The physical Meaning of Replica Symmetry Breaking

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Abstract

In this talk I will presente the physical meaning of replica symmetry breaking stressing the physical concepts. After introducing the theoretical framework and the experimental evidence for replica symmetry breaking, I will describe some of the basic ideas using a probabilistic language. The predictions for off-equilibrium dynamics will be shortly outlined.

1 Introduction

In this talk I will underline the physical meaning of replica symmetry breaking [1, 2, 3]. I will stress the physical concepts and I will skip most of the technical details. It is an hard job because the field has grown in a significant way in the last twenty years and many results are available.

I will try to make a selection of the most significant results, which is however partly arbitrary and incomplete. The main points I would like to discuss are:

• Complex Systems: the coexistence of many phases.
• The definition of the overlap and its probabilities distributions.
• Experimental evidence of replica symmetry breaking.
• High level statistical mechanics.
• Stochastic stability.
• Overlap equivalence and ultrametricity.
• The algebraic replica approach.
• Off-equilibrium dynamics.

As you can see from the previous list, in this talk I will try to connect rather different topics which can be studied using an unified approach in the replica framework.
2 Complex Systems: the coexistence of many phases

Boltzmann statistical mechanics can be considered an example of a successful reductionistic program in the sense that it gives an microscopic derivation of the presence of emergent (collective) behaviour of a system which has many variables. This phenomenon is known as phase transition.

If the different phases are separated by a first order transition, just at the phase transition point a very interesting phenomenon is present: phase coexistence. This usually happens if we tune one parameter: the gas liquid coexistence is present on a line in the pressure-volume plane, while the liquid-gas-solid triple point is just a point in this plane. This behaviour is summarized by the Gibbs rule which states that, in absence of symmetries, we have to tune $n$ parameters in order to have the coexistence of $n + 1$ phases.

The Gibbs rule is appropriate for many systems, however in the case of complex systems we have that the opposite situation is valid: the number of phases is very large (infinite) for a generic choice of parameters. This last property may be taken as a definition of a complex system. It is usual to assume that all these states are globally very similar: translational invariant quantities (e.g. energy) have the same value in all the phases (apart from corrections proportional to $N^{-1/2}$), this last properties being called phase democracy. These states cannot be separated by external parameters coupled to translational invariant quantities, but only by comparing one state with an other.

An example of this phenomenon would be a very long heteropolymer, e.g. a protein or RNA, which may folds in many different structures. However quite different foldings may have a very similar density. Of course you will discover that two proteins have folded in two different structures if we compare them.

In order to be precise we should consider a large but (finite) system. We want to decompose the phase state in valleys (phases, states) separated by barriers. If the free energy as function of the configuration space has many minima (a corrugated free energy landscape, as shown in Figure 1: An artistic view of the free energy of a complex system as function of the configuration space.

*For a discussion of the meaning of finite volume states, which are different from infinite volume states, see [1].*
the number of states will be very large. An analytic and quantitative study of the properties of the free energy landscape in a particular model can be found in [5].

Let us consider for definitiveness a spin system with \( N \) points (spins are labeled by \( i \), which in some cases will be a lattice point).

States (labeled by \( \alpha \)) are characterized by different local magnetizations:

\[
m_{\alpha}(i) = \langle \sigma(i) \rangle_{\alpha},
\]

where \( \langle \cdot \rangle_{\alpha} \) is the expectation value in the valley labeled by \( \alpha \). The average done with the Boltzmann distribution is denoted as \( \langle \cdot \rangle \) and it can be written as linear combinations of the averages inside the valleys. We have the relation:

\[
\langle \cdot \rangle \approx \sum_{\alpha} w_{\alpha} \langle \cdot \rangle_{\alpha}.
\]

We can write that the relation

\[
w_{\alpha} \propto \exp(-\beta F_{\alpha}),
\]

where by definition \( F_{\alpha} \) is the free energy of the valley labeled by \( \alpha \).

