Light bending by a Coulomb field and the Aichelburg–Sexl ultraboost

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Abstract
We use light deflection by a Coulomb field, due to nonlinear quantum electrodynamics effects, as an opportunity for a pedagogical discussion of the electrodynamical analogue of the Aichelburg–Sexl ultraboost.

1. Introduction
Gravitational light deflection, predicted by general relativity, is a fascinating phenomenon with numerous important applications in astronomy, astrophysics and cosmology [1–3].

At first sight, there is no analogous effect in electrodynamics because Maxwell’s equations are linear and, therefore, a photon does not interact with the electromagnetic field of an alleged deflector charge. However, quantum electrodynamical corrections bring nonlinearities in the theory [4, 5]. As a result, in a external electromagnetic field, the vacuum acquires an effective refractive index [6, 7]

\[ n \approx 1 + \epsilon \vec{Q}^2, \] (1)

where

\[ \epsilon = \frac{a \alpha^2 \hbar^3}{45 m^4 c^5} \] (2)

and

\[ \vec{Q} = \vec{r} \times \vec{E} + \vec{r} \times (\vec{r} \times \vec{B}). \] (3)

Here, \( a = 14 \) or \( a = 8 \) depending on the polarization mode of the photon, \( \vec{r} \) is a unit vector in the direction of light propagation, \( m \) is the electron mass and \( \alpha = e^2/4\pi \hbar c \) is the fine structure constant.

One immediate consequence of this effective refractive index is the deflection of light in a Coulomb field [8, 9]. In this paper, we use light bending by a Coulomb field as a pedagogical tool for a discussion of the electrodynamical analogue of the general relativistic Aichelburg–Sexl ultraboost [10].
2. Light bending in a Coulomb field

In geometrical optics, the light trajectory in an inhomogeneous medium is determined by the equation \[ \frac{d}{ds} (n \vec{\tau}) = \nabla n, \] (4)
where \( s \) represents the arc length of the trajectory, \( n \) is the index of refraction and \( \vec{\tau} = \frac{d\vec{r}}{ds} \) is the unit tangent vector to the light ray. In the Coulomb field, equations (1) and (3) give
\[ n \approx 1 + \epsilon (\vec{\tau} \times \vec{E})^2, \]
(5)
where
\[ \vec{E} = \frac{Ze^4}{4\pi r^2} \vec{r} \]
is the Coulomb field of a nucleus with electric charge Ze (in the Heaviside–Lorentz rationalized natural unit system). For reasonable impact parameters, the nonlinear effects are very small, \( \epsilon E^2 \ll 1 \), and the index of refraction is only slightly different from unity.

It is clear from the symmetry of the problem that the light trajectory is a planar curve and therefore we can assume \( ds = \sqrt{dx^2 + dy^2} \) in Cartesian coordinates. The tiny light deflection angle can be found as follows. From (4) we obtain
\[ n \frac{d\tau_y}{ds} + \tau_y \frac{dn}{ds} = (\nabla n)_y = \frac{\partial n}{\partial y}. \]
(6)
However, both \( \tau_y \) and \( \frac{dn}{ds} \) are small quantities of order \( \epsilon \) (we assume that the incident light ray was in the \( x \)-direction). Therefore, neglecting the terms that are of the second order in the small parameter \( \epsilon \), (6) can be replaced by
\[ \frac{d\tau_y}{dx} = \frac{\partial n}{\partial y}, \]
(7)
and, hence,
\[ \tau_y = \int \frac{\partial n}{\partial y} \, ds, \]
(8)
where the integration contour \( \gamma \) is the light trajectory. But for small deflection angles (and note that \( \frac{\partial n}{\partial y} \) in (8) is very small), we can assume a rectilinear light trajectory while calculating the integral in (8) (the impulse approximation) and, finally,
\[ \tau_y \approx \int_{-\infty}^{\infty} \frac{\partial n}{\partial y} \, dx. \]
(9)
In the vicinity of the rectilinear light trajectory,

$$(\vec{r} \times \vec{E})^2 = E_y^2 \frac{y^2}{r^2} = \frac{Z^2 e^2}{16\pi^2} \frac{y^2}{(x^2 + y^2)^3}$$

and we obtain, on the trajectory,

$$\frac{\partial n}{\partial y} = \frac{\epsilon Z^2 e^2}{8\pi y^2} \left( \frac{b}{(x^2 + b^2)^3} - \frac{3b^3}{(x^2 + b^2)^4} \right),$$

