STOCHASTIC DIFFERENCE-OF-CONVEX ALGORITHMS FOR SOLVING NONCONVEX OPTIMIZATION PROBLEMS

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Abstract. The paper deals with stochastic difference-of-convex functions programs, that is, optimization problems whose cost function is a sum of a lower semicontinuous difference-of-convex function and the expectation of a stochastic difference-of-convex function with respect to a probability distribution. This class of nonsmooth and nonconvex stochastic optimization problems plays a central role in many practical applications. While in the literature there are many contributions dealing with convex and/or smooth stochastic optimizations problems, there is still a few algorithms dealing with nonconvex and nonsmooth programs. In deterministic optimization literature, the Difference-of-Convex functions Algorithm (DCA) is recognized to be one of a few algorithms to solve effectively nonconvex and nonsmooth optimization problems. The main purpose of this paper is to present some new stochastic variants of DCA for solving stochastic difference-of-convex functions programs. The convergence analysis of the proposed algorithms are carefully studied.

Key words. DC program, Stochastic DC program, Stochastic DC function, DCA, Stochastic DCA, subdifferential.

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1. Introduction. We consider the single stage stochastic optimization problems of the form:

\begin{equation}
\alpha = \inf \{ f(x) := \Phi(x) + r(x) : \ x \in \mathbb{R}^n \}.
\end{equation}

Here, \((\Omega, \Sigma_\Omega, \mathbb{P})\) is a complete probability space; \(\Phi : \mathbb{R}^n \to \mathbb{R}\) is the expectation of a stochastic loss function with respect to the probability distribution \(\mathbb{P}\):

\begin{equation}
\Phi(x) := \mathbb{E}_{s \sim \mathbb{P}} \varphi(x, s) = \int_{\Omega} \varphi(x, s) d\mathbb{P},
\end{equation}

and \(r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) is an extended real valued lower semicontinuous function. In general the probability distribution \(\mathbb{P}\) is unknown.

A particular case, when \(r\) is the indicator function of a closed convex set \(C \subseteq \mathbb{R}^n\), the problem reduces to the one of minimizing an expected loss function \(\Phi\) over a closed convex set:

\begin{equation}
\inf \{ \Phi(x) = \mathbb{E}_{s \sim \mathbb{P}} \varphi(x, s) : \ x \in C \}.
\end{equation}

Stochastic optimization problems play a key role in many fields of applied science: Statistics, signal processing, finance, machine learning, and data science,... (see e.g., \[3, 6, 19, 22, 48, 49, 58, 62\] and references given therein). Since the pioneering work by Robbin-Monro in 1951 (\[51\]) for solving stochastic programs with smooth and strongly convex data, a huge of publications on the methods for solving \(1.1\) have been produced in both theoretical aspects and applications. Generally, there are the two principal approaches to stochastic optimization problems, and some variants of combinations of these two approaches have been exploited:

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1. Approximate the stochastic loss function by a deterministic function in some appropriate stochastic ways, and investigate deterministic (or stochastic, as well) optimization methods for solving the resulting approximate deterministic optimization problem. Solutions of approximate problems found out are then regarded as approximate solutions of the original problem ([18, 23, 39, 58, 57, 59]). A popular approximation method is the Monte-Carlo sample average approximation, described briefly as follows. Let $s_1, s_2, ..., s_m$ are independent identically distributed realizations of the probability distribution $P$. The expected loss function is approximated by

$$\Phi(x) \approx \Phi_m(x) := \frac{1}{m} \sum_{i=1}^{m} \varphi(x, s_i),$$

and the approximate optimization problem is formulated as

$$\min \{ \Phi_m(x) + r(x) : x \in \mathbb{R}^n \}.$$ 

In this approach, for approximate problems, although the cost function is known, due to its complexity in nature when the sample size $m$ is large, deterministic approaches to (1.5) may be still prohibitively expensive, while some stochastic approaches may be more effective for solving it ([4, 35, 37, 55]).

2. Develop some stochastic versions of iterative approximation methods inspired from deterministic optimization for solving directly the original stochastic optimization problem. Among such methods, the stochastic subgradient-based method and the stochastic proximal (possibly subgradient-based combined) method (mainly for smooth or/and convex optimization problems) are the most attractive and widely used, see for instance, [5, 11, 19, 20, 21, 50, 56, 63, 67] and references therein. An advantage of these methods is that the computational cost per iteration is cheap, however, the practical convergence rates are relatively slow since the method variance is large. Recently, some variance reduction techniques have been made for these methods (e.g., [2, 37]).

In our knowledge, up to now almost stochastic optimization methods in both two approaches in the literature have been developed for essentially solving smooth and/or convex stochastic programs. There is still a few algorithms dealing with nonsmooth and nonconvex stochastic optimization problems. Among that algorithms, a stochastic generalized gradient method for nonsmooth nonconvex optimization with a special generalized differentiability has been presented in [16, 17]. Recently, some versions of the stochastic proximal subgradient-based method have been developed for solving weakly convex and composite convex optimization problems ([11, 14, 13]).

In this contribution, we are interested in Stochastic Difference-of-Convex functions (SDC in short) optimization problems, that is, the problem (1.1) with the functions $\varphi(\cdot, s)$ being given by:

$$\varphi(x, s) = g(x, s) - h(x, s), \quad (x, s) \in \mathbb{R}^n \times \Omega,$$

where $g(\cdot, s)$ and $h(\cdot, s)$ are convex functions, and $r$ is an extended real valued lower semicontinuous DC function. As $\varphi(\cdot, s)$ are Difference of Convex functions (DC), the expected loss $\Phi$ is DC too, and therefore the original problem (1.1) is naturally a
DC program. However, this expected loss function is usually either unknown since the probability distribution $P$ is unknown or too expensive for computations when working directly on it by deterministic optimization methods.

In the deterministic optimization literature, DC optimization problems appear in many practical situations, and DC programming plays a central role in nonconvex programming. The (deterministic) Difference-of-Convex Algorithm (DCA) was introduced in 1985 by Pham Dinh Tao [46] in the preliminary state and extensively developed throughout various joint works of Le Thi Hoai An and Pham Dinh Tao (see [29] and references therein) to become now classic and increasingly popular. The standard DCA solves a DC program of the form

$$f(x) := g(x) - h(x), \quad x \in \mathbb{R}^n,$$

where $g$ and $h$, called DC components of $f$, are convex functions on $\mathbb{R}^n$. The main idea of DCA is quite simple: at each iteration $k$, DCA approximates the second DC component $h(x)$ by its affine minorization $h_k(x) := h(x^k) + \langle x - x^k, y^k \rangle$, with $y^k \in \partial h(x^k)$, and minimizes the resulting convex function.

Nowadays it is recognized that (DCA) is one of a few algorithms to solve effectively nonconvex and nonsmooth programs, and there is a large range of applications of DCA in various fields of applied sciences. The DCA was successfully applied to a lot of different optimization problems, and many nonconvex programs to which it gave almost always global solutions and proved to be more robust and more efficient than related standard methods, especially in the large-scale setting. It is worth to notice that (see [29]) with appropriate DC decompositions, and suitably equivalent DC reformulations, DCA makes it possible to recover all (resp. most) standard methods in convex (resp. nonconvex) programming. For instance, the readers are referred to [25, 26, 27, 28, 32, 33, 34, 42, 43, 44], as well as [29] dealing with a survey on thirty years of developments of DC programming and DCA and references therein, and very recent papers (e.g. [1, 9, 36, 40, 41, 47]), for the nice properties of DC programming, DCA and their fruitful applications.

A natural question raised is how to construct the DC type algorithms in the stochastic setting? The first response to this question was stated in [35], where the authors have proposed a stochastic DC approach to the DC problems of the average approximation form (1.5). However, the stochastic DCA version proposed in [35] does not work longer for the general stochastic DC programs of the form (1.1). Very recently, a stochastic algorithm based on DCA in combining regularization, convexification, and sample average approximation for solving of two-stage stochastic programs with a linearly bi-parameterized recourse function defined by a convex quadratic program, has been proposed by the authors Liu, Cui, Pang and Sen in [36]. In [66], the authors proposed some stochastic algorithms for DC programs. Basing on a similar idea of deterministic DCA, say, iteratively replace $h$ by its convex majorization (but quadratic instead to affine majorization in DCA), and then solve the resulting convex program, the authors investigated a stochastic proximal subgradient type method for the convex subproblems. In fact, the "stochastic nature" of their algorithms is attached only on the solution method for convex subproblems.

This paper aims to develop stochastic DC algorithms for solving stochastic DC optimization problems of the form (1.1) with $\phi$ being given in (1.6). This class of problems is very broad to cover almost all stochastic programs appeared in practice, and this is the first time in the literature that such a common model is being considered. In fact, even if there is a few works studying (1.1) with $\phi(\cdot, s)$ being nonconvex
and nonsmooth, these works often assume that $r$ is convex. Basing on DCA, our main idea is to iteratively randomly approximate the second DC component of the objective function as well as its subgradient while maintaining (or also approximating) the first DC component. The major feature of our algorithms is completely different from \[66\] and others in the literature. We propose the following three variants of the Stochastic DC Algorithms (SCDA in short):

- SDCA with the storage of the past samples and subgradients;
- SDCA with the storage of the past samples but updating subgradients; and
- SDCA with the regularization technique.

For the first variant, we develop four algorithms. In the first (resp. the second) algorithm, we iteratively randomly approximate the second DC component of the objective function as well as its subgradient while maintaining (resp. approximating) the first DC component. To accelerate possibly the convergence rate of these algorithms we propose their two generalized versions. For the second variant, two versions are investigated which differs from one of other by the fact that the first DC component is approximated or not. In sum, seven algorithms are developed.

The paper is organized as follows. In Section 2, we recall some basic notions from Convex, Nonsmooth, and Variational Analysis which will be used in the subsequent sections. Furthermore, we give a brief presentation on DC programming and DCA. In Section 3, we present the SCDAs and the convergence results of these proposed algorithms. Some conclusion remarks and further researches are discussed in the final section.

2. Preliminaries.

2.1. Tools from Convex and Variational Analysis. Firstly we recall some notions from Convex Analysis and Nonsmooth Analysis, which will be needed thereafter (see, e.g., \[38\], \[53\], \[54\]). In the sequel, the space $\mathbb{R}^n$ is equipped with the canonical inner product $\langle \cdot \rangle$. Its dual space is identified with $\mathbb{R}^n$ itself. The open and closed balls with the center $x \in \mathbb{R}^n$ and radius $\varepsilon > 0$ are denoted, respectively, by $B(x, \varepsilon)$ and $B[x, \varepsilon]$, while the closed unit ball is denoted by $B[0]$. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called $\rho$-convex for some $\rho \geq 0$, if for all $x, y \in \mathbb{R}^n$, $\lambda \in [0,1]$ one has

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \frac{\rho}{2}\lambda(1 - \lambda)\|x - y\|^2.$$ 

The supremum of all $\rho \geq 0$ such that the above inequality is verified is called the convex modulus of $f$, which is denoted by $\rho(f)$.

The conjugate of a convex function $f$ is denoted $f^*$ and is defined by

$$f^*(y) := \sup\{\langle x, y \rangle - g(x) : x \in \mathbb{R}^n\}, \ y \in \mathbb{R}^n.$$ 

The effective domain of $f$, denoted $\text{Dom} f$, is given by $\text{Dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The subdifferential of a convex function $f \in \mathcal{S}(\mathbb{R}^n)$ at $x \in \text{Dom} f$ is defined by

$$\partial f(x) = \{x^* \in \mathbb{R}^n : \langle x^*, y - x \rangle \leq f(y) - f(x) \quad \forall y \in \mathbb{R}^n\}.$$ 

We set $\partial f(x) = \emptyset$ if $x \notin \text{Dom} f$. For a lower semicontinuous real extended valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the Fréchet subdifferential of $f$ at $x \in \text{Dom} f$ is defined by

$$\partial F f(x) = \left\{x^* \in \mathbb{R}^n : \liminf_{h \to 0} \frac{f(x + h) - f(x) - \langle x^*, h \rangle}{\|h\|} \geq 0 \right\}.$$
For $x \notin \text{Dom} \, f$, we set $\partial^F f(x) = \emptyset$. The \textit{limiting subdifferential} of $f$ at $x \in \mathbb{R}^n$ is

$$\partial f(x) = \{x^* \in \mathbb{R}^n : \exists (x_k, f(x_k)) \rightarrow (x, f(x)), \, x_k^* \in \partial^F f(x_k), \, (x_k^*) \rightarrow x^*\}.$$  

It is worth to mention that $\partial^F f(x)$ is not necessarily closed, although $\hat{\partial} f(x)$ is closed, for any $x \in \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is called a Fréchet (resp. limiting) critical point for the function $f$, if $0 \in \partial^F f(x)$ (resp. $0 \in \hat{\partial} f(x)$).

