Two Constructions for Minimal Ternary Linear Codes

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Abstract

Recently, minimal linear codes have been extensively studied due to their applications in secret sharing schemes, two-party computations, and so on. Constructing minimal linear codes violating the Ashikhmin-Barg condition and determining their weight distributions have been an interesting research topic in coding theory and cryptography. In this paper, basing on exponential sums and Krawtchouk polynomials, we first prove that $g(m,k)$ in [18], which is the characteristic function of some subset in $\mathbb{F}_3^m$, can be generalized to be $f(m,k)$ for obtaining a minimal linear code violating the Ashikhmin-Barg condition; secondly, we employ $g(m,k)$ to construct a class of ternary minimal linear codes violating the Ashikhmin-Barg condition, whose minimal distance is better than that of codes in [18].

Keywords Linear code, minimal code, minimal vector, weight distribution.

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1 Introduction

Throughout the whole paper, let $p$ be a prime and $q = p^m$ for some positive integer $m$, denote $\mathbb{F}_p$ to be the finite field with $p$ elements. An $[n,k,d]$ code $C$ over $\mathbb{F}_p$ is a $k$-dimensional subspace of $\mathbb{F}_p^n$ with minimum (Hamming) distance $d$. For any $i = 0, 1, \cdots, n$, $A_i$ denotes the number of codewords with Hamming weight $i$ in $C$ of length $n$. The weight enumerator of $C$ is defined by

$$1 + A_1 z + A_2 z^2 + \cdots + A_n z^n.$$  

The weight distribution $(1, A_1, A_2, \cdots, A_n)$ is an important research topic in coding theory, since it contains some crucial information as to estimate the error-correcting capability and the probability of error-detection and correction with respect to some algorithms [12]. $C$ is said to be a $t$-weight code if the number of nonzero $A_i$ in the sequence $(A_1, A_2, \cdots, A_n)$ is equal to $t$.

For a vector $a = (a_1, a_2, \cdots, a_n) \in \mathbb{F}_p^n$, the support of $a$ is defined by

$$\text{Supp}(a) = \{1 \leq i \leq n : a_i \neq 0\}.$$  

Let $wt(a)$ be the Hamming weight of $a$, then $wt(a) = |\text{Supp}(a)|$. We say that a vector $a \in \mathbb{F}_p^m$ covers a vector $b \in \mathbb{F}_p^m$ if $\text{Supp}(b) \subseteq \text{Supp}(a)$. A codeword $c$ in a linear code $C$ is minimal if $c$ covers only those codewords $uc$ ($u \in \mathbb{F}_p^*$). A linear code $C$ is said to be minimal if every codeword in $C$ is minimal.

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It’s well-known that minimal linear codes are wildly applied, especially in secret sharing schemes \cite{6} \cite{24}. A sufficient condition for a linear code to be minimal is given in the following lemma \cite{1}.

**Lemma 1.1. (Ashikhmin-Barg)** A linear code $C$ over $\mathbb{F}_p$ is minimal if

$$\frac{w_{\text{min}}}{w_{\text{max}}} > \frac{p-1}{p},$$

where $w_{\text{min}}$ and $w_{\text{max}}$ denote the minimum and maximum nonzero Hamming weights in the code $C$, respectively.

With the help of Lemma 1.1, many minimal linear codes are constructed from linear codes with a few weights \cite{9} \cite{10} \cite{12} \cite{19} \cite{22}. The sufficient condition in Lemma 1.1 is not usually necessary for minimal codes. Recently, searching for minimal linear codes with $\frac{w_{\text{min}}}{w_{\text{max}}} \leq \frac{p-1}{p}$ has been an interesting research topic. Chang and Hyun \cite{7} made a breakthrough and constructed an infinite family of minimal binary linear codes with $\frac{w_{\text{min}}}{w_{\text{max}}} < \frac{1}{2}$ by the following generic construction

$$C_f = \{(uf(x) + v \cdot x) : u \in \mathbb{F}_p, v \in \mathbb{F}_p^m\}. \quad (1.1)$$

Basing on this generic construction, Ding et al. \cite{11} gave a necessary and sufficient condition for a binary linear code to be minimal, and employed special Boolean functions to obtain three classes of minimal binary linear codes with $\frac{w_{\text{min}}}{w_{\text{max}}} < \frac{1}{2}$, they \cite{18} also used the characteristic function of the subset in $\mathbb{F}_3^m$ to construct a class of minimal ternary linear codes with $\frac{w_{\text{min}}}{w_{\text{max}}} < \frac{2}{3}$. Bartoli and Bonini \cite{2} generalized the construction of minimal linear codes in \cite{18} from ternary case to be odd characteristic case. Bonini and Borello \cite{4} presented a family of codes arising from cutting blocking sets, many of these codes do not satisfy the Ashikhmin-Barg condition.

Tang \cite{20} gave two classes of codes arising from cutting blocking sets, which don’t satisfy the Ashikhmin-Barg condition either. In \cite{4}, an inductive construction of minimal codes was presented. Li and Yue \cite{15} obtained some minimal binary linear codes basing on Boolean functions. Xu and Qu \cite{23} constructed minimal $p$-ary linear codes from some special functions, where $p$ is an odd prime. Lu et al. \cite{17} studied the existence of minimal linear codes. Tao et al. \cite{21} obtained three-weight or four-weight minimal linear codes not satisfying the Ashikhmin-Barg condition by using partial difference sets.

