On the average order of the gcd-sum function over arbitrary sets of integers

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Abstract: Let \( \mathbb{N} \) denote the set of all positive integers and for \( j, n \in \mathbb{N} \), let \((j, n)\) denote their greatest common divisor. For any \( S \subseteq \mathbb{N} \), we define \( P_S(n) \) to be the sum of those \((j, n) \in S\), where \( j \in \{1, 2, 3, \ldots, n\} \). An asymptotic formula for the summatory function of \( P_S(n) \) is obtained in this paper which is applicable to a variety of sets \( S \). Also the formula given by Bordellès for the summatory function of \( P_{\mathbb{N}}(n) \) can be derived from our result. Further, depending on the structure of \( S \), the asymptotic formulae obtained from our theorem give better error terms than those deducible from a theorem of Bordellès (see Remark 4.4).

Keywords: Pillai function, gcd-sum function, Asymptotic formula, Möbius function of \( S \), Dirichlet product, \( r \)-free integer, Semi-\( r \)-free integer, \((k, r)\)-integer, Unitary divisor.

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1 Introduction

Let \( \mathbb{N} \) denote the set of all positive integers. For \( j, n \in \mathbb{N} \), let \((j, n)\) denote their greatest common divisor (gcd).

If \( S \subseteq \mathbb{N} \), then define

\[ P_S(n) = \sum_{\substack{j=1 \\ (j,n) \in S}}^{n} (j, n) \quad \text{for } n \in \mathbb{N}. \] (1.1)
Observe that $P_n(n) = P(n)$, the arithmetic function studied by Pillai [9]. Possibly unaware of this work, Broughan [5] considered the same function (under a different notation) and obtained an asymptotic formula for $\sum_{n \leq x} P(n)$. Later, Bordellès [2] improved the error term in that asymptotic formula.

Also Bordellès [3] introduced a more general situation of

$$P_f(n) = \sum_{j=1}^{n} f((j,n)),$$

where $f$ is any arithmetic function and gave a proof of the Cesàro formula:

$$P_f(n) = (f \ast \varphi)(n) \quad \text{for any } n \in \mathbb{N},$$

in which $\varphi$ is the Euler totient function and $\ast$ is the classical Dirichlet product of arithmetic functions. Moreover in the same paper unified asymptotic formulae for $\sum_{n \leq x} P_f(n)$ are obtained for multiplicative arithmetic functions that lie in certain special classes.

A very informative survey on the gcd-sum functions by Tóth [14] and the paper on the weighted gcd-sum function (which is yet another general situation) by the same author [15] are worth to be mentioned here.

The purpose of this paper is to estimate $\sum_{n \leq x} P_S(n)$, for $S \subseteq \mathbb{N}$ which satisfy a condition; and to show that the formula of Bordellès [2] is deducible from our result. Further the formula is applicable to a variety of sets of integers such as the set of $r$-free integers, the set of semi-$r$-free integers and the set of $(k,r)$-integers studied by earlier researchers, in different contexts. The error terms in these asymptotic formulae are better than those deducible from a theorem of Bordellès ([3], Theorem 4, Part 4).

## 2 Notation and Preliminaries

For $S \subseteq \mathbb{N}$, let $\chi_S(n)$ be its characteristic function. (That is, $\chi_S(n) = 1$ or 0, respectively, as $n \in S$ or $n \notin S$.) Following Cohen [6], the Möbius function of $S$, denoted by $\mu_S(n)$, is defined by

$$\mu_S(n) = \sum_{d \mid n} \mu(d) \chi_S\left(\frac{n}{d}\right) = (\mu \ast \chi_S)(n) \quad \text{for } n \in \mathbb{N},$$

where $\mu(n)$ is the well-known Möbius function.

Several properties of $\ast$ are studied in [1] (Chapter 2) some of which we use in this paper. For example, if $u(n) = 1$ for all $n \in \mathbb{N}$ and $\varepsilon_0(n) = 1$ or 0, respectively, as $n = 1$ or $n > 1$, then

$$\mu \ast u = \varepsilon_0$$

and $f \ast \varepsilon_0 = f$ for any arithmetic function $f$.

It follows from (2.1) and (2.2), that

$$\mu_{\{1\}} = \mu \quad \text{and} \quad \mu_{\mathbb{N}} = \varepsilon_0,$$

since $\chi_{\{1\}} = \varepsilon_0$ and $\chi_{\mathbb{N}} = u$; and that

$$\chi_S = u \ast \mu_S \quad \text{or equivalently} \quad \chi_S(n) = \sum_{d \mid n} \mu_S(d) \quad \text{for any } S \subseteq \mathbb{N} \text{ and } n \in \mathbb{N}.$$
Also if \( I(n) = n \) for all \( n \in \mathbb{N} \) then \( I(f * g) = If * Ig \) for arithmetic functions \( f \) and \( g \). Further, it is clear that
\[
(u * u)(n) = \tau(n), \text{ the number of positive divisors of } n \in \mathbb{N}. \tag{2.5}
\]
A well-known identity is
\[
\varphi(n) = \sum_{d | n} \frac{n}{d} \quad \text{or equivalently} \quad \varphi = I * \mu. \tag{2.6}
\]

Now we express below \( P_S \) as a Dirichlet product of some of the functions mentioned above.

