A Formal Privacy Framework for Partially Private Data
(working paper)

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Abstract

Formal privacy, in particular differential privacy (DP) and its variants, is a useful framework for mathematically provable privacy guarantees in a transparent manner. Despite its many useful theoretical properties, DP has one substantial blind spot: any release that non-trivially depends on confidential data without additional privacy-preserving randomization fails to satisfy DP. Such a restriction is rarely met in practice, as most data releases under DP are only "partially private" (borrowing the language from "partially synthetic" data), due to legal or practical requirements, for example. This poses a significant barrier to accounting for privacy risk and utility holistically under logistical constraints frequently imposed on data curators, especially for those working with survey data, administrative data, or other official statistics. In this paper, we propose a privacy definition, $\epsilon$-DPZ, that extends Pufferfish privacy to accommodate public information. We prove that it maintains similar desirable privacy properties as standard $\epsilon$-DP. Next, we propose a generic release mechanism for PPD, the Wasserstein Exponential Mechanism, with a new sensitivity based on optimal transport measures of distributional distance. We prove that this new mechanism satisfies $\epsilon$-DPZ and provide example implementations of this mechanism. In particular, we demonstrate the importance of base measure choice as it pertains to congeniality of the resulting statistic with public information, proving results on inferential improvements using these congenial base measures. We also derive algorithms for valid statistical inference on mixtures of private and public information, and demonstrate their improved performance over methods based on post-processing more commonly used in the DP literature. Finally, we explore and empirically demonstrate all these ideas in a case study by generating PPD from US Census and CDC to investigate private COVID-19 infection rates. In doing so, we show how data curating agencies can use our framework to overcome barriers to operationalizing formal privacy while providing more transparency and accountability to data subjects and users in negotiating between privacy and utility.

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1 Introduction

1.1 Differential Privacy (DP)

In the history of statistical disclosure limitation (SDL) methodology [Matthews and Harel, 2011, Willenborg and De Waal, 2012, Hundepool et al., 2012], [Desfontaines and Pejó, 2019] notes that the introduction of differential privacy (DP) in [Dwork et al., 2006] has marked an important inflection point. Prior to DP, most SDL methods defined notions of privacy in terms of the properties of individual data sets. This includes definitions like k-anonymity [Sweeney, 2002], ℓ-diversity [Machanavajjhala et al., 2007], and t-closeness [Li et al., 2007]. Despite these advances, it was shown these methods do not prevent against database reconstruction [Dinur and Nissim, 2003, Kifer, 2009, Abowd, 2018a]. This revelation motivated new desiderata for SDL methods. First, it shifted the focus away from provably impossible absolute guarantees to relative guarantees, requiring less stringent assumptions about adversarial knowledge. Second, it motivated methods for which being transparent about SDL methodology does not pose itself as a privacy risk. Differential privacy, first formulated by [Dwork et al., 2006], responded to these calls with these desired properties. Since then, hundreds of variants have emerged with different intended use cases and properties (see [Desfontaines and Pejó, 2019] for a taxonomy of such definitions and their related properties). Notable definitions include concentrated and zero-concentrated DP, [Dwork and Rothblum, 2016, Bun and Steinke, 2016], Rényi DP [Mironov, 2017], Pufferfish privacy [Kifer and Machanavajjhala, 2014], and f-DP [Dong et al., 2019].

We recall the original definition here, but recast in an explicitly probabilistic language. For the purposes of this paper, we limit ourselves to “bounded” DP, in which the pairs of databases
being compared have the same size but differ in one individual’s contribution (as opposed to “unbounded” DP, which compares databases that differ in the addition or deletion of exactly one individual’s contribution). This is especially important for analyzing finite-sample properties of these algorithms. We also restrict ourselves to the central model of differential privacy, in which we assume the presence of a trusted data curator with access to the confidential data (as opposed to the “local” model, in which no trusted data curator processes the confidential data) [Evfimievski et al., 2003, Kasiviswanathan et al., 2011].

Definition 1 ((\(\epsilon, \delta\))-differential privacy [Dwork et al., 2006]). Let \(\mathcal{M} \triangleq \{\mu_x \mid x \in X^n\}\) be a collection of probability distributions on \((Y, F_Y)\), for which the mechanism releases a random variable \(Y \sim \mu_x\) where \(x \in X^n\) is the confidential data. We say \(\mathcal{M}\) satisfies \((\epsilon, \delta)\)-differential privacy (\(\epsilon\)-DP) if for all \(B \in F_Y\) and all \(x, x' \in X^n\) that differ on exactly one entry, we have:

\[\mu_x(B) \leq \mu_{x'}(B) e^\epsilon + \delta\]

When \(\delta = 0\), we refer to the result as “pure” differential privacy or \(\epsilon\)-DP.

This definition admits some desirable properties that previous privacy formalisms lacked. First, knowledge of \(\mathcal{M}\) does not affect the privacy guarantees associated with the release, meaning data practitioners can release details of privacy-preserving methodologies otherwise kept secret. We will refer to this property as the “transparency” property. Second, releasing any function of \(Y\) maintains the same privacy guarantees as \(Y\) itself, meaning private data releases can be “post-processed” without concerns of further privacy degradation. Finally, the parameter \(\epsilon\) can be interpreted as a “privacy-loss budget,” where results from multiple releases can be composed and maintain \(\epsilon\)-DP guarantees at a larger privacy budget.

In recent years, there has been a truly massive proliferation of DP methods. While we cannot enumerate all such methods, there have been some notable developments in private selection problems in machine learning [McSherry and Talwar, 2007, Reimherr and Awan, 2019, Asi and Duchi, 2020], statistical inference tasks such as hypothesis testing and confidence intervals [Awan and Slavković, 2018, Canonne et al., 2019, Biswas et al., 2020], and synthetic data generation [Snoke and Slavković, 2018, Torkzadehmahani et al., 2019, McKenna et al., 2018]. Additionally, some large-scale data curators such as those at the US Census [Abowd, 2018b] and Facebook [Messing et al., 2020] have begun releasing DP synthetic data for general usage.

1.2 Public Information and Partially Private Data

1.2.1 Definitions and examples

Essential to the original definition of DP is the necessity of randomization in releasing any statistic that depends on the confidential data, each of which naturally consumes some confidentiality protections as measured by the privacy budget. To demonstrate this necessity, let \(g : X^n \rightarrow \mathbb{R}\) be any non-constant function of the confidential data. Then releasing \(Y = g(x)\) is not \(\epsilon\)-DP for any \(\epsilon < \infty\), meaning any function of the confidential data cannot be released while satisfying \(\epsilon\)-DP. Such a property also holds for relaxations of \(\epsilon\)-DP. We will also see later that complications can arise if we release a random variable \(Z\) correlated with the random response. For the purposes of this paper, we define “public information” as any collection of statistics that depends on the confidential data, regardless of whether this dependence is deterministic or probabilistic. This idea will be formalized later.

In order to precisely satisfy the properties of \(\epsilon\)-DP, one would hope any statistic dependent on the confidential data would have additional randomization due to privacy preservation. This
restriction imposes a theoretical imaginary where the central data curator has perfect control over all statistical results derived from its database contents. Furthermore, DP’s claim of protections against “arbitrary background information” only applies to adversary priors which do not depend on any aspect of the confidential data [Kasiviswanathan and Smith, 2008]. Such rosy assumptions are not only overly restrictive but frequently violated in practice. As a result, many private data releases are only partially private. This paper will study the problems associated with partially private data (PPD), data releases which contain both public and private components.

To further motivate the problem of PPD, we first describe the myriad ways in which PPD naturally arises in data curation and dissemination. The first broad group of PPD incidences concerns cases when data curators intentionally release public information along with their private releases. Data curators are frequently compelled by user-requested or methodological standards to release metadata, such as information about data collection methods or summary statistics, that are exact descriptors of the confidential data. This is often true with contingency table data or other data sets with microdata requirements [Abowd et al., 2019b]; it also concerns survey data where methodological details are public [Seeman and Brummet, 2021]. Such restrictions also apply to tuning parameters that may arise in optimizing fitness for use on a particular statistical task, such as non-privately selecting regularization parameters in a machine learning problem [Kusner et al., 2015]. Alternatively, aspects of the privacy algorithm itself may be selected non-privately, such as privacy budgets or sensitivities chosen to satisfy a particular pre-existing fitness for use goal. This is true for the US Census [Abowd et al., 2019b].

The second broad group of PPD incidences concerns cases where information about the population of interest already exists prior to choosing to implement DP. Most existing data curators do not use formal privacy methods, and choosing to do so when previous releases have not had such protections poses a challenge in reconciling future releases with past ones. Such a problem arises frequently in releasing private results subject to temporal autocorrelation [Quick, 2021]. Furthermore, multiple data curators may possess data about the same population of interest, and unless all data curators choose to use privacy simultaneously, the guarantees desired by DP may not hold in practice. Additionally, many descriptors of a data set can be fairly presumed as common knowledge, and it would be unreasonable to justify that a privacy violation has occurred given common knowledge. This most commonly arises when DP produces results falling outside a “common knowledge” domain (e.g., negative counts). However, such issues can also arise with more complex structures, such as data living on manifolds [Soto and Reimherr, 2021].

1.2.2 Technical Challenges in PPD

DP and its variants are built axiomatically around the desirable properties described in section 1.2.1. Introducing public information largely disrupts the operation of these properties. Robustness to post-processing now only holds for particular functions with no probabilistic dependence on the public information [Gong and Meng, 2020]. Transparency of the privacy becomes muddled when considering if the mechanism’s properties depend on the confidential data. Additionally, privacy budget accounting can ambiguously situate public information, with different interpretations depending on whether the information constitutes a data release or whether it frames the space of possible databases. Ideally, we want a privacy formalism to be explicit about these assumptions, maintain similar properties as DP, and avoid the ambiguities raised by forcing problems of public information into existing frameworks.

