Abstract: Quantum communication is one of the cutting-edge research areas today, where the scheme of remote state preparation (RSP) has drawn significant attention of researchers. The authors propose here a hierarchical RSP protocol for sending a two-qubit entangled state using a seven-qubit highly entangled state derived from Brown et al. state. They have also studied the effects of two well-known noise models namely amplitude damping (AD) and phase damping (PD). An investigation on the variation of fidelity of the state with respect to the noise operator and receiver is made. PD noise is found to affect the fidelity more than the AD noise. Furthermore, the higher power receiver obtains the state with higher fidelity than the lower power receiver under the effect of noise. To the best of their knowledge, they believe that they have achieved the highest fidelity of the higher power receiver, 0.89 in the presence of maximum AD noise and 0.72 in the presence of maximum PD noise, compared to all the previously proposed RSP protocols in noisy environments. The study of noise is described in a very pedagogical manner for a better understanding of the application of noise models to a communication protocol.

1 Introduction

Remote state preparation (RSP) is considered to be a more efficient scheme of teleportation of a known state than the usual teleportation scheme as it involves a comparatively lesser number of classical bits [1]. Ever since the introduction of RSP, many different modified schemes of RSP [2–16] have been reported such as hierarchical RSP [13], joint RSP [14] etc. Researchers have also reported schemes such as controlled bidirectional RSP [15], hierarchical joint RSP [16]. Since no quantum channel is practically error-free, in general, every scheme works in a noisy environment. A detailed analysis of the effect of noise on a quantum channel has also been reported in the last few years [16, 17]. Motivated from these schemes, we have designed a protocol for hierarchical RSP of a two-qubit entangled state using a maximally entangled seven-qubit state obtained from Brown et al. state. The proposed scheme is also analysed in the presence of a noisy environment.

Hierarchical RSP (HRSP) refers to the RSP of a quantum state at the end of multiple receivers such that there exists a hierarchy among the receivers. The hierarchy decides which receiver would retrieve the quantum state more easily than the others. Briefly, we can describe our HRSP protocol as follows. Alice prepares a five-qubit maximally entangled state called Brown et al. state. Then with the addition of two ancilla qubits and by operating CNOT operation on them, she prepares a seven-qubit entangled state. Out of these seven entangled qubits, she keeps the first one with her and sends the rest to the others; two qubits to each of Bob, Charlie, and David. This entangled state can be represented now in such a fashion that to obtain the desired state, Bob would need to know the measurement outcomes of Alice and Charlie/David, whereas Charlie (David) would need to know the measurement outcomes of Alice, Bob, and David (Charlie). Once the receiver obtains the information about others’ measurement outcomes his task is to apply certain Pauli and CNOT operations to retrieve the desired state. Hence, it is easier for Bob to retrieve the information than Charlie/David. This forms the hierarchy between the two. The details of the protocol up until here are discussed in Section 2.

Practically, it is inevitable to do quantum communication in a noisy environment. Thus, we should also see how noise affects our HRSP protocol. In Section 3, we make a detailed discussion on how noise affects our quantum channel through which entanglement is shared. Finally, we give the explanations for the observed effects of noise and conclude in Section 4.

2 Scheme of HRSP using a seven-qubit entangled state

Following the convention in quantum communication, we assume that the participants involved in this scheme are Alice, Bob, Charlie, and David. Furthermore, Alice wants to prepare a known state remotely at Bob’s, Charlie’s, or David’s end. We can divide our scheme into the following steps.

2.1 Step 1. Preparation of the entangled state

Alice starts with a five-qubit entangled state called Brown et al. state, which is considered to be a maximally entangled state showing a high degree of entanglement. This state has the following form:

\[
|\psi\rangle_B = \frac{1}{\sqrt{2}} (|001\rangle|\phi^+\rangle + |010\rangle|\phi^-angle + |100\rangle|\phi^0\rangle + |111\rangle|\phi^+\rangle)_{AB,BC,C_2}
\]

(1)

where \(|\phi^\pm\rangle = (1/\sqrt{2})(|00\rangle \pm |11\rangle)\) and \(|\phi^0\rangle = (1/\sqrt{2})(|01\rangle \pm |10\rangle)\).

