Remarks on the tensor product structure of no-signaling theories

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In the quantum logic framework we show that the no-signaling box model is a particular type of tensor product of the logics of single boxes. Such notion of tensor product is too strong to apply in the category of logics of quantum mechanical systems. Consequently, we show that the no-signaling box models cannot be considered as generalizations of quantum mechanical models.

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I. INTRODUCTION

Let us consider the following very simple model: a composite system consisting of two parties, each of them is a “black box”, a device that produce an output value \( \alpha \) when it is provided with an input value \( a \). An input can be physically identified with an observable on the box, while the output is the outcome of that measurement. Any such device is characterized by the input set and a family of sets, indexed by input values, of allowed outputs. It is assumed that all these sets are finite. A state of such device defines a probability \( P(\alpha|a) \) of getting \( \alpha \) given \( a \). The system of two such devices produces a pair of output values \((\alpha, \beta)\) upon pair of input values \((a, b)\). Again, the state of composite system defines the probability \( P(\alpha\beta|ab) \) of getting \((\alpha, \beta)\) given \((a, b)\). Additionally, we assume certain independence conditions, called a no-signaling hold:

\[
\sum_\alpha P(\alpha\beta|ab) = \sum_\alpha P(\alpha\beta|cb), \quad \sum_\beta P(\alpha\beta|ab) = \sum_\beta P(\alpha\beta|ac).
\]

This is an expression of Einstein’s causality principle that forbids instantaneous interactions (thus also information transfer); in plain words: “what happens in one box does not influence the other.” Intuitively, a no-signaling model is a composite system in which all components are coupled in minimal physically sensible way.

The above described models were introduced by Popescu and Rohrlich (thus their alternative name is Popescu-Rohrlich or PR-boxes) as an example of a system violating quantum bound of Bell-type inequalities, while still being compatible with Einstein’s causality principle. Since then, such models found many applications in quantum information theory, ranging from security of communication and distributed computing, communication complexity, to quantifying randomness. It is also a widely used tool in discussions related to foundations of physics.

Despite numerous applications, a rigorous mathematical treatment of no-signaling models is rather scarce. The need for such is two-fold. Firstly, since there are no physical realizations of no-signaling models, properties of such systems cannot be verified experimentally but only on the basis of rigorous mathematical framework. Secondly, since no-signaling models actually generalize probability beyond quantum probability, we should expect new qualitative changes, in analogy to passing from classical to quantum probability. Only a systematic study of mathematical structure can reveal such new non-intuitive properties.
As far as the authors are concerned, up to date there are only two rigorous mathematical treatments of no-signaling theories. One is based on the convex set approach and is usually called a \textit{Generalized} or \textit{Generic Probability Theory (GPT)} (see Ref. \cite{20} and references therein). The second approach is due to us \cite{21,22} and describes no-signaling models in the framework of quantum logics (in the sense of Ref. \cite{23}). Ours approach is less general than GPT, but covers standard definition of no-signaling models and allows to study more fine-grained structures of the theory.

One of the greatest arguments against quantum logic framework is the problem of defining suitable tensor product. From that point of view the convex set approach seems to be better, as the notion of tensor products (since they are not unique; see e.g. Ref. \cite{24} or Ref. \cite{25}) is well studied for them. It can be shown that the state space of no-signaling model is a maximal tensor product (as a tensor product of ordered linear spaces) of state spaces of its components, while the subset of classically correlated states is a minimal tensor product. In this paper we show how the quantum logics of components of no-signaling model combine together to form a quantum logic of the whole system. Due to the simple structure of the components, there exists a suitable notion of tensor product of quantum logics \cite{26} that fits this scheme.

\section{Quantum Structures}

Let us recall the most important definitions and facts that will be used in the sequel.

\textbf{Definition 1} A \textit{quantum logic} is a partially ordered set \( L \) with a map \( c : L \to L \) such that

\begin{enumerate}
  \item there exists the greatest (denoted by 1) and the least (denoted by 0) element in \( L \),
  \item map \( p \mapsto p^c \) is order reversing, i.e. \( p \leq q \) implies that \( q^c \leq p^c \),
  \item map \( p \mapsto p^c \) is idempotent, i.e. \((p^c)^c = p\),
  \item for a countable family \( \{p_i\} \), s.t. \( p_i \leq p_j^c \) for \( i \neq j \), the supremum \( \bigvee \{p_i\} \) exists,
  \item if \( p \leq q \) then \( q = p \lor (q \land p^c) \) (orthomodular law),
\end{enumerate}

where \( p \lor q \) is the least upper bound and \( p \land q \) the greatest lower bound of \( p \) and \( q \).
Two elements $p, q$ of quantum logic $L$ are called disjoint whenever $p \leq q^c$. An element $p$ is said to cover $q$ whenever $q \leq r \leq p$ implies $r = q$ or $r = p$. Elements covering 0 are called atoms and $L$ is called atomistic whenever any element $q \in L$ is a supremum of all atoms less than $q$. In a typical way we define a sublogic $K$ of a quantum logic $L$ as a subset $K \subset L$ closed under orthocompletion and countable sums of disjoint elements.