In the rest of this talk we will call \( J \) the control parameters of the systems. The average over \( J \) will be denoted by a bar (e.g. \( \bar{F} \)). In the cases I will consider here a quenched disorder is present: the variables \( J \) parametrize the quenched disorder.

The general problem we face is to find those quantities which do not depend on \( J \) and to find the probability distribution of those quantities which do depend on \( J \).

### 3 The overlap and its probabilities

As we have already remarked in the case of heteropolymers folding, states may be separated making a comparison among them. At this end it is convenient to consider their mutual overlap. Given two configurations (\( \sigma \) and \( \tau \)), we define their overlap:

\[
q[\sigma, \tau] = \frac{1}{N} \sum_{i=1}^{N} \sigma(i)\tau(i).
\]

The overlap among the states is defined as

\[
q(\alpha, \gamma) = \frac{1}{N} \sum_{i=1}^{N} m_{\alpha}(i)m_{\gamma}(i) \approx q[\sigma, \tau],
\]

where \( \sigma \) and \( \tau \) are two generic configurations that belong to the states \( \alpha \) and \( \gamma \) respectively.

We define \( P_{J}(q) \) as probability distribution of the overlap \( q \) at given \( J \), i.e. the histogram of \( q[\sigma, \tau] \) where \( \sigma \) and \( \tau \) are two equilibrium configurations. Using eq. (2), one finds that

\[
P_{J}(q) = \sum_{\alpha,\gamma} w_{\alpha} w_{\gamma} \delta(q - q_{\alpha,\gamma}),
\]

where in a finite volume system the delta functions are smoothed. If there is more than one state, \( P_{J}(q) \) is not a single delta function

\[
P_{J}(q) \neq \delta(q - q_{EA}).
\]

If this happens we say that the replica symmetry is broken: two identical replicas of the same system may stay in a quite different state.
There are many models where the function $P_J(q)$ is non-trivial: a well known example is given by Ising spin glasses [1, 6, 7]. In this case the Hamiltonian is given by

$$H = -\sum_{i,k} J_{i,k} \sigma_i \sigma_k - \sum_i h_i \sigma_i,$$

where $\sigma = \pm 1$ are the spins. The variables $J$ are random couplings (e.g. Gaussian or $\pm 1$) and the variables $h_i$ are the magnetic fields, which may be point dependent.

Let us consider two different models for spin glasses:

- The Sherrington Kirkpatrick model (infinite range): all $N$ points are connected: $J_{i,k} = O(N^{-1/2})$. Eventually $N$ goes to infinity.
- Short range models: $i$ belongs to a $L^D$ lattice. The interaction is nearest neighbour (the variables $J$ are or zero or of order 1) and eventually $L$ goes to infinity at fixed $D$ (e.g. $D = 3$).

Analytic studies have been done in the case of the SK model, where one can prove rigorously that the function $P_J(q)$ is non-trivial. In the finite dimensional case no theorem has been proved and in order to answer to the question if the function $P_J(q)$ is trivial we must resort to numerical simulations [8] or to experiments.

In fig. (2) we show the numerical simulations for 4 different systems (i.e. different choices of the $J$ extracted with the same probability) of size $16^3$ [10]. The slightly asymmetry of the

Figure 2: The function $P_J(q)$ for four different samples (i.e. different choices of $J$) for $D = 3$, $L = 16$ ($16^3$ spins).
functions is an effect of the finite simulation time. It is evident that the function \( P_J(q) \) is non-trivial and it looks like a sum of a smoothed delta functions. It is also evident that the function \( P_J(q) \) changes dramatically from system to systems.

