The AdS-CFT correspondence, consistent truncations and gauge invariance

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Abstract

We give arguments for a conjecture made in a previous paper, that one has to use only the gauged sugra action for the calculation of correlators of certain operators via the AdS-CFT correspondence. The existence of consistent truncations implies that the massive modes decouple, and gauged supergravity is sufficient for computing n-point functions of CFT operators coupled to the massless (sugra) sector. The action obtained from the linear ansatz, of the type \( \phi(x, y) = \phi_I(x)Y^I(y) \) gives only part of the gauged sugra. This means that there is a difference for the correlators on the boundary of AdS space. We find, studying examples of correlators, that the right prescription is to use the full gauged sugra, which implies using the full nonlinear KK ansatz. To this purpose, we analyze 3 point functions of various gauge fields in 5 and 7 dimensions, and the R-current anomaly in the corresponding CFT. We also show that the nonlinear rotation in the tower of scalar fields of Lee et al., Corrado et al. and Bastianelli and Zucchini produces a consistent truncation to the massless level and coincides with the Taylor expansion of the nonlinear KK ansatz in massless scalar fluctuations. Finally, we speculate about the way to do the full nonlinear rotation for the massive tower.

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1 Introduction

In two previous papers together with Peter van Nieuwenhuizen [1, 2], we showed that there exists a nonlinear embedding of 7d maximal gauged sugra into 11d sugra and proved the consistency of this truncation 11 dimensional fields to the 7 dimensional fields in the $AdS_7 \times S_4$ background. For the $AdS_4 \times S_7$ KK reduction of 11d sugra (to maximal d=4 gauged sugra), de Wit and Nicolai [3] proved the consistency of the truncation indirectly (starting from another formulation of 11d sugra, with SU(8) invariance). For the $AdS_5 \times S_5$ case presumably one can find also a consistent truncation of 10d IIB sugra to 5d maximal gauged sugra.

Based on the existence of these consistent truncations we conjectured in [1] that for the computation of correlators via the AdS-CFT correspondence [4, 5, 6], if we are interested in operators corresponding to gauged sugra fields, it is enough to take the gauged sugra action. This eliminates an ambiguity in the formulation of the correspondence. Let’s explain this further: a priori, there are two ways of dealing with the computation of correlators. The prescription says to take string theory on the $AdS_p \times S_{D-p}$ background, and compute the effective action as a function of the boundary fields. One way could be to take the linear KK expansion in spherical harmonics (given in [7])

$$\phi_{Ai}(x, y) = \sum_I \phi^I_A(x) Y^I_i(y)$$

(1.1)

and plug it into the 11d sugra action.

Then the truncation to the subset of fields of interest, $\{\phi^I_A\}$ is not consistent in general, because there are terms in the action linear in the fields set to zero, $\{\phi^I_A\}$. That means that their equation of motion, $\delta S/\delta \phi^I_A(x) = 0$, contains the fields $\phi^{I_0}_A(x)$ as sources, which gives a contradiction. For the AdS-CFT correspondence, the inconsistency implies that for 4- and higher-point functions of $\phi^{I_0}_A(x)$, all $\phi^{I_0}_A$ will contribute through Witten diagrams involving the troublesome couplings:

Another possibility appears when we can have a nonlinear ansatz relating the $\{\phi_{Ai}(x, y)\}$ to $\{\phi^{I_0}_A(x)\}$ such that the truncation is consistent (implying in particular that the $\{\phi^{I_0}\}$ don’t appear in Witten diagrams for $\{\phi^{I_0}\}$).

A priori, we don’t know which one to take. We need a physical principle to decide. In the cases we study, 11d sugra on $AdS_7 \times S_4$ truncated to 7d gauged sugra and 10d IIB sugra on $AdS_5 \times S_5$ truncated to 5d gauged sugra, we will argue by examples that it is correct to take the ansatz giving a consistent truncation, and not the linear ansatz.
In other words, the gauged sugra action gives the correct CFT correlators, whereas the action coming from the linear ansatz doesn’t. At this moment, it becomes clear what is the sought-for physical principle. Or rather physical principles: gauge symmetry and susy. Indeed, by taking the linearized action for the gauged sugra and imposing gauge invariance and susy (by the Noether procedure) you obtain the gauged sugra action. In fact, this is how 7d (and 5d) gauged sugra were obtained in \[8, 9\].

One might think that taking the gauged sugra action for the calculation of correlators is the natural thing to do, but this procedure is available only if there exists a consistent truncation. If there would exist an inconsistent truncation to gauged sugra, that would mean that for 4 point correlators one would have to consider the contribution of the whole tower of massive fields.

So the procedure one needs in order to obtain the gauged sugra action is to modify the linearized ansatz in such a way that the action one obtains is gauge invariant and susy. This procedure can be easily generalized. The parent action was invariant under local “gauge” transformations, with parameter \(\xi_{\mu} = \xi^{AB}(x)V^{AB}_{\mu}(y)\) (where \(V^{AB}_{\mu}\) is a Killing vector). After the “nonlinear redefinition” (by nonlinear redefinition we understand a nonlinear KK ansatz as opposed to a linear one) of the massless fields, this invariance is lost, and so we need a corresponding nonlinear rotation for the massive fields in order to restore it. It is not clear whether this can be done multiplet by multiplet or for the whole tower at once. We conjecture that this nonlinear ansatz, which we get after performing the rotation, is the one needed for the AdS-CFT correspondence.

We have described how to obtain the nonlinear ansatz to be used for the AdS-CFT conjecture. One possible objection to this procedure is that a nonlinear redefinition of fields which doesn’t change the quadratic action, like the one from the linearized ansatz, \(\phi_{A}(x,y) = \sum_{I} \phi_{A}^{I}(x)Y^{I}_{I}(y)\) to the full nonlinear ansatz, will not change the S matrices of fields. This is so in usual field theory, but for the AdS-CFT correspondence there is one important difference: the S matrices are for the sources on the boundary. And boundary terms, which are usually neglected, become important. We will show that with a very simple example, of a \(\lambda \phi^{3}\) theory in the bulk.

Now, to show that taking the gauged sugra is the correct procedure for the AdS-CFT correspondence as opposed to taking the action coming from the linearized ansatz, we will analyze several n-point functions coming from both approaches. We will first analyze in section 2.1 some relevant 3-point functions, listing all the possible ones and discussing in particular the ones involving gauge fields. Then in section 2.2 we will analyze the CS terms and what we can say for the field theory anomalies. Finally, we will discuss the scalar 3-point functions (corresponding to CPOs) from the work of Lee et al \[10\], Corrado et al. \[11\], and Bastianelli and Zucchini \[12\] and how the nonlinear rotation they found is needed to obtain a consistent truncation to gauged sugra. We also give arguments on why this is just a Taylor expansion in fluctuations of the full nonlinear rotation.

## 2 3-point functions of gauge fields
2.1 General considerations

In this section we will make some general remarks about relevant 3-point functions, in particular about gauge fields correlators.

**7d gauged sugra**

**Bosonic fields:** gauge fields $B^{AB}_\alpha$ with gauge group $SO(5)_g$, antisymmetric tensors $S_{\alpha\beta\gamma,A}$, graviton $e^{i}_\alpha$, scalars $\Pi^{A}$ in the coset $Sl(5,R)/SO(5)_c$.

