NUMERICAL ANALYSIS OF AN ODE AND A LEVEL SET METHODS FOR EVOLVING SPIRALS BY CRYSTALLINE EIKONAL-CURVATURE FLOW

TETSUYA ISHIWATA
Department of Mathematical Sciences, Shibaura Institute of Technology
Fukasaku 309, Minuma-ku, Saitama 337-8570, Japan

TAKESHI OHTSUKA∗
Division of Pure and Applied Science, Faculty of Science and Technology
Gunma University
Aramaki-machi 4-2, Maebashi, 371-8510 Gunma, Japan

Abstract. In this paper, the evolution of a polygonal spiral curve by the crystalline curvature flow with a pinned center is considered from two viewpoints; a discrete model consisting of an ODE system describing facet lengths and another using level set method. We investigate the difference of these models numerically by calculating the area of an interposed region by their spiral curves. The area difference is calculated by the normalized $L^1$ norm of the difference of step-like functions which are branches of $\arg(x)$ whose discontinuities are on the spirals. We find that the differences in the numerical results are small, even though the model equations around the center and the farthest facet are slightly different.

1. Introduction. The crystalline curvature of a curve $\Gamma$, which is denoted by $H_\gamma$, is defined by the first variation of an anisotropic surface energy functional

$$E_\gamma(\Gamma) = \int_{\Gamma} \gamma(n) d\sigma$$

for a singular density function $\gamma: \mathbb{R}^2 \to [0, \infty)$ with respect to the volume of the region enclosed by $\Gamma$, where $n$ is a continuous unit normal vector field of $\Gamma$ and $d\sigma$ is the line element. Here, denoting the 1-sphere by $S^1$, singular means that the Wulff shape

$$\mathcal{W}_\gamma = \{ p \in \mathbb{R}^2; \ p \cdot q \leq \gamma(q) \text{ for } q \in S^1 \},$$

which satisfies $H_\gamma = 1$ on $\partial \mathcal{W}_\gamma$, is a convex polygon. See [13] for details of the crystalline curvature. This singular energy expresses the surface energy of the polygonal structure of interfaces such as crystal surface. A typical example of $\gamma$ is the $\ell^1$ norm.

For describing general settings, we here assume that

(A1) $\gamma$ is convex,
(A2) $\gamma$ is positively homogeneous of degree 1, i.e., $\gamma(\lambda p) = \lambda \gamma(p)$ for $p \in \mathbb{R}^2$ and $\lambda > 0$,
(A3) $\gamma > 0$ on $S^1$.

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* Corresponding author: Takeshi Ohtsuka.
(A4) $\gamma$ is piecewise linear.

Note that (A2) is for the level set formulation of curves mentioned later. Moreover, (A4) is a sufficient condition for obtaining the situation such that $\mathcal{W}$ is a convex polygon, since $\mathcal{W} = \{ p \in \mathbb{R}^2; \gamma^\circ(p) \leq 1 \}$ and $(\gamma^\circ)^\circ = \gamma$ whenever $\gamma$ is convex. Here, $\gamma^\circ(p) := \sup\{ p \cdot q; \gamma(q) \leq 1 \}$ is a support function of $\gamma$. See [22] for details of the properties of $\gamma$ and $\gamma^\circ$.

In this paper we consider the evolution of a convex polygonal spiral by
\[
\beta V_\gamma = U - \rho_c H_\gamma \quad \text{on } \Gamma_t,
\]
where $V_\gamma$ is an anisotropic normal velocity under the Finsler metric defined by $\gamma^\circ(x,y) = \gamma^\circ(x-y)$, and $U > 0$ and $\rho_c > 0$ are assumed to be constants. (Note that we do not assume the symmetry of this metric.) For the evolution of a pinned spiral, the authors of this paper introduced a discrete model by an ODE system of the facet lengths in [15], due to the ideas in [3, 23, 14], see also [13] for details.

On the other hand, Tsai, Giga and the second author [18, 17] introduced a level set formulation for evolving spirals with fixed centers. According to their formulation, an evolving spiral curve with a fixed center at the origin is given as
\[
\Gamma_L(t) = \{ x; u(t, x) - \theta(x) \equiv 0 \mod 2\pi \mathbb{Z} \}, \quad n = -\frac{\nabla(u - \theta)}{|\nabla(u - \theta)|},
\]
with an auxiliary function $u(t, x)$ and a pre-defined multivalued function $\theta(x) = \arg(\gamma^\circ(x,y))$, which is an argument of $x \in \mathbb{R}^2 \setminus \{0\}$, i.e., $\theta$ satisfies $x/|x| = (\cos \theta, \sin \theta)$. When $\gamma$ is smooth, then $V_\gamma$ and $H_\gamma$ are interpreted as
\[
V_\gamma = \frac{u_t}{\gamma(-\nabla(u - \theta))}, \quad H_\gamma = -\text{div}\{\xi(-\nabla(u - \theta))\},
\]
where $\xi = D\gamma$. Hence, we formally obtain the level set equation for (1) of the form
\[
\hat{\beta}(\nabla(u - \theta))u_t - \hat{\gamma}(\nabla(u - \theta)) \left[ \text{div}\{\xi(\nabla(u - \theta))\} + U \right] = 0, \quad (2)
\]
where $\hat{\beta}(p) = \beta(-p)$, $\hat{\gamma}(p) = \gamma(-p)$, and $\xi(p) = \xi(-p)$. The crucial problem to the computation of (2) under (A4) is that $\xi = D\gamma$ is not defined. For this problem, we have an option approximating the singular diffusion term due to the stability result in the theory of viscosity solution due to [9]. In fact, the derivative of $\gamma$ under (A4) can be defined almost everywhere as a step function. Then, we consider an approximation of $\xi$ using the idea as in [8]. Note that the viscosity solution theory for the usual level set equation of (2) (i.e. $\theta \equiv 0$) under (A1)–(A4) is recently extended to higher spatial dimension case by [11]. On the other hand, a variational approach due to [2] is one of the powerful option. This idea is extended to the crystalline motion of interfaces by [1]. Chambolle [5] combined the variational approach and a level set method with signed distance function of interfaces. This work is extended to higher spatial dimensions by [7, 6]. Oberman, Osher, Takei and Tsai [16] introduced a numerical method for the Chambolle’s algorithm. Recently, Tsai and the second author extended this numerical algorithm to the motion of spirals; see [19, 20].

