Energy conditions in $f(T)$ gravity with higher-derivative torsion terms

Tahereh Azizi and Miysam Gorjizadeh

Department of Physics, Faculty of Basic Sciences,
University of Mazandaran,
P. O. Box 47416-95447, Babolsar, IRAN

Abstract
We study the energy conditions in the framework of the modified gravity with higher-derivative torsional terms in the action. We discuss the viability of the model by studying the energy conditions in terms of the cosmographical parameters like Hubble, deceleration, jerk, snap and lerk parameters. In particular, We consider two specific models that are proposed in literature and examine the viability bounds imposed by the weak energy condition.

Key Words: Teleparallel gravity, higher-derivative torsion terms, Energy conditions

1 Introduction
Recent astronomical observations have shown that the current universe experiences an accelerated expansion. To explain this unexpected phenomenon, two remarkable approaches have been suggested. In the first approach, an exotic matter component with negative pressure is considered in the right hand side of the Einstein equations dubbed as, dark energy in the literature. There are several candidates for the dark energy proposal such as the cosmological constant, canonical scalar field (quintessence), phantom field, chaplygin gas and so on (for a review see [8]). The second
approach is based on the modifying the left hand side of the field equations dubbed as, dark gravity. Some examples of the modified gravity models are \( f(R) \) theories, string inspired gravity, braneworld gravity, etc (for review see [9–11] and references therein). One of the interesting modified gravity model is the \( f(T) \) gravity which \( T \) is the torsion scalar. This scenario is based on the teleparallel equivalent of General Relativity (TEGR) [12, 13] which uses the Weitzenböck connection that has no curvature but only torsion. Note that the Lagrangian density of the Einstein gravity is constructed from the curvature defined via the Levi-Civita connection. In the context of the TEGR, the dynamical object is a vierbein field \( e_i(x^\mu), i = 0, 1, 2, 3 \), which form the orthogonal bases for the tangent space at each point of spacetime. The metric tensor is obtained from the dual vierbein as \( g_{\mu\nu}(x) = \eta_{ij}e^i_\mu(x)e^j_\nu(x) \) where \( \eta_{ij} = e_i,e_j \) is the Minkowski metric and \( e^i_\mu \) is the component of the vector \( e_i \) in a coordinate basis. Note that the Greek indices are refer to the coordinates on the manifold while Latin indices label the tangent space. The Lagrangian density of the teleparallel gravity is constructed from the torsion tensor which is defined as

\[
T^\lambda_{\mu\nu} = e^i_\lambda(\partial_\mu e^j_\nu - \partial_\nu e^j_\mu).
\]

One can write down the torsion scalar \( T \equiv S^\lambda_{\mu\nu}T^\lambda_{\nu\mu} \) where

\[
S^\lambda_{\mu\nu} = \frac{1}{2}(K^\mu_{\lambda\nu} + \delta^\mu_{\lambda}T^\sigma_{\nu} - \delta^\nu_{\lambda}T^\sigma_{\mu})
\]

and \( K^\mu_{\lambda\nu} \) is the contorsion tensor defined as

\[
K^\mu_{\lambda\nu} = -\frac{1}{2}(T^\mu_{\lambda\nu} - T^\nu_{\lambda\mu} - T^\lambda_{\mu\nu}).
\]

Using the torsion scalar as the teleparallel Lagrangian leads to the same gravitational equations of the general relativity.

Similar to the \( f(R) \) modified gravity, one can modify teleparallel gravity by considering an arbitrary function of the torsion scalar in the action of the theory, which leads to the \( f(T) \) theories of gravity [14–16]. It is worth noticing that the field equations of the \( f(T) \) gravity are second order differential equations and so it is more manageable compared to the \( f(R) \) theories whose the field equations are 4nd order equations. Consequently, the modified TEGR models have attracted a lot of interest in literature (see [17] and references therein ). Recently, a further modification of the teleparallel gravity has been proposed with constructing a torsional gravitational
modifications using higher-derivative terms such as \((\nabla T)^2\) and \(\Box T\) terms in the Lagrangian of the theory \([18]\). In this regard, the dynamical system of this model has been studied via performing the phase-space analysis of the cosmological scenario and consequently, an effective dark energy sector that comprises of the novel torsional contributions is obtained.

