SOME TOPICS IN COALGEBRA CALCULUS

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Abstract

We study a coderivation from a cobimodule into a coalgebra. Vector cofields are defined by the action of a codual bicomodule on a coalgebra. This action is induced by a codifferential. A construction of a codual object in the category of bicomodules over a given coalgebra has been proposed. Various notions of duality have been analysed in this context.

1 Introduction

It is well known, that the notion of coalgebra and comodule is dual to the concept of algebra and module: the dual vector space to a linear finite dimensional algebra inherits a canonical coalgebra structure (see e.g. [1, 6]). Similarly, dualizing the concept of derivation one can obtain the notion of coderivation, etc.

A general vector field formalism for a first order differential calculus over a given algebra $A$ has been recently proposed in [2, 3]. Our aim in the present note is to introduce a dual concept of vector cofields for a codifferential calculus over a linear coalgebra $C$.

To clarify the above idea, one should notice that there are several notions of duality. E.g. the already mentioned duality between algebras and coalgebras, modules and comodules holds in the category of (finite dimensional) vector spaces over a fixed field $k$ [4, 5]. Also one-forms and vector fields are dual concepts, but this time duality holds in the category of bimodules over the algebra $A$ [4, 5]. Similarly, a major step in our program relies upon introducing a notion of duality in the category of bicomodules over the linear coalgebra $C$. The present letter has a preliminary character. More detailed study of this subject will be presented elsewhere. Some proofs are left to the reader.

As explained above, we shall work mainly in the category $\mathcal{V}ect_k$ of finite dimensional vector $k$-spaces. However, the authors believe that all results can be extended after adding suitable topological conditions or/and to the case of rational comodules (cf. [4, 5]).

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Throughout this letter we let $k$ be a field. All objects considered here are finite dimensional vector spaces over the field $k$ and all maps are $k$-linear maps. Tensor products are also assumed over $k$. Given vector spaces $U$ and $W$, we denote by $\text{Hom}(U, W)$ the space of linear maps from $U$ to $W$. For any space $U$, let us denote by $U^\bullet = \text{Hom}(U, k)$ the linear dual of $U$.

Due to finite dimensionality the following canonical isomorphisms:

\[ V \otimes W \cong W \otimes V, \quad \text{Hom}(V, W) \cong V^\bullet \otimes W, \quad \text{Hom}(V, W \otimes U) \cong \text{Hom}(V, W) \otimes U \]

will be utilized without special mention.

For the same reason, we found it also appropriate to adopt the covariant index notation, together with the Einstein summation convention over repeated (contravariant) and down (covariant) indices, borrowed from tensor analysis on manifolds and General Relativity. For example, an element $u \in U$ will be written as $u = u^i e_i$, where $\{e_k\}_{k=1}^{\dim U}$ denotes some basis in $U$. Thus our results, although formally formulated in a basis-dependent way, are in fact basis-independent.

Our main references to coalgebras and comodules are [1, 6]. For notational convenience, we fix some coassociative and counital coalgebra $\mathcal{C} \equiv (\mathcal{C}, \triangle, \varepsilon)$ in $\text{Vect}_k$. All comodules will be considered over the coalgebra $\mathcal{C}$.

\section{Bicomodules and their coduals}

If $(U, \triangle^L)$ is a left comodule with a (finite dimensional) carrier space $U$ then in an arbitrary basis $\{e_k\}$ for $U$ one can set

\[ \triangle^L(e_k) = L^i_k \otimes e_i \quad (\text{summation convention }!), \]

where $\mathcal{C}$-valued matrix elements $L^i_k$ have to satisfy the following relations (see e.g. [1]):

\[ \triangle(L^i_k) = L^m_k \otimes L^i_m, \quad \varepsilon(L^i_k) = \delta^i_k. \]  

Similarly, a right comodule structure $\triangle^R$ on $U$ can be encoded by

\[ \triangle^R(e_k) = e_i \otimes R^i_k, \]

with

\[ \triangle(R^i_k) = R^i_m \otimes R^m_k, \quad \varepsilon(R^i_k) = \delta^i_k. \]

