Determination the optimum orbit for low Earth satellites by changing the eccentricity

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Abstract. The main objective of this paper is to determine an acceptable value of eccentricity for the satellites in a Low Earth Orbit LEO that are affected by drag perturbation only. The method of converting the orbital elements into state vectors was presented. Perturbed equation of motion was numerically integrated using 4th order Runge-Kutta’s method and the perturbation in orbital elements for different altitudes and eccentricities were tested and analysed during 84.23 days. The results indicated to the value of semi major axis and eccentricity at altitude 200 km and eccentricity 0.001 are more stable. As well, at altitude 600 km and eccentricity 0.01, but at 800 km and eccentricities (0.01, 0.05 and 0.1) the stability of the orbit was not depending strongly on eccentricity value, because the effect of the drag is too small.

1. Introduction
The rotation of two bodies about each other is described by two-body problem; the mass of the satellite is usually ignored, because it is small as compared with the mass of the Earth. To describe an orbit, six parameters are required that are: semi-major axis (a) describes the size, eccentricity (e) describes the shape, inclination (i) describes the orientation of an orbit, Right Assentation of Ascending Node (RAAN), Argument of Perigee (w), and the true anomaly (f), which gives the position of the satellite in its orbit at a specific time [1].

The existence of gas molecules and atoms in the Earth's upper atmosphere causes drag force that disturbs the satellite's motion and change the orbit shape. It is the main non-conservative force that effects on a satellite in LEO in the direction opposite to the velocity vector and removes energy from the orbit in the form of friction on the satellite. Also, the energy decreasing produces an orbit decaying until the satellite re-enters the atmosphere. The effect of drag is the main perturbation during the last few revolutions of a satellite's lifetime [2]. Different analytic, semi-analytic and numerical techniques are used for solving perturbed equations of motion [3]. Many researchers over the past few decades were interest to study the motion of the satellite and its lifetime. Eshagh and Najafi were evaluated the perturbations in orbital elements for a satellite LEO by integrating equation of motion [4]. Asma was developed a simulation for the orbit propagator for LEO satellite including dominant perturbations for it. This simulation involves modeling the perturbing forces to optimize the propagation [5]. Mishra, S. etal. were determined orbit of the satellite by using a technique to measure position, velocity and behavior of orbital elements under the effect of the drag [6]. Ahmed H. etal were evaluated the perturbations in CubeSat’s orbital elements for LEO satellite using Cowell’s method [7]. Mohammed and Abdul-Rahman were studied extensively the behavior of orbital elements by using different A/M, altitudes and eccentricities [1].

2. Methods
The following steps are essential to achieve the requirements of this study:
2.1. The orbital elements calculation

- The semi-major axis is defined as [8]:
  \[ a = \left( \frac{2}{r} - \frac{v^2}{\mu} \right)^{-1} \]  
  (1)
  Where: \( \mu \) is the gravitational constant for the Earth equal to 398600 km³/sec², \( r \) is the position vector and \( v \) is the velocity vector. Both the position vector and velocity vector will discuss in details in state vector calculation section.

- The eccentricity is defined as below [8]:
  \[ e = \frac{r_a - r_p}{2a} \]  
  (2)
  Where: \( r_p \) is the distance from perigee point equal to a \((1-e)\) and \( r_a \) is the distance from apogee point equal to a \((1+e)\).

- The inclination is represented as follows [8]:
  \[ \tan i = \frac{\sqrt{h_x^2 + h_y^2}}{h_z} \]  
  (3)
  Where \( h \) is angular momentum vector can be stated as [9]:
  \[ h = r \times v \]  
  (4)
  Hence, \( h_x = yv_z - zv_y, h_y = zv_x - xv_z, h_z = xv_y - yv_x \)
  Their magnitude is \( h = \sqrt{h_x^2 + h_y^2 + h_z^2} \)

- The Right Assentation of Ascending Node is [8]:
  \[ \tan RAAN = \frac{h_x}{-h_y} \]  
  (5)

- The Argument of Perigee is stated as below [9]:
  \[ w = u - f \]  
  (6)
  Where: \( u \) is the argument of the latitude can be given by [9]:
  \[ \tan u = -\frac{xh_y + yh_x}{zh} \]  
  (7)

\( f \) is the true anomaly can be calculated as [8]:
\[ \tan \frac{f}{2} = \sqrt{1+e} \tan \frac{e}{2} \]  
(8)

