On a Notion of Nonlocal Curvature Tensor

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Dedicated to Roger L. Fosdick in recognition of his lifelong invaluable service to our scientific community

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Abstract

In the literature various notions of nonlocal curvature can be found. Here we propose a notion of nonlocal curvature tensor. This we do by generalizing an appropriate representation of the classical curvature tensor and by exploiting some analogies with certain fractional differential operators.

Keywords

Nonlocal curvatures · Nonlocal curvature tensor · Fractional perimeter functional · Fractional area functional

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1 Introduction

The objects of classical differential geometry are curves, surfaces, and curves on surfaces. Their study is local, in terms of certain first- and second-order differential characters, which for surfaces are the surface metric tensor and the curvature tensor, respectively. The latter tensor carries the information of interest for most applications, via its trace, the mean curvature, and its determinant, the Gaussian curvature. The classical notion of mean curvature offers a well known bridge between geometry and analysis, in that it is deducible from the stationarity condition of the area functional, a condition crucial to solve a central problem in modern calculus of variations, the minimal surface problem.
To date, various notions of nonlocal curvature have been proposed for surfaces. To our
knowledge, the only proposition for curves has been made in [15] (see [16] for an updated,
corrected version of this work). The common fundamental idea of the numerous papers
dealing with surfaces is to introduce a fractional notion of area and find what condition(s)
must hold point-wise on the surface to ensure that the fractional area has a minimum subject
to some boundary condition. In this paper we propose a notion of nonlocal curvature tensor.
Our present work is a follow-up of [14], a paper prompted by [13], inspired by our reading
of [1], and written in the wake of a quite abundant literature, part of which we briefly review
in the next section (see also [2, 3, 5, 6, 8, 10, 12]).

In Sect. 2 we discuss the notions of nonlocal mean and directional curvatures at a typical
point of a surface, both when the surface in question is the boundary of a bounded and com-
pact set and when it is not. In Sect. 3 we motivate and introduce a new notion of nonlocal
curvature tensor, which we illustrate by computing it for a $n$-dimensional sphere. In Sect. 4
we set forth some conjectures about representing such nonlocal curvature tensor in terms of
fractional differential operators. The mathematical machinery from fractional calculus we
employ is partly standard, quickly recapped in Sect. 4.1; the representation of a second frac-
tional gradient of a scalar-valued function exposed in Sect. 4.2 is new. We finish with a con-
jecture, namely that the nonlocal curvature tensor should be defined in terms of a—at the mo-
moment of this writing—nonexisting notion of fractional surface gradient of the normal field.

2 Notions of Nonlocal Curvature for a Surface

In this section we summarize some information about the nonlocal geometry of surfaces,
both when they bound a set and when they do not, and hence have a boundary.

The perimeter measure of a bounded set $E$ with nice boundary $\partial E$ is the same as the
area measure of $\partial E$. An identical, at bottom purely terminological, alternative occurs when
fractional counterparts of these notions are introduced. For each $\sigma$ between $0$ and $1$, $\sigma$-Per,
the fractional perimeter, also called $\sigma$-perimeter, of a measurable set $E \subseteq \mathbb{R}^n$ relative to a
bounded set $\Omega$, is delivered by the functional

$$
\sigma\text{-Per}(E, \Omega) := \mathcal{I}(E \cap \Omega, C E \cap \Omega) + \mathcal{I}(E \cap \Omega, C E \cap C \Omega) + \mathcal{I}(E \cap C \Omega, C E \cap \Omega),
$$

(1)

where the integral

$$
\mathcal{I}(A, B) := \frac{1}{\alpha_{n-1}} \int_A \int_B \frac{1}{|x - y|^{n+\sigma}} \, dx \, dy
$$

(2)
can be interpreted as a geometric distance interaction between the sets $A$ and $B$; here, $\alpha_{n-1}$
denotes the volume of the unit ball in $\mathbb{R}^{n-1}$. Caffarelli and Valdinoci [9] showed that if
$\partial E \cap B_R$ is $C^{1, \beta}$ for $\beta \in (0, 1)$ and $B_R$ a ball of radius $R$, then

$$
\lim_{\sigma \to 1^-} (1 - \sigma)\sigma\text{-Per}(E, B_r) = \text{Per}(E, B_r)
$$

(3)

for almost every $r \in (0, R)$.

A set $E$ is a minimizer of the $\sigma$-perimeter relative to $\Omega$ if

$$
\sigma\text{-Per}(E, \Omega) \leq \sigma\text{-Per}(F, \Omega)
$$

(4)
for all measurable sets $F$ such that $E \setminus \Omega = F \setminus \Omega$, meaning that $E$ and $F$ agree outside of $\Omega$. It was shown by Caffarelli, Roquejoffre, and Savin [7] that if $E$ is a minimizer, then it must satisfy
\begin{equation}
\int_{\mathbb{R}^n} \tilde{\chi}_E(y) \left\| \frac{z - y}{|z - y|^{n+\sigma}} \right\| dy = 0 \quad \text{for all } z \in \partial E, \tag{5}
\end{equation}
where
\begin{equation}
\tilde{\chi}_E := \chi_E - \chi_{\mathbb{R}^n} \tag{6}
\end{equation}
is a difference of characteristic functions.\(^1\) Consideration of the relationship in (3) between $\sigma$-perimeter and classical perimeter and of the well known fact that surfaces minimizing their perimeter have boundaries with zero mean curvature, motivates defining a fractional, or nonlocal, mean curvature at $z \in \partial E$ by
\begin{equation}
H_\sigma(z) := \frac{1}{\omega_{n-2}} \int_{\mathbb{R}^n} \tilde{\chi}_E(y) \left\| \frac{z - y}{|z - y|^{n+\sigma}} \right\| dy, \tag{7}
\end{equation}
where $\omega_{n-2}$ is the $(n - 2)$-dimensional measure of the unit sphere in $\mathbb{R}^{n-1}$. The factor used in this definition is suggested by a result of Abatangelo and Valdinoci’s [1]: if $E$ has a smooth boundary, then
\begin{equation}
\lim_{\sigma \to 1^-} (1 - \sigma) H_\sigma(z) = H(z), \tag{8}
\end{equation}
where $H(z)$ is the classical mean curvature at $z \in \partial E$.

The above discussion involves $(n - 1)$-dimensional surfaces that are the boundary of a set. However, these ideas can be extended to orientable surfaces that are not the boundary of a set.

Let $S$ be a compact $C^1$ surface with or without boundary in $\mathbb{R}^n$, and let a unit-vector valued function $n$ defined on $S$ define its orientation. For $0 < \sigma < 1$ and $\Omega$ a bounded open set, the $\sigma$-Area of $S$ relative to $\Omega$ was defined in [14] to be
\begin{equation}
\sigma\text{-Area}(S, \Omega) := \frac{1}{2\omega_{n-1}} \int_{\mathcal{X}(S)} \max\{|\chi_\Omega(x), \chi_\Omega(y)| \left\| \frac{x - y}{|x - y|^{n+\sigma}} \right\| dxdy, \tag{9}
\end{equation}
where $\mathcal{X}(S)$ is the set of all pairs $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that the line segment joining $x$ and $y$ crosses $S$ an odd number of times.

