ON THE TRAJECTORIES OF O(1)-KEPLER PROBLEMS

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Abstract. The trajectories of the O(1)-Kepler problem at level \(n \geq 2\) are completely determined. It is found in particular that a non-colliding trajectory is an ellipse, a parabola or a branch of hyperbola according as the total energy is negative, zero or positive. Moreover, it is shown that the group \(\text{GL}(n, \mathbb{R})/\text{O}(1)\) acts transitively on both the set of elliptic trajectories and the set of parabolic trajectories. The method employed here is similar to the one used by Levi-Civita in the study of planar Kepler problem in 1920.

1. Introduction

The O(1)-Kepler problem at level \(n \geq 2\) is a generalized Kepler problem \([1]\) whose configuration space is the space \(C_1\) of rank one semi-positive elements in the euclidean Jordan algebra \(H_n(\mathbb{R})\) of real symmetric matrices of order \(n\), and Lagrangian is

\[
L = \frac{1}{2}||\dot{x}||^2 + \frac{n}{\text{tr} x}.
\]

Here \(\text{tr} x\) is the trace of \(x\), hence always positive because \(x \in C_1\). The length square of the velocity vector \(\dot{x}\) is not calculated with the euclidean structure on \(H_n(\mathbb{R})\). To describe it, we note that the tangent space \(T_x C_1\) is the subspace \(\{x\} \times \text{Range} L_x\) of \(T_x H_n(\mathbb{R})\) where \(L_x\) is the Jordan multiplication by \(x\). We also note that \(\bar{L}_x : H_n(\mathbb{R}) \rightarrow H_n(\mathbb{R})\) maps \(\text{Range} L_x\) isomorphically onto \(\text{Range} \bar{L}_x\), so we have an automorphism \(\bar{L}_x\) of \(T_x C_1\). (Both of these two statements can be easily verified by assuming that \(x\) is in diagonal form.) By definition

\[
||\dot{x}||^2 = \frac{1}{n} \text{tr} x \langle \dot{x}, \bar{L}_x^{-1}(\dot{x}) \rangle
\]

where \(\bar{L}_x^{-1}\) is the inverse of \(\bar{L}_x\), and \(\langle , \rangle\) is the inner product on \(T_x H_n(\mathbb{R})\): for \((x, u), (x, v)\) in \(T_x H_n(\mathbb{R})\), we have

\[
\langle (x, u), (x, v) \rangle = \frac{1}{n} \text{tr} (uv)
\]

where \(uv\) is the matrix (or Jordan) multiplication of \(u\) with \(v\).

The bound state analysis for O(1)-Kepler problems has been done in Ref. \([2]\). Here we analyze the trajectories of O(1)-Kepler problems. We shall show that a trajectory is always the intersection of a 2D plane with \(C_1\), consequently, since

\[C_1 = \{x \in H_n(\mathbb{R}) \mid x^2 = \text{tr} x x, \text{ tr} x > 0\}\]

a trajectory must be a quadratic plane curve. In fact, we shall show that a non-colliding trajectory is an ellipse, a parabola or a branch of a hyperbola according as the total energy

\[
E = \frac{1}{2}||\dot{x}||^2 - \frac{n}{\text{tr} x}
\]
is negative, zero or positive. It will also be shown that the group \( \text{GL}(n, \mathbb{R})/\text{O}(1) \) acts transitively on both the set of elliptic trajectories and the set of parabolic trajectories.

### 2. Solving Equation of Motion via an Idea of Levi-Civita

The equation of motion for the planar Kepler problem was ingeniously solved by Levi-Civita in Ref. [3] in which the nonlinear equation of motion was transformed into a linear ordinary differential equation (ODE). This transformation, referred to as the Levi-Civita transformation in literatures, is based on the quadratic map from \( \mathbb{C} \to \mathbb{C} : z \mapsto z^2 \).

