On the Milnor number of non-isolated singularities of holomorphic foliations and its topological invariance

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Abstract
We define the Milnor number of a one-dimensional holomorphic foliation $\mathcal{F}$ as the intersection number of two holomorphic sections with respect to a compact connected component $C$ of its singular set. Under certain conditions, we prove that the Milnor number of $\mathcal{F}$ on a three-dimensional manifold with respect to $C$ is invariant by $C^1$ topological equivalences.

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1 | INTRODUCTION

One of the most studied invariants in Singularity theory is the Milnor number of a complex hypersurface, cf. [15]. In Foliation theory, for a holomorphic vector field $v = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$ in $\mathbb{C}^2$, the Milnor number arises initially as the intersection number of the curves $P(x, y) = 0$ and $Q(x, y) = 0$ in Seidenberg [18] and Van den Essen [20]; however, these authors did not call it Milnor number. The first authors to establish the terminology of Milnor number for holomorphic foliations have been Camacho, Lins Neto, and Sad in [6]. They proved that the Milnor number of a one-dimensional holomorphic foliation with an isolated singularity is a topological invariant, see [6, Theorem A, p. 149].
The proposal to investigate the Milnor number of foliations with non-isolated singularities arises naturally. In the case of complex hypersurfaces, such a study was done by Parusiński [17] and later generalized to the category of schemes by Aluffi [2]. Motivated by these studies, in this paper we will adapt Parusiński’s definition to approach to define the Milnor number of a one-dimensional holomorphic foliation with non-isolated singularities. For foliations on smooth algebraic varieties, we will apply Fulton’s intersection theory [12] to obtain an explicit formula for the Milnor number of a foliation in terms of the Chern and Segre classes. In a similar way to Parusiński and Aluffi, our definition is given by the intersection number of two sections of a holomorphic vector bundle associated to foliation. The appropriate intersection number is known in Fulton’s theory as an excess intersection.

Before establishing the objectives of this paper, we will give some notations and results. Let $\mathcal{F}$ be a one-dimensional holomorphic foliation in an open subset $U$ of $\mathbb{C}^n$ induced by a holomorphic vector field $v$ in $U$. The Milnor number of $\mathcal{F}$ at $p \in U$ is

$$\mu(\mathcal{F}, p) = \dim_{\mathcal{C}} \mathcal{O}_{n,p} / (P_1, ..., P_n),$$  \hspace{1cm} (1.1)

where $\mathcal{O}_{n,p}$ is the ring of germs of holomorphic functions at $p$ and $(P_1, ..., P_n)$ is the ideal generated by the germs at $p \in U$ of the coordinate functions of $v$. Note that $\mu(\mathcal{F}, p)$ is finite if and only if $p \in U$ is an isolated singularity of $v$. Moreover, it follows from [12, p. 123] that the Milnor number of $\mathcal{F}$ at $p$ agrees with the intersection number at $p$ of the divisors $D_i = \{ P_i = 0 \} \forall i = 1, ..., n$, that is,

$$\mu(\mathcal{F}, p) = i_p(D_1, ..., D_n).$$

When $p$ is an isolated singularity of $\mathcal{F}$, $\mu(\mathcal{F}, p)$ is a topological invariant of $\mathcal{F}$, provided that $n \geq 2$ as proved in [6, Theorem A]. More specifically, one-dimensional holomorphic foliations $\mathcal{F}$ and $\mathcal{F}'$ locally topologically equivalent at $p$ and $p'$, respectively, whenever there is a homeomorphism $\phi$ between neighborhoods of $p$ and $p'$ such that $\phi(p) = p'$ and taking leaves of $\mathcal{F}$ to leaves of $\mathcal{F}'$. Then

$$\mu(\mathcal{F}, p) = \mu(\mathcal{F}', p').$$

Our first aim is to define the Milnor number of foliations with non-isolated singularities. We will give a definition that works for any compact connected component of the singular set of such a foliation (see Definition 3.1), and we will show that it is a generalization of the usual Milnor number of a foliation with an isolated singularity.

Second, motivated by [6], we study the topological invariance problem for the Milnor number of a one-dimensional foliation with a non-isolated singularity. Under some conditions, we solve the problem for one-dimensional foliations on three-dimensional complex manifolds (see Theorem 6.1), and moreover, we explain the reason why our proof does not adapt to arbitrary dimensions (see Remark 6.6).

We remark that if $X$ is a complex hypersurface with non-isolated singularities on a complex manifold $M$, the difference $c_i^{SM}(X) - c_i^{FJ}(X)$ of the Schwartz–MacPherson and Fulton–Johnson classes is, for each $i$, a homology class with support in the homology $H_2(Sing(X))$ of the singular set of $X$. This difference is called the Milnor class of degree $i$ of $X$, see, for instance, [4] and [5, p. 194]. The $0$-degree class of $c_1^{SM}(X) - c_1^{FJ}(X)$ coincides with the Milnor–Parusiński–Aluffi’s number of a
complex hypersurface defined in [2] and [17]. In a future paper, we hope to study and describe the Milnor class of degree \( i \geq 1 \) of a one-dimensional holomorphic foliation on a complex manifold.

The paper is organized as follows: in Section 2, we define the concept of one-dimensional holomorphic foliations on complex manifolds. Section 3 is devoted to the definition of the Milnor number of a one-dimensional foliation with respect to a compact connected component of its singular set as an intersection number. In Section 4, we give two examples of foliations on the three-dimensional complex projective space with non-isolated singularities where its Milnor number is exhibited. In Section 5, we explain the reason why the Milnor number of foliations along non-isolated singularities cannot be defined as the Poincaré–Hopf index of any vector field. In Section 6, we prove the main result of the paper, Theorem 6.1. The theorem asserts, under certain conditions, that the Milnor number of a foliation is a topological invariant.

