Supersymmetric Two-Time Physics

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Abstract

We construct an Sp(2,R) gauge invariant particle action which possesses manifest space-time SO(d,2) symmetry, global supersymmetry and kappa supersymmetry. The global and local supersymmetries are non-abelian generalizations of Poincare type supersymmetries and are consistent with the presence of two timelike dimensions. In particular, this action provides a unified and explicit superparticle representation of the superconformal groups OSp(N/4), SU(2,2/N) and OSp(8∗/N) which underlie various AdS/CFT dualities in M/string theory. By making diverse Sp(2,R) gauge choices our action reduces to diverse one-time physics systems, one of which is the ordinary (one-time) massless superparticle with superconformal symmetry that we discuss explicitly. We show how to generalize our approach to the case of superalgebras, such as OSp(1/32), which do not have direct space-time interpretations in terms of only zero branes, but may be realizable in the presence of p-branes.

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1 Introduction: what is two-time physics?

Recently, a surprising connection between $SO(d, 2)$ and point particle physics has been uncovered in [1] and explored further in [2]-[5]. The motivation for this work comes from the algebraic approach to M-theory and its extensions to F-theory and S-theory. The work in [2]-[5] has been called Two-Time Physics (although it can be viewed as a reformulation of one-time physics) because it involves the higher symmetry $SO(d, 2)$ and because this symmetry is realized linearly on a vector $X^M(\tau)$ with one extra time-like and one extra space-like dimension as compared to the usual relativistic vector $x^\mu(\tau)$ in ordinary spacetime in $(d-1) + 1$ dimensions. Thanks to an $Sp(2, R)$ gauge symmetry, which includes $\tau$-reparametrization as a subgroup, the extra 1+1 space-time dimensions can be related to $Sp(2, R)$ gauge degrees of freedom at the cost of losing the manifest $SO(d, 2)$ symmetry. This is analogous to relating $x^0(\tau)$ to $\tau$ by making a gauge choice at the cost of losing manifest $SO(d-1, 1)$ symmetry.

The most conservative way of understanding two-time physics is to view it as a reformulation of ordinary one-time physics in $(d-1) + 1$ dimensions that displays higher symmetries. Through this reformulation one learns that the action functional for ordinary one-time physical systems, for free or interacting particles, have a hidden space-time $SO(d, 2)$ symmetry that is realized non-linearly. In the case of the ordinary massless relativistic particle the symmetry is none other than the usual conformal symmetry $SO(d, 2)$ of massless systems. However, the same global symmetry $SO(d, 2)$ is also realized in unusual ways in many other systems [3], all of which correspond to different $Sp(2)$ gauge choices. The other realizations of $SO(d, 2)$ cannot be interpreted as conformal symmetries; they are simply hidden symmetries of ordinary one-time systems that escaped detection before (except for the well know $SO(4, 2)$ dynamical symmetry of the hydrogen atom). Some specific examples that were treated in [1]-[5] are: massive non-relativistic particle, massive relativistic particle, harmonic oscillator, hydrogen atom, particle in black hole background, particle in an $AdS_D \times S^n$ background (with $D + n = d$), etc.. All of these have their analogs with spinning particles with spin $n/2$. In that case $Sp(2)$ is replaced by $OSp(n/2)$ [3].

A remarkable connection between all these one-time systems is that they are all connected to each other by $Sp(2, R)$ or $OSp(n/2)$ gauge transformations that play a role analogous to duality in M-theory. In this context one may also make a connection to canonical transformations in which the time is also transformed. The analogy to duality in M-theory may be described as follows: usually one thinks of various dual string theories as different corners of the moduli space of M-theory that are connected to each other by duality (gauge) transformations. Each one of the string theories is considered to be a perturbative expansion of the full theory and somehow capable of capturing the full M-theory by including all non-perturbative sectors. Thus, different looking string theories are supposed to be just different starting points for the same M-theory. In our case there is an analogous and possibly stronger notion of duality.
We called it “duality” to emphasize the analogy for the benefit of the reader, but in the long term it probably should be given a different name to avoid confusion. Namely, different one-time physics models appear as different gauge choices of the same theory and therefore they can be transformed into each other by gauge transformations. These systems are unified together in one parent theory that has both $\text{Sp}(2, R)$ gauge symmetry (duality) and a higher global symmetry $\text{SO}(d, 2)$. Any version can be used to compute the gauge invariant quantities of the parent theory. There is a single action that describes all of them and thus unifies them. The unified theory is the theory that we called Two-Time Physics for a zero brane. We have indications that the same approach can be extended to p-branes.

The two previous paragraphs give a description of two-time physics from a conservative point of view, involving only one-time physics, hidden symmetries, and dualities that relate different systems. A more radical point of view is offered by the manifestly $\text{Sp}(2)$ (generally $\text{OSp}(n/2)$) local gauge invariant and $\text{SO}(d, 2)$ global invariant formulation. In this case the dynamical degrees of freedom are the position-momentum doublet $X_i^M = (X_i^M, P_i^M)$ which transforms linearly as an $\text{SO}(d, 2)$ vector. The vector $X^M(\tau)$ is most directly interpreted as the position of a particle in a space-time that includes two timelike dimensions. The meaning of $\text{SO}(d, 2)$ is simply the Lorentz symmetry of the system in a flat spacetime with two times. This suggests a very radical view of the meaning of “time” in one-time physics. To begin with there are two timelike dimensions; which one is the physical time in one-time physics? This is answered quite explicitly in the examples that have been worked out in detail. Namely, the physical “time” in one-time physics is not determined until one makes an $\text{Sp}(2, R)$ gauge choice. From the point of view of two-time physics the “time” of one-time physics appears as a gauge artifact. One-time physics seems different from one $\text{Sp}(2, R)$ gauge choice to another because the choice of time determines a corresponding one-time Hamiltonian (time evolution generator for that choice of time) that looks different for every gauge choice. However, in reality these very different looking one-time-physics systems describe the same gauge invariant sector of the two-time physics system.

