The unified soliton system as the AdS$_2$ system

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Abstract

We study the Riemann geometric approach to be aimed at unifying soliton systems. The general two-dimensional Einstein equation with a constant negative scalar curvature becomes an integrable differential equation. We show that such Einstein equation includes KdV/mKdV/sine-Gordon equations.

1. Introduction

The discovery of the soliton [1–3] has given the breakthrough to exactly solvable non-linear equations. There have been many interesting developments to understand soliton systems such as the AKNS formulation [4–6], the Bäcklund transformation [7–9], the Hirota equation [10, 11], the Sato theory [12], the vertex construction of the soliton solution [13, 14], and the Schwarzian type mKdV/KdV equation [15].

Our characterization of the soliton system is that it is a system of integrable non-linear differential equations, which have not only some exact solutions but also $N$-soliton solutions i.e. systematically obtained infinitely many solutions. The KdV/mKdV/sine-Gordon systems are such ones, and we study common structures for these soliton systems.

In our previous paper [16, 17], we have characterized such a soliton system as it has the local GL$(2, \mathbb{R})$ self gauge symmetry, where the local gauge parameter is connected with the gauge potential. We have pointed out that a special local self gauge transformation becomes the Bäcklund transformation. Combining various Bäcklund transformations, we have the algebraic relation, which becomes the addition formula to construct the $N$-soliton solution from various 1-soliton solutions. In our approach, the mechanism to be able to construct $N$-soliton solutions for the non-linear soliton systems comes from the GL$(2, \mathbb{R})$ group (= Mobius group) structure for such systems.

So far almost all soliton systems, which admit $N$-soliton solutions, are restricted only to two-dimensional models, and the Lie group structure of such soliton systems is restricted to the rank one Lie group GL$(2, \mathbb{R})/\mathbb{Z}_2 \cong \text{SO}(2, 1) \cong SU(1, 1)/\mathbb{Z}_2$. We will study in this paper the reasons of the restrictions above from the Riemann geometric approach.

The sine-Gordon equation is a well-known integrable system from old days [18]. In the theory of the curved surface, the fundamental equation is known as the Gauss-Weingarten equation. The integrability condition of this Gauss-Weingarten equation is given as the Gauss-Codazzi formula. The sine-Gordon equation comes from this Gauss-Codazzi formula for the pseudo-sphere. However, the approach from the curved surface is more complicated than that from the Riemann geometry, so that the approach from the curved surface seems to be difficult to generalize it into the higher dimensional and higher Lie group symmetric soliton systems. Then we use the Riemann geometric approach to the soliton systems in this paper.

We will show in this paper, (i) the general two-dimensional Einstein equation with a constant negative scalar curvature, that is AdS$_2$ system, becomes an integrable differential equation. (ii) such Einstein equation includes the unified soliton systems of the KdV/mKdV/sine-Gordon equations.
2. Riemann geometric approach to two-dimensional soliton system as AdS$_2$ system

We start with the general two-dimensional metric in the form
\[ ds^2 = f(x, t)dt^2 + g(x, t)dx^2 + 2h(x, t)dt \; dx, \]
where $g_{\theta\theta}(x, t) = f(x, t), g_{tt}(x, t) = g(x, t), g_{01}(x, t) = g_{10}(x, t) = h(x, t)$. In two dimensions, due to the first Bianchi identity of the Riemann tensor, we have
\[ R_{ij}(x, t) = K(x, t)g_{ij}(x, t), \]
with
\[ K(x, t) = \frac{L(x, t)}{\det(g)}, \quad L(x, t) = R_{0101} = \tilde{R}_{0101} = -R_{0110} = -\tilde{R}_{0001}. \]

We consider the case of the constant scalar curvature $R$, i.e. $R = 2K(x, t) = \text{constant}$, and we take $K(x, t) = -1$ for simplicity. In this case, the metric becomes the Einstein metric which satisfies $R_{ij} = -g_{ij}$. Computing the scalar curvature with $R = -2$, we obtain the following differential equation
\[
2(fg - h^2) (R + 2) = -2(fg - h^2) (f_{xx} + g_{tt} - 2h_{xt}) + f_x^2 g + f_y g_x f - 2f_y h_x h + g_x^2 f + f_y g_x g - 2g_x h_x h - f_x g_y h + f_y g_y h - 2f_x h_y g - 2g_x h_y f + 4h_x h_y + 4(fg - h^2)^2 = 0. \]

