REDUCTION OF ABELIAN VARIETIES

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1. Introduction

In this paper we study the reduction of abelian varieties. We assume $F$ is a field with a discrete valuation $v$, $X$ is an abelian variety over $F$, and $n$ is an integer not divisible by the residue characteristic.

In Part 1 we give criteria for semistable reduction. Suppose $n \geq 5$. In Theorem 4.4 we show that $X$ has semistable reduction if and only if $(\sigma - 1)^2 = 0$ on the $n$-torsion in $X$, for every $\sigma$ in the absolute inertia group. In Theorem 4.5 we show (using Theorem 4.4) that $X$ has semistable reduction if and only if there exists a subgroup of $n$-torsion points such that the absolute inertia group acts trivially on both it and its orthogonal complement with respect to the $e_n$-pairing. We deduce as special cases both Raynaud’s criterion (that the abelian variety have full level $n$ structure for $n \geq 3$; see Theorem 4.2) and the criterion of [7] (that the abelian variety have partial level $n$ structure for $n \geq 5$; see Theorem 4.6). We also obtain a (near) converse to the criterion of [7]. The proofs are based on the fundamental results of Grothendieck on semistable reduction of abelian varieties (see [3]). In §5 we allow $n < 5$. In §6 we give a measure of potentially good reduction. We discuss other measures of potentially good reduction in Part 2.

In Part 2 we study Néron models of abelian varieties with potentially good reduction and torsion points of small order. Suppose that the valuation ring is henselian and the residue field is algebraically closed. If $X$ has good reduction, then $X_n \subseteq X(F)$ (this is an immediate corollary of the existence of Néron models; see Lemma [4.3] below). On the other hand, if $X_n \subseteq X(F)$ and $n \geq 3$, then by virtue of Raynaud’s criterion for semistable reduction, $X$ has good reduction. Notice that the failure of $X$ to have good reduction is measured by the dimension $u$ of the unipotent radical of the special fiber of the Néron model of $X$. In particular, $u = 0$ if and only if $X$ has good reduction. In general, $0 \leq u \leq \dim(X)$. The equality $u = \dim(X)$ says that $X$ has purely additive reduction. Another measure of the deviation from good reduction is the (finite) group of connected components $\Phi$ of the special fiber of the Néron model. If $X$ has good reduction then $\Phi = \{0\}$, but the converse statement is not true in general.

The aim of §8 is to connect explicitly the invariants $u$ and $\Phi$ with the failure of $X(F)$ to contain all the $n$-torsion points. This failure can be measured by the index $[X_n : X_n(F)]$. We assume that at least “half” of the $n$-torsion points are rational over $F$. More precisely, we assume that there exists an $F$-rational polarization $\lambda$ on $X$ and a maximal isotropic (with respect to the pairing $e_{\lambda,n}$ induced from the Weil $e_n$-pairing by $\lambda$) subgroup of $X_n$ consisting of $F$-rational points. If in addition $n \geq 5$, then $X$ has good reduction (see Theorem 7.4 of [3]), and therefore $u = 0$, $\Phi = \{0\}$, and $X_n = X_n(F)$. Therefore, we have to investigate only the cases

\begin{itemize}
  \item $u = 0$,
  \item $u > 0$, and $\Phi = \{0\}$,
  \item $u > 0$, and $\Phi \neq \{0\}$.
\end{itemize}
n = 2, 3, and 4. Let $\Phi'$ denote the prime-to-$p$ part of $\Phi$, where $p$ is the residue characteristic (with $\Phi' = \Phi$ if $p = 0$).

We show that if $n = 2$ then $\Phi'$ is an elementary abelian 2-group and $[X_2 : X_2(F)] \# \Phi' = 4^n$, if $n = 3$ then $[X_3 : X_3(F)] = 3^n$ and $\Phi' \cong (\mathbb{Z}/3\mathbb{Z})^u$, and if $n = 4$ then $X_2 \subseteq X(F)$, $[X_4 : X_4(F)] = 4^n$, and $\Phi' \cong (\mathbb{Z}/2\mathbb{Z})^{2u}$. If instead of assuming partial level $n$ structure we assume that all the points of order 2 on $X$ are defined over $F$, then $[X_4 : X_4(F)] = 4^n$ and $\Phi' \cong (\mathbb{Z}/2\mathbb{Z})^{2u}$.

Earlier work on abelian varieties with potentially good reduction and on groups of connected components of Néron models has been done by Serre and Tate [6], Silverman [13], Lenstra and Oort [1], Lorenzini [14], and Edixhoven [1].

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2. Notation and definitions

If $F$ is a field, let $F^s$ denote a separable closure. Suppose that $X$ is an abelian variety defined over $F$, and $n$ is a positive integer not divisible by the characteristic of $F$. Let $X^*$ denote the dual abelian variety of $X$, let $X_n$ denote the kernel of multiplication by $n$ in $X(F^s)$, let $X_n^*$ denote the kernel of multiplication by $n$ in $X^*(F^s)$, and let $\mu_n$ denote the $\text{Gal}(F^s/F)$-module of $n$-th roots of unity in $F^s$. The $e_n$-pairing

$$e_n : X_n \times X_n^* \to \mu_n$$

is a $\text{Gal}(F^s/F)$-equivariant nondegenerate pairing (see §74 of [14]). If $S$ is a subgroup of $X_n$, let

$$S^\perp = \{ y \in X_n^* : e_n(x, y) = 1 \text{ for every } x \in S \} \subseteq X_n^*.$$  

If $\lambda$ is a polarization on $X$, define

$$e_{\lambda,n} : X_n \times X_n \to \mu_n$$

by $e_{\lambda,n}(x, y) = e_n(x, \lambda(y))$ (see §75 of [14]). Then

$$\sigma(e_{\lambda,n}(x_1, x_2)) = e_{\sigma(\lambda),n}(\sigma(x_1), \sigma(x_2))$$

for every $\sigma \in \text{Gal}(F^s/F)$ and $x_1, x_2 \in X_n$. If $n$ is relatively prime to the degree of the polarization $\lambda$, then the pairing $e_{\lambda,n}$ is nondegenerate. If $\ell$ is a prime not equal to the characteristic of $F$, and $d = \dim(X)$, let

$$\rho_{\ell,X} : \text{Gal}(F^s/F) \to \text{Aut}(T_\ell(X)) \cong M_{2d}(\mathbb{Z}_\ell)$$

denote the $\ell$-adic representation on the Tate module $T_\ell(X)$ of $X$, and let $V_\ell(X) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Let $I$ denote the identity matrix in $M_{2d}(\mathbb{Z}_\ell)$.