What would be considered tests of two time physics? Whether one takes the conservative or radical point of view some tests can provide evidence of the unification of different one-time physics systems that occurs anyway. For now we concentrate on theoretical tests and hope that we will produce some experimental tests when we consider interacting two-time-physics systems. The tests must concentrate on $\text{Sp}(2, R)$ gauge invariants while exploring the global $\text{SO}(d, 2)$ symmetry. For two-time physics the only gauge invariant observables are the generators of the global $\text{SO}(d, 2)$ symmetry $L_{MN} = X^M P_N - X^N P_M$. All physical states or other gauge invariant quantities can be computed fully covariantly without choosing a gauge, or non-covariantly by choosing any gauge that defines a one-time system. At the end, all gauge invariant quantities, including the spectrum of the theory, are determined by
the representation theory of $SO(d, 2)$. It was demonstrated in [1]-[5] that indeed the quantum spectrum of the theory is the same in several different gauges and that it corresponds to the unitary representation of $SO(d, 2)$ with Casimir eigenvalues fully determined. In particular $C_2(SO(d, 2)) = 1 - d^2/4$ for the spinless system and $C_2(SO(d, 2)) = \frac{1}{8}(n - 2)(d + 2)(d + n - 2)$ for the spinning system with spin $n/2$. The difference between Hilbert spaces of diverse but dual one-time models is only in the choice of basis labelled by different quantum numbers. These bases are related to each other by unitary transformations (duality at the quantum level) within the same $SO(d, 2)$ representation. This test was verified in several different gauges. For example the H-atom and free massless particle (and all other cases) correspond to the same $SO(d, 2)$ representation with the same Casimir eigenvalues. This was not known before the advent of two-time physics. This is a test at the quantum level. A similar test at the classical level is trivial because all Casimir eigenvalues vanish due to constraints; the test becomes non-trivial at the quantum level due to quantum ordering.

Whether one takes the conservative or the radical point of view, the unified approach is also useful in uncovering and understanding the symmetries of one-time physics. The fact that previously unknown symmetries have been found in very familiar systems is a success of the approach [3]. The hidden symmetries arise naturally in fixed gauges as the natural $SO(d, 2)$ symmetry of the original theory. Much of this symmetry remains hidden in the one-time physics formulation. For example, it was discovered that the field theory that describes a free scalar particle in an $AdS_D \times S^n$ background has $SO(D + n, 2)$ symmetry, which is larger than the previously known Killing symmetry $SO(D - 1, 2) \times SO(n + 1)$. Thus, for $AdS_5 \times S^5$ [2] the symmetry is $SO(10, 2)$ not just $SO(4, 2) \times SO(6)$. For the symmetry to be valid the particle must have a quantized mass as shown in [3].

In this paper we generalize the two-time zero brane system to spacetime supersymmetry. We construct an $Sp(2, R)$ gauge invariant particle action which possesses a manifest space-time superconformal symmetry. In particular, we demonstrate that the superconformal groups $OSp(N/4)$, $SU(2, 2/N)$ and $OSp(8*/N)$ which underlie various $AdS/CFT$ dualities in M/string theory are given unified and explicit superparticle representations. Our approach can be generalized for the case of superconformal algebras, such as $OSp(1/32)$, which do not have direct space-time interpretation in terms of only zero branes, but could be realized in the presence of p-branes.

As in [1] we emphasize that the $Sp(2, R)$ symmetry that is gauged may also be viewed via $Sp(2, R) = SO(1, 2)$ as the conformal group in $0 + 1$ dimensions. From this point of view, many $0 + 1$ quantum gravity systems (relativistic massless and massive particle, non-relativistic massless and massive particle, particle moving in the $AdS_D \times S^n$ background etc.) can be viewed as different gauge choices of the same $0 + 1$ conformal quantum gravity theory.

One gauge choice that we will investigate in detail corresponds to the standard superparticle in one-time physics. It has been known for a long time [9] that the mass-
less superparticle has actually non-linearly realized hidden superconformal symmetry, extending the conformal symmetry $SO(d, 2)$ of massless systems for $d = 3, 4, 6$. In our approach the hidden conformal supersymmetries of the superparticle action $OSp(N/4)$, $SU(2, 2/N)$, $OSp(8^*/N)$ are just the explicit supersymmetries in the two-time physics formulation. This is another example of the utility of our approach, now in a supersymmetric setting. This also gives a two-time meaning to the superparticle system as well as to supersymmetry itself.

Finally, there are other possible gauge choices of our Lagrangian, which would provide explicit unusual realizations of the various space-time supersymmetries $OSp(N/4)$, $SU(2, 2/N)$, $OSp(8^*/2N)$. Guided by the analogy with the bosonic case, we list just a few: superparticles moving in the background of $AdS_D \times S^n$, supersymmetric H-atom, supersymmetric particle moving in the black hole background, etc.. The case of supersymmetric particle in $AdS_5 \times S^5$ would be of special current interest. We predict, following the bosonic case investigated in [5], that the field theory that describes the particle in the supersymmetric $AdS_5 \times S^5$ background [6] has $OSp(8^*/8)$ supersymmetry and not just the symmetry of its subgroup $SU(2, 2/4)$; similarly, the supersymmetric $AdS_3 \times S^3$ background [7] has $OSp(4/4)$ supersymmetry and not just the symmetry of its subgroup $SU(1, 1/2)$.

2 Lagrangian: why two time-like dimensions?

The dynamical variables are the bosons $X^M_i (\tau)$ and fermions $\Theta^{\alpha a} (\tau)$. The $X^M_i$ form an $Sp(2, R)$ doublet, $i = 1, 2$, while the $\Theta^{\alpha a} (\tau)$ are $Sp(2, R)$ singlets. The $X^M_i (\tau)$ are position-momentum vectors in $d + 2$ dimensions, $M = 1, \cdots, d + 2$. The $\Theta^{\alpha a} (\tau)$ are spinors of $SO(d, 2)$, $\alpha = 1, \cdots, s (d)$, and there are $N$ of them $a = 1, \cdots, N$. To construct a covariant derivative $D_\tau X^M_i$ as an $Sp(2, R)$ doublet we use the $Sp(2, R)$ gauge field $A^{ij} = A^{ji}$ (a triplet) as in [1] (there is only the time component $A^{ij}_\tau = A^{ij}_\tau$ since we are on the worldline)

$$D_\tau X^M_i = \partial_\tau X^M_i - \varepsilon_{ij} A^{jk} X^M_k - \Omega^{MN} X^M_{iN}.$$ (1)

The new $\Omega^{MN}$ (an $Sp(2, R)$ singlet) will play a role for supersymmetry as explained below. The antisymmetric $\varepsilon_{ij}$ is the $Sp(2, R)$ metric that is used to lower or raise indices.

The $Sp(2, R)$ gauge invariant Lagrangian is

$$L = \frac{1}{2} D_\tau X^M_i X^N_j \varepsilon^{ij} \eta_{MN}.$$

$$= \partial_\tau X_1 \cdot X_2 - \frac{1}{2} A^{ij} X_i \cdot X_j - \frac{1}{2} \Omega^{MN} L_{MN}.$$ (3)

The first term in the Lagrangian is simplified by doing an integration by parts so that one may identify $X^M_1 = X^M$ as position and $X^M_2 = P^M = \partial L/\partial (\partial_\tau X_1 M)$ as
momentum. Then the gauge invariant
\[ L_{MN} = X^M_i X^N_j \varepsilon^{ij} = X^M P^N - X^N P^M \]  \hspace{1cm} (4)
is the orbital angular momentum that generates SO\((d, 2)\) on the bosons \((X^M, P^M)\).

