MORSE MATCHING METHOD FOR CONFORMAL COHOMOLOGIES

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Abstract. We apply discrete algebraic Morse theory to the computation of Hochschild cohomologies of associative conformal algebras. As an example, we evaluate the dimensions of the universal associative conformal envelope $U(3)$ of the Virasoro Lie conformal algebra relative to the associative locality $N = 3$ on the generator with scalar coefficients.

1. Introduction

1.1. Conformal algebras and their cohomologies. The notion of a conformal Lie algebra (also known as vertex Lie algebra) emerged in [12] as an algebraic tool formalizing the properties of coefficients in the singular part of the operator product expansion (OPE) formula for chiral fields in 2-dimensional conformal field theory. Namely, if $V$ is a vertex algebra, $Y(a, z)$ and $Y(b, z)$ are vertex operators corresponding to $a, b \in V$ then the commutator $[Y(a, w), Y(b, z)] \in \text{End} V[[z, z^{-1}, w, w^{-1}]]$ may be expressed as a finite distribution with respect to derivatives of the formal delta-function:

$$[Y(a, w), Y(b, z)] = \sum_{s=0}^{N(a, b)-1} Y(c_s, z) \frac{\partial^s \delta(w - z)}{s! \partial z^s},$$

$$\delta(w - z) = \sum_{t \in \mathbb{Z}} w^t z^{-t-1}.$$

The properties of binary algebraic operations $(\cdot(s))$ on $V$ given by $(a(s)b) = c_s, s \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}, a, b \in V$, along with the translation operator on $V$ lie in the background of the formal definition of a (Lie) conformal algebra.

Lie conformal algebras also appear naturally (see [22]) from the Hamiltonian formalism in the theory of differential equations of hydrodynamic type [10]. Namely, every Novikov algebra (or, more generally, a Gelfand–Dorfman algebra) gives rise to a conformal Lie algebra. In particular, the simplest Poisson bracket on the phase space with one field function $v(x)$ given by

$$\{v(x), v(y)\} = 2v(x)\delta'(x - y) + v'(x)\delta(x - y)$$

corresponds to the Virasoro conformal Lie algebra $\text{Vir}$.

In a more general context, the family of operations $(\cdot(s))$, $s \in \mathbb{Z}_+$, may be defined on a space of pairwise local formal distributions over an arbitrary (not necessarily Lie) algebra. This leads to the definition of what is an associative (commutative, Jordan, etc.) conformal algebra, see [19].

A categorial approach to the theory of conformal algebras was proposed in [5]. In this way, a conformal algebra is an algebra in the pseudo-tensor category of modules over the bialgebra $\mathbb{C}[\partial]$ of polynomials in one variable (with respect to the standard bialgebra structure). This categorial definition also makes clear how to define cohomology of conformal algebras (c. f. with [4]).

The structure theory of Lie conformal algebras that are finite modules over $\mathbb{C}[\partial]$ was established in [9]. In particular, every finite-dimensional simple Lie algebra $\mathfrak{g}$ gives rise to a simple

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conformal algebra $\text{Cur} \mathfrak{g}$ embedded into the space of formal distributions over $\mathfrak{g}[t, t^{-1}]$, and there is one exceptional simple Lie conformal algebra $\text{Vir}$ mentioned above.

Irreducible modules over $\text{Cur} \mathfrak{g}$ and $\text{Vir}$ were described in [8], and the corresponding cohomologies were computed in [4]. Conformal cohomologies have the same relations to derivations, extensions, and deformations of conformal algebras as the “ordinary” ones.

1.2. Statement of the problem and main results. It is well-known (see, e.g., [7] Ch. XIII) that for a Lie algebra $\mathfrak{g}$ acting on a $\mathfrak{g}$-module $V$ the cohomology groups $H^\mathfrak{g}(\mathfrak{g}, V)$ coincide with the Hochschild cohomology groups $H^n(U(\mathfrak{g}), V)$ of the universal associative enveloping algebra $U(\mathfrak{g})$ with coefficients in the same $V$ equipped with the induced structure of a $U(\mathfrak{g})$-module. The situation is different in the case for conformal algebras.

Given a conformal Lie algebra $L$ with a conformal $L$-module $M$, one may construct a series of universal enveloping associative conformal algebras corresponding to different associative locality functions on the generators [20]. For example, consider the Virasoro conformal algebra $\text{Vir}$. One may fix a natural number $N$ and construct the associative conformal algebra $U(N)$ generated by a single element $v$ such that $(v (n) v) = 0$ for $n \geq N$, and the commutation relations of $\text{Vir}$ hold. Obviously, $U(1) = 0$; the algebra $U(2)$ is known as the Weyl conformal algebra (also denoted $\text{Cend}_1$, [3]). The structure of $U(3) = U(4)$ was studied in [15] by means of the Gröbner–Shirshov bases method.

It was shown in [17] that the second Hochschild cohomology groups $H^2(U(2), M)$ are trivial for every conformal (bi)-module $M$.

If $M$ is a conformal $\text{Vir}$-module then it is not true in general that $M$ is a $U(N)$-module. A representation of $\text{Vir}$ on $M$ is determined by the image $\rho(v)$ of $v$ in the space of conformal endomorphisms $\text{Cend} M$ (see [12] Section 2.10). For $M$ to be a $U(N)$-module, we need $\rho(v)(n) \rho(v) = 0$ in $\text{Cend} M$ for $n \geq N$. The trivial $1$-dimensional $\text{Vir}$-module $M = \mathbb{C}$ is always a module over $U(N)$ since we have $\rho(v)(n) \rho(v) = 0$ for all $n \geq 0$. It is not hard to note (see [11]) that $H^n(U(2), \mathbb{C}) = 0$ for all $n \geq 1$.

Hence, even in the case of scalar coefficients there is no coincidence between $H^n(\text{Vir}, \mathbb{C})$ and $H^n(U(2), \mathbb{C})$. The purpose of this note is to study the Hochschild cohomologies of $U(3)$, the next envelope in the series. In [1], it was found that $\dim H^2(U(3), \mathbb{C}) = 1$, but for higher Hochschild cohomologies the direct computation becomes too complicated since $U(3)$ is of quadratic growth (Gelfand–Kirillov dimension $= 2$).

The purpose of this work is to develop a modification of the Morse matching method for calculation of Hochschild cohomologies of associative conformal algebras. As an application, we find higher Hochschild cohomologies of $U(3)$ with coefficients in $\mathbb{C}$.

One more reason to study $U(3)$ rather than the smallest nonzero envelope $U(2)$ is the following. If $M$ is a finite irreducible $\text{Vir}$-module (one of those described in [8]) then $\rho(v)(n) \rho(v) = 0$ in $\text{Cend} M$ for $n \geq 3$. Hence, $U(3)$ is a more adequate associative conformal envelope of $\text{Vir}$ than the Weyl conformal algebra $U(2)$.

The calculations of conformal cohomologies in [5] is performed in an indirect way; they rest upon deep and nontrivial auxiliary construction. There is a natural question: whether one can arrive at these results in a more universal and natural fashion?

The aim of this paper is to develop a modification of the algebraic discrete Morse theory machinery (we call it the Morse matching method) to calculate cohomology of (associative) conformal algebras. We believe that calculation of homology of conformal algebras should be obtained in a natural manner; they should be deduced from an intrinsic structure of a combinatorial presentation of algebras (i.e., presentation via generators and relations). In this case this machinery looks natural and powerful and we demonstrate it in some examples.
2. The basics of the algebraic discrete Morse theory

In this section, we recall the basic definitions of the algebraic discrete Morse theory and show how to apply this theory to the computation of Anick resolutions.

2.1. The Morse matching method. Algebraic discrete Morse theory is an algebraic version of discrete Morse theory developed independently by Sköldberg [21], and by Jöllenbeck and Welker [11]. It allows us to construct, starting from a chain complex, a new homotopically equivalent smaller complex using directed graphs. Here, for the convenience of the reader, we present a short version of this machinery. We follow closely [11, Chapter 2], with minor simplifications and variations in notation.

Let \((B_\bullet, d_\bullet)\) be a chain complex of free \(R\)-modules over a ring \(R\),

\[
B_0 \xleftarrow{d_1} B_1 \xleftarrow{d_2} B_2 \xleftarrow{d_3} \ldots
\]

Suppose \(X_n, n \geq 0\), are some bases of the free \(R\)-modules \(B_n\). Consider the coordinates of the differentials \(d_n : B_n \to B_{n-1}\) with respect to these bases:

\[
d_n(c) = \sum_{c' \in X_{n-1}} [c : c'] \cdot c',
\]

where \(c \in X_n\), and \([c : c']\) are coefficients from \(R\).

From these data, we construct a directed weighted graph \(\Gamma(B_\bullet) = (V, E)\). The set of vertices \(V\) of \(\Gamma(B_\bullet)\) is the union of all bases, \(V = \bigcup_{n \geq 0} X_n\), and the set \(E\) of weighted edges \(c \to c'\) consists of triples

\[
[(c, c', [c : c']) | c \in X_n, c' \in X_{n-1}, [c : c'] \neq 0].
\]

Here \(c\) and \(c'\) are starting and ending vertices of an edge, \([c : c'] \in R\) is its weight.

A subset \(M \subseteq E\) of the set of edges is called an acyclic matching, if it satisfies the following three conditions:

1. (Matching) Each vertex \(v \in V\) lies in at most one edge in \(M\).
2. (Invertibility) For all edges \((c, c', [c : c']) \in M\) the weight \([c : c']\) lies in the center of \(R\) and is a unit (invertible) in \(R\).
3. (Acyclicity) The graph \(\Gamma^M(B_\bullet) = (V, E^M)\) constructed from the graph \(\Gamma(B_\bullet)\) has no directed cycles, where

\[
E^M = (E \setminus M) \cup \{(c', c, -[c : c']^{-1}) | (c, c', [c : c']) \in M\}.
\]

For an acyclic matching \(M\) on the graph \(\Gamma(B_\bullet)\), we introduce the following notation:

- Define \(X^M_n = \{c \in X_n | c\) does not lie in any edge in \(M\}\).

The vertices in \(X^M_n\) are called critical cells of homological degree \(n\). Let \(B^M_n\) be the subspace of \(B_n\) spanned by \(X^M_n\).

- \(\mathcal{P}(c, c')\) is the set of paths from \(c\) to \(c'\) in \(\Gamma^M(B_\bullet)\).

- The weight \(\omega(p)\) of a path \(p = c_1 \to \ldots \to c_r \in \mathcal{P}(c_1, c_r)\) is defined as the product of all weights of the edges \(c_k \to c_{k+1}\), \(k = 1, \ldots, r-1\). Note that all these weights except one belong to the center of \(R\), so the order of multiplication is not essential.