On expanding the state \(|\psi\rangle_B\) fully, we obtain a five-qubit cluster state containing eight terms

\[
|\psi\rangle_B = \frac{1}{2\sqrt{2}} (|000101\rangle + |001100\rangle + |101000\rangle + |010110\rangle + |100011\rangle + |001001\rangle + |111100\rangle + |111111\rangle)_{AB,BC,C_2}
\]

(2)
She then uses two ancilla qubits each prepared in state $|0\rangle$ and makes them entangled with $|\psi\rangle_{B_2}$ by applying two CNOT operations taking the fourth and fifth qubits of $|\psi\rangle_{B_2}$ as control qubits and the ancilla qubits as the target qubits.

$$|\Psi\rangle = \text{CNOT}(|\psi\rangle_{B_2} \otimes |00\rangle_{D_1D_2})$$  (3)

This is a very straightforward operation and after its execution, Alice finally prepares the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|0010101\rangle + |0111010\rangle + |0100000\rangle + |0010111\rangle + |1000101\rangle + |1001010\rangle)$$  (4)

We believe this prepared state is highly entangled, if not maximally entangled, because firstly it is prepared from a maximally entangled Brown et al. state, which itself incorporates two-qubit maximally entangled Bell states, and secondly it is prepared by the implementation of CNOT operation, which is believed to add entanglement to a state. After obtaining this state, Alice keeps the first qubit with herself and sends the second and third qubits to Bob, fourth and fifth qubits to Charlie, and sixth and seventh qubits to David through 'different quantum channels'. The word 'different' signifies some kind of separation as discussed in Section 3. This state is further used by Alice for the execution of HRSP of a particular two-qubit state to Bob or Charlie or David as described below.

### 2.2 Step 2. Factorisation of $|\Psi\rangle$ into different bases

Let us assume that Alice wants to communicate the state $|\xi\rangle = \alpha|00\rangle + \beta|11\rangle$, where the relation $|\alpha|^2 + |\beta|^2 = 1$ takes care of the normalisation of the state $|\xi\rangle$. The HRSP of known state $|\xi\rangle$ mainly requires the factorisation of $|\Psi\rangle$ in a particular fashion, which ensures that only the sender (in this case Alice) uses a basis constructed out of the known parameters, i.e. by using the following known state. It can be shown that $|\Psi\rangle$ can be written in the following fashion:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\xi_1\rangle |\xi_2\rangle + |\xi_1\rangle |\xi_3\rangle)_{AB,B_3,C_3,D_3,D_2}$$  (5)

where $|\xi_1\rangle_{AB,B_3,C_3,D_3,D_2}$ and $|\xi_2\rangle_{AB,B_3,C_3,D_3,D_2}$ are given as follows:

$$|\xi_1\rangle = \frac{1}{2} (|\alpha|01\rangle + |\beta|00\rangle)_{AB} |01\rangle_{C_3D_3} |01\rangle_{D_1D_2} + (|\beta|10\rangle - |\alpha|01\rangle)_{AB} |10\rangle_{C_3D_3} |10\rangle_{D_1D_2} + (|\alpha|10\rangle + |\beta|11\rangle)_{AB} |00\rangle_{C_3D_3} |00\rangle_{D_1D_2} + (|\alpha|11\rangle - |\beta|10\rangle)_{AB} |11\rangle_{C_3D_3} |11\rangle_{D_1D_2}$$  (6)

