EXISTENCE AND STABILITY OF THE LAMB DIPOLES TYPE FOR THE QUASI-GEOSTROPHIC SHALLOW-WATER EQUATIONS

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Abstract. In this paper, we construct a family of traveling wave vortex pairs with specific forms like Lamb Dipoles for the quasi-geostrophic shallow-water (QGSW) equations. The solutions are obtained by maximization of a penalized energy with multiple constraints. We establish the uniqueness of maximizers and compactness of maximizing sequences in our variational setting. Based on the vorticity method, we prove the orbital stability of the counter-rotating vortex pairs for the QGSW equations.

Keywords: Lamb dipole type, QGSW equations, orbital stability, variational problems

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1. Introduction and main results

In this paper we investigate the quasi-geostrophic shallow-water equation which is a non-linear and nonlocal transport equation generalizing 2D Euler equations and used to describe large scale motion for the atmosphere and the ocean circulation.

1.1. The quasi-geostrophic shallow-water equations. The quasi-geostrophic shallow water (QGSW) equations are derived from the rotating shallow water equations, in the limit of rapid rotation and weak variations of the free surface [33], which is given by

\[
\begin{aligned}
\partial_t q + v \cdot \nabla q &= 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\
v &= \nabla^\perp \psi \\
\psi &= (-\Delta + \varepsilon^2)^{-1} q \\
q|_{t=0} &= q_0
\end{aligned}
\]  

(1.1)

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where \( q \) refers to potential vorticity, \( v \) is the velocity field, \( \psi \) is the stream function, \( \nabla^\perp = (\partial_2, -\partial_1) \) and \( \varepsilon \in \mathbb{R} \).

When the parameter \( \varepsilon \) is positive, it is called the inverse Rossby deformation length, which is the natural length scale resulting from the balance between rotation and stratification. For \( \varepsilon = 0 \), we can recover the two-dimensional Euler equations.

In this paper, we will mainly consider the case when \( \varepsilon = 1 \) for equations (1.1), and the other cases also have corresponding results through similar discussions. So we can rewrite equation (1.1) as follows

\[
\partial_t q + v \cdot \nabla q = 0, \quad v = k \ast q \quad \text{in } \mathbb{R}^2 \times (0, \infty),
\]

\[
q = q_0 \quad \text{on } \mathbb{R}^2 \times \{t = 0\},
\]

with the kernel \( k(x) = \nabla^\perp G(x, 0) \), where \( G \) is the Green function of the Bessel operator \( -\Delta + I \) with the Dirichlet boundary condition in the whole plane. The equation (1.2) admit a vortex pair, which has the form

\[
v(x,t) = u(x + u_\infty t) - u_\infty,
\]

\[
q(x,t) = \omega(x + u_\infty t),
\]

with a constant velocity \( u_\infty \in \mathbb{R}^2 \) vanishing at space infinity.

A vortex pair is a symmetrical dipole with two vorticity compactly supported, which have opposite signs and shift in the same direction. They are theoretical models of coherent vortex structures in large-scale geophysical flows, and some experimental work can be referred to [34, 16].

Without losing generality, we may assume that \( u_\infty = (-W, 0), W > 0 \) by rotation invariance of (1.2). By substituting \((v, q)\) for equation (1.2), we can obtain the steady-state problem in the following form in the half plane \( \Pi = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\} \)

\[
u \cdot \nabla \omega = 0, \quad \text{in } \Pi,
\]

\[
u \to u_\infty \quad \text{as } |x| \to \infty.
\]

Historically, the Lamb vortex type has been studied more the Euler equations than the QGSW equations. The Lamb dipole introduced by H. Lamb [24] in 1906, which is a solution \( \omega_C = \lambda \max\{\Psi_C, 0\}, v_C = \nabla^\perp \Psi_C, 0 < \lambda < \infty \), with the form

\[
\Psi_C(x) = \begin{cases}
C_C J_1(\lambda^{1/2} r) \sin \theta, & r \leq a, \\
-W \left( r - \frac{a^2}{r} \right) \sin \theta, & r > a
\end{cases}
\]

with the constants

\[
C_C = \frac{2W}{\lambda^{1/2} J_0(c_0)}, \quad a = c_0 \lambda^{-1/2},
\]

where \((r, \theta)\) is the polar coordinate, \( J_m(r) \) is the \( m \)-th order Bessel function of the first kind and the constant \( c_0 \) is the first zero point of \( J_1 \), i.e., \( J_1(c_0) = 0, c_0 = 3.8317 \cdots, J_0(c_0) < 0 \). The Lamb dipole (1.4) is a special case of non-symmetric Chaplygin dipoles, independently founded by S. A. Chaplygin [12, 13, 27], so it also be called Chaplygin–Lamb dipole. The stream function \( \Psi_C \) is a solution of

\[
\begin{cases}
-\Delta \Psi = \lambda (\Psi - W x_2)_+ \quad \text{in } \Pi := \{x \in \mathbb{R}^2 \mid x_2 > 0\} \\
\Psi \to 0 \quad \text{as } r \to \infty, \quad \Psi = 0 \quad \text{on } \partial \Pi \\
\Psi(x_1, x_2) = -\Psi(x_1, -x_2), \quad \forall x \in \mathbb{R}^2
\end{cases}
\]
where \( f_+ \) denotes the positive part of \( f \). The Chaplygin-Lamb dipole \( \omega_C \) has the following form

\[
\omega_C(x_1, x_2) = -\omega_C(x_1, -x_2) = \lambda (\Psi_C(x) - Wx_2)_+, \quad \forall x \in \Pi.
\]

The Lamb dipole is regarded as a steady vortex structure, and there are some numerical and experimental studies on stability, see [20, 16]. Recently, its mathematical stability had been studied by K. Abe and K. Choi [1]. For the existence of the traveling-wave vortex pair problem of the Euler equations, see [10, 6], and the related stability results refer to [4, 8, 5].

It is worth noting that there are some mathematical and numerical studies of the vortex patch solution of the QGSW equation. Polvani [28] and Polvani, Zabusky and Flierl [29] computed the generalizations of Kirchhoff ellipses under various \( \varepsilon \) values, including doubly-connected patches and multi-layer flows. Later, H. Plotka and D. G. Dritschel [30] studied the equilibrium form and stability of the steadily rotating simply-connected vortex patches for the quasi-geostrophic shallow water equations numerically. There has been some recent work on the mathematical aspects of (1.1). D. G. Dritschel, T. Hmidi, and C. Renault [15] investigated analytical and numerical aspects of the bifurcation diagram of simply-connected rotating vortex patch equilibria for the quasi-geostrophic shallow-water (QGSW) equations.

### 1.2. Variational formulation

In this paper, we consider traveling solution for the quasi-geostrophic shallow-water (QGSW) equations, which has the fixed form and traveling with a constant velocity. We take the form of the traveling solution \( q(x,t) \)

\[
q(x,t) = \omega(x - Wte_1), \quad \forall \, t \in \mathbb{R}
\]

for some profile function \( \omega(x) \) defined on \( \mathbb{R}^2 \), where \( e_1 = (1,0) \) and \( W \geq 0 \) is the traveling speed. Let \( \Psi = (-\Delta + I)^{-1}\omega \), we can use (1.5) to rewrite the first equation in (1.2) as

\[
(\nabla \perp \Psi - W e_1) \cdot \nabla \omega = 0.
\]

This is equivalent to

\[
\nabla \perp (\Psi - Wx_2) \cdot \nabla \omega = 0.
\]

As described by V. I. Arnol’d [3], a natural approach to the solution of the stationary problem (1.6) is to impose that \( \Psi - Wx_2 \) and \( \omega \) are locally function dependent. In order to obtain traveling wave vortex pairs similar to the Lamb dipole type, we can assume that vorticity has the following functional correlation properties

\[
\omega = \lambda (\Psi - Wx_2)_+ \quad \text{in } \Pi,
\]

where \( \lambda \) will be chosen suitably. Then the stream function \( \Psi \) satisfies

\[
\begin{cases}
-\Delta \Psi + \Psi = \lambda (\Psi - Wx_2)_+ \quad \text{in } \Pi, \\
\Psi \to 0 \text{ as } r \to \infty, \quad \Psi = 0 \text{ on } \partial \Pi, \\
\Psi(x_1,x_2) = -\Psi(x_1,-x_2), \quad \forall x \in \mathbb{R}^2.
\end{cases}
\]

We observe that the flow is symmetric with respect to the \( x_1 \)-axis, and that its far field approximates the uniform flow in the \( x_1 \)-direction. We will prove the existence and uniqueness of the solution to problem (1.7) with the following expression

\[
\Psi_L(x) = \begin{cases}
(C_LJ_1((\lambda - 1)^{1/2}r) + \frac{W}{\lambda - 1}r) \sin(\theta), & r \leq a \\
(-W + \frac{W_0}{\lambda_{1(a)}}K_1(r)) \sin(\theta), & r > a,
\end{cases}
\]

\( 3 \)
where $J_1(r)$ is the Bessel function of the first kind of order one, $K_1(r)$ is the modified Bessel functions of the second kind of order one,

$$C_L = -\frac{Wa}{\lambda - 1} \cdot \frac{1}{J_1((\lambda - 1)^{1/2}a)}$$

and $a$ satisfies

$$a \left( \frac{K'_1(a)}{K_1(a)} + \frac{1}{(\lambda - 1)^{1/2} \cdot J'_1((\lambda - 1)^{1/2}a)} \right) = \frac{\lambda}{\lambda - 1}.$$

When $\lambda > 1$ is given, $a$ is the first positive zero of the function $W'$, where $W$ be defined on $A = \{ t \in \mathbb{R}^+ | J_1 \left( \frac{((\lambda - 1)^{1/2}t)}{t^{\lambda/(\lambda-1)}} \right) \neq 0 \}$, with the form

$$W(t) = \ln \frac{K_1(t)}{t^{\lambda/(\lambda-1)}} |J_1((\lambda - 1)^{1/2}t)|^{1/((\lambda-1)}), \quad t \in A. \quad (1.9)$$

It is easy to see that $\omega_L = \lambda (\Psi_L - Wx_2)_+$ is actually the type of the Chaplygin–Lamb dipole for two-dimensional Euler equations and Hill’s vortex (see [2, 14, 22]) introduced by Hill in 1894 for three-dimensional axisymmetric Euler equations.

We use the vortex method to construct the solution, which is usually used to construct the solution of the steady-state problem, as shown in [10, 9]. Compared with the stream function method, the vortex method can not only construct the solution, but also obtain the stability of the solution through uniqueness and compactness theorem.