**Theorem 2.1** ([11, Theorem 2.2]). The chain complex \((B_\bullet, d_\bullet)\) is homotopy equivalent to the complex \((B^M_\bullet, d^M_\bullet)\), where the differential \(d^M_n : B^M_n \to B^M_{n-1}\) is defined as

\[
d^M_n(c) = \sum_{c' \in X^M_{n-1}} \Gamma^M_{B\bullet}(c, c'), \quad \Gamma^M_{B\bullet}(c, c') = \sum_{p \in \mathcal{P}(c, c')} \omega(p)c',
\]

where \(c \in B^M_n\).
2.2. The Anick resolution for associative algebras. Let $\Lambda$ be an associative unital algebra over a field $k$, the cokernel of the embedding map $\eta : \mathbb{k} \to \Lambda$ is denoted $\Lambda/\mathbb{k}$. Assume further that $X$ is a set of generators of $\Lambda$ (as of an associative algebra with identity). Suppose that a Gröbner–Shirshov basis of $\Lambda$ with respect to some ordering of the free monoid $W(X)$ generated by $X$ is known.

As usual we denote by $\Lambda^e := \Lambda \otimes (\Lambda/\mathbb{k})^\text{op}$ the enveloping algebra for the algebra $\Lambda$. This algebra plays the role of $R$ in the previous subsection. Following [11] and [21] we will see how to construct a free $\Lambda^e$-resolution for $\Lambda$.

Let us start with the two-sided bar resolution $B_\ast(\Lambda, \Lambda)$ which is a $\Lambda^e$-free resolution of $\Lambda$, where

$$B_n(\Lambda, \Lambda) := \Lambda \otimes (\Lambda/\mathbb{k})^\otimes n \otimes \Lambda \cong \Lambda^e \otimes (\Lambda/\mathbb{k})^\otimes n.$$

The basis of $B_n(\Lambda, \Lambda)$ as of $\Lambda^e$-module is presented by $[a_1|a_2|\ldots|a_n]$, where $a_i$ are nontrivial (i.e., not equal to 1) reduced normal forms relative to the fixed Gröbner–Shirshov basis.

The differential $d_n : B_n(\Lambda, \Lambda) \to B_{n-1}(\Lambda, \Lambda)$ is defined as follows:

$$d_n([a_1|\ldots|a_n]) = (a_1 \otimes 1)[a_2|\ldots|a_n] + \sum_{i=1}^{n-1} (-1)^i[a_1|\ldots|N(a_ia_{i+1})|\ldots|a_n]$$

$$+ (-1)^n(1 \otimes a_n)[a_1|\ldots|a_{n-1}].$$

Here $N(a_ia_{i+1})$ is the reduced normal form of the product $a_ia_{i+1}$.

In [2], Anick showed how to construct a free resolution $A_\ast(\Lambda, \Lambda)$ which is essentially smaller than the bar-resolution but homotopically equivalent to the latter. The linear bases $\Lambda^{(n-1)}$ of $A_n(\Lambda, \Lambda)$ consist of so-called Anick chains related with the chosen Gröbner–Shirshov basis of $\Lambda$. The computation of a differential in the Anick resolution may be simplified by means of the Morse matching method.

For $w \in W(X)$, let $\Lambda_{w, p}$ be set of all the vertices $[w_1|\ldots|w_n]$ in $\Gamma(B_\ast(\Lambda, \Lambda))$ such that $w = w_1 \cdots w_n$ and $p$ is the largest integer $p \geq -1$ for which $w_1 \cdots w_{p+1} \in \Lambda^{(p)}$ is an Anick $p$-chain. Let $\Lambda_w := \bigcup_{p \geq -1} \Lambda_{w, p}$.

Define a partial matching $M_w$ on $\Gamma(B_\ast(\Lambda, \Lambda))|_{\Lambda_w}$ by letting $M_w$ consist of all edges

$$[w_1|\ldots|w_{p+2}|w'_{p+2}|\ldots|w_n] \to [w_1|\ldots|w_{p+2}|\ldots|w_m]$$

where $w'_{p+2} w''_{p+2} = w_{p+2}$, $[w_1|\ldots|w_m] \in \Lambda_{w, p}$, and $[w_1|\ldots|w_{p+1}|w'_{p+2}] \in \Lambda^{(p+1)}$ is an Anick $(p + 1)$-chain.

Theorem 2.2 (Jöllenbeck–Scöldberg–Welker). The set of edges $M = \bigcup_w M_w$ is a Morse matching on $\Gamma(B_\ast(\Lambda, \Lambda))$, the critical cells of homological degree $p \geq 0$ are exactly $(p - 1)$th Anick chains $\Lambda^{(p-1)}$.

For the matching $M = \bigcup_M M_w$ we have the following result.

Proposition 2.3 ([11] Chapter 5, [21] Lemma 9 and Theorem 5). The set of edges $M = \bigcup_M M_w$ is a Morse matching on $\Gamma(B_\ast(\Lambda, \Lambda))$, with Anick chains as critical cells. Moreover, the complex $A_\ast(\Lambda, \Lambda) = B_\ast(M, \Lambda)$ which is defined as follows:

$$A_{n+1}(\Lambda, \Lambda) = \Lambda^e \otimes \mathbb{k} \Lambda^{(n)}, \quad d_{n+1}(v) = \sum_{v' \in \Lambda^{(n-1)}} \Gamma_B(v, v')v',$$

is a free $\Lambda^e$-resolution of $\Lambda$.

The Morse matching method provides us a powerful tool to compute differentials in the Anick resolution $A_\ast(\Lambda, \Lambda)$ and in the complex $A_\ast = A_\ast(\Lambda, \Lambda) \otimes \mathbb{k}$. 

Example 2.4. Let $\Lambda = U(\mathfrak{g})$ be the universal enveloping associative algebra of a Lie algebra $\mathfrak{g}$ with an ordered basis $X$. Then $\{xy - yx - [x,y] \mid x, y \in X, x > y\}$ is a Gröbner–Shirshov basis of $\Lambda$ with respect to the deg-lex ordering, and the corresponding Anick resolution is exactly the Chevalley–Eilenberg complex of $\mathfrak{g}$.

Example 2.5. Let $\Lambda$ be the unital associative algebra over a field $\mathbb{k}$ generated by infinite family of elements $v(n), n \geq 0$, relative to the defining relations $v(n)v(m) - v(m)v(n) - (n-m)v(n+m-1)$ and $v(n+2)v(m) - 2v(n+1)v(m+1) + v(n)v(m+2), n, m \geq 0$. The Gröbner–Shirshov basis consists of relations $v(n)v(m) - v(0)v(n+m) - nv(n+m-1), n \geq 1, m \geq 0$.

Then $\Lambda^{(n)} = \{[v(p_1)] \cdots [v(p_n)]v(p_{n+1})\mid p_1, \ldots, p_n \geq 1\}$. In order to find $d_2(v)$ for $v = [v(n)]v(m))$ we have to apply the formulas from Proposition 2.3

with $\Gamma(v, v')$ calculated via the graph on Figure 1.

\[
d_2([v(n)]v(m)) = v(n) \otimes [v(m)] - v(0) \otimes [v(n+m)] - [v(0)] \otimes v(n+m) - n[v(n+m-1)] + [v(n)] \otimes v(m).
\]

The calculation of $d_3([v(n)]v(m))v(p)) \in A_3(\Lambda, \Lambda)$ by means of Proposition 2.3 leads to the following formula:

\[
d_3([v(n)]v(m))v(p)) = v(n) \otimes [v(m)]v(p)) - v(0) \otimes [v(n+m)]v(p)) - n[v(n+m-1)]v(p)) + m[v(n)]v(m + p - 1)) + [v(n)]v(0)) \otimes v(m + p) - [v(n)]v(m)) \otimes v(p) + n[v(n-1)]v(m + p)] + v(0) \otimes [v(n)]v(m + p)).
\]

(2.1)

Indeed, the corresponding segment of $\Gamma^{\Lambda}(B_4(\Lambda, \Lambda))$ is stated on Figure 2. The edges from $M$ in the initial graph are drawn dashed, the vertices corresponding to the Anick chains are boxed. They are critical cells corresponding to $M$, but the vertices of the form $[v(0)]v(k)$ are not critical cells since they lie on the matching edges of the “smaller level”, see Figure 3. This is why such vertices are “dead ends” of the calculation process, they do not contribute to the expression for $d_n$.

In order to find the value of $d_n$ on a particular Anick chain we do not need to expand vertices in $(\Lambda/\mathbb{k})^{\otimes n}$ that contain two or more components $v(0)$ since neither of them may produce an Anick chain from $(\Lambda/\mathbb{k})^{\otimes (n-1)}$. The same observation will be used in Section 4.

In the following, we will omit the tensor sign to make expressions shorter.

In this paper, we will adjust the Morse matching method for associative conformal algebras and, as an application, calculate all Hochschild cohomology groups for $U(3)$ with coefficients in the scalar module $M = \mathbb{k}$. Note that $H^2(U(3), \mathbb{k})$ was earlier found in [1] by “elementary”
Definition and examples of conformal algebras.

3.1. 

methods which do not work even for the third cohomology since the computations become too bulky.

3. Conformal algebras

Let $k$ be a field of characteristic zero, and let $\mathbb{Z}_+$ stand for the set of nonnegative integers.

3.1. Definition and examples of conformal algebras.
**Definition 3.1** ([12]). A conformal algebra is a linear space $C$ equipped with a linear operator $\partial : C \rightarrow C$ and with a polynomial-valued map ($\lambda$-product)

$$(\cdot_{(\lambda)} \cdot) : C \otimes C \rightarrow \mathbb{K}[\partial, \lambda] \otimes_{\mathbb{K}[\partial]} C \cong \mathbb{K}[\lambda] \otimes C,$$

where $\lambda$ is a formal variable, satisfying the following axioms:

$$(\partial a_{(\lambda)} b) = -\lambda (a_{(\lambda)} b), \quad (3.1)$$

$$(a_{(\lambda)} \partial b) = (\partial + \lambda)(a_{(\lambda)} b), \quad (3.2)$$

for all $a, b \in C$.

The coefficients of the polynomial $(a_{(\lambda)} b)$ at $\lambda^n/n!$ are denoted $a_{(n)} b, n \in \mathbb{Z}_+$. The definition of a conformal algebra may be stated equivalently in terms of multiple binary operations $(\cdot_{(\lambda)} \cdot)$, for all non-negative integers $n$.

Note that the torsion of $C$ as of $\mathbb{K}[\partial]$-module has the annihilating property: $\text{tor} C_{(\lambda)} C = C_{(\lambda)} \text{tor} C = 0$.

For every conformal algebra $C$ one may construct an ordinary algebra $A = \mathcal{A}(C)$ in the following way. As a linear space,

$$A = \mathbb{K}[t, t^{-1}] \otimes_{\mathbb{K}[\partial]} C,$$

where $\partial$ acts on $\mathbb{K}[t, t^{-1}]$ as $-d/dt$. Denote $t^n \otimes_{\mathbb{K}[\partial]} a, a \in C, n \in \mathbb{Z}$, by $a(n)$. The operation on $A$ is given by

$$a(n) \cdot b(m) = \sum_{s \geq 0} \left( \begin{array}{c} n \\ s \end{array} \right) a_{(s)} b(n + m - s).$$

The map

$$\partial : a(n) \mapsto na(n - 1) \quad (3.3)$$

is a well-defined derivation of $A$.

