An identity for two integral transforms applied to the uniqueness of a distribution via its Laplace–Stieltjes transform

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ABSTRACT

It is well known that the Laplace–Stieltjes transform of a nonnegative random variable (or random vector) uniquely determines its distribution function. We extend this uniqueness theorem by using the Müntz–Szász Theorem and the identity for the Laplace–Stieltjes and Laplace–Carson transforms of a distribution function. The latter appears for the first time to the best of our knowledge. In particular, if $X$ and $Y$ are two nonnegative random variables with joint distribution $H$, then $H$ can be characterized by a suitable set of \textit{countably many values} of its bivariate Laplace–Stieltjes transform. The general high-dimensional case is also investigated. Besides, Lerch’s uniqueness theorem for conventional Laplace transforms is extended as well. The identity can be used to simplify the calculation of Laplace–Stieltjes transforms when the underlying distributions have singular parts. Finally, some examples are given to illustrate the characterization results via the uniqueness theorem.

1. Introduction

The Laplace–Stieltjes transform of nonnegative random variables (or their distribution functions) plays a crucial role in the areas of theoretical and applied probability, especially in survival analysis and reliability theory. The most important fundamental property of the transform might be the uniqueness theorem; namely, any two nonnegative random variables (or random vectors) sharing the same Laplace–Stieltjes transform should have the same distribution function (see, e.g., Farrell\textsuperscript{[1]}, pp. 16–18).

Another related integral transform of functions is the conventional Laplace transform which is a powerful tool in solving systems of both ordinary and partial differential equations through the uniqueness theorem – Lerch’s Theorem (see Theorem A below, Aghili and Parsa Moghaddam\textsuperscript{[2]} or van der Pol and Bremmer\textsuperscript{[3]}). In this paper, we prove \textit{the identity for the Laplace–Stieltjes and Laplace–Carson transforms of a distribution function} (the latter is in terms of the conventional Laplace transform; see Theorems E, F and 4.1). We will apply this identity to extend the previous uniqueness theorem for Laplace–Stieltjes transforms with the help of Müntz–Szász Theorem (for the latter, see...
Theorems B and C). Moreover, the identity can be used to simplify the calculation of Laplace–Stieltjes transforms, especially, when the underlying distributions have singular parts.

In Section 2, we formally define the integral transforms in question and provide the needed preliminary results. In Section 3, the main results for one- and two-dimensional cases are proved (Theorems 3.1–3.3). In particular, let \( X \) and \( Y \) be two nonnegative random variables with joint distribution \( H \), and let \( \{ m_i \} \) and \( \{ n_j \} \) be two sequences of strictly increasing positive real numbers satisfying the conditions:

\[
\sum_{i=1}^{\infty} \frac{1}{m_i} = \infty, \quad \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty.
\]

Then the joint distribution \( H \) is uniquely determined by the countably many values of bivariate Laplace–Stieltjes transform:

\[
\{ \mathbb{E} \exp(-m_i X - n_j Y) : i, j = 1, 2, \ldots \}.
\]

The general high-dimensional case is investigated in Section 4. Section 5 shows the advantage of the identity when calculating Laplace–Stieltjes transforms of distributions with singular parts. We give in Section 6 some examples to illustrate the characterization results via the uniqueness theorem. Appendix provides tedious calculations for the proof of the identity.

2. Preliminaries

For a real-valued measurable function \( f \) on \( \mathbb{R}_+ := [0, \infty) \), we denote by \( L_f \) the (conventional) Laplace transform of \( f \):

\[
L_f(s) = \int_0^\infty f(x) e^{-sx} \, dx, \quad s > 0,
\]

provided the integral exists. The following uniqueness theorem is due to Lerch [4]; see, e.g., Cohen [5], p. 23.

**Theorem A (Lerch’s Theorem):** Suppose \( f \) and \( g \) are continuous on \( \mathbb{R}_+ \) and of exponential type \( a \geq 0 \) at infinity, namely, \( |f(x)| = \mathcal{O}(e^{ax}) \) and \( |g(x)| = \mathcal{O}(e^{ax}) \) as \( x \to \infty \). If \( L_f(s) = L_g(s) \) for all \( s > a \), then \( f(x) = g(x) \) on \( \mathbb{R}_+ \).

In other words, the Laplace transform \( L_f \) uniquely determines the continuous function \( f \) of exponential type on \( \mathbb{R}_+ \). Note that to recover \( f \) from \( L_f \), we can apply the Post–Widder Inversion Formula:

\[
f(x) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left( \frac{n}{x} \right)^{n+1} L_f^{(n)} \left( \frac{n}{x} \right), \quad x > 0,
\]

where \( L_f^{(n)} \) denotes the \( n \)th derivative of \( L_f \) (see, e.g., Cohen [5], p. 37). Conversely, from Watson’s lemma it follows the asymptotic equivalence

\[
L_f(s) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{s^{n+1}} \quad \text{as} \quad s \to \infty,
\]

provided that \( f \in C^\infty \) (infinitely continuously differentiable) in a neighbourhood of zero (see, e.g., Miller [6], p. 53). Interestingly, if \( f(x) = e^{-x} \), then we can replace the above approximation (2) by equality, namely, \( L_f(s) = (1 + s)^{-1} = \sum_{n=0}^{\infty} f^{(n)}(0) / s^{n+1} \) for all \( s > 1 \).
One of our main purposes in this paper is to extend Lerch’s Theorem (Theorem A) to the case of measurable functions (see Theorem D). To do this, we need the following lemmas and Müntz–Szász Theorem (Theorems B and C; Müntz [7], Szász [8]). The latter extends the famous Weierstrass approximation theorem in C[0,1], the space of all continuous functions on [0, 1] with supremum norm (see, e.g., Pérez and Quintana [9]). We provide detailed proofs here for completeness, although some of them might be known with different approaches in the literature.

**Lemma A:** Let $f$ and $g$ be two Lebesgue integrable nonnegative functions on $(0, \infty)$. Assume further that $\int_0^\infty f(x)e^{-sx}dx = \int_0^\infty g(x)e^{-sx}dx$ for all $s > 0$. Then $f(x) = g(x)$ a.e. (almost everywhere) on $(0, \infty)$.

**Proof:** Note that $\int_0^\infty f(x)dx = \int_0^\infty g(x)dx =: A \in [0, \infty)$ by the Dominated Convergence Theorem. If $A = 0$, then $f(x) = g(x) = 0$ a.e. on $(0, \infty)$, due to the assumptions $f, g \geq 0$. Suppose now that $A \in (0, \infty)$. Let us define two absolutely continuous distributions:

$$F(x) = \frac{1}{A} \int_0^x f(t)dt, \quad x \geq 0, \quad G(y) = \frac{1}{A} \int_y^\infty g(t)dt, \quad y \geq 0.$$  

Moreover, let $X$ and $Y$ obey the distributions $F$ and $G$, respectively, denoted $X \sim F$ and $Y \sim G$. Then $X$ and $Y$ have the same Laplace transforms by assumptions:

$$E[\exp(-sX)] = E[\exp(-sY)], \quad s \geq 0.$$  

This means that $F = G$ and hence the difference between their densities $f(x)/A - g(x)/A = 0$ a.e. on $(0, \infty)$. Equivalently, $f(x) = g(x)$ a.e. on $(0, \infty)$. The proof is complete.  

