An analytical approach to the cooling of a flat plate

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Abstract. The present work deals with the cooling of a flat plate having an initial non-homogeneous temperature distribution, which undergoes a sudden temperature drop at its surfaces. Hence, starting from an arbitrary steady-state temperature distribution, the two surfaces of the slab are suddenly cooled down to the value \( \theta_0 \), resulting in a strong perturbation of the initial temperature distribution. A general analytical solution is provided by adopting the variable separation technique to the Fourier equation in transient conditions, thus obtaining a general solution in terms of a very fast converging series, which includes a definite integral of the initial steady-state distribution. Common engineering applications are investigated and the related results, obtained thanks to an extremely low computational effort, are presented and discussed.

1. Introduction
The transient cooling of a flat plate is of interest in many engineering applications, including the cooling of foundry products, nuclear fuel elements, electronic devices and building walls.

Although the transient conduction problems are usually solved by means of numerical methods [1-4], analytical solutions are still preferable due to their lower computational burden and higher accuracy, thus providing useful tools for the validation of numerical models [5,6]. Some of the commonly employed analytical methods are the Laplace transform method [7,8], the orthogonal and quasi-orthogonal expansion technique [9,10], the finite integral function technique [11], the Green’s function method [12] and the variable separation technique [13].

The aim of the present work is to supply a general analytical solution for the transient Fourier equation in a flat plate subjected to a sudden cooling of its surfaces, having an initial arbitrary temperature distribution. The study was carried out by adopting a one-dimensional approach, where the spatial coordinate \( \xi \) varies along the slab width of thickness \( L \), in transient conditions; the properties of the material were assumed to be constant. The analysed solid slab can be pretended as a solid with internal heat generation, used in electrical or nuclear engineering, or a solid hit by radiations (\( \gamma \) heating). The considered slab can be also assimilated as a product of foundry, an engine component or a building wall with a linear, initial temperature distribution, subjected to a sudden drop of its surface temperature. In fact, at the transient occurrence, the heating source is switched off, while the two surfaces are cooled down to a constant temperature \( \theta_0 \). Under the given assumptions, the governing Fourier equation is:

\[
\frac{\partial^2 \theta}{\partial \xi^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial \tau} \tag{1}
\]

with initial boundary condition:

\[
\theta(\xi, 0) = \gamma(\xi) \tag{2}
\]
and, to simplify the analytical solution, the Dirichlet boundary condition is assumed on the two surfaces:

\[ \theta(0, \tau) = \theta_0 \]  
\[ \theta(L, \tau) = \theta_0 \]  

By assuming dimensionless quantities, eq. (1) becomes:

\[ \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial F_o} \]  

where \( x = \xi/L, F_o = \alpha x/L^2 \), and \( T = (\theta - \theta_0)/(\theta_{\text{max}} - \theta_0) \). The dimensionless initial condition is:

\[ T(x, 0) = \varphi(x) = \frac{[\gamma(x) - \theta_0]/(\theta_{\text{max}} - \theta_0)} \]

and the dimensionless boundary conditions are:

\[ T(0, F_o) = 0 \]  
\[ T(1, F_o) = 0 \]  

2. General analytical solution

Firstly, \( T(x,F_o) \) is set, in eq. (5), equal to the product \( X(x) F(F_o) \) [14], and all the homogeneous conditions of the problem, i.e. \( X(0) = X(1) = 0 \), are imposed on each separable integral. By dividing each term of the equation by \( T \):

\[ \frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{F} \frac{dF}{dF_o} = \beta^2 \]

The first term of eq. (9) can depend only on \( x \), while the second one can depend only on \( F_o \); hence, \( \beta \) can be but a constant. \( \beta \) is an eigenvalue which differs from 0, in order to avoid the obvious solution \( T=0 \), having no physical meaning. The two unknown functions \( X(x) \) and \( F(F_o) \) are easily determined by the differential equation solutions:

\[ F(F_o) = \exp(\beta^2 F_o) \]  
\[ X(x) = A \exp(-\beta x) + B \exp(\beta x) \]

where the function \( F \) is defined unless an arbitrary multiplicative constant.

The boundary conditions are satisfied when \( A = -B \) and \( \exp(\beta) - \exp(-\beta) = 0 \). Hence, the eigenvalues are infinite and countable, \( \beta = n \pi i (n = 1,2,3,...) \).

