POSITIVE MAPS, STATES, ENTANGLEMENT AND ALL THAT; 

some old and new problems.

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Abstract. We outline a new approach to the characterization as well as to the classification of positive maps. This approach is based on the facial structures of the set of states and of the cone of positive maps. In particular, the equivalence between Schroedinger’s and Heisenberg’s pictures is reviewed in this more general setting. Furthermore, we discuss in detail the structure of positive maps for two and three dimensional systems. In particular, the explicit form of decomposition of a positive map and the uniqueness of this decomposition for extremal positive maps for 2 dimensional case are described. The difference of the structure of positive maps between 2 dimensional and 3 dimensional cases is clarified. The resulting characterization of positive maps is applied to the study of quantum correlations and entanglement.

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1. INTRODUCTION

The aim of this paper is to bring together two areas, the theory of positive maps on $C^*$-algebras and the abstract characterization of the set of states on a $C^*$-algebra. The present paper concerns crucial aspects of the quantization procedure; as such it is an extension of our recent publication [31]. The classification of positive maps and the full characterization of states are at the heart of quantum theory, with particular reference to the foundations of quantum information theory [1], [22]. To be more precise, the full description of the set of states of a physical system, the complete characterization of distinct types of states, and the detailed account of the properties of maps of states into states (i.e. the Schrödinger approach to dynamical maps) involve various aspects of the affine structure of the convex set of (all) states. The standard tool for a study of convex compact sets is Krein-Milman theorem which states that such a set is the closure of the convex hull of its extreme points. In particular, this idea was used by Størmer [42] for a classification of positive maps. But, such an approach depends on the description of extreme positive maps. The characterization of extreme positive maps is available only for the $M_2(\mathbb{C})$-case ($M_n(\mathbb{C})$ stands for the set of all $n$ by $n$ matrices with complex entries). Hence it is natural to go one step further and consider the set of positive maps or of states as a (convex compact) subset of an ordered Banach space. The characterization of certain subsets of the set of all states would provide a nice illustration of such method. In particular, using the Krein approach to the geometric version of Hahn-Banach theorem, one can introduce special functionals as basic tools for a characterization of some subsets of states. As instance of this is witnesses of entanglement (cf. page 452 in [22]) to study entanglement of states.

Further, we can combine ordered Banach space techniques (defined for the state space) with the theory of linear positive unital maps and exploit the algebraic structure of the underlying algebra. In this manner we can get a better understanding of the structure of subsets of states. This is due to the fact, that the “plain” language of ordered Banach spaces, in general, does not “feel” the non-commutativity of the underlying algebra. Consequently, there is a need for a supplementary geometrical structure of the set of states to establish rigorous relations between the theory given in terms of states and the algebraic structure of the set of observables, i.e. to get a more complete
understanding of the nature of “the equivalence” between Schroedinger and Heisenberg’s pictures. We note that the need for a clarification of such an equivalence is not new; Dirac working within the context of quantum electrodynamics had already noted some problems connected with it. As another example, we wish to point out problems emerging from the description of quantum chaos. In this paper we will provide the mathematical argument showing the necessity for a new look upon the discussed equivalence (see Section 3).

Our approach will stem from the so called Kadison question: under what conditions a convex set is affinely isomorphic to the set of states on a Jordan \((C^*, W^*)\) algebra? (see the Glossary in the Appendix). We recall that the abstract characterization of the set of observables based on Jordan algebras (Heisenberg picture) was established by von Neumann, Jordan and Wigner some seventy years ago (see [17] and [38]) while the essence of the abstract characterization of the set of states (Schroedinger picture) is contained in the Kadison question. The full answer to the Kadison question, provided recently by Alfsen and Shultz in [2] and [3], will be our starting point.

We will show that pure quantum features of non-commutative dynamical systems such as the peculiar behaviour of positive maps, quantum correlations and entanglement can be more easily understood within the mathematical framework which will be introduced in the subsequent sections. The main idea of our approach to the description of distinct types of states as well as to that of positive maps is to replace small boundary subsets (extreme points) by larger subsets for which we have an explicit description. In particular, we propose a modification of Størmer’s approach to the classification of positive maps; namely to replace extreme positive maps by maximal faces of (n-) positive maps; for the appropriate definitions see Section 2. Here, we only note that an extreme point is a face, thus it is contained in a maximal face. As we will see in Section 4, Kye gave [25] - [27] the complete characterization of appropriate (i.e. maximal) faces. This points to the choice of maximal faces as a powerful tool for our purpose.

The paper is organized as follows. In Section 2 we review some of the standard facts on theory of convex sets and set up notation and terminology. Our presentation is entirely based on two fundamental books by Alfsen-Shultz [2] and [3]. Section 3 is devoted to the study of positive
maps from the physical point of view. Again, using Alfsen-Shultz monographs we will compare
Schroedinger’s and Heisenberg’s picture to show that there is a one-to-one correspondence in the
description of decomposable maps (in both) pictures if and only if one equips the Schroedinger
picture with additional geometrical structure. The relations between the facial structures of states
and positive maps are given in Section 4. Section 5 presents basic properties of positive maps
while Section 6 concerns low dimensional cases. This section is based on a joint work with Marcin
Marciniak and its aim is to get a deeper understanding of the reason why the theory of positive
maps changes so dramatically when one goes from 2-level systems to 3-level ones. We present
the explicit form of decomposition of a positive map and the uniqueness of this decomposition
for extremal positive maps, for 2 dimensional case, are described. Furthermore, the difference of
the structure of positive maps between 2 dimensional and 3 dimensional cases is clarified from
the geometrical point of view. The last section contains a brief discussion of applications of our
results to the description of entangled states of quantum systems. It is worth pointing out that our
approach sheds new light on the construction of non-decomposable maps which is an important
issue in any attempt to classifying entangled states.

Finally, we want to stress that, in order to make the paper more accessible to a quantum com-
puting audience, we shall deliberately not address the problem in its full generality. Consequently,
although the theory may be formulated in general $C^*$-algebraic terms, we will be interested mainly
in $\mathcal{B}(\mathcal{H})$ (i.e. in the $C^*$-algebra of all linear bounded operators on a Hilbert space $\mathcal{H}$).

2. GEOMETRY OF STATE SPACES

Let $F$ be a convex subset of a convex set $\mathcal{S}$ in some Banach space. $F$ is said to be a face of $\mathcal{S}$
if the following property holds:

\begin{equation}
(2.1) \quad x, y \in \mathcal{S}, (1 - t)x + ty \in F \quad \text{for some} \quad t \in (0, 1) \quad \Rightarrow \quad x, y \in F.
\end{equation}

A proper face $F$ is a face of $\mathcal{S}$ which is neither $\mathcal{S}$ itself nor the empty set. Note that a face
of a face of a convex set $\mathcal{S}$ is a face. It is also clear that the intersection of faces is again a
face. Therefore, there is a unique smallest face contained in a given subset. Also, given a family
\( \{F_i; i \in I\} \) of faces, we denote by $\bigvee_{i \in I} F_i$, the smallest face containing every $F_i$. Hence, the set
\( \mathcal{F}(S) \) of all faces of a convex set \( S \) is a complete lattice (see A17) with respect to the partial order induced by the set of inclusions.

Here and below, \( S \) will denote the set of all states on a \( C^* \)-algebra \( A = \mathcal{B}(\mathcal{H}) \), namely the convex set of all normalized, positive linear maps \( \varrho : A \rightarrow \mathbb{C} \), on \( A \). \( \text{face}(\varrho) \) will stand for the face generated by the state \( \varrho \), i.e. the smallest non-trivial convex set of convex decompositions of \( \varrho \), \( \varrho = \sum \lambda_i \sigma_i \), \( \lambda_i \geq 0 \), \( \sum \lambda_i = 1 \), into other states \( \sigma_i \). Let \( (\mathcal{H}_\varrho, \pi_\varrho, \Omega_\varrho) \) be the GNS triple (see A9) associated with a state \( \varrho \) on a \( C^* \)-algebra \( A \). Then, one has the following nice characterization of the face \( \text{face}(\varrho) \). Namely, for every positive functional \( \sigma \in \text{face}(\varrho) \) there exists a unique positive element \( b \in \pi_\varrho(A)' \) such that

\[
\sigma(a) = (\Omega_\varrho, b\pi_\varrho(a)\Omega_\varrho) \quad \text{for all} \quad a \in A.
\]

Here, \( \pi_\varrho(A)' \) stands for the commutant of \( \pi_\varrho(A) \); (see A4). Moreover, the map \( \phi : \sigma \mapsto b \) is an order preserving affine isomorphism (see A1) of \( \text{face}(\varrho) \) onto \( (\pi_\varrho(A)')_+ = \{ a \in \pi_\varrho(A)', a \geq 0 \} \), i.e. \( \phi \) is the affine isomorphism such that \( \sigma \leq \sigma' \) implies \( \phi(\sigma) = b \leq b' = \phi(\sigma') \).

To proceed with the discussion of the geometry of state space we will need two concepts. The first one is the so called projective face which can be characterized as follows. Let \( F \) be a norm closed face in \( S \). If \( p \) is the carrier projection of \( F \) (the smallest projection \( p \) such that \( \sigma(p) = ||\sigma|| \) for all \( \sigma \in F \)) then \( F \equiv F_p \) where

\[
F_p = \{ \sigma \in S; \sigma(p) = 1 \}.
\]

A face of the form \( F_p \), where \( p \) is a projection in \( A \), will be called a projective face. This concept can be defined in much more general setting, namely for a pair of ordered unit space and base norm space, for details see [3] and A16 for the terminology.

Let \( p \in \mathcal{B}(\mathcal{H}) \) be an orthogonal projection. Then the map \( p \mapsto F_p \) determines an isomorphism from the lattice of closed subspaces of \( \mathcal{H} \) to the lattice of norm closed (projective) faces of \( S \). The closed faces associated with a projection have another interesting property which will be useful to fully understand the Alfsen-Shultz result. Namely, if \( p \) is a projection onto the closed subspace

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1In the following, reference marks like this in the text refer to definitions and basic facts listed in the Glossary at the Appendix given at the end of the paper.
spanned by a family of unit vectors \( \{ \eta_i \}_{i \in I} \subset \mathcal{H} \), then the norm closed face \( F \) associated with \( p \) is the smallest face of \( S \) that contains the vector states \( (\omega_i)_{i \in I} \) where \( \omega_i(\cdot) = (\eta_i, \cdot \eta_i) \).

