Constructing a parasupersymmetric Virasoro algebra

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Abstract. We construct a para SUSY Virasoro algebra by generalizing the ordinary fermion in SUSY Virasoro algebra (Ramond or Neveu-Schwarz algebra) to the parafermion. First, we obtain a polynomial relation (PR) between different-mode parafermion $f_i$'s by generalizing the corresponding single-mode PR to such that is invariant under the unitary transformation of $f_i$ (Green's condition). Differently from a usual context, where the Green's condition is imposed only on the defining relation of $f_i$ (degree three with respect to $f_i$ and $f_i^\dagger$), we impose it on any degree of PR. For the case of order-two parafermion (the simplest case of para SUSY), we calculate a PR between the parasupercharge $G_0$, the bosonic hamiltonian $L_B^0$ and parafermionic one $L_F^0$, although it is difficult to obtain a PR between $G_0$ and the total hamiltonian $L_0 (= L_B^0 + L_F^0)$. Finally, we construct a para SUSY Virasoro algebra by generalizing $L_0$ to the $L_n$'s such that form a Virasoro algebra.

1. Introduction
Parafermion (SUSY) conformal field theory has been first introduced as an exactly solvable model in statistical systems, where a two-dimensional $\mathbb{Z}_n$ symmetric model is developed [1] by generalizing an ordinary spin to fractional spins (parafermions). In a usual context of para SUSY field theory, parafermions are formulated using the paraGrassmann variables [2, 3]. Along this line, para SUSY Virasoro algebra has been constructed using paraGrassmann variables [4]. However, the paraGrassmann algebra is not necessarily equivalent to the parafermion algebra in its original sense of the word. Recalling that in quantizing a relativistic quantum field, the most general statistics is parastatistics [5], we should use the paraparticles themselves to have a physically realizable relativistic field theory model.

To construct a Virasoro algebra using (para) particle creation and annihilation operators $c_i$'s (Feigin-Fuks realization), it is necessary to obtain a polynomial relation between the $c_i$'s of different modes, although it is not so trivial a matter to obtain the polynomial relation. To obtain it, we use the Green's condition [6], where the polynomial relation is required to be invariant under the unitary transformation $c_i \to c_i' = \sum_j U_{ij} c_j$. However, it is found that the Green's condition is incompatible with the paraboson-including Virasoro algebra. This is because the commutation relation between the multi-mode parabose fields cannot be obtained, under the Green's condition, from the corresponding single-mode commutation relation (Calogero-Vasiliev oscillator algebra [7]). Considering that the Green's condition guarantees the realization of a Virasoro algebra, we may state that it is difficult to construct a paraboson-including Virasoro algebra. In this sense, we keep bosons intact, while generalizing fermions to parafermions.

The aim of this article is to construct the para SUSY Virasoro algebra by generalizing the ordinary fermion in SUSY Virasoro algebra to the parafermion $f_i$. First, we obtain the multi-
mode parafermi polynomial relation from the corresponding single-mode relation under the Green’s condition. Differently from a usual context, where the Green’s condition is imposed only on the defining relation of $f_i$ (degree three with respect to $f_i$ and $f_i^\dagger$), we apply the Green’s condition to the polynomial relation of any degree. Then we calculate the polynomial relation between the parasupercharge $G_0$, bosonic hamiltonian $L_0^B$, and parafermionic one $L_0^F$.

As opposed to the ordinary SUSY case, it seems to be impossible to obtain the polynomial relation between $G_0$ and the total hamiltonian $L_0 (= L_0^B + L_0^F)$. Finally, we obtain the para SUSY Virasoro algebra by generalizing $L_0$ to the $L_n$’s (for $n \in \mathbb{Z}$) which form a Virasoro algebra.

2. Para SUSY Virasoro Algebra

In this section, we first review the Green’s condition, under which the parabose single-mode commutation relation cannot be generalized to a multi-mode one. Next, we obtain the multi-mode parafermi polynomial relation for the parafermionic order $p_f = 2$, followed by the calculation of the polynomial relation between $G_0^i$, $L_0^B$, and $L_0^F$. Finally, we generalize $L_0$ to the $L_n$’s that form a Virasoro algebra, as in the ordinary SUSY case.