When $f$ is a convex function, then the Fréchet subdifferential and the limiting subdifferential coincide with the subdifferential in the sense of Convex Analysis. Moreover, if $f$ is a DC function, i.e., $f := g - h$, where $g, h$ are convex functions, then

$$\partial^F f(x) \subseteq \partial f(x) \subseteq \partial g(x) - \partial h(x),$$

wherever $h$ is continuous at $x$, especially, when $h$ is differentiable at $x$, then one has the equalities. The limiting subdifferential enjoys the following sum rule:

For two lower semicontinuous functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ such that either $f$ or $g$ is locally Lipchitz at $\bar{x} \in \text{Dom} \, f \cap \text{Dom} \, g$, one has

$$\partial (f + g)(\bar{x}) \subseteq \partial f(\bar{x}) + \partial g(\bar{x}).$$

Let us recall the well-known subdifferential characterizations of the $\rho$–convexity.

\textsc{Theorem 2.1.} Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. For $\rho \geq 0$, the following three statements are equivalent.

(i) $f$ is a $\rho$–convex function.

(ii) For all $x, y \in \mathbb{R}^n$, $x^* \in \partial^F f(x)$, one has

$$\langle x^*, y - x \rangle \leq f(y) - f(x) - \frac{\rho}{2} \|y - x\|^2.$$  

(iii) The sudifferential operator of $f$, $\partial^F f$, is a $\rho$–monotone operator: for all $x, y \in \mathbb{R}^n$, $x^* \in \partial^F f(x)$, $y^* \in \partial^F f(y)$,

$$\langle x^* - y^*, x - y \rangle \geq \rho \|x - y\|^2.$$  

\textsc{2.2. A brief presentation on DC programming and DCA.} Let $\Gamma_0(\mathbb{R}^n)$ denote the convex cone of all lower semicontinuous proper convex functions on $\mathbb{R}^n$. The vector space of DC functions is denoted by $DC(\mathbb{R}^n) = \Gamma_0(\mathbb{R}^n) - \Gamma_0(\mathbb{R}^n)$, that is quite large to contain almost real life objective functions and is closed under all the operations usually considered in Optimization (see, e.g., [27]). We consider now a standard DC program, that is, an optimization problem of the form:

$$\begin{align*}
(P) \quad \alpha &= \inf \{f(x) := g(x) - h(x) : \, x \in \mathbb{R}^n\},

d \quad h^*(y) - g^*(y) : \, y \in \mathbb{R}^n\}.
\end{align*}$$

where $g, h$ belong to $\Gamma_0(\mathbb{R}^n)$. Recall the natural convention $+\infty - (+\infty) = +\infty$ and the fact that $\alpha \in \mathbb{R}$ implies $\text{Dom} \, g \subset \text{Dom} \, h$, The dual problem of $(P)$ is defined by

$$\begin{align*}
(D) \quad \inf \{h^*(y) - g^*(y) : \, y \in \mathbb{R}^n\}.
\end{align*}$$

where $g^*, h^*$ are the conjugate functions of $g, h$, respectively. Due to the dual result by Toland ([24]), the optimal values of the primal and dual problems coincide and there is the perfect symmetry between primal and dual programs $(P)$ and $(D)$: the dual program to $(D)$ is exactly $(P)$. 

\textsc{Stochastic DC algorithms} 

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A point \( x^* \in \mathbb{R}^n \) is called a DC critical point of DC problem \((P)\) if \( 0 \in \partial g(x^*) - \partial h(x^*) \), or equivalently \( 0 \in \partial g(x^*) \cap \partial h(x^*) \). In the framework of DC programming, the terminology "critical point" is referred to the notion of DC criticality. As was mentioned in the previous subsection, when \( h \) is a differentiable convex function, all three notions of criticality, say DC, Fréchet, limiting, coincide.

For a DC optimization problem with a closed convex set constraint:

\[
\inf \{ f(x) : x \in C \},
\]

where \( f \) is a DC function and \( C \subseteq \mathbb{R}^n \) is a nonempty convex set, we can equivalently transform it into a standard DC program by using the indicator function of \( C \) as follows.

\[
\inf \{ f(x) + \chi_C(x) : x \in \mathbb{R}^n \},
\]

where, \( \chi_C \) stands for the indicator function of \( C \), that is, \( \chi_C(x) = 0 \) if \( x \in C \), \(+\infty\), otherwise. For general DC programs with equality/inequality constraints defined by DC functions, some penalty techniques have been used to transform them to standard DC programs (see \([20, 31]\)).

The DC algorithm (DCA) which is based on local optimality and DC duality, consists in the construction of the two sequences \( \{x^k\} \) and \( \{y^k\} \) (candidates for being primal and dual solutions, respectively) such that the sequences of values of the primal and dual objective functions \( \{g(x^k) - h(x^k)\} \), \( \{h^*(y^k) - g^*(y^k)\} \) are decreasing, and their corresponding limits \( x^\infty \) and \( y^\infty \) satisfy local optimality conditions (see, e.g., \([20, 27, 44, 45]\)). Briefly, the standard DC Algorithm (DCA) is described as follows. Starting a given \( x^0 \in \text{Dom} \ g \), and for \( k = 0, 1, \ldots \), set

\[
\text{(DCA) } y^k \in \partial h(x^k); \quad x^{k+1} = \arg\min \{ g(x) - (y^k, x) : x \in \mathbb{R}^n \}.
\]

3. Stochastic DC Algorithms and convergence analysis. Let \((\Omega, \Sigma_\Omega, \mathbb{P})\) is a probability space. Consider the stochastic DC program:

\[
\alpha = \min \{ f(x) := \Phi(x) + r(x) : x \in \mathbb{R}^n \},
\]

where, \( r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is a lower semicontinuous DC function, namely

\[
r(x) := r_1(x) - r_2(x), \quad x \in \mathbb{R}^n,
\]

where \( r_1, r_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) are lower semicontinuous convex functions, and the expected loss function

\[
\Phi(x) = \mathbb{E}_\alpha[\varphi(x, s) = g(x, s) - h(x, s)] = \int_\Omega [g(x, s) - h(x, s)] d\mathbb{P},
\]

with respect to continuous convex functions \( g(\cdot, s) \), \( h(\cdot, s) \), \( s \in \Omega \), defined on \( \mathbb{R}^n \). Throughout the paper, we assume that the expectations of \( g(x, \cdot) \) and \( h(x, \cdot) \) are finite for all \( x \in \mathbb{R}^n \), and denote by

\[
G(x) := \mathbb{E}_\alpha[g(x, s)] = \int_\Omega g(x, s) d\mathbb{P}, \quad H(x) := \mathbb{E}_\alpha[h(x, s)] = \int_\Omega h(x, s) d\mathbb{P}.
\]

Then, \( G, H \) are continuous convex functions on the whole space \( \mathbb{R}^n \), and therefore the objective function of problem \((3.1)\) admits a DC decomposition: \( f = (G + r_1) - (H + r_2) \). In what follows we will make use of the following assumptions:
(A1) For each \( x \in \mathbb{R}^n \), the function \( h(x, \cdot) \) is bounded on \( \Omega \) and the functions \( h(\cdot, s) \) are uniformly continuous at each \( x \in \mathbb{R}^n \), for \( s \in \Omega \).

(A2) For each \( x \in \mathbb{R}^n \), the functions \( g(x, \cdot), h(x, \cdot) \) are bounded on \( \Omega \) and the functions \( g(\cdot, s), h(\cdot, s) \) are uniformly continuous at each \( x \in \mathbb{R}^n \), for \( s \in \Omega \).

Recall that a critical point \( x^* \in \mathbb{R}^n \) of problem (3.3) is such that

\[
0 \in \partial(G + r_1)(x^*) - \partial(H + r_2)(x^*) = \partial G(x^*) + \partial r_1(x^*) - \partial H(x^*) - \partial r_2(x^*).
\]

So for \( x^* \in \mathbb{R}^n \), the distance

\[
d(0, \partial G(x^*) + \partial r_1(x^*) - \partial H(x^*) - \partial r_2(x^*)) = \inf \{ \| y \| : y \in \partial G(x^*) + \partial r_1(x^*) - \partial H(x^*) - \partial r_2(x^*) \},
\]

serves as a measure of "proximity to criticality".

Many practical optimization models in various fields of science and engineering can be formulated as stochastic difference-of-convex optimization problems. Note that this class of programs (SDC) contains convex-composite and weakly convex optimization problems. For the sake of illustration, let us give just an example of the following robust real phase retrieval problem (see e.g., [7, 11, 14]):

\[
\min \{ \mathbb{E}_{a, b} |\langle a, x \rangle|^2 - b : x \in \mathbb{R}^n \},
\]

where, \( a \in \mathbb{R}^n \), \( b \in \mathbb{R} \) are independent random variables with given probability distributions. Usually, \( a \) is a standard Gaussian random vector in \( \mathbb{R}^n \), and \( b \) is defined by \( b = \langle a, x \rangle^2 + \eta \), with a noise \( \eta \). Obviously, the functions \( \varphi(\cdot, a, b) := |\langle \cdot, a \rangle|^2 - b \) are DC functions, therefore problem (3.5) belongs to the class of stochastic DC programs. In a particular case when \( a, b \) are random variables of the uniform distribution on a finite set of \( m \) elements with the values respectively \( \{ a_1, ..., a_m \} \) and \( \{ b_1, ..., b_m \} \), problem (3.5) reduces to the following optimization problem

\[
\min \left\{ f(x) = \frac{1}{m} \sum_{i=1}^{m} |\langle a_i, x \rangle|^2 - b_i : x \in \mathbb{R}^n \right\}.
\]

Problem (3.8) can be regarded as an approximate problem of (3.5), when \( a_i \)'s and \( b_i \)'s are realizations respectively of \( a \) and \( b \). This latter problem is equivalent to the following: Find \( x \in \mathbb{R}^n \) such that \( b_i = \langle a_i, x \rangle^2, \ i = 1, 2, ..., m \). By noticing the functions \( |\langle a_i, x \rangle|^2 - b_i \) are convex functions, for each \( i = 1, ..., m \), the function \( |\langle a_i, x \rangle|^2 - b_i \) admits a DC decomposition as follows.

\[
|\langle a_i, x \rangle|^2 - b_i = \max \{ \langle a_i, x \rangle^2 - b_i, 0 \} - 2\langle a_i, x \rangle^2 - b_i := g_i(x) - h_i(x), \ x \in \mathbb{R}^n,
\]

where, for \( i = 1, ..., m \),

\[
g_i(x) := \max \{ \langle a_i, x \rangle^2 - b_i, 0 \}; \ h_i(x) := 2\langle a_i, x \rangle^2 - b_i, \ x \in \mathbb{R}^n.
\]

Then, a DC decomposition of the objective function is

\[
f(x) = \frac{1}{m} \sum_{i=1}^{m} g_i(x) - \frac{1}{m} \sum_{i=1}^{m} h_i(x) := G(x) - H(x).
\]

We are now going to present the proposed Stochastic DC Algorithms.
3.1. Algorithms with the storage the past samples and subgradients.
Firstly we propose the following two algorithms in which at each iteration, the realized samples as well as subgradients from the past iterations are inherited. It is worth to notice that in these first algorithms, at each iterations, we have to compute only one subgradient of the function $h(\cdot, s)$ with respect to one current realization of $s$.

Algorithm 1: Stochastic DC Algorithm 1 (SDCA1)

Initialization: Initial data: $x^0 \in \text{Dom } r_1$, draw $s^0 \overset{iid}{\sim} P$. Set $k = 0$.

Repeat: For $k = 0, 1, \ldots$, draw $s^k \overset{iid}{\sim} P$, which is independent of the past.
1. Compute $z^k \in \partial h(x^k, s^k); u^k \in \partial r_2(x^k)$.
2. Set $y^k = \frac{1}{k+1} \sum_{i=0}^{k} z^i = \frac{k u^{k-1} + y^k}{k+1}; w^k = \frac{1}{k+1} \sum_{i=0}^{k} u^i = \frac{k w^{k-1} + w^k}{k+1}$.
3. Compute a solution $x^{k+1}$ of the convex program

$$
\min \{ G(x) + r_1(x) - \langle y^k + w^k, x \rangle : x \in \mathbb{R}^n \}.
$$

4. Set $k := k + 1$ and draw $s^{k+1} \overset{iid}{\sim} P$.
Until Stopping criterion.

In the second algorithm, the first DC component of the expected loss function is also randomly approximated at each iteration.