The aim of this paper is to explore the energy conditions of this modified teleparallel gravity by taking into account the further degrees of freedom related to the higher derivative terms. Indeed, one procedure for analyzing the viability of the modified gravity models is studying the energy conditions to constrain the free parameters of them. In this respect, one can impose the null, weak, dominant and strong energy conditions, which arise from the Raychaudhuri equation for the expansion \([19, 20]\), to the modified gravity model. In the literature, this approach has been extensively studied to evaluate the possible ranges of the free parameter of the generalized gravity models. For instance, the energy bound have been explored to constrain \(f(R)\) theories of gravity \([21, 24]\) and some extensions of \(f(R)\) gravity \([25, 33]\), modified Gauss-Bonnet gravity \([34, 37]\) and scalar-tensor gravity \([38, 39]\). The energy condition have also been analysed in \(f(T)\) gravity \([40, 42]\) and generalized models of \(f(T)\) gravity \([43, 46]\).

To study the energy conditions in this modified teleparallel gravity, we consider the flat FRW universe model with perfect fluid matter and define an effective energy density and pressure originates from the higher-derivative torsion terms. Then we discuss the energy conditions in term of the cosmographical parameters such as the Hubble, deceleration, jerk, snap and lerk parameters. Particularly, we consider two specific models that are proposed in literature and using the present-day values of the cosmographical parameters, we analyze the weak energy condition to determine the possible constraints on the free parameter of the presented models.

The paper is organized as follows: In section 2 we review the modified \(f(T)\) gravity with higher-derivative torsion terms, the equations of motion and the resulted modified Friedmann equations related to the model. Section 3 is devoted to the energy conditions in this modified teleparallel gravity. In section 4, we explore the weak energy condition in two specific models of the scenario by using present-day values of the cosmographic quantities. Finally, our conclusion will be appeared in section 4.
2 The field equations

The action of the modified teleparallel gravity with higher-derivative torsion terms is defined as \[18\]

\[
S = \frac{1}{2} \int d^4 x |e F(T, (\nabla T)^2, \Box T) + S_m(e^A_{\mu}, \Psi_m), \quad (1)
\]

where \( e = \det(e^A_{\mu}) = \sqrt{-g} \), \((\nabla T)^2 = \eta^{AB} e^A_{\mu} e^B_{\nu} \nabla_\mu T \nabla_\nu T = g^{\mu\nu} \nabla_\mu T \nabla_\nu T \) and \( \Box T = \eta^{AB} e^A_{\mu} e^B_{\nu} \nabla_\mu \nabla_\nu T = g^{\mu\nu} \nabla_\mu \nabla_\nu T \), and for simplicity we have set \( \kappa^2 = 8\pi G = 1 \). The \( S_m(e^A_{\mu}, \Psi_m) \) is the matter action includes the general matter field which can, in general, have an arbitrary coupling to the vierbein field. If the matter couples to the metric in the standard form, then varying the action (1) with respect to the vierbein yields the generalized field equations as follows \[18\]

\[
\frac{1}{e} \partial_\mu \left( e F_T e^A_{\mu} S^\rho_{\sigma} \right) - F_T e^A_{\nu} S^\mu\rho_{\sigma\nu} + \frac{1}{4} e^\rho_{\mu} F - \frac{1}{4} e^\rho_{\nu} F = \frac{1}{e} \partial_\lambda \partial_\mu \partial_\nu \left( e F_{X_2} \partial_\lambda e^A_{\rho} \right) - \frac{1}{4} \sum_{a=1}^{2} \left\{ F_{X_a} \frac{\partial X_a}{\partial e^A_{\rho}} - \frac{1}{e} \left[ \partial_\mu \left( e F_{X_a} \frac{\partial X_a}{\partial \partial_\mu e^A_{\rho}} \right) - \partial_\nu \left( e F_{X_a} \frac{\partial X_a}{\partial \partial_\nu e^A_{\rho}} \right) \right] \right\} = \frac{1}{2} e^A_{\mu} T^{(m)\rho}_{\sigma}, \quad (2)
\]

