Scaling theory for hydrodynamic lubrication, with application to non-Newtonian lubricants

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Scaling arguments are developed for the load balance in hydrodynamic lubrication, and applied to non-Newtonian lubricants with a shear-thinning rheology typical of a structured liquid. It is argued that the shear thinning regime may be mechanically unstable in lubrication flow, and consequently the Stribeck (friction) curve should be discontinuous, with possible hysteresis. Further analysis suggests that normal stress and flow transience (stress overshoot) do not destroy this basic picture, although they may provide stabilising mechanisms at higher shear rates. Extensional viscosity is also expected to be insignificant unless the Trouton ratio is large. A possible application to recent theories of shear thickening in non-Brownian particulate suspensions is indicated.

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I. INTRODUCTION

Lubrication is usually considered to be a complex, multi-scale, mechanical engineering problem [1–3], demanding sophisticated numerical approaches for its solution [4, 5]. Indeed, with few exceptions [6], pure hydrodynamic lubrication is rarely studied; rather the focus is more usually on elasto-hydrodynamic lubrication (EHL) or boundary layer lubrication. Given this, the utility of a simplified scaling theory for pure hydrodynamic lubrication may be questioned. However there are numerous areas to which lubrication is additionally relevant, for example in understanding the origins of ‘mouth-feel’ and the sensory properties of foods [7], and in ‘psychorheology’ and the perception of skin care products [8]. These diverse applications differ from the traditional engineering problem in that the lubricants are often structured liquids or soft solids with a non-Newtonian rheology, and the gaps are large enough that pure hydrodynamic lubrication may be relevant. This motivates the development of the present scaling theory, which aims to provide insights and guidance for these unconventional application areas, and perhaps even reveal qualitatively new phenomena.

To begin with, let me outline the fundamental mechanism of pure hydrodynamic lubrication [9, 10]. In a fully lubricated conjunction (Fig. 1), mass conservation within the converging and diverging wedges induces a Poiseuille-like contribution to the entrainment flow, superimposed on a Couette-type shearing motion. There is a corresponding emergent pressure distribution (Fig. 1 inset), which supports the load. The ratio of the load to the lateral sliding force defines the friction coefficient $\mu$.

As we shortly shall see, Reynolds lubrication theory predicts $\mu(S)$ behaviour is often summarised in the semi-empirical Stribeck curve (Fig. 2). Note that pure hydrodynamic scaling breaks down as $S \to 0$ since the surfaces come more and more into close contact where EHL becomes increasingly important. This is reflected in the behaviour observed empirically in the Stribeck curve. The present scaling theory should be applied, and interpreted, within this wider context.

Before embarking on the detailed development, one further general remark should be made. In a geometry which possesses reflection symmetry, Reynolds lubrication theory predicts the lubrication pressure should be skew-symmetric about the minimum gap (Fig. 1 inset). Therefore, the integrated lubrication pressure should vanish. In actuality it has long been recognised that some additional physics intervenes to knock down this result. For instance in the trailing edge where the lubrication pressure is sub-ambient (shaded area in Fig. 1 inset), the free surface may separate [11], or the fluid may cavitate [12]. These considerations make the exact solution to the problem dependent on the nature of the additional curvature of the surfaces, and $W$ is the normal load. The overall $\mu(S)$ behaviour is often summarised in the semi-empirical Stribeck curve (Fig. 2). Note that pure hydrodynamic scaling breaks down as $S \to 0$ since the surfaces come more and more into close contact where EHL becomes increasingly important. This is reflected in the behaviour observed empirically in the Stribeck curve. The present scaling theory should be applied, and interpreted, within this wider context.

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physics. However a simple and commonly used prescription is to discard the contribution from the sub-ambient pressure region (the so-called half-Sommerfeld boundary condition [2]). I shall tacitly use this assumption below.

Keeping all this in mind, in the next section I shall develop the scaling theory for Newtonian hydrodynamic lubrication, and in the following section I shall explore the implications for non-Newtonian liquids.