It is interesting to see what happens if we average over the samples. We can define

\[
P(q) = \overline{P_J(q)} .
\]

Of course, if \( P_J(q) \) depends on \( J \), we have that

\[
\overline{P_J(q_1)P_J(q_2)} \equiv P(q_1, q_2) \neq P(q_1)P(q_2) .
\]

In fig. (3) we show the average over many samples of \( P_J(q) \) in the \( D = 4 \) case (a similar picture holds in \( D = 3 \)). In this way we obtain a smooth function, with two peaks which are slightly shifted and becomes sharper and sharper when the size of the system becomes larger. It seems quite reasonable that when the system becomes infinite this peak evolves toward a delta function which corresponds to the contribution coming from two configurations \( \sigma \) and \( \tau \) which belongs to the same state.

### 4 Experimental evidence of replica symmetry breaking

Replica symmetry breaking affects the equilibrium properties of the system and in particular the magnetic susceptibility. For example let us consider a system in presence of an external constant magnetic field, with Hamiltonian given by:

\[
H[\sigma] = H_0[\sigma] + \sum_i h\sigma_i .
\]

As soon as replica symmetry is broken we can define two magnetic susceptibilities which are different:
The magnetic susceptibility that we obtain when the system is constrained to remain in a valley. In the limit of zero magnetic field this susceptibility is given by $\chi_{LR} = \beta (1 - q_{EA})$.

The total susceptibility magnetic susceptibility (the system is allowed to change state as an effect of the magnetic field). In the limit of zero magnetic field this susceptibility is given by $\chi_{eq} = \beta \int dq P(q)(1 - q) \leq \beta (1 - q_{EA})$.

Both susceptibilities are experimentally observable.

- The first susceptibility is the susceptibility that you measure if you add an very small magnetic field at low temperature. The field should be small enough in order to neglect non-linear effects. In this situation, when we change the magnetic field, the system remains inside a given state and it is not forced to jump from a state to another state and we measure the ZFC (zero field cooled) susceptibility, that corresponds to $\chi_{LR}$.

- The second susceptibility can be approximately measured doing a cooling in presence of a small field: in this case the system has the ability to chose the state which is most appropriate in presence of the applied field. This susceptibility, the so called FC (field cooled) susceptibility is nearly independent from the temperature and corresponds to $\chi_{eq}$.

Therefore one can identify $\chi_{LR}$ and $\chi_{eq}$ with the ZFC susceptibility and with the FC susceptibility respectively. The experimental plot of the two susceptibilities is shown in fig. (4). They are clearly equal in the high temperature phase while they differ in the low temperature phase.

The difference among the two susceptibilities is a crucial signature of replica symmetry breaking and, as far as I known, can explained only in this framework. This phenomenon is due to the fact that a small change in the magnetic field pushes the system in a slightly metastable state, which may decay only with a very long time scale. This may happens only if there are many states which differs one from the other by a very small amount in free energy.
5 The theoretical framework

The general theoretical problem we face is to find out which is the probability distribution of the set of all $q_{\alpha, \gamma}$ and $F_\alpha$ (or equivalently $w_\alpha$). More precisely for each given $N$ and $J$ we call $P$ the set of all $q_{\alpha, \gamma}$ and $F_\alpha$: as we have seen this quantity has strong variations when we change the system. We now face the problem of compute the probability distribution of $P$, that we call $\mathcal{P}(P)$. Moreover it should be clear that also when $\mathcal{P}(P)$ is known the computation of the average of some quantities over this distribution is non-trivial because for large systems $P$ contains an unbounded number of variables. The task of doing these kind of averages we can regarded as a sort of high level (macroscopic) statistical mechanics [12], where the basic entities are the phases of the system, while the usual statistical mechanics can be thought as low level (microscopic) statistical mechanics.

The number of possible form of the probability distribution $\mathcal{P}(P)$ is very high ($P$ is an infinite dimensional vector). In order to reduce the number of possible distributions one usually uses some general guiding principles. The simplest theory is based on two principles:

- Stochastic stability [13, 14, 15].
- Overlap equivalence [15, 16].

Stochastic stability is nearly automatically implemented in the algebraic replica approach that will be described in the next section and it seems to be a rather compulsory property in equilibrium statistical mechanics. Overlap equivalence is usually implemented in the algebraic replica approach, but is certainly less compulsory than stochastic stability.

