On the $p$-version of FEM in one dimension: The known and unknown features

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Abstract

The paper analyses the convergence of the $p$-version of the FEM when the solution is piecewise analytic function. It focuses on pointwise convergence of the gradient. It shows that at boundary the rate is different than inside the element and there is a Gibbs phenomenon in the neighborhood of the point where the solution is not analytic. The major result is a conjecture written in the form of a theorem. The conjecture is based on careful numerical computations. The known theoretical results are stated.

Keywords:

1. Introduction

There are two basic versions of the Finite Element Method; The $h$-version and the $p$-version. In addition the versions can be combined into the $hp$-version. The first papers on the $p$-version and the $hp$-version were [1] [2]. In these papers the major basic theorems on the errors in the energy norm were established. The $h$-version with its roots in engineering problems is
much older, however. The *mathematical* theory was developed in the early 1970’s. We refer to \[3, 4\] where the history up to 1970 is presented. For the basic (older) monographs on the theory of the \(h\)-version we refer for example to \[5, 6\]. Since then the theory of the \(h\)-version has been developed in many different ways and there is a large body of literature on this subject. Especially we would like to mention the theory of the error estimates in norms other than the energy norm. For the local behavior of the \(h\)-version and pointwise estimates we refer to the excellent survey \[7\] and literature mentioned therein. All theoretical results for the \(h\)-version are independent of the degree of the elements.

Situation is different for the \(p\)-version. The domain is divided into elements as in the \(h\)-version but the number of elements is fixed and the convergence is obtained by increasing the degrees \(p\) either uniformly or adaptively. In the \(hp\)-version the convergence is obtained by the simultaneous refinement of the finite element mesh. For computation with the \(hp\)-version we refer to the books \[8, 9, 10, 11\] and citations therein, and the commercial software STRESSCHECK. The existing theory addresses only the error estimates in the energy and \(L^2\)-type norms. Thus, general theory of the local behavior is not available.

In one dimension the \(p\)-version is closely related to the expansion of functions in the Legendre series. Indeed, the subject of approximation and interpolation of a function by polynomials is a classical one. We refer here to the excellent books of the approximation theory and orthogonal polynomials in one dimension \[12, 13\]. There is large literature on orthogonal polynomials, see e.g. \[14\], and polynomials in general, e.g. \[15\]. For the history of the numerical problems we refer to the second chapter of \[16\] and to \[17\]. For more recent books on the \(p\)-version also called the spectral method we refer to \[18, 19\], and the recent book \[20\] with citations.

Nevertheless, despite all available results there are still open theoretical problems about the local behavior and the pointwise convergence of the \(p\)-version in one and higher dimensions. This paper addresses the behavior of the \(p\)-version when the solution is piecewise analytic. Such class of solutions is typical in the applications. Here the focus is on one dimensional problems. The paper is a survey of known results and it formulates new theorems as conjectures based on the detailed computations. It also serves as an introduction to the analysis of the \(p\)-version in two and three dimensions which will be addressed in the next papers. There are various similarities between the behavior of the \(p\)-version in one and higher dimensions as well significant
differences. For example in one dimension there was no pollution from one element into the neighboring ones. This is important when the elements are refined in the neighborhood of a singularity. For some results in this direction we refer to [8, p192] and [9, p190]. In higher dimensions solving the system of linear equations became nontrivial task. Recently various iterative methods were analyzed. Recently in [21] a multigrid method for solving the problem in one three dimensional cube with $p = 16$. Nevertheless this paper and the next ones are concentrating only on the problem of the convergence, Gibbs phenomenon, and element boundary behavior among others.

Through a series of numerical experiments with highly accurate results we arrive at the main theoretical result of this paper, the theorem (conjecture) on the error of the Legendre expansion of piecewise analytic functions,

$$u(x) = \sum_{i=1}^{m} c_i \cdot |x - a_i|^{\beta_i} + v(x), |a_i| < 1, \beta_i > -1$$

and $v(x)$ analytic function on $[-1,1]$. This conjecture is fully stated in Section 8 below. The stated convergence rates are optimal in the sense that they cannot be improved. We emphasize that the existing mathematical theory does not cover all parts of the conjecture and thus hopefully will lead to more refined mathematical analysis in the future.

The paper is organized as follows. In the Section 2 we define the one dimensional problem and its solution $u$. Section 3 shows that the $p$-version in this simple setting is the Legendre expansion of $u'$. In the Section 4 we analyze the Legendre expansion of $u'$, the error of its partial sum, and comment on the known and not available results. Section 5 analyzes the Legendre expansion of the solution $u$. Section 6 addresses the $p$-version of the solution which is the partial Legendre sum with a constraint at the boundary points. Section 7 generalizes the problem addressed in the Section 5. In Section 8 we summarize the result and formulate the theorem (conjectures) on the behavior of the $p$-version when the solution is piecewise analytic function.

