Relationship between the linear entropy, the von Neumann entropy and the atomic Wehrl entropy for the Jaynes-Cummings model

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The linear entropy, the von Neumann entropy and the atomic Wehrl entropy are frequently used to quantify entanglement in the quantum systems. These relations provide typical information on the entanglement in the Jaynes-Cummings model (JCM). In this Letter, we explain the origin of this analytically and derive a closed form for the atomic Wehrl entropy. Moreover, we show that it is more convenient to use the Bloch sphere radius for quantifying entanglement in the JCM instead of these entropic relations.

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Quantum entanglement is an extensively-studied topic in the recent years in the advent of growing realizations and applications in the quantum information processing such as quantum computing [1], teleportation [2], cryptographic [3], dense coding [4] and entanglement swapping [5]. Entanglement gives new insights for understanding many physical phenomena including super-radiance [6], superconductivity [7], disordered systems [8], etc. Various types of experiments have been dealt with the entanglement, e.g. long-distance entanglement [9], ion-photon entanglement [10], many photons entanglement [11], etc. For recent review reader can consult [12].

The notion of entropy, originating from thermodynamics, has been reconsidered in the context of classical information theory [13] and quantum information theory [14]. There are several definitions for entropy. For instance, the von Neumann entropy [14], the relative entropy [15], the generalized entropy [16], the Renyi entropy [17], the linear entropy, and the Wehrl entropy [18]. The Wehrl entropy has been successfully applied in the description

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of different properties of the quantum optical fields such as phase-space uncertainty [19, 20], quantum interference [20], decoherence [21, 22], a measure of both noise (phase-space uncertainty) and phase randomization [23], etc. Also it has been applied to the evolution of the radiation field with the Kerr-like medium [24] as well as with the two-level atom [22]. Quite recently, the Wehrl entropy has been used in quantifying the entanglement of pure states of $N \times N$ bipartite quantum systems [25]. Moreover, the concept of the atomic Wehrl entropy has been developed [26] and applied to the atom-field interaction [27].

One of the elementary models in quantum optics, which describes the interaction between the radiation field and the matter, is the Jaynes-Cummings model (JCM) [28]. The JCM is a rich source for the nonclassical effects, e.g., [29]. Most importantly the JCM has been experimentally implemented by several means, e.g. one-atom mazer [30], the NMR refocusing [31], a Rydberg atom in a superconducting cavity [32] and the trapped ion [33]. The JCM is a subject of continuous studies. Previous investigations for the JCM have been shown that the von Neumann entropy [34], the linear entropy, [35] and the atomic Wehrl entropy [27] provide typical dynamical behaviors (N.B. The references given here are just examples, however, there are a large number of articles dealt this issue). In all these studies the attention is focused on the numerical investigations only and hence there was no clear answer to the question: Why these quantities give typical dynamical behaviors? In this Letter we answer this question using straightforward calculations. Moreover, we derive a closed form for the atomic Wehrl entropy. Additionally, we show that for quantifying entanglement in the JCM one should use the Bloch sphere radius (, i.e., the length of the Bloch sphere vector [36] ) instead of these three quantities. These are interesting results and will be useful for the scientific community.

We start the investigation by describing the system under consideration and giving the basic relations and equations, which will be frequently used in this Letter. The simplest form of the JCM is the two-level atom interacting with the single cavity mode. In the rotating wave approximation and dipole approximation the Hamiltonian controlling this system is:

$$\hat{H} = \hat{H}_0 + \hat{H}_i$$

$$\hat{H}_0 = \omega_0 \hat{a}^\dagger \hat{a} + \frac{1}{2} \omega_n \hat{\sigma}_z, \quad \hat{H}_i = \lambda (\hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-),$$

