The Sine-Gordon Solitons as a N-Body Problem.

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Abstract.
We consider the N-soliton solutions in the sine-Gordon model as a N-body problem. This leads to a relativistic generalization of the Calogero model first introduced by Ruijsenaars. We show that the fundamental Poisson bracket of the Lax matrix is quadratic, and the $r$-matrix is a dynamical one. This is in contrast to the Calogero model where the fundamental Poisson bracket of the Lax matrix is linear.

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1 Introduction.

We reconsider the relation between the sine-Gordon solitons and the relativistically invariant N-body problem introduced by Ruijsenaars. One of the motivations consists in extracting coordinates suitable for quantization. Since in the quantum theory, the asymptotic solitons are of primary importance [1, 2], soliton related coordinates seem appropriate. As illustrated by eq. (10), the sine-Gordon field is expressed in a simple way in terms of these coordinates. This kind of formula could be useful for computing N-soliton contribution to e.g. Casimir energy, form factors, etc...

The Ruijsenaars models [3] are relativistic generalizations of the Calogero models [4]. The later are integrable and even exactly solved: the spectrum and more remarkably the wave functions are known. Recently their rich algebraic structure began to appear [5]. It is tempting to believe that a similar structure will still exist in these relativistic models.

The feature of the Calogero models that we want to extend to the relativistic case is the existence of a $r$-matrix [6]. We find that the Poisson bracket for the Lax matrix is quadratic, in constrast to the non-relativistic case where it is linear. The existence of this quadratic Poisson bracket, eq. (19), is probably a manifestation of the quantum affine symmetry of the sine-Gordon field theory [7]. Despite the fact that our motivation is the quantum theory, in this Letter we restrict ourselves to the classical theory. We illustrate the use of the $r$-matrix by providing simple proofs for the commuting property of the Hamiltonians and for the symplectic property for a remarkable coordinate transformation. It is interesting to notice that after quantization, the Hamiltonians eq. (21, 22) will lead to finite difference problems instead of differential equations.

We would like to point out that elliptic generalizations of these models exist. It was recently shown independently by Sklyanin, that a quadratic Poisson bracket also exists for their Lax matrix [8].

2 Tau functions and solitons.

We first introduce few notations for the sine-Gordon equation and its solutions. Let $z_{\pm} = x \pm t$ be the light cone coordinates and $\partial_{\pm} = \frac{1}{2}(\partial_x \pm \partial_t)$. The sine-Gordon equation is:

$$\partial_{\pm} \partial_{\mp} \phi = 2 \sin(2\phi)$$

It is convenient to introduce the two tau functions $\tau_{\pm}$, which satisfy the Hirota equations:

$$\tau_{\pm}(\partial_{\mp} \partial_{\mp} \tau_{\pm}) - (\partial_{\mp} \tau_{\pm})(\partial_{\mp} \tau_{\pm}) = \tau_{\pm}^2 - \tau_{\mp}^2$$

The sine-Gordon field $\phi$ is related to the tau functions by

$$\frac{\tau_+}{\tau_-} = \exp(-i\phi).$$

The tau functions of the $N$-soliton solutions of the sine-Gordon equation are given by

$$\tau_{\pm}^{(N)}(z_+, z_-) = \det (1 \pm V)$$

with $V$ a $N \times N$ matrix with elements:

$$V_{ij} = 2 \frac{\sqrt{\mu_i \mu_j}}{\mu_i + \mu_j} \sqrt{X_i X_j} \quad \text{with} \quad X_i = a_i \exp \left(2 \left(\mu_i z_+ + \mu_i^{-1} z_-\right)\right)$$
The quantities $a_i$ and $\mu_i$ are the parameters of the solitons: $\mu_i$ are the rapidities and $a_i$ are related to the positions. For the sine-Gordon equation, they satisfy specific reality conditions. For solitons or antisolitons, the rapidity $\mu$ is real and $a$ is purely imaginary, i.e. $a = i\epsilon e^\gamma$ with $\epsilon = +1$ for a soliton and $\epsilon = -1$ for an antisoliton. The “breathers” correspond to pairs of complex conjugated rapidities $(\mu, \overline{\mu})$ and positions $(a, -\overline{a})$. Notice that these conditions are preserved by the dynamics.