Algorithm 2: Stochastic DC Algorithm 2 (SDCA2)

Initialization: Initial data: $x^0 \in \text{Dom } r_1$, draw $s^0 \overset{iid}{\sim} P$. Set $k = 0$.

Repeat: For $k = 0, 1, \ldots$, draw $s^k \overset{iid}{\sim} P$, which is independent of the past.
1. Compute $z^k \in \partial h(x^k, s^k); u^k \in \partial r_2(x^k)$.
2. Set $y^k = \frac{1}{k+1} \sum_{i=0}^{k} z^i = \frac{k u^{k-1} + y^k}{k+1}; w^k = \frac{1}{k+1} \sum_{i=0}^{k} u^i = \frac{k w^{k-1} + w^k}{k+1}$.
3. Compute a solution $x^{k+1}$ of the convex program

$$
\min \left\{ \frac{1}{k+1} \sum_{i=0}^{k} g(x, s^i) + r_1(x) - \langle y^k + w^k, x \rangle : x \in \mathbb{R}^n \right\}.
$$

4. Set $k := k + 1$ and draw $s^{k+1} \overset{iid}{\sim} P$.
Until Stopping criterion.

The convergence of the two algorithms is given in the following theorem. For these two algorithms, we establish the almost sure convergence of a subsequence to a critical point. The convergence rate will be given by means of the following measure of proximity to criticality:

$$
d_k := \min_{i=0, \ldots, k} \mathbb{E} d(0, \partial G(x^i) + \partial r_1(x^i) - \partial H(x^i) - \partial r_2(x^i)), \ k \in \mathbb{N},
$$

where the notation $d(x, A)$, for a point $x \in \mathbb{R}^n$, and a subset $A \subseteq \mathbb{R}^n$, stands for the distance from $x$ to $A$.

**Theorem 3.1.** Suppose that for Algorithm 1, assumption (A1) holds, and for Algorithm 2, (A2) holds. Assume that $\rho := \inf_{s \in \Omega} \rho(h(\cdot, s)) + \rho(r_2) > 0$. Let $\{x^k\}$ be
a sequence generated by either Algorithm 1 or Algorithm 2. Suppose that the optimal value \( \alpha \) of problem (3.7) is finite and with probability 1, \( \lim \sup_{k \to \infty} \| x^k \| < \infty \) and \( \lim \sup_{k \to \infty} \| u^k \| < \infty \). Then one has

(i) There exists a subsequence \( \{ x^k \} \) converging almost surely to a critical point of problem (3.7).

(ii) Assume further that, \( r_2 \) is differentiable with locally Lipschitz derivative, and that for Algorithm 1, \( h(\cdot, s) \), \( s \in \Omega \) is differentiable such that for each \( x \in \mathbb{R}^n \), \( \nabla h(x, \cdot) \) is bounded and the derivatives \( \nabla h(\cdot, s) \) is uniformly locally Lipschitz; for Algorithm 2, both \( g(\cdot, s) \), \( h(\cdot, s) \) are differentiable such that their derivatives are uniformly locally Lipschitz and \( \nabla g(x, \cdot) \), \( \nabla h(x, \cdot) \) are bounded for each \( x \in \mathbb{R}^n \). Then one has \( d_k = O(1/\sqrt{\ln k}) \) as \( k \to \infty \).

Proof. We consider the case where the sequence \( \{ x^k \} \) is generated by Algorithm 1. Denote by \( F_k = \sigma(x^0, s^0, \ldots, s^{k-1}) \), \( k \in \mathbb{N} \), the increasing \( \sigma \)–field generated by random variables \( x^0, s^0, \ldots, s^{k-1} \). Define the functions \( V_k : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \}, k \in \mathbb{N} \) by

\[
V_k(x) = G(x) + r_1(x) - \frac{1}{k+1} \sum_{i=0}^{k} (z^i + u^i, x - x^i) - \frac{1}{k+1} \sum_{i=0}^{k} [h(x^i, s^i) + r_2(x^i)].
\]

As \( x^{k+1} \) is a solution of the problem (3.9), one has

\[
(3.11) \quad V_k(x^{k+1}) \leq V_k(x^k) = G(x^k) + r_1(x^k) - \frac{1}{k+1} \sum_{i=0}^{k} (z^i + u^i, x^k - x^i) - \frac{1}{k+1} \sum_{i=0}^{k} h(x^i, s^i) + r_2(x^i).
\]

Next, one has the following estimate, due to the strong convexity of the function \( h(\cdot, s) \) and \( r_2 \), \( s \in \Omega \) with modulus at least \( \rho \), Theorem 2.1 implies

\[
(3.12) \quad V_{k-1}(x^k) - V_k(x^k) = \frac{1}{2(k+1)} h(x^k, s^k) + r_2(x^k) - \frac{1}{k+1} \sum_{i=0}^{k-1} \left( (z^i + u^i, x^k - x^i) + h(x^i, s^i) + r_2(x^i) \right) \geq \frac{1}{2(k+1)} \left( h(x^k, s^k) - \frac{1}{\rho} \sum_{i=0}^{k-1} h(x^i, s^i) \right) + \frac{\rho}{2k(k+1)} \sum_{i=0}^{k-1} \| x^k - x^i \|^2.
\]

From the preceding two inequalities, one has

\[
(3.13) \quad V_k(x^{k+1}) \leq V_{k-1}(x^k) - (V_{k-1}(x^k) - V_k(x^k)) \leq V_{k-1}(x^k) - \frac{1}{k+1} \left( h(x^k, s^k) - \frac{1}{\rho} \sum_{i=0}^{k-1} h(x^i, s^i) \right) - \frac{\rho}{2k(k+1)} \sum_{i=0}^{k-1} \| x^k - x^i \|^2.
\]

For \( x \in \mathbb{R}^n, k \in \mathbb{N} \), define \( Z_k(x) = H(x) - \frac{1}{\rho} \sum_{i=0}^{k-1} h(x, s^i) \). Taking expectations with respect to \( F_k \) on both sides of the inequality above, one obtains

\[
(3.14) \quad \mathbb{E}_{F_k} V_k(x^{k+1}) \leq V_{k-1}(x^k) - \xi_k - \frac{\rho}{2k(k+1)} \sum_{i=0}^{k-1} \| x^k - x^i \|^2,
\]

where \( \xi_k := \mathbb{E}_k Z_k(x^k) / (k+1) \). Since \( \{ x^k \} \) is assumed to be bounded almost everywhere (a.e.), we can assume that there is a compact set \( B \subseteq \mathbb{R}^n \) such that \( x^k \in B \) for all \( k \in \mathbb{N} \). By assumption (A1), \( h \) is bounded on \( B \times \Omega \). Moreover, as \( h(\cdot, s) \) are convex functions, the uniform continuity implies the uniformly Lipschitz property of \( h(\cdot, s) \) on the compact set \( B \), with a Lipschitz constant \( L \). In view of the uniform law of large numbers in mean (Lemma B.2) (see also, \([6], ([20])\), there is some \( c > 0 \) such that

\[
(3.15) \quad \mathbb{E} |\xi_k| = \frac{1}{k+1} \mathbb{E} \left| \sum_{i=0}^{k-1} |Z_i(x)| \right| \leq \mathbb{E} \frac{1}{k+1} \sup_{x \in B} |Z_k(x)| \leq \frac{c(1 + \sqrt{\ln k})}{\sqrt{k(k+1)}} \quad \text{for all } k \in \mathbb{N}.
\]
Thus $\sum_{k=1}^{\infty} \mathbb{E}|\xi_k| < +\infty$, and from relation (3.14), thanks to the supermartingale convergence (11), (52), almost surely the sequence $\{V_k(x^{k+1})\}$ converges and

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=0}^{k-1} \|x^k - x^i\|^2 < +\infty.$$  

Hence, there is a subsequence of $\delta_k := \frac{1}{k} \sum_{i=0}^{k-1} \|x^k - x^i\|^2$, converging almost surely to 0. Next, by picking a subsequence and relabeling if necessary, without loss of generality, we can assume that $\delta_k \to 0$. By the Cauchy inequality,

$$\left( \frac{1}{k} \sum_{i=0}^{k-1} \|x^k - x^i\| \right)^2 \leq \frac{1}{k} \sum_{i=0}^{k-1} \|x^k - x^i\|^2,$$

which implies $\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \|x^k - x^i\| = 0$. Let $x^* \in \mathbb{R}^n$ be a limit point of the sequence $\{x^k\}$, say, there is a subsequence $\{x^{l_k}\}$ converges to $x^*$. The preceding relation yields immediately

$$\text{a.s., } \lim_{k \to \infty} \frac{1}{l_k} \sum_{i=0}^{l_k-1} x^i = \lim_{k \to \infty} x^{l_k} = x^*.$$

By the uniformly Lipschitzian property of $h(\cdot, s)$ with a Lipschitz constant $L$ on $B$, $\|z^i\| \leq L$, $(i = 0, 1, \ldots)$ and $H$ is also Lipschitz on $B$ with the same constant $L$. Therefore one has

$$\frac{1}{l_k} \left| \sum_{i=0}^{l_k-1} \langle z^i, x^{l_k} - x^i \rangle \right| \leq \frac{L}{l_k} \sum_{i=0}^{l_k-1} \|x^{l_k} - x^i\|,$$

and

$$\left| \frac{1}{l_k} \sum_{i=0}^{l_k-1} h(x^i, s^i) - H(x^{l_k}) \right| \leq \left| \frac{1}{l_k} \sum_{i=0}^{l_k-1} h(x^*, s^i) - H(x^*) \right| + \frac{L}{l_k} \sum_{i=0}^{l_k-1} \|x^* - x^i\|.$$ 

As $\lim_{k \to \infty} \|x^{l_k} - x^*\| = \lim_{k \to \infty} \frac{1}{l_k} \sum_{i=0}^{l_k-1} \|x^{l_k} - x^i\| = 0$, and by the strong law of large numbers,

$$\lim_{k \to \infty} \left[ \frac{1}{l_k} \sum_{i=0}^{l_k-1} h(x^*, s^i) - H(x^*) \right] = 0 \text{ a.s.,}$$

one has almost surely,

$$\lim_{k \to \infty} \frac{1}{l_k} \sum_{i=0}^{l_k-1} \langle z^i, x^{l_k} - x^i \rangle = \lim_{k \to \infty} \left[ \frac{1}{l_k} \sum_{i=0}^{l_k-1} h(x^i, s^i) - H(x^{l_k}) \right] = 0;$$

By passing to a subsequence if necessary, assume that

$$\lim_{k \to \infty} y^{l_k} = \lim_{k \to \infty} \frac{1}{l_k} \sum_{i=0}^{l_k-1} z^i = y^*, \text{ and } \lim_{k \to \infty} w^{l_k} = \lim_{k \to \infty} \frac{1}{l_k} \sum_{i=0}^{l_k-1} u^i = w^*.$$
Since \( y^{k-1} + w^{k-1} \in \partial G(x^{k}) + \partial r_{1}(x^{k}) \), passing to the limit, one obtains \( y^{*} + w^{*} \in \partial G(x^{*}) + \partial r(x^{*}). \) Next, \( w^{k-1} \in \frac{1}{l_{k}} \sum_{i=0}^{l_{k}-1} \partial r_{2}(x^{i}) \) and the Jensen inequality implies, for some \( M > 0 \) such that \( \|u^{k}\| \leq M \), for all \( k \in \mathbb{N} \),

\[
\langle u^{l-1}, x - x^{l-1} \rangle = \frac{1}{l} \sum_{i=0}^{l-1} \langle u^{i}, x - x^{i} \rangle + \frac{1}{l} \sum_{i=0}^{l-1} \langle u^{i}, x^{i} - x \rangle \leq r_{2}(x) - \frac{1}{l} \left( \sum_{i=0}^{l-1} r_{2}(x^{i}) + \frac{l}{l} \sum_{i=0}^{l-1} \|u^{i}\| \|x^{i} - x\| \right) \leq r_{2}(x) - r_{2} \left( \frac{1}{l} \sum_{i=0}^{l-1} x^{i} \right) + \frac{M}{l} \sum_{i=0}^{l-1} \|x^{i} - x\|. \forall x \in \mathbb{R}^{n}.
\]