Where: \( E \) is the Eccentric anomaly can be obtained by solving Kepler’s equation using four methods like Danby’s, Mikkola’s, Halley’s and Newton’s-Raphson Method. In this work Newton’s-Raphson method was used to calculate the value of eccentric anomaly from the mean anomaly \((M)\), which can be represented by [10, 11]:
\[ M = n(t - t_p) \]  
(9)
Where: \( t_p \) is the time for an object to pass through the perigee, \( n \) is the mean motion calculated by [11]:
\[ n = \sqrt{\frac{\mu}{a^3}} \]  
(10)

2.2. The state vectors calculation

Orbit determination is an important in the case of several satellite missions, which is specified by the satellite’s state vectors at any time. These state vectors can be completely described by six independent parameters in the equatorial plane, the position \((X, Y, Z)\) and the velocity \((v_X, v_Y, v_Z)\). The Cartesian coordinates \((x, y \text{ and } z)\) for satellite in its orbit, are stated as [8, 9]:
\[ x = a(\cos(E) - e) \]  
(11)
\[ y = a\sqrt{1 - e^2} \sin(E) \]  
(12)
\[ z = 0 \]  
(13)
\[ v_x = -\frac{\mu}{p} \sin(f) \]  
(14)

\[ v_y = \frac{\mu}{p} (e + \cos(f)) \]  
(15)

\[ v_z = 0 \]  
(16)

Where: \( p \) is the semi latus rectum equal to \( a(1 - e^2) \). To transformation of the position and velocity components from the satellite plane to the equatorial plane could be achieved by using a Gaussian vector (conversion matrix), which is given by [9]:

\[
\begin{bmatrix}
X \\
Y \\
Z \\
\dot{X}
\end{bmatrix} = R^{-1}
\begin{bmatrix}
\dot{X} \\
\dot{Y} \\
\dot{Z} \\
\dot{V}_x
\end{bmatrix}
\]  
(17)

\[
\begin{bmatrix}
V_x \\
V_y \\
V_z
\end{bmatrix} = R^{-1}
\begin{bmatrix}
\dot{V}_x \\
\dot{V}_y \\
\dot{V}_z
\end{bmatrix}
\]  
(18)

Where: \( R^{-1} \) is the inverse of Gauss matrix, which contents Euler angles (\( i, \omega \) and \( \Omega \)), as below:

\[
R^{-1} = \begin{bmatrix}
P_x & Q_x & W_x \\
P_y & Q_y & W_y \\
P_z & Q_z & W_z
\end{bmatrix}
\]  
(19)

\[
P_x = \cos(\omega) \cos(\Omega) - \sin(\omega) \sin(\Omega) \cos(i), \quad P_y = \cos(\omega) \sin(\Omega) + \sin(\omega) \cos(\Omega) \cos(i), \quad P_z = \sin(\omega) \sin(i), \quad Q_x = -\sin(\omega) \cos(\Omega) - \cos(\omega) \sin(\Omega) \cos(i), \quad Q_y = -\sin(\omega) \sin(\Omega) + \cos(\omega) \cos(\Omega) \cos(i), \quad Q_z = \cos(\omega) \sin(i), \quad W_x = \sin(\Omega) \sin(i), \quad W_y = -\cos(\Omega) \sin(i), \quad W_z = \cos(i)
\]

Eqs.(9-11) and Eqs.(12-14) can be written as the following:

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} = \begin{bmatrix}
P_x X_w + Q_x Y_w + W_x Z_w \\
P_y X_w + Q_y Y_w + W_y Z_w \\
P_z X_w + Q_z Y_w + W_z Z_w
\end{bmatrix}
\]  
(20)

\[
\begin{bmatrix}
V_x \\
V_y \\
V_z
\end{bmatrix} = \begin{bmatrix}
P_x V_x + Q_x V_y + W_x V_z \\
P_y V_x + Q_y V_y + W_y V_z \\
P_z V_x + Q_z V_y + W_z V_z
\end{bmatrix}
\]  
(21)

Where: \( X, Y, Z, v_X, v_Y, \) and \( v_Z \) represent the state vectors (position and velocity components) of the satellite in the equatorial plane at time \( t \). Hence, the state vectors are completely described by:

- The position vector \( \mathbf{r} \) is equal to \( X+Y+j Z \), its magnitude is given by [8]:

\[
\mathbf{r} = \sqrt{X^2 + Y^2 + Z^2}
\]  
(22)

- The velocity vector \( \mathbf{v} \) is equal to \( v_X+i v_Y + v_Z k \), its magnitude is given by [8]:

\[
\mathbf{v} = \sqrt{v_X^2 + v_Y^2 + v_Z^2}
\]  
(23)