II. SCALING THEORY: NEWTONIAN CASE

The gap between the lubricated surfaces, as shown in Fig. 1, for example, can always be represented in the conjunction region by a parabolic profile

\[ h = h_0 + \frac{r^2}{2R}. \]

In this \( h_0 \) is the minimum gap, \( r \) is the radial distance from the minimum gap, and \( R \) is a measure of the radius of curvature of the surfaces (e.g., the sphere radius in Fig. 1). The radius within which \( h \) remains of order \( h_0 \) defines a length scale \( r_0 \sim (Rh_0)^{1/2} \). This length scale sets the size of the conjunction region and is the key to the development of the scaling theory.

As already alluded to, and described in more detail in the Appendix, mass conservation induces a Poiseuille-type flow of magnitude \( U \) superimposed on the Couette-type shearing motion. The matching pressure distribution is such that \( U \sim (h_0/\eta)\nabla p \) where \( \nabla p \sim p/r_0 \) is pressure gradient. Hence the Reynolds lubrication pressure has a magnitude

\[ p \sim \frac{\eta Ur_0}{h_0^2}. \]

The integrated pressure (e.g., the unshaded area in the Fig. 1 inset) then corresponds to a normal force

\[ pr_0^2 \sim \frac{\eta Ur_0^3}{h_0^2} \sim \eta UR^{3/2}h_0^{-1/2}. \]

In this I have used \( r_0 \sim (Rh_0)^{1/2} \) for the second step. This integrated pressure balances the load when \( pr_0^2 \sim W \), or alternatively, using Eq. (3), when

\[ \frac{h_0}{R} \sim \left( \frac{\eta Ur}{W} \right)^2 (\equiv S^2). \]

More careful treatments restore a prefactor to this, for example Kapitza [1] derived \( h_0 = (72\pi^2/25)RS^2 \approx 28.4 RS^2 \), and Hamrock [2] has \( h_0 \approx 34.8 RS^2 \).

A point not often stressed, but which will be critical in the sequel, is that the lubricated conjunction is mechanically stable at the load balance condition. To see this suppose that the minimum gap \( h_0 \) has not yet taken its steady state value. The load \( W \) presses down on the conjunction, and is opposed by the lubrication pressure which exerts a force of order \( pr_0^2 \) given by Eq. (3), which I note is a decreasing function of \( h_0 \). Therefore if the gap is too large, the lubrication pressure does not support the load and the gap closes. Conversely, if the gap is too small, the lubrication pressure overcompensates for the load and the gap opens up. This stabilising mechanism is shown schematically in Fig. 3 where the solid line is the normal force arising from the lubrication pressure. In the language of dynamical systems theory, the filled circle in this diagram is a stable fixed point.

I turn now to the Stribeck curve. The tangential wall stress in the conjunction zone is of order \( \eta U/h_0 \) (see Appendix). Multiplying by the area of the conjunction \( (r_0^2) \) gives the tangential force \( T \sim \eta Ur_0^2/h_0 \). Hence the friction coefficient

\[ \mu \equiv \frac{T}{W} \sim \frac{\eta Ur_0^2}{W h_0} \sim \frac{\eta UR}{W} (\equiv S) \]

(5)

(4)

(3)

(2)

(1)

(4)

(3)

(2)

(1)

(4)

(3)

(2)

(1)

(4)

(3)

(2)

(1)
FIG. 4. (a) Load curves in the non-Newtonian strong shear thinning case. The solid line is the normal force generated by the lubrication pressure. A filled (open) circle is a stable (unstable) fixed point, under a given load. The fixed points disappear in a saddle-node bifurcation as the load is increased. (b) The corresponding Stribeck curve is predicted to jump discontinuously at the critical load, perhaps into a boundary lubrication regime.

III. NON-NEWTONIAN CASE

I now propose the obvious extensions of the above scaling arguments to incorporate non-linear rheological effects of interest for non-Newtonian lubricants. For the time being I shall leave aside the question of normal stress, flow transience, and extensional flow, and focus first on the typical shear thinning behaviour found in structured liquids.

