Abstract
Linear Logic was introduced by Girard as a resource-sensitive refinement of classical logic. It turned out that full propositional Linear Logic is undecidable (Lincoln, Mitchell, Secegov, and Shankar) and, hence, it is more expressive than (modalized) classical or intuitionistic logic. In this paper we focus on the study of the simplest fragments of Linear Logic, such as the one-literal and constant-only fragments (the latter contains no literals at all).

Here we demonstrate that all these extremely simple fragments of Linear Logic (one-literal, \(\bot\)-only, and even unit-only) are exactly of the same expressive power as the corresponding full versions:

(a) On the level of the *multiplicatives* \(\otimes, \otimes, \circ\) we get \(NP\)-completeness.

(b) Enriching this basic set of connectives by *additives* \(\&, \oplus\) yields \(PSPACE\)-completeness.

(c) Using in addition the *storage* operator \(!,\), we can prove the undecidability of all these three fragments.

We present also a complete computational interpretation (in terms of *acyclic programs with stack*) for \(\bot\)-free Intuitionistic Linear Logic. Based on this interpretation, we prove the *fairness* of our encodings and establish the foregoing complexity results.
1 Introduction and Summary

Linear Logic was introduced by J.-Y. Girard [23] as a resource-sensitive refinement of classical logic. It turned out that full propositional Linear Logic is undecidable [89], and, hence, it is more expressive than (modalized) classical or intuitionistic logic. Moreover, an exact correspondence between natural fragments of propositional Linear Logic and natural complexity classes can be established [89, 48]. In this paper we focus on the study of the simplest fragments of Linear Logic, such as one-literal and constant-only fragments (the latter contains no literals at all) and demonstrate that these extremely simple fragments are of the same expressive power as the corresponding full versions.

Formulas of propositional Linear Logic are built up of literals and constants \((\bot, \|)\) by the following connectives:

\(\otimes, \otimes, \neg, \&\), and \(\lnot\).

According to the well-known approaches, the hierarchy of natural fragments of Linear Logic can be developed in the following three directions:

(1) We start from the basic set of connectives, the multiplicatives: \(\otimes, \otimes\), and proceed to enrich it either by additives: \(&, \oplus\), or by exponentials: \(\lnot, \lnot\), or by both additives and exponentials.

(1a) Thus we can start with the Multiplicative Fragment \(\mathbf{LL}(\otimes, \otimes, \bot, \|)\) (which is proved to be \(NP\)-complete [48]),

(1b) and go either to the Multiplicative-Additive Fragment \(\mathbf{LL}(\otimes, \otimes, \&), \otimes, \bot, \|)\) (which is \(PSPACE\)-complete [89]),

(1c) or to the Multiplicative-Exponential Fragment \(\mathbf{LL}(\otimes, \otimes, \lnot, \lnot, \otimes, \bot, \|)\) (its exact complexity level is unknown),

(1d) and finish in the full propositional Linear Logic \(\mathbf{LL}(\otimes, \otimes, \&), \otimes, \bot, \|)\) (which is undecidable [89]).

(2) We can confine ourselves to formulas of a certain simple structure. E.g., it is typical of many logical systems to limit the depth of nesting of implications. In particular, it leads to the consideration of the so-called \(Horn\) formulas having the form \((X \rightarrow Y)\).

As a rule, the \(Horn\) fragments are essentially simpler than their corresponding full versions.

Contrary, for Linear Logic we have the following results [48] demonstrating the maximum possible expressive power of \(Horn\) fragments:

(2a) the purely \(Horn\) fragment \(\mathbf{HLL}(\otimes, \neg, \otimes)\) consisting of \(Horn\) implications \((X \rightarrow Y)\), is already \(NP\)-complete,

(2b) the \((\otimes, \&)-Horn\) fragment \(\mathbf{HLL}(\otimes, \neg, \&), \otimes\), that contains \(\otimes-Horn\) implications \((X \rightarrow (Y_1 \lor Y_2))\) and \(\&-Horn\) implications \(((X_1 \rightarrow Y_1) \lor (X_2 \rightarrow Y_2))\), is already \(PSPACE\)-complete,

(2c) the \(!-Horn\) fragment \(\mathbf{HLL}(\otimes, \neg, \lnot)\) is still decidable (it is polynomially equivalent to Petri nets),

(2d) and the \((!, \otimes)-Horn\) fragment \(\mathbf{HLL}(\otimes, \neg, \otimes, \lnot)\) can simulate many-counter Minsky machines.

Theorem 2.1 shows the collapse of this hierarchy on the next step when we introduce \emph{elementary embedded implications} \((U \rightarrow V) \rightarrow Y)\).

(3) Finally, for a given fragment of Linear Logic \(\mathbf{LL}(\sigma)\) (its formulas are built up of literals and constants by connectives from the set \(\sigma\), constants are also taken from \(\sigma\)), we can reduce the number of the literals used to a fixed number \(k\) and study the corresponding fragment \(\mathbf{LL}^k(\sigma)\). Following such a \emph{bottom-up} approach, we will start with the simplest cases when \(k\) is small, namely, we will study the \emph{one-literal} fragment \(\mathbf{LL}^1(\sigma)\) and \emph{constant-only} fragment \(\mathbf{LL}^0(\sigma)\).

Actually, this approach is also quite traditional.

E.g., consideration of the one-literal fragment of intuitionistic propositional logic allows us to obtain the full characterization of this fragment and shed light on the true nature of intuitionistic logic as a whole [105, 42].

As for the expressive power of constant-only fragments of traditional logical systems, it is equal to zero: the entire problem boils down to primitive Boolean calculations over constants.

The intricate story for Linear Logic began with the following unexpected results:

(a) The simplest one-literal fragment \(\mathbf{LL}^1(\neg)\) is \(NP\)-complete [49],

(b) The simplest constant-only fragment \(\mathbf{LL}^0(\otimes, \otimes, \neg, \bot, \|)\) is \(NP\)-complete [93].

As for one-literal and constant-only fragments enriched by additives and/or exponentials, \emph{a priori} we could indicate both \emph{pro} and \emph{contra} arguments for their expressive power to be of high level.
In particular, we could point out that all known proofs of the PSPACE-completeness of implicational fragment of intuitionistic propositional logic as well as of quantified Boolean propositional formulas are essentially based on an unbounded number of variables used.

Regarding to the expressive power of connectives involved, the \( \bot \)-only case seemed to be easier to consideration, because we could use at least the functionally complete set of connectives including negation. The only problem was to wipe out the influence of the inference rules specified for \( \bot \) and, as a result, cause \( \bot \) to be thought of as an ordinary positive literal.

The one-literal and unit-only cases met a problem at this point because, in the absence of \( \bot \), the whole system of connectives

\[
\otimes, \otimes, \bot, \&,
\]

is functionally incomplete (even in the Boolean sense).

The unit-only case was the most complicated one because it is quite hard to conceive of the unit \( \mathbb{1} \) as a literal.

Nevertheless, Corollary 4.3 validates the following.

Theorem 1.1 For one-literal fragments of Linear Logic, we prove that

1. \( \text{LL}^1(\otimes, \otimes, \bot) \) is NP-complete.
2. Moreover, the purely implicational one-literal fragment \( \text{LL}^1(\bot) \) is already NP-complete.
3. \( \text{LL}^1(\bot, \&) \) is PSPACE-complete.
4. \( \text{LL}^1(\bot, !) \) can polynomially simulate the whole \( \bot \)-free Intuitionistic Linear Logic \( \text{ILL}(\otimes, \bot, !) \).

In particular, the reachability problem for Petri Nets can be encoded in this one-literal fragment.

5. \( \text{LL}^1(\bot, !, !) \) can directly simulate many-counter Minsky machines, and, hence, it is undecidable.

Theorem 1.2 For \( \bot \)-only fragments of Linear Logic, we have

1. \( \text{LL}^0(\otimes, \otimes, \bot) \) is NP-complete [93].
2. Moreover, the purely implicational \( \bot \)-only fragment \( \text{LL}^0(\bot, \bot) \) is already NP-complete.

\[\text{The latter consists of sequents of the form } \Sigma \vdash A \] where multiset \( \Sigma \) and formula \( A \) belong to the language of \( \text{LL}(\otimes, \bot, !) \) (containing neither \( \bot \) nor \( \otimes \)).

\( [93] \)

Theorem 1.3 Finally, for unit-only fragments of Linear Logic, we prove that

1. \( \text{LL}^0(\otimes, \otimes, \mathbb{1}) \) is trivial.
2. Nevertheless, \( \text{LL}^0(\otimes, \otimes, \bot, \mathbb{1}) \) is NP-complete [93].
3. \( \text{LL}^0(\otimes, \otimes, \bot, \&) \) is proved to be PSPACE-complete.
4. \( \text{LL}^0(\otimes, \otimes, \bot, !) \) can polynomially simulate \( \text{ILL}(\otimes, \otimes, !) \) and, hence, the complexity level of this Unit-Only Fragment is not less than the level of the whole Multiplicative-Exponential Fragment of \( \bot \)-free Intuitionistic Linear Logic.

5. \( \text{LL}^0(\otimes, \otimes, \bot, \&!, \mathbb{1}) \) can directly simulate many-counter Minsky machines, and, hence, it is undecidable.

The plan of the paper is as follows:

(a) We present a complete computational interpretation of the Normalized Intuitionistic Linear Logic \( \text{NLL}(\otimes, \bot, \&!, \mathbb{1}) \) in terms of acyclic programs with stack.

(b) Then we encode all normalized sequents by one-literal, \( \bot \)-only, and unit-only sequents, and prove the fairness of these encodings.

(c) Finally, based on the uniformity of our encodings, we establish the foregoing complexity results for the natural fragments of one-literal and constant-only Linear Logic.

2 Normalized Sequents

Here we consider formulas of propositional Linear Logic that are built up of positive literals

\[ p_1, p_2, \ldots, p_m, \ldots, p_m, \ldots \]

and constants

\[ \bot, \mathbb{1} \]
|   |   |
|---|---|
| I | $A \vdash A$ |
| L-o | $\Sigma_1 \vdash A, \Phi_1$, $B, \Sigma_2 \vdash \Phi_2$  
$\Sigma_1, (A \rightarrow B), \Sigma_2 \vdash \Phi_1, \Phi_2$  
$\Sigma, A \vdash B, \Phi$  
$\Sigma \vdash (A \rightarrow B), \Phi$ |
| R-o | $\Sigma_1 \vdash A, \Phi_1$, $\Sigma_2 \vdash B, \Phi_2$  
$\Sigma_1, \Sigma_2 \vdash (A \rightarrow B), \Phi_1, \Phi_2$  
$\Sigma \vdash (A \rightarrow B), \Phi$ |
| L\$ \otimes \Sigma, (A \otimes B) \vdash \Phi$  
$\Sigma_1, \Sigma_2, (A \otimes B) \vdash \Phi_1, \Phi_2$  
$\Sigma \vdash A, \Phi$  
$\Sigma \vdash (A \otimes B), \Phi$ |
| R\$ \otimes \Sigma \vdash A, \Phi$  
$\Sigma \vdash (A \otimes B), \Phi$ |
| L\$ \otimes \Sigma, (A \oplus B) \vdash \Phi$  
$\Sigma \vdash A \oplus B)$  
$\Phi$  
$\Sigma \vdash B, \Phi$  
$\Sigma \vdash (A \oplus B), \Phi$ |
| R\$ \oplus \Sigma \vdash A, \Phi$  
$\Sigma \vdash (A \oplus B), \Phi$ |
| L\& | $\Sigma, A \vdash \Phi$  
$\Sigma, (A \& B) \vdash \Phi$  
$\Sigma \vdash A, \Phi$  
$\Sigma \vdash (A \& B), \Phi$ |
| R\& | $\Sigma \vdash A, \Phi$  
$\Sigma \vdash B, \Phi$  
$\Sigma \vdash (A \& B), \Phi$ |
| L| $\Sigma, A \vdash \Phi$  
$\Sigma, !A \vdash \Phi$  
$\Sigma \vdash !A, !A !A \vdash \Phi$ |
| R| $\Sigma \vdash !A, !A \vdash \Phi$ |
| W| $\Sigma, !A \vdash \Phi$  
$\Sigma \vdash !A, !A \vdash \Phi$ |
| C| $\Sigma \vdash A, !A \vdash \Phi$  
$\Sigma \vdash !A, !A \vdash \Phi$ |
| L| $\bot \vdash$  
$\bot \vdash$  
$\bot \vdash$  
$\bot \vdash$ |
| R| $\Sigma \vdash \Phi, \bot$  
$\Sigma \vdash \Phi, \bot$  
$\Sigma \vdash \Phi, \bot$  
$\Sigma \vdash \Phi, \bot$ |

Table 1: The Inference Rules of Linear Logic.
by the following connectives:
\[ \otimes, \exists, \neg, \& , \oplus, \text{ and } \neg \]

The inference rules for these connectives are given in Table 1.

Without loss of generality, we can confine ourselves to normalized sequents, i.e., sequents of the form\(^2\)

\[ W, \Delta, \Gamma \vdash Z \]

where \( W \) and \( Z \) are non-empty tensor products of positive literals,\(^3\) \( \Gamma \) and \( \Delta \) are multisets consisting of Horn implications

\[ (X \rightarrow Y), \]

\( \oplus \)-Horn implications

\[ (X \rightarrow (Y_1 \oplus Y_2)), \]

\( \& \)-Horn implications

\[ ((X_1 \rightarrow Y_1) \& (X_2 \rightarrow Y_2)), \]

and elementary embedded implications

\[ ((U \rightarrow V) \rightarrow Y), \]

here (and henceforth) \( X, X_1, X_2, Y, Y_1, Y_2, U, \) and \( V \) are simple products.

**Definition 2.1** The tensor product of a positive number of positive literals is called a simple product. A single literal \( q \) is also called a simple product.

**Definition 2.2** Taking into account the associativity and commutativity laws, we use a natural isomorphism between non-empty finite multisets of positive literals and simple products:

A multiset

\[ \{q_1, q_2, \ldots, q_k\} \]

is represented by the simple product

\[ (q_1 \otimes q_2 \otimes \cdots \otimes q_k), \]

and vice versa.

**Definition 2.3** We will write

\[ X \cong Y \]

to indicate that \( X \) and \( Y \) represent one and the same multiset \( M \).

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\(^2\)Where \( \Gamma \) stands for the multiset resulting from putting the modal storage operator \( ! \) before each formula in \( \Gamma \).

\(^3\)Henceforth, such products will be called simple products.

**Definition 2.4** Normalized formulas are defined as follows:

(a) A Horn implication is a formula of the form

\[ (X \rightarrow Y). \]

(b) A \( \oplus \)-Horn implication is a formula of the form

\[ (X \rightarrow (Y_1 \oplus Y_2)). \]

(c) An \&-Horn implication is a formula of the form

\[ ((X_1 \rightarrow Y_1) \& (X_2 \rightarrow Y_2)). \]

(d) An elementary embedded implication is a formula of the form

\[ ((U \rightarrow V) \rightarrow Y). \]

Here \( X, X_1, X_2, Y, Y_1, Y_2, U, \) and \( V \) are simple products.

We will consider the Normalized Fragment of Linear Logic NLL (\( \otimes, \neg, \& , \oplus, \neg \)) that consists of such normalized sequents.

The most interesting case is as follows.

**Theorem 2.1** The whole Multiplicative-Exponential Fragment of \( \bot \)-free Intuitionistic Linear Logic ILL (\( \otimes, \neg, \& , \oplus, \neg \)) is polynomial-time reducible to its Normalized Fragment NLL (\( \otimes, \neg, \& , \oplus, \neg \)) containing only Horn implications and elementary embedded implications. Moreover, under our reduction the depth of implication nesting does not increase.

As a first attempt to determine the complexity level of the whole \( \text{LL}(\otimes, \exists, !, ?, \bot, \bot) \), we have:

**Corollary 2.1** The derivability problem is decidable for the Multiplicative-Exponential Fragment of \( \bot \)-free Intuitionistic Linear Logic ILL (\( \otimes, \neg, \neg \)) consisting of sequents of the form \( \Sigma \vdash Z \) where \( \Sigma \) contains no embedded implications (unbounded nesting of the storage operator \( ! \) is allowed!).

### 3 Acyclic Programs with Stack

Acyclic programs with stack will be considered as computational counterparts of Linear Logic sequents.

From the computational point of view, when we intend to use an elementary embedded implication

\[ ((U \rightarrow V) \rightarrow Y), \]

before involving \( Y \) in the computational process, we should solve the subtask of producing \( V \) for the given \( U \). It is complicated additionally because we have to keep in mind the resource problems related to the current value: one part of it should
be suspended together with \( Y \), the rest should be incorporated in a solution of the foregoing subtask.

For these purposes we will use the standard stack operations push and pop [5] in a resource-fair manner:

(a) While pushing, we should indicate explicitly the part of the current value that will be involved in a further active computation, the remaining part is suspended in our stack. More precisely, the command \( \text{PUSH}(Y_1; X_2, Y_2) \) will mean: split the current value into two parts \( X_2 \) and, say \( X_1 \), add the value \( (X_1 \odot Y_1) \) to the top of our pushdown store, and place the value \( (X_2 \odot Y_2) \) as a new active input for a further computation.

(b) While popping, we should indicate explicitly that the desired result has been obtained in our active computation and, hence, the active memory can be cleaned up. Formally, the command \( \text{POP}(V) \) will mean: remove the topmost value \( Y \) from our pushdown store and place this \( Y \) as a new active input for a further computation, provided that the desired target \( V \) has been obtained at the current point.

Without loss of generality, we can confine ourselves to studying programs with the following peculiarities:

**Definition 3.1** An acyclic program with stack is a rooted binary tree such that

(a) Every edge of it is labelled either by a Horn implication of the form \( (X \multimap Y) \), or by a push command of the form \( \text{PUSH}(Y_1; X_2, Y_2) \), or by a pop command of the form \( \text{POP}(V) \).

(b) The root of the tree is specified as the input vertex. A vertex with no outgoing edges will be specified as an output one.

(c) A vertex \( v \) with exactly two outgoing edges \( (v, w_1) \) and \( (v, w_2) \) will be called divergent. These two outgoing edges should be labelled by Horn implications with one and the same antecedent, say \( (X \multimap Y_1) \) and \( (X \multimap Y_2) \), respectively.

(d) On each path \( b \) leading from the input vertex to an output vertex, the sequence of push’s and pop’s should be well-blocked: each of push’s has the unique partner pop.\(^4\)

Now, we should explain how such a program \( P \) runs for a given input \( W \).

\(^4\)A push-edge may have different pop-partners which must belong to different paths.

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**Definition 3.2** For a given program \( P \) and any simple product \( W \), a strong computation is defined by induction as follows: we assign a simple product \( \text{Value}[v] \) and a stack \( \text{Stack}[v] \) to each vertex \( v \) of \( P \) in such a way that

(a) For the input vertex \( v \), \( \text{Stack}[v] \) is empty and

\[
\text{Value}[v] = W.
\]

(b) For any vertex \( v \) and its son \( w \) with the edge \( (v, w) \) labelled by a Horn implication \( (X \multimap Y) \), if \( \text{Value}[v] \) is defined and, for some simple product \( V \):

\[
\text{Value}[v] \cong (X \otimes V),
\]

then

\[
\text{Value}[w] = (Y \otimes V)
\]

and

\[
\text{Stack}[w] = \text{Stack}[v].
\]

(c) For any edge \( (v, w) \) labelled by a push command of the form \( \text{PUSH}(Y_1; X_2, Y_2) \), if \( \text{Value}[v] \) is defined and, for some simple product \( X_1 \):

\[
\text{Value}[v] \cong (X_1 \otimes X_2),
\]

then \( \text{Stack}[w] \) is the result of pushing \( (X_1 \otimes Y_1) \) onto the \( \text{Stack}[v] \) and

\[
\text{Value}[w] = (X_2 \otimes Y_2).
\]

(d) For any edge \( (v, w) \) labelled by a pop command of the form \( \text{POP}(V) \), if

\[
\text{Value}[v] \cong V
\]

then \( \text{Stack}[w] \) is the result of popping a product \( Y \) from the top of the stack \( \text{Stack}[v] \) and

\[
\text{Value}[w] = Y.
\]

Otherwise, \( \text{Value}[w] \) is declared to be undefined.

**Definition 3.3** For a program \( P \) and a simple product \( W \), we say that

\[
P(W) = Z
\]

if and only if, for each output vertex \( v \) of \( P \), the stack \( \text{Stack}[v] \) is empty and

\[
\text{Value}[v] = Z.
\]

These definitions fall within the paradigm of Linear Logic, ensuring that
(a) the execution of a program does not allow for its operators to share their inputs,
(b) after the program has been executed, the push-down memory that was occupied by temporary and auxiliary objects is free.

We will describe each of our program constructs by Linear Logic formulas. Namely, we will associate a certain formula $A$ to each edge $e$ of a given program $P$, and say that

“This formula $A$ is used on the edge e.”

**Definition 3.4** Let $P$ be a one-stack acyclic program.

(a) Let $v$ be a non-divergent vertex of $P$ with an outgoing edge $e$ labelled by a Horn implication $A$. Then we will say that either

“Formula $A$ itself is used on the edge e.”

or

“Formula $(A \& B)$ is used on the edge e.”

or

“Formula $(B \& A)$ is used on the edge e.”

where $B$ is an arbitrary Horn implication.

