, \ldots, p_k^0) \times$$

$$\left\{ \begin{array}{l}
p_1^0, \ldots, p_{k-1}^0 \\
p_k^0 \end{array} \right\} \left\{ \begin{array}{l}
p_1^0 \quad \text{with} \\
\frac{p_k^0}{Q_0} \quad \sqrt{s_0(T)} \end{array} \right\}, \quad \cdots,$$

$$\left\{ \begin{array}{l}
p_k^0 \quad \text{with} \\
\frac{p_k^0}{Q_0} \quad \sqrt{s_0(T)} \end{array} \right\} = \frac{Q_0}{\sqrt{s_0(T)}}. \quad (27)$$

With

$$\lambda = \sqrt{\frac{s_0(T)}{s_0(0)}}, \quad (28)$$

and from comparison of Eq. (26) and Eq. (27) one easily obtains

$$\langle 0|O|0\rangle_T (\lambda Q_0) = \lambda^d \langle 0|O|0\rangle(Q_0). \quad (29)$$

Eq. (29) relates the average of a diagonal operator $O$ with mass dimension $d$ taken in a temperature distorted vacuum and renormalized at $\lambda Q_0$ to the $T = 0$ vacuum aver-
age renormalized at \( Q_0 \). This scaling relation should embody a good approximation for the thermal vacuum average, provided that the above premises of massless quarks, \( T \) considerably smaller than \( T \), and \( Q_0 \) of the order of 1 GeV are fulfilled.

For an RGI operator \( \mathcal{O} (\delta_\sigma=0) \) the rescaling from \( \lambda Q_0 \) to \( Q_0 \) is trivial (for example \( \mathcal{O}^2 \)). The situation becomes more involved for the scalar operators \( \mathcal{O}_{b,1}^2 \) and \( \mathcal{O}_{b,2}^2 \) of mass-dimension six given in Eq. (17), and we will focus on them now.

The perturbative one-loop renormalization group rescaling from \( \lambda Q_0 \) to \( Q_0 \) for the diagonal operator \( \mathcal{O} \) is given by

\[
(0|\mathcal{O}|0)_{\mathcal{T}}(Q_0) = \left( \frac{\log(\lambda Q_0)^2/A^2}{\log Q_0^2/A^2} \right)^{-\delta_\sigma/b} (0|\mathcal{O}|0)(\lambda Q_0),
\]

where \( b \) is defined in Eq. (24). In the OPE, operators renormalized at \( Q_0 \) are expressed by operators renormalized at a common reference point \( \mu \). Eq. (24) implies that first we rescale from \( \lambda Q_0 \) to \( Q_0 \) which is accomplished by

\[
(0|\mathcal{O}|0)_{\mathcal{T}}(Q_0) = \lambda^d \left( \frac{\log(\lambda Q_0)^2/A^2}{\log Q_0^2/A^2} \right)^{-\delta_\sigma/b} (0|\mathcal{O}|0)(\mu),
\]

and then express \( (0|\mathcal{O}|0)(\mu) \) by \( (0|\mathcal{O}|0)(\mu) \) as

\[
(0|\mathcal{O}|0)_{\mathcal{T}}(Q_0) = \lambda^d \left( \frac{\log(\lambda Q_0)^2/A^2}{\log Q_0^2/A^2} \right)^{-\delta_\sigma/b} \left( \frac{\log Q_0^2/A^2}{\log \mu^2/A^2} \right) (0|\mathcal{O}|0)(\mu).
\]

In order to apply the following identity for the Borel transformation \( \lambda^M \) [14]

\[
\lambda^M \left[ \left( \frac{1}{Q_0^2} \right)^k \left( \frac{1}{\ln Q_0^2/A^2} \right)^\varepsilon \right] = \left( \frac{1}{M^2} \right)^k \left( \frac{1}{\ln (M^2/A^2)} \right)^\varepsilon \left[ 1 + O \left( \frac{1}{\ln (M^2/A^2)} \right) \right],
\]

we expand \( \lambda^M \) \( (0|\mathcal{O}|0)(\mu) \) about \( \lambda^2 = 1 \) up to quadratic order and perform afterwards the Borel transformation indicated in Eq. (33). The result is

\[
\lambda^M \left[ \left( \frac{1}{Q_0^2} \right)^k \langle 0|\mathcal{O}|0 \rangle_{\mathcal{T}}(Q_0) \right] \approx \frac{\lambda^{2k}}{\Gamma(k)} \left( \frac{1}{M^2} \right)^k \left\{ \left( \ln (M^2/A^2) \right)^{\delta_\sigma/b-1} - \frac{\delta_\sigma}{2b^2} \left[ \delta_\sigma + b \left( 1 + \ln (\mu^2/A^2) \right) \right] \left( \ln (M^2/A^2) \right)^{\delta_\sigma/b-3} \left( \ln (\mu^2/A^2) \right)^{\delta_\sigma/b-3} \left( \ln (\mu^2/A^2) \right)^{\delta_\sigma/b-3} \left( \ln (\mu^2/A^2) \right)^{\delta_\sigma/b-3} \right\} (0|\mathcal{O}|0)(\mu) + O((\lambda^2 - 1)^3). \]

With \( M^2 \approx 0.75 \text{ GeV}^2 \) and \( A^2 = 0.04 \text{ GeV}^2 \) the coefficient \( c_2 \) in the expansion of Eq. (34) is roughly given by \( 1/2c_1 \). So even for a value of \( \lambda^2 \) as low as \( 1/2 \) we obtain a suppression for the quadratic as compared to the linear term in Eq. (34) by a factor of four.