where for simplicity, we have used the notation \( X_1 \equiv (\nabla T)^2 \) and \( X_2 \equiv \Box T \). Note that \( F_T \) and \( F_{X_a} \) denote derivative with respect to the torsion scalar and \( X_a \), with \( a = 1, 2 \), respectively and \( T^{(m)\rho}_{\sigma} \) is the matter energy momentum tensor which is defined as \( e^A_{\mu} T^{(m)\rho}_{\sigma} \equiv -\frac{1}{e} \frac{\delta S_m}{\delta e^A_{\rho}} \). In order to study the cosmological implication of the model, we consider a spatially flat Friedmann-Robertson-Walker (FRW) universe with metric \( ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j \). This metric arises from the following diagonal vierbein

\[
e^A_{\mu} = \text{diag}(1, a(t), a(t), a(t)), \quad (3)
\]

where \( a(t) \) is the scale factor. Now we assume that the matter content of the universe is given by the perfect fluid with the energy density \( \rho_m \) and pressure \( p_m \). Thus using the field equations (2), we obtain the generalized Friedmann
equations as follows

\[ F_T H^2 + (24H^2 F_{X_1} + F_{X_2}) \left( 3H \dot{H} + \dot{H} \right) H + F_{X_2} \ddot{H}^2 + \left( 3H^2 - \dot{H} \right) H \dot{F}_{X_2} + 24H^3 \dot{H} \dot{F}_{X_1} + H^2 \ddot{F}_{X_2} + \frac{F}{12} = \frac{\rho_m}{6}, \] (4)

\[ F_T \dot{H} + H \ddot{F}_T + 24H \left[ 2H \ddot{H} + 3(\dot{H} + H^2) \dot{H} \right] \dot{F}_{X_1} + 12H \dot{H} \dot{F}_{X_2} + 24H^2 \dot{H} \ddot{F}_{X_1} + \left( \dot{H} + 3H^2 \right) \ddot{F}_{X_2} + 24H^2 \dot{H} \dot{F}_{X_1} + 24H^2 \dot{H} \dot{F}_{X_2} + 12 \dot{H} \ddot{F}_{X_1} \left( 12H^2 + \dot{H} \right) \] (5)

where dot denotes the derivative with respect to cosmic time and \( H = \frac{\dot{a}}{a} \) is the Hubble parameter. With the definition of the vierbien \( T \), the torsion scalar and the functions \( X_1 \) and \( X_2 \) respectively are given by

\[ T = -6H^2, \] (6)

\[ X_1 = 144H^2 \dot{H}^2, \] (7)

\[ X_2 = -12 \left[ \dot{H} (\dot{H} + 3H^2) + H \ddot{H} \right]. \] (8)

Substituting (6), (7) and (8) in the equations (4) and (5), we can rewrite the Friedmann equations in the usual form as general relativity

\[ 3H^2 = \rho_{de} + \rho_m, \] (9)

\[ -2\dot{H} = \rho_m + p_m + \rho_{de} + p_{de}, \] (10)

where the energy density and pressure of the effective dark energy sector are respectively defined as

\[ \rho_{de} \equiv -\frac{F}{2} - 6H^2 F_T + 3H^2 - 6F_{X_2} \dot{H}^2 - 6H (3H^2 - \dot{H}) \dot{F}_{X_2} - 144H^3 \dot{H} \dot{F}_{X_1} - 6H \left( F_{X_2} + 24H^2 F_{X_1} \right) \left( 3H \dot{H} + \dot{H} \right) - 6H^2 \ddot{F}_{X_2}, \] (11)

\[ p_{de} \equiv -3H^2 - 2\dot{H} + 2 \left[ F_T \dot{H} + H \ddot{F}_T + 24H \left( 2H \ddot{H} + 3(\dot{H} + H^2) \dot{H} \right) \dot{F}_{X_1} + 12H \dot{H} \ddot{F}_{X_2} + 24H^2 \dot{H} \ddot{F}_{X_1} + (\dot{H} + 3H^2) \ddot{F}_{X_2} + 24H^2 F_{X_1} \ddot{H} + H \ddot{F}_{X_2} + 24F_{X_1} \ddot{H}(12H^2 + \dot{H}) + 24HF_{X_1}(4\ddot{H} + 3H^2) \ddot{H} \right]. \] (12)
Since we have assumed a minimally coupled matter to the veirbien field, the standard matter is satisfied in continuity equation, i.e. $\dot{\rho}_m + 3H(p_m + \rho_m) = 0$. So from the friedmann equations (9) and (10), one can easily verified that the dark energy density and pressure satisfy the conservation equation

$$\dot{\rho}_{de} + 3H(\rho_{de} + p_{de}) = 0. \quad (13)$$

In the rest of this paper we analyze the viability of this modified teleparallel gravity scenario by studying the energy conditions to constrain the free parameters of the model.