**Bosonic action:**

$$e^{-1}\mathcal{L} = -\frac{1}{2}R + \frac{1}{4}m^2(T^2 - 2T_{ij}T^{ij}) - \frac{1}{2}P_{\alpha ij}P^{\alpha ij} - \frac{1}{4}(\Pi^{i}_A\Pi^{j}_B F^{AB}_\alpha)^2$$

$$+ \frac{1}{2}(\Pi^{-1}_{\alpha i} S_{\alpha\beta\gamma,A})^2 + \frac{1}{48}m e^{-1}\epsilon^{\alpha\beta\gamma\delta\epsilon\eta\zeta} \delta^{AB} S_{\alpha\beta\gamma,A} F_{\delta\epsilon\eta\zeta,B}$$

$$+ \frac{ie^{-1}}{16\sqrt{3}}\epsilon^{\alpha\beta\gamma\delta\epsilon\eta\zeta} \epsilon_{ABCDE}^G S_{\alpha\beta\gamma,G} F_{BC}^{DE} F_{\delta\epsilon\eta\zeta}$$

$$+ \frac{m^{-1}}{8}e^{-1}\Omega_{5}[B] - \frac{m^{-1}}{16}e^{-1}\Omega_{3}[B] \quad (2.1)$$

The first remark is that gravity will appear in the correct way just because of general coordinate invariance, both in the linearized ansatz and in the nonlinear one. (or rather, the 11d graviton will be nonlinearly redefined – Weyl rescaled – but only by the scalars: $e^{i}_\alpha \rightarrow e^{i}_\alpha [\det E^{\mu}]^{-1/5}$). So we will disregard the 3-point functions involving the graviton. Also, the $(\delta \Pi^{A}_{i})^{3}$ 3-point function will be analyzed in the last section.

The remaining 3-point functions are:

- Involving scalars: $B^{AB}_{\alpha} \delta \Pi^{A}_{i} \delta \Pi^{B}_{i}$, from $[P_{\alpha ij}]^2$ term, $BB\delta\Pi$, from the $dB \wedge *dB\delta\Pi$ piece in $(\Pi^{i}_A\Pi^{j}_B F^{AB}_\alpha)^2$ and $SS\delta\Pi$, from the $[(\Pi^{-1}_{i})^{A} S_{\alpha\beta\gamma,A}]$ term.

- Involving no scalars: $(B^{AB}_{\alpha})^{3}$, from the kinetic term, $2 * dB \wedge B \wedge B$; SSB, from the $S$ kinetic term, i.e. $\frac{1}{12}m e^{-1}\epsilon^{\alpha\beta\gamma\delta\epsilon\eta\zeta} \delta^{AB} S_{\alpha\beta\gamma,A} F_{\delta\epsilon\eta\zeta,B}$, and BBS, from the term $\frac{ie^{-1}}{16\sqrt{3}}\epsilon^{\alpha\beta\gamma\delta\epsilon\eta\zeta} \epsilon_{ABCDE}^G S_{\alpha\beta\gamma,G} F_{BC}^{DE} F_{\delta\epsilon\eta\zeta}$.

Let’s look at the $B^{3}$ term. The calculation of $AdS$ space correlators of gauge fields was done in [13, 14].

Let’s see what would happen if we took the linearized ansatz:

For the $B^{3}$ term the $AdS$ space, the calculation of correlators was done in [13, 14]. In 7d, the $*dB \wedge B \wedge B$ term comes in the nonlinear ansatz in part from the kinetic term $F^{2}_{\alpha\beta\mu\nu}$ in d=11, and so this piece will be absent if we take the linearized ansatz. But there is also a piece coming from $\int \sqrt{G^{(11)}} R^{(11)}$, which will remain. So the coefficient of the CFT correlator of 3 R-currents would get modified. Although the CFT has no lagrangean formulation, one can think of making a free field calculation, as it was done for the correlators of stress tensors in [14]. The coefficient would not be fixed, but it can be fixed by taking susy variations on the stress tensor correlator in [14]. One should obtain the result matching the $AdS$ 3-point function in [13, 14].

So the correct result is the one coming from the nonlinear ansatz. Moreover, we clearly see that imposing gauge invariance on $dB \wedge *dB$ we get the usual $dF \wedge *dF$ action, so gauge invariance here is clearly the physical principle needed to modify the linearized ansatz.
The same comment applies to the BBS correlator: The $\varepsilon S F F$ term in the action comes from two sources: the $\varepsilon^{\mu\nu\rho\sigma}e_{(151)}F_{\mu\nu\rho\sigma}F_{\alpha\beta\gamma\delta}$ and $\varepsilon^{\mu\nu\rho\sigma}e_{(151)}A_{\mu\nu\rho\sigma}F_{\alpha\beta\gamma\delta}$. If we use the linear ansatz, the last term would give the correct piece, $\varepsilon S \partial B \partial B$, but the former would not contribute, and so the normalization of the $S \partial B \partial B$ correlator would be wrong. Here one would have to compute the AdS correlator first, which we leave for future work [10].

5d gauged sugra

Bosonic fields: -ungauged model: gravitons $e^r_\mu$, gauge fields $A^\mu_{AB}$ scalars $V_{AB} \, ab$ (27-bein), global symmetry group $E_6(6)$, composite symmetry USp(8).

-gauged model: $e^r_\mu$, gauge fields $A^I_{\mu}$, $B^I_\alpha$, scalars $V^{IJ\mu}$, $V^I_{\mu\alpha}$, ab, gauge group: $SO(p, 6-p) \times SL(2, R)$, composite symmetry USp(8). Under $27 \rightarrow (15, 1) \oplus (6, 12), AB \rightarrow IJ \oplus I\alpha$.

Bosonic action:

$$e^{-1}L_{bosonic} = -\frac{1}{4} R + \frac{1}{24} P_{\mu\nu\rho\sigma}P^{\mu\nu\rho\sigma}$$

$$-\frac{1}{8} (F_{\mu\nu} + B_{\mu\nu})^2 - \frac{1}{96} \varepsilon^{\mu\nu\rho\sigma} \eta_{IJK\lambda\alpha\beta} B^I_{\mu\nu} D^J_{\rho\sigma} B^\lambda_{\tau\sigma}$$

$$+ g^2 \left[ \frac{6}{45} T_{ab}^2 - \frac{1}{96} A^2_{abcd} \right] + \frac{1}{12} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{IJKLMN}$$

$$(F^I_{\mu\nu} F_{KLM\rho\sigma} A_{MN\tau} + g\eta_{PQ} F_{IJ\mu\nu} A_{KLM\rho\sigma} A_{PN\tau})$$

$$+ \frac{1}{2} g^2 \eta_{PQ} \eta_{RS} A_{IJ\mu\nu} A_{KLP\sigma} A_{MR\tau} A_{SN\tau}$$  \hspace{1cm} (2.2)

where in the ungauged model $\tilde{V}_{\alpha\beta} \, AB \partial^\mu V_{AB} \, ab = 2 Q_\mu^{[\alpha} \delta_{\beta]} + P^I_{\mu} \, \delta_{cd}$, which becomes in the gauged model $\tilde{V} D^\mu_\nu V = 2 Q_\mu + P^I_{\mu}$, and $\tilde{V} = V^{-1}$, $D^\mu_\nu$ is the $SO(p, 6-p)$ covariant derivative, $Q_\mu$ is the USp(8) connection, $D_\mu$ is the full $USp(8) \times SO(p, 6-p)$-covariant connection. Also,

$$A_{abcd} = T_{[abcd]} \quad T_{ab} = T_{c}^{\ \abc}$$

$$T^{a}_{\ bcde} = \gamma^{ac} \gamma_{bced} (2 V^{IJK\alpha \beta} \tilde{V}_{\alpha\beta JK - V_{IJK} \, \gamma\beta \gamma \lambda \alpha}) \eta^{IJKLM} \tilde{V}_{\alpha\gamma\lambda\beta}$$

$$F^{ab}_{\mu\nu} = V^{IJ\mu\nu} F_{IJ} \quad B^{I\alpha}_{\mu} = V^{I\alpha}_{\mu} B^{I\alpha}_{\mu}$$  \hspace{1cm} (2.3)

and $F^{I\alpha}_{\mu\nu}$ and $B^{I\alpha}_{\mu}$ are the field strengths of $A^{I\alpha}_{\mu}$ and $B^{I\alpha}_{\mu}$, respectively.

Bosonic 3-point functions- except the ones involving the graviton, for the same reasons as in 7d, and the ones involving only scalars, which are treated in the last section.

-Involving no scalars: AAA, from the $* dA \wedge A \wedge A$ and $dA \wedge dA \wedge A$ terms in the action, BBA from the $* dA \wedge B \wedge B$ and $dB \wedge dB \wedge A$ term in the action (namely from the $\varepsilon B B B$ term), BBB from the $* dB \wedge B \wedge B$ term, and BAA from the $* dB \wedge A \wedge A$ term.