where $\hat{H}_0$ ($\hat{H}_i$) is the free (interaction) part, $\hat{\sigma}_\pm$ and $\hat{\sigma}_z$ are the Pauli spin operators; $\omega_0$ and
ωa are the frequencies of the cavity mode and the atomic transition, respectively, \( \hat{a} \) (\( \hat{a}^\dagger \)) is the annihilation (creation) of the cavity mode, and \( \lambda \) is the atom-field coupling constant. We assume that the field and the atom are initially prepared in the coherent state \( |\alpha\rangle \) and the excited atomic state \( |e\rangle \), respectively, and \( \omega_0 = \omega_a \) (i.e., the resonance case). Under these conditions the dynamical wave function of the system in the interaction picture can be expressed as:

\[
|\Psi(T)\rangle = \sum_{n=0}^{\infty} C_n \left[ \cos(T\sqrt{n+1})|e, n\rangle - i \sin(T\sqrt{n+1})|g, n+1\rangle \right],
\]

where

\[
C_n = \frac{\alpha^n}{\sqrt{n!}} \exp\left(-\frac{1}{2} \alpha^2\right), \quad \alpha = |\alpha| \exp(i\vartheta), \quad T = t\lambda,
\]

\( |g\rangle \) denotes the atomic ground state and \( \vartheta \) is a phase. Information about the bipartite (i.e., atom and field) is involved in the wavefunction \( |\Psi(T)\rangle\langle\Psi(T)| \). Nevertheless, the information on the atomic system solely can be obtained from the atomic reduced density matrix \( \hat{\rho}_a(T) \) having the form

\[
\hat{\rho}_a(T) = \text{Tr}_f \hat{\rho}(T),
\]

\[
\hat{\rho}_a(T) = \hat{\rho}_{ee}(T)|e\rangle\langle e| + \hat{\rho}_{gg}(T)|g\rangle\langle g| + \hat{\rho}_{eg}(T)|e\rangle\langle g| + \hat{\rho}_{eg}^*(T)|g\rangle\langle e|,
\]

where the subscript \( f \) means that we trace out the field and \( \hat{\rho}_{ij}(T) = \langle i|\hat{\rho}_a(T)|j\rangle \) with \( i, j = e, g \). Form \( |2\rangle \) the coefficients \( \hat{\rho}_{ij}(T) \) can be expressed as:

\[
\rho_{ee}(T) = \sum_{n=0}^{\infty} |C_n|^2 \cos^2(T\sqrt{n+1}), \quad \rho_{gg}(T) = \sum_{n=0}^{\infty} |C_n|^2 \sin^2(T\sqrt{n+1}),
\]

\[
\rho_{eg}(T) = i \exp(i\vartheta) \sum_{n=0}^{\infty} |C_{n+1}C_n| \cos(T\sqrt{n+2}) \sin(T\sqrt{n+1}).
\]

Additionally, for the atomic set operators \( \{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\} \) we obtain:

\[
\langle \hat{\sigma}_z(T) \rangle = \rho_{ee}(T) - \rho_{gg}(T), \quad \langle \hat{\sigma}_x(T) \rangle = 2\text{Re}[\rho_{eg}(T)],
\]

\[
\langle \hat{\sigma}_y(T) \rangle = 2\text{Im}[\rho_{eg}(T)], \quad \rho_{ee}(T) + \rho_{gg}(T) = 1.
\]

Now the linear entropy is defined as:

\[
\xi(T) = 1 - \text{Tr}\rho_a^2(T).
\]
For the system under consideration the relation (7) by means of (6) can be easily evaluated as:

$$\xi(T) = 1 - \rho_{ee}^2(T) - \rho_{gg}^2(T) - 2|\rho_{eg}(T)|^2$$

(8)

$$= \frac{1}{2}[1 - \eta^2(T)], \quad \eta^2(T) = \langle \hat{\sigma}_x(T) \rangle^2 + \langle \hat{\sigma}_y(T) \rangle^2 + \langle \hat{\sigma}_z(T) \rangle^2,$$

