One family, six distributions – A flexible model for insurance claim severity

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Abstract

We propose a new class of claim severity distributions with six parameters, that has the standard two-parameter distributions, the log-normal, the log-Gamma, the Weibull, the Gamma and the Pareto, as special cases. This distribution is much more flexible than its special cases, and therefore more able to capture important characteristics of claim severity data. Further, we have investigated how increased parameter uncertainty due to a larger number of parameters affects the estimate of the reserve. This is done in a large simulation study, where both the characteristics of the claim size distributions and the sample size are varied. We have also tried our model on a set of motor insurance claims from a Norwegian insurance company. The results from the study show that as long as the amount of data is reasonable, the five- and six-parameter versions of our model provide very good estimates of both the quantiles of the claim severity distribution and the reserves, for claim size distributions ranging from medium to very heavy tailed. However, when the sample size is small, our model appears to struggle with heavy-tailed data, but is still adequate for data with more moderate tails.

Keywords: Automization, extended Pareto, power transformation, reserve estimation, unimodal families

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1. Introduction

The Burr family of loss distributions goes back to Burr (1942) and is also called extended or generalized Pareto and Beta prime. Its probability density functions are all unimodal (or monotonically decreasing). Further, there are three parameters which capture distributions of widely different shapes that range from heavy-tailed Paretos to light-tailed Gammas. The latter are on the boundary of the parameter space and are only reached as two of the parameters approach infinity. An even more versatile parametric family of distributions is proposed in this paper by applying parametrised power transformations to Burr-distributed random variables. We have used the classical BoxCox approach in Box and Cox (1964) with the modification that there are two free parameters. The resulting PowerBurr family has five parameters. It does again lead to unimodal probability density functions, and includes ten of the most commonly applied and quoted loss models in actuarial science, for example the log-normal and the Weibull in addition to the parent Burr distribution itself. This will be verified in the next section where an even wider class with a sixth parameter also will be introduced.

Constructions such as the preceding one raise the intriguing question of whether loss modelling can be carried out by fitting one of these many-parameter families and simply use the distribution found without any more ado, thus avoiding an additional model selection step. In situations where an underlying heterogeneity of unknown or unquantifiable source that cannot be linked to an observed covariate and expressed into a regression relationship, more complex modelling may be needed. In such cases the natural model is a mixture distribution as in Lee and Lin (2010), Bakar et al. (2015) and Miljkovic and Grün (2016), or even a traditional kernel density estimate based on, say Gaussian mixtures, as in Scott (1992). Yet, the most common of all types of variations is undoubtedly the unimodal one, and here PowerBurr fitting is an alternative to the traditional approach of trying different two-parameter families and choosing between them by Q-Q plotting, formal selection criteria like AIC or BIC or goodness-of-fit testing.

Our aim is not to decide whether PowerBurr fitting is superior or inferior to such methods in terms of the error in the final solution. However, we will argue that its conceptual simplicity is attractive and offers a useful potential for automating the entire process from historical data to statements of risk in the computer. With a standard Poisson model for claim frequency, 99% or 99.5% solvency capital may be be evaluated by Monte Carlo, after estimating
claim frequency and fitting loss data. The entire process may then be carried out with no or almost no human intervention at all. Such a program is not hard to implement in the computer and would draw on the simple algorithm for simulating PowerBurr variables in Section 2, but it does require a robust numerical procedure for estimating the parameters, which is a challenge. The criterion might be quite flat in some of the parameters, perhaps with multiple optima which are important practical obstacles to be addressed below.

A flat likelihood function means that many alternative distributions are almost equally likely given historical losses. Loss distributions are in reality no more than tools for evaluating risk. Whether they reduce to simple families or not and the interpretation of their parameters may not necessarily be issues of primary importance. However, what is important is that the tail behaviour may be quite different for distributions that are almost equally well-fitting in the central domain containing most of the data. A traditional attempt to deal with this is extreme value mixing, drawing on the result due to Pickands (1975) that over-threshold distributions always become Pareto or exponentially distributed as the threshold becomes infinite; consult Embrechts et al. (1997) for a review of such methods. A more recent contribution to such modelling is Lee et al. (2012). PowerBurr fitting may be an alternative even here, perhaps by replacing the likelihood by a criterion that emphasizes more strongly the fit in the extreme right tail, for example using a weighted likelihood approach. This will however not be studied here. What will be investigated, is how errors in statements of risk, like the solvency capital, are linked to estimation errors in the PowerBurr family parameters. In particular, estimation uncertainty will increase with the number of parameters, which in the PowerBurr is quite large, as the amount of available data decreases. An extensive numerical experiment will be conducted, where the error in 99% solvency capital will be examined. <Litt mer her>.

2. The PowerBurr family

2.1. Definition

The most convenient route to the Burr family is to start with Gamma variables $G_\alpha$ and $G_\theta$ with expectation 1 and shape parameters $\alpha > 0$ and $\theta > 0$. Their standard deviations are $\text{sd}(G_\alpha) = 1/\sqrt{\alpha}$ and $\text{sd}(G_\theta) = 1/\sqrt{\theta}$ from which it follows that $G_\alpha \xrightarrow{p} 1$ and $G_\theta \xrightarrow{p} 1$ as $\alpha \to \infty$ and $\theta \to \infty$. The
Figure 1: Examples of PowerBurr probability density functions. The parameter vectors are \( \Phi_1 = (5, 1.5, 1, 1, 1) \), \( \Phi_2 = (5, 2, 0.75, 1, 1, 1) \), \( \Phi_3 = (1000, 4, 1.5, 1, 1, 1) \), \( \Phi_4 = (1000, 4, 1000, 1000, 1.5, 1) \), \( \Phi_5 = (1000, 4, 1, 3250, 1000, 1) \), \( \Phi_6 = (1e6, 1000, \exp(-10.6), 1, 10, 100) \), \( \Phi_7 = (1000, 1, 0.4, 2, 1, 1) \), \( \Phi_8 = (5, 2, 1.5, 2, 16, 1.2) \) and \( \Phi_9 = (5, 4, 1.8, 2, 16, 1.2) \).

The probability density function of \( X \) is given by

\[
g(x) = \frac{\Gamma(\alpha + \theta)}{\Gamma(\alpha) \Gamma(\theta)} \left( \frac{\alpha}{\theta} \right)^\alpha \frac{x^{\theta-1}}{(\alpha/\theta + x)^{\theta + \alpha}}, \quad x > 0
\]

where \( \Gamma(\cdot) \) is the Gamma function. That was how the Burr family was defined originally in [Burr (1942)]. The proof of (2) is elementary; consult Section 9.7 in [Bølviken (2014)].

For the Burr model to be workable in practice it must be augmented with a parameter of scale \( \beta > 0 \) so that it becomes \( Z = \beta X \). Our proposal is to

\[
X = \frac{G_\theta}{G_\alpha}
\]
replace this relationship with the more general

\[ Z = \beta \{(1 + X/\tau)^\gamma - 1\} \quad \text{or even} \quad Z = \beta \{(1 + X^\eta/\tau)^\gamma - 1\} \quad (3) \]

with \( \gamma > 0, \tau > 0 \) and \( \eta > 0 \) as additional parameters. The version on the left with parameter vector \( \Phi = (\alpha, \theta, \beta, \tau, \gamma) \) will be referred to as PowerBurr\(5\), and represents the main thrust of this paper. The other, PowerBurr\(6\), has \( \eta \) as a sixth parameter, and now \( \Phi = (\alpha, \theta, \beta, \tau, \gamma, \eta) \). In either case \( Z > 0 \), and pure Burr reappears when \( \gamma = \tau = \eta = 1 \).

\[ \gamma > 0, \tau > 0 \text{ and } \eta > 0 \text{ as additional parameters.} \]

\[ \Phi = (\alpha, \theta, \beta, \tau, \gamma) \]

\[ \Phi = (\alpha, \theta, \beta, \tau, \gamma, \eta) \]

\[ \tau = \gamma \text{ and } \eta = 1. \]

The construction is inspired by the classical power transformation in Box and Cox (1964), but there \( \tau = \gamma \).

2.2. Properties

Unimodality The power transformation in (3) is strictly increasing everywhere. This is however not a guarantee that when one uses it to transform a random variable, the probability density function of the new variable has a shape suitable for modelling. Yet, that is largely the case here, as the following proposition shows.

**Proposition 1.** (i) All PowerBurr\(5\) and PowerBurr\(6\) distributions are unimodal if \( \gamma \geq 1 \). (ii) The PowerBurr\(5\) distributions are unimodal if \( \max(\theta, \gamma) \geq 1 \).

The calculations when the proposition is proved in Section 2.4 shows that the precise condition for PowerBurr\(5\) distributions being unimodal is

\[ \theta \geq 1 \quad \text{or} \quad \alpha(1 - \gamma/\theta) - \tau(1 + \alpha) \leq 2\sqrt{|1 - \theta|\tau(\alpha + \gamma)\alpha/\theta} \quad (4) \]

which may be too complicated to be of much practical value. If this fails to hold, the density function has both a local minimum and a maximum. Examples of probability density functions and their shapes are offered in Figure 1.

Moments and percentiles Moments do not always exist. It emerges from expression (2) that \( E(X^r) \) is finite if \( r < \alpha \), and since \( X^{\eta\gamma} \) is the leading term in (3), the condition for \( E(Z^r) \) being finite is \( r\eta\gamma < \alpha \). Closed mathematical expressions for these moments are not available unless \( \theta = 1 \), and numerical methods are needed. Standard one-dimensional quadrature using (2) is straightforward, and so is Monte Carlo (see below). Similarly, the percentiles of \( X \) lack closed formulae unless \( \theta = 1 \), and again numerical methods

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have to be used.

**Sampling** It is easy to draw from a PowerBurr distribution when a Gamma sampler is available. One simply generates Monte Carlo variates of $G_\theta$ and $G_\alpha$, compute their ratio $X$ and inserts that into (3).

**Special cases** Most standard loss distributions in property insurance are included as special cases, which is summarized in the following proposition, verified in Section 2.4.

**Proposition 2.** The PowerBurr family contains the Burr, Pareto, Gamma, Inverse Gamma, Log-gamma, Logistic, Log-logistic, Weibull, Fréchet, and the Log-normal families when the parameters $\Phi = (\alpha, \theta, \beta, \tau, \gamma)$ take the values shown in Table 1.