(10)

where $b$ is the impact parameter for the incoming light ray (that is, the equation of the trajectory is $y = b$). Substituting (10) into (9), we obtain

$$\tau_y = \frac{\epsilon Z^2 e^2}{8\pi} \int_{-\infty}^{\infty} \left( \frac{b}{(x^2 + b^2)^3} - \frac{3b^3}{(x^2 + b^2)^4} \right) dx.$$  

(11)

The integral equals $-9\pi/16b^4$. Therefore,

$$\tau_y = -\frac{9\epsilon Z^2 e^2}{128\pi b^4}.$$  

(12)

Substituting here $\epsilon$ from (2), we obtain for the light deflection angle, in agreement with [8],

$$\theta \approx \sin \theta = |\tau_y| = \frac{a \alpha Z^2}{160} \left( \frac{\lambda_e}{b} \right)^4,$$  

(13)

where we have introduced the Compton wavelength of the electron $\lambda_e = \frac{\hbar}{mc}$.

### 3. Aichelburg–Sexl ultraboost for a Coulomb field

What is the electromagnetic field of a massless charged particle? This is a classic textbook problem [13] with an elegant and interesting solution. Considered in a number of publications [14–22] at various levels of mathematical rigor, this problem, however, has been largely ignored in classical electrodynamics textbooks (the third edition of Jackson’s classic [23] has it).

It is plausible to assume that the electromagnetic field of a massless charged particle is a limiting case of the field of an ultrarelativistic charged particle with finite mass. In the rest frame $S'$ of a charge $Ze$ we have the Coulomb field

$$\vec{E}' = \frac{Ze}{4\pi r^2} \vec{r}', \quad \vec{B}' = 0,$$  

(14)

In the laboratory frame $S$, where the charge moves with velocity $v$ along the $x$-axis, the electromagnetic field is given by [23, 24] (we will assume $c = 1$ for the light velocity in the rest of the paper)

$$E_x = E_x', \quad B_x = B_x', \quad E_y = \gamma(E_y' - \vec{v} \times \vec{B}'), \quad B_y = \gamma(B_y' + \vec{v} \times \vec{E}'),$$  

$$E_z = \gamma(E_z' - \vec{v} \times \vec{B}'), \quad B_z = \gamma(B_z' + \vec{v} \times \vec{E}').$$  

(15)

In combination with the Lorentz transformation

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx), \quad y' = y, \quad z' = z,$$  

(16)

(14) and (15) give

$$E_x = \frac{Ze}{4\pi y^2 R^3} (x - vt), \quad E_y = \frac{Ze}{4\pi y^2 R^3} y, \quad E_z = \frac{Ze}{4\pi y^2 R^3} z,$$  

$$\vec{B} = \vec{v} \times \vec{E},$$  

(17)

where

$$R = \sqrt{(x - vt)^2 + y^2 + z^2}.$$  

(18)
These are the well-known fields of a point charge in uniform motion, first obtained by Heaviside in 1888.

We need the limiting case of (17) when \( v \to 1 \). In the gravitational case, an analogous problem was considered by Aichelburg and Sexl in their seminal paper [10]. Therefore, usually such a limit is called the Aichelburg–Sexl ultraboost.

Note that
\[
\lim_{\gamma \to \infty} \frac{\gamma^{-2}}{R^3} = \begin{cases} 
\infty, & \text{if } x - t = 0, \\
0, & \text{if } x - t \neq 0,
\end{cases}
\]
but
\[
\int_{-\infty}^{\infty} \frac{\gamma}{(\gamma^2 x^2 + \rho^2)^{3/2}} \, dx = \frac{2}{\rho^2}.
\]
Therefore, we conclude that
\[
\lim_{\gamma \to \infty} \frac{\gamma^{-2}}{R^3} = \frac{2}{y^2 + z^2} \delta(x - t).
\] (19)

Using (19) and the identity \((x - t) \delta(x - t) = 0\), we obtain from (17) the electromagnetic field after the Aichelburg–Sexl ultraboost
\[
E_x = 0, \quad E_y = \frac{Ze}{2\pi} \frac{y}{y^2 + z^2} \delta(x - t), \quad E_z = \frac{Ze}{2\pi} \frac{z}{y^2 + z^2} \delta(x - t), \quad \vec{B} = \vec{i} \times \vec{E},
\] (20)
where \( \vec{i} \) is the unit vector in the \( x \)-direction. As we see, the fields are confined to a plane perpendicular to the motion, that sweeps along with the charge.