The velocity at perigee point can be represented as:

\[
v_p = \frac{\mu (1+e)}{\sqrt{a (1-e)}}
\]  
(24)

\subsection*{2.3 Atmospheric drag and perturbed equation of motion calculation}

The perturbing acceleration of the satellite according to atmospheric drag is [3]:

\[
a_{\text{Drag}} = -\frac{1}{2} \frac{C_D}{M} \rho v_r^2
\]  
(25)

Where: \( C_D \) is drag coefficient the used value 2.2, \( A \) is cross sectional area of the satellite perpendicular to velocity vector the used value 5.1m², \( M \) is the mass of satellite, the used value 900 kg and \( v_r \) is satellite velocity vector relative to the atmosphere, \( \rho \) is atmospheric density, the density is calculated
using NLRMSISE-00. We assume that the atmosphere rotates at the same angular speed as the Earth with this assumption the relative velocity vector is given by [3]:

\[ \mathbf{v}_r = \mathbf{v}_{ln} - \mathbf{W}_{Earth} \times \mathbf{r} \]  

Where: \( \mathbf{v}_{ln} \) is the inertial velocity vector for satellite, \( \mathbf{W}_{Earth} \) is the Earth’s rotational velocity vector and \( \mathbf{r} \) is the inertial satellite position vector. Further broken down into vector components equation (26) can be represented as [3]:

\[ \begin{align*}
\mathbf{v}_x &= \mathbf{v}_x + \mathbf{W}_{Earth} \times Y \\
\mathbf{v}_y &= \mathbf{v}_y - \mathbf{W}_{Earth} \times X \\
\mathbf{v}_z &= \mathbf{v}_z 
\end{align*} \]  

Their magnitude is \( \mathbf{v}_r = \sqrt{\mathbf{v}_x^2 + \mathbf{v}_y^2 + \mathbf{v}_z^2} \)

\[ \begin{align*}
\mathbf{v}_{x(unit)} &= \mathbf{v}_x / v_r \\
\mathbf{v}_{y(unit)} &= \mathbf{v}_y / v_r \\
\mathbf{v}_{z(unit)} &= \mathbf{v}_z / v_r \\
\mathbf{a}_{drag_x} &= \mathbf{a}_{drag} \times \mathbf{v}_{x(unit)} \\
\mathbf{a}_{drag_y} &= \mathbf{a}_{drag} \times \mathbf{v}_{y(unit)} \\
\mathbf{a}_{drag_z} &= \mathbf{a}_{drag} \times \mathbf{v}_{z(unit)} 
\end{align*} \]  

The total acceleration is stated as \( \mathbf{a}_{drag} = \sqrt{(\mathbf{a}_{drag_x})^2 + (\mathbf{a}_{drag_y})^2 + (\mathbf{a}_{drag_z})^2} \)

The equation of motion with drag, which can be represented as [9, 12]:

\[ \ddot{\mathbf{r}} = -\frac{\mu}{r^3} \mathbf{r} + \mathbf{a}_{Drag} \]  

2.4. The numerical integration method

Perturbed equation of motion was integrated numerically by using 4th order Runge-Kutta’s method to update the state vectors as below [3, 9]:

\[ \mathbf{v}_{x1} = \mathbf{v}_{x0} + \frac{\sigma_{step}}{6} (k_1 + 2k_2 + 2k_3 + k_4) \]  

Where: \( \mathbf{v}_{x0} \) is the initial velocity at epoch, \( \mathbf{v}_{x1} \) is the predicated velocity, \( k_1 = \alpha x_o, k_2 = \alpha x_o + \frac{\sigma_{step}}{2} k_1, k_3 = \alpha x_o + \sigma_{step} k_2, k_4 = \alpha x_o + \frac{\sigma_{step}}{2} k_3, \alpha x_o \) is the acceleration at epoch, \( \sigma_{step} \) is the step’s method equal to \( \frac{T_p}{m_{Sub-step}} \), \( t_{step} = \frac{T_p}{m_{step}}, m_{Sub-step} \) is the sub steps number the used value equal to 50 , \( T_p \) is the satellite’s orbital period, the used value 5317.5 second, and \( m_{step} \) is the step number during 1300 revolution, the used value is 1000.

\[ X_1 = X_0 + \frac{\sigma_{step}}{6} (k_{k_1} + 2k_{k_2} + 2k_{k_3} + k_{k_4}) \]  

Where \( X_0 \) is the initial position at epoch, \( X_1 \) is the predicated position, \( k_{k_1} = \nu x_o, k_{k_2} = \nu x_o + \frac{\sigma_{step}}{2} k_{k_1}, k_{k_3} = \nu x_o + \sigma_{step} k_{k_2}, k_{k_4} = \nu x_o + \frac{\sigma_{step}}{2} k_{k_3}. \) Also, Equations (32) and (33) can be used to calculate the other velocities (\( \mathbf{v}_{y1} \) and \( \mathbf{v}_{z1} \)) and positions (\( Y_1 \) and \( Z_1 \)) components by the same way.

