IMPROVEMENTS IN $L^2$ RESTRICTION BOUNDS FOR NEUMANN DATA ALONG HYPERSURFACES

XIANCHAO WU

Abstract. We seek to improve the restriction bounds of Neumann data of semiclassical Schrödinger eigenfunctions $u_h$ considered by Christianson-Hassell-Toth [CHT15] and Tacy [Tac17] by studying the $L^2$ restriction bounds of eigenfunctions and their $L^2$ concentration as measured by defect measures. Let $\Gamma$ be a smooth hypersurface with unit exterior normal $\nu$. Our main result says that $\|h\partial_\nu u_h\|_{L^2(\Gamma)} = o(1)$ when $\{u_h\}$ is admissible.

1. Introduction

Let $(M, g)$ be a compact, smooth $n$-dimensional Riemannian manifold without boundary. Consider semiclassical Schrödinger operator $P(h) = -h^2 \Delta_g + V(x)$ with $V(x) \in C^\infty(M; \mathbb{R})$. Let $u_h$ be $L^2$-normalized eigenfunction solving

$$P(h)u_h = E(h)u_h \quad \text{on } M,$$

here $E(h) = E + o(1)$, $E > \min V$.

Christianson-Hassell-Toth [CHT15] and Tacy [Tac17] showed the boundedness of the Neumann data restricted to a smooth oriented separating hypersurface $\Gamma \subset M$ with $V(x) < E$ for $x \in \Gamma$. That is

$$\|h\partial_\nu u_h\|_{L^2(\Gamma)} = O(1).$$

(1.2)

This estimate can be seen as a statement of non-concentration. Note that by [CHT15] we know that when $V \equiv 0$, (1.2) is saturated by considering a sequence of spherical harmonics.

In this paper we consider the problem when the upper bound (1.2) can be improved. Motivated by [GT18] and [CGT18] which studied the relationship between $L^\infty$ growth (and averages on hypersurfaces) of Laplace eigenfunctions and their $L^2$ concentration as measured by defect measures, we link the $L^2$ restriction bound of semiclassical Schrödinger eigenfunctions and their $L^2$ concentration as measured by defect measures to show that if a defect measure which is too diffuse in the sense of (1.6), the corresponding sequence of eigenfunctions is incompatible with maximal eigenfunction growth (1.2).

Any sequence $\{u_h\}$ of solutions to (1.1) has a subsequence $\{u_{h_k}\}$ with a defect measure $\mu$ in the sense that for $a \in C^\infty_0(T^*M)$

$$\langle a(x, hD)u_h, u_h \rangle \to \int_{T^*M} ad\mu.$$
exterior normal vector to $M \setminus \Omega_\Gamma$ with base point at $\Gamma$, and $U_\Gamma$ is a Fermi collar neighborhood of $\Gamma$,

$$U_\Gamma = \{(x', x_n) : x' \in \Gamma \text{ and } x_n \in (-c, c)\}$$

for some $c > 0$, and $\Gamma = \{ (x', 0) \}$.

Define respectively the flow out and time $T$ flow out from $A \subset \Sigma_{T^*M}$ by

$$\Lambda_A := \bigcup_T \Lambda_{A,T}, \quad \Lambda_{A,T} := \bigcup_{t=-T}^{T} \varphi_t(A)$$

where

$$\Sigma_{T^*M} = \{(x, \xi) \in T^*M \mid |\xi|^2_{g} + V(x) = E\}.$$ 

We write $\Sigma_{\Gamma} \subset \Sigma_{T^*M}$ for the space of covectors with foot-points in $\Gamma$.

**Definition 1.1.** Let $\mathcal{H}^{2n-1}$ be $n$-dimensional Hausdorff measure on $T^*M$ induced by the Sasaki metric on $T^*M$ (see for example [Bla10, Chapter 9] for a treatment of the Sasaki metric). We say that the subsequence $u_{h_j}, j = 1, 2, \ldots$ is admissible on $\Gamma$ if

$$\mathcal{H}^{2n-1}(\text{supp } \mu|_{\Sigma_{\Gamma}}) = 0. \quad (1.6)$$

**Theorem 1.** Let $\Gamma \subset M$ be a smooth oriented separating hypersurface $\Gamma \subset M$ with $V(x) < E$ for $x \in \Gamma$. Let $\{u_h\}$ be a sequence of $L^2$-normalized eigenfunctions of (1.1) that is admissible in the sense of (1.6). Then,

$$\|h\partial_{\nu}u_h\|_{L^2(\Gamma)} = o(1). \quad (1.7)$$

**Definition 1.2.** We say that an eigenfunction subsequence is strongly scarring provided $\text{supp } \mu$ is a finite union of periodic trajectory of Hamiltonian vector field.