$$|\xi_2\rangle = \frac{1}{2} (|\beta|01\rangle - |\alpha|00\rangle)_{AB} |01\rangle_{C_3D_3} |01\rangle_{D_1D_2} - (|\alpha|00\rangle + |\beta|01\rangle)_{AB} |10\rangle_{C_3D_3} |10\rangle_{D_1D_2} + (|\beta|10\rangle - |\alpha|11\rangle)_{AB} |00\rangle_{C_3D_3} |00\rangle_{D_1D_2} - (|\alpha|11\rangle + |\beta|10\rangle)_{AB} |11\rangle_{C_3D_3} |11\rangle_{D_1D_2}$$  (7)

and further $|\xi_1\rangle$ and $|\xi_2\rangle$ (belonging to Alice) are simply

$$|\xi_1\rangle_{AB} = (|\alpha|00\rangle + |\beta|11\rangle)_A$$

$$|\xi_2\rangle_{AB} = (|\beta|01\rangle - |\alpha|00\rangle)_A$$  (8)

The subscripts A, B, C, D, $|\xi_1\rangle$ and $|\xi_2\rangle$ belonging to Alice, Bob, Charlie, and David, respectively. In a different fashion, $|\Psi\rangle$ can also be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\xi_1\rangle |\phi_1\rangle + |\xi_2\rangle |\phi_2\rangle)_{AB,B_3,C_3,D_3,D_2}$$  (9)

where $|\xi_1\rangle_{AC}$ and $|\xi_2\rangle_{AC}$ are the same as given in (8) (belonging to Alice) and $|\phi_1\rangle$ and $|\phi_2\rangle$ having the same meaning as mentioned previously.

### 2.3 Step 3. Measuring in specific basis and executing HRSP

In previous subsection, we can see two different factorisation of $|\Psi\rangle$ given by (5) and (9). Referring to the factorisation given in (5), Alice measures in $|\xi_1\rangle$ and $|\xi_2\rangle$ basis and Charlie and David measure in $|\xi_1\rangle$ and $|\xi_2\rangle$ basis then Bob can easily retrieve the state $|\phi_1\rangle$ by knowing the outcomes of Alice, Charlie, and David. For example, suppose Alice measures her qubit to be in state $|\xi_1\rangle$ and convey her results to Bob using some classical communication. After knowing Alice’s outcome, Bob figures
that his qubits along with Charlie's and David's have collapsed to the six-qubit state $\Xi_\psi$. Now, we can see that in both state $\Xi_\psi$ and state $\Xi_{\psi'}$ there is a direct correlation between Charlie's and David's measurement outcomes for all possibilities, i.e. every time Charlie and Bob would have the same measurement outcomes. Suppose now, Charlie measures his qubits to be in state $|01\rangle$ and conveys the result to Bob over some classical communication channel.

**Table 1** Operations that Bob have to operate on his qubits to retrieve state $|\xi_{\psi}\rangle$ depending on the outcomes of Alice and Charlie/David is listed here. It can be conspicuously seen that Charlie and David get the same outcome for their measurement each time. This is the reason it suffices for David to know the outcome of only one of the two parties, i.e. either Bob or Charlie in addition to the outcome of Alice. In other words, Bob would need the collaboration of only two parties (Alice plus Charlie/David). The notation $O_i$ (where $O \in \{H, X, iY, Z, CX, RCX\}$) indicates that Bob has to apply the operation $O$ on his first qubit and the notation $O_2$ indicates that Bob has to apply the operation $O$ on his second qubit. Similarly, $O_{1,2}$ indicates that Bob has to apply the operation $O$ on both his qubits. The operator $O_{1,1}$ indicates that Bob has to apply the operator $O$ taking the first one as the controlled qubit and the second one as the target qubit. CX and RCX refers to CNOT and ReverseCNOT operations, respectively.

