NOISE INDUCED PHENOMENA IN LOTKA-VOLTERRA SYSTEMS

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We study the time evolution of two ecosystems in the presence of external noise and climatic periodical forcing by a generalized Lotka-Volterra (LV) model. In the first ecosystem, composed by two competing species, we find noise induced phenomena such as: (i) quasi deterministic oscillations, (ii) stochastic resonance, (iii) noise delayed extinction and (iv) spatial patterns. In the second ecosystem, composed by three interacting species (one predator and two preys), using a discrete model of the LV equations we find that the time evolution of the spatial patterns is strongly dependent on the initial conditions of the three species.

Keywords: Statistical mechanics; population dynamics; noise induced effects; Lotka-Volterra equations.

1. Introduction

The study of the effects of noise is a well established subject in several different disciplines ranging from physics, to chemistry and to biology [1]. The essential role of the noise in theoretical ecology however has been recently recognized. Some key questions in population ecology are related to the understanding of the role that noise, climatic forcing and nonlinear interactions between individuals of the same or different species play on the dynamics of the ecosystems [2,3,4,5,6,7,8,9,10,11,12]. More recently noise induced effects on population dynamics have been investigated [13,14,15,16,17,18]. In this paper we study the dynamics of two and three interacting species in the presence of multiplicative noise and a periodic driving force. Specifically we consider: (a) two competing species and (b) three interacting species. The multiplicative noise models the interaction between the species and the environment, and the driving force mimics the climatic temperature oscillations. In case (a) the interaction parameter between the species is a stochastic
process which obeys an Ito stochastic differential equation. The noise induces spatio-temporal patterns, quasi-deterministic oscillations, stochastic resonance (SR) phenomenon \cite{19,20} and noise delayed extinction, which is akin of the noise enhanced stability \cite{21,22}. In case (b) we find noise induced spatial patterns whose time evolution is strongly dependent on the initial conditions. Specifically we consider two different initial conditions: (i) an uniform initial distribution which gives rise to anticorrelated spatio-temporal patterns between the two preys, and (ii) a peaked initial distribution which gives rise periodically to a strong correlation between the patterns of the three species.

2. The Model

2.1. Two competing species

Time evolution of two competing species is obtained within the formalism of the Lotka-Volterra equations \cite{23} using the Ito scheme

\[
\frac{dx}{dt} = \mu x (\alpha - x - \beta_x(t)y) + x \xi_x(t), \quad (1)
\]
\[
\frac{dy}{dt} = \mu y (\alpha - y - \beta_y(t)x) + y \xi_y(t), \quad (2)
\]

where \(\xi_i(t)\) is Gaussian white noise with zero mean and \(\langle \xi_i(t)\xi_j(t') \rangle = \sigma \delta(t-t')\delta_{ij}\). The multiplicative noise models the interaction between the environment and the species. The interaction parameters \(\beta_x\) and \(\beta_y\) are characterized by a critical value corresponding to \(\beta_c = 1\). We choose \(\beta_x = \beta_y = \beta\). For \(\beta < \beta_c\) a coexistence regime of the two species is established, while for \(\beta > \beta_c\) an exclusion regime takes place, i.e. in a finite time one of the two species extinguishes.

Fig 1. The bistable potential \(U(\beta)\) of the interaction parameter \(\beta(t)\), centered at \(\beta = 0.99\) (coexistence region). The values of the parameters are \(h = 10^{-3}\), \(\rho = 10^{-2}\), \(\eta = 0.05\). The initial value of \(\beta(t)\) is \(\beta(0) = 0.94\) (bottom of left well).

The two regimes correspond to stable states of the deterministic Lotka-Volterra model. It is then interesting to investigate the time evolution of the ecosystem for \(\beta\) varying around the critical value \(\beta_c\) in the presence of fluctuations, due to the significant interaction with the environment. The interaction parameter \(\beta\) therefore can be described by a stochastic process which obeys the following differential equation.
\[
d\frac{d\beta(t)}{dt} = -\frac{dU(\beta)}{d\beta} + \gamma \cos \omega_0 t + \xi(t),
\]

where \(U(\beta)\) is a bistable potential (see Fig.1)

\[
U(\beta) = h(\beta - (1 + \rho))^4/\eta^4 - 2h(\beta - (1 + \rho))^2/\eta^2,
\]

