DYNAMICS OF ASYMPTOTICALLY HOLOMORPHIC POLYNOMIAL-LIKE MAPS

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Abstract. The purpose of this paper is to initiate a theory concerning the dynamics of asymptotically holomorphic polynomial-like maps. Our maps arise naturally as deep renormalizations of asymptotically holomorphic extensions of $C^r$ ($r > 3$) unimodal maps that are infinitely renormalizable of bounded type. Here we prove a version of the Fatou-Julia-Sullivan theorem and a topological straightening theorem in this setting. In particular, these maps do not have wandering domains and their Julia sets are locally connected.

1. Introduction

Over the last decades many remarkable results were obtained for rational maps of the Riemann sphere, and somewhat surprisingly it turned out that quite a few of these have an analogue in the case of smooth interval maps. For example, the celebrated Julia-Fatou-Sullivan structure theorem for rational maps establishes the absence of wandering domains, showing that each component of the Fatou set is eventually periodic, and moreover gives a simple classification of the possible dynamics on a periodic component of the Fatou set, see [59]. For smooth interval maps analogous results were obtained, starting with Denjoy’s results for $C^2$ circle diffeomorphisms dating back to 1932. We now know that $C^2$ interval or circle maps cannot have wandering intervals provided all their critical points are non-flat, proved in increasing generality in [26, 39, 7, 48, 45, 49, 58]. Interestingly, although the statements for the Julia-Fatou-Sullivan structure theorem for rational maps and the generalised Denjoy theorems for interval and circle maps are analogous, the proofs use entirely different ideas. In the former case, they rely on the Measurable Riemann Mapping Theorem (MRMT) while in the latter case the proofs rely on real bounds coming from $C^2$ distortion estimates together with arguments relating to the order structure of the real line.

However, overall, not only the results but also the techniques used in the fields of holomorphic dynamics and interval dynamics have become increasingly intertwined over the last decades. Indeed, within the literature of real one-dimensional dynamics a growing number of results are obtained under the additional assumption that the maps are real analytic rather than smooth. The reason for this is that a real analytic map (obviously) has

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a complex extension to a small neighbourhood in $\mathbb{C}$ of the dynamical interval, and therefore many tools from complex analysis can be applied to such a real map. For instance, many results in the theory of renormalization of interval maps are either not known in the smooth category, or were only obtained with a significant amount of additional effort. Specifically, the Feigenbaum-Coullet-Tresser conjectures were first obtained using computer supported proofs, e.g. [36] and later using conceptual proofs for real analytic unimodal interval maps in [60, 47, 42, 5], for real analytic circle homeomorphisms with critical points in [14, 15, 61, 32], and for certain multimodal maps in [53, 54, 55, 56]. All these later results heavily use complex analytic machinery, and in particular rely on the complex analytic extensions of interval maps.

Within the literature on holomorphic dynamics one sees a similar development: many conjectures about iterations of general polynomials are only solved in the context of polynomials with real coefficients. An example of such a conjecture is density of hyperbolicity which is unsolved in the general case but was proved for real quadratic maps independently by Lyubich and Graczyk - Swiatek and in the general case by Kozlovski, Shen and van Strien, see [24, 39, 34, 35]. These results heavily rely on the existence of so-called real and complex bounds, [38, 23, 43, 52, 11] but such complex bounds do not hold for general non-real polynomials or rational maps. Indeed they hold for non-renormalizable polynomial maps [62, 27, 33] but in general not for non-real infinitely renormalizable quadratic maps, see for example [51, 57].

Of course there are plenty of results on renormalization and towards density of hyperbolicity in the setting of non-real polynomials [41, 29, 30, 31, 28, 9] and similarly there are plenty impressive results on interval maps which do not use complex tools, on for example invariant measures, thermodynamic formalism and stochastic stability. Nevertheless it is fair to say that a growing number of results within the field of real one-dimensional dynamics crucially rely on complex analytic tools, and vice versa many results about polynomial maps are only known when these preserve the real line.

When studying real one-dimensional maps, it is unnatural to restrict attention to maps which are real analytic. Indeed, in certain cases renormalization results for real analytic interval maps can be extended to $C^3$ or $C^4$ maps. This was done using a functional analytic approach in [17] for unimodal interval maps and heavily exploiting what is known for real analytic circle homeomorphisms in [21]. A purely real approach which gives existence of periodic points of the renormalization operator for unimodal maps of the form $g(|x|^{\ell})$, $\ell > 1$, was obtained by Martens [44].

The purpose of this paper is to initiate a theory for $C^{3+}$ interval maps showing that these have extensions to the complex plane with properties analogous to those of real polynomial maps. Thus the eventual aim of this theory is to show that $C^{3+}$ maps can be treated with techniques which are very similar to the complex analytic techniques which were so fruitful in the case of polynomial and real-analytic maps.

In this paper we will establish the first cornerstone of this theory by showing that one has a Julia-Fatou-Sullivan type description for such maps in a very important situation, namely for infinitely renormalizable maps of bounded type.

Let us be more precise and consider a $C^r$ map $f : I \to \mathbb{R}$. Such a map $f$ has an extension to a $C^r$ map $F : \mathbb{C} \to \mathbb{C}$ which is asymptotically holomorphic of order $r$, i.e., $\frac{\partial}{\partial \bar{z}} F(z) = 0$ when $\text{Im } z = 0$ and $\frac{\partial}{\partial \bar{z}} F(z) = O(|\text{Im } z|^{r-1})$ uniformly, see [25]. The notion of
asymptotically holomorphic maps goes back at least to [8]. In dynamics this notion was used in [40], [60], [11], [21], [4], [10] (see also [19], [20] for related material on the more restrictive notion of uniformly asymptotically conformal (UAC) map). Note that $F$ is not conformal outside the real line, and so in principle periodic points can be of saddle type. Even if a periodic point is repelling, in general the linearization at such a point will not be conformal. It follows that $F$ cannot be quasiconformally conjugate to a polynomial-like map (the pullbacks of a small circle in a small neighbourhood of a non-conformal repelling point become badly distorted, but this is not the case in a small neighbourhood of a conformal repelling point). For this reason, the absence of wandering domains for $F$ cannot be obtained via Sullivan’s Nonwandering Domains Theorem [60].

**Main Theorem.** Let $f \in C^{3+\alpha}$ ($\alpha > 0$) be a unimodal, infinitely renormalizable interval map of bounded type whose critical point has criticality given by an even integer $d$. Then every $C^{3+\alpha}$ extension $F$ of $f$ to a map defined on a neighborhood of the interval in the complex plane is such that there exist a sequence of domains $U_n \subset V_n \subset \mathbb{C}$ containing the critical point of $f$ and iterates $q^n$ with the following properties.

1. The map $G := F^{q^n} : U_n \rightarrow V_n$ is a degree $d$, quasi-regular polynomial-like map.
2. For large enough $n$, each periodic point in the filled Julia set $K_G := \{ z \in U_n; G^i(z) \in U_n \ \forall \ i \geq 0 \}$ is repelling.
3. The Julia $J_G := \partial K_G$ and filled-in Julia set of $G$ coincide, i.e., $J_G = K_G$.
4. The map $G$ is topologically conjugate to a polynomial mapping in a neighbourhood of its Julia set. In particular, $G$ has no wandering domains.
5. The Julia set $J_G$ is locally connected.

A more precise statement of this theorem can be found in Corollary 6.8 where we use the notion of controlled AHPL-maps, see Definition 5.1. We expect a similar result to hold in much greater generality, for example for general $C^{3+\alpha}$ asymptotically holomorphic interval maps with finitely many critical points of integer order.

Our plan is to build on the results in this paper to prove absence of invariant line fields for asymptotically holomorphic maps extending the methods of [47]. In addition, rather than using functional analytic tools as in [17], we plan to prove renormalization results for $C^r$ maps through the McMullen tower construction directly following the ideas in [47], or more ambitiously following the approach of Avila-Lyubich [5]. Thus our ultimate goal is to establish a closer analogy between real and complex one-dimensional dynamics along the lines suggested in the table below.

| setting         | real polynomials on the complex plane | $C^4$ asympt. hol. maps |
|-----------------|--------------------------------------|-------------------------|
| analogy         | Julia-Fatou-Sullivan Theory           | Yes (this paper)        |
|                 | McMullen tower construction           | ?                       |
|                 | Schwarz contraction                   | ?                       |
|                 | Hyperbolicity of Renormalization      | ?                       |
|                 | Deformation theory (through MRMT)     | ?                       |

1.1. **Object of study.** We shall study the dynamics of certain quasi-regular maps in the complex plane that are generalizations of standard (holomorphic) polynomial-like maps, as defined by Douady-Hubbard in [12]. Such generalized polynomial-like maps arise as deep renormalizations of unimodal interval maps that admit an asymptotically holomorphic...
extension to a complex neighborhood of their real domain. Let \( \varphi : U \to V \) be a \( C^1 \) map between two domains in the complex plane, and assume that \( U \cap \mathbb{R} \neq \emptyset \). We say that \( \varphi \) is \textit{asymptotically holomorphic of order} \( r > 1 \) if \( \varphi \) is quasi-regular and its complex dilatation \( \mu_{\varphi} \) satisfies \( |\mu_{\varphi}(z)| \leq C|\text{Im} z|^{r-1} \) for all \( z \in U \) and some constant \( C > 0 \) (in particular, \( \mu_{\varphi} \) vanishes on the real axis, i.e., \( \varphi \) is conformal there). As mentioned above, every \( C^r \) map of the real line admits an extension to a neighborhood of the real axis which is asymptotically holomorphic of order \( r \). (The notion of asymptotically holomorphic maps can even be defined for maps which are merely quasiconformal on \( \mathbb{C} \). It can be shown that if such a map is asymptotically holomorphic of order \( r \) then its restriction to the real line is actually \( C^r \), see \cite{24}.)

We may now formally define the class of dynamical systems we intend to study. Please note that in what follows we only consider maps having a unique critical point of finite even order \( d \geq 2 \).

**Definition 1.1.** Let \( U, V \subset \mathbb{C} \) be Jordan domains symmetric about the real axis, and suppose \( U \) is compactly contained in \( V \). A \( C^r \) (\( r \geq 3 \)) map \( f : U \to V \) is said to be an asymptotically holomorphic polynomial-like map, or \textit{AHPL-map} for short, if

(i) \( f \) is a degree \( d \geq 2 \) proper branched covering map of \( U \) onto \( V \), branched at a unique critical point \( c \in U \cap \mathbb{R} \) of criticality given by \( d \);

(ii) \( f \) is symmetric about the real axis, i.e., \( f(\overline{z}) = \overline{f(z)} \) for all \( z \in U \);

(iii) \( f \) is asymptotically holomorphic of order \( r \).

It follows from the well-known \textit{Stoilow Factorization Theorem} (see \cite{3}, Cor. 5.5.3) that an AHPL-map \( f \) as above can be written as \( f = \phi \circ g \), where \( g : U \to V \) is a (holomorphic) polynomial-like map and \( \phi : V \to V \) is a \( C^r \) quasiconformal diffeomorphism which is also asymptotically holomorphic of order \( r \).

Just as in the case of standard polynomial-like maps, we define the \textit{filled-in Julia set} of an AHPL-map \( f : U \to V \) to be the closure of the set of points which never escape under iteration, namely

\[
K_f = \bigcap_{n \geq 0} f^{-n}(V) = \bigcap_{n \geq 0} f^{-n}(\overline{U}) .
\]

This is a compact, totally \( f \)-invariant subset of \( U \). Its boundary \( J_f = \partial K_f \) is called the \textit{Julia set} of \( f \). By simple analogy with the case of holomorphic polynomial-like maps, there are natural questions to be asked about AHPL-maps and their Julia sets, to wit:

1. Are the (expanding) periodic points dense in \( J_f \)?
2. When is \( J_f \) locally connected?
3. What is the classification of stable components of \( K_f \setminus J_f \)?
4. Can \( f \) have non-wandering domains?
5. Is there a (topological) straightening theorem for AHPL-maps?

These questions do not have obvious answers. For instance, in the holomorphic case, the first question has an affirmative answer whose proof is easy thanks to Montel’s theorem – a tool which is not useful here. Likewise, in the holomorphic case question (4) has a negative answer thanks to Sullivan’s non-wandering domains theorem, whose proof uses quasiconformal deformations of \( f \) in a way that is not immediately available here, because in general the iterates of an AHPL-map are not uniformly quasiconformal.
Rather than studying very general AHPL-maps, in this paper we will restrict our attention to those which can be renormalized, in fact infinitely many times. The definition of renormalization in the present context is the same as the one for polynomial-like mappings: an AHPL-map \( f \) is renormalizable if there exists a topological disk \( D \) containing the critical point of \( f \) and an integer \( p > 1 \) so that \( D \) is compactly contained in \( f^p(D) \) and \( f^p : D \to f^p(D) \) is again an AHPL-map. Thanks to a theorem proved in [11], every sufficiently deep renormalization of an asymptotically holomorphic map whose restriction to the real line is an infinitely renormalizable map (in the usual real sense) is an (infinitely renormalizable) AHPL-map with a priori bounds.

One of our goals in the present paper is to provide answers to (some of) the above questions under the assumption that the AHPL-map \( f \) is infinitely renormalizable of bounded type. Another goal will be to prove \( C^2 \) a priori bounds for the renormalizations of such an \( f \), under the same bounded type assumption.

1.2. Summary. Here is a brief description of the contents of this paper. We start by revisiting the real bounds for \( C^3 \) unimodal maps in §2. In §3 we prove that the successive renormalizations of a \( C^3 \) infinitely renormalizable AHPL-map of bounded type are uniformly bounded in the \( C^2 \) topology, and that such bounds are beau in the sense of Sullivan. In proving these bounds, we employ as a tool the matrix form of the chain rule for the second derivative of a composition of maps. This tool does not seem to have been used at all in the literature on low-dimensional dynamics. The key ingredient that allows us to prove our Main Theorem is a result that, roughly speaking, states that (a deep renormalization of) an AHPL-map is an infinitesimal expansion of the hyperbolic metric on its co-domain minus the real axis. This is the main result in §5.1, namely Theorem 5.4.

In §4 we introduce techniques which are crucial in establishing Theorem 5.4, namely Proposition 4.14 and Theorem 4.15. Specifically, we give a bound for the hyperbolic Jacobian of a \( C^2 \) quasiconformal map in terms of its local quasiconformal distortion in two situations: for maps with small dilatation and for maps which are asymptotically holomorphic. These bounds are applied to the diffeomorphic part of our AHPL-map, which therefore needs to be at least \( C^2 \) with good bounds. This is the main reason why we need the \( C^2 \) bounds developed in §3. This infinitesimal expansion of the hyperbolic metric has several consequences, e.g., the fact that every periodic point of (a sufficiently deep renormalization of) an AHPL-map is expanding – once again, see Theorem 5.4.

Finally, in §6 we go further and construct puzzle pieces for such AHPL-maps, and show with the help of Theorem 5.4 that the puzzle pieces containing any given point of the Julia set of an infinitely renormalizable AHPL-map shrink around that point. This implies that the Julia set of such a map is always locally connected. Even more, as a consequence, such a map is in fact topologically conjugate to an actual (holomorphic) polynomial-like map and therefore does not have wandering domains.

2. Revisiting the real bounds

In this section we will recall some basic facts about renormalization of real unimodal maps.

2.1. Renormalization of unimodal maps. We need to recall some definitions and a few facts concerning the renormalization theory of interval maps. Let us consider a \( C^3 \)
unimodal map $f : I \to I$ defined on the interval $I = [-1, 1] \subset \mathbb{R}$, with its unique critical point at 0 and corresponding critical value at 1, i.e., with $f'(0) = 0$ and $f(0) = 1$. From the viewpoint of renormalization, to be defined below, there is no loss of generality in assuming that $f$ is even, i.e., that $f(-x) = f(x)$ for all $x \in I$. We also assume that the critical point of $f$ has finite even order $d \geq 2$. Hence we oftentimes refer to $f$ as a $d$-unimodal map.

We say that such an $f$ is renormalizable if there exist an integer $p = p(f) > 1$ and $\lambda = \lambda(f) = f^p(0)$ such that $f^p([-|\lambda|, |\lambda|]$ is unimodal and maps $[-|\lambda|, |\lambda|]$ into itself. Taking $p$ the smallest possible, we define the first renormalization of $f$ to be the map $Rf : I \to I$ given by

$$Rf(x) = \frac{1}{\lambda} f^p(\lambda x) .$$

The intervals $\Delta_j = f^j([-|\lambda|, |\lambda|])$, for $0 \leq j \leq p - 1$, have pairwise disjoint interiors, and their relative order inside $I_0$ determines a unimodal permutation $\theta$ of $\{0, 1, \ldots, p - 1\}$. Thus, renormalization consists of a first return map to a small neighbourhood of the critical point rescaled to unit size via a linear rescale.

It makes sense to ask whether $Rf$ is also renormalizable, since $Rf$ is certainly a normalized unimodal map. If the answer is yes then one can define $R^2 f = R(Rf)$, and so on. In particular, it may be the case that the unimodal map $f$ is infinitely renormalizable, in the sense that the entire sequence of renormalizations $f, Rf, R^2 f, \ldots$ is well-defined.

We assume from now on that $f$ is infinitely renormalizable. Let us denote by $P(f) \subseteq I$ the closure of the forward orbit of the critical point under $f$ (the post-critical set of $f$). The set $P(f)$ is a Cantor set with zero Lebesgue measure, see below. It can be shown also that $P(f)$ is the global attractor of $f$ both from the topological and metric points of view.

Note that for each $n \geq 0$, we can write

$$R^n f(x) = \frac{1}{\lambda_n} f^{q_n}(\lambda_n x) ,$$

where $q_0 = 1$, $\lambda_0 = 1$, $q_n = \prod_{i=0}^{n-1} p(R^i f)$ and $\lambda_n = \prod_{i=0}^{n-1} \lambda(R^i f) = f^{q_n}(0)$. The positive integers $a_i = p(R^i f) \geq 2$ are called the renormalization periods of $f$, and the $q_n$’s are the closest return times of the orbit of the critical point. Note that $q_{n+1} = a_n q_n = \prod_{i=0}^{n} a_i \geq 2^{n+1}$; in particular, the sequence $q_n$ goes to infinity at least exponentially fast.

It will be important to consider the renormalization intervals of $f$ at level $n$, namely $\Delta_{0,n} = [-|\lambda_n|, |\lambda_n|] \subset I_0$, and $\Delta_{i,n} = f^i(\Delta_{0,n})$ for $i = 0, 1, \ldots, q_n - 1$. The collection $\mathcal{C}_n = \{\Delta_{0,n}, \ldots, \Delta_{q_n-1,n}\}$ consists of pairwise disjoint intervals. Moreover, $\bigcup \{\Delta : \Delta \in \mathcal{C}_n\} \subseteq \bigcup \{\Delta : \Delta \in \mathcal{C}_{n+1}\}$ for all $n \geq 0$ and we have

$$P(f) = \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{q_n-1} \Delta_{i,n} .$$

Once we know that $\max_{0 \leq i \leq q_{n-1}} |\Delta_{i,n}| \to 0$ as $n \to \infty$, it follows that $P(f)$ is, indeed, a Cantor set. This (and much more) follows from the so-called real a priori bounds proved by Sullivan in [60]. The following form of the real bounds is not the most general, but it will be quite sufficient for our purposes. We say that an infinitely renormalizable map $f$ as above has combinatorial type bounded by $N$ if its renormalization periods are bounded by $N$, i.e., $a_n \leq N$ for all $n \in \mathbb{N}$.
Theorem 2.1 (Real Bounds). Let $f : I \to I$ be a $C^3$ unimodal map as above, and suppose that $f$ is infinitely renormalizable with combinatorial type bounded by $N > 1$. Then there exist constants $K_f > 0$ and $0 < \alpha_f < \beta_f < 1$ such that the following holds for all $n \in \mathbb{N}$.

(i) If $\Delta \in \mathcal{C}_{n+1}$, $\Delta^* \in \mathcal{C}_n$ and $\Delta \subset \Delta^*$, then $\alpha_f|\Delta^*| \leq |\Delta| \leq \beta_f|\Delta^*|.$

(ii) For all $1 \leq i < j \leq q_n - 1$ and each $x \in \Delta_{i,n}$, we have

$$\frac{1}{K_f} |\Delta_{j,n}| \leq |(f^{j-i})(x)| \leq K_f |\Delta_{j,n}|.$$  

(iii) We have $\|R^nf\|_{C^1(I)} \leq K_f$.

Moreover, there exist positive constants $K = K(N)$, $\alpha = \alpha(N)$, $\beta = \beta(N)$, with $0 < \alpha < \beta < 1$, and $n_0 = n_0(f) \in \mathbb{N}$ such that, for all $n \geq n_0$, the constants $K_f$, $\alpha_f$ and $\beta_f$ in (i), (ii) and (iii) above can be replaced by $K$, $\alpha$ and $\beta$, respectively.

For a complete proof of this theorem, see [49]. In informal terms, the theorem states three things. First, that the post-critical set $P(f)$ of an infinitely renormalizable $d$-unimodal map with bounded combinatorics is a Cantor set with bounded geometry. Second, that the successive renormalizations of such a map are uniformly bounded in the $C^1$ topology. Third, that the bounds on the geometry of the Cantor set and on the $C^1$ norms of the renormalizations become universal at sufficiently deep levels (such bounds are called beau by Sullivan in [60] – see also [49]).

Further analysis of the non-linearity of renormalizations yields the following consequence of the real bounds.