As a corollary one has the following result. The face generated by two distinct pure states of the normal state space of \( \mathcal{B}(\mathcal{H}) \) is an Euclidean 3-ball, i.e. the face is affinely isomorphic to the closed unit ball in Euclidean 3-dimensional space.

The second concept, orientation, is another important ingredient of the affine structure of the set \( S \). For the sake of conciseness, we present this concept only briefly, as it is a necessary "tool" for understanding the relations between the affine structure of the state space \( S \) and the Jordan and Lie products of the corresponding algebra.

To illustrate this idea let us consider the algebra of all 2 by 2 matrices with complex entries, \( M_2(\mathbb{C}) \), so a very special example of \( \mathcal{B}(\mathcal{H}) \). It can be shown \( \footnote{If \( r, s, t \) are \( e \)-symmetries then a Cartesian triple of \( e \)-symmetries is a Cartesian triple of symmetries in the von Neumann algebra \( e\mathcal{B}(\mathcal{H})e \).} \) that the affine structure of the state space determines the Jordan product on \( M_2(\mathbb{C}) \) uniquely. There are two possible \( \mathbb{C}^* \)-products, both being well defined Jordan products: the usual one \( M_2(\mathbb{C}) \times M_2(\mathbb{C}) \ni < a, b > \mapsto ab \in M_2(\mathbb{C}) \) and the opposite one \( M_2(\mathbb{C}) \times M_2(\mathbb{C}) \ni < a, b > \mapsto ba \in M_2(\mathbb{C}) \). However, the definition of positive elements in \( M_2(\mathbb{C}) \) is not affected by the above ambiguity. This clearly shows that an additional concept is necessary to determine the form of the associative product in the algebra.

Let us be more formal and provide some further tools necessary to the geometrical characterization of the state space. A self-adjoint operator \( s \in \mathcal{B}(\mathcal{H}) \) is called (e-) symmetry if \( s^2 = 1 \) (\( s^2 = e, e \) a projector, respectively). Then, to each symmetry with canonical (spectral) decomposition \( s = p - q \) (\( p, q \) are orthogonal projectors, \( pq = 0 \), and \( p + q = 1 \) or \( p + q = e \) respectively) we assign the pair of projective faces \( (F_p, F_q) \) called the associated generalized axis. Having the concept of (e-) symmetries one can generalize the idea of orthogonal frame of axes in the state space \( S(M_2(\mathbb{C})) \) which, we recall, is affine isomorphic to the ball in 3 dimensional Euclidean space. In general, a triple of symmetries \( (r, s, t) \) is called a Cartesian triple \( \footnote{If \( r, s, t \) are \( e \)-symmetries then a Cartesian triple of \( e \)-symmetries is a Cartesian triple of symmetries in the von Neumann algebra \( e\mathcal{B}(\mathcal{H})e \).} \) if the following conditions are satisfied:

\[ \bullet \quad r \circ s = s \circ t = t \circ r = 0, \] where \( \circ \) stands for the Jordan product.
• $U_r U_s U_t = id$, $id$ stands for the identity operator while $U_v a \equiv v a v$ for any $a \in B(\mathcal{H})$ and any symmetry $v \in \{r, s, t\}$.

A nice example of a Cartesian triple of symmetries for $M_2(\mathbb{C})$ is provided by Pauli spin matrices. The question of existence of Cartesian triples is settled by the following result: a von Neumann algebra $\mathcal{M}$ (so in particular, $B(\mathcal{H})$) contains a Cartesian triple of $e$-symmetries if and only if $e$ is a halvable projector, i.e. $e$ is a sum of two equivalent (in the von Neumann sense, see A13) projectors. Then, $F_e$ is affinely isomorphic to the normal space of the local algebra $eM_e$.

**Example 2.1.** $M_4(\mathbb{C})$ contains two families of Cartesian triples of $e$-symmetries. One for $e$ of rank 2 and the one for identity. Note, that for $M_2(\mathbb{C})$ one has only one family of discussed triples: the one which is associated with $I$.

Now we can define the previously mentioned concept of orientation for a von Neumann algebra $B(\mathcal{H})$. First, the local orientation of $B(\mathcal{H})$ is a unitary equivalence class of Cartesian triples in $eB(\mathcal{H})e$ where $e$ is a halvable projection in $B(\mathcal{H})$. Then the global orientation is defined as a ”continuous choice” of local orientations. It can be proved (cf. [2]) that there is one-to-one correspondence between global orientations of $B(\mathcal{H})$ and Jordan compatible associative products in $B(\mathcal{H})$, i.e. the Jordan product associated with the geometry of states coincides with the original product of $B(\mathcal{H})$.

After these preliminaries we are in position to give the answer to Kadison question and to discuss, in the next Section, the equivalence between Heisenberg and Schroedinger’s pictures. Here, we will do it for $B(\mathcal{H})$ only (for a general treatment see [3]; see also A16, A10, A11, A15 for the terminology).

**Theorem 2.2** (Alfsen, Shultz, [3]). Let $K$ be the base of a complete base norm space. Then $K$ is affine isomorphic to the normal state space, $S_0$, of $B(\mathcal{H})$ with $\mathcal{H}$ a complex Hilbert space if and only if the following conditions hold:

• every norm exposed face is projective;

• the $\sigma$-convex hull of extreme points of $K$ equals $K$;
• the face generated by every pair of extreme points of $K$ is a 3-ball and is norm exposed.

It is worth pointing out that since the 3-ball constitutes the so called Bloch sphere which coincides with the state space for the standard two-level system the last condition of Theorem 2.2 clearly indicates the fundamental role of qubits. In other words, the set of “two dimensional states” plays “locally” a crucial role in the general characterization of the set of all normal states over $B(H)$. However, it should be stressed that for a general $C^*$-algebra the face generated by a pair of pure states is either a 3-ball or a line segment. Thus, the above simple picture for $B(H)$ turns to be more complicated for a general $C^*$-algebra.

3. POSITIVE MAPS AND THEIR DUALS.

Let $P_0$ denote the convex set of all $\sigma$-weakly continuous unital positive linear maps (so such maps $\alpha$ that the state $\varphi \circ \alpha$ is determined by a density matrix whenever $\varphi$ has this property) from the von Neumann algebra $B(H)$ into itself; the subscript “_0” stands for unital. We emphasize that contrary to the standard conjecture saying that only completely positive maps have a direct interpretation as dynamical maps, it seems that some maps in the class $P_0$ of plain positive maps could also be relevant for description of time evolution (see [18] for a recent discussion of this question; a recent survey on dynamics of open quantum systems can be found in the lecture notes [24]). Moreover, $P_0$ contains large subsets of positive maps which are directly connected with a characterization of various types of entangled states which provides the additional motivation for our interest in this class.

Let us turn to the question of dual (transposed) maps, i.e. maps defined on the set of states. Suppose $T \in P_0$ and define $(T^* \omega)(a) = \omega(Ta) where $a \in B(H)$ and $\omega$ is a normal state on $B(H)$. Then

**Theorem 3.1.** There is a one-to-one correspondence between $\sigma$-weakly continuous positive unital linear maps from $B(H)$ into itself, and affine maps from the normal state space of $B(H)$ into itself.
We now denote by $D_0$ the more specialized family, $D_0 \subset P_0$, of positive maps consisting of the so-called decomposable maps. The general form of such maps, in the Heisenberg picture is defined by the following relations

\[(3.1) \quad T(a) = \sum_i W_i^* \tau_i(a) W_i \quad (\equiv T_W(a)),\]

where $W_i \in B(H)$, $\sum_i W_i^* W_i = I$, while $\tau_i$ stands for a unital Jordan homomorphism, i.e. $\tau_i$ is a linear map preserving the Jordan structure $\tau_i(\{a, b\}) = \{\tau_i(a), \tau_i(b)\}$, with $\{\cdot, \cdot\}$ standing for the anticommutator. In (3.1), $W$ denote the subspace of $B(H)$ spanned by $\{W_1, \ldots, W_n\}$. The special case when the $\tau_i$’s are $^*$-morphisms leads to the important class of completely positive maps, $CP_0$.

To pass to the dual picture (so to go to Schroedinger’s picture), we need (see also A18, A17)

**Theorem 3.2** (Alfsen, Shultz [40, 3]). Consider $B(H_1)$, $B(H_2)$ with normal state spaces $S_1$ and $S_2$ respectively and let $T^*_0 : S_2 \rightarrow S_1$ be an affine map. Let $T : B(H_1) \rightarrow B(H_2)$ be the unital positive $\sigma$-weakly continuous map such that $T^*|_{S_2} = T^*_0$ where $T^*$ is defined by the formula:

$$(T^* \omega)(a) \equiv \omega(T(a)) \quad \text{for any} \quad a \in B(H_1) \quad \text{where} \quad \omega \quad \text{is any linear normal functional on} \quad B(H_2).$$

Then, the following statements are equivalent:

- $T$ is a unital Jordan homomorphism from $B(H_1)$ into $B(H_2)$.
- $(T^*_0)^{-1}$ preserves complements of projective faces.
- $(T^*_0)^{-1}$ as a map from the lattice of projective faces of $S_1$ into the lattice of projective faces of $S_2$ preserves lattice operations and complements.

Consequently one has

**Corollary 3.3.** There is a one-to-one correspondence between the set $D_0$ of decomposable maps and the set of affine maps of the form

\[(3.2) \quad \omega \mapsto \sum_i U_{W_i}^* (T^*_0 i \omega)(\equiv \sum_i \omega(W_i^* T(\cdot) W_i))\]

where $U_{W_i}^* \omega(\cdot) = \omega(W_i^* \cdot W_i)$, $\omega$ any normal state, and $T^*_0 i$ satisfies one of the conditions given in Theorem 3.2.
Finally, we want to describe the most "regular" case - the case of invertible maps; note that the hamiltonian time evolution is the best known example of such maps.

