Superstring sigma models from spin chains:  
the SU(1, 1|1) case

S. Bellucci, P.-Y. Casteill  
*INFN – Laboratori Nazionali di Frascati,*  
*Via E. Fermi 40, 00044 Frascati, Italy*  

J.F. Morales  
*Department of Physics, CERN Theory Division*  
*1211 Geneva 23, Switzerland*

August 16, 2018

Abstract

We derive the coherent state representation of the integrable spin chain Hamiltonian with non-compact supersymmetry group $G = \text{SU}(1, 1|1)$. By passing to the continuous limit, we find a spin chain sigma model describing a string moving on the supercoset $G/H$, $H$ being the stabilizer group. The action is written in a manifestly $G$-invariant form in terms of the Cartan forms and the string coordinates in the supercoset.

The spin chain sigma model is shown to agree with that following from the Green-Schwarz action describing two-charged string spinning on AdS$_5 \times S^5$. 

1 Introduction

The AdS/CFT correspondence \cite{1,2,3} between strings on anti-de Sitter (AdS) spaces and boundary gauge theories is now of common use. The typical example relates string theory on $\text{AdS}_5 \times S^5$ to $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM). String states in the bulk are in correspondence to gauge invariant operators in the boundary and likewise correlation functions in the two theories are related by a well established holographic dictionary. There are several tests of the correspondence at the supergravity level (see \cite{4} for a review and a complete list of references), but few ones beyond this limit. Waiting for a better understanding of the string physics on AdS one can explore particular limits of the AdS geometry where physics simplifies itself. In \cite{5,6,7,8} the spectrum of strings on AdS and SYM operators was shown to agree at the higher symmetry enhancement point. In \cite{9}, the authors explore the holographic correspondence in the neighborhood of null geodesics of $\text{AdS}_5 \times S^5$, where the geometry looks like a pp-wave \cite{10}. String theory on pp-wave geometries is known to be solvable \cite{11,12}. On the gauge theory side this limit corresponds to focusing on SYM operators with large $R$-symmetry charge $J$.

In a similar spirit, fluctuations around semiclassical spinning strings were studied in \cite{13}-\cite{22}. Once again, energies of classical string solutions were shown to match the anomalous dimensions of SYM operators with large charges. On the gauge theory side, the planar one-loop anomalous dimensions in $\mathcal{N} = 4$ SYM are governed by integrable spin chain Hamiltonians \cite{23,24,25}. Non planar corrections were computed in \cite{26} - \cite{28} in terms of a joining-splitting spin chain operator mimicking string interactions. An alternative approach to the description of non planar corrections can be found in \cite{29}. Moreover, the analysis of \cite{27} - \cite{28} was extended in \cite{30} to the two loop level of SYM perturbation theory, by considering the SYM anomalous dimension/mixing matrix to two loops and applying to it the map to the spin bit system. In the large $N$ limit, the corresponding SU(2) spin bit model was shown to be reduced to the two loop planar integrable spin chain \cite{30}.

In the continuous (BMN) limit, i.e. for SYM operators with large $J$, the spin chain can be identified with the worldsheet of a closed string with spin chain excitations de-
scribing the string profile in the symmetry group taken as a target space. The spin chain Hamiltonian describes the dynamics of this string. As for the BMN case, the perturbative regime of SYM is accessible to this limit and accordingly the string and spin chain sigma model actions should agree. This was shown to be the case in [31] - [35] for $SO(6)$ and its compact subgroups, in [36, 37, 38] for $SL(2)$ and recently in [39] for the compact supergroup $SU(1|2)$. In all cases, semiclassical spinning string states were identified with coherent states made out of spin chain eigenstates of the symmetry group (see [40] - [48] for further developments in this subject).

The aim of this note is to extend this result to the simplest non-compact supergroup, namely $G = SU(1, 1|1)$. This sector corresponds to SYM operators made out of a single scalar, a fermion and its derivatives along a fixed direction. It gives the minimal supersymmetric extension of the SL(2) spin chain. On the string side they describe supersymmetric excitations around a string spinning in both $S^5$ and AdS$_5$. We first derive the coherent state representation of the spin chain Hamiltonian. $SU(1, 1|1)$ is non-compact and its representations are infinite dimensional. This makes the analysis of $SU(1, 1|1)$ more involved than for the $SU(1|2)$ case and leads to a non-linear form for the Hamiltonian. Remarkably, like in the SL(2) bosonic case [36, 37], the infinite series of “higher derivative” terms can be summed into a simple Log dependence. By passing to the continuous limit, the spin chain action reduces to a linear sigma model for a string moving on the supercoset $SU(1, 1|1)/SU(1|1) \times U(1)$. The results in this limit (and only in this limit) are related to those for $SU(1|2)$ found in [39] via an analytic continuation. The spin chain sigma model actions will be written in a manifestly $G$-invariant form in terms of the Cartan forms and the string coordinates on the supercoset $G/H$.

The paper is organized as follows. In Section 2 we build the $SU(1, 1|1)$ coherent state. In Section 3 we evaluate the spin chain Hamiltonian in the coherent state basis. By passing to the continuous limit and expanding in derivatives we find a linear sigma model on the group manifold $G/H$. In Subsection 3.1 we rewrite the action in a manifest $SU(1, 1|1)$ form in terms of Cartan forms and the string coordinates in the supercoset. In Section 4 we show that the same sigma model arises by considering a superstring spinning fast on $S^1_\phi \times S^1_\varphi$ on AdS$_5 \times S^5$. Finally in Section 5 we summarize our results. Appendices A, B collect technical details and useful formulas.

## 2 The coherent state

In this section we derive a coherent state representation for the spin chain Hamiltonian with symmetry group $SU(1, 1|1)$. Coherent states are defined by the choice of a group $G$ and a state $\ket{S}$ in a representation $\mathcal{R}$ of the group. We denote by $H$ the stabilizer subgroup, i.e. the group of elements of $G$ that leave invariant $\ket{S}$ up to a phase. The coherent state is then defined by the action of a finite group element of $g \in G/H$ on $\ket{S}$.

We will take $\ket{S}$ to be the physical vacuum $\ket{\phi_0}$ and denote by $G$ the rank two supergroup $SU(1, 1|1)$. The generators for the algebra $\mathfrak{g}$ are taken to be

$$T_A = (P_0, J_0, P, K, Q, \bar{Q}, \bar{S}).$$

Conventions and details about the algebra and its singleton representation are given in
appendix A. The stabilizer subgroup $H$ is generated by

$$H = \{ P_0, J_0, Q, S \} = SU(1|1) \times U(1)$$

The coherent state is defined by acting with an element $g \in G/H$ on the physical vacuum

$$|\vec{n}\rangle = g(\vec{n}) |\phi_0\rangle = e^{zP-\bar{z}K} e^{-\xi Q-\bar{\xi} S} |\phi_0\rangle \quad z = \rho e^{2i\phi} \tag{2.1}$$

and it is parameterized by two real parameters $\rho$ and $\phi$ and one complex grassmanian $\xi$.