If $L$ is a Galois extension of $F$, $v$ is a discrete valuation on $F$, and $w$ is an extension of $v$ to $L$, let $I_w$ denote the inertia subgroup at $w$ of $\text{Gal}(L/F)$. If $X$ is an abelian variety over $F$, let $X_v$ denote the special fiber of the Néron model of $X$ at $v$ and let $X_v^{\text{al}}$ denote its identity connected component. Let $a, u$, and $t$ denote, respectively, the abelian, unipotent, and toric ranks of $X_v$. Then $a + u + t = \dim(X)$. If $p$ ($\geq 0$) is the residue characteristic of $v$, let $\Phi_v'$ denote the
prime-to-$p$ part of the group of connected components of the special fiber of the Néron model of $X$ at $v$ (with $\Phi$ the full group of components if $p = 0$).

**Definition 2.1.** If $v$ is a discrete valuation on a field $F$, we say the valuation ring is *strictly henselian* if the valuation ring is henselian and the residue field is algebraically closed.

**Definition 2.2.** Suppose $L/F$ is an extension of fields, $w$ is a discrete valuation on $L$, and $v$ is the restriction of $w$ to $F$. We say that $w/v$ is *unramified* if a uniformizing element of the valuation ring for $v$ induces a uniformizing element of the valuation ring for $w$ and the residue field extension is separable (see Definition 1 on page 78 of [1]).

**Remark 2.3** (Remark 5.3 of [7]). Suppose $v$ is a discrete valuation on a field $F$, and $m$ is a positive integer not divisible by the residue characteristic. Then every degree $m$ Galois extension of $F$ totally ramified at $v$ is cyclic. If $F(\zeta_m) = F$, then $F$ has a cyclic totally ramified extension of degree $m$. (See Remark 5.3 of [7].) Note also that $F$ has no non-trivial unramified extensions if and only if the valuation ring is henselian and the residue field is separably closed.

**Part 1. Semistable reduction of abelian varieties**

**3. Preliminaries**

**Definition 3.1.** If $k$ is a positive integer, define a finite set of prime powers $N(k)$ by

$$N(k) = \{ \text{prime powers } \ell^m : 0 \leq m(\ell - 1) \leq k \}.$$ 

For example,

$$N(1) = \{1, 2\}, \quad N(2) = \{1, 2, 3, 4\},$$

$$N(3) = \{1, 2, 3, 4, 8\}, \quad N(4) = \{1, 2, 3, 4, 5, 8, 9, 16\}.$$ 

**Theorem 3.2.** Suppose $n$ and $k$ are positive integers, $\mathcal{O}$ is an integral domain of characteristic zero such that no rational prime which divides $n$ is a unit in $\mathcal{O}$, $\alpha \in \mathcal{O}$, $\alpha$ has finite multiplicative order, and $(\alpha - 1)^k \in n\mathcal{O}$. If $n \notin N(k)$, then $\alpha = 1$. In particular, if $(\alpha - 1)^2 \in n\mathcal{O}$ and $n \geq 5$, then $\alpha = 1$.

**Proof.** See Corollary 3.3 of [4].

**Lemma 3.3** (Lemma 5.2 of [4]). Suppose that $d$ and $n$ are positive integers, and for each prime $\ell$ which divides $n$ we have a matrix $A_\ell \in M_{2d}(\mathbb{Z}_\ell)$ such that the characteristic polynomials of the $A_\ell$ have integral coefficients independent of $\ell$, and such that $(A_\ell - I)^2 \in nM_{2d}(\mathbb{Z}_\ell)$. Then for every eigenvalue $\alpha$ of $A_\ell$, $(\alpha - 1)/\sqrt{n}$ satisfies a monic polynomial with integer coefficients.
Lemma 3.4 (Lemma 4.2 of [10]). Suppose \( v \) is a discrete valuation on a field \( F \) with residue characteristic \( p \geq 0 \), \( m \) is a positive integer, \( \ell \) is a prime, \( p \) does not divide \( ml \), \( K \) is a degree \( m \) extension of \( F \) which is totally ramified above \( v \), and \( \bar{v} \) is an extension of \( v \) to a separable closure \( K^s \) of \( K \). Suppose that \( X \) is an abelian variety over \( F \), and for every \( \sigma \in \mathcal{I}(\bar{v}/v) \), all the eigenvalues of \( \rho_{\ell,X}(\sigma) \) are \( m \)-th roots of unity. Then \( X \) has semistable reduction at the extension of \( v \) to \( K \).

4. Criteria for semistable reduction

Theorem 4.1 (Galois Criterion for Semistable Reduction). Suppose \( X \) is an abelian variety over a field \( F \), \( v \) is a discrete valuation on \( F \), \( \ell \) is a prime not equal to the residue characteristic of \( v \), \( \bar{v} \) is an extension of \( v \) to \( F^s \), and \( \mathcal{I} = \mathcal{I}(\bar{v}/v) \). Then the following are equivalent:

(i) \( X \) has semistable reduction at \( v \),
(ii) \( \mathcal{I} \) acts unipotently on \( T_\ell(X) \); i.e., all the eigenvalues of \( \rho_{\ell,X}(\sigma) \) are 1, for every \( \sigma \in \mathcal{I} \),
(iii) for every \( \sigma \in \mathcal{I} \), \((\rho_{\ell,X}(\sigma) - I)^2 = 0\).

Proof. See Proposition 3.5 and Corollaire 3.8 of [3] and Theorem 6 on p. 184 of [1].

Theorem 4.2 (Raynaud Criterion for Semistable Reduction). Suppose \( X \) is an abelian variety over a field \( F \) with a discrete valuation \( v \), \( m \) is a positive integer not divisible by the residue characteristic of \( v \), and the points of \( X_m \) are defined over an extension of \( F \) which is unramified over \( v \). If \( m \geq 3 \), then \( X \) has semistable reduction at \( v \).

Proof. See Proposition 4.7 of [3].