| Alice's outcome | Charlie/David's outcome | Bob's operations |
|----------------|-------------------------|-----------------|
| $|01\rangle$   | $|01\rangle$            | $X_1CX_{1,2}$  |
| $|00\rangle$   | $|00\rangle$            | $Y_1CX_{1,2}$  |
| $|11\rangle$   | $|11\rangle$            | $-iY_1X_1CX_{1,2}$ |
| $|00\rangle$   | $|00\rangle$            | $iY_1X_1CX_{1,2}$ |
| $|01\rangle$   | $|01\rangle$            | $iY_1X_1CX_{1,2}$ |
| $|10\rangle$   | $|10\rangle$            | $-iY_1RCX_{1,2}$ |
| $|00\rangle$   | $|00\rangle$            | $Z_1X_1Z_1CX_{1,2}$ |
| $|11\rangle$   | $|11\rangle$            | $iY_1X_1CX_{1,2}$ |

**Table 2** Operations that David have to operate on his qubits to retrieve state $|\zeta_{\psi}\rangle$ when Alice gets the outcome $|\xi_{\psi}\rangle$ listed here. It can be clearly seen that Bob and Charlie get random outcomes for their measurements with respect to each other. This is the reason David would require the collaboration of all other parties unlike that in Table 1 to retrieve state $|\zeta_{\psi}\rangle$. The notations $O_1$, $O_2$, $O_{1,2}$, $O_{1,2,3}$, $O_{2,1,2}$, and $O_{1,2,3}$ have the same meaning as described in Table 1 with the only difference that here David would apply those operations instead of Bob.

| Alice's outcome | Bob/Charlie's outcome | David's operations |
|----------------|-----------------------|-------------------|
| $|1\rangle$    | $|1\rangle$           | $H_1X_1CX_{1,2}$  |
| $|0\rangle$    | $|0\rangle$           | $H_1Z_1RCX_{1,2}$ |
| $|1\rangle$    | $|1\rangle$           | $H_1Z_1RCX_{1,2}$ |
| $|1\rangle$    | $|1\rangle$           | $H_1Y_1CX_{1,2}$  |
| $|0\rangle$    | $|0\rangle$           | $H_1Y_1RX_{1,2}$  |
| $|1\rangle$    | $|1\rangle$           | $H_1Y_1RCX_{1,2}$ |
| $|0\rangle$    | $|0\rangle$           | $H_1X_1Z_1CX_{1,2}$ |
| $|1\rangle$    | $|1\rangle$           | $H_1X_1Z_1CX_{1,2}$ |
| $|0\rangle$    | $|0\rangle$           | $H_1X_1Z_1RCX_{1,2}$ |
| $|1\rangle$    | $|1\rangle$           | $H_1X_1Z_1RCX_{1,2}$ |

Knowing Charlie's outcome, Bob immediately gets to know that David would also get the same outcome as per the factorisation of state $\Xi_\psi$. This ensures Bob that he has got the state $a|01\rangle + \beta|00\rangle$ and using some simple one-qubit and two-qubit operations on it he can retrieve the required state $|\xi_{\psi}^B\rangle = a|00\rangle + \beta|11\rangle$. Thus, Bob needs to know the measurement outcomes of only two parties; Alice's outcome plus either Charlie's or David's outcome. The specific set of operations that Bob has to apply on his obtained state to retrieve $|\xi_{\psi}^B\rangle$ after knowing the outcomes from Alice and Charlie/David are given in Table 1.

Let us now refer to the factorisation given in (9). We can see that here, the state is finally retrieved by David. Suppose, Alice measures in the $\{|\zeta_{\psi}, \zeta_{\psi}'\rangle\}$ basis and finds her qubit to be in state $|\zeta_{\psi}\rangle$. Once David obtains Alice's measurement outcome over a classical channel he gets to know that his qubits along with Charlie's and David's have collapsed to the six-qubit state $|\psi_{\phi}\rangle$. However, we do not see any kind of correlation between the measurement outcomes of Bob and Charlie for states $|\phi_{\psi}\rangle$ and $|\psi_{\phi}\rangle$ (as we saw between Charlie and David for the states $|\xi_{\psi}\rangle$ and $|\xi_{\psi'}\rangle$). That means to figure out which state he has obtained, David needs the measurement outcomes of both Bob and Charlie. Suppose, Bob and Charlie individually measure in the two-qubit Hadamard basis and find their states to be $|++\rangle$ and $|+\rangle$, respectively. Once David receives this information over a classical channel from Bob and Charlie he gets to know that his state has collapsed to state $|\alpha|+\rangle - \beta|+\rangle$ as it can be seen from (10) and accordingly he will apply some simple one-qubit and two-qubit operations on his qubits to obtain the desired state $|\xi_{\psi}^B\rangle = a|00\rangle + \beta|11\rangle$. The exact set of operations David has to apply based on the outcomes $|\alpha\rangle$, Bob, and Charlie given in Tables 2 and 3. In a similar fashion, we can also factorise $|\phi_{\psi}\rangle$ and $|\psi_{\phi}\rangle$ in such a way that David receives the desired state for which he will have to know the outcomes of Alice, Bob, and David.