## 2 ONE-DIMENSIONAL HOLOMORPHIC FOLIATIONS

Let \( M \) be an \( n \)-dimensional complex manifold. A one-dimensional holomorphic foliation \( \mathcal{F} \) on \( M \) may be defined as follows: we take an open covering \( \{ U_j \}_{j \in I} \) of \( M \) and on each \( U_j \), a holomorphic vector field \( v_j \) with zeros set of codimension at least 2, and we require that on \( U_j \cap U_i \), the vector fields \( v_j \) and \( v_i \) coincide up to multiplication by a nowhere vanishing holomorphic function:

\[
v_i = g_{ij} v_j \quad \text{on } U_i \cap U_j, \quad g_{ij} \in \mathcal{O}_M^*(U_i \cap U_j).
\]

This means that the local integral curves of \( v_i \) and \( v_j \) glue together, up to reparametrization, giving the so-called leaves of \( \mathcal{F} \). Then \( \mathcal{F} \) is an equivalence class of collection \( \{ U_j, v_j \}_{j \in I} \), where the equivalence relation is given by: \( \{ U_j, v_j \}_{j \in I} \sim \{ U'_j, v'_j \}_{j \in I'} \) if \( v_j \) and \( v'_j \) coincide on \( U_j \cap U'_j \) up to multiplication by a nowhere vanishing holomorphic function. The singular set \( \text{Sing}(\mathcal{F}) \) of \( \mathcal{F} \) is the complex subvariety of \( M \) defined by

\[
\text{Sing}(\mathcal{F}) \cap U_j := \text{Sing}(v_j), \quad \forall j \in I.
\]

The functions \( g_{ij} \in \mathcal{O}_M^*(U_i \cap U_j) \) form a multiplicative cocycle and hence give a cohomology class in \( H^1(M, \mathcal{O}_M^*) \), that is, a line bundle on \( M \), the so-called cotangent bundle of \( \mathcal{F} \), and denoted by \( T^*_\mathcal{F} \). Its dual \( T_\mathcal{F} \), which is represented by the inverse cocycle \( \{ g_{ij}^{-1} \} \), is called tangent bundle of \( \mathcal{F} \).

The relations \( v_i = g_{ij} v_j \) on \( U_i \cap U_j \) can be glued to a global holomorphic section \( s \) of \( TM \otimes T^*_\mathcal{F} \). Since each \( v_j \) has zeros set of codimension at least 2, \( s \) also has zeros set of codimension at least 2. Note that \( s \) is not entirely intrinsically defined by \( \mathcal{F} \): if we change from \( \{ U_j, v_j \}_{j \in I} \) to \( \{ U_j, f v_j \}_{j \in I} \), where \( f \in \mathcal{O}_M^*(M) \), then \( s \) will be replaced by \( f s \). But this ambiguity causes no problem to the definition of \( \mathcal{F} \) by global sections.

A complex hypersurface \( V \) in \( M \) is said to be invariant by \( \mathcal{F} \) if

\[
v_j(f_j) = h_j f_j \quad \forall j \in I,
\]
where \( V \cap U_j = \{ f_j = 0 \} \) and \( h_j \in \mathcal{O}(U_j) \).
3 | THE MILNOR NUMBER AS INTERSECTION NUMBER

Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $M$. Suppose that $\text{Sing}(\mathcal{F})$ has complex codimension at least 2. By definition, $\mathcal{F}$ is given by a section $s : M \to E := TM \otimes T^*_{\mathcal{F}}$ with zero set $\text{Sing}(\mathcal{F})$. Let $C$ be a compact connected component of $\text{Sing}(\mathcal{F})$. We have the fiber square

$$
\begin{array}{ccc}
C & \xrightarrow{i} & M \\
\downarrow & & \downarrow s \\
M & \xrightarrow{s_0} & E
\end{array}
$$

where $i$ is the canonical inclusion and $s_0$ is the zero section of $E$. Let $U$ be a small neighborhood of $C$. We follow Parusiński [17, p. 248] to consider $\text{ind}_U(s)$ the intersection number over $U$ of $s$ and the zero section $s_0$ of $E$. Parusiński remarked that if $s'$ is a small perturbation of $s$ transversal to the zero section, then $\text{ind}_U(s)$ equals the number of zeros of $s'$ counted with signs (local indices). Moreover, $\text{ind}_U(s)$ depends only on the homotopy class of $s|_{\partial U}$ in the space of nowhere zero section of $E|_{\partial U}$ and if $E$ is trivial, this definition agrees with that of the topological degree (see, for instance, [1]). Using standard homotopy arguments, it is easy to prove that $\text{ind}_U(s)$ depends only on $s$ and $C$, so in order to standardize the notation with the intersection theory in Algebraic Geometry, we shall denote $i_C(s, s_0)$ the number $\text{ind}_U(s)$.

**Definition 3.1.** We define the Milnor number of $\mathcal{F}$ at $C$ by

$$
\mu(\mathcal{F}, C) = i_C(s, s_0).
$$

**Remark 3.2.** If $M$ is a smooth $n$-dimensional algebraic variety and $C$ is smooth, we can apply Fulton’s intersection theory (see, for instance, [12, Proposition 6.1] or [2, p. 328]) to define the intersection number between the sections $s_0$ and $s$ along $C$ as

$$
i_C(s, s_0) = \{c(TM \otimes T^*_{\mathcal{F}}) \cap s(C, M)\}_0 \in A_0(C) \tag{3.1}
$$

where $A_0(C)$ is the Chow group of $S$ of degree zero. Here and in the following $c$ denotes total Chern class and $s$ is the Segre class (in the sense of [12]); moreover, pullback notations are omitted when there is no ambiguity. According to Fulton [12, p. 153] (see also Eisenbud–Harris [10, p. 458]), this definition of Minor number for $\mathcal{F}$ works for compact connected components of $\text{Sing}(\mathcal{F})$.

**Remark 3.3.** If $C = \{p\}$ is an isolated singularity of $\mathcal{F}$, it follows from [12, Proposition 8.2] that

$$
\mu(\mathcal{F}, p) = \text{length}(\mathcal{O}_n_{\mathcal{F}}(0), p).
$$

If $s$ is locally generated at $p$ by $v = P_1(z_1, \ldots, z_n) \frac{\partial}{\partial z_1} + \cdots + P_n(z_1, \ldots, z_n) \frac{\partial}{\partial z_n}$, where $P_1, \ldots, P_n \in \mathcal{O}_{n, p}$, we get

$$
\mu(\mathcal{F}, p) = \dim_c \frac{\mathcal{O}_{n, p}}{(P_1, \ldots, P_n)}.
$$
If the singularities of \( \mathcal{F} \) are all isolated, the Baum–Bott formula \([3]\) says

\[
\sum_{p \in \text{Sing}(\mathcal{F})} \mu(\mathcal{F}, p) = c_n(TM \otimes T^*_{\mathcal{F}}) \cap [M],
\]

where \( c_n \) denotes the top Chern class.