Corollary 2.2 ($C^2$ real bounds). Under the assumptions of Theorem 2.1, the successive renormalizations of $f$ are uniformly bounded in the $C^2$ topology, and the bound is beau in the sense of Sullivan.

The following consequence of the real bounds, namely Lemma 2.3 below, is adapted from [14, Lemma A.5, page 379], and also from [18, §2.1].

Let $f : I \to I$ be a $C^3$ unimodal map as defined above, and suppose $f$ is infinitely renormalizable with renormalization periods bounded by $N$. For each $n \geq 1$, let $\mathcal{C}_n = \{\Delta_{i,n} : 0 \leq i \leq q_n - 1\}$ denote the collection of renormalization intervals of $f$ at level $n$. For each $n \geq 1$, we define

$$S_n = \sum_{\mathcal{C}_n \ni \Delta \neq \Delta_{0,n}} \frac{|\Delta|}{d(c, \Delta)},$$

where $d(c, \Delta)$ denotes the Euclidean distance between $\Delta \subset I$ and the critical point $c = 0$. Roughly speaking, the result states that the for each infinitely renormalizable unimodal map of bounded type, the sequence $\{S_n\}_{n \geq 1}$ is bounded, and the bound is beau in the sense of Sullivan.

Lemma 2.3. There exists a constant $B_1 = B_1(N) > 0$ with the following property. For each infinitely renormalizable unimodal map $f$ of combinatorial type bounded by $N$, there exists $n_1 = n_1(f) \in \mathbb{N}$ such that, for all $n \geq n_1$, we have $S_n \leq B_1$.

Proof. The desired bound can be proved by a recursive estimate. Note that we can write

$$S_{n+1} = \sum_{\mathcal{C}_{n+1} \ni J \subset \Delta_{0,n} \setminus \Delta_{0,n+1}} \frac{|J|}{d(c, J)} + \sum_{\mathcal{C}_n \ni \Delta \neq \Delta_{0,n}} \left( \sum_{\mathcal{C}_{n+1} \ni J \subset \Delta} \frac{|J|}{d(c, J)} \right) \tag{2.2}$$
Now, since $d(c, J) > \frac{1}{2} |\Delta_{0,n+1}|$ for each $J \in C_{n+1}$, we certainly have
\[
\sum_{c_{n+1} \ni J \subset \Delta_{0,n} \setminus \Delta_{0,n+1}} \frac{|J|}{d(c, J)} \leq 2 \frac{|\Delta_{0,n}|}{|\Delta_{0,n+1}|}.
\] (2.3)

From the real bounds, Theorem 2.1 we know that there exists a constant $0 < \alpha = \alpha(N) < 1$ such that $|\Delta_{0,n}| \leq \alpha^{-1}|\Delta_{0,n+1}|$ for all sufficiently large $n$. For each $\Delta \in C_n$, let $J_1, J_2, \ldots, J_n \in C_{n+1}$ be all the intervals at level $n+1$ which are contained in $\Delta$. Then, again from the real bounds, we have $\sum_{i=1}^{\alpha n} |J_i| \leq \beta|\Delta|$, where $0 < \beta = \beta(N) < 1$, provided the renormalization level $n$ is sufficiently large. Moreover, $d(c, J_i) \geq d(c, \Delta)$ for all $i$. Hence we have, for all $n$ sufficiently large,
\[
\sum_{c_n \ni \Delta \not\ni \Delta_{0,n}} \left( \sum_{c_{n+1} \ni J \subset \Delta} \frac{|J|}{d(c, J)} \right) \leq \sum_{c_n \ni \Delta \not\ni \Delta_{0,n}} \left( \sum_{c_{n+1} \ni J \subset \Delta} \frac{|J|}{d(c, \Delta)} \right)
\leq \beta \sum_{c_n \ni \Delta \not\ni \Delta_{0,n}} \frac{|\Delta|}{d(c, \Delta)} = \beta S_n.
\] (2.4)

Putting (2.3) and (2.4) back into (2.2), we deduce that there exists $n_0 = n_0(f)$ such that $S_{n+1} \leq \beta S_n + \alpha^{-1}$ for all $n \geq n_0$. By induction, it follows that $S_{n+k} \leq \beta^k S_{n_0} + \alpha^{-1}(1 + \beta + \cdots + \beta^{k-1})$ for all $k \geq 0$. Since $\beta < 1$, this shows that the sequence $(S_n)_{n \geq 1}$ is bounded, and eventually universally so.

What we will need is in fact a consequence of this lemma. Given $f$ as in Lemma 2.3, write for all $n \geq 1$
\[
S^*_n = \sum_{i=1}^{\alpha n} |\Delta_{i,n}|^2 |d(c, \Delta_{i,n})|^{-d-2}
\] (2.5)

where $d$ is the order of $f$ at the critical point $c$.

**Lemma 2.4.** There exists a constant $B_2 = B_2(N) > 0$ with the following property. For each infinitely renormalizable unimodal map $f$ of combinatorial type bounded by $N$, there exists $n_2 = n_2(f) \in \mathbb{N}$ such that, for all $n \geq n_2$, we have $S^*_n \leq B_2$.

**Proof.** Since $f$ has a critical point of order $d$ at $c$, we have $|f'(x)| \geq C_0|x-c|^{d-1}$ for all $x \in I$, for some $C_0 = C_0(f) > 0$. Replacing, if necessary, $f$ by $R^k f$ for sufficiently large $k$, we can assume that $C_0$ depends in fact only on $N$. Now, for each $i$ we can write $|\Delta_{i+1,n}|/|\Delta_{i,n}| = |f'(x_{i,n})|$ for some $x_{i,n} \in \Delta_{i,n}$, by the mean-value theorem. Hence, using that $|x_{i,n} - c| \geq d(c, \Delta_{i,n})$, we have
\[
\frac{|\Delta_{i,n}|^2}{|\Delta_{i+1,n}|} [d(c, \Delta_{i,n})]^{-d-2} = \frac{|\Delta_{i,n}|}{|f'(x_{i,n})|} [d(c, \Delta_{i,n})]^{-d-2} \leq C_0^{-1} \frac{|\Delta_{i,n}|}{|x_{i,n} - c|} \leq C_0^{-1} \frac{|\Delta_{i,n}|}{d(c, \Delta_{i,n})}
\]
This shows that $S^*_n \leq C_0^{-1} S_n$ for all (sufficiently large) $n$, and the desired result follows from Lemma 2.3. \qed
3. The $C^2$ bounds for AHPL-maps

In this section we prove that the successive renormalizations of an infinitely renormalizable AHPL-map of bounded combinatorial type are uniformly bounded in the $C^2$ topology, and the bound are $beau$. Such bounds will be required when we study the diffeomorphic part of a AHPL-map.

The main result of this section can be stated more precisely as follows.

**Theorem 3.1.** Let $f : U \to V$ be an infinitely renormalizable, $C^2$, AHPL-map of combinatorial type bounded by $N \in \mathbb{N}$, and let $R^n(f) : U_n \to V_n$, $n \geq 1$, be the sequence of renormalizations of $f$. There exists a constant $C_f > 0$ such that $\|R^n(f)\|_{C^2(U_n)} \leq C_f$. Moreover, there exist $C = C(N) > 0$ and $m = m(f) \in \mathbb{N}$ such that $\|R^n(f)\|_{C^2(U_n)} \leq C$ for all $n \geq m$.

The proof will use the real bounds as formulated in [2.1, Lemma 2.4], as well as the complex bounds established in [11], in the form stated in [3.1] below. In fact, the complex bounds are essential even to make sure that the renormalizations $R^n f$ appearing in Theorem 3.1 are well-defined AHPL-maps (see Remark 3.3 below).

3.1. The complex bounds. We conform with the notation introduced earlier when dealing with infinitely renormalizable interval maps, and with AHPL-maps.

**Theorem 3.2 (Complex bounds).** Let $f : U \to V$ be an AHPL-map and suppose that $f|_I : I \to I$ is an infinitely renormalizable quadratic unimodal map with combinatorial type bounded by $N$. There exist $C = C(N) > 1$ and $n_3 = n_3(f) \in \mathbb{N}$ such that the following statements hold true for all $n \geq n_3$.

(i) For each $0 \leq i \leq q_n - 1$ there exist Jordan domains $U_{i,n}, V_{i,n}$, with piecewise smooth boundaries and symmetric about the real axis, such that $\Delta_{i,n} \subset U_{i,n} \subset V_{i,n}$, the $V_{i,n}$ are pairwise disjoint, and we have the sequence of surjections

$$U_{0,n} \xrightarrow{f} U_{1,n} \xrightarrow{f} \cdots \xrightarrow{f} U_{q_n-1,n} \xrightarrow{f} V_{0,n} \xrightarrow{f} V_{1,n} \xrightarrow{f} \cdots \xrightarrow{f} V_{q_n-1,n}.$$  

(ii) For each $0 \leq i \leq q_n - 1$, $f_{i,n} = f g_{i,n} : U_{i,n} \to V_{i,n}$ is a well-defined AHPL-map with critical point at $f^i(c)$.

(iii) We have $\text{mod}(V_{i,n} \setminus U_{i,n}) \geq C^{-1}$ and $\text{diam}(V_{i,n}) \leq C |\Delta_{i,n}|$, for all $0 \leq i \leq q_n - 1$.

(iv) The map $f_{i,n} : U_{i,n} \to V_{i,n}$ has a Stoiow decomposition $f_{i,n} = \phi_{i,n} \circ g_{i,n}$ such that $K(\phi_{i,n}) \leq 1 + C |\Delta_{0,n}|$, for each $0 \leq i \leq q_n - 1$.

This theorem is a straightforward consequence of (a special case of) the complex bounds proved in [11].

**Remark 3.3.** For each $n \geq 1$, consider the linear map $\Lambda_n(z) = |\Delta_{0,n}| z$, and consider the Jordan domains $U_n = \Lambda_n^{-1}(U_{0,n}) \subset \mathbb{C}$ and $V_n = \Lambda_n^{-1}(V_{0,n}) \subset \mathbb{C}$. Note that $I \subset U_n \subset V_n$. We define $R^n f : U_n \to V_n$ by $R^n f = \Lambda_n^{-1} \circ f_{0,n} \circ \Lambda_n$. This is the $n$-th renormalization of $f$ that appears in the statement of Theorem 3.1. Note that the complex bounds given by this theorem guarantee that $\text{diam}(V_n) \asymp |I|$; in particular, the $C^0$ norms $\|R^n f\|_{C^0(U_n)}$ are uniformly bounded (by a beau constant).
3.2. **Digression on the chain rule.** Let \( \phi : U \to \mathbb{R}^n \) be a \( C^2 \) map defined on an open set \( U \subset \mathbb{R}^n \). In matrix form, the second derivative \( D^2 \phi \) of \( \phi \) is a \( n \times n^2 \) matrix obtained by the juxtaposition of the Hessian matrices of each of the \( n \) scalar components of \( \phi \). For instance, in dimension \( n = 2 \), the second derivative of a map \( \phi = u + iv \) is given by the \( 2 \times 4 \) matrix \( D^2 \phi = \begin{bmatrix} u_{xx} & u_{xy} & v_{xx} & v_{xy} \\ u_{yx} & u_{yy} & v_{yx} & v_{yy} \end{bmatrix} \) obtained by adjoining the Hessian matrices of the two components of \( \phi \).

Now, if \( U, V, W \subset \mathbb{R}^n \) are open sets with \( V \subset W \), and if \( \psi : U \to V \) and \( \phi : W \to \mathbb{R}^n \) are both \( C^2 \), then the composition \( \phi \circ \psi \) is \( C^2 \), and

\[
D^2(\phi \circ \psi) = D^2 \phi \circ \psi \cdot D\psi \otimes D\psi + D\phi \circ \psi \cdot D^2 \psi .
\]

(3.1)

This is the chain rule for the second derivative of a composition in matrix form. Here, we denote by \( A \otimes B \) the tensor (or Kronecker) product of two square matrices \( A, B \) of the same size; thus, in our case \( D\psi \otimes D\psi \) is a square \( n^2 \times n^2 \) matrix. For a proof of this formula, see [46].

We will need in fact a formula for the second derivative of an (arbitrarily high) iterate of a given map. We formulate it as a lemma.

**Lemma 3.4.** Let \( \phi : U \to \mathbb{R}^n, U \subset \mathbb{R}^n \) open, be a \( C^2 \) map. Then for each \( k \geq 0 \) we have

\[
D^2 \phi^k = D^2 \phi \circ \phi^{k-1} \cdot (D\phi^{k-1}) \otimes \cdots \otimes (D\phi^{k-1}) + \sum_{j=1}^{k-1} D\phi^{k-j} \circ \phi^j \cdot D^2 \phi \circ \phi^{j-1} \cdot (D\phi^{j-1}) \otimes \cdots \otimes (D\phi^{j-1}) ,
\]

wherever the \( k \)-th iterate \( \phi^k \) is defined.

**Proof.** This easily established from (3.1) by induction (write \( \phi^{k+1} = \phi \circ \phi^k \) for the induction step).

Of course, in this paper we will only need these formulas in dimension \( n = 2 \).

3.3. **Proof of Theorem 3.1.** Here we prove our first main result, namely Theorem 3.1.

It is natural to divide the proof into two steps: in the first step we bound the \( C^1 \) norms of renormalizations, and in the second step we bound the \( C^2 \) norms. Throughout the proof, we shall successively denote by \( C_0, C_1, C_2, \ldots \) positive constants that are either absolute or depend only on the constants given by the real and complex bounds. Also, in the estimates to follow we use the operator norm on matrices; to wit, we define \( \|A\| = \sup_{|v|=1} |Av| \) (here, \( |v| \) denotes the euclidean norm of the vector \( v \)). This norm has the advantage of being sub-multiplicative, which is to say that \( \|AB\| \leq \|A\| \cdot \|B\| \) whenever the product \( AB \) is well-defined. It also satisfies \( \|A \otimes B\| \leq \|A\| \cdot \|B\| \).

**Bounding the \( C^1 \) norms.** First we prove that the successive renormalizations of \( f \) are uniformly bounded in the \( C^1 \) topology, with \textit{beau} bounds. We will prove a bit more than what is required. Let us fix \( n \in \mathbb{N} \) so large that the real and complex bounds given by Theorem 2.1 and Theorem 3.2 hold true for \( R^n f \). We divide our argument into a series of steps.

\[\text{1}^\text{We use the abbreviation } A^\otimes m = A \otimes A \otimes \cdots \otimes A \text{ (} m \text{ times).}\]
(i) Replacing \( f \) by a sufficiently high renormalization we may assume, using Corollary 2.2 that the \( C^2 \) norm of \( f|_I \) is bounded by a \emph{beau} constant (that depends only on \( N \)). In particular, there exists an open complex neighborhood \( \mathcal{O} \) of the dynamical interval \( I \subset \mathbb{R} \), with \( \mathcal{O} \cap U, \) such that \( \|f\|_{C^2(\mathcal{O})} \leq C_0 \). And, because the critical point \( c \) has order \( d \), we may also assume that \( \|Df(y)\| \leq C_0|y - c|^{d-1} \) and \( \|D^2f(y)\| \leq C_0|y - c|^{d-2} \) for all \( y \in \mathcal{O} \).

(ii) We may assume that \( n \) is so large that \( V_{i,n} \subset \mathcal{O} \) for all \( i \). This is possible because, by the complex bounds (Theorem 3.2), \( \text{diam}(V_{i,n}) \approx |\Delta_{i,n}| \), and therefore the \( V_{i,n} \) shrink exponentially fast as \( n \to \infty \), by the real bounds.

(iii) Let \( j, k \) be positive integers such that \( 1 \leq j < j + k \leq q_n \). Then for each \( x \in \Delta_{j,n} \) we have, by Theorem 2.2

\[
C_1^{-1} \frac{|\Delta_{j+k,n}|}{|\Delta_{j,n}|} \leq \|Df^k(x)\| = |(f^k)'(x)| \leq C_1 \frac{|\Delta_{j+k,n}|}{|\Delta_{j,n}|}.
\]

(iv) Given \( x \in \Delta_{j,n} \) and \( y \in U_{j,n} \), let us write \( x_i = f^i(x), y_i = f^i(y) \) for all \( i = 0, 1, \ldots, k \). By step (i), and since \( f \) has a critical point at \( c \) of order \( d \), we have

\[
\frac{\|Df(x_i) - Df(y_i)\|}{|d(c, \Delta_{i,j,n})|^{d-2}} \leq C_2 |x_i - y_i| \leq C_3 |\Delta_{i,j,n}|
\]

for \( i = 0, 1, \ldots, k - 1 \). From (3.3) we obviously have

\[
\|Df(y_i)\| \leq \|Df(x_i)\| + C_3 |\Delta_{i,j,n}| \cdot |d(c, \Delta_{i,j,n})|^{d-2},
\]

for \( i = 0, 1, \ldots, k - 1 \).

(v) By the chain rule for first derivatives, we have

\[
\|Df^k(y)\| \leq \prod_{i=0}^{k-1} \|Df(y_i)\| \leq \prod_{i=0}^{k-1} \|Df(y_i)\|.
\]

(vi) Using (3.4) and (3.5) we get

\[
\|Df^k(y)\| \leq \prod_{i=0}^{k-1} \left( \|Df(x_i)\| + C_3 |\Delta_{i,j,n}| \cdot |d(c, \Delta_{i,j,n})|^{d-2} \right)
\]

\[
\leq \prod_{i=0}^{k-1} \|Df(x_i)\| \cdot \prod_{i=0}^{k-1} \left( 1 + C_3 \frac{|\Delta_{i,j,n}|}{\|Df(x_i)\| |d(c, \Delta_{i,j,n})|^{d-2}} \right).
\]

(vii) But since \( x_i \) is real (and \( f \) preserves the real line), we have

\[
\prod_{i=0}^{k-1} \|Df(x_i)\| = \prod_{i=0}^{k-1} \|Df^k(x)\| = \|Df^k(x)\|.
\]

Moreover, for each \( i = 0, 1, \ldots, k \) we have

\[
\|Df(x_i)\| = |f'(x_i)| \times \frac{|\Delta_{i,j+1,n}|}{|\Delta_{i,j,n}|}.
\]

(viii) Putting (3.7) and (3.8) back into (3.6), we get

\[
\|Df^k(y)\| \leq \|Df^k(x)\| \cdot \prod_{i=0}^{k-1} \left( 1 + C_3 \frac{|\Delta_{i,j+1,n}|^2}{|\Delta_{i,j+1,n}| d(c, \Delta_{i,j+1,n})^{d-2}} \right).
\]
But now, using Lemma 2.4, we see that the product in the right-hand side of (3.9) is uniformly bounded, because

\[
\prod_{i=0}^{k-1} \left( 1 + C_4 \frac{\Delta_{i+j,n}^2}{\Delta_{i+j+1,n}} [d(c, \Delta_{i+j,n})]^{d-2} \right)
\]

\[
\leq \exp \left\{ C_4 \sum_{i=0}^{k-1} \frac{\Delta_{i+j,n}^2}{\Delta_{i+j+1,n}} [d(c, \Delta_{i+j,n})]^{d-2} \right\}
\]

\[
\leq \exp \left\{ C_4 \sum_{i=1}^{q_n-1} \frac{\Delta_{i,n}^2}{\Delta_{i+1,n}} [d(c, \Delta_{i,n})]^{d-2} \right\}
\]

\[
= \exp\{C_4 S^k\} \leq \exp\{B_2 C_4\} .
\] (3.10)

(ix) Hence we have proved that \( \|Df^k(y)\| \leq C_5 \|Df^k(x)\| \), for all \( y \in U_{j,n} \) and all \( x \in \Delta_{j,n} \). From (3.2), it follows that

\[
\|Df^k(y)\| \leq C_6 |\Delta_{j,k,n}|, \quad \text{for all} \quad y \in U_{j,n} .
\] (3.11)

In particular, taking \( j = 1 \) and \( k = q_n - 1 \), we see that the first derivative of the map \( f^{q_n-1}|_{U_{1,n}} : U_{1,n} \to V_{0,n} \) satisfies

\[
\|Df^{q_n-1}(y)\| \leq C_6 \frac{|\Delta_{0,n}|}{|\Delta_{j,n}|}, \quad \text{for all} \quad y \in U_{1,n} .
\] (3.12)

(x) On the other hand, since \( f \) has a critical point of order \( d \) at \( c = 0 \), the restriction \( f|_{U_{0,n}} : U_{0,n} \to U_{1,n} \) satisfies \( \|Df(y)\| \leq C_7 |y|^{d-1} \leq C_8 |\Delta_{0,n}|^{d-1} \) for all \( y \in U_{0,n} \) (we are implicitly using step (i) here). Combining this fact with step (ix), (3.12), and using the chain rule, we see that the first derivative of the map

\[
f_{0,n} = f^{q_n}|_{U_{0,n}} = f^{q_n-1}|_{U_{1,n}} \circ f|_{U_{0,n}} : U_{0,n} \to V_{0,n}
\]

satisfies

\[
\|Df^{q_n}(y)\| \leq C_9 \frac{|\Delta_{0,n}|^d}{|\Delta_{1,n}|}, \quad \text{for all} \quad y \in U_{0,n} .
\] (3.13)

But, again using that the critical point has order \( d \), we have \( |\Delta_{1,n}| \asymp |\Delta_{0,n}|^d \). Putting this information back in (3.13), we deduce that

\[
\|Df_0,n\|_{C^0(U_{0,n})} = \|Df^{q_n}\|_{C^0(U_{0,n})} \leq C_{10} .
\]

Therefore \( \|DR^n f\|_{C^0(U_{n})} \leq C_{10} \) also, since \( R^n f \) is a simply a linearly rescaled copy of \( f_{0,n} \). This shows that the successive renormalizations of \( f \) around the critical point are indeed uniformly bounded in the \( C^1 \) topology, and the bounds are beau.

\[\footnote{Recall that \( \Delta_{q_n,n} = \Delta_{0,n} \).}\]
Bounding the $C^2$ norms. We now move to the task of bounding the second derivatives of the renormalizations of $f$. Here we use the chain rule for the second derivative of a (long) composition, as given by Lemma 3.4. Once again, we break the proof into a series of (short) steps.

