A Normal Form for the Onset of Collapse:
the Prototypical Example of the Nonlinear Schrödinger Equation

S Jon Chapman
Mathematical Institute, University of Oxford, AWB, ROQ, Woodstock Road, Oxford OX2 6GG

M. Kavousanakis
School of Chemical Engineering, National Technical University of Athens, 15780, Athens, Greece

I.G. Kevrekidis
Department of Chemical and Biomolecular Engineering &
Department of Applied Mathematics and Statistics,
Johns Hopkins University, Baltimore, MD 21218, USA

P.G. Kevrekidis
Department of Mathematics and Statistics, University of Massachusetts, Amherst MA 01003-1515, USA and
Mathematical Institute, University of Oxford, AWB, ROQ, Woodstock Road, Oxford OX2 6GG
(Dated: August 21, 2020)

The study of nonlinear waves that collapse in finite time is a theme of universal interest, e.g. within optical, atomic, plasma physics, and nonlinear dynamics. Here we revisit the quintessential example of the nonlinear Schrödinger equation and systematically derive a normal form for the emergence of blowup solutions from stationary ones. While this is an extensively studied problem, such a normal form, based on the methodology of asymptotics beyond all algebraic orders, unifies both the dimension-dependent and power-law-dependent bifurcations previously studied; it yields excellent agreement with numerics in both leading and higher-order effects; it is applicable to both infinite and finite domains; and it is valid in all (subcritical, critical and supercritical) regimes.

Introduction. The nonlinear Schrödinger (NLS) model [1–4] has, arguably, been one of the most central nonlinear partial differential equations (PDEs) within Mathematical Physics for the last few decades. Its wide appeal stems from the fact that it is a ubiquitous envelope wave equation arising in a variety of diverse physical contexts. Its applications span water waves [5–7], nonlinear optical media [8, 9], plasma physics [10] and more recently the atomic physics realm of Bose-Einstein condensates and their variants [11, 12].

The solitonic waveforms of the NLS model have been central to all of the above investigations. A similarly prominent feature of the NLS model is its finite-time, self-similar blowup in higher (integer) dimensions or for higher powers of the associated nonlinearity. Indeed, the latter manifestation of lack of well-posedness has been central to both books [3, 13, 14] and reviews [15–17] and has been the objective of continued study not only in the physical literature, but also in the mathematical one; see, e.g., [18–20] and [21, 22] for only some recent examples (and also references therein). Importantly for our purposes, these focusing aspects have become accessible to physical experiments. On the one hand, there is the well-developed field of nonlinear optics, where not only the well-known, two-dimensional collapsing waveform of the Townes soliton has been observed [23], but also more elaborate themes have been touched upon including the collapse of optical vortices [24], the loss of phase information of collapsing filaments [25] or the manipulation of the medium to avert optical collapse [26].

On the other hand, there is the flourishing area of Bose-Einstein condensates where the Townes soliton has recently been announced [27]. Here, collapsing waveforms in higher dimensions had been experimentally identified earlier [28, 29] and the ability to manipulate the nonlinearity [30] and the initial conditions [31] has continued to improve in recent times.

The emergence of collapsing solutions out of solitonic ones is a topic that has been long studied since the early works of [32, 33] and summarized in numerous reviews and books [3, 13, 14]. Nevertheless, remarkably, a normal form—a prototypical model equation compactly describing the relevant bifurcation, namely the onset of collapsing solutions out of non-collapsing ones—does not exist, to the best of our knowledge. Recent attempts to capture even the well-known log-log law of the critical case and its corrections [18] will confirm that. It is known that at the critical point at which collapse emerges, $\sigma d = 2$, where $\sigma$ is the exponent of the nonlinearity and $d$ the spatial dimension of the NLS model, a symmetry enabling self-similar rescaling of the solution towards becoming singular at a finite time (the so-called pseudo-conformal invariance) arises. Beyond this critical point, solitary waves become unstable and, in a form somewhat reminiscent of the traditional pitchfork bifurcation, two collapsing branches of solutions emerge [34]. Yet, this is no ordinary pitchfork like, e.g., the one experimentally probed in BECs in double-well potentials [35]. Here, pseudo-
conformal symmetry breaks and, thus, collapse phenomena will not follow the standard cubic pitchfork normal form, but rather are associated with the exponentially-small, beyond-all-algebraic-orders phenomenology of the relevant symmetry breaking. Our aim is to go beyond the heuristic (steady state only) arguments of earlier studies \cite{32, 33} and present a systematic derivation of the associated normal form. Key features of our analysis are:

- We unify the case of general nonlinearity exponent and that of arbitrary dimension, offering a result broadly applicable in the above physical settings of interest.
- Our analysis captures both the case of the critical log-log collapse and the supercritical $t^{-1/2}$ collapse.
- Crucially, we capture not only the leading collapse order but also systematically the higher-order corrections.
- We find excellent agreement with computations of the stationary solutions and of the dynamical evolution.