We recall why one must have two timelike dimensions as in [1]. The equations of motion of the gauge field \(A^{ij}\) demand the constraints (for related work see [10][11])
\[ X_i \cdot X_j = 0, \quad \text{or} \quad X^2 = P^2 = X \cdot P = 0, \]  \hspace{1cm} (5)
which require that the two light-like vectors \((X^M, P^M)\) are orthogonal. In a spacetime with only one timelike dimension, \(X^M\) and \(P^M\) must be parallel lightlike vectors with zero angular momentum \(L_{MN} = 0\). Since \(L_{MN}\) is the only gauge invariant observable, this becomes a trivial theory if spacetime contains only one timelike dimension. To have solutions of the constraints with non-trivial angular momentum \(L_{MN}\) one must admit a spacetime metric \(\eta_{MN}\) that has two time-like dimensions. The Sp\((2, R)\) gauge symmetry is just enough to remove all the ghosts introduced by the two timelike dimensions and provide a unitary theory. One cannot have more than two time-like dimensions because the theory would be non-unitary due to ghosts that cannot be removed.

There are ways of having more than two timelike dimensions in more complicated multi-particle (or multi-string) systems [8], but not in the system of one zero brane.

3 Fermions and spin connection
In the covariant derivative (1) we have included a new term involving \(\Omega^{MN} (\Theta, \partial_\tau \Theta)\) which is constructed from the fermions \(\Theta^{\alpha a} (\tau)\) and their derivative. The role of this term is to insure a covariant derivative \(D_\tau X^M_i\) under global supersymmetry. For this to be possible, we will see that \(\Omega^{MN}\) transforms like a spin connection under field dependent local SO\((d, 2)\) transformations induced by global supersymmetry transformations. \(\Omega^{MN}\) is given as a supertrace
\[ \Omega^{MN} = \text{Str} \left( \begin{pmatrix} \Gamma^{MN} & 0 \\ 0 & 0 \end{pmatrix} \right) (\partial_\tau t) t^{-1}, \quad t (\Theta) \in G/H \]  \hspace{1cm} (6)
where \(t (\Theta)\) is an element of the coset \(G/H\) parametrized by the fermions \(\Theta^{\alpha a}\). An explicit parametrization of \(t (\Theta)\) and \(\Omega^{MN} (\Theta)\) are given below, but only the general properties of cosets rather than explicit expressions are essential for most of the dis-
cussion. The coset \( G/H \) is given by one of the following three cases\(^2\): 

\[
\begin{align*}
OSp(N/4) & \sim [SO(N) \times Sp(4, R)], \\
SU(2, 2/N) & \sim [SU(N) \times SU(2, 2)], \\
OSp(8^*/N) & \sim [Sp(N) \times Spin(8^*)], \quad N = \text{even}
\end{align*}
\]

Zoom

Notice that precisely these supergroups occur naturally in the AdS/CFT correspondence in M/IIB string theory. The subgroup \( H \) includes the spacetime Lorentz group \( SO(d, 2) \), with two timelike dimensions for \( d + 2 = 5, 6, 8 \), (or the conformal group in \( d \) dimensions), namely

\[
Sp(4, R) = SO(3, 2), \quad SU(2, 2) = SO(4, 2), \quad Spin(8^*) = SO(6, 2).
\]

The number of supersymmetries is \( N \) (real), \( N \) (complex), \( N \) (complex and even) respectively in the three cases. The generators of \( SO(d, 2) \) acting on the spinors (\( \alpha \) index) is constructed from gamma matrices in the form \( \Gamma^{MN} = \frac{1}{2} [\Gamma^M, \Gamma^N] \), and it is embedded in the spacetime block of the matrix representation of the groups above, as seen in (6).

\[
t(\Theta) = \exp (\vartheta^{\alpha a} Q_{\alpha a}) \quad \text{is given by exponentiation the off-diagonal coset generators } \\
Q_{\alpha a} \text{ (represented by the } s(d) \times N \text{ rectangular blocks) combined with the parameters } \\
\vartheta^{\alpha a}.
\]

\[
t(\Theta) = \exp \left( \begin{array}{cc} 0 & \vartheta \\ -\vartheta & 0 \end{array} \right) = \left( \begin{array}{cc} \frac{1}{\sqrt{1+\Theta}} & \Theta \frac{1}{\sqrt{1+\Theta}} \\ -1 & \frac{1}{\sqrt{1+\Theta}} \end{array} \right)
\]

where \( \Theta = \vartheta \tan \sqrt{\frac{1}{\vartheta \vartheta}} \). The square roots or \( \tan \sqrt{\frac{1}{\vartheta \vartheta}} \) are understood in the sense of infinite series. The \( \bar{\vartheta}_{\alpha a}, \bar{\Theta}_{\alpha a} \) are obtained by transposition or hermitian conjugation combined with multiplication by appropriate metrics or charge conjugation matrices \( C \) for the spinors,

\[
\bar{\Theta}_{\alpha a} = (\Theta^T C)_{\alpha a}, \quad (\Theta^T C)_{\alpha a}, \quad (K \Theta^T C)_{\alpha a}, \quad (12)
\]

respectively for the three cases. Here the unit matrix \( \delta_{ab} \) is the metric for \( SO(N) \), the unit matrix \( \delta_a^b = \delta_{ab} \) is the metric for \( SU(N) \), and the antisymmetric matrix \( K_{ab} \) is the metric for \( Sp(N) \).

According to a general theorem, an element \( g \) of any (super)group can always be decomposed into the product of an element of the subgroup \( h \) times an element of the

\(^2\)The reason for considering only these supergroups is that the spacetime subgroup is precisely \( SO(d, 2) \) for \( d = 3, 4, 6 \) respectively. For these cases the orbital part of \( SO(d, 2) \), i.e. \( L^{MN} \), can be constructed from dynamical variables \( X^N(\tau) \) that are zero branes, as in (6). For other supergroups, such as \( OSp(1/32) \), additional dynamical bosonic variables that are possibly related to p-branes must be included. Except for comments on the general construction of the Lagrangian found in the discussion section, we leave the study of such cases to future research.
coset $t$, thus $g = ht$. Following this, a basic relation that we will use in our case is ($t_1t_2$ is a group element $g$)

$$t(\Theta_1)t(\Theta_2) = h(\Theta_1, \Theta_2)t(\Theta_{12})$$

(13)

where both $h(\Theta_1, \Theta_2)$ and $t(\Theta_{12})$ are functions of $\Theta_1, \Theta_2$. Explicit forms for $h(\Theta_1, \Theta_2)$ and $t(\Theta_{12})$ are given in [12] and in the appendix, but these details are not needed in most of our discussion. In our case $t(\Theta_{12})$ depends on $\Theta_{12}$ which takes values in the off-diagonal blocks and has the form (11) while $h(\Theta_1, \Theta_2)$ takes values in the diagonal blocks with the upper block being the spinor representation of $\text{SO}(d, 2)$ and the lower block the fundamental representation of $\text{SO}(N)$, $\text{SU}(N)$, $\text{Sp}(N)$ respectively.