2.1. Green’s condition

Denote by $c_i$ and $c_i^\dagger$ the annihilation and creation operators, respectively. The Green’s condition is given by [6]

\[
\begin{align*}
\{c_i, [c_j^\dagger, c_k]_{\pm}\} &= 2\delta_{ij}c_k, \\
\{c_i, [c_j^\dagger, c_k]_{\pm}\} &= 0,
\end{align*}
\]

for parabosons (upper sign) and parafermion (lower sign). As was pointed out in [8], the relation of Eq. (1) can be derived from the requirement that the relation

\[
[c_i, h_j] = \delta_{ij}c_i, \quad \text{where} \quad h_j = \frac{1}{2}[c_j^\dagger, c_j]_{\pm}
\]

is invariant under the unitary transformation of $c_i \rightarrow c_i' = \sum_j U_{ij}c_j$, where $\sum_j U_{ij}U_{kj}^* = \sum_j U_{ji}^*U_{jk} = \delta_{ik}$. This requirement seems reasonable in the sense that the total hamiltonian $\sum_i h_i$ is invariant under the unitary transformation of the $c_i$’s.

First we deal with the parabosons for a parabosonic order $p_b$. For the single mode, the commutation relation can be given by the Calogero-Vasiliev oscillator algebra [7]. This algebra can be generalized to such that satisfies the relation of Eq. (2) as

\[
\begin{align*}
\{c_i, c_j\} &= 0, \\
\{c_i, c_j^\dagger\} &= (1 + \nu R_i)\delta_{ij},
\end{align*}
\]

where $\nu = p_b - 1$ and $R_i = e^{i\pi(h_i - \frac{p_b}{2})} = \cos(i\pi(h_i - \frac{p_b}{2}))$. More explicitly, $c_i$ can be written as [9, 10] $c_i = \frac{1}{\sqrt{2}}(x_i + i\frac{d}{dx_i} - \frac{\nu R_i}{2x_i})$, where $[R_i, x_i]_+ = [R_i, \frac{d}{dx_i}]_+ = 0$. However, it is found that Eq. (3) is not compatible with the first relation of Eq. (1) unless $\nu = 0$, due to

\[
[c_i, R_j]_+ = 0 \quad (i \neq j)
\]

by Eq. (2). If Eq. (3) were compatible with the first relation of Eq. (1), it would be required that $\nu c_i, R_j]_+ = 0$ (for all $i, j$), which, together with Eq. (4), leads to $\nu c_i = 0$ (due to $R_i^2 = 1$). Thus, Eq. (3) is incompatible with the first relation of Eq. (1) unless $\nu = 0$. Considering that Eq. (1) is sufficient for the realization of a Virasoro algebra, as in the ordinary bosonic case, we may state that it is difficult to realize a Virasoro algebra constructed from the paraboson (with a definite value of $p_b$). In this sense, we concentrate on the ordinary bosonic case, where $p_b = 1$. 
Next, we deal with the parafermion. As opposed to the paraboson, the polynomial relation between the $c_i$’s and $c_i^\dagger$’s can be obtained from the corresponding single-mode polynomial relation by requiring that the polynomial relation is invariant under the unitary transformation of the $c_i$’s (we call this requirement the generalized Green’s condition, or the Green’s condition for short). Some useful anti-commutators are introduced.

$$[A, B, C]_+ = ABC + CBA,$$
$$[A, B, C, D]_+ = [A, [B, C]_+, D]_+.$$

**Notation 1** For the parafermions, $f_i = c_i$ and $f_{-i} = c_i^\dagger$ (for $i > 0$).

In what follows, we concentrate on the case of $p_f = 2$, for simplicity.

**Lemma 2** For $p_f = 2$, the polynomial relations between the $f_i$’s are given under the generalized Green’s condition by

$$[f_i, f_j, f_k]_+ = 2(\delta_{i+j,0}f_k + (i \leftrightarrow k)),$$
$$[f_i, f_j, f_k, f_\ell]_+ = (F_{ijkt} + (j \leftrightarrow k)) + (i \leftrightarrow \ell),$$
where $F_{ijkt} = 2\delta_{i+j,0}[f_k, f_\ell]_+ - \delta_{i+\ell,0}[f_j, f_k]_+ - 4\delta_{i+j,0}\delta_{k+\ell,0} + 4\delta_{i+\ell,0}\delta_{j+k,0}$.