A. Shear thinning

I shall represent the non-linear viscosity in this case by the simple model, cf. [5],

\[ \eta = \frac{\eta_0}{1 + (\dot{\gamma} \tau)^{\alpha}} \]  

where \( \eta_0 \) is the low shear viscosity (first Newtonian plateau), \( \dot{\gamma} \) is the shear rate, \( \tau \) is a characteristic relaxation time, and \( \alpha \) is an exponent characterising the rate of decrease of the viscosity in the shear-thinning regime. The dimensionless quantity \( \dot{\gamma} \tau \) is the Weissenberg number and is a measure of the extent to which the non-linear regime has been penetrated. For concreteness, a typical value of the exponent for polymer melt rheology is \( \alpha \approx 0.8–1 \) [14–16], although I should note that \( \alpha > 1 \) would correspond to shear banding. Eq. (6) omits the second Newtonian plateau which is expected to obtain at very high shear rates; this will be added informally to the discussion at the end.

Inserting Eq. (6) into Eq. (3) gives a revised estimate for the normal force produced by the lubrication pressure,

\[ p r_0^2 \sim \frac{\eta_0}{1 + (\dot{\gamma} \tau)^{\alpha}} \times UR^{3/2} h_0^{-1/2} \]  

(7)

where the shear rate is estimated by \( \dot{\gamma} \sim U/h_0 \). The perhaps naive and rather drastic assumptions being made here will be reviewed later. Since the shear rate is a decreasing function of the minimum gap, there are two regimes. For large gap, the Weissenberg number is small (\( \dot{\gamma} \tau \ll 1 \)) and one recovers the Newtonian behaviour seen already, with \( \eta = \eta_0 \) being the viscosity in the first Newtonian plateau.

For small gap, one enters the high Weissenberg number regime (\( \dot{\gamma} \tau \gg 1 \)) in which case Eq. (7) becomes

\[ p r_0^2 \sim \eta_0 (\dot{\gamma} \tau)^{-\alpha} \times UR^{3/2} h_0^{-1/2} \sim h_0^{\alpha-1/2} \]  

(8)

Only the \( h_0 \)-dependence has been retained in the final step. If \( \alpha < 1/2 \) (weak shear thinning), then Eq. (8) is a decreasing function of \( h_0 \) and one would again expect the gap to be mechanically stable, albeit with a dependence on a modified or instantaneous Sommerfeld number. On the other hand, if \( \alpha > 1/2 \) (strong shear thinning), \( p r_0^2 \) in Eq. (8) is an increasing function of \( h_0 \) and therefore the load balance in this regime will be unstable.
The implication for the Strubeck curve is as follows. In the low Weissenberg number regime, the analysis goes through as for the Newtonian case, with \eta_0 featuring as the viscosity. The Weissenberg number \gamma \tau \sim S^{-2}, and increases as the Sommerfeld number shrinks. Mechanical stability is lost at the point where one enters the shear thinning regime (i.e. at the above saddle-node bifurcation). At this point the surfaces should jump into (near) contact. Correspondingly the Strubeck curve should show a discontinuous jump, most likely into a boundary lubrication regime (Fig. 4).

If one envisages that a second Newtonian plateau appears at very high shear rates, then a restabilisation mechanism emerges naturally. In this situation, the expected behaviour of the lubrication pressure is shown in Fig. 4b.

I shall take this latter case (strong shear thinning) to be the one of most interest (for polymer melts, for instance). The two regimes in the load balance are illustrated in Fig. 4. The left hand branch is the high Weissenberg number regime where shear thinning takes place. In this branch, the open circle corresponds to an unstable fixed point at load \( W_1 \). To the left of the open circle, the lubrication pressure is insufficient to support the load and the gap closes until interrupted by some additional physics. To the right of the open circle, the lubrication pressure forces the surfaces to move apart until one reaches the stable fixed point (filled circle) in the low Weissenberg number regime. Now consider increasing the load to \( W_2 \).

In this situation, the lubrication pressure is never sufficient to keep the surfaces apart and the gap closes until one enters the mixed or boundary layer lubrication regime. Somewhere in between \( W_1 \) and \( W_2 \) is a critical load at which the stable and unstable fixed points merge in a saddle-node bifurcation.