We shall use a similar idea to solve the equation of motion for \( \text{O}(1) \)-Kepler problems. The similar transformation that we shall use, which turns the equation of motion into a linear ODE, is based on the following quadratic map

\[
q : \mathbb{R}^n \to \mathbb{H}_n(\mathbb{R}) \\
X \mapsto nXX^t
\]

where \( X^t \) is the transpose of the column vector \( X \) and \( XX^t \) is the matrix multiplication of \( X \) with \( X^t \). Map \( q \), when restricted to \( \mathbb{R}^n_* := \mathbb{R}^n \setminus \{0\} \), becomes a two-to-one covering map onto \( \mathbb{C}_1 \):

\[
\bar{q} : \mathbb{R}^n_* \xrightarrow{2:1} \mathbb{C}_1.
\]

Consequently the tangent map \( T\bar{q} : T\mathbb{R}^n_* \to T\mathbb{C}_1 \) is a two-to-one covering map, too.

**Proposition 2.1.** The composition of Lagrangian \( L \) in Eq. (1.1) with the tangent map \( T\bar{q} \), denoted by \( \tilde{L} : T\mathbb{R}^n_* \to \mathbb{R} \), is of the form

\[
\tilde{L} = 2X^2 \dot{X}^2 + \frac{1}{X^2}
\]

where \( X^2 = X \cdot X \) and \( \dot{X}^2 = \dot{X} \cdot \dot{X} \). Also, \( \tilde{E} := E \circ T\bar{q} : T\mathbb{R}^n_* \to \mathbb{R} \) is of the form

\[
\tilde{E} = 2X^2 \dot{X}^2 - \frac{1}{X^2}.
\]

**Proof.** It is clear that \( \text{tr} \left( XX^t \right) = X^2 \), if suffices to show that

\[
\text{tr} \left( \frac{d}{dt} \left( XX^t \right) \right) = 4\dot{X}^2.
\]

Since \( \bar{q} \) is \( O(n) \)-equivariant, and both \( L \) and \( \tilde{L} \) are \( O(n) \)-invariant, it suffices to verify the proposition at the point \( X = [a, 0, \ldots, 0]^t \) with \( a > 0 \). Let \( \dot{X} = [y_1, \ldots, y_n]^t \). Then

\[
XX^t = a^2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},
\]
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\[
\frac{d}{dt}(XX^t) = a \begin{bmatrix}
2y_1 & y_2 & \cdots & y_n \\
y_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
y_n & 0 & \cdots & 0
\end{bmatrix}
\]

and

\[
\tilde{L}^{-1}_{XX^t} \left( \frac{d}{dt}(XX^t) \right) = \frac{1}{a} \begin{bmatrix}
2y_1 & 2y_2 & \cdots & 2y_n \\
2y_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
2y_n & 0 & \cdots & 0
\end{bmatrix}.
\]

Then we have

\[
\text{tr} \left( \frac{d}{dt}(XX^t) \tilde{L}^{-1}_{XX^t} \left( \frac{d}{dt}(XX^t) \right) \right) = 4 \sum_{i=1}^{n} y_i^2 = 4 \dot{X}^2.
\]

\[\square\]

Remark 2.1. The dynamical problem with configuration space $\mathbb{R}^n$ and Lagrangian $\tilde{L}$ in Proposition 2.1 is a conformal Kepler problem in the sense of T. Iwai [4].

In view of the fact that a smooth covering map is a local diffeomorphism, the following proposition, one of the two reasons for the success of Levi-Civita’s approach to planar Kepler problem, is almost evident.

Proposition 2.2. Let $p : E \to X$ be a covering map from manifold $E$ onto manifold $X$, $L : TX \to \mathbb{R}$ be a smooth function, $\alpha : I \to X$ be a smooth map from interval $I$ to $X$. Then $\alpha$ is a solution to the Euler-Lagrange equation associated with Lagrangian $L$ if and only if any lifting $\tilde{\alpha}$ of $\alpha$ is a solution to the Euler-Lagrange equation associated with Lagrangian $L \circ T_p$.

Remark 2.2. There always exists a lifting for $\alpha$ because $I$ is topological trivial.

In view of Proposition 2.2, by finding all solutions to the Euler-Lagrange equation associated with Lagrange $\tilde{L}$ in Proposition 2.1 and then composing them with $\tilde{q}$, we get all solutions to the equation of motion for the $O(1)$-Kepler problem at level $n$.