**Definition 2** A state $\rho$ on a quantum logic $L$ is a map $\rho: L \to [0,1]$, s.t.

S1 $\rho(\mathbb{I}) = 1$,

S2 for a countable family $\{p_i\}$, s.t. $p_i \leq p_j^c \rho(\bigvee \{p_i\}) = \sum_i p_i$.

We will be denoted by $\mathcal{S}(\mathcal{L})$ the set of all states on a quantum logic $L$.

**Definition 3** Elements $p, q \in L$ of a quantum logic $L$ are compatible, what we denote by $p \leftrightarrow q$, whenever there exist pairwise disjoint elements $p_1, q_1, r$ such that $p = p_1 \lor r, q = q_1 \lor r$.

More generally, a subset $A \subset L$ is said to be compatible whenever for any finite subset $\{p_1, \ldots, p_n\} \subset A$ there exist finite subset $G \subset L$, such that (i) elements of $G$ are mutually disjoint, (ii) any $p_i$ is supremum of some subset of $G$.

It is easier to think about compatibility in terms of the following property:

**Theorem 4 (Ref. 23, Thm. 1.3.23)** Let $A \subset \mathcal{L}$ be a compatible subset of quantum logic. Then there exists a Boolean sublogic $K \subset \mathcal{L}$, s.t. $A \subset K$.

For orthomodular lattices, compatibility of a set $A$ is equivalent to pairwise compatibility of elements. For general quantum logic it is no longer true. However, if quantum logic is regular, i.e. for any triple of mutually compatible elements $\{a, b, c\}$, $a \leftrightarrow b \lor c$, a set $A$ is compatible if and only if all elements are pairwise compatible (see Ref. 23, Def. 1.3.26 and Prop. 1.3.27.)

**Definition 5 (see Ref. 23, Sec. 1.1)** Let $\Delta$ be a family of subsets of some set $\Omega$ with partial order relation given by set inclusion and $A^c = \Omega \setminus A$ satisfying:

C1 $\emptyset \in \Delta$,

C2 $A \in \Delta$ implies $\Omega \setminus A \in \Delta$, 

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C3 for any countable family \( \{ A_i \} \subset \Delta \) of mutually disjoint sets \( \bigcup \{ A_i \} \in \Delta \).

Then \((\Omega, \Delta)\) is called a concrete (quantum) logic.

**Definition 6 (see Def. 44 in Ref. 27)** Let \( \mathcal{L} \) be a quantum logic. The set of states \( \mathcal{S} \) is said to be rich whenever:

\[
\{ \mu \in \mathcal{S} \mid \mu(a) = 1 \} \subset \{ \mu \in \mathcal{S} \mid \mu(b) = 1 \} \implies a \leq b.
\]

We say that \( \mathcal{L} \) is rich whenever it has a rich subset of states.

**Theorem 7 (see Thm. 48 in Ref. 27)** A quantum logic \( \mathcal{L} \) is set-representable, i.e. there exists order preserving isomorphism between \( \mathcal{L} \) and some concrete logic \((\Omega, \Delta)\), if and only if \( \mathcal{L} \) has a rich set of two-valued states (i.e. states with a property that \( \forall q \in \mathcal{L}, \sigma(q) = 1 \) or \( \sigma(q) = 0 \)).

**Definition 8 (see Ref. 28 p. 53)** Let \( S \subset \mathcal{S} \) be subset of the set of states \( \mathcal{S} \) of a quantum logic \( \mathcal{L} \). We say that \( \mu \in \mathcal{S} \) is a superposition of states in \( S \) whenever

\[
\nu(a) = 0, \forall \nu \in S \implies \mu(a) = 0.
\]

Let us denote by \( \overline{S} = \{ \mu \in \mathcal{S} \mid \mu \text{ is superposition of states in } S \} \).