**Proof.** The corresponding single-mode relations are given by

$$\begin{cases} f f^\dagger f = 2f, & [f, f, f^\dagger]_+ = 2f, f^3 = 0, \\ f f^\dagger f f = 2[f, f^\dagger]_+ - 4, \end{cases}$$

where the other tetralinear relations such as $f^4 = 0$, $[f^3, f^\dagger]_+ = 0$, $f[f, f^\dagger]_+ f = 4f^2$, $[f, f, f^\dagger, f^\dagger]_+ = 8$ can be derived from the remaining relations. First we show Eq. (5). Under the generalized Green’s condition, $[f_i, f_j, f_k]_+$ can be written as in the form $[f_i, f_j, f_k]_+ = a \delta_{i+k,0}f_j + b(\delta_{i+j,0}f_k + (i \leftrightarrow k))$, where $a, b \in \mathbb{R}$. The coefficients $a$ and $b$ can be obtained from the single-mode relations as $2b = 4$ and $a + b = 2$. Hence we get $a = 0, b = 2$. Eq. (6) can be obtained in a similar way. □

The relation of Eq. (5), which is found in many references [5, 11], is often derived from the assumption of the Green’s ansatz [6], in which the $f_i$’s for a parafemion order $p_f$ is decomposed as $f_i = \sum_{\alpha=1}^{p_f} f_i^{(\alpha)}$, where the (anti)commutation relations between the $f_i^{(\alpha)}$’s and $f_i^{(\alpha)*}$’s are so chosen that $f_i$ satisfies the relation of Eq. (1). However, the representation of Eq. (1) under the Green’s ansatz is not an irreducible representation. In the case of $p_f = 2$, for example, the vector space $V$ associated with the representation under the Green’s ansatz can be decomposed into $V_1 \oplus V_2$, where $V_1$ and $V_2$ are associated with the irreducible representation of $f_i$ for $p_f = 2$ and that for $p_f = 0$, respectively. Actually, the relation of Eq. (5) is satisfied by the irreducible representation of $f_i$ for $p_f = 0$, that is, $f_i = 0$. On the other hand, the relation of Eq. (6) cannot be obtained from the Green’s ansatz, because the relation of $f_i = 0$ does not satisfy Eq. (6).

As will be found in the next subparagraph, the relation of Eq. (6) is indispensable to the para SUSY relation for $p_f = 2$.

### 2.2. Para SUSY

**Notation 3** For the (para)bosons, let the $b_k$’s be given by $b_k = \sqrt{k}c_k$ and $b_{-k} = \sqrt{k}c_k^\dagger$ (for $k > 0$).

**Notation 4** Let $G_r$, $L^B_n$, and $L^F_n$ ($r \in \mathbb{Z} + \kappa$, $n \in \mathbb{Z}$) be given by $G_r = \sum_{k \in \mathbb{Z}} b_k f_{r-k}$, $L^B_n = 1/4 \sum_{k \in \mathbb{Z}} [b_{n-k}, b_k]_+$, and $L^F_n = 1/4 \sum_{r \in \mathbb{Z} + \kappa} (r - \frac{\kappa}{2})[f_{n-r}, f_r]$, where $\kappa = 0$ (Ramond type), $\frac{1}{2}$ (Neveu-Schwarz type).
To obtain the polynomial relation between the $G_r$'s, $L_0^B$'s, and $L_0^F$'s, we should specify the polynomial relations between the $b_k$'s and $f_r$'s. Considering the Green's condition of Eq. (1), one may point out that it seems natural to choose the trilinear relations between the $b_k$'s and $f_r$'s, as is found in Ref. [11]. However, we adopt a simpler one: the bilinear commutation relation

$$[b_k, f_r] = 0 \quad \text{(for all } k \in \mathbb{Z}, \ r \in \mathbb{Z} + \kappa),$$

as in the ordinary SUSY.

In the rest of this subsection, we deal with the Ramond type ($\kappa = 0$) for somewhat simplicity. The Neveu-Schwarz type ($\kappa = \frac{1}{2}$) can be dealt with in quite an analogous way.

**Notation 5** \(\mathcal{G}_0 = \frac{1}{2} \sum_{k \in \mathbb{Z}} k f_{-k} G_0 f_k\) and \(N = \sum_{k \in \mathbb{Z}} k^2 f_{-k} f_k\).

