AN INTRODUCTION TO MAXIMAL REGULARITY
FOR PARABOLIC EVOLUTION EQUATIONS

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Abstract. In this note, we give an introduction to the concept of maximal $L^p$-regularity as a method to solve nonlinear partial differential equations. We first define maximal regularity for autonomous and non-autonomous problems and describe the connection to Fourier multipliers and $\mathcal{R}$-boundedness. The abstract results are applied to a large class of parabolic systems in the whole space and to general parabolic boundary value problems. For this, both the construction of solution operators for boundary value problems and a characterization of trace spaces of Sobolev spaces are discussed. For the nonlinear equation, we obtain local in time well-posedness in appropriately chosen Sobolev spaces. This manuscript is based on known results and consists of an extended version of lecture notes on this topic.

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1. Introduction

In this survey, we give an introduction to the method of maximal $L^p$-regularity which has turned out to be useful for the analysis of nonlinear (in particular, quasilinear) partial differential equations. The aim of this note is to present an overview on the main ideas and tools for this approach. Therefore, we are not trying to present the state of the art but restrict ourselves to relatively simple situations. At the same time, we focus on the mathematical presentation and not on the historical development of this successful branch of analysis. So we do not give detailed bibliographical remarks but refer to some nowadays standard literature, where more details on the history and on the bibliography can be found. This survey could serve as a basis for an advanced lecture course in partial differential equations, for instance for Ph.D. students. In fact, the present paper is based on a series of lectures given in July 2017 at the Tohoku University in Sendai, Japan, and on an advanced course for master students at the University of Konstanz, Germany, in the summer term 2019.

Although the concept of maximal regularity is classical, some main achievements for the abstract theory were obtained in the 1990’s and in the first decade of the present century by, e.g., Amann (see [Ama04], [Ama95]) and Prüss (see [Pru02]). The basic idea of maximal regularity is to solve nonlinear partial differential equations by a linearization approach. Let us consider an abstract quasilinear equation of the form

\[
\partial_t u(t) - A(u(t))u(t) = F(u(t)),
\]
\[
 u(0) = u_0.
\]

The linearization of (1-1) at some fixed function $u$ is given by

\[
\partial_t v(t) - A(u(t))v(t) = F(u(t)),
\]
\[
 v(0) = u_0.
\]

In the maximal regularity approach, one tries to solve the linear equation in appropriate function spaces and to show that the solution has the optimal regularity one could expect. In this case, let $v = S_u[F(u), u_0]$ denote the ($u$-dependent) solution operator of the linear equation (1-2). If $S_u$ induces an isomorphism between appropriately chosen pairs of Banach spaces, then the solvability of the nonlinear equation (1-1) can be reduced to a fixed-point equation of the form $u = S_u(F(u), u_0)$. In many situations, the contraction mapping principle can be applied to obtain a unique solution of the fixed point equation and, consequently, of the nonlinear equation (1-1). In this way, typically short-time existence or existence for small data can be shown. For the long-time asymptotics and the stability of the solution, different methods have to be used. Here, we mention the...
monograph by Prüss and Simonett \cite{PS16}, which covers the abstract theory of maximal regularity, stability results, and many examples in fluid mechanics and geometry.

As mentioned above, one key ingredient in the maximal regularity approach is the choice of appropriate function spaces for the right-hand sides and the solution of the nonlinear equation. In the present note, we restrict ourselves to the $L^p$-setting, where the basic spaces are $L^p$-Sobolev spaces. (For maximal regularity in Hölder spaces, we mention the monograph by Lunardi \cite{Lun95}.) Maximal $L^p$-regularity is closely related to the question of Fourier multipliers, as we will see in Section 3 below. Therefore, it was a breakthrough for the application of this concept, when an equivalent description for maximal regularity in terms of vector-valued Fourier multipliers and $\mathcal{R}$-sectoriality was found by Weis \cite{Wei01} in the year 2001.

The description of maximal $L^p$-regularity by $\mathcal{R}$-boundedness made it possible to show that a large class of parabolic boundary value problems have this property. As standard references for $\mathcal{R}$-boundedness and applications to partial differential operators, we mention \cite{DHP03} and \cite{KW04}. For boundary value problems, also the question of appropriate function spaces on the boundary appears, which leads to the characterization of trace spaces. Here the trace can be taken with respect to time (for the initial value at time 0) or with respect to the space variable (for inhomogeneous boundary data). It turns out that the theory of trace spaces is highly nontrivial and connected with interpolation properties of intersections of Sobolev spaces. In this way, modern theory of vector-valued Sobolev spaces with non-integer order of differentiability enters. Results on trace spaces can be found, e.g., in \cite{DHP07}, for a survey on vector-valued Sobolev spaces we refer to \cite{Ama19} and \cite{HvNVW16}.

The plan of the present survey follows the topics just mentioned. In Section 2, we state the idea and the formal definition of maximal regularity, mentioning the graphical mean curvature flow as a prototype example. The connection to vector-valued Fourier multipliers and $\mathcal{R}$-boundedness is given in Section 3. In Section 4, we briefly summarize the main definitions of the different types of (non-integer) Sobolev spaces and give some key references. The application of the abstract concept to parabolic partial differential equations in the whole space is given in Section 5, the application to parabolic boundary value problems in Section 6. Finally, we return to nonlinear evolution equations in Section 7, where local well-posedness and higher regularity for the solution are discussed.

There are, of course, many topics in the context of maximal $L^p$-regularity which are not covered here. First, we want to mention the application of maximal regularity to stochastic partial differential equations, which leads to the notion of stochastic maximal regularity. Here, the class of radonifying operators plays an important role. A survey on stochastic maximal regularity can be found, e.g., in \cite{vNVW15}, for random sums and radonifying operators see also \cite{HvNVW17}. Another development that could be mentioned is the maximal $L^p$-regularity approach for boundary value problems which are not parabolic in a classical sense (as defined in Sections 5 and 6 below). Some main applications are free boundary value problems from fluid mechanics or problems describing phase transitions like the Stefan problem. Here, the related symbols are not quasi-homogeneous, and the theory described below cannot be applied. One concept to show maximal $L^p$-regularity for such problems uses the Newton polygon, and we refer to \cite{DK13} for more details.
2. Maximal regularity and $L^p$-Sobolev spaces

2.1. Linearization and maximal regularity. We start with an example of a quasilinear parabolic equation.

**Example 2.1** (Graphical mean curvature flow). Let $T_0 \in (0, \infty]$, let $M$ denote an $n$-dimensional parameter space, and let $X(t, \cdot): M \to \mathbb{R}^{n+1}$, $t \in [0, T_0)$, be a family of regular maps. Here, regular means that the Jacobian $D_x X(t, x)$ with respect to $x \in M$ is injective for all $x \in M$ and $t \in [0, T_0)$. We set $M_t := X(t, M)$. Then the vectors $\partial_{x_1} X(t, x), \ldots, \partial_{x_n} X(t, x)$ form a basis for the tangent space $T_x M_t$ at the point $X(t, x)$. In particular, we are interested in the graphical situation where $M = \mathbb{R}^n$ (or some domain in $\mathbb{R}^n$) and where $X$ is given as the graph of some function $u: [0, T_0) \times \mathbb{R}^n \to \mathbb{R}$, so we have $X(t, x) = (x, u(t, x))$ for $x \in \mathbb{R}^n$ and $t \in [0, T_0)$.

Let $\nu: [0, T_0) \times M \to \mathbb{R}^{n+1}$ be one choice of the normal vector to $M_t$, so $\nu(t, x)$ is a unit vector which is orthogonal to the tangent space $T_x M$. For each $i = 1, \ldots, n$, the vector $\partial_{x_i} \nu(t, x)$ is an element of $T_x M_t$, and therefore we can write

$$\partial_{x_i} \nu(t, x) = \sum_{i=1}^n S_{ij}(t, x) \partial_{x_j} X(t, x).$$

The matrix $S(t, x) := (S_{ij}(t, x))_{i,j=1,\ldots,n}$ is called the shape operator at the point $X(t, x)$, its eigenvalues are called the principal curvatures, and its trace $H(t, x) := \text{tr} S(t, x)$ is called the mean curvature.

The family of hypersurfaces $(M_t)_{t \in [0, T_0)}$ is said to move according to the mean curvature flow (see, e.g., [CMIP15] for a survey) if

$$\partial_t X(t, x) \cdot \nu(t, x) = -H(t, x) \nu(t, x) \quad ((t, x) \in [0, T_0) \times M^n).$$

In the graphical situation, one choice of the normal vector is given by

$$\nu(t, x) = \frac{1}{\sqrt{1 + |\nabla u(t, x)|^2}} \begin{pmatrix} -\nabla u(t, x) \\ 1 \end{pmatrix}.$$

From this, we obtain for the mean curvature

$$H(t, x) = -\text{div} \left( \frac{\nabla u(t, x)}{\sqrt{1 + |\nabla u(t, x)|^2}} \right),$$

and the equation for the graphical mean curvature flow is given by

$$\partial_t u - \left( \Delta u - \sum_{i,j=1}^n \frac{\partial_i u \partial_j u}{1 + |\nabla u|^2} \partial_i \partial_j u \right) = 0 \quad \text{in } (0, T_0),$$

$$u(0) = u_0.$$

Here, $u_0$ is the initial value at time $t = 0$, so $M_0$ is given as the $X(0, \mathbb{R}^n)$ with $X(0, x) = (x, u_0(x))$. As the coefficients of the second derivatives of $u$ depend on $u$ itself, this is an example of a quasilinear parabolic equation.

The above example can be written in the abstract form

$$\partial_t u + F(u)u = G(u),$$

$$u(0) = u_0,$$
where $F(u)$ is a linear operator depending on $u$ and $G(u)$ (which equals zero in the example) is, in general, some nonlinear function depending on $u$. For the linearization of (2-2), we fix some function $u$ and are looking for a solution of the Cauchy problem

$$
\begin{align*}
\partial_t v + F(u)v &= G(u), \\
v(0) &= u_0.
\end{align*}
$$

(2-3)

Note that (2-3) is a linear equation with respect to $v$, and therefore it can be treated with methods from linear operator theory and semigroup theory. In general, (2-3) is a non-autonomous problem, as $u$ and therefore also $F(u)$ still depend on time. Setting $A(t) := F(u(t))$ and $f(t) := G(u(t))$, we obtain

$$
\begin{align*}
\partial_t v(t) - A(t)v &= f(t) \quad (t > 0), \\
v(0) &= u_0.
\end{align*}
$$

(2-4)

The idea of maximal regularity consists in showing “optimal” regularity for the linearized equation. Roughly speaking, one should not lose any regularity when solving the linear equation, as the solution will be inserted into the equation in the next step of some iteration process. Considering (2-4) in an operator theoretic sense, we want to have good mapping properties of the solution operator who maps the right-hand side data $f$ and $u_0$ to the solution $v$. For this, we have to fix function spaces for the right-hand side and the solution. So we have to choose the basic space $\mathbb{F}$ for the right-hand side $f$ and a solution space $\mathbb{E}$ for $v$. The choice of the space $\gamma_t \mathbb{E}$ for the initial value $u_0$ will then be canonical, see below.