Because of the dimensional reason, one has

**Corollary 1.1.** Let $\{u_h\}$ be a strongly scarring sequence of solutions to (1.1). Then

$$\|h\partial_{\nu}u_h\|_{L^2(\Gamma)} = o(1).$$

This paper is organized in the following way. In Section 2, we give the proof of Theorem 1. In Section 3, we specialized to the case of Laplace eigenfunctions with relaxing the condition in Theorem 1 and one example is provided to Theorem 2.

**2. Proof of theorem**

Let $\mathcal{H}^{2n-1}$ be the $2n - 1$-dimensional Hausdorff measure on the flow out $\Lambda_{\Sigma_{\Gamma}}$. By assumption, $\mathcal{H}^{2n-1}(\text{supp } \mu|_{\Lambda_{\Sigma_{\Gamma}}}) = 0$. Now consider subset $\Lambda_{\Sigma_{\Gamma},\delta}$. For any $\varepsilon > 0$, there exist $2n - 1$-dimensional balls $B(r_j), j = 1, 2, \ldots$ with radii $r_j > 0$, $j = 1, 2, \ldots$ such that

$$\text{supp } \mu|_{\Lambda_{\Sigma_{\Gamma},\delta}} \subset \bigcup_{j=1}^{\infty} B(r_j), \quad \mathcal{H}^{2n-1}\left(\bigcup_{j=1}^{\infty} B(r_j)\right) < \varepsilon. \quad (2.1)$$

By the $C^\infty$ Uryshon lemma, there exists $\chi_{\Gamma} \in C^\infty_{0}(T_\Gamma^*M; [0, 1])$ with

$$\chi_{\Gamma}|_{\Lambda_{\Sigma_{\Gamma},\delta}} = 1, \quad \text{supp } \chi_{\Gamma} \subset \bigcup_{j=1}^{\infty} B(r_j). \quad (2.2)$$
Note that the canonical projection $\pi : T^*M \to M$ induces
\[
\pi : \Lambda_{\Sigma_\Gamma, \delta} \to \{ x \in M ; \ d(x, \Gamma) \leq C\delta \}. \tag{2.3}
\]

We construct a cut off function $\chi_\alpha \in C^\infty_0(\Omega_\Gamma ; [0,1])$ near $\Gamma$ in $\Omega_\Gamma$ such that
\[
\chi_\alpha(x_n) = \begin{cases} 
0 & \text{if } |x_n| \geq 2\alpha \\
1 & \text{if } |x_n| \leq \alpha/2,
\end{cases}
\tag{2.4}
\]
with $|\chi'_\alpha(x_n)| \leq 3/\alpha$ for all $x_n \in \mathbb{R}$.

Let $R(x', \xi') = \sigma(-h^2\Delta_\Gamma)(x', \xi')$ be the principal symbol of the induced hypersurface Laplacian $-h^2\Delta_\Gamma : C^\infty(\Gamma) \to C^\infty(\Gamma)$. We use a Rellich identity, involving the commutator of $-h^2\Delta + V(x) - E(h)$ with the operator $\chi_\alpha(x_n)hD_n$. Integrating over $\Omega_\Gamma$, using Green’s formula, we have
\[
i \int_{\Omega_\Gamma} \left[ -h^2\Delta + V(x) - E(h) , \chi_\alpha(x_n)hD_n \right]u_h \overline{u_h} dx 
= \int_{\Gamma} ((hD_n)^2u_h) \overline{u_h} d\sigma_\Gamma + \int_{\Gamma} (hD_n u_h) \overline{hD_n u_h} d\sigma_\Gamma 
\geq \int_{\Gamma} (E(h) - V(x) + h^2\Delta_\Gamma) u_h \overline{u_h} d\sigma_\Gamma + \frac{1}{2} \int_{\Gamma} |hD_n u_h|^2 d\sigma_\Gamma, \tag{2.5}
\]

since $u_h$ is an eigenfunction, $(hD_n)^2u_h$ is equal to $(E(h) - V(x) + h^2\Delta_\Gamma) u_h$ up to an error term $h^2Lu_h|_{\Gamma}$, where $L$ is a first order differential operator about $\partial_{x_n}$, which can be bounded by $\frac{1}{2} \int_{\Gamma} |hD_n u_h|^2 d\sigma_\Gamma$ using Cauchy-Schwarz inequality and $O(h^{-1/4})$ bound [Tac10] for $u_h|_{\Gamma}$.