When $S = \partial E$, we have that
\[ \mathcal{X}(S) = (E \times CE) \cup (CE \times E) \]
up to a set of measure zero in dimension $2n$, see [14]. Moreover, we deduce from (9) that
\[ \sigma\text{-Area}(S, \Omega) = \mathcal{I}(E \cap \Omega, CE \cap \Omega) + \mathcal{I}(E \cap \Omega, CE \cap C\Omega) + \mathcal{I}(E \cap C\Omega, CE \cap \Omega) \]
and hence it follows from (1) that
\[ \sigma\text{-Area}(S, \Omega) = \sigma\text{-Per}(E, \Omega); \]
\(^1\)The integral in (5) must be understood in the principal value sense as the integrand has a singularity at $y = z$.\]
The solid line depicts $S$. The set $A_e(z)$ is shown in light grey, the set $A_i(z)$ in dark grey. The dashed lines depict the part of the boundary between $A_e(z)$ and $A_i(z)$ that is not part of $S$

therefore, the $\sigma$-Area generalizes the $\sigma$-Per.

The condition necessary and sufficient for the vanishing of the first variation of the $\sigma$-Area functional with respect to surfaces with the same boundary is a pointwise condition which was used in [14] to motivate the following notion of nonlocal mean curvature:

$$H_\sigma(z) := \frac{1}{\omega_{n-2}} \int_{\mathbb{R}^n} \frac{\tilde{\chi}_S(z, y)}{|z - y|^{n+\sigma}} dy \quad \text{for all } z \in \mathcal{S}, \quad (10)$$

where

$$\tilde{\chi}_S(z, y) := \begin{cases} 
1 & y \in A_i(z), \\
0 & y \notin A_i(z) \cup A_e(z), \\
-1 & y \in A_e(z),
\end{cases} \quad (11)$$

and

$$A_e(z) := \left\{ y \in \mathbb{R}^n \mid ((z, y) \in \mathcal{X}(S) \text{ and } (z - y) \cdot n(z) > 0) \right\} \text{ or } ((z, y) \in \mathcal{X}(S)^c \text{ and } (z - y) \cdot n(z) < 0) \right\},$$

$$A_i(z) := \left\{ y \in \mathbb{R}^n \mid ((z, y) \in \mathcal{X}(S)^c \text{ and } (z - y) \cdot n(z) > 0) \right\} \text{ or } ((z, y) \in \mathcal{X}(S) \text{ and } (z - y) \cdot n(z) < 0) \right\}.$$

As with the integral in (7), the integral in (10) must be understood in the principal value sense. Note that this definition does not require the surface $S$ to be compact, so it could be unbounded; for a depiction of the sets $A_e(z)$ and $A_i(z)$, see Fig. 1.

On making use of the identity

$$|z - y|^{-(n+\sigma)} = \frac{1}{\sigma} \text{div}_y \left[ |z - y|^{-(n+\sigma)} (z - y) \right] \quad (12)$$

(see [4]) and the divergence theorem, it is possible to write the nonlocal mean curvature as an integral over $S$:

$$H_\sigma(z) = \frac{2}{\sigma \omega_{n-2}} \int_S \frac{(z - y) \cdot n_{A_i(z)}(y)}{|z - y|^{n+\sigma}} dy. \quad (13)$$

In [1], Abatangelo and Valdinoci introduced a notion of nonlocal directional curvature for surfaces being the complete boundaries of sets. Their definition was extended to surfaces with boundary in [14], in the way described here below.

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For any point $z \in S$ and any unit vector $e$ tangent to $S$ at $z$, let
\[ \pi(z, e) := \{ y \in \mathbb{R}^n \mid y = z + \rho e + h n(z), \; \rho > 0, \; h \in \mathbb{R} \} \]
be the half-plane through $z$ defined by the unit vector $e$ and the normal $n(z)$ (see Fig. 2). Moreover, given any point $y \in \pi(z, e)$, let $y'$ denote the projection of this point onto the line through $z$ in the direction of $e$. The nonlocal directional curvature at $z$ in the direction of $e$,
\[ k_{\sigma, e}(z) := \int_{\pi(z, e)} \frac{|z - y'|^{n-2} \hat{\chi}_S(z, y)}{|z - y|^{n+\sigma}} \, dy, \]
satisfies a limit relation analogous to (8), namely
\[ \lim_{\sigma \to 1^-} (1 - \sigma) k_{\sigma, e}(z) = k_e(z). \]

The nonlocal mean and directional curvatures at $z$ are related through
\[ H_\sigma(z) = \frac{1}{\omega_{n-2}} \int_{\mathcal{U}(T_z S)} k_{\sigma, e}(z) \, de, \]
where $\mathcal{U}(T_z S)$ is the set of unit vectors in the tangent space $T_z S$ to $S$ at $z$. This means that the nonlocal mean curvature is the average of the nonlocal directional curvatures, just as the local mean curvature is the average of the local directional curvatures.

### 3 A Notion of Nonlocal Curvature Tensor

Our definition of a nonlocal curvature tensor will be motivated by a new representation, that we now derive, for its local counterpart.

The curvature tensor $L(z)$ at a point $z \in S$ is defined, see [11], as minus the surface gradient of the normal $n$ at $z$:
\[ L(z) = -^s \nabla n(z). \]

Given a unit vector $e \in T_z S$, the directional curvature in the direction of $e$ is
\[ k_e(z) := e \cdot L(z) e. \]

The following result shows that the curvature tensor is completely determined by the set of the directional curvatures.
Proposition 3.1 For all \( z \in S \),
\[
\mathbf{L}(z) = \frac{n-1}{2\omega_{n-2}} \int_{U(T_zS)} k_e(z) ((n+1)e \otimes e - \mathbf{1}_{T_zS}) \, de.
\] (19)

Proof Let \( B(T_zS) \) denote the unit ball in \( T_zS \), and let \( x = re \), with \( r \in [0, 1] \) and \( e \in U(T_zS) \), denote a typical point of \( B(T_zS) \). Using spherical coordinates we find that
\[
\int_{B(T_zS)} x \otimes x \, dx = \int_{U(T_zS)} \int_0^1 r^{n-2} e \otimes r e \, dr \, de = \frac{1}{n+1} \int_{U(T_zS)} e \otimes e \, de
\] (20)
\[
= \frac{1}{n+1} \int_{B(T_zS)} \nabla x \, dx = \frac{\alpha_{n-1}}{n+1} \mathbf{1}_{T_zS},
\] (21)

where the first identity of the second line is obtained by observing that \( U(T_zS) \) can be identified as the set of unit normals to \( \partial B(T_zS) \) and by applying the divergence theorem in the form
\[
\int_{B(T_zS)} \nabla x \, dx = \int_{\partial B(T_zS)} n \otimes n \, dn.
\]