In case of maximal regularity, we expect a unique solution of (2-4) and a continuous solution operator $S_u$ (depending on $A(t)$ and therefore on $u$)

$$
S_u: \mathbb{F} \times \gamma_t \mathbb{E} \to \mathbb{E}, (f, u_0) \mapsto v
$$

of the linear equation (2-4). Then the nonlinear Cauchy problem is uniquely solvable if and only if the fixed point equation

$$
u = S_u(G(u), u_0)
$$

has a unique solution $u \in \mathbb{E}$.

In many cases, one can show that the right-hand side of this fixed point equation defines a contraction, and therefore Banach’s fixed point theorem (contraction mapping principle) gives a unique solution. To obtain the contraction property, one usually has to choose a small time interval or small initial data $u_0$. Typical applications for this method are

- the graphical mean curvature flow or more general geometric equations,
- Stefan problems describing phase transitions with a free boundary,
- Cahn-Hilliard equations,
- variants of the Navier-Stokes equation.

For a survey on the idea of maximal regularity and on the above applications, we mention the monographs [Ama95], [Pru02], and [PS16].

The notion of maximal regularity depends on the function spaces in which the equation is considered. Typical function spaces for partial differential equations are Hölder spaces and $L^p$-Sobolev spaces. In the present survey, we restrict ourselves to $L^p$-Sobolev spaces, i.e., we are considering maximal $L^p$-regularity. Here, the basic function space for the right-hand side of (2-4) will be $f \in L^p((0, T); X)$, where $X$ is some Banach space. In the
\(L^p\)-setting, one will typically choose \(X = L^p(G)\) for some domain \(G \subset \mathbb{R}^n\). The aim is to show that the operator \(A(t) := F(u(t))\) has, for every fixed \(u\), maximal regularity in the sense specified below.

2.2. Definition of maximal \(L^p\)-regularity. We start with the notion of maximal \(L^p\)-regularity in the autonomous setting, i.e. for an operator \(A\) independent of \(t\). Let \(X\) be a Banach space, and let \(A : X \supset D(A) \rightarrow X\) be a closed and densely defined linear operator. Let \(J = (0,T)\) with \(T \in (0,\infty)\). We consider the initial value problem

\[
\begin{align*}
\partial_t u(t) - Au(t) &= f(t) \quad (t \in J), \\
u(0) &= u_0.
\end{align*}
\]

Here, the right-hand side of (2-5) belongs to \(E := L^p(J; X)\). For optimal regularity, we will expect \(\partial_t u \in L^p(J; X)\) and (consequently) \(Au \in L^p(J; X)\). An even stronger assumption would include \(u \in L^p(J; X)\), too, so that the “optimal” space for the solution \(u\) is given by

\[
E := W^{1,p}_p(J; X) \cap L^p(J; D(A)).
\]

Here, for \(k \in \mathbb{N}_0\) the vector-valued Sobolev space \(W^{k,p}_p(J; X)\) is defined as the space of all \(X\)-valued distributions \(u\) for which \(\partial^\alpha u \in L^p(J; X)\) for all \(|\alpha| \leq k\), see Section 4 (cf. also [HvNVW16], Section 2.5).

For the initial value \(u_0\), we define the trace space:

**Definition 2.2.** a) The trace space \(\gamma_t E\) is defined by \(\gamma_t E := \{\gamma_t u : u \in E\}\), where \(\gamma_t u := u|_{t=0}\) stands for the time trace of the function \(u\) at time \(t = 0\). We endow \(\gamma_t E\) with its canonical norm

\[
\|x\|_{\gamma_t E} := \inf\{\|u\|_E : u \in E, \gamma_t u = x\}.
\]

b) We set \(0 E := \{u \in E : \gamma_t u = 0\}\) for the space of all functions in \(E\) with vanishing time trace at \(t = 0\).

**Remark 2.3.** a) Note in the above definition that, by Sobolev’s embedding theorem, one has the continuous embedding

\[
W^{1,p}_p((0,T); X) \subset C([0,T], X)
\]

for every finite \(T\), where the right-hand side stands for the space of continuous \(X\)-valued functions. Therefore, the value \(\gamma_t u = u(0)\) is well defined as an element of \(X\) for every \(u \in E\).

b) Let \(T \in (0,\infty)\) again. By a), we obtain for \(x \in \gamma_t E\) and for every \(u \in E\) with \(\gamma_t u = x\),

\[
\|x\|_X = \|\gamma_t u\|_X \leq \max_{t \in [0,T]} \|u(t)\|_X \leq C \|u\|_{W^{1,p}_p(J; X)} \leq C \|u\|_E.
\]

Therefore, \(\gamma_t E \subset X\) with continuous embedding. On the other hand, if \(x \in D(A)\), then the function \(u(t) := e^{-t} x\) belongs to \(E\) with \(\|u\|_E \leq C \|x\|_X\) and satisfies \(\gamma_t u = x\). Therefore, also the continuous embedding \(D(A) \subset \gamma_t E\) holds.

The following theorem is a deep result in the theory of interpolation of Banach spaces. Here, the real interpolation functor \(\cdot,\cdot \gamma_{\theta,p}\) appears. We refer to [Lun18] and [Tri95] for an introduction and survey on interpolation spaces.
Lemma 2.4. Let $A$ be a closed and densely defined operator, and let $E$ be defined by (2-7).

a) The trace space $\gamma_t E$ coincides with the real interpolation space with parameters $1 - \frac{1}{p}$ and $p$, i.e., we have

$$\gamma_t E = (X, D(A))_{1-1/p, p}$$

in the sense of equivalent norms.

b) We have the continuous embedding $E \subset C([0,T]; \gamma_t E)$. In particular, the time trace $\gamma_t: E \to \gamma_t E, u \mapsto u(0)$ is well defined, and $\gamma_t E$ is independent of $T$.

c) The norm of the continuous embedding $E \subset C([0,T]; \gamma_t E)$ depends, in general, on $T$ and grows for decreasing $T$. On the subspace $0E$, however, this norm can be chosen independently of $T > 0$, i.e., there exists a constant $C_1$ independent of $T$ such that

$$\|u\|_{C([0,T]; \gamma_t E)} \leq C_1 \|u\|_E \quad (u \in 0E).$$

Definition 2.5. Let $T \in (0, \infty]$, $J := (0, T)$, and $p \in [1, \infty]$.

a) We say that $A$ has maximal $L^p$-regularity ($A \in \text{MR}_p(J; X)$) if for each $f \in F$ and $u_0 \in \gamma_t E$ there exists a unique solution $u \in E$ of (2-5). Here, a function $u \in E$ is called a solution of (2-5)–(2-6) if equality in (2-5) holds in the space $L^p(J; X)$ (i.e., for almost all $t \in (0,T)$), and equality (2-6) holds in $X$.

b) We write $A \in \text{MR}_p(J; X)$ if for each $f \in F$ and $u_0 \in \gamma_t E$ there exists a function $u: [0,T] \to X$ satisfying $\partial_t u \in L^p(J; X)$ and $Au \in L^p(J; X)$ such that (2-5) holds for almost all $t \in (0,T)$ and (2-6) holds as equality in $X$, and if for all $f \in F$ and $u_0 \in \gamma_t E$ the inequality

$$\|\partial_t u\|_{L^p(J; X)} + \|Au\|_{L^p(J; X)} \leq C(\|f\|_{L^p(J; X)} + \|u_0\|_{\gamma_t E})$$

holds with a constant $C = C(J)$ independent of $f$ and $u_0$.

c) We set $\text{MR}_p(X) := \text{MR}_p((0, \infty); X)$ and $0\text{MR}_p(X) := 0\text{MR}_p((0, \infty); X)$.

Remark 2.6. a) By the definition of the spaces, the map

$$\begin{pmatrix} \partial_t - A \\ \gamma_t \end{pmatrix}: E \to F \times \gamma_t E, u \mapsto \begin{pmatrix} \partial_t u - Au \\ \gamma_t u \end{pmatrix}$$

is continuous. If $A \in \text{MR}_p(J; X)$, then, due to the definition of maximal regularity, this map is a bijection and therefore, by the open mapping theorem, an isomorphism. In particular, we obtain the a priori estimate

$$\|u\|_{L^p(J; X)} + \|\partial_t u\|_{L^p(J; X)} + \|Au\|_{L^p(J; X)} \leq C(\|f\|_{L^p(J; X)} + \|u_0\|_{\gamma_t E}),$$

which is stronger than (2-8).

b) If $A \in \text{MR}_p(J; X)$, then (2-5)–(2-6) with $u_0 := 0$ is uniquely solvable for all $f \in F$. On the other hand, for a given $u_0 \in \gamma_t E$, there exists an extension $u_1 \in E$ with $\gamma_t u_1 = u_0$ by the definition of the trace space. Setting $u = u_1 + u_2$, then we see that $u_2$ has to satisfy

$$\begin{cases} \partial_t u_2(t) - Au_2(t) = \tilde{f}(t) \quad (t > 0), \\ u_2(0) = 0, \end{cases}$$

where $\tilde{f} := f - Au_1 \in F$. Therefore, the operator $A$ has maximal regularity if and only if the Cauchy problem (2-10) is uniquely solvable for all $\tilde{f} \in F$. 
c) Let the time interval J be finite, and assume \( A \in _0\text{MR}_p(J; X) \). Then the Cauchy problem (2-10) has a unique solution \( u \) for all \( \tilde{f} \in F \) with \( \partial_t u \in L^p(J; X) \). As \( u(0) = 0 \), we can apply Poincaré’s inequality in the vector-valued Sobolev space \( W^1_p((0, T); X) \) and obtain \( u \in L^p(J; X) \). This yields \( u \in \mathcal{E} \), and by part b) of this remark, we see that \( A \in \text{MR}_p(J, X) \). Therefore, \( _0\text{MR}_p(J; X) = \text{MR}_p(J; X) \) for finite time intervals. Similarly, if \( A \in _0\text{MR}_p((0, \infty); X) \) and if \( A \) is invertible, we can estimate \( \|u\|_{L^p(J; X)} \leq C \|Au\|_{L^p(J; X)} \) and obtain \( u \in \mathcal{E} \) again, which implies \( A \in \text{MR}_p((0, \infty); X) \).

