Abstract. We consider the structure of the Goldman Lie algebra for the closed torus, and show that it is finitely generated over the rationals. We also consider other traditional Lie algebra structures and determine that the Goldman Lie algebra for the torus is not nilpotent or solvable, and we compute the derived Lie algebra.

1. The Goldman Lie Algebra

The Goldman Lie algebra is an algebra over the module generated by the set of free homotopy classes of loops on a surface, described by intersection and concatenation, that was introduced by William M. Goldman in 1986 [Go].

Throughout this paper, let $\Sigma_{g,n}$ denote an oriented, genus $g$ surface with $n \geq 0$ boundary components. Denote $\hat{\pi}(\Sigma_{g,n})$ to be the set of free homotopy classes of loops on $\Sigma_{g,n}$, where we use $\hat{\pi}$ when it is clear from the context which fixed surface we are discussing. Recall the following.

Lemma 1.1. The set of free homotopy classes of loops on a surface $\Sigma_{g,n}$ is in one-to-one correspondence with conjugacy classes of $\pi_1(\Sigma_{g,n})$.

Remark 1.2. We can represent homotopy classes of loops by cyclically reduced words with letters the generators of the fundamental group.

Definition 1.3. Fix a surface $\Sigma_{g,n}$ and an orientation of $\Sigma_{g,n}$. Let $\alpha, \beta \in \hat{\pi}$. The Goldman bracket of $\alpha$ and $\beta$ is defined to be

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \epsilon(p)\alpha * p \beta,$$

where $\alpha$ and $\beta$ intersect in transverse double points $p$, and $\epsilon(p)$ is the sign of the intersection, or $\epsilon(p) = 1$ if the ordered vectors in the tangent space to $\Sigma_{g,n}$ tangent to loop $\alpha$ and $\beta$ match the orientation of the surface, and $\epsilon(p) = -1$ otherwise [Go].

Example 1.5. We compute $[aab, b]$ on the surface $\Sigma_{1,1}$. The loops represented by words $aab$ and $b$ are shown in Figure 1 with intersection points $p_1$ and $p_2$. At the intersection point $p_1$, we smooth the intersection by creating a new loop, $aabb$, by following the red loop $aab$ in the direction of its orientation at $p_1$, and when returning to $p_1$, we now follow the blue loop $b$ in the direction of its orientation. When we return back to $p_1$, we close the loop. At the intersection $p_2$, we do the same, and create the loop $abab$. We get that $[aab, b] = \pm(aabb + abab)$ where the sign depends on the chosen orientation of $\Sigma_{1,1}$. 

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Figure 1. Loops $aab$ and $b$ on the torus with one boundary component.

Theorem 1.6. (Goldman) The Goldman bracket is well defined, skew-symmetric, and satisfies the Jacobi identity [Go].

We can extend the bracket linearly to $\mathbb{Z}[\pi]$ (or $\mathbb{Q}[\pi]$), the free module over $\mathbb{Z}$ (or $\mathbb{Q}$) with basis $\pi$, to get a bilinear map
\[\langle -, - \rangle : \mathbb{Z}[\pi] \times \mathbb{Z}[\pi] \to \mathbb{Z}[\pi].\]
Thus, $\mathbb{Z}[\pi]$ is a Lie algebra with bracket $\langle -, - \rangle$, which we call the Goldman Lie Algebra, denoted by $\mathfrak{g}$ throughout the rest of this chapter. When it is unclear what the surface we are referring to, we use $\mathfrak{g}_{\Sigma_{g,n}}$.

2. Goldman Lie Algebra Structure

So far, the center of the Goldman Lie algebra is known, but other parts of the structure is still unknown. The following theorems are results on the center of the Goldman Lie algebra for closed surfaces, and surfaces with boundary, respectively.

Theorem 2.1. (Etingof) The center of $\mathfrak{g}_{\Sigma_{g,0}}$ is spanned by the contractible loop [Et].