The contribution that normal stress makes to the load balance can be found by multiplying by the area of the conjunction, \( r_0^2 \sim R h_0 \). Alternatively we can just compare the normal stress with the Reynolds lubrication pressure. In the low Weissenberg number regime (\( \gamma \tau \lesssim 1 \)) the ratio of these two is

\[
\frac{N}{p} \sim \frac{\psi_0 \gamma^2}{\eta_0 U r_0 / h_0^2} \sim \gamma \tau \times \frac{h_0}{r_0},
\]

where the pressure has been taken from Eq. 2, and \( U \sim \gamma h_0 \) and \( \psi_0 \sim \eta_0 \gamma \) have been substituted in the final step. The last factor \( h_0 / r_0 \sim (h_0 / R)^{1/2} \) is expected to be a small number (e.g. \( \lesssim 0.1 \), say), so the ratio \( N / p \) is expected to remain small in the \( \gamma \tau \lesssim 1 \) regime. In particular normal stress should not affect the situation in the stable lubrication flow up until the loss of stability at the saddle-node bifurcation in Fig. 4. Therefore the overall picture shown in Fig. 4 remains unchanged.

In the high Weissenberg number regime (\( \gamma \tau \gtrsim 1 \)) one has \( Nr_0^2 \sim h_0^{\beta - 1} \). As long as \( \beta > 1 \) (which is the case for polymer melts), this is an increasing function of \( h_0 \), and the gap remains mechanically unstable in this regime. Only if \( \beta < 1 \) will normal stress rescue the conjunction from complete mechanical collapse by providing enough normal force to support the load in a stable sliding configuration. However the above argument shows this should not happen until well into the non-linear regime.

### C. Flow transience

Another factor that can be considered is that the flow is transient on a time scale of order the transit time \( t_s \sim r_0 / U \). The dimensionless ratio \( \tau / t_s \sim U \tau / r_0 \) is known as the Deborah number, and its magnitude gives an indication of the importance of transient flow effects, such as stress overshoot. Substituting \( U \sim \gamma h_0 \) shows that the Deborah number is of the order \( \gamma \times h_0 / r_0 \). This is the same as the ratio \( N / p \) for the normal stress above, and the same argument goes through so that the overall picture remains unchanged. Like normal stress, it is possible that transient flow effects may grow to become significant in the non-linear regime since the Deborah number scales as \( h_0^{-1/2} \). Hence this may provide another mechanism to re-stabilise the gap at a smaller distance.

### D. Extensional viscosity

In the Appendix, the ratio of the extension to the shear components in the lubrication flow is estimated to be of the order \( (h_0 / R)^{1/2} \). This is the same small number encountered above, and here should be compared to the

\[
N = \frac{\psi_0 \gamma^2}{1 + (\gamma \tau)^\beta}
\]
**Trouton ratio** (i.e., between extensional and shear viscosity). For example, for polymer melts the Trouton ratio is of the order 3 in the low Weissenberg number regime (where the gap is stable), and does not increase much in the non-linear regime \[14\]. On these grounds one would not expect extensional viscosity to affect the friction curve, but the situation may have to be re-evaluated if the Trouton ratio is large.

### IV. DISCUSSION

The main result is that hydrodynamic lubrication by non-Newtonian liquids which exhibit strong shear thinning, in the sense that \( \eta \sim \dot{\gamma}^{-\alpha} \) with \( \alpha > 1/2 \), is predicted to become mechanically unstable as the shear thinning regime is entered, with the surfaces closing to near contact. This would be marked by a discontinuous jump in the Stribeck (friction) curve as the load is increased (Fig. 4b). Non-linear effects such as normal stress and flow transience are estimated to be subdominant at the point of entry into the non-linear regime, so it is unlikely they would destroy the basic picture although they may provide stabilising mechanisms at high shear rates. Extensional viscosity is also estimated to be insignificant unless the Trouton ratio is large (say, \( \gtrsim 10 \)). Depending on the nature of the restabilising mechanism at small gaps, the Stribeck curve may show hysteresis (Fig. 5b).