-Involving scalars: VVA terms from the $\partial V \partial V A$ piece of $P^2_{\mu\nu} (A^I_{\mu} \, \delta_{I\mu}$ coming from $D^I_{\mu}$), and BBV from $\frac{1}{96} \varepsilon^{\mu\nu\rho\sigma} \eta_{IJK\alpha\beta} B^I_{\mu\nu} D_\rho B^\lambda_{\sigma\tau} (V \text{ coming from the } Q_\mu \text{ term in } D_\lambda)$.

The AAA term was computed in [13, 14] and gives the correct CFT correlators (we should stress once again that the agreement between the AdS and CFT computations holds as long as one uses the gauged sugra interactions). We can easily extend this result to all the 3 point functions of gauge fields (BBA, BAA and BBB), and all that changes are the coefficients of the terms in the action involving gauge fields, and
the combinatorial factors (coming from differentiating with respect to the boundary sources of the gauge fields).

On the other hand, if we take the 10d IIB sugra action,

\[ S_{IIB,10d} = \frac{1}{(2\pi)^3\alpha'} \int d^{10}x \sqrt{-g^{(10)}} e^{-2\phi} (R^{(10)} + 4|d\phi|^2 - \frac{1}{3}|H|^2) - 2|dl|^2 - \frac{1}{3}|H' - lH|^2 - \frac{1}{60}|M^+|^2 - \frac{1}{48}C^+ \right) \]

(2.4)

and plug in the linearized ansatz, we will again miss some terms of the type \(*dA \wedge A \wedge A\) coming from the kinetic term \(|H'| - lH|^2\) of the antisymmetric tensors. The same comment applies to the \(*dB \wedge B \wedge B\) term for the \(BBB\) correlator, the \(*dB \wedge A \wedge A\) term for the \(BAA\) correlator, and \(*dA \wedge B \wedge B\) for the \(ABB\) correlator. The fact that we don’t know the nonlinear KK embedding it’s not relevant, because we know that if we have a consistent truncation, the prescription we suggest is to use the gauged sugra action. We also know the linearized KK reduction of [17]. So we can say that the correlators obtained from the linearized ansatz will differ from the ones obtained from the nonlinear ansatz, which we know to be correct (i.e. in agreement with \(N = 4\) SYM results). Once again, the nonlinear ansatz is seen to be the correct one to take.

2.2 Anomalies

5 dimensions

For the relation between the CS term in maximal 5d gauged sugra and the R-current anomaly in 4d N=4 SYM, Witten gave a very elegant argument in his original paper on the AdS-CFT correspondence [1]. The argument goes as follows:

If we vary the bulk gauge fields (in 5d) by \(\delta A_\mu^a(x) = (D_\mu A)^a(x)\) the only nonzero term in the variation of the action \(\delta_A S_{cl}[A_\mu^a(x)[A_\mu^a(x)]\] will be a boundary term coming from the CS term,

\[
\delta_A S_{cl} = \delta_A S_{CS} = \int d^4x \Lambda^a(\vec{x}) \left( \frac{-ik}{96\pi^2} d^{abc} \epsilon^{ijkl} \partial_i (A_j^a \partial_l A_k^c) + \frac{1}{4} f^{cde} A_j^b A_k^d A_l^c \right)
\]

(2.5)

And the conjecture implies that \(S_{cl}[A_\mu^a(x)]\) is equal to \(W[A_\mu^a(x)]\), the generating functional of connected Green’s functions on the boundary. Since also \(J^a_i(x) = \delta W[A]/\delta A^a_i(x)\), we get

\[
\delta_A S_{cl} = \delta_A W = - \int d^4x \Lambda^a(\vec{x}) D_i J_i(\vec{x})
\]

(2.6)

which implies that

\[
< D_i J_i(\vec{x}) > = \delta_A S_{CS}
\]

(2.7)

and gives a concrete physical interpretation to the known mathematical fact that the consistent anomaly in n dimensions is obtained by a descent equation from the 2n+1 dimensional CS action.

That implies that if one takes the full 1-loop 3-point function of R-currents in SYM and one takes a divergence, it should reproduce the result for the ‘Witten diagram’ of
3 gauge fields in AdS, with a divergence taken. (That is because the anomaly is only 1 loop, by the Adler-Bardeen theorem.) It is indeed so, as noted in [13, 14]; but it also implies a similar result for the 4 point function. In 4d N=4 SYM, the box diagram and its anomalous part are also nonzero. A priori, there seems to be another diagram contributing an anomaly, but it actually gives only a renormalization. (The wavy lines denote gauge vector propagators while the straight lines indicate fermion propagators.)

Only by summing all the one loop anomalous diagrams we get a gauge covariant anomaly

\[ (D^\mu J_\mu)^a = \frac{N^2 - 1}{384\pi^2} \epsilon^{abc} \epsilon_{\mu\nu\rho\sigma} F^{b\mu}_{\nu\rho} F^{c\rho}_{\sigma} \]

(2.8)

(the triangle anomaly anomaly alone yields just the \(dA \wedge dA\) term on the right hand side).

On the AdS side, there are two diagrams, the 4 point vertex for the CS term and the exchange diagrams, with 3-point vertices coming also from the CS term. The naive expectation, that the AdS 4-point vertex equals the box diagram from field theory, and the AdS exchange diagram equals the diagram with two triangles glued, is wrong, because the AdS exchange diagram gives a genuine contribution, not just a renormalization.

However, Witten’s argument tells us that the sum of the AdS diagrams should be equal to the sum of the field theory diagrams, so we don’t need to worry.

What would happen if we take the linearized ansatz instead? Even if we don’t know the nonlinear ansatz, we know that using the nonlinear ansatz, the action for the massless sector will be 5d maximal gauged sugra, which has a CS term, and therefore generates a R-current anomaly. On the other hand, using the linear ansatz [17], and substituting it in the 10d IIB sugra action, we notice that there is no surviving CS term! That is so because the linear ansatz of the 3-form field strengths \(H\) and \(H'\) does not contain the massless gauge vector fields. So, by using the linear ansatz, one would conclude that there is no R-current anomaly.

And again, gauge and supersymmetry invariance tells us that we should take the nonlinear ansatz, because by imposing them both on the linearized action we get the 5d maximal gauged sugra, with a CS term.

**7 dimensions** Witten’s argument applies equally well in all odd dimensions, relating the anomaly in 2n dimensions to a CS term in a gauged sugra in 2n+1 dimensions. However, the only other example of maximal gauged sugra in 2n+1 dimensions is d=7.

Again, by varying

\[ S_{CS} = \frac{m^{-1}}{8} e^{-1}\Omega_5[B] - \frac{m^{-1}}{16} e^{-1}\Omega_3[B] \]

(2.9)
where $\Omega_3[B]$ and $\Omega_5[B]$ are the Chern-Simons forms for $B^A B$ (normalized to $d\Omega_3[B] = (Tr F^2)^2$ and $d\Omega_5[B] = (Tr F^4)$), we should get the chiral anomaly in 6d. But now it is unclear how to compute this anomaly, since the dual 6d theory is a nontrivial (0,2) CFT without a lagrangean formulation. The anomaly means that the $SO(5)$ R-symmetry, which is part of the susy algebra, is broken by the fact that correlators of R-currents are anomalous. In 6d, the first anomalous correlator is the 4-point function (corresponding to the 4-point CS coupling $dB \wedge dB \wedge dB \wedge B$). And it is easy to see from the nonlinear ansatz [1,2] that the 4-point CS coupling (in fact, all the CS term!) will be missing for the linearized ansatz (the 7 dimensional CS has terms with at least four fields, $Tr (dB \wedge dB \wedge dB \wedge B)$ and $Tr (dB \wedge dB) \wedge Tr (dB \wedge B)$, while the 11 dimensional CS $dF(4) \wedge dF(4) \wedge A(3)$ cannot generate them after substituting a linearized ansatz).