**Theorem 3.4** (Kadison [20]). Let $T^*$ be an affine invertible map from the state space $S$ of a C*-algebra $A$ (so also $B(\mathcal{H})$) onto itself. It follows that there exists a unique Jordan automorphism $T$ of $A$ such that

\begin{equation}
(T^*\omega)(a) = \omega(Ta)
\end{equation}

for all $\omega \in S$ and $a \in A$.

Recall that any Jordan isomorphism can be split into the sum of *-isomorphism and *-anti-isomorphism. However, there is a possibility to distinguish between *-isomorphism and *-anti-isomorphism on the Schroedinger picture level. Namely, employing the geometrical structure introduced in Section 2, one has:

**Proposition 3.5.** (Alfsen, Shultz [3]) Let $\Phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ be a Jordan isomorphism. Then $\Phi$ is a *-isomorphism if and only if it preserves orientation, and $\Phi$ is a *-anti-isomorphism if and only if it reverses orientation.

Summarizing this section one has:

1. Plain positive maps as well as invertible dynamical maps are not sufficiently sensitive to the facial structure of states when one passes from Heisenberg’s picture to Schroedinger’s. This follows from the fact that for plain positive maps, the ordered Banach space framework was used while for invertible maps, in Kadison theorem, the Jordan structure was indispensable ingredient.

2. On the other hand, for decomposable positive maps the facial structure is essential.

3. *-morphisms and *-anti-morphisms can be distinguished on the set of states; in this case the geometrical structure of states plays again crucial role.

4. Let the time evolution be given in terms of a group. Then, the continuity properties of the group strengthen the conclusions stemming from Kadison’s result for hamiltonian type dynamics (cf. Theorem 3.4). Namely, one parameter group of affine maps on $S$ with
suitably strong continuity properties gives rise to a group of \(*\)-automorphisms and not merely to Jordan automorphisms (see \cite{6}). Consequently, to guarantee the equivalence of the description of positive maps in both pictures the Schroedinger picture should be equipped with the additional geometrical structure described in Section 2. This conclusion is all the more interesting in view of the fact that quantum computing needs decomposable maps. In particular, a description of entangled states may appeal to the specific geometrical features of the set of states. Finally, to comment (3) and (4) we note that to have the equivalence between Schroedinger’s and Heisenberg’s pictures one should be able to determine the associative product on the set observables not merely the Jordan structure. But, the observables in quantum mechanics are (quantum) random variables with a specified probability distribution for each state. However, to determine evolution of observables (so to define non-commutative derivations as for example in the Heisenberg equation) one needs the Lie product. On the other hand, the Lie product with the Jordan product determine the associative product of the algebra of observables. This clarifies the role of orientations (cf also Proposition 3.5). As in this paper, the evolution of quantum systems will be not studied we skip the details.

4. FACIAL STRUCTURES FOR STATES AND POSITIVE MAPS.

Having noted that the facial structure plays an essential role in the characterization of the set of all states, we turn to discussing the facial structures of positive maps and their relations to the corresponding structures of states. Throughout this Section we assume finite dimensionality of \( \mathcal{H} \) and consider \( B(\mathcal{H}) \), i.e. \( M_n(\mathbb{C}) \) where \( n = \text{dim}\mathcal{H} \). First, we wish to indicate relations between projections in the set of observables, so projections in \( M_n(\mathbb{C}) \) for the considered case, and faces of positive and completely positive maps, so faces in \( \mathcal{P} \) and \( \mathcal{C}\mathcal{P} \), i.e. throughout this section we are going to consider positive maps which are not necessary unital. The relations between facial structures of positive maps and of states are expected due to Theorem 2.2 and Corollary 3.3. We begin with the following result:

**Theorem 4.1** (Kye \cite{27}). Denote by \( \mathcal{V} \) the complete lattice of all subspaces of the \( n \)-dimensional vector space \( \mathbb{C}^n \), and by \( \mathcal{J}(\mathcal{V}) \) the complete lattice of all homomorphisms from \( \mathcal{V} \) into itself. Then,
there is a well-defined homomorphism

\[ \phi : \mathcal{F}(\mathcal{P}) \to \mathcal{J}(\mathcal{V}) \]

where \( \mathcal{F}(\mathcal{P}) \) is the complete lattice of all faces of \( \mathcal{P} \).

To give a more specialized result we need the concept of matricially convex faces in \( \mathcal{CP} \). Let \( T_i \in \mathcal{CP} \) and \( b_i \in M_n(\mathbb{C}) \) for \( 1 \leq i \leq p \). Then, a completely positive map \( \sum_{i=1}^{p} b_i^* \cdot T_i \cdot b_i \) may be defined on \( M_n(\mathbb{C}) \) by

\[
\sum_{i=1}^{p} b_i^* \cdot T_i \cdot b_i(a) = \sum_{i=1}^{p} b_i^* T_i(a) b_i
\]

for all \( a \in M_n(\mathbb{C}) \).

**Definition 4.2.** A subset \( \mathcal{V} \subset \mathcal{CP} \) is called matricially convex (see [5], [30]) if for \( T_i \in \mathcal{V}, 1 \leq i \leq p \) and for all \( b_i \in M_n(\mathbb{C}) \) such that \( \sum_{i=1}^{p} b_i^* b_i = I \) it follows that

\[
\sum_{i=1}^{p} b_i^* T_i b_i \in \mathcal{V}.
\]

One has

**Theorem 4.3** (Smith, Ward [41]). There is a one-to-one correspondence between matricially convex faces in \( \mathcal{CP} \) and faces in the state space \( \mathcal{S} \).

Now it is clear that this result combined with Theorem 2.2 says that there are “more” faces in \( \mathcal{CP} \) than projectors in \( M_n(\mathbb{C}) \). Hence, it is natural to restrict the class of faces which we are interested in. Following this idea we turn to a characterization of all maximal faces of \( \mathcal{P} \) and \( \mathcal{CP} \).

We begin with

**Proposition 4.4.** (Kye [27]) Every maximal face of \( \mathcal{P} \) is of the form

\[
F_{\text{max}}(p_\xi, \eta) = \{ T \in \mathcal{P}; T(p_\xi) \eta = 0 \}.
\]

where \( p_\xi \) is a one dimensional projection on \( \xi \) and \( \eta \) is another nonzero vector. Moreover, if \( F_1 \) and \( F_2 \) are two maximal faces of \( \mathcal{P} \) then they are affine isomorphic to each other.
Consequently, any maximal face of $\mathcal{P}$ corresponds to a pair of one dimensional subspaces in $\mathbb{C}^n$. The maximal faces in $\mathcal{CP}$ are characterized by

**Proposition 4.5. (Kye [28])** Every maximal face of $\mathcal{CP}$ is of the form

\[ F_{\text{max}}(V) = \{TW \in \mathcal{CP}; W \subset V^\perp \}. \] \hspace{1cm} (4.3)

where $V \in M_n(\mathbb{C})$ and $\perp$ is understood in the sense of the inner product $<V, W> = Tr(W^*V)$. $T_W$ and $W$ are defined via relation (3.1) with $\tau_i$ *-homomorphisms.

Consequently, in the finite dimensional case, there is a complete characterization of maximal faces in $\mathcal{P}$ and $\mathcal{CP}$. Moreover, in these cases every face of $\mathcal{D}$ is the convex hull of a face of $\mathcal{CP}$ and a face of completely copositive maps $^{3}$ co-$\mathcal{CP}$ (cf [26]). Hence, also one has as a corollary

**Corollary 4.6.** $\mathcal{D}$ is a convex hull of $\{ F_{\text{max}}(V_1), \tau \circ F_{\text{max}}(V_2); V_1, V_2 \in M_n(\mathbb{C}) \}$ where $\tau$ stands for the transposition.

The relation between maximal faces (4.2) of $\mathcal{P}$ and maximal faces (4.3) of $\mathcal{CP}$ is given by

**Proposition 4.7. (Kye [28])** Let $V = |\xi><\eta|$ with unit vectors $\xi, \eta \in \mathbb{C}^n$. Then one has the identity

\[ F_{\text{max}}(V) = F_{\text{max}}(p_{\xi}, \eta) \cap \partial \mathcal{CP}. \] \hspace{1cm} (4.4)

where $\partial \mathcal{CP}$ stands for the boundary of $\mathcal{CP}$. Moreover, for such $V$

\[ F_{\text{max}}(V) \subseteq F_{\text{max}}(p_{\xi}, \eta). \] \hspace{1cm} (4.5)

Again, there is a more specialized result, see [25]. Namely, denote by $\mathcal{P}_k$ the convex cone of all $k$-positive maps from $M_n(\mathbb{C})$ into $M_n(\mathbb{C})$. Kye has shown that every maximal face of $\mathcal{P}_k$ corresponds to an $n \times n$ matrix whose rank is less or equal to $k$. Hence, the number of maximal faces of $\mathcal{P}_k$ grows with $k$. However, the number of maximal faces of $\mathcal{P}_k$ which are contained in the boundary of $\mathcal{P}$ is constant and determined by matrices of rank one (see Corollary 3.2 in [25]).

We end this section with another Kye’s result:

---

$^{3}$Completely copositive map is the composition of transposition with a CP map.
Proposition 4.8. (Kye [27]) For a positive linear map $T \in \mathcal{P}$, the following are equivalent:

- $T$ is an interior point of $\mathcal{P}$.
- $T(p_\xi)$ is nonsingular for each one-dimensional projection $p_\xi \in \mathcal{B}(\mathbb{C}^n) \equiv M_n(\mathbb{C})$.

Consequently, interior points of $\mathcal{P}$ are “far” from $F_{\text{max}}(p_\xi, \eta)$.

5. POSITIVE MAPS AND LOCALLY DECOMPOSABLE MAPS

In this Section we outline briefly the general construction of a linear positive map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ with an emphasis on the local decomposability and extreme positive maps. Here, again, $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional Hilbert spaces of dimension greater than 1.

For any $x \in \mathcal{H}$ we define the linear operator $V_x : \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ by $V_x z = x \otimes z$ for $z \in \mathcal{K}$. By $e_{x,y}$, where $x, y \in \mathcal{H}$, we denote the one dimensional operator on $\mathcal{H}$ defined by $e_{x,y} u = (y, u)x$ for $u \in \mathcal{H}$, i.e. $e_{x,y} \equiv |x><y|$. For simplicity reasons, if $\{v_i\}_{i=1}^n$ is a basis in $\mathcal{H}$, we will write $V_i$ and $e_{i,j}$ instead of $V_{v_i}$ and $e_{v_i,v_j}$ for any $i, j = 1, 2, ..., n$ when no confusion can arise.