In this paper, using the generic construction (1.1), we first present a class of ternary minimal linear codes violating the Ashikhmin-Barg condition, in which $f_{(m,k)}$ is not a characteristic function of any subset in $\mathbb{F}_3^m$. Secondly, basing on $f_{(m,k)}$, we obtain another class of ternary minimal linear codes violating the Ashikhmin-Barg condition, whose minimal distance is better than that in \cite{18}. The paper is organized as follows. Section 2 provides some properties for Krawtchouk polynomials and some results about ternary minimal codes, which will be needed in the sequel. Section 3 presents two classes of ternary minimal codes violating the Ashikhmin-Barg condition, and determines their weight distributions. Section 4 concludes the whole paper.

## 2 Preliminaries
Krawtchouk polynomials introduced by Lloyd in 1957 [16] are applied in coding theory [3] [13], cryptography [8] and combinatorics [14]. Here we only give a brief introduction to Krawtchouk polynomials with their essential properties. For more details, readers are referred to [3] [13] [14] [16].

Let $m$, $h$ be positive integers and $x$ be a variable taking nonnegative values. The Krawtchouk polynomial (of degree $t$ with parameters $h$ and $m$) is defined by

$$K_t(x, m) = \sum_{j=0}^{t} (-1)^j (h-1)^{t-j} \binom{x}{j} \binom{m-x}{t-j}.$$ 

Accordingly, the Lloyd polynomial $\Psi_k(x, m)$ (of degree $t$ with parameters $h$ and $m$) is given by

$$\Psi_k(x, m) = \sum_{t=0}^{k} K_t(x, m).$$

The following Lemma 2.1 will be useful in the sequel.

**Lemma 2.1.** [18] Let symbols and notations be as above, suppose that $1 \leq x \leq m$, $1 \leq k \leq m-1$, $u \in \mathbb{Z}_q^m$ with Hamming weight $\text{wt}(u) = i$, then the following hold:

1. $\Psi_k(x, m) = K_k(x-1, m-1)$;
2. $K_t(0, m) = (h-1)^t \binom{m-1}{t}$;
3. $|\Psi_k(x, m)| \leq (h-1)^k \binom{m-1}{k}$;
4. $\sum_{\nu \in \mathbb{Z}_q^m, \text{wt}(\nu) = i} \zeta_q^{u \cdot \nu} = K_t(i, m),$

where $\zeta_q$ denotes the $q$-th primitive root of complex unity, and the inner product $u \cdot v$ in $\mathbb{Z}_q^m$ is defined by $u \cdot v = u_1v_1 + \cdots + u_mv_m$.

Note that the upper bound for $|\Psi_k(x, m)|$ in Lemma 2.1 is tight since

$$\Psi_k(1, m) = K_k(0, m-1) = (h-1)^k \binom{m-1}{k}.$$ 

Assume $f(x)$ is a function from $\mathbb{F}_p^m$ to $\mathbb{F}_p$ such that $f(0) = 0$ and $f(b) \neq 0$ for at least one $b \in \mathbb{F}_p^m$. Recall that the Walsh transform of $f$ is given by

$$\hat{f}(w) = \sum_{x \in \mathbb{F}_p^m} \zeta_p^{f(x) - w \cdot x} \quad (w \in \mathbb{F}_p^m).$$

The following result shows that for $p = 3$ the weight distribution of the ternary code $C_f$ can be determined by the Walsh spectrum of $f$.

**Lemma 2.2.** [18] For $p = 3$, assume that $f(x) \neq w \cdot x$ for any $w \in \mathbb{F}_3^m$. Then the linear code $C_f$ in (1.1) has length $3^m - 1$ and dimension $m + 1$. In addition, the weight distribution of $C_f$ is given by the following multiset union:

$$\left\{ \{2(3^m-1) - \text{Re}(\hat{f}(v))\} : u \in \mathbb{F}_3^x, v \in \mathbb{F}_3^m \right\} \cup$$

$$\left\{ \{3^m - 3^{m-1} : u = 0, v \in \mathbb{F}_3^m \} \right\} \cup \{ \{0\} \}.$$

**Herein and hereafter, Re(x) denotes the real part of the complex number x.**

The following Lemma 2.3 gives a sufficient and necessary condition for $C_f$ to be minimal in terms of the Walsh spectrum of $f$. 

**Lemma 2.3.** [18] For $p = 3$, assume that $f(x) \neq w \cdot x$ for any $w \in \mathbb{F}_3^m$. Then the linear code $C_f$ in (1.1) has length $3^m - 1$ and dimension $m + 1$. In addition, the weight distribution of $C_f$ is given by the following multiset union:

$$\left\{ \{2(3^m-1) - \text{Re}(\hat{f}(v))\} : u \in \mathbb{F}_3^x, v \in \mathbb{F}_3^m \right\} \cup$$

$$\left\{ \{3^m - 3^{m-1} : u = 0, v \in \mathbb{F}_3^m \} \right\} \cup \{ \{0\} \}.$$

**Herein and hereafter, Re(x) denotes the real part of the complex number x.**
Lemma 2.3. \[18\] Let $C_f$ be the ternary code in Lemma 2.2. Assume that $f(x) \neq v \cdot x$ for any $x \in \mathbb{F}_3^m$. Then $C_f$ is a $[3^m - 1, m + 1]$ minimal code if and only if
\[
\text{Re}(\hat{f}(w_1)) + \text{Re}(\hat{f}(w_2)) - 2\text{Re}(\hat{f}(w_3)) \neq 3^m,
\]
and
\[
\text{Re}(\hat{f}(w_1)) + \text{Re}(\hat{f}(w_2)) + \text{Re}(\hat{f}(w_3)) \neq 3^m
\]
for any pairwise distinct vectors $w_1, w_2$, and $w_3$ in $\mathbb{F}_3^m$ satisfying $w_1 + w_2 + w_3 = 0$.