The left comodule structure $\triangle^L$ on $U$ induces by the transpose coaction a canonical right comodule structure $\triangle^L_\bullet$ on $U^\bullet$. Taking the dual basis $\{e^k\}$ in $U^\bullet$ one defines:

\[ \triangle^L_\bullet(e^k) \doteq e^m \otimes L^k_m = (\text{id} \otimes e^k) \circ \triangle^L, \]

as equalities in $\text{Hom}(U, \mathcal{C})$. Similarly, $\triangle^R$ generates

\[ \triangle^R_\bullet(e^k) \doteq R^i_k \otimes e^m = (e^k \otimes \text{id}) \circ \triangle^R \]

a canonical left comodule structure on $U^\bullet$.

\textbf{Remark 2.1.} It should, however, be noticed that the interpretation of $\triangle^L_\bullet$ as a right
comultiplication in $U^\bullet$ requires the identification $\text{Hom}(U, C) \cong U^\bullet \otimes C$. Similarly, to see $\Delta_R^\bullet$ as a left comultiplication one has to identify $\text{Hom}(U, C)$ with $C \otimes U^\bullet$ instead.

If $(U, \Delta_L, \Delta_R)$ is a bicomodule, then the commutation relation $(\text{id} \otimes \Delta_R) \circ \Delta_L = (\Delta_L \otimes \text{id}) \circ \Delta_R$ gives rise to the following identity in $C \otimes C$

$$L_i^i \otimes R_i^m = L_j^m \otimes R_i^j.$$  

(7)

In the above situation $(U^\bullet, \Delta_L^\bullet, \Delta_R^\bullet)$ becomes automatically a bicomodule too.

**Example 2.2.** Choosing an arbitrary basis $\{c_\alpha\}$ in the vector space $C$, one can define

$$\Delta(c_\alpha) = \Omega^\beta_\alpha c_\beta \otimes c_\gamma \cong l_\alpha^i \otimes c_\gamma \cong c_\beta \otimes r_\alpha^\beta,$$

(8)

where $\Omega^\beta_\alpha \in \mathbb{k}$ are structure constants for $C$; $l_\alpha^i \triangleq \Omega^\beta_\alpha c_\beta \in C$ and $r_\alpha^\beta \triangleq \Omega^\beta_\gamma c_\gamma \in C$ do define a canonical bicomodule structure $(C, \Delta_i, \Delta_r)$ with $\Delta \equiv \Delta_i \equiv \Delta_r$ being mappings from $C$ into $C \otimes C$. This means, that its dual $(C^\bullet, \Delta^\bullet_L, \Delta^\bullet_R)$ is a bicomodule too (but not a coalgebra!). Of course, $C^\bullet$ possesses a canonical (unital, associative) algebra structure with a multiplication table given by the same structure constants:

$$m(c^\alpha \otimes c^\beta) \equiv c^\alpha \cdot c^\beta \equiv \Omega^\gamma_\alpha c^\gamma.$$  

(9)

The counit $\varepsilon \in C^\bullet$ is a unit for this algebraic structure and

$$\Delta^\bullet_L(\varepsilon) = \Delta^\bullet_R(\varepsilon) = c^\alpha \otimes c_\alpha = \text{id}_C.$$  

(10)

Notice, as a side remark, that algebra and comodule structures on $C^\bullet$ are related by two compatibility conditions:

$$\Delta^\bullet_L \circ m = (\text{id} \otimes m) \circ (\Delta^\bullet_L \otimes \text{id}) , \quad \Delta^\bullet_R \circ m = (m \otimes \text{id}) \circ (\text{id} \otimes \Delta^\bullet_R),$$

(11)

i.e. $C^\bullet$ carries a double dimodule structure in the terminology of $[5]$.

