Higher Spin Gauge Theories in Any Dimension

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Abstract
Some general properties of higher spin gauge theories are summarized with the emphasize on the nonlinear theories in any dimension.

1 Introduction
First of all, I would like to thank the organizers for the invitation to talk on higher spin gauge theory at Strings 2004. Although a relationship of the higher spin (HS) gauge theory to superstring theory is not yet completely clear, the impressive convergency that took place during recent years indicates that HS theories and Superstring might be different faces of the same fundamental theory to be found. (For an interesting new argument in the same direction see [1] and the talk of M.Bianchi at this conference).

1.1 Symmetric Massless Free Fields
Simplest free HS gauge fields are so called totally symmetric fields. They can be described by rank $s$ totally symmetric tensors $\varphi_{n_1...n_s}(x)$ subject to the double-tracelessness condition $\varphi^{m}_{m k n_5...n_s}(x) = 0 \, \Box (m, n, \ldots = 0, \ldots, d - 1)$. The Abelian HS gauge symmetries $\delta \varphi_{n_1...n_s}(x) = \partial_{\{n} \varepsilon_{n_2...n_s\}}(x)$ with rank $s - 1$ symmetric traceless gauge parameters $\varepsilon_{n_1...n_{s-1}}(x)$ ($\varepsilon'_{r n_3...n_{s-1}}(x) = 0$) leave invariant the quadratic action

\begin{equation}
S^s = \frac{1}{2} (-1)^s \int d^d x \left\{ \partial_n \varphi_{m_1...m_s} \partial^n \varphi^{m_1...m_s} \right. \\
- \left( s(s - 1) \right) \partial_n \varphi_{r m_1...m_{s-2}} \partial^n \varphi^{k m_1...m_{s-2}} + s(s - 1) \partial_n \varphi_{r m_1...m_{s-2}} \partial_k \varphi^{n k m_1...m_{s-2}} \\
- \left. s \partial_n \varphi_{m_1...m_{s-1}} \partial_r \varphi^{r m_1...m_{s-1}} - \frac{s(s - 1)(s - 2)}{4} \partial_n \varphi_{r m_1...m_{s-2}} \partial_k \varphi^{t k m_1...m_{s-3}} \right\}.
\end{equation}
This action describes a spin $s$ massless field $[2]$ and generalizes the spin 1 Maxwell action and spin 2 (linearized) Einstein action to any integer spin. The key question is what is a fundamental unifying symmetry principle underlying HS gauge fields. Even at the free field level, the existence of elegant metric-like $[3]$ and frame-like $[4, 5]$ “geometric formulations” indicates that there must be some deep reason for HS theories to exist.

1.2 Higher Spin Problem

The problem is to find a nonlinear HS gauge theory such that it has

- Correct free field limit
- Unbroken HS gauge symmetries
- Non-Abelian global HS symmetry of a vacuum solution

The first condition demands the theory to be free of ghosts, that is to be equivalent to the Fronsdal theory for the case of totally symmetric fields. The third condition avoids trivial possibility of Abelian interactions built of Abelian gauge invariant HS field strengths like nonlinear terms built of higher powers of the spin 1 Abelian field strength instead of Yang-Mills interactions.

The HS problem is of interest on its own right. An additional stringy motivation is, in the first place, that it is tempting to interpret massive HS modes in Superstring as resulting from breaking of HS gauge symmetries. In that case, superstring should exhibit higher symmetries in the high-energy limit as was argued long ago by Gross $[6]$. A more recent argument came from the $AdS/CFT$ side after it was realized $[7]$ that HS symmetries should be unbroken in the Sundborg–Witten limit $\lambda = g^2 N \rightarrow 0$, $l_{str}^2 \Lambda_{AdS} \rightarrow \infty$ just because the boundary conformal theory becomes free. A dual string theory in the highly curved $AdS$ space-time is therefore going to be a HS theory.