From the physical point of view, it is essential to have a tool to describe composite systems. There are some arguments in the direction that it should always be a kind of (categorical) tensor product\(^9\). Unfortunately, this notion presents some difficulties in the theory of quantum logics when one tries to define it universally (cf. Ref. 30, chapter 4, and references therein). Let us recall only these notions that we will discuss in the sequel. The following definition is a direct generalization of Def. 3 of Ref. 31 to a category of regular quantum logics.

**Definition 9** Let \( \mathcal{L}_1, \mathcal{L}_2 \) be regular quantum logics. The \textit{free orthodistributive product} of \( \mathcal{L}_1, \mathcal{L}_2 \) is a triple \((\mathcal{L}, u_1, u_2)\), where \( \mathcal{L} \) is a quantum logic and

(i) \( u_i : \mathcal{L}_i \to \mathcal{L} \) are monomorphisms,

(ii) \( u_1(\mathcal{L}_1) \cup u_2(\mathcal{L}_2) \) generates \( \mathcal{L} \),

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(iii) \( u_1(a) \land u_2(b) = 0 \) iff \( a = 0 \) or \( b = 0 \),

(iv) \( u_1(a) \leftrightarrow u_2(b) \) for any \( a \in L_1, b \in L_2 \).

Pulmannová\textsuperscript{32} has shown that the existence of free orthodistributive product in the category of atomistic \( \sigma \)-lattices is quite an exceptional case:

**Theorem 10** (Ref. \textsuperscript{32}, Thm. 2) Let \( L, L_1, L_2 \) be complete atomistic \( \sigma \)-orthomodular lattices, and let \( (L, u_1, u_2) \) be free orthodistributive product of \( L_1, L_2 \). Let \( A, A_1, A_2 \) be atoms of \( L, L_1, L_2 \) respectively. If

\[
A = \{ u_1(a) \land u_2(b) \mid a \in A_1, b \in A_2 \}
\]  

then at least one of \( L_1, L_2 \) is a Boolean algebra.

Let us remark that the property (\textsuperscript{i1}) is not satisfied for a pair of lattices of projections on a Hilbert space, thus this theorem is not applicable to the tensor product of Hilbert spaces (which obviously is a free orthodistributive product of lattices of projections).

In order to define a *tensor product* of quantum logics (which is required to be a free orthodistributive product from the category-theoretical standpoint) we want to specify not only how the set of propositions behave but also how sets of states combine together. Pulmannová proposed the following two definitions\textsuperscript{26,32}:

**Definition 11** Let \( L, K \) be quantum logics with \( S, R \) being a state spaces of \( L \) and \( K \) respectively. A quantum logic \( T \) with a state space \( U \) will be called a *strong tensor product* of \( L \) and \( K \) whenever there are mappings \( \alpha: L \times K \to T, \beta: S, R \to U \) such that:

(i) \( \beta(\mu, \nu)(\alpha(a, b)) = \mu(a)\nu(b) \),

(ii) set \( \beta(S, R) \) is rich for \( T \),

(iii) \( T \) is generated by \( \alpha(L, K) \).

if instead of (ii) following two weaker conditions are satisfied

(ii’) \( \{ \chi \in U \mid \chi(c) = 1 \} = \{ \beta(\mu, \nu) \mid \beta(\mu, \nu)(c) = 1 \} \), for all \( c \) of the form:

(a) \( c = \land_i \alpha(a_i, b_i) \), for all \( a_i \in L, b_i \in K \) or

(b) \( c = \alpha(a, \mathbb{1}) \) for all \( a \in L \) or
\[(c) \ c = \alpha(1, b) \text{ for all } b \in L, \]

(ii”) \( \beta(S, R) = \mathcal{U} \)

then we say that \( T \) is a weak tensor product of \( L \) and \( K \).

From the previously quoted Thm. 10 it follows\(^{32}\) that in the category of atomistic \( \sigma \)-orthomodular lattices the strong tensor product exists only if at least one of the components is a Boolean algebra. In particular, it does not exist for lattices of projections on separable Hilbert spaces. On the other hand, if \( L_1, L_2 \) are lattices of projections on Hilbert spaces \( \mathcal{H}_1, \mathcal{H}_2 \), then the weak tensor product exists and coincides with the lattice of projections on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) (see Ref. \(^{32}\) Sec. IV; actually, there is second, inequivalent possibility - the lattice of projections on \( \mathcal{H}_1^* \otimes \mathcal{H}_2 \), but this is not relevant for our considerations here).