2. The Problem

Consider the boundary value problem

$$-u''(x) = \delta_a(x), \ x \in I = (-1, 1),$$

(1)
with the boundary conditions
\[ u(\pm 1) = 0. \]  

Here \( \delta_a(x) \) is the Dirac function acting at \( x = a \).

Obviously
\[ u(x) = \begin{cases} 
  c(x + 1), & \text{if } -1 < x < a, \\
  c(a + 1) + (1 + c)(x - a), & \text{if } a \leq x < 1,
\end{cases} \]  

where \( c = -\frac{1-a}{2} \). The solution \( u \notin H^{3/2}_0(I) \), but \( u \in H^1_0(I) \cap B^{3/2}_{2,\infty}(I) \) where \( B^{3/2}_{2,\infty}(I) \) is the Besov space.

The weak solution \( u \in H^1_0(I) \) satisfies
\[ B(u, v) = \int_{-1}^{1} u'v' dx = v(a), \forall v \in H^1_0(I). \]  

Let us denote by \( \mathcal{E}_0(I) = H^1_0(I) \) the energy space with the product \( (u, v)_E \).

3. The \( h \)- and \( p \)-versions of the Finite Element Method

Let \( \mathcal{M} \) be a uniform FEM mesh with the nodes \( x_i^{(h)} = ih - 1, i = 0, 1, \ldots, n, n = \frac{2}{h} \), and the (open) elements \( \omega_i^{(h)} = (x_{i-1}^{(h)}, x_i^{(h)}) \). Let us assume that point \( a \in \omega_s^{(h)} \). Further let \( S_p^{(h)}(I) \subset \mathcal{E}_0 \) be the space of the continuous piecewise continuous polynomials of degree \( p \) on every element \( \omega_i^{(h)} \) and \( u_p^{(h)} \in S_p^{(h)} \), the \( hp \)-FEM solution of the Problem 1 which satisfies
\[ B(u_p^{(h)}, v) = v(a), \forall v \in S_p^{(h)}. \]  

From the classical FEM theory we have \( u_j^{(h)} = u \) on every \( \omega_j^{(h)}, j \neq s \), i.e., on every element which does not contain the point \( x = a \). Further denote \( \hat{u}^{(h)} \in S_p^{(h)} \) the piecewise linear function which coincides with the exact solution \( u \) at the nodes. Let us denote the difference \( w^{(h)} = u - \hat{u}^{(h)} \). Then \( w^{(h)} = 0 \) on every \( \omega_i^{(h)}, i \neq s \), and on the boundary of the element \( \omega_s^{(h)} \) containing the point \( x = a \). Further let \( w_p^{(h)} = u_p^{(h)} - \hat{u}^{(h)} \). Then \( w_p^{(h)} = 0 \) on all \( \omega_j^{(h)}, j \neq s \), and \( (w_p^{(h)})' \) is the \( p \)-partial sum of the Legendre expansion of the function \( (w^{(h)})' \) on \( \omega_s^{(h)} \). Fixing the \( p \) and letting \( h \to 0 \) we get the \( h \)-version of the FEM with the elements of order \( p \). Fixing \( h \) and letting \( p \to \infty \) we get the \( p \)-version.
For small fixed $p$ the error $\varepsilon_p^{(h)}(x) = u(x) - u_p^{(h)}(x)$ can be easily estimated. For the $h$-version there are many available results. Not so for the $p$-version. Here in this paper we shall be especially interested to study the pointwise convergence for $\varepsilon_p^{(h)}$ and $(\varepsilon_p^{(h)})'$ for the $p$-version of the FEM.

Thus, without any loss of the generality, we can concentrate only on the element $\omega_h^{(h)}$ which contains $x = a$ and assume that $h = 2$, i.e., to analyze the error of the Legendre expansion when using $p$-terms in the Legendre expansion.

4. The Legendre expansion of the function $(u^{(h)})'(x)$ with $h = 2$

Here we write $\varepsilon_p$ resp. $u$ instead of $\varepsilon_p^{(h)}$ resp. $u^{(h)}(x)$ and $x \in I = (-1,1)$.

Obviously we have

$$ u'(x) = \begin{cases} 
  c, & \text{if } -1 \leq x < a, \\
  1/2 + c, & \text{if } x = a, \\
  1 + c, & \text{if } a \leq x \leq 1, 
\end{cases} \quad (6) $$

with $\int_{-1}^{1} u'(x) dx = 0$. From it we get $c = 2^{-1}$.

