RESULTS ON GENERAL QUADRATIC GROUPS

RABEYA BASU AND KUNTAL CHAKRABORTY

2020 Mathematics Subject Classification:
11E57, 13A02, 13C10, 15A63, 19B14, 19D35, 20F18

Key words: Graded rings, Form rings, Bak’s unitary groups, Bass’ nil groups

Abstract: In the first part of this article we discuss the relative cases of Quillen–Suslin’s local-global principle for the general quadratic (Bak’s unitary) groups, and its applications for the (relative) stable and unstable $K_1$-groups. The second part is dedicated to the graded version of the local-global principle for the general quadratic groups and its application to deduce a result for Bass’ nil groups.

1. Introduction

In 1966–67, A. Bak introduced the general quadratic (widely known as Bak’s unitary) groups to give a uniform definition of classical groups, and to overcome the difficulties arise to handle classical groups over fields of characteristic 2. In [7] and [8], the first author has discussed many results in classical K-theory for the absolute cases, related to the Serre’s problem on projective modules. In this article, following the trick used in [10], we are going to consider those problems for the relative cases; viz Quillen–Suslin’s local-global principle for the transvection subgroups, $K_1$-stabilization and the structure of unstable $K_1$-groups for the general quadratic (Bak’s unitary) group over associative rings which are finite over the center. For previous results on these problems we refer works of Bak–Basu–Rao–Khanna for the local-global principle (L-G principle) in [9], [4], [10], Bak–Petrov–Tang for $K_1$-stabilization in [5], and Bak–Harzat–Vavilov for solvability of unstable $K_1$-groups in [2], [19], [18].

For the linear case, the graded version of L-G principle was studied by Chouinard in [13], and by Gubeladze in [14], [15]. In [11], the first author and M.K. Singh deduced an analog for the traditional classical groups. In the second part of the article we have deduced an analog for the transvection subgroups of the general quadratic groups over graded rings. As an application, by using a recent result on Higman linearization due to V. Kepoeiko (cf. [21], [22]), we could revisit a problem on absence of torsion in Bass’ nil group for the general quadratic groups, and its graded analog. For previous results in this direction we refer [20], [12], [23], [24], and [7].

2. Preliminaries

Let us recall necessary definitions and the key lemmas.

---

Research by the first author was supported by SERB-MATRICS grant for the financial year 2020–2021.
Research by the second author was supported by IISER (Pune) post-doctoral research grant, 2020–2021.
Definition 2.1. (cf. [1]) Let $R$ be an (not necessarily commutative) associative ring with identity, and with involution $- : R \to R$, $a \mapsto \overline{a}$. Let $\lambda \in C(R) = \text{center of } R$ be an element with the property $\lambda \overline{\lambda} = 1$. We define additive subgroups of $R$: $$\Lambda_{\text{max}} = \{ a \in R \mid a = -\lambda \overline{a} \} \quad \text{and} \quad \Lambda_{\text{min}} = \{ a - \lambda \overline{a} \mid a \in R \}.$$ One checks that $\Lambda_{\text{max}}$ and $\Lambda_{\text{min}}$ are closed under the conjugation operation $a \mapsto \overline{a}$ for any $x \in R$. A $\lambda$-form parameter on $R$ is an additive subgroup $\Lambda$ of $R$ such that $\Lambda_{\text{min}} \subset \Lambda \subset \Lambda_{\text{max}}$, and $\overline{x} \Lambda x \subset \Lambda$ for all $x \in R$. A pair $(R, \Lambda)$ is called a form ring.

To define Bak’s unitary group or the general quadratic group, we fix a (non-zero) central element $\lambda \in R$ with $\lambda \overline{\lambda} = 1$, and then consider the form $$\psi_n = \begin{pmatrix} 0 & I_n \\ \lambda I_n & 0 \end{pmatrix}.$$ Bak’s Unitary or General Quadratic Groups $GQ$:

$$GQ(2n, R, \Lambda) = \{ \sigma \in \text{GL}(2n, R) \mid \overline{\psi}_n \sigma = \psi_n \}.$$ 

Elementary Quadratic Matrices: Let $\rho$ be the permutation, defined by $\rho(i) = n + i$ for $i = 1, \ldots, n$. Let $e_{ij}$ be the matrix with 1 in the $ij$-th position and 0’s elsewhere. For $a \in R$, and $1 \leq i, j \leq n$, we define

$$q_{\varepsilon_{ij}}(a) = I_{2n} + ae_{ij} - \overline{\lambda}e_{j\rho(i)} \quad \text{for } i \neq j,$$

$$q_{r_{ij}}(a) = \begin{cases} I_{2n} + ae_{ji} - \lambda \overline{e}_{j\rho(i)} & \text{for } i \neq j, \\ I_{2n} + ae_{\rho(i)j} & \text{for } i = j, \end{cases}$$

$$q_{l_{ij}}(a) = \begin{cases} I_{2n} + ae_{i\rho(j)} - \overline{\lambda}e_{j\rho(i)} & \text{for } i \neq j, \\ I_{2n} + ae_{\rho(i)j} & \text{for } i = j. \end{cases}$$

(Note that for the second and third type of elementary matrices, if $i = j$, then we get $a = -\lambda \overline{a}$, and hence it forces that $a \in \Lambda_{\text{max}}(R)$. One checks that these above matrices belong to $GQ(2n, R, \Lambda)$; cf. [1].)

$n$-th Elementary Quadratic Group $EQ(2n, R, \Lambda)$: The subgroup generated by $q_{\varepsilon_{ij}}(a), q_{r_{ij}}(a)$ and $q_{l_{ij}}(a)$, for $a \in R$ and $1 \leq i, j \leq n$. For uniformity we denote the elementary generators of $EQ(2n, R, \Lambda)$ by $\eta_{ij}(\ast)$.

It is clear that the stabilization map takes generators of $EQ(2n, R, \Lambda)$ to the generators of $EQ(2(n+1), R, \Lambda)$.

Definition 2.2. The relative general quadratic subgroups of $GQ(2n, R, \Lambda)$ with respect to the ideal $J$ is defined by the set $\{ \alpha \in GQ(2n, R, \Lambda) : \alpha \equiv I_{2n} \pmod{J} \}$ and it is denoted by $GQ(2n, R, \Lambda, J)$.

Definition 2.3. Let $(R, \Lambda)$ be a form ring and $J \subset R$ be an ideal. The subgroup of $GQ(2n, R, \Lambda)$ generated by the matrices of the form $\eta_{ij}(x)\eta_{ji}(a)\eta_{ij}(x)^{-1}$, where $x \in R$ and $a \in J$, is called the relative elementary subgroup and is denoted by $EQ(2n, R, \Lambda, J)$.

Notation 2.4. A row $(a_1, a_2, \ldots, a_n) \in R^n$ is said to be unimodular if there exists $(b_1, b_2, \ldots, b_n) \in R^n$ such that $\sum_{i=1}^n a_i b_i = 1$. The set of all unimodular rows of length $n$ is denoted by $\text{Um}_n(R)$. By the set $\text{Um}_n(R, J)$ we define the set of all unimodular rows of length $n$ which are congruent to $e_1 = (1, 0, \ldots, 0)$ modulo the ideal $J$. For
an ideal \( J \subset R \) the extended ideal \( J \otimes_R R[X] \) of \( R[X] \) is denoted by \( J[X] \). We will mostly use localizations with respect to two types of multiplicatively closed subsets of \( R \). \textit{viz.} \( S = \{1, s, s^2, \ldots\} \), where \( s \in R \) is a non-nilpotent, non-zero divisor, and \( S = R \setminus \mathfrak{m} \) for \( \mathfrak{m} = \text{Max}(R) \). By \( J_s[X] \) and \( J_m[X] \) we shall mean the extension of \( J[X] \) in \( R_s[X] \) and \( R_m[X] \) respectively.