The aim of this paper is to show the numerical difference between the spirals calculated by the discrete model due to [15] and the level set method due to [17]. To measure the difference between these spirals, we calculate the area of an interposed region by their spirals. Let $\Gamma_D(t)$ and $\Gamma_L(t)$ be the spirals in $W = \{ x \in \Omega; |x| > \rho \}$.
for a bounded domain $\Omega$ and $\rho > 0$. Our proposed difference is calculated by

$$D(t) = \frac{1}{|W|} \int_W \frac{|\theta_D(t, x) - \theta_L(t, x)|}{2\pi} dx,$$

where $\theta_D$ and $\theta_L$ are branches of $\theta$ whose discontinuities are only on the spiral curves $\Gamma_D(t) = \sum_{j=0}^k L_j(t)$ obtained by the discrete algorithm and $\Gamma_L(t)$ by the level set method, respectively. See Figure 1 for the details of the interposed region. A practical way to construct $\theta_L$ from solution $u$ of the level set equation is provided in [17]. Thus, we shall give a way to construct $\theta_D$ in §3.2. Note that the discrete model in [15] is constructed from $W_\gamma$. Then, we shall give a way to construct $\gamma$ from $\gamma^0$ in §3.1 to obtain the level set equation corresponding to the discrete model.

2. Numerical methods. In this section, we recall the discrete model due to [15] and the level set method due to [17]. To compare the evolving spiral curves from these models, we have to give a Wulff shape $W_\gamma$ for the discrete model and a corresponding surface energy density $\gamma$ for the level set method. In this section, we consider the situation where $W_\gamma$ and the corresponding $\gamma$ are already given. A practical way to obtain $\gamma$ from $W_\gamma$ will be discussed in §3.1. We briefly review mathematical results on these models.

2.1. Discrete model. We recall the ODE model in [15].

We first prepare some notations for $W_\gamma$. Let $W_\gamma$ be a $N_\gamma$ sided convex polygon. The $j$-th facet of $W_\gamma$ has an outer unit normal vector $N_j$ with angle $\varphi_j$ for $j = 0, 1, 2, \ldots, N_\gamma - 1$. Set the unit tangential vector $T_j$ of the $j$-th facet as well as the definition of the Frenet frame, i.e.,

$$N_j = (\cos \varphi_j, \sin \varphi_j), \quad T_j = (\sin \varphi_j, -\cos \varphi_j).$$

We assume the following for expressing the convexity of $W_\gamma$.

\begin{itemize}
  \item [(W1)] $\varphi_0 < \varphi_1 < \varphi_2 < \cdots < \varphi_{N_\gamma - 1} < \varphi_0 + 2\pi$.
  \item [(W2)] $\varphi_j < \varphi_{j+1} < \varphi_j + \pi$ for $j = 0, 1, 2, \ldots, N_\gamma - 1$.
\end{itemize}

Note that $\varphi_{N_\gamma} = \varphi_0$. We denote the length of the $j$-th facet of $W_\gamma$ by $\ell_j > 0$.

We next prepare notation for describing evolving polygonal spirals. We denote a polygonal spiral curve evolving by (1) by $\Gamma_D(t) = \bigcup_{j=0}^k L_j(t)$. According to [15],
we here consider the evolution of a positive convex polygonal spiral. Assume that the $j$-th facet $L_j(t)$ is given as

$$L_j(t) = \begin{cases} 
\{\lambda y_j(t) + (1 - \lambda)y_{j-1}(t) ; \ \lambda \in [0, 1]\} & \text{for } j = k, k - 1, \ldots, 1 \\
\{y_0(t) + \lambda T_0 ; \ \lambda > 0\} & \text{if } j = 0
\end{cases}$$

with vertices $y_j(t)$ ($j = 0, 1, 2, \ldots, k - 1$) and the center $y_k(t) = O$. Assume that

$$T_j = \frac{y_{j-1}(t) - y_j(t)}{|y_{j-1}(t) - y_j(t)|}.$$ 

We have extended the number $j$ of $T_j$ from $j = 0, 1, 2, \ldots, N_\gamma - 1$ to $\mathbb{Z}$; let $T_{j+nN_\gamma} = T_j$ for $j = 0, 1, 2, \ldots, N_\gamma - 1$ and $n \in \mathbb{Z}$. Then, the evolution of $\Gamma_D(t)$ by (1) with fixed center $y_k(t) = O$ is expressed by an ODE system for $d_j(t) = |y_j(t) - y_{j-1}(t)|$ of the form