Using two degrees of freedom of the general coordinate transformation in two dimensions, $t \rightarrow T(t, x)$ and $x \rightarrow X(t, x)$, two of $g_{00}, g_{11}, g_{01}$ can be set to be constant. Then the Einstein metric, which satisfies $R_{ij} = -g_{ij}$, can be transformed into that of the pseudo-sphere $x^2 + y^2 - z^2 = -1$ [18], which has the symmetry of SO(2, 1).

We parametrize it in the form $x = \sinh \theta \cos \phi, y = \sinh \theta \sin \phi, z = \cosh \theta$, which gives
\[
\begin{align*}
\frac{dx}{d\theta} &= \cosh \theta \cos \phi, \\
\frac{dy}{d\theta} &= \cosh \theta \sin \phi, \\
\frac{dz}{d\theta} &= \sinh \theta, \\
\end{align*}
\]
These lead the metric
\[ ds^2 = dx^2 + dy^2 - dz^2 = d\theta^2 + \sinh^2 \theta \; d\phi^2, \]
which is one of the AdS$_2$ parametrizations. In this case, $g_{00} = 1, g_{01} = 0$ are constants. For this metric, we have
\[ R_{00} = -1 = -g_{00}, \quad R_1 = -\sinh^2 \theta = -g_{11}, \quad R_{01} = 0, \quad R = -2. \]

In general, the Lie group symmetry of the general two-dimensional surface with the negative constant scalar curvature is that of the rank one Lie group $GL(2, \mathbb{R})/\mathbb{Z}_2 \cong SO(2, 1) \cong SU(1, 1)/\mathbb{Z}_2$. The metric equation (2.5) is the real two-dimensional metric without complex structure and the Lie group symmetry becomes the Möbius group symmetry. If the system has the real two-dimensional metric with complex structure instead, the Lie group symmetry of that system becomes the infinite dimensional conformal symmetry.

3. Integrable condition of surface in three-dimensional euclidean space

3.1. Point in three-dimensional Euclidean space

We formulate the three-dimensional geometry by the Maurer–Cartan formalism with the exterior differential form [19, 20]. For any point $x$, we attach the moving orthonormal basis $e_1, e_2, e_3$. The differential 1-form $dx$ leads the structure equation given in the form
\[ dx = \sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3 = \sigma e, \quad \text{(3.1)} \]
\[ de = \Omega \; e, \quad \text{(3.2)} \]
with
\[ \sigma = (\sigma_1 \sigma_2 \sigma_3), \quad e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & -\omega_1 \\ \omega_2 & \omega_1 & 0 \end{pmatrix}. \]

If we choose $K(x, t) = 1$, we cannot connected differential equations of $R = 2$ with the soliton equations. In other words, the two-dimensional sphere with $K(x, t) = 1$ has too simple structure ("No hair") to construct the N-soliton solutions.
The torsion free condition is defined by
\[ ds = \sigma \wedge \Omega, \] (3.3)
and the integrability conditions of equation (3.3) gives
\[ 0 = ds \wedge \Omega - \sigma \wedge d\Omega = \sigma \wedge (\Omega \wedge \Omega - d\Omega). \] (3.4)
By using the first Bianchi identity, the general solution of equation (3.4) is given by
\[ d\Omega = \Omega \wedge \Omega = \tilde{R} \sigma \wedge \sigma, \] (3.5)
where \( \tilde{R}_{ijkl} \) is the Riemann tensor. This is called as the curvature condition.