Theorem 2.2. (Kabiraj) The center of $\mathfrak{g}_{\Sigma_{g,n}}$ is generated by peripheral loops [Ka].

A question posed by Chas [Ch] is whether or not $\mathfrak{g}$ is finitely generated. In Goldman’s paper [Go], he also introduces what is called the homological Goldman Lie algebra. This Lie algebra is defined on intersection form on the first homology group of a surface. It is known that this Lie algebra is indeed finitely generated [KKT], the ideals are known [To], and the center for the surface of infinite genus has been found [KK].
3. The Goldman Lie Algebra Structure for the Torus

The closed torus is a special case as $\mathcal{G}_{\Sigma_{1,0}}$ is finitely generated. Recall that we can represent free homotopy classes of loops on $\Sigma_{1,0}$ by cyclically reduced words in two letters, $a$ and $b$, and we can represent all homotopy classes of loops on the torus by the word $a^i b^k$ for $k,l \in \mathbb{Z}$. The following theorem by Chas says that the Goldman bracket gives the intersection number of the free homotopy classes of loops as a coefficient of the new concatenated loop.

Proposition 3.1. (Chas [Ch1]) The Goldman bracket structure of $\mathcal{G}_{\Sigma_{0,1}}$ is given by
\[ [a^i b^j, a^k b^l] = (il - jk)a^{i+k}b^{i+l}. \]

Theorem 3.2. $\mathcal{G}_{\Sigma_{1,0}}$ is finitely generated when considered as a Lie algebra over $\mathbb{Q}$.

Proof. We denote a contractible loop by 1. We claim that $\mathcal{G}_{\Sigma_{1,0}}$ is generated by \{a, b, a^{-1}, b^{-1}\}. This will take many steps. We will first show that we can generate certain homotopy classes of loops. Below, we assume $n \neq 0$.

(1) $a^n b^1 = [a, a^{-1} b], \text{ which we get inductively,}$
(2) $a^n = [b^{-1}, -\frac{1}{n} a^n b]\text{,}$
(3) $ab^n = [b, -ab^{-1}]\text{,}$
(4) $b^n = [a^{-1}, -\frac{1}{n} a b^n]$, 
(5) $a^n b^n = [a^n, \frac{1}{n} b^n]$, 
(6) $a^{-n} b = [-a^{-1}, a^{-n} b]$, 
(7) $a^{-n} = [b^{-1}, \frac{1}{n} a^{-n} b]$, 
(8) $a^{-1} b^n = [-a^{-1}, -\frac{1}{n} b^n]$, 
(9) $a^{-n} b^n = [a^{-n}, -\frac{1}{n^2} b^n]$, 
(10) $ab^{-n} = [b^{-1}, ab^{n+1}] \text{ which we get inductively,}$
(11) $b^{-1} = [a^{-1}, \frac{1}{n} a b^{-n}]$, 
(12) $a^{-n} b^{-n} = [a^{-n}, \frac{1}{n^2} b^{-n}]$, 
(13) $a^n b^{-n} = [a^n, -\frac{1}{n^2} b^{-n}]$, 
(14) From 13. and 9. for $n = 1$, we get $a^0 b^0 = 1 = [a b^{-1}, \frac{1}{n} a^{-1} b]$. 

We still have a few more cases to show, namely how to generate the homotopy class of the loop $a^ib^j$ in the following cases.

Case 1: Suppose $i, j > 0$.
(a) Suppose $i < j$, then $n = i + r$ for some $r \in \mathbb{Z} - \{0\}$. Then $\frac{1}{ar} [a^{i} b^{j}, a^{r}] = a^{i+j} b^{j}$.
(b) Suppose $i > j$, then $n = j + r$ for $r \in \mathbb{Z} - \{0\}$. Then $-\frac{1}{ar} [a^{i} b^{j}, a^{r}] = a^{i+j} b^{j}$.