**Lemma B:** Let $f$ be a Lebesgue integrable function on $(0, \infty)$ and let $\int_0^\infty f(x)e^{-sx}dx = 0$ for all $s > 0$. Then $f(x) = 0$ a.e. on $(0, \infty)$.

**Proof:** Define $f_+(x) = \max\{0, f(x)\}$ and $f_-(x) = -\min\{0, f(x)\}$, $x > 0$. Then $f = f_+ - f_-$ and both $f_+$ and $f_-$ are nonnegative Lebesgue integrable functions on $(0, \infty)$ satisfying

$$\int_0^\infty f_+(x)e^{-sx}dx = \int_0^\infty f_-(x)e^{-sx}dx, \quad s > 0.$$  

It then follows from Lemma A that $f_+(x) = f_-(x)$ a.e. on $(0, \infty)$ and hence $f(x) = f_+(x) - f_-(x) = 0$ a.e. on $(0, \infty)$. The proof is complete.

**Lemma C:** Let $f$ be a measurable function on $(0, \infty)$ such that $\int_0^\infty f(x)e^{-sx}dx = 0$, $s > a$ for some $a \geq 0$. Then $f(x) = 0$ a.e. on $(0, \infty)$.

**Proof:** Define $f_a(x) = f(x)e^{-ax}$, $x > 0$. Then $f_a$ is a Lebesgue integrable function on $(0, \infty)$ satisfying

$$\int_0^\infty f_a(x)e^{-sx}dx = 0, \quad s > 0.$$  

By Lemma B, we have $f_a(x) = 0$ a.e. on $(0, \infty)$ and so is $f$. This completes the proof.
Theorem B: \textit{(M"{u}ntz–Sz"{a}sz Theorem in C[0, 1]).} Suppose that $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ is a sequence of positive and distinct real numbers satisfying $\inf \Lambda > 0$ and $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$. Then the span\{1, $x^{\lambda_1}, x^{\lambda_2}, \ldots$\} is dense in C[0, 1]. Equivalently, the collection of all finite linear combinations of the functions \{1, $x^{\lambda_1}, x^{\lambda_2}, \ldots$\} is dense in C[0, 1], or, we also say that the set \{1, $x^{\lambda_1}, x^{\lambda_2}, \ldots$\} is total in C[0, 1].

Proof: Case (I): $\sup \Lambda = \infty$. Without loss of generality we may assume that $\lim_{k \to \infty} \lambda_k = \infty$. We will elaborate von Golitschek’s \cite{10} elementary constructive proof. Let $\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ be a sequence of positive and distinct real numbers satisfying $\lambda_1 > 1/\lambda_1 > \lambda_2 > 1/\lambda_2 > \lambda_3 > \ldots > 1/\lambda_n > \ldots$.

Define a sequence of functions

$$Q_n(x) = (\lambda_n - q)x^{\lambda_n} \int_x^1 Q_{n-1}(t)t^{-(1+\lambda_n)} \, dt, \quad x \in [0, 1].$$

Then we have explicitly

$$Q_1(x) = x^q - x^{\lambda_1} = x^q - a_{1,1} x^{\lambda_1}, \quad x \in [0, 1],$$

and

$$Q_2(x) = x^q - \frac{\lambda_2 - q}{\lambda_2 - \lambda_1} x^{\lambda_1} - \left(1 - \frac{\lambda_2 - q}{\lambda_2 - \lambda_1}\right) x^{\lambda_2} = x^q - a_{1,2} x^{\lambda_1} - a_{2,2} x^{\lambda_2}, \quad x \in [0, 1],$$

where $a_{2,2} = 1 - a_{1,2}$. In general, for $n \geq 2$, suppose $Q_{n-1}(x)$ is of the form:

$$Q_{n-1}(x) = x^q - \sum_{k=1}^{n-1} a_{k,n-1} x^{\lambda_k} =: x^q - P_{n-1}(x), \quad x \in [0, 1],$$

where $P_{n-1}(x)$ is a finite linear combination of $x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_{n-1}}$. Then we carry out $Q_n(x)$ by definition and obtain

$$Q_n(x) = (\lambda_n - q)x^{\lambda_n} \int_x^1 \left(x^q - \sum_{k=1}^{n-1} a_{k,n-1} x^{\lambda_k}\right) t^{-(1+\lambda_n)} \, dt$$

$$= x^q - \sum_{k=1}^{n} a_{k,n} x^{\lambda_k} =: x^q - P_n(x), \quad x \in [0, 1],$$

where $P_n(x)$ is a finite linear combination of $x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_n}$ with coefficients

$$a_{k,n} = a_{k,n-1} \frac{\lambda_n - q}{\lambda_n - \lambda_k}, \quad k = 1, 2, \ldots, n-1, \quad a_{n,n} = 1 - \sum_{k=1}^{n-1} a_{k,n}.$$ 

Hence, the coefficients $\{a_{k,n}\}$ in $Q_n(x)$ can be derived from $\{a_{k,n-1}\}$ iteratively. We now estimate the supremum of $|Q_n(x)|$ on $[0, 1]$, denoted $\|Q_n\|$. By the definition of $Q_n$,

$$\|Q_0\| = 1, \quad \|Q_1\| \leq \left|1 - \frac{q}{\lambda_1}\right|,$$

$$\|Q_n\| \leq \left|1 - \frac{q}{\lambda_n}\right| \|Q_{n-1}\| \sup_{0 \leq x \leq 1} |x^{\lambda_n} [1 - x^{-\lambda_n}]|$$

$$\leq \left|1 - \frac{q}{\lambda_n}\right| \|Q_{n-1}\| \leq \prod_{k=1}^{n} \left|1 - \frac{q}{\lambda_k}\right| \to 0 \quad \text{as} \quad n \to \infty.$$
The last conclusion is due to the assumptions that \( \lim_{k \to \infty} \lambda_k = \infty \) and \( \sum_{k=1}^\infty 1/\lambda_k = \infty \) [11, p. 209]. Namely, for any real \( q > 0 \), the monomials \( x^q \) belong to the closure of the span\( \{x^{\lambda_k} : k = 1, 2, \ldots \} \). Therefore, the span\( \{1, x^{\lambda_1}, x^{\lambda_2}, \ldots \} \) is dense in \( C[0, 1] \) by the Weierstrass approximation theorem. The latter asserts that every function \( f \in C[0, 1] \) is a uniform limit of polynomials [9, 12].

