The $K$-Server Dual 
and
Loose Competitiveness for Paging

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Abstract

Weighted caching is a generalization of paging in which the cost to evict an item depends on the item. We give two results concerning strategies for these problems that incur a cost within a factor of the minimum possible on each input.

We explore the linear programming structure of the more general $k$-server problem. We obtain the surprising insight that the well-known “least recently used” and “balance” algorithms are primal-dual algorithms. We generalize them both, obtaining a single $\frac{k}{k-h+1}$-competitive, primal-dual strategy for weighted caching.

We introduce loose competitiveness, motivated by Sleator and Tarjan’s complaint [ST85] that the standard competitive ratios for paging strategies are too high. A $k$-server strategy is loosely $c(k)$-competitive if, for any sequence, for almost all $k$, the cost incurred by the strategy with $k$ servers either is no more than $c(k)$ times the minimum cost or is insignificant. We show that $k$-competitive paging strategies including “least recently used” and “first in first out” are loosely $c(k)$-competitive provided $c(k)/\ln k \to \infty$. We show that the $(2 \ln k)$-competitive, randomized “marking algorithm” of Fiat et al. [FKL+91] is loosely $c(k)$-competitive provided $c(k) - 2 \ln \ln k \to \infty$.

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The body of this paper consists of four sections. In §1, the introduction, we describe background, our results, and related work. In §2, we give our weighted caching strategy. In §3, we show loose competitiveness of various paging strategies. We conclude with comments about further research in §4.

1 Introduction

Many real problems must be solved on-line — decisions that restrict possible solutions must be made before the entire problem is known. Generally, one can not guarantee an optimal solution if one must solve a problem on-line. Thus a natural question for such a problem is whether a strategy exists that guarantees an approximately optimal solution.

In this paper we study the \( k \)-server problem \cite{McG87, MMS90}. Various definitions of the problem exist in the literature; we take the following definition, which is technically convenient and essentially equivalent to the other definitions: One is given a complete directed graph with edge lengths \( d(u, v) \), a number, \( k \), of identical, mobile servers, and a sequence \( r \) of requests, each to some node. In response to the first request, all servers are placed on the requested node. In response to each subsequent request \( v \), if no server is on \( v \), some server must be chosen to move from its current node \( u \) to \( v \) at a cost of \( d(u, v) \). A strategy for solving the problem is on-line if it chooses the server to move independently of later requests. The goal is to minimize the total cost.

As many authors (e.g. Chrobak and Larmore \cite{CL91}) have pointed out, the \( k \)-server problem is an abstraction of a number of practical on-line problems, including linear search, paging, font caching, and motion planning for two-headed disks.

We focus on two special cases of the \( k \)-server problem: the weighted caching problem \cite{MMS90}, in which \( d(u, v) = w(u) \) for \( u \neq v \) (the cost to move a server from a node depends only on the node), and the paging problem \cite{ST85}, in which the cost to move a server is uniformly 1.

Traditionally, paging is described as the problem of managing a fast memory, or cache, capable of holding \( k \) items: items are requested; if a requested item is not in the fast memory, it must be placed in the fast memory, possibly evicting some other item to make room. The goal is to minimize the fault rate — the number of evictions per request. Weighted caching is similar, except that the cost to evict an item depends on the item.

For these two problems, for technical reasons and without loss of gener-
ality, we replace the assumption that all servers begin on the first requested node with the assumption that initially no servers reside on nodes, and, in response to any request, any server that has not yet served a request may be placed on the requested node at no cost.

Following a number of authors (Sleator and Tarjan [ST85]; Borodin, Linial, and Saks [BLS87]; and Manasse, McGeoch, and Sleator [MMS90]), we are interested in strategies that are competitive, that is, strategies that on any sequence incur a cost bounded by some constant times the minimum cost possible for that sequence. Formally,

- \( r \) denotes an arbitrary sequence of requests.
- \( X \) denotes some on-line \( k \)-server strategy.
- \( k \) denotes the number of servers given to the on-line strategy.
- \( \text{OPT} \) denotes the (off-line) strategy producing a minimum cost solution.
- \( h \) denotes the number of servers given to \( \text{OPT} \).
- \( C_r(X,k) \) denotes the (expected) cost incurred by the (randomized\(^1\)) strategy \( X \) with \( k \) servers on request sequence \( r \).
- A strategy \( X \) is \( c \)-competitive for a given \( r \), \( h \), and \( k \), when
  \[
  C_r(X,k) \leq c \cdot C_r(\text{OPT},h) + b,
  \]
  where \( b \) depends on the initial positions of the optimal and on-line servers, but is otherwise independent of \( r \).

- A strategy \( X \) is \( c(h,k) \)-competitive when \( X \) is \( c(h,k) \)-competitive for all \( r \), \( h \) and \( k \). \( C(h,k) \) is then called a competitive ratio of \( X \).

Note that the competitiveness of a strategy is unrelated to its computational complexity.

Before we describe our results, here is a summary of the strategies relevant to our work.