We recall a known fact from the theory of modules: if $A$ is a linear algebra and $M, N$ are $A$-bimodules then $\text{Hom}(M, N)$ is a quadruple $A$-module. It means that $\text{Hom}(M, N)$ has an $A$-module structure in four different ways: two induced by the multiplications in $M$ and another two induced by the multiplications in $N$. Moreover, these different $A$-module structures are pair-wise commuting.

One needs a comodule analogue of this. Let us begin with a simple example.

**Example 2.3.** Given two comodules $(U, \Delta_L)$ and $(W, \Delta_R)$. The vector space $U \otimes W \cong W \otimes U$ can be endowed with an obvious bicomodule structure by

$$\Delta^U_{\otimes W} = \Delta_L \otimes \text{id}_W, \quad \Delta^U_{R \otimes W} = \text{id}_U \otimes \Delta_R.$$  

(12)

Let now $(U, \Delta_L, \Delta_R)$ and $(W, \Delta_L, \Delta_R)$ be two bicomodules. In the vector space $\text{Hom}(W, U)$ one can introduce four different comodule structures induced correspondingly by $\Delta_R, \Delta_L, \Delta^\bullet_L, \Delta^\bullet_R$:  

$$\Delta^1_L(\Phi) = (\Phi \otimes \text{id}) \circ \Delta_R \in \text{Hom}(W, U \otimes C), \quad \Delta^1_R(\Phi) = (\text{id} \otimes \Phi) \circ \Delta_L \in \text{Hom}(W, C \otimes U);$$  

(13)

$$\Delta^2_L(\Phi) = \Delta_L \circ \Phi \in \text{Hom}(W, C \otimes U), \quad \Delta^2_R(\Phi) = \Delta_R \circ \Phi \in \text{Hom}(W, U \otimes C).$$  

(14)
**Remark 2.4.** Again, in order to ensure a proper interpretation for the corresponding comultiplications, one has to identify \( \text{Hom}(W, U \otimes C) \cong C \otimes \text{Hom}(W, U) \) and \( \text{Hom}(W, C \otimes U) \cong \text{Hom}(W, U) \otimes C \) in (13). On the contrary, (14) requires that \( \text{Hom}(W, C \otimes U) \cong C \otimes \text{Hom}(W, U) \) and \( \text{Hom}(W, U \otimes C) \cong \text{Hom}(W, U) \otimes C \) instead.

According to the Example 2.3, after making the identification \( \text{Hom}(W, U) \cong W \otimes W^* \) one can also write:

\[
\begin{align*}
\Delta^1_L &= \tilde{\Delta}_R^* \otimes \text{id}_U, \\
\Delta^1_R &= \text{id}_U \otimes \tilde{\Delta}_L^*, \\
\Delta^2_L &= \Delta_L \otimes \text{id}_{W^*}, \\
\Delta^2_R &= \text{id}_{W^*} \otimes \Delta_R.
\end{align*}
\]

This argue in favour of the following

**Proposition 2.5.** Above four comodule structures are pairwise commuting, i.e. \((\text{Hom}(W, U), \Delta^1_L, \Delta^1_R, \Delta^2_L, \Delta^2_R)\) is a quadruple-comodule.

**Remark 2.6.** In particular, \( \text{End}C \) is a quadruple-comodule too. In what follows, we shall use \((\text{End}C, \Delta^1_L, \Delta^1_R)\) as a default bicomodule structure on \( \text{End}C \). Thus, for \( \xi \in \text{End}C \)

\[
\Delta^1_L(\xi) = (\xi \otimes \text{id}) \circ \Delta, \\
\Delta^1_R(\xi) = (\text{id} \otimes \xi) \circ \Delta.
\]

If \( \Phi \in \text{Hom}(W, U) \) is a left comodule map from \((W, \Delta_L)\) into \((U, \Delta_L)\) then

\[
\Delta_L \circ \Phi = (\text{id} \otimes \Phi) \circ \tilde{\Delta}_L.
\]

In our notation it reads \( \Delta^2_L(\Phi) = \Delta^1_R(\Phi) \) in \( \text{Hom}(W, C \otimes U) \). Select the subspace \( \text{Com}^{(-C)}(W, U) \subset \text{Hom}(W, U) \) of all left comodule maps from \( W \) into \( U \). It turns out that this subspace becomes a subbicomodule with respect to \((\Delta^1_L, \Delta^2_R)\). Similarly, the space of right comodule maps \( \text{Com}^{C}(W, U), \Delta^2_L, \Delta^1_R) \) is also a bicomodule.