1.3 Difficulties

Although the formulation of the HS problem may look rather flexible, the conditions on the HS interactions is so hard to satisfy that many believed it admits no solution at all. One difficulty is due to the $S$-matrix argument $a la$ Coleman-Mandula $[8]$ that, if $S$-matrix has too many symmetries carrying
nontrivial representations of the Lorentz symmetry as HS symmetries do, then
\( S = I \), i.e. there is no interactions.

Another one is the HS-gravity interaction problem as was originally pointed
out by Aragone and Deser in [9]. The point is that covariantization of deriva-
tives \( \partial \rightarrow D_n = \partial_n - \Gamma_n, \delta \varphi_{nm} \rightarrow D_n \varepsilon_m \) changes the situation drastically
because they do not commute, \([D_n, D_m] = \mathcal{R}_{nm} \ldots\), if the Riemann tensor
\( \mathcal{R}_{nm,pq} \) is nonzero. As a result, the variation of the covariantized HS action
under covariantized HS gauge theories is not any longer zero

\[
\delta S^{\text{cov}}_s = \int \mathcal{R}_{\ldots}(\varepsilon_{\ldots}D\varphi_{\ldots}) \neq 0 ?!
\] (2)

Most important is that the Weyl tensor part of the Riemann tensor contributes
to this variation for \( s > 2 \), which contribution seems to be hard to compensate
by any modification of the action and/or field transformations\(^1\).

1.4 Resolution

Despite the difficulties with HS interactions, in the important works [10, 11]
it was shown that some consistent (i.e., gauge invariant) interactions of HS
gauge fields with matter fields and with themselves do exist at least at the
cubic level

\[
S = S^2 + S^3 + \ldots \quad S^3 = \sum_{p,q,r} (D^p\varphi)(D^q\varphi)(D^r\varphi)\ell^{p+q+r+\frac{d-3}{2}},
\] (3)

where \( \ell \) is a parameter of dimension of length. These authors discovered that
interactions of HS fields contain higher space-time derivatives: the higher in-
teracting spins are, the more (but finite) number of derivatives appear in their
interactions.

Another important observation was [12] that the situation with HS interac-
tions and, in particular, with HS-gravitational interactions, improves once the
problem is reconsidered in the \((A)dS\) background. The dimensionful parame-
ter \( \ell \) then identifies with the radius of \((A)dS\) space \( \ell = \Lambda^{-\frac{1}{2}} = R_{AdS} \). The key
difference between flat and \((A)dS\) background is that, in the latter case, back-
ground covariant derivatives do not commute \([D_n, D_m] \sim \Lambda \sim O(1)\). This has

\(^1\)Recall that, for spin 3/2, analogous terms can be compensated by the modification of
the transformation of the metric tensor under local SUSY transformation because only Ricci
tensor contributes in this case, that opens a way towards supergravity.
an important consequence that the terms of different orders in derivatives in
the action \((3)\) do talk to each other. As a result, there exists such a unique
(modulo field redefinitions) combination of higher derivative interaction terms
that a contribution from their gauge variation cancels the problematic terms
\((2)\) of the flat space analysis. Since the coupling constants of the interaction
terms contain positive powers of the \((A)dS\) radius, they blow up in the flat
limit in agreement with the flat space no-go results. In \((A)dS\) space, the no-go
arguments do not work (in particular, no \(S\)-matrix). HS theories suggest deep
analogy between the \(AdS\) scale and string length scale \(\ell \sim \sqrt{\alpha'}\), although the
precise identification is far from being clear at the moment. Note that the role
of \((A)dS\) background in HS theories was realized years before the discovery of
the \(AdS/CFT\) correspondence \([13]\).