Thus, we see here that to retrieve the required state $|\xi_{\psi}^B\rangle$ Bob needs the information of only two people (Alice and Charlie/ David), whereas Charlie (David) needs the information of three people (Alice, Bob, and David (Charlie)). Thus it is easier for Alice to prepare a known state remotely at Bob's end than that in Charlie/ David's end. In other words, Bob here is the higher power receiver whereas Charlie and David are the lower power receivers having the same power. This forms a hierarchy between Bob, Charlie, and David, where Bob lies at the top level followed by Charlie and David lying in the same level. Hence, this scheme becomes an example of HRSP.

### 3 Effect of noise on the quantum channel

Practically, it is impossible to do quantum communication through a noiseless channel. At best what we can do is to study the sources and effects of noise and try to minimise them. We can conveniently study the effects of different kinds of noises for the HRSP of state $|\xi_{\psi}\rangle$ by studying the evolution of $p = |\Psi\rangle\langle\Psi|$ under the effects of different noise operators (more commonly called Kraus operators). For the sake of simplicity, we study here the effects of only two prominent kinds of noises namely amplitude damping (AD) and phase damping (PD). Fortunately, there exist the defined forms of noise operators for AD and PD noise for which we would not have to separately construct them here [18, 19]. The noise operators usually evolve $\rho$ in the following way:

$$\rho \rightarrow \sum K_i \rho K_i^\dagger$$

where $K_i$ is any particular noise operator. The noise operators are constructed following the operator-sum representation and the set of noise operators, $(K_\alpha, K_\gamma)$ for AD can be given as [16, 17]

$$K_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\eta_\alpha} \end{bmatrix} \quad K_\gamma = \begin{bmatrix} 0 & \sqrt{\eta_\alpha} \\ 0 & 0 \end{bmatrix}$$

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51
where \( \eta_A \) describes the probability of error due to AD noise. Similarly, the set of noise operators, \( \{ E_\alpha, E_\beta, E_\gamma \} \) for PD can be given as [16, 17]

\[
E_\alpha = \begin{pmatrix} 1 - \eta_p & 0 \\ 0 & 1 - \eta_p \end{pmatrix}, \quad E_\beta = \begin{pmatrix} \sqrt{\eta_p} & 0 \\ 0 & \sqrt{\eta_p} \end{pmatrix}, \quad E_\gamma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

(14)

where \( \eta_p \) describes the probability of error due to PD noise. Furthermore, \( \eta_A \) and \( \eta_p \) can take values between 0 and 1. Next, we see the effects of AD noise and PD noise one by one.