Proposition 4.3. Suppose \( X \) is an abelian variety over a field \( F \), \( v \) is a discrete valuation on \( F \), \( n \) is an integer not divisible by the residue characteristic of \( v \), \( \bar{v} \) is an extension of \( v \) to \( F^s \), and \( \mathcal{I} = \mathcal{I}(\bar{v}/v) \). Let \( S = X_m^2 \), the elements of \( X_n \) on which \( \mathcal{I} \) acts as the identity. If \( X \) has semistable reduction at \( v \), then

(i) \((\sigma - 1)^2 X_n = 0\) for every \( \sigma \in \mathcal{I} \), and
(ii) \( \mathcal{I} \) acts as the identity on \( S^{1,n} \).

Proof. Suppose \( X \) has semistable reduction at \( v \). By Theorem 4.1 we have (i). It follows that \( \sigma^n = 1 \) on \( X_n \). Since \( n \) is not divisible by the residue characteristic, \( X_n \) is tamely ramified over \( F \). Let \( \mathcal{J} \) denote the first ramification group. Then the action of \( \mathcal{I} \) on \( X_n \) factors through \( \mathcal{I}/\mathcal{J} \). Let \( \tau \) denote a lift to \( \mathcal{I} \) of a topological generator of the pro-cyclic group \( \mathcal{I}/\mathcal{J} \). Since

\[
e_n((\tau - 1)X_n, (X_n^*)^\mathcal{I}) = 1,
\]

we have

\[
\#((X_n^*)^\mathcal{I}) \#((\tau - 1)X_n) \leq \#X_n^*.\]

The map from \( X_n \) to \((\tau - 1)X_n \) defined by \( y \mapsto (\tau - 1)y \) defines a short exact sequence

\[0 \to S \to X_n \to (\tau - 1)X_n \to 0.\]

Therefore,

\[\#S \#((\tau - 1)X_n) = \#X_n = \#S \#S^{1,n}.\]
Similarly,
\[ \#((X_n^*)^T)\#((\tau - 1)X_n^*) = \#X_n^*. \]
Therefore,
\[ \#S^{1,n} = \#((\tau - 1)X_n) \leq \#((\tau - 1)X_n^*). \]
Since \((\tau - 1)X_n^* \subseteq S^{1,n}\), we conclude that
\[ S^{1,n} = (\tau - 1)X_n^*. \]
By (i), we have \((\tau - 1)^2X_n = 0\). It follows from the natural Gal\((F^s/F)\)-equivariant isomorphism \(X_n^* \cong \text{Hom}(X_n, \mu_n)\) that \((\tau - 1)^2X_n^* = 0\), and therefore \(I\) acts as the identity on \(S^{1,n}\).

**Theorem 4.4.** Suppose \(X\) is an abelian variety over a field \(F\), \(v\) is a discrete valuation on \(F\), \(n\) is an integer not divisible by the residue characteristic of \(v\), \(n \geq 5\), \(\bar{v}\) is an extension of \(v\) to \(F^s\), and \(I = I(\bar{v}/v)\). Then \(X\) has semistable reduction at \(v\) if and only if \((\sigma - 1)^2X_n = 0\) for every \(\sigma \in I\).

**Proof.** If \(X\) has semistable reduction at \(v\) then for every \(\sigma \in I\) we have \((\sigma - 1)^2X_n = 0\), by Proposition 1.3.

Conversely, suppose \(n \geq 5\) and \((\sigma - 1)^2X_n = 0\) for every \(\sigma \in I\). Let \(\mathcal{I}' \subseteq \mathcal{I}\) be the inertia group for the prime below \(\bar{v}\) in a finite Galois extension of \(F\) over which \(X\) has semistable reduction. Take \(\sigma \in \mathcal{I}\). Then \(\sigma^m \in \mathcal{I}'\) for some \(m\). Let \(\ell\) be a prime divisor of \(n\). Theorem 4.1 implies that \((\rho_{\ell,X}(\sigma)^m - I)^2 = 0\). Let \(\alpha\) be an eigenvalue of \(\rho_{\ell,X}(\sigma)\). Then \((\alpha^m - 1)^2 = 0\). Therefore, \(\alpha^m = 1\). By our hypothesis,
\[ (\rho_{\ell,X}(\sigma) - I)^2 \in nM_{2d}(\mathbb{Z}), \]
where \(d = \dim(X)\). By Theorem 4.3 on p. 359 of [3], the characteristic polynomial of \(\rho_{\ell,X}(\sigma)\) has integer coefficients which are independent of \(\ell\). By Lemma 3.3, \((\alpha - 1)^2 \in n\mathbb{Z}\), where \(\mathbb{Z}\) denotes the ring of algebraic integers. Since \(n \geq 5\), by Theorem 3.2 we have \(\alpha = 1\) (i.e., \(I\) acts unipotently on \(T_\ell(X)\)). By Theorem 4.1, \(X\) has semistable reduction at \(v\).

**Theorem 4.5.** Suppose \(X\) is an abelian variety over a field \(F\), \(v\) is a discrete valuation on \(F\), \(n\) is an integer not divisible by the residue characteristic of \(v\), \(n \geq 5\), \(\bar{v}\) is an extension of \(v\) to \(F^s\), and \(I = I(\bar{v}/v)\). Then \(X\) has semistable reduction at \(v\) if and only if there exists a subgroup \(S\) of \(X_n\) such that \(I\) acts as the identity on \(S\) and on \(S^{1,n}\).

**Proof.** Suppose there exists a subgroup \(S\) as in the statement of the theorem. The map \(x \mapsto (y \mapsto e_n(x, y))\) induces a Gal\((F^s/F)\)-equivariant isomorphism from \(X_n/S\) onto \(\text{Hom}(S^{1,n}, \mu_n)\). Suppose \(\sigma \in I\). Then \(\sigma = 1\) on \(S^{1,n}\) and on \(\mu_n\). Therefore, \(\sigma = 1\) on \(X_n/S\). Thus, \((\sigma - 1)^2X_n \subseteq (\sigma - 1)S = 0\). By Theorem 4.4, \(X\) has semistable reduction at \(v\).

Conversely, suppose \(X\) has semistable reduction at \(v\). Let \(S = X_n^\mathcal{I}\), and apply Proposition 1.3.