Case 2: Suppose $i < 0 < j$.
(a) Suppose $|i| < |j|$, then $n = -i + r$ for $r \in \mathbb{Z} - \{0\}$. Then $\frac{1}{ar} [a^{i} b^{j}, a^{r}] = a^{i+j} b^{j}$.
(b) Suppose $|a| > |b|$, then $n = j + r$ for $r \in \mathbb{Z} - \{0\}$. Then $-\frac{1}{ar} [a^{i} b^{j}, a^{r}] = a^{i+j} b^{j}$.

Case 3: The case $i, j < 0$, and $i \neq j$ is similar to Case 1.

Case 4: The case $b < 0 < a$ is similar to Case 2.

Thus, everything in $\mathcal{G}_{\Sigma_{1,0}}$ can be generated as a Lie algebra over $\mathbb{Q}$. \qed

Corollary 3.3. We can refine the generators of $\mathcal{G}_{\Sigma_{0,1}}$ to a smaller basis, namely \{a, a^{-1} b^{-1} + b + 1, b\}. 

Case 2: Suppose that

\[ [a^{-1}b^{-1} + b + 1, b] = -a^{-1}, \]

\[ [a^{-1}b^{-1} + b + 1, a] = b^{-1} - ab, \]

\[ [a, b] = ab. \]

\[ \square \]

**Corollary 3.4.** $\mathfrak{G}_{\Sigma_{1,0}}$ as a Lie algebra over $\mathbb{Q}$ is not nilpotent, nor solvable, since $[\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}] = \mathfrak{G}_{\Sigma_{1,0}}$.

**Remark 3.5.** $\mathfrak{G}_{\Sigma_{1,0}}$ is not finitely generated as a Lie algebra over $\mathbb{Z}$.

**Proof.** We will show that the set $\{(n-1)a^n\}_{n \geq 2, n \in \mathbb{Z}}$ cannot be generated. Suppose to the contrary that we can generate $(n-1)a^n$, so there exists $(i_s, j_s), (k_s, l_s) \in \mathbb{Z}^2$ such that

\[ \sum_{s=1}^{t} \pm[a^{i_s}b^{j_s}, a^{k_s}, b^{l_s}] = (n-1)a^n. \]

Then, as in Proposition 3.1

\[ \sum_{s=1}^{t} \pm[a^{i_s}b^{j_s}, a^{k_s}, b^{l_s}] = \sum_{s=1}^{t} \pm(i_s l_s - j_s k_s) a^{i_s+k_s} b^{l_s}. \]

We need that $i_s + k_s = n$ and $j_s + l_s = 0$, so $i_s l_s - j_s k_s = -i_s j_s - j_s n + j_s s = -j_s n$. So $n \mid i_s l_s - j_s k_s$, and $n \mid \sum_{s=1}^{t} \pm(i_s l_s - j_s k_s)$, so $n \mid (n-1)$, which is a contradiction. \( \square \)

**Conjecture 3.6.** We conjecture that $\mathfrak{G}_{\Sigma_{g,n}}$ for $g \geq 1$ and $n > 1$ is not finitely generated. For the particular case for a punctured torus, the peripheral loop is given by a commutator word. We noticed in using Chas’ program for computing the bracket seems to not generate a commutator word, nor products of commutators. This needs more work, but this would mean we have a set $\{(aba^{-1}b^{-1})^n\}_{n \in \mathbb{Z}}$ of infinitely many homotopy classes of loops that each cannot be generated by any other homotopy classes of loops.

**Proposition 3.7.** The derived Lie algebra for $\mathfrak{G}_{\Sigma_{1,0}}$ is given by

\[ [\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}] = \langle d(a^l b^j), na^n, nb^n \rangle, \]

for $d = \text{gcd}(i, j)$ and $n \in \mathbb{Z} - \{0\}$.

**Proof.** We first show $[\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}] \subset \langle d(a^l b^j), na^n, nb^n \rangle$.

Case 1: Suppose $d = \text{gcd}(i, j)$, $i, j \neq 0$ and $ma^l b^j \in [\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}]$ for some $m \in \mathbb{Z}$.