It turns out that the property of maximal \( L^p \)-regularity is independent of \( p \). For a proof of the following result, we refer to [Dor93], Theorem 4.2.

**Lemma 2.7.** If \( A \in \text{MR}_p(X) \) holds for some \( p \in (1, \infty) \), then \( A \in \text{MR}_p(X) \) holds for every \( p \in (1, \infty) \).

Based on this, we write \( \text{MR}(X) \) instead of \( \text{MR}_p(X) \). Note that the constant \( C \) in (2-8) still depends on \( p \).

By Definition 2.5 and Remark 2.6 b), the operator \( A \) has maximal \( L^p \)-regularity in \( J = (0, \infty) \) if and only if the Cauchy problem

\[
\partial_t u(t) - Au(t) = f(t) \quad (t \in (0, \infty)),
\]

\( u(0) = 0 \)

(2-11)

has a unique solution \( u \in W^1_p(J; X) \). We can extend \( f \) and \( u \) by zero to the whole line \( t \in \mathbb{R} \) and obtain functions \( f \in L^p(\mathbb{R}; X) \) and \( u \in W^1_p(\mathbb{R}; X) \) (for this, we need \( u(0) = 0 \)). After this, we apply the Fourier transform in \( t \), which is defined for smooth functions by

\[
(\mathcal{F}_t u)(\tau) := (2\pi)^{-1/2} \int_{\mathbb{R}} u(t)e^{-it\tau} dt.
\]

For tempered distributions, we define \( \mathcal{F}_t \) by duality. Note that \( [\mathcal{F}_t(\partial_t u)](\tau) = i\tau(\mathcal{F}_t u)(\tau) \). Therefore, (2-11) is equivalent to

\[
(i\tau - A)(\mathcal{F}_t u)(\tau) = (\mathcal{F}_t f)(\tau) \quad (\tau \in \mathbb{R}).
\]

**Theorem 2.8.** Let \( J = (0, \infty) \) and \( A \) be a closed densely defined operator. Then \( A \in _0\text{MR}_p(J; X) \) if and only if the operator

\[
\mathcal{F}_t^{-1}i\tau (i\tau - A)^{-1}\mathcal{F}_t
\]

defines a continuous operator in \( L^p(\mathbb{R}; X) \).

**Proof.** By definition, \( A \in _0\text{MR}_p(J; X) \) if and only if (2-11) has a unique solution \( u \) with \( \partial_t u \in L^p(\mathbb{R}; X) \) (again extending the functions by zero to the whole line), and if we have an estimate of \( \partial_t u \). This is equivalent to unique solvability of the Fourier transformed problem (2-12), i.e., the existence of \( (i\tau - A)^{-1} \) for almost all \( \tau \in \mathbb{R} \) such that the solution \( u \) satisfies

\[
\partial_t u = \mathcal{F}_t^{-1}i\tau (i\tau - A)^{-1}\mathcal{F}_t f \in L^p(\mathbb{R}; X),
\]

and the estimate of \( \partial_t u \) is equivalent to the condition \( \mathcal{F}_t^{-1}i\tau (i\tau - A)^{-1}\mathcal{F}_t \in L(L^p(\mathbb{R}; X)) \).

\( \square \)
2.3. Maximal regularity for non-autonomous problems. With respect to the non-linear equation (2.3) and its linearization (2.4), it makes sense to define maximal regularity also for non-autonomous problems. So we consider
\[ \partial_t u(t) - A(t)u(t) = f(t) \quad (t \in (0, T)), \]
\[ u(0) = u_0. \]

Here we assume that all operators \( A(t) \) are closed and densely defined operators in some Banach space \( X \) and have the same domain \( D_A \). We also assume that we have a norm \( \| \cdot \|_A \) on \( D(A) \) which is, for every \( t \in (0, T) \), equivalent to the graph norm of \( A(t) \), which is given by \( \| \cdot \|_X + \| A(t) \cdot \|_X \). In this way, we can identify the unbounded operator \( A(t) : X \supset D_A \to X \) with the bounded operator \( A(t) \in L(D_A, X) \). Moreover, we assume that \( A \in L^\infty((0, T); L(D_A, X)) \).

Analogously to the autonomous case, we consider the basic space for the right-hand side \( F := L^p(J; X) \) with \( J := (0, T) \) and the solution space
\[ E := W^1_p(J; X) \cap L^p(J; D_A). \]

We identify \( A : (0, T) \to L(D_A, X) \) with a function on \( E \) by setting
\[ (Au)(t) := A(t)u(t) \quad (t \in (0, T), \, u \in E). \]

The trace space \( \gamma_t E \) is defined as in Definition 2.2 a).

**Definition 2.9.** a) Let \( f \in F \) and \( u_0 \in \gamma_t E \). Then a function \( u : (0, T) \to X \) is called a strong \( (L^p) \)-solution of (2.13)–(2.14) if \( u \in E \) and if (2.13) holds for almost all \( t \in (0, T) \) and (2.14) holds in \( X \).

b) We say that \( A \in L^\infty((0, T); L(D_A, X)) \) has maximal \( L^p \)-regularity on \( (0, T) \) if for all \( f \in F \) and \( u_0 \in \gamma_t E \) there exists a unique strong solution \( u \in E \) of (2.13)–(2.14).

**Remark 2.10.** Similarly to the autonomous case, the operator \( A \in L^\infty((0, T); L(D_A, X)) \) has maximal regularity if and only if
\[ (\partial_t - A, \gamma_t) : E \to F \times \gamma_t E \]
is an isomorphism of Banach spaces. By trace results, this is equivalent to the condition that (2.13)–(2.14) with \( u_0 = 0 \) has a unique solution \( u \in E \) for every \( f \in F \).

The following result shows that maximal regularity for the non-autonomous operator family \( (A(t))_{t \in (0, T)} \) can be reduced to maximal regularity for each \( A(t) \) if the operator depends continuously on time.

**Theorem 2.11.** Let \( T \in (0, \infty) \) and \( A \in C([0, T], L(D_A, X)) \). Then \( A \) has maximal \( L^p \)-regularity in the sense of Definition 2.9 if and only if for every \( t \in [0, T] \) we have \( A(t) \in \text{MR}((0, T); X) \).

This is shown, using perturbation arguments, in [Ama04], Theorem 7.1.

3. The concept of \( R \)-boundedness and the theorem of Mikhlin

In Theorem 2.8, we have seen that maximal regularity of the operator \( A \) is equivalent to the boundedness of the operator
\[ \mathcal{F}_t^{-1} m \mathcal{F}_t : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X), \]
where the operator-valued symbol \( m : \mathbb{R} \to L(X) \) is given by \( m(\tau) := i\tau(i\tau - A)^{-1} \).

The classical theorem of Mikhlin gives sufficient conditions for a scalar-valued symbol to induce a bounded operator in \( L^p(\mathbb{R}^n) \). For the operator-valued analogue, the concept of \( \mathcal{R} \)-boundedness can be used. Therefore, we discuss in this section the notion of an \( \mathcal{R} \)-bounded family and vector-valued variants of Mikhlin’s theorem. As references for this section, we mention [DHP03], Section 3, and [KW04], Section 2.

3.1. \( \mathcal{R} \)-bounded operator families. Let \( X \) and \( Y \) be Banach spaces.

**Definition 3.1.** A family \( \mathcal{T} \subset L(X, Y) \) is called \( \mathcal{R} \)-bounded if there exists a constant \( C > 0 \) and some \( p \in [1, \infty) \) such that for all \( N \in \mathbb{N} \), \( T_j \in \mathcal{T} \), \( x_j \in X \) \((j = 1, \ldots, N)\) and all sequences \((\varepsilon_j)_{j \in \mathbb{N}}\) of independent and identically distributed \([-1,1]\)-valued and symmetric random variables on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) we have

\[
\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|_{L^p(\Omega, Y)} \leq C \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|_{L^p(\Omega, X)}.
\]

In this case, \( \mathcal{R}_p(\mathcal{T}) := \inf\{C > 0 : (3.1) \text{ holds}\} \) is called the \( \mathcal{R} \)-bound of \( \mathcal{T} \).

**Remark 3.2.** a) For the sequence of random variables as above, we have \( \mathbb{P}(\{\varepsilon_j = 1\}) = \mathbb{P}(\{\varepsilon_j = -1\}) = \frac{1}{2} \). As the measure \( \mathbb{P} \circ (\varepsilon_1, \ldots, \varepsilon_N)^{-1} \) is discrete, the independence of the sequence is equivalent to the condition

\[
\mathbb{P}(\{\varepsilon_1 = z_1, \ldots, \varepsilon_N = z_N\}) = 2^{-N} \quad ((z_1, \ldots, z_N) \in \{-1,1\}^N, \ N \in \mathbb{N}).
\]

Therefore, \( \mathcal{R} \)-boundedness is equivalent to the condition

\[
\exists \ C > 0 \ \forall \ N \in \mathbb{N} \ \forall \ T_1, \ldots, T_N \in \mathcal{T} \ \forall \ x_1, \ldots, x_N \in X
\]

\[
(3.2) \quad \left( \sum_{z_1,\ldots,z_N=\pm 1} \left\| \sum_{j=1}^{N} z_j T_j x_j \right\|_Y^p \right)^{1/p} \leq C \left( \sum_{z_1,\ldots,z_N=\pm 1} \left\| \sum_{j=1}^{N} z_j x_j \right\|_X^p \right)^{1/p}.
\]

However, the stochastic description is advantageous, in particular, one can choose the probability space \((\Omega, \mathcal{A}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)\), where \( \mathcal{B}([0,1]) \) stands for the Borel \( \sigma \)-algebra, \( \lambda \) for the Lebesgue measure, and the random variables \( \varepsilon_j \) are given by the Rademacher functions (see below). It seems to be unclear if the notation “\( \mathcal{R} \)” stands for “randomized” or for “Rademacher”.

**Definition 3.3.** The Rademacher functions \( r_n : [0,1] \to \{-1,1\} \) are defined by

\[
r_n(t) := \text{sign} \sin(2^n \pi t) \quad (t \in [0,1]).
\]

By definition, we have

\[
r_1(t) = \begin{cases} 1, & t \in (0, \frac{1}{2}), \\ -1, & t \in (\frac{1}{2}, 1). \end{cases}
\]

The function \( r_2 \) has value 1 on the intervals \((0, \frac{1}{4})\) and \((\frac{1}{2}, \frac{3}{4})\). An immediate calculation yields

\[
\int_0^1 r_n(t)r_m(t)dt = \delta_{nm} \quad (n, m \in \mathbb{N}).
\]
Moreover,

$$\lambda\{t \in [0,1] : r_{n_1}(t) = z_1, \ldots, r_{n_M}(t) = z_M\} = \frac{1}{2M} = \prod_{j=1}^{M} \lambda\{t \in [0,1] : r_{n_j}(t) = z_j\}.$$

Therefore, the sequence $(r_n)_{n \in \mathbb{N}}$ is independent and identically distributed on the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ as in Definition 3.1. As all properties of $(\varepsilon_j)_j$ which are needed in this definition only depend on the joint probability distribution, we can always choose $\varepsilon_n = r_n$.