Nine of the ten families in Table 1 are defined in terms of shape parameters $a$ and $c$ and a parameter of scale $b$, some of them through stochastic representation, others through the survival function $\bar{F}(z) = \Pr(Z > z)$. Only the definition of the Log-normal is different. The table shows how $\alpha$, $\theta$, $\beta$, $\tau$ and $\gamma$ must be defined for the various special cases to appear. Many of them are located at the boundary of the parameter space and are only reached in the limit. As an example consider the Log-gamma family on row 5 which appears when $\tau = \gamma/b$, $\gamma \to \infty$ and $\alpha \to \infty$. This means that if PowerBurr is fitted Log-gamma losses with enough data, the computer will return large values for all three of $\tau$, $\gamma$ and $\alpha$ with the scale parameter $b$ approximately the ratio $\gamma/\tau$.

**Identifiability** A stochastic model is identifiable if different parameter vectors $\Phi$ always lead to different distributions. That is not the case here. For example, if $\gamma = 1$, (3) shows that $\beta$ and $\tau$ affect $Z$ through their ratio $\beta/\tau$. It follows that if $\gamma = 1$, the likelihood method in the next section will pick a pair $(\beta, \tau)$ with the right ratio. Which one does not matter since we in applications rarely are interested in the parameters themselves.

**2.3. Fitting**

Models have in this paper been fitted by maximum likelihood, which requires a mathematical expression for the probability density function $f(z; \Phi)$ of $Z$. Using the logarithmic form in (8) below and inserting (2) and (3),
For the log-normal limit to be true $\alpha$ must tend to $\infty$ faster than $\theta$ in the sense that $\theta/\alpha \to 0$.

| Definition | $\alpha$ | $\theta$ | $\tau$ | $\gamma$ | $\beta$ |
|------------|---------|---------|-------|-------|-------|
| 1 Burr     | $Z = bG_c/G_a$ | $a$ | $c$ | $1$ | $1$ | $b$ |
| 2 Pareto   | $\bar{F}(z) = 1/(1 + z/b)^a$ | $a$ | $1$ | $1$ | $1$ | $b$ |
| 3 Gamma    | $Z = bG_c$ | $\to \infty$ | $c$ | $1$ | $1$ | $b$ |
| 4 Inverse Gamma | $Z = b/G_a$ | $a$ | $\to \infty$ | $1$ | $1$ | $b$ |
| 5 Log-gamma | $\log(1 + Z) = bG_c$ | $\to \infty$ | $c$ | $\gamma/b$ | $\to \infty$ | $1$ |
| 6 Logistic | $\bar{F}(z) = 2/(1 - a + ae^{z/b})$ | $1$ | $\to \infty$ | $a$ | $\to 0$ | $b/\gamma$ |
| 7 Log-logistic | $\bar{F}(z) = 1/[1 + (z/b)^a]$ | $\to \infty$ | $1$ | $\to 0$ | $1/a$ | $b\tau^\gamma$ |
| 8 Weibull  | $Z = bG^a_1$ | $1$ | $\to \infty$ | $\to 0$ | $a$ | $b\tau^\gamma$ |
| 9 Fréchet  | $\bar{F}(z) = 1 - e^{-(z/b)^a}$ | $\to \infty$ | $1$ | $\to 0$ | $a$ | $b\tau^\gamma$ |
| 10 Log-normal* | $\log(Z) \sim N(\xi, \sigma)$ | $\to \infty$ | $\to \infty$ | $\sqrt{\theta\sigma}$ | $\theta\sigma^2$ | $e^{-\sqrt{\theta\sigma}+\xi+1/2}$ |

Table 1: Special cases of PowerBurr showing how its parameters are defined in terms of shape ($a$ and $c$) and scale ($b$) of the parent distribution with the log-normal defined differently.

Straightforward calculations lead to

$$
\log\{f(z; \Phi)\} = \log\{C(\Phi)\} + (\theta - \eta)\log(x) - (\alpha + \theta)\log(\alpha/\theta + x)
+ (\gamma - 1)\log(1 + x^\eta/\tau)
$$

(5)

where

$$
C(\Phi) = \frac{\Gamma(\alpha + \theta)\tau(\alpha/\theta)^a}{\Gamma(\alpha)\Gamma(\theta)\gamma\beta\eta}
$$

and

$$
x = \{(z/\beta + 1)^{1/\gamma} - 1\}^{1/\eta}.
$$

(6)

The relationship between $z$ and $x$ is the inverse of (3). When historical losses $z_1, \ldots, z_n$ are given, $\log\{f(z_i; \Phi)\}$ must be computed for each observation and added for the log-likelihood function

$$
\mathcal{L}(\Phi; z_1, \ldots, z_n) = \log\{f(z_1; \Phi)\} + \cdots + \log\{f(z_n; \Phi)\}
$$

(7)

Optimization was in this paper carried out by the quasi-Newton option in the R-function optim(). This is straightforward for two-parameter models, but more challenging when there are five or even six parameters and a log-likelihood function that may be quite flat, especially when $n$ is moderate. This means that widely different parameter values result in rather similar distributions. Consequently, the optimisation of the likelihood is sometimes challenging, but this is not that problematic when the resulting distributions

* For the log-normal limit to be true $\alpha$ must tend to $\infty$ faster than $\theta$ in the sense that $\theta/\alpha \to 0$. 

are sensible. Still, we have derived the derivatives of the log-likelihood functions with respect to all parameters, and supply those to \texttt{optim()}. These are given in the Appendix. To ease the optimisation further, it is performed on the log-transformed parameters as all the parameters $\alpha, \theta, \beta, \eta, \tau, \gamma$ are positive. Finally, we do the optimisation with several sets of start values for the parameters, and choose the parameter estimates that give the highest likelihood value. More numerical details are given in Appendix 2.

2.4. Proofs of propositions

Proof of Proposition 1. It is convenient to drop the parameter vector and let $f(z)$ be the probability density function for $Z$ rather than $f(z; \Phi)$ as above. Its relationship to the probability density $g(x)$ for $X$ is through

$$ f(z) = g(x) \frac{dx}{dz} $$

and since $dx/dz = (dz/dx)^{-1}$, it follows that

$$ \log \{ f(z) \} = \log \{ g(x) \} - \log \left( \frac{dz}{dx} \right) \quad (8) $$

so that

$$ \frac{d \log \{ f(z) \} }{dz} = \frac{dx}{dz} \left( \frac{d \log \{ g(x) \} }{dx} - \frac{d \log (dz/dx)}{dx} \right) \quad (9) $$

Recall that $g(x) = c x^{\theta-1}/(\alpha/\theta + x)^{\theta+\alpha}$ where $c$ is a constant. Hence

$$ \frac{d \log \{ g(x) \} }{dx} = \frac{\theta - 1}{x} - \frac{\theta + \alpha}{\alpha/\theta + x} \quad (10) $$

and moreover $z = \beta \{ (1 + x^n/\tau)^\gamma - 1 \}$ so that

$$ \frac{dz}{dx} = \frac{\beta \gamma \eta}{\tau} (1 + x^n/\tau)^{\gamma-1} x^{n-1} $$

and

$$ \log \left( \frac{dz}{dx} \right) = \log \left( \frac{\beta \gamma \eta}{\tau} \right) + (\gamma - 1) \log(1 + x^n/\tau) + (\eta - 1) \log(x). $$

Differentiating this yields

$$ \frac{d \log (dz/dx)}{dx} = \frac{(\gamma - 1) \eta x^{n-1}/\tau}{1 + x^n/\tau} + \frac{\eta - 1}{x} \quad (11) $$


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which is with (10)) to be inserted into (9). Then
\[
\frac{d \log \{ f(z) \}}{dz} = \frac{dx}{dz} \left( \frac{\theta - \eta}{x} - \frac{\theta + \alpha}{\alpha/\theta + x} + \frac{(1 - \gamma)\eta x^{\gamma-1}/\tau}{\tau + x^{\eta}} \right),
\]
and after some straightforward manipulations this can be rewritten
\[
\frac{d \log \{ f(z) \}}{dz} = \frac{dx}{dz} \frac{P(x)}{x(\alpha/\theta + x)(\tau + x^{\eta})}
\]
where
\[
P(x) = -(\alpha + \gamma \eta)x^{\eta+1} + \alpha(1 - \gamma \eta/\theta)x^{\eta} - \tau(\eta + \alpha)x + \tau(\theta - \eta)\alpha/\theta.
\]
The sign of the derivative is determined by \(P(x)\) since the all other factors are positive. A more useful form is
\[
P(x) = \alpha(x^{\eta} + \tau)(1 - x) - \eta(\gamma x^{\eta} + \tau)(x + \alpha/\theta)
\]
which is established by multiplying out both factors and collecting the terms. A solution of the equation \(P(x) = 0\) must therefore satisfy
\[
\frac{\gamma x^{\eta} + \tau}{x^{\eta} + \tau} = \alpha \frac{1 - x}{x + \alpha/\theta}.
\]
(12)
When \(\gamma \geq 1\), the left hand side of (12) is monotonically increasing in \(x\) whereas the right hand side is monotonically decreasing, and there is at most one solution to the equation. This means that \(d \log \{ f(z) \}/dz\) has zero or one root, and \(f(z)\) is monotonically decreasing everywhere or has a single mode. The latter applies when the maximum of the right hand side of (12) is larger than the minimum on the left, i.e. when \(\theta > \eta\). This verifies the first part of Proposition 1.

To address the second part concerning PowerBurr5 let \(\eta = 1\). Now \(P(x)\) becomes
\[
P(x) = -(\alpha + \gamma)x^{2} + \{\alpha(1 - \gamma/\theta) - \tau(1 + \alpha)\}x + \tau(\theta - 1)\alpha/\theta.
\]
which is a second order polynomial with negative quadratic term. Such functions have a single maximum, and if \(\theta \geq 1\) so that \(P(0) \geq 0\), there can be no more than one solution of the equation \(P(x) = 0\) so that unimodality
still holds. The condition can be expressed as \( \max(\theta, \gamma) \geq 1 \), as claimed in part (ii) of Proposition 1.

To derive the precise algebraic condition \( \underline{4} \) we must examine \( P(x) \) in detail. Elementary calculations show that its maximum occurs at

\[ x_m = \frac{\alpha(1 - \gamma / \theta) - \tau(1 + \alpha)}{2(\alpha + \gamma)} \]

with maximizing value

\[ P(x_m) = \frac{\{\alpha(1 - \gamma / \theta) - \tau(1 + \alpha)\}^2}{4(\alpha + \gamma)} + \tau(\theta - 1)\alpha / \theta. \]

The condition for the equation \( P(x) = 0 \) to have a single, positive root or no positive root at all is that either \( P(0) \geq 0 \) so that \( \theta \geq 1 \) as in \( \underline{4} \) left or either of \( x_m \leq 0 \) and \( P(x_m) \leq 0 \) being true. The latter pair of conditions can be expressed jointly as

\[ \alpha(1 - \gamma / \theta) - \tau(1 + \alpha) \leq 2\sqrt{|1 - \theta|\tau(\alpha + \gamma)\alpha / \theta} \]

as in \( \underline{4} \) right.