For \(p_f = 2\) and \(p_b = 1\), the anticommutation of \(G_0\) with \(\mathcal{G}_0\) (and with \(N\)) gives (see Appendix)

$$\begin{cases}
[\mathcal{G}_0, G_0]_+ = -5N + [[L_0^F, G_0], G_0] - 2(L_0^F)^2 + 12 \zeta(-2) \\
[N, G_0]_+ = 2[[G_0, L_0^F], L_0^F] + 4 \zeta(-2)G_0
\end{cases} \quad \text{(for } p_f = 2, p_b = 1), \quad (8)
$$

where \(\zeta(-2) = \frac{1}{2} \sum_{k \in \mathbb{Z}} k^2\). Although \(\zeta(-2)\) can be chosen as vanishing through the analytic continuation of the zeta function, we keep it remaining to check that the following relation is correct in the single-mode limit.

**Proposition 6** For \(p_b = 2\) and \(p_b = 1\), \(G_0, L_0^B, \) and \(L_0^F\) satisfy the polynomial relation

$$\varphi(G_0, L_0^B + L_0^F) = \Delta(G_0, L_0^F : \zeta(-2)),
$$

where \(\varphi(x, y) = x^5 - 4x^3y\) and \(\Delta(x, y; \lambda) = -[x, xyx]_+ - 3[x, y^2]_+ + 5xy + \lambda x\).

**Proof.** Substituting \(b_i b_j b_{-k}\) with both sides of Eq. (5) and summing over \(i, j, k \in \mathbb{Z}\), we obtain

$$G_0^3 - 4G_0(L_0^B + L_0^F) = \mathcal{G}_0 - [G_0, L_0^F]_+.$$

Applying \([\cdot, [\cdot, G_0]]_+\) to both sides of the above relation, and using Eq. (8), we obtain the relation to be proven, where use has been made of \([G_0, L_0^B + L_0^F] = 0\).

Now we check the validity of Prop. 6 by showing that it has a correct relation in the single-mode limit, where the sum over \(k\) in \(G_0, L_0^B, \) and \(L_0^F\) is restricted to \(\pm 1\). Even in this limit, the polynomial relation remains the same form as the original relation, so that it follows from Prop. 6 that the corresponding single-mode relation satisfies

$$\varphi(Q, H) = \Delta(Q, H_f; 1),$$

where we have replaced as \(G_0 \rightarrow Q, (L_0^B + L_0^F) \rightarrow H, L_0^F \rightarrow H_f,\) and \(\zeta(-2) \rightarrow 1\). For \(p_f = 2, p_b = 1\), it is found that \(\Delta(Q, H_f; 1) = 0\) from \([Q, QH_f]_+ = -2Q, [Q, H_f^2]_+ = Q, \) and \(H_f QH_f = 0\). As a consequence, we get \(\varphi(Q, H) = 0\), which is equivalent to

$$Q(Q^2 - 4H) = 0,$$

where the hermiticity of \(Q\) and \(H\) has been used. The relation of \(Q(Q^2 - 4H) = 0\) coincides with a well-known relation found in the \(p_f = 2\) para SUSY quantum mechanics [12]. Thus, the relation of Prop. 6 turns out to be correct in the single-mode limit.
2.3. Conformality

**Notation 7** Let \( L_n \ (n \in \mathbb{Z}) \) be given by

\[
L_n = (L_n^B - \langle L_n^B \rangle) + (L_n^F - \langle L_n^F \rangle + \alpha \delta_{n,0}),
\]

where \( \langle \ldots \rangle \) denotes the vacuum expectation value, and \( \alpha = \frac{1}{16}p_f \) (for \( \kappa = 0 \)) and \( 0 \) (for \( \kappa = \frac{1}{2} \)).

In the ordinary SUSY, \( L_n^S - \langle L_n^S \rangle \) (for \( S = B, F \)) can be replaced by the creation-annihilation normal ordering \( \hat{S}_L \). For the para SUSY case, however, the difference between \( L_n^S \) and \( \hat{S}_L \) does not necessarily amount to a constant, so we use instead the vacuum expectation value to avoid the ambiguity caused from \(-\infty \rightarrow \infty\) in the calculation of \([L_m, L_n]\).

