Relativistic energy analysis for D-Dimensional Dirac equation with Eckart plus Hulthen central potential coupled by modified Yukawa tensor potential using Romanovski polynomial method

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Abstract. Romanovski Polynomial method was used to analyze D-dimensional Dirac equation with Eckart plus Hulthen central potential coupled by modified Yukawa tensor potential in the case of spin symmetry and pseudospin symmetry. By using parameter and variable substitution, the Dirac equation was reduced into one dimensional Schrodinger like equation with centrifugal approximation that can be solved using Romanovski polynomial. The D-dimensional relativistic energy are obtained from D-dimensional relativistic energy equation by using Matlab R2008b software. The relativistic energy spectra with dimensional variation were obtained.

1. Introduction

In quantum mechanics, Dirac equation is used to explain dynamic particle with spin ½ [1]. Shape invariance central potential or non-central potential are used to discribe relativistic effect on vibration energy which its complex stucture of molecule [2]. By assumption that scalar potential equal to vector potential, then Dirac equation was reduced to be resemble equation with Schrodinger equation and one dimension Dirac equation can be solved with method used to solved Schrodinger equation [3].

In Dirac equation, there are two cases with depend on the relation between scalar and vector potential. Spin symmetry occurs when $S(r) = V(r)$ and pseudospin symmetry (p-spin) occurs when $S(r) = -V(r)$[4]. The concepts of spin symmetry and pseudospin symmetry are used to study the change of core in nuclear physics. Spin symmetry is applicable at meson and antinucleon spectrum [5], while the concept of pseudospin symmetry used to explain degeneration-quation from nucleon doublet [6].

Romanovski polynomial method can be used for some central potentials, especially the spin symmetry and pseudospin symmetry [7]. This method is traditional method consisting differentials Schrodinger equation by variable change from general hypergeometric equation [8-9].

The aim of the research is to calculate relativistic energy spectra as a function of dimension of system with Matlab R2008b Software from relativistic energy equation.

In this paper, there are several sections are in the basic theory of Dirac equation in this section, Romanovski polynomial method, result and discussion, and the last section is conclusion.
Basic Theory Dirac Equation

The motion of a particle with mass \( M \) in a repulsive vector potential \( V(\mathbf{r}) \) and an attractive scalar potential \( S(\mathbf{r}) \) and coupled by a tensor potential \( U(\mathbf{r}) \) is discribed by the Dirac equation is \([10]\):

\[
\left[ \bar{\alpha} \cdot \mathbf{p} + \beta (M + S(\mathbf{r})) - i \beta \bar{\alpha} \cdot \mathbf{U}(\mathbf{r}) \right] \psi(\mathbf{r}) = [E - V(\mathbf{r})] \psi(\mathbf{r})
\]

(1)

where \( E \) is relativistic energy system and \( \mathbf{p} \) is the three dimensional momentum operator, \( -i \nabla \)

\[
\bar{\alpha} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(2)

The D-dimensional Dirac spinor equation is \([11]\):

\[
\psi_{nk}(\mathbf{r}) = \begin{pmatrix} \zeta_{nk}(\mathbf{r}) \\ \Omega_{nk}(\mathbf{r}) \end{pmatrix} = \frac{1}{r^{D-1}} \left( \begin{array}{c} F_{nk}(\mathbf{r}) Y_{jm}(\theta, \varphi) \\ i G_{nk}(\mathbf{r}) Y_{jm}(\theta, \varphi) \end{array} \right)
\]

(3)

Inserting equation (3) and (2) into equation (1), we obtain :

\[
\begin{align*}
\left[ \sigma \cdot \mathbf{p} - i \sigma \cdot \mathbf{U}(\mathbf{r}) \right] \Omega_{nk}(\mathbf{r}) &= [E - M - V(\mathbf{r}) - S(\mathbf{r})] \zeta_{nk}(\mathbf{r}) \\
\left[ \sigma \cdot \mathbf{p} - i \sigma \cdot \mathbf{U}(\mathbf{r}) \right] \zeta_{nk}(\mathbf{r}) &= [E + M - V(\mathbf{r}) + S(\mathbf{r})] \Omega_{nk}(\mathbf{r})
\end{align*}
\]

(4) \hspace{1cm} (5)