Proof of Proposition 3. One immediately sees that the Burr and Pareto distributions themselves are included in the PowerBurr family. The Gamma distribution is obtained by letting \( \alpha \to \infty \) so that \( G_{\alpha} \overset{p}{\to} 1 \). By Slutsky’s theorem this implies \( X \overset{d}{\to} \beta G_{\theta} \) which defines the Gamma family. The Inverse Gamma is obtained in a similar way. Now \( G_{\theta} \overset{p}{\to} 1 \) as \( \theta \to \infty \) and \( X \overset{d}{\to} \beta / G_{\alpha} \) on row 4 in Table 1.

In order to get the Log-Gamma or row 5 insert \( \beta = 1 \) and \( \tau = \gamma / b \) into \( \underline{3} \). Then

\[ Z = (1 + bX/\gamma)^{\gamma} - 1 \overset{p}{\to} e^{bX} - 1 \quad \text{as} \quad \gamma \to \infty, \]

and it follows that \( \log(1 + Z) \overset{p}{\to} bX \). Since \( X \overset{p}{\to} G_{\theta} \) as \( \alpha \to \infty \), \( \log(1 + Z) \overset{p}{\to} bG_{\theta} \) and \( \log(1 + Z) \overset{d}{\to} bG_{\theta} \).

The logistic distribution on row 6 emerges when \( \beta = b / \gamma \) so that by an elementary application of l’Hôpital’s rule

\[ Z = b \frac{(1 + X/\tau)^{\gamma} - 1}{\gamma} \overset{p}{\to} b \log(1 + X / \tau) \quad \text{as} \quad \gamma \to 0, \]

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which yields

$$\Pr(Z > z) \to \Pr(\log(1 + X/\tau) > z/b) = \Pr(X > \tau(e^{z/b} - 1)).$$

Let $\alpha = \theta = 1$ so that $X$ follows an ordinary Pareto distribution with parameters $a = b = 1$. As remarked in in Section 2.1, this is also the distribution of $1/X$ which means that the limit on the right is $<Er det en 1– for mye i frste likhet?>$

$$\Pr(1/X > \tau(e^{z/b} - 1)) = 1 - \frac{1}{1 + \tau^{-1}(e^{z/b} - 1)^{-1}} = \frac{1}{1 - \tau + \tau(e^{z/b} - 1)}$$

and in conclusion

$$\Pr(Z > z) \to \frac{1}{1 - \tau + \tau(e^{z/b} - 1)} \quad \text{as} \quad \gamma \to 0.$$ 

The logistic model on row 6 follows when $\tau = 1/2$. This is the definition used in [Beirlant et al. (1996)], but one could also contemplate an extended version with $\tau$ as a general parameter.

To obtain the log-logistic, the Weibull and the Frechét on rows 7, 8 and 9 one starts with $\beta = b\tau^\gamma$. Then

$$Z = b\tau^\gamma \{(1 + X/\tau)^\gamma - 1\} \to bX^\gamma \quad \text{as} \quad \tau \to 0.$$ 

When $\theta = \alpha = 1$, $X$ follows an ordinary Pareto distribution, such that

$$\Pr(Z > z) \to \Pr(X > (z/b)^\gamma) = \frac{1}{1 + (z/b)^{1/\gamma}}$$

which is the expression on row 7 when $\gamma = 1/a$.

The Weibull model is defined as $Z = bG_1^a$, where $G_1$ is the exponential distribution with mean 1, and emerges when $\theta = 1$, $\alpha \to \infty$, $\gamma = a$ and $\tau \to 0$.

The Frechét is obtained with the same specifications except that now $\theta \to \infty$ and $\alpha = 1$. Then

$$\Pr(Z > z) \to \Pr(1/G_1 > (z/b)^\gamma) = 1 - e^{-(z/b)^{-a}}$$

as on row 9.
Finally, to get the Log-normal on row 10, let $\beta = e^{\xi + 1/2 - \sqrt{\gamma}}$ and $\tau = \sqrt{\gamma}$ so that
\[ Z = \beta (1 + X/\tau)^{\gamma} - \beta = e^{\xi + 1/2 - \sqrt{\gamma} \log(1 + X/\sqrt{\gamma})} - \beta. \]

Taylor’s formula with two terms and remainder yields
\[ \log(1 + X/\sqrt{\gamma}) = X/\sqrt{\gamma} - X^2/(2\gamma) + R \quad \text{where} \quad 0 < R < \frac{1}{6} \frac{X^3}{\gamma^{3/2}}. \]

The bounds for $R$ are consequences of $X > 0$ and imply $\gamma R \xrightarrow{p} 0$ as $\gamma \to \infty$. It now follows after a little reorganization that
\[ Z = e^{\xi + \sqrt{\gamma}(X-1) + (X^2-1)/2 + \gamma R} - \beta. \] (13)

Insert $\gamma = \theta \sigma^2$ so that
\[ \sqrt{\gamma}(X - 1) = \sigma \sqrt{\theta} \left( \frac{G_\theta}{G_\alpha} - 1 \right) = \sigma \frac{\sqrt{\theta}(G_\theta - 1) + \sqrt{\theta/\alpha}\sqrt{\alpha}(G_\alpha - 1)}{G_\alpha}. \]

As $\theta \to \infty$ and $\alpha \to \infty$, $\sqrt{\theta}(G_\theta - 1) \xrightarrow{d} N(0,1)$ and $\sqrt{\alpha}(G_\alpha - 1) \xrightarrow{d} N(0,1)$. In the denominator, $G_\alpha \xrightarrow{p} 1$ from which it follows by Slutsky’s theorem that $\sqrt{\gamma}(X - 1) \xrightarrow{d} N(0,\sigma)$ if both $\theta$ and $\alpha \to \infty$ and $\theta/\alpha \to 0$. Further, $\beta \to 0$ as $\gamma \to \infty$. Since $X^2 - 1 \xrightarrow{p} 0$ and $\gamma R \xrightarrow{p} 0$, the exponent in 13 tends to $N(\xi,\sigma)$ in distribution, as was to be proved.

3. Total loss model

As mentioned in the introduction, we assume the classic collective model for the total loss $X$, given by (??), where the claim sizes $Z_i$ are independent and identically distributed, and also independent of the number $N$ of claims. Assuming that $Z_i$s from different policies are identically distributed is obviously a simplification. However, it may be argued that when the main interest is measuring risk in terms of the total loss $X$, and not pricing of individual contracts, differences between policies will be averaged out.

Further, we assume that the number of claims follows a Poisson distribution with fixed intensity $\mu$ for all policies, i.e. $N \sim Poisson(\mu A)$, where $A$ is the total risk exposure of the portfolio, i.e. the number of policy years in the portfolio during the period over which the claims are counted. This is also a simplification, but as noted earlier, the main focus of our paper is on the claim size distribution.
Finally, the claims sizes are assumed to follow a distribution with parameters $\theta$ and pdf $f_Z(z; \theta)$, more specifically the six-parameter distribution or one of its special cases. The model for the total loss is then such that there is no explicit expression for the reserve for any of the claim severity distributions we consider. Therefore, these have to be estimated with Monte Carlo methods. The first step is then to generate $m$ independent samples $X^*_1, \ldots, X^*_m$ from the model, as specified in Algorithm 1. Subsequently, the reserve is estimated by

$$q_\epsilon^* = X^*_{(1-(\epsilon) m)},$$

where $X^*_{(1)} \leq \ldots \leq X^*_{(m)}$. In this setting, the total risk exposure is $A = JT$, where $J$ is the current number of policies in the portfolio and $T = 1$ year. In addition, we are interested in the $100 \cdot (1 - \epsilon)\%$ quantile $q^Z_\epsilon$ of the claim size distribution, given by

$$P(Z > q^Z_\epsilon) = \epsilon.$$

This will indicate how well our model captures the tail properties of the claim severity distribution and may help us to understand the performance of our model for estimating the reserve. For the extended Pareto distribution, we compute $q^Z_\epsilon$ with a numeric search algorithm and for the six-parameter distribution, it is computed by (??).

Even though the six-parameter distribution is very flexible, it will never be the exact true data generating process $g(\cdot)$. Hence, there will always be some model error in estimates of the reserve from this distribution. If the parameters $\theta$ are estimated by the maximum likelihood, the estimate $\hat{\theta}$ will converge to the value of $\theta_0$ that minimises the Kullback-Leibler divergence

$$KL(g,f) = \int_0^\infty \log \left( \frac{g(z)}{f(z; \theta)} \right) g(z) dz$$

from the true $g$ to the model $f$. Now, let $q_\epsilon(\theta)$ be the reserve based on $f(\cdot; \theta)$, and $q_\epsilon^*(\hat{\theta})$ the corresponding Monte Carlo estimate. Then we have:

$$q_\epsilon^*(\hat{\theta}) - q_\epsilon = (q_\epsilon^*(\hat{\theta}) - q_\epsilon(\hat{\theta})) + (q_\epsilon(\hat{\theta}) - q_\epsilon(\theta_0)) + (q_\epsilon(\theta_0) - q_\epsilon) = E_1 + E_2 + E_3,$$

where $E_1$ is the Monte Carlo error, $E_2$ is the estimation error and $E_3$ is the model error. The former, $E_1$, depends on the number of simulations $m$, and will converge to 0 as $m$ tends to $\infty$. Hence, the Monte Carlo error will be quite small as long as one chooses a large enough $m$. Further, the
model error $E_3$ depends on the Kullback-Leibler divergence from the true distribution to the assumed distribution. As the six-parameter distribution is very malleable, this error is also likely to be small. We may thus assume that the estimation error $E_2$, which is a function of the sample size, is the main source of error.

Algorithm 1

| Line | Code |
|------|------|
| 1    | for $i = 1, \ldots, m$ do |
| 2    | $X_i^* = 0$ |
| 3    | Draw $N^* \sim \text{Poisson}(J \mu T)$ |
| 4    | for $j = 1, \ldots, N^*$ do |
| 5    | Draw $Z^*$ from $f_{Z^*}(z; \theta)$ |
| 6    | $X_i^* = X_i^* + Z^*$ |
| 7    | end for |
| 8    | end for |
| 9    | Return $X_1^*, \ldots, X_m^*$ |

4. Simulation study

The proposed claim size distribution is more flexible than its two-parameter special cases. The question is how well it estimates the reserve, and how this estimate is affected by the increased parameter uncertainty due to the larger number of parameters. In order to investigate this, we conduct a large simulation study. The parameter settings for the study are presented in Section 4.1, some remarks about parameter estimation are given in Section 4.2 and the results from the study are given in Section 4.3.