The sine-Gordon equation is a Hamiltonian system. The symplectic form is the canonical one:

$$\Omega_{SG} = \int_{-\infty}^{+\infty} dx \, \delta \pi(x) \wedge \delta \phi(x)$$

with $\pi(x)$ the momentum conjugated to the field $\phi(x)$. Above, $\delta$ denote the differential on the phase space. One can take the restriction of this symplectic form to the $N$-soliton subspace of the phase space. In the coordinates $a_i$ and $\mu_i$, we find that this restriction is:

$$\omega = \sum_{i=1}^{N} \frac{\delta a_i}{a_i} \wedge \frac{\delta \mu_i}{\mu_i} + \sum_{i<j} \left( \frac{4\mu_i \mu_j}{\mu_i^2 - \mu_j^2} \right) \frac{\delta \mu_i}{\mu_i} \wedge \frac{\delta \mu_j}{\mu_j} \quad (4)$$

This two-form is non degenerate. It therefore defines a symplectic structure on the restricted phase space. The corresponding Poisson brackets are found by inverting the symplectic form (5):

$$\{\mu_i, \mu_j\} = 0$$

$$\{a_i, \mu_j\} = a_i \mu_j \delta_{ij}$$

$$\{a_i, a_j\} = -\left( \frac{4\mu_i \mu_j}{\mu_i^2 - \mu_j^2} \right) a_i a_j$$

How can Eq.(4) be proved? First, we consider the one and two soliton solutions. In these cases, the computation can be done directly using the formula for the field $\phi(x)$ and the momentum $\pi(x) = \partial_t \phi(x)$. We find:

$$\Omega_{SG} \big|_{\text{restricted}} = -2 \int_{-\infty}^{+\infty} dx \partial_x \left( \frac{1}{\tau_+ \tau_-} \right) \cdot \omega \quad (6)$$

Thanks to the behaviour of the $\tau$-functions at infinity, the prefactor is an irrelevant numerical constant. Then, we consider the general case with an arbitrary number of solitons with parameters $(\mu_i, a_i)$. Since the dynamics is Hamiltonian, the symplectic form can be computed at any time; in particular at $t \to \pm \infty$. In this in- or out-limit, the sine-Gordon field becomes asymptotically equal to the sum of the one-soliton solutions with parameters $(\mu_{i\text{ in}}, a_{i\text{ in}})$ and $(\mu_{i\text{ out}}, a_{i\text{ out}})$ [10]:

$$\mu_{i\text{ in}} = \mu_{i\text{ out}} = \mu_i$$

$$a_{i\text{ in}} = a_i \prod_{|\mu_j| \leq |\mu_i|} \left( \frac{\mu_i - \mu_j}{\mu_i + \mu_j} \right)^2$$

Since asymptotically the solitons decouple, in the symplectic form the crossed terms vanish (the overlap integrals are zero). Therefore the symplectic form reduces to the sum of the one-soliton expressions, but with the shifted in and out parameters:

$$\omega = \sum_{i=1}^{N} \frac{\delta a_{i\text{ in}}}{a_{i\text{ in}}} \wedge \frac{\delta \mu_i}{\mu_i} = \sum_{i=1}^{N} \frac{\delta a_{i\text{ out}}}{a_{i\text{ out}}} \wedge \frac{\delta \mu_i}{\mu_i}$$

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This is equal to eq. (4). If breathers are present the proof is identical since we checked eq. (6) up to two solitons.

As a by product, we see that the transformation from the in-variables \((\mu_i^{\text{in}}, a_i^{\text{in}})\) to the out-variables \((\mu_i^{\text{out}}, a_i^{\text{out}})\) is symplectic. The classical \(S\)-matrix is the generating function for this transformation.

Now that we have specified the Poisson brackets between the parameters \((\mu_i, a_i)\), we can compute the Poisson brackets between the matrix elements \(V_{ij}\). Remarkably, the latter can be written with the help of a \(d\)-matrix depending on the dynamical variables.

