Superradiant emission spectra of a two-qubit system in circuit quantum electrodynamics

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Abstract. In this paper, we study the spontaneous emission spectra and the emission decay rates of a simplest atom system that exhibits sub- and superradiant properties: a system which consists of two artificial atoms (superconducting qubits) embedded in a one-dimensional open waveguide. The calculations are based on the method of the transition operator which was firstly introduced by R. H. Lehmberg to theoretically describe the spontaneous emission of two-level atoms in a free space. We obtain the explicit expressions for the photon radiation spectra and the emission decay rates for different initial two-qubit configurations with one and two excitations. For every initial state we calculate the radiation spectra and the emission decay rates for different effective distances between qubits. In every case, a decay rate is compared with a single qubit decay to show the superradiant or subradiant nature of a two-qubit decay with a given initial state.

1 Introduction

The control of spontaneous emission in the multi-atom (or qubit) system that interacts with a quantized radiation field in restricted geometries has received a great deal of attention in recent years (see review paper [1] and references therein). This can be achieved in various physical set-ups, for example, by putting two-level atoms in an optical cavity [2], by embedding them in a nanophotonic waveguide [3] or by coupling superconducting qubits to a transmission line resonator [4–6]. Due to spatial confinement, these set-ups allow one to achieve an almost ideal mode matching which results in a strong coupling regime when the interaction strength overwhelms the relaxation rates [7] and even an ultra-strong coupling regime when the coupling and energy level splitting are of the same order of magnitude [8]. These experimental conditions are very challenging to obtain for regular atoms in the optical domain.

The interaction of excited atoms with the continuum of the environment modes leads to spontaneous emission which is one of the major sources of decoherence. The spontaneous emission results in irreversible loss of the information encoded in the internal states of the system and thus is regarded as the main obstacle in practical implementations of quantum processing.

The early studies of spontaneous emission in multi- and two-atom systems deal mainly with atoms placed in a free space environment [9–14]. In this case, the interaction of atoms with the vacuum modes gives rise to short range dipole-dipole interaction between the atoms [9]. The spontaneous emission of excited atom in free space is the result of interaction of the atom with the continuum modes of the vacuum. A collection of $N$ identical two level excited atoms undergoes a spontaneous coherent transition to the ground state, which is accompanied by the emission of $N$ photons, the intensity of which scales as $N^2$, and the decay rate of which is $N \Gamma$, where $\Gamma$ is the decay rate of an isolated atom [15]. Therefore, the $N$ excited atoms decay $N$ times faster than an isolated atom. This property of $N$ atom system was named superradiance [16,17].

However, the behavior of atoms in a confined geometry is quite different from that in a free space. For example, a spontaneous decay rate of an atom embedded in a resonator may significantly differ from its decay rate in a free space (so called Purcell effect) [18]. The exchange of the virtual photons between the identical equally spaced atoms in a one-dimensional waveguide results in an infinite-range inter-atomic interaction the strength of which periodically depends on the ratio $d/\lambda$, where $d$ is the distance between neighbor atoms, $\lambda$ is the wavelength of the guiding mode. Furthermore, this system exhibits collective excitations with lifetimes from extremely sub- to superradiant values relative to the radiative lifetime of the individual atom [19–23].

Many of these effects have experimentally been realized within the frame of circuit quantum electrodynamics (QED) with superconducting qubits as artificial atoms [4]. A significant difference between natural atoms and superconducting qubits is that in cir-
cuit QED we can create artificial atoms with desirable parameters some of which can be tuned, e.g. resonant frequency or coupling strength [24]. What is more important, we can address and manipulate the artificial atoms individually [25]. This makes it possible to in-depth study of different types of interaction between a few qubits which can be far more interesting and complex than just one qubit in a cavity [11,26]. Because of the perfect mode matching the exchange interaction between the qubits is very strong and critically depends on the effective distance between them which can be tuned by changing the wavelength via qubit resonant frequency. Different spatial arrangements of qubits in a chain can lead to significant modification of decay rates [27,28]. This collective effect may lead to both the enhancement of a decay rate, which is called superradiance [29,30], and the reduction of the decay rate, which corresponds to subradiance [19,20].

The arrangement consisting of two interacting atoms is a simplest system which exhibits sub- and superradiant properties. This system has been extensively studied in the frame of 1D circuit quantum electrodynamics [21,23,26–29,31,32].

In this paper, we consider a system which consists of two artificial atoms (superconducting qubits) in a one-dimensional open waveguide. It is worth noting that our formalism can also be applied to real atoms in the optical domain. However, the advantage of superconducting qubits which work in the microwave domain is that they technologically can be located in the prescribed positions. Moreover, their frequency can externally be tuned to a desired value that allows for changing the effective distance between neighbor qubits. In contrast to previous studies, we obtain here the general explicit expression which allows us to calculate the radiation photon spectra and the emission decay rates (decay rates of the energy loss) for different initial two-qubit configurations with one and two excitations and for different values of the ratio $d/\lambda$. We systematically compare two-qubit spontaneous emission spectra with those of a single qubit. We show that depending on the ratio $d/\lambda$, there exist both superradiant states when a decay rate of initial two-qubit configuration exceeds that of a single qubit and subradiant states when a decay rate of initial two-qubit configuration is less than that of a single qubit. Our results for $d/\lambda \ll 1$ are consistent with those which has experimentally been observed for two superconducting qubits in a low quality cavity [33].

There exist two approaches which allow us to perform these calculations. The most common method uses the master equation for the reduced density matrix in the Lindblad form [27]. The second approach uses the Heisenberg equation of motion for arbitrary system operator [9,10]. Two approaches are, of course, equivalent. The choice in favour of either of these methods depends on the problem at hand.

Here we choose the second method and take as a system operator the so-called transition operator, firstly introduced by Lehmerg [34] to theoretically describe the spontaneous emission of two-level atoms in free space [9,10]. The exact expressions for the matrix elements of the transition operator can be obtained within standard quantum mechanics formalism using Heisenberg equations. Tracing out the photonic modes from the equations of motion allows us to obtain equations only for atomic operators independent of the photon number. As distinct from conventional density matrix approach, differential equations for the matrix elements of the transition operator are linear in a basis set of a spin system and, therefore, it is easier to solve them analytically. Moreover, because the solutions of equations are operator functions, they are independent of the specific initial state of the system. It means that we don’t need to find a new solution for every new initial state like in the case of equations for the density matrix. Once the exact solutions for transition operators are found, we can use them not only to obtain transition probabilities but also for the calculation of the photonic emission spectrum for an arbitrary initial density matrix. As we show here, transition operators and elements of the density matrix are very closely related, so one can easily switch from one approach to the other one if needed.

The paper is organized as follows. In Sect. 2 we define the transition operator, describe its general properties, and establish its connection to a density matrix. In Sect. 3 we show the application of the method for the description of N-qubit system in a 1D transmission line and obtain the general equation of motion for the matrix elements of transition operators. In Sect. 4 we present a general expression for the spectral density of photons in terms of two-time correlation functions of spin operators and show how these types of correlation functions can be calculated in terms of the vacuum average of transition operators and initial density matrix. In Sect. 5 we apply our method to a two-qubit system and find the diagonal and off-diagonal matrix elements of the transition operator. In Sect. 6 we calculate the probabilities of different transitions between the states in a two-qubit system, which contribute to the radiation spectra and the photon emission decay rates. The main results of our paper are presented in Sect. 7. In this section we obtain the general expressions which allow us to calculate both the radiation spectra and the emission decay rates for arbitrary initial states of a two-qubit system. For every initial state we consider in Sect. 7, we calculate the radiation spectra and the emission decay rates for different effective distances between qubits. In every case, a decay rate is compared with a single qubit decay to show the superradiant or subradiant nature of a two-qubit decay with a given initial state. A conclusion of our work is presented in Sect. 8.

2 General properties of transition operator

We consider a system of $N$ identical qubits with eigenstates $|i\rangle$ coupled to a continuum of photon modes $|\nu\rangle$ in a one-dimensional open waveguide. Our main interest is the probability of transition from some arbitrary initial state of a qubit system $|\Psi_0\rangle$ with no photons to some
final state \( |\Psi_1\rangle \) with \( \nu \) photons in the field. According to the general principles of quantum mechanics, the probability amplitude of such transition is given by the following matrix element:

\[
\langle \Psi_1, \nu | e^{-iHt} | \Psi_0, 0 \rangle,
\]

(1)

where \( H \) is the complete Hamiltonian which includes the N-qubit system, photon field, and their interaction, \( |0\rangle, |\nu\rangle \) are the Fock states with zero and \( \nu \) photons, respectively. To find the total transition probability we must find the squared modulus of this amplitude and sum it over the complete set of possible final photon states \( |\mu\rangle \) for the field.

\[
W_{0\rightarrow1} = \sum_\mu \left| \langle \Psi_1, \nu | e^{-iHt} | \Psi_0, 0 \rangle \right|^2
\]

(2)

Using the completeness of the set of the photon states \( \sum_\mu \langle \mu | \mu \rangle = 1 \), we rewrite (2) as follows:

\[
W_{0\rightarrow1} = \langle \Psi_0 | 0 \rangle e^{iHt} | \langle \Psi_1 | e^{-iHt} 0 \rangle | \Psi_0 \rangle
\]

(3)

By expanding wave function \( |\Psi_1\rangle \) over the complete set of qubit states \( |i\rangle \) with some coefficients \( c_i \) we can rewrite (3) in the following form:

\[
W_{0\rightarrow1} = \sum_{i,j} c_i c_j^* \langle \Psi_0 | e^{iHt} | i \rangle | \langle j | e^{-iHt} 0 \rangle | \Psi_0 \rangle
\]

(4)

where \( \langle \ldots \rangle_0 \) is the average over photon vacuum \( \langle \ldots \rangle_0 = \langle 0 \ldots 0 \rangle \).