### 3.1 Effect of AD noise

Referring to (12) we can describe the evolution of \( \rho \) under the effect of AD noise as follows:

\[
\rho' = \sum_{i,j} (i^2 \otimes K_{Ri} \otimes K_{Rj} \otimes K_{Cj} \otimes K_{Dj}) \times \rho (i^2 \otimes K_{Ri} \otimes K_{Rj} \otimes K_{Cj} \otimes K_{Dj})^\dagger
\]

(15)

Clearly, \( \rho \) is a 128 \( \times \) 128 (or \( 2^7 \times 2^7 \)) matrix as \( |\Psi\rangle \) defines a seven-qubit system. Accordingly, we have seven one-qubit operators on the left and seven on the right of \( \rho \). The superscripts on the noise operators denote the particular qubit on which the operator would operate. Furthermore, we should be clear that \( B_i \) and \( B_j \) denote the two qubits of Bob, \( C_i \) and \( C_j \) denote the two qubits of Charlie and David, and \( D_i \) denote the two qubits of David. The subscript, however, denotes the two different AD operators, i.e., \( i, j \in \{0, 1, \ldots, 7\} \). It is assumed that Alice prepares the entangled state and then distributes the qubits to Bob, Charlie, and David through some quantum channels. Thus, we can practically assume that the noise would affect all qubits except for Alice's qubit. Furthermore, assuming that the two qubits belonging to the same receiver are communicated through the same quantum channel, we believe that they are affected by the same noise operators and hence we put the same index on the subscript of noise operators that act on the qubits of the same receiver. For example, the two noise operators \( K_{Ri} \) and \( K_{Rj} \) acting on the two qubits of Bob have the same subscript ‘i’. Following similar argument, the noise operators acting on the qubits of two different receivers bear a different index on their subscripts.

We then compute \( \rho' \) using MATLAB and obtain a 128 \( \times \) 128 matrix, which turns out to be a function of \( \eta_A \). It is important here to notice that we take the noise parameters to be the same for different channels while doing the computation. This is practically inaccurate for real environments but we have taken the assumption to simplify the calculation drastically. Certainly, now \( \rho' \) is the state, which evolves as per the measurements made by Alice, Bob, and Charlie as depicted by the protocol in Section 2 to some state \( \rho'' \) (say). We construct this unitary evolution operator by carefully choosing the measurement operators that would act on each qubit.

Hence

\[
\rho'' = U\rho' U^\dagger
\]

(16)

where \( U \) is the following operator:

\[
U = M_A \otimes M_{Bj} \otimes M_{Bi} \otimes M_{Ci} \otimes M_{Di} \otimes M_{Dj}
\]

(17)

with \( M_x \) representing the measurement operator that would act on qubit \( X \). After obtaining \( \rho'' \), we have to normalise it, just the way states are normalised after measurement. Thus, we obtain our normalised \( \rho''' \) (say, \( \rho'' \)) by dividing it by the trace of \( \rho'' \)

\[
\rho''' = \frac{\rho''}{\text{Tr}(\rho'')} = \frac{U\rho' U^\dagger}{\text{Tr}(U\rho' U^\dagger)}
\]

(18)

\( \rho''' \) itself is a 128 \( \times \) 128 matrix and to know what state the receiver gets we must reduce it to a \( 4 \times 4 \) matrix (since the receiver obtains a two-qubit state here). If Bob is the receiver then we have to trace out the qubits of Alice, Charlie, and David (corresponding to \( A, C_i, D_i, D_j \))

\[
\rho_{Bob} = Tr_{A,C_i,D_j,D_i}(\rho''')
\]

(19)

Once the receiver receives this state, he uses the appropriate unitary operator, say \( O \), to obtain the desired state \( \xi \)

\[
\rho_0 = O\rho_{Bob} O^\dagger
\]

(20)

Hence, \( \rho_0 \) is the state received through the noisy channel. As noise here is parameterised by \( \eta_A \) we can see \( \rho_0 \) as a function of \( \eta_A \). Now, to see the effect of noise on our desired state \( \rho_{\xi} = |\xi\rangle\langle\xi| \) we calculate the fidelity between \( \rho_0 \) and \( \rho_{\xi} \) since fidelity would show how close is \( \rho_0 \) to \( \rho_{\xi} \).

\[
F = \text{Tr}(\sqrt{\sqrt{\rho_0} \rho_{\xi} \sqrt{\rho_0}})
\]

(21)