4 Global supersymmetry

The global supersymmetry transformation on the fermions $\Theta \to \Theta_\varepsilon$ is defined by using the relation (13) for $\Theta_1 = \Theta, \Theta_2 = \varepsilon, \Theta_{12} = \Theta_\varepsilon$. Since $\Theta$ appears only in the form $t(\Theta)$ the SUSY transformation may be written as the following product of super matrices

$$t(\Theta) \to t(\Theta_\varepsilon) = h^{-1}(\Theta, \varepsilon)t(\Theta)t(\varepsilon) .$$

(14)

Inserting this in (8) we find

$$\Omega^{MN}(\Theta_\varepsilon) = \text{Str} \left( \begin{pmatrix} \Gamma^{MN} & 0 \\ 0 & 0 \end{pmatrix} \partial_\tau \left[ h^{-1}(\Theta, \varepsilon)t(\Theta)t(\varepsilon) \right] t^{-1}(\varepsilon)t^{-1}(\Theta)h(\Theta, \varepsilon) \right)$$

(15)

$$= \text{Str} \left( \begin{pmatrix} \Gamma^{MN} & 0 \\ 0 & 0 \end{pmatrix} h^{-1}((\partial_\tau t)^{-1} - \partial_\tau)t \right)$$

(16)

$$= \left[ \Lambda^{-1}(\Omega(\Theta) - \partial_\tau) \Lambda \right]^{MN}$$

(17)

where $\Lambda^M_N(\Theta, \varepsilon)$ is a field dependent local $\text{SO}(d, 2)$ transformation and $\Omega^{MN}(\Theta)$ transforms like a spin connection. In arriving at this result we used that $t(\varepsilon)$ is independent of $\tau$, and noted that $\Lambda^M_N(\Theta, \varepsilon)$ is the vector representation of $\text{SO}(d, 2)$ whose spinor representation corresponds to the upper block in $h(\Theta, \varepsilon)$. Note that $\Omega^{MN}$ is invariant under the internal symmetry transformations contained in the lower block of $h(\Theta, \varepsilon)$.

We now turn to the transformations of $X^M_i$ and the covariant derivative $D_\tau X^M_i$ in (8). We see that, if we take the supersymmetry transformation for $X^M_i$ to be the induced local Lorentz transformation

$$X^M_i \to X^M_i\varepsilon = \left( \Lambda^{-1} \right)^M_N(\Theta, \varepsilon)X^N_i,$$

(18)

then the covariant derivative also transforms covariantly under the local Lorentz transformation since $\Omega^{MN}$ transforms like a spin connection

$$D_\tau X^M_i \to \left( \Lambda^{-1} \right)^M_N(\Theta, \varepsilon) \left( D_\tau X^N_i \right) .$$

(19)
Since the Lagrangian is already invariant under Lorentz transformations \( \text{SO}(d, 2) \), it is also invariant under the local transformation \( \Lambda^M_N (\Theta, \varepsilon) \). Hence it is invariant under the global supersymmetry transformation generated by \( t (\varepsilon) \) as given by eqs. (11) and (14).

The global symmetry transformations with the subgroup \( H \) including \( \text{SO}(d, 2) \) and \( \text{SO}(N), \text{SU}(N), \text{Sp}(N) \) proceeds in the same way. Instead of eq. (13) consider the product \( t (\Theta) h \) where is global \( h \in H \). Using the general theorem we can write \( t (\Theta) h = h t (\Theta_h) \) where \( \Theta_h \) is the transformed \( \Theta \) so that

\[
t (\Theta) \to t (\Theta_h) = h^{-1} t (\Theta) h \tag{20}
\]

In the present case the same global \( h \) must appear on both sides of \( t (\Theta) \) since this is the only way for \( t (\Theta) \) to maintain the matrix form in (11). Following the same steps we learn that \( \Omega^{MN} \) transforms like (17) but with a global \( \Lambda \). Therefore \( X^M_i \) also transforms under the global internal symmetry. Also \( A^{ij} \) does not transform under any of the global symmetries.

This establishes that the global symmetries of the Lagrangian are all the transformations of the supergroup \( G = \text{OSp}(N/4), \text{SU}(2, 2/N), \text{OSp}(8^*/N) \) respectively for the various dimensions \( d + 2 = 5, 6, 8 \). All global transformations originate from the multiplication of \( t (\Theta) \) from the right, \( t (\Theta) g \), where \( g \) is a global group element \( g \in G \). By construction it is seen that these transformations close to form the respective supergroups. The conserved generators are easily derived by using Noether’s theorem, we find

\[
\Psi_{aa} = \left( \frac{1}{\sqrt{1 + \Theta \tilde{\Theta}}} \left( \frac{1}{2} \Gamma^{KL} L_{KL} \right) \Theta \frac{1}{\sqrt{1 + \tilde{\Theta} \Theta}} \right)_{aa} \tag{21}
\]

\[
J^{MN} = \text{Tr} \left( \Gamma^{MN} \frac{1}{\sqrt{1 + \Theta \tilde{\Theta}}} \left( \frac{1}{2} \Gamma^{KL} L_{KL} \right) \frac{1}{\sqrt{1 + \tilde{\Theta} \Theta}} \right) \tag{22}
\]

\[
T^A = \frac{1}{2} \left( \tilde{\Theta}_a t^A \Psi^a + h.c. \right), \tag{23}
\]

where \( (t^A)^a_b \) are the hermitian matrix representations for the generators of the internal group \( \text{SO}(N), \text{SU}(N), \text{or Sp}(N) \) in the fundamental representation.