But we know that the 6d (0,2) CFT should have a chiral anomaly, because it is obtained as the IR limit of the M5-brane theory, which has an anomaly. Moreover, in [19], a brane calculation of the anomaly was performed, and it was found that the anomaly has the expected functional dependence, i.e. coming by descent formalism from the 7d Chern-Simons term. The only nontrivial aspect is the coefficient in front of this anomaly, which was found to be proportional to $N^3$. The same $N^3$ dependence is found also from the AdS calculation, because the sugra coupling has this dependence. One would like to understand the $N^3$ dependence in a field theory context, because the calculation in [19] uses M theory, as does the AdS-CFT calculation. However, that was not done yet. The free-field calculation in [15] for the stress-tensor correlators (trying to match with the anomaly calculation of [20] on the AdS side), and the free-field calculation in [21] for the R-current anomaly, both impose by hand the $N^3$ dependence. In [22], the [13] calculation was extended to other gauge groups, and in [23] the calculation was related to Witten’s [24] calculation in type IIA string theory, but a real field theory explanation is still lacking.

So again, since we want an anomaly in 7d, because we know it should be there by the AdS-CFT correspondence, the nonlinear ansatz is the correct one. As before, we need to impose both susy and gauge invariance on the linearized action obtained by compactification in order to recover the correct result. (The absence of a CS term respects gauge invariance alone.)

### 3 Scalar 3-point functions

Let’s start by giving the $\lambda \phi^3$ example as promised in the introduction. † For a $\lambda \phi^3$ theory in the bulk, a redefinition of fields in the bulk doesn’t change the bulk S matrices, but does the ones computed on the boundary (via the AdS-CFT-type correspondence), because of the presence of a boundary term. Let’s start with the lagrangean

$$L = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \lambda \phi^3$$

(3.1)

The bulk 3-point function will be equal to $\lambda$. Let’s now redefine $\phi = \tilde{\phi} + a \tilde{\phi}^2$. Then,

$$L = \frac{1}{2} (\partial_\mu \tilde{\phi})^2 + \frac{1}{2} m^2 \tilde{\phi}^2 + \lambda \tilde{\phi}^3 + 2a \tilde{\phi}(\partial_\mu \tilde{\phi})^2 + m^2 a \tilde{\phi}^3$$

†This example emerged in a discussion we had with Fiorenzo Bastianelli.
\[ +2a^2 \tilde{\phi}^2 (\partial_\mu \tilde{\phi})^2 + \frac{1}{2} m^2 \tilde{\phi}^4 + 3 \lambda a \tilde{\phi}^4 + 3 \lambda a^2 \tilde{\phi}^6 + \lambda a^3 \tilde{\phi}^6 \]  

(3.2)

But

\[ \int [2a \tilde{\phi} (\partial_\mu \tilde{\phi})^2 + m^2 a \tilde{\phi}] = -a \int \tilde{\phi}^2 \Box \tilde{\phi} + \int m^2 a \tilde{\phi}^3 + 2a \int \partial_\mu (\tilde{\phi}^2 \partial_\mu \tilde{\phi}) \]  

(3.3)

On shell, \((\Box - m^2) \tilde{\phi} = 0\), so on-shell, the 3-point correlator is obtained from

\[ \int \lambda \tilde{\phi}^3 + a \int \tilde{\phi}^2 (\Box - m^2) \tilde{\phi} = \lambda \int \tilde{\phi}^3 \]  

(3.4)

and therefore the correlator is still equal to \(\lambda\) (For the 4-point correlator, the calculation is a bit more involved, but the result is the same.). But we see that if we compute \(S_3[\tilde{\phi}|_{bd}]\) to get the 3-point correlator of the boundary theory, it will differ from \(S_3[\phi|_{bd}]\) by \(\int \partial_\mu (\tilde{\phi}^2 \partial_\mu \tilde{\phi})\). So a nonlinear redefinition of bulk fields, which doesn’t change the masses, does change the boundary correlators.

However, the following observation was made in [25]. The extra term

\[ \int_M \partial_\mu (\phi^2 \partial^\mu \phi) \]  

will contribute only a contact term to correlators, because we have

\[ \frac{\delta S}{\delta \phi(x) \delta \phi(y) \delta \phi(z)} \propto \delta(x - y) \]  

(3.6)

where \(\phi(x)\) here lives on the boundary. Still, if we have the correct action from the start, no unwanted contact terms will appear. But we had only contact terms due to the simplicity of the example. The field redefinition used in [10, 11] involves also derivatives, and is of the type \(\phi = \tilde{\phi} + a \tilde{\phi}^2 + b (\partial_\mu \tilde{\phi})^2\). Then, substituting into (3.1), we get an extra cubic term to be added to (3.2) (and some higher order terms too)

\[ \int [2b \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \partial_\nu \tilde{\phi} + m^2 b \tilde{\phi} (\partial_\mu \tilde{\phi})^2 ] \]  

(3.7)

which can be rewritten by partial integration as

\[ b \int (\partial_\mu \tilde{\phi})^2 (m^2 - \Box) \tilde{\phi} - b \int \partial_\nu ((\partial_\mu \tilde{\phi})^2 \partial_\nu \tilde{\phi}) \]  

(3.8)

so again, on shell we get only a boundary term contribution to the 3-point correlator. But this time it is not just a contact term, as it was also noticed in [25]. We will come back to the discussion of [25] at the end of this section.

Let’s now turn to the 3-point functions of scalars.

**7 dimensions** The gauged sugra scalar fields in 7 dimensions are described by a coset element \(\Pi^i_A \in SL(5, R)/SO(5)_c\). In the physical gauge, it is symmetric and traceless. In terms of scalar fluctuations, \(\delta \pi_{Ai}\), we can write:

\[ \Pi^i_A = e^{\delta \pi_{Ai}} = [1 + \delta \pi + \frac{\delta \pi^2}{2} + \frac{\delta \pi^3}{3!} + ...]_{Ai} \]

\[ (\Pi^{-1})^i_A = e^{-\delta \pi_{Ai}} = [1 - \delta \pi + \frac{\delta \pi^2}{2} - \frac{\delta \pi^3}{3!} + ...]_{Ai} \]  

(3.9)
And so

\[ T_{ij} = [1 - 2\delta\pi + 2\delta\pi^2 - \frac{4}{3}\delta\pi^3 + \ldots]_{ij} \]

\[ TrT = 5 + 2Tr\delta\pi^2 - \frac{4}{3}Tr\delta\pi^3 + \ldots \] (3.10)

where \( T_{ij} = (\Pi^{-1})_{i}^{A}(\Pi^{-1})_{j}^{A} \). From \( P_{\alpha ij} = [\Pi^{-1}\partial_{\alpha}\Pi]_{ij} \), we get by expansion

\[ P_{\alpha ij}P^{\alpha ij} = Tr(\partial_{\alpha}\pi^2) + 0 - \frac{5}{6}Tr((\bigtriangledown\pi)\pi^3) - \frac{1}{2}Tr[(\partial_{\alpha}\pi^2)\pi^3] \] (3.11)

We notice that the cubic terms in \( P_{\alpha ij}P^{\alpha ij} \) cancel, but the quartic ones don’t.

So, the cubic action for the scalars in the 7d gauged sugra is

\[ -\frac{1}{2}P_{\alpha ij}P^{\alpha ij} + \frac{1}{4}(T^2 - 2T_{ij}T^{ij}) = -\frac{1}{2}Tr(\partial_{\alpha}\pi^2)^2 + Tr\delta\pi^2 + 2Tr\delta\pi^3 \] (3.12)

Let us describe the work Corrado et al. [1] and Bastianelli et al. [2] in 7 dimensions for computing correlators of CPOs in the boundary CFT from the scalar fields correlators in AdS space and see what one can learn from this (This procedure was introduced for the first time by Lee et al. [10] for the study of AdS_5/N=4 SYM correspondence.).