(xi) Since $R^n f = \Lambda_n^{-1} \circ f_{0,n} \circ \Lambda_n$, with $\Lambda_n(z) = |\Delta_{0,n}| z$, we have
\[
\|D^2 R^n f\|_{C^0(U_n)} \leq |\Delta_{0,n}| \cdot \|D^2 f_{0,n}\|_{C^0(U_{0,n})}.
\] (3.14)

We need to bound the norm on the right-hand side of (3.14).

(xii) Recall from step (x) the decomposition $f_{0,n} = f_{q-n}^{-1} |\varphi_{1,n} \circ f|_{\varphi_{0,n}}$. By the chain rule for second derivatives, for each $y \in U_{0,n}$ we have
\[
D^2 f_{0,n}(y) = D^2 f_{q-n}^{-1}(f(y)) Df(y) \otimes Df(y) + Df_{q-n}^{-1}(f(y)) D^2 f(y).
\] (3.15)

Note from step (i) that $\|D^2 f(y)\| \leq C_0 |y-c|^{d-2} \leq C_{11} |\Delta_{0,n}|^{d-2}$. Moreover, applying (3.12) with $y$ replaced by $f(y)$, we have
\[
\|D f_{q-n}^{-1}(f(y))\| \leq C_6 \frac{|\Delta_{0,n}|}{|\Delta_{1,n}|}.
\] (3.16)

These two estimates combined yield an upper bound for the matrix norm of the second summand in the right-hand side of (3.15), namely
\[
\|D f_{q-n}^{-1}(f(y))D^2 f(y)\| \leq C_{12} \frac{|\Delta_{0,n}|^{d-1}}{|\Delta_{1,n}|},
\] (3.17)

where $C_{12} = C_6 C_{11}$.

(xiii) It remains to bound the matrix norm of the first summand in the right-hand side of (3.15). Applying Lemma 3.4 with $\phi = f$ and $k = q_n - 1$ to any point $z \in U_{1,n}$, we have
\[
D^2 f_{q-n}^{-1}(z) = D^2 f(\delta_{q-n}^{-1}(z))\delta_{q-n}^{-1}(z)) \otimes Df(\delta_{q-n}^{-1}(z))\delta_{q-n}^{-1}(z)) \otimes
\]
\[
+ \sum_{j=1}^{q_n-2} Df_{q-n-j}^{-1}(f(z)) D^2 f(f^{-1}(z)) Df(f^{-1}(z)) = \delta_{q-n}^{-1}(z)) \otimes
\]
\[
,\]

Note that $\|D^2 f(\delta_{q-n}^{-1}(z))\| \leq C_0$, by step (i). Since $f^{-1}(z) \in U_{j,n} \subset O$, it also follows from step (i) that
\[
\|D^2 f(f^{-1}(z))\| \leq C_0 |f^{-1}(z) - c|^{d-2} \leq C_{13} [d(c, \Delta_{j,n})]^{d-2},
\]
for all $j \leq q_n$. Using this information in (3.18), we get
\[
\|D^2 f_{q-n}^{-1}(z)\| \leq C_0 \|D f_{q-n}^{-1}(z)\|^2
\]
\[
+ C_{13} \sum_{j=1}^{q_n-2} \|D f_{q-n-j}^{-1}(f(z))\| \|D f_{j-1}^{-1}(z)\|^2 [d(c, \Delta_{j,n})]^{d-2}.
\] (3.19)

(xiv) We now need to bound the norms on the right-hand side of (3.19). Using the estimate (3.11) given in step (ix), we have
\[
\|D f_{q-n}^{-1}(z)\| \leq C_6 \frac{|\Delta_{q-n,1,n}|}{|\Delta_{1,n}|},
\] (3.20)
as well as

$$
\|Df^{q_n-j-1}(f^j(z))\| \leq C_6 \frac{\|\Delta_{q_n-1,n}\|}{\|\Delta_{j+1,n}\|},
$$

(3.21)

and

$$
\|Df^j(z)\| \leq C_6 \frac{\|\Delta_{j,n}\|}{\|\Delta_{1,n}\|},
$$

(3.22)

for all $j \leq q_n - 1$. Putting (3.20), (3.21) and (3.22) back in (3.19), we get

$$
\|D^2f^{q_n-1}(z)\| \leq C_{14} \left[ \frac{\|\Delta_{q_n-1,n}\|^2}{\|\Delta_{1,n}\|^2} + \sum_{j=1}^{q_n-2} \frac{\|\Delta_{q_n-1,n}\| \|\Delta_{j,n}\|^2}{\|\Delta_{j+1,n}\| \|\Delta_{1,n}\|^2} [d(c, \Delta_{j,n})]^{d-2} \right].
$$

(3.23)

(xv) Now we note that $|\Delta_{q_n-1,n}| \approx |\Delta_{0,n}|$, by the real bounds. Using this information in (3.23), we deduce that

$$
\|D^2f^{q_n-1}(z)\| \leq C_{15} \frac{|\Delta_{0,n}|}{\|\Delta_{1,n}\|^2} \left[ |\Delta_{0,n}| + \sum_{j=1}^{q_n-2} \frac{|\Delta_{j,n}|^2}{\|\Delta_{j+1,n}\| \|\Delta_{1,n}\|^2} [d(c, \Delta_{j,n})]^{d-2} \right].
$$

(3.24)

Applying Lemma 2.4, we see that the sum inside square-brackets in the right-hand side of (3.24) is bounded (by a beau constant). Hence we have established that

$$
\|D^2f^{q_n-1}(z)\| \leq C_{16} \frac{|\Delta_{0,n}|}{\|\Delta_{1,n}\|^2}.
$$

(3.25)

(xvi) Carrying the estimates (3.17) and (3.25) back into (3.15), we deduce that

$$
\|D^2f_{0,n}(y)\| \leq C_{17} \left( \frac{|\Delta_{0,n}|^{2d-1}}{|\Delta_{1,n}|^2} + \frac{|\Delta_{0,n}|^{d-1}}{|\Delta_{1,n}|} \right).
$$

(3.26)

This inequality is established for all $y \in U_{0,n}$.

(xvii) Finally, combining (3.26) with (3.14), we get

$$
\|D^2R^n f\|_{C^0(U_n)} \leq C_{18} \left( \frac{|\Delta_{0,n}|^{2d}}{|\Delta_{1,n}|^2} + \frac{|\Delta_{0,n}|^d}{|\Delta_{1,n}|} \right).
$$

Using once again the fact that $|\Delta_{1,n}| \approx |\Delta_{0,n}|^d$, we deduce at last the inequality

$$
\|D^2R^n f\|_{C^0(U_n)} \leq C_{20}.
$$

Hence the successive renormalizations of $f$ are uniformly bounded in the $C^2$ topology, as claimed (and the bounds are beau).

This finishes the proof of Theorem 3.1.

**Remark 3.5.** If we consider the Stoilow decomposition $R^n f = \phi_n \circ g_n$ coming from Theorem 3.2(iv), where $g_n : U_n \to V_n$ is a $d$-to-1 holomorphic branched covering map, and $\phi_n : V_n \to V_n$ is an asymptotically holomorphic diffeomorphism, then it is possible to prove, using similar estimates, that $\|\phi_n\|_{C^2(V_n)}$, $\|\phi_n^{-1}\|_{C^2(V_n)}$ and $\|g_n\|_{C^2(U_n)}$ are uniformly bounded, and the bounds are beau.

---

3We have $|\Delta_{0,n}| = |f'(\xi)||\Delta_{q_n-1,n}|$ for some $\xi \in \Delta_{q_n-1,n}$, by the mean value theorem, so $|\Delta_{0,n}| \leq C_6|\Delta_{q_n-1,n}|$ (where $C_6$ is the constant of step (i)). An inequality in the opposite direction follows from the fact, due to Guckenheimer (and using [35] Theorem IV.B) if $f$ is not symmetric), that when $f|_I$ has negative Schwarzian derivative, the renormalization interval containing the critical point is the largest among all renormalization intervals at its level. Here we have not assumed the negative Schwarzian property for $f$, but it can be proved that $R^n f|_I$ has this property for all sufficiently large $n$. For details, see [17] p. 760].
This section is a conformal/quasiconformal intermezzo. Here we develop the distortion tools that will be used in the proof of Theorem 5.4 in §5. We believe that these tools – especially those concerning the control of infinitesimal distortion of hyperbolic metric by an asymptotically conformal diffeomorphism, see Proposition 4.14 (for self-maps of the disk) and Theorem 4.15 (for other domains) – are of independent interest, and may find applications in other topics of study, such as Riemann surface theory.

4.1. Comparison of hyperbolic metrics. We view any non-empty open set \( Y \subset \mathbb{C} \) whose complement has at least two points as a hyperbolic Riemann surface. As such, \( Y \) admits a conformal metric of constant negative curvature equal to \(-1\), the so-called hyperbolic or Poincaré metric of \( Y \). We denote by \( \rho_Y(z)dz \) this metric; \( \rho_Y(z) \) is the Poincaré density at \( z \in Y \). Integrating this metric along a given rectifiable path \( \gamma \subset Y \), we get its hyperbolic length \( \ell_Y(\gamma) \). This gives rise to a distance \( d_Y \) in the usual way: for any given pair of points \( z,w \in Y \), we set \( d_Y(z,w) = \inf_\gamma \ell_Y(\gamma) \), where \( \gamma \) ranges over all paths joining \( z \) to \( w \) (this will be equal to \( \infty \) if \( z \) and \( w \) lie in distinct components of \( Y \)). We call \( d_Y \) the hyperbolic distance of \( Y \). Accordingly, given \( E \subset Y \), we denote by \( \text{diam}_Y(E) \) the hyperbolic diameter of \( E \). We also use the following notation: if \( z \in Y \) and \( v \in T_zY \) is a tangent vector to \( Y \) at \( z \), then we write \( |v|_Y \) for the hyperbolic length of \( v \) (i.e., the length of \( v \) in the above infinitesimal conformal metric).

Thus, when \( Y \) is the upper or lower half-plane, we have \( \rho_Y(z) = |\text{Im } z|^{-1} \). When \( Y \) is the disk of center \( z_0 \in \mathbb{C} \) and radius \( R > 0 \), we have

\[
\rho_Y(z) = \frac{2R}{R^2 - |z - z_0|^2}.
\]

In the case of the unit disk, one can easily compute that

\[
d_{\mathbb{D}}(0,z) = \log \frac{1 + |z|}{1 - |z|}.
\]

This yields the following elementary estimate which will be used in §5.1 (see Remark 5.2).

**Lemma 4.1.** Let \( 0 \in E \subset \mathbb{D} \) and \( 0 < \delta \leq 1 \). If \( z \in \mathbb{D} \) is any point whose distance to the boundary of \( \mathbb{D} \) is at least \( \delta \), and if \( w \in E \), then

\[
d_{\mathbb{D}}(z,w) \leq \text{diam}_{\mathbb{D}}(E) + \log \frac{1}{\delta}.
\]

The well-known **Schwarz lemma** states that any holomorphic map \( \varphi : X \to Y \) between two hyperbolic Riemann surfaces weakly contracts the underlying hyperbolic metrics. In other words, \( |D\varphi(z)v|_Y \leq |v|_X \) for all \( z \in X \) and every tangent vector \( v \in T_zX \). If equality holds for some \( z \) even at a single non-zero vector \( v \in T_zX \), then \( \varphi \) is a local isometry between (a component of) \( X \) and (a component of) \( Y \). In particular, if \( X \) is connected and \( X \subset Y \) is a strict inclusion, and \( \varphi : X \to Y \) is the inclusion map, then \( \varphi \) is a strict contraction of the hyperbolic metrics. This leads, in the case when \( X \) is connected and \( X \subset Y \subset \mathbb{C} \), to the strict monotonicity of Poincaré densities: \( \rho_X(z) > \rho_Y(z) \) for all \( z \in X \). The following comparison of Poincaré densities follows from monotonicity and will prove useful later.
Lemma 4.2. Let $Y \subseteq C \setminus \mathbb{R}$ be an non-empty open set, and let $z, w \in Y$ be such that $\text{Re } z = \text{Re } w$ and $|\text{Im } z| \leq |\text{Im } w|$. If $z \in D(w, |\text{Im } w|) \subseteq Y$, then
\[
\frac{1}{|\text{Im } z|} \leq \rho_Y(z) \leq \frac{1}{|\text{Im } z|} \left(1 - \frac{1}{2} \frac{|\text{Im } z|}{|\text{Im } w|}\right)^{-1}.
\]

Proof. Look at the inclusions $D(w, |\text{Im } w|) \subseteq Y \subseteq C \setminus \mathbb{R}$ and use (4.1) with $z_0 = w$ and $R = |\text{Im } w|$. □

4.2. Expansion of hyperbolic metric. It so happens that contraction sometimes leads to expansion. If $\psi : X \to Y$ is a bi-holomorphic map between two hyperbolic Riemann surfaces and $X \subset Y$, then the inverse $\psi^{-1}$, viewed as a map from $Y$ into $X$, can be written as a composition of $\psi^{-1} : Y \to X$ with the inclusion $X \subset Y$. The first map in the composition is an isometry between the underlying hyperbolic metrics, whereas the second map is a contraction. Therefore $\psi$ expands the hyperbolic metric of $Y$. In the present paper, we shall need a more quantitative version of this fact. This is given by the following lemma due to McMullen (see [47]).

Lemma 4.3. Let $X, Y$ be hyperbolic Riemann surfaces with $X \subset Y$, and let $\psi : X \to Y$ be holomorphic univalent and onto. Then for all $x \in X$ and each tangent vector $v \in T_xX$ we have
\[
|D\psi(x)v|_Y \geq \Phi(s_{X,Y}(x))^{-1}|v|_X,
\]
where $s_{X,Y}(x) = d_Y(x, Y \setminus X)$ and $\Phi(\cdot)$ is the universal function given by
\[
\Phi(s) = \sinh(s) \log \left(\frac{1+e^{-s}}{1-e^{-s}}\right).
\]

We remark that $\Phi(s)$ is a continuous monotone increasing function with $\Phi(0) = 0$ and $\Phi(\infty) = 1$. Instead of (4.4), we shall need merely the estimate
\[
\Phi(s) < 1 - \frac{1}{3}e^{-2s}.
\]
This estimate is valid provided $s > \frac{1}{2} \log 2$, and is easily proved with the help of Taylor’s formula.

4.3. Non-linearity and conformal distortion. We will also need certain well-known results concerning the geometric distortion of holomorphic univalent maps. For details and some background, we recommend [16] §3.8.

Let $\varphi : V \to \mathbb{C}$ be a holomorphic univalent map defined on an open set $V \subset \mathbb{C}$. Then we have Koebe’s pointwise estimate on the non-linearity $\varphi''/\varphi'$; to wit, for every $z \in V$ we have
\[
\left|\frac{\varphi''(z)}{\varphi'(z)}\right| \leq \frac{4}{\text{dist}(z, \partial V)},
\]
where $\text{dist}(\cdot, \cdot)$ denotes euclidean distance. This form of pointwise control of the non-linearity of $\varphi$ has the following geometric consequence. Suppose $D \subset V$ is a compact

\[\text{In [17] McMullen gives } \Phi(s) = 2^{t \log t}, \text{ where } 0 \leq t < 1 \text{ is such that } s = d_D(0, t). \text{ Eliminating } t \text{ yields (4.4).} \]
convex subset, and write

\[ N_\varphi(D) = \text{diam}(D) \sup_{z \in D} \left| \frac{\varphi''(z)}{\varphi'(z)} \right|. \]  (4.7)

Then for all \( z, w \in D \) we have

\[ e^{-N_\varphi(D)} \leq \frac{\varphi'(z)}{\varphi'(w)} \leq e^{N_\varphi(D)}. \]  (4.8)

When \( D \) is not convex, we can still get an estimate like (4.8) by covering \( D \) with small disks. The following result is by no means the sharpest of its kind, but it will be quite sufficient for our purposes.

**Lemma 4.4.** Let \( \varphi : V \to \mathbb{C} \) be holomorphic univalent, and let \( W \subset V \) be a non-empty compact connected set. Suppose \( M > 1 \) is such that \( 1 \leq \text{diam}(V) \leq M \) and \( \text{dist}(\partial V, \partial W) \geq M^{-1} \). Also, let \( z_0 \in W \) be given. Then the following assertions hold.

(i) There exists \( K_1 = K_1(M) > 1 \) such that, for all \( z, w \in W \), we have

\[ \frac{1}{K_1} \leq \left| \frac{\varphi'(z)}{\varphi'(w)} \right| \leq K_1. \]  (4.9)

In fact, we can take \( K_1 = e^{32\pi M^4} \).

(ii) There exists \( K_2 = K_2(M) > 0 \) such that \( \max \{ \| \varphi' \|_{\mathcal{C}^0}, \| \varphi'' \|_{\mathcal{C}^0} \} \leq K_2 |\varphi'(z_0)| \).

**Proof.** Cover \( W \) with a finite number \( m \) of non-overlapping closed squares \( Q_j, 1 \leq j \leq m \), each \( Q_j \) having the same side \( \ell = (2\sqrt{2}M)^{-1} \), and take \( m \) to be the smallest possible. Then \( Q_j \cap W \neq \emptyset \), the diameter of \( Q_j \) is \( (2M)^{-1} \), and \( \text{dist}(Q_j, \partial V) \geq (2M)^{-1} \), for each \( 1 \leq j \leq m \). Since the total area of these squares cannot exceed the area of \( V \), which is less than \( \pi M^2 \), we see that \( m < 8\pi M^4 \). Moreover, from Koebe’s estimate (4.7) we have for each \( j \)

\[ N_{\varphi}(Q_j) \leq (2M)^{-1} \cdot \frac{4}{(2M)^{-1}} = 4. \]

Now, since \( W \) is connected, given any pair of points \( z, w \in W \), we can join them by a chain of pairwise distinct squares \( Q_{j_1}, Q_{j_2}, \ldots, Q_{j_n} \) such that \( Q_{j_k} \cap Q_{j_{k+1}} \neq \emptyset \), with \( z \in Q_{j_1} \) and \( w \in Q_{j_n} \), say. Choose \( z_k \in Q_{j_k} \cap Q_{j_{k+1}} \) for \( k = 1, 2, \ldots, n-1 \), and set \( z_0 = z, z_n = w \). Use (4.8) to get

\[ \left| \frac{\varphi'(z)}{\varphi'(w)} \right| = \prod_{k=0}^{n-1} \left| \frac{\varphi'(z_k)}{\varphi'(z_{k+1})} \right| \leq \exp \left( \sum_{k=1}^{n} N_{\varphi}(Q_{j_k}) \right) \leq e^{4m}. \]

This establishes the upper bound in (4.9); the lower bound is obtained in the same way, or simply interchanging \( z \) and \( w \). Hence assertion (i) is proved. Assertion (ii) follows from assertion (i) and the inequality (4.9). \( \square \)
4.4. **Quasiconformality and holomorphic motions.** We need some non-trivial facts from the theory of quasiconformal mappings. Good references for what follows are [1] and [3]. Given a quasiconformal homeomorphism $\phi$, we write $\mu_\phi(z)$ for the Beltrami form of $\phi$ at $z$, and $K_\phi(z) = (1 + |\mu_\phi(z)|)/(1 - |\mu_\phi(z)|)$ for the dilatation of $\phi$ at $z$. We also denote by $K_\phi$ the maximal dilatation of $\phi$, namely the supremum of $K_\phi(z)$ over all $z$ in the domain of $\phi$.

**Lemma 4.5.** Let $\phi : \mathbb{C} \to \mathbb{C}$ be a $K$-quasiconformal homeomorphism. Then for each $z \in \mathbb{C}$ and all $r > 0$ and $s > 0$ we have

$$\max_{|\zeta - z| = rs} |\phi(\zeta) - \phi(z)| \leq \exp(\pi K) \max \{r^K, r^{1/K}\}.$$

For a proof of this lemma, see [3] pp. 312-313.

**Lemma 4.6.** Let $\phi : \mathbb{D} \to \mathbb{C}$ be a quasiconformal embedding of the disk with $\phi(0) = 0$, and let $0 < r < 1$. Then the restriction $\phi_{|\mathbb{D}(0,r)}$ admits a homeomorphic $K$-quasiconformal extension to the entire plane, where $K = \frac{1+r}{1-r} K_\phi$.

This lemma and its proof can be found in [3] p. 310]. We shall need also the following rather non-trivial result due to Slodkowski. Recall that a **holomorphic motion** of a set $E \subseteq \hat{\mathbb{C}}$ is a map $F : \Delta \times E \to \hat{\mathbb{C}}$, where $\Delta \subseteq \mathbb{C}$ is a disk, such that (i) for each $z \in E$, the map $t \mapsto F(t,z)$ is holomorphic in $\Delta$; (ii) for each $t \in \Delta$, the map $\varphi_t : E \to \hat{\mathbb{C}}$ given by $\varphi_t(z) = F(t,z)$ is injective; (iii) for a certain $t_0 \in \Delta$ we have $\varphi_{t_0}(z) = z$ for all $z \in E$. The point $t_0$ is called the base point of the motion.