For a positive integer $k$ with $1 \leq k \leq m$. Let $S(m, k)$ denote the set of vectors in $\mathbb{F}_3^m \setminus \{0\}$ with Hamming weight at most $k$. It’s clear that
\[
|S(m, k)| = \sum_{j=1}^{k} 2^j \binom{m}{j}.
\]

Define the function $g(m, k) : \mathbb{F}_3^m \rightarrow \mathbb{F}_3$ as
\[
g(m, k) = \begin{cases} 
1, & \text{if } x \in S(m, k); \\
0, & \text{otherwise}.
\end{cases}
\]

Using $g(m, k)$ to replace the function $f$ in \[1, 1\], one can obtain a ternary linear code $C_{g(m, k)}$, which is a minimal linear code violating the Ashikhmin-Barg condition. The parameters and weight distribution of $C_{g(m, k)}$ are given as follows, which will be needed in next section.

Lemma 2.4. \[18\] Let $m, k$ be integer with $m \geq 5$ and $2 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$. Then the linear code $C_{g(m, k)}$ is minimal and has parameters
\[
[3^m - 1, m + 1, \sum_{j=1}^{k} 2^j \binom{m}{j}]
\]
the weight distribution is given in Table 1. Furthermore, $\frac{w_{\text{min}}}{w_{\text{max}}} \leq \frac{2}{3}$ if and only if
\[
3 \sum_{j=1}^{k} 2^j \binom{m}{j} \leq 2(3^m - 3^{m-1}) + 2^{k+1} \binom{m-1}{k} - 2.
\]

| Weight $w$ | Multiplicity $A_w$ | condition |
|------------|-------------------|------------|
| 0          | 1                 | $u = 0, v = 0$ |
| $3^m - 3^{m-1} + \Psi_k(i, m) - 1$ | $2^{i+1} \binom{m}{i}$ | $1 \leq i \leq m$; $u \in \mathbb{F}_3^*; \text{wt}(v) = i$ with $\text{Re}(\hat{g}(m, k)(v)) = -\frac{2}{3}(\Psi_k(i, m) - 1)$ |
| $\sum_{j=1}^{k} 2^j \binom{m}{j}$ | 2 | $u \in \mathbb{F}_3^*, v = 0$ with $\text{Re}(\hat{g}(m, k)(0)) = 3^m - \frac{2}{3} \sum_{j=1}^{k} 2^j \binom{m}{j}$ |
| $3^m - 3^{m-1}$ | $3^{m-1}$ | $u = 0, v \in \mathbb{F}_3^*$ |

Let $D \subseteq \mathbb{F}_q^*$, and $\overline{D} = \mathbb{F}_q^* \setminus D$, the characteristic function of $D$ is defined to be
\[
f_D(x) = \begin{cases} 
1, & \text{if } x \in D; \\
0, & \text{otherwise}.
\end{cases}
\]

The relationship between $\hat{f}_D(w)$ and $\hat{f}_{\overline{D}}(w)$ is given by the following lemma.
Lemma 2.5. Let $D \subseteq \mathbb{F}_q^*$ and $\overline{D} = \mathbb{F}_q^* \setminus D$. Then
\[
\hat{f}_D(w) + \hat{f}_{\overline{D}}(w) = \begin{cases} 
(q-1)\zeta_p + q + 1, & \text{if } w = 0; \\
1 - \zeta_p, & \text{otherwise}.
\end{cases}
\]

3 Main Results and Their Proofs

In this section, we first prove that the characteristic function $g_{(m,k)}$ in Lemma 2.4 can be generalized to be $f_{(m,k)}$ for obtaining the same minimal linear code violating the Ashikhmin-Barg condition.

Let $S$ be a nonempty subset of $[k] = \{1, 2, 3, \cdots, k\}$, define the function $f_{(m,k)}$ from $\mathbb{F}_3^m$ to $\mathbb{F}_3$ as
\[
f_{(m,k)} = \begin{cases} -1, & \text{if } \text{wt}(x) \in S; \\
1, & \text{if } \text{wt}(x) \in [k] \setminus S; \\
0, & \text{otherwise}.
\end{cases}
\]

Theorem 3.1. Let $m, k$ be integers with $m \geq 5$ and $2 \leq k \leq \left\lfloor \frac{m-1}{2} \right\rfloor$. Then the linear code $C_{f_{(m,k)}}$ is minimal and has parameters
\[
[3^m - 1, m + 1, \sum_{j=1}^{k} 2^j \binom{m}{j}],
\]
whose distribution is given in Table II. Furthermore, $\frac{w_{\text{max}}}{w_{\text{min}}} < \frac{2}{3}$ if and only if
\[
3 \sum_{j=1}^{k} 2^j \binom{m}{j} \leq 2(3^m - 3^{m-1}) + 2^{k+1} \binom{m-1}{k} - 2.
\]

Table II: the weight distribution of $C_{f_{(m,k)}}$ in lemma 3.1

| Weight $w$ | Multiplicity $A_w$ | Condition |
|------------|-------------------|-----------|
| 0          | 1                 | $u = 0, v = 0$ |
| $3^m - 3^{m-1} + \Psi_k(i, m) - 1$ | $2^{i+1} \binom{m}{i} (1 \leq i \leq m)$ | $u \in \mathbb{F}_3^*, \text{wt}(v) = i$ with $\text{Re}(\hat{f}_{(m,k)}(v)) = -\frac{2}{3}(\Psi_k(i, m) - 1)$ |
| $\sum_{j=1}^{k} 2^j \binom{m}{j}$ | 2                 | $u \in \mathbb{F}_3^*, v = 0$ with $\text{Re}(\hat{f}_{(m,k)}(0)) = 3^m - \frac{2}{3} \sum_{j=1}^{k} 2^j \binom{m}{j}$ |
| $3^m - 3^{m-1}$ | $3^m - 1$         | $u = 0, v \in \mathbb{F}_3^{m*}$ |