**Blanket assumption:** Throughout the paper we shall assume that \( 2n \geq 6 \).

**Definition 2.5.** Let \((R, \Lambda)\) be a form ring and \( J \subset R \) be an ideal of \( R \). The excision ring of \( R \) with respect to the ideal \( J \) is denoted by \( R \oplus J \) and is defined by the set \( \{(r, i) : r \in R, i \in J\} \) with the addition defined by \((r, i) + (s, j) = (r + s, i + j)\), and the multiplication defined by \((r, i)(s, j) = (rs, rj + si + ij)\). We can extend the involution “\( \sim \)” to the ring \( R \oplus J \) by setting \((r, i) = (\bar{r}, i)\). The element \((\lambda, 0) \in R \oplus J\) satisfies \((\lambda, 0)(\lambda, 0) = (1, 0)\), where \((1, 0)\) is the identity element of \( R \oplus J \). Thus the additive subgroup \( \Lambda \oplus J \) will be the natural choice for the \( \lambda \)-parameter on \( R \oplus J \). Hence we get the form ring \((R \oplus J, \Lambda \oplus J)\).

Recall that there is a natural map \( f : R \oplus J \to R \) given by \( f(r, i) = r + i \). This map induces a canonical homomorphism on \( \text{GQ}(2n, R \oplus J, \Lambda \oplus J) \). We shall use the same notation \( f \) to denote this map.

The following two key lemmas are proved in (cf.\cite{10}) for the traditional classical groups. Proofs are same for the general quadratic groups.

**Lemma 2.6.** For \( \alpha \in \text{EQ}(2n, R, \Lambda, J) \), there exists a matrix \( \bar{\alpha} \in \text{EQ}(2n, R \oplus J, \Lambda \oplus J) \) such that \( f(\bar{\alpha}) = \alpha \).

**Lemma 2.7.** For \( \alpha \in \text{GQ}(2n, R, \Lambda, J) \), there exists a matrix \( \bar{\alpha} \in \text{GQ}(2n, R \oplus J, \Lambda \oplus J) \) such that \( f(\bar{\alpha}) = \alpha \).

3. **Relative L-G Principle for the Transvection Subgroups**

It is known that any module finite ring \( \text{i.e.} \) finite over its center \( R \) as a direct limit of its finitely generated subrings. Also, \( G(R, \Lambda) = \lim \to G(R_i, \Lambda_i) \), where the limit is taken over all finitely generated subring of \( R \). Hence we can assume that \( C(R) \) is Noetherian. For the rest of this section we shall consider \( R \) to be module finite ring with identity.

The local-global principle for the transvection subgroups (absolute cases) of the full automorphism groups was established in \cite{4} for the traditional classical groups. Then, in \cite{3} it was generalized for the general quadratic groups. In this section we deduce the relative L-G principle for the transvection subgroups.

Following results are proved in \cite{10} for traditional classical groups of free modules, and the steps of the proof for the general quadratic groups are identical. Therefore, we state these results without proof.

For any column vector \( v \in (R^{2n})^t \) we consider the row vector \( \bar{v} = \mathfrak{T}^t \psi_n \).

**Definition 3.1.** We define a map \( M : (R^{2n})^t \times (R^{2n})^t \to M(2n, R) \) and the inner product \( \langle \cdot, \cdot \rangle \) as follows:

\[
M(v, w) = v.\bar{w} - \bar{w}v, \quad \langle v, w \rangle = \bar{v}.w
\]

**Lemma 3.2.** Let \( (R, \Lambda) \) be a form ring and \( v \in \text{EQ}(2n, R, \Lambda, I) e_1 \). Let \( w \in I^{2n} \) be a column vector such that \( \langle v, w \rangle = 0 \). Then \( I_{2n} + M(v, w) \in \text{EQ}(2n, R, \Lambda, I) \).
Theorem 3.3. (Relative L-G principle) Let $R$ be a ring and $J \subset R$ be an ideal of $R$. Let $\alpha(X) \in GQ(2n, R[X], \Lambda[X], [J])$, with $\alpha(0) = I_{2n}$ be such that for every maximal ideal $m \in \text{Max}(C(R))$, $\alpha_m(X) \in \text{EQ}(2n, R_m[X], \Lambda_m[X], [J_m])$. Then $\alpha(X) \in \text{EQ}(2n, R[X], \Lambda[X], [J])$.

We recall some definitions and fix notations.

Definition 3.4. Let $(R, \Lambda)$ be a form ring and $P$ be a right $R$-module. A map $f : P \times P \to R$ is said to be a sesquilinear form if $f(pa, qb) = \overline{pf(p, q)b}$ for all $p, q \in P$ and $a, b \in R$. A map $q : P \to R/\Lambda$ is said to be a quadratic form if $q(p) = f(p, p) + \Lambda$, where $f$ is a sesquilinear form on $P$. With respect to a sesquilinear form on $P$, we can define an associated $\lambda$-Hermitian form $h : P \times P \to R$ by $h(p, q) = f(p, q) + \lambda \overline{f(q, p)}$. The triplet $(P, h, q)$ is called a quadratic module.

Definition 3.5. Let $(P, q, h)$ be a quadratic module and $\text{GL}(P)$ be the full automorphism group of $P$. The quadratic module $P$ is said to be non-singular if $P$ is a projective $R$-module and the associated $\lambda$-Hermitian form is non-singular. For a non-singular $P$, the general quadratic group of $P$ is defined as follows:

$\text{GQ}(P, q, h) = \{ \sigma \in \text{GL}(V) : h(\sigma u, \sigma v) = h(u, v), q(\sigma u) = q(u) \}$

Definition 3.6. Let $(P, h, q)$ be a quadratic module over $(R, \Lambda)$ and $J \subset R$ be an ideal of $R$. Let $u, v \in P$ and $a \in R$ be such that $f(u, u) \in \Lambda, h(u, v) = 0$ and $f(v, v) = a$ (mod $\Lambda$). Then the transvection map $\sigma = \sigma_{u,v,a} : P \to P$ is defined by

$\sigma(x) = x + uh(v, x) - v \overline{h(u, x)} - u \overline{ah(u, x)}$.

The set of all transvections of $P$ will be denoted as $T(P)$. A map $\sigma \in T(P)$ is said to be a transvection relative to $J$ if either $u$ or $v$ belongs to the submodule $JP$. The set of all transvections relative to the ideal $J$ will be denoted by $T(P, J)$.

Definition 3.7. Let $(P, h, q)$ be a quadratic module over a form ring $(R, \Lambda)$ and $J \subset R$ be an ideal of $R$. Let $M$ be the quadratic module $P \perp H(R)$, where $H(R)$ denotes the hyperbolic form $R \perp R^*$. Then the transvections of the form

$q = (p, a, b) \mapsto (p - aq, a, b + h(p, q))$,

or,

$q = (p, a, b) \mapsto (p - bq, a + h(p, q), b)$,

where $a \in R, b \in R^*, p, q \in P$, are called elementary transvections. The set of all elementary transvections is denoted by $\text{ET}(M)$. An elementary transvection is said to be elementary transvection relative to $J$ if $q \in JM$. The subgroup of $\text{ET}(M)$ generated by elementary transvections relative to $J$ is denoted by $\text{ET}(JM)$. And we denote $\text{ET}(M, JM)$ by the normal closure of $\text{ET}(JM)$ in $\text{ET}(M)$.

Notation 3.8. Let $P$ be a finitely generated quadratic $R$ module of rank $2n$ with a fixed form $\langle , \rangle$. And $M$ will denote $P \perp H(R)$. We will use the notation $M[X]$ to denote $(P \perp H(R))[X]$. We assume that the rank of the quadratic module is $2n \geq 6$.

We shall also assume the following two hypotheses:

$(H1)$ For every maximal ideal $m$ of $R$, the quadratic module $M_m$ is isomorphic to $R^{2n+2}$ for the standard bilinear form $H(R_m^{n+1})$.

$(H2)$ For every non-nilpotent $s \in R$, if the projective module $M_s$ is free $R_s$-module, then the quadratic module $M_s$ is isomorphic to $R^{2n+2}$ for the standard bilinear form $H(R^{n+1})$. 

4
Lemma 3.9. ([7, Lemma 5.7], [10, Lemma 3.6]) The group $\mathrm{GQ}(2n, R[X], \Lambda[X], (X)) \cap \mathrm{EQ}(2n, R[X], \Lambda[X], J[X])$ is generated by the elements of the type $\varepsilon\eta_1(\chi h(X))\varepsilon^{-1}$, where $\varepsilon \in \mathrm{EQ}(2n, R, \Lambda)$ and $h(X) \in J[X]$.