5.1 Stochastic stability

In the nutshell stochastic stability states that the system we are considering behaves like a generic random system. Technically speaking in order to formulate stochastic stability we have to consider the statistical properties of the system with Hamiltonian given by the original Hamiltonian ($H$) plus a random perturbation ($H_R$):

$$H(\epsilon) = H + \epsilon H_R .$$

Stochastic stability states that all the properties of the system are smooth functions of $\epsilon$ around $\epsilon = 0$, after doing the appropriate averages over the original Hamiltonian and the random Hamiltonian.

Typical examples of random perturbations perturbations (we can chose the value of $r$ in an arbitrary way):

$$H_R^{(r)} = N^{(r-1)/2} \sum_{i_1 \ldots i_r} R(i_1 \ldots i_r)\sigma(i_1)\ldots\sigma(i_r) ,$$

where for simplicity we can restrict ourselves to the case where the variables $R$ are random uncorrelated Gaussian variables. For $r = 1$ this perturbation corresponds to adding a random magnetic field:

$$H_R^{(1)} = \sum_{i_1} R(i_1)\sigma(i_1) .$$

Stochastic stability is non-trivial statement: when we add a perturbation the weight of the states changes of an amount that diverges when $N$ goes to infinity at fixed $\epsilon$. Indeed the variation in the individual free energies is given by $\delta F_\alpha = O(\epsilon N^{1/2})$.

\footnote{The words “low level” and “high level” are used in the same spirit as “low level language” and “high level language” in computer science.}
Figure 5: The quantity $\langle q^2 \rangle^2 - \left( \frac{1}{3} \langle q^4 \rangle + \frac{2}{3} \langle q^2 \rangle^2 \right)$ as function of the temperature for different values of $L$ in $D = 3$.

It is useful to remark that if a symmetry is present, a system cannot be stochastically stable. Indeed spin glasses may be stochastically stable only in the presence of a finite, non-zero magnetic field which breaks the $\sigma \leftrightarrow -\sigma$ symmetry. If a symmetry is present, stochastic stability may be valid only for those quantities which are invariant under the action of the symmetry group. It is also remarkable that the union of two non-trivial uncoupled stochastically stable systems is not stochastically stable. Therefore a non-trivial stochastically stable system cannot be decomposed as the union of two or more parts whose interaction can be neglected.

We will now give an example of the predictive power of stochastic stability [18].

There are systems in which the replica symmetry is broken at one-step. In other words in this kind of systems the overlap may take only two values:

$$q_{\alpha,\alpha} = q_1 = q_{EA}, \quad q_{\alpha,\gamma} = q_0 \quad \text{for} \quad \alpha \neq \gamma.$$  

This is the simplest situation: all pairs of different states have the same (i.e. $q_0$) mutual overlap, which for simplicity we will take equal to zero. The only quantity we have to determine is the probability of the free energies. The free energies are assumed to be random uncorrelated variables and the probability of having a state with total free energy in the interval $[F, F+dF]$ is

$$\rho(F)dF.$$  

Stochastic stability implies that in the region which is dominant for the thermodynamic quantities, i.e. for the states having not too high free energy:

$$\rho(F) \propto \exp(\beta m(F - F_0)),$$  

where $F_0$ a system dependent reference free energy, proportional to $N$. As a byproduct the function $P(q)$ can be magnetisations computed and is given by

$$P(q) = m\delta(q) + (1 - m)\delta(q - q_{EA}).$$  

The proof of the previous statement is rather simple [18]. Stochastic stability imposes that the form of the function $\rho(F)$ remains unchanged (apart from a possible shift in $F_0$) when one
adds a small random perturbation. Let us consider the effect of a perturbation of strength $\epsilon$ on the free energy of a state, say $\alpha$. The unperturbed value of the free energy is denoted by $F_\alpha$. The new value of the free energy $G_\alpha$ is given by

$$G_\alpha = F_\alpha + \epsilon r_\alpha,$$

where $r_\alpha$ are identically distributed uncorrelated random numbers. Stochastic stability implies that the distribution $\rho(G)$ is the same as $\rho(F)$. Expanding to second order in $\epsilon$ we get:

$$\frac{d\rho}{dF} \propto \frac{d^2\rho}{dF^2}.$$  

Therefore $\rho(F) \propto \exp(\beta m(F - F_0))$.