It is easy to see that the linear span of all $a(n)$ with $n \in \mathbb{Z}_+$ is a $\partial$-invariant subalgebra of $A$ denoted $\mathcal{A}_c(C)$. This algebra plays an important role in the cohomology theory of conformal algebras.

The algebra $A = \mathcal{A}(C)$ constructed has the following properties [19]. First, $C$ embeds into a formal distribution conformal algebra over $A$, i.e., the map $\iota : C \rightarrow A[[z, z^{-1}]], a \mapsto \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$, is injective, $\iota(\partial a) = dt(a)/dz$, and

$$\iota(a_{(\lambda)} b)(z) = \text{Res}_{w=0} (\exp^{\lambda(w-z)} \iota(a)(w) \cdot \iota(b)(z)).$$

Second, the algebra $A$ is universal among all ordinary algebras $B$ such that $C$ maps to a formal distribution conformal algebra over $B$.

The correspondence $C \mapsto \mathcal{A}(C)$ is functorial and provides a foundation for a definition of what is a Lie or associative conformal algebra. A conformal algebra $C$ is said to be associative (Lie, etc.) if so is $\mathcal{A}(C)$. In terms of $\lambda$-products, the associativity may be expressed as

$$(a_{(\lambda)} (b_{(\mu)} c)) = ((a_{(\lambda)} b)_{(\mu + \lambda)} c) \in C[\lambda, \mu], \quad a, b, c \in C. \quad (3.4)$$

The anti-commutativity and Jacobi identity are equivalent to

$$(a_{(\lambda)} b) = -(b_{(-\lambda-\partial)} a), \quad (3.5)$$

$$(a_{(\lambda)} (b_{(\mu)} c)) - (b_{(\mu)} (a_{(\lambda)} c)) = ((a_{(\lambda)} b)_{(\mu + \lambda)} c), \quad (3.6)$$

respectively.

An associative conformal algebra $C$ turns into a Lie conformal algebra $C^{(-)}$ under a new $\lambda$-product corresponding to the ordinary commutator in $\mathcal{A}(C)$:

$$[a_{(\lambda)} b] = a_{(\lambda)} b - (b_{(-\lambda-\partial)} a), \quad a, b \in C.$$
Example 3.3. Let $A$ be an ordinary algebra over $\mathbb{k}$. Then the free $\mathbb{k}[\partial]$-module $C = \mathbb{k}[\partial] \otimes A$ equipped with

$$(\partial^n \otimes a) (\partial^m \otimes b) = -\lambda^n (\lambda + \partial)^m \otimes ab, \quad a, b \in A,$$

is a conformal algebra denoted $\text{Cur} A$, the current conformal algebra over $A$.

Clearly, it is enough to define the $\lambda$-product only on the generators of a free $\mathbb{k}[\partial]$-module due to (3.1), (3.2).

If $A$ is associative or Lie then so is $\text{Cur} A$ as a conformal algebra: $\mathcal{A}_+(\text{Cur} A) = A[t, t^{-1}], \mathcal{A}_+(\text{Cur} A) = A[t]$.

Example 3.4. Let $A$ be an associative algebra with a locally nilpotent derivation $D : A \to A$. Then the free $\mathbb{k}[\partial]$-module $C = \mathbb{k}[\partial] \otimes A$ equipped with

$$(1 \otimes a) (\partial) (1 \otimes b) = \sum_{s \geq 0} \frac{\lambda^s}{s!} (1 \otimes aD^s(b)), \quad a, b \in A,$$

is an associative conformal algebra denoted $\text{Diff}(A, D)$.

If $A = M_n(\mathbb{k}[x])$ with $D = d/dx$ then $\text{Diff}(A, D)$ is denoted $\text{Cend}_n$, the algebra of conformal endomorphisms of the free $\mathbb{k}[\partial]$-module of rank $n$.

If $A = \mathbb{k}[x]$ is the augmentation ideal of the polynomial algebra and $D = d/dx$ as above then $\text{Diff}(A, D)$ is known as the Weyl conformal algebra $\text{Cend}_{1,1}$.

Example 3.5. The rank 1 free $\mathbb{k}[\partial]$-module $V = \mathbb{k}[\partial]v$ is a Lie conformal algebra with respect to the $\lambda$-product

$$[v, (\partial) v] = (\partial + 2\lambda)v.$$

This structure is known as the Virasoro conformal algebra $\text{Vir}$, the exceptional simple finite Lie conformal algebra [9].

The coefficient algebra for $\text{Vir}$ is the Witt algebra of vector fields on a circle: $\mathcal{A}(\text{Vir}) = \text{Der} \mathbb{k}[t, t^{-1}], \mathcal{A}_+(\text{Vir}) = \text{Der} \mathbb{k}[t]$.

Let $\mathfrak{g}$ be a Lie algebra. If $U(\mathfrak{g})_0$ stands for the augmentation ideal of the universal enveloping associative algebra $U(\mathfrak{g})$ then $\text{Cur} U(\mathfrak{g})_0$ is the universal enveloping associative conformal algebra of $L = \text{Cur} \mathfrak{g}$ in the class of all those associative conformal envelopes $C$ of $L$ for which $N_C(\mathfrak{g}, \mathfrak{g}) \leq 1$. Let us denote this conformal algebra by $U(\text{Cur} \mathfrak{g}; N = 1)$. The structures of $U(\text{Cur} \mathfrak{g}; N = 2)$ and $U(\text{Cur} \mathfrak{g}; N = 3)$ are more complicated, they were studied in [16] by means of the Gröbner–Shirshov bases technique which was developed for associative conformal algebras in [6].

Given an integer $N \geq 2$, one may construct an associative conformal envelope $U(N)$ for the Virasoro Lie conformal algebra $\text{Vir}$ with a generator $v$ which is universal in the class of all such envelopes $C$ that $N_C(v, v) \leq N$. For example, $U(2) = U(\text{Vir}; N = 2)$ is the Weyl conformal
conformal modules over Lie conformal algebras are defined. In brief, a bimodule \( M \) over an associative conformal algebra \( C \) is a linear space equipped with a linear operator \( \partial \) (denoted by the same symbol as in \( C \)) and with left and right \( \lambda \)-actions

\[
C \otimes M \rightarrow M[\lambda], \quad M \otimes C \rightarrow M[\lambda],
\]

satisfying the analogues of (3.1), (3.2), and (3.4). Obviously, these conditions are equivalent to

\[\text{Example 2.5.}\]

For every finite conformal Vir-module there exists a composition series of submodules in \( C \) and with left and right \( \lambda \)-actions

\[\text{Example 3.6.}\]

The action of Vir on \( M \) may be considered as a bimodule over the ordinary associative algebra \( A_+(C) \). Namely,

\[\text{a(n)}u = a(n)u, \quad uM(n) = [u(n), a], \quad u \in M, \quad a \in C, \quad n \geq 0,
\]

where

\[\{u(n), a\} = (-1)^n \sum_{s \geq 0} \frac{(-\partial)^s}{s!}(u(n+s), a).
\]

\[\text{Example 3.6.}\]

Let \( L = \text{Vir}, \ M = \mathbb{C}[\partial]u \) is a 1-generated free \( \mathbb{C}[\partial] \)-module equipped with a left action of Vir via

\[(\partial \cdot u) = (\alpha + \partial + \Delta \lambda)u,
\]

where \( \alpha, \Delta \in \mathbb{C} \). These are all irreducible (for \( \Delta \neq 0 \)) conformal modules over Vir (see \( \text{[8]} \)).

For every finite conformal Vir-module there exists a composition series of submodules in which all quotients are either \( M(\alpha, \Delta) \) or just 1-dimensional \( \mathbb{C} \) considered as a trivial module \( \text{[8]} \).

The action of Vir on \( M(\alpha, \Delta) \) corresponds to the following homomorphism of conformal algebras (representation)

\[\rho : \text{Vir} \rightarrow (\text{Cend}_1)^{(-)} = \text{gc}_1,
\]

\[v \mapsto f = x - \Delta \partial + \alpha.
\]

It is easy to see that if \( \Delta \neq 0 \) then \( \deg(f) = 2 \), so for \( C = \text{Cend}_1 \) we have \( N_C(\rho(v), \rho(v)) = 3 \). Therefore, the structure of a conformal Vir-module on \( M(\alpha, \Delta) \) may not be extended to \( U(2) \) but may be extended to \( U(3) \). This is a motivation to study \( U(3) \) rather than a more simple Weyl conformal algebra. One more reason is related with the cohomology theory of these algebras.

3.2. Conformal cohomologies. The study of cohomologies for conformal algebras was initiated in \( \text{[4]} \). Let us state the main definitions and results concerning the Hochschild cohomologies of associative conformal algebras. We will focus on the reduced complex \( \text{[5]} \) since its cohomologies have the expected relations to derivations, extensions, and deformations of associative conformal algebras. Moreover, the reduced complex coincides with the construction arising when one considers conformal algebras and their modules in the framework of pseudo-tensor categories \( \text{[5]} \), see also \( \text{[13]} \).

Let \( C \) be an associative conformal algebra, and let \( M \) be a conformal bimodule over \( C \). The Hochschild complex \( C^*(C, M) \) consists of the cochain spaces \( C^n(C, M), n = 1, 2, \ldots \), each of them is the space of all maps

\[\varphi_n : C^n \rightarrow M[\lambda_1, \ldots, \lambda_{n-1}],
\]
where \( \bar{\lambda} = (\lambda_1, \ldots, \lambda_{n-1}) \), satisfying the analogues of (3.1) and (3.2):
\[
\varphi_{\bar{\lambda}}(a_1, \ldots, \bar{\partial} a_i, \ldots, a_n) = -\lambda_i \varphi_{\bar{\lambda}}(a_1, \ldots, a_n), \quad i = 1, \ldots, n - 1,
\]
\[
\varphi_{\bar{\lambda}}(a_1, \ldots, \bar{\partial} a_n) = (\bar{\partial} + \lambda_1 + \cdots + \lambda_{n-1}) \varphi_{\bar{\lambda}}(a_1, \ldots, a_n).
\]
The Hochschild differential
\[
d_n : C^n(C, M) \to C^{n+1}(C, M)
\]
is given by
\[
(d_n \varphi_{\bar{\lambda}})(a_1, \ldots, a_{n+1}) = a_1 \varphi_{\bar{\lambda}_0}(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi_{\bar{\lambda}_i}(a_1, \ldots, \bar{\partial} a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} \varphi_{\bar{\lambda}_{n+1}}(a_1, \ldots, a_n) \varphi_{\bar{\lambda}_{i+\cdots+i_0}} a_{n+1},
\]
where \( \bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \), \( \bar{\lambda}_0 = (\lambda_2, \ldots, \lambda_n) \), \( \bar{\lambda}_i = (\lambda_1, \ldots, \lambda_i + \lambda_{i+1}, \ldots, \lambda_n) \), \( \bar{\lambda}_{n+1} = (\lambda_1, \ldots, \lambda_{n-1}) \).