4.1. Parameter settings

In all, we consider 10 different claim size distributions: the log-normal, the log-Gamma, the Weibull, the Pareto, the Gamma, the extended Pareto, the four-parameter special case with $\beta = \tau = 1$, the two five-parameter special cases with $\eta = 1$ and $\tau = 1$, respectively, and the full six-parameter distribution. Each experiment is performed as described in Algorithm 2. One of the 10 distributions is the true claim size distribution with pdf $f_{Z^*}(\cdot; \theta)$. First, a sample $z_1, \ldots, z_n$ of size $n$ is drawn from this distribution. Then, the parameters of each of the 10 distributions considered in this study are estimated based on the sample, resulting in the estimates $\hat{\theta}_1, \ldots, \hat{\theta}_{10}$. Finally,
one estimates the reserve $q_e$ and the quantile $q_e^Z$ of the claim size distribution using Algorithm 1 with each $\hat{\theta}_i$ and corresponding $f_{Z,i}(\cdot;\hat{\theta}_i)$. This procedure is repeated $N$ times. Note that the parameter of the claim frequency distribution is not estimated, but set to its true value. This is done in order to isolate the uncertainty in the reserve stemming from the claim size distribution.

\begin{algorithm}
\begin{algorithmic}[1]
\STATE Input: $J$, $\mu$, $\theta$, $f_Z(\cdot;\theta)$, $m$
\FOR{$k = 1, \ldots, N$}
\FOR{$j = 1, \ldots, n$}
\STATE Draw $Z^*$ from $f_Z(z;\theta)$
\ENDFOR
\FOR{$i = 1, \ldots, 10$}
\STATE Estimate $\hat{\theta}_i$
\STATE Estimate $(\hat{q}_{e,k,i}, \hat{q}_{e,k,i}^Z)$ using Algorithm 1 and (14) with $\theta = \hat{\theta}_i$
\ENDFOR
\ENDFOR
\STATE Return $(\hat{q}_{e,k,i}, \hat{q}_{e,k,i}^Z)$, $i = 1, \ldots, 10$, $k = 1, \ldots, N$.
\end{algorithmic}
\end{algorithm}

The parameter values used in the study are shown in Table 2. These are chosen so that $E(Z) \approx 1$, except for the log-Gamma distribution, for which this was impossible without making $\text{sd}(Z)$ very small. Further, the parameter values are set so that the corresponding distributions range from medium to very heavy-tailed. Figure 2 shows the corresponding pdfs. For the claim frequency distribution (which is Poisson), we let $\lambda = J\mu T$ take each of the values 10, 100, 1,000, to represent variations in portfolio size and claim intensity.

To assess the effect of the sample size, we have run the experiments for each of $n = 5,000, 500, 50$, which corresponds to data sets ranging from rather large to rather small, but realistic, depending on the line of business. Further, we used $m = 100,000$ in the Monte Carlo estimation of the reserve. Finally, we let $N = 1,000$ in all experiments.

4.2. Parameter estimation

All the distribution parameters are estimated by maximum likelihood estimation, using the R function \texttt{optim()} to do quasi-Newton optimisation.
Distribution | **θ** | E(Z) | sd(Z)
---|---|---|---
Log - normal(ξ, σ) | (-0.5, 1) | 1.00 | 1.31
Log - Gamma(ξ, θ) | (0.75, 5) | 1.25 | 0.93
Weibull(β, η) | (2, 1.13) | 1.00 | 0.52
Pareto(α, β) | (3, 2) | 1.00 | 1.77
Gamma(ξ, α) | (1, 2) | 1.00 | 0.71
Ext. Pareto(α, θ, β) | (3, 2, 1) | 1.50 | 2.22
Four - parameter(α, θ, β, τ, γ) | (4, 2, 0.6, 1.3) | 1.02 | 1.81
Five - parameter(α, θ, β, τ, γ) | (4, 2, 0.7, 5, 1.3) | 1.00 | 1.27
Five - parameter2(α, θ, β, η, γ) | (4, 2, 0.5, 1.2, 1.1) | 0.94 | 1.60
Four - parameter(α, θ, β, η, τ, γ) | (4, 2, 1.3, 10, 1.2) | 0.86 | 1.85

Table 2: Parameters used for the 10 distributions in the simulation study, as well as the mean and standard deviation of the corresponding distributions.

For the two-parameter families, this optimisation is straightforward. However, especially the five- and six-parameter distributions appear to have likelihood functions that are rather flat in some of the parameters. This means that widely different parameter values result in rather similar distributions. Consequently, the optimisation of the likelihood is sometimes challenging, but this is not that problematic when the resulting distributions are sensible. Still, we have derived the derivatives of the log-likelihood functions with respect to all parameters, and supply those to `optim()`. These are given in the Appendix. To ease the optimisation further, it is performed on the log-transformed parameters as all the parameters α, θ, β, η, τ, γ are positive. Finally, we do the optimisation with several sets of start values for the parameters, and choose the parameter estimates that give the highest likelihood value. More specifically, we use the moment estimates (\(\hat{\alpha}_{ep}, \hat{\theta}_{ep}, \hat{\beta}_{ep}\)), as well as (10\(^{10}\), \(\hat{\alpha}_{ga}, \hat{\xi}_{ga}\)) and (\(\hat{\alpha}^{pa}, 1, \hat{\beta}^{pa}\)) for the extended Pareto distribution, where (\(\hat{\xi}_{ga}, \hat{\alpha}_{ga}\)) and (\(\hat{\alpha}^{pa}, \hat{\beta}^{pa}\)) are the maximum likelihood estimates obtained for the Gamma and Pareto distributions, respectively. For the four-parameter distribution, we employ (\(\hat{\alpha}_{ep}, \hat{\theta}_{ep}, \hat{\beta}_{ep}, 1\)) and (10\(^{10}\), 1, \(\hat{\eta}_{we}, 1/\hat{\beta}_{we}\)), where (\(\hat{\alpha}_{ep}, \hat{\theta}_{ep}, \hat{\beta}_{ep}\)) and (\(\hat{\beta}_{we}, \hat{\eta}_{we}\)) are the estimates from the extended Pareto and Weibull distributions. The start values used for the five-parameter distribution are (\(\hat{\alpha}_{ep}, \hat{\theta}_{ep}, \hat{\beta}_{ep}, 1, 1\)) and (10\(^{10}\), \(\hat{\theta}_{ga}, 1, 10^{10}, \hat{\xi}_{ga}, 10^{10}\)), where (\(\hat{\xi}_{ga}, \hat{\theta}_{ga}\)) is the estimate from the log-Gamma distribution. For the five-parameter 2 distribution, the start values are (\(\hat{\alpha}_{fp}, \hat{\theta}_{fp}, \hat{\beta}_{fp}, \hat{\eta}_{fp}, 1\)),...
where \((\hat{\alpha}_{fp}, \hat{\theta}_{fp}, \hat{\beta}_{fp}, \hat{\eta}_{fp})\) are the estimates from the four-parameter distribution. Finally, the start values for the six-parameter distribution are
\((\hat{\alpha}_{fip}, \hat{\theta}_{fip}, \hat{\beta}_{fip}, 1, \hat{\tau}_{fip}, \hat{\gamma}_{fip})\) and \((\hat{\alpha}_{fip2}, \hat{\theta}_{fip2}, \hat{\beta}_{fip2}, \hat{\eta}_{fip2}, 1, \hat{\gamma}_{fip2})\), where
\((\hat{\alpha}_{fip}, \hat{\theta}_{fip}, \hat{\beta}_{fip}, \hat{\tau}_{fip}, \hat{\gamma}_{fip})\) and \((\hat{\alpha}_{fip2}, \hat{\theta}_{fip2}, \hat{\beta}_{fip2}, \hat{\eta}_{fip2}, \hat{\gamma}_{fip2})\) are the estimates from the five-parameter and five-parameter 2 distributions.

4.3. Results

The results from the simulations in terms of the bias and root mean squared error (RMSE) of the quantile and reserve estimates are shown in Tables 3 to 14. The relative performance of the models with \(n = 500\) is rather similar to that with \(n = 5,000\), though with larger biases and RMSEs. Hence, the results from the corresponding simulations are not shown here. Furthermore, our main interest is the performance of the six-parameter distribution and its five- and four-parameter versions when the true distribution is one of the six special cases from Table 1, i.e. whether our model is able to capture the behaviour of these different distributions in practice for finite sample sizes, as it has been shown to do in theory. Hence, we only report the results from the simulations with one of the Table 1 distributions as the true distribution.

Tables 3 and 4 display the bias and RMSE in the estimates of \(q_{t}^{Z}\) for \(\epsilon = 0.05\) and 0.01, respectively, when the sample size is 5,000. Corresponding results for sample size 50 are shown in Tables 5 and 6. We see that when
the sample size is large, the quantile estimates from the six-parameter, five-
parameter 2 and four-parameter distribution are just as good as the ones from
the true distribution, both in terms of bias and RMSE. The performance of
the five-parameter distribution is good for the 95% quantile, but inferior to
the others for the 99% quantile. As mentioned above, the impression from
the simulations with \( n = 500 \) are the same. Hence, our model performs
well also for medium sample sizes. When the sample size is reduced to 50,
we see that the estimates of \( q_{0.05}^Z \) from our model are still rather good in
terms of bias, but with comparatively higher RMSE, especially for the most
heavy-tailed distributions. As one moves further out in the tail of the claim
size distribution, the quality of the estimates is reasonable except when the
true distribution is log-normal or log-Gamma. As mentioned in Section 3,
the main source of error in the reserve estimates from the six-parameter
distribution is likely to be the estimation error, while the Monte Carlo and
model error should be small. Hence, the decreased performance is probably
due to increased parameter uncertainty for the smaller sample size.

Tables 7 and 8 show the bias and RMSE in the estimates of the reserve \( q_\epsilon \)
for \( \epsilon = 0.05 \) and 0.01, respectively, when the parameter of the claim frequency
distribution is \( \lambda = 10 \) and \( n = 5,000 \). Corresponding results for \( \lambda = 1,000 \)
are displayed in Tables 9 and 10. We see that the reserve estimates from our
model are very good, at least for all distributions but the log-normal. The
six- and five-parameter 2 distributions perform rather well also for the log-
normal claims. The four- and five-parameter distributions do not perform
that well in this case, especially for the 99% reserve. This may be due to the
fact that these two distributions do not have the log-normal as a special case.
When \( \lambda \) increases to 1,000, the relative performance of the applied models is
similar, though the difference between them is smaller, as one would expect
when one sums many independent, identically distributed variables. The
results for \( \lambda = 100 \) are somewhere between those for \( \lambda = 10 \) and \( \lambda = 1,000 \),
and are therefore not shown here.