Following Lehmberg [34], we define a transition operator:

\[
P_{i,j}(t) = e^{iHt} | i \rangle | \langle j | e^{-iHt}.
\]

(5)

The expression (4) can then be written in a form:

\[
W_{0\rightarrow1} = \sum_{i,j} c_i c_j^* \langle \Psi_0 | \langle P_{i,j}(t) \rangle_0 | \Psi_0 \rangle.
\]

(6)

Thus, the probability of transition from the eigenstate \( |n\rangle \) to the eigenstate \( |m\rangle \) can be calculated by using the corresponding matrix element of the transition operator \( P_{m,m} \) averaged over photon vacuum:

\[
W_{n\rightarrow m} = \langle n | \langle P_{m,m}(t) \rangle_0 | n \rangle.
\]

(7)

It follows from the completeness of the qubits states \( |i\rangle \) that the sum of diagonal elements of the transition operator is equal to one:

\[
\sum_i P_{i,i} = 1.
\]

(8)

As follows from the definition (5), the expression for the elements of the transition operator satisfies the Heisenberg equation:

\[
\frac{d}{dt} P_{i,j}(t) = i [H, P_{i,j}(t)],
\]

(9)

with the initial conditions \( P_{i,j}(0) = |i \rangle \langle j | \).

Unlike the equations for spin operators or elements of the density matrix, equations for \( P_{i,j} \) are linear: they contain only the first degrees of the same operators. The number of equations for the transition operator is determined by the number of states from the complete set. For example, for one qubit there are only two states, the excited state \( |e\rangle \) and the ground state \( |g\rangle \). Accordingly, there are four matrix elements of the transition operator: \( P_{ge}, P_{eg}, P_{eg}, P_{ge} \). However, there are only three independent equations since the \( P_{eg} \) is a complex conjugate to a \( P_{ge} \). For two qubits there are four states and, consequently, ten independent equations: four equations for the diagonal elements of the transition operator and six equations for off-diagonal ones (excluding complex conjugates).

In general case, the matrix elements of the transition operator averaged over photon vacuum have the following form:

\[
\langle P_{i,j}(t) \rangle_0 = \sum_{m,n} c_{ij}^{mn}(t) |m\rangle \langle n|,
\]

(10)

where \( c_{ij}^{mn}(t) \) are c-numbers.

In principle, we may choose the basis states of a spin system in the following way:

\[
\langle P_{i,j}(t) \rangle_0 = \sum_{m,n} c_{ij}^{mn}(t) |m\rangle \langle n|; \ i \neq j; \ m \neq n
\]

(11)

\[
\langle P_{i,i}(t) \rangle_0 = \sum_q c_{i}^q(t) |q\rangle \langle q|
\]

(12)

In this case, the off-diagonal element, Eq. 11, of transition operator provides the transitions between different states of a spin system, while the diagonal element of transition operator has only diagonal matrix elements in the Hilbert space of a spin system.

By the definition, the qubit density matrix of a spin system, \( \rho_{S,j,i}(t) = \langle j | \rho(t) | i \rangle \equiv \langle j | \rho(t) | i \rangle \), where \( \rho(t) \) is the density matrix of the whole system:

\[
\rho(t) = e^{-iHt} \rho(0) e^{iHt},
\]

(13)

can be expressed in terms of the average value of the transition operator:

\[
\rho_{S,j,i}(t) = Tr_{\nu} (\rho(0) P_{j,i}(t)) \equiv \langle P_{j,i}(t) \rangle,
\]

(14)

where the trace is taken over both the qubit system and the photon field, and \( \rho(0) \) is the initial density matrix of the whole system.

If we assume that initially the field is in a photon vacuum:

\[
\rho(0) = \rho_S(0) \otimes \rho_{\nu}(0) = \rho_S(0) \otimes |0 \rangle \langle 0|,
\]

(15)
we then obtain from (14):

$$\langle l | \rho_S(t) | m \rangle = \sum_{n \neq n'} \langle n | \rho_S(0) | q \rangle \langle q | P_{m,l}(t) \rangle_0 | m \rangle.$$  \hspace{1cm} (16)

Therefore, the matrix elements of a reduced density matrix can be expressed in terms of the matrix elements of the vacuum average of the transition operator if the initial density matrix $\rho_S(0)$ is known. It should be noted that in contrast to the elements of the density matrix, which are numerical functions, the matrix elements of $P_{m,l}(t)$ are the operator functions. If the system is initially in one of its basis states $|s\rangle (\langle s | \rho_S(0) | s \rangle = 1)$, then it follows from (16):

$$\langle l | \rho_S(t) | m \rangle = \langle s | \langle P_{m,l}(t) \rangle_0 | s \rangle$$  \hspace{1cm} (17)

For basis set with the properties (11) and (12) it follows from (17) that the off-diagonal elements of reduced density matrix are zero. For the diagonal elements of reduced density matrix, that is, for the populations, we obtain:

$$\langle m | \rho_S(t) | m \rangle = \langle s | \langle P_{m,m}(t) \rangle_0 | s \rangle.$$  \hspace{1cm} (18)

Therefore, if $m \neq s$ the population $\langle m | \rho_S(t) | m \rangle$ can be understood as the transition amplitude from the initial state $|s\rangle$ to the state $|m\rangle$.

3 Multi-qubit system

Consider a system consisting of $N$ qubits in a one-dimensional infinite waveguide. This system can be described by a Jaynes-Cummings Hamiltonian (from now on we use the units $\hbar = 1$, therefore, all energies are expressed in frequency units):

$$H = \frac{1}{2} \sum_{n=1}^{N} \left( 1 + \sigma^{(n)}_z \right) \Omega_n + \sum_k \omega_k a_k^\dagger a_k$$

$$+ \sum_k \left( a_k^\dagger S_k^- + S_k^+ a_k \right),$$  \hspace{1cm} (19)

where we introduced collective atomic spin operators:

$$S_k^- = \sum_{n=1}^{N} g_k(n) e^{-i k n x} \sigma^{(n)}_-, \quad S_k^+ = \sum_{n=1}^{N} g_k^{*(n)} e^{i k n x} \sigma^{(n)}_+.$$  \hspace{1cm} (20)

Here $\sigma^{(n)}_z$ is a Pauli spin operator, $\Omega_n$ is a resonant frequency of $n$th qubit, $a_k^\dagger(a_k)$ are creation (annihilation) operators for a $k$ mode photon, $\omega_k$ is a photon frequency, $\sigma^{(n)}_z = |g|_{n} \langle e |$ and $\sigma^{(n)}_- = |e|_{n} \langle g |$ are the lowering and raising atomic operators which lower or raise a state of the $n$th qubit, $g_k$ is a coupling strength between the qubit and the field, $x_n$ is a spatial coordinate of the $n$th qubit.

Here we assume the qubit system losses energy only into waveguide mode and does not include any non-radiative channels. The conditions in which non-radiative decay and dephasing are much smaller than the field coupling and spontaneous emission rate are necessary to study super- and sub-radiant effects and can be realized in experiment [33].

From (9) and (19) we obtain the equation of motion for the transition operator:

$$\frac{d}{dt} P_{ij} = \frac{i}{2} \left[ \sum_{n=1}^{N} \Omega_n \left( e^{i \Omega t} \sigma^{(n)}_z \right) |i \rangle \langle j | e^{-i \Omega t} - e^{i \Omega t} |j \rangle \langle i | \sigma^{(n)}_z e^{-i \Omega t} \right)$$

$$+ i \sum_k \left( e^{i \Omega t} S_k^- |i \rangle \langle j | e^{-i \Omega t} - e^{i \Omega t} |j \rangle S_k^- e^{-i \Omega t} \right)$$

$$+ i \sum_k \left( e^{i \Omega t} S_k^+ |i \rangle \langle j | e^{-i \Omega t} - e^{i \Omega t} |j \rangle S_k^+ e^{-i \Omega t} \right) a_k(t),$$  \hspace{1cm} (21)

where the photon operators are in the Heisenberg representation:

$$a_k^\dagger(t) = e^{i \Omega t} a_k^\dagger e^{-i \Omega t}, \quad a_k(t) = e^{i \Omega t} a_k e^{-i \Omega t}.$$  \hspace{1cm} (22)

For photon operators the equations of motion are as follows:

$$i \frac{da_k}{dt} = \left[ a_k(t), H \right] = \omega_k a_k(t) + S_k^-(t),$$  \hspace{1cm} (23a)

$$i \frac{da_k^\dagger}{dt} = \left[ a_k^\dagger(t), H \right] = -\omega_k a_k^\dagger(t) - S_k^+(t).$$  \hspace{1cm} (23b)

where $S_k^\pm(t)$ are collective spin operators in the Heisenberg picture. The formal solution of these equations is given by:

$$a_k(t) = a_k(0) e^{-i \omega_k t} - i \int_{0}^{t} e^{-i \omega_k (t-\tau)} S_k^-(\tau) d\tau,$$  \hspace{1cm} (24a)

$$a_k^\dagger(t) = a_k^\dagger(0) e^{i \omega_k t} + i \int_{0}^{t} e^{i \omega_k (t-\tau)} S_k^+(\tau) d\tau.$$  \hspace{1cm} (24b)

where the first term in the right hand side of (24) is a free field part and the second term is a part radiated by atoms.

Since we considering a 1D waveguide, $k$ takes only two directions, $k = \pm \omega/v_g$. From (20) it follows that positive $k$, $|k| = +\omega/v_g$, corresponds to the right (forward) propagating modes, while the negative $k$, $|k| = -\omega/v_g$, corresponds to the left (backward) propagating modes. Therefore, the operators $a_{\pm|k|}$, $a_{\mp|k|}$, and $a_{\pm|k|}$, $a_{\mp|k|}$ correspond to right and left propagating photons, respectively.