Again, if one compactifies the 11d sugra action on \( AdS_7 \times S_4 \) as in [3, 12], one can write an ansatz for the fields as

\[
\begin{align*}
G_{\Lambda\Pi} &= \tilde{g}_{\Lambda\Pi} + h_{\Lambda\Pi} \\
h_{\alpha\beta} &= h'_{(\alpha\beta)} + \left( \frac{h'_2}{7} - \frac{h_2}{5} \right) \\
h_{\mu\nu} &= h_{(\mu\nu)} + \frac{h_2}{4}g_{\mu\nu} \\
F^{(4)} &= F^{(4)} + da^{(3)}
\end{align*}
\] (3.13)

where the notation \( \tilde{g}_{\Lambda\Pi} \) means background metric and \( h_{(\mu\nu)} \) stands for symmetric and traceless. The indices \( \Lambda, \ldots \) are 11d, \( \alpha, \ldots \) are \( AdS_7 \) and \( \mu, \ldots \) are \( S_4 \) indices. One decomposes the sugra fields (in the gauge \( D_{\mu}h_{(\mu\nu)} = D_{\mu}h_{(\mu\alpha)} = D_{\alpha}h_{(\mu\alpha)} = 0 \)) in spherical harmonics:

\[
\begin{align*}
h'_{\alpha\beta} &= \sum h'_{(\alpha\beta)I}Y^I \\
h_{(\mu\nu)} &= \sum \phi^IY^I_{(\mu\nu)} \\
h' &= \sum h'^{I}Y^{I} \\
h_2 &= \sum h_{2I}Y^{I} \\
a_{\mu\rho\sigma} &= \sum b'_{\epsilon_{\mu\rho\sigma}}D^{\sigma}Y_{I}
\end{align*}
\] (3.14)

where \( \Box_{2}Y^{I}(x) = -k(k + 3)Y^{I}(x) \). The scalar kinetic term is diagonalized by the eigenvectors

\[ s' = \frac{k_1}{2k_1 + 3}(h'_2 + 32/\sqrt{2}(k + 3)b') \]
\[
\begin{align*}
I^I &= \frac{k + 3}{2k + 3} \left( h_I^2 - 32/\sqrt{2kb_I} \right) \\
\Box s^I &= k(k - 3), k \geq 2 \\
\Box t^I &= (k + 3)(k + 6), k \geq 0 
\end{align*}
\]

Now the cubic action gives the equations of motion

\[
(\Box - k(k + 3)) \phi^I = D^I D_k (s^I - s^I) + E^I D_k D \phi^I + F^I D_k D \phi^I + G^I D_k D \phi^I + ...
\]

The condition of getting rid of nonlinear terms with derivatives in the equations of motion suggests the rotation:

\[
\begin{align*}
\phi^I &= \tilde{\phi}^I + \phi^I D_k \tilde{s}^I + \tilde{L}^I D_k \tilde{s}^I + L^I D_k \tilde{s}^I \\
\phi^I &= \tilde{\phi}^I + \phi^I D_k \tilde{s}^I + \tilde{L}^I D_k \tilde{s}^I + L^I D_k \tilde{s}^I \\
\lambda_1^I &- \lambda_2^I + \lambda_3^I = \lambda_4^I + \lambda_5^I + \lambda_6^I + \lambda_7^I + ... + \lambda_9^I
\end{align*}
\]

In terms of the redefined fields, the equations of motion become

\[
\begin{align*}
(\Box - k(k + 3)) \phi^I &= \lambda_1^I \phi^I \tilde{s}^I + \lambda_2^I \phi^I \tilde{s}^I + \lambda_3^I \phi^I \tilde{s}^I + \lambda_4^I \phi^I \tilde{s}^I \\
(\Box - k(k - 3)) \phi^I &= \lambda_5^I \phi^I \tilde{s}^I + \lambda_6^I \phi^I \tilde{s}^I + \lambda_7^I \phi^I \tilde{s}^I + \lambda_8^I \phi^I \tilde{s}^I
\end{align*}
\]

where

\[
\lambda_1^I = -\frac{9k_2!k_3!\Gamma(k_1 + 5/2)2^{3k_1 - 3\Sigma/2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)\Gamma(\alpha_3 + 1)\Gamma(\Sigma/2)2k_2(2k_2 + 1)k_3(2k_2 + 1)} \frac{(2\alpha_1 - 3)\Sigma(\Sigma + 1)(\Sigma + 3)}{\alpha_1(\alpha_1 - 1)} - T^I C^I D^I C^I > 
\]

\[

\lambda_2^I = \frac{9k_2!k_3!\Gamma(k_1 + 5/2)2^{3k_1 - 3\Sigma/2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)\Gamma(\alpha_3 + 1)\Gamma(\Sigma/2)2k_2(2k_2 + 1)k_3(2k_3 + 1)} \frac{(2\alpha_1 - 3)\Sigma(\Sigma + 1)(\Sigma + 3)}{\alpha_1(\alpha_1 - 1)} - T^I C^I D^I C^I > 
\]

and where $< C^I C^I C^I >$ and $< T^I C^I D^I C^I >$ are similar expressions.

We used the notation:

\[
\alpha_i = \frac{1}{2} \sum_{j=1}^{3} k_j - k_i \\
\Sigma = k_1 + k_2 + k_3
\]
The expression $< T^{I_1} C^{I_2} C^{I_3} >$ is non-zero if the 'modified triangle inequalities' are satisfied: $\alpha_1 \geq 1, \alpha_2 \geq 0, \alpha_3 \geq 0$, together with the condition that $\Sigma$ be even.

For the s-s-s vertex, if $k_2 = k_3 = 2$, we have two possibilities: $k_1 = 2, 4$. The only massive coupling ($k_1 = 4$) is extremal, and it vanishes due to the factor of $\alpha_1 = 0$, (this fact also signals the possibility of a consistent truncation to the massless sector) but the corresponding CFT correlator is finite, because it is obtained after multiplying with a factor of

$$\frac{\Gamma(2\alpha_1)\Gamma(2\alpha_2)\Gamma(2\alpha_3)}{\Gamma(2k_1 - 3)\Gamma(2k_2 - 3)\Gamma(2k_3 - 3)} \quad (3.23)$$

and we use that $\Gamma(2\alpha_1)/\Gamma(\alpha_1) \to 1/2$ when $\alpha_1 \to 0$. This analytical continuation procedure was discussed by Liu and Tseytlin [20] who noticed that although the coupling dilaton-dilaton-massive singlet ($M^2 = 32$) vanishes, the 3-point function of associated CPOs does not. For the $\phi - s - s$ vertex, the rescaling factor is finite, namely

$$\frac{\Gamma(2\alpha_3 + 3)\Gamma(2\alpha_1 - 3)\Gamma(2\alpha_2 + 3)}{\Gamma(2k_2)\Gamma(2k_3)\Gamma(2k_1 + 6)} \quad (3.24)$$

so the CFT correlator computed from $\lambda_{222}^\phi$ remains zero.