Function $u'(x)$ is discontinuous at $x = a$ and in [6] we define $u'(a) = \frac{1}{2}(u'(x + 0) + u'(x - 0))$

Denoting the Legendre polynomials of degree $k$ with $P_k(x)$ we have

$$ u'(x) = \sum_{k=1}^{\infty} a_k P_k(x), \quad (7) $$

where the coefficients

$$ a_k = (k + \frac{1}{2}) \int_{-1}^{1} u'(x) P_k(x) dx = \frac{1}{2} (P_{k-1}(a) - P_{k+1}(a)), k = 1, 2, \ldots \quad (8) $$

For the error

$$ \varepsilon'_p(x) = u'(x) - u'_p(x) = u'(x) - \sum_{k=1}^{p} a_k P_k(x). \quad (9) $$

In the following we analyze $\varepsilon'_p(x)$ as a function of $p$ for different $x$.

First let us mention the major theorem about the error of the Legendre expansion.
Theorem 1 (22). Let \( f(x) \) be a function of bounded variation on \([-1,1]\). Let 
\[ f(x) = \sum_{k=0}^{\infty} a_k(f)P_k(x) \] 
and 
\[ S_p(f,x) = \sum_{k=0}^{p} a_k(f)P_k(x) \] 
be the \( p \)-partial sum of the Legendre series of \( f \). One has 
\[ a_k(f) = \frac{(k+\frac{1}{2})}{\pi} \int_{-1}^{1} f(t)P_k(t)dt. \]

Then for \( x \in (-1,1) \) and \( p \geq 2 \)
\[
| S_p(f,x) - \frac{1}{2} (f(x+0) - f(x-0)) | \leq 
\frac{28}{p} (1-x^2)^{-3/2} \sum_{k=1}^{p} V_{x+1/2}^{(1-x)/k} (g_x) + \frac{1}{\pi p} (1-x^2)^{-1} | f(x+0) - f(x-0) |, \tag{10}
\]
where
\[
g_x(t) = \begin{cases} 
  f(t) - f(x-0), & \text{if } -1 \leq t < x, \\
  0, & \text{if } t = x \\
  f(t) - f(x+0), & \text{if } x < t \leq 1,
\end{cases} \tag{11}
\]
and \( V_{a,b}^b(g) \) is the total variation of \( g \) on \([a,b]\).

Remark 2. In the proof of the above theorem the inequality
\[
| P_p(x) | \leq \left( \frac{2}{\pi} \right)^{1/2} (1-x^2)^{-1/2} p^{-1/2} \tag{12}
\]
is used. Nevertheless later results [13] proved
\[
| P_p(x) | \leq (1-x^2)^{1/4} \sqrt{\frac{2}{\pi p}}. \tag{13}
\]
By this improved estimate we get the powers of \((1-x^2)\) as \(-5/4\) and \(-1/2\) in the first and second terms, respectively.

Remark 3. With \( a = 0 \) in (6) the following holds:

Theorem 4 (23). We have
\[
| \varepsilon'_p(x) | \leq \frac{C}{p((1-x^2)^{1/2} + p^{-1})^{1/2}}
\]
for \( 2\delta < |x| \leq 1, \ 0 < \delta < 1/4, \) and \( C \) independent of \( p \).
The paper [24] is only a slight generalization of the Theorem 1 and the paper [25] addresses Legendre expansion of functions which are analytic on [-1,1]. We are not aware of any other general theorems similar to Theorem 1.

Let us now apply the Theorem 1 to our example and consider the different cases separately:

1. \( x = a \). Then the second term in (10) is \((\pi p (1 - a^2))^{-1}\) and the first term is

\[
\frac{28}{p} \left(1-a^2\right)^{-3/2} \sum_{k=1}^{p} 1 + \frac{1}{\pi p} \left(1-x^2\right)^{-1} = 28\left(1-a^2\right)^{-3/2} + \frac{1}{\pi p} \left(1-a^2\right)^{-1}
\]

(14)

Theorem does not indicate the convergence, however, we can compute the error directly. We have

\[
| \varepsilon'_p(a) | = | u(a) - \frac{1}{2} \sum_{k=1}^{m} (P_{k-1}(a) - P_{k+1}(a)) P_k(a) | = \frac{1}{2} P_{p+1}(a) P_p(a) | \leq \frac{1}{\pi p} \left(1-x^2\right)^{-1/2}
\]

(15)

As we will see below \( \| \varepsilon'_p(x) \|_{L^\infty(I)} \not\to 0 \) as \( p \to \infty \).

2. \( x \neq a, \pm 1 \). Then the second term in (10) is zero and we have

\[
| \varepsilon'_p(x) | = \frac{28}{p} \left(1-x^2\right)^{-3/2} \sum_{k=1}^{m} 1 = \frac{28}{p} \left(1-x^2\right)^{-3/2} m,
\]

(16)

where for \( x < a \), \( m \) is the largest integer such that \( x + (1-x)/k \geq 1+a \), and for \( x > a \), \( m \) is the largest integer such that \( x - (1+x)/k \geq (1-a) \).