Let us consider \( M = \mathbb{P}^n \) and let \( \mathcal{F} \) be a one-dimensional foliation of degree \( d \) in \( \mathbb{P}^n \). The degree \( d \) of \( \mathcal{F} \) is the number of tangencies between \( \mathcal{F} \) and a generic hyperplane. It is not difficult to prove that \( T^*_{\mathcal{F}} = \mathcal{O}(d - 1) \) so that \( \mathcal{F} \) is given by a global section \( s \) of \( T\mathbb{P}^n(d - 1) \). When all the singularities of \( \mathcal{F} \) are isolated, we get from equality \( 3.2 \) that

\[
\sum_{p \in \text{Sing}(\mathcal{F})} \mu(\mathcal{F}, p) = \sum_{i=0}^{n} d^i.
\]

On the other hand, when the singular scheme of \( \mathcal{F} \) is formed by a disjoint union of proper smooth subschemes \( C \) and \( F \), with \( F \) finite, then it follows from Vainsencher \([19, \text{p. 81}]\) that

\[
\mu(\mathcal{F}, C) = \sum_{i \geq 0} \int_{\mathbb{P}^n} c_{n-(1+i)}(T\mathbb{P}^n(d - 1))s_{1+i}(C, \mathbb{P}^n)
\]

and

\[
\sum_{p \in F} \mu(\mathcal{F}, p) + \mu(\mathcal{F}, C) = \sum_{i=0}^{n} d^i.
\]

For a study on the number of residual isolated singularities of foliations on complex projective spaces, see \([8, 11, 16]\) and the interested reader may consult \([7, 9]\) for results about the classification of one-dimensional foliations of low degree on threefolds.

## 4 | EXAMPLES

In this section, we give some examples of one-dimensional holomorphic foliations on \( \mathbb{P}^3 \) with non-isolated singularities where its Milnor number is computed.

**Example 4.1.** Let us consider the foliation \( \mathcal{F}_0 \) of degree 2 in \( \mathbb{P}^3 \) defined in the open affine set \( U_3 = \{[\xi_0 : \xi_1 : \xi_2 : \xi_3] \in \mathbb{P}^3 : \xi_3 \neq 0 \} \) by the vector field

\[
X_0(z) = z_1^2 \frac{\partial}{\partial z_1} + z_2^2 \frac{\partial}{\partial z_2} + z_3^2 \frac{\partial}{\partial z_3},
\]

where \( z_i = \xi_{i-1}/\xi_3 \) for \( i = 1, 2, 3 \). Let \( C = \{\xi_0 = \xi_1 = 0\} \), then the singular set of \( \mathcal{F}_0 \) is

\[
\text{Sing}(\mathcal{F}_0) = C \cup \{p\},
\]

where \( p = [1 : 1 : 1 : 0] \). It is not difficult to see \( \mu(\mathcal{F}_0, p) = 1 \) which implies that

\[
\mu(\mathcal{F}_0, C) = 14.
\]
In fact, let $\tilde{F}_t$ be a generic perturbation of $F_0$, $0 < |t| < \varepsilon$, with $\varepsilon$ sufficiently small, described in $U_3$ by the vector field $X_t$ as follows:

$$X_t = X_0(z) + t \sum_{i=1}^{3} \sum_{j=0}^{2} P_{ij}(z_1, z_2, z_3) \frac{\partial}{\partial z_i},$$

where $P_{ij}$ are homogeneous polynomials of degree $j$ for all $i = 1, 2, 3$. Note that $\tilde{F}_t$ has degree 2, for all $t \neq 0$. For $P_{ij}$ generic polynomials, Bézout’s theorem implies that $\tilde{F}_t|U_3$ contains 8 isolated points, counted with multiplicities. Let $z'_k = (z'_{1k}, z'_{2k}, z'_{3k})$ be one these points, where $k = 1, \ldots, 8$. Therefore, we have

$$\lim_{t \to 0} \left( (z'_{1k})^2 + t \sum_{j=0}^{2} P_{1j}(z'_k) \right) = \lim_{t \to 0} (z'_{1k})^2 = 0,$$

which implies that

$$\lim_{t \to 0} z'_{1k} = 0.$$

In the same way, we can conclude

$$\lim_{t \to 0} z'_{2k} = 0,$$

that is,

$$\lim_{t \to 0} z'_{k} \in \mathbb{C}, \quad \forall \ k = 1, \ldots, 8.$$

The hyperplane at infinite $H_3 = \mathbb{P}^3 \setminus U_3$ is an invariant hypersurface by $\tilde{F}_t$ which on it is described by the vector field

$$Y_t = (u_t^2 - u_1 u_2 + t Q_1(u)) \frac{\partial}{\partial u_1} + (u_1^2 - u_2^3 + t Q_2(u)) \frac{\partial}{\partial u_2},$$

where $Q_i(u) = P_{i2}(u_1, u_2, 1) - u_i P_{32}(u_1, u_2, 1)$, $u_i = \xi_{i-1}/\xi_2$ for $i = 1, 2$. On $H_3$, there are seven singular points of $\tilde{F}_t$, counted with multiplicities. Let $u'_k = (u'_{1k}, u'_{2k})$ be one these points. In order to compute these singular points, we must solve the following system:

$$\begin{cases}
(u'_{1k})^2 - (u'_{1k}) (u'_{2k})^2 + t Q_1(u'_{k}) = 0 \\
(u'_{1k})^2 - (u'_{2k})^3 + t Q_2(u'_{k}) = 0.
\end{cases}$$

With these two equations, we get

$$(u'_{2k})^3 - u'_{1k} (u'_{2k})^2 + t (Q_1(u'_{k}) - Q_2(u'_{k})) = 0,$$

where we obtain an expression for $u'_{1k}$. By replacing this expression of $u'_{1k}$ in the second equation of the above system, we get the following equation:

$$(u'_{2k})^6 - (u'_{2k})^7 + t Q_2(u'_{k})(u'_{2k})^4 + 2t(u'_{2k})^3(Q_1(u'_{k}) - Q_2(u'_{k})) + t^2(Q_1(u'_{k}) - Q_2(u'_{k}))^2 = 0.$$
Let \((u_{1k}, u_{2k}) = \lim_{t \to 0} u_k^t = \lim_{t \to 0}(u_{1k}^t, u_{2k}^t)\). We get
\[
(u_{2k})^6 - (u_{2k})^7 = 0,
\]
which implies that either \(u_{2k} = 0\) or \(u_{2k} = 1\). If \(u_{2k} = 0\), then \(u_{1k} = 0\) and if \(u_{2k} = 1\), then \(u_{1k} = 1\) since \(u_{1k}^2 - u_{1k}u_{2k}^2 = u_{1k}^2 - u_{2k}^2 = 0\). Hence, we have two possibilities: either \(\lim_{t \to 0} u_k^t = (0, 0)\) or \(\lim_{t \to 0} u_k^t = (1, 1)\). The point \(p = [1 : 1 : 1 : 0]\) corresponds to \((1,1)\) and \(q = [0 : 0 : 1 : 0] \in C\) corresponds to \((0,0)\). Finally, it is not difficult to see that \(\mu(F_t|_{H_3}, q) = 6\) which implies
\[
\mu(F_0, C) = (8 + 6) = 14.
\]