4. Small-angle scattering of a charged particle in a Coulomb field

As an application of the limiting electromagnetic field of the previous section, let us consider the small angle scattering of a highly relativistic charge \( e' \) on a heavy nucleus carrying a charge \( Ze \) [26].

In the rest frame \( S \) of the nucleus, the charge \( Ze \) is at the spatial origin and the charge \( e' \) moves, before the collision, with an ultra-relativistic speed \( v \approx 1 \) in the positive \( x \)-direction, and the orientations of the \( y \) and \( z \) axes are chosen in such a way that we have \( y = b \) and \( z = 0 \), \( b \) being the impact parameter.

In the rest frame \( S' \) of the projectile charge \( e' \), the nucleus appears to be travelling with the ultrarelativistic speed \( v \approx 1 \) in the negative \( x \)-direction, while the charge \( e' \) is sitting at the point \( x' = z' = 0 \), \( y' = b \). Therefore, the electromagnetic field of the nucleus in this frame is a plane impulsive electromagnetic wave given by (20) (with obvious substitution \( x \to -x \) because now the wave is travelling in the negative \( x \)-direction). When this impulsive wave meets the motionless charge \( e' \) at the time \( t' = 0 \), it will give the charge \( e' \) a kick in the \( y \)-direction because the only nonzero component of the electric field of the electromagnetic wave, on the line \( y' = b \), \( z' = 0 \), is
\[
E'_y = \frac{Ze}{2\pi b} \delta(x' + t').
\]
Therefore, after the kick the charge \( e' \) acquires a small momentum in the \( y \)-direction
\[
p'_y = p'_z = 0, \quad p'_y = \Delta p'_y = \int_{-\infty}^{\infty} e'e' \, dt' = \frac{Ze'e'}{2\pi} \frac{1}{b}
\] (in the impulse approximation, we can assume \( x' = 0 \) while calculating the integral).
Let us now return to the laboratory frame $S$ via the Lorentz transformations

$$p_x = \gamma(p'_x + v E') \approx \gamma m,$$

$$p_y = p'_y = \frac{Ze' e}{2\pi} \frac{1}{b'},$$

$$p_z = p'_z = 0,$$  \hspace{1cm} (22)

where in the first equation we have used $v \approx 1$ and $E' = \sqrt{m^2 + p'^2_z} \approx m$. Therefore, we get the following deflection angle in the laboratory frame:

$$\alpha \approx \tan \alpha = \frac{p_y}{p_x} = \frac{Ze' e}{2\pi} \frac{1}{mb'}. \quad \hspace{1cm} (23)$$

To check that the result (23) is correct, let us calculate it in the standard way [27]. We have

$$\Delta \tilde{p} = \int_{-\infty}^{\infty} \hat{F} \, dt = \int_{\gamma} \hat{F} \frac{dl}{v},$$

where the integration is along the projectile trajectory. For small angle scattering, we can assume that the projectile moves along the straight line while calculating the integral (24) (the impulse approximation). Therefore, for $v \approx 1$, we obtain for the $y$-component of the projectile momentum after the scattering

$$p_y = \Delta p_y = \int_{-\infty}^{\infty} F_y \, dx = \frac{Zee'}{4\pi} \int_{\infty}^{\infty} b \, dx \int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)^{3/2}}.$$  \hspace{1cm} (25)

The integral in (25) is $2/b$. Therefore,

$$p_y \approx \frac{Zee'}{2\pi b}$$

and, since $p = mv \gamma \approx m \gamma$,

$$\alpha \approx \sin \alpha = \frac{p_y}{p} = \frac{Ze' e}{2\pi} \frac{1}{mb' \gamma},$$

which coincides with (23).

5. Light deflection in a Coulomb field as a Cheshire cat’s smile

If we try to describe the light bending by a Coulomb field in the manner of the first method of the previous section, we encounter an immediate obstacle. The refractive index (5) depends quadratically on the electromagnetic field strength. However, the limiting field (20) is proportional to the delta function. Therefore, while calculating the refractive index, we get the square of the delta function which is notoriously an ill-defined quantity.

On the other hand, the electromagnetic energy–momentum tensor is quadratic in the fields too and, therefore, it also will contain squares of delta functions for the limiting field (20). This fact casts serious doubts on the physical reality of the limit implied in (20) [18]. However, more careful analysis of physical premises of the Aichelburg–Sexl limit indicates a way out of this dilemma.