For completeness of this paper, we shall follow the steps in [CHT15], using the exterior mass estimates and the $h^{-1/4}$ bound [Tac10] to show that
\[
\int_{\Gamma} (E(h) - V(x) + h^2\Delta_\Gamma) u_h \overline{u_h} d\sigma_\Gamma \geq -C h^\sigma, \tag{2.6}
\]
where $\sigma > 0$. Firstly let $\chi \in C^\infty_0(\mathbb{R}; [0,1])$ with $\chi(u) = 1$ for $|u| \leq 1/2$ and $\chi(u) = 0$ for $|u| > 1$, $\chi_- \in C^\infty(\mathbb{R})$ with $\chi_-(u) = 1$ when $u < -1$ and $\chi_+ \in C^\infty(\mathbb{R})$ with $\chi_+(u) = 1$ when $u > 1$. In addition we require that
\[
\chi_-(u) + \chi(u) + \chi_+(u) = 1, \quad u \in \mathbb{R}.
\]

We use small scale decomposition such that
\[
(\chi_{\text{in}})^w_{h, \delta} + (\chi_{\text{tan}})^w_{h, \delta} + (\chi_{\text{out}})^w_{h, \delta} = 1,
\]
where $(\chi_{\text{in}})^w_{h, \delta}(x', \xi') = \chi_-(h^{-\delta}(R(x', \xi') + V(x) - E(h)))$, $(\chi_{\text{tan}})^w_{h, \delta}(x', \xi') = \chi(h^{-\delta}(R(x', \xi') + V(x) - E(h)))$, and $(\chi_{\text{out}})^w_{h, \delta}(x', \xi') = \chi_+(h^{-\delta}(R(x', \xi') + V(x) - E(h)))$. In the following, we set $u^*_h = \gamma_\Gamma u_h$, here $\gamma_\Gamma : C^\infty(M) \to C^\infty(\Gamma)$ is the canonical restriction map.

We also need following relevant 2-microlocal algebra of $h$-pseudodifferential operators localized on small scale $\sim h^\delta$ where $\delta \in (1/2, 1)$.

**Definition 2.1.** Let $\Gamma \subset M$ be a hypersurface with $V(x) < E$ for $x \in H$. We say that a smiclasical symbol $b$ is 2-microlocalized along $\Omega_\Gamma$ and write $b \in S^m_{\Omega_\Gamma, \delta}(T^*\Gamma \times (0, h_0))$ provided there exists $\chi \in C^\infty_0(\mathbb{R})$, $a \in S^0(\Omega_\Gamma \times (0, h_0))$ such that
\[
b(x', \xi'; h) = h^{-m}a(x', \xi'; h) \cdot \chi \left( \frac{R(x', x_n = 0, \xi') + V(x) - E(h)}{h^\delta} \right), \quad 0 \leq \delta < 1.
\]
for all \((x', \xi')\) in \(T^* \Gamma\).

**Proposition 2.1 (CHT15).** Given \(a^w(x, hD_x) \in \text{Op}_h(S_{\Omega_r, \delta}^{m_1})\) and \(b^w(x, hD_x) \in \text{Op}_h(S_{\Omega_r, \delta}^{m_2})\) it follows that

\[
a^w(x, hD_x) \circ b^w(x, hD_x) = c^w(x, hD_x) \in \text{Op}_h(S_{\Omega_r, \delta}^{m_1 + m_2})
\]

with

\[
c(x, \xi; h) = a(x, \xi; h) b(x, \xi; h).
\]

Furthermore, we have a version of the Gåding inequality

**Lemma 2.2 (CHT15).** Suppose \(a \in S_{\Omega_r, \delta}^0\) is real valued and \(a \geq 0\). Then

\[
\langle a^w u, u \rangle \geq -Ch^{1-\delta} \| u \|^2.
\]

In particular,

\[
\langle a^w \Gamma u_h^{\Gamma}, u_h^{\Gamma} \rangle_{L^2(\Gamma)} \geq -Ch^{1-\delta} \| u_h^{\Gamma} \|^2_{L^2(\Gamma)}.
\]