Under the Green’s condition of Eq. (1), together with the commutativity of Eq. (7), it is found that the \( L_n \)'s \( (n \in \mathbb{Z}) \) and \( G_r \)'s \( (r \in \mathbb{Z} + \kappa) \) satisfy the following commutation relations:

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \\
[G_r, L_n] &= (r - \frac{9}{2})G_{r+n}
\end{align*}
\]

(9)

where \( c = p_b + \frac{1}{2}p_f \). In deriving Eq. (9), we have used

\[
\begin{align*}
[b_m, L_n] &= m b_{m+n}, \\
|f_r, L_n] &= (r + \frac{9}{2})f_{r+n}, \\
\langle b_{n-k}, b_k \rangle &= p_b |k|\delta_{n,0}, \\
\langle f_{n-r}, f_r \rangle &= -p_f \text{sgn}(r)\delta_{n,0},
\end{align*}
\]

where the second set of the relations is given by \( \forall i, j > 0, c_i c_j |0\rangle = p \delta_{i,j} |0\rangle \) with \( c_i |0\rangle = 0 \) (where \( p = p_b, p_f \) for \( c_i = b_i, f_i \), respectively). Rearranging the terms in \( L_n \), and using the formula \( \sum_{k=1}^{\infty} (k - \kappa) = \frac{1}{21} - \frac{1}{8}(2\kappa - 1)^2 \) (for all \( \kappa \in \mathbb{C} \)), we can formally rewrite \( L_n \) as

\[
L_n = L_n^B + L_n^F + \frac{c}{12}\delta_{n,0}
\]

\[
= \sum_{S=B,F} \left( L_n^S + \frac{c^S}{12}\delta_{n,0} \right)
\]

\[
= \frac{c^B}{12} b_{n,0} + \frac{c^F}{12} f_{n,0}
\]

(10)

which indicates that

\[
[L_m, L_n^S] = (m - n)L_{m+n}^S + \frac{c^S}{12}m^3\delta_{m+n,0} \quad (S = B, F).
\]

Now we are in a position to obtain the polynomial relation between the \( G_r \)'s, \( L_n^B \)'s, and \( L_n^F \)'s. For the Ramond type \( (\kappa = 0) \), it is sufficient to obtain the polynomial relation between \( G_0 \), \( L_0^B \), and \( L_0^F \). The repeated application of the commutator with the \( L_n \)'s to the polynomial relation \( \phi^R(G_0, L_0^B, L_0^F) = 0 \), yields from Eq. (10) a result that we want. For \( p_f = 2 \) and \( p_b = 1 \), the result is given by

\[
\phi_{r_1 \ldots r_5} + (\text{perm})_{r_1 \ldots r_5} = 0 \quad (\text{for } p_f = 2, p_b = 1),
\]

(11)

where \( (\text{perm})_{r_1 \ldots r_5} \) represents the permutation with respect to \( r_1, \ldots, r_5 \), and

\[
\phi_{r_1 \ldots r_5} = G_{r_1} G_{r_2} G_{r_3} G_{r_4} G_{r_5} - 4 G_{r_1} G_{r_2} G_{r_3} (L_{r_4,r_5}^B + L_{r_4,r_5}^F) + [G_{r_1}, G_{r_2} L_{r_4,r_5}^F G_{r_3}] + 3 [G_{r_1}, L_{r_2,r_3} L_{r_4,r_5}^F] + 5 L_{r_2,r_3} L_{r_4,r_5}^F
\]

(12)

with \( L_{r,s}^S := L_{r+s}^S + \frac{c^S}{6} \delta_{r+s,0} \ (S = B, F) \).
For the Neveu-Schwarz type ($\kappa = \frac{1}{2}$), on the other hand, the starting polynomial relation would be of the form $\phi^{NS}(G_{\frac{1}{2}}, L_{B}^{1}, L_{F}^{1}) = 0$. However, this is not sufficient to obtain the polynomial relation between the $G_{r}$’s, $L_{B}^{n}$’s, and $L_{F}^{n}$’s by the repeated application of the commutator with the $L_{m}$’s. The polynomial relation of the form $\phi^{NS}_{(n)}(G_{\frac{1}{2}+n}, L_{B}^{1+2n}, L_{F}^{1+2n}) = 0$ (for $n \in \mathbb{Z}$) is furthermore necessary. Fortunately, the functional form of $\phi^{NS}(x, y, z)$, if chosen as $x$-monic, is independent of $n$, and is the same as $\phi^{R}(x, y, z)$, as is observed in the ordinary SUSY case. Eventually, the relation of Eq. (11) with (12) holds for all $r_{i} \in \mathbb{Z} + \kappa$ ($i = 1, \ldots, 5$).