In the general case stochastic stability implies that

$$P(q_1, q_2) = P_J(q_1)P_J(q_2) = \frac{2}{3}P(q_1)P(q_2) + \frac{1}{3}P(q_1)\delta(q_1 - q_2).$$

A particular case of the previous relation is the following one

$$\langle q^2 \rangle^2 = \frac{1}{3}\langle q^4 \rangle + \frac{2}{3}\langle q^2 \rangle^2.$$  

We have tested the previous relations in three dimensions as function of the temperature at different values of $L$. In fig. we plot the quantity $\langle q^2 \rangle^2 - \left(\frac{1}{3}\langle q^4 \rangle + \frac{2}{3}\langle q^2 \rangle^2\right)$, which should be equal to zero. Indeed it is very small and its values decreases with $L$. In order to give a more precise idea of the accuracy of stochastic stability in fig. we plot separately the quantities $\langle q^2 \rangle^2$ and $\frac{1}{3}\langle q^4 \rangle + \frac{2}{3}\langle q^2 \rangle^2$. The two quantities cannot be distinguished on this scale and in the low temperature region each of them is a factor of $10^3$ bigger of their difference. I believe that there should be few doubts on the fact that stochastic stability is satisfied for three dimensional spin glasses.
5.2 Overlap equivalence and ultrametricity

In the case in which the overlap may take three or more values, stochastic stability apparently does not fix the probability distribution. A further general principle may be useful in order to get new constraints. This principle is overlap equivalence.

In order to formulate the principle of overlap equivalence it is convenient to introduce a generalized overlap. Let \( A(i) \) be a local quantity. We define:

\[
q_{\alpha,\gamma}^A = N^{-1} \sum_i \langle A(i) \rangle_\alpha \langle A(i) \rangle_\gamma .
\]  

(23)

Let us consider two possibilities:

- For \( A(i) = \sigma(i) \) we get the usual overlap: \( q^A = q \).
- For \( A(i) = H(i) \) we get the usual energy overlap: \( q^A = q^E \).

If we consider also the generalized overlaps we have that the description of a system is much more involved: we have to specify the \( w_\alpha \) and the \( q_{\alpha,\gamma}^A \) for all possible choices of \( A \). In the general case an infinite number of quantities (i.e. \( q_{\alpha,\gamma}^A \), for all choices of \( A \)) characterizes the mutual relations among the state \( \alpha \) and the state \( \gamma \) [12, 3].

Overlap equivalence states that this infinite number of different overlaps is reduced to one (the usual overlap) [15, 16, 18]. There is only one significant overlap and all overlaps (depending on the operator \( A \)) are given functions of the spin overlap. For any choices of \( A \) there is a corresponding function \( f^A(q) \) such that \( q_{\alpha,\gamma}^A = f^A(q_{\alpha,\gamma}) \).

Overlap equivalence may be also formulated if we define the \( q \)-restricted ensemble:

\[
\langle f(\sigma, \tau) \rangle_q \propto \sum_{\sigma,\tau} f(\sigma, \tau) \delta(Nq - q[\sigma, \tau]).
\]

(24)

Overlap equivalence implies the validity of cluster decomposition in the \( q \)-restricted ensembles.