$$\dot{d}_k = c_k^-(U - \frac{\rho_c \ell_k-1}{d_{k-1}}),$$

$$\dot{d}_{k-1} = -b_{k-1} \left(U - \frac{\rho_c \ell_{k-1}}{d_{k-1}}\right) + c_{k-1}^- \left(U - \frac{\rho_c \ell_{k-2}}{d_{k-2}}\right),$$

$$\dot{d}_j = -b_j \left(U - \frac{\rho_c \ell_j}{d_j}\right) + c_j^+ \left(U - \frac{\rho_c \ell_{j+1}}{d_{j+1}}\right) + c_j^- \left(U - \frac{\rho_c \ell_{j-1}}{d_{j-1}}\right)$$

for $j = 2, 3, \ldots, k - 2$,

$$\dot{d}_1 = -b_1 \left(U - \frac{\rho_c \ell_1}{d_1}\right) + c_1^+ \left(U - \frac{\rho_c \ell_2}{d_2}\right) + c_1^- U,$$

where $b_j \in \mathbb{R}$ and $c_j^\pm > 0$ are numerical constants defined by

$$b_j = \frac{1}{\beta_j} \left(\frac{1}{\tan (\varphi_{j+1} - \varphi_j)} + \frac{1}{\tan (\varphi_j - \varphi_{j-1})}\right), \quad c_j^\pm = \pm \frac{1}{\beta_j \pm 1} \sin (\varphi_{j \pm 1} - \varphi_j)$$

and $\beta_j = \beta(N_j)$. Tracking the evolution of $\Gamma_D(t)$ is established by drawing $\Gamma_D(t)$ with setting

$$y_k(t) = O, \quad y_{j-1}(t) = y_j(t) + d_j(t)T_j \text{ for } j = k, k - 1, k - 2, \ldots, 1.$$ 

See Figure 2 for the details of $\Gamma_D(t)$ described with the above notations.

**Figure 2.** Description of $\Gamma_D = \bigcup_{j=0}^k L_j(t)$. Note that, for the simplicity, the variable $t$ of $L_j$ and $y_j$ is omitted in the above figure.
In this paper, we give an initial curve as \( k = 1 \) with \( d_1(0) = 0 \), i.e., \( y_1(0) = y_0(0) = O \) and
\[
\Gamma_D(0) = L_1(0) \cup L_0(0) = \{ \lambda T_0; \ \lambda \geq 0 \}. \tag{5}
\]
For an evolving spiral, a new facet should be generated as the result of the evolution of present facets. Let \( T_1 = 0 \) and inductively set the generation time of facet \( L_{k+1}(t) \) as
\[
T_{k+1} = \sup \{ T > T_k; \ d_k(t) < \rho c \ell_k/U \text{ for } t \in [T_k, T] \}.
\]
When \( t = T_{k+1} \), we add a new facet \( L_{k+1}(T_{k+1}) \) with \( y_{k+1}(T_{k+1}) = O \) and \( d_{k+1}(T_{k+1}) = 0 \). Then, change the spiral center to \( y_{k+1}(t) \) from \( y_k(t) \).

In summary, the algorithm describing our discrete model of polygonal spirals evolving by (1) is as follows:

(I) Input the generation time \( T_k \) and curve \( \Gamma_D(T_k) = \bigcup_{j=0}^{k} L_j(T_k) \) (with \( d_k(T_k) = 0 \)).

(II) Solve (3)–(4) on \( [T_k, T_{k+1}] \) to obtain the evolution of \( \Gamma_D(t) \).

(III) When \( t = T_{k+1} \), add a new facet \( L_{k+1}(T_{k+1}) \) with \( y_{k+1}(T_{k+1}) = O \) (then \( d_{k+1}(T_{k+1}) = 0 \)) as the fixed center of \( \Gamma_D(t) \). Then, return to (I).

The existence and uniqueness of the solution to (3)–(4), the existence of the sequence \( \{ T_k \}_{k=1}^{\infty} \) of the generation times, \( \lim_{k \to \infty} T_k = \infty \), and the intersection-free result of \( \Gamma_D(t) \) has been obtained in [15].

2.2. Level set method. We recall the level set method [17] for a evolving spiral corresponding to the discrete model explained in the previous section. We first derive the level set equation of (1) by formal discussion and assuming enough regularity for \( \gamma \). For piecewise linear \( \gamma \), since \( \xi = D\gamma \) can be defined almost everywhere as a step function, we provide its approximation.