**Theorem 4.6.** Suppose \(X\) is an abelian variety over a field \(F\), \(v\) is a discrete valuation on \(F\), \(\lambda\) is a polarization on \(X\) defined over an extension of \(F\) which is unramified over \(v\), \(n\) is a positive integer not divisible by the residue characteristic of \(v\), and \(n \geq 5\).
(i) If $\bar{X}_n$ is a maximal isotropic subgroup of $X_n$ with respect to $e_{\lambda,n}$, and the points of $\bar{X}_n$ are defined over an extension of $F$ which is unramified over $v$, then $X$ has semistable reduction at $v$.

(ii) Conversely, if $X$ has semistable reduction at $v$, and the degree of the polarization $\lambda$ is relatively prime to $n$, then there exists a maximal isotropic subgroup of $X_n$ with respect to $e_{\lambda,n}$, all of whose points are defined over an extension of $F$ which is unramified over $v$.

**Proof.** Under the hypotheses in (i), let $S = \bar{X}_n$. Then $S^{\perp,n} = \lambda(S)$, and $X$ has semistable reduction at $v$ by applying Theorem 4.5.

Conversely, suppose $X$ has semistable reduction at $v$. Let $\bar{v}$ be an extension of $v$ to $F^s$ and let $\bar{I} = \bar{I}(\bar{v}/v)$. Let $S = X^{\perp,n}_n$. If $G$ is a subgroup of $X_n$, let

$$G^{\perp,\lambda,n} = \{ y \in X_n : e_{\lambda,n}(x, y) = 1 \text{ for every } x \in G \}.$$ 

Since the degree of $\lambda$ is relatively prime to $n$, $\lambda$ induces an isomorphism between $S^{\perp,\lambda,n}$ and $S^{\perp,n}$. Since $\lambda$ is defined over an unramified extension, $\bar{I}$ acts as the identity on $S^{\perp,\lambda,n}$ by Proposition 4.3ii. Therefore, $S^{\perp,\lambda,n} \subseteq S^{\perp,n}$. Let $H$ be the inverse image in $S$ (under the natural projection) of a maximal isotropic subgroup of $S/S^{\perp,\lambda,n}$. It is easy to check that $H$ is a maximal isotropic subgroup of $X_n$ with respect to $e_{\lambda,n}$, proving (ii).

**Remarks 4.7.** Raynaud’s criterion (Theorem 4.2) follows from Theorem 4.5 by letting $n = m^2$ and $S = X_m \subset X_n$ (since then $S^{\perp,n} = X^{\ast}_m$, the dual Galois module of $X_m$, and $n \geq 5$ whenever $m \geq 3$). The converse of Raynaud’s criterion is clearly false, i.e., semistable reduction does not imply that the $n$-torsion points are unramified (for $n \geq 3$ and $n$ not divisible by the residue characteristic), as can be seen, for example, by comparing Raynaud’s criterion with the Néron-Ogg-Shafarevich criterion for good reduction, and considering an abelian variety with semistable but not good reduction.

Theorem 4.6i is Theorem 6.2 of [7]. Similarly, the other results of [6] and of §3 of [5] can readily be generalized to the setting of Theorem 4.5. Theorem 4.6i shows that the sufficient condition for semistability given in Theorem 6.2 of [7] comes close to being a necessary condition. Note that Theorem 4.6i would be false if the condition on the degree of the polarization were omitted.

**Definition 4.8.** Suppose $v$ is a discrete valuation on $F$ of residue characteristic $p$. We say $v$ satisfies (*) if at least one of the following conditions is satisfied:

(a) $p \neq 2$,

(b) the valuation ring is henselian and the residue field is separably closed.

The techniques of the above proofs can be extended to prove the following result. The proof will appear in Section 14.

**Theorem 4.9.** Suppose $X$ is an abelian variety over a field $F$, and $v$ is a discrete valuation on $F$ of residue characteristic $p \geq 0$. Suppose $k \in \mathbb{Z}$, and $0 < k < 2\dim(X)$.

(i) If either $X$ has semistable reduction at $v$, or $k$ is even and $X$ has purely additive reduction at $v$ which becomes semistable over a quadratic extension of $F$, then

$$(\sigma - 1)^{k+1} H^k_F(X \times_F F^s, \mathbb{Z}_\ell) = 0$$
for every $\sigma \in \mathcal{I}$ and every prime $\ell \neq p$, and
\[(\sigma - 1)^{k+1} H^k_{\text{et}}(X \times_F F^*, \mathbb{Z}/n\mathbb{Z}) = 0\]
for every $\sigma \in \mathcal{I}$ and every prime $\ell \neq p$.

(ii) Suppose $n$ is a positive integer not divisible by $p$, and
\[(\sigma - 1)^{k+1} H^k_{\text{et}}(X \times_F F^*, \mathbb{Z}/n\mathbb{Z}) = 0\]
for every $\sigma \in \mathcal{I}$. Suppose $L$ is a degree $R(k+1, n)$ extension of $F$ which is totally ramified above $v$, and let $w$ be the extension of $v$ to $L$. If $k$ is odd, then $X$ has semistable reduction at $w$. If $k$ is even and $v$ satisfies (*), then either $X$ has semistable reduction at $w$, or $X$ has purely additive reduction at $w$ which becomes semistable over a quadratic extension of $L$.

If we restrict to the case where $n \notin N(k+1)$, we obtain the following result. This result gives necessary and sufficient conditions for semistable reduction, and also necessary and sufficient conditions for $X$ to have either semistable reduction or purely additive reduction which becomes semistable over a quadratic extension.

**Corollary 4.10.** Suppose $X$ is an abelian variety over a field $F$, $v$ is a discrete valuation on $F$ of residue characteristic $p \geq 0$, $k$ and $n$ are positive integers, $\ell$ is a prime number, $k < 2\dim(X)$, $n$ and $\ell$ are not divisible by $p$, and $n \notin N(k+1)$.

(i) Suppose $k$ is odd. Then the following are equivalent:
(a) $X$ has semistable reduction at $v$,
(b) for every $\sigma \in \mathcal{I}$,
\[(\sigma - 1)^{k+1} H^k_{\text{et}}(X \times_F F^*, \mathbb{Z}/\ell\mathbb{Z}) = 0,\]
(c) for every $\sigma \in \mathcal{I}$,
\[(\sigma - 1)^{k+1} H^k_{\text{et}}(X \times_F F^*, \mathbb{Z}/n\mathbb{Z}) = 0.\]

(ii) Suppose $k$ is even and $v$ satisfies (*). Then the following are equivalent:
(a) either $X$ has semistable reduction at $v$, or $X$ has purely additive reduction at $v$ which becomes semistable over a quadratic extension of $F$,
(b) for every $\sigma \in \mathcal{I}$,
\[(\sigma - 1)^{k+1} H^k_{\text{et}}(X \times_F F^*, \mathbb{Z}/\ell\mathbb{Z}) = 0,\]
(c) for every $\sigma \in \mathcal{I}$,
\[(\sigma - 1)^{k+1} H^k_{\text{et}}(X \times_F F^*, \mathbb{Z}/n\mathbb{Z}) = 0.\]

5. Exceptional $n$

In this section we discuss briefly the “exceptional” cases $n = 2, 3, 4$. For the proofs, and for examples which show the results are sharp, we refer the reader to [10].