2 HS Fields as Gauge Connections

It is well-known that gauge fields of supergravity result from gauging the SUSY
algebra:

\[
\begin{array}{cccc}
o(d-1,2) & o(N) & T^{AB} & Q^{\alpha}_{\alpha} \\
\omega_{nAB} & \Phi_{n\alpha} & \eta^{pq} & A, B, \ldots = 0, \ldots d, \quad p, q, \ldots = 1, \ldots N \\
\end{array}
\]

In particular, \(s = 2\) gravitational field is described in supergravity by the frame
field \(e_{n}^{a}\) which along with the Lorentz connection \(\omega_{n}^{ab}\) can be interpreted
as components of the gauge field \(\omega_{n}^{AB}\) of the \(AdS_{d}\) algebra \(o(d-1,2)\)

\[
g_{nm} \rightarrow e_{n}^{a} \rightarrow \{e_{n}^{a}, \omega_{n}^{ab}\} \rightarrow \omega_{n}^{AB}
\]

Analogously, a totally symmetric field \(\varphi_{n1...ns}\) in the Fronsdal formulation
admits an equivalent description in terms of the gauge 1-form \(\omega_{n}^{A_{1}...A_{s-1},B_{1}...B_{s-1}}\)

\[
\varphi_{n1...ns} \rightarrow e_{n}^{a_{1}...a_{s-1}} \rightarrow \{e_{n}^{a_{1}...a_{s-1}}, \omega_{n}^{a_{1}...a_{s-1},b_{1}...b_{s-1}}\} \rightarrow \omega_{n}^{A_{1}...A_{s-1},B_{1}...B_{s-1}},
\]

which takes values in the irreducible representation of the \(AdS_{d}\) algebra \(o(d-1,2)\) depicted by the (traceless) rectangular two-row Young tableau

\[
\begin{array}{cccc}
\omega_{n}^{\{A_{1}...A_{s-1},A_{s}\}} B_{2}...B_{s-1} & = 0 \\
\omega_{n}^{A_{1}...A_{s-3} C, B_{1}...B_{s-1}} & = 0 \\
\end{array}
\]

\[
\frac{s-1}{o(d-1,2)}
\]
Let an $AdS$ vector $V_A$ define the Lorentz subalgebra $o(d-1,1) \subset o(d-1,2)$ as its stability subalgebra. The simplest choice is $V_B = \delta^d_B$ where $\hat{d}$ denotes the $(d+1)^{th}$ Lorentz invariant direction of an $AdS$ vector. The HS dynamical frame-like field is then identified with the components of the HS connection with a maximal possible number of $AdS_d$ vector components along the extra direction $V_A$

\[ e_n^{a_1...a_{s-1}} = \omega_n^{a_1...a_{s-1}} B_1...B_{s-1} V_{B_1} \ldots V_{B_{s-1}}. \]  

(5)

Analogously to the spin 2 metric case, Fronsdal field is the totally symmetric part of the frame field $\varphi_{n_1n_2...n_s} = e_{\{n_1,n_2...n_s\}}$. Generalized Lorentz connections identify with those components of the connection that carry more Lorentz indices

\[ \omega_n^{a_1...a_{s-1}, b_1...b_t} \sim \left( \frac{1}{\sqrt{\Lambda}} \frac{\partial}{\partial x} \right)^t (e) \]  so that every additional Lorentz index brings one derivative along with one power of $\ell = \frac{1}{\sqrt{\Lambda}}$. In this formalism, the higher derivatives of HS interactions, as well as negative powers of the cosmological constant, result from the dependence of the nonlinear terms in the HS actions on the higher generalized Lorentz connections.

### 3 Higher Spin Algebra $hu(1|2;[d−1,2])$

The structure of HS gauge connections suggests that they result from gauging a HS algebra that contains the $AdS_d$ algebra $o(d−1,2)$ as a subalgebra and decomposes under adjoint action of the latter into a sum of representations described by traceless two-row Young tableaux. In other words, generators $T_{A_1...A_n,B_1...B_n}$ of a HS algebra should satisfy the conditions $T_{\{A_1...A_n,A_{n+1}\}B_2...B_n} = 0$ and $T_{C,A_3...A_n,B_1...B_n} = 0$. Such an algebra was originally found by Eastwood [17] as conformal HS algebra of a scalar field in $d−1$ dimensions. For our purpose it is most convenient to use its oscillator realization suggested in [18].