Since the desired flows are odd symmetric about the $x_1$-axis, we can restrict our attention henceforth to the upper half-plane $\Pi$. Let $\bar{x} = (x_1, -x_2)$ be the reflection of $x$ in the $x_1$-axis. Denote

$$G_\Pi (x, y) = G (x, y) - G (\bar{x}, y), \forall x, y \in \Pi, \quad (1.10)$$

where

$$G (x, y) = \begin{cases} C \left( \ln \frac{2}{|x|} + 1 + O \left( |x - y|^2 \right) \right), & \text{if } |x - y| \leq 2, \\ Ce^{-|x-y|^2}, & \text{if } |x - y| > 2, \end{cases}$$

and

$$G_\omega (x) = \int_\Pi G_\Pi (x, y) \omega (y) \, dy, \forall x \in \Pi. \quad (1.11)$$

We introduced the kinetic energy of the fluid as follows

$$E (\omega) = \frac{1}{2} \int_\Pi \omega (x) G_\omega (x) \, dx,$$

and its impulse

$$I (\omega) = \int_{\mathbb{R}^2} x_2 \omega (x) \, dx.$$

To prove the existence theorem for solutions in section 2, we need a space of admissible functions

$$\mathcal{A}_{\mu, \nu} := \{ \omega \in L^2 (\Pi) \mid |\omega| \geq 0, \int_\Pi x_2 \omega \, dx = \mu, \int_\Pi \omega \, dx \leq \nu \},$$

and the energy functional $\mathcal{E}_\lambda$ corresponding to the flows

$$\mathcal{E}_\lambda (\omega) = E (\omega) - \frac{1}{2\lambda} \int_\Pi \omega^2 \, dx, \quad \omega \in \mathcal{A}_{\mu, \nu}.$$
We will consider the maximization of the energy functional $\mathcal{E}_\lambda$ relative to $A_{\mu,\nu}$ and denote that
\[
S_{\mu,\nu,\lambda} := \sup_{\omega \in A_{\mu,\nu}} \mathcal{E}_\lambda(\omega) \,. (1.12)
\]
Let the set of maximizers of (1.12) be as follows
\[
\Sigma_{\mu,\nu,\lambda} := \{ \omega \in A_{\mu,\nu} | \mathcal{E}_\lambda(\omega) = S_{\mu,\nu,\lambda} \} \,. (1.13)
\]
In literatures [1, 11], the scalability properties of the Laplace operator or the GSQG operator allow the parameters $\mu, \nu$ and $\lambda$ to be simplified. However, in this article, the bessel operator has no scaling properties resulting in our inability to simplify the parameters. Then we have to make some constraints on the parameters so that our proof can continue. Fortunately, inspired by Literature [4], we can obtain the existence of the solution through some translational properties, the specific process of which can be found in section 2.

1.3. Presentation of the main result. Our first result is about the existence of a traveling-wave vortex pair solution for Equation (1.1).

**Theorem 1.1.** There exists a Lipschitz smooth traveling wave vortex pair $\omega_L$, which has the following expression
\[
\omega_L(x_1, x_2) = \lambda (\Psi_L - Wx_2) \_ \text{ in } \Pi, (1.14)
\]
where $\lambda$ is a positive number greater than 1 and $\Psi_L$ is defined at (1.8). Moreover, the vorticity $\omega_L$ satisfies the following properties
\[
\begin{cases}
\omega_L(x_1, x_2) = \omega_L(x_1, -x_2), \text{ for } x = (x_1, x_2) \in \mathbb{R}^2, \\
\text{supp}(\omega_L) = B_a(0) \text{ is the ball with radius } a \text{ in the plane,} \\
q(x, t) = \omega_L(x - Wte_1) \text{ is a solution of (1.1), for some constant } W > 0,
\end{cases} (1.15)
\]
where $a$ is the first positive zero of the function $W'$, where $W$ be defined at (1.9), $\text{supp}(\cdot)$ denotes the support of a function.

In history, there are few examples with exact expressions for traveling wave solutions. Theorem 1.1 gives a solution similar to Lamb dipole, and it has exact expressions, which makes it convenient for us to study its uniqueness and stability.

The following theorem proves that the solution $\omega_L$ in Theorem 1.1, when the parameters satisfy some constraints, we can obtain its unique properties.

**Theorem 1.2.** There exists a constant $\mu_0$ such that if $0 < \mu \leq \mu_0$. Let $\omega_L$ be defined by (1.14). Then the set of maximizers of (1.12) are translation of $\omega_L$, i.e.,
\[
\Sigma_{\mu,\nu,\lambda} = \{ \omega_L(\cdot + ce_1) | c \in \mathbb{R} \}
\]
In [2], C. Amick and L. Fraenkel transform the Hill problem from two-dimensional space to five-dimensional space by transforming $\psi = r^2 \nu$, and obtain the existence and uniqueness results by using variational method and moving plane method respectively. Later, Burton [7] transformed the uniqueness of Chaplygin-Lamb dipole from two-dimensional space to four-dimensional space by transforming $\psi = r\nu$, and obtained the uniqueness by using the moving plane method.

In this paper, to prove Theorem 1.2, we continue Burton’s thought and obtain the uniqueness theorem of solutions in four-dimensional space. These uniqueness theorems, combined with the characterization of the energy of the solution, are fundamental to establishing the
compactness of the maximized sequence and the stability of the solution. Interested readers may refer to [14, 5, 8, 4] and their references for the stability of vortex solutions of Euler equations.

For QGSW equations, there are some numerical stability results [30, 23], but there seem to be very few results on mathematical stability. However, due to the special energy characteristics of our constructed solution, we can prove its orbital stability. Similar to Burton [8], we need to introduce the following definition of $L^p$-regular solution.

**Definition 1.3.** For the function $\zeta \in L^\infty_{loc}([0, \infty), L^1(\mathbb{R}^2)) \cap L^\infty_{loc}([0, \infty), L^p(\mathbb{R}^2))$ is called a $L^p$-regular solution of (1.1), if $\zeta$ satisfies (1.1) in the sense of distributions, such that $E(\zeta(t, \cdot)), I(\zeta(t, \cdot))$ and $\|\zeta(t, \cdot)\|_s$ for $1 \leq s \leq p$ are constant for $t \in [0, \infty)$. Moreover, if $\zeta_0$ is non-negative and odd symmetric in $x_2$, then we require that $\zeta(t, \cdot)$ is also non-negative and odd symmetric in $x_2$.

Roughly speaking, the $L^p$-regular solution is a weak solution of (1.1), whose kinetic energy, impulse, and $L^p$ norm are conserved when $1 \leq s \leq p$. This is true for sufficiently smooth solutions. By using the transport characteristics of the Euler equation, Burton [8], K. Abe and K. Choi [1] obtains the existence of $L^p$-regular solutions for the Euler equation.

Since the QGSW equations is also a transport equation, the existence of $L^p$-regular solution in Definition 1.3 above can be proved by making some modifications through the method in [8, 1] as long as the existence of sufficiently smooth solutions to the Cauchy problem of the QGSW equations is a priori known.

Next, we will prove that the solution in Theorem 1.2 has the following stability theorem.

**Theorem 1.4.** For $0 < \mu \leq \mu_0$, the circular vortex-pair $\omega_L$ in Theorem 1.2 is orbitally stable in the sense that, for any $M > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for non-negative function $\zeta_0 \in L^1 \cap L^2(\Pi)$ with $\|\zeta_0\|_1 \leq \nu, \|\zeta_0\|_2 < M$ and

$$\inf_{c \in \mathbb{R}} \{\|\zeta_0 - \omega_L (\cdot + ce_1)\|_2 + \|x_2 (\zeta_0 - \omega_L (\cdot + ce_1))\|_1\} \leq \delta,$$

if there exists a $L^2$-regular solution $\zeta(t)$ with initial data $\zeta_0$ for $t \in [0, \infty)$, then

$$\inf_{c \in \mathbb{R}} \{\|\zeta(t) - \omega_L (\cdot + ce_1)\|_2 + \|x_2 (\zeta(t) - \omega_L (\cdot + ce_1))\|_1\} \leq \varepsilon, \quad \forall t \in [0, \infty).$$

The organization of this paper is as follows. In Section 2, we prove some basic lemmas that are fundamental to proving the existence of solutions, and the existence of maximizers of $\mathcal{E}_\lambda$ with respect to $\mathcal{A}_{\mu, \nu}$. In Section 3, we established the uniqueness of the maximizer by direct calculation. Then we prove the compactness of maximized sequences in Section 4 by using Lions’ [25] theorem of concentrated compactness. Finally, we proved orbital stability in section 5.

2. Existence of Maximizers

In this section, we need some basic estimates to prove the existence of maximizers, which will be used frequently later.

**Lemma 2.1.** There exists a positive constant $C$ such that if $0 \leq \omega \in L^1(\Pi) \cap L^2(\Pi)$, then

$$G_\omega \leq C(x_2 + 1)^{1/2} \|\omega\|_1^{1/2} \|\omega\|_2^{1/2},$$

and

$$E(\omega) \leq C\|(x_2 + 1)\omega\|_1^{1/2} \|\omega\|_1 \|\omega\|_2^{1/2}.$$
Proof. Let us first prove (2.1). By the Hölder’s inequality, we have
\[
\int_{\Pi} G_{\Pi}(x, y)\omega(y)dy = \int_{|x - y| \leq 2} G_{\Pi}(x, y)\omega(y)dy + \int_{|x - y| \geq 2} G_{\Pi}(x, y)\omega(y)dy
\]
\[
\leq C \left( \int_{|x - y| < 2} \left( \ln(1 + \frac{4|x_2y_2|}{|x - y|^2}) \right)^4 dy \right)^{1/4} \|\omega\|_{4/3} + C \|\omega\|_{4/3}
\]
\[
\leq C(x_2 + 1)^{1/2}\|\omega\|_1^{1/2}\|\omega\|_2^{1/2}.
\]
By the definition of \(E(\omega)\), (2.1) and the Hölder’s inequality, we get
\[
E(\omega) \leq C\|\omega\|_1^{1/2}\|\omega\|_2^{1/2} \int_{\Pi} (x_2 + 1)^{1/2}\omega(x)dx
\]
\[
\leq C\|(x_2 + 1)\omega\|_1^{1/2}\|\omega\|_1\|\omega\|_2^{1/2}
\]
\(\square\)