\(^1\)We implicitly assume that the input requests are independent of the random choices made by the strategy; for other models see [BDBK+90].
- LRU, FIFO, and FWF are, respectively the “least recently used”, “first in first out”, and “flush when full” paging strategies. LRU moves the server from the least recently requested, served node. FIFO, which can be obtained from LRU by ignoring served requests, moves the least recently moved server. FWF evicts all items from the fast memory (removes all servers from the graph at a cost of $k$) when the fast memory is full and the requested item is not in the fast memory.

- MARK is the marking algorithm, a randomized paging strategy. MARK may be described as follows: if the requested node has no server, mark all servers if none are marked, and then move and unmark a marked server chosen uniformly at random; if the requested node has a server, unmark that server.

- BALANCE is the balance algorithm, a $k$-server strategy. In response to request $r$, BALANCE moves the server from served node $u$ minimizing $d(u, r) + W(u)$, where $W(u)$ denotes the net distance traveled so far by the server on $u$. BALANCE generalizes FIFO.

### 1.1 A primal-dual strategy for weighted caching

In §2 we introduce and analyze GREEDYDUAL, a new, primal-dual, deterministic, on-line weighted-caching strategy that is (optimally) $\frac{k}{k-h+1}$-competitive. Figure 1 contains a direct description of GREEDYDUAL.

GREEDYDUAL is of practical interest because it generalizes LRU, one of the best paging strategies, to weighted caching. GREEDYDUAL also generalizes BALANCE for weighted caching, and thus FIFO.$^2$

GREEDYDUAL is of theoretical interest because its analysis is the first primal-dual analysis$^3$ of an on-line algorithm and because the analysis, which shows an (optimal) competitive ratio of $\frac{k}{k-h+1}$, is the first to show a ratio less than $k$ when $h < k$ for any $k$-server problem more general than paging. A consequence of this reduced ratio is that GREEDYDUAL has a constant competitive ratio provided $h$ is any fraction of $k$.

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$^2$The natural generalization of LRU for weighted caching can be obtained by ignoring the $L[\cdot]$ values and lowering as much as possible in the RELABEL step. BALANCE, as it specializes for weighted caching, can be obtained by ignoring the $H[\cdot]$ values and lowering as little as possible in the RELABEL step.

$^3$We will assume familiarity with linear programming primal-dual techniques. For an introduction, see [PS82].
GreedyDual

Maintain a pair of real values $L[s] \leq H[s]$ with each server $s$.
In response to each request, let $v$ be the requested node:

if some server $s$ is on $v$ then
  let $H[s] \leftarrow w(v)$
else if some server $s$ has yet to serve a request then
  let $L[s] \leftarrow H[s] \leftarrow w(v)$
  Place $s$ on $v$.
else

ReLabel: Uniformly lower $L[s]$ and $H[s]$ for all $s$ so that

$$\min_s L[s] \leq 0 \leq \min_s H[s].$$

Move any server $s'$ such that $L[s'] \leq 0$ to $v$.
let $L[s'] \leftarrow H[s'] \leftarrow w(v)$

Figure 1: The weighted caching algorithm GreedyDual

We feel that the primal-dual approach, well developed for exact optimization problems, is also important for approximation problems, including on-line problems, because primal-dual considerations help reveal combinatoric structure, especially how to bound optimal costs. The primal-dual approach also has the potential to unify the arguably ad hoc existing on-line analyses. For instance, the analyses of LRU and FIFO [ST85], of Balance for weighted caching [CKPV91], and of Mark [FKL+91] can all be cast as closely related primal-dual analyses. The primal-dual approach can also reveal connections to existing optimization theory. For these reasons, we take pains to make explicit the primal-dual framework behind our analysis.

Here is a sketch of our primal-dual approach. The $k$-server problem has a natural formulation as an integer linear program (IP) that is essentially a minimum-weight matching problem. Relaxing the integrality constraints of IP yields a linear program (LP) (which, incidentally, has optimal integer solutions). GreedyDual implicitly generates a solution to the dual program (DP) of LP. The dual solution serves two purposes: GreedyDual
uses the structural information that the solution provides about the problem instance to guide its choices, and GreedyDual uses the cost of the dual solution as a lower bound on \( C_r(Opt, h) \) to certify competitiveness.

Related work includes the following. Sleator and Tarjan [ST85] show that LRU and FIFO are \( \frac{k}{k-h+1} \)-competitive, and that this ratio is optimal for deterministic, on-line paging strategies. A similar analysis shows that FWF is also \( \frac{k}{k-h+1} \)-competitive.

Fiat et al. [FKL+91; You91a] introduce and analyze MARK, showing that it is \( 2H_k \)-competitive (\( H_k \approx \ln k \)), and showing that no randomized paging strategy is better than \( H_k \)-competitive when \( h = k \). McGeoch and Sleator [MSS90] subsequently give a \( H_k \)-competitive randomized paging strategy. Young [You91b] shows that MARK is roughly \( 2 \ln \frac{k}{k-h} \)-competitive when \( h < k \) and that no randomized strategy is better than roughly \( \ln \frac{k}{k-h} \)-competitive.