Passing to the limit as \( k \to \infty \), by \( \frac{1}{l_{k}} \sum_{i=0}^{l_{k}-1} x^{i} \to x^{*} \) as well as \( \frac{M}{l_{k}} \sum_{i=0}^{l_{k}-1} \|x^{i} - x\| \to 0 \), one derives \( w^{*} \in \partial r_{2}(x^{*}). \) On the other hand, \( y^{k-1} \in \frac{1}{l_{k}} \sum_{i=0}^{l_{k}-1} \partial h(x^{i}, s^{i}) \), then

\[
\langle y^{l-1}, x - x^{l} \rangle = \frac{1}{l} \sum_{i=0}^{l-1} \langle y^{i}, x - x^{i} \rangle + \frac{l}{l} \sum_{i=0}^{l-1} \langle y^{i}, x^{i} - x \rangle \leq \frac{1}{l} \left( \sum_{i=0}^{l-1} h(x^{i}, s^{i}) + \frac{1}{l} \sum_{i=0}^{l-1} h(x^{i}, s^{i}) \right) + \frac{M}{l} \sum_{i=0}^{l-1} \|x^{i} - x\| \quad \forall x \in \mathbb{R}^{n}.
\]

Noticing that \( \frac{1}{l} \sum_{i=0}^{l-1} \|x^{i} - x\| \to 0 \), and by (3.17), \( \frac{1}{l} \sum_{i=0}^{l-1} h(x^{i}, s^{i}) \to H(x^{*}) \), almost surely, so to finish the proof of part (i), we need to show that, for any \( \delta > 0 \), almost surely, for all \( x \in B[x^{*}, \delta] \), one has

\[
\sigma_{k}(x) := \frac{1}{l} \sum_{i=0}^{l-1} h(x^{i}, s^{i}) \to H(x).
\]

Indeed, let \( \mathbb{Q}^{n} \subseteq \mathbb{R}^{n} \) the set of points with all rational coordinates. Then \( \mathbb{Q}^{n} \cap B[x^{*}, \delta] \) is a countable set which is dense in \( B[x^{*}, \delta] \). Denote \( \mathbb{Q}^{n} \cap B[x^{*}, \delta] := \{ z^{1}, z^{2}, ..., z^{k}, ... \} \), then for each \( l = 1, 2, ..., \mathbb{P}\{ \sigma_{k}(z^{l}) \to H(z^{l}) \} = 0 \). Hence, denoting by \( S \) the event

\[
S := \bigcap_{l=1}^{\infty} \{ \sigma_{k}(z^{l}) \to H(z^{l}) \},
\]

one has

\[
1 \geq \mathbb{P}(S) := \mathbb{P} \left( \bigcap_{l=1}^{\infty} \{ \sigma_{k}(z^{l}) \to H(z^{l}) \} \right) \geq 1 - \sum_{l=1}^{\infty} \mathbb{P}(\{ \sigma_{k}(z^{l}) \nrightarrow H(z^{l}) \}) = 1,
\]

so \( \mathbb{P}(S) = 1 \). Let \( x \in B[x^{*}, \delta] \) be given. Let \( \{ \varepsilon_{l}\}_{l \in \mathbb{N}} \) be a sequence of positive reals converging to 0. Then there is a subsequence \( \{ z^{k_{l}} \} \subseteq \mathbb{Q}^{n} \cap B[x^{*}, \delta] \) such that \( |H(x) - H(z^{k_{l}})| < \varepsilon_{l} \) as well as \( |h(x, s) - h(z^{k_{l}}, s)| < \varepsilon_{l} \) for all \( l \in \mathbb{N} \), all \( s \in \mathbb{S} \). For \( \{ z^{k_{l}} \}_{k_{l} \in \mathbb{N}} \in S \), and for \( l \in \mathbb{N} \), since \( \sigma_{k_{l}}(z^{k_{l}}) \to h(z^{k_{l}}) \), there is an index \( K_{l} \) such that \( |\sigma_{k_{l}}(z^{l}) - h(z^{l})| < \varepsilon_{l} \) for all \( k_{l} \geq K_{l} \), hence,

\[
|\sigma_{k}(x) - H(x)| \leq |\sigma_{k}(z^{l}) - H(z^{l})| + |\sigma_{k}(x) - \sigma_{k}(z^{l})| + |H(x) - H(z^{l})| < 3\varepsilon_{l}, \quad \text{for all} \ k \geq K_{l},
\]

which shows that \( \sigma_{k}(x) \to H(x) \) on \( S \). Now, by letting \( k \to \infty \) in relation (3.18), one obtains \( y^{*} \in \partial H(x^{*}) \), consequently \( y^{*} + v^{*} \in \partial G(x^{*}) + \partial r_{1}(x^{*}) \cap \partial H(x^{*}) + \partial r_{2}(x^{*}) \). Thus, \( x^{*} \) is a critical point of problem (3.1).

We now prove the part (ii). Assume the derivative \( \nabla r_{2} \) and the derivatives \( \nabla h(\cdot, s), s \in \mathbb{S} \), are uniformly Lipschitz with the same modulus \( L/2 \) on a compact set containing \( \{x^{i}\} \). Then, \( H \) is differentiable and \( \nabla H(x) = \mathbb{E}_{x} \nabla h(x, s) \), for all \( x \in \mathbb{R}^{n} \).
For $k \in \mathbb{N} \setminus \{0\}$,
\[
  y^{k-1} + w^{k-1} = \frac{1}{k} \sum_{i=0}^{k-1} [\nabla b(x^i, s^i) + \nabla r_2(x^i)] \\
  \in \frac{1}{k} \sum_{i=0}^{k-1} \nabla h(x^k, s^i) + \nabla r_2(x^k) + \frac{1}{k} \sum_{i=0}^{k-1} \|x^k - x^i\|B \\
  \subseteq \nabla H(x^k) + \nabla r_2(x^k) + (\eta_k + L\mu_k)B,
\]
where, $\mu_k = \frac{1}{k} \sum_{i=0}^{k-1} \|x^k - x^i\|$ and $\eta_k = \left\|\frac{1}{k} \sum_{i=0}^{k-1} \nabla h(x^k, s^i) - \nabla H(x^k)\right\|$. As $y^{k-1} + w^{k-1} \in \partial G(x^k) + \partial r_1(x^k)$, we derive that
\[
d(0, \partial G(x^k) + \partial r_1(x^k) - \nabla H(x^k) - \nabla r_2(x^k)) \leq \eta_k + L\mu_k.
\]
By using Lemma B.3 [18], there is, say, the same constant $c > 0$ above, such that for all $k \in \mathbb{N}_*$,
\[
  \mathbb{E}\eta_k = \mathbb{E}\left\|\frac{1}{k} \sum_{i=0}^{k-1} \nabla h(x^k, s^i) - \nabla H(x^k)\right\| \leq \frac{c(1 + \sqrt{\ln k})}{\sqrt{k}},
\]
which together with the preceding relation yielding
\[
  \tau_k := \mathbb{E}d(0, \partial G(x^k) + \partial r_1(x^k) - \nabla H(x^k) - \nabla r_2(x^k)) \leq L\mu_k + \frac{c(1 + \sqrt{\ln k})}{\sqrt{k}}.
\]
Consequently, by using the Cauchy inequality, $(a + b)^2 \leq 2(a^2 + b^2)$ for $a, b \geq 0$,
\[
  (\mathbb{E}\eta_k)^2 \geq \tau_k^2 / (2L^2) - c^2(1 + \sqrt{\ln k})^2 / (L^2k),
\]
and as $\mu_k^2 \leq \delta_k$, from (3.15) and (3.13), one obtains
\[
  \mathbb{E}V_k(x^{k+1}) - \mathbb{E}V_k(x^k) \leq \mathbb{E}d(0, \partial G(x^k) + \partial r_1(x^k) - \nabla H(x^k) - \nabla r_2(x^k)) \leq L\mu_k + \frac{c(1 + \sqrt{\ln k})}{\sqrt{k}}.
\]
By adding these inequalities with $k = 1, 2, ..., \eta$ one derives for $k \in \mathbb{N}_*$,
\[
  \sum_{i=1}^{k} \frac{\rho\mathbb{E}\tau_i^2}{4L^2(i+1)} \leq \mathbb{E}V_0(x^1) - \mathbb{E}V_k(x^{k+1}) + \sum_{i=1}^{k} \left[ \frac{\rho c^2(1 + \sqrt{\ln i})^2}{2L^2i(i+1)} + \frac{c(1 + \sqrt{\ln i})}{\sqrt{i(i+1)}} \right].
\]
Since $d_k \leq \tau_i$, for $i = 1, ..., k$, and $\{V_k(x^{k+1})\}$ is bounded a.s., say $V_k(x^{k+1}) \geq V^*$ for some $V^* \in \mathbb{R}$, for all $k \in \mathbb{N}$, the inequality above implies
\[
  \frac{\rho}{4L^2} d_k^2 \leq \mathbb{E}V_0(x^1) - V^* + \sum_{i=1}^{k} \left[ \frac{\rho c^2(1 + \sqrt{\ln i})^2}{2L^2i(i+1)} + \frac{c(1 + \sqrt{\ln i})}{\sqrt{i(i+1)}} \right] = O \left( \frac{1}{\sum_{i=1}^{k} \frac{1}{i+1}} \right) = O \left( \frac{1}{\ln k} \right).
\]
Consider now the sequence $\{x_k\}$ generated by Algorithm 2. The proof is almost similar to the one for Algorithm 1, so we sketch it. Here consider the function $V_k : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}, k \in \mathbb{N}$, defined by
\[
  V_k(x) = \frac{1}{k+1} \sum_{i=0}^{k} g(x, s^i) + r_1(x) - \frac{1}{k+1} \sum_{i=0}^{k} \langle z^k + u^k, x - x^i \rangle - \frac{1}{k+1} \sum_{i=0}^{k} \|h(x^i, s^i) + r_2(x^i)\|.$
We arrive at
\[ V_k(x^{k+1}) \leq V_{k-1}(x^k) - \delta_k/(k+1) + \frac{1}{k+1} \left[ h(x^k, s^k) - g(x^k, s^k) - \frac{1}{k} \sum_{i=0}^{k-1} (h(x, s^i) - g(x, s^i)) \right] , \]
where \( \delta_k := \frac{1}{k} \sum_{i=0}^{k-1} \|x^i - x^k\|^2 \). Taking expectations with respect to \( F_k \), one obtains
\[ \mathbb{E}_{F_k} V_k(x^{k+1}) \leq V_{k-1}(x^k) - \frac{\delta_k}{k+1} + \xi_k, \]
where
\[ \xi_k := \frac{1}{k+1} \left[ H(x^k) - G(x^k) - \frac{1}{k} \sum_{i=0}^{k-1} (h(x, s^i) - g(x, s^i)) \right] ; \]
The rest is completely similar to the preceding one, here instead of \( h \), we make use of assumption (A2) to obtain the uniformly Lipschit property of the functions \( g(\cdot, s) - h(\cdot, s) \), and then the existence of a subsequence of \( \{\delta_k\} \) converging to 0.

Remark 1. For \( \alpha \in (0, 1) \), set
\[ \gamma_k := \min_{i=[k^\alpha], \ldots, k} \frac{1}{i+1} \sum_{j=0}^{i} \|x^j - x^i\|^2; \quad l_k := \arg\min_{i=[k^\alpha], \ldots, k} \frac{1}{i+1} \sum_{j=0}^{i} \|x^j - x^i\|^2. \]
Observe from the proof of the part (i) of the theorem that almost surely \( \gamma_k \to 0 \) as \( k \to \infty \) and any limit point of the sequence \( \{x_k\} \) is a critical point of problem (3.1). So in the general case where without the smoothness of the functions \( h(\cdot, s) \) and/or \( g(\cdot, s) \), reasonably the quantity \( \gamma_k \) can serve as a measure of “proximity to criticality”, and the convergence rate of the sequence \( \{\gamma_k\} \) serves as a convergence rate of the algorithms.

Theoretically, as was shown in Theorem 3.1, the convergence rate of Algorithms 1 and 2 is of order \( O(1/\sqrt{\ln k}) \) that is relatively slow. To accelerate possibly the convergence rate, next we propose the following generalized schemata of Algorithms 1 and 2. The generalized version differs from the original algorithm only in the step 2, i.e., the way to determine \( y^k, w^k \). Pick a sequence of positive reals \( \{\alpha_k\} \) with \( \sum_{k=0}^{\infty} \alpha_k = +\infty \).

**Algorithm 3: Generalized SDCA1 (SDCA3)**

Apply Algorithm 1 in which the step 2 is replaced by setting
\[ y^k = \frac{1}{\sum_{i=0}^{k} \alpha_i} \sum_{i=0}^{k} \alpha_i z^i, \quad w^k = \frac{1}{\sum_{i=0}^{k} \alpha_i} \sum_{i=0}^{k} \alpha_i u^i. \]

**Algorithm 4: Generalized SDCA2 (SDCA4)**

Apply Algorithm 2 in which the step 2 is replaced by setting
\[ y^k = \frac{1}{\sum_{i=0}^{k} \alpha_i} \sum_{i=0}^{k} \alpha_i z^i, \quad w^k = \frac{1}{\sum_{i=0}^{k} \alpha_i} \sum_{i=0}^{k} \alpha_i u^i; \]
and in the step 3, the function $G$ is replaced by

$$G_k(x) := \frac{1}{\sum_{i=0}^{k} \alpha_i} \sum_{i=0}^{k} \alpha_i g(x, s^i), \ x \in \mathbb{R}^n.$$ 

The convergence of Algorithms 3 and 4 is stated in the next theorem. We need the following lemmas.