For exact spin symmetry case, happen when the scalar potential is equal to vector potential \( S(\mathbf{r}) = V(\mathbf{r}) \), and for exact pseudospin symmetry case, happen when \( S(\mathbf{r}) = -V(\mathbf{r}) \), then the upper Dirac spinor obtained from equation (4) and (5), and

\[
\begin{align*}
\left[ \sigma \cdot \mathbf{p} - i \sigma \cdot \mathbf{U}(\mathbf{r}) \right] \zeta_{nk}(\mathbf{r}) &= [E^2 - M^2] \Omega_{nk}(\mathbf{r}) \\
\left[ \sigma \cdot \mathbf{p} - i \sigma \cdot \mathbf{U}(\mathbf{r}) \right] \Omega_{nk}(\mathbf{r}) &= [-E^2 + M^2] \zeta_{nk}(\mathbf{r})
\end{align*}
\]

(6) \hspace{1cm} (7)

where \( \mathbf{p}^2 = -\nabla^2 \) with the Laplacian, eigen value of \( L_{D-1} \) is \( l(l + D - 2) \) and the angular momentum operator is \( \frac{L_{D-1}}{r^2} \) \([11-12]\). So that, from equation (6) and (7), we obtain :

\[
\begin{align*}
\frac{\partial^2}{\partial r^2} + \left( \frac{(D-1)}{r} \right) \left( \frac{\partial}{\partial r} \right) + \left( \frac{L_{D-1}}{r^2} \right) - 2V(\mathbf{r})[M + E] + (U(\mathbf{r}))^2 \zeta_{nk}(\mathbf{r}) \\
\frac{\partial^2}{\partial r^2} + \left( \frac{(D-1)}{r} \right) \left( \frac{\partial}{\partial r} \right) + \left( \frac{L_{D-1}}{r^2} \right) + 2V(\mathbf{r})[M - E] + (U(\mathbf{r}))^2 \Omega_{nk}(\mathbf{r})
\end{align*}
\]

(8) \hspace{1cm} (9)

In this paper, \( V(r) \) is Eckart plus Hulthen central potential and \( U(r) \) is modified Yukawa tensor potential.

2. Method

This researcher used finite Romanovski polynomials method used to solve the Dirac equation in the spin symmetry and pseudospin symmetry cases. The Dirac equation for spin symmetry and pseudospin symmetry was reduced into one dimensional Schrodinger like equation. One dimensional Schrodinger equation for the potential of interest was reduced into differential equation of Romanovski polynomial by approximate variable and wave function substitution.

Generalized hypergeometric equation is \([9]\) :
\[
\sigma(s) \frac{d^2 y_n}{ds^2} + \tau(s) \frac{dy_n(s)}{ds} + \lambda_n y_n(s) = 0
\]  
(10)

where:
\[
\sigma(s) = as^2 + bs + c, \quad \tau(s) = ds + e, \quad \text{dan} \quad \lambda_n = -\{n(n-1) + 2n(1-p)\}
\]  
(11)

The Pearson differential equation is:

\[
\frac{d(\sigma(s)w(s))}{ds} = \sigma(s)w(s)
\]  
(12)

Weight function \(w(s)\) can be determined from the solution Person differential equation:

\[
w(s) = \exp\left(\int \frac{(d-2a)s+(e-b)}{as^2+bs+c} ds\right)
\]  
(13)

The appropriate polynomial for weight function equation (13) from Rodrigues representation:

\[
y_n = \frac{1}{w(s)} \frac{d^n}{ds^n}((as^2 + bs + c)w(s))
\]  
(14)

For Romanovski polynomials, the value from parameter in equation (13) are:

\[
a = 1, b = 0, c = 1, d = 2(1 - p), \quad \text{and} \quad e = q \quad \text{with} \quad p > 0
\]  
(15)

Substitution of equation (15) into (13) it can be obtained weight function:

\[
w(s) = (1 + s^2)^{-p}e^{qtan^{-1}(s)}
\]  
(16)

The required condition that the differential Romanovski equation is convergent when \(m' + m < 2p - 1\) and obeys to the integral equation:

\[
\int_{-\infty}^{\infty} w(p,q) R_{m}^{(p,q)}(s) R_{m'}^{(p,q)}(s) ds
\]  
(17)

The differential Romanovski polynomials equation determined with substitution equation (11) and (15) into (10) obtained as:

\[
(1 + s^2) \frac{d^2 R_n^{(p,q)}(s)}{ds^2} + 2s(-p + 1 + q) \frac{d R_n^{(p,q)}(s)}{ds} - \{n(n-1) + 2n(1-p)\} R_n^{(p,q)}(s)
\]  
(18)