3.2. Point on surface in three-dimensional Euclidean space
We formulate the unified two-dimensional soliton system as the system of the negative constant surface in this Maurer–Cartan geometry [5, 6]. We choose the normal vector of the surface as \( e_3 \). Then the point on the surface is given by \( \sigma_3 = 0 \). Further, we have \( \omega_1 = \omega_2 = 0 \) from equation (3.3). The structure equation now becomes
\[ ds = \omega_3, \quad d\omega_3 = \omega_3 \wedge \omega_3. \] (3.6)
The integrability condition turns to be
\[ d\omega_1 = -\omega_1 \wedge \omega_1, \quad d\omega_2 = -\omega_2 \wedge \omega_2, \quad d\omega_3 = -\omega_3 \wedge \omega_3. \] (3.7)
where we denote \( K = -\tilde{R}_{0010} \) as the Gauss curvature. Here we consider the pseudo-sphere, that is, the constant negative curvature and we take \( K = -1 \) for simplicity. Then we have the integrability condition in the form
\[ d\sigma_1 = \omega_3 \wedge \sigma_1, \quad d\sigma_2 = -\omega_3 \wedge \sigma_2, \quad d\sigma_3 = \sigma_1 \wedge \sigma_2. \] (3.8)
This integrability condition can be expressed in the Sasaki’s 2 \( \times \) 2 matrix form [6]
\[ d\hat{\Omega} = \hat{\Omega} \wedge \hat{\Omega}, \quad \text{with} \quad \hat{\Omega} = \frac{1}{2} \begin{pmatrix} -\omega_3 + \sigma_1 & -\omega_3 + \sigma_2 \\ -\omega_3 + \sigma_1 & -\omega_3 + \sigma_2 \end{pmatrix}. \] (3.11)

3.3. General integrable differential equation
We start with the general 1-forms \( \sigma_1, \sigma_2 \), and we construct \( \omega_3 \) from equations (3.8) and (3.9). Then equation (3.10) gives the following integrable differential equation. We put
\[ \sigma_1 = f_1 dt + f_2 dx, \quad \sigma_2 = g_1 dt + g_2 dx, \quad \omega_3 = h_1 dt + h_2 dx. \] (3.12)
From equations (3.8) and (3.9), we have
\[ \omega_3 = \left( \frac{-f_{1,3} + f_{2,1} f_1 + (-g_{1,3} + g_{2,1}) g_1}{f_1 g_2 - f_2 g_1} \right) dt + \left( \frac{-f_{1,3} + f_{2,1} f_2 + (-g_{1,3} + g_{2,1}) g_2}{f_1 g_2 - f_2 g_1} \right) dx. \] (3.13)
Then equation (3.10) gives the following integrable differential equation
\[ \partial_x \left( \frac{(-f_{1,3} + f_{2,1}) f_1 + (-g_{1,3} + g_{2,1}) g_1}{f_1 g_2 - f_2 g_1} \right) + \partial_n \left( \frac{(-f_{1,3} + f_{2,1}) f_2 + (-g_{1,3} + g_{2,1}) g_2}{f_1 g_2 - f_2 g_1} \right) = f_1 g_2 - f_2 g_1 = 0. \] (3.14)
In this case, the metric is given by
\[ ds^2 = (dx)^2 = (\sigma_1 e_1 + \sigma_2 e_2)^2 = \sigma_1^2 + \sigma_2^2 = (f_1^2 + g_1^2) dt^2 + (f_2^2 + g_2^2) dx^2 + 2(f_1 f_2 + g_1 g_2) dt dx, \] (3.15)
where \( f = f_1^2 + g_1^2, \; g = f_2^2 + g_2^2, \; h = f_1 f_2 + g_1 g_2 \). If we compare equations (2.3) with (3.14), it gives the same differential equation, which means that equation (2.3) is the general integrable differential equation.
4. KdV/mKdV/sine-Gordon equations as AdS$_2$ differential equation

4.1. Sine-Gordon equation

The AKNS formalism of the sine-Gordon equation

\[ u_{xx} = \sin(u), \]  

(4.1)

is given in the form

\[ d\hat{\Omega} = \hat{\Omega} \wedge \hat{\Omega}, \]

with

\[ \hat{\Omega} = \left(\begin{array}{c}
\eta dx + \cos(u)dt/4\eta \\
u_x dx/2 + \sin(u)dt/4\eta \\
u_x dx/2 - \sin(u)dt/4\eta \\
-\eta dx - \cos(u)dt/4\eta
\end{array}\right). \]

(4.2)