One has

\[
\int_{\Gamma} (E(h) - V(x) + h^2 \Delta_{\Gamma}) u_h \overline{u_h} d\sigma_{\Gamma} = \int_{\Gamma} (E(h) - V(x) + h^2 \Delta_{\Gamma}) (\chi_{\text{in}})_{h, \delta} u_h \overline{u_h} d\sigma_{\Gamma} + \int_{\Gamma} (E(h) - V(x) + h^2 \Delta_{\Gamma}) (\chi_{\text{tan}})_{h, \delta} u_h \overline{u_h} d\sigma_{\Gamma} + \frac{1}{\delta^2} \int_{\Gamma} (E(h) - V(x) + h^2 \Delta_{\Gamma}) (\chi_{\text{out}})_{h, \delta} u_h \overline{u_h} d\sigma_{\Gamma} + O(h^\infty)
\]

where we used exterior mass estimate [CHT15] in the last line.

On the support of \(\chi_{\text{in}}\), one has \(E(h) - V(x) - R(x', 0, \xi') \geq h^\delta\). Assume \(\chi_{\text{in}} = \psi^2\) for some \(\psi \geq 0\),

\[
\psi = \psi \left( (R(x', 0, \xi') - E(h) + V(x)) / h^\delta \right) \in S_{\Omega_r, \delta}^0.
\]

Take \(\bar{\chi} \in S_{\Omega_r, \delta}^0\) satisfying \(\bar{\chi} \equiv 1\) on \(\text{supp} \chi_{\text{in}}\) with slightly larger support, say on a set where \(E(h) - V(x) - R(x', 0, \xi') \geq h^\delta / M\) for some large \(M\). Then

\[
\ell = (1 - \bar{\chi}) + h^{-\delta} \bar{\chi} \cdot (E(h) - V(x) - R(x', 0, \xi')) \in S_{\Omega_r, \delta}^\delta
\]

satisfies \(\ell \geq c_0 > 0\). If \(L = \ell^w\), then the Gåding inequality implies

\[
\langle Lu, u \rangle \geq (c_0 - Ch^{1-\delta}) \| u \|^2.
\]
Then by definition of \( L \) and Weyl calculus
\[
\langle (E(h) - V(x) + h^2 \Delta) \chi_{\text{tr}} \rangle_{h,\delta} u_h^\Gamma, u_h^\Gamma
\]
\[
= h^\delta \langle L(\chi_{\text{tr}}) u_h^\Gamma, u_h^\Gamma \rangle + O(h^\infty) \| u_h^\Gamma \|^2
\]
\[
= h^\delta \langle L(\psi^w) u_h^\Gamma, u_h^\Gamma \rangle + O(h^\infty) \| u_h^\Gamma \|^2
\]
\[
= h^\delta \langle L(\psi^w) L\psi^w u_h^\Gamma, u_h^\Gamma \rangle + h^\delta \langle \langle L, (\psi^w) \rangle \psi^w u_h^\Gamma, u_h^\Gamma \rangle + O(h^\infty) \| u_h^\Gamma \|^2
\]
\[
= h^\delta \langle (\psi^w) L\psi^w u_h^\Gamma, u_h^\Gamma \rangle + O(h^\infty) \| u_h^\Gamma \|^2
\]
\[
\geq h^\delta (c_0 - Ch^{1-\delta}) \| \psi^w u_h^\Gamma \|^2 - Ch^{3-\delta} \| u_h^\Gamma \|^2
\]
\[
\geq -Ch^{3-\delta} \| u_h^\Gamma \|^2
\]

Notice on the support of \( \chi_{\text{tan}} \), we have \(|E(h) - V(x) - R(x',0,\xi)| \leq C_2 h^\delta \), so
\[
\left| \int \Gamma \left( (E(h) - V(x) + h^2 \Delta) \chi_{\text{tan}} \right)_{h,\delta} u_h^\Gamma d\sigma \right| \leq C_2 h^\delta \| u_h^\Gamma \|^2.
\]

Putting these two estimates together, we have
\[
\int \Gamma \left( (E(h) - V(x) + h^2 \Delta) u_h^\Gamma \right) d\sigma \geq -C_1 h^{3-\delta} \| u_h^\Gamma \|^2 - C_2 h^\delta \| u_h^\Gamma \|^2 + O(h^\infty).
\]

Since \( \delta \in (1/2, 2/3) \) and employing the \( h^{-1/4} \) bound of \( \| u_h^\Gamma \| \) \cite{Tac10}, we have
\[
\int \Gamma \left( (E(h) - V(x) + h^2 \Delta) u_h^\Gamma \right) d\sigma \geq -C h^{\delta-1/2}
\]
which implies \( (2.6) \).

