THE ADDITIVE STRUCTURE OF CARTESIAN PRODUCTS SPANNING FEW DISTINCT DISTANCES

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Received July 12, 2016
Online First February 13, 2017

Guth and Katz proved that any point set \( P \) in the plane determines \( \Omega(|P|/\log |P|) \) distinct distances. We show that when near to this lower bound, a point set \( P \) of the form \( A \times A \) must satisfy \( |A-A| \ll |A|^{2-1/8} \).

1. Introduction

If \( P \) is a set of \( N \) points in the plane, let \( \Delta(P) \) denote the set of (squared) distances spanned by \( P \), that is

\[
\Delta(P) = \{(p_1 - q_1)^2 + (p_2 - q_2)^2 : (p_1, p_2), (q_1, q_2) \in P\}.
\]

Erdős famously conjectured [3] that any set of points needs to determine at least a \( \Omega(N/\sqrt{\log N}) \) distinct numbers as distances. This lower bound occurs when the points are arranged in a square grid in the integer lattice, for instance. In a landmark paper [5], Guth and Katz very nearly established Erdős’ conjecture, by proving the lower bound:

**Theorem 1 (Guth-Katz).** If \( P \) is a set of \( N \) points in the plane, then \( |\Delta(P)| \gg \frac{N}{\log N} \).

Though the problem of estimating the number of distinct distances is now almost resolved, what remains to be shown is a characterization of those point sets for which the lower bound of Theorem 1 is sharp. In this note, we think of an extremal configuration as a set of points \( P \) spanning \( O(|P|^{1+\epsilon}) \)
distances. The known examples of such sets all appear to come from sets with algebraic structure - in the grid example above, the points are coming from a lattice. Another example is to take the vertices of a regular \( n \)-gon, or \( n \) equally spaced points on a line. One might ask if this algebraic structure is a necessary feature of such point sets - in the listed examples, a very strong rotational or translational symmetry is present. Such a conjecture was made by Erdős in [4], where he states that all extremal configurations should exhibit a lattice-like structure. Some progress towards such a result is given in [8]. Erdős is pretty vague about what he means by lattice like, but we are going to take a lattice to mean a discrete subgroup of the plane. Here, we will prove a theorem in further support of this fact. Let \( A \) be a set of real numbers such that \( P = A \times A \) determines few distances. Theorem 1 says that there are at least \( c|A|^2/\log |A| \) distinct distances for some positive, absolute constant \( c \). We will show that when near this lower bound, the difference set

\[
D = A - A = \{a_1 - a_2 : a_1, a_2 \in A\}
\]

has to be somewhat small, thus showing that the set \( P \) is to some extent additively structured. It can certainly be argued that the assumption that \( P \) is a cartesian product means that our point set is already lattice-like, but I think that the additive behaviour of \( A \) is essential to be considered truly lattice-like.

**Theorem (Main theorem).** Suppose \( A \) is a finite set of real numbers and let \( \Delta(A \times A) \) be the set of distances spanned by \( A \times A \). Then

\[
|A - A| \ll |\Delta(A \times A)||A|^{-1/8}.
\]

In [4], it was conjectured that in extremal point configurations there should be many points (\( |P|^{1/2} \) is conjectured, but \( |P|^\varepsilon \) is already interesting) on a line or circle. In the case of a cartesian product, there are trivially many points on axis parallel lines, or the main diagonal. However, this theorem also shows that there are non-trivial lines which contain many points.

**Theorem (Rich lines).** Suppose \( A \) is a finite set of real numbers such that \( \Delta(A \times A) = O(|A|^2) \). There is a non-trivial line (in fact many) which contains \( \Omega(|A|^{1/8}) \) points of \( A \times A \).

**Proof.** Since

\[
|A|^2 = \sum_{d \in A - A} \sum_{a,b \in A} \sum_{a-b=d} 1,
\]
one of the inner sums is at least $\Omega(|A|^{1/8})$. Thus, there are $\Omega(|A|^{1/8})$ points $(a, b) \in A \times A$ with $a - b = d$, and this equation defines the desired line.  

From an arithmetic combinatorial point of view, our main theorem ought to be true because the set of distances spanned by $P$ is the set of numbers $\Delta = D^2 + D^2$, where $D^2 = \{d^2 : d \in D\}$. Heuristically, the squaring of an additively structured set (in this case $D$, a difference set) should result in a set (in this case $D^2$) which is not additively structured. The fact that squaring, or in fact any convex function, meddles with the additive properties of a set of numbers is captured in a theorem of Elekes-Nathanson-Ruzsa from [2], later improved by Li and Roche-Newton [6].

**Theorem 2 (Convexity and Sumsets).** Let $S$ be a finite set of real numbers, let $f$ be a strictly convex function and let $\varepsilon > 0$. Then

$$\max\{|S + S|, |f(S) + f(S)|\} \gg |S|^{14/11 - \varepsilon}.$$

In particular,

$$\max\{|S + S|, |S^2 + S^2|\} \gg |S|^{14/11 - \varepsilon}.$$

Viewed in this way, the main theorem is really one in arithmetic combinatorics. Recently, Shkredov has proved similar theorems, examining the multiplicative structure (or lack thereof) of difference sets, see [9].