### 3 Energy conditions

The energy conditions are originated from the Raychaudhuri equation together with the requirement that gravity is attractive for a space-time manifold which is endowed by a metric $g_{\mu\nu}$. In the case of a congruence of timelike and null geodesics with tangent vector field $u^\mu$ and $k^\mu$ respectively, the Raychaudhuri equation is given by the temporal variation of expansion for the respective curve as follows [19, 20, 24]

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu, \quad (14)$$

$$\frac{d\theta}{d\tau} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu. \quad (15)$$

Here $R_{\mu\nu}$ is the Ricci tensor and $\theta$, $\sigma^{\mu\nu}$ and $\omega_{\mu\nu}$ are, respectively, the expansion scalar, shear tensor and rotation tensor associated with the congruence of timelike or null geodesics. Note that the Raychaudhuri equation is a purely geometric equation hence, it makes no reference to a specific theory of gravitation. Since the shear is a purely spatial tensor ($\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$), for any hypersurface of orthogonal congruence ($\omega_{\mu\nu} = 0$), the conditions for attractive gravity reduces to

$$R_{\mu\nu}u^\mu u^\nu \geq 0, \quad R_{\mu\nu}k^\mu k^\nu \geq 0, \quad (16)$$

where the first condition is refer to the strong energy condition (SEC) and the second condition is named null energy condition (NEC). From the field equations in general relativity and its modifications, the Ricci tensor is related
to the energy-momentum tensor of the matter contents. Thus inequalities (16) give rise to the respective physical conditions on the energy-momentum tensor as

\[(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}) u^\mu u^\nu \geq 0, \quad (T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}) k^\mu k^\nu \geq 0,\]  

(17)

where \( T \) is the trace of the energy-momentum tensor. For perfect fluid with energy density \( \rho_m \) and pressure \( p_m \), the SEC and NEC are defined by \( \rho_m + 3p_m \geq 0 \) and \( \rho_m + p_m \geq 0 \) respectively, while the dominant energy condition (DEC) and weak energy condition (WEC) are defined respectively, by \( \rho_m \pm p_m \geq 0 \) and \( \rho \geq 0 \). Note that the violation of the NEC leads to the violation of all other conditions. Since the Raychaudhuri equation is a purely geometric equation, the concept of energy conditions can be extended to the case of modified theories of gravity with the assumption that the total matter contents of the universe act like a perfect fluid. Hence, the respective conditions can be defined by replacing the energy density and pressure with an effective energy density and effective pressure, respectively as follows

\[\text{NEC} : \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0,\]
\[\text{SEC} : \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \quad \rho_{\text{eff}} + 3p_{\text{eff}} \geq 0,\]
\[\text{DEC} : \quad \rho_{\text{eff}} \geq 0, \quad \rho_{\text{eff}} \pm p_{\text{eff}} \geq 0,\]
\[\text{WEC} : \quad \rho_{\text{eff}} \geq 0, \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0.\]  

(18)

Using Eqs. (9) and (10), the energy conditions in modified teleparallel gravity model with higher-derivative torsion terms are obtained as

\[\text{NEC} : \quad \rho_{\text{eff}} + p_{\text{eff}} = \rho_m + p_m - \frac{F}{2} - 6H^2F_T - 6F_X_2 \dot{H}^2 - 6HF_X_2(3H \dot{H} + \ddot{H}) + 30H \dot{H}F_X_2 - 432H^4F_X_1 \dot{H} - 18H^3F_X_2 - 2\dot{H} + 2 \left[ F_T \dot{H} + H \ddot{F}_T \right] + 24H^2 \dot{H}F_X_1 + \dot{H}F_X_2H \ddot{F}_X_2 + 24H(2H \dot{H} + 3\dot{H}^2)F_X_1 + 24H^2F_X_1 \ddot{H} + 96HF_X_1 \dot{H}^2 + 24F_X_1 \dot{H}^2(12H^2 + \dot{H}) \right] \geq 0,\]  