\(\xi(t)\) is a Gaussian white noise with the usual statistical properties: \(\langle \xi_\beta(t) \rangle = 0\) and \(\langle \xi_\beta(t)\xi_\beta(t') \rangle = \sigma \delta(t - t')\). We perform simulations of Eqs. (1) and (2) by using a bistable potential \(U(\beta)\) centered at \(\beta = 0.99 < 1\), i.e. in the coexistence region. We consider as initial value of the interaction parameter \(\beta(0) = 0.94\), which corresponds to the left minimum of the potential well (see Fig.1). We fix the intensity of the multiplicative noise at a constant value \(\sigma = \sigma_x = \sigma_y = 10^{-9}\), and we vary the intensity of the additive noise. For low noise intensity the two species coexist all the time, while by increasing noise one species dominates the other one for a
random time and quasi-deterministic oscillations take place. By increasing even more the noise intensity the coherent response to environment variations is lost (see Figs. 2(a),(b) and (c)). These time variations of competing populations are due to noise induced phenomena, namely quasi-deterministic oscillations and stochastic resonance. In Fig. 2(d) we report the SNR of the squared difference of population densities \((x - y)^2\). At a finite noise intensity \(\sigma_\beta = 5 \cdot 10^{-4}\) the SNR exhibits a maximum, which characterizes the SR phenomenon. Now we consider the effect of the additive noise \(\sigma_\beta\) in Eq.(3) on the extinction time of the species and we neglect the multiplicative noise and the periodical forcing. Because of the initial condition, the ecosystem is in the coexistence region, then the deterministic extinction time of both species is infinity. By introducing noise, exclusion takes place and a finite mean extinction time (MET) appears. By increasing the noise intensity we obtain a noise delayed extinction: both species survive because of the noise. Then the average extinction time grows reaching a saturation value, which corresponds to the absence of the barrier of the potential \(U(\beta)\). The MET as a function of the noise intensity \(\sigma_\beta\) takes a minimum at a finite noise intensity \(\sigma_\beta \approx 7.5 \cdot 10^{-4}\). We perform 1000 realizations of Eqs. (2) and the results are shown in Fig. 3.

![Graph showing mean extinction time of one species as a function of noise intensity](image)

Fig 3. Mean extinction time of one species as a function of the noise intensity \(\sigma_\beta\) of Eq.(3). The potential \(U(\beta)\) and the initial condition for \(\beta(t)\) are the same of Fig.1.

3. Spatio-Temporal Patterns

In this section we study the effects of the noise on the spatial distribution of interacting species. In particular we analyze the spatial distribution of two different ecosystems: (a) two competing species, (b) three interacting species, one predator and two preys. To study the spatial effects we consider a discrete time evolution model, which is the discrete version of Eqs.(1) and (2) with diffusive terms, namely a coupled map lattice \[\box{13}\].
3.1. Two competing species

The time evolution of the spatial distribution for the two species is given by the following equations

\[
x_{i,j}^{n+1} = \mu x_{i,j}^{n} (1 - x_{i,j}^{n} - \beta^{n} y_{i,j}^{n}) + \sqrt{\sigma_{x}} x_{i,j}^{n} X_{i,j}^{n} + D \sum_{\gamma} (x_{\gamma}^{n} - x_{i,j}^{n}),
\]

\[
y_{i,j}^{n+1} = \mu y_{i,j}^{n} (1 - y_{i,j}^{n} - \beta^{n} x_{i,j}^{n}) + \sqrt{\sigma_{y}} y_{i,j}^{n} Y_{i,j}^{n} + D \sum_{\gamma} (y_{\gamma}^{n} - y_{i,j}^{n}),
\]

where \(x_{i,j}^{n}\) and \(y_{i,j}^{n}\) are the densities of the species in the site \((i,j)\) at the time step \(n\), \(\beta^{n}\) is the interaction parameter at the same time, \(D\) is the diffusion constant and \(\sum_{\gamma}\) indicates the sum over the four nearest neighbors. The \(X_{i,j}^{n}\) and \(Y_{i,j}^{n}\) terms are independent Gaussian random variables with zero mean and variance unit. The interaction parameter \(\beta^{n}\) is a stochastic process obtained by solving iteratively Eq. 3. The bistable potential is centered at \(\beta = 1\) and \(\beta(0) = 0.95\), which corresponds to the minimum of the left well. The results of our simulations are shown in Fig. 4.