Case (II): \( \Lambda < \infty \). Without loss of generality, we may assume \( \lim_{k \to \infty} \lambda_k = \lambda_* \in (0, \infty) \). Recall that \( C[0, 1] \) is dense in \( L(0, 1) \), the space of all Lebesgue integrable functions on \( (0, 1) \). Suppose, on the contrary, that the span\( \{1, x^{\lambda_1}, x^{\lambda_2}, \ldots \} \) is not dense in \( C[0, 1] \), and hence not dense in \( L(0, 1) \) either. Then by the Hahn–Banach Theorem there exists a bounded nonzero measurable function \( g \) such that \( \int_0^1 x^{\lambda_k} g(x) \, dx = 0 \) for all \( k \). Define the complex-valued function

\[
h(z) = \int_0^1 x^z g(x) \, dx, \quad z \in \Pi = \{z : \text{Re } z > 0\}.
\]

Then \( h \) is a bounded analytic function on the right half-plane \( \Pi \) and has zero value at the points \( \lambda_k \) and the limit \( \lambda_* \in \Pi \). This implies that \( h(z) = 0 \) on \( \Pi \). By the uniqueness theorem for Mellin transforms (or using Lemma C by changing variables), \( g(x) = 0 \) a.e. on \( (0, 1) \), a contradiction. Therefore, the span\( \{1, x^{\lambda_1}, x^{\lambda_2}, \ldots \} \) is dense in \( C[0, 1] \). The proof is complete.

We need some more notations. Let \( 0 \leq a < b \leq \infty \) and denote by \( L(a, b) \) the space of all Lebesgue integrable functions on \( (a, b) \). We say that the set of functions \( \{f_n\}_{n=1}^\infty \) is complete in the space \( L(a, b) \) if for any function \( g \in L(a, b) \), the equalities

\[
\int_a^b f_n(x) g(x) \, dx = 0, \quad n \in \mathbb{N} := \{1, 2, \ldots \},
\]

together imply that \( g(x) = 0 \) a.e. on \( (a, b) \) [13, p. 234].

We have the following ramification of Müntz–Szász Theorem.

**Theorem C:** Let \( \Lambda = \{\lambda_k\}_{k=1}^\infty \) be a sequence of positive and distinct real numbers satisfying \( \inf \Lambda > 0 \) and \( \sum_{k=1}^\infty 1/\lambda_k = \infty \). Then the set of functions \( \{x^{\lambda_k}\}_{k=1}^\infty \) is complete in \( L(0, 1) \).

**Proof:** Suppose \( g \in L(0, 1) \) and for \( \lambda_k \), we have the equalities

\[
\int_0^1 t^{\lambda_k} g(t) \, dt = 0, \quad k \in \mathbb{N}.
\]

Then we want to prove that \( g(t) = 0 \) a.e. on \( (0, 1) \). By changing variables \( x = - \log t \), we rewrite the above equalities in the form

\[
0 = \int_0^\infty e^{-\lambda_k x} g(e^{-x}) e^{-x} \, dx =: \int_0^\infty e^{-\lambda_k x} h(x) \, dx = L_h(\lambda_k), \quad k \in \mathbb{N},
\]

where \( h(x) = g(e^{-x}) e^{-x} \in L(0, \infty) \) with Laplace transform \( L_h(\lambda) = \int_0^\infty e^{-\lambda x} h(x) \, dx, \lambda > 0 \). Define the complex-valued function

\[
L_h(z) = \int_0^\infty e^{-z x} h(x) \, dx, \quad z \in \Pi = \{z : \text{Re } z > 0\}.
\]
Then \( L_h \) is bounded and analytic on the right half-plane \( \Pi \) and vanishes at \( \lambda_k \) for all \( k \in \mathbb{N} \).

Set the function

\[
H(z) = L_h\left(\frac{1+z}{1-z}\right) \quad \text{for} \quad z \in U = \{z : |z| < 1\}.
\]

Then \( H \) is bounded and analytic in \( U \). Letting \( \alpha_k = (\lambda_k - 1)/(\lambda_k + 1) \), we have that \( H(\alpha_k) = 0 \) and \( \sum_{k=1}^{\infty} (1 - |\alpha_k|) = \infty \). Therefore, \( H(z) = 0 \) on \( U \) \([14, p.312]\), or, equivalently, \( L_h(z) = 0 \) on \( \Pi \). Particularly, \( L_h(\lambda) = 0 \) on \( (0, \infty) \). By the uniqueness theorem for Laplace transforms of measurable functions (see Lemma C above), we conclude that \( h(x) = 0 \) a.e. on \( (0, \infty) \) and hence \( g(t) = 0 \) a.e. on \( (0, 1) \). This completes the proof. \( \blacksquare \)

It is seen that the next theorem improves significantly both Theorem A and Lemma C.

**Theorem D:** Let \( f \) be a measurable function on \( (0, \infty) \) and let \( \{n_j\}_{j=1}^{\infty} \) be a sequence of positive and distinct increasing real numbers satisfying \( \sum_{j=1}^{\infty} 1/n_j = \infty \). Assume further that \( \int_0^{\infty} f(x)e^{-n_jx}dx = 0 \) for all \( j \in \mathbb{N} \). Then \( f(x) = 0 \) a.e. on \( (0, \infty) \).

**Proof:** We rewrite, by changing variables \( t = e^{-x} \),

\[
\int_0^{\infty} f(x)e^{-n_jx}dx = \int_0^{\infty} [f(x)e^{-n_1x}]e^{-(n_j-n_1)x}dx = \int_0^1 h(t)t^{n_j-n_1}dt = 0, \quad j = 2, 3, \ldots,
\]

where the function \( h(t) = f(-\ln t)t^{n_1-1} \in L(0, 1) \) since \( \int_0^{\infty} f(x)e^{-n_1x}dx = 0 \). Then by Theorem C, \( h(t) = 0 \) a.e. on \( (0, 1) \) because \( \sum_{j=2}^{\infty} 1/(n_j - n_1) \geq \sum_{j=2}^{\infty} 1/n_j = \infty \), and hence \( f(x) = 0 \) a.e. on \( (0, \infty) \). The proof is complete. \( \blacksquare \)

For recent developments on the Müntz–Szász Theorem, see Erdélyi and Johnson \([15]\), Erdélyi \([16]\), Almira \([17]\) and the references therein.

We now consider two nonnegative random variables \( X \) and \( Y \) having joint distribution \( H \) with marginals \( F \) and \( G \), that is, \( (X, Y) \sim H, X \sim F \) and \( Y \sim G \). Denote by \( \mathcal{L}_F \) and \( \mathcal{L}_H \) the Laplace–Stieltjes transforms of \( X \sim F \) and \( (X, Y) \sim H \), respectively. Formally,

\[
\mathcal{L}_F(s) = \mathbb{E}[\exp(-sX)] = \int_0^{\infty} e^{-sx}dF(x) = \int_{[0, \infty)} e^{-sx}dF(x) \quad (3)
\]

\[
= \lim_{\varepsilon \downarrow 0} \int_{-\varepsilon}^{\infty} e^{-sx}dF(x) = F(0) + \int_{(0, \infty)} e^{-sx}dF(x), \quad s > 0,
\]

\[
\mathcal{L}_H(s, t) = \mathbb{E}[\exp(-sX - tY)] = \int_0^{\infty} \int_0^{\infty} e^{-sx-ty}dH(x,y), \quad s, t > 0. \quad (4)
\]

Also, analogously to (1), denote by \( L_H \) the (conventional) Laplace transform of bivariate \( H \):

\[
L_H(s, t) = \int_0^{\infty} \int_0^{\infty} H(x,y)e^{-sx-ty}dxdy, \quad s, t > 0. \quad (5)
\]

It is known that \( \mathcal{L}_F \in C^\infty((0, \infty)) \) is completely monotone and that for each continuity point \( x > 0 \) of \( F \),

\[
F(x) = \lim_{n \to \infty} \sum_{k \leq nx} (-1)^k \frac{n^k}{k!} \mathcal{L}_F^{(k)}(n) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left(\frac{n}{x}\right)^k \mathcal{L}_F^{(k)}\left(\frac{n}{x}\right) \quad (6)
\]
The second equality in (6) also follows from the facts: (i) Theorem E below, (ii) the Post–Widder Inversion Formula and (iii) Leibniz’s rule. Besides, there are several interesting and useful relationships between the Laplace–Stieltjes transform and the conventional Laplace transform. Note first that if \( F \) has a density \( f \), then \( \mathcal{L}_F = L_f \) by definition, and that if the bivariate \( H \) has a joint density \( h \), then \( \mathcal{L}_H = L_h \).