(b) Let $v$ be a divergent vertex of $P$ with two outgoing edges $e_1$ and $e_2$ labelled by Horn implications $(X \rightarrow Y_1)$ and $(X \rightarrow Y_2)$, respectively, and let $A$ be the $\oplus$-Horn implication

$$(X \rightarrow (Y_1 \oplus Y_2)).$$

Then we will say that

“Formula $A$ is used on $e_1$."

and

“Formula $A$ is used on $e_2$."

(c) Let $v$ be a non-divergent vertex of $P$ with an outgoing edge $e$ labelled by a push command of the form $PUSH(Y_1; X_2, Y_2)$, and let $A$ be a formula of the form

$$(Y_2 \rightarrow V) \rightarrow Y_1).$$

We will say that

“Formula $A$ is used on the edge e.”

if each of pop-partners of our push-edge $e$ is labelled by a pop command of the form $POP(V)$.

**Definition 3.5** A one-stack acyclic program $P$ is said to be a strong solution to a sequent of the form

$$W_0, W_1, \ldots, W_n, \Delta, \Gamma \vdash Z$$

if

(a) $P((W_0 \otimes W_1 \otimes \cdots \otimes W_n)) = Z.$

(b) For every (non-pop) edge $e$ in $P$, the formula $A$ used on $e$ is drawn either from $\Gamma$ or from $\Delta$.

(c) Whatever path $b$ leading from the input vertex to an output vertex we take, each formula $A$ from $\Delta$ is used once and exactly once on this path $b$.

**Theorem 3.1** (Completeness) Let $\Gamma$ and $\Delta$ be multisets consisting of normalized formulas.

Any sequent of the form

$$W, \Delta, \Gamma \vdash Z$$

is derivable in Linear Logic if and only if one can construct a one-stack acyclic program $P$ which is a strong solution to the given sequent.

**Proof.** For a given strong solution $P$, running from its leaves to its root, we can assemble a derivation of our sequent.

In the other direction we can apply Theorem 4.1, Lemma 4.1, and Theorem 4.2. \hfill $\blacksquare$

4 The Main Encoding

Now we demonstrate how to encode normalized sequents into one-literal, $\perp$-only, and unit-only fragments of Linear Logic.

**Definition 4.1** We will use the abbreviation:

$$A^n = (A \otimes A \otimes \cdots \otimes A).$$

$n$ times

For $n = 0$, $A^0 = \perp$.

Dually, we will define:

$$A[\overline{n}] = (A \overline{\otimes} A \overline{\otimes} \cdots \overline{\otimes} A).$$

$n$ times

For $n = 0$, $A[\overline{0}] = \perp$.

**Definition 4.2** We define

$$A^{(n)} \rightarrow B$$

by induction:

$$A^{(0)} \rightarrow B = B,$$

$$A^{(n+1)} \rightarrow B = (A \rightarrow (A^{(n)} \rightarrow B)).$$

For a given integer $N$, let

$$p_1, p_2, \ldots, p_m, \ldots, p_{N-1}$$

be the list of all literals that will be used here and henceforth in Linear Logic formulas.
\[ \tilde{H}_0(p) = ((p^{(2)} - p) \circ (p^{(N+2)} - p)), \]
\[ \tilde{C}_0(p) = ((p^{(3)} - p) \circ ((p^{(3)} - p) \circ (\tilde{H}_0(p)^{(2)} - p) \circ p)), \]
\[ \tilde{H}_1(p) = ((p^{(N)} - p) \circ (\tilde{C}_0(p)^{(4)} - p)), \]
\[ H_{00} = ((\perp N + 2 - \perp 2)), \]
\[ C_{00} = ((H_{00}^2 - \perp 3) - \perp 3), \]
\[ H_1 = (C_{00}^4 - \perp N), \]
\[ \#_{\perp}(H_{00}) = -N, \]
\[ \#_{\perp}(C_{00}) = -2N, \]
\[ \#_{\perp}(H_1) = 9N = 0 \pmod{9N}, \]
\[ H_{01} = ([2] - \perp [N + 2]), \]
\[ C_{01} = ([3] - ([3] \otimes H_{01} \otimes H_{01})). \]

Table 2: The basic one-literal and constant-only formulas.

For literal \( p_m \), we set
\[
\tilde{D}_{p_m}(p) = ((p^{(m+4)} - p) \circ ((p^{(m+4)} - p) \circ (\tilde{H}_1(p) - p) \circ p)),
\]
\[ D_{p_m} = ((H_1 - \perp m + 4) - \perp m + 4), \]
\[ G_{p_m} = ([m+4] \otimes ([m+4] - \perp ([N] \otimes C_{01}^4))). \]

Let a simple product \( X \) be of the form
\[ X = (q_1 \otimes q_2 \otimes \cdots \otimes q_{n-1} \otimes q_n). \]
Then we set
\[ \tilde{G}_X(p) = (\tilde{D}_{q_1}(p) - (\tilde{D}_{q_2}(p) - \cdots (\tilde{D}_{q_{n-1}}(p) - \circ (\tilde{D}_{q_n}(p) - \circ p)) \cdots)), \]
\[ D_X = (D_{q_1} \otimes D_{q_2} \otimes \cdots \otimes D_{q_n}), \]
\[ G_X = (G_{q_1} \otimes G_{q_2} \otimes \cdots \otimes G_{q_n}). \]

Table 3: The encoding of literals and tensor products.
\[\widetilde{E}_X(p) = (\widetilde{C}_0(p)(6) \rightarrow \widetilde{G}_X(p)),\]
\[\widetilde{F}_Y(p) = \widetilde{E}_Y(p) = (\widetilde{E}_Y(p) \rightarrow \widetilde{E}_Y(p) \rightarrow \widetilde{E}_Y(p))\]
\[\widetilde{F}_Y(p) = (\widetilde{E}_Y(p) \rightarrow \widetilde{E}_Y(p)).\]

\[\widetilde{F}_Y((U \rightarrow V) \rightarrow Y)(p) = ((\widetilde{F}_Y(p) \rightarrow p) \rightarrow (\widetilde{F}_Y(U \rightarrow V)(p) \rightarrow p)),\]
\[\widetilde{F}_Y(X \rightarrow (Y_1 \oplus Y_2))(p) = ((\widetilde{E}_Y(p \otimes Y_1)(p) \& \widetilde{E}_Y(p \otimes Y_2)(p)) \rightarrow \widetilde{E}_Y(p \otimes X)(p)),\]
\[\widetilde{F}_Y((X_1 \rightarrow Y_1) \& (X_2 \rightarrow Y_2))(p) = (\widetilde{F}_Y(X_1 \rightarrow Y_1)(p) \& \widetilde{F}_Y(X_2 \rightarrow Y_2)(p)),\]

\[E_X = (C_{00}^6 \otimes D_X),\]
\[F(Y) = (E(p \otimes Y) \rightarrow E(p \otimes Y)),\]
\[F(U \rightarrow V) = (F(U \rightarrow V) \rightarrow F_Y),\]
\[F(X \rightarrow (Y_1 \oplus Y_2)) = (E(p \otimes X) \rightarrow (E(p \otimes Y_1) \oplus E(p \otimes Y_2))),\]
\[F((X_1 \rightarrow Y_1) \& (X_2 \rightarrow Y_2)) = (F(X_1 \rightarrow Y_1) \& F(X_2 \rightarrow Y_2)),\]

\[E^1_X = (C_{00}^6 \rightarrow G_X),\]
\[F^1(Y) = (E^1_Y(p \otimes Y) \rightarrow E^1_Y(p \otimes X)),\]
\[F^1(U \rightarrow V) = (F^1(U \rightarrow V) \rightarrow F^1_Y),\]
\[F^1(X \rightarrow (Y_1 \oplus Y_2)) = ((E^1_Y(p \otimes Y_1) \& E^1_Y(p \otimes Y_2)) \rightarrow E^1_Y(p \otimes X)),\]
\[F^1((X_1 \rightarrow Y_1) \& (X_2 \rightarrow Y_2)) = (F^1(X_1 \rightarrow Y_1) \& F^1(X_2 \rightarrow Y_2)).\]

Table 4: Encoding \textbf{ILL}(\otimes, \rightarrow, \oplus, \& \&) into the one-literal, $\bot$-only, and unit-only fragments of Linear Logic.
In particular, we assume that this list includes a certain literal $p$. This leading literal $p$ will be involved in our encodings only for a more reasonable representation of embedded implications

$$((U \rightarrow V) \rightarrow Y)$$

by embedded implications with non-empty antecedents:

$$(((p \otimes U) \rightarrow (p \otimes V)) \rightarrow (p \rightarrow (p \otimes Y))).$$

First of all, we specify certain one-literal, $\bot$-only, and unit-only formulas by Table 2.

**Definition 4.3** We will encode each simple tensor product $X$ by one-literal, $\bot$-only, and unit-only formulas $\bar{G}_X(p), D_X,$ and $G_X,$ respectively. (See Table 3)

According to Table 4, we encode each normalized formula $A$ by one-literal, $\bot$-only, and unit-only formulas $\bar{F}_A(p), F_A,$ and $F_A^\bot,$ respectively.

Let $\Gamma$ be a multiset consisting of normalized formulas. By $\bar{F}_\Gamma(p), F_\Gamma,$ and $F_\Gamma^\bot,$ we will denote multisets that are obtained from $\Gamma$ by replacing each formula $A$ with formulas $\bar{F}_A(p), F_A,$ and $F_A^\bot,$ respectively.

**Lemma 4.1** For any normalized formula $A$, formulas $F_A, \bar{F}_A(\bot)$, and $F_A^\bot$ are equivalent pairwise in Linear Logic.

As a corollary, the following three sentences are equivalent:

(i) An auxiliary $\bot$-only sequent of the form

$$E(p \otimes W), \bar{F}_\Delta, \neg F_\Gamma \vdash E(p \otimes Z)$$

is derivable in Linear Logic.

(ii) A $\bot$-only sequent of the form

$$\bar{E}(p \otimes Z)(\bot), \bar{F}_\Delta(\bot), \neg F_\Gamma(\bot) \vdash \bar{E}(p \otimes W)(\bot)$$

is derivable in Linear Logic, as well.

(iii) The unit-only sequent

$$E_1^1(p \otimes Z), \bar{F}_\Delta^1, \neg F_\Gamma^1 \vdash E_1^1(p \otimes W)$$

is also derivable in Linear Logic.

**Theorem 4.1** Let $\Gamma$ and $\Delta$ be multisets consisting of normalized formulas.

If a sequent of the form

$$W, \Delta, \neg \Gamma \vdash Z$$

is derivable in Linear Logic then the following three sequents, the one-literal sequent

$$\bar{E}(p \otimes Z)(p), \bar{F}_\Delta(p), \neg F_\Gamma(p) \vdash \bar{E}(p \otimes W)(p),$$

the $\bot$-only sequent

$$\bar{E}(p \otimes Z)(\bot), \bar{F}_\Delta(\bot), \neg F_\Gamma(\bot) \vdash \bar{E}(p \otimes W)(\bot),$$

and the unit-only sequent

$$E_1^1(p \otimes Z), \bar{F}_\Delta^1, \neg F_\Gamma^1 \vdash E_1^1(p \otimes W)$$

are also derivable in Linear Logic.

**Proof.** By induction on derivations. \[\blacksquare\]

Now we should prove the fairness of our encodings. We will kill three birds (one-literal, $\bot$-only, and unit-only ones) with one stone.

Namely, we will prove that all derivations of an auxiliary $\bot$-only sequent of the form

$$E(p \otimes W), \bar{F}_\Delta, \neg F_\Gamma \vdash E(p \otimes Z)$$

must be regular: Due to the following key technical lemmas, any derivation cannot develop in undesired directions. Let us demonstrate the crucial point of our construction:

**Lemma 4.2** Let $\Gamma$ and $\Delta$ be multisets consisting of normalized formulas.

Let a sequent of the form

$$(C_{00}^6 \otimes D_W), F_\Gamma, \neg F_\Delta \vdash (C_{00}^6 \otimes D_Z)$$

be derivable in Linear Logic, and let the last step in some cut-free derivation of it be performed according to rule $R_\otimes$.

Then, as a matter of fact, we meet a trivial axiom situation:

This multiset $\Gamma$ must be empty, $\neg F_\Delta$ can be produced by rules $W!$ and $C!$ only (there is no applications of rule $L!$ in the given derivation), and, moreover,

$$W \simeq Z.$$  

**Proof.** See Case of an Axiom in the proof of Theorem 4.2 below. \[\blacksquare\]

The detailed proof of Lemma 4.2 involves a huge number of technical lemmas related to derivations of specific sequents. All this technical stuff is contained in section 5.

In our proof we exploit the well-known idea that all derivable sequents should be well-balanced.
Definition 4.4 The total number \(#_\perp(A)\) of positive and negative occurrences of \(\perp\) in \(A\) is defined as follows:
\[
\begin{align*}
#_\perp(q) &= 0, \text{for every literal } q, \\
#_\perp(\perp) &= 1, \\
#_\perp(\bot) &= 0.
\end{align*}
\]
For any formulas \(A\) and \(B\),
\[
\begin{align*}
#_\perp(A \otimes B) &= #_\perp(A) + #_\perp(B), \\
#_\perp(A \otimes B) &= #_\perp(A) - #_\perp(B), \\
#_\perp(A \& B) &= \min\{#_\perp(A), #_\perp(B)\}, \\
#_\perp(A \oplus B) &= \max\{#_\perp(A), #_\perp(B)\}.
\end{align*}
\]
Lemma 4.3 For basic \(\perp\)-only formulas we have:
\[
\begin{align*}
#_\perp(H_0) &= -N, \\
#_\perp(C_0) &= -2N, \\
#_\perp(H_1) &= 9N = 0 \pmod{9N}.
\end{align*}
\]
For any simple products \(X, Y_1,\) and \(Y_2:\)
\[
\begin{align*}
#_\perp(D_X) &= 0 \pmod{9N}, \\
#_\perp(E_X) &= 6N \pmod{9N}, \\
#_\perp(E_{Y_1} \oplus E_{Y_2}) &= 6N \pmod{9N}.
\end{align*}
\]
For any simple product \(Y:\)
\[
#_\perp(F_Y) = 0 \pmod{9N}.
\]
For any normalized formula \(A:\)
\[
#_\perp(F_A) = 0 \pmod{9N}.
\]
Lemma 4.4 Let \(\Gamma\) and \(\Delta\) be multisets consisting of normalized formulas, and let \(A_1, A_2, \ldots, A_k\) and \(B_1, B_2, \ldots, B_m\) be formulas built up of constant \(\perp\) by connectives from the set \(\{\otimes, -\}\). In addition, some of \(A_i\) is allowed to be of the form \((E_{Y_1} \oplus E_{Y_2})\).

If a sequent of the form
\[
\Gamma, A_1, A_2, \ldots, A_k, F_\Gamma, !F_\Delta \vdash B_1, B_2, \ldots, B_m
\]
is derivable in Linear Logic then the following holds:
\[
\sum_{i=1}^{k} #_\perp(A_i) = 1 - m + \sum_{j=1}^{m} #_\perp(B_j) \pmod{9N}.
\]
In particular, for the empty right-hand side \((m = 0):\)
\[
\sum_{i=1}^{k} #_\perp(A_i) = 1 \pmod{9N}.
\]

Proof. By induction on cut-free derivations.

The key fairness theorem is as follows:

Theorem 4.2 Let \(\Gamma\) and \(\Delta\) be multisets consisting of normalized formulas that do not contain a certain literal \(p\).

Let all simple products
\[
T_1, T_2, \ldots, T_n, Z
\]
do not contain this flat literal \(p\), either.

Let \(K\) be a multiset of the form
\[
K = T_1, T_2, \ldots, T_n.
\]

(a) If a \(\perp\)-only sequent of the form
\[
E_{W'}, F_K, F_\Delta, !F_\Gamma \vdash E_p \otimes Z
\]
is derivable in Linear Logic then
\[
W' \equiv (p \otimes W),
\]

(a2) one can construct a one-stack program \(P\)

that is a strong solution to the original sequent

\[
W, K, \Delta, \Gamma \vdash Z.
\]

(b) For the case of the ‘empty’ \(Z:\)

If a \(\perp\)-only sequent of the form
\[
E_{W'}, F_K, F_\Delta, !F_\Gamma \vdash E_p
\]
is derivable in Linear Logic then
\[
W' = p,
\]

(b2) both multisets \(K\) and \(\Delta\) must be empty, and \(!F_\Gamma\) must be degenerate. \(!F_\Gamma\) can be produced by rules \(W!\) and \(C!\) only (there is no applications of rule \(L!\) in the derivation above this sequent).

Proof. We assemble the desired program \(P\) by induction on a given derivation in Linear Logic.

First of all, regarding to the form of the principal formula at a current point of the derivation, we demonstrate inconsistency of the following undesirable cases.

Assume that the principal formula belongs to the left-hand side, and it is of the form
\[
D_q = ((H_1 \circ \perp b) \circ \perp b)
\]
and, according to rule $\textbf{L}\neg\neg$, our sequent is derived from two sequents of the form
\[ C_{00}^{k1}, D^T, F_K, F_{\Delta_1}, 1F_{\Gamma_1} \vdash (H_1 \rightarrow \bot^b) \]
and
\[ C_{00}^{k2}, D_{W_2}, F_{K_2}, \bot^b, F_{\Delta_2}, 1F_{\Gamma_2} \vdash E(p \otimes Z) \]
where
\[
\begin{align*}
6 &= k_1 + k_2, \\
W' &= (g \otimes T' \otimes W_2), \\
K &= K_1, K_2, \\
\Delta &= \Delta_1, \Delta_2, \\
\Gamma &= \Gamma_1, \Gamma_2.
\end{align*}
\]
Then Lemma 4.4 and Lemma 4.3 yield a contradiction:
\[
\begin{align*}
-2Nk_1 &= b \pmod {9N}, \\
-2Nk_2 + b &= 6N \pmod {9N}.
\end{align*}
\]
If our sequent were derived from two sequents of the form
\[ C_{00}^{k1}, D^T, F_K, F_{\Delta_1}, 1F_{\Gamma_1} \vdash (H_1 \rightarrow \bot^b), E(p \otimes Z) \]
and
\[ C_{00}^{k2}, D_{W_2}, F_{K_2}, \bot^b, F_{\Delta_2}, 1F_{\Gamma_2} \vdash \]
then we had a contradiction as well:
\[
\begin{align*}
-2Nk_1 &= b + 6N - 1 \pmod {9N}, \\
-2Nk_2 + b &= 1 \pmod {9N}.
\end{align*}
\]
Assume that the left-hand principal formula is of the form
\[ C_{00} = (H_{00}^2 \rightarrow \bot^3) \neg \bot^3, \]
and, according to rule $\textbf{L}\neg\neg$, our sequent is derived from two sequents of the form
\[ C_{00}^{k1}, D^T', F_K, F_{\Delta_1}, 1F_{\Gamma_1} \vdash (H_{00}^{2} \rightarrow \bot^3) \]
and
\[ C_{00}^{k2}, D_{W_2}, F_{K_2}, \bot^3, F_{\Delta_2}, 1F_{\Gamma_2} \vdash E(p \otimes Z) \]
where
\[
\begin{align*}
5 &= k_1 + k_2, \\
W' &= (T' \otimes W_2), \\
K &= K_1, K_2, \\
\Delta &= \Delta_1, \Delta_2, \\
\Gamma &= \Gamma_1, \Gamma_2.
\end{align*}
\]
Then, by Lemma 4.4 and Lemma 4.3, the following contradiction is immediate:
\[
\begin{align*}
-2Nk_1 &= 2N + 3 \pmod {9N}, \\
-2Nk_2 + 3 &= 6N \pmod {9N}.
\end{align*}
\]
If our sequent were derived from two sequents of the form
\[ C_{00}^{k1}, D^T, F_K, F_{\Delta_1}, 1F_{\Gamma_1} \vdash (H_{00}^2 \rightarrow \bot^3), E(p \otimes Z) \]
and
\[ C_{00}^{k2}, D_{W_2}, F_{K_2}, \bot^3, F_{\Delta_2}, 1F_{\Gamma_2} \vdash \]
then we had also a contradiction:
\[
\begin{align*}
-2Nk_1 &= 8N + 2 \pmod {9N}, \\
-2Nk_2 + 3 &= 1 \pmod {9N}.
\end{align*}
\]
Thus Lemma 4.4 and Lemma 4.3 contract eventually the set of all possible cases to the following set. **Case of a formula from $F_K$.** Suppose that the principal formula is from $F_K$, and it is of the form
\[ (E_p \neg E_{(p \otimes Y)}), \]
and, according to rule $\textbf{L}\neg\neg$, our sequent is derived from two sequents of the form
\[ C_{00}^{k1}, D^T, F_K, F_{\Delta_1}, 1F_{\Gamma_1} \vdash E_p \]
and
\[ C_{00}^{k2}, D_{W_2}, F_{K_2}, E(p \otimes Y), F_{\Delta_2}, 1F_{\Gamma_2} \vdash E(p \otimes Z) \]
where
\[
\begin{align*}
6 &= k_1 + k_2, \\
W' &= (T' \otimes W_2), \\
K &= K_1, K_2, (E_p \neg E_{(p \otimes Y)}), \\
\Delta &= \Delta_1, \Delta_2, \\
\Gamma &= \Gamma_1, \Gamma_2.
\end{align*}
\]
According to Lemma 4.4 and Lemma 4.3:
\[
\begin{align*}
-2Nk_1 &= 6N \pmod {9N}, \\
-2Nk_2 + 6N &= 6N \pmod {9N},
\end{align*}
\]
and, hence,
\[
\begin{align*}
k_1 &= 6, \\
k_2 &= 0.
\end{align*}
\]
In the case of item (b) we have a contradiction that, by the inductive hypothesis:
\[ (W_2 \otimes p \otimes Y) = p \]
for non-empty $Y$.