We note here that DP does not provide protection against model-based statistical inference generalized or transferred from one target population to another, as has been noted in recent miscommunications about the Census use case [Ullman, 2021]. Our goal in studying PPD is not to
offer such protections, as they suffer from the same reconstruction vulnerabilities as described in section 1.1. However, organizations hoping to share DP results frequently release results in a non-private manner about the same population, even if the exact correspondence is not deterministic. For example, suppose an organization has released statistics about a population at multiple points in time and suddenly chooses to use DP at one particular point in time. Then, the DP release does not account for potential autocorrelations between existing public results and results present in the population at the time the data curators choose to use DP. We argue that this case warrants special attention because unlike the examples given in [Ullman, 2021], here the data curator is directly in charge of multiple releases on the same target population, only some of which may satisfy some formal privacy protection. Therefore including probabilistic public information allows us to more holistically incorporate existing results that can reasonably be assumed to be about the same population, in hopes that DP’s guarantees can better reflect a more honest picture of extant public information. Though our methods take a different general approach, we remain true to the spirit of DP in that our goal is to limit the additional information an adversary can gain from using new private results relative to priors that may be informed by public releases; in other words, we are still providing a relative and not absolute guarantee. Note that whenever possible, we still advocate for avoiding the complications induced by releasing public information, as we will show how it fundamentally weakens the privacy guarantees. This means we still advocate for using DP as intended to provide stronger privacy guarantees. Still, this work addresses real logistical barriers to the creation and dissemination of private statistics, especially official statistics and other important results subject to intense scrutiny and regulation.

Note that the presence of public information not only affects the privacy guarantees of any resulting analysis, but also the power of any statistical inference. Regardless of the privacy guarantees, or privacy formalism alone for that matter, a data user should use all available information to perform the best statistical inference given the private and public information. However, these data sources may not be directly available, as for many DP algorithms, public information is combined with private releases through deterministic post-processing into a single new release. For example, the US Census post-processes its privatized contingency tables into a single self-consistent database that conforms with public descriptors of the entire underlying saturated table [Abowd et al., 2019a]. While robustness to post-processing is a theoretical property of DP [Dwork et al., 2014], it does not generically hold in the presence of private information [Gong and Meng, 2020]. Furthermore, we show that post-processing both theoretically and empirically degrades the quality of inference as opposed to inference directly on confidential data and public information. This was empirically shown in our previous case study work [Seeman et al., 2020]; in this paper, we extend these results both theoretically and empirically.

1.3 Contributions

The examples discussed in section 1.2.1 suggest a few desiderata for reasoning about PPD. First, we need to formally characterize what privacy guarantees exist when releasing private and public information simultaneously. Second, we need to think about how new privacy mechanisms accommodate these joint releases of private and public information. In order to achieve the goals above, we turn to inferential notions of privacy instead of the traditional notion of DP. In doing so, we specify our guarantees not over a particular set of databases but instead over a collection of data generating distributions and of events in the data’s sample space for which we want to limit additional information gained by releasing a private statistic.

Our contributions in this paper are as follows:
1. We propose a mathematical formalism, $\epsilon$-DP relative to a random variable $Z$ (abbreviated as $\epsilon$-DPZ), which explicitly describes the privacy guarantees of PPD given an observation from the “public” random variable $Z$.

2. We propose a generic release mechanism for PPD, the Wasserstein Exponential Mechanism, that satisfies $\epsilon$-DPZ and provide example implementations of this mechanism. In particular, we demonstrate the importance of base measure choice as it pertains to congeniality of the resulting statistic with public information.

3. We provide sampling algorithms for statistically valid inference on parameters of interest given PPD, and we demonstrate the theoretical and computational benefits of these algorithms over post-processing approaches more common in the computer science literature on private mechanism design.

4. We demonstrate the effects above on simulation studies and data analyses using COVID-19 infection rates synthesized from a mixture of private and public sources.

1.4 Related literature

There are many existing frameworks that can be used to generalize DP to accommodate certain kinds of public information. Pufferfish privacy [Kifer and Machanavajjhala, 2014] is an extremely generic framework based on defining data generating distributions for the confidential data and events for which we wish to protect adversarial differences. Similar work, such as Blowfish privacy [He et al., 2014], coupled worlds privacy [Bassily et al., 2013], and dependent differential privacy [Liu et al., 2016], can be viewed as instances of Pufferfish due to its generality. While our formalism could be massaged into any of the abstract frameworks above, our primary requirement is that we treat public information as a random variable, regardless of which “abstraction” language we choose. This gives us two unique benefits: first, we can allow for more general dependencies between public and private information besides those induced by deterministic functions of the confidential data. Second, treating public information as random is necessary for valid inference from the joint release of private and public statistics.

Existing works on DP in the presence of public information focus on consistency, or agreement between results derived from private and public information. Consistency problems were introduced by [Barak et al., 2007], and since then, numerous optimization-based procedures have been introduced to ensure consistency [Hay et al., 2010, Ding et al., 2011, Abowd et al., 2019a]. The effect of public information on privacy guarantees for consistent $\epsilon$-DP releases has been studied in [Ashmead et al., 2019] and [Gong and Meng, 2020]. Our work is more general in that we do not require our private releases and our public information to be congenial, as both can contribute different kinds of information based on how dependencies between private and public information are specified. We will compare our results to those based on the conditional mechanism in [Gong and Meng, 2020]. Furthermore, congeniality only makes sense when reasoning about deterministic public information in a few specific contexts. However, we will discuss some particular benefits of congeniality later on.

The problem of doing inference on mixtures of private and public information has not been well studied in the existing literature. Statistical learning via mechanisms which input both private and public samples have been proposed by [Bassily et al., 2020]. While this may be helpful from a mechanism design perspective, the authors require that the private and public samples are independently and identically distributed. We are interested in cases when privacy guarantees degrade specifically because of dependencies between private and public information, and the authors
assume a priori no such dependencies exist. While there has been plenty of existing work on DP versions of statistical inference procedures, there’s been less work on ensuring their validity, by which we mean specifically accounting for randomness due to privacy in releasing descriptors of an inferential procedure (for example, confidence intervals). As an example of an empirical case study, we perform posterior inference given private and public information in our previous results [Seeman et al., 2020] by extending an exact sampling algorithm [Gong, 2019] (a similar algorithm based on Metropolized independent sampling appears in [Gong and Meng, 2020]). The empirical results from this case study motivated the theoretical results derived in this paper.

Robustness to post-processing is a hallmark property of DP and its variants [Dwork et al., 2014]. However, this property has been shown to fail in the presence of public information [Gong and Meng, 2020]. The statistical impact of post-processing, particularly the Census TopDown Algorithm [Abowd et al., 2019a], has been widely studied in a number of different contexts; see [Santos-Lozada et al., 2020] and [Cohen et al., 2021] for just a few of many examples.

2 Privacy guarantees of joint private and public releases

2.1 Pufferfish privacy and $\epsilon$-DP relative to $Z$

As motivated by the introduction, we need definitions of privacy that accommodate existing information more generically. We will write these as instances of Pufferfish privacy, a framework which generalizes inferential notions of privacy [Kifer and Machanavajjhala, 2014]:

Definition 2 (Pufferfish [Kifer and Machanavajjhala, 2014]). Let $\mathcal{D}$ be a collection of probability distributions on $(\mathcal{X}^n, \mathcal{F}_X)$ indexed by a parameter $\theta \in \Theta$ (called “data evolution scenarios” ¹). Let $\mathcal{S} \subset \mathcal{F}_X$ be a collection of events (called “secrets”), and let $\mathcal{S}_{\text{pairs}} \subset \mathcal{S} \times \mathcal{S}$ be a collection of pairs of disjoint events in $\mathcal{S}$ (called “discriminative pairs”). Then a mechanism $\{\mu_{\theta} \mid \theta \in \mathcal{D}\}$ as a collection of distributions on $(\mathcal{Y}, \mathcal{F}_Y)$ satisfies $\epsilon$-Pufferfish($\mathcal{S}, \mathcal{S}_{\text{pairs}}, \mathcal{D}$) privacy if for all $B \in \mathcal{F}_Y$, $\theta \in \mathcal{D}$, and $(s_1, s_2) \in \mathcal{S}_{\text{pairs}}$ such that $P(s_1 \mid \theta) \notin \{0, 1\}$ for $i = 1, 2$:

$$
\begin{align*}
\left\{ \begin{array}{l}
P(Y \in B \mid s_1, \theta) \leq e^\epsilon P(Y \in B \mid s_2, \theta) \\
P(Y \in B \mid s_2, \theta) \leq e^\epsilon P(Y \in B \mid s_1, \theta)
\end{array} \right.
\end{align*}
$$

Pufferfish has a nice Bayesian interpretation in that it limits the power of Bayesian inference to allow an adversary to distinguish between any discriminative pair $(s_1, s_2) \in \mathcal{S}_{\text{pairs}}$. Specifically, if $H_0 : X \in s_1$ and $H_1 : X \in s_2$, then the Bayes factor for the test is bounded in the interval $[e^{-\epsilon}, e^\epsilon]$. This is formalized by a lemma in their original paper:

Lemma 1 (Bayesian semantics of Pufferfish [Kifer and Machanavajjhala, 2014]). In the setup for Definition 2, Equation 1 equivalently implies, for all $\theta \in \mathcal{D}$, and $(s_1, s_2) \in \mathcal{S}_{\text{pairs}}$ such that $P(s_1 \mid \theta) \notin \{0, 1\}$ for $i = 1, 2$:

$$
\begin{align*}
e^{-\epsilon} \leq \frac{P(s_1 \mid y, \theta) / P(s_2 \mid y, \theta)}{P(s_1 \mid \theta) / P(s_2 \mid \theta)} \leq e^{\epsilon}
\end{align*}
$$