Even though the photon operators $a_k^\dagger(t)$, $a_k(t)$ commute with collective spin operators $S_k^\pm(t)$, each term
in (24) does not commute with $S_k^\pm(t)$. This explains the position of these operators in the third term of Hamiltonian (19). They should be placed in such a way that the final expression the creation operators $a_k^+(t)$ were placed on the left of the transition operator $\hat{P}_{ij}$, while the annihilation operators $a_k(t)$ were placed on the right. This is necessary for the terms with initial photons to be dropped out upon averaging the transition operator over the photon vacuum.

Substituting the expressions (24) into the equation of motion (21) we obtain:

\[
\frac{d\hat{P}_{ij}}{dt} = i \sum_{n=1}^{N} \Omega_n \left( \langle i | e^{-iHt} | j \rangle e^{-iHt} - e^{iHt} \langle j | e^{-iHt} | i \rangle \sigma_z^{(n)} e^{-iHt} \right) + i \sum_{k} a_k^+(t) e^{i\omega_k t} e^{iHt} S_k^+ |i\rangle \langle j| e^{-iHt} - i \sum_{k} a_k(t) e^{-i\omega_k t} e^{iHt} S_k^- |i\rangle \langle j| e^{-iHt}
\]

\[
+ i \sum_{k} e^{iHt} S_k^+ |i\rangle \langle j| e^{-iHt} a_k(t) e^{-i\omega_k t} - i \sum_{k} e^{iHt} |j\rangle \langle i| S_k^+ e^{-i\omega_k t} a_k(t) e^{-iHt}
\]

\[
+ \sum_{k} \int_{-\infty}^{t} e^{-i\omega_k (t-t')} S_k^+(t')d\tau e^{iHt} \left[ |i\rangle \langle j|, S_k^- \right] e^{-iHt}
\]

\[
+ \sum_{k} e^{iHt} \left[ S_k^+ |i\rangle \langle j| \right] e^{-iHt} \int_{0}^{t} e^{-i\omega_k (t-t')} S_k^-(t) d\tau
\]

(25)

Here we write the equation in such a manner that the action of atomic operators on system states is clearly visible. Equation (25) can be rewritten in terms of transition operators and atomic operators in the Heisenberg picture, since $e^{iHt} \sigma_z^{(n)} |i\rangle \langle j| e^{-iHt} = \sigma_z^{(n)}(t)\hat{P}_{ij}(t)$ (the same procedure applies to the terms with $S_k^\pm$ as well).

Up to now, we have not made any approximations: the above expressions are exact. To solve equation (25), the following assumptions are made:

\[
\sigma_z^{(n)}(\tau) \approx \sigma_z^{(n)}(t) e^{-i\Omega_n(\tau-t)}, \quad \sigma_z^{(n)}(\tau) \approx \sigma_z^{(n)}(t) e^{-i\Omega_n(\tau-t)},
\]

(26a)

(26b)

Assuming that all qubit frequencies are identical and equal to some value $\Omega$, we then obtain:

\[
S_k^+(\tau) \approx S_k^+(t) e^{-i\Omega(\tau-t)},
\]

\[
S_k^-(\tau) \approx S_k^-(t) e^{i\Omega(\tau-t)},
\]

(27a)

(27b)

The assumptions (26), (27) are equivalent to Wigner-Weisskopf or Markov approximations [9,27]. It allows us to take $S_k^\pm(\tau)$ out of the integrand in the last line of (25). We then rewrite the rest of the integrals by taking into account the resonant approximation, i.e. assuming that the main contribution to the integral is near the resonance frequency $\Omega$. This allows us to take the upper limit to infinity, and we get:

\[
\int_{0}^{t} e^{i(\omega-\Omega)(\tau-t')}d\tau' \approx \int_{0}^{\infty} e^{i(\omega-\Omega)\tau} d\tau
\]

\[
= \pi\delta(\omega-\Omega) + iP.v. \left( \frac{1}{\omega-\Omega} \right),
\]

(28)

where $\delta(\omega)$ is a Dirac delta function and $P.v.$ is a Cauchy principal value.

According to all assumptions above, we obtain from (25) the equation of motion for the transition operator in the following form:

\[
\frac{d\hat{P}_{ij}}{dt} = \sum_{n=1}^{N} \Omega_n \left( e^{iHt} \sigma_z^{(n)} |i\rangle \langle j| e^{-iHt} - e^{iHt} \sigma_z^{(n)} e^{-iHt} \right) + i \sum_{k} a_k^+(0) e^{i\omega_k t} e^{iHt} S_k^+ |i\rangle \langle j| e^{-iHt}
\]

\[
+ i \sum_{k} e^{iHt} S_k^+ |i\rangle \langle j| e^{-iHt} a_k(0) e^{-i\omega_k t}
\]

\[
+ \sum_{n,m} \frac{\Gamma_{n,m}}{2} e^{iHt} \left( 2\sigma_+^{(n)} |i\rangle \langle j| \sigma_-^{(m)} - \sigma_-^{(n)} |i\rangle \langle j| - |i\rangle \langle j| \sigma_+^{(n)} \sigma_-^{(m)} \right) e^{-iHt}
\]

\[
+ i \sum_{n,m} \alpha_{n,m} e^{iHt} \left( |i\rangle \langle j| \sigma_+^{(n)} \sigma_-^{(m)} - \sigma_-^{(n)} \sigma_+^{(m)} \right) e^{-iHt}
\]

(29)

where according to the Fermi Golden rule we have introduced a decay rate $\Gamma_{n,m}$:

\[
\Gamma_{n,m} = \sum_{k} 2\pi \delta (\omega_k - \Omega) g_k^{(n)} g_k^{(m)} e^{-ik(x_n-x_m)},
\]

(30)

and a frequency shift $\alpha_{n,m}$:

\[
\alpha_{n,m} = \sum_{k} g_k^{(n)} g_k^{(m)} e^{-ik(x_n-x_m)} P.v. \left( \frac{1}{\omega_k - \Omega} \right).
\]

(31)

Equation (29) is the most general case of the equation of motion for the matrix elements of the transition operator (5) for N qubits with identical resonant frequencies. When equation (29) is averaged over the initial photon vacuum, the second and the third lines in (29) can be dropped out.

For a long 1D waveguide we can replace the summation over $k$ by the integration:

\[
\sum_{k} f(k) = \frac{L}{2\pi} \int_{-\infty}^{+\infty} f(k) d|k| = \frac{L}{\pi v_g} \int_{0}^{\infty} f(|k|) + f(-|k|) d\omega
\]

(32)
where $L$ is the quantization length in the propagation direction and $v_g$ is the photon group velocity.

If we assume that the coupling strength is the same for all qubits, $g_k^{(n)} = g_k^{(m)} \equiv g_k$ and is symmetrical, i.e. $g_k = g_{-k}$, and it contributes mainly near the resonance $g_k \approx g_{k_0}$, where $k_0 = \Omega/v_g$, we then obtain for (30) and (31):

$$\Gamma_{n,m} = \Gamma \cos(k_0 |d_{n,m}|),$$ \hspace{1cm} (33a) \\
$$\alpha_{n,m} = \frac{\Gamma}{2} \sin(k_0 |d_{n,m}|),$$ \hspace{1cm} (33b)

where $d_{n,m} = (x_n - x_m)$ is the distance between $n$th and $m$th qubit, $\Gamma$ is the single-qubit emission rate into the waveguide mode:

$$\Gamma = \frac{2L}{v_g |g_{k_0}|^2},$$ \hspace{1cm} (34)

The expression (33b) is obtained with the help of the following relation:

$$p.v. \int_0^\infty \frac{\cos(\omega d_{n,m}/v_g)}{\omega - \Omega} d\omega = -\pi \sin(k_0 |d_{n,m}|).$$ \hspace{1cm} (35)

The expression (35) is exact if counter-rotating terms in the qubit-field interaction are taken into account (Suppl. in [35]). Nevertheless, within a rotating wave approximation the Eq. 35 provides a good accuracy for $d > \lambda/4$ [21].

The quantities $\Gamma_{n,m}$ and $\alpha_{n,m}$ denote the dissipative and coherent interaction rates, respectively. The coherent interaction results from the exchange of virtual photons between qubits at all continuum frequencies except for a single frequency $\omega = \Omega$. It gives rise to the shift of the qubit frequencies. In contrast to the case of a free space, these inter-qubit interactions have an infinite range. In addition, the interaction between any two qubits can be easily switched off by a proper choice of the value $k_0|d_{n,m}|$. In the simple case for which a distance between any two neighbor qubits is $d$, the coherent inter qubit interaction vanishes if $k_0d = n\pi$ where $n$ is any positive integer.

### 4 Radiation spectrum and calculation of the correlation functions

In circuit quantum electrodynamics it is possible to experimentally measure both the full photon spectrum $\langle a_k^+(t)a_k(t) \rangle$ and one-time mean values of single-photon operators $\langle a_k(t) \rangle$ and $\langle a_k^+(t) \rangle$ [36,37]. Moreover, we can construct more complex photon correlation functions with a higher order of photon operators and experimentally measure them as well [38]. In this paper, the main attention is paid to the quantity $\langle a_k^+(t)a_k(t) \rangle$ which defines the photon radiation spectrum.

From the expressions for photon operators (24) we obtain:

$$\langle a_k^+(t)a_k(t) \rangle = \int_0^t \int_0^t e^{-i\omega_k(\tau - \tau')} \langle S_k^+(\tau)S_k^-(\tau') \rangle d\tau d\tau'.$$ \hspace{1cm} (36)

From (36) we obtain the total photon emission rate, that is, the rate of the energy loss:

$$W(t) = \frac{d}{dt} \sum_k \langle a_k^+(t)a_k(t) \rangle = \langle S_k^+(t)S_k^-(t) \rangle$$ \hspace{1cm} (37)

The frequency dependent photon radiation spectrum is defined as the limit of (36) when $t$ tends to infinity:

$$S(\omega) = \int_0^\infty \int_0^\infty e^{-i\omega_k(\tau - \tau')} \langle S_k^+(\tau)S_k^-(\tau') \rangle d\tau d\tau'.$$ \hspace{1cm} (38)

The averaging in the above equations is understood as the tracing over both the states $S$ of a spin system and the states $\nu$ of a photon field:

$$\langle S_k^+(\tau)S_k^-(\tau') \rangle = \text{Tr}_{S,\nu} \langle S_k^+(\tau)S_k^-(\tau')\rho(0) \rangle$$ \hspace{1cm} (39)

where $\rho(0)$ is the initial density matrix of the whole system.