Overlap equivalence (plus stochastic stability) seems [16] to imply the ultrametricity condition

\[
q_{\alpha,\gamma} \geq \min(q_{\alpha,\beta}, q_{\beta,\gamma}) \quad \forall \beta.
\]

(25)

If ultrametricity is valid, one finds that

\[
P^{12,31,23}(q_{12}, q_{23}, q_{31}) = 0 ,
\]

(26)

as soon as one of the three ultrametricity relations,

\[
q_{12} \geq \min(q_{23}, q_{31}) , \quad q_{23} \geq \min(q_{31}, q_{12}) , \quad q_{31} \geq \min(q_{12}, q_{23}) ,
\]

(27)

is not satisfied.

It is remarkable that, given the function \( P(q) \), the ultrametricity condition completely determines the probability \( P^{12,23,31} \), if we assume stochastic stability. Overlap equivalence may be less compulsory of stochastic stability. There are some indications [17] for its validity in the finite dimensional case (i.e. beyond mean field theory), but they are no so strong as for stochastic stability.

6 The algebraic replica approach

The algebraic replica approach is a compact way to code all the previous information into a matrix and also to compute the free energy [1, 2, 15].
A crucial role is played by a matrix $Q_{a,b}$ which is said to be a $0 \times 0$ matrix. The direct definition of a $0 \times 0$ matrix may be not too easy. Instead we can consider a family $Q^{(n)}_{a,b}$ of $n \times n$ matrices which depend analytically on $n$: they are defined for $n$ multiple than $M$ in such a way that the analytic continuation of some scalar functions of these matrices at $n = 0$ is well defined.

In this formulation the probability (after average over the permutation of lines and columns of the matrix) that an element of the matrix $Q_{a,b}$ with $a \neq b$ is equal to $q$, coincides with the function $P(q) \equiv P_J(q)$:

$$P(q) = \sum_{\alpha,\gamma} w_{\alpha} w_{\gamma} \delta(q - q_{\alpha,\gamma}) = \lim_{n \to 0} \sum_{a,b} \frac{\delta(Q_{a,b} - q)}{n(n-1)} .$$  

(28)

In the same way the probability that an element of the matrix $(Q_{a,b})$ is equal to $q_1$ and an other element of the matrix $(Q_{c,d})$ is equal to $q_2$ (with $a$, $b$, $c$ and $d$ different) give us $P(q_1, q_2) \equiv P_J(q_1)P_J(q_2)$:

$$P(q_1, q_2) = \sum_{\alpha,\beta,\gamma,\epsilon} w_{\alpha} w_{\beta} w_{\gamma} w_{\epsilon} \delta(q_1 - q_{\alpha,\beta}) \delta(q_2 - q_{\gamma,\epsilon}) = \lim_{n \to 0} \frac{\sum_{a,b,c,d} \delta(Q_{a,b} - q_1) \delta(Q_{c,d} - q_2)}{n(n-1)(n-2)(n-3)} .$$  

(29)

In this approach probability statements become algebraic statements.

- Stochastic stability becomes the statement that one line of the matrix is a permutation of an other line of the matrix.

- Overlap equivalence is a equivalent to more complex statement: if there are four indices $(a, b, c$ and $d$) such that $Q_{a,b} = Q_{c,d}$, there is a permutation $\pi$ that leaves the matrix unchanged ($Q_{a,b} = Q_{\pi(a),\pi(b)}$) and brings $a$ in $c$ and $b$ in $d$ (i.e. $\pi(a) = c$ and $\pi(b) = d$).

As you see in the algebraic approach one uses a quite different (and more abstract) language from the probabilistic approach. Using this language computations are often more simple and compact.

### 7 Off-equilibrium dynamics

The general problem that we face is to find what happens if the system is carried in a slightly off equilibrium situation. The are are two ways in which this can be done.

- We rapidly cool the system starting from a random (high temperature) configuration at time zero and we wait a time much larger than the microscopical one. The system orders at distances smaller that a coherence distance $\xi(t)$ (which eventually diverges when $t$ goes to infinity) but remains always disordered at distances larger than $\xi(t)$.