Namely we introduce a canonical pair of $AdS$ vectors $Y_i^A$,

\[ [Y_i^A, Y_j^B] = \epsilon_{ij} \eta^{AB}, \]  

(7)

where $\eta_{AB} = \eta_{BA}$ is the $AdS_d$ invariant metric and $\epsilon_{ij} = -\epsilon_{ji}, i,j = 1,2$ is the $sp(2)$ invariant form. Here we use the star product notation for the
oscillator algebra defined by the relations (7) (for its precise definition see eq.(25) for $Z$-independent functions). The bilinear combinations of oscillators, $T^{AB}$ and $t_{ij}$,

$$T^{A,B} = -T^{B,A} = \frac{1}{2}\{Y_i^A, Y_j^B\}_\ast \epsilon^{ij}, \quad t_{ij} = t_{ji} = \frac{1}{2}\{Y_i^A, Y_j^A\}_\ast,$$

(8)

form, respectively, the $o(d-1,2)$ generators, which rotate $AdS_d$ vector indices $A, B$, and $sp(2)$ generators, which rotate symplectic indices $i, j$. They commute to each other, $[t_{ij}, T^{A,B}]_\ast = 0$, thus being Howe dual. Let us note that the $sp(2)$ plays here a role analogous to that in the description of dynamical models in the conformal framework (two-time physics) [15, 16].

Now it is easy to define the simplest HS algebra $hu(1|2;[d-1, 2])$ by virtue of a sort of Hamiltonian reduction. First one considers the Lie algebra of functions of oscillators with the Lie bracket $[f(Y), g(Y)]_\ast$. Then one considers its subalgebra $S$ spanned by $sp(2)$ invariants

$$[f(Y), t_{ij}]_\ast = 0$$

(9)

and next its quotient $S/I$ where the ideal $I$ is spanned by the elements proportional to the $sp(2)$ generators i.e., $\{f \in I : f(Y) = t^{i\ell}_j \ast f_{i\ell j} = f_{r i j} \ast t^{\ell j} \sim 0.\}$ The algebra $S/I$ we call $hu(1|2;[d-1, 2])$ (upon imposing appropriate reality conditions [18]).

The gauge fields of $hu(1|2;[d-1, 2])$ are

$$\omega(Y|x) = \sum_{n=0}^{\infty} dx^n \omega_{m A_1 \ldots A_n, B_1 \ldots B_n}(x) Y_1^{A_1} \ldots Y_n^{A_n} Y_1^{B_1} \ldots Y_n^{B_n}.$$

(10)

It is easy to see that the condition (9) (which can be written in the covariant form $Dt_{ij} = dt_{ij} + [\omega, t_{ij}]_\ast = 0$ taking into account $dt_{ij} = 0$) imposes the Young properties

$$\omega_{m \{A_1 \ldots A_n, A_{n+1}\} B_2 \ldots B_n} = 0$$

(including the property that the r.h.s. of (10) contains equal numbers of oscillators $Y_1^A$ and $Y_2^A$). According to (8), the factorization over the terms proportional to $t_{ij}$ is equivalent to factorization of traces of the gauge field components in (10) that gives rise precisely to the set of gauge fields associated with different spins as explained in section 2.

6
The non-Abelian HS field strength is
\[ R = d\omega(Y|x) + \omega(Y|x) \wedge *\omega(Y|x), \tag{11} \]
where terms on the r.h.s., which take values in the ideal \( I \), are factored out. The infinite-dimensional HS algebra contains the maximal finite-dimensional subalgebra \( o(d-1,2) \oplus u(1) \) spanned by bilinears in oscillators and constants, respectively. Different spins correspond to homogeneous polynomials \( \omega(\mu Y|x) = \mu^{2(s-1)}\omega(Y|x) \). The gauge fields of \( o(d-1,2) \oplus u(1) \) carry spin 2 and spin 1 respectively. That \( o(d-1,2) \oplus u(1) \) is a maximal finite-dimensional subalgebra of \( hu(1|2;d-1,2) \) is a consequence of the fact that the commutator of degree \( p \) and degree \( q \) polynomials of oscillators gives a degree \( p + q - 2 \) polynomial. For example, if spin 3 associated with degree 4 polynomials in oscillators appears, polynomials of all higher degrees appear in the closure of its generators. Thus, beyond the barrier of spin 2, the systems of HS fields are necessarily infinite. Let us note that there exists a generalization of the non-linear HS gauge theory to the case with HS gauge connection carrying matrix indices \( \omega \rightarrow \omega_{pq}(Y|x) \quad p, q = 1, \ldots, n \) so that the spin 1 Yang-Mills algebra is promoted to \( u(n) \) (HS models with the Yang-Mills gauge algebras \( o(n) \) and \( u_{sp}(n) \) also exist [18]).