From definition (20) we obtain:

$$\langle S_k^+(\tau)S_k^-(\tau') \rangle = \sum_{n,m} g_k^{(m)} g_k^{(n)} e^{ik(x_n - x_m)} \langle \sigma_+^{(n)}(\tau)\sigma_-^{(m)}(\tau') \rangle.$$ \hspace{1cm} (40)

The expression for two-time correlation function $\langle \sigma_+^{(n)}(\tau)\sigma_-^{(m)}(\tau') \rangle$ can be written in two ways depending on the relation between $\tau$ and $\tau'$. If $\tau > \tau'$, then:

$$\langle \sigma_+^{(n)}(\tau)\sigma_-^{(m)}(\tau') \rangle = \text{Tr}_{S,\nu} \left[ \rho(\tau')\sigma_+^{(n)}(\tau - \tau')\sigma_-^{(m)}(0) \right].$$ \hspace{1cm} (41a)

If $\tau < \tau'$, then:

$$\langle \sigma_+^{(n)}(\tau)\sigma_-^{(m)}(\tau') \rangle = \text{Tr}_{S,\nu} \left[ \sigma_+^{(m)}(0)\sigma_-^{(n)}(\tau' - \tau)\rho(\tau) \right].$$ \hspace{1cm} (41b)

These prescriptions come from the requirement for the time argument of a spin operator to be always positive.

The expressions (41) are exact. If we assume the system is always in a photon vacuum state (15) we obtain for (41a), (41b):
more complex ones as well. In (43) we reduce the average value of two operators to the average of one operator multiplied by the second operator at zero time. The same principle can be used to reduce, say, a four-time correlation function to a three-time correlation function, which can also be reduced the same way to a two-time correlation function, and so on. Thus, we can find higher-order correlation functions using only transition operators found from (29).

5 Transition operators for two-qubit system

For two-qubit system there are four basis states:

\[
|1\rangle = |gg\rangle ; \quad |2\rangle = |ee\rangle ; \quad |3\rangle = |ge\rangle ; \quad |4\rangle = |eg\rangle
\]

(45)

However, we will use a so-called Dicke basis consisting of states |1\rangle, |2\rangle and symmetrical and asymmetrical superpositions of states |3\rangle and |4\rangle:

\[
|G\rangle = |gg\rangle , \quad |E\rangle = |ee\rangle , \quad |S\rangle = \frac{1}{\sqrt{2}}(|ge\rangle + |eg\rangle) , \quad |A\rangle = \frac{1}{\sqrt{2}}(|ge\rangle - |eg\rangle).
\]

(46)

The advantage of basis states (46) over (45) is that the equations of motion for diagonal matrix elements of transition operator are independent of the off-diagonal ones.

Using the definition of lowering and raising operators for the regular basis (45), it is easy to show how they act on the Dicke basis (46):

\[
\sigma^{(1,2)}_+ |G\rangle = \frac{1}{\sqrt{2}} (|S\rangle \mp |A\rangle) , \quad \sigma^{(1,2)}_- |G\rangle = 0 ,
\]

(47a)

\[
\sigma^{(1,2)}_+ |E\rangle = 0 , \quad \sigma^{(1,2)}_- |E\rangle = \frac{1}{\sqrt{2}} (|S\rangle \pm |A\rangle) ,
\]

(47b)

The same can be easily done for Pauli spin operators:

\[
\sigma^{(1,2)}_Z |G\rangle = -|G\rangle , \quad \sigma^{(1,2)}_Z |E\rangle = |E\rangle , \quad \sigma^{(1,2)}_Z |A\rangle = \mp |S\rangle , \quad \sigma^{(1,2)}_Z |S\rangle = \mp |A\rangle.
\]

(48)

Next, we apply the equation (29) for \(N = 2\) and assume for the decay rates, \(\Gamma_{11} = \Gamma_{22} = \Gamma, \quad \Gamma_{12} = \Gamma_{21} = \Gamma \cos(k_{0}d),\) and for the frequency shifts \(\alpha_{11} = \alpha_{22} = 0, \quad \alpha_{12} = \alpha_{21} = \Gamma \sin(k_{0}d)/2,\) where \(d\) is the distance between two qubits. By averaging the equation (29) over the photon vacuum state |0\rangle, the terms including photon operators in the second and the third lines
in (29) will be dropped out, and we can obtain equations for the matrix elements of the transition operator. For the basis (46) we have sixteen equations in total, but since the off-diagonal transition operators $P_{i,j}$ is a Hermitian conjugate of $P_{j,i}$, it is sufficient to find the solution only for ten matrix elements of the transition operator. For the diagonal matrix elements of transition operator (which we will refer to as populations by analogy with diagonal elements of density matrix), we find:

$$
\frac{d}{dt} \langle E \rangle = -2\Gamma \langle E \rangle, \quad (49a)
$$

$$
\frac{d}{dt} \langle S \rangle = -\Gamma \langle S \rangle, \quad (49b)
$$

$$
\frac{d}{dt} \langle A \rangle = -\Gamma \langle A \rangle, \quad (49c)
$$

$$
\frac{d}{dt} \langle P \rangle = -\Gamma \langle P \rangle, \quad (49d)
$$

For the off-diagonal matrix elements of the transition operator (which we will refer to as coherences) we obtain:

$$
\frac{d}{dt} \langle P_{E} \rangle = -\left(2i\Omega + \Gamma \right) \langle P_{E} \rangle, \quad (50a)
$$

$$
\frac{d}{dt} \langle P_{S} \rangle = -\Gamma \langle P_{S} \rangle, \quad (50b)
$$

$$
\frac{d}{dt} \langle P_{A} \rangle = -\Gamma \langle P_{A} \rangle, \quad (50c)
$$

$$
\frac{d}{dt} \langle P_{P} \rangle = -\Gamma \langle P_{P} \rangle, \quad (50d)
$$

$$
\frac{d}{dt} \langle P_{G} \rangle = -\Gamma \langle P_{G} \rangle, \quad (50e)
$$

$$
\frac{d}{dt} \langle P_{S} \rangle = -\Gamma \langle P_{S} \rangle, \quad (50f)
$$

For coherences we obtain:

$$
\langle P_{E} \rangle = e^{-2\Gamma t} \langle E \rangle, \quad (51a)
$$

$$
\langle P_{S} \rangle = e^{-\Gamma t} \langle S \rangle, \quad (51b)
$$

$$
\langle P_{A} \rangle = e^{-\Gamma t} \langle A \rangle, \quad (51c)
$$

$$
\langle P_{P} \rangle = e^{-\Gamma t} \langle P \rangle, \quad (51d)
$$

Thus in the basis (46), the equations for populations are decoupled from those for the coherences. Moreover, the first four equations for the coherences are fully independent and related only to their corresponding matrix elements. These equations can be solved without any problems since the initial conditions, which are based on the definition of transition operator (5), are always unique: $P_{ij}(0) = |i \rangle \langle j |$.

By solving two groups of equations we find all matrix elements for the transition operator for a two-qubit system in an open waveguide. For the populations we obtain the following solutions:

$$
\langle P_{E} \rangle = e^{-2\Gamma t} |E \rangle \langle E |, \quad (52a)
$$

$$
\langle P_{S} \rangle = e^{-\Gamma t} |S \rangle \langle S |, \quad (52b)
$$

$$
\langle P_{A} \rangle = e^{-\Gamma t} |A \rangle \langle A |, \quad (52c)
$$

$$
\langle P_{P} \rangle = e^{-\Gamma t} |P \rangle \langle P |, \quad (52d)
$$

Here for simplification, we introduce the shifted resonant frequencies and modified decay rates:

$$\Omega_+ = \Omega + \frac{\Gamma}{2} \sin k_0 d; \quad \Omega_- = \Omega - \frac{\Gamma}{2} \sin k_0 d; \quad (53a)$$

$$\Gamma_+ = \Gamma (1 + \cos k_0 d); \quad \Gamma_- = \Gamma (1 - \cos k_0 d); \quad (53b)$$

which depend on the effective distance between the qubits.
6 Transition probabilities for two qubits

As was noted in Sect. 2, the probability of a system to transit from one state to another can be found with the aid of transition operators (see Eq. 7). Here we calculate the probabilities which contribute to the total rate of superradiant emission which will be given below in Sect. 7.

For both qubits initially in an excited state \(|\Psi_0\rangle = |ee\rangle = |E\rangle\) we can find the probability that at time \(t\) the system remains in the initial state:

\[
W_{E\rightarrow E} = \langle E | P_{EE} | E \rangle = e^{-2\Gamma t}. \tag{54}
\]

The probabilities for both qubits to decay to symmetric and asymmetric states are as follows:

\[
W_{E\rightarrow S} = \langle E | P_{SS} | E \rangle = \frac{1 + \cos k\Omega d}{1 - \cos k\Omega d} \left( e^{-2\Gamma t} - e^{-\Gamma(1 + \cos k\Omega d)t} \right). \tag{55}
\]

\[
W_{E\rightarrow A} = \langle E | P_{AA} | E \rangle = \frac{1 - \cos k\Omega d}{1 + \cos k\Omega d} \left( e^{-2\Gamma t} - e^{-\Gamma(1 - \cos k\Omega d)t} \right). \tag{56}
\]

When qubits are initially in a symmetric, \(|\Psi_0\rangle = (|ge\rangle + |eg\rangle)/\sqrt{2} = |S\rangle\), or asymmetric, \(|\Psi_0\rangle = (|ge\rangle - |eg\rangle)/\sqrt{2} = |A\rangle\) states, the probabilities of the system to remain in the initial states are given by:

\[
W_{S\rightarrow S} = \langle S | P_{SS} | S \rangle = e^{-\Gamma(1 + \cos k\Omega d)t}. \tag{57}
\]

\[
W_{A\rightarrow A} = \langle A | P_{AA} | A \rangle = e^{-\Gamma(1 - \cos k\Omega d)t}. \tag{58}
\]

As is clear from (51b) and (51c) the transitions between symmetric and asymmetric states are forbidden:

\[
W_{A\rightarrow S} = \langle A | P_{SS} | A \rangle = 0; \quad W_{S\rightarrow A} = \langle S | P_{AA} | S \rangle = 0. \tag{59}
\]

Therefore, the \(|A\rangle\) and \(|S\rangle\) states are completely decoupled from each other no matter what is the value of \(k\Omega d\).