**Proposition.** Denote by \( E_{ij} = |i\rangle \langle j| \) the canonical basis of the \(N \times N\) matrices. Put \( V = \sum_{ij} V_{ij} E_{ij} \). Then:

\[
\{ V_1 \otimes V_2 \} = [d_{12}, V_1] - [d_{21}, V_2] \tag{7}
\]

where \( d_{ij} = \sum_{ijkl} d_{ij;kl} E_{ij} \otimes E_{kl} \) with,

\[
d_{ij;kl} = -\frac{1}{8} \left( \frac{\mu_i + \mu_j}{\mu_i - \mu_j} \right) (V_{jk} \delta_{il} + V_{jl} \delta_{ik} + V_{ik} \delta_{jl} + V_{il} \delta_{jk}) \tag{8}
\]

In Eq. (8), we used the standard notation in which the lower indices refer to the space on which the matrices are acting and \( d_{21} = P d_{12} P \) with \( P \) the flip operator. Eq. (7) can be proved by computing both sides and comparing them.

Alternatively, eq. (7) can be written in a quadratic form. Let \( M_{12} = \sum_{ijkl} M_{ij;kl} E_{ij} \otimes E_{kl} \) with, \( M_{ij;kl} = -\frac{1}{8} \left( \frac{\mu_i + \mu_j}{\mu_i - \mu_j} \right) \delta_{jk} \delta_{il} \), and define

\[
\begin{align*}
    r_{12} &= M_{12} - M_{21} - (M_{12}^t - M_{21}^t) \\
    s_{12} &= M_{12} + M_{21} - (M_{12}^t + M_{21}^t)
\end{align*}
\]

Then, we have:

\[
\{ V_1 \otimes V_2 \} = r_{12} V_1 V_2 + V_2 s_{12} V_1 - V_1 s_{12} V_2 - V_1 V_2 r_{12} \tag{9}
\]

Examples of this type of quadratic brackets first appeared in [12]. Finally, we recall that the existence of a \( d\)-matrix is equivalent to the property that the eigenvalues of the matrix \( V \) are Poisson commuting [13].

### 3 The N-soliton solution as a N-body problem.

From the definition of the sine-Gordon field in terms of the \(\tau\)-functions, we see that only the eigenvalues \(Q_i\) of the matrix \(V\) are important. We have

\[
e^{-i\phi} = \prod_{i=1}^{N} \left( \frac{1 + Q_i}{1 - Q_i} \right) \tag{10}
\]

In this section, we rewrite the soliton dynamics in terms of the variables \(Q_i\). This leads to a relativistic generalization of the Calogero model, first introduced by Ruijsenaars. We consider for definiteness the evolution with respect to the light cone variable \(z_+ = x + t\). Let \( U \) be the matrix which diagonalizes \(V\):

\[
V = U^{-1} QU \tag{11}
\]
where $Q$ denotes the diagonal matrix $\text{diag}(Q_i)$. We define

$$L = U\mu U^{-1}$$

(12)

where $\mu$ denotes the diagonal matrix $\text{diag}(\mu_i)$. As above, $\mu_i$ are the rapidities of the solitons. The matrix $L$ plays the role of a Lax operator. Obviously, the quantities $tr(L^n) = \sum_{i=1}^{N} \mu_i^n$ are conserved during the evolution of the solitons. Moreover, they are in involution under the Poisson bracket (3). Finally, we have

**Proposition.** The “time” evolution of $L$ is given by a Lax equation

$$\dot{L} = [M, L], \quad M = \dot{U}U^{-1}$$

Here the dot means $\frac{\partial}{\partial z^+}$. Remarkably, $L$ and $M$ can be expressed in terms of the quantities $Q_i$ and $\dot{Q}_i$:

$$L_{ij} = 2\sqrt{\dot{Q}_i\dot{Q}_j} / (Q_i + Q_j), \quad \text{and} \quad M_{ij} = \frac{\sqrt{\dot{Q}_i\dot{Q}_j}}{Q_i - Q_j}(1 - \delta_{ij})$$

(13)

