ON GROMOV’S DIHEDRAL EXTREMALITY AND RIGIDITY CONJECTURES

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Abstract. In this paper, we prove Gromov’s dihedral extremality conjecture and dihedral rigidity conjecture regarding comparisons of scalar curvatures, mean curvatures and dihedral angles between two compact manifolds with corners possibly of different dimensions. As a consequence, we answer positively Gromov’s dihedral extremality conjecture for convex polyhedra in all dimensions, and Gromov’s dihedral rigidity conjecture for convex polyhedra in dimension three. A key ingredient of our proofs is a new index theory for manifolds with corners.

1. Introduction

In the past several years, Gromov has formulated an extensive list of conjectures and open questions on scalar curvature [15, 16, 17, 18]. This has given rise to new perspectives on scalar curvature and inspired a wave of recent activity in this area [6, 7, 11, 17, 18, 19, 20, 24, 27, 30, 31, 33, 34, 35]. In particular, Gromov proposed two rigidity conjectures: the dihedral extremality conjecture (Conjecture 1.2) [16] and the dihedral rigidity conjecture (Conjecture 1.3) [15] about comparisons of scalar curvatures, mean curvatures and dihedral angles for compact manifolds with corners, which can be viewed as scalar curvature analogues of the Alexandrov’s triangle comparisons for spaces whose sectional curvature is bounded below [1, 2]. These two conjectures have profound implications in geometry and mathematical physics such as the positive mass theorem, a foundational result in general relativity and differential geometry [28, 29] [32] (cf. [23, Section 5] and also the discussion after Theorem 1.9). In this paper, we answer positively Gromov’s dihedral extremality conjecture (Conjecture 1.2) for convex polyhedra in all dimensions, and Gromov’s dihedral rigidity conjecture (Conjecture 1.3) for convex polyhedra in dimension 3. In fact, we shall prove a more general theorem (Theorem 1.8) on comparisons of scalar curvatures, mean curvatures and dihedral angles between two compact manifolds with corners possibly of different dimensions, which includes Conjecture 1.2 and Conjecture 1.3 as its special cases.

Given a Riemannian metric $g$ on an oriented manifold $M$ with corners, we shall denote the scalar curvature of $g$ by $\text{Sc}(g)$, the mean curvature of each face $F_i$ of
Angle in $(0, \pi)$. Angle in $(\pi, 2\pi)$.

Figure 1. Dihedral angles.

$M$ by $H_g(F_i)$, and the dihedral angle function of two adjacent faces $F_i$ and $F_j$ by $\theta_{ij}(g)$. Here the dihedral angle $\theta_{ij}(g)_x$ at a point $x \in F_i \cap F_j$ is defined as follows.

**Definition 1.1.** Write $F_{ij} = F_i \cap F_j$. Let $u$ and $v$ be the unit inner normal vector of $F_{ij}$ with respect to $F_i$ and $F_j$ at $x \in F_{ij}$, respectively. Let $\theta_{ij}(g)_x$ be either the angle of $u$ and $v$, or $\pi$ plus this angle, depending on the vector $(u + v)/2$ points inward or outward, respectively. See Figure 1.

Here the angle $\theta_{ij}(g)_x$ takes value in $(0, \pi) \cup (\pi, 2\pi)$. Roughly speaking, if $M$ is convex at $x$, then $\theta_{ij}(g)_x < \pi$; and if $M$ is concave at $x$, then $\theta_{ij}(g)_x > \pi$.

We remark that if one allows the corner structure of the manifold $M$ to have degeneracy, then $\theta_{ij}(g)_x = \pi$ if $u = -v$, and $\theta_{ij}(g)_x = 0$ if $u = v$.

Furthermore, our sign convention for the mean curvature is that the mean curvature of the standard round sphere viewed as the boundary of a Euclidean ball is positive.

**Conjecture 1.2** (Gromov’s dihedral extremal conjecture for convex polyhedra, [16, Section 7]). Let $P$ be a convex polyhedron in $\mathbb{R}^n$ and $g_0$ the Euclidean metric on $P$. If $g$ is a smooth Riemannian metric on $P$ such that

1. $\text{Sc}(g) \geq \text{Sc}(g_0) = 0$,
2. $H_g(F_i) \geq H_{g_0}(F_i) = 0$ for each face $F_i$ of $P$, and
3. $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ on each $F_{ij} = F_i \cap F_j$,

then we have

$\text{Sc}(g) = 0, H_g(F_i) = 0$ and $\theta_{ij}(g) = \theta_{ij}(g_0)$

for all $i$ and all $j \neq i$.

In other words, the dihedral extremal conjecture states that on a convex polyhedron, one cannot simultaneously increase the scalar curvature of the metric and the mean curvature of the faces while decreasing the dihedral angles at the corners.

**Conjecture 1.3** (Gromov’s dihedral rigidity conjecture for convex polyhedra, [15, Section 2.2]). Let $P$ be a convex polyhedron in $\mathbb{R}^n$ and $g_0$ the Euclidean metric on $P$. If $g$ is a smooth Riemannian metric on $P$ such that

1. $\text{Sc}(g) \geq \text{Sc}(g_0) = 0$,
2. $H_g(F_i) \geq H_{g_0}(F_i) = 0$ for each face $F_i$ of $P$, and
(3) $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ on each $F_{ij} = F_i \cap F_j$, then $g$ is also a flat metric.

In dimension two, both conjectures are immediate consequences of the classical Gauss–Bonnet theorem for compact surfaces with boundary. In higher dimensions, Gromov showed that the dihedral extremal conjecture holds for a class of convex polyhedra, under some extra restrictions on dihedral angles [15]. In particular, Gromov showed that the dihedral extremal conjecture holds for the standard Euclidean cube $[0,1]^n$ for all $n \geq 2$ [15]. More recently, Li proved the dihedral rigidity conjecture for a class of convex polyhedra [23, 24], which includes:

(1) cone type 3-dimensional polyhedra satisfying some extra dihedral angle conditions,

(2) prism-type polyhedra of dimension $n \leq 7$, that is, $P$ is the Cartesian product $P_0 \times [0,1]^{n-2}$, where $P_0$ is 2-dimensional polygon with non-obtuse dihedral angles.

In this paper, we shall prove Gromov’s dihedral extremality conjecture for convex polyhedra (Conjecture 1.2) in complete generality. Furthermore, our proof also simultaneously implies a weaker version of Gromov’s dihedral rigidity conjecture for convex polyhedra in all dimensions, where we show that $g$ is Ricci flat instead of showing $g$ is flat. As Ricci flatness coincides with flatness in dimension three, our theorem completely settles Gromov’s dihedral rigidity conjecture for all 3-dimensional convex polyhedra. In fact, we shall derive the dihedral extremality conjecture and dihedral rigidity conjecture for polyhedra as a consequence of a more general theorem (Theorem 1.7 or Theorem 1.8) on comparisons of scalar curvatures, mean curvatures and dihedral angles between two compact manifolds with corners, which are possibly of different dimensions.

Before we state the theorems, let us first fix some notation. Let $f : (N, \overline{g}) \rightarrow (M,g)$ be a smooth map between two Riemannian manifolds with corners. We have the linear maps

$df : TN \rightarrow TM$ and more generally $\wedge^k df : \wedge^k TN \rightarrow \wedge^k TM$ where $df$ is the tangent map and $\wedge^k TN$ is the $k$-th exterior product of $TN$.

**Definition 1.4.** We define $\|df\|_x$ to be the norm of the map

$df : T_xN \rightarrow T_{f(x)}M$

and more generally $\|\wedge^k df\|_x$ to be the norm of the map

$\wedge^k df : \wedge^k T_xN \rightarrow \wedge^k T_{f(x)}M$.

We say $f$ is distance-non-increasing if $\|df\|_x \leq 1$ for all $x \in N$. Similarly, we say $f$ is area-non-increasing if $\|\wedge^2 df\|_x \leq 1$ for all $x \in N$.

**Definition 1.5.** A smooth map $f : N \rightarrow M$ is called a spin map if the second Stiefel–Whitney classes of $TM$ and $TN$ are related by

$w_2(TN) = f^*(w_2(TM))$. 
Equivalently, \( f: N \to M \) is a spin map if \( TN \oplus f^*TM \) admits a spin structure.

Note that here we do not require either \( M \) or \( N \) to be a spin manifold. On the other hand, if \( M \) happens to be a spin manifold, then \( f \) being a spin map implies that \( N \) is also a spin manifold.

**Definition 1.6.** A map \( f: N \to M \) between manifolds with corners is called a corner map if

1. \( f \) is smooth map\(^1\) between manifolds with corners;
2. \( f \) maps faces to faces, that is, for each codimension one face \( F_i \) of \( N \), we have \( f(F_i) \subseteq F_i \) for some codimension one face \( F_i \) of \( M \);
3. for any collection of codimension one faces, say, \( \{ F_1, \ldots, F_k \} \) and any \( x \in \cap_{j=1}^k F_j \), the tangent map
   \[
   f_*: T_xN \to T_{f(x)}M
   \]
   restricted to the linear subspace \( \mathcal{N}_x \subset T_xN \) is injective and
   \[
   \cap_{j=1}^k TF_j \cap f_*(\mathcal{N}_x) = 0,
   \]
   where \( \mathcal{N}_x \) is the linear subspace of \( T_xN \) spanned by the normal vectors of \( \{ F_j \}_{j=1}^k \) at \( x \), and \( F_j \) is the corresponding codimension one face in \( M \) such that \( f(F_j) \subseteq F_j \).

We have the following main theorem of the paper.

**Theorem 1.7.** Let \((N, \overline{g})\) and \((M, g)\) be compact oriented Riemannian manifolds with corners. Suppose

(a) the curvature operator of \( g \) is non-negative,
(b) each codimension one face \( F_i \) of \( M \) is convex, that is, the second fundamental form of \( F_i \) is non-negative,
(c) all dihedral angles \( \theta_{ij}(g) \) of \( M \) are \( \leq \pi \).

Let \( f: (N, \overline{g}) \to (M, g) \) be a corner map of manifolds with corners. Suppose \( f \) is a spin map and

1. \( f \) is area-non-increasing on \( N \), and \( f \) is distance-non-increasing on the boundary \( \partial N \),
2. \( \text{Sc}(\overline{g})_x \geq \text{Sc}(g)_{f(x)} \) for all \( x \in N \),
3. \( H_{\overline{g}}(\overline{F}_i)_y \geq H_g(F_i)_{f(y)} \) for all \( y \) in each codimension one face\(^2\) \( \overline{F}_i \) of \( N \),
4. \( \theta_{ij}(\overline{g})_z \leq \theta_{ij}(g)_{f(z)} \) for all \( \overline{F}_i, \overline{F}_j \) and all \( z \in \overline{F}_i \cap \overline{F}_j \),
5. \( M \) has nonzero Euler characteristic,

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\(^1\)Recall that \( f \) is a smooth map between manifolds with corners if and only if \( f \) is the restriction of a smooth map \( \varphi: N' \to M' \), where \( N' \) and \( M' \) are two open manifolds which respectively contain \( N \) and \( M \) as their submanifolds.

\(^2\)Here the notations \( \overline{F}_i \) and \( F_i \) are chosen so that the map \( f \) takes the face \( \overline{F}_i \) of \( N \) to the face \( F_i \) of \( M \).
(6) the $\hat{A}$-degree $\text{deg}_{\hat{A}}(f)$ of $f$ is nonzero, that is, 

$$
\text{deg}_{\hat{A}}(f) := \int_N \hat{A}(N) \wedge f^*[M] \neq 0.
$$

Then we have

(i) $\text{Sc}(\pi) x = \text{Sc}(g)_{f(x)}$ for all $x \in N$,

(ii) $H_\pi(F_i)_y = H_g(F_i)_{f(y)}$ for all $y \in F_i$,

(iii) $\theta_{ij}(\pi)_z = \theta_{ij}(g)_{f(z)}$ for all $F_i, F_j$ and all $z \in F_i \cap F_j$,

and the following are true.

(I) If $\text{Ric}(g) > 0$ and $f$ is distance-non-increasing on the whole $N$, then $f$ is a Riemannian submersion. Here $\text{Ric}(g)$ is the Ricci curvature of the metric $g$ on $M$.

(II) If $0 < \text{Ric}(g) < \frac{1}{2} \text{Sc}(g) \cdot g$, then $f$ is a Riemannian submersion.

(III) If $(M, g)$ is flat, then $(N, \pi)$ is Ricci flat. Consequently, $(N, \pi)$ is flat if $

\dim N = 3.

In the above, $\hat{A}(N)$ is the $\hat{A}$-class of $N$ and $[M] \in H^{\dim M}(M, \partial M)$ is the fundamental class of $M$ (with respect to the given orientation on $M$). In fact, with a little more care, we can improve the estimates to obtain the following strengthening of Theorem 1.7.

**Theorem 1.8.** Let $(N, \pi)$ and $(M, g)$ be compact oriented Riemannian manifolds with corners. Suppose

(a) the curvature operator of $g$ is non-negative,

(b) each codimension one face $F_i$ of $M$ is convex, that is, the second fundamental form of $F_i$ is non-negative,

(c) all dihedral angles $\theta_{ij}(g)$ of $M$ are $\leq \pi$.

Let $f : (N, \pi) \to (M, g)$ be a corner map of manifolds with corners. Suppose $f$ is a spin map and

(1) $\text{Sc}(\pi)_x \geq \| \wedge^2 df \| \cdot \text{Sc}(g)_{f(x)}$ for all $x \in N$,

(2) $H_\pi(F_i)_y \geq \| df \| \cdot H_g(F_i)_{f(y)}$ for all codimension one faces $F_i$ of $N$ and all $y \in F_i$,

(3) $\theta_{ij}(\pi)_z \leq \theta_{ij}(g)_{f(z)}$ for all $F_i, F_j$ and all $z \in F_i \cap F_j$,

(4) $M$ has nonzero Euler characteristic,

(5) the $\hat{A}$-degree $\text{deg}_{\hat{A}}(f)$ of $f$ is nonzero,

Then we have

(i) $\text{Sc}(\pi)_x = \| \wedge^2 df \| \cdot \text{Sc}(g)_{f(x)}$ for all $x \in N$,

(ii) $H_\pi(F_i)_y = \| df \| \cdot H_g(F_i)_{f(y)}$ for all $y \in F_i$,

(iii) $\theta_{ij}(\pi)_z = \theta_{ij}(g)_{f(z)}$ for all $F_i, F_j$ and all $z \in F_i \cap F_j$,

and the following are true.
(I) Suppose $\dim M = \dim N$. If $\text{Ric}(g) > 0$ and
$$\text{Sc}(\bar{g}) \geq \|df\|^2 \cdot \text{Sc}(g)_{f(x)}$$
for all $x \in N$, then $\|df\| \equiv a$ for some constant $a > 0$ and $f : (N, a \cdot \bar{g}) \to (M, g)$ is a Riemannian covering map.

(II) If $\dim M = \dim N$ and $0 < \text{Ric}(g) < \frac{1}{2} \text{Sc}(g) \cdot g$, then $\|\wedge^2 df\| \equiv c$ for some constant $c > 0$ and $f : (N, \sqrt{c} \cdot \bar{g}) \to (M, g)$ is a Riemannian covering map.

(III) If $(M, g)$ is flat, then $(N, \bar{g})$ is Ricci flat. Consequently, $(N, \bar{g})$ is flat if $\dim N = 3$.

We point out that both Theorem 1.7 and Theorem 1.8 hold for manifolds with more generalized corner structures, for example, manifolds with polyhedron-like boundary. A key step of the proof of Theorem 1.8 is to analyze Dirac type operators that arise from asymptotically conical metrics (cf. Proposition 2.6, Theorem 3.8 and Theorem 3.19). Since a Riemannian metric on a manifold with polyhedron-like boundary is also asymptotically conical near its singular points, the same exact proofs of Proposition 2.6, Theorem 3.8 and Theorem 3.19 also apply to Dirac type operators on manifolds with polyhedron-like boundary. In particular, Theorem 1.7 and Theorem 1.8 hold when $(M, g)$ is a convex polyhedron in $\mathbb{R}^n$. As an immediate consequence, we obtain the following theorem, which solves Gromov’s dihedral extremality conjecture (Conjecture 1.2) for convex polyhedra in all dimensions, and Gromov’s dihedral rigidity conjecture (Conjecture 1.3) for convex polyhedra in dimension three.

**Theorem 1.9.** Let $P$ be a convex polyhedron in $\mathbb{R}^n$ and $g_0$ the Euclidean metric on $P$. If $g$ is a smooth Riemannian metric on $P$ such that

1. $\text{Sc}(g) \geq \text{Sc}(g_0) = 0$,
2. $H_g(F_i) \geq H_{g_0}(F_i) = 0$ for each face $F_i$ of $P$, and
3. $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ on each $F_{ij} = F_i \cap F_j$,

then we have
$$\text{Sc}(g) = 0, H_g(F_i) = 0 \text{ and } \theta_{ij}(g) = \theta_{ij}(g_0)$$
for all $i$ and all $j \neq i$. Furthermore, $g$ is Ricci flat. Consequently, $g$ is flat if $\dim P = 3$.

Theorem 1.7 and Theorem 1.8 can be viewed as a localization of the well-known positive mass theorem (for spin manifolds). Recall that the positive mass theorem states that if $(X, g)$ is a complete asymptotically Euclidean manifold of dimension $n \geq 3$ such that its scalar curvature is non-negative, then the ADM mass of each end of $X$ is non-negative. We refer the reader to [28, 29] [32] for the precise meanings of “asymptotically Euclidean” and ADM mass. Here we shall briefly indicate how one can deduce the positive mass theorem for spin manifolds from Theorem 1.8 (cf. [23, Section 5] for a similar discussion). Indeed, by a result of Lohkamp [26, Lemma 6.2], if the ADM mass of an end of $X$ is negative, then one can reduce to the case where $X$ has only one end and the ADM mass of that end is negative. In this case, again by a result of Lohkamp [26, Proposition 6.1],
there exists another complete Riemannian metric $g_1$ on $X$ with $\text{Sc}(g_1) \geq 0$ and $\text{Sc}(g_1)_x > 0$ for some point $x \in X$ such that there is a compact set $K \subset X$ with $(X - K, g_1)$ being isometric to the standard Euclidean space minus a ball. Choose a large flat cube in $\mathbb{R}^n$ and denote it by $M$. Let $Z$ be the isometric copy of $\mathbb{R}^n - M$ in $X - K$, and define $N$ to be $X - Z$. Note that $N$ is a spin manifold, since we have assumed $X$ is spin. The boundary $\partial N$ of $N$ is isometric to the boundary $\partial M$ by construction. Furthermore, since $M$ is a flat cube, clearly there exists a smooth map $f : N \to M$ such that $f$ equals the identity near the boundary and all conditions for $f$ in Theorem 1.8 are satisfied. For example, take $f : N \to M$ to be a map that is identity near the boundary and crashes $K \subset N$ to a point in $M$. However the scalar curvature on $N$ is strictly positive somewhere, we arrive at a contradiction. Therefore, we see that Theorem 1.8 implies the positive mass theorem for spin manifolds.

Our strategy to prove Theorem 1.7 and Theorem 1.8 is to use Dirac type operators with appropriate elliptic boundary conditions. Let us briefly outline the key steps. First, we shall find a suitable elliptic boundary condition for the relevant twisted Dirac operator that naturally arises in our geometric setup, and use it to construct a self-adjoint Fredholm operator. As we are dealing with manifolds with corners, the Riemannian metric on the boundary is not smooth but with singularities. In general, the analysis for elliptic boundary problems of differential operators on manifolds with singularities is rather delicate. The approach that we develop in the current paper takes advantage of the special features of both the operators and the underlying geometry, and should be of independent interest on its own. Due to the presence of corners, we are naturally led to the analysis of operators that arise from Riemannian metrics of conical type. There is an extensive literature on this type of analysis since the work of Cheeger [9, 8, 10]. For general Dirac type operators, one usually needs to impose extra ideal boundary conditions (on the links of cones) in order to have a self-adjoint extension of the given operator. However, this would make it rather difficult to keep track of the Fredholm index when we deform the underlying Riemannian metric, which is a step that is needed to assure that we are working with a Fredholm operator with nonzero index. A key observation that greatly simplifies the computation of the Fredholm index in the geometric setup of the current paper is the following. Near the conical singularities (of the Riemannian metric), the Dirac type operator that appears in our geometric setting can be viewed as the product of a Dirac type operator on a closed manifold and the de Rham operator on a certain cone (more precisely, the cone over a sector of some sphere). Furthermore, the assumption on dihedral angles in Theorem 1.7 and 1.8 ensures a natural choice of an elliptic boundary condition that makes the operator essentially self-adjoint, and consequently produces a Fredholm operator whose index is precisely the Euler characteristic of $M$ multiplied by the $\hat{A}$-degree of $f$.

Now the next key step is to extract the geometric information such as the scalar curvature, mean curvature and dihedral angles from the index. While it is
relatively straightforward to identify the contributions to the index of the scalar curvature and mean curvature, it is trickier to identify the contribution of the dihedral angles. Intuitively speaking, one can interpret the dihedral angle of two faces as the distributional mean curvature at the intersection where the two faces meet. To make this precise, we shall carefully choose a family of approximations of $N$ via manifolds with smooth boundary so that the scalar curvature and mean curvature contributions calculated on these approximations eventually converge to the contributions of scalar curvature, mean curvature and dihedral angles of $N$, from which we deduce Theorem 1.7 and Theorem 1.8.

We point out that Lott proved Theorem 1.7 for even dimensional manifolds with smooth boundary (in this case, there are no dihedral angles) [27]. In the case of odd dimensional manifolds with smooth boundary, a natural approach is to reduce it to the even dimensional case by taking direct products of the manifolds with the closed unit interval. But such a product construction results in manifolds with corners. This explains the importance of studying manifolds with corners.

This paper is organized as follows. In Section 2, we establish some key estimates for scalar curvature, mean curvature and dihedral angles on manifolds with corners. In Section 3, we introduce some natural elliptic boundary conditions for Dirac operators (twisted with coefficients) on manifolds with corners. We use these boundary conditions to prove that the relevant Dirac operators are essentially self-adjoint under a suitable condition on the dihedral angles. We then prove an index theorem for these Dirac operators (subject to local boundary conditions) on manifolds with corners. Finally in Section 4, we prove the main results of this paper.

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2. Estimates of scalar curvature, mean curvature and dihedral angles on manifolds with corners

In this section, we establish some estimates for scalar curvature, mean curvature and dihedral angles on manifolds with corners. These estimates constitute a key ingredient of the proof of our main theorem.

2.1. Estimates of scalar curvature and mean curvature on manifolds with smooth boundary. Let us first review some estimates for scalar curvature and mean curvature on manifolds with smooth boundary, cf. [14, 25, 27].

Let $(M, g)$ and $(N, \overline{g})$ be two oriented compact Riemannian manifolds with smooth boundary. Suppose $f : N \to M$ is a spin map. So the bundle $TN \oplus f^* TM$ admits a spin structure. Let us denote by $S_N \otimes f^* S_M$ the associated spinor bundle over $N$. The vector bundle $S_N \otimes f^* S_M$ carries a natural Hermitian metric and a unitary connection $\nabla$ compatible with Clifford multiplication by elements of
the Bochner–Lichnerowicz–Weitzenbock formula, we have

\[
\sum_{i} c(e_i)\nabla e_i, \quad (2.1)
\]

where \(\{e_i\}\) is a local orthonormal basis with respect to the metric \(\bar{g}\) on \(TN\). By the Bochner–Lichnerowicz–Weitzenbock formula, we have

\[
D^2 = \nabla^* \nabla + \frac{\overline{Sc}}{4} + \frac{1}{8} \sum_{i,j} \sum_{k,l} \langle f^* R_{e_i,e_j}^M e_k, e_l \rangle_M \bar{c}(e_i)\bar{c}(e_j) \otimes c(e_k)c(e_l), \quad (2.2)
\]

where \(\overline{Sc} := \text{Sc}(\bar{g})\) is the scalar curvature of \(N\), \(\{e_i\}\) is a local orthonormal basis of \(f^*TM\), and \(f^*R^M\) is the curvature form of \(f^*TM\).

Let us define Clifford multiplication by 2-forms to be

\[
\bar{c}(u \wedge v) = \bar{c}(u)\bar{c}(v), \quad c(u \wedge v) = c(u)c(v),
\]

for all \(\bar{u}, \bar{v} \in TN\) and \(u, v \in (f^*TM)_x\) with \(\bar{g}(\bar{u}, \bar{v}) = 0 = g(u, v)\). If \(\{\pi_j\}\) and \(\{w_i\}\) are local orthonormal bases of \(\Lambda^2TN\) and \(f^*\Lambda^2TM\) respectively, then we can rewrite (2.2) as

\[
D^2 = \nabla^* \nabla + \frac{\overline{Sc}}{4} - \frac{1}{2} \sum_{i,j} \langle R^M f_*(\pi_j), w_i \rangle_M \bar{c}(\pi_j) \otimes c(w_i), \quad (2.4)
\]

Let \(\varphi\) be a smooth section of \(S_N \otimes f^*S_M\) over \(N\). By the Stokes formula, we have

\[
\int_N \langle D\varphi, D\varphi \rangle = \int_N \langle D^2\varphi, \varphi \rangle + \int_{\partial N} \langle \bar{c}(n) D\varphi, \varphi \rangle, \quad (2.5)
\]

where \(\bar{c}_n\) denotes the unit inner normal vector to \(\partial N\). From line (2.4), we have

\[
\int_N \langle D^2\varphi, \varphi \rangle = \int_N \langle \nabla^* \nabla \varphi, \varphi \rangle + \int_N \frac{\overline{Sc}}{4} |\varphi|^2
\]

\[
- \frac{1}{2} \int N \sum_{i,j} \langle R^M f_*(\pi_j), w_i \rangle_M \bar{c}(\pi_j) \otimes c(w_i) \varphi, \varphi \rangle.
\]

We have the following lemma (cf. [14, Section 1.1]).