The symmetric and asymmetric states have different decay rates which depend on the value of \(k\Omega d\). As is seen from (57), (58) for a given value of \(k\Omega d\) the decay rate for the state \(|S\rangle\) is always greater than that for the state \(|A\rangle\). In addition, for \(k\Omega d = 2n\pi\), where \(n\) is a positive integer or 0, the population of the \(|A\rangle\) remains constant (\(W_{A\rightarrow A} = 1\)). In this case, the state \(|A\rangle\) is called the dark state since it does not interact with the electromagnetic field, while the state \(|S\rangle\) is called a bright state. If \(k\Omega d = (2n+1)\pi\) the situation is reversed: the state \(|S\rangle\) becomes a dark state, while the state \(|A\rangle\) becomes a bright state.

Finally, for the calculation of \(W(t)\) we will need the off-diagonal matrix elements:

\[
\langle A | (P_{PS}(t))_0 | S \rangle = e^{-\Gamma(1 + i\sin(k\Omega d))t} \tag{60}
\]

The transitions (55, 56) depend on the effective distance between the qubits \(k\Omega d\). For example, for \(k\Omega d = \pi/2\) there are equal probabilities of transitions to symmetric and asymmetric states, \(W_{E\rightarrow S} = W_{E\rightarrow A} = e^{-2\Gamma t} - e^{-\Gamma t}\).

We should separately consider the case when \(k\Omega d = n\pi\). For this case, both the numerator and denominator in (55) and (56) tend to zero. A correct solution can be obtained if we put \(k\Omega d = n\pi\) directly in the equations (49b) and (49c), or by expanding \(\cos(k\Omega d)\) near \(k\Omega d \approx n\pi + \epsilon\) where \(\epsilon\) is a small value. Both approaches give the same result. For \(k\Omega d = 2n\pi\) we obtain the following transition probabilities:

\[
W_{E\rightarrow S} = 2\Gamma t e^{-2\Gamma t}, \quad W_{E\rightarrow A} = 0 \tag{61}
\]

As it is clearly seen from (61), for an even number of \(n = 0, 2, 4, \ldots\), the transition from state \(|E\rangle\) to asymmetric entangled state \(|A\rangle\) is forbidden. For an odd number of \(n = 1, 3, \ldots\) the situation is reversed: the transition to symmetric state is now forbidden, and for transition to asymmetric state we get the relation \(W_{E\rightarrow A} = 2\Gamma t e^{-2\Gamma t}\).

7 Superradiant spectra of two qubits in a waveguide

Now we switch to the calculation of radiation spectrum for a two-qubit system. As was shown in (36), the spectrum can be found using a set of atomic correlation functions. For \(N = 2\) we obtain:

\[
\langle a^+_k(t)a_k(t) \rangle = \langle a^+_k(t)a_k(t) \rangle_+ + \langle a^+_k(t)a_k(t) \rangle_- \tag{62a}
\]

\[
\langle a^+_k(t)a_k(t) \rangle_+ = |g_k|^2 \int_0^{\tau} d\tau' e^{-i\omega(\tau' - \tau)} \Theta_k(\tau, \tau'), \tag{62b}
\]

\[
\langle a^+_k(t)a_k(t) \rangle_- = |g_k|^2 \int_0^{\tau} d\tau' e^{-i\omega(\tau' - \tau)} \Theta_k(\tau, \tau'). \tag{62c}
\]

where:

\[
\Theta_k(\tau, \tau') = \langle \sigma_+^{(1)}(\tau)\sigma_-^{(1)}(\tau') \rangle + \langle \sigma_+^{(2)}(\tau)\sigma_-^{(2)}(\tau') \rangle + e^{-ikd} \langle \sigma_+^{(1)}(\tau)\sigma_-^{(2)}(\tau') \rangle + e^{ikd} \langle \sigma_+^{(2)}(\tau)\sigma_-^{(1)}(\tau') \rangle. \tag{63}
\]
To correctly calculate the two-time spin correlation functions we subdivide the whole spectrum (62a) into two parts, for positive time difference (62b) (when $\tau > \tau'$) and negative time difference (62c) (when $\tau < \tau'$).

We want to remind again that in (62) the positive $k = +\omega/v_g$ corresponds to a forward, right moving wave, while the negative $k = -\omega/v_g$ corresponds to a backward, left moving wave.

To find a complete spectra (62a) one should calculate the two-time correlation functions using (43). From (47) we can express the lowering and raising spin operators in terms of basis set (46):

\[
s^{(1,2)}_+ = \frac{1}{\sqrt{2}} \left( |S⟩⟨G| + |A⟩⟨G| + |E⟩⟨S| \pm |E⟩⟨A| \right),
\]

\[
s^{(1,2)}_- = \frac{1}{\sqrt{2}} \left( |G⟩⟨S| \mp |G⟩⟨A| + |S⟩⟨E| \pm |A⟩⟨E| \right),
\]

and by switching to a Heisenberg picture, we find:

\[
\sigma^{(1,2)}_+(t) = \frac{1}{\sqrt{2}} \left( P_{SG}(t) \mp P_{AG}(t) + P_{ES}(t) \pm P_{EA}(t) \right),
\]

\[
\sigma^{(1,2)}_-(t) = \frac{1}{\sqrt{2}} \left( P_{GS}(t) \mp P_{GA}(t) + P_{SE}(t) \pm P_{AE}(t) \right),
\]

Thus, one can find the complete spectra (62a) by calculating four two-time correlation functions with the already obtained transition operators (51) and (52).

For the positive time difference $\tau > \tau'$ we obtain the following general expression:

\[
\langle a_k^+(t) a_k(t) \rangle = |g_k|^2 \int_0^t \int_{\tau}^t dt' e^{-i\omega(t-t')} \times \left[ (1 + \cos(kd)) \langle |E⟩⟨P_{SG}(\tau - t')⟩_0 |S⟩⟨E⟩⟨P_{EE}(\tau')⟩_0 |E⟩ \right.
\]
\[
+ \langle |E⟩⟨P_{ES}(\tau - t')⟩_0 |S⟩⟨E⟩⟨P_{EE}(\tau')⟩_0 |E⟩ \right]
\]
\[
+ \langle |S⟩⟨P_{SG}(\tau - t')⟩_0 |G⟩⟨E⟩⟨P_{SS}(\tau')⟩_0 |E⟩\langle E⟩ P_{\rho S}(0) |E⟩ \right]
\]
\[
+ \langle |1 - \cos(kd)⟩ \langle |E⟩⟨P_{EA}(\tau - t')⟩_0 |A⟩⟨E⟩⟨P_{EE}(\tau')⟩_0 |E⟩ \right]
\]
\[
- \langle |E⟩⟨P_{AG}(\tau - t')⟩_0 |A⟩⟨E⟩⟨P_{EE}(\tau')⟩_0 |E⟩ \right]
\]
\[
+ \langle |A⟩⟨P_{AG}(\tau - t')⟩_0 |G⟩⟨E⟩⟨P_{AA}(\tau')⟩_0 |E⟩ \right] \langle E⟩ P_{\rho S}(0) |E⟩ \right]
\]
\[
+ \langle |1 + \cos(kd)⟩ \langle |S⟩⟨P_{SG}(\tau - t')⟩_0 |G⟩⟨S⟩⟨P_{SS}(\tau')⟩_0 |S⟩ \right]
\]
\[
\left. \times \langle S⟩ P_{\rho S}(0) |S⟩ \right]
\]
\[
+ \langle |1 - \cos(kd)⟩ \langle |A⟩⟨P_{AG}(\tau - t')⟩_0 |G⟩⟨A⟩⟨P_{AA}(\tau')⟩_0 |A⟩ \right]
\]
\[
\left. \times \langle A⟩ P_{\rho S}(0) |A⟩ \right]
\]
\[
+ i \sin(kd) \langle |A⟩⟨P_{AG}(\tau - t')⟩_0 |G⟩⟨A⟩⟨P_{AS}(\tau')⟩_0 |A⟩ \right]
\]
\[
\left. \times \langle A⟩ P_{\rho S}(0) |A⟩ \right]
\]
\[
- \langle |S⟩⟨P_{SG}(\tau - t')⟩_0 |G⟩⟨S⟩⟨P_{SS}(\tau')⟩_0 |S⟩ \right]
\]
\[
\left. \times \langle S⟩ P_{\rho S}(0) |S⟩ \right] \right];
\]

and for the negative time difference $\tau < \tau'$ we obtain:

\[
\langle a_k^+(t) a_k(t) \rangle = |g_k|^2 \int_0^t \int_{\tau}^t dt' e^{-i\omega(t-t')} \times \left[ (1 + \cos(kd)) \langle |S⟩⟨P_{GS}(\tau - t')⟩_0 |E⟩⟨E⟩⟨P_{EE}(\tau')⟩_0 |E⟩ \right.
\]
\[
+ \langle |S⟩⟨P_{SE}(\tau - t')⟩_0 |E⟩⟨E⟩⟨P_{EE}(\tau')⟩_0 |E⟩ \right]
\]
\[
+ \langle |G⟩⟨P_{GS}(\tau - t')⟩_0 |S⟩⟨E⟩⟨P_{SS}(\tau')⟩_0 |E⟩\langle E⟩ P_{\rho S}(0) |E⟩ \right]
\]
\[
+ \langle |1 - \cos(kd)⟩ \langle |A⟩⟨P_{AE}(\tau - t')⟩_0 |E⟩⟨E⟩⟨P_{EE}(\tau')⟩_0 |E⟩ \right]
\]
\[
- \langle |A⟩⟨P_{GA}(\tau - t')⟩_0 |E⟩⟨E⟩⟨P_{EE}(\tau')⟩_0 |E⟩ \right]
\]
\[
+ \langle |G⟩⟨P_{GA}(\tau - t')⟩_0 |A⟩⟨E⟩⟨P_{AA}(\tau')⟩_0 |E⟩ \right] \langle E⟩ P_{\rho S}(0) |E⟩ \right]
\]
\[
+ \langle |1 + \cos(kd)⟩ \langle |G⟩⟨P_{GS}(\tau - t')⟩_0 |S⟩⟨S⟩⟨P_{SS}(\tau')⟩_0 |S⟩ \right]
\]
\[
\left. \times \langle S⟩ P_{\rho S}(0) |S⟩ \right]
\]