**Theorem 4.7.** Let $F : \Delta \times E \to \hat{\mathbb{C}}$ be a holomorphic motion of a set $E \subseteq \hat{\mathbb{C}}$ with base point $t_0 \in \Delta$. Then there exists a continuous map $\hat{F} : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with the following properties.

(i) The map $\hat{F}$ is a holomorphic motion of $\hat{\mathbb{C}}$ which extends $F$ (in the sense that $\hat{F}(t,z) = F(t,z)$ for all $z \in E$ and all $t \in \Delta$).

(ii) For each $t \in \Delta$, the map $\psi_t(z) = \hat{F}(t,z)$ is a global $K_t$-quasiconformal homeomorphism with $K_t \leq \exp\{d_\Delta(t,t_0)\}$ (where $d_\Delta$ denotes the hyperbolic metric of $\Delta$).

The following lemma contains a well-known result stating that every quasiconformal homeomorphism can be embedded in a holomorphic motion (see [3] ch. 12]). It will be used in combination with Slodkowski’s theorem.

**Lemma 4.8.** Let $\psi : \mathbb{C} \to \mathbb{C}$ be a quasiconformal homeomorphism with $k = \|\mu_\psi\|_\infty \neq 0$, and let $z_0 \in \mathbb{C}$ be such that $\psi(z_0) = z_0$.

(i) There exists a holomorphic motion $\psi_t : \mathbb{C} \to \mathbb{C}$, $t \in \mathbb{D}$, such that $\psi_k = \psi$ and $\psi_t(z_0) = z_0$ for all $t$.

(ii) If $0 < r_0 < 1$ and $M > 1$ are such that $\psi(D(z_0,r_0)) \subseteq D(z_0,Mr_0)$, then for all $0 \leq r < 1$ and all $t$ with $|t| < \frac{1}{2}$ we have $\psi_t(D(z_0,r)) \subseteq D(z_0,R)$, where

$$R = \frac{2Me^{6\pi r^{1/3}}}{kr_0^{1/3}}.$$  

(4.10)
Proof. We may assume that $z_0 = 0$ (otherwise we simply conjugate $\psi$ by the translation $z \mapsto z - z_0$ and work with the resulting map, which fixes 0). For each $t \in \mathbb{D}$, let $\varphi_t : \mathbb{C} \to \mathbb{C}$ be the unique solution to the Beltrami equation

$$\overline{\partial} \varphi_t = \frac{t}{k} \mu \psi \partial \varphi_t,$$

normalized so that $\varphi_t$ fixes 0, 1 and $\infty$. Define $\psi_t : \mathbb{C} \to \mathbb{C}$ by the formula

$$\psi_t(\zeta) = \left[1 + \frac{t}{k} (\psi(1) - 1)\right] \varphi_t(\zeta). \quad (4.11)$$

Note that $\psi_t(0) = 0$ for all $t$. Also, for $t = k$, we have $\psi_k(\zeta) = \psi(1) \varphi_k(\zeta)$, so $\psi_k(1) = \psi(1)$. Since the Beltrami form of $\psi_k$ is the same as the Beltrami form of $\varphi_k$, which is $\mu \psi$, it follows from uniqueness of normalized solutions to the Beltrami equation that $\psi_k = \psi$. This proves (i).

Applying Lemma 4.5 to $\phi = \varphi_t$, $z = 0$ and $s = 1$, we see that for all $0 < r < 1$

$$\max_{|\zeta| = r} |\varphi_t(\zeta)| \leq e^{\pi K_t r 1/K_t},$$

where $K_t$ is the maximal dilatation of $\varphi_t$, which satisfies

$$K_t \leq \frac{1 + |t|}{1 - |t|}.$$

In particular, since $K_t < 3$ for all $t$ with $|t| < \frac{1}{2}$, we have

$$\varphi_t(D(0, r)) \subseteq D(0, e^{3\pi r 1/3}) \quad (4.12)$$

Let us now estimate the scaling factor multiplying $\varphi_t(\zeta)$ on the right-hand side of (4.11). Applying Lemma 4.5 with $\phi = \psi$, $z = 0$, $s = r_0$ and $r = r_0^{-1}$, and taking into account that the maximal dilatation of $\psi$ is less than 3, we get

$$\max_{|\zeta| = 1} |\psi(\zeta)| \leq e^{3\pi} \frac{1}{r_0^3} \min_{|\zeta| = r_0} |\psi(\zeta)| \leq e^{3\pi} \frac{1}{r_0^3} (Mr_0) = \frac{Me^{3\pi}}{r_0^2}.$$

In particular, $|\psi(1) - 1| \leq 2Me^{3\pi r_0^{-2}}$, and therefore

$$\left|1 + \frac{t}{k} (\psi(1) - 1)\right| \leq \frac{2Me^{3\pi}}{kr_0^2}$$

for all $t$ with $|t| \leq \frac{1}{2}$. Combining this fact with (4.12), it follows that for all such $t$ we have

$$\max_{|\zeta| \leq r} |\psi_t(\zeta)| \leq \frac{2Me^{6\pi r 1/3}}{kr_0^2}.$$

Therefore $\psi_t(D(0, r)) \subseteq D(0, R)$ for all $t$ with $|t| \leq \frac{1}{2}$ and all $0 < r < 1$, where $R$ is given by (4.10). This proves (ii).
4.5. **Quasi-isometry estimates for almost conformal maps.** Our goal in this subsection is to make more precise a somewhat vague but intuitive assertion, namely that if a self-map of a hyperbolic domain (or Riemann surface) is almost conformal, then it is an almost isometry of the hyperbolic metric. For the sake of the dynamical applications we have in mind, what is needed is an infinitesimal version of this statement.

The desired infinitesimal quasi-isometry property will be presented in two versions. In the first version we deal with the case when the quasiconformal map has small dilatation everywhere, and the quasi-isometry bounds we get are in terms of this global small dilatation. In the second version we deal with the situation when the map is $K$-quasiconformal (with $K$ not necessarily small) but the quasi-isometry bounds we get are local, near any point $z \in \mathbb{D}$ where the dilatation is bounded by some fixed power of the distance between $z$ and $\partial \mathbb{D}$. This last version is precisely what we need when studying the metric distortion properties of maps which are asymptotically holomorphic. Both versions are first established for quasiconformal diffeomorphisms of the unit disk, but at the end of this subsection we show how to transfer these results to the kind of simply-connected regions that matter to us.

First, let us introduce some notation. We denote by $\rho_{\mathbb{D}}(z) = 2(1 - |z|^2)^{-1}$ the Poincaré density of the unit disk, as before. We also denote by $\Delta_z \subset \mathbb{D}$ the closed euclidean disk $\{ \zeta : |\zeta - z| \leq \frac{1}{2}(1 - |z|) \}$. Given a $C^2$ map $\phi : \mathbb{D} \to \mathbb{D}$, we denote by $m_\phi(z)$ the $C^2$ norm of $\phi|_{\Delta_z}$. We write $J_\phi(z) = \det D\phi(z)$ for the euclidean Jacobian of $\phi$ at $z$, and

$$J^h_\phi(z) = J_\phi(z) \left( \frac{\rho_{\mathbb{D}}(\phi(z))}{\rho_{\mathbb{D}}(z)} \right)^2$$

for the hyperbolic Jacobian of $\phi$ at $z$.

**Proposition 4.9.** For each $0 < \theta < 1$, there exists a universal continuous function $A_\theta : (1, \infty) \times \mathbb{R}^+ \to \mathbb{R}^+$ for which the following holds. Let $0 < \epsilon < 1$ and $\alpha > 1$ be given, and suppose $\phi : \mathbb{D} \to \mathbb{D}$ is a $C^2$ quasiconformal diffeomorphism with $K_\phi \leq 1 + \epsilon$. If $z \in \mathbb{D}$ is such that

$$\alpha^{-1} \leq \frac{\rho_{\mathbb{D}}(\phi(z))}{\rho_{\mathbb{D}}(z)} \leq \alpha,$$ (4.13)

then

$$J^h_\phi(z) \leq 1 + A_\theta(\alpha, m_\phi(z)) \epsilon^{1-\theta}.$$  

The proof, given later in this subsection, will use the following three lemmas.

**Lemma 4.10.** Let $z \in \mathbb{D}$ and let $0 < r < 1 - |z|$. Then

$$\text{mod}(\mathbb{D} \setminus D(z, r)) \leq \log \left( \frac{1 - |z|^2 + |z|r}{r} \right).$$ (4.14)

**Proof.** We may assume that $z$ is real and non-negative, say $z = x \in [0, 1)$. Let $\varphi \in \text{Aut}(\mathbb{D})$ be given by

$$\varphi(\zeta) = \frac{\zeta - x}{1 - x \zeta},$$

and define

$$\alpha = \varphi(x - r) = \frac{-r}{1 - x^2 + rx}; \quad \beta = \varphi(x + r) = \frac{r}{1 - x^2 - rx}.$$
Then $D'_r = \varphi(D(x, r))$ is a disk with diameter $(\alpha, \beta) \subset (-1, 1)$. Since $|\alpha| \leq \beta$, we see that $D'_r \supseteq D(0, |\alpha|)$. Therefore
\[
\mod(\mathbb{D} \setminus D(x, r)) = \mod(\mathbb{D} \setminus D'_r)
\leq \mod(\mathbb{D} \setminus D(0, |\alpha|)) = \log \frac{1}{|\alpha|}
= \log \frac{1-x^2+rx}{r},
\]
and this finishes the proof. \hfill \square

**Remark 4.11.** It follows from (4.14) that $\mod(\mathbb{D} \setminus D(z, r)) \leq \log \left( \frac{2}{r} \right)$. This estimate will be useful when $r$ is small compared to the distance from $z$ to $\partial \mathbb{D}$. If $r = \frac{1}{2}\delta(1-|z|)$ with $0 < \delta \leq 1$, then an easy manipulation of the right-hand side of (4.14) yields the estimate $\mod(\mathbb{D} \setminus D(z, r)) \leq \log \left( \frac{5}{\delta} \right)$. This remark will be used in the proof of Lemma 4.13 below.

**Lemma 4.12.** Let $\alpha > 1$ and suppose $z, w \in \mathbb{D}$ are such that
\[
\alpha^{-1} \leq \frac{\rho_\mathbb{D}(z)}{\rho_\mathbb{D}(w)} \leq \alpha,
\]
Then there exists $\psi \in \text{Aut}(\mathbb{D})$ with $\psi(z) = w$ such that the following inequalities hold for all $\zeta \in \Delta_z$:
\[
\text{(i) } \frac{1}{2\alpha} \leq |\psi'(\zeta)| \leq 4\alpha^2; \quad \text{(ii) } |\psi''(\zeta)| \leq 16\alpha^3.
\]

**Proof.** Write $a = |z|$ and $b = |w|$, so that $0 \leq a, b < 1$. We have $1-a^2 = \rho_\mathbb{D}(z)^{-1}$ and $1-b^2 = \rho_\mathbb{D}(w)^{-1}$, so (4.15) tells us that
\[
\alpha^{-1} \leq \frac{1-a^2}{1-b^2} \leq \alpha.
\]
Let $\varphi \in \text{Aut}(\mathbb{D})$ be the hyperbolic translation with axis $(-1, 1) \subset \mathbb{D}$ such that $\varphi(a) = b$. Then
\[
\varphi(\zeta) = \frac{\zeta - c}{1 - c\zeta},
\]
where $c = (a-b)/(1-ab) \in (-1, 1)$, as a simple calculation shows. Moreover, we have
\[
\varphi'(\zeta) = \frac{1-c^2}{(1-c\zeta)^2},
\]
as well as
\[
\varphi''(\zeta) = \frac{2c(1-c^2)}{(1-c\zeta)^3},
\]
Since $1-c^2 = (1-a^2)(1-b^2)/(1-ab)^2$, and since $\min\{1-a^2, 1-b^2\} \leq 1-ab \leq \max\{1-a^2, 1-b^2\}$, it follows from (4.16) that
\[
\alpha^{-1} \leq 1-c^2 \leq 1.
\]
Now, if \( \zeta \in \Delta_a \), then \(|\zeta| \leq (1 + a)/2\). Hence
\[
|1 - c\zeta| \geq 1 - |c| \left( \frac{1 + a}{2} \right) = \frac{1 - |c|}{2} + \frac{1 - |c|a}{2} > \frac{1 - |c|a}{2}.
\]

Here, there are two cases to consider. If \( a \geq b \), then \( c \geq 0 \) and \( 1 - |c|a = (1 - a^2)/(1 - ab) \), so from (4.16) we deduce that \( 1 - |c|a \geq \alpha^{-1} \). If however \( a < b \), then \( c < 0 \), and in this case we see that
\[
1 - |c|a = \frac{1 - b^2 + (b - a)^2}{1 - ab} > \frac{1 - b^2}{1 - a^2} \geq \alpha^{-1},
\]
where once again we have used (4.16). Thus, in either case we have
\[
\frac{1}{2\alpha} \leq |1 - c\zeta| < 2, \quad \text{for all } \zeta \in \Delta_a. \tag{4.20}
\]

Using both (4.19) and (4.20) in (4.17) and (4.18), we easily arrive at inequalities (i) and (ii) with \( \phi \) replacing \( \psi \) (and \( \Delta_a \) replacing \( \Delta_z \)). Finally, we define \( \psi = R_b \circ \phi \circ R_a \), where \( R_a \) is the rigid rotation around 0 with \( R_a(z) = a \), and \( R_b \) is the rigid rotation around 0 with \( R_b(b) = w \). Then \( \psi(z) = w \), and since \( R_a, R_b \) are euclidean isometries and \( R_a(\Delta_z) = \Delta_a \), the inequalities (i) and (ii) for \( \psi \) follow from the corresponding inequalities for \( \phi \). \( \square \)

For our final lemma, we introduce further notation. Given a \( C^2 \) map \( \phi : \mathbb{D} \to \mathbb{D} \), a point \( z \in \mathbb{D} \) and \( 0 < \delta \leq 1 \), we denote by \( m_\phi(z, \delta) \) the \( C^2 \) norm of the restriction of \( \phi \) to the disk \( \{ \zeta : |\zeta - z| \leq \delta r_z \} \), where \( r_z = \frac{1}{2}(1 - |z|) \). In particular, \( m_\phi(z, 1) = m_\phi(z) \).

**Lemma 4.13.** For each \( 0 < \theta < 1 \) there exists a universal, continuous monotone function \( B_\theta : \mathbb{R}^+ \to \mathbb{R}^+ \) such that the following holds. Given \( 0 < \epsilon < 1 \), let \( \phi : \mathbb{D} \to \mathbb{D} \) be a \( C^2 \) quasiconformal diffeomorphism with \( K_\phi \leq 1 + \epsilon \), and suppose that \( z \in \mathbb{D} \) is a fixed point of \( \phi \). Then for each \( 0 < \delta \leq 1 \) we have
\[
J_\phi^b(z) \leq 1 + B_\theta \left( \frac{m_\phi(z, \delta)}{\delta} \right) \epsilon^{1-\theta}. \tag{4.21}
\]

**Proof.** The basic geometric idea behind the proof is to use macroscopic estimates on the moduli of certain annuli in order to bound a microscopic quantity, namely the hyperbolic Jacobian at \( z \). Rotating the coordinate axes if necessary, we may also assume that \( D\phi(z) = S \cdot T \), where \( S = \rho I = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \), for some \( \rho > 0 \), and \( T = \begin{pmatrix} \lambda & b \\ 0 & \lambda^{-1} \end{pmatrix} \), where \( \lambda \geq 1 \) and \( b \in \mathbb{R} \). Here we obviously have \( \rho^2 = \det D\phi(z) = J_\phi(z) = J_\phi^b(z) \). We shall prove the lemma only in the case when \( b = 0 \) and \( \lambda > 1 \). The cases when \( b \neq 0 \) and/or \( \lambda = 1 \) are similarly handled. Note that the linear map \( D\phi(z) \) maps the circle of radius 1 about the origin onto an ellipse with major axis \( \rho \lambda \) and minor axis \( \rho / \lambda \). Since \( \phi \) is \((1+\epsilon)\)-qc, we have \( \lambda^2 \leq 1 + \epsilon \). In what follows, we assume that \( \rho > \lambda + \epsilon \), as otherwise \( \rho^2 \leq (\lambda + \epsilon)^2 \leq 1 + 6\epsilon \) and there is nothing to prove.

If \( \zeta \) is such that \( |\zeta - z| \leq \delta r_z \) we can write, using Taylor’s formula and the fact that \( \phi(z) = z \),
\[
\phi(\zeta) = z + D\phi(z) \cdot (\zeta - z) + R_\phi(\zeta), \tag{4.22}
\]
where the remainder \( R_\phi(\zeta) \) satisfies \(|R_\phi(\zeta)| \leq C|\zeta - z|^2\), with \( C = C_0 m_\phi(z, \delta) > 0 \) (and \( C_0 > 0 \) an absolute constant). Let us choose \( 0 < r \leq \delta r_z \) so small that

\[
\frac{\rho}{\lambda} r - Cr^2 > \frac{\rho}{\lambda + \epsilon} r.
\] (4.23)

For definiteness, we take

\[
r = \min \left\{ \delta r_z, \frac{\rho \epsilon}{C \lambda^2 (\lambda + \epsilon)} \right\}.
\] (4.24)

Then (4.22) and (4.23) tell us that \( \phi \) maps the disk \( D(z, r) \) onto a Jordan domain \( V_r \) which contains that disk and also the round annulus \( \Omega = \{ \zeta : r < |\zeta - z| < \frac{\rho}{\lambda + \epsilon} r \} \). Setting \( \Omega_0 = V_r \setminus D(z, r) \), we have \( \Omega_0 \supseteq \Omega \), and so

\[
\text{mod}(\Omega_0) \geq \text{mod}(\Omega) = \log \left( \frac{\rho}{\lambda + \epsilon} \right).
\] (4.25)

Consider the images of \( \Omega_0 \) under the forward iterates of \( \phi \), i.e., \( \Omega_n = \phi^n(\Omega_0) \), \( n \geq 0 \). The annuli \( \Omega_n \) are pairwise disjoint, and \( \bigcup_{n=0}^{\infty} \Omega_n \subset \mathbb{D} \setminus D(z, r) \). By sub-additivity of the modulus, we have

\[
\sum_{n=0}^{\infty} \text{mod}(\Omega_n) \leq \mu_r = \text{mod}(\mathbb{D} \setminus D(z, r)) .
\] (4.26)

Now, since \( \phi \) is \((1 + \epsilon)-qc\), we know that \( \phi^n \) is \((1 + \epsilon)^n\)-qc, and therefore

\[
\text{mod}(\Omega_n) \geq \frac{\text{mod}(\Omega_0)}{(1 + \epsilon)^n} .
\] (4.27)

Putting together (4.25), (4.26) and (4.27), we get

\[
\log \left( \frac{\rho}{\lambda + \epsilon} \right) \sum_{n=0}^{\infty} \frac{1}{(1 + \epsilon)^n} \leq \mu_r .
\] (4.28)

Applying Lemma 4.10 and Remark 4.11 to our \( r \) as defined in (4.24), we see that

\[
\mu_r \leq \begin{cases} 
\log \left( \frac{5}{\delta} \right), \quad & \text{when } r = \delta r_z ; \\
\log \left( \frac{2C \lambda^2 (\lambda + \epsilon)}{\rho \epsilon} \right), \quad & \text{when } r = \frac{\rho \epsilon}{C \lambda^2 (\lambda + \epsilon)} .
\end{cases}
\] (4.29)

Regardless of which of the two cases occur, we certainly have

\[
\mu_r \leq \log \left( \frac{10C \lambda^2 (\lambda + \epsilon)}{\delta \rho \epsilon} \right) < \log \left( \frac{60C}{\delta \epsilon} \right),
\] (4.30)

where in the last step we have used that \( \lambda^2 (\lambda + \epsilon) < 6 \) and \( \rho > 1 \). Combining (4.28) and (4.30), we deduce that

\[
\log \left( \frac{\rho}{\lambda + \epsilon} \right) \leq \frac{\epsilon}{1 + \epsilon} \log \left( \frac{60C}{\delta \epsilon} \right) < \epsilon \log \left( \frac{60C}{\delta} \right) + \epsilon \log \frac{1}{\epsilon} .
\] (4.31)
Since $0 < \epsilon < 1$, we have $\epsilon < e^{1-\theta}$ and $e^\theta \log \frac{1}{\epsilon} \leq (\theta e)^{-1}$. Using these facts in (4.31), we get
\[
\rho \leq (\lambda + \epsilon) \exp \left\{ \left( \frac{1}{\theta e} + \log \frac{60C'}{\delta} \right) e^{1-\theta} \right\} \leq 1 + \left( 2 + 180 e^{1/\theta e} C' \frac{m_\phi(z, \delta)}{\delta} \right) e^{1-\theta},
\]
where we have used that $\lambda + \epsilon \leq 1 + 2\epsilon$. From this, and the fact that $C = C_0 m_\phi(z, \delta)$, it readily follows that
\[
J^h_\phi(z) = \rho^2 \leq 1 + 3 \left( 2 + 180 e^{1/\theta e} C_0 \frac{m_\phi(z, \delta)}{\delta} \right)^2 e^{1-\theta}.
\]
This proves (4.21), provided we take $B_\theta(t) = 3 \left( 2 + 180 e^{1/\theta e} C_0 t \right)^2$.

We are now ready for the proof of the first main result of this subsection.