Proof. For any $v \in \mathbb{F}_3^m$, to prove Theorem 3.1, it's enough to show that $\text{Re}(\hat{f}_{(m,k)}(v))$ equals to $\text{Re}(\hat{g}_{(m,k)}(v))$. In fact, from the definition of $f_{(m,k)}$, we have
\[
\hat{f}_{(m,k)}(v) = \sum_{x \in \mathbb{F}_3^m} \zeta_3^{f_{(m,k)}(x)v} = \sum_{\text{wt}(x) \in S} \zeta_3^{1-v_x} + \sum_{\text{wt}(x) \in [k] \setminus S} \zeta_3^{1-v_x} + \sum_{x \in \mathbb{F}_3^m \setminus S(m,k)} \zeta_3^{v_x} = \sum_{x \in \mathbb{F}_3^m} \zeta_3^{v_x} + \sum_{x \in [k] \setminus S} (\zeta_3^{1-v_x} - \zeta_3^{-v_x}) + \sum_{x \in \mathbb{F}_3^m \setminus S(m,k)} (\zeta_3^{1-v_x} - \zeta_3^{-v_x}) = \sum_{x \in \mathbb{F}_3^m} \zeta_3^{v_x} + \sum_{x \in [k]} (\zeta_3 - 1)\zeta_3^{-v_x} + \sum_{x \in \mathbb{F}_3^m \setminus S(m,k)} (\zeta_3 - 1)\zeta_3^{-v_x} = \sum_{x \in \mathbb{F}_3^m} \zeta_3^{v_x} + \sum_{x \in [k]} (\zeta_3 - 1)\zeta_3^{-v_x} + \sum_{x \in \mathbb{F}_3^m \setminus S(m,k)} (\zeta_3 - 1)\zeta_3^{-v_x}.
\]
for \( v \in \mathbb{F}_3^m \). If \( v = 0 \), then we have

\[
\hat{f}(m,k)(0) = 3^m + (\zeta_3 - 1) \sum_{j=1}^{k} 2^{j} \binom{m}{j} + (\zeta_3^{-1} - \zeta_3)|S|
\]

and

\[
Re(\hat{f}(m,k)(0)) = 3^m - \frac{3}{2} \sum_{j=1}^{k} 2^{j} \binom{m}{j}.
\]

If \( v \neq 0 \) with \( wt(v) = i \), then by Lemma 2.1 we have

\[
\hat{f}(m,k)(v) = (\zeta_3 - 1) \sum_{i=1}^{k} K_i(i,m) + (\zeta_3^{-1} - \zeta_3) \sum_{i \in S} K_i(i,m) = (\zeta_3 - 1)(\Psi_k(i,m) - 1) + (\zeta_3^{-1} - \zeta_3) \sum_{i \in S} K_i(i,m).
\]

Thus

\[
Re(\hat{f}(m,k)(v)) = -\frac{3}{2}(\Psi_k(i,m) - 1)).
\]

Then the weight distribution of \( C_{\hat{f}(m,k)} \) follows from Lemma 2.2 and for \( v \in \mathbb{F}_3^m \), \( Re(\hat{f}(m,k)(v)) = Re(\widehat{g}(m,k)(v)) \), thus the desired results follow from Lemma 2.4.

This completes the proof for Theorem 3.1.

**Remark 3.1.** For any odd prime \( p \), constructing minimal \( p \)-ary linear codes and determining their weight distributions based on \( f \), \( f \) is usually taken to be the characteristic function of some subset in \( \mathbb{F}_p^m \). There are a few results when \( f \) is not the characteristic function for some subset in \( \mathbb{F}_p^m \). And it is interesting to present minimal \( p \)-ary linear codes based on non-characteristic functions. Although the minimal linear code violating the Ashikhmin-Barg condition in Theorem 3.1 is the same as that in Lemma 2.4, \( f(m,k) \) is not the characteristic function of some subset in \( \mathbb{F}_3^m \), which partially gives a result for the case mentioned above.

Next, we construct a new class of minimal linear codes violating the Ashikhmin-Barg condition basing on Lemma 2.2 and Lemma 2.4. Recall that \( S(m,k) \) denote the set of vectors in \( \mathbb{F}_3^{m-*} \) with Hamming weight at most \( k \), let \( \overline{S}(m,k) = \mathbb{F}_3^{m-*} \setminus S(m,k) \), define the function \( \overline{g}(m,k) : \mathbb{F}_3^m \rightarrow \mathbb{F}_3 \) as

\[
\overline{g}(m,k) = \begin{cases} 
1, & \text{if } x \in \overline{S}(m,k); \\
0, & \text{otherwise}.
\end{cases}
\]

**Theorem 3.2.** The ternary code \( C_{\overline{g}(m,k)} \) has length \( 3^m - 1 \), and dimension \( m+1 \), the weight distribution is given in Table III, where \( \Psi_k(x,m) \) is the Lloyd polynomial.