Let $H \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Define $T_H : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ as follows

\begin{equation}
T_H(e_{x,y}) = V_x^* HV_y,
\end{equation}

where $x, y \in \mathcal{H}$. It was Choi [7], who firstly discovered correspondences among various types of $H \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ and classes of linear positive maps $T_H$ (see also [16]). We will need the following result (cf. [16], [35])

**Theorem 5.1.** If $H = H^*$ and $(x \otimes y, Hx \otimes y) \geq 0$ for any $x \in \mathcal{H}$ and $y \in \mathcal{K}$ then $T_H$ is a positive map. Moreover, for any positive map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ there exists uniquely determined selfadjoint operator $H \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ with the property $(x \otimes y, Hx \otimes y) \geq 0$ for any $x \in \mathcal{H}$ and $y \in \mathcal{K}$, such that $T = T_H$.

It should be mentioned that Choi [7] proved the following remarkable result concerning complete positive maps: $T_H$ is a completely positive map if and only if $H$ is a positive operator; so not only “block-positive” as in Theorem 5.1.
The important point to note here is that there is an explicit relation between $H$ and $T$. Namely, (cf. [16], [35]) suppose $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is any positive map and define

\[ H = (\text{id} \otimes T) \left( \sum_{kl} |\xi_k><\xi_l| \otimes |\xi_k><\xi_l| \right), \]

for a basis $\{\xi_j\}$ in $\mathcal{H}$. For any $y, w \in \mathcal{K}$ we have

\[ (y, T_H(e_{ij})w) = (y, V_{\xi_i}^*HV_{\xi_j}w) = \sum_{kl} (\xi_i, |\xi_k><\xi_l|)(y, T(|\xi_k><\xi_l|w) = (y, T(e_{ij})w), \]

where $e_{ij} \equiv |\xi_i><\xi_j|$.

In the sequel, we will need another very important property of positive maps. This property, called local decomposability, is defined as follows (cf. [12]):

**Definition 5.2.** A linear map $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is locally decomposable if for $0 \neq x \in \mathcal{H}$, there exists a Hilbert space $\mathcal{K}_x$, a bounded operator $W_x : \mathcal{K}_x \to \mathcal{H}$ and a $\mathbb{C}^*$-homomorphism (equivalently, Jordan homomorphism) $\pi_x$ of $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K}_x)$ such that

\[ W_x\pi_x(a)W_x^*x = \tau(a)x, \]

for all $a \in \mathcal{B}(\mathcal{H})$.

It was Størmer who proved

**Theorem 5.3.** (Størmer [12]) Every bounded positive linear map of a C*-algebra $A$ into the bounded operators on a Hilbert space $\mathcal{H}$ is locally decomposable. Moreover, if $a$ in (5.4) is selfadjoint, then $\pi$ (in (5.4)) can be taken to be $^*$-morphism.

Hence, every positive linear map is locally decomposable, but in 2D-case (two dimensional) the notions of decomposability and local decomposability are the same as exactly for this case every positive map is decomposable one, see [10], [12]. Going to higher dimensions, so for nD-case (n dimensional) with $n > 2$, there are non-decomposable maps which are only locally decomposable. This explains our remark given in Introduction that properties of positive maps for 2-level systems and 3-level systems are dramatically different. We claim that to understand this difference we should use the facial geometry of the underlying convex structures (presented above). This will be done in the next Section.
However, we want to close this Section with another Størmer’s result. He obtained in 2D-case (and only for this case) the classification of all extreme points in $\mathcal{P}$.

**Theorem 5.4.** (Størmer [42]) Let $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a positive map. Then $T$ is extreme if and only if $T$ is unitarily equivalent to a map of the form

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \mapsto \begin{pmatrix}
a & \alpha b + \beta c \\
\alpha c + \beta b & \gamma a + \epsilon b + \tau c + \delta d
\end{pmatrix}
$$

where $|\epsilon|^2 = 2\gamma(\delta - |\alpha|^2 - |\beta|^2)$ in the case when $\gamma \neq 0$, and $|\alpha|$ or $|\beta|$ equals 1 when $\gamma = 0$. In the former case, $|\alpha| + |\beta| = \delta^{1/2}$.

6. LOW DIMENSIONAL CASES: $M_2(\mathbb{C})$ AND $M_3(\mathbb{C})$

To understand the phenomenon of non-decomposable maps we should firstly recognize the meaning of locally decomposable maps, see Definition 5.2 and Theorem 5.3. To this end we will compare the facial structure of 2D-case with that for 3D-case. Let us start with 2D-case. The maximal faces of $\mathcal{P}$ are characterized by Proposition 4.4. We wish to combine the general form of a maximal face of $\mathcal{P}$ with the local decomposability. Assume that a unital positive map $T$ is in a fixed arbitrary maximal face, i.e. $T \in F_{\text{max}}(p_\xi, \eta)$. Define a functional $\phi_T : M_2(\mathbb{C}) \to \mathbb{C}$ such that

$$
\phi_T(\cdot) = (\eta, T(\cdot)\eta).
$$

Following the GNS recipe one has

$$
\mathcal{H}_{\phi_T} = M_2(\mathbb{C})/\mathcal{L}_{\phi_T}
$$

where $\mathcal{L}_{\phi_T} = \{ a \in M_2(\mathbb{C}) : \phi_T(a^*a) = 0 \} = M_2(\mathbb{C})p$ for an orthogonal projector $p \in M_2(\mathbb{C})$. The definition of $F_{\text{max}}(p_\xi, \eta)$ implies that $p = p_\xi$. Hence $\mathcal{H}_{\phi_T} = \mathcal{H}_{\phi_{T'}}$, provided that $T, T' \in F_{\text{max}}(p_\xi, \eta)$. Furthermore, the $C^*$-homomorphism $\pi$ (cf. Definition 5.2) is the same map for all positive maps in the fixed face. More precisely, one can define $\mathcal{R}_{\phi_T} = \{ a \in M_2(\mathbb{C}) : \phi_T(aa^*) = 0 \}$. $\mathcal{R}_{\phi_T}$ is a right ideal. By $\mathcal{H}^0_{\phi_T}$ we denote the quotient space $M_2(\mathbb{C})/\mathcal{R}_{\phi_T}$. For any $a \in M_2(\mathbb{C})$ we write $[a]$ and $[a]_r$ the equivalence classes (see A20) of $a$ in $\mathcal{H}_{\phi_T}$ and $\mathcal{H}^0_{\phi_T}$ respectively. For simplicity we will write $[a]$ instead of $[a] \oplus [a]_r$ for $a \in M_2(\mathbb{C})$. Next, let $\mathcal{K}_\eta = \mathcal{H}_{\phi_T} \oplus \mathcal{H}^0_{\phi_T}$. $W_\eta$
and $\pi_\eta$ are given by

\begin{equation}
\pi_\eta(a)([b_1]_l \oplus [b_2]_r) = [ab_1]_l \oplus [b_2a]_r, \quad a, b_1, b_2 \in M_2(\mathbb{C});
\end{equation}

\begin{equation}
W_\eta\pi_\eta(a)[I] = T(a)\eta.
\end{equation}

Consequently, we are able to write all ingredients of local decomposability in explicit way. However, to obtain decomposability within the Størmer construction one should add the additional condition (see [36]). To present this result we need some notations. If $\xi$ and $\eta$ are arbitrary unit vectors in $\mathbb{C}^2$ then let $\xi_1, \xi_2$ be an orthonormal basis in $\mathbb{C}^2$ such that $\xi_1 = \xi$, $\xi_2 = \xi^\perp$ and similarly $\eta_1, \eta_2$ be a basis such that $\eta_1 = \eta$. Again, by $e_{ij}$ we denote the operator $|\xi_i><\xi_j|$ for $i, j = 1, 2$.

**Proposition 6.1.** Suppose a unital positive map $T \in F_{\text{max}}(p_\xi, \eta)$. Let $K_\eta$, $W_\eta$ and $\pi_\eta$ be as in (6.3) (and described by (6.3) - (6.4)). Then the condition for decomposability

\begin{equation}
T(a) = W_\eta\pi_\eta(a)W_\eta^*, \quad a \in M_2(\mathbb{C})
\end{equation}

is satisfied if and only if

\begin{equation}
\text{Tr}\{T(e_{12})\} = \text{Tr}\{T(e_{21})\} = 0, \quad \text{Tr}\{T(e_{22})\} = 1,
\end{equation}

\begin{equation}
\text{Tr}\{T(e_{11})\} = 2 \left( |<\eta_2, T(e_{12})\eta_1|^2 + |<\eta_2, T(e_{21})\eta_1|^2 \right).
\end{equation}

This result clearly shows that even in the simple 2D case, local decomposability does not lead directly to decomposability (we recall that in 2D case each positive map is decomposable). However, for the considered case one can go one step further (see [36]). Namely, easy calculations lead to the explicit form $H_T = \sum_{i,j} e_{ij} \otimes T(e_{ij})$ (cf. (5.2)) in the basis $\{\xi_i \otimes \eta_k\}$. One has

\begin{equation}
H_T = \begin{pmatrix}
0 & 0 & 0 & y \\
0 & \lambda & \frac{z}{\lambda} & t \\
0 & z & 1 & 0 \\
\frac{y}{\lambda} & \frac{z}{t} & 0 & 1 - \lambda \\
\end{pmatrix}
\end{equation}

where $\lambda \in [0, 1]$ and for any $x, y \in \mathbb{C}^2$.
\[ \begin{align*}
\lambda &|\langle \xi_1, x \rangle^2|(|v, \eta_2|)^2 + |\langle \xi_2, x \rangle^2|(|v, \eta_1|)^2 \\
+ (1 - \lambda) &|\langle \xi_2, x \rangle^2|(|v, \eta_2|)^2 + 2\Re \{ \langle x, \xi_1 \rangle \langle \xi_2, x \rangle \langle v, \eta_2 \rangle \langle v, \eta_1 \rangle \} \\
\geq & -2\Re \{ \langle x, \xi_1 \rangle \langle \xi_2, x \rangle \langle y, v, \eta_1 \rangle \langle \eta_2, v \rangle + 2\Re \{ \langle x, \xi_1 \rangle \langle \xi_2, x \rangle \langle v, \eta_1 \rangle \langle \eta_2, v \rangle \} \}.
\end{align*}\]

Moreover, these calculations give the following explicit form of a map in the maximal face:

\[ \begin{align*}
T(|\xi_1 \rangle &|\xi_1 \rangle) = \lambda|\eta_2 \rangle|\eta_2 \rangle, \\
T(|\xi_1 \rangle &|\xi_2 \rangle) = y|\eta_1 \rangle|\eta_2 \rangle + \Re \{ \langle \xi_2, x \rangle \langle v, \eta_1 \rangle \langle y, v \rangle \} + t|\eta_2 \rangle|\eta_2 \rangle, \\
T(|\xi_2 \rangle &|\xi_2 \rangle) = |\eta_1 \rangle|\eta_1 \rangle + (1 - \lambda)|\eta_2 \rangle|\eta_2 \rangle.
\end{align*}\]

where, we recall, \( \xi_1 \equiv \xi, \xi_2 \equiv \xi^\perp \), analogously for \( \eta \)'s. Numbers \( \lambda, z, y, \) and \( t \) satisfy a condition of the type (6.9).