The following two identities can be derived by using the existing results and will be used in the sequel.

**Theorem E:** For any distribution \( F \) on \( \mathbb{R}_+ \),

\[
\mathcal{L}_F(s) = sL_F(s) \quad \text{for all } s > 0.
\]

**Proof:** For \( s > 0 \), using integration by parts we have

\[
\mathcal{L}_F(s) = \int_0^\infty e^{-sx}dF(x) = F(0) + \int_{(0,\infty)} e^{-sx}dF(x)
\]

\[
= F(0) + e^{-sx}F(x)|_0^\infty + s \int_{(0,\infty)} F(x)e^{-sx}dx = sL_F(s).
\]

(7)

Also, note that the identity (7) has an equivalent form in terms of the survival function \( \overline{F} \):

\[
\left(1 - \mathcal{L}_F(s)\right)/s = \int_0^\infty e^{-sx}\overline{F}(x)\,dx, \quad s > 0,
\]

where \( \overline{F}(x) = P(X > x) = 1 - F(x), \ x \geq 0 \) (see Lin [19], Lemma 1, or Feller [18], p. 435).

**Theorem F:** For any bivariate distribution \( H \) on \( \mathbb{R}^2_+ \),

\[
\mathcal{L}_H(s, t) = stL_H(s, t) \quad \text{for all } s, t > 0.
\]

**Proof:** From Theorem E it follows that the identity in question is equivalent to

\[
\mathcal{L}_H(s, t) = st \int_0^\infty \int_0^\infty \overline{H}(x, y)e^{-sx-ty}\,dxdy - 1 + \mathcal{L}_F(s) + \mathcal{L}_G(t), \quad s, t \geq 0
\]

(8)

(by an expansion of the double integral on the RHS of (8)), where the joint survival function

\[
\overline{H}(x, y) = P(X > x, Y > y) = 1 - F(x) + G(y) + H(x, y), \quad x, y \geq 0.
\]

The identity (8) is exactly Corollary 2 in Lin et al. [20]. The proof is complete.

The RHS \( (sL_F(s)) \) of the identity in Theorem E is called the Laplace–Carson transform of a function \( F \) (compare (1) and (3)). For definition of the latter transform, see, e.g., Carson [21], Ditkin and Prudnikov [22] and Donolato [23]. In other words, Theorems E and F claim the identity for the Laplace–Stieltjes and the Laplace–Carson transforms of a distribution function in the first two dimensions. To the best of our knowledge, at least the bivariate identity in Theorem F appears for the first time.
3. Main results: one- and two-dimensional cases

We restate Theorem D as follows.

**Theorem 3.1:** Let \( f, g \) be two measurable functions on \((0, \infty)\) and let \( \{n_j\}_{j=1}^{\infty} \) be a sequence of positive and distinct increasing real numbers satisfying \( \sum_{j=1}^{\infty} 1/n_j = \infty \). If \( \int_{0}^{\infty} f(x)e^{-n_jx} \, dx = \int_{0}^{\infty} g(x)e^{-n_jx} \, dx \) (finite) for all \( j \in \mathbb{N} \), then \( f(x) = g(x) \) a.e. on \((0, \infty)\).

**Theorem 3.2:** Let \( 0 \leq X \sim F \) and let the real numbers \( 0 < m_1 < m_2 < \cdots \) satisfy \( \sum_{i=1}^{\infty} 1/m_i = \infty \). Then the distribution \( F \) is uniquely determined by the countable set of values \( \{L_F(m_i) \}_{i=1}^{\infty} \) of its Laplace–Stieltjes transform.

**Proof:** Suppose \( 0 \leq X_1 \sim F_1, 0 \leq X_2 \sim F_2 \) and \( L_{F_1}(m_i) = L_{F_2}(m_i), i \in \mathbb{N} \). Then we want to show that \( F_1 = F_2 \) under the condition \( \sum_{i=1}^{\infty} 1/m_i = \infty \). By Theorem E and the assumptions, we have the equalities:

\[
\int_{0}^{\infty} F_1(x)e^{-m_ix} \, dx = \int_{0}^{\infty} F_2(x)e^{-m_ix} \, dx \text{ (finite)} \quad \forall i \in \mathbb{N}.
\]

It then follows from Theorem 3.1 that \( F_1(x) = F_2(x) \) a.e. on \([0, \infty)\), and hence \( F_1 = F_2 \) due to the right continuity of distributions. The proof is complete.

An alternative proof of Theorem 3.2 was given in Lin ([24], Lemma 4), in which two other sufficient conditions were provided:

(i) \( \lim_{i \to \infty} m_i = m_0 \in (0, \infty) \), and
(ii) \( \lim_{i \to \infty} m_i = 0, \sum_{i=1}^{\infty} m_i = \infty \).

These three conditions also apply to the high-dimensional cases, but for simplicity, we consider only the strictly monotone sequence below.

**Theorem 3.3:** Let \( 0 \leq X \sim F, 0 \leq Y \sim G \) and \( (X, Y) \sim H \). Assume further that the two sequences \( \{m_i\}_{i=1}^{\infty} \) and \( \{n_j\}_{j=1}^{\infty} \) of real numbers satisfy

(i) \( 0 < m_1 < m_2 < \cdots \) with \( \sum_{i=1}^{\infty} 1/m_i = \infty \), and
(ii) \( 0 < n_1 < n_2 < \cdots \) with \( \sum_{j=1}^{\infty} 1/n_j = \infty \).

Then the bivariate distribution \( H \) is uniquely determined by the countably many values of its Laplace–Stieltjes transform: \( \{L_H(m_i, n_j) : i, j = 1, 2, \ldots \} \).