We will argue that the nonlinear field redefinition (3.17) is not just a matter of conveniently getting rid of unwanted higher-derivative terms in the scalar field action, but it is precisely (when truncated to the massless sector) a Taylor expansion of the nonlinear KK ansatz [1, 2] in the transverse fluctuation gauge. So, in this respect, it is not an unnatural redefinition, but it is the one which gives the gauged susy action. For instance a nonlinear redefinition of the 3-index antisymmetric tensor $a_{\alpha\beta\gamma} \to a_{\alpha\beta\gamma} + \epsilon_{ABCDEF} F_{[\alpha \beta}^{AB} B_{C]}^{D} Y^{E} + \text{more}$ will generate part of the 7d CS terms, which we previously argued that are absent when one uses the linearized ansatz. Start with the nonlinear KK metric ansatz

$$G_{\alpha\beta}(x, y) = \Delta^{-2/5}(x, y) g_{\alpha\beta}(y) \quad (3.25)$$
$$G_{\mu\nu}(x, y) = \Delta^{4/5}(x, y) C^{A}(x) T_{AB}^{-1}(y) C^{B}(x) \quad (3.26)$$
$$G_{\mu\alpha} = 2\Delta^{4/5}(x, y) B^{AB}(y) Y^{B}(x) C^{C}_{\mu}(x) T_{AC}^{-1}(y) \quad (3.27)$$

where $\Delta^{-6/5} = Y \cdot T \cdot Y$, and the spherical harmonic satisfy the following identities: $\Box Y^{A}(x) = -4Y^{A}(x)$, $Y(x) \cdot Y(x) = \sum_{A=1}^{3} Y^{A} Y^{A} = 1$, $\partial_{\mu} Y^{A}(x) = C^{A}_{\mu}(x)$ is the conformal Killing vector and $C^{A}_{\mu} C^{B}_{\rho} = \delta^{AB} - Y^{A} Y^{B}$; indices are raised and lowered with Kronecker delta. Set the gauge fields to zero and expand in linear order in the scalar fluctuations $\delta \pi_{AB}$. Then

$$h_{\mu\nu} = 2C_{\mu} \cdot \delta \pi \cdot C_{\nu} + \frac{4}{3} Y \cdot \delta \pi \cdot Y \cdot \delta_{\mu\nu} \quad (3.28)$$

will not be in the transverse gauge, and we need a compensating Einstein transformation with parameter $\xi_{\nu} = -C_{\nu} \cdot \delta \pi \cdot Y$ to satisfy the gauge condition $\tilde{D}^{\mu} h_{i(\mu)} = 0$. Thus, in the transverse gauge, and up to quadratic order in fluctuations $h_{i(\mu)} = 0$. The gauge condition $\tilde{D}^{\mu} h_{\mu\alpha} = 0$ also implies the need of another Einstein transformation

\[ \tilde{g} = \sum_{n=1}^{\infty} (L_{\xi})^{n} \tilde{g} \] where $L_{\xi}$ is the Lie derivative along $\xi$, and $\tilde{g}$ is the transformed metric
with parameter $\xi_\alpha = 1/2Y \cdot \hat{D}_\alpha \delta \pi \cdot Y$ and one gets $h_{\mu \nu} = 0$, after performing the Einstein transformations. To second order in fluctuations, and in the transverse gauge we have

$$h_{(\mu \nu)} = \frac{4}{9} C(\mu \cdot \delta \pi^2 \cdot C_\nu) - \frac{4}{3} C(\mu \cdot \delta \pi \cdot C_\nu Y \cdot Y \cdot Y \cdot Y + \frac{4}{3} C(\mu \cdot \delta \pi \cdot Y C_\nu) \cdot \delta \pi \cdot Y + \frac{1}{9} C(\mu \cdot \hat{D}_\alpha \delta \pi \cdot \hat{D}^{\alpha} \cdot \delta \pi \cdot Y C_\nu) \cdot \delta \pi \cdot Y - \frac{1}{3} C(\mu \cdot \hat{D}_\alpha \delta \pi \cdot Y C_\nu) \cdot \hat{D}^{\alpha} \cdot \delta \pi \cdot Y$$

(3.29)

where, again we needed a compensating Einstein transformation with parameter

$$\tilde{\xi}_\nu = -\frac{20}{9} C_\nu \cdot (\delta \pi)^2 \cdot Y - \frac{1}{12} Y \cdot \tilde{D}_\alpha \delta \pi \cdot Y C_\nu \cdot \tilde{D}^{\alpha} \cdot \delta \pi \cdot Y + \frac{1}{18} Y \cdot \tilde{D}_\alpha \delta \pi \cdot \tilde{D}^{\alpha} \cdot \delta \pi \cdot C_\nu$$

(3.30)

Using now that the first massive mode in $h_{\mu \nu}(y, x) = \phi^I(y) Y_I(x)$ has a spherical harmonic

$$Y_{\mu \nu}^{ABCD} = C(\mu C_\nu)^B \cdot Y(C Y^D) - \frac{1}{2} C(\mu C_\nu)^B Y(A Y^D) - \frac{1}{2} C(\mu C_\nu)^B Y(B Y^D)$$

$$- \frac{1}{2} C(\mu C_\nu)^B Y(B Y^D) - \frac{1}{2} \delta^{AB} C(\mu C_\nu)^B + \delta^{CD} C(\mu C_\nu)^B + \delta^{BC} C(\mu C_\nu)^B - \delta^{AB} C(\mu C_\nu)^B$$

(3.31)

where the symmetry in the $\{ABCD\}$ indices is given by the box Young tableau (therefore it is traceless in any pair of indices) and $\square_y Y_{\mu \nu}^{ABCD} = -8 Y_{\mu \nu}^{ABCD}$, we notice that (3.29) can be rewritten as

$$h_{(\mu \nu)} = \frac{1}{6} (-4 \delta \pi^{AB} \delta \pi^{CD} + \hat{D}_\alpha \delta \pi^{AB} \hat{D}^{\alpha} \delta \pi^{CD}) Y_{\mu \nu}^{ABCD}$$

(3.32)

For the same massive mode the nonlinear redefinition reads:

$$0 = \phi^{\mu \nu} \cdot (\hat{D}_\alpha s^{AB} \hat{D}^{\alpha} s^{CD} + 4 R^{-2} s^{AB} s^{CD})$$

(3.33)

where $R$ is the $S_4$ radius (for us, $R = 1$, while in $R = 1/2$).

Thus we explicitly showed that the nonlinear redefinition coincides with the nonlinear KK ansatz in the transverse gauge.

In conclusion, since after the nonlinear rotation we get a consistent truncation, and moreover, we get the correct gauged sugra terms (up to cubic order in fluctuations), we can say that the nonlinear ansatz in the $\{\}$ is the correct one to use in the AdS-CFT correspondence.

---

\[ \text{Footnote 5: The constraint } (h^I - 9/10 h^I) \hat{D}(\alpha \hat{D}^I) Y^I = 0 \text{ is trivially satisfied on-shell if we restrict ourselves to the massless scalar sector, as it yields the scalar field eq. } Y \cdot (\square + 2 \delta \pi) \cdot Y = 0. \]

\[ \text{Footnote 4: One cannot directly read off from here the relationship between } \delta \pi^{AB} \text{ and } s^{AB}, \text{ one of the reasons being that the spherical harmonics were normalized differently in } \{\} \text{ and in } [\]. \]
Let us now discuss the 5-dimensional case. In fact, it is a characteristic of gauged sugras that the kinetic term is \( P_{\alpha ij} P^{\alpha ij} \), with \( P_{\alpha ij} = [\Pi^{-1} \partial_{\alpha} \Pi]_{(ij)} \), with \( \Pi \) the scalar coset vielbein. It follows that the kinetic term has always two derivatives, and we also find no cubic term in \( P^2_{\alpha ij} \). Now, we would like to see that the gauged sugra action is the correct one to use for the AdS-CFT correspondence.

Lee et al. [10] looked at the 10d IIB action compactified on \( AdS_5 \times S_5 \). The fluctuations are written as:

\[
G_{mn} = g_{mn} + h_{mn}, \quad h_{\alpha\beta} = h(\alpha\beta) + \frac{h_2}{5}, \quad h_{\mu\nu} = h'(\mu\nu) - \frac{h_2}{3} g_{\mu\nu} + \frac{h'}{5} g_{\mu\nu}, \quad F = \bar{F} + 5 \nabla_{[\alpha jklm]} \]

(3.34)

If one decomposes linearly in spherical harmonics as:

\[
h'_{\mu\nu} = \sum h'_{\mu\nu} Y^I, \quad h_2 = \sum h_{2I} Y^I, \quad a_{\alpha_1\alpha_2\alpha_3\alpha_4} = \sum \nabla^a \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4} b_I Y^I, \quad a_{\mu_1\mu_2\mu_3\mu_4} = \sum a_{\mu_1\mu_2\mu_3\mu_4} Y^I \]

(3.35)

the constraints on the fields can be solved and the \( h_I^I, B^I \) system is diagonalized by

\[
s^{I} = \frac{1}{20(k+2)}[h_2^I - 10(k+4)b^I] \]

\[
t^{I} = \frac{1}{20(k+2)}[h_2^I + 10kb^I] \]