Lemma 2.2. Suppose that \(0 \leq \omega \in L^1(\Pi) \cap L^2(\Pi)\), we have
\[
G\omega(x) \to 0 \text{ as } |x| \to \infty. \tag{2.3}
\]

Proof. For \(|x|\) large and \(x_2 \leq 4\), by (2.1) we have
\[
0 \leq G\omega(x) \leq \int_{|y| \leq |x|/2} G_{\Pi}(x, y)\omega(y)dy + \int_{|y| \geq |x|/2} G_{\Pi}(x, y)\omega(y)dy
\]
\[
\leq C \left( e^{-|x|/4}\|\omega\|_1 + \|\omega\|_{11} B_{|x|/2}(0) \right) + 1\|\omega\|_{11} B_{|x|/2}(0) \right) \leq o(1).
\]
If \(x_2 > 4\), by the following decomposition
\[
\int_{\Pi} G_{\Pi}(x, y)\omega(y)dy = \int_{|x - y| \geq x_2/2} + \int_{|x - y| < x_2/2}.
\]
We have
\[
\int_{|x - y| \geq x_2/2} G_{\Pi}(x, y)\omega(y)dy \leq Ce^{-x_2/4}\|\omega\|_1.
\]
By the Hölder’s inequality, \(1/q + 1/q' = 1\), \(1/q = \theta + (1 - \theta)/2\),
\[
\int_{|x - y| < x_2/2} G_{\Pi}(x, y)\omega(y)dy \leq \left( \int_{|x - y| < x_2/2} G_{\Pi}(x, y)^q dy \right)^{1/q'} \left( \int_{|x - y| < x_2/2} \omega(y)^q dy \right)^{1/q}
\]
\[
\leq C(x_2 + 1)^{2/q'} \left( \int_{|x - y| < x_2/2} \omega(y)^q dy \right)^{\theta} \left( \int_{|x - y| < x_2/2} \omega(y)^2 dy \right)^{(1-\theta)/2}.
\]
Since
\[
\int_{|x - y| < x_2/2} \omega(y)dy \leq \frac{2}{(x_2 + 1)}((y_2 + 1)\omega)_1,
\]
we have
\[
\int_{|x - y| < x_2/2} G_{\Pi}(x, y)\omega(y)dy \leq \frac{C}{(x_2 + 1)^{4/q-3}}((y_2 + 1)\omega)_1\|\omega\|_1^{\theta}\|\omega\|_2^{1-\theta}.
\]
For $\delta \in (0,1)$, we take $q \in (1,2]$ small enough, then
\[
\int_{\Pi} G_\Pi(x,y)\omega(y)dy \leq \frac{C_\delta}{(x_2+1)^{1-\delta}} (\|y_2\omega\|_1 + \|\omega\|_{L^1_{x_1}L^2}) + C e^{-x_2/4}\|\omega\|_1. \quad (2.4)
\]
We take a sequence $\{\omega_n\}_{n=1}^\infty \subset C^\infty_c(\Pi)$ such that $\omega_n \to \omega$ in $L^1 \cap L^2(\Pi)$ and $y_2\omega_n \to y_2\omega$ in $L^1(\Pi)$. By (2.4), it holds
\[
\int_{\Pi} G_\Pi(x,y)\omega(y)dy = \int_{\Pi} G_\Pi(x,y)(\omega(y) - \omega_n(y))dy + \int_{\Pi} G_\Pi(x,y)\omega_n(y)dy
\leq C(\|y_2(\omega - \omega_n)\|_1 + \|\omega - \omega_n\|_{L^1_{x_1}L^2})
+ \frac{C x_2}{\inf_{y \in \text{supp}(\omega_n)}|x - y|^2} \|y_2\omega_n\|_1 + Ce^{-\inf_{y \in \text{supp}(\omega_n)}|x - y|^2/2}\|\omega_n\|_1.
\]
Letting $|x| \to \infty$ and then $n \to \infty$, we get (2.3). \hfill \Box

Since the energy $E_\lambda$ is invariant under translations in the $x_1$-direction, to control maximizers, we shall take the Steiner symmetrization in the $x_1$-variable.

We have the following result, which can be found in [17, 31, 32].

**Lemma 2.3.** For $\omega \geq 0$ satisfying $\omega \in L^1 \cap L^2(\Pi)$ and $x_2\omega \in L^1(\Pi)$, there exists $\omega^* \geq 0$ such that
\[
\omega^*(x_1, x_2) = \omega^*(-x_1, x_2),
\omega^*(x_1, x_2) \text{ is non-increasing for } x_1 > 0
\]
and
\[
\|\omega^*\|_q = \|\omega\|_q \quad 1 \leq q \leq 2,
\|x_2\omega^*\|_1 = \|x_2\omega^*\|_1,
E(\omega^*) \geq E(\omega).
\]

For a Steiner symmetric function, we have the decay estimate for the stream function for the $x_1$-variable.

**Lemma 2.4.** For $0 \leq \omega \in L^1 \cap L^2(\Pi)$ that are Steiner symmetric in the $x_1$-variable, we have
\[
G\omega \leq C \left( (x_2 + 1)^{1/2} |x_1|^{-3/8} \|\omega\|_1^{1/2} \|\omega\|_2^{1/2} + \frac{x_2}{|x_1|} \|x_2\omega\|_1 + e^{-\frac{x_2}{2\sqrt{|x_1|}}} \|\omega\|_1 \right), \quad x \in \Pi, \quad (2.6)
\]

**Proof.** For $x \in \Pi$ fixed, let
\[
\omega_1(y) = \begin{cases} 
\omega(y), & \text{if } |y_1 - x_1| < \sqrt{|x_1|}, \\
0, & \text{if } |y_1 - x_1| \geq \sqrt{|x_1|}.
\end{cases}
\]
Using equation (2.11) in [6], it is easy to see that
\[
\|\omega_1\|_p \leq |x_1|^{-\frac{3p}{8}} \|\omega\|_p, \quad 1 \leq p \leq \infty.
\]
Hence, by (2.1), we have

\[ G\omega_1(x) \leq C(x_2 + 1)^{1/2}\|\omega_1\|_1^{1/2}\|\omega_1\|_2^{1/2} \leq C(x_2 + 1)^{1/2}|x_1|^{-3/8}\|\omega\|_1^{1/2}\|\omega\|_2^{1/2}. \]  

(2.7)

Letting \( \omega_2 = \omega - \omega_1 \), we have

\[ G\omega_2(x) = \int_{|x_1 - y_1| > \sqrt{|x_1|}} G\Pi(x, y)\omega_2(y)dy \leq C \int_{|x_1 - y_1| > \sqrt{|x_1|}} \left( \frac{x_2y_2}{|x - y|^2} + e^{-|x-y|^2} - e^{-|x-y|^2} \right) \omega(y)dy \leq C \frac{x_2}{|x_1|} \|x_2\omega\|_1 + C e^{-\frac{|x_1|^2}{4}} \|\omega\|_1, \]  

(2.8)

Combining with (2.7) and (2.8), we get (2.6). \( \square \)

For compactness estimation, we need the maximum to be positive and not infinite.

**Lemma 2.5.** If \( 0 < \nu < \infty, 0 < \mu < \min\{1, \frac{2}{\pi^2} c_0^{3/2} \nu^{3/2} \} \) and \( \lambda \mu^{3/2} > c \) for some constant \( c \), then

\[ 0 < S_{\mu, \nu} < \infty, \]  

(2.9)

**Proof.** By (2.2), we have for \( \omega \in A_{\mu, \nu} \)

\[ E_{\lambda}(\omega) = E(\omega) - \frac{1}{2\lambda} \int_{\Pi} \omega^2 dx \leq C\|(x_2 + 1)\omega\|_1^{1/2}\|\omega\|_1\|\omega\|_2^{1/2} - \frac{1}{2\lambda} \int_{\Pi} \omega^2 dx \leq C(\mu + \nu)^{1/2}\|\omega\|_2^{1/2} - \frac{1}{2\lambda} \|\omega\|_2. \]

From \( \|\omega\|_2 \in [0, \infty) \), we get \( S_{\mu, \nu} < \infty \).

Let \( \omega_1 = 1_B \) with \( B = \Pi \cap B_r(0) \), \( r > 0 \) satisfying

\[ \int_{\Pi} x_2 \omega_1 dx = \mu, \int_{\Pi} \omega_1 dx \leq \nu, \]  

(2.10)

then \( \omega_1 \in A_{\mu, \nu} \). From the first integral of equation (2.10) above, we get

\[ \int_{\Pi} x_2 \omega_1 dx = \int_B x_2 \omega_1 dx = r^3 \int_{\Pi \cap B_1(0)} y_2 dy = c_0 r^3 = \mu. \]

This implies

\[ r = c_0^{-\frac{1}{3}} \mu^{\frac{1}{3}}. \]

From the second integral of equation (2.10), we have

\[ \int_{\Pi} \omega_1 dx = \frac{\pi}{2} r^2 = \frac{\pi}{2} c_0^{-\frac{2}{3}} \mu^{\frac{2}{3}}, \quad \int_{\Pi} \omega_1^2 dx = \frac{\pi}{2} c_0^{-\frac{2}{3}} \mu^{\frac{1}{3}}. \]
If \( \mu \) is sufficiently small (\( \mu << 1 \)), by expanding the Bessel kernel, we obtain

\[
\int_{\Pi} \int_{\Pi} G_{\Pi}(x,y) \omega_1(x) \omega_1(y) \, dx \, dy = \int_{\Pi \cap B_r(0)} G_{\Pi}(x,y) \, dx \, dy
\]

\[
= \frac{1}{2\pi} \int_{\Pi \cap B_r(0)} \left( \ln \frac{|x-y|}{|x-y|} + o(1) \right) \, dx \, dy
\]

\[
= \frac{1}{2\pi} \int_{\Pi \cap B_r(0)} \ln \frac{|x-y|}{|x-y|} \, dx \, dy + o(1) \left( \int_{\Pi} \omega_1 \, dx \right)^2
\]

\[
= \frac{r^4}{2\pi} \int_{\Pi \cap B_r(0)} \ln \frac{|x-y|}{|x-y|} \, dx \, dy + o \left( \mu^\frac{4}{3} \right)
\]

\[
= c_1 \mu^\frac{4}{3} + o \left( \mu^\frac{4}{3} \right).
\]

We choose \( \lambda, \mu \) such that

\[
\lambda \mu^\frac{2}{3} > \frac{\pi c_0}{4} \frac{-\frac{2}{3}}{c_1} := c
\]

Hence

\[
E_\lambda(\omega_1) = \int_{\Pi} \int_{\Pi} G_{\Pi}(x,y) \omega_1(x) \omega_1(y) \, dx \, dy - \frac{1}{2\lambda} \int_{\Pi} \omega_1^2(x) \, dx
\]

\[
= c_1 \mu^\frac{4}{3} - \frac{1}{2\lambda} \cdot \frac{\pi}{2} c_0^{-\frac{2}{3}} \mu^\frac{4}{3} + o \left( \mu^\frac{4}{3} \right) > 0.
\]

We obtain \( E_\lambda(\omega_1) > 0 \), that is \( S_{\mu,\nu} > 0 \).