**Problem Formulation & Asymptotic Analysis.** Upon exposing the general formulation of the problem, we will solve it separately in the near and far fields. The far field has a turning point, resulting in an exponentially-small reflection back towards the near field \cite{36}. Matching with the near field solution yields our onset of collapse normal form, bearing this exponentially small contribution.

We start with the NLS in dimension $d$ and nonlinearity power determined by the exponent $\sigma$ as:

$$
\frac{1}{i} \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial \xi^2} + \frac{(d-1)}{r} \frac{\partial \psi}{\partial r} + |\psi|^{2\sigma} \psi = 0. 
$$

(1)

We will perturb around the critical (radially symmetric) case $d \sigma = 2$ \cite{3, 13}. Introducing the well-known stretched variables \cite{3, 13, 34}

$$
\xi = \frac{r}{L}, \quad \tau = \int_0^t \frac{dt'}{L^{2/(t')}}.
$$

leads to

$$
\frac{i}{\tau} \frac{\partial v}{\partial \tau} + \frac{\partial^2 v}{\partial \xi^2} + \frac{(d-1)}{\xi} \frac{\partial v}{\partial \xi} + |v|^{2\sigma} v - vG \left( \frac{\partial v}{\partial \xi} + \frac{1}{\sigma} v \right) = 0,
$$

(2)

where the blowup rate $G = -LL_t = -L_{\tau}/L$. In this dynamic change of variables, and in order to close the dynamics in this “co-explosive” frame (upon determining $G(\tau)$), we impose a pinning condition of the form \cite{34}

$$
\int_{-\infty}^{\infty} \text{Re}(v(\xi, \tau))T(\xi) \, d\xi = C,
$$

for some constant $C$ and some (essentially arbitrary) “template function” $T$, to enable us to uniquely identify the solution $v$ and the blowup rate $G$. In our numerical examples we choose $T = \delta(\xi - 2)$ \cite{37}. Finally, we write

$$
v(\xi, \tau) = V(\xi, \tau) e^{-iG(\tau)\xi^2}/4
$$

to give (using $G' = dG/d\tau$)

$$
\int(3) \text{can be neglected for now. We look for a solution:}

$$
V = e^{i\Phi(\tau)} (V_0(\xi, \tau; G(\tau)) + V_{\exp}(\xi, \tau))
$$

(4)

where $V_0$ is the (real) regular algebraic expansion in $G$, $V_{\exp}$ is exponentially small in $G$, and the exponentially-slowly-varying phase $\Phi$ is determined by the pinning condition.
The solution of which is the critical ground-state soliton. The next order
Far Field. The above expansion (5) breaks down at long distances. In the far field we rescale \( \xi = \rho/G \) to give
\[
\frac{\partial^2 V_0}{\partial \xi^2} + \frac{(d_c - 1)}{\xi} \frac{\partial V_0}{\partial \xi} + V_0^{2n} - V_0 = 0, \tag{6}
\]
the solution of which is the critical ground-state soliton.

At leading order this gives the eikonal equation:
\[
(\phi')^2 = 1 - \frac{\rho^2}{4} \quad \Rightarrow \quad \phi = -\int_0^\rho \left(1 - \frac{\rho^2}{4}\right)^{1/2} \, d\bar{\rho} \tag{8}
\]
for some constant \( a_0 \), which we will determine by matching with the near field. As \( \rho \to 0 \), the far field yields:
\[
G^k e^{\phi(\rho)/G} A_0 \sim \frac{a_0 G^k e^{-\rho/G}}{\rho^{(d_c-1)/2}}. \tag{9}
\]
As \( \xi \to \infty \), the near field expression is dominated by:
\[
V_0(\xi) \sim \frac{A_d e^{-\xi}}{\xi^{(d_c-1)/2}} = \frac{A_d G^{(d_c-1)/2} e^{-\rho/G}}{\rho^{(d_c-1)/2}}, \tag{10}
\]
for some dimension-dependent constant \( A_d \). We note, in particular, the values \( A_1 = 12^{1/4} \) (from the quintic NLS exact soliton solution [3]), while \( A_2 \approx 3.518 \) [18]. Matching (9) with (10) gives \( k = (d_c-1)/2 \) and \( a_0 = A_d \).