Lemma 3.10. ([10, Corollary 3.8]) If $\eta = \eta_1\eta_2 \ldots \eta_r$, where each $\eta_i$ is an elementary generator, and $h(Y) \in J[Y]$, then there are $b_i(X, Y) \in J[X, Y]$ such that

$$\eta\eta_p(\chi^{2^m}h(Y))\eta^{-1} = \prod_{i=1}^k \eta_{p_i}\eta_i(\chi^{m}h(X, Y)).$$

Here we recall following standard facts:

Lemma 3.11. Let $R$ be a ring and $K$ a finitely presented left (right) $R$-module, and let $L$ be any left (right) $R$-module. Then we have a natural isomorphism:

$$f : \mathrm{Hom}_R(K, L)[X] \to \mathrm{Hom}_R(K[X], L[X]).$$

Lemma 3.12. Let $S$ be a multiplicative close subset of a ring $R$. Let $K$ be a finitely presented $R$-module and $L$ be any $R$ module. Then we have a natural isomorphism

$$g : S^{-1}(\mathrm{Hom}_R(K, L)) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}K, S^{-1}L).$$

Following lemma is used frequently (sometime in a subtle way) in the proof of the main results.

Lemma 3.13. (cf. [19, Lemma 5.1]) Let $A$ be Noetherian ring and $s \in A$. Let $s \in A$ and $s \neq 0$. Then there exists a natural number $k$ such that the homomorphism $G(A, s^kA, s^k\Lambda) \to G(\Lambda_s, \Lambda_s)$ (induced by the localization homomorphism $A \to \Lambda_s$) is injective.

Now we prove the relative L-G principle by using Lemma 2.6.

Proposition 3.14 (Relative Dilation Principle). Let $R$ be an almost commutative ring (i.e., a commutative ring which is finite over its center $C(R)$) and $J \subset R$ be an ideal. Let $P$ and $M$ be as in [13, 16]. Let $s$ be a non-nilpotent in $R$ such that $P_s$ is free, and let $\sigma(X) \in \mathrm{GQ}(Q[X], J[X])$ with $\sigma(0) = \mathrm{Id}$. Suppose $\sigma_s(X) \in \mathrm{EQ}(2n + 2, R_s[X], J_s[X])$, then there exists $\tilde{\sigma}(X) \in \mathrm{ET}(Q[X], J[X])$ and $l > 0$ such that $\tilde{\sigma}(X)$ localizes to $\sigma(bX)$ for some $b \in (s^l)$ and $\tilde{\sigma}(0) = \mathrm{Id}$.

**Proof**: Since elementary transvections can always be lifted, then we may assume that $R$ is reduced. We will show that there exists $l > 0$ such that $\sigma(bX) \in \mathrm{ET}(M[X], J[X])$ for all $b \in (s^k)$, for all $k \geq l$.

As $\sigma(0) = \mathrm{Id}$, by Lemma 3.9 we can write $\sigma_s(X) = \prod_k \gamma_k \eta_{s,j_k}(X\lambda_k(X))\gamma_k^{-1}$, where $\gamma_k \in \mathrm{EQ}(2n, R_s, \Lambda_s)$, and $\lambda_k(X) \in J_s[X]$. Hence by the proof of Lemma 2.6 there exists $\tilde{\sigma}_{(s,0)}(X) \in \mathrm{EQ}(2n + 2, (R_s \oplus J_s)[X], (\Lambda_s \oplus J_s)[X])$ such that

$$\tilde{\sigma}_{(s,0)}(X) = \prod_k \gamma_k \eta_{s,j_k}(0, X\lambda_k(X))\gamma_k^{-1},$$

where $\phi(\gamma_k) = \gamma_k$, $(0, X\lambda_k(X)) \in (R \oplus J)_{(s,0)}[X]$ and $\phi : R_s \oplus J_s \to R_s$ is defined by $\phi((a, i)) = a + i$. Hence for $d > 0$, $\tilde{\sigma}_{(s,0)}(X^{2^d}) = \prod_k \gamma_k \eta_{s,j_k}(0, XT^{2d}\lambda_k(XT^{2d}))\gamma_k^{-1}$, for some $\gamma_k \in \mathrm{EQ}(2n + 2, R_s, \Lambda_s)$. Using Lemma 3.10 and standard commutator formulas ([14, Lemma 3.16, pg. 43]) we get $\tilde{\sigma}_{(s,0)}(X^{2^d}) = \prod_i \eta_{p_i}\eta_i(T\mu_i(X))$, for some $\mu_i(X) \in (R \oplus J)_{(s,0)}[X]$ with $p_i = 1$ or $q_i = 1$.

Since $P_s$ is a free $R_s$-module, then we have

$$(P \oplus J)_{(s,0)}[X, T] \cong (R \oplus J)_{(s,0)}[X, T] \cong (P \oplus J)_{(s,0)}[X, T]^*$$
Thus using the isomorphism, polynomials in \((P \oplus J)_{(s,0)}[X,T]\) can be regarded as linear forms.

First we consider the case: \(p_i = 1\). Let \(p_1^{*}, p_2^{*}, \ldots, p_n^{*}, p_{-1}^{*}, \ldots, p_{-n}^{*}\) be the standard basis of \((P \oplus J)_{(s,0)}\). Let \(s^m p_i^{*} \in P \oplus J\) for some \(m > 0\) and \(i = \pm 1, \pm 2, \ldots, \pm n\). Let \(e_{i}^{(s)}\) be the standard basis of \((R \oplus J)^{2^n}\). Then for \(q_i = \pm i\), consider the element \(T_{\mu_i}(X)e_{i}^{(s)}\) as an element in \((P \oplus J)_{(s,0)}[X,T]^*\). As \((P \oplus J)_{(s,0)}\) is free, by Lemma 3.11 we can say \(T_{\mu_i}(X)e_{i}^{(s)}\) is a polynomial in \(T\). Again by Lemma 3.12 there exists \(k_1 > 0\) such that \(k_1\) is the maximum power of \((s,0)\) occurring in the denominator of \(\mu_{t}(X)e_{i}^{(s)}\). Choose \(l_1 \geq \max(k, m)\).

Now consider the case \(q_i = 1\). Then, for \(p_i = \pm j\), \(T_{\mu_i}(X)e_{i,j}^{(s)} \in (P \oplus J)_{(s,0)}[X,T]\). By similar argument we can consider \(T_{\mu_i}(X)e_{i,j}^{(s)}\) as a polynomial in \(T\) and hence there exists \(k_2 > 0\) such that \(k_2\) is the maximum power of \((s,0)\) occurring in \(\mu_{i,j}^{(s)}\). Now choose \(l_2 \geq \max(k_2, m)\). For \(l \geq \max(l_1, l_2)\), under the transformation \(T \mapsto (s,0)^{s}(T)\), \(\tilde{\sigma}_{(s,0)}((b,0)XT^{2d})\) is defined over \((Q \oplus J)[\tilde{X},T], i.e., there exists some \(\tilde{\sigma}(X,T) \in \text{ET}((M \oplus J)[X,T])\) such that \(\tilde{\sigma}_{(s,0)}((b,0)XT^{2d}) = \tilde{\sigma}((b,0)X)\in \text{ET}(M[X] \oplus J[X], 0 \oplus J[X])\).

Hence, as before the result follows by applying \(\phi\).

Consequence: Relative L-G principle for the transvection subgroups:

**Theorem 3.15.** Let \(R\) be an almost commutative ring and \(J \subset R\) be an ideal. Let \(P\) and \(Q\) be as in \(\text{[7]}\). Let \(\sigma(X) \in \text{GG}(M[X], J[X])\) with \(\sigma(0) = \text{Id}\). If \(\sigma_m \in \text{EQ}(2n + 2, R_m[X], J_m[X])\) for all \(m \in \text{Max}(C(R))\), then \(\sigma(X) \in \text{ET}(M[X], J[X])\).

**Proof:** Follows by arguing as in the proof of \([8\text{, Lemma 3.10}]\), and using the Proposition \(3.14\).

\[\square\]

4. Relative Stability for Quadratic K1

The aim of this section is to establish the \(K_1\)-stability of the relative transvection groups as an application of Theorem \(3.15\). For the absolute case we refer \(\text{[5]}\) and \(\text{[8]}\).