Manasse, McGeoch, and Sleator [MMS90] show that Balance is \( k \)-competitive for the general problem provided only \( k + 1 \) distinct nodes are requested, and that no deterministic algorithm is better than \( \frac{k}{k-h+1} \)-competitive in any graph with at least \( k + 1 \) distinct nodes.

Chrobak, Karloff, Payne, and Vishwanathan [CKPV91] show that Balance is \( k \)-competitive for weighted caching. Independently of their analysis of Balance, Chrobak, Karloff, Payne, and Vishwanathan [CKPV91] formulate the \( k \)-server problem as an integral-capacity minimum-cost maximum-flow problem and use this formulation to give a polynomial time algorithm to find a minimum-cost solution.

The primal-dual approach has been used extensively for exact optimization problems [PS82], and is used implicitly in a number of recent analyses of approximation algorithms. Goemans and Williamson [GW92] explicitly use the approach for finding approximate solutions to NP-hard connectivity problems.

1.2 A more realistic \( k \)-server model

In §3 we give the second contribution of this paper: loose competitiveness. Loose competitiveness is motivated by Sleator and Tarjan’s [ST85] complaint that (when \( h = k \)) the competitive ratios of paging strategies are too high to be of practical interest. We have done simulations that suggest that in practice good paging strategies usually incur a cost within a small constant factor of minimum. The graph in Figure 3 plots competitive ratio \( C_r(X, k) / C_r(Opt, k) \) versus \( k \) for a number of paging strategies on a typical
Figure 2: Competitiveness \( \left( \frac{C_r(\cdot, k)}{C_r(OPT, k)} \right) \) vs. \( k \) for typical \( r \) sequence\(^4\).

We would like to keep the worst-case character of competitive analysis but somehow show more realistic competitive ratios.

- A strategy \( X \) is loosely \( c(k) \)-competitive when, for all \( d > 0 \), for all \( n \in \mathcal{N} \), for any request sequence \( r \), only \( o(n) \) values of \( k \) in \( \{1, \ldots, n\} \) satisfy

\[
C_r(X, k) \geq \max\{c(k) \cdot C_r(OPT, k), C_r(OPT, 1)/n^d\} + b,
\]

where \( b \) depends only on the starting configurations of \( X \) and \( OPT \), and the \( o(n) \) is independent of \( r \).

That is, \( X \) is loosely \( c(k) \)-competitive when, for any sequence, at all but a vanishing fraction of the values of \( k \) in any range \( \{1, \ldots, n\} \), either \( X \) is \( c(k) \)-competitive in the usual sense, or the cost to \( X \) with \( k \) servers is insignificant.

\(^4\)The input sequence, traced by Dick Sites [SAS88], consists of 692,057 requests to 642 distinct pages of 1024 bytes each. The sequence was generated by two X-windows network processes, a “make” (program compilation), and a disk copy running concurrently. The requests include data reads and writes and instruction fetches.
(or both). For instance, if a paging strategy is loosely $3 \ln \ln k$-competitive, then, for any fixed $d > 0$, on any sequence, for almost any choice of $k$ in any range $\{1, 2, ..., n\}$, the fault rate will be either at most $1/n^d$ or at most $3 \ln \ln k$ times the minimum possible using a cache of size $k$.

This model is realistic provided input sequences are not critically correlated with $k$ and provided we are only concerned about being near-optimal when the cost is significant. Both criteria are arguably true for most paging applications.

- A paging strategy is *conservative* if it moves no server from a node until all servers have been placed on the graph, and it moves servers at most $k$ times during any consecutive subsequence requesting $k$ or fewer distinct items.

**LRU, FIFO, MARK, and even FWF are conservative.** Any conservative paging strategy is $\frac{k}{\ln k + 1}$-competitive [You91].

The results we obtain are as follows: any conservative paging strategy is loosely $c(k)$-competitive provided $c(k)/\ln k \to \infty$ and both $c(k)$
and $k/c(k)$ are non-decreasing; $\text{MARK}$ is loosely $c(k)$-competitive provided $c(k) - 2 \ln \ln k \to \infty$ and both $c(k)$ and $2 \ln k - c(k)$ are non-decreasing.

Loose competitive ratios are thus shown to be exponentially lower than standard competitive ratios.

Borodin, Irani, Raghavan and Scheiber [BIRS91] give a related work, in which the possible request sequences are quantified by the degree to which they exhibit a certain kind of locality of reference, and competitive ratios are considered as a function of this parameter. The work is extended by Irani, Karlin, and Phillips [IKP92]. The ratios shown in their model are, in most cases, much higher than the loose competitive ratios established in this paper.

2. GreedyDual

In this section we develop and analyse GreedyDual. We first develop a linear programming framework for the general $k$-server problem, and then we present and analyze GreedyDual as a primal-dual algorithm within this framework.

2.1 The $k$-server dual

Fix a request sequence $r_0, r_1, ..., r_N$, so that request $i$ is to node $r_i$.