(19)

\[\text{WEC} : \quad \text{NEC},\]
\[\rho_{\text{eff}} = \rho_m - \frac{F}{2} + 3H^2 - 6H^2F_T - 6F_X_2 \dot{H}^2 - 6H(3H^2 - \dot{H})F_X_2 - 144H^3 \dot{H}F_X_1 - 6H^2F_X_2 - 6H(F_X_2 + 24H^2F_X_1)(3H \dot{H} + \ddot{H}) \geq 0,\]  

(20)
\[ \text{SEC : NEC,} \]
\[ \rho_{\text{eff}} + 3p_{\text{eff}} = \rho_m + 3p_m - \frac{F}{2} - 6H^2(1 + F_T) - 6F_{X_2}\dot{H}^2 - 6HF_{X_2}(3H\ddot{H} + \dot{H}) \]
\[ - 432H^4F_{X_1}\dot{H} + 288H^3\dot{H}F_{X_1} + 432H^3\dot{H}F_{X_1} + 78H\dot{H}\ddot{F}_{X_2} - 18H^3\dddot{F}_{X_2} \]
\[ - 6H^2\dddot{F}_{X_2} - 6\dddot{H} + 6\left[ F_T\dot{H} + H\dddot{F}_T + 24H(2H\dddot{H} + 3\dot{H}^2)\dddot{F}_{X_1} + 24H^2\dddot{H}\dddot{F}_{X_1} \right] \]
\[ + \dot{H}\dddot{F}_{X_2} + 24H^2F_{X_1}\dddot{H} + H\dddot{F}_{X_2} + 24F_{X_1}\dot{H}^2(12H^2 + \dot{H}) + 96HF_{X_1}\dddot{H}\dddot{H} \geq 0, \tag{21} \]

\[ \text{DEC : WEC,} \]
\[ \rho_{\text{eff}} - p_{\text{eff}} = \rho_m - p_m - \frac{F}{2} + 6H^2(1 - F_T) - 6F_{X_2}\dot{H}^2 - 6HF_{X_2}(3H\dddot{H} + \dot{H}) \]
\[ - 432H^4F_{X_1}\dot{H} - 288H^3F_{X_1}\dot{H} - 288H^3\dot{H}F_{X_1} - 6H(3H^2 - \dot{H})\dddot{F}_{X_2} + 2\dddot{H} \]
\[ - 2\left[ F_T\dot{H} + H\dddot{F}_T + 24H(2H\dddot{H} + 3\dot{H}^2)\dddot{F}_{X_1} - 6H^2\dddot{F}_{X_2} + 24H^2\dddot{H}\dddot{F}_{X_1} \right] \]
\[ + \dot{H}\dddot{F}_{X_2} + 24H^2F_{X_1}\dddot{H} + H\dddot{F}_{X_2} + 24F_{X_1}\dot{H}^2(12H^2 + \dot{H}) \]
\[ + 96HF_{X_1}\dddot{H}\dddot{H} \geq 0. \tag{22} \]

To get some insight on the meaning of the above energy conditions, in the next section, we consider two specific functions for the Lagrangian (II), to obtain the constraints on the parametric space of the model.

## 4 Constraints on specific Models

In order to analyze the torsional modified gravity model with higher-derivative terms from the point of view of energy conditions, we use the standard terminology in studying energy conditions for modified gravity theories. To this end, we investigate such energy bounds in terms of the cosmographic parameters \[47\], i.e. the Hubble, deceleration, jerk, snap and lerk parameters, defined respectively as

\[ q = -\frac{1}{H^2}\frac{a^{(2)}}{a}, \quad j = \frac{1}{H^3}\frac{a^{(3)}}{a}, \quad s = \frac{1}{H^4}\frac{a^{(4)}}{a}, \quad l = \frac{1}{H^5}\frac{a^{(5)}}{a}, \tag{23} \]

where the superscripts represent the derivative with respect to time. In terms of these parameters, the Hubble parameter as well as its higher time
derivatives are given by
\begin{align*}
H^{(1)} &= -H^2(1 + q), \\
H^{(2)} &= H^3(j + 3q + 2), \\
H^{(3)} &= H^4(s - 2j - 5q - 3), \\
H^{(4)} &= H^5[l - 5s + 10(q + 2)j + 30(q + 2)q + 24]
\end{align*}
respectively. The results of energy conditions in terms of cosmographic parameters for our modified \( f(T) \) gravity model can be achieved from the constraints (19)-(22).