For \( \rho > 2 \) only the solution of (8) in which
\[
\phi = i \left(\frac{\rho^2}{4} - 1\right)^{1/2}
\]
has a finite Hamiltonian. Thus for \( \rho > 2 \),
\[
V_G = \alpha G^k e^{\phi_2(\rho)/G} \sum_{n=0}^{\infty} B_n(\rho)(iG)^n, \tag{11}
\]
for some constant \( \alpha \), where
\[
\phi_2 = \int_2^\rho \left(\frac{\rho^2}{4} - 1\right)^{1/2} \, d\bar{\rho}, \quad B_0(\rho) = \frac{2^{1/2} a_0}{\rho^{(d_c-1)/2}(\rho^2 - 4)^{1/4}}.
\]
The fact that only one of the oscillatory exponentials is present in \( \rho > 2 \) forces an exponential small reflection back towards the near field, which we will obtain by analysing the turning point region. This is a key feature of our exponential asymptotics analysis.

Turning Point. Writing \( \rho = 2 + G^{2/3}s \), the equation near the turning point becomes, to leading order,
\[
\frac{d^2 V_G}{ds^2} + s V_G = 0,
\]
with solution \( V_G = \lambda \text{Ai}(-s) + \mu \text{Bi}(-s) \), where \( \lambda \) and \( \mu \) are Airy functions of the first and second kind, respectively. The asymptotic expansions of \( \text{Ai} \) and \( \text{Bi} \) give:

\[
V_G \sim \frac{\lambda e^{-3/2} s^{3/2}}{2 \sqrt{\pi} s^{1/4}} + \frac{\mu e^{3/2} s^{3/2}}{\sqrt{\pi} s^{1/4}} \quad \text{as } s \to -\infty, \quad (12)
\]

\[
V_G \sim \frac{e^{2i\pi/3}}{2 \sqrt{\pi} s^{1/4}} \left( \frac{\lambda e^{i\pi/4} + \mu e^{i\pi/4}}{2} \right) + \frac{e^{-2i\pi/3}}{2 \sqrt{\pi} s^{1/4}} \left( \frac{\lambda e^{-i\pi/4} + \mu e^{-i\pi/4}}{2} \right) \quad \text{as } s \to \infty. \quad (13)
\]

Matching with (7) and (11) gives \( \alpha = e^{i\pi/4} \) and \( \lambda = i \mu = \frac{a_0 \nu}{G^{1/6}} \).

Including both WKB solutions in \( \rho < 2 \) replaces (7) with

\[
V_G \sim \left( e^{(\rho)/G} + \gamma e^{-\phi(\rho)/G} \right) G^n \sum_{n=0}^{\infty} A_n G^n, \quad (14)
\]

where matching with (12) gives

\[
\gamma = \frac{i}{2} 2^{\phi(2)/G} = \frac{i}{2} e^{-\pi/G}.
\]

Exponentially small correction to the near field. As \( \rho \to 0 \),

\[
\gamma e^{-\phi(\rho)/G} \sum_{n=0}^{\infty} A_n G^n \sim \frac{a_0 \gamma G^{(d_r-1)/2} e^{\phi(2)/G}}{\rho^{(d_r-1)/2}}. \quad (15)
\]

This term will match with the exponentially small correction to the near field. In the original near-field scaling, using Eq. (4) neglecting time derivatives and quadratic terms in \( V_{\exp} \), but keeping all the exponentially-small terms, gives

\[
\frac{\partial^2 V_{\exp}}{\partial \xi^2} + \frac{(d_e - 1) \partial V_{\exp}}{\xi} + V_{\exp} G^{2\sigma} \left( \sigma_c V_{\exp} + (\sigma_c + 1) V_{\exp} \right) - V_{\exp} + \frac{G^2 \xi^2}{4} V_{\exp} = - \frac{(d_e - d_c) \partial^2 V_{\exp}}{\xi} - i \frac{\partial V_{\exp}}{\partial \tau} + \Phi V_{\exp} - \frac{G^2 \xi^2}{4} V_{\exp} - 2(\sigma - \sigma_c) V_{\exp} G^{2\sigma+1} \log V_{\exp} + \frac{i(\sigma - 2) G}{2\sigma} V_{\exp},
\]