**Proof.** We start from the relation

$$\mu V + V \mu = 2|e><e|$$

where $|e>$ is the column vector with components $e_i = \sqrt{X_i \mu_i}$. This relation remains true when we go to the basis where $V$ is diagonal. Using the definitions eqs.(11,12), we get

$$QL + LQ = 2|\tilde{e}><\tilde{e}|, \quad \text{with} \quad |\tilde{e}>= U|e>$$

Since $Q$ is diagonal, we immediatly obtain from this relation

$$L_{ij} = 2\tilde{e}_i\tilde{e}_j / (Q_i + Q_j)$$

Next, we remark that ($\sqrt{X_i} = \mu_i\sqrt{X_i}$)

$$\dot{V} = \mu V + V \mu = |e><e| = U^{-1} \left( \dot{Q} + [Q, M] \right) U$$

Multiplying this equation on the right by $U^{-1}$ and by $U$ on the left, and using the definition of $|\tilde{e}>$, we get $|\tilde{e}><\tilde{e}| = \dot{Q} + [Q, M]$. In components, this reads

$$\tilde{e}_i\tilde{e}_j = \dot{Q}_i\delta_{ij} + (Q_i - Q_j)M_{ij}$$

If $i = j$, we find $\tilde{e}_i = \sqrt{\dot{Q}_i}$, and if $i \neq j$, we find the value of $M_{ij}$ in terms of $Q_i$, $\dot{Q}_i$. This completely determines the matrix $M$ since we can always ensure that $M$ is antisymmetric by a suitable choice of basis. This ends the proof.
4 A canonical transformation.

We now come to the description of the Poisson structure in terms of the variables \( Q_i, \dot{Q}_i \). Instead of the variables \( \dot{Q}_i \), let us introduce the variables \( \rho_i = \dot{Q}_i / Q_i \). The Lax matrix becomes

\[
L_{ij} = 2 \sqrt{Q_i Q_j} / \sqrt{Q_i + Q_j} \rho_i \rho_j
\]

Comparing with eq.(3), we see that \( L \) has exactly the same form as \( V \) with the change of variables \( (\mu_i, a_i) \rightarrow (Q_i, \rho_i) \). This symmetry extends at the level of the symplectic structure.

We have

**Proposition.** The transformation \( (\mu_i, a_i) \rightarrow (Q_i, \rho_i) \) is a symplectic transformation.

**Proof.** We want to show that

\[
\omega = \sum_{i=1}^{N} \frac{\delta a_i}{a_i} \wedge \frac{\delta \mu_i}{\mu_i} + \sum_{i<j} \left( \frac{4 \mu_i \mu_j}{\mu_i^2 - \mu_j^2} \right) \frac{\delta \mu_i}{\mu_i} \wedge \frac{\delta \mu_j}{\mu_j}
\]

(14)

Let us take as independent variables the pairs \( (Q_i, \mu_i) \), and expand both lines of eq.(14) in these variables. The equality of the two-forms then requires that:

\[
\Lambda_{ij} = \left( \frac{\mu_j}{a_j} \frac{\partial a_i}{\partial \mu_j} - \frac{\mu_i}{a_i} \frac{\partial a_j}{\partial \mu_i} + \frac{4 \mu_i \mu_j}{\mu_i^2 - \mu_j^2} \right) = 0
\]

(15)

\[
\tilde{\Lambda}_{ij} = \left( \frac{Q_j}{\rho_j} \frac{\partial \rho_i}{\partial Q_j} - \frac{Q_i}{\rho_i} \frac{\partial \rho_j}{\partial Q_i} + \frac{4 Q_i Q_j}{Q_i^2 - Q_j^2} \right) = 0
\]

Let us first prove the third identity. By differentiating at \( \mu_i \) fixed the matrix \( V \) written in either forms eq.(3) or eq.(11), we obtain:

\[
dQ + [Q, dU U^{-1}] = U dV U^{-1} = \frac{1}{2} \left( (U a^{-1} da U^{-1}) Q + Q (U a^{-1} da U^{-1}) \right)
\]

with \( a \) the diagonal matrix \( \text{diag}(a_i) \). Using the fact that both \( Q \) and \( a \) are diagonal matrices, we derive that:

\[
\frac{dQ_i}{Q_i} = \sum_{j=1}^{N} U_{ij} \frac{da_j}{a_j} U_{ji}^{-1}
\]