4.1 Model I: \( F(T, X_1, X_2) = T + \alpha_1 X_1 + \alpha_2 e^{\delta X_1} \)

In this subsection, we adopt a specific function for the modified teleparallel Lagrangian \(^{(1)}\) as \(^{(18)}\)
\begin{equation}
F(T, X_1, X_2) = T + \frac{\alpha_1 X_1}{T^2} + \alpha_2 e^{\frac{\delta X_1}{T^4}},
\end{equation}
where \(\alpha_1, \alpha_2\) and \(\delta\) are constants. It has been shown that for a wide range of the model parameters the universe can result in a dark-energy dominated, accelerating universe and the model can describe the thermal history of the universe, i.e. the successive sequence of radiation, matter and dark energy epochs, which is a necessary requirement for any realistic scenario. Since for a theoretical model to be cosmologically viable, it should satisfy at least the weak energy condition, we examine specially the weak energy condition in our analysis. More ever, for simplicity we consider vacuum, i.e. \(\rho_m = p_m = 0\).

Inserting the cosmographical parameters \(^{(23)}\) in Equ. \(^{(20)}\), the bounds on the model parameters imposed by the weak energy condition are given by
\begin{align*}
\rho_{\text{eff}} &= -10\alpha_1 H_0^2(1 + q_0)^2 - \frac{\alpha_2}{2} e^{\delta(1 + q_0)^2} \left[1 + \frac{24\delta(1 + q_0)^2}{27H_0^2}\right] \\
- 9H_0^2(j_0 - 1) &\left[36\alpha_1 + \alpha_2 \delta e^{\frac{\delta(1 + q_0)^2}{9H_0^2}}\right] + 16H_0^2\alpha_1(1 + q_0)^2 \\
+ \left(\frac{\alpha_2\delta^2}{1164H_0^4} e^{\frac{\delta(1 + q_0)^2}{9H_0^2}}\right) &\left[-288H_0^7(1 + q_0)^2(j_0 + 3q_0 + 2) \\
- 288H_0^3(1 + q_0)^4 + \frac{8}{9} \frac{(1 + q_0)^4}{H_0} + \frac{72H_0^4}{\delta}(1 + q_0)\right] \geq 0,
\end{align*}
(26)
4.2 Model II: \( F(T, X_1, X_2) = T + \frac{\beta_1 X_2}{T} + \frac{\beta_2 X_2^2}{T^3} + \beta_3 e^{\frac{\sigma X_2}{T^4}} \)

In second case, we consider a class of models in which the action does not depend on \( X_1 \equiv (\nabla T)^2 \) but only on \( X_2 \equiv \Box T \) given by the following functional form [18]

\[
F(T, X_1, X_2) = T + \frac{\beta_1 X_2}{T} + \frac{\beta_2 X_2^2}{T^3} + \beta_3 e^{\frac{\sigma X_2}{T^4}},
\]  

where \( \beta_1, \beta_2, \beta_3 \) and \( \sigma \) are constants. It has been found that in this model the universe will be led to a dark energy dominated, accelerating phase for a wide region of the parameter space. The scale factor behaves asymptotically either as a power law or as an exponential law, while for large parameter regions the exact value of the dark-energy equation-of-state parameter can be in
great agreement with observations [18]. To examine the model via energy conditions, in a similar procedure to the previous subsection, we consider the vacuum. Using the cosmographical parameters as before, the condition to justification of the WEC are obtained as

$$\rho_{\text{eff}} = -\frac{\beta_3}{54H_0^2} e^{\frac{\sigma(\delta^2 + j_0^2 + 2\sigma^2)}{4\mu^2}} \left(27H_0^2 + 12\sigma^2q_0^4 + 36\sigma^2q_0^3 + 36\sigma^2q_0^2 + 4\sigma^2j_0 + 12\sigma^2q_0 + 2\sigma^2s_0 + 252\sigma H_0^2 + 9\sigma q_0^2 + 9\sigma j_0 + 18\sigma q_0 + 6\sigma^2j_0q_0^2 + 216\sigma H_0^2q_0^2 + 10\sigma^2j_0q_0 + 2\sigma^2q_0s_0 - 36\sigma H_0^2j_0 + 432\sigma H_0^2q_0^2 \right)$$