(3.36)

such that

\[
\Box s^{I} = k(k-4) s^{I}, \quad k \geq 2 \]

\[
\Box t^{I} = (k+4)(k+8) t^{I}, \quad k \geq 0 \]

(3.37)

If one compactifies the type IIB action in 10d one obtains the following equations of motion for the \( s \) fields (up to quadratic order in fluctuations)

\[
(\Box - m_{I_1}^2) s^{I_1} = \sum_{I_2, I_3} \left( D_{I_1 I_2 I_3} s^{I_2} s^{I_3} + E_{I_1 I_2 I_3} \nabla_{\mu} s^{I_2} \nabla^{\mu} s^{I_3} + F_{I_1 I_2 I_3} \nabla^{(\mu} \nabla^{\nu)} s^{I_2} \nabla_{(\mu} \nabla^{\nu)} s^{I_3} \right) \]

(3.38)

where \( D_{I_1 I_2 I_3}, E_{I_1 I_2 I_3} \) and \( F_{I_1 I_2 I_3} \) are constants depending on \( k_1, k_2, k_3 \). But in order to get rid of the terms in the equations of motion nonlinear in \( s^I \) and involving derivatives,
one needs to make a nonlinear redefinition of fields,

\[ s^{I_1} = s^{I_1} + \sum_{I_2, I_3} (J_{I_1 I_2 I_3} s^{I_2} S^{I_3} + L_{I_1 I_2 I_3} \nabla^\mu s^{I_2} \nabla_\mu s^{I_3}) \]

\[ J_{I_1 I_2 I_3} = \frac{1}{2} E_{I_1 I_2 I_3} + \frac{1}{4} F_{I_1 I_2 I_3} (m_{I_1}^2 - m_{I_2}^2 - m_{I_3}^2 - 8) \]

\[ L_{I_1 I_2 I_3} = \frac{1}{2} F_{I_1 I_2 I_3} \]

which modifies the equations of motion to:

\[ (\Box - m_{I_1}^2) s^{I_1} = \sum_{I_2, I_3} \lambda_{I_1 I_2 I_3} s^{I_2} s^{I_3} \]

where

\[ \lambda_{I_1 I_2 I_3} = \frac{k_1 k_2 k_3}{\alpha_1! \alpha_2! \alpha_3!} \frac{(k + 1)2^{k+2-\Sigma/2}\Sigma((\Sigma/2)^2 - 1)((\Sigma/2)^2 - 4)}{k_1(k_1 - 1)(k_2 + 1)(k_3 + 1)(\Sigma/2 + 2)!} \alpha_1 \alpha_2 \alpha_3 < C^{I_1 I_2 I_3} > \]

We notice that the invariant tensor \(< C^{I_1 I_2 I_3} >\) is nonzero (one can contract the indices correctly) only if the ‘triangle inequalities’ \(\alpha_i \geq 0\) are satisfied, together with the the condition that \(\Sigma\) is even. If the coupling \(\lambda_{I_1 I_2 I_3}\) corresponds to a massive mode and two massless modes, i.e. \(k_2 = k_3 = 2\), then \(k_1\) is restricted to be 2 (massless) or 4 (massive). The latter case is ‘extremal’, in the sense that \(\alpha_1\) takes the extreme value zero. But in the extremal case \(\lambda_{I_1 I_2 I_3} = 0\) because the \(\alpha_1\) factor vanishes.

The fact that after the nonlinear redefinition one has a consistent truncation, both in 5 and in 7 dimensions, was noticed already in a paper we wrote with Peter van Nieuwenhuizen \[\text{[1]}\], but we gave no details there. Afterwards, Aryutunov and Frolov wrote a series of papers where they found similar results. In \[\text{[27, 28]}\], the cubic and quartic terms in the action were calculated by expanding the 10d IIB action in scalar fluctuations, and after a nonlinear redefinition of fields they also found a consistent truncation to the massless sector (the calculation also involves more fields, not just the \(s^I\) scalars), i.e. all couplings between massless scalars and one massive scalar vanish.

Moreover, in \[\text{[29]}\] they find that the corresponding action for the massless scalars (to quartic order) coincides with the terms in the gauged sugra action.

After finding the coupling \(\lambda_{I_1 I_2 I_3}\), one can use it to compute correlators of CPO’s in the boundary CFT using the formulas in \[\text{[3]}\] which are found to agree with the weak coupling result. But for that one needed a nonlinear redefinition of fields, that is effectively modifying the linear ansatz in \[\text{[17]}\] to a nonlinear one. This nonlinear ansatz gives a consistent truncation, because one can consistently put all the massive \(s^I\) in \[\text{[17]}\] to zero. We notice that when going from a 3-point coupling in AdS space to a correlator on the boundary, one picks up a factor

\[ \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(k_1 - 2)\Gamma(k_2 - 2)\Gamma(k_3 - 2)} \]

The fact that the denominator becomes infinite is absorbed in the normalization of the operators \[\text{[3]}\]. We notice that the \(\Gamma(\alpha_1)\) in the numerator becomes infinite, so that the

\[ \Gamma(\alpha_1) \approx (k - 2)^{\alpha_1} \]

The 2-point function of CPOs coupled to the \(k = 2\) scalar fields behaves \(\sim (k - 2)^2\), and therefore vanishes. To get a nonvanishing result (in accord with the CFT calculation) one has to rescale the supergravity fields with the infinite factor \(1/(k - 2)\). This can be interpreted as another example of analytical continuation for the ‘extremal’ correlator \(k_1 = k_2 = 2\).
'extremal' correlator becomes nonzero. We will say more about that at the end of this section.

In the $AdS_5 \times S^5$ case, no fully consistent KK truncation is known, but it is generally believed that one exists. If this is the case, the procedure of taking the consistent truncation is seen to be the correct one.

Another point to be stressed is that before the rotation, the action has a term cubic in scalars, but with two derivatives. We have seen that in the gauged sugra action we don't have such a term, so the fact that the nonlinear rotation removes it is another confirmation of our procedure.

Finally, we shall comment on the calculation in [25]. This paper tried to address the following puzzle raised by the calculation in [10]. If one takes the limit when $k_1 \rightarrow k_2 + k_3$ in the calculation of [10], the coefficient of the cubic action for the scalars tend to zero, but the integration diverges in such a way that the 3-point function becomes zero.

To address this issue, D'Hoker et al. study the $t\phi\phi$ three point function, and instead of using the nonlinear redefinition of fields used in [10] (as we argue that is the correct procedure), use equations of motion and partial integrations to arrive at

$$2k_5^2 S_{cubic} = -8 \frac{(\Sigma + 4)\alpha_1(\alpha_2 + 2)(\alpha_3 + 2)}{(k_1 + 3)} \int_{AdS_5} a(k_1, k_2, k_3) t^{k_1} \phi^{k_2} \phi^{k_3} +$$

$$\int_{\partial(AdS_5)} \frac{a(k_1, k_2, k_3)}{k_1 + 3} (-D_n \phi^{k_3} D^\mu \phi^{k_1} D^\mu \phi^{k_2} - D_n \phi^{k_2} D^\mu \phi^{k_1} t^{k_1} D^\mu \phi^{k_3}$$

$$+ D_n t^{k_1} D_{mu} \phi^{k_2} D^\mu \phi^{k_3}) + \text{contact terms}$$

(3.43)

where $\Sigma = \frac{1}{2}(k_1 + k_2 + k_3), \alpha_1 = \frac{1}{2}(k_2 + k_3 - k_1), \text{etc.}$, obtaining what we described at the beginning of this section, namely that the difference between making a nonlinear redefinition of fields and using equations of motion and partial integrations is given by boundary terms. The boundary terms of the type in (3.3) are contact terms which were dropped, and the boundary terms in (3.3) are of the same type as the ones in (3.43). We indeed notice that the coefficient of the bulk integral in (3.43) becomes equal to zero for $k_1 = k_2 + k_3$. The point of view adopted in [25] is the following. At $k_1 < k_2 + k_3$ only the bulk integral contributes, and the boundary one doesn’t. But at $k_1 = k_2 + k_3$, the situation is reversed: only the boundary integral contributes, and the boundary one doesn’t. Moreover, the result for $k_1 = k_2 + k_3$ coincides with the one from the limit $k_1 \rightarrow k_2 + k_3$.