**Definition 4.1.** Let \(R\) be an associative ring with unity and \(J \subset R\) an ideal. Consider the ring \(D = \{(a, b) \in R \times R : a - b \in J\}\) with addition and multiplication defined by component wise. We call it the double ring relative to the ideal \(J\). For a form ring \((R, \Lambda)\), one extends the involution of \(- : R \rightarrow R\) to the ring \(D\) by defining \(- : D \rightarrow D\) by \(\tilde{\sigma}(a,b) = (\tilde{a}, \tilde{b})\). We fix the element \((\Lambda, \Lambda)\) and define \(\Lambda' = \{(a,b) \in \Lambda \times \Lambda : a - b \in J\}\). Then one can show that \((D, \Lambda')\) is a form ring.

**Definition 4.2.** Let \((R, \Lambda, \lambda)\) and \((S, \Lambda', \lambda')\) be two form rings. A ring homomorphism \(f : R \rightarrow S\) is said to be a morphism of form rings if \(f(\tilde{r}) = \tilde{f(\tilde{r})}\), \(f(\Lambda) \subset \Lambda'\) and \(f(\lambda) = \lambda'\).

**Lemma 4.3.** Let \((R, \Lambda)\) be a form ring and \(J \subset R\) be a two sided ideal. Then the form rings \((D, \Lambda')\) and \((R \oplus J, \Lambda \oplus J)\) are isomorphic.

**Proof:** Consider the homomorphisms \(f : D \rightarrow R \oplus J\) defined by \(f((a,b)) = (a,b - a)\) and \(g : R \oplus J \rightarrow D\) defined by \(g((a,i)) = (a,a + i)\). It can be checked that both \(f\) and \(g\) are form homomorphisms. They serve as inverse of each other.

**Lemma 4.4.** Let \(A\) be a commutative Noetherian ring of (Krull) dimension \(d\) and \(J \subset A\) be an ideal. Then the ring \(D_A\) is also a commutative ring of dimension \(d\).

**Proof:** Clearly \(D_A\) is a commutative Noetherian ring. We first prove that dimension of \(A \oplus J = d\), and the rest follows from Lemma 4.3. Clearly \(A\) can be identified
as a sub-ring \( \{ (r, 0) : r \in A \} \) of \( A \oplus J \). The element \((0, 0)\) is integral over \( A \) since \((0, i)^2 - (i, 0)(0, i) = (0, 0)\). Hence, every element of \( A \oplus J \) is integral over \( R \). Hence \( \dim(A) = \dim(A \oplus J) \). \( \square \)

Recall \( K_1 \)-stability result of Bak–Petrov–Tang (cf. [5]) for general quadratic groups in the absolute case.

**Theorem 4.5.** ([5]) Let \( R \) be an almost commutative with \( \dim(C(R)) = d \). Consider the form ring \((R, \Lambda)\). Then the stabilization map
\[
\frac{GQ(2n, R, \Lambda)}{EQ(2n, R, \Lambda)} \rightarrow \frac{GQ(2n + 2, R, \Lambda)}{EQ(2n + 2, R, \Lambda)}
\]
is an isomorphism for \( n \geq \max(3, d + 2) \).

We prove the above result in the relative case.

**Theorem 4.6.** Let \( R \) be a form ring finitely generated over its center \( C(R) \) with \( \dim(C(R)) = d \) and \( J \subset R \) be an two-sided ideal of \( R \). Then the stabilization map
\[
\frac{GQ(2n, R, J, \Lambda)}{EQ(2n, R, J, \Lambda)} \rightarrow \frac{GQ(2n + 2, R, J, \Lambda)}{EQ(2n + 2, R, J, \Lambda)}
\]
is an isomorphism for \( n \geq \max(3, d + 2) \).

**Proof:** Consider the stabilization map \( \phi : KQ_{1,n} \rightarrow KQ_{1,n+1} \), where
\[
KQ_{1,n}(R, J, \Lambda) = \frac{GQ(2n, R, J, \Lambda)}{EQ(2n, R, J, \Lambda)}.
\]
By [17, 5.3.22], we have the following sequences are exact
\[
1 \rightarrow GQ(2n, R, J, \Lambda) \overset{i}{\rightarrow} GQ(2n, D, \Lambda') \overset{p_2}{\rightarrow} GQ(2n, R, \Lambda) \rightarrow 1
\]
\[
1 \rightarrow EQ(2n, R, J, \Lambda) \overset{i}{\rightarrow} EQ(2n, D, \Lambda') \overset{p_2}{\rightarrow} EQ(2n, R, \Lambda) \rightarrow 1
\]
where the map \( i \) is induced from the map \( M_{2n}(J) \rightarrow M_{2n}(D) \) given by \( i(\alpha) = (\alpha, I_{2n}) \) and \( p_2 \) is induced from the projection map \( p_2 : D \rightarrow R \) given by \( p_2((a, b)) = b \). Thus we have the following commutative diagram:

\[
\begin{array}{cccccc}
1 & \rightarrow & GQ(2n, R, J, \Lambda) & \overset{i}{\rightarrow} & GQ(2n, D, \Lambda') & \overset{p_2}{\rightarrow} & GQ(2n, R, \Lambda) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & EQ(2n, R, J, \Lambda) & \overset{i}{\rightarrow} & EQ(2n, D, \Lambda') & \overset{p_2}{\rightarrow} & EQ(2n, R, \Lambda) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & GQ(2n, R, J, \Lambda) & \overset{i}{\rightarrow} & GQ(2n, D, \Lambda') & \overset{p_2}{\rightarrow} & GQ(2n, R, \Lambda) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & KQ_{1,n}(R, J, \Lambda) & \overset{i}{\rightarrow} & KQ_{1,n}(D, \Lambda') & \overset{p_2}{\rightarrow} & KQ_{1,n}(R, \Lambda) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1
\end{array}
\]

Taking limits over \( n \) in the lower row, we get an exact sequence
\[
1 \rightarrow KQ_1(R, J, \Lambda) \rightarrow KQ_1(D, \Lambda') \rightarrow KQ_1(R, \Lambda) \rightarrow 1
\]
and for any $n$, we get a homomorphism of exact sequences:

\[
1 \longrightarrow \text{KQ}_{1,n}(R, J, \Lambda) \longrightarrow \text{KQ}_{1,n}(D, \Lambda') \longrightarrow \text{KQ}_{1,n+1}(R, \Lambda) \longrightarrow 1
\]

\[
1 \longrightarrow \text{KQ}_1(R, J, \Lambda) \longrightarrow \text{KQ}_1(D, \Lambda') \longrightarrow \text{KQ}_1(R, \Lambda) \longrightarrow 1
\]

For $n \geq d + 2$ the right most map is an isomorphism by Theorem 4.5. Since $R$ is finitely generated over $C(R)$, then it can be checked that $D$ is finitely generated over $C(D)$. Since $C(D)$ is a double ring of $C(R)$ relative to the ideal $C(R) \cap J$, by Lemma 4.2 we get $\dim(C(D)) = d$. Hence for $n \geq d + 2$, the middle homomorphism is also an isomorphism by Theorem 4.5. Hence it follows that the left most map is an isomorphism.

Now we prove the $K_1$-stability of relative transvection groups as an application of relative local-global principle of transvection groups. For this we need to recall the following result of Vaserstein.

**Lemma 4.7.** Let $(R, \Lambda)$ be a form ring finitely generated over $C(R)$ with Krull dimension of $C(R) = d$ and $J \subset R$ be a two-sided ideal of $R$. Let $P, M$ be as in Notation 3.8. Let the rank of $M$ be $2n \geq \max(6, 2d + 2)$. Then the group of elementary transvection $\text{ET}(M \perp \mathbb{H}(R), I)$ acts transitively on the set $\text{Um}(M \perp \mathbb{H}(R), I)$ of unimodular elements which are congruent to $(0, \ldots, 1, \ldots, 0)$ modulo $I$.

**Theorem 4.8.** Let $(R, \Lambda)$ be a form ring finitely generated over $(C(R))$ with Krull dimension of $C(R) = d$ and $J \subset R$ be a two-sided ideal. Let $P, M$ be as defined in the Notation 3.8. Let the rank of $M$ be $2n \geq \max(6, 2d + 4)$. Then the stabilization map

\[ i_n : \text{KQ}_{1,n}GQ(M, J) \rightarrow \text{KQ}_{1,n+1}GQ(M \perp \mathbb{H}(R), J) \]

is an isomorphism.