**Lemma 2.1.** If the curvature operator of \(M\) is non-negative, then

\[
- \frac{1}{2} \sum_{i,j} \langle R^M f_*(\pi_j), w_i \rangle_M \bar{c}(\pi_j) \otimes c(w_i) \geq -\|\Lambda^2 df\|^2 \cdot \frac{f^*\overline{Sc}}{4}. \quad (2.6)
\]
Proof. As the curvature operator $\mathcal{R}^M$ is non-negative, there exists a self-adjoint $L \in \text{End}(\Lambda^2\mathcal{T}M)$ such that $\mathcal{R}^M = L^2$, that is, $\langle \mathcal{R}^M w_j, w_i \rangle_M = \langle Lw_j, Lw_i \rangle_M$.

Set $\overline{L}w_k := \sum_i \langle Lw_k, f_\ast \overline{w}_i \rangle_M \overline{w}_i \in \Lambda^2\mathcal{T}N$.

If $\| \Lambda^2 df \|$ is zero, that is, $\Lambda^2 df = 0$, then both sides of line (2.6) are zero. We assume that $\| \Lambda^2 df \| > 0$ and set $\alpha = \sqrt{\| \Lambda^2 df \|}$. The left hand side of line (2.6) can be written as

$$\frac{1}{2} \sum_{i,j,k} \langle R^M f_\ast \overline{w}_j, w_i \rangle_M \overline{c}(\overline{w}_j) \otimes c(w_i)$$

$$= -\frac{1}{2} \sum_{i,j,k} \langle L(f_\ast \overline{w}_j), w_k \rangle_M \cdot \langle Lw_i, w_k \rangle_M \cdot \overline{c}(\overline{w}_j) \otimes c(w_i)$$

$$= -\frac{1}{2} \sum_k \overline{c}(\overline{L}w_k) \otimes c(Lw_k)$$

$$= \frac{1}{4} \sum_k \left( \alpha^{-2} \overline{c}(\overline{L}w_k)^2 \otimes 1 + 1 \otimes \alpha^2 c(Lw_k)^2 - \left( \alpha^{-1} \overline{c}(\overline{L}w_k) \otimes 1 + 1 \otimes \alpha c(Lw_k) \right)^2 \right)$$

$$\geq \frac{1}{4} \sum_k \alpha^{-2} \overline{c}(\overline{L}w_k)^2 \otimes 1 + \frac{1}{4} \sum_k 1 \otimes \alpha^2 c(Lw_k)^2,$$

where the last inequality follows from the fact that the element

$$\alpha^{-1} \overline{c}(\overline{L}w_k) \otimes 1 + 1 \otimes \alpha c(Lw_k)$$

is skew-symmetric, hence its square is non-positive.

Now we consider the terms $\sum_k \alpha^{-2} \overline{c}(\overline{L}w_k)^2 \otimes 1$ and $\sum_k 1 \otimes \alpha^2 c(Lw_k)^2$. The same proof for the Lichnerowicz formula (cf. [21, Theorem II.8.8]) shows that

$$\sum_k \alpha^2 c(Lw_k)^2 = -\alpha^2 \cdot \frac{f^*Sc}{2} = -\| \Lambda^2 df \| \cdot \frac{f^*Sc}{2}.$$  

Similarly, by the definition of $\overline{L}$, we have

$$\sum_k \overline{c}(\overline{L}w_k)^2 = \sum_{i,j,k} \langle \overline{L}w_k, f_\ast \overline{w}_i \rangle_M \cdot \langle \overline{L}w_k, f_\ast \overline{w}_j \rangle_M \cdot \overline{c}(\overline{w}_i) \otimes c(\overline{w}_j)$$

$$= \sum_{i,j,k} \langle \mathcal{R}^M f_\ast \overline{w}_i, f_\ast \overline{w}_j \rangle_M \cdot \overline{c}(\overline{w}_i) \overline{c}(\overline{w}_j).$$

We choose a local $\overline{\mathfrak{g}}$-orthonormal frame $\overline{e}_1, \ldots, \overline{e}_n$ of $\mathcal{T}N$ and a local $g$-orthonormal frame $e_1, \ldots, e_m$ of $\mathcal{T}M$ such that $f_\ast \overline{e}_i = \mu_i e_i$ with $\mu_i \geq 0$ for any $i \leq \min\{m, n\}$, and $f_\ast \overline{e}_i = 0$ otherwise. This can be done by diagonalizing $f^*g$ with respect to the metric $\overline{\mathfrak{g}}$. Then we have $f_\ast (\overline{e}_i \wedge \overline{e}_j) = \mu_i \mu_j e_i \wedge e_j$. Clearly, we have $\mu_i \mu_j \leq \| \Lambda^2 df \|$ for all $i, j$ with $i \neq j$. Therefore we have

$$\sum_k \alpha^{-2} \overline{c}(\overline{L}w_k)^2 = -\alpha^{-2} \sum_{i,j} \mu_i^2 \mu_j^2 (f^*R^M_{ijji}) \geq -\| \Lambda^2 df \| \cdot \frac{f^*Sc}{2}.$$
This finishes the proof.

Next we shall review a comparison formula for mean curvature. By the Stokes formula, we have

$$\int_N \langle \nabla^* \nabla \phi, \phi \rangle = \int_N |\nabla \phi|^2 + \int_{\partial N} \langle \nabla_{\nabla_n} \phi, \phi \rangle$$

(2.7)

for all smooth sections $\phi$ of $S_N \otimes f^* S_M$.

**Definition 2.2.** We define $c_{\partial}$ to be the Clifford action of $T(\partial N)$ on the bundle $S_N \otimes f^* S_M$ given by

$$c_{\partial}(e_{\lambda}) = \overline{c}(e_n) c(e_{\lambda}),$$

for all $e_{\lambda} \in T(\partial N)$, where $e_n$ is the unit inner normal vector to $\partial N$. Similarly, we define $c_{\partial}$ to be the Clifford action of $f^* T(\partial M)$ on the bundle $S_N \otimes f^* S_M$ given by

$$c_{\partial}(e_{\lambda}) = c(e_n) c(e_{\lambda}),$$

where $e_n$ is the unit inner normal vector to $\partial M$ in $M$.

The boundary Dirac operator $D^\partial$ acting on $S_N \otimes f^* S_M$ over $\partial N$ is given by

$$D^\partial := \sum_{\lambda} \overline{c}_{\partial}(e_{\lambda}) \nabla^\partial_{e_{\lambda}},$$

(2.8)

where $\nabla^\partial$ is the connection on $S_N \otimes f^* S_M$ over $\partial N$ defined by

$$\nabla^\partial = \nabla - \frac{1}{2} \sum_{\mu} \langle N \nabla e_n, e_\mu \rangle_N \cdot \overline{c}(e_n) c(e_\mu) \otimes 1$$

$$- \frac{1}{2} \sum_{\mu} \langle M \nabla e_n, e_\mu \rangle_M \cdot 1 \otimes c(e_n) c(e_\mu),$$

(2.9)

where $^N \nabla$ and $^M \nabla$ are the Levi–Civita connections on $N$ and $M$ respectively (cf. [4, Theorem 2.7]). Here we use Greek symbols $\lambda$ and $\mu$ to indicate that the summation is taken over basis vectors tangential to the boundary. We have

$$\overline{c}(e_n) D + \nabla_{e_n} = \overline{c}(e_n) \sum_{\lambda} \overline{c}(e_{\lambda}) \nabla_{e_{\lambda}} = D^\partial + \sum_{\lambda} \overline{c}_{\partial}(e_{\lambda}) (\nabla_{e_{\lambda}} - \nabla^\partial_{e_{\lambda}}).$$

(2.10)

Let $\overline{A}$ be the second fundamental form of $\partial N$ in $N$, that is,

$$^N \nabla_{e_{\lambda}} e_\mu - ^{\partial N} \nabla_{e_{\lambda}} e_\mu = \overline{A}_{\lambda,\mu} e_n.$$ 

Let $\overline{H}$ be the mean curvature of $\partial N$, that is, \(^3 \overline{H} := \text{tr} \overline{A} \). Similarly, we define on $M$

$$^M \nabla_{e_{\lambda}} e_\mu - ^{\partial M} \nabla_{e_{\lambda}} e_\mu = \overline{A}_{\lambda,\mu} e_n.$$ 

and $\overline{H} = \text{tr} \overline{A}$. Note that

$$\sum_{\lambda,\mu} \overline{c}_{\partial}(e_{\lambda}) \langle ^N \nabla_{e_{\lambda}} e_n, e_\mu \rangle_N \cdot \overline{c}(e_n) c(e_\mu) = - \sum_{\lambda,\mu} \overline{A}_{\lambda,\mu} \overline{c}_{\partial}(e_{\lambda}) \overline{c}_{\partial}(e_\mu) = \overline{H},$$

\(^3\text{Our convention of the mean curvature is that the mean curvature is the trace of the second fundamental form, or equivalently the sum of all principal curvatures.}\)
and
\[
\sum_{\lambda, \mu} \tilde{c}_{\partial}(\tilde{e}_\lambda) \otimes \left( \langle M \nabla_{f^* \tilde{e}_\lambda} e_\mu, e_\nu \rangle_M c(e_\mu) c(e_\nu) \right) = - \sum_{\lambda, \mu} A(f^* \tilde{e}_\lambda, e_\mu) \tilde{c}_{\partial}(\tilde{e}_\lambda) \otimes c_\partial(e_\mu).
\]

We obtain that
\[
\sum_{\lambda} \tilde{c}_{\partial}(\tilde{e}_\lambda) (\nabla_{\tilde{e}_\lambda} - \nabla^{\partial}_{\tilde{e}_\lambda}) = \frac{\Pi}{2} - \frac{1}{2} \sum_{\lambda, \mu} A(f^* \tilde{e}_\lambda, e_\mu) \tilde{c}_{\partial}(\tilde{e}_\lambda) \otimes c_\partial(e_\mu). \tag{2.11}
\]

For the last term on the right hand side of the above equation, we have the following lemma (cf. [27, Lemma 2.1]).

**Lemma 2.3.** If the second fundamental form \( A \) of \( \partial M \) is non-negative, then
\[
-\frac{1}{2} \sum_{\lambda, \mu} A(f^* \tilde{e}_\lambda, e_\mu) \tilde{c}_{\partial}(\tilde{e}_\lambda) \otimes c_\partial(e_\mu) \geq -\|df\| \cdot \frac{f^* H}{2}. \tag{2.12}
\]

**Proof.** The strategy is similar to that of Lemma 2.1. As the second fundamental form \( A \) is non-negative, there exists a self-adjoint operator \( L \in \text{End}(TM) \) such that \( A = L^2 \), that is,
\[
A(e_\lambda, e_\mu) = \langle Le_\lambda, Le_\mu \rangle_M.
\]

Let us define
\[
\mathcal{L}e_\nu := \sum_{\lambda} \langle Le_\nu, f^* \tilde{e}_\lambda \rangle_M \cdot \tilde{e}_\lambda.
\]

If \( \|df\| = 0 \), that is, \( df = 0 \), then both sides of line (2.12) are zero. If \( \|df\| > 0 \), we set \( \alpha = \sqrt{\|df\|} \) and rewrite the left hand side of line (2.12) as
\[
-\frac{1}{2} \sum_{\lambda, \mu} A(f^* \tilde{e}_\lambda, e_\mu) \tilde{c}_{\partial}(\tilde{e}_\lambda) \otimes c_\partial(e_\mu)
\]
\[
= -\frac{1}{2} \sum_{\lambda, \mu, \nu} \langle L(f^* \tilde{e}_\lambda), e_\nu \rangle_M \langle L(e_\mu), e_\nu \rangle_M \tilde{c}_{\partial}(\tilde{e}_\lambda) \otimes c_\partial(e_\mu)
\]
\[
= -\frac{1}{2} \sum_{\nu} \tilde{c}_{\partial}(\mathcal{L}e_\nu) \otimes c_\partial(Le_\nu)
\]
\[
= \frac{1}{4} \sum_{\nu} \left( \alpha^{-2} \tilde{c}_{\partial}(\mathcal{L}e_\nu)^2 \otimes 1 + 1 \otimes \alpha^2 c_\partial(Le_\nu)^2 - \left( \alpha^{-1} \tilde{c}_{\partial}(\mathcal{L}e_\nu) \otimes 1 + 1 \otimes \alpha c_\partial(Le_\nu) \right)^2 \right)
\]
\[
\geq \frac{1}{4} \sum_{\nu} \alpha^{-2} \tilde{c}_{\partial}(\mathcal{L}e_\nu)^2 \otimes 1 + \frac{1}{4} \sum_{\nu} 1 \otimes \alpha^2 c_\partial(Le_\nu)^2,
\]

where the last inequality follows from the fact that the element
\[
(\alpha^{-1} \tilde{c}_{\partial}(\mathcal{L}e_\nu) \otimes 1 + 1 \otimes \alpha c_\partial(Le_\nu))
\]
is skew-symmetric, hence its square is non-positive.
If we write \( Le_\nu = \sum_\lambda L_{\nu\lambda} \cdot e_\lambda \), then we have

\[
\alpha^2 \sum_\nu c_\partial(Le_\nu)^2 = \alpha^2 \sum_\nu L_{\nu\lambda} L_{\nu\lambda} \cdot c_\partial(e_\lambda)^2 + \alpha^2 \sum_\nu L_{\nu\lambda} L_{\nu\mu} \cdot c_\partial(e_\lambda) c_\partial(e_\mu)
\]

\[
= - \alpha^2 \sum_\lambda A_{\lambda\lambda} + \alpha^2 \sum_{\lambda\neq \mu} A_{\lambda\mu} \cdot c_\partial(e_\lambda) c_\partial(e_\mu)
\]

\[
= - \alpha^2 \text{tr}(A) = -\|df\| \cdot H.
\]

Similarly, since \( \langle f_\ast \xi_\nu, f_\ast \xi_\mu \rangle_M \leq \|df\|^2 \cdot \langle \xi_\nu, \xi_\mu \rangle \), we have

\[
\alpha^{-2} \sum_\nu c_\partial(Le_\nu)^2 = \alpha^{-2} \sum_{\lambda, \mu} A(f_\ast \xi_\lambda, f_\ast \xi_\mu) \cdot c_\partial(e_\lambda) c_\partial(e_\mu)
\]

\[
= - \alpha^{-2} \sum_\lambda A(f_\ast \xi_\lambda, f_\ast \xi_\lambda)
\]

\[
\geq - \alpha^{-2}\|df\|^2 \cdot \text{tr}(A) = -\|df\| \cdot H.
\]

This finishes the proof. \( \square \)

By combining Lemma 2.1 and Lemma 2.3, we obtain the following proposition.

**Proposition 2.4.** Let \((M, g)\) and \((N, \mathbf{g})\) be two oriented compact Riemannian manifolds with smooth boundary and \(f: N \to M\) is a spin map. Assume that both the curvature operator of \(M\) and the second fundamental form of \(\partial M\) are non-negative. Then for a smooth section \(\varphi\) of \(S_N \otimes f^\ast S_M\) over \(N\), we have

\[
\int_N |D_\varphi|^2 \geq \int_N |\nabla_\varphi|^2 + \int_N \frac{\mathbf{S}_\mathbf{c}}{4} |\varphi|^2 - \int N \| \wedge^2 df \| \cdot \frac{f_\ast \mathbf{S}_\mathbf{c}}{4} |\varphi|^2
\]

\[
+ \int_{\partial N} \langle D^0 \varphi, \varphi \rangle + \int_{\partial N} \frac{H}{2} |\varphi|^2 - \int_{\partial N} \| df \| \cdot \frac{f_\ast H}{2} |\varphi|^2.
\]

**Proof.** By the Stokes formula, we have

\[
\int_N \langle D_\varphi, D_\varphi \rangle = \int_N \langle D^2_\varphi, \varphi \rangle + \int_{\partial N} \langle \overline{c}(\overline{\varphi}) D_\varphi, \varphi \rangle.
\]

From Equation (2.4) and Lemma 2.1, we have

\[
\int_N \langle D^2_\varphi, \varphi \rangle \geq \int_N \langle \nabla_\varphi, \varphi \rangle + \int_N \frac{\mathbf{S}_\mathbf{c}}{4} |\varphi|^2 - \int N \| \wedge^2 df \| \cdot \frac{f_\ast \mathbf{S}_\mathbf{c}}{4} |\varphi|^2 + \int_{\partial N} \langle \nabla_{\overline{\varphi}} \varphi, \varphi \rangle
\]

\[
\geq \int N \frac{\mathbf{S}_\mathbf{c}}{4} |\varphi|^2 - \int N \| \wedge^2 df \| \cdot \frac{f_\ast \mathbf{S}_\mathbf{c}}{4} |\varphi|^2 + \int_{\partial N} \langle \nabla_{\overline{\varphi}} \varphi, \varphi \rangle.
\]

By applying line (2.10), line (2.11) and Lemma 2.3, we obtain

\[
\int_{\partial N} \langle \nabla_{\overline{\varphi}} \varphi, \varphi \rangle + \int_{\partial N} \langle \overline{c}(\overline{\varphi}) D_\varphi, \varphi \rangle \geq \int_{\partial N} \langle D^0 \varphi, \varphi \rangle + \int_{\partial N} \frac{H}{2} |\varphi|^2 - \int_{\partial N} \| df \| \cdot \frac{f_\ast H}{2} |\varphi|^2.
\]

\( \square \)
2.2. Estimates of scalar curvature, mean curvature and dihedral angles on manifolds with corners. In this subsection, we shall extend the estimates in Section 2.1 and obtain estimates for scalar curvature, mean curvature and dihedral angles on manifolds with corners.

**Geometric Setup 2.5.** Let \((M, g)\) and \((N, \overline{g})\) be two oriented compact Riemannian manifolds with corners. Suppose \(f : N \to M\) is a spin map (Definition 1.5) and also a corner map (Definition 1.6).

Let us retain the same notation from Section 2.1. In particular, \(\overline{Sc}\) (resp. \(Sc\)) is the scalar curvatures of \(N\) (resp. \(M\)), and \(\overline{H}\) (resp. \(H\)) is the mean curvatures of \(\partial N\) (resp. \(\partial M\)).

Let \(\{F_i\}\) be the collection of codimension one faces of \(N\), and denote the intersection \(F_i \cap F_j\) by \(F_{ij}\). Similar, let \(\{F_i\}\) be the collection of codimension one faces of \(M\) and write \(F_{ij} = F_i \cap F_j\). We denote \(\tilde{\theta}_{ij}\) and \(\theta_{ij}\) to be the dihedral angle functions at \(F_{ij}\) and \(\overline{F}_{ij}\) respectively. In the following, whenever we write \(F_i\) and \(F_i\), we mean that \(f\) maps \(F_i\) to \(F_i\). The same applies to \(F_{ij}\) and \(\theta_{ij}\). This should not cause any confusion. The main goal of this subsection is to prove the following proposition.

**Proposition 2.6.** Let \((M, g)\) and \((N, \overline{g})\) be two oriented compact Riemannian manifolds with corners. Suppose we are given a spin map \(f : N \to M\) that is also a corner map. Let \(D\) be the Dirac operator on \(S_N \otimes f^*S_M\). If both the curvature operator of \(M\) and the second fundamental form of \(\partial M\) are non-negative, then we have

\[
\int_N |D\varphi|^2 \geq \int_N |\nabla \varphi|^2 + \int_N \frac{\overline{Sc}}{4} \varphi^2 - \int_N \Lambda^2 \langle f^*Sc, \varphi \rangle + \int_{\partial N} \| \Lambda \varphi \|^2 + \int_{\overline{F}_{ij}} \frac{f^*Sc}{4} \varphi^2 - \int_{\partial N} \| \Lambda \varphi \|^2 + \frac{f^*H}{2} \varphi^2
\]

(2.14)

for all smooth sections \(\varphi\) of \(S_N \otimes f^*S_M\), where \(\tilde{\theta}_{ij}\) (resp. \(\theta_{ij}\)) are the dihedral angles of \(N\) (resp. \(M\)), cf. Definition 1.1.

The exact same proof in the following actually shows that Proposition 2.6 holds for manifolds with more generalized corner structures, for example, manifolds with polyhedron-like boundary (in particular, polyhedra in \(\mathbb{R}^n\)). For notational simplicity, we shall focus on the case of manifolds with corners.

**Proof of Proposition 2.6.** Our general strategy is to approximate \(N\) by a family of manifolds with smooth boundary, then apply the computation in Section 2.1 on each of these approximations, and finally obtain the inequality in line (2.14) by taking the limit.
The choice of these approximations, which are manifolds with smooth boundary, has to be made in a careful way. As a matter of fact, sometimes an approximation of $N$ may not be contained inside $N$. For this reason, we introduce the following ambient manifolds. Let $N^\dagger$ be a manifold that contains $N$ in its interior. We extend the Riemannian metric $\overline{g}$ of $N$ to a smooth Riemannian metric on $N^\dagger$. Similarly let $M^\dagger$ be a manifold that contains $M$ in its interior. We also extend the Riemannian metric of $M$ to a smooth Riemannian metric on $M^\dagger$. For example, we can choose $N^\dagger$ to be the space obtained by attaching a small cylinder $\partial N \times [0, \varepsilon]$ to $N$ along $\partial N$, and similarly choose $M^\dagger = M \cup_{\partial M} (\partial M \times [0, \varepsilon])$. In this case, $f: N \to M$ also extends to a smooth spin map from $N^\dagger$ to $M^\dagger$.

We will construct a family of manifolds $\{N_r\}_{r > 0}$ so that the following holds:

- $N_r$ is a manifold with smooth boundary,
- $N_r$ coincides with $N$ away from the $r$-neighborhood of the codimension two faces of $N$,
- Within the $r$-neighborhood of each codimension two face of $N$, the second fundamental form of $\partial N_r$ is dominated by the curvature of a family of curves that are asymptotically orthogonal to the tangent space of the given codimension two face,
- For each curve above, the signed curvature of the image curve in $M^\dagger$ does not change its sign.

For each $r > 0$, we construct $N_r$ inductively as follows.

1. Away from the $r$-neighborhood of all codimension two faces, we simply choose $N_r$ to coincide with $N$.

2. Now we shall smooth out the part of codimension two faces that is away from the codimension three faces. Intuitively speaking, we will smooth out the dihedral angle at each point by a nice smooth curve while the dihedral angle will be remembered as the integral of the signed curvature over this curve. Here the signed curvature at a point of a curve is positive if the normal vector of the curve at that point is inner (i.e., this normal vector points inward of $N$).

If $F_{ij} = F_i \cap F_j$ is a codimension two face of $N$, we denote its interior by $\overline{F}_{ij}$, that is, all points in $\overline{F}_{ij}$, but not in any codimension three faces. For each $x \in \overline{F}_{ij}$, let $\mathfrak{H}_x \subset T_x N$ be the linear subspace of $T_x N$ spanned by the normal vectors to $F_i$ and $F_j$ at $x$. For any smooth curve $\alpha$ in the $r$-disc of $\mathfrak{H}_x$, we denote by $f_\ast(\alpha)$ its image in $\mathfrak{H}_x \subset T_{f(x)} M$. Furthermore, we denote by $\alpha_N$ the image curve of $\alpha$ in $N$ under the exponential map $\exp: T_x N \to N$. Similarly, $\alpha_M$ is the image of $f_\ast(\alpha)$ in $M$ under the exponential map $\exp: T_{f(x)} M \to M$.

Since $f$ is a corner map (Definition 1.6), we can choose a curve $\gamma$ in the $r$-disc of $\mathfrak{H}_x$ such that the corresponding curve $\gamma_M$ in $M$ intersects the faces $F_i$ and $F_j$ tangentially, and such that the signed curvature $k_{\gamma_M}$ of $\gamma_M$ does not change sign, i.e., either $k_{\gamma_M} \geq 0$ on the whole curve or $k_{\gamma_M} \leq 0$ on the whole curve. Furthermore, we can assume without loss of
generality that $|k_{\gamma_M}| \leq 1/r$. The curve $\gamma_N$ could lie inside of $N$ or outside of $N$ depending on the dihedral angle at $f(x)$. See Figure 2 for when the curve $\gamma_N$ lies inside of $N$; and see Figure 3 for when the curve $\gamma_N$ lies in $N^\dagger$ but outside of $N$. Let us also denote $\gamma_N$ by $\gamma_N^x$ if we want to specify its relevance to $x$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{\(\gamma_N\) lies inside of $N$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{\(\gamma_N\) lies outside of $N$}
\end{figure}

In the above choice of $\gamma_N^x$, we can in fact choose the curves $\gamma_N^x$ (as $x$ varies in $F_{ij}^0$) so that these curves together form a smooth codimension one submanifold $Y_{ij}$ of $N^\dagger$ (cf. Figure 4). Roughly speaking, $Y_{ij}$ is the boundary of some tubular neighborhood of $F_{ij}^0$ in $N$. Furthermore, we can arrange the curves so that the second fundamental form $A_r$ of $Y_{ij}$ is of the form:

$$A_r = \bar{k}_r \cdot \bar{e}_1 + \sum_{\lambda > 1 \text{ or } \mu > 1} O(1) \bar{e}_\lambda \bar{e}_\mu,$$

where $\{\bar{e}_\lambda\}$ is a local orthonormal basis of $T\bar{Y}_{ij}$ with $\bar{e}_1$ being the unit tangent vector of the curve $\gamma_N^x$ and $\bar{k}_r$ is the signed curvature of $\gamma_N^x$. Here $O(1)$ means a quantity that is uniformly bounded as $r \to 0$.