In the case of the two sets \( P_1, ..., P_4 \) and \( P_5, P_6 \) of Eq. (38) the mixing under renormalization results in

\[
\langle 0|\pi \alpha_s P_1 P_4 |0 \rangle_{\mathcal{T}}(M) = \mathbf{T} \mathbf{D}^{-1} \langle 0|\pi \alpha_s P_1 P_4 |0 \rangle (\mu),
\]

\[
\langle 0|\pi \alpha_s P_5 P_6 |0 \rangle_{\mathcal{T}}(M) = \tilde{\mathbf{T}} \tilde{\mathbf{D}}^{-1} \langle 0|\pi \alpha_s P_5 P_6 |0 \rangle (\mu),
\]

where the matrices \( \mathbf{T} \) and \( \tilde{\mathbf{T}} \), transforming the matrices \( \delta \) and \( \tilde{\delta} \) of Eq. (22), respectively, are given by

\[
\mathbf{T} = \left( \begin{array}{cccc}
0.1965 & -0.0556 & 0.0496 & 0.6838 \\
0.9224 & -0.1298 & -0.0955 & -0.5003 \\
-0.0008 & 0.4786 & -0.5481 & 0.3554 \\
-0.3324 & -0.8666 & -0.8295 & 0.3947
\end{array} \right),
\]

\[
\tilde{\mathbf{T}} = \left( \begin{array}{cccc}
8/9 & -16/27 & 0 & 0 \\
1/6 & 8/9 & 0 & 0
\end{array} \right).
\]

The diagonal matrices \( \mathbf{D} = \text{diag}(D_1, D_2, D_3, D_4) \) and \( \tilde{\mathbf{D}} = \text{diag}(D_5, D_6) \) have matrix elements of the form

\[
D_i = \lambda^6 \left\{ \left( \ln (M^2/A^2) \right)^{(\delta_i/b-1)} - \frac{\delta_i}{2b^2} \left( \ln (M^2/A^2) \right)^{(\delta_i/b-3)} \left( \ln (M^2/A^2) \right)^{(\delta_i/b-3)} \left( \ln (M^2/A^2) \right)^{(\delta_i/b-3)} \right\} (0|\mathcal{O}|0)(\mu),
\]

where \( \delta_1, ..., \delta_6 \) are given in Eq. (23). Note that the one-loop scaling of \( \alpha_s \) is also included in \( \mathbf{D} \) and \( \tilde{\mathbf{D}} \). When evaluating the sum rule numerically we will consider a truncation of \( D_i \) after terms of zeroth and second order in \( (\lambda^2 - 1) \) to test the sensitivity of the \( T \)-evolution of \( m_{\rho} \), \( s_0 \), and the gluon condensate.

Making use of the vacuum saturation hypothesis of Shifman, Vainshtein, and Zakharov (SVZ) [14]

\[
\langle 0|\psi \Gamma_1 \bar{\psi} \Gamma_2 \psi |0 \rangle = N^{-2} \text{Tr} \Gamma_1 \Gamma_2 - \text{Tr} \Gamma_1 \Gamma_2 |0|\psi \psi |0 \rangle^2,
\]

where \( N = 4 \times N_f \times N_c \), and the \( \Gamma_i \), (\( i = 1, 2 \)), are direct products of Dirac, flavor, and color matrices, one can easily verify [14] that

\[
\langle 0|P_2 |0 \rangle (\mu) = \frac{16}{3} \langle 0|P_1 |0 \rangle (\mu),
\]
(0|P_3|0)(μ) = (0|P_4|0)(μ) = 0 ,
(0|P_5|0)(μ) = \frac{16}{3}(0|P_6|0)(μ) . \quad (39)

With Eqs. (21), (35), (37), and (39) we obtain a description for the thermal vacuum contributions of scalar operators to the Gibbs averaged OPE of Eq. (13).

