Essential core of the Hawking–Ellis types

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Abstract

The Hawking–Ellis (Segre–Plebański) classification of possible stress–energy tensors is an essential tool in analyzing the implications of the Einstein field equations in a more-or-less model-independent manner. In the current article the basic idea is to simplify the Hawking–Ellis type I, II, III, and IV classification by isolating the ‘essential core’ of the type II, type III, and type IV stress–energy tensors; this being done by subtracting (special cases of) type I to simplify the (Lorentz invariant) eigenvalue structure as much as possible without disturbing the eigenvector structure. We will denote these ‘simplified cores’ type II₀, type III₀, and type IV₀. These ‘simplified cores’ have very nice and simple algebraic properties. Furthermore, types I and II₀ have very simple classical interpretations, while type IV₀ is known to arise semi-classically (in renormalized expectation values of standard stress–energy tensors). In contrast type III₀ stands out in that it has neither a simple classical interpretation, nor even a simple semi-classical interpretation. We will also consider the robustness of this classification considering the stability of the different Hawking–Ellis types under perturbations. We argue that types II and III are definitively unstable, whereas types I and IV are stable.

Keywords: stress–energy classification, energy conditions, Hawking–Ellis

1. Introduction

The classification of stress–energy tensors popularized by Hawking and Ellis [1] (which was in turn based on earlier work by Segre [2] and Plebański [3]) has become an essential standard tool in developing more-or-less model-independent analyses of the implications of the Einstein field equations in astrophysical and cosmological contexts. (See for example [4–6].) The Hawking–Ellis classification in turn feeds into (and to a large extent underlies) the formulation of the classical energy conditions [1], and their non-linear and semi-classical quantum
generalizations. (See for example [4–13], and [14–19], and related work on the Rainich conditions [6, 20–31].)

Note that in setting up the Hawking–Ellis classification, we are working in a local orthonormal frame with \( \eta^{ab} = \text{diag}(-1, 1, 1, 1) \) and looking for Lorentz-invariant eigenvalues and Lorentz-covariant eigenvectors:

\[
(T^{ab} - \lambda \eta^{ab}) V_b = 0.
\]  
(1)

(This is what mathematicians would call a ‘generalized eigenvector problem’ [32, 33].)

Equivalently one could raise and lower indices

\[
(T^a_b - \lambda \delta^a_b) V^b = 0.
\]  
(2)

(This is what mathematicians would call an ‘ordinary eigenvector problem’, but it is the fact that \( T^a_b \) is now generically not symmetric that renders the classification programme mathematically non-trivial [32, 33].) We will use \( \sim_L \) to denote similarity under Lorentz transformations; and \( \sim \) to denote similarity under generic non-singular transformations. Note that it is the Lorentz transformations that are used in the Hawking–Ellis classification, which is based on diagonalizing the stress–energy tensor (as much as possible) in a physical orthonormal basis, with one timelike and three spacelike basis vectors. In contrast, in order to get the Jordan normal form, one instead considers generic non-singular (possibly complex) transformations. The Jordan form is particularly useful in the classification based on the minimal polynomials [6]. However, whenever the basis in which the stress–energy tensor is expressed in its Jordan normal form does not contain a timelike eigenvector, (that is, it cannot be co-moving with any physical observer), then the Jordan form does not provide us with anywhere near so clear a physical interpretation. We will (mostly) work in \((3+1)\) signature.

The basic idea we describe below is to ‘simplify’ the Hawking–Ellis type I, II, III, IV classification as much as possible by isolating what we shall call the ‘essential core’ of the type II, type III, and type IV stress–energy tensors; this being done by subtracting (special cases of) type I to simplify the (Lorentz invariant) eigenvalue structure as much as possible without disturbing the eigenvector structure. We shall see that these ‘simplified cores’ have very nice and simple algebraic properties; and very straightforward mathematical characterizations. Furthermore, physically types I and II0 have very simple classical interpretations, while type IV0 is known to arise semi-classically (in renormalized expectation values of standard stress–energy tensors). In contrast type III0 stands out in that physically it has neither a simple classical interpretation, nor even a simple semi-classical interpretation. Because of this, we shall spend some extra effort analyzing type III0.