As is seen from (66), (67) only the initial density matrix of the form $\rho_S(0) = a|E⟩⟨E| + b|A⟩⟨A| + c|S⟩⟨S| + d|A⟩⟨S| + f|S⟩⟨A|$, where $a, b, c, d, f$ are arbitrary complex values, contributes to the radiation spectrum. We can also calculate a total emission rate (37) for $N = 2$:

\[
W(t) = \frac{\Gamma}{2} \left( \langle \sigma^{(1)}_+(t) \sigma^{(1)}_-(t) \rangle + \langle \sigma^{(2)}_+(t) \sigma^{(2)}_-(t) \rangle \right)
\]
\[
+ e^{-ikd} \langle \sigma^{(1)}_+(t) \sigma^{(2)}_-(t) \rangle + e^{ikd} \langle \sigma^{(2)}_+(t) \sigma^{(1)}_-(t) \rangle \right)
\]

where $\langle \sigma^{(i)}_+(t) \sigma^{(j)}_-(t) \rangle$ can be found with $\tau = \tau' = t$ in either of equations (43). Thus, the emission rate can easily be calculated since it is proportional only to single-time correlation functions. With the help of expressions for spin operators (65) we can express (68) in terms of transition operators:

\[
W(t) = \left( \frac{\Gamma}{2} \langle |E⟩⟨P_{EE}(\tau')⟩_0 |E⟩ \right) \left( \frac{\Gamma}{2} \langle |E⟩⟨P_{SS}(\tau')⟩_0 |E⟩ \right)
\]
\[
+ \frac{\Gamma}{2} \langle |E⟩⟨P_{AA}(\tau')⟩_0 |A⟩ \langle A⟩ P_{\rho S}(0) |A⟩ \right]
\]
\[
+ \frac{\Gamma}{2} \langle |A⟩⟨P_{AG}(\tau - t')⟩_0 |G⟩ \langle G⟩ ⟨P_{AA}(\tau')⟩_0 |A⟩ \right]
\]
\[
\left. \times \langle A⟩ P_{\rho S}(0) |A⟩ \right]
\]
\[
- \langle |S⟩⟨P_{SG}(\tau - t')⟩_0 |G⟩ ⟨S⟩ ⟨P_{SS}(\tau')⟩_0 |S⟩ \right]
\]
\[
\left. \times \langle S⟩ P_{\rho S}(0) |S⟩ \right] \right);
From (69) we see that there are two types of contributions to the total emission rate. First, the populations (see Eq. 18 where the population is expressed in terms of the transition operator) of all states, except for $|G\rangle$, contribute to the emission rate according to their fraction in the initial density matrix. The second contribution comes from off-diagonal matrix elements $\langle A|\rho_S(0)|S\rangle$, $\langle S|\rho_S(0)|A\rangle$. This contribution is due to the coherent interaction which results from the exchange of virtual photons between qubits at all continuum frequencies except for a single frequency $\omega - \Omega$.

The expressions (66), (67), and (69) are the central result that we use in the following to calculate the super- and subradiant spectra and emission rates in the two-qubit system for various initial configurations. They can be applied to any initial density matrix $\rho_S(0)$.

Below we consider several excited configurations of the two-qubit system. For every configuration we calculate the emission photon spectrum and the total rate of photon emission for different values of $k_0d$. To make evident the influence of the second qubit on the radiation spectrum, we compare these quantities with those for a single qubit in the system.

In all figures to this section the radiation spectral densities and the emission rates are given in dimensionless units $S(\omega)2\Omega/v_g$ and $W(t)/\Gamma$, respectively. All calculations are made for $\Gamma/\Omega = 0.05$.

7.1 Initial symmetric and asymmetric states

We start with initially prepared entangled states in the form of a symmetrical state $|\Psi(0)\rangle = (|e_1g_2\rangle + |g_1e_2\rangle)/\sqrt{2} = |S\rangle$ and an asymmetrical state $|\Psi(0)\rangle = (|g_1e_2\rangle - |e_1g_2\rangle)/\sqrt{2} = |A\rangle$. The experimental technique for the preparation of these entangled states is widely known in the circuit QED field and can be implemented by the sequence of Hadamard and CNOT gates [5].

As the qubit-photon coupling is efficient at $\omega \approx \Omega$, we perform subsequent calculations for $k \approx \pm k_0$ where $k_0 = \Omega/v_g$.

As is seen from (66), (67) the contribution of the symmetric and asymmetric initial states, $\rho_S(0) = |S\rangle\langle S|$, $\rho_A(0) = |A\rangle\langle A|$ are the even function of $k$. Therefore, the corresponding spectra are the same in both directions. From (66), (67), and (51b, 52f) we obtain:

$$\langle a_k^+(t)a_k(t)\rangle_S = \frac{v_g\Gamma+}{2L} \left( e^{(i\delta_+-\Gamma_+^t)t} - 1 \right) \left( e^{-(i\delta_+^t+\Gamma_+^t)t} - 1 \right),$$

where we introduce the detuning parameters:

$$\delta_+ = \omega - \Omega_+; \quad \delta_- = \omega - \Omega_-;$$

For the total emission rate we obtain from (69):

$$W(t) = \frac{\Gamma}{2} e^{-\Gamma_+t}$$

The calculation procedure for asymmetric state $\rho_A(0) = |A\rangle\langle A|$ is very similar. For this initial state, we obtain for the spectrum, spectral density, and emission rate, respectively:

$$\langle a_k^+(t)a_k(t)\rangle_A = \frac{v_g\Gamma-}{2L} \left( e^{(i\delta_-^t-R_-^t) - 1} \right) \left( e^{-(i\delta_+^t+\Gamma_+^t) - 1} \right),$$

$$S_A(\omega) = \frac{v_g\Gamma-}{2L} \left( \frac{\Gamma-}{\delta_-^2 + \Gamma_+^2/4} \right),$$

$$W_A(t) = \frac{\Gamma}{2} e^{-\Gamma^-t}$$

Below we compare these quantities with those for an initially excited single qubit with the same frequency $\Omega$ and the decay rate $\Gamma$. For this case, the spectral density and the total emission rate are as follows:

$$S_1(\omega) = \frac{v_g\Gamma}{2L(\omega - \Omega^2 + \Gamma^2/4)},$$

$$W_1(t) = \frac{\Gamma}{2} e^{-\Gamma_1t}$$

As is seen from (72) and (75) both spectra are Lorentzian lines whose central frequencies and widths depend on $k_0d$. The total emission rates (73) and (76) also depend on $k_0d$. For Dicke case, $k_0d = 0$, we obtain $W_A = 0$, $W_S = \Gamma e^{-2\Gamma t}$. The comparison of this result with (78) shows that the mere presence of a second unexcited qubit significantly alters the photon emission from excited qubit: its initial amplitude is twice as large as that for a single qubit and its decay proceeds at a twofold rate. This phenomenon is called a single photon superradiance [15, 39] that can occur when an identical two level atoms are in a symmetrical superposition of states with one excited atom and $N-1$ atoms in the ground state. In this case, the decay rate of a single photon is also equal to $\Gamma N$. The total radiated energy must be the same for both cases: $\int_0^\infty W_1(t)dt = \int_0^\infty W_S(t)dt$.

In relation to our problem it is important to note that contrary to free space in a one-dimensional geometry the Dicke case $k_0d = 0$ occurs also for any $k_0d = 2n\pi$, where $n$ is a positive integer.

The radiation spectra for initially states $|S\rangle$ and $|A\rangle$, together with the corresponding values of photon emis-
exist a range of subradiant states with \( \Gamma \ll \Gamma_0 \) we may expect that in the vicinity of this value there is a collective decay rate which is identical to the one for a single qubit but is shifted to the left by \( \Gamma/2 \). A distinctive manifestation of the subradiant decay of asymmetric state is seen for \( k_0d = \pi/4 \) (green dashed line in Fig. 2b). This decay is noticeably slower than the superradiant decay (purple dashed line in Fig. 2b). The width of its spectral line is much smaller than that of a single qubit (purple dashed line in Fig. 2a). If \( k_0d = 2\pi \) the state \(|A\rangle\) does not radiate. Here we also may expect the range of subradiant states with \( \Gamma_{sub} \ll \Gamma \) in the vicinity of \( k_0d = 2\pi \).