From Proposition 2.1 one can see that the Euler-Lagrange equation associated with $\tilde{L}$ is

\[
\frac{d}{dt}(4X^2 \dot{X}) = \tilde{E} \frac{2X \dot{X}}{2X} \text{ or } \left( X^2 \frac{d}{dt} \right)^2 X = \tilde{E} \frac{2X}{2X}.
\]

(2.4)

Let us fix a solution to the equation of motion for the $O(1)$-Kepler problem at level $n$, then the total energy $E$ is a constant of motion, and so is $\tilde{E} = E$. With this in mind we can solve Eq. (2.4) in three cases according as the total energy $E$ is negative, zero or positive.
2.1. The case $E < 0$. In this case we introduce variable

$$\tau = \sqrt{\frac{-E}{2}} \int_0^t \frac{d\tilde{\tau}}{X(t)^2}.$$

Then $\tau$ is an increasing smooth function of $t$ and Eq. (2.4) becomes

$$\frac{d^2X}{d\tau^2} + X = 0,$$

so $X$ is of the form

$$X(t(\tau)) = \cos \tau \ u + \sin \tau \ v \quad (2.5)$$

for some $u \in \mathbb{R}^n_*$ and $v \in \mathbb{R}^n$. Substituting this solution to equation $0 = E = 2X^2X^2 - \frac{1}{X^2}$, we get

$$E = -\frac{1}{u^2 + v^2}.$$ 

Although $\tau$ is a complicated function of $t$, $t$ is actually quiet a simple increasing function of $\tau$:

$$\begin{align*}
t &= \sqrt{\frac{-2}{E}} \int_0^\tau X(t(\tilde{\tau}))^2 \, d\tilde{\tau} \\
 &= \sqrt{\frac{-2}{E}} \int_0^\tau (\cos \tilde{\tau} \ u + \sin \tilde{\tau} \ v)^2 \, d\tilde{\tau} \\
 &= \sqrt{2(u^2 + v^2)} \left( \frac{u^2 + v^2}{2} \tau + \frac{u^2 - v^2}{4} \sin(2\tau) + \frac{u \cdot v}{2} (1 - \cos(2\tau)) \right).
\end{align*}$$

2.2. The case $E = 0$. In this case we introduce variable

$$\tau = \int_0^t \frac{ds}{X(s)^2}.$$

Then $\tau$ is an increasing smooth function of $t$ and Eq. (2.4) becomes

$$\frac{d^2X}{d\tau^2} = 0,$$

so $X$ is of the form

$$X(t(\tau)) = u + \tau v \quad (2.6)$$

for some $u \in \mathbb{R}^n_*$ and $v \in \mathbb{R}^n$. Substituting this solution to equation $0 = E = 2X^2X^2 - \frac{1}{X^2}$, we get $v^2 = \frac{1}{2}$. Again $t$ is a simple increasing function of $\tau$:

$$t = u^2 \tau + u \cdot v \tau^2 + \frac{1}{6} \tau^3.$$ 

2.3. The case $E > 0$. In this case we introduce variable

$$\tau = \sqrt{\frac{E}{2}} \int_0^t \frac{ds}{X(s)^2}.$$

Then $\tau$ is an increasing smooth function of $t$ and Eq. (2.4) becomes

$$\frac{d^2X}{d\tau^2} - X = 0,$$

so $X$ is of the form

$$X(t(\tau)) = \cosh \tau \ u + \sinh \tau \ v \quad (2.7)$$
for some $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. Substituting this solution to equation $E = 2X^2 \dot{X}^2 - \frac{1}{X^2}$, we get

$$E = \frac{1}{v^2 - u^2}.$$  

Since $E > 0$, we must have $v^2 > u^2$ in solution (2.7). Again, $t$ is a simple increasing function of $\tau$:

$$t = \sqrt{2(v^2 - u^2)} \left( \frac{u^2 - v^2}{2} \tau + \frac{u^2 + v^2}{4} \sinh(2\tau) + \frac{u \cdot v}{2} (\cosh(2\tau) - 1) \right).$$

The above analysis, when combined with Proposition 2.2, yields all solutions to the equation of motion of the O(1)-Kepler problem at level $n$, though the dependence on time $t$ is only implicitly given. Moreover, for any solution $X(t)$ to Eq. (2.4) we have obtained above, one can check that the total trace of $q(X(t))$ always lies inside an affine plane of $H_n(\mathbb{R})$. Therefore, combining with Proposition 2.2 the above analysis implies

**Theorem 1.** For the O(1)-Kepler problem at level $n$, the followings are true.

1) A trajectory is always the intersection of the space $C_1$ with a plane, and it is bounded or unbounded according as the total energy $E$ is negative or not.