2. Essential core of the Hawking–Ellis classification

Let us consider the Hawking–Ellis types I, II, III, and IV [1]. (This classification is discussed in many places, including [4–13].)

**type I:** Under a Lorentz transformation we can set

\[
T^{\mu\nu} \sim_L \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{bmatrix}; \quad T^a_b \sim_L \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{bmatrix}.
\]  
(3)
As in this case the stress–energy tensor is fully diagonalizable, one obtains the same matrix by expressing this tensor in the orthonormal basis formed by its eigenvectors

\[
T^\mu_\nu \sim \begin{pmatrix}
-\rho & 0 & 0 & 0 \\
0 & p_1 & 0 & 0 \\
0 & 0 & p_2 & 0 \\
0 & 0 & 0 & p_3
\end{pmatrix}.
\]

(4)

The eigenvalues are \{-\rho, p_1, p_2, p_3\}. This is as simple as type I gets. In \((3+1)\) dimensions type I is invariantly characterized by the existence of a unique timelike eigenvector, implying the existence of three spacelike eigenvectors. (In Euclidean signature, since \(\eta^{ab} \rightarrow \delta^{ab}\), all stress–energy tensors are type I.)

type II: Under a Lorentz transformation we can set

\[
T^\mu_\nu \sim_L \begin{pmatrix}
\mu + f & f & 0 & 0 \\
f & -\mu + f & 0 & 0 \\
0 & 0 & p_2 & 0 \\
0 & 0 & 0 & p_3
\end{pmatrix}:
T^\mu_\nu \sim_L \begin{pmatrix}
-\mu - f & f & 0 & 0 \\
-f & -\mu + f & 0 & 0 \\
0 & 0 & p_2 & 0 \\
0 & 0 & 0 & p_3
\end{pmatrix}:
\]

(5)

while under a generic non-singular similarity transformation we can set

\[
T^\mu_\nu \sim \begin{pmatrix}
-\mu & 1 & 0 & 0 \\
0 & -\mu & 0 & 0 \\
0 & 0 & p_2 & 0 \\
0 & 0 & 0 & p_3
\end{pmatrix}.
\]

(6)

The eigenvalues are \{-\mu, -\mu, p_2, p_3\}. The non-trivial structure of the Jordan form is due to the null eigenvector associated to the double eigenvalue. That is, it is not expressed in a basis with a timelike eigenvector, though there are two spacelike eigenvectors associated to the \(p_i\) eigenvalues. As a matrix, type II is defective, there are only three eigenvectors, not four. Now from type II subtract out as much of type I as possible; and call what is left type II\(_0\). (Make the eigenvalues as simple as possible; but do not disturb the eigenvectors.) Consider this:

\[
(T^\mu_\nu)_{II\_0} \sim_L \begin{pmatrix}
f & f & 0 & 0 \\
f & f & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}:
(T^\mu_\nu)_{II\_0} \sim_L \begin{pmatrix}
-\mu & 0 & 0 \\
0 & -\mu & 0 \\
0 & 0 & p_2 \\
0 & 0 & 0
\end{pmatrix}:
\]

(7)

and

\[
(T^\mu_\nu)_{II\_0} \sim \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(8)

The eigenvalues are now as simple as possible, \(\{0, 0, 0, 0\}\), and there is still one null and two spacelike eigenvectors. Observe that type II\(_0\) can be invariantly characterized by the equation \(\{(T^\mu_\nu)_{II\_0}\}^2 = 0\). (So it is nilpotent of order 2.) We can also write \((T^\mu_\nu)_{II\_0} = f\ell^a\bar{\ell}^b\) where \(\ell\) is a null vector with unit time component. On the other hand, type II\(_0\) tensors have a generalized timelike eigenvector, such that \(\{(T^\mu_\nu)_{II\_0}\}^{\ell}_b = f\ell^a\), implying \(\{(T^\mu_\nu)_{II\_0}\}^{\ell}_b = 0\). The minimum dimensionality for a type II\(_0\) stress–energy tensor is \((1+1)\).
**type III:** Under a Lorentz transformation we can set