**Lemma 2.1.** \( P_S = (I\mu_S) * (I\tau * \mu) \), for any \( S \subseteq \mathbb{N} \).

**Proof.** First observe that \( P_S(n) = \sum_{j=1}^{n} I((j, n)) \chi_S((j, n)) = \mathcal{P}_{I\chi_S}(n) \), so that, in view of (1.3), (2.4), (2.6) and (2.5),
\[
P_S = \mathcal{P}_{I\chi_S} = (I\chi_S) * \varphi = I(\mu_S * u) * (I * \mu) = I\mu_S * Iu * (Iu * \mu)
\]
proving the lemma. \( \square \)

One can observe that if \( S = \mathbb{N} \) then Lemma 2.1 gives \( P = I \tau * \mu \), a result proved by Bordellès ([2], Lemma 2.1).

If \( M(x) = \sum_{n \leq x} \mu(n) \), then its exact order of magnitude is not known. The best estimate given by Walfisz ([16], p.191) is that
\[
M(x) = O(x^{\delta(x)}) \text{ for } x > 1, \tag{2.7}
\]
where
\[
\delta(x) = \exp\{-A(\log x)^{3/2}(\log \log x)^{\frac{1}{5}}\}, \tag{2.8}
\]
in which \( A \) is a positive constant.

Note that \( \delta(x) \) is a monotonic decreasing function.

Using (2.7), Suryanarayana and Siva Rama Prasad [13] proved that, when \( x > 1 \),
\[
\sum_{n \leq x} \frac{\mu(n)}{n^t} = \frac{1}{\zeta(t)} + O\left(\frac{\delta(x)}{x^{t-1}}\right) \text{ for } t > 1 \quad ([13], \text{Lemma 2.2}) \tag{2.9}
\]
and
\[
\sum_{n \leq x} \frac{\mu(n) \log n}{n^t} = \frac{\zeta'(t)}{\zeta^2(t)} + O\left(\frac{\delta(x) \log x}{x^{t-1}}\right) \text{ for } t > 1 \quad ([13], \text{Lemma 2.3}), \tag{2.10}
\]
where \( \zeta(t) \) is the Riemann-zeta function.

The classical Dirichlet divisor problem seeks the least value of \( \theta \) for which the asymptotic formula
\[
\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(x^\theta) \tag{2.11}
\]
holds, where \( \gamma \) is the Euler constant. It is known that \( \frac{1}{4} \leq \theta \leq \frac{517}{1648} \). The lower bound for \( \theta \) is due to Hardy [8] while the upper bound is obtained recently by Bourgain and Watt [4].
Now using (2.11) and the Abel’s identity ([1], Theorem 4.2), it is easy to prove
\[ \sum_{n \leq x} I(n)\tau(n) = \frac{1}{2}x^2 \left( \log x + 2\gamma - \frac{1}{2} \right) + O \left( x^{1+\theta+\varepsilon} \right), \quad (2.12) \]
where \( \varepsilon > 0 \).

3 Main result

In this section we prove the theorem given below:

**Theorem 3.1.** Suppose \( S \subseteq \mathbb{N} \) is such that the infinite series \( \sum_{n=1}^{\infty} \frac{\mu_S(n) \log n}{n} \) converges absolutely. Then for \( x \geq 1 \), we have
\[
\sum_{n \leq x} P_S(n) = \frac{x^2}{2\zeta(2)} \left\{ \alpha_S \left( \log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) - \beta_S \right\} + \Delta_S(x),
\]
where
\[
\Delta_S(x) = \frac{x}{2\zeta(2)} \left( \beta_S(x) - \alpha_S(x) \right) + O \left( x^{1+\theta+\varepsilon} \gamma_S(x) \right), \quad (3.1)
\]
\[
\alpha_S = \sum_{n=1}^{\infty} \frac{\mu_S(n)}{n}, \quad (3.2)
\]
\[
\beta_S = \sum_{n=1}^{\infty} \frac{\mu_S(n) \log n}{n}, \quad (3.3)
\]
\[
\alpha_S(x) = \sum_{n>x} \frac{\mu_S(n)}{n}, \quad (3.4)
\]
\[
\beta_S(x) = \sum_{n>x} \frac{\mu_S(n) \log n}{n}, \quad (3.5)
\]
and
\[
\gamma_S(x) = \sum_{n \leq x} \frac{|\mu_S(n)|}{n^{\theta+\varepsilon}}, \quad (3.6)
\]
in which \( \varepsilon > 0 \).