**Proof of Proposition 4.9.** The idea, of course, is to reduce the required estimate to the case treated in Lemma 4.13. Let $\psi \in \text{Aut}(\mathbb{D})$ be the conformal automorphism given by Lemma 4.12 with $\psi(z) = w = \phi(z)$. Then the diffeomorphism $F = \psi^{-1} \circ \phi : \mathbb{D} \to \mathbb{D}$ has a fixed point at $z$. Since $\psi^{-1}$ is an isometry of the hyperbolic metric, we certainly have $J^h_F(z) = J^h_\phi(z)$. We would like to estimate $J^h_\phi(z)$ using Lemma 4.13. For this, we need an estimate on the $C^2$ norm of the composition $\psi^{-1} \circ \phi$ in a suitable disk around $z$. By Koebe’s one-quarter theorem, $\psi(D_\Delta)$ contains the disk
\[
D = \left\{ \zeta : |\zeta - w| < \frac{1}{4} |\psi'(z)| \cdot r_z \right\}.
\]
Since we know from Lemma 4.12(i) that $|\psi'(z)| \geq (2\alpha)^{-1}$, it follows that $\psi(D_\Delta) \supset D(w, R)$, where $R = r_z/8\alpha$. Now let us define
\[
\delta = \frac{1}{8\alpha m_\phi(z)} \quad \text{and} \quad M = \sup_{\zeta \in D_\Delta} |\phi(\zeta)| \leq m_\phi(z).
\]
Then we have $\phi(D(z, \delta r_z)) \subset D(w, M\delta r_z) \subset D(w, R) \subset \psi(D_\Delta)$. We can now estimate the $C^2$ norm of $F$ restricted to the disk $D(z, \delta r_z)$, i.e. we can estimate $m_F(z, \delta)$, with the help of Lemma 4.12. We do this by means of the following two steps.

(i) By the chain rule for first derivatives, we have $DF = D\psi^{-1} \circ \phi \cdot D\phi$. Since $\psi^{-1}$ is holomorphic, for each $\zeta \in D(z, \delta r_z)$ we have
\[
||DF(\phi(\zeta))|| \leq |(\psi^{-1})'(\phi(\zeta))| = |\psi'(\psi^{-1} \circ \phi(\zeta))|^{-1} \leq 2\alpha.
\]
Hence the $C^0$ norm of $DF$ in $D(z, \delta r_z)$ is bounded by $2\alpha m_\phi(z)$.

(ii) By the chain rule for second derivatives, we have
\[
D^2 F = (D^2 \psi^{-1} \circ \phi) \cdot (D\phi \otimes D\phi) + D\psi^{-1} \circ \phi \cdot D^2 \phi.
\]
Again, since $\psi^{-1}$ is holomorphic, a simple calculation shows that
\[
(\psi^{-1})'' = -\frac{\psi'' \circ \psi^{-1}}{(\psi' \circ \psi^{-1})^3}.
\]
Therefore, for each $\zeta \in D(z, \delta r_z)$ we have, with the help of Lemma 4.12
\[
\|D^2\psi^{-1}(\phi(\zeta))\| \leq \| (\psi^{-1})''(\phi(\zeta)) \| \leq 128\alpha^6 .
\] (4.36)

Using (1.31), (4.36) and the fact that $\|D\phi \otimes D\phi\| \leq \|D\phi\|^2$ in (1.33), we deduce that the $C^0$ norm of $D^2 F$ in the disk $D(z, \delta r_z)$ is bounded by $(128\alpha^6 + 2\alpha)m_\phi(z) < 130\alpha^6 m_\phi(z)$.

From steps (i) and (ii) above we deduce that $m_F(z, \delta) \leq 130\alpha^6 m_\phi(z)$. Therefore, applying Lemma 4.13 for $F$ yields
\[
J^h_F(z) = J^h_D(z) = 1 + B_\theta \left( \frac{m_F(z, \delta)}{\delta} \right) \epsilon^{1-\theta} \leq 1 + B_\theta \left( 1040\alpha^7(m_\phi(z))^2 \right) \epsilon^{1-\theta} .
\]

This completes the proof of our theorem, provided we take $A_\theta(s, t) = B_\theta(1040\alpha^7 t^2)$.

\[\Box\]

**Proposition 4.14.** For each $0 < \theta < 1$, there exists a universal continuous function $C_\theta : (1, \infty) \times (1, \infty) \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ for which the following holds. Let $\alpha > 1$ and $\beta > 1$ be given, and suppose $\phi : \mathbb{D} \rightarrow \mathbb{D}$ is a $C^2$ quasiconformal diffeomorphism. If $z \in \mathbb{D}$ is such that
\[
\alpha^{-1} \leq \frac{\rho_\mathbb{D}(\phi(z))}{\rho_\mathbb{D}(z)} \leq \alpha ,
\] (4.37)

and
\[
\sup_{\zeta \in \Delta_z} |\mu_\phi(\zeta)| \leq b_0(1 - |z|)^{\beta} ,
\] (4.38)

then
\[
J^h_\phi(z) \leq 1 + C_\theta(\alpha, \beta, b_0, m_\phi(z))(1 - |z|)^{\beta(1-\theta)} .
\] (4.39)

**Proof.** We present the proof of the required estimate under the additional assumption that $z$ is a fixed-point of $\phi$. The general case can be reduced to this one by post-composing $\phi$ with a suitable conformal automorphism of the unit disk, and proceeding just as in the proof of Proposition 4.9 mutatis mutandis. For the sake of clarity of exposition, we divide the proof into a series of steps.

(i) First we introduce some notation. Throughout the proof we denote by $c_0, c_1, \ldots$ positive constants that are either absolute or depend on the given constants $\alpha, \beta, b_0, M$, where $M = m_\phi(z)$. Let us write $\epsilon = b_0(1 - |z|)^{\beta} = (b_02^\beta)r_z^\beta$. Also, let $k_0 = \sup_{\zeta \in \Delta_z} |\mu_\phi(\zeta)| \leq \epsilon$, and set $r_0 = \epsilon r_z$. We may assume without loss of generality that $\epsilon$ is small, say $\epsilon < 1/32$.

(ii) The restricted map $\phi|_{\Delta_z} : \Delta_z \rightarrow \mathbb{D}$ is a $\frac{1+k_0}{k_0}$ quasiconformal embedding. By Lemma 4.6 the further restriction $\phi|_{D(z, r_0)}$ can be extended to a global quasiconformal homeomorphism $\psi : \mathbb{C} \rightarrow \mathbb{C}$ with $k = \|\mu_\psi\|_\infty$ satisfying
\[
\frac{1+k}{1-k} \leq \frac{1+\epsilon}{1-\epsilon} .
\]

(iii) In particular, $k \leq 16\epsilon < \frac{1}{2}$ (by our assumption on $\epsilon$ in (i)). We may assume that $k \neq 0$ (if this is not the case, it is easy to perturb $\psi$ slightly in a neighborhood of
infinity). By Lemma 4.8(i), there exists a global holomorphic motion \( \psi_t : \mathbb{C} \to \mathbb{C} \) with \( \psi_k = \psi \) and \( \psi_t(z) = z \) for all \( t \in \mathbb{D} \). Now choose \( r_1 > 0 \) so small that

\[
R = \frac{2M e^{6\pi}}{k_0 r_0^{1/3}} < r_z.
\]

For definiteness, take \( r_1 = c_1 k^3 r_0^{6\beta + 9} \), where \( c_1 = b_0^6/(M^3 e^{18\pi}) \). Then, by Lemma 4.8(ii), we have \( \psi_t(D(z, r_1)) \subset D(z, R) \) for all \( t \) with \( |t| < \frac{1}{2} \) (note that this includes the time \( t = k \)).

(iv) We may now define, for each \( t \in D(0, \frac{1}{2}) \), the map \( \tilde{\psi}_t : D(z, r_1) \cup (\mathbb{C} \setminus \mathbb{D}) \to \mathbb{C} \) by

\[
\tilde{\psi}_t(\zeta) = \begin{cases} 
\psi_t(\zeta) & \text{for } \zeta \in D(z, r_1), \\
\zeta & \text{for } \zeta \in \mathbb{C} \setminus \mathbb{D}.
\end{cases}
\]

Since \( D(z, R) \subset \mathbb{D} \), we have from step (iii) that \( \psi_t(D(z, r_1)) \cap \mathbb{C} \setminus \mathbb{D} = \emptyset \). Hence \( \tilde{\psi}_t, |t| < \frac{1}{2} \), is a holomorphic family of injections, i.e., a holomorphic motion of the set \( D(z, r_1) \cup (\mathbb{C} \setminus \mathbb{D}) \).

(v) Now apply Slodkowski’s Theorem 4.7 to get a global extension \( \hat{\psi}_t : \mathbb{C} \to \mathbb{C} \) of the motion \( \psi_t \), with time parameter \( t \in D(0, \frac{1}{2}) \). In particular, the map \( \hat{\psi} = \hat{\psi}_k \) is \( K \)-quasiconformal with \( K = \frac{1+2k}{1-2k} \), and it maps the unit disk onto itself. Moreover, we have

\[
\hat{\psi}|_{D(z, r_1)} = \psi|_{D(z, r_1)} = \phi|_{D(z, r_1)}.
\]

Thus, \( \hat{\psi} \) is the desired modification of \( \phi \) away from \( z \).

(vi) We are now in a position to use the same annulus trick we employed in the proof of Lemma 4.13. Let \( \rho > 0, \lambda > 1 \) and the absolute constant \( C_0 > 0 \) be as in the proof of that Lemma. In particular, \( \rho^2 = J^b_\phi(z) = J^b_\psi(z) \), and thus our goal is to bound \( \rho \) from above. We have \( \lambda \leq 1 + \epsilon \), and we may assume that \( \rho > \lambda + \epsilon \); otherwise there is nothing to prove. Now let \( r_2 > 0 \) be given by

\[
r_2 = \frac{\epsilon}{3C_0 M} < \frac{\rho \epsilon}{C_0 M \lambda^2 (\lambda + \epsilon)}.
\]

Then for all \( r \leq r_2 \) the inequality (4.23) holds. Let us choose \( r = \min\{r_1, r_2\} \). With this choice of \( r \), using the Taylor expansion (4.22) as in the proof of Lemma 4.13 we see that \( \Omega_0 = \hat{\psi}(D(z, r)) \setminus D(z, r) = \phi(D(z, r)) \setminus D(z, r) \) is a conformal annulus, with

\[
\text{mod} (\Omega_0) \geq \log \frac{\rho}{\lambda + \epsilon}.
\]

(vii) Now define \( \Omega_n = \hat{\psi}^n(\Omega_0) \) for all \( n \geq 0 \), and note that

\[
\text{mod} (\Omega_n) \geq \left( \frac{1 - 2k}{1 + 2k} \right)^n \text{mod} (\Omega_0).
\]

Since \( \cup_{n \geq 0} \Omega_n \subset \mathbb{D} \setminus D(z, r) \), we deduce from (4.40) and (4.41) that

\[
\log \left( \frac{\rho}{\lambda + \epsilon} \right) \sum_{n=0}^{\infty} \left( \frac{1 - 2k}{1 + 2k} \right)^n \leq \log \frac{2}{r},
\]

(4.42)
where we have used the estimate on mod $\left( \mathbb{D} \setminus D(z, r) \right)$ given by Lemma 4.10 (and Remark 4.11). From (4.42) it follows that
\[
\log \left( \frac{\rho}{\lambda + \epsilon} \right) \leq \frac{4k}{1+2k} \log \frac{2}{r} < 4k \log \frac{2}{r} \quad (4.43)
\]
(vii) But from our choices of $r_1$ and $r_2$, we see that $r = \min\{r_1, r_2\} = c_2 k^3 r_z^{6\beta+9}$, for some constant $c_2 > 0$. Hence
\[
\log \frac{2}{r} \leq \log \frac{2}{c_2} + 3 \log \frac{1}{k} + (6\beta + 9) \log \frac{1}{r_z}.
\]
Putting this back into (4.43) and using that $k \leq (\text{const.}) r_z^\beta$, we deduce that, for each $0 < \theta < 1$,
\[
\log \left( \frac{\rho}{\lambda + \epsilon} \right) \leq c_3 k + c_4 k \log \frac{1}{k} + c_5 k \log \frac{1}{r_z} \\
\leq c_6 r_z^{\beta(1-\theta)} + c_7 r_z^{\beta} \log \frac{1}{r_z} \\
\leq c_8 r_z^{\beta(1-\theta)}.
\]
Here the constants $c_6, c_7, c_8$ depend on $M, \beta, b_0$ and also on $\theta$. From this it follows that
\[
\rho \leq 1 + c_9 r_z^{\beta(1-\theta)},
\]
and therefore
\[
J_h^\phi(z) = \rho^2 \leq 1 + c_{10} r_z^{\beta(1-\theta)},
\]
where the constant $c_{10}$ depends on $M, \beta, b_0$ and $\theta$.

Hence we have established (4.39), with $c_{10}$ playing the role of $C_\theta$, in the case when $z$ is a fixed-point of $\phi$. As we already remarked, the general case follows from this one by post-composition of $\phi$ with a suitable automorphism of the disk, using the same procedure given in the proof of Proposition 4.9. It is here, and only here, that (4.37) is used. Hence the final constant $C_\theta$ indeed depends on $M, \alpha, \beta, b_0$, and of course also on $\theta$. This finishes the proof.

As we informally said in the beginning of this subsection, our goal is to develop bounds on the infinitesimal distortion, by a self-map (diffeomorphism) of a hyperbolic Riemann surface, of the underlying hyperbolic metric in terms of the local quasiconformal distortion of the map. So far we have only shown how to bound in such terms the hyperbolic Jacobian of these maps. Can we use such estimates on the Jacobian to bound the infinitesimal distortion of the hyperbolic metric? The answer is yes, and the reason lies in the fact that there is a simple relationship between the two concepts. More precisely, let $\phi : Y \to Y$ be a quasiconformal diffeomorphism. Then for each $z \in Y$ and each non-zero tangent vector $v \in T_z Y$, we have
\[
\frac{1}{K_\phi(z)} J^h_\phi(z) \leq \left( \frac{|D\phi(z)v|_Y}{|v|_Y} \right)^2 \leq K_\phi(z) J^h_\phi(z) \quad (4.44)
\]
This fact is classical (see for instance [47, p. 17]).
**Theorem 4.15.** Let $U, V \subset \mathbb{C}$ be Jordan domains, symmetric about the real axis, with $\overline{U} \subset V$, and let $Y = V \setminus \mathbb{R}$. Let $\phi : V \to V$ be a $C^r$ diffeomorphism which is symmetric about the real axis, and write

$$M = \max \{ \text{diam}(V), (\text{dist}(\partial V, \partial U))^{-1}, \|\phi\|_{C^2}, \|\phi^{-1}\|_{C^2} \} > 0$$

Then the following facts hold true for each $0 < \theta < 1$.

(i) If $\phi$ is $(1 + \delta)$-quasiconformal ($\delta > 0$), then for each $z \in U \cap Y$ with $\phi(z) \in U \cap Y$ and all non-zero tangent vectors $v \in T_z Y$ we have

$$(1 + C_\theta \delta^{1-\theta})^{-1} \leq \frac{|D\phi(z)v|_Y}{|v|_Y} \leq 1 + C_\theta \delta^{1-\theta}, \quad (4.45)$$

where $C_\theta > 0$ depends only on $\theta$ and $M$.

(ii) If $\phi$ is asymptotically holomorphic of order $r$, so that $|\mu_\phi(z)| \leq b_0|\text{Im } z|^{r-1}$ for all $z \in Y$, then for each $z \in U \cap Y$ with $\phi(z) \in U \cap Y$ and all non-zero tangent vectors $v \in T_z Y$ we have

$$(1 + C_\theta |\text{Im } z|^{(r-1)(1-\theta)})^{-1} \leq \frac{|D\phi(z)v|_Y}{|v|_Y} \leq 1 + C_\theta |\text{Im } z|^{(r-1)(1-\theta)} \quad (4.46)$$

where $C_\theta > 0$ depends only on $\theta$, $M$ and $b_0$.

**Proof.** The hard work has already been done in Propositions 4.9 and 4.14, and all we have to do is to show, with the help of (4.44), how to reduce the present theorem to the situation in those auxiliary results. There is no loss of generality in assuming that $\phi$ preserves $Y^+ = Y \cap \mathbb{C}^+$ (and therefore also $Y^- = Y \cap \mathbb{C}^-$). Also, it suffices to establish the upper estimates in (4.45) and (4.46), since the lower estimates follow by replacing $\phi$ with its inverse. Moreover, by symmetry we only need to establish these upper estimates for points $z \in U \cap Y^+$.

Let $(a, b) = V \cap \mathbb{R}$, and let $\varphi : V \to \hat{\mathbb{C}}$ be a holomorphic univalent map with $\varphi(Y^+) = \mathbb{D}$, $\varphi(Y^-) = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, normalized so that $\varphi(a) = -1$, $\varphi(b) = +1$. Let $W^* = \bigcup_{z \in \varphi(V^+)} \Delta_z \subset \mathbb{D}$, and consider $W = \varphi^{-1}(W^*) \subset Y^+$. Note that $W \supset U^+$. By Lemma 4.13 (ii), the $C^2$ norms of the restrictions $\varphi|_W$ and $\varphi^{-1}|_{\varphi(W^*)}$ are both bounded by a constant that depends only on $\text{dist}(\partial V, \partial W)$, and it is not difficult (albeit a bit laborious) to see that this last distance is bounded by a constant that depends only on $M$. These bounds also imply that there exists a constant $K_1 > 1$ depending only on $M$ such that

$$\frac{1}{K_1} (1 - |\varphi(z)|) \leq |\text{Im } z| \leq K_1 (1 - |\varphi(z)|) \quad (4.47)$$

for all $z \in W$.

Now consider the $C^2$ diffeomorphism $\psi : \mathbb{D} \to \mathbb{D}$ given by $\psi = \varphi \circ \phi \circ \varphi^{-1}$. Note that, by the chain rule and the bounds on $\varphi$, $\varphi^{-1}$ stated above, the $C^2$ norm of $\psi|_{W^*}$ is also bounded by a constant that depends only on $M$.

Given a point $z \in Y^+$ and a vector $v \in T_z Y^+ \equiv T_z Y$, let $\zeta = \varphi(z) \in \mathbb{D}$ and $w = D\varphi(z)v \in T_{\zeta} \mathbb{D}$. Since $\varphi$ yields an isometry between the hyperbolic metric of $Y^+$ (i.e., of $Y$) and the hyperbolic metric of $\mathbb{D}$, we have $|v|_Y = |w|_D$. Moreover, by the chain rule we have

$$|D\phi(z)v|_Y = |D\varphi^{-1}(\psi(\zeta)) D\psi(\zeta) w|_Y = |D\psi(\zeta) w|_D,$$
where in the last step we have used that \( \varphi^{-1} \) yields an isometry between the hyperbolic metric of \( \mathbb{D} \) and the hyperbolic metric of \( Y^+ \) (and therefore the derivative \( D\varphi^{-1}(\psi(\zeta)) \) is an infinitesimal isometry between corresponding tangent spaces). This shows that for each \( z \in Y^+ \) and each non-zero tangent vector \( v \in T_z \mathbb{D} \), we have

\[
\frac{|D\phi(z)v|_Y}{|v|_Y} = \frac{|D\psi(\zeta)w|_\mathbb{D}}{|w|_\mathbb{D}}. \tag{4.48}
\]

In addition, since \( \varphi \) and \( \varphi^{-1} \) are conformal, we have that \( \psi \) and \( \phi \) have the same dilatation at corresponding points, \textit{i.e.}, \( K_\psi(\zeta) = K_\phi(z) \) for all \( z \in Y^+ \). Also, since \( \varphi \) and \( \varphi^{-1} \) are hyperbolic isometries, the hyperbolic Jacobians of \( \psi \) and \( \phi \) agree on corresponding points, \textit{i.e.}, \( J^h_\psi(\zeta) = J^h_\phi(z) \).

Putting these facts together, we see that the assertions (i) and (ii) in the statement (\textit{i.e.}, the estimates in (4.45) and (4.46)) will be proved for \( \phi \) as soon as the corresponding assertions for \( \psi \) are proved. But assertion (i) for \( \psi \) follows by putting together Proposition 4.9 and (4.44), whereas assertion (ii) for \( \psi \) follows by putting together Proposition 4.14 and (4.44). To see why this is so, we need to check that, in each case, the hypotheses of the corresponding propositions are satisfied by \( \psi \).

