Pizzetti formulae and the Radon Transform on the Sphere

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Abstract

In this paper, we obtain Pizzetti-type formulae on regions of the unit sphere $S^{m-1}$ of $\mathbb{R}^m$, and study their applications to the problem of inverting the spherical Radon transform. In particular, we approach integration over $(m-2)$-dimensional sub-spheres of $S^{m-1}$, $(m-1)$-dimensional sub-balls, and over $(m-1)$-dimensional spherical caps as the action of suitable concentrated delta distributions. In turn, this leads to Pizzetti formulae that express such integrals in terms of the action of $\text{SO}(m-1)$-invariant differential operators. In the last section of the paper, we use some of these expressions to derive the inversion formulae for the Radon transform on $S^{m-1}$ in a direct way.

Keywords. Pizzetti formula, Radon transform, spherical harmonics, integration, delta distributions

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1 Introduction

The classical Pizzetti formula expresses the integral over the unit sphere $S^{m-1} \subset \mathbb{R}^m$ as a certain power series of the Euclidean Laplacian operator acting on the integrand [12]. This fits in the larger framework of Stiefel manifolds recently studied by Coulembier and Kieburg in [3], as $S^{m-1} \cong \text{SO}(m)/\text{SO}(m-1)$ is the Stiefel manifold of order $k = 1$ in $\mathbb{R}^m$.

In [3, 8], Pizzetti formulae for the Stiefel manifold

$$\text{St}(m, k) := \left\{ (\omega_1, \ldots, \omega_k) \in (\mathbb{R}^m)^k : \langle \omega_j, \omega_\ell \rangle = \delta_{j,\ell} \right\} \cong \text{SO}(m)/\text{SO}(m-k)$$

were derived in terms of a recursive application of $k$ Pizzetti formulae over geodesic sub-spheres of $S^{m-1}$ of respective codimensions $1, 2, \ldots, k$ ($1 \leq k \leq m-2$). These formulae turn out the be closely related to the Radon transform on $S^{m-1}$. This relation becomes evident from the fact that spherical Radon transforms can be considered as functions on Stiefel manifolds, and reciprocally, integrals on Stiefel manifolds can be written in terms of Radon transforms.

Indeed, we recall that the spherical Radon transform maps integrable functions on $S^{m-1}$ to functions defined on the set $\Xi$ of $(m-k-1)$-dimensional totally geodesic sub-spheres of $S^{m-1}$. This transformation takes place by means of the correspondence

$$f \mapsto \hat{f}, \quad \text{with} \quad \hat{f}(\xi) = \int_{\Xi} f(\varphi) dS_{\Xi},$$

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where $f \in L^2(\mathbb{S}^{m-1})$, $\xi \in \Xi$, and $dS$ is the $(m-k-1)$-dimensional Euclidean surface measure on $\xi$. It is easily seen that $\hat{f}$ can be written as a function on $\text{St}(m,k)$ as follows

$$\hat{f}(\omega_1, \ldots, \omega_k) = 2 \int_{\mathbb{R}^m} \delta(||x||^2 - 1) \delta(\langle x, \omega_1 \rangle) \cdots \delta(\langle x, \omega_k \rangle) f(x) \, dV_x, \quad (\omega_1, \ldots, \omega_k) \in \text{St}(m,k),$$

where $dV_x = dx_1 \cdots dx_m$ is the classical Lebesgue measure in $\mathbb{R}^m$. Here we have made use of the concentrated delta distribution $\delta(||x||^2 - 1) \delta(\langle x, \omega_1 \rangle) \cdots \delta(\langle x, \omega_k \rangle)$ (see Section 2.3), and of the mapping $\varphi : \text{St}(m,k) \rightarrow \Xi$ given by

$$(\omega_1, \ldots, \omega_k) \mapsto \xi = \{ \xi \in \mathbb{S}^{m-1} : \langle x, \omega_1 \rangle = \cdots = \langle x, \omega_k \rangle = 0 \}.$$ 

In (1) we have abused of the notation $\hat{f}$ to refer to the function $\hat{f} \circ \varphi$. As a matter of fact, following the same procedure, (2) allows one to write any function on $\Xi$ as a function defined on $\text{St}(m,k)$.

Similarly, it is possible to connect integration over $\text{St}(m,k)$ with $(m-k)$-dimensional spherical Radon transforms. Indeed, given a function $\Phi(\omega_1, \ldots, \omega_k)$ of $k$ vector variables in $\mathbb{R}^m$, one easily obtains

$$\int_{\text{St}(m,k)} \Phi(\omega_1, \ldots, \omega_k) \, dS_{\omega_1, \ldots, \omega_k} = \int_{\text{St}(m,k-1)} \hat{\Phi}(\omega_1, \ldots, \omega_{k-1}) \, dS_{\omega_1, \ldots, \omega_{k-1}},$$

where $\hat{\Phi}(\omega_1, \ldots, \omega_{k-1})$ denotes the $(m-k)$-dimensional Radon transform of the function $\Phi(\omega_1, \ldots, \omega_{k-1}, \cdot)$ evaluated at the sub-sphere orthogonal to the $(k-1)$ orthonormal frame $(\omega_1, \ldots, \omega_{k-1})$, i.e.

$$\hat{\Phi}(\omega_1, \ldots, \omega_{k-1}) = 2 \int_{\mathbb{R}^m} \delta(\|w_k\|^2 - 1) \delta(\langle w_1, \omega_1 \rangle) \cdots \delta(\langle w_{k-1}, \omega_{k-1} \rangle) \Phi(\omega_1, \ldots, \omega_k) \, dV_{w_k}.$$ 

This paper is intended to be the first of a series of manuscripts dedicated to study the applications of Pizzetti-type formulas to the problem of inverting spherical Radon transforms, by exploiting the relations (1) and (3). An additional motivation for this idea is the fact that the dual of the Radon transform, which we recall that this set is invariant under the action of the group of rotations around $\xi$. In view of the mapping (2), the above integral can be rewritten as the following integral over $\text{St}(m-1,k)$, seeing the latter as the set of all orthonormal $k$-frames orthogonal to $\xi$, i.e.

$$\hat{\phi}(\xi) = \frac{1}{\text{vol}(\text{St}(m-1,k))} \int_{\text{St}(m,k)} \delta(\langle x, \omega_1 \rangle) \cdots \delta(\langle x, \omega_k \rangle) \phi(x) \, dS_{\omega_1, \ldots, \omega_k}.$$ 

In particular, it is our goal to find alternative direct proofs for the following inversion theorems for the Radon transform on $\mathbb{S}^{m-1}$, which have been proved in [9, 10].

**Theorem 1.** [10, Thm. 1.17 Ch. 3.1]. Let $m, k \in \mathbb{N}$ be natural numbers such that $1 \leq k \leq m-2$. Assume $m-k-1$ even and let $P_{m-k-1}$ be the polynomial

$$P_{m-k-1}(z) = \prod_{j=1}^{\frac{m-k-1}{2}} \left(z - (m-k-2j)(k+2j-2)\right),$$

of degree $\frac{m-k-1}{2}$. The $(m-k-1)$-dimensional Radon transform on $\mathbb{S}^{m-1}$ $f \mapsto \hat{f}$ is, for even functions $f$, inverted by the formula

$$2(-4\pi)^{\frac{m-k}{2}} \Gamma\left(\frac{m-k}{2}\right) \Gamma\left(\frac{m}{2}\right) f = P_{m-k-1}(\Delta_{LB}) (\hat{f})^{-},$$

where $\Delta_{LB}$ is the Laplace-Beltrami operator on $\mathbb{S}^{m-1}$ (see Section 3).
Theorem 2. [10, Thm. 1.22 Ch. 3.1]. The \((m - k - 1)\)-dimensional Radon transform \(f \mapsto \hat{f}\) on \(S^{m-1}\) is, for even functions \(f\), inverted by
\[
f(x) = \frac{2^{m-k-1}}{(m-k-1)!\sigma_{m-k}} \left(\frac{d}{dt}\right)^{m-k-1} \left[ \int_0^t (\hat{f})(\cos^{-1}(q))(z) \, q^{m-k-1} (1 - q^2)^{m-k-1} \, dq \right]_{t=1},
\]
where by \(\hat{\phi}_r(x)\) we denote the generalization of the dual Radon transform defined in \((\text{IV})\) in Section \((\text{IV})\).

In this paper, we will restrict our attention to the case of codimension \(k = 1\). We will use this case as a starting point in our study because it is the simplest-case scenario to consider from a computational point of view. Indeed, formula \((\text{I})\) shows that the \((m - 2)\)-dimensional Radon transform can be seen as an even function defined on \(S^{m-1}\). Moreover, formula \((\text{II})\) shows that the dual of this transform is also an integral over \((m - 2)\)-dimensional geodesic sub-spheres, which makes it (up to a constant) identical to the Radon transform. Hence, in order to deal with the inversion formulae given in Theorems \((\text{I})\) and \((\text{II})\) for \(k = 1\), we only need to establish Pizzetti-type ones for all (not necessary geodesic) sub-spheres on \(S^{m-1}\), which is achieved in Theorem \((\text{III})\). In a forthcoming paper, we shall study the more general case of arbitrary codimension \(1 \leq k \leq m - 2\).

The advantage of using Pizzetti formulae to prove these inversion results is that they offer a direct, straightforward way of computing the expressions on the right hand sides of \((\text{II})\) and \((\text{III})\). However, it is worth mentioning that this computational method hides, up to some extend, the geometric properties used by Helgason in his original proofs in [9, 10]. This is due to the fact that the Pizzetti formulae implicitly encode the group invariance of the sub-spheres under the action of suitable rotation subgroups of \(SO(m)\).

The paper is structured as follows. In Section \(2\) we briefly present the notation, definitions and preliminary results needed in the sequel. We pay particular attention to the notions spherical harmonics in \(\mathbb{R}^m\), Clifford algebras, and the use of concentrated delta distributions in integration. Section \(3\) is devoted to the study of Pizzetti-type formulae describing integration on certain regions of the unit sphere \(S^{m-1}\). In particular, we consider the integral over \((m - 2)\)-dimensional sub-spheres, \(m\)-dimensional sub-balls, and \((m - 1)\)-dimensional spherical caps. These results are all summarized in Theorem \(4\). The integral results found for the spherical caps yield a new expression for the classical Pizzetti formula on \(S^{m-1}\) which is discussed in more detail in Appendices \(\text{I} \& \text{II}\). Finally, in Section \(4\) we apply the Pizzetti formulae found for the \((m - 2)\)-dimensional sub-spheres to prove Helgason’s Radon inversion results given in Theorems \((\text{I})\) and \((\text{II})\) for \(k = 1\).

## 2 Preliminaries

In this section we provide a few preliminaries on the topics of spherical harmonics, Clifford algebras, and integration over manifolds using distributions; which shall be useful in the sequel.

### 2.1 Spherical harmonics

Let \(\mathcal{P}(\mathbb{R}^m) = \mathbb{C}[x_1, \ldots, x_m]\) be the space of complex-valued polynomials in the vector variable \(\mathbf{x} = (x_1, \ldots, x_m)^T\) of \(\mathbb{R}^m\). On \(\mathbb{R}^m\) we consider the standard Euclidean inner-product \(\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^m x_j y_j\) and its associated norm \(|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}\).

A polynomial \(P \in \mathcal{P}(\mathbb{R}^m)\) is called homogeneous of degree \(k\in\mathbb{N}_0 := \mathbb{N} \cup \{0\}\) (or \(k\)-homogeneous) if it holds for every \(\mathbf{x} \in \mathbb{R}^m\setminus\{0\}\) that
\[
P(\mathbf{x}) = |\mathbf{x}|^k P \left( \frac{\mathbf{x}}{|\mathbf{x}|} \right),
\]
where \(\left|\mathbf{x}\right|\) is the restriction of \(\mathbf{x}\) to the unit sphere \(S^{m-1} := \{\mathbf{w} \in \mathbb{R}^m : |\mathbf{w}| = 1\}\).