where the quantity $\eta(T)$ is the Bloch sphere radius. The Bloch sphere is a well-known tool in quantum optics, where the simple qubit state is faithfully represented, up to an overall phase, by a point on a standard sphere with radius unity, whose coordinates are expectation values of the atomic set operators of the system. In the language of entanglement $\xi(T)$ ranges from 0 (i.e., $\eta(T) = 1$) for disentangled and/or pure states to 0.5 (i.e., $\eta(T) = 0$) for maximally entangled bipartite [37]. On the other hand, the von Neumann entropy is defined as [14]:

$$\gamma(T) = -\text{Tr}\{\rho_a(T)\ln\rho_a(T)\},$$

(9)

$$= -\mu_-(T)\ln\mu_-(T) - \mu_+(T)\ln\mu_+(T),$$

where $\mu_\pm(T)$ are the eigenvalues of the $\rho_a(T)$, which for (11) can be expressed as:

$$\mu_\pm(T) = \frac{1}{2}\{1 \pm \sqrt{1 - 4[\rho_{ee}(T)\rho_{gg}(T) - |\rho_{eg}(T)|^2]}\},$$

(10)

$$= \frac{1}{2}\{1 \pm \eta(T)\}.$$  

The second line in (10) has been evaluated by means of (5) and (6). From the limitations on the $\eta(T)$ one can prove $0 \leq \gamma(T) \leq 0.693$. Finally, the atomic Wehrl entropy has been defined as [26]:

$$W_a(T) = -\int_0^{2\pi} \int_0^\pi Q_a(\Theta, \Phi, T) \ln Q_a(\Theta, \Phi, T) \sin \Theta d\Theta d\Phi,$$

(11)

where $Q_a(\Theta, \Phi, T)$ is the atomic $Q$-function defined as:

$$Q_a(\Theta, \Phi, T) = \frac{1}{2\pi} \langle \Theta, \Phi | \hat{\rho}_a(T) | \Theta, \Phi \rangle$$

(12)

and $|\Theta, \Phi\rangle$ is the atomic coherent state having the form [38]:

$$|\Theta, \Phi\rangle = \cos(\Theta/2) |\epsilon\rangle + \sin(\Theta/2) \exp(i\Phi) |g\rangle$$

(13)
FIG. 1: The linear entropy (a), the von Neumann entropy (b), the atomic Wehrl entropy (c) and the Bloch sphere radius $\eta(T)$ (d) for $(|\alpha|, \phi) = (7, 0)$.

with $0 \leq \Theta \leq \pi, 0 \leq \Phi \leq 2\pi$. For the wavefunction (2) the atomic $Q_a$ function can be evaluated as

$$Q_a(\Theta, \Phi, T) = \frac{1}{4\pi} [1 + \beta(T)]$$

$$\beta(T) = \langle \hat{\sigma}_z(T) \rangle \cos \Theta + [\langle \hat{\sigma}_x(T) \rangle \cos \Phi + \langle \hat{\sigma}_y(T) \rangle \sin \Phi] \sin \Theta.$$  

One can easily check that $Q_a$ given by (14) is normalized. On substituting (14) into (11) and carrying out the integration we obtain

$$W_a(T) = \ln(4\pi) - \frac{1}{4\pi} \int_0^{2\pi} [1 + \beta(T)]\ln[1 + \beta(T)]\sin \Theta d\Phi d\Theta,$$

$$= \ln(4\pi) - \sum_{n=1}^{\infty} \sum_{r=0}^{n} \sum_{s=0}^{r} \frac{(2n)!((-1)^r \langle \hat{\sigma}_x(T) \rangle ^2 (n-r)\langle \hat{\sigma}_y(T) \rangle ^2 + \langle \hat{\sigma}_y(T) \rangle ^2 \langle \hat{\sigma}_y(T) \rangle ^2)^r}{2n(2n-1)(2n-2r)!4^r (r-s)!s!2(2n-r-1)}.$$  