Results corresponding to the ones in Tables 7 to 10 for \( n = 50 \) are dis-
played in Tables 11 to 14. The five- and six-parameter versions of our model
still produce good estimates of the 95% reserve when \( \lambda = 10 \). The quality of
the 99% reserve estimates is also relatively good, except for the log-normal
and log-Gamma distributed claims. The four-parameter distribution does
not perform well in this case. When \( \lambda \) increases to 1,000, our model still
gives reasonable estimates of the 95 and 99% reserves of the medium-tailed
Weibull and Gamma claims. However, it does not perform well for claims
from heavy-tailed distributions.

Thus, all in all, our model appears to capture the tail behaviour of claims from widely different distributions, ranging from the moderate-tailed Weibull and Gamma to the very heavy-tailed Pareto and extended Pareto when the amount of data is large or medium. When the sample size is small, however, our model should be used with care. For medium-tailed claim sizes, the performance seems to be reasonable, but not for heavy-tailed ones. Very large RMSEs in such cases indicate that the parameter uncertainty is huge. Further, the performance of the five-parameter 2 and six-parameter distributions is consistently better than that of the four- and five-parameter ones, even for small sample sizes. Also, the five-parameter 2 gives very similar results to the six-parameter. This indicates that the sixth parameter $\tau$ may not be needed, but that the fifth parameter $\gamma$ makes a significant contribution to the model.

5. Real data example

In this section, we will apply our model as a claim severity distribution to a set of motor insurance losses from a Norwegian insurance company. These are discussed in Bølviken (2014), and consist of 6,446 claims from the years ... to ... The deductible is subtracted from the claims, which are given in NOK. Further, the average claim intensity is 5.7%. We have fitted each of the ten distributions considered in the simulation study, and estimated the 95 and 99% reserves for each of them, when we assume an expected number of $\lambda = 1,000$ claims during the next year. We have also estimated the 95 and 99% quantiles of the claim severity distribution for interpretation purposes. Each quantile estimate is accompanied by a 95% confidence interval obtained from $m_b = 1,000$ bootstrap samples from the given distribution. These are compared to the corresponding empirical estimates from the data. The results are shown in Table 15.

We see that the reserve estimates from the five-parameter 2 and the extended Pareto distribution are the ones that are the closest to the empirical ones. However, the empirical estimate is within the confidence interval from almost all the distributions. The five- and six-parameter distributions all have confidence intervals that cover the empirical estimates, but the five-parameter 2 gives the estimate with the smallest uncertainty. The four-parameter distribution, on the other hand, over-estimates the two reserves. For the quantiles of the claim size distribution, the log-normal and the five-
parameter 2 distribution give the estimates that are the closest to the empirical ones. All the versions of our model have confidence intervals that cover the empirical estimate of the 95% quantile, but the four-parameter distributions significantly over-estimates the 99% quantile, which may explain why its estimates of the reserve are too large. Further, the quantiles are over-estimated by the log-Gamma and partly by the Pareto, whereas they are under-estimated by the Weibull, Gamma and extended Pareto distributions.

To test the quality of the quantile estimates, we also perform a binomial back-test. The results from the test are shown in Table 16. These confirm the impression that the five- and six-parameter distributions manage to capture the tail behaviour of the data. All in all, the five- and six-parameter distributions therefore seem to be good models for this data set, and particularly the five-parameter 2, which is consistent with our findings from Section 4.

6. Conclusion

We propose a new class of claim severity distributions with six parameters, and show that this model has the standard two-parameter distributions, the log-normal, the log-Gamma, the Weibull, the Gamma and the Pareto, as special cases. This distribution is much more flexible than its special cases, and therefore more able to capture important characteristics of claim severity data. On the other hand, an increased number of parameters usually leads to a larger uncertainty in parameter estimates. Therefore, we have investigated how this parameter uncertainty affects the estimate of the reserve, which is one of the most important risk measures within non-life insurance. This is done in a large simulation study, where we vary both the characteristics of the claim size distributions and the sample size. We have also tried our model on a set of motor insurance claims from a Norwegian insurance company.

In all we simulate claim size data from ten different distributions, including the six special cases and four versions of our model, each time fitting all the ten distributions to the data. Further, we estimate the corresponding 95 and 99% quantiles of the fitted distributions, and also the 95 and 99% reserves, where the total loss distributions follows the collective risk model with Poisson distributed claim numbers. The quality of the resulting estimates is assessed by the bias and the RMSE.
The results from the study show that as long as the amount of data is reasonable, the five- and six-parameter versions of our model provide very good estimates of both the quantiles of the claim severity distribution and the reserves, for claim size distributions ranging from medium to very heavy tailed. When the sample size is small, our model appears to struggle with heavy-tailed data, but is still adequate for data with more moderate tails. The impression from the fit to the real data set is the same. Further, the performance of the five-parameter 2 distribution, obtained when the parameter \( \tau \) is set to 1, is overall just as good as the six-parameter one. This sixth parameter therefore does not seem to provide a flexibility that is needed. On the other hand, the performance of the four-parameter distribution, where the parameter \( \gamma \) has also been set to 1, is not as consistently good, not even for smaller sample sizes. Hence, this fifth parameter really seems to improve the distribution.

The poor performance of our model for small data sets following heavy-tailed distributions is worth a closer look. Of course, estimates of quantiles far out in the right tail will inevitably be quite uncertain for such data. Still, one might obtain more reliable parameter estimates in these cases by adapting the estimation procedure. One could for instance try to put more emphasis on the right tail by using a weighted maximum likelihood estimator with an adequate weight function. This is a subject for further work.

7. Acknowledgements

Appendix 1

Let \( D_\alpha(z), D_\theta(z), D_\beta(z), D_\tau(z), D_\gamma(z), D_\eta(z) \) be the partial derivatives of \( \log \{ f(z) \} \) with respect to \( \alpha, \theta, \beta, \tau, \gamma, \eta \) which is straightforward to derive from in (??) and (??). Adding their values \( D_\alpha(z_i), D_\theta(z_i) \) and so forth over all historical losses \( z_1, \ldots, z_n \) yield the gradient vector of the log-likelihood function that can assist optimization. Introduce

\[
v = 1 + \frac{z}{\beta} \quad \text{and} \quad w = v^{\frac{1}{\gamma}}.
\]

and write \( \psi(x) = d \log \{ \Gamma(x) \} / dx \) for the so-called Digamma function. Then,

\[
D_\alpha(z) = \psi(\alpha+\theta) - \psi(\alpha) - \frac{\theta}{\alpha} \log \left( \frac{\theta}{\alpha} \frac{1}{\tau^\gamma} (w - 1)^{\frac{1}{\gamma}} + 1 \right) + \frac{\alpha + \theta}{\alpha} \cdot \frac{(w - 1)^{\frac{1}{\gamma}}}{(w - 1)^{\frac{1}{\gamma}} + \alpha/\theta \tau^{\frac{1}{\gamma}}}.
\]
\[ D_\theta(z) = \psi(\alpha + \theta) - \psi(\theta) + \log(\theta) + 1 + \frac{\log(\tau)}{\eta} - \log(\alpha) + \frac{\log(w - 1)}{\eta} \]
\[ \quad - \log\left(\frac{\theta}{\alpha} \frac{1}{\tau^\frac{1}{\eta}} (w - 1)^{\frac{1}{\eta}} + 1\right) - \frac{\alpha + \theta}{\theta} \cdot \frac{(w - 1)^{\frac{1}{\eta}}}{(w - 1)^{\frac{1}{\eta}} + \alpha/\theta \tau^{\frac{1}{\eta}}} \]

\[ D_\beta(z) = -\frac{1}{\beta} \cdot \frac{(1 - \gamma)z}{\beta^2 \gamma v} - \left(\frac{\theta}{\eta} - 1\right) \frac{zw}{\beta^2 \gamma v (w - 1)} + \frac{\alpha + \theta}{\beta^2 \eta \gamma} \cdot \frac{zw(w - 1)^{\frac{1}{\eta} - 1}}{v \left((w - 1)^{\frac{1}{\eta}} + \alpha/\theta \tau^{\frac{1}{\eta}}\right)} \]

\[ D_\tau(z) = \frac{\theta}{\eta \tau} - \frac{\alpha + \theta}{\eta \tau} \cdot \frac{(w - 1)^{\frac{1}{\eta}}}{(w - 1)^{\frac{1}{\eta}} + \alpha/\theta \tau^{\frac{1}{\eta}}} \]

\[ D_\gamma(z) = -\frac{1}{\gamma} \cdot \frac{\log(v)}{\gamma^2} - \left(\frac{\theta}{\eta} - 1\right) \frac{w}{\gamma^2 (w - 1)} \log(v) + \frac{\alpha + \theta}{\gamma^2 \eta} \cdot \frac{(w - 1)^{\frac{1}{\eta} - 1}}{v \left((w - 1)^{\frac{1}{\eta}} + \alpha/\theta \tau^{\frac{1}{\eta}}\right)} w \log(v) \]

\[ D_\eta(z) = -\frac{\theta \log(\tau)}{\eta^2} - \frac{1}{\eta} \cdot \frac{\theta}{\eta^2} \log(w - 1) + \frac{\alpha + \theta}{\eta^2} \cdot \frac{(\log(w - 1) + \log(\tau))}{(w - 1)^{\frac{1}{\eta}} + \alpha/\theta \tau^{\frac{1}{\eta}}} \]

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### Bias

|     | L-N  | L-G  | We   | Pa   | Ga   | E. Pa. |
|-----|------|------|------|------|------|--------|
| L-N | 0.004| 0.023| 0.476| 1.250| 0.487| 0.338  |
| L-G | -0.164| 0.000| 0.385| 0.309| 0.341| 0.176  |
| We  | -0.143| -0.056| 0.000| -0.016| -0.047| -0.041 |
| Pa  | 0.043| 0.809| 1.045| 0.001| 0.624| 0.205  |
| Ga  | -0.275| -0.167| 0.119| -0.096| 0.000| -0.237 |
| E. Pa.| 0.004| -0.001| 0.120| 0.002| 0.001| 0.004  |
| 4-par.| 0.008| -0.001| 0.001| 0.001| -0.001| 0.004  |
| 5-par.| 0.007| -0.001| 0.039| 0.001| 0.000| 0.003  |
| 5-par. 2 | -0.009| -0.001| 0.001| 0.002| -0.001| 0.004  |
| 6-par | -0.004| -0.001| 0.001| 0.000| -0.001| 0.003  |

### RMSE

|     | L-N  | L-G  | We   | Pa   | Ga   | E. Pa. |
|-----|------|------|------|------|------|--------|
| L-N | 0.071| 0.049| 0.477| 1.258| 0.489| 0.356  |
| L-G | 0.177| 0.043| 0.386| 0.324| 0.343| 0.207  |
| We  | 0.162| 0.077| 0.015| 0.097| 0.055| 0.139  |
| Pa  | 0.084| 0.810| 1.045| 0.088| 0.625| 0.232  |
| Ga  | 0.284| 0.172| 0.121| 0.137| 0.028| 0.265  |
| E. Pa.| 0.080| 0.047| 0.121| 0.089| 0.029| 0.112  |
| 4-par.| 0.077| 0.047| 0.016| 0.091| 0.030| 0.113  |
| 5-par.| 0.080| 0.047| 0.057| 0.090| 0.029| 0.112  |
| 5-par. 2 | 0.077| 0.047| 0.016| 0.090| 0.030| 0.112  |
| 6-par | 0.077| 0.047| 0.016| 0.091| 0.029| 0.112  |

Table 3: Bias and RMSE on the upper 95% quantile of the distribution of $Z$ from each of the applied (A) distributions and each distribution from Table 1 as the true (T) one when $n = 5,000$. 