**Example 4.2.** Let \(F\) be the holomorphic foliation on \(\mathbb{P}^3\) defined in the affine open set \(U_3 = \{[\xi_0 : \xi_1 : \xi_2 : \xi_3] \in \mathbb{P}^3 : \xi_3 \neq 0\}\) by the vector field
\[
X_0 = (a_0 z_1(z_3 - 1) + a_1 z_2(z_1 - 1)) \frac{\partial}{\partial z_1} + (b_0 z_1(z_1 - 1) + b_1 z_2(z_3 - 1)) \frac{\partial}{\partial z_2}
\]
\[
+ z_1(c_0(z_1 - 1) + c_1(z_3 - 1)) \frac{\partial}{\partial z_3},
\]
where \(z_i = \xi_{i-1}/\xi_3\) for \(i = 1, 2, 3\) and \(a_i, b_i, c_i\) are non-null complex numbers. One checks that the singular set of \(F\) consists of two curves \(C_1\) and \(C_2\) defined by
\[
C_1 = \{\xi_0 = \xi_1 = 0\}, \quad C_2 = \{\xi_0 - \xi_3 = \xi_2 - \xi_3 = 0\}
\]
and four isolated points on the hyperplane \(H_3 = \mathbb{P}^3 \setminus U_3\). In order to compute the Milnor numbers of \(C_1\) and \(C_2\), we will use the perturbation \(F_t\) of \(F\) which is described in \(U_3\) by the vector field \(X_t\) given by
\[
X_t = X_0 + t \left( A(z) \frac{\partial}{\partial z_2} + B(z) \frac{\partial}{\partial z_3} \right),
\]
where \(A(z) = \alpha_0 z_1^2 + \alpha_1 z_1 z_2 + \alpha_2 z_2^2\) and \(B(z) = \beta_0(z_1 - 1)^2 + \beta_1(z_1 - 1)(z_3 - 1) + \beta_2(z_3 - 1)^2\) are generic quadratic functions. Note that \(C_1\) and \(C_2\) are invariant curves of \(F_t\) for \(t \neq 0\). By Baum–Bott’s formula, there are three isolated points of \(\text{Sing}(F_t)\) on \(C_1\) and \(C_2\) for \(t \neq 0\). In fact, the points \(p_1 = [0 : 0 : 1 : 0] \in H_3, p_2 = [0 : 0 : z_{31} : 1]\) and \(p_3 = [0 : 0 : z_{32} : 1]\) belong to \(C_1\), with \(B(0,0, z_{3i}) = 0, i = 1, 2;\) and the points \(p_4 = [0 : 1 : 0 : 0] \in H_3, p_5 = [1 : z_{21} : 1 : 1]\) and \(p_6 = [1 : z_{22} : 1 : 1]\) belong to \(C_2\), with \(A(1, z_{2i}, 1) = 0, i = 1, 2.\) Therefore, the singular set of \(F_t\) contains eight isolated points in the affine open set \(U_3\), counting with multiplicities. More precisely, two of these eight points are on \(C_1\), namely, \(p_2\) and \(p_3\); two these points are on \(C_2\), namely, \(p_5\) and \(p_6\). Furthermore, two of these eight points converge to \(C_1\) and two these points converge to \(C_2\) when \(t\) tends to \(0\). In fact, let \(z_t = (z_{1t}, z_{2t}, z_{3t}) \in (\text{Sing}(F_t) \setminus \{C_1 \cup C_2\}) \cap U_3\). Thus, we can write \(z_{1t} = \lambda_t z_{2t}\) and \(z_{3t}^{-1} - 1 = \eta_t(z_{1t} - 1)\) which yields \(a_0 \lambda_t \eta_t + a_1 = 0\) and
\[
\begin{cases}
(b_0 \lambda_t + b_1 \eta_t)(z_{1t} - 1) + ta(\lambda_t)z_{2t} = 0, \\
\lambda_t(c_0 + c_1 \eta_t)z_{2t} + tb(\eta_t)(z_{1t} - 1) = 0,
\end{cases}
\]
(4.1)
where \( a(\lambda) = \alpha_0 \lambda^2 + \alpha_1 \lambda + \alpha_2 \) and \( b(\eta) = \beta_0 + \beta_1 \eta + \beta_2 \eta^2 \). Given that \( z_{1t} - 1 \neq 0 \) and \( z_{2t} \neq 0 \) we get
\[
\lambda_t (b_0 \lambda_t + b_1 \eta_t) (c_0 + c_1 \eta_t) - t^2 a(\lambda_t) b(\eta_t) = a_0 \lambda_t \eta_t + a_1 = 0.
\]
Let \( \lambda^{(i)}_t \) be the roots of this last equations, for \( i = 1, 2, 3, 4 \). Reordering, if necessary, we can admit that
\[
\lim_{t \to 0} \lambda^{(1)}_t = 0, \quad \lim_{t \to 0} \lambda^{(2)}_t = \frac{a_1 c_1}{a_0 c_0}, \quad \lim_{t \to 0} \lambda^{(3)}_t = \sqrt{\frac{a_1 b_1}{a_0 b_0}}, \quad \lim_{t \to 0} \lambda^{(4)}_t = -\sqrt{\frac{a_1 b_1}{a_0 b_0}}.
\]
Solving the system (4.1), we get
\[
z^{(i)}_t = \left( \frac{b_0 (\lambda^{(i)}_t)^2 + b_1 \eta^{(i)}_t \lambda^{(i)}_t}{b_0 (\lambda^{(i)}_t)^2 + b_1 \eta^{(i)}_t \lambda^{(i)}_t + t a(\lambda^{(i)}_t)}, 1 + \eta^{(i)}_t (z^{(i)}_{1t} - 1) \right).
\]
For \( i = 1, 2 \) we obtain
\[
\lim_{t \to 0} z^{(i)}_{1t} = 1,
\]
which implies that
\[
\lim_{t \to 0} z^{(i)}_t \in C_2.
\]
More precisely,
\[
\lim_{t \to 0} z^{(1)}_t = p_4, \quad \lim_{t \to 0} z^{(2)}_t = (1, 1 / \lambda^{(2)}_0, 1), \quad \lambda^{(2)}_0 = \frac{a_1 c_1}{a_0 c_0}.
\]
Now, from the second equation of (4.1), we get
\[
z_{2t} = \frac{tb(\eta_t)}{\lambda_t [(c_0 + c_1 \eta_t) + tb(\eta_t)]},
\]
which yields
\[
\lim_{t \to 0} z^{(i)}_{2t} = 0
\]
for \( i = 3, 4 \). Thus, in this situation,
\[
\lim_{t \to 0} z^{(i)}_t \in C_1.
\]
In the hyperplane \( H_3 \), the singular set of \( \mathcal{F}_t \) contains seven more points, \( p_1 \in C_1 \) and \( p_4 \in C_2 \) being two of those seven points. However, given that the Milnor numbers \( \mu(\mathcal{F}|_{H_3}, p_1) = 1 \) and \( \mu(\mathcal{F}|_{H_3}, p_4) = 2 \) we get \( \mu(\mathcal{F}, C_1) = 5 \) and \( \mu(\mathcal{F}, C_2) = 6 \).