\[
T_{\mu\nu}^\ell \sim_L \begin{bmatrix}
\rho & f & 0 & 0 \\
0 & \rho & f & 0 \\
f & -\rho & f & 0 \\
0 & f & -\rho & 0 \\
0 & 0 & 0 & p_3
\end{bmatrix} ; \quad T_{\mu\nu}^s \sim_L \begin{bmatrix}
-\rho & f & 0 & 0 \\
0 & -\rho & f & 0 \\
-f & -\rho & f & 0 \\
0 & f & -\rho & 0 \\
0 & 0 & 0 & p_3
\end{bmatrix}
\]

where the parameter \( f \) is unnecessarily set to 1 in reference [1]. The Jordan normal form of this tensor is

\[
T_{\mu\nu}^\ell \sim \begin{bmatrix}
-\rho & 1 & 0 & 0 \\
0 & -\rho & 1 & 0 \\
0 & 0 & -\rho & 0 \\
0 & 0 & 0 & p_3
\end{bmatrix} .
\]

The eigenvalues are \( \{-\rho, -\rho, -\rho, p_3\} \), and there is a single null eigenvector associated to the triple eigenvalue, plus a single spacelike eigenvector associated to the eigenvalue \( p_3 \).

As a matrix type III is defective, there are only two eigenvectors, not four.

Now from type III subtract out as much of type I as possible; call what is left type III_0. (Make the eigenvalues as simple as possible; but do not disturb the eigenvectors.) Consider this:

\[
(T_{\mu\nu})_{\text{III}_0} \sim_L \begin{bmatrix}
0 & f & 0 & 0 \\
0 & 0 & f & 0 \\
f & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} ; \quad (T_{\mu\nu})_{\text{III}_0} \sim_L \begin{bmatrix}
0 & f & 0 & 0 \\
0 & 0 & f & 0 \\
-f & 0 & f & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

with Jordan form

\[
(T_{\mu\nu})_{\text{III}_0} \sim \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} .
\]

The eigenvalues are now as simple as possible, \( \{0, 0, 0, 0\} \), and we still have one null and one spacelike eigenvector. Observe that type III_0 can be invariantly characterized by the equation \([T_{\mu\nu}^{a\ell})_{\text{III}_0}]^3 = 0\), so that type III_0 tensors are nilpotent of order 3. We can also write \((T_{ab})_{\text{III}_0} = f(\ell^a s^b + s^a \ell^b)\); where \( \ell \) is a null vector, and \( s \) is spacelike and orthogonal to \( \ell \). For example, take \( \ell^a \sim_L (1, 0, 1, 0) \), and \( s^a \sim_L (0, 1, 0, 0) \). To verify the invariant characterization \([T_{\mu\nu}^{a\ell})_{\text{III}_0}]^3 = 0\), one can first check that \((\ell^a s^b + s^a \ell^b)_{\text{III}_0} = (\ell^a s^b + s^a \ell^b) = \ell^a \ell^b\). This implies that \([T_{\mu\nu}^{a\ell})_{\text{III}_0}]^2 = f^2 \ell^a \ell^b \in (T_{\mu\nu})_{\text{III}_0}\) and, therefore, the square of a type III_0 tensor is a type II_0 tensor. Note that as \( \ell \) is a null vector and \( s \) is orthogonal to \( \ell \), one obtains \([T_{\mu\nu}^{a\ell})_{\text{III}_0}]^3 = 0\).