In order to show \( \| hD_n u_h \|_{L^2(\Gamma)} = o(1) \), we only need to show that
\[
\lim_{\alpha \to 0} \frac{i}{h} \int_{\Gamma} [-h^2 \Delta + V(x) - E(h), \chi_{\alpha}(x) h D_n] u_h^\Gamma dx = o(1). \tag{2.7}
\]

Before proving \( (2.7) \), we shall state some results of decomposition of defect measures following \cite{CGT18}. Let \( N \) be a smooth manifold, \( \mathcal{V} \) be a vector field on \( N \) and write \( \varphi^\mathcal{V}_t : N \to N \) for the flow map generated by \( \mathcal{V} \) at time \( t \). Let \( \Sigma \subset M \) be a smooth manifold transverse to \( \mathcal{V} \). Then for \( \varepsilon > 0 \) small enough, the map \( \iota : (-2\varepsilon, 2\varepsilon) \times \Sigma \to N \)
\[
\iota(t, q) = \varphi^\mathcal{V}_t(q)
\]
is a diffeomorphism onto its image and we may use \( (-2\varepsilon, 2\varepsilon) \times \Sigma \) as coordinates on \( N \) near \( \Sigma \).

**Lemma 2.3** (\cite{CGT18}). Suppose that \( \mu \) is a finite Borel measure on \( N \) and that \( \mathcal{V} \mu = 0 \) i.e. \( (\varphi^\mathcal{V}_t)_* \mu = \mu \). Then, for a Borel \( A \subset [-\varepsilon, \varepsilon] \times \Sigma \),
\[
\iota^* \mu(A) = d\tau d\mu_{\Sigma}(A)
\]
where \( d\mu_{\Sigma} \) is a finite Borel measure on \( \Sigma \).
We now apply Lemma 2.3 to the special case of defect measures, using the fact that they are invariant under the Hamiltonian flow. In what follows we write $|\xi'|_{x'} := |\xi'|_{g_T(x')}$, where $g_T$ is the Riemannian metric on $T^*$ induced by $g$. Let

$$G_\Gamma(\delta) := \{(x, \xi) \in \Sigma_T : |\xi'|_{x'}^2 \geq E - V - \delta^2\},$$

and define the set of non-glancing directions $\Sigma_\delta := \Sigma_T \setminus G_\Gamma(\delta)

**Lemma 2.4.** Suppose $\mu$ is a defect measure associated to a sequence of Schrödinger eigenfunctions. Then, for all $\delta > 0$ there exists $\varepsilon > 0$ small enough so that

$$\nu^* \mu = dt d\mu_{\Sigma_\delta} \quad \text{on } (-\varepsilon, \varepsilon) \times \Sigma_\delta$$

where

$$\nu : (-\varepsilon, \varepsilon) \times \Sigma_\delta \to \bigcup_{|s| < \varepsilon} \varphi_t(\varphi_t(\Sigma_\delta)), \quad \nu(t, q) = \varphi_t(q),$$

is a diffeomorphism and $d\mu_{\Sigma_\delta}$ is a finite Borel measure on $\Sigma_\delta$.

**Proof.** We shall use Lemma 2.3 with $N = \Sigma_T^* \setminus M$, $V = H_p$ the Hamiltonian vector field for $p = |\xi|^2 + V(x)$, and $\varphi_t^V = \varphi_t$ the Hamiltonian flow. Note that since $\mu$ is a defect measure for a sequence of Schrödinger eigenfunctions, it is invariant under the Hamiltonian flow $\varphi_t$. Then, for $q \in \Sigma_\delta$,

$$|H_p x_n(q)| = |\{p(q), x_n(q)\}| > c\delta > 0$$

and hence $\Sigma_\delta$ is transverse to $\varphi_t$. There exists $\varepsilon > 0$ such that $\nu : (-2\varepsilon, 2\varepsilon) \times \Sigma_\delta \to \Sigma_T^* \setminus M$, with $\nu(t, q) = \varphi_t(q)$, is a coordinate map. \(\square\)