**Proof.** Under the hypothesis of the theorem, note that \( \beta_S \) and hence \( \alpha_S \) are both well-defined.

By Lemma 2.1, we have \( P_S = f \ast g \), where \( f = I\mu_S \) and \( g = I\tau \ast \mu \), so that
\[
\sum_{n \leq x} P_S(n) = \sum_{n \leq x} f(u) \left\{ \sum_{v \leq \frac{x}{u}} g(v) \right\}. \quad (3.7)
\]

To estimate the inner sum on the right of (3.7), we use (2.12), (2.9) and (2.10) to get
\[
\sum_{n \leq x} g(n) = \sum_{d \leq x} \mu(d) \left\{ \sum_{t \leq \frac{x}{d}} I(t)\tau(t) \right\}
= \sum_{d \leq x} \mu(d) \left\{ \frac{(x/d)^2}{2} \left( \log \left( \frac{x}{d} \right) + 2\gamma - \frac{1}{2} \right) + O \left( \left( \frac{x}{d} \right)^{1+\theta+\varepsilon} \right) \right\}
\]

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Proof. In view of (2.3), the condition of Theorem 3.1 holds if $S = \mathbb{N}$. Also since $\alpha_1 = 1$, $\beta_0 = 0$, $\alpha_N(x) = \beta_N(x) = 0$ and $\gamma_N(x) = 1$ for $x \geq 1$, the corollary follows. \qed
Recall that, for \( t > 1 \),

\[
\zeta(t) = \sum_{n=1}^{\infty} \frac{1}{n^t}
\]  

(3.11)

and

\[
\frac{1}{\zeta(t)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^t}.
\]  

(3.12)

Both the series on the right of (3.11) and (3.12) converge absolutely and, therefore, by Theorem 11.2 of [1], they can be differentiated term by term with respect to \( t \), to get

\[
\zeta'(t) = -\sum_{n=1}^{\infty} \frac{\log n}{n^t} \quad \text{for } t > 1.
\]  

(3.13)

and

\[
\frac{\zeta'(t)}{\zeta^2(t)} = \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n^t} \quad \text{for } t > 1.
\]  

(3.14)

Now (2.9) and (3.12) give

\[
\sum_{n>x} \frac{\mu(n)}{n^t} = O\left(\frac{\delta(x)}{x^{t-1}}\right) \quad \text{for } t > 1;
\]  

(3.15)

while (2.10) and (3.14) show

\[
\sum_{n>x} \frac{\mu(n) \log n}{n^t} = O\left(\frac{\delta(x) \log x}{x^{t-1}}\right) \quad \text{for } t > 1.
\]  

(3.16)

4. Application to some special subsets of \( \mathbb{N} \)

In the rest of this paper \( n \in \mathbb{N} \) with \( n > 1 \) is of the form \( n = \prod_{i=1}^{l} p_i^{\alpha_i} \), where \( p_1, p_2, \ldots, p_l \) are distinct primes and integers \( \alpha_i \) are \( \geq 1 \) for \( 1 \leq i \leq l \).

To show the richness of the sets \( S \subseteq \mathbb{N} \) for which Theorem 3.1 is applicable, first we make a brief study of the \( M \)-free integers introduced by Rieger [10].

Let \( M \) be a set of positive integers with the minimal element \( r \), where \( r > 1 \). A number \( n \geq 1 \) is said to be \( M \)-free if \( \alpha_i \notin M \) for \( i = 1, 2, \ldots, l \). The set of all \( M \)-free integers will be denoted by \( Q_M \).

Clearly \( 1 \in Q_M \) for every \( M \subseteq \mathbb{N} \). Also \( \chi_{Q_M} \) is a multiplicative function (that is, \( \chi_{Q_M}(ab) = \chi_{Q_M}(a) \chi_{Q_M}(b) \) whenever \( (a, b) = 1 \)). Then, by (2.1), \( \mu_{Q_M} \) is a multiplicative function. Further for any prime \( p \) and \( \alpha \in \mathbb{N} \) we have

\[
\mu_{Q_M}(p^\alpha) = \chi_{Q_M}(p^\alpha) - \chi_{Q_M}(p^{\alpha-1})
\]

\[
= \begin{cases} 
-1 & \text{if } \alpha \in M^* = \{ \alpha \in \mathbb{N} : \alpha \in M \text{ and } \alpha - 1 \notin M \} \\
1 & \text{if } \alpha \in M^{**} = \{ \alpha \in \mathbb{N} : \alpha \notin M \text{ and } \alpha - 1 \in M \} \\
0 & \text{otherwise.} 
\end{cases}
\]  

(4.1)

Hence, for \( n > 1 \), the value \( \mu_{Q_M}(n) \) is non-zero if and only if (shortly, iff) \( n \) can be written as

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\( n = n^*, n^{**} \), where \( n^* = \prod_{\alpha_i \in M^*} p_i^{\alpha_i} \) and \( n^{**} = \prod_{\alpha_i \in M^{**}} p_i^{\alpha_i} \) which are such that \((n^*, n^{**}) = 1\) (since \( M^* \cap M^{**} = \emptyset \)). Also in this case
\[
\mu_{Q_M}(n) = (-1)^{\omega(n^*)} \cdot 1^{\omega(n^{**})} = (-1)^{\omega(n^*)},
\]
where \( \omega(m) \) is the number of distinct prime factors of \( m \).