From the above, the minimum dimensionality for type III is \((2+1)\). In a certain technical sense type III_0 is a ‘square root’ of type II_0, but to take the ‘square root’ one (at a minimum) has to go to one higher dimension than was needed for type II. (This is vaguely similar to what happens for real \( \rightarrow \) complex.) Observe that the vector \( \ell \) is the (unique) null eigenvector, since \([T_{\mu\nu}^{a\ell})_{\text{III}_0}]^3 = 0\). In contrast the vector \( s \) is not an eigenvector, since \([T_{\mu\nu}^{a\ell})_{\text{III}_0}]^3 = f s^a \). However \( s \) is a generalized (spacelike) eigenvector of order 2, that is \([T_{\mu\nu}^{a\ell})_{\text{III}_0}]^2 s^a = 0\). Moreover, for the generalized (timelike) eigenvector of order 3, one has \([T_{\mu\nu}^{a\ell})_{\text{III}_0}]^3 = f s^a \). However \( s \) is a generalized (spacelike) eigenvector of order 2, that is \([T_{\mu\nu}^{a\ell})_{\text{III}_0}]^2 s^a = 0\). Moreover, for the generalized (timelike) eigenvector of order 3, one has \([T_{\mu\nu}^{a\ell})_{\text{III}_0}]^3 = f s^a \).
**Type IV:** Under a Lorentz transformation we can set
\[
T_{\mu\nu} \sim L \begin{pmatrix} \rho & f & 0 & 0 \\ -\rho & -\rho & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}; \quad T_{\mu\nu} \sim L \begin{pmatrix} -\rho & f & 0 & 0 \\ -f & -\rho & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}; \quad (13)
\]
while under a generic non-singular similarity transformation one can set
\[
T_{\mu\nu} \sim L \begin{pmatrix} -\rho + if & 0 & 0 & 0 \\ 0 & -\rho - if & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}. \quad (14)
\]
The eigenvalues are \{-\rho + if, -\rho - if, p_2, p_3\}, and there are no causal eigenvectors; there are only two spacelike eigenvectors associated with the \(p_i\) eigenvalues. (There are also two complex eigenvectors, that do not fit into the timelike/null/spacelike classification. The matrix is diagonalizable, but there are only two real eigenvectors, not four.)

Now from type IV subtract out as much of type I as possible; call what is left type IV 0. (Make the eigenvalues as simple as possible; but do not disturb the eigenvectors.) Since the eigenvalues of type I are all real, we will not be able to disturb the imaginary parts of the type IV eigenvalues. Consider this:
\[
(T_{\mu
u})_{IV0} \sim L \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad (T_{\mu
u})_{IV0} \sim L \begin{pmatrix} 0 & f & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad (15)
\]
with Jordan form
\[
(T_{\mu
u})_{IV0} \sim L \begin{pmatrix} +if & 0 & 0 & 0 \\ 0 & -if & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (16)
\]
The eigenvalues are now \{+if, -if, 0, 0\}. Observe that a type IV 0 stress–energy tensor can be invariantly characterized either as \(((T_{\mu\nu})_{IV0})^2 = -f^2[I_2 \oplus 0_2]^a_b\), or, if one prefers, as \([T_{\mu\nu}]_{IV_0}^{-1}[T_{\mu\nu}]_{IV_0} = -f^2[I_2 \oplus 0_2]\).

We can also write \((T^{ab})_{IV0} = f(t^as^b + s^at^b)\); were \(t\) is a timelike vector, and the vector \(s\) is spacelike and orthogonal to \(t\). For example, take \(t^a \sim L (1, 0, 0, 0)\), and \(s^a \sim L (0, 1, 0, 0)\). Then, it is easy to see that \(((T_{\mu\nu})_{IV0})^2 = f^2(t^a_h - s^a_i)\) and, therefore, \([[(T^{ab})_{IV0}]^3 = -f^2(T^{ab})_{IV0}\), as happens with pure imaginary numbers. So, in a vague sense type IV 0 is the imaginary version of type I. The minimum dimensionality for type IV 0 is \(1+1\).

### 3. General lessons

What general lessons can we draw?