5 | THE MILNOR NUMBER AND THE POINCARÉ–HOPF INDEX

Let \( \mathcal{F} \) be a one-dimensional holomorphic foliation on a neighborhood of \( (\mathbb{C}^n, 0) \), \( n \geq 2 \), with an isolated singularity at the origin. It is well known (see [6]) that the Milnor number \( \mu(\mathcal{F}, 0) \) coincides with the Poincaré–Hopf index at \( 0 \in \mathbb{C}^n \) of any holomorphic vector field generating \( \mathcal{F} \). In
particular, the Poincaré–Hopf index of a holomorphic vector field tangent to $\mathcal{F}$ depends only on the foliation $\mathcal{F}$. This property holds also for non-holomorphic vector fields and will be important to give a brief explanation of this fact. Let $\mathcal{F}$ and $\mathcal{F}$ be two continuous vector fields tangent to $\mathcal{F}$, both with an isolated singularity at $0 \in \mathbb{C}^n$. Then, if $B$ is a small ball centered at the origin, there exists a continuous $f : \partial B \rightarrow \mathbb{C}^*$ such that $\mathcal{F} = f \mathcal{F}$ on $\partial B$. But it happens that any such map $f$ is necessarily homotopic to a constant map. This implies that the vector fields $\mathcal{F}$ and $\mathcal{F}$ are homotopic as nowhere zero sections of $T\mathbb{C}^n|_{\partial B}$ and therefore their Poincaré–Hopf indexes coincide. These arguments also work for defining a Milnor number for isolated singularities of continuous orientable two-dimensional distributions on manifolds, as we can see in [13].

The Poincaré–Hopf index for real vector fields is well understood, even for non-isolated singularities. Nevertheless, this fact cannot be directly used to define a Milnor number for non-isolated singularities of holomorphic foliations: a foliation is not always defined only by a vector field. Furthermore, even if the foliation is defined by a vector field, in general, this vector field will not be unique, as we have seen in the case of isolated singularities above. In that case, the Milnor number is well defined because any continuous map from $\partial B$ to $\mathbb{C}^*$ is homotopically trivial. This fact is a particular property of the sphere $\partial B$ and need not to be true if $B$ is a neighborhood of a general connected component of the singular set of the foliation.

We recall the definition of the Poincaré–Hopf index for non-isolated singularities of real vector fields. Let $\mathcal{F}$ be a continuous vector field on a manifold $M$ and let $S$ be a compact connected component of the singular set of $\mathcal{F}$. Let $T$ be a compact neighborhood of $S$ such that there are no singularities on $T \setminus S$ and take a vector field $\mathcal{F}$ on $T$ with isolated singularities and such that $\mathcal{F} = \mathcal{F}$ near $\partial T$. Then the Poincaré–Hopf index of $\mathcal{F}$ at $S$ is defined as

$$\text{Ind}(\mathcal{F}, S) = \sum_{p \in \text{Sing}(\mathcal{F})} \text{Ind}(\mathcal{F}, p),$$

where $\text{Ind}(\mathcal{F}, p)$ denotes the Poincaré–Hopf index of $\mathcal{F}$ at the isolated singularity $p$. Of course, this definition is based on the fact that the sum of indexes of $\mathcal{F}$ on $T$ depends only on the vector field $\mathcal{F}|_{\partial T} = \mathcal{F}|_{\partial T}$, that is, only depends on $\mathcal{F}$. In fact, that sum depends only on the homotopy class of $\mathcal{F}$ in the space of nowhere zero sections of $TM|_{\partial T}$. Unfortunately, this property is no longer true for holomorphic foliations, as shown in the following examples.

**Example 5.1.** Let $\mathcal{F}$ and $\mathcal{F}$ be two polynomial vector fields on $\mathbb{C}^2$ with isolated singularities, both with the same linear part

$$(ax + by) \frac{\partial}{\partial x} + (cx + dy) \frac{\partial}{\partial y},$$

where $ad - bc \neq 0$. Let $B$ be a small ball centered at the origin and let $T$ be the complement of $B$ in the projective complex plane $\mathbb{C}P^2$. The vector fields $\mathcal{F}$ and $\mathcal{F}$ define two holomorphic foliations $\mathcal{F}$ and $\mathcal{F}$ with isolated singularities on $T$. Since $\mathcal{F}$ and $\mathcal{F}$ have the same non-degenerated linear part at $0 \in \mathbb{C}^2$, for $B$ small enough the vector fields $\mathcal{F}$ and $\mathcal{F}$ are homotopic as nowhere zero sections of $T\mathbb{C}^2|_{\partial B}$. This implies that, viewed as continuous distributions, $\mathcal{F}$ and $\mathcal{F}$ are homotopic on $\partial T$. Nevertheless, the sum of Milnor numbers on $T$ for the foliations $\mathcal{F}$ and $\mathcal{F}$ are not necessarily the same, because this sum depends on the degree of the corresponding foliation. In fact, by deforming these foliations we easily obtain an example of two continuous distributions on $T$, coinciding on $\partial T$, but with different sums of Milnor numbers on $T$. This shows that a possible definition of the
Milnor number for non-isolated singularities of two-dimensional distributions does not work in the same way as in the case of vector fields, at least in a general setting.

**Remark 5.2.** Example 5.1 also shows that the Poincaré–Hopf Index Theorem for holomorphic vector fields proved in [14, Theorem 1] cannot be extended to one-dimensional holomorphic foliations.

**Example 5.3.** Consider the holomorphic vector field

\[ v = x^2 \frac{\partial}{\partial x} + y^n \frac{\partial}{\partial y} \]

on \( \mathbb{C} \times \mathbb{C} \). This vector field extends to a vector field on \( \mathbb{C} \times \mathbb{C} \) with a unique singularity at \((0,0)\) of Milnor number \(2n\). It is easy to see that the vector field \( v \) is transverse to the boundary of the compact domain

\[ D = \mathbb{C} \times \overline{\mathbb{D}}. \]

Nevertheless, the sum of Poincaré–Hopf indexes of \( v \) on \( D \) — exactly \( 2n \) — is not always equal to \( \chi(D) = 2 \). Thus, this example shows that the surjectivity of the natural morphism \( H^1(D, \mathbb{Z}) \rightarrow H^1(\partial D, \mathbb{Z}) \), assumed in [14] as a hypothesis, cannot be removed.