### 5 Local kappa symmetry

The local kappa supersymmetry transformation on the fermions \( \Theta \to \Theta_\kappa \) is defined by using the relation (13) for \( \Theta_1 = \kappa, \Theta_2 = \Theta, \Theta_{12} = \Theta_\kappa \). We will see below that
the local spinors $\kappa^{aa}$ must be constructed from a specific combination of $X_i^M$ and two independent local spinors $\kappa^{i\alphaa} (\tau), i = 1, 2,$ as follows

$$
\kappa^{aa} = (X_i^a)^{\alphaa}.
$$

(24)

This transformation is realized by the product of super matrices

$$
t (\Theta) \rightarrow t (\Theta \kappa) = h^{-1} (\kappa, \Theta) t (\kappa) t (\Theta).
$$

(25)

Comparing to the global transformation (14) we emphasize that the global $t (\varepsilon)$ is a right multiplication while the local $t (\kappa)$ is a left multiplication\textsuperscript{3} with $t (\Theta)$. We also emphasize that the order of the arguments in $h (\kappa, \Theta)$ and $h (\Theta, \varepsilon)$ are interchanged because of the same reason. Inserting this form in (6) we find

$$
\Omega^{MN} (\Theta \kappa) = \text{Str} \left( \left( \Gamma^{MN} 0 0 0 \right) \partial_\tau \left[ h^{-1} (\kappa, \Theta) g (\kappa, \Theta) \right] g^{-1} (\kappa, \Theta) h (\kappa, \Theta) \right)
$$

(26)

$$
= \text{Str} \left( \left( \Gamma^{MN} 0 0 0 \right) h^{-1} \left[ (\partial_\tau g) g^{-1} - \partial_\tau \right] h \right)
$$

(27)

where we have defined

$$
g (\kappa, \Theta) = t (\kappa) t (\Theta).
$$

(28)

As in (17) we can rewrite

$$
\Omega^{MN} (\Theta \kappa) = \left[ \Lambda^{-1} (\Omega_g - \partial_\tau) \Lambda \right]^{MN},
$$

(29)

where $\Lambda^{MN} (\kappa, \Theta)$ is a field dependent local SO($d, 2$) transformation similar to the previous case, except for interchanging the orders of the two fermions and substituting $\kappa$ instead of $\varepsilon$. Furthermore, $\Omega_g^{MN}$ is not $\Omega^{MN} (\Theta)$, since it is constructed from $g (\kappa, \Theta)$ not from $t (\Theta)$ as in (8).

We now turn to the kappa transformations of $X_i^M$. We see that, if we take the supersymmetry transformation for $X_i^M$ to be the induced local Lorentz transformation

$$
X_i^M \rightarrow X_i^M_{\kappa} = \left( \Lambda^{-1} \right)_N^M (\kappa, \Theta) X_i^N,
$$

(30)

then the covariant derivative transforms into

$$
D_\tau X_i^M \rightarrow \left( \Lambda^{-1} \right)_N^M (\kappa, \Theta) \left( \partial_\tau X_i^N - \varepsilon_{ij} A_{jk} X_j^N - \Omega_g^{NK} X_k^K \right).
$$

(31)

\textsuperscript{3}The group theoretical origin of kappa supersymmetry for the ordinary superparticle or super p-branes has remained obscure. In our treatment it is seen to originate from a group transformation similar to the global supersymmetry, except that it acts on the other side of the coset element $t (\Theta)$. Indeed the ordinary superparticle can be reformulated in precisely the same way. For previous approaches to kappa supersymmetry see [13][14][15].
The right hand side is not the covariant derivative since it contains $\Omega_{g}^{NK}$ instead of $\Omega_{g}^{NK}(\Theta)$, and the transformed $A_{\kappa}^{jk}$ instead of $A^{ij}$. Inserting these transformations into the Lorentz invariant Lagrangian we see that $\Lambda_{M}^{N}(\kappa, \Theta)$ drops out, and we find the result

$$L(X_\kappa, \Theta_\kappa, A_\kappa) = L(X, \Theta, A) - \frac{1}{2} L_{MN} \left[ \Omega_{g}^{MN} - \Omega_{\Theta}^{MN} \right] - \frac{1}{2} \left( A_{\kappa}^{ij} - A^{ij} \right) X_i \cdot X_j.$$  \hspace{1cm} (32)

The condition for kappa invariance is the vanishing of the last two terms which may be written in the form

$$Str \left( \left( \begin{array}{cc} L & 0 \\ 0 & 0 \end{array} \right) \left[ (\partial_\tau g) g^{-1} - (\partial_\tau t) t^{-1} \right] \right) = - \frac{1}{2} \left( A_{\kappa}^{ij} - A^{ij} \right) X_i \cdot X_j.$$  \hspace{1cm} (33)

with $L = \frac{1}{2} L_{MN} \Gamma^{MN}$. We can examine this equation for infinitesimal $\kappa$ by using

$$g(\kappa, \Theta) = \left( 1 + \left( \begin{array}{cc} 0 & \kappa \\ \tilde{\kappa} & 0 \end{array} \right) + \cdots \right) t(\Theta).$$  \hspace{1cm} (34)

$$(\partial_\tau g)^{-1} - (\partial_\tau t) t^{-1} = \left( \begin{array}{cc} 0 & \partial_\tau \kappa \\ \partial_\tau \tilde{\kappa} & 0 \end{array} \right) + \left[ \left( \begin{array}{cc} 0 & \kappa \\ \tilde{\kappa} & 0 \end{array} \right), (\partial_\tau t) t^{-1} \right] + \cdots.$$  \hspace{1cm} (35)

Therefore (33) becomes

$$Str \left( \left( \begin{array}{cc} 0 & L_{\kappa} \\ -\tilde{\kappa} L & 0 \end{array} \right) (\partial_\tau t) t^{-1} \right) + \frac{1}{2} \delta_\kappa A^{ij} X_i \cdot X_j = 0.$$  \hspace{1cm} (36)

This will vanish only if $L_{\kappa}$ is proportional to $X_i \cdot X_j$ since then $\delta_\kappa A^{ij}$ can be chosen to cancel its coefficient. Fortunately this is easily arranged by taking $\kappa^{\alpha \alpha} = (X_i^{\alpha} \kappa^i)^{\alpha \alpha}$ where $\kappa^{\alpha \alpha}(\tau)$ are two independent local fermions $i = 1, 2$. Indeed, we then find

$$L_{\kappa} = \frac{1}{2} \varepsilon^{ij} X_k^{j} X_j^{N} X_i^{K} \left( \Gamma_{MN} \Gamma_{K} \kappa^i \right) = - X_i \cdot X_j \left( X_k^{k} \kappa^i \right) \varepsilon^{jk}. \hspace{1cm} (37)$$

A three gamma term $\Gamma_{MNK}$ that could have appeared on the right-hand side vanishes because it imposes antisymmetry in $M, N, K$ which is impossible to have with only two independent vectors $(X^M, P^M)$. Therefore the kappa transformation for $A^{ij}$ must be

$$\delta_\kappa A^{ij} = \frac{1}{2} Str \left( \left( \begin{array}{cc} 0 & X_k^{k} \kappa^i \varepsilon^{jk} \\ -\tilde{\kappa}^{(i} X_k^{(j)} k \varepsilon^{jk} \end{array} \right) \left( \partial_\tau t(\Theta) t^{-1}(\Theta) \right) \right). \hspace{1cm} (38)$$