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a smooth boundary. Consider the evolution of a single spiral by (1), and set the center of the spiral at the origin. We give such a spiral curve and its direction of evolution, which is denoted by \( \xi \). Let \( \Omega \) be a bounded domain with a smooth boundary. Consider the evolution of \( \Gamma_L(t) \) as
\[
\Gamma_L(t) = \{ x \in \mathbb{W}_\gamma; \ u(t, x) - \theta(x) \equiv 0 \ \text{mod} \ 2\pi \mathbb{Z} \}, \quad \mathbb{n} = -\frac{\nabla (u - \theta)}{|\nabla (u - \theta)|},
\]
where \( \mathbb{W}_\gamma = \{ x \in \Omega; \ |x| > \rho \} \) for a constant \( \rho > 0 \), and \( \theta = \arg(x) \). According to [10], we obtain the anisotropic curvature \( H_\gamma \) of \( \Gamma_L(t) \) as
\[
H_\gamma = -\text{div}_{\xi} (-\nabla (u - \theta))
\]
with \( \xi = D\gamma \) when \( \gamma \in C^2(\mathbb{R}^2 \ \setminus \ {0}) \) satisfying (A1)–(A3). It is well-known that
\[
\mathcal{W}_\gamma = \{ p \in \mathbb{R}^2; \ \gamma^0(p) \leq 1 \},
\]
and \( H_\gamma = 1 \) on \( \mathcal{W}_\gamma \). Moreover, from the derivation of (3)–(4) in [15], one can find that \( \{ p \in \mathbb{R}^2; \ \gamma^0(p) \leq t + 1 \} \) is a self-similar solution to the equation of \( d_j \) in (4) with \( \rho_c = 0 \) and \( U = 1 \), i.e., \( \{ p \in \mathbb{R}^2; \ \gamma^0(p) \leq t + 1 \} \) describes the evolution of interfacial curve by \( V_\gamma = 1 \) with initial curve \( \mathcal{W}_\gamma \). In light of the above fact, we measure the normal velocity by the Finsler metric
\[
\begin{align*}
d_\gamma(x, y) &= \gamma^0(x - y) \\
\text{with} \quad \gamma^0(N_j) &= 1 \quad \text{for} \ j = 0, 1, 2, \ldots, N_\gamma - 1. \tag{6}
\end{align*}
\]
Then, the normal velocity in this case should be given by
\[
V_\gamma = \frac{u_t}{\gamma(-\nabla (u - \theta))}
\]
since $\gamma(D\gamma^\circ(p)) = 1$ for $p \in \mathbb{R}^2 \setminus \{0\}$ under some additional regularity and convexity assumptions on $\gamma$ and $\gamma^\circ$; see [4] for the details.

As a boundary condition for evolution under (1), we impose the right angle condition between $\Gamma_L(t)$ and $\partial W$. Then the level set equation for the motion of spirals by (1) is of the form
\[
\dot{\beta}(\nabla(u-\theta))u_t - \dot{\gamma}(\nabla(u-\theta)) \left\{ \rho_c \text{div}\xi(\nabla(u-\theta)) + U \right\} = 0 \text{ in } (0,T) \times W,
\]
\[
\bar{v} \cdot \nabla(u-\theta) = 0 \text{ on } (0,T) \times \partial W,
\]
where $\bar{v} \in S^1$ is the outer unit normal vector field of $\partial W$, $\dot{\beta}(p) = \beta(-p)$, $\dot{\gamma}(p) = \gamma(-p)$ and $\dot{\xi}(p) = \xi(-p)$. See [10] for the details of the level set method.

A mathematical analysis of equations (7)–(8) with $\gamma \in C^2(\mathbb{R}^2 \setminus \{0\})$ and $\beta \in C(\mathbb{R}^2 \setminus \{0\})$ has been established in [18]. For given initial data $u_0 \in C(\overline{W})$, there exists a unique global viscosity solution $u \in C([0,\infty) \times \overline{W})$ to (7)–(8) with $u(0,\cdot) = u_0$. Moreover, the uniqueness of evolution of $\Gamma_L(t)$ has been established in [12]; if there are continuous viscosity solutions $u$ and $v$ to (7)–(8) satisfying $\Gamma_L^u(0) = \Gamma_L^v(0)$ with the same orientations, then $\Gamma_L^u(t) = \Gamma_L^v(t)$ for $t > 0$, where $\Gamma_L^u(t) = \{x \in \overline{W}; u(t,x) - \theta(x) \equiv 0 \mod 2\pi\}$. Hence, we may give an arbitrary $u_0 \in C(\overline{W})$ to obtain the motion of $\Gamma_L(t)$. In this paper, we give $u_0$ for (5) as $u_0 \equiv \varphi_0 - \pi/2$ due to [17].

Recall that we consider the situation such that $W_\gamma$ is a convex polygon. The assumption (A4) is imposed in such a situation. Then, $\gamma$ is given as
\[
\gamma(p) = \max_{0 \leq j \leq N, -1} n_j \cdot p = \sum_{j=0}^{N-1} (n_j \cdot p) \chi_{Q_j}(p)
\]
with some $Q_j \subset \mathbb{R}^2$ for $j = 0, 1, 2, \ldots, N-1$, where
\[
\chi_{Q}(x) = \begin{cases} 1 & \text{if } x \in Q, \\ 0 & \text{otherwise} \end{cases}
\]
for $Q \subset \mathbb{R}^2$. The crucial problem for solving (7)–(8) is how to treat $\text{div}\xi(\nabla(u-\theta))$. For this problem, approximation of $\xi$ by the analogy of the stability result in [9] is a simple option. From (9), we formally obtain
\[
\xi(p) = \sum_{j=0}^{N-1} \chi_{Q_j}(p)n_j,
\]
so that we approximate $\chi_{Q_j}$ with the method in [8] to remove the singularities. More precisely, we use the function
\[
\sigma(z; q_1, q_2) := \begin{cases} \frac{z}{\sqrt{\varepsilon^2 + \varepsilon^2(|q_1| + |q_2|)^2}} & \text{if } z \neq 0, \\ 0 & \text{otherwise} \end{cases}
\]
with $\varepsilon \ll 1$ to approximate the sign function $z/|z|$. This approximation is also used in [15] or §4.1 and §4.2 when we approximate $\xi = D\gamma$ of $\gamma(p) = ||p||_1 = |p_1| + |p_2|$ and $\gamma(p) = ||p||_\infty = \max\{|p_1|, |p_2|\}$ for $p = (p_1, p_2)$; see [15] for the details. In general, consider the case when $Q$ is given as a level set of a continuous function $f$, i.e., $Q = \{x \in \mathbb{R}^2; f(x) > 0\}$ and $\mathbb{R}^2 \setminus \overline{Q} = \{x \in \mathbb{R}^2; f(x) < 0\}$. Then, we approximate $\chi_Q$ by
\[
\chi_Q(p) \approx \zeta(f(p); q_1, q_2), \quad \text{with } \zeta(z; q_1, q_2) := \frac{\sigma(z; q_1, q_2) + 1}{2}
\]
for a suitable parameter \((q_1, q_2)\). (We often choose \((q_1, q_2) = p\) as in [8], or \((q_1, q_2) = (1, 0)\) for simplicity.) Hence, we obtain the approximation