The important point to note here is the rather striking similarity between (6.8) and the Størmer result (5.5). Namely, the Choi’s matrix for extreme positive map has the form

\[
\begin{pmatrix}
1 & 0 & 0 & \alpha \\
0 & \gamma & \beta & \epsilon \\
0 & \beta & 0 & 0 \\
\Re & \gamma & 0 & \delta
\end{pmatrix}
\]

obviously, with the same conditions for \( \alpha, \beta, \gamma, \delta, \epsilon \) as these stated in Theorem 5.4.

Secondly, we note that LHS (6.8) does not depend on phases of the complex numbers \( \langle v, \eta_k \rangle \), \( k = 1, 2 \) while RHS (6.9) does. In particular, there are many vectors \( v \in \mathbb{C}^2 \) with the property that the coefficient of \( z \) (\( y \) respectively) in RHS (6.9) is equal to 0. This suggests the possibility of splitting the family of matrices (6.8) into two classes

\[ \begin{align*}
T(|\xi_1 \rangle &|\xi_1 \rangle) = \lambda|\eta_2 \rangle|\eta_2 \rangle, \\
T(|\xi_1 \rangle &|\xi_2 \rangle) = y|\eta_1 \rangle|\eta_2 \rangle + \Re \{ \langle \xi_2, x \rangle \langle v, \eta_1 \rangle \langle y, v \rangle \} + t|\eta_2 \rangle|\eta_2 \rangle, \\
T(|\xi_2 \rangle &|\xi_2 \rangle) = |\eta_1 \rangle|\eta_1 \rangle + (1 - \lambda)|\eta_2 \rangle|\eta_2 \rangle.
\end{align*}\]

with \( \lambda' \in [0, 1] \) and for any \( x, v \in \mathbb{C}^2 \)

\[ \begin{align*}
\lambda' &|\langle \xi_1, x \rangle^2|(|v, \eta_2|)^2 + q' |\langle \xi_2, x \rangle^2|(|v, \eta_1|)^2 + \left( \frac{1}{2} - \lambda' \right) |\langle \xi_2, x \rangle^2|(|v, \eta_2|)^2 \\
+ 2\Re \{ \langle x, \xi_1 \rangle \langle \xi_2, x \rangle \langle v, \eta_2 \rangle \langle 2t' \rangle \} &\geq -2\Re \{ \langle x, \xi_1 \rangle \langle \xi_2, x \rangle \langle y, v, \eta_1 \rangle \langle \eta_2, v \rangle \}.
\end{align*}\]
Ando-Choi, inequality leads to the following condition on $T$:

$$\lambda''(\xi_1, x)^2|((v, \eta_1)|^2 + q''((\xi_2, x)|^2((v, \eta_2))|((v, \eta_2))|((\xi_2, x))|((v, \eta_2))|^2$$

$$+2\text{Re}\{(x, \xi_1)((\xi_2, x)|^2|v'|^2\} \geq -2\text{Re}\{(x, \xi_1)((\xi_2, x)|^2|v^2\}$$

(6.17)

and $\lambda' + \lambda'' = \lambda$, $t' + t'' = t$, $q' + q'' = 1$.

The maps determined by matrices of the form (6.16) have a very interesting property. To describe this feature of the corresponding positive maps we recall Choi’s result saying (see Section 5) that a map determined by a positive matrix is completely positive, i.e., $T$ is a completely positive map if and only if the $2 \times 2$ operator matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
\lambda'' & 0 & 0 & 0 \\
0 & z & q'' & 0 \\
0 & \frac{1}{2} - \lambda'' & 0 & 0
\end{pmatrix}$$

where $\lambda'' \in [0, 1]$, for any $x, v \in \mathbb{C}^2$

is positive. We recall $e_{ij} = |\xi_i| < \xi_j|$. On the other hand, it is well-known [3], [8] (see also [4] where the matrix version of this inequality is described) that any matrix of the form (6.16) is positive if and only if $T(e_{11}), T(e_{22})$ are positive and $T(e_{11}) \geq T(e_{12})T(e_{22})^{-1}T(e_{12})^*$. Here if $T(e_{22})$ is not invertible $T(e_{22})^{-1}$ is understood to be its generalized inverse. The latter, Ando-Choi, inequality leads to the following condition on $\lambda'', z''$ and $t''$:

$$\lambda'' \geq (q'')^{-1}|z|^2 + \left(\frac{1}{2} - \lambda''\right)^{-1}|t''|^2.$$

(6.19)

On the other hand, Corollary 8.4 in [32] implies that the map $T_H$ (i.e. the map determined by the matrix $H$ of the form (6.10) is positive if and only if

$$|\gamma(x, \eta_2)(\eta_1, x) + t''|((\eta_2, x)|^2|^2 \leq \lambda''((\eta_2, x)|^2((q'')(\eta_1, x)|^2 + \left(\frac{1}{2} - \lambda''\right)((\eta_2, x)|^2$$

(6.20)

for any $x \in \mathbb{C}^2$. In particular

$$((\eta_1, x)||z| + (\eta_2, x)||t''|)^2 \leq \lambda''(q'')(\eta_1, x)|^2 + \left(\frac{1}{2} - \lambda''\right)((\eta_2, x)|^2$$

(6.21)

Without loss of generality we can assume $|(\eta_1, x)| \neq 0$. Let us define $\sigma = \frac{|(\eta_2, x)|}{|(\eta_1, x)|}$. Then

$$|z| + \sigma|(t''|)^2 \leq \lambda''(q'' + \left(\frac{1}{2} - \lambda''\right)\sigma^2$$

(6.22)
The only admissible case is when the discriminant of the quadratic equation (6.23) is negative, i.e. \( \Delta \leq 0 \). However, this implies

\[
q''|t''|^2 + (\frac{1}{2} - \lambda'')|z|^2 \leq q''\lambda''(\frac{1}{2} - \lambda'').
\]

But this means (cf. 6.19) that for the studied class of maps, positivity implies complete positivity.

Now, let us turn to maps determined by matrices of the form (6.14). The first easy observation says that an application of partial transposition to matrices of the form (6.14) leads to matrices of the form (6.16). But then combining the argument given in the preceding paragraph with the relation between the matrix \( H \) and the positive map \( T_H \) given in Section 5 one can conclude that matrices of the form (6.14) correspond to co-completely positive maps. Therefore, the considered splitting of matrix (6.8) corresponds to decomposition of a positive map into the sum of completely positive and completely co-positive maps provided that conditions 6.15 and 6.17 are satisfied.

Consequently, any unital positive map \( T \) in the face \( F_{max}(p_\xi, \eta) \) (cf Section 5) is decomposable one and the decomposition can be written explicitly. Clearly, this extends to a map \( S \in \mathcal{P} \) since such \( S \) is a convex combination of maps having the form 6.10 - 6.12. The important point to note here is the form of maps which constitute the discussed decomposition: both are not normalized, i.e. the summands do not preserve the identity. However, the summands are in the same face. Consequently, we got an indication that for explicit splitting of decomposable map the face structure is appearing as the natural one.

The presented decomposition of positive maps for 2D-case is not conclusive as there is still one unanswered question whether condition 6.9 implies 6.15 and 6.17 (the converse implication is easy). In other words, we wish to decompose any matrix (6.8) satisfying the general conditions (6.9). This can be done but, as we just learnt, there is a price to pay. Namely, the normalization is lost, i.e. in general, the summands in the decomposition do not preserve identity.
More precisely, we will show that (6.25) implies (6.14) and (6.17). To this end, multiplying (6.25) by $|y|\lambda^{-\frac{1}{2}}$ and assuming that $|y| + |z| = \lambda^\frac{1}{2}$ (this is the property characterizing an extremal positive map, cf Theorem 5.3), one can show that the matrix

\[
H_{T_1} = \begin{pmatrix}
0 & 0 & 0 & y \\
0 & \lambda_1 & 0 & t_1 \\
0 & 0 & a_1 & c \\
\overline{y} & \overline{t_1} & \overline{c} & b_1
\end{pmatrix}
\]

corresponds to the positive map $T_1$, as the following condition holds

\[
\lambda_1 |(\xi_1, x)|^2 |(v, \eta_2)|^2 + a_1 |(\xi_2, x)|^2 |(v, \eta_1)|^2
\]

\[
+ b_1 |(\xi_2, x)|^2 |(v, \eta_2)|^2 + 2 \text{Re} \{(x, \xi_1)(\xi_2, x)|(v, \eta_2)|^2 t_1 \}
\]

\[
\geq -2 \text{Re} \{(x, \xi_1)(\xi_2, x)y(v, \eta_1)(\eta_2, v) + c|(x, \xi_2)|^2(v, \eta_1)(\eta_2, v)\}.\]

(6.26)

where we put $a_1 = |y|\lambda^{-\frac{1}{2}}, \lambda_1 = |y|\lambda^\frac{1}{2}, t_1 = \frac{t}{2}, b_1 = |z|\lambda^{-\frac{1}{2}}(1 - \lambda)$, and finally $c \in \mathbb{C}$.