- Type I is the most generic case, corresponding to perfect fluids and anisotropic perfect fluids—and even anisotropic solids. Any stress–energy tensor that has a ‘natural rest frame’ (meaning a timelike eigenvector) is of type I.
• Type II$_0$ corresponds to pure (coherent) radiation travelling at the speed of light.
• Type III$_0$ corresponds to no known source of stress–energy, neither classical nor quantum.
• Type IV$_0$ corresponds to no known source of classical stress–energy, though in the semi-classical quantum realm renormalized expectation values of the stress–energy tensor are often of type IV$_0$. See for instance [5–9].
• Note that all the essential core types, II$_0$, III$_0$, and IV$_0$, are traceless, $T_{aa} = 0$.
• Types II$_0$ and III$_0$ have $\det(T_{ab}) = 0$, whereas in contrast for type IV$_0$ one has $\det(T_{ab}) = (T_{ab}T_{ba}) = -2f^2 < 0$.
• The absolutely simplest form of type III requires at least $(2+1)$ dimensions and corresponds to

$$\begin{pmatrix} 0 & f & 0 \\ f & 0 & f \\ 0 & f & 0 \end{pmatrix} \sim L \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with eigenvalues: $\{0, 0, 0\}$. Since the simplest forms of types I, II, and IV can be defined in $(1+1)$ dimensions, we see that type III stands out in this regard. (This implies, for instance that type III is incompatible with either planar or spherical symmetry. See for instance [5–9].)
• According to the classification based on considering the minimal polynomial [6], type II$_0$ is degree 2.I (meaning that the minimal polynomial has degree 2 and the matrix has only one distinct eigenvalue). Similarly type III$_0$ is degree 3.I, and type IV$_0$ is degree 3.III.
• When converting type II–IV stress-energy tensors to their essential cores, it is perhaps useful to note that one does not have to subtract the most general form of type I. The type I piece being subtracted is degenerate, in that it has either 2 or 3 of its Lorentz-invariant eigenvalues equal to the eigenvalue of its timelike eigenvector.

4. Energy conditions

4.1. Type I

The energy conditions for type I stress–energy are discussed in very many places, from the well-known Hawking–Ellis text [1], to many secondary sources. (See for instance [4–13]). With regards to type I, we have nothing substantial to add.

4.2. Type II$_0$

For type II$_0$ stress–energy we have $(T^{ab})_{II_0} = f\epsilon^{ab}$. For the standard energy conditions (NEC, WEC, SEC, DEC) this implies the straightforward results:

• The null energy condition (NEC) is satisfied iff $f \geq 0$.
• The weak energy condition (WEC) is satisfied iff $f \geq 0$.
• Since $T = 0$, the strong energy condition (SEC) is satisfied iff $f \geq 0$.
• The dominant energy condition (DEC) is satisfied iff $f \geq 0$.

(Note that DEC = WEC + FEC, and see comments on the flux energy condition (FEC) below.)
For the less standard energy conditions (FEC, TEC, TOSEC, DETEC [8]) we have:

- **FEC** (flux energy condition) is marginally satisfied. For any timelike observer $V$, the flux $F^a = T^{ab}V_b = (f^bV_b)\ell^a$ is always null.
- **TEC** (trace energy condition) is marginally satisfied ($T = 0$).
- **TOSEC** (Trace-of-square energy condition) is marginally satisfied ($\text{tr}[T^2] = 0$).
- **DETEC** (Determinant energy condition) is marginally satisfied ($\text{det}[T] = 0$).

### 4.3. Type III$_0$

For type III$_0$, that is $(T^{ab})_{III_0} = f(\ell^a s^b + s^a \ell^b)$, the situation becomes more subtle. For the standard energy conditions:

- **NEC** is violated. For any null vector $k$ we have $T_{ab}k^a k^b = 2f(k \cdot \ell)(k \cdot s)$. We can always choose signs to enforce $k \cdot \ell \geq 0$, but $k \cdot s$ can easily flip sign. Thus type III$_0$ cannot satisfy the inequality on which the NEC is based for all null vectors.
- **WEC** is violated. For any timelike vector $V$ we have $T_{ab}V^a V^b = 2f(V \cdot \ell)(V \cdot s)$. We can always choose signs to enforce $V \cdot \ell > 0$, but $V \cdot s$ can easily flip sign. Thus type III$_0$ cannot satisfy the inequality on which WEC is based for all timelike vectors.
- SEC is violated. SEC = WEC because $T = 0$.
- **DEC** is violated. (DEC = WEC + FEC, and FEC is violated; see below.)