\[
\xi(p) \approx \sum_{j=0}^{N_\gamma-1} \zeta(f_j(p); q_1, q_2)n_j
\]

when \(Q_j = \{p \in \mathbb{R}^2; f_j(p) > 0\}\) for some \(f_j \in C(\mathbb{R}^2)\). This approximation is used in §4.3.

3. Measuring difference.

3.1. Crystalline energy density. We give a way to reconstruct a convex and piecewise linear \(\gamma: [0, \infty) \to [0, \infty)\) from \(\gamma^o\) and its some remarks.

Let us consider the situation such that the Wulff shape \(W\) and a support function \(\gamma^o: \mathbb{R}^2 \to [0, \infty)\) satisfying

\[
W_\gamma = \{p \in \mathbb{R}^2; \gamma^o(p) \leq 1\}
\]

are given. Note that \(\gamma^o = \sup\{p \cdot q; \gamma(q) \leq 1\}\) is convex and positively homogeneous of degree 1. According to these facts and since \(W_\gamma\) is a convex polygon, we assume that \(\gamma^o\) is given as

\[
\gamma^o(p) := \max_{0 \leq j \leq N_\gamma-1} m_j \cdot p, \quad m_j = \eta_j (\cos \psi_j, \sin \psi_j)
\]

with \(\eta_j > 0\) and \(\psi_j \in \mathbb{R}\). Assume that

\[
(\gamma 1) \psi_0 < \psi_1 < \psi_2 \cdots < \psi_{N_\gamma-1} < \psi_0 + 2\pi,
\]

\[
(\gamma 2) \psi_j < \psi_{j+1} < \psi_j + \pi \quad \text{for} \ j = 0, 1, 2, \ldots, N_\gamma - 1.
\]

\[
(\gamma 3) \text{The set} \{m_0, m_1, \ldots, m_{N_\gamma-1}\} \text{is minimal,} \quad \text{i.e.,} \quad \gamma^o(p) \neq \max\{m_j \cdot p; j \neq k\} \text{for any} \ k \in \{0, 1, \ldots, N_\gamma - 1\}. \text{This is equivalent to the following:}
\]

\[
P_j = \{p \in \mathbb{R}^2; m_j \cdot p > m_k \cdot p \text{ for} \ k = 0, 1, 2, \ldots, N_\gamma - 1\} = \Xi_{j,j-1} \cap \Xi_{j,j+1} \neq \emptyset
\]

for \(j = 0, 1, 2, \ldots, N_\gamma - 1\), where \(\Xi_{j,k} = \{p \in \mathbb{R}^2; m_j \cdot p > m_k \cdot p\}\).

(Nota that \(\psi_{j+nN_\gamma} = \psi_j \text{ for} \ n \in \mathbb{Z}\).) Note that we do not impose the normalizing assumption (6) in this section. Then, \(W_\gamma\) is given as the convex hull of the vertices \(x_j\) of \(W_\gamma\), i.e.,

\[
W_\gamma = \left\{ \sum_{j=0}^{N_\gamma-1} c_j x_j \mid 0 \leq c_j \leq 1 \text{ for} \ j = 0, 1, \ldots, N_\gamma - 1, \sum_{j=0}^{N_\gamma-1} c_j = 1 \right\},
\]

where the vertex \(x_j\) is which satisfies \(m_j \cdot x_j = m_{j+1} \cdot x_j = 1\), and \(m_k \cdot x_j \leq 1\) for \(k = 0, \ldots, N_\gamma - 1\) by \((\gamma 3)\). Then, one can find that

\[
\gamma(p) = (\gamma^o)^o(p) = \max_{x \in W_\gamma} x \cdot p = \max_{0 \leq j \leq N_\gamma-1} x_j \cdot p.
\]

Remark 1. (i) When we only give the parameters of \(\ell_j\) and \(\psi_j\) for \(W_\gamma\), then we have to set the location of the origin \(O \in W_\gamma\) to determine \(\gamma^o\). Note that \(\eta_j\) depends on the location of the origin in \(W_\gamma\).

(ii) When we do not assume \((\gamma 3)\), then there is a case that \(P_j = \emptyset\) and thus \(P_k \neq \Xi_{k,k-1} \cap \Xi_{k,k+1}\) for some \(j, k \in \{0, 1, 2, \ldots, N_\gamma - 1\}\) when \(N_\gamma \geq 4\), even if \(\psi_j\) satisfies \((\gamma 1)\) and \((\gamma 2)\). In fact, \(\gamma^o(p) = \max_{0 \leq j \leq 3} m_j \cdot p\) with

\[
m_0 = (3, 0), \ m_1 = (1, 1), \ m_2 = (0, 2), \ m_3 = (-1, -1)
\]
Hence, we can define the difference $D$ of the stepwise surface at $Γ$. $Γ$ will describe a practical way to construct $θ$.

Consequently, we set $L$.