The function \( X(x) \) is then:

\[ X(x) = B \left[ \exp(i n \pi x) - \exp(-i n \pi x) \right] = \frac{B}{2} \sin(n \pi x) \]

The resulting general solution to eq. (4) is:

\[ T(x, F_o) = \sum_{n=1}^{\infty} C_n \sin(n \pi x) \exp(-n^2 \pi^2 F_o) \]

where the constants \( C_n \) depend on the initial condition and \( B_n \).

The boundary conditions are thus satisfied; however, the initial condition \( (F_o = 0) \) is verified only if:

\[ \varphi(x) = \sum_{n=1}^{\infty} C_n \sin(n \pi x) \]

In the domain \( 0 \div 1 \), the sine system is complete. By multiplying each term of the equation by \( \sin(n \pi x) \) and integrating in the whole domain, results:

\[ C_n = 2 \int_0^1 \varphi(x) \sin(n \pi x) \, dx \]

Finally, by substituting eqs. (14-15) into eq. (13):

\[ T(x, F_o) = \sum_{n=1}^{\infty} \sin(n \pi x) \exp(-n^2 \pi^2 F_o) \int_0^1 \sin(n \pi x) \varphi(x) \, dx \]

Moreover, the dimensionless internal energy \( U = U'/[L \alpha \rho c(\theta_{\text{max}}-\theta_0)] \) of the slab can be evaluated as a function of \( F_o \); due to the non-null heat exchanged on the surfaces, the internal energy decreases exponentially as:
\[ U(Fo) = \int_{0}^{1} T(x) \, dx = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \int_{0}^{1} \sin(n\pi x) \, \varphi(x) \, dx \exp(-n^2\pi^2 Fo) \]  

The ratio \[ R(Fo) = \frac{U(Fo)}{U(Fo = 0)} \] represents the drop rate of the slab internal energy, being:

\[ U(Fo = 0) = \int_{0}^{1} \varphi(x) \, dx \]

The definite integral of \( \varphi(x) \sin(n\pi x) \), for many common cases, can be found in the literature [15,16]. The provided general analytical solution was therefore adopted in the study of different cases that may occur in civil or industrial engineering, depending on the initial temperature distribution \( \varphi(x) \):

- \( \varphi(x) = 1 \) simulates the cooling of a hot component in foundry, metallurgy or food processing.
- \( \varphi(x) = x \) simulates the cooling of a wall separating two zones at different temperature, as occurring in many situations, e.g. in cold plate heat exchangers, where thin, stacked plates sort the hot fluid from the cold one, or in power electronic cooling [17].
- \( \varphi(x) = -4 \, x(1-x) \) simulates a parabolic temperature profile, typical of electronic devices with internal heat generation [5,18] or fissile materials in nuclear engineering [1-4,19].
- \( \varphi(x) = \exp(-x) \) corresponds to a solid hit in \( x = 0 \) by an energy flux of either \( X \) or \( \gamma \) radiations, partially absorbed in the slab [20,21].

3. Initial constant temperature

According to eq. (16), when \( \varphi(x) = 1 \), the temperature transient temperature reads as:

\[ T(x,Fo) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin(n\pi x) \exp(-n^2\pi^2 Fo) \]  

The dimensionless thermal heat fluxes \( q = q' \lambda / (\theta_{max} - \theta_0) \partial T / \partial x \), exchanged on the surfaces, are:

\[ q(x = 0, Fo) = 2 \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \exp(-n^2\pi^2 Fo) \]  
\[ q(x = 1, Fo) = -q(x = 0, Fo) \]

Figure 1. Dimensionless temperature distribution along the dimensionless coordinate \( x \) (a) and exchanged heat flux on the slab surfaces over the dimensionless time \( Fo \) (b), for \( \varphi(x) = 1 \).

In Fig 1, both the temperature distribution and the exchanged heat fluxes are plotted versus the dimensionless space coordinate and time \( 0 \leq Fo \leq 0.2 \), respectively. For \( Fo = 0 \), the temperature distribution is always symmetrical with respect to the median axis of the plate, it assumes a constant value \( T = 1 \), then decreases with the increasing of \( Fo \). As physically expected, the temperature profiles
are always symmetrical with respect to the center of the slab (x = 0.5), in which T is maximum for each Fo, and the highest drop in temperature can be appreciated near the surfaces. On the other hand, the dimensionless heat fluxes are very high in module for low Fo, then decrease with an exponential law.