The case of item (a) is handled in the following way.

By applying the inductive hypothesis from item (b), we have:
Let us note that if, according to rule $L$ to the sequent

$$P(2)$$

there exists a one-stack program $P$ that is a strong solution to the sequent

$$(W_2 \otimes Y), K_2, \Delta_2, \not\Gamma_2 \vdash Z.$$  

Just the same program $P$ will be also a strong solution to the sequent

$$W_2, K, \Delta, \not\Gamma \vdash Z.$$  

Let us note that if, according to rule $L \rightarrow o$, our sequent were derived from two sequents of the form

$$C_{k1}^{k1}, D_{T''}, F_{K_1}, F_{\Delta_1}, !F_{\Gamma_1} \vdash E_{p \otimes Z}$$

and

$$C_{00}^{k2}, D_{W_2}, F_{K_2}, E_{p \otimes Y}, !F_{\Delta_2}, !F_{\Gamma_2} \vdash$$

then we got the following contradiction:

$$\begin{cases} 
-2Nk_1 &= 12N - 1 \quad \text{(mod } 9N), \\
-2Nk_2 + 6N &= 1 \quad \text{(mod } 9N).
\end{cases}$$

**Case of a Horn implication.** Our sequent is of the form

$$E_{W''}, F_{K}, F_{(X \rightarrow o Y)}, F_{\Delta}, !F_{\Gamma} \vdash E_{p \otimes Z}$$

and, according to rule $L \rightarrow o$, it is derived from the following two sequents:

$$C_{00}^{k1}, D_{T''}, F_{K_1}, E_{p \otimes X}, !F_{\Gamma_1} \vdash E_{p \otimes Z}$$

and

$$C_{00}^{k2}, D_{W_2}, F_{K_2}, E_{p \otimes Y}, F_{\Delta_2}, !F_{\Gamma_2} \vdash E_{p \otimes Z}$$

According to Lemma 4.4 and Lemma 4.3:

$$\begin{cases} 
6 &= k_1 + k_2, \\
W' &= (T' \otimes W_2), \\
K &= K_1, K_2, \\
\Delta &= \Delta_1, \Delta_2, \\
\Gamma &= \Gamma_1, \Gamma_2.
\end{cases}$$

and, hence,

$$\begin{cases} 
k_1 &= 6, \\
k_2 &= 0.
\end{cases}$$

In the case of item (b) we have a contradiction that, by the inductive hypothesis:

$$(W_2 \otimes p \otimes Y) = p$$

for non-empty $Y$.

For the case of item (a), both these sequents can be rewritten as

$$E_{T''}, F_{K_1}, F_{\Delta_1}, !F_{\Gamma_1} \vdash E_{p \otimes X}$$

and

$$E_{(W_2 \otimes p \otimes Y)}, F_{K_2}, F_{\Delta_2}, !F_{\Gamma_2} \vdash E_{p \otimes Z},$$

respectively.
By the inductive hypothesis, for some $W_1$:
\[ T' \cong (p \otimes W_1), \]
and $(W_1 \otimes W_2)$ does not contain literal $p$.

According to the inductive hypothesis, suppose that $P_1$ is a strong solution to a sequent of the form
\[ W_1, K_1, \Delta_1, \Gamma_1 \vdash X, \]
and $P_2$ is a strong solution to a sequent of the form
\[ (W_2 \otimes Y), K_2, \Delta_2, \Gamma_2 \vdash Z. \]

Now a program $P$ is assembled by the following (see Figure 1):

(a) For each output vertex, say $v_1$, of program $P_1$, we connect this vertex $v_1$ with the root $v_2$ of $P_2$ by a new edge $(v_1, v_2)$ and label this edge by the Horn implication $(X \rightarrow Y)$.

It is easily verified that our program $P$ is a strong solution to the sequent
\[ (W_1 \otimes W_2), K, (X \rightarrow Y), \Delta, \Gamma \vdash Z. \]

If our sequent were derived from two sequents of the form
\[ C_{k_1}^{p_1}, D_{T'}, F_{K_1}, F_{\Delta_1}, \Gamma_1 \vdash E(p \otimes X), E(p \otimes Z) \]
and
\[ C_{k_2}^{p_2}, D_{W_2}, F_{K_2}, E(p \otimes Y), F_{\Delta_2}, \Gamma_2 \vdash \]
then, by Lemma 4.4 and Lemma 4.3, we had a contradiction:
\[
\begin{cases}
-2Nk_1 &= 12N - 1 \pmod{9N}, \\
-2Nk_2 + 6N &= 1 \pmod{9N}.
\end{cases}
\]

Case of an &-Horn implication. Our sequent is of the form
\[ E_{W'}, K, F((X_1 \rightarrow Y_1) \& (X_2 \rightarrow Y_2)), F_{\Delta}, \Gamma \vdash E(p \otimes Z) \]
and it is derived either from the sequent
\[ E_{W'}, K, F(X_1 \rightarrow Y_1), F_{\Delta}, \Gamma \vdash E(p \otimes Z) \]
or from the sequent
\[ E_{W'}, K, F(X_2 \rightarrow Y_2), F_{\Delta}, \Gamma \vdash E(p \otimes Z) \]
In item (a) by the inductive hypothesis, for some $W$: $W' \cong (p \otimes W)$, and we have a program $P$ that is a strong solution $\vdash$ one of the following sequents:
\[ W, K, \Delta, (X_1 \rightarrow Y_1), \Gamma \vdash Z \]
or
\[ W, K, \Delta, (X_2 \rightarrow Y_2), \Gamma \vdash Z. \]
This $P$ will be also a strong solution to the sequent
\[ W, K, ((X_1 \rightarrow Y_1) \& (X_2 \rightarrow Y_2)), \Delta, \Gamma \vdash Z. \]
For the case of item (b) we have a contradiction that, by the inductive hypothesis, one of these non-empty multisets
\[ \Delta, (X_1 \rightarrow Y_1) \]
and
\[ \Delta, (X_2 \rightarrow Y_2), \]
must be empty.

Case of an embedded implication. Our sequent is of the form
\[ E_{W'}, F_K, F((U \rightarrow V) \rightarrow Y), F_{\Delta}, \Gamma \vdash E(p \otimes Z) \]
and it is derived from the following two sequents:
\[ C_{k_1}^{p_1}, D_{T'}, F_{K_1}, F_{Y}, F_{\Delta_1}, \Gamma_1 \vdash E(p \otimes Z) \]
and
\[ C_{k_2}^{p_2}, D_{W_2}, F_{K_2}, F_{\Delta_2}, \Gamma_2 \vdash F(U \rightarrow V) \]
where
\[
\begin{cases}
6 &= k_1 + k_2, \\
W' &= (T' \otimes W_2), \\
K &= K_1, K_2, \\
\Delta &= \Delta_1, \Delta_2, \\
\Gamma &= \Gamma_1, \Gamma_2.
\end{cases}
\]

By Lemma 4.4 and Lemma 4.3, we have
\[
\begin{cases}
-2Nk_1 &= 6N \pmod{9N}, \\
-2Nk_2 &= 0 \pmod{9N},
\end{cases}
\]
and
\[ \begin{cases}
k_1 &= 6, \\
k_2 &= 0.
\end{cases} \]

In the case of item (b) we have a contradiction that, by the inductive hypothesis, the non-empty multiset
\[ F_Y, F_{\Delta_1} \]
must be empty.
For item (a), our sequent can be derived from the following two sequents:

\[ E_{T'}, F_{K_1}, F_Y, F_{\Delta_1}, !F_{\Gamma_1} \vdash E(p \otimes Z) \]

and

\[ E(W_2 \otimes p \otimes U), F_{K_2}, F_{\Delta_2}, !F_{\Gamma_2} \vdash E(p \otimes V). \]

By the inductive hypothesis, for some \( W_1 \):

\[ T' \equiv (p \otimes W_1), \]

and \((W_1 \otimes W_2)\) does not contain literal \( p \).

According to the inductive hypothesis, suppose that \( P_1 \) and \( P_2 \) are strong solutions to sequents of the form

\[ W_1, Y, K_1, \Delta_1, !\Gamma_1 \vdash Z, \]

and

\[ (W_2 \otimes U), K_2, \Delta_2, !\Gamma_2 \vdash V, \]

respectively.

Let us set

\[
\begin{cases}
  X_1 = (W_1 \otimes K_1), \\
  X_2 = (W_2 \otimes K_2).
\end{cases}
\]

Now a one-stack program \( P \) can be assembled as follows (see Figure 2):

(a) First, we create a new input vertex \( v_0 \).

(b) After that, we connect this input vertex \( v_0 \) with the root \( v_1 \) of \( P_2 \) by a new edge \((v_0, v_1)\) and label this edge by the push operation \( PUSH(Y; X_2, U) \).

(c) Finally, we connect each output vertex \( w_k \) of program \( P_2 \) with the root \( t_k \) of \( k \)-th copy of program \( P_1 \) by a new edge \((w_k, t_k)\) and label this edge by the pop operation \( POP(V) \).

We can verify that our program \( P \) is a strong solution to the sequent

\[ (W_1 \otimes W_2), K, ((U \circ V) \circ Y), \Delta, !\Gamma \vdash Z. \]

If our sequent were derived from two sequents of the form

\[ C_{k_1}^{k_1}, D_{T'}, F_{K_1}, F_Y, F_{\Delta_1}, !F_{\Gamma_1} \vdash \]

and

\[ C_{k_2}^{k_2}, D_{W_2}, F_{K_2}, F_{\Delta_2}, !F_{\Gamma} \vdash F(U \circ V), E(p \otimes Z) \]
Figure 3: Strong Forking.
then we got an immediate contradiction:

\[
\begin{aligned}
-2Nk_1 &= 1 \pmod{9N}, \\
-2Nk_2 &= 6N - 1 \pmod{9N}.
\end{aligned}
\]

**Case of a \( \oplus \)-Horn implication.** Our sequent is of the form

\[E_{W'}, F_{K}, F_{(X \circ (Y_1 \oplus Y_2))}, F_{\Delta}, !F_{\Gamma} \vdash E(p \otimes X)\]

and, taking into account Lemma 4.4 and Lemma 4.3, it is derived from the following three sequents:

\[E_{(W_2 \otimes p \otimes Y_1)}, F_{K_2}, F_{\Delta_2}, !F_{\Gamma_2} \vdash E(p \otimes Z),\]

and

\[E_{W_2 \otimes p \otimes Y_2}, F_{K_2}, F_{\Delta_2}, !F_{\Gamma_2} \vdash E(p \otimes Z),\]

where

\[
\begin{aligned}
W' &= (T' \otimes W_2), \\
K &= K_1, K_2, \\
\Delta &= \Delta_1, \Delta_2, \\
\Gamma &= \Gamma_1, \Gamma_2.
\end{aligned}
\]

In the case of item (b) we have a contradiction that, by the inductive hypothesis:

\[(W_2 \otimes p \otimes Y_1) = p\]

for non-empty \( Y_1 \).

For item (a), by the inductive hypothesis, for some \( W_1 \):

\[T' \cong (p \otimes W_1),\]

and \((W_1 \otimes W_2)\) does not contain literal \( p \).

Suppose that \( P_0 \) is a strong solution to a sequent of the form

\[W_1, K_1, \Delta_1, \! \! \Gamma_1 \vdash X,\]

and \( P_1 \) and \( P_2 \) are strong solutions to sequents of the form

\[(Y_1 \otimes W_2), K_2, \Delta_2, \! \! \Gamma_2 \vdash Z\]

and

\[(Y_2 \otimes W_2), K_2, \Delta_2, \! \! \Gamma_2 \vdash Z,\]

respectively.

Now a program \( P \) can be assembled by the following **Strong Forking** (see Figure 3):

(a) For each output vertex, say \( v_0 \), of program \( P_0 \), we connect this vertex \( v_0 \) with the root \( v_1 \) of \( P_1 \) by a new edge \((v_0, v_1)\) and label this edge by the Horn implication \((X \circ Y_1)\).

(b) In its turn, we connect this vertex \( v_0 \) with the root \( v_2 \) of \( P_2 \) by a new edge \((v_0, v_2)\) and label this edge by the Horn implication \((X \circ Y_2)\).

It is easily verified that our program \( P \) is a strong solution to the sequent

\[(W_1 \otimes W_2), K, (X \circ (Y_1 \oplus Y_2)), \Delta, \! \! \Gamma \vdash Z.\]

**Case of a formula from \( !F_{\Gamma} \).** Suppose that the principal formula belongs to \( !F_{\Gamma} \), and it is of the form

\[!F_A.\]

Assume that it is produced by rule \( \text{L!} \), and our sequent is derived from a sequent of the form

\[E_{W'}, F_{K}, F_{\Delta}, F_A, !F_{\Gamma'} \vdash E(p \otimes Z).\]

Then item (a) can be completed by the inductive hypothesis. As for item (b), in this case we have a contradiction that the non-empty multiset

\[\Delta, A\]

must be empty.

The remaining cases of rules \( \text{W!} \) and \( \text{C!} \) are readily completed by the inductive hypothesis.

![Figure 4: The Axiom Case.](image-url)
According to Lemma 5.2, we have:

\[
\begin{align*}
W_1 & \text{ must be empty,} \\
K_1 & \text{ must be empty,} \\
\Delta_1 & \text{ must be empty,} \\
!F_{\Gamma_1} & \text{ must be degenerate.}
\end{align*}
\]

Lemma 5.4 demonstrates that:

\[
\begin{align*}
T' & \cong (p \otimes Z), \\
K_2 & \text{ must be empty,} \\
\Delta_2 & \text{ must be empty,} \\
!F_{\Gamma_2} & \text{ must be degenerate.}
\end{align*}
\]

Hence, for item (a) we can conclude that

(a) \( W' \cong (p \otimes Z) \),

(b) and the most trivial program \( P \) consisting of single vertex (see Figure 4) will be a strong solution to the corresponding sequent

\[
Z, K, \Delta, !\Gamma \vdash Z.
\]

In the case of item (b) we have the desired:

(a) \( W' = p \),

(b) and the whole multisets \( K \) and \( \Delta \) are empty, and the whole \( !F_{\Gamma} \) is degenerate.

Finally, bringing together all the cases considered, we can complete the proof of Theorem 4.2.

\[\square\]

**Corollary 4.1** Let \( \Gamma \) and \( \Delta \) be multisets consisting of normalized formulas.
Let literal \( p \) do not occur in a sequent of the form

\[
W, \Delta, !\Gamma \vdash Z.
\]

Then the sequent

\[
W, \Delta, !\Gamma \vdash Z
\]

is derivable in Linear Logic if and only if the auxiliary \( \bot \)-only sequent

\[
E(p \otimes W), F_{\Delta}, !F_{\Gamma} \vdash E(p \otimes Z)
\]

is also derivable in Linear Logic.

**Proof.** The first implication (from the left to the right) can be proved by induction on derivations.

In the other direction, by applying Theorem 4.2, we construct a strong solution \( P \) to the sequent

\[
W, \Delta, !\Gamma \vdash Z.
\]

After that, for such a program \( P \), running from its leaves to its root, we assemble a derivation of this sequent.

\[\square\]

**Corollary 4.2 (Fairness)** Let \( \Gamma \) and \( \Delta \) be multisets consisting of normalized formulas.

Let literal \( p \) do not occur in a sequent of the form

\[
W, \Delta, !\Gamma \vdash Z.
\]

The following sentences are equivalent pairwise:

(a) A sequent of the form

\[
W', \Delta, !\Gamma \vdash Z
\]

is derivable in Linear Logic.

(b) The one-literal sequent

\[
E(p \otimes Z), F_{\Delta}(p), !F_{\Gamma}(p) \vdash E(p \otimes W)(p)
\]

is also derivable in Linear Logic.

(c) A \( \bot \)-only sequent of the form

\[
E(p \otimes Z)(\bot), F_{\Delta}(\bot), !F_{\Gamma}(\bot) \vdash E(p \otimes W)(\bot)
\]

is derivable in Linear Logic, as well.

(d) The following unit-only sequent

\[
E_1(p \otimes Z), F_1_{\Delta}, !F_1_{\Gamma} \vdash E_1(p \otimes W)
\]

is derivable in Linear Logic.

**Proof.** One direction is provided by Theorem 4.1. The most complicated implications are provided by Corollary 4.1 and Lemma 4.1.

**Remark.** In our proof we use also the fact that the derivable sequents in question must be well-balanced with respect to the leading literal \( p \) as well. In fact, we need this leading literal \( p \) only for simulating embedded implications

\[
((U \rightarrow V) \rightarrow Y)
\]

by embedded implications with non-empty antecedents:

\[
(((p \otimes U) \rightarrow (p \otimes V)) \rightarrow (p \rightarrow (p \otimes Y))).
\]

**Corollary 4.3** Theorems 1.1, 1.2, and 1.3 are valid.

**Proof.** According to Corollary 4.2, we can apply all complexity constructions from [48] and Theorem 2.1.

\[\square\]
5 The Proof of Key Technical Lemmas

5.1 Lemma 5.1

Lemma 5.1 Let \( a \) be an integer such that

\[ 1 \leq a \leq N + 2. \]

Let \( \Delta \) consist of formulas of the form \( F_A \), and \( \Gamma \) consist of formulas of the form \( H, D_X, F_A \), and \( F_Y \).

(a) Let \( B \) be a formula either of the form \( \bot^a \), or of the form \( (H_1 \rightarrow \bot^a) \), or of the form \( C_{00}^4 \).

If a sequent of the form

\[ \bot^a, \Gamma, !\Delta \vdash B \]

occurs in a cut-free derivation in Linear Logic then this formula \( B \) must be exactly equal to \( \bot^a \), multiset \( \Gamma \) must be empty, and \( !\Delta \) can be produced by rules \( W \) and \( C \) only (there is no applications of rule \( L \) in the derivation above this sequent).

(b) For the case of the 'empty' formula \( B \):

If a sequent of the form

\[ \bot, \Gamma, !\Delta \vdash \]

occurs in a cut-free derivation in Linear Logic then \( \Gamma \) must be empty, and \( !\Delta \) can be produced by rules \( W \) and \( C \) only.

(c) Let \( B \) be a formula of the form \( C_{00}^5 \).

Any sequent of the form

\[ C_{00}, \bot^a, \Gamma, !\Delta \vdash B \]

does not occur in any derivations in Linear Logic.

Proof. We use induction on a given derivation. Regarding to the form of the principal formula at a current point of the derivation, we will demonstrate that each of the undesirable cases is inconsistent.

Case 0 The principal formula belongs to \( !\Delta \).

Assume that it is produced by rule \( L! \), and our sequent is derived from a sequent of the form

\[ \bot^a, \Gamma, F_A, !\Delta' \vdash B. \]

Then, by the inductive hypothesis, the multiset

\[ \Gamma, F_A \]

must be empty, which is a contradiction.

---

Hence, the only possibility is to apply either \( W^q \) or \( C! \). It remains to use the inductive hypothesis for completing this case.

Item (c) is handled similarly.

Case 1 Formula \( B \) is principal.

There are the following subcases to be considered.

Case 1.1 Formula \( B \) is of the form \( \bot^a \), and, according to rule \( R\circ \), our sequent is derived from two sequents of the form

\[ \bot^{a_1}, \Gamma_1, !\Delta_1 \vdash \]

and

\[ \bot^{a_2}, \Gamma_2, !\Delta_2 \vdash \bot^{a - 1} \]

where

\[
\begin{align*}
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

By Lemma 4.4 we have

\[
\begin{align*}
a_1 &= 1 \pmod{9N}, \\
a_2 &= a - 1 \pmod{9N},
\end{align*}
\]

and, therefore,

\[
\begin{align*}
a_1 &= 1, \\
a_2 &= a - 1.
\end{align*}
\]

By applying the inductive hypothesis to both sequents, we get the emptiness of both \( \Gamma_1 \) and \( \Gamma_2 \), and the degeneracy of both \( !\Delta_1 \) and \( !\Delta_2 \), which results in the desired emptiness of the whole \( \Gamma \) and the degeneracy of the whole \( !\Delta \).

Case 1.2 Assume that formula \( B \) is of the form \( (H_1 \rightarrow \bot^a) \), and, by rule \( R\circ \), our sequent is derived from the sequent

\[ \bot^a, \Gamma, !\Delta, H_1 \vdash \bot^a. \]

Then, according to the inductive hypothesis, the multiset

\[ \Gamma, H_1 \]

must be empty, which is a contradiction.