It is common to complete the complex \( C^*(C, M) \) described above with \( C^0(C, M) = M / \partial M \) and \( d_0 : C^0(C, M) \to C^1(C, M) \), where
\[
d_0(u + \partial M) : a \mapsto \{ a(\partial) u \} - u(\partial) a.
\]
The elements of the second cohomology group \( H^2(C, M) = \text{Ker} d_2 / \text{Im} d_1 \) are in one-to-one correspondence with the classes of equivalent null extensions \( E \),
\[
0 \to M \to E \to C \to 0, \quad (M \langle \partial \rangle M) = 0.
\]
The similar statement holds for the cohomologies of Lie conformal algebras \([4]\) and their central extensions. Let us state two examples to demonstrate this relation.

**Example 3.7.** The Virasoro Lie conformal algebra \( \text{Vir} \) has a unique (up to a scalar multiple) central extension
\[
0 \to \mathbb{k} \to \text{Vir}_c \to \text{Vir} \to 0, \quad c \in \mathbb{k},
\]
where
\[
[v(\partial), v] = (2\lambda + \partial)v + \lambda^3 c \in \text{Vir}_c = \text{Vir} \oplus \mathbb{k}.
\]
So \( H^2(\text{Vir}, \mathbb{k}) \) is 1-dimensional.

**Example 3.8.** Let \( \mathfrak{g} \) be a finite-dimensional semisimple Lie algebra with the Killing form \( \langle \cdot, \cdot \rangle \). Then \( \text{Cur} \mathfrak{g} \) has a nontrivial central extension \( \text{Cur} \mathfrak{g} \oplus \mathbb{k} \), where
\[
[a(\partial), b] = ab + \lambda \langle a, b \rangle, \quad a, b \in \mathfrak{g}.
\]
The Hochschild complex \( C^*(C, M) \) for an associative conformal algebra \( C \) and a conformal \( C \)-bimodule \( M \) may also be constructed as follows. Let us consider \( M \) as a bimodule over the associative algebra \( \Lambda = \mathcal{A}_+(C) \oplus \mathbb{K}1 \) (with external identity). Both \( M \) and \( \mathcal{A}_+(C) \) carry linear operators (both denoted \( \partial \)) such that their sum is a derivation of the split null extension \( \mathcal{A}_+(C) \oplus M \). Then define
\[
\partial_n^* : C^n(\mathcal{A}_+(C), M) \to C^n(\mathcal{A}_+(C), M)
\]
as follows:
\[
(\partial_n^* f)(\alpha_1, \ldots, \alpha_n) = \partial f(\alpha_1, \ldots, \alpha_n) + \sum_{i=1}^{n} f(\alpha_1, \ldots, \partial \alpha_i, \ldots, \alpha_n),
\]
where \( \partial(a(n)) = na(n-1) \) for \( a \in \mathcal{A}_+, n \geq 0 \). The maps \( \partial_n^* \) form a morphism of complexes \( \partial_n^* : C^*(\mathcal{A}_+(C), M) \to C^*(\mathcal{A}_+(C), M) \), and the following statement holds.

**Proposition 3.9** ([4] Theorem 6.1). \( C^*(C, M) \cong C^*(\mathcal{A}_+(C), M) / \partial_n^* C^*(\mathcal{A}_+(C), M) \).

The latter complex is called a reduced Hochschild complex of an associative conformal algebra \( C \) with coefficients in \( M \).
3.3. Reduced complex and Hochschild cohomologies of current conformal algebras. Let \( \Lambda = A_\ast(C) \oplus \mathbb{k}1 \), with \( \varepsilon(a(n)) = 0 \) for all \( a \in C, n \geq 0 \). Assume \( \Lambda \) acts trivially on the 1-dimensional space \( \mathbb{k} \), i.e., \( a \beta = \varepsilon(a) \beta \) for \( a, \beta \in \Lambda \). In order to calculate Hochschild cohomologies of \( \Lambda \) it is enough to apply the \( \text{Hom} \) functor to the complex \( B_\ast = B_\ast(\Lambda, \Lambda) \otimes \Lambda \mathbb{k} \), where \( B_\ast(\Lambda, \Lambda) \) is the two-sided bar resolution for \( \Lambda \), \( \Lambda^e \) stands for the enveloping algebra \( \Lambda \otimes \Lambda^{op} \). Apparently,

\[
B_n = (\Lambda/\mathbb{k})^{\otimes n}
\]

Denote by \( d_n : B_n \to B_{n-1} \) the differential of \( B_\ast \) induced by the bar differential. The dual map \( d_n^* \) is the Hochschild differential \( C^{n-1} \to C^n, C^i = C^i(A_\ast(C), \mathbb{k}) \).

Recall that \( \Lambda \) is equipped with a derivation \( \partial \) such that \( \partial(a(n)) = n a(n - 1) \) for \( a \in C, n \geq 0 \), and the 1-dimensional bimodule carries trivial derivation. Let us extend \( \partial \) to the \( \Lambda^e \)-linear map \( \partial_n : B_n \to B_n \) by the rule

\[
\partial_n : [\lambda_1] \cdots [\lambda_n] \mapsto \sum_{i=1}^n [\lambda_1] \cdots [\partial(\lambda_i)] \cdots [\lambda_n].
\]

Then the dual map \( \partial_n^* \) is the morphism mentioned in Proposition 3.9. Since \( C^n/\partial_n^* C^n \simeq (\ker \partial_n)^* \) by the Fredholm principle, we reduce the problem of computing conformal cohomologies \( H^p(C, \mathbb{k}) \), \( n \geq 1 \), to the application of the \( \text{Hom} \) functor to the complex

\[
\cdots \leftarrow \ker \partial_n \leftarrow \ker \partial_{n+1} \leftarrow \cdots.
\]

The arrows here are restrictions of \( \partial \) onto \( \ker \partial_i \subset B_i \).

Example 3.10. Let \( A \) be an associative algebra, and let \( C = \text{Cur } A \) be the current conformal algebra over \( \mathbb{k} \). Then \( \mathbb{k}/\mathbb{k} = A_\ast(C) = A[t] \) is spanned by \( a(m) \), \( a \in A, m \geq 0 \), so that

\[
a(n)b(m) = (ab)(n + m).
\]

Note that for the current conformal algebra the kernel of \( \partial_n \) is easy to find. There is an isomorphism of linear spaces

\[
B_n = A[t]^{\otimes n} \cong \mathbb{k}[x_1, \ldots, x_n] \otimes A^{\otimes n},
\]

where \( [a_1(m_1)] \cdots [a_n(m_n)] \) corresponds to \( x_1^{m_1} \cdots x_n^{m_n} \otimes (a_1| \cdots |a_n) \). The map \( \partial_n \) acts on the space \( \mathbb{k}[x_1, \ldots, x_n] \otimes A^{\otimes n} \) as the differential operator

\[
D = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}
\]

Thus we may identify \( \ker \partial_n \) with the subspace \( K_n \otimes A^{\otimes n} \) where \( K_n \) is the kernel of \( D \) in \( \mathbb{k}[x_1, \ldots, x_n] \). It is not hard to note that \( K_n \) is the subalgebra generated by \( y_i = x_{i+1} - x_i, i = 1, \ldots, n - 1 \).

With these notations, a value of \( d_n \) on \( \ker \partial_n \) may be expressed as follows:

\[
d_n(f \otimes v) = \sum_{i=1}^{n-1} d_n^{(i)}(f) \otimes v^{(j)}, \quad f \in K_n, \ v \in A^{\otimes n},
\]

where

\[
d_n^{(i)}(f(y_1, \ldots, y_{n-1})) = (-1)^i f(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_{n-2}),
\]

and \( v^{(j)} = a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \) for \( v = a_1 \otimes \cdots \otimes a_n \).

In general, we may deduce the following statement on the Hochschild cohomologies of current associative conformal algebras with trivial coefficients.
Theorem 3.11. Let $A$ be an associative algebra acting trivially on $\mathbb{k}$, and let $\Lambda = A \oplus \mathbb{k}$ be the augmented algebra with $e(A) = 0$. Then $H^1(\text{Cur} A, \mathbb{k}) = H^1(A, \mathbb{k}) = (A/A^2)^*$. For every $n \geq 2$, if $\dim H^1(\Lambda, \mathbb{k}) = \cdots = \dim H^{n-1}(\Lambda, \mathbb{k}) = 0$ then

$$H^n(\text{Cur} A, \mathbb{k}) = H^n(\Lambda, \mathbb{k}).$$

Proof. Let $\delta_\Lambda : A^{\otimes n} \to A^{\otimes (n-1)}$ stand for the derived differential of the bar resolution of $\Lambda$, i.e., $H^n(\Lambda, \mathbb{k}) = \text{Ker } \delta_{n+1}/\text{Im } \delta_n$.

Let us start with $n = 1$. By definition, Ker $\delta_1$ consists of $[x(0)] = 1 \otimes x \in K_n \otimes A$, $x \in A$, and it is easy to see that Ker $\delta_2$ is spanned by

$$e_m(a, b) = y_1^m \otimes (a \otimes b) = \sum_{s=0}^m (-1)^s \binom{m}{s} [a(m-s)]b(s)]$$

for $a, b \in A$, $m \geq 0$.

Obviously,

$$d_2 e_m(a, b) = \begin{cases} [ab(0)], & m = 0, \\ 0, & m > 0. \end{cases}$$

Therefore, Ker $d_1/\text{Im } d_2$ in the complex (3.8) is isomorphic to $A/A^2$. Hence, $H^1(C, \mathbb{k}) = H^1(\Lambda, \mathbb{k}) = (A/A^2)^*$.

Proceed to $n = 2$. Note that

$$d_2 \left( \sum_{n \geq 0} y_1^n \otimes b_n \right) = -b^{(1)}_0, \quad d_3 \left( \sum_{n \geq 0} y_1^n y_2^m \otimes c_{nm} \right) = \sum_{n \geq 0} y_1^n (c^{(2)}_n - c^{(1)}_{0,n}).$$

Suppose $u \in \text{Ker } d_2$, $u = \sum y_1^n \otimes b_n$. Then $A = A^2$ for every $n > 0$ we may find $c_n \in A^{\otimes 3}$ such that $c^{(1)}_n = b_n$. Then for $v = \sum y_2^n \otimes c_n$ we have $u - d_3(v) \in 1 \otimes A^{\otimes 2}$. On the other hand, Ker $d_2 \cap (1 \otimes A^{\otimes 2}) = 1 \otimes \text{Ker } \delta_2$, as well as Im $d_3 \cap (1 \otimes A^{\otimes 2}) = 1 \otimes \text{Im } \delta_3$. Therefore,

$$\text{Ker } d_2/\text{Im } d_3 = \text{Ker } \delta_2/\text{Im } \delta_3,$$

as desired.