With the chosen transformations for $\Theta^{\alpha \alpha}, X_i^{M}, A^{ij}$ the Lagrangian is invariant under the kappa supersymmetry with the local parameters $\kappa^{\alpha \alpha}(\tau)$ which form an Sp(2, $R$) doublet $i = 1, 2$. Due to the local symmetry there is a corresponding constraint on the canonical degrees of freedom that takes the form

$$X_i \sqrt{1 + \Theta \tilde{\Theta} \Psi} = 0.$$  \hspace{1cm} (39)

This is easily verified by using (21), the manipulations of (37), and the constraints (3).
6 Gauge fixing to superparticle

Consider the gamma matrices $\gamma^\mu$ appropriate for SO($d-1,1$) and construct the gamma matrices appropriate for SO($d,2$) by taking direct products with $\tau_3$ and $\tau^\pm = \frac{1}{\sqrt{2}}(\tau_1 \pm i\tau_2)$, as follows

$$ M = \begin{pmatrix} \pm' & \mu \end{pmatrix} $$

$$ \Gamma^M = \left(\pm \tau^\pm \times 1, \ \tau_3 \times \gamma^\mu \right) $$

$$ \left\{ \Gamma^M, \Gamma^N \right\} = 2\eta^{MN}, \ \eta^{+\prime} = -1, \ \eta^{\mu\nu} = \text{Minkowski} $$

For our purposes we need to construct $\Gamma^{MN} = \frac{1}{2} \left[ \Gamma^M, \Gamma^N \right]$, which takes the form

$$ \Gamma^{MN} : \Gamma^{+\prime} = -\tau_3 \times 1, \ \Gamma^{\pm\mu} = -\tau^\pm \times \gamma^\mu, \ \Gamma^{\mu\nu} = 1 \times \gamma^{\mu\nu}. $$

We now fix two of the Sp($2, R$) and half of the kappa gauge symmetries as follows

$$ X^{+'} = 1, \ P^{+'} = 0, \ \Gamma^{+'}\Theta^a = 0. $$

After solving the two constraints $X^2 = X \cdot P = 0$, the gauge fixed forms of $X, P$ are

$$ M = \begin{pmatrix} +' & -' & \mu \end{pmatrix} $$

$$ X^M = \begin{pmatrix} 1 & x^2/2 & x^\mu \end{pmatrix} $$

$$ P^M = \begin{pmatrix} 0 & x \cdot p & p^\mu \end{pmatrix}. $$

Using $\Gamma^{+'} = \tau^+ \times 1$ and $C = \tau_1 \times c$, we can also write explicitly the gauge fixed form of $\Theta^a, \tilde{\Theta}_a$

$$ \Theta^a = \begin{pmatrix} \theta^a & 0 \end{pmatrix}, \ \tilde{\Theta}_a = \begin{pmatrix} 0 & \tilde{\theta}_a \end{pmatrix} $$

Before gauge fixing $\Omega^{MN}$ is computed by using (3) and (4)

$$ \Omega^{MN} = Tr \left( \Gamma^{MN} \frac{1}{\sqrt{1 + \Theta\tilde{\Theta}}} \left( \partial_r \Theta \tilde{\Theta} + \frac{1}{\sqrt{1 + \Theta\tilde{\Theta}}} - \partial_r \sqrt{1 + \Theta\tilde{\Theta}} \right) \right) $$

Then for the gauge fixed $\Theta$ we obtain $(\tilde{\Theta} \Theta)^b_a = (\tilde{\Theta} \partial_r \Theta)^b_a = 0$, etc.. Therefore, by the series expansion of the square roots, the expression for $\Omega^{MN}$ simplifies to $\Omega^{MN} = -\tilde{\Theta}_a \Gamma^{MN} \partial_r \Theta^a$, although still all of its components are zero except for the one that contains $\Gamma^{-\mu} = -\tau^- \times \gamma^\mu$. Thus,

$$ \Omega^{-\mu} = -\tilde{\theta}_a \Gamma^{-\mu} \partial_r \Theta^a = -\tilde{\theta}_a \gamma^\mu \partial_r \theta^a, $$

$$ \frac{1}{2} \Omega^{MN} L_{MN} = \Omega^{-\mu} L^{-\mu}_{\mu} = -\tilde{\theta}_a \gamma^\mu \partial_r \theta^a p_\mu, $$

12
where we have used $L_{\mu} \eta_{\nu} \eta_{\alpha} \mu_{\beta} \nu_{\gamma} = -p_{\mu}$. Therefore, in this gauge, the invariant Lagrangian \( L \) collapses to the superparticle Lagrangian

\[
L = \dot{x} \cdot p - \frac{1}{2} A_{\alpha} p^2 + \dot{\theta}_a \dot{p}_a \theta^a, \tag{52}
\]

\[
= \frac{1}{2 A^2} \left( \dot{x}^\mu + \dot{\theta}_a \gamma^\mu \partial_\mu \theta^a \right)^2, \tag{53}
\]

where the last form is obtained by integrating out $p^\alpha$.

The superparticle Lagrangian is well known to have local symmetries that correspond to $\tau$ reparametrization and kappa supersymmetry. Because of $\tau$ reparametrization (part of Sp(2, $R$) that has not been gauge fixed) there remains one bosonic constraint $P^2 = p^2 = 0$, and because of the remaining kappa supersymmetry there remains a fermionic constraint $\dot{p} \mathcal{Q} = 0$ where $\mathcal{Q}$ is the global supersymmetry generator $\mathcal{Q} = \dot{\theta} \theta$.

It has been known for some time \cite{9} that this Lagrangian also has hidden superconformal symmetries. In our approach the presence of the hidden superconformal symmetry is a natural consequence of the manifestly supersymmetric Lagrangian we started from, which had global OSp($N/4$), $SU(2, 2/N)$ or OSp($8^*/N$) symmetries. These global symmetries are not lost by gauge fixing, since the original Lagrangian is both gauge invariant and globally symmetric.