With all these and keeping in mind an anlogous situation in the category of bimodules over an algebra, described in \[3, 3\], we are ready to introduce a codual object in the category \( \mathcal{C} \) of \( C \)-bicomodules. Let \((U, \Delta_L, \Delta_R)\) be a bicomodule.

**Definition 2.7.** A bicomodule \((U \equiv \text{Com}^{(-C)}(C, U), \Delta^1_L, \Delta^2_R)\) is called a left \( C \)-codual of \( U \). In a similar manner, one defines \((U^\dagger \equiv \text{Com}^{C}(C, U), \Delta^2_L, \Delta^1_R)\) to be a right \( C \)-codual of \( U \).

Consider a generic element \( X \in U^\dagger \). Since \( X : C \to U \) is a left module map thus from (18)

\[
\Delta_L \circ X = (\text{id} \otimes X) \circ \Delta.
\]

Due to (13,14) the left and the right comultiplications on \( X \) reads as follows

\[
\Delta^1_L(X) = (X \otimes \text{id}) \circ \Delta, \\
\Delta^2_R(X) = \Delta_R \circ X
\]

modulo the identifications indicated in Remark 2.4.

**Proposition 2.8.** For a counital coassociative coalgebra \( C \) the bicomodules \( \mathcal{C} \) and \( C^* \) are isomorphic.

Instead of presenting the proof, we only notice that the required isomorphism \( \mathcal{C} \cong C^* \) can be realized by a mapping \( \varepsilon : \mathcal{C} \ni X \mapsto X^\varepsilon = \varepsilon \circ X \in C^* \) induced by the counit \( \varepsilon \) (see Example 2.1 in this context).
3 First order codifferential calculus

According to the general scheme \[1\], the definition of a codifferential calculus can be obtained by systematical reversing of all arrows in the diagrams defining the first order differential calculus on the algebra A. One has also to replace the words: algebra, bimodule and multiplication by its dual counterparts: coalgebra, bicomodule and comultiplication, etc. An interesting notion of braided Hopf algebra biderivation has been introduced in \[1\].

**Definition 3.1.** A first order codifferential calculus on a coalgebra \( C \) (FOCC in short) consists of a \( C \)-bicomodule \((U, \Delta_L, \Delta_R)\) and a \( \mathbb{K} \)-linear map (called a coderivation) \( \delta \in \text{Hom}(U, C) \) such that

\[
\Delta \circ \delta = (\text{id} \otimes \delta) \circ \Delta_L + (\delta \otimes \text{id}) \circ \Delta_R.
\]

Writing down in the coordinate language \( \delta = e^k \otimes \delta_k \) with \( \delta_k \in C \) uniquely defined for a given basis \( \{e_k\} \) in \( U \) one deduces from (21)

\[
\Delta(\delta_k) = L^i_k \otimes \delta_i + \delta_i \otimes R^i_k
\]

as the restriction on the coefficients \( \delta_k \). In particular, we can consider a coderivation \( \xi \in \text{Coder}(C) \) from the coalgebra to itself. That is

\[
\Delta \circ \xi = (\text{id} \otimes \xi + \xi \otimes \text{id}) \circ \Delta.
\]

Let us fix some FOCC \( \delta : U \to C \). In analogy with the algebra case, the elements of \( U \) can be called 1-form cofields. Let us take the left codual \( ^\mathbb{L}U \) of \( U \): its elements can be called (left) vector cofields. Now, with any vector cofield \( X = X^i \otimes e_i \in ^\mathbb{L}U \) one can associate an endomorphism \( X^\delta \in \text{End} C \)

\[
X^\delta = \delta \circ X = X^i \otimes \delta_i.
\]