Notice that unless the elements of \( M \) are known explicitly, we cannot find \( M^* \) and \( M^{**} \); and thereby we cannot determine those \( n \) for which \( \mu_{Q_M}(n) \neq 0 \). Therefore we take some special sets for \( M \) and the corresponding \( Q_M \) below.

### 4.1 The set of \( r \)-free integers

Suppose \( A = \{r, r + 1, r + 2, \ldots\} \), where \( r \in \mathbb{N} \) and \( r > 1 \). Then \( n > 1 \) is in \( Q_A \) iff \( 1 \leq \alpha_i \leq r - 1 \) for \( i = 1, 2, \ldots, l \). In other words, an integer \( n > 1 \) is in \( Q_A \) iff \( p^r \) is not a divisor of \( n \) for any prime \( p \). Such numbers are called \( r \)-free integers in the literature. In fact, 2-free integers are well-known as square-free integers. Clearly \( n \) is square-free iff \( \mu^2(n) = 1 \). Thus \( Q_A \) is the set of all \( r \)-free integers.

For this set \( A \), we find \( A^* = \{\alpha \in \mathbb{N} : \alpha \in A \text{ and } \alpha - 1 \notin A\} = \{r\} \) and \( A^{**} = \{\alpha \in \mathbb{N} : \alpha \notin A \text{ and } \alpha - 1 \in A\} = \emptyset \), so that \( n^* = \prod_{\alpha_i \in A^*} p_i^{\alpha_i} = \alpha^r \), where \( \alpha = p_1 p_2 \cdots p_l \)
is square-free and \( n^{**} = 1 \). Therefore \( \mu_{Q_A}(n) \) is non-zero iff \( n = \alpha^r \), for some square-free \( \alpha \). Also, by (4.2), for such \( n \), \( \mu_{Q_A}(n) = (-1)^{\omega(\alpha)} = \mu(\alpha) \).

Hence by (3.12) and (3.14), we get
\[
\alpha_{Q_A} = \sum_{a=1}^{\infty} \frac{\mu(a)}{a^r} = \frac{1}{\zeta(r)},
\]
and
\[
\beta_{Q_A} = r \cdot \sum_{a=1}^{\infty} \frac{\mu(a) \log a}{a^r} = r \cdot \frac{\zeta'(r)}{\zeta^2(r)}.
\]
Also, by (3.15) and (3.16), we have
\[
\alpha_{Q_A}(x) = \sum_{a>x^{1/r}} \frac{\mu(a)}{a^r} = O \left( \frac{\delta(x^{1/r})}{x^{1-\frac{1}{r}}} \right)
\]
and
\[
\beta_{Q_A}(x) = r \cdot \sum_{a>x^{1/r}} \frac{\mu(a) \log a}{a^r} = O \left( \frac{\delta(x^{1/r}) \log x}{x^{1-\frac{1}{r}}} \right),
\]
so that
\[
x^2 \cdot |\beta_{Q_A}(x) - \alpha_{Q_A}(x)| = O \left( x^{1+\frac{1}{r}} \delta(x^{1/r}) \log x \right). \tag{4.5}
\]
Further \( \gamma_{Q_A}(x) = \sum_{a \leq x^{1/r}} \left| \frac{\mu(a)}{a^{(r+\varepsilon)}(\theta+\varepsilon)} \right| \) in which \( r(\theta + \varepsilon) \leq 3 \left( \frac{517}{1048} + \varepsilon \right) < 1 \) for sufficiently small \( \varepsilon > 0 \) in case \( r = 2 \) or 3; and that \( r(\theta + \varepsilon) > 1 \) if \( r \geq 4 \). Therefore
\[
\gamma_{Q_A}(x) = \begin{cases} 
O \left( x^{\frac{1}{2}-\theta-\varepsilon} \right), & \text{if } r = 2 \text{ or } 3 \\
O(1), & \text{if } r \geq 4.
\end{cases} \tag{4.6}
\]
Hence, by (4.5) and (4.6), we find

\[
\Delta_{Q_A}(x) = \begin{cases} 
O \left( x^{1+\frac{1}{r}} \delta(x^{1/r}) \log x \right) + O \left( x^{1+\frac{1}{r}} \right), & \text{if } r = 2 \text{ or } 3 \\
O \left( x^{1+\frac{1}{r}} \delta(x^{1/r}) \log x \right) + O \left( x^{1+\theta+\varepsilon} \right), & \text{if } r \geq 4
\end{cases}, \quad (4.7)
\]

In view of (4.4), the condition of Theorem 3.1 holds for \( S = Q_A \). Hence by (4.3), (4.4) and (4.7) we have a new asymptotic formula given below:

**Corollary 4.1.** For \( x \geq 1 \),

\[
\sum_{n \leq x} P_{Q_A}(n) = \frac{x^2}{2\zeta(2)\zeta(r)} \left( \log x + 2\gamma - \frac{1}{2} \frac{\zeta'(2)}{\zeta(2)} - r \frac{\zeta'(r)}{\zeta(r)} \right) + \Delta_{Q_A}(x),
\]

where \( \Delta_{Q_A}(x) \) is as in (4.7).