For the less-standard energy conditions:

- **FEC** is violated. For timelike 4-velocities $V$, the observed flux is

$$F_a = T_{ab}V^b = f(s_a(\ell_b V^b) + \ell_a (s_b V^b)).$$  \hspace{1cm} (18)

But then $F_a F^a = f^2 (\ell_a V^b)^2 > 0$, so the flux vector is always spacelike, indicating that type III$_0$ corresponds to tachyonic matter.
- **TEC** is marginally satisfied ($T = 0$).
- **TOSEC** is marginally satisfied ($\text{tr}[T^2] = 0$).
- **DETEC** is marginally satisfied ($\text{det}[T] = 0$).

### 4.4. Type IV$_0$

For type IV$_0$, $(T^{ab})_{IV_0} = f(t^a s^b + s^a t^b)$. So, for the standard energy conditions:

- **NEC** is violated. For any null vector $k^a$ we have $T_{ab}k^a k^b = 2f(t \cdot k)(s \cdot k)$. We can always choose signs so that $(t \cdot k) > 0$, but $(s \cdot k)$ can easily flip sign. Thus type IV$_0$ cannot satisfy the inequality on which NEC is based for all null vectors.
- **WEC** is violated.
For any timelike vector $V^a$ we have $T_{ab}V^aV^b = 2f(t \cdot V)(s \cdot V)$. We can always choose signs so that $(t \cdot V) > 0$, but $(s \cdot V)$ can easily flip sign. Thus type IV$_0$ cannot satisfy the inequality on which WEC is based for all timelike vectors.

- SEC is violated.
  - SEC = WEC because $T = 0$.
- DEC is violated.
  - DEC = WEC + FEC, and FEC is violated; see below.

For the less-standard energy conditions:

- FEC is violated.
  - For any timelike 4-velocity $V$ we have the flux vector
    \[ F_a = T_{ab}V^b = f[t_a(s \cdot V) + s_a(t \cdot V)]. \] (19)
  - But then $F_aF^a = f^2[(t \cdot V)^2 - (s \cdot V)^2] > 0$ where the last step takes into account that the projection of a timelike vector along a timelike direction is larger than its projection along a spacelike direction. So the flux vector is always spacelike, indicating that type IV$_0$ corresponds to \textit{tachyonic matter}.
- TEC is marginally satisfied ($T = 0$).
- TOSEC is violated ($\text{tr}[T^2] = -2f^2 < 0$).
- DETEC is satisfied ($\text{det}[T] = f^2 > 0$).

5. Stability of the Hawking–Ellis types

Now let us consider how stable the Hawking–Ellis types are under infinitesimal perturbations. One reason for being particularly interested in this is to understand the potential pitfalls of relying on numerical estimates and calculations of semi-classical renormalized stress–energy tensors; there will always be numerical approximation and round-off issues, and we would like to understand how these issues affect the Hawking–Ellis classification.

We find it useful to first step back to consider standard purely mathematical matrix results regarding diagonalizable versus defective matrices [32, 33]:

Generic perturbations of any matrix, (regardless of whether the original matrix is diagonalizable under similarity transformations or not), will lead to a diagonalizable matrix.

The point is that generic perturbations will lift any eigenvalue degeneracy which might be present, while preserving or inducing an eigenvalue degeneracy would require a very non-generic perturbation [32, 33]. Since distinct eigenvalues are a sufficient condition for diagonalizability, generic perturbations of any matrix will lead to a diagonalizable (under similarity transformations) matrix—any matrix is infinitesimally close to a diagonalizable matrix [32, 33].

Within the context of the Hawking–Ellis classification, the diagonalizable stress–energy tensors correspond to types I and IV, whereas types II and III are non-diagonalizable. So we have:

- Perturbing generic type I, (with all eigenvalues unequal), generically leads to type I.
- Perturbing degenerate type I, (with some eigenvalues equal), generically leads to either type I or type IV.
- Perturbing type II generically leads to either type I or type IV.
• Perturbing type III generically leads to either type I or type IV.
• Perturbing type IV generically leads to type IV.