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| A \ T  | L-N  | L-G  | We  | Pa  | Ga  | E. Pa. |
|--------|------|------|-----|-----|-----|--------|
| L-N    | 0.010 | -0.060 | 1.340 | 5.105 | 1.622 | 0.615  |
| L-G    | -0.305 | 0.000 | 1.185 | 2.131 | 1.362 | 0.911  |
| We     | -1.600 | -0.862 | 0.000 | -1.453 | -0.212 | -2.243 |
| Pa     | -0.664 | 1.073 | 2.188 | 0.007 | 1.287 | -0.982 |
| Ga     | -1.886 | -0.911 | 0.333 | -1.914 | -0.001 | -2.667 |
| E. Pa. | 0.735 | 0.013 | 0.333 | 0.008 | 0.007 | 0.020  |
| 4-par. | 0.296 | 0.013 | 0.004 | 0.009 | -0.001 | 0.021  |
| 5-par. | 0.668 | 0.008 | 0.105 | 0.018 | 0.005 | 0.014  |
| 5-par. 2 | 0.081 | 0.009 | 0.003 | 0.007 | -0.001 | 0.009  |
| 6-par  | 0.028 | 0.005 | 0.003 | 0.006 | 0.000 | 0.006  |

| A \ T  | L-N  | L-G  | We  | Pa  | Ga  | E. Pa. |
|--------|------|------|-----|-----|-----|--------|
| L-N    | 0.175 | 0.104 | 1.342 | 5.132 | 1.625 | 0.680  |
| L-G    | 0.354 | 0.091 | 1.186 | 2.161 | 1.366 | 0.969  |
| We     | 1.606 | 0.867 | 0.022 | 1.467 | 0.216 | 2.258  |
| Pa     | 0.692 | 1.075 | 2.188 | 0.320 | 1.288 | 1.025  |
| Ga     | 1.890 | 0.913 | 0.334 | 1.922 | 0.044 | 2.674  |
| E. Pa. | 0.797 | 0.126 | 0.334 | 0.325 | 0.051 | 0.384  |
| 4-par. | 0.393 | 0.127 | 0.026 | 0.338 | 0.057 | 0.399  |
| 5-par. | 0.744 | 0.123 | 0.154 | 0.336 | 0.052 | 0.392  |
| 5-par. 2 | 0.279 | 0.127 | 0.026 | 0.339 | 0.057 | 0.403  |
| 6-par  | 0.268 | 0.126 | 0.026 | 0.342 | 0.055 | 0.406  |

Table 4: Bias and RMSE on the upper 99% quantile of the distribution of Z from each of the applied (A) distributions and each distribution from Table 1 as the true (T) one when \( n = 5,000 \).
|       | A\T       | L-N | L-G | We   | Pa   | Ga   | E. Pa. |       |
|-------|-----------|-----|-----|------|------|------|--------|-------|
| L-N   | 0.113     | 0.049 | 0.462 | 1.312 | 0.501 | 0.316 |        |
| L-G   | -0.094    | -0.005 | 0.355 | 0.276 | 0.324 | 0.103 |        |
| We    | -0.111    | -0.100 | -0.017 | -0.120 | -0.071 | -0.242 |        |
| Pa    | 0.102     | 0.825 | 1.042 | -0.050 | 0.629 | 0.084 |        |
| Ga    | -0.210    | -0.172 | 0.100 | -0.170 | -0.016 | -0.349 |        |
| E. Pa.| 0.145     | -0.002 | 0.102 | -0.034 | 0.010 | -0.075 |        |
| 4-par.| 0.123     | -0.009 | -0.020 | -0.043 | -0.041 | -0.077 |        |
| 5-par.| 0.079     | -0.029 | -0.003 | -0.035 | -0.031 | -0.099 |        |
| 5-par. 2 | 0.066 | -0.018 | -0.021 | -0.102 | -0.042 | -0.125 |        |
| 6-par.| 0.059     | -0.016 | -0.018 | -0.078 | -0.038 | -0.131 |        |

|       | A\T       | L-N | L-G | We   | Pa   | Ga   | E. Pa. |       |
|-------|-----------|-----|-----|------|------|------|--------|-------|
| L-N   | 0.719     | 0.448 | 0.551 | 1.962 | 0.666 | 1.131 |        |
| L-G   | 0.684     | 0.439 | 0.435 | 1.005 | 0.494 | 1.060 |        |
| We    | 0.722     | 0.507 | 0.155 | 0.881 | 0.278 | 1.111 |        |
| Pa    | 0.688     | 0.925 | 1.064 | 0.898 | 0.694 | 0.991 |        |
| Ga    | 0.730     | 0.463 | 0.196 | 0.923 | 0.266 | 1.122 |        |
| E. Pa.| 0.861     | 0.532 | 0.197 | 0.927 | 0.284 | 1.168 |        |
| 4-par.| 0.827     | 0.529 | 0.160 | 0.949 | 0.290 | 1.148 |        |
| 5-par.| 0.785     | 0.497 | 0.160 | 0.933 | 0.285 | 1.111 |        |
| 5-par. 2 | 0.787 | 0.522 | 0.160 | 0.921 | 0.289 | 1.121 |        |
| 6-par.| 0.779     | 0.515 | 0.161 | 0.933 | 0.289 | 1.112 |        |

Table 5: Bias and RMSE on the upper 95% quantile of the distribution of Z from each of the applied (A) distributions and each distribution from Table 1 as the true (T) one when \( n = 50 \).
Table 6: Bias and RMSE on the upper 99% quantile of the distribution of $Z$ from each of the applied (A) distributions and each distribution from Table 1 as the true (T) one when $n = 50$. 

|       | L-N | L-G | We  | Pa  | Ga  | E. Pa. |
|-------|-----|-----|-----|-----|-----|--------|
| L-N   | 0.325 | 0.007 | 1.329 | 5.675 | 1.681 | 0.643  |
| L-G   | -0.052 | 0.007 | 1.132 | 2.215 | 1.340 | 0.825  |
| We    | -1.536 | -0.933 | -0.026 | -1.652 | -0.253 | -2.610  |
| Pa    | -0.433 | 1.117 | 2.183 | 0.127 | 1.296 | -1.180 |
| Ga    | -1.776 | -0.918 | 0.300 | -2.042 | -0.030 | -2.852 |
| E. Pa.| 1.725 | 0.140 | 0.310 | 0.394 | 0.112 | 0.231 |
| 4-par.| 1.452 | 0.266 | -0.021 | 0.594 | -0.016 | 0.445 |
| 5-par.| 0.691 | -0.006 | 0.030 | 0.280 | -0.014 | -0.187 |
| 5-par. 2 | 0.676 | 0.132 | -0.021 | -0.160 | -0.025 | -0.230 |
| 6-par | 0.318 | 0.094 | -0.013 | -0.124 | -0.023 | -0.421 |

|       | L-N | L-G | We  | Pa  | Ga  | E. Pa. |
|-------|-----|-----|-----|-----|-----|--------|
| L-N   | 1.841 | 0.893 | 1.491 | 8.173 | 1.993 | 2.947  |
| L-G   | 1.891 | 0.939 | 1.261 | 4.298 | 1.620 | 3.392  |
| We    | 2.032 | 1.235 | 0.224 | 2.443 | 0.492 | 3.317  |
| Pa    | 1.941 | 1.316 | 2.209 | 3.612 | 1.371 | 2.951  |
| Ga    | 2.130 | 1.137 | 0.397 | 2.595 | 0.417 | 3.377  |
| E. Pa.| 4.173 | 1.615 | 0.410 | 4.359 | 0.668 | 4.901  |
| 4-par.| 4.042 | 1.752 | 0.274 | 5.033 | 0.660 | 5.019  |
| 5-par.| 2.762 | 1.341 | 0.271 | 3.900 | 0.583 | 3.893  |
| 5-par. 2 | 2.809 | 1.494 | 0.274 | 3.444 | 0.638 | 3.935 |
| 6-par | 2.523 | 1.445 | 0.279 | 3.592 | 0.606 | 3.701  |
|     | L-N   | L-G   | We    | Pa    | Ga    | E. Pa. |
|-----|-------|-------|-------|-------|-------|--------|
| **Bias** |       |       |       |       |       |        |
| L-N  | 0.025 | -0.054| 1.344 | 9.265 | 2.012 | 0.976  |
| L-G  | -0.399| 0.001 | 1.073 | 3.835 | 1.568 | 1.496  |
| We   | -1.437| -0.215| 0.000 | -1.875| -0.041| -2.104 |
| Pa   | -0.662| 1.315 | 1.853 | 0.006 | 1.224 | -1.136 |
| Ga   | -1.719| -0.549| 0.137 | -2.064| -0.004| -2.520 |
| E. Pa.| 1.321 | 0.188 | 0.138 | 0.009 | -0.001| 0.038  |
| 4-par.| 0.523 | 0.018 | 0.000 | 0.024 | -0.006| 0.046  |
| 5-par.| 1.191 | 0.012 | 0.043 | 0.028 | -0.005| 0.034  |
| 5-par. 2| 0.193 | 0.012 | 0.000 | 0.006 | -0.005| 0.026  |
| 6-par | 0.097 | 0.006 | -0.001| 0.009 | -0.005| 0.029  |
| **RMSE** |       |       |       |       |       |        |
| L-N  | 0.419 | 0.250 | 1.353 | 9.329 | 2.027 | 1.181  |
| L-G  | 0.587 | 0.253 | 1.082 | 3.917 | 1.585 | 1.672  |
| We   | 1.487 | 0.334 | 0.120 | 1.934 | 0.182 | 2.194  |
| Pa   | 0.789 | 1.337 | 1.858 | 0.652 | 1.239 | 1.302  |
| Ga   | 1.760 | 0.599 | 0.184 | 2.121 | 0.177 | 2.593  |
| E. Pa.| 1.461 | 0.273 | 0.184 | 0.655 | 0.178 | 0.781  |
| 4-par.| 0.748 | 0.269 | 0.123 | 0.676 | 0.179 | 0.798  |
| 5-par.| 1.358 | 0.270 | 0.138 | 0.676 | 0.179 | 0.795  |
| 5-par. 2| 0.562 | 0.273 | 0.121 | 0.678 | 0.179 | 0.808  |
| 6-par | 0.540 | 0.273 | 0.121 | 0.682 | 0.180 | 0.817  |