| Weight \( w \) | Multiplicity \( A_w \) | Condition |
|----------------|-----------------|-----------|
| 0              | 1               | \( u = 0, v = 0 \) |
| \( 3^m - 3^{m-1} - \Psi_k(i,m) \) | \( 2^{i+i^*} \binom{m}{i} \) (\( 1 \leq i \leq m \)) | \( u \in \mathbb{F}_3^m, wt(v) = i \) with \( Re(\overline{g}(m,k)(v)) = \frac{2}{3}(\Psi_k(i,m)) \) |
| \( 3^m - \sum_{j=0}^{k} 2^j \binom{m}{j} \) | 2 | \( u \in \mathbb{F}_3^m, v = 0 \) with \( Re(\overline{g}(m,k)(0)) = -\frac{3^m}{2} + \frac{2}{3} \sum_{j=0}^{k} 2^j \binom{m}{j} \) |
| \( 3^m - 3^{m-1} \) | \( 3^m - 1 \) | \( u = 0, v \in \mathbb{F}_3^{m-*} \) |
Proof. Note that \( S(m, k) = \mathbb{P}_3^m \setminus S(m, k) \), if \( \mathbf{v} = 0 \), we have

\[
\hat{g}(m, k)(0) = (3^m - 1)\zeta_3 + 3^m + 1 - 3^m - (\zeta_3 - 1) \sum_{j=1}^{k} 2^j \binom{m}{j} + 1
\]

by Lemma 2.5 for \( p = 3 \) and Lemma 2.4. Then

\[
\text{Re}(\hat{g}(m, k)(0)) = -\frac{1}{2}(3^m - 1) + \frac{3}{2} \sum_{j=1}^{k} 2^j \binom{m}{j} + 1
\]

If \( \mathbf{v} \neq 0 \) and \( wt(\mathbf{v}) = i \), then by Lemma 2.5 for \( p = 3 \) and Lemma 2.4, we have

\[
\hat{g}(m, k)(\mathbf{v}) = 1 - \zeta_3 - (\zeta_3 - 1)(\Psi_k(i, m) - 1) = (1 - \zeta_3)\Psi_k(i, m).
\]

Thus

\[
\text{Re}(\hat{g}(m, k)(\mathbf{v})) = \frac{3}{2}\Psi_k(i, m).
\]

The weight distribution of \( C_{\hat{g}(m, k)} \) follows from Lemma 2.2.

This completes the proof for Theorem 3.2.

We need the following lemmas to obtain the parameters of the linear code in Theorem 3.2.

Lemma 3.1. Let \( m \) and \( t \) be integers with \( m \geq 16 \), \( t = \left\lfloor \frac{m - 1}{2} \right\rfloor \), then

\[
\binom{m+1}{t} - 3 \binom{m-1}{t} > 3 \binom{m-1}{t}.
\]

Proof. Note that

\[
\binom{m+1}{t} - 3 \binom{m-1}{t} = \frac{(m-1)(m-2)\cdots(m-t)t!}{(m-t+1)(m-t)} - \frac{(m+1)m}{(m-t+1)(m-t)} - 3,
\]

so we can consider (3.1) in two cases.

Case 1. If \( m = 2m_1 + 1 \) and \( t = \left\lfloor \frac{m-1}{2} \right\rfloor = m_1 \) with \( m_1 \geq 8 \), then we have

\[
\frac{(m+1)m}{(m-t+1)(m-t)} - 3 = \frac{(2m_1 + 2)(2m_1 + 1)}{(m_1 + 2)(m_1 + 1)} - 3 = 4m_1 + 2 - 3 = 4 - \frac{6}{m_1 + 2} - 3 = 1 - \frac{6}{m_1 + 2} > 0.
\]

Case 2. If \( m = 2m_1 \) and \( t = \left\lfloor \frac{m-1}{2} \right\rfloor = m_1 - 1 \) with \( m_1 \geq 8 \), then we have

\[
\frac{(m+1)m}{(m-t+1)(m-t)} - 3 = \frac{(2m_1 + 1)2m_1}{(m_1 + 2)(m_1 + 1)} - 3 = 1 - \frac{10m_1 + 8}{(m_1 + 2)(m_1 + 1)} = 1 - \frac{2}{m_1 + 2} \frac{5m_1 + 4}{m_1 + 1} = 1 - \frac{2}{m_1 + 2} (5 - \frac{1}{m_1 + 1}) > 1 - \frac{10}{m_1 + 2} \geq 0.
\]

Thus we complete the proof for Lemma 3.1.
Lemma 3.2. Let \( m, k \) be integers with \( m \geq 5 \) and \( 1 \leq k \leq \lfloor \frac{m-1}{2} \rfloor \), then
\[
3^{m-1} - 2^k \binom{m-1}{k} - \sum_{j=0}^{k} 2^j \binom{m}{j} > 0. \tag{3.2}
\]

Proof. Note that the value of formula
\[
3^{m-1} - 2^k \binom{m-1}{k} - \sum_{j=0}^{k} 2^j \binom{m}{j}
\]
decreases with \( k \) increasing, so for \( t = \lfloor \frac{m-1}{2} \rfloor \), we have
\[
3^{m-1} - 2^k \binom{m-1}{k} - \sum_{j=0}^{k} 2^j \binom{m}{j} \\
\geq (1 + 2)^{m-1} - 2^t \binom{m-1}{t} - \sum_{j=0}^{t} 2^j \binom{m}{j} \\
= \sum_{j=0}^{m-1} 2^j \binom{m-1}{j} - 2^t \binom{m-1}{t} - \sum_{j=1}^{t} 2^j \binom{m-1}{j} + (m-1) \binom{m-1}{t} - 1 \\
= \sum_{j=1}^{m-1} 2^j \binom{m-1}{j} - \sum_{j=1}^{t} 2^j \binom{m-1}{j} - \sum_{j=0}^{t-1} 2^{j+1} \binom{m-1}{j} - 2^t \binom{m-1}{t} \\
= \sum_{j=t+1}^{m-1} 2^{m-1-j} \binom{m-1}{j} - \sum_{j=0}^{t-1} 2^{j+1} \binom{m-1}{j} - 2^t \binom{m-1}{t} \\
\geq \sum_{j=0}^{t-1} 2^{j+1} (2^{m-2j-2} - 1) \binom{m-1}{j} - 2^t \binom{m-1}{t} \\
> \sum_{j=0}^{t-1} 2^{j+1} \binom{m-1}{j} - 2^t \binom{m-1}{t} \\
> 2^{t-1} \binom{m-1}{t-2} + 2^t \binom{m-1}{t-1} - 2^t \binom{m-1}{t-2} \\
= 2^{t-1} \left( \binom{m+1}{t} - 3 \binom{m-1}{t} \right).
\]

By lemma 3.1, the inequality \( 3.2 \) holds when \( m \geq 16 \), and for \( 5 \leq m \leq 15 \) one can easily verify \( 3.2 \) holds, here we omit the details.