Next, a use of (18) and (21) and an application of the divergence theorem give
\[
\int_{U(T_zS)} k_e(z) e \otimes e \, de = \int_{U(T_zS)} (e \otimes \mathbf{L}(z)e)e \otimes e \, de
\]
\[
= \int_{B(T_zS)} (x \otimes \nabla x \otimes (x \cdot \mathbf{L}(z)x)x) \, dx
\]
\[
= \int_{B(T_zS)} [(x \cdot \mathbf{L}(z)x)1_{T_zS} + 2x \otimes \mathbf{L}(z)x] \, dx
\]
\[
= \left( \int_{B(T_zS)} x \otimes x \, dx \cdot \mathbf{L}(z) \right) 1_{T_zS} + 2 \left( \int_{B(T_zS)} x \otimes x \, dx \right) \mathbf{L}(z)
\]
\[
= \frac{\alpha_{n-1}}{n+1} \left( \text{tr} \mathbf{L}(z) \right) 1_{T_zS} + 2 \mathbf{L}(z).
\] (22)

Solving for \( \mathbf{L}(z) \) shows that
\[
\mathbf{L}(z) = \frac{n+1}{2\alpha_{n-1}} \int_{U(T_zS)} k_e(z)e \otimes e \, de - \frac{1}{2} \left( \text{tr} \mathbf{L}(z) \right) \mathbf{1}_{T_zS}.
\] (23)

Taking the trace of (23), we find
\[
\text{tr} \mathbf{L}(z) = \frac{1}{\alpha_{n-1}} \int_{U(T_zS)} k_e(z) \, de;
\] (24)

the desired result follows by substituting (24) in (23) and using the fact that \( \omega_{n-2} = (n-1)\alpha_{n-1} \).
\[
\square
\]

The definition of the nonlocal curvature tensor we propose is modeled on (19), with
(i) the directional curvature \( k_e(z) \) replaced by the nonlocal analog \( k_{\sigma_e}(z) \) defined by (14);
(ii) the 2 appearing in the denominator replaced by $1 + \sigma$;
(iii) the $n + 1$ appearing in the integrand replaced by $n + \sigma$.

Precisely, we define the nonlocal curvature tensor as
\[
L_\sigma(z) := \frac{n - 1}{(1 + \sigma)\omega_{n-2}} \int_{U(T_zS)} k_{\sigma,e}(z)((n + \sigma)e \otimes e - 1_{T_zS}) \, de.
\] (25)

Notice that $L_\sigma(z)$ is a symmetric tensor which takes tangent vectors into tangent vectors, just like the classical curvature tensor does, moreover, it follows from (15) and (19) that
\[
\lim_{\sigma \to 1^-} (1 - \sigma)L_\sigma(z) = L(z).
\] (26)

By taking the trace of (25) and making use of (16) we arrive at
\[
H_\sigma(z) = \frac{1}{n - 1} \text{tr} L_\sigma(z),
\] (27)

that is, the nonlocal mean curvature is the mean of the terms on the diagonal appearing in any matrix representing, with respect to an orthonormal basis, the nonlocal curvature tensor. This identity would not hold as stated without the modification above indicated by (ii). The reason for the modification indicated by (iii) will be explained in Sect. 4.

Finally, we define the nonlocal Gaussian curvature at $z \in S$ as
\[
K_\sigma(z) := \det L_\sigma(z).
\] (28)

We conclude this section with two results related to the nonlocal curvature tensor we just introduced.

(i) $L_\sigma(z)$ could be written in terms of $\hat{\chi}(z, \cdot)$ rather than $k_{\sigma,e}(z)$. Indeed, on using (14) and a change of variables, $L_\sigma(z)$ can be expressed as
\[
L_\sigma(z) = \frac{n - 1}{(1 + \sigma)\omega_{n-2}} \int_{\mathbb{R}^n} \frac{\hat{\chi}_S(z, y)}{|z - y|^{n+\sigma}} ((n + \sigma)\hat{e}_z(y) \otimes \hat{e}_z(y) - 1_{T_zS}) \, dy,
\] (29)

where
\[
\hat{e}_z(y) := \frac{y' - z}{|y' - z|}.
\] (30)

Using the fact that
\[
(\nabla_y \hat{e}_z(y))(z - y) = 0 \quad \text{for all } y \in \mathbb{R}^n \setminus \{z\}
\] (31)

and arguments similar to those leading to (13), one can show that $L_\sigma(z)$ also has the representation
\[
L_\sigma(z) = \frac{2(n - 1)}{(1 + \sigma)\sigma\omega_{n-2}} \int_{S} \frac{(z - y) \cdot n_{A_k(z)}(y)}{|z - y|^{n+\sigma}} ((n + \sigma)\hat{e}_z(y) \otimes \hat{e}_z(y) - 1_{T_zS}) \, dy
\] (32)

involving an integral over the surface $S$.

(ii) Consider the case $n = 3$. Then, (25) gives
\[
L_\sigma(z) = \frac{1}{(1 + \sigma)\pi} \int_{U(T_zS)} k_{\sigma,e}(z)((3 + \sigma)e \otimes e - 1_{T_zS}) \, de.
\] (33)
For any unit vector $e \in T_z S$, let

$$ e^\bot := n(z) \times e; $$

note that $e \cdot e^\bot = -e^\bot \cdot e$, for any unit vectors $e$ and $\tilde{e}$. Furthermore, a calculation shows that, for any constant $c$ and any unit vector $e$,

$$ \text{cof}(c e \otimes e - 1_{T_z S}) = c e^\bot \otimes e^\bot - 1_{T_z S}, $$

where “cof” denotes the cofactor of a tensor, i.e., the transpose of its adjugate. Since

$$ (\det L_\sigma(z)) 1_{T_z S} = L_\sigma(z) (\text{cof} L_\sigma(z))^T, $$

we have that

$$ 2 \det L_\sigma = L_\sigma \cdot \text{cof} L_\sigma $$

$$ = L_\sigma \cdot \text{cof} \int_{U(T_z S)} k_{\sigma, e}(z)((3 + \sigma)e \otimes e - 1_{T_z S}) \, de \quad (k_{\sigma, e} := \frac{1}{(1 + \sigma)\pi} k_{\sigma, e}) $$

$$ = L_\sigma \cdot \int_{U(T_z S)} k_{\sigma, e}(z)((3 + \sigma)e^\bot \otimes e^\bot - 1_{T_z S}) \, de, $$

where the last equality follows from the linearity of the cofactor map in dimension 2. Next, by the use of (33), we find that

$$ 2 \det L_\sigma = \int_{U(T_z S)} k_{\sigma, \tilde{e}}((3 + \sigma)\tilde{e} \otimes \tilde{e} - 1_{T_z S}) \, d\tilde{e} \cdot \int_{U(T_z S)} k_{\sigma, e}((3 + \sigma)e^\bot \otimes e^\bot - 1_{T_z S}) \, de $$

$$ = \int_{U(T_z S)} \int_{U(T_z S)} \tilde{k}_{\sigma, e} k_{\sigma, e} ((3 + \sigma)^2 (\tilde{e} \cdot e^\bot)^2 - 2(3 + \sigma) + 2) \, d\tilde{e} \, de $$

$$ = \int_{U(T_z S)} \int_{U(T_z S)} \tilde{k}_{\sigma, e} k_{\sigma, e} ((3 + \sigma)^2 (\tilde{e} \cdot e^\bot)^2 - 2(2 + \sigma)) \, d\tilde{e} \, de. $$