It is well-known that \(k\)-homogeneous polynomials are the eigenfunctions of the Euler operator \(E = \sum_{j=0}^m x_j \partial_{x_j}\) corresponding to the eigenvalue \(k\). Therefore one can define the space of homogeneous polynomials of degree \(k\) as
\[
\mathcal{P}_k(\mathbb{R}^m) = \{ P \in \mathcal{P}(\mathbb{R}^m) : E[P] = kP \},
\]
which allows for the decomposition
\[ \mathcal{P}(\mathbb{R}^m) = \bigoplus_{k=0}^{\infty} \mathcal{P}_k(\mathbb{R}^m). \]  

**Definition 1.** The space of \( k \)-homogeneous polynomials that are also harmonic, (i.e. null-solutions of the Laplace operator \( \Delta_x = \sum_{j=1}^{\infty} \ell_j^2 \)), is denoted by
\[ \mathcal{H}_k = \{ P \in \mathcal{P}_k(\mathbb{R}^m) : \Delta_x[P] = 0 \}. \]

The restriction of such \( k \)-homogeneous, harmonic polynomials to the unit sphere \( S^{m-1} \) are called spherical harmonics of degree \( k \). We denote the space of all spherical harmonics of degree \( k \) by \( \mathcal{H}_k(S^{m-1}) \).

The spaces of spherical harmonics \( \mathcal{H}_k(S^{m-1}) \) are eigenspaces of the Laplace-Beltrami operator on the sphere \( S^{m-1} \)
\[ \Delta_{LB} = \|x\|^2 \Delta_x - (m - 2 + \mu)E. \]

When acting on \( \mathcal{P}(\mathbb{R}^m) \), the operators \( \Delta_x, \|x\|^2 \) and \( E \) generate a representation of the special linear Lie algebra \( \mathfrak{sl}_2 \). Indeed, it can be easily verified that
\[ \left[ \frac{\Delta_x}{2}, \frac{\|x\|^2}{2} \right] = E + \frac{m}{2}, \quad \left[ \frac{\Delta_x}{2}, E + \frac{m}{2} \right] = \Delta_x, \quad \left[ \frac{\|x\|^2}{2}, E + \frac{m}{2} \right] = -\|x\|^2, \]
where \([a, b] := ab - ba\).

The Laplacian \( \Delta_x \) also plays an essential role when considering \( \mathcal{P}(\mathbb{R}^m) \) as a representation of the special orthogonal group \( SO(m) \) under the natural action \( H : SO(m) \to \text{Aut}(\mathcal{P}(\mathbb{R}^m)) \) given by
\[ M[P](x) = P(M^{-1}x), \quad M \in SO(m), \ P \in \mathcal{P}(\mathbb{R}^m). \]
Indeed, it is easily seen that \( \Delta_x \) commutes with the above action of \( SO(m) \), which implies that \( \mathcal{H}_k \) is a \( SO(m) \)-invariant subspace of \( \mathcal{P}_k(\mathbb{R}^m) \). The main assertions of the classical theory of spherical harmonics can be summarized as follows (see, for example [3, 9]).

**Proposition 1.** [Fischer decomposition]

i) The spaces \( \mathcal{H}_k \) \( (k \in \mathbb{N}_0) \) are irreducible representations of \( SO(m) \).

ii) \( \mathcal{P}_k(\mathbb{R}^m) = \mathcal{H}_k \oplus \|x\|^2 \mathcal{P}_{k-2}(\mathbb{R}^m) \).

iii) \( \mathcal{P}_k(\mathbb{R}^m) = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} \|x\|^{2j} \mathcal{H}_{k-2j} \).

When considering the restrictions of polynomials to \( S^{m-1} \), the statement iii) provides an orthogonal decomposition with respect to the inner product in \( L^2(S^{m-1}) \)
\[ \langle P(x), Q(x) \rangle_S = \frac{1}{\sigma_m} \int_{S^{m-1}} P(x)^c Q(x) dS_x, \]
where \( dS_x \) is the Lebesgue measure on \( S^{m-1} \), \( \sigma_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} \) is the surface area of \( S^{m-1} \) and \( (\cdot)^c \) stands for the complex conjugation. More in general, the decomposition in Proposition iii) can be extended to \( L^2(S^{m-1}) \) as follows (see e.g. [9, Ch. 1]).

**Proposition 2.** Any function \( f \in L^2(S^{m-1}) \) admits a unique decomposition into spherical harmonics,
\[ f(x) = \sum_{k=0}^{\infty} H_k(x), \quad H_k \in \mathcal{H}_k(S^{m-1}). \]
2.2 Clifford algebras

Throughout this paper, we shall use the language of Clifford algebras as a tool to compute some of the geometric quantities needed in the sequel. Let \( \{e_1, ..., e_m\} \) be an orthonormal basis of \( \mathbb{R}^m \). The real Clifford algebra \( \mathbb{R}_m \) is the real associative algebra with generators \( e_1, ..., e_m \) satisfying the defining relations \( e_j e_\ell + e_\ell e_j = -2\delta_{j\ell} \), for \( j, \ell = 1, ..., m \), where \( \delta_{j\ell} \) is the Kronecker symbol. Every element \( a \in \mathbb{R}_m \) can be written in the form

\[
a = \sum_{A \subseteq M} a_A e_A,
\]

where \( a_A \in \mathbb{R} \), \( M := \{1, ..., m\} \) and for any multi-index \( A = \{j_1, ..., j_k\} \subseteq M \) with \( j_1 < ... < j_k \) we put \( e_A = e_{j_1} \cdots e_{j_k} \) and \( |A| = k \). Every \( a \in \mathbb{R}_m \) admits a multi-vector decomposition

\[
a = \sum_{k=0}^{m} [a]_k, \quad \text{where} \quad [a]_k = \sum_{|A|=k} a_A e_A.
\]

Here \([\cdot]_k : \mathbb{R}_m \to \mathbb{R}^{(k)}\) denotes the canonical projection of \( \mathbb{R}_m \) onto the space of \( k \)-vectors \( \mathbb{R}^{(k)}_m = \text{span}_\mathbb{R}\{e_A : |A| = k\} \). Note that \( \mathbb{R}^{(0)}_m = \mathbb{R} \) is the set of scalars, while the space of \( 1 \)-vectors \( \mathbb{R}^{(1)}_m \) is isomorphic to \( \mathbb{R}^m \) via the identification \( \psi = (v_1, ..., v_m)^T \to \sum_{j=1}^m v_j e_j \). When necessary, we shall use this interpretation of a column vector in \( \mathbb{R}^m \) as a \( 1 \)-vector in \( \mathbb{R}^{(1)}_m \).

An important automorphism on \( \mathbb{R}_m \) leaving the multivector structure invariant is the Clifford conjugation \( \tau \), which is defined as the linear mapping satisfying

\[
\overline{ab} = \overline{b} \overline{a}, \quad a, b \in \mathbb{R}_m, \quad \text{and} \quad \overline{v_j} = -e_j, \quad j \in M.
\]

This leads to the norm on the Clifford algebra defined by \( \|a\|^2 = \langle a, a \rangle_0 = \sum_{A \subseteq M} a_A^2 \) for \( a \in \mathbb{R}_m \).

The Clifford product of two vectors \( \underline{v}, \underline{u} \in \mathbb{R}^{(1)}_m \) may be written as \( \underline{v} \underline{u} = \underline{v} \cdot \underline{u} + \underline{v} \wedge \underline{u} \) where

\[
\underline{v} \cdot \underline{u} = \frac{1}{2}(\underline{v} \underline{u} + \underline{u} \underline{v}) = -\sum_{j=1}^m u_j v_j, \quad \text{and} \quad \underline{v} \wedge \underline{u} = \frac{1}{2}(\underline{v} \underline{u} - \underline{u} \underline{v}) = \sum_{j<\ell} (v_j u_\ell - u_j v_\ell) e_\ell e_\ell
\]

are the so-called dot and wedge products respectively. Note that \( \underline{v} \cdot \underline{u} = -\langle \underline{u}, \underline{v} \rangle \in \mathbb{R} \), while \( \underline{v} \wedge \underline{u} \in \mathbb{R}^{(2)}_m \).

In particular, for \( \underline{v} = \underline{u} \) one obtains \( \|\underline{v}\|^2 = -\underline{v}^2 \). Furthermore, for \( \underline{u}_1, ..., \underline{u}_k \in \mathbb{R}^{(1)}_m \) we define the wedge (or exterior) product in terms of the Clifford product by

\[
\underline{u}_1 \wedge \cdots \wedge \underline{u}_k = \frac{1}{k!} \sum_{\pi \in \text{Sym}(k)} \text{sgn}(\pi) \underline{u}_{\pi(1)} \wedge \cdots \wedge \underline{u}_{\pi(k)} \in \mathbb{R}^{(k)}_m.
\]

Among the most important properties of the wedge product we have

- \( \underline{u}_{\pi(1)} \wedge \cdots \wedge \underline{u}_{\pi(k)} = \text{sgn}(\pi) \underline{u}_1 \wedge \cdots \wedge \underline{u}_k \);
- \( \underline{u}_1 \cdots \underline{u}_k \) is a set of orthogonal vectors, then \( \underline{u}_1 \wedge \cdots \wedge \underline{u}_k = \underline{u}_1 \cdots \underline{u}_k \);
- \( \underline{u}_1 \wedge \cdots \wedge \underline{u}_k = 0 \) if and only if the vectors \( \underline{u}_1, ..., \underline{u}_k \) are linearly dependent.

We shall also make use of the so-called Gram matrix of the vectors \( \underline{u}_1, ..., \underline{u}_k \in \mathbb{R}^m \), which is defined as

\[
G(\underline{u}_1, ..., \underline{u}_k) = \begin{pmatrix}
\langle \underline{u}_1, \underline{u}_1 \rangle & \langle \underline{u}_1, \underline{u}_2 \rangle & \cdots & \langle \underline{u}_1, \underline{u}_k \rangle \\
\langle \underline{u}_2, \underline{u}_1 \rangle & \langle \underline{u}_2, \underline{u}_2 \rangle & \cdots & \langle \underline{u}_2, \underline{u}_k \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \underline{u}_k, \underline{u}_1 \rangle & \langle \underline{u}_k, \underline{u}_2 \rangle & \cdots & \langle \underline{u}_k, \underline{u}_k \rangle
\end{pmatrix}.
\]

The Gram determinant is the determinant of \( G(\underline{u}_1, ..., \underline{u}_k) \) and can be expressed in terms of the wedge product of vectors by

\[
\det G(\underline{u}_1, ..., \underline{u}_k) = \|\underline{u}_1 \wedge \cdots \wedge \underline{u}_k\|^2.
\]
This determinant coincides with the square of the volume of the parallelepiped spanned by the vectors \( \Sigma_1, \ldots, \Sigma_k \).

The most important operator in the theory of Clifford-valued functions is the so-called Dirac operator (or gradient) defined as

\[
\pounds = e_1 \pounds_{x_1} + \cdots + e_m \pounds_{x_m}.
\]

The function theory centred around the null solutions of \( \pounds \) constitutes a natural and successful extension of classical complex analysis to higher dimensions. As the Dirac operator factorizes the Laplace operator

\[
\Delta = -\pounds^2,
\]

this theory is also a refinement of harmonic analysis. Standard references on this setting are [2, 6, 7].

### 2.3 A distributional approach to integration

The central idea in the proofs of most of our Pizzetti formulae relies in the use of the concentrated delta distribution to describe integration on submanifolds of \( \mathbb{R}^m \). Therefore, we now recall some basic notions related to this distribution, see [6, Ch. 3] for more details.