**Proof:** For \( k = 1, 2 \), suppose \( 0 \leq X_k \sim F_k, 0 \leq Y_k \sim G_k \), \( (X_k, Y_k) \sim H_k \), and \( L_{H_1}(m_i, n_j) = L_{H_2}(m_i, n_j), i, j \in \mathbb{N} \). Then we want to prove that \( H_1 = H_2 \) under the conditions \( \sum_{i=1}^{\infty} 1/m_i = \infty \) and \( \sum_{j=1}^{\infty} 1/n_j = \infty \). By Theorem F and the assumptions, we have the
equalities:

\[(m_i; n_j) \int_0^\infty \left[ \int_0^\infty H^*(x, y) e^{-m_i x} dx \right] e^{-n_j y} dy = 0, \quad i, j \in \mathbb{N}, \quad (9)\]

where the function \(H^* = H_1 - H_2\). For fixed \(y\), let \(H^*_y\) denote a function \(H^*(x, y)\) of \(x\). It then follows from (9) and Theorem D that

\[L_{H^*_y}(m_i) := \int_0^\infty H^*(x, y) e^{-m_i x} dx = 0 \quad \forall \ y \geq 0, \quad i \in \mathbb{N}, \quad (10)\]

because \(\sum_{j=1}^\infty 1/n_j = \infty\) and \(H^*(x, y)\) is right continuous in \(y\). By (10) and Theorem D again, we have that \(H^*(x, y) = 0 \ \forall \ x, y \geq 0\), due to the assumption \(\sum_{i=1}^\infty 1/m_i = \infty\) and the right continuity of \(H^*(x, y)\) in \(x\). Therefore, \(H_1 = H_2\). This completes the proof. ■

It is seen that the identity for two integral transforms plays a crucial role in the proofs of Theorems 3.2 and 3.3. Motivated by the results of the one- and two-dimensional cases, we next consider the \(n\)-dimensional case with \(n \geq 3\), which is much more complicated.

4. The general result: \(n\)-dimensional case with \(n \geq 3\)

Consider the random variables \(0 \leq X_i \sim H_i, \quad i = 1, 2, \ldots, n (\geq 3)\), and suppose that they have the joint distribution \(H\) on \(\mathbb{R}_+^n\). For \(s_i > 0, \quad i = 1, 2, \ldots, n\), denote the Laplace–Stieltjes transform of \(H\) by

\[L_H(s_1, s_2, \ldots, s_n) = \mathbb{E}\left[\exp\left(-\sum_{i=1}^n s_i X_i\right)\right] = \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^n s_i x_i\right) dH(x_1, x_2, \ldots, x_n),\]

and the (conventional) Laplace transform of \(H\) by

\[L_H(s_1, s_2, \ldots, s_n) = \int_0^\infty \cdots \int_0^\infty H(x_1, x_2, \ldots, x_n) \exp\left(-\sum_{i=1}^n s_i x_i\right) dx_1 dx_2 \cdots dx_n.\]

Under the above setting, we extend Theorems E and F to the following identity in dimension \(n\), where \(n \geq 3\).

**Theorem 4.1:** The Laplace–Stieltjes transform of the joint distribution \(H\) on \(\mathbb{R}_+^n\) satisfies

\[L_H(s_1, s_2, \ldots, s_n) = \left(\prod_{i=1}^n s_i\right) L_H(s_1, s_2, \ldots, s_n), \quad s_i > 0, \quad i = 1, 2, \ldots, n. \quad (11)\]

**Proof:** (I) We first consider the special case: \(H\) is absolutely continuous with positive marginal densities on \((0, \infty)\), because its proof is simpler and easier to understand than that in the general case. In this case, we prove the identity (11) by induction on \(n\). It follows from Theorems E and F that the identity (11) holds true for \(n = 1\) and \(n = 2\). Now,
suppose it holds true for \( n = m \geq 2 \), then we want to prove the identity for \( n = m + 1 \). Denote the density of \( H \) by \( h \) and its \( j \)th marginal density by \( h_j \). Then the marginal density of the first \( m + 1 \) components \( X_1, X_2, \ldots, X_m, X_{m+1} \) is written as follows:

\[
h(x_1, x_2, \ldots, x_m, x_{m+1}) = h(x_1, x_2, \ldots, x_m|x_{m+1})h_{m+1}(x_{m+1}),
\]

and by the absolutely continuous condition on \( H \),

\[
\mathcal{L}_H(s_1, s_2, \ldots, s_{m+1}) = L_h(s_1, s_2, \ldots, s_{m+1})
\]

\[
= \int_0^\infty \cdots \int_0^\infty \exp \left( - \sum_{i=1}^{m+1} s_i x_i \right) h(x_1, x_2, \ldots, x_m|x_{m+1})h_{m+1}(x_{m+1}) \, dx_1 \, dx_2 \cdots dx_{m+1}
\]

\[
= \int_0^\infty \left[ \int_0^\infty \cdots \int_0^\infty \exp \left( - \sum_{i=1}^{m} s_i x_i \right) h(x_1, x_2, \ldots, x_m|x_{m+1}) \, dx_1 \, dx_2 \cdots dx_m \right] \, dx_{m+1}.
\]

By the assumption on the case \( n = m \), the factor in square brackets is equal to

\[
\left( \prod_{i=1}^{m} s_i \right) \int_0^\infty \cdots \int_0^\infty H(x_1, x_2, \ldots, x_m|X_{m+1} = x_{m+1}) \exp \left( - \sum_{i=1}^{m} s_i x_i \right) \, dx_1 \, dx_2 \cdots dx_m.
\]

Next, rewrite

\[
H(x_1, x_2, \ldots, x_m|X_{m+1} = x_{m+1}) = h_{m+1}(x_{m+1}|X_1 \leq x_1, \ldots, X_m \leq x_m)P(X_1 \leq x_1, \ldots, X_m \leq x_m).
\]

Then we have, by Theorem E,

\[
\int_0^\infty e^{-s_{m+1}x_{m+1}} h_{m+1}(x_{m+1}|X_1 \leq x_1, \ldots, X_m \leq x_m) \, dx_{m+1}
\]

\[
= s_{m+1} \int_0^\infty P(X_{m+1} \leq x_{m+1}|X_1 \leq x_1, \ldots, X_m \leq x_m) e^{-s_{m+1}x_{m+1}} \, dx_{m+1},
\]

and hence the required conclusion follows by Fubini’s theorem, because

\[
P(X_{m+1} \leq x_{m+1}|X_1 \leq x_1, \ldots, X_m \leq x_m)P(X_1 \leq x_1, \ldots, X_m \leq x_m)
\]

\[
= H(x_1, x_2, \ldots, x_{m+1}).
\]

(II) We next treat the general case, in which no smoothness conditions on distributions are assumed. Again, we prove the identity (11) by induction on \( n \). Suppose it holds true for \( n = 1, 2, \ldots, m \geq 2 \), then we want to prove the identity for \( n = m + 1 \).
For fixed $s_i > 0$, $i = 1, 2, \ldots, m + 1$, define the function $K$ on $\mathbb{R}^{m+1}_+$ by

$$K(x_1, x_2, \ldots, x_{m+1}) = \prod_{i=1}^{m+1} (1 - \exp(-s_i x_i)), \quad x_i \geq 0, \quad i = 1, 2, \ldots, m + 1.$$  