Let us explicitly calculate the fidelity for a particular set of measurements. Let us consider the first set of measurements (the first line in the table) from Table 1, where Bob is the receiver of the desired state. It should be noted that for AD, \( \rho' \) remains the same for all the sets of measurements given in Tables 1–3. However, the operators \( U \) and \( O \) differ from set to set. The measurement operators of Alice, Charlie, David, and Bob to be used in (10) can be given as

Table 3 Operations that David have to operate on his qubits to retrieve state \( \xi \) when Alice gets the outcome \( \xi \) listed here. Similar to our observation in Table 2, Bob and Charlie get random outcomes for their measurements with respect to each other and consequently David would require the collaboration of all other parties to retrieve state \( \xi \). The notations \( O_i, O_{\alpha}, O_{\beta}, O_{\gamma}, CX \) and \( RCX \) have the same meaning as described in Table 2

| Alice’s outcome | Bob/Charlie’s outcome | David’s operations |
|----------------|----------------------|--------------------|
| \( \xi_2 \) | \( O_1 + O_0 \) \( O_1 + O_0 \) \( O_1 + O_0 \) | \( H_1.XZ.CX_{1-2} \) |
| \( \xi_3 \) | \( O_1 - O_0 \) \( O_1 + O_0 \) \( O_1 - O_0 \) | \( H_1.X1.CX_{1-2} \) |
| \( \xi_4 \) | \( O_1 - O_0 \) \( O_1 + O_0 \) \( O_1 - O_0 \) | \( H_1.X1.CX_{1-2} \) |
| \( \xi_5 \) | \( O_1 + O_0 \) \( O_1 + O_0 \) \( O_1 + O_0 \) | \( H_1.X1.CX_{1-2} \) |
| \( \xi_6 \) | \( O_1 + O_0 \) \( O_1 + O_0 \) \( O_1 + O_0 \) | \( H_1.X1.CX_{1-2} \) |
| \( \xi_7 \) | \( O_1 + O_0 \) \( O_1 + O_0 \) \( O_1 + O_0 \) | \( H_1.X1.CX_{1-2} \) |
| \( \xi_8 \) | \( O_1 - O_0 \) \( O_1 + O_0 \) \( O_1 - O_0 \) | \( H_1.X1.CX_{1-2} \) |
| \( \xi_9 \) | \( O_1 - O_0 \) \( O_1 + O_0 \) \( O_1 - O_0 \) | \( H_1.X1.CX_{1-2} \) |
| \( \xi_{10} \) | \( O_1 + O_0 \) \( O_1 + O_0 \) \( O_1 + O_0 \) | \( H_1.X1.CX_{1-2} \) |
| \( \xi_{11} \) | \( O_1 + O_0 \) \( O_1 + O_0 \) \( O_1 + O_0 \) | \( H_1.X1.CX_{1-2} \) |
| \( \xi_{12} \) | \( O_1 + O_0 \) \( O_1 + O_0 \) \( O_1 + O_0 \) | \( H_1.X1.CX_{1-2} \) |
To simplify the calculations, we have taken $\alpha = \beta = 1/\sqrt{2}$ in $M_A = (\alpha|0\rangle + \beta|1\rangle)(\alpha^*|0\rangle + \beta^*|1\rangle)$ in (15). Furthermore, we can see that Bob's measurement operator is an identity matrix. This is because, while the state is evolving under the noise Bob performs no measurement. More precisely, Bob performs a local measurement only after he gets to know the outcomes of others, i.e. only after tracing out the qubits of Alice, Charlie, and David. Consequently, we take Bob's measurement into consideration only while calculating $\rho_D$. For the particular set of measurements that we have considered, Bob's measurement operator ($O$) as used in (20) becomes

$$O = M_{B_0} \otimes M_{B_1} = CX_{2\ldots4} (I \otimes X) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}$$