Let us consider an \((m-k)\)-surface \( \Sigma \subset \mathbb{R}^m \) defined by means of \( k \) equations of the form

\[
\varphi_1(x_1, \ldots, x_m) = 0, \quad \varphi_2(x_1, \ldots, x_m) = 0, \ldots, \quad \varphi_k(x_1, \ldots, x_m) = 0,
\]

where the so-called defining phase functions \( \varphi_1, \ldots, \varphi_k \in C^\infty(\mathbb{R}^m) \) are independent, i.e.

\[
\pounds_{\Sigma}[\varphi_1] \wedge \cdots \wedge \pounds_{\Sigma}[\varphi_k] \neq 0 \quad \text{on} \quad \Sigma,
\]

or equivalently, the gradients \( \pounds_{\Sigma}[\varphi_1], \ldots, \pounds_{\Sigma}[\varphi_k] \) are linearly independent on every point of \( \Sigma \).

The previous condition means that, at any point of \( \Sigma \), there is a \( k \)-blade orthogonal to \( \Sigma \) and therefore a \((m-k)\)-dimensional tangent plane. We thus have have that, in an \( m \)-dimensional neighborhood \( U \) of any point of \( \Sigma \), there exists a \( C^\infty \)-local coordinate system in which the first \( k \) coordinates are \( u_1 = \varphi_1, \ldots, u_k = \varphi_k \), and the remaining \( u_{k+1}, \ldots, u_m \) can be chosen so that \( J(\frac{x}{u}) > 0 \). Then, for any test function \( \phi \in C^\infty(\mathbb{R}^m) \) with support in \( U \), i.e. \( \text{supp} \phi \subset U \), one has

\[
\int_{\mathbb{R}^m} \delta(\varphi_1) \cdots \delta(\varphi_k) \phi(\Sigma) \, dV_\Sigma = \int_{\mathbb{R}^m} \delta(u_1) \cdots \delta(u_k) \psi(\Sigma) \, du_1 \cdots du_m,
\]

where \( \psi(\Sigma) = \phi(\Sigma(u)) J(\frac{x}{u}) \) and \( dV_\Sigma = dx_1 \cdots dx_m \) is the classical Lebesgue measure in \( \mathbb{R}^m \). It is thus natural to define \( \delta(\varphi_1) \cdots \delta(\varphi_k) \) as

\[
(\delta(\varphi_1) \cdots \delta(\varphi_k), \phi) = \int_{\mathbb{R}^m} \delta(\varphi_1) \cdots \delta(\varphi_k) \phi(\Sigma) \, dV_\Sigma = \int_{\mathbb{R}^m} \psi(0, \ldots, 0, u_{k+1}, \ldots, u_m) \, du_{k+1} \cdots du_m.
\]

This last integral is taken over the surface \( \Sigma \cap U \), which is why the generalized function \( \delta(\varphi_1) \cdots \delta(\varphi_k) \) is said to be concentrated on this surface. This definition is extended to smooth functions \( \phi \) with compact support on \( \Sigma \), i.e. \( \text{supp} \phi \cap \Sigma \) compact, by considering a finite sum of integrals over local parametrizations using partitions of unity, see e.g. [13, Ch. 6].

In [3], the following result was obtained to describe non-oriented integration of functions over the \((m-k)\)-dimensional surface \( \Sigma = \{ \Sigma \in \mathbb{R}^m : \varphi_1(\Sigma) = \cdots = \varphi_k(\Sigma) = 0 \} \) in terms of the generalized function \( \delta(\varphi_1) \cdots \delta(\varphi_k) \). Let us consider an open region \( \Omega \subset \mathbb{R}^m \), then the result reads as follows.

**Theorem 3.** Let \( \Sigma \subset \Omega \) be an \((m-k)\)-surface defined as in [9] by means of the independent phase functions \( \varphi_1, \ldots, \varphi_k \in C^\infty(\mathbb{R}^m) \). Then for any function \( f \in C^\infty(\Omega) \), with \( \text{supp} f \cap \Sigma \) compact, we have

\[
\int_{\Sigma} f \, dS = \int_{\mathbb{R}^m} \delta(\varphi_1) \cdots \delta(\varphi_k) \| \pounds_{\Sigma}[\varphi_1] \wedge \cdots \wedge \pounds_{\Sigma}[\varphi_k] \| \, f \, dV.
\]

where \( dS \) is the \((m-k)\)-dimensional Lebesgue measure on \( \Sigma \).
Remark 2.1. Theorem 3 is a generalization to higher co-dimensions of Theorem 6.1.5 in [11].

The above distributional approach can be easily adapted if we want to integrate over suitable regions of the \((m - k)\)-surface \(\Sigma\). Indeed, let \(C = \{ x \in \mathbb{R}^m : \varphi(x) \leq 0 \}\) be an \(m\)-dimensional region of \(\mathbb{R}^m\) with \(\varphi \in C^\infty(\mathbb{R}^m)\) and \(\partial_2[\varphi] \wedge \partial_3[\varphi] \wedge \ldots \wedge \partial_m[\varphi] \neq 0\) on \(\Sigma \cap \partial C := \{ x \in \Sigma : \varphi(x) = 0 \}\). Hence, the integral over the region \(\Sigma \cap C\) can be written as

\[
\int_{\Sigma \cap C} f \, dS = \int_{\mathbb{R}^m} H(-\varphi) \delta(\varphi_1) \ldots \delta(\varphi_k) \| \partial_2[\varphi_1] \wedge \ldots \wedge \partial_m[\varphi] \| \, f \, dV,
\]

where \(H(-\varphi) = \begin{cases} 1, & \varphi \leq 0, \\ 0, & \varphi > 0, \end{cases}\) is the Heaviside distribution.

As it is expected, the generalized function \(\delta(\varphi_1) \ldots \delta(\varphi_k) \| \partial_2[\varphi_1] \wedge \ldots \wedge \partial_m[\varphi] \|\) is independent of the system of equations defining \(\Sigma\). To see this, let us transform the equations \(\varphi_1 = \ldots = \varphi_k = 0\) to \(\psi_1 = \ldots = \psi_k = 0\) where

\[
\psi_\ell(x) = \sum_{j=1}^k \alpha_{\ell,j}(x) \varphi_j(x), \quad \ell = 1, \ldots, k.
\]

Here the functions \(\alpha_{\ell,j} \in C^\infty(\mathbb{R}^m)\) are such that the matrix they form is nonsingular, i.e. \(\det\{\alpha_{\ell,j}\} \neq 0\) for every \(x \in \mathbb{R}^m\). Obviously both sets of equations define the same manifold \(\Sigma\). In these cases, we have the following result (see [3]).

Proposition 3. Let \(\Sigma \subset \mathbb{R}^m\) be a \((m - k)\)-surface defined by means of the independent phase functions \(\varphi_1, \ldots, \varphi_k \in C^\infty(\mathbb{R}^m)\) and let \(\psi_\ell = \sum_{j=1}^k \alpha_{\ell,j}(x) \varphi_j(x), \ell = 1, \ldots, k,\) be new functions such that \(\alpha_{\ell,j} \in C^\infty(\mathbb{R}^m)\) and \(\det\{\alpha_{\ell,j}\} \neq 0\) for every \(x \in \mathbb{R}^m\). We then have

\[
\delta(\psi_1) \ldots \delta(\psi_k) \| \partial_2[\psi_1] \wedge \ldots \wedge \partial_m[\psi] \|= \delta(\varphi_1) \ldots \delta(\varphi_k) \| \partial_2[\varphi_1] \wedge \ldots \wedge \partial_m[\varphi] \|.
\]

2.4 Integral over \(S^{m-1}\)

A good example to illustrate the invariance property established in Proposition 3 is the integral over the sphere \(rS^{m-1} := \{ x \in \mathbb{R}^m : \|x\| = r \}, \ r > 0\). Indeed, \(rS^{m-1}\) can be defined by means of any of the two equations \(\|x\|^2 - r^2 = 0\) or \(\|x\| - r = 0\). Both definitions can be used in (11) without changing the result of the integral since

\[
\|x\|^2 - r^2 = (\|x\| + r)(\|x\| - r), \quad \text{and} \quad \|x\| + r \neq 0, \text{ for all } x \in \mathbb{R}^m.
\]

Using these two phase functions defining \(S^{m-1}\), and making use of the corresponding gradients

\[
\partial_2[\|x\|^2 - r^2] = 2x, \quad \text{and} \quad \partial_2[\|x\| - r] = \frac{x}{\|x\|},
\]

we obtain from (11) that

\[
\int_{rS^{m-1}} f(x) dS_x = 2r \int_{\mathbb{R}^m} \delta(|x|^2 - r^2) f(x) dV_x = \int_{\mathbb{R}^m} \delta(|x| - r) f(x) dV_x.
\]

Pizzetti’s formula provides a method to compute integrals over the unit sphere \(S^{m-1}\) by acting with a certain power series of the Laplacian operator on the integrand, see [12]. For any polynomial \(P : \mathbb{R}^m \to \mathbb{C}\), this formula reads as

\[
\int_{S^{m-1}} P(x) \, dS_x = \sum_{k=0}^m \frac{2^m}{2k!} \Gamma(k + m/2) \Delta^k_{\|x\|} P(0).
\]

From formula (14) we can easily derive the following Pizzetti-type formulae for the sphere \(rS^{m-1}\), and for the ball \(B(0, r) := \{ x \in \mathbb{R}^m : \|x\| < r \}, \text{ with } r > 0\).
Proposition 4. Let $R$ be a polynomial in $\mathcal{P}(\mathbb{R}^m)$ and $r > 0$. Then

\begin{align}
\int_{S^{m-1}} R(\underline{x}) \, dS_{\underline{x}} &= \sum_{k=0}^{\infty} \frac{2 \pi^\frac{m}{2}}{2^{k+1}k! \left( \frac{m}{2} + k \right)} \Delta_k^2 [R](0) \, r^{2k}, \\
\int_{B(0,r)} R(\underline{x}) \, dV_{\underline{x}} &= \sum_{k=0}^{\infty} \frac{\pi^\frac{m}{2}}{2^{k+1}k! \left( \frac{m}{2} + 1 + k \right)} \Delta_k^k [R](0) \, r^{2k+m}, \\
\int_{rS^{m-1}} R(\underline{x}) \, dS_{\underline{x}} &= \sum_{k=0}^{\infty} \frac{2 \pi^\frac{m}{2}}{2^{k+1}k! \left( \frac{m}{2} + k \right)} \Delta_k^2 [R](0) \, r^{2k+m-1}.
\end{align}

Proof. Formula (15) directly follows from (14) and from the identity $\Delta_2 [R(r\underline{x})] = r^2 \Delta_2 [R](r\underline{x})$. To prove formula (16), we use spherical coordinates, i.e., $\underline{x} = t \underline{\omega}$ with $t = |\underline{x}|$ and $\underline{\omega} \in S^{m-1}$. Hence

\begin{align}
\int_{B(0,r)} R(\underline{x}) \, dV_{\underline{x}} &= \int_0^r \int_{S^{m-1}} R(t \underline{\omega}) t^{m-1} \, dS_{\underline{\omega}} \, dt \\
&= \int_0^r t^{m-1} \left( \int_{S^{m-1}} R(t \underline{\omega}) \, dS_{\underline{\omega}} \right) \, dt \\
&= \sum_{k=0}^{\infty} \frac{2 \pi^\frac{m}{2}}{2^{k+1}k! \left( \frac{m}{2} + k \right)} \Delta_k^k [R](0) \left( \int_0^r t^{2k+m-1} \, dt \right) \\
&= \sum_{k=0}^{\infty} \frac{\pi^\frac{m}{2}}{2^{k+1}k! \left( \frac{m}{2} + 1 + k \right)} \Delta_k^k [R](0) \, r^{2k+m}.
\end{align}

Finally, formula (17) directly follows from the change of coordinates $\underline{x} = r \underline{\omega}$, $\underline{\omega} \in S^{m-1}$, which implies that $dS_{\underline{x}} = r^{m-1} \, dS_{\underline{\omega}}$. \hfill \qed

3 Pizzetti Formulae on regions of the sphere

In this section we prove a handful of Pizetti-type formulae for integration of certain submanifolds of the unit sphere $S^{m-1}$ and the unit ball $B(0,1)$. In particular, we are interested on the sub-spheres, sub-balls and spherical caps that arise when one intersects $S^{m-1}$ and $B(0,1)$ with hyperplanes on $\mathbb{R}^m$ (see figures below). In order to write the integrals over these regions as the actions of the corresponding invariant differential operators, we will see that the language offered by the concentrated $\delta$ distribution becomes quite useful.