It is interesting to show how the gauge fixed form of $\Theta, X^M_i$ provide a basis for these global symmetries. Since the global symmetry tends to change the gauge fixed form of $\Theta, X^M_i$, there must be some compensation from the gauge transformations kappa and Sp(2, $R$) to restore the gauge fixed forms. In particular let us examine the form of $\Theta$. The global and local supersymmetry transformations $\delta_{\varepsilon + \kappa} \Theta$ given in the appendix simplify in this gauge due to \( (\tilde{\Theta} \Theta)_{a} = 0 \)

\[
\delta_{\varepsilon + \kappa} \Theta = \varepsilon + X^i_i \kappa^i + \Theta \tilde{\varepsilon} \Theta + \frac{1}{2} \Theta \tilde{\Theta} X^i_i \kappa^i. \tag{54}
\]

The global $\varepsilon$ has a lower component. But the lower component of $\Theta$ must be maintained at zero by the combined $\varepsilon$ and $\kappa$ transformations. Indeed this requirement is satisfied by taking \( (\Gamma^+ \kappa^a)^{\alpha} = 0 \) with

\[
\varepsilon = \begin{pmatrix} \epsilon \\ \lambda \end{pmatrix}, \quad \kappa^1 = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \quad \kappa^2 = \begin{pmatrix} k(\tau) \\ 0 \end{pmatrix}, \tag{55}
\]

\[
\kappa = X^i_i \kappa^i = \begin{pmatrix} \dot{p} k(\tau) + \dot{\theta} \lambda \\ -\lambda \end{pmatrix}, \quad \delta_{\varepsilon + \kappa} \Theta = \begin{pmatrix} \delta \theta \\ 0 \end{pmatrix}, \tag{56}
\]

\[
\delta \theta = \varepsilon + \dot{\theta} k(\tau) + \dot{\theta} \lambda + \theta \left( \lambda \dot{\theta} - \frac{1}{2} \tilde{\theta} \lambda \right). \tag{57}
\]

where we have also used the gauge fixed form of $X^M_i$. The $-\lambda$ in the second line of $\kappa$ comes from $-X^+ \Gamma^{-\kappa^1}$. The upper component $\epsilon$ is the standard global SUSY parameter for the superparticle. Similarly, $k(\tau)$ is the standard local kappa symmetry.
parameter. The lower component of the global supersymmetry tends to add $\lambda$ to the lower component of $\Theta$, shifting it away from the gauge fixed form, but by taking the upper component of $\kappa^1$ equal to the global $\lambda$ this is cancelled and the gauge fixed form of $\Theta$ is maintained. From our fully covariant construction we know that $\lambda$ is the global parameter of the special superconformal transformation. Therefore the special superconformal transformation in some sense probes the hidden fermionic dimensions that were gauge fixed to zero. This is similar to what happens in the purely bosonic theory with special conformal transformations. There the naive Lorentz boosts that mix $x^\mu$ with the extra dimensions $X^{\pm'}$ are compensated by the Sp(2, $R$) gauge transformations. The combined Lorentz boost and Sp(2, $R$) transformation is the special conformal transformation, so that it may be viewed as a hidden symmetry that probes the extra dimensions.

7 Superalgebra generators in superparticle gauge

The generators given by eqs. (21)-(23) $\Psi^{\alpha a}$, $J^{MN}$, $T^A$ are gauge invariant under both the local Sp(2, $R$) and kappa supersymmetry transformations and therefore they are observables. Together with the action, they can be evaluated in any fixed gauge simply by inserting any gauge fixed form of $X^M_i$, $\Theta^{\alpha a}$.

In the superparticle gauge the fermionic generators take the form (after simplifications that follow from the series expansion of the square root and $(\bar{\Theta}\Theta)^b_a = 0$)

$$\Psi^a = \frac{1}{2} L_{MN} \left( \Gamma^{MN} \Theta - \frac{1}{2} \Theta \bar{\Theta} \Gamma^{MN} \Theta \right)^a = \left( \frac{S^a}{Q^a} \right),$$  \hspace{1cm} (58)

$$S^a = \not{x} \not{\theta} \theta^a + \frac{1}{2} \theta^b (\bar{\theta}_b \not{x} \theta^a),$$  \hspace{1cm} (59)

$$Q^a = \sqrt{2} (\not{x} \theta^a)^\alpha.$$  \hspace{1cm} (60)

where the interpretation of the upper/lower components are

$$S^a = \text{special superconformal generator}$$  \hspace{1cm} (61)

$$Q^a = \text{global SUSY generator in d dimensions}$$  \hspace{1cm} (62)

The bosonic generators $J^{MN}$, $T^A$ in eq.(24-23) take the form (dilatations $D$, translations $P^\mu$, special conformal transformations $K^\mu$, Lorentz transformations $J^{\mu\nu}$, internal symmetries $T^A$)

$$D = J^{+'}_{-'} = L^{+'}_{-'} + 0,$$  \hspace{1cm} (63)

$$P^\mu = J^{+'}^\mu = L^{+'}^\mu + 0,$$  \hspace{1cm} (64)

$$K^\mu = J^{-'}^\mu = L^{-'}^\mu - \frac{1}{2} \left( \bar{\theta}_a \gamma^\mu \not{x} Q^a + h.c. \right) - \frac{1}{2 \sqrt{2}} \left( \bar{\theta}_a \gamma^\mu \theta^b \right) \left( \bar{\theta}_b \not{x} \theta^a \right).$$  \hspace{1cm} (65)
The explicit form of the purely bosonic part $L^M$ is given in this gauge by

$$L^+ = p^\mu, \quad L^\mu = x^\mu p^\nu - x^\nu p^\mu$$

(68)

$$L^- = \frac{1}{2} x^2 p^\mu - x \cdot p x^\mu, \quad L^{+\cdot-} = x \cdot p,$$

(69)

The fermionic part in $J^{\prime\mu}$ vanishes, thus yielding just the momentum $J^{\prime\mu} = p^\mu$. In addition, the fermionic part in $D$ vanishes due to $i\theta a \gamma^\mu \theta a = 0$ for $d = 3, 4, 6$.

The fermionic parts in $K^\mu, J^{\mu\nu}, T^A$ can be expressed in terms of $\theta$ by replacing $Q^a, S^a$ from (60,59). We then find that for $d = 3, 4$ our results agree with the formulas in [9] (3.20c, 3.20e for $d=3$) and (4.14c and 4.14g for $d=4$) after a Fierz rearrangement. Our compact expressions for $J^{MN}$ and $T^A$ also agree with the rest of the formulas (3.20 for $d=3$) and (4.14 for $d=4$) from [9]. On the other hand, our gauge fixed expressions provide new formulas for the $d = 6$ superparticle. This result can also be obtained with the methods of [9] [17].

### 8 Discussion and future directions

In this paper we have generalized the results of [1] to target space supersymmetry. We have constructed an Sp(2, $R$) gauge invariant particle action which possesses manifest space-time SO(d,2) symmetry, global supersymmetry and kappa supersymmetry. In particular, we have demonstrated that the superconformal groups $OSp(N/4)$, $SU(2, 2/N)$ and $OSp(8*/N)$ familiar in the context of $AdS/CFT$ duality framework can be given unified and explicit superparticle representations.

One way of gauge fixing produced the superparticle Lagrangian with superconformal symmetry, thus showing that the results of [1] appear only as a particular gauge in our framework. As pointed out in the introduction there are other possible gauge choices for our Lagrangian, which would provide explicit realizations of the various space-time superconformal symmetries, such as superparticles moving in the background of $AdS_D \times S^n$, supersymmetric H-atom, supersymmetric particle moving in the black hole background, etc., to name a few. It may be interesting to investigate the details of the theory in other gauges.