**Definition 3.4.** Let $X$ be a Banach space and $1 \leq p < \infty$. Then $\text{Rad}_p(X)$ is defined as the Banach space of all sequences $(x_n)_{n \in \mathbb{N}} \subset X$ for which the limit $\lim_{N \to \infty} \sum_{n=1}^{N} r_n(t) x_n = f(t)$ exists for almost all $t \in [0,1]$ and defines a function $f \in L^p([0,1];X)$. For $(x_n)_{n \in \mathbb{N}} \in \text{Rad}_p(X)$, we define

$$\| (x_n)_{n \in \mathbb{N}} \|_{\text{Rad}_p(X)} := \left\| \sum_{n=1}^{\infty} r_n x_n \right\|_{L^p([0,1];X)}.$$

**Remark 3.5.** a) It can be shown that for any sequence $(x_n)_{n \in \mathbb{N}} \subset X$, the sequence $(\| \sum_{n=1}^{N} r_n x_n \|_{L^p([0,1];X)})_{N \in \mathbb{N}}$ is increasing, and therefore $\text{Rad}_p(X)$ is the space of all sequences $(x_n)_{n \in \mathbb{N}}$ such that

$$\left\| \sum_{n=1}^{\infty} r_n x_n \right\|_{L^p([0,1];X)} < \infty.$$

b) By definition, the map $J : \text{Rad}_p(X) \to L^p([0,1];X)$, $(x_n)_{n} \mapsto \sum_{n=1}^{\infty} r_n x_n$ is well-defined. Assume that $J([x_n]) = 0$, i.e., $\sum_{n} r_n x_n = 0$ holds in $L^p([0,1];X)$. Then $\sum_{n} r_n f(x_n) = 0$ for all $f \in X'$. Taking the inner product in $L^2$ with $r_{n_0}$ for some fixed $n_0$, we get, using the orthogonality, $f(x_{n_0}) = 0$ for all $f \in X'$ and therefore $x_{n_0} = 0$. As $n_0$ was arbitrary, we obtain $x_n = 0$ for all $n \in \mathbb{N}$, which shows that $J$ is injective. Therefore, $\text{Rad}_p(X)$ can be considered as a subspace of $L^p([0,1];X)$, and the norm in $\text{Rad}_p(X)$ is the restriction of the norm in $L^p([0,1];X)$.

**Theorem 3.6** (Kahane–Khintchine inequality). The spaces $\text{Rad}_p(X)$ are isomorphic for all $1 \leq p < \infty$, i.e., there exist constants $C_p > 0$ with

$$\frac{1}{C_p} \left\| \sum_{n=1}^{\infty} r_n x_n \right\|_{L^2([0,1];X)} \leq \left\| \sum_{n=1}^{\infty} r_n x_n \right\|_{L^p([0,1];X)} \leq C_p \left\| \sum_{n=1}^{\infty} r_n x_n \right\|_{L^2([0,1];X)}.$$

In the scalar case $X = \mathbb{C}$, the proof of this inequality is elementary, for arbitrary Banach spaces, however, rather complicated. In the scalar case Theorem 3.6 is known as Khintchine’s inequality, in the Banach space valued case as Kahane’s inequality. We omit the proof which can be found, e.g., in [HvNVW16], Theorem 3.2.23.

**Lemma 3.7.** a) If condition (3-1) in Definition 3.1 holds for some $p \in [1, \infty)$, then it holds for all $p \in [1, \infty)$. For the corresponding $\mathcal{R}$-bounds $\mathcal{R}_p(T)$ the inequality

$$\frac{1}{C_p^2} \mathcal{R}_2(T) \leq \mathcal{R}_p(T) \leq C_p^2 \mathcal{R}_2(T)$$

holds, where the constants $C_p$ are from Theorem 3.6.
b) A family $\mathcal{T} \subset L(X,Y)$ is $\mathcal{R}$-bounded with $\mathcal{R}_2(\mathcal{T}) \leq C$ if and only if for all $N \in \mathbb{N}$ and all $T_1, \ldots, T_N \in \mathcal{T}$, the map

$$T((x_n)_{n \in \mathbb{N}}) := (y_n)_{n \in \mathbb{N}}, \quad y_n := \begin{cases} T_n x_n, & n \leq N, \\ 0, & n > N \end{cases}$$

defines a bounded linear operator $T \in L(\text{Rad}_2(X))$ with norm $\|T\| \leq C$.

Proof. Part a) follows directly from Kahane’s inequality, and part b) is a reformulation of the definition of $\mathcal{R}$-boundedness and an application of the $p$-independence from a). □

**Remark 3.8.**

a) If $\mathcal{T} \subset L(X,Y)$ is $\mathcal{R}$-bounded, then $\mathcal{T}$ is uniformly bounded with $\sup_{T \in \mathcal{T}} \|T\| \leq \mathcal{R}(\mathcal{T})$. This follows immediately if we set $N = 1$ in the definition of $\mathcal{R}$-boundedness.

b) If $X$ and $Y$ are Hilbert spaces, then $\mathcal{R}$-boundedness is equivalent to uniform boundedness. In fact, in this situation also the spaces $L^2([0,1]; X)$ and $L^2([0,1]; Y)$ are Hilbert spaces, and $(r_n x_n)_{n \in \mathbb{N}} \subset L^2([0,1]; X)$ and $(r_n T_n x_n)_{n \in \mathbb{N}} \subset L^2([0,1]; Y)$ are orthogonal sequences. If $\|T\| \leq C_T$ for all $T \in \mathcal{T} \subset L(X,Y)$, then

$$\left\| \sum_{n=1}^{N} r_n T_n x_n \right\|^2_{L^2([0,1]; Y)} = \sum_{n=1}^{N} \left\| r_n T_n x_n \right\|^2_{L^2([0,1]; Y)} \leq C_T^2 \sum_{n=1}^{N} \|x_n\|^2_{X}$$

Remark 3.9. Let $X, Y, Z$ be Banach spaces, and $\mathcal{T}, \mathcal{S} \subset L(X,Y)$ and $\mathcal{U} \subset L(Y,Z)$ be $\mathcal{R}$-bounded. Then the families

$$\mathcal{T} + \mathcal{S} := \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$

and

$$\mathcal{U} \mathcal{T} := \{UT : U \in \mathcal{U}, T \in \mathcal{T}\}$$

are $\mathcal{R}$-bounded, too, with

$$\mathcal{R}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S}), \quad \mathcal{R}(\mathcal{U} \mathcal{T}) \leq \mathcal{R}(\mathcal{U}) \mathcal{R}(\mathcal{T}).$$

To see this, let $S_n \in \mathcal{S}, T_n \in \mathcal{T}$ and $U_n \in \mathcal{U}$ for $n = 1, \ldots, N$. Then the statement follows from

$$\left\| \sum_{n=1}^{N} r_n (T_n + S_n) x_n \right\|_{L^1([0,1]; Y)} \leq \left\| \sum_{n=1}^{N} r_n T_n x_n \right\|_{L^1([0,1]; Y)} + \left\| \sum_{n=1}^{N} r_n S_n x_n \right\|_{L^1([0,1]; Y)}$$

and

$$\left\| \sum_{n=1}^{N} r_n U_n T_n x_n \right\|_{L^1([0,1]; Z)} \leq \mathcal{R}(\mathcal{U}) \left\| \sum_{n=1}^{N} r_n T_n x_n \right\|_{L^1([0,1]; Y)}.$$

The following result turns out to be useful for showing $\mathcal{R}$-boundedness.

**Lemma 3.10** (Kahane’s contraction principle). Let $1 \leq p < \infty$. Then for all $N \in \mathbb{N}$, for all $x_j \in X$ and all $a_j, b_j \in \mathbb{C}$ with $|a_j| \leq |b_j|$, $j = 1, \ldots, N$ we have

$$\left\| \sum_{j=1}^{N} a_j r_j x_j \right\|_{L^p([0,1]; X)} \leq 2 \left\| \sum_{j=1}^{N} b_j r_j x_j \right\|_{L^p([0,1]; X)}.$$

(3-3)
Proof. Considering $\bar{x}_j := b_j x_j$, we may assume without loss of generality that $b_j = 1$ and $|a_j| \leq 1$ for all $j = 1, \ldots, N$. Treating $\text{Re} a_j$ and $\text{Im} a_j$ separately, we only have to show that for real $a_j$ with $|a_j| \leq 1$ the inequality

$$
(3.4) \quad \left\| \sum_{j=1}^{N} a_j r_j x_j \right\|_{L^p([0,1];X)} \leq \left\| \sum_{j=1}^{N} r_j x_j \right\|_{L^p([0,1];X)}
$$

holds. For this, let $\{e^{(k)}\}_{k=1}^{2^N}$ be a numbering of all vertices of the cube $[-1,1]^N$. Because of $a := (a_1, \ldots, a_N)^T \in [-1,1]^N$, the vector $a$ can be written as a convex combination of all $e^{(k)}$, i.e., there exist $\lambda_k \in [0,1]$ with

$$
\sum_{k=1}^{2^N} \lambda_k = 1 \quad \text{and} \quad a = \sum_{k=1}^{2^N} \lambda_k e^{(k)}.
$$

Therefore, for $e^{(k)} = (e_1^{(k)}, \ldots, e_N^{(k)})^T$ we see that

$$
\left\| \sum_{j=1}^{N} a_j r_j x_j \right\|_{L^p([0,1];X)} \leq \sum_{k=1}^{2^N} \lambda_k \left\| \sum_{j=1}^{N} r_j e^{(k)}_j x_j \right\|_{L^p([0,1];X)}
$$

$$
\leq \max_{1 \leq k \leq 2^N} \left\| \sum_{j=1}^{N} r_j e^{(k)}_j x_j \right\|_{L^p([0,1];X)} = \left\| \sum_{j=1}^{N} r_j x_j \right\|_{L^p([0,1];X)}.
$$

In the last equality we used the fact that $\{r_j : j = 1, \ldots, N\}$ and $\{r_j e^{(k)}_j : j = 1, \ldots, N\}$ have the same joint probability distribution. \qed

**Theorem 3.11.** Let $T \subseteq L(X,Y)$ be $\mathcal{R}$-bounded. Then also the convex hull

$$
\text{conv} T := \left\{ \sum_{k=1}^{n} \lambda_k T_k : n \in \mathbb{N}, T_k \in T, \lambda_k \in [0,1], \sum_{k=1}^{n} \lambda_k = 1 \right\}
$$

and the absolute convex hull

$$
\text{aconv} T := \left\{ \sum_{k=1}^{n} \lambda_k T_k : n \in \mathbb{N}, T_k \in T, \lambda_k \in \mathbb{C}, \sum_{k=1}^{n} |\lambda_k| = 1 \right\}
$$

are $\mathcal{R}$-bounded. The same holds for the closures $\text{conv}^\ast T$ of $\text{conv} T$ and $\text{aconv}^\ast T$ of $\text{aconv} T$ with respect to the strong operator topology. We have $\mathcal{R}(\text{conv}^\ast T) \leq \mathcal{R}(T)$ and $\mathcal{R}(\text{aconv}^\ast T) \leq 2\mathcal{R}(T)$.