**Proof:** The surjectivity part will follow from Lemma 4.7. To prove the injectivity, let $\alpha \in \text{GQ}(M, J)$ be such that $\tilde{\alpha} = \alpha \perp \text{Id}$ lies in $\text{ET}(M \perp \mathbb{H}(R), J)$. Let $\varphi(X)$ be the isotropy between $\tilde{\alpha}$ and $\text{Id}$. Now by similar argument as given in [theorem 4.4], we can get an isotropy $\varphi(X) \in \text{GQ}(M[X], J[X])$ between $\alpha$ and $\text{Id}$. Now by localising $\varphi$ at a maximal ideal $m \in \text{Max}(C(R))$, $\varphi_m(X)$ becomes (relative) stably elementary. By Theorem 4.6 it follows that $\varphi_m(X)$ is actually (relative) elementary. Therefore, by relative L-G principle (Theorem 3.8.14) one gets $\tilde{\varphi}(X) \in \text{ET}(M[X], J[X])$. Hence $\alpha = \tilde{\varphi}(1) \in \text{ET}(Q, J)$.

5. **Structure of Unstable Quadratic $K_1$ Groups; Relative Case**

We devote this section to discuss the study of nilpotent property of (relative) unstable $K_1$-groups. Throughout this section we assume $R$ is a commutative ring with identity, i.e., we are considering trivial involution and $n \geq 3$. By $\text{SQ}(2n, R, \Lambda)$ we shall denote the subgroup of $\text{GQ}(2n, R, \Lambda)$ with matrices of determinant 1. And analogously for the relative cases.

Let $G$ be a group. Define $Z^r$ inductively by $Z^0 = G$, $Z^1 = [G, G]$ and $Z^i = [G, G^{i-1}]$. $G$ is said to be nilpotent if $Z^r = \{e\}$ for some $r > 0$.

**Definition 5.1.** A group $G$ is called nilpotent-by-abelian if it has a normal subgroup $H$ such that $H$ is nilpotent and $G/H$ is abelian.
In (cf. [2]), A.Bak proved that the unstable $K_1(R)$ group of $GL_n(R)$ is nilpotent-by-abelian for $n \geq 3$, and hence $K_1(R)$ is solvable. Later it was generalized by R. Hazrat the general quadratic groups over module finite rings; (cf. [18]). In [19] Hazrat–Vavilov revisited this problem for ordinary classical Chevalley groups (that is types A, C, and D) and finally extends it further to the exceptional Chevalley groups (that is types E, F, and G). A simpler and shorter proof is given in [4] for the linear, symplectic and orthogonal groups (absolute cases). In [7], the first author proved this result for the general Hermitian groups and the same proof also works for the general quadratic groups. The relative cases are proved in [4] for the traditional classical groups. In this article we prove the result for the relative general quadratic groups, and as a consequence we get the result for the module cases as an application of (relative) L-G principle for the transvection subgroups.

**Theorem 5.2.** The quotient group $SQ(2n, R, J)/EQ(2n, R, J)$ is nilpotent for $n \geq 3$. The class of nilpotency is at most $\max(1, d + 2 - n)$ where $d = \dim R$.

**Proof:** Note that for $n \geq d + 2$, then the quotient group $SQ(2n, R, J)/EQ(2n, R, J)$ is abelian and hence nilpotent. So we consider the case $n \leq d + 2$. Let us fix a $n$. We prove the theorem by induction on $d = \dim R$. Let $G = SQ(2n, R, J)/EQ(2n, R, J)$, $m = d + 2 - n$ and $\alpha = [\beta, \gamma]$ for some $\beta \in G$ and $\gamma \in Z^{m - 1}$. Clearly the result is true for $d = 0$. Let $\tilde{\beta}$ be the pre-image of $\beta$ and $\tilde{\gamma}$ be the pre-image of $\gamma$ under the map

$$SQ(2n, R, J) \rightarrow SQ(2n, R, J)/EQ(2n, R, J).$$

Consider the double ring $(D', \Lambda')$ as defined in [4]. By [17, 5.3.22], we have the following exact sequences:

$$1 \longrightarrow SQ(2n, R, J, \Lambda) \stackrel{i}{\longrightarrow} SQ(2n, D, \Lambda') \stackrel{p_2}{\longrightarrow} SQ(2n, R, \Lambda) \longrightarrow 1$$

$$1 \longrightarrow EQ(2n, R, J, \Lambda) \stackrel{i}{\longrightarrow} EQ(2n, D, \Lambda') \stackrel{p_2}{\longrightarrow} EQ(2n, R, \Lambda) \longrightarrow 1$$

where the map $i$ is induced from the map $M_{2n}(J) \rightarrow M_{2n}(D)$ given by $i(\alpha) = (\alpha, I_{2n})$ and $p_2$ is induced from the projection map $p_2 : D \rightarrow R$ given by $p_2((a, b)) = b$. Now in the group $H = SQ(2n, D, \Lambda')/EQ(2n, D, \Lambda')$ we have $i(\tilde{\gamma}) \in H^m$. Since $\dim(R) = \dim(D)$ (by Lemma [4.3]), by the result of the absolute cases, we get $[i(\tilde{\beta}), i(\tilde{\gamma})] \in EQ(2n, D, \Lambda')$. It can be checked that $i([\tilde{\beta}, \tilde{\gamma}]) = [i(\tilde{\beta}), i(\tilde{\gamma})]$ and the image of this element under $p_2$ is identity. Hence by the above diagram one gets $[\tilde{\beta}, \tilde{\gamma}] \in EQ(2n, R, J)$, consequently $[\beta, \gamma]$ is trivial in $G$. This completes the proof.

Next consider the pre-images $\tilde{\beta}_1$ of $\tilde{\beta}$ and $\tilde{\gamma}_1$ of $\tilde{\gamma}$ under the map $SQ(2n, R \oplus J) \rightarrow SQ(2n, R, J)$. Now by the result of absolute cases we get $[\beta_1, \gamma_1] \in EQ(2n, R \oplus J)$. Also it can be checked that $[\beta_1, \gamma_1] \in EQ(2n, R \oplus J, 0 \oplus J)$. By projecting $R \oplus J$ onto $R$, one gets $[\tilde{\beta}, \tilde{\gamma}] \in EQ(2n, R, J)$, and hence $\alpha = \{1\}$ in $G$. □

**Corollary 5.3.** Let $(R, \Lambda)$ be a commutative form ring and $J \subset R$ be an ideal of $R$. Then the quotient group $GQ(2n, R, J)/ET(2n, R, J)$ is nilpotent-by-abelian for $n \geq 3$.

**Corollary 5.4.** Let $(R, \Lambda)$ be a commutative form ring and $J \subset R$ be an ideal of $R$. Consider the notation as in [4.8] Let $d = \dim(R)$ and $t = \text{the local rank of } M$. The quotient group $T(M, J)/ET(M, J)$ is nilpotent of class at most $\max(1, d + 3 - t/2)$.

**Proof:** The proof is same as [4, Theorem 4.1]. □
6. Bass’ Nil Group \( NKQ_1(R) \)

The Bass nil-group \( NK_1(R) = \ker(K_1(R[X]) \to K_1(R)); \ X = 0 \). i.e., the subgroup consisting of elements \( [\alpha(X)] \in K_1(R[X]) \) such that \( [\alpha(0)] = [I] \). Hence \( K_1(R[X]) \cong NK_1(R) \otimes K_1(R) \). The aim of the next sections is to study some properties of Bass nil-groups \( NK_1 \) for the general quadratic groups or Bak’s unitary groups.

In this section recall some basic definitions and properties of the representatives of \( NKQ_1(R) \). We represent any element of \( M_{2n}(R) \) as \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), where \( a, b, c, d \in M_n(R) \).

For \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we call \( (a, b) \) the upper half of \( \alpha \). Let \( (R, \lambda, \Lambda) \) be a form ring. By setting \( \Lambda = \{ \bar{a} : a \in \Lambda \} \) we get another form ring \( (R, \bar{\Lambda}, \Lambda) \). We can extend the involution of \( R \) to \( M_n(R) \) by setting \( (a_{ij})^* = (\pi_{ji}) \). For details, cf. [21], [22].

**Definition 6.1.** Let \( (R, \lambda, \Lambda) \) be a form ring. A matrix \( \alpha = (a_{ij}) \in M_n(R) \) is said to be \( \Lambda \)-Hermitian if \( \alpha = -\lambda \alpha^* \) and all the diagonal entries of \( \alpha \) are contained in \( \Lambda \). A matrix \( \beta \in M_n(R) \) is said to be \( \bar{\Lambda} \)-Hermitian if \( \beta = -\lambda \beta^* \) and all the diagonal entries of \( \beta \) are contained in \( \Lambda \).