In the derivation of (15) we have used the series expansion of the logarithmic function, the binomial expansion and the identity [39]:

$$\int_0^{2\pi} (c_1 \sin x + c_2 \cos x)^k dx = \begin{cases} 0 & \text{for } k = 2m + 1, \\ 2\pi \frac{(2m)!}{4^m (m)!^2} (c_1^2 + c_2^2)^m & \text{for } k = 2m, \end{cases}$$

where $c_1, c_2$ are c-numbers and $k$ is a positive integer. Expression (15) is relevant for the numerical investigation. In Figs. 1(a), (b) and (c) we have plotted the linear entropy (8), the von Neumann entropy (9) and the atomic Wehrl entropy (15), respectively, for the given values of the interaction parameters. Comparison between these figures is instructive and shows that the three entropic relations provide typical information (with different scales) on
the entanglement in the JCM. Now we’ll explain why this occurs. We start by expressing (9) in a series form using the Taylor expansion for the logarithmic functions, where we obtain

\[ \gamma(T) = \ln 2 - \sum_{n=1}^{\infty} \frac{\eta^{2n}(T)}{2n(2n-1)}. \]  

(17)

From (8) and (17) one can realize that \( \xi(T) \) and \( \gamma(T) \) are functions in \( \eta^2(T) \) and hence they exhibit similar behaviors. Now we draw the attention to (15), which is rather complicated. Nevertheless, by expanding the first few terms in this expression one can obtain:

\[ W_a(T) = \ln(4\pi) - \left\{ \frac{\eta^2(T)}{1 \times 2 \times 3} + \frac{\eta^4(T)}{3 \times 4 \times 5} + \frac{\eta^6(T)}{5 \times 6 \times 7} + \ldots \right\} \]

(18)

The series in the second line of (18) has been obtained from the first one via mathematical induction. From (18) it is obvious that \( W_a(T) \) is a function in \( \eta^2(T) \) and this is the reason for having behavior similar to those of \( \xi(T) \) and \( \gamma(T) \). Also from the available information on \( \eta(T) \) and (18) one can explore the limitations of \( W_a(T) \). For instance, for maximal entangled bipartite we have \( \eta(T) = 0 \) and hence the upper bound of the \( W_a(T) \) is \( \ln(4\pi) \), however, for disentangled bipartite (i.e. \( \eta(T) = 1 \)) the lower bound can be evaluated as:

\[ W_a(T) = \ln(4\pi) - \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)(2n+1)} \approx \ln(4\pi) - 0.19315 \approx 2.3379. \]  

(19)

The exact value of the series in (19) has been evaluated numerically. All the analytical facts obtained above are remarkable in Fig. 1(c).

We conclude this Letter by deriving a closed form for (18). By means of the partial fraction one can prove obtain

\[ \frac{1}{2n(2n-1)(2n+1)} = \frac{1}{2n(2n-1)} - \frac{1}{2(2n-1)} + \frac{1}{2(2n+1)}. \]  

(20)

Substituting (20) into (18) and using the identities:

\[ \sum_{n=1}^{\infty} \frac{\eta^{2n}(T)}{2n(2n-1)} = \frac{1}{2} \ln[1 - \eta^2(T)] + \frac{\eta(T)}{2} \ln \left[ \frac{1 + \eta(T)}{1 - \eta(T)} \right], \]  

(21)

\[ \sum_{n=1}^{\infty} \frac{\eta^{2n}(T)}{(2n-1)} = \frac{\eta(T)}{2} \ln \left[ \frac{1 + \eta(T)}{1 - \eta(T)} \right], \quad \sum_{n=1}^{\infty} \frac{\eta^{2n}(T)}{(2n+1)} = -1 + \frac{1}{2\eta(T)} \ln \left[ \frac{1 + \eta(T)}{1 - \eta(T)} \right], \]  

(22)
we obtain

\[ W_a(T) = \frac{1}{2} + \ln(4\pi) - \frac{1}{2} \ln(1 - \eta^2(T)) + \frac{1}{4} \left[ \eta(T) + \frac{1}{\eta(T)} \right] \ln \left[ \frac{1 - \eta(T)}{1 + \eta(T)} \right]. \quad (23) \]

The identity (21) is obtained from (9) and (17), however, those in (22) are related to the logarithmic form of the \( \tanh^{-1}(.) \). The expression (23) is valid for all values of \( \eta(T) \) except \( \eta(T) = 0 \). We have checked the validity of the expression (23) by getting the typical behavior shown in Fig. 1(c).