(3) Now we shall smooth out the part of codimension three faces that is away from codimension four faces. In order to make our exposition more transparent, we will in fact first choose an approximation that roughly speaking turns codimension three singularities in $N$ into codimension two singularities.
Let $F^{0}_{ijk}$ be the interior of a codimension three face $F_{ijk} = F_i \cap F_j \cap F_k$ of $N$. Roughly speaking, we shall choose $Y_{ijk}$ to be the boundary of some tubular neighborhood of $F^0_{ijk}$ in $N$. More precisely, for each $x \in F^0_{ijk}$, let $N_x$ be the linear subspace of $T_xN$ spanned by $\{v_i, v_j, v_k\}$, where $v_i, v_j, v_k$ are the inner normal vectors to $F_i, F_j$ and $F_k$ respectively. Let $S_r$ be the sphere of radius $r$ in $N_x$. Let $\exp(S_r)$ be the image of $S_r$ under the exponential map $\exp : T_yN \to N$. Then the intersection of $\exp(S_r)$ with $Y_{ij} \cup Y_{jk} \cup Y_{ik}$ is a smooth curve $\beta$. Let $R_x$ be the part of $\exp(S_r)$ that is enclosed by the curve $\beta$. As $x$ varies in $F^0_{ijk}$, these $R_x$’s trace out a smooth codimension one submanifold in $N^\dagger$, which we shall denote by $Y_{ijk}$. Note that in general the exterior angle is non-trivial at a point where $Y_{ijk}$ and $Y_{ij} \cup Y_{jk} \cup Y_{ik}$ intersect. In any case, the intersection of $Y_{ijk}$ and $Y_{ij} \cup Y_{jk} \cup Y_{ik}$ contains only singularities of codimension $\leq 2$, which will eventually be smoothed out in a subsequent step. In the following, we shall use $Y_\Theta$ as a generic notation for referring to $Y_{ij}$ or $Y_{ijk}$ without specifying the actual sub-index.

Roughly speaking, we have partially resolved the codimension three singularities in the original space by codimension two singularities. We will eventually smooth out these new codimension two singularities by repeating the construction in Step (2). But for the moment, let us continue this partial resolution process. Inductively, suppose we have partially resolved the codimension $\ell$ singularities by codimension $(\ell - 1)$ singularities. Let $F^0_{\Lambda}$ be the interior of a codimension $(\ell + 1)$ face $F_{\Lambda}$ of $N$. Again, for each $x \in F^0_{\Lambda}$, let $S_s$ be the sphere of sufficiently small radius $s$ in the linear subspace $N_x \subset T_xN$ spanned by the normal vectors of all codimension one faces that intersect $F^0_{\Lambda}$ nonempty. Let $\exp(S_s)$ be the image of $S_s$ under the exponential map $\exp : T_yN \to N$. Then the intersection of $\exp(S_s)$ with the previously constructed $Y_\Theta$’s is a closed $(\ell - 1)$-dimensional manifold $\Sigma$. In general the inherited Riemannian metric on $\Sigma$ is not smooth, but with singularities. Let $R_x$ be the part of $\exp(S_s)$ that is enclosed by
Step (2)

Step (3)

Step (4)

Figure 5. An illustration of the steps in the construction of the smooth approximation $N_r$.

As $x$ varies in $F^{\Lambda}$, these $R_x$'s trace out a codimension one submanifold in $N^\dagger$, which we shall denote by $Y_\Lambda$. Again, the dihedral angle is non-trivial at a point where $Y_\Lambda$ and the previous $Y_\Theta$'s intersect. On the other hand, the intersection of $Y_\Lambda$ and the previous $Y_\Theta$'s contains only singularities of codimension $\leq \ell$. By repeating this partial resolution process inductively, we eventually obtain an approximation of $N$ by another $n$-dimensional manifold $N_{(1)}$ with corners such that $N_{(1)}$ only has singularities of codimension $\leq (n - 1)$. That is, $N_{(1)}$ does not have codimension $n$ faces, or equivalently, any collection of $n$ distinct codimension one faces of $N_{(1)}$ has empty intersection.

(4) We repeat the same process above on $N_{(1)}$ to obtain an approximation of $N$ by an $n$-dimensional manifold $N_{(2)}$ with corners such that $N_{(2)}$ only has singularities of codimension $\leq (n - 2)$. Now by induction, we finally arrive at an approximation of $N$ by a manifold $N_r$ with smooth boundary. See Figure 5 for the effect of these smoothing steps when performed near a vertex of a 3-dimensional manifold with corners.

Set $C_r = \partial N_r - \partial N$, i.e., $C_r$ is the part of $\partial N_r$ that is not in $\partial N$. We denote by $H_r$ the mean curvature of $C_r$. Given a smooth section $\varphi$ of $S_N \otimes f^*S_M$ over $N$, we extend it to a smooth section of the corresponding bundle over $N^\dagger$, which will still be denoted by $\varphi$. By the proof of Proposition 2.4, we have

\[
\int_{N_r} |D\varphi|^2 \geq \int_{N_r} |\nabla \varphi|^2 + \int_{\partial N_r} \langle \Phi^\varphi, \varphi \rangle + o(1)
+ \int_{N_r \cap N} \frac{Sc}{4} |\varphi|^2 - \int_{N_r \cap N} \| \wedge^2 \alpha \| \cdot \frac{f^*Sc}{4} |\varphi|^2
+ \int_{\partial N \cap \partial N_r} \frac{H}{2} |\varphi|^2 - \int_{\partial N \cap \partial N_r} \| \alpha \| \cdot \frac{f^*H}{2} |\varphi|^2
+ \int_{C_r} \sum_{\lambda} \langle \pi_{\lambda} \pi_\lambda, \varphi \rangle,
\]

where $\Phi^\varphi$ is an appropriate substitute of $D^\varphi$ and $o(1)$ means a quantity that goes to zero, as $r \to 0$. Recall that in our definition of the boundary Dirac operator $D^\varphi$ from line (2.8), we used both the normal vectors to $\partial N$ and $\partial M$. Since the
image \( f(\partial N_r) \) of \( \partial N_r \) may not be a manifold in general, the notion of normal vectors to \( f(\partial N_r) \) does not quite make sense. However, there exists a natural substitute \( \mathcal{B}^\partial \) of \( D^\partial \) that makes all the estimates work. The precise definition of \( \mathcal{B}^\partial \) will be given in line (2.19). Also, the reason for having the extra term \( o(1) \) is the following. The curvature operator on \( f(N_r) - M \) (i.e., the part of \( f(N_r) \) that is outside of \( M \)) is generally not non-negative, hence the curvature estimate in Lemma 2.1 does not apply. This results in an extra error term coming from a certain integral over \( N_r - N \). On the other hand, the volume of \( N_r - N \) goes to zero, as \( r \to 0 \). We see this extra error term goes to zero, as \( r \to 0 \).

We need to show that each term on the right hand side of the inequality (2.16) converge to the corresponding term of line (2.14). This is relatively straightforward for all terms except the last term. So we shall mainly focus on estimating the last term

\[
\int_{\mathcal{C}_r} \sum_\lambda \langle \tau_\partial(\overline{e}_\lambda) \nabla_\overline{e}_\lambda \varphi, \varphi \rangle.
\]

To illustrate the key idea of our estimation, we first prove the special case where \( \dim N = \dim M = 2 \). In this case, by applying Equation (2.11), we have

\[
\int_{\mathcal{C}_r} \langle \tau_\partial(\overline{e}_1) \nabla_\overline{e}_1 \varphi, \varphi \rangle = \int_{\mathcal{C}_r} \langle D^\partial \varphi, \varphi \rangle + \int_{\mathcal{C}_r} \frac{H_r}{2} |\varphi|^2 \\
- \frac{1}{2} \int_{\mathcal{C}_r} A_r(e_1, f_*\overline{e}_1) \cdot (\tau_\partial(\overline{e}_1) \otimes c_\partial(e_1) \varphi, \varphi),
\]

where \( A_r \) is the second fundamental form of the curve \( C_r = f(\overline{C}_r) \) in \( M^1 \), and \( e_1 \) is the unit tangent vector of \( C_r \). We denote by \( \overline{v}_{ij} \) the vertices of \( N \). Then we have \( \overline{C}_r = \cup \overline{C}_{ij,r} \), where \( \overline{C}_{ij,r} \) is the curve near \( \overline{v}_{ij} \) coming from the construction of the approximation \( N_r \). In this case, the mean curvature \( H_r \) of \( \overline{C}_{ij,r} \) is simply the signed curvature of \( \overline{C}_{ij,r} \), which measures how fast the angle of the tangent vector of the curve changes with respect to the arc length of the curve. Therefore we have

\[
\int_{\overline{C}_{ij,r}} H_r |\varphi|^2 = (\pi - \overline{\theta}_{ij}) \cdot |\varphi(\overline{v}_{ij})|^2 + o(1) \text{ as } r \to 0,
\]

where \( \overline{\theta}_{ij} \) is the dihedral angle at the vertex \( \overline{v}_{ij} \), and \( (\pi - \overline{\theta}_{ij}) \) is the jump angle of the tangent vector.

Set \( C_{ij,r} = f(\overline{C}_{ij,r}) \), which is a smooth curve near the vertex \( v_{ij} = f(\overline{v}_{ij}) \) in \( M^1 \). We have \( A_r(u, v) = k_r \cdot \langle u, v \rangle_M \), where \( k_r \) is the signed curvature of the curve \( C_{ij,r} \). Therefore, when restricted on each curve \( \overline{C}_{ij,r} \), the last term on the right hand side of (2.17) becomes

\[
- \frac{1}{2} \int_{\overline{C}_{ij,r}} k_r \cdot \langle e_1, f_*\overline{e}_1 \rangle_M \cdot \langle \tau_\partial(\overline{e}_1) \otimes c_\partial(e_1) \varphi, \varphi \rangle d\overline{s},
\]

where \( d\overline{s} \) is the infinitesimal arc length of \( \overline{C}_{ij,r} \). Since \( e_1 \) is a unit vector in \( TM^1 \) and is parallel to \( f_*\overline{e}_1 \), we have

\[
f_*\overline{e}_1 = \sqrt{\langle f_*\overline{e}_1, f_*\overline{e}_1 \rangle_M} \cdot e_1.
\]
Therefore
\[ \langle e_1, f_\ast \bar{e}_1 \rangle_M \cdot ds = \langle e_1, e_1 \rangle_M \cdot ds = ds, \]
where \( ds \) is the infinitesimal arc length of \( C_{ij,r} \).

Furthermore, since \( \bar{c}_\partial(\bar{e}_1) \otimes 1 + 1 \otimes c_\partial(e_1) \) is skew-symmetric, it follows that
\[ -\bar{c}_\partial(\bar{e}_1) \otimes c_\partial(e_1) = \frac{1}{2} \left(- (\bar{c}_\partial(\bar{e}_1) \otimes 1 + 1 \otimes c_\partial(e_1))^2 + \bar{c}_\partial(\bar{e}_1)^2 \otimes 1 + 1 \otimes c_\partial(e_1)^2 \right) \geq -1. \]
Therefore, as \( r \to 0 \), we have
\[ -\int_{C_{ij,r}} k_r \langle e_1, f_\ast \bar{e}_1 \rangle_M \cdot \langle \bar{c}_\partial(\bar{e}_1) \otimes c_\partial(e_1)\varphi, \varphi \rangle ds \geq -\int_{C_{ij,r}} |k_r| \cdot |\varphi, \varphi| ds \]
\[ = -|\pi - f^* \theta_{ij}| \cdot |\varphi(v_{ij})|^2 + o(1), \]
where the last equality follows from the fact that \( k_r \) does not change sign and that \( \varphi \) is smooth.

Now we derive a similar estimate for the general case. Consider the decomposition
\[ \overline{C}_r = \bigcup_{ij} \overline{C}_{ij,r} \cup \nabla_r, \]
where \( \nabla_r \) is the part of \( \partial N_r \) that is sufficiently close to the codimension three faces of \( N \), and \( \overline{C}_{ij,r} \) is the part that lies in the complement of \( \nabla_r \) and near the codimension two face \( F_{ij} \) of \( N \). A main observation in the estimation for the general case is that only the mean curvature of \( \overline{C}_{ij,r} \) will eventually contribute to the final limit, the integral of which converges to become the dihedral angle contribution.

Recall that in our definition of the boundary Dirac operator \( D^\partial \) from line (2.8), we used both the normal vectors to \( \partial N \) and \( \partial M \). Since the image \( f(\partial N_r) \) of \( \partial N_r \) may not be a manifold in general, the notion of normal vectors to \( f(\partial N_r) \) does not quite make sense. Hence our first step is to make sense of the "boundary Dirac operator" in this general case. Indeed, there is a natural substitute for the normal vector even at points of \( f(\partial N_r) \) where it fails to have a manifold structure.

First, for any point \( z \in \partial N_r \cap \partial N \), we have \( f(z) \in F_{ij}^0 \) for some codimension one face \( F_j \) of \( M \). In this case, we simply choose \( e_n \in T_{f(z)} M \) to be the unit inner normal vector to \( F_j \).

Now consider a point \( z \in \gamma_N \subset \overline{\gamma}_{ij} \), where \( x \in F_{ij}^0 \), and \( \gamma_N^r \) and \( \overline{\gamma}_{ij} \) are the curve and submanifold constructed in Step (2) above. At \( f(x) \in F_{ij}^0 \subset M \), let \( T_{f(x)} F_{ij} \) be the tangent space of \( F_{ij} \) at \( f(x) \). We identify \( T_{f(x)} F_{ij} \) with a linear subspace \( \mathcal{M}_{f(z)} \) of \( T_{f(z)} M \) via the exponential map \( \exp: T_{f(x)} M \to M \). Let \( \bar{e}_1 \) be the unit tangent vector of the curve \( \gamma_N^r \) at the point \( z \in \gamma_N^r \). Since by assumption \( f \) is a corner map, it follows that \( f_\ast(\bar{e}_1) \) is non-zero and does not lie in \( \mathcal{M}_{f(z)} \). Therefore, there exists a unique unit inner vector in \( T_{f(z)} M \) that is orthogonal to the linear span of \( \mathcal{M}_{f(z)} \) and \( f_\ast(\bar{e}_1) \). Again, we denote this unique unit inner vector by \( e_n \). More generally, for a point \( z \in \overline{\gamma}_N \), we simply replace
$F_{ij}$ and $\gamma_N^x$ by $F_\lambda$ and $R_\lambda$ as in Step (3), and carry out the same construction as above. This produces the desired unit vector $e_n \in T_{f(z)}M^\dagger$.

Here we shall also introduce an extra notation to be used in later part of the proof. We define $e_1 \in T_{f(z)}M^\dagger$ to be

$$e_1 := \frac{f_*(\bar{e}_1) - v}{\|f_*(\bar{e}_1) - v\|}$$

(2.18)

where $v$ is the projection of $f_*(\bar{e}_1)$ onto $\mathfrak{M}_{f(z)}$. In particular, $e_1$ is orthogonal to $\mathfrak{M}_{f(z)}$ in $T_{f(z)}M^\dagger$.

Now let $e_n \in TN_r$ be the unit inner normal vector to $\partial N_r$ and $e_n \in TM^\dagger$ the unit vector chosen above. Similar to line (2.9), we define the following boundary Dirac operator over $\partial N_r$:

$$\mathcal{D}^\partial := \sum_\lambda \bar{e}_\partial(\bar{e}_\lambda) \nabla_{\bar{e}_\lambda} - \frac{1}{2} \sum_{\lambda,\mu} \langle N \nabla_{\bar{e}_\lambda} \bar{e}_n, \bar{e}_\mu, N \bar{e}_\partial(\bar{e}_\lambda) \bar{e}_\partial(\bar{e}_\mu) \rangle \otimes 1$$

$$- \frac{1}{2} \sum_{\lambda,\mu} \bar{e}_\partial(\bar{e}_\lambda) \otimes \langle M \nabla_{f_*(\bar{e}_\lambda)} e_n, e_\mu \rangle_M c_\partial(e_\mu)$$

(2.19)

where $\{e_\lambda\}$ is a local orthonormal basis of $T(\partial N_r)$ and $\{e_\lambda\}$ is local orthonormal basis of $f^*(\mathbb{R}e_1 + \mathfrak{M}) \subset f^*(TM^\dagger)$. Set $\overline{H}_r$ to be the mean curvature of $\partial N_r$ and $A_r$ to be as follows

$$A_r(e_\lambda, e_\mu) := \langle M \nabla_{e_\lambda} e_\mu, e_n \rangle_M = -\langle M \nabla_{e_\lambda} e_n, e_\mu \rangle_M. $$

(2.20)

for $e_\lambda, e_\mu$ in $f^*TM^\dagger$ that are orthogonal to $e_n$. Then we have

$$\sum_\lambda \bar{e}_\partial(\bar{e}_\lambda) \nabla_{\bar{e}_\lambda} = \mathcal{D}^\partial + \frac{\overline{H}_r}{2} - \frac{1}{2} \sum_{\lambda,\mu} A_r(f_*(\bar{e}_\lambda), e_\mu) \bar{e}_\partial(\bar{e}_\lambda) \otimes c_\partial(e_\mu).$$

(2.21)

By construction, the second fundamental form of $\overline{C}_{ij,r}$ is of the form (2.15). It follows that essentially only the curvatures of the curves $\gamma_N^x$ in Step (2) contribute to the mean curvature $\overline{H}_r$. In particular, as $r \to 0$, we have

$$\int_{C_{ij,r}} \overline{H}_r |\varphi|^2 = \int_{F_{ij}} (\pi - \bar{e}_{ij}) \cdot |\varphi|^2 + o(1).$$

Again, consider a point $z \in \gamma_N^x \subset Y_{ij}$, where $x \in F_{ij}^0$ and $\gamma_N^x$ and $Y_{ij}$ are the curve and submanifold constructed in Step (2) above. Since $f: N \to M$ is a corner map, the tangent map $f_*: T_xN \to T_{f(x)}M$ maps $T_xF_{ij}$ to $T_{f(x)}F_{ij}$. By construction, $\mathfrak{M}_{f(z)}$ is a copy of $T_{f(x)}F_{ij}$ in $T_{f(z)}M^\dagger$ via the exponential map $\exp: T_{f(x)}M^\dagger \to M^\dagger$. Since the second fundamental form $\mathfrak{A}_r$ of $\partial N_r$ is of the form (2.15), it follows that $A_r$ defined in line (2.20) is also of the form

$$A_r = k_r \cdot de_1^2 + \sum_{\lambda > 1 \text{ or } \mu > 1} O(1) de_\lambda de_\mu,$$

(2.22)

where $k_r$ is the signed curvature of local flow curves in $M^\dagger$ generated by the vector field $e_1$ from line (2.18).
Recall that by construction \( C_{ij,r} \) is part of \( Y_{ij} \), where \( Y_{ij} \) can be viewed as a fiber bundle over \( F_{ij}^0 \) with fibers being the curves \( \gamma_x^N \), cf. Step (2). Let \( d\overline{\sigma} \) be the infinitesimal volume element of \( C_{ij,r} \). Then as \( r \to 0 \), we have asymptotically
\[
d\overline{\sigma} = ds \, d\overline{\sigma},
\]
where \( d\overline{\sigma} \) is the infinitesimal volume element of \( F_{ij}^0 \), and \( ds \) is the infinitesimal length element along the curves \( \gamma_x^N \).

By the definition of \( e_1 \) in line (2.18), when \( r \to 0 \), the local flow curves generated by the vector field \( e_1 \) asymptotically coincide with the curves obtained by projecting \( f(\gamma_x^N) \) to \( \exp(\mathfrak{L}_f(x)) \), where \( \mathfrak{L}_f(x) \) is the linear subspace of \( T_f(x)M \) spanned by the normal vectors of \( F_i \) and \( F_j \), and \( \exp(\mathfrak{L}_f(x)) \) is its image in \( M^\perp \) under the exponential map. For simplicity, let us denote this local flow curve by \( \zeta_{f}^{x_M} \). See Figure 6. In particular, if \( ds \) is the infinitesimal length element of the curve \( \zeta_{f}^{x_M} \), then we have
\[
ds = \langle f_*(\overline{e}_1), e_1 \rangle_{M} \cdot ds.
\]
Since the total length of each curve \( \gamma_x^N \) goes to zero as \( r \to 0 \), it follows that, as \( r \to 0 \), we have
\[
- \int_{r_{ij,r}} \sum_{\lambda,\mu} A_r(e_\lambda, f_*e_\mu) \langle \overline{e}_\theta(\overline{e}_\mu) \otimes c_\phi(e_\lambda), \varphi \rangle d\overline{\sigma} \leq - \int_{r_{ij,r}} \int_{\gamma_{f(x)}^M} \langle \varphi, \varphi \rangle \cdot |k_r| \cdot ds \, d\overline{\sigma} + o(1) \]
\[
= - \int_{r_{ij}} |\pi - f^*\theta_{ij}| \cdot |\varphi|^2 d\sigma + o(1),
\]
where the last equality follows from the fact that \( \varphi \) is smooth and that \( k_r \) does not change its sign along the curve \( \zeta_{f(x)}^M \).

It remains to estimate the following integral over \( V_r \):
\[
\int_{V_r} \langle \overline{e}_\theta(\overline{e}_1) \nabla_\pi \varphi, \varphi \rangle = \int_{V_r} \langle P^\theta \varphi, \varphi \rangle + \int_{V_r} \frac{H_r}{2} |\varphi|^2 - \frac{1}{2} \int_{V_r} \sum_{\lambda,\mu} A_r(e_\lambda, f_*e_\mu) \langle \overline{e}_\theta(\overline{e}_\mu) \otimes c_\phi(e_\lambda), \varphi \rangle.
\]
By repetitively applying the same estimation on \( C_{ij,r} \) above to the last two terms of the right hand side of Equation (2.24), it follows that, as \( r \to 0 \),
\[
\int_{V_r} \frac{H_r}{2} |\varphi|^2 \to 0,
\]
\[
\frac{1}{2} \int_{V_r} \sum_{\lambda,\mu} A_r(e_\lambda, f_*e_\mu) \langle \overline{e}_\theta(\overline{e}_\mu) \otimes c_\phi(e_\lambda), \varphi \rangle \to 0.
\]
This is because the decay rate of the volume of $V_r$ is greater than the growth rate of the mean curvature function $H_r$ of $V_r$, as $r \to 0$.

To summarize, for each smooth section $\varphi$ of $S_{N^*} \otimes f^* S_{M^*}$, we have proved that

$$
\int_{N_r} |D\varphi|^2 \geq \int_{N_r} |\nabla \varphi|^2 + \int_{\partial N_r} \langle \mathcal{D}_\varphi, \varphi \rangle - o(1) \\
+ \int_{N_r} \frac{Sc}{4} |\varphi|^2 - \int_{N_r} \|\nabla H \cdot f^* \frac{Sc}{4} |\varphi|^2 \\
+ \int_{\partial N \cap \partial N_r} H |\varphi|^2 - \int_{\partial N \cap \partial N_r} \|df \| \cdot \frac{f^* H}{2} |\varphi|^2 \\
+ \frac{1}{2} \sum_{i,j} \int_{F_{ij}} (\pi - \theta_{ij}) \cdot |\varphi|^2 - \frac{1}{2} \sum_{i,j} \int_{F_{ij}} |\pi - f^* \theta_{ij}| \cdot |\varphi|^2.
$$

The proof will be complete once we show that each term in line (2.25) converges to the corresponding term in line (2.14). The only term that needs an explanation is

$$
\int_{\partial N_r} \langle \mathcal{D}_\varphi, \varphi \rangle.
$$

That is, we want to show that

$$
\int_{\partial N_r} \langle \mathcal{D}_\varphi, \varphi \rangle \to \int_{\partial N} \langle D_\varphi, \varphi \rangle
$$

as $r \to 0$. This will be an immediate consequence of the following claim and the fact that the coefficients of $\mathcal{D}_\varphi$ converges to the coefficients of $D_\varphi$ (as differential operators).
Claim. For any smooth section $\varphi$ of $S_{N^\dagger} \otimes f^* S_{M^\dagger}$, the supreme norm $|\mathcal{D}^\vartheta \varphi|$ of $\mathcal{D}^\vartheta \varphi$ is uniformly bounded (i.e. independent of $r$).