where \( \Phi' = d\Phi/d\tau \). We now use \( V_{\exp} = U_{\exp} + i W_{\exp} \) and separate into real and imaginary parts. Since \( V_{\exp} \) satisfies the homogeneous version of the equation for \( W_{\exp} \), this enables a solvability condition: multiplying that equation by \( e^{d_e-1} V_{\exp} \), integrating from 0 to \( R \), and using (6), we obtain:

\[
\xi^{d_e-1} V_{\exp}(R) \frac{\partial W_{\exp}}{\partial \xi}(R) - \xi^{d_e-1} W_{\exp}(R) \frac{\partial V_{\exp}}{\partial \xi}(R) = - \int_0^R \xi^{d_e-1} V_{\exp} \frac{\partial V_{\exp}}{\partial \tau} - \xi^{d_e-1} \frac{(\sigma - 2) G}{2\sigma} V_{\exp}^2 d\xi \quad (16)
\]

As \( R \to \infty \) we evaluate the boundary terms by matching using (15), giving

\[
\lim_{R \to \infty} \xi^{d_e-1} V_{\exp}(R) \frac{\partial W_{\exp}}{\partial \xi}(R) - \xi^{d_e-1} W_{\exp}(R) \frac{\partial V_{\exp}}{\partial \xi}(R) \sim \lim_{R \to \infty} a_0 e^{-R} (a_0 \text{Im}(\gamma) e^R) - (a_0 \text{Im}(\gamma) e^R) (-a_0 e^{-R}) = 2a_0^2 \text{Im}(\gamma).
\]

Now

\[
\int_0^\infty \xi^{d_e-1} V_0^2 d\xi \sim \int_0^\infty \xi^{d_e-1} (V_0 + G^2 V_1 + \cdots)^2 d\xi \sim b_0 + 2G^2 c_0 + \cdots,
\]

say, where

\[
b_0 = \int_0^\infty \xi^{d_e-1} V_0^2 d\xi, \quad c_0 = \int_0^\infty \xi^{d_e-1} V_0 V_1 d\xi.
\]

Thus the solvability condition (16) ultimately results in:

\[
2a_0 \frac{dG}{d\tau} = \frac{d\sigma - 2) b_0}{2\sigma} - A_{d_e}^2 \frac{e^{-\pi/G}}{G}. \quad (17)
\]

which is the normal form for the onset of collapse. In principle \( a_0 \), \( b_0 \), and \( c_0 \) are leading terms in power series expansions in \( G \) above, and we can calculate the full power series. In some of our numerical examples we include the \( O(G^2) \) and \( O(G^4) \) corrections to these terms. One can discern similarities of Eq. (17) with the pitchfork bifurcation normal form: the natural bifurcation parameter is \( r = (\sigma - 2) \). Multiplying both sides by \( G \), it can be seen that for all \( r < 0 \), \( G = 0 \) is the only equilibrium branch of solutions. When \( r > 0 \), the dynamics tends towards the non-trivial (stable, collapsing) steady state solution of Eq. (17). Changing the sign of \( G \) and \( \tau \) and of the imaginary part \( W \) (in Eq. (3)), we obtain the final branch of this unusual pitchfork bifurcation diagram, a solution that is a mirror image but is stably collapsing in negative (rather than positive) time, i.e., “coming back from infinity”. These are some of the intriguing by-products of unfolding the original Hamiltonian dynamical system of Eq. (1) into the dissipative renormalized frame of Eq. (3). Moreover, a key feature of this collapse nor-
mal form is its exponentially small (large) nonlinear term (rather than the usual cubic in the standard pitchfork), yielding a nearly vertical bifurcation for \( G = G(\sigma) \), as shown in Fig. 1. Notice that our analysis is still valid for \( r = 0 \).

**Finite Domain.** Usually, when numerically simulating (1) or (3) the domain is truncated to some large but finite domain \([0, K]\). In that case both oscillatory WKB solutions are present in \( \rho > 2 \), and the ratio of their amplitudes is determined by the position of the boundary and the nature of the boundary condition. A similar analysis can be performed, and the result is a more complicated expression for the coefficient \( \gamma \), the prefactor of the reflection term at the turning point. For example, imposing a Neumann condition on \( v \) at \( \xi = K \) results in

\[
\text{Im}(\gamma) = \frac{(1-\nu_0^2)e^{-\pi G}}{2(1-\nu_0^2\sin(2\phi_2(KG)/G)) + \nu_0^2},
\]

(18)

where

\[
\nu_0 = \sqrt{(KG)^2 - 4 - KG} / \sqrt{(KG)^2 - 4 + KG}.
\]

We see that as \( K \to \infty, \nu_0 \to 0 \) and \( \text{Im}(\gamma) \to e^{-\pi G}/2 \).