or equivalently,

\[
\sum_{j=1}^{N} U_{ij} U_{ji}^{-1} \left( \frac{Q_k}{a_j} \frac{\partial a_j}{\partial Q_k} \right) = \delta_{ik}
\]
Introducing the matrices $M_{ij} = U_{ij} U^{-1}_{ji}$ and $A_{jk} = Q_k \frac{\partial a_j}{\partial Q_k}$, we can rewrite this equation in matrix form:

$$MA = \text{Id} \quad (16)$$

Similarly, using the expression of $L$ in terms of $\rho_i$ and $Q_j$, we find, by differentiating the relation $L = U \mu U^{-1}$ at $Q_j$ fixed, that:

$$RM = \text{Id} \quad (17)$$

where $R_{kj} = \mu_k \rho_j \frac{\partial \rho_j}{\partial \mu_k}$. Comparing eqs. (16) and (17) proves the third relation in (15) since the inverse of $M$ is unique.

Let us now prove that $\Lambda_{ij} = 0$. Expanding the first line of eq.(14) in the variables $(\mu_i, Q_i)$ we have

$$\omega = \sum_{ij} A_{ji} \frac{\delta Q_i}{Q_i} \wedge \frac{\delta \mu_j}{\mu_j} + \sum_{i<j} \Lambda_{ij} \frac{\delta \mu_i}{\mu_i} \wedge \frac{\delta \mu_j}{\mu_j}$$

The Poisson brackets $\{Q_i, Q_j\}$ can be computed by inverting the symplectic form. We find

$$\{Q_i, Q_j\} = \left( A^{-1} \Lambda^t A^{-1} \right)_{ij}$$

We know from the existence of the $d$-matrix (7), that these Poisson brackets vanish. Therefore $\Lambda_{ij} = 0$. The relations $\Lambda_{ij} = 0$ are functional relations between the variables $\mu_i, a_i$ at $Q_i$ fixed. Since the variables $(\mu_i, a_i)$ and $(Q_i, \rho_i)$ play a completely symmetric role, the conditions $\Lambda_{ij} = 0$ are also satisfied. This ends the proof that the transformation $(\mu_i, a_i) \rightarrow (Q_i, \rho_i)$ is symplectic.

Explicitly, the Poisson brackets between $(Q_i, \rho_i)$ are:

$$\{Q_i, Q_j\} = 0$$
$$\{\rho_i, Q_j\} = Q_j \rho_i \delta_{ij}$$
$$\{\rho_i, \rho_j\} = -\left( \frac{4Q_i Q_j}{Q_i^2 - Q_j^2} \right) \rho_i \rho_j$$

It will be convenient to introduce a new set of variables $p_i$, which are conjugated to the coordinates $Q_i$. We define them by:

$$\rho_i = \exp(p_i) \prod_{k \neq i} \left( \frac{Q_k + Q_i}{Q_k - Q_i} \right) \quad (18)$$

The Poisson brackets are then canonical:

$$\{Q_i, Q_j\} = \{p_i, p_j\} = 0$$
$$\{p_i, Q_j\} = Q_j \delta_{ij}$$
5 Hamiltonians and equations of motion.

The result of the previous section has an immediate consequence. The Poisson brackets of the Lax matrix $L$, given in eq.(13), are quadratic with a dynamical $R$-matrix:

$$\{L_1 \otimes L_2\} = R_{12}L_1L_2 + L_2S_{12}L_1 - L_1S_{12}L_2 - L_1L_2R_{12} \tag{19}$$

where $R_{12}$ and $S_{12}$ are defined as $r_{12}$ and $s_{12}$ in section 2, but with $\mu_i$ changed into $Q_i$. This follows directly from the fact that the transformation $(\mu_i, a_i)$ to $(Q_i, \rho_i)$ is symplectic, and the fact that the Lax matrix $L$ is identical to the matrix $V$ but with $(\mu_i, a_i)$ changed into $(Q_i, \rho_i)$.