The Pizetti-type formulae to be obtained in this section are summarized in the following theorem.

Theorem 4. Given $\underline{\omega} \in S^{m-1}$ and $p \in \mathbb{R}$, consider the hyperplane $H_{\underline{\omega},p} := \{ \underline{x} \in \mathbb{R}^m : \langle \underline{x}, \underline{\omega} \rangle = p \}$. Then following statements hold for $0 \leq p < 1$ and $R \in \mathcal{P}(\mathbb{R}^m)$.

i) Let $S_{\underline{\omega},p} := S^{m-1} \cap H_{\underline{\omega},p}$ be the sub-sphere of $S^{m-1}$ contained in the hyperplane $H_{\underline{\omega},p}$. Then

\begin{equation}
\int_{S_{\underline{\omega},p}} R(\underline{x}) \, dS_{\underline{x}} = \sum_{k=0}^{\infty} \frac{2 \pi^\frac{m}{2}}{2^{k+1}k! \left( \frac{m}{2} + k \right)} \left( \Delta_k^2 - \langle \underline{\omega}, \nabla \Delta_k^2 \rangle^2 \right) [R]_{\underline{x}=\underline{\omega}} (1 - p^2)^{k+\frac{m}{2} - 1}.
\end{equation}

ii) Let $B_{\underline{\omega},p} := B(0,1) \cap H_{\underline{\omega},p}$ be the sub-ball of $B(0,1)$ contained in the hyperplane $H_{\underline{\omega},p}$. Then

\begin{equation}
\int_{B_{\underline{\omega},p}} R(\underline{x}) \, dS_{\underline{x}} = \sum_{k=0}^{\infty} \frac{\pi^\frac{m}{2}}{2^{k+1}k! \left( \frac{m}{2} + k \right)} \left( \Delta_k^k - \langle \underline{\omega}, \nabla \Delta_k^k \rangle^2 \right) [R]_{\underline{x}=\underline{\omega}} (1 - p^2)^{k+\frac{m}{2} - 1}.
\end{equation}

iii) Let $C_{\underline{\omega},p} := \{ \underline{x} \in S^{m-1} : \langle \underline{x}, \underlink{\omega} \rangle > p \}$ be the spherical cap in $S^{m-1}$ located in the upper-half of $\mathbb{R}^{m+1}$ relative to $H_{\underline{\omega},p}$. Then, for $R \in \mathcal{P}(\mathbb{R}^m)$ one has,

\begin{equation}
\int_{C_{\underline{\omega},p}} R(\underline{x}) \, dS_{\underline{x}} = \sum_{k=0}^{\infty} \frac{2 \pi^\frac{m}{2}}{2^{k+1}k! \left( \frac{m}{2} + k \right)} \epsilon_k (p) \left( \Delta_k^k - \langle \underline{\omega}, \nabla \Delta_k^k \rangle^2 \right) [R]_{\underline{x}=\underline{\omega}}.
\end{equation}
where the coefficients $c_{k,\ell}(p)$ are given by
\[ c_{k,\ell}(p) = \int_p^1 y_1^{\ell-2k} (1 - y_1^2)^{k + \frac{m-1}{2}} dy_1 = \frac{\Gamma \left( \frac{\ell+1}{2} - k \right) \Gamma \left( k + \frac{m-1}{2} \right)}{2\Gamma \left( \frac{\ell+m+1}{2} \right)} - \frac{p^{\ell-2k+1}}{\ell - 2k + 1} F_1 \left( -k - \frac{m-3}{2}, \frac{\ell+1}{2} - k; \frac{\ell+3}{2} - k, p^2 \right), \]

iv) Similarly, we obtain a Pizzetti formula for the spherical cap located in the lower-half of $\mathbb{R}^{m+1}$ relative to $H_{p,\omega}$, i.e. $C_{p,\omega}$ := \{ $x \in S^{m-1} : \langle x, \omega \rangle < p$ \}. In this case we obtain for $R \in \mathcal{P}_s(\mathbb{R}^m)$ that
\[ \int_{C_{p,\omega}} R_\ell(x) dS_x = \sum_{k=0}^{\left\lceil \frac{\ell}{2} \right\rceil} \frac{2\pi^{m-1}}{2k! \Gamma \left( \frac{m+1}{2} + k \right)} c_{k,\ell}^\prime(p) \left( \Delta_x - \langle \partial \omega, \partial \omega \rangle \right)^k \left[ R_x \right] \bigg|_{x=\omega}, \]

where the coefficients $c_{k,\ell}^\prime(p)$ are given by
\[ c_{k,\ell}^\prime(p) = \int_{-1}^p y_1^{\ell-2k} (1 - y_1^2)^{k + \frac{m-1}{2}} dy_1 = \frac{p^{\ell-2k+1}}{\ell - 2k + 1} F_1 \left( -k - \frac{m-3}{2}, \frac{\ell+3}{2} - k; \frac{\ell+1}{2} - k, p^2 \right) + \frac{(-1)^k \Gamma \left( \frac{\ell+1}{2} - k \right) \Gamma \left( k + \frac{m+1}{2} \right)}{2\Gamma \left( \frac{\ell+m+1}{2} \right)}. \]

In Figure 1, we depict (for the case $m = 3$) the intersection of the sphere $S^{m-1}$ with the hyperplane $H_{p,\omega}$. The blue contour represents the sub-sphere $S_{p,\omega}$ in part i) of the theorem, while the red shaded area represents the sub-ball $B_{p,\omega}$ in part ii).

![Figure 1: Intersection of the sphere with a hyperplane](image)

Similarly, Figure 2 depicts the two complementary caps $C_{p,\omega}$ and $C_{p,\omega}'$ in purple and green, respectively.

![Figure 2: Spherical Caps](image)
We note that the two formulae (20) and (21) describe integration over complementary caps on $S^{m-1}$, therefore their sum yields an alternative Pizzetti formula on $S^{m-1}$. Indeed, from the sum of these two we obtain:

$$
\int_{S^{m-1}} R_t(x) dS_x = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{2 \pi \frac{m-k}{2}}{2^{2k} k! \Gamma \left( \frac{m-k}{2} + k \right)} \left( \int_{-1}^{1} y^{t} (1-y^{2})^{k} \, dy \right) \left( \Delta_{x} - \langle \omega, \hat{\omega} \rangle^2 \right)^{k} \left[ R_t \right]_{\omega = \omega}.
$$

If $t$ is odd, it is clear that both sides of (22) vanish. On the other hand, for $t = 2s$, we obtain the following consequence of Theorem II

**Corollary 1.** Let $R_{2s} \in \mathcal{P}_{2s}(\mathbb{R}^m)$. Then

$$
\int_{S^{m-1}} R_{2s}(x) dS_x = \frac{2 \pi \frac{m-k}{2}}{\Gamma \left( s + \frac{m}{2} \right)} \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( s - k + \frac{1}{2} \right) \left( \Delta_{x} - \langle \omega, \hat{\omega} \rangle^2 \right)^{k} \left[ R_{2s} \right]_{\omega = \omega},
$$

where $\omega \in S^{m-1}$ is an arbitrary unit vector. When compared with the classical Pizzetti formula (14) on $S^{m-1}$, the above formula yields the identity

$$
\Delta_{x}^{s} \left[ R_{2s} \right]_{\omega = \omega} = \frac{2^{2s} \pi \frac{m-k}{2}}{\pi^{s} (2s)!} \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( s - k + \frac{1}{2} \right) \left( \Delta_{x} - \langle \omega, \hat{\omega} \rangle^2 \right)^{k} \left[ R_{2s} \right]_{\omega = \omega}.
$$

This result shows that the expression on the right hand side of (23) is independent of the unit vector $\omega$ and in Appendix A we provide a direct proof for this identity. Now, we proceed to prove the formulae in Theorem III

### 3.1 Integration on the subsphere $S_{p,\omega}$

Our first goal is to integrate over the intersection $S_{p,\omega}$ of the unit sphere $S^{m-1}$ with the hyperplane $H_{\omega, p}$, i.e.:

$$S_{p,\omega} = \{ x \in \mathbb{R}^m : \langle x, \omega \rangle = p, \| x \| = 1 \}, \quad \omega \in S^{m-1}, \quad 0 \leq p < 1. $$

This is an $(m-2)$-dimensional sphere on the hyperplane $H_{\omega, p}$ centered at $p\omega$ and with radius $(1-p^2)^{\frac{1}{2}}$; see Figure 1. A simple computation shows that this is indeed the case

$$\| x - p\omega \|^2 = \langle x - p\omega, x - p\omega \rangle = \| x \|^2 - 2p \langle x, \omega \rangle + p^2 \| \omega \|^2 = 1 - 2p^2 + p^2 = 1 - p^2. $$

The subsphere $S_{p,\omega}$ is defined by the pair of smooth functions $\varphi_1(x) = \| x \|^2 - 1$ and $\varphi_2(x) = \langle x, \omega \rangle - p$, whose gradients are given by $\hat{\varphi}_1[\varphi_1] = 2x$ and $\hat{\varphi}_2[\varphi_2] = \omega$ respectively. We thus obtain

$$\hat{\varphi}_1[\varphi_1] \wedge \hat{\varphi}_2[\varphi_2] = 2x \wedge \omega = 2(x \omega + \langle x, \omega \rangle) = 2(\omega x + p),$$

which yields

$$\| \hat{\varphi}_1[\varphi_1] \wedge \hat{\varphi}_2[\varphi_2] \|^2 = 4(\omega x + p)(\omega x + p) = 4(\omega x + p)(\omega x + p) = 4(1 - p^2).$$

**Remark 3.1.** We can see this in a more geometric way. Indeed, we can write $x = \cos(\theta)\omega + \sin(\theta)\xi$ with $\xi \perp \omega$. Hence $x \wedge \omega = \xi \wedge \omega \sin(\theta) = \xi \omega \sin(\theta)$, which implies $\| x \wedge \omega \| = \| \sin(\theta) \| = (1 - \cos(\theta)^2)^{\frac{1}{2}}$, where $\cos(\theta) = \langle x, \omega \rangle = p$.  