4. Initial linear temperature

Here, the initial temperature is \( \varphi(x) = x \). Remembering that [15,16]:

\[
\int_0^1 x \sin(n\pi x) \, dx = \frac{1}{(n\pi)^2} \left[ \sin(n\pi x) - n\pi x \cos(n\pi x) \right]_0^1 = \frac{(-1)^{n+1}}{n\pi} \tag{22}
\]

the transient temperature distribution is expressed by eq. (23).

\[
T(x, Fo) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x) \exp(-n^2\pi^2 Fo) \int_0^1 x \sin(n\pi x) \, dx = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x) \exp(-n^2\pi^2 Fo) \tag{23}
\]

The dimensionless heat fluxes are instead evaluated by means of eqs. (24-25).

\[
q(x = 0, Fo) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \exp(-n^2\pi^2 Fo) \tag{24}
\]

\[
q(x = 1, Fo) = 2 \sum_{n=1}^{\infty} (-1)^{2n+1} \exp(-n^2\pi^2 Fo) = -2 \sum_{n=1}^{\infty} \exp(-n^2\pi^2 Fo) \tag{25}
\]

In Fig.2, solutions for eqs. (23-25) are shown: the initial temperature distribution is linear for Fo = 0, then strongly decreases near the hot surface (x = 1), thus flattening towards the center of the slab. The heat flux on the hot surface is higher in module with respect to that on the cold surface; on the contrary, for Fo > 0.1, both fluxes present a similar trend.

5. Initial parabolic temperature

The initial temperature is defined by the function \( \varphi(x) = -4x(x-1) \). Remembering that [15,16]:

\[
\int_0^1 x^2 \sin(n\pi x) \, dx = \frac{1}{n^3\pi^3} \left[ 2n\pi \sin(n\pi x) - (n\pi x)^2 \cos(n\pi x) + 2 \cos(n\pi x) \right]_0^1 = \frac{(-1)^{n+1}}{n\pi} + 2 \left(\frac{(-1)^n}{n^3\pi^3} - 1 \right) \tag{26}
\]
the transient temperature reads as:

\[
T(x, Fo) = -8 \sum_{n=1}^{\infty} \sin(n\pi x) \exp(-n^2\pi^2 Fo) \left[ \int_0^1 x^2 \sin(n\pi x) \, dx - \int_0^1 x \sin(n\pi x) \, dx \right] = \\
= \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin(n\pi x) \exp(-n^2\pi^2 Fo)
\]

(27)

The dimensionless heat fluxes are:

\[
q(x = 0, Fo) = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} \exp(-n^2\pi^2 Fo)
\]

(28)

\[
q(x = 1, Fo) = -q(x = 0, Fo)
\]

(29)

During the transient, the temperature distribution (Fig.3a) is always symmetric and progressively flattened towards the steady state temperature \(T = 0\). The dimensionless heat fluxes (Fig.3b) present same module over the entire transient condition and decrease with an exponential law.

Figure 3. Dimensionless temperature distribution along the dimensionless coordinate \(x\) (a) and exchanged heat flux on the slab surfaces over the dimensionless time \(Fo\) (b), for parabolic initial temperature distribution \(\varphi(x) = -4x(x-1)\).

6. Initial exponential temperature

The initial temperature is defined by the function \(\varphi(x) = \exp(-x)\).

Remembering that [15,16]:

\[
\int_0^1 \sin(n\pi x) \exp(-x) \, dx = \frac{-\exp(-x)}{1 + n^2\pi^2} \left[ \sin(n\pi x) + n\pi \cos(n\pi x) \right]_0^1
\]

\[
= \frac{n\pi}{1 + n^2\pi^2} \left[ 1 - \frac{(-1)^n}{e} \right]
\]

(30)

the transient temperature reads as:

\[
T(x, Fo) = 2 \sum_{n=1}^{\infty} \sin(n\pi x) \exp(-n^2\pi^2 Fo) \int_0^1 \sin(n\pi x) \exp(-x) \, dx = \\
= 2 \pi \sum_{n=1}^{\infty} \frac{n}{n^2\pi^2 + 1} \left[ 1 - \frac{(-1)^n}{e} \right] \sin(n\pi x) \exp(-n^2\pi^2 Fo)
\]