Through the previous estimation, we can obtain the existence of the maximizers.

**Lemma 2.6.** For \( \mu, \nu, \lambda \) satisfies the assumption of Lemma 2.5, then there exist \( \omega_{\mu,\nu} \in A_{\mu,\nu} \) such that

\[
E_\lambda(\omega_{\mu,\nu}) = \sup_{\omega \in A_{\mu,\nu}} E_\lambda(\omega).
\]

**Proof.** Let \( \{\omega_j\}_{j=1}^\infty \subset A_{\mu,\nu} \) be a maximizing sequence. By Lemma 2.5, it can be seen that \( E(\omega_j) > 0 \) for all large \( j \). Using the definition of \( E_\lambda \) and (2.2), we have

\[
\|\omega_j\|_2^2 \leq 2\lambda \left( E(\omega_j) - E(\omega_j) \right) < 2\lambda E(\omega_j) \leq C \left( (x_2 + 1)|\omega_j|_1 \right)^{1/2} \|\omega_j\|_1 \|\omega_j\|_2^{1/2} \leq C \|\omega_j\|_2^{1/2}.
\]

Hence \( \|\omega_j\|_2 \) is bounded by a constant independent of \( j \).

According to Lemma 2.3, we may assume that \( \omega_j \) is Steiner symmetric by replacing \( \omega_j \) with its Steiner symmetrisation. We assume \( \omega_j \rightharpoonup \omega \) weekly in \( L^2(\Pi) \) as \( j \to \infty \) by passing to a sub-sequence if necessary (still denoted by \( \{\omega_j\}_{j=1}^\infty \)). By weak lower semi-continuity, one can verify that

\[
\int_{\Pi} x_2 \omega \, dx \leq \mu \text{ and } \int_{\Pi} \omega \, dx \leq \nu.
\]
On the one hand, by Lemmas 2.1 and 2.4, we note that
\[ 2E(\omega_j) = \int_{\Pi} \int_{\Pi} \omega_j(x)G\omega_j(y) \, dx \, dy \]
\[ \leq \int_{|x_1|<R, 0<x_2<R \cap |y_1|<R, 0<x_2<R} \omega_j(x)G\omega_j(y) \, dx \, dy \]
\[ \leq \int_{|x_1|<R, 0<x_2<R} \int_{|y_1|<R, 0<x_2<R} \omega_j(x)G\omega_j(y) \, dx \, dy \]
\[ + C \left( R^{-3/8} \|\omega_j\|_1 \|\omega_j\|_2^{1/2} \left( \|x_2\omega_j\|_1 + \|\omega_j\|_1 \right) + R^{-1} \|x_2\omega_j\|_1^2 + e^{-\frac{\omega_j}{\lambda}} \|\omega_j\|_1^2 \right) \]
\[ + 2R^{-1/2} C \|\omega_j\|_1 \int_{x_2>R} \omega_j(x+1) \, dx \]
\[ \leq \int_{|x_1|<R, 0<x_2<R} \int_{|y_1|<R, 0<x_2<R} \omega_j(x)G\omega_j(y) \, dx \, dy \]
\[ + C \left( R^{-3/8} \|\omega_j\|_1 \|\omega_j\|_2^{1/2} \left( \|x_2\omega_j\|_1 + \|\omega_j\|_1 \right) + R^{-1} \|x_2\omega_j\|_1^2 + e^{-\frac{\omega_j}{\lambda}} \|\omega_j\|_1^2 \right) \]
\[ + 2R^{-1/2} C \|\omega_j\|_1 \int_{x_2>R} (\|x_2\omega_j\|_1 + \|\omega_j\|_1) \]

Due to $G_\Pi \in L^2_{\text{loc}}(\overline{\Pi} \times \overline{\Pi})$, we get
\[ \limsup_{j \to \infty} E(\omega_j) \leq E(\omega) \]

by first letting $j \to \infty$ and then $R \to \infty$.

On the other hand, we have
\[ 2E(\omega_j) = \int_{\Pi} \omega_j G\omega_j \, dx \geq \int_{|x_1|<R, 0<x_2<R \cap |y_1|<R, 0<x_2<R} \omega_j(x)G\omega_j(y) \, dx \, dy, \]
it implies that
\[ \liminf_{j \to \infty} E(\omega_j) \geq E(\omega) \]

by first letting $j \to \infty$ and then $R \to \infty$.

Hence, we obtain that
\[ \lim_{j \to \infty} E(\omega_j) = E(\omega) \]

and
\[ E_\lambda(\omega) = E(\omega) - \frac{1}{2\lambda} \int_\Pi \omega^2 \, dx \geq \lim_{j \to \infty} E(\omega_j) - \liminf_{j \to \infty} \frac{1}{2\lambda} \int_\Pi \omega_j^2 \, dx = S_{\mu,\nu}. \]

We now check that $\int_\Pi x_2 \omega \, dx = \mu$. Indeed, if not, then there exist some $\tau > 0$ such that
\[ \omega_\tau(x_1, x_2) := \begin{cases} \omega(x_1, x_2 - \tau), & \text{if } x_2 > \tau, \\ 0, & \text{if } x_2 \leq \tau, \end{cases} \]

belongs to $A_{\mu,\nu}$. Since $G_\Pi(x, y) = G(|x - y|) - G(|\overline{x} - y|)$ and $G(r)$ is decreasing at $(0, \infty)$, it yields that
\[ S_{\mu,\nu} = E_\lambda(\omega) < E_\lambda(\omega_\tau) \leq S_{\mu,\nu}. \]

This is a contradiction and the proof is thus complete. \qed
By looking at the latter part of the Lemma 2.6, we can obtain the increasing property of $S_{\mu,\nu}$ with respect to $\mu$.

**Lemma 2.7.** If $0 < \alpha, \mu, \nu < \infty$, the parameter triples $(\alpha, \nu, \lambda)$ and $(\mu, \nu, \lambda)$ satisfies the assumption of Lemma 2.5 then

$$S_{\alpha,\nu} < S_{\mu,\nu}, \text{ for } \alpha < \mu.$$ (2.11)

**Proof.** By the proof of Lemma 2.6, we can also obtain Lemma 2.7. \qed

3. **Uniqueness of Maximizers**

At the end of the previous section, we proved the existence of maximizers for $E_\lambda$ over $A_{\mu,\nu}$. To prove the compactness theorem, we also need the uniqueness of maximizers. The proof idea of the following lemma was first used in [19, 18].

**Lemma 3.1.** Each $\omega \in \Sigma_{\mu,\nu}$ satisfies

$$\omega = \lambda(G\omega - W x_2 - \gamma)_+$$ (3.1)

for some constants $W, \gamma \geq 0$, uniquely determined by $\omega$.

**Proof.** By Lemma 2.5, $S_{\mu,\nu} > 0$. There exists a constant $\delta_0 > 0$ such that $\text{meas}\{\delta_0 < \omega\} > 0$. We take functions $h_1, h_2 \in L^\infty(\Pi)$ with compact support and satisfying

$$\begin{align*}
&\text{supp}(h_1), \text{supp}(h_2) \subset \{\delta_0 \leq \omega\}, \\
&\int_{\Pi} h_1(x)dx = 1, \int_{\Pi} x_2 h_1(x)dx = 0, \\
&\int_{\Pi} h_2(x)dx = 0, \int_{\Pi} x_2 h_2(x)dx = 1,
\end{align*}$$

We take any arbitrary $\delta \in (0, \delta_0)$ and compactly supported $h \in L^\infty(\Pi)$, $h \geq 0$ on $\{0 \leq \omega \leq \delta\}$. We set the test function

$$\omega_\epsilon = \omega + \epsilon \eta, \epsilon > 0,$$

where

$$\eta = h - \left(\int_{\Pi} hdx\right) h_1 - \left(\int_{\Pi} x_2 hdx\right) h_2.$$

If $\epsilon$ is small enough, one can verify that $\omega_\epsilon \in A_{\mu,\nu}$. Since $\omega$ is a maximizer,

$$0 \geq \left.\frac{dE(\omega_\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{\Pi} (G\omega - \frac{1}{\lambda}\omega) \eta dx.$$

We define

$$\gamma := \int_{\Pi} (G\omega - \frac{1}{\lambda}\omega) h_1 dx, \ W := \int_{\Pi} (G\omega - \frac{1}{\lambda}\omega) h_2 dx,$$

and

$$\Psi := G\omega - W x_2 - \gamma.$$

Hence we get

$$0 \geq \int_{\Pi} (G\omega - \frac{1}{\lambda}\omega) \eta$$

$$= \int_{\Pi} (\Psi - \frac{1}{\lambda}) h dx.$$
Since the arbitrariness of \( h \), we have
\[
\Psi - \frac{1}{\lambda} \omega = 0 \quad \text{on} \quad \{ \omega > \delta \},
\]
\[
\Psi - \frac{1}{\lambda} \omega = 0 \quad \text{on} \quad \{ 0 \leq \omega \leq \delta \}.
\]
by letting \( \delta \to 0 \), we obtain \( \omega = \lambda \Psi_+ \).

According to \( \int_{\Omega} \omega \, dx \leq \nu \), we can take a sequence \( \{ x_i \}_{i=1}^{\infty} \) with \( x_i = (x_{1i}, x_{2i}) \), such that \( x_{1i} \to \infty, x_{2i} \to 0 \) and \( \omega(x_i) \to 0 \) as \( i \to \infty \). By (2.3) in Lemma 2.2, we have
\[
\lim_{n \to \infty} \sup(G_\omega(x_i) - Wx_{2i} - \gamma) \leq 0.
\]
Hence \( \gamma \geq 0 \). Similarly, we can take another sequence \( \{ x_j \}_{j=1}^{\infty} \) with \( x_j = (x_{1j}, x_{2j}) \), such that \( x_{1j} \to 0, x_{2j} \to \infty \) and \( \omega(x_j) \to 0 \) as \( j \to \infty \). By (2.3) in Lemma 2.2, we have
\[
0 = \lim_{j \to \infty} (G_\omega(x_j) - Wx_{2j} - \gamma)_+ = \lim_{j \to \infty} (-Wx_{2j} - \gamma)_+,
\]
which implies \( W \geq 0 \).