Using (A.2), coherent states can be expanded in the basis $\{|\phi_m\rangle, |\lambda_m\rangle\}$, with $\phi_0$ a scalar field, $\lambda_0$ a fermion, and $m = 0, 1, 2, \ldots$ labelling the number of derivatives. The expansion coefficients are given by

$$|\vec{n}\rangle = \sum_{m=0}^{\infty} e^{2i m \phi} \tanh^m \rho \left[ (1 + \frac{1}{2} \xi \bar{\xi}) |\phi_m\rangle - \frac{\xi}{\cosh \rho} |\lambda_m\rangle \right]. \tag{2.2}$$

The expansion is such that

$$\langle \vec{n} | \vec{n} \rangle = 1,$$

and the coherent states are over-complete

$$\mathbb{I} = \frac{2j + F}{\pi} \int_0^\pi d\phi \int_0^\infty \sinh 2\rho d\rho \int d\xi d\bar{\xi} |\vec{n}\rangle \langle \vec{n} |,$$

with $F = 0, 1$ the supersymmetric grading of the state on which $\mathbb{I}$ acts. In the singleton representation $j = 0$, formula (2.3) is defined only in the limit $j \to 0$ (see appendix A and [37] for details).

Conversely, to each coherent state $|\vec{n}\rangle$ we can associate a point $n_A$ in the superspace

$$n_A \equiv \langle \vec{n} | T_A | \vec{n} \rangle. \tag{2.4}$$

Evaluating (2.3) (charges $T_A$ are displayed in (A.2)) one finds

$$\begin{align*}
n_{P_0} &= \frac{1}{2} (1 - \xi \bar{\xi}) \cosh 2\rho \\
n_{J_0} &= \frac{1}{2} (1 + \xi \bar{\xi}) \\
n_P &= n_K = \frac{1}{2} e^{-i2\phi} (1 - \xi \bar{\xi}) \sinh 2\rho \\
n_Q &= n_S = e^{-i2\phi} \xi \sinh \rho \\
n_{\bar{Q}} &= n_{\bar{S}} = \bar{\xi} \cosh \rho
\end{align*} \tag{2.5}$$

The resulting vector is null $n^A n_A = 0$ with respect to the Killing metric $g_{AB}$ defined by\footnote{More precisely $g_{AB} = \sum_{C,D} (-1)^F_{AB} f_{AC}^D f_{BD}^C$ with $F_C = 0, 1$ depending whether the generator $C$ is even or odd with respect to the supersymmetric grading. The inverse of the Killing metric also defines the Casimir as $\hat{C}_2 = g_{AB} T_A T_B$ (see (A.4)).}

$$n_A m^A = g^{AB} n_A m_B = \frac{1}{2} n_{P_0} m_{P_0} - \frac{1}{2} n_{J_0} m_{J_0} - \frac{1}{2} n_P m_K - \frac{1}{2} n_Q m_S - \frac{1}{2} n_{\bar{Q}} m_{\bar{S}} + \text{h.c.} \tag{2.6}$$

with $g^{AB}$ denoting the inverse of $g_{AB}$.
## 3 Hamiltonian in the coherent state basis

Here we compute the average of the spin chain Hamiltonian over a coherent spin chain configuration

$$|\textbf{n}\rangle = |\vec{n}_1\rangle \otimes |\vec{n}_2\rangle \ldots |\vec{n}_J\rangle$$

with $|\vec{n}_k\rangle$ denoting the coherent state describing the spin chain excitation at site $k$ and time $t$.

The spin chain action is given in terms of the spin chain Hamiltonian $H$ by

$$S = -\int dt \left( i \langle \textbf{n} | \partial_t | \textbf{n}\rangle + \lambda \langle \textbf{n} | H | \textbf{n}\rangle \right).$$

The first (Wess-Zumino) term can be easily evaluated by taking the derivative of (2.2) and then performing the infinite sum. It has the simple form

$$i \langle \textbf{n} | \partial_t | \textbf{n}\rangle = \sum_k \left[ -C_t + i \frac{1}{2} (\xi D_t \xi + \xi D_t \bar{\xi}) \right]_k$$

$$D_t \equiv \partial_t + i C_t, \quad C_t \equiv 2 \sinh^2 \rho \partial_t \phi.$$ 

Evaluating the second term requires more work. The first task is to rewrite the SU(1,1|1) two-site harmonic Hamiltonian in our ($\phi_m, \lambda_m$) basis. One finds

$$H = \sum_{k=1}^J H_{k,k+1}$$

with

$$H_{12} |A_k, B_l\rangle = \left( h(k + \alpha) + h(l + \beta) \right) |A_k, B_l\rangle - \left( \frac{\alpha(1-\alpha)}{k+1} + \frac{\beta(1-\beta)}{l+1} \right) |B_k, A_l\rangle$$

$$- \sum_{i=1}^l \left( \frac{1}{i} - \frac{\beta}{i+1} \right) |A_{k-i}, B_{l+i}\rangle - \sum_{i=1}^l \frac{\alpha(1-\beta)}{i+1} |B_{k-i}, A_{l+i}\rangle$$

$$- \sum_{i=1}^l \left( \frac{1}{i} - \frac{\alpha}{k+i+1} \right) |A_{k+i}, B_{l-i}\rangle - \sum_{i=1}^l \frac{\beta(1-\alpha)}{k+i+1} |B_{k+i}, A_{l-i}\rangle,$$ 

and

$$|A, B\rangle \equiv |A\rangle \otimes |B\rangle, \quad \alpha \equiv \delta_{A=\lambda}, \quad \beta \equiv \delta_{B=\lambda}, \quad h(m) = \sum_{i=1}^m \frac{1}{i}.$$ 

The Hamiltonian has a nice representation in the coherent state basis $|\vec{n}\rangle$. We first compute the average of $H_{k,k+1}$ over two-site coherent states $|\vec{n}_k \vec{n}_{k+1}\rangle \equiv |\vec{n}_k\rangle \otimes |\vec{n}_{k+1}\rangle$, then we sum up over the spin chain sites $k = 1, \ldots J$. The algebra is extremely long but the result can be written in the remarkably simple form

$$\langle \textbf{n} | H | \textbf{n}\rangle = \sum_{k=1}^J \langle \vec{n}_k \vec{n}_{k+1} | H_{k,k+1} | \vec{n}_k \vec{n}_{k+1}\rangle = \sum_{k=1}^J \log \left[ 1 + 2 \vec{n}_k \vec{n}_{k+1} \right]$$

$$= \sum_{k=1}^J \log \left[ 1 - (\vec{n}_k - \vec{n}_{k+1})^2 \right].$$

5
As before, we use the Killing metric (2.6) to compute the scalar products in (3.6).

The coherent state representation (3.6) is the main result of this section. Note that in terms of $\vec{n}_k$, it takes exactly the same form as for the $sl(2)$ case, but now in terms of the $SU(1,1|1)$ vector and the corresponding Killing metric.