Proof. a) Let $T_1, \ldots, T_N \in \text{conv}(T)$. Then there exist $\lambda_{k,j} \in [0,1]$ and $T_{k,j} \in T$ with $\sum_{j=1}^{m_k} \lambda_{k,j} = 1$ and $T_k = \sum_{j=1}^{m_k} \lambda_{k,j} T_{k,j}$.

Define $\lambda_{k,j} := 0$ and $T_{k,j} := 0$ for $j \in \mathbb{N}$ with $j > m_k$ and $k = 1, \ldots, N$. For $\ell \in \mathbb{N}^N$ we define $\lambda_\ell := \prod_{k=1}^{N} \lambda_{k,\ell_k}$ and $T_{k,\ell} := T_{k,\ell_k}$ for $k = 1, \ldots, N$. Then $\lambda_\ell \in [0,1]$ as well as

$$
\sum_{\ell \in \mathbb{N}^N} \lambda_\ell = \sum_{\ell_1 \in \mathbb{N}} \cdots \sum_{\ell_N \in \mathbb{N}} \lambda_1, \ell_1 \cdot \cdots \cdot \lambda_N, \ell_N = 1.
$$

For all $k = 1, \ldots, N$ we obtain

$$
\sum_{\ell \in \mathbb{N}^N} \lambda_\ell T_{k,\ell} = \sum_{\ell \in \mathbb{N}^N} \lambda_\ell T_{k,\ell_k} = \left( \sum_{\ell_k \in \mathbb{N}} \lambda_{k,\ell_k} T_{k,\ell_k} \right) \prod_{j \neq k} \left( \sum_{\ell_j \in \mathbb{N}} \lambda_{j,\ell_j} \right)
$$
Note that these sums are finite. We get
\[
\left\| \sum_{k=1}^{N} r_k T_k x_k \right\|_{L^p([0,1]; Y)} \leq \left\| \sum_{k=1}^{N} \sum_{\ell \in \mathbb{N}} r_k \lambda_{\ell} T_k \ell x_k \right\|_{L^p([0,1]; Y)} \leq R(T) \left\| \sum_{\ell \in \mathbb{N}} \lambda_{\ell} \sum_{k=1}^{N} r_k x_k \right\|_{L^p([0,1]; X)}
\]
Consequently, \( R(\text{conv } T) \leq R(T) \).

b) By Kahane's contraction principle, \( R(T_0) \leq 2R(T) \), where we define
\[
T_0 := \{ \lambda T : T \in T, \lambda \in \mathbb{C}, |\lambda| \leq 1 \}.
\]
Because of \( \text{conv } T_0 = a\text{conv } T \), we get \( R(a\text{conv } T) \leq 2R(T) \) due to a).

c) The closedness with respect to the strong operator topology follows directly from the definition of \( R \)-boundedness.

The above results are useful to prove \( R \)-boundedness in general Banach spaces. In the special situation that \( X \) is some \( L^q \)-space, there is a helpful description of \( R \)-boundedness:

**Lemma 3.12** (Square function estimate). Let \((G, \mathcal{A}, \mu)\) be a \( \sigma \)-finite measure space, \( X = L^q(G) \), and let \( 1 \leq q < \infty \). Then \( T \subset L(X) \) is \( R \)-bounded if and only if there exists an \( M > 0 \) with
\[
\left\| \left( \sum_{j=1}^{N} |T_n f_n|^2 \right)^{1/2} \right\|_{L^q(G)} \leq M \left\| \left( \sum_{j=1}^{N} |f_n|^2 \right)^{1/2} \right\|_{L^q(G)}
\]
for all \( N \in \mathbb{N} \), \( T_n \in T \) and \( f_n \in L^q(G) \).

**Proof.** We write \( f \approx g \) if there are constants \( C_1, C_2 > 0 \) with \( C_1 |f| \leq |g| \leq C_2 |f| \). To show \( R \)-boundedness, by Kahane's inequality, we can consider the \( R_q \)-bound. For this, we can calculate
\[
\left\| \sum_{n=1}^{N} r_n f_n \right\|_{L^q([0,1]; L^q(G))}^q = \int_{0}^{1} \left\| \sum_{n=1}^{N} r_n(t) f_n(\cdot) \right\|_{L^q(G)}^q dt = \int_{0}^{1} \int_{G} \left( \sum_{n=1}^{N} r_n(t) f_n(\omega) \right)^q d\mu(\omega) dt
\]
\[
= \int_{G} \left( \sum_{n=1}^{N} |f_n(\omega)|^2 \right)^{q/2} d\mu(\omega) = \left\| \left( \sum_{n=1}^{N} |f_n|^2 \right)^{1/2} \right\|_{L^q(G)}^q
\]
Here, Fubini’s theorem and the inequality of Khintchine were used. Now the statement follows by considering the above calculation for both sides of the definition of \( R \)-boundedness.
Example 3.13. Using the square function estimate, it is easy to construct an example of a uniformly bounded operator family which is not $\mathcal{R}$-bounded. Let $p \in [1, \infty) \setminus \{2\}$. Then the family $\{T_n : n \in \mathbb{N}_0\} \subset L(L^p(\mathbb{R}))$, $T_n f(\cdot) := f(\cdot - n)$ of translations is not $\mathcal{R}$-bounded, as for $f_n = \chi_{[0,1]}$ we have

$$\left\| \left( \sum_{n=0}^{N-1} |T_n f_n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} = \|\chi_{[0,N]}\|_{L^p(\mathbb{R})} = N^{1/p},$$

$$\left\| \left( \sum_{n=0}^{N-1} |f_n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} = N^{1/2} \|\chi_{[0,1]}\|_{L^p(\mathbb{R})} = N^{1/2}.$$ 

For $1 \leq p < 2$, we use the fact that $\frac{N^{1/p}}{N^{1/2}} \to \infty$ for $N \to \infty$. The proof for $p > 2$ is similar.

Lemma 3.14. a) Let $G \subset \mathbb{R}^n$ be open and $1 \leq p < \infty$. For $\varphi \in L^\infty(G)$, define $m_\varphi \in L(L^p(G; X))$ by $(m_\varphi f)(x) := \varphi(x)f(x)$. Then for $r > 0$ one obtains

$$\mathcal{R}_p \left( \{ m_\varphi : \varphi \in L^\infty(G), \|\varphi\|_\infty \leq r \} \right) \leq 2r.$$ 

b) Let $1 \leq p < \infty$, $G \subset \mathbb{R}^n$ be open, and $\mathcal{T} \subset L(L^p(G; X), L^p(G; Y))$ be $\mathcal{R}$-bounded. Then

$$\mathcal{R}_p \left( \{ m_\varphi T m_\psi : T \in \mathcal{T}, \varphi, \psi \in L^\infty(G), \|\varphi\|_\infty \leq r, \|\psi\|_\infty \leq s \} \right) \leq 4rs \mathcal{R}_p(\mathcal{T}).$$

Proof. a) By the theorem of Fubini and Kahane’s contraction principle,

$$\left\| \sum_{k=1}^{N} r_k m_\varphi f_k \right\|_{L^p([0,1]; L^p(G; X))} = \left\| \sum_{k=1}^{N} r_k \varphi_k f_k \right\|_{L^p(G; L^p([0,1]; X))} \leq 2r \left\| \sum_{k=1}^{N} r_k f_k \right\|_{L^p(G; L^p([0,1]; X))} = 2r \left\| \sum_{k=1}^{N} r_k f_k \right\|_{L^p([0,1]; L^p(G; X))}.$$

b) follows from a) and Remark 3.9. \(\square\)

In the following corollary, we consider strongly measurable function. Note that a function $N : G \to L(X, Y)$ is called strongly measurable if there exists a $\mu$-zero set $A \in \mathcal{A}$ such that $N|_{G \setminus A}$ is measurable and $N(G \setminus A)$ is separable.

Corollary 3.15. Let $(G, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and $\mathcal{T} \subset L(X, Y)$ be $\mathcal{R}$-bounded. Let

$$\mathcal{N} := \{ N : G \to L(X, Y) \mid N \text{ strongly measurable with } N(G) \subset \mathcal{T} \}.$$ 

For $h \in L^1(G, \mu)$ and $N \in \mathcal{N}$ define

$$T_{N,h} x := \int_G h(\omega) N(\omega) x d\mu(\omega) \quad (x \in X).$$

Then

$$\mathcal{R} \left( \{ T_{N,h} : \|h\|_{L^1(G, \mu)} \leq 1, N \in \mathcal{N} \} \right) \leq 2\mathcal{R}(\mathcal{T}).$$

Proof. Let $\varepsilon > 0$. For $x_1, \ldots, x_N \in X$, $h \in L^1(G, \mu)$ and $N \in \mathcal{N}$ we consider the measurable map

$$M : G \to Y^N, \quad M(\omega) := (N(\omega)x_j)_{j=1,\ldots,N}.$$
Then $M \in L^\infty(G; Y^N)$ is strongly measurable, and therefore there exist a measurable partition $G = \bigcup_{j=1}^\infty G_j$, $G_i \cap G_j = \emptyset$ for $i \neq j$, and $\omega_j \in G_j$ with

$$\|N(\omega)x_k - N(\omega_j)x_k\|_Y < \varepsilon$$

for almost all $\omega \in G_j$ and all $k = 1, \ldots, N$.