### 4 Lower Spin Examples

To illustrate the idea of the approach that allows us to formulate nonlinear HS dynamics let us start with lower spin examples.

The nonlinear \( s = 2 \) equations are equivalent to zero-torsion condition \( R^a = 0 \) together with the Einstein equation in the form
\[ R^{ab} = e_c \wedge e_d \, C^{ac, \, bd}, \tag{12} \]
where \( e \) is the frame 1-form and \( C^{ac, \, bd} \) has algebraic properties of the Weyl tensor, i.e. it is traceless and symmetrization over three indices gives zero\(^2\). The equation (12) tells us that \( C^{ac, \, db} \) is indeed the Weyl tensor and, because it is traceless, that the Ricci tensor is zero. Bianchi identities then imply at

\({^2}\)For the future convenience we use symmetric basis with \( C^{ac, \, bd} = C^{ca, \, bd} = C^{ac, \, db} \). The relationship with the standard antisymmetric Weyl tensor \( \bar{C}^{[ac], \, [bd]} \) is \( C^{ab, \, cd} = \frac{1}{2} \left( \bar{C}^{[ac], \, [bd]} + \bar{C}^{[bc], \, [ad]} \right) \).
the linearized level that non-zero components of order $k$ derivatives of the Weyl tensor $\partial_{n_1} \ldots \partial_{n_k} C_{a_1a_2b_1b_2}$ form Lorentz tensors $C_{c_1 \ldots c_{k+2},d_1d_2}$ described by the Young tableaux $\begin{array}{c} \begin{array}{c} k+2 \\ 11 \end{array} \end{array}$. Einstein equations imply that all $C_{a_1 \ldots a_{k+2},b_1b_2}$ are traceless.

In terms of the quantities $C_{c_1 \ldots c_{k+2},d_1d_2}$, consequences of the linearized $s = 2$ equations in flat space can be written in the form of covariant constancy conditions

$$dC_{a_1 \ldots a_l b_1 b_2} = e^c_0 \left( l C_{a_1 \ldots a_l c b_1 b_2} + 2 C_{a_1 \ldots a_l b_1 b_2 c} \right),$$

where $e^a_0$ is the flat Minkowski frame $e^a_0 = dx^a$.

Analogously, $s = 1$ Maxwell equations can be reformulated as

$$F = e^c_0 \wedge e^d_0 C_{c_1 d_1},$$

$$dC_{a_1 \ldots a_l b} = e^0_0 \left( (l+1) C_{a_1 \ldots a_l c b} + C_{a_1 \ldots a_l c b c} \right)$$

with the 0-forms $C_{a_1 \ldots a_l b}$ described by the traceless Young tableaux $\begin{array}{c} \begin{array}{c} k+1 \\ 11 \end{array} \end{array}$.

$s = 0$ Klein-Gordon dynamics is reformulated in the form

$$dC_{a_1 \ldots a_k} = e^c_0 (k+2) C_{a_1 \ldots a_k c}$$

in terms of symmetric traceless 0-forms $C_{a_1 \ldots a_k}$, which parametrize all on-mass-shell nontrivial combination of derivatives of the scalar field $C$ and are described by the Young tableaux $\begin{array}{c} \begin{array}{c} k \\ 11 \end{array} \end{array}$.