Our point of view is that we need to start with only the bulk integral in (3.43) (in other words make the nonlinear redefinition of fields). The analytic continuation $k_1 \rightarrow k_2 + k_3$ gives the correct result. With the linearized ansatz one also gets the boundary integral. If one considers it to be nonzero as [25] does, then one can only spoil the result by a factor of 2, in this example.

We note that for this case of scalar fields, this boundary terms seem to contribute only for 'extremal correlators' ($k_1 = k_2 + k_3$), with a singular limit needed to be taken, but for general fields (gauge fields, for instance) the same will probably not happen.

Indeed, as an example, for gauge fields we saw that the Chern-Simons term is completely missed by the linearized ansatz. A nonlinear redefinition in 7d which would give it would have to involve $\epsilon_{\alpha_1...\alpha_7}$. And for such a redefinition the $A_\mu$ equation of motion ($\Box A_\mu - \partial_\mu \partial_\nu A_\nu = 0$) is not very useful either in terms of creating the wanted
Chern-Simons term by partial integration and use of the equations of motion. Neither
the natural redefinition one would think of, $A_{\alpha 1} \rightarrow A_{\alpha 1} + \epsilon_{\alpha 1 \ldots \alpha 7} \partial^{\alpha 2} A^{\alpha 3} \partial^{\alpha 4} A^{\alpha 5} \partial^{\alpha 6} A^{\alpha 7}$, nor any other combination is able to reproduce the required Chern-Simons term. To
be explicit, the nonlinear redefinition has to involve massive fields being redefined too:
masse$\rightarrow$ massive + (massless)$^n$. Similarly, for the use of equations of motion one
needs to use the massive equations of motion to obtain a Chern-Simons term.

But if one takes as a starting point the linearized ansatz, one will obtain no anomalous CFT correlators. That is because for that one needs an $\epsilon$ symbol. In 5 dimensions, the anomaly should be in the 3 point function already, but clearly the 3 point vertex is non-anomalous (because the $\epsilon$ symbol comes only from the Chern-Simons which is now absent). In 7 dimensions, the anomaly starts at the 4 point function. So even though there is no Chern-Simons term, one might hope that the exchange diagram, where massive fields are exchanged, could contribute. But using the linearized ansatz we don’t get any couplings involving the $\epsilon$ symbol of the type gauge field-gauge field-massive field. Therefore again, no anomaly on the CFT side. From this discussion, one concludes that, even if one does the steps in [25], namely partial integration and the use of the bulk equations of motion, if one obtains the Chern-Simons term, it will be together with extra boundary terms canceling the effect of the anomaly in the boundary correlators!

So we have an example where, even if the methods of [25] can be applied, the boundary terms which are generated will contribute not only to ‘extremal’ correlators, but to the ‘massless’ ones as well.

Therefore, here (for the scalar field case) it is somewhat a matter of taste which philosophy one takes, maybe the one of D’Hoker et al. looks more attractive, however one has to consider a more general case. As we have seen, we have an argument that taking the nonlinear ansatz from the start produces the correct anomaly, the correct gauge invariant AdS correlators (so correct $R$-invariant CFT correlators) and gets rid of unwanted contact terms. Therefore, it is better to have a clear physical principle to deal with all of the above.

For the ‘massless’ (gauged sugra) AdS fields, we know that gauge invariance and susy forces us to take the gauged sugra action. For the massive fields, we don’t know the nonlinear ansatz. But then we can do what Seiberg et al. and Corrado et al. did, namely to find order by order and ansatz which removes unwanted terms in the action (with too many derivatives, for instance). Ideally, one would have to do what we sketched in the introduction, namely to use gauge invariance and susy to fix the nonlinear ansatz for the tower of massive fields.

4 Conclusions

In this paper, we gave arguments for the previous conjecture that one has to use the gauged sugra action, as obtained from 11d, or 10d IIB sugra through a nonlinear KK ansatz, for the computation of n-point functions of CFT operators coupled to the massless (sugra) sector. A similar nonlinear rotation is needed for the massive tower in order to restore the gauge invariance of the action for the whole KK tower after performing the “nonlinear rotation” at the massless level. Part of this rotation, namely up to quadratic order in scalar fluctuations was already introduced by Lee et
al. [10], Corrado et al. [11] and Bastianelli and Zucchini [12]. But their reason for doing this was to eliminate certain higher derivative couplings from the reduced action.

Our arguments are based on:

- previously computed R-current correlators. In particular, we noticed that the linear KK ansatz completely misses the CS term (in both 5 and 7 dimensions) which corresponds to the R-current anomaly.

- we explicitly showed that the nonlinear rotation of [11, 12] corresponds to a Taylor expansion of the nonlinear KK ansatz [1, 2] (in the transverse gauge) in massless scalar fluctuations.

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References

[1] H. Nastase, D. Vaman and P. van Nieuwenhuizen, Phys. Lett. B 469 (1999) 96 and hep-th/9905075
[2] H. Nastase, D. Vaman and P. van Nieuwenhuizen, hep-th/9911238
[3] B.de Wit and H. Nicolai, Nucl. Phys. B281 (1987) 211
[4] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200
[5] S. Gubser, I.R. Klebanov, A.M. Polyakov, Phys. Lett. B428 (1998) 105, hep-th/9802109
[6] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150
[7] P. van Nieuwenhuizen, Class. Qu. Gr. 2 (1985) 1
[8] M. Pernici, K. Pilch and P. van Nieuwenhuizen, Phys.Lett. 143 B (1984) 103
[9] M. Pernici, K. Pilch and P.van Nieuwenhuizen, Nucl.Phys. B259 (1985) 460; M.Gunaydin, L.J.Romans and N.P.Warner, Phys.Lett. 154 B (1985) 268
[10] S. Lee, S. Minwalla, M. Rangamani, N. Seiberg, Adv. Theor. Math. Phys. 2 (1998) 697
[11] R. Corrado, B. Florea, R. McNees, Phys. Rev. D60 (1999) 085011
[12] F. Bastianelli and R. Zucchini, hep-th/9903161, hep-th/9907047 and hep-th/9909179
[13] D.Z. Freedman, S.D. Mathur, A. Matusis, L. Rastelli, Nucl. Phys. B546 (1999) 96
[14] G. Chalmers, H. Nastase, K. Schalm, R. Siebelink, Nucl.Phys. B540 (1999) 247
[15] F. Bastianelli, S. Frolov and A.A. Tseytlin, hep-th/9911135
[16] H. Nastase and D. Vaman, work in progress
[17] H.J. Kim, L.J. Romans, P. van Nieuwenhuizen, Phys. Rev. D32 (1985) 389
[18] O. Aharony, S. S. Gubser, J. Maldacena and H. Ooguri, hep-th/9905111
[19] J. A. Harvey, R. Minasian, G. Moore, hep-th/9808060, D. Freed, J. A. Harvey, R. Minasian and G. Moore, hep-th/9803205
[20] M. Henningson and K. Skenderis, JHEP 9906:012, 1999, hep-th/9905163
[21] R. Manvelyan, A.C. Petkou, hep-th/0003017
[22] K. Intriligator, hep-th/0001205
[23] K. Becker, M. Becker, hep-th/9911138
[24] E. Witten, J. Geom. Phys. 22 (1997) 1, hep-th/9610234
[25] E. D’Hoker, D.Z. Freedman, S.M. Mathur, A. Matusis and L. Rastelli, hep-th/9908160
[26] H. Liu, A.A.Tseytlin, JHEP 9910:003, hep-th/9906151
[27] G. Arutyunov, S. Frolov, hep-th/9907085
[28] G. Arutyunov, S. Frolov, hep-th/9912210
[29] G. Arutyunov, S. Frolov, hep-th/0002170