### 6 | TOPOLOGICAL INVARIANCE OF THE MILNOR NUMBER

It is well known [6, 13] that the Milnor number of an isolated singularity of a holomorphic foliation is a topological invariant. Essentially, this theorem is based upon the following two facts.

1. A holomorphic foliation on a complex surface \( V \) near an isolated singularity \( p \in V \) is always generated by a vector field.
2. The isolated singularity \( p \) has an arbitrarily small neighborhood \( B \) in \( V \) such that the set \( B^* = B \setminus \{p\} \) has the following property: Any continuous map \( f : B^* \rightarrow \mathbb{C}^* \) is homotopically null. In fact, if \( B \) is a ball, then \( B \) is homotopically equivalent to the three-dimensional sphere and we know that \( \pi_3(\mathbb{C}^*) = 0 \).

We give a sketch of the proof of the topological invariance of the Milnor number of an isolated singularity. Let \( \phi \) be a homeomorphism between a neighborhood of \( p \) in \( V \) to a neighborhood of \( p' \) in \( V' \) conjugating the foliations \( \mathcal{F} \) and \( \mathcal{F}' \). Let \( Z \) be a holomorphic vector field defining \( \mathcal{F} \) near \( p \). For the sake of simplicity, we assume that \( \phi \) is a \( C^1 \) diffeomorphism. Then the vector field \( Z' = d\phi \cdot Z \) is a continuous vector field tangent to \( \mathcal{F}' \) with an isolated singularity at \( p' \). As we have seen in Section 5, the Milnor number of \( \mathcal{F}' \) at \( p' \) is equal to the Poincaré–Hopf index of \( Z' \). So the topological invariance of the Milnor number follows from the topological invariance of the Poincaré–Hopf index. If \( \phi \) is only a homeomorphism, we still can define \( Z' \) as a local continuous real flow with an isolated singularity at \( p' \). In this case, we need to extend the definition of the Poincaré–Hopf index for isolated singularities of real continuous flows (see [13]), and the proof follows essentially in the same way. The importance of the first fact above is evident because it permits us to reduce the Milnor number to a Poincaré–Hopf index. On the other hand, the second
fact above is of capital importance, because it guarantees that the Milnor number of a foliation coincides with the Poincaré–Hopf index of any continuous flow tangent to the foliation.

In general, if $\mathcal{F}$ is a holomorphic foliation on a complex manifold $M$ and $C$ is a connected component of $\text{Sing}(\mathcal{F})$, the two facts above are not necessarily true, so the topological invariance of the Milnor number $\mu(\mathcal{F}, C)$ seems to be a non-trivial problem if $C$ has positive dimension. We present a partial solution to this problem.

**Theorem 6.1.** Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on a complex three-dimensional manifold $M$ such that $\text{Sing}(\mathcal{F})$ has codimension bigger than one. Let $\mathcal{F}'$ be another holomorphic foliation on a complex three-dimensional manifold $M'$ topologically equivalent to $\mathcal{F}$ by an orientation-preserving $C^1$ diffeomorphism $\phi : M \rightarrow M'$. We assume that $\phi$ preserves the natural orientation of the leaves. Let $C$ be a compact connected component of $\text{Sing}(\mathcal{F})$. Suppose that $C$ has arbitrarily small neighborhoods $V$ with $H^1(V, \mathbb{Z}) = 0$. Then

$$\mu(\mathcal{F}, C) = \mu(\mathcal{F}', \phi(C)).$$

Observe that in the statement of Theorem 6.1 the equivalence $\phi$ is globally defined on $M$. Nevertheless, the manifold $M$ need not be closed and, in particular, $M$ could be a small neighborhood of $C$. We also note that the existence of arbitrarily small neighborhoods of $C$ with vanishing first cohomology group is obviously fulfilled if $C$ is an isolated singularity.

It is easy to see that Theorem 6.1 is a direct consequence of the following two propositions.

**Proposition 6.2.** Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on a complex three-dimensional manifold $M$ such that $\text{Sing}(\mathcal{F})$ has codimension bigger than one. Let $\mathcal{F}'$ be another holomorphic foliation on a complex three-dimensional manifold $M'$ topologically equivalent to $\mathcal{F}$ by a $C^1$ diffeomorphism $\phi : M \rightarrow M'$. We assume that $\phi$ preserves the natural orientation of the leaves. Let $C$ be a compact connected component of $\text{Sing}(\mathcal{F})$. Then there exist a neighborhood $\Omega$ of $C$ and an isomorphism $g : TM|_\Omega \rightarrow TM'|_{\phi(\Omega)}$ of complex vector bundles with the following properties:

1. $g$ covers the homeomorphism $\phi|_\Omega : \Omega \rightarrow \phi(\Omega)$;
2. if $x \in \Omega \setminus \text{Sing}(\mathcal{F})$, then $g(T_x \mathcal{F}) = T_{\phi(x)} \mathcal{F}'$.

**Proposition 6.3.** Let $M$ be a complex manifold such that $H^1(M, \mathbb{Z}) = 0$. Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $M$ such that $\text{Sing}(\mathcal{F})$ has codimension bigger than one. Let $\mathcal{F}'$ be another holomorphic foliation on a complex manifold $M'$ topologically equivalent to $\mathcal{F}$ by an orientation-preserving homeomorphism $\phi : M \rightarrow M'$. Let $C$ be a compact connected component of $\text{Sing}(\mathcal{F})$. Suppose that there exists an isomorphism $g : TM \rightarrow TM'$ of complex vector bundles with the following properties:

1. $g$ covers the homeomorphism $\phi : M \rightarrow M'$;
2. if $x \in M \setminus \text{Sing}(\mathcal{F})$, then $g(T_x \mathcal{F}) = T_{\phi(x)} \mathcal{F}'$.