(iii) Set

$$\Theta_k(x) = \arg(x) \in [\varphi_k + 2\pi n - \pi/2, \varphi_k + 2\pi (n+1) - \pi/2);$$
a branch of arg $x$ whose discontinuity is only on $L_k(t) = \{rT_k; \ r > 0\}$.

(See Figure 3(2).)

(iii) Set

$$R_{k-1}(t) := \{x \in \mathbb{R}^2; x \cdot N_k < s_k(t), x \cdot N_{k-1} \geq s_{k-1}(t)\}$$

(denoted by the gray regions in Figure 3(3)). To remove a discontinuity on a dashed line in $∂R_{k-1}(t)$, we set

$$\Theta_{k-1}(x) = \Theta_k(x) - 2\pi \chi_{R_{k-1}(t)}(x).$$

(iv) We inductively set

$$\Theta_{k-\ell}(x) = \Theta_{k-\ell+1}(x) - 2\pi \chi_{R_{k-\ell}(t)}(x)$$

$$= \Theta_k(x) - 2\pi \sum_{j=1}^{\ell} \chi_{R_{k-j}(t)}(x)$$

to remove illegal discontinuities of $Θ_{k-1}$ from $\ell = 1$ to $\ell = k$, where

$$R_j(t) := \{x \in \mathbb{R}^2; x \cdot N_{j+1} < s_{j+1}(t), x \cdot N_j \geq s_j(t)\}$$

for $j = 0, 1, \ldots, k-1$ (see Figure 3(4) for $R_{k-2}(t)$).

Consequently, we set

$$\theta_D(t,x) = \Theta_{k,0}(x) = \Theta_k(x) - 2\pi \sum_{j=0}^{k-1} \chi_{R_j(t)}(x), \ h_D(t,x) = \frac{1}{2\pi} \theta_D(t,x).$$

Hence, we can define the difference $D(t)$ between $Γ_D(t)$ and $Γ_L(t)$ as

$$D(t) = \frac{1}{|W|} \int_W |h_D(t,x) - h_L(t,x)| dx. \quad (10)$$
Figure 3. Construction of $\theta_D(t,x)$: we construct a branch of arg$(x)$ whose discontinuities are only on $\Gamma(t)(\text{the dashed line in (1)})$. For this purpose we first construct $\vartheta(x) = \text{arg}(x)$ whose discontinuities are only on $L_k(t)$ (the solid line in (2)). Then, we make go down the height of $\vartheta(x)$ on $R_{j}(t)$ (the gray region in (3) or (4)) with the jump-height $2\pi$ from $j = k-1$ to $j = 0$ inductively to remove illegal discontinuities. The solid line in figure (3) or (4) denotes the discontinuity of $\Theta_{k,k-1}$ or $\Theta_{k,k-2}$, respectively.

4. Numerical results. In this section, we present some numerical simulations measuring the difference between $\Gamma_D(t)$ and $\Gamma_L(t)$ evolving by

$$V_\gamma = 1 - \rho_c H_\gamma,$$

i.e., (1) with $\beta \equiv 1$ and $U = 1$ for several $\gamma$. The initial curve is chosen as

$$\Gamma_D(0) = \Gamma_L(0) = L_0(0) = \{x \in \Omega; \lambda \geq 0\},$$

using (5) for the discrete model, and $u(0,x) = \varphi_0 - \pi/2$ for the level set method. Throughout this section, we set

$$\Omega = [-1.5,1.5]^2, \quad D_s = \{x_{i,j} = (i\Delta x,j\Delta x) \in W; \ -75s \leq i,j \leq 75s\}$$

for some $s \in \mathbb{N}$, and $\Delta x = 0.02/s$. In the following subsections, we will choose time intervals of the numerical simulations so that $\Gamma_D(t) \setminus L_0(t)$ does not touch the outer boundary $\partial \Omega$. In other words, we avoid the situation that the boundary condition on $\partial \Omega$ imparts a difference in $\Gamma_D(t)$ and $\Gamma_L(t)$. Note that, however, the difference of the boundary condition at the center and the evolution law of the first facet $L_0(t)$ still remains.

We solve the ODE system (3)–(4) by the 4-th order Runge-Kutta method with the time step $\Delta t = 10^{-6}$. From the numerical results, we construct $h_D(t,x)$ on each numerical mesh $D_s$ to compare the results with those from the level set method. On the other hand, the level set equation (7)–(8) is calculated using the explicit finite differences, as in [17] with the time step $\Delta t = 0.1 \times \Delta x^2$. When we calculate (7) at $x_{i,j}$ which is close to the boundary $\partial W$, then we need to construct the data of $u - \theta$ at $x_{i+k,j+t} \notin W$ by the finite difference equation of (8). See also [17] for the way to construct $h_L(t,x)$ with the step height $h_0 = 1$. To draw a graph of $D(t)$, we pick up the data $D(t_k) = D(kT/20)$ for $0 \leq k \leq 20$ on the time interval $[0,T]$. 

We now recall the differences between the discrete model in §2.1 and the level set method in §2.2.

(i) The domain of the level set method has a center $B_{\rho} = \{ x \in \mathbb{R}^2; |x| \leq \rho \}$ with a finite radius $\rho > 0$. However, the discrete model has the center at the origin as a point (null set).

(ii) The boundary conditions are different:

- [Discrete model] $L_0(t)$ evolves by $V = 1$ since $d_0(t) = \infty$. On the other hand, the behavior of the facets associated with the center is imposed by fixing and the generation rule for new facets.