Using the operators given in (22) we first calculate $U$ as per (17) and then $\rho''$ as per (18). Then we trace out the qubits corresponding to $A$, $C_\alpha$, $C_\beta$, $D_\alpha$, $D_\beta$ using a MATLAB programme to find out $\rho^{Bob}$ which is a $4 \times 4$ matrix. After that using the operator $O (a 4 \times 4$ matrix as we would expect it to be multiplied with $\rho^{Bob}$) we calculate $\rho_D$ as per formula (20). We have taken $\alpha = \beta = 1/\sqrt{2}$, thus the intended state that Bob should obtain in the absence of noise is

$$\rho_D = \frac{1}{\sqrt{2}}\langle 00 \rangle + \langle 11 \rangle \frac{1}{\sqrt{2}}(\langle 00 \rangle + \langle 11 \rangle)$$

$$\rho_D = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

Then, finally, we calculate the fidelity by plugging in $\rho$ and $\rho_D$ in (21). We obtain fidelity as a function of $\eta_A$. Furthermore, as $\eta_A$ can take values between 0 and 1 only we calculate and plot fidelity as a function of $\eta_A$ by varying $\eta_A$ in steps of 0.1. This plot is shown in Fig. 1. To see the effect of noise based on the power of the receiver we calculate the fidelity once taking Bob (higher power receiver) as the receiver using the first set of measurements given in the top line of Table 1, and once taking David (lower power receiver) as the receiver using the first set of measurements given in the top line of Table 2. The behaviour of Bob's and David's fidelities with $\eta_{AD}$ is depicted by the blue line and red line in Fig. 1, respectively. It can be understood that the behaviour of fidelities with $\eta_{AD}$ would be the same for other sets of measurements of the particular table as long as we calculate it for the same receiver.
behaviour of Bob's and David's fidelities with \( \eta_{\text{hrsp}} \) is depicted by the blue line and red line in Fig. 1, respectively. The behaviour of fidelities with \( \eta_{\text{hrsp}} \) would be the same for the other sets of measurements of the particular table as long as we calculate it for the same receiver.

4 Discussion

We have proposed here a scheme for the HRSP of a two-qubit entangled state using a seven-qubit entangled state generated from Brown et al. state. Emphasis has been placed on describing a generic HRSP protocol in the most pedagogic way for a better understanding of the protocol. We believe the way we have presented the protocol here would help readers to come up with the HRSP of even more complicated states (involving a maximum number of qubits, say \( n \)) using the factorisation of a multipartite-entangled state involving a maximum number of qubits, say \( n \). The efficiency of the protocol would depend upon the ratio \( n/m \); higher being more efficient.

We have also shown the effects of AD and PD noises on our protocol considering the noise to only affect the travelling qubits (the ones Alice sends to others). From the two figures, it can be seen that the least value of fidelity for AD noise is 0.73 and for PD noise is 0.5. There are publications in the recent past where people have reported a variety of RSP protocols. These protocols are also analysed in noisy channels wherein the fidelities have been plotted [16, 17, 20–26]. Our approach, in comparison with the aforementioned protocols appears to be giving better fidelity with the effects of AD and PD. This proves our scheme to be quite tractable for practical implementation. We believe, the reason behind the higher fidelity is the highly entangled seven-qubit state that we have taken. It would be interesting to know the exact amount of entanglement present in this seven-qubit state which is generated from the maximally entangled Brown et al. state. We would like to leave it to the future researchers to measure the amount of entanglement (in any plausible measure of entanglement) in this seven-qubit entangled state. We have seen that the PD noise has more effect on the fidelity than the AD noise.

Another important conclusion from our study of noise is that the higher power agent (Bob here) would receive a high fidelity state than the low power agent (David here) as it can be very conspicuously seen in Figs. 1 and 2. However, to generalise this observation to all different kinds of noises and all different kinds of hierarchical protocols, we need a much detailed study of different kinds of hierarchical protocols in different noisy environments. Thus, we feel that further research in this area may lead to more concrete results which would help to design more efficient protocols for practical implementations in the future.

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