Then

$$\mu(\mathcal{F}, C) = \mu(\mathcal{F}', \phi(C)).$$

In order to prove Proposition 6.2 we need two lemmas that we state and prove below. Let $E$ and $E'$ be finite-dimensional complex vector spaces and let $\sigma : E \rightarrow E'$ be a real-linear map. We
say that $\sigma$ is complex-antilinear if $\sigma(\alpha x) = \overline{\alpha} \sigma(x)$ for all $\alpha \in \mathbb{C}$, $x \in E$. It is well known that any real-linear map $\sigma : E \to E'$ can be expressed in a unique way as

$$\sigma = \partial \sigma + \bar{\sigma},$$

were $\partial \sigma : E \to E'$ is complex-linear and $\bar{\sigma} : E \to E'$ is complex-antilinear. For each $t \in [0, 1]$ define the real linear map

$$H^t_\sigma = \partial \sigma + t\bar{\sigma}.$$ 

The family $\{H^t_\sigma\}_{t \in [0, 1]}$ will be called the canonical deformation of $\sigma$. Since $H^t_\sigma$ depends continuously on $\sigma$, this canonical deformation will be useful in the construction of deformations of real isomorphism of complex bundles. Let $e_1, \ldots, e_n$ be a base of $E$. As a real vector space, $E$ can be endowed with the natural orientation defined by the basis $e_1, ie_1, \ldots, e_n, ie_n$. We do the same with $E'$. A subspace $L \subset E$ is called a complex line if $\dim_{\mathbb{C}} L = 1$; in this case, $L = \mathbb{C}v$ for any non-zero element $v \in L$ and, as a real vector space, the complex line $L$ has the natural orientation defined by the basis $\{v, iv\}$.

**Lemma 6.4.** Let $\sigma : E \to E'$ be any real-linear map between the complex vector spaces $E$ and $E'$. Consider the canonical deformation $H^t_\sigma$ of $\sigma$. Let $L$ and $L'$ be complex lines in $E$ and $E'$, respectively, and suppose that $\sigma(L) \subset L'$. Then, for each $t \in [0, 1]$ we have $H^t_\sigma(L) \subset L'$.

**Proof.** Let $v \in L$. Since $L \subset L'$, we have that

$$\sigma(v) = \partial \sigma(v) + \bar{\sigma}(v)$$

and

$$\sigma(iv) = i\partial \sigma(v) - i\bar{\sigma}(v)$$

are contained in $L'$. From this we obtain that $\partial \sigma(v)$ and $\bar{\sigma}(v)$ are contained in $L'$, and therefore, given $t \in [0, 1]$, we have that

$$H^t_\sigma(v) = \partial \sigma(v) + t\bar{\sigma}(v)$$

is contained in $L'$.

**Lemma 6.5.** Assume that $\dim_{\mathbb{C}} E = \dim_{\mathbb{C}} E' = n \in \mathbb{N}$, let $\sigma : E \to E'$ be an orientation-preserving real-linear isomorphism, and consider the canonical deformation $H^t_\sigma$ of $\sigma$. Suppose that there exist $n - 1$ linearly independent complex lines $L_1, \ldots, L_{n-1}$ in $E$ such that each $L_j$ is mapped by $\sigma$ onto a complex line in $E'$ preserving the natural orientations of complex lines. Then, for each $t \in [0, 1]$, the map $H^t_\sigma : E \to E'$ is an orientation-preserving real-linear isomorphism. In particular, $H^0_\sigma = \partial \sigma$ is a complex-linear isomorphism.

**Proof.** Without loss of generality we can assume that $E = E' = \mathbb{C}^n$ and that, for $j = 1, \ldots, n - 1$, both $L_j$ and $\sigma(L_j)$ are equal to the $j$th complex axis of $\mathbb{C}^n$. Then $\sigma$ preserves each of the first $n - 1$
axes and we can express \( \sigma \) as a \( n \times n \) matrix

\[
\sigma = \begin{bmatrix}
A_1 & 0 & 0 & \ldots & 0 & B_1 \\
0 & A_2 & 0 & \ldots & 0 & B_2 \\
0 & 0 & A_3 & \ldots & 0 & B_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & A_{n-1} & B_{n-1} \\
0 & 0 & 0 & \ldots & 0 & A_n
\end{bmatrix},
\]

whose entries are real \( 2 \times 2 \) matrices. So, it is easy to see that

\[
H_t' = \begin{bmatrix}
\partial A_1 + t \bar{\partial} A_1 & 0 & \ldots & 0 & \partial B_1 + t \bar{\partial} B_1 \\
0 & \partial A_2 + t \bar{\partial} A_2 & \ldots & 0 & \partial B_2 + t \bar{\partial} B_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \partial A_{n-1} + t \bar{\partial} A_{n-1} & \partial B_{n-1} + t \bar{\partial} B_{n-1} \\
0 & 0 & \ldots & 0 & \partial A_n + t \bar{\partial} A_n
\end{bmatrix},
\]

hence

\[
\det H_t' = \det(\partial A_1 + t \bar{\partial} A_1) \ldots \det(\partial A_n + t \bar{\partial} A_n).
\]

Therefore, it suffices to show that

\[
\det(\partial A_j + t \bar{\partial} A_j) > 0
\]

for all \( t \in [0, 1] \), \( j = 1, \ldots, n \). Since \( \sigma \) preserves the orientation of each of the first \( n-1 \) axes, we have that \( \det A_1, \ldots, \det A_{n-1} > 0 \). Then, since by hypothesis \( \det \sigma \) is positive, we also have \( \det A_n > 0 \). Given \( j = 1, \ldots, n \), there are constants \( a, b \in \mathbb{C} \) such that \( \partial A_j(z) = az, \bar{\partial} A_j(z) = b \bar{z} \), so by a direct computation we obtain

\[
\det(\partial A_j + t \bar{\partial} A_j) = |a|^2 - t^2 |b|^2 \geq |a|^2 - |b|^2 = \det(A_j) > 0.
\]

6.1 Proof of Proposition 6.2

We start the proof with the following assertion.

**Assertion.** If \( p \in C \), then there exist infinitely many complex lines \( L \) in \( T_p M \) such that \( d\phi(p)(L) \) is a complex line in \( T_{\phi(p)} M' \). In fact, consider the set \( A \) of the complex lines \( L \) in \( T_p M \) such that \( L = \lim T_{p_n} F \) for some sequence \( (p_n) \) in \( M \setminus \text{Sing}(F) \) with \( p_n \to p \). Given any such complex line \( L \) in \( A \), since \( \phi \) maps leaves of \( F \) to leaves of \( F' \) we have

\[
d\phi(p_n)
\big(T_{p_n} F\big) = T_{\phi(p_n)} F' \quad \text{for all } n \in \mathbb{N}.
\]

Thus, since \( \phi \in C^1 \) and the space of complex lines in \( TM' \) is closed, the real linear space

\[
d\phi(p)(L) = \lim d\phi(p_n)
\big(T_{p_n} F\big) = \lim T_{\phi(p_n)} F'
\]
is in fact a complex line in $T_{\phi(p)}M'$. So it suffices to prove that the set $A$ is infinite. It is easy to see that the set $A$ is non-empty and connected; thus, if $A$ is finite, we need to assume that it is unitary, say $A = \{ L \}$. From this we obtain the following implication:

$$
\zeta \to p, \ \zeta \in M \setminus \text{Sing}(\mathcal{F}) \Rightarrow \lim T_{\zeta} \mathcal{F} = L. \tag{6.1}
$$