Write $xi + yj = d$ for some $x, y \in \mathbb{Z}$ and $ma^l b^j = [a^k b^l, a^p, b^q]$ for $k, l, p, q \in \mathbb{Z}$. But

\[ [a^k b^l, a^p, b^q] = (kq - lp)a^{k+p} b^{l+q}, \]

so we have that $k + p = i$, $l + q = j$, and $kq - lp = kj - li = d(k(\frac{i}{d}) - l(\frac{j}{d})) = m$. Thus $d \mid m$.

Case 2: Suppose that $n \neq 0$ and that $[a^l b^j, a^k b^l] = ma^n$ for $i, j, k, l, m \in \mathbb{Z}$. Then $i + k = n, j + l = 0$, so

\[ [a^l b^j, a^k b^l] = -jna^n, \]

so $n \mid m$. 

\[ (3.8) \]

\[ \text{for } d = \text{gcd}(i, j) \text{ and } n \in \mathbb{Z} - \{0\}. \]
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Case 3: Showing that for \( n \neq 0 \) and \( n \mid m \) for \( mb^n \in [\mathfrak{G}_{\Sigma_1,0}, \mathfrak{G}_{\Sigma_1,0}] \) is similar to Case 2.

To show the other containment, we can consider the equality [3.8] with \( k = y \) and \( l = -x \) for Case 1, we can consider the equality [3.9] with \( j = -1 \), and we can do something similar for Case 3.

\[ \square \]

**Proposition 3.10.** The lower central series for \( \mathfrak{G}_{\Sigma_1,0} \) stabilizes, i.e.

\[ [\mathfrak{G}_{\Sigma_1,0}, G_i] = \langle d(a^i b^j), na^n, nb^n \rangle \]

where \( d = \gcd(i, j) \), \( n \in \mathbb{Z} - \{0\} \), for all \( i \geq 0 \), and \( G_i = [\mathfrak{G}_{\Sigma_1,0}, G_{i-1}] \) defined inductively, where \( G_0 = \mathfrak{G}_{\Sigma_1,0} \).

**Proof.** For \( i = 1 \), this is just Proposition 3.7. For \( i = 2 \), we need to show that

\[ [\mathfrak{G}_{\Sigma_1,0}, [\mathfrak{G}_{\Sigma_1,0}, \mathfrak{G}_{\Sigma_1,0}]] = \langle d(a^i b^j), na^n, nb^n \rangle. \]

The "\( \subseteq \)" containment is clear. First, consider \( a^{i+1} b^j - 1 \in \mathfrak{G}_{\Sigma_1,0} \) and \( a^{n-i} b \in \langle d(a^i b^j), na^n, nb^n \rangle \) (since \( \gcd(n-i, 1) = 1 \)). We have that

\[ [a^{i+1} b^j - 1, a^{n-i} b] = na^n. \]

In a similar way, we can show that \( nb^n \in [\mathfrak{G}_{\Sigma_1,0}, [\mathfrak{G}_{\Sigma_1,0}, \mathfrak{G}_{\Sigma_1,0}]] \).

Now consider \( d = \gcd(i, j) \), so we can write \( d = xi + yj \). Consider \( a^{i+1} b^{j-x} \in \mathfrak{G}_{\Sigma_1,0} \) and \( a^{-y} b^x \in [\mathfrak{G}_{\Sigma_1,0}, \mathfrak{G}_{\Sigma_1,0}] \) (since \( 1 = \frac{x}{d} x + \frac{y}{d} y \) implies \( \gcd(x, y) = 1 \)). We have

\[ [a^{i+1} b^{j-x}, a^{-y} b^x] = da^i b^j. \]

Thus, it follows that the lower central series stabilizes. \[ \square \]

**Corollary 3.11.** As a Lie algebra over \( \mathbb{Z} \), \( \mathfrak{G}_{\Sigma_1,0} \) is not nilpotent.

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