This implies that the function $T(x) = \det(1 + xL)$ is a generating function of commuting Hamiltonians:

$$\{T(x), T(y)\} = 0 \tag{20}$$

Expanding it in power of $x$, we find:

$$T(x) = 1 + \sum_{p=1}^{N} x^p H_p$$

$$= 1 + \sum_{p=1}^{N} x^p \sum_{k_1 < \cdots < k_p} \rho_{\rho_{k_1}} \cdots \rho_{\rho_{k_p}} \prod_{k_i < k_j} \left(\frac{Q_{k_i} - Q_{k_j}}{Q_{k_i} + Q_{k_j}}\right)^2$$

As is well known, the existence of a $R$-matrix ensures that all the flow generated by these Hamiltonians admit a Lax pair formulation. In particular, the Hamiltonian generating the evolution in the light cone coordinate $z_+$ is $H_+ = \text{tr}(L)$:

$$H_+ = \sum_{j=1}^{N} \rho_j = \sum_{j=1}^{N} \exp(p_j) \prod_{k \neq j} \left(\frac{Q_j + Q_k}{Q_j - Q_k}\right) \tag{21}$$

The evolution in the other light cone coordinate $z_-$ is generated by the inverse of the Lax matrix. Its Hamiltonian is $H_- = \text{tr}(L^{-1})$. Using $\text{tr}(L^{-1}) = H_{N-1}/\det(L)$, we find:

$$H_- = \sum_{j=1}^{N} \rho_j^{-1} \prod_{k \neq j} \left(\frac{Q_j + Q_k}{Q_j - Q_k}\right)^2 = \sum_{j=1}^{N} \exp(-p_j) \prod_{k \neq j} \left(\frac{Q_j + Q_k}{Q_j - Q_k}\right) \tag{22}$$

Notice that one goes from $H_+$ to $H_-$ by changing the sign of $p_j$.

The equations of motion follow from these Hamiltonians. For the light cone coordinate $z_+$, we have

$$\dot{Q}_i = \rho_i Q_i$$

$$\dot{\rho}_i = \sum_{k \neq i} \left(\frac{4Q_i Q_k}{Q_i^2 - Q_k^2}\right) \rho_i \rho_k$$

where the dot still denotes $\frac{\partial}{\partial z_+}$. The flow is the $z_-$ direction is very similar: one just formally changes the dot by the derivative $\frac{\partial}{\partial z_-}$. These are the equations of motion written by Ruijsenaars [3].
To write explicitly the equations of motion for the (anti-) solitons and the breathers, we have to disentangle the reality conditions on \((Q_i, p_i)\). For the sine-Gordon field \((1)\) and the Hamiltonians \((2,22)\) to be real, the coordinates \((Q_i, p_i)\) have to come by pairs \((j, \overline{j})\), with \(Q_j = -Q_{\overline{j}}\) and \(p_j = p_{\overline{j}}\). The case \(j = \overline{j}\) corresponds to a soliton or an antisoliton, and the case \(j \neq \overline{j}\) to a breather. Therefore, we introduce new coordinates \(q_j\) by \(Q_j = ie^{q_j}\), such that:

\[
\begin{align*}
\text{Im } q_s &= 0 \quad \text{for } s \text{ a soliton} \\
\text{Im } q_\overline{s} &= \pi \quad \text{for } \overline{s} \text{ an antisoliton} \\
q_b &= q_b \quad \text{for } b \text{ a breather}
\end{align*}
\]

Similarly the momenta \(p_s\) and \(p_\overline{s}\) are real and \(p_b\) is complex with \(p_{\overline{b}} = \overline{p_b}\). In these coordinates, the Hamiltonians \(H_{\pm}\) become:

\[
H_{\pm} = \sum_j e^{\pm p_j} \prod_{k \neq j} \coth \left( \frac{q_j - q_k}{2} \right)
\]

The equations of motion for the flow generated by \(H_+\) read:

\[
\begin{align*}
\rho_i &= \dot{q}_i = e^{p_i} \prod_{k \neq i} \coth \left( \frac{q_i - q_k}{2} \right) \\
\ddot{q}_i &= \sum_{k \neq i} \frac{2\dot{q}_i \dot{q}_k}{\sinh(q_i - q_k)}
\end{align*}
\]

By construction, these evolution equations are linearized in the variables \((\mu_i, a_i)\).

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