To clarify this point further, note that for all of the Hawking–Ellis types there is always an orthonormal basis where any stress–energy tensor can be written as

$$
T^\mu_\nu \sim \begin{bmatrix}
-\rho & f_1 & f_2 & 0 \\
-f_1 & p_1 & s & 0 \\
-f_2 & s & p_2 & 0 \\
0 & 0 & 0 & p_3
\end{bmatrix}.
$$

(20)

The characteristic polynomial of this tensor is of the form

$$
c(\lambda) = (p_3 - \lambda)[\lambda^3 + b\lambda^2 + c\lambda + d].
$$

(21)

Apart from the eigenvalue $\lambda = p_3$, the multiplicity of the other eigenvalues (which is closely related to the Hawking–Ellis type of the stress–energy tensor) will depend on the roots of the cubic equation resulting from equating to zero the square bracket of the characteristic polynomial. Defining the characteristic $\Delta = 18bcd - 4b^3d + b^2c^2 - 4c^3 - 27d^2$ of the cubic polynomial, those roots can be classified as follows:

- $\Delta > 0$: There are three distinct real roots; thus, we have type I stress–energy.
- $\Delta = 0$: There is at least one multiple eigenvalue; we have either degenerate type I stress energy, or alternatively type II or type III stress–energy.
- $\Delta < 0$: There is one real root and two complex roots; we have type IV stress–energy.

A small perturbation of the stress–energy tensor implies a small perturbation of the value of $\Delta$, disturbing its value from zero if it is initially zero. Thus, degenerate type I, and types II and III, are indeed unstable.

Numerical calculations of renormalized stress–energy tensors in spherical symmetry often lead to thick spherical regions of type I and (in the Unruh quantum vacuum state) thick spherical regions of type IV, see [5–9, 11–13]. From the above so far very general discussion, such regions must be separated by a region of either degenerate type I, or type II, but since that region is numerically unstable, it can at best be a thin zero-thickness shell whose location cannot be estimated to better than numerical precision. But we can actually do better that this—since type IV in spherical symmetry requires $f = T_{\hat{\mu}\hat{\nu}} \neq 0$, (so that we must be in the Unruh quantum vacuum state, and conservation of stress energy then implies $f \neq 0$ everywhere throughout the spacetime), we can write the interesting part of the stress–energy tensor for any spherically symmetric scenario as

$$
(T^{\mu\nu}) = \begin{bmatrix}
\rho & f \\
f & p
\end{bmatrix} = \begin{bmatrix}
f + \mu + \delta & f \\
f & f - \mu + \delta
\end{bmatrix}.
$$

(22)

The Lorentz invariant eigenvalues are easily determined to be $-\mu \pm \sqrt{\delta(\delta + 2f)}$. Then the region where $\delta(\delta + 2f) > 0$ is type I, while the region where $\delta(\delta + 2f) < 0$ is type IV. At the transition layer, we have either $\delta = 0$ or $\delta = -2f$. This corresponds to

$$
(T^{\mu\nu}) = \begin{bmatrix}
\mu \pm f & f \\
f & -\mu \pm f
\end{bmatrix},
$$

(23)

both of which are type II. So the transition layer is guaranteed to be type II. (This is indeed what seems to happen in practice.)
Particular examples of perturbations of degenerate type I, type II, and type III are most effectively dealt with by directly applying perturbation arguments within the context of the type II₀ and type III₀ classifications.

- Consider this perturbation $\epsilon$ of a degenerate type I stress–energy tensor
  \[
  (T^\mu_\nu) \sim \begin{bmatrix}
  \rho + \epsilon & 0 & 0 \\
  0 & -\rho + \epsilon & 0 \\
  0 & 0 & p_2
  \end{bmatrix};
  \]
  \[
  (T^\nu_\mu) \sim \begin{bmatrix}
  -\rho - \epsilon & 0 & 0 \\
  0 & -\rho + \epsilon & 0 \\
  0 & 0 & p_2
  \end{bmatrix}.
  \]