3. Results and Discussion

A program in matlab was designed to solve the equations above and to plot the results. The variations of orbital elements were studied at altitudes (200, 600 and 800) km and eccencricities (0.001, 0.01, 0.05 and 0.1) to conclude an appropriate eccentricity for a satellite in LEO due to drag within 84.23 days. According to figure (1-a) the semi major axis has a secular dropping behaviour. It drops from (6580-6460) km during 84.23 days. Also, the eccentricity has secular dropping behaviour. It drops from about (10×10^{-5}-9.985×10^{-4}) as the time passes. The inclination is a secular with simple
periodic, it reduces from (63.5-63.485)° with time. The Right Assentation of Ascending Node behaves periodically with low amplitude as the time passing, it changes from (20-20.001)°. The argument of perigee has a secular growing difference. This element grows quickly form (120-148)°. For the true anomaly of the satellite, it grows from (0-360)° as the time passes. One can note from figure (1-b), the distance at perigee behaves secularly, it falls from (6580-6450) km also the velocity at perigee has the same behaviour, but it grows form (7.79-7.88) km/sec. Orbital period has a secular behaviour as the time passes. It falls from (5320-5170) sec.

At altitude 200 km and eccentricity 0.01, the semi major axis has a secular decreasing behaviour as well as a simple periodic. It decreases from (6645-6625) km. The eccentricity seems to have a secular falling behaviour with simple periodic; it falls from about \((10\times10^{-3}-9.972\times10^{-3})\). The inclination has the same performance like the semi major axis and the eccentricity as the time passes, it decreases from (63.5-63.498)°. The Right Assentation of Ascending Node appears to have a secular increasing behaviour, besides a simple periodic in this element one can note, it increases too little from (20-20.001)°. Similarly, the argument of perigee is secular, but with too fast rising form (120-149)°. The true anomaly increases from (0-360)° for the duration of 84.23 days, as displayed in figure (2-a). In figure (2-b) the distance and velocity at perigee behave secularly with simple periodic as the time passing. The distance reduces from (6578-6560) km while the velocity grows from (7.823-7.835) km/sec. Orbital period has the same behaviour such as the distance at perigee, it decreases from (5390-5368) sec.

It can note from figure (3-a) at altitude 600 and eccentricity 0.001, the semi major axis behaves secularly; it reduces gradually and too slowly from about (6985.3-6984.95) km. Also, the eccentricity has the same performance like the semi major axis; it decreases from (10\times10^{-4}-9.9997\times10^{-4}). The inclination is secular with time, it falls too little from (63.5-63.4997)°. For Right Assentation of Ascending Node is secular, but growing with time, it grows slowly from about (20-20.0003)°. The argument of perigee has a secular rising as the time passing, it rises frequently form (120-149)°. The true anomaly gives an impression of being increased from (0-360)° through 84.23 days. From figure (3-b) the distance at perigee, the velocity at perigee and orbital period have a secular variation. They distance decreases from (6978.15-6977.99) km, the velocity increases from (7.56164-7.56173) km/sec and period reduces from (5809.9-5809.7) sec as the time passing.

At the same altitude 600 km and eccentricity 0.01, the semi major axis, the eccentricity and the inclination have a secular falling behaviour, the semi major axis falls too slowly from about (7048.62-7048.47) km, the eccentricity from (10\times10^{-3}-9.9997\times10^{-3}) and the inclination from (63.5-63.4997)° respectively through the period of 84.23 days whereas the Right Assentation of Ascending Node and the argument of perigee have a secular growing behaviour. The Right Assentation of Ascending Node grows too slowly from about (20-20.0003)° and the argument of perigee from (120-149)°. The true anomaly increases from (0-360)° as the time passes, as figure (4-a). As we note from figure (4-b), the distance at perigee becomes less than the starting value, it decreases from (6978.15-6978) km while the velocity increases as the time passes. Orbital period reduces too slowly from (5889.35-5889.15) sec.