Using (24) and Theorem III we can now write the integral of a polynomial $R$ on $S_{p,\omega}$ as

$$
\int_{S_{p,\omega}} R(x) dS_x = \int_{\mathbb{R}^m} \delta(\varphi_1)\delta(\varphi_2)\| \hat{\varphi}_1[\varphi_1] \wedge \hat{\varphi}_2[\varphi_2] \| R(x) dV_x = 2\sqrt{1 - p^2} \int_{\mathbb{R}^m} \delta(\| x \|^2 - 1)\delta(\langle x, \omega \rangle - p) R(x) dV_x.
$$

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Let us now consider the change of coordinates \( y = M \omega \) where \( M \in \text{SO}(m) \) is a rotation matrix whose first row is given by the unit vector \( \omega \in S^{m-1} \), i.e. the first component of \( \omega \) is \( y_1 = \langle \omega, \omega \rangle \). Under this transformation, the above integral transforms into:

\[
\int_{S_p} R(z) \, dS_p = 2 \sqrt{1-p^2} \int_{m=1}^\infty \delta(||y||^2 - 1)\delta(y_1 - p)R(M^{-1}y) \, dV_y
\]

\[
= 2 \sqrt{1-p^2} \int_{m=1}^\infty \delta(y_2^2 + \cdots + y_n^2 - (1-p^2)) \, R(M^{-1}(p, y_2, \ldots, y_m)^T) \, dy_2 \ldots dy_m. \tag{25}
\]

The last expression corresponds to an integral over the sphere \((1-p^2)^{\frac{n}{2}}S^{m-2} \subset \mathbb{R}^{m-1}\) with respect to the vector \((y_2, \ldots, y_m)\), see [15]. Thus, applying the Pizzetti formula [17] on this sphere, we obtain:

\[
\int_{S_p} R(z) \, dS_p = \sum_{k=0}^\infty \frac{(2\pi)^{\frac{m-1}{2}}}{2^{2k}k!\Gamma(\frac{m-1}{2}+k)} (\Delta_y - \hat{\beta}_{y_1})^k R(M^{-1}(p, y_2, \ldots, y_m)^T) \bigg|_{y_2, \ldots, y_m=0} (1-p^2)^{k+\frac{m-1}{2}}.
\]

Returning to the original coordinate system \( z = M^{-1}y \) and making use of the identities \( \Delta_z = \Delta_y \) and \( \hat{\beta}_{y_1} = \langle \omega, \hat{\beta}_x \rangle = \sum_{j=1}^m \omega_j \hat{x}_j \), we have:

\[
(\Delta_y - \hat{\beta}_{y_1})^k R(M^{-1}(p, y_2, \ldots, y_m)^T) \bigg|_{y_2, \ldots, y_m=0} = (\Delta_z - \langle \omega, \hat{\beta}_x \rangle^2)^k [R] \big|_{z=M^{-1}(p, 0, \ldots, 0)^T}.
\]

Finally, observe that the point at which the last expression is evaluated is given by \( z = p \omega \). Indeed, it is easily seen that \( M \omega = (p, 0, \ldots, 0)^T \), and the Pizzetti formula for \( S_{p \omega} \) reads as:

\[
\int_{S_{p \omega}} R(z) \, dS_p = \sum_{k=0}^\infty \frac{(2\pi)^{\frac{m-1}{2}}}{2^{2k}k!\Gamma(\frac{m-1}{2}+k)} (\Delta_z - \langle \omega, \hat{\beta}_x \rangle^2)^k [R] \big|_{z=p \omega} (1-p^2)^{k+\frac{m-1}{2}},
\]

which proves the first part of the Theorem [3 ii).

### 3.2 Integration on the sub-ball \( \mathbb{B}_{p \omega} \)

We now turn our attention to the integral over the intersection \( \mathbb{B}_{p \omega} \) of the unit ball \( B(0, 1) \) with the hyperplane \( H_{p \omega} \), i.e.

\[
\mathbb{B}_{p \omega} = \{ x \in \mathbb{R}^m : \langle x, \omega \rangle = p, \| x \| < 1 \}, \quad \omega \in S^{m-1}, \quad 0 \leq p < 1.
\]

Similarly to the previous case, \( \mathbb{B}_{p \omega} \) is an \((m-1)\)-dimensional ball on the hyperplane \( H_{p \omega} \) centered at \( p \omega \) and with radius \((1-p^2)^{\frac{1}{2}}\), see Figure 1.

From [12] we obtain that

\[
\int_{\mathbb{B}_{p \omega}} R(z) \, dS_p = \int_{\mathbb{R}^m} H(1-\|x\|)\delta(\langle x, \omega \rangle - p)R(x) \, dV_x. \tag{26}
\]

Just as before, we consider a rotation \( y = M \omega \) where the first row of matrix \( M \in \text{SO}(m) \) is given by \( \omega \in S^{m-1} \), i.e. \( y_1 = \langle \omega, \omega \rangle \). With this change of coordinates in the above integral we obtain:

\[
\int_{\mathbb{B}_{p \omega}} R(z) \, dS_p = \int_{\mathbb{R}^m} H(1-\|y\|)\delta(y_1 - p)R(M^{-1}y) \, dV_y
\]

\[
= \int_{m=1}^\infty H(1-p^2 - (y_2^2 + \cdots + y_m^2)) \, R(M^{-1}(p, y_2, \ldots, y_m)^T) \, dy_2 \ldots dy_m,
\]

where this last expression corresponds to the integral on the ball \( \mathbb{B}(0, (1-p^2)^{\frac{1}{2}}) \) in \( \mathbb{R}^{m-1} \) with respect to the vector \((y_2, \ldots, y_m)\). Applying the Pizzetti formula [16] on this ball and following a similar reasoning as in the previous case, we obtain:

\[
\int_{\mathbb{B}_{p \omega}} R(z) \, dS_p = \sum_{k=0}^\infty \frac{(2\pi)^{\frac{m-1}{2}}}{2^{2k}k!\Gamma(\frac{m-1}{2}+1+k)} (\Delta_y - \hat{\beta}_{y_1})^k R(M^{-1}(p, y_2, \ldots, y_m)^T) \bigg|_{y_2, \ldots, y_m=0} (1-p^2)^{k+\frac{m-1}{2}}
\]

\[
= \sum_{k=0}^\infty \frac{(2\pi)^{\frac{m-1}{2}}}{2^{2k}k!\Gamma(\frac{m-1}{2}+1+k)} (\Delta_z - \langle \omega, \hat{\beta}_x \rangle^2)^k [R] \bigg|_{z=p \omega} (1-p^2)^{k+\frac{m-1}{2}},
\]

which proves the second part of Theorem [3 ii).
3.3 Integration on the Spherical Caps $C_{p,\varpi}$ and $C'_{p,\varpi}$

We now discuss the cases of the complementary spherical caps $C_{p,\varpi}$ and $C'_{p,\varpi}$, given by the intersection of $S^{m-1}$ with the upper and lower half of $\mathbb{R}^{m+1}$ with respect to $H_{\varpi}$ respectively, see Figure 2. It is enough to explain the details of the case of $C_{p,\varpi}$, since integration on $C'_{p,\varpi}$ can be treated in an analogous way. We first recall that:

$$C_{p,\varpi} = \{ x \in \mathbb{R}^m : \langle x, \varpi \rangle > p, \| x \| = 1 \}, \quad \varpi \in S^{m-1}, \quad 0 < p < 1.$$  

Clearly $C_{p,\varpi}$ is a $(m-1)$-dimensional surface on $S^{m-1}$ and, by virtue of (12), we obtain the following integration formula:

$$\int_{C_{p,\varpi}} R(x) dS_{\varpi} = 2 \int_{\mathbb{R}^m} H(\langle x, \varpi \rangle - p) \delta(\| x \|^2 - 1) R(x) dV_x.$$  

The same type of rotation $y = Mx$ (where $M \in SO(m)$ is such that $y_1 = \langle x, \varpi \rangle$) yields:

$$\int_{C_{p,\varpi}} R(x) dS_{\varpi} = 2 \int_{\mathbb{R}^m} H(y_1 - p) \delta(\| y \|^2 - 1) R(M^{-1}y) dV_y = \int_0^1 \left( 2 \int_{\mathbb{R}^m} \delta(y_2^2 + \cdots + y_m^2 - (1 - y_1^2)) R(M^{-1}(y_1, y_2, \ldots, y_m)^T) dy_2 \cdots dy_m \right) dy_1.$$  

The expression between the brackets above has been already computed in (25). This expression coincides with the integral over the sphere $(1 - y_1^2)^{\frac{m-2}{2}} \subset \mathbb{R}^{m-1}$ with respect to $(y_2, \ldots, y_m)$, up to the factor $(1 - y_1^2)^{-\frac{1}{2}}$ (see (13)). Pizzetti’s formula (17) then yields:

$$\int_{C_{p,\varpi}} R(x) dS_{\varpi} = (1 - y_1^2)^{-\frac{1}{2}} \int_0^1 \left( \int_{(1 - y_1^2)^{\frac{m-2}{2}}} R(M^{-1}(y_1, y_2, \ldots, y_m)^T) dS_{(y_2, \ldots, y_m)} \right) dy_1$$

$$= \sum_{k=0}^{\infty} 2\pi \frac{\pi^{\frac{m-1}{2}}}{k!} \Gamma \left( \frac{m}{2} + k \right) \int_0^1 (\Delta_{\varpi} - \omega, \hat{\partial}_\omega^2)^k \left[ R \right]_{y_1 = \varpi, y_2 = \cdots = y_m = 0} (1 - y_1^2)^{k + \frac{m-3}{2}} dy_1.$$

Formula (27) provides a Pizzetti formula for integrating arbitrary polynomials $R \in \mathcal{P}(\mathbb{R}^m)$ over the spherical cap $C_{p,\varpi}$. Contrary to the previous Pizzetti-type formulae, the above expression does not depend only on the action of invariant differential operators on the integrand. Instead, it combines this action with a 1-dimensional integral with respect to the component $y_1 = \langle x, \varpi \rangle$. This integral can be further simplified if we restrict these actions to the homogeneous components of $R$.

Indeed, let us assume that $R = R_\ell$ is an homogeneous polynomial of degree $\ell \in \mathbb{N}_0$. This assumption involves no loss of generality since we can always decompose polynomials into their homogeneous components, see (7). In this case, we have that $(\Delta_{\varpi} - \langle \omega, \hat{\partial}_\omega^2 \rangle)^k [R_\ell]$ is homogeneous of degree $\ell - 2k$ for $\ell \geq 2k$. Then one obtains:

$$(\Delta_{\varpi} - \langle \omega, \hat{\partial}_\omega^2 \rangle)^k [R_\ell]_{y_1 = \varpi} = \begin{cases} y_1^{\ell-2k} (\Delta_{\varpi} - \langle \omega, \hat{\partial}_\omega^2 \rangle)^k [R_\ell]_{y_1 = \varpi}, & \ell \geq 2k, \\ 0, & \ell < 2k. \end{cases}$$

Therefore, on the space $\mathcal{P}_\ell(\mathbb{R}^m)$ of homogeneous polynomials of degree $\ell$, we can re-write (27) as

$$\int_{C_{p,\varpi}} R_\ell(x) dS_{\varpi} = \sum_{k=0}^{\infty} 2\pi \frac{\pi^{\frac{m-1}{2}}}{k!} \Gamma \left( \frac{m}{2} + k \right) c_{k,\ell}(p) (\Delta_{\varpi} - \langle \omega, \hat{\partial}_\omega^2 \rangle)^k [R_\ell]_{y_1 = \varpi},$$

where $c_{k,\ell}(p) := \int_0^1 y_1^{\ell-2k} (1 - y_2^2)^{k + \frac{m-3}{2}} dy_1$. 