  For $\epsilon = 0$ this is degenerate type I. For $\epsilon \neq 0$ this is type I. The eigenvalues are \{\(\rho \pm \epsilon, 0, 0\)\}. Thus perturbations of degenerate type I can easily lead to type I.
- Consider this (different) perturbation $\epsilon$ of a degenerate type I stress–energy tensor
  \[
  (T^\mu_\nu) \sim \begin{bmatrix}
  \rho & \epsilon & 0 \\
  \epsilon & -\rho & 0 \\
  0 & 0 & p_2
  \end{bmatrix};
  \]
  \[
  (T^\nu_\mu) \sim \begin{bmatrix}
  -\rho & \epsilon & 0 \\
  -\epsilon & -\rho & 0 \\
  0 & 0 & p_2
  \end{bmatrix}.
  \]

  For $\epsilon = 0$ this is degenerate type I. For $\epsilon \neq 0$ this is type IV. The eigenvalues are \{\(\rho \pm i\epsilon, 0, 0\)\}. Thus perturbations of degenerate type I can easily lead to type IV.
- Consider this perturbation $\epsilon$ of a type II₀ stress–energy tensor
  \[
  (T^\mu_\nu) \sim \begin{bmatrix}
  f & f + \epsilon & 0 \\
  f + \epsilon & f & 0 \\
  0 & 0 & 0
  \end{bmatrix};
  \]
  \[
  (T^\nu_\mu) \sim \begin{bmatrix}
  -f & f + \epsilon & 0 \\
  -f + \epsilon & f & 0 \\
  0 & 0 & 0
  \end{bmatrix}.
  \]

  For $\epsilon = 0$ this is type II₀. For $\epsilon \neq 0$ this is, (depending on the sign of $\epsilon f$), either type I or type IV. The eigenvalues are \{\(\pm \sqrt{-2\epsilon f} + O(\epsilon^{3/2}), 0, 0\)\}. Thus perturbations of type II₀ (and so mutatis mutandi type II) can lead to either type I or type IV.
- Consider finally this perturbation $\epsilon$ of a type III₀ stress–energy tensor
  \[
  (T^\mu_\nu) \sim \begin{bmatrix}
  0 & f & 0 \\
  f & 0 & f + \epsilon \\
  0 & f + \epsilon & 0
  \end{bmatrix};
  \]
  \[
  (T^\nu_\mu) \sim \begin{bmatrix}
  0 & f & 0 \\
  -f & 0 & f + \epsilon \\
  0 & f + \epsilon & 0
  \end{bmatrix}.
  \]

  For $\epsilon = 0$ this is type III₀. For $\epsilon \neq 0$ this is, (depending on the sign of $\epsilon f$), either type I or type IV. The eigenvalues are \{\(\pm \sqrt{2\epsilon f} + O(\epsilon^{3/2}), 0, 0\)\}. Thus perturbations of type III₀ (and so mutatis mutandi type III) can lead to either type I or type IV.

Overall, we see that non-degenerate type I and type IV are stable under perturbations, while degenerate type I, and types II and III, are unstable.

6. Discussion

Note that all the essential core types, II₀, III₀, and IV₀ are significantly simpler to work with than the full type II, III, IV, stress–energy tensors, and have rather nice algebraic properties.
Physically they correspond to subtracting as much of type I as possible, to always make the eigenvalues as simple as possible (while preserving eigenvector structure).

We have also seen that the energy conditions are easier to deal with using essential core types, II$_0$, III$_0$, and IV$_0$, and that it is easier to get a grasp of perturbative stability of the Hawking–Ellis classification using the essential core II$_0$ and III$_0$ types.

Furthermore, focussing on type III$_0$, (rather than the full type III), makes it a little clearer just how physically odd type III really is. Type III stands out in that it does not seem to have any straightforward physical interpretation in either classical or quantum physics, a point we plan to address in future work [34].

Finally, we have discussed the stability of the Hawking–Ellis classification under infinitesimal perturbations. We have shown that types II and III, which are those energy tensors that are non-diagonalizable, are unstable and generically decay into either type I or type IV. In contrast non-degenerate type I and generic type IV remain so under generic perturbations. Note that degenerate type I, which has a multiple eigenvalue but only one timelike eigenvector, is also unstable.

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