\textbf{Case (i).} If \( \phi \) is \((1 + \delta)\)-quasiconformal, as in (i), then \( \psi \) is \((1 + \delta)\)-quasiconformal as well. The hypotheses on \( \phi \) imply that there exists a constant \( K_2 > 1 \) depending only on \( M \) such that

\[
\frac{1}{K_2} \leq \frac{|\text{Im} \, z|}{|\text{Im} \, \phi(z)|} \leq K_2 \tag{4.49}
\]

for all \( z \in W \). Applying this with \( z = \varphi^{-1}(\zeta) \) for \( \zeta \in W^* \) and using (4.47), we deduce that there exists \( K_3 > 1 \) depending only on \( M \) such that

\[
\frac{1}{K_3} \leq \frac{\rho_{\mathbb{D}}(\zeta)}{\rho_{\mathbb{D}}(\psi(\zeta))} \leq K_2
\]

for all \( \zeta \in W^* \). This shows that the inequality (4.13) in the hypothesis of Proposition 4.9 is satisfied for \( \psi \). Moreover, we have for each \( \zeta \in \varphi(U^+) \) we have \( \Delta_\zeta \subset W^* \), and so, in the notation introduced before, \( m_\psi(\zeta) \leq \|\psi|_{W^*}\|_{C^2} \leq K_4 \), where \( K_4 > 0 \) is a constant that depends only on \( M \). Hence all the hypotheses of Proposition 4.9 are satisfied by \( \psi \). It follows that, for each \( 0 < \theta < 1 \), there exists a constant \( K_\theta \) depending only on \( \theta \) and \( M \) such that

\[
J^h_\psi(\zeta) \leq 1 + K_\theta \delta^{1-\theta}, \tag{4.50}
\]

for all \( \zeta \in \varphi(U^+) \). Combining (4.50) with the general upper estimate in (4.47) (for \( \psi \)), we see that for each \( 0 < \theta < 1 \) there exists a constant \( C_\theta > 0 \) depending only on \( \theta \) and \( M \) such that

\[
\frac{|D\psi(\zeta)w|_\mathbb{D}}{|w|_\mathbb{D}} \leq 1 + C_\theta \delta^{1-\theta}, \tag{4.51}
\]

for all \( \zeta \in \varphi(U^+) \) and each non-zero tangent vector \( w \in T_\zeta \mathbb{D} \). Putting (4.51) together with (4.48) for \( z = \varphi^{-1}(\zeta) \in U^+ \) and \( v = D\varphi^{-1}(\zeta)w \in T_z Y^+ \), we deduce the upper estimate in (4.45), as desired.

\textbf{Case (ii).} If \( \phi \) is asymptotically holomorphic (near the real axis) then so is \( \psi \) (near the boundary of the unit disk). Verifying the hypotheses of Proposition 4.14 for \( \psi \) in this case is similar to what was done in case (i), hence we omit the details.

\[\square\]
Remark 4.16. In the application we have in mind, namely Theorem 5.4 below, the diffeomorphism $\phi$ will be the asymptotically holomorphic diffeomorphism appearing in the Stoilow decomposition of a high renormalization of an (infinitely renormalizable) AHPL-map. For such maps, we can always assume that the constant $b_0$ appearing in assertion (ii) is equal to one. The reason for this is embedded in the proof of a slightly improved version of the complex bounds (see Theorem 3.2 (iv)).

5. Recurrence and expansion

This section contains a crucial step towards the proof of our Main Theorem (as stated in the introduction), namely Theorem 5.4 below. We show that every AHPL-map arising as a deep renormalization of an infinitely renormalizable $C^r$ unimodal map with bounded combinatorics expands the hyperbolic metric of its co-domain minus the real axis. From this we deduce a few basic properties concerning the global dynamics of these AHPL-maps – such as the fact that all of their periodic points are expanding. The expansion property proved here will lead to much stronger results in §6, including, of course, the proof of the Main Theorem.

5.1. Controlled AHPL-maps. In order to establish the desired expansion property, we need to assume that our AHPL-maps satisfy certain geometric constraints. We call such maps controlled AHPL-maps. These geometric constraints may seem artificial, but the point is that they are always verified once we renormalize a given AHPL-map a sufficient number of times.

Let us proceed with the formal definition. First, we need some notation. Given $z = x + iy \in \mathbb{C} \setminus \mathbb{R}$ and $\alpha > 1$, let $z^\alpha = x + i\alpha y$.

**Definition 5.1.** Let $\alpha, M > 1$ and $0 < \delta, \theta < 1$ be real constants, and let $n_0 \in \mathbb{N}$. An AHPL-map $f : U \rightarrow V$ of class $C^r$, $r \geq 3$, is said to be $(\alpha, \delta, \theta, M, n_0)$-controlled if the following conditions are satisfied.

(i) We have $\text{diam}(V) \leq M$ and $\text{mod}(V \setminus U) \geq M^{-1};$

(ii) If $f = \phi \circ g$ is the Stoilow decomposition of $f$, with $\phi : V \rightarrow V$ a $C^r$-diffeomorphism and $g : U \rightarrow V$ holomorphic, then $\|\phi\|_{C^2}, \|\phi^{-1}\|_{C^2} \leq M;$

(iii) $\phi$ is $(1 + \delta)$-quasiconformal on $V$;

(iv) The dilatation $\mu_\phi$ satisfies $|\mu_\phi(z)| \leq M|\text{Im} z|^{r-1};$

(v) For all $z \in U_\alpha = U \cap \{w : |\text{Im} w| \leq (\alpha M)^{-1}\}$, we have $D(z_\alpha, |\text{Im} z_\alpha|) \subset Y = V \setminus \mathbb{R};$

(vi) For all $z \in U \setminus \mathbb{R}$ we have $M^{-1} \leq |\text{Im} z|/|\text{Im} \phi(z)| \leq M$, as well as $M^{-1} \leq \rho_Y(z)/\rho_Y(\phi(z)) \leq M$;

(vii) We have

$$\Phi(\text{diam}_Y(U \setminus U_\alpha) + 2n_0 \log M) < 1 - C_\theta \delta^{1-\theta},$$

where $\Phi$ is McMullen’s universal function (4.4) and $C_\theta = C_\theta(M)$ is the constant appearing in Theorem 4.15 (i).

**Remark 5.2.** It is possible to prove, with the help of Lemma 4.1 and the Riemann mapping theorem, that $\text{diam}_Y(U \setminus U_\alpha) \leq C + \log \alpha$ for some positive constant $C = C(M)$.

The following result is a straightforward consequence of the complex bounds, as given by Theorem 3.2 together with the $C^2$ bounds, as given by Theorem 3.1 and Remark 3.5.
Theorem 5.3. For each positive integer $N$ there exists $M = M(N) > 1$ such that the following holds. Let $f : U \to V$ be an AHPL-map of class $C^r$, $r \geq 3$, whose restriction to the real line is an infinitely renormalizable unimodal map with combinatorics bounded by $N$. Then for each $\alpha > 1$ and $0 < \theta < 1$ and each $n_0 \in \mathbb{N}$, there exist $0 < \delta < 1$ and $n_1 = n_1(f, \alpha, \theta, n_0) \in \mathbb{N}$ such that, for all $n \geq n_1$, the $n$-th renormalization $R^n f : U_n \to V_n$ is an $(\alpha, \delta, \theta, M, n_0)$-controlled AHPL map.

Now, we have the following main theorem.

Theorem 5.4. Given $M > 1$, $r > 3$ and $0 < \theta < 1$ so small that $(r - 1)(1 - \theta) > 2$, there exists $\alpha_0 > 1$ such that the following holds for all $\alpha > \alpha_0$. Let $f : U \to V$ be an AHPL-map of class $C^r$ and assume that $f$ is $(\alpha, \delta, \theta, M, n_0)$-controlled for some $0 < \delta < 1$ and some $n_0 \in \mathbb{N}$. Suppose also that $r$, $\alpha$, $\theta$ and $n_0$ are such that

$$r > 1 + \frac{4n_0 \alpha}{(n_0 - 1)(1 - \theta)(2\alpha - 1)}.$$  \hspace{1cm} (5.1)

Then the following assertions hold true.

(a) There exists a constant $0 < \eta < 1$ such that $|Df^n(z)|_Y \geq \eta |v|_Y$, for all $z \in Y \cap U$ such that $f^i(z) \in Y$ for $0 \leq i \leq n$ and all $v \in T_z Y$.

(b) If $z$ is a point in the filled-in Julia set of $f$ and its $\omega$-limit set is not contained in the real axis, we have $|Df^n(z)|_Y / |v|_Y \to \infty$ as $n \to \infty$, for each non-zero tangent vector $v \in T_z Y$.

(c) Every periodic orbit of $f$ is expanding.

(d) The expanding periodic points are dense in the set of all recurrent points.

Proof. First we give an informal description of the argument. For a suitable constant $0 < \lambda < 1$, we partition the domain of $f = \phi \circ g$ into a sequence of scales, the $n$-th scale being the set of points in the domain (off the real axis) whose distance to the real axis is of the order $\lambda^n$. The rough idea then is that at each level the worst expansion of the hyperbolic metric of $Y$ by $g$ beats the best contraction of that metric by $\phi$. In this, we are aided by Theorem 4.15 and Lemma 4.3. We warn the reader that, in what follows, whenever invoking Theorem 4.15 we denote by $C_\theta$ the largest of the two constants with that name appearing in assertions (i) and (ii) of said theorem.

Let us now present the formal proof. Let us assume we are given a large number $\alpha > 1$. How large $\alpha$ must be will be determined in the course of the argument.

To start with, note that by (4.2) in Lemma 4.2 we have, for all $z \in U_1$,

$$\frac{1}{|\text{Im } z|} \leq \rho_Y(z) \leq \frac{1}{|\text{Im } z|} \left(1 - \frac{1}{2\alpha}\right)^{-1}. \hspace{1cm} (5.2)$$

Let us fix for the time being a real number $0 < \lambda < 1$, which we will use to define the scales we mentioned above. For definiteness, we take $\lambda = M^{-1}$. For each $n \geq 1$ we define

$$W_n = \left\{ z \in U_1 : \frac{\lambda^n}{\alpha M} \leq |\text{Im } z| < \frac{\lambda^{n-1}}{\alpha M} \right\}.$$

Also, we set $W_0 = U \setminus U_1 \subset Y$. Then we have, of course, $U \setminus \mathbb{R} = \bigcup_{n=0}^{\infty} W_n$. 
Claim. There exists a sequence of numbers \( \xi_n > 1, \ n \geq 0 \), with \( \xi_n \to 1 \) as \( n \to \infty \), having the following property: For each \( z \in W_n \) and each tangent vector \( v \in T_z Y \), we have
\[
|D(g \circ \phi)(z)v|_Y \geq \xi_n |v|_Y . \tag{5.3}
\]

Proof of Claim. In order to prove this claim, we analyse separately the expansion of the conformal map \( g \) and the (possible) contraction of the quasi-conformal diffeomorphism \( \phi \). We proceed through the following steps.

(i) Let \( X \subset Y \) be the open set containing \( \phi(z) \) such that \( g \) maps \( X \) univalently onto \( Y \). Writing \( w = D\phi(z)v \in T_{\phi(z)}Y \), and applying Lemma 4.3 together with the estimate (4.5), we deduce that
\[
|Dg(\phi(z))w|_Y \geq \left(1 + \frac{1}{3}e^{-2s_{X,Y}(\phi(z))}\right)|w|_Y . \tag{5.4}
\]

Now we need to estimate \( s_{X,Y}(\phi(z)) \).

(ii) Let us write \( p = \phi(z) = x + iy \) and let \( q = x + i(\alpha M)^{-1}\frac{y}{|y|} \in U \setminus U_\alpha \), which lies in the same vertical as \( p \). There are two cases to consider:

(1) We have \( p \in X \) but \( q \notin X \). In this case, we have \( d_Y(p, Y \setminus X) \leq d_Y(p, q) \).

Using (5.2), we get
\[
s_{X,Y}(\phi(z)) \leq d_Y(p, q) \leq \left(1 - \frac{1}{2\alpha}\right)^{-1} \log \frac{1}{|\text{Im} \phi(z)|} .
\]

But by property (vi) of Definition 5.1 we have \( |\text{Im} \phi(z)| \geq M^{-1} \lambda^n(\alpha M)^{-1} \).

Hence
\[
s_{X,Y}(\phi(z)) \leq \left(1 - \frac{1}{2\alpha}\right)^{-1} \left[n \log \frac{1}{\lambda} + \log M \right]. \tag{5.5}
\]

(2) We have \( p \in X \) and \( q \in X \). In this case we have
\[
d_Y(p, Y \setminus X) \leq d_Y(p, q) + d_Y(q, Y \setminus X) \leq d_Y(p, q) + \text{diam}_Y(U \setminus U_\alpha).
\]

Therefore
\[
s_{X,Y}(\phi(z)) \leq C_\alpha + \left(1 - \frac{1}{2\alpha}\right)^{-1} \left[n \log \frac{1}{\lambda} + \log M \right], \tag{5.6}
\]

where \( C_\alpha = \text{diam}_Y(U \setminus U_\alpha) \).

Whichever case occurs, we see that (5.6) always holds. Combining these facts with (5.4) we deduce that
\[
|Dg(\phi(z))w|_Y \geq \left(1 + K_1 \lambda^{2n(1-\frac{1}{2\alpha})} \right)|w|_Y , \tag{5.7}
\]

where \( K_1 = K_1(\alpha, M) \) is the constant given by
\[
K_1 = \frac{1}{3}e^{-2C_\alpha} \exp \left\{ -2 \left(1 - \frac{1}{2\alpha}\right)^{-1} \log M \right\} < 1 . \tag{5.8}
\]
This gives us a lower bound on the amount of expansion of the hyperbolic metric of $Y$ by the conformal map $g$ for points at level $n$.

(iii) Let us now bound the amount of contraction of the hyperbolic metric by the quasi-conformal diffeomorphism $\phi$ at $z \in W_n$. First we assume that $n \geq n_0$. Applying Theorem $4.15(ii)$, we have for all $v \in T_z Y$ the estimate
\[
|D\phi(z)v|_Y \geq (1 - C_\theta \Im z)^{(r-1)/(1-\theta)} |v|_Y,
\]
But since $z \in W_n$, we know that $|\Im z| \leq (\alpha M)^{-1} \lambda^{n-1}$. Carrying this information back into (5.9), we deduce that
\[
|D\phi(z)v|_Y \geq (1 - K_2 \lambda^{(n-1)(r-1)/(1-\theta)}) |v|_Y,
\]
where $K_2 = K_2(\alpha, \theta, r, M)$ is the constant given by
\[
K_2 = C_\theta (\alpha M)^{(1-r)/(1-\theta)}.
\]

(iv) Note that both constants $K_1$ and $K_2$ depend on $\alpha$. We claim that the ratio $K_2/K_1$ goes to zero as $\alpha \to \infty$. From (5.8) and (5.11), we see that
\[
\frac{K_2}{K_1} < C_1 e^{2C_\alpha (1-r)(1-\theta)},
\]
where $C_1 = 3C_\theta M^{(1-r)/(1-\theta)} M^4$ is independent of $\alpha$. By Remark 5.2, we have $C_\alpha < C_2 + \log \alpha$, for some constant $C_2$ depending only on $M$. Hence
\[
\frac{K_2}{K_1} < C_3 e^{2(1-r)(1-\theta)},
\]
where $C_3 = C_1 e^{2C_2}$. Since by hypothesis $(r-1)(1-\theta) > 2$, it follows that the right-hand side of (5.12) indeed goes to zero as $\alpha \to \infty$. Hence we assume from now on that $\alpha$ is so large that $2K_2 < K_1$.

(v) Thus, if for each $n \geq n_0$ we let $\xi_n$ be given by
\[
\xi_n = \left(1 + K_1 \lambda^{2(n-1)(1-\frac{1-\theta}{\alpha})}\right) (1 - K_2 \lambda^{(n-1)(r-1)/(1-\theta)}),
\]
then we have $|D(g \circ \phi)(z)v|_Y \geq \xi_n |v|_Y$ for all $z \in W_n$ and each $v \in T_z Y$. Note that $\xi_n \to 1$ as $n \to \infty$, because $\lambda < 1$. We still need to check that $\xi_n > 1$ for all $n \geq n_0$. This will be true provided
\[
K_1 \lambda^{2(n-1)(1-\frac{1}{\alpha})} > 2K_2 \lambda^{(n-1)(r-1)/(1-\theta)},
\]
for all $n \geq n_0$. Note that both sides of (5.14) are indeed smaller than 1, because from (5.8) and step (iv) we have $2K_2 < K_1 < 1$, and $\lambda < 1$. Extracting logarithms from both sides of (5.14), we get
\[
2n \left(1 - \frac{1}{2\alpha}\right)^{-1} \log \lambda > (n-1)(r-1)(1-\theta) \log \lambda + \log (2K_1^{-1} K_2).
\]
Dividing both sides of the above inequality by $(n-1)(1-\theta) \log \lambda < 0$, we arrive at
\[
r > 1 + \frac{2n}{(n-1)(1-\theta) \left(1 - \frac{1}{2\alpha}\right)} + \frac{\log (2K_1^{-1} K_2)}{(n-1)(1-\theta) \log \frac{1}{\lambda}}.
\]
But since $2K^{-1}K_2 < 1$ (by our choice of $\alpha$ at the end of step (iv)), the third term on the right-hand side of (5.15) is negative and therefore can be safely ignored. Moreover, since $n \geq n_0$ we have $2n/(n-1) \leq 2n_0/(n_0-1)$. Therefore the inequality \[ r > 1 + \frac{2n_0}{(n_0-1)(1-\theta)} \left( 1 - \frac{1}{2\alpha} \right) \] will hold for all $n \geq n_0$ provided

$$r > 1 + \frac{2n_0}{(n_0-1)(1-\theta)} \left( 1 - \frac{1}{2\alpha} \right).$$

But this is nothing but (5.1) in disguise! Hence we have established that the $\xi_n$’s given by (5.13) satisfy $\xi_n > 1$, for all $n \geq n_0$.

(vi) In order to establish the claim, it remains to analyse what happens when $z \in W_0 \cup W_1 \cup \cdots \cup W_{n_0-1}$. On the one hand, since $\phi$ is $(1+\delta)$-quasiconformal throughout, applying Theorem 4.15 for such $z$ and any $v \in T_z Y$ yields the lower bound

$$|D\phi(z)v|_Y \geq (1 - C_\theta \delta^{1-\theta}) |v|_Y. \quad (5.16)$$

On the other hand, using the estimate (5.6) above with $n = n_0$ we deduce that

$$s_{X,Y}(\phi(z)) \leq C_\alpha + 2(n_0 - 1) \log \frac{1}{\lambda} + 2 \log M = C_\alpha + 2n_0 \log M.$$

Therefore, by McMullen’s Lemma 4.3, we have for all $w \in T_{\phi(z)} Y$,

$$|Dg(\phi(z))w|_Y \geq \Phi(s_{X,Y}(\phi(z))^{-1}) |w|_Y \geq \Phi \left( C_\alpha + 2n_0 \log M \right)^{-1} |w|_Y. \quad (5.18)$$

Combining (5.16) and (5.17) (with $w = D\phi(z)v$), we deduce that

$$|D(\phi \circ \phi)(z)v|_Y \geq \Phi \left( C_\alpha + 2n_0 \log M \right)^{-1} (1 - C_\theta \delta^{1-\theta}) |v|_Y. \quad (5.18)$$

Hence we can take

$$\xi_0 = \xi_1 = \cdots = \xi_{n_0-1} = \Phi \left( C_\alpha + 2n_0 \log M \right)^{-1} (1 - C_\theta \delta^{1-\theta}) > 1.$$

This establishes (5.3) for all $z \in W_n$, for all $n \geq 0$, and completes the proof of our claim.

With the Claim at hand, we proceed to the proof of the assertions in the statement of our theorem. Let $z \in K_f$ be a point whose iterates up to time $n > 1$ stay off the real axis – in other words, $f^i(z) \in Y$ for all $0 \leq i \leq n$. Note that, since $f = \phi \circ g$, we have $f^n = \phi \circ (g \circ \phi)^{n-1} \circ g$. Write $z_1 = g(z)$ and define inductively $z_{j+1} = g \circ \phi(z_j)$, for $j = 1, \ldots, n-1$. Then for each non-zero tangent vector $v \in T_z Y$, we have by the chain rule

$$Df^n(z)v = D\phi(z_n) \left[ \prod_{j=1}^{n-1} Dg(\phi(z_j))D\phi(z_j) \right] Dg(z)v. \quad (5.19)$$

Now, since the holomorphic map $g$ expands the hyperbolic metric of $Y$, we have that $|Dg(z)v|_Y > |v|_Y$. Moreover, the amount of possible contraction of the hyperbolic metric by the $(1+\delta)$-quasiconformal diffeomorphism $\phi$ is bounded from below. Indeed, we have $|D\phi(\zeta)w|_Y \geq (1 - C_\theta \delta^{1-\theta})|w|_Y$ for all $\zeta \in Y$ and all $w \in T_\zeta Y$. Moreover, writing $v_1 =$
$Dg(z)v \in T_zY$ and $v_{j+1} = D(g \circ \phi)(z_j)v_j \in T_{z_{j+1}}Y$ for $j = 1, \ldots, n - 1$, and applying the above Claim, we get

$$|v_{j+1}|_Y = |D(g \circ \phi)(z_j)v_j|_Y \geq \xi_{k_j}|v_j|_Y,$$

where $k_j \geq 0$ is the unique integer such that $z_j \in W_{k_j}$. Setting $\eta = 1 - C_\theta \delta^{1-\theta} < 1$ and carrying these facts back into (5.19), we deduce that

$$|Df^n(z)v|_Y > \eta \left[ \prod_{k=1}^{\infty} \xi_{k_{N_{k,n}}(z)}^N \right] |v|_Y,$$

(5.20)

where $N_{k,n}(z)$ is the total number of $j$’s in the range $1 \leq j \leq n - 1$ such that $z_j \in W_k$ (in particular, the product appearing in the right-hand side is actually finite). This proves assertion (a). Now suppose that $z$ is such that its $\omega$-limit set accumulates at a point off the real axis, say $p \in Y$. This is the case, for instance, if $z$ is a recurrent or periodic point for $f$. Then there exist $k \geq 0$ and a sequence $j_\nu \to \infty$ such that $z_{j_\nu} \to p$ as $\nu \to \infty$ and $z_{j_\nu} \in W_k$ for all $\nu$. But this tells us that $N_{k,n}(z) \to \infty$ as $n \to \infty$, and therefore, from (5.20), we deduce at last that $|Df^n(z)v|_Y/|v|_Y \to \infty$ as $n \to \infty$. This proves the desired expansion property stated in assertion (b), and it also proves assertion (c). Hence it remains to prove assertion (d).