In the continuous limit, (3.6) reduces to

$$\langle \vec{n} | H | \vec{n} \rangle = -\frac{1}{J} \int d\sigma \ g^{AB} \partial_{\sigma} n_{A} \partial_{\sigma} n_{B}$$

$$= \frac{1}{J} \int d\sigma \left( \bar{D}_{\sigma} \xi D_{\sigma} \xi + (1 + \xi \xi) \left[ (\partial_{\sigma} \rho)^2 + \sinh^2 2\rho \ (\partial_{\sigma} \phi)^2 \right] \right)$$

(3.7)

and coincides with the Hamiltonian of a string moving on the supercoset manifold $G/H$.

Finally, plugging (3.3) and (3.7) into (3.2), one finds for the spin chain action the result

$$S = -J \int d\sigma dt \left( -C_{I} + i \bar{\xi} D_{I} \xi + \frac{\lambda}{J^2} \left( e^2 + \bar{D}_{\sigma} \xi D_{\sigma} \xi + \bar{\xi} \xi e^2 \right) \right),$$

(3.8)

with

$$e^2 = (\partial_{\sigma} \rho)^2 + \sinh^2 2\rho \ (\partial_{\sigma} \phi)^2 .$$

$$D_a = \partial_a + i C_a \quad C_a \equiv 2 \sinh^2 \rho \partial_a \phi .$$

(3.9)

In Section 4 we will find that the same sigma model describe strings spinning fast on $S^1_\phi \times S^1_\rho$ inside $AdS_5 \times S^5$.

### 3.1 Cartan forms

The result (3.8) can be written in a manifestly $G$-invariant form in terms of the Cartan forms $L^A$ and the string coordinates $n_A$ in the supercoset $G/H$. This is the aim of this subsection. Readers not interested in these details can skip this section.

The Cartan forms $L^A$ are defined by

$$dg g^{-1} = L^A T_A = L^A_a T_A \ d\sigma^a , \quad \sigma^{0,1} \equiv (t, \sigma) ,$$

(3.10)

with $g$ given by (2.4). They parameterize the gradient $dn_A = \partial_a n_A \ d\sigma^a$ of the string position on the supercoset along its worldsheet coordinates $\sigma^a$. The explicit relation can be determined as follows:

$$dn_A = \langle 0 | g^{-1} T_A g | 0 \rangle + \langle 0 | g^{-1} T_A dg | 0 \rangle$$

$$= -L^B \langle \vec{n} | \{ T_B, T_A \} | \vec{n} \rangle$$

$$= -L^B \ f_{BA}^C n_C ,$$

(3.11)

with $\{ \}$ denoting commutators or anticommutators according to spin of the generator. The first term in (3.11) can also be written in terms of $L^A$ and $n_A$

$$\langle \vec{n} | \partial_{|} \vec{n} \rangle = \langle \vec{n} | \partial_t g g^{-1} | \vec{n} \rangle = L^A_i n_A .$$

(3.12)

\[\text{2}\text{Notice that the scalar product here is not positively defined since the group is non-compact.}\]
Plugging (3.11,3.12) in (3.2) one can finally rewrite the spin chain action in a manifestly $G$-invariant form

$$S = -J \int d^2 \sigma \left[ i L^A n_A - \lambda \frac{\dot{\lambda}}{J^2} (L^B f_{BA}^C n_C)^2 \right].$$

(3.13)

This formula is the main result of this section and it is valid for any spin chain with Hamiltonian given by the first line in (3.7)!

In the present case one finds

$$\begin{cases}
L^Q = L^S = e^{2i\phi} \sinh \rho \, d\overline{\xi} \\
L^Q = L^S = - \cosh \rho \, d\xi \\
L^P = -L^K = e^{2i\phi} \left[ d\rho + \sinh 2\rho \left( i \, d\phi - \frac{1}{4} d\overline{\xi} \, \xi + \frac{1}{4} \overline{\xi} d\xi \right) \right] \\
L^P = -4i \sinh^2 \rho \, d\phi - \frac{1}{2} \cosh 2\rho \left( \overline{\xi} d\xi - d\overline{\xi} \right) \\
L^J = \frac{1}{2} \left( d\overline{\xi} \, \xi - \overline{\xi} \, d\xi \right)
\end{cases}$$

(3.14)

and the result (3.8) follows.

## 4 String action

Here we describe the string duals of the SU(1,1|1) spin chain system. We follow the strategy sketched in [39] for superstrings spinning on a sphere. The results will be related to that case via analytic continuation to AdS.

To second order in the fermionic excitations, the string action on AdS$_5 \times S^5$ is given by

$$S = \frac{R^2}{4\pi \alpha'} \int d\sigma d\tau \left[ g_{MN} \partial^a X^M \partial_a X^N - 2i \partial (\rho^a \partial_a + \frac{i}{2} \epsilon^{ab} \rho_a \Gamma_{*} \rho_b) \bar{\theta} \right]$$

(4.1)

with

$$\rho_a = \partial_a X^M E^A_M \Gamma_A, \quad D_a \equiv \partial_a + A_a, \quad A_a \equiv \frac{1}{4} \partial_a X^M \omega^{AB}_M \Gamma_{AB}.$$ 

(4.2)

Here $E^A_M$ is the Zehnbein, $\Gamma^A$ are the usual flat ten-dimensional gamma matrices, $\Gamma_{*}$ is the chirality operator along AdS, $\rho_a$ is the induced gamma matrices and $\omega^{AB}_M$ is the spin connection. We first write the metric for the SL(2) spinning string as

$$ds^2 = - \cosh^2 \rho \, d\tilde{t}^2 + d\rho^2 + d\phi^2 + \sinh^2 \rho \, d\tilde{\phi}^2$$

(4.3)

with $\tilde{t}, \rho, \tilde{\phi}$ denoting three coordinates inside AdS$_5$ and $\varphi_3$ being an angle on $S^5$. Tangent space labels $A = 0, 1, 2, 3$ will be associated with the coordinates $\tilde{t}, \rho, \tilde{\phi}, \phi$ respectively. In terms of such variables, one has $\Gamma_{*} \equiv i \Gamma_{01345}$. We introduce the notation $\Pi \equiv \Gamma_{145}$ and choose our ten-dimensional spinors such that $\overline{\Pi} = i$.