It should be noticed that, in obtaining $\phi_{r_{1}, \ldots, r_{5}}$, we have only to replace the $G_{r}$’s and $L_{2n}$’s in $\phi(G_{\kappa}, L_{2n}^{B}, L_{2n}^{F})$ ($\phi = \phi^{R}$ for $\kappa = 0$ and $\phi = \phi^{NS}$ for $\kappa = \frac{1}{2}$) by

$$
\begin{align*}
(G_{n})^{n} & \rightarrow G_{s_{1}} \ldots G_{s_{n}}, \\
(L_{2n}^{r})^{m} & \rightarrow L_{t_{1}, t_{2}}^{S} \ldots L_{t_{m-1}, t_{2m}}^{S} (S = B, F),
\end{align*}
$$

where the set of the subscripts in each term of $\phi_{r_{1}, \ldots, r_{5}}$, $\{s_{1}, \ldots, s_{n}; t_{1}, \ldots, t_{2m}, \ldots\}$, is so chosen that it is equal to the set of the subscripts of $\phi_{r_{1}, \ldots, r_{5}}$ itself, $\{r_{1}, \ldots, r_{5}\}$. This substitution is valid for an arbitrary $\phi(x, y, z)$, provided that $\phi$ satisfies the homogeneity of the form

$$
\forall \lambda \in \mathbb{C}, \phi(\lambda x, \lambda^{2} y, \lambda^{3} z) = \lambda^{n} \phi(x, y, z),
$$

where $n = \deg_{x} \phi(x, y, z)$ for an $x$-monic $\phi(x, y, z)$. The relation of Eq. (14) guarantees that all the numbers of the subscripts in each term of $\phi_{r_{1}, \ldots, r_{n}}$ are the same (as $n$).

At the end of this subsection, we verify the validity of the substitution of Eq. (13) for an arbitrary $\phi(x, y, z)$ such that Eq. (14) is satisfied. Let $\psi_{r_{1}, \ldots, r_{n}}$ be defined by

$$
\psi_{r_{1}, \ldots, r_{n}} := \phi_{r_{1}, \ldots, r_{n}} + (\text{perm})_{r_{1}, \ldots, r_{n}},
$$

where $\phi_{r_{1}, \ldots, r_{n}}$ is given by $\phi^{R}(G_{0}, L_{B}^{0}, L_{F}^{0})$ or $\phi^{NS}(G_{\frac{1}{2}}, L_{B}^{1}, L_{F}^{1})$ under the substitution of Eq. (13). In particular, $\phi(G_{\kappa}, L_{2n}^{B}, L_{2n}^{F}) = 0$ corresponds to

$$
\phi_{\kappa \ldots \kappa} = 0 \quad (\kappa = 0, \frac{1}{2}).
$$

We verify the validity of the substitution of Eq. (13) by showing that $\psi_{r_{1}, \ldots, r_{n}} = 0$ for all $r_{i} \in \mathbb{Z} + \kappa$ ($i = 1, \ldots, n$). To show it, we first calculate the commutation relation of $[L_{m}, \psi_{r_{1}, \ldots, r_{n}}]$. For this purpose, it may be convenient to align the parameters $s_{i}$’s and $t_{i}$’s as

$$(s_{1}, \ldots, s_{n}, \ldots; t_{1}, \ldots, t_{2m}, \ldots) = (r_{1}, \ldots, r_{n}).$$

In this case, the term including $L_{t_{i}, t_{j}}^{S}$ in $\phi_{r_{1} \ldots r_{n}}$ is symmetric under $(r_{n-1} \leftrightarrow r_{n})$, $(r_{n-3} \leftrightarrow r_{n-2})$, and so on, due to $L_{t_{1}, t_{2}}^{S} = L_{t_{2}, t_{1}}^{S}$. To make the other terms in $\phi_{r_{1}, r_{n}}$ preserve the same symmetry, we rewrite $\psi_{r_{1}, \ldots, r_{n}}$ by symmetrizing $\phi_{r_{1}, \ldots, r_{n}}$ as

$$
\psi_{r_{1}, \ldots, r_{n}} = \phi'_{r_{1}, \ldots, r_{n}} + (\text{perm})_{r_{1}, \ldots, r_{n}}, \quad \phi'_{r_{1}, \ldots, r_{n}} = \begin{cases} 
\phi_{r_{1} r_{2}} \{r_{3} r_{4} \ldots r_{n-1} r_{n}\} & (n \in 2N), \\
\phi_{r_{1} \{r_{2} r_{3} \ldots \} r_{n-1} r_{n}} & (n \in 2N + 1),
\end{cases}
$$