Then the Lebesgue–Stieltjes integral

$$E[K(X_1, X_2, \ldots, X_{m+1})]$$

$$\begin{align*}
&= \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} K(x_1, x_2, \ldots, x_{m+1}) \, dH(x_1, x_2, \ldots, x_{m+1}) \\
&= \lim_{R \to \infty} \int_{0}^{R} \int_{0}^{R} \cdots \int_{0}^{R} K(x_1, x_2, \ldots, x_{m+1}) \, dH(x_1, x_2, \ldots, x_{m+1}) \\
&= (-1)^{m+1} \lim_{R \to \infty} \int_{0}^{R} \int_{0}^{R} \cdots \int_{0}^{R} K(x_1, x_2, \ldots, x_{m+1}) \, d\bar{H}(x_1, x_2, \ldots, x_{m+1}).
\end{align*}$$

Here the joint survival function is

$$\bar{H}(x_1, x_2, \ldots, x_{m+1}) = P(X_1 > x_1, X_2 > x_2, \ldots, X_{m+1} > x_{m+1})$$

$$\begin{align*}
&= 1 - \sum_{i=1}^{m+1} H_i(x_i) + \sum_{1 \leq i_1 < i_2 \leq m+1} H_{i_1i_2}(x_{i_1}, x_{i_2}) - \sum_{1 \leq i_1 < i_2 < i_3 \leq m+1} H_{i_1i_2i_3}(x_{i_1}, x_{i_2}, x_{i_3}) \\
&\quad + \cdots + (-1)^m H(x_1, x_2, \ldots, x_{m+1}), \quad \text{(12)}
\end{align*}$$

and the $k$-dimensional marginal distribution, that is, the joint distribution of $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$, is

$$H_{i_1i_2\ldots i_k}(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = P(X_{i_1} \leq x_{i_1}, X_{i_2} \leq x_{i_2}, \ldots, X_{i_k} \leq x_{i_k}). \quad \text{(13)}$$

Formula (12) for $\bar{H}$ is valid for any collection of random variables. The proof is standard, it is based on the well-known inclusion–exclusion representation for the union of arbitrary collection of random events, then we take probability and follow induction arguments. Details can be seen in many books in Probability Theory; see, e.g., Ross [25], p. 6.

Using the multidimensional integration by parts (see Young [26], Section 9, or Zaremba [27], Proposition 2) and proceeding in a similar way as in Lin et al. [20], pp. 3–4, we have

$$\begin{align*}
&= (-1)^{m+1} \lim_{R \to \infty} \int_{0}^{R} \int_{0}^{R} \cdots \int_{0}^{R} K(x_1, x_2, \ldots, x_{m+1}) \, d\bar{H}(x_1, x_2, \ldots, x_{m+1}) \\
&= (-1)^{2(m+1)} \lim_{R \to \infty} \int_{0}^{R} \int_{0}^{R} \cdots \int_{0}^{R} \bar{H}(x_1, x_2, \ldots, x_{m+1}) \, dK(x_1, x_2, \ldots, x_{m+1}) \\
&= \left(\prod_{i=1}^{m+1} s_i\right) \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \bar{H}(x_1, x_2, \ldots, x_{m+1}) \exp\left(-\sum_{i=1}^{m+1} s_j x_j\right) \, dx_1 \, dx_2 \cdots \, dx_{m+1}, \quad \text{(14)}
\end{align*}$$

in which to derive the identity (14), we apply the facts that $K(x_1, x_2, \ldots, x_{m+1}) = 0$ if $x_j = 0$ for some $i$ and that

$$\bar{H}(x_1, x_2, \ldots, x_{m+1}) \to 0 \quad \text{if} \quad x_j \to \infty \quad \text{for some} \ j.$$
Therefore, for $s_i > 0$, $i = 1, 2, \ldots, m + 1$, the above Lebesgue–Stieltjes integral becomes

$$
E\left[ \prod_{i=1}^{m+1} \left( 1 - \exp \left( -s_i X_i \right) \right) \right] = \left( \prod_{i=1}^{m+1} s_i \right) \int_0^\infty \int_0^\infty \cdots \int_0^\infty H(x_1, x_2, \ldots, x_{m+1}) \exp \left( - \sum_{i=1}^{m+1} s_i x_i \right) \, dx_1 \, dx_2 \cdots \, dx_{m+1}.
$$

(15)

The last identity reduces to the following one by the assumption on induction and by a tedious calculation (using (12) and canceling the same terms on both sides of (15)):

$$
E\left[ \exp \left( - \sum_{i=1}^{m+1} s_i X_i \right) \right] = \left( \prod_{i=1}^{m+1} s_i \right) \int_0^\infty \int_0^\infty \cdots \int_0^\infty H(x_1, x_2, \ldots, x_{m+1}) \exp \left( - \sum_{i=1}^{m+1} s_i x_i \right) \, dx_1 \, dx_2 \cdots \, dx_{m+1}.
$$

(16)

(See Appendix for a detailed proof of the identity (16).) The proof is complete.

Using the crucial identity (11) and mimicking the proof of Theorem 3.3, we have, by induction, the following uniqueness theorem for Laplace–Stieltjes transforms in the high-dimensional case. The proof is omitted.

**Theorem 4.2:** Let $\{m_{ij}\}_{j=1}^\infty$, $i = 1, 2, \ldots, n$, be the $n$ ($\geq 3$) sequences of real numbers satisfying

$$
0 < m_{i,1} < m_{i,2} < \cdots \quad \text{and} \quad \sum_{j=1}^\infty \frac{1}{m_{ij}} = \infty \quad \text{for all} \quad i = 1, 2, \ldots, n.
$$

Then any $n$-dimensional distribution $H$ on $\mathbb{R}_+^n$ is uniquely determined by the countably many values of its Laplace–Stieltjes transform:

$$
\{ \mathcal{L}_H(s_1, s_2, \ldots, s_n) : s_i = m_{ij}, \quad i = 1, 2, \ldots, n, \, j \in \mathbb{N} \}.
$$

The following is a special case of Theorems 3.2, 3.3 and 4.2, because $\sum_{j=1}^\infty 1/j = \infty$.

**Corollary 4.3:** Any $n$-dimensional distribution $H$ on $\mathbb{R}_+^n$ is uniquely determined by the countably many values of its Laplace–Stieltjes transform: $\{ \mathcal{L}_H(s_1, s_2, \ldots, s_n) : s_i = j, \quad i = 1, 2, \ldots, n, \, j \in \mathbb{N} \}$. In other words, the set of values $\mathcal{L}_H(s_1, s_2, \ldots, s_n)$ at the lattice points in $\mathbb{N}^n$ characterizes the distribution $H$. 
5. Application to calculation of Laplace–Stieltjes transforms

In general, the calculation of the Laplace–Stieltjes transform \( L_H \) is much more complicated than that of the conventional Laplace transform \( L_H \), especially, when the underlying distribution \( H \) has a singular part, which often happens in survival analysis. The identity (11) then can be used to simplify the calculation. To illustrate this advantage, let us consider the bivariate-lack-of-memory (BLM) distribution defined below.