Next, we show the uniqueness of \( W \) and \( \gamma \). Suppose there are \( W_1, \gamma_1 \geq 0 \) such that (3.1) holds. Hence, we have
\[
G_\omega(x) - W_1x_2 - \gamma_1 = G_\omega(x) - Wx_2 - \gamma,
\]
for all \( x \in \Omega \) such that \( \omega > 0 \). Then,
\[
(W_1 - W)x_2 = \gamma - \gamma_1,
\]
which implies \( W_1 = W \) and \( \gamma_1 = \gamma \).

Since \( \omega \in \Sigma_{\mu, \nu} \) and \( \gamma > 0 \), the \( \supp(\omega) \) and \( x_1 \)-axis have a positive distance. We shall show that if \( \mu \) is sufficiently small, then \( W > 0 \) and \( \gamma = 0 \).

**Lemma 3.2.** For \( 0 < \nu < \infty, 0 < \mu \leq \min\{1, \mu_0 \nu, (\frac{2}{\pi})^\frac{3}{2} c_0 \nu^\frac{3}{2} \} \) and \( c \mu^{-\frac{3}{4}} < \lambda < c \nu \mu^{-1} \), then the constants \( W > 0, \gamma = 0 \) in Lemma 3.1.

**Proof.** Let \( \omega_1 \in \Sigma_{\mu_1, \nu} \), we start to prove \( \gamma = 0 \) for small \( \frac{\mu}{\nu} < 1 \). We define \( \mu = \frac{\mu}{\nu}, \omega = \frac{\omega_1}{\nu} \). Since
\[
\mu = \int_\Omega x_2 \omega \, dx \geq 2\mu \int_{x_2 \geq 2\mu} \omega \, dx,
\]
we have
\[
\int_{x_2 \geq 2\mu} \omega \, dx \leq \frac{1}{2}.
\]

On the one hand, by Lemma 3.1, \( \omega \leq \lambda G_\omega \), then
\[
\int_{0<x_2<2\mu} \omega \, dx \leq \int_\Omega \int_{0<y_2<2\mu} \lambda G_\Omega(x,y) \omega(x) \, dy \, dx = \int_{0<x_2<4\mu} \int_{0<y_2<2\mu} + \int_{x_2 \geq 4\mu} \int_{0<y_2<2\mu}.
\]
For \( x_2 \geq 4\mu, 0 < y_2 < 2\mu \), by \( x_2 - y_2 \geq x_2/2 \), we have
\[
\int_{0<y_2<2\mu} G_\Omega(x,y) \, dy \leq C \int_{0<y_2<2\mu} \left( \frac{x_2 y_2}{|x-y|^2} + e^{-\frac{|x-y|}{\mu}} (1 - e^{-y_2}) \right) \, dy
\]
\[
\leq C (\mu^2 + (1 - e^{-2\mu})).
\]
Hence we have
\[ \int_{x_2 \geq 4\mu} \omega(x) \int_{0 < y_2 < 2\mu} G_\Pi(x, y) dy dx \leq C(\mu^2 + (1 - e^{-2\mu})). \] (3.3)

For \(0 < x_2 < 4\mu < 4\),
\[ \int_{0 < y_2 < 2\mu} G_\Pi(x, y) dy = \int_{0 < y_2 < 2\mu, |x-y| < x_2/2} + \int_{0 < y_2 < 2\mu, |x-y| \geq x_2/2}. \]

We estimate
\[ \int_{0 < y_2 < 2\mu, |x-y| > x_2/2} G_\Pi(x, y) dy \leq C \left( \int_{0 < y_2 < 2\mu, |x-y| \geq x_2/2} \frac{x_2 y_2}{|x-y|^2} + e^{-\frac{|x-y|}{2}} (1 - e^{-y_2}) dy \right) \leq C(\mu^2 + (1 - e^{-2\mu})). \]

and
\[ \int_{0 < y_2 < 2\mu, |x-y| < x_2/2} G_\Pi(x, y) dy \leq C \int_{|x-y| < x_2/2} \ln \left( 1 + \frac{4 x_2 y_2}{|x-y|^2} \right) dy \leq C \mu^2. \]

Hence we get
\[ \int_{0 < x_2 < 4\mu} \int_{0 < y_2 < 2\mu} \omega(x) G_\Pi(x, y) dy dx \leq C(\mu^2 + (1 - e^{-2\mu})). \] (3.4)

Combining (3.2), (3.3) and (3.4), we obtain
\[ \int_\Pi \omega dx \leq \frac{1}{2} + C\lambda(\mu^2 + (1 - e^{-2\mu})). \] (3.5)

Then there exists \(\mu_0 = \mu_0(\nu) > 0\) small such that for \(0 < \mu < \mu_0\), we have
\[ \int_\Pi \omega_1 dx < \nu. \]

Hence, we can take
\[ \eta = h - \left( \int_\Pi x_2 h dx \right) h_2. \]

As the test function \(\omega_1 + \varepsilon \eta\) for sufficiently small \(\varepsilon > 0\) in the proof of Lemma 3.1, we obtain
\[ \omega_1 = \lambda (G \omega_1 - Wx_2)_+, \]
which implies \(\gamma = 0\).

Now, we start to prove \(W > 0\). By (3.1), we have
\[ 0 < \int_\Pi \omega_1 G \omega_1 - \frac{1}{\lambda} \int_\Pi \omega_1^2 dx \]
\[ = \int_\Pi \omega_1 G \omega_1 - \int_\Pi \omega_1 (G \omega_1 - Wx_2)_+ dx \]
\[ \leq \int_\Pi \omega_1 G \omega_1 - \int_\Pi \omega_1 (G \omega_1 - Wx_2) dx \]
\[ = W \mu_1. \]

Then we get \(W > 0\) and the proof of Lemma 3.2 is thus finished. \(\square\)

The positivities of \(W > 0\) or \(\gamma > 0\) implies compactness of support for maximizers.
Lemma 3.3. Suppose that \( \omega \in \Sigma_{\mu, \nu} \), then supp(\( \omega \)) is a compact set in \( \Pi \).

Proof. Let \( \omega \in \Sigma_{\mu, \nu} \). By (3.1), we have supp(\( \omega \)) = \( \{ x \in \Pi \mid \mathcal{G}_\omega - Wx_2 - \gamma > 0 \} \) for \( W \geq 0 \) and \( \gamma \geq 0 \). If \( \gamma > 0 \), the conclusion follows easily from (2.3). If \( \gamma = 0 \), by the proof of Lemma 3.2, we have \( W > 0 \).

Since \( \omega \in L^1 \cap L^2 \), it implies \( \nabla^2 \mathcal{G}_\omega \in L^p \), \( p \in (1, 2) \) and \( \nabla \mathcal{G}_\omega \in L^q \), \( 1/q = 1/p - 1/2 \). By (2.3) and (3.1), \( \mathcal{G}_\omega \) satisfies the following elliptic equation

\[-\Delta \psi + \psi = \lambda (\psi - Wx_2 - \gamma) \quad \text{in } \Pi , \]

\[\psi = 0 \quad \text{on } \partial \Pi , \]

\[\psi \to 0 \quad \text{as } |x| \to \infty .\]

By the Sobolev embedding, we have \( \mathcal{G}_\omega \in BUC^{2, \alpha} (\Pi) \). Since \( \mathcal{G}_\omega (x_1, 0) = 0 \) and

\[\frac{\mathcal{G}_\omega}{x_2} = \int_0^1 (\partial_2 \mathcal{G}_\omega)(x_1, x_2 s) ds ,\]

hence \( \mathcal{G}_\omega / x_2 \in BUC^{1, \alpha} (\Pi) \). According to the Hardy’s inequality [26],

\[\| \mathcal{G}_\omega / x_2 \|_2 \leq 2 \| \nabla \mathcal{G}_\omega \|_2 ,\]

then \( \mathcal{G}_\omega / x_2 \in BUC (\Pi) \cap L^2 (\Pi) \), we have

\[\frac{\mathcal{G}_\omega}{x_2} \to 0 \quad \text{as } |x| \to \infty ,\]

it implies that supp(\( \omega \)) is a compact set. \( \Box \)

Next, we consider positive solutions \( \psi > 0 \) of the problem

\[-\Delta \psi + \psi = \lambda (\psi - Wx_2) \quad \text{in } \Pi , \]

\[\psi = 0 \quad \text{on } \partial \Pi , \]

\[\psi \to 0 \quad \text{as } |x| \to \infty .\]  \hspace{1cm} (3.6)

Lemma 3.4. Let \( \psi \in BUC^{2, \alpha} \), \( 0 < \alpha < 1 \), be a positive solution of (3.6) for some \( W > 0 \) and \( \lambda > \lambda_0 \). Then \( \psi(x) = \Psi_L(x + ce_1) \) for some \( c \in \mathbb{R} \), where \( \Psi_L = \Psi_L + Wx_2 \), \( \Psi_L \) is defined by (1.8).

Proof. For \( y = (y', y_4) \in \mathbb{R}^4 \), \( y' = (y_1, y_2, y_3) \), we set \( x_1 = y_4, x_2 = |y'| \) and

\[\phi(y) = \frac{\psi(x_1, x_2)}{x_2} .\] \hspace{1cm} (3.7)

By a direct calculation, we have

\[\Delta_y \phi + \phi = \lambda (\phi - W) \quad \text{in } \mathbb{R}^4 ,\]

\[\phi \to 0 \quad \text{as } |y| \to \infty .\]

Thus \( \phi \) satisfies the integral equation

\[\phi(x) = \int_{\mathbb{R}^4} G(x - y) \lambda (\phi(y) - W) dy .\] \hspace{1cm} (3.8)

where \( G \) is the fundamental solution of the Bessel equation in \( \mathbb{R}^4 \).