First, let us state the following “one-way” generalization of Theorem 4.5.

**Theorem 5.1.** Suppose $X$ is an abelian variety over a field $F$, $v$ is a discrete valuation on $F$, and $n$ is an integer greater than 1 which is not divisible by the residue characteristic of $v$. Suppose there exists a subgroup $S$ of $X_n$ such that $\mathcal{I}$ acts as the identity on $S$ and on $S^{\perp_n}$. Then $X$ has semistable reduction over every degree $R(n)$ extension of $F$ totally ramified above $v$. 


It turns out that the converse statement is not true. However, the following result gives an “approximate converse”.

**Theorem 5.2.** Suppose \( n = 2, 3, \) or \( 4 \), respectively. Suppose \( X \) is an abelian variety over a field \( F \), and \( v \) is a discrete valuation on \( F \) whose residue characteristic does not divide \( n \). Suppose \( L \) is an extension of \( F \) of degree \( 4, 3, \) or \( 2 \), respectively, which is totally ramified above \( v \). Then the following are equivalent:

(i) \( X \) has semistable reduction over \( L \) above \( v \),

(ii) there exist an abelian variety \( Y \) over a finite extension \( K \) of \( F \) unramified above \( v \), a separable \( K \)-isogeny \( \varphi : X \to Y \), and a subgroup \( S \) of \( Y_n \) such that \( \bar{I} \) acts as the identity on \( S \) and on \( S^{|\cdot|} \).

Further, \( \varphi \) can be taken so that its kernel is killed by \( 8, 9, \) or \( 4 \), respectively. If \( X \) has potentially good reduction at \( v \), then \( \varphi \) can be taken so that its kernel is killed by \( 2, 3, \) or \( 2 \), respectively.

In the case of low-dimensional \( X \) this result may be improved as follows.

**Theorem 5.3.** In Theorem 5.2, with \( d = \dim(X) \), \( \varphi \) can be taken so that its kernel is killed by \( 4 \) if \( d = 3 \) and \( n = 2 \), by \( 3 \) if \( d = 2 \) and \( n = 3 \), and by \( 2 \) if \( d = n = 2 \). If \( d = 1 \), then we can take \( Y = X \) and \( \varphi \) the identity map.

In the case of elliptic curves this implies the following statement.

**Corollary 5.4.** Suppose \( X \) is an elliptic curve over a field \( F \), and \( v \) is a discrete valuation on \( F \) of residue characteristic \( p \geq 0 \).

(a) If \( p \neq 2 \), then \( X \) has semistable reduction above \( v \) over a totally ramified quartic extension of \( F \) if and only if \( X \) has an \( \bar{I} \)-invariant point of order 2.

(b) If \( p \neq 3 \), then \( X \) has semistable reduction above \( v \) over a totally ramified cubic extension of \( F \) if and only if \( X \) has an \( \bar{I} \)-invariant point of order 3.

(c) If \( p \neq 2 \), then \( X \) has semistable reduction above \( v \) over a quadratic extension of \( F \) if and only if either \( X \) has an \( \bar{I} \)-invariant point of order 4, or all the points of order 2 on \( X \) are \( \bar{I} \)-invariant.

(d) If \( p \neq 2 \) and \( X \) has bad but potentially good reduction at \( v \), then \( X \) has good reduction above \( v \) over a quadratic extension of \( F \) if and only if \( X \) has no \( \bar{I} \)-invariant point of order 4 and all its points of order 2 are \( \bar{I} \)-invariant.

(e) Suppose \( p \) is not 2 or 3. Then the following are equivalent:

(i) \( X \) has no \( \bar{I} \)-invariant points of order 2 or 3,

(ii) there does not exist a finite separable extension \( L \) of \( F \) of degree less than 6 such that \( X \) has semistable reduction at the restriction of \( \bar{v} \) to \( L \).

(f) Suppose \( p \) is not 2 or 3. Then the following are equivalent:

(i) \( X \) has no \( \bar{I} \)-invariant points of order 4 or 3 and not all the points of order 2 are \( \bar{I} \)-invariant,

(ii) there does not exist a finite separable extension \( L \) of \( F \) of degree less than 4 such that \( X \) has semistable reduction at the restriction of \( \bar{v} \) to \( L \).

In the case of potentially good reduction the following statement holds true.

**Theorem 5.5.** Suppose \( X \) is an abelian variety over a field \( F \), \( v \) is a discrete valuation on \( F \) of residue characteristic \( p \geq 0 \), and \( X \) has purely additive and potentially good reduction at \( v \).

(a) If \( p \neq 2 \), then \( X \) has good reduction above \( v \) over a quadratic extension of \( F \) if and only if there exists a subgroup \( S \) of \( X_4 \) such that \( \bar{I} \) acts as the identity on \( S \) and on \( S^{|\cdot|} \).
(b) If $p \neq 3$, then $X$ has good reduction above $v$ over a totally ramified cubic extension of $F$ if and only if there exists a subgroup $S$ of $X_3$ such that $I$ acts as the identity on $S$ and on $S^{1/3}$.

(c) Suppose $p \neq 2$, and $L/F$ is a degree 4 extension, totally ramified above $v$, which has a quadratic subextension over which $X$ has purely additive reduction. Then $X$ has good reduction above $v$ over $L$ if and only if there exists a subgroup $S$ of $X_2$ such that $I$ acts as the identity on $S$ and on $S^{1/2}$.