As already mentioned in the beginning of this subsection and following Ref. [2] we restrict the thermal trace of Eq. (1) to the contribution of one-particle pion states (the spectral integral over the pion energy can safely be extended to infinity because of the strong Boltzmann suppression of the integrand). We use the results of Ref. [2] for which \( O_3^a \) were obtained by appealing to the trace anomaly of the QCD energy-momentum-tensor. In the chiral limit there is no explicit temperature dependence of the thermal average of the two-gluon operator \( O_4^a \) due to pion contributions. Applying the soft pion theorem twice and using the vacuum saturation hypothesis of Eq. (38), according to Hatsuda et al. [2] one obtains for the pionic part of the Gibbs average of the operators \( O_{6,1}^a \) and \( O_{6,2}^a \)

\[
\sum_{a=1}^{3} \int \frac{d^3p}{2|p|} \left( \pi^a(p) \right) \frac{1}{2} O_{6,1}^a + \frac{1}{9} O_{6,2}^a \pi^a(p) \left. n_B(|p|/T) \right|_{\pi^a(p)} = \frac{128}{81} \pi \alpha_s(q\bar{q})^2 \left( 1 - \frac{3 T^2}{8 f_{\pi}^2} \right) . \quad (40)
\]

In the chiral limit we have \( f_\pi = 88 \text{ MeV} \) [1], \( n_B \) is the Bose distribution, and \( q \) indicates a single light flavor quark field. As in Ref. [2] we only consider perfect vacuum saturation since for the finite temperature evaluation we are only interested in relative changes as compared to the \( T = 0 \) case.

### 4.2 O(3) invariant Operators

The O(3) invariants of the OPE of Eq. (13) have vanishing vacuum expectation values [1] and therefore we only have to consider their pionic matrix elements. In the chiral limit we encountered the following \( O(3) \) invariant operators of mass-dimension four in the OPE of Eq. (13)

\[
\text{dim4} : \quad O_{4,1}^{O(3)} = \theta_{00} , \quad O_{4,2}^{O(3)} = \frac{16}{3} \theta_{00}^{f} - \theta_{00}^{g} . \quad (41)
\]

The operators \( O_{4,1}^{O(3)} \) and \( O_{4,2}^{O(3)} \) are diagonal combinations with respect to the anomalous mixing matrix of \( \theta_{00}^{f} \) and \( \theta_{00}^{g} \) [1]. The pion matrix elements of the quark part \( N_f \theta_{00}^{f} \) (\( N_f \) denotes the number of quark flavors considered) and the gluon contribution \( \theta_{00}^{g} \) to the total energy density \( \theta_{00} \) are estimated to be equal and can be calculated from the valence parton distribution in the pion as found by Glueck et al. [2][3]. Thereby a low-energy \( (Q^2 = 0.25 \text{ GeV}^2) \) valence like parton distribution fitted to experimental data (direct photon production and Drell-Yan) is evaluated at one and two loop order to the sum rule scale of about \( Q^2 = 1.0 \text{ GeV}^2 \). One obtains

\[
\sum_{a=1}^{3} \int \frac{d^3p}{2|p|} \left( \pi^a(p) \right) \theta_{00}^{f} \left| \pi^a(p) \right| (Q) n_B(|p|/T) = \frac{\pi^2 T^4}{120} \sum_{a=1}^{3} A_2^{(u+d)}(Q) , \quad (42)
\]

with

\[
A_2^{(u+d)}(1 \text{ GeV}) = 0.972 , \forall a . \quad (43)
\]

At mass-dimension six we consider according to Ref. [3] only the following twist two \( O(3) \) operator

\[
\text{dim6} : \quad O_{6}^{O(3)} = i (\bar{u} \gamma_\mu D_\mu D_\nu u + (u \rightarrow d)) . \quad (44)
\]

Again with the model of Glueck et al. and in the chiral limit the result for the pionic contribution to the Gibbs average of \( O_{6}^{O(3)} \) has been determined in Ref. [3] as

\[
\sum_{a=1}^{3} \int \frac{d^3p}{2|p|} \left( \pi^a(p) \right) O_{6}^{O(3)} \left| \pi^a(p) \right| (Q) n_B(|p|/T) = -\frac{2}{5} \left( \frac{\pi T^6}{63} \sum_{a=1}^{3} A_4^{(u+d)}(Q) \right) , \quad (45)
\]

where

\[
A_4^{(u+d)}(1 \text{ GeV}) = 0.255 , \forall a . \quad (46)
\]

The pion matrix elements of the operators \( O_{4,1}^{O(3)} \), \( O_{4,2}^{O(3)} \), \( O_{6,1}^{O(3)} \), \( O_{6,2}^{O(3)} \), \( O_{8}^{O(3)} \), and \( O(3) \) of order \( T^4 \) or higher which led the authors of Ref. [2] to omit them in the sum rule since otherwise higher orders in \( T \) would also have to be calculated for the scalar contributions which a priori are of order \( T^2 \). We do not agree with this power counting argument. The fundamental approximation for the inclusion of pions in the Gibbs average is that of a dilute pion gas. Higher orders in \( T \) would come in by considering interactions between pions as it was shown in Refs. [13][22]. Since the pionic matrix elements of the above nonscalar operators happen to be of order \( T^4 \) and higher already when interactions are switched off the omission of these operators would have the effect of artificially introducing pionic interactions. Hence we do not omit the nonscalar operators in the sum rule.