Define

$$S := \sum_{j=1}^\infty \left( \int_{G_j} h(\omega)d\mu(\omega) \right) N(\omega_j).$$

Then $\|T_{N,h}x_k - Sx_k\|_Y < \varepsilon$ for all $k = 1, \ldots, N$. Therefore, $T_{N,h}$ is a subset of the neighbourhood of $S$ given by $x_1, \ldots, x_N$ and $\varepsilon$ with respect to the strong operator topology. Because of $S \in \operatorname{aconv} T$, we obtain $T_{N,h} \in \operatorname{aconv} T$. Now the statement follows from Theorem 3.11. \hfill \Box

**Corollary 3.16.** Let $N : \Sigma_{\theta'} \to L(X, Y)$ be holomorphic and bounded, and let $N(\partial \Sigma_{\theta}) \setminus \{0\}$ be $\mathcal{R}$-bounded for some $\theta < \theta'$. Then $N(\Sigma_{\theta})$ is $\mathcal{R}$-bounded, and for every $\theta_1 < \theta$ the family $\{\lambda \frac{\partial}{\partial \lambda} N(\lambda) : \lambda \in \Sigma_{\theta_1}\}$ is $\mathcal{R}$-bounded.

**Proof.** Considering $M(\lambda) := N(\lambda^{\theta'/\pi})$, we may assume $\theta = \frac{\pi}{2}$. Now we use Poisson’s formula

$$N(\alpha + i\beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2 + (s-\beta)^2} N(is)ds \quad (\alpha > 0).$$

Because of $\|\frac{1}{\pi} \frac{\alpha}{\alpha^2 + (s-\beta)^2}\|_{L^1(\mathbb{R})} = 1$, the first assertion follows from Corollary 3.15.

By Cauchy’s integral formula, we have

$$\lambda \frac{\partial}{\partial \lambda} N(\lambda) = \int_{\partial \Sigma_{\theta}} h(\mu)N(\mu)d\mu \quad (\lambda \in \Sigma_{\theta_1})$$

for $h(\lambda) := \frac{1}{2\pi i} \frac{\lambda}{\mu - \lambda \beta}$. Because of $\sup_{\lambda \in \Sigma_{\theta_1}} \|h_\lambda\|_{L^1(\partial \Sigma_{\theta})} < \infty$, the second assertion follows from Corollary 3.15, too. \hfill \Box

**Lemma 3.17.** Let $G \subset \mathbb{C}$ be open, $K \subset G$ be compact, and $H : G \to L(X, Y)$ be holomorphic. Then $H(K)$ is $\mathcal{R}$-bounded.

**Proof.** Let $z_0 \in K$. Then there exists an $r > 0$ with

$$H(z) = \sum_{k=0}^\infty H^{(k)}(z_0) \frac{(z - z_0)^k}{k!} \quad (|z - z_0| \leq r).$$

Here the series converges in $L(X, Y)$ and

$$\rho_0 := \sum_{k=0}^\infty \|H^{(k)}(z_0)\|_{L(X, Y)} \frac{1}{k!} < \infty.$$ 

As a set with one element, $\{H^{(k)}(z_0)\}$ is $\mathcal{R}$-bounded with $\mathcal{R}$-bound $\|H^{(k)}(z_0)\|_{L(X, Y)}$. By Kahane’s contraction principle, the family $\{H^{(k)}(z_0) \frac{(z - z_0)^k}{k!} : z \in B(z_0, r)\}$ is $\mathcal{R}$-bounded, too, with $\mathcal{R}$-bound not greater than $2\rho_0 \|H^{(k)}(z_0)\|_{L(X, Y)}$. Therefore, we obtain for all finite partial sums the $\mathcal{R}$-bound $2\rho_0$. Taking the closure with respect to the strong operator topology, the same holds for the infinite sum. By a finite covering of $K$, we obtain the statement of the lemma. \hfill \Box
Theorem 3.18. Let $G \subset \mathbb{R}^n$ be open and $1 < p < \infty$. Let $\Lambda$ be a set and $\{k_\lambda : \lambda \in \Lambda\}$ be a family of measurable kernels $k_\lambda : G \times G \rightarrow L(X,Y)$ with

$$\mathcal{R}_p\left(\{k_\lambda(z,z') : \lambda \in \Lambda\}\right) \leq k_0(z,z') \quad (z,z' \in G).$$

Assume that for the corresponding scalar integral operator

$$(K_0f)(z) = \int_G k_0(z,z') f(z') dz' \quad (f \in L^p(G))$$

one has $K_0 \in L(L^p(G))$. Define

$$(K_\lambda f)(z) = \int_G k_\lambda(z,z') f(z') dz' \quad (f \in L^p(G;X)).$$

Then $K_\lambda \in L(L^p(G;X), L^p(G;Y))$ with

$$\mathcal{R}_p\left(\{K_\lambda : \lambda \in \Lambda\}\right) \leq \|K_0\|_{L(L^p(G))}.$$

Proof. We use the definition of $\mathcal{R}$-boundedness and get

$$\left\| \sum_{j=1}^N r_j K_{\lambda_j} f_j \right\|_{L^p([0,1];L^p(G;Y))} = \left( \int_0^1 \left\| \sum_{j=1}^N r_j(t) \int_G k_{\lambda_j}(\cdot,z') f_j(z') dz' \right\|_{L^p(G;Y)}^p dt \right)^{1/p}$$

$$= \left( \int_0^1 \left\| \int_G \sum_{j=1}^N r_j(t) k_{\lambda_j}(\cdot,z') f_j(z') dz' \right\|_{L^p(G;Y)}^p dt \right)^{1/p}$$

$$= \left( \int_0^1 \int_G \left\| \sum_{j=1}^N r_j(t) k_{\lambda_j}(z,z') f_j(z') dz' \right\|^p_Y dz \ dt \right)^{1/p}$$

$$= \left( \int_G \int_0^1 \left\| \sum_{j=1}^N r_j(t) k_{\lambda_j}(z,z') f_j(z') dz' \right\|^p_Y dt \ dz \right)^{1/p}.$$

Setting $\varphi(t,z,z') := \sum_{j=1}^N r_j(t) k_{\lambda_j}(z,z') f_j(z')$, the integral with respect to $t$ in the last term equals $\left\| \int_G \varphi(\cdot,z,z') dz' \right\|_{L^p([0,1])}^p$. Now we apply the inequality

$$\left\| \int_G \varphi(\cdot,z,z') dz' \right\|_{L^p([0,1])} \leq \int_G \left\| \varphi(\cdot,z,z') \right\|_{L^p([0,1])} dz'$$

for Bochner integrals and obtain, using the assumption of $\mathcal{R}$-boundedness,

$$\left\| \sum_{j=1}^N r_j K_{\lambda_j} f_j \right\|_{L^p([0,1];L^p(G;Y))} \leq \left( \int_G \left[ \int_G \left\| \sum_{j=1}^N r_j(\cdot) k_{\lambda_j}(z,z') f_j(z') \right\|_{L^p([0,1];Y)}^p dz' \right] dz \right)^{1/p}$$

$$\leq \left( \int_G \left[ \int_G \left\| k_0(z,z') \sum_{j=1}^N r_j(\cdot) f_j(z') \right\|_{L^p([0,1];X)}^p dz' \right] \right)^{1/p}$$

$$= \left\| K_0 \left( \left\| \sum_{j=1}^N r_j f_j(\cdot) \right\|_{L^p([0,1];X)} \right) \right\|_{L^p(G)}.$$
\[
\leq \|K_0\|_{L(L^p(G))}\left(\left\|\sum_{j=1}^{N} r_j f_j(\cdot)\right\|_{L^p([0,1];X)}\right)_{L^p(G)}
= \|K_0\|_{L(L^p(G))}\left\|\sum_{j=1}^{N} r_j f_j\right\|_{L^p([0,1];L^p(G;X))}.
\]

3.2. **Fourier multipliers and Mikhlin’s theorem.** We have already seen in Theorem 2.8 that maximal regularity is equivalent to the \(L^p(\mathbb{R};X)\)-boundedness of the operator \(\mathcal{F}_t^{-1}i\tau(i\tau - A)^{-1}\mathcal{F}_t\). This is a typical example of a (vector-valued) Fourier multiplier. In the analysis of partial differential equations and boundary value problems in \(L^p\)-spaces, the question of Fourier multipliers play a central role. The answer is given by the classical theorem of Mikhlin and by its Banach space valued variants.

In the following, we use the standard notation \(D := -i(\partial_{x_1}, \ldots, \partial_{x_n})\) as well as the standard multi-index notation \(D^\alpha = (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}\). We start with a simple example.

**Example 3.19.** Consider the Laplacian \(\Delta\) in \(L^p(\mathbb{R}^n)\) with maximal domain \(D(\Delta) := \{u \in L^p(\mathbb{R}^n) : \Delta u \in L^p(\mathbb{R}^n)\}\). Obviously we have \(D(\Delta) \supset W^2_p(\mathbb{R}^n)\). To show that we even have equality, we consider \(u \in D(\Delta)\) and \(f := u - \Delta u \in L^p(\mathbb{R}^n)\). Let \(|\alpha| \leq 2\). Then
\[
D^\alpha u = \mathcal{F}_t^{-1} \frac{\xi^\alpha}{1 + |\xi|^2}\mathcal{F} f
\]
holds as equality in \(\mathcal{S}'(\mathbb{R}^n)\), where \(\mathcal{F}\) stands for the \(n\)-dimensional Fourier transform (see below). To obtain \(D^\alpha u \in L^p(\mathbb{R}^n)\), we have to show \(\mathcal{F}_t^{-1}m_\alpha \mathcal{F} f \in L^p(\mathbb{R}^n)\), where \(m_\alpha := \frac{\xi^\alpha}{1 + |\xi|^2}\). So we have to prove that
\[
f \mapsto \mathcal{F}_t^{-1}m_\alpha \mathcal{F} f
\]
defines a bounded linear operator on \(L^p(\mathbb{R}^n)\). This is in fact the case, as we will see from the classical version of Mikhlin’s theorem, Theorem 3.22 below.

In contrast to the above example, we will also need vector-valued versions of Mikhlin’s theorem. For this, we need some preparation, starting with the vector-valued Fourier transform. Let \(X\) be a Banach space. Then the Schwartz space \(\mathcal{S}(\mathbb{R}^n;X)\) is defined as the space of all \(\mathcal{S}(\mathbb{R}^n)\) valued \(X\)-valued tempered distributions. The space of all \(X\)-valued tempered distributions is defined by
\[
\mathcal{S}'(\mathbb{R}^n;X) := L(\mathcal{S}(\mathbb{R}^n), X).
\]

On \(\mathcal{S}(\mathbb{R}^n;X)\), we consider the family of seminorms
\[
\pi_\varphi : \mathcal{S}(\mathbb{R}^n;X) \to [0, \infty), \ u \mapsto \|u(\varphi)\|_X \ (\varphi \in \mathcal{S}(\mathbb{R}^n)).
\]
Then the family \(\{\pi_\varphi : \varphi \in \mathcal{S}(\mathbb{R}^n)\}\) defines a locally convex topology on \(\mathcal{S}'(\mathbb{R}^n;X)\). Note that in the scalar case \(X = \mathbb{C}\), this is the weak*-topology. One can see as in the scalar case that the Fourier transform, defined for \(\varphi \in \mathcal{S}(\mathbb{R}^n;X)\) by
\[
(\mathcal{F}\varphi)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\varphi(x)dx \ (\xi \in \mathbb{R}^n, \varphi \in \mathcal{S}(\mathbb{R}^n;X)),
\]
can be extended by duality to an isomorphism $\mathcal{F}: \mathcal{S}'(\mathbb{R}^n; X) \to \mathcal{S}'(\mathbb{R}^n; X)$.