These results were established on the basis of a scaling analysis which involves perhaps naive and rather drastic assumptions. For example, the lubrication flow is assumed to remain essentially the same as in the Newtonian case, but with a substituted shear-rate dependent viscosity \( \eta(\dot{\gamma}) \). If this breaks down the analysis would be invalid. Furthermore, in Eqs. (6) and (9), very simple models were taken for the shear viscosity and normal stress as a function of shear rate, and additivity of the normal stress and lubrication pressure was assumed. There are certainly available much more sophisticated (tensor) constitutive models, such as the Rolie-Poly equation for polymer melts \[10\]. A major and important avenue for further work is to investigate the properties of Reynolds lubrication flow using these constitutive models.

The possible hysteresis in the Stribeck curve makes an interesting connection with recent theories of the microscopic origins of shear thickening in non-Brownian suspensions \[17\] \[18\]. In these theories the shear thickening transition is associated with a ‘breakthrough’ to frictional contacts, and indeed the simulations in Fernandez et al. \[17\] are based on a discontinuous friction curve of exactly the form shown in Fig. 4b. Of course, the non-Brownian particles that show the phenomenon are more typically suspended in a Newtonian solvent, and not a structured liquid, and therefore one can question the relevance of the present analysis. Nevertheless, experimental systems are often prepared with polymer additives (as in Ref. \[17\]), and as such they may perhaps exhibit non-linear frictional behaviour of the kind described here if significant polymer adsorption occurs.

I thank M. J. Adams and S. A. Johnson for useful discussions.

### Appendix A: Aspects of Reynolds lubrication flow

Lubrication flow is a superimposition of Couette and Poiseuille contributions. For a conjunction between non-conformal surfaces, such as that shown in Fig. 1, the full solution requires numerics \[2\] but basic insights can be gained by considering the quasi-one-dimensional case \[10\]. In that case the flow is

\[
v_x = -\frac{U}{h}(h-y) - \frac{1}{2\eta} \frac{dp}{dx} y(h-y).
\]

Here \( x \) (cf. \( r \)) measures the distance along the gap, \( y \) measures the distance from the lower surface (moving at velocity \(-U\)), \( h \) is the gap, \( p \) is the lubrication pressure, and \( \eta \) the viscosity. This velocity field corresponds to a net material flux

\[
Q = \int_0^h v_x dy = -\frac{U h}{2} - \frac{h^3}{12 \eta} \frac{dp}{dx}.
\]

This has to be constant even though \( h \) varies, and is of order \( U h_0 \). The implication is that \( (h_0^3/12\eta) \frac{dp}{dx} \sim U h_0 \), which gives an order of magnitude estimate of the pressure gradient. The associated Poiseuille flow velocity (along the centerline for instance) is then \( (h_0^2/\eta) \frac{dp}{dx} \sim U \), as utilised in the main discussion.

The shear rate at the lower surface is

\[
\frac{\partial v_x}{\partial y} \bigg|_{y=0} = \frac{U}{h} - \frac{h}{2\eta} \frac{dp}{dx}.
\]

Since \( dp/dx \sim \eta U/h_0^2 \), the two terms are comparable, and the wall stress can be estimated by \( \eta U/h_0 \).

The flow field in Eq. (A1) is predominantly in the longitudinal direction. Differentiating with respect to this direction yields an estimate for the extensional component. This yields a number of terms, all of which are of the order \( U/h \times dh/\dot{x} \). Since \( h = h_0 + x^2/2R \) (cf. Eq. (1)), one has \( dh/\dot{x} = x/R \), and setting \( x \sim x_0 \sim (h_0 R)^{1/2} \) and \( h \sim h_0 \) gives \( U/x_0 \) for the magnitude of the extensional flow component. This agrees with the simple picture that entainment involves of the order 100% extensional strain, in a distance of the order \( x_0 \), on a time scale of the order the transit time \( x_0/U \). Hence the extensional strain rate \( U/x_0 \sim (U/h_0) \times (h_0/x_0) \sim \dot{\gamma} \times (h_0/R)^{1/2} \). This is the origin of the estimate used in the main text.
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