Recall the definition of $\mathcal{D}^\vartheta$ from line (2.19). We first estimate the first term in line (2.19). By the construction of the curves $\gamma^\vartheta_x$ in Step (2), we have

$$N \nabla_{e_1} e_1 = \overline{k}_r e_n + O(1),$$

where $\overline{k}_r$ is the signed curvature of $\gamma^\vartheta_x$. Similarly, we have

$$M \nabla_{e_1} e_1 = k_r e_n + O(1).$$

Since the tangent map $f_* : T_x N \to T_{f(x)} M$ maps $T_x \overline{F}_{ij}$ to $T_{f(x)} F_{ij}$, we see that asymptotically $e_1$ is the only basis vector in $T_z (\partial N_r)$ such that $\langle f_*(e_1), e_1 \rangle \neq 0$. In other words, if $\lambda > 1$, then

$$\langle f_*(e_\lambda), e_1 \rangle = o(1)$$

as $r \to 0$, where $\{e_\lambda\}$ are basis vectors of $T_z (\partial N_r)$ chosen above. Therefore, we have

$$\sum_\lambda \overline{c}_\vartheta (e_\lambda) \nabla_{e_\lambda} \varphi \overline{c}_\vartheta (e_1) \cdot \left( \frac{1}{2} N \nabla_{e_1} e_1, e_n \right)_N \cdot \left( \overline{c}(e_1) \overline{c}(e_n) \otimes 1 \right) \varphi$$

$$+ \frac{1}{2} M \nabla_{f_*(e_1)} e_1, e_n \right)_M \cdot \left( \overline{c}_\vartheta (e_1) \otimes c(e_1) c(e_n) \right) \varphi + O(1)$$

$$= \frac{\overline{k}_r}{2} \varphi - \frac{1}{2} k_r \langle f_*(e_1), e_1 \rangle_M \cdot \left( \overline{c}_\vartheta (e_1) \otimes c_\vartheta (e_1) \right) \varphi + O(1).$$

Here the first equality follows from the explicit formula of the spinor connection $\nabla$ [4, Theorem 2.7], and the $O(1)$ term depends on the $C^1$-norm of $\varphi$. Furthermore, by the expressions in line (2.15) and line (2.22), the last two terms of the right hand side of Equation (2.19) become

$$- \frac{\overline{k}_r}{2} \varphi + \frac{1}{2} k_r \langle f_*(e_1), e_1 \rangle_M \cdot \overline{c}_\vartheta (e_1) \otimes c_\vartheta (e_1) + O(1).$$

To summarize, we have

$$\langle \mathcal{D}^\vartheta \varphi, \varphi \rangle = O(1).$$

This proves the claim, hence completes the proof of the proposition.

\[\square\]

3. INDEX THEORY ON MANIFOLDS WITH CORNERS

While the theory of elliptic boundary conditions for operators on manifolds with smooth boundary is quite well developed, the corresponding theory for operators on manifolds with corners has been relatively unknown. In this section, we investigate a class of twisted Dirac operators on manifolds with corners and their elliptic boundary conditions. Under suitable assumptions on dihedral angles (such as those appearing in Theorem 1.7 and Theorem 1.8), we show that this class of twisted Dirac operators become essentially self-adjoint with respect to a natural class of local boundary conditions. We then prove an index theorem for the twisted Dirac operator on manifolds with corners. In Section 4, we shall use this index theorem to prove our main theorems.
Throughout this section, let us assume the Geometric Setup 2.5 and that both \( N \) and \( M \) are even dimensional. As already mentioned in the introduction, although the results in this section are stated for manifolds with corners, they actually hold for manifolds with more generalized corner structures, for example, manifolds with polyhedron-like boundary. A key ingredient of this section is to analyze Dirac type operators that arise from asymptotically conical metrics. Since a Riemannian metric on a manifold with polyhedron-like boundary is also asymptotically conical near its singular points, the same exact proofs also apply to Dirac type operators on manifolds with polyhedron-like boundary.

### 3.1. Local boundary condition on codimension one faces.

In this subsection, we introduce a local boundary condition on the codimension one faces of \( N \), for the Dirac operator \( D \) associated to \( S_N \otimes f^*S_M \).

Let \( \{ F_i \} \) be the collection of all codimension one faces of \( N \). We denote the unit inner normal vector to \( F_i \) in \( N \) by \( \tau_n \). Similarly, we denote by \( e_n \) the unit inner normal vector to codimension one faces \( F_i \) of \( M \). Let \( \tau \) and \( e \) be the grading operators on \( S_N \) and \( f^*S_M \), respectively.

**Definition 3.1.** We say a section \( \varphi \) of \( S_N \otimes f^*S_M \) over \( N \) satisfies the local boundary condition \( B \) if \( \varphi|_{\partial N} \) lies in \( \ker(1 + (\epsilon \otimes e)(\tau_n \otimes c(e_n))) \) on every codimension one face \( F_i \) of \( N \).

Equivalently, if locally we write
\[
\varphi = \varphi_{++} + \varphi_{+-} + \varphi_{-+} + \varphi_{--}
\]
with respect to the grading operators \( \tau \) and \( e \) on \( S_N \otimes f^*S_M \), then \( \varphi \) satisfies the boundary condition \( B \) if
\[
\varphi_{--} = - (\tau(\tau_n) \otimes c(e_n))\varphi_{++}, \quad \varphi_{-+} = (\tau(\tau_n) \otimes c(e_n))\varphi_{+-}
\]
when restricted to each face \( F_i \).

Let \( H^1(N; S_N \otimes f^*S_M) \) be the Sobolev \( H^1 \)-space of sections of \( S_N \otimes f^*S_M \), and \( H^1(N; S_N \otimes f^*S_M; B) \) the subspace of sections of \( S_N \otimes f^*S_M \) that satisfy the boundary condition \( B \).

**Lemma 3.2.** The operator \( D \) is symmetric with respect to the above boundary condition \( B \). More precisely, if \( \varphi \) and \( \psi \) are smooth sections of \( S_N \otimes f^*S_M \) over \( N \) satisfying the boundary condition \( B \), then \( \langle D\varphi, \psi \rangle = \langle \varphi, D\psi \rangle \).

**Proof.** By the Stokes formula, we have
\[
\int_N \langle D\varphi, \psi \rangle - \int_N \langle \varphi, D\psi \rangle = \int_{\partial N} \langle \varphi, \tau(\tau_n)\psi \rangle = \sum_i \int_{F_i} \langle \varphi, \tau(\tau_n)\psi \rangle.
\]

Due to the boundary condition \( B \), we have
\[
\varphi = - (\tau \otimes e)(\tau(\tau_n) \otimes c(e_n))\varphi, \quad \psi = - (\tau \otimes e)(\tau(\tau_n) \otimes c(e_n))\psi.
\]

Note that \( \tau(\tau_n) \) commutes with \( \tau(\tau_n) \otimes c(e_n) \), but anti-commutes with \( \tau \otimes e \). It follows that
\[
\tau(\tau_n)\psi = (\tau \otimes e)(\tau(\tau_n) \otimes c(e_n))(\tau(\tau_n)\psi).
\]
Hence $\mathcal{C}(\mathcal{E}_n)\psi$ lies in $\ker(1 - (\mathcal{E} \otimes \epsilon)(\mathcal{E}(\mathcal{E}_n) \otimes c(e_n)))$, which is orthogonal to $B = \ker(1 + (\mathcal{E} \otimes \epsilon)(\mathcal{E}(\mathcal{E}_n) \otimes c(e_n)))$. This finishes the proof.

Recall that we define the boundary Dirac operator $D^\partial$ on $S_N \otimes f^* S_M$ over $\partial N$ by

$$D^\partial := \sum_{\lambda} \mathcal{C}(\mathcal{E}_\lambda) \nabla^\partial_{\mathcal{E}_\lambda},$$

where $\nabla^\partial$ is given by

$$\nabla^\partial = \nabla - \frac{1}{2} \sum_{\mu} \langle N \nabla e_n, \mathcal{E}_\mu \rangle N \mathcal{E}(\mathcal{E}_\mu) \otimes c(e_n) - \frac{1}{2} \sum_{\mu} 1 \otimes \langle M \nabla e_n, e_\mu \rangle M c(e_n)c(e_\mu).$$

Here $N \nabla$ and $M \nabla$ denote the Levi–Civita connections on $N$ and $M$ respectively. We will need the following lemma in Section 4.

**Lemma 3.3.** If $\varphi$ satisfies the boundary condition $B$, then $\langle D^\partial \varphi, \varphi \rangle = 0$ on the boundary $\partial N$.

**Proof.** For brevity, let us write $\gamma = \mathcal{C}(\mathcal{E}_n) \otimes c(e_n)$. We first show that $\gamma$ is parallel with respect to the connection $\nabla^\partial$.

Note that

$$\nabla^\partial \gamma - \gamma \nabla^\partial = (\nabla \gamma - \gamma \nabla) - (\nabla - \nabla^\partial) \gamma + \gamma (\nabla - \nabla^\partial) = \nabla \gamma - \gamma \nabla + 2\gamma (\nabla - \nabla^\partial),$$

since $\gamma$ anti-commutes with $(\nabla - \nabla^\partial)$. By the definition of $\nabla^\partial$, we have

$$2\gamma (\nabla_{\mathcal{E}_\lambda} - \nabla^\partial_{\mathcal{E}_\lambda})$$

$$= - \sum_{\mu} \langle N \nabla_{\mathcal{E}_\lambda} e_n, \mathcal{E}_\mu \rangle N \mathcal{E}(\mathcal{E}_\mu) \otimes c(e_n) - \sum_{\mu} \mathcal{E}(\mathcal{E}_\mu) \otimes \langle M \nabla_{f^* \mathcal{E}_\lambda} e_n, e_\mu \rangle M c(e_\mu)$$

$$= - \sum_{\mu} \mathcal{E}(\mathcal{E}_\mu) \otimes c(e_n) - \sum_{\mu} \mathcal{E}(\mathcal{E}_\mu) \otimes c(M \nabla_{f^* \mathcal{E}_\lambda} e_n)$$

$$= - (\nabla_{\mathcal{E}_\lambda} \gamma - \gamma \nabla_{\mathcal{E}_\lambda}),$$

where the last identity follows from the Leibniz rule of the spinor connection $\nabla$.

It is easy to see that the grading operator $\mathcal{E} \otimes \epsilon$ also commutes with $\nabla^\partial$. Note that $\mathcal{C}(\mathcal{E}_\lambda) := \mathcal{C}(\mathcal{E}_n) \mathcal{C}(\mathcal{E}_\lambda)$ commutes with $\mathcal{E}$, but anti-commutes with $\mathcal{C}(\mathcal{E}_n)$. Therefore, if $\varphi$ satisfies the boundary condition $B$, that is,

$$\varphi = - (\mathcal{E} \otimes \epsilon)(\mathcal{C}(\mathcal{E}_n) \otimes c(e_n)) \varphi,$$

then we have

$$D^\partial \varphi = (\mathcal{E} \otimes \epsilon)(\mathcal{C}(\mathcal{E}_n) \otimes c(e_n)) D^\partial \varphi.$$

It follows that $D^\partial \varphi$ lies in $\ker(1 - (\mathcal{E} \otimes \epsilon)(\mathcal{C}(\mathcal{E}_n) \otimes c(e_n)))$, which is orthogonal to the boundary condition $B = \ker(1 + (\mathcal{E} \otimes \epsilon)(\mathcal{C}(\mathcal{E}_n) \otimes c(e_n)))$. This finishes the proof.

To prepare for the next proposition, let us introduce the following notation.
Definition 3.4.

(1) A section of a vector bundle over a manifold with corners is called smooth if it can be extended to a larger manifold that contains the original one in the interior. Equivalently, it is smooth if all of its derivatives and higher derivatives are uniformly bounded.

(2) Let $C^\infty_0(N, S_N \otimes f^* S_M; B)$ be the collection of smooth sections that satisfy the boundary condition $B$ at each codimension one face and vanishes near all faces with codimension $\geq 2$.

(3) Let $H^1(N, S_N \otimes f^* S_M; B)$ be the completion of $C^\infty_0(N, S_N \otimes f^* S_M; B)$ with respect to the $H^1$-norm

$$\|\varphi\|_1 := (\|\varphi\|^2 + \|
abla\varphi\|^2)^{1/2}.$$ 

Remark 3.5. In fact, the Sobolev space $H^1(N, S_N \otimes f^* S_M; B)$ defined above coincides with the usual $H^1$ Sobolev norm completion of smooth sections satisfying boundary condition $B$ but not necessarily vanishing near codimension two faces. This is because removing a subspace of codimension $\geq 2$ does not affect the $H^1$ Sobolev space.

Proposition 3.6. Let $f : N \to M$ be a spin corner map between two manifold with corners $N$ and $M$. Let $D$ be the Dirac operator on $S_N \otimes f^* S_M$. Then for sections in $H^1(N, S_N \otimes f^* S_M; B)$, the $H^1$-norm is equivalent to the following norm

$$\|\varphi\|_D := (\|\varphi\|^2 + \|D\varphi\|^2)^{1/2}.$$ 

Proof. Clearly, there exists a constant $C > 0$ such that $\|\varphi\|_D \leq C\|\varphi\|_1$ for all $\varphi \in H^1(N, S_N \otimes f^* S_M; B)$. Hence to prove the proposition, it suffices to show the reversed inequality for smooth sections $\varphi \in C^\infty_0(N, S_N \otimes f^* S_M; B)$.

The same proof of Proposition 2.6 implies that there exist $C_1 >$ and $C_2 > 0$ such that

$$\|D\varphi\|^2 \geq \|\nabla\varphi\|^2 - C_1\|\varphi\|^2 - C_2\int_{\partial N} |\varphi|^2$$

for all $\varphi \in C^\infty_0(N, S_N \otimes f^* S_M; B)$. We remark that the above inequality does not require the curvature operator of $M$ or the second fundamental form of $\partial M$ to be non-negative.

Let $H^+_{4}(N, S_N \otimes f^* S_M)$ be the $H^+_{4}$ Sobolev space over $N$ with norm $\|\cdot\|_{4}$, and $H^+_{4}(\partial N, S_N \otimes f^* S_M)$ the $H^+_{4}$ Sobolev space over $\partial N$ with norm $\|\cdot\|_{4}^\beta$. By the trace theorem (cf. [12]), there exists a constant $C_3 > 0$ such that

$$\|\varphi|_{\partial N}\|_{4} \leq C_3\|\varphi\|_{4}^\beta.$$ 

\textsuperscript{4}If $\Omega$ is a Lipschitz domain in $\mathbb{R}^n$ and $\partial \Omega$ is the boundary of $\Omega$, then the Sobolev space $H^s(\partial \Omega)$ over $\partial \Omega$ is well-defined for all $s$ with $|s| \leq 1$. Since $\partial N$ is the boundary of a manifold with corners, we see that the Sobolev space $H^s(\partial N, S_N \otimes f^* S_M)$ over $\partial N$ is well-defined for all $s$ with $|s| \leq 1.$
for all \( \varphi \in H^{\frac{3}{4}}(N, S_N \otimes f^*S_M) \). In particular, it follows that
\[
\int_{\partial N} |\varphi|^2 \leq C_3^2 \|\varphi\|_{\frac{3}{4}}^2
\]
for all \( \varphi \in H^{\frac{3}{4}}(N, S_N \otimes f^*S_M) \). Furthermore, it follows from the interpolation theorem that there exists \( C_4 > 0 \) such that
\[
\|\varphi\|_{\frac{3}{4}} \leq C_4 \|\varphi\|_{\frac{3}{4}}^{3/4} \|\varphi\|_1^{1/4} \leq C_4 \left( \frac{3}{4} \epsilon^{4/3} \|\varphi\|_1 + \frac{1}{4} \epsilon^{-4} \|\varphi\| \right)
\]
for all \( \epsilon > 0 \). For a sufficiently small \( \epsilon \), we see that there exists \( C_5 > 0 \) such that
\[
\|\varphi\|_D \geq C_5 \|\varphi\|_1
\]
for all \( \varphi \in H^1(N, S_N \otimes f^*S_M; B) \). This finishes the proof. \( \Box \)

Remark 3.7. Note that Proposition 3.6 is only a statement about the equivalence of \( H^1 \)-norm and the graph norm of \( D \) on elements within \( H^1(N, S_N \otimes f^*S_M; B) \). It does not imply that the maximal domain of \( D \) is \( H^1(N, S_N \otimes f^*S_M; B) \) in general. In Section 3.2 below, we shall prove that, under extra assumptions on the dihedral angles, the operator \( D \) with the boundary condition \( B \) becomes essentially self-adjoint (cf. Theorem 3.8). In particular, under these extra assumptions, the maximal domain of \( D_B \) equals \( H^1(N, S_N \otimes f^*S_M; B) \) in this case.

3.2. Conical singularities and self-adjoint extensions of twisted Dirac operators on manifolds with corners. In this subsection, under suitable assumptions on dihedral angles (such as those appearing in Theorem 1.7 and Theorem 1.8), we show that the local boundary condition introduced in the previous subsection make the corresponding twisted Dirac operator essentially self-adjoint.

Recall that \( \{F_i\} \) is the family of codimension one faces of \( N \). We also denote the codimension two faces of \( N \) by \( F_{ij} = F_i \cap F_j \). Similarly, higher codimensional faces of \( N \) will be denoted by \( F_{ijk} = F_i \cap F_j \cap F_k \) and so on. Let
\[
U = \{U, U_i, U_{ij}, U_{ijk}, \ldots\}
\]
be an open cover of \( N \) such that each member of \( U \) is localized near a face of certain codimension in \( N \). More precisely, we assume that there exists \( \epsilon > 0 \) such that \( U \) is an open cover of \( N \) satisfying the following:

- \( U \) lies in the interior of \( N \),
- \( U_i \) is the \( \epsilon \)-neighborhood of \( V_i \), where \( V_i \) an open subset of \( F_i \) such that \( V_i \) is disjoint from the \( (\epsilon/2) \)-neighborhood of \( F_{ij} \) for all \( j \neq i \),
- \( U_{ij} \) is the \( \epsilon \)-neighborhood of \( V_{ij} \), where \( V_{ij} \) is an open subset of \( F_{ij} \) such that \( V_{ij} \) is disjoint from the \( (\epsilon/2) \)-neighborhood of \( F_{ijk} \) for all \( k \) with \( k \neq i \) and \( k \neq j \),
- Similarly, near a codimension-\( \ell \) face \( F_{i_1i_2\ldots i_\ell} \) of \( N \), \( U_{i_1i_2\ldots i_\ell} \) is the \( \epsilon \)-neighborhood of \( V_{i_1i_2\ldots i_\ell} \), where \( V_{i_1i_2\ldots i_\ell} \) is an open subset of \( F_{i_1i_2\ldots i_\ell} \) such that \( V_{i_1i_2\ldots i_\ell} \) is disjoint from the \( (\epsilon/2) \)-neighborhood of all codimension-(\( \ell + 1 \)) faces.
Note that $U_i$ is diffeomorphic to $V_i \times [0, \varepsilon)$ and $U_{ij}$ is diffeomorphic to $V_{ij} \times [0, \varepsilon)^2$. Similarly, near a codimension-$\ell$ face $F_A$, the open set $U_A$ is of the form

$$V_A \times [0, \varepsilon)^\ell.$$ 

Roughly speaking, the Riemannian metric $\bar{g}$ of $U_A$ restricted on each fiber $[0, \varepsilon)^\ell$ is asymptotically conical, that is,

$$\bar{g} \big|_{[0, \varepsilon)^\ell} = dr^2 + r^2 h + \text{error terms},$$ 

where $r$ is the radial coordinate and $h$ is some Riemannian metric on the link of $[0, \varepsilon)^\ell$. Here the link of $[0, \varepsilon)^\ell$ is the part of the unit sphere $S^{\ell-1}$ that lies in $[0, \infty)^\ell$, cf. Figure 9. Instead of giving the precise definition of asymptotically conical metrics, we shall first carry out our analysis in the case of actual conical metrics, then show how to reduce the asymptotically conical case to the conical case. For the details on this reduction step, see the discussion in the proof of Theorem 3.8 at the end of this subsection.

Without loss of generality, we can assume each $V_A$ is an open subset of a closed manifold $W_A$ of dimension $(n - \ell)$. By applying a smooth partition of unity subordinate to $U$, we can localize our construction of self-adjoint extensions of $D$ to each member of $U$. In particular, it suffices to carry out our analysis on the following model spaces

$$W \times [0, \infty)^\ell$$

equipped with a Riemannian metric that is fiberwise conical, where $W$ is a closed smooth manifold. Of course, the choice of $W$ depends on $V_A$.

For each model space $W \times [0, \infty)^\ell$, let $F_x$ be the fiber $[0, \infty)^\ell$ over $x \in W$. We denote the link of $F_x$ by $L_x$. Here again the link of $F_x$ is the part of the unit sphere $S^{\ell-1}$ that lies in $[0, \infty)^\ell$, cf. Figure 9. If no confusion is likely to arise, we will write $W \times F$ in place of $W \times [0, \infty)^\ell$. We have the following decomposition of the local spinor bundle $S_{W \times F}$ of $W \times F$:

$$S_{W \times F} \cong \begin{cases} S_F \otimes S_W, & \text{if } \ell = \dim F \text{ is even} \\ (S_F \otimes S_W) \oplus (S_F \otimes S_W), & \text{if } \ell = \dim F \text{ is odd}. \end{cases}$$

(3.4)

The Clifford actions of $v \in TF$ and $w \in TW$ are given by $c(v) \otimes 1$ and $1 \otimes c(w)$ if $\ell$ is even, and $\begin{pmatrix} c(v) & -c(v) \\ c(w) & c(w) \end{pmatrix}$ if $\ell$ is odd. Now suppose $\ell = \dim F$ is even for the moment. Then we have the local decomposition

$$S_N = S_{W \times F} \cong S_F \otimes S_W.$$

By assumption, $f : N \to M$ is a corner map, cf. Definition 1.6. In terms of the model space $W \times F$, this means that $f$ is diffeomorphic along the fiber $F$. Consider a similar local decomposition

$$S_M \cong S_{f(F)} \otimes S_{f(F)}^\perp,$$

where $S_{f(F)}^\perp$ is the (local) spinor bundle generated by vectors of $TM$ that are orthogonal to $Tf(F)$. When restricted to each fiber, $D$ simply becomes the Dirac
operator $D_F$ associated to the spinor bundle $S_F \otimes f^*S_f(\mathbb{F})$. Let $r$ be the radial coordinate of the conical metric on the fiber $\mathbb{F}$. There is a natural unitary that transform $D_F$ into an elliptic operator in terms of the cylindrical metric (i.e., product metric) on $[0, \infty) \times \mathbb{L}$, cf. [22, Section 5]. In the current setup, there is a rather explicit description of this unitary. Indeed, up to a zero or der term, the Dirac operator $D_F$ can be naturally identified with the de Rham operator $D_{dR}$ on $F$, cf. Appendix A. For the de Rham operator $D_{dR}$, we have the following identification. The unitary that naturally identifies the space of differential form $\Omega^*(\mathbb{F})$ with $\Omega^*([0, \infty) \times \mathbb{L}) \cong C^\infty((0, \varepsilon), \Omega^*\mathbb{L}) \oplus C^\infty((0, \varepsilon), \Omega^*\mathbb{L})$

is given as follows:

$$\Psi_{\text{even}}: C^\infty((0, \varepsilon), \Omega^*\mathbb{L}) \to \Omega^\text{even}\mathbb{F}, \omega_p \mapsto \begin{cases} r^{p-\frac{\ell-1}{2}}\omega_p, & \text{if } p \text{ is even} \\ r^{p-\frac{\ell-1}{2}}\omega_p \wedge dr, & \text{if } p \text{ is odd} \end{cases} \quad (3.5)$$

and

$$\Psi_{\text{odd}}: C^\infty((0, \varepsilon), \Omega^*\mathbb{L}) \to \Omega^\text{odd}\mathbb{F}, \omega_p \mapsto \begin{cases} r^{p-\frac{\ell-1}{2}}\omega_p, & \text{if } p \text{ is odd} \\ r^{p-\frac{\ell-1}{2}}\omega_p \wedge dr, & \text{if } p \text{ is even} \end{cases} \quad (3.6)$$

With the even/odd grading of differential forms, we have

$$D_{dR}^F = \begin{pmatrix} D_{dR,+}^F \\ D_{dR,-}^F \end{pmatrix},$$

where $D_{dR,+}^F: \Omega^{\text{odd}}\mathbb{F} \to \Omega^{\text{even}}\mathbb{F}$ and $D_{dR,-}^F: \Omega^{\text{even}}\mathbb{F} \to \Omega^{\text{odd}}\mathbb{F}$. Let us define

$$P := \begin{pmatrix} c_0 & d^* \\ d & e_1 \\ \vdots & \ddots \\ c_{\ell-2} & d^* \\ d & c_{\ell-1} \end{pmatrix} \quad (3.7)$$

where $d$ is the de Rham differential on $\Omega^*\mathbb{L}$, $d^*$ is the adjoint of $d$, and

$$c_p = (-1)^p(p - \frac{\ell-1}{2}).$$

A straightforward computation shows that (cf. [5, Section 5] [22, Proposition 5.3])

$$\Psi_{\text{odd}}^{-1} D_{dR,+}^F \Psi_{\text{even}} = \frac{\partial}{\partial r} + \frac{1}{r} P: C^\infty((0, \varepsilon), \Omega^*\mathbb{L}) \to C^\infty((0, \varepsilon), \Omega^*\mathbb{L}), \quad (3.8)$$

and

$$\Psi_{\text{even}}^{-1} D_{dR,-}^F \Psi_{\text{odd}} = -\frac{\partial}{\partial r} + \frac{1}{r} P: C^\infty((0, \varepsilon), \Omega^*\mathbb{L}) \to C^\infty((0, \varepsilon), \Omega^*\mathbb{L}). \quad (3.9)$$

So far, we have analyzed the associated Dirac operator along each fiber when the fiber has even dimension. The case where the fiber has odd dimension is completely similar. We omit the details.
Our first main theorem of this section is the following.

**Theorem 3.8.** Assume the geometric setup 2.5. Let $D_B$ be the Dirac operator $D$ on $S_N \otimes f^*S_M$ subject to the boundary condition $B$, whose domain is (Definition 3.4)

$$\text{dom}(D_B) = C_0^\infty(N, S_N \otimes f^*S_M; B).$$

If the dihedral angles $\theta_{ij}(\gamma)$ of $N$ and $\theta_{ij}(g)$ of $M$ satisfy

$$\theta_{ij}(\gamma)_z \leq \theta_{ij}(g)_f(z) \leq \pi$$

for all codimension one faces $F_i, F_j$ of $N$ and all $z \in \overline{F_i} \cap \overline{F_j}$, then $D_B$ is essentially self-adjoint. Furthermore, its self-adjoint extension $\overline{D_B}$ is Fredholm and the domain $\text{dom}(\overline{D_B})$ of $\overline{D_B}$ is $H^1(N, S_N \otimes f^*S_M; B)$.