**Numerical Verification.** Equation (17) predicts the existence of a stable branch of solutions bifurcating from \( d\sigma = 2 \). We compare this prediction with direct numerical simulations of (2) by fixing \( d = 1 \) and varying \( \sigma \) close to \( \sigma_c = 2 \). The relevant bifurcation diagram can be seen in Fig. 1. Here, we compare the PDE results obtained directly from Eq. (2) with the normal form of Eq. (17) finding excellent agreement between the two. The definitive comparison of the full NLS results with those of our normal form is illustrated in Fig. 2. The top panel clearly showcases the exponential nature of the relevant bifurcation over 8 orders of magnitude of the associated ODE and PDE data in excellent agreement between the two. Notice that the finite nature of the computation leads to some nearly imperceptible oscillations in the top panel of the figure, also observed but not commented in earlier works [32, 34]. The full power of our methodology is revealed when factoring out the exponentially small leading order by rescaling through \( e^{\pi G}/G \) as shown in the bottom panel of Fig. 2. In addition to the leading-order behavior we present the first- and second-order corrections, illustrating how they progressively match in a remarkably quantitative fashion the PDE results. To complement the quality of the match, we show in Fig. 3 how we can capture not only the rate of collapse, but also near perfectly both the real and the imaginary parts of the profile of the associated solution \( U + iW \).

Lastly, we note that our methodology not only offers a tool for capturing the statics (i.e., the equilibrium collapse branch and its spatial profile), but also enables an excellent capturing of the associated dynamics as shown in Fig. 4. Here, in addition to the spatio-temporal evolution of the field in the \((\xi, \tau)\) variables, the evolution of the collapse rate \( G(\tau) \) towards its stable asymptotic value is observed in the inset, and compared against the numerical solution showing excellent agreement.

**Conclusions.** In the present work we have revisited the fundamental problem of the collapse of a nonlinear Schrödinger equation. We have offered a unified perspective of the emergence of the self-similar solutions via a mathematically compact, yet quantitatively accurate normal form that combines the famous log-log behavior at the critical point, the emergence of a stable self-similarly collapsing branch past that point, the exponentially small (large) breaking of the pseudo-conformal invariance of the critical point, the Hamiltonian nature of the original model and the dissipative features of the renormalized dynamics. In our view this constitutes a
generic and broadly applicable (in optics, BECs and beyond) normal form associated with the onset of collapse. The identification of this normal form prompts numerous exciting questions for the future, such as, e.g., the examination of the stability of the collapsing solutions or the examination of a potential normal form for generalized Korteweg-de Vries equations [38] and their traveling waves that are of broad relevance to water waves and plasmas. This analysis may also pave the way for the study of self-similar periodic orbits that have recently emerged in interfacial hydrodynamics [39].

This material is based upon work supported by the US National Science Foundation under Grants No. PHY-1602994 and DMS-1809074 (PGK) and by the US ARO MURI (IGK). PGK also acknowledges support from the Leverhulme Trust via a Visiting Fellowship and thanks the Mathematical Institute of the University of Oxford for its hospitality during part of this work.