**Remark 1.** For each $A \subset \Sigma_T$ with $\overline{A} \subset \Sigma_T \setminus \Sigma_T^*$, there exists $\delta_0 > 0$ so that

$$d\mu_{\Sigma_\delta}(A) = \lim_{t \to 0^+} \frac{1}{2t} \mu \left( \bigcup_{|s| \leq t} \varphi_t(A) \right)$$

for all $0 < \delta \leq \delta_0$. Indeed, since $\overline{A}$ is compact, there exists $\delta_0 = \delta(\overline{A}) > 0$ such that $\overline{A} \subset \Sigma_\delta_0$. Then, by Lemma 2.4, there exists $\varepsilon = \varepsilon(A) > 0$ so that if $|t| \leq \varepsilon$, then

$$\mu \left( \bigcup_{|s| \leq t} \varphi_t(A) \right) = 2td\mu_{\Sigma_\delta}(A).$$

In particular, we conclude that the quotient $\frac{1}{2t} \mu \left( \bigcup_{|s| \leq t} \varphi_t(A) \right)$ is independent of $t$ as long as $|t| \leq \varepsilon$.

**Lemma 2.5.** Suppose $\mu$ is a defect measure associated to a sequence of Schrödinger eigenfunctions, and let $\delta > 0$. Then, in the notation of Lemma 2.4 there exists $\varepsilon_0 > 0$ small enough so that

$$\mu = |\xi_n|^{-1} d\mu_{\Sigma_\delta(x', \xi', \xi_n)} dx_n,$$

for $(x', x_n, \xi', \xi_n) \in \nu((-\varepsilon_0, \varepsilon_0) \times \Sigma_\delta)$. 

Now coming back to the proof of (2.7), note that
\[
\left\langle \frac{i}{\hbar} \left[ -\hbar^2 \Delta + V(x) - E(h), \chi_\alpha(x_n) hD_n \right] u_h, u_h \right\rangle_{L^2(\Omega_r)} = \langle Op_h \left\{ \sigma(-\hbar^2 \Delta + V(x) - E(h)), \sigma(\chi_\alpha(x_n) hD_n) \right\} u_h, u_h \rangle_{L^2(\Omega_r)} + O(h),
\] (2.8)
where the Poisson bracket
\[
\{ |(\xi', \xi_n)|_x^2 + V(x) - E(h), \chi_\alpha(x_n) \xi_n \} = 2\chi'_\alpha(x_n) \xi_n^2 - \chi_\alpha(x_n) \partial_{x_n} (R + V),
\] (2.9)
here in Fermi coordinate system we have
\[
|(\xi', \xi_n)|_x^2 = \xi_n^2 + R(x', x_n, \xi'),
\] (2.10)
where $R(x', 0, \xi') = |\xi'|_x^2$, for all $(x', \xi') \in T^*\Gamma$.

Then it follows that
\[
\left\langle \frac{i}{\hbar} \left[ -\hbar^2 \Delta + V(x) - E(h), \chi_\alpha(x_n) hD_n \right] u_h, u_h \right\rangle_{L^2(\Omega_r)} = \int_{\Sigma_{\Omega_r}} (2\chi'_\alpha(x_n) \xi_n^2 - \chi_\alpha(x_n) \partial_{x_n} (R + V)) \, d\mu + o(1)
\] (2.11)
where
\[
I_1 := \int_{\Sigma_{\Omega_r}} 2\chi'_\alpha(x_n) \xi_n^2 \, d\mu, \quad I_2 := \int_{\Sigma_{\Omega_r}} \chi_\alpha(x_n) \partial_{x_n} (R + V) \, d\mu.
\]

Firstly we estimate $I_2$. Note that since $\mathcal{H}^{2n-1} (\supp \chi_\Gamma) \leq \mathcal{H}^{2n-1} \left( \bigcup_{j=1}^{\infty} B(r_j) \right) \leq C\varepsilon$, it follows that $\|\chi_\Gamma\|_{L^2(\Sigma_r, d\mu_{\Sigma_r})} \leq C\varepsilon$. Then by the dominated convergence theorem and Cauchy-Schwarz inequality,
\[
\lim_{\alpha \to 0} I_2 = \lim_{\alpha \to 0} \int_{\Sigma_{\Omega_r}} \chi_\alpha(x_n) \chi_\Gamma \partial_{x_n} (R + V) \, d\mu + \lim_{\alpha \to 0} \int_{\Sigma_{\Omega_r}} \chi_\alpha(x_n) (1 - \chi_\Gamma) \partial_{x_n} (R + V) \, d\mu
\]
\[
\leq C\|\chi_\Gamma\|_{L^2(\Sigma_r, d\mu_{\Sigma_r})} + \lim_{\alpha \to 0} \int_{\Sigma_{\Omega_r}} \chi_\alpha(x_n) (1 - \chi_\Gamma) \partial_{x_n} (R + V) \, d\mu
\]
\[
\leq o(1) + \lim_{\alpha \to 0} \int_{\Sigma_{\Omega_r}} \chi_\alpha(x_n) (1 - \chi_\Gamma) \partial_{x_n} (R + V) \, d\mu.
\] (2.12)