Note that

\[
P_{Q_A}(n) = \sum_{(j,n) \text{ is } r\text{-free}} (j,n).
\]

### 4.2 The set of semi-\( r \)-free integers

Suppose \( B = \{r\} \), where \( r \in \mathbb{N} \) and \( r > 1 \). Then \( n > 1 \) is in \( Q_B \) iff \( \alpha_i \neq r \) for \( i = 1, 2, \ldots, l \). In other words, \( n \in Q_B \) iff \( p^i \) is not a unitary divisor of \( n \) for any prime \( p \). (Recall that a divisor \( d \) of \( n \) is said to be *unitary* if \( (d, \frac{n}{d}) = 1 \).) Such \( n \) is called a *semi-\( r \)-free integer* in [12]. Thus \( Q_B \) is the set of all *semi-\( r \)-free integers*.

For this set \( B \), we note \( B^* = \{ \alpha \in \mathbb{N} : \alpha \in B \text{ and } \alpha - 1 \notin B \} = \{r\} \), while \( B^{**} = \{ \alpha \in \mathbb{N} : \alpha \notin B \text{ and } \alpha - 1 \in B \} = \{r+1\} \), so that \( n^* = \prod_{\alpha_i \in B^*} p_i^{\alpha_i} = a^r \) and \( n^{**} = \prod_{\alpha_i \in B^{**}} p_i^{\alpha_i} = b^{r+1} \), where \( a \) and \( b \) are both square-free. Thus \( \mu_{Q_B}(n) \neq 0 \) iff \( n = a^r b^{r+1} \),

where \( a \) and \( b \) are both square-free; and \( (a, b) = 1 \). For such \( n \), we have, by (4.2), that

\[
\mu_{Q_B}(n) = (-1)^{\omega(a)} \mu^2(b) = \mu(a) \mu^2(b).
\]

Hence

\[
\alpha_{Q_B} = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\mu(a) \mu^2(b)}{a^r b^{r+1}} = \left( \sum_{a=1}^{\infty} \frac{\mu(a)}{a^r} \right) \left( \sum_{b=1}^{\infty} \frac{\mu^2(b)}{b^{r+1}} \right) = \frac{1}{\zeta(r)} \frac{\zeta(r+1)}{\zeta(2r+2)}, \quad (4.8)
\]

by (3.12) and the fact that \( \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^t} = \frac{\zeta(t)}{\zeta(2t)} \), which can be proved by Euler product representation theorem ([11], Theorem 11.6).

Also using this fact, Theorem 11.12 of [1], (3.12) and (3.14), we get
\[ \beta_{Q_B} = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\mu(a)\mu^2(b)}{a^r b^{r+1}} \{ r \log a + (r+1) \log b \} \]

\[ = r \left( \sum_{a=1}^{\infty} \frac{\mu(a) \log a}{a^r} \right) \left( \sum_{b=1}^{\infty} \frac{\mu^2(b)}{b^{r+1}} \right) + (r + 1) \left( \sum_{a=1}^{\infty} \frac{\mu(a)}{a^r} \right) \left( \sum_{b=1}^{\infty} \frac{\mu^2(b) \log b}{b^{r+1}} \right) \]

\[ = \frac{\zeta'(r)}{\zeta(r)} \zeta(r+1) - (r + 1) \frac{1}{\zeta(r)} \cdot \frac{d}{dr} \left[ \frac{\zeta(r+1)}{\zeta(2r+2)} \right] \]

Further, by (3.15) and (3.16), we have

\[ \alpha_{Q_B}(x) = \sum_{a=r^{1/r+1}}^{\infty} \frac{\mu(a)\mu^2(b)}{a^r b^{r+1}} = \sum_{b=1}^{\infty} \frac{\mu^2(b)}{b^{r+1}} \left\{ \sum_{a=1}^{\infty} \frac{\mu(a)}{a^r (a^{1/r})^{1/r}} \right\} \]

\[ = O \left( \sum_{b=1}^{\infty} \frac{\mu^2(b)}{b^{r+1}} \right) \cdot \frac{\delta(x^{1/r}) b^{1+\frac{1}{r}}}{x^{1-\frac{1}{r}}} = O \left( \frac{\delta(x^{1/r})}{x^{1-\frac{1}{r}}} \cdot \sum_{b=1}^{\infty} \frac{\mu^2(b)}{b^{1+\frac{1}{r}}} \right) = O \left( \frac{\delta(x^{1/r})}{x^{1-\frac{1}{r}}} \right) \quad (4.10) \]