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The integral coefficients \( c_{k,ℓ}(p) \) in the above summation can be computed explicitly. Indeed, let us first recall that a primitive for the function \( f(t) = t^a(1 - t^2)^b \) is given by

\[
\frac{t^{a+1}}{a+1} \, 2F_1 \left( -b, \frac{a+1}{2}; \frac{a+3}{2}; t^2 \right),
\]

where \( 2F_1(a; b; c, z) \) is the hypergeometric function in the variable \( z \) of parameters \((a, b)\) and \(c\). Combining this fact with the well-known Gauss summation theorem for hypergeometric functions (see Theorem 2.2.2 in [1]), i.e.

\[
2F_1(a; b; c, 1) = \frac{\Gamma(c)\Gamma(c - b - a)}{\Gamma(c - a)\Gamma(c - b)}, \quad c > a + b, \quad a, b, c \in \mathbb{R},
\]

we obtain that

\[
\int_{\mathbb{R}} t^n(1 - t^2)^b \, dt = \frac{t^{a+1}}{a+1} \, 2F_1 \left( -b, \frac{a+1}{2}; \frac{a+3}{2}; t^2 \right) \bigg|_{t=p}^{t=1} = \frac{\Gamma \left( \frac{a+1}{2} \right) \Gamma (b+1)}{2\Gamma \left( \frac{a+3}{2} \right) + b} - \frac{p^{a+1}}{a+1} \, 2F_1 \left( -b, \frac{a+1}{2}; \frac{a+3}{2}; p^2 \right).
\]

Finally substituting \( a = l - 2k \) and \( b = k + \frac{m-3}{2} \) we obtain that

\[
c_{k,ℓ}(p) := \int_{\mathbb{R}} y^{l-2k}(1 - y^2)^{\ell-k+\frac{m-2}{2}} \, dy = \frac{\Gamma \left( \frac{l+1-k}{2} \right) \Gamma \left( k + \frac{m-1}{2} \right)}{2\Gamma \left( \frac{l+3}{2} \right)} - \frac{p^{l-2k+1}}{\ell - 2k + 1} \, 2F_1 \left( -k - \frac{m-3}{2}, \frac{\ell+1}{2} - k; \frac{\ell+3}{2} - k; p^2 \right),
\]

which proves the third part of the Theorem 3 (iii). As mentioned above, similar arguments apply to the proof of the fourth part \( iv \).

### 4 Application: Inversion Formulae for the Spherical Radon Transform

In this section we provide alternative proofs to the inversion formulae of the \((m-2)\)-dimensional spherical Radon transform (also known as the Funk transform). In particular, we will prove Theorems 1 and 2 for \( k = 1 \) by means of direct computations using the Pizzetti formulae in Theorem 4.

The spherical Radon transform in question is an integral transformation that maps functions in \( L^2(\mathbb{S}^{m-1}) \) to functions in \( L^2(\Xi) \), where \( \Xi \) is the manifold of all \((m-2)\)-dimensional totally geodesic sub-spheres of \( \mathbb{S}^{m-1} \). Given a function \( f \in L^2(\mathbb{S}^{m-1}) \), its spherical Radon transform \( \hat{f} \) is defined as

\[
\hat{f}(\xi) = \int_{\mathbb{S}^{m-1}} f(\varpi)dS_{\varpi}, \quad \xi \in \Xi.
\]

Since the above transform is taken over sub-spheres of co-dimension 1 with respect to \( \mathbb{S}^{m-1} \), the formula above can be rewritten in simpler terms. Indeed, it is easily seen that for every \( \xi \in \Xi \) there exists \( \varpi \in \mathbb{S}^{m-1} \) such that \( \varpi \perp \xi \), i.e.

\[
\xi = \mathbb{S}_{0,\varpi} = \{\varpi \in \mathbb{S}^{m-1} : \langle \varpi, \xi \rangle = 0\}.
\]

In other words, in the case \( k = 1 \), the map [2] defines a double covering of \( \Xi \) by \( \mathbb{S}^{m-1} \) by means of the identification

\[
\varpi \mapsto \mathbb{S}_{0,\varpi} := \{\varpi \in \mathbb{S}^{m-1} : \langle \varpi, \xi \rangle = 0\}.
\]

It is readily seen that every pair of antipodal unit vectors \( \varpi \) and \( -\varpi \) in \( \mathbb{S}^{m-1} \) define the same \((m-2)\)-dimensional geodesic sphere \( \mathbb{S}^{n-2}_{\varpi} \).
Remark 4.1. In the general case $1 \leq k \leq m - 2$, the map $\xi \rightarrow \hat{f}(\omega)$ defines a $O(k)$-invariant mapping from $St(m,k)$ onto $\Xi$. To see this, it is enough to consider the natural group action of $O(k)$ on $St(m,k)$:

$$St(m,k) \times O(k) \rightarrow St(m,k) : M \mapsto Mg, \quad M \in St(m,k), \quad g \in O(k).$$

Here we are seeing elements in $St(m,k)$ as matrices of $k$ orthonormal column vectors in $\mathbb{R}^m$, i.e. $St(m,k) = \{ M \in \mathbb{R}^{m \times k} : M^T M = I_k \}$. It is clear that any element of the orbit $\{ Mg : g \in O(k) \}$ of $M \in St(m,k)$ defines the same $(m-k-1)$-dimensional subshpere via the mapping $\xi \rightarrow \hat{f}(\omega)$, which establishes the $O(k)$-invariance.

In view of the double covering $\Xi \rightarrow \mathbb{R}$, we can see any function $\phi$ in $\Xi$ as an even function defined on $S^{m-1}$. By virtue of this reasoning, and by abuse of notation, we can rewrite formula (29) (see also (4)) as

$$\hat{f}(\omega) = \int_{S^{m-1}} f(\xi) dS_{\omega}, \quad \omega \in S^{m-1}. \quad (31)$$

Along with the transformation $f \mapsto \hat{f}$ we shall consider its dual transform $\phi \mapsto \tilde{\phi}$, which to a function $\phi \in L^2(\Xi)$ associates the function $\tilde{\phi} \in L^2(S^{m-1})$ given by

$$\tilde{\phi}(\omega) = \int_{\Xi} \phi(\xi) d\mu(\xi),$$

where $d\mu$ is the normalized invariant measure on the set $\{ \xi \in \Xi : \xi \in \xi \}$. In view of the double-covering $\Xi \rightarrow \mathbb{R}$, we can repeat a reasoning similar to the one used for the Radon transform. In particular, it is easily seen that set of $(m - 2)$-dimensional geodesic spheres passing through $\xi \in S^{m-1}$ is double covered by the geodesic sub-sphere $S_{\omega}^{m-1}$ orthogonal to $\omega$. Thus the dual transform of $\phi$ can be written as (see (4)):

$$\tilde{\phi}(\omega) = \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} \phi(\omega) dS_{\omega}. \quad (32)$$

Remark 4.2. Comparing formulae (31) and (32), it is easily seen that the spherical Radon transform, defined on geodesic sub-spheres of $S^{m-1}$ of codimension one, is (up to a constant factor) identical to its dual transform.

Before we proceed to the inversion formulae for the spherical Radon transform, we need the following observation. Each geodesic sub-sphere in $S^{m-1}$ through a point $\xi$ also passes through the antipodal point $A\xi = -\xi$. It is thus clear that $f = f \circ A$ and, as a result, $\hat{f} = 0$ if $f$ is an odd function on $S^{m-1}$, i.e. if $f + f \circ A = 0$. In [5, Thm 4.7], it was shown that the kernel of $\hat{\hat{f}}$ is exactly the set of odd functions on $S^{m-1}$. So, when considering the problem of inverting this transform, it is natural to confine our attention to the case of even functions.

4.1 Inversion formula, case $m$ even

The inversion formula provided in Theorem 4 was established by Helgason (see e.g. [10, Thm. 1.17 Ch. 3.1] and [3, Thm 4.7]) under the assumption that $m - k - 1$ is even. In the particular case where $k = 1$, this result reads as follows:

Theorem 5. Let $m \in \mathbb{N}$ be even. The spherical Radon transform $f \mapsto \hat{f}$ (for an even function $f$) is inverted by the formula

$$2(-4\pi)^{m-1} \frac{\Gamma \left( \frac{m-1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} f = P_{m-2}(\Delta_{LB}) (\hat{f})^\sim, \quad (33)$$

where $\Delta_{LB} = \Delta - (m - 2 + \mathbb{E})\mathbb{E}$ is the Laplace-Beltrami operator on $S^{m-1}$ and $P_{m-2}$ is the polynomial

$$P_{m-2}(z) = \prod_{j=1}^{m-2} \left[ z - (m - 2j - 1)(2j - 1) \right].$$
We shall now use Theorem 4 to directly verify (33). To that end, we first recall that \( L^2(S^{m-1}) \) admits the following orthogonal decomposition into spaces of spherical harmonics (see Proposition 2):
\[
L^2(S^{m-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(S^{m-1}).
\]
In view of this decomposition, it is enough to prove that Theorem 5 holds for elements of \( \mathcal{H}_k(S^{m-1}) \), \( k \in \mathbb{N} \). Let us consider then \( \mathcal{H}_k \subset \mathcal{H}_k \). Using Pizzetti’s formula (13) for \( p = 0 \), we have:
\[
\hat{H}_{2k}(\varrho) = \int_{S^m_{-\varrho}} H_{2k}(\varrho) dS_k
\]
\[
= \frac{2 \pi^{m-1}}{2^{2k} k! \Gamma \left( \frac{m+1}{2} \right)} (\Delta_{\varrho} - \langle \varrho, \varrho \rangle)^2 \left[ H_{2k}(\varrho) \right]
\]
\[
= \frac{2 \pi^{m-1}}{2^{2k} k! \Gamma \left( \frac{m+1}{2} \right)} (-1)^k \langle \varrho, \varrho \rangle^{2k} H_{2k}(\varrho).
\]
(34)

We now shall make use of the following result:

**Lemma 1.** Let \( j, \ell \in \mathbb{N}_0 \) be such that \( j \leq \ell \), and let \( P_\ell \in \mathcal{P}_\ell(\mathbb{R}^m) \). Then
\[
\langle \varrho, \varrho \rangle^{j}[P_\ell]\big|_{\varrho=\varrho} = \frac{\ell!}{(\ell-j)!} P_\ell(\varrho).
\]
(35)

**Proof of Lemma 1.** It is enough to prove this result for the basis elements \( \varrho^\beta := x_1^{\beta_1} \cdots x_m^{\beta_m} \) of \( \mathcal{P}_\ell(\mathbb{R}^m) \), where \( \beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}_0^m \) is a multi-index such that \( |\beta| := \beta_1 + \cdots + \beta_m = \ell \). For these elements, we have
\[
\langle \varrho, \varrho \rangle^{j}[\varrho^\beta]\big|_{\varrho=\varrho} = \sum_{|\alpha|=j} \binom{j}{\alpha} \varrho^\alpha \varrho^{\alpha_1} \cdots \varrho^{\alpha_m} [\varrho^\beta]\big|_{\varrho=\varrho}
\]
\[
= \sum_{|\alpha|=j} \binom{j}{\alpha} \frac{\beta_1! (\beta_1 - \alpha_1)! \cdots (\beta_m! (\beta_m - \alpha_m)! \varrho^{\beta_1 - \alpha_1} \cdots \varrho^{\beta_m - \alpha_m}}{\alpha_1! \cdots \alpha_m!}
\]
\[
= j! \binom{\beta_1 + \cdots + \beta_m}{\alpha_1 + \cdots + \alpha_m} \varrho^\beta
\]
\[
= \frac{\ell!}{(\ell-j)!} \varrho^\beta.
\]

**Lemma 1** clearly yields the identity \( \langle \varrho, \varrho \rangle^{2k} H_{2k}(\varrho) = (2k)! H_{2k}(\varrho) \). Substituting this into the formula (34) we obtain
\[
\hat{H}_{2k}(\varrho) = 2 \pi^{m-1} \frac{(-1)^k (2k)!}{2^{2k} k! \Gamma \left( \frac{m+1}{2} \right)} H_{2k}(\varrho) = 2 \pi^{m-1} \frac{(-1)^k \Gamma \left( k + \frac{1}{2} \right)}{\Gamma \left( \frac{m+1}{2} \right) + k} H_{2k}(\varrho),
\]
(36)

where in the last equality we have used the identity \( \frac{(2k)!}{2^{2k} k!} = \frac{\Gamma \left( k + \frac{1}{2} \right)}{\pi^{\frac{1}{2}}} \). Repeating the same reasoning with the dual transform \( \phi \mapsto \phi \), we obtain:
\[
(\hat{H}_{2k})^\vee(\varrho) = \frac{4 \pi^{m-2}}{\sigma_{m-1}} \frac{\Gamma \left( k + \frac{1}{2} \right)}{\sigma_{m-1} \Gamma \left( \frac{m+1}{2} \right)} \varrho^2 H_{2k}(\varrho).
\]
(37)