Consider the general case $n \geq 2$. Suppose $u \in \text{Ker } d_n$, $u \notin 1 \otimes A^{\otimes r}$. Let us present $u$ as

$$u = y_1^{m_1} \cdots y_{n-1}^{m_{n-1}} \otimes b_0 + \ldots,$$

where the first (“leading”) term is chosen in the following way. Find maximal $i$ such that a positive power of $y_i$ appears in $u$. Among all such monomials (with $y_i$), choose the maximal one in the lexicographic sense.

If $i = n - 1$ then find $c \in A^{\otimes (n+1)}$ such that $c^{(n)} = b_0$, and note that for $v = y_1^{m_1} \cdots y_{n-1}^{m_{n-1}} \otimes c$ we have

$$d_{n+1}(v) = (-1)^n y_1^{m_1} \cdots y_{n-1}^{m_{n-1}} \otimes c^{(n)} + \ldots$$

where all remaining terms do not contain $y_{n-1}$. Hence, $u - (-1)^n d_{n+1}(v)$ has a smaller leading term than $u$.

If $i < n - 1$ then consider the coefficient at $y_1^{m_1} \cdots y_i^{m_i}$ in $d_n(u)$. By the choice of the leading term in $u$, we have

$$d_n(u) = y_1^{m_1} \cdots y_i^{m_i} \otimes ((-1)^{i+1} b_0^{(i+1)} + (-1)^{i+2} b_0^{(i+2)} + \ldots + (-1)^{n-1} b_0^{(n-1)} + \ldots),$$

where all remaining terms either do not contain $y_i$ or lexicographically smaller than $y_1^{m_1} \cdots y_i^{m_i}$. Therefore,

$$(\text{id}^{\otimes i} \otimes \delta_{n-1})(b_0) = 0$$

and since $H^{n-1}(\Lambda, \mathbb{k}) = 0$ we may find $c \in A^{\otimes (n+1)}$ such that

$$(\text{id}^{\otimes i} \otimes \delta_{n+1})(c) = b_0.$$
As in the previous case, we may reduce the leading term of $u$ by considering $u - d_{n+1}(v)$ for

$$v = (-1)^i y_1^{m_1} \cdots y_i^{m_i} \otimes c.$$  

The reduction of the leading term described above shows that every class in $\text{Ker } d_n / \text{Im } d_{n+1}$ contains an element from $1 \otimes A^{\otimes n}$. The rest of the proof is completely similar to the $n = 2$ case.  

**Corollary 3.12.** If $A = M_n(\kappa)$ then $H^n(\text{Cur } A, \kappa) = 0$ for all $n \geq 1$.

The result obtained for Hochschild cohomologies of current associative conformal algebras looks different from what was proved in [4] for current Lie conformal algebras: if $\mathfrak{g}$ is a semisimple finite-dimensional Lie conformal algebra then $H^n(\text{Cur } \mathfrak{g}, \kappa) = H^n(\mathfrak{g}, \kappa) + H^{n+1}(\mathfrak{g}, \kappa)$ for all $n \geq 0$.

In particular, if $A = \mathfrak{g}U(\mathfrak{g})$ then Cur $A$ is the universal enveloping associative conformal algebra for Cur $\mathfrak{g}$ relative to the locality bound $N = 1$ on the elements of $\mathfrak{g}$, $\Lambda = U(\mathfrak{g})$. If $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ then $A^2 = A$, so by Theorem 3.11 $H^2(\text{Cur } A, \kappa) = H^2(\Lambda, \kappa) = H^2(\mathfrak{g}, \kappa) = 0$ in contrast to the 1-dimensional Lie conformal cohomology group $H^2(\text{Cur } \mathfrak{g}, \kappa)$.

A similar picture appears when we consider cohomology groups with trivial coefficients for the Virasoro Lie conformal algebra Vir and its universal enveloping associative conformal algebras $U(N), N = 2, 3, \ldots$. It was proved in [17] that $H^2(U(2), M) = 0$ for every conformal bimodule over the Weyl conformal algebra $U(2)$ in contrast to 1-dimensional $H^2(\text{Vir}, \kappa)$. This is the reason to study $H^2(U(3), \kappa)$ which is the aim of Section 4.

### 3.4. On the Anick resolution for differential algebras

In the previous section we exploited the bar resolution $B_n(\Lambda, \Lambda)$ and the complex $B_* \otimes \Lambda$, $\kappa$ for the augmented associative algebra $\Lambda = A_n(C) \oplus \kappa$ to compute Hochschild cohomology groups of a conformal algebra $C$ in the case when $C$ is the current associative conformal algebra. In more complicated cases, the computation with bar resolution becomes much harder, so it is reasonable to replace the bar resolution with a more compact Anick (two-sided) resolution $A_n(\Lambda, \Lambda)$. The first problem in this route is to translate the mapping $d_n$ from (3.7) to the complex $A_n = A_n(\Lambda, \Lambda) \otimes A^x, \kappa$.

The mapping $d_n : B_n \to B_n$, defined by (3.7) is a morphism of complexes, $d_n \circ d_{n-1} = d_{n-1} \circ d_n$. (This is not the case when we consider a nontrivial conformal module.) Suppose $A_n = A_n(\Lambda, \Lambda) \otimes A^x, \kappa$, $A_n = \Lambda^{(n-1)}$.

The space $B_n$ is spanned by elements of the form $[u_1] \ldots [u_n]$, where $u_i$ are nontrivial reduced words in the generators of $\Lambda$. Then $A_n$ is a subspace of $B_n$ spanned by the Anick $(n-1)$-chains. Let us define the linear projection $\pi_{A_n} : B_n \to A_n$ assuming

$$\pi_{A_n}(a) = \begin{cases} 0, & \text{if } a \text{ is not an Anick chain,} \\ a, & \text{otherwise.} \end{cases}$$

**Proposition 3.13.** Let $d_n : B_n \to B_n$ be defined by (3.7) then in terms of the complex $A_n$ we have

$$\tilde{d}(a_n) = \sum_{b_n \in B_n} \Gamma_{A_n}(a_n, b_n) \cdot \pi_{A_n}(d_n(b_n)).$$

**Proof.** Since the Anick complex can be obtained from the bar complex by Morse matching machinery then by [11, Appendix B, (B2), (B3)] for any $n > 0$ we have the following commutative diagram

$$\begin{array}{ccc}
A_n & \xrightarrow{d_n} & A_n \\
\downarrow{g_n} & & \downarrow{i_n} \\
B_n & \xrightarrow{\partial_n} & B_n
\end{array}$$

13
where the maps $g_\bullet, f_\bullet$ are defined as follows

\[ f_n(b_n) = \sum_{a_n \in A_\bullet} \Gamma_{B_\bullet}(b_n, a_n)a_n, \quad g_n(a_n) = \sum_{b_n \in B_\bullet} \Gamma_{A_\bullet}(a_n, b_n)b_n. \]

By Lemma [11, Appendix B, Lemma B.3] these maps define a chain homotopy between the resolutions $A_\bullet$ and $B_\bullet$. Therefore we can define $\tilde{\partial}$ as follows: \( \tilde{\partial}(a_n) = (f_n \circ \partial_n \circ g_n)(a_n) \) for every $n > 0$. We thus have

\[ \tilde{\partial}(a_n) = \sum_{b_n \in B_\bullet} \sum_{a_n \in A_\bullet} \Gamma_{A_\bullet}(a_n, b_n)(\partial_n(b_n), a_n') \cdot a_n'. \tag{3.9} \]

On the other hand,

\[ \Gamma_{B_\bullet}(b_n, a_n) = \begin{cases} 1 & \text{if } b_n = a_n \in A_\bullet, \\ 0 & \text{otherwise}, \end{cases} \]

and the statement follows.

\[ \square \]

**Example 3.14.** Let $\mathfrak{g}$ be a Lie algebra with a linearly ordered basis $X$. Then consider

\[ A = \mathbb{k}(a(n), a \in X, n \geq 0 \mid a(n)b(m) - b(m)a(n) - [a, b](n + m)). \]

Then $\partial : a(n) \mapsto na(n-1)$ is a derivation of $A$. The corresponding augmented algebra $\Lambda = A \oplus \mathbb{k}1$ with $\varepsilon(a(n)) = 0$ is just the universal enveloping associative algebra of $\mathfrak{g}[t]$.

Let us order the generators of $A$ as follows:

\[ a(n) > b(m) \iff n > m \text{ or } n = m \text{ and } a > b. \]

Then $[a(1)b(0)] \in \Lambda^{(1)}$ for $a < b$. It is not hard to construct the graph $\Gamma(A_\bullet)$ (similar to Example 2.5) and find $g_2([a(1)b(0)]) = [a(1)b(0)] - [b(0)a(1)]$. Since $[a(0)b(0)]$ is not a chain for $a < b$, we obtain

\[ \tilde{\partial}_2([a(1)b(0)]) = -[b(0)a(0)]. \]

**Example 3.15.** Let $\Lambda$ be the algebra from Example 2.5 acting trivially on the scalar module $M = \mathbb{k}$. Note that $\partial : v(n) \mapsto nv(n-1)$ defines a derivation on $\Lambda$. We will denote an element of the form $[v(n_1), \ldots, v(n_k)]$ by $[n_1, \ldots, n_k]$ for brevity.

Consider the complex $A_\bullet = A_\bullet(\Lambda, \Lambda) \oplus \Lambda \cdot \mathbb{k}$. Then $d_3([2|1|0]) = 2[1|1] - [2|0]$ and $d_3([1|1|0]) = 0$ by (2.1). Following Figure 2 we may see that $g_3([2|1|0]) = [2|1|0] - [0|3|0] + 3[0|0|3] - [2|0|1] + [0|2|1] - [0|0|3]$. Hence,

\[ \tilde{\partial}_3([2|1|0]) = 2[1|1|0], \]

and by (2.1) we have $d_3([1|1|0]) = 0$. In a similar way, one may calculate $\tilde{\partial}_2(2[1|1] - [2|0]) = 2[1|0] - 2[1|0] = 0$ in compliance with $d_3 \tilde{\partial}_3 = \tilde{\partial}_2 d_3$.

Therefore, in order to calculate conformal cohomologies $H^p(C, \mathbb{k})$ following the scheme of Section 3.3 we have to study the complex

\[ \cdots \leftarrow \text{Ker } \tilde{\partial}_{n-1} \leftarrow \text{Ker } \tilde{\partial}_n \leftarrow \cdots, \]

where the arrows are restrictions of the Anick differential $d_n : A_n \to A_{n-1}$ to the kernel of $\tilde{\partial}_n$. 

14
4. Hochschild cohomologies of the $N = 3$ universal associative envelope of the Virasoro conformal algebra

4.1. Gröbner–Shirshov basis of $\mathcal{A}_+(U(3))$. By definition, the universal enveloping associative conformal algebra $U(3)$ of the Virasoro Lie conformal algebraVir relative to the locality bound $N = 3$ is generated by a single element $v$ such that $v(n)v = 0$ for $n \geq 3$. The remaining defining relation of $U(3)$ is

$$2v(1)v - \partial(v(2)v) = 2v.$$ (4.1)

The algebra $\mathcal{A}_+(U(3))$ is generated by the elements $v(n)$, $n \geq 0$, relative to the following relations (see [20]):

$$v(n)v(m) - 3v(n - 1)v(m + 1) + 3v(n - 2)v(m + 2) - v(n - 3)v(m + 3) = 0, \quad n \geq 3, m \geq 0,$$ (4.2)

$$v(n)v(m) - v(m)v(n) = (n - m)v(n + m - 1), \quad n > m \geq 0.$$ (4.3)

Let us fix the deg-lex order on the set of words of the form $v(n_1)\ldots v(n_k)$ assuming $v(n) > v(m)$ iff $n > m$.