Table 7: Bias and RMSE on the upper 95% reserve from each of the applied (A) distributions and each distribution from Table 1 as the true (T) one when $\lambda = 10$ and $n = 5,000$. 
Table 8: Bias and RMSE on the upper 99% reserve from each of the applied (A) distributions and each distribution from Table 1 as the true (T) one when $\lambda = 10$ and $n = 5,000$. 

|     | A/T  | L-N   | L-G   | We   | Pa   | Ga   | E. Pa. |
|-----|------|-------|-------|------|------|------|--------|
| Bias|      |       |       |      |      |      |        |
| L-N | 0.054| -0.199| 2.112 | 20.726| 3.478| 0.423|
| L-G | -0.079| 0.006| 1.760 | 12.505| 2.967| 4.066|
| We  | -3.958| -0.781| 0.008| -6.153| -0.115| -7.145|
| Pa  | -2.140| 1.936| 3.162 | 0.042| 2.089| -4.802|
| Ga  | -4.444| -1.221| 0.252| -6.763| -0.004| -7.870|
| E. Pa.| 5.122| 0.054| 0.253| 0.039| 0.007| 0.133|
| 4-par.| 1.809| 0.057| 0.009| 0.095| -0.005| 0.169|
| 5-par.| 4.536| 0.039| 0.082| 0.135| -0.004| 0.125|
| 5-par. 2| 0.814| 0.050| 0.004| 0.057| 0.000| 0.091|
| 6-par| 0.459| 0.033| 0.006| 0.076| 0.001| 0.106|

|     | A/T  | L-N   | L-G   | We   | Pa   | Ga   | E. Pa. |
|-----|------|-------|-------|------|------|------|--------|
| RMSE|      |       |       |      |      |      |        |
| L-N | 0.663| 0.380| 2.122 | 20.867| 3.496| 1.166|
| L-G | 0.764| 0.346| 1.771 | 12.669| 2.988| 4.335|
| We  | 3.991| 0.846| 0.150 | 6.189| 0.249| 7.194|
| Pa  | 2.241| 1.962| 3.168 | 1.464| 2.103| 4.903|
| Ga  | 4.471| 1.258| 0.295 | 6.793| 0.225| 7.909|
| E. Pa.| 5.363| 0.405| 0.296| 1.523| 0.228| 1.679|
| 4-par.| 2.142| 0.404| 0.150| 1.736| 0.225| 1.867|
| 5-par.| 4.871| 0.398| 0.194| 1.761| 0.223| 1.763|
| 5-par. 2| 1.388| 0.412| 0.150| 1.764| 0.225| 1.909|
| 6-par| 1.232| 0.407| 0.150| 1.776| 0.226| 1.964|
Table 9: Bias and RMSE on the upper 95% reserve from each of the applied (A) distributions and each distribution from Table 1 as the true (T) one when $\lambda = 1000$ and $n = 5,000$. 

|     | L-N   | L-G   | We     | Pa     | Ga     | E. Pa. |
|-----|-------|-------|--------|--------|--------|--------|
| L-N | 1.26  | -0.49 | 46.57  | 319.30 | 66.01  | 29.99  |
| L-G | -9.65 | 0.02  | 34.58  | 161.65 | 47.61  | 53.50  |
| We  | -12.18| 8.62  | 0.03   | -39.49 | 2.79   | -26.22 |
| Pa  | -11.98| 11.10 | 15.36  | 0.99   | 10.32  | -31.01 |
| Ga  | -14.39| -4.37 | 1.10   | -25.48 | 0.14   | -29.39 |
| E. Pa. | 55.84 | 0.47  | 1.02   | 1.17   | 0.08   | 1.76   |
| 4-par. | 19.15 | 0.52  | -0.11  | 1.91   | 0.07   | 2.32   |
| 5-par. | 48.58 | 0.31  | 0.49   | 2.32   | -0.01  | 1.63   |
| 5-par. 2 | 8.61  | 0.39  | -0.13  | 1.66   | 0.07   | 1.58   |
| 6-par  | 5.25  | 0.23  | -0.14  | 1.80   | 0.00   | 1.49   |

|     | L-N   | L-G   | We     | Pa     | Ga     | E. Pa. |
|-----|-------|-------|--------|--------|--------|--------|
| L-N | 20.12 | 13.65 | 47.34  | 322.21 | 67.19  | 42.76  |
| L-G | 23.02 | 13.85 | 35.57  | 166.01 | 49.17  | 63.76  |
| We  | 23.60 | 16.73 | 7.84   | 45.75  | 11.27  | 40.92  |
| Pa  | 23.50 | 17.85 | 17.32  | 27.93  | 15.07  | 43.01  |
| Ga  | 25.362| 14.62 | 7.93   | 36.20  | 10.89  | 44.28  |
| E. Pa. | 62.72 | 14.10 | 7.93   | 28.00  | 10.91  | 34.87  |
| 4-par. | 30.11 | 14.10 | 7.96   | 29.30  | 10.92  | 35.83  |
| 5-par. | 57.06 | 14.05 | 8.13   | 29.40  | 10.92  | 35.25  |
| 5-par. 2 | 24.55 | 14.12 | 7.98   | 29.37  | 10.93  | 36.04  |
| 6-par  | 23.68 | 14.11 | 8.00   | 29.54  | 10.96  | 36.09  |
Table 10: Bias and RMSE on the upper 99% reserve from each of the applied (A) distributions and each distribution from Table T1 as the true (T) one when \( \lambda = 1000 \) and \( n = 5,000 \).
### Table 11: Bias and RMSE on the upper 95% reserve from each of the applied (A) distributions and each distribution from Table II as the true (T) one when $\lambda = 10$ and $n = 50.$

|   | L-N | L-G | We  | Pa  | Ga  | E. Pa. |
|---|-----|-----|-----|-----|-----|--------|
| L-N | 0.79 | 0.14 | 1.37 | 10.56 | 2.22 | 1.08   |
| L-G | 0.23 | 0.08 | 1.02 | 4.14  | 1.60 | 1.40   |
| We  | -1.13 | -0.23 | -0.03 | -2.24 | -0.05 | -2.79  |
| Pa  | -0.05 | 1.40 | 1.84 | 0.54  | 1.26 | -1.48  |
| Ga  | -1.32 | -0.50 | 0.10 | -2.37 | -0.00 | -2.98  |
| E. Pa. | 3.66 | 0.40 | 0.11 | 1.20  | 0.13 | 0.97   |
| 4-par. | 3.10 | 0.56 | -0.02 | 1.64  | 0.06 | 1.21   |
| 5-par. | 1.59 | 0.15 | -0.02 | 0.88  | 0.04 | -0.02  |
| 5-par. 2 | 1.46 | 0.31 | -0.02 | -0.11 | 0.05 | -0.20  |
| 6-par | 0.90 | 0.26 | -0.01 | 0.06  | 0.05 | -0.52  |

|   | L-N | L-G | We  | Pa  | Ga  | E. Pa. |
|---|-----|-----|-----|-----|-----|--------|
| L-N | 4.31 | 2.51 | 2.03 | 16.10 | 3.24 | 6.59   |
| L-G | 4.44 | 2.56 | 1.72 | 9.17  | 2.71 | 7.52   |
| We  | 3.81 | 2.57 | 1.15 | 5.01  | 1.65 | 6.10   |
| Pa  | 4.29 | 2.87 | 2.26 | 7.67  | 2.17 | 6.36   |
| Ga  | 3.90 | 2.50 | 1.17 | 5.21  | 1.66 | 6.29   |
| E. Pa. | 8.79 | 3.33 | 1.17 | 9.53  | 1.80 | 10.08  |
| 4-par. | 8.71 | 3.51 | 1.15 | 11.56 | 1.72 | 10.66  |
| 5-par. | 5.95 | 2.96 | 1.16 | 8.34  | 1.69 | 8.26   |
| 5-par. 2 | 5.96 | 3.11 | 1.15 | 7.22  | 1.71 | 8.24   |
| 6-par | 5.41 | 3.08 | 1.16 | 7.64  | 1.69 | 7.77   |