Repeating this argument, i.e. multiplying (6.25) by $|z|\lambda^{-\frac{1}{2}}$ and again assuming that $|y| + |z| = \lambda^\frac{1}{2}$, one can show that the matrix

\[
H_{T_2} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & t_2 \\
0 & z & a_2 & -c \\
0 & \overline{t_2} & \overline{-c} & b_2
\end{pmatrix}
\]

corresponds to the positive map $T_2$, as the following condition holds

\[
\lambda_2 |(\xi_1, x)|^2 |(v, \eta_2)|^2 + a_2 |(\xi_2, x)|^2 |(v, \eta_1)|^2
\]

\[
+ b_2 |(\xi_2, x)|^2 |(v, \eta_2)|^2 + 2 \text{Re} \{(x, \xi_1)(\xi_2, x)|(v, \eta_2)|^2 t_2 \}
\]

\[
\geq -2 \text{Re} \{(x, \xi_1)(\xi_2, x)z(v, \eta_2)(\eta_1, v) - c|(x, \xi_2)|^2(v, \eta_1)(\eta_2, v)\}.\]

(6.28)

where we put $a_2 = |z|\lambda^{-\frac{1}{2}}, \lambda_2 = |z|\lambda^\frac{1}{2}, t_2 = \frac{t}{2}, b_2 = |y|\lambda^{-\frac{1}{2}}(1 - \lambda)$, and finally $c \in \mathbb{C}$.

Clearly, $\lambda_1 + \lambda_2 = \lambda, a_1 + a_2 = 1, t_1 + t_2 = t$, and $b_1 + b_2 = 1 - \lambda$. Consequently, (6.25) and (6.27) gives the desired splitting of (6.8). Furthermore, if $c$ satisfies

\[
\frac{\overline{y_1} \lambda^{-\frac{1}{2}} t_1}{y_1} = y_1 \lambda^\frac{1}{2} \sigma,
\]

(6.29)
where \( y_1^2 = y \) then the matrix 6.27 is positive, thus \( T_2 \) is CP map. Similarly, for the special choice of \( c \), the matrix 6.25 corresponds to co-CP map. As a result, whenever \( \lambda, y, z \) are not equal to 0 we obtained the unique decomposition of an extremal positive map into the sum of CP and co-CP maps but both, in general, are not normalized. This is indicated by the fact that both matrices contain, in general, \( c \neq 0 \), and the sum of diagonal elements does not need to be 2.

Now, before turning to 3D-case, let us consider unital positive maps from \( M_2(\mathbb{C}) \to M_3(\mathbb{C}) \). Again, our starting point is the explicit form of maximal faces \( F_{\max}(p_\xi, \eta) \) in \( \mathcal{P} \) (cf Section 4). Let us take a unital positive map \( T \in F_{\max}(p_\xi, \eta) \) and pick up two bases \( \{\xi_k\}_{k=1}^2 \) and \( \{\eta_l\}_{l=1}^3 \) in \( \mathbb{C}^2 \) and \( \mathbb{C}^3 \) such that \( \xi_1 \equiv \xi \) and \( \eta_1 \equiv \eta \) respectively. We observe

\[
\eta_1 = T(I)\eta_1 = \sum_{k=1}^2 T(p_\xi_k)\eta_1 = T(p_{\xi_1})\eta_1.
\]

We can conclude from (6.30) as well as from the given description of maximal face that the explicit form of Choi’s matrix \( H_T \) (for the normalized map \( T \)) is:

\[
H_T = \begin{pmatrix}
0 & 0 & 0 & 0 & v_{12} & v_{13} \\
0 & a & c & v_{21} & v_{22} & v_{23} \\
0 & b & v_{31} & v_{32} & v_{33} \\
\frac{v_{12}}{v_{11}} & \frac{v_{22}}{v_{11}} & \frac{v_{32}}{v_{11}} & 1 & 0 & 0 \\
\frac{v_{13}}{v_{11}} & \frac{v_{23}}{v_{11}} & \frac{v_{33}}{v_{11}} & 0 & 1 - a & -c \\
\end{pmatrix} \equiv \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22} \\
\end{pmatrix}
\]

where \( a, b \) are non-negative numbers, \( c, v_{ij} \) are in \( \mathbb{C} \), \( v_{11} = 0 \) due to the block-positivity, \( |c|^2 \leq ab \) and \( v_{ij} \) satisfy the condition of the type 6.39 (now a little bit complicated). Finally, the last equality says that we partitioned the matrix \( H_T \), i.e. \( H_T \) is written as a \( 2 \times 2 \) square matrix with entries \( A_{ij} \in M_3(\mathbb{C}) \). We note that \( A_{21} = A_{12}^* \). We recall that the decomposability of any unital positive map holds for this case (see 40). Furthermore, in terms of Choi’s matrix, it means that there are two block-matrices splitting 6.31. The first matrix has the form

\[
\begin{pmatrix}
A_{11}^f & A_{12}^f \\
A_{21}^f & A_{22}^f \\
\end{pmatrix}
\]

where \( A_{11}^f \) and \( A_{22}^f \) are positive semidefinite matrices with \( A_{12}^f \) satisfying

\[
|(x, A_{12}^f y)| \leq \|(A_{11}^f)^{1/2} x\| \cdot \|(A_{22}^f)^{1/2} y\|,
\]
for any \( x, y \in \mathbb{C}^3 \). The second block-matrix is of the form

\[
\begin{pmatrix}
A_{11}^{II} & A_{12}^{II} \\
A_{21}^{II} & A_{22}^{II}
\end{pmatrix}
\]

where \( A_{11}^{II} \) and \( A_{22}^{II} \) are positive semidefinite matrices. Furthermore, \( A_{12}^{II} \) satisfies

\[
|\langle x, (A_{12}^{II})^* y \rangle| \leq \|(A_{11}^{II})^{\frac{1}{2}} x\| \cdot \|(A_{22}^{II})^{\frac{1}{2}} y\|,
\]

for any \( x, y \in \mathbb{C}^3 \), and \( A_{ij} = A_{ij}^I + A_{ij}^{II} \).

Now, we are in position to consider 3D-case. Let us consider a unital positive map \( T \in \mathcal{F}_{\text{max}}(p_\xi, \eta) \) and pick up two bases \( \{\xi_k\}_{k=1}^3 \) and \( \{\eta_l\}_{l=1}^3 \) in \( \mathbb{C}^3 \) such that \( \xi_1 \equiv \xi \) and \( \eta_1 \equiv \eta \) respectively. We observe

\[
\eta_1 = T(1)\eta_1 = \sum_{k=1}^3 T(p_{\xi_k})\eta_1 = T(p_{\xi_2})\eta_1 + T(p_{\xi_3})\eta_1.
\]

We can only conclude from (6.36) that \( T(p_{\xi_k}) \), \( k = 2, 3 \) are positive operators in \( \mathcal{B}(\mathbb{C}^3) \) such that their sum \( T(p_{\xi_2}) + T(p_{\xi_3}) \) has \( \eta_1 \) as its eigenvector. The Choi’s matrix \( H_T \) (for the considered map \( T \)) is given by:

\[
H_T = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{pmatrix}
\]

where \( A_{ij}, i, j = 1, 2, 3 \), are \( 3 \times 3 \) matrices such that

\[
\begin{pmatrix}
\langle x, A_{11} x \rangle & \langle x, A_{12} x \rangle & \langle x, A_{13} x \rangle \\
\langle x, A_{21} x \rangle & \langle x, A_{22} x \rangle & \langle x, A_{23} x \rangle \\
\langle x, A_{31} x \rangle & \langle x, A_{32} x \rangle & \langle x, A_{33} x \rangle
\end{pmatrix}
\]

is positive semidefinite matrix for any \( x \in \mathbb{C}^3 \) (in particular, \( A_{kk} \) are positive semidefinite matrices). The formulae (6.37) and (6.31) suggest that the method only based on matricial analysis of the Choi operator becomes too complicated to be effectively used for understanding the nature of non-decomposable maps; there are too many variables. Hence, we will exploit the geometrical structure described in Sections 2 and 3. To this end we begin with remark that the indicated partitioning of the matrix (6.37) corresponds to the separation of the matrix \( \{A_{ij}\}_{i,j=1}^2 \) which can be attributed to family of all maps: \( M_2(\mathbb{C}) \to M_3(\mathbb{C}) \). As it was mentioned, unital positive maps:
M_2(\mathbb{C}) \to M_3(\mathbb{C}) are decomposable with the transposition associated to the basis \( \xi_i \otimes \eta_k \), where \( i = 1, 2 \) and \( k = 1, 2, 3 \). If one could expect decomposability for 3D-case (we know that there are counterexamples, see [7, 46]) the corresponding transposition would be associated with the basis \( \xi_i \otimes \eta_k \), where \( i = 1, 2, 3 \) and \( k = 1, 2, 3 \). In general these transpositions do not need to co-operate well. Hence, one guesses that decomposable maps are, somehow, more regular than plain positive maps. Let us examine this question in detail.