In figure (5-a), the semi major axis, the eccentricity, the inclination, the Right Assentation of Ascending Node, and the argument of perigee are secular growing except the inclination, it is decreasing as the time passes. The semi major axis grows from about (7753.49-7753.58) km, the eccentricity from (0.1-0.1000012), the Right Assentation of Ascending Node from (20-20.00033)°, and the argument of perigee from (120-155)°, the inclination decreases from (63.5-63.4996). The true anomaly has preserved its behaviour and value like previous altitudes and eccentricities. According to figure (5-b) distance and velocity at perigee seem to have a secular behaviour. The distance grows too little from about (6978.14-6978.21) km, in contrast the velocity reduces too slowly from about (7.9268-7.92672) km/sec. Orbital period has the same behaviour as the distance, but with different magnitudes. It grows step by step from about (6794.48-6794.62) sec.

Figures (6, 7 and 8) present the behaviour of the semi major axis, the eccentricity, the distance at perigee and the period for satellite at altitude 800 km and eccentricities 0.01, 0.05 and 0.1. All the
parameters in these figures have a secular growing with time. They are growing too small with time. The difference for semi major axis is approximately the same for different eccentricities, but the eccentricity at this altitude has not obvious changing as a value with time. Table (1) explains the difference of orbital elements. At altitude 200 km, the semi major axis, the eccentricity and the inclination have a large changing and influence by the value of the eccentricity whereas the right assentation of ascending node and the argument of perigee there is a too small changing. Increasing the altitude to 600 km gave a noticeable effect only on the semi major axis and the eccentricity. Finally, at altitude 800 km the semi major axis remains changeable with time; on the contrary the eccentricity has not. The rest of the orbital elements are not taking into account, because they are not influence by the eccentricity’s value.

| Altitude (km) | Eccentricity | Orbital Elements | Orbital Elements |
|--------------|--------------|------------------|------------------|
|              |              | Δa (km) | Δe | Δi (deg) | ΔRAAN (deg) | Δw (deg) |
| 200          | 0.001        | 0.0185 | 0.015 | 0.236 | 0.00005 | 0.233 |
|              | 0.01         | 0.03   | 0.028 | 0.0031 | 0.00055 | 0.241 |
|              | 0.01         | 0.005  | 0.0003 | 0.000047 | 0.0015 | 0.241 |
|              | 0.1          | 0.0011 | 0.0001 | 0.000062 | 0.000165 | 0.291 |
| 600          | 0.1          | 0.277  | 0 | / | / | / |
|              | 0.05         | 0.358  | 0 | / | / | / |
|              | 0.1          | 0.361  | 0 | / | / | / |
| 800          | 0.1          | 0.361  | 0 | / | / | / |

Figure 1-a. represents the orbital elements variation at altitude 200 km and eccentricity 0.001.
Figure 1-b. Depicts the distance, velocity and orbital period variation at altitude 200 km and eccentricity 0.001.

Figure 2-a. Shows the orbital elements variation at altitude 200 km and eccentricity 0.01.
Figure 2-b. Illustrates the distance, velocity and orbital period variation at altitude 200 km and eccentricity 0.01.

Figure 3-a. Describes the orbital elements variation at altitude 600 km and eccentricity 0.001.
Figure 3-b. Clarifies the distance, velocity and orbital period variation at altitude 600 km and eccentricity 0.001.

Figure 4-a. Represents the orbital elements variation at altitude 600 km and eccentricity 0.01.
Figure 4-b. Shows the distance, velocity and orbital period variation at altitude 600 km and eccentricity 0.001

Figure 5-a. Illustrates the orbital elements variation at altitude 600 km and eccentricity 0.1.
Figure 5-b. Clarifies the distance, velocity and orbital period variation at altitude 600 km and eccentricity 0.1.

Figure 6. Shows the semi major axis, eccentricity, distance, and orbital period variation at altitude 800 km and eccentricity 0.01.
Figure 7. Represents the semi major axis, eccentricity, distance, and orbital period variation at altitude 800 km and eccentricity 0.05.

Figure 8. Illustrates the semi major axis, eccentricity, distance, and orbital period variation at altitude 800 km and eccentricity 0.1.
4. Conclusion

The results showed to the value of semi major axis and eccentricity at altitude 200 km and eccentricity 0.001 are more acceptable. Likewise, at altitude 600 km and eccentricity 0.01, but at altitude 800 km and eccentricities (0.01, 0.05 and 0.1), the orbit was not depending intensely on the eccentricity’s value, because the drag has a small influence on the orbit at this altitude. Also, the distances, velocities at perigee and period of the satellite.

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