Since \( \phi \) is continuous and the support of \( (\phi(y) - W)_+ \) is compact, one can apply the standard method of moving planes in integral form to deduce that \( \phi \) is radially symmetric.
with respect to some point \( y^0 = (0, c) \in \mathbb{R}^4 \) and hence unique up to translations in \( y_4 \) by \( |y| = |x| \), we have

\[
\frac{\psi(x_1 + c, x_2)}{x_2} = \varphi(|x|).
\]

By translation of \( \psi \) for the \( x_1 \)-variable, we may assume that \( c = 0 \). By the polar coordinate \( x_1 = r \cos(\theta), x_2 = r \sin(\theta) \), we define

\[
\Psi(x) = \psi(x) - Wx_2 = (\varphi(r) - W)r \sin(\theta) =: \eta(r) \sin(\theta).
\]

By (3.6), \( \Psi \) satisfies

\[
-\Delta \Psi + \Psi = \lambda \Psi + Wx_2 \quad \text{in } \Omega,
\]

\[
-\Delta \Psi + \Psi = 0 \quad \text{in } \Pi \setminus \Omega,
\]

\[
\Psi = 0 \quad \text{on } \partial \Pi \cup \partial \Omega,
\]

\[
\partial x_1 \Psi \to 0, \partial x_2 \Psi \to -W \quad \text{as } |x| \to \infty.
\]

where \( \Omega = B_a(0) \cap \Pi \) for some \( a > 0 \). Using (3.9), we have

\[
r^2 \eta'' + r \eta' + ((\lambda - 1)r^2 - 1)\eta - Wr^3 = 0, \quad \eta > 0, \quad 0 < r < a,
\]

\[
\eta(a) = 0.
\]

We take \( \eta_0 = \eta - \frac{W}{\lambda - 1} r \), then \( \eta_0 \) satisfies

\[
r^2 \eta_0'' + r \eta_0' + ((\lambda - 1)r^2 - 1)\eta_0 = 0, \quad \eta_0(r) > -\frac{W}{\lambda - 1} r, \quad 0 < r < a,
\]

\[
\eta_0(a) = -\frac{W}{\lambda - 1} a.
\]

Since \( \eta(0) \) is bounded, we have \( \eta_0 = CJ_1((\lambda - 1)^{1/2} r) \) and \( a \) is the first lowest zero of the following equation at \( (0, \infty) \)

\[
CJ_1((\lambda - 1)^{1/2} r) + \frac{W}{\lambda - 1} r = 0.
\]

Similarly, in \( \Pi \setminus \Omega \), \( \eta \) satisfies

\[
r^2 \eta'' + r \eta' - (r^2 + 1)\eta - Wr^3 = 0.
\]

We take \( \eta_1 = \eta + Wr \), then \( \eta_1 \) satisfies

\[
r^2 \eta_1'' + r \eta_1' - (r^2 + 1)\eta_1 = 0.
\]

Since \( \eta \) is decaying at \( \infty \) and \( \eta(a) = 0 \), so we get \( \eta_1 = \frac{Wa}{K_1(a)} K_1(r) \). By the continuity of \( \partial_x \Psi \) at \( a \), we obtain

\[
C_1 = -\frac{Wa}{\lambda - 1} \cdot \frac{1}{J_1((\lambda - 1)^{1/2} a)}
\]

and \( a \) satisfies

\[
a \left( \frac{K_1'(a)}{K_1(a)} \right) + \frac{1}{(\lambda - 1)^{1/2}} \cdot \frac{J_1((\lambda - 1)^{1/2} a)}{J_1((\lambda - 1)^{1/2} a)} = \frac{\lambda}{\lambda - 1}.
\]

(3.12)
Hence we get
\[ \Psi(x) = \Psi_L(x) = \begin{cases} (C_1 J_1((\lambda - 1)^{1/2}r) + \frac{W}{\lambda - 1}r) \sin(\theta), & r \leq a, \\ (-W_r + \frac{W a_1}{K_1(a)} K_1(r)) \sin(\theta), & r > a. \end{cases} \]

\[ \square \]

**Remark 3.5.** We want to show that equation (3.12) is solvable. Define the set as follows
\[ A = \{ t \in \mathbb{R}_+ \mid J_1((\lambda - 1)^{1/2}t) \neq 0 \} \]
and the function
\[ W(t) = \ln \frac{K_1(t) \cdot |J_1((\lambda - 1)^{1/2}t)|^{1/(\lambda - 1)}}{t^{\lambda/(\lambda - 1)}}, t \in A. \]  
(3.13)

By the properties of \( J_1 \), we know that \( \mathbb{R}_+ \setminus A \) is at most countable. Suppose
\[ \mathbb{R}_+ \setminus A = \{ x_1, x_2, \ldots, x_n, \ldots \}, \text{ for } x_{i+1} > x_i > 0, i \in \{ 1, 2, 3, \ldots \}. \]

We find that
\[ \lim_{t \to x_i} W(t) = -\infty, \]
and
\[ W(t) > -\infty, \text{ for } t \in (x_i, x_{i+1}), \]
where \( i \in \{ 1, 2, 3, \ldots \}. \) Therefore, on each interval \( (x_i, x_{i+1}) \), \( W \) has at least one extreme point, then (3.12) is solvable. We choose \( a \) to be the first least positive solution of the equation (3.12).

### 4. Compactness of Maximizing Sequences

In this section, we shall prove the compactness of a maximizing sequence up to translations for the \( x_1 \)-variable by using a concentration compactness principle.

**Theorem 4.1.** Let \( \mu, \nu, \lambda \) satisfy the assumptions of Lemma 3.2. Suppose that \( \{ \omega_n \}_{n=1}^\infty \) is a maximizing sequence in the sense that
\[ \omega_n \geq 0, \omega_n \in L^1 \cap L^2, \int_\Pi \omega_n dx \leq \nu, \| \omega_n \|_2 \leq C, \forall n \geq 1, \]  
(4.1)
\[ \mu_n = \int_\Pi x_2 \omega_n dx \to \mu, \text{ as } n \to \infty, \]  
(4.2)
and
\[ E(\omega_n) \to S_{\mu, \nu}, \text{ as } n \to \infty. \]  
(4.3)

Then there exists \( \omega \in \Sigma_{\mu, \nu} \), a sub-sequence \( \{ \omega_{n_k} \}_{k=1}^\infty \) and a sequence of real numbers \( \{ c_k \}_{k=1}^\infty \) such that as \( k \to \infty \), it holds
\[ \omega_{n_k}(\cdot + c_k e_1) \to \omega \text{ in } L^2(\Pi), \]  
(4.4)
and
\[ x_2 \omega_{n_k}(\cdot + c_k e_1) \to x_2 \omega \text{ in } L^1(\Pi). \]  
(4.5)

We need the following concentration compactness lemma (see [25]) to prove Theorem 4.1.
Lemma 4.2. Let \( \{\xi_n\}_{n=1}^\infty \) be a sequence of nonnegative functions in \( L^1(\Pi) \) satisfying
\[
\limsup_{n \to \infty} \int_\Pi \xi_n \, dx \to \mu,
\]
for some \( 0 < \mu < \infty \). Then, after passing to a subsequence, one of the following holds:

(i) (Compactness) There exists a sequence \( \{y_n\}_{n=1}^\infty \) in \( \bar{\Pi} \) such that for arbitrary \( \varepsilon > 0 \), there exists \( R > 0 \) satisfying
\[
\int_{\Pi \cap B_R(y_n)} \xi_n \, dx \geq \mu - \varepsilon, \quad \forall n \geq 1.
\]

(ii) (Vanishing) For each \( R > 0 \),
\[
\limsup_{n \to \infty} \int_{B_R(y) \cap \Pi} \xi_n \, dx = 0.
\]

(iii) (Dichotomy) There exists a constant \( 0 < \alpha < \mu \) such that for any \( \varepsilon > 0 \), there exist \( N = N(\varepsilon) \geq 1 \) and \( 0 \leq \xi_{i,n} \leq \xi_n \), \( i = 1, 2 \) satisfying
\[
\left\{ \begin{array}{l}
\|\xi_n - \xi_{1,n} - \xi_{2,n}\|_1 + |\alpha - \int_\Pi \xi_{1,n} \, dx| + |\mu - \alpha - \int_\Pi \xi_{2,n} \, dx| < \varepsilon, \quad \text{for } n \geq N,
\end{array} \right.
\]
\[
d_n := \text{dist} (\text{supp } (\xi_{1,n}), \text{supp } (\xi_{2,n})) \to \infty, \quad \text{as } n \to \infty.
\]

Proof of Theorem 4.1. Let \( \xi_n = x_2 \omega_n \). Using Lemma 4.2, we find that for a certain subsequence, still denoted by \( \{\omega_n\}_{n=1}^\infty \), one of the three cases in Lemma 4.2 should occur. To deal with the three cases, we divide the proof into three steps.

Step 1. (Vanishing excluded) Suppose that for each fixed \( R > 0 \),
\[
\limsup_{n \to \infty} \int_{B_R(y) \cap \Pi} x_2 \omega_n \, dx = 0.
\]

To get a contradiction for \( S_{\mu,\nu} > 0 \), we’re going to prove \( \lim_{n \to \infty} E(\omega_n) = 0 \).

We set
\[
2E(\omega_n) = \int_\Pi \int_\Pi \omega_n(x) G_\Pi(x, y) \omega_n(y) \, dx \, dy = \int_{|x-y| \geq R} + \int_{2<|x-y|<R} + \int_{|x-y| \leq 2}.
\]

If \( |x-y| \geq R > 2 \), we note that \( G_\Pi(x, y) \leq Ce^{-\frac{|x-y|^2}{2}} \),
\[
\int_{|x-y| \geq R} G_\Pi(x, y) \omega_n(x) \omega_n(y) \, dx \, dy \leq Ce^{-\frac{R^2}{2}} \nu^2.
\]

If \( 2 < |x-y| < R \), we find that \( G_\Pi \leq Ce^{-\frac{|x-y|^2}{2}} (1 - e^{-x^2}) \),
\[
\int_{2<|x-y|<R} G_\Pi(x, y) \omega_n(x) \omega_n(y) \, dx \, dy \leq C\nu \left( \sup_{y \in \Pi} \int_{B_R(y) \cap \Pi} x_2 \omega_n(x) \, dx \right) \to 0 \text{ as } n \to \infty.
\]

If \( |x-y| \leq 2 \), we set
\[
\int_{|x-y| \leq 2} G_\Pi(x, y) \omega_n(x) \omega_n(y) \, dx \, dy = \int_{|x-y| \leq 2, \ G_\Pi < R x_2 y_2} \int_{|x-y| \leq 2, \ G_\Pi \geq R x_2 y_2}.
\]

and observe that
\[
\int_{|x-y| \leq 2, \ G_\Pi < R x_2 y_2} G_\Pi(x, y) \omega_n(x) \omega_n(y) \, dx \, dy \leq R\mu \left( \sup_{y \in \Pi} \int_{B_R(y) \cap \Pi} x_2 \omega_n(x) \, dx \right) \to 0 \text{ as } n \to \infty.
\]
The condition $G_\Pi \geq Rx_2y_2$ implies $|x - y| \leq CR^{-1/2}$, 
$$G_\Pi(x, y) \leq C(|\ln |x - y|| + x_2),$$
and
$$\left( \int_{|x - y| < CR^{-1/2}} G_\Pi(x, y)^2 dy \right)^{1/2} \leq C(R)(1 + x_2).$$
where $C(R) \to 0$ as $R \to \infty$. Hence
$$\iint_{|x - y| \leq 2, G_\Pi \geq Rx_2y_2} G_\Pi(x, y)\omega_n(x)\omega_n(y)dxdy$$
$$\leq \iint_{|x - y| \leq CR^{-1/2}} G_\Pi(x, y)\omega_n(x)\omega_n(y)dxdy$$
$$\leq \|\omega_n\|_2 \int_{\Pi} \omega_n(x) \left( \int_{|x - y| < CR^{-1/2}} G_\Pi(x, y)^2 dy \right)^{1/2} dx$$
$$\leq C(R)' \to 0 \text{ as } R \to \infty.$$ 
Letting $n \to \infty$ and then $R \to \infty$ implies \( \lim_{n \to \infty} E(\omega_n) = 0 \).