Case 1.3 Assume that formula \( B \) is of the form \( C_{00}^m \), \( m = 4, 5 \), and, according to rule \( R\circ \), our sequent is derived from two sequents of the form

\[ C_{00}^{k_1}, \bot^{a_1}, \Gamma_1, !\Delta_1 \vdash C_{00} \]

and

\[ C_{00}^{k_2}, \bot^{a_2}, \Gamma_2, !\Delta_2 \vdash C_{00}^m - 1 \]
The only solution of this system is the following:

\[
\begin{cases}
1 & \geq k_1 + k_2, \\
a & = a_1 + a_2, \\
\Gamma & = \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{cases}
\]

Then Lemma 4.4 and Lemma 4.3 show:

\[
\begin{cases}
-2Nk_1 + a_1 = -2N \pmod{9N}, \\
-2Nk_2 + a_2 = -2N(m-1) \pmod{9N}.
\end{cases}
\]

The only solution of this system is the following:

\[
\begin{cases}
k_2 &= 0, \\
a_2 &= N, \\
m &= 5,
\end{cases}
\]

which yield a contradiction because, according to the inductive hypothesis, the latter sequent cannot occur in our derivation.

**Case 2** Assume that the principal formula belongs to \(\Gamma\), and it is of the form \(F_A\) (or \(F_Y\)).

The following subcases are to be considered.

**Case 2.0** The principal formula is of the form \((F_A \& F_A)\), and, by rule \(L\&\), our sequent is derived either from the sequent

\[ \perp^a, \Gamma', F_{A_1}, \Delta \vdash B \]

or from the sequent

\[ \perp^a, \Gamma', F_{A_2}, \Delta \vdash B. \]

Then, according to the inductive hypothesis, either the multiset

\[ \Gamma', F_{A_1} \]

or the multiset

\[ \Gamma', F_{A_2} \]

must be empty, which is a contradiction.

**Item (c)** is handled similarly.

**Case 2.1** Assume that the principal formula is of the form \((E_X \rightarrow E_Y)\), and, according to rule \(L\rightarrow\), our sequent is derived from two sequents of the form

\[ C_{00}^{k_1}, \perp^a_1, \Gamma_1, \Delta_1 \vdash E_X \]

and

\[ C_{00}^{k_2}, \perp^a_2, E_Y, \Gamma_2, \Delta_2 \vdash B \]

where

\[
\begin{cases}
1 & \geq k_1 + k_2, \\
a & = a_1 + a_2, \\
\Gamma & \supset \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{cases}
\]

Then Lemma 4.4 and Lemma 4.3 yield:

\[
\begin{cases}
-2Nk_1 + a_1 &= 6N \pmod{9N}, \\
-2Nk_2 + a_2 + 6N &= \#_\perp(B) \pmod{9N},
\end{cases}
\]

which is a contradiction.

**Case 2.2** Assume that the principal formula is of the form \((E_X \rightarrow E_Y)\), and, by rule \(L\rightarrow\), our sequent is derived from two sequents of the form

\[ C_{00}^{k_1}, \perp^a_1, \Gamma_1, \Delta_1 \vdash E_X, B \]

and

\[ C_{00}^{k_2}, \perp^a_2, E_Y, \Gamma_2, \Delta_2 \vdash \]

Then Lemma 4.4 and Lemma 4.3 yield:

\[
\begin{cases}
-2Nk_1 + a_1 &= 0 \pmod{9N}, \\
-2Nk_2 + a_2 &= \#_\perp(B) \pmod{9N},
\end{cases}
\]

Hence,

\[ k_1 = 0, \]

and either

\[ a_2 = a, \]

or (for the case of the 'empty' \(B\))

\[ a_2 = 1. \]

By the inductive hypothesis, we can get a contradiction that the non-empty multiset

\[ F_Y, \Gamma_2 \]

must be empty.
Case 2.4  Assume that the principal formula is of the form \( (F_A \leftarrow F_Y) \), and, according to rule L\(\leftarrow\)o, our sequent is derived from two sequents of the form
\[
C_{00}^{k_1}, \bot a_1, \Gamma_1, !\Delta_1 \vdash F_A, B
\]
and
\[
C_{00}^{k_2}, \bot a_2, F_Y, \Gamma_2, !\Delta_2 \vdash
\]
where
\[
\begin{cases}
1 & \geq k_1 + k_2, \\
a & = a_1 + a_2, \\
\Gamma & \supset \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{cases}
\]
Then Lemma 4.4 and Lemma 4.3 yield:
\[
\begin{cases}
-2Nk_1 + a_1 = \#_\bot(B) - 1 \pmod{9N}, \\
-2Nk_2 + a_2 = 1 \pmod{9N}.
\end{cases}
\]
Therefore,
\[
\begin{cases}
k_2 = 0, \\
a_2 = 1,
\end{cases}
\]
and, by the inductive hypothesis, the multiset
\[
F_Y, \Gamma_2
\]
must be empty, which is a contradiction as well.

Case 2.5  Case of the principal formula of the form \( (E_X \leftarrow (E_X \oplus E_Y)) \) is handled similarly to Case 2.1 and Case 2.2.

Case 3  Assume that the principal formula belongs to \( \Gamma \), and it is of the form
\[
D_q = ((H_1 \leftarrow \bot^b) \leftarrow \bot^b)
\]
where
\[
4 \leq b \leq N - 3.
\]

Case 3.1  According to rule L\(\leftarrow\)o, let our sequent be derived from two sequents of the form
\[
C_{00}^{k_1}, \bot a_1, \Gamma_1, !\Delta_1 \vdash (H_1 \leftarrow \bot^b)
\]
and
\[
C_{00}^{k_2}, \bot a_2, \Gamma_2, !\Delta_2 \vdash B
\]
where
\[
\begin{cases}
1 & \geq k_1 + k_2, \\
a & = a_1 + a_2, \\
\Gamma & \supset \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{cases}
\]
Then Lemma 4.4 and Lemma 4.3 yield:
\[
\begin{cases}
-2Nk_1 + a_1 = b \pmod{9N}, \\
-2Nk_2 + a_2 + b = \#_\bot(B) \pmod{9N},
\end{cases}
\]
and, hence,
\[
\begin{cases}
k_1 = 0, \\
a_1 = b.
\end{cases}
\]
According to the inductive hypothesis, the right-hand side of the first sequent must be exactly \( \bot^b \), and, therefore, such a sequent cannot occur in our derivation.

Case 3.2  Let our sequent be derived from two sequents of the form
\[
C_{00}^{k_1}, \bot a_1, \Gamma_1, !\Delta_1 \vdash (H_1 \leftarrow \bot^b), B
\]
and
\[
C_{00}^{k_2}, \bot a_2 + b, \Gamma_2, !\Delta_2 \vdash
\]
where
\[
\begin{cases}
1 & \geq k_1 + k_2, \\
a & = a_1 + a_2, \\
\Gamma & \supset \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{cases}
\]
Then, by Lemma 4.4 and Lemma 4.3 we have:
\[
\begin{cases}
-2Nk_1 + a_1 = b + \#_\bot(B) - 1 \pmod{9N}, \\
-2Nk_2 + a_2 + b = 1 \pmod{9N},
\end{cases}
\]
which is also a contradiction because of
\[
4 \leq a_2 + b \leq 2N - 1.
\]

Case 4  Assume that the principal formula belongs to \( \Gamma \), and it is of the form
\[
H_1 = (C_{00}^4 \leftarrow \bot^N).
\]

Case 4.1  According to rule L\(\leftarrow\)o, let our sequent be derived from two sequents of the form
\[
C_{00}^{k_1}, \bot a_1, \Gamma_1, !\Delta_1 \vdash C_{00}^4
\]
and
\[
C_{00}^{k_2}, \bot a_2 + N, \Gamma_2, !\Delta_2 \vdash B
\]
where
\[
\begin{cases}
1 & \geq k_1 + k_2, \\
a & = a_1 + a_2, \\
\Gamma & \supset \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{cases}
\]
Then Lemma 4.4 and Lemma 4.3 show:
\[
\begin{cases}
-2Nk_1 + a_1 = -8N \pmod{9N}, \\
-2Nk_2 + a_2 + N = \#_\bot(B) \pmod{9N}.
\end{cases}
\]
Therefore,
\[
\begin{cases}
k_1 = 0, \\
a_1 = N.
\end{cases}
\]
and, according to the inductive hypothesis, the first sequent with its wrong right-hand side cannot occur in our derivation.

**Case 4.2** Let our sequent be derived from two sequents of the form

\[ C_{00}^k, \bot a_1, \Gamma_1, \Delta_1 \vdash C_{00}^4, B \]

and

\[ C_{00}^k, \bot a_2 + N, \Gamma_2, \Delta_2 \vdash \]

where

\[
\begin{align*}
1 & \geq k_1 + k_2, \\
a & = a_1 + a_2, \\
\Gamma & \supset \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{align*}
\]

By Lemma 4.4 and Lemma 4.3 we have:

\[
\begin{align*}
-2N(k_1 + k_2) + a & = \# \bot (B) \ (\text{mod } 9N), \\
-2Nk_2 + a_2 + N & = 1 \ (\text{mod } 9N).
\end{align*}
\]

Assuming that \( k_2 = 0 \), we get a contradiction:

\[ a_2 + (N - 1) = 0. \]

For \( k_2 = 1 \), we get also a contradiction as follows:

\[
\begin{align*}
B & = C_{00}^5, \\
\# \bot (B) & = -10N, \\
a & = N, \\
a_2 & = N + 1.
\end{align*}
\]

**Case 5** Finally, for item (c), assume that the left-hand principal formula is of the form

\[ C_{00} = ((H_{00}^2 \circ \bot^3) \circ \bot^3), \]

and, according to rule \( L\circ \), our sequent of item (c) is derived from two sequents of the form

\[ \bot a_1, \Gamma_1, \Delta_1 \vdash (H_{00}^2 \circ \bot^3) \]

and

\[ \bot a_2 + 3, \Gamma_2, \Delta_2 \vdash B \]

where

\[
\begin{align*}
a & = a_1 + a_2, \\
\Gamma & = \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{align*}
\]

Then, by Lemma 4.4 and Lemma 4.3, the following contradiction is immediate:

\[
\begin{align*}
a_1 & = 2N + 3 \ (\text{mod } 9N), \\
a_2 + 3 & = -10N \ (\text{mod } 9N).
\end{align*}
\]

If our sequent of item (c) were derived from two sequents of the form

\[ \bot a_1, \Gamma_1, \Delta_1 \vdash (H_{00}^2 \circ \bot^3), B \]

and

\[ \bot a_2 + 3, \Gamma_2, \Delta_2 \vdash \]

then we got a contradiction as well:

\[ \left\{ \begin{align*}
a_1 & = -8N + 2 \ (\text{mod } 9N), \\
a_2 + 3 & = 1 \ (\text{mod } 9N).
\end{align*} \right. \]

Now, bringing together all the cases considered, we can complete the proof of Lemma 5.1.

**5.2 Lemma 5.2**

**Lemma 5.2** Let \( \Delta \) consist of formulas of the form \( F_A \), and \( \Gamma \) consist of formulas of the form \( H_1, D_X, F_A, \) and \( F_Y \).

(a) Let \( K \) be a multiset either of the form

\[ H_{00}, \]

or of the form

\[ H_{00}, H_{00}, \]

or of the form

\[ C_{00}, C_{00}, \ldots, C_{00} \]

\[ \text{k times} \]

where \( 1 \leq k \leq 5 \).

Let \( B \) be a formula either of the form \( H_{00}^2, \) or of the form \( H_{00}^2, \) or of the form \( C_{00} \), where \( 1 \leq m \leq 5 \).

If a sequent of the form

\[ K, \Gamma, \Delta \vdash B \]

occurs in a cut-free derivation in Linear Logic then \( \Gamma \) must be empty, and \( \Delta \) can be produced by rules \( W! \) and \( C! \) only (there is no applications of rule \( L! \) in the derivation above this sequent).\(^6\)

(b) If a sequent of the form

\[ (H_{00}^2 \circ \bot^3), \Gamma, \Delta \vdash (H_{00}^2 \circ \bot^3) \]

occurs in a cut-free derivation in Linear Logic then \( \Gamma \) must be empty, and \( \Delta \) can be produced by rules \( W! \) and \( C! \) only.

\(^6\)We say that this \( \Delta \) is degenerate.
(c) Let $K$ be a multiset either of the form
\[ C_{00} \]
or of the form
\[ H_{00}, H_{00}. \]
If a sequent of the form
\[ K, (H_{00} \circ \perp^3), \Gamma, !\Delta \vdash \perp^3 \]
occurs in a cut-free derivation in Linear Logic then $\Gamma$ must be empty, and $!\Delta$ can be produced by rules $W!$ and $C!$ only.

(d) Let $a$ be an integer such that
\[ 1 \leq a \leq 2. \]
If a sequent of the form
\[ H_{00}, \perp^a + N, \Gamma, !\Delta \vdash \perp^a \]
occurs in a cut-free derivation in Linear Logic then this integer $a$ must be equal exactly to 2, multiset $\Gamma$ must be empty, and $!\Delta$ can be produced by rules $W!$ and $C!$ only.

(e) Any sequent of the form
\[ H_{00}, \perp + 1, \Gamma, !\Delta \vdash \]
does not occur in derivations in Linear Logic.

Proof. We use induction on a given derivation. Regarding to the form of the principal formula at a current point of the derivation, we will demonstrate that each of the undesirable cases is inconsistent.

Case 0  The principal formula belongs to $!\Delta$.
Assume that it is produced by rule $L!$, and our sequent of the form
\[ \ldots, \Gamma, !\Delta \vdash \ldots \]
is derived from a sequent of the form
\[ \ldots, \Gamma, F_A, !\Delta' \vdash \ldots \]
Then, by the inductive hypothesis, the multiset
\[ \Gamma, F_A \]
must be empty, which is a contradiction.
Hence, the only possibility is to apply either $W!$ or $C!$. It remains to use the inductive hypothesis for completing this case.

Case 1  The right-side formula is principal.
There are the following cases to be considered.

Case 1.a  For item (a), let us note that for any subset $K'$ of multiset $K$:
\[ \text{either } \#\perp(K') = -N, \]
or \[ \text{or } \#\perp(K') = -2Nk', \text{ for some } k' \leq 5. \]
Let us consider four possible versions of the principal formula $B$.

Case 1.a.1  The principal formula $B$ is of the form
\[ H_{00} = (\perp N + 2 \circ \perp^2), \]
and, according to rule $R_{\circ}$, our sequent of item (a) is derived from the sequent
\[ H_{00}, \perp N + 2, \Gamma, !\Delta \vdash \perp^2. \]
Here we have accounted that, by Lemma 4.4,
\[ \#\perp(K) = \#\perp(B) = -N. \]
Now we can apply the inductive hypothesis from item (d).

Case 1.a.2  The principal formula $B$ is of the form $H_{00}^2$, and, according to rule $R_{\circ}$ and Lemmas 4.3 and 4.4, our sequent of item (a) is derived from two sequents of the form
\[ H_{00}, \Gamma_1, !\Delta_1 \vdash H_{00} \]
and
\[ H_{00}, \Gamma_2, !\Delta_2 \vdash H_{00} \]
where
\[ \begin{cases} \Gamma = \Gamma_1, \Gamma_2, \\ \Delta = \Delta_1, \Delta_2. \end{cases} \]
By applying the inductive hypothesis from item (c), we get the emptiness of both $\Gamma_1$ and $\Gamma_2$, and the degeneracy of both $!\Delta_1$ and $!\Delta_2$, which results in the desired emptiness of the whole $\Gamma$ and the degeneracy of the whole $!\Delta$.

Case 1.a.3  The principal formula $B$ is of the form
\[ C_{00} = ((H_{00}^2 \circ \perp^3) \circ \perp^3), \]
and, according to rule $R_{\circ}$, our sequent of item (a) is derived from the sequent
\[ K, (H_{00}^2 \circ \perp^3), \Gamma, !\Delta \vdash \perp^3. \]
By Lemma 4.4
\[ \#\perp(K) = \#\perp(B) = -2N. \]
Therefore, we can complete this case by applying the inductive hypothesis from item (c).
Case 1.a.4  The principal formula is of the form $C_{m_0}^n$, and, according to rule $R\circ$, our sequent of item (a) is derived from two sequents of the form

$$K_1, \Gamma_1, !\Delta_1 \vdash C_{m_0}^n$$

and

$$K_2, \Gamma_2, !\Delta_2 \vdash C_{m_0}^n - 1$$

where

$$\begin{cases} K = K_1, K_2, \\ \Gamma = \Gamma_1, \Gamma_2, \\ \Delta = \Delta_1, \Delta_2. \end{cases}$$

By applying the inductive hypothesis, we can get the emptiness of both $\Gamma_1$ and $\Gamma_2$, and the degeneracy of both $!\Delta_1$ and $!\Delta_2$, which results in the desired emptiness of the whole $\Gamma$ and the degeneracy of the whole $!\Delta$.

Case 1.b  The principal formula is of the form $(H_{m_0}^2 \circ \perp^3)$, and, according to rule $R\circ$, the corresponding sequent of item (b) is derived from the sequent

$$(H_{m_0}^2 \circ \perp^3), \Gamma, !\Delta \vdash \perp^3.$$  

It remains to apply the inductive hypothesis from item (c).

Case 1.c  Assume that the corresponding sequent of item (c) is derived from two sequents of the form

$$K_1, (H_{m_0}^2 \circ \perp^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash \perp$$

and

$$K_2, (H_{m_0}^2 \circ \perp^3)^{h_2}, \Gamma_2, !\Delta_2 \vdash \perp^2$$

where

$$\begin{cases} K = K_1, K_2, \\ 1 = h_1 + h_2, \\ \Gamma = \Gamma_1, \Gamma_2, \\ \Delta = \Delta_1, \Delta_2. \end{cases}$$

By Lemma 4.4 and Lemma 4.3 we have:

$$\begin{cases} \#_{\perp}(K_1) + (2N + 3)h_1 = 1 \pmod{9N}, \\ \#_{\perp}(K_2) + (2N + 3)h_2 = 2 \pmod{9N}, \\ \#_{\perp}(K_1) = -nk', \text{ for some } k' \leq 2, \end{cases}$$

which is a contradiction.

Case 1.d.2  Assume that $a = 2$, and, by rule $R_{24}$, our sequent of item (d) is derived from two sequents of the form

$$H_{m_0}, \perp^{a_1}, \Gamma_1, !\Delta_1 \vdash \perp$$

and

$$\perp^{a_2}, \Gamma_2, !\Delta_2 \vdash \perp$$

where

$$\begin{cases} N + 2 = a_1 + a_2, \\ \Gamma = \Gamma_1, \Gamma_2, \\ \Delta = \Delta_1, \Delta_2. \end{cases}$$

By Lemma 4.4 and Lemma 4.3 we have:

$$\begin{cases} -N + a_1 = 1 \pmod{9N}, \\ a_2 = 1 \pmod{9N}, \end{cases}$$

and, therefore,

$$a_1 = N + 1,$$

which leads to a contradiction because, according to the inductive hypothesis, the first sequent cannot occur in our derivation.

Case 2  Assume that the principal formula belongs to $\Gamma$, and it is of the form $FA$ (or $FY$). The following subcases are to be considered.

Case 2.0  The principal formula is of the form $(FA_1 \& FA_2)$, and, by rule $L\&$, the corresponding sequent of the form

$$\ldots, \Gamma, !\Delta \vdash \ldots$$

is derived either from the sequent

$$\ldots, \Gamma', FA_1, !\Delta \vdash \ldots$$

or from the sequent

$$\ldots, \Gamma', FA_2, !\Delta \vdash \ldots$$

Then, according to the inductive hypothesis, either the multiset

$$\Gamma', FA_1$$

or the multiset

$$\Gamma', FA_2$$

must be empty, which is a contradiction.

Case 2.1.a  Assume that the principal formula is of the form $(EX \circ FY)$, and, according to rule $L\circ$, the sequent of item (a) is derived from two sequents of the form

$$K_1, \Gamma_1, !\Delta_1 \vdash EX$$

and

$$K_2, EY, \Gamma_2, !\Delta_2 \vdash B$$
Then Lemma 4.4 and Lemma 4.3 yield:
\[
\begin{align*}
\#(K_1) &= 6N \pmod{9N}, \\
\#(K_2) + 6N &= \#(B) \pmod{9N}, \\
\text{either } \#(K_1) &= -N, \\
\text{or } \#(K_2) &= -2Nk', \text{ for some } k' \leq 5,
\end{align*}
\]
which is a contradiction.

**Case 2.2.a** Assume that the principal formula is of the form \((E_X \rightarrow E_Y)\), and, by rule \(L \rightarrow \), our sequent of item (a) is derived from two sequents of the form
\[
K_1, \Gamma_1, \Delta_1 \vdash E_X, B
\]
and
\[
K_2, E_Y, \Gamma_2, \Delta_2 \vdash
\]
Then Lemma 4.4 and Lemma 4.3 yield:
\[
\begin{align*}
\#(K_1) &= 6N + \#(B) - 1 \pmod{9N}, \\
\#(K_2) + 6N &= 1 \pmod{9N}, \\
\text{either } \#(K_2) &= -N, \\
\text{or } \#(K_2) &= -2Nk', \text{ for some } k' \leq 5,
\end{align*}
\]
which is also a contradiction.