While considering the $v \to 1$ limit, if we want the energy of the nucleus to remain finite, we must rescale its mass as follows [10]: $m \to \gamma^{-1} m$. If the mass were of purely electromagnetic origin, we would have $m \sim (Ze)^2$, where $Ze$ is the nuclear charge. This suggests that the physically interesting Aichelburg–Sexl limit may require not only $m \to \gamma^{-1} m$, but also the rescaling of the nuclear charge [18]

$$(Ze)^2 \to (Z e)^2 = \gamma^{-1} (Ze)^2.$$  \hspace{1cm} (26)

Note that this is exactly the kind of charge rescaling used in considerations of the Aichelburg–Sexl ultraboost for a Reissner–Nordström black hole [28].
Now, if we use the rescaling (26) in (17), we obtain
\[
E_1^2 = \left( \frac{\tilde{Z} e}{4\pi} \right)^2 \frac{\gamma^{-5}}{R^6} (x - vt)^2, \quad E_2^2 = \left( \frac{\tilde{Z} e}{4\pi} \right)^2 \frac{\gamma^{-5}}{R^6} y^2, \quad E_3^2 = \left( \frac{\tilde{Z} e}{4\pi} \right)^2 \frac{\gamma^{-5}}{R^6} z^2,
\]
and we should consider the limit of (27) when \( v \to 1 \). Note that
\[
(x - vt)^2 \frac{\gamma^{-5}}{R^6} = \frac{\gamma^{-5}}{R^6} [(x - t)^2 + (1 - v)t[2x - t(1 + v)]],
\]
and the second term leads to the well-defined limit (except the singular point \( x - t = y = z = 0 \))
\[
\lim_{v \to 1} \frac{\gamma^{-5}}{R^6} = 0.
\]
Indeed, this is evident if \( x - t \neq 0 \). But if \( x - t = 0 \), we have
\[
\lim_{v \to 1} \frac{\gamma^{-5}}{R^6} = \lim_{v \to 1} \frac{1}{(y^2 + z^2)} \sqrt{\frac{1 - v}{1 + v}} = 0.
\]
Next, we have to consider the limit
\[
\lim_{\gamma \to \infty} \frac{\gamma^{-5}}{R^6} = \begin{cases} \infty, & \text{if } x - t = 0, \\ 0, & \text{if } x - t \neq 0. \end{cases}
\]
But
\[
\int_{-\infty}^{\infty} \frac{\gamma^{-5} dx}{[x^2 + \gamma^{-2} \rho^2]^3} = \int_{-\infty}^{\infty} \frac{dx}{(x^2 + \rho^2)^3} = \frac{3\pi}{8\rho^5}.
\]
Therefore, we see that [18]
\[
\lim_{\gamma \to \infty} \frac{\gamma^{-5}}{R^6} = \frac{3\pi}{8(y^2 + z^2)^{5/2}} \delta(x - t),
\]
and since \((x - t)^2 \delta(x - t) = 0\), we obtain the following limiting field squares:
\[
E_1^2 = 0, \quad E_2^2 = \left( \frac{\tilde{Z} e}{4\pi} \right)^2 \frac{3\pi}{8} \frac{y^2}{(y^2 + z^2)^{5/2}} \delta(x - t), \quad E_3^2 = \left( \frac{\tilde{Z} e}{4\pi} \right)^2 \frac{3\pi}{8} \frac{z^2}{(y^2 + z^2)^{5/2}} \delta(x - t).
\]
The fields themselves, however, tend to zero under this Aichelburg–Sexl limit and we are left with a strange situation, which is as bizarre as the Cheshire cat’s smile\(^3\). The cat (electromagnetic field) disappears but its grin (electromagnetic effects quadratic in the fields) remains.

Nevertheless, this Cheshire cat’s smile, the limiting field squares (29), can be used to re-derive the light deflection formula (13) in the Coulomb field.

Suppose in the laboratory frame \( \mathcal{S} \) a light ray propagates along the line \( y = b, z = 0 \) in the positive \( x \)-direction and encounters a motionless nucleus of charge \( Ze \) situated at the spatial origin.