**Theorem 4.1.** The Gröbner–Shirshov basis of $\mathcal{A}_+(U(3))$ consists of the relations

$$v(1)v(0) = v(0)v(1) + v(0),$$ (4.4)

$$v(n)v(m) = \frac{nm}{n + m - 1}v(1)v(n + m - 1) - \frac{(n - 1)(m - 1)}{n + m - 1}v(0)v(n + m) + \frac{n(n - 1)}{n + m - 1}v(n + m - 1), \quad n \geq 2.$$ (4.5)

**Proof.** First, let us prove that (4.5) hold on $U(3)$ for all $n \geq 2$ and $m \geq 0$. For $m = 0, 1$ and for every $n \geq 2$ this is just (4.3). Proceed to the case $n = 2$. Calculate the $(m + 1)$th Fourier coefficient of (4.1):

$$2(v(1)v(m + 1) - v(0)v(m + 2)) + (m + 1)(v(2)v(m) - 2v(1)v(m + 1) + v(0)v(m + 2)) = 2v(m + 1),$$

i.e., (4.5) holds for $n = 2$ and $m \geq 0$. It remains to apply induction on $n \geq 2$ using (4.3) for the induction step. For example, in the generic case $n \geq 5$ we have

$$v(n)v(m) = 3v(n - 1)v(m + 1) - 3v(n - 2)v(m + 2) + v(n - 3)v(m + 3)$$

$$= \frac{1}{n + m - 1}(3(n - 1)(m + 1)v(1)v(n + m - 1) - 3(n - 2)(m + 1)v(0)v(n + m) + 3(n - 1)(n - 2)v(n + m - 1) - 3(n - 2)(m + 2)v(1)v(n + m - 1) + 3(n - 3)(m + 1)v(0)v(n + m - 1) - (n - 4)(m + 2)v(0)v(n + m) + (n - 3)(n - 4)v(n + m - 1))$$

$$= \frac{1}{n + m - 1}(nmv(n + m - 1) - (n - 1)(m - 1)v(0)v(n + m) + n(n - 1)v(n + m - 1)),$$

as desired. (For $n = 3, 4$ the only difference is that we should not expand some terms via (4.5).)

Next, let us make sure that the set of relations (4.4), (4.5) is closed under composition. This may be done in a straightforward way, but we may simply note that the reduced words are linearly independent in $\mathcal{A}_+(U(3))$. Indeed, the words reduced modulo (4.4), (4.5) are of the form

$$v_{k,p,m} = v(0)^k v(1)^p v(m),$$

where $n \geq 1$ for $p > 0$ or $n \geq 0$ for $p = 0$. On the other hand, the basis of $U(3)$ as of an $H$-module was found in [14], it consists of

$$v_k = (v(0))^k v, \quad v_{k,p} = (v(0))^k v(1)^p v(2)^p.$$
for \(k, p \geq 0\). Here \(v_{(n)}\) stands for the operator of \(n\)th conformal multiplication \((v_{(n)} \cdot)\) on \(U(3)\).

It is easy to calculate the principal terms of the Fourier coefficients for \(v_k, v_{k,p}\):

\[
v_k(m) = v_{k,0m}, \quad v_{k,p}(m) = v_{k,p+1; m}\.
\]

The linear independence of \(v_k, v_{k,p}\) over \(H = \mathbb{k}[\partial]\) implies linear independence of the reduced words \(v_{k,p;m}\). \(\square\)

4.2. The Anick complex for \(A_\ast(U(3))\). Throughout the rest of the paper \(\Lambda\) stands for the augmented algebra \(\Lambda = A_\ast(U(3)) \oplus \mathbb{k}\) with \(\varepsilon(v(n)) = 0\) for all \(n \geq 0\).

**Corollary 4.2.** The Anick chains \(\Lambda^{(n-1)}\) are of the following form:

\[
[v(m_1)]v(m_2)[v(m_{n-1})]v(m_n)],
\]

where \(m_1, \ldots, m_{n-2} \geq 2\) and either \(m_{n-1} \geq 2\) or \((m_{n-1}, m_n) = (1, 0)\).

We will write \([m_1 \ldots m_n]\) instead of \([v(m_1) \ldots v(m_n)]\) for the sake of simplicity.

Let \(A_\ast = A_\ast(\Lambda, \Lambda) \otimes A' \mathbb{k}\). In order to calculate the differential in \(A_\ast\) it is enough to construct a Morse matching in the graph \(\Gamma_{B_n}\). In Figures 4–7 below, we add extra edges corresponding to the rewriting of a non-reduced monomial into reduced form. This makes it easier to track paths in the graph \(\Gamma_{B_n}\).

**Theorem 4.3.** For \(n \geq 1\), the differential \(d_{n+1} : A_{n+1} \to A_n\) is given by

\[
d_{n+1}[i_1i_2\ldots i_n i_{n+1}] = \sum_{j=1}^{n} (-1)^j \frac{i_j(i_j - 1)}{i_j + i_{j+1} - 1}[i_1i_2\ldots i_j + i_{j+1} - 1]\ldots[i_{n+1}]
\]

\[
+ \sum_{j=2}^{n} \sum_{i=1}^{j-1} (-1)^i \frac{i_i(i_i - 1)}{i_i + i_{i+1} - 1}[i_1\ldots i_{i-1}\ldots i_j + i_{j+1}\ldots i_{n+1}]
\]

\[
+ \sum_{j=2}^{n} \sum_{i=1}^{j-1} (-1)^i \frac{i_i j_{j+1}}{i_j + i_{j+1} - 1}(i_{i-1}[i_1i_2\ldots i_j + i_{j+1} - 1]\ldots[i_{n+1}]], \quad (4.6)
\]

for \(i_n > 1\), and

\[
d_{n+1}[i_1i_2\ldots i_{n-1}10] = \sum_{j=1}^{n-2} (-1)^j \frac{i_j(i_j - 1)}{i_j + i_{j+1} - 1}[i_1i_2\ldots i_j + i_{j+1} - 1]\ldots[i_{n-1}10]
\]

\[
+ \sum_{j=2}^{n-2} \sum_{i=1}^{j-1} (-1)^i \frac{i_i(i_i - 1)}{i_i + i_{i+1} - 1}[i_1\ldots i_{i-1}\ldots i_j + i_{j+1}\ldots i_{n-1}10]
\]

\[
+ \sum_{j=2}^{n-2} \sum_{i=1}^{j-1} (-1)^i \frac{i_i j_{j+1}}{i_j + i_{j+1} - 1}(i_{i-1}[i_1i_2\ldots i_j + i_{j+1} - 1]\ldots[i_{n-1}10]
\]

\[
+ \sum_{j=1}^{n-1} (-1)^i [i_1\ldots i_{j-1}\ldots i_{n-1}1] + \sum_{j=1}^{n-1} (-1)^{n-1}(i_j - 1)[i_1\ldots i_{n-1}0] + (-1)^{n}[i_1\ldots i_{n-1}0]. \quad (4.7)
\]

**Proof.** Let us draw the segment of \(\Gamma_{B_n}\) with the matched edges. First, draw the edges of \(\Gamma_{B_n}\) starting at \(b = [i_1\ldots i_n i_{n+1}]\), \(i_n \geq 2\), by means of Theorem 4.1 (Fig. 4).

Next, choose the matching for those vertices that are not Anick chains, i.e., those that contain \(v(1)v(p)\) (Fig. 5) or \(v(0)v(p+1)\) (Fig. 6) at \(t\)th position, for \(t = 2, \ldots, j\). In this way, we continue obtaining vertices of the same form until \(t = 1\).
Example 4.4. Let \([n|m]\) be an Anick 1-chain in \(\Lambda\) which is not equal to \([1|0]\). Then
\[
d_3([n|m]) = -\frac{n(n-1)}{n+m-1}[n + m - 1].
\]
For the remaining 1-chain we have
\[
d_3([1|0]) = -[0].
\]

Example 4.5. Let \([n|m|p]\) be an Anick 2-chain in \(\Lambda\) with \(m \geq 2\). Then \(d_3([n|m|p])\) is equal to
\[
\frac{m(p(n-1) + m - 1)}{m + p - 1}[n|m + p - 1] - \frac{n(n-1)}{n+m-1}[n + m - 1|p] - \frac{n(m-1)(p-1)}{m + p - 1}[n - 1|m + p].
\]
(For \(n = 2\), the last summand is missing.) In a similar way,
\[
d_3([1|0]) = (2 - n)[n|0] + n[n - 1|1], \quad n \geq 2.
\]
Recall that \( \Lambda \) carries a derivation \( \partial \) given by \( \partial(v(n)) = nv(n-1) \). Let us describe the corresponding map \( \tilde{\partial}_n \) on \( A_n \). According to the general scheme (3.9), in order to calculate \( \tilde{\partial}_n([m_1] \ldots [m_n]) \) one has to evaluate \( g_n([m_1] \ldots [m_n]) \in B_n \), apply \( \partial_n \) by (3.7), and remove all those summands that are not Anick chains (i.e., apply \( f_n \)).

Note that if \( [m_1] \ldots [m_n] \in B_n \) is not an Anick chain then the expression for \( \partial_n([m_1] \ldots [m_n]) \) does not contain Anick chains except for the case when \( m_{n-1} = m_n = 1 \), \( m_1, \ldots, m_{n-2} \geq 2 \). It can be seen from the proof of Theorem 4.3 that such a summand \( u = [m_1] \ldots [m_{n-2}] [1] [1] \) \( \in B_n \) may appear only in \( g_n([m_1] \ldots [m_{n-2}] [2] [0]) \), but even in this case the coefficient at \( u \) is equal to zero since \( v(2)v(0) = v(0)v(2) + 2v(1) \), does not contain \( v(1)v(1) \).

Hence, the calculation of \( \tilde{\partial}_n \) on \( A_n \) becomes quite simple: one has to differentiate an Anick chain \( [m_1] \ldots [m_n] \) as it was an element of \( B_n \), and then remove all those summands that are not Anick chains. For example, \( \tilde{\partial}_n([2][p]) = p[2][p-1] \), \( \tilde{\partial}_n([n][2][1]) = n[n-1][2][1] + [n][2][0] \) if \( n > 2 \), \( \tilde{\partial}_n([2][2] \ldots [2][1][0]) = 0 \), etc.
For every $n \geq 1$, the space $A_n$ has the following grading:

$$A_n^{(d)} = \{ a \in A_n \mid \deg a = d \},$$

where $\deg [i_1 \ldots i_n] = i_1 + \cdots + i_n$. Theorem 4.3 shows $d_n$ to be a degree $-1$ map: $d_n(A_n) \subseteq A_n^{(d-1)}$.