**Definition 12** (cf. Ref. \(^{23}\)) Let \( \{L_i\}_i \) be a countable family of quantum logics. A 0-1-pasting of \( \{L_i\} \) is quantum logics \( L \) defined as quotient of disjoint union of \( L_i \)’s:

\[
L = \coprod_i L_i / \sim,
\]

where \( a \sim b \) iff \( a, b \) are both the least elements or the greatest elements in any of \( L_i \)’s. In other words, \( L \) is the disjoint union of logics \( L_i \) glued together at 0 and 1.

**III. LOGIC OF NON-SIGNALLING BOXES AS A TENSOR PRODUCT**

Let us briefly recall the structure of the logic of arbitrary two box system; cf. Ref. \(^{21}\) and \(^{22}\) for detailed discussion and proof that the construction below indeed yields the logic of arbitrary no-signaling box model. To fix the notation, let the first box accept \( N \) distinct inputs, labelled by \( 1, \ldots, N \). For each input \( a = 1, \ldots, N \) let \( \mathcal{U}_a \) denote the set of possible outcomes (also of finite cardinality). Denote by \( \mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_N) \). Similarly, let the second box accept \( M \) distinct inputs, again labelled by \( 1, \ldots, M \) and set of outcomes for input \( b \) will be denoted by \( \mathcal{V}_b \). Let \( \mathcal{V} = (\mathcal{V}_1, \ldots, \mathcal{V}_M) \). We call such two box system a \( (\mathcal{U}, \mathcal{V}) \)-box world.

Denote by \( \Gamma_1 = \{(x_1, \ldots, x_N) \mid x_a \in \mathcal{U}_a \} \) and \( \Gamma_2 = \{(y_1, \ldots, y_M) \mid y_b \in \mathcal{V}_b \} \) the classical phase space that can be associated with the first and the second box, respectively. The composite (classical) system will be described a classical phase space \( \Gamma = \Gamma_1 \times \Gamma_2 \). An
experimental question “does pair of inputs \((a, b)\) results in a pair of outputs \((\alpha, \beta)\)” can be associated with the following subset of the phase space \(\Gamma\):
\[
[a\alpha, b\beta] := \{ (x, y) \in \Gamma \mid x_a = \alpha, y_b = \beta \}
\]
The logic \(L\) of \((U, V)\)-box world is the quantum logic generated in the Boolean algebra \(2^\Gamma\) by all questions of the above form (see Ref. [21] and [22] for details):
\[
A = \{ [a\alpha, b\beta] \mid (x, y) \in \Gamma \mid x_a = \alpha, y_b = \beta \}
\]

Of course, \(L\) does not have to be (and is not) a Boolean algebra anymore.

In a similar way we can assign a logic \(L_U\) to the first box. It is the quantum logic \(L_U\) generated in \(2^U\) by elements \([a\alpha] := \{ x \in \Gamma_1 \mid x_a = \alpha \}\). Observe that \(L_U\) is a 0-1-pasting of the family of Boolean logics \(\{2^U_a\}_{a=1,\ldots,N}\). Intuitively speaking, this means that we do not impose any relations between outputs for different inputs. In the same manner we define logic \(L_V\) of the second box.

**Theorem 13** The quantum logic \(L\) of \((U, V)\)-box system is an orthodistributive product of logics \(L_U\) and \(L_V\).

**Proof** Since \(L_U\) is a 0-1 pasting of Boolean algebras \(2^U\), any nonzero element of \(L_U\) is of the form:
\[
[a \in A] = \bigoplus_{a \in A} [a\alpha]
\]
where \(a = 1, \ldots, N\) and \(A \subset U_a\). The same is true for \(L_V\). Maps
\[
u: L_U \rightarrow L, \quad \nu([a \in A]) = \{ (x, y) \in \Gamma \mid x \in A \} \equiv [a \in A, 1],
\]
\[
u: L_V \rightarrow L, \quad \nu([b \in B]) = \{ (x, y) \in \Gamma \mid y \in B \} \equiv [1, b \in B]
\]
are clearly injective mappings from \(L_U\) and \(L_V\) to \(L\). Since any atom \([a\alpha, b\beta]\) of \(L\) equals to \([a\alpha, 1] \land [1, b\beta]\), union of images \(u(L_U) \cup v(L_V)\) generates \(L\). Clearly \(u([a \in A]) \land v([b \in B])\) exists and in Ref. [22] we have shown that \(u([a \in A]) \leftrightarrow v([b \in B]).\)

**Corollary 14** Thm. 10 is no longer valid in the category of regular quantum logics.

Actually, we can show even more:
Theorem 15 The logic $L$ of $(\mathcal{U}, \mathcal{V})$-box world is a strong tensor product of single box logics $L_1, L_2$.