This formulation extends naturally to any spin $s$ described by “Weyl 0-forms” $C_{a_1 \ldots a_{s+k}, b_1 \ldots b_s}$ with the symmetry properties of the traceless Young tableaux $\begin{array}{c} \begin{array}{c} s+k \\ s \end{array} \end{array}$. The meaning of the set of 0-forms $C_{a_1 \ldots a_{s+k}, b_1 \ldots b_s}$ is that they form a basis in the space of gauge invariant on-mass-shell nontrivial derivatives of a massless field under consideration. As a result, the space of $C_{a_1 \ldots a_{s+k}, b_1 \ldots b_s}(x)$ at any given $x$ is analogous (in fact, dual by a nonunitary Bogolyubov transform) to the space $H$ of spin $s$ single-particle states. Thus, Weyl 0-forms are sections of the fiber bundle over space-time with the fiber space dual of the space of single-particle quantum states in the system.
5 Central On-Mass-Shell Theorem

The next step is to observe that free massless field equations in $(A)dS_d$ space can be concisely formulated in terms of the star product algebra. To this end one describes the background gravitational field as flat connection $\omega_0 \in o(d - 1, 2)$, $R^{AB}(\omega_0) = 0$ with $\omega_0^{A B} \neq 0$, and $\omega_0^{A_1...A_{s-1},B_1...B_{s-1}} = 0$ for $s > 2$. In the linearized approximation one sets $\omega(Y|x) = \omega_0(Y|x) + \omega_1(Y|x)$ where $\omega_1(Y|x)$ describes dynamical fluctuations. Then the generalization of the free lower spin equations (12)-(16) to the free equations for massless fields of all spins (plus constraints on auxiliary fields) is \[5, 18\]

\[R_1(Y|x) = \frac{1}{2} e^a_0 \wedge e^b_0 \frac{\partial^2}{\partial Y^a_0 \partial Y^b_0} \varepsilon_{ij} C(Y|x) \bigg|_{V_A Y^A_i = 0}, \quad (17)\]

\[\tilde{D}_0(C) = 0, \quad t_{ij} * C = C * t_{ij}, \quad D(t_{ij}) = 0, \quad (18)\]

where $R_1$ is the linearized HS field strength (11) and $\tilde{D}_0(C)$ is the covariant derivative in the twisted adjoint representation,

\[R_1 = d\omega + \omega_0 * \omega_1 + \omega_1 * \omega_0, \quad \tilde{D}_0(C) = dC + \omega_0 * C - C * \tilde{\omega}_0, \quad \tilde{f}(Y) = f(\tilde{Y}), \quad \tilde{Y}^A_i = Y^A_i - \frac{2}{V^2} V^A V^B Y^B_i. \quad (19)\]

6 Nonlinear Construction

6.1 General idea

The form of the equations (17), (18) suggests the idea \[19\] to search nonlinear HS equations in the “unfolded” form of generalized flatness conditions

\[d\omega^\Phi = F^\Phi(\omega) \quad \quad d = dx^n \frac{\partial}{\partial x^n}, \quad (20)\]

where $\omega^\Phi(x)$ is a set of differential forms (including 0-forms) and the function $F^\Phi(\omega)$ contains only wedge products of $\omega^\Phi(x)$ and is such that the consistency condition

\[F^\Phi \wedge \frac{\delta F^\Omega}{\delta \omega^\Phi} = 0 \quad (21)\]

is true for any $\omega^\Phi(x)$. (Once this is the case, the function $F^\Phi(\omega)$ defines a free differential algebra \[20\]).

The unfolded form (20) of the field equations has several nice properties:
• It is manifestly invariant under gauge transformations

\[ \delta \Omega^{\Phi} = d\varepsilon^{\Phi} - \varepsilon^{\Omega} \frac{\delta F^{\Phi}}{\delta \omega^{\Omega}}, \quad \deg \varepsilon^{\Phi}(x) = \deg \omega^{\Phi}(x) - 1. \]  

(22)

• Invariant under diffeomorphisms.

• Interactions: a nonlinear deformation of \( F^{\Phi}(\omega) \).

• Degrees of freedom are in the 0-form fields which form an infinite-dimensional module dual to the space of single-particle states.

• Universality: any dynamical system can be reformulated in the unfolded form.