Note that we will use $\mathcal{D}_{\text{DP}}$, $\mathcal{S}_{\text{DP}}$, and $\mathcal{S}_{\text{pairs,DP}}$ to refer to the Pufferfish components that define $\epsilon$-DP’s guarantees in the language of Pufferfish. In particular, let $\mathcal{H} \triangleq \{h_i\}_{i=1}^N$ be a population of $N$ individuals and $\mathcal{R} = \{x_i\}_{i=1}^n$ be a sample of records. Define the events:

$$
\begin{align*}
\sigma_i & \triangleq \text{“record } r_i \text{ belongs to individual } h_i \text{ in the data”} \\
\sigma_{i,x} & \triangleq \text{“record } r_i \text{ belongs to individual } h_i \text{ in the data and has value } x \in \mathcal{X}”
\end{align*}
$$

¹more commonly referred to as “data generating distributions” in statistics
Then:

\[
\begin{align*}
\mathcal{S}_{\text{DP}} & \triangleq \{ \sigma_{i,x} \mid i \in [N], x \in \mathcal{X} \} \cup \{ \sigma^c_i \mid i \in [N] \} \\
\mathcal{S}_{\text{pairs,DP}} & \triangleq \{ (\sigma_{i,x}, \sigma^c_i) \mid i \in [N], x \in \mathcal{X} \}
\end{align*}
\]

The corresponding data model depends on \( \pi_i \triangleq \mathbb{P}(h_i \in \mathcal{R}) \) and independent densities \( f_i(r_i) \) on \((\mathcal{X}, \mathcal{F}_X)\), yielding \( \mathbb{D}_{\text{DP}} \) as the set of all distributions with densities given by:

\[
\mathbb{D}_{\text{DP}} \triangleq \left\{ \mathbb{P}_X \mid f_X(x) = \prod_{r_i \in \mathcal{R}} \pi_i f_i(r_i) \prod_{r_i \notin \mathcal{R}} (1 - \pi_i) \right\}
\]

As expected, [Kifer and Machanavajjhala, 2014] showed that \( \epsilon\text{-DP} \) is equivalent to \( \epsilon\text{-Pufferfish}(\mathcal{S}_{\text{DP}}, \mathcal{S}_{\text{pairs,DP}}, \mathbb{D}_{\text{DP}}) \).

Next, we introduce our privacy formalism. Our goal with this approach is to extend existing \( \epsilon\text{-DP} \) as little as possible while accommodating generic public statistical releases as random variables \( Z \). Within this setup, our target data evolution scenarios are those conditioned on the existing public results \( Z \). This yields the intuition for our definition:

**Definition 3** (Differential Privacy relative to a random variable \( Z \)). Let \( Z \) be a random variable on \((\mathcal{Z}, \mathcal{F}_Z)\) that depends on \( X \). For each \( z \in \mathcal{Z} \), let \( \mathbb{D}_z \) be a collection of conditional distributions for \( X \mid Z = z \) indexed by \( \theta_z \in \Theta_z \). We say that the mechanism that releases a random variable \( Y \) satisfies \( \epsilon\text{-DPZ} \) relative to \( Z \) (which we will call \( \epsilon\text{-DPZ} \)) if:

1. For all \( z \in \mathcal{Z} \) and \( B \in \mathcal{F}_Y \),
2. For all distributions \( \theta_z \in \mathbb{D}_z \): 
3. For all \( (s_1, s_2) \in \mathcal{S}_{\text{pairs,DPz}} \) where:

\[
\mathcal{S}_{\text{pairs,DPz}} \triangleq \{(s_1, s_2) \in \mathcal{S}_{\text{pairs,DP}} \mid \mathbb{P}(s_1 \mid \theta_z) \notin \{0, 1\} \quad \forall i \in \{1, 2\}, \theta_z \in \mathbb{D}_z\},
\]

we have:

\[
\begin{align*}
\mathbb{P}(Y \in B \mid s_1, \theta_z) & \leq e^\epsilon \mathbb{P}(Y \in B \mid s_2, \theta_z) \\
\mathbb{P}(Y \in B \mid s_2, \theta_z) & \leq e^\epsilon \mathbb{P}(Y \in B \mid s_1, \theta_z).
\end{align*}
\]

(3)

Note that this defines a collection of Pufferfish mechanisms, each of which induced by a particular realization of the public information \( z \in \mathcal{Z} \). We show that this definition has a similar, but not identical, Bayesian interpretation like that in Lemma 1:

**Lemma 2** (Bayesian semantics of \( \epsilon\text{-DPZ} \)). In the setup for Definition 3, Equation 3 equivalently implies, for all \( B \in \mathcal{F}_Y \), \( z \in \mathcal{Z} \), and associated \( \theta \in \mathbb{D}_z \) and \( (s_1, s_2) \in \mathcal{S}_{\text{pairs,DPz}} \):

\[
e^{-\epsilon} \leq \frac{\mathbb{P}(s_1 \mid Y \in B, \theta_z) / \mathbb{P}(s_2 \mid Y \in B, \theta_z)}{\mathbb{P}(s_1 \mid \theta_z) / \mathbb{P}(s_2 \mid \theta_z)} \leq e^\epsilon.
\]

(4)

**Lemma 3** (\( \epsilon\text{-DPZ} \) post-processing). Let \( h : \mathcal{Y} \times \mathcal{Z} \rightarrow \mathcal{Y}^* \) be a measurable function of \( Y \) and \( Z \). Then releasing \( Y^* \triangleq h(Y, Z) \) is \( \epsilon\text{-DPZ} \) relative to \( Z \).

Although \( \epsilon\text{-DPZ} \) inherits these similar properties as \( \epsilon\text{-DP} \), the relationship to composition is more complex. In general, we only consider composition from the perspective of two different releases \( Y_1 \) and \( Y_2 \), with privacy budgets \( \epsilon_1 \) and \( \epsilon_2 \), relative to the same public information realization \( z \in \mathcal{Z} \). No such general composition properties will exist for multiple public random variables \( Z_1, Z_2 \).

**Lemma 4** (Sequential composition for \( \epsilon\text{-DPZ} \)). Suppose \( Y_1 \) is \( \epsilon_1\text{-DPZ} \), \( Y_2 \) is \( \epsilon_2\text{-DPZ} \), and:

\[
Y_1 \perp Y_2 \mid X, Z.
\]

Then the random vector \((Y_1, Y_2)\) is \((\epsilon_1 + \epsilon_2)\text{-DPZ} \).
2.2 Mechanisms that satisfy $\epsilon$-DPZ

Translating $\epsilon$-DP guarantees into $\epsilon$-DPZ guarantees can sometimes be difficult. However, in the common case where $Z \triangleq \mathbb{1}_{\{X \in \mathcal{X}^n\}}$ where $\mathcal{X}^n \subseteq \mathcal{X}^n$, we have the following result that re-expresses the conditional mechanism established by [Gong and Meng, 2020]:

**Lemma 5.** Let $Z \triangleq \mathbb{1}_{\{X \in \mathcal{X}^n\}}$ and let $Y$ be an $\epsilon$-DP release where $Y = X^n$. Then $Y \mid Y \in \mathcal{X}^n$ is $\epsilon^*$-DP relative to $Z$ for some $\epsilon^* \in [0, 2\epsilon]$.

Next, we need to define a generic class of mechanisms that satisfies $\epsilon$-DP relative to $Z$. In one dimension defined by some output function $h : \mathcal{Y} \mapsto \mathbb{R}$, [Song et al., 2017] defines an equivalent sensitivity for an arbitrary instance of Pufferfish:

$$\Delta \triangleq \sup_{s_1, s_2 \in \mathcal{S} \text{pairs}} \sup_{\theta \in \Theta} W_{\infty}(P(h(\cdot) \mid s_1, \theta), P(h(\cdot) \mid s_2, \theta))$$

Where $W_{\infty}$ is the Wasserstein-$\infty$ metric:

$$W_{\infty}(\mu, \nu) \triangleq \inf_{\gamma \in \Gamma(\mu, \nu)} \text{esssup}_{\gamma} \|\mu - \nu\|_1.$$  

In the above definition, $\mu$ and $\nu$ are measures on $(\mathcal{Y}, \mathcal{F}_Y)$, and $\gamma$ is the set of all joint distributions on $(\mathcal{Y} \times \mathcal{Y}, \mathcal{F}_Y \times \mathcal{F}_Y)$. Then the Wasserstein mechanism releases:

$$Y \triangleq h(X) + \epsilon, \quad \epsilon \sim \text{Laplace}(\Delta/\epsilon)$$

Note that in the $\epsilon$-DP definition, $\Delta$ reduces to the sensitivity of the Laplace mechanism [Dwork et al., 2006]. The first natural extension for 1-dimensional real-valued public releases is to extend $\Delta$ to depend on $z \in Z$:

$$\Delta_z \triangleq \sup_{s_1, s_2 \in \mathcal{S} \text{pairs},\mathcal{D}_{\text{Pz}}} \sup_{\theta \in \Theta_z} W_{\infty}(P(h(\cdot) \mid s_1, \theta_z), P(h(\cdot) \mid s_2, \theta_z))$$

**Corollary 1.** The Wasserstein mechanism with $\Delta_z$ for $z \in Z$ satisfies $\epsilon$-DPZ.

We propose the following generalization of the Wasserstein mechanism for arbitrary loss functions. This relaxes the restrictions on the output space and extends $\mathcal{X}$ to be an arbitrary metric space.

**Theorem 1 (Wasserstein Exponential Mechanism).** Let $(\mathcal{X}, d)$ be a metric space and $L_x : \mathcal{X} \times \mathcal{Y} \mapsto [0, \infty]$ be a loss function. Fix $z \in Z$. Define:

$$\Delta_z \triangleq \sup_{\theta \in \Theta_z} \sup_{s_1, s_2 \in \mathcal{S}_{\text{pairs,DPz}}} W_{\infty}(P(\cdot \mid \theta_z, s_1), P(\cdot \mid \theta_z, s_1))$$

and:

$$\sigma(\Delta_z) = \sup \{ |L_x(y) - L_{x'}(y)| \mid x, x' \in \mathcal{X}^n, d(x, x') \leq \Delta_z \}$$

Then releasing a sample $Y$ with density given by:

$$f_X(Y) \propto \exp \left( -\frac{\epsilon L_x(y)}{\sigma(\Delta_z)} \right) \nu(y)$$

Where $\nu$ is an arbitrary base measure, satisfies $\epsilon$-DP relative to $Z$. 