Then, we make the change of coordinates

$$\tilde{t} \to t - \varphi, \quad \tilde{\phi} \to t, \quad \tilde{\phi} \to t - \varphi + 2\phi$$

7
in order to bring the metric into a form with $g_{tt} = 0$ where a BMN like limit (see below) is well defined. We take
\[ t = \kappa \tau, \quad \dot{X}^M \equiv \partial_t X^M, \quad X'^M \equiv \partial_\sigma X^M \] (4.4)
and consider the limit $\kappa \to \infty$ keeping $\kappa^2 \dot{X}^M \neq t$ fixed. We look for superstring excitations satisfying the Virasoro constraints
\[
g_{MN} \partial_\tau X^M \partial_\sigma X^N - i \bar{\vartheta} (\rho_\sigma \mathcal{D}_\tau + \rho_\tau \mathcal{D}_\sigma) \vartheta = 0 \]
\[
g_{MN} (\partial_\tau X^M \partial_\sigma X^N + \partial_\sigma X^M \partial_\tau X^N) - 2 i \bar{\vartheta} (\rho_\tau \mathcal{D}_\tau + \rho_\sigma \mathcal{D}_\sigma) \vartheta = 0.\]

They can be used to solve for $\varphi$ in favor of the remaining variables. To leading order in $\kappa$ one finds
\[
\varphi' = -2 \sinh^2 \rho \phi' + \mathcal{O}(\vartheta^2) \\
\varphi = -2 \sinh^2 \rho \phi - \frac{e^2}{2 \kappa^2} + \mathcal{O}(\vartheta^2) \] (4.5)
with $e^2 = \rho^2 + \sinh^2 2\rho \phi'^2$. One can easily see that fermionic terms in (4.3) contribute to the lagrangian either as quartic terms in the fermions or to subleading terms in the $\frac{1}{\kappa}$-expansion and therefore can be discarded.

Using the first of these equations we can write the bosonic part of the action as
\[
S_B = -\frac{R^2 \kappa}{2 \pi \alpha'} \int dt \left( -\dot{\varphi} - C_t + \frac{e^2}{2 \kappa^2} \right) \] (4.6)

Now let us consider the fermionic Lagrangian. Evaluating (4.2) one finds
\[
\rho_0 = -\kappa \left[ (1 - \varphi) \cosh \rho \Gamma_0 - \dot{\rho} \Gamma_1 - \Gamma_2 - \left( 1 + 2 \dot{\phi'} - \dot{\phi} \right) \sinh \rho \Gamma_3 \right] \] (4.7)
\[
\rho_1 = \varphi' \cosh \rho \Gamma_0 + \rho' \Gamma_1 + \left( 2 \dot{\phi'} - \varphi' \right) \sinh \rho \Gamma_3 \]
\[
A_0 = \frac{\kappa}{2} \left[ (-1 + \varphi) \sinh \rho \Gamma_{01} - \left( 1 + 2 \dot{\phi} - \dot{\varphi} \right) \cosh \rho \Gamma_{13} \right] \]
\[
A_1 = \frac{1}{2} \varphi' \sinh \rho \Gamma_{01} - \left( \varphi' - \frac{1}{2} \dot{\varphi} \right) \cosh \rho \Gamma_{13} \]

In addition, eqs. (4.5) can be used to show that the matrices $\rho_a$ satisfy the Clifford algebra
\[
- \{ \rho_0, \rho_0 \} = \{ \rho_1, \rho_1 \} = 2 \partial_\sigma X^M \partial_\sigma X_M = 2 e^2 \\
\{ \rho_1, \rho_0 \} = 2 \partial_\sigma X^M \partial_\tau X_M = 0 \] (4.8)
and therefore can be put in the form
\[
\rho_0 = -e \Gamma_0, \quad \rho_1 = e \Gamma_3, \] (4.9)
via a spinor rotation $\rho_a \to S \rho_a S^{-1}$. The precise form of $S$ and its derivation are given in Appendix B. In the new basis the fermionic string action reads
\[
S_F = i \frac{R^2 \kappa}{4 \pi \alpha'} \int dt \bar{\Psi} \left( \Gamma_0 \mathcal{D}_t + \frac{1}{\kappa} \Gamma_3 \mathcal{D}_\sigma + \left( 1 + \frac{e^2}{2 \kappa^2} \right) \right) \Psi \] (4.10)

\[^3\text{We will not need the explicit form of this metric here. Spin connections will be computed using the starting metric (4.3).}\]
\[^4\text{As it can be seen from the coefficient in front of } \Gamma_0 \text{ in eq. (4.10), } \rho_0 \text{ gets a negative sign.}\]
where
\[ D_a = \partial_a + C_a \Gamma_{123}, \]
with \( C_a \) defined as in (3.9). In the derivation of (4.10) the field \( \phi \) is taken to be on the mass shell up to order \( \frac{1}{k^2} \). This is consistent with the fact that the limit \( \kappa \to \infty \) in (4.10) corresponds to the semiclassical expansion around \( \hbar \sim \frac{1}{\kappa} \to 0 \) where fields are put on the mass shell.

Following [39] we choose our spinor as a four dimensional Majorana spinor:
\[
\Psi = \begin{pmatrix} e^{-it} \xi_1 \\ e^{it} \bar{\xi} \end{pmatrix} \quad \bar{\Psi} = \Psi^\dagger \Gamma_0 \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \quad \bar{\xi} = \begin{pmatrix} i \xi_2^* \\ -i \xi_1^* \end{pmatrix}.
\]
The factor \( e^{it} \) is included, in order to remove the fast mode fermionic oscillations. For the Gamma matrices we take:
\[
\Gamma_0 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \quad \Gamma_1 = \begin{pmatrix} 0 & i \sigma_2 \\ -i \sigma_2 & 0 \end{pmatrix} \quad \Gamma_2 = \begin{pmatrix} 0 & -i \sigma_3 \\ i \sigma_3 & 0 \end{pmatrix} \quad \Gamma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\]
Plugging into (4.10) one finds the fermionic action:
\[
S_F = -\frac{R^2 \kappa}{2\pi \alpha'} \int d\sigma dt \left[ i \xi_1^*(D_t \xi_1 + \frac{1}{\kappa} D_\sigma \xi_2) + i \xi_2^*(D_t \xi_2 + \frac{1}{\kappa} D_\sigma \xi_1) \\
+ \frac{e^2}{2 \kappa^2} \xi_1^* \xi_1 - (2 + \frac{e^2}{2 \kappa^2}) \xi_2^* \xi_2 \right] \quad (4.11)
\]
with covariant derivatives given by (3.9). To leading order in \( \frac{1}{k} \) the field \( \xi_2 \) is non-dynamical and can be solved via its equation of motion in favor of \( \xi_1 \).
\[
\xi_2 = \frac{i}{2\kappa} D_\sigma \xi_1.
\]
Plugging into the action one finally finds
\[
S_F = -\frac{R^2 \kappa}{2\pi \alpha'} \int d\sigma dt \left[ i \xi_1^* D_t \xi_1 + \frac{1}{2 \kappa^2} \left( D_\sigma \xi_1^* D_\sigma \xi_1 + e^2 \xi_1^* \xi_1 \right) \right]. \quad (4.12)
\]
One can easily see that the string action \( S = S_B + S_F \) following from (4.6) + (4.12) perfectly matches the spin chain result (3.8) after the identifications [37]
\[
J = \frac{R^2 \kappa}{2\pi \alpha'} \quad \lambda = \frac{R^4}{8\pi^2 \alpha'^2} \quad (4.13)
\]