(31)

The dimensionless thermal heat fluxes on the surfaces of the slab are:
q(x = 0, Fo) = 2 \pi^2 \sum_{n=1}^{\infty} \frac{n^2}{n^2 \pi^2} \left[ 1 - \frac{(-1)^n}{e} \right] \exp(-n^2 \pi^2 Fo) \quad (32)

q(x = 1, Fo) = 2 \pi^2 \sum_{n=1}^{\infty} \frac{n^2}{n^2 \pi^2 + 1} \left[ (-1)^n - \frac{1}{e} \right] \exp(-n^2 \pi^2 Fo) \quad (33)

The results are shown in Fig.4. It has to be pointed out that the center of the slab (x = 0.5) present no perturbation for Fo < 0.02. The dimensionless heat flux on the hot surface (x = 0) is higher than that on the cold one (x = 1).

**Figure 4.** Dimensionless temperature distribution along the dimensionless coordinate x (a) and exchanged heat flux on the slab surfaces over the dimensionless time Fo (b), for exponential initial temperature distribution \( \varphi(x) = \exp(-x) \).

Finally, for every considered initial temperature distribution \( \varphi(x) \), the drop rate R of internal energy were evaluated by means of eqs. (17-18) and shown in Fig. 5. For \( \varphi(x) = 1 \), \( \varphi(x) = x \), and \( \varphi(x) = \exp(-x) \) the drop rates are almost coincident, while \( \varphi(x) = -4x(x-1) \) denotes a greater drop rate, although it assumes more and more similar values to those related to the other study cases, as Fo increases.

**Figure 5.** Drop rate R of the slab internal energy for each initial temperature distribution \( \varphi(x) \).
7. Conclusions
The cooling of a flat slab was studied by adopting an analytical approach. The geometry was first simplified as a one-dimensional domain having a known initial temperature distribution. The transient Fourier equation, which describes the conduction phenomenon occurring in the slab during the transient conditions, was then solved by means of the variable separation technique, thus providing a general analytical solution. Such solution was finally adapted to different cases occurring in civil and industrial engineering, by considering different initial temperature distributions (cooling in foundry or food processing, plate heat exchangers, flat plate nuclear fuel elements used in research reactors, γ heating). All the results can represent practical cases in engineering applications. When the initial temperature distribution is symmetrical, T is similarly symmetrical with respect to the median axis of the plate. On the contrary, for initial strongly asymmetric distributions, T tends asymptotically to the symmetric stationary distribution T = 0 as Fo increases, although it seems to reach a quasi-symmetric distribution for Fo > 0.1. These rigorous solutions can be used as reliable benchmarks for the widely used numerical methods, implemented on general purpose CFD modules. To investigate the role of the coolant flow and the Biot number, future works could be devoted to the solution of a similar problem by considering more general convective conditions instead of the here adopted Dirichlet boundary conditions on the surfaces.

Nomenclature

| Symbol | Quantity | SI unit |
|--------|----------|---------|
| A      | Slab surface | m²      |
| C      | Specific heat | J/kgK   |
| E      | Euler’s number | -       |
| F      | Auxiliary time function | -       |
| Fo     | Fourier number | -       |
| I      | Imaginary unit | -       |
| L      | Slab width | M       |
| Q      | Dimensionless thermal heat flux | -       |
| q'     | Thermal heat flux | W/m²    |
| R      | Drop rate of internal energy | -       |
| U      | Dimensionless internal energy | -       |
| U’     | Internal energy | J       |
| X      | Auxiliary spatial function | -       |
| X’     | Dimensionless coordinate | -       |
| T      | Dimensionless temperature | -       |
| α      | Thermal diffusivity | m²/s    |
| β      | Eigenvalue | -       |
| γ      | Initial temperature distribution | K       |
| φ      | Dimensionless initial temperature distribution | -       |
| λ      | Thermal conductivity | W/mK    |
| θ      | Slab temperature | K       |
| \(θ_{\text{max}}\) | Maximum slab temperature at \(τ = 0\) | K       |
| \(θ_0\) | Constant wall temperature at \(τ > 0\) | K       |
| ρ      | Density | kg/m³    |
| τ      | Time | S       |
| ξ      | Spatial coordinate | M       |

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