We now recall that \( H_{2k} \) is an eigenvector of the Laplace-Beltrami operator with eigenvalue \(-2k(m-2+2k)\). Therefore the action of \( P_{m-2}(\Delta_{LB}) \) on both sides of (37) yields:
\[
P_{m-2}(\Delta_{LB}) (\hat{H}_{2k})^\vee = \frac{4 \pi^{m-2}}{\sigma_{m-1}} \frac{\Gamma \left( k + \frac{1}{2} \right)}{\sigma_{m-1} \Gamma \left( \frac{m+1}{2} \right) + k} C_{k,m} H_{2k},
\]
(38)
where the constant $C_{k,m}$ is determined by the action of $P_{m-2}(\Delta_{LB})$ on $H_{2k}$, or equivalently,

$$C_{k,m} = P_{m-2}(-2k(m-2+2k)) = \prod_{j=1}^{m-1} [-2k(m - 2 + 2k) - (m - 2j - 1)(2j - 1)].$$

To explicitly compute $C_{k,m}$ we first note that

$$P_{m-2}(z) = \prod_{j=1}^{m-2} [z - (m - 2j - 1)(2j - 1)]$$

$$= 4^{\frac{m-2}{2}} \prod_{j=1}^{m-2} \left(j^2 - \frac{m}{2}j + \frac{z + m - 1}{4}\right)$$

$$= 4^{\frac{m}{2}-1} \prod_{j=1}^{m-2} (j-j_+)(j-j_-)$$

$$= 4^{\frac{m}{2}-1} \frac{\Gamma \left( \frac{m}{2} - j_+ \right) \Gamma \left( \frac{m}{2} - j_- \right)}{\Gamma(1-j_+) \Gamma(1-j_-)}, \quad (39)$$

where

$$j_\pm = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} - z + m - 1} = \frac{1}{4} \left( m \pm \sqrt{m^2 - 4(z + m - 1)} \right)$$

are the roots of the quadratic polynomial $j^2 - \frac{m}{2}j + \frac{z + m - 1}{4}$ in the variable $j$.

In the case where $z = -2k(m-2+2k)$, we have that $m^2 - 4(z + m - 1) = (m + 4k - 2)^2$. Thus the roots $j_\pm$ can be computed in this case as

$$j_+ = \frac{m}{2} + k - \frac{1}{2} \quad \text{and} \quad j_- = -k + \frac{1}{2},$$

respectively. Therefore

$$C_{k,m} = 4^{\frac{m}{2}-1} \frac{\Gamma \left( \frac{m}{2} - k \right) \Gamma \left( \frac{m}{2} + k \right)}{\Gamma \left( \frac{m}{2} - k \right) \Gamma \left( \frac{m}{2} + k \right)},$$

which yields the following equality when substituted in (38),

$$P_{m-2}(\Delta_{LB})(\widetilde{H}_{2k})^\vee = \frac{4^{\frac{m}{2}} \sigma_{m-2}}{\sigma_{m-1}} \frac{\Gamma(k + \frac{1}{2}) \Gamma(\frac{1}{2} - k)}{\Gamma(\frac{m}{2} + k) \Gamma(\frac{m}{2} - k)} \frac{H_{2k}}{H_{2k}}. \quad (40)$$

Using the identities $\Gamma \left( z + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} - z \right) = \frac{\pi}{\cos(\pi z)}$ and $\Gamma(z)\Gamma(-z + 1) = \frac{\pi}{\sin(\pi z)}$, we easily obtain that

$$\Gamma \left( k + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} - k \right) = (-1)^k \pi \quad \text{and} \quad \Gamma \left( \frac{3}{2} - \frac{m}{2} - k \right) \Gamma \left( \frac{m}{2} + k - \frac{1}{2} \right) = (-1)^{\frac{m}{2}+k-1} \pi,$$

respectively. Finally, substituting these identities into (40) yields

$$P_{m-2}(\Delta_{LB})(\widetilde{H}_{2k})^\vee = (-1)^{\frac{m}{2}} \frac{4^{\frac{m}{2}} \sigma_{m-2}}{\sigma_{m-1}} \frac{H_{2k}}{H_{2k}} = 2(-4\pi)^{\frac{m}{2} - 1} \frac{\Gamma \left( \frac{m}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} H_{2k},$$

which completes the proof of Theorem 5.
4.2 Inversion formula, the general case

In this section, we further apply Pizzetti formulae to the problem of inverting the spherical Radon transform. In particular, we will prove Theorem 2 for the case \( k = 1 \), which constitutes an extension of Theorem 5 to any arbitrary dimension \( m \in \mathbb{N} \).

To that end, we first introduce the following generalization of the dual Radon transform \( (32) \). Given \( \phi \in L^2(\Xi) \) and \( 0 \leq r < \frac{\pi}{2} \), we denote by \( \tilde{\phi}_r(x) \) the average of \( \phi \) over all \( (m-2) \)-dimensional geodesic sub-spheres passing at a distance \( r \) from \( x \in S^{m-1} \) i.e.

\[
\tilde{\phi}_r(x) = \int_{d(x,\xi) = r} \phi(\xi) \, d\mu(\xi), \quad \phi \in L^2(\Xi),
\]

where \( d(x,\xi) \) is the geodesic distance between \( x \) and the sub-sphere \( \xi \) and \( d\mu \) is the normalized invariant measure on the set \( \{ \xi \in \Xi : d(x,\xi) = r \} \). Observe that taking \( r = 0 \) yields the dual transform \( \tilde{\phi} \) defined in \( (32) \). We can now state the following version of Theorem 2 for the particular case \( k = 1 \).

**Theorem 6.** The Spherical Radon transform \( f \mapsto \hat{f} \) is, for even functions \( f \), inverted by

\[
f(x) = \frac{2^{m-2}}{(m-3)!} \sigma_{m-1} \left( \frac{d}{dt} \right)^{m-2} \left[ \int_0^t (\hat{f})_{\cos^{-1}(y)}(x) q_{m-2} \left( t^2 - q^2 \right)^{\frac{m-4}{2}} \, dq \right]_{t=1}.
\]

In order to prove this result using Pizzetti formulae, we need the following crucial observation. In view of the double covering \( (30) \) of \( \Xi \) by \( S^{m-1} \), formula \( (41) \) can be rewritten in terms of a spherical integral. Indeed, let \( y \in S^{m-1} \) be such that \( d(y,x) = r \), and let \( \xi \in \Xi \) be the sub-sphere passing through \( y \) such that \( d(\xi,x) = r \), see Figure 3. It is easily seen that one of the two normal vectors defining \( \xi \) satisfies

\[
d(y,\omega) = \frac{\pi}{2} - r. \quad \text{(43)}
\]

Conversely, any unit vector \( \omega \) satisfying \( (43) \) defines, by virtue of the double covering \( (30) \), a geodesic sub-sphere \( \xi \) such that \( d(\xi,x) = r \). Then the set \( \{ \xi \in \Xi : d(\xi,x) = r \} \) is determined by the set \( \{ \omega \in S^{m-1} : d(\omega,x) = \frac{\pi}{2} - r \} \), which in turn coincides with the sub-sphere

\[
S_{\sin(r),x} = \{ \omega \in S^{m-1} : \langle \omega, \omega \rangle = \sin(r) \},
\]

defined in Theorem \( (31) i) \). This geometric argument thus yields

\[
\tilde{\phi}_r(x) = \frac{1}{\cos^{m-2}(r) \sigma_{m-1}} \int_{S_{\sin(r),x}} \phi(\omega) \, dS_{\omega}.
\]

**Proof of Theorem 6:**
Due to the decomposition of $L^2(S^{m-1})$-functions into spherical harmonics, it is enough to prove the theorem for $f = H_{2k} \in H_{2k}(S^{n-1})$. From (46) we know that

$$\widehat{H_{2k}}(\omega) = d_{m,k} H_{2k}(\omega), \quad \text{with} \quad d_{m,k} = 2\pi^{m-1} \frac{(-1)^k \Gamma(k + \frac{1}{2})}{\Gamma(\frac{m-1}{2} + k)}.$$

Hence, from (44) we obtain

$$\left(\widehat{H_{2k}}(\omega)\right)' = \frac{d_{m,k}}{\cos^{m-2}(\varphi) \sigma_{m-1}} \int_{S_{m-1}} H_{2k}(\omega) dS_\omega.$$

Taking $p = \sin(\varphi)$ and $q = \cos(\varphi)$, i.e. $p^2 + q^2 = 1$, by virtue of the Pizzetti formula (18), we obtain

$$\left(\frac{\partial}{\partial \varphi} \widehat{H_{2k}}(\omega)\right)_{\cos^{-1}(q)} = \frac{d_{m,k}}{q^{m-2} \sigma_{m-1}} \int_{S_{m-1}} H_{2k}(\omega) dS_\omega =$$

$$= \frac{d_{m,k}}{\sigma_{m-1}} \int_{S_{m-1}} \sum_{j=0}^{k} \frac{2\pi^{m-1} (-1)^j}{2^{j+1} j! \Gamma(\frac{m-1}{2} + j)} \left(\Delta_{\varphi} - \langle \varphi, \partial_{\varphi} \omega \rangle^2\right)^j [H_{2k}](\omega) \bigg|_{\omega = \varphi = q^{2j} \omega} dS_\omega.$$

We now recall that $\langle \varphi, \partial_{\varphi} \omega \rangle^2 H_{2k}$ is a homogeneous polynomial of degree $2k - 2j$ in the variable $\omega$, therefore:

$$\langle \varphi, \partial_{\varphi} \omega \rangle^2 H_{2k} \bigg|_{\omega = \varphi = q^{2j} \omega} = p^{2k-2j} \langle \varphi, \partial_{\varphi} \omega \rangle^2 H_{2k} \bigg|_{\omega = \varphi = q^{2j} \omega},$$

and, applying Lemma (11) we obtain:

$$\langle \varphi, \partial_{\varphi} \omega \rangle^2 H_{2k} \bigg|_{\omega = \varphi = q^{2j} \omega} = \frac{(2k)!}{(2k - 2j)!} p^{2k-2j} H_{2k}(\varphi).$$

Substituting the equality above into (45) yields:

$$\left(\frac{\partial}{\partial \varphi} \widehat{H_{2k}}(\omega)\right)_{\cos^{-1}(q)} = \frac{d_{m,k}}{\sigma_{m-1}} \sum_{j=0}^{k} \frac{2\pi^{m-1} (-1)^j}{2^{j+1} j! \Gamma(\frac{m-1}{2} + j)} \frac{(2k)!}{(2k - 2j)!} p^{2k-2j} H_{2k}(\varphi) q^{2j}$$

$$= h_{m,k} H_{2k}(\varphi) \sum_{j=0}^{k} \frac{2\pi^{m-1} (-1)^j}{2^{j+1} j! \Gamma(\frac{m-1}{2} + j)} \frac{(2k)!}{(2k - 2j)!} q^{2j}$$

where we have introduced the new constant

$$h_{m,k} = 2\pi^{m-1} (2k)! \frac{d_{m,k}}{\sigma_{m-1}} = 2\pi^{m-1} (-1)^k (2k)! \frac{\Gamma(\frac{m-1}{2} + k)}{\Gamma(\frac{m-1}{2} + k)}.$$