We observe periodical formation of correlated patterns. After 600 time steps the spatial distributions of the two species appear almost independent. After 3000 time steps the anticorrelation between the two species densities increases. A further evolution of the system produces again spatial distributions which are independent after 4800 time steps and anticorrelated after 5400 time steps. The two competing species tend to occupy different spatial places periodically without overlap.
3.2. Three interacting species: two preys and one predator

Finally we analyzed an ecosystem composed by three species: two preys and one predator. We use the same coupled map lattice model of the previous section

\[ x_{i,j}^{n+1} = \mu x_{i,j}^n (1 - x_{i,j}^n - \beta x_{i,j}^n - \gamma z_{i,j}^n) + \sqrt{\sigma_x} x_{i,j}^n X_{i,j}^n + D \sum_{\delta} (x_{\delta}^n - x_{i,j}^n), \]

\[ y_{i,j}^{n+1} = \mu y_{i,j}^n (1 - y_{i,j}^n - \beta y_{i,j}^n - \gamma z_{i,j}^n) + \sqrt{\sigma_y} y_{i,j}^n Y_{i,j}^n + D \sum_{\delta} (y_{\delta}^n - y_{i,j}^n), \]

\[ z_{i,j}^{n+1} = z_{i,j}^n [-\beta z + \gamma z (x_{i,j}^n + y_{i,j}^n)] + \sqrt{\sigma_z} z_{i,j}^n Z_{i,j}^n + D \sum_{\delta} (z_{\delta}^n - z_{i,j}^n), \]

where \( x_{i,j}^n, y_{i,j}^n \) and \( z_{i,j}^n \) are respectively the densities of preys \( x, y \) and of the predator \( z \) in the site \((i,j)\) at the time step \( n \), \( \gamma \) and \( \gamma_z \) are the interaction parameters between preys and predator and \( D \) is the diffusion coefficient. The interaction parameter \( \beta \) between the two preys is a periodical function whose value, after \( n \) time steps, is given by

\[ \beta(t) = 1 + \epsilon + \alpha \cos(\omega_0 t), \]

with \( \epsilon = -0.01, \alpha = 0.1 \) and \( \nu_0 = (\omega_0/2\pi) = 10^{-3} \). We consider two different initial conditions: (i) a homogeneous initial distribution and (ii) a peaked initial distribution. In the first case we find exactly anticorrelated spatial patterns of the two preys, while the spatial patterns of the predator show correlations with both the spatial distributions of the preys (see Fig. 5). The preys tend to occupy different positions as in the case of two competing species. In the second case we use delta-like initial distributions for the two preys and a homogeneous distribution for the predator. After 800 steps we find strongly correlated spatial patterns of the preys which almost overlap each other. The maximum of spatial distribution of
the predator is just at the boundary of the spatial concentrations of the preys, so that the predator surrounds the preys (see Fig.6). The preys now tend to overlap spatially as it occurs in real ecosystems when preys tend to defend themselves against the predator attacks.

Fig 6. Spatial patterns induced by the noise for three interacting species (two preys and one predator) with delta-like initial distributions of the preys and a homogeneous distribution of the predator: (a) initial conditions, (b) spatial patterns after 800 time steps. Here we set $\epsilon = -0.05$, $D = 0.1$, $\sigma_x = \sigma_y = \sigma_z = 10^{-3}$ and the other parameters are the same as in Fig.5.

4. Conclusions

We analyzed the role of the noise on the spatio-temporal behaviors of two ecosystems composed by two and three interacting species. We use Lotka-Volterra generalized equations with multiplicative noise, climatic periodic forcing and random interaction parameter. The main conclusions of this work are: (i) in the case of two interacting species the appearance of noise induced phenomena such as quasi-deterministic oscillations, stochastic resonance, noise delayed extinction and spatial patterns; (ii) in the case of three interacting species the formation of dynamical spatial patterns exhibiting correlations which are strongly dependent on the initial conditions. Our population dynamical model could be useful to explain spatio-temporal experimental data of interacting species strongly interacting with the noisy environment [24, 25, 26]. A more detailed study of the dynamics of spatial patterns through stochastic partial differential equations will be the subject of future investigations.

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