We now have all the ingredients to the Gibbs averaged OPE of Eq. (13).

### 5 Sum Rule

Performing a Borel transformation of the OPE of Eq. (13), we obtain in the chiral limit the thermal sum rule for the
invariant $T_1$:

$$T_1(M) = \frac{1}{8\pi^2}(1 + \frac{\alpha_s(\mu)}{\pi}) + \frac{1}{M^2} \left\{ \frac{\alpha_s}{2\pi} F^a_{\mu\nu} F^{\mu\nu a} \right\}_T(\mu) + \frac{2}{11} \left[ \frac{\delta b}{\delta \theta_0} T(\mu) \right] - \frac{1}{M^2} \left[ \frac{\alpha_s(\mu^2)}{\alpha_s(M^2)} \right] - \frac{1}{18} \left( P_4 + 2P_2 \right)(\mu) + \frac{4\pi i}{3} \left\langle \bar{q} \gamma_0 D_0 D_0 q \right\rangle_T(\mu),$$

(47)

where the dispersion integral for $T_1(M)$ appearing on the left-hand side of Eq. (17) is given by Eqs. (12) and (11). The scalar mass-dimension six operators $P_1, P_5, P_4$ renormalized at $M$ can be expressed by the operators $P_2$ and $P_3$ renormalized at $\mu = 1 \text{ GeV}$ with the help of Eqs. (55) and (8)  

In order to eliminate the coupling $f_\rho$ we solve Eq. (17) for the $\rho$-meson term

$$R \equiv m_\rho^2 e^{-m_\rho^2/M^2},$$

substitute $\tau = 1/M^2$, and perform the logarithmic derivative of $R$ to obtain the ratio of moments 

$$m_\rho^2(T, s_0, \tau) = -\frac{\partial}{\partial \tau} \log R(T, s_0, \tau) \tag{48}$$

which we will consider.

6 Numerical evaluation

In this section we discuss the numerical evaluation of the sum rule of Eq. (18). Thereby, we employ the following values for the one-flavor quark and the gluon condensate $\bar{\rho}$:

$$\left\langle 0 \bar{q} q(0) \right\rangle (\mu = 1 \text{ GeV}) = -(250 \text{ MeV})^3,$$

$$\left\langle 0 \frac{\alpha_s}{\pi} F^a_{\mu\nu} F^{\mu\nu a} \right\rangle (\mu = 1 \text{ GeV}) = 0.012 \text{ GeV}^4.$$

(49)

As a result of vacuum sum rule calculations in a variety of channels the central value for the gluon condensate is actually believed to be twice as high as the value stated in Eq. (18) with an error of about 50%. The central value for the quark condensate indicated here is rather up to date and exhibits a smaller error of about 30%  

However, the higher values for the gluon condensate have been obtained using numerical correction factors $\kappa$ of about $\kappa = 2$ which multiply the four-quark condensate derived from the vacuum saturation hypothesis  

For exact vacuum saturation ($\kappa = 1$) the values of Eq. (19) for the quark and gluon condensates nicely reproduce the $\rho$-meson mass in the vacuum ($m_\rho(T = 0) \approx 760 \text{ MeV}$). Since we are only interested in relative changes of the spectral parameters and condensates for $T > 0$ as compared to the zero temperature case we will stick to the values of Eq. (48). Our strategy in evaluating the sum rule of Eq. (18) is as follows:

Since $\tau$ is not a direct physical observable, it is chosen as the stationary point $\tau_s$ of $m_\rho^2(0, s_0(0), \tau)$ for a given $s_0$ at $T = 0$.

For a value of $s_0(0) = 1.5 \text{ GeV}^2$ an additional contribution to the spectral function due to $a_1 - \pi$ production is regarded as a part of the pQCD continuum. This is reasonable as long as the Borel parameter $\tau$ is larger than 1 GeV$^{-2}$ since structures stemming from the $a_1 - \pi$ production and possible radial excitations of the $\rho$-meson are then sufficiently suppressed in the spectral integral  

There is a pronounced minimum for a numerically obtained value of $M_s^2 \equiv 1/\tau_s \approx 0.73 \text{ GeV}^2$ corresponding to a $\rho$-meson mass of 745 MeV in the chiral limit considered here.

For finite temperatures $T$ we determine the value of $s_0(T)$ from the stationary point of $m_\rho^2(T, s_0(T), \tau)$ at the same value $\tau = \tau_s$ as in the zero temperature case. Performing this calculation at a sufficient number of $T$ points yields the temperature evolution of $m_\rho^2$, $s_0$ and, with the parametrization of Eq. (24), also the temperature dependence of the gluon condensate. The usefulness of the procedure to determine the $T$-evolution for a fixed $\tau_s$ was checked by applying this technique to the $\rho^0$ sum rule given in Eq. (4.4) of Ref. 2 ($s_0(0) = 1.5 \text{ GeV}^2$). Our method yields the same results as found in Ref. 4, where a more elaborate evaluation analysis (averaging over a $T$-dependent Borel window) was used.