One also has
\[
\int_{\Sigma_{\Omega_r}} \chi_\alpha(x_n) (1 - \chi_\Gamma) \partial_{x_n} (R + V) \, d\mu = 0
\] (2.13)
since by construction $\pi : \supp (1 - \chi_\Gamma) \to \{ x \in M; |x_n| > C\delta \}$. Take $\delta$ large,
\[
\supp \chi_\alpha \bigcap \supp (1 - \chi_\Gamma) = \emptyset.
\] (2.14)
Next we deal with $I_1$. Since $\text{supp} \left( \chi'_\alpha \right) \subseteq (-\alpha, 0)$, by the Fubini theorem

$$I_1 = \int_{\Sigma_{\Gamma}} 2\chi'_\alpha(x_n) \chi_\tau(\xi_n) \xi_n^2 d\mu + \int_{\Sigma_{\Gamma}} 2\chi'_\alpha(x_n)(1 - \chi_\tau(\xi_n)) \xi_n^2 d\mu$$

$$\leq C\tau^2/\alpha + \int_{-\alpha}^0 2\chi'_\alpha(x_n) \left( \int_{\Sigma_{\Gamma}} (1 - \chi_\tau(\xi_n)) |\xi_n|^{-1} d\mu_{\Sigma_{\Gamma}}(x', \xi', \xi_n) \right) dx_n$$

$$= C\tau^2/\alpha + 2 \int_{\Sigma_{\Gamma}} (1 - \chi_\tau(\xi_n)) |\xi_n| d\mu_{\Sigma_{\Gamma}}(x', \xi', \xi_n) \tag{2.15}$$

Following the steps in estimating the term $I_2$, one has

$$\int_{\Sigma_{\Gamma}} (1 - \chi_\tau(\xi_n)) |\xi_n| d\mu_{\Sigma_{\Gamma}}$$

$$= \int_{\Sigma_{\Gamma}} (1 - \chi_\tau(\xi_n)) |\xi_n| \chi_\Gamma d\mu_{\Sigma_{\Gamma}} + \int_{\Sigma_{\Gamma}} (1 - \chi_\tau(\xi_n)) |\xi_n|(1 - \chi_\Gamma) d\mu_{\Sigma_{\Gamma}}$$

$$\leq C\|\chi_\Gamma\|_{L^2(\Sigma_{\Gamma}, d\mu_{\Sigma_{\Gamma}})}^2 + 0 \leq C\varepsilon \tag{2.16}$$

with noticing $\text{supp} \left( 1 - \chi_\Gamma \right) |_{\Sigma_{\Gamma}} = \emptyset$ in the last line.

Finally, combine (2.15) and (2.16) with taking $\tau = \alpha$ and $\varepsilon$ small to get

$$I_1 = o(1).$$

Hence one has

$$\lim_{\alpha \to 0} \left\langle \frac{i}{\hbar} \left[ -\hbar^2 \Delta + V(x) - E(h), \chi_\alpha(x_n) h D_n u_h, u_h \right] \right\rangle_{L^2(\Omega_{\Gamma})} = o(1). \tag{2.17}$$

3. LAPLACE EIGENFUNCTIONS AND AN EXAMPLE

In this section, we consider normalized Laplace eigenfunctions $\{u_\lambda\}$,

$$-\Delta_g u_\lambda = \lambda^2 u_\lambda.$$

Without the contribution from the potential term one shall see that the Neumann data can be improved with relaxing the condition (1.6).

3.1. Laplace eigenfunctions.

**Definition 3.1.** We say that the subsequence $u_{\lambda_j}$, $j = 1, 2, \ldots$ is tangentially concentrated with respect to $\Gamma$ if

$$\frac{1}{2\ell} \mu(\Lambda_{S_{\Gamma}^* M \setminus S_{\Gamma} \cap \tau}) = 0. \tag{3.1}$$

**Remark 2.** Notice our definition is stronger than [CGT18, Definition 1].

**Theorem 2.** Let $\Gamma \subset M$ be a closed smooth hypersurface. Let $\{u_\lambda\}$ be a sequence of $L^2$-normalized Laplace eigenfunctions associated to a defect measure $\mu$ that is tangentially concentrated with respect to $\Gamma$. Then

$$\|\lambda_j^{-1} \partial_\nu u_{\lambda_j}\|_{L^2(\Gamma)} = o(1). \tag{3.2}$$
We should point out that [Tat98], [CHT15] and [Tac17] independently used quite different proof to obtain the $O(1)$ upper bound of $\| \lambda - 1 \partial \nu u_\lambda \|_{L^2(\Gamma)}$.