**Remark 6.2.** A matrix \( \alpha \in M_n(R) \) is \( \Lambda \)-Hermitian if and only if \( \alpha^* \) is \( \bar{\Lambda} \)-Hermitian.

**Lemma 6.3.** [21] Example 2] Let \( \beta \in GL_n(R) \) be a \( \Lambda \)-Hermitian matrix. Then the matrix \( \alpha^* \beta \alpha \) is \( \Lambda \)-Hermitian for every \( \alpha \in GL_n(R) \).

**Definition 6.4.** Let \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2n}(R) \) be a matrix. Then \( \alpha \) is said to be a \( \Lambda \)-quadratic matrix if one of the following equivalent conditions holds:

1. \( \alpha \in GQ(2n, R, \Lambda) \) and the diagonal entries of the matrices \( a^* c, b^* d \) are in \( \Lambda \),
2. \( a^* d + \lambda c^* d = I_n \) and the matrices \( a^* c, b^* d \) are \( \Lambda \)-Hermitian,
3. \( \alpha \in GQ(2n, R, \Lambda) \) and the diagonal entries of the matrices \( ab^*, cd^* \) are in \( \Lambda \),
4. \( ad^* + \lambda bc^* = I_n \) and the matrices \( ab^*, cd^* \) are \( \Lambda \)-Hermitian.

**Lemma 6.5.** Let \( \alpha = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_{2n}(R) \). Then \( \alpha \in GQ^\lambda(2n, R, \Lambda) \) if and only if \( a \in GL_n(R) \) and \( d = (a^*)^{-1} \).

**Proof:** Let \( \alpha \in GQ^\lambda(2n, R, \Lambda) \). In view of (2) of Definition 6.3 we have, \( a^* d = I_n \). Hence \( a \) is invertible and \( d = (a^*)^{-1} \). Converse holds by (2) of Definition 6.3.

**Definition 6.6.** Let \( \alpha \in GL_n(R) \) be a matrix. A matrix of the form \( \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^*)^{-1} \end{pmatrix} \) is denoted by \( \mathbb{H}(\alpha) \) and is said to be hyperbolic.

**Remark 6.7.** In a similar way we can show that matrices of the form \( T_{12}(\beta) := \begin{pmatrix} I_n & \beta \\ 0 & I_n \end{pmatrix} \) is \( \Lambda \)-quadratic matrix if and only if \( \beta \) is \( \bar{\Lambda} \)-Hermitian. And the matrix of the form \( T_{21}(\gamma) := \begin{pmatrix} I_n & 0 \\ \gamma & I_n \end{pmatrix} \) is \( \Lambda \)-quadratic matrix if and only if \( \gamma \) is \( \Lambda \)-Hermitian.

Likewise in the quadratic case we can define the notion of \( \Lambda \)-elementary quadratic groups in the following way:

**Definition 6.8.** The \( \Lambda \)-elementary quadratic group is denoted by \( EQ^\lambda(2n, R, \Lambda) \) and defined by the group generated by \( 2n \times 2n \) matrices of the form \( \mathbb{H}(\alpha) \), where \( \alpha \in E_n(R) \), \( T_{12}(\beta) \) and \( \beta \) is \( \bar{\Lambda} \)-Hermitian and \( T_{21}(\gamma) \) is \( \gamma \) \( \Lambda \)-Hermitian.
Lemma 6.9. Let $A = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in M_{2n}(R)$. Then $A \in GQ^\lambda(2n, R, \Lambda)$ if and only if $\alpha \in GL_n(R)$, $\delta = (\alpha^*)^{-1}$ and $\alpha^{-1}\beta$ is $\bar{\Lambda}$-Hermitian. In this case $A \equiv H(\alpha)$ (mod $E \! Q^\lambda(2n, R, \Lambda)$).

**Proof:** Let $A \in GQ^\lambda(2n, R, \Lambda)$. Then by (4) of Definition 6.3 we have $\alpha \delta^* = I_n$ and $\alpha \beta^*$ is $\Lambda$-Hermitian. Hence $\alpha$ is invertible and $\delta = (\alpha^*)^{-1}$. For $\alpha^{-1}\beta$, we get

$$
(\alpha^{-1}\beta)^* = \beta^*(\alpha^{-1})^* = \alpha^{-1}(\alpha \beta^*)(\alpha^{-1})^*,
$$

which is $\Lambda$-Hermitian by Lemma 6.3. Hence $\alpha^{-1}\beta$ is $\bar{\Lambda}$-Hermitian. Conversely, the condition on $A$ will fulfill the condition (4) of Definition 6.4. Hence $A$ is $\Lambda$-quadratic. Now let $T_{12}(-\alpha^{-1}\beta) \in E \! Q^\lambda(2n, R, \Lambda)$, since $\alpha^{-1}\beta$ is $\Lambda$-Hermitian, and $AT_{12}(\alpha^{-1}\beta) = H(\alpha)$. Thus $A \equiv H(\alpha)$ (mod $E \! Q^\lambda(2n, R, \Lambda)$).

Arguing similarly one gets the following:

Lemma 6.10. Let $B = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in M_{2n}(R)$. Then $B \in GQ^\lambda(2n, R, \Lambda)$ if and only if $\alpha \in GL_n(R)$, $\delta = (\alpha^*)^{-1}$ and $\gamma$ is $\Lambda$-Hermitian. In this case

$$
B \equiv H(\alpha) \pmod{E \! Q^\lambda(2n, R, \Lambda)}.
$$

Lemma 6.11. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GQ^\lambda(2n, R, \Lambda)$. If $a \in GL_n(R)$ then

$$
\alpha \equiv H(a) \pmod{E \! Q^\lambda(2n, R, \Lambda)}.
$$

**Proof:** By same argument as given in Lemma 6.9 we have $a^{-1}b$ is $\bar{\Lambda}$-Hermitian. Hence $T_{12}(-a^{-1}b) \in E \! Q^\lambda(2n, R, \Lambda)$, and consequently $\alpha T_{12}(-a^{-1}b) = \begin{pmatrix} a & 0 \\ c & d' \end{pmatrix} \in GQ^\lambda(2n, R, \Lambda)$ for some $d' \in GL_n(R)$. Hence by Lemma 6.10 we get

$$
\alpha T_{12}(-a^{-1}b) \equiv H(a) \pmod{E \! Q^\lambda(2n, R, \Lambda)}.
$$

Hence $\alpha \equiv H(a)$ (mod $E \! Q^\lambda(2n, R, \Lambda)$).

Definition 6.12. Consider the embedding:

$$
GQ^\lambda(2n, R, \Lambda) \to GQ^\lambda(2n + 2, R, \Lambda),
$$

We denote $GQ^\lambda(R, \Lambda) = \bigcup_{n=1}^{\infty} GQ^\lambda(2n, R, \Lambda)$ and $E \! Q^\lambda(R, \Lambda) = \bigcup_{n=1}^{\infty} E \! Q^\lambda(2n, R, \Lambda)$.

In view of quadratic analog of Whitehead Lemma, we have the group $E \! Q^\lambda(R, \Lambda)$ coincides with the commutator of $GQ^\lambda(R, \Lambda)$. Therefore the group

$$
K \! Q^\lambda(R, \Lambda) := \frac{GQ^\lambda(R, \Lambda)}{E \! Q^\lambda(R, \Lambda)}
$$

is well-defined. The class of a matrix $\alpha \in GQ^\lambda(R, \Lambda)$ in the group $K \! Q^\lambda(R, \Lambda)$ is denoted by $[\alpha]$. In this way we obtain a $K_1$-functor $K \! Q^\lambda(R, \Lambda)$ acting form the category of form rings to the category of abelian groups.

Definition 6.13. In the quadratic case, the kernel of the group homomorphism

$$
K \! Q^\lambda(R[X], \Lambda[X]) \to K \! Q^\lambda(R, \Lambda)
$$

induced from the form ring homomorphism $(R[X], \Lambda[X]) \to (R, \Lambda); X \mapsto 0$ is denoted by $N K \! Q^\lambda(R, \Lambda)$. 