Finally, for $\theta(e, \tilde{e})$ the angle between the vectors $e$ and $\tilde{e}$,

$$ (\tilde{e} \cdot e^\bot)^2 = ((n \times e) \cdot \tilde{e})^2 = ((e \times \tilde{e}) \cdot n)^2 = |e \times \tilde{e}|^2 = \sin^2 \theta(e, \tilde{e}). $$

In view of definition (28), we conclude that

$$ \kappa_{\sigma}(z) = \frac{1}{2(1 + \sigma)^2 \pi^2} \int_{U(T_z S)} \int_{U(T_z S)} k_{\sigma, e}(z) k_{\sigma, e}(z) ((3 + \sigma)^2 \sin^2 \theta(e, \tilde{e}) - 2(2 + \sigma)) \, d\tilde{e} \, de. $$

### 3.1 The Case of a Sphere

Our present goal is to compute the nonlocal curvature tensor of a sphere $S$ of radius $\rho$ in $\mathbb{R}^n$; without loss of generality we take $S$ centered at the origin, as depicted in Fig. 3, the version of Fig. 2 appropriate to the present context.
In view of the symmetry intrinsic to a sphere, we specialize definition (25)

\[
L_\sigma(z) = k_{\sigma,e}(z) \frac{n-1}{(1+\sigma)\omega_{n-2}} \int_{U(T_zS)} (n+\sigma)\tilde{e} \otimes \tilde{e} - 1_{T_zS}) \, d\tilde{e}
\]

(34)

\[
= k_{\sigma,e}(z)1_{T_zS},
\]

(35)

where the second equality follows from (20) and (21). We now concentrate on computing the nonlocal directional derivative according to definition (14), that is, on computing

\[
k_{\sigma,e}(z) = \lim_{\epsilon \to 0} \int_{\pi(z,e) \cap B_\epsilon(z) \cap A_i(z)} \frac{|y' - z|^{n-2} \tilde{\chi}_S(y,z)}{|y - z|^{n+\sigma}} \, dy.
\]

(36)

We begin by observing that

\[
\int_{\pi(z,e) \cap B_\epsilon(z) \cap A_i(z)} \frac{|y' - z|^{n-2} \tilde{\chi}_S(y,z)}{|y - z|^{n+\sigma}} \, dy
\]

\[
= \int_{(\pi(z,e) \cap B_\epsilon(z) \cap A_i(z)} \frac{|y' - z|^{n-2}}{|y - z|^{n+\sigma}} \, dy - \int_{(\pi(z,e) \cap B_\epsilon(z) \cap A_c(z)} \frac{|y' - z|^{n-2}}{|y - z|^{n+\sigma}} \, dy
\]

\[
= \int_{(\pi(z,e) \cap B_\epsilon(z) \cap A_i(z)} \frac{|y' - z|^{n-2}}{|y - z|^{n+\sigma}} \, dy - \int_{(\pi(z,e) \cap B_\epsilon(z) \cap A_c(z)} \frac{|y' - z|^{n-2}}{|y - z|^{n+\sigma}} \, dy
\]

(37)

\[
+ \int_{(\pi(z,e) \cap B_\epsilon(z) \cap A_c(z)} \frac{|y' - z|^{n-2}}{|y - z|^{n+\sigma}} \, dy
\]

\[
= -\int_{\pi(z,e) \cap B_\epsilon(z)} \frac{|y' - z|^{n-2}}{|y - z|^{n+\sigma}} \, dy + 2 \int_{(\pi(z,e) \cap A_i(z) \cap B_\epsilon(z))} \frac{|y' - z|^{n-2}}{|y - z|^{n+\sigma}} \, dy,
\]

where the first identity follows from (11), \(A_c(z)\) denotes the complement of \(A_i(z)\), and the last identity follows since the set \((\pi(z,e) \cap B_\epsilon(z)) \cap (A_c(z) \cap A_i(z))\) has measure zero. As to the first of the two integrals on the right side, on setting

\[
y - z = r a(\varphi), \quad a(\varphi) := \cos \varphi \, n(z) + \sin \varphi \, e, \quad (r, \varphi) \in (\epsilon, \infty) \times (0, \pi),
\]

(38)
we find that
\[ \int_{\pi(z,e) \setminus B_r(z)} \frac{|y' - z|^{n-2}}{|y - z|^{n+\sigma}} dy = \int_{\epsilon}^{\pi} \int_{0}^{\pi/2} r^{-1-\sigma} \sin^{n-2} \varphi d\varphi dr \]
\[ = \frac{2}{\sigma \epsilon^\sigma} \int_{0}^{\pi/2} \sin^{n-2}(2\phi) d\phi \]
\[ = \frac{2^{n-1}}{\sigma \epsilon^\sigma} \int_{0}^{\pi/2} \sin^{n-2}(\phi) \cos^{n-2}(\phi) d\phi \]
\[ = \frac{2^{n-2}}{\sigma \epsilon^\sigma} B\left(\frac{n-1}{2}, \frac{n+1}{2}\right), \tag{39} \]

where we have utilized (86), one of the many representations of the beta function.\(^2\) Using (83), (84), and (88), we find that (39) can be rewritten as
\[ \int_{\pi(z,e) \setminus B_r(z)} \frac{|y' - z|^{n-2}}{|y - z|^{n+\sigma}} dy = \frac{2^{n-2} \Gamma\left(\frac{n+1}{2}\right)^2}{\sigma \epsilon^\sigma \Gamma(n-1)} = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\sigma \epsilon^\sigma \Gamma\left(\frac{n+1}{2}\right)} = \frac{1}{\sigma \epsilon^\sigma} B\left(\frac{n-1}{2}, \frac{1}{2}\right). \tag{40} \]

The calculation of the second integral on the right-hand side of (37) is a bit more complicated. Note that, for \(r \in (\epsilon, 2\rho)\), the spheres \(B_r(z)\) and \(S\) intersect in the plane \(\pi(z,e)\) at the point
\[ p(r) = z + r \sqrt{1 - \left(\frac{r}{2\rho}\right)^2} e - \frac{r^2}{2\rho} n(z); \tag{41} \]
the angle between \((p(r) - z)\) and \(n(z)\) is
\[ \varphi(r) = \pi - \arctan \frac{\sqrt{1 - \left(\frac{r}{2\rho}\right)^2}}{\frac{r}{2\rho}}; \tag{42} \]
in particular,
\[ \varphi(\epsilon) = \pi - \arctan \frac{\sqrt{1 - \left(\frac{\epsilon}{2\rho}\right)^2}}{\frac{\epsilon}{2\rho}} \quad \text{and} \quad \lim_{\epsilon \to 0} \varphi(\epsilon) = \pi/2. \tag{43} \]