We next define IP, an integer linear program whose feasible solutions correspond to solutions of the $k$-server problem given by $r$. The variables of IP are $\{x_{ij} : 0 \leq i < j \leq N\}$, where $x_{ij} \in \{0, 1\}$ is 1 if and only if the request served by the server of request $j$ before serving request $j$ is request $i$.

After defining IP, we construct its fractional relaxation LP, and the dual DP of LP.
• IP(k) (or just IP, if k is determined by context) denotes the integer linear program

\[
\begin{align*}
\text{minimize} & \quad \sum_{0 \leq i < j \leq N} d(r_i, r_j)x_{ij} \\
\text{subject to} & \quad \begin{cases}
x(\text{out}(0)) \leq k \\
x(\text{out}(i)) \leq 1 & (1 \leq i \leq N - 1) \\
x(\text{in}(i)) = 1 & (1 \leq i \leq N) \\
x_{ij} \in \{0, 1\} & (0 \leq i < j \leq N)
\end{cases}
\end{align*}
\]

where out(i) denotes the set \{(i, j) : i < j \leq N\}, in(i) denotes the set \{(j, i) : 0 \leq j < i\}, and \(x(S) = \sum_{(i,j) \in S} x_{ij}\).

For the weighted caching and paging problems (where initially no servers reside on the graph, and each server is allowed to serve its first request by being placed on the requested node at no cost), IP is defined as above, but we stipulate that request 0 is to an artificial node that is never requested again and that is at distance 0 to all later requests. With this stipulation the initial conditions for the general problem reduce to the initial conditions for weighted caching and paging.

• LP(k) (or just LP) denotes the relaxation of IP (obtained by replacing each constraint \(x_{ij} \in \{0, 1\}\) with the constraint \(0 \leq x_{ij} \leq 1\)).

• DP(h) (or just DP) denotes the dual of LP(h):

\[
\begin{align*}
\text{maximize} & \quad -ha_0 - \sum_{1 \leq i \leq N - 1} a_i + \sum_{1 \leq i \leq N} b_i \\
\text{subject to} & \quad \begin{cases}
b_j - a_i \leq d(i, j) & (0 \leq i < j \leq N) \\
a_i \geq 0 & (0 \leq i \leq N)
\end{cases}
\end{align*}
\]

• \(\|(a, b)\|_h\) denotes the cost, \(-ha_0 - \sum_{i \geq 1} a_i + \sum_i b_i\), of a feasible solution to DP(h).

Note that the dual constraints are independent of h, so that a dual solution is feasible independently of h.

By duality, for any feasible dual solution \((a, b)\),

\[
C_r(\text{Opt}, h) \geq \|(a, b)\|_h
\]
Incidentally, a standard transformation shows that IP is equivalent to a minimum-weight, bipartite, perfect matching problem\(^5\) and thus that LP has optimal integer solutions, so that, for given \(r\) and \(h\), the above bound is tight for some \((a, b)\).

2.2 The algorithm

Here are the definitions and notations specific to GreedyDual:

- A request has a server if the request has been served and the server has not subsequently served any other request.
- The notation \(i^-\) denotes the most recent request (up to and including request \(i\)) that resulted in the server of request \(i\) moving. We define \(0^- = 0\).
- \(S\) denotes the set \(\{i: \text{request } i \text{ has a server}\}\). (More correctly, \(S\) is a multiset, as \(0\) occurs in \(S\) once for each server on node \(r_0\). Any \(i > 0\) can occur only once.)
- \((a, b)\) denotes a feasible dual solution maintained by GreedyDual.

GreedyDual responds to each request as follows. If the requested node has a server, it does nothing. Otherwise, it uniformly raises a subset of the dual variables enough to account for the cost of moving some server, but not so much that feasibility is violated. It then moves a server whose movement cost can be accounted for. The full algorithm is given in Figure 4.

2.3 Analysis of the algorithm

A simple proof by induction on \(n\) shows that every \(b_i \geq b_{i+1}\), that \(b_{i+1} \leq w(r_i)\) for \(i \in S\) (two facts that we use again later), and that the RELABEL

\(^5\) It may be useful for the reader in understanding LP to study the equivalent minimum-weight perfect matching problem, so we briefly outline it here. Construct a weighted bipartite graph \(G = (U, W, E)\), with \(U = \{A_0, A_1,...,A_{N-1}\}\), \(W = \{B_1,...,B_N\}\), \(E = \{(A_i, B_j) \in U \times W: i < j\}\), and \(w(A_i, B_j) = d(r_i, r_j)\). Each solution \(x\) to IP corresponds to the subset \(\{e : x_e = 1\}\) of \(E\). The cost of \(x\) equals the net weight of edges in the subset. Such subsets are exactly those such that every vertex in \(W\) touches one edge in the subset, every vertex in \(U\) except \(A_0\) touches at most one edge, and \(A_0\) touches at most \(k\) edges. We can leave the problem in this form, or we can convert it into a true perfect matching problem by duplicating \(A_0\) with its edges \(k\) times and adding \(k\) copies of a new node \(B_{\infty}\) with zero-cost edges from every \(A_i\).
Comment: moves servers in response to requests \( r_0, r_1, ..., r_N \), maintaining \((a, b)\), a dual solution, and \( S \), a multiset containing the currently served requests, such that \((a, b)\) is feasible and the distance traveled by servers is at most \( \frac{k}{k-h+1} \| (a, b) \|_h - \sum_{i \in S} b_{i+1} \).