We think it would be very exciting to extend the same treatment for the case of superstrings (some results in the same framework have been obtained for various bosonic models [21]). It would be particularly exciting to see if various superstring theories can be indeed obtained as different gauge choices of the same theory.

Another possible direction for exploration is to consider superconformal algebras that do not have an obvious space-time interpretation, such as $OSp(1/32)$ [13], $OSp(16/2)$...
OSp(1/32) \times OSp(1/32) [14], OSp(1/64) [20], which have been argued from various points of view to be important in M-theory or S-theory. Our method can be generalized to construct Lagrangians that are invariant under these supergroups. For example, for OSp(1/32) denote the generators of OSp(1/32) as $L_{MN}, L_{M_{1}...M_{6}}$, and $\Psi_{\alpha}$ in the SO(10, 2) basis with parameters $l_{MN}, l_{M_{1}...M_{6}}$, and $\Theta^{\alpha}$ (32 component spinor). Consider the coset element $t(l_{+}, \Theta)$ in OSp(1/32)/SO(10, 2), where $l_{+}$ now represent extra bosonic degrees of freedom, and then repeat the steps of our construction to obtain $\Omega_{MN}(l_{+}, \Theta)$ and a Lagrangian invariant under global OSp(1/32) with transformations $t(l_{+}, \Theta) \rightarrow t(l_{+}', \Theta') = h^{-1} t(l_{+}, \Theta) g \, (70)$

Here $g \in OSp(1/32)$ global, and $h(l_{+}, \Theta; g) \in SO(10, 2)$ is the induced local Lorentz transformation which also transforms $X^{M}_{i}(\tau)$. There is also local symmetry that originates with transformations on the left side of $t(l_{+}, \Theta)$ with fermionic as well as bosonic parameters $b_{+M_{1}...M_{6}}(\tau), \kappa^{\alpha}(\tau)$ that form a generalization of kappa supersymmetry. The transformation is $t(l_{+}, \Theta) \rightarrow t(b_{+}', \Theta') = h^{-1} t(b_{+}, \kappa) \, t(l_{+}, \Theta) \, (71)$

where again $h(b_{+}, \kappa; l_{+}, \Theta)$ is the induced local Lorentz transformation. The dynamical degrees of freedom in this construction are $X^{M}_{i}(\tau), l_{+M_{1}...M_{6}}(\tau), \Theta^{\alpha}(\tau)$. The interpretation of the extra bosonic degrees of freedom $l_{+}$ is not obvious from the space-time point of view and we suspect that these are related to p-brane degrees of freedom. In any case the Lagrangian thus constructed may be an interesting toy model to explore these types of supersymmetries, involving two timelike dimensions that are covariantly included as part of the SO(10,2) particle position-momentum vectors $X^{M}(\tau), P^{M}(\tau)$. In addition to OSp(1/32), the supergroups not included in Nahm’s classification thus become relevant. We hope to return to these problems and give more details in the future [22].

9 Appendix: some details of the construction

Using (11) the left-hand side and the right-hand side of (13) become respectively

$$t(\Theta_{1}) \, t(\Theta_{2}) = \left( \begin{array}{ccc} \frac{1}{\sqrt{1+\Theta_{1} \bar{\Theta}_{1}}} (1 - \Theta_{1} \bar{\Theta}_{2}) \frac{1}{\sqrt{1+\Theta_{2} \bar{\Theta}_{2}}} & \frac{1}{\sqrt{1+\Theta_{1} \bar{\Theta}_{1}}} (\Theta_{1} + \Theta_{2}) \frac{1}{\sqrt{1+\Theta_{2} \bar{\Theta}_{2}}} \\ -\frac{1}{\sqrt{1+\Theta_{1} \bar{\Theta}_{1}}} (\bar{\Theta}_{1} + \bar{\Theta}_{2}) \frac{1}{\sqrt{1+\Theta_{2} \bar{\Theta}_{2}}} & \frac{1}{\sqrt{1+\Theta_{1} \bar{\Theta}_{1}}} (1 - \bar{\Theta}_{1} \Theta_{2}) \frac{1}{\sqrt{1+\Theta_{2} \bar{\Theta}_{2}}} \end{array} \right) \, (72)$$

and

$$h(\Theta_{1}, \Theta_{2}) \, t(\Theta_{12})$$
\[
\begin{pmatrix}
h_{12} & 0 \\
0 & \tilde{h}_{12}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{1+\Theta_{12}\tilde{\Theta}_{12}}} & \frac{1}{\sqrt{1+\Theta_{12}\tilde{\Theta}_{12}}} \\
-\frac{1}{\sqrt{1+\Theta_{12}\tilde{\Theta}_{12}}} & \frac{1}{\sqrt{1+\Theta_{12}\tilde{\Theta}_{12}}}
\end{pmatrix}
\]

(73)

By comparing these two equations we infer

\[
\Theta_{12} = \sqrt{1 + \tilde{\Theta}_{2}}(1 - \Theta_{1}\tilde{\Theta}_{2})^{-1}(\Theta_{1} + \Theta_{2})\frac{1}{\sqrt{1+\tilde{\Theta}_{2}\Theta_{2}}}
\]

(74)

\[
h_{12} = \frac{1}{\sqrt{1+\Theta_{1}\tilde{\Theta}_{1}}}(1 - \Theta_{1}\tilde{\Theta}_{2})\frac{1}{\sqrt{1+\tilde{\Theta}_{2}\Theta_{2}}}\frac{1}{\sqrt{1+\Theta_{12}\tilde{\Theta}_{12}}}
\]

(75)

\[
\tilde{h}_{12} = \frac{1}{\sqrt{1+\Theta_{1}\tilde{\Theta}_{1}}}(1 - \tilde{\Theta}_{1}\Theta_{2})\frac{1}{\sqrt{1+\tilde{\Theta}_{2}\Theta_{2}}}\frac{1}{\sqrt{1+\tilde{\Theta}_{12}\Theta_{12}}}
\]

(76)

Then we can specialize to the infinitesimal transformations (\(\Theta_{1} = \Theta, \Theta_{2} = \varepsilon, \) and small) and find

\[
\delta_{\varepsilon}\Theta \approx \varepsilon + \Theta\varepsilon\Theta,
\]

(77)

If we set (\(\Theta_{2} = \Theta, \Theta_{1} = \kappa, \) and small) we get

\[
\delta_{\kappa}\Theta = \sqrt{1 + \Theta\tilde{\Theta}\kappa}\sqrt{1 + \tilde{\Theta}\Theta}.
\]

(78)

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