6. A Measure of Potentially Good Reduction

Suppose $v$ is a discrete valuation on a field $F$, and $X$ is an abelian variety over $F$ which has potentially good reduction at $v$. Let $F_{vnr}$ denote the maximal unramified extension of the completion of $F$ at $v$, let $L$ denote the smallest extension of $F_{vnr}$ over which $X$ has good reduction, and let

$$G_{v,X} = \text{Gal}(L/F_{vnr}).$$

Then $G_{v,X}$ can also be characterized as the inertia group of the extension $F(X_n)/F$, where $n$ is any integer greater than 2 and not divisible by the residue characteristic of $v$ (see Corollary 2 on p. 497 of [6]). Clearly, $X$ has good reduction at $v$ if and only if $G_{v,X} = 1$. The finite group $G_{v,X}$ is a measure of how far $X$ is from having good reduction at $v$.

If $A$ is a matrix, let $P_A$ denote its characteristic polynomial. The following result gives constraints on the group $G_{v,X}$.

**Theorem 6.1.** Suppose $v$ is a discrete valuation on a field $F$, and $X$ is a $d$-dimensional abelian variety over $F$ which has potentially good reduction at $v$. Let $G = G_{v,X}$. Suppose $\ell$ is a prime number not equal to the residue characteristic of $v$. Then the action of $\text{Gal}(F^s/F)$ on the $\ell$-adic Tate module $V_\ell(X)$ induces an embedding

$$f : G \hookrightarrow \text{Sp}_{2d}(\mathbb{Q}_\ell)$$

which satisfies the following properties.

(i) For every $\sigma \in G$, the coefficients of $P_{f(\sigma)}$ are integers which are independent of $\ell$. If $X$ has an $F$-polarization of degree not divisible by $\ell$, then one may choose $f$ so that its image lies in $\text{Sp}_{2d}(\mathbb{Z}_\ell)$.

(ii) If either $(\ell, \#G) = 1$ or $\ell > d + 1$, then there exists an embedding

$$g : G \hookrightarrow \text{Sp}_{2d}(\mathbb{Z}_\ell)$$

such that $P_{g(\sigma)} = P_{f(\sigma)}$ for every $\sigma \in G$.

(iii) If $\ell \geq 5$ then there exists an embedding

$$h : G \hookrightarrow \text{Sp}_{2d}(\mathbb{F}_\ell)$$

such that $P_{h(\sigma)} \equiv P_{f(\sigma)} \pmod{\ell}$ for every $\sigma \in G$.

Further, if $\ell \geq 5$ then there exists an embedding

$$G \hookrightarrow \text{Sp}_{2d}(\mathbb{Z}_\ell)$$

(which does not necessarily “preserve” the characteristic polynomials obtained from the embedding $f$).
See [1] for (i), and see [11] for the case \( (\ell, \#G) = 1 \) of (ii). The remainder of Theorem 6.1 follows from results whose proofs will appear elsewhere (along with examples which show that the results are sharp). Those results apply more generally to measure how far an abelian variety (not necessarily with potentially good reduction) is from having semistable reduction. In some cases, these results apply to more general finite groups than those obtained as \( G_v, X \)'s.

Part 2. Néron models of abelian varieties with potentially good reduction

7. Preliminaries

In [3], the following result was obtained as a corollary of Theorem 3.2 above.

Proposition 7.1 (Theorem 6.10a of [9]). Suppose \( \ell \) is a prime, \( m \) and \( r \) are positive integers, \( \mathcal{O} \) is an integral domain of characteristic zero with no non-zero infinitely \( \ell \)-divisible elements, \( \mathcal{I}\mathcal{O} \) is a maximal ideal of \( \mathcal{O} \), \( M \) is a free \( \mathcal{O} \)-module of finite rank, and \( \Phi \) is an endomorphism of \( M \) of finite multiplicative order such that \( (A-1)^m(\ell-1)^{-1} \in \ell^m \text{End}(M) \). If \( r > 1 \), then the torsion subgroup of \( M/(A-1)M \) is killed by \( \ell^{r-1} \).

Proposition 7.2 (see Proposition 6.1i and Corollary 7.1 of [11]). Suppose \( X \) is a \( d \)-dimensional abelian variety over a field \( F \), \( v \) is a discrete valuation on \( F \) with residue characteristic not equal to \( 2 \), \( \lambda \) is a polarization on \( X \), \( \bar{X}_2 \) is a maximal isotropic subgroup of \( X_2 \) with respect to \( e_{\lambda,2} \), \( \lambda \) and the points of \( \bar{X}_2 \) are defined over an extension of \( F \) which is unramified over \( v \), \( \bar{v} \) is an extension of \( v \) to a separable closure of \( F \), and \( \sigma \in \mathcal{I}(\bar{v}/v) \). Then \( (\rho_{2,X}(\sigma)-I)^2 \in 2M_{2d}(\mathbb{Z}_2) \), and \( X \) has semistable reduction above \( v \) over every totally ramified Galois (necessarily cyclic) extension of \( F \) of degree 4.

Recall that \( u \) denotes the unipotent rank of \( X_v \), \( a \) denotes the abelian rank, and \( \Phi' \) denotes the prime-to-\( p \) part of the group of connected components of the special fiber of the Néron model of \( X \) at \( v \), where \( p \) is the residue characteristic of the discrete valuation \( v \). If \( X \) has potentially good reduction, then \( \dim(X) = a + u \).

Theorem 7.3 (Theorem 7.5 of [11]). Suppose \( v \) is a discrete valuation on a field \( F \) with strictly henselian valuation ring, \( X \) is an abelian variety over \( F \) which has potentially good reduction at \( v \), and either

(a) \( n = 2 \) and the points of \( X_2 \) are defined over \( F \), or

(b) \( n = 3 \) or \( 4 \), \( \lambda \) is a polarization on \( X \) defined over \( F \), and the points of a maximal isotropic subgroup of \( X_n \) with respect to \( e_{\lambda,n} \) are defined over \( F \).

Suppose the residue characteristic \( p \ (\geq 0) \) of \( v \) does not divide \( n \). Then \( \Phi' \cong (\mathbb{Z}/2\mathbb{Z})^{2a} \) if \( n = 2 \) or \( 4 \), and \( \Phi' \cong (\mathbb{Z}/3\mathbb{Z})^u \) if \( n = 3 \).