Let \( X \sim F \) and \( Y \sim G \) be two positive random variables having joint distribution \( H \). Namely, \( (X, Y) \sim H \) with marginals \( F \) and \( G \). We say that \( (X, Y) \) has a BLM distribution if it satisfies the BLM property:

\[
\Pr(X > x + t, Y > y + t | X > t, Y > t) = \Pr(X > x, Y > y), \quad x, y, t \geq 0,
\]
or, equivalently, if its survival function \( \overline{H} \) satisfies the functional equation:

\[
\overline{H}(x + t, y + t) = \overline{H}(x, y)\overline{H}(t, t), \quad \forall x, y, t \geq 0.
\]

Explicitly, the survival function \( \overline{H} \) can be written in the form

\[
\overline{H}(x, y) = \begin{cases} 
  e^{-\theta y} \overline{F}(x - y), & x \geq y \geq 0 \\
  e^{-\theta x} \overline{G}(y - x), & y \geq x \geq 0,
\end{cases}
\]

where \( \theta \) is a positive constant (see Marshall and Olkin [28], or Barlow and Proschan [29], p. 130). For convenience, we denote \( H = BLM(F, G, \theta) \), which has a singular part on the line \( x = y \) with probability \( p(\theta) := (f(0) + g(0)) / \theta - 1 \geq 0 \), where \( f(0) = \lim_{\varepsilon \to 0^+} F(\varepsilon) / \varepsilon \), and \( g(0) = \lim_{\varepsilon \to 0^+} G(\varepsilon) / \varepsilon \) (see Remark 2 in Lin et al. [30]).

When \( p(\theta) > 0 \), \( H \) has a singular part and it is hard to calculate directly the Laplace–Stieltjes transform \( L_H(s, t) = \mathbb{E} [\exp(-sX - tY)] \), \( s, t > 0 \). Fortunately, applying the helpful identity, it suffices to carry out

\[
stL_H(s, t) = st \int_0^\infty \int_0^\infty H(x, y) \exp(-sx - ty) \, dx \, dy
\]

\[
= st \int_0^\infty \int_0^\infty [\overline{H}(x, y) - 1 + F(x) + G(y)] \exp(-sx - ty) \, dx \, dy.
\]

The advantage is that we now can ignore the singular part of \( H \) (or \( \overline{H} \)), because its two-dimensional Lebesgue measure is zero. The final result is

\[
L_H(s, t) = \frac{1}{\theta + s + t} [((\theta + s)L_F(s) + (\theta + t)L_G(t))] - \frac{\theta}{\theta + s + t}, \quad s, t > 0.
\]

(See Lin et al. [30], Theorem 2, for a different proof.)

6. Application to the characterization of distributions

To illustrate the use of Theorem 4.2, we will consider some frequently used Laplace–Stieltjes transforms below. Denote by \( p_j \) the \( j \)th prime number (we have \( p_1 = 2 \), \( p_2 = 3 \), \( p_3 = 5 \), \ldots) and denote by \( \mathcal{P} := \{p_j\}_{j=1}^\infty \) the sequence of all prime numbers.
Then it is known that $\sum_{j=1}^{\infty} 1/p_j = \infty$ because $p_j \sim j \ln j$ as $j \to \infty$ (see, e.g., Apostol [31], p. 80).

Now we present eight examples in which $F$ is a one-dimensional distribution and $H$ is a two-dimensional distribution in Examples 3–7, while three-dimensional distribution in Example 8. More examples can be found in Balakrishnan and Lai [32]. Notice that based on our results, we are in a position to formulate each example as a direct statement involving a specific distribution.

**Example 1:** The Laplace–Stieltjes transform $L_F$ satisfies

$$L_F(p_j) = \frac{\lambda}{\lambda + p_j}, \quad j \in \mathbb{N}, \text{ where } \lambda > 0,$$

if and only if $F$ is the exponential distribution with mean $1/\lambda$. More generally,

$$L_F(p_j) = \left[ \frac{\lambda}{\lambda + p_j} \right]^q, \quad j \in \mathbb{N}, \text{ where } \lambda, q > 0,$$

if and only if $F$ is the Gamma distribution with mean $q/\lambda$ and shape parameter $q$, namely, $F$ has a density $f(x) = \left[ \frac{\lambda^q}{\Gamma(q)} \right] x^{q-1} \exp(-\lambda x), \quad x \geq 0$.

**Example 2:** The Laplace–Stieltjes transform $L_F$ satisfies

$$L_F(p_j) = \exp[-p_j^\alpha], \quad j \in \mathbb{N}, \text{ where } \alpha \in (0, 1),$$

if and only if $F$ is the positive stable distribution with density function

$$f_\alpha(x) = -\frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{\Gamma(ak+1)}{k!} \left( -x^{-\alpha} \right)^k \sin(\alpha k \pi), \quad x > 0$$

(see Feller [18], p. 583, and Hougaard [33]).

**Example 3:** The Laplace–Stieltjes transform $L_H$ satisfies

$$L_H(p_i,p_j) = \frac{(\lambda + p_i + p_j)(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_12) + p_i p_j \lambda_12}{(\lambda + p_i + p_j)(\lambda_1 + \lambda_12 + p_i)(\lambda_2 + \lambda_12 + p_j)}, \quad i,j \in \mathbb{N},$$

where $\lambda_1, \lambda_2, \lambda_12$ are positive constants and $\lambda = \lambda_1 + \lambda_2 + \lambda_12$, if and only if $H$ is the Marshall–Olkin bivariate exponential (BVE) distribution with survival function $\overline{H}(x,y) = \exp[-\lambda_1 x - \lambda_2 y - \lambda_12 \max\{x,y\}], \quad x,y \geq 0$ (see Marshall and Olkin [28]).

**Example 4:** The Laplace–Stieltjes transform $L_H$ satisfies

$$L_H(p_i,p_j) = \frac{1}{\alpha + \beta + p_i + p_j} \left[ \frac{\alpha' \beta}{\alpha' + p_i} + \frac{\alpha \beta'}{\beta' + p_j} \right], \quad i,j \in \mathbb{N}, \text{ where } \alpha, \alpha', \beta, \beta' > 0,$$

if and only if $H$ is the Freund BVE distribution with joint density

$$h(x,y) = \begin{cases} \alpha' \beta e^{-(\alpha + \beta - \alpha') x - \alpha' y}, & x > y \geq 0 \\ \alpha \beta' e^{-(\alpha + \beta - \beta') x - \beta' y}, & y \geq x \geq 0 \end{cases}$$

(see Freund [34]).
Example 5: The Laplace–Stieltjes transform \( \mathcal{L}_H \) satisfies

\[
\mathcal{L}_H(p_i, p_j) = \frac{1}{(1 + p_i)(1 + p_j) - r p_i p_j}, \quad i, j \in \mathbb{N}, \text{ where } r \in [0, 1),
\]

if and only if \( H \) is the Moran–Downton BVE distribution with density function

\[
h(x, y) = \frac{1}{1-r} I_0 \left( \frac{2 \sqrt{rxy}}{1-r} \right) e^{-(x+y)/(1-r)}, \quad x, y > 0,
\]

in which \( r \) is the correlation coefficient and \( I_0(t) = \sum_{k=0}^{\infty} (t/2)^{2k}/(k!)^2, \ t \in \mathbb{R} := (-\infty, \infty) \), is the modified Bessel function of the first kind of order zero (see Moran [35] and Downton [36]). Using \( \mathcal{L}_H \), we see that the marginal distributions of \( H \) have finite second moments.