With this notation the second integral on the right-hand side of (37) can be written as
\[ \int_{\pi(z,e) \cap (A_\epsilon(z), B_r(z))} \frac{|y' - z|^{2-n}}{|y - z|^{n+\sigma}} dy = \int_{\varphi(\epsilon)}^{\pi} \int_{\epsilon}^{\pi/2} r^{-1-\sigma} \sin^{n-2} \phi d\phi dr d\phi \]
\[ = \frac{1}{\sigma} \int_{\varphi(\epsilon)}^{\pi} (\epsilon^{-\sigma} \sin^{n-2} \phi - (-2\rho \cos \phi)^{-\sigma} \sin^{n-2} \phi) d\phi. \tag{44} \]

Now, utilizing the change of variables \(\phi \mapsto \phi - \pi/2\) and (86) we find that

\(^2\)For the reader’s convenience, we have collected in the Appendix the definitions of the gamma and beta functions, as well as the properties of those functions we here use.
\[
\int_{\pi(z, e) \cap (A_i(z) \setminus B_\varepsilon(z))} |y' - z|^{2-n} dy = \frac{1}{\sigma} \int_{\phi(e) - \pi/2}^{\pi/2} (\varepsilon^{-\sigma} \cos^{n-2} \phi - (2\rho \sin \phi)^{-\sigma} \cos^{n-2} \phi) d\phi
\]
\[
= B\left(\frac{1}{2}, \frac{n-1}{2}\right) + \frac{1}{\sigma e^\sigma} \int_{\phi(e) - \pi/2}^{\pi/2} \cos^{n-2} \phi d\phi
\]
\[
- \frac{1}{\sigma} \int_{\phi(e) - \pi/2}^{\pi/2} (2\rho \sin \phi)^{-\sigma} \cos^{n-2} \phi d\phi.
\]

Substituting (40) and (45) into (37) and using (86) again results in

\[
\begin{align*}
\lim_{\varepsilon \to 0} \left( \frac{2}{\sigma e^\sigma} \int_{\phi(e) - \pi/2}^{\pi/2} \cos^{n-2} \varphi d\varphi - \frac{2}{\sigma} \int_{\phi(e) - \pi/2}^{\pi/2} (2\rho \sin \phi)^{-\sigma} \cos^{n-2} \phi d\phi \right).
\end{align*}
\]

Using de L'Hôpital's rule, one finds that

\[
\lim_{\varepsilon \to 0} \int_{\phi(e) - \pi/2}^{\pi/2} \cos^{n-2} \varphi d\varphi = \lim_{\varepsilon \to 0} -\varepsilon^{1-\sigma} \sin^{n-2} \varphi(\varepsilon) \varphi'(\varepsilon) = 0.
\]

Moreover, using (43)_2 and (86) we have

\[
\lim_{\varepsilon \to 0} \frac{2}{\sigma} \int_{\phi(e) - \pi/2}^{\pi/2} (2\rho \sin \phi)^{-\sigma} \cos^{n-2} \phi d\phi = \frac{B\left(\frac{1}{2}, \frac{n-1}{2}\right)}{\sigma (2\rho)^\sigma}.
\]

Thus, the nonlocal directional curvature vector for a sphere is

\[
k_{s, e}(z) = -B\left(\frac{1}{2}, \frac{n-1}{2}\right) \sigma (2\rho)^\sigma.
\]

In conclusion, in view of (35), the nonlocal curvature tensor for a sphere is

\[
L_{\sigma}(z) = -B\left(\frac{1}{2}, \frac{n-1}{2}\right) \sigma (2\rho)^\sigma 1_{T_z S}.
\]

### 4 Possible Connections with Fractional Operators

Here we outline a possible connection between the notion of nonlocal curvature tensor proposed in definition (25) and certain fractional differential operators (see Sect. 4.1). This connection is motivated by the following lines of thinking.

In the case where the surface under attention is the boundary of a set \(E\), the classical curvature tensor is the opposite of the surface gradient of the normal (recall (17)); moreover, the normal can be obtained by looking at the jump part of the distributional derivative of the characteristic function of \(E\). Thus, the curvature tensor is related to a second derivative of \(\tilde{\chi}_E\) or, equivalently, to the function \(\hat{\chi}_S\) defined in (6). When the surface is not the boundary of a set, the appropriate generalization of \(\tilde{\chi}_E\) is the function \(\tilde{\chi}_S\) defined in (11). If we compare (10)-(11) with (56), we are driven to relate the nonlocal mean curvature to the fractional Laplacian:

\[
H_{\sigma}(z) = \frac{1}{V_\sigma \omega_{n-2}} (-\Delta)^{\sigma/2} \tilde{\chi}_S(z, \cdot).
\]
On using the identities (58) and (59) in Sect. 4.1, it follows from this relation that

\[ H_\sigma(z) = \frac{1}{\nu_\sigma \omega_{n-2}} \text{tr}(\nabla^{\sigma/2} \nabla^{\sigma/2} \hat{\mathcal{K}}_S(z, \cdot)). \] (52)

But this is not the only reason why we argue that perhaps a second fractional gradient of \( \hat{\chi}_S \) may be related to the nonlocal curvature tensor we propose. In fact, utilizing (60), a result proved in Sect. 4.2, we see that

\[ \nabla^{\sigma/2} \nabla^{\sigma/2} \hat{\mathcal{K}}_S(z, \cdot) = -\frac{\nu_\sigma}{\sigma} \int_{\mathbb{R}^n} \hat{\mathcal{K}}_S(z, z + v) \left( \frac{n + \sigma}{|v|^2} v \otimes v - 1_{\mathbb{R}^n} \right) dv. \] (53)

Let us compare the right side of (53) and (29), a relation we repeat here for the reader’s convenience,

\[ L_\sigma(z) = \frac{n - 1}{(1 + \sigma)\omega_{n-2}} \int_{\mathbb{R}^n} \hat{\mathcal{K}}_S(z, y) ((n + \sigma)\hat{e}_z(y) \otimes \hat{e}_z(y) - 1_{T_z S}) dy. \]

The similarity is striking.\(^3\) However, unfortunately, although the trace of (53) does give the nonlocal mean curvature up to a multiplicative constant, it does not converge to the classical curvature tensor in the limit as \( \sigma \) goes to 1. This negative outcome leaves us with a final, admittedly vague conjecture. While the classical curvature tensor is defined using a surface gradient, to our knowledge no fractional analog of surface gradient has been developed yet. Perhaps an appropriate notion of fractional surface gradient might establish that there is a direct connection with the nonlocal curvature tensor we have proposed.