In response to request 0:

1. let \( a_{i-1} \leftarrow b_i \leftarrow 0 \) for \( i = 1, ..., N \)
2. let \( S \leftarrow \) the multiset containing request 0 with multiplicity \( k \)

In response to each subsequent request \( n > 0 \):

- if node \( r_n \) has a server then
  - **STAY**: choose \( i \in S \) such that \( r_i = r_n \)
    - let \( S \leftarrow S \cup \{n\} - \{i\} \), satisfying request \( n \)
- else
- **RELABEL**: Uniformly raise the dual variables in the set
  \[ \{a_i : 0 \leq i \leq n-1, i \notin S\} \cup \{b_i : 1 \leq i \leq n\} \]
  so that \((\forall i \in S) b_{i+1} \leq w(r_i) \) but \((\exists i \in S) b_{i-1} \geq w(r_i)\).
- **MOVE**: choose \( i \in S \) such that \( b_{i-1} \geq w(r_i) \)
  - let \( S \leftarrow S \cup \{n\} - \{i\} \), satisfying request \( n \)

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**Figure 4:** \textsc{GreedyDual} as a primal-dual algorithm

step can in fact be performed. All other steps can be seen to be well-defined by inspection, and clearly \textsc{GreedyDual} produces an appropriate sequence of server movements. This establishes the correctness of \textsc{GreedyDual}.

To establish that \textsc{GreedyDual} is \( \frac{k}{k-h+1} \)-competitive, we show two invariants: that the dual solution \((a, b)\) is feasible, and that the distance traveled by servers is bounded by \( \frac{k}{k-h+1} \| (a, b) \|_h - \sum_{i \in S} b_{i+1} \). Since every \( b_i \) is nonnegative and \( \| (a, b) \|_h \) is a lower bound on \( C_r(Opt, h) \), this gives the result.

**Lemma 2.1** \textsc{GreedyDual} maintains the invariant that \((a, b)\) is feasible.

**Proof:** By induction on \( n \).
Clearly \((a, b)\) is initially feasible.
The only step that changes \((a, b)\) is RELABEL.
Clearly RELABEL maintains that every \(a_i\) is nonnegative.
Thus the only dual constraint that RELABEL might violate is of the form
\[
b_j - a_i \leq d(r_i, r_j)
\]
for some \(0 \leq i < j \leq N\).
By inspection of the RELABEL step, such a constraint can only be violated if \(i \in S\) and \(j \leq n\).
In this case, \(a_i = 0\) and \(r_i \neq r_j\) because \(i\) has a server, so the constraint reduces to \(b_j \leq w(r_i)\).
Since \(b_{i+1} \leq w(r_i)\) after the step, and \(b_{i+1} \geq b_j\) (since \(j > i\) and we have already established that every \(b_i \geq b_{i+1}\)), the constraint is maintained. 

**Lemma 2.2** GreedyDual maintains the invariant that the net distance traveled by servers is bounded by
\[
\frac{k}{k - h + 1}||(a, b)||_h - \sum_{i \in S} b_{i-1} + 1
\]
**Proof:** By induction on \(n\).
Clearly the invariant is initially true.
The Stay step leaves the net distance and the bound unchanged.
The RELABEL step also leaves the net distance and the bound unchanged. If \(0 \notin S\), that the bound remains unchanged can be seen by inspecting the definition of \(||(a, b)||_h||\), and noting that when the dual variables are raised, \(n - k\) of the \(a_i\)'s, including \(a_0\), and \(n\) of the \(b_i\)'s increase. Consequently \(||(a, b)||_h||\) is increased by \(k - h + 1\) times as much as any individual term, and, in the bound, the increase in the minuend exactly counterbalances the increase in the subtrahend.
If \(0 \in S\), the bound remains unchanged because the constraint \(b_1 \leq w(r_0) = 0\) ensures that the raise is degenerate — that the dual variables are in fact unchanged.
The Move step increases the distance traveled by \(w(r_i)\), and increases the bound by \(b_{i-1} - b_{n-1}\). Since \(n^- = n\), \(b_{n+1} = 0\), and \(b_{i-1} \geq w(r_i)\), the bound is increased by at least \(w(r_i)\), and the invariant is maintained. 

\(\square\)
Corollary 2.3 GreedyDual is \( \frac{k}{k-h+1} \)-competitive.