Now, the different scales in the above entropic relations can be treated by redefining them as follows:

\[ \tilde{\gamma}(T) = \frac{\gamma(T)}{0.693}, \quad \tilde{W}_a(T) = \frac{\ln(4\pi) - W_a(T)}{0.19315}. \quad (24) \]

In this case, the entropic relations \( \xi(T), \tilde{\gamma}(T) \) and \( \tilde{W}_a(T) \) yield typical information on the JCM. As all forms of the two-level JCM (i.e. off-resonance JCM, multimode JCM, etc.) can be described by the atomic density matrix (4), the results obtained in this Letter are universal. From the above investigation the quantities \( \xi(T), \gamma(T) \) and \( W_a(T) \) depend only on \( \eta(T) \) and hence it would be more convenient to use the Bloch sphere radius \( \eta(T) \) directly for getting information on the entanglement in the JCM (see Fig. 1(d)). In this case, for maximum (minimum) entanglement we have \( \eta(T) = 0 \) (1), i.e., \( 0 \leq \eta(T) \leq 1 \). It is worth mentioning that the concept of the Bloch sphere has been used recently for the JCM with different types of initial states for analyzing the correlation between entropy exchange and entanglement [40]. The final remark, the linear entropy, the von Neumann entropy and the atomic Wehrl operator of the JCM are invariant quantities under unitary transformations, as they depend only on the eigenvalues of the density operator.

In conclusion in this Letter we have analytically explained why the linear entropy, the von Neumann entropy and the atomic Wehrl entropy of the JCM have similar behaviors. We have shown that the Bloch sphere radius has to be used for quantifying the entanglement in the JCM instead of the entropic relations. Finally, the results obtained in this Letter are universal.

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[1] Benenti G, Casati G and Strini G 2005 "Principle of Quantum Computation and Information" (World Scientific, Singapore).

[2] Bennet C H, Brassard G, Crepeau C, Jozsa R, Peresand A and Wootters W K 1993 Phys. Rev. Lett. 70 1895.

[3] Ekert A 1991 Phys. Rev. Lett. 67 661; Cirac J I and Gisin N 1997 Phys. Lett. A 229 1; Fuchs C A, Gisin N, Griffiths R B, Niu C-S and Peres A 1997 Phys. Rev. A 56 1163.

[4] Ye L and Guo G-C 2005 Phy. Rev. A 71 034304; Mozes S, Oppenheim J and Reznik B 2005 Phys. Rev. A 71 012311.

[5] Glöckl O, Lorenz S, Marquardt C, Heersink J, Brownnutt M, Silberhorn C, Pan Q, Loock P V, Korolkova N and Leuchs G 2003 Phys. Rev. A 68 012319; Yang M, Song W and Cao Z-L 2005 Phys. Rev. A 71 034312; Li H-R, Li F-L, Yang Y and Zhang Q 2005 Phys. Rev. A 71 022314.

[6] Lambert N, Emary C and Brandes T 2004 Phys. Rev. Lett 92 073602.

[7] Vedral V 2004 New. J. Phys. 6 102.

[8] Dür W, Hartmann L, Hein M, Lewenstein M and Briegel H J 2005 Phys. Rev. Lett 94 097203.

[9] Peng C-Z, et al 2005 Phys. Rev. Lett. 94 150501.

[10] Volz J, Weber M, Schlenk D, Rosenfeld W J, Vrana K, Saucke C, Kurtsiefer, and Wefinfurter H 2006 Phys. Rev. Lett. 96 030404.

[11] Zhao Z, Chen Y-A, Zhang A-N, Yang T, Briegel H and Pan J-W 2004 Nature 430 54.

[12] Horodecki R, Horodecki P, Horodecki M and Horodecki K quant-ph/0702225.