### 4.1 Fractional Gradient, Laplacian, and Divergence

For each \( \alpha \in (0, 1) \), let

\[ \mu_\alpha := \frac{2^\alpha \Gamma \left( \frac{n+\alpha+1}{2} \right)}{\pi^{n/2} \Gamma \left( \frac{1-\alpha}{2} \right)} \quad \text{and} \quad \nu_\alpha := \frac{2^\alpha \Gamma \left( \frac{n+\alpha}{2} \right)}{\pi^{n/2} \Gamma \left( \frac{-\alpha}{2} \right)}. \] (54)

The fractional gradient \( \nabla^\alpha \) and the Laplacian \( (-\Delta)^{\alpha/2} \) of a function \( f \) are

\[ \nabla^\alpha f(x) := \mu_\alpha \int_{\mathbb{R}^n} \frac{f(y) \otimes (y - x)}{|x - y|^{n+\alpha+1}} dy, \] (55)

\[ (-\Delta)^{\alpha/2} f(x) := \nu_\alpha \int_{\mathbb{R}^n} \frac{f(y) - f(x)}{|x - y|^{n+\alpha}} dy. \] (56)

Notice that these definitions make sense if \( f \) is scalar-, vector-, or tensor-valued (in the first instance, \( f(y) \otimes (y - x) \) must be understood as \( f(y)(y - x) \)).

The fractional divergence of a vector-valued function \( w \) is

\[ \text{div}^\alpha w(x) := \mu_\alpha \int_{\mathbb{R}^n} \frac{(w(y) - w(x)) \cdot (y - x)}{|x - y|^{n+\alpha+1}} dy. \] (57)

\(^3\) It was the \( n + \sigma \) factor appearing in front of \( v \otimes v \) in the integrand in (53) which prompted us to introduce in (29) the modification (iii) stated in Sect. 3.
One can readily see that
\[ \text{tr}(\nabla^\alpha w) = \text{div}^\alpha w; \] (58)
verifying the identity
\[ \text{div}^\alpha (\nabla^\beta f) = -(-\Delta)^{(\alpha+\beta)/2} f \] (59)
is harder (for a proof, see [17]).

4.2 A Characterization of the Fractional Hessian

Here we characterize the second fractional gradient of a scalar-valued function. The main result is the following theorem.

Theorem 4.1 Let \( \alpha, \beta \in (0, 1) \). If \( f \) is a scalar-valued function, then
\[ \nabla^\alpha (\nabla^\beta f)(x) = \frac{-\nu_{\alpha+\beta}}{(\alpha+\beta)} \int_{\mathbb{R}^n} \frac{f(x+v) - f(x)}{|v|^{n+\alpha+\beta}} \left( \frac{n + \alpha + \beta}{|v|^2} v \otimes v - I_n \right) dv. \] (60)

To prove it, we make use of the Fourier transform, that for a function \( f \) is defined by
\[ \mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx. \] (61)
The following identities are classical, see for instance [18],
\[ f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi, \] (62)
\[ \mathcal{F}(\partial^\gamma f)(\xi) = (2\pi i \xi)^\gamma \hat{f}(\xi), \] (63)
\[ \partial^\gamma \hat{f}(\xi) = \mathcal{F}((-2\pi i)^\gamma f(x))(\xi), \] (64)
where \( \gamma \) is a multi-index.

The Gauss–Weierstrass kernel defined as
\[ g_t(u) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|u|^2}{4t}}, \] (65)
satisfies the following identities:
\[ g_t(u) = \mathcal{F}(e^{-4\pi^2 t|x|^2})(u), \] (66)
and
\[ \frac{1}{|u|^{n+\alpha+1}} = \frac{\pi^{n/2}}{2^{\alpha+1} \Gamma\left(\frac{n+\alpha+1}{2}\right)} \int_0^\infty g_t(u)t^{-(\alpha+3)/2} dt \]
\[ = \frac{1}{2\mu_{\alpha} \Gamma\left(\frac{1-\alpha}{2}\right)} \int_0^\infty g_t(u)t^{-(\alpha+3)/2} dt, \] (67)
where \( \Gamma \) denotes the gamma function defined in (81).

The proof of Theorem 4.1 will be achieved through two Lemmas.
Lemma 4.2 Let γ be a multi-index and

\[ G^\gamma_{s,t}(v) = \int_{\mathbb{R}^n} u^\gamma g_t(v-u) g_s(u) du. \]

Then, if \(|\gamma| = 1\) or \(|\gamma| = 2\) and \(\max_k \gamma_k = 1\)

\[ G^\gamma_{s,t}(v) = \left( \frac{iv}{s+t} \right)^\gamma g_{s+t}(v), \]

while if \(|\gamma| = 2\) and \(\max_k \gamma_k = 2\)

\[ G^\gamma_{s,t}(v) = \left( \frac{iv}{s+t} + \frac{2st}{s+t} \right) g_{s+t}(v). \]

**Proof** By using (61) and (66) we have

\[
G^\gamma_{s,t}(v) = \int_{\mathbb{R}^n} u^\gamma g_t(u) \int_{\mathbb{R}^n} e^{-2\pi i x \cdot (v-u)} e^{-4\pi^2 s|x|^2} dx du
\]

\[
= \frac{1}{(2\pi i)^\gamma} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot v} e^{-4\pi^2 s|x|^2} \int_{\mathbb{R}^n} e^{2\pi i x \cdot u} (2\pi i u)^\gamma g_t(u) du dx
\]

and thanks to (63) and (66) we find

\[
G^\gamma_{s,t}(v) = \frac{1}{(2\pi i)^\gamma} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot v} e^{-4\pi^2 s|x|^2} \int_{\mathbb{R}^n} e^{2\pi i x \cdot u} F(\partial^\gamma e^{-4\pi^2 t|x|^2})(u) du dx
\]

\[
= \frac{1}{(2\pi i)^\gamma} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot v} e^{-4\pi^2 s|x|^2} \partial^\gamma e^{-4\pi^2 t|x|^2} dx,
\]

where the last equality follows from (62). If \(|\gamma| = 1\),

\[
\partial^\gamma e^{-4\pi^2 t|x|^2} = -8\pi^2 t x^\gamma e^{-4\pi^2 t|x|^2},
\]

if \(|\gamma| = 2\) and \(\max_k \gamma_k = 1\), i.e., \(\gamma = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)\), then

\[
\partial^\gamma e^{-4\pi^2 t|x|^2} = (-8\pi^2 t x) e^{-4\pi^2 t|x|^2},
\]

while if \(|\gamma| = 2\) and \(\max_k \gamma_k = 2\), i.e., \(\gamma = (0, \ldots, 0, 2, 0, \ldots, 0)\), then

\[
\partial^\gamma e^{-4\pi^2 t|x|^2} = (-8\pi^2 t x)^2 e^{-4\pi^2 t|x|^2} - 8\pi^2 t e^{-4\pi^2 t|x|^2}.
\]