A decomposable map \( \alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) can be written as the composition of a Jordan morphism \( \tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) with the map \( U_W, U_W(b) \equiv W^*bW, b \in \mathcal{B}(\mathcal{K}) \) where \( W : \mathcal{H} \to \mathcal{K} \), i.e. \( \alpha = U_W \circ \tau \) (see also Section 3)\(^4\). We know that Jordan morphisms are regular in the sense that they respect certain properties of the facial structure, see Theorem 3.2 and Corollary 3.3. Thus, to examine the regularity of decomposable maps we should carefully study the composition of \( U_W \)-maps with Jordan morphisms. Denote \( \tau(1) \) by \( Q \in \mathcal{B}(\mathcal{K}) \). As Jordan morphisms send (orthogonal) projectors into (orthogonal) projectors, \( Q \) is a projector. We put \( K_0 \equiv QK \) and observe

\[
(6.39) \quad \tau(a) = \tau(1 \cdot a) = \frac{1}{2} \tau(\{1, a\}) = \frac{1}{2} \{\tau(1), \tau(a)\} = \frac{1}{2} Q \tau(a) + \frac{1}{2} \tau(a) Q.
\]

Hence \( \tau(a)Q = Q \tau(a)Q \) and \( Q \tau(a) = Q \tau(a)Q \). Thus \( [Q, \tau(a)] = 0 \) for any \( a \in \mathcal{B}(\mathcal{H}) \) and one can restrict oneself to unital Jordan morphism, which will be also denoted by \( \tau \). To see this, we note that, for any \( f \in \mathcal{H} \) and \( a \in \mathcal{B}(\mathcal{H}) \) one has

\[
\alpha(a)f = W^*\tau(a)Wf = \alpha(a \cdot 1)f = W^*\tau(a \cdot 1)Wf = \frac{1}{2} W^*\tau(\{a, 1\})Wf
\]

\[
(6.40) \quad = \frac{1}{2} W^*\{\tau(a), Q\}Wf = W^*Q \tau(a)QWf = W^*Q \tau(a)Qf,
\]

where \( W_Q \equiv QW \) is an isometry. Consequently, it is enough to consider \( \tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(K_0), W : \mathcal{H} \to K_0 \) such that \( W^*W = 1 \) (so for simplicity of notation we drop the subscript \( Q \) and put \( Q = 1 \)).

Having well described the facial structure of the set of all states (see Sections 2 and 3) we wish to examine the regularity of decomposable map, in the Schroedinger picture, with respect to this structure. To this end, we denote by \( \rho \in \mathcal{B}(\mathcal{H}) \) the density matrix determining a state \( \phi \in \mathcal{B}(\mathcal{H})^* \).

\[^4\] At first sight this form of decomposable maps can be taken as the particular case of definition given in Section 3. However, this is not true as the Hilbert space \( \mathcal{K} \) can be taken large enough to take into account all summands given in right hand side of formula \(^{3,1} \).
One has

\[(6.41) \quad U_W : \mathcal{B}(\mathcal{K}_0) \to \mathcal{B}(\mathcal{H}); \quad U^*_W : \mathcal{B}(\mathcal{H})^* \to \mathcal{B}(\mathcal{K}_0)^* \]

and

\[(6.42) \quad (\phi \circ U_W)(b) = \text{Tr}_H(\rho W^* b W) = \text{Tr}_{\mathcal{K}_0}(W \rho W^* b) = (U^*_W \phi)(b) \]

where \(b \in \mathcal{B}(\mathcal{K}_0)\).

Let \(\rho \in F\) where \(F \subset \mathcal{S}(\mathcal{B}(\mathcal{H}))\) is a face. We recall (see Section 2) that each face in the set of all states (for the finite dimensional case each state is a normal one) is a projective face, i.e. there exists an (orthogonal) projector \(p\) such that (see Section 2 or \[3\] for a recent account of the theory):

\[(6.43) \quad F \equiv F_p = \{ \phi \in \mathcal{S}(\mathcal{B}(\mathcal{H})); \quad \phi(p) = 1. \} \]

On the other hand, let \(p\) be a one dimensional projector, i.e. \(p = |f><f| \equiv p_f\) for some \(f \in \mathcal{H}\). We observe

\[(6.44) \quad W p W^* = |W f><W f| \equiv |\xi><\xi| \equiv p \xi \]

and

\[(6.45) \quad ||W f||^2 = (W f, W f) = (W^* W f, f) = ||f||^2 \]

Consequently, \(U^*_W(p_f)\) is a pure state.

Now, let us turn to an analysis of Jordan morphism. We begin with the Heisenberg picture and recall (see \[6, 21\]) for a Jordan morphism \(\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}_0)\) there exists a central projection \(z \in \tau(\mathcal{B}(\mathcal{H}))' \cap \tau(\mathcal{B}(\mathcal{H}))''\) such that

\[a \mapsto \tau(a) z\]

is a morphism, and

\[a \mapsto \tau(a)(1 - z)\]

is an antimorphism.
Hence, passing to the Schroedinger picture, for a pure state \( \varrho = p_f \equiv |f><f|, f \in \mathcal{H} \), and any \( a \in B(\mathcal{H}) \) one has

\[
Tr\{ p_f \alpha(a) \} = Tr\{ p_f U_W \circ \tau(a) \} = Tr\{ W p_f W^* \tau(a)z \} + Tr\{ W p_f W^* \tau(a)(\mathbb{I} - z) \}
\]

\[
= Tr\{ z p_\xi z \tau(a) \} + Tr\{ (\mathbb{I} - z) p_\xi (\mathbb{I} - z) \tau(a) \} = \lambda Tr\{ \tau_1^*(p_\xi_1) a \} + \lambda' Tr\{ \tau_2^*(p_\xi_2) a \}
\]

(6.46)

where \( \xi_1 \equiv ||z\xi||^{-1}z\xi, \xi_2 \equiv ||(\mathbb{I} - z)\xi||^{-1}(\mathbb{I} - z)\xi, \tau_1(\cdot) \equiv \tau(\cdot)z, \tau_2(\cdot) \equiv \tau(\cdot)(\mathbb{I} - z), \lambda \equiv ||z\xi||^2 \), and \( \lambda' \equiv ||(\mathbb{I} - z)\xi||^2 \). We note that the Pythagorean theorem implies

(6.47)

\( \lambda + \lambda' = 1 \).

Having this explicit form of decomposable maps we can examine regularity properties of these maps with respect to the facial structure (cf Section 2). In particular, we are in position to compare 2D and 3D cases and to explain the striking difference between these two cases. We begin with 2D case for which there is only one non-trivial family of projectors: one dimensional ones. Moreover, each non-trivial face (in the set of all states) is determined by such projector (cf. 6.43). On the other hand, 6.46 and 6.47 imply that in the Schroedinger picture any genuine decomposable map, i.e., neither plain morphism nor plain antimorphism, sends a non-trivial face (one dimensional projector) to the convex combination of two projectors. We know (cf. Section 2) that the smallest face containing two (one dimensional) projectors is a 3-ball. But a 3-ball for 2D case is just the set of all states! Therefore, there is no room for other maps and we arrive to the geometrical explanation why any positive map, in 2D case, is decomposable.

Turning to 3D-case, our first observation is that this case is equipped with much “richer” geometry. Namely, there are two non-trivial families of projectors: one and two dimensional ones. Therefore, the facial structure of 3D system is richer. A repetition of the argument based on 6.46 and 6.47 says that in the Schroedinger picture, the non-trivial decomposable map sends one-dimensional projective faces into the 3-balls, which is not the set of all states for this case! Similar analysis performed, now, for projective faces determined by two dimensional projections shows global invariance of the family of projective faces (what is expected! cf Theorem 3.2). Therefore, a certain ingredients of the facial structure are preserved what is not expected for any plain positive
map. Consequently, there is room for more general maps than decomposable ones. This explains, from the geometrical point of view why non-decomposable maps can appear in 3D-case.

7. POSITIVE MAPS VERSUS ENTANGLEMENT

Positive maps as well as quantum correlations exhibit their non-trivial features only when they are defined on non-commutative structures, so in the quantum mechanics setting. Hence, it is not surprising that the concept of entanglement, strictly related to quantum correlations (see [32]) plays an important role in quantum computing [11], [22]. Its analysis indicates that there is a need for an operational measure of entanglement. This demand is strengthened by the observation that the number of states that can be used for quantum information is measured by the entanglement. On the other hand, the programme of classification of entanglement (so quantum correlations) seems to be a very difficult task. In particular, it was realized that the first step must presumably take the full classification of all positive maps, see [15]: as a consequence this fact has revitalized the theory of positive maps in Physics. This topic has always been studied in Mathematics as can be seen from the literature (e.g. see [7]-[9], [19], [25]-[27], [41]-[46]).

To see the relation between positive maps and entanglement, from a physical point of view, let us take a positive map $\alpha_{1,t} : A_1 \to A_1$, ($A_1 \equiv \mathcal{B}(\mathcal{H}_1)$, $t$ being identified as a time parameter, and consider the evolution of a density matrix $\varrho$ (where $\varrho$ determines the state in $\mathcal{S}(A_1 \otimes A_2)$).

In other words, we wish to study $(\alpha_{1,t} \otimes id_2)^d \varrho$. Here $(\alpha_{1,t} \otimes id_2)^d$ stands for the dual map, i.e. for the dynamical map in the Schrödinger picture. Then, if $\varrho$ is an entangled state, $(\alpha_{1,t} \otimes id_2)^d \varrho$ may develop negative eigenvalues and thus lose consistency as a physical state. That observation was the origin of rediscovery, now in a physical context, of Stinespring’s argument saying that the tensor product of transposition with the identity map can distinguish various cones in the tensor product structure (see [44], [45]). This led to the criterion of separability ([39], [15]) saying that only separable states are globally invariant with respect to the family of all positive maps.

It is known ([46], [71]) that for the case $M_k(\mathbb{C}) \to M_l(\mathbb{C})$ with $k = 2 = l$ and $k = 2, l = 3$ all positive maps are decomposable. A new argument clarifying this phenomenon was presented in Section 6. Here we note only that for this low dimensional case the criterion for separability
simplifies significantly. Namely, to verify separability of a state $\phi$ it is enough to analyse $(\tau \otimes \text{id})^d \phi$, with $\tau$ being the transposition, as other positive maps are just convex combinations of $\text{CP}$ maps (they always map states into states) and the composition of $\text{CP}$ maps with $\tau \otimes \text{id}$. This observation is the essence of the Peres-Horodecki criterion. We want to add that the lack of normalization for the summands of decomposition of a positive map (see Section 6) does not affect this criterion.

The situation changes dramatically, as we have seen in Section 6, when both $k$ and $l$ are larger than 2. In that case there are plenty of non-decomposable maps (see [23] and the references given there as well as see the preceding Section) and to analyse entanglement one cannot restrict oneself to study $\tau \otimes \text{id}$. Thus, a full description of positive maps is needed. In particular, one wishes to have a canonical form of non-decomposable maps. We note that in Section 6 we obtained only some clarification of the nature of decomposable maps. The importance of the former follows from the observation saying that this class of maps does not contain transposition. On the other hand, the theory of non-decomposable maps offers a nice construction of examples of entangled states (see [13]). However, the classification of non-decomposable maps is a difficult task which is still not completed ([43], [29]). Nevertheless, it seems that such classification is an indispensable step for an operational generalization of the Peres-Horodecki criterion.