Lemma 7.4. Suppose \( v \) is a discrete valuation on a field \( F \) such that the valuation ring is strictly henselian. Suppose \( X \) is an abelian variety over \( F \) which has potentially good reduction at \( v \), and suppose \( n \) is a positive integer not divisible by the residue characteristic of \( v \). Let \( \Phi_n \) denote the subgroup of \( X_v/X_v^0 \) of points of order dividing \( n \). Then:

(i) \( (X_v)_n \cong X_n(F) \),

(ii) \( (X_v^0)_n \cong (\mathbb{Z}/n\mathbb{Z})^{2a} \).
(iii) $\Phi_n \cong (X_\nu)_n/(X_\nu^0)_n$, and
(iv) if $X_n(F) \cong (\mathbb{Z}/n\mathbb{Z})^b$, then $\Phi_n \cong (\mathbb{Z}/n\mathbb{Z})^{b-2a}$.

Proof. By Lemma 2 of [3], the reduction map defines an isomorphism of $X^T_n$ onto $(X_\nu)_n$, where $\mathcal{I} = \mathcal{I}(\overline{v}/v)$ for some extension $\overline{v}$ of $v$ to $F^s$. Under our hypotheses on $v$, we have $X^T_n \cong X_n(F)$. Therefore, $(X_\nu)_n \cong X_n(F)$. As shown in the proof of Lemma 1 of [3], $(X_\nu^0)_n \cong (\mathbb{Z}/n\mathbb{Z})^{2a+t}$, where $t$ denotes the toric rank of $X_\nu$. Since $X$ has potentially good reduction at $v$, $t = 0$. Since $X^0_\nu$ is $n$-divisible, we have $\Phi_n \cong (X_\nu)_n/(X_\nu^0)_n$. Part (iv) follows easily from (i), (ii), and (iii). □

8. Néron models

In Theorem 6.3, we generalize Theorem 7.3 to the case of partial level 2 structure. We can recover Theorem 7.3 as a special case. Recall that $u$ denotes the unipotent rank of $X_\nu$, $a$ denotes the abelian rank, and $\Phi'$ denotes the prime-to-$p$ part of the group of connected components of the special fiber of the Néron model of $X$ at $v$, where $p$ is the residue characteristic of $v$ (with $\Phi'$ the full group of components if $p = 0$).

Theorem 8.1. Suppose $v$ is a discrete valuation on a field $F$, suppose the valuation ring is strictly henselian, and suppose the residue field has characteristic $p \neq 2$. Suppose $(X, \lambda)$ is a $d$-dimensional polarized abelian variety over $F$, $X$ has potentially good reduction at $v$, and the points of a maximal isotropic subgroup of $X_2$ with respect to $e_{X,2}$ are defined over $F$. Then:

(i) $\Phi' \cong (\mathbb{Z}/2\mathbb{Z})^{b-2a} = (\mathbb{Z}/2\mathbb{Z})^{b + 2u - 2d}$, where $b$ is defined by $X_2(F) \cong (\mathbb{Z}/2\mathbb{Z})^b$,
(ii) $[X_2 : X_2(F)] \# \Phi' = 2^{2u}$, and
(iii) $X$ has good reduction at $v$ if and only if $\Phi' = \{0\}$ and $X_2 \subseteq X(F)$.

Proof. Let $\overline{v}$ be an extension of $v$ to a separable closure of $F$, let $\mathcal{I} = \mathcal{I}(\overline{v}/v)$, let $k$ be the residue field of $v$, and let $\mathcal{J}$ be the first ramification group (i.e., $\mathcal{J}$ is trivial if $p = 0$ and $\mathcal{J}$ is the pro-$p$-Sylow subgroup of $\mathcal{I}$ if $p > 0$). Suppose $q$ is a prime not equal to $p$, and let $\Phi_q$ denote the $q$-part of the group of connected components of the special fiber of the Néron model of $X$. Since $X$ has potentially good reduction at $v$, $\rho_{q,X}(\sigma)$ has finite multiplicative order for every $\sigma \in \mathcal{I}$. Let $\tau$ be a lift to $\mathcal{I}$ of a generator of the pro-cyclic group $\mathcal{I}/\mathcal{J}$. By §II of [3] (see Lemma 2.1 of [3]),

$\Phi_q$ is isomorphic to the torsion subgroup of $T_q(X)^\mathcal{J}/(\tau - 1)T_q(X)^\mathcal{J}$.

By Proposition 7.2 and Remark 2.3, $X$ has semistable reduction (and therefore good reduction) above $v$ over a totally ramified cyclic Galois extension of $F$ of degree 4. Therefore $\mathcal{I}$ acts on $T_q(X)$ through a cyclic quotient of order 4, so $\rho_{q,X}(\sigma)^4 = I$ for every $\sigma \in \mathcal{I}$. Since $p \neq 2$, we have $\rho_{q,X}(\sigma) = I$ for every $\sigma \in \mathcal{J}$. Therefore, $T_q(X)^\mathcal{J} = T_q(X)$. If $q \neq 2$, then $T_q(X)/(\rho_{q,X}(\tau) - I)T_q(X)$ is torsion-free, so $\Phi_q$ is trivial. Further,

$\Phi_2$ is isomorphic to the torsion subgroup of $T_2(X)/(\tau - 1)T_2(X)$.

We have $(\rho_{2,X}(\tau) - I)^2 \in 2M_{2d}(\mathbb{Z}_2)$, by Proposition 7.2. By Proposition 7.4 with $\ell = 2$, $r = 2$, $m = 1$, and $\mathcal{O} = \mathbb{Z}_2$, $\Phi_2$ is annihilated by 2. Therefore, $\Phi_2$ is an elementary abelian 2-group. By Lemma 7.4, $\Phi' \cong (\mathbb{Z}/2\mathbb{Z})^{b-2a}$. Parts (ii) and (iii) follow immediately. Note that Theorem 7.3b is a special case of Theorem 8.1. □
Theorem 8.2. Suppose \( v \) is a discrete valuation on a field \( F \), suppose the valuation ring is strictly henselian, and suppose the residue field has characteristic \( p \neq 3 \). Suppose \( (X, \lambda) \) is a \( d \)-dimensional polarized abelian variety over \( F \), \( X \) has potentially good reduction at \( v \), and the points of a maximal isotropic subgroup of \( X_3 \) with respect to \( e_{\lambda, 3} \) are defined over \( F \). Then:

(i) \( X_3(F) \cong (\mathbb{Z}/3\mathbb{Z})^{2d-u} \),
(ii) \( X \) has good reduction at \( v \) if and only if \( X_3(F) = X_3 \), and
(iii) \( X \) has purely additive reduction at \( v \) if and only if \( X_3(F) \cong (\mathbb{Z}/3\mathbb{Z})^d \).