9
2.3 Relationship to $\epsilon$-DP

Here we emphasize that the relationship between $\epsilon$-DP and $\epsilon$-DPZ is not hierarchical. Even ignoring the conceptual differences between the guarantees provided by the two frameworks, $\epsilon$-DPZ is neither a strengthening nor a relaxation of $\epsilon$-DP. First, if the collection of distributions $\mathcal{D}_{\text{DPZ}}$ is sufficiently rich for each $z \in Z$, then in general the mechanism defined in Theorem 1 induces a sensitivity at least as large as that in the $\epsilon$-DP analogue.

To formalize this we have the following lemma:

**Lemma 6.** Define $\mathcal{D}_X \triangleq \{\xi_x \mid x \in \mathcal{X}^n\}$. Suppose for every distribution $\theta_x \in \mathcal{D}_X$, there exists a sequence of distributions $\{\theta_{z,x}^{(m)}\}_{m=1}^\infty \subseteq \mathcal{D}_{\text{DPZ}}$ and a pair of secrets $(s_1, s_2) \in \mathcal{S}_{\text{pairs},\text{DPZ}}$ such that:

$$\theta_{z,x}^{(m)} \rightarrow D \xi_x \quad \forall x \in \mathcal{X}^n, \quad \mathbb{P}(\cdot \mid s_1, \theta_{z,x}), \mathbb{P}(\cdot \mid s_2, \theta_{z,x}) \text{ well defined}$$

Then:

$$\sup_{x, x'} \sup_{d_H(x, x')} \|L_x(y) - L_{x'}(y)\| \leq \sigma(\Delta_z)$$

The above results have a few important implications. First, they shows that $\epsilon$-DP mechanisms can be $\epsilon$-DPZ for a new, usually larger privacy budget. Second, this budget inflation factor for binary event public information is upper bounded at 2, consistent with results from [Gong and Meng, 2020]. Third, the reason this relationship is neither a strengthening nor weakening of $\epsilon$-DP is because the privacy guarantees are relatively stronger in $\epsilon$-DPZ at the same privacy budget, but over a restricted set of secret pairs as opposed to the uniform guarantee in $\epsilon$-DP.

2.4 Congeniality

In implementing the Wasserstein exponential mechanism, we do not require any assumptions on the base measure $\nu$. However, in the presence of public information, we can be more judicious in how $\nu$ is chosen. Independent of whether $\nu$ encodes any prior belief about $Y$, we already saw that any particular realizations of $Z$ can restrict which values of $X$ are supported, with some works already addressing this problem [Gong and Meng, 2020, Gao et al., 2021, Soto and Reimherr, 2021]. In particular, the manifold approach of Theorem 4 in [Soto and Reimherr, 2021] can yield important results for the choice on base measure and loss function:

**Proposition 1.** Let $Y \triangleq \mathbb{R}^{d_1}$ be the output space for an instance of the Wasserstein exponential mechanism, and let $Z \triangleq \text{Span}(A)$ for some matrix $A$ of rank $d_2 < d_1$ (this may correspond to the results of summation queries on the confidential data, for example). Let $Y_1$ and $Y_2$ be two $\epsilon$-DPZ releases estimating a mean $\mathcal{X}$ using Laplace noise, supported on $\mathbb{R}^{d_1}$ and $\text{Span}(A)$, respectively. Then under the regularity conditions of Theorem 3 in [Soto and Reimherr, 2021]:

\[
\begin{align*}
\mathbb{E}[\|\text{Proj}_A(Y_1) - \mathcal{X}\|^2_2] &= O\left(\frac{d_2^2}{n^2 \epsilon^2}\right) \\
\mathbb{E}[\|Y_2 - \mathcal{X}\|^2_2] &= O\left(\frac{d_2^2}{n^2 \epsilon^2}\right)
\end{align*}
\]

These results can be extended to nonlinear manifolds under the regularity assumptions in [Soto and Reimherr, 2021] due to the uniqueness of the local linear approximation as $n \to \infty$. 

10
3 Examples

The flexibility of ϵ-DPZ is simultaneously a blessing and a curse, as it defers power largely to how distributions are defined on Z and their relationship to the underlying database X through \( D_{\text{DPZ}} \). To demonstrate these effects, we work through some examples showing how different choices of distributions on Z affect implementation of the exponential mechanism. Throughout this section, we consider estimating a univariate mean \( \mu \) from a multivariate normal distribution with covariance as a nuisance parameter. We choose these examples because they allow for closed-form calculations of all the quantities we need above, including satisfying the conditions of Lemma 6. Additionally, they explicitly motivate ϵ-DP relative to Z and the importance of quantifying randomness in public information.

**Example 1** (Multivariate normal: analogues between ϵ-DP and Pufferfish). In the realm of standard ϵ-DP, if we assume \( X_1, \ldots, X_n \in [-\Delta/2, \Delta/2] \) then \( \sup_{X \sim X'} |\bar{X} - \bar{X}'| \leq \frac{\Delta}{n} \). Then we can sample \( Y \) satisfying ϵ-DP from the density:

\[
f_x(y) \propto \exp\left(-\frac{ne}{2\Delta}|y - \bar{X}|\right)1_{\{y \in [-\Delta/2, \Delta/2]\}}.
\]

As a first example of Pufferfish, we can use it to mimic what we would do with standard ϵ-DP and get identical results. First define:

\[
\{X_1, \ldots, X_n \mid \text{iid } N(\mu, \sigma^2) \mid \mu \in [-\Delta/2, \Delta/2], \sigma \in [0, \infty) \}
\]

\( S_{\text{pairs}} \triangleq \{\{\omega \in \Omega \mid X_i(\omega) = v\} \mid i \in [n], v \in [-\Delta/2, \Delta/2]\}. \)

(Note that if we wanted to, we could be more generic here and consider arbitrary distributions under an independence assumption between entries. However, this reduces trivially to the ϵ-DP sensitivity, since the distributions that maximize the Wasserstein distances are degenerate point masses at the boundaries of the sample space).

By our iid assumption, we only consider conditioning on \( X_1 \) without loss of generality. Then:

\[
\left(\frac{X}{X_1}\right) \sim N\left(\left(\frac{\mu}{\sigma^2}, \frac{\sigma^2}{\sigma^2}, \frac{n}{\sigma^2}\right)\right) \implies \bar{X} \mid X_1 = v \sim N\left(\mu + \frac{1}{n}(v - \mu), \frac{n - 1}{n^2}\sigma^2\right)
\]

Define \( \mathbb{P}_{\theta,v_1}(B) \triangleq \mathbb{P}_{\theta}(X \in B \mid X_1 = v_1) \). By definition:

\[
W_\infty(\mathbb{P}_{\theta,v_1}, \mathbb{P}_{\theta,v_2}) \triangleq \inf_{\gamma \in \Gamma(\mathbb{P}_{\theta,v_1}, \mathbb{P}_{\theta,v_2})} \text{ess sup}_{(y_1, y_2) \in \Omega} |y_1 - y_2|.
\]

For this particular problem setup, \( W_\infty \) is explicitly calculable. If we fix \( \theta \), then \( \{\mathbb{P}_{\theta,v} \mid v \in \mathbb{R}\} \) is a location family with a location parameter linear in \( v \). Therefore in the \( W_\infty \) optimal transport solution, each infinitesimal piece of probability mass travels the same constant distance as specified by the difference in conditional means. In other words, if \( V_1 \sim \mathbb{P}_{\theta,v_1} \) and \( V_2 \sim \mathbb{P}_{\theta,v_2} \), then the optimal transport solution takes the joint distribution where \( V_2 = V_1 + (v_2 - v_1)/n \) with probability 1. Any other solution of \( W_\infty \) would require a larger essential supremum. (Fun fact: this is analogous to the classical result in Bayesian inference wherein if a Bayes estimator has constant risk, then it is minimax).

This implies that the distance above is independent of both \( \mu \) and \( \sigma \), and therefore only depends on the secret pairs \((s_1, s_2) \in S_{\text{pairs}}\) for which:

\[
\sup_{\theta \in \mathbb{D}} \sup_{(s_1, s_2) \in S_{\text{pairs}}} W_\infty(\mathbb{P}_{\theta,v_1}, \mathbb{P}_{\theta,v_2}) = \frac{\Delta}{n}
\]
This allows us to instantiate the Wasserstein exponential mechanism. First, for all databases whose \( L_1 \) distance is bounded by \( \Delta/n \) we have:

\[
\sup_{X,X'} \sup_{\|x-x'\|_{L_1}} \frac{|L_X(y) - L_{X'}(y)|}{\Delta/n} \leq \Delta/n
\]

Therefore the Wasserstein exponential mechanism gives us exactly the same result as the \( \epsilon \)-DP exponential mechanism.