Note that in order to implement GreedyDual, only the values \( L[s_i] = w(r_i) - b_{i+1} \) and \( H[s_i] = w(r_i) - b_i + 1 \) (for each server \( s_i \) of a request \( i \in S \)) need to be maintained, and that the artificial first request may be dropped, instead placing the servers on nodes when they first truly serve a request. We leave it to the reader to verify that these modifications lead to the direct description of GreedyDual given in Figure 1.

3 Loose Competitiveness

In this section we give our analyses of loose competitiveness of paging strategies. The theorems and lemmas in this section, except as noted, first appeared in [You91b, You91a].

Recall that a \( k \)-server strategy is loosely \( c(k) \)-competitive if, for any \( d \), for any \( n \), for any request sequence \( r \), only \( o(n) \) values of \( k \in \{1, \ldots, n\} \) satisfy

\[
C_r(X, k) \geq \max\{c(k)C_r(OPT, k), C_r(OPT, 1)/n^d\} + b.
\]

The following terminology is essentially from Fiat et al.'s [FKL+91, You91a, You91b] analysis of the marking algorithm. Given a sequence \( r \) and a positive integer \( k \),

- The \( k \)-phases of \( r \) are defined as follows. The first \( k \)-phase is the maximum prefix of \( r \) containing requests to at most \( k \) distinct nodes. In general, the \( i \)th \( k \)-phase is the maximum substring\(^6\) of \( r \) beginning with the request, if any, following the \( i-1 \)st \( k \)-phase and containing requests to at most \( k \) distinct nodes.

Thus the \( i+1 \)st \( k \)-phase begins with the (new) request that would cause FWF to flush its fast memory for the \( i \)th time.

- \( P_r(k) \) denotes the number of \( k \)-phases, minus 1.

- A new request (for a given \( k \)) in a \( k \)-phase (other than the first) is a request to a node that is not requested previously in the \( k \)-phase or in the previous \( k \)-phase. Thus in two consecutive \( k \)-phases, the number of distinct nodes requested is \( k \) plus the number of new requests in the second \( k \)-phase.

\(^6\)By “substring” we mean a subsequence of consecutive items.
• \(N_r(k)\) denotes the *average* number of new requests per \(k\)-phase of \(r\) other than the first.

Thus the total number of new requests in \(r\) for a given \(k\) is \(N_r(k) \cdot P_r(k)\).

Our analysis has two parts. In the first part (Theorem 3.2) we show that, for any sequence, few values of \(k\) yield both a large number of \(k\)-phases and a low average number of new requests per \(k\)-phase.

In the second part, we show (Lemma 3.4) that, for the paging strategies that interested us, for a given sequence and \(k\), the cost incurred by the strategy is proportional to the number of \(k\)-phases, and the competitiveness is inversely related to the average number of new requests per \(k\)-phase. Consequently (Corollary 3.5), by the first part of the analysis, few values of \(k\) yield both a high cost and a high competitiveness.

The key technical insight (Lemma 3.1) for the first part of the analysis is that if, for a given \(k\), the average number of new requests per \(k\)-phase is low, then, for \(k'\) just slightly larger than \(k\), the number of \(k'\)-phases is a fraction of the number of \(k\)-phases.

**Lemma 3.1** Fix a sequence \(r\). For any \(k\), and any \(k' \geq k + 2N_r(k)\),
\[
P_r(k') \leq \frac{3}{4} P_r(k).
\]

**Proof:** Let \(p_0, \ldots, p_{P_r(k)}\) denote the \(k\)-phase partitioning of \(r\).

At least half (and thus at least \(\lceil P_r(k)/2 \rceil\)) of the \(P_r(k)\) \(k\)-phases \(p_1, \ldots, p_{P_r(k)}\) have a number of new requests not exceeding \(2N_r(k)\). Denote these by \(p_i_1, \ldots, p_{\lceil P_r(k)/2 \rceil}\).

If we modify the \(k\)-phase partitioning of \(r\) by joining \(p_{i_{j-1}}\) and \(p_{i_j}\) for odd \(j\), we obtain a coarser partitioning of \(r\) into at most \(P_r(k) - \lceil P_r(k)/4 \rceil\) pieces. In the coarser partitioning, each piece resulting from a join references at most \(k + 2N_r(k) \leq k'\) distinct nodes, while each pieces remaining unchanged from the \(k\)-phase partitioning references at most \(k \leq k'\) distinct nodes.

If we now consider the \(k'\)-phase partitioning, we find that each \(k'\)-phase must contain the final request of at least one of the pieces in the coarser partition, because if a \(k'\)-phase begins at or after the beginning of a subsequence of requests to at most \(k + 2N_r(k)\) distinct nodes, it will continue at least through the end of the subsequence.

Thus \(P_r(k') \leq P_r(k) - \lceil P_r(k)/4 \rceil \leq \frac{3}{4} P_r(k)\). \(\square\)

From this we show that there are not too many values of \(k\) yielding both a low average number of new requests per \(k\)-phase and a significant number of \(k\)-phases:
Theorem 3.2  For any $\epsilon > 0, M > 0$, and any sequence $r$, the number of $k$ satisfying
\[ N_r(k) \leq M \quad \text{and} \quad P_r(k) \geq \epsilon P_r(1) \] (1)
is $O(M \ln \frac{1}{\epsilon})$.