11
Remark 6.14. Since the $\Lambda$-quadratic groups are subclass of the quadratic groups, the local-global principle holds for $\Lambda$-quadratic groups. We use this fact throughout for the next section.

7. Absence of torsion in $NK_\Lambda^1(R, \Lambda)$

In [20], J. Stienstra showed that $NK_1(R)$ is a $W(R)$-module, where $W(R)$ is the ring of big Witt vectors (cf. [12] and [21]). Consequently, in (20, S3), C. Weibel observed that if $k$ is a unit in $R$, then $SK_1(R[X])$ has no $k$-torsion, when $R$ is a commutative local ring. Note that if $R$ is a commutative local ring then $SK_1(R[X])$ coincides with $NK_1(R)$; indeed, if $R$ is a local ring then $SL_n(R) = E_n(R)$ for all $n > 0$. Therefore, we may replace $\alpha(X)$ by $\alpha(X)\alpha(0)^{-1}$ and assume that $[\alpha(0)] = [I]$. In [6], the first author extended Weibel’s result for arbitrary associative rings. In this section we prove the analog result for $\lambda$-unitary Bass nil-groups, viz. $NK_1GQ^\lambda(R, \Lambda)$, where $(R, \Lambda)$ is the form ring as introduced by A. Bak in [1]. The main ingredient for our proof is an analog of Higman linearisation (for a subclass of Bak’s unitary group) due to V. Kopeiko; cf. [21]. For the general linear groups, Higman linearisation (cf. [1]) allows us to show that $NK_1(R)$ has a unipotent representative. The same result is not true in general for the unitary nil-groups. Kopeiko’s results in [21], [22] explain a complete description of the elements of $NK_1GQ^\lambda(R, \Lambda)$ that have (unitary) unipotent representatives.

Definition 7.1. For a associative ring $R$ with unity we consider the truncated polynomial ring

$$R_t = \frac{R[X]}{(X^{t+1})}.$$ 

Lemma 7.2. (cf. [6], Lemma 4.1) Let $P(X) \in R[X]$ be any polynomial. Then the following identity holds in the ring $R_t$:

$$(1 + X^r P(X)) = (1 + X^r P(0))(1 + X^{r+1}Q(X)),$$

where $r > 0$ and $Q(X) \in R[X]$, with $\text{deg}(Q(X)) < t - r$.

Proof: Let us write $P(X) = a_0 + a_1 X + \cdots + a_t X^t$. Then we can write $P(X) = P(0) + XP'(X)$ for some $P'(X) \in R[X]$. Now, in $R_t$

$$(1 + X^r P(X))(1 + X^r P(0))^{-1} = (1 + X^r P(0) + X^{r+1}P'(X))(1 + X^r P(0))^{-1}$$

$$= 1 + X^{r+1}P'(X)(1 - X^r P(0) + X^{2r}(P(0))^2 - \cdots)$$

$$= 1 + X^{r+1}Q(X)$$

where $Q(X) \in R[X]$ with $\text{deg}(Q(X)) < t - r$. Hence the lemma follows. \qed

Remark. Iterating the above process we can write for any polynomial $P(X) \in R[X]$,

$$(1 + XP(X)) = \Pi_{i=1}^t (1 + a_i X^i)$$

in $R_t$, for some $a_i \in R$. By ascending induction it will follow that the $a_i$’s are uniquely determined.

Lemma 7.3. Let $R$ be an associative ring, and $k \in \mathbb{Z}$ such that $kR = R$. Let $P(X)$ be a polynomial in $R[X]$. Assume $P(0)$ lies in the center of $R$. Then

$$(1 + X^r P(X))^{k^r} = 1 \Rightarrow (1 + X^r P(X)) = (1 + X^{r+1}Q(X))$$

in the ring $R_t$ for some $r > 0$, and $Q(X) \in R[X]$ with $\text{deg}(Q(X)) < t - r$.
Following result is due to V. Kopeiko, cf. [21]. This is an analog of Higman linearization for this special case.

**Proposition 7.4.** Let \((R, \Lambda)\) be a form ring. Then, every element of the group \(\text{NKQ}^{\Lambda}(R, \Lambda)\) has a representative of the form

\[
[a; b, c]_n = \begin{pmatrix}
I_r - aX & bX \\
-cX^n & I_r + a^*X + \cdots + (a^*)^nX^n
\end{pmatrix} \in \text{GQ}^\Lambda(2r, R[X], \Lambda[X])
\]

for some positive integers \(r\) and \(n\), where \(a, b, c \in M_r(R)\) satisfy the following conditions:

1. the matrices \(b\) and \(ab\) are Hermitian and also \(ab = ba^*\),
2. the matrices \(c\) and \(ca\) are Hermitian and also \(ca = a^*c\),
3. \(bc = a^{n+1}\) and \(cb = (a^*)^{n+1}\).

**Corollary 7.5.** Let \(R\) be an associative ring. Let \([\alpha] \in \text{NKQ}^{\Lambda}(R, \Lambda)\) has the representation \([a; b, c]_n\) for some \(a, b, c \in M_n(R)\) according to Proposition 7.4. Then \([\alpha] = [\mathbb{H}(I_r - aX)]\) in \(\text{NKQ}^{\Lambda}(R, \Lambda)\) if \((I_r - aX) \in \text{GL}_r(R)\).

**Proof:** By Lemma 6.11 we have \([a; b, c]_n = [\mathbb{H}(I_r - aX)] \pmod{\text{EQ}^\Lambda(2r, R[X], \Lambda[X])}\). Hence \([\alpha] = [\mathbb{H}(I_r - aX)]\) in \(\text{NKQ}^{\Lambda}(R, \Lambda)\). □

**Theorem 7.6.** Let \((R, \Lambda)\) be a form ring, where \(R\) is an associative ring with 1 and \(k\) is an invertible integer, i.e. \(kR = R\). Let

\[\alpha(X) = [\begin{pmatrix} A(X) & B(X) \\ C(X) & D(X) \end{pmatrix}] \in \text{NKQ}^{\Lambda}(R, \Lambda)\]

with \(A(X) \in \text{GL}_r(R[X])\) for some \(r \in \mathbb{N}\). Then \([\alpha(X)]\) has no \(k\)-torsion.

**Proof:** By Theorem 7.4 \([\alpha(X)] = [[a; b, c]_n]\) for some \(a, b, c \in M_n(R)\) and for some natural numbers \(n\) and \(s\). Note that in the Step 1 of the Proposition 7.4 the invertibility of the first corner of the matrix \(\alpha\) will not be changed during the linearization process. Also the invertibility of the first corner is preserved in the remaining steps of the Proposition 7.4. Therefore since the first corner matrix \(A(X) \in \text{GL}_r(R[X])\), we get \((I_r - aX) \in \text{GL}_r(R[X])\). Using Corollary 7.5 one gets \([\alpha(X)] = [\mathbb{H}(I_r - aX)]\). Now let \([\alpha]\) be a \(k\)-torsion. Hence \([\mathbb{H}(I_r - aX)]\) is a \(k\)-torsion. Since \((I_r - aX)\) is invertible, it follows that \(a\) is nilpotent. Let \(a^{k+1} = 0\). Since \([I_r - aX]^{k+1} = [I]\) in \(\text{NKQ}^{\Lambda}(R[X], \Lambda[X])\), by arguing as given in 7, we get \([I_r - aX] = [I]\) in \(\text{NKQ}^{\Lambda}(R[X], \Lambda[X])\). This completes the proof. □

### 8. Graded Analog

We recall the well-known “Swan–Weibel homotopy trick”, which is the main ingredient to handle the graded case. Let \(R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots\) be a graded ring. An element \(a \in R\) will be denoted by \(a = a_0 + a_1 + a_2 + \cdots\), where \(a_i \in R_i\) for each \(i\), and all but finitely many \(a_i\)’s are zero. Let \(R_+ = R_1 \oplus R_2 \oplus \cdots\). Graded structure of \(R\) induces a graded structure on \(M_n(R)\) (ring of \(n \times n\) matrices).