For \(|\gamma| = 1\) or \(|\gamma| = 2\) and \(\max_k \gamma_k = 1\) we find

\[
G^\gamma_{s,t}(v) = \frac{1}{(2\pi i)^\gamma} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot v} e^{-4\pi^2 (s+t)|x|^2} (-8\pi^2 t x)^\gamma dx
\]

\[
= \frac{1}{(2\pi i)^\gamma} \left( \frac{t}{s+t} \right)^\gamma \int_{\mathbb{R}^n} e^{-2\pi i x \cdot v} e^{-4\pi^2 (s+t)|x|^2} (-8\pi^2 (s+t) x)^\gamma dx
\]

\[
= \frac{1}{(2\pi i)^\gamma} \left( \frac{t}{s+t} \right)^\gamma \int_{\mathbb{R}^n} e^{-2\pi i x \cdot v} \partial^\gamma e^{-4\pi^2 (s+t)|x|^2} dx,
\]
where we used (68) or (69), according to \( \gamma \). Hence, from (61), (63), and (66) it follows that

\[
G_{s,t}^\gamma(v) = \frac{1}{(2\pi i)^r} \left( \frac{t}{s + t} \right)^r \mathcal{F}(\partial^\gamma e^{-4\pi^2(s + t)|x|^2})(v)
\]

\[
= \frac{1}{(2\pi i)^r} \left( \frac{t}{s + t} \right)^r (2\pi iv)^r \mathcal{F}(e^{-4\pi^2(s + t)|x|^2})(v)
\]

\[
= \left( \frac{iv}{s + t} \right)^r g_{s+t}(v). \tag{72}
\]

Instead, if \( |\gamma| = 2 \) and \( \max_k \gamma_k = 2 \) we have

\[
G_{s,t}^\gamma(v) = \frac{1}{(2\pi i)^r} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot v} e^{-4\pi^2(s + t)|x|^2} ((-8\pi^2tx)^\gamma - 8\pi^2t) \, dx.
\]

By the identity

\[
(-8\pi^2tx)^\gamma - 8\pi^2t = \left( \frac{t}{s + t} \right)^2 (\alpha_1 - \nu) - \frac{8\pi^2st}{s + t}
\]

we find

\[
G_{s,t}^\gamma(v) = \frac{1}{(2\pi i)^r} \left( \frac{t}{s + t} \right)^2 \int_{\mathbb{R}^n} e^{-2\pi i x \cdot v} \partial^\gamma e^{-4\pi^2(s + t)|x|^2} \, dx
\]

\[
- \frac{8\pi^2st}{s + t} \frac{1}{(2\pi i)^r} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot v} e^{-4\pi^2(s + t)|x|^2} \, dx.
\]

The first line of this equation is equal to (71) and hence to (72), while the second may be simplified by means of (66). It is found that

\[
G_{s,t}^\gamma(v) = \left( \frac{iv}{s + t} \right)^r g_{s+t}(v) + \frac{2st}{s + t} g_{s+t}(v). \tag{73}
\]

**Lemma 4.3** We have

\[
\int_{\mathbb{R}^n} \frac{(v - u) \otimes u}{|u - v|^{n+\beta+1}|u|^{\gamma+\alpha+1}} \, du = \frac{-v_{\alpha+\beta}}{\mu_\alpha \mu_\beta (\alpha + \beta)} |v|^{\gamma+\alpha+\beta} \left( n + \alpha + \beta \right) \left( \frac{v \otimes v - \mathbf{1}_{\mathbb{R}^n}}{|v|^2} \right). \tag{73}
\]

**Proof** With (67) we find

\[
4\mu_\alpha \mu_\beta \Gamma \left( \frac{1-\alpha}{2} \right) \Gamma \left( \frac{1-\beta}{2} \right) \int_{\mathbb{R}^n} \frac{(v - u) \otimes u}{|u - v|^{n+\beta+1}|u|^{\gamma+\alpha+1}} \, du
\]

\[
= \int_{\mathbb{R}^n} (v - u) \otimes u \int_0^\infty g_s(v - u) s^{-(\beta+3)/2} \, ds \int_0^\infty g_t(u) t^{-(\alpha+3)/2} \, dt \, du
\]

\[
= \int_0^\infty \int_0^\infty s^{-(\beta+3)/2} t^{-(\alpha+3)/2} \int_{\mathbb{R}^n} (v - u) \otimes u g_s(v - u) g_t(u) \, du \, ds \, dt
\]

\[
= \int_0^\infty \int_0^\infty s^{-(\beta+3)/2} t^{-(\alpha+3)/2} \mathcal{H}(v, s, t) \, ds \, dt, \tag{74}
\]
where we have set
\[ \mathbf{H}(v, s, t) = \int_{\mathbb{R}^n} (v - u) \otimes u \, g_s(v - u) g_t(u) \, du. \]

The component \( pq \), with \( p \neq q \), is
\[ H_{pq}(v, s, t) = \int_{\mathbb{R}^n} (v_p - u_p) u_q \, g_s(v - u) g_t(u) \, du = \int_{\mathbb{R}^n} (v_p \, \bar{u}^\gamma - \bar{u}^\gamma) \, g_s(v - u) g_t(u) \, du, \]
where
- \( \bar{u}^\gamma = (0, \ldots, 0, 1, 0, \ldots, 0) \) with the one in the \( q \) position
- \( \gamma = (0, \ldots, 0, 1, 0, \ldots, 0) \) with the ones in the \( p \) and \( q \) positions.

By Lemma 4.2, for \( p \neq q \) we have
\[ H_{pq}(v, s, t) = v_p G_{s,t}^{\bar{u}^\gamma}(v) - G_{s,t}^{\bar{u}^\gamma}(v) = v_p \left( \frac{tv}{s + t} \right)^\gamma g_{s+t}(v) - \left( \frac{tv}{s + t} \right)^\gamma g_{s+t}(v) \]
\[ = \frac{t}{s + t} v_p G_{s,t}^{\bar{u}^\gamma}(v) - \left( \frac{t}{s + t} \right)^2 v_p v_q g_{s+t}(v) = \frac{st}{(s + t)^2} v_p v_q g_{s+t}(v), \quad (75) \]
while the \( qq \) component is
\[ H_{qq}(v, s, t) = \int_{\mathbb{R}^n} (v_q u_p - u_q^2) \, g_s(v - u) g_t(u) \, du = \int_{\mathbb{R}^n} (v_p u^\gamma - u^\gamma) \, g_s(v - u) g_t(u) \, du, \]
where
- \( u^\gamma = (0, \ldots, 0, 2, 0, \ldots, 0) \) with the two in the \( q \) position.