The space $K_n = \text{Ker} \tilde{d}_n \subseteq A_n$ inherits the grading from $A_n$: $K_n^{(d)} = K_n \cap A_n^{(d)}$.

4.3. Low-dimensional Hochschild conformal cohomology of $U(3)$. Let us explicitly calculate the dimensions of the first, second, and third Hochschild cohomology groups for the associative conformal algebra $U(3)$ with coefficients in the scalar module $\mathbb{k}$.

Denote $K_n = \text{Ker} \tilde{d}_n \subseteq A_n$ for $n \geq 1$. According to the general principle from Section 3.3, $H^n(U(3), \mathbb{k}) = (\text{Ker} d_n/\text{Im} \tilde{d}_{n+1})^*$, where $d_n$ is the restriction of $d_n : A_n \rightarrow A_{n-1}$ onto $K_n$.

For example, $\dim K_1 = 1$, the basis is $[0]$, and $d_2([1|0]) = -[0]$, where $[1|0] \in K_2$. Hence, $H^1(U(3), \mathbb{k}) = 0$.

An arbitrary element $a \in A_n$ may be uniquely written as

$$a = [a_0 | 0] + [a_1 | 1] + \ldots,$$

where $a_i \in A_{n-1}$.

**Lemma 4.6.** Every element $a \in K_n$ is completely determined by $a_0 \in A_{n-1}$ which satisfies the following properties: (1) $(a_0)_0 = 0$; (2) $(\tilde{d}_{n-1} a_0)_1 = 0$.

**Proof.** Suppose $b = \tilde{d}_n a, b = [b_0 | 0] + [b_1 | 1] + \ldots$. Then

$$[b_1 | i] = f_n(\tilde{d}_{n-1}(a_i|i) + (i + 1)(a_{i+1}|i]), \quad i = 0, 1, \ldots$$

Note that $f_n([a_{i+1}|i]) = [a_{i+1}|i]$ for $i \geq 0$.

Then $a \in K_n$ if and only if $b_i = 0$ for all $i \geq 0$. Hence, $a_{i+1}$ is completely determined by $f_n(\tilde{d}_{n-1}(a_i)|i))$. The only problem emerges for $i = 0$: $f_n(\tilde{d}_{n-1}(a_0)|0)$ may contain a chain of the form $[\cdots | 1|0]$, but $a_1$ may not contain a chain ending with $[\cdots | 1]$. Therefore, $a$ is completely defined by $a_0$ such that $[a_0 | 0]$ is a combination of chains (i.e., $a_0 \in A_{n-1}$ does not contain zeros) and $\tilde{d}_{n-1}(a_0)$ does not contain units. \hfill \Box

Let $A_n^{(d)}$ stand for the space spanned by *regular* chains, i.e., those that have all components $\geq 2$. Similarly, denote $A_n^{(d,0)} = A_n^{(d)} \cap A_n^{(d)}$.

**Corollary 4.7.** For $n \geq 3$ we have $K_n \cong A_n^{(d-1,0)} \oplus A_n^{(d-3,0)} \oplus A_n^{(d-4,0)} \oplus \ldots$

**Proof.** Assume $n \geq 3$ and $v = a_0 \in K_n$ meets the conditions (1), (2) of Lemma 4.6. Then $v \in A_{n-1}$ may be presented in the same form

$$v = [v_1 | 1] + [v_2 | 2] + \ldots,$$

where $v_1 \in A_n^{(d-1,0)}$.

The condition $(\tilde{d}_{n-1} v)_1 = 0$ is equivalent to

$$f_{n-1}(\tilde{d}_{n-2}(v_1)|1] + 2[v_2|1] = 0.$$

Hence, $v_1$ uniquely determines $v_2$ and, together with $v_k, k \geq 3$, uniquely define $a \in K_n$. \hfill \Box

**Corollary 4.8.** The linear basis of $K_2$ consists of

$$e_1 = [1|0], \quad e_3 = [3|0] - 3[2|1], \quad \ldots, \quad e_d = \sum_{s=0}^{d-2} (-1)^s \binom{d}{s} [d - s|s],$$

for $d \geq 3$.

**Proposition 4.9** (c.f. [1]). $\dim H^2(U(3), \mathbb{k}) = 1$. 

19
Proof. Note that $e_d$ for $d \geq 3$ belong to $\text{Ker } d_2$. One may either apply Theorem 4.3 or just note that $d_2(K^{(d)}_2) \subseteq K^{(d-1)}_1$ and $K^{(d-1)}_1 = 0$ for $d \neq 1$.

On the other hand, for every $d \geq 3$ there exists $f_{d+1} \in K^{(d+2)}_3$ such that

$$f_{d+1} = [2|d|0] + [\cdots |1] + \ldots .$$

Then $d_3(f_{d+1}) = - \frac{2}{d+1}[d + 1|0] + [\cdots |1] + \cdots \in K^{(d+1)}_2$, so $e_{d+1} \in \text{Im } \tilde{d}_3$. Therefore, it remains to compare $\text{Ker } \tilde{d}_3$ and $\text{Im } \tilde{d}_3$ in $K^{(3)}_3$. The kernel is 1-dimensional, but the image is zero. Indeed, the only regular chain in $A_2^{(3)}$ is $[2|2]$, so $\dim K^{(d)}_3 = 1$ and the basis (e.g., recovered by Lemma 4.6) is

$$f_3 = [2|2|0] - \frac{2}{3}[3|1|0].$$

It is straightforward to compute that $d_3(f_3) = 0$, so $\text{Im } \tilde{d}_3 = 0$. □

Proposition 4.10. $\dim H^3(U(3), k) = 1$.

Proof. Let $\tilde{d}_n^d$ stand for the differential $K_n^{(d)} \rightarrow K_{n-1}^{(d-1)}$. By Corollary 4.7, $\dim K_3^{(d)} = d - 3$ for $d \geq 4$. On the other hand, $\text{Im } \tilde{d}_3^{(d)} = K_2^{(d-1)}$ for $d - 1 \geq 4$, as we have seen in the proof of Proposition 4.9. Hence, $\dim \text{Ker } \tilde{d}_3^{(d)} = d - 4$ for $d \geq 5$.

Suppose $d \geq 5$. Choose the following elements $a_i^{(i)} \in K_{d+1}^{(d+1)}$, $i = 2, \ldots, d - 4$, by defining their zero components

$$a_0^{(i)} = [2|i]d - 1 - i].$$

One more element $a_{d-3}^{(d-3)} \in K_{d+1}^{(d+1)}$ is defined by

$$a_0^{(d-3)} = [2|d - 3|2] - \frac{2}{d-2}[2|d - 2|1].$$

Let us calculate the leading (in the lexicographic sense) terms of $\tilde{d}_4^{(d+1)}(a_i^{(i)})$ by Theorem 4.3

$$\tilde{d}_4^{(d+1)}(a_i^{(i)}) = \alpha_i[i + 1|d - i - 1|0] + \ldots , \quad i = 2, 3, \ldots, d - 3,$$

where $\alpha_i \in k$, $\alpha_i \neq 0$. Hence, $\dim \text{Im } \tilde{d}_4^{(d+1)} = d - 4 = \dim \text{Ker } \tilde{d}_3^{(d)}$.

Therefore, nontrivial cohomology may appear in $K_3^{(d)}$ for $d < 5$ only. Recall that $K_3^{(4)}$ is spanned by $[2|3|1|0] - 3[2|2|0]$, and is proportional to the image of $[2|2|1|0] \in K_4^{(5)}$ under $d_4$. For $d = 3$, we have only $[2|1|0] \in \text{Ker } \tilde{d}_3^{(3)}$ which may not be an image of $\tilde{d}_4^{(3)}$ since there are no Anick chains of degree 4 in $\Lambda^{(3)}$. □

4.4. Higher Hochschild cohomology of $U(3)$. There is a filtration

$$A_{n,0} \supset A_{n,1} \supset A_{n,2} \supset \ldots ,$$

where

$$A_{n,k} = \text{span}[i_1|\ldots|i_n] \in \Lambda^{(n-1)} \mid i_n \geq k].$$

It follows from Theorem 4.3 that $d_n(A_{n,k}) \subseteq A_{n-1,k}$.

Proposition 4.11. Let $n \geq 2$, $u \in A_{n,2}$, and $d_n u = 0$. Then there exists $w \in A_{n+1,2}$ such that $d_{n+1} w = u$.

Proof. We may suppose that $u \in A_{n}^{(d)}$, $d \geq 5$ (since $[2|2]$ is the only regular chain of degree 4). For $n = 2$, the proof is straightforward. Indeed, assume the converse and choose

$$u = \sum_{i=0}^{d-4} \alpha_i[d - i - 2|i + 2] \in (\text{Ker } d_2 \cap A_{n+1,2}^{(d)} \setminus d_3(A_{n,2}^{(d)})) \quad (4.8)$$
with minimal \( l \) such that \( \alpha_i \neq 0 \). Note that \( l < d - 4 \) since \( d_2([2|d - 2]) \neq 0 \). Then we may build

\[
\mathbf{u}' = \mathbf{u} - \gamma d_{1}[d - l - 1][i + 2]
\]

for \( \gamma = -((d - l - 1)(d - l - 2)/(d - l))^{-1} \) which has the same property as \( \mathbf{u} \) in (4.8), but its presentation has nonzero coefficients for \( i > l \), which is a contradiction.

Suppose \( n > 2 \) and assume the statement is true for \( \mathbf{v} \in \mathcal{A}_{m, 2} \) with \( m < n \). Then

\[
\mathbf{u} = [\mathbf{u}_2][2] + \mathbf{v}, \quad \mathbf{v} \in \mathcal{A}_{n, 3},
\]

and \( d_n \mathbf{u} = 0 \) implies \( d_{n-1} \mathbf{u}_2 = 0 \) since \( d_n \mathbf{v} \in \mathcal{A}_{n-1, 3} \). Hence there exists \( \mathbf{w}_2 \in \mathcal{A}_{n, 2} \) such that \( d_n \mathbf{w}_2 = \mathbf{u}_2 \). Consider \( \mathbf{u}' = \mathbf{u} - d_{n+1}[\mathbf{w}_2][2] \). Since \( d_{n+1}[\mathbf{w}_2][2] \in [d_n \mathbf{w}_2][2] + \mathcal{A}_{n, 3} \) by Theorem 4.3 we obtain

\[
\mathbf{u}' \in \text{Ker} d_n \cap \mathcal{A}_{n, 3}.
\]

In a similar way, we may find \( \mathbf{w}_3 \in \mathcal{A}_{n, 2} \) such that \( \mathbf{u}' - d_{n+1}[\mathbf{w}_3][3] \in \mathcal{A}_{n, 4} \), and so on. Since \( \mathcal{A}^{\text{(d)}}_{n,k} \) = 0 for sufficiently large \( k \), the result follows.