and

\[ \beta_{Q_B}(x) = \sum_{a=r^{1/r+1}}^{\infty} \frac{\mu(a)\mu^2(b)}{a^r b^{r+1}} \{ r \log a + (r+1) \log b \} \]

\[ = r \sum_{b=1}^{\infty} \frac{\mu^2(b)}{b^{r+1}} \left\{ \sum_{a=1}^{\infty} \frac{\mu(a) \log a}{a^r (a^{1/r})^{1/r}} \right\} + (r + 1) \sum_{b=1}^{\infty} \frac{\mu^2(b) \log b}{b^{r+1}} \left\{ \sum_{a=1}^{\infty} \frac{\mu(a)}{a^r (a^{1/r})^{1/r}} \right\} \]

\[ = O \left( \sum_{b=1}^{\infty} \frac{\mu^2(b)}{b^{r+1}} \right) \cdot \frac{\delta(x^{1/r}) \log x}{x^{1-\frac{1}{r}}} + O \left( \sum_{b=1}^{\infty} \frac{\mu^2(b) \log b}{b^{r+1}} \right) \cdot \frac{\delta(x^{1/r})}{(x/b^{r+1})^{1-\frac{1}{r}}} \]

\[ = O \left( \frac{\delta(x^{1/r}) \log x}{x^{1-\frac{1}{r}}} \right), \quad (4.11) \]

so that, by (4.10) and (4.11), we get

\[ x^2 |\beta_{Q_B}(x) - \alpha_{Q_B}(x)| = O \left( x^{1+\frac{1}{r}} \delta(x^{1/r}) \log x \right). \quad (4.12) \]

Also

\[ \gamma_{Q_B}(x) = \sum_{a \leq x^{1/r}} \frac{|\mu(a)|}{a^{\theta+\epsilon}} \left( \sum_{b \leq (\frac{x}{a})^{1/r+1}} 1 \right) \]

\[ = \begin{cases} O \left( \sum_{a \leq x^{1/r}} \frac{|\mu(a)|}{a^{\theta+\epsilon}} \cdot \left( \frac{x}{a^{1+\theta+\epsilon}} \right) \right), & \text{if } r = 2 \\ O \left( \sum_{a \leq x^{1/r}} \frac{|\mu(a)|}{a^{\theta+\epsilon}} \right), & \text{if } r = 3 \\ O(1), & \text{if } r \geq 4 \end{cases} \]

\[ = \begin{cases} O \left( x^{\frac{1}{r} - \theta - \epsilon} \right), & \text{if } r = 2 \text{ or } 3 \\ O(1), & \text{if } r \geq 4. \end{cases} \quad (4.13) \]
Now (4.12) and (4.13) give

\[ \Delta_{Q_B}(x) = \begin{cases} O\left(x^{1+\frac{1}{m}}\right), & \text{if } r = 2 \text{ or } 3 \\ O\left(x^{1+\theta+x}\right), & \text{if } r \geq 4. \end{cases} \tag{4.14} \]

Here the condition of Theorem 3.1 holds for \( S = Q_B \) in view of (4.9). Therefore using (4.8), (4.9) and (4.14) in Theorem 3.1 we get another asymptotic formula given below:

**Corollary 4.2.** For \( x \geq 1 \),

\[
\sum_{n \leq x} P_{Q_B}(n) = \frac{\zeta(r+1)}{2\zeta(2)\zeta(r)\zeta(2r+2)}x^2 \left( \log x + 2\gamma - \frac{1}{2} \frac{\zeta''(2)}{\zeta(2)} - F(r) \right) + \Delta_{Q_B}(x),
\]

where \( F(r) = r\frac{\zeta(r)}{\zeta(r)} - (r+1)\frac{\zeta'(r+1)}{\zeta(r+1)} + (2r+2)\frac{\zeta(2r+2)}{\zeta(2r+2)} \) and \( \Delta_{Q_B}(x) \) is as given in (4.14).

Note that

\[
P_{Q_B}(n) = \sum_{\substack{j=1 \\
 (j,n) \text{ is semi-} r \text{-free}}}^n (j,n).
\]

### 4.3 The set of \((k, r)\)-integers

Let \( r, k \in \mathbb{N} \) be such that \( 2 \leq r < k \). Suppose \( C = \{ \alpha \in \mathbb{N} : \alpha \geq r \text{ and } \alpha \equiv j \pmod{k} \} \) for some \( j \) with \( r \leq j \leq k-1 \).