**Lemma 3.2.** For any increasing sequence of positive reals $\{\gamma_k\}$ with $\lim_{k \to \infty} \gamma_k = +\infty$, one has $\sum_{k=1}^{\infty}(\gamma_k - \gamma_{k-1})/\gamma_k = +\infty$.

**Proof.** Assume to contrary that $\sum_{k=0}^{\infty}(\gamma_k - \gamma_{k-1})/\gamma_k < +\infty$. Then $\lim_{k \to \infty} \gamma_k - \gamma_{k-1}/\gamma_k = 1$, and since $\lim_{t \to 1}(t - 1)/\ln t = 1$, one has

$$\lim_{k \to \infty} \frac{1 - \gamma_k - 1/\gamma_k}{\ln(\gamma_k/\gamma_{k-1})} = 1.$$ 

As $\sum_{k=0}^{\infty}\ln(\gamma_k/\gamma_{k-1}) = \lim_{k \to \infty}(\ln \gamma_k - \ln \gamma_0) = +\infty$, we derive the conclusion of the lemma. \qed

The second is a variant of the strong law of large numbers. The proofs of this lemma and the next lemma will be given in Appendix.

**Lemma 3.3.** Let $\{\alpha_k\}$ be a sequence of positive reals such that

$$\frac{\sum_{i=0}^{k} \alpha_i^2}{\left(\sum_{i=0}^{k} \alpha_i\right)^2} \leq \frac{N}{k^\gamma}, \ \text{for all} \ k \in \mathbb{N}, \ \text{for some} \ N > 0, \ \gamma > 0.$$ 

Let $\{X_k\}$ be a sequence of independent identically distributed (i.i.d for short) random variables with $\mathbb{E}X_k = \mu$. If either $\mathbb{E}X_k^4 < +\infty$ and $\gamma > 1/2$ or $\mathbb{E}X_k^2 < +\infty$ and $\lim_{k \to \infty} k^{1/2}/k = 1$, then almost surely $\sum_{i=0}^{k} \alpha_i X_k / \sum_{i=0}^{k} \alpha_i \to \mu$ as $k \to \infty$.

The next lemma is a weighted variant of the uniform law of large numbers ([13], Lemma B.2).

**Lemma 3.4.** Let a compact set $X \subseteq \mathbb{R}^n$ and let $(\Omega, \Sigma, \mu)$ be a complete probability space. Let $f: X \times \Omega \to \mathbb{R}$ be a function such that functions $f(\cdot, \omega)$ is uniformly bounded and Hölder continuous on $X$, that is, there are $M, L > 0$ and $\gamma \in (0, 1)$ such that

$$|f(x, \omega)| \leq M, \ |f(x, \omega) - f(y, \omega)| \leq L \|x - y\|^{\gamma}, \ \forall x \in X, \ \omega \in \Omega.$$ 

Then there exists some constant $c > 0$ such that for any sequence of positive reals $\{\alpha_k\}$ and any sequence of independent identically distributed random variables with the probability distribution $\mathbb{P}, \ \{s^k\}$, one has

$$\mathbb{E} \max_{x \in X} \left(\frac{1}{\sum_{i=0}^{k} \alpha_i} \sum_{i=0}^{k} \alpha_i f(x, s^i) - \mathbb{E}_\omega f(x, s)\right) \leq \frac{c(1 + \sqrt{\ln \beta_k})}{\beta_k}, \ \text{for all} \ k \in \mathbb{N}_*,$$

where $\beta_k := \left(\frac{\sum_{i=0}^{k} \alpha_i^2}{\sum_{i=0}^{k} \alpha_i}\right)^{1/\gamma}, \ k \in \mathbb{N}$.

**Theorem 3.5.** Let a sequence of positive reals $\{\alpha_k\}$ such that $\sum_{k=0}^{\infty} \alpha_k = +\infty$, and for some $N, \gamma > 0$; for $\beta_k$ as in Lemma 3.4.

$$\sum_{i=0}^{k} \alpha_i^2 \sum_{k=0}^{\infty} \frac{\alpha_i^2}{k^\gamma} \leq \frac{N}{k^\gamma}, \ \text{for all} \ k \in \mathbb{N}_* \ \text{and} \ \sum_{k=1}^{\infty} \beta_k \sum_{i=0}^{k} \alpha_i < +\infty.$$
Assume further that either $\gamma > 1/2$ or
\begin{equation}
\lim_{k,l \to \infty, l/k \to 1} \sum_{i=0}^{k} \alpha_i = 1, \tag{3.24}
\end{equation}

Suppose that (A1) holds for Algorithm 3; (A2) holds for Algorithm 4, and $\rho := \inf_{x \in \Omega} \rho(h(\cdot, s)) + \rho(r_2) > 0$. Let $\{x^k\}$ be a sequence generated by either Algorithm 3 or Algorithm 4. Suppose that the optimal value $\alpha$ of problem (3.1) is finite and with probability 1, \( \limsup_{k \to \infty} \|x^k\| < \infty \), and \( \limsup_{k \to \infty} \|u^k\| < \infty \). Then one has

(i) There exists a subsequence $\{x^{k_i}\}$ converging almost surely to a critical point of problem (3.1).

(ii) With the additional assumptions as in Theorem 3.4 (ii), one has
\begin{equation}
d_k = O \left( \sqrt{k} \sum_{i=0}^{k} \alpha_i / A_k \right) \quad \text{as} \quad k \to \infty, \quad \text{where} \quad A_k = \sum_{i=0}^{k} \alpha_i, \quad k = 0, 1, \ldots \tag{3.25}
\end{equation}

Proof. Let the sequence $\{x^k\}$ be generated by Algorithm 3, and let the functions $V_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $k \in \mathbb{N}$ be defined by
\begin{equation}
V_k(x) = G(x) + r_2(x) - \frac{1}{A_k} \sum_{i=0}^{k} \alpha_i (z^i + u^i, x - x^i) - \frac{1}{A_k} \sum_{i=0}^{k} \alpha_i [h(x^i, s^i) + r_2(x^i)]. \tag{3.26}
\end{equation}

By the same argument as in the proof of Theorem 3.1 one has
\begin{equation}
V_k(x^{k+1}) \leq V_{k-1}(x^k) - \frac{\alpha_k}{A_k} \left[ h(x^k, s^k) - \frac{1}{A_{k-1}} \sum_{i=0}^{k-1} \alpha_i h(x^i, s^i) \right] - \frac{\rho \alpha_k}{2A_k A_{k-1}} \sum_{i=0}^{k-1} \alpha_i \|x^k - x^i\|^2, \tag{3.27}
\end{equation}

and therefore,
\begin{equation}
\mathbb{E}[\xi_k V_k(x^{k+1})] \leq V_{k-1}(x^k) - \xi_k - \frac{\rho \alpha_k}{2A_k A_{k-1}} \sum_{i=0}^{k-1} \alpha_i \|x^k - x^i\|^2, \tag{3.28}
\end{equation}

where, $\xi_k = \frac{1}{A_k} \left[ H(x^k) - \frac{1}{A_{k-1}} \sum_{i=0}^{k-1} \alpha_i h(x^i, s^i) \right]$. By virtue of of Lemma 3.3 for some $c, c_1 > 0$,
\begin{equation}
\mathbb{E}[\xi_k] \leq \frac{c \alpha_k (1 + \sqrt{\ln b_k})}{b_k - 1} \leq \frac{c \alpha_k (1 + \sqrt{\ln b_k})}{b_k - 1} \leq \frac{c_1 \alpha_k (1 + \sqrt{\ln b_k})}{b_k - 1}, \quad \text{for all} \quad k = 1, 2, \ldots, \tag{3.29}
\end{equation}

Thus by the second relation of (3.28), $\sum_{k=1}^{\infty} \mathbb{E}[\xi_k] < +\infty$, and therefore by (3.20), almost surely the sequence $\{V_k(x^{k+1})\}$ converges and
\begin{equation}
\sum_{k=1}^{\infty} \frac{\alpha_k}{A_k A_{k-1}} \sum_{i=0}^{k-1} \alpha_i \|x^k - x^i\|^2 < +\infty. \tag{3.30}
\end{equation}

In view of Lemma 3.2, $\sum_{k=1}^{\infty} \alpha_k / A_k = +\infty$, consequently there is a subsequence of $\delta_k := \frac{1}{A_{k-1}} \sum_{i=0}^{k-1} \alpha_i \|x^k - x^i\|^2$, converging almost surely to 0. Next, by picking a subsequence and relabeling if necessary, without loss of generality, we can assume
that $\delta_k \to 0$. From the Jensen inequality due to the convexity of the function $\| \cdot \|^2$, one has
\[
\left( \frac{1}{A_{k-1}} \sum_{i=0}^{k-1} \alpha_i \| x^k - x^i \| \right)^2 \leq \frac{1}{A_{k-1}} \sum_{i=0}^{k-1} \alpha_i \| x^k - x^i \|^2,
\]
which implies $\lim_k \frac{1}{A_{k-1}} \sum_{i=0}^{k-1} \alpha_i \| x^k - x^i \| = 0$. Let $x^* \in \mathbb{R}^n$ be a limit point of the sequence $\{x^k\}$, say, there is a subsequence $\{x^{k'}\}$ converges to $x^*$. Then, from the above relation,
a.s., $\lim \frac{1}{A_{k-1}} \sum_{i=0}^{l_k-1} \alpha_i x^i = \lim k \alpha x^{k} = x^*$.

Note that with the assumption (A1) and either $\gamma > 1/2$ or (3.24), Lemma 3.3 holds for the sequence $h(x^*, s_0), ..., h(x^*, s^k), ...$. With the same argument as in the proof of the preceding theorem, one has
\[
(3.28)
\lim_{k \to \infty} \frac{1}{A_{k-1}} \sum_{i=0}^{l_k-1} \alpha_i (z^i, x^{k} - x^i) = 0; \quad \lim_{k \to \infty} \left[ \frac{1}{A_{k-1}} \sum_{i=0}^{l_k-1} \alpha_i h(x^i, s^i) - H(x^{k}) \right] = 0.
\]
Furthermore, when $\gamma > 1/2$ or (3.24) is verified, the strong law of large numbers with weighted averages stated in Lemma 3.3 also holds for the sequence $\{h(x, s^k)\}$ for each $x \in \mathbb{R}^n$. So repeat the respective part in the proof of the preceding theorem, one shows that almost surely, for all $x \in B[x^*, \delta]$ ($\delta > 0$),
\[
\frac{1}{A_k} \sum_{i=0}^{k} \alpha_i h(x, s^i) \to H(x).
\]

Now, similarly to the proof of Theorem 3.1, we derive the desired of the part $(i)$. The proof of the part $(ii)$ is also similar, so we omit it, just noting that here we make use of (3.27) instead of (3.16). For Algorithm 4, the same argument with the function $V$ defined by
\[
V_k(x) := \frac{1}{A_k} \sum_{i=0}^{k} \alpha_i g(x, s^i) + r_1(x) - \frac{1}{A_k} \sum_{i=0}^{k} \alpha_i (z^i + u^i, x-x^i) - \frac{1}{A_k} \sum_{i=0}^{k} \alpha_i [h(x^i, s^i) + r_2(x^i)].
\]

Remark 2. $(i)$ Observe from the proof that to obtain the convergence rate in the part $(ii)$ of the theorem, it needs only the convergence of the series
\[
\sum_{k=1}^{\infty} \frac{\alpha_k}{\beta_k} \frac{\alpha_k}{\beta_k} \sum_{i=0}^{k} \alpha_i.
\]

$(ii)$ Obviously, the sequence $\alpha_k := k^\alpha$, $k \in \mathbb{N}_*$ with $\alpha > -1/2$ verifies all three conditions (3.20) and (3.24) of Theorem 3.5. Let us take another example of the sequence $\{\alpha_k\}$ which produces an asymptotic convergence rate better than the one by the sequence $\{k^\alpha\}$ ($\alpha > -1/2$). For $\alpha > 1$ and $\alpha \in (0, 1/2)$, let $\{\alpha_k\}$ be a sequence of
positive reals such that 0 < \lim_{k \to \infty} \frac{4k}{k^\alpha} < +\infty, where \( A_k = \sum_{i=0}^{k} \alpha_i \). For example, one can take the sequence \( \alpha_k = a^{(k+1)\alpha} - a^{k\alpha}, \ k \in \mathbb{N} \). One sees obviously that \( \alpha_k = O(a^{k\alpha}k^{\alpha-1}) \), therefore

\[
\alpha_k/A_k = O\left(k^{\alpha-1}\right), \quad \text{and} \quad \sum_{i=0}^{k} \alpha_k/A_k = O(k^\alpha).
\]

One has the following estimate

\[
\sum_{i=1}^{k} \alpha_i^2 \leq C \sum_{i=1}^{k} a^{2\alpha_i} \leq Ca^2k^\alpha \sum_{i=1}^{k} \alpha^2 - (a-1) \leq Ca^2k^\alpha k^{2\alpha}-1,
\]

for some \( C > 0 \). It implies the first relation of condition \( \ref{3.32} \) is verified with \( \gamma = 1 - 2\alpha \), and moreover

\[
\sum_{k=0}^{\infty} \frac{\alpha_k \sqrt{\ln b_k}}{\beta_k A_k} = \sum_{k=0}^{\infty} O\left(\frac{a^{k\alpha-1} \sqrt{\ln k^{1/2-\alpha} a^{k\alpha}}}{k^{3/2-2\alpha}}\right) = \sum_{k=0}^{\infty} O\left(\frac{\sqrt{\ln k}}{k^{3/2-2\alpha}}\right) < +\infty,
\]

when \( \alpha \in (0, 1/4) \).