We consider the following two cases: (a) naive rescaling and (b) renormalization group rescaling when including terms up to quadratic order in $\lambda^2 - 1$ in the expansion of $D_i$ given in Eq. (37).

The temperature evolution is determined in steps of $\Delta T = 3 \text{ MeV}$. For $s_0(0) = 1.5 \text{ GeV}^2$ and in the case (b) the $\rho$-meson mass exhibits an increase of about 17.5% from its zero temperature value at a ‘critical temperature’ of $T_c = 157 \text{ MeV}$ which is in contrast to the result of Ref. 2. Thereby, $T_c$ is obtained by demanding the following: If a variation of $s_0(T)$ with respect to $s_0(T - \Delta T)$ is larger than 0.3 GeV$^2$ to produce the minimum of the Borel curve for $m_\rho$ at $\tau_s$, we set $T = T_c$. This critical value merely indicates the breakdown of our sum rule analysis and may not coincide with the critical temperature of a deconfinement phase transition. There is no strong dependence of $T_c$ on the initial value of $s_0$ for the case (b) $- T_c(s_0(0) = 1.2 \text{ GeV}^2) = 157 \text{ MeV}$, $T_c(s_0(0) = 1.5 \text{ GeV}^2) = 157 \text{ MeV}$, and $T_c(s_0(0) = 1.8 \text{ GeV}^2) = 163 \text{ MeV}$. Fig. 1 shows the results for $m_\rho$ as a function of the temperature, where for $s_0(0) = 1.2 \text{ GeV}^2$ and $s_0(0) = 1.8 \text{ GeV}^2$ only the case (b) has been considered. There is practically no difference in the evolution of $m_\rho$ when including the quadratic terms in $\lambda^2 - 1$ for the $D_i$ of Eq. (35) as compared to the linear and to the zeroth order approximation. The deviation at $T = 160 \text{ MeV}$ is then at most 5 MeV.

Naive rescaling means, that, according to Eq. (18), the anomalous renormalization group rescaling of the operators (powers of logarithms) are suppressed, leaving only the factor $\lambda^6$ at mass-dimension six.
Fig. 2 indicates the temperature dependence of the pQCD threshold \( s_0(T) \) for the three initial values and the case (b). In addition, we show the case (a) choosing \( s_0(0) = 1.5 \text{ GeV}^2 \). Again, there is practically no difference between the results for truncations of \( D_i \) in zeroth, linear, and quadratic order. Hence the logarithmic corrections in the Borel parameter \( \tau \) seem to have a much greater effect on the \( T \)-evolution of \( m_\rho \) and \( s_0 \) than the logarithmic corrections in \( \chi^2 \). For \( s_0(0) = 1.5 \text{ GeV}^2 \) the behavior in the case (b) is qualitatively similar to that found in Ref. 2. However, the influence of \( s_0(T) \) on the sum rule is quite different in our work. Besides the truncation of the hadronic part of the spectral function, \( s_0(T) \) also scales the vacuum part of the Gibbs averages in the OPE. In the case (a) we obtain an increase of \( s_0(T) \) and hence a rise of the gluon condensate with increasing temperature which is in contradiction to the results obtained on the lattice and by effective meson models 8,9,10.

In Fig. 3 we finally show the temperature evolution of the gluon condensate normalized to its \( T = 0 \) value which due to the scaling relation of Eq. (29), the invariance under a change of the renormalization point, and the explicit \( T \)-independence in the chiral limit (according to Eq. (29)) is proportional to \( s_0(T)^2 \). We indicate the same combinations of initial values \( s_0(0) \) and cases (a) and (b) as for the \( T \)-evolution of \( s_0 \). Fig. 3 indicates a universality of the \( T \) evolution in a sense, that for different values of \( s_0(0) \) almost identical results are obtained.

7 Summary and Discussion

The main concern of this paper was the investigation of the consequences of a temperature dependent vacuum for the \( T \)-evolution of the \( \rho \)-meson mass and the gluon condensate in the chiral limit. Thereby, we used the method of thermal QCD sum rules. The results obtained in Refs. 2, 11 for the thermal Operator Product Expansion containing also \( O(3) \) invariant contributions and for the thermal \( \rho^0 \) spectral function have been combined in our calculations. Following Ref. 4, the Gibbs averages of local, gauge invariant operators contributing to the thermal OPE of the \( \rho^0 \) current-current correlator were saturated by vacuum and one-pion matrix elements. For lack of better knowledge as far as the \( T \)-dependence of pionic twist two matrix elements is concerned we had to work with the \( T = 0 \) parton distributions of Ref. 21. Consistency then demanded the use of a \( T \)-independent \( f_\pi \), although the \( T \)-dependence of this quantity is known for low temperatures from chiral perturbation theory 22.