Again, by Lemma 4.2, we have
\[ H_{qq}(v, s, t) = v_q G_{s,t}^{u^\gamma}(v) - G_{s,t}^{u^\gamma}(v) = v_q \left( \frac{tv}{s + t} \right)^\gamma g_{s+t}(v) - \left( \frac{tv}{s + t} \right)^\gamma g_{s+t}(v) \]
\[ = \frac{t}{s + t} v_q G_{s,t}^{u^\gamma}(v) - \left( \frac{t}{s + t} \right)^2 v_q v_q g_{s+t}(v) - \frac{2st}{s + t} g_{s+t}(v) \]
\[ = \frac{st}{(s + t)^2} v_q v_q g_{s+t}(v) - \frac{2st}{s + t} g_{s+t}(v). \quad (76) \]

By (75) and (76) we deduce that
\[ \mathbf{H}(v, s, t) = \left( \frac{st}{(s + t)^2} v \otimes v - \frac{2st}{s + t} 1_{\mathbb{R}^n} \right) g_{s+t}(v). \]

We therefore find
\[
\begin{align*}
&\int_0^\infty \int_0^\infty s^{-(\beta+3)/2} t^{-(\alpha+3)/2} \mathbf{H}(v, s, t) \, ds \, dt \\
&= \int_0^\infty \int_0^\infty s^{-(\beta+1)/2} t^{-(\alpha+1)/2} \left( \frac{1}{s + t} v \otimes v - 2 \mathbf{1}_{\mathbb{R}^n} \right) g_{s+t}(v) \, ds \, dt.
\end{align*}
\]
Setting \( p = s + t \) and \( r = t/(s + t) \) we have that \( s = p(1 - r) \) and \( t = pr \) and the Jacobian of the transformation is \( 1/(s + t) \). Thus

\[
\int_0^\infty \int_0^\infty s^{-(\beta+3)/2} t^{-(\alpha+3)/2} H(v, s, t) ds dt
\]

\[
= \int_0^\infty \int_0^1 (p(1 - r))^{-(\beta+1)/2} (pr)^{-(\alpha+1)/2} \left( \frac{1}{p} v \otimes v - 21_{\mathbb{R^n}} \right) g_p(v) dr dp
\]

\[
= \int_0^\infty p^{-(\alpha+\beta+2)/2} \left( \frac{1}{p} v \otimes v - 21_{\mathbb{R^n}} \right) g_p(v) dp \int_0^1 (1 - r)^{-(\beta+1)/2} r^{-(\alpha+1)/2} dr
\]

\[
= B\left( \frac{1-a}{2}, \frac{1-b}{2} \right) \int_0^\infty p^{-(\alpha+\beta+2)/2} \left( \frac{1}{p} v \otimes v - 21_{\mathbb{R^n}} \right) g_p(v) dp,
\]

where to obtain the last identity we used (87). By using (67), we find that

\[
\int_0^\infty \int_0^\infty s^{-(\beta+3)/2} t^{-(\alpha+3)/2} H(v, s, t) ds dt
\]

\[
= B\left( \frac{1-a}{2}, \frac{1-b}{2} \right) \left( \frac{2^{\alpha+\beta+2} \Gamma\left( \frac{n+\alpha+\beta+2}{2} \right)}{\pi^{n/2} |\mathbf{v}|^{n+\alpha+\beta+2}} \mathbf{v} \otimes \mathbf{v} - 2 \frac{2^{\alpha+\beta} \Gamma\left( \frac{n+\alpha+\beta}{2} \right)}{\pi^{n/2} |\mathbf{v}|^{n+\alpha+\beta}} 1_{\mathbb{R^n}} \right)
\]

\[
= B\left( \frac{1-a}{2}, \frac{1-b}{2} \right) 2^{\alpha+\beta+1} \Gamma\left( \frac{n+\alpha+\beta}{2} \right) \left( \frac{n + \alpha + \beta}{|\mathbf{v}|^2} \mathbf{v} \otimes \mathbf{v} - 1_{\mathbb{R^n}} \right),
\]

where we used (82). From this identity, (88), and (74) we deduce that

\[
\int_{\mathbb{R^n}} \frac{(v - u) \otimes u}{|u - v|^{\alpha+\beta+1}|u|^{n+\alpha+1}} du
\]

\[
= \frac{1}{4 \mu_{\alpha} \mu_{\beta} \Gamma(1 - \frac{\alpha+\beta}{2})} 2^{\alpha+\beta+1} \Gamma\left( \frac{n+\alpha+\beta}{2} \right) \left( \frac{n + \alpha + \beta}{|\mathbf{v}|^2} \mathbf{v} \otimes \mathbf{v} - 1_{\mathbb{R^n}} \right)
\]

\[
= \frac{-1}{\mu_{\alpha} \mu_{\beta} |\mathbf{v}|^{n+\alpha+\beta} (\alpha + \beta)} 2^{\alpha+\beta} \Gamma\left( \frac{n+\alpha+\beta}{2} \right) \left( \frac{n + \alpha + \beta}{|\mathbf{v}|^2} \mathbf{v} \otimes \mathbf{v} - 1_{\mathbb{R^n}} \right)
\]

from which the statement of the Lemma follows. \( \square \)

**Proof of Theorem 4.1** By applying (55) twice, we deduce that

\[
\nabla^\alpha (\nabla^\beta f)(x) = \mu_{\alpha} \mu_{\beta} \int_{\mathbb{R^n}} \int_{\mathbb{R^n}} \frac{f(z)(z - y) \otimes (y - x)}{|y - z|^{n+\beta+1}|x - y|^{n+\alpha+1}}dz dy,
\]

and making the change of variables \( u = y - x \) and \( v = z - x \), we obtain

\[
\nabla^\alpha (\nabla^\beta f)(x) = \mu_{\alpha} \mu_{\beta} \int_{\mathbb{R^n}} f(x + v) \left[ \int_{\mathbb{R^n}} \frac{(v - u) \otimes u}{|u - v|^{n+\beta+1}|u|^{n+\alpha+1}} du \right] dv.
\]

The proof of Theorem 4.1 readily follows from (80) and (73). \( \square \)
Appendix. Some Properties of the Gamma and Beta Functions

The gamma function $\Gamma$ is defined for all positive numbers $x$ by

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy;$$

(81)

it can be viewed as a generalization of the factorial function in that it satisfies

$$\Gamma(x + 1) = x \Gamma(x).$$

(82)

While $\Gamma(n) = (n - 1)!$ for all $n \in \mathbb{N}$, $n \neq 0$, perhaps the most famous value of the gamma function on a non natural number is

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

(83)

Of the many identities involving the gamma function, the one we use is the Legendre duplication formula

$$\Gamma(x) \Gamma\left(x + \frac{1}{2}\right) = 2^{1-2x} \sqrt{\pi} \Gamma(2x),$$

(84)

and Euler’s reflection formula

$$\Gamma\left(\frac{1}{2} - x\right) \Gamma\left(\frac{1}{2} + x\right) = \frac{\pi}{\cos(\pi x)}.$$

(85)

The beta function $B$ has many equivalent definitions, of which the two useful here are

$$B(x, y) = 2 \int_0^{\pi/2} \sin^{2x-1} \theta \cos^{2y-1} \theta \, d\theta$$

$$= \int_0^1 t^{x-1} (1 - t)^{y-1} \, dt, \quad x, y > 0.$$

(86)

(87)

$B$ is related to $\Gamma$ by the following relation:

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}.$$

(88)

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Declarations

Competing interests The authors declare no competing interests.
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