### 3.2. Algorithms with the storage the past samples but updating sub-gradients

In Algorithms 1 and 2 (and their generalized versions, Algorithms 3 and 4), at each \( k \)th iteration, we need only to compute a subgradient of the function \( h(\cdot, s^k) \) at \( x^k \). In this subsection, we propose the following two algorithms, in which all subgradients of the functions \( h(\cdot, s^i), i = 0, 1, ..., k \) are computed at the current iteration \( x^k \).

**Algorithm 5: Stochastic DC Algorithm 5 (SDCA5)**

**Initialization:** Initial data: \( x^0 \in \text{Dom} r_1 \), draw \( s^0 \overset{iid}{\sim} P \). Set \( k = 0 \).

**Repeat:** For \( k = 0, 1, ..., \) draw \( s^k \overset{iid}{\sim} P \), which is independent of the past.
1. Compute \( z^k_i \in \partial h(x^k, s^i), \ i = 0, ..., k \), and \( w^k \in \partial r_2(x^k) \).
2. Set \( y^k = \frac{1}{k+1} \sum_{i=0}^{k} z^k_i \).
3. Compute a solution \( x^{k+1} \) of the convex program

\[
\min \{ G(x) + r_1(x) - \langle y^k + w^k, x \rangle : \ x \in \mathbb{R}^n \}.
\]

4. Set \( k := k + 1 \) and draw \( s^{k+1} \overset{iid}{\sim} P \).

**Algorithm 6: Stochastic DC Algorithm 6 (SDCA6)**

**Initialization:** Initial data: \( x^0 \in \text{Dom} r_1 \), draw \( s^0 \overset{iid}{\sim} P \). Set \( k = 0 \).

**Repeat:** For \( k = 0, 1, ..., \) draw \( s^k \overset{iid}{\sim} P \), which is independent of the past.
1. Compute \( z^k_i \in \partial h(x^k, s^i), \ i = 0, ..., k \), and \( w^k \in \partial r_2(x^k) \).
2. Set \( y^k = \frac{1}{k+1} \sum_{i=0}^{k} z^k_i \).
3. Compute a solution \( x^{k+1} \) of the convex program

\[
\min \left\{ \frac{1}{k+1} \sum_{i=0}^{k} g(x, s^i) + r_1(x) - \langle y^k + w^k, x \rangle : \ x \in \mathbb{R}^n \right\}.
\]
4. Set $k := k + 1$ and draw $s_{k+1} \sim P$.

Remark 3. The principal difference between these algorithms and Algorithms 1 and 2 is that at each $k$th iteration, the subgradients of $h(\cdot, s_i)$ $(i = 1, ..., k-1)$ with respect to the past realizations of the sample are all updated. For Algorithms 5 and 6, we obtain a stronger convergence result that almost surely the sequence $\{f(x^k)\}$ converges and all limit points are critical points. Moreover, with the added same assumption as for Algorithms 1 and 2, the convergence rate of $d_k$ is of order $O(\ln k^{1/\sqrt{k}})$.

Theorem 3.6. Suppose that (A1) holds for Algorithm 5 and (A2) holds for Algorithm 6. Let $\{x^k\}$ be a sequence generated by either Algorithm 5 or Algorithm 6. Suppose that the optimal value $\alpha$ of problem (3.4) is finite and with probability 1, $\limsup_{k \to \infty} \|x^k\| < \infty$. Assuming $\rho := \inf_{s \in \Omega} \rho(h(\cdot, s)) + \rho(r_2) > 0$, then one has

(i) Almost surely the sequence of function values $\{f(x^k)\}$ converges; $\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 < +\infty$, and all limit points of $(x^k)$ are critical points of (3.1).

(ii) With the additional assumptions as in Theorem 3.1 (ii), one has $d_k = O(\ln k^{1/\sqrt{k}})$ as $k \to \infty$.

Proof. (i). Consider the sequence $\{x^k\}$ generated by Algorithm 5. The proof is similar for Algorithm 6. As before, denote by $F_k = \sigma(x^0, s^0, ..., s^{k-1})$, $k \in \mathbb{N}$, the increasing $\sigma-$ field generated by random variables $x^0, s^0, ..., s^{k-1}$. For $k = 1, 2, ...$, define the function $V_k: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $k \in \mathbb{N}$,

$$V_k(x) = G(x) + r_1(x) - \left(\frac{1}{k} \sum_{i=0}^{k} z_i^k + w^k, x - x^k\right) - \frac{1}{k+1} \sum_{i=0}^{k} h(x^i, s^i) - r_2(x^k).$$

As $x^{k+1}$ is a solution of the problem (3.30), one has

$$V_k(x^{k+1}) \leq V_k(x^k) = G(x^k) + r_1(x^k) - \frac{1}{k+1} \sum_{i=0}^{k} h(x^i, s^i) - r_2(x^k).$$

In virtue of the strong convexity of the function $h(\cdot, s) + r_2$, $s \in \Omega$ with modulus at least $\rho$,

$$V_{k-1}(x^k) - V_k(x^k) = \frac{1}{k+1} \sum_{i=0}^{k} h(x^i, s^i) - \frac{1}{k} \left[\sum_{i=0}^{k-1} \langle z_i^{k-1}, x^k - x^{k-1}\rangle + h(x^{k-1}, s^i)\right]$$

$$+ [r_2(x^k) - \langle w_i^{k-1}, x^k - x^{k-1}\rangle - r_2(x^{k-1})]$$

$$\geq \frac{1}{k+1} \left[\sum_{i=0}^{k-1} h(x^i, s^i) - \frac{1}{k} \sum_{i=0}^{k-1} h(x^i, s^i)\right] + \frac{\rho}{2} \|x^k - x^{k-1}\|^2.$$

Thus

$$V_k(x^{k+1}) \leq V_{k-1}(x^k) - (V_{k-1}(x^k) - V_k(x^k))$$

$$\leq V_{k-1}(x^k) - \frac{1}{k+1} \left[\sum_{i=0}^{k-1} h(x^i, s^i) - \frac{1}{k} \sum_{i=0}^{k-1} h(x^i, s^i)\right] - \frac{\rho}{2} \|x^k - x^{k-1}\|^2.$$

By taking expectations with respect to $F_k$ on both sides of the inequality above,

$$(3.33) \quad \mathbb{E}_{F_k} V_k(x^{k+1}) \leq V_{k-1}(x^k) - \frac{1}{k+1} \left[\sum_{i=0}^{k-1} h(x^i, s^i) - \frac{\rho}{2} \|x^k - x^{k-1}\|^2\right]$$

where $\xi_k = \frac{1}{k+1} \left[H(x^k) - \frac{1}{k} \sum_{i=0}^{k-1} h(x^i, s^i)\right]$. As in the proof of Theorem 3.1 one has for some $c > 0$,

$$(3.34) \quad \mathbb{E}\xi_k \leq c \left(1 + \sqrt{\ln k}\right) \sqrt{k(k+1)}, \quad \text{for all } k \in \mathbb{N} \setminus \{0\}.$$
Hence, \( \sum_{k=0}^{\infty} \mathbb{E} |\xi_k| < +\infty \), then thanks to the supermartingale convergence, we arrive at the conclusion that almost surely the sequence \( \{V_k(x^{k+1})\} \) converges, and \( \sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 < +\infty \). The convergence of \( \{V_k(x^{k+1})\} \) implies obviously the convergence of \( \{V_k(x^k)\} \), which follows so is \( (f(x^k)) \), since

\[
V_k(x^k) = f(x^k) + \left[ H(x^k) - \frac{1}{k+1} \sum_{i=0}^{k} h(x^k, s^i) \right],
\]

in which, the second term converges almost surely to 0. The convergence of any subsequence to a critical point is proved similarly as for Theorem 3.1, since \( \lim_{k \to \infty} \|x^k - x^{k-1}\| = 0 \).

(ii). Assume now \( \nabla r_2 \) and the derivatives \( \nabla h(\cdot, s), s \in \Omega \), are uniformly Lipschitz with modulus \( L/2 \) on a compact set containing \( (x^k) \). For \( k \in \mathbb{N} \),

\[
y^{k-1} + w^{k-1} = \frac{1}{k} \sum_{i=0}^{k-1} \nabla h(x^{k-1}, s^i) + \partial r_2(x^{k-1})
\]

(3.3.5)

where, \( \eta_k = \left\| \frac{1}{k} \sum_{i=0}^{k-1} \nabla h(x^i, s^i) - \nabla H(x^k) \right\| \). Noticing \( y^{k-1} + w^{k-1} \in \partial G(x^k) + \partial r_1(x^k) \), one has

\[
d(0, \partial G(x^k) + \partial r_1(x^k) - \nabla H(x^k)) - \partial r_2(x^k) \leq \eta_k + L \|x^k - x^{k-1}\|.
\]

Relation 3.2.1 in the proof of Theorem 3.1 implies

(3.3.6)

\[
\tau_k := \mathbb{E} \left( 0, \partial G(x^k) + \partial r_1(x^k) - \nabla H(x^k) - \nabla r_2(x^k) \right) \leq LE \|x^k - x^{k-1}\| + \frac{c(1 + \ln k)}{\sqrt{k}}
\]

Thus \( \mathbb{E} \|x^k - x^{k-1}\|^2 \geq \tau_k^2/(2L^2) - \frac{2(1 + \ln k)^2}{4L^2k} \), and by (3.3.5),

\[
EV_k(x^{k+1}) - EV_k(x^k) \leq \mathbb{E} \|x^k - x^{k-1}\|^2 + \mathbb{E} |\xi_k| \leq \frac{\rho \tau_k^2}{4L^2} + \frac{c(1 + \ln k)}{2L^2k} + \frac{c(1 + \ln k)}{\sqrt{k(k+1)}}
\]

Therefore, for \( k \in \mathbb{N} \),

\[
\frac{\rho}{4L^2} \sum_{i=1}^{k} \mathbb{E} r_i^2 \leq EV_0(x^1) - EV_k(x^{k+1}) + \sum_{i=1}^{k} \frac{c(1 + \ln i)^2}{2L^2i} + \sum_{i=1}^{k} \frac{c(1 + \ln i)}{\sqrt{i(i+1)}},
\]

implying, for some \( V^* > 0 \),

\[
\frac{\rho}{4L^2} d_k^2 \leq EV_0(x^1) - V^* + \sum_{i=1}^{k} \frac{c(1 + \ln i)^2}{2L^2i} + \sum_{i=1}^{k} \frac{c(1 + \ln i)}{\sqrt{i(i+1)}},
\]

By noticing that

\[
\sum_{i=1}^{k} \frac{(1 + \ln i)^2}{i} = O(\ln^2 k), \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1 + \ln i}{\sqrt{i(i+1)}} < +\infty,
\]

we conclude \( d_k = O(\ln k/\sqrt{k}) \).
3.3. Regularized Stochastic DC Algorithm. In this final subsection, we consider a stochastic DC algorithm with proximal regularization process for solving the DC problem \((3.1)\). The algorithm is given as follows.