We point out that Theorem 3.8 also holds for manifolds with more generalized corner structures, for example, manifolds with polyhedron-like boundary. As we shall see, Theorem 3.8 is proved by analyzing Dirac type operators that arise from asymptotically conical metrics. Since a Riemannian metric on a manifold with polyhedron-like boundary is also asymptotically conical near its singular points, the same exact proof also applies to Dirac type operators on manifolds with polyhedron-like boundary. For simplicity, we shall only focus on the case of manifolds with corners.

We will decompose the proof of the above theorem into several steps. The goal is to show that the deficiency indices $\dim E_\pm(D_B)$ of $D_B$ are zero. Observe that, by localizing via a partition of unity (cf. [22, Theorem 2.1]), we reduce the computation to the model spaces $W \times F$, where $W$ is a closed manifold, $F = [0, \infty)^{\ell}$ and fiberwise $f$ becomes asymptotically a linear map. Let us first start with the simplest case where $\dim N = \dim M = 2$ and $f$ is the identity map. Then we generalize our computation to the case where $\dim N = \dim M = 2$ and $f$ is a linear map in general. Finally, we prove the higher dimensional case by induction.

Let us review the boundary condition for each model space $W \times F$. In the codimension one case, we simply impose the boundary condition $B$ from Definition 3.1 on $W \times \{0\} \subset W \times [0, \infty)$. For the codimension two case, the model space is

$$W \times [0, \infty)^2$$

equipped with a Riemannian metric that is fiberwise (asymptotically) conical. In this case, the fiber $F_x \cong [0, \infty)^2$ over $x \in W$ is simply a circular sector. The link $L_x$ of $F_x$ is an arc of a certain angle. Let $P$ be the operator along the link $L$ given in line (3.7). At the two end points of each arc $L_x$, we impose the same local boundary condition $B$ (for the spinor bundle $S_L \otimes f^*S_f(L)$) from Definition 3.1, which turns $P$ into a self-adjoint Fredholm operator $P_B$. When viewed as a subset of $N$, the two end points of each arc $L_x$ lie in the interior of codimension one faces of $N$. We remark that the boundary condition for the operator $P$ is chosen so that it coincides with the local boundary condition $B$ imposed at each codimension one face of $N$, cf. Figure 7.
Figure 7. Local boundary condition for the operator $P$ along each link.

For higher codimensional cases, the corresponding model space is

$$W \times [0, \infty)^k$$

equipped with a Riemannian metric that is fiberwise (asymptotically) conic. In this case, the link $L_x$ of a fiber $F_x$ is a spherical sector. See Figure 9 for the case where $k = 3$. Now on each $L$, we impose the boundary condition $B$ (for the spinor bundle $S_L \otimes f^*S_f(L)$) from Definition 3.1. The operator $P$ from line (3.7) on the link is a symmetric operator under the boundary condition $B$ and will be denoted by $P_B$.

On each fiber $F$, let us denote by $D_{dR}^{dR}F,B$ the de Rham operator of $F$ defined on differential forms that are supported away from the vertex of $F$, subject to the local boundary condition $B$. The following lemma characterizes when $D_{dR}^{dR}F,B$ is essentially self-adjoint in terms of the spectrum the operator $P_B$ on the link.

**Lemma 3.9** (cf. [5, Theorem 3.1]). Let $P_B$ be the operator $P$ on the link subject to the induced boundary $B$ from above. Assume that $P_B$ is essentially self-adjoint. Then $D_{dR}^{dR}F,B$ is essentially self-adjoint if and only if $|P_B| \geq 1/2$.

**Proof.** Set $E_\pm(D_{dR}^{dR}F,B) := \ker(D_{dR}^{dR}F,B \mp i)$. The von Neumann deficiency indices theorem states that $D_{dR}^{dR}F,B$ is essentially self-adjoint if and only if

$$E_+(D_{dR}^{dR}F,B) = E_-(D_{dR}^{dR}F,B) = 0.$$

By assumption, the $L^2$-space of differential forms on the link $L$ admits an orthonormal basis $\{\phi_\lambda\}$, where each $\phi_\lambda$ is the eigenvector of $P_B$ with eigenvalue $\lambda$.

Suppose that $\varphi = \varphi_{\text{even}} \oplus \varphi_{\text{odd}}$ lies in the kernel of $D_{dR}^{dR}F,B - i$. Write $\psi_0 = \Psi_{\text{even}}^{-1}(\varphi_{\text{even}})$ and $\psi_1 = \Psi_{\text{odd}}^{-1}(\varphi_{\text{odd}})$. We have

$$\begin{pmatrix} 0 & -\partial_r \varphi \\ \partial_r & 0 \end{pmatrix} + \frac{1}{r} \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} - i \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = 0.$$

It follows that

$$\begin{cases} -\frac{\partial^2}{\partial r^2} \psi_0 - \frac{1}{r^2} P \psi_0 + \frac{1}{r^2} P^2 \psi_0 + \psi_0 = 0, \\
 i \psi_1 = \frac{\partial}{\partial r} \psi_0 + \frac{1}{r} P \psi_0. \end{cases}$$
We solve the first equation by splitting into the eigenspaces of $P_B$. If $\psi_0 = h(r) \cdot \phi_\lambda$ for some $\lambda$-eigenvector $\phi_\lambda$ of $P_B$, then the first equation becomes the following differential equation

$$-h'' - \frac{1}{r^2} \lambda h + \frac{1}{r^2} \lambda^2 h + h = 0$$

whose solutions consist of modified Bessel functions. More precisely, we have

$$\begin{cases}
\psi_0 = c_1 \sqrt{r} \cdot K_{\lambda-1/2}(r) \phi_\lambda + c_2 \sqrt{r} \cdot I_{\lambda-1/2}(r) \phi_\lambda \\
\psi_1 = c_1 \sqrt{r} \cdot K_{\lambda+1/2}(r) \phi_\lambda + c_2 \sqrt{r} \cdot K_{\lambda+1/2}(r) \phi_\lambda
\end{cases} \quad (3.10)$$

where $I_\nu$ and $K_\nu$ are modified Bessel functions of the first and the second kind, respectively. It follows that $\psi_0 \oplus \psi_1$ is an $L^2$ solution if and only if $-1/2 < \lambda < 1/2$ (cf. [22, Lemma 4.2]). This finishes the proof. □

**Remark 3.10.** In general, if $\dim E_+(D_{F,B}^{\text{dr}}) = \dim E_-(D_{F,B}^{\text{dr}})$ (but not necessarily zero), then $D_{F,B}^{\text{dr}}$ admits self-adjoint extensions. The self-adjoint extensions of $D_{F,B}^{\text{dr}}$ are in one-to-one correspondence to isometries between $E_+(D_{F,B}^{\text{dr}})$ to $E_-(D_{F,B}^{\text{dr}})$. More precisely, the domain of a self-adjoint extension of $D_{F,B}^{\text{dr}}$ has the following form

$$\text{dom}((D_{F,B}^{\text{dr}})_\text{min}) + \{ x - \Phi x : x \in E_+(D_{F,B}^{\text{dr}}) \}, \quad (3.11)$$

where $\Phi$ is a unitary operator from $E_+(D_{F,B}^{\text{dr}})$ to $E_-(D_{F,B}^{\text{dr}})$ and $(D_{F,B}^{\text{dr}})_\text{min}$ is the closure of $D_{F,B}^{\text{dr}}$.

Now we shall proceed to prove Theorem 3.8. Let us first consider the simplest case where $\dim F = 2$ and $f$ is the identity map on $F$.

**Lemma 3.11.** Let $D_B^{\text{dr}}$ be the de Rham operator acting on the differential forms over the two dimensional sector as in Figure 7, subject to the boundary condition $B$ (cf. Definition 3.1 and Proposition A.2). Then $D_B^{\text{dr}}$ is essentially self-adjoint if and only if the angle of the sector is less than or equal to $\pi$.

**Proof.** Using the unitaries given in line (3.5) and (3.6), the de Rham operator is conjugate to

$$\begin{pmatrix}
0 & -\frac{\partial}{\partial r} \\
\frac{\partial}{\partial r} & 0
\end{pmatrix} + \frac{1}{r} \begin{pmatrix}
P & 0 \\
0 & 0
\end{pmatrix}$$

where

$$P = \begin{pmatrix}
-1/2 & -\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial \theta} & -1/2
\end{pmatrix}$$

Let $\alpha$ be the angle of the sector. Let $\phi = \phi_0(\theta) + \phi_1(\theta) d\theta$ be a differential form on the link satisfying the boundary condition $B$, that is, $\phi_1(0) = \phi_1(\alpha) = 0$.

Let $D_{\theta}^{\text{dr}} = \begin{pmatrix}
0 & -\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial \theta} & 0
\end{pmatrix}$ be the de Rham operator on the link. If $D_{\theta}^{\text{dr}} \phi = \lambda \phi$, then

$$-\phi'_1 = \lambda \phi_0, \quad \text{and} \quad \phi'_0 = \lambda \phi_1. \quad (3.12)$$
\[ u_1 = v_1 \]

\[ v_2 \]

Figure 8. The boundary condition at the two edge of \( \mathbb{F} \).

Hence \( \phi_1'' = -\lambda^2 \phi_1 \). By the boundary condition \( \phi_1(0) = \phi_1(\alpha) = 0 \), we obtain that

\[ \phi_1(\theta) = \rho \cdot \sin(k\pi\theta/\alpha), \]

for some \( k \in \mathbb{Z} \) and some constant \( \rho \). Note that if \( k = 0 \), then \( \phi_1 = 0 \) and \( \phi_0 \) is a constant function. In any case, the spectrum of \( D^\text{dR} \) subject to the boundary condition \( B \) is precisely the set \{\( k\pi/\alpha \)\}_{k \in \mathbb{Z}}. Therefore, the spectrum of \( P = -1/2 + D^\text{dR} \) (subject to the boundary condition \( B \)) is

\[ \left\{ -\frac{1}{2} + \frac{k\pi}{\alpha} \right\}_{k \in \mathbb{Z}}. \]

In particular, we have \( |P_B| \geq 1/2 \) if and only if \( \alpha \leq \pi \). By Lemma 3.9, this finishes the proof. \( \square \)

Remark 3.12. Suppose we impose the following different boundary conditions on the two boundary edges in Figure 7, that is, the boundary condition \( B \) on one edge and \( B^\perp \) on the other edge, where \( B^\perp \) is the orthogonal complement of \( B \). Then the same argument from Lemma 3.11 above shows that the de Rham operator \( D^\text{dR} \) with these mixed boundary conditions is essentially self-adjoint if and only if the angle of the sector is less than or equal to \( \pi/2 \).

Now let us consider the two dimensional case but the map \( f \) is a general invertible linear map instead of the identity map.

Lemma 3.13. Let \( \mathbb{F} \) and \( \mathbb{G} \) be two sectors in \( \mathbb{R}^2 \) as in Figure 8. Suppose \( f: \mathbb{F} \to \mathbb{G} \) is the restriction of an invertible linear map on \( \mathbb{R}^2 \). Let \( D_B \) be the Dirac operator \( D \) acting on the vector bundle \( S_{\mathbb{F}} \otimes f^*S_{\mathbb{G}} \) with the boundary condition \( B \) (cf. Definition 3.1). Assume the angle \( \alpha \) of \( \mathbb{F} \) and the angle \( \beta \) of \( \mathbb{G} \) are less than or equal to \( \pi \). Then \( D_B \) is essentially self-adjoint if and only if \( \alpha \leq \beta \).

Proof. As \( \mathbb{F} \) and \( \mathbb{G} \) are flat, both the vector bundles \( S_{\mathbb{F}} \) over \( \mathbb{F} \) and \( S_{\mathbb{G}} \) over \( \mathbb{G} \) are trivial bundles with flat metric. Hence the vector bundle \( f^*S_{\mathbb{G}} \) is also a trivial bundle over \( \mathbb{F} \) with flat metric, which can be canonically identified with \( S_{\mathbb{F}} \). In particular, this induces a natural isomorphism \( S_{\mathbb{F}} \otimes f^*S_{\mathbb{G}} \cong S_{\mathbb{F}} \otimes S_{\mathbb{F}}^* \cong \bigwedge^*\mathbb{R}^2 \) in this case, where \( \bigwedge^*\mathbb{R}^2 = \bigwedge^*T^*\mathbb{R}^2 \), cf. [21, I.5.18]. We emphasize that here the identification between \( S_{\mathbb{F}} \) and \( f^*S_{\mathbb{G}} \) is not induced by the map \( f \) in general.
With the isometry given above, the Dirac operator is identified with the de Rham operator acting on $\Lambda^* \mathbb{R}^2$, cf. [21, I.5.12]. Let $u_1$ and $u_2$ be the unit inner normal vectors of the two edges of $F$, and $v_1$ and $v_2$ the unit inner normal vectors of the two edges of $G$, respectively. By rotating $G$ in $\mathbb{R}^2$ if necessary, we may assume $u_1 = v_1$ under the isometry between $f^* S_G$ and $S_F$. Then the vector $v_2$ differs from $u_2$ by an angle $(\beta - \alpha)$ counterclockwise from $u_2$ to $v_2$. For brevity, let us write $\delta = \beta - \alpha$. More precisely, if we choose unit vectors $u^\perp_2$ and $v^\perp_2$ such that $u^\perp_2$ is orthogonal to $u_2$, $v^\perp_2$ is orthogonal to $v_2$, and $u_2 \wedge u^\perp_2 = v_2 \wedge v^\perp_2$ is the volume form of $\mathbb{R}^2$, then we have

$$
\begin{pmatrix}
v_2 \\
v_2^\perp
\end{pmatrix} = \begin{pmatrix}
\cos \delta & \sin \delta \\
-\sin \delta & \cos \delta
\end{pmatrix} \begin{pmatrix}
u_2 \\
u_2^\perp
\end{pmatrix} \quad \text{and} \quad
\begin{pmatrix}
u_2 \\
u_2^\perp
\end{pmatrix} = \begin{pmatrix}
\cos \delta & -\sin \delta \\
\sin \delta & \cos \delta
\end{pmatrix} \begin{pmatrix}
v_2 \\
v_2^\perp
\end{pmatrix}
$$

(3.13)

On each edge of $F$, a section $w$ of $S_F \otimes f^* S_G$ satisfies the boundary condition $B$ from Definition 3.1 if on the given edge we have

$$(\tau \otimes \epsilon)(\tau(u_i) \otimes c(v_i))w = -w$$

for $i = 1, 2$, respectively. Under the isomorphism $S_F \otimes f^* S_G \cong \Lambda^* \mathbb{R}^2$, the Clifford action on differential forms is given by (cf. [21, I.3.9])

$$\tau(u)w = u \wedge w - u_{w}w \quad \text{and} \quad c(v)w = (-1)^{\deg w}(v \wedge w + w_{\perp}w),$$

where $\perp$ is the contraction operator. In particular, the $\mathbb{Z}_2$-grading operator $\tau \otimes \epsilon$ is identified with the even-odd grading operator on $\Lambda^* \mathbb{R}^2$.

To summarize, the Dirac operator $D$ on $S_F \otimes f^* S_G$ with boundary condition $B$ is identified with the de Rham operator acting on $\Lambda^* \mathbb{R}^2$ with mixed boundary conditions on two edges as follows. On one edge, it is the usual absolute boundary condition, that is, if we decompose a differential form as

$$w = w_1 + w_2 dx$$

where $dx$ is the differential of the normal direction and $w_j$ are tangential differential forms, then $w$ satisfies the absolute boundary condition at a edge if $w_2$ vanishes on the given edge. At the other edge, we have

$$(\tau \otimes \epsilon)(\tau(u_2) \otimes c(v_2))w = -w.$$

For a given differential form $w$, we have the following decompositions:

$$w = \varphi_1 + \varphi_2 u_2 + \varphi_3 u^\perp_2 + \varphi_4 u_2 \wedge u^\perp_2,$$

and

$$(\tau \otimes \epsilon)(\tau(u_2) \otimes c(v_2))w = \psi_1 + \psi_2 u_2 + \psi_3 u^\perp_2 + \psi_4 u_2 \wedge u^\perp_2.$$

A direct computation shows that

$$
\begin{pmatrix}
\psi_1 \\
\psi_4 \\
\psi_2 \\
\psi_3
\end{pmatrix} = \begin{pmatrix}
-\cos \delta & \sin \delta \\
\sin \delta & \cos \delta \\
\cos \delta & \sin \delta \\
\sin \delta & -\cos \delta
\end{pmatrix} \begin{pmatrix}
\varphi_1 \\
\varphi_4 \\
\varphi_2 \\
\varphi_3
\end{pmatrix}
$$

(continued)
Therefore, if \((\tilde{\tau} \otimes \epsilon) (\tau(u_2) \otimes c(v_2)) w = -w\), then we have
\[ \varphi_1 \sin \frac{\delta}{2} + \varphi_4 \cos \frac{\delta}{2} = 0, \quad \text{and} \quad \varphi_2 \sin \frac{\delta}{2} + \varphi_3 \cos \frac{\delta}{2} = 0 \]
(3.14)
at the given edge. Note that this new boundary condition does not mix even and odd degree differential forms.

Under the unitaries given in line (3.5) and (3.6), the de Rham operator on \(\mathbb{F}\) is conjugate to the operator
\[ \begin{pmatrix} 0 & -\frac{\partial}{\partial r} \\ \frac{\partial}{\partial r} & 0 \end{pmatrix} + \frac{1}{r} \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \]
(3.15)
where
\[ P = \begin{pmatrix} -1/2 & -\frac{\partial}{\partial \theta} \\ -\frac{\partial}{\partial \theta} & -1/2 \end{pmatrix}. \]

Let \(\phi(\theta) = \phi_0(\theta) + \phi_1(\theta) d\theta\) be a differential form on the link \(\mathbb{L}\) of \(\mathbb{F}\). Under the same conjugation above, the boundary condition \(B\) becomes the following boundary condition: \(\phi_1(0) = 0\), and
\[ -\phi_0(\alpha) \sin \frac{\delta}{2} + \phi_1(\alpha) \cos \frac{\delta}{2} = 0. \]
(3.16)
Furthermore, the explicit formula in line (3.14) shows that the boundary conditions for the two copies of \(P\) (appearing in the matrix from line (3.15)) coincide. It is easy to see that the operator \(P\) with this boundary condition becomes an essentially self-adjoint Fredholm operator, which will be denoted by \(P_B\).

Let \(D^{\text{dr}}_\theta = \begin{pmatrix} 0 & -\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \theta} & 0 \end{pmatrix}\) be the de Rham operator on the link. If \(D^{\text{dr}}_\theta \phi = \lambda \phi\), then
\[ -\phi_1' = \lambda f_0, \quad \text{and} \quad \phi_0' = \lambda \phi_1. \]
(3.17)
Hence \(\phi_1'' = -\lambda^2 \phi_1\). By the boundary condition \(\phi_1(0) = 0\), we see that \(\phi_1(\theta) = \rho \cdot \sin(\lambda \theta)\) for some constant \(\rho\). It follows that \(\phi_0(\theta) = -\rho \cdot \cos(\lambda \theta)\). The boundary condition at \(\theta = \alpha\) implies that
\[ \sin(\lambda \alpha) \cos \frac{\delta}{2} + \cos(\lambda \alpha) \sin \frac{\delta}{2} = 0, \]
(3.18)
that is, \(\sin(\lambda \alpha + \delta/2) = 0\). Therefore the spectrum of the operator \(D^{\text{dr}}_\theta\) with this mixed boundary condition is
\[ \left\{ -\frac{\delta}{2\alpha} + \frac{k\pi}{\alpha} \right\}_{k \in \mathbb{Z}}. \]
Hence the spectrum of \(P_B = -1/2 + D^{\text{dr}}_\theta\) with this mixed boundary condition is given by
\[ \left\{ -\frac{\beta}{2\alpha} + \frac{k\pi}{\alpha} \right\}_{k \in \mathbb{Z}}. \]
Note that when \(k = 1\), we always have
\[ -\frac{\beta}{2\alpha} + \frac{\pi}{\alpha} = \frac{2\pi - \beta}{2\alpha} \geq \frac{1}{2}. \]
since we have assumed that $\alpha + \beta \leq 2\pi$. Moreover, since by assumption $\alpha \leq \pi$, it follows that $|P_B| \geq 1/2$ if and only if $-\frac{\beta}{2\pi} \leq -\frac{1}{2}$, that is, $\alpha \leq \beta$. By Lemma 3.9, this finishes the proof. 

Remark 3.14. Here is an observation from the above proof of Lemma 3.13 that will be useful for proving the higher dimensional case, i.e., Lemma 3.15. As already pointed out in the above proof, the identification between $S_F$ and $f^*S_G$ is independent of the linear map $f$. The only place where the linear map $f$ enters into the proof is to determine the unit vectors $v_1$ and $v_2$ that are used in the construction of the boundary condition. In other words, one can actually restate Lemma 3.13 solely in terms of assumptions on the boundary condition $B$ for the de Rham operator $D_{\mathbb{R}}^F$ of $F$, with no mentioning of $G$ or the map $f$. Furthermore, we point out that it is not necessary to have $u_1$ and $u_2$ (in Figure 8) to be exactly the normal vectors. What is important here is that both the dihedral angle $\alpha$ determined by $u_1$ and $u_2$ and the dihedral angle $\beta$ determined by $v_1$ and $v_2$ are less than or equal to $\pi$ and also $\alpha \leq \beta$.

Now we turn to the higher dimensional case.

Lemma 3.15. Let $F$ and $G$ be two regions in $\mathbb{R}^n$ that are enclosed by $n$ hyperplanes through the origin, respectively. Suppose $f : F \to G$ is the restriction of an invertible linear map on $\mathbb{R}^n$. Let $D_B$ be the Dirac operator acting on the vector bundle $S_F \otimes f^*S_G$ with the boundary condition $B$. If all the dihedral angles of $F$ and $G$ are $\leq \pi$, and $f$ is non-increasing for dihedral angles, then $D_B$ is essentially self-adjoint.

Proof. We prove this by induction on $n$. The case where $n = 2$ has been proved in Lemma 3.11 and Lemma 3.13. Assume that the lemma holds for dimensions $1, 2, \ldots, (n - 1)$.

If $n$ is even, as in the proof of Lemma 3.13, there is a canonical isometry $S_F \otimes f^*S_G \cong \Lambda^n\mathbb{R}^n$. We emphasize this canonical isometry is induced by the canonical identification of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself. In particular, it is not induced by the map $f$. Similarly, if $n$ is odd, the direct sum of two copies of $S_F \otimes f^*S_G$ can be canonically identified with $\Lambda^n\mathbb{R}^n$. 

Figure 9. A two dimensional link of a three dimensional cone.
Under the unitaries given in line (3.5) and (3.6), the de Rham operator on $F$ is conjugate to the operator
\[
\begin{pmatrix}
0 & -\frac{\partial}{\partial r} \\
\frac{\partial}{\partial r} & 0
\end{pmatrix} + \frac{1}{r} \begin{pmatrix}
0 & P \\
P & 0
\end{pmatrix}.
\]

Recall again that the boundary condition $B$ (as given in Definition 3.1) for codimensional faces of $F$ does not mix the even and odd degree differential forms. In particular, the boundary condition $B$ induces a boundary condition, denoted by $B$, for the operator $P$ on the link $L$ of $F$.

The link $L$ of $F$ is a sector of the standard unit sphere $S^{n-1}$, which in particular is itself a manifold with corners.

**Claim 3.16.** Under the assumptions on the dihedral angles of $F$ and $G$, the operator $P$ on the link with boundary condition $B$ is essentially self-adjoint.

We shall prove both the claim and the lemma by inductively alternating between the two statements. More precisely, we shall first prove Claim 3.16 for the case of $\dim F = 3$, which will in turn be used to prove Lemma 3.15 for the case of $\dim F = 3$. We will then use this proven case of Lemma 3.15 to prove Claim 3.16 for the case of $\dim F = 4$, and so on.

Let us first prove the above claim in the case where $\dim F = 3$. By the explicit formula of the operator $P$ from line (3.7), $P$ with the boundary condition $B$ is essentially self-adjoint if and only if the de Rham operator $D_{dR}^L$ on the link with boundary condition $B$ is essentially self-adjoint, since $P$ and $D_{dR}^L$ only differ by a bounded zeroth order term. Now $F$ is a region of $\mathbb{R}^3$ enclosed by three planes $H_1, H_2$ and $H_3$ that go through the origin. The link $L$ of $F$ is a sector of $S^2$, and the dihedral angles of $L$ coincide with those of $F$. By localizing via a partition of unity, it is clear that $D_{dR}^L$ with boundary condition $B$ is essentially self-adjoint away from the vertices. Now near the vertices, in terms of the viewpoint given in Remark 3.14, it suffices to verify the boundary condition $B$ for the operator $D_{dR}^L$ satisfies the conditions of Lemma 3.13 so that $D_{dR}^L,B$ is essentially self-adjoint near the vertices.

Consider the dihedral angle of $L$ at a vertex $y_0$ that lies in the intersection of the planes $H_1$ and $H_2$ (cf. Figure 9). By rotating $G$ if necessary, we can assume the boundary plane $f(H_1)$ of $G$ coincides with $H_1$. Let $u_1$ and $u_2$ be the unit inner normal vectors of $H_1$ and $H_2$ at $y_0$. Similarly, let $v_1$ and $v_2$ be the unit inner normal vectors of $f(H_1)$ and $f(H_2)$ at $y_1$. Clearly, due to the current relative position of $F$ and $G$, all four vectors $u_1, u_2, v_1, v_2$ lie in the same plane, more precisely, in the tangent plane $T_{y_0}L$ of the link at $y_0$. In particular, under this identification, the boundary condition $B$ (for the operator $D_{dR}^L$) becomes precisely the boundary condition that is used in Lemma 3.13. Moreover, the assumptions on the dihedral angles of $F$ and $G$ translate into exactly those assumptions on the dihedral angles of Lemma 3.13. Therefore, it follows from Lemma 3.13 that the operator $D_{dR}^L$ with boundary condition $B$ is essentially self-adjoint near the vertex $y_0$, and similarly essentially self-adjoint near the other vertices as well. This proves that $P_B$ is essentially self-adjoint in the case where $\dim F = 3$. 
Now we shall prove Lemma 3.15 for the case where \( \dim \mathbb{F} = 3 \). Since we have already shown the operator \( P_B \) on the link is essentially self-adjoint in this case, it suffices to show that the spectrum of \( P_B \) satisfies \(|P_B| \geq 1/2\) (cf. Lemma 3.9).