We want to close the section with another important remark concerning the relation between quantum correlations and entanglement. Following the idea of coefficients of independence from classical probability calculus and working within the framework of non-commutative integration theory one can define (see [32] and [33]) the coefficient of quantum correlations. If the coefficient of quantum correlations is equal to zero for any $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$ then, using the description of locally decomposable maps, we proved that the state $\phi$ is separable. These observations provide the complementary approach to entanglement and the just quoted result shows how strong is the interplay between separability and certain subtle features of positive maps. However, this is not unexpected as the indicated correspondence between Schrödinger’s and Heisenberg’s picture relies on the underlying algebraic structure and geometry of the state space, see Sections 1 and 2 as well as [2], [10], and [12]. Nevertheless, it should be stressed that the complete description of quantum
correlations as well as the full classification of all positive maps are still open and challenging problems.

8. Appendix: Glossary

In order to make the paper more accessible to readers not really familiar with abstract mathematical terminology we add a glossary, in which the basic notions are defined and some basic facts are noted. The theory of \( \mathbb{C}^* \)-algebras can be found in the books of [6], [21] while the geometry of states is described in [2] and [3].

- A1. Let \( K (K') \) denote a convex set of a real vector space \( X (X' \) respectively). A map \( \alpha : K \rightarrow K' \) is called affine if the following property holds: 
  \[
  \alpha(\lambda k_1 + (1-\lambda)k_2) = \lambda \alpha(k_1) + (1-\lambda)\alpha(k_2)
  \]
  for all \( k_1, k_2 \in K \) and \( 0 \leq \lambda \leq 1 \).

- A2. A Jordan algebra over \( \mathbb{R} \) is a real vector space \( \mathcal{A} \) equipped with a commutative bilinear product \( \circ \) that satisfies the identity
  \[
  (a^2 \circ b) \circ a = a^2 \circ (b \circ a)
  \]
  for all \( a, b \in \mathcal{A} \).

- A3. An associative algebra \( \mathcal{A} \) (linear space equipped with associative multiplication) with involution \( ^* \) is called \( ^* \)-algebra. When on \( \mathcal{A} \) is defined a norm and \( \mathcal{A} \) is complete with respect to this norm, \( \mathcal{A} \) is called a Banach \( ^* \)-algebra. Finally, a Banach \( ^* \)-algebra \( \mathcal{A} \) is called a \( \mathbb{C}^* \)-algebra if it satisfies \( ||a^*a|| = ||a||^2 \) for \( a \in \mathcal{A} \).

- A4. For a subset \( Y \) of \( \mathcal{B}(\mathcal{H}) \) the set of operators in \( \mathcal{B}(\mathcal{H}) \) that commute with all operators in \( Y \) is called the commutant of \( Y \) and is denoted by \( Y' \).

- A5. A von Neumann algebra on \( \mathcal{H} \) is a \( ^* \)-algebra \( \mathcal{M} \) of \( \mathcal{B}(\mathcal{H}) \) such that \( \mathcal{M} = \mathcal{M}'' \) (\( ^\prime \prime \) stands for the double commutant). Another name for an (abstract) von Neumann algebra is \( W^* \)-algebra.

- A6. A homomorphism between two \( \mathbb{C}^* \)-algebras, \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), is a map \( \Phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) preserving the algebraic structures, i.e. \( \Phi(a^*b) = \Phi(a)\Phi(b) \) and \( \Phi(a^*) = \Phi(a)^* \). If an inverse homomorphism exists, \( \Phi \) is called isomorphism.
A7. A state on a $C^*$-algebra $A$ is a linear functional $\omega : A \to \mathbb{C}$ which is positive (i.e. $a \geq 0$ implies $\omega(a) \geq 0$) and normalized (i.e. $\omega(1) = 1$). The set of all states of $A$ will be called the state space and denoted by $S$. The state space $S$ of $A$ is a ($\omega^*$-compact) face of the unit ball of $A^*$.  

A8. Normal states $\omega$ on $B(H)$ are of the form $\omega(a) = \text{Tr}(\varrho a)$ where $\varrho$ is a uniquely determined positive operator on $H$ having the trace $\text{Tr}$ equal to 1. The set of all normal states on $B(H)$ will be called the normal state space. There is no useful compact topology on the normal state space of $B(H)$ for a general $H$. However, the normal state space of a von Neumann algebra $M$ is a face of the state space $S$ of $M$.  

A9. GNS-representation: if $A$ is a $C^*$-algebra and $\omega$ is a state on $A$, then there exists a Hilbert space $H_\omega$, a cyclic unit vector $\Omega \in H_\omega$ and a representation $\pi_\omega$ of $A$ on $H_\omega$ such that $\omega(a) = (\Omega, \pi_\omega(a)\Omega)$.  

A10. A face $F$ of the state space $S$ of $C^*$-algebra $A$ is exposed iff there exists an $a \in A$ and an $\alpha \in \mathbb{R}$ such that $x(a) = \alpha$ for all $x \in F$ and $x(a) > \alpha$ for all $x \in S \setminus F$.  

A11. A face $F$ of the normal state space $K$ of $B(H)$ is said to be norm exposed if there exists an $a \in B(H)$, positive, such that $F = \{\sigma \in K; \sigma(a) = 0\}$. A norm closed face $F$ of the normal state space of $B(H)$ is norm exposed. In general, a face $F$ of a compact subset $K$ in a vector space $V$ is norm exposed if there is a positive bounded affine functional $\phi$ on $K$ whose zero set equals $F$.  

A12. An element $p \in A$ is called a projector if $p^* = p$ and $p = p^2$.  

A13. Two projections $e$ and $f$ in $B(H)$ are said to be equivalent if there exists $v \in B(H)$ such that $v^*v = e$ and $vv^* = f$.  

A14. The convex hull of a subset $E$ of a real vector space $X$ consists of all elements of the form $\sum_{i=1}^n \lambda_i x_i$ where $x_i \in E, \lambda_i \geq 0$ for $i = 1, ..., n$ and $\sum_{i=1}^n \lambda_i = 1$. It will be denoted by $\text{co}(E)$.  

A15. The $\sigma$-convex hull of a bounded set $F$ of elements in a Banach space is the set of all sums $\sum \lambda_i x_i$ where $\lambda_1, ...$ are positive scalars with sum 1 and $x_1, ...$ are elements of $F$. 
A16. An ordered normed vector space $V$ with a generating cone $V^+$ is said to be a base norm space if $V^+$ has a base $K$ located on a hyperplane $H$ ($0 \notin H$) such that the closed unit ball of $V$ is $co(K \cup -K)$. The convex set $K$ is called the (distinguished) base of $V$. An order unit space is an ordered normed vector space $V$ over $\mathbb{R}$ with a closed positive cone and an element $e$, satisfying

$$||a|| = \inf\{\lambda > 0; -\lambda e \leq a \leq \lambda e\}$$

for any $a \in V$.

A17. A lattice is a set with an order relation such that every pair of elements $p, q$ has a least upper bound (denoted by $p \lor q$) and a greatest lower bound (denoted by $p \land q$). A lattice $L$ is complete if every subset has a least upper bound.

A18. Let $\mathcal{F}$ stand for the set of projective faces of the normal state space of $B(H)$, ordered by inclusion. Define the map $F \mapsto F'$ on $\mathcal{F}$ by $(F_p)' = F_{p'}$ for each projector $p$ ($p' \equiv 1 - p$). $F'$ is called the complementary face of $F$.

A19. Let $\mathcal{M} \subset B(H)$ be a *-algebra. The set $Z \equiv \mathcal{M}' \cap \mathcal{M}''$ is called the centrum of the von Neumann algebra $\mathcal{M}''$. A projector $p \in Z$ is called a central projector.

A20. Let $Y$ be a subspace of a vector space $X$. The equivalence class (coset) of an element $x \in X$ with respect to $Y$ is denoted by $x + Y$ and is defined to be the set $x + Y = \{v; v = x + y, y \in Y\}$. The name equivalence stems from the definition saying that two elements $x, z$ of $X$ are $Y$-equivalent whenever $x - z \in Y$. It can be shown that under algebraic operations defined by $(w + Y) + (x + Y) = (w + x) + Y$ and $\lambda(x + Y) = \lambda x + Y$, $\lambda \in \mathbb{C}$, these classes constitute the elements of vector space. This space is called the quotient space and it is denoted by $X/Y$.

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[37] A. W., Majewski, M. Marciniak, Decomposability of extremal positive maps on $M_2(\mathbb{C})$, Banach Center Publications, to appear.

[38] J. von Neumann, Mathematische Grundlagen der Quantenmechanik, Berlin, Julius Springer, 1932; english, Mathematical Foundation of Quantum Mechanics, Princeton University Press, Princeton (1955).

[39] A. Peres, Separability criterion for density matrices, Phys. Rev. Lett 77, 1413 (1996)

[40] F. W. Shultz, Duals maps of Jordan homomorphisms and $*$-homomorphisms between $C^*$-algebras, Pacific J. Math. 93, 435-441 (1981)

[41] R.R. Smith, J. D. Ward, The geometric structure of generalized state spaces, J. Funct. Anal. 40 (1981) 170-184

[42] E. Størmer, Positive linear maps of operator algebras, Acta Math. 110 (1963), 233–278.

[43] E. Størmer, Decomposable positive maps on $C^*$-algebras, Proc. Amer. Math. Soc. 86 (1980), 402–404.

[44] W. Stinespring, Positive functions on $C^*$-algebras, Proc. Amer. Math. Soc. 6 (1955), 211-216

[45] G. Wittstock, Ordered Normed Tensor Products in “Foundations of Quantum Mechanics and Ordered Linear Spaces” (Advanced Study Institute held in Marburg) A. Hartkämper and H. Neumann eds. Lecture Notes in Physics vol. 29, Springer Verlag 1974.

[46] S. L. Woronowicz, Positive maps of low dimensional matrix algebras, Rep. Math. Phys. 10 (1976), 165–183.