2) A bounded trajectory can be parametrized as $\alpha(\tau) = q(\cos \tau u + \sin \tau v)$ for some $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. Moreover, any parametrized curve of this form is a bounded trajectory with negative total energy $E = \frac{1}{\sqrt{v^2 - u^2}}$.

3) An unbounded trajectory with zero total energy can be parametrized as $\alpha(\tau) = q(u + \tau v)$ for some $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ with $v^2 = \frac{1}{2}$. Moreover, any parametrized curve of this form is a trajectory with zero total energy.

4) An unbounded trajectory with positive total energy can be parametrized as $\alpha(\tau) = q(\cosh \tau u + \sinh \tau v)$ for some $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ with $v^2 > u^2$. Moreover, any parametrized curve of this form is a trajectory with positive total energy $E = \frac{1}{v^2 - u^2}$.

3. **Non-colliding trajectories**

The interesting trajectories are the non-colliding ones, i.e., the ones such that in their parametrization $\alpha(\tau)$ given in theorem 1 $\alpha(\tau) \neq 0 \in H_n(\mathbb{R})$ for any $\tau \in \mathbb{R}$. It is evident that if $v$ is a scalar multiple of $u$ in theorem 1 then $\alpha(\tau) = 0$ for some finite value of $\tau$ and it is not hard to check that the converse is also true. Therefore, applied to non-colliding trajectories only, theorem 1 becomes

**Theorem 2.** For a non-colliding trajectory of the O(1)-Kepler problem at level $n$, the followings are true.

1) It is an ellipse, a parabola or a branch of hyperbola according as the total energy $E$ is negative, zero or positive.

(We assume in the next three statements that the variable $\tau$ runs over the entire $\mathbb{R}$.)

2) If it is an ellipse then it can be parametrized as $\alpha(\tau) = q(\cos \tau u + \sin \tau v)$ for some linearly independent $u, v \in \mathbb{R}^n$. Moreover, any parametrized curve of this form is an elliptic trajectory with negative total energy $E = -\frac{1}{u^2 + v^2}$.

3) If it is a parabola then it can be parametrized as $\alpha(\tau) = q(u + \tau v)$ for some linearly independent $u, v \in \mathbb{R}^n$. Moreover, any parametrized curve of this form is a parabolic trajectory with zero total energy.

4) If it is a branch of hyperbola then it can be parametrized as $\alpha(\tau) = q(\cosh \tau u + \sinh \tau v)$ for some linearly independent $u, v \in \mathbb{R}^n$ with $v^2 > u^2$. Moreover, any parametrized curve of this form is a hyperbolic trajectory with positive total energy $E = \frac{1}{v^2 - u^2}$.
Note that, in statement 3) of Theorem 2 the condition $v^2 = \frac{1}{2}$ is no longer needed because one can rescale $v$ due to the fact that $\tau \in \mathbb{R}$. Since the standard linear action of $\text{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ ($n \geq 2$) acts transitively on the set of linearly independent pairs of vectors in $\mathbb{R}^n$, Theorem 2 implies the following

**Theorem 3.** For the O(1)-Kepler problem at level $n$, the group $\text{GL}(n, \mathbb{R})/\text{O}(1)$ acts transitively on both set of elliptic trajectories and the set of parabolic trajectories.

This theorem is a direct analogue of parts 3) and 4) in Theorem 2 of Ref. [5]. Note that the O(1)-Kepler problem at level 2 is just the planar Kepler problem and $\text{GL}(n, \mathbb{R})/\text{O}(1)$ is the orientation-preserving automorphism group of $\mathbb{R}^n$.

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