**Definition 3.20.** Let $X, Y$ be Banach spaces, $1 \leq p < \infty$, and let $m: \mathbb{R}^n \to L(X, Y)$ be a bounded and strongly measurable function. Because of $\mathcal{F}^{-1} \in L(L^1(\mathbb{R}^n; X), L^\infty(\mathbb{R}^n; Y))$, the function $m$ induces a map $T_m: \mathcal{S}(\mathbb{R}^n; X) \to L^\infty(\mathbb{R}^n; Y)$ by

$$T_m f := \mathcal{F}^{-1} m \mathcal{F} f \quad (f \in \mathcal{S}(\mathbb{R}^n; X)).$$

The function $m$ is called a Fourier multiplier (more precisely, an $L^p$-Fourier multiplier) if

$$\|T_m f\|_{L^p(\mathbb{R}^n; Y)} \leq C \|f\|_{L^p(\mathbb{R}^n; X)} \quad (f \in \mathcal{S}(\mathbb{R}^n; X)).$$

As $\mathcal{S}(\mathbb{R}^n; X)$ is dense in $L^p(\mathbb{R}^n; X)$ for $p \in [1, \infty)$, this implies that $T_m$ has a unique extension to a bounded linear operator $T_m \in L(L^p(\mathbb{R}^n; X), L^p(\mathbb{R}^n; Y))$. In this case, $m$ is called the symbol of the operator $T_m$, and we write $\text{op}[m] := \mathcal{F} m \mathcal{F}^{-1} := T_m$ and $\text{symb}[T_m] := m$.

We start with the scalar case $X = Y = \mathbb{C}$.

**Remark 3.21.** In the Hilbert space case $p = 2$, one can apply Plancherel’s theorem. Therefore, we have $\text{op}[m] \in L(L^2(\mathbb{R}^n))$ if and only if the multiplication operator $g \mapsto mg$ is a bounded operator in $L^2(\mathbb{R}^n)$. This is equivalent to the condition $m \in L^\infty(\mathbb{R}^n)$.

In fact, if $m \in L^\infty(\mathbb{R}^n)$, then $\|mg\|_{L^2(\mathbb{R}^n)} \leq \|m\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}$. On the other hand, if $m \notin L^\infty(\mathbb{R}^n)$, then there exists a sequence $(A_k)_{k \in \mathbb{N}}$ of measurable subsets of $\mathbb{R}^n$ such that $0 < \lambda(A_k) < \infty$ and $|m(x)| \geq k$ for $x \in A_k$. For the characteristic function $g_k := \chi_{A_k}$ we obtain $g_k \in L^2(\mathbb{R}^n)$ and

$$\|mg_k\|^2_{L^2(\mathbb{R}^n)} = \int |m(\xi)g_k(\xi)|^2 d\xi \geq k^2 \lambda(A_k) = k^2 \|g_k\|^2_{L^2(\mathbb{R}^n)}.$$ 

Therefore, $\text{op}[m]$ cannot be a bounded operator in $L^2(\mathbb{R}^n)$.

The following classical theorem gives a sufficient condition for a function to be a (scalar) Fourier multiplier and has many applications in the theory of partial differential equations. In the following, $\lfloor \frac{n}{2} \rfloor$ denotes the largest integer not greater than $\frac{n}{2}$. We state this result in two variants.

**Theorem 3.22 (Mikhlin’s multiplier theorem).** Let $1 < p < \infty$ and $m: \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$. If one of the two conditions

(i) $m \in C^{\lfloor \frac{n}{2} \rfloor + 1}(\mathbb{R}^n \setminus \{0\})$ and

$$|\xi|^\beta |\partial^\beta m(\xi)| \leq C_M \quad (\xi \in \mathbb{R}^n \setminus \{0\}, |\beta| \leq \lfloor \frac{n}{2} \rfloor + 1),$$

(ii) $m \in C^n(\mathbb{R}^n \setminus \{0\})$ and

$$|\xi^\beta \partial^\beta m(\xi)| \leq C_M \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \beta \in \{0,1\}^n)$$

holds with a constant $C_M > 0$, then $m$ is an $L^p$-Fourier multiplier with

$$\|\text{op}[m]\|_{L(L^p(\mathbb{R}^n))} \leq c(n, p)C_M,$$

with a constant $c(n, p)$ depending only on $n$ and $p$. 
A proof of this theorem (which is also called Mikhlin-Hörmander theorem) can be found, e.g., in [Gra14], Section 6.2.3. Condition (i) is sometimes called the Mikhlin condition, whereas condition (ii) is called the Lizorkin condition. For the $L^p$-continuity of singular integral operators, we also refer to [Ste93], Section 6.5.

For the following result, note that a function $m: \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called (positively) homogeneous with respect to $\xi$ of degree $d \in \mathbb{R}$ if

$$m(\rho \xi) = \rho^d m(\xi) \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \rho > 0).$$

**Lemma 3.23.** Let $m \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree 0. Then $m$ satisfies the Mikhlin condition.

**Proof.** If a function $m \in C^k(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree $d$, then its derivative $\partial^\beta m(\xi)$ is homogeneous of degree $d - |\beta|$ for all $|\beta| \leq k$. This follows from the identities

$$\partial^\beta [m(\rho \xi)] = \rho^{|\beta|} (\partial^\beta m)(\rho \xi)$$

and

$$\partial^\beta [\rho^d m(\xi)] = \rho^d (\partial^\beta m)(\xi).$$

Now let $m \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree 0, and let $|\beta| \leq \left[\frac{n}{2}\right] + 1$. Then $m_\beta(\xi) := \xi^{|\beta|} \partial^\beta m(\xi)$ is homogeneous of degree 0 and continuous. Therefore,

$$|m_\beta(\xi)| = \left| m_\beta \left( \frac{\xi}{\xi} \right) \right| \leq \max_{|\eta| = 1} \left| m_\beta(\eta) \right| < \infty \quad (\xi \in \mathbb{R}^n \setminus \{0\}).$$

As a first application of Mikhlin’s theorem, we can now answer the question from Example 3.19.

**Corollary 3.24.** Let $1 < p < \infty$. Then \( \{ u \in L^p(\mathbb{R}^n) : \Delta u \in L^p(\mathbb{R}^n) \} = W^2_p(\mathbb{R}^n) \).

**Proof.** As we have seen in Example 3.19, we have to show that the function $m_\alpha(\xi) := \frac{\xi^{\alpha}}{1+|\xi|^2}$ satisfies the Mikhlin condition for all $|\alpha| \leq 2$. For this, we write $m_\alpha(\xi) = \tilde{m}_\alpha(\xi, 1)$ where the function $\tilde{m}_\alpha: \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{C}$ is defined by

$$\tilde{m}_\alpha(\xi, \mu) := \frac{\xi^{\alpha}}{\mu^2 + |\xi|^2}.$$

As the function $\tilde{m}_\alpha$ is smooth and homogeneous of degree 0, it satisfies the Mikhlin condition by Lemma 3.23. Setting $\mu = 1$, we see that also $m_\alpha$ satisfies the Mikhlin condition.

As mentioned above, we also need vector-valued variants of Mikhlin’s theorem. The following results assume some geometric conditions on the Banach space $X$. For a detailed discussion of these properties, see, e.g., [HvNW16], Chapter 4.

**Definition 3.25.** a) A Banach space $X$ is called a UMD space or a space of class HT if the symbol $m(\xi) := -i \text{sgn}(\xi) \text{id}_X$ yields a bounded operator $\text{op}[m] \in L(L^p(\mathbb{R}; X))$. The operator $\text{op}[m]$ is called the Hilbert transform.

b) A Banach space $X$ is said to have property $(\alpha)$ if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$, all i.i.d. symmetric $\{-1, 1\}$-valued random variables $\varepsilon_1, \ldots, \varepsilon_N$ on $\Omega$ and $\varepsilon_1', \ldots, \varepsilon_N'$ on $\Omega'$, all $\alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \leq 1$, and all $x_{ij} \in X$ we have

$$\left\| \sum_{i,j=1}^N \alpha_{ij} \varepsilon_i \varepsilon_j' x_{ij} \right\|_{L^2(\Omega \times \Omega'; X)} \leq C \left\| \sum_{i,j=1}^N \varepsilon_i \varepsilon_j' x_{ij} \right\|_{L^2(\Omega \times \Omega'; X)}.$$
Remark 3.26. a) Every UMD space is reflexive. In particular, $L^1(G)$ and $L^\infty(G)$ are no UMD spaces. However, $L^2(G)$ has property $(\alpha)$.

b) Every Hilbert space is a UMD space with property $(\alpha)$. If $E$ is a UMD space with property $(\alpha)$ and if $(S, \sigma, \mu)$ is a $\sigma$-finite measure space, then also $L^p(S; E)$ is a UMD space with property $(\alpha)$ for all $p \in (1, \infty)$.

c) More generally, if $G \subseteq \mathbb{R}^n$ is a domain, $E$ is a UMD space with property $(\alpha)$ and $p, q \in (1, \infty)$, then the vector-valued Besov space $B^s_{pq}(G; E)$ and the vector-valued Triebel-Lizorkin space $F^s_{pq}(G; E)$ are again UMD spaces with property $(\alpha)$. In particular, this holds in the scalar case $E = \mathbb{C}$.

The following result is the vector-valued analog of Mikhlin’s theorem and was central in the development of the theory and application of maximal $L^p$-regularity.

Theorem 3.27. Let $X$ and $Y$ be UMD Banach spaces, and let $1 < p < \infty$. Assume $m \in C^n(\mathbb{R}^n \setminus \{0\}; L(X, Y))$ with
\[
\mathcal{R}\left(\{\langle |\xi|^\alpha \partial^\alpha m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \alpha \in \{0, 1\}^n\}\right) =: \kappa < \infty.
\]
Then $m$ is a vector-valued Fourier multiplier with
\[
\|\operatorname{op}[m]\|_{L(\ell^p(\mathbb{R}^n), \ell^p(\mathbb{R}^n; Y))} \leq C\kappa,
\]
where the constant $C$ depends only on $n, p, X$, and $Y$.