- [Level set method] The right angle conditions, in particular, $\Gamma_L(t) \perp \partial B_{\rho}$ and $\Gamma_D(t) \perp \partial \Omega$ are imposed by (8).

Because of the above differences, we have no conjectures for the convergence of $\Gamma_L(t)$ and $\Gamma_D(t)$ now. Moreover, the numerical results of the isotropic case in [17, 21] shows that taking the approximation parameters to zero, as well as letting $\rho \to 0$ is required for numerical accuracy. We therefore check the numerical results with fixed radius $0 < \rho \ll 1$ and reducing radius $\rho = O(\Delta x)$. In the following sections, we choose $\rho = M \Delta x$ for some $M \in \mathbb{N}$. However, because of implementation problems of the boundary condition (8) numerically, we check if $x_{i,j} \in B_{\rho}$ by $|x_{i,j}| < \rho$.

4.1. **Square spiral.** The first examination is the square spiral case, i.e.,

$$ \mathcal{W}_\gamma = \{ p = (p_1, p_2); \max\{|p_1|, |p_2|\} \leq 1 \}.$$ 

Thus, we define the parameters of $\mathcal{W}_\gamma$ for the discrete model as

$$ \varphi_j = \frac{\pi(j + 1)}{2}, \quad \ell_j = 2 \quad \text{for } j = 0, 1, 2, 3. $$

For the level set equation, since $\gamma^\delta(p) = \max\{|p_1|, |p_2|\}$ for $p = (p_1, p_2)$, we observe that

$$ \gamma(p) = |p_1| + |p_2|, \quad \text{and} \quad \xi(p) = (\text{sgn}(p_1), \text{sgn}(p_2)). $$

We calculate the ODE system (3)–(4) and the level set equation (7)–(8) for $V = 1 - 0.02H_\gamma$ (i.e., $\beta \equiv 1$, $U = 1$ and $\rho_c = 0.02$) on the time interval $[0, 1]$. See [15, §4] for details on approximating $\xi$ of the above $\gamma$. Figure 4 shows profiles of the square spiral at $t = 1$ under the above setting. Note that, in this and the following sections, the profile of spirals by the level set method is calculated using $\rho = 0.02$ and $\Delta x = 0.0050$ ($s = 4$).

![Figure 4](image-url)  

**Figure 4.** Profiles of the square spiral at $t = 1$. The level set method is calculated using $\rho = 0.02$ and $\Delta x = 0.0050$. 

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The left figure of Figure 5 represents the graph of $D(t)$ for $s = 2, 3, 4, 5, 6$ with a fixed center radius $\rho = 0.02$. One can find that the differences are less than 4% of the area $|W|$ for all cases, although the value of $D(t)$ becomes worse when we choose smaller $\Delta x$. The best one is the case with $\Delta x = 0.010$ ($s = 2$).

On the other hand, we obtain better results when $\rho = O(\Delta x)$. The right figure of Figure 5 represents the graph of $D(t)$ for $s = 2, 3, 4, 5, 6$ with the center size $\rho = 2\Delta x$, i.e., the setting $\rho \rightarrow 0$ as $\Delta x \rightarrow 0$. Note that the cases of $\Delta x = 0.010$ ($s = 2$) in both figures of Figure 5 are the same. One can find that the differences are less than 2.5% of the area $|W|$ for all cases, and that $D(t)$ in the cases where $s \geq 3$ are smaller than that of $s = 2$, although the smallest $D(t)$ is the case $\Delta x = 0.0067$ ($s = 3$).

4.2. Diagonal spiral. Our second examination is the diagonal spiral case, i.e., the $\pi/4$ rotation of the first case. Here

$$\varphi_j = \frac{\pi j}{2} + \frac{\pi}{4}, \quad \ell_j = 2 \quad \text{for } j = 0, 1, 2, 3$$

for the discrete model. In this case, one can find that

$$\mathcal{W}_\gamma = \{p = (p_1, p_2); |p_1| + |p_2| \leq \sqrt{2}\},$$

and thus

$$\gamma^\circ(p) = \frac{|p_1| + |p_2|}{\sqrt{2}}.$$  

For the level set equation, we set

$$\gamma(p) = \sqrt{2} \max\{|p_1|, |p_2|\}.$$  

According to [15], this function can be represented as

$$\gamma(p) = \frac{|p_1 + p_2|}{\sqrt{2}} + \frac{|p_1 - p_2|}{\sqrt{2}}$$

and thus

$$\xi(p) = \frac{1}{\sqrt{2}}(\text{sgn}(p_1 + p_2) + \text{sgn}(p_1 - p_2), \text{sgn}(p_1 + p_2) - \text{sgn}(p_1 - p_2)).$$
Figure 6. Profiles of the diagonal spiral at $t = 1$. The level set method is calculated using $\rho = 0.02$ and $\Delta x = 0.0050$.

See [15] for the approximation of the above $\xi$. We calculate the ODE system (3)–(4) and the level set equation (7)–(8) for (11) on the time interval $[0, 1]$. Figure 6 shows profiles of the diagonal spiral at $t = 1$ under the above setting.

The left figure of Figure 7 shows graphs of $D(t)$ for $s = 2, 3, 4, 5, 6$ with a fixed center radius $\rho = 0.02$. One can find that $D(t)$ is reduced by choosing smaller $\Delta x$, and the smallest $D(t)$ is obtained in the case $\Delta x = 0.0033$ ($s = 6$). Our numerical simulations show that the differences are less than 4% of $|W|$ when $s \geq 4$. Note that $\rho \approx 4\Delta x$ when $s = 4$.