[13] Shannon C E 1948 Bell Syst. Tech. J. 27 379.

[14] von Neumann J 1955 "Mathematical Foundations of Quantum Mechanics" (Princeton University Press, Princeton, NJ).

[15] Vedral V 2002 Rev. Mod. Phys. 74 197.

[16] Bastiaanns M J 1984 J. Opt. Soc. Am. A 1 711;Tsallis C 1988 J. Stat. Phys. 55 479.

[17] Renyi A 1970 "Probability Theory" (North Holland, Amsterdam, 1970).

[18] Wehrl A 1978 Rev. Mod. Phys. 50 221; Wehrl A 1991 Rep. Math. Phys. 30 119.

[19] Bužek V, Keitel C H and Knight P L 1995 Phys. Rev. A 51 2575; Vaccaro J A and Orlowski
A 1995 Phys. Rev. A 51 4172; Watson J B, Keitel C H, Knight P L and Burnett K 1996 Phys. Rev. A 729.

[20] Bužek V, Keitel C H and Knight P L 1995 Phys. Rev. A 51 2594.
[21] Anderson A and Halliwell J J 1993 Phys. Rev. D 48 2753.
[22] Orlowski A, Paul H and Kastelowicz G 1995 Phys. Rev. A 52 1621.
[23] Miranowicz A, Matsueda H and Wahiddin M R B 2000 J. Phys. A: Math. Gen. 33 51519.
[24] Jex I and Orlowski A 1994 J. Mod. Opt. 41 2301.
[25] Minter F and Zyczkowski K 2003 quant-ph/0307169.
[26] Zyczkowski K 2001 Physica E 9 583.
[27] Obada A-S and Abdel-Khalek S 2004 J. Phys. A: Math. Gen. 37 6573; El-Orany F A A, Abdel-Khalek S, Abd-Aty M and Wahiddin M R B quant-ph/0703043.
[28] Jaynes E T and Cummings F W 1963 Proc. IEEE 51 89.
[29] Shore B W and Knight P L 1993 J. Mod. Opt. 40 1195.
[30] Rempe G, Walther H and Klein N 1987 Phys. Rev. Lett. 57 353.
[31] Meunier T, Gleyzes S, Maioli P, Auffeves A, Nogues G, Brune M, Raimond J M and Haroche S 2005 Phys. Rev. Lett. 94 010401.
[32] Yeazell J A, Mallalieu M and Stroud C R Jr 1990 Phys. Rev. Lett. 64 2007; Brune M, Schmidt-Kaler F, Maali A, Dreyer J, Hagley E, Raimond J M and Haroche S 1996 Phys. Rev. Lett. 76 1800.
[33] Vogel W and De Matos Filho R L 1995 Phys. Rev. A 52 4214.
[34] Phoenix S J D and Knight P L 1988 Ann. Phys. (N. Y.) 186 381.
[35] El-Orany F A A and Obada A-S 2003 J. Opt. B: Quant. Semiclass. Opt. 5 60.
[36] Ekert A K, Alves C M, Daniel K L O, Horodecki M, Horodecki P and Kwek L C 2002 Phys. Rev. Lett. 88 217901.
[37] Gea-Banacloche J Phy. Rev. Lett. 65 (1990) 3385.
[38] Vieira V R and Sacramento P D 1995 Ann. Phys. (N.Y.) 242 (1995) 188.
[39] Gradshteyn S and Ryzhik I M "Table of Integrals, Series, and Products" Ed. Jeffrey A, Fifth edition (Academic Press, Inc. 1994) P. 424,425.
[40] Boukobza E and Tannor D J 2005 Phys. Rev. A 71 063821.
[41] Wootters W K 1998 Phys. Rev. Lett. 80 2245; Hill S and Wootters W K 1997 Phys. Rev. Lett. 78 5022.
[42] Peres A 1996 *Phys. Rev. Lett.* 77 1413; Horodecki P 1997 *Phys. Lett. A* 232 333.