where $(\text{perm})_{r_{1}, \ldots, r_{n}}$ represents the $(r_{1} \ldots r_{n})$-permutation where the cases of $(r_{n-1} \leftrightarrow r_{n})$, $(r_{n-3} \leftrightarrow r_{n-2})$, . . . are excluded; and $\phi_{r_{1} r_{2}} = \phi_{r_{1} r_{2}} + \phi_{r_{2} r_{1}}$ and the like. Hence, it is found
that \([L_m, \psi_{r_1...r_n}]\) can be written as a linear combination of the \(\psi_{r_1...r_n}\)'s:

\[
[L_m, \psi_{r_1...r_n}] = [L_m, (\phi'_{r_1...r_n} + (\text{perm})_{r_1...r_n})]
= [L_m, \phi'_{r_1...r_n}] + (\text{perm})_{r_1...r_n}
\]

\[
= \sum_{i=1}^{n} \left( \frac{m}{2} - r_i \right) \phi'_{r_1...r_{i-1}, r_i, r_{i+1}...r_n} + (\text{perm})_{r_1...r_n}
\]

\[
= \sum_{i=1}^{n} \left( \frac{m}{2} - r_i \right) \psi_{r_1...r_{i-1}, r_i, r_{i+1}...r_n},
\]

where in the third line, we have used the identity

\[
\frac{1}{6} \left( \frac{m}{2} - r \right) (m + r)^2 + \frac{1}{6} \left( \frac{m}{2} - s \right) (m + s)^2 = \frac{1}{12} m^3 \quad \text{(for } m + r + s = 0).\]

The point of the definition of Eq. (15) is that under the map \([L_m, .], \psi_{r_1...r_n}\) transforms in a same way as \(G_{r_1} \ldots G_{r_n} + (\text{perm})_{r_1...r_n}\).

Then the repeated application of the commutator with the \(L_m\)'s to Eq. (16) (for the Neveu-Schwarz type, together with \(\psi_{\kappa_n...\kappa_n} = 0 (\kappa_n = n + \frac{1}{2})\)) yields from Eq. (17) the relation

\[
\psi_{r_1...r_n} = 0 \quad \text{(for } r_i \in \mathbb{Z} + \kappa).\]

This ends the verification of the validity of Eq. (13) by showing the vanishing of \(\psi_{r_1...r_n}\).

3. Summary
We have constructed the para SUSY Virasoro algebra under Eq. (7) and the Green’s condition, where the polynomial relations between the \(c_i\)'s are invariant under the unitary transformation of the \(c_i\)'s. However, we neglect the paraboson-including Virasoro algebra, because the commutation relation between the parabosonic operators \(b_i\)'s cannot be obtained from the corresponding single-mode commutation relation (Calogero-Vasiliev oscillator algebra, which has a definite value of \(p_b\)) under the Green’s condition. Even if we assume that the Green’s condition is applied only to the defining relation of the \(b_i\)'s, that is, Eq. (1), it is found that the paraboson-including para SUSY Virasoro algebra is not favorable by the following reason.

Consider, for example, the parabose trilinear relation found in many literatures [5, 11] as

\[
[b_i, b_j, b_k] = 2(i \delta_{i+j+k} b_k - (i \leftrightarrow k)),
\]