Proof:
Let $s$ be the number of $k$ satisfying the condition.
We can choose $l = \lceil s/\lceil 2M \rceil \rceil$ such that for each $i$
\[ N_r(k_i) \leq M, \quad k_{i+1} - k_i \geq 2M, \quad \text{and} \quad P_r(k_i) \geq \epsilon P_r(1). \] (2) (3) (4)

Then for any $i$, by (2) and (3), $k_{i+1} - k_i \geq 2M$, so, by Lemma 3.1,
\[ P_r(k_{i+1}) \leq (3/4)P_r(k_i). \]
Inductively, $P_r(k_l) \leq (3/4)^{l-1}P_r(1)$.

This, and (4), imply $(3/4)^{l-1} \geq \epsilon$, so
\[ \lceil s/\lceil 2M \rceil \rceil - 1 = l - 1 \leq \ln_{4/3} \frac{1}{\epsilon}. \]
This implies the bound on $s$. \qed

This establishes the first part of the analysis.

We begin the second part by showing that OPT’s cost per $k$-phase is at least proportional to the average number of new requests per $k$-phase:

Lemma 3.3  For arbitrary paging request sequence $r$, and arbitrary $k, h > 0$,
\[ C_r(\text{OPT}, h)/P_r(k) \geq (k - h + N_r(k))/2 \] (5)
(We use only the case $k = h$, but prove the general case.)

Proof:  Let $m_i (1 < i \leq P_r(k))$ denote the number of new requests in the $i$th $k$-phase, so that $N_r(k)P_r(k) = \sum_{i\geq 1} m_i$. During the $i - 1$st and $i$th $k$-phases, $k + m_i$ distinct nodes are referenced. Consequently, any strategy for $r$ with $h$ servers makes at least $k + m_i - h$ server movements during the two phases. Thus the total cost for the strategy is at least
\[ \max \left\{ \sum_{i\geq 1} (k - h + m_{2i+1}), \sum_{i\geq 1} (k - h + m_{2i}) \right\} \geq (k - h + N_r(k))P_r(k)/2. \]
We next show that the strategies that interest us incur a cost proportional to $k$ times the number of $k$-phases, and (using the above lemma) that the strategies have competitiveness inversely related to the average number of new requests per $k$-phase:

**Lemma 3.4** Let $X$ denote any conservative paging strategy. Let $\text{Mark}$ denote the marking algorithm. Then

\[
\begin{align*}
\mathcal{P}_r(k) & \geq \frac{C_r(X, k)}{k} \quad \text{(6)} \\
\mathcal{N}_r(k) & \leq 2k \frac{C_r(\text{Opt}, k)}{C_r(X, k)} \quad \text{(7)} \\
\mathcal{N}_r(k) & \leq k \exp \left(1 - \frac{1}{2} \frac{C_r(\text{Mark}, k)}{C_r(\text{Opt}, k)}\right) \quad \text{(8)}
\end{align*}
\]

**Proof:** Bound (6) follows directly from the definition of conservativeness.

Bound (7) follows from Bound (6) and Bound (5) of Lemma 3.3, applied with $h = k$.

Finally, we prove Bound (8). Fix a request sequence $r$, and let $m_i$ ($1 < i \leq \mathcal{P}_r(k)$) denote the number of new requests in the $i$th $k$-phase. Fiat et al. [FKL+91] showootnote{Mark moves a server chosen uniformly at random from those on nodes not yet requested in the current $k$-phase.} that $C_r(\text{Mark}, k) \leq \sum_i m_i(H_k - H_{m_i} + 1)$.

Since $H_a - H_b = \sum_{i=a+1}^{b} \frac{1}{i} \leq \int_{a}^{b} \frac{1}{x} \, dx = \ln \frac{b}{a}$, letting $f(m) = m(1 + \ln \frac{k}{m})$, $C_r(\text{Mark}, k) \leq \sum_i f(m_i)$. Since $f$ is convex, $C_r(\text{Mark}, k) \leq \mathcal{P}_r(k) f(\mathcal{N}_r(k))$. Applying Bound (5), $C_r(\text{Mark}, k) \leq 2C_r(\text{Opt}, k) f(\mathcal{N}_r(k))/\mathcal{N}_r(k)$, which is equivalent to Bound (8).

We have established (Theorem 3.2) that for any sequence there are few values of $k$ yielding many $k$-phases and a low average number of new requests per $k$-phase.

We have established (Lemma 3.4) that for the strategies we are interested in, the cost they incur with $k$ servers on a sequence is proportional to the number of $k$-phases, and the expected cost per old request is bounded by $m/(k - i + 1)$ because at least $k - m - i + 1$ of the $k - i + 1$ nodes requested in the previous phase but not yet in this phase are served, each with equal probability.
number of $k$-phases, while the competitiveness is inversely related to the average number of new requests per $k$-phase.