**Case 2.3.a** Assume that the principal formula is of the form \((F_A \rightarrow F_Y)\), and, according to rule \(L \rightarrow \), our sequent of item (a) is derived from two sequents of the form
\[
K_1, \Gamma_1, \Delta_1 \vdash F_A
\]
and
\[
K_2, E_Y, \Gamma_2, \Delta_2 \vdash B
\]
where
\[
\begin{align*}
K &= K_1, K_2, \\
\Gamma &\supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]
Then, by Lemma 4.4 and Lemma 4.3, we have:
\[
\begin{align*}
\#(K) &= \#(B) \pmod{9N}, \\
\#(K_1) &= 0 \pmod{9N}, \\
\#(K_2) &= \#(B) \pmod{9N},
\end{align*}
\]
Hence,
\[
K_2 = K,
\]
and, by the inductive hypothesis, the multiset
\[
F_Y, \Gamma_2
\]
must be empty, which is a contradiction.

**Case 2.4.a** Assume that the principal formula is of the form \((F_A \rightarrow F_Y)\), and, according to rule \(L \rightarrow \), our sequent of item (a) is derived from two sequents of the form
\[
K_1, \Gamma_1, \Delta_1 \vdash F_A, B
\]
and
\[
K_2, F_Y, \Gamma_2, \Delta_2 \vdash
\]
Then Lemma 4.4 and Lemma 4.3 show that
\[
\begin{align*}
\#(K_1) &= \#(B) - 1 \pmod{9N}, \\
\#(K_2) &= 1 \pmod{9N}, \\
\text{either } \#(K_2) &= -N, \\
\text{or } \#(K_2) &= -2Nk', \text{ for some } k' \leq 5,
\end{align*}
\]
which is a contradiction as well.

**Case 2.1.b** Assume that the principal formula is of the form \((E_X \rightarrow E_Y)\), and our sequent of item (b) is derived from two sequents of the form
\[
(H_{20}^2 \rightarrow \bot^3)^{h_1}, \Gamma_1, \Delta_1 \vdash E_X
\]
and
\[
(H_{20}^2 \rightarrow \bot^3)^{h_2}, E_Y, \Gamma_2, \Delta_2 \vdash (H_{20}^2 \rightarrow \bot^3)
\]
where
\[
\begin{align*}
1 &= h_1 + h_2, \\
\Gamma &\supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]
By Lemma 4.4 and Lemma 4.3 we have:
\[
\begin{align*}
(2N + 3)h_1 &= 6N \pmod{9N}, \\
(2N + 3)h_2 + 6N &= 2N + 3 \pmod{9N},
\end{align*}
\]
which is a contradiction.

**Case 2.2.b** Assume that the principal formula is of the form \((E_X \rightarrow E_Y)\), and now our sequent of item (b) is derived from two sequents of the form
\[
(H_{20}^2 \rightarrow \bot^3)^{h_1}, \Gamma_1, \Delta_1 \vdash E_X, (H_{20}^2 \rightarrow \bot^3)
\]
and
\[
(H_{20}^2 \rightarrow \bot^3)^{h_2}, E_Y, \Gamma_2, \Delta_2 \vdash
\]
Then Lemma 4.4 and Lemma 4.3 yield:
\[
\begin{align*}
(2N + 3)h_1 &= 8N + 2 \pmod{9N}, \\
(2N + 3)h_2 + 6N &= 1 \pmod{9N},
\end{align*}
\]
which is also a contradiction.

**Case 2.3.b** Assume that the principal formula is of the form \((F_A \rightarrow F_Y)\), and our sequent of item (b) is derived from two sequents of the form
\[
(H_{20}^2 \rightarrow \bot^3)^{h_1}, \Gamma_1, \Delta_1 \vdash F_A
\]
Hence, and by the inductive hypothesis, the multiset
\[ (H_{\theta_0}^2 \circ \perp^3)^{h_2}, F_Y, \Gamma_2, !\Delta_2 \vdash (H_{\theta_0}^2 \circ \perp^3) \]
where
\[ \begin{cases} 1 & h_1 + h_2, \\ \Gamma & \Gamma_1, \Gamma_2 \\ \Delta & = \Delta_1, \Delta_2 \end{cases} \]
According to Lemma 4.4 and Lemma 4.3, we have:
\[ \begin{cases} (2N + 3)h_1 = 0 \pmod{9N}, \\ (2N + 3)h_2 = 2N + 3 \pmod{9N}. \end{cases} \]
Hence, \[ h_2 = 1, \]
and, by the inductive hypothesis, the multiset
\[ F_Y, \Gamma_2 \]
must be empty, which is a contradiction.

**Case 2.4.b** Assume that the principal formula is of the form \( (F_A \circ \perp F_Y) \), and now our sequent of item (b) is derived from two sequents of the form
\[ (H_{\theta_0}^2 \circ \perp^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash F_A, (H_{\theta_0}^2 \circ \perp^3) \]
and
\[ (H_{\theta_0}^2 \circ \perp^3)^{h_2}, F_Y, \Gamma_2, !\Delta_2 \vdash \]
Then Lemma 4.4 and Lemma 4.3 yield:
\[ \begin{cases} (2N + 3)h_1 = 2N + 2 \pmod{9N}, \\ (2N + 3)h_2 = 1 \pmod{9N}, \end{cases} \]
which is a contradiction.

**Case 2.1.c** Assume that the principal formula is of the form \( (E_X \circ \perp E_Y) \), and our sequent of item (c) is derived from two sequents of the form
\[ K_1, (H_{\theta_0}^2 \circ \perp^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash E_X \]
and
\[ K_2, (H_{\theta_0}^2 \circ \perp^3)^{h_2}, E_Y, \Gamma_2, !\Delta_2 \vdash \]
where
\[ \begin{cases} K = K_1, K_2, \\ 1 = h_1 + h_2, \\ \Gamma \supset \Gamma_1, \Gamma_2, \\ \Delta = \Delta_1, \Delta_2. \end{cases} \]
By Lemma 4.4 and Lemma 4.3 we have:
\[ \begin{cases} \#(K_1) + (2N + 3)h_1 = 6N \pmod{9N}, \\ \#(K_2) + (2N + 3)h_2 + 6N = 3 \pmod{9N}, \\ \#(K_1) = -Nk', \text{ for some } k' \leq 2, \end{cases} \]
which is a contradiction.

**Case 2.2.c** Assume that the principal formula is of the form \( (E_X \circ \perp E_Y) \), and now our sequent of item (c) is derived from two sequents of the form
\[ K_1, (H_{\theta_0}^2 \circ \perp^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash E_X, \perp^3 \]
and
\[ K_2, (H_{\theta_0}^2 \circ \perp^3)^{h_2}, E_Y, \Gamma_2, !\Delta_2 \vdash \]
Then Lemma 4.4 and Lemma 4.3 yield:
\[ \begin{cases} \#(K_1) + (2N + 3)h_1 = 6N + 2 \pmod{9N}, \\ \#(K_2) + (2N + 3)h_2 + 6N = 1 \pmod{9N}, \\ \#(K_2) = -Nk', \text{ for some } k' \leq 2, \end{cases} \]
which is also a contradiction.

**Case 2.3.c** Assume that the principal formula is of the form \( (F_A \circ \perp F_Y) \), and our sequent of item (c) is derived from two sequents of the form
\[ K_1, (H_{\theta_0}^2 \circ \perp^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash F_A \]
and
\[ K_2, (H_{\theta_0}^2 \circ \perp^3)^{h_2}, F_Y, \Gamma_2, !\Delta_2 \vdash \]
where
\[ \begin{cases} K = K_1, K_2, \\ 1 = h_1 + h_2, \\ \Gamma \supset \Gamma_1, \Gamma_2, \\ \Delta = \Delta_1, \Delta_2. \end{cases} \]
Then Lemma 4.4 and Lemma 4.3 show:
\[ \begin{cases} \#(K_1) + (2N + 3)h_1 = 3 \pmod{9N}, \\ \#(K_1) + (2N + 3)h_1 = 0 \pmod{9N}, \\ \#(K_2) + (2N + 3)h_2 = 3 \pmod{9N}, \\ \#(K_2) = -Nk', \text{ for some } k' \leq 2. \end{cases} \]
Hence,
\[ \begin{cases} h_2 = 1, \\ K_2 = K, \end{cases} \]
and, by the inductive hypothesis, the multiset
\[ F_Y, \Gamma_2 \]
must be empty, which is a contradiction.

**Case 2.4.c** Assume that the principal formula is of the form \( (F_A \circ \perp F_Y) \), and now our sequent of item (c) is derived from two sequents of the form
\[ K_1, (H_{\theta_0}^2 \circ \perp^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash F_A, \perp^3 \]
and
\[ K_2, (H_{\theta_0}^2 \circ \perp^3)^{h_2}, F_Y, \Gamma_2, !\Delta_2 \vdash \]
Then Lemma 4.4 and Lemma 4.3 yield:

\[
\begin{align*}
\#_+(K_1) + (2N + 3)h_1 &= 2 \pmod{9N}, \\
\#_+(K_2) + (2N + 3)h_2 &= 1 \pmod{9N}, \\
\#_+(K_2) &= -Nh'_2, \text{ for some } k' \leq 2,
\end{align*}
\]

which is a contradiction.

**Case 2.1.d+e** Assume that the principal formula is of the form \((EX \rightarrow EY)\), and the corresponding sequent of item (d) or (e) is derived from two sequents of the form

\[
H_{00}^{h_1}, \bot a_1, \Gamma_1, !\Delta_1 \vdash E_X
\]

and

\[
H_{00}^{h_2}, \bot a_2, E_Y, \Gamma_2, !\Delta_2 \vdash \bot a
\]

where

\[
\begin{align*}
1 &= h_1 + h_2, \\
N + a &= a_1 + a_2, \\
\Gamma &\supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

By Lemma 4.4 and Lemma 4.3 we have:

\[
\begin{align*}
-Nh_1 + a_1 &= 6N \pmod{9N}, \\
-Nh_2 + a_2 + 6N &= a \pmod{9N},
\end{align*}
\]

which is a contradiction.

**Case 2.2.d** Assume that the principal formula is of the form \((EX \rightarrow EY)\), and now our sequent of item (d) is derived from two sequents of the form

\[
H_{00}^{h_1}, \bot a_1, \Gamma_1, !\Delta_1 \vdash E_X, \bot a
\]

and

\[
H_{00}^{h_2}, \bot a_2, E_Y, \Gamma_2, !\Delta_2 \vdash
\]

Then Lemma 4.4 and Lemma 4.3 yield:

\[
\begin{align*}
-Nh_1 + a_1 &= 6N + a - 1 \pmod{9N}, \\
-Nh_2 + a_2 + 6N &= 1 \pmod{9N},
\end{align*}
\]

which is also a contradiction.

**Case 2.3.d+e** Let the principal formula be of the form \((FA \rightarrow FY)\), and let our sequent of item (d) or (e) be derived from two sequents of the form

\[
H_{00}^{h_1}, \bot a_1, \Gamma_1, !\Delta_1 \vdash FA
\]

and

\[
H_{00}^{h_2}, \bot a_2, E_Y, \Gamma_2, !\Delta_2 \vdash \bot a
\]

where

\[
\begin{align*}
1 &= h_1 + h_2, \\
N + a &= a_1 + a_2, \\
\Gamma &\supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

Then Lemma 4.4 and Lemma 4.3 show:

\[
\begin{align*}
-Nh_1 + a_1 &= 0 \pmod{9N}, \\
-Nh_2 + a_2 &= a \pmod{9N}.
\end{align*}
\]

Assume that \(h_2 = 0\).

Then \(a_2 = a\), and, according to Lemma 5.1, the multiset

\[
FY, \Gamma_2
\]

must be empty, which is a contradiction.

Assume that \(h_2 = 1\).

Then \(a_2 = N + a\). Now the non-empty multiset

\[
FY, \Gamma_2
\]

must be empty by the inductive hypothesis.

**Case 2.4.d** Let the principal formula be of the form \((FA \rightarrow FY)\), and let our sequent of item (d) be derived from two sequents of the form

\[
H_{00}^{h_1}, \bot a_1, \Gamma_1, !\Delta_1 \vdash FA, \bot a
\]

and

\[
H_{00}^{h_2}, \bot a_2, E_Y, \Gamma_2, !\Delta_2 \vdash
\]

where

\[
\begin{align*}
1 &= h_1 + h_2, \\
N + a &= a_1 + a_2, \\
\Gamma &\supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

Then Lemma 4.4 and Lemma 4.3 yield:

\[
\begin{align*}
-Nh_1 + a_1 &= a - 1 \pmod{9N}, \\
-Nh_2 + a_2 &= 1 \pmod{9N}.
\end{align*}
\]

Assume that \(h_2 = 0\).

Then \(a_2 = 1\), and, according to Lemma 5.1, the multiset

\[
FY, \Gamma_2
\]

must be empty, which is a contradiction.

Assume that \(h_2 = 1\).

Then \(a_2 = N + 1\), and we get a contradiction to item (e).

**Case 2.5** Case of the principal formula of the form \((EX \rightarrow (FY_1 \oplus FY_2))\) is handled similarly to **Cases 2.1.abcd** and **Cases 2.2.abcd**.

**Case 3** Assume that the principal formula belongs to \(\Gamma\), and it is of the form

\[
D_q = ((H_1 \rightarrow \bot b) \rightarrow \bot b)
\]

where

\[
4 \leq b \leq N - 3.
\]
Case 3.1.a  According to rule L─o, let our sequent of item (a) be derived from two sequents of the form

$$K_1, \Gamma_1, !\Delta_1 \vdash (H_1 \circ \bot^b)$$

and

$$K_2, \bot^b, \Gamma_2, !\Delta_2 \vdash B$$

where

$$\begin{cases} K = K_1, K_2, \\
\Gamma \supset \Gamma_1, \Gamma_2, \\
\Delta = \Delta_1, \Delta_2. \end{cases}$$

Then Lemma 4.4 and Lemma 4.3 yield:

$$\begin{cases} \#_\bot(K_1) = b \pmod{9N}, \\
\#_\bot(K_2) + b = \#_\bot(B) \pmod{9N}, \\
either \#_\bot(K_1) = -N, \\
or \#_\bot(K_2) = -2Nk', \text{ for some } k' \leq 5, \end{cases}$$

which is a contradiction.

Case 3.2.a  Now let our sequent of item (a) be derived from two sequents of the form

$$K_1, \Gamma_1, !\Delta_1 \vdash (H_1 \circ \bot^b), B$$

and

$$K_2, \bot^b, \Gamma_2, !\Delta_2 \vdash$$

Then, by Lemma 4.4 and Lemma 4.3, we have:

$$\begin{cases} \#_\bot(K_1) = b \pmod{9N}, \\
\#_\bot(K_2) + b = 1 \pmod{9N}, \\
either \#_\bot(K_2) = -N, \end{cases}$$

which is a contradiction as well.

Case 3.1.b  According to rule L─o, assume that the corresponding sequent of item (b) is derived from two sequents of the form

$$(H_{00}^2 \circ \bot^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash (H_1 \circ \bot^b)$$

and

$$(H_{00}^2 \circ \bot^3)^{h_2}, \bot^b, \Gamma_2, !\Delta_2 \vdash (H_{00}^2 \circ \bot^3)$$

where

$$\begin{cases} 1 = h_1 + h_2, \\
\Gamma \supset \Gamma_1, \Gamma_2, \\
\Delta = \Delta_1, \Delta_2. \end{cases}$$

Then Lemma 4.4 and Lemma 4.3 yield the following contradiction:

$$\begin{cases} (2N + 3)h_1 = b \pmod{9N}, \\
(2N + 3)h_2 + b = 2N + 3 \pmod{9N}. \end{cases}$$

Case 3.2.b  Now assume that our sequent of item (b) is derived from two sequents of the form

$$(H_{00}^2 \circ \bot^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash (H_1 \circ \bot^b), (H_{00}^2 \circ \bot^3)$$

and

$$(H_{00}^2 \circ \bot^3)^{h_2}, \bot^b, \Gamma_2, !\Delta_2 \vdash$$

Then, by Lemma 4.4 and Lemma 4.3, we have:

$$\begin{cases} (2N + 3)h_1 = b + 2N + 2 \pmod{9N}, \\
(2N + 3)h_2 + b = 1 \pmod{9N}, \end{cases}$$

which is a contradiction as well.

Case 3.1.c  According to rule L─o, let our sequent of item (c) be derived from two sequents of the form

$$K_1, (H_{00}^2 \circ \bot^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash (H_1 \circ \bot^b)$$

and

$$K_2, (H_{00}^2 \circ \bot^3)^{h_2}, \bot^b, \Gamma_2, !\Delta_2 \vdash \bot^3$$

where

$$\begin{cases} K = K_1, K_2, \\
1 = h_1 + h_2, \\
\Gamma \supset \Gamma_1, \Gamma_2, \\
\Delta = \Delta_1, \Delta_2. \end{cases}$$

By Lemma 4.4 and Lemma 4.3 we have:

$$\begin{cases} \#_\bot(K_1) + (2N + 3)h_1 = b \pmod{9N}, \\
\#_\bot(K_2) + (2N + 3)h_2 + b = 3 \pmod{9N}, \\
\#_\bot(K_1) = -Nk', \text{ for some } k' \leq 2, \end{cases}$$

which is a contradiction.

Case 3.2.c  Now assume that our sequent of item (c) is derived from two sequents of the form

$$K_1, (H_{00}^2 \circ \bot^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash (H_1 \circ \bot^b), \bot^3$$

and

$$K_2, (H_{00}^2 \circ \bot^3)^{h_2}, \bot^b, \Gamma_2, !\Delta_2 \vdash$$

Then Lemma 4.4 and Lemma 4.3 yield:

$$\begin{cases} \#_\bot(K_1) + (2N + 3)h_1 = b + 2 \pmod{9N}, \\
\#_\bot(K_2) + (2N + 3)h_2 + b = 1 \pmod{9N}, \\
\#_\bot(K_2) = -Nk', \text{ for some } k' \leq 2, \end{cases}$$

which is also a contradiction.

Case 3.1.d+e  Let the corresponding sequent of item (d) or (e) be derived from two sequents of the form

$$H_{00}^{h_1}, \bot^a, \Gamma_1, !\Delta_1 \vdash (H_1 \circ \bot^b)$$
where
\[ \begin{align*}
1 &= h_1 + h_2, \\
N + a &= a_1 + a_2, \\
\Gamma &\supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*} \]

By Lemma 4.4 and Lemma 4.3 we have:
\[ \begin{align*}
-Nh_1 + a_1 &= b \pmod{9N}, \\
-Nh_2 + a_2 + b &= a \pmod{9N}.
\end{align*} \]

Assuming that \( h_1 = 0 \), we can conclude that \( a_1 = b \), which gives a contradiction because, by Lemma 5.1, the first sequent with its wrong right-hand side cannot occur in our derivation.

For \( h_2 = 0 \), we can get also a contradiction because of
\[ 2 \leq a_2 + (b - a) \leq 2N - 1. \]

**Case 3.2.d** Let the corresponding sequent of item (d) be derived from two sequents of the form
\[ H^h_{00}, \perp a_1, \Gamma_1, \! \Delta_1 \vdash (H \circ \perp b), \perp a \]
and
\[ H^h_{00}, \perp a_2 + b, \Gamma_2, \! \Delta_2 \vdash \]
where
\[ \begin{align*}
1 &= h_1 + h_2, \\
N + a &= a_1 + a_2, \\
\Gamma &\supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*} \]

By Lemma 4.4 and Lemma 4.3 we have:
\[ \begin{align*}
-Nh_1 + a_1 &= b + a - 1 \pmod{9N}, \\
-Nh_2 + a_2 + b &= 1 \pmod{9N}.
\end{align*} \]

Assuming that \( h_2 = 0 \), we get a contradiction because of
\[ 3 \leq a_2 + (b - 1) \leq 2N - 2. \]

For \( h_2 = 1 \), we have that \( a_2 + b = N + 1 \), which yields a contradiction because, according to the inductive hypothesis from item (e), the latter sequent cannot occur in our derivation.

**Case 4** Assume that the principal formula belongs to \( \Gamma \), and it is of the form
\[ H_1 = (C^4_{00} \circ \perp N). \]

**Case 4.1.a** According to rule \( \text{L} \circ \perp \), assume that the corresponding sequent of item (a) is derived from two sequents of the form
\[ K_1, \Gamma_1, \! \Delta_1 \vdash \]
and
\[ K_2, \perp N, \Gamma_2, \! \Delta_2 \vdash B \]
where
\[ \begin{align*}
K &= K_1, K_2, \\
\Gamma &\supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*} \]

Then Lemma 4.4 and Lemma 4.3 yield:
\[ \begin{align*}
\#_\perp(K) &= \#_\perp(B) \pmod{9N}, \\
\#_\perp(K_1) &= -8N \pmod{9N}, \\
\text{either } \#_\perp(K_1) &= -N, \\
\text{or } \#_\perp(K_1) &= -2Nk', \text{ for some } k' \leq 5.
\end{align*} \]

The only solution of this system is as follows:
\[ \begin{align*}
K_1 &= C_{00}, C_{00}, C_{00}, C_{00}, \\
B &= C_{m0}^{m0} \text{ (where } m = 4, 5). \end{align*} \]

Hence, the latter sequent is of the following form:
\[ C_{00}^{m0} - 4, \perp N, \Gamma, \! \Delta \vdash C_{00}^{m0}. \]

According to Lemma 5.1, such a sequent cannot occur in any derivation in Linear Logic.