In light of the above results, we choose $\rho \approx 4\Delta x$ for accurate simulations with a reduced center radius $\rho = O(\Delta x)$. The right figure of Figure 7 represents graphs of $D(t)$ for $s = 2, 3, 4, 5, 6$ with $\rho = 4\Delta x$. One can find that the differences are less than 5% for all cases, although the worst one is obtained when $\Delta x = 0.0033$ ($s = 6$). Note that the results when $\Delta x = 0.0050$ ($s = 4$) are the same in both figures of Figure 7.

4.3. Triangle spiral. Finally, we examine a triangle spiral as an asymmetric case of $\gamma$ and $\gamma^\circ$. We first describe $\gamma^\circ$. Because of the normalizing assumption (6), we
Then, \( \mathcal{W}_\gamma = \{ p \in \mathbb{R}^2; \gamma^\circ(p) \leq 1 \} \) implies that

\[
\varphi_j = \frac{2\pi j}{3}, \quad \ell_j = 2\sqrt{3}
\]

since \( \mathcal{W}_\gamma \) is an equilateral triangle whose vertices are at \((1, \pm \sqrt{3})\) and \((-2, 0)\). On the other hand, from the computation in \(\S 3.1\) we obtain

\[
\gamma(p) = \max_{0 \leq j \leq 2} n_j \cdot p \quad \text{with} \quad n_j = 2 \left( \cos \left( \frac{(2j + 1)\pi}{3} \right), \sin \left( \frac{(2j + 1)\pi}{3} \right) \right),
\]

and

\[
\tilde{\gamma}(p) = \gamma(-p) = \max_{0 \leq j \leq 2} \tilde{n}_j \cdot p \quad \text{with} \quad \tilde{n}_j = 2 \left( \cos \left( \frac{2\pi j}{3} \right), \sin \left( \frac{2\pi j}{3} \right) \right).
\]

Note that \( Q_j \) in (9) is given as

\[
Q_j = \{ p \in \mathbb{R}^2; g_j(p) \geq g_k(p) \text{ for } k \neq j \} = \{ p \in \mathbb{R}^2; \min_{k \neq j} (g_j(p) - g_k(p)) \geq 0 \}
\]

with \( g_0(p) = 2p_1, \ g_1(p) = -p_1 + \sqrt{3}p_2, \ g_2(p) = -p_1 + \sqrt{3}p_2. \)

Then, we obtain

\[
\tilde{\gamma}(p) \approx \sum_{j=0}^{2} (\tilde{n}_j \cdot p) \zeta(f_j(p); 1, 0), \quad \tilde{\xi}(p) \approx \sum_{j=0}^{2} \zeta(f_j(p); 1, 0) \tilde{n}_j,
\]

where \( f_j(p) = \min_{k \neq j} (g_j(p) - g_k(p)) \). We calculate (3)–(4) and (7)–(8) for the evolution equation

\[
V_\gamma = 1 - 0.01 H_\gamma \quad (\beta = 1, \ U = 1, \ \rho_c = 0.01)
\]

on the time interval \([0, 0.8]\) with the above anisotropic setting. This time interval is chosen so that \( \Gamma_D(t) \setminus \Lambda_0(t) \) does not touch \( \partial \Omega \) for \( t \in [0, 0.8] \). Figure 8 displays profiles of the diagonal spiral at \( t = 0.8 \) with the above setting.

![ODE approach](image1.png) ![Level set method](image2.png)

**Figure 8.** Profiles of the triangle spiral at \( t = 0.8 \). The level set method is calculated using \( \rho = 0.02 \) and \( \Delta x = 0.0050 \).

The left figure of Figure 9 shows graphs of \( D(t) \) for \( s = 2, 3, 4, 5, 6 \) with a fixed center radius \( \rho = 0.02 \). One can find that the differences are less than 4% for all cases except \( \Delta x = 0.0067 \) (\( s = 3 \)), and the best one is obtained when \( \Delta x = 0.0040 \) (\( s = 5 \)).
From the analogy of the diagonal spiral case, we choose $\rho \approx 4\Delta x$ as a reducing center radius $\rho = O(\Delta x)$. The right figure of Figure 9 again shows graphs of $D(t)$ for $s = 2, 3, 4, 5, 6$ with a fixed center radius $\rho = 4\Delta x$. One can find that the differences are less than 5% when $\Delta x \leq 0.0050$ ($s \geq 4$). Note that the results when $\Delta x = 0.0050 (s = 4)$ are the same in both figures of Figure 9.

5. Conclusion. We have compared a discrete model from [15] and a level set method from [17] for evolving spirals by the crystalline eikonal-curvature flow (1). The level set equation includes the derivative of a piecewise linear energy density function and so we introduced an approximation of level set equation for the crystalline curvature flow. This was established with the approximation of the characteristic function as in [8]. To measure the difference between the two curves obtained by the discrete model and the level set method, we introduced an area difference function defined by (10). It consists of the $L^1$ difference of the height function as in [17] with the step-height $h_0 = 1$. When the resolution of numerical lattice is high enough, and the radius of the center for the level set method is suitably small, we found that, although the discrete and level set methods differ in their boundary conditions at the center and outer boundary of the domain, the area differences obtained from our results differ by less than 5% in the case of square, diagonal, and triangle spirals.

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E-mail address: tisiwata@shibaura-it.ac.jp
E-mail address: tohtsuka@gunma-u.ac.jp