Originally it was shown by direct inspection that a nonlinear deformation of the \( d = 4 \) unfolded free massless field equations [18] exists in the lowest orders [19]. To go beyond lowest orders some more sophisticated approach was needed. The useful idea was [22] to find an appropriate generalization \( g' \) of the HS algebra \( g \) such that a substitution

\[ \omega \to W = \omega + \omega C + \omega C^2 + \ldots \]  

(23)

into the \( g' \) zero curvature equation \( dW + W \wedge W = 0 \) reconstructs nonlinear HS equations. The key issue is of course to find restrictions on \( W \) that reconstruct \( g' \) in all orders. The guiding principle is [21, 18] to preserve \( sp(2) \) at the nonlinear level. Before going into details of the construction let us mention that the resulting interactions are unique up to field redefinitions. The only dimensionless coupling constant is the YM constant \( g^2 = |\Lambda|^{-\frac{d-2}{2}} \kappa^2 \) which, however, is artificial in the classical pure gauge HS theory because it can be rescaled away just as in the classical pure Yang-Mills theory.

### 6.2 Nonlinear HS Equations

In [18] it was shown that the appropriate extension \( g \to g' \) is achieved by the doubling of oscillators \( Y_i^A \to (Z_i^A, Y_i^A) \) so that the HS fields extend to \( \omega(Y|x) \to W(Z, Y|x), C(Y|x) \to B(Z, Y|x) \). In addition we introduce the \( S \) connection along \( Z_i^A \) that together with \( W \) form a noncommutative connection

\[ W = d + W + S, \quad S(Z, Y|x) = dZ_i^A S_i^A. \]  

(24)
The star product in $g'$ is

$$(f \ast g)(Z, Y) = \int dSdT f(Z + S, Y + S)g(Z - T, Y + T) \exp 2S^i_A T^A_i. \quad (25)$$

One can see that this is the oscillator algebra with the nonzero basis relations
\[ [Y^A_i, Y^B_j]_* = -[Z^A_i, Z^B_j]_* = \varepsilon_{ij} \eta^{AB}. \]
It is not however the Moyal star product, being the normal ordered star product with respect to $Z - Y$:

normal ordering because the left star multiplication by $Z - Y$ and right star multiplication by $Z + Y$ are equivalent to the usual pointwise multiplication.

An important property of this star product is that it admits the Klein operator $K$ that generates the automorphism (19)

\[ K = \exp 2V^2 V^A_i Z^A_i, \quad K \ast f = \tilde{f} \ast K, \quad K \ast K = 1. \quad (26) \]

The nonlinear HS field equations can be concise formulated in the form (18)

\[ \mathcal{W} \ast \mathcal{W} = \frac{1}{2}(dZ^A_i dZ^A_i + 4\Lambda^{-1} dz^i d_z^i B \ast \mathcal{K}) \quad \mathcal{W} \ast B = B \ast \tilde{\mathcal{W}}, \quad (27) \]

where $\tilde{\mathcal{W}}(dZ, Z, Y) = \mathcal{W}(\tilde{d}Z, \tilde{Z}, \tilde{Y})$ and $dz^i = \frac{1}{\sqrt{V^2}} V_B dZ^B_i$. This system is manifestly gauge invariant under the gauge transformations

\[ \delta \mathcal{W} = [\varepsilon, \mathcal{W}]_*, \quad \delta B = \varepsilon \ast B - B \ast \tilde{\varepsilon} \quad (28) \]

with an arbitrary gauge parameter $\varepsilon = \varepsilon(Z, Y|x)$. One of the most important properties of the system (28) is (18) that it admits $sp(2)$ such that its generators $\tau_{ij}$ form a nonlinear deformation of the $sp(2)$ algebra (8) and single out the physical sector of HS fields by the $sp(2)$ invariance condition $D \tau_{ij} = 0$ (followed by factorization of the terms proportional to $\tau_{ij}$). The nonlinearly realized $sp(2)$ can be interpreted as a symmetry of a two-dimensional fuzzy hyperboloid in the noncommutative space of $Y^A_i$ and $Z^A_j$. A radius of the fuzzy hyperboloid depends on $B(Z, Y|x)$ which is the generating function for the Weyl 0-forms.