**Claim 3.17.** If the operator \( P_B \) acting on the differential forms \( \wedge^* \mathbb{L} \) over \( \mathbb{L} \) is essentially self-adjoint and \( n = \dim \mathbb{F} \geq 3 \), then \(|P_B| \geq 1/2\).

In fact, we shall prove a stronger version of Claim 3.17. That is, under the same assumption, we will show that \(|P_B| \geq \sqrt{(n-1)(n-2)/2}\), where again \( \dim \mathbb{F} = n \) and thus \( \dim \mathbb{L} = n - 1 \). Before we prove Claim 3.17 above, let us fix some notation. Recall that \( P = D + \mathcal{E} A \),

where \( D = D^{dR}_L \) is the de Rham operator on \( \wedge^* \mathbb{L} \), \( \mathcal{E} \) is the even-odd grading operator, and \( A \) is a diagonal matrix with diagonal entries \( (p - n - 1/2) \) on \( p \)-forms.

Since \( P_B \) is essentially self-adjoint and \( \mathcal{E} A \) is a zeroth order operator, we see that \( D \) is also essentially self-adjoint with respect to the boundary condition \( B \). In particular, the domain of \( D_B \) is \( H^1(\mathbb{L}, \wedge^* \mathbb{L}; B) \), the closure of \( C_0^\infty(\mathbb{L}, \wedge^* \mathbb{L}; B) \) with respect to the Sobolev \( H^1 \)-norm, cf. Definition 3.4.

Let \( \{F_i\} \) be the set of codimension one faces of \( \mathbb{F} \). Recall that on each codimension one face \( \mathbb{L}_i = F_i \cap \mathbb{L} \) of \( \mathbb{L} \), the boundary condition \( B \) is given by

\[
\ker \left( (\mathcal{E} \otimes \mathcal{E})(\mathcal{E}(u_i) \otimes c(v_i)) + 1 \right),
\]

where \( u_i \) is the inner unit normal vector field of \( F_i \) and \( v_i \) is the unit inner normal vector field of the corresponding codimension one face \( G_i \) of \( G \). Observe that, in the current setup of the lemma, the vector fields \( u_i \) and \( v_i \) are constant along \( F_i \), hence in particular constant along \( \mathbb{L}_i \).

Let \( \alpha \) be a differential form in \( C_0^\infty(\mathbb{L}, \wedge^* \mathbb{L}; B) \). By applying similar smooth approximations as those in the proof of Proposition 2.6, we have

\[
\int_{\mathbb{L}} \langle D\alpha, D\alpha \rangle = \int_{\mathbb{L}} \langle D^2\alpha, \alpha \rangle + \sum_i \int_{\mathbb{L}_i} \langle \mathcal{E}(u_i) D\alpha, \alpha \rangle
\]

\[
= \int_{\mathbb{L}} \langle (\nabla^* \nabla + \mathcal{R}) \alpha, \alpha \rangle + \sum_i \int_{\mathbb{L}_i} \langle \mathcal{E}(u_i) D\alpha, \alpha \rangle
\]

\[
= \int_{\mathbb{L}} \langle \nabla^2 \alpha, \nabla \alpha \rangle + \int_{\mathbb{L}} \langle \mathcal{R} \alpha, \alpha \rangle + \sum_i \sum_j \int_{\mathbb{L}_i} \langle \mathcal{E}(u_i) \mathcal{E}(e^j_i) \nabla e^j_i \alpha, \alpha \rangle,
\]

where in the last summation \( \{e^j_i\} \) is an orthonormal basis of tangent vectors of \( \mathbb{L}_i \). Here \( \mathcal{R} \) is the curvature term appearing in the following Weitzenböck formula

\[
\Delta = D^2 = \nabla^* \nabla + \mathcal{R}.
\]
Observe that the dihedral angles do not appear in Equation (3.19), since the element $\alpha$ vanishes near the corners.

The normal vectors $u_i$ and $v_i$ are constant along each $L_i$, hence commute with $\nabla$. It follows that the grading operator $(\mathcal{C} \otimes \mathcal{C})(u_i \otimes c(v_j))$ anti-commutes with $\mathcal{C}(u_i)\mathcal{C}(e^i_\lambda)\nabla e^i_j$ for all $j$ on each $L_i$. We conclude that the boundary term (i.e., the last term) of Equation (3.19) vanishes, since the element $\alpha$ satisfies the boundary condition $B$. Therefore, we have

$$\int_L \langle D\alpha, D\alpha \rangle \geq \int_L (|\nabla \alpha|^2 + \langle \mathcal{R}\alpha, \alpha \rangle)$$

(3.20)

for all $\alpha \in C^\infty_0(\mathbb{L}, \Lambda^\ast \mathbb{L}; B)$. By completion with respect to the Sobolev $H^1$-norm, inequality (3.20) continues to hold for all $\alpha \in \text{dom}(D_B) = H^1(\mathbb{L}, \Lambda^\ast \mathbb{L}; B)$.

Recall that the Gallot–Meyer estimate [13] states that the curvature term $\mathcal{R}$ from above satisfies the inequality

$$\mathcal{R} \geq p(n - 1 - p)\gamma$$

(3.21)

on all $p$-forms, where $\gamma$ is the (pointwise) minimal eigenvalue of the curvature operator of $L$, cf. line (2.3). Since the link $L$ is a sector of the unit sphere $S^{n-1}$ with $n \geq 3$, we have $\gamma = 1$.

Now let us prove Claim 3.17. Suppose that $P_B(\varphi) = \lambda \varphi$ for some $\lambda \in \mathbb{R}$ and $\varphi \in H^1(\mathbb{L}, \Lambda^\ast \mathbb{L}; B)$ with $\|\varphi\|^2 = 1$. Then we have

$$\lambda^2 = \int_L \langle (D + \mathcal{C}A)\varphi, (D + \mathcal{C}A)\varphi \rangle$$

$$= \int_L \langle (D + \mathcal{C}A)\varphi, D\varphi \rangle + \int_L \langle (D + \mathcal{C}A)\varphi, \mathcal{C}A\varphi \rangle$$

Since $(D + \mathcal{C}A)\varphi = \lambda \varphi$ satisfies the boundary condition $B$, we apply the Stokes formula to the first term of the above equation. Again by using the fact $\varphi$ satisfies the boundary condition $B$ and the fact $\mathcal{C}D = -D\mathcal{C}$, it follows from Equation (3.19) that

$$\lambda^2 = \int_L \left( |\nabla \varphi|^2 + \langle \mathcal{R}\varphi, \varphi \rangle + |A\varphi|^2 + \langle \mathcal{C}(AD - DA)\varphi, \varphi \rangle \right).$$

(3.22)

Since $A$ acts as multiplication by $(p - \frac{n-1}{2})$ on $p$-forms, a direct computation shows that

$$\mathcal{C}(AD - DA) = \sum_{j=1}^{n-1} (1 \otimes c(e_j))\nabla e_j,$$

(3.23)

where $\{e_j\}_{1 \leq j \leq n-1}$ is an orthonormal basis of the tangent bundle $T\mathbb{L}$ of $\mathbb{L}$. Recall that $\|\varphi\| = 1$. By the Cauchy–Schwartz inequality, it follows that

$$\int_L |\langle \mathcal{C}(AD - DA)\varphi, \varphi \rangle| \leq \sqrt{\|\mathcal{C}(AD - DA)\varphi\|} \leq \sqrt{n - 1} \cdot \|\nabla \varphi\|.$$  

(3.24)

On the other hand, we have

$$\mathcal{R} + A^2 \geq p(n - 1 - p) + (p - \frac{n-1}{2})^2 = \frac{(n-1)^2}{4}.$$
on $p$ forms. Since the right hand side of the above inequality is independent of $p$, we see that
\[ \Re + A \geq \frac{(n-1)^2}{4} \]  
(3.25)
on $L^2(L, \Lambda^* L)$. By combining (3.22), (3.24), and (3.25), we obtain that
\[ \lambda^2 \geq \|\nabla \phi\|^2 - \sqrt{n-1}\|\nabla \phi\| + \frac{(n-1)^2}{4} \]
\[ = \left(\|\nabla \phi\| - \frac{\sqrt{n-1}}{2}\right)^2 + \frac{(n-1)(n-2)}{4}, \]
which implies
\[ |\lambda| \geq \frac{\sqrt{(n-1)(n-2)}}{2}. \]
(3.26)
In particular, we conclude that $|\lambda| \geq 1/2$ if $n \geq 3$. This proves Claim 3.17.

Now the proofs of Claim 3.16 and Lemma 3.15 are finished by inductively alternating between Claim 3.16 and Lemma 3.15. \hfill \square

Now let us complete the proof of Theorem 3.8.

**Proof of Theorem 3.8.** By localizing via a partition of unity (cf. [22, Theorem 2.1]), we reduce the computation near neighborhoods of singular points. Let $U$ be a neighborhood of a point in a codimension $\ell$ face of $N$. By definition of manifolds with corners, there is a smooth diffeomorphism $\rho: U \to \overline{W} \times \overline{F}$, where $\overline{W}$ is an open subset of a closed manifold and $\overline{F}$ is a convex region in $\mathbb{R}^\ell$ enclosed by $\ell$ hyperplanes that go through the origin. Without loss of generality, we further assume that the diffeomorphism $\rho: U \to \overline{W} \times \overline{F}$ is asymptotically unitary. More precisely, this means the following. Let us still denote by $T_U$ the tangent map $T_{\overline{W} \times \overline{F}}$ induce by $\rho$. Let $\rho^*: T(\overline{W} \times \overline{F}) \to T_U$ be the adjoint map of $\rho$. The diffeomorphism $\rho$ is asymptotically unitary if both $\rho^* \rho$ and $\rho \rho^*$ are of the form $1 + O(r)$, where $r$ is the distance to the base $\overline{W}$. In this case, $\rho$ preserves the relative positions of the normal vectors at each point of the codimension $\ell$ face, in particular $\rho$ preserves the dihedral angles. Let $U$ be a neighborhood of a singular point in $M$ that contains $f(U)$. The same reasoning above applies to give an asymptotic unitary $\rho: TU \to T(\overline{W} \times \overline{F})$.

Let $S_N \otimes f^* S_M$ be the spinor bundle associated to the spinor structure of $T_N \otimes f^* T_M$ over $N$, and $D$ the Dirac operator acting on $S_N \otimes f^* S_M$. The bundle maps $\rho$ and $\rho$ induces a bundle map on the spinors
\[ \rho \otimes \rho: S_N \otimes f^* S_M \to S_{\overline{W} \times \overline{F}} \otimes (\rho f \rho^{-1})^* S_{W \times F}, \]
where $\rho f \rho^{-1}$ is the composition $\overline{W} \times \overline{F} \xrightarrow{\rho^{-1}} \overline{U} \xrightarrow{f} U \xrightarrow{\rho} W \times F$.

Write for short $E = S_{\overline{W} \times \overline{F}} \otimes (\rho f \rho^{-1})^* S_{W \times F}$. Then the differential operator
\[ (\rho \otimes \rho) D(\rho \otimes \rho)^* \]
acting on $E$ over $\overline{W} \times \overline{F}$ is of the form
\[ D^{dR} + S_1 + S_0 \]  
(3.27)
where $D_{\text{dR}}$ is a Dirac-type operator that is fiberwise the de Rham operator along $\mathbb{F}$, $S_{1}$ is a first order differential operator whose coefficients are $O(r)$ as $r \to 0$, and $S_{0}$ is a zeroth order differential operator whose coefficients are uniformly bounded. Since $\rho$ and $\bar{\rho}$ are smooth, conjugation by $\rho \otimes \rho$ preserves the maximal domain and the minimal domain of $D$. It follows from [5, Theorem 3.1 & Lemma 3.2] that the essential self-adjointness of $(\rho \otimes \rho)D(\rho \otimes \rho)^{*}$, hence that of $D$, is equivalent to the essential self-adjointness of $D_{\text{dR}}$, since $D_{\text{dR}}$ is the leading term in the expression (3.27), cf. [5, Section 1, condition (RS4)]. This also follows from Lemma 3.18.

Let $B^{\dagger}$ be the boundary condition as given in Definition 3.1 on sections of $E$. Note that the normal vector field of a codimension one face of $\overline{W} \times F$ is parallel along each fiber $F$. As $\bar{\rho}$ is asymptotically unitary, it maps asymptotically the normal vectors of $N$ to the normal vectors of $\overline{W} \times \mathbb{F}$. Similarly, $\rho$ maps asymptotically the normal vectors of $M$ to the normal vectors of $W \times \mathbb{F}$. Therefore the boundary condition $(\rho \otimes \rho)B$ asymptotically coincides with the boundary condition $B^{\dagger}$. We shall show that (locally) the essential self-adjointness of the original Dirac operator $D$ is equivalent to the essential self-adjointness of $D_{\text{dR}}$ with respect to $B^{\dagger}$.

Let $L_{r}$ be the link of $\mathbb{F}$ at radius $r$, that is, $L_{r}$ is the spherical sector of $\mathbb{F}$ at radius $r$. We first consider the codimension two case, that is, the case where $\dim \mathbb{F} = \ell = 2$. When $r$ is sufficiently small, there is a unitary $V_{r}$ acting on $E$ over $L_{r}$ that maps the boundary condition $(\rho \otimes \rho)B$ to $B^{\dagger}$ (e.g. by using the fact that if two orthogonal projections are sufficiently close, then they are unitarily equivalent). These $V_{r}$ together form a unitary $V$ acting on $E$ over $\overline{W} \times \mathbb{F}$. We may choose $V$ so that it is smooth along the base $W$. Although $V$ may not be smooth at the cone point (i.e., the origin of $\mathbb{F}$) along the fiber $\mathbb{F}$, we can choose $V$ so that it is almost flat along each link $L_{r}$ and is Lipschitz along the $r$-direction, that is,

$$
\| \nabla_{L_{r}} V_{r} \| \leq C \text{ for some } C > 0 \text{ and } \| \nabla^{2}_{L_{r}} V_{r} \| = O(r), \text{ as } r \to 0,
$$

where $\nabla^{2}_{L_{r}}$ is the gradient operator on $L_{r}$.

By the change of coordinates given in line (3.5) and (3.6), the fiber part of the Dirac operator $D_{\text{dR}}$ along $\mathbb{F}$ becomes

$$
D_{\text{dR}} = \left( \begin{array}{cc} 0 & -\partial \rho \partial r \\ \partial r & 0 \end{array} \right) + \left( \begin{array}{cc} 0 & P \\ P & 0 \end{array} \right),
$$

where $P$ is a differential operator defined along each link (cf. the proof of Lemma 3.11). By the above discussion, we have

$$
\left\| \left[ \frac{\partial}{\partial r}, V_{r} \right] \right\| \leq C \text{ and } \|[P, V_{r}]\| = O(r).
$$

Therefore $V D_{\text{dR}} V^{*}$ only differs from $D_{\text{dR}}$ by a bounded term. Since $V$ preserves the space of $L^{2}$ sections of $E$, the essential self-adjointness of $D_{\text{dR}}$ with respect to the two boundary conditions $(\rho \otimes \rho)B$ and $B^{\dagger}$ are equivalent.
Now we have reduced the computation to the model cases that were handled by Lemma 3.11 and Lemma 3.13. It follows that \( D \) with respect to the boundary condition \( B \) is essentially self-adjoint near the codimension two faces.

Now we consider the case where \( \dim F = \ell = 3 \). The link \( \mathbb{L}_r \) at radius \( r \) is a 2-dimensional spherical sector with corners. Again, the fiber part of the Dirac operator is given by

\[
D_{\mathbb{F}}^{\text{Dir}} = \begin{pmatrix}
0 & -\frac{\partial}{\partial r} \\
\frac{\partial}{\partial s} & 0
\end{pmatrix} + \frac{1}{r} \begin{pmatrix}
P & 0 \\
0 & P
\end{pmatrix},
\]

after the change of coordinates given in line (3.5) and (3.6).

Since we have already verified the codimension two case, it follows that the Dirac operator \( D_{\mathbb{L}_r}^{\text{Dir}} \) along the link \( \mathbb{L}_r \) with respect to the boundary condition \((\overline{\mathbb{F}} \otimes \rho)B\) is essentially self-adjoint. Again, when \( r \) is sufficiently small, there exists a unitary \( \mathcal{V}_r \) that maps \((\overline{\mathbb{F}} \otimes \rho)B\) to \( B^\dagger \) such that

\[
\left\| \left[ \frac{\partial}{\partial r}, \mathcal{V}_r \right] \right\| \leq C_1 \text{ for some } C_1 > 0.
\]

These \( \mathcal{V}_r \) together form a unitary \( \mathcal{V} \) acting on \( E \) over \( \overline{W} \times \overline{F} \). However, in the current case, since the dihedral angles may vary along the edges of \( \overline{F} \), the coefficients of the operator \([P, \mathcal{V}_r]\) are unbounded in general. More precisely, we denote by \( s \) the radial coordinate near a vertex of the link \( \mathbb{L}_r \). Note that the operator \( P \) differs from the Dirac operator \( D_{\mathbb{L}_r}^{\text{Dir}} \) along \( \mathbb{L}_r \) by a bounded term, cf. line (3.7). Near a vertex of \( \mathbb{L}_r \), we have

\[
D_{\mathbb{L}_r}^{\text{Dir}} = \begin{pmatrix}
0 & -\frac{\partial}{\partial s} \\
\frac{\partial}{\partial s} & 0
\end{pmatrix} + \frac{1}{s} \begin{pmatrix}
P & Q \\
Q & 0
\end{pmatrix},
\]

after the change of coordinates given in line (3.5) and (3.6). The unitary \( \mathcal{V}_r \) commutes with \( \frac{\partial}{\partial s} \) and \( Q \) up to bounded endomorphisms. If we denote by \( \left\| [Q, \mathcal{V}_r] \right\| \) the pointwise norm of the operator \([Q, \mathcal{V}_r]\), then clearly \( \left\| [Q, \mathcal{V}_r] \right\| \) is determined by the size of the difference between \((\overline{\mathbb{F}} \otimes \rho)B\) and \( B^\dagger \), which goes to zero as \( r \) goes to zero. To summarize, we have that

\[
\left\| [P, \mathcal{V}_r] \right\| \leq \frac{C_2 \cdot r}{s}
\]

for some \( C_2 > 0 \), where \( \left\| [P, \mathcal{V}_r] \right\| \) is the pointwise norm of the operator \([P, \mathcal{V}_r]\).

The boundary condition \( B^\dagger \) restricted on \( \mathbb{L}_r \) also satisfies the dihedral angle conditions in Lemma 3.11 and Lemma 3.13, so \( D_{\mathbb{L}_r}^{\text{Dir}} \) (or equivalently \( P \)) acting on \( E|_{\mathbb{L}_r} \) with respect to \( B^\dagger \) is also essentially self-adjoint, and its domain is the Sobolev \( H^1 \) space \( H^1(\mathbb{L}_r, E; B^\dagger) \). By Lemma 3.9, the essential self-adjointness of \( D_{\mathbb{L}_r}^{\text{Dir}} \) implies that \( |Q| \geq 1/2 \). It follows that

\[
\left\| \frac{1}{s} \varphi \right\| \leq 2 \left\| \frac{1}{s} Q \varphi \right\|, \forall \varphi \in H^1(\mathbb{L}_r, E; B^\dagger).
\]
Now the Gårding inequality (cf. Proposition 3.6) implies that there exists $C_3 > 0$ (independent of $r$) such that
\[
\left\| \frac{1}{s} \varphi \right\| \leq 2 \left\| \frac{1}{s} Q \varphi \right\| \leq 2 C_3 \| D_{\mathbb{L}_r} \varphi \|, \quad \forall \varphi \in H^1(\mathbb{L}_r, E; B^\dagger).
\]
It follows that there exists $C_4 > 0$ such that
\[
\| [P, V_r] \varphi \| \leq C_4 \cdot r \| P \varphi \|
\]
for all $\varphi \in H^1(\mathbb{L}_r, E; B^\dagger)$. To summarize, there exists $C_5 > 0$ such that
\[
\| V_r P V_r^* \varphi \| \leq C_5 \| P \varphi \|
\]
for all $\varphi \in H^1(\mathbb{L}_r, E; B^\dagger)$. To summarize, there exists a family of (unbounded) bundle endomorphisms $A_r$ such that
\[
\mathcal{V} D_{\mathbb{F}}^B \varphi = \begin{pmatrix} 0 & -\varphi \\ -\varphi & 0 \end{pmatrix} + \frac{1}{r} \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_r \\ A_r & 0 \end{pmatrix}
\] (3.28)
with the following properties:

1. both $P$ and $P + r A_r$ are essentially self-adjoint with respect to the boundary condition $B^\dagger$ and have the same domain $H^1(\mathbb{L}_r, E; B^\dagger)$;
2. along each link $\mathbb{L}_r$, $P^{-1} A_r$ and $A_r P^{-1}$ are bounded operators, and their operator norms are uniformly bounded by a constant that is independent of $r$.

It follows from Lemma 3.18 below that $\mathcal{V} D_{\mathbb{F}}^B \varphi$ is essentially self-adjoint subject to the boundary condition $B^\dagger$, hence $D_{\mathbb{F}}^B \varphi$ is also essentially self-adjoint subject to $(\mathbb{F} \otimes \rho) B$. This completes the proof of the codimension three case.

The higher codimensional cases can now be proved by induction via the same argument above. In conclusion, the Dirac operator $D$ on $S_N \otimes f^* S_M$ subject to the boundary condition $B$ is essentially self-adjoint.

Let $\mathcal{D}_B$ be the closure of $D_B$. By construction, $\mathcal{D}_B$ is Fredholm, cf. [5, Section 3]. By definition, $\text{dom}(\mathcal{D}_B)$ is the closure of $C^\infty_0 (N, S_N \otimes f^* S_M; B)$ with respect to the graph norm
\[
\| \varphi \|_B := (\| \varphi \|^2 + \| D \varphi \|^2)^{1/2},
\]
cf. Definition 3.4. It follows from Proposition 3.6 that
\[
\text{dom}(\mathcal{D}_B) = H^1(N, S_N \otimes f^* S_M; B).
\]
This finishes the proof. \[\square\]
The following lemma is roughly speaking a generalization of Lemma 3.9 to the case of asymptotically conical metrics. It follows from the analysis in [5, Section 3]. For the convenience of the reader, we sketch a proof here.

**Lemma 3.18.** Assume the same notation as in the proof of Theorem 3.8. In particular, let \( \mathbb{F} \) be a convex region in \( \mathbb{R}^\ell \) with \( \ell \geq 3 \) enclosed by \( \ell \) hyperplanes that go through the origin. Let \( \bigwedge^* \mathbb{R}^\ell \) be the bundle of forms over \( \mathbb{F} \) and \( \mathbb{B}^\dagger \) the boundary condition given by
\[
(\bar{\epsilon} \otimes \epsilon)(\bar{v} \otimes c(v))\varphi = -\varphi
\]
on each codimension one face of \( \mathbb{F} \), where \( \bar{v} \) and \( v \) unit vector fields that are parallel along each face and satisfy the angle conditions of Theorem 3.8. Consider the differential operator
\[
D = V D_{dR}^\mathbb{F} V^*,
\]
which in cylindrical coordinates becomes (cf. line (3.28))
\[
\begin{pmatrix}
0 & \frac{-\partial}{\partial r} \\
\frac{\partial}{\partial r} & 0
\end{pmatrix} + \frac{1}{r} \begin{pmatrix}
P & 0 \\
0 & A_r
\end{pmatrix}
\]
where \( A_r \) is a family of self-adjoint (possibly unbounded) endomorphisms. Suppose that
\begin{enumerate}
\item both the operators \( P \) and \( P + r A_r \) (along each link) are essentially self-adjoint with respect to the boundary condition \( \mathbb{B}^\dagger \) and have the same domain \( H^1(\mathbb{L}_r, E; \mathbb{B}^\dagger) \),
\item \( |P| \geq 1/2 \),
\item \( P^{-1} A_r \) and \( A_r P^{-1} \) are bounded operators, and their operator norms are uniformly bounded by a constant that is independent of \( r \),
\end{enumerate}
then \( D \) is essentially self-adjoint with respect to the boundary condition \( \mathbb{B}^\dagger \) and its domain is \( H^1(\mathbb{F}, \bigwedge^* \mathbb{R}^\ell; \mathbb{B}^\dagger) \).