We check how the integrability conditions equations (3.8)–(3.10) are satisfied. First, equations (3.8) and (3.9) are identically satisfied, and equation (3.10) gives the sine-Gordon equation $u_{xx} = \sin(u)$. Then the integrability conditions of the sine-Gordon equation are in the standard form as equations (3.12)–(3.14). In this case, we have

\[ \sigma_1 = \sin(u)dt/2, \sigma_2 = -2dx - \cos(u)dt \]

and the metric becomes

\[ ds^2 = (dx)^2 = (\sigma_1 e_1 + \sigma_2 e_2)^2 = \sigma_1^2 + \sigma_2^2 \]

\[ = \frac{(dr)^2}{4} + 4dx^2 + 2\cos(u)dt \ dx \]

\[ = (dr)^2 + (d\bar{r})^2 + 2\cos(u)d\bar{r} \ dx. \]

(4.3)

where $\bar{r} = t/2, \bar{x} = 2x$. This is the standard metric of the sine-Gordon equation. Calculating the scalar curvature of this metric, and taking the negative constant curvature, we have

\[ K = \frac{R}{2} = -1 = -\frac{u_{xx}}{\sin(u)}, \]

(4.4)

which gives the sine-Gordon equation equation (4.1).

4.2. mKdV equation

The AKNS formalism of the mKdV equation

\[ u_t + 6u_xu_x + u_{xxx} = 0, \]

(4.5)

is given by

\[ d\hat{\Omega} = \hat{\Omega} \wedge \hat{\Omega}, \]

with

\[ \hat{\Omega} = \left(\begin{array}{c}
\eta dx - (4\eta^2 + 2\eta u_x)dt \\
u_x dx - (2\eta u_x + 4\eta^2 u + 2u^3)dt \\
-\eta dx + (4\eta^2 + 2\eta u_x) dt
\end{array}\right). \]

(4.6)

We check how the integrability conditions equations (3.8)–(3.10) are satisfied. First, equations (3.8) and (3.9) are identically satisfied, and equation (3.10) gives the mKdV equation $u_t + 6u_xu_x + u_{xxx} = 0$. Then the integrability conditions of the mKdV equation are in the standard form. We take $\eta = 1$ for simplicity and we have $\sigma_1 = -4u_x dt, \sigma_2 = -2dx + 4(u^2 + 2) dt$ and we have the metric

\[ ds^2 = \sigma_1^2 + \sigma_2^2 = 4(4u_x^2 + (u^2 + 2)^2) dt^2 + dx^2 - 4(u^2 + 2) dt \ dx. \]

(4.7)

Calculating the scalar curvature of this metric, and taking the negative constant curvature, we have

\[ K = \frac{R}{2} = -1 = -\frac{2u_x + 32(u_t + 6u_xu_x + u_{xxx})}{2u_x}, \]

(4.8)

which gives the mKdV equation equation (4.5).

4.3. KdV equation

For the AKNS formalism of the KdV equation

\[ u_t + 6u_xu_x + u_{xxx} = 0, \]

(4.9)

we use the Sasaki’s form [6] to obtain the simple expression of the integrable geometrical differential equation. Then we take

\[ d\hat{\Omega} = \hat{\Omega} \wedge \hat{\Omega}, \]
with
\[
\Omega = \begin{pmatrix}
-u_x dt \\
-dx + (2u + 4\eta^2) dt \\
u_x dt
\end{pmatrix}
\begin{pmatrix}
(u - \eta^2) dx - (u_{xx} + 2u^2 + 2\eta^2 u - 4\eta^4) dt \\
-s \\
\end{pmatrix}.
\] (4.10)

We check how the integrability conditions equations (3.8)–(3.10) are satisfied. First, equation (3.8) is identically satisfied. While equation (3.9) is not identically satisfied but gives the KdV equation \(u_t + 6u u_x + u_{xxx} = 0\). Equation (3.10) gives the KdV equation. Then the integrability conditions of the KdV equation in the standard form.