In equation (18), \(y_n = R_n^{(p,q)}(s)\). When \(r = f(s)\), eigen function in Romanovski polynomials is:

\[
\Psi(r) = g_n(s) = (1 + s^2)^{\frac{p}{2}}e^{\frac{-q}{2}tan^{-1}(s)} D_n^{(\beta,\alpha)}(s)
\]  
(19)

where:

\[
D_n^{(\beta,\alpha)}(s) = R_n^{(p,q)}(s)
\]  
(20)

Romanovski polynomial from Rodrigues equation with weight function in equation (14) is:

\[
D_n^{(\beta,\alpha)}(s) = R_n^{(p,q)}(s) = \frac{1}{(1 + s^2)^{-p}e^{qtan^{-1}(s)}} \frac{d^n}{ds^n} (1 + s^2)^{n} (1 + s^2)^{-p}e^{qtan^{-1}(s)}
\]  
(21)

Wave function in equation (19) to be:

\[
\psi_n(r) = \frac{1}{\sqrt{\frac{d(f(s))}{ds)}} (1 + s^2)^{\frac{p}{2}}e^{\frac{-q}{2}tan^{-1}(s)} R_n^{(p,q)}(s)
\]  
(22)

Orthogonal integral from Romanovski polynomial is:

\[
\int_{0}^{\infty} \psi_n(r) \psi_m(r) dr = \int_{-\infty}^{\infty} w(p,q) R_n^{(p,q)}(s) R_m^{(p,q)}(s) ds
\]  
(23)
In this case, the value of \( p \) and \( q \) are not \( n \) dependent where \( n \) is the degree of the polynomials.

### 3. Result and Discussion

Equation for Eckart plus Hulthen central potential and equation modified Yukawa like tensor potential can be written [13]:

\[
V(r) = -V_0 \frac{\cosh(\alpha r) - \sinh(\alpha r)}{2 \sinh(\alpha r)} + 4V_1 \cosh^2(\alpha r) - V_2 \coth(\alpha r)
\]

\[
U(r) = -V_3 \frac{(1 + e^{-\alpha r})}{r}
\]

#### 3.1. Spin symmetry case

The D-dimensional Dirac equation for spin case in equation (8), can be written:

\[
\left[ \frac{\partial^2}{\partial r^2} - \left( \frac{L_{D-1} + (D-1) \gamma}{2} \right)^2 \right] F_{nk}(r) = -[E^2 - M^2] F_{nk}(r)
\]

From equation (27), use centrifugal approximation to the \( \frac{1}{r^2} \) term. For \( \alpha r \ll 1 \) is:

\[
\frac{1}{r^2} \approx \frac{a^2}{\sinh^2(\alpha r)}
\]

Substitution equation (27) into equation (26), we have:

\[
\left[ \frac{\partial^2}{\partial r^2} - \frac{a^2}{\sinh^2(\alpha r)} \left( \frac{L_{D-1} + (D-1) \gamma}{2} \right)^2 \right] F_{nk}(r) = -\frac{a^2}{r^2} E_{s}' F_{nk}(r)
\]

We can setting equation (28) become:

\[
A_s = \left( \frac{L_{D-1} + (D-1) \gamma}{2} \right)^2 - V_2[M + E] + a^2V_3^2 \cosh(\alpha r) - a^2V_3^2 \coth(\alpha r) - a^2V_3^2
\]

\[
B_s = -\frac{\left( V_0^2[M + E] + \frac{V_1^2}{2} \cosh(\alpha r) + a^2V_3^2 \coth(\alpha r) - 2a^2V_3^2 \right)}{a^2}
\]

\[
E_{s}' = -\frac{\left( V_0^2[M + E] - [E^2 - M^2] - a^2V_3^2 \right)}{a^2}
\]

In equation (27), we can reduce to one dimensional Schrodinger type equation:

\[
\left( \frac{d^2}{dr^2} - \frac{a^2 A_s}{\sinh^2(\alpha r)} - a^2B_s \coth(\alpha r) \right) F_{nk}(r) = -a^2 E_{s}' F_{nk}(r)
\]