**Algorithm 7: Regularized Stochastic DC Algorithm (RSDCA)**

**Initialization:** \(x^0 \in \text{Dom} \, r_1\), a sequence of positive reals \(\{\alpha_k\}\), and set \(k := 0\).

**Repeat:** \(k = 0, 1, \ldots\),

1. Generating a random vector \(\zeta^k \in \mathbb{R}^n\), \(y^k \in \partial H(x^k)\), and \(w^k \in \partial r_2(x^k)\).
2. Compute a minimizer \(x^{k+1}\) of the problem
   \[
   \min \left\{ G(x) + r_1(x) - (y^k + w^k + \zeta^k, x) + \frac{1}{2\alpha_k} \|x - x^k\|^2 : \ x \in \mathbb{R}^n \right\}.
   \]

**Remark 4.** The stochastic proximal (sub)gradient type method for smooth/convex or weakly convex functions can be regarded as particular cases of this stochastic algorithm. In fact, the presence of the proximal regularization term in the subproblems \((3.38)\) can be regarded by the "eye" of DCA as applying DCA on the following DC decomposition (see e.g., [27]):

\[
\begin{align*}
    f(x) & := G(x) + r_1(x) - H(x) - r_2(x) \\
    & = \left[ G(x) + r_1(x) + \frac{1}{2\alpha_k}\|x\|^2 \right] - \left[ H(x) + r_2(x) + \frac{1}{2\alpha_k}\|x\|^2 \right].
\end{align*}
\]

**Theorem 3.7.** Let \(\{x^k\}\) be a sequence generated by Algorithm 7. Denote by \(\mathcal{F}_k = \mathcal{F}(\zeta^0, \ldots, \zeta^{k-1})\), \(k \in \mathbb{N}\), the increasing \(\sigma\)-field generated by random vectors \(\zeta^0, \ldots, \zeta^{k-1}\). Suppose that the bounded sequence \(\{\alpha_k\}\) and the sequence of random vectors \(\{\zeta_k\}\) satisfies

\[
\sum_{k=0}^{\infty} \alpha_k = +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha_k \mathbb{E}_{\mathcal{F}_k} \|\zeta_k\|^2 < +\infty.
\]

Assume further that almost surely \(\limsup_{k \to \infty} \|x^k\| < +\infty\) and \(\limsup_{k \to \infty} \|w^k\| < +\infty\). Then one has

(i) \((\text{Convergence of the sequence of the objective values})\) The sequence \(\{f(x^k)\}\) converges and \(\sum_{k=0}^{\infty} \alpha_k \|x^{k+1} - x^k\|^2 < +\infty\). As a result, there is a subsequence of \(\{x^k\}\) converging almost surely to a critical point of problem \((3.1)\).

(ii) \((\text{The rate of the convergence to critical points})\) Suppose that either \(G + r_1\) or \(H + r_2\) is continuously differentiable with a Lipschitz derivative with a modulus \(L > 0\) on a compact set containing the sequence \(\{x^k\}\). The one has the following estimate, almost surely for each \(k \in \mathbb{N}\), for some \(c > 0\),

\[
\min_{i=1, \ldots, k+1} \mathbb{E}_{\mathcal{F}_i} d_i^2 \leq \frac{c}{\sum_{i=0}^{k} \alpha_i / (1 + \alpha_i L)^2},
\]

where \(d_k := d(0, \partial G(x^k) + \partial r_1(x^k) - \partial H(x^k) - \partial r_2(x^k))\), \(k \in \mathbb{N}\).
Proof. Define the functions $V_k: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$V_k(x) := G(x) + r_1(x) - (y^k + w^k + \zeta^k, x - x^k) - H(x^k) - r_2(x^k) + \frac{1}{2\alpha_k} \|x - x^k\|^2, \ x \in \mathbb{R}^n.$$ 

As $x^{k+1}$ is a solution of (3.38), and by the convexity of the functions $H, r_2$, for $k \in \mathbb{N}$,

$$V_k(x^{k+1}) \leq V_k(x^k) = V_{k-1}(x^k) - [H(x^k) + r_2(x^k) - H(x^{k-1}) - r_2(x^{k-1}) - (y^{k-1} + w^{k-1}, x^k - x^{k-1})] + \langle \zeta^{k-1}, x^k - x^{k-1} \rangle - \frac{1}{2\alpha_{k-1}} \|x^k - x^{k-1}\|^2 \leq V_{k-1}(x^k) + \|\zeta^{k-1}\| \|x^k - x^{k-1}\| - \frac{1}{2\alpha_{k-1}} \|x^k - x^{k-1}\|^2.$$ 

Therefore, invoking the inequality $ab - \alpha^2b^2/2 \leq a^2/(2\alpha)$ for $a, b > 0$, one obtains

$$(3.41) \quad V_k(x^{k+1}) \leq V_k(x^k) \leq V_{k-1}(x^k) + \alpha_k \|\zeta^k\|^2 - \frac{1}{4\alpha_{k-1}} \|x^k - x^{k-1}\|^2,$$

consequently,

$$\mathbb{E}_{\mathcal{F}_k} V_k(x^{k+1}) \leq V_{k-1}(x^k) + \alpha_k \mathbb{E}_{\mathcal{F}_k} \|\zeta^k\|^2 - \frac{1}{4\alpha_{k-1}} \|x^k - x^{k-1}\|^2.$$ 

Since $\sum_{k=0}^{\infty} \alpha_k \mathbb{E}_{\mathcal{F}_k} \|\zeta^k\|^2 < \infty$, thanks to the supermartingale convergence theorem, the sequence $\{V_k(x^{k+1})\}$ converges, and $\sum_{k=0}^{\infty} \frac{1}{\alpha_k} \|x^{k+1} - x^k\|^2 < \infty$. From (3.41), and the second condition of (3.39), one obtains the convergence of the sequence $\{V_k(x^k)\} = \{f(x^k)\}$. Next, since $\sum_{k=0}^{\infty} \frac{1}{\alpha_k} \|x^{k+1} - x^k\|^2 < \infty$, and the second condition of (3.39),

$$\sum_{k=0}^{\infty} \alpha_k \left(\|x^{k+1} - x^k\|/\alpha_k\right)^2 + \mathbb{E}_{\mathcal{F}_k} \|\zeta^k\|^2 < +\infty.$$ 

By the definition of the sequence $\{\alpha_k\}$ that, $\sum_{k=0}^{\infty} \alpha_k = +\infty$, the preceding relation implies that there are subsequences of $\|x^{k+1} - x^k\|/\alpha_k$ and of $\{\zeta^k\}$ with the same indices, say, $\|x^{k+1} - x^k\|/\alpha_k$ and $\{\zeta^k\}$ converging to 0 almost surely. Let $\{x^k\}$ converge to some $x^\infty$ and let $\{y^k\}, \{w^k\}$ converge to some $y^\infty, w^\infty$, respectively almost surely. Then almost surely $y^\infty + w^\infty \in \partial H(x^\infty) + \partial r_2(x^\infty)$. Since

$$y^k + w^k + \zeta^k \in \partial G(x^{k+1}) + \partial r_1(x^{k+1}) + (x^{k+1} - x^k)/\alpha_k,$$

one obtains $y^\infty + w^\infty \in \partial G(x^\infty) + \partial r_1(x^\infty)$ a.s., that is

$$(\partial G(x^\infty) + \partial r(x^\infty)) \cap (\partial H(x^\infty) + \partial r_2(x^\infty)) \neq \emptyset,$$

showing $x^\infty$ is a critical point of $f$.

For (ii), assume now $H + r_2$ is differentiable with Lipschitz derivative with modulus $L$ on a set containing the sequence $\{x^k\}$ (the case of such the assumption on $G + r_1$ is proved similarly). Then for $k \in \mathbb{N}$, for each $i = 0, 1, ..., k$, since

$$y^i + w^i = \nabla H(x^i) + \nabla r_2(x^i) \in \nabla H(x^{i+1}) + \nabla r_2(x^{i+1}) + L\|x^{i+1} - x^i\| \mathbb{B};$$

$$y^i + w^i + \zeta^i \in \partial G(x^{i+1}) + \partial r_1(x^{i+1}) + (x^{i+1} - x^i)/\alpha_i,$$

one has

$$d_{i+1} := d(0, \partial G(x^{i+1}) + \partial r(x^{i+1}) - \nabla H(x^{i+1}) - r_2(x^{i+1})) \leq \|\zeta_i\| + (L+1/\alpha_i)\|x^{i+1} - x^i\|.$$
Thus, by using the inequality \((a + b)^2 \leq 2(a^2 + b^2)\) with \(a, b \in \mathbb{R}\),
\[
\mathbb{E}_F, d_{i+1}^2 \leq 2\mathbb{E}_F, |\xi_i|^2 + 2(L + 1/\alpha_i)^2 \mathbb{E}_F, \|x^{i+1} - x^i\|^2,
\]
consequently, for some \(c > 0\),
\[
\sum_{i=1}^{k+1} \alpha_i \mathbb{E}_F, d_i^2 (1 + L\alpha_i)^2 \leq 2 \sum_{i=0}^{k} \alpha_i \mathbb{E}_F, |\xi_i|^2 / (1 + L\alpha_i)^2 + 2 \sum_{i=0}^{k} \frac{1}{\alpha_i} \mathbb{E}_F, \|x^{i+1} - x^i\|^2 \leq 2 \sum_{i=0}^{k} \frac{1}{\alpha_i} \mathbb{E}_F, \|x^{i+1} - x^i\|^2 \leq c,
\]
and \(\|x\| = \sqrt{\mathbb{E}_F, \|x\|^2} = \sqrt{\sum_{i=1}^{n} \alpha_i \mathbb{E}_F, d_i^2} \leq \sqrt{\sum_{i=0}^{n} \alpha_i / (1 + L\alpha_i)^2} \cdot \sqrt{c} \cdot \). \(\square\)

We consider a particular way to generate \(y^k\) and \(\zeta_k\) at the iteration \(k\), \(k \in \mathbb{N}\). Let \(y(x, s)\) be a measurable selection of \(\partial h(x, s)\), \((x, s) \in \mathbb{R}^n \times \Omega\). By the measurable selection theorem (see e.g., [54] - Cor. 14.6, P. 647), such a measurable selection exists. Then \(E_s y(x, s) \in \partial H(x)\). At each iteration \(k\) of Algorithm 7, generate \(n_k\) \((n_k \in \mathbb{N}_+\) independent samples \(s_1, ..., s_{n_k} \simeq P\) and set
\[
y^k := E_s y(x^k, s), \quad y^k + \zeta^k := \frac{1}{n_k} \sum_{i=1}^{n_k} y(x^k, s^i).
\]
Then \(E_{F_k} \zeta^k = 0\), and \(E_{F_k} \|\zeta^k\|^2 = E_s \|y(x^k, s)\|^2 / n_k\). Hence, the second condition of (3.30) becomes
\[
\sum_{k=0}^{\infty} \alpha_k E_s \|y(x^k, s)\|^2 / n_k < +\infty.
\]
Obviously, when the sequence \(\{x^k\}\) is assumed to be bounded, this condition is satisfied if the following two conditions are verified:

(a) For each \(x \in \mathbb{R}^n, s \in \Omega, h(\cdot, s)\) is locally Lipschitz around \(x\) with a Lipschitz modulus \(L(x, s) > 0\) such that \(E_s L^2(x, s) < \infty\);

(b) The sequences of regularized parameters \(\{\alpha_k\}\) and of sample sizes \(\{n_k\}\) are picked such that
\[
\sum_{k=0}^{\infty} \alpha_k n_k^{-1} < +\infty.
\]

4. Conclusion remarks. In this paper, we have proposed the three variants of SDCA (including 7 algorithms) and have established the convergence results for these algorithms. From the theoretical viewpoint, as shown in the convergence theorems, the convergence rate of Algorithms 1 and 2 is relatively slow. To accelerate the convergence rate, the generalized versions of these algorithms in making use of weighted averages are proposed. The convergence rate of Algorithms 5 and 6 in which the subgradients with respect to all past samples are updated at the current iteration, is considerably faster than Algorithms 1 and 2, while the convergence rate of the regularized SDCA (Algorithm 7) depends on the sequence of regularized parameters \(\{\alpha_k\}\) and the sizes of samples at each iterations. In particular, if in Algorithm 7, the constant parameter is used, then its convergence rate is of \(O(1/\sqrt{k})\). An important procedure in the proposed SDCA is to solve convex optimization subproblems. For this purpose, existing stochastic convex optimization approaches, such as the stochastic
proximal subgradient methods, could be used. The practical convergence rate of the algorithms depends strictly on the methods dealing with those convex subprograms. The convergence analysis of the proposed SDCA according to the stochastic methods for solving the convex subproblems could be a challenge for further researches. The applications of the proposed SDCA on real-world optimization problems will be the forthcoming works as well.