Denoting \(P = u_t + 6u u_x + u_{xxx}, Q = P_x = u_{xt} + 6u u_{xx} + 6u_x^2 + u_{xxxx}\), and after calculating the scalar curvature, we have
\[
2u_x^3 (u - 2)^3 \times (R + 2)
\]
\[
= \{ -u_{xxx} u^2 + 4u_{xx} u_x - 4u_{xxx} u_x u_x - 4u_{xxx} u^2 u + 2u_{xxx} u_x + u_x^2 u - 2u_x^2 \\
+ u_{xx} u^2 - 4u_{xx} u^3 + 20u_{xx} u^2 - 32u_{xx} u + 16u_{xx} u - 6u_x^2 u^2 + 24u_x^2 u - 24u_x^2 \} P \\
+ \{ -u_{xx} u_x u + 2u_{xx} u_x - 2u_x u^3 + 4u_x u^2 + 8u_x u - 16u_x \} Q.
\] (4.12)

Then if the KdV equation equation (4.9) is satisfied, we have \(P = 0\) and \(Q = 0\), which gives \(R/2 = K = -1\).

However, the opposite is not always true, that is, even if \(R/2 = K = -1\) is satisfied, we have more general differential equation which contains the KdV equation as the special case. The reason of this property will come from the fact that equation (3.8) is not identically satisfied but gives the KdV equation itself. In the non-linear system, it is generally difficult to have the one-to-one correspondence between two integrable systems. For example, KdV system and mKdV system is connected by the Miura transformation \(u = \pm i \psi + v^3\),
\[
u_t + 6u u_x + u_{xxx} = (\pm i \partial_x + 2v)(\psi_t + 6\psi^2 \psi_x + v_{xxx}),
\] (4.13)

which means that if \(v\) satisfies the mKdV equation, \(u\) satisfies the KdV equation. But the opposite is not always true. That is, if \(u\) satisfies the KdV equation, \(v\) satisfies more general differential equation which contains the mKdV equation as the special case. In this KdV case, the integrability conditions equations (3.8)–(3.10) are not in the standard form. After the two-dimensional general coordinate transformation, we will be able to make the integrability conditions of the KdV equation in the standard form.

5. Summary and discussions

We have studied the Riemann geometric approach to the unified soliton systems, KdV/mKdV/sine-Gordon equations. We have found that the general two-dimensional Einstein equation with a constant negative scalar curvature becomes the integrable differential equation. Furthermore, we have explicitly shown that such Einstein equation includes KdV/mKdV/sine-Gordon equations.

We consider that a common nature of the soliton systems is the Lie group structure of them. Through the Moëbius group, many interesting approaches to the soliton systems are connected. The conventional geometrical approach to the soliton systems is to formulate the integrable systems in context of two-dimensional curved surface. In order to use that formalism, one must construct the complicated detailed AKNS operators equations (4.2), (4.6) and (4.10) for each soliton systems. In this approach, it seems difficult to find common features of the soliton systems, so that it is difficult to generalize to the higher dimensional and higher symmetric soliton systems. Our geometrical approach in this paper is the Riemann geometric approach, which is easier to understand a common nature of the soliton systems in terms of the constant negative scalar curvature of the two-dimensional Einstein manifold, which is the simple realization of the GL(2, \(\mathbb{R}\))\(/\mathbb{Z}_2 \cong SO(2, 1) \cong \text{Mobius group}\). One can find literatures in which authors use the Lie group symmetry to obtain the new solution for the differential equations such as the modified Klein–Gordon equation [21], which is similar to our approach. See also the related papers [22, 23].

The reason why the two-dimensional Einstein equation with the negative constant scalar curvature becomes the integrable equation might be that such Einstein equation has the Einstein metric. Even in the higher dimensional and higher Lie group symmetric soliton equations, the existence of the Einstein metric may be essential. Then the Einstein equation for the AdS\(p = SO(2, n - 1)/SO(1, n - 1)\) system and/or the Einstein equation for the Kähler-Einstein manifold, both of these systems have the Einstein metric, may be the candidate of the higher dimensional soliton system. Our Riemann geometric approach to the soliton systems may give the
non-pertubative approach to the superstring theory through the Kähler-Einstein manifolds or the Calabi-Yau varieties.

The complex one-dimensional Einstein equation for the Kähler-Einstein manifold becomes the real two-dimensional Liouville equation

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u(x, t) = \exp(u(x, t)),
\]

which is the integrable differential equation [24].

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