Let $z \in Y \cap \mathcal{K}_f$ be a recurrent point. Let $N \geq 1$ be such that $|Df^N(z)v|_Y \geq 3\eta^{-1}|v|_Y$ for all $v \in T_zY$, where $\eta$ is the constant of assertion (a). Such an $N$ exists because of assertion (b). By continuity of $\zeta \mapsto Df^N(\zeta)$, we can find $\epsilon_0 > 0$ such that $|Df^N(\zeta)v|_Y \geq 2\eta^{-1}|v|_Y$ for all $\zeta \in B_Y(z, \epsilon_0)$ and each $v \in T_zY$. Now, given $0 < \epsilon < \frac{1}{4} \eta \epsilon_0$, choose $m > N$ such that $f^m(z) \in B_Y(z, \epsilon)$; this is possible because $z$ is recurrent. Write $\mathcal{O} = B_Y(f^m(z), 2\epsilon) \subset B_y(z, \epsilon_0)$, and let $\mathcal{O}' \subset Y$ be the component of $f^{-m}(\mathcal{O})$ that contains $z$. Then $f^m|_{\mathcal{O}'} : \mathcal{O}' \to \mathcal{O}$ is a diffeomorphism. By assertion (a), the inverse diffeomorphism $f^{-m}|_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}'$ is Lipschitz with constant $\eta^{-1}$ in the hyperbolic metric of $Y$. Therefore

$$\mathcal{O}' \subset B_Y(z, \eta^{-1}, (2\epsilon)) \subset B_Y(z, \epsilon_0).$$

Now that we know this fact, writing $f^m = f^{m-N} \circ f^N$ we see that, for all $\zeta \in \mathcal{O}'$ and each non-zero $v \in T_\zeta Y$,

$$\frac{|Df^m(\zeta)v|_Y}{|v|_Y} = \frac{|Df^{m-N}(f^N(\zeta))Df^N(\zeta)v|_Y}{|Df^N(\zeta)v|_Y} \cdot \frac{|Df^N(\zeta)v|_Y}{|v|_Y} \geq \eta \cdot (2\eta^{-1}) = \frac{1}{2}.$$

Equivalently, we have shown that $|Df^{-m}(\zeta)v|_Y \leq \frac{1}{2}|v|_Y$ for all $\zeta \in \mathcal{O}$ and each $v \in T_\zeta Y$. In other words, $f^{-m}|_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}'$ is, in fact, a contraction of the hyperbolic metric of $Y$, with contraction constant $\frac{1}{2}$. In particular,

$$\mathcal{O}' = f^{-m}|_{\mathcal{O}}(\mathcal{O}) \subset B_Y(z, \epsilon) \subset B_Y(f^m(z), 2\epsilon) = \mathcal{O}.$$

This means that $f^{-m}|_{\mathcal{O}}$ maps the hyperbolic ball $\mathcal{O}$ strictly inside itself (and it is a contraction of the hyperbolic metric). Hence there exists $z_\ast \in \mathcal{O}'$ such that $f^m(z_\ast) = z_\ast$, and this periodic point is necessarily expanding, by assertion (c). Thus, we have proved that for each $\epsilon > 0$ there exists an expanding periodic point $\epsilon$-close to $z$. This establishes assertion (d) and completes the proof of our theorem.

□
It is worth pointing out that, combining Theorem [5.4] with Theorem [5.3], we already deduce the following simple properties of the dynamics of all sufficiently deep renormalizations of a given AHPL-map. Considerably stronger results will be proved in §6 below.

**Corollary 5.5.** Let $f : U \to V$ be an AHPL-map of class $C^r$, with $r > 3$, whose restriction to the real line is an infinitely renormalizable unimodal map with bounded combinatorics. There exists $n_1 = n_1(f) \in \mathbb{N}$ such that, for all $n \geq n_0$, the $n$-th renormalization $f_n = R^a f : U_n \to V_n$ is an AHPL-map with the following properties.

(a) Every periodic orbit of $f_n$ is expanding.

(b) The expanding periodic points are dense in the set of all recurrent points.

(c) There are no stable components of $\text{int}(K_{f_n})$ whose closures intersect the real axis.

**Proof.** Choose $0 < \theta < 1$, as well as $n_0 \in \mathbb{N}$ and $\alpha > 1$ large enough so that (5.1) holds true. This is possible because $r > 3$. Then, by Theorem [5.3], there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, the $n$-th renormalization $f_n$ of $f$ is an $(\alpha, \delta, \theta, M, n_0)$-controlled AHPL map, for some $0 < \delta < 1$. Hence assertions (a) and (b) follow from the corresponding assertions in Theorem [5.4]. To prove (c), suppose $\Omega \subset Y_n = V_n \setminus \mathbb{R}$ is a stable component of $\text{int}(K_{f_n})$ such that $\Omega \cap \mathbb{R} = \emptyset$. Let $p \geq 1$ be such that $f_n^p(\Omega) = \Omega$. Also, consider the decomposition of the domain of $f_n$ into scales as in Theorem [5.4]. Since $\Omega \subset U_n \setminus \mathbb{R} \subset Y_n$ is compact, it is contained in the union of finitely many scales. In each scale $f_n$ expands the hyperbolic metric of $Y_n$ by a definite amount. Hence so does $f_n^p$ on $\Omega$. But this is impossible, because $\Omega$ has finite hyperbolic area. 

6. **Topological conjugacy to polynomials and local connectivity of Julia sets**

In this section, we will prove that a $(\alpha, \delta, \theta, M, n_0)$-controlled AHPL-mapping $f : U \to V$, which is infinitely renormalizable of bounded type, is topologically conjugate to a real polynomial in a neighbourhood of its filled Julia set, so that from the topological point of view, the dynamics of these mappings are the same as those of polynomials; in particular, such mappings do not have wandering domains. We will also prove that the Julia set of such an AHPL-mapping is locally connected. Specifically, we will assume that $f$ satisfies the conditions of Theorem [5.4]. In particular, we assume that $f : U \to V$ is a $C^r$ asymptotically holomorphic polynomial-like mapping that is $(\alpha, \delta, \theta, M, n_0)$-controlled,

$$r > 1 + \frac{4n_0\alpha}{(n_0 - 1)(1 - \theta)(2\alpha - 1)},$$

and that the conclusions of Theorem [5.4] all hold. By Theorems [3.2] and [5.3], for any $r > 3$, if $g$ is a $C^r$ mapping of the interval, which is infinitely renormalizable of bounded type, then for any $n$ sufficiently large, there is a renormalization, $R^ng : U_n \to V_n$ of $g$, which is an AHPL-mapping that satisfies these assumptions.

6.1. **Dilatation and expansion.** The proof of the following lemma is implicit in the proof of Theorem [5.4]; it makes the lower bound in Equation (5.3) explicit.

**Lemma 6.1.** Let $\xi_n$ be the constant defined in Equation (5.13). There exists $N \geq n_0$ such that if $n \geq N$, then

$$1 + M(\frac{\lambda^{n-1}}{\alpha M})^{r-1} \leq \xi_n.$$  

(6.1)
Proof. It is sufficient to show that
\[ M \left( \frac{\lambda^{n-1}}{\alpha M} \right)^{r-1} \leq K_1 \lambda^{2(n-1)} \left(1 - \frac{1}{2\alpha} \right)^{n-1} - 2K_2 \lambda^{(n-1)(r-1)(1-\theta)}, \]
see Equation (5.14). Factoring out \( \lambda^{(n-1)(r-1)} \) on the right and cancelling it with the same term on the left, this is equivalent to:
\[ \frac{M}{(\alpha M)^{r-1}} \leq K_1 \lambda^{2n(1-\frac{1}{2\alpha})-1} - (n-1)(r-1) - 2K_2 \lambda^{-\theta(n-1)(r-1)}. \] (6.2)
Since \( n > n_0 \), we have that
\[ 4n_0\alpha(n-1) - 4n\alpha(n_0 - 1) = 4\alpha(n_0(n-1) - n(n_0 - 1)) > 0 \] (6.3)
Now, since
\[ r \geq 1 + \frac{4n_0\alpha}{(n_0 - 1)(1-\theta)(2\alpha - 1)}, \]
we have that
\[ (r - 1)(1-\theta)(n-1) \geq \frac{4n_0\alpha(n-1)}{(n_0 - 1)(2\alpha - 1)}. \] (6.4)
So
\[ (r - 1)(1-\theta)(n-1) - 2n(1-\frac{1}{2\alpha})^{-1} \geq \frac{4n_0\alpha(n-1)}{(n_0 - 1)(2\alpha - 1)} - 2n \frac{2\alpha}{2\alpha - 1} \]
\[ = \frac{4n_0\alpha(n-1) - 4n\alpha(n_0 - 1)}{(n_0 - 1)(2\alpha - 1)} > 0, \]
where the first inequality follows from (6.4) and the last inequality follows from (6.3). Thus we have
\[ 2n(1-\frac{1}{2\alpha})^{-1} - (n-1)(r-1) \leq -\theta(n-1)(r-1), \]
since both exponents on the right hand side of (6.2):
\[ 2n(1-\frac{1}{2\alpha})^{-1} - \theta(n-1)(r-1) \]
are negative, equation (6.1) holds for \( n \) sufficiently large. \( \Box \)

Let
\[ K_{f^n}(z) = \frac{1 + |\mu_{f^n}(z)|}{1 - |\mu_{f^n}(z)|}, \]
be the quasiconformal distortion of \( f^n \) at \( z \). A chain of domains is a sequence of domains \( \{B_j\}_{j=0}^n \) where \( B_j \) is a component of \( f^{-1}(B_{j+1}) \) for all \( j = 0, 1, 2, \ldots, n-1 \) and \( B_n \) is a domain in \( \mathbb{C} \). To a mapping \( f^n : A \to B \), we associate the chain of domains \( \{B_j\}_{j=0}^n \) where \( B_n = B \) and \( B_j = \text{Comp}_{f(B)} f^{-(n-j)}(B) \) for \( j = 0, \ldots, n-1 \).

Recall that \( W_k \) is the strip
\[ W_k = \left\{ z \in U_{\alpha} : \frac{\lambda^k}{\alpha M} \leq |\text{Im} z| < \frac{\lambda^{k-1}}{\alpha M} \right\}. \]
Corollary 6.2. For each $N \in \mathbb{N}$ there exists $c > 0$ such that the following holds. Let $A$ be an open domain in $\mathbb{C}$. Suppose that $f^n : A \to B$ is onto and let $\{B_j\}_{j=0}^n$ be the chain with $B_0 = A$ and $B_n = B$. Assume that for each $0 \leq j \leq n$ that
\[
\#\{k : B_j \cap W_k \neq \emptyset\} \leq N.
\]
Then
\[
c \cdot \sup_{z \in A} \log K_{f^n}(z) \leq \inf_{z \in A} \log |Df^n(z)|_Y,
\]
for each unit tangent vector $v \in T_Y$.

Proof. Let us express $f^n : B_0 \to B_n$ as $\phi \circ (g \circ \cdots \circ g \circ \phi) \circ g$. For each $0 \leq j < n$, $g : B_j \to \phi^{-1}(B_{j+1})$. Since $\phi$ is a $(1 + \varepsilon(\delta))$-quasi-isometry in the hyperbolic metric on $Y$ where $\varepsilon(\delta) \to 0$ as $\delta \to 0$, we have that there exists $N_1$, depending only on $N$, so that $\phi^{-1}(B_j)$ intersects at most $N_1$ strips $W_k$.

For each $B_j$, let $n_j$ be minimal so that $\phi^{-1}(B_j) \cap W_{n_j} \neq \emptyset$. Then for any $g(z) \in \phi^{-1}(B_j)$, $1 \leq j < n$, we have that
\[
\left| \frac{\partial (g \circ \phi)}{\partial (g \circ \phi)}(g(z)) \right| = \left| \frac{\partial \phi}{\partial \phi}(g(z)) \right| \leq M \left( \frac{\lambda^{n_j-1}0}{\alpha M} \right)^{r-1}.
\]

By equation (5.3) and Lemma 6.1, we have that for all $v \in T_Y$, with $|v|_Y = 1$,
\[
|D(g \circ \phi)(z)|_Y \geq 1 + M \left( \frac{\lambda^{n_j-1}0}{\alpha M} \right)^{r-1} = 1 + M \lambda^{N_1}(r-1) \left( \frac{\lambda^{n_j-1}}{\alpha M} \right)^{r-1},
\]
so that
\[
|D(g \circ \phi)(z)|_Y \geq 1 + \lambda^{N_1(r-1)} \sup_{z \in B_j} \left| \frac{\partial (g \circ \phi)}{\partial (g \circ \phi)}(g(z)) \right| |v|_Y.
\]
Thus we have that
\[
\inf_{z \in B_j} |D(g \circ \phi)(z)|_Y \geq 1 + \lambda^{N_1(r-1)} \sup_{z \in B_j} \left| \frac{\partial (g \circ \phi)}{\partial (g \circ \phi)}(g(z)) \right| |v|_Y.
\]

For each $i$, let
\[
k_i = \#\{j : B_j \cap W_i \neq \emptyset\}, \text{ and for all } i' < i, B_{i'} \cap W_i = \emptyset,
\]
and let us reindex the $B_j$ as follows: For each $i \in \mathbb{N} \cup \{0\}$, let $B_{i_0}, \ldots, B_{i_{k_i}}$ be an enumeration of all $B_j$ so that $B_j \cap W_i \neq \emptyset$ and for all $0 \leq i' < i$, $B_{i'} \cap W_i = \emptyset$. Notice that $n = \sum_{i=0}^{\infty} k_i$.

By the chain rule and Theorem 3.1 we have that there exists a constant $c_1 > 0$ so that
\[
\inf_{z \in B_0} |Df^n(z)|_Y \geq c_1 \prod_{i=0}^{\infty} \prod_{j=0}^{k_i} \left( 1 + \lambda^{N_1(r-1)} \sup_{z \in B_{ij}} |\mu_f(z)| \right)
\]
Now, there exists a constant $c_2 > 0$ such that
\[
\log \prod_{i=0}^{\infty} \prod_{j=0}^{k_i} \left( 1 + \lambda^{N_1(r-1)} \sup_{z \in B_{ij}} |\mu_f(z)| \right) = \sum_{i=0}^{\infty} \sum_{j=0}^{k_i} \log \left( 1 + \lambda^{N_1(r-1)} \sup_{z \in B_{ij}} |\mu_f(z)| \right)
\]
\[ \geq c_2 \sum_{i=0}^{\infty} \sum_{j=0}^{k_i} \lambda^{N_i(r-1)} \sup_{z \in B_{i,j}} |\mu_f(z)| \]
\[ = c_2 \frac{\lambda^{N_i(r-1)}}{2} \sum_{i=0}^{\infty} \sum_{j=0}^{k_i} \left( \sup_{z \in B_{i,j}} \left( |\mu_f(z)| - (-|\mu_f(z)|) \right) \right) \]
\[ \geq c_2 \frac{\lambda^{N_i(r-1)}}{2} \sup_{\sum_{i=0}^{\infty} \sum_{j=0}^{k_i} \sup_{z \in B_{i,j}} \left( \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|} \right) } \]
\[ = c_2 \frac{\lambda^{N_i(r-1)}}{2} \log \prod_{i=0}^{\infty} \prod_{j=0}^{k_i} \sup_{z \in B_{i,j}} \left( \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|} \right). \]

Hence there exists a constant \( c \) so that,
\[ \inf_{z \in \partial B_0} \log |Df^n(z)|_Y \geq c \cdot \log \sup_{z \in B_0} K_{f^n}(z). \]

6.2. Puzzle pieces. Let us construct external rays for \( f \). These will allow us to construct Yoccoz puzzle pieces for \( f \) where the role of equipotentials is played by the curves \( f^{-i} \partial V \). To construct these rays, we use a method analogous to the one used by Levin-Przytycki in [37] to construct external rays for holomorphic polynomial-like maps.

First, we associate to \( f \) an external map, \( h_f \) as follows: Let \( X_0 = V \) and for \( i \in \mathbb{N} \), set \( X_{i+1} = f^{-1}(X_i) \). Notice that since \( U \in V \), \( f : U \to V \) is a branched covering of \( V \), ramified at a single point, \( 0 \), and \( f'(0) \in U \) for all \( i \), we have that \( X_i = f^{-i}(V) \) is a connected and simply connected topological disk for all \( i \in \mathbb{N} \cup \{0\} \), and \( X_{i+1} \subseteq X_i \). Let \( M = \mod(V \setminus K_f) \), and let
\[ \phi : D(0, e^M) \setminus \overline{D} \to V \setminus K_f \]
be the uniformization of \( V \setminus K_f \) by a round annulus. Let \( D_i = \phi^{-1}(X_i) \), we have that each annulus \( D_i \setminus D_{i+1} \) is mapped as a \( d \)-to-1 covering map onto \( D_{i-1} \setminus D_i \) by \( h_f = \phi^{-1} \circ f \circ \phi \). The mapping \( h_f \) extends continuously to \( \partial D \), and by Schwarz reflection, \( h_f \) can be defined as a mapping between annuli \( W' \subseteq W \), each with the same core curve, \( \partial D \). We have that \( h_f \) is a \( C^3 \) expanding mapping of \( S^1 \) (see the proof of [11] Lemma 10.17) and that the dilatation \( h_f \) on \( W' \) is the same as the dilatation of \( f \). Foliate \( W \setminus W' \) by \( C \), \( h_f \) invariant rays, connecting \( \partial W' \) and \( \partial W \). and pull them back by \( h_f \). We obtain a foliation by \( C^n \) rays of \( W' \setminus \partial D \) that is continuous on \( W' \). Pulling back this foliation of \( W' \) by \( \phi \), we obtain a foliation of \( V \setminus K_f \). The leaves of this foliation are the external rays of \( f \).

Remark 6.3. Observe that since \( h_f|S_1 \) is a degree \( d \) expanding mapping of the circle, it is topologically conjugate to \( z \mapsto z^d \) on a neighbourhood of \( S^1 \). Consequently, one can carry out this construction simultaneously for two mappings \( f : U \to V \) and \( \tilde{f} : \tilde{U} \to \tilde{V} \) to obtain a mapping \( H : V \to \tilde{V} \) such that \( H \circ F(z) = \tilde{F} \circ H(z) \) for any \( z \in U \) contained in an equipotential or ray.

For each \( z \in V \setminus K_f \), we let \( R_z \) denote the ray through \( z \). Let us parameterize \( R_z \) by \( R_z(t), t \geq 0 \), such that for each \( n \in \mathbb{N} \) we have that \( R_z(n) \) is the unique point on \( R_z \) that passes through \( \partial X_n \). We say that a ray \( R_z \) lands at a point \( p \) if \( \lim_{t \to \infty} R_z(t) = p \).
To prove that certain rays land, we will need the following lemma.

**Lemma 6.4.** [6] Lemma 2.3 Let $Ω \subset \mathbb{C}$ be a hyperbolic region. Let $γ_n : [0, 1] \to Ω$ be a family of curves with uniformly bounded hyperbolic length and such that $γ_n(0) \to ∂Ω$. Then $\text{diam}(γ_n) \to 0$.

**Lemma 6.5.** If $R_z$ accumulates on a real repelling periodic point $p$, then $R_z$ lands at $p$.

**Proof.** Compare [37] Lemma 2.1 and [6]. Suppose that $p$ is a real repelling periodic point of period $s$. Then one can repeat the proof of linearization near repelling periodic points of holomorphic maps to prove that there exists a neighbourhood $B$ of $p$ such that $f^s$ is conjugate to $z \mapsto λz$ near $p$, where $λ = Df^s(p)$, see [50].

Let $R_z([n-1,n])$ be the segment of the ray connecting $∂X_{n-1}$ and $∂X_n$. Let us show that $\text{diam}(R_z([n-1,n])) \to 0$ as $n \to ∞$. By Lemma 6.4 and since $φ$ is an isometry in the hyperbolic metric, it is sufficient to show that the curves $φ^{-1}(R_z([n-1,n]))$ have uniformly bounded hyperbolic lengths. This follows from the fact that $∥Dφ(z)∥ > 1$ in the hyperbolic metric for $z$ sufficiently close to $∂\mathbb{D}$, which was proved in the proof of [11] Lemma 10.17. Thus we have that $\text{diam}(R_z([n-1,n])) \to 0$ as $n \to ∞$. So there exists $n_0 \in \mathbb{N}$ such that for all $n ≥ n_0$, we have that $R_z([n,n+1]) \subset (f^s|_B)^{-s(n-n_0)}(B)$. Since $f^s|_B$ is qc-conjugate to $z \mapsto λz$ with $λ > 1$ in a neighbourhood of 0, we have that $\bigcap_{n=n_0}^{∞} (f^s|_B)^{-s(n-n_0)}(B) = \{p\}$. So the only accumulation point of the ray is $p$. □

We define puzzle pieces for $f$ as follows. Let us index the renormalizations $R^nf : U_n \to V_n$ of $f$ by $f_n : U_n \to V_n$, so that $f_n = f^{q_n}|_{U_n}$. Let $I_n = K_{f_n} \cap \mathbb{R}$ denote the invariant interval for $f_n$. Let $τ : I_0 \to I_0$ be the even, dynamical symmetry about the even critical point at 0. Let $β_n \in ∂I_n$ be the orientation preserving fixed point of $f_n$ in $∂I_n$. By realsymmetry, there exist two rays, labeled $R_{β_n}$ and $R'_{β_n}$ that land at $β_n$. Let $R_{τ(β_n)}$ and $R'_{τ(β_n)}$ denote the preimages under $f^{q_n}$ of $R_{β_n}$ and $R'_{β_n}$, respectively, which land at $τ(β_n)$. For each $n \in \mathbb{N}$, the initial configuration of puzzle pieces at level $n$ are the components of $V \setminus (R_{β_n} \cup R'_{β_n} \cup R_{τ(β_n)} \cup R'_{τ(β_n)} \cup \{β_n, τ(β_n)\})$. We denote this union of puzzle pieces by $Y^{(n)}_0$. Given an initial configuration, $Y^{(n)}_j$, for $j \in \mathbb{N} \cup \{0\}$, we define $Y^{(n)}_j$ to be the union of the connected components of $f^{-j}(Y^{(n)}_0)$. Given any $z \in K_f$, we let $Y^{(n)}_j(z)$ denote the component of $Y^{(n)}_j$ that contains $z$, and we let $Y^{(n)}_j(0)$ be the component that contains the critical point.