To analyse the HS field equations perturbatively one sets

\[ W = W_0 + W_1, \quad S = dZ^A_i Z^A_i + S_1, \quad B = B_1, \quad (29) \]

where $W_0 = \frac{1}{2} \omega^{AB}_0(x) Y^i_A Y^i_B$ with $\omega^{AB}_0(x)$ describing background $AdS_4$ gravitational field. It is not hard to see that central on-mass-shell theorem (18) is reproduced in the lowest order (18).
Let us note that the form of the noncommutative connection $S$ in (29) implies that, because of the first term in $S$, the HS symmetry (28) is spontaneously broken down to the HS symmetry with $Z$-independent parameters $\varepsilon(Y|x)$ (The $Z$-dependent components in $\varepsilon(Z,Y|x)$ are used to gauge fix the noncommutative $Z$–connection $S$). Because of the $B$-dependent term in (28), the leftover HS symmetries with the HS gauge parameters $\varepsilon(Y|x)$ acquire $B$-dependent nonlinear corrections. As a result, HS gauge symmetries in the nonlinear HS theory are different from the Yang-Mills gauging of the global HS symmetry of a free theory one starts with.

7 Singletons in any Dimension

The simplest HS algebra $hu(1|2;[d−1,2])$ admits the fermionic generalization $hu(1|(1,2);[d−1,2])$

$$sp(2) \to osp(1,2) \quad Y^A_i \to (Y^A_i, \phi^A), \quad \{\phi^A, \phi^B\} = -2\eta^{AB}. \quad (30)$$

The Fock-type modules of $hu(1|2;[M,2])$ and $hu(1|(1,2);[M,2])$ describe massless scalar $S_M$ and spinor $F_M$ in $M$ dimensions [23] (closely related analysis of $M$-dimensional field equations in terms of conformal algebra and dual $sp(2)$ and $osp(1,2)$ was given in [15, 16]). This gives realization of the HS algebras as conformal HS algebras acting on the scalar and spinor conformal fields (i.e., singletons) in $M$ dimensions.

The following generalization of the 4d Flato-Fronsdal theorem [24] takes place [23]:

$$S_{d−1} \otimes S_{d−1} = \sum_{s=0}^{\infty} \oplus \begin{array}{c} s \end{array} \quad m = 0 \quad \text{bosons in } AdS_d \quad (31)$$

(note that related statements were discussed in [25])

$$F_{d−1} \otimes S_{d−1} = \sum_{s=-\frac{1}{2}}^{\infty} \oplus \begin{array}{c} s - 1/2 \end{array} \quad m = 0 \quad \text{fermions in } AdS_d \quad (32)$$

$$F_{d−1} \otimes F_{d−1} = \sum_{p,q} \oplus \begin{array}{c} g \end{array} \quad m = 0 \quad \text{bosons in } AdS_d \quad (31)$$
These results show precise matching between spectra of gauge fields in HS theories and appropriate UIRs of HS algebras, indicating the existence of HS gauge theories with fermions and mixed symmetry fields. Moreover, HS superalgebras exist in any $d$. There is no contradiction here with the absence of usual supersymmetries in higher dimensions because HS superalgebras contain usual finite-dimensional subsuperalgebras only for some lower dimensions like $d = 3, 4, 5$.

8 Conclusions

The main conclusion is that nonlinear HS theories exist in any dimension. Note that HS gauge symmetries in the nonlinear HS theory differ from the Yang-Mills gauging of the global HS symmetry of a free theory one starts with by HS field strength dependent nonlinear corrections resulting from the partial gauge fixing of spontaneously broken HS symmetries in the extended noncommutative space.

The HS geometry is that of the fuzzy hyperboloid in the auxiliary (fiber) noncommutative space. Its radius depends on the Weyl 0-forms which take values in the infinite-dimensional module dual to the space of single-particle states in the system.

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