### 5 Summary of results

In this note we derive a coherent state representation for the integrable spin chain Hamiltonian with symmetry group \( G = SU(1,1|1) \). The result can be cast in the remarkably compact and simple form
\[
\langle n | H | n \rangle = \sum_{k=1}^{L} \log \left[ 1 - (\bar{n}_{k+1} - \bar{n}_k)^2 \right]
\]

\footnote{We believe that this is also the case for the \( SU(1|2) \) spin chain considered in [39].}
with \( \vec{n}_k \) parameterizing a point in the supercoset \( G/H \), \( H = SU(1\mid1) \times U(1) \) being the stabilizer group. The scalar product is defined in terms of the Killing metric on \( G \).

By passing to the continuous limit \( \partial_a \vec{n} \equiv (\vec{n}_{k+1} - \vec{n}_k) J \), we find a spin chain sigma model describing a string moving on the group manifold \( G/H \). The result can be written in a manifestly \( G \)-invariant form in terms of the Cartan forms \( L^A \) and the string coordinates \( n_A = \langle n \mid T_A \mid n \rangle \) on the supercoset

\[
S = -\int d^2\sigma \left[ i \langle n \mid \partial_t \mid n \rangle + \hat{\lambda} \langle n \mid H \mid n \rangle \right] = -\int d^2\sigma \left[ i L_A^A n_A - \frac{\hat{\lambda}}{J^2} g^{AB} \partial_\sigma n_A \partial_\sigma n_A \right].
\]

Here \( n_A(\sigma, t) \) describes the profile of the string evolving in time and \( g_{AB} \) denotes the Killing metric of \( SU(1,1\mid1) \). The same formula applies to \( SU(2\mid3) \). In components one finds

\[
S = -J \int d\sigma dt \left[ -C_t + i \bar{\xi} D_t \xi + \frac{\hat{\lambda}}{J^2} \left( e^\sigma + \bar{D}_\sigma \bar{\xi} D_\sigma \xi + \bar{\xi} \xi e^\sigma \right) \right].
\]

The same sigma model was found by considering the Green-Schwarz (GS) action of a superstring spinning fast on a \( S^1_\phi \times S^1_\phi \) torus inside \( \text{AdS}_5 \times S^5 \). This establishes a precise map between coherent states in the \( SU(1,1\mid1) \) sector and string states and matches their dynamics.

It is worth to stress that, unlike the GS action, the spin sigma model Lagrangian is built out of the \( SU(1,1\mid1) \) invariant Killing metric of the supergroup. The two actions are related in a highly non-trivial way in the limit \( J \to \infty \) where the string becomes semiclassical and fields come near the mass shell. The analysis here provide us with a detailed dictionary between the two descriptions. In particular the agreement found here between the two actions implies a similar match between their classical solutions. It would be nice to explore the simpler spin chain sigma model description as a possible bottom-up definition for the study of more general string configurations on \( \text{AdS}_5 \times S^5 \).

**Acknowledgements**

We thank R. Hernandez, A. Marrani and C. Sochichiu for discussions. This work was partially supported by INTAS-00-00254 grant, INTAS-00-00262, RF Presidential grants MD-252.2003.02, NS-1252.2003.2, INTAS grant 03-51-6346, RFBR-DFG grant 436 RYS 113/669/0-2, RFBR grant 03-02-16193 and the European Community’s ”Marie Curie Research Training Network under contract MRTN-CT-2004-005104 Forces Universe”.

**Appendix**

In this appendix we collect the commutation relations and details on the “singleton” representations of the superalgebra \( g = su(1,1\mid1) \). A singleton corresponds to a subsector of the \( N = 4 \) SYM multiplet that closes under \( g \). Here we adopt the oscillator description (see [25] for details). In this formalism, elementary SYM fields (the singleton of

\footnote{This can be easily seen for the \( SU(1\mid2) \) case by replacing hyperbolic functions in \( \text{Pan} \) by their trigonometric analogs and comparing the resulting action with \( \text{S9} \).}
\( \mathfrak{psu}(2, 2|4) \) are represented by acting on a Fock vacuum \(|0\rangle\) with bosonic \((a_\alpha, b_\dot{\alpha})\) and fermionic oscillators \(c_A\), \((\alpha, \dot{\alpha} = 1, 2, A = 1, \ldots 4)\). Physical states satisfy the condition

\[
C = n_a - n_b + n_c = 2
\]

with \(n_a, n_b, n_c\) denoting the number of oscillators of a given type.

The closed subalgebras of \(\mathfrak{su}(2, 2|4)\) are defined by restricting the range of \(\alpha, \dot{\alpha}, A\).

### A \( \mathfrak{su}(1, 1|1) \) algebra

The algebra \(\mathfrak{su}(1, 1|1)\) is built in terms of bilinears of two bosonic \((a, b)\) and one fermionic \((c_3)\) oscillators. The physical vacuum can be taken to be \(|\phi_0\rangle = c_1^\dagger c_2^\dagger|0\rangle\). States ("letters") in the singleton representation are given by

\[
|\phi_m\rangle = \frac{1}{m!}(a^\dagger b^\dagger)^m|\phi_0\rangle \quad \Leftrightarrow \quad \frac{1}{m!}D^m\phi
\]

\[
|\lambda_m\rangle = \frac{1}{m!}(a^\dagger b^\dagger)^m b^\dagger c_3^\dagger|\phi_0\rangle \quad \Leftrightarrow \quad \frac{1}{m!}D^m\lambda
\]

and correspond to a scalar field \(\phi_0\), a fermion \(\lambda_0\) and their \(m\)-derivatives along a fixed direction. The algebra in this case is non-compact and the representations are infinite-dimensional.