**Lemma 6.6.** For each $n \in \mathbb{N}$, there exists $j$, so that $K_{f_n} \subset Y^{(n)}_j \subset U_n$.

**Proof.** For all $j \in \mathbb{N}$, $K_{f_n} \subset \overline{Y}^{(n)}_j$. Let $q_n$ be the period of the renormalization $f_n$ of $f$. Let $K_j = \text{comp}_0 f^{-q_n}j(Y^{(n)}_0)$. Since $K_j \subset K_{j-1}$ and $f^{q_n} : K_j \to K_{j-1}$, and $\bigcap_{j=0}^{∞} K_j$ is a compact connected set, we have that $K_{f_n} \subset \bigcap_{j=0}^{∞} K_j \subset U_n$. □

**Proposition 6.7.** Suppose that $z \in K_f$. Then there exist arbitrarily small neighbourhoods $P$ of $z$ such that $P$ is a union of puzzle pieces.

**Proof.** Observe that Lemma 6.6 implies that there are arbitrarily small puzzle pieces containing the critical point of $f$. Let us start by spreading this information throughout the filled Julia set of $f$. Let $z \in K_f$. 
Case 1: Assume that $0 \in \omega(z)$. For each $n$, let $C_n \subset U_n$ be the puzzle piece given by Lemma 6.6. Let $r_n$ be minimal so that $f^{r_n}(x) \in C_n$ and let $C_0^n = \text{comp}_x f^{-r_n}(C_n)$. Each $C_n$ is contained in the topological disk, $\Gamma_n$, bounded by the core curve $\gamma_n$ of the annulus $V_n \setminus \overline{U}_n$. By Theorem 3.2 there exists $C > 0$ such that for all $n \in \mathbb{N}$, we have that $\text{mod}(V_n \setminus \overline{U}_n) \geq C^{-1}$. Thus the domain $\Gamma_n$ is a $K = K(C)$-quasidisk. Let $V_0^n = \text{Comp}_x f^{-r_n}(V_n)$, and $\Gamma_0^n = \text{Comp}_x f^{-r_n}(\Gamma_n)$. It is not hard to see that $f^{r_n} : V_0^n \rightarrow V_n$ is a diffeomorphism: Suppose that there exists $0 < j < r_n$ so that $f^j(V_0^n) \supseteq 0$, but $f^{j+1}(V_0^n)$ is not contained in $U_n$, so that $f^j(V_0^n) \cap \partial U \neq \emptyset$. Since $f^{s_n} : U_n \rightarrow V_n$ is a first return mapping to $V_n$, for all $k \in \mathbb{N}$, $f^{j+k s_n}(V_0^n)$ intersects both $K_{f_s}$ and $\partial V_n$, and we have that there exists no $j_1 \in \mathbb{N}$ such that $f^{j_1}(V_0^n) = V_n$. Thus we have that if for some $j$, $f^j(V_0^n) \supseteq 0$, then $f^j(V_0^n) \subset U_n$, but then since for all $k \in \mathbb{N}$, $f^{-k s_n}(C_n) \cap (V_n \setminus C_n) = \emptyset$, $j = r_n$, and so $f^{r_n} : V_0^n \rightarrow V_n$ is a diffeomorphism.

Case 1a: Suppose that $0 \in \omega(z)$ and $z \in \mathbb{R} \cap K_f$. Then, by the complex bounds, we have that there exists $K > 1$ for each $n$, the mapping $f^{r_n} : V_0^n \rightarrow V_n$ is a diffeomorphism with quasiconformal distortion bounded by $K$. Hence there exists $m > 0$ depending only on $K$ and $M$ such that for all $n$, mod $(\Gamma_0^n \setminus \Gamma_{n+1}) > m$. Thus the puzzle pieces $C_0^n$ have diameters converging to 0.

Case 1b: Suppose that $0 \in \omega(z)$, $\omega(z) \subset \mathbb{R}$, and for all $j$, $f^j(z) \not\in \mathbb{R}$. We consider the case when the mappings $f^j$ have uniformly bounded quasiconformal distortion near $z$, and the case when they have unbounded quasiconformal distortion near $z$, separately. First, suppose that there exists $K_x \geq 1$ such that for each $n$ the mapping $f^{r_n} : C_0^n \rightarrow C_n$ extends to a mapping from $V_0^n$ onto $V_n$ with quasiconformal distortion bounded by $K_x$. We have that each $\Gamma_0^n$ is a $K_1$-quasidisk, for some $K_1 > 1$ depending on $x$, and there exists a constant $m > 0$ such that for all $n$, mod $(\Gamma_0^n \setminus \Gamma_{n+1}) \geq m$, and so the puzzle pieces $C_0^n$ shrink to $z$.

Suppose now that the quasiconformal distortion of $f^{r_n} : V_0^n \rightarrow V_n$ tends to infinity as $n$ tends to infinity. For each $n$, let $\{V_j^n\}_{j=0}^{r_n}$ be the chain with $V_0^n = V_n$ and $V_j^n = \text{Comp}_x f^{-r_n}(V_{j-1}^n)$, and let $\{\Gamma_j^n\}_{j=0}^{r_n}$ be the chain with $\Gamma_0^n = \Gamma_n$ and $\Gamma_j^n = \text{Comp}_x f^{-r_n}(\Gamma_{j-1}^n)$ for all $n$ sufficiently large, there exists $0 \leq j_n < r_n$ maximal so that the set $V_{j_n}^n = \text{Comp}_{f^{j_n}} f^{-(r_n-j_n)}(V_0^n)$ does not intersect the real line (see case 2a below). Let $\Gamma_{j_n}^n = \text{Comp}_{f^{j_n}} f^{-(r_n-j_n)}(\Gamma_0^n)$. Since $\partial \Gamma_{j_n}^n$ is the core curve $V_n \setminus U_n$, and $f^{r_n-j_n}$ has bounded quasiconformal distortion, we have that there exists $m_1 > 0$ such that $\text{mod}(V_{j_n}^n \setminus \Gamma_{j_n}^n) > m_1$. But this implies that there exists $m_2 > 0$ such that dist $(\partial V_{j_n}^n, \Gamma_{j_n}^n) > m_2 \text{diam}(\Gamma_{j_n}^n)$, which immediately gives us that there exists $m_3 > 0$ so that dist $(\Gamma_{j_n}^n, \mathbb{R}) > m_3 \text{diam}(\Gamma_{j_n}^n)$. It follows that there exists $\xi > 0$ such that for all $n$, diam$_{\mathbb{R}}(\Gamma_{j_n}^n) < \xi$.

Let us inductively choose a subsequence $V_{n_i}$ of of the levels $V_n$ so that the landing maps from $V_{n_i}^{j_{n_i}}$ to $V_{n_{i+1}}^{j_{n_{i+1}}}$ all have definite expansion. Let $\eta \in (0,1)$ be the constant from Theorem 3.4 so that $|Df^i(z)||_{Y} \geq \eta |v||_{Y}$. Then we have that if $X$, a component of $f^{-i_0}(V_{n_n}^n)$, is a pullback of $V_{n_n}^{j_{n_n}}$ such that the quasiconformal distortion of $f^{i_0}|_X$ is bounded by $2(1+\delta)/\eta$, then there exists $N \in \mathbb{N}$ such that for each $i \leq i_0$, for each element $X_i = f^i(X)$ in the chain associated to the pullback, $X_i$ intersects at most $N$ of the strips $W_k$. Let $c > 0$ be the constant associated to $N$ from Corollary 6.2. Let $k_0 > 0$ be minimal
so that
\[ \sup_{z \in \Gamma_{j_0}^k} K f^j(z)^c \geq \frac{2}{\eta}. \]
Let \( 0 \leq j'_0 < j_0 \) be maximal so that
\[ \sup_{z \in \Gamma_{j_0}^k} \sup_{z \in \Gamma_{j_0}^k} K f^{j_0-j'_0}(z)^c \geq \frac{2}{\eta}. \]
Then, since \( f \) is \((1+\delta)-qc\), we have that
\[ \sup_{z \in \Gamma_{j_0}^k} K f^{j_0-j'_0}(z)^c \leq (1+\delta)\frac{2}{\eta}. \]
Thus by Corollary 6.2 we have that
\[ \text{diam}_Y(\Gamma_{j_0}^k) \leq \frac{\eta}{2} \text{diam}_Y(\Gamma_{j_0}^k) \leq \frac{\eta \xi}{2}, \]
and by Theorem 5.4 we have that
\[ \text{diam}_Y(\Gamma_{j_0}^0) < \frac{\xi}{2}. \]
We now repeat the argument: let \( k_1 > k_0 \) be minimal so that so that
\[ \sup_{z \in \Gamma_{j_0}^k} K f^{-(j_1-j_0)}(z)^c \geq \frac{2}{\eta}, \]
and let \( 0 \leq j'_1 < j_1 \) be maximal so that
\[ \sup_{z \in \Gamma_{j_1}^k} K f^{j_1-j'_1}(z)^c \geq \frac{2}{\eta}. \]
Then, since \( f \) is \((1+\delta)-qc\), we have that
\[ \sup_{z \in \Gamma_{j_1}^k} K f^{j_1-j'_1}(z)^c \leq (1+\delta)\frac{2}{\eta}. \]
Again by Corollary 6.2 we have that
\[ \text{diam}_Y(\Gamma_{n}^{j_1}) \leq \frac{\eta}{2} \text{diam}_Y(\Gamma_{j_1}^{j_1}) \leq \frac{\eta \xi}{2}, \]
and by Theorem 5.4 we have that
\[ \text{diam}_Y(\Gamma_{k_1}^{j_1}) < \frac{\xi}{2}. \]
Combining this with the first step, we have that
\[ \text{diam}_Y(\Gamma_{k_1}^{j_1}) < \frac{\xi}{4}. \]
If the quasiconformal distortion of \( f^n \) diverges at \( x \), we see that we can repeat this argument infinitely many times to obtain a nest of puzzle pieces \( \{C_{k_i}^0\} \) about \( z \) such that \( \text{diam}_Y(C_{k_i}^0) \to 0. \)
Combining Cases (1a) and (1b), we have that for all \( z \) such that \( 0 \in \omega(z) \), that there are arbitrarily small puzzle pieces \( P \ni z \). Now we treat the cases when \( 0 \notin \omega(z) \).

**Case 2a:** Suppose that there exists \( n \in \mathbb{N} \) such that \( \omega(z) \subset \mathbb{R} \setminus V_n \). Let \( \mathcal{Y}_0^{(n)} \), be the initial configuration of puzzle pieces at level \( n \). Let \( x_0 \in \omega(z) \), then, since the real traces of puzzle pieces shrink to points, there exist \( m_0 > 0 \) and a union of (closed) puzzle pieces of \( \mathcal{Y}_m^{(n)} \), denoted by \( Q_0 \), such that \( Q_0 \cap \omega(0) = \emptyset \) and \( x_0 \in \text{int}(Q_0) \). Let \( \mathcal{Y}_j^{(n)}(x_0) \) denote the closure of the set of puzzle pieces \( P \) in \( \mathcal{Y}_j^{(n)} \) with \( x_0 \in \mathcal{T} \). Let \( Q = \cap_{j=0}^{\infty} \mathcal{Y}_j^{(n)}(x_0) \).

Let us show that \( Q = \{x_0\} \). If \( \text{diam}(Q) > 0 \), then, since \( \cup_n f^n(Q) \) is a bounded set, there exists \( C > 0 \), \( x \in Q \) and a vector \( v \in T_x \mathcal{C} \) such that \( |Df^k(x)v| < C \). If \( \omega(x) \) is not contained in the real-line, then in a small neighbourhood of \( x \), the hyperbolic metric on \( Y \) is comparable to the Euclidean metric, but now \( |Df^k(x)v| < C \) contradicts Theorem 5.4 (b). So we can assume that \( \omega(x) \subset \mathbb{R} \), but then \( \omega(x) \) is contained in the hyperbolic set of points that avoid \( V_n \), and we have that \( |Df^k(x)v| \to \infty \) for any \( v \in T_x \mathcal{C} \), and so \( \text{diam}(Q) = 0 \). Let us point out that this argument shows that if \( z \in \mathbb{R} \) is contained in a hyperbolic set, then for any \( n \) sufficiently big, \( \text{diam}(\mathcal{Y}_j^{(n)}(z)) \to 0 \) as \( j \to \infty \), and indeed that \( f_j \) is locally connected at any point in \( f_j \cap \mathbb{R} \) that is contained in a hyperbolic set.

Suppose that for all \( j \in \mathbb{N} \cup \{0\} \), \( f_j(z) \notin \mathbb{R} \). Let \( r_0 \) be the first return time of \( x_0 \) to \( Q_0 \), and let \( Q_1 = \text{Comp}_{x_0} f^{-r_0}(Q_0) \). Inductively define \( Q_{i+1} \) by taking \( r_i \) to be the first return time of \( x_0 \) to \( Q_i \) and setting \( Q_{i+1} = \text{Comp}_{x_0} f^{-r_i}(Q_i) \). Let \( \varepsilon > 0 \) be so small that if \( z = x + iy \) satisfies \( \text{dist}(z, \mathbb{R}) < \varepsilon \) and \( z \notin V_n \), then \( \text{dist}(x, 0) > \text{diam}(V_n)/2 \). Since \( x_0 \in \omega(z) \), there exist \( n_i \to \infty \) with the property that \( n_i \) is minimal with \( f^{n_i}(z) \in Q_i \). It is sufficient to show that there exists a constant \( c > 0 \) so that for all \( i \), \( |Df^{n_i}(z)| \geq c \). Fix some \( i \in \mathbb{N} \). Let \( j_0 \geq n_i \) be minimal so that \( \text{dist}(f^{j_0}(z), \mathbb{R}) > \varepsilon \), and let \( j_1 \leq n_i \) be maximal so that \( \text{dist}(f^{j_1}(z), \mathbb{R}) > \varepsilon \), then there exists a constant \( c_1 > 0 \) so that

\[
|Df^{j_0-n_i}(z_0)| \geq c_1 \eta |Df^{j_0-j_1}(f^{j_1}(z_0))||Df^{j_0-n_0}(f^{n_0}(z))| |Df^{n_0}(z)|.
\]

Thus it suffices to bound \( |Df^{j_0-j_1}(f^{j_1}(z_0))| \) and \( |Df^{j_0-n_0}(f^{n_0}(z))| \) from below. Let \( z_0 = f^{n_0}(z) \) and define \( z_i = f^{j_i}(z_0) \), \( x_i = f^{j_i}(x_0) \). Then there exist constants \( c_2, c_3 \) so that

\[
|Df^{j_0-n_0}(z_0)| \geq c_2 \prod_{i=0}^{j_0-n_0} |Df(x_i)| \prod_{i=0}^{j_0-n_0} (1 - c_3 \frac{|z_i - x_i|}{|Df(x_i)|}).
\]

By our choice of \( \varepsilon \), and since \( x_0 \) is contained in a hyperbolic Cantor set, we have that there exists a constant \( c_4 > 0 \) and \( \Lambda > 1 \) so that

\[
\sum_{i=0}^{j_0-n_0} |z_i - x_i| \leq \frac{1}{2\text{diam}(V_n)} \sum_{i=0}^{j_0-n_0} |z_i - x_i| \leq \frac{1}{2\text{diam}(V_n)} \frac{c_4 \varepsilon}{1 - \Lambda^{-1}}.
\]

Thus we have that \( |Df^{j_0-n_0}(z_0)| \) is bounded from below. The proof that \( |Df^{n_i-j_1}(f^{j_1}(z_0))| \) is bounded from below is similar.

**Case 2b:** Suppose that \( \omega(z) \notin \mathbb{R} \). Let \( z_0 \) be an accumulation point of \( \omega(z) \) that is not contained in \( \mathbb{R} \). Since the real puzzle pieces shrink to points, there exist \( n \) and \( m \) and a union \( Q \) of puzzle pieces in \( \mathcal{Y}_m^{(n)} \) and a sequence \( k_i \to \infty \) such that \( Q \cap \mathbb{R} = \emptyset \), and \( f^{k_i}(z) \in Q \) for all \( i \). By Theorem 5.4 (b), we have that

\[
\text{diam}(\text{Comp}_{f^{k_0}(z)}(f^{-(k_i-k_0)}(Q))) \to 0 \text{ as } i \to \infty.
\]
Thus by Theorem 5.4(a),
\[
\text{diam}(\text{Comp}_z(f^{-ki}(Q))) \rightarrow 0 \text{ as } i \to \infty.
\]

\[\square\]

Proposition 6.7 has several important consequences.

**Corollary 6.8.** Suppose that \( f \in C^r \) is an asymptotically holomorphic polynomial-like mapping, which is \((\alpha, \delta, \theta, M, n_0)\)-controlled, and that
\[
\frac{4n_0\alpha}{(n_0-1)(1-\theta)(2\alpha-1)} > 1.
\]

Then the following hold:

1. \( J_f = K_f \).
2. \( f : U \to V \) is topologically conjugate to a polynomial mapping in a neighbourhood of its Julia set. In particular, \( f : U \to V \) has no wandering domains.
3. \( J_f \) is locally connected.

**Proof.** (1). To see that \( J_f = K_f \) observe that for each \( z \in K_f \), there are arbitrarily small puzzle pieces containing \( z \), so \( z \) is a limit of points whose orbits eventually land in \( V \setminus U \). Thus \( z \in J_f \). In particular, \( K_f \) has empty interior.

(2). Let us now show that \( f : U \to V \) is topologically conjugate to a polynomial mapping in a neighbourhood of its Julia set. Let \( I \subset U \cap \mathbb{R} \) denote the invariant interval for \( f \). Since \( f\vert_I \) has negative Schwarzian derivative, there exists a real polynomial \( p \) with a critical point of the same degree as the critical point of \( f \) such that \( f \) is topologically conjugate to \( p \) on \( I \). Let \( \tilde{h} : I \to I \) be the continuous mapping such that \( \tilde{h} \circ f\vert_I = p \circ \tilde{h} \). Let \( \tilde{V} \) be a domain containing \( J_p \) that is bounded by some level set of the Green’s function for \( p \). Let \( \tilde{V} = p^{-1}(V) \).

Let \( H_0 : V \to \tilde{V} \) be a homeomorphism such that
- for each \( z \in \partial U \), \( H_0 \circ F(z) = p \circ H_0(z) \),
- for each \( z \in \bigcup_n (R_{\beta_n} \cup \overline{R}_{\tau(\beta_n)}) \), we have that \( H_0(z) \circ f = p \circ H_0(z) \), and
- \( H_0|_I = \tilde{h} \).

See Remark 6.3 for a description of how to construct such an \( H_0 \).

Given that \( H_i \) is defined, define \( H_{i+1} \) by \( H_i \circ f = p \circ H_{i+1} \). Since each \( H_i \) is conjugacy on \( J \) between \( f \) and \( p \) that maps that critical value of \( f \) to the critical value of \( p \), this pullback is always well-defined and continuous. Observe that for each \( z \in U \setminus K_f \), \( H_i \) eventually stabilizes. Let \( H : V \to \tilde{V} \) be a limit of the \( H_i \). To see that \( H \) is continuous, take any \( z \in U \) and let \( \{z_n\} \) be a sequence of points such that \( z_n \to z \). If \( z \notin K_f \), then there exists a neighbourhood \( W \) of \( z \) and \( i_0 \in \mathbb{N} \), large such that for all \( i \geq i_0 \) and \( w \in W \), \( H_i(w) = H_{i_0}(w) \). Hence \( H(z_n) \to H(z) \). So suppose that \( z \in K_f \), then since the nests of puzzle pieces about \( z \) and \( H(z) \) both shrink to points and \( H \) maps puzzle pieces for \( f \) to corresponding puzzle pieces for \( p \), \( H(z_n) \to H(z) \). Also, since for each \( z \in U \setminus K_f \), \( H_i \) eventually stabilizes, \( H : U \to \tilde{U} \) satisfies \( H \circ F(z) = p \circ H(z) \) for all \( z \in U \setminus K_f \) and since \( K_f \) has empty interior, we have that \( H \) is a conjugacy between \( f \) and \( p \) on \( U \).

(3). Finally, let us show that \( J_f \) is locally connected. Let \( z \in J_f \), and let \( B \) be any open set that contains \( z \), by Proposition 6.7 there exists a neighbourhood \( Q \subset B \) of \( z \), such
that $Q$ is a union of puzzle pieces. Since $J_f \cap P$ is connected for any puzzle piece $P$, we have that $J_f \cap Q$ is connected too.

Let us remark that since $f$ is topologically conjugate to a polynomial, we obtain that the repelling periodic points of $f$ are dense in $J_f$. We also point out that this implies that $f$ has no wandering domains, but that this fact can be deduced immediately from the fact that the puzzle pieces shrink to points.

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