Example 6: The Laplace–Stieltjes transform \( \mathcal{L}_H \) satisfies

\[
\mathcal{L}_H(p_i, p_j) = \frac{1}{\theta + p_i + p_j} \left[ (\theta + p_i) \mathcal{L}_F(p_i) + (\theta + p_j) \mathcal{L}_G(p_j) \right] - \frac{\theta}{\theta + p_i + p_j}, \quad i, j \in \mathbb{N},
\]

where \( \theta > 0 \) is a constant, if and only if \( H \) is the bivariate lack-of-memory distribution \( BLM(F, G, \theta) \) with survival function (see Section 5 or Lin et al. [30]):

\[
H(x, y) = \begin{cases} 
  e^{-\theta y} F(x - y), & x \geq y \geq 0 \\
  e^{-\theta x} G(y - x), & y \geq x \geq 0.
\end{cases}
\]

Example 7: The Laplace–Stieltjes transform \( \mathcal{L}_H \) satisfies

\[
\mathcal{L}_H(p_i, p_j) = \frac{(1 - r)^q}{(1 - r + p_i + p_j + p_i p_j)^q}, \quad i, j \in \mathbb{N}, \text{ where } r \in [0, 1) \text{ and } q > 0,
\]

if and only if \( H \) is the standard bivariate Gamma distribution with density function

\[
h(x, y) = (1 - r)^q \frac{(xy)^{q-1}}{\Gamma(q)} f_q(rxy) e^{-x-y}, \quad x, y > 0,
\]

in which \( f_q(z) = \sum_{n=0}^{\infty} z^n/[n! \Gamma(q + n)] \), and \( r, q > 0 \) are the correlation coefficient and the shape parameter, respectively ( [37, p. 14]; [38, Theorem 1.1]).

Example 8: Let \( H \) be the joint distribution of three nonnegative variables \((X, Y, Z)\). The Laplace–Stieltjes transform \( \mathcal{L}_H \) satisfies

\[
\mathcal{L}_H(p_i, p_j, p_k) = \left[ \frac{1 - a^2 - b^2}{(1 + p_i)(1 + p_j)(1 + p_k)} \right]^\alpha \\
\times \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)n!} \binom{n}{\ell} \frac{a^2\ell b^{2(n-\ell)}}{(1 + p_i)^\ell (1 + p_j)^n (1 + p_k)^{n-\ell}}, \quad i, j, k \in \mathbb{N},
\]
where $\alpha, a, b > 0$ and $a^2 + b^2 < 1$, if and only if $H$ is the three-variate Gamma distribution with density function (see Marcus [38, Example 2.3]):

$$
\begin{aligned}
    h(x, y, z) &= (1 - a^2 - b^2)^{\alpha} (xyz)^{\alpha - 1} e^{-(x+y+z)} \\
    &\times \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \frac{y^n}{\Gamma(\alpha) n!} \left( \begin{array}{c} n \\ \ell \end{array} \right) \frac{(a^2 x)^{\ell} (b^2 z)^{n-\ell}}{\Gamma(\ell + \alpha) \Gamma(n - \ell + \alpha)}, \\
    x, y, z &> 0.
\end{aligned}
$$

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Appendix

Proof of the identity (16). Rewrite the LHS of the Lebesgue–Stieltjes integral in (15) as follows:

\[
E \left[ \prod_{i=1}^{m+1} \left( 1 - \exp(-s_i X_i) \right) \right]
\]

\[
= 1 + \sum_{k=1}^{m+1} (-1)^k E \left[ \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m+1} \exp(-s_{i_1} X_{i_1} - s_{i_2} X_{i_2} - \cdots - s_{i_k} X_{i_k}) \right]
\]

\[
= 1 + \sum_{k=1}^{m+1} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m+1} E[\exp(-s_{i_1} X_{i_1} - s_{i_2} X_{i_2} - \cdots - s_{i_k} X_{i_k})]
\]

\[
= 1 + \sum_{k=1}^{m+1} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m+1} \sum_{j=1}^{m} s_{i_j} L_{H_{i_1 \cdots i_k}} (s_{i_1}, s_{i_2}, \ldots, s_{i_k})
\]

\[
= 1 + \sum_{k=1}^{m} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m+1} \left( \prod_{j=1}^{k} s_{i_j} \right) L_{H_{i_1 \cdots i_k}} (s_{i_1}, s_{i_2}, \ldots, s_{i_k})
\]

\[
+ (-1)^{m+1} L_H (s_1, s_2, \ldots, s_{m+1}). \tag{A1}
\]

In the last equality, we use the assumption on induction: the identity (11) holds true for \( n = 1, 2, \ldots, m \). Namely, for \( k = 1, 2, \ldots, m \),

\[
L_{H_{i_1 \cdots i_k}} (s_{i_1}, s_{i_2}, \ldots, s_{i_k}) = \left( \prod_{j=1}^{k} s_{i_j} \right) L_{H_{i_1 \cdots i_k}} (s_{i_1}, s_{i_2}, \ldots, s_{i_k}).
\]

On the other hand, using (12) we rewrite the RHS of the identity (15) as

\[
\left( \prod_{i=1}^{m+1} s_i \right) \int_0^\infty \int_0^\infty \cdots \int_0^\infty H(x_1, x_2, \ldots, x_{m+1}) \exp \left( - \sum_{i=1}^{m+1} s_i x_i \right) \prod_{i=1}^{m+1} dx_i
\]

\[
= 1 + \sum_{k=1}^{m+1} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m+1} \left( \prod_{i=1}^{m+1} s_i \right)
\]

\[
\times \int_0^\infty \int_0^\infty \cdots \int_0^\infty H_{i_1 \cdots i_k} (x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \exp \left( - \sum_{i=1}^{m+1} s_i x_i \right) \prod_{i=1}^{m+1} dx_i \tag{A2}
\]

\[
= 1 + \sum_{k=1}^{m} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m+1} \left( \prod_{j=1}^{k} s_{i_j} \right) L_{H_{i_1 \cdots i_k}} (s_{i_1}, s_{i_2}, \ldots, s_{i_k})
\]

\[
+ (-1)^{m+1} \left( \prod_{i=1}^{m+1} s_i \right) L_H (s_1, s_2, \ldots, s_{m+1}). \tag{A3}
\]

In Equation (A2), we use the fact that

\[
\left( \prod_{i=1}^{m+1} s_i \right) \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp \left( - \sum_{i=1}^{m+1} s_i x_i \right) \prod_{i=1}^{m+1} dx_i = 1,
\]
while in Equation (A3), we apply the result
\[
\left( \prod_{i \in J_k} s_i \right) \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp\left( -\sum_{i \in J_k} s_i x_i \right) \prod_{i \in J_k} dx_i = 1,
\]
where \( J_k = \{1, 2, \ldots, m + 1\} \setminus \{i_1, i_2, \ldots, i_k\}, k = 1, 2, \ldots, m. \)

Finally, comparing Equations (15), (A1) and (A3), we claim the identity (11), or, equivalently, the identity (16). This completes the proof.