**Remark 5.** From (16) we see that the error \( | \varepsilon'_p(x) | \) increases as \( x \to a \) and for \( x = a - \frac{1}{p} \) we have \( | \varepsilon'_p(x) | \sim 1 \) which relates to the Gibbs phenomenon discussed below. This is also the reason of exclusion of \( x = 0 \) in the Theorem 4.

3. \( x = \pm 1 \). Theorem 1 is not applicable to this case. Nevertheless in our example we can compute the error directly. We get

\[
| \varepsilon'_p(\pm 1) | \leq \frac{1}{2} (| P_p(a) | + | P_{p+1}(a) |) \leq (1-a^2)^{-1/4} \left( \frac{2}{\pi p} \right)^{1/2}
\]

(17)
Let us now show the computational results for the specific values $a = 1/2$ and $c = -\frac{1}{4}$. From the above theory we expect that

$$| \varepsilon'_p(x) | \leq C(x) p^{-\alpha(x)}$$

with $\alpha$ and $C$ depending on $x$. We shall compute for $p \in (1, 2200)$ and present the results in the loglog-scale. Notice that the upper limit 2200 is chosen so that in double precision floating point arithmetic, the rounding errors do not pollute the results in those cases where the series coefficients are known in advance. In some instances, however, we are forced to compute using symbolic computations in exact arithmetic. Naturally, this requires significant computing resources in terms of time and memory.

**Figure 1:** $\varepsilon'(x)$: Absolute value vs polynomial order, $p = 1, \ldots, 2200$.

Different behavior of the error for different values of $x$ will be seen. Figures 1a–1b show the error for $x = \pm 1$. We see that the form (18) leads to very accurate upper estimate with $\alpha(\pm 1) = 1/2$, $C(\pm 1) = 0.44194$ for $x = -1$ and for $x = 1$ we have $C(1) = 0.75$. From (17) we get $C = 0.85738$.

There is slightly larger error for $x = 1$ than for $x = -1$ because the discontinuity is closer. The preasymptotic behavior is very similar as well. It is interesting to notice that we also have $| \varepsilon'(\pm 1) | \geq C^* p^{-1/2}$ with $C^* = 0.15$.

**Figure 2** shows the error for $x = a$. We see $\alpha(1) = 1$ and $C(a) = 0.25293$. The estimate (15) leads to $C(a) = 0.31830$. Once more the rates agree but the observed constant is lower.

**Figure 3a** shows the error for $x = 0.1$ with convergence rate $\alpha(0.1) = 1$ and $C(0.1) = 0.84622$. This rate of convergence follows from the Theorem.
Figure 2: $\varepsilon'(x)$: Error at $x = a$, $\alpha(a) = 1$, $C(a) = 0.25293$. Absolute value vs polynomial order, $p = 1, \ldots, 2200$.

By the Theorem we get $C = 32.793$, i.e., largely overestimated error estimate although with correct rate. Of course we have to have in mind that the estimate in Theorem covers much larger class of functions and hence the constants have to be larger. Moreover, the overall convergence pattern has a very different character than previously and the lower bound seems not have the same rate as the upper bound. We are not able to make any hypothesis about the lower bound and there are no theoretical results available that are addressing the lower bound.

We see different rate of convergence for $x = 0.1$ and $x = \pm 1$. Hence let us address the error close to the boundary. Figures 3b–3d shows the error for $x_m = -1 + 10^{-m}, m = 2, 4, 6$. We see that the pre-asymptotic range increases with $m$. We have rate $\alpha = 1$ for all $x_m$ and $C$ increasing with $m$. $C(x_2) = 0.889506, C(x_4) = 2.733292, C(x_6) = 7.765878$. Figure 4 shows very different pattern of the error $|\varepsilon'_p(x)|$ for $x = 1$ and $x = 0.1$ for $1000 \leq p \leq 1500$.

The growth of $C$ is caused by the different rates $\alpha(-1) < \alpha(x_m)$ for all $m$. In the Theorem we have seen the term $(1 - x^2)^{-3/2}$ which indicates the growth of the rate. In the Figure 5a we show in the log log scale the growth of the constant $C(x)$. We see that $C(-1 + \xi) \sim D(-1 + \xi)^\beta$ with $\beta = -1/4$ and small $\xi > 0$. From the theory we have $\beta = -3/2$.

In the point $x = a$ the function $u'$ is discontinuous. Nevertheless convergence to $\frac{1}{2}(u'(a + 0) + u'(a - 0))$ and elsewhere in the neighborhood of $x = a$ has the same rate namely $\alpha = 1$. In the Figure 6 we see a typical error overshoot which is independent of $p$. Denoting by $y$ the position of the maximal error we have $|y - a| = Dp^{-1}$ with $D = 2.7777$. This is the well known
(a) Error at $x = 1/10$, $\alpha(1/10) = 1$, $C(1/10) = 0.84622$.