The only common feature of the decay of the states \(|S\rangle\) and \(|A\rangle\) is observed for \( k_0d = (n + 1/2)\pi \) which corresponds to \( \lambda_0 = 2d/(n + 1/2) \), where \( \lambda_0 = 2\pi \Omega/v_g \). For this case, the radiating spectra for both symmetric and asymmetric states have the same linewidth \( \Gamma_+ = \Gamma_- = \Gamma \), but their peaks are shifted in opposite directions because of the frequency shift \( \delta_s = \omega - (\Omega \pm \Gamma/2) \). The evolution of their decay rates coincides with that for a single qubit.

We see from the Figs. 1 and 2 that there are many subradiant states in the vicinity of \( k_0d = \pi \) and \( k_0d = 2\pi \) for initial \(|S\rangle\) and \(|A\rangle\) states, respectively. As the example, two subradiant states are shown in Fig. 3 for initial state \(|S\rangle\). The widths of the emission spectra in Fig. 3a are equal to the corresponding decay rates in Fig. 3b.

### 7.2 Initial state with one excited qubit

Now we consider the initial state when only the first qubit is excited \( |\psi(0)\rangle = |e_1g_2\rangle \). The corresponding initial density matrix is given by:

\[
\rho_S(0) = |eg\rangle \langle eg| = \frac{1}{2} |S - A\rangle \langle S - A| \tag{79}
\]
From (79) it is seen, that two terms proportional to $|S⟩⟨S|$ and $|A⟩⟨A|$ provide the same result we calculated in the previous section. Hence, we only need to find the contribution of off-diagonal elements of $ρ(0)$.

Using (66), (67), and explicit expressions for the transition operators (51a)-(51d), (52a)-(52f) we obtain the following expression:

\[
\left\langle a_\text{eg}^\dagger(t) a_\text{eg}(t) \right\rangle_{eg} = \frac{1}{2} \left\langle a_\text{eg}^\dagger(t) a_\text{eg}(t) \right\rangle_S + \frac{1}{2} \left\langle a_\text{eg}^\dagger(t) a_\text{eg}(t) \right\rangle_A \\
+ \frac{\nu_\text{g} \Gamma}{4L} \sin(kd) \left[ \frac{\left( e^{-(i\delta_-+\Gamma_-/2)t} - 1 \right)}{(i\delta_-+\Gamma_-/2)(i\delta_- - \Gamma_-/2)} - \frac{\left( e^{-(i\delta_+ + \Gamma_+/2)t} - 1 \right)}{(i\delta_+ + \Gamma_+/2)(i\delta_+ - \Gamma_+/2)} \right] 
\]

where the first two terms are given in (70) and (74). In the limit $t \to \infty$, we find frequency-dependent spectrum density:

\[
S_{eg}(\omega) = \frac{1}{2} S_S(\omega) + \frac{1}{2} S_A(\omega) \\
- \frac{\nu_\text{g} \Gamma}{4L} \sin(kd) \left[ \frac{\left( \delta_- \Gamma_+ - \delta_+ \Gamma_- \right)}{(\delta_+^2 + \Gamma_+^2/4)(\delta_-^2 + \Gamma_-^2/4)} \right] 
\]

where $S_S(\omega)$ and $S_A(\omega)$ are given in (72) and (75). Finally, for the total emission rate we obtain:

\[
W_{eg}(t) = \frac{\Gamma_+}{4} e^{-\Gamma_+ t} + \frac{\Gamma_-}{4} e^{-\Gamma_- t} \\
- \frac{\Gamma \sin(kd)}{2} e^{-\Gamma t} \sin(\Gamma \sin(k_0 d) t) 
\]

where $\Gamma_\pm = \frac{\Gamma}{\pi} \pm \frac{\Gamma_0}{\pi}$.

In the expressions (80), (81), and (82) first two terms correspond to the contribution from the states $|S⟩$ and $|A⟩$, while the second term results from the contribution of the off-diagonal transition elements of the transition operator (last two lines in equations (66), (67), (69)). The backward and forward radiation corresponds to $k \approx -k_0$ and $k \approx +k_0$, respectively.

The backward radiation spectra and the forward emission decay rates are shown in Fig. 4a, b. Here, the interference terms in equations (81), and (82) significantly alter the picture. If $kd_0$ is an integer multiple of $\pi$, then the interference term is zero. Therefore, for $kd_0 = 2\pi$ there is a superradiant state (red, solid line in Fig. 4a, b) with the decay rate being equal to $2\Gamma$, $W_{eg}^{\text{d}}(t) = 0.5\Gamma e^{-2\Gamma t}$. In this case, the spectral line is a Lorentzian as shown in Fig. 4a. This effect which is known as single-atom [15] or single-photon [39] superradiance predicts the decay of the excited atom at an enhanced rate in the presence of a second atom even though that second atom is in its ground state.

However, for other values of $kd_0$ for which the interference term is not zero, the decay rate may also be as fast as the superradiance decay for $kd_0 = 2\pi$. For example, for $kd_0 = \pi/2$ we obtain from (82) $W_{eg}^{\text{d}}(t) = 0.5\Gamma e^{-\Gamma t}(1 - \sin(\Gamma t))$ (dashed blue line in Fig. 4b). Therefore, with respect to a single qubit case, all decay plots in Fig. 4b within the initial time scale, $0 < \Gamma t < 1$, may be considered as superradiant ones. In addition, the spectral line for $kd_0 = \pi/2$, dashed blue line in Fig. 4a, has a double peak symmetrical structure. A distance between the peaks is a measure of the coherent exchange interaction between qubits mediated by the continuum spectra of virtual photons. For this case, the inter-peak distance, which is determined by numerics, is $1.45\Gamma$. By taking intermediate values of $kd_0$ we can break this symmetry of interaction between the qubits. The plot of such asymmetric structure, which can be a signature of Fano resonance, is shown in Fig. 4a for $kd_0 = \pi/4$. It is also worth mentioning the absence of forward radiation at the qubit frequency for $kd_0 = \pi/2$ (dashed blue line in Fig. 4a). In this case, the radiation propagates from left to right, from the first, excited qubit, to the second, unexcited qubit, and does not penetrate behind the second qubit. The second qubit acts as an ideal mirror at this frequency. The same result was obtained in [32] by a different method.

The radiation spectra and the emission rate for backward scattering can be obtained from (81) and (82) for $k \approx -k_0$. The corresponding plots are shown in Fig. 5a, b. The inter-peak distance (dashed blue line in Fig. 5a), which is a measure of the photon mediated coupling between qubits, is approximately $0.68\Gamma$. Here, a superradiant decay also takes place for $kd_0 = 2\pi$, where the interference terms in (81) and (82) are equal to zero. In this case, backward radiation is the same as that in the forward direction. However, for other values of $kd_0$ for which the interference terms are not zero, the backward radiation is significantly different from the forward radi-
The backward radiation spectra \(S_{eg}^b(\omega) = S_{eg}^b(\omega)2\Omega/v_g\) for initially excited first qubit. 

For comparison a single-qubit case is shown by black thin line. \(\Gamma/\Omega = 0.05\)

7.3 Initial state with two excited qubits

Here we consider the spectrum for the initial state with both qubits being excited, \(|\Psi(0)\rangle = |e_1e_2\rangle = |E\rangle\). The corresponding density matrix is \(\rho_S(0) = |E\rangle \langle E|\). From (66), (67) we obtain the following result for the radiation spectrum:

\[
\langle a_0^\dagger(t)a_k(t)\rangle_E = \frac{\Gamma_+^2 + \Gamma_-^2}{\Gamma_+^2 + \Gamma_-^2} \left[ \frac{e^{-ik_0d}}{1 + i\sin(k_0d)} \right] \langle a_0^\dagger(t)a_k(t)\rangle_S + \frac{\Gamma_+^2 + \Gamma_-^2}{2\Gamma} \left( e^{-2\Gamma \tau} - 1 \right) \langle a_0^\dagger(t)a_k(t)\rangle_A
\]

By taking time in (83) to infinity, we get the radiation spectral density:

\[
S_E(\omega) = \frac{\Gamma_+}{\Gamma_+ + \Gamma_-} S_S(\omega) + \frac{\Gamma_-}{\Gamma_+ + \Gamma_-} S_A(\omega) + \frac{v_g}{2L} \left( \frac{2\Gamma^2}{(1 + \sin^2(k_0d))} \right) \times \left[ \frac{e^{ik_0d}(1 + i\sin(k_0d))}{\Gamma_+} \right] \frac{\Gamma_+^2 + \Gamma_-^2}{\Gamma_+} \left( \frac{1}{1 + i\sin(k_0d)} \right) \langle a_0^\dagger(t)a_k(t)\rangle_S + \frac{\Gamma_+^2 + \Gamma_-^2}{2\Gamma} \left( e^{-2\Gamma \tau} - 1 \right) \langle a_0^\dagger(t)a_k(t)\rangle_A
\]

For the emission rate we obtain from (69):

\[
W_E(t) = \frac{1}{2} \frac{\Gamma_+^2}{\Gamma_-} e^{-\Gamma_+ t} + \frac{1}{2} \frac{\Gamma_-^2}{\Gamma_+} e^{-\Gamma_- t} - \frac{4\Gamma \cos^2(k_0d)}{\Gamma - \cos^2(k_0d)} e^{-2\Gamma \tau}
\]
Obtain the right solution if we put points the numerator is also zero. As previously, we can close inspection of these equations reveals that at these values of \( \Omega = 0 \).