This completes the proof for Lemma 3.2.

Lemma 3.3. Let \( m \) be an integer with \( m \geq 2 \), then
\[
\sum_{j=\lceil \frac{m}{2} \rceil + 1}^{m-1} 2^j \binom{m-1}{j} - \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} 2^{j+1} \binom{m-1}{j} > 0.
\]
Proof. We prove Lemma 3.3 in two cases. 
If \( m = 2m_1(m_1 \geq 1) \) is even, then we have 
\[
\begin{align*}
\sum_{j=\lfloor m \div 2 \rfloor + 1}^{m-1} 2^j \binom{m-1}{j} &- \sum_{j=0}^{\lfloor m \div 2 \rfloor - 1} 2^{j+1} \binom{m-1}{j} \\
= \sum_{j=0}^{m_1-1} 2^j \binom{2m_1 - 1}{j} &- \sum_{j=0}^{m_1-2} 2^{j+1} \binom{2m_1 - 1}{j} \\
= \sum_{j=0}^{m_1-1} 2^{2m_1-1-j} \binom{2m_1 - 1}{j} &- \sum_{j=0}^{m_1-2} 2^{j+1} \binom{2m_1 - 1}{j} \\
= \sum_{j=0}^{m_1-1} 2^{2m_1-1-j} \binom{2m_1 - 1}{j} &- \sum_{j=0}^{m_1-2} 2^{j+1} \binom{2m_1 - 1}{j} \\
> \sum_{j=0}^{m_1-2} \binom{2m_1 - 1}{j} (2^{2m_1-1-j} - 2^{j+1}) > 0.
\end{align*}
\]
If \( m = 2m_1 + 1(m_1 \geq 1) \) is odd, then we have 
\[
\begin{align*}
\sum_{j=\lfloor m \div 2 \rfloor + 1}^{m-1} 2^j \binom{m-1}{j} &- \sum_{j=0}^{\lfloor m \div 2 \rfloor - 1} 2^{j+1} \binom{m-1}{j} \\
= \sum_{j=0}^{2m_1+1} 2^j \binom{2m_1}{j} &- \sum_{j=0}^{m_1-1} 2^{j+1} \binom{2m_1}{j} \\
= \sum_{j=0}^{m_1-1} 2^{2m_1-j} \binom{2m_1}{j} &- \sum_{j=0}^{m_1-1} 2^{j+1} \binom{2m_1}{j} \\
= \sum_{j=0}^{m_1-1} 2^{2m_1-j} \binom{2m_1}{j} &- \sum_{j=0}^{m_1-1} 2^{j+1} \binom{2m_1}{j} \\
> \sum_{j=0}^{m_1-1} \binom{2m_1}{j} (2^{2m_1-j} - 2^{j+1}) > 0.
\end{align*}
\]

By the discussions above, we complete the proof for this lemma.

Now, we give the parameters of the code in Theorem 3.2 as follows.

**Corollary 3.1.** Let \( m, k \) be integers with \( m \geq 5 \) and \( 2 \leq k \leq \lfloor \frac{m-1}{2} \rfloor \). Then the linear code \( C_{\mathbf{e}(m,k)} \) in Theorem 3.2 has parameters 
\[
[3^{m-1} - 1, m + 1, 3^m - 3^{m-1} - 2^k \binom{m-1}{k}].
\]
Furthermore, \( \frac{w_{\text{min}}}{\text{w}_{\text{max}}} \leq \frac{2}{3} \) if and only if 
\[
2 \sum_{j=1}^{k} 2^j \binom{m}{j} \leq 3 \cdot 2^k \binom{m-1}{k}.
\]
Proof. By Table III, we divide the nonzero weights in $C_{g_{(m,k)}}$ into the following three cases.

$$\begin{cases}
  w(i) = 3^m - 3^{m-1} - \Psi_k(i, m), \\
  w' = 3^m - \sum_{j=0}^{k} 2^j \binom{m}{j}, \\
  w'' = 3^m - 3^{m-1}.
\end{cases}$$

Then we have

$$w' - w'' = 3^m - \sum_{j=0}^{k} 2^j \binom{m}{j} - 3^m + 3^{m-1} = 3^{m-1} - \sum_{j=0}^{k} 2^j \binom{m}{j} = \sum_{j=0}^{m-1} 2^j \binom{m-1}{j} + \sum_{j=0}^{k} 2^j \binom{m-1}{j} - \sum_{j=0}^{k-1} 2^j \binom{m-1}{j}.$$  

Note that the value of

$$\sum_{j=k+1}^{m-1} 2^j \binom{m-1}{j} - \sum_{j=0}^{k-1} 2^j \binom{m-1}{j}$$

decreases with $k$ increasing, thus we have

$$w' - w'' \geq \sum_{j=\lfloor \frac{m}{2} \rfloor+1}^{m-1} 2^j \binom{m-1}{j} - \sum_{j=0}^{k-1} 2^j \binom{m-1}{j} > 0$$

by Lemma 3.3. Note that

$$w' - w(i) = 3^m - \sum_{j=0}^{k} 2^j \binom{m}{j} - 3^m + 3^{m-1} + \Psi_k(i, m) = 3^{m-1} + \Psi_k(i, m) - \sum_{j=0}^{k} 2^j \binom{m}{j}.$$  