**Proof.** By Lemma 3.15, the standard de Rham operator \( D_{dR} \) on \( \mathbb{F} \) is essentially self-adjoint with respect to \( \mathbb{B}^\dagger \) and its domain is \( H^1(\mathbb{F}, \bigwedge^* \mathbb{R}^\ell; \mathbb{B}^\dagger) \). In cylindrical coordinates (that is, after the change of coordinates given in line (3.5) and (3.6)), we have
\[
D_{dR} = \begin{pmatrix}
0 & \frac{-\partial}{\partial r} \\
\frac{\partial}{\partial r} & 0
\end{pmatrix} + \frac{1}{r} \begin{pmatrix}
P & 0 \\
0 & A_r
\end{pmatrix}
\]
It suffices to show that the maximal domain of \( D \) is contained in the maximal domain of \( D_{dR} \). Denote by \( \mathbb{L} \) the spherical sector of the unit sphere in \( \mathbb{F} \). It suffices to show that if \( u \in L^2([0, 1] \times \mathbb{L}, \bigwedge^* \mathbb{L}) \) and
\[
h := \frac{\partial}{\partial r} u + \frac{1}{r} P u + A_r u \in L^2([0, 1] \times \mathbb{L}, \bigwedge^* \mathbb{L})
\]
where \( h \) is defined in the weak sense, then \( u \) lies in the (maximal) domain of
\[
\frac{\partial}{\partial r} + \frac{1}{r} P.
\]
Let \( \{\psi_j\}_{j \in \mathbb{N}^+} \) be a partition of unity subordinate to the open cover
\[
\{(2^{-j-1}, 2^{-j+1})\}_{j \in \mathbb{N}^+}
\]
of \((0, 1)\). Note that each \(\psi_j\) is supported away from the origin. From the condition (1), we see that \(\psi_j u\) lies in \(H^1((2^{-j-1}, 2^{-j+1}) \times \mathbb{L}, \Lambda^\ast \mathbb{L})\). Therefore, for any \(\varepsilon > 0\), there exists a smooth section \(u_j\) supported on \((2^{-j-1}, 2^{-j+1}) \times \mathbb{L}\) such that

\[
\|\psi_j u - u_j\|_D \leq \frac{\varepsilon}{2^j},
\]

where \(\| \cdot \|_D\) is the graph norm of \(D\). Since the partition of unity is locally finite, the summation \(\sum_{j=1}^{\infty} u_j\) is well-defined and furthermore we have

\[
\left\| \sum_{j=1}^{\infty} u_j - u \right\|_D \leq \varepsilon.
\]

Therefore, by replacing \(u\) with \(\sum_{j=1}^{\infty} u_j\), we may assume that \(u\) is smooth on \((0, 1] \times \mathbb{L}\). Note that in general \(u\) is not smooth at \(\{0\} \times \mathbb{L}\), and we do not have much control of how \(u\) behaves at \(\{0\} \times \mathbb{L}\). In any case, by multiplying a smooth cut-off function, we assume without loss of generality that \(u\) is supported on \([0, \delta) \times \mathbb{L}\) for some sufficiently small \(\delta > 0\).

Since \(P_{B^\ast}\) is essentially self-adjoint, there is an orthonormal basis \(\{\varphi_\lambda\}_{\lambda \in \Lambda}\) of \(L^2(\mathbb{L}, \Lambda^\ast \mathbb{L})\) such that \(\varphi_\lambda \in H^1(\mathbb{L}, \Lambda^\ast \mathbb{L}; B^\ast)\) and \(P\varphi_\lambda = \lambda \varphi_\lambda\). Note that we have \(|\lambda| \geq 1/2\) by assumption. Since \(P^{-1}A_r\) is bounded and its operator norm is uniformly bounded (independent of \(r\)), there exists \(C > 0\) such that

\[
\|A_r^\ast \varphi_\lambda\| \leq C|\lambda|
\]

for all \(\varphi_\lambda\).

Denote by \(u(r)\) (resp. \(h(r)\)) the restriction of \(u\) (resp. \(h\)) on \(\{r\} \times \mathbb{L}\). We define

\[
g(r) := h(r) - A_r u(r).
\]

Let us set

\[
u_\lambda(r) := \langle u(r), \varphi_\lambda \rangle, \quad h_\lambda(r) := \langle h(r), \varphi_\lambda \rangle \quad \text{and} \quad g_\lambda(r) := \langle g(r), \varphi_\lambda \rangle.
\]

Since both \(u_\lambda\) and \(h_\lambda\) are in \(L^2[0, 1]\), it follows that \(g_\lambda \in L^2[0, 1]\). Moreover, by construction, we have

\[
g_\lambda = \frac{d}{dr} u_\lambda + \lambda \frac{r}{r} u_\lambda
\]

in the weak sense. In particular, we have

\[
u_\lambda(r) = \int_1^r \left( \frac{t}{r} \right)^\lambda g_\lambda(t) dt,
\]

since \(u_\lambda\) vanishes if \(r > \delta\). In the case where \(\lambda \geq 1/2\), we also have

\[
u_\lambda(r) = \int_0^r \left( \frac{t}{r} \right)^\lambda g_\lambda(t) dt.
\]
This leads us to define the following operator:

$$T_\lambda(\xi) = \begin{cases} \int_1^r (\frac{t}{r})^\lambda \xi(t) dt & \text{if } \lambda \leq -1/2, \\
\int_0^r (\frac{t}{r})^\lambda \xi(t) dt & \text{if } \lambda \geq 1/2. \end{cases}$$

for all $\xi \in L^2[0, \delta]$.

Moreover, we define

$$T(\zeta) := \sum_{\lambda \in \Lambda} T_\lambda((\zeta, \varphi_\lambda)) \cdot \varphi_\lambda$$

for all $\zeta \in L^2([0, \delta] \times \mathbb{L}, \Lambda^* \mathbb{L})$.

By the same proofs of [5, Lemma 2.2] and [5, Lemma 2.3], we see that $T$ maps $L^2([0, \delta] \times \mathbb{L}, \Lambda^* \mathbb{L})$ to the domain of $\frac{\partial}{\partial r} + \frac{1}{r} P$. Strictly speaking, [5, Lemma 2.2] and [5, Lemma 2.3] were only stated for the case where the link $\mathbb{L}$ is a closed manifold. However, since the proofs of [5, Lemma 2.2] and [5, Lemma 2.3] were carried out by some spectral computation, the same proofs also apply in our current setting where $\mathbb{L}$ is a manifold with corners. More precisely, an element $v \in L^2([0, \delta] \times \mathbb{L}, \Lambda^* \mathbb{L})$ lies in the domain of $\frac{\partial}{\partial r} + \frac{1}{r} P$ if and only if

$$\sum_{\lambda} \|\frac{\lambda}{r} (v, \varphi_\lambda)\|_{L^2[0,1]}^2 < \infty, \quad \text{and} \quad \sum_{\lambda} \left\| \frac{d}{dr} (v, \varphi_\lambda) \right\|_{L^2[0,1]}^2 < \infty.$$ 

In particular, whether an element $v$ lies in the domain of $\frac{\partial}{\partial r} + \frac{1}{r} P$ or not is completely determined by $\langle v, \varphi_\lambda \rangle$ as functions over $[0, 1]$. Therefore we can apply the same proofs of [5, Lemma 2.2 & Lemma 2.3] and Schur’s test to show that $T$ maps $L^2([0, \delta] \times \mathbb{L}, \Lambda^* \mathbb{L})$ to the domain of $\frac{\partial}{\partial r} + \frac{1}{r} P$.

Let $A$ be the (unbounded) operator on $L^2([0, \delta] \times \mathbb{L}, \Lambda^* \mathbb{L})$ defined by setting

$$A(\xi)_r = A_r(\xi)_r$$

for $\xi$ in the domain of $\frac{\partial}{\partial r} + \frac{1}{r} P$, where $\xi_r$ (resp. $A(\xi)_r$) is the restriction of $\xi$ (resp. $A(\xi)$) on $\mathbb{L}_r = \{r\} \times \mathbb{L}$. A straightforward computation shows that $AT$ and $TA$ are bounded operators such that the operator norms $\|AT\|$ and $\|TA\|$ are $\leq C(\delta)$, where $C(\delta)$ is a positive number that goes to zero as $\delta \to 0$, cf. [5, Lemma 2.2].

Since by construction $h_\lambda(r) = g_\lambda(r) + \langle A_r u(r), \varphi_\lambda \rangle$ and $T g_\lambda = u_\lambda$, we see that

$$Th = u + TAu.$$ 

By choosing a sufficiently small $\delta$, we can assume without loss of generality that $C(\delta) < 1$. It follows that $(1 + TA)$ is invertible bounded operator in this case and

$$(1 + TA)^{-1} = \sum_{j=0}^{\infty} (-1)^j (TA)^j,$$

which implies that

$$u = (1 + TA)^{-1} Th = \sum_{j=0}^{\infty} (-1)^j (TA)^j Th = T \left( \sum_{j=0}^{\infty} (-1)^j (AT)^j h \right).$$
Note that \( \sum_{j=0}^{\infty} (-1)^j (\mathcal{A}T)^j h \) lies in \( L^2([0, \delta] \times \mathbb{L}, \Lambda^* \mathbb{L}) \), since \( h \) does. We have already observed that \( T \) maps \( L^2([0, \delta] \times \mathbb{L}, \Lambda^* \mathbb{L}) \) to the domain of \( \frac{\partial}{\partial r} + \frac{1}{r} P \). Therefore, we conclude that \( u \) lies in the maximal domain of \( D^dR \). This finishes the proof.

3.3. The Fredholm index of twisted Dirac operator on manifolds with corners. We shall devote the rest of this section to computing the Fredholm index of the Dirac operator \( D \) on \( S_N \otimes f^* S_M \) subject to the boundary condition \( B \). More precisely, our second main theorem of this section is as follows.

Theorem 3.19. Assume the same setup as Theorem 3.8. Let \( \overline{\mathcal{D}}_B \) be the unique self-adjoint extension of \( \mathcal{D}_B \) as in Theorem 3.8. Then the Fredholm index of \( \overline{\mathcal{D}}_B \) is

\[
\text{Ind}(\overline{\mathcal{D}}_B) = \deg \hat{A}(f) \cdot \chi(M),
\]

where \( \deg \hat{A}(f) \) is \( \hat{A} \)-degree of \( f \) and \( \chi(M) \) is Euler characteristic of \( M \).

We point out that the same proof below shows that Theorem 3.19 also holds for manifolds with more generalized corner structures, for example, manifolds with polyhedron-like boundary. A key ingredient of the proof of Theorem 3.19 is the following theorem on the homotopy invariance of Fredholm index of the Dirac type operators that arise in the geometric setup of the current paper.

Theorem 3.20. Let \( \{F_i\} \) be the set of codimension one faces of \( N \) and \( B_i(t), t \in [0, 1] \), a continuous family\(^5\) of smooth sub-bundles of \( (S_N \otimes f^* S_M)|_{F_i} \) over \( F_i \). Let \( D_t, t \in [0, 1] \), be a continuous family of Dirac operators. Denote by \( D_t,B_t \) the operator \( D_t \) with respect to the boundary condition \( B_t \). Suppose the operator \( D_t,B_t \colon L^2(N, S_N \otimes f^* S_M) \to L^2(N, S_N \otimes f^* S_M) \)
is Fredholm and essentially self-adjoint for every \( t \in [0, 1] \), and the domain of (the closure of) \( D_t,B_t \) is \( H^1(N, S_N \otimes f^* S_M; B_t) \). Then we have

\[
\text{Ind}(D_{0,B_0}) = \text{Ind}(D_{1,B_1}).
\]

Proof. We first use the unitaries \( \Psi_{\text{even}} \) and \( \Psi_{\text{odd}} \) (which are defined near the corners of \( N \)) in line (3.5) and (3.6) to change from (iterated) conical metrics near the corners to (iterated) cylindrical metrics. More precisely, we first apply the unitaries \( \Psi_{\text{even}} \) and \( \Psi_{\text{odd}} \) as in line (3.5) and (3.6) near codimension \( n \) faces, where \( n = \dim N \). Then we iteratively apply the same type of unitaries near codimension \( (n - 1) \) faces and so on, all the way to codimension 2 faces. See Figure 10 for the codimension two case and Figure 11 for the iterated process in the codimension three case. Let us denote the resulting new manifold by \( \hat{N} \). We point out that this change from conical metric to cylindrical metric is only for convenience so that we carry out the analysis in polar coordinates.

\(^5\) We say a family of sub-bundles is continuous, if the corresponding family of orthogonal projections is norm-continuous.
We shall denote the canonical map from $\hat{N}$ to $N$ by $\Psi$. For brevity, let us write $E = S_N \otimes f^*S_M$. Let $\Psi^*E$ be the pullback bundle of $E$ by $\Psi$. Since no confusion is likely to arise, we will also denote the bundle map from $\Psi^*E$ to $E$ by $\Psi$. We denote by $\hat{F}_i$ the copy of $F_i$ in $\hat{N}$. By construction, the faces $\hat{F}_i$ are separated from each other (cf. Figure 10 and 11). In particular, there exists $\rho > 0$ such that the $\rho$-neighborhoods of the faces $\hat{F}_i$ are disjoint from each other.

Let us define $\hat{D}_t = \Psi^*D_t\Psi$ and $\hat{B}_t = \Psi^*B_t$. Since $D_{t,B_t}$ is essentially self-adjoint and Fredholm, it follows that

$$\hat{D}_{t,\hat{B}_t} : L^2(\hat{N}, \Psi^*E) \to L^2(\hat{N}, \Psi^*E)$$

is Fredholm and essentially self-adjoint, $\text{dom}(\hat{D}_{t,\hat{B}_t}) = \Psi^*H^1(N, E; B_t)$ and

$$\text{Ind}(\hat{D}_{t,\hat{B}_t}) = \text{Ind}(D_{t,B_t}).$$

Furthermore, the graph norm of $\hat{D}_{t,\hat{B}_t}$ is equivalent to the following norm

$$\|\varphi\|_{\Psi^* \nabla^N} := \|\varphi\| + \|\Psi^* \nabla^N \Psi \varphi\|$$

for elements $\varphi \in \Psi^*H^1(N, E; B_t)$, where $\nabla^N$ is the connection on $N$. Note that $\Psi^*C^\infty_0(N, E; B_t)$ consists of smooth sections that satisfy the boundary condition $B_t$ near the codimension one faces $\hat{F}_i$ and vanish near the rest of the codimension one faces of $\hat{N}$, where $C^\infty_0(N, E; B_t)$ is defined in Definition 3.4. By construction, $\Psi^*C^\infty_0(N, E; B_t)$ is dense in $\Psi^*H^1(N, E; B_t)$ with respect to the norm $\| \cdot \|_{\Psi^* \nabla^N}$ given in line (3.29). The reader should not confuse $\Psi^*H^1(N, E; B_t)$ with the $H^1$-space $H^1(\hat{N}, \Psi^*E; \hat{B}_t)$ of $\hat{N}$.

Since the family of boundary conditions $B_t$ is continuous, there exists a continuous family of isometries

$$U_{i,t} : E|_{\hat{F}_i} \to E|_{\hat{F}_i}$$
on each $\overline{F}_i$ such that $U_{i,t}$ is a smooth bundle isomorphism and maps $B_0$ isometrically to $B_t$ (e.g. by using the fact that if two orthogonal projections are sufficiently close, then they are unitarily equivalent). By the compactness of $[0,1]$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that
\[ \|U_{i,s}U_{i,t}^{-1} - 1\| < \varepsilon \] for all $s, t$ with $|s - t| \leq \delta$.

Let us now fix $s_0 \in [0,1]$ for the moment. As long as $\varepsilon$ is sufficiently small, for each $t \in [0,1]$ satisfying $|t - s_0| < \delta$, we can extend $U_{i,s_0}U_{i,t}^{-1}$ from the face $\overline{F}_i$ to an bundle isometry on its $\rho$-neighborhood such that it becomes the identity bundle map on the inner boundary of this $\rho$-neighborhood. Then we can further extend these bundle isometries (by the identity map) over the entire $\overline{N}$ and together they form a continuous path of unitaries
\[ \tilde{V}_i : \Psi^* E \to \Psi^* E \] with $t \in [s_0 - \delta, s_0 + \delta]$

such that $\tilde{V}_i$ is a smooth bundle isometry mapping $\tilde{B}_t$ to $\tilde{B}_{s_0}$. We remark that $\Psi\tilde{V}_i\Psi^*$, the pushforward of $\tilde{V}_i$ onto $N$, is generally not well-defined on $E$ over the entire $N$, as it has singularities at the codimension two faces in general.

It is clear that the operator
\[ \tilde{V}_i \tilde{D}_{t,\tilde{B}_t} \tilde{V}_i^* : L^2(\tilde{N}, \Psi^* E) \to L^2(\tilde{N}, \Psi^* E) \]
with domain $\text{dom}(\tilde{V}_i \tilde{D}_{t,\tilde{B}_t} \tilde{V}_i^*) := \tilde{V}_i(\text{dom}(\tilde{D}_{t,\tilde{B}_t}))$ is Fredholm and essentially self-adjoint for all $t \in [s_0 - \delta, s_0 + \delta]$, and
\[ \text{Ind}(\tilde{V}_i \tilde{D}_{t,\tilde{B}_t} \tilde{V}_i^*) = \text{Ind}(\tilde{D}_{t,\tilde{B}_t}) = \text{Ind}(D_{t,B_t}). \]

For any $\varphi \in \text{dom}(\tilde{D}_{s_0,\tilde{B}_{s_0}})$, we have
\[ \|\tilde{V}_i \tilde{D}_{t,\tilde{B}_t} \tilde{V}_i^* \varphi\| \leq \|\tilde{D}_t \varphi\| + \|\tilde{D}_t, \tilde{V}_i\| \varphi\| \]

First note that there exist $C_1 > 0$ and $C_2 > 0$ such that
\[ \|\tilde{D}_t \varphi\| = \|D_t \Psi \varphi\| \leq C_1 \|\nabla \Psi \varphi\| \leq C_2 \|D_{s_0,B_{s_0}} \Psi \varphi\| = C_2 \|\tilde{D}_{s_0,\tilde{B}_{s_0}} \varphi\| \]

for all $\varphi \in \text{dom}(\tilde{D}_{s_0,\tilde{B}_{s_0}})$. To estimate $\|\tilde{D}_t, \tilde{V}_i\| \varphi\|$, it suffices to carry out the estimation near codimension two faces. Similar to the proof of Theorem 3.8, we shall carry out the analysis fiberwise. Recall that near a conical singularity, along the fiber $\mathbb{F}$, we have
\[ \tilde{D}_t = \begin{pmatrix} 0 & -\frac{\partial}{\partial r} \\ -\frac{\partial}{\partial r} & 0 \end{pmatrix} + \frac{1}{r} \begin{pmatrix} 0 & P_t \\ P_t & 0 \end{pmatrix} + R_t, \]

as given in line (3.8) and (3.9) from Section 3.2. Here $P_t$ is a first-order elliptic differential operator on the link $\mathbb{L}$ of $\mathbb{F}$, and the remainder term $R_t$ is a first-order differential operator with uniformly bounded coefficients. As the Riemannian metric of $E$ is smooth near the corners of $N$, the boundary condition $\tilde{B}_t$ (for a fixed $t$) on each codimension one face is constant, up to a $O(r)$ term, along the

\footnote{Since our metric is only asymptotically conical along the fiber $\mathbb{F}$, there is a remainder term here.}
direction that is orthogonal to codimension two faces. In particular, it follows that, on each codimension one face, \( \hat{V}_t \) (for a fixed \( t \)) is constant, up to a \( O(r) \) term, along the direction that is orthogonal to codimension two faces. Therefore both the commutators

\[
[\hat{V}_t, \mathcal{R}_t] \text{ and } [\hat{V}_t, \left( \begin{array}{cc} 0 & -\frac{\partial}{\partial s} \\ -\alpha & 0 \end{array} \right) ]
\]

are zeroth order differential operators with uniformly bounded coefficients. In particular, there exists \( C_3 > 0 \) such that

\[
[\hat{V}_t, \mathcal{R}_t] \varphi \leq C_3 \| \varphi \| \text{ and } [\hat{V}_t, \left( \begin{array}{cc} 0 & -\frac{\partial}{\partial s} \\ -\alpha & 0 \end{array} \right) ] \varphi \leq C_3 \| \varphi \|
\]

for \( L^2 \) sections \( \varphi \). The essential self-adjointness of \( \hat{D}_{s_0, \hat{B}_{s_0}} \) implies that \( |P_{s_0}| \geq 1/2 \), where \( |P_{s_0}| \) is the absolute value of the operator \( P_{s_0} \) (subject to the corresponding boundary condition) on the link \( \mathbb{L} \) (see Lemma 3.9). Hence

\[
\| \varphi \|_L \leq 2 \| P_{s_0} \varphi \|_L, \quad (3.31)
\]

for all smooth sections \( \varphi \in \text{dom}(\hat{D}_{s_0, \hat{B}_{s_0}}) \), where \( \| \cdot \|_L \) is the \( L^2 \)-norm taken along the link \( \mathbb{L} \). Therefore, there exists \( C_4 > 0 \) such that

\[
\left\| \frac{1}{r} \left[ \begin{array}{cc} 0 & P_t \\ 0 & 0 \end{array} \right], \hat{V}_t \right\|_L \leq C_4 \left\| \frac{1}{r} \left[ \begin{array}{cc} 0 & P_{s_0} \\ 0 & 0 \end{array} \right] \varphi \right\|_L
\]

for all smooth sections \( \varphi \in \text{dom}(\hat{D}_{s_0, \hat{B}_{s_0}}) \). By integrating along the \( r \) direction and combining with the inequalities above, we see that there exists \( C > 0 \) such that

\[
\| \hat{V}_t \hat{D}_{t, \hat{B}_t} \hat{V}_t^* \varphi \| \leq C (\| \varphi \| + \| \hat{D}_{s_0, \hat{B}_{s_0}} \varphi \|)
\]

(3.32) for all \( \varphi \in \text{dom}(\hat{D}_{s_0, \hat{B}_{s_0}}) \) and all \( t \in [s_0 - \delta, s_0 + \delta] \). Therefore, we have

\[
\text{dom}(\hat{V}_t \hat{D}_{t, \hat{B}_t} \hat{V}_t^*) = \hat{V}_t(\text{dom}(\hat{D}_{t, \hat{B}_t})) \subset \text{dom}(\hat{D}_{s_0, \hat{B}_{s_0}})
\]

for each \( t \in [s_0 - \delta, s_0 + \delta] \). Now we switch the roles of \( \hat{D}_{t, \hat{B}_t} \) and \( \hat{D}_{s_0, \hat{B}_{s_0}} \); and apply the same argument above with \( \hat{V}_t \) replaced by \( \hat{V}_t^* \) to obtain

\[
\hat{V}_t^* \text{dom}(\hat{D}_{s_0, \hat{B}_{s_0}}) \subset \text{dom}(\hat{D}_{t, \hat{B}_t})
\]

To summarize, we have

\[
\text{dom}(\hat{V}_t \hat{D}_{t, \hat{B}_t} \hat{V}_t^*) = \text{dom}(\hat{D}_{s_0, \hat{B}_{s_0}})
\]

(3.33) for all \( t \in [s_0 - \delta, s_0 + \delta] \).

In general, the path of bounded operators

\[
\hat{V}_t \hat{D}_{t, \hat{B}_t} \hat{V}_t^*: \text{dom}(\hat{D}_{s_0, \hat{B}_{s_0}}) = \Psi^* H^1(N, E; B_{s_0}) \rightarrow L^2(\hat{N}, \Psi^* E)
\]

may not be norm-continuous in \( t \). However, we note that

\[
\hat{V}_t \hat{D}_{t, \hat{B}_t} \hat{V}_t^* - \hat{D}_{s_0, \hat{B}_{s_0}} = \hat{V}_t[\hat{D}_{t, \hat{B}_t}, \hat{V}_t^*] + (\hat{D}_{t, \hat{B}_t} - \hat{D}_{s_0, \hat{B}_{s_0}}).
\]
By assumption, the operator $D_t$ is a first order differential operator whose coefficients vary continuously with respect to $t$. Since $\tilde{D}_{s_0,B_{s_0}}$ is essentially self-adjoint and satisfies the Gårding inequality on its domain, it follows that there exists $a_1(\delta) > 0$ such that

$$\| \tilde{D}_{t,B_t} - \tilde{D}_{s_0,B_{s_0}} \| \leq a_1(\delta)\| \tilde{D}_{s_0,B_{s_0}} \|$$

for all $t \in [s_0 - \delta, s_0 + \delta]$, where $\| \cdot \|$ refers to the operator norm of the relevant operator from $\text{dom}(\tilde{D}_{s_0,B_{s_0}}) = \Psi^*H^1(N, E; B_{s_0})$ to $L^2(\tilde{N}, \Psi^*E)$. Note that $a_1(\delta)$ goes to zero as $\delta \to 0$.

By the Rellich Theorem, the embedding map

$$H^1(N, E; B_t) \to L^2(N, S_N \otimes f^*S_M)$$

is compact. Therefore the embedding map

$$\Psi^*H^1(N, E; B_t) \to L^2(\tilde{N}, \Psi^*E)$$

is also compact. We use line (3.30) again to estimate $[\tilde{D}_{t,B_t}, \tilde{V}_t^*]$ near the codimension two faces. Similarly as before, the commutators

$$[\tilde{V}_t^*, \mathcal{R}] \text{ and } [\tilde{V}_t^*, \left( \begin{array}{cc} 0 & -\frac{\partial}{\partial r} \\ \frac{\partial}{\partial r} & 0 \end{array} \right)]$$

are zeroth-order differential operators with uniformly bounded coefficients, thus compact operators when viewed as operators from $\Psi^*H^1(N, E; B_{s_0})$ to $L^2(\tilde{N}, \Psi^*E)$. Recall that the boundary condition $B_t$ (for a fixed $t$) on each codimension one face is constant along the direction that is orthogonal to codimension two faces, up to a $O(r)$ term. In particular, it follows that $U_{i,s_0}U_{i,t}^{-1}$ and $\tilde{V}_t^*$ (for a fixed $t$) on each codimension one face is constant along the direction that is orthogonal to codimension two faces, up to a $O(r)$ term. Therefore

$$\frac{1}{r}[P_t, \tilde{V}_t^*] = \frac{1}{r}\nabla_{n_i}\tilde{V}_t^* + \mathcal{Q}_t$$

near the codimension one face $\tilde{F}_i$, where $n_i$ is the unit inner normal vector of $\tilde{F}_i$ (with respect to $\tilde{N}$) and $\mathcal{Q}_t$ is an order-zero differential operator with uniformly bounded coefficients. By the previous argument in line (3.31),

$$\left\| \frac{1}{r}\nabla_{n_i}\tilde{V}_t^* \right\| \leq C \cdot \| \nabla_{n_i}\tilde{V}_t^* \|_{\infty} \cdot \| \tilde{D}_{t_{i+1}, B_{i+1}} \|,$$

where $\| \nabla_{n_i}\tilde{V}_t^* \|_{\infty}$ is the supremum norm of $\nabla_{n_i}\tilde{V}_t^*$.