RMSE

|   | L-N | L-G | We  | Pa  | Ga  | E. Pa. |
|---|-----|-----|-----|-----|-----|--------|
| L-N | 4.31 | 2.51 | 2.03 | 16.10 | 3.24 | 6.59   |
| L-G | 4.44 | 2.56 | 1.72 | 9.17  | 2.71 | 7.52   |
| We  | 3.81 | 2.57 | 1.15 | 5.01  | 1.65 | 6.10   |
| Pa  | 4.29 | 2.87 | 2.26 | 7.67  | 2.17 | 6.36   |
| Ga  | 3.90 | 2.50 | 1.17 | 5.21  | 1.66 | 6.29   |
| E. Pa. | 8.79 | 3.33 | 1.17 | 9.53  | 1.80 | 10.08  |
| 4-par. | 8.71 | 3.51 | 1.15 | 11.56 | 1.72 | 10.66  |
| 5-par. | 5.95 | 2.96 | 1.16 | 8.34  | 1.69 | 8.26   |
| 5-par. 2 | 5.96 | 3.11 | 1.15 | 7.22  | 1.71 | 8.24   |
| 6-par | 5.41 | 3.08 | 1.16 | 7.64  | 1.69 | 7.77   |
|       | L-N | L-G | We   | Pa   | Ga   | E. Pa. |
|-------|-----|-----|------|------|------|--------|
| L-N   | 1.41| 0.10| 2.18 | 25.13| 3.85 | 0.89   |
| L-G   | 1.38| 0.16| 1.70 | 14.84| 3.09 | 4.68   |
| We    | -3.56| -0.83| -0.04| -6.63| -0.14| -8.07  |
| Pa    | -0.66| 2.07| 3.14 | 3.86 | 2.13 | -4.64  |
| Ga    | -3.94| -1.17| 0.20 | -7.16| -0.01| -8.46  |
| E. Pa.| 14.70| 1.25| 0.21 | 7.89 | 0.32 | 6.33   |
| 4-par.| 12.82| 2.03| -0.02| 11.57| 0.20 | 8.24   |
| 5-par.| 6.07 | 0.61| -0.010| 5.89 | 0.09 | 2.45   |
| 5-par. 2 | 5.53 | 1.09| -0.02| 2.37 | 0.16 | 2.05   |
| 6-par | 3.82 | 1.08| -0.01| 2.93 | 0.14 | 1.02   |

|       | L-N | L-G | We   | Pa   | Ga   | E. Pa. |
|-------|-----|-----|------|------|------|--------|
| L-N   | 6.97| 3.29| 2.99 | 39.46| 5.22 | 10.85  |
| L-G   | 8.32| 3.50| 2.51 | 27.20| 4.50 | 16.41  |
| We    | 5.95| 3.31| 1.36 | 9.05 | 2.01 | 10.80  |
| Pa    | 7.67| 3.79| 3.54 | 24.29| 3.06 | 11.92  |
| Ga    | 6.13| 3.24| 1.40 | 9.37 | 2.02 | 11.06  |
| E. Pa.| 30.94| 6.36| 1.41 | 42.30| 2.71 | 32.95  |
| 4-par.| 35.87| 8.03| 1.36 | 62.22| 2.37 | 39.93  |
| 5-par. | 18.65| 4.92| 1.37 | 31.50| 2.13 | 25.01  |
| 5-par. 2 | 17.54 | 5.56| 1.36 | 23.48| 2.35 | 23.98  |
| 6-par | 16.97| 6.25| 1.37 | 27.72| 2.28 | 22.07  |

Table 12: Bias and RMSE on the upper 99% reserve from each of the applied (A) distributions and each distribution from Table 1 as the true (T) one when $\lambda = 10$ and $n = 50$. 

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### Bias

|     | L-N | L-G | We  | Pa  | Ga  | E. Pa. |
|-----|-----|-----|-----|-----|-----|--------|
| L-N | 35.95 | 9.37 | 47.76 | 379.00 | 75.69 | 33.02 |
| L-G | 21.65 | 4.37 | 32.34 | 203.38 | 50.69 | 55.15 |
| We  | 6.11  | 10.80 | -0.91 | -55.01 | 3.86  | -56.26 |
| Pa  | 16.91 | 14.76 | 14.28 | 63.59  | 12.09 | -47.55 |
| Ga  | 8.57  | -0.46 | -0.15 | -40.20 | 1.65  | -52.97 |
| E. Pa. | 230.25 | 16.97 | 0.02  | 164.58 | 4.95  | 100.62 |
| 4-par. | 221.97 | 25.83 | -0.47 | 279.04 | 4.26  | 143.03 |
| 5-par. | 107.65 | 7.68  | -1.03 | 113.62 | 2.71  | 82.51 |
| 5-par. 2 | 88.75  | 12.74 | -0.48 | 57.37  | 3.86  | 38.23 |
| 6-par | 97.79  | 18.13 | -0.25 | 112.87 | 3.84  | 31.50 |

### RMSE

|     | L-N | L-G | We  | Pa  | Ga  | E. Pa. |
|-----|-----|-----|-----|-----|-----|--------|
| L-N | 205.64 | 142.70 | 97.14 | 632.16 | 142.68 | 304.37 |
| L-G | 217.06 | 143.74 | 88.36 | 489.20 | 127.64 | 360.71 |
| We  | 194.86 | 149.82 | 76.73 | 233.29 | 103.96 | 292.84 |
| Pa  | 209.12 | 147.00 | 79.63 | 493.96 | 105.37 | 305.72 |
| Ga  | 204.40 | 146.70 | 76.95 | 251.56 | 103.61 | 310.20 |
| E. Pa. | 583.73 | 172.87 | 76.95 | 1,255.97 | 106.77 | 671.77 |
| 4-par. | 802.98 | 187.32 | 76.86 | 2,046.01 | 105.02 | 877.81 |
| 5-par. | 445.44 | 159.55 | 77.27 | 719.91 | 104.17 | 994.71 |
| 5-par. 2 | 378.09 | 164.44 | 76.85 | 552.24 | 105.09 | 564.65 |
| 6-par | 611.29 | 202.13 | 77.06 | 1,254.26 | 105.26 | 502.11 |

Table 13: Bias and RMSE on the upper 95% reserve from each of the applied (A) distributions and each distribution from Table 4 as the true (T) one when $\lambda = 1000$ and $n = 50$.  

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### Bias

| A\T | L-N | L-G | We  | Pa  | Ga  | E. Pa. |
|-----|-----|-----|-----|-----|-----|--------|
| L-N | 38.71 | 9.44 | 51.42 | 464.55 | 82.16 | 27.93  |
| L-G | 32.59 | 4.59 | 35.23 | 401.23 | 56.32 | 91.35  |
| We  | -67.00 | 9.34 | -0.99 | -74.62 | 3.56 | -7.84  |
| Pa  | 19.52 | 19.65 | 21.08 | 197.96 | 16.69 | -58.51 |
| Ga  | 96.10 | -2.60 | 0.23 | -60.93 | 1.61 | -76.01 |
| E. Pa. | 532.12 | 23.82 | 0.44 | 640.07 | 6.21 | 303.66 |
| 4-par. | 603.33 | 44.13 | -0.53 | 1,307.18 | 5.05 | 482.09 |
| 5-par. | 331.77 | 11.94 | -1.06 | 401.44 | 2.86 | 1,632.20 |
| 5-par. 2 | 203.55 | 19.97 | -0.54 | 308.52 | 5.01 | 217.32 |
| 6-par | 831.41 | 68.95 | -0.29 | 1,279.52 | 5.52 | 250.79 |

### RMSE

| A\T | L-N | L-G | We  | Pa  | Ga  | E. Pa. |
|-----|-----|-----|-----|-----|-----|--------|
| L-N | 217.17 | 147.23 | 101.21 | 799.52 | 150.22 | 322.07 |
| L-G | 244.53 | 148.72 | 91.57 | 913.80 | 134.35 | 452.60 |
| We  | 201.83 | 154.23 | 78.41 | 247.45 | 106.58 | 308.23 |
| Pa  | 230.12 | 152.05 | 83.37 | 1,346.81 | 109.04 | 341.41 |
| Ga  | 211.16 | 150.92 | 78.65 | 264.53 | 106.25 | 325.22 |
| E. Pa. | 1,523.81 | 203.41 | 78.68 | 6,470.25 | 112.34 | 1,911.60 |
| 4-par. | 3,027.09 | 262.63 | 78.54 | 12,924.03 | 108.72 | 2.84  |
| 5-par. | 2,842.04 | 173.23 | 78.96 | 2,667.00 | 106.85 | 31,310.46 |
| 5-par. 2 | 1,161.21 | 185.25 | 78.52 | 2,268.97 | 111.26 | 2,020.67 |
| 6-par | 16,704.91 | 813.31 | 78.72 | 34,387.69 | 120.78 | 1,767.23 |

Table 14: Bias and RMSE on the upper 99% reserve from each of the applied (A) distributions and each distribution from Table 1 as the true (T) one when \( \lambda = 1000 \) and \( n = 50 \).
Table 15: Estimated 95 and 99% quantiles of the claim size distribution and reserves from each of the ten distributions considered in the simulation study, as well as empirical estimates, along with 95% confidence intervals in the first two columns. The reserves are divided by 1,000.

| Distr. | 95%     | 99%     | Reserve 95% | Reserve 99% |
|--------|---------|---------|-------------|-------------|
| Emp    | 72.5    | 139.9   | 25.9        | 26.7        |
| L-N    | 73.9 (71.3, 76.5) | 140.8 (134.7, 146.9) | 26.4 (25.6, 27.2) | 27.3 (26.5, 28.1) |
| L-G    | 91.7 (87.7, 96.2) | 231.4 (217.4, 247.2) | 32.8 (31.2, 34.4) | 35.8 (33.8, 38.0) |
| We     | 68.5 (66.7, 70.2) | 102.2 (99.1, 105.2) | 25.7 (25.2, 26.3) | 26.5 (25.9, 27.0) |
| Pa     | 74.3 (71.9, 76.7) | 122.9 (117.0, 128.9) | 25.7 (25.0, 26.3) | 26.5 (25.8, 27.2) |
| Ga     | 65.1 (63.5, 66.6) | 96.3 (93.7, 98.7) | 25.6 (25.0, 26.1) | 26.3 (25.7, 26.8) |
| E. Pa. | 68.5 (65.6, 71.0) | 131.1 (121.8, 139.1) | 25.7 (24.9, 26.5) | 26.6 (25.8, 27.5) |
| 4-par. | 69.1 (65.5, 72.6) | 160.5 (144.5, 179.1) | 28.2 (26.6, 29.9) | 31.3 (28.8, 34.5) |
| 5-par. | 69.4 (65.9, 86.9) | 146.4 (130.9, 192.7) | 26.5 (24.8, 30.3) | 27.8 (25.7, 32.5) |
| 5-par. 2| 70.3 (67.1, 73.7) | 143.0 (131.4, 154.2) | 26.0 (25.1, 27.0) | 27.0 (26.0, 28.1) |
| 6-par. | 69.6 (65.8, 84.6) | 147.9 (131.0, 185.8) | 26.5 (24.7, 29.6) | 27.8 (25.7, 31.7) |

Table 16: Numbers of rejections and p-values from the back-test of the quantile estimates.

| Distr. | No. of exc. 95% | No. of exc. 99% | P-value 95% | P-value 99% |
|--------|----------------|----------------|-------------|-------------|
| L-N    | 314            | 64             | 0.668       | 1.000       |
| L-G    | 212            | 11             | 0.000       | 0.000       |
| We     | 361            | 162            | 0.030       | 0.000       |
| Pa     | 310            | 102            | 0.511       | 0.000       |
| Ga     | 393            | 195            | 0.000       | 0.000       |
| E. Pa. | 361            | 85             | 0.030       | 0.012       |
| 4-par. | 355            | 43             | 0.063       | 0.006       |
| 5-par. | 351            | 58             | 0.103       | 0.453       |
| 5-par. 2| 346            | 61             | 0.179       | 0.754       |
| 6-par. | 348            | 57             | 0.145       | 0.381       |

Table 16: Numbers of rejections and p-values from the back-test of the quantile estimates.