Proof. By Theorem 7.3, \( \Phi' \cong (\mathbb{Z}/3\mathbb{Z})^a \). Write \( X_3(F) \cong (\mathbb{Z}/3\mathbb{Z})^b \). By Lemma 7.4, \( \Phi' \cong (\mathbb{Z}/3\mathbb{Z})^{b-2d+2a} \). Therefore, \( b = 2d - u \), and we obtain the desired result. \( \square \)

Theorem 8.3. Suppose \( v \) is a discrete valuation on a field \( F \) with strictly henselian valuation ring, \( X \) is an abelian variety over \( F \) which has potentially good reduction at \( v \), the residue field has characteristic \( p \neq 2 \), and either

(a) the points of \( X_2 \) are defined over \( F \), or
(b) \( \lambda \) is a polarization on \( X \) defined over \( F \), and the points of a maximal isotropic subgroup of \( X_4 \) with respect to \( e_{\lambda, 4} \) are defined over \( F \).

Then

\[ X_4(F) \cong (\mathbb{Z}/4\mathbb{Z})^{2a} \times (\mathbb{Z}/2\mathbb{Z})^{2u}. \]

In particular:

(i) \( X_2 \subseteq X_4(F) \subseteq X_4 \), \( [X_4 : X_4(F)] = 2^{2u} \), \( [X_4(F) : X_2] = 2^{2a} \),
(ii) \( X \) has good reduction at \( v \) if and only if \( X_4(F) = X_4 \), and
(iii) \( X \) has purely additive reduction at \( v \) if and only if \( X_4(F) = X_2 \).

Proof. By Theorem 7.3, we have \( \Phi' \cong (\mathbb{Z}/2\mathbb{Z})^{2u} \). By Lemma 7.4, we have a short exact sequence

\[ 0 \to (\mathbb{Z}/4\mathbb{Z})^{2a} \to X_4(F) \to (\mathbb{Z}/2\mathbb{Z})^{2u} \to 0. \]

Let \( d = \dim(X) \). Since \( X_4(F) \subseteq X_4 \cong (\mathbb{Z}/4\mathbb{Z})^{2d} \), we conclude that \( X_4(F) \cong (\mathbb{Z}/4\mathbb{Z})^{2a} \times (\mathbb{Z}/2\mathbb{Z})^{2u} \). Note that \( X_2 \cong (\mathbb{Z}/2\mathbb{Z})^{2d} = (\mathbb{Z}/2\mathbb{Z})^{2a+2u} \). The rest of the result follows immediately. \( \square \)

As an example, let \( X \) be the elliptic curve defined by the equation \( y^2 = x^3 - 9x \), and let \( F \) be the maximal unramified extension of \( \mathbb{Q} \). Then \( X_2(F) = X_2 = X_4(F) \), \( X \) has additive and potentially good reduction, and \( \Phi' \cong (\mathbb{Z}/2\mathbb{Z})^2 \).

Remarks 8.4. If \( X \) has a polarization \( \lambda \) of odd degree, then \( X_2 \) is a maximal isotropic subgroup of \( X_4 \) with respect to \( e_{\lambda, 4} \).

As stated in the Introduction, Theorems 8.2i and 8.3i are immediate corollaries of Raynaud’s criterion for semistable reduction.

If \( X \) has purely additive reduction, then \( X_4(F) \cong \Phi_p \) (see [4]).

Suppose \( v \) is a discrete valuation on a field \( F \), \( X \) is an abelian variety over \( F \) with potentially good reduction at \( v \), the valuation ring is strictly henselian, \( \ell = 2 \) or \( 3 \), and the residue characteristic is not equal to \( \ell \). Then Theorem 6.1 of [5] implies that if \( \Phi' \) is an elementary abelian \( \ell \)-group, then \( \Phi' \) is a subgroup of \( (\mathbb{Z}/2\mathbb{Z})^{2u} \) if \( \ell = 2 \) or of \( (\mathbb{Z}/3\mathbb{Z})^a \) if \( \ell = 3 \).

For simplicity of exposition, we do not generalize the results of §8 (or the prerequisite results from [5], or related results in §3 of [5]) to the setting of Theorem §5 but leave such generalizations as a straightforward exercise for the reader.
REFERENCES

[1] S. Bosch, W. Lütkébohmert, M. Raynaud, Néron models, Springer, Berlin-Heidelberg-New York, 1990.
[2] B. Edixhoven, On the prime-to-p part of the group of connected components of Néron models, Comp. Math. 97 (1995), 29–49.
[3] A. Grothendieck, Modèles de Néron et monodromie, in Groupes de monodromie en géométrie algébrique, SGA7 I, A. Grothendieck, ed., Lecture Notes in Math. 288, Springer, Berlin-Heidelberg-New York, 1972, pp. 313–523.
[4] H. W. Lenstra, Jr., F. Oort, Abelian varieties having purely additive reduction, J. Pure and Applied Algebra 36 (1985), 281–298.
[5] D. Lorenzini, On the group of components of a Néron model, J. Reine Angew. Math. 445 (1993), 109–160.
[6] J-P. Serre, J. Tate. Good reduction of abelian varieties, Ann. of Math. 88 (1968), 492–517.
[7] A. Silverberg, Yu. G. Zarhin, Semistable reduction and torsion subgroups of abelian varieties, Ann. Inst. Fourier 45, no. 2 (1995), 403–420.
[8] A. Silverberg, Yu. G. Zarhin, Connectedness results for ℓ-adic representations associated to abelian varieties, Comp. math. 97 (1995), 273–284.
[9] A. Silverberg, Yu. G. Zarhin, Variations on a theme of Minkowski and Serre, J. Pure and Applied Algebra 111 (1996), 285–302.
[10] A. Silverberg, Yu. G. Zarhin, Semistable reduction of abelian varieties over extensions of small degree, J. Pure and Applied Algebra, to appear.
[11] A. Silverberg, Yu. G. Zarhin, Subgroups of inertia groups arising from abelian varieties, J. Algebra, to appear.
[12] A. Silverberg, Yu. G. Zarhin, Étale cohomology and reduction of abelian varieties, preprint.
[13] J. H. Silverman, The Néron fiber of abelian varieties with potential good reduction, Math. Ann. 264 (1983), 1–3.
[14] A. Weil, Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948.

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