Thus according to Lemma 2.1, one can get

$$3^{m-1} + \Psi_k(i, m) - \sum_{j=0}^{k} 2^j \binom{m}{j} \geq 3^{m-1} - 2^k \binom{m-1}{k} - \sum_{j=0}^{k} 2^j \binom{m}{j}.$$  

Hence by Lemma 3.2 we have

$$w' - w(i) = 3^{m-1} + \Psi_k(i, m) - \sum_{j=0}^{k} 2^j \binom{m}{j} \geq 3^{m-1} - 2^k \binom{m-1}{k} - \sum_{j=0}^{k} 2^j \binom{m}{j} > 0.$$  

From the discussions above, the maximum Hamming weight of $C_{g_{(m,k)}}$ is given by $w_{max} = w'$. According to Lemma 2.1, the minimum Hamming weight of $C_{g_{(m,k)}}$ is given by

$$w_{min} = w(1) = 3^m - 3^{m-1} - \Psi_k(1, m) = 3^m - 3^{m-1} - 2^k \binom{m-1}{k}.$$  

This completes the proof.
Remark 3.2. The minimal distance of $C_{g(m,k)}$ is $\sum_{j=1}^{k} 2^j (m)$, and the minimal distance of $C_{g(m,k)}$ is $3^m - 3^{m-1} - 2^k (m-1)$, so by Lemma 3.2 we have

$$3^m - 3^{m-1} - 2^k \left(\frac{m-1}{k}\right) - \sum_{j=1}^{k} 2^j \left(\frac{m}{j}\right) > 3^m - 2^k \left(\frac{m-1}{k}\right) - \sum_{j=0}^{k} 2^j \left(\frac{m}{j}\right) > 0.$$  

This means that the minimal distance of the code in Theorem 3.2 is better than that of codes in [3].

Theorem 3.3. Let $m, k$ be integers with $m \geq 5$ and $2 \leq k \leq \lceil \frac{m-1}{2} \rceil$. Then $C_{g(m,k)}$ is a minimal code and has parameters

$$[3^m-1, m+1, 3^m-3^{m-1}-2^k \left(\frac{m-1}{k}\right)].$$

Furthermore, $\frac{w_{\min}}{w_{\max}} \leq \frac{2}{3}$ if and only if

$$2 \sum_{j=1}^{k} 2^j \left(\frac{m}{j}\right) \leq 3 \cdot 2^k \left(\frac{m-1}{k}\right).$$

Proof. According to Corollary 3.1 we only need to prove that $C_{g(m,k)}$ is minimal. From the proof of Theorem 3.2 we have

$$\text{Re}(\widehat{g}(m,k)(v)) = \begin{cases} -\frac{3^m}{2} + \frac{3}{2} \sum_{j=0}^{k} 2^j \left(\frac{m}{j}\right), & \text{if } v = 0; \\ \frac{3}{2} \Psi_k(i, m), & \text{wt}(v) = i > 0. \end{cases} \quad (3.3)$$

On the other hand Lemma 2.3 implies that $C_{g(m,k)}$ is minimal if and only if both

$$\text{Re}(\widehat{g}(m,k)(v_1)) + \text{Re}(\widehat{g}(m,k)(v_2)) \neq 2 \text{Re}(\widehat{g}(m,k)(v_3)) \neq 3^m, \quad (3.4)$$

and

$$\text{Re}(\widehat{g}(m,k)(v_1)) + \text{Re}(\widehat{g}(m,k)(v_2)) + \text{Re}(\widehat{g}(m,k)(v_3)) \neq 3^m \quad (3.5)$$

hold for any pairwise distinct vectors $v_1, v_2, v_3 \in \mathbb{F}_3^m$ with $v_1 + v_2 + v_3 = 0$. We distinguish the following two cases to show that both (3.4) and (3.5) hold for the claimed vectors.

Case 1. One of $v_1, v_2, v_3$ is 0.

We first consider the inequality (3.4). Without loss of the generality, we assume that $v_1 = 0$ and then $v_2 = -v_3 \neq 0$, where $\text{wt}(v_2) = \text{wt}(v_3) = i \ (1 \leq i \leq m)$. Then by (3.3), (3.5) is equivalent to

$$-\frac{3^m}{2} + \frac{3}{2} \sum_{j=0}^{k} 2^j \left(\frac{m}{j}\right) + 3 \Psi_k(i, m) \neq 3^m \Leftrightarrow \frac{3}{2} \sum_{j=0}^{k} 2^j \left(\frac{m}{j}\right) + 3 \Psi_k(i, m) \neq \frac{3}{2} \cdot 3^m.$$  

Note that $\sum_{j=0}^{k} 2^j \left(\frac{m}{j}\right) < 3^{m-1}$ for $2 \leq k \leq \lceil \frac{m-1}{2} \rceil$, then $\frac{3}{2} \sum_{j=0}^{k} 2^j \left(\frac{m}{j}\right) < \frac{3^m}{2}$. Thus we have

$$\frac{3}{2} \sum_{j=0}^{k} 2^j \left(\frac{m}{j}\right) + 3 \Psi_k(i, m) < \frac{3^m}{2} + 3 \Psi_k(i, m) < \frac{3^m}{2} + 3 \cdot 2^k \left(\frac{m-1}{k}\right)$$

$$< \frac{3^m}{2} + 3 \cdot \sum_{j=0}^{m-1} 2^j \left(\frac{m-1}{j}\right) = \frac{3}{2} \cdot 3^m.$$
by Lemma 2.1. Next we prove (3.4) in two cases.