**Case 4.2.a** Now assume that our sequent of item (a) is derived from two sequents of the form
\[ K_1, \Gamma_1, \! \Delta_1 \vdash C_{00}^{4}, B \]
and
\[ K_2, \perp N, \Gamma_2, \! \Delta_2 \vdash \]
Then Lemma 4.4 and Lemma 4.3 yield:
\[ \begin{align*}
\#_\perp(K_2) + N &= 1 \pmod{9N}, \\
\text{either } \#_\perp(K_2) &= -N, \\
\text{or } \#_\perp(K_2) &= -2Nk', \text{ for some } k' \leq 5,
\end{align*} \]
which is a contradiction as well.

**Case 4.1.b** Assume that the corresponding sequent of item (b) is derived from two sequents of the form
\[ (H^2_{00} \circ \perp 3)^h_1, \Gamma_1, \! \Delta_1 \vdash C_{00}^{4} \]
and
\[ (H^2_{00} \circ \perp 3)^{h_2}, \perp N, \Gamma_2, \! \Delta_2 \vdash (H^2_{00} \circ \perp 3) \]
where
\[ \begin{align*}
1 &= h_1 + h_2, \\
\Gamma &\supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*} \]

Then Lemma 4.4 and Lemma 4.3 yield a contradiction as follows:
\[ \begin{align*}
(2N + 3)h_1 &= -8N \pmod{9N}, \\
(2N + 3)h_2 + N &= 2N + 3 \pmod{9N}. \end{align*} \]
Then Lemma 4.4 and Lemma 4.3 yield:

\[(H^2_{00} \rightarrow \bot^3)^{h_1}, \Gamma_1, \Delta_1 \vdash C^4_{00}, (H^2_{00} \rightarrow \bot^3)\]

and

\[(H^2_{00} \rightarrow \bot^3)^{h_2}, \bot, \Gamma_2, \Delta_2 \vdash \]

Then, by Lemma 4.4 and Lemma 4.3, we have a contradiction as well:

\[
\begin{cases}
(2N + 3)h_1 & = -6N + 2 \pmod{9N}, \\
(2N + 3)h_2 + N & = 1 \pmod{9N}.
\end{cases}
\]

**Case 4.1.c** According to rule L–o, let our sequent of item (c) be derived from two sequents of the form

\[K_1, (H^2_{00} \rightarrow \bot^3)^{h_1}, \Gamma_1, \Delta_1 \vdash C^4_{00}\]

and

\[K_2, (H^2_{00} \rightarrow \bot^3)^{h_2}, \bot, \Gamma_2, \Delta_2 \vdash \bot^3\]

where

\[
\begin{cases}
K & = K_1, K_2, \\
1 & = h_1 + h_2, \\
\Gamma & \supset \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{cases}
\]

By Lemma 4.4 and Lemma 4.3 we have a contradiction:

\[
\begin{cases}
\#_\bot(K_1) + (2N + 3)h_1 & = -8N \pmod{9N}, \\
\#_\bot(K_2) + (2N + 3)h_2 + N & = 3 \pmod{9N}, \\
\#_\bot(K_1) & = -Nk', \text{ for some } k' \leq 2.
\end{cases}
\]

**Case 4.2.c** Now assume that our sequent of item (c) is derived from two sequents of the form

\[K_1, (H^2_{00} \rightarrow \bot^3)^{h_1}, \Gamma_1, \Delta_1 \vdash C^4_{00}, \bot^3\]

and

\[K_2, (H^2_{00} \rightarrow \bot^3)^{h_2}, \bot, \Gamma_2, \Delta_2 \vdash \]

Then Lemma 4.4 and Lemma 4.3 yield:

\[
\begin{cases}
\#_\bot(K_1) + (2N + 3)h_1 & = -8N + 2 \pmod{9N}, \\
\#_\bot(K_2) + (2N + 3)h_2 + N & = 1 \pmod{9N}, \\
\#_\bot(K_2) & = -Nk', \text{ for some } k' \leq 2,
\end{cases}
\]

which is also a contradiction.

**Case 4.1.d+e** Let the corresponding sequent of item (d) or (e) be derived from two sequents of the form

\[H^{h_1}_{00}, \bot^a, \Gamma_1, \Delta_1 \vdash C^4_{00}\]

and

\[H^{h_2}_{00}, \bot^a + N, \Gamma_2, \Delta_2 \vdash \]

where

\[
\begin{cases}
1 & = h_1 + h_2, \\
N + a & = a_1 + a_2, \\
\Gamma & \supset \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{cases}
\]

By Lemma 4.4 and Lemma 4.3 we have:

\[
\begin{cases}
-Nh_1 + a_1 & = -8N \pmod{9N}, \\
-Nh_2 + a_2 + N & = a \pmod{9N}.
\end{cases}
\]

Assuming that \(h_1 = 0\), we can conclude that \(a_1 = N\), which gives a contradiction because, by Lemma 5.1, the first sequent with its \textit{wrong} right-hand side cannot occur in our derivation.

For \(h_2 = 0\), we can get also a contradiction because of

\[N - 2 \leq a_2 + (N - a) \leq 2N + 1.
\]

**Case 4.2.d** Let the corresponding sequent of item (d) be derived from two sequents of the form

\[H^{h_1}_{00}, \bot^a, \Gamma_1, \Delta_1 \vdash C^4_{00}, \bot^a\]

and

\[H^{h_2}_{00}, \bot^a + N, \Gamma_2, \Delta_2 \vdash \]

where

\[
\begin{cases}
1 & = h_1 + h_2, \\
N + a & = a_1 + a_2, \\
\Gamma & \supset \Gamma_1, \Gamma_2, \\
\Delta & = \Delta_1, \Delta_2.
\end{cases}
\]

By Lemma 4.4 and Lemma 4.3 we have:

\[
\begin{cases}
-Nh_1 + a_1 & = -8N + a - 1 \pmod{9N}, \\
-Nh_2 + a_2 + N & = 1 \pmod{9N}.
\end{cases}
\]

Assuming that \(h_2 = 0\), we get a contradiction because of

\[N - 1 \leq a_2 + (N - 1) \leq 2N + 1.
\]

For \(h_2 = 1\), we have that \(a_2 + N = N + 1\), which gives a contradiction because, according to the inductive hypothesis from item (e), the latter sequent cannot occur in our derivation.

**Case 5** Lastly, let the left-hand \textit{principal} formula belong neither to \(\Gamma\) nor to \(\Delta\).

**Case 5.a.1** Assume that the \textit{principal} formula is of the form

\[C^0_{00} = ((H^2_{00} \rightarrow \bot^3) \rightarrow \bot^3),\]
and, according to rule L-\circ, the corresponding sequent of item (a) is derived from two sequents of the form

$$K_1, \Gamma_1, !\Delta_1 \vdash (H_{00}^2 \circ \bot^3)$$

and

$$K_2, \bot^3, \Gamma_2, !\Delta_2 \vdash B$$

where

$$\left\{ \begin{array}{l} K \supset K_1, K_2, \\ \Gamma = \Gamma_1, \Gamma_2, \\ \Delta = \Delta_1, \Delta_2. \end{array} \right.$$ Then, by Lemma 4.4 and Lemma 4.3, the following contradiction is immediate:

$$\left\{ \begin{array}{l} \#_\bot(K_1) = 2N + 3 \pmod{9N}, \\ \#_\bot(K_2) + 3 = \#_\bot(B) \pmod{9N}, \\ \text{either } \#_\bot(K_1) = -N, \\ \text{or } \#_\bot(K_1) = -2Nk', \text{ for some } k' \leq 4. \end{array} \right.$$ If our sequent of item (a) were derived from two sequents of the form

$$K_1, \Gamma_1, !\Delta_1 \vdash (H_{00}^2 \circ \bot^3), B$$

and

$$K_2, \bot^3, \Gamma_2, !\Delta_2 \vdash$$

then we had a contradiction as well:

$$\left\{ \begin{array}{l} \#_\bot(K_1) = 2N + 2 + \#_\bot(B) \pmod{9N}, \\ \#_\bot(K_2) + 3 = 1 \pmod{9N}, \\ \text{either } \#_\bot(K_2) = -N, \\ \text{or } \#_\bot(K_2) = -2Nk', \text{ for some } k' \leq 4. \end{array} \right.$$ **Case 5.a.2** Assume that the principal formula is of the form

$$H_{00} = (\bot^N \circ - \circ \bot^N),$$

and our sequent of item (a) is derived from two sequents of the form

$$H_{00}^{h_1}, \Gamma_1, !\Delta_1 \vdash \bot^N + 2$$

and

$$H_{00}^{h_2}, \bot^3, \Gamma_2, !\Delta_2 \vdash B$$

where

$$\left\{ \begin{array}{l} 1 \geq h_1 + h_2, \\ \Gamma = \Gamma_1, \Gamma_2, \\ \Delta = \Delta_1, \Delta_2. \end{array} \right.$$ Then Lemma 4.4 and Lemma 4.3 yield the following contradiction:

$$\left\{ \begin{array}{l} -Nh_1 = N + 2 \pmod{9N}, \\ -Nh_2 + 2 = \#_\bot(B) \pmod{9N}. \end{array} \right.$$ If our sequent of item (a) were derived from two sequents of the form

$$H_{00}^{h_1}, \Gamma_1, !\Delta_1 \vdash \bot^N + 2, B$$

and

$$H_{00}^{h_2}, \bot^3, \Gamma_2, !\Delta_2 \vdash$$

then we got a contradiction as follows:

$$\left\{ \begin{array}{l} -Nh_1 = N + 1 + \#_\bot(B) \pmod{9N}, \\ -Nh_2 + 2 = 1 \pmod{9N}. \end{array} \right.$$ **Case 5.b** Assume that the principal formula is of the form

$$(H_{00}^2 \circ \bot^3),$$

and the corresponding sequent of item (b) is derived from two sequents of the form

$$\Gamma_1, !\Delta_1 \vdash H_{00}^2$$

and

$$\bot^3, \Gamma_2, !\Delta_2 \vdash (H_{00}^2 \circ \bot^3)$$

where

$$\left\{ \begin{array}{l} \Gamma = \Gamma_1, \Gamma_2, \\ \Delta = \Delta_1, \Delta_2. \end{array} \right.$$ Then, by Lemma 4.4 and Lemma 4.3, a contradiction is immediate:

$$\left\{ \begin{array}{l} 0 = -2N \pmod{9N}, \\ 3 = 2N + 3 \pmod{9N}. \end{array} \right.$$ If our sequent of item (b) is derived from two sequents of the form

$$\Gamma_1, !\Delta_1 \vdash H_{00}^2, (H_{00}^2 \circ \bot^3)$$

and

$$\bot^3, \Gamma_2, !\Delta_2 \vdash$$

then we get also a contradiction:

$$\left\{ \begin{array}{l} 0 = 2 \pmod{9N}, \\ 3 = 1 \pmod{9N}. \end{array} \right.$$ **Case 5.c.1** Suppose that the principal formula is of the form

$$C_{00} = ((H_{00}^2 \circ \bot^3) \circ \bot^3),$$

and, according to rule L-\circ, the corresponding sequent of item (c) is derived from two sequents of the form

$$(H_{00}^2 \circ \bot^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash (H_{00}^2 \circ \bot^3)$$
Then Lemma 4.4 and Lemma 4.3 yield the following contradiction:
\[
\begin{cases}
-Nk_1 + (2N + 3)h_1 &= N + 2 \pmod{9N}, \\
-Nk_2 + (2N + 3)h_2 + 2 &= 3 \pmod{9N}.
\end{cases}
\]
If our sequent of item (c) were derived from two sequents of the form
\[
H_{00}^{k_1}, (H_{00}^2 \rightarrow \bot^3)^{h_1}, \Gamma_1, !\Delta_1 \vdash \bot + N + 2, \bot^3
\]
and
\[
H_{00}^{k_2}, (H_{00}^2 \rightarrow \bot^3)^{h_2}, \bot^2, \Gamma_2, !\Delta_2 \vdash
\]
then we got a contradiction as follows:
\[
\begin{cases}
-Nk_1 + (2N + 3)h_1 &= N + 4 \pmod{9N}, \\
-Nk_2 + (2N + 3)h_2 + 2 &= 1 \pmod{9N}.
\end{cases}
\]
**Case 5.c.3** Suppose that the principal formula is of the form
\[(H_{00}^2 \rightarrow \bot^3),\]
and our sequent of item (c) is derived from two sequents of the form
\[
K_1, \Gamma_1, !\Delta_1 \vdash H_{00}^2
\]
and
\[
K_2, \bot^3, \Gamma_2, !\Delta_2 \vdash \bot^3
\]
where
\[
\begin{cases}
K &= K_1, K_2, \\
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{cases}
\]
Then Lemma 4.4 and Lemma 4.3 yield:
\[
\begin{cases}
\#_\bot(K) + 2N + 3 &= 3 \pmod{9N}, \\
\#_\bot(K_1) &= -2N \pmod{9N}, \\
\#_\bot(K_1) &= -Nk', \text{ for some } k' \leq 2.
\end{cases}
\]
The only solution of this system is the following:
\[K_1 = K.\]
By applying the inductive hypothesis from item (a) and Lemma 5.1 to our both sequents, we can get the emptiness of both \(\Gamma_1\) and \(\Gamma_2\), and the degeneracy of both \(!\Delta_1\) and \(!\Delta_2\), which results in the desired emptiness of the whole \(\Gamma\) and the degeneracy of the whole \(!\Delta\).
If our sequent of item (c) were derived from two sequents of the form
\[K_1, \Gamma_1, !\Delta_1 \vdash H_{00}^2, \bot^3\]
and
\[ K_2, \bot^3, \Gamma_2, \!\Delta_2 \vdash \]
then we had an immediate contradiction:
\[
\begin{align*}
\#(K_1) &= -2N + 2 \pmod{9N}, \\
\#(K_2) + 3 &= 1 \pmod{9N}, \\
\#(K_2) &= -Nk', \text{ for some } k' \leq 2.
\end{align*}
\]

**Case 5.d+e** Suppose that the principal formula is of the form
\[ H_{00} = (\bot N + 2 - \cdots N^2), \]
and our sequent of item (d) or (e) is derived from two sequents of the form
\[ \bot^a_1, \Gamma_1, \!\Delta_1 \vdash \bot N + 2 \]
and
\[ \bot^a_2, \Gamma_2, \!\Delta_2 \vdash \bot^a \]
where
\[
\begin{align*}
N + a &= a_1 + a_2, \\
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]
By Lemma 4.4 and Lemma 4.3, we have:
\[
\begin{align*}
a_1 &= N + 2 \pmod{9N}, \\
a_2 + 2 &= a \pmod{9N}.
\end{align*}
\]
The only solution is as follows:
\[
\begin{align*}
a &= 2, \\
a_1 &= N + 2, \\
a_2 + 2 &= a.
\end{align*}
\]
Then, by applying Lemma 5.1 to our both sequents, we can get the desired emptiness of the whole \( \Gamma \) and the degeneracy of the whole \( \!\Delta \).

If our sequent of item (d) were derived from two sequents of the form
\[ \bot^a_1, \Gamma_1, \!\Delta_1 \vdash \bot N + 2, \bot^a \]
and
\[ \bot^a_2, \Gamma_2, \!\Delta_2 \vdash \bot^a \]
then we got an immediate contradiction:
\[
\begin{align*}
a_1 &= N + 1 + a \pmod{9N}, \\
a_2 + 2 &= 1 \pmod{9N}.
\end{align*}
\]

Now, extracting the possible cases from this huge amount of inconsistency, we can complete Lemma 5.2.

### 5.3 Lemma 5.3

**Lemma 5.3** Let \( \Delta \) consist of formulas of the form \( F_A \), and \( \Gamma \) consist of formulas of the form \( H_1, D_q, F_A, \) and \( F_Y \).

Let \( a \) be an integer such that
\[ 4 \leq a \leq N - 3. \]

(a) Let \( B \) be a formula either of the form
\[ H_1 = (C^4_{00} - \bot^N) \]
or of the form
\[ ((H_1 - \bot^a) - \bot^a) \]
If a sequent of the form
\[ \Gamma, \!\Delta \vdash B \]
occurs in a cut-free derivation in Linear Logic then \( \Gamma \) must be a singleton of the form either
\[ \Gamma = (C^4_{00} - \bot^N) \]
or
\[ \Gamma = ((H_1 - \bot^a) - \bot^a), \]
and \( \!\Delta \) can be produced by rules \( W! \) and \( C! \) only (there is no applications of rule \( L! \) in the derivation above this sequent).\(^7\)

(b) If a sequent of the form
\[ (H_1 - \bot^a), \Gamma, \!\Delta \vdash (H_1 - \bot^a) \]
occurs in a cut-free derivation in Linear Logic then \( \Gamma \) must be empty, and \( \!\Delta \) can be produced by rules \( W! \) and \( C! \) only.

(c) If a sequent of the form
\[ (H_1 - \bot^a), \Gamma, \!\Delta \vdash \bot^a \]
occurs in a cut-free derivation in Linear Logic then \( \Gamma \) must be a singleton of the form either
\[ \Gamma = (C^4_{00} - \bot^N) \]
or
\[ \Gamma = ((H_1 - \bot^a) - \bot^a), \]
and \( \!\Delta \) can be produced by rules \( W! \) and \( C! \) only.

\(^7\)We say that such a \( \!\Delta \) is degenerate.
(d) If a sequent of the form
\[ C^4_{00}, \Gamma, !\Delta \vdash \bot^N \]
occurs in a cut-free derivation in Linear Logic then \( \Gamma \) must be a singleton of the form
\[ \Gamma = H_1 = (C^4_{00} \to \bot^N), \]
and \( !\Delta \) can be produced by rules \( W! \) and \( C! \) only.

**Proof.** We use induction on a given derivation. Regarding to the form of the principal formula at a current point of the derivation, we will demonstrate that each of the undesirable cases is inconsistent.

**Case 0** The principal formula belongs to \( !\Delta \).
Assume that it is produced by rule \( L! \), and our sequent of the form
\[ \ldots, \Gamma, !\Delta \vdash \ldots \]
is derived from a sequent of the form
\[ \ldots, \Gamma, F_A, !\Delta' \vdash \ldots \]
Then we get a contradiction because, according to the inductive hypothesis, the form of the multiset
\[ \Gamma, F_A \]
is not correct.
Hence, the only possibility is to apply either \( W! \) or \( C! \). It remains to use the inductive hypothesis for completing this case.

**Case 1** The right-side formula is principal.
There are the following cases to be considered.

**Case 1.a** For item (a), let us consider two possible versions of the principal formula \( B \).

**Case 1.a.1** The principal formula \( B \) is of the form
\[ H_1 = (C^4_{00} \to \bot^N) \]
and, according to rule \( R\to \), our sequent of item (a) is derived from the sequent
\[ C^4_{00}, \Gamma, !\Delta \vdash \bot^N, \]
Now we can apply the inductive hypothesis from item (d).

**Case 1.a.2** The principal formula \( B \) is of the form
\[ ((H_1 \to \bot^a) \to \bot^a) \]
and, according to rule \( R\to \), our sequent of item (a) is derived from the sequent
\[ (H_1 \to \bot^a), \Gamma, !\Delta \vdash \bot^a. \]
It remains to apply the inductive hypothesis from item (c).

**Case 1.b** The principal formula is of the form \( (H_1 \to \bot^a) \), and, according to rule \( R\to \), the corresponding sequent of item (b) is derived from the sequent
\[ H_1, (H_1 \to \bot^a), \Gamma, !\Delta \vdash \bot^a. \]
By applying the inductive hypothesis from item (c), we prove that \( !\Delta \) is degenerate and that the multiset
\[ H_1, \Gamma \]
should be a singleton that means the emptiness of \( \Gamma \).

**Case 1.c** Assume that the corresponding sequent of item (c) is derived from two sequents of the form
\[ (H_1 \to \bot^a)^{h_1}, \Gamma_1, !\Delta_1 \vdash \bot \]
and
\[ (H_1 \to \bot^a)^{h_2}, \Gamma_2, !\Delta_2 \vdash \bot^a - 1 \]
where
\[
\begin{align*}
1 &= h_1 + h_2, \\
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]
By Lemma 4.4 and Lemma 4.3 we have a contradiction:
\[
\begin{cases}
ah_1 &= 1 \pmod{9N}, \\
ah_2 &= a - 1 \pmod{9N}.
\end{cases}
\]

**Case 1.d** Assume that the corresponding sequent of item (d) is derived from two sequents of the form
\[ C^{k_1}_{00}, \Gamma_1, !\Delta_1 \vdash \bot \]
and
\[ C^{k_2}_{00}, \Gamma_2, !\Delta_2 \vdash \bot^N - 1 \]
where
\[
\begin{cases}
4 &= k_1 + k_2, \\
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{cases}
\]
Then a contradiction is immediate:
\[
\begin{cases}
-2Nk_1 &= 1 \pmod{9N}, \\
-2Nk_2 &= N - 1 \pmod{9N}.
\end{cases}
\]

**Case 2** Assume that the principal formula belongs to \( \Gamma \), and it is of the form \( F_A \) (or \( F_Y \)).
The following subcases are to be considered.