$$-\frac{1}{3} \left(-288\beta_2 H_0^4q_0^4 + 144\beta_1 H_0^4q_0^2 + 216\beta_1 H_0^4 - 288\beta_1 H_0^4q_0 - 912H_0^3q_0^2 \right)$$

$$- 144\beta_2 H_0^4j_0 - 384\beta_2 H_0^4q_0 - 48\beta_2 H_0^4s_0 - 864\beta_2 H_0^4q_0^3 + 48\beta_2 H_0^4j_0^2 + 9\beta_1 H_0^2q_0^2 + 9\beta_1 H_0^2j_0 + 18\beta_1 H_0^2q_0 - 7\beta_2 H_0^2q_0^4 - 28\beta_2 H_0^2q_0^3 - 7\beta_2 H_0^2j_0^2 - 28\beta_2 H_0^2q_0^2 - 96\beta_2 H_0^2j_0q_0 - 48\beta_2 H_0^2q_0s_0 - 72\beta_1 H_0^2j_0$$

$$- 14\beta_2 H_0^2j_0q_0^2 - 28\beta_2 H_0^2j_0q_0 - 144\beta_2 H_0^4j_0q_0 \right) \geq 0,$$

Figure 1: Plot of the weak energy condition for the specific form given by Equ. [25]. Plot (a) represents $\rho_{\text{eff}} \geq 0$ versus $\alpha_2$ and $\delta$ with $\alpha_1 = -2$. Plot (b) shows $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$ with the same parameters as plot (a). The positivity requirement of the weak energy condition is satisfied in the plots for the parameter range considered.
\( \rho_{\text{eff}} + p_{\text{eff}} = \frac{\beta_3}{1458H_0^4} e^{\frac{\sigma^3.q_0^7 + 180\sigma^3.q_0^6 + 414\sigma^2H_0^2q_0^5 - 729H_0^4}{18H_0^2}} \left( + 36\sigma^3.q_0^7 + 180\sigma^3.q_0^6 + 414\sigma^2H_0^2q_0^5 - 729H_0^4 \right) \)

\[ (30) \]

From Eqs. (29) and (30) it is clear that the WEC is dependent on a wide range of parameters, i.e. \( \beta_1, \beta_2, \beta_3 \) and \( \sigma \). So to analyse the energy conditions, we consider specific values for some of the parameters. In particular, we take \( \beta_1 = -4 \) and \( \beta_3 = 1 \) and plot the WEC as a function of \( \beta_2 \) and \( \sigma \). Figure 2 shows the behavior of the \( \rho_{\text{eff}} \) and \( \rho_{\text{eff}} + p_{\text{eff}} \) versus the free parameters \( \beta_2 \) and \( \sigma \). As the figure shows, the specific model II is consistent with the WEC inequalities in the subspaces of the model parametric space.
Figure 2: The plots depict the weak energy condition versus $\beta_2$ and $\sigma$ for the specific form $g$. Plot (a) corresponds to $\rho_{\text{eff}} \geq 0$; Plot (b) is depicted for $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$. We have considered the values $\beta_1 = -4$ and $\beta_3 = 1$.

5 Conclusion

The modified teleparallel gravity with higher-derivative torsion terms is a generalization of $f(T)$ gravity theories constructed from the higher-derivative, $(\nabla T)^2$ and $\Box T$ terms, i.e. theories that are characterized by the Lagrangian $F(T, (\nabla T)^2, \Box T)$. It has been shown that this model can result in an effective dark energy sector that comprises of the novel torsional contributions due to the higher-order derivative terms.

In this paper, we have developed some constraints on this generalized $f(T)$ model by examining the respective energy conditions. In this respect, we have written the WEC, DEC, NEC and SEC in terms of an effective energy-momentum tensor which arises from the further degrees of freedom related to the higher-derivative terms. In order to illustrate how these conditions can constrain the model, we have considered two specific functional forms of the Lagrangian that account for accelerating phase of the universe for a wide range of their parameter spaces.

Specially, we have examined the validity of WEC to obtain bounds on the free parameters of both models. Rephrasing the WEC inequalities in terms of the present-day values of the cosmographic parameters, we have found the constraints on the free parameter of the presented models. As a result, the WEC is fulfilled in both models in some subspaces of the model parametric
6 Acknowledgment

The authors would like to thank professor Kourosh Nozari for useful comments and discussions.

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