**Example 2** (Multivariate normal with probabilistic public information). Next, we consider a second example that motivates \( \epsilon \)-DP relative to \( Z \). Suppose we observe \( Z \), another multivariate normal with some dependencies with \( X \). The joint distribution of \( \overline{X} \), our secret value \( X_1 \), and our public information \( Z \) looks like (in block matrix notation):

\[
\begin{pmatrix}
\overline{X} \\
X_1 \\
Z
\end{pmatrix} \sim N \left( \begin{pmatrix}
\mu \\
\mu^Z
\end{pmatrix}, \begin{pmatrix}
\Sigma_{XVZ} & \Sigma_{XVZ}^T \\
\Sigma_{XVZ}^T & \Sigma_{VZ}
\end{pmatrix} \right).
\]

This implies:

\[
\overline{X} | X_1 = v, Z = z \sim N \left( \mu + \Sigma_{XVZ} \Sigma_{VZ}^{-1} \begin{pmatrix} v - \mu \\ z - \mu^Z \end{pmatrix}, \frac{\sigma^2}{n} - \Sigma_{XVZ} \Sigma_{VZ}^{-1} \Sigma_{XVZ} \right).
\]

Again, by properties of the conditional multivariate normal, the resulting family is still a location family parameterized by a (more complicated) linear change in \( v \). However, this indicates that the more complicated dependence affects the Wasserstein distance through the nuisance parameter, which canceled in the previous case:

\[
\sup_{\theta \in \mathcal{D}} \sup_{(s_1,s_2) \in \mathcal{S}_{\text{pairs}}} W_\infty(\mathbb{P}_{\theta,v_1}, \mathbb{P}_{\theta,v_2}) = \sup_{\theta \in \mathcal{D}} \left| \Sigma_{XVZ} \Sigma_{VZ}^{-1} \left( \begin{pmatrix} \Delta \\ z - \mu^Z \end{pmatrix} \right) \right|.
\]

Therefore in this situation, the “sensitivity” of the Wasserstein exponential mechanism loss depends explicitly on what kinds of dependence we allow between the private and public information, i.e. how we choose \( \mathcal{D} \).

**Example 3** (Count distributions). Let \( X \in \{0,1\}^n \), and our goal is to implement the Wasserstein exponential mechanism with \( L_X(y) = |y - \sum_{i=1}^n X_i| \). The secret pairs are of the form:

\[
s_{ij} = \{\omega \in \Omega \mid X_i(\omega) = j\}, \quad \mathcal{S}_{\text{pairs}} \triangleq \{(s_{i0}, s_{i1}) \mid i \in [n]\}
\]

Possible distributions conditioned on the secrets take the form:

\[
\mathbb{P} \left( \sum_{i=1}^n X_i = k \mid s_{i0}, Z = z \right) = f_{\theta_{z,0}}(k), \quad \mathbb{P} \left( \sum_{i=1}^n X_i = k \mid s_{i1}, Z = z \right) = f_{\theta_{z,1}}(k),
\]

Where \( f_{\theta_{z,0}}(k) \) and \( f_{\theta_{z,1}}(k) \) are mass functions supported on \( \{0, \ldots, n-1\} \) and \( \{1, \ldots, n\} \), respectively. In the simplest possible case, \( X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Bernoulli}(p) \) for \( p \in (0,1) \) and \( X \perp Z \). In this case:

\[
f_{\theta_{z,0}}(k-1) = f_{\theta_{z,1}}(k) \quad \forall \ k \in \{1, \ldots, n\} \implies W_\infty(\mathbb{P}_{X|s_{i0}}, \mathbb{P}_{X|s_{i0}}) = 1
\]

Since this is true for all \( p \in (0,1) \), we have \( \Delta_z = 1 \). More generically, we need to consider:

\[
\Delta_z = \sup_{\theta_z \in \Theta_z} W_\infty(\mathbb{P}_{\theta_z,0}, \mathbb{P}_{\theta_z,1}) = \sup_{\theta_z \in \Theta_z} \sup_{t \in T} |F_{\theta_z,0}^-(t) - F_{\theta_z,1}^-(t)| \in \{1, \ldots, n\}
\]
Where:

\[ T = \left\{ P\left( \sum_{i=1}^{n} X_i \leq k \mid s_{j0}, Z = z \right) \mid j \in \{1, \ldots, n\} \right\} \]

As an example demonstrating scaling towards extreme dependence (i.e. \( \Delta_z = n \)), let \( i \neq j \) and assume \( Z \) defines the following probabilistic public information:

\[ P(X_i = 1 \mid s_{j1}, Z = z) = z_1 \in [0, 1], \quad P(X_i = 1 \mid s_{j0}, Z = z) = z_0 \in [0, 1] \]

So, under the conservative assumption that the remaining unknown confidential responses are mutually independent:

\[ P\left( \sum_{i=1}^{n} X_i \mid s_{i1}, Z = z \right) \sim \text{Binom}(n - 1, z_1) + 1, \quad P\left( \sum_{i=1}^{n} X_i \mid s_{i0}, Z = z \right) \sim \text{Binom}(n - 1, z_0) \]

In other words \( \theta \) quantifies the probability of \( X_i \) agreeing with the secret data \( s_{j0} \) or \( s_{j1} \), in which case \( \Delta_z \) can be numerically evaluated, increasing as \( z_1 \) and \( z_0 \) become more extreme, as we see in Figure 1.

4 Inference using PPD

To set up the inferential problem, we need to establish the relationships between a parameter of interest \( \theta \), the confidential data \( X \), any publicly released information \( Z \), and any information released with privacy preservation, \( Y \). One common practice is to exploit known dependencies between \( Y \) and \( Z \), since \( \epsilon \)-DP is preserved through post-processing [Dwork et al., 2014] (note that this property is not generically true for Pufferfish!). Many common post-processing operations, such as truncation, rescaling, etc. are forms of a projection operator where, for some function \( h: \mathcal{Y} \mapsto \mathcal{Z} \) and distance function \( d \) between elements of \( \mathcal{Y} \):

\[ Y^* = \text{Proj}(Y, Z) \triangleq \arg\min_{y' \in \mathcal{Y}: h(y') = Z} d(y', Y) \]
This is the approach taken by the Geometric mechanism [Ghosh et al., 2012], the Top-Down algorithm [Abowd et al., 2019b], private hypothesis testing [Canonne et al., 2019], among many other methods. The full probability model is specified in Figure 2, where \( Z \triangleq g(X) \) is our public information and \( Y \) is \( \epsilon \)-DP relative to \( Z \).

\[
\begin{align*}
\theta & \triangleq \text{parameter of interest in } (\Theta, \mathcal{F}_\theta) \\
X & \triangleq \text{confidential data in } (\mathcal{X}^n, \mathcal{F}_X) \\
Z = g(X) & \triangleq \text{public release in } (Z, \mathcal{F}_Z) \\
Y & \triangleq \epsilon\text{-DP release relative to } g \text{ in } (Y, \mathcal{F}_Y) \\
Y^* = \text{Proj}(Y, Z) & \triangleq \text{projected public information in } (Y^*, \mathcal{F}_{Y^*})
\end{align*}
\]

Figure 2: Graphical model for PPD

4.1 General inferential utility

The underlying question is to best perform inference for \( \theta \). In the absence of a model for the data generating process, post-processing is commonly used to reduce the expected noise of a fixed statistical result relative to a function of the data. For example, if the mechanism privatizes a statistic \( T(X) \), then post-processing can often be used to minimize \( \mathbb{E}[(T(X) - Y)^2] \) at \( Y^* \). However, when the goal instead becomes inference on \( \theta \), the conditioning approach offers a few distinct advantages.

From an information theoretic perspective, since \( Y^* \) is a deterministic function of \( Z \) and \( Y \), then the Shannon entropy of \( Y^* \) is bounded above by the Shannon entropy of \( Z \) and \( Y \) by construction. More generally, we can use Blackwell’s classical comparison of experiments theorem [Blackwell, 1953]. This trivially yields the following lemma:

**Lemma 7** (Expected performance of post-processed Bayes estimators). For any Bayesian decision problem estimating some \( \tau(\theta) \) with loss function \( L : \Theta \times \Theta \mapsto [0, \infty) \), the Bayes estimators based on \( Z, Y \) and \( Y^* \), \( \delta_{Z,Y} \) and \( \delta_{Y^*} \) respectively, are related by:

\[
\mathbb{E}[L(\delta_{Z,Y}, \tau(\theta))] \leq \mathbb{E}[L(\delta_{Y^*}, \tau(\theta))].
\]

In particular, choosing \( L(\theta, \delta) = (\theta - \delta)^2 \) and \( \tau(\theta) = \theta \) implies:

\[
\mathbb{E}[	ext{Var}(\theta \mid Z, Y)] \leq \mathbb{E}[	ext{Var}(\theta \mid Y^*)]
\]
Both the results above are statements about the estimator performance in expectation. Additionally, for a broader class of problems, inference can be improved in a way that stochastically dominates using $Y^*$:

**Theorem 2** (Exponential family inference). Let $X_1, \ldots, X_n \sim f_\theta$ with density:

$$f_\theta(x) = h(x) \exp(\eta(\theta)T(x) - A(\eta(\theta))).$$

Let $Y$ be an instance of the exponential mechanism [McSherry and Talwar, 2007] with density:

$$g_x(y) \propto \exp\left(-\frac{\epsilon}{2\Delta L}(\|y - T(x)\|)\right)$$

For some loss function $L$ that depends only on a norm $\|y - T(x)\|$. Let $Z = h(X)$ and let $Y^* = \text{Proj}(Y, Z)$. Then:

1. $Y$ has a monotone likelihood ratio in $\theta$.
2. Define the test:

   $$H_0 : \theta \leq \theta_0, \quad H_1 : \theta > \theta_0.$$  

   For any unbiased test $\phi : Y^* \mapsto [0, 1]$ for $\theta$ based on $Y^*$, there exists a uniformly more powerful test $\phi' : Y \times Z \mapsto [0, 1]$. If $P_\theta(Y \neq Y^*) > 0$ for all $\theta \in \Theta$, then this improvement is strict.

4.2 Auxiliary Information Lost Due to Postprocessing

Although post-processing to conform with public information is often easy to detect, the effects of post-processing when contained within subroutines of existing DP algorithms are more elusive. Asymptotic arguments often fail to detect these issues because for large samples these effects are provably negligible:

**Proposition 2** (Asymptotic negligibility of set restriction). Let $\{(X_n, Y_n, Y^*_n)\}_{n=1}^\infty$ be a sequence of random variables obeying the model in Figure for all $n \in \mathbb{N}$. Suppose $Z \triangleq 1_{\{x \in X^*\}}$ where $X^* \subseteq X$ is a connected set, $h(Y) = 1_{\{Y \in X^*\}}$, and $P(Z = 1 \mid \theta) = 1$ for all $\theta \in \Theta$. Suppose $\sqrt{n}(Y - X) \rightarrow_P 0$. Then by construction:

$$\lim_{n \rightarrow \infty} P(Y^*_n \neq Y_n) = 0$$

Proposition 2 applies to many algorithms where the magnitude of the errors due to privacy are asymptotically dominated by errors due to sampling. This is true for sufficient statistic perturbation using the Laplace mechanism [Foulds et al., 2016] and for regression in the $K$-norm gradient mechanism [Reimherr and Awan, 2019].