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5. Appendix. Proofs of Lemmas 3.3 and 3.4 Proof of Lemma 3.3 The idea of the proof is standard, as the one for the classical strong law of large number (see 15). We prove the lemma firstly for the case where $EX_i > +\infty$ and $\gamma > 1/2$. By consider $X_i - \mu$ instead of $X_i$, we can assume that $EX_i = 0$, for all $i \in \mathbb{N}_\ast$. Setting $S_k := \sum_{i=0}^{k-1} \alpha_i X_i$, since $\sum_{k=0}^{\infty} \alpha_i X_i$, and by independence, $\mathbb{E} \left( X_i^2 \right) = \mathbb{E} \left( X_i^2 \right) \mathbb{E} \left( X_j^2 \right)$, for all $0 \leq i < j < k$; $1 \leq i \neq j \neq l \neq m \leq k$, one has

$$
\mathbb{E} S_k^4 = \mathbb{E} X_i^4 \sum_{i=0}^{k-1} \alpha_i X_i + (\mathbb{E} X_i^2)^2 \sum_{0 \leq i \neq j \leq k} \alpha_i^2 \alpha_j^2 \alpha_i \alpha_j
$$

$$
= \mathbb{E} X_i^4 \sum_{i=0}^{k-1} \alpha_i^4 + (\mathbb{E} X_i^2)^2 \left( \sum_{i=0}^{k-1} \alpha_i^2 \right)^2 - \sum_{i=0}^{k-1} \alpha_i^4 \leq C \left( \sum_{i=0}^{k-1} \alpha_i^2 \right)^2
$$

for some $C > 0$. Therefore, the Chebyshev inequality implies, for any $\varepsilon > 0$,

$$
\mathbb{P} \left( |S_k| \geq \varepsilon \sum_{i=0}^{k-1} \alpha_i \right) \leq \varepsilon^{-4} \mathbb{E} S_k^4 \left( \sum_{i=0}^{k-1} \alpha_i^2 \right)^2 \leq C \varepsilon^{-4} \left( \sum_{i=0}^{k-1} \alpha_i^2 \right)^2 \left( \sum_{i=0}^{k-1} \alpha_i^4 \right)^2 \leq C \varepsilon^{-4} / k^{2\gamma}
$$

Consequently, $\sum_{k=0}^{\infty} \mathbb{P} \left( |S_k| \geq \varepsilon \sum_{i=0}^{k-1} \alpha_i \right) \leq C \varepsilon^{-4} \sum_{k=0}^{\infty} 1/k^{2\gamma} < +\infty$. As $\varepsilon > 0$ is arbitrary, by the Borel-Cantelli Lemma, one has $S_k \rightarrow 0$ a.s.

For the second case, by setting $X^+_k = \max \{ x_k, 0 \}$ and $X^-_k = \max \{ -X_k, 0 \}$, then $X^+_k, X^-_k \geq 0$ and $X_k = X^+_k - X^-_k$, so it is enough to prove the lemma for the case $X_k \geq 0$. Let $A_k = \sum_{i=0}^{k-1} \alpha_i$ and $S_k = \sum_{i=0}^{k-1} \alpha_i X_i$, $k = 0, 1, \ldots$. Then
\[ \sum_{i=0}^{k} \alpha_i \mathbb{E} X_i = A_k \mu, \text{ and } \mathbb{D} S_k = \sum_{i=0}^{k} \alpha_i^2 \mathbb{D} X_0. \] The Chebyshev inequality implies, for any \( \varepsilon > 0 \),
\[
\mathbb{P} \left( |S_k - A_k \mu| > \varepsilon A_k \right) \leq \varepsilon^{-2} \mathbb{D} S_k \frac{\sum_{i=0}^{k} \alpha_i^2}{A_k^2} \leq \varepsilon^{-2} \mathbb{D} X_0 \frac{N}{k^\gamma}.
\]
Let \( l_k = \lceil k^{2/\alpha} \rceil, k \in \mathbb{N} \), the integer part of \( k^{2/\alpha} \). One has
\[
\sum_{k=1}^{\infty} \mathbb{P} \left( |S_k - A_k \mu| > \varepsilon A_k \right) \leq \varepsilon^{-2} \mathbb{D} X_1 \sum_{k=1}^{\infty} 1/|k^{2/\gamma}| < +\infty.
\]
Thanks to the Borel-Cantelli Lemma, since \( \varepsilon > 0 \) is arbitrary, one obtains \( S_k / A_k \rightarrow \mu \) almost surely. To show \( S_k / A_k \rightarrow \mu \) a.s., for each \( k = 0, 1, \ldots, \) picking \( l_m(k) \) such that
\[ l_m(k) \leq k < l_m(k)+1, \]
\[
\frac{S_{l_m(k)}}{A_{l_m(k)}} \leq \frac{S_k}{A_k} \leq \frac{S_{l_m(k)+1}}{A_{l_m(k)}},
\]
Since \( l_m(k)+1 \rightarrow 1, A_{l_m(k)+1} \rightarrow 1 \), therefore \( \lim_{k \rightarrow \infty} \frac{S_{l_m(k)}}{A_{l_m(k)}} = \frac{S_{l_m(k)+1}}{A_{l_m(k)}} = \mu \) a.s., implying the desired conclusion \( S_k / A_k \rightarrow \mu \) a.s. as \( k \rightarrow \infty \). \( \square \)

**Proof of Lemma 5.1.** The proof is similar to the one of Lemma 2.2 in [18]. By using symmetrization arguments and Rademacher averages as in Lemma 2.3.6 [60], one obtains the following estimate, for \( k \in \mathbb{N} \),
\[
\mathbb{E} \max_{x \in X} \left| \frac{1}{\sum_{i=0}^{k} \alpha_i} \sum_{i=0}^{k} \alpha_i f(x, s^i) - \mathbb{E} f(x, s) \right| \leq 2 \mathbb{E} R_k(f, \alpha, X),
\]
where,
\[ R_k(f, \alpha, X) := \mathbb{E} \sup_{x \in X} \frac{1}{\sum_{i=0}^{k} \alpha_i} \left| \sum_{i=0}^{k} \alpha_i f(x, s^i) \right|; \]
\[ \{\alpha_i\} \text{ are i.i.d random variables with values } \pm 1 \text{ with probability } 1/2. \]
For a set \( A \subseteq \mathbb{R}^{k+1} \), denote by \( R_k(A) \) the Rademacher average of \( A \) with respect to \( \{\alpha_i\} \):
\[ R_k(\alpha, A) := \mathbb{E} \sup_{a \in A} \frac{1}{\sum_{i=0}^{k} \alpha_i} \left| \sum_{i=0}^{k} \sigma_i \alpha_i a_i \right|, \]
where \( a = (a_0, a_1, \ldots, a_k) \in \mathbb{R}^{k+1} \). The following lemma gives an upper bound of \( R_k(\alpha, A) \), which generalizes the one in Theorem 3.3 in [6].

**Lemma 5.1.** Let \( A \subseteq \mathbb{R}^{k+1} \) be a finite set of \( N \) elements. One has
\[
R_k(A) \leq \max_{a \in A} \max_{i=0, \ldots, k} |a_i| \sqrt{\frac{2 \ln(2N)}{\sum_{i=0}^{k} \alpha_i^2}} \frac{\sqrt{2 \ln(2N)}}{\sum_{i=0}^{k} \alpha_i^2}. \]

**Proof.** Set \( A_k = \sum_{i=0}^{k} \alpha_i \). By using the Hoeffding inequality, stating that for a zero-mean random variable with values in \([t_1, t_2]\), \( \mathbb{E} e^x \leq e^{(t_2-t_1)^2/8} \), for any \( s > 0 \), one has
\[
\mathbb{E} e^{2 \sum_{i=0}^{k} \sigma_i \alpha_i a_i} = \prod_{i=0}^{k} \mathbb{E} e^{2 \alpha_i \sigma_i a_i} \leq \prod_{i=0}^{k} e^{\frac{2 \alpha_i^2 s^2}{2 \alpha_i^2}} \leq e^{\frac{2 \alpha_i^2 s^2}{\sum_{i=0}^{k} \alpha_i^2}}.
\]
where the first equality is by independence. Hence, by the Jensen inequality,
\[
e^{\sum_{a \in A} \max_{\varepsilon}(1/A) \sum_{i=0}^k \sigma_i a_i} \leq \mathbb{E} e^{\sum_{a \in A} \max_{\varepsilon}(1/A) \sum_{i=0}^k \sigma_i a_i} \leq \sum_{a \in A} \mathbb{E} e^{\sum_{i=0}^k \sigma_i a_i} \leq N \max_{\varepsilon} e^{\frac{2}{\alpha_i^2} \sum_{i=0}^k \sigma_i a_i^2},
\]
which implies \( \mathbb{E} \max_{\varepsilon}(1/A) \sum_{i=0}^k \sigma_i a_i \leq \frac{\ln N}{\varepsilon} + \max_{\varepsilon} \sum_{i=0}^k \sigma_i a_i^2. \) By noting that \( R_k(\alpha, A) = \mathbb{E} \max_{\varepsilon}(1/A) \sum_{i=0}^k \sigma_i a_i, \) the preceding inequality yields, by considering the set \( A \cup (-A) \) instead of \( A, \)
\[
R_k(\alpha, A) \leq \frac{\ln(2N)}{s} + \max_{\varepsilon} \frac{s}{2A^2} \sum_{i=0}^k \sigma_i a_i^2,
\]
and by setting \( s = A_k \sqrt{2\ln(2N)/\max_{\varepsilon}(1/A)} \sum_{i=0}^k \sigma_i a_i^2, \) it implies the desired estimate:
\[
R_k(\alpha, A) \leq \max_{\varepsilon} \frac{2\ln(2N)}{A_k} \sum_{i=0}^k \sigma_i a_i^2 \leq \max_{\varepsilon} \max_{\alpha, A} \frac{2\ln(2N)}{\sum_{i=0}^k \sigma_i a_i^2} |a_i| \sqrt{2\ln(2N)} \sum_{i=0}^k \alpha_i^2.
\]

To end the proof of Lemma 3.34 in view of estimate 5.1, we shall show that for some \( c > 0, \) for all \( k \in \mathbb{N}, \)
\[
(5.2) \quad R_k(\alpha, X) \leq \frac{c(1 + \sqrt{\ln \beta_k})}{\beta_k},
\]
where \( \beta_k := \frac{\sum_{i=0}^k \alpha_i}{(\sum_{i=0}^k \alpha_i^2)^{1/2}}. \) Indeed, assume that \( X \subseteq B_R \subseteq \mathbb{R}^n, \) a Euclidean ball centered at 0 with radius \( R > 0. \) For any \( \varepsilon > 0, \) there are \( N := N(\varepsilon) \leq 4Re^n/\varepsilon \) balls with radius \( \varepsilon : B(y^i, \varepsilon), \) \( y^i \in X, \) \( i = 1, \ldots, N, \) which covers \( X, \) that is, \( X \subseteq \bigcup_{i=1}^N B(y^i, \varepsilon) \) (see, e.g., [3]). Setting \( Y_\varepsilon := \{y^1, \ldots, y^N\}, \) Since \( f(\cdot, s) \) are Hölder continuous on \( X \) with constants \( L, \gamma >, \) one has
\[
|R_k(f, \alpha, X) - R_k(f, \alpha, Y_\varepsilon)| \leq L\varepsilon^\gamma.
\]
Defining the finite set \( A \subseteq \mathbb{R}^{k+1} \) by
\[
A := \{a^i = (f(y^i, s_0), \ldots, f(y^i, s_k)) : i = 0, \ldots, N\},
\]
we see that \( R_k(f, \alpha, Y_\varepsilon) = R(\alpha, A), \) therefore Lemma 5.1 implies, noting that \( |f(x, s)| \leq M \) for all \( (x, s) \in X \times \Omega, \)
\[
R_k(f, \alpha, Y_\varepsilon) = R(\alpha, A) \leq \max_{a \in A} \max_{i=0, \ldots, k} |a_i| \sqrt{2\ln(8R/\varepsilon)} \sum_{i=0}^k \alpha_i^2 \leq M \sqrt{2\ln(8R/\varepsilon)} \sum_{i=0}^k \alpha_i^2.
\]
This together with the preceding inequality imply
\[
R_k(f, \alpha, X) \leq L\varepsilon^\gamma + M \sqrt{2\ln(8R/\varepsilon)} \sum_{i=0}^k \alpha_i^2,
\]
and by taking \( \varepsilon = \left( \frac{\sum_{i=0}^k \alpha_i^2}{\beta_k} \right)^{1/\gamma}, \) we derive the desired estimate.