where \([A, B, C] := ABC - CBA\). Using Eq. (18), which indeed satisfies the relation of Eq. (1), and Eq. (5), we obtain a para SUSY relation of \(G_0^3 = 4G_0^3 (L_0^B + L_0^F)\), leading to another para SUSY Virasoro algebra. However, the \(b_i\)'s in Eq. (18) does not have a definite parabosonic order \(p_b\), because the relation of Eq. (18) is different from the trilinear commutation relation constructed from the \(c_i\)'s and \(c_i^\dagger\)'s in Eq. (3), where the \(c_i\)'s and \(c_i^\dagger\)'s have a definite value of \(p_b\). Recalling that the parabosonic central charge \(c_i^B\) is given by \(p_b\), we find that the Virasoro algebra constructed from Eq. (18) has not a well-defined central charge. If we assume that a Virasoro algebra should have a well-defined central charge, it follows that the relation of Eq. (18) is not favorable in constructing a Virasoro algebra, although Eq. (18) in itself may be available as a parabosonic relation.
For $p_f = 2$ (with $p_b = 1$), we calculate the polynomial relation between the $G_r$’s, $L^B_r$’s, and $L^F_r$’s. The key point of the calculation is that we have not only Eq. (5) but also Eq. (6) available; without Eq. (6), we would not obtain the first relation of Eq. (8); without Eq. (8), it would be difficult to obtain the polynomial relation between $G_0$, $L^B_0$, and $L^F_0$. As contrasted with the ordinary SUSY case, it is difficult to write the polynomial relation in terms of the $G_r$’s and $(L^B_n + L^F_n)$’s only. For $p_f = 3$, it is still more difficult to obtain the polynomial relation between the $G_r$’s, $L^B_r$’s, and $L^F_r$’s, due to the lack of the (symmetric) trilinear relation corresponding to Eq. (5). Bearing these in mind, we find it natural to conjecture that there is a polynomial relation between the $G_r$’s and $(L^B_n + L^F_n)$’s if and only if $p_f = 1$, the ordinary SUSY case.

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Appendix. Derivation of Eq. (8)

To begin with, we show the first relation of Eq. (8). From $[b_i, b_j] = i\delta_{i+j,0}$, $G_0 f_k G_0$ can be rewritten as

$$G_0 f_k G_0 = \frac{1}{2} \sum_{\ell,m \in \mathbb{Z}} b_{-\ell} b_{-m} [f_\ell, f_k, f_m]_+, + \frac{1}{2} \sum_{\ell \in \mathbb{Z}} \ell f_{-\ell} f_k f_\ell$$

$$= [b_k, G_0]_+ + \frac{1}{2} \sum_{\ell \in \mathbb{Z}} \ell f_{-\ell} f_k f_\ell,$$

where in the second equality, use has been made of Eq. (5). Then it is found that

$$2[G_0, G_0]_+ = \sum_{k \in \mathbb{Z}} k f_{-k} \left([b_k, G_0]_+ + \frac{1}{2} \sum_{\ell \in \mathbb{Z}} \ell f_{-\ell} f_k f_\ell \right) + (h.c.)$$

$$= \left(2[L^F_0, G_0] G_0 - N + \frac{1}{2} (-8N - 4(L^F_0)^2 + 24 \zeta(-2)) \right) + (h.c.)$$

$$= 2[F^F_0, G_0, G_0] - 10N - 4(L^F_0)^2 + 24 \zeta(-2),$$

where we have used $[b_k, G_0]_+ = 2 b_k G_0 - k f_k b_k$ by $[b_i, b_j] = i\delta_{i+j,0}$, $\sum_{k \in \mathbb{Z}} k f_{-k} b_k = [L^F_0, G_0]$ by $[f_r, L^F_0] = r f_r$ (for $r \in \mathbb{Z} + \kappa$), and $\sum_{k,\ell \in \mathbb{Z}} k \ell f_{-k} f_{-\ell} f_k f_\ell = -8N - 4(L^F_0)^2 + 24 \zeta(-2)$ by Eq. (6) with $i = -k, j = -\ell$. This ends the proof of the first relation of Eq. (8).

The second relation of Eq. (8) can be derived from Eq. (5).

References

[1] Zamolodchikov A B and Fateev V A 1985 Sov. Phys. JETP 62 215
[2] Beckers J and Debergh N 1993 Int. J. Mod. Phys. A 8 5041
[3] Nikitin A G and Gal’kin A V 2001 Nucl. Phys. B 102&103 322
[4] Durand S, Floreanini R and Vinet M 1989 Phys. Lett. B 233 158
[5] Ohnuki Y and Kametani S 1982 Quantum Field Theory and Parastatistics (Tokyo/Berlin: Univ. of Tokyo/Springer)
[6] Green H S 1953 Phys. Rev. 90 270
[7] Vasiliev M A 1991 Int. J. Mod. Phys. A 6 1115
[8] Bialynicki-Birula I 1963 Nucl. Phys. 49 605
[9] Wigner E 1950 Phys. Rev. 77 711
[10] Yang L M 1951 Phys. Rev. 84 788
[11] Greenberg O W and Messiah A M L 1965 Phys. Rev. 138B 1155
[12] Ruvakov V A and Spiridonov V P 1988 Mod. Phys. Lett. A 3 1337