The generators can be written as bilinears in the oscillators

\[
P = a^\dagger b^\dagger \quad Q = a^\dagger c_3
\]

\[
K = ab \quad \bar{Q} = b^\dagger c_3^\dagger
\]

\[
P_0 = \frac{1}{2}(1 + a^\dagger a + b^\dagger b) \quad S = ac_3^\dagger
\]

\[
\bar{S} = bc_3
\]

The charges \(P_0\) and \(J_0\) give the Cartan of the group. Non-vanishing commutation relations are given by

\[
[K, P] = 2P_0 \quad [S, P] = \bar{Q} \quad [K, \bar{Q}] = S
\]

\[
\{Q, S\} = P_0 - J_0 \quad \{\bar{S}, P\} = Q \quad [K, Q] = \bar{S}
\]

\[
\{Q, \bar{S}\} = P_0 + J_0 \quad \{\bar{Q}, Q\} = P \quad \{S, \bar{S}\} = K
\]

\[
[P_0, \bar{Q}] = \frac{1}{2}Q \quad [J_0, \bar{Q}] = \frac{1}{2}Q
\]

\[
[P_0, K] = -K \quad [P_0, \bar{S}] = -\frac{1}{2}S \quad [J_0, \bar{S}] = -\frac{1}{2}S
\]

The action of the generators on the states (A.1) is given by

\[
P_0 |\phi_m\rangle = (m + \frac{1}{2}) |\phi_m\rangle \quad P_0 |\lambda_m\rangle = (m + 1) |\lambda_m\rangle
\]

\[
J_0 |\phi_m\rangle = \frac{1}{2} \quad J_0 |\lambda_m\rangle = 0
\]

\[
P |\phi_m\rangle = (m + 1) |\phi_{m+1}\rangle \quad P |\lambda_m\rangle = (m + 1) |\lambda_{m+1}\rangle
\]

\[
K |\phi_m\rangle = m |\phi_{m-1}\rangle \quad K |\lambda_m\rangle = (m + 1) |\lambda_{m-1}\rangle
\]

\[
Q |\phi_m\rangle = 0 \quad Q |\lambda_m\rangle = (m + 1) |\phi_{m+1}\rangle
\]

\[
S |\phi_m\rangle = |\lambda_{m-1}\rangle \quad S |\lambda_m\rangle = 0
\]

\[
\bar{Q} |\phi_m\rangle = |\lambda_m\rangle \quad \bar{Q} |\lambda_m\rangle = 0
\]

\[
\bar{S} |\phi_m\rangle = 0 \quad \bar{S} |\lambda_m\rangle = (m + 1) |\phi_m\rangle
\]

(A.2)
For later convenience we choose the normalization
\[
\langle \phi_m | \phi_n \rangle = \delta_{mn}, \quad \langle \lambda_m | \lambda_n \rangle = (m + 1)\delta_{mn}, \quad \langle \phi_m | \lambda_n \rangle = 0.
\]

SYM operators in the $SU(1,1|1)$ sector are given by tensor products of $J$ singletons “words made out of letters”, i.e. we take $J$ copies of the considered oscillators $a, b, c$ and impose the condition (A.0) at each site. The symmetry algebra is taken to be the diagonal $SU(1,1|1)$ algebra
\[
T^A = \sum_{k=1}^{J} T_k^A
\]
with $T_k^A$ acting on the $k^{th}$ site.

It is not difficult to verify that the quadratic operator
\[
\hat{C}_2 = g^{AB} T_A T_B = P_0^2 - J_0^2 - \frac{1}{2} \{P, K\} - \frac{1}{2} [Q, S] - \frac{1}{2} [\bar{Q}, \bar{S}]
\]
(A.4)
commutes with all generators, i.e. it is a Casimir of the algebra. Therefore, it is proportional to a unit matrix:
\[
\hat{C}_2 = j (j + 1) \mathbb{I}.
\]

The number $j$ labels the irreducible representations of the algebra. In particular, for $j = 0$ the Casimir vanishes. The defining representation $j = 0$ is the so called “singleton representation” and is generated by acting with the lowering charges on the highest weight $|\phi_0\rangle \equiv c_1^\dagger c_2^\dagger |0\rangle$.

All spin $j$ representations arise already in the tensor product of two singletons. The spin $j$ highest weight state spin $j$ representation can be written as follows:
\[
|j\rangle_{k_1 k_2} = \sum_{n=0}^{j} (-1)^n \binom{j}{n} |\phi_{j-n}\rangle_{k_1} \otimes |\phi_n\rangle_{k_2}.
\]
(A.5)

B Spinor rotations

Here we derive the spinor rotation $S$ and string action in the new spinor basis. These results are relevant for the analysis of Section 4. $S$ is defined by
\[
S \rho_0 S^{-1} = -e \Gamma_0 \quad S \rho_1 S^{-1} = e \Gamma_3
\]
As in [39] we write $S$ as a product of rotations of the type
\[
S_{ij}(p) = e^{\frac{ip}{2} \Gamma_{ij}} = \cos \frac{p}{2} + \sin \frac{p}{2} \Gamma_{ij}
\]
\[
S_{0i}(p) = e^{\frac{ip}{2} \Gamma_{0i}} = \cosh \frac{p}{2} + \sinh \frac{p}{2} \Gamma_{ij}.
\]
The spinor rotation can be written as
\[
S = S_{13}(p_4) S_{02}(p_3) S_{01}(p_2) S_{03}(p_1)
\]
(B.1)
with

\[ p_1 = \rho + \sinh 2\rho \dot{\phi} \quad \quad p_2 = \dot{\rho} \]
\[ \cosh p_3 = \frac{\kappa}{e} + \frac{e}{2\kappa} \quad \quad \cos p_4 = \sinh 2\rho \frac{\phi'}{e} \]
\[ \sinh p_3 = \frac{\kappa}{e} \quad \quad \sin p_4 = -\frac{\rho'}{e}. \quad \quad \text{(B.2)} \]

The transformed matrices can be computed with the help of

\[ S_{ij} \Gamma_i S_{ij}^{-1} = \cos p \Gamma_i - \sin p \Gamma_j \]
\[ S_{ij} \Gamma_j S_{ij}^{-1} = \sin p \Gamma_i + \cos p \Gamma_j \]
\[ S_{0i} \Gamma_0 S_{0i}^{-1} = \cosh p \Gamma_0 + \sinh p \Gamma_i \]
\[ S_{0i} \Gamma_0 S_{0i}^{-1} = \sinh p \Gamma_0 + \cosh p \Gamma_i. \quad \quad \text{(B.3)} \]

In the process, we use the Virasoro constraints

\[ \varphi' + 2 \sinh^2 \rho \phi' = 0, \quad \phi' + 2 \sinh^2 \rho \phi = -\frac{e^2}{2\kappa^2} \quad \quad \text{(B.4)} \]

in order to solve for \( \varphi \) in favor of \( \phi \). We also use the equality given by the cross derivative condition \( \partial_t \phi' = \partial_\sigma \dot{\phi} \) to solve for \( \rho'' \).