In the ultrarelativistic frame \( \mathcal{S}' \), which moves with the velocity \( v \approx 1 \) in the same direction as the incident light ray, the contracted electromagnetic field of the nucleus induces an effective index of refraction only in a thin layer moving with the speed \(-v\). According to relations (1)

\(^3\) ‘All right,’ said the Cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone. ‘Well! I’ve often seen a cat without a grin,’ thought Alice; ‘but a grin without a cat! It’s the most curious thing I ever saw in all my life!’ [29].
and (3), when this layer meets the incident light ray, the index of refraction at the encounter point equals (for some time, we will not use the primed notation, although we assume that we are in the frame $S'$)

$$n = 1 + 4\epsilon E^2,$$  

(30)

because for large $\gamma$ the contracted electromagnetic field of the nucleus looks like a plane impulsive electromagnetic wave in which $E^2 = B^2$, $E \perp B \perp \tau$ and, since the wave and the photon are in a head-on collision, $E \times B \parallel (-\tau)$.

We cannot replace the square of the nucleus electric field in (30) by the limiting square field (29) because the limit implies the rescaling of the charge while in our case the charge is not actually rescaled. However, if $\gamma$ is large, the field square of the rescaled charge $(Z\gamma e)^2 = (Ze)^2 \gamma^{-1}$ will be close to the limiting square field. Therefore, we can replace $E^2$ in (30) by $\gamma$ times the limiting square field with $Z = Z$ and the refractive index on the photon’s trajectory will take the form

$$n = 1 + 4\epsilon \gamma \left(\frac{Ze}{4\pi}\right)^2 \frac{3\pi}{8} \frac{\delta(x + t)}{\gamma^3}.$$

(31)

As before, we obtain

$$\tau_y \approx \int_{-\infty}^{\infty} \frac{\partial n}{\partial y} \, dx$$

and, while calculating the integral, we can assume the unperturbed photon trajectory $x = t$, $y = b$, $z = 0$, so that $\delta(x + t) = \delta(2x) = \frac{1}{2} \delta(x)$ (we are using again the impulse approximation). Thanks to the delta function, the integration is elementary and we obtain

$$\tau_y \approx \frac{9\pi}{4} \epsilon \gamma \left(\frac{Ze}{4\pi}\right)^2 \frac{1}{b^2}.$$

Therefore, in the $S'$ system, the deflection angle of the photon is

$$\alpha' \approx \sin \alpha' = |\tau_y| \approx \frac{9\pi}{4} \epsilon \gamma \left(\frac{Ze}{4\pi}\right)^2 \frac{1}{b^2}.$$  

(32)

In the laboratory frame $S$, the deflection angle can be obtained via the aberration formula (see, for example, [30]) which follows from the relativistic velocity addition law. Namely, when $v \approx 1$, we obtain

$$\sin \alpha = \frac{\sin \alpha'}{\gamma(1 + v \cos \alpha')} \approx \frac{1}{\gamma} \frac{\sin \alpha'}{1 + \cos \alpha'} = \frac{1}{\gamma} \tan \frac{\alpha'}{2}.$$  

(33)

Therefore,

$$\alpha \approx \frac{\alpha'}{2\gamma} \approx \frac{9\epsilon Z^2 e^2}{128\pi b^4},$$

(34)

which is exactly the result implied by (13).

6. Concluding remarks

We believe that the material presented above will be useful for students who have taken at least a semester of electrodynamics, or (more realistically) for graduate students towards the end of a full-year course in the subject. While, on the one hand, it is simple enough to follow with limited mathematical background, on the other hand, the argument is rather sophisticated and illustrates some difficulties of using generalized functions in nonlinear physical theories such as general relativity [31].
The Aichelburg–Sexl ultrarelativistic limit of the Coulomb field is subtle. We have a surprising result that the electrodynamics allows as a limit a massless ‘charged’ particle which creates no electromagnetic field, but has a nonzero electromagnetic energy–momentum tensor [18], and thus induces electromagnetic light deflection. Physically, this situation may seem unsatisfactory, but mathematically the Aichelburg–Sexl limit is perfectly well defined [21, 31]. We have seen in the previous section that, when appropriately used, this limit can produce physically reasonable results.

An interesting question remains whether a massless charge can really exist in nature. Up to now, no massless elementary particle with non-zero electric charge was ever found experimentally. It was argued that massless electric charges cannot exist in nature as they are completely locally screened in the process of formation [32]. However, that screening occurs only at very large distances and meanwhile the massless charge, born in the hard collisional process, may interact with the electromagnetic field [33]. We feel that, although massless charged particles are undoubtedly peculiar objects [34], the final word has not yet been said on the delicate issue of whether they really exist in nature.

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