- A second possibility consists in forcing the system in on off-equilibrium state by gently shaking it. This can be done for example by adding a small time dependent magnetic field, which should however strong enough to force a large scale rearrangement of the system \cite{22}.

In the first case we have the phenomenon of ageing. This effects may be evidenziated if we define a two time correlation function and two time relaxation functions (we cool the system at time 0) \cite{20,21}. The correlation function is defined to be

$$C(t, t_w) \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i(t_w) \sigma_i(t_w + t) \rangle ,$$  

(30)
Figure 7: The correlation function for spin glasses as function of time $t$ at different $t_w$. which is equal to the overlap $q(t_w, t_w + t)$ among a configuration at time $t_w$ and one at time $t_w + t$. The relaxation function $S(t, t_w)$ is thus given by

$$\beta^{-1} \lim_{\delta h \to 0} \frac{\delta m(t + t_w)}{\delta h},$$

where $\delta m$ is the variation of the magnetization when we add a magnetic field $\delta h$ starting from time $t_w$. More generally we can introduce the time dependent Hamiltonian:

$$H = H_0 + \theta(t - t_w) \sum_i h_i \sigma_i.$$  

The relaxation function is thus defined as:

$$\beta S(t, t_w) \equiv \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\partial \sigma_i(t + t_w)}{\partial h_i} \right).$$

We can distinguish two situations

- For $t << t_w$ we stay in the quasi-equilibrium” regime, $C(t, t_w) \simeq C_{eq}(t)$, where $C_{eq}(t)$ is the equilibrium correlation function; in this case $q_{EA} \equiv \lim_{t \to \infty} \lim_{t_w \to \infty} C(t, t_w)$.

- For $t = O(t_w)$ or larger we stay in the aging regime. In the case where simple aging holds, $C(t, t_w) \propto C(t/t_w)$. A plot of the correlation function for spin glasses at different $t_w$ is shown in fig. (7).

In the equilibrium regime, if we plot parametrically the relaxation function as function of the correlation, we find that

$$\frac{dS}{dC} = -1,$$

which is a compact way of writing the fluctuation-dissipation theorem.

We can distinguish two situations
Generally speaking the fluctuation-dissipation theorem is not valid in the off-equilibrium regime. In this case one can use stochastic stability to derive a relation among statics properties and the form of the function $S(C)$ measured in off-equilibrium:  

$$ \frac{dS}{dC} = \int_0^C dq P(q) \equiv X(C). $$  

(35)

In fig. (8) we show three main different kinds of dynamical response $S(C)$, that correspond to different shapes of the static $P(q)$ (which in the case of spin glasses at zero magnetic field should be replaced by $P(|q|)$). Case A correspond to systems where replica symmetry is not broken, case B to one step replica symmetry breaking, which should be present in structural glasses and case C to continuous replica symmetry breaking, which is present in spin glasses.

The validity of these relation has been intensively checked in numerical experiments (see for example fig. (9)).

In spin glasses the relaxation function has been experimentally measured many times in the aging regime, while the correlation function has not yet been measured: it would be a much more difficult experiment in which one has to measure thermal fluctuations. Fortunately enough measurements of both quantities for spin glasses are in now progress. It would be extremely interesting to see if they agree with the theoretical predictions.

For reasons of time I shall not discuss the generalization of the previous arguments to the case of a spin glass in presence of a time dependent magnetic field. I only remark that in this case the correlation function is directly related to the Birkhausen noise, which as far as I know, has never been measured in spin glasses.
8 Conclusions

In this talk I have presented a review of the basic ideas in the mean field approach to spin glasses. There are many points which I have not covered and are very important. Let me just mention some of them;

• The analytic studies of the corrections to mean field theory.
• The purely dynamical approach which can be used without any reference to equilibrium.
• The extension of these ideas to other disorder systems, to neural network and in general to the problem of learning.
• The relevance of this approach for biological systems, both at the molecular level and at the systemic level.
• The extension of these ideas to systems in which quenched disorder is absent, e.g. structural glasses [26].

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