Proof Let

$$\Phi([a \in A], [b \in B]) = [a \in A, b \in B] = \bigoplus_{\alpha \in A, \beta \in B} [a\alpha, b\beta]$$

and $\Psi(\mu, \nu)([a\alpha, b\beta]) = \mu(a\alpha)\nu(b\beta)$ (and extend by orthogonal sums to all $L_i$). It is clear that

$$\Psi(\mu, \nu)(\alpha([a \in A, b \in B])) = \mu([a \in A])\nu([b \in B]).$$

and $L$ is generated (by definition) by elements $\phi([a \in A], [b \in B])$. It remains to show that the set $\Psi(S_U, S_V)$, where $S_i$ is set of states on $L_i$, is rich for $L$.

Firstly, observe that since $L$ is a concrete logic, it has a rich set of two-valued states. Any such state $\chi$ on $L$ can be characterized by the set of atoms $\mathcal{O}_\chi$ on which it obtains value 1 (logic is atomistic, so the value of a state on atoms fully describe the state). Moreover, if $[p, q], [r, s] \in \mathcal{O}_\chi$, then also $[p, s], [r, q] \in \mathcal{O}_\chi$. Indeed, it follows from

$$1 = \mu([p, q]) = \mu([p, q]) + \mu([p^c, q]) = \mu([1, q]) = \mu([r, q]) + \mu([r^c, q]) = 1$$

and

$$1 = \mu([r, s]) = \mu([r, s]) + \mu([r, s^c]) = \mu([r, 1]) = \mu([r, q]) + \mu([r, q^c]) = 1$$

that either $\mu([r, q])) = 1$ or $\mu([r^c, s]) = 1 = \mu([r, s^c])$. But the latter cannot be true since $[r^c, s] \perp [r, s^c]$. The proof that $[p, s] \in \mathcal{O}_\chi$ is completely analogous. Consequently, we can select subsets $\mathcal{O}_\chi^1, \mathcal{O}_\chi^2$ of atoms in $L_1$ and $L_2$ respectively, such that:

$$\mathcal{O}_\chi = \{[p, q] \mid p \in \mathcal{O}_\chi^1, q \in \mathcal{O}_\chi^2\}$$

In the next step we define two functions $\mu, \nu$ on $L_1$ and $L_2$ respectively, by:

$$\mu(p) = 1 \text{ if } p \in \mathcal{O}_\chi^1, \text{ otherwise } 0,$$

$$\nu(q) = 1 \text{ if } q \in \mathcal{O}_\chi^2, \text{ otherwise } 0,$$

on atoms $p, q$ in $L_1, L_2$ and extended by orthogonal sums to the whole $L_1, L_2$. Clearly $\chi([p, q]) = \mu(p)\nu(q)$ for any $p \in \mathcal{L}_1$ and $q \in \mathcal{L}_2$. Moreover, since there can be no disjoint
pair in $O^1_\chi$ (otherwise, $\chi$ would not be a state), we infer that $\mu$ is a state on $L_1$. The same is true for $\nu$. Consequently, we have shown that any two-valued state on $L$ is a product state of two two-valued states on single box logics. Thus the set of product states is rich in $L$. ■

This fact has remarkable consequences. As it was mentioned in the introduction, for lattices of projections on Hilbert spaces, strong tensor product does not exist. It is a consequence of the fact that pure entangled states on the composite quantum system are not mixtures but superpositions of pure product states. Thus to describe properly a composed quantum mechanical systems we have to use weaker notion of the weak tensor product.

One the other hand, the logic of two non-signalling boxes is a strong tensor product of logics of single boxes, so, contrary to the quantum mechanical states, the set of product states fully describes the physical structure of a box system. It might suggest that what is called “entanglement” or “non-local” property of certain states on box-world system is in fact a weaker notion than the quantum mechanical entanglement (despite the fact that it allows stronger violation of Bell-type inequalities). Giving precise meaning to this statement is an interesting topic of a further research.

Our example suggests also that no-signaling box models are not the best tools to investigate the question of what distinguishes quantum mechanics from other no-signaling theories. Their super-quantum properties are the result of a rather trivial structure, allowing for strong tensor product to exists. By no means one can state that the no-signaling boxes are more general than the quantum mechanics, even restricted to the finite dimensional Hilbert spaces.

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