(b) Error at $x_2 = -1 + 10^{-2}$, $\alpha(x_2) = 1$, $C(x_2) = 0.889506$.

(c) Error at $x_4 = -1 + 10^{-4}$, $\alpha(x_4) = 1$, $C(x_4) = 2.733292$.

(d) Error at $x_6 = -1 + 10^{-6}$, $\alpha(x_6) = 1$, $C(x_6) = 7.765878$.

Figure 3: $\varepsilon'(x)$: Absolute value vs polynomial order, $p = 1, \ldots, 2200$, except for (d), $p = 1, \ldots, 10000$.

Gibbs phenomenon. We have $\varepsilon'(a + \xi) \leq D \xi^{-1} p^{-1}$ with $D$ independent of $p$ and small $\xi$. In the Figure 7, we show the convergence of $\| \varepsilon'_p \|_{L^2} = \| \varepsilon_p \| \varepsilon$. As expected, since $u' \in B_{2,\infty}^{1/2}$, it follows from the theory that $\| \varepsilon_p \| \varepsilon \leq C p^{-1/2}$ because of the regularity of $u$. This coincides very well with the numerical computations.

Remark 6. Approximation of the function $u'(x)$ was first addressed already more than 100 years ago [17]. It has been used as an example for the Legendre expansion in the most simple setting in various books. Let us mention for example [12, p. 163] and [26, p. 58].

Let us summarize our results
Figure 4: Patterns: Absolute value vs polynomial order.

(a) Pattern at \( x = -1 \)

(b) Pattern at \( x = 1/10 \)

Figure 5: \( \varepsilon'(x) \): Value of the coefficient vs distance to the point of interest.

(a) Growth of the constant \( C(-1 + \xi) \sim D \xi^{-1/4} \).

(b) Growth of the constant \( C(a + \xi) \sim D \xi^{-1} \).

1. The classical error estimate in the energy norm is in a very good agreement with the numerical results

2. The rate of convergence for \( x \in I, x \neq -1, 1, a \) does follow from the general theory based on only the total variation of the function but the constant is very inaccurate. The classical Gibbs phenomenon is clearly visible in the Figure 6.

We see that the convergence of the Legendre polynomials in a point \( x \) is not governed by the smoothness of the function in that point and its neighborhood. In our case the function was constant in the neighborhood. There is a strong pollution effect. We note that this pollution could be removed by a postprocessing \([16]\). It is characteristic
for the $p$-version that the pollution in the boundary points is larger than in its neighbors which leads to the boundary layer in the convergence. This is a significant difference in comparison to the $h$-version. For the analysis of the pollution in the $h$-version we refer to [7, Sect 9].

3. We have seen that in the neighborhood of the points $x = \pm 1$ and $x = a$ the error $\varepsilon'_p(x)$ behaves differently. This behavior can be described by using weighted space $L^\infty_w$ with the norm \[ \| \varepsilon'_p \|_{L^\infty_w} = \max_{x \in (-1, 1)} | \varepsilon'_p(x)w(x) | \] with \[ w(x) = | 1 - x |^\alpha | 1 + x |^\beta | x - a |^\gamma \]. Particularly in our case we have \[ \| \varepsilon'_p \|_{L^\infty_w} \leq C p^{-1} \] with $\alpha = \beta = 1/2$ and $\gamma = 1$. Nevertheless this characterization gives no information about the behavior in the singular points.
5. Legendre expansion of the solution $u$ given in (3)

Let $u(x) = \sum_{k=0}^{\infty} c_k P_k(x)$ be the Legendre expansion of the solution $u(x)$ of (2.1). Then $\sum_{k=0}^{p} c_k P_k(x)$ is not the $p$-version approximate solution of $u(x)$ because the constraint $u(\pm 1) = 0$ would be not satisfied. To prevent any misunderstanding we will write $w(x)$ instead of $u(x)$ and $w_p(x) = \sum_{k=1}^{p} c_k P_k(x)$ with $\eta_p(x) = w(x) - w_p(x)$ the error. Obviously

$$u(x) = w(x) = |x - a| + \text{linear function}$$

and hence $\eta_p(x)$ is the error of the Legendre expansion of the function $|x - a|$. The error of the $p$-version will be analyzed in the next section.