**Corollary 4.12.** Let \( \mathbf{u} \in \mathcal{A}_{n, 1}, \ n \geq 3 \), and \( d_n \mathbf{u} = 0 \). Then there exists \( \mathbf{w} \in \mathcal{A}_{n+1, 1} \) such that \( d_{n+1} \mathbf{w} = \mathbf{u} \).

**Proof.** Let \( \mathbf{u} = [\mathbf{u}_1][1] + \mathbf{u}' \), \( \mathbf{u}' \in \mathcal{A}_{n, 2} \). Then \( d_n \mathbf{u} = 0 \) implies \( d_{n-1} \mathbf{u}_1 = 0 \). Since \( \mathbf{u}_1 \in \mathcal{A}_{n-1, 2} \) and \( n - 1 \geq 2 \), there exists \( \mathbf{w}_1 \in \mathcal{A}_{n, 2} \) such that \( d_n \mathbf{w}_1 = \mathbf{u}_1 \). Then \( \mathbf{u} - d_{n+1}[\mathbf{w}_1][1] \in \mathcal{A}_{n, 2} \) may be presented as \( d_{n+1} \mathbf{w} \) for an appropriate \( \mathbf{w} \in \mathcal{A}_{n, 2} \).

**Corollary 4.13.** Let \( \mathbf{u} \in \mathcal{A}_{n}, \ n \geq 4 \), and \( d_n \mathbf{u} = 0 \). Then there exists \( \mathbf{w} \in \mathcal{A}_{n+1} \) such that \( d_{n+1} \mathbf{w} = \mathbf{u} \).

**Proof.** If \( \mathbf{u} \in \mathcal{A}_{n, 1} \) then we are done. Hence, consider \( \mathbf{u} = [\mathbf{u}_0][0] + \mathbf{u}', \ \mathbf{u}' \in \mathcal{A}_{n, 1} \). Then \( \mathbf{u}_0 \in \mathcal{A}_{n-1, 1} \) and \( d_{n-1} \mathbf{u}_0 = 0 \). By Corollary 4.12 there exists \( \mathbf{w}_0 \in \mathcal{A}_{n, 1} \) such that \( d_n \mathbf{w}_0 = \mathbf{u}_0 \). Then \( \mathbf{u} - d_{n+1}[\mathbf{w}_0][0] \in \mathcal{A}_{n, 1} \) and thus \( \mathbf{u} = d_{n+1} \mathbf{w} \) for an appropriate \( \mathbf{w} \in \mathcal{A}_{n, 1} \).

**Theorem 4.14.** \( H^n(U(3), \mathbb{k}) = 0 \) for \( n \geq 4 \).

**Proof.** Corollary 4.13 implies the cohomology groups of the non-restricted complex \( \widetilde{C}^* \) of the conformal algebra \( U(3) \) with coefficients in \( M = \mathbb{k} \) are trivial for \( n \geq 4 \). The short exact sequence of complexes

\[
0 \to \partial \widetilde{C}^* \to \widetilde{C}^* \to C^* \to 0
\]

leads to the long exact sequence of cohomology groups

\[
\ldots \to H^n(\partial \widetilde{C}^*) \to H^n(\widetilde{C}^*) \to H(C^*)
\]

\[
\to H^{n+1}(\partial \widetilde{C}^*) \to \ldots .
\]

By [4] Proposition 2.1 \( \partial \widetilde{C}^* = \widetilde{C}^* \) in degrees \( n \geq 1 \), so \( H^n(\widetilde{C}^*) = H^n(\partial \widetilde{C}^*) = 0 \) for \( n \geq 4 \). Hence, the restricted complex \( C^* \) has zero cohomologies for \( n \geq 4 \).

4.5. **Final remarks.** We have shown that the Hochschild cohomology groups for the universal associative conformal envelope \( U(3) = U(Vir; N = 3) \) with coefficients in the scalar module are the same as for the Virasoro Lie conformal algebra Vir. However, this is not true for the cohomologies with coefficients in a non-trivial irreducible module.

Recall that all irreducible modules over Vir are of the form \( M(\alpha, \Delta) \), \( \Delta \neq 0 \) [8], see also Example 3.6. It was found in [4] that

\[
\dim H^1(\text{Vir}, M(0, \Delta)) = \begin{cases} 2, & \Delta = 1, \\ 1, & \Delta = -1, 0, \\ 0, & \text{otherwise}. \end{cases}
\]
All these representations may be extended to the representations of $U(3)$, so, in particular, $M(0, \Delta)$ is a left $U(3)$-module for every scalar $\Delta$. (Note that the envelope $U(2)$ inherits only those representations that correspond to $\Delta = 0, 1$.)

Let us compute $H^1(U(3), M)$ for $M = M(0, \Delta)$, $\Delta \in \mathbb{k}$, in a straightforward way.

Let $\varphi \in Z^1(U(3), M)$. This is a conformal module map, i.e., a $\mathbb{k}[\partial]$-linear map from $U(3)$ to $M$ such that

$$\varphi(a \cdot \partial b) = a \cdot \varphi(b), \quad a, b \in U(3).$$

The value $\varphi(v) = f(\partial)u$, $f \in \mathbb{k}[\partial]$, on the generator of $U(3)$ determines the cocycle completely.

In order to preserve the defining relation \([4.1]\), the polynomial $f(\partial)$ has to meet the following conditions:

$$2v(1)f(\partial)u - \partial(v(2)f(\partial)u) = 2f(\partial)u,$$

$$v(n)f(\partial)u = 0, \quad n \geq 3.$$ \([4.2]\)

The sesqui-linearity implies $v(n)f(\partial)u = (\partial f^{(n)}(\partial) + n\Delta f^{(n-1)}(\partial))u$, for $n \geq 0$. Hence, for $\Delta \neq 0$ we have $f^{(2)} = f^{(3)} = \cdots = 0$ from the locality condition, so $f(\partial) = c_0 + \partial c_1$. The latter meets

$$\partial f' + \Delta f = f,$$

which turns into $(\Delta - 1)c_0 = 0$. Hence, $\dim Z^1(U(3), M) = 1$ for $\Delta \neq 1, 0$ and $\dim Z^1(U(3), M) = 2$ for $\Delta = 1$.

If $\Delta = 0$ then we obtain $f^{(3)} = f^{(4)} = \cdots = 0$ from the locality condition, thus $f = c_0 + c_1\partial + c_2\partial^2$ in this case. Then the defining relation turns into

$$2\partial f' - \partial^2 f'' = 2f,$$

which implies the only condition $c_0 = 0$. Therefore, $\dim Z^1(U(3), M) = 2$ for $\Delta = 0$.

The space of coboundaries $B^1(U(3), M)$ consists of those $\varphi \in Z^1(U(3), M)$ that can be presented as $\varphi(a) = a(0)u = (a \cdot \partial)u$, $a \in U(3)$. For $a = v$, we obtain $\varphi(v) = f(\partial)u = (\partial \Delta + \partial)u$. Hence, $\dim B^1(U(3), M) = 1$ for $\Delta \neq 1$ and $B^1(U(3), M) = 0$ for $\Delta = 1$.

As a result, for the 1st Hochschild cohomologies of $U(3)$ we have

$$\dim H^1(U(3), M(0, \Delta)) = \begin{cases} 2, & \Delta = 1, \\ 1, & \Delta = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, there is a difference between cohomologies of $\text{Vir}$ and its universal associative envelope $U(3)$. An interesting task is to calculate all Hochschild cohomology groups for $U(3)$ with coefficients in $M(\alpha, \Delta)$.

**References**

[1] Alhussein, H., Kolesnikov, P.S.: On the Hochschild cohomology of universal enveloping associative conformal algebras. J. Math. Phys. 62, 121701 (2021).
[2] Anick, D. J.: On the homology of associative algebras. Trans. Amer. Math. Soc. 296, no. 2, 641–659 (1986).
[3] Boyallian, C., Kac, V.G., Liberati, J.-I.: On the classification of subalgebras of $\text{Cend}_N$ and $\text{gcd}_N$. J. Algebra 260, 32–63 (2003).
[4] Bakalov, B., Kac, V.G., Voronov A.: Cohomology of conformal algebras. Comm. Math. Phys. 200, 561–589 (1999).
[5] Bakalov, B., D’Andrea, A., Kac, V. G.: Theory of finite pseudoalgebras. Adv. Math. 162, 1–140 (2001).
[6] Bokut, L. A., Fong, Y., Ke, W.-F.: Composition-Diamond lemma for associative conformal algebras. J. Algebra 272, 739–774 (2004).
[7] Cartan, H., Eilenberg, S.: Homological Algebra. Princeton University. Press, Princeton, NJ, 1956.
[8] Cheng, S.-J., Kac, V. G.: Conformal modules. Asian J. Math. 1, 181-193 (1997).
[9] D’Andrea, A., Kac, V. G.: Structure theory of finite conformal algebras. Sel. Math., New Ser. 4, 377–418 (1998).
[10] Dubrovin, B. A., Novikov, S. P.: Poisson brackets of hydrodynamic type. Dokl. Akad. Nauk SSSR 279, no. 2, 294–297 (1984).
[11] Jöllenbeck, M., Welker, V.: Minimal resolutions via algebraic discrete Morse theory. Mem. Am. Math. Soc. 197, no. 929 (2009).
[12] Kac, V. G.: Vertex Algebras for Beginners. Univ. Lect. Ser., 10, Am. Math. Soc., Providence, RI (1998).
[13] Kolesnikov, P. S., Kozlov, R. A.: On the Hochschild cohomologies of associative conformal algebras with a finite faithful representation. Comm. Math. Phys. 369, no. 1, 351–370 (2019).
[14] Kolesnikov, P. S.: Gröbner–Shirshov bases for associative conformal algebras with arbitrary locality function. New Trends in Algebra and Combinatorics, Proceedings of the 3rd International Congress 15in Algebra and Combinatorics (K. P. Shum et al, Eds.), World Scientific (2020), 255–267.
[15] Kolesnikov, P.S.: Universal enveloping Poisson conformal algebras. Internat. J. Algebra and Computation 30(5) 1015–1034 (2020).
[16] Kolesnikov, P.S., Kozlov, R.A.: Standard bases for the universal associative conformal envelopes of Kac–Moody conformal algebras. (to appear) Algebras and Representation Theory (to appear), DOI: 10.1007/s10468-021-10050-0.
[17] Kozlov, R.A.: Hochshild cohomology of the associative conformal algebra Cend_{1,\omega}. Algebra and Logic 58, 36–47 (2019).
[18] Maclane S., Homology. Springer Verl., Berlin–Göttingen–Heidelberg, 1963.
[19] Roitman, M.: On free conformal and vertex algebras. J. Algebra 217, 496–527 (1999).
[20] Roitman, M.: Universal enveloping conformal algebras, Sel. Math. New Ser. 6, 319–345 (2000).
[21] Sköldberg, E.: Morse theory from an algebraic viewpoint. Trans. Amer. Math. Soc. 358, no. 1, 115–129 (2006).
[22] Xu, X.: Quadratic conformal superalgebras. J. Algebra 231, 1–38 (2000).

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