In contrast to previous sum rule calculations we suggested to account for a temperature dependence of the vacuum part of the Gibbs average. This was motivated by the observation, that the gluon condensate remains practically \( T \)-independent up to temperatures of 200 MeV in the sum rule analysis of Ref. 2, while a lattice measurement yields a drastic decrease at a critical temperature \( T_c \) of about \( T_c = 140 \text{ MeV} \). To resolve this contradiction, we derived a scaling relation for the thermal vacuum average of scalar operators with the \( T \)-dependent spectral pQCD continuum threshold \( s_0 \). Thereby, the chiral limit, temperatures considerably smaller than the QCD scale \( \Lambda \), and a renormalization point of the order of 1 GeV were assumed. The implementation of the above scaling relation is trivial for renormalization group invariant operators and becomes rather involved for operators which mix and scale anomalously under a change of the renormalization point. On the spectral side we used a narrow width approximation for the \( \rho^0 \) resonance since including a finite width \( \Gamma \) leads to ambiguities in the determination of \( m_\rho \) and \( \Gamma \) as was shown in Ref. 4 for the case of finite density. One would expect, that this is also true for finite temperatures. On the other hand, the effect of the scaling of the vacuum averages can only be isolated if one compares the results with previous calculations also using the narrow width approximation. In analogy to a previous sum rule calculation in the \( \rho^0 \)-channel (see Ref. 14) we employed a thermal pion continuum for the hadronic part, while the high energy tail was approximated by a \( T \)-independent pQCD continuum.

As compared to the vacuum case our sum rule calculations indicate a rise of the \( \rho \)-meson mass (17.5% at \( T_c \)) and a decrease of \( s_0 \) and of the gluon condensate with temperature. Thereby, the critical temperature \( T_c \), where the sum rule analysis breaks down, is \( T_c \approx 160 \text{ MeV} \). This "critical" temperature is rather close to the QCD scale \( \Lambda \), and therefore we do not expect the scaling relation of Eq. (29) to be a good approximation for the thermal vacuum average of scalar operators. The increase of the \( \rho \)-meson mass contradicts the sum rule results obtained in Refs. 11, 12, 23, where a monotonic decrease of this quantity is obtained. In the difference sum rule analysis of Ref. 13 the \( \rho \)-meson mass was also found to increase slightly (\( \Delta m_\rho \approx 5 \text{ MeV} \)) up to a temperature of about 125 MeV while decreasing thereafter. There are indications for a slight rise of the \( \rho \)-meson mass up to \( T = 150 \text{ MeV} \) (\( \Delta m_\rho \approx 30 \text{ MeV} \) at \( T = 150 \text{ MeV} \)) from a microscopical calculation of the spectral function in the \( \rho^0 \) channel using an effective \( \rho - \pi \) Lagrangian 22. We emphasize again, that the critical temperature of our calculation does only indicate the breakdown of the sum rule analysis based on a scaling relation for the scalar condensates and may not coincide with the critical value for a deconfinement phase transition. It is interesting to note, that a calculation in the Nambu-Jona-Lasinio model yields at \( T \approx 160 \text{ MeV} \) and already at nuclear saturation density a relative increase of the dynamically generated constituent light-quark mass which is of the same order as that of the \( \rho \)-meson mass in our calculation 25,26. In Ref. 13, where by means of PCAC, current algebra, and the LSZ reduction formula a mixing of the vector with the axial vector correlator for space-like \( q^2 \) was found for small temperatures (no OPE was used), a difference sum rule analysis yields an increase of the \( \rho \)-meson mass and a decrease of the mass of the \( a_1 \)-meson with temperature. This was interpreted as the process of chiral symmetry restoration. The analysis of Ref. 13 is meaningful for small temperatures (\( T < f_\pi \)) and for a \( T \)-independent \( f_\pi \) consistent with the recent lattice simulation of Ref. 19 indicating no sign of a \( T \) dependence up...
to $T \approx T_c$ of the chiral transition. Our result for the $T$-evolution of the $\rho$-meson mass is consistent with the one obtained in the analysis of Ref. [13]. In addition, qualitative arguments based on the instanton model [30] imply a cancellation of the effects of a decreasing constituent quark mass and a lowering of the instanton mediated attraction between constituent quarks for the $T$-evolution of the $\rho$-meson mass.

Depending on $s_0(0)$, our result for the gluon condensate exhibits a 25–30% decrease at $T \approx 160$ MeV which is compatible with the lattice data for two quark flavor QCD of Ref. [8]. The analysis showed, that the results are sensitive to the anomalous scaling of mass-dimension six operators under a change of the renormalization point (see Fig. 3). As Fig. 3 indicates the $T$-evolution of the gluon condensate is universal in a sense, that it is practically independent of the initial value $s_0(0)$.

There are still open questions concerning the inclusion of nonscalar, mixed operators at mass-dimension six in the thermal OPE. So far we must be content with the hope, that in the future their pionic matrix elements can be estimated from deep inelastic lepton scattering off the pion target [8].