Consider holomorphic coordinates $(x, y, z)$ at $p$ such that $p = (0, 0, 0), L = [0 : 0 : 1]$ and take a holomorphic vector field

$$a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}$$

generating $\mathcal{F}$ at $p$. The property (6.1) implies that for $\zeta \in M \setminus \text{Sing}(\mathcal{F})$ close enough to $p$ we have $c(\zeta) \neq 0$. Thus, since $\text{Sing}(\mathcal{F})$ has codimension $\geq 2$, we deduce that $c(\zeta) \neq 0$ for all $\zeta$ in a neighborhood of $p$; hence, $\mathcal{F}$ is regular at $p$, which is a contradiction and the assertion is proved.

By the assertion above we can take two different complex lines $L_1$ and $L_2$ in $T_p M$ such that $d\phi(p)(L_1)$ and $d\phi(p)(L_2)$ are complex lines in $T_{\phi(p)} M'$. Observe that $L_1 \neq L_2$ implies that $L_1, L_2$ are linearly independent. So, it follows by Lemma 6.5 that, for any $p \in S$, the canonical deformation $H_t^{d\phi(p)}$ of $d\phi(p)$ is an orientation-preserving isomorphism for each $t \in [0, 1]$. Since $S$ is compact and $d\phi$ is continuous, we can find a neighborhood $\Omega$ of $S$ in $M$ such that the canonical deformation $H_t^{d\phi(x)}$ of $d\phi(x)$ is an orientation-preserving isomorphism for each $t \in [0, 1] x \in \Omega$. This allows us to deform isotonically the real isomorphism of complex bundles

$$d\phi : TM|_\Omega \to TM'|_{\phi(\Omega)}$$

into the complex isomorphism

$$g : TM|_\Omega \to TM'|_{\phi(\Omega)}$$

defined by $g|_{T_x M} = \partial(d\phi(x))$ for all $x \in \Omega$. The first statement of Proposition 6.3 is clearly satisfied by $g$. Let $x \in \Omega \setminus \text{Sing}(\mathcal{F})$. Since $d\phi(x)(T_x \mathcal{F}) = T_{\phi(x)} \mathcal{F}'$, it follows from Lemma 6.4 that $g(T_x \mathcal{F}) = T_{\phi(x)} \mathcal{F}'$, so the second statement is proved.

**Remark 6.6.** Observe that three pairwise distinct complex lines can be linearly dependent. For this reason, our proof of Proposition 6.2 only works in dimension three.

### 6.2 Proof of Proposition 6.3

By [13, Theorem 5.1], the isomorphism

$$\phi^* : H^2(M, \mathbb{Z}) \to H^2(M', \mathbb{Z})$$

induced by $\phi$ maps the Chern class of $T_{\mathcal{F}}$ onto the Chern class of $T_{\mathcal{F}'}$. Since the Chern Class is a complete invariant in the classification of complex line bundles up to isomorphism, we have that there exists an isomorphism $\xi : T_{\mathcal{F}} \to T_{\mathcal{F}'}$, covering the map $\phi : M \to M'$. Consider the dual of
the inverse isomorphism $\xi^{-1}$,

$$f = (\xi^{-1})^* : T^*_\mathcal{F} \to T^*_\mathcal{F},$$

and

$$h := g \otimes f : TM \otimes T^*_\mathcal{F} \to TM' \otimes T^*_\mathcal{F}.$$

Let $s$ be a section of $TM \otimes T^*_\mathcal{F}$ defining $\mathcal{F}$. Denote by $s_0$ and $s'_0$ the zero sections of $TM \otimes T^*_\mathcal{F}$ and $TM' \otimes T^*_\mathcal{F}$, respectively. By the topological invariance of the intersection number we have that

$$\mu(\mathcal{F}, C) = i_C(s, s_0) = i_{C'}(h \circ s, s'_0), \quad (6.2)$$

where $C' = \phi(C)$.

**Assertion.** Let $s'$ be a section of $TM' \otimes T^*_\mathcal{F}'$ defining $\mathcal{F}'$. Then there exists $\theta : M \setminus C \to \mathbb{C}^*$ continuous such that

$$h(s(x)) = \theta(x)s'(\phi(x)),$$

for all $x \in M \setminus C$. Fix $x \in M \setminus C$. Since $\mathcal{F}$ is defined by the section $s$, there exists $\zeta \in (T_{\mathcal{F}})_x$ and $v \in T_x\mathcal{F} \subset T_x M$ such that $s(x) = v \otimes \zeta$, so

$$h(s(x)) = g(v) \otimes f(\zeta).$$

Since $\mathcal{F}'$ is defined by the section $s'$, there exist $\zeta' \in (T_{\mathcal{F}'})_{\phi(x)}$ and $v' \in T_{\phi(x)}\mathcal{F}' \subset T_{\phi(x)} M'$ such that $s'(\phi(x)) = v' \otimes \zeta'$. Since $v \in T_x\mathcal{F}$, by hypothesis we have that $g(v) \in T_{\phi(x)}\mathcal{F}'$, so there exists $\alpha \in \mathbb{C}^*$ such that $g(v) = \alpha v'$. Therefore

$$h(s(x)) = g(v) \otimes f(\zeta) = (\alpha v') \otimes f(\zeta) = \alpha(v' \otimes f(\zeta)) = \theta(v' \otimes \zeta') = \theta s'(\phi(x))$$

for some $\theta \in \mathbb{C}^*$ (here $\alpha f(\zeta) = \theta \zeta'$). It is easy to see that $\theta$ depends continuously on $x \in M \setminus C$.

Since $C$ has complex codimension at least 2 in $M$, we have $H^1(M \setminus C, \mathbb{Z}) = H^1(M, \mathbb{Z}) = 0$. From this fact, it is easy to prove that the map $\theta : M \setminus C \to \mathbb{C}^*$ is homotopic to a constant map. Then, far from $C'$, the section $h(s)$ of $TM' \otimes T^*_\mathcal{F}$ can be deformed to coincide with $s'$ with no variation in the intersection number with the zero section. Thus, we deduce that

$$i_{C'}(h \circ s, s'_0) = i_{C'}(s', s'_0) = \mu(\mathcal{F}', C')$$

and therefore $\mu(\mathcal{F}', C') = \mu(\mathcal{F}, C)$, by Equation (6.2).

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