**Case 2.0** The principal formula is of the form \( (F_{A_1} & F_{A_2}) \), and, by rule \( L\& \), the corresponding sequent of the form
\[ \ldots, \Gamma, !\Delta \vdash \ldots \]
is derived either from the sequent
\[ \ldots, \Gamma', F_A, !\Delta \vdash \ldots \]
or from the sequent
\[ \ldots, \Gamma', F_A, !\Delta \vdash \ldots \]
Then we have a contradiction because, according to the inductive hypothesis, either the form of the multiset
\[ \Gamma', F_A \]
is not correct, or the form of the multiset
\[ \Gamma', F_A \]
is not correct.

**Case 2.1.a** Assume that the principal formula is of the form \( (E_X \rightarrow E_Y) \), and, according to rule L→, the sequent of item (a) is derived from two sequents of the form
\[ \Gamma, !\Delta_1 \vdash E_X \]
and
\[ E_Y, \Gamma_2, !\Delta_2 \vdash B \]
where
\[
\begin{align*}
0 &= 6N \pmod{9N}, \\
6N &= \#_\perp(B) \pmod{9N},
\end{align*}
\]
which is a contradiction.

**Case 2.2.a** Assume that the principal formula is of the form \( (E_X \leftarrow E_Y) \), and, our sequent of item (a) is derived from two sequents of the form
\[ \Gamma, !\Delta_1 \vdash E_X, B \]
and
\[ E_Y, \Gamma_2, !\Delta_2 \vdash \]
Then Lemma 4.4 and Lemma 4.3 yield a contradiction:
\[
\begin{align*}
0 &= 6N + \#_\perp(B) - 1 \pmod{9N}, \\
6N &= 1 \pmod{9N},
\end{align*}
\]

**Case 2.3.a** Assume that the principal formula is of the form \( (F_A \leftarrow F_Y) \), and, according to rule L←, our sequent of item (a) is derived from two sequents of the form
\[ \Gamma, !\Delta_1 \vdash F_A \]
and
\[ F_Y, \Gamma_2, !\Delta_2 \vdash B \]
where
\[
\begin{align*}
\{ \Gamma &
\rightarrow \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\}
\]
Then we can get a contradiction because, according to the inductive hypothesis, the multiset
\[ F_Y, \Gamma_2 \]
must be a singleton of the differing form.

**Case 2.4.a** Assume that the principal formula is of the form \( (F_A \rightarrow F_Y) \), and, our sequent of item (a) is derived from two sequents of the form
\[ \Gamma, !\Delta_1 \vdash F_A, B \]
and
\[ F_Y, \Gamma_2, !\Delta_2 \vdash \]
where
\[
\begin{align*}
\{ \Gamma &
\rightarrow \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\}
\]
Then Lemma 4.4 and Lemma 4.3 show the following contradiction:
\[
\begin{align*}
0 &= \#_\perp(B) - 1 \pmod{9N}, \\
0 &= 1 \pmod{9N}.
\end{align*}
\]

**Case 2.1.b** Assume that the principal formula is of the form \( (E_X \leftarrow E_Y) \), and, our sequent of item (b) is derived from two sequents of the form
\[ (H_1 \rightarrow \perp \ a)^{h_1}, \Gamma, !\Delta_1 \vdash E_X \]
and
\[ (H_1 \rightarrow \perp \ a)^{h_2}, E_Y, \Gamma_2, !\Delta_2 \vdash (H_1 \rightarrow \perp \ a) \]
where
\[
\begin{align*}
1 &= h_1 + h_2, \\
\Gamma &
\rightarrow \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]
Then Lemma 4.4 and Lemma 4.3 yield a contradiction:
\[
\begin{align*}
ah_1 &= 6N \pmod{9N}, \\
ah_2 + 6N &= a \pmod{9N},
\end{align*}
\]
which is a contradiction.

**Case 2.2.b** Assume that the principal formula is of the form \( (E_X \rightarrow E_Y) \), and, our sequent of item (b) is derived from two sequents of the form
\[ (H_1 \rightarrow \perp \ a)^{h_1}, \Gamma, !\Delta_1 \vdash E_X, (H_1 \rightarrow \perp \ a) \]
and
\[ (H_1 \rightarrow \perp \ a)^{h_2}, E_Y, \Gamma_2, !\Delta_2 \vdash \]
Then Lemma 4.4 and Lemma 4.3 yield also a contradiction:

\[
\begin{cases}
    ah_1 &= 6N + a - 1 \pmod{9N}, \\
    ah_2 + 6N &= 1 \pmod{9N}.
\end{cases}
\]

**Case 2.3.b** Assume that the principal formula is of the form \((FA \cdash \neg FY)\), and our sequent of item (b) is derived from two sequents of the form

\[
(H_1 \neg \perp^a)^{h_1}, \Gamma_1, \neg \Delta_1 \vdash FA
\]

and

\[
(H_1 \neg \perp^a)^{h_2}, FY, \Gamma_2, \neg \Delta_2 \vdash (H_1 \neg \perp^a)
\]

where

\[
\begin{cases}
    \Gamma &\supset \Gamma_1, \Gamma_2, \\
    \Delta &\supset \Delta_1, \Delta_2.
\end{cases}
\]

According to Lemma 4.4 and Lemma 4.3, we have:

\[
\begin{cases}
    ah_1 &= 0 \pmod{9N}, \\
    ah_2 &= a \pmod{9N}.
\end{cases}
\]

Hence,

\[h_2 = 1,\]

and we can get a contradiction because, by the inductive hypothesis, the non-empty multiset

\[FY, \Gamma_2\]

must be empty.

**Case 2.4.b** Assume that the principal formula is of the form \((FA \cdash \neg FY)\), and now our sequent of item (b) is derived from two sequents of the form

\[
(H_1 \neg \perp^a)^{h_1}, \Gamma_1, \neg \Delta_1 \vdash FA, (H_1 \neg \perp^a)
\]

and

\[
(H_1 \neg \perp^a)^{h_2}, FY, \Gamma_2, \neg \Delta_2 \vdash 
\]

Then Lemma 4.4 and Lemma 4.3 yield a contradiction as well:

\[
\begin{cases}
    ah_1 &= a - 1 \pmod{9N}, \\
    ah_2 &= 1 \pmod{9N}.
\end{cases}
\]

**Case 2.1.c** Assume that the principal formula is of the form \((EX \neg \neg FY)\), and our sequent of item (c) is derived from two sequents of the form

\[
(H_1 \neg \perp^a)^{h_1}, \Gamma_1, \neg \Delta_1 \vdash EX
\]

and

\[
(H_1 \neg \perp^a)^{h_2}, FY, \Gamma_2, \neg \Delta_2 \vdash \perp^a
\]

where

\[
\begin{cases}
    1 &= h_1 + h_2, \\
    \Gamma &\supset \Gamma_1, \Gamma_2, \\
    \Delta &\supset \Delta_1, \Delta_2.
\end{cases}
\]

Then a contradiction is as follows:

\[
\begin{cases}
    ah_1 &= 6N \pmod{9N}, \\
    ah_2 + 6N &= a \pmod{9N}.
\end{cases}
\]

**Case 2.2.c** Assume that the principal formula is of the form \((EX \neg \neg FY)\), and our sequent of item (c) is derived from two sequents of the form

\[
(H_1 \neg \perp^a)^{h_1}, \Gamma_1, \neg \Delta_1 \vdash EX, \perp^a
\]

and

\[
(H_1 \neg \perp^a)^{h_2}, FY, \Gamma_2, \neg \Delta_2 \vdash
\]

Then we have an immediate contradiction:

\[
\begin{cases}
    ah_1 &= 6N + a - 1 \pmod{9N}, \\
    ah_2 + 6N &= 1 \pmod{9N}.
\end{cases}
\]

**Case 2.3.c** Assume that the principal formula is of the form \((FA \neg \neg FY)\), and our sequent of item (c) is derived from two sequents of the form

\[
(H_1 \neg \perp^a)^{h_1}, \Gamma_1, \neg \Delta_1 \vdash FA
\]

and

\[
(H_1 \neg \perp^a)^{h_2}, FY, \Gamma_2, \neg \Delta_2 \vdash \perp^a
\]

where

\[
\begin{cases}
    1 &= h_1 + h_2, \\
    \Gamma &\supset \Gamma_1, \Gamma_2, \\
    \Delta &\supset \Delta_1, \Delta_2.
\end{cases}
\]

According to Lemma 4.4 and Lemma 4.3, we have:

\[
\begin{cases}
    ah_1 &= 0 \pmod{9N}, \\
    ah_2 &= a \pmod{9N}.
\end{cases}
\]

Then

\[h_2 = 1,\]

and we get a contradiction because, according to the inductive hypothesis, the multiset

\[FY, \Gamma_2\]

must be a singleton of the differing form.

**Case 2.4.c** Assume that the principal formula is of the form \((FA \neg \neg FY)\), and our sequent of item (c) is derived from two sequents of the form

\[
(H_1 \neg \perp^a)^{h_1}, \Gamma_1, \neg \Delta_1 \vdash FA, \perp^a
\]
and
\[(H_1 \rightarrow \bot^a)^b_2, F_Y, \Gamma_2, !\Delta_2 \vdash\]

Then a contradiction is immediate:
\[
\begin{cases}
ah_1 = a - 1 \pmod{9N}, \\
ah_2 = 1 \pmod{9N}.
\end{cases}
\]

**Case 2.1.d** Assume that the principal formula is of the form \((E_X \rightarrow E_Y)\), and the corresponding sequent of item (d) is derived from two sequents of the form
\[
C_{00}^k, \Gamma_1, !\Delta_1 \vdash E_X
\]
and
\[
C_{00}^k, E_Y, \Gamma_2, !\Delta_2 \vdash \bot^N
\]
where
\[
\begin{cases}
4 = k_1 + k_2, \\
\Gamma \supset \Gamma_1, \Gamma_2, \\
\Delta = \Delta_1, \Delta_2.
\end{cases}
\]

Then, by Lemma 4.4 and Lemma 4.3 we have:
\[
\begin{cases}
-2Nk_1 = 6N \pmod{9N}, \\
-2Nk_2 + 6N = N \pmod{9N},
\end{cases}
\]
which is a contradiction.

**Case 2.2.d** Assume that the principal formula is of the form \((E_X \rightarrow E_Y)\), and our sequent of item (d) is derived from two sequents of the form
\[
C_{00}^k, \Gamma_1, !\Delta_1 \vdash E_X, \bot^N
\]
and
\[
C_{00}^k, E_Y, \Gamma_2, !\Delta_2 \vdash
\]
Then we get also a contradiction:
\[
\begin{cases}
-2Nk_1 = 7N - 1 \pmod{9N}, \\
-2Nk_2 + 6N = 1 \pmod{9N}.
\end{cases}
\]

**Case 2.3.d** Assume that the principal formula is of the form \((F_A \rightarrow F_Y)\), and our sequent of item (d) is derived from two sequents of the form
\[
C_{00}^k, \Gamma_1, !\Delta_1 \vdash F_A
\]
and
\[
C_{00}^k, F_Y, \Gamma_2, !\Delta_2 \vdash \bot^N
\]
where
\[
\begin{cases}
4 = k_1 + k_2, \\
\Gamma \supset \Gamma_1, \Gamma_2, \\
\Delta = \Delta_1, \Delta_2.
\end{cases}
\]

According to Lemma 4.4 and Lemma 4.3, we have:
\[
\begin{cases}
-2Nk_1 = 0 \pmod{9N}, \\
-2Nk_2 = N \pmod{9N}.
\end{cases}
\]

Then
\[k_2 = 4,\]
and we get a contradiction because, according to the inductive hypothesis, the multiset
\[F_Y, \Gamma_2\]
must be a singleton of the differing form.

**Case 2.4.d** Assume that the principal formula is of the form \((F_A \rightarrow F_Y)\), and our sequent of item (d) is derived from two sequents of the form
\[
C_{00}^k, \Gamma_1, !\Delta_1 \vdash F_A, \bot^N
\]
and
\[
C_{00}^k, F_Y, \Gamma_2, !\Delta_2 \vdash
\]
Then we can get a contradiction as follows:
\[
\begin{cases}
-2Nk_1 = N - 1 \pmod{9N}, \\
-2Nk_2 = 1 \pmod{9N}.
\end{cases}
\]

**Case 2.5** Case of the principal formula of the form \((E_X \rightarrow (E_Y \oplus E_Y))\) is handled similarly to cases 2.1.abcd and cases 2.2.abcd.

**Case 3** Assume that the principal formula belongs to \(\Gamma\), and it is of the form
\[D_q = ((H_1 \rightarrow \bot^b) \rightarrow \bot^b)\]
where
\[4 \leq b \leq N - 3.\]

**Case 3.1.a** According to rule \(L\rightarrow\), let our sequent of item (a) be derived from two sequents of the form
\[\Gamma_1, !\Delta_1 \vdash (H_1 \rightarrow \bot^b)\]
and
\[\bot^b, \Gamma_2, !\Delta_2 \vdash B\]
where
\[
\begin{cases}
\Gamma \supset \Gamma_1, \Gamma_2, \\
\Delta = \Delta_1, \Delta_2.
\end{cases}
\]

Then Lemma 4.4 and Lemma 4.3 yield a contradiction:
\[
\begin{cases}
b = b \pmod{9N}, \\
b = \#_1(B) \pmod{9N}.
\end{cases}
\]

**Case 3.2.a** Now let our sequent of item (a) be derived from two sequents of the form
\[\Gamma_1, !\Delta_1 \vdash (H_1 \rightarrow \bot^b), B\]
and
\[\bot^b, \Gamma_2, !\Delta_2 \vdash\]
Then we have the following contradiction:

\[
\begin{aligned}
0 &= b + \#(B) - 1 \pmod{9N}, \\
B &= 1 \pmod{9N}.
\end{aligned}
\]

**Case 3.1.b** According to rule L→, assume that the corresponding sequent of item (b) is derived from two sequents of the form

\[
(H_1 \rightarrow \bot a)^{h_1}, \Gamma_1, !\Delta_1 \vdash (H_1 \rightarrow \bot b)
\]

and

\[
(H_1 \rightarrow \bot a)^{h_2}, \bot b, \Gamma_2, !\Delta_2 \vdash (H_1 \rightarrow \bot a)
\]

where

\[
\begin{aligned}
1 &= h_1 + h_2, \\
\Gamma &= \Gamma_1, \Gamma_2, ((H_1 \rightarrow \bot b) \rightarrow \bot b), \\
\Delta &= \Delta_1, \Delta_2.
\end{aligned}
\]

By Lemma 4.4 and Lemma 4.3 we have:

\[
\begin{aligned}
a h_1 &= b \pmod{9N}, \\
ah_2 + b &= a \pmod{9N}.
\end{aligned}
\]

Hence,

\[
\begin{aligned}
h_2 &= 0, \\
b &= a.
\end{aligned}
\]

And we have a contradiction because, according to Lemma 5.1, the latter sequent with its wrong right-hand side cannot occur in our derivation.

**Case 3.2.c** Assume that our sequent of item (c) is derived from two sequents of the form

\[
(H_1 \rightarrow \bot a)^{h_1}, \Gamma_1, !\Delta_1 \vdash (H_1 \rightarrow \bot b), \bot a
\]

and

\[
(H_1 \rightarrow \bot a)^{h_2}, \bot b, \Gamma_2, !\Delta_2 \vdash
\]

A contradiction is immediate:

\[
\begin{aligned}
ah_1 &= b + a - 1 \pmod{9N}, \\
ah_2 + b &= 1 \pmod{9N}.
\end{aligned}
\]

**Case 3.1.c** According to rule L→, suppose that the corresponding sequent of item (c) is derived from two sequents of the form

\[
(H_1 \rightarrow \bot a)^{h_1}, \Gamma_1, !\Delta_1 \vdash (H_1 \rightarrow \bot b)
\]

and

\[
(H_1 \rightarrow \bot a)^{h_2}, \bot b, \Gamma_2, !\Delta_2 \vdash \bot a
\]

where

\[
\begin{aligned}
1 &= h_1 + h_2, \\
\Gamma &= \Gamma_1, \Gamma_2, ((H_1 \rightarrow \bot b) \rightarrow \bot b), \\
\Delta &= \Delta_1, \Delta_2.
\end{aligned}
\]

By Lemma 4.4 and Lemma 4.3 we have:

\[
\begin{aligned}
a h_1 &= b \pmod{9N}, \\
ah_2 + b &= a \pmod{9N}.
\end{aligned}
\]

Hence,

\[
\begin{aligned}
h_2 &= 0, \\
b &= a.
\end{aligned}
\]

By applying the inductive hypothesis from item (b) and Lemma 5.1 to both sequents, we prove the *emptiness* of both \( \Gamma_1 \) and \( \Gamma_2 \), and the *degeneracy* of both \( !\Delta_1 \) and \( !\Delta_2 \). Therefore, the whole \( !\Delta \) is degenerate, and the whole \( \Gamma \) is a *singleton* of the form

\[
\Gamma = ((H_1 \rightarrow \bot b) \rightarrow \bot b).
\]

**Case 3.2.d** Let the corresponding sequent of item (d) be derived from two sequents of the form

\[
C_{k_1}^{k_0}, \Gamma_1, !\Delta_1 \vdash (H_1 \rightarrow \bot b)
\]

and

\[
C_{k_2}^{k_0}, \bot b, \Gamma_2, !\Delta_2 \vdash \bot N
\]

where

\[
\begin{aligned}
k_1 &= k_1 + k_2, \\
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{aligned}
\]

Then, by Lemma 4.4 and Lemma 4.3 we have:

\[
\begin{aligned}
-2N k_1 &= b \pmod{9N}, \\
-2N k_2 + b &= N \pmod{9N},
\end{aligned}
\]

which is a contradiction.

**Case 3.2.d** Let our sequent of item (d) be derived from two sequents of the form

\[
C_{k_1}^{k_0}, \Gamma_1, !\Delta_1 \vdash (H_1 \rightarrow \bot b), \bot N
\]

and

\[
C_{k_2}^{k_0}, \bot b, \Gamma_2, !\Delta_2 \vdash
\]
Then we get also a contradiction:

\[
\begin{align*}
-2Nk_1 &= b + N - 1 \pmod{9N}, \\
-2Nk_2 + b &= 1 \pmod{9N}.
\end{align*}
\]

**Case 4** Assume that the principal formula belongs to \( \Gamma \), and it is of the form

\( H_1 = (C^4_{00} \circ \bot N) \).

**Case 4.1.a** According to rule \( \text{L}\!-\!\circ \), assume that the corresponding sequent of item (a) is derived from two sequents of the form

\( \Gamma_1, \bot N, \Gamma_2, \bot \Delta_2 \vdash B \)

and

\( \bot N, \Gamma_2, \bot \Delta_2 \vdash \bot \)

Then Lemma 4.4 and Lemma 4.3 yield a contradiction

\[
\begin{align*}
0 &= -8N \pmod{9N}, \\
N &= \#_\bot(B) \pmod{9N},
\end{align*}
\]

which is a contradiction.

**Case 4.2.a** Now assume that our sequent of item (a) is derived from two sequents of the form

\( \Gamma_1, \bot \Delta_1 \vdash C^4_{00}, B \)

and

\( \bot N, \Gamma_2, \bot \Delta_2 \vdash \bot \)

Then we get contradiction as well:

\[
\begin{align*}
0 &= -8N + \#_\bot(B) - 1 \pmod{9N}, \\
N &= 1 \pmod{9N}.
\end{align*}
\]

**Case 4.1.b** Assume that the corresponding sequent of item (b) is derived from two sequents of the form

\( (H_1 \circ \bot a)^{h_1}, \Gamma_1, \bot \Delta_1 \vdash C^4_{00} \)

and

\( (H_1 \circ \bot a)^{h_2}, \bot N, \Gamma_2, \bot \Delta_2 \vdash (H_1 \circ \bot a) \)

where

\[
\begin{align*}
1 &= h_1 + h_2, \\
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

Then Lemma 4.4 and Lemma 4.3 yield a contradiction as follows:

\[
\begin{align*}
ah_1 &= -8N \pmod{9N}, \\
ah_2 + N &= a \pmod{9N}.
\end{align*}
\]

**Case 4.2.b** Now assume that our sequent of item (b) is derived from two sequents of the form

\( (H_1 \circ \bot a)^{h_1}, \Gamma_1, \bot \Delta_1 \vdash C^4_{00}, (H_1 \circ \bot a) \)

and

\( (H_1 \circ \bot a)^{h_2}, \bot N, \Gamma_2, \bot \Delta_2 \vdash \)

Then, by Lemma 4.4 and Lemma 4.3, we have a contradiction as well:

\[
\begin{align*}
ah_1 &= -8N + a - 1 \pmod{9N}, \\
ah_2 + N &= 1 \pmod{9N}.
\end{align*}
\]

**Case 4.1.c** According to rule \( \text{L}\!-\!\circ \), let our sequent of item (c) be derived from two sequents of the form

\( (H_1 \circ \bot a)^{h_1}, \Gamma_1, \bot \Delta_1 \vdash C^4_{00} \)

and

\( (H_1 \circ \bot a)^{h_2}, \bot N, \Gamma_2, \bot \Delta_2 \vdash \bot a \)

where

\[
\begin{align*}
1 &= h_1 + h_2, \\
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

By Lemma 4.4 and Lemma 4.3 we have a contradiction:

\[
\begin{align*}
ah_1 &= -8N \pmod{9N}, \\
ah_2 + N &= a \pmod{9N}.
\end{align*}
\]

**Case 4.2.c** Now assume that our sequent of item (c) is derived from two sequents of the form

\( (H_1 \circ \bot a)^{h_1}, \Gamma_1, \bot \Delta_1 \vdash C^4_{00}, \bot a \)

and

\( (H_1 \circ \bot a)^{h_2}, \bot N, \Gamma_2, \bot \Delta_2 \vdash \)

Then Lemma 4.4 and Lemma 4.3 yield:

\[
\begin{align*}
ah_1 &= -8N + a - 1 \pmod{9N}, \\
ah_2 + N &= 1 \pmod{9N},
\end{align*}
\]

which is also a contradiction.