Step 2. (Dichotomy excluded) Suppose that there exists some $\alpha \in (0, \mu)$ such that
$$\begin{cases}
\omega_n = \omega_{1,n} + \omega_{2,n} + \omega_{3,n}, & 0 \leq \omega_{i,n} \leq \omega_n, i = 1, 2, 3, \\
\|x_2\omega_{3,n}\|_1 + |\alpha - \alpha_n| + |\mu - \alpha - \beta_n| \to 0, & \text{as } n \to \infty, \\
\|\omega_n\|_2 = \text{dist} \left( \text{supp } (\omega_{1,n}), \text{supp } (\omega_{2,n}) \right) \to \infty, & \text{as } n \to \infty. 
\end{cases}$$ (4.7)

According to the symmetry of $E$, we have
$$2E(\omega_n) = 2E(\omega_{1,n} + \omega_{2,n} + \omega_{3,n})$$
$$= \int_{\Pi} \int_{\Pi} \omega_{1,n}(x)G_\Pi(x, y)\omega_{1,n}(y)dxdy$$
$$+ \int_{\Pi} \int_{\Pi} \omega_{2,n}(x)G_\Pi(x, y)\omega_{2,n}(y)dxdy + 2 \int_{\Pi} \int_{\Pi} \omega_{1,n}(x)G_\Pi(x, y)\omega_{2,n}(y)dxdy$$
$$+ \int_{\Pi} \int_{\Pi} (2\omega_n - \omega_{3,n}(x)) G_\Pi(x, y)\omega_{3,n}(y)dxdy.$$ 

For fixed $R > 0$,
$$\int_{\Pi} \int_{\Pi} (2\omega_n - \omega_{3,n}(x)) G_\Pi(x, y)\omega_{3,n}(y)dxdy$$
$$\leq \int_{G_\Pi(x, y) < Rx_2y_2} 2\omega_n(x)G_\Pi(x, y)\omega_{3,n}(y)dxdy$$
$$+ \int_{G_\Pi(x, y) \geq Rx_2y_2} 2\omega_n(x)G_\Pi(x, y)\omega_{3,n}(y)dxdy$$
$$\leq 2R\mu_n \|x_2\omega_{3,n}\|_1 + 2\|\omega_{3,n}\|_2 \int_{\Pi} \omega_n(x) \left( \int_{|x - y| < CR^{-1/2}} G_\Pi(x, y)^2 dy \right)^{1/2} dx$$
$$\leq 2R\mu_n(1) + CR^{-1/2}.$$

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Since \( G_\Pi(x, y) \leq C(x_2 y_2 |x - y|^2 + e^{-|x-y|/2}) \),
\[
\int_\Pi \int_\Pi \omega_{1,n}(x)G_\Pi(x, y)\omega_{2,n}(y)dx \, dy \leq C(\frac{\mu^2}{d_n^2} + e^{-d_n/2})
\]
Hence we obtain
\[
\mathcal{E}_\lambda(\omega_n) = E(\omega_n) - \frac{1}{2\lambda} \int_\Pi \omega_n^2 dx
\leq \mathcal{E}_\lambda(\omega_{1,n}) + \mathcal{E}_\lambda(\omega_{2,n}) + C(\frac{\mu^2}{d_n^2} + e^{-d_n/2}) + 2R_0(1) + CR^{-1/2}.
\]
Taking Steiner symmetrization \( \omega_{i,n}^* \) of \( \omega_{i,n} \) for \( i = 1, 2 \), we get
\[
\begin{cases}
E(\omega_n) \leq E(\omega_{1,n}^*) + E(\omega_{2,n}^*) + C(\frac{\mu^2}{d_n^2} + e^{-d_n/2}) + 2R_0(1) + CR^{-1/2} \\
\|\omega_{1,n}^*\|_1 + \|\omega_{2,n}^*\|_1 \leq \nu_1, \|\omega_{1,n}^*\|_2 + \|\omega_{2,n}^*\|_2 \leq C \\
x_2\omega_{1,n}^* \leq \alpha_n, ||x_2\omega_{2,n}^*||_1 = \beta_n.
\end{cases}
\]
We assume that \( \omega_{i,n}^* \to \omega_i^* \) weakly in \( L^2(\Pi) \) as \( n \to \infty \) for \( i = 1, 2 \). Similar to the proof of Lemma 2.6, we can obtain the convergence of the kinetic energy
\[
\lim_{n \to \infty} E(\omega_{i,n}^*) = E(\omega_i^*), \text{ for } i = 1, 2.
\]
Letting \( n \to \infty \), then \( R \to \infty \), we obtain
\[
\begin{cases}
S_{\mu,\nu} \leq \mathcal{E}_\lambda(\omega_1^*) + \mathcal{E}_\lambda(\omega_2^*) , \\
\|\omega_1^*\|_1 + \|\omega_2^*\|_1 \leq \nu, \|\omega_1^*\|_2 + \|\omega_2^*\|_2 \leq C, \\
x_2\omega_1^* \leq \alpha, ||x_2\omega_2^*||_1 = \mu - \alpha.
\end{cases}
\]
We set \( \alpha_1 = \|x_2\omega_1^*\|_1 \leq \alpha, \nu_1 = \|\omega_1^*\|_1, \beta_1 = ||x_2\omega_2^*||_1 \leq \mu - \alpha \) and \( \nu_2 = \|\omega_2^*\|_1 \). It holds
\[
\alpha_1 > 0, \beta_1 > 0.
\]
In fact, suppose that \( \alpha_1 = 0 \), then we have \( \omega_1^* \equiv 0 \), and hence
\[
S_{\mu,\nu} \leq \mathcal{E}_\lambda(\omega_1^*) + \mathcal{E}_\lambda(\omega_2^*) \leq \mathcal{E}_\lambda(\omega_2^*) \leq S_{\beta_1,\nu}.
\]
This is a contradiction to Lemma 2.7. Similarly, one can verify \( \beta_1 > 0 \). We choose \( \hat{\omega}_1 \in \Sigma_{\alpha_1,\nu_1}, \hat{\omega}_2 \in \Sigma_{\beta_1,\nu_2} \). Moreover, We have that supports of \( \hat{\omega}_i, i = 1, 2 \) are bounded by Lemma 3.3. Therefore, we may assume that \( \text{supp}(\hat{\omega}_1) \cap \text{supp}(\hat{\omega}_2) = \emptyset \) by suitable translations in \( x_1 \)-direction. Letting \( \hat{\omega} = \hat{\omega}_1 + \hat{\omega}_2 \), then we have
\[
\begin{cases}
\int_\Pi \hat{\omega} dx = \int_\Pi \hat{\omega}_1 dx + \int_\Pi \hat{\omega}_2 dx \leq \nu, \\
\int_\Pi x_2\hat{\omega} dx = \int_\Pi x_2\hat{\omega}_1 dx + \int_\Pi x_2\hat{\omega}_2 dx = \alpha_1 + \beta_1 \leq \mu,
\end{cases}
\]
it implies that \( \hat{\omega} \in A_{\alpha_1+\beta_1,\nu} \). Observing that \( \hat{\omega}_1 \neq 0 \) and \( \hat{\omega}_2 \neq 0 \), we have
\[
S_{\mu,\nu} \leq \mathcal{E}_\lambda(\omega_1^*) + \mathcal{E}_\lambda(\omega_2^*) \leq \mathcal{E}_\lambda(\hat{\omega}_1) + \mathcal{E}_\lambda(\hat{\omega}_2) = \mathcal{E}_\lambda(\hat{\omega}) - \int_\Pi \int_\Pi \hat{\omega}_1(x)G_\Pi(x, y)\hat{\omega}_2(y)dx \, dy < S_{\alpha_1+\beta_1,\nu} \leq S_{\mu,\nu},
\]
which is a contradiction.
Step 3. (Compactness) Assume that there is a sequence \( \{y_n\}_{n=1}^\infty \) in \( \Pi \) such that for arbitrary \( \varepsilon > 0 \), there exists \( R > 0 \) satisfying
\[
\int_{\Pi \cap B_R(y_n)} x_2 \omega_n \, dx \geq \mu - \varepsilon, \quad \forall n \geq 1. \tag{4.8}
\]
We may assume that \( y_n = (0, y_{n,2}) \) after a suitable \( x_1 \)-translation. We claim that
\[
\sup_{n \geq 1} y_{n,2} < \infty. \tag{4.9}
\]
Indeed, if (4.9) is false, then there exists a subsequence, still denoted by \( \{y_{n,2}\} \), such that
\[
\lim_{n \to \infty} y_{n,2} = \infty.
\]
By direct calculation, we have
\[
2E(\omega_n) = \int_{\Pi} \omega_n(x) G_n(x) dx
= \int_{\Pi \cap B_R(y_n)} \omega_n(x) G_n(x) dx + \int_{\Pi \backslash B_R(y_n)} \omega_n(x) G_n(x) dx.
\]
Since \( \{\omega_n\}_{n=1}^{\infty} \) is uniformly bounded in \( L^2(\Pi) \), \( \|x_2 \omega_n\|_1 \leq \mu + o(1) \) and (2.6), we have
\[
\int_{\Pi \cap B_R(y_n)} \omega_n(x) G_n(x) dx \leq \frac{C\mu}{(y_{n,2} + 1 - R)^{1/2}} \to 0 \text{ as } n \to \infty.
\]
For any fixed \( M > 0 \) large, we have
\[
\int_{\Pi \backslash B_R(y_n)} \omega_n(x) G_n(x) dx
= \int \int_{x \in \Pi \backslash B_R(y_n), G_n(x,y) \leq Mx_2y_2} + \int \int_{x \in \Pi \backslash B_R(y_n), G_n(x,y) \geq Mx_2y_2}
\leq M\mu_n (\mu_n - \mu + \varepsilon) + C M^{-1/2}.
\]
Hence, by first letting \( n \to \infty \), then \( \varepsilon \to 0 \) and lastly \( M \to \infty \), we obtain
\[
0 < S_{\mu,\nu} \leq \lim_{n \to \infty} E(\omega_n) = 0.
\]
Thus, we have proved claim (4.9) and assume that \( y_{n,2} = 0 \) by taking \( R \) larger. Therefore, we have
\[
\int_{\Pi \cap B_R(0)} x_2 \omega_n \, dx \geq \mu - \varepsilon, \quad \forall n \geq 1.
\]
Since \( \{\omega_n\} \) is uniformly bounded in \( L^2 \), by choosing a subsequence, \( \omega_n \to \omega \) weekly in \( L^2 \) for some \( \omega \). By sending \( n \to \infty \),
\[
\int_{\Pi} \omega dx \leq \nu, \quad \int_{\Pi} x_2 \omega dx = \mu.
\]
Hence \( \omega \in A_{\mu,\nu} \). We shall show that
\[
\lim_{n \to \infty} E(\omega_n) = E(\omega). \tag{4.11}
\]
This implies that
\[ S_{\mu,\nu} = \lim_{n \to \infty} \mathcal{E}_\lambda (\omega_n) \]
\[ \leq \lim_{n \to \infty} E (\omega_n) - \frac{1}{2\lambda} \lim_{n \to \infty} \|\omega_n\|_2^2 \]
\[ \leq \mathcal{E}_\lambda (\omega) \leq S_{\mu,\nu}. \]