To conclude, our analysis indicates, that in the Gibbs average the $T$-dependence of the vacuum matrix elements of scalar operators, as introduced via a scaling relation in $s_0$, produces a sizable decrease of the gluon condensate and a moderate increase of the $\rho$-meson mass up to temperatures of about 160 MeV, where the analysis breaks down (and the confidence in a good accuracy of the approximations made is not high anymore).

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A

Under the premises that $T$ be considerably less than the fundamental QCD scale $\Lambda$, and that the renormalization scale $Q_0$ is taken of the order of 1 GeV we may argue in favor for the assumption of an implicit dependence of the scalar condensates on $T$ via $s_0(T)$ in the chiral limit.

At $T = 0$ the most fundamental description of the QCD vacuum structure by the appearance of nonvanishing vacuum averages of local, scalar, and gauge invariant operators may in the chiral limit only involve the scale $\Lambda$ and a renormalization scale $Q_0$. With the number of independent parameters kept fixed, we hypothesize that there is a linear relation between $\Lambda$ and $\sqrt{s_0(0)}$.

In this picture the constant of proportionality expresses this channel dependence. In the following we will argue that for $T > 0$ and provided that $T$ is considerably smaller than $\Lambda$ it is approximately possible to define a $T$-dependent fundamental scale $A_T$ such that the thermal vacuum averages of scalar operators have the same functional dependence on $A_T$ and $Q_0$ as in the case of $T = 0$, where they depend on $\Lambda$ and $Q_0$. Then there should be the identical linear relation between $A_T$ and $\sqrt{s_0(T)}$ as between $A$ and $\sqrt{s_0(0)}$ at $T = 0$.

For the thermal vacuum average of $O$ renormalized at $Q_0$ we may write

$$
\langle 0 | O | 0 \rangle_T (Q_0) = f(T/\Lambda, T/Q_0, A/Q_0) \times Q_0^d,
$$

where $f$ denotes a dimensionless function of its dimensionless arguments, and $d$ indicates the mass dimension of $O$. We define $A_T$ by demanding

$$
f(0, 0, A_T/Q_0) = f(T/\Lambda, T/Q_0, A/Q_0).
$$

Solving Eq. (51) for $A_T$ yields

$$
A_T = A_T(A, Q_0, T) = \tilde{A}_T(T/\Lambda, T/Q_0, A/Q_0) \times A,
$$

where $\tilde{A}_T$ is a dimensionless function of its dimensionless arguments. Since in our applications the renormalization scale $Q_0$ is of the order of 1 GeV and with $T$ considerably less than $\Lambda$, we can neglect the dependence of $\tilde{A}_T$ on $T/Q_0$

$$
A_T(A, Q_0, T) \approx A_T(T/\Lambda, A/Q_0) \times A,
$$

where $A_T$ again is a dimensionless function of its dimensionless arguments. Let us now expand $\tilde{A}_T$ in powers of $A/Q_0$

$$
A_T = \Lambda \times (c_0(T/\Lambda) + c_1(T/\Lambda)A/Q_0 + \cdots).
$$

Since $A_{T=0} = 0$ it follows that

$$
c_0(0) = 1, \quad c_i(0) = 0, \quad (i \geq 1).
$$

Then the expansion of Eq. (54) should be well approximated by its first term if $T$ is considerably less than $\Lambda$, namely by

$$
A_T \approx \Lambda \times c_0(T/\Lambda).
$$

Hence we obtain a well defined (since $Q_0$-independent) $T$-dependent scale $A_T$ which yields the same functional dependence of the thermal vacuum average on $A_T$ and $Q_0$ as that of the $T = 0$ vacuum average on $\Lambda$ and $Q_0$.

B

Here we derive the decomposition of Eq. (24). For the operator $O_{6,2}^a$ one simply uses

$$
\bar{q}\gamma_\alpha t^a q = \bar{q}_L\gamma_\alpha t^a q_L + \bar{q}_R\gamma_\alpha t^a q_R
$$

to obtain

$$
O_{6,2}^a = \pi\alpha_s (P_4 + 2 P_2).
$$

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7 There is then no other, independent scale to allow for a different behavior since $s_0$ cannot depend on the external variable $Q_0$. 

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Using
\[ \bar{q} \gamma_\alpha \gamma_5 t^a q = \bar{q} L \gamma_\alpha t^a q L - \bar{q} R \gamma_\alpha t^a q R , \]
yields the following decomposition for \( O_{6,1} \)
\[
O_{6,1} = \pi \alpha_s \left[ (\bar{L} \gamma_\alpha t^a \gamma_5 \psi_L)^2 - 2 \bar{\psi}_L \gamma_\alpha t^a \gamma_5 \psi_R \gamma_\alpha t^a \psi_R + (\bar{\psi}_R \gamma_\alpha t^a \gamma_5 \psi_R)^2 \right].
\]
(58)
We are only interested in the flavor singlet part of the right-hand side, for which we also write \( O_{6,1} \)
\[
O_{6,1}^s = \frac{1}{3} \pi \alpha_s \left[ (\bar{\psi}_L \gamma_\alpha t^a \gamma_5 \psi_L)^2 - 2 \bar{\psi}_L \gamma_\alpha t^a \gamma_5 \psi_R \gamma_\alpha t^a \psi_R + (\bar{\psi}_R \gamma_\alpha t^a \gamma_5 \psi_R)^2 \right].
\]
(59)
For the sake of brevity we only consider the following operator
\[ O_{L(R)} = (\bar{\psi}_L(R) \gamma_\alpha t^a \gamma_5 \psi_L(R))^2 . \]
(60)
Writing out color and flavor indices (in this order), using
\[
\tau^c_{ij} \tau^c_{km} = 2(\delta_{im} \delta_{jm} - \frac{1}{N} \delta_{ij} \delta_{mn}) ,
\]
with \( N_c = 3 \) and \( N_f = 2 \) one obtains
\[
O_{L(R)} =
4 \left[ \bar{\psi}_L(R), i \mu \gamma_\alpha \delta_{im} \delta_{j\nu} \psi_L(R), j \nu \bar{\psi}_L(R), jk \gamma_\alpha \delta_{i\nu} \delta_{\nu k} \psi_L(R), m \lambda - \bar{\psi}_L(R), i \mu \gamma_\alpha \delta_{im} \delta_{j\mu} \psi_L(R), j \nu \bar{\psi}_L(R), jk \gamma_\alpha \delta_{i\nu} \delta_{\mu k} \psi_L(R), m \lambda - \bar{\psi}_L(R), i \mu \gamma_\alpha \delta_{ij} \delta_{\mu \lambda} \psi_L(R), j \nu \bar{\psi}_L(R), jk \gamma_\alpha \delta_{i\nu} \delta_{\nu k} \psi_L(R), m \lambda + \frac{1}{3} \bar{\psi}_L(R), i \mu \gamma_\alpha \delta_{ij} \delta_{\mu \lambda} \psi_L(R), j \nu \bar{\psi}_L(R), jk \gamma_\alpha \delta_{i\nu} \delta_{\nu k} \psi_L(R), m \lambda \right] .
\]
(61)
Applying the Fierz transformation
\[ \bar{\psi}_L(R), i \gamma_\alpha \psi_L(R), 2 \bar{\psi}_L(R), 3 \gamma_\alpha \psi_L(R), 4 = \bar{\psi}_L(R), i \gamma_\alpha \psi_L(R), 4 \bar{\psi}_L(R), 3 \gamma_\alpha \psi_L(R), 2 \]
(62)
to the first and the third line of Eq. (61) and relabelling \( m \leftrightarrow j \) in the second line, the sum of the second and the third line of Eq. (2) reads
\[ - \frac{5}{6} \bar{\psi}_L(R), i \gamma_\alpha \psi_L(R), j \bar{\psi}_L(R), j \gamma_\alpha \psi_L(R), i , \]
(63)
where the summation over flavor indices is implicit. This can easily be rewritten as
\[ - \frac{5}{12} \bar{\psi}_L(R) \gamma_\alpha t^a \psi_L(R) \gamma_\alpha t^a \psi_L(R) - \frac{5}{18} \bar{\psi}_L(R) \gamma_\alpha \psi_L(R) \gamma_\alpha \psi_L(R) . \]
(64)
Putting everything together we finally obtain
\[ O_{6,1}^s = \frac{1}{3} \pi \alpha_s \left( - \frac{5}{3} P_4 + \frac{32}{9} P_3 - 2 P_5 \right) . \]
(65)
Figure 1: Temperature evolution of the $\rho$-meson mass. The solid lines correspond to the case (b) for $s_0(0) = 1.2 \text{ GeV}^2$ ($m_\rho(T = 0) = 694 \text{ MeV}$), $s_0(0) = 1.5 \text{ GeV}^2$ ($m_\rho(T = 0) = 745 \text{ MeV}$), and $s_0(0) = 1.8 \text{ GeV}^2$ ($m_\rho(T = 0) = 776 \text{ MeV}$). For $s_0(0) = 1.5 \text{ GeV}^2$ the case (a) is associated with a dashed line.

Figure 2: Temperature evolution of the pQCD threshold $s_\rho$. The solid lines correspond to the case (b). For $s_0(0) = 1.5 \text{ GeV}^2$ the case (a) is associated with the dashed line.

Figure 3: Temperature evolution of the gluon condensate normalized to its zero temperature value. For the case (b) we take $s_0(0) = 1.2 \text{ GeV}^2$, $s_0(0) = 1.5 \text{ GeV}^2$, and $s_0(0) = 1.8 \text{ GeV}^2$ corresponding to dot-dashed, solid, and long dashed lines, respectively. For $s_0(0) = 1.5 \text{ GeV}^2$ the case (a) is associated with a dotted line.
Fig. 1.
Fig. 2.
Fig. 3.