Finally, we combine the two parts to show that, for any sequence, there are few values of $k$ for which our strategies incur high cost and high competitiveness. This establishes loose competitiveness.

**Corollary 3.5** Let $X$ denote any conservative paging strategy and $C : \mathbb{N}^+ \to \mathbb{R}^+$ a nondecreasing function.

$X$ is loosely $c(k)$-competitive provided that $k/c(k)$ is nondecreasing and

$$\frac{c(k)}{\ln k} \to \infty,$$

while MARK is loosely $c(k)$-competitive provided $2\ln k - c(k)$ is nondecreasing and

$$c(k) - 2\ln \ln k \to \infty.$$

**Proof:** Let $X$ denote either any conservative paging strategy, in which case we assume condition (9) and that $k/c(k)$ is nondecreasing, or MARK, in which case we instead assume condition (10) and that $2\ln k - c(k)$ is nondecreasing.

We show that, for any $d > 0$, $n > 0$, and request sequence $r$, the number of violators $k \in \{1, \ldots, n\}$ is $o(n)$, where a violator is a $k$ such that

$$C_r(X, k) \geq \max\{c(k)C_r(\text{OPT}, k), C_r(\text{OPT}, 1)/n^d\}.$$

Let $k$ be a violator. Then bound (3) implies

$$P_r(k) \geq \frac{C_r(X, k)}{k} \geq \frac{C_r(\text{OPT}, 1)}{n^{d+1}} = \frac{1}{n^{d+1}}P_r(1).$$

Bound (4) and the monotonicity of $k/c(k)$ imply

$$N_r(k) \leq 2k \frac{C_r(\text{OPT}, k)}{C_r(X, k)} \leq \frac{2k}{c(k)} \leq \frac{2n}{c(n)}.$$

Since each violator $k$ satisfies (11) and (12), by Theorem 3.2, the number of violators is $O\left((\ln n^{d+1})/n/c(n)\right)$. This is $o(n)$ by assumption (1).

If $X = \text{MARK}$, then bound (8) and the monotonicity of $2\ln k - c(k)$ imply, for each violator $k$, that

$$N_r(k) \leq k \exp\left(1 - \frac{1}{2} \frac{C_r(\text{MARK}, k)}{C_r(\text{OPT}, k)}\right) \leq k \exp(1 - c(k)/2) \leq n \exp(1 - c(n)/2).$$
so that by bounds \([11]\) and \([13]\) and Theorem 3.2 the number of violators is \(O\left( (\ln n^{d+1}) n \exp(1 - c(n)/2) \right)\). This is \(o(n)\) by assumption \([10]\).

\[\blacksquare\]

4 Concluding Remarks

We conclude in this section with comments about further avenues of research.

Historically, the role of duality in solving optimization problems is well-explored: dual solutions are used to guide the construction of primal solutions and to certify optimality. For on-line problems such as the \(k\)-server problem, duality can serve a similar role; the differences are that the solutions we seek are approximate, and that the problem we want to solve is on-line. For on-line problems, it seems natural to seek a sequence of closely related dual solutions, one for each prefix of the request sequence.

For those interested in extending our approach to the general \(k\)-server problem we give the following brief hints. Solutions with monotonic \(b_i\)'s are not sufficient to give good bounds: add constraints \(b_{i+1} \leq b_i\) to the dual problem and reformulate the primal; in the new primal request sequences can be much cheaper than in the old. Raising all of the \(b_i\)'s is probably not a good idea: consider a request sequence with requests from two infinitely separate metric spaces; a \(b_i\) should change only when a request is made to the metric space of \(r_i\). Finally, a promising experimental approach: if \(r\) is a worst-case sequence for a \(k\)-competitive algorithm \(X\), and the bound \(C_r(X, k) \leq k\|(a, b)\|_k\) is sufficient to establish competitiveness, then \((a, b)\) must be optimal; thus by examining optimal dual solutions for worst-case sequences, we may discover the special properties of the (generally non-optimal) dual solutions that we seek for such an analysis. A similar technique has been tried for potential functions, but in that case each experiment is much less informative: it reveals only a single number, not an entire dual solution.

There is a suggestive similarity between potential function and primal-dual techniques \([You91a]\). Briefly, both can be viewed as transforming the costs associated with operations so that a sum of local inequalities gives the necessary global bound. This connection might yield some insight into the special nature of primal-dual analyses for on-line problems.

Open questions remain concerning loose competitiveness for paging. In
Theorem 3.2 is shown to be tight, and consequently the analysis of loose competitiveness for FWF is shown to be tight. No lower bounds on the loose competitive ratios of LRU, FIFO, or MARK have been shown.

Finally, two challenges: find a randomized algorithm for weighted caching that is better than $k$-competitive, and show reduced loose competitiveness for a weighted-caching algorithm. A possible hint: the concept of “new requests” used in analyzing MARK and showing loose competitiveness of paging strategies may be captured by an algorithm that mimics GreedyDual, but increases each $b_i$ at only half the rate that GreedyDual does, and increases each $a_i$ only as much as necessary to maintain the dual constraints.

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