The task is now to compute:

$$F(\varphi) := \left( \frac{d}{d\varphi} \right)^{m-2} \left[ \int_0^t \left( \frac{\partial}{\partial \varphi} \widehat{H_{2k}}(\omega)\right)_{\cos^{-1}(q)}(\varphi) q^{m-2} (t^2 - q^2)^{\frac{m-1}{2}} dq \right]_{t=1}.$$

From (46) we obtain

$$F(\varphi) = h_{m,k} H_{2k}(\varphi) \sum_{j=0}^{k} \frac{(-1)^j}{2^{j+1} j! \Gamma(\frac{m-1}{2} + j)(2k - 2j)!} \left( \frac{d}{d\varphi} \right)^{m-2} \left[ I_{k,j}(t) \right]_{t=1},$$

where

$$I_{k,j}(t) := \int_0^t (1 - q^2)^{k-j} q^{2j+m-2} (t^2 - q^2)^{\frac{m-1}{2}} dq.$$
Applying the change of coordinates \( q = t \sin(\theta) \), we obtain \( dq = t \cos(\theta)d\theta \), and
\[
I_{k,j}(t) = t^{2j+2m-5} \int_0^\pi \sin^j(\theta) \sin^{2j+2\ell+2m-5}(\theta) \cos^{m-3}(\theta) d\theta.
\]
Expanding \((1 - t^2 \sin^2(\theta))^{k-j}\) we now get
\[
I_{k,j}(t) = \sum_{\ell=0}^{k-j} \binom{k-j}{\ell} (-1)^\ell t^{2j+2\ell+2m-5} \int_0^\pi \sin^{2j+2\ell+m-2}(\theta) \cos^{m-3}(\theta) d\theta.
\]
Using the known identity
\[
\int_0^\pi \sin^a(\theta) \cos^b(\theta) d\theta = \frac{\Gamma \left( \frac{a+1}{2} \right) \Gamma \left( \frac{b+1}{2} \right)}{2 \Gamma \left( \frac{a+b+1}{2} \right)},
\]
for the values \( a = 2j + 2\ell + m - 2 \) and \( b = m - 3 \), we have:
\[
I_{k,j}(t) = \sum_{\ell=0}^{k-j} \binom{k-j}{\ell} (-1)^\ell \frac{\Gamma \left( j + \ell + \frac{m-1}{2} \right) \Gamma \left( \frac{m-2}{2} \right)}{2 \Gamma \left( j + \ell + m - \frac{3}{2} \right)}.
\]
Moreover, from the identity \( \left( \frac{d}{dt} \right)^n t^a \) we immediately obtain:
\[
\left( \frac{d}{dt} \right)^{m-2} t^{2j+2\ell+2m-5} \bigg|_{t=1} = \frac{\Gamma (j + \ell + m - \frac{3}{2})}{\Gamma (j + \ell + \frac{1}{2})},
\]
which yields:
\[
\left( \frac{d}{dt} \right)^{m-2} [I_{k,j}(t)] \bigg|_{t=1} = \frac{\Gamma \left( \frac{m-2}{2} \right)}{2} \sum_{\ell=0}^{k-j} \binom{k-j}{\ell} (-1)^\ell \frac{\Gamma (j + \ell + m - \frac{1}{2})}{\Gamma \left( j + \ell + \frac{1}{2} \right)}.
\]
From the definition of the hypergeometric function \( _2F_1(a; b; c; z) \), we now recall that
\[
_2F_1(-n, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{\Gamma(b + \ell)}{\Gamma(c + \ell)} z^\ell, \quad n \in \mathbb{N}.
\]
Replacing \( n = k - j, b = j + \frac{m-1}{2}, c = j + \frac{1}{2}, z = 1 \), and comparing the resulting expression with \( 48 \), yields:
\[
\left( \frac{d}{dt} \right)^{m-2} [I_{k,j}(t)] \bigg|_{t=1} = \frac{\Gamma \left( \frac{m-2}{2} \right)}{2} \frac{\Gamma (j + \frac{m-1}{2})}{\Gamma \left( j + \frac{1}{2} \right)} _2F_1 \left( j-k, j + \frac{m-1}{2}; j + \frac{1}{2}, 1 \right).
\]
Applying the Gauss summation formula \( 25 \) we now obtain:
\[
\left( \frac{d}{dt} \right)^{m-2} [I_{k,j}(t)] \bigg|_{t=1} = \frac{\Gamma \left( \frac{m-2}{2} \right)}{2} \frac{\Gamma (j + \frac{m-1}{2})}{\Gamma (k + \frac{1}{2})} \frac{\Gamma (k - j + \frac{2-m}{2})}{\Gamma \left( \frac{2-m}{2} \right)}.
\]
Substituting this result back into \( 17 \) we have:
\[
F(x) = \frac{h_{m,k}}{2 \Gamma \left( k + \frac{1}{2} \right)} \frac{\Gamma \left( \frac{m-2}{2} \right)}{\Gamma \left( \frac{2-m}{2} \right)} H_{2k}(x) \sum_{j=0}^{k} \frac{(-1)^j \Gamma \left( k - j + \frac{2-m}{2} \right)}{(2j)! (2k-2j)!}.
\]
Applying a similar reasoning as before, we can identify the above summation with a hypergeometric function. Indeed, it is easily seen that
\[
\sum_{j=0}^{k} \frac{(-1)^j \Gamma \left( k - j + \frac{2-m}{2} \right)}{(2j)! (2k-2j)!} = \frac{\Gamma \left( k + \frac{m-1}{2} \right)}{\Gamma (k)!} _2F_1 \left( -k, -k + \frac{1}{2}; \frac{m}{2} - k, 1 \right).
\]
In this appendix section, our goal is to provide a direct proof of formula (23). To that end, let us denote the right hand side of (23) by

\[
A \overset{\text{def}}{=} \frac{\Gamma(k + \frac{m}{2} - \frac{1}{2})}{(2k)! \Gamma(m/2)} \sum_{j=0}^{k} \frac{(-1)^j \Gamma(k-j + \frac{2-j}{2})}{2^{2j} j! \Gamma(m/2)} \Gamma(k+j+\frac{2-j}{2}).
\]

Using the identity \( \frac{\Gamma(n-k)}{\Gamma(n)} = (-1)^k \frac{\Gamma(n-m)}{\Gamma(n-k+1)} \) in the previous formula we obtain:

\[
A = \sum_{j=0}^{k} \frac{(-1)^j \Gamma(k-j + \frac{2-m}{2})}{2^{2j} j! \Gamma(m/2)} \Gamma(k+j+\frac{2-m}{2}),
\]

which in turn allows us to rewrite (19) as

\[
F(x) = \frac{h_{m,k}}{2 \Gamma(k+\frac{1}{2})} \frac{\Gamma(m/2 - 1)}{\Gamma(m/2)} (-1)^k \frac{\Gamma(k + m/2 - 1)}{(2k)!} H_{2k}(x)
\]

\[
= \pi^{\frac{-m}{2}} \Gamma\left(\frac{m-2}{2}\right) H_{2k}(x).
\]

Finally, applying the Legendre duplication formula for the Gamma function we have that

\[
\Gamma\left(\frac{m-2}{2}\right) \Gamma\left(\frac{m-1}{2}\right) = \frac{\pi^{\frac{3}{2}}}{2^{m-3}} (m-3)!.
\]

Hence, the formula (50) can be rewritten as

\[
F(x) = \frac{2^m \pi^{\frac{-m}{2}}}{\Gamma\left(\frac{m-1}{2}\right)} (m-3)! H_{2k}(x) = \frac{\sigma_{m-1} (m-3)!}{2^{m-2}} H_{2k}(x),
\]

which completes the proof of Theorem 6. \(\square\)

\section{A Recovering classical Pizzetti formula from integration on spherical caps}

In this appendix section, our goal is to provide a direct proof of formula (24). To that end, let us denote the right hand side of (23) by

\[
S := \frac{2^{2s} s!}{\pi^{\frac{1}{2}}} \sum_{k=0}^{s} \sum_{\ell=0}^{k} \frac{\Gamma(s - k + \frac{1}{2})}{2^{2\ell} (2k)!} \left\langle \omega, \hat{\partial}_{\omega} \right\rangle^2 (\Delta_{\omega} - \left\langle \omega, \hat{\partial}_{\omega} \right\rangle^2)^k \left[ R_{2s} \right]_{\omega = \omega}
\]

\[
= \frac{2^{2s} s!}{\pi^{\frac{1}{2}}} \sum_{k=0}^{s} \sum_{\ell=0}^{k} \frac{\Gamma(s - k + \frac{1}{2})}{2^{2\ell} (2k)!} \left\langle \omega, \hat{\partial}_{\omega} \right\rangle^2 \left( (-1)^{k-\ell} \left\langle \omega, \hat{\partial}_{\omega} \right\rangle^2 \Delta^{k-\ell}_{\omega} \right) \left[ R_{2s} \right]_{\omega = \omega}.
\]

Applying Lemma 4 to the polynomial \( \Delta^{k-\ell}_{\omega} [R_{2s}] \in \mathcal{P}_{2s+2\ell - 2k}(\mathbb{R}^m) \) we obtain

\[
\left\langle \omega, \hat{\partial}_{\omega} \right\rangle^2 \Delta^{k-\ell}_{\omega} [R_{2s}] = \left\langle \omega, \hat{\partial}_{\omega} \right\rangle^2 \Delta^{k-\ell}_{\omega} [R_{2s}] = \frac{(2s + 2\ell - 2k)!}{(2s - 2k)!} \Delta^{k-\ell}_{\omega} [R_{2s}] = \left[ \Delta^{k-\ell}_{\omega} [R_{2s}] \right]_{\omega = \omega}.
\]

Hence

\[
S = \frac{2^{2s} s!}{\pi^{\frac{1}{2}}} \sum_{k=0}^{s} \sum_{\ell=0}^{k} \frac{(-1)^{k-\ell} \Gamma(s - k + \frac{1}{2}) (2s + 2\ell - 2k)!}{2^{2\ell} \ell! (2s - 2k)!} \Delta^{k-\ell}_{\omega} [R_{2s}] = \left[ \Delta^{k-\ell}_{\omega} [R_{2s}] \right]_{\omega = \omega}.
\]
Taking now \( j = k - \ell \) and changing the order of summation yields

\[
S = \frac{2^{2s}s!}{\pi^s} \sum_{k=0}^{s} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{2^{2k}} \frac{\Gamma(s-k+j+\frac{1}{2}) (2s-2j)!}{(2s-2k)!} \Delta_s^j [R_{2s}]|_{z=\omega} = \frac{2^{2s}s!}{\pi^s} \sum_{j=0}^{s-1} \frac{(-1)^{k-j}}{2^{2s-2j}} \frac{\Gamma(s-k+j+\frac{1}{2}) (2s-2j)!}{2^{2s}k!(2s-2k-2j)!} \Delta_s^j [R_{2s}]|_{z=\omega}.
\]

Using the identity \( \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} = \frac{\pi^{\frac{1}{2}}}{2^{n-1}n!} \) for \( n = s-k-j \), and by virtue of the binomial theorem, we obtain

\[
\sum_{k=0}^{s-j} \frac{(-1)^{k}}{2^{2s-2j}} \frac{\Gamma(s-k-j+\frac{1}{2})}{k!(s-j-k)!} = \frac{\pi^{\frac{1}{2}}}{2^{2s-2j}} \sum_{k=0}^{s-j} \frac{(-1)^{k}}{2^{2s}k!(2s-2k-2j)!} (1-1)^{s-j}.
\]

Thus the only non-vanishing term in the sum \([51]\) is the term corresponding to \( j = s \). Therefore,

\[
S = \Delta_s^s [R_{2s}]|_{z=\omega} = \Delta_s^s [R_{2s}]|_{z=0},
\]

where the second equality follows from the fact the \( \Delta_s^s [R_{2s}] \) is a constant polynomial. This establishes formula \([51]\).

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