This becomes the dual counterpart of the famous Cartan formula (cf. \[2\], \[3\]) Our task here is to investigate properties of the map \( \tilde{\delta} : ^\mathbb{L}U \ni X \mapsto X^\delta \in \text{End} C \). This is done by the following

**Theorem 3.2.** The map \( \tilde{\delta} : ^\mathbb{L}U \to \text{End} C \) is a left comodule map. Moreover, for any \( X \in ^\mathbb{L}U \) one has:

\[
\Delta \circ X^\delta = (\text{id} \otimes X^\delta) \circ \Delta + \Delta_R(X)^{(\delta \otimes \text{id})}.
\]

**Proof:** The first claim requires the equality

\[
\Delta^{\text{End} C} \circ \tilde{\delta} = (\text{id} \otimes \tilde{\delta}) \circ \Delta^U_L
\]

which follows easily from (17) and (20). To show the second, compose both sides of (21) with \( X \). Thus

\[
\Delta \circ X^\delta = (\text{id} \otimes \delta) \circ \Delta_L \circ X + (\delta \otimes \text{id}) \circ \Delta_R \circ X.
\]

From (19) one calculates \((\text{id} \otimes \delta) \circ \Delta_L \circ X = (\text{id} \otimes \delta) \circ (\text{id} \otimes X) \circ \Delta = (\text{id} \otimes X^\delta) \circ \Delta \).

Applying (20) to the second term gives \((\delta \otimes \text{id}) \circ \Delta_R \circ X = (\delta \otimes \text{id}) \circ \Delta_R^2(X)\). This, obviously can be rewritten as \( \Delta_R^2(X)^{(\delta \otimes \text{id})} \). The proof is done. \( \square \)
Because of the second term on the RHS in (25) and comparing with (23), one can say that $X^\delta$ is a deformed coderivation (see also [2, 3] for a similar situation concerning vector fields). We are going to explain how to get an undeformed co-Leibniz rule (23).

Assume now that the coalgebra $\mathcal{C}$ is cocommutative. There exists a well-known one-to-one correspondence between comodules and cobimodules having the same left and right comultiplication, i.e. $\Delta_R = SW \circ \Delta_L$, where $SW : \mathcal{C} \otimes U \to U \otimes \mathcal{C}$ denotes the canonical switch. In this case, one can recalculate the second term in (26):

$$(\delta \otimes \text{id}) \circ \Delta_R \circ X = (\delta \otimes \text{id}) \circ SW \circ \Delta_L \circ X = (\delta \otimes \text{id}) \circ SW \circ (\text{id} \otimes X) \circ \Delta = (X^\delta \otimes \text{id}) \circ \Delta.$$

This proves

**Theorem 3.3.** Let $U$ be a comodule and let $\delta : U \to \mathcal{C}$ be a FOCC over a cocommutative coalgebra $\mathcal{C}$. Then for each $X \in \mathcal{U}$ one has $X^\delta \in \text{Coder}(\mathcal{C})$.

Let us finish by posing an unsolved question: Does there exist a canonical (Kähler type) codifferential calculus $\delta_0 : U_0 \to \mathcal{C}$ on an arbitrary cocommutative coalgebra $\mathcal{C}$ such that $U_0 = \text{Coder}(\mathcal{C})$?

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**References**

[1] E. Abe: *Hopf Algebras*. Cambridge University Press, Cambridge, 1980.

[2] A. Borowiec: Czech. J. Phys. 46 (1996) 1197 [q-alg/9609011]

[3] A. Borowiec: Czech. J. Phys. 47 (1997) 1093 [q-alg/9609011]

[4] Z. Oziewicz, E. Paal and J. Rózański: Acta Physica Polonica B 26 (1995) 1253.

[5] B. Pareigis: *On Lie Algebras in the Category of Yetter-Drinfeld modules*, [q-alg/9612023]

[6] M.E. Sweedler: *Hopf Algebras*. Benjamin, New York, 1969.