(1) If $v_3 = 0$, then $wt(v_1) = wt(v_2) = i$ ($1 \leq i \leq m$). Thus by (3.3), (3.4) is equivalent to

$$3\Psi_k(i, m) + 3^m - 3 \sum_{j=0}^{k} 2^j \binom{m}{j} \neq 3^m \Rightarrow \Psi_k(i, m) \neq \sum_{j=0}^{k} 2^j \binom{m}{j}.$$ 

Note that

$$\Psi_k(i, m) < 2^k \binom{m - 1}{k} < 2^k \binom{m}{k} < \sum_{j=0}^{k} 2^j \binom{m}{j}$$

by Lemma 2.1 and so (3.4) holds.

(2) If one of $v_1$ and $v_2$ is 0, assume that $v_1 = 0$ without loss of the generality. Then $wt(v_2) = wt(v_3) = i$ ($1 \leq i \leq m$). Thus by (3.3), (3.4) is equivalent to

$$-\frac{3^m}{2} + \frac{3}{2} \sum_{j=0}^{k} 2^j \binom{m}{j} + \frac{3}{2} \Psi_k(i, m) - 3\Psi_k(i, m) \neq 3^m,$$

which holds if and only if

$$\frac{3}{2} \sum_{j=0}^{k} 2^j \binom{m}{j} - \frac{3}{2} \Psi_k(i, m) \neq \frac{3}{2} 3^m,$$

i.e.,

$$\sum_{j=0}^{k} 2^j \binom{m}{j} - \Psi_k(i, m) \neq 3^m.$$

Note that

$$|\sum_{j=0}^{k} 2^j \binom{m}{j} - \Psi_k(i, m)| \leq \sum_{j=0}^{k} 2^j \binom{m}{j} + |\Psi_k(i, m)| \leq \sum_{j=0}^{k} 2^j \binom{m}{j} + 2^k \binom{m - 1}{k}$$

$$< \sum_{j=0}^{k} 2^j \binom{m}{j} + 2^k \binom{m - 1}{k} + 2^k \binom{m - 1}{k + 1} = \sum_{j=0}^{k} 2^j \binom{m}{j} + 2^k \binom{m}{k + 1}$$

$$< \sum_{j=0}^{k} 2^j \binom{m}{j} + 2^{k+1} \binom{m}{k + 1} < \sum_{j=0}^{m} 2^j \binom{m}{j} = 3^m.$$ 

Thus (3.4) holds.

In this case, both (3.4) and (3.5) follow from the discussions above.

Case 2. $v_1, v_2, v_3$ are all nonzero.

Due to Lemma 2.1 and (3.3), we derive that

$$|Re(\overline{\gamma}_{(m,k)}(v_1)) + Re(\overline{\gamma}_{(m,k)}(v_2)) - 2Re(\overline{\gamma}_{(m,k)}(v_3))| \leq 6 \times 2^k \binom{m - 1}{k},$$

and

$$|Re(\overline{\gamma}_{(m,k)}(v_1)) + Re(\overline{\gamma}_{(m,k)}(v_2)) + Re(\overline{\gamma}_{(m,k)}(v_3))| \leq \frac{9}{2} \times 2^k \binom{m - 1}{k}.$$
To show that both (3.4) and (3.5) hold, it is sufficient to show that
\[ 6 \times 2^k \binom{m-1}{k} < 3^m, \]
i.e.,
\[ 2^{k+1} \binom{m-1}{k} < 3^{m-1}. \]
In fact, by \(2 \leq k \leq \lfloor \frac{m-1}{2} \rfloor\), we have
\[
2^{k+1} \binom{m-1}{k} = 2^k \binom{m-1}{k} + 2^k \binom{m-1}{k+1} \\
< 2^k \binom{m-1}{k} + 2^{k+1} \binom{m-1}{k+1} \\
< \sum_{j=0}^{m-1} 2^j \binom{m-1}{j} = 3^{m-1}.
\]
Thus both (3.4) and (3.5) hold in this case.

Summarizing the discussions above, we complete the proof for Theorem 3.3.

Based on Magma’s software, the following example is presented to support our main results.

**Example 3.1.** Let \(m = 9\) and \(k = 2\), then the code \(C_{g(m,k)}\) in Theorem 3.2 is a minimal code with parameters \([19682, 10, 13010]\) and weight enumerator
\[
1 + 36z^{13010} + 288z^{13052} + 1344z^{13085} + 1024z^{13094} + 4032z^{13109} + 4608z^{13115} \\
+ 19682z^{13122} + 8064z^{13124} + 10752z^{13130} + 9216z^{13133} + 2z^{19520}.
\]
Thus \(\frac{\text{\(w\)min}}{\text{\(w\)max}} = \frac{13010}{19520} < \frac{2}{3}\).

4 Conclusions

For an odd prime \(p\), using the generic construction (1.1) to obtain minimal codes violating the Ashikhmin-Barg condition, \(f\) usually is taken to be the characteristic function of a subset in \(\mathbb{F}_p^m\). There are a few known results when \(f\) is a non-characteristic function. In this paper, we first present a class of ternary minimal linear codes violating the Ashikhmin-Barg condition, where \(f_{(m,k)}\) is not characteristic function of any subset in \(\mathbb{F}_3^m\). Secondly, basing on \(\overline{g}_{(m,k)}\), we construct another kind of ternary minimal linear codes violating the Ashikhmin-Barg condition, whose distance is better than those of \(C_{g(m,k)}\) in [18]. For odd prime \(p(p \geq 5)\), constructing minimal codes violating the Ashikhmin-Barg condition and determining their weight distributions are difficult by (1.1), due to the difficulty of computing exponential sum of \(f\).

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