The proof of Theorem 3.27 uses Paley-Littlewood decompositions, see [KW04], Theorem 4.6, or [HvNVW16], Theorem 5.3.18.

In the last result, we had one symbol $m$ and the related operator $\operatorname{op}[m]$. The following theorem shows that for a family of symbols satisfying uniform Mikhlin type estimates, also the related operator family is $\mathcal{R}$-bounded.

Theorem 3.28. Let $X$ and $Y$ be UMD Banach spaces with property $(\alpha)$. Let $\mathcal{T} \subset L(X, Y)$ be $\mathcal{R}$-bounded. Consider the set
\[ M := \left\{ m \in C^n(\mathbb{R}^n \setminus \{0\}; L(X, Y)) : \xi^\alpha D^\alpha m(\xi) \in \mathcal{T} \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \alpha \in \{0, 1\}^n) \right\}. \]
Then $\{\operatorname{op}[m] : m \in M\} \subset L(\ell^p(\mathbb{R}^n; X), \ell^p(\mathbb{R}^n; Y))$ is $\mathcal{R}$-bounded with $\mathcal{R}_p(\{\operatorname{op}[m] : m \in M\}) \leq \mathcal{C}\mathcal{R}_p(\mathcal{T})$, where the constant $\mathcal{C}$ depends only on $p, m, X$, and $Y$.

For a proof of this result, we refer to [GW03], Theorem 3.2. Theorem 3.28 is also the basis for an iteration process: $\mathcal{R}$-bounded symbol families yield $\mathcal{R}$-bounded operator families. For an application to pseudodifferential operators with $\mathcal{R}$-bounded symbols, we also refer to [DK07].

Note that Theorem 3.28 also gives a strong result in the scalar case $X = \mathbb{C}$. As $\mathbb{C}$ is a Hilbert space, boundedness in $\mathbb{C}$ equals $\mathcal{R}$-boundedness. Therefore, boundedness of a family of scalar symbols implies $\mathcal{R}$-boundedness of the corresponding operator family. The same holds if $X$ is a general Hilbert space. We give a simple but useful example.

Corollary 3.29. Let $\{m_\lambda : \lambda \in \Lambda\}$ be a family of matrix valued functions $m_\lambda \in C^n(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{N \times N})$ with
\[ |\xi^\alpha D^\alpha m_\lambda(\xi)|_{\mathbb{C}^{N \times N}} \leq C_0 \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \alpha \in \{0, 1\}^n, \lambda \in \Lambda). \]
Then \( \{ \text{op}[m_\lambda] : \lambda \in \Lambda \} \subset L(L^p(\mathbb{R}^n; \mathbb{C}^N)) \) is \( \mathcal{R} \)-bounded with \( \mathcal{R} \)-bound \( C \cdot C_0 \), where \( C \) only depends on \( p \) and \( N \).

**Proof.** As a Hilbert space, \( X = \mathbb{C}^N \) is a UMD space with property \((\alpha)\). By assumption, we know that

\[
\{ \xi^\alpha D_\xi^\alpha m_\lambda(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \alpha \in \{0,1\}^n, \lambda \in \Lambda \} \subset L(X)
\]

is norm bounded and consequently, as \( X \) is a Hilbert space, also \( \mathcal{R} \)-bounded. Choosing \( T := \{ A \in \mathbb{C}^{N \times N} : |A| \leq C_0 \} \) in Theorem 3.28, we obtain the \( \mathcal{R} \)-boundedness of \( \{ \text{op}[m_\lambda] : \lambda \in \Lambda \} \subset L(L^p(\mathbb{R}^n; \mathbb{C}^N)) \). \( \square \)

### 3.3. \( \mathcal{R} \)-sectorial operators

Now we come back to the question of maximal \( L^p \)-regularity. As we have seen in Theorem 2.11, maximal regularity holds if and only if the operator-valued symbol \( m(\lambda) := \lambda(\lambda - A)^{-1} \) for \( \lambda \in i\mathbb{R} \) is a bounded operator in \( L^p(\mathbb{R}; X) \). So we can apply the one-dimensional case of Theorem 3.27. We start with a notion from operator theory.

In the following, let

\[
\Sigma_\varphi := \left\{ z \in \mathbb{C} \setminus \{0\} : \arg(z) < \varphi \right\}
\]

for \( \varphi \in (0, \pi] \). We denote the spectrum and the resolvent set of an operator \( A \) by \( \sigma(A) \) and \( \rho(A) \), respectively.

**Definition 3.30.** Let \( A : D(A) \rightarrow X \) be a linear and densely defined operator. Then \( A \) is called sectorial if there exists an angle \( \varphi > 0 \) such that \( \rho(A) \supset \Sigma_\varphi \) and

\[
\sup_{\lambda \in \Sigma_\varphi} \| \lambda(\lambda - A)^{-1} \|_{L(X)} < \infty.
\]

If this is the case, we call

\[
\varphi_A := \sup\{ \varphi : \rho(A) \supset \Sigma_\varphi, \sup_{\lambda \in \Sigma_\varphi} \| \lambda(\lambda - A)^{-1} \|_{L(X)} < \infty \}
\]

the spectral angle of \( A \).

The following theorem is an important result from the theory of semigroups of operators (see, e.g., [EN00], Theorem II.4.6).

**Theorem 3.31.** Let \( A : D(A) \rightarrow X \) be a linear and densely defined. Then the following statements are equivalent:

(i) \( A \) generates a bounded holomorphic \( C_0 \)-semigroup on \( X \) with angle \( \vartheta \in (0, \frac{\pi}{2}] \).

(ii) \( A \) is sectorial with spectral angle \( \varphi_A \geq \vartheta + \frac{\pi}{2} \).

It turns out that a similar condition characterizes operators with maximal \( L^p \)-regularity. For the following result, cf. [DHP03], Theorem 4.4, [Wei01], Theorem 4.2, and [KW04], Theorem 1.11.

**Theorem 3.32** (Theorem of Weis). Let \( X \) be a UMD Banach space, \( 1 < p < \infty \), and \( A \) be a sectorial operator with spectral angle \( \varphi_A > \frac{\pi}{2} \). Then \( A \in \text{MR}((0, \infty); X) \) if the family

\[
\{ \lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\varphi \} \subset L(X)
\]

is \( \mathcal{R} \)-bounded for some \( \varphi > \frac{\pi}{2} \).
With respect to the last theorem, one defines $\mathcal{R}$-sectorial operators:

**Definition 3.33.** Let $A: D(A) \to X$ be a linear and densely defined operator. Then $A$ is called $\mathcal{R}$-sectorial if there exists an angle $\varphi > 0$ with $\rho(A) \supset \Sigma_\varphi$ and

$$\mathcal{R}\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\varphi\} < \infty.$$  

The $\mathcal{R}$-angle of $A$ is defined as the supremum of all angles for which the above $\mathcal{R}$-bound is finite.

By Theorem 3.32, a sectorial operator has maximal regularity if it is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle larger than $\frac{\pi}{2}$. In fact, one has the following equivalences.

**Theorem 3.34.** Let $A$ be the generator of a bounded holomorphic $C_0$-semigroup. Then the following statements are equivalent:

(i) There exists a $\delta > 0$ such that $A$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\varphi_\mathcal{R} = \frac{\pi}{2} + \delta$.

(ii) There exists an $n \in \mathbb{N}$ such that $\{t^n(i(t - A))^{-n} : t \in \mathbb{R} \setminus \{0\}\}$ is $\mathcal{R}$-bounded.

(iii) There exists a $\delta > 0$ such that the family $\{T_z : z \in \Sigma_\delta\}$ is $\mathcal{R}$-bounded.

(iv) The family $\{T_t, tAT_t : t > 0\}$ is $\mathcal{R}$-bounded.

**Proof.** We only give a sketch of proof, for the full version see [KW04], Theorem 1.11.

(i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(i). We write

$$(it - A)^{-n+1} = (n-1)i \int_t^\infty (is - A)^{-n} ds$$

and obtain

$$(it)^{n-1}(it - A)^{-n+1} = \int_0^\infty h_t(s)[(is)^n(is - A)^{-n}] ds$$

for the function $h_t(s) := (n-1)t^{n-1}s^{-n}\chi_{[t,\infty)}$. We have $\int_0^\infty h_t(s)ds = 1$, and Corollary 3.15 yields (ii) for $n-1$ instead of $n$. Iteratively, we see that (ii) holds for $n = 1$. Now we use Corollary 3.16 to show the $\mathcal{R}$-boundedness of $\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_{\pi/2}\}$. By considering power series expansion, one can show that $\lambda(\lambda - A)^{-1}$ is in fact $\mathcal{R}$-bounded on some larger sector.

(iii)$\Rightarrow$(i). This follows from Corollary 3.15, too, with help of the representation

$$(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t}T_t dt.$$  

(i)$\Rightarrow$(iii) follows similarly by

$$T_z = \frac{1}{2\pi i} \int_{\Gamma_t} e^{\lambda z}(\lambda - A)^{-1} d\lambda.$$  

(iii)$\iff$(iv) can be shown using Corollary 3.16. □
4. $L^p$-Sobolev spaces

In the definition of maximal regularity, the vector-valued Sobolev space $W^1_p(J;X)$ appears. In many cases, $X = L^p(G)$ for some domain $G \subset \mathbb{R}^n$, and it would be desirable to obtain a more explicit description of the space $\gamma_l^E$ of time traces in this situation. Note that Lemma 2.4 tells us that this is connected with real interpolation. A similar question arises if the operator $A$ is a differential operator in some domain $G \subset \mathbb{R}^n$. In this case, the domain $D(A)$ is described by boundary operators, and the spaces for the boundary traces will be non-integer Sobolev spaces. For $p \neq 2$, there are different scales of non-integer Sobolev spaces: Besov spaces, Triebel-Lizorkin spaces, and Bessel potential spaces. A modern definition of these scales is based on dyadic decomposition and on the Fourier transform. A classical reference for this is the book by Triebel ([Tri95], Section 2.3), where the scalar case is discussed. For a modern presentation, including the vector-valued situation, we mention the monograph by Amann ([Ama19], Chapter VII). Note that in the vector-valued situation, the related integrals are Bochner integrals, and we refer to [HvNVW16], Section 1, and [ABHN11], Section 1.1, for an introduction to vector-valued integration.

Definition 4.1. A sequence $(\varphi_k)_{k \in \mathbb{N}_0}$ of $C^\infty$-functions $(\varphi_k)_{k \in \mathbb{N}_0}$ is called a dyadic decomposition if

(i) $\varphi_k \geