S-matrix singularities and CFT correlation functions

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Abstract: In this note, we explore the correspondence between four-dimensional flat space S-matrix and two-dimensional CFT proposed by Pasterski et al. We demonstrate that the factorization singularities of an $n$-point cubic diagram reproduces the AdS Witten diagrams if mass conservation is imposed at each vertex. Such configuration arises naturally if we consider the 4-dimensional S-matrix as a compactified massless 5-dimensional theory. This identification allows us to rewrite the massless S-matrix in the CHY formulation, where the factorization singularities are re-interpreted as factorization limits of a Riemann sphere. In this light, the map is recast into a form of $2d/2d$ correspondence.
1 Introduction

The isomorphism between the Lorentz group in four dimensions and the Mobius group of conformal transformations in two dimensions, have been used in several theoretical approaches for the computation of scattering amplitudes in the past. It is indeed the corner stone of the realization of Penrose’s Twistor Space [1] and the impressive subsequent development of scattering amplitudes of massless particles in Twistor space, started by Witten more than a decade ago [2]. A generalisation of this isomorphism is the Embedding Formalism for the $d$-dimensional conformal group, built upon the work of Dirac [3] and which has been particularly useful in the context of the AdS/CFT correspondence.

Recently, there has been a growing effort in writing the dynamics of four-dimensional Minkowski space in terms of observables in two-dimensional conformal field theory, greatly motivated by the renewed interest on the asymptotic BMS symmetries in gravitational theories [4, 5] (for a more recent discussion see [6]), based on the observation that the Lorentz group in four-dimension acts as the Mobius group in two dimensions over the null-infinity boundary of the three-dimensional space, recently baptized as the
celestial sphere $CS$ [7]. From this point of view, it is expected that scattering amplitudes in four-dimensions can be recasted in terms of some sort of correlator in a certain two-dimensional conformal field theory. It has indeed been shown that soft theorems can be rewritten as Ward identities in a two-dimensional conformal field theory [8, 9] and hence, they should be somehow related to two-dimensional current algebras [10, 11]. The conserved currents and the stress-tensor of the corresponding two-dimensional field theory has been discussed in [7, 12].

More recently, Pasterski, Shao and Strominger considered S-matrix for conformal primary wave functions. The wave functions are constructed by convoluting plane waves with bulk-to-boundary propagators in $AdS_3$, and thus transforms covariantly under the Mobius group [13]. Using this transformation allows one to transform Lorentz invariant (Little group covariant) scattering amplitudes into Mobius covariant quantities, which can be considered as the correlation functions of some 2d CFT. The precise map is:

$$\tilde{A}(\Delta_i, w_i, \bar{w}_i) \equiv \prod_{i=1}^{n} \left( \int d^2 z_i \frac{dy_i}{y_i^3} G_{\Delta}(y_i, z_i, \bar{z}_i; w, \bar{w}) \right) A(m_j \hat{p}_j) ,$$

(1.1)

where we have $n$ copies of integration over $AdS_3$ coordinates, and $G_{\Delta}$ are the bulk to boundary propagators. The $AdS_3$ coordinates are imbedded in the four-dimensional momenta satisfying the massive on-shell constraint $p^2 = -m^2$. Explicit results were obtained for the three-point function of $\phi^3$ theory, which is fixed by symmetries.

In this note we intend to explore this relation further by studying the convolution of factorization singularities of flat space scattering amplitudes. We consider the factorization singularities of massive scattering amplitudes that admit a cubic diagram expansion. By ensuring that the mass is conserved at each vertex, we show that the factorization singularity for massive poles, enforces that the $n$-point kinematics can be mapped to a configuration of a contact $AdS$ Witten diagram. Once dressed with bulk to boundary propagators and integrate over the whole $AdS$ space, one reproduce an $n$-point correlation function, i.e.:

$$\tilde{A}(\Delta_i, w_i, \bar{w}_i) = \prod_{i=1}^{n} \left( \int d^2 z_i \frac{dy_i}{y_i^3} G_{\Delta}(y_i, z_i, \bar{z}_i; w, \bar{w}) \right) A(m_j \hat{p}_j) ,$$

(1.2)

where on the cubic diagram on the RHS we identify the pole singularities. Since all tree-level scalar Witten diagrams can be re-expressed in terms of finite sums of contact diagrams [14–16], with the scaling dimensions of external operators being shifted,
this implies that the tree-level S-matrix singularities reproduces all tree-level Witten diagrams.

Note that mass conservation can be interpreted as momentum conservation in higher dimensions, where the propagators are massless. The embedding of the previous construction in terms of a massless theory allows us to introduce the Cachazo-He-Yuan (CHY) representation for the propagator singularities. In particular, as discussed in [17], by parameterising the moduli space of \(n\)-punctures by a single parameter \(\tau\) such that

\[
\sigma_i = \frac{v_i}{\tau} \quad i \in L, \quad \sigma_i = \tau u_i \quad i \in R.
\]

where \(\sigma_i\) are the coordinates of the punctures and we’ve separated it into a left and right set, the parameter \(\tau\) then encodes the pinch limit of the Reimann sphere. Moreover, in such parameterization, one can identify that one of the scattering equation constraints becomes

\[
\delta(\tau^2 F - p^2_I)
\]

where \(p_I\) would be the associated momenta of the internal particle and \(F\) is some \(\tau\) independent polynomial. Thus by inserting a factor of \(\delta(\tau)\) in the CHY integrand, we reproduce the factorization constraint. This combined with the fact that there the CHY formula can be naturally computed from a worldsheet chiral string theory [18, 19], give an interesting 2\(d\)/2\(d\) correspondence:

\[
2d \ CFT \leftrightarrow \prod_{i=1}^{n} \left( \int d^2 z_i \frac{dy_i}{y_i^3} G_{\Delta}(y_i, z_i, \bar{z}_i; w, \bar{w}) \right) \int d^{2n} \sigma_i \ \text{CHY} \left( \prod_j \tau_j \delta(\tau_j) \right),
\]

where \(\tau_i\) are moduli for the degenerate limits of punctures on the Riemann sphere.

The remaining of this paper is organized as follows: in section 2 we quickly review the Pasterski-Shao-Strominger (PSS) proposal for the transformation of scattering amplitudes in flat space into correlation functions of conformal field theory at co-dimension two, then in section 3 we move to the study of the transform at the factorization singularities of the S-matrix. Finally, at section 4 we propose a duality relation between correlation functions in two-dimensional chiral string theory in the CHY representation of the S-matrix and correlation functions from the PSS transformation.

During the preparation of this draft, the authors were aware of the coming work by Dhritiman Nandan, Anastasia Volovich, Congkao Wen, and Michael Zlotnikov, which has some overlap.
2 The PSS proposal

The fact that $d$-dimensional conformal symmetry can be linearly realised as a $d + 2$-dimensional Lorentz symmetry has a long history of applications that dates back to Dirac (see [20] for references and review). It is then natural to ask whether or not observables on both sides of the relation can also be mapped. For CFTs, the natural physical observables are correlation functions, which transform covariantly under conformal transformations. On the $d + 2$-dimensional side, the natural Lorentz covariant physical observable is the S-matrix, which upon Lorentz transformation generates a little transformation. The next task is then to find a map between the variables on the two sides.

Recently Pasterski, Shao and Strominger (PSS) [13] presented a proposal for such a map, where as a first step, the four-dimensional massive momenta $\mathbb{R}^{1,3}$ is mapped into the coordinates of hyperbolic space $H_3$. Taking the metric of $H_3$ as,

$$ds_{H_3}^2 = \frac{dy^2 + dz d\bar{z}}{y^2}, \quad (2.1)$$

the three-dimensional hyperbolic space is mapped to the four-dimensional momenta in a SL(2,C) covariant form as:

$$\hat{p}_{a\dot{a}}(y, z) \equiv \hat{p}^\mu(y, z)\sigma_\mu = \frac{i}{y} \left[ \begin{array}{c} 1 \\ \bar{z} \\
\end{array} \right]$$

$$\left[ \begin{array}{c} y^2 + |z|^2 \\
\end{array} \right] \quad (2.2)$$

such that $\hat{p}^2 = \text{Det}[\hat{p}_{a\dot{a}}] = -1$. For a particle of mass $m$, its momenta is given as $p = m\hat{p}$.

It was conjectured that the correlation function of some 2D CFT can be related to the S-matrix in four-dimensions, where the external states are conformal primary wave-functions, defined as:

$$\phi_{\Delta,m}^\pm(X^\mu; w, \bar{w}) = \int_0^\infty \frac{dy}{y^3} \int dz d\bar{z} G_\Delta(y, z, \bar{z}; w, \bar{w}) \exp \left[ \pm im \hat{p}^\mu(y, z, \bar{z}) X_\mu \right] \quad (2.3)$$

where the $\pm$ in the exponent indicates the incoming and outgoing states and $G_\Delta(y, z, \bar{z}; w, \bar{w})$ be the scalar bulk-to-boundary propagator in $H_3$ of conformal dimension $\Delta$ [21],

$$G_\Delta(y, z, \bar{z}; w, \bar{w}) = \left( \frac{y}{y^2 + |z - w|^2} \right)^\Delta. \quad (2.4)$$

Note that we can write the bulk to boundary propagator as

$$G_\Delta(y, z; w) = \left( -i \left[ w \begin{array}{c} 1 \\
\end{array} \right] \hat{p}(y, -z)_{a\dot{a}} \left[ \bar{w} \begin{array}{c} 1 \\
\end{array} \right] \right)^{-\Delta}.$$  

$$\quad (2.5)$$

\[ -4 - \]
which manifests it’s covariant property under the $SL(2, \mathbb{C})$ transformation $w' = (aw + b)/(cw + d)$, \(^1\),

$$G_\Delta(y', z', \bar{z}'; w', \bar{w}') = |cw + d|^2 \Delta_\Delta (y, z, \bar{z}; w, \bar{w}) . \quad (2.6)$$

The correlation function $\tilde{A}_{\Delta_1, \cdots, \Delta_n}(w_i, \bar{w}_i)$ is then related to the flat space S-matrix as defined as $A(m\hat{p})$ by:

$$\tilde{A}(\Delta_i, w_i, \bar{w}_i) \equiv \int d^4X \prod_{i=1}^{n} \phi_{\Delta_i, m_i}^\pm (X^\mu; w_i, \bar{w}_i) A(m\hat{p}) . \quad (2.7)$$

The four-dimensional integral $d^4X$ simply produces the momentum conservation delta function. The plausibility of eq.(2.7) stems from the two sides sharing the same symmetry, as verified in [13], for the case of $\phi^2 \phi$ interaction, where $m_\phi \sim 2m_\phi$.

### 2.1 The three point contact term

The explicit example shown in [13] is a three-point contact term, and the mass of one particle is near-extremal, i.e. it is near the sum of the other two. Here we present a brief review, since most of the details will be utilised for the $n$-point construction.

Taking the mass of the first particle $\phi$ to be $2(1 + \epsilon)m$ and the masses of the other two particles be $m$ and evaluating the $X^\mu$-integral, we arrive at the following expression for the scalar three-point amplitude,

$$\tilde{A}(w_i, \bar{w}_i) = i(2\pi)^4 \lambda m^{-4} \left( \prod_{i=1}^{3} \int_0^\infty dy_i \int d\bar{z}_i d\bar{z}_i \right) \times \prod_{i=1}^{3} G_\Delta(y_i, z_i, \bar{z}_i; w_i, \bar{w}_i) \delta^{(4)}(-2(1 + \epsilon)\hat{p}_1 + \hat{p}_2 + \hat{p}_3) , \quad (2.8)$$

where we used $-p_1 + p_2 + p_3 = -2(1 + \epsilon)\hat{p}_1 + \hat{p}_2 + \hat{p}_3$. In general, three of the four momentum conservation delta function solves one of the momenta in terms of others, while the remaining one simply enforces

$$p_n^2 = \left( \sum_{i=1}^{n-1} p_i \right)^2 = -m_n^2 \quad (2.9)$$

For the current case the integral of $(y_3, z_3)$ is straightforwardly localized, leaving a Jacobian factor of

$$- \left( -2(1 + \epsilon)\frac{y_1^2 + |z_1|^2}{y_1} + \frac{y_2^2 + |z_2|^2}{y_2} \right)^{-1} \quad (2.10)$$

\(^1\)Here and in what follows we have only used the dependence on variable $\bar{z}$, both for simplify the notation and because we are not taken $\bar{z}$ to be an independent variable, but instead as the conjugate of $z$. 

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and we are left with the final delta function, which is proportional to the on-shell condition,
\[
\delta \left( 2(1+\epsilon) \frac{1}{y_1} - \frac{1}{y_2} - \frac{1}{y_3} \right) = \left( -2(1+\epsilon) \frac{y_2^2 + |z_1|^2}{y_1} + \frac{y_2^2 + |z_2|^2}{y_2} \right) \delta (\hat{p}_1 \cdot \hat{p}_2 + (1+\epsilon))
\]
Note that the prefactor on the RHS exactly cancels the previous Jacobian factor. Thus one concludes that momentum conservation fixes the position of one leg:
\[
z_n = -y_n \left( \sum_i m_i \frac{z_i}{y_i} \right), \quad y_n = -\frac{\sum_i m_i y_i^2 + |z_i|^2}{1 + \left( \sum_{ij} m_i m_j z_i z_j / y_i y_j \right)}
\tag{2.11}
\]
where the summation range is \(i = 1, \cdots, n-1\) and appropriate additional signs for each term depending on whether it is incoming or outgoing.

It is convenient to parameterize the remaining two AdS_3 points as
\[
y_2 = y_1 + R \cos \theta, \quad z_2 = z_1 + R \sin \theta e^{i\phi},
\tag{2.12}
\]
for which the argument of the delta function becomes
\[
\delta (\hat{p}_1 \cdot \hat{p}_2 + (1+\epsilon)) = 2y_1 y_2 \delta (2\epsilon y_1 y_2 - R^2).
\tag{2.13}
\]
Thus the final constraint forces \((y_2, z_2) \to (y_1, z_1)\), and through eq.(2.11), so is \((y_3, z_3)\), arriving at a contact diagram in AdS_3. Note that if \(\epsilon < 0\), there will be no solution for the delta function since it is below the production threshold.

The support of the delta function in equation (2.13), at leading order in \(\epsilon\), is given by,
\[
R = \sqrt{2\epsilon} y_1 = \tilde{R} \sqrt{\epsilon},
\tag{2.14}
\]
where \(\tilde{R} = y_1 \sqrt{2}\). Therefore, the integration measure translate into the coordinate \((\tilde{R}, \theta, \phi)\) as,
\[
\int d^2 z_1 \, dy_1 \to \epsilon^{3/2} 2\pi \int \tilde{R}^2 d\tilde{R} d\Omega.
\tag{2.15}
\]
So more precisely, we are retaining the leading order in small \(\epsilon\) expansion. In the following sections we are going to use heavily the condition (2.13), therefore the final transformation should be understood as the leading piece in \(\epsilon\). More explicitly for the three-point function considered in this section we write,
Figure 1. The factorization singularity of s-channel diagram leads to a contact Witten diagram in AdS3.

\[ \tilde{A}_3(w_i, \bar{w}_i) \approx 2i(2\pi)^5 \frac{\lambda^{1/2}}{m^4} \int_0^\infty \frac{dy_2}{(y_2)^4} \int d^2 z_2 \times \int R^2 dR \int d\Omega \delta(R^2 - 2y_2^2) \prod_{i=1}^3 G_{\Delta_i}(y_i, z_i, \bar{z}_i; w_i, \bar{w}_i) \]

\[ \sim \lambda^{1/2} \int_0^\infty \frac{dy_2}{y_2^3} \int d^2 z_2 \prod_{i=1}^3 G_{\Delta_i}(y_i, z_i, \bar{z}_i; w_i, \bar{w}_i) . \] (2.16)

3 Contact diagrams from factorization singularities

As mentioned previously, an interesting questions one can immediately pose for such a correspondence is how consistency conditions are mapped. The S-matrix is subject to locality and unitarity constraints which translate into the presence of propagator singularities as well as consistent residues. Here we will explore the implication of factorization singularities on the n-copy AdS integral.

3.1 A four-point example

Let’s assume that (1, 2) are incoming and (3, 4) are outgoing with identical absolute mass. Mass conservation restricts the s-channel pole to be massive. Thus the complete kinematic constraint for s-channel factorization is given by:

\[ \delta^{(4)}(\sum_i p_i)\delta((p_1 + p_2)^2 + 4m^2) \] (3.1)

In terms of \( \hat{p}_i \), the argument of the delta functions can be written as

\[ \hat{p}_1 \cdot \hat{p}_2 + 1, \quad \text{or} \quad \hat{p}_3 \cdot \hat{p}_4 + 1 . \] (3.2)

As seen from the previous section whenever the AdS coordinates appear under the constraint

\[ \hat{p}_i \cdot \hat{p}_j + 1 = 0 , \] (3.3)
points $i$ and $j$ will be forced to be coincidental. Indeed here if one performs a change of variable,

$$
y_{12} = R_{12} \cos \theta_{12}, \quad z_{12} = R_{12} \sin \theta_{12} e^{i \phi_{12}}
$$

$$
y_{34} = R_{34} \cos \theta_{34}, \quad z_{34} = R_{34} \sin \theta_{34} e^{i \phi_{34}}.
$$

(3.4)

where $y_{ij} \equiv y_i - y_j$ and $z_{ij} \equiv z_i - z_j$, it is straightforward to see that this constraint simply forces points 1, 2 and 3, 4 to be coincidental respectively. Finally, three of the four momentum conservation delta function solves one of the momenta in terms of others, and the remaining constraint is (2.9), which in this case leads to:

$$
\hat{p}_1 \cdot \hat{p}_3 + 1 = 0.
$$

(3.5)

Therefore momentum conservation forces all points to be coincidental. In other words, we end up with a contact term in AdS$_3$ as indicated in fig.(1).

Let us put all the pieces together in this simple example, to illustrate how this will work in general. From equation (2.7) for four-particles we have,

$$
\tilde{A}_4(w_i, \bar{w}_i) = i(2\pi)^4 \frac{\lambda^2}{4m^4} \left( \prod_{i=1}^{3} \int_0^\infty \frac{dy_i}{y_i^3} \int d\bar{z}_i d\bar{z}_i \right) \prod_{i=1}^{4} G_{\Delta_i}(y_i, z_i, \bar{z}_i; w_i, \bar{w}_i) \times
$$

$$
\delta(\hat{p}_1 \cdot \hat{p}_2 + 1 + \epsilon) \delta(\hat{p}_1 \cdot \hat{p}_3 + 1),
$$

(3.6)

where should be understood that $p_4$ has been fixed by momentum conservation, the first delta function in the second line comes from the singularity pole (3.1) whereas the second delta function comes from the remaining delta from momentum conservation (2.9), namely $\delta(p_2^2 + m^2)$. From the explicit computation for the vertex (2.16), we have also learned that upon integration, every localization delta function $\delta(\hat{p}_i \cdot \hat{p}_j + 1 + \epsilon)$, give us a leading scaling $\epsilon^{1/2}$, so up to a numerical factor and at leading order in $\epsilon$ we find,

$$
\tilde{A}_4(w_i, \bar{w}_i) \sim \lambda^2 \frac{\epsilon^{1/2}}{m^4} \int_0^\infty \frac{dy_1}{y_1^3} \int d^2 z_1 \prod_{i=1}^{4} G_{\Delta_i}(y_i, z_i, \bar{z}_i; w_i, \bar{w}_i).
$$

(3.7)

This is the contact four-point Witten diagram in AdS$_3$.

Note that even though we’ve only reproduced contact Witten diagrams, factorization Witten diagrams can be represented in a similar fashion. It was shown [14–16] that the latter can be rewritten as a finite series expansion in terms of contact quartic
Witten diagrams,

\[
\frac{2\Delta_0 - \Delta}{2} \sum_{k=1}^{\frac{2\Delta_0 - \Delta}{2}} \frac{a_k}{|w_{13}|^k} \int dp_1 G_k(p_1; w_1, \bar{w}_1) G_k(p_1; w_3, \bar{w}_3) \times \\
G_{\Delta_0}(p_1; w_2, \bar{w}_2) G_{\Delta_0}(p_1; w_4, \bar{w}_4),
\]

\[
a_k = \frac{\left(\Delta_0\right)^2}{4 \left(\frac{2\Delta_0 - \Delta}{2}\right)^{1-k} \left(\frac{2\Delta_0 + \Delta - d}{2}\right)^{1-k}}.
\]

(3.8)

where \(\Delta_0\) denotes the conformal dimension of the external fields, which in this note have been taken to be identical, and \(\Delta\) denotes the conformal dimension of the exchanged field. Taking \(w_1 = 0, w_3 = 1\), we find that factorization Witten diagrams can be identified as the residue of

\[
A(\hat{p}) = \sum_{k=1}^{\frac{2\Delta_0 - \Delta}{2}} \frac{a_k}{(\hat{p}_1 - \hat{p}_4)}.
\]

(3.9)

Note that the coefficients \(a_k\) are all positive definite which is consistent from the viewpoint of S-matrix unitarity.

Finally, since the constraint localizes all AdS points, the only possible Lorentz invariant is \(\hat{p}_i \cdot \hat{p}_j = \hat{p}_i^2 = -m_i^2\), i.e. the residue degenerates to a number. In other words, exchanging different higher spin states amounts to a trivial normalization constant for the contact AdS\(_3\) diagram.\(^2\)

### 3.2 The \(n\)-point generalization

We are now ready to make the full fledged proposal. We will consider all cubic graphs whose mass is conserved. The constraint arising from all massive propagators going on-shell, along with all momentum conservation, will reduce the \(n\) copies of AdS integrals to a single copy, thus corresponding to a contact Witten diagram.

Consider the five-point cubic diagram represented in fig.(2), where the arrows represent the moment flow for \(p_1 + p_2 = p_3 + p_4 + p_5\). The associated pole singularities are,

\[
\delta \left((p_1 + p_2)^2 + 4m_{in}^2\right) \delta \left((p_3 + p_4)^2 + 4m_{out}^2\right)
\]

where \(m_1 = m_2 = m_{in}\) and \(m_3 = m_4 = m_5 = m_{out}\). This localizes \(\hat{p}_1 \to \hat{p}_2\) and \(\hat{p}_3 \to \hat{p}_4\). Once again, momentum conservation can be used to solve for \(\hat{p}_5\) and as in

\(^2\)This can also be understood from the four-point kinematics \(s\) and \(t\). When \(s = 4m^2\),

\[
t = \frac{s - 4m^2}{2} (1 - \cos(\theta)) = 0,
\]

and thus the only possible residue is a number.
previous sections, the remaining on-shell condition becomes,
\[ p_5^2 = -m_{out}^2 = [(p_1 + p_2) - (p_3 + p_4)]^2 = 4(-m_{in}^2 - m_{out}^2 - 2m_{out}m_{in}\hat{p}_1 \cdot \hat{p}_3) \] (3.10)
Mass conservation ensures that \( m_{in} = 3m_{out}/2 \), and one again finds,
\[ \hat{p}_1 \cdot \hat{p}_3 + 1 = 0 \]
which localizes \( \hat{p}_1 \rightarrow \hat{p}_3 \), leading us to a single contact term in \( AdS_3 \).

\textit{n-point ladder diagrams}

The above analysis can be straightforwardly generalized to the scattering of \( n \)–particles for the half-ladder diagram shown at fig.(3). Let us start considering an even number of particles such as, half of them are incoming with identical mass \( m_{in} \) and the rest are out-going with also the same mass \( m_{out} \), i.e \( m_{in} = m_{out} \). Let the incoming particles to be adjacent. As before, mass conservation is imposed on each vertex. Momentum conservation allow us to fix one momentum, namely \( p_n \). Following the order of the particle labels as in figure 3, the analysis is quite simple. The delta function singularity coming from the first propagator \( \delta ((p_1 + p_2)^2 + 4m_o^2) \), forces \( \hat{p}_1 \rightarrow \hat{p}_2 \) as we have shown before. Then the singularity from the next vertex, namely,
\[ \delta ((p_1 + p_2 + p_3)^2 + 9m_o^2) = \delta ((2p_1 + p_3)^2 + 9m_o^2) \]
lead us to \( \delta(\hat{p}_1 \cdot \hat{p}_3 + 1) \), or \( \hat{p}_1 \rightarrow \hat{p}_3 \). Continuing in this order, the singularity corresponding to the propagator at the right hand side of the vertex attached to \( p_r \) with
$r < n/2$ is given by,

$$
\delta \left( \left( \sum_{j=1}^{r} p_j \right)^2 + \left( \sum_{j=1}^{r} m_0 \right)^2 \right) = \delta \left( \left( (r - 1)p_1 + p_r \right)^2 + (r m_0)^2 \right),
$$

(3.11)

which once again lead us to $\hat{p}_1 \rightarrow \hat{p}_r$. We keep going until we reach the last propagator at the final vertex and we use it to localize the leg $p_{n-2}$ along with the previous localized $n - 3$ particles at the right of it \footnote{We can as well keep going until reach half of the diagram at leg $n/2$ and then perform the same analysis from right-to left until we reach the leg corresponding to particle $\frac{n}{2} + 2$, in any case, we will arrive to the same conclusion.}. Finally, the condition (2.9) plus mass conservation, lead us to

$$
p_n^2 = -m_0^2 = \left( \sum_{j=1}^{n/2} p_j - \sum_{j=\frac{n}{2}+1}^{n-1} p_j \right)^2 = \left( \frac{n}{2}p_1 - \left( \frac{n}{2} - 2 \right) p_1 - p_{n-1} \right)^2,
$$

(3.12)

or equivalently

$$
\hat{p}_1 \cdot \hat{p}_{n-1} + 1 = 0,
$$

(3.13)

which implies $\hat{p}_{n-1} \rightarrow \hat{p}_1$ and therefore we end up again with a single contact term in $AdS_3$.

Now considering an odd-number of particles, with $(n - 1)/2$ incoming and $(n + 1)/2$ outgoing. Mass conservation tell us now

$$
m_{in} = \frac{n + 1}{n - 1} m_{out}
$$

(3.14)

we perform a similar procedure as before, starting from left-to-right until we reach the leg $(n - 1)/2$ and then the other way around until we reach the leg $\frac{n+1}{2} + 1$. So finally the condition (2.9), lead us to

$$
p_n^2 = \left( \sum_{j=1}^{(n-1)/2} p_j - \sum_{j=\frac{n}{2}+1}^{n-1} p_j \right)^2 = \left( \frac{n-1}{2} \right)^2 (p_1 - p_{n-1})^2,
$$

(3.15)

which after some algebra is equivalent to

$$
\hat{p}_1 \cdot \hat{p}_{n-1} + 1 = 0,
$$

(3.16)

where we have used mass conservation condition (3.14). Let us recall that on the final transform we are retaining the leading order in an $\epsilon$ expansion, and as we saw above,
every delta function forcing a localization will contribute to a $\epsilon^{1/2}$ scaling, therefore, the Witten contact diagram coming from the $n-$particle half-ladder appear at order $\epsilon^{(n-3)/2}$, coming from $(n - 3)$ propagators.\footnote{\textsuperscript{4}It is worth to recall that we are using $\epsilon$ to slightly get off from the singularity since on the singularity the transformation actually vanishes, so the role of the $\epsilon$ is to keep track of the order of the given zero.}

\textit{Benzene diagram}

Now we consider the Benzene-type diagram represented in figure 4 for six-particles scattering with $p_1 + p_2 = p_3 + p_4 + p_5 + p_6$. Looking at the following pole singularity,

$$\prod_{j=\{1,3,5\}} \delta ((p_j + p_{j+1})^2 + 4m_j^2)$$

where $m_1 = m_2 = m_{in}$ and $m_3 = m_4 = m_5 = m_6 = m_{out}$. It localizes $\hat{p}_j \rightarrow \hat{p}_{j+1}$ for $j = \{1,3,5\}$. Momentum conservation can be used to solve for $p_6$ and the additional condition (2.9) lead us to,

$$4p_6^2 = -4m_{out}^2 = [2p_1 - 2p_3]^2 = 4(-m_{in}^2 - m_{out}^2 - 2m_{out}m_{in}\hat{p}_1 \cdot \hat{p}_3)$$

by mass conservation we have that $m_{in} = 2m_{out}$, so, replacing it in the above equation we end up with

$$\hat{p}_1 \cdot \hat{p}_3 + 1 = 0$$

which is again the localization condition to $\hat{p}_1 \rightarrow \hat{p}_3$.

\textit{One-loop on-shell graphs}

Finally, let us consider the one-loop on-shell triangle and box graphs displayed in figure 5. For the triangle, after imposing mass-conservation on every vertex, we can

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{benzene.png}
\caption{Benzene-type diagram}
\end{figure}
write the amplitude as (notice that $m_{out} = 2m_{in}$)

$$A(p_i) = \int dp_\ell \delta (p_\ell^2 + m_\ell^2) \delta ( (p_1 + p_\ell)^2 + (m_{in} + m_\ell)^2 ) \delta ( (p_2 - p_\ell)^2 + (m_\ell - m_{in})^2 )$$

The first delta function constraint the loop momentum to be on-shell, and hence the integration $dp_\ell$ above corresponds to a new $AdS_3$ point. After some algebra we can see that the second delta and third delta functions enforce the localization of $\hat{p}_1 \rightarrow \hat{p}_\ell$ and $\hat{p}_2 \rightarrow \hat{p}_\ell$ respectively, and momentum conservation fix $p_3 = p_1 + p_2 = 2m_{in} \hat{p}_\ell$.

The remaining condition from momentum conservation,

$$p_3^2 = -4m_{in}^2 = (p_1 + p_2)^2 = -4m_{in}^2 ,$$

it is satisfied by mass conservation and therefore does not impose any additional constraint.

The box is slightly more involved, but the argument applies similarly. The on-shell loop momenta condition and vertex mass conservation (taking all the external masses to be the same magnitude), allow us to write the amplitude as,

$$A(p_i) = \int dp_\ell \delta (p_\ell^2 + m_\ell^2) \delta ( (p_1 + p_\ell)^2 + (m_{in} + m_\ell)^2 ) \times \delta ( (p_1 + p_\ell - p_3)^2 - m_\ell^2 ) \delta ( (p_2 - p_\ell)^2 + (m_\ell - m_{in})^2 ) ,$$

as before, the first delta function forces the loop momenta to be on-shell, the second on the first line and the second on the second line, enforces $\hat{p}_1 \rightarrow \hat{p}_\ell$ and $\hat{p}_2 \rightarrow \hat{p}_\ell$ respectively. The first one on the second line can be carried out similarly leading us to,

$$\frac{1}{(2m_{in}^2 + 2m_{in}m_\ell)} \delta (1 + \hat{p}_\ell \cdot \hat{p}_3) ,$$

hence implying $\hat{p}_3 \rightarrow \hat{p}_\ell$. The remaining momentum conservation delta function reads then,

$$p_3^2 = -m_{out}^2 = (p_1 + p_2 - p_3)^2 = p_1^2 = -m_{in}^2 ,$$

\textbf{Figure 5.} One-shell one-loop diagrams
which is trivially satisfied by mass conservation. Notice that we can extend
this procedure for general polygons at one-loop, i.e. beyond four-points unlike as for uni-
tarity cuts, since the delta functions are constraining the external data instead of the
loop-momentum.

In summary, imposing the mass conservation condition coming from momentum
conservation in five dimensions, plus momentum conservation in 4d lead us to a Witten
contact term in \( \text{AdS}_3 \) for the massive exchanges. At this point, is worth to make a couple
of remarks. Contact Witten diagrams are also known as \( D \)-functions, as has been
defined in [23], and they can be elegantly represented in terms of Mellin amplitudes
[16]. So far we have considered all incoming particles to be adjacent and similarly for
all the outgoing particles. This has been done in order to guarantee that we always
have massive propagators, in the case where there are alternate incoming and outgoing
particles, it is possible to produce massless propagators due to mass conservation. This
case deserves special attention and is going to be discussed later on this note.

4 A 2d–2d duality

We see that in the above construction, mass conservation plays an essential role. While
this appears unnatural in a four-dimensional point of view, it arises naturally if we
consider it as a five-dimensional massless \( \phi^3 \) theory, where the fifth momenta can be
identified as the four-dimensional mass: \( \vec{p}^{(5)} = (p, m) \). Momentum conservation in
fifth dimension will then ensure that all masses are conserved at each vertex. At this
point, this appears to be mere cosmetics. However, by reinterpreting the kinematics as
massless, we can utilise the Cachazo, He, and Yuan (CHY) representation to reproduce
the factorization singularity.

Recall that for \( \phi^3 \) theory, we can reconstruct its S-matrix by integrating over the
moduli space of punctured points on the Riemann sphere [17], where the integrand is
given by double cycles

\[
m^{\phi^3}(\alpha, \beta) = \int \prod_{a=1}^{n} \frac{d\sigma_a}{\text{SL}(2, \mathbb{C})} \frac{1}{(\sigma_{\alpha_1,\alpha_2} \cdots \sigma_{\alpha_n,\alpha_1})(\sigma_{\beta_1,\beta_2} \cdots \sigma_{\beta_n,\beta_1})} \prod_i \delta \left( \sum_{j \neq i} \frac{s_{ij}}{\sigma_{ij}} \right), \tag{4.1}
\]

where \((\alpha, \beta)\) are any couple of permutations over the set of labels \(\{1, 2, \cdots, n\}\), \(\sigma_i\) are
the positions of the punctures on the sphere, \(\sigma_{ij} = \sigma_i - \sigma_j\) and \(s_{ij} = (p_i + p_j)^2\). By
relabeling the puncture locations on the sphere as,

\[
\sigma_a = \frac{\tau}{u_a}, \quad a \in L, \quad \sigma_a = \frac{v_a}{r}, \quad a \in R, \tag{4.2}
\]
where $L$ and $R$ define the subset of index $L = \{1, \cdots, n_L\}$ and $R = \{n_L + 1, \cdots, n\}$, it can be shown that the product of scattering equation delta functions factorises and the integrand in (4.1) takes the form:

$$-(u_{1,2}u_{1,2}v_{n-1,n}v_{n-1,v_n}) \frac{d\tau^2}{\tau^2} m^{\phi_3}(\alpha_L, L | \beta_L, L)(\{u\}) \delta \left(p_R^2 - \tau^2 F\right) m^{\phi_3}(\alpha_R, R | \beta_R, R)(\{v\}).$$

Here $F$ is some $\tau$ independent polynomial, and

$$m^{\phi_3}(\alpha_L, L | \beta_L, L)(\{u\}) = \int \prod_{a=3}^{n_L} \frac{du_a}{SL(2, \mathbb{C})} \frac{1}{(u_1 u_{\alpha_L(2)} \cdots u_{\alpha_L(n_L-1)} u_{\alpha_L(n_L)} u_{\alpha_L(n_L)})} \times \frac{1}{(u_1 u_{\beta_L(2)} \cdots u_{\beta_L(n_L-1)} u_{\beta_L(n_L)})} \prod_{a \in L \setminus \{1,2\}} \delta(f^L_a) \quad (4.4)$$

where $(\alpha_L, \beta_L)$ are any couple of permutations over the set of labels $L \setminus \{1\}$ and $f^L_a$ are the scattering equations for the subset of punctures in $L$,

$$f^L_a = \sum_{b \in L \setminus a} \frac{s_{a,b}}{u_{a,b}}. \quad (4.5)$$

with a similar expression for $m^{\phi_3}(\alpha_R, R | \beta_R, R)(\{v\})$ but for the subset of elements $R$ and variables $v_a$.

On the factorization kinematics $p_R^2 = 0$, the delta function $\delta(p_R^2 - \tau^2 F)$ localises $\tau^2$ to vanish and hence correspond to the pinch limit of the Riemann sphere. Now we can easily turn this around and consider the pinch limit directly in the CHY integrand, which via the scattering equations will enforce the kinematics be in the factorization limit. For example, at four-points, one would have:

$$\delta(s_{12}) = \int \frac{d\tau^2}{\tau^2} \delta(p_R^2 - \tau^2 F) \tau \delta(\tau). \quad (4.6)$$

At higher points, one can reproduce all factorization singularities by successively pinching the Riemann sphere, i.e. for $n$-factorization singularities one simply performs a change of variable to make $n$-pinch parameters manifest, and insert $n$-factors of delta functions that localise these $n$ parameters. The result of the integration would simply a product of $n$ factorization delta functions.

Finally we parameterise the five dimensional momenta using our AdS coordinates, where there is freedom in the embedding of four-dimensions. Depending on the embedding, the internal moment can be massive or massless. Will will discuss the massless case in the next section. For the massive case, one finally establishes the following
correspondence:

\[ 2d \ CFT \leftrightarrow \prod_{i=1}^{n} \left( \int d^2 z_i \frac{dy_i}{y_i^3} G_\Delta(y_i, z_i, \bar{z}_i; w, \bar{w}) \right) \int d^{2n} \sigma_i \ \text{CHY} \ \left( \prod_j \tau_j \delta(\tau_j) \right) . \] (4.7)

where \( \tau_j \) are the moduli of the Riemann sphere whose zero limit correspond to pinched limit.

5 Massless singularities

In reducing the five-dimensional representation to four-dimensions, one also naturally recover cases where the propagators are massless. For massless singularities the AdS points are no-longer localised to a single point. Rather we recover configurations with multiple bulk points similar to the factorization Witten diagrams. We will demonstrate this with the previous four-point example.

Consider again the kinematics of the external lines in fig1 but with factorization in the \( t \) -channel. Now mass conservation implies that the internal propagator is massless, and hence

\[ \delta((\hat{p}_1 - \hat{p}_4)^2) = \frac{1}{2} \delta(\hat{p}_1 \cdot \hat{p}_4 + 1) \] (5.1)

The factorization singularity now identifies \( \hat{p}_1 \) to \( \hat{p}_4 \) and \( \hat{p}_2 \) to \( \hat{p}_3 \). However as momentum conservation is simply \( \hat{p}_1 + \hat{p}_2 - \hat{p}_3 - \hat{p}_4 = 0 \), it is trivially satisfied in this limit, and hence one ends up with two free points in AdS\(_3\), i.e. putting this together we got

\[ \tilde{A}_4(w_i, \bar{w}_i) \sim \frac{\lambda^2 \epsilon^{1/2}}{m^4} \int_0^{\infty} \frac{dy_1}{y_1^3} \int d^2 z_1 \int_0^{\infty} \frac{dy_3}{y_3^3} \int d^2 z_3 \prod_{i=1}^{4} G_\Delta_i(y_i, z_i, \bar{z}_i; w_i, \bar{w}_i) \delta(0) \tilde{A}(p) , \] (5.2)

where \( \tilde{A}(p) \) represents the residue of the flat space amplitude around the singularity in the \( t \)– channel, whose form now depends on the spin of the internal particle. This expression is a factorization Witten-like diagram, as shown in fig.(6).

The residue of the massless singularity is now sensitive to the spin of the exchanged particle. For spin-0 the residue is simply 1, while for spin-1 one has \( \hat{p}_3 \cdot \hat{p}_1 \). These do not appear to correspond to any bulk to bulk propagators in AdS which are non-rational functions of \( \hat{p}_3 \cdot \hat{p}_1 \), and thus cannot correspond to any single operator exchange. We will further comment on this in the conclusions.
Figure 6. The factorization singularity of u-channel diagram leads to a factorization Witten-like diagram in AdS

6 Conclusions and outlook

In this note, we have considered the correspondence between four-dimensional S-matrix singularities and two dimensional CFT. This is in the context of the Pasterski, Shao and Strominger construction, where one replaces the usual S-matrix plane waves with conformal primary wave function. The kinematic space is conveniently parametrized by AdS\(_3\) coordinates, and thus correspond to \(n\)-AdS bulk points. The focus on the factorization singularities is then the first step to understand the implications of the dynamic properties of the S-matrix.

We show that for massive scalar theories, if mass conservation is implemented at each cubic vertex, the massive factorization singularities along with over all momentum conservation, will localizes the \(n\)-copy of AdS bulk points to a single point thus forming a contact Witten diagram. Note that in this case, the different spin exchanges simply degenerates to an overall normalisation constant. For massless singularities, the result is a factorization Witten diagram with polynomial residues whose degree depends on the spin of the exchange particle, which are rational polynomials. However, these do not correspond to the usual bulk to bulk massless spin-1 exchanges which would require non-rational functions of \(\hat{p}_i \cdot \hat{p}_j\). One possibility is that the dimension of the exchange operator is continuous, and one is required to perform an integral along some contour for the dimensions. Another would be that this contributions no longer corresponds to single operator exchange. Finally, in this note we only consider massive scalars. For massive higher-spin amplitudes, the S-matrix will transform covariantly under the massive SU(2) little group. The meaning of this SU(2) on the CFT is not clear to us, and we leave these interesting questions to future studies.

The fact that mass must be conserved at each vertex indicates that the constraints are more naturally embedded in massless constraint of five-dimensional kinematics, where the conservation is simply the extra dimensional Poincare symmetry. This identification also allows us to use the CHY representation for massless S-matrix, and reinterpret the factorization singularities as pinch limits of the Reimann sphere. This
estabhlishes an interesting 2d/2d correspondence.

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