Recall that $\tilde{V}_t$ is a bundle isometry obtained by extending $U_{i,s_0}U_{i,t}^{-1}$ from $\tilde{F}_i$ to its $\rho$-neighborhood of $\tilde{F}_i$ by connecting $U_{i,s_0}U_{i,t}^{-1}$ to the identity bundle map near the inner boundary of this $\rho$-neighborhood. Since $\| U_{i,s_0}U_{i,t}^{-1} - 1 \| < \varepsilon$, we can choose to connect $U_{i,s_0}U_{i,t}^{-1}$ to 1 via a linear path (and then smooth it near the two end points) along the inner normal direction $n_i$, provided that $\varepsilon$ is sufficiently small. In particular, we can assume without loss of generality that there exists $a_2(\varepsilon) > 0$ such that

$$\| \nabla_{n_i}\tilde{V}_t^* \|_{\infty} < a_2(\varepsilon).$$
Here \( a_2(\varepsilon) \) goes to zero as \( \varepsilon \to 0 \).

To summarize, we have shown that
\[
\hat{\mathcal{V}}_t \hat{\mathcal{D}}_t, \hat{\mathcal{B}}_t \cdot \hat{\mathcal{V}}_t^\ast - \hat{\mathcal{D}}_{s_0, \hat{\beta}_{s_0}} = \mathcal{T}_t + \mathcal{K}_t: \Psi^* H^1(N, E; B_{s_0}) \to L^2(\hat{N}, \Psi^* E)
\]
where \( \mathcal{T}_t \) is an operator with \( ||\mathcal{T}_t|| \leq a_1(\delta) + a_2(\varepsilon) \) and \( \mathcal{K}_t \) is a compact operator for \( t \in [s_0 - \delta, s_0 + \delta] \). Recall that \( a_1(\delta) \) goes to zero as \( \delta \to 0 \), and \( a_2(\varepsilon) \) goes to zero as \( \varepsilon \to 0 \). In particular, as long as \( \varepsilon \) and \( \delta \) are sufficiently small, we have
\[
\text{Ind}(\hat{\mathcal{D}}_{s_0, \hat{\beta}_{s_0}}) = \text{Ind}(\hat{\mathcal{V}}_t \hat{\mathcal{D}}_t, \hat{\mathcal{V}}_t^\ast) = \text{Ind}(\hat{\mathcal{D}}_{t, \hat{\beta}_t}),
\]
or equivalently,
\[
\text{Ind}(D_{s_0, B_{s_0}}) = \text{Ind}(D_{t, B_t})
\]
for \( t \in [s_0 - \delta, s_0 + \delta] \). Now by the compactness of the interval \([0, 1]\), the theorem follows. This finishes the proof.

Now let us complete the proof of Theorem 3.19.

**Proof of Theorem 3.19.** By Theorem 3.20, we can deform the Riemannian metrics on \( N \) and \( M \), and the map \( f \) without changing the Fredholm index of the associated Dirac operator, as long as the dihedral angle conditions in Theorem 3.8 are satisfied. Note that the conical type analysis used in Theorem 3.8 and Theorem 3.20 is performed near the corners of \( N \), but does not include the corners themselves. In other words, the analysis is performed on the complement \( N_o \) of the codimension two faces in \( N \). Furthermore, on this complement \( N_o \), both Theorem 3.8 and Theorem 3.20 still apply when the dihedral angles are \( \pi \), as long as the dihedral angle conditions in Theorem 3.8 are satisfied. Observe that we can deform \( N \) (resp. \( M \)) into a manifold with smooth boundary that is homeomorphic to \( N \) (resp. \( M \)), and the map \( f \) into a spin map between manifolds with smooth boundary, while preserving the dihedral angle conditions in Theorem 3.8 and the \( \tilde{A} \)-degree of \( f \). In particular, the Fredholm index remains unchanged for the associated Dirac operator subject to the corresponding local boundary condition. Let us denote the resulting manifolds with smooth boundary by \( N^\# \) and \( M^\# \), and the map by \( f^\#: N^\# \to M^\# \). For manifolds with smooth boundary, one can compute \( \text{Ind}(D_B^\#) \) (e.g. via a standard doubling argument) and conclude that
\[
\text{Ind}(D_B^\#) = \text{deg}_{\tilde{A}}(f^\#) \cdot \chi(M^\#)
\]
It is clear from our construction that \( \text{deg}_{\tilde{A}}(f^\#) \cdot \chi(M^\#) = \text{deg}_{\tilde{A}}(f) \cdot \chi(M) \). It follows that
\[
\text{Ind}(D_B) = \text{deg}_{\tilde{A}}(f) \cdot \chi(M).
\]
This finishes the proof.

4. **Proof of Theorem 1.7 and Theorem 1.8**

In this section, we prove the main theorems of the paper (Theorem 1.7 and Theorem 1.8). As pointed out in the introduction, Theorem 1.9 is an immediate consequence of Theorem 1.8.
Proof of Theorem 1.8. The assumption that deg_\tilde{\Lambda}(f) \neq 0 implies that dim N and dim M are of the same parity.

The odd dimension can be reduced to the even dimensional case as follows. Suppose both N and M are odd dimensional. For part (i), (ii), (iii) and (III), we simply consider the map \( f \times \text{id}: N \times [0, 1] \rightarrow M \times [0, 1] \), where \( \text{id}: [0, 1] \rightarrow [0, 1] \) is the identity map on \([0, 1]\), and both \( N \times [0, 1] \) and \( M \times [0, 1] \) are equipped with the product metric. For part (I) and (II), let \( Q \) be a contractible polyhedron in the 3-dimensional unit round sphere \( S^3 \) such that every codimension one face of \( Q \) is totally geodesic. For example, let \( Q \) be the intersection of \( S^3 \) with the first hyperoctant of \( \mathbb{R}^4 \). Consider the map \( f \times \text{id}: N \times Q \rightarrow M \times Q \), where \( \text{id}: Q \rightarrow Q \) is the identity map on \( Q \), and both \( N \times Q \) and \( M \times Q \) are equipped with the product metric. Hence without loss of generality, we assume that \( N \) and \( M \) are even dimensional.

Let \( D \) be the Dirac operator on \( S_N \otimes f^*S_M \) over \( N \) and \( B \) the boundary condition from Definition 3.1. Let \( D_B \) be the operator \( D \) with the boundary condition \( B \). By Theorem 3.8, \( D_B \) is essentially self-adjoint and Fredholm. We shall denote the closure of \( D_B \) still by \( D_B \). Note that the domain of \( D_B \) is

\[ \text{dom}(D_B) = H^1(N, S_N \otimes f^*S_M; B). \]

By Theorem 3.19, we have

\[ \text{Ind}(D_B) = \deg_\tilde{\Lambda}(f) \cdot \chi(M), \]

where the right hand side is nonzero by assumption. Therefore, there exists a nonzero element \( \varphi \) in \( H^1(N, S_N \otimes f^*S_M; B) \) such that \( D\varphi = 0 \). Then for each positive integer \( n \), there exists a smooth section \( \varphi_n \in C_0^\infty(N, S_N \otimes f^*S_M; B) \) such that \( \| \varphi - \varphi_n \|_1 \leq 1/n \). By Proposition 2.6 and the assumptions that \( \text{Sc} \geq \| \wedge^2 df \| \cdot f^*\text{Sc} \) and \( \text{H} \geq \| df \| \cdot f^*\text{H} \), we see that there exists \( C > 0 \) such that

\[ \int_N |\nabla \varphi_n|^2 \leq C/n^2 \]

for all \( n \). It follows that \( |\nabla \varphi| = 0 \), that is, \( \varphi \) is parallel. We identify sections of \( S_N \otimes f^*S_M \) over \( N \) with functions from \( N \) to \( \mathbb{R}^L \) by choosing a smooth embedding of \( S_N \otimes f^*S_M \) into a trivial bundle over \( N \) with rank \( L \). Since \( \varphi \) is parallel, \( \varphi: N \rightarrow \mathbb{R}^L \) is a continuous function in the interior of \( N \), and is uniformly continuous since \( N \) is compact and the metric tensor of \( S_N \otimes f^*S_M \) over \( N \) is smooth. Therefore, \( \varphi \) is a smooth section in the sense of Definition 3.4. Furthermore, as the connection \( \nabla \) preserves the metric, \( |\varphi| \) is non-zero everywhere.

We apply again Proposition 2.6 to \( \varphi \). This proves items (i), (ii) and (iii) of the theorem.

Now let us prove the rigidity part, that is, (I), (II) and (III) of the theorem. To prove (I), we first make the following observation (cf. [25, Proposition 1]).

Claim. Assume the conditions in (I) hold, that is, \( \dim M = \dim N \), \( \text{Ric}(g) > 0 \) and

\[ \text{Sc}(\bar{g})_x \geq \| df \|^2 \cdot \text{Sc}(g)_{f(x)} \]
for all \( x \in N \), then for each \( x \in N \), either \((df)_x: T_xN \to T_{f(x)}M\) is a homothety or \((df)_x = 0\).

We have already proved the equality \( \overline{S c} = \| \wedge^2 df \| \cdot f^*S c \) in the above. Since clearly \( \| df \|^2 \geq \| \wedge^2 df \| \), the condition \( \overline{S c} \geq \| df \|^2 \cdot f^*S c \) implies that
\[
\overline{S c} = \| df \|^2 \cdot f^*S c = \| \wedge^2 df \| \cdot f^*S c,
\]
Choose a local \( \overline{g} \)-orthonormal frame \( e_1, \ldots, e_n \) of \( TN \) and a local \( g \)-orthonormal frame \( e_1, \ldots, e_n \) of \( TM \) such that
\[
f^*e_i = \mu_{i} e_i \quad \text{with} \quad \mu_i \geq 0.
\]
For any \( i \), the condition \( \text{Ric}(g) > 0 \) implies that
\[
\sum_j R^M_{ijji} > 0.
\]
That is, for any \( i \), there is a \( j \neq i \) such that \( R^M_{ijji} > 0 \). The proof of Lemma 2.1 shows that
\[
\sum_{i,j} (\| df \|^2 - \mu_i^2 \mu_j^2) R^M_{ijji} = 0.
\]
Hence for any \( i \), there exists \( j \neq i \) such that
\[
\| df \|^2 = \mu_i \mu_j.
\]
It follows that \( \mu_i = \| df \| \) for all \( 1 \leq i \leq n \). This proves the claim.

Now let us define \( h := \| df \| \) and \( U \) the open subset of \( N \) consisting of points where \( h > 0 \). Since by assumption \( \text{deg} f \) is nonzero, we see that \( U \) is nonempty. On \( U \), we have \( f^*g = h^2 \cdot \overline{g} \). Therefore on \( U \) we have
\[
f^*S c = \frac{S c}{h^2} - \frac{2(n-1)}{h^3} \Delta h - \frac{(n-1)(n-4)}{h^4} |dh|^2.
\]
Since we have \( \overline{S c} = \| \wedge^2 df \| \cdot f^*S c \) and that \( h = 0 \) on \( N - U \), it follows from the above equation that
\[
2h^k \Delta h = -(n-4)h^{k-1}|dh|^2 \quad (4.1)
\]
on the whole \( N \), for all \( k \geq 1 \). Furthermore, on \( U \cap (\partial N) \), we have
\[
f^*H = \frac{\overline{H}}{h} - (n-1) \frac{1}{h^2} \frac{\partial h}{\partial e_n},
\]
where \( e_n \) is the unit inner normal vector of the codimension one faces of \( N \). Since we have \( \overline{H} = \| df \| \cdot f^*H \) on \( \partial N \) and that \( h = 0 \) on \( N - U \), it follows that
\[
\frac{\partial h}{\partial e_n} \equiv 0 \text{ on } \partial N.
\]
Hence it follows from the Stokes’ theorem that
\[
0 = \int_N h^k \Delta h + \int_N \langle d(h^k), dh \rangle = \int_N h^k \Delta h + k \int_N h^{k-1}|dh|^2.
\]
Applying Equation (4.1), we obtain
\[
(k - \frac{n-4}{2}) \int_N h^{k-1}|dh|^2 = 0
\]
for all \( k \geq 1 \). Therefore, \( dh \equiv 0 \) on \( N \). Recall that \( U \) is nonempty. It follows that \( h = \| df \| \) is a non-zero constant, say \( a > 0 \), on \( N \) and \( f: (N, a \cdot \overline{g}) \to (M, g) \) is a Riemannian covering map. This finishes of the proof of part (I).
Now let us prove part (II). Let $U'$ be the open subset of $N$ consisting of points where $\| \wedge^2 df \| \neq 0$. If we have $\text{Ric}(g) < Sc \cdot g/2$, then for any fixed $k$, we have

$$\sum_j R^M_{kjjk} + \sum_i R^M_{ikki} < \sum_{i,j} R^M_{ijji}.$$ 

Therefore

$$\sum_{i \neq k, j \neq k} R^M_{ijji} > 0.$$ 

Hence, for any $k$, there exists at least one pair $7 (i, j)$ with $i \neq k$ and $j \neq k$ such that $R^M_{ijji} > 0$. Since we have the equality $\overline{Sc} = \| \wedge^2 df \| \cdot f^*Sc$, it follows from the proof of Lemma 2.1 that

$$\sum_{i,j} (\| \wedge^2 df \| - \mu_i^2 \mu_j^2) R^M_{ijji} = 0.$$ 

Therefore, for each fixed $k$, there exist $i \neq k$ and $j \neq k$ such that $\mu_i \mu_j = \| \wedge^2 df \|$. Since $\| \wedge^2 df \| \neq 0$ at $x \in U'$, it follows that $\mu_i \neq 0$ and $\mu_j \neq 0$. Note that $\mu_i \mu_k \leq \| \wedge^2 df \|$ and $\mu_i \mu_k \leq \| \wedge^2 df \|$. Thus $\mu_k \leq \| \wedge^2 df \|^{1/2}$ for all $1 \leq k \leq n$. This shows that $\| df \|^2 = \| \wedge^2 df \|$ on $U'$. Now the same proof for part (I) shows that $\text{Ric}(g) > 0$ implies $\mu_k = \| \wedge^2 df \|^{1/2}$ for all $1 \leq k \leq n$. Now let $h := \sqrt{\| \wedge^2 df \|}$ on $N$. Then by the above discussion, we have $h = 0$ on $N - U'$ and $h = \| df \|$ on $U'$. The same proof for part (I) shows that $h = \sqrt{\| \wedge^2 df \|}$ is a nonzero constant, say $c > 0$, on $N$ and $f: (N, \sqrt{c \cdot g}) \to (M, g)$ is a Riemannian covering map. This finishes the proof of part (II).

Now let us prove part (III). If $M$ is and even dimensional flat manifold, then the connection $\nabla^M$ on $S_M$ is flat. Hence locally we can write

$$\varphi = \sum_{\alpha} \varphi_\alpha \otimes s_\alpha,$$

where $\{s_\alpha\}$ is a parallel basis of $f^*S_M$ and $\varphi_\alpha$ are local sections of $S_N$. Since there exists a nonzero section $\varphi$ of $S_N \otimes f^*S_M$ such that $\nabla \varphi = 0$. It follows that

$$0 = \sum_{i,j} R_{ijkl} \overline{\varepsilon(\varepsilon_i)(\varepsilon_j)} \varphi_\alpha = -\frac{1}{2} \text{Ric}_{kl} \varphi_\alpha$$

for any $\alpha, k, l$ in this case, cf. [4, Corollary 2.8]. Hence $N$ is Ricci flat. If $M$ is an odd dimensional flat manifold, then $M \times [0, 1]$ is even dimensional and flat. The above discussion shows that $N \times [0, 1]$ is Ricci-flat, which implies that $N$ is also Ricci-flat.

Note that in dimension three, Ricci-flatness coincides with flatness. So for 3-dimensional manifolds, the flatness of $M$ implies the flatness of $N$. This finishes the proof of part (III), hence completes the proof of the theorem.

Now let us prove Theorem 1.7.

---

7Here we need the fact that $n$ is at least 3. Indeed, this is implied by the assumption that $\text{Ric}(g) < Sc \cdot g/2$ in (II), as in dimension two the Ricci curvature is equal to $K \cdot g = Sc \cdot g/2$, where $K$ is the Gaussian curvature.
Proof of Theorem 1.7. Except part (I) and (II), Theorem 1.7 is simply a special case of Theorem 1.8.

Now for part (I), choose a local $\overline{g}$-orthonormal frame $\overline{e}_1, \ldots, \overline{e}_n$ of $TN$ and a local $g$-orthonormal frame $e_1, \ldots, e_m$ of $TM$ such that

$$f_* \overline{e}_i = \begin{cases} \mu_i e_i & \text{if } i \leq \min(m, n) \\ 0 & \text{otherwise} \end{cases}$$

with $\mu_i \geq 0$. For any $1 \leq i \leq m$, we see that $\text{Ric}(g) > 0$ implies that $\sum_j R^M_{ijji} > 0$. That is, for any $1 \leq i \leq m$, there is a $j \neq i$ such that $R^M_{ijji} > 0$. The proof of Lemma 2.1 shows that

$$\sum_{i,j} (1 - \mu_i^2 \mu_j^2) R^M_{ijji} = 0.$$ 

Hence for any $1 \leq i \leq m$, there exists $j \neq i$ such that

$$1 = \mu_i \mu_j.$$ 

It follows that $\mu_i = 1$ for all $1 \leq i \leq m$. This shows that $m = \dim M \leq n = \dim N$ and $f$ is a Riemannian submersion. The proof for part (II) is completely similar. This finishes the proof.

□

Remark 4.1. Note that part (I) and (II) of Theorem 1.8 requires the extra assumption that $\dim N = \dim M$, while part (I) and (II) of Theorem 1.7 hold even if the dimensions of $N$ and $M$ are different. In general, one needs extra geometric assumptions on the metrics $\overline{g}$ on $N$ in order to have a version of part (I) or (II) of Theorem 1.8 for the case where $N$ and $M$ have different dimensions (cf. [25, Section 4]).

Appendix A. Clifford bundles

In this appendix, we review some standard identifications of spinor bundles and Clifford bundles.

Let $X$ be an $n$-dimensional smooth spin manifold with boundary and $S$ the complexified spinor bundle over $X$. We assume that $n$ is even. The Clifford bundle of $TX$ acts on $S$ by left multiplication. On the dual bundle $S^*$, there is naturally a Clifford right action given by

$$\langle a \cdot c^*(v), b \rangle := \langle a, c(v) \cdot b \rangle, \quad \text{for all } a \in S^*, \ b \in S.$$ (A.1)

The left action of $\mathbb{C}\ell(TX)$ on $S$ induces a natural bundle isomorphism

$$\mathbb{C}\ell(TX) \longrightarrow S \otimes S^* \cong \text{End}(S, S)$$

$$\alpha \mapsto \sum_j (\alpha s_j) \otimes s^*_j,$$ (A.2)

where $\{s_j\}$ is a local orthonormal basis of $S$ and $\{s^*_j\}$ the corresponding dual basis of $S^*$ (cf. [21, I.5.18]).
The spinor bundle $S$ admits a natural $\mathbb{Z}_2$-grading given by Clifford multiplication of the volume element

$$(\sqrt{-1})^{n/2}e_1 \cdots e_n.$$  

We denote the grading operators on $S$ and $S^*$ by $\epsilon$ and $\epsilon^*$, respectively. Let $S_{\pm}$ and $S_{\pm}^*$ be the $\pm$-part of $S$ and $S^*$, respectively. The two grading operators induce a bi-grading on $S \otimes S^*$. In particular, the $\mathbb{Z}_2$-grading on $S \otimes S^*$ given by $\epsilon \otimes \epsilon^*$ corresponds to the even-odd grading on $\mathbb{C}\ell(TX)$, under the isomorphism in line (A.2).

Furthermore, under the natural isomorphism $\mathbb{C}\ell(TX) \cong \wedge^* X = \wedge^*(T^*X)$, the even/odd grading on $\mathbb{C}\ell(TX)$ corresponds to the usual even/odd grading (with respect to the degrees of differential forms) on $\wedge^* X$. In particular, the Dirac operator on $S \otimes S^*$ with respect to the $\mathbb{Z}_2$-grading given by $\epsilon \otimes \epsilon^*$ can be naturally identified with the de Rham operator of $X$ (cf. [21, II.5.12]).

Let $e_n$ be the unit inner normal vector field of $\partial X$. As $c(e_n)$ anti-commutes with $\epsilon$, multiplication by $c(e_n)$ switches the $+$ and $-$ parts of $S$. Similarly, multiplication by $c^*(e_n)$ switches the $+$ and $-$ parts of $S^*$.

**Definition A.1.** Let us restrict the bundle $S \otimes S^*$ over $\partial X$. We define $B$ to be the sub-bundle of $S \otimes S^*$ over $\partial X$ consisting of the $(-1)$-eigenspace of the operator $(\epsilon \otimes \epsilon^*)(c(e_n) \otimes c^*(e_n))$, that is,

$$B := \ker(1 + (\epsilon \otimes \epsilon^*)(c(e_n) \otimes c^*(e_n))).$$

Equivalently, if a sections $\varphi$ decomposed into

$$\varphi = \varphi_{++} + \varphi_{+-} + \varphi_{-+} + \varphi_{--}$$

with respect to the bi-grading on $S \otimes S^*$, then $\varphi$ lies in $B$ if and only if

$$\varphi_{--} = -(c(e_n) \otimes c^*(e_n))\varphi_{++} \text{ and } \varphi_{-+} = (c(e_n) \otimes c^*(e_n))\varphi_{++}.$$  

**Proposition A.2.** Under the natural bundle isomorphisms

$$S \otimes S^* \cong \mathbb{C}\ell(TX) \cong \wedge^* X$$

over $\partial X$, the sub-bundle $B$ corresponds to the sub-bundle of $\wedge^* X$ generated by forms that are tangential to $\partial X$.

**Proof.** Choose a local orthonormal basis $\{s_i\}$ of $S_+$. Then $\{s_i, -c(\overline{e}_n)s_i\}$ is a local orthonormal basis of $S$. By definition, we have $s_i^*c^*(e_n) = (-c(e_n)s_i)^*$. For simplicity, let us write $c_n$ and $c_n^*$ in place of $c(e_n)$ and $c^*(e_n)$. The map in line (A.2) from $\mathbb{C}\ell\mathcal{N}$ to $S_N \otimes S_N^*$ becomes

$$\alpha \mapsto \sum_i \alpha s_i \otimes s_i^* + \sum_i \alpha\overline{s}_i \otimes s_i^* c_n^*.$$  

Let $\alpha$ be an element in $\mathbb{C}\ell(TN)$ consisting of vectors tangential to $\partial X$. Let us assume first that $\alpha$ has even degree. In this case, $\alpha$ commutes with the grading operator $\epsilon$ on $S$ and preserves the grading. Equivalently, when viewed as an
element $\mathcal{S} \otimes \mathcal{S}^*$, $\alpha$ has only $++$ and $---$ components. Since $\alpha$ does not contain the vector $e_n$, we see that $c_n$ commutes with $\alpha$. Therefore, we have
\[
\alpha(-c_n s_i) \otimes s_i^* c_n^* = -c_n \alpha(s_i) \otimes s_i^* c_n^* = -(c_n \otimes c_n^*)(\alpha(s_i) \otimes s_i).
\]
In other words, we have
\[
\alpha_{--} = -(c_n \otimes c_n^*) \alpha_{++}.
\]
Similarly, now assume that $\alpha$ has odd degree. In this case, $\alpha$ anti-commutes with the grading operator $\epsilon$ on $\mathcal{S}$, and as an element in $\mathcal{S} \otimes \mathcal{S}^*$, $\alpha$ has only $+-$ and $-+$ components. Also, $c_n$ anti-commutes with $\alpha$. Therefore, we have
\[
\alpha(-c_n s_i) \otimes s_i^* c_n^* = c_n \alpha(s_i) \otimes s_i^* c_n^* = (c_n \otimes c_n^*)(\alpha(s_i) \otimes s_i).
\]
In other words, we have
\[
\alpha_{+--} = (c_n \otimes c_n^*) \alpha_{-+-}
\]
in this case.

In conclusion, we have shown that $B$ is mapped to the sub-bundle\(^8\) $\Lambda^*(\partial X)$ of $\Lambda^* \mathcal{X}$ generated by forms tangential to $\partial X$. Since $B$ and $\Lambda^*(\partial X)$ have the same rank, the bundle isomorphism $\mathcal{S} \otimes \mathcal{S}^* \sim \Lambda^* \mathcal{X}$ over $\partial X$ restricts to an isomorphism between $B$ and $\Lambda^*(\partial X)$. This finishes the proof. □

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\(^8\) The bundle $\Lambda^*(\partial X)$ over $\partial X$ is a subbundle of $\Lambda^* \mathcal{X}$ over $\partial X$ in a canonical way.
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