**Case 4.1.d** Suppose that the corresponding sequent of item (d) is derived from two sequents of the form

\( C^k_{00}, \Gamma_1, \bot \Delta_1 \vdash C^4_{00} \)

and

\( C^k_{00}, \bot N, \Gamma_2, \bot \Delta_2 \vdash \bot N \)

where

\[
\begin{align*}
k_1 &= k_1 + k_2, \\
k_2 &= \Gamma, \Gamma_2, (C^4_{00} \circ \bot N), \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]
By Lemma 4.4 and Lemma 4.3 we have:
\[
\begin{align*}
-2Nk_1 &= -8N \pmod{9N}, \\
-2Nk_2 + N &= N \pmod{9N}.
\end{align*}
\]
The only solution of this system is as follows:
\[
\begin{align*}
k_1 &= 4, \\
k_2 &= 0.
\end{align*}
\]
Then, by applying Lemma 5.2 and Lemma 5.1 to both sequents, we prove the emptiness of both \(\Gamma_1\) and \(\Gamma_2\), and the degeneracy of both \(!\Delta_1\) and \(!\Delta_2\). Therefore, the whole \(\Delta\) is degenerate, and the whole \(\Gamma\) is a singleton of the form
\[
\Gamma = (C_{00}^4 \circ \perp N).
\]

**Case 4.2.d** Assuming that our sequent of item (d) is derived from two sequents of the form
\[
C_{00}^{k_1}, \Gamma_1, \!\Delta_1 \vdash C_{00}^4, \perp N
\]
and
\[
C_{00}^{k_2}, \perp N, \Gamma_2, \!\Delta_2 \vdash
\]
we have a contradiction:
\[
\begin{align*}
-2Nk_1 &= -7N - 1 \pmod{9N}, \\
-2Nk_2 + N &= 1 \pmod{9N}.
\end{align*}
\]

**Case 5** Finally, let us consider the case where the left-hand principal formula belongs neither to \(\Gamma\) nor to \(!\Delta\).

**Case 5.b** Assume that the principal formula is of the form
\[(H_1 \circ \perp^a),\]
and the corresponding sequent of item (b) is derived from two sequents of the form
\[\Gamma_1, \!\Delta_1 \vdash H_1\]
and
\[\perp^a, \Gamma_2, \!\Delta_2 \vdash (H_1 \circ \perp^a)\]
where
\[
\begin{align*}
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]
And we have a contradiction because, according to Lemma 5.1, the latter sequent with its wrong right-hand side cannot occur in our derivation.

If our sequent of item (b) were derived from two sequents of the form
\[\Gamma_1, \!\Delta_1 \vdash H_1, (H_1 \circ \perp^a)\]
and
\[\perp^a, \Gamma_2, \!\Delta_2 \vdash\]
then, by Lemma 4.4 and Lemma 4.3, we got a contradiction as well:
\[
\begin{align*}
0 &= a - 1 \pmod{9N}, \\
a &= 1 \pmod{9N}.
\end{align*}
\]

**Case 5.c** Suppose that the principal formula is of the form
\[(H_1 \circ \perp^a),\]
and the corresponding sequent of item (c) is derived from two sequents of the form
\[\Gamma_1, \!\Delta_1 \vdash H_1\]
and
\[\perp^a, \Gamma_2, \!\Delta_2 \vdash \perp^a\]
where
\[
\begin{align*}
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]
Then, by applying the inductive hypothesis from item (a) and Lemma 5.1 to our both sequents, we can prove that

1. multiset \(\Gamma_1\) must be a singleton of the form either
\[\Gamma_1 = (C_{00}^4 \circ \perp N)\]
or
\[\Gamma_1 = ((H_1 \circ \perp^a) \circ \perp^a),\]
2. multiset \(\Gamma_2\) must be empty,
3. both \(!\Delta_1\) and \(!\Delta_2\) must be degenerate, which results in the desired degeneracy of the whole \(!\Delta\).

Hence, the whole \(\Gamma\) is of the required form
\[\Gamma = \Gamma_1, \Gamma_2\]
If our sequent of item (c) were derived from two sequents of the form
\[\Gamma_1, \!\Delta_1 \vdash H_1, \perp^a\]
and
\[\perp^a, \Gamma_2, \!\Delta_2 \vdash\]
then, by Lemma 4.4 and Lemma 4.3, we had a contradiction:
\[
\begin{align*}
0 &= a - 1 \pmod{9N}, \\
a &= 1 \pmod{9N}.
\end{align*}
\]
Case 5.d  Assume that the principal formula is of the form

\[ C_{00} = (H_{00}^2 \multimap \bot^3) \multimap \bot^3, \]

and, according to rule L\multimap, the corresponding sequent of item (d) is derived from two sequents of the form

\[ C_{00}^{k_1}, \Gamma_1, !\Delta_1 \vdash (H_{00}^2 \multimap \bot^3) \]

and

\[ C_{00}^{k_2}, \bot^3, \Gamma_2, !\Delta_2 \vdash \bot^N \]

where

\[
\begin{align*}
3 &= k_1 + k_2, \\
\Gamma &= \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

Then, by Lemma 4.4 and Lemma 4.3, the following contradiction is immediate:

\[
\begin{align*}
-2Nk_1 &= 2N + 3 \pmod{9N}, \\
-2Nk_2 + 3 &= N \pmod{9N}.
\end{align*}
\]

If our sequent of item (d) were derived from two sequents of the form

\[ C_{00}^{k_1}, \Gamma_1, !\Delta_1 \vdash (H_{00}^2 \multimap \bot^3), \bot^N \]

and

\[ C_{00}^{k_2}, \bot^3, \Gamma_2, !\Delta_2 \vdash \]

then we got a contradiction as well:

\[
\begin{align*}
-2Nk_1 &= 3N + 2 \pmod{9N}, \\
-2Nk_2 + 3 &= 1 \pmod{9N}.
\end{align*}
\]

Now, bringing together all the cases considered, we can complete the proof of Lemma 5.3.

5.4 Lemma 5.4

Lemma 5.4 Let \( \Delta \) consist of formulas of the form \( FA \), and \( \Gamma \) consist of formulas of the form \( Dq, FA, FY \).

Let \( k \) and \( m \) be integers such that

\[
\begin{align*}
0 &\leq k \leq 6, \\
0 &\leq m \leq 5.
\end{align*}
\]

Let \( K \) be a multiset of the form

\[
\overbrace{C_{00}, C_{00}, \ldots, C_{00}}^{k \text{ times}}.
\]

(For \( k = 0 \), \( K \) is the empty multiset.)

Let \( B \) be a formula of the form

\[ B = (C_{00}^m \otimes DZ). \]

If a sequent of the form

\[ K, \Gamma, !\Delta \vdash B \]

occurs in a cut-free derivation in Linear Logic then

1. \( k = m \),
2. this \( \Gamma \) must be an \( n \)-element multiset of the following form

\[ \Gamma = Dq_1, Dq_2, \ldots, Dq_n, \]
3. and \( !\Delta \) can be produced by rules \( W! \) and \( C! \) only (there is no applications of rule \( L! \) in the derivation above this sequent).\(^8\)

Proof. First of all, by Lemma 4.4 and Lemma 4.3:

\[ -2Nk = -2Nm \pmod{9N}. \]

Hence,

\[ k = m \leq 5. \]

Now we will develop induction on a given derivation. Regarding to the form of the principal formula at a current point of the derivation, we will demonstrate that each of the undesirable cases is inconsistent.

Case 0 The principal formula belongs to \( !\Delta \).

Assume that it is produced by rule \( L! \), and our sequent is derived from a sequent of the form

\[ K, \Gamma, FA, !\Delta' \vdash B \]

Then we can get a contradiction because, according to the inductive hypothesis, the multiset

\[ FA, \Gamma \]

must be a multiset of the differing form.

Hence, the only possibility is to apply either \( W! \) or \( C! \). It remains to use the inductive hypothesis for completing this case.

Case 1 The right-side formula \( B \) is principal. Let us consider four possible versions of the principal formula \( B \).

Case 1.1 The principal formula \( B \) is of the form

\[ B = (C_{00}^m \otimes DZ) \]

where

\[ DZ = (Dq_1 \otimes Dq_2 \otimes \cdots \otimes Dq_n), \]

and, according to rule \( R\otimes \), our sequent is derived from two sequents of the form

\[ C_{00}^{k_1}, \Gamma_1, !\Delta_1 \vdash C_{00} \]

and

\[ C_{00}^{k_2}, \Gamma_2, !\Delta_2 \vdash (C_{00}^m - 1 \otimes DZ) \]

\( ^8 \)We will say that such a \( !\Delta \) is degenerate.
where
\[
\begin{align*}
  k &= k_1 + k_2, \\
  \Gamma &= \Gamma_1, \Gamma_2, \\
  \Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

Then, by applying Lemma 5.2 and the inductive hypothesis to our both sequents, we can prove that

1. multiset $\Gamma_1$ must be empty,
2. multiset $\Gamma_2$ must be a multiset of the form $\Gamma_2 = Dq_1, Dq_2, \ldots, Dq_n$,
3. both $\Delta_1$ and $\Delta_2$ must be degenerate, which results in the desired degeneracy of the whole $\Delta$.

Hence, the whole $\Gamma$ is of the required form $\Gamma = Dq_1, Dq_2, \ldots, Dq_n$.

**Case 1.2** The principal formula $B$ is of the form $B = (Dq_1 \otimes DZ')$ where $DZ' = (Dq_2 \otimes \cdots \otimes Dq_n)$, and, according to rule $R \otimes$, our sequent is derived from two sequents of the form $\Gamma_1, \Delta_1 \vdash Dq_1$ and $\Gamma_2, \Delta_2 \vdash DZ'$ where
\[
\begin{align*}
  \Gamma &= \Gamma_1, \Gamma_2, \\
  \Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

Then, by applying Lemma 5.3 and the inductive hypothesis to our both sequents, we can prove that

1. multiset $\Gamma_1$ must be a singleton of the form $\Gamma = Dq_1$,
2. multiset $\Gamma_2$ must be a multiset of the form $\Gamma_2 = Dq_2, \ldots, Dq_n$,
3. both $\Delta_1$ and $\Delta_2$ must be degenerate, which results in the desired degeneracy of the whole $\Delta$.

Hence, the whole $\Gamma$ is of the required form $\Gamma = Dq_1, Dq_2, \ldots, Dq_n$.

**Case 1.3** The principal formula $B$ is of the form $Dq = ((H_1 \circ \perp a) \circ \perp a)$, and, according to rule $R \circ$, our sequent is derived from the sequent $(H_1 \circ \perp a), \Gamma, \Delta \vdash \perp a$.

By Lemma 5.3, $\Gamma$ must be a singleton of the form $\Gamma = Dq_1, Dq_2, \ldots, Dq_n$.

**Case 2** Assume that the principal formula belongs to $\Gamma$, and it is of the form $FA$ (or $FY$).

The following subcases are to be considered.

**Case 2.0** The principal formula is of the form $(FA_1 \& FA_2)$, and, by rule $L \&$, our sequent is derived either from a sequent of the form $K, \Gamma_1, FA_1, \Delta_1 \vdash B$ or from a sequent of the form $K, \Gamma_2, FA_2, \Delta_2 \vdash B$.

Then we get a contradiction because, according to the inductive hypothesis, either the multiset $\Gamma_1, FA_1$ or the multiset $\Gamma_2, FA_2$ is in the wrong form.

**Case 2.1** Assume that the principal formula is of the form $(EX \circ EY)$, and, according to rule $L \circ$, our sequent is derived from two sequents of the form $C_{00}, \Gamma_1, \Delta_1 \vdash EX$ and $C_{00}, \Gamma_2, \Delta_2 \vdash EY$, where
\[
\begin{align*}
  k &= k_1 + k_2, \\
  \Gamma &\supset \Gamma_1, \Gamma_2, \\
  \Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

Then Lemma 4.4 and Lemma 4.3 yield:
\[
\begin{align*}
  -2k_1 &= 6N \pmod{9N}, \\
  -2k_2 + 6N &= -2Nm \pmod{9N},
\end{align*}
\]

9Take into account that $\Gamma$ does not contain any $H_1$. 

**Case 2.0** The principal formula is of the form $(FA_1 \& FA_2)$, and, by rule $L \&$, our sequent is derived either from a sequent of the form $K, \Gamma_1, FA_1, \Delta_1 \vdash B$ or from a sequent of the form $K, \Gamma_2, FA_2, \Delta_2 \vdash B$.

Then we get a contradiction because, according to the inductive hypothesis, either the multiset $\Gamma_1, FA_1$ or the multiset $\Gamma_2, FA_2$ is in the wrong form.

**Case 2.1** Assume that the principal formula is of the form $(EX \circ EY)$, and, according to rule $L \circ$, our sequent is derived from two sequents of the form $C_{00}, \Gamma_1, \Delta_1 \vdash EX$ and $C_{00}, \Gamma_2, \Delta_2 \vdash EY$, where
\[
\begin{align*}
  k &= k_1 + k_2, \\
  \Gamma &\supset \Gamma_1, \Gamma_2, \\
  \Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

Then Lemma 4.4 and Lemma 4.3 yield:
\[
\begin{align*}
  -2k_1 &= 6N \pmod{9N}, \\
  -2k_2 + 6N &= -2Nm \pmod{9N},
\end{align*}
\]

9Take into account that $\Gamma$ does not contain any $H_1$. 

**Case 2.0** The principal formula is of the form $(FA_1 \& FA_2)$, and, by rule $L \&$, our sequent is derived either from a sequent of the form $K, \Gamma_1, FA_1, \Delta_1 \vdash B$ or from a sequent of the form $K, \Gamma_2, FA_2, \Delta_2 \vdash B$.

Then we get a contradiction because, according to the inductive hypothesis, either the multiset $\Gamma_1, FA_1$ or the multiset $\Gamma_2, FA_2$ is in the wrong form.

**Case 2.1** Assume that the principal formula is of the form $(EX \circ EY)$, and, according to rule $L \circ$, our sequent is derived from two sequents of the form $C_{00}, \Gamma_1, \Delta_1 \vdash EX$ and $C_{00}, \Gamma_2, \Delta_2 \vdash EY$, where
\[
\begin{align*}
  k &= k_1 + k_2, \\
  \Gamma &\supset \Gamma_1, \Gamma_2, \\
  \Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

Then Lemma 4.4 and Lemma 4.3 yield:
\[
\begin{align*}
  -2k_1 &= 6N \pmod{9N}, \\
  -2k_2 + 6N &= -2Nm \pmod{9N},
\end{align*}
\]
which is a contradiction.

**Case 2.2** Assume that the principal formula is of the form \((E_X \rightarrow E_Y )\), and our sequent is derived from two sequents of the form

\[
C_{00}^{k_1}, \Gamma_1, !\Delta_1 \vdash E_X, B
\]

and

\[
C_{00}^{k_2}, E_Y, \Gamma_2, !\Delta_2 \vdash
\]

Then Lemma 4.4 and Lemma 4.3 yield:

\[
\begin{cases}
-2Nk_1 &= 6N - 2Nm - 1 \pmod{9N}, \\
-2Nk_2 + 6N &= 1 \pmod{9N},
\end{cases}
\]

which is also a contradiction.

**Case 2.3** Assume that the principal formula is of the form \((F_A \rightarrow F_Y )\), and, according to rule \(L\rightarrow\), our sequent is derived from two sequents of the form

\[
C_{00}^{k_1}, \Gamma_1, !\Delta_1 \vdash F_A
\]

and

\[
C_{00}^{k_2}, F_Y, \Gamma_2, !\Delta_2 \vdash B
\]

where

\[
\begin{cases}
k &= k_1 + k_2, \\
\Gamma \supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{cases}
\]

Then, by Lemma 4.4 and Lemma 4.3, we have:

\[
\begin{cases}
-2Nk_1 &= 0 \pmod{9N}, \\
-2Nk_2 &= -2Nm \pmod{9N}.
\end{cases}
\]

Hence,

\[k_2 = m,\]

and we get a contradiction because, according to the inductive hypothesis, the multiset

\[F_Y, \Gamma_2\]

is in the wrong form.

**Case 2.4** Assume that the principal formula is of the form \((F_A \rightarrow \neg F_Y )\), and our sequent is derived from two sequents of the form

\[
C_{00}^{k_1}, \Gamma_1, !\Delta_1 \vdash F_A, B
\]

and

\[
C_{00}^{k_2}, F_Y, \Gamma_2, !\Delta_2 \vdash
\]

Then Lemma 4.4 and Lemma 4.3 show a contradiction as well:

\[
\begin{cases}
-2Nk_1 &= -2Nm - 1 \pmod{9N}, \\
-2Nk_2 &= 1 \pmod{9N}.
\end{cases}
\]

**Case 2.5** Case of the principal formula of the form \((E_X \rightarrow (E_Y_1 \oplus E_Y_2))\) is handled similarly to Cases 2.1 and Cases 2.2.

**Case 3** Assume that the principal formula belongs to \(\Gamma\), and it is of the form

\[
D_q = ((H_1 \rightarrow \bot^b) \rightarrow \bot^b)
\]

where

\[4 \leq b \leq N - 3.\]

**Case 3.1** According to rule \(L\rightarrow\), let our sequent be derived from two sequents of the form

\[
C_{00}^{k_1}, \Gamma_1, !\Delta_1 \vdash (H_1 \rightarrow \bot^b)
\]

and

\[
C_{00}^{k_2}, \bot^b, \Gamma_2, !\Delta_2 \vdash B
\]

where

\[
\begin{cases}
k &= k_1 + k_2, \\
\Gamma \supset \Gamma_1, \Gamma_2, \\
\Delta &= \Delta_1, \Delta_2.
\end{cases}
\]

Then Lemma 4.4 and Lemma 4.3 yield a contradiction:

\[
\begin{cases}
-2Nk_1 &= b \pmod{9N}, \\
-2Nk_2 + b &= -2Nm \pmod{9N}.
\end{cases}
\]

**Case 3.2** Now let our sequent of be derived from two sequents of the form

\[
C_{00}^{k_1}, \Gamma_1, !\Delta_1 \vdash (H_1 \rightarrow \bot^b), B
\]

and

\[
C_{00}^{k_2}, \bot^b, \Gamma_2, !\Delta_2 \vdash
\]

Then, by Lemma 4.4 and Lemma 4.3, we have:

\[
\begin{cases}
-2Nk_1 &= b - 2Nm - 1 \pmod{9N}, \\
-2Nk_2 + b &= 1 \pmod{9N},
\end{cases}
\]

which is a contradiction as well.

**Case 4** Finally, let the left-hand principal formula belong neither to \(\Gamma\) nor to !\(\Delta\). Hence, it is of the form

\[
C_{00} = ((H_{00}^2 \rightarrow \bot^3) \rightarrow \bot^3),
\]

and, according to rule \(L\rightarrow\), our sequent is derived from two sequents of the form

\[
C_{00}^{k_1}, \Gamma_1, !\Delta_1 \vdash (H_{00}^2 \rightarrow \bot^3)
\]

and

\[
C_{00}^{k_2}, \bot^3, \Gamma_2, !\Delta_2 \vdash B
\]
where
\[
\begin{align*}
    k - 1 &= k_1 + k_2, \\
    \Gamma &= \Gamma_1, \Gamma_2, \\
    \Delta &= \Delta_1, \Delta_2.
\end{align*}
\]

Then, by Lemma 4.4 and Lemma 4.3, the following contradiction is immediate:
\[
\begin{cases}
    -2Nk_1 &= 2N + 3 \pmod{9N}, \\
    -2Nk_2 + 3 &= -2Nm \pmod{9N}.
\end{cases}
\]

If our sequent were derived from two sequents of the form
\[
C_{00}^{k_1}, \Gamma_1, !\Delta_1 \vdash (H_{00}^2 \rightarrow \bot^3), B
\]
and
\[
C_{00}^{k_2}, \bot^3, \Gamma_2, !\Delta_2 \vdash
\]
then we had a contradiction as well:
\[
\begin{cases}
    -2Nk_1 &= 2N + 2 - 2Nm \pmod{9N}, \\
    -2Nk_2 + 3 &= 1 \pmod{9N}.
\end{cases}
\]

Now, bringing together all the cases considered, we can complete the proof of Lemma 5.4.

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