It is easy to see that the coefficients of the expansion of $u(x)$ are given in (8)

$$c_0 = -a_1/3, c_k = -a_{k+1}/(2k + 3) + a_{k-1}/(2k - 1), k = 1, 2, 3, \ldots$$

(19)

The function $u$ is smoother than $u'$ of the previous section. Numerically we see some analogous behavior with $\alpha(x) = 2$ for $x \neq a, \pm 1$, $\alpha(\pm 1) = \frac{3}{2}$ and $\alpha(a) = 1$. Once more we see the different rates of convergence for $x \neq a, \pm 1$. At $x = \pm 1$ the rate is by $\frac{1}{2}$ smaller than in its neighboring points and we have similar increase of $C$ (Figure 10). In fact, the observed growth rate is exactly the same, $\beta = -1/4$, yet as expected, the value of the constant is smaller here. Figure 8 shows the $|\eta_p(x)|$ for $x = -1, 0.99, 0.1$. We have no Gibbs phenomenon in its classical form but we have different rates of convergence in $x = a$ and its neighboring points. The difference of the rates is exactly 1. This difference in the rates is stronger here than at the boundary points. Figure 9 shows the error for $x = a$ and $x = a + 0.01$ and the Gibbs phenomenon is different. We have $\eta(a + \xi) \sim \xi^{-1} p^{-2}$.

Application of the Theorem 1 leads to $\alpha(x) = 1$ for $x \neq \pm 1, a$. For $x = a$ we get the error estimate $|\eta_p(a)| \leq C p^{-1} \lg p$, the right rate up to the log term. Notice, that in Figure 9a the series has been evaluated up to $p = 10000$ and there is no evidence of the log term affecting the convergence.

Typical theorem related to $\eta_p$ is

**Theorem 7** (12). Let $f(x)$ satisfy on $[-1,1]$ the Lipschitz condition with $\gamma > 1/2$. Then we have

$$|f(x) - \sum_{k=0}^{p} c_k P_k(x)| \leq c(x)/p^{\gamma - 1/2}$$

(20)
(a) Error at $x = -1$, $\alpha(-1) = 3/2$, $C(-1) = 0.42625$.
(b) Error at $x = -99/100$, $\alpha(-99/100) = 2$, $C(-99/100) = 0.76483$.
(c) Error at $x = 1/10$, $\alpha(1/10) = 2$, $C(1/10) = 0.73185$.

Figure 8: $\eta_p(x)$: Absolute value vs polynomial order, $p = 1, \ldots, 2200$.

In our case we have $\gamma = 1$ and the Theorem [1] predicts the rate $\alpha = 1/2$ while we have seen $\alpha = 1$.

In the Figure [11] we show the convergence rate $\| \eta_p \|_{L^2} \leq C p^{-3/2}$ as expected because of the regularity of the function $u$.

Let us summarize the results.

1. The results are very similar as in the previous case – only the rate is increased by one except for $x = 1$ because we addressed the convergence to the average $(u'(a + 0) + u'(a - 0))/2$. This increase was caused by the increase of the smoothness of the expanded function.

2. We have seen that the theory gives more pessimistic results than observed. The reason is that the theory deals with the functions having
only bounded variation. In addition the above theorems are addressing the $L^\infty$ norm which does not distinguish between the interior points and the points at the boundary.

6. The error $\varepsilon_p$ of the $p$-version

We addressed in the Section 5 the error of the partial Legendre expansion $w_p(x)$. We underlined that $w_p(x)$ is not the $p$-version solution of the problem (2.1) because the constraint of the boundary condition was not used.

The $p$-version solution $u_p$ is a modification of $w_p$. We get

$$| u_p(x) | = \sum_{k=0}^{p+1} b_k P_k(x)$$

(21)

where

$$b_k = a_{k-1}/(2k-1), \quad (\text{instead of } c_k = a_{k-1}/(2k-1) - a_{k+1}/(2k+3)),$$

$$b_1 = -a_2/5, \quad (\text{instead of } (c_1 = a_1/3 - a_3/7)),$$

$$b_0 = -a_1/3, \quad (\text{unchanged}).$$

(22)

Above in the parentheses we list the coefficients $c_k$ of the direct expansion of $w$ addressed in the previous section.

The results are very similar as before except that now there is no error at the boundary. Figure 12 shows the error for $x = -1+10^{-6}, x = -1+0.01, x =$
(a) Growth of the constant $C(-1 + \xi) \sim D \xi^{-1/4}$.

(b) Growth of the constant $C(a + \xi) \sim D \xi^{-1}$.

Figure 10: $\eta_p(x)$: Value of the coefficient vs distance to the point of interest.

Figure 11: Convergence in the norm: $\| \eta_p \|_{L^2} \leq C p^{-3/2}$. Value vs polynomial order, $p = 1, \ldots, 100$.

$a, x = a + 0.01$. We have $\alpha(x) = 2$ for $x \neq a, \alpha(a) = 1$. Comparing with the results in the Section 6 we see the same rates. It should be emphasized that the solution $u_p(x)$ is not the partial sum of the Legendre expansion of the solution $u$.