At intermediate steps, one gets

\[ \frac{i}{2} e^{ab} S \rho_a \Gamma_\ast \rho_b S^{-1} = -\kappa e \left( 1 + \frac{e^2}{2\kappa^2} \right) \Gamma_1 \Pi \]
\[ S A_r S^{-1} = \frac{\kappa^2}{2e^2} \left( \dot{\rho} \rho' + 2 \sinh^2 2\rho \dot{\phi} \phi' \right) (\Gamma_{01} - \Gamma_{12}) \]
\[ \quad - \frac{\kappa^2}{2e^2} \sinh 2\rho \left( \dot{\phi} \phi' - 2 \dot{\phi} \rho' \right) (\Gamma_{03} + \Gamma_{23}) \]
\[ \quad - \frac{\kappa}{2} \left( 1 + \frac{e^2}{2\kappa^2} + 2 \cosh 2\rho \dot{\phi} \right) \Gamma_{13} \]
\[ S A_\sigma S^{-1} = \frac{\kappa}{e^2} \left[ \cosh 2\rho \left( \dot{\rho} \rho' + \sinh^2 2\rho \dot{\phi} \phi' \right) \phi' + \frac{1}{2} \sinh^2 2\rho \phi'^2 \right] (\Gamma_{01} - \Gamma_{12}) \]
\[ \quad + \frac{\kappa}{e^2} \sinh 2\rho \phi' \left[ \frac{1}{2} \rho' + \cosh 2\rho \left( \dot{\phi} \rho' - \dot{\rho} \phi' \right) \right] (\Gamma_{03} + \Gamma_{23}) \]
\[ \quad - \cosh 2\rho \phi' \Gamma_{13} + \frac{1}{4\kappa} \sinh^2 2\rho \phi'^2 \Gamma_{01} + \frac{1}{4\kappa} \sinh 2\rho \rho' \phi' \Gamma_{03} \quad \quad \text{(B.5)} \]
and

\[ S \partial_\tau S^{-1} = \frac{k^2}{2} \dot{\rho} \rho' (\Gamma_0 - \Gamma_1) - \frac{k^2}{2} \sinh2\rho \dot{\rho} \dot{\rho}' (\Gamma_0 + \Gamma_2) + \left( 2 \cosh2\rho \dot{\rho} \dot{\rho}' + \left( \dot{\rho}' - \dot{\rho}' \dot{\rho}' \right) \sinh2\rho \right) \Gamma_{13} \]

\[ + \frac{k}{2} \left( \dot{\rho} \dot{\rho}' + \sinh2\rho \dot{\rho} \dot{\rho}' + \sinh4\rho \dot{\rho} \dot{\rho}'^2 \right) \Gamma_{02} \]

\[ S \partial_\sigma S^{-1} = \frac{1}{4k} \rho' (\rho' \Gamma_0 - \sinh2\rho \dot{\rho} \dot{\rho}' \Gamma_0) \]

\[ - \frac{k^2}{e^2} \left( \dot{\rho} \rho' - \dot{\rho} \rho' \right) \sinh2\rho \left( \Gamma_{02} + \sinh2\rho \frac{\dot{\rho}'}{\rho} \Gamma_{13} \right) \]

\[ - \left( \cosh2\rho \dot{\rho}' + \frac{1}{2} \sinh2\rho \frac{\dot{\rho}'}{\rho} \right) \Gamma_{13} \]

\[ + \frac{k}{2} \left( \rho'^2 + 2 \cosh2\rho \dot{\rho} \rho'^2 + \sinh2\rho \left( \dot{\rho}' - \dot{\rho}' \dot{\rho}' \right) \right) (\Gamma_0 - \Gamma_1) \]

\[ - \frac{k}{2} \left( \dot{\rho}' \dot{\rho}' + \sinh2\rho \dot{\rho} \dot{\rho}' \left( \rho' + 2 \cosh2\rho \dot{\rho} \rho' + \sinh2\rho \dot{\rho} \right) \right) \]

\[ \times (\Gamma_{03} + \Gamma_{23}) \]

(B.6)

Rewriting the action in terms of

\[ \vartheta \equiv \sqrt{\frac{k}{2e}} S^{-1} \Psi, \quad \bar{\vartheta} = \sqrt{\frac{k}{2e}} \bar{\Psi} S, \]

one finds

\[ S_F = i \frac{R^2 \kappa}{4 \pi \alpha'} \int d\sigma d\tau \bar{\Psi} \left[ \Gamma_0 \left( \partial_\tau + SA_\tau S^{-1} + S \partial_\tau S^{-1} + \sqrt{e} \partial_\tau \frac{1}{\sqrt{e}} \right) \right. \]

\[ + \Gamma_3 \left( \partial_\sigma + SA_\sigma S^{-1} + S \partial_\sigma S^{-1} + \sqrt{e} \partial_\sigma \frac{1}{\sqrt{e}} \right) + k \left( 1 + \frac{e^2}{2 \kappa^2} \right) \Gamma_1 \bar{\Pi} \left. \right] \Psi \]

\[ = i \frac{R^2 \kappa}{4 \pi \alpha'} \int d\sigma dt \bar{\Psi} \left[ \Gamma_0 \left( \partial_t + \left( C_t + \frac{1}{2} \rho_4 + \dot{\rho} + \frac{1}{2} \right) \Gamma_{0123} \right) \right. \]

\[ + \frac{1}{k} \Gamma_3 \left( \partial_\sigma + \left( C_\sigma + \frac{1}{2} \rho_4 + \dot{\rho}' + X \right) \Gamma_{0123} + X \Gamma_{13} \right) + \left( 1 + \frac{e^2}{2 \kappa^2} \right) i \Gamma_1 \left. \right] \Psi \]

with

\[ X = k^2 \dot{\rho} \rho' - 2 \cosh2\rho \dot{\rho}' - \sinh2\rho \frac{\dot{\rho}'}{\rho} = 0 + O \left( \frac{1}{\kappa^2} \right). \]

The left hand side here is proportional to the equation of motion for \( \phi \) that should be satisfied to order \( \frac{1}{\kappa^2} \) in the semiclassical limit \( \hbar = \frac{1}{\kappa} \to 0 \).

In order to get rid of full derivatives in the connections and obtain (4.10), one can finally make the following change in the spinors:

\[ \Psi \longrightarrow e^{-\frac{1}{2}(t+p_4+2\phi)\Gamma_{0123}} \Psi. \]
References

[1] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2 (1998) 231–252 [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. B428 (1998) 105–114 [hep-th/9802109].

[3] E. Witten, *Anti-de sitter space and holography*, Adv. Theor. Math. Phys. 2 (1998) 253–291 [hep-th/9802150].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rept. 323 (2000) 183–386 [hep-th/9905111].

[5] M. Bianchi, J. F. Morales and H. Samtleben, *On stringy AdS$_5 \times S^5$ and higher spin holography*, [hep-th/0305052].

[6] N. Beisert, M. Bianchi, J. F. Morales and H. Samtleben, *On the spectrum of AdS/CFT beyond supergravity*, JHEP 02 (2004) 001 [hep-th/0310292].