This can yield problems for finite sample inference even when asymptotically efficient procedures exist:

**Example 4** (Generic two-distribution hypothesis testing). Let $X_1, \ldots, X_n$ be an iid sample from one of two distributions, $P$ or $Q$, and suppose our goal is to perform inference on the hypotheses $H_0 : P$ versus $H_1 : Q$. [Canonne et al., 2019] demonstrate that asymptotically optimal DP simple hypothesis tests can be constructed as functions of the clamped log-likelihood ratio:

$$\text{cLLR}_{[a, b]} \triangleq \sum_{j=1}^{n} \text{Trunc} \left( \log \left( \frac{Q(X_j)}{P(X_j)} \right), a, b \right)$$
By construction, the cLLR has a sensitivity of \( b - a \), hence we can release \( Y \triangleq \text{ncLLR}_{[a,b]} \triangleq \text{cLLR}_{[a,b]} + \text{Lap}(b - a)/\epsilon \) with \( \epsilon \)-DP. In the asymptotic regime, the relative magnitude of the Laplace noise decays faster to 0 than the error due to sampling in the Neyman-Pearson test. As a result, the test is asymptotically efficient up to a constant compared to the non-private test under the assumption that:

\[
\epsilon \geq \max_{x \in \mathcal{X}} \left| \frac{Q(x)}{P(x)} \right|
\]

In particular the authors discuss \( Y^* \triangleq \phi(Y) \triangleq 1_{\{Y > 0\}} \) which shares the asymptotic optimality of the non-private test under the condition above. However, there are more opportunities for finite-sample tuning when releasing \( Y \) instead of \( \phi(Y) \):

1. By only observing \( \phi(Y) \) we have insufficient information power to determine Type I and Type II errors, or what an appropriate cutoff value for a binary decision would be to bound either of these errors.

2. Unless \( \epsilon \) is chosen rather liberally to meet the assumption, the probability of truncating the log-likelihood ratio is non-ignorable for finite samples, i.e. \( \text{cLLR}_{[a,b]} \) has non-negligible point masses at \( na \) and \( nb \). The exact distributions of \( \text{cLLR}_{[a,b]} \) can be calculated under \( P \) and \( Q \) respectively, and could be used to choose \( a \) and \( b \) more optimally for a given \( \alpha \) and class of randomized tests \( \phi \in \Phi \):

\[
a = \arg \max_{a \in \mathbb{R}} \arg \max_{\phi \in \Phi} \mathbb{E}_Q[\phi(\text{ncLLR}_{[a,a+\epsilon]})]
\]

This optimal choice need not be symmetric (as considered originally by the authors), and the form of \( \Phi \) need not be sigmoidal due to the finite point masses due to truncation.

### 4.3 Posterior inference from the target distribution

In the previous section, we motivated the construction of estimators directly using \( Y \) and \( Z \) instead of relying on a post-processed \( Y^* \). Because the sampling distribution of estimators from two potentially dependent sources can be difficult to quantify, we turn to computational approaches from the Bayesian perspective. This section is largely motivated by our previous case study work studying these effects empirically in small area estimation from partially private Census and CDC data [Seeman et al., 2020]; here, we theoretically and empirically extend our case study results. Details about this case study can be found in the appendix.

Our first goal will be to find an exact sampling algorithm for \( \theta \mid Y, Z \), as motivated by the previous section. In [Fearnhead and Prangle, 2012], the authors demonstrate an important connection between approximate Bayesian computation (ABC) and inference under measurement error. They noted that samples drawn from an approximate posterior distribution based on an observed summary statistic can be alternatively interpreted as exact samples from a posterior distribution conditional on a summary statistic measured with noise. [Gong, 2019] noted that this agrees with the sufficient statistic perturbation setup for private Bayesian inference (e.g., [Foulds et al., 2016]), and applied the algorithm for exact inference from a private posterior. We extend this algorithm to additionally condition on \( Z \) in our inference, presented here:

**Theorem 3.** Algorithm 1 samples from \( \theta \mid Y, Z \)
Result: One sample from $\theta \mid Y = y, Z = z$
Sample $\theta^* \sim \pi(\theta \mid Z = z)$;
Sample $X^* \sim \pi(\cdot \mid \Theta = \theta^*, Z = z)$;
Sample $U \sim \text{Unif}(0, 1)$;
if $\frac{\pi(\cdot \mid X = x^*, Z = z)}{\sup_{y \in Y} \pi(\cdot \mid X = x^*, Z = z)} \leq U$
then
| Return $\theta^*$.
else
| Go to beginning.
end

Algorithm 1: Posterior rejection sampling conditional on public information

5 Data Analysis

5.1 Data Description

The Pennsylvania Department of Health collects data on the total number of reported COVID-19 cases and deaths per county [Pennsylvania Department of Health, 2022]. Data from the US Census Bureau’s Current Population Survey (CPS) provided estimated population counts by demographic strata, accessed through IPUMS [Ruggles et al., 2022]. For the purposes of our analysis, we will treat these values as fixed population totals; however, additional work is required to incorporate errors due to sampling and weighting schemes.

In our work, we will examine the effect of both deterministic and probabilistic public information on our ability to perform inference from these data sources. Throughout this section, we use the following notation, with $[n] \triangleq \{1, 2, \ldots, n\}$:

\[
\begin{align*}
    \{ j \in \{1, \ldots, J\} \} & \triangleq \text{Pennsylvania counties, } J = 67 \\
    t \in \{1, \ldots, T\} & \triangleq \text{Year-month periods, } T = 24 \\
    X_{j,t}^{(c)} & \triangleq \text{Number of COVID-19 cases in county } j \text{ at time } t \\
    X_{j,t}^{(d)} & \triangleq \text{Number of COVID-19 deaths in county } j \text{ at time } t
\end{align*}
\]

Our goal will be to privatize $X_{j,t}^{(c)}$ and $X_{j,t}^{(d)}$ under the following public information scenarios:

\[
Z_t = \begin{cases} 
    X_{j,t-1}^{(c)} = x_{j,t-1}^{(c)} \\
    X_{j,t-1}^{(d)} = x_{j,t-1}^{(d)} \\
    \sum_{j=1}^{J} X_{j,t} = s^{(c)}_{j,t} \\
    \mathbb{P}(X_{j,t}^{(c)} \geq X_{j,t}^{(d)}) = 1
\end{cases}
\]

For this analysis, we will consider the following synthesis procedures for case counts:

1. **Post-processing**: first, we synthesize:

\[
\begin{align*}
    Y_{j,t}^{(c)} &= X_{j,t}^{(c)} + \varepsilon_{j,t}^{(c)} \\
    Y_{j,t}^{(d)} &= X_{j,t}^{(d)} + \varepsilon_{j,t}^{(d)} \\
    \varepsilon_{j,t}^{(c)}, \varepsilon_{j,t}^{(d)} &\sim \text{DiscreteLaplace} \left( \frac{\epsilon}{\Delta} \right)
\end{align*}
\]
Figure 3: COVID-19 data (from Pennsylvania Department of Health [Pennsylvania Department of Health, 2022])
nism provides expected improvements over the naive choice of base measure in every situation, as processing. In particular, the choice of base measure will greatly affect the loss of these mechanisms. Such a method fails to consider the optimality of these mechanisms under other forms of post-processing methods based solely on total counts as public information are provably optimal for minimizing the expected value of any convex loss function [Ghosh et al., 2012].

5.2 Effect of base measure choice on mechanism performance

Then, we will perform deterministic two-stage post-processing:

\[
\left( \frac{\hat{Y}_t^{(c)}}{\hat{Y}_t^{(d)}} \right) = \arg \min_{y \in \mathbb{R}^{2J}} \left\| \left( \begin{array}{c} y_c \vspace{1mm} \\ y_d \end{array} \right) - \left( \begin{array}{c} Y_t^{(c)} \\
Y_t^{(d)} \end{array} \right) \right\|^2_{L_2} \quad \text{s.t.} \quad \begin{cases} \sum_{j=1}^{J} y_j^{(c)} = s_{j,t} \\
y_j \geq 0 \forall j \in [J] \end{cases}
\]

Next:

\[
\left( \begin{array}{c} Y_t^{(c)} \\
Y_t^{(d)} \end{array} \right) = \left( \begin{array}{c} \hat{Y}_t^{(c)} \\
\hat{Y}_t^{(d)} \end{array} \right) + \arg \min_{y \in \{0,1\}^{2J}} \left\| \left( \begin{array}{c} y_c \\
Y_t^{(d)} \end{array} \right) - \left( \begin{array}{c} \hat{Y}_t^{(c)} \\
\hat{Y}_t^{(d)} \end{array} \right) \right\|_{L_1} \quad \text{s.t.} \quad \begin{cases} \sum_{j=1}^{J} y_j^{(c)} = s_{j,t} \\
y_j^{(c)} \geq 0 \forall j \in [J] \end{cases}
\]

2. Wasserstein Mechanism

Alternatively, we’ll compare this method to the following version of the Wasserstein exponential mechanism with loss function:

\[
L_X(Y) = \left\| \left( \begin{array}{c} y_c \\
Y_t^{(d)} \end{array} \right) - \left( \begin{array}{c} Y_t^{(c)} \\
Y_t^{(d)} \end{array} \right) \right\|_{L_2}
\]

Notably, we’ll use the public information to inform the base measure in three different scenarios:

(a) Naive base measure:

\[
\nu_{Z}(Y) \propto 1_{\{Y \in \mathbb{Z}^J\}}
\]

(b) Deterministic congenial base measure:

\[
\nu_{Z}(Y) \propto 1_{\{Y \in \mathbb{Z}^J, \sum_{i=1}^{J} Y_{j,i}^{(c)} = s_{j,t}, Y_{j,i}^{(c)} \geq Y_{j,i}^{(d)} \geq 0\}}
\]

(c) Prior congenial base measure:

\[
\nu_{Z}(Y) \propto \phi(Y; s_{t-1}^{(c)}, X_{t-1}^{(c)}) 1_{\{Y \in \mathbb{Z}^J, \sum_{i=1}^{J} Y_{j,i}^{(c)} = s_{j,t}, Y_{j,i}^{(c)} \geq Y_{j,i}^{(d)} \geq 0\}}
\]

Where \( \phi \) is the PMF of the Dirichlet-Multinomial distribution:

\[
\phi(Y^{(c)}; s_t^{(c)}, \alpha X_{t-1}^{(c)}) = \frac{\Gamma(\alpha s_t^{(c)}) \Gamma(s_t^{(c)} + 1)}{\Gamma((\alpha + 1)s_t^{(c)})} \prod_{j=1}^{J} \frac{\Gamma(Y_{j,t}^{(c)} + \alpha X_{j,t-1}^{(c)})}{\Gamma(Y_{j,t}^{(c)}) \Gamma(\alpha X_{j,t-1}^{(c)} + 1)}
\]

For consistent and fair comparisons between the two approaches, we will only compare the mechanisms with \( \Delta_Z \) equal. In doing so, both mechanisms offer the same privacy guarantees under \( \epsilon \)-DPZ. The only differences are how both mechanisms systematically use public information \( Z \).

5.2 Effect of base measure choice on mechanism performance

Because the post-processed counts along 1-way marginals were designed to minimize the expected value of any convex loss function [Ghosh et al., 2012], post-processing methods based solely on total counts as public information are provably optimal for minimizing the expected \( L_1 \) loss. However, such a method fails to consider the optimality of these mechanisms under other forms of post-processing. In particular, the choice of base measure will greatly affect the loss of these mechanisms.

In terms of total loss, we see a couple effects from Figure 4. First, the post-processing mechanism provides expected improvements over the naive choice of base measure in every situation, as
expected. However, the effect of the probabilistic public information introduces a new hyperparameter $\alpha$ which dictates the effect of the base measure on regularization. While some regularization can improve the total loss value, too much regularization can produce results that simply resample from the posterior predictive of the county-level cases at the previous data collection time.

What’s important to know, however, is that the effect of this shrinkage is especially important for managing relative errors induced by privacy. In Figure 5, we see that the prior regularization saves up to an entire order of magnitude on relative errors due to privacy for small counties. As one such example, we look at the synthesis results for Cameron County, PA, the smallest county in Pennsylvania by 2020 ACS population estimates. In this county, the effect of distortion due to post-processing is exacerbated relative to the total counts, but this effect is tempered when using the regularizing base measure.

5.3 Inference improvements over post-processing

Comparing the inferential improvements of results directly is intractable for one primary reason, in that we cannot easily compute the Bayes estimator $\delta_Y^*$ for our data, nor do we have an exact parameter estimate with which to compare. However, in the proof of Theorem 2, we used the fact
Figure 6: Comparison of synthesis estimates for Cameron county COVID-19 cases.
that post-processing maps multiple values of $Y$ with different inferential interpretations to the same value of $Y^*$. Therefore we can investigate how frequently this effect occurs at different privacy budget configurations. This extends the analysis of [Santos-Lozada et al., 2020] by specifically looking at the effect post-processing has on health disparities in the context of small-area estimates for COVID-19 prevalence.

In particular, the post-processing as defined earlier produces the following effects:

- **ContractedCaseZeros**: cases where multiple potential imputations of the private COVID-19 case data are contracted to 0
- **ContractedDeathZeros**: cases where multiple potential imputations of the private COVID-19 death data are contracted to 0
- **ContractedRates**: cases where the constrait that COVID-19 cases is bounded below by COVID-19 deaths contracts imputations of the COVID-19 survival rate to 0.

We calculate the prevalence of these effects in our synthesis and plot the results in Figure 7. We only show the results for the cases with the largest and smallest numbers of total COVID-19 cases; full results are available in the appendix. In general, we see that degradation of inference due to post-processing occurs most frequently in the high-privacy, low-sample-size regime. These effects disproportionately fall on the least populous counties, demonstrating that post-processing exacerbates the inequitable inference on larger counties versus smaller counties present in smaller data sets. This problem is tempered not only by using the Wasserstein exponential mechanism, but also by performing regularization with a well-chosen base measure.

6 Discussion and Broader Impacts

Our work provides a statistically reasoned framework for understanding formally private data guarantees for partially private data. In summary, public information increases the effective sensitivity of formal privacy mechanisms and restricts the space of possible databases we can compare in providing DP-style privacy guarantees. Here we re-emphasize the crucial difference between standard DP and our framework, in that the guarantees are no longer uniform across databases. Understanding the effect of this public information from a privacy-utility trade-off perspective will require some adversarial modeling assumptions, echoing the “no free lunch” property of data privacy [Kifer and Machanavajjhala, 2011]. Such is the realistic cost of moving to a perspective that accommodates public data.

Additionally, our work re-evaluates the role of post-processing in DP algorithms. We have demonstrated, theoretically and empirically, that the information lost due to post-processing is not trivially recoverable, and that working directly with PPP from a Bayesian perspective offers more powerful statistically valid inferences. This yields immediate policy implications for privacy mechanism designers, in that we encourage privacy engineers to release their results without post-processing applied whenever possible.

Note that our work is designed as a way to accommodate public information, but this itself is not a unilateral endorsement of releasing public information to accommodate DP results. The privacy affordances offered by DP are kept closest to their idealized guarantees with little to no public information released outside the confines of the framework. Whenever possible, data curators should attempt to release results that accommodate randomization to satisfy FP, and carefully consider when certain cases that do not logistically accommodate this randomization occur.
Figure 7: Frequencies of post-processing induced inference effects on private synthetic data.
Still, we believe this work is valuable because situations that necessitate public information arise frequently in the social sciences, in particular for the production of official statistics, administrative data, and survey data. The US Census Bureau’s public information use case was mandated by public law [94th Congress of the United States of America, 1975], but other cases, such as public disclosure of survey methodology [Seeman and Brummet, 2021], help more generally to quantitatively establish trust in the quality of data created by the curator. In cases like these, disclosing public information not only satisfies an important logistical requirement, but also respects a context specific normative practice on how information flows between parties. This context-specificity is an essential component in how privacy is sociologically conceived [Nissenbaum, 2009]. As a result, our framework serves to align important technical and sociological goals in privacy preserving data sharing and analysis.

As threats to data privacy become more prevalent in the age of mass data collection and surveillance, we will need to ask ourselves important normative questions about whether using public information to respect these contexts remains valuable. Implementing changes to these practices to accommodate will be no easy task, as existing data privacy regulations largely fail to grapple with the probabilistic nature of disclosure and disclosure limitation, should we even choose to pursue this option. Until such a time comes, data curators in the social sciences will need to reconcile the privacy risks inherent in disseminating data for public use with the legal, ethical, and normative demands of the citizens and organizations they serve. Our framework offers these data curators an avenue to pursue this essential work.

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### A Proofs

#### A.1 Proof of Lemma 1

Let $z \in Z$, $B \in \mathcal{F}_Y$, $\theta_z \in \mathcal{D}_{DPz}$, and $(s_1, s_2) \in \mathcal{S}_{pairs,DPz}$. By Bayes rule:

\[
\frac{\mathbb{P}(s_1 | Y \in B, \theta_z)}{\mathbb{P}(s_2 | Y \in B, \theta_z)} = \frac{\mathbb{P}(Y \in B | s_1, \theta_z)}{\mathbb{P}(Y \in B | s_2, \theta_z)}
\]

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Note that we avoid dividing by zero in either quantity because we assume \( \theta_z \in \mathbb{D}_{\text{DP}_z} \) and \((s_1, s_2) \in S_{\text{pairs,DP}_z} \). The ratio on the right is within \([e^{-\epsilon}, e^\epsilon]\) if and only if Equation 3 holds.

### A.2 Proof of Lemma 3

Fix \( z \in \mathcal{Z} \), \((s_1, s_2) \in S_{\text{pairs,DP}_z} \), and \( \theta_z \in \mathbb{D}_{\text{DP}_z} \). For \( B \in \mathcal{F}_Y \), let \( B^*_z \in \mathcal{F}_{Y^* \times \mathcal{Z}} \) and define \( B^*_z = \{ y \in \mathcal{Y} \mid h(y, z) = Y^* \} \). Then \( B^*_z \in \mathcal{F}_Y \) for all \( z \in \mathcal{Z} \) by measurability. This implies:

\[
\mathbb{P}(Y^* \in B^*_z \mid s_1, \theta_z) = \mathbb{P}(Y \in B^*_z \mid s_1, \theta_z) \leq \mathbb{P}(Y \in B^*_z \mid s_2, \theta_z) e^\epsilon = \mathbb{P}(Y^* \in B^*_z \mid s_2, \theta_z) e^\epsilon.
\]

Proof. Let \( Y \overset{\Delta}{=} (Y_1, Y_2) \) on \((\mathcal{Y}, \mathcal{F}_Y)\). By the conditional independence assumption, there exists \( B_1, B_2 \in \mathcal{F}_{Y_1} \times \mathcal{F}_{Y_2} \) such that:

\[
\mathbb{P}(Y \in B \mid s_1, \theta_z) = \mathbb{P}(Y_1 \in B_1 \mid s_1, \theta_z) \mathbb{P}(Y_2 \in B_2 \mid s_1, \theta_z) \\
\leq (e^{\epsilon_1} \mathbb{P}(Y_1 \in B_1 \mid s_2, \theta_z)) (e^{\epsilon_2} \mathbb{P}(Y_2 \in B_2 \mid s_2, \theta_z)) \\
= e^{(\epsilon_1 + \epsilon_2)} \mathbb{P}(Y \in B \mid s_2, \theta_z).
\]

\[\square\]

### A.3 Proof of Lemma 5

Without loss of generality, let \( z = 1 \) (otherwise, replace \( \mathcal{X}^{n*} \) with \( \mathcal{X}^{n} \setminus \mathcal{X}^{n*} \)). Then the desired guarantee becomes, for all \( x, x' \in \mathcal{X}^{n*} \) such that \( d_H(x, x') = 1 \):

\[
\begin{cases} 
\mathbb{P}(Y \in B \mid X = x, Y \in \mathcal{X}^{n*}) \leq e^\epsilon \mathbb{P}(Y \in B \mid X = x', Y \in \mathcal{X}^{n*}) \\
\mathbb{P}(Y \in B \mid X = x', Y \in \mathcal{X}^{n*}) \leq e^\epsilon \mathbb{P}(Y \in B \mid X = x, Y \in \mathcal{X}^{n*})
\end{cases}
\]

The rest of the proof follows [Gong and Meng, 2020] setting \( k = 1 \) and \( \gamma = 1 \).