The proof of Theorem 2 also uses the Rellich identity, involving the commutator of $-h^2 \Delta - 1$ with the operator $\chi_\alpha(x_n) h D_n$. Integrating over $\Omega$, one has

$$i \int_{\Omega} [-h^2 \Delta - 1, \chi_\alpha(x_n) h D_n] u_h \bar{u}_h dx$$

$$= \int_{\Gamma} ((h D_n)^2 u_h) \bar{u}_h d\sigma + \int_{\Gamma} (h D_n u_h) \bar{h} D_n u_h d\sigma$$

$$\geq \int_{\Gamma} (1 + h^2 \Delta) u_h \bar{u}_h d\sigma + \frac{1}{2} \int_{\Gamma} |h D_n u_h|^2 d\sigma,$$

(3.3)

since $u_h$ is an eigenfunction, $(h D_n)^2 u_h$ is equal to $(1 + h^2 \Delta) u_h$ up to an error term $h^2 Lu_h \Gamma$, where $L$ is a first order differential operator about $\partial x_n$, which can be bounded by $\frac{1}{2} \int_{\Gamma} |h D_n u_h|^2 d\sigma$ using Cauchy-Schwarz inequality and the $h^{-1/4}$ bound of Burq-Gérard-Tzvetkov [BGT07] for $u_h$.

In [CHT15] the authors, using the exterior mass estimates and the $h^{1/4}$ bound of Burq-Gérard-Tzvetkov [BGT07], showed that

$$\int_{\Gamma} (1 + h^2 \Delta) u_h \bar{u}_h d\sigma \geq -Ch^\sigma,$$

(3.4)

where $\sigma > 0$.

Like (2.11), without the extra term from potential $V(x)$, [GZ21] shows that

$$\lim_{a \to 0} \int_{S^*_M} \chi_\alpha(x_n) \partial x_n R d\mu = 0$$

and

$$\lim_{h \to 0} \lim_{a \to 0} \frac{i}{h} \int_{\Omega} [-h^2 \Delta - 1, \chi_\alpha(x_n) h D_n] u_h \bar{u}_h dx = \int_{S^*_M \setminus S^* \Gamma} |\xi_n| d\mu^\perp,$$

here

$$\mu^\perp(A) := \lim_{t \to 0^+} \frac{1}{2t} \mu \left( \bigcup_{|s| \leq t} \exp(s H_p)(A) \right).$$

Since $u_{\lambda_j}, j = 1, 2, \ldots$ is tangentially concentrated with respect to $\Gamma$, we can get

$$\| \lambda_j^{-1} \partial \nu u_{\lambda_j} \|_{L^2(\Gamma)} = o(1).$$

3.2. Tangentially concentrated on the torus. Let $\mathbb{T}^2$ be the 2-dimensional square flat torus which is identified with $\{ (x_1, x_2) : (x_1, x_2) \in [0, 1) \times [0, 1) \}$. Consider the sequence of eigenfunctions

$$\varphi_h(x_1, x_2) = e^{h x_1}, \quad h^{-1} \in 2\pi \mathbb{Z}.$$

(3.5)

As shown in the [CGT18 Section 5.1], the associated defect measure is

$$\mu(x_1, x_2, \xi_1, \xi_2) = \delta_{(1,0)}(\xi_1, \xi_2) dx_1 dx_2.$$

(3.6)
Now consider the curve $\Gamma \subset T^2$ defined as $\Gamma = \{(x_1, x_2) : x_2 = 0\}$. Since $S^*_{1^\perp}T^2 \setminus S^* \Gamma = \{(x_1, x_2, \xi_1, \xi_2) \in S^*T^2 : \xi_2 > 0\}$, so we have
\[
\frac{1}{2T} \mu(\Lambda_{S^*_{1^\perp}T^2 \setminus S^* \Gamma}) = 0.
\] (3.7)

And it’s straightforward to get that
\[
\|h \partial_{x_2} \varphi_h(x_1, x_2)\|_{L^2(\Gamma)} = 0
\] (3.8)
which is consistent with Theorem 2.

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