[7] N. Beisert, M. Bianchi, J. F. Morales and H. Samtleben, *Higher spin symmetry and $\mathcal{N} = 4$ SYM*, JHEP 07 (2004) 058 [hep-th/0405057].

[8] J. F. Morales and H. Samtleben, *Higher spin holography for sym in d dimensions*, Phys. Lett. B607 (2005) 286–293 [hep-th/0411246].

[9] D. Berenstein, J. M. Maldacena and H. Nastase, *Strings in flat space and pp waves from N=4 super Yang Mills*, JHEP 04 (2002) 013 [hep-th/0202021].

[10] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, JHEP 01 (2002) 047 [hep-th/0110242].

[11] R. R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, Nucl. Phys. B625 (2002) 70–96 [hep-th/0112044].

[12] R. R. Metsaev and A. A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond–Ramond background*, Phys. Rev. D65 (2002) 126004 [hep-th/0202109].

[13] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, Nucl. Phys. B636 (2002) 99–114 [hep-th/0204051].

[14] S. Frolov and A. A. Tseytlin, *Semiclassical quantization of rotating superstring in AdS$_5 \times S^5$*, JHEP 06 (2002) 007 [hep-th/0204226].

[15] A. A. Tseytlin, *On semiclassical approximation and spinning string vertex operators in AdS$_5 \times S^5$*, Nucl. Phys. B664 (2003) 247–275 [hep-th/0304139].
[16] S. Frolov and A. A. Tseytlin, Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors, Phys. Lett. B570 (2003) 96–104 [hep-th/0306143].

[17] S. Frolov and A. A. Tseytlin, Multi-spin string solutions in AdS5 × S5, Nucl. Phys. B668 (2003) 77–110 [hep-th/0304255].

[18] S. Frolov and A. A. Tseytlin, Quantizing three-spin string solution in AdS5 × S5, JHEP 07 (2003) 016 [hep-th/0306130].

[19] S. Frolov, I. Y. Park and A. A. Tseytlin, On one-loop correction to energy of spinning strings in S(5), Phys. Rev. D71 (2005) 026006 [hep-th/0408187].

[20] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, Spinning strings in AdS5 × S5 and integrable systems, Nucl. Phys. B671 (2003) 3–50 [hep-th/0307191].

[21] N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, Precision spectroscopy of AdS/CFT, JHEP 10 (2003) 037 [hep-th/0308117].

[22] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, Stringing spins and spinning strings, JHEP 09 (2003) 010 [hep-th/0306139].

[23] J. A. Minahan and K. Zarembo, The Bethe-ansatz for N = 4 super Yang–Mills, JHEP 03 (2003) 013 [hep-th/0212208].

[24] N. Beisert and M. Staudacher, The N = 4 SYM integrable super spin chain, [hep-th/0307042].

[25] N. Beisert, The complete one-loop dilatation operator of N = 4 super yang-mills theory, [hep-th/0307015].

[26] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, BMN gauge theory as a quantum mechanical system, Phys. Lett. B558 (2003) 229–237 [hep-th/0212269].

[27] S. Bellucci, P. Y. Casteill, J. F. Morales and C. Sochichiu, Spin bit models from non-planar N = 4 SYM, Nucl. Phys. B699 (2004) 151–173 [hep-th/0404066].

[28] S. Bellucci, P. Y. Casteill, J. F. Morales and C. Sochichiu, Chaining spins from (super)Yang–Mills, [hep-th/0408102].

[29] S. Bellucci and C. Sochichiu, On matrix models for anomalous dimensions of super Yang-Mills theory, [hep-th/0410010].

[30] S. Bellucci, P. Y. Casteill, A. Marrani and C. Sochichiu, Spin bits at two loops, Phys. Lett. B607 (2005) 180–187 [hep-th/0411261].

[31] M. Kruczenski, Spin chains and string theory, [hep-th/0311203].

[32] M. Kruczenski and A. A. Tseytlin, Semiclassical relativistic strings in S5 and long coherent operators in N = 4 SYM theory, [hep-th/0406189].

[33] R. Hernandez and E. Lopez, The su(3) spin chain sigma model and string theory, JHEP 04 (2004) 052 [hep-th/0403139].
[34] C. Kristjansen and T. Mansson, The circular, elliptic three-spin string from the $su(3)$ spin chain, Phys. Lett. B596 (2004) 265–276 [hep-th/0406176].

[35] M. Kruczenski, A. V. Ryzhov and A. A. Tseytlin, Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains, Nucl. Phys. B692 (2004) 3–49 [hep-th/0403120].

[36] J. Stefanski, B. and A. A. Tseytlin, Large spin limits of $AdS/CFT$ and generalized landau- lifshitz equations, JHEP 05 (2004) 042 [hep-th/0404133].

[37] S. Bellucci, P. Y. Casteill, J. F. Morales and C. Sochichiu, $SL(2)$ spin chain and spinning strings on $AdS_5 \times S^5$, Nucl. Phys. B707 (2005) 303–320 [hep-th/0409086].

[38] S. Ryang, Circular and folded multi-spin strings in spin chain sigma models, JHEP 10 (2004) 059 [hep-th/0409217].

[39] R. Hernandez and E. Lopez, Spin chain sigma models with fermions, JHEP 11 (2004) 079 [hep-th/0410022].

[40] V. A. Kazakov and K. Zarembo, Classical / quantum integrability in non-compact sector of $AdS/CFT$, JHEP 10 (2004) 060 [hep-th/0410105].

[41] M. Kruczenski, Spiky strings and single trace operators in gauge theories, [hep-th/0410226].

[42] G. Arutyunov and S. Frolov, Integrable hamiltonian for classical strings on $AdS_5 \times S^5$, JHEP 02 (2005) 059 [hep-th/0411089].

[43] I. Y. Park, A. Tirziu and A. A. Tseytlin, Spinning strings in $AdS_5 \times S^5$: One-loop correction to energy in $sl(2)$ sector, [hep-th/0501203].

[44] A. Khan and A. L. Larsen, Improved stability for pulsating multi-spin string solitons, [hep-th/0502063].

[45] D. Berenstein, D. H. Correa and S. E. Vazquez, Quantizing open spin chains with variable length: An example from giant gravitons, [hep-th/0502172].

[46] L. Freyhult and C. Kristjansen, Finite size corrections to three-spin string duals, [hep-th/0502122].

[47] N. Beisert, A. A. Tseytlin and K. Zarembo, Matching quantum strings to quantum spins: One-loop vs. finite-size corrections, [hep-th/0502173].

[48] R. Hernandez, E. Lopez, A. Perianez and G. Sierra, Finite size effects in ferromagnetic spin chains and quantum corrections to classical strings, [hep-th/0502188].

[49] R. R. Metsaev and A. A. Tseytlin, Type iib superstring action in $AdS_5 \times S^5$ background, Nucl. Phys. B533 (1998) 109–126 [hep-th/9805028].