Now \( n > 1 \) is in \( Q_C \) iff for each \( i \) \((1 \leq i \leq l)\) we have either \( \alpha_i < r \) or \( \alpha_i \equiv v_i \pmod{k} \) for some \( v_i \) with \( 0 \leq v_i \leq r-1 \) in which case we can write \( n \) as

\[
n = \prod_{i=1}^l p_i^{k_u_i+v_i} \cdot \prod_{\substack{i=1 \\
 \alpha_i < r}}^l p_i^{\alpha_i},
\]

where \( u_i \in \mathbb{N} \). Thus \( n \in Q_C \) iff \( n = a^k.b.c \), where \( a = \prod_{\alpha_i \geq r} p_i^{u_i} \), \( b = \prod_{\alpha_i \geq r} p_i^{v_i} \), and \( c = \prod_{\alpha_i < r} p_i^{\alpha_i} \).

Here \((ab,c) = 1\); and \( b, c \) are both \( r\)-free giving \( bc \) is \( r\)-free. Hence \( n \in Q_C \) iff \( n \) is of the form \( n = a^k \cdot m \), where \( a \in \mathbb{N} \) and \( m = bc \in Q_A \) (the set of \( r\)-free integers). Such numbers are called \((k, r)\)-integers in [11]; and the same numbers were considered by Cohen [7], under a different notation. Since \((\infty, r)\)-integers are \( r\)-free integers, the notion of a \((k, r)\)-integer may be regarded as a generalization of an \( r\)-free integer. Thus \( Q_C \) is the set of all \((k, r)\)-integers.

For this set \( C \), the set \( C^* = \{ \alpha \in \mathbb{N} : \alpha \in C \text{ and } \alpha - 1 \notin C \} = \{ \alpha \in \mathbb{N} : \alpha \equiv 0 \pmod{k} \} \) and \( C^{**} = \{ \alpha \in \mathbb{N} : \alpha \notin C \text{ and } \alpha - 1 \in C \} = \{ \alpha \in \mathbb{N} : \alpha \equiv 0 \pmod{k} \} \). Therefore, by (4.2), for \( n > 1 \), writing \( \alpha_i = k u_i + r \) if \( \alpha_i \in C^* \) and \( \alpha_i = k v_i \) if \( \alpha_i \in C^{**} \), we have \( n = a^k b^c \cdot k \), where \( a = \prod_{\alpha_i \in C^*} p_i^{u_i} \), \( b = \prod_{\alpha_i \in C^*} p_i^{v_i} \), and \( c = \prod_{\alpha_i \in C^{**}} p_i^{v_i} \). Also for such \( n \) the value of \( \mu_{Q_C}(n) \) is non-zero and is given by \( \mu_{Q_C}(n) = \mu(b) \), since \( b \) is square-free.

Hence

\[
\alpha_{Q_C} = \sum_{a=1}^\infty \sum_{b=1}^\infty \sum_{c=1}^\infty \frac{\mu(b)}{a^k b^c c^k} = \left( \sum_{a=1}^\infty \frac{1}{a^k} \right) \left( \sum_{b=1}^\infty \frac{\mu(b)}{b^r} \right) \left( \sum_{c=1}^\infty \frac{1}{c^k} \right) = \frac{\zeta^2(k)}{\zeta(r)} \tag{4.15}
\]
and

\[
\beta_{Qc} = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \mu(b) \left\{ k \log a + r \log b + k \log c \right\} \\
= k \left( \sum_{a=1}^{\infty} \log a \right) \left( \sum_{b=1}^{\infty} \frac{\mu(b)}{b^r} \right) \left( \sum_{c=1}^{\infty} \frac{1}{c^k} \right) + r \left( \sum_{a=1}^{\infty} \frac{1}{a^k} \right) \left( \sum_{b=1}^{\infty} \frac{\mu(b) \log b}{b^r} \right) \left( \sum_{c=1}^{\infty} \frac{1}{c^k} \right) \\
+ k \left( \sum_{a=1}^{\infty} \frac{1}{a^k} \right) \left( \sum_{b=1}^{\infty} \frac{\mu(b)}{b^r} \right) \left( \sum_{c=1}^{\infty} \log c \right) \\
= -k \frac{\zeta'(k) \zeta(k)}{\zeta(r)} + r \frac{\zeta^2(k) \zeta'(r)}{\zeta^2(r)} - k \frac{\zeta(k) \zeta'(k)}{\zeta(r)} \\
= \frac{\zeta(k)}{\zeta^2(r)} \left\{ r \zeta(k) \zeta'(r) - 2k \zeta(r) \zeta'(k) \right\},
\]

(4.16)

wherein we used (3.11), (3.12), (3.13) and (3.14).

Also, by (3.15)

\[
\alpha_{Qc}(x) = \sum_{u^k b^r > x} \frac{\mu(b)}{u^k b^r} = \sum_{u=1}^{\infty} \frac{1}{u^k} \left\{ \sum_{b=\left(\frac{x}{u^k}\right)^{1/r}}^{\infty} \frac{\mu(b)}{b^r} \right\} \\
= O \left( \sum_{u=1}^{\infty} \frac{1}{u^k} \frac{\delta(x^{1/r})}{(x/u^k)^{1-\frac{r}{k}}} \right) = O \left( \frac{\delta(x^{1/r})}{x^{1-\frac{r}{k}}} \cdot \sum_{u=1}^{\infty} \frac{1}{u^k} \right) \\
= O \left( \frac{\delta(x^{1/r})}{x^{1-\frac{r}{k}}} \right),
\]

(4.17)

because $2 \leq r < k$ implies that the series in the order term is convergent.