We close this introduction by remarking that if a well-known conjecture of Rudin holds (this form is due to Ruzsa, [1]), then a very strong improvement of the main theorem can be made to points with integral co-ordinates.

**Conjecture (Rudin/Ruzsa).** For $\varepsilon > 0$, if $S$ is a set of perfect squares, then $|S + S| \gg_{\varepsilon} |S|^{2 - \varepsilon}$.

Since $D = A - A \subset \mathbb{Z}$, $D^2$ is a set of perfect squares. It follows that $\Delta(A \times A) \gg_{\varepsilon} |D|^{2 - \varepsilon}$. If $\Delta(A \times A) = O(|A|^2)$ then $|D| = O(|A|^{1+\delta})$ for some $\delta \to 0$ as $\varepsilon \to 0$.

**2. Proofs**

The fundamental observation in this paper is a simple one. By taking particular elements of

$$2D^2 - 2D^2 = \{d_1^2 + d_2^2 - d_3^2 - d_4^2 : d_1, d_2, d_3, d_4 \in D\}$$

we can find a dilated copy of $D \cdot D$. 
Lemma 1. If $D$ is a difference set, then $2D^2 - 2D^2$ contains a dilate of the set $D \cdot D$.

Proof. Let $D = A - A$ and suppose $d_1 = a_2 - a_1$ and $d_2 = b_1 - b_2 \in D$. Then

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 - (b_1 - a_2)^2 - (b_2 - a_1)^2 = 2d_1d_2.$$ 

The left hand side clearly shows this element of $2D \cdot D$ belongs to $2D^2 - 2D^2$. 

The other essential ingredient is the Plünnecke-Ruzsa Theorem, which is nicely proved in [7].

Theorem 3 (Plünnecke-Ruzsa). Suppose $S$ is a finite subset of an abelian group. Then

$$|mS - nS| \leq \left( \frac{|S + S|}{|S|} \right)^{n+m} |S|.$$ 

Combining these facts gives a number of options, and I am certain that there is a more efficient idea that could lead to a quantitative improvement of the main theorem. In particular, avoiding such a large exponent in the use of Theorem 3 could drastically improve the final result. In a first iteration of this paper, we would have used that $2D^2 - 2D^2$ contains a dilate of $D$. Then because of Theorem 2, $4D^2 - 4D^2$ (which contains a dilate of $D + D$) is bounded from below in terms of $|D|$. We can get a better exponent by instead using an idea of Solymosi from his well-known work on the Sum-Product problem, [10].

Lemma 2. Let $\mathcal{L}$ be a collection of $L$ lines in the plane which pass through the origin. Let $\mathcal{P}$ be a collection of points lying in a single quadrant in the plane, and such that each line $l \in \mathcal{L}$ contains at least $n$ points from $\mathcal{P}$. Then

$$|\mathcal{P} + \mathcal{P}| \geq (L - 1)n^2.$$ 

Proof. Sort the lines in $\mathcal{L}$ by increasing slope. Let $l_1$ and $l_2$ be adjacent lines. Then the (vector) sum of any point on $l_1$ and any point on $l_2$ lies between $l_1$ and $l_2$. This means that all sums of points in $\mathcal{P}$ coming from $l_1$ and $l_2$ are distinct from the sums coming from any other two adjacent lines in $\mathcal{L}$. Moreover, all such sums are distinct by linear independence, so any two adjacent lines produce $n^2$ sums. Since there are $L - 1$ adjacent pairs, the lemma follows. 

Corollary 1. Let $S$ be a set of real numbers and let $S/S = \{s_1/s_2 : s_1, s_2 \in S\}$. Then

$$|S \cdot S + S \cdot S|^2 \gg |S/S||S|^2.$$
Theorem 4. Let \( \mathcal{P} \) be a finite set of points, not all on a line. Then the set
\[
\left\{ \frac{y_1 - y_2}{x_1 - x_2} : (x_1, y_1), (x_2, y_2) \in \mathcal{P} \right\}
\]
has size at least \(|\mathcal{P}| - 1\).

Proof of Main theorem. Set \( D = A - A \) and \( \Delta = \Delta(\mathcal{P}) = D^2 + D^2 \). From Lemma 1 and Theorem 3, we see that
\[
|D^2 \cdot D + D \cdot D| \leq |4D^2 - 4D^2| \leq \left( \frac{|D^2 + D^2|}{|D^2|} \right)^8 |D^2| = \frac{|\Delta|^8}{|D|^7}.
\]
From Corollary 1, we have that
\[
|D^2 \cdot D + D \cdot D| \gg |D||D/D|^{1/2}.
\]
But \( D/D \) is the set of slopes from the set \( A \times A \), so Theorem 4 gives
\[
|D^2 \cdot D + D \cdot D| \gg |D||A|.
\]
Combining these estimates gives that
\[
|D| \ll |\Delta||A|^{-1/8}.
\]
The theorem follows.
Acknowledgments. I thank Oliver Roche-Newton and Adam Sheffer for much helpful discussion. I became interested in this problem while attending the IPAM reunion conference for the program Algebraic Techniques for Combinatorial and Computational Geometry.

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