[1] M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press (Cambridge, 1991).
[2] M.J. Ablowitz, B. Prinari and A.D. Trubatch, Discrete and Continuous Nonlinear Schrödinger Systems, Cambridge University Press (Cambridge, 2004).
[3] C. Sulem and P.L. Sulem, The Nonlinear Schrödinger Equation, Springer-Verlag (New York, 1999).
[4] P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González, The defocusing nonlinear Schrödinger equation: from dark solitons and vortices to vortex rings (SIAM, Philadelphia, 2015).
[5] E. Infeld and G. Rowlands, Nonlinear Waves, Solitons and Chaos (Cambridge University Press Cambridge, 1990).
[6] M. J. Ablowitz, Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons (Cambridge University Press, Cambridge, 2011).
[7] Th. Dauxois, M. Peyrard, Physics of Solitons, Cambridge University Press (Cambridge, 2006).
[8] A. Hasegawa, Solitons in Optical Communications, Clarendon Press (Oxford, NY 1995).
[9] Yu.S. Kivshar and G.P. Agrawal, Optical solitons: from fibers to photonic crystals, Academic Press (San Diego, 2003).
[10] M. Kono and M. M. Skorić, Nonlinear Physics of Plasmas, Springer-Verlag (Heidelberg, 2010).
[11] L. P. Pitaevskii and S. Stringari, Bose-Einstein Condensation, Oxford University Press (Oxford, 2003).
[12] C.J. Pethick and H. Smith, Bose-Einstein condensation in dilute gases, Cambridge University Press (Cambridge, 2002).
[13] G. Fibich, The nonlinear Schrödinger equation, singular solutions and optical collapse, Springer-Verlag (New York, 2015).
[14] R.W. Boyd, S.G. Lukishova, Y.R. Shen, Self-focusing: Past and Present, Springer-Verlag (New York, 2009).
[15] G. Fibich and G. Papanicolaou, SIAM J. Appl. Math. 60, 183 (1999).
[16] L. Bergé, Phys. Rep. 303, 259 (1998).
[17] Yu.S. Kivshar, D.E. Pelinovsky, Phys. Rep. 331, 117 (2000).
[18] P.M. Lushnikov, S.A. Dyachenko, N. Vladimirova, Phys. Rev. A 88, 013845 (2013).
[19] P.M. Lushnikov, N. Vladimirova, Opt. Express 23, 31120 (2015).
[20] B. Shim, S.E. Schrauth, A.L. Gaeta, M. Klein, G. Fibich, Phys. Rev. Lett. 108, 043902 (2012).
[21] H. Koch, Nonlinearity 28, 545 (2015).
[22] K. Yang, S. Roudenko, Y. Zhao, Nonlinearity 31, 4354 (2018).
[23] K.D. Moll, A.L. Gaeta, G. Fibich, Phys. Rev. Lett. 90, 203902 (2003).
[24] L.T. Vuong, T.D. Grow, A. Ishaaya, A.L. Gaeta, G.W.’t Hooft, E.R. Eliel, G. Fibich, Phys. Rev. Lett. 96, 133901 (2006).
[25] A. Sagiv, A. Ditkowskvi, G. Fibich, Opt. Express 25, 24387 (2017).
[26] M. Centurion, M.A. Porter, P. G. Kevrekidis, and D. Psaltis Phys. Rev. Lett. 97, 033903 (2006).
[27] C.-A. Chen and C.-L. Hung, arXiv:1907.12550 (2019).
[28] E.A. Donley, N.R. Claussen, S.L. Cornish, J.L. Roberts, E.A. Cornell and C.E. Wieman, Nature 412, 295 (2001).
[29] S.L. Cornish, S.T. Thompson, and C.E. Wieman Phys. Rev. Lett. 96, 170401 (2006).
[30] A. Di Carli, C.D. Colquhoun, G. Henderson, S. Flammig, G.-L. Oppo, A.J. Daley, S. Kuhr, E. Haller, Phys. Rev. Lett. 123, 123602 (2019).
[31] D. Luo, Y. Jin, J.H. V. Nguyen, B.A. Malomed, O.V. Marchukov, V.A. Yurovsky, V. Dunjko, M. Olszani, R. G. Hulet, arXiv:2007.03766.
[32] M.J. Landman, G.C. Papanicolaou, C. Sulem, P.L. Sulem, Phys. Rev. A 38, 3837 (1988).
[33] B.J. LeMesurier, G. Papanicolaou, C. Sulem, P.L. Sulem, Physica D 31, 78 (1988).
[34] C.I. Siettos, I.G. Kevrekidis, P.G. Kevrekidis, Nonlinearity 16, 497 (2003).
[35] T. Zibold, E. Nicklas, C. Gross, and M.K. Oberthaler Phys. Rev. Lett. 105, 204101 (2010).
[36] J.P. Boyd, Acta Applicandae 56, 1 (1999).
[37] Naturally, there are numerous choices of the template condition. However, the specifics of such a choice will not affect the normal form obtained herein. Differentiating this template condition and substituting Eq. (2), we obtain the algebraic equation that completes the differential-algebraic system for v(ξ, τ) and G.
[38] J.L. Bona, V.A. Dougalis, O.A. Karakashian and W.R. McKinney, J. Comp. Appl. Math. 74, 127 (1996).
[39] M.C. Dallaston, M.A. Fontelos, D. Tseluiko, and S. Kalliadasis Phys. Rev. Lett. 120, 034505 (2018).