So that, substitution \( \coth(\alpha r) = ix \) in equation (32), we obtain:

\[
\left( (x^2 + 1) \frac{d^2}{dx^2} F_{nk}(r) + 2x \frac{d}{dx} F_{nk}(r) \right) - \left( A_s - B_s \frac{\coth(\alpha r)}{(x^2 + 1)} + \frac{E_{s}'}{(x^2 + 1)} \right) F_{nk}(r) = 0
\]
From equation (33) and wave equation in equation (19), we have new wave function, then we get:

\[(x^2 + 1) \frac{d^2}{dx^2} (1 + x^2) \frac{\beta}{2e} \frac{\alpha}{2} \tan^{-1}(x) \frac{D_n}{D_n}(\beta, \alpha)(x) + 2x \frac{d}{dx} (1 + x^2) \frac{\beta}{2e} \frac{\alpha}{2} \tan^{-1}(x) \frac{D_n}{D_n}(\beta, \alpha)(x) - \left[A_s - B_s^0 \coth(\alpha r) + E_s^0 \right] (1 + x^2) \frac{\beta}{2e} \frac{\alpha}{2} \tan^{-1}(x) \frac{D_n}{D_n}(\beta, \alpha)(x) = 0 \] (34)

If the coefficient \( \frac{1}{(1 + x^2)} \) term to zero, we have:

\[- \frac{\alpha^2}{4} + \frac{\beta^2}{4} - E_s^0 = 0 \quad \text{and} \quad \alpha \beta + iB_s = 0 \] (35)

So that, equation (34) become:

\[(1 + x^2) \frac{d^2}{dx^2} + [2x(\beta + 1) - \alpha] \frac{d}{dx} - [A_s - \beta^2 - \beta] D = 0 \] (36)

And the comparing the parameter between equation (18) and (36), we get relations \( \alpha = -q \) and \( p = -\beta \). From equation (35), we get new equation is:

\[\beta = \sqrt{A_s + \frac{1}{4} - n - \frac{1}{2}} \quad \text{and} \quad \alpha = \frac{iB_s}{-\sqrt{A_s + \frac{1}{4} + n + \frac{1}{2}}} \] (37)

From equation (37), we can obtain the value of \( \beta \) and \( \alpha \), there is:

\[\beta = \sqrt{\left(\frac{L_{D-1} + (D-1)}{2}\right)\left(\frac{L_{D-1} + (D-3)}{2}\right) + 4V_1[M + E] - V_3^2 + 2V_3^2 \sinh(\alpha r) + \frac{1}{4}} \]

\[-n - \frac{1}{2} \] (38)

\[\alpha = \frac{iB_s}{\sqrt{-\left(\frac{L_{D-1} + (D-1)}{2}\right)\left(\frac{L_{D-1} + (D-3)}{2}\right) + 4V_1[M + E] - V_3^2 + 2V_3^2 \sinh(\alpha r) + \frac{1}{4}}} + n + \frac{1}{2} \] (39)

Equation (35), we can setting:

\[\beta^4 - E_s^0 \beta^2 + \frac{B_s^2}{4} = 0 \] (40)

Using quadratic equation, energy eigenvalues from equation (40) and (37), we obtain:

\[\sqrt{A_s + \frac{1}{4} - n - \frac{1}{2}} = \frac{E_s^0 \pm \sqrt{E_s^0^2 - B_s^2}}{2} \] (41)

Substitution \( E_s \) from equation (31), we get energy eigenvalue is:

\[(M^2 - E_s^0) = \frac{\alpha^2}{\left(\frac{4}{a^2} A_s + \frac{1}{4} \frac{4n+2}{a^2}\right)^2} - \frac{4}{a^2} \left[\frac{4}{a^2} A_s + \frac{1}{4} \frac{4n+2}{a^2}\right] + \frac{a^2}{2} [M + E] \] (42)

The relativistic energy for spin symmetry case is given in equation (42).

3.2. Pseudospin symmetry case

The D-dimensional Dirac equation for pseudospin case in equation (9), can be written:
The solution for pseudospin symmetry case in equation (43) is similar method for spin symmetry case. Therefore, by repeating the steps in equation (27)-(41), we have:

\[
\frac{\partial^2}{\partial r^2} - \frac{a^2}{\sinh^2(\alpha r)} \left( \left( L_{D-1} - \frac{(D-1)}{2} \right)^2 - 4V_1[M - E] - V_3^2 + 2V_3^2 \sinh(\alpha r) \right) + \left( -\frac{V_0}{2} [M - E] - V_2[M - E] + 2a^2 V_3^2 \cosech(\alpha r) + a^2 V_3^2 \coth(\alpha r) - a^2 V_3^2 \right) \coth(\alpha r) + \frac{V_0}{2} [M - E] + \frac{a^2}{a^2} \] 
\[
G_{nk}(r) = \left[ -E^2 - M^2 \right] G_{nk}(r) \tag{44}
\]