**Definition 8.1.** Let \(a \in R_0\) be a fixed element. We fix an element \(b = b_0 + b_1 + \cdots\) in \(R\) and define a ring homomorphism \(\epsilon: R \rightarrow R[X]\) as follows:

\[\epsilon(b) = \epsilon(b_0 + b_1 + \cdots) = b_0 + b_1X + b_2X^2 + \cdots + b_iX^i + \cdots.\]

Then we evaluate the polynomial \(\epsilon(b)(X)\) at \(X = a\) and denote the image by \(b^+(a)\) i.e., \(b^+(a) = \epsilon(b)(a)\). Note that \((b^+(x))^+ (y) = b^+(xy)\). Observe, \(b_0 = b^+(0)\). We shall use this fact frequently.
The above ring homomorphism $\epsilon$ induces a group homomorphism for $GL(2n, R)$ at the $2n$-th level for every $n \geq 1$, i.e., for $\alpha \in GL(2n, R)$ we get a map

$$\epsilon : GL(2n, R, \Lambda) \to GL(2n, R[X], \Lambda[X])$$

defined by

$$\alpha = a_0 \oplus a_1 \oplus a_2 \oplus \cdots \mapsto a_0 \oplus a_1 X \oplus a_2 X^2 \cdots,$$

where $a_i \in M(2n, R_i)$. As above for $a \in R_0$, we define $\alpha^+(a)$ as $\alpha^+(a) = \epsilon(\alpha)(a)$.

The graded dilation lemma and graded local-global principle is proved in [11] for linear, symplectic and orthogonal groups. Arguing in similar manner one gets:

**Theorem 8.2. (Graded Local-Global Principle)** Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be an almost commutative graded ring with identity $1$. Let $\alpha \in GQ(2n, R, \Lambda)$ be such that $\alpha \equiv I_{2n} \pmod{R_+}$. If $\alpha_m \in EQ(2n, R_m, \Lambda_m)$, for every maximal ideal $m \in \text{Max}(C(R_0))$, then $\alpha \in EQ(2n, R, \Lambda)$.

Moreover, the L-G principle for the elementary subgroups and their normality property are equivalent.

**Theorem 8.3.** Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be an almost commutative graded ring with identity $1$. Then the followings are equivalent for $2n \geq 6$:

1. $EQ(2n, R, \Lambda)$ is a normal subgroup of $GQ(2n, R, \Lambda)$.
2. If $\alpha \in GQ(2n, R, \Lambda)$ with $\alpha^+(0) = I_{2n}$, and $\alpha_m \in EQ(2n, R_m, \Lambda_m)$ for every maximal ideal $m \in \text{Max}(C(R_0))$, then $\alpha \in EQ(2n, R, \Lambda)$.

As an application of Theorem 8.2 and the Theorem 7.3, we obtain the following:

**Theorem 8.4.** Let $R = R_0 \oplus R_1 \oplus \cdots$ be an almost commutative graded (i.e., finite over its center) ring with $1$. Let $N = N_0 + N_1 + \cdots + N_r \in M_r(R)$ be a nilpotent matrix, and $I$ denote the identity matrix. Let $k \in \mathbb{Z}$ be a unit in $R_0$. If $[(I+N)]^k = [I]$ in $NKQ^1(R, \Lambda)$, then $[I+N] = [I+N_0]$.

**Proof:** Consider the ring homomorphism $f : R \to R[X]$ defined by

$$f(a_0 + a_1 + \ldots) = a_0 + a_1 X + \ldots.$$

Then

$$[(I+N)^k] = [I] \Rightarrow f([(I+N)^k]) = f([I+N])^k = [I]$$

$$\Rightarrow [(I+N_0 + N_1 X + \cdots + N_r X^r)]^k = [I].$$

Let $m$ be a maximal ideal in $C(R_0)$. By Theorem 7.6 we have

$$[(I + N_0 + N_1 X + \cdots + N_r X^r)] = [I]$$

in $NKQ^1((R)_m, \Lambda_m)$. Hence by using the local-global principle we conclude

$$[(I+N)] = [I+N_0]$$

in $NK_1GQ^1(R, \Lambda)$, as required. $\square$

**References**

[1] A. Bak; K-Theory of forms. Annals of Mathematics Studies, 98. Princeton University Press, Princeton, N.J. University of Tokyo Press, Tokyo, (1981).

[2] A. Bak; Nonabelian K-theory: the nilpotent class of $K_1$ and general stability. K-Theory 4 (1991), no. 4, 363–397.

[3] H. Bass; K-Theory and stable algebra, Publ. Math. I.H.E.S. No. 22 (1964), 5–60.

[4] A. Bak, R. Basu, R.A. Rao; Local-global principle for transvection groups. Proceedings of The American Mathematical Society 138 (2010), no. 4, 1191–1204.
A. Bak, V. Petrov, G. Tang; Stability for Quadratic $K_1$, K-Theory(30) (2003), 1–11.

R. Basu; Absence of torsion for $NK_1(R)$ over associative rings, J. Algebra Appl. 10(4) (2011), 793–799.

R. Basu; Local-global principle for general quadratic and general Hermitian groups and the nilpotence of $KH_1$. Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) 452 (2016), Voprosy Teorii PredstavleniUı Algebr i Grupp. 30, 5–31. translation in J. Math. Sci. (N.Y.) 232 (2018), no. 5, 591–609.

R. Basu; A note on general quadratic groups. J. Algebra Appl. 17 (2018), no. 11, 1850217, 13.

R. Basu, R.A. Rao, R. Khanna; On Quillen’s local global principle, Commutative algebra and algebraic geometry, 17–30, Contemp. Math., 390, Amer. Math. Soc., Providence, RI, (2005).

R. Basu, R.A. Rao, R. Khanna; Pillars of relative Quillen-Suslin Theory. “Leavitt Path Algebra”, ISI Series, Springer, (2020), 211–223.

R. Basu, Manish Kumar Singh; On Quillen–Suslin Theory for Classical Groups; Revisited over Graded Rings. Contemp. Math. Amer. Math. Soc., Vol. 751, (2020), 5–18.

S. Bloch; Algebraic K-Theory and crystalline cohomology, Publ. Math. I.H.E.S. 47 (1977), 187–268. itary $K_1$-group of unitary ring, Journal of Mathematical Sciences, Vol 240, No 4 (2019), 459–473.

Chouinard L.G.; L.G., Projective modules over Krull semigroups, Michigan Math.J.29, 143-148, 1982.

J. Gubeladze; Classical algebraic K-theory of monoid algebras, K-theory and homological algebra, Lecture Notes in Mathematics 137, (1990), 36–94.

J. Gubeladze; Anderson’s conjecture and the maximal monoid class over which projective modules are free, Math. USSR-Sb. 63 (1998), 165–180.

A. Gupta; Optimal injective stability for the symplectic $K_1Sp$ group, J. Pure Appl. Algebra 219 (2015), 1336–1348.

A.J. Han, O.T. O’Meara; The Classical Groups and K-Theory, Springer-Verlag, Berlin et all (1989).

R. Hazrat; Dimension theory and nonstable $K_1$ of quadratic modules. K-Theory 27 (2002), no. 4, 293–328.

R. Hazrat, N. Vavilov; $K_1$ of Chevalley groups are nilpotent. Journal of Pure and Applied Algebra 179 (2003), no. 1-2, 99–116. birthday. Journal of K-Theory 4, no. 1 (2009), 1–65.

J. Stienstra; Operation in the linear K-theory of endomorphisms, Current Trends in Algebraic Topology, Conf. Proc. Can. Math. Soc. 2 (1982).

V.I. Kopeiko; Bass nilpotent unitary $K_1$ group of unitary ring. (Russian) Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) 460 (2017), Voprosy Teorii PredstavleniUı Algebr i Grupp. 32, 134–157; translation in J. Math. Sci. (N.Y.) 240 (2019), no. 4, 459–473.

V.I. Kopeiko; Unitary symbols and the factorization of hyperbolic matrices. (Russian) Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) 470 (2018), Voprosy Teorii PredstavleniUı Algebr i Grupp. 33, 111–119; translation in J. Math. Sci. (N.Y.) 243 (2019), no. 4, 577–582.

C. Weibel; Mayer-Vietoris Sequence and module structure on $K_0$, Springer Lecture Notes in Mathematics 854 (1981), 466–498.

C. Weibel; Module Structures on the K-theory of Graded Rings, Journal of Algebra 105 (1987), 465–483.

Indian Institute of Science Education and Research (IISER) Pune, India
Email address: rabeya.basu@gmail.com, rbasu@iiserpune.ac.in

Indian Institute of Science Education and Research (IISER) Pune, India
Email address: kuntal.math@gmail.com