Hence \( \lim_{n \to \infty} \|\omega_n\|_2 = \|\omega\|_2 \) and \( \omega_n \to \omega \) in \( L^2 \) follows. By
\[
\int_\Pi x_2 |\omega_n - \omega| \, dx = \int_{\Pi \cap B_R(0)} x_2 |\omega_n - \omega| \, dx + \int_{\Pi \setminus B_R(0)} x_2 |\omega_n - \omega| \, dx
\]
\[ \leq CR^2 \|\omega_n - \omega\|_2 + \int_{\Pi \setminus B_R(0)} x_2 (\omega_n + \omega) \, dx
\]
\[ \leq CR^2 \|\omega_n - \omega\|_2 + \mu_n - \mu + 2\varepsilon. \]

sending \( n \to \infty \) and then \( \varepsilon \to 0 \) implies \( x_2 \omega_n \to x_2 \omega \) in \( L^1(\Pi) \). Since \( \mathcal{E}_\lambda (\omega_n) \to \mathcal{E}_\lambda (\omega) \), the limit \( \omega \in A_{\mu,\nu} \) is a maximizer of \( S_{\mu,\nu} \).

Next, it remains to show (4.11). On the one hand, for any fixed \( M > 0 \) large, we have
\[
2E(\omega_n) = \int_\Pi \int_\Pi \omega_n(x)G_\Pi(x,y)\omega_n(y) \, dx \, dy
\]
\[ \leq \int_{\Pi \cap B_R(0)} \int_{\Pi \cap B_R(0)} \omega_n(x)G_\Pi(x,y)\omega_n(y) \, dx \, dy
\]
\[ + 2 \int_{\Pi \setminus B_R(0)} \int_{\Pi \setminus B_R(0)} \omega_n(x)G_\Pi(x,y)\omega_n(y) \, dx \, dy
\]
\[ \leq \int_{\Pi \cap B_L(0)} \int_{\Pi \cap B_L(0)} \omega_n(x)G_\Pi(x,y)\omega_n(y) \, dx \, dy
\]
\[ + M\mu_n(\mu_n - \mu + \varepsilon) + CM^{-1/2}. \]

Letting \( n \to \infty \), then \( \varepsilon \to 0 \) and lastly \( M \to \infty \), we get
\[ \limsup_{n \to \infty} E(\omega_n) \leq E(\omega) \]

On the other hand, for any \( L > 0 \), we have
\[
2E(\omega_n) = \int_\Pi \int_\Pi \omega_n(x)G_\Pi(x,y)\omega_n(y) \, dx \, dy
\]
\[ \geq \int_{\Pi \cap B_L(0)} \int_{\Pi \cap B_L(0)} \omega_n(x)G_\Pi(x,y)\omega_n(y) \, dx \, dy,
\]
which implies
\[ \liminf_{n \to \infty} E(\omega_n) \geq E(\omega). \]

Hence we obtain (4.11). \( \square \)
5. Orbital Stability

In this section, we want to use the compactness theorem of the previous section to obtain the orbital stability of the solution.

**Theorem 5.1.** Let $\mu, \nu, \lambda$ satisfy the assumptions of Lemma 3.2. Then for any $M, \varepsilon > 0$, there exists $\delta > 0$ such that for non-negative function $\zeta_0 \in L^1 \cap L^\infty(\Pi)$, $\|\zeta_0\|_1 \leq \nu$, $\|\zeta_0\|_2 \leq M$ and

$$\inf_{\omega \in \Sigma_{\mu, \nu}} \{ \|\zeta_0 - \omega\|_2 + \|x_2(\zeta_0 - \omega)\|_1 \} \leq \delta,$$

there exists a global weak solution $\zeta(t)$ of (1.1) with the initial data $\zeta_0$, then

$$\inf_{\omega \in \Sigma_{\mu, \nu}} \{ \|\zeta(t) - \omega\|_2 + \|x_2(\zeta(t) - \omega)\|_1 \} \leq \varepsilon, \quad \text{for all } t \geq 0. \quad (5.2)$$

**Proof.** We argue by contradiction. Suppose that (5.2) were false. Then there exists $\varepsilon_0 > 0$ such that for $n \geq 1$, there exist $\zeta_{0,n} \in L^2 \cap L^1$ satisfying $\zeta_{0,n} \geq 0$, $\|\zeta_{0,n}\|_1 \leq \nu$ and $t_n \geq 0$ such that

$$\inf_{\omega \in \Sigma_{\mu, \nu}} \{ \|\zeta_{0,n} - \omega\|_2 + \|x_2(\zeta_{0,n} - \omega)\|_1 \} \leq \frac{1}{n}$$

and

$$\inf_{\omega \in \Sigma_{\mu, \nu}} \{ \|\zeta_n(t_n) - \omega\|_2 + \|x_2(\zeta_n(t_n) - \omega)\|_1 \} \geq \varepsilon_0, \quad (5.3)$$

where $\zeta_n(t)$ is a $L^2$-regularity solution with the initial data $\zeta_{0,n}$. We take $\omega_n \in \Sigma_{\mu, \nu}$ such that

$$\|\zeta_{0,n} - \omega_n\|_2 + \|x_2(\zeta_{0,n} - \omega_n)\|_1 \to 0 \text{ as } n \to \infty.$$ 

By Hölder’s inequality, we have

$$|E(\zeta_{0,n}) - E(\omega_n)| = \left| \int_{\Pi} \int_{\Pi} (\zeta_{0,n} - \omega_n) G_{\Pi}(x, y) (\zeta_{0,n} + \omega_n) \, dx \, dy \right|$$

$$\leq \int_{G_{\Pi}(x, y) > R \times y_2} |\zeta_{0,n} - \omega_n| (x) G_{\Pi}(x, y) (\zeta_{0,n} + \omega_n) (y) \, dx \, dy$$

$$+ \int_{G_{\Pi}(x, y) \leq R \times y_2} |\zeta_{0,n} - \omega_n| G_{\Pi}(x, y) (\zeta_{0,n} + \omega_n) \, dx \, dy$$

$$\leq \int_{|x - y| < CR^{-1/2}} |\zeta_{0,n} - \omega_n| (x) G_{\Pi}(x, y) (\zeta_{0,n} + \omega_n) (y) \, dx \, dy$$

$$+ R \int_{\Pi} \int_{\Pi} |\zeta_{0,n} - \omega_n| (x) x_2(y_2) (\zeta_{0,n} + \omega_n) (y) \, dx \, dy$$

$$\leq CR^{-1/2} + CR \|x_2(\zeta_{0,n} - \omega_n)\|_1.$$ 

Therefore, we obtain

$$|E_\lambda(\zeta_{0,n}) - S_{\mu, \nu}| = |E_\lambda(\zeta_{0,n}) - E_\lambda(\omega_n)| \leq \frac{M}{\lambda} \|\zeta_{0,n} - \omega_n\|_2 + CR \|x_2(\zeta_{0,n} - \omega_n)\|_1 + CR^{-1/2},$$

which implies

$$E(\zeta_{0,n}) \to S_{\mu, \nu},$$

by letting $n \to \infty$ and then $R \to \infty.$
We write $\zeta_n = \zeta_n(t_n)$ by suppressing $t_n$. By the conservation laws, one has

$$
\begin{cases}
\zeta_n \geq 0, \zeta_n \in L^1(\Pi) \cap L^2(\Pi), \int_\Pi \zeta_n dx \leq \nu, \|\zeta_n\|_2 \leq M \\
\mu_n = \int_\Pi x_2 \zeta_n dx \to \mu, \text{ as } n \to \infty \\
E_\lambda(\zeta_n) \to S_{\mu, \nu}, \text{ as } n \to \infty
\end{cases}
$$

By Theorem 4.1, there exists $\omega \in \Sigma_{\mu, \nu}$, a subsequence $\{\zeta_{n_k}\}_{k=1}^\infty$ and a sequence of real numbers $\{c_k\}_{k=1}^\infty$ such that

$$
\|\zeta_{n_k}(\cdot + c_k e_1) - \omega\|_2 + \|x_2(\zeta_{n_k}(\cdot + c_k e_1) - \omega)\|_1 \to 0, \text{ as } k \to \infty,
$$

which is contrary to (5.3), and the proof of Theorem 5.1 is completed.

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