For \( k_0 d = n \pi \) directly in the equations (49) and (50), or by expanding \( \cos(k_0 d) \) near \( k_0 d = n \pi \) in (83-85). For example, for \( k_0 d = 2 \pi \) we find:

\[
\left( a_k^\dagger(t) a_k(t) \right)_{E, k_0 d = 2\pi} = \frac{4 v_g \Gamma \left( e^{-(\delta+\Gamma)t} - 1 \right)}{2L} \frac{e^{i\delta t} - 1}{\delta^2 + \Gamma^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (\delta - \Gamma)^2 (i\delta - 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta+2\Gamma)t}}{i\delta (i\delta - 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta+\Gamma)t}}{(i\delta - \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (i\delta - 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta+\Gamma)t}}{(i\delta - \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta+\Gamma)t}}{(i\delta + \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (i\delta + 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta-2\Gamma)t}}{(i\delta + 2\Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta-\Gamma)t}}{(i\delta - \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (i\delta - 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta-2\Gamma)t}}{(i\delta + 2\Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta-\Gamma)t}}{(i\delta - \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (i\delta + 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta+\Gamma)t}}{(i\delta + \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (i\delta - 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta-\Gamma)t}}{(i\delta - \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (i\delta + 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta-2\Gamma)t}}{(i\delta + 2\Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta+\Gamma)t}}{(i\delta + \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (i\delta - 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta-\Gamma)t}}{(i\delta - \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (i\delta + 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta+\Gamma)t}}{(i\delta + \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (i\delta - 2\Gamma)} + \frac{2 v_g \Gamma}{2L} \frac{e^{-(i\delta-\Gamma)t}}{(i\delta - \Gamma)^2} + \frac{2 v_g \Gamma}{2L} \frac{e^{-2\Gamma t}}{i\delta (i\delta + 2\Gamma)} \right.
\]

The corresponding initial density matrix is given by:

\[
\rho_S(0) = \frac{1}{4} \left( \langle S \rangle S + |A\rangle \langle A| - |S\rangle \langle A| - |A\rangle \langle S| \right) + \frac{1}{2} \left( |S\rangle \langle G| - |A\rangle \langle G| + |G\rangle \langle S| - |G\rangle \langle A| \right) + \frac{1}{2} \langle G | G \rangle
\]

The first line in (90) is a half of the density matrix of the state \( |eg\rangle \) (79), therefore, we get the same spectrum and other related parameters similar to those for the first excited qubit (80-82), but reduced by the factor of two:

\[
\left( a_k^\dagger(t) a_k(t) \right)_{S_1 g_2} = \frac{1}{2} \left( a_k^\dagger(t) a_k(t) \right)_{eg} ;
\]

\[
S_{s_{1} g_{2}}(\omega) = \frac{1}{2} S_{s_{eg}}(\omega) ; \quad W_{s_{1} g_{2}}(t) = \frac{1}{2} W_{s_{eg}}(t);
\]

The second line in (90) which describes the transitions to the ground state does not contribute to \( \left( a_k^\dagger(t) a_k(t) \right) \) (expressions (66), (67)).

Therefore the state with the first qubit prepared in a superposition state and the second one in a ground state shows the spectral properties identical to those shown in Figs. 4 and 5, but on a smaller scale.

### 7.4 Initial states with qubits superposition

#### 7.4.1 First qubit is in a superposition state, a second qubit is in a ground state

\[
|\Psi(0)\rangle = |s_1\rangle \otimes |g_2\rangle = \frac{1}{\sqrt{2}} (|e_1\rangle + |g_1\rangle) \otimes |g_2\rangle
\]

\[
= \frac{1}{2} |S\rangle - \frac{1}{2} |A\rangle + \frac{1}{\sqrt{2}} (|G\rangle) \quad (89)
\]

The plots of emission decay rate for these values of \( k_0 d \) are shown in Fig. 6a. It is seen that the emission rates for two initially excited qubits are noticeably faster than the decay rate of a single qubit. Obviously, this is a signature of superradiant emission.

#### 7.4.2 First qubit is in a superposition state, a second qubit is in an excited state

\[
|\Psi(0)\rangle = |s_1\rangle \otimes |e_2\rangle = \frac{1}{\sqrt{2}} (|e_1\rangle + |g_1\rangle) \otimes |e_2\rangle
\]

\[
= \frac{1}{\sqrt{2}} |E\rangle + \frac{1}{2} |S\rangle + \frac{1}{2} |A\rangle \quad (92)
\]
and the corresponding initial density matrix is:

$$\rho_S(0) = \frac{1}{2} \left( |E\rangle \langle E| + |S\rangle \langle S| + |A\rangle \langle A| + |S\rangle \langle A| + |A\rangle \langle S| \right)$$

$$+ \frac{1}{2} \left( |E\rangle \langle S| + |S\rangle \langle E| + |A\rangle \langle A| \right)$$

(93)

As in the previous example, the first line in (93) corresponds to the initial states already considered above. Therefore, we can construct the spectrum and emission rate using (83) and |ge⟩ counterpart of (80) (see the last paragraph in Sect. 7.2):

$$\langle a^\dagger_k(t) a_k(t) \rangle_{s_1e_2} = \frac{1}{2} \langle a^\dagger_k a_k \rangle_E + \frac{1}{2} \langle a^\dagger_k a_k \rangle_{ge},$$

$$S_{s_1e_2}(\omega) = \frac{1}{2} S_E(\omega) + \frac{1}{2} S_{ge}(\omega),$$

$$W_{s_1e_2}(t) = \frac{1}{2} W_E(t) + \frac{1}{2} W_{ge}(t),$$

(94)

where $$\langle a^\dagger_k a_k \rangle_{ge}, S_{ge}(\omega), and W_{ge}(t)$$ are given by the equations (80), (81), and (82) with the sign of the interference term in these equations being changed.

Thus, the spectrum for the initial state (92) is a combination of the spectrum of two-excited qubits and that of a first excited qubit. Here, the probabilities for the photon to be emitted in left and right directions are different. The forward and backward radiation spectra and emission rates for this case are presented in Figs. 7a, 8a, and Figs. 7b, 8b, respectively.

7.4.3 Both qubits are initially prepared in a superposition state

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (|e_1\rangle + |g_1\rangle) \otimes \frac{1}{\sqrt{2}} (|e_2\rangle + |g_2\rangle)$$

$$= \frac{1}{2} |E\rangle + \frac{1}{2} |S\rangle + \frac{1}{2} |G\rangle$$

(95)

with the initial density matrix:

$$\rho_S(0) = \frac{1}{4} \left( |E\rangle \langle E| + |S\rangle \langle S| \right)$$

$$+ \frac{1}{4} \left( |E\rangle \langle G| + |G\rangle \langle E| + |G\rangle \langle G| \right)$$

$$+ \frac{1}{2\sqrt{2}} \left( |E\rangle \langle S| + |S\rangle \langle E| + |S\rangle \langle G| + |G\rangle \langle S| \right)$$

(96)

As it follows from (66), (67), only the first line in (96) contributes to the radiation spectrum, which can be presented as a combination of a two-excited qubit state (83-85) and a symmetrical state (70, 72, 73):

$$\langle a^\dagger_k(t) a_k(t) \rangle_{s_1s_2} = \frac{1}{4} \langle a^\dagger_k a_k \rangle_E + \frac{1}{2} \langle a^\dagger_k a_k \rangle_S,$$

$$S_{s_1s_2}(\omega) = \frac{1}{4} S_E(\omega) + \frac{1}{2} S_S(\omega),$$

$$W_{s_1s_2}(t) = \frac{1}{4} W_E(t) + \frac{1}{2} W_S(t).$$

(97)

In this case, the probabilities to find the photon in left or right detectors are the same. The characteristic plots
for this case are presented in Fig. 9a, b. We see from Fig. 9b that the plot for $k_0d = \pi/2$ is superimposed on a single qubit plot. It means that the areas of corresponding spectral lines (Fig. 9a) are equal to each other, although their line shapes are different. Another feature is the existence of a subradiant state for $k_0d = \pi$, dashed purple line in Fig. 9a, b.

As a concluding remarks to this subsection we note that as can be seen from Fig. 7a, the radiation spectra for the initial state with the first qubit being in a superposition state and the other one being in an excited state are very similar to the spectrum of two excited qubits shown in Fig. 6a. On the other hand, when both qubits are prepared in a superposition state, the spectrum changes significantly, Fig. 9, and only for $k_0d = n\pi$ similarity is retained. This difference comes from the different initial density matrix for both cases. For the case of Figs. 7 and 8 off-diagonal elements of density matrix (93), namely contributes to the spectrum. In this case, the initial wavefunction (92) is not symmetric under permutation of qubits. Therefore, every qubit feels the different phase shift which results in the difference between forward and backward scattering. For Fig. 9 only diagonal elements of initial density matrix (96) contribute to the spectrum. For this case, the initial wavefunction (95) is symmetric, therefore, the forward and backward spectra are the same. Note that the line width of both spectrum (94) and (97) for $k_0d = 2\pi$ is identical. Moreover, it matches the line width of the spectrum of two excited qubits (87) (red line in Fig. 6a).

8 Conclusion

In this paper, we investigate superradiant and subradiant properties of the photon emission spectra for a two-qubit system coupled to a one-dimensional open waveguide. We obtain the general expression which allows us to calculate the radiation spectra for arbitrary initial configuration of a two-qubit system. We obtain the explicit expressions for the photon radiation spectra and the emission decay rates for different initial two-qubit configurations with one and two excitations. We show that the line shape of the photon radiation spectra and the emission decay rate, that is, the rate of the energy loss depend significantly on the effective distance between qubits, $k_0d$.

Here we mainly focused on a two-qubit system because of the possibility to directly compare theoretical results with known experimental data. Nonetheless, the used formalism is not limited to a two-qubit system and can be extended to many atoms as well. But the analytical solution of such systems quickly becomes cumbersome since the dimension of the transition operator matrix grows like $2^N$, though one could potentially use numerical calculations for this task.

We believe that the results obtained in this paper may have practical applications in quantum information technologies including a control and optimization of the two-qubit entangling gates necessary for the realization of arbitrary unitary operations needed for quantum computation.

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Author contributions

YSG wrote the manuscript and contributed to its theoretical interpretation. OAC performed analytical calculations and computer simulations. All authors discussed the results and commented on the manuscript. The authors declare that they have no competing interests.

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