Let us summarize the results

1. We see very similar results except in the neighborhood of the boundary points. In the cases when no constraint was used we had $\varepsilon'(-1 + \xi) \sim (\xi)^{-1/4} p^{-1}$ and $\eta(-1 + \xi) \sim (\xi)^{-1/4} p^{-2}$. In the case with the constraint we have $\varepsilon(-1 + \xi) \sim (\xi)^{1/4} p^{-2}$, see Figure 13.
7. Generalizations

7.1. Legendre expansion of the function $|x|^{\beta}$, $-1 < \beta < 0$.

In the Sections 4 and 5 we addressed Legendre expansion of the function $|x-a|$ with $a = 1/2$ and $\beta = 0$ resp $\beta = 1$. The behavior of the error is completely analogous for $-1 < \alpha < 1$. In the case $|a| > 1$ the function is analytic and the convergence is exponential. For the theory of Legendre expansion for analytic functions we refer to [17]. In this section we will address the expansion of the function $f(\beta,x) = |x|^{\beta}$, $-1 < \beta < 0$. Obviously we have $f(\beta,x) \in L^1$ for $-1 < \beta$, and $f(\beta,x) \in L^2$ for $\beta > -1/2$. The case $\beta = -1/2$ is a special one because $f$ is “almost” in $L^2$ and hence its integral is in $H^1$. This case is important in two dimensions because the Green’s function is a function of this type. As we have already stated, the analysis

Figure 12: $u_p(x)$: Absolute value vs polynomial order, $p = 1, \ldots, 2200.$
of the two dimensional case is in preparation.

As shown in Figure 14, the observed rate \( \alpha = 1 + \beta \) for all \( -1 < \beta < 0 \).
Because of the strong singularity in \( x = 0 \) the rate \( \alpha \) has to be understood so that

\[
\left| \sum_{k=0}^{k=p} c_k P_k(0) \right| = Q(p) < C p^\alpha,
\]
with \( \alpha \) negative and \( C \) independent of \( p \).

Then we have

1. \( \alpha(\pm 1) = \beta + 1/2 \). Hence for \( \beta > -1/2 \) we have convergence; for \( \beta < -1/2 \) we have divergence with the growth \( p^{|\alpha|} \). The case \( \beta = -1/2 \) is a special one. We see divergence with bounded partial sums. More specifically we have \( c_{2p}(\beta) P_{2p}(\pm 1) = (-1)^p C(\beta) p^{\beta-1/2} \left[ 1 + O(1/p) \right] \). In general the rate \( \alpha = 0 \) could mean one of two possibilities: Either convergence to a wrong limit or divergence with the partial sums bounded.

2. \( \alpha(0) = \beta \). For all \( -1 < \beta < 0 \) there is divergence bounded by \( p^{|\beta|} \).

3. \( \alpha(x) = \beta + 1 \). For all \( 0 < |x| < 1 \) we see convergence with the error \( p^{-|\beta+1|} \). Further we have for \( x = (-1 + \xi) \), small \( \xi > 0 \), \( |\varepsilon(x)| \sim \xi^{-1/4} p^{-|\beta+1|} \) and \( Q(p) \sim \xi^{-1} p^{-|\beta+1|} \).

7.2. Legendre expansion of the function \( |x + 1|^\beta, -1 < \beta \).

Using the conventions of the previous section we have (Figure 15) for \( \beta > -1 \)

1. \( \alpha(-1) = 2\beta \), naturally only for \( \beta > 0 \).
Figure 14: Legendre expansion of the function $|x|^\beta$. The observed rate $\alpha = 1 + \beta$ in all cases. Absolute value of the error vs polynomial order, $p = 1, \ldots, 2200$.

2. $\alpha(1) = 2\beta + 1$.

3. $\alpha(x) = 2\beta + 3/2$. Further we have for $x = (-1 + \xi)$, small $\xi > 0$, $|\varepsilon(x)| \sim \xi^{-3/4}p^{-(2\beta+3/2)}$ and $x = (1 - \xi)$, small $\xi > 0$, $|\varepsilon(x)| \sim \xi^{-1/4}p^{-(2\beta+3/2)}$.

Notice, that unlike in the previous examples, the coefficients of the Legendre expansion are not known a priori. The construction used in the computations is given in the Appendix.
8. Summary and Conjecture

We have analyzed the behavior of the $p$-version in one dimension on $[-1,1]$ when the solution $u(x)$ is a piecewise analytic function,

$$u(x) = \sum_{i=1}^{m} c_i \left| x - a_i \right|^{\beta_i} + \gamma(x), \left| a_i \right| < 1, \beta_i > -1$$

and $\gamma(x)$ analytic function on $[-1,1]$. We have shown that the approximate solution is the partial sum of the Legendre expansion of $u$.