### A.4 Proof of Theorem 1

Let \((s_1, s_2) \in S_{\text{pairs,DP}_z} \) and \( B \in \mathcal{F}_Y \). Let \( \mu_i, \theta_z \overset{\Delta}{=} \mathbb{P}(\cdot \mid \theta_z, s_i) \). Let \( \gamma^* \) be the joint distribution that achieves the Wasserstein distance bound. Then:

\[
\frac{\mathbb{P}(Y \in B \mid s_1, \theta_z)}{\mathbb{P}(Y \in B \mid s_1, \theta_z)} = \frac{\int_X \mathbb{P}(Y \in B \mid s_1, \theta_z, X) \, d\mu_1, \theta_z}{\int_X \mathbb{P}(Y \in B \mid s_1, \theta_z, X) \, d\mu_2, \theta_z}
\]

\[
= \frac{\int_X 1_{\{y \in B\}} \exp \left( -\frac{L_x(y)}{\sigma(\Delta_z)} \right) \, d\mu_1, \theta_z}{\int_X 1_{\{y \in B\}} \exp \left( -\frac{L_x(y)}{\sigma(\Delta_z)} \right) \, d\mu_2, \theta_z}
\]

\[
= \frac{\int_X \int_{X'} 1_{\{y \in B\}} \exp \left( -\frac{L_x(y)}{\sigma(\Delta_z)} \right) \, d\gamma(x, x')}{\int_X \int_{X'} 1_{\{y \in B\}} \exp \left( -\frac{L_{x'}(y)}{\sigma(\Delta_z)} \right) \, d\gamma(x, x')}
\]

Let \( B_{\Delta_z}(x) \overset{\Delta}{=} \{ x' \in X \mid d(x, x') \leq \Delta_z \} \). Then by construction and definition of the Wasserstein distance:

\[
\frac{\mathbb{P}(Y \in B \mid s_1, \theta_z)}{\mathbb{P}(Y \in B \mid s_1, \theta_z)} = \frac{\int_X \int_{B_{\Delta_z}(x)} 1_{\{y \in B\}} \exp \left( -\frac{L_x(y)}{\sigma(\Delta_z)} \right) \, d\gamma(x, x')}{\int_X \int_{B_{\Delta_z}(x')} 1_{\{y \in B\}} \exp \left( -\frac{L_{x'}(y)}{\sigma(\Delta_z)} \right) \, d\gamma(x, x')} \leq e^\epsilon
\]

Where in the last line, we used the fact that \(|L_x(y) - L_{x'}(y)| \leq \sigma(\Delta_z)\) uniformly.
A.5 Proof of Lemma 6

Fix $\delta > 0, x \in X$, and let $d_H$ be a metric on $X$. Then for sufficiently large $m$:

$$\sup_{\theta \in \Theta} \sup_{(s_1, s_2) \in S_{\text{pairs}, DP}} W_\infty \left\{ \mathbb{P}(\cdot \mid \theta, s_1), \mathbb{P}(\cdot \mid \theta, s_2) \right\} \geq \sup_{(s_1, s_2) \in S_{\text{pairs}, DP}} W_\infty \left\{ \mathbb{P}(\cdot \mid \theta^{(m)}_{\phi, x}, s_1), \mathbb{P}(\cdot \mid \theta^{(m)}_{\phi, x}, s_2) \right\} \geq 1 - \delta$$

Since this is true for arbitrary $\delta$, then we can ensure $\Delta_x \geq 1$, yielding the desired result.

A.6 Proof of Theorem 2

First we show that $Y$ has a monotone likelihood in $\theta$. Let $\Xi \triangleq Y - T(X)$. Then the marginal density of $Y$, $m_\theta(y)$ can be expressed as a function of the density of $\Xi$, $g_\Xi(\xi)$ as a convolution:

$$g_\Xi(\xi) \propto \exp \left( - \frac{\epsilon}{2\Delta_L} L(\xi) \right) \implies m_\theta(y) \triangleq \int_X g_\Xi(y - x) f_\theta(x) \, dx.$$  

Let $y_1, y_2 \in \mathcal{Y}$ such that $y_1 < y_2$ and $\theta_1, \theta_2 \in \Theta$ such that $\theta_1 < \theta_2$. Suppose first that $n = 1$. Note that $f_\theta(x)$ has a monotone likelihood because $f_\theta$ belongs to an exponential family. Similarly, $g_\Xi$ has a monotone likelihood ratio by construction. Then using the main result from [Wijsman, 1985]:

$$\int_X g_\Xi(y_1 - x) f_\theta_1(x) \, dx \int_X g_\Xi(y_2 - x) f_\theta_2(x) \, dx \geq \int_X g_\Xi(y_2 - x) f_\theta_1(x) \, dx \int_X g_\Xi(y_2 - x) f_\theta_1(x) \, dx \implies m_{\theta_1}(y_1) m_{\theta_2}(y_2) \geq m_{\theta_2}(y_1) m_{\theta_1}(y_2)$$

$$\implies \frac{m_{\theta_2}(y_1)}{m_{\theta_1}(y_1)} \leq \frac{m_{\theta_2}(y_2)}{m_{\theta_1}(y_2)}$$

The results extends to arbitrary $n$ by the iid assumption, and therefore $Y$ has a monotone likelihood ratio in $\theta$.

Next, let $\phi : \mathcal{Y}^* \mapsto [0, 1]$ be an unbiased test for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Fix the type I error as $\alpha$. Using the fact that $Y$ has a monotone likelihood ratio in $\theta$ and the unbiasedness of $\phi$, we know $\beta_\phi(\theta_0) \leq \beta_\phi(\theta_1)$ for all $\theta_0, \theta_1 \in \Theta$ such that $\theta_0 \leq \theta_1$, where:

$$\beta_{Y^*}(\theta) \triangleq \mathbb{E}_\theta[\phi(Y^*)] = \mathbb{E}_\theta[\phi(Y^*) (1_{Y = Y^*} + 1_{Y < Y^*} + 1_{Y > Y^*})].$$

When $\theta = \theta_0$ and $Y < Y^*$, using the MLR property, there exists a test $\phi' : \mathcal{Y} \times \mathcal{Z} \mapsto [0, 1]$ such that $\phi'(Y, Z) = \phi(Y^*)$ whenever $Y = Y^*$, and:

$$\mathbb{E}_{\theta_0}[\phi'(Y, Z) \mid Y < Y^*] \leq \mathbb{E}_{\theta_0}[\phi(Y^*) \mid Y < Y^*]$$

Let $\gamma \triangleq \mathbb{E}_{\theta_0}[\phi(Y^*) - \phi'(Y, Z)] \in [0, \alpha]$. We can modify $\phi'$ such that, for $Y > Y^*$:

$$\mathbb{E}_{\theta_0}[\phi'(Y, Z) - \phi(Y^*) \mid Y > Y^*] = \gamma$$

This yields a final test $\phi'(Y, Z)$ which dominates $\phi(Y^*)$ and is still level $\alpha$. As long as $\mathbb{P}_\theta(Y \neq Y^*) > 0$, the dominance is non-trivial; the new test may have a smaller type I error, type II error, or both.
A.7 Proof of Theorem 3

Throughout this proof, we take \( \pi \) to refer to the density of the variables in question. The target density has the form:

\[
\pi(\theta \mid y, z) = \int \pi(\theta, y \mid z) \, d\nu_X 
= \int \pi(y \mid z) \pi(\theta \mid z) \, d\nu_X 
= \int \pi(y \mid z) \pi(\theta \mid z) \pi(\theta, x \mid x, z) \, d\nu_X \, d\nu_\Theta 
\]

Following [Gong, 2019], let \( A \) be a Bernoulli variable with success probability associated with accepting one proposal in the algorithm with public information \( z \) and proposed samples \( \theta^* \in \Theta \) and \( x^* \in X^n \). Define:

\[
C \triangleq \sup_{y \in Y} \pi(y \mid x^*, z) 
\]

Then:

\[
\pi(\theta^* \mid A = 1, z) = \int \frac{\pi(\theta^*, x^*, A = 1)}{\pi(A = 1)} \, d\nu_X 
= \frac{C \int \pi(\theta \mid z) \pi(x \mid \theta, z) \pi(y \mid x, z) \, d\nu_X}{C \int \pi(\theta \mid z) \pi(x \mid \theta, z) \pi(y \mid x, z) \, d\nu_X \, d\nu_\Theta} 
= \pi(\theta \mid y, z) 
\]

Therefore accepted samples in Algorithm 1 are exact draws from \( \theta \mid Y, Z \).