We can setting equation (44) become:

\[ A_{ps} = \left( L_{D-1} - \frac{(D-1)}{2} \right)^2 - 4V_1[M - E] - V_3^2 + 2V_3^2 \sinh(\alpha r) \] \tag{45}

\[ B_{ps} = -\left( -\frac{V_0}{2} [M - E] - V_2[M - E] + 2a^2 V_3^2 \cosech(\alpha r) + a^2 V_3^2 \coth(\alpha r) - a^2 V_3^2 \right) \frac{a^2}{a^2} \tag{46}

\[ E_{ps} = -\left( -\frac{V_0}{2} [M - E] - [E^2 - M^2] - a^2 V_3^2 \right) \tag{47}

we can obtain the value of \( \beta \) and \( \alpha \), there is:

\[
\beta = \sqrt{\left( L_{D-1} - \frac{(D-1)}{2} \right)^2 - 4V_1[M - E] - V_3^2 + 2V_3^2 \sinh(\alpha r) + \frac{1}{4}} \tag{48}
\]

\[
-\frac{1}{2} - n = \frac{iB_{ps}}{\sqrt{-\left( L_{D-1} - \frac{(D-1)}{2} \right)^2 - 4V_1[M - E] - V_3^2 + 2V_3^2 \sinh(\alpha r) + \frac{1}{4}}} + n + \frac{1}{2} \tag{49}
\]

Equation (35), we can setting:

\[
\beta^4 - E_{ps}' \beta^2 + \frac{B_{ps}^2}{4} = 0 \tag{50}
\]

Using quadratic equation, energy eigenvalues from equation (50) and (37), we obtain:

\[
\sqrt{A_{ps} + \frac{1}{4}} - n = \frac{E_{ps}' \pm \sqrt{E_{ps}'^2 - B_{ps}^2}}{2} \tag{51}
\]

Substitution \( E_{ps} \) from equation (47), we get energy eigenvalue is:

\[
(M^2 - E^2) = \frac{\alpha^2 \left( \frac{4}{A_{ps} + \frac{1}{4}} \right)^2 - (2n + 1)^2 - B_{ps}^2 - 4A_{ps} + \frac{1}{4}}{2} + a^2 V_3^2 + \frac{V_0}{2} [M - E] \tag{52}
\]

The relativistic energy equation for pseudospin symmetry case is given in equation (52).
The result from this research is relativistic energy was numerically calculated from relativistic energy equation with Matlab R2008b software. The result of relativistic energy spectra is shown in Table 1 for spin symmetry case \((E_s)\) and pseudospin symmetry case \((E_{ps})\) with variation of \(D\) value.

**Table 1.** The Relativistic Energy for Spin and pseudospin Symmetry Cases with \(M = 5 fm^{-1}\), \(a = 0.5 fm^{-1}\), \(V_0 = 2 fm^{-1}\), \(V_1 = 0.6 fm^{-1}\), \(V_2 = 0.5 fm^{-1}\), \(V_3 = 0.4 fm^{-1}\)

| \(D\) | \(L_{D-1}\) | \(E_s\)    | \(E_{ps}\) |
|-------|-------------|------------|------------|
| 3     | 2           | -6.76693   | 4.41158    |
| 4     | 3           | -8.35021   | 4.53653    |
| 5     | 4           | -10.34434  | 4.65566    |
| 6     | 5           | -12.71695  | 4.73661    |

The relativistic energy spectra for Eckart plus Hulthen central potential coupled by modified Yukawa tensor potential for each parameter \(D\) and \(L_{D-1}\) for spin symmetry \((E_s)\) and pseudospin symmetry \((E_{ps})\) cases. For spin symmetry case, the increase value of \(D\) causes the decrease of relativistic energy value, while for the pseudospin symmetry case is otherwise.

**4. Conclusion**

The D-dimensional Dirac equation for Eckart plus Hulthen central potential is coupled by modified Yukawa like tensor potential can be solved by using Romanovski polynomial method for spin symmetry and pseudospin symmetry cases. The numerical relativistic energy for D-dimensional is obtained from the relativistic energy equation with Matlab R2008 software. The result is relativistic energy data with D-dimensional variation from 3 to 6, is for spin symmetry case, the increase value of \(D\) causes the decrease of relativistic energy value, while for the pseudospin symmetry case is otherwise.

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