We focus on the asymptotic behavior of the $p$ partial Legendre expansion of the functions $\left| x - a \right|^{\beta}$ leading to the algebraic convergence rate while the convergence rate for an analytic function is exponential.
We concentrated first on the case $\beta = 0, 1$ in connection with solving a simple typical second order boundary value problem by the $p-$version. In this context we mentioned all theoretical estimates known to us and compared the computational results with their theoretical predictions. Then we addressed the case for general $\beta$. Based on these results we formulate now the conjecture on the error of the Legendre expansion.

**Conjecture 8.** Let $w(x) = \sum_{k=0}^{\infty} c_k P_k(x)$ be the Legendre expansion of the function $w(x) = |x - a|^{\beta}, |a| < 1, -1 < \beta$, and $w_p(x) = \sum_{k=0}^{p} c_k P_k(x)$ be its partial sum. Denote by $\varepsilon(x) = |w(x) - w_p(x)|$ the error. Then $\varepsilon(x)$ has the following properties.

1. $x \in (-1, a) \cup (a, 1):$ Then $\varepsilon(x) \leq C(x) p^{-\alpha}, \alpha = - (\beta + 1)$, where $C(x)$ independent of $p$, the rate $\alpha = (\beta + 1)$ is optimal, i.e., it cannot be improved.

2. We have $C(-1 + \xi) \leq D\xi^{-\rho}, C(1 - \xi) \leq D\xi^{-\rho}, \rho = 1/4, 0 < \xi < \delta, 0 < \delta$ with $D$ and $\delta$ independent of $p$ and the rate $\rho = 1/4$ is optimal.

3. We have $C(a + \xi) \leq D\xi^{-\sigma}, C(a - \xi) \leq D\xi^{-\sigma}, \sigma = 1, 0 < |\xi| \leq \delta$, with $D$ and $\delta$ independent of $p$ and the rate $\sigma = 1$ is optimal.

4. $x = \pm 1:$ Then $\varepsilon(\pm 1) \leq C p^{-\alpha}, \alpha = (\beta + 1/2)$ where $C$ is independent of $p$. The rate $\alpha = (\beta + 1/2)$ is optimal. For $\beta < -1/2$ there is divergence and for $\beta = -1/2$ there is convergence to a limit which is not $w(\pm 1)$. In general the rate $\alpha = 0$ will be understood as indicating a bounded sequence.

5. $x = a:$ Then $\varepsilon(a) \leq C p^{-\alpha}, \alpha = 1$ is independent of $p$. The rate $\alpha = 0$ is special because the function $w$ is then discontinuous. Denoting $\bar{w}(a) = \frac{1}{2}(w(a + 0) + w(a - 0))$. Then there is convergence to $\bar{w}$ with the rate $\alpha = 1$.

Some comments: Statement 2 is related to the boundary layer because the rates in $x = -1$ and $x > -1$ are different. Statement 3 is related to the Gibbs phenomenon which appears for all $\beta$. Note that in the above statements the term $\lg p$ is not present.

We have seen typical features of the $p-$version for piecewise analytic solutions.
1. The statements in the above theorems are not covered by the available mathematical theory. They are conjectures based on the careful computations and their generalization.

2. The errors are highly oscillatory and the pattern is different for different values of $x$.

3. The preasymptotic range is large and the practical computations likely outside the asymptotic range.

4. The error behavior in the energy norm and the $L^2$ is well covered by the theory, is not oscillatory, and the preasymptotic range is much shorter.

5. When the solution is very smooth, precisely analytic on the entire domain $[-1.1]$, the convergence is exponential.

6. The behavior of the $p$-version has some but not all characteristics in higher dimensions. We shall address these issues in the future.

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Appendix A. Legendre Coefficients of $|x + 1|^\beta$, $-1 < \beta$.

Consider the identity

$$I_k = \int_{-1}^{1} |x + 1|^\beta x^k dx = \frac{2F_1(k + 1, -\beta; k + 2; -1)}{k + 1} + \frac{(-1)^k \Gamma(\beta + 1) \Gamma(k + 1)}{\Gamma(k + \beta + 2)}, \quad \beta > -1, \ k \in \mathbb{Z}, k \geq 0,$$

(A.1)

where $2F_1(a, b; c, z)$ denotes the hypergeometric function. Notice that every term converges for $-1 < \beta$, since $2F_1(a, b; c, z)$ converges for $z = \pm 1$, if $c > a + b$, that is, $k + 2 > k + 1 - \beta$.

Expanding the Legendre polynomial $P_k(x) = \sum_{m=0}^{k} a_m x^m$, the standard expansion

$$c_k = \int_{-1}^{1} |x + 1|^\beta P_k(x) dx (2k + 1)/2,$$
becomes using the identity of (A.1)

\[ c_k = \frac{\left( \sum_{m=0}^{k} a_m I_m \right)(2k + 1)}{2}. \quad (A.2) \]

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