Again, using (3.15) and (3.16), we find that

\[
\beta_{Qc}(x) = \sum_{u^k b^r > x} \frac{\mu(b) \log(u^k b^r)}{u^k b^r} \\
= k \sum_{u^k b^r > x} \frac{\mu(b) \log u}{u^k b^r} + r \sum_{u^k b^r > x} \frac{\mu(b) \log b}{u^k b^r} \\
= k \left( \sum_{u=1}^{\infty} \log u \frac{1}{u^k} \left\{ \sum_{b=\left(\frac{x}{u^k}\right)^{1/r}}^{\infty} \frac{\mu(b) \log b}{b^r} \right\} \right) + r \sum_{u=1}^{\infty} \frac{1}{u^k} \left\{ \sum_{b=\left(\frac{x}{u^k}\right)^{1/r}}^{\infty} \frac{\mu(b)}{b^r} \right\} \\
= O \left( \sum_{u=1}^{\infty} \log u \frac{\delta(x^{1/r})}{(x/u^k)^{1-\frac{r}{k}}} \right) + O \left( \sum_{u=1}^{\infty} \frac{1}{u^k} \delta(x^{1/r}) \log x \right) \\
= O \left( \frac{\delta(x^{1/r})}{x^{1-\frac{r}{k}} \cdot \sum_{u=1}^{\infty} \frac{1}{u^k}} \right) + O \left( \frac{\delta(x^{1/r}) \log x}{x^{1-\frac{r}{k}} \cdot \sum_{u=1}^{\infty} \frac{1}{u^k}} \right) \\
= O \left( \frac{\delta(x^{1/r}) \log x}{x^{1-\frac{r}{k}}} \right),
\]

(4.18)

because $2 \leq r < k$ implies that both the series in the order terms are convergent.
Hence
\[ x^2 \cdot |\beta_{Q_C}(x) - \alpha_{Q_C}(x)| = O \left( x^{1 + \frac{1}{r} \delta (x^{1/r}) \log x} \right). \]  (4.19)

Further
\[
\gamma_{Q_C}(x) = \sum_{u^k b^r \leq x} \frac{\mu(b)}{u^{k(\theta + \varepsilon) b^{r(\theta + \varepsilon)}}} = \sum_{u \leq x^{1/k}} \frac{1}{u^{k(\theta + \varepsilon)}} \left( \sum_{b \leq \left( \frac{x}{u} \right)^{1/k}} \frac{|\mu(b)|}{b^{r(\theta + \varepsilon)}} \right)
\]
\[
= \begin{cases} 
O \left( x^{\frac{1}{r} - \theta - \varepsilon} \right), & \text{if } r = 2 \text{ or } 3 \\
O(1), & \text{if } r \geq 4. 
\end{cases} \tag{4.20}
\]

Now (4.19) and (4.20) give
\[
\Delta_{Q_C}(x) = \begin{cases} 
O \left( x^{1 + \frac{1}{r}} \right), & \text{if } r = 2 \text{ or } 3 \\
O(x^{1 + \theta + \varepsilon}), & \text{if } r \geq 4. 
\end{cases} \tag{4.21}
\]

In view of (4.16), the condition of Theorem 3.1 holds if \( S = Q_C \). Therefore using (4.15), (4.16) and (4.21) in Theorem 3.1, we get yet another asymptotic formula.

**Corollary 4.3.** For \( x \geq 1 \),
\[
\sum_{n \leq x} P_{Q_C}(n) = \frac{\zeta(k) x^2}{2 \zeta(2) \zeta(r)} \left\{ \zeta(k) \left( \log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) - \frac{1}{\zeta(r)} \left( r \zeta'(r) \zeta(k) - 2 k \zeta(r) \zeta'(k) \right) \right\} + \Delta_{Q_C}(x),
\]
where \( \Delta_{Q_C}(x) \) is as in (4.21).

Note that
\[ P_{Q_C}(n) = \sum_{(j, n) \text{ is a } (k, r)-\text{integer}} (j, n). \]

**Remark 4.4.** Any \( f \in \{ I_{\chi_{Q_A}}, I_{\chi_{Q_B}}, I_{\chi_{Q_C}} \} \) lies in the class of multiplicative functions discussed in the case 4 of Theorem 4 in [3], wherein asymptotic formula with error term \( O(x^2) \) is given for \( \sum_{n \leq x} P_f(n) \). That is, the asymptotic formulae established in Corollaries 4.1, 4.2 and 4.3 are deducible from case 4 of Theorem 4 in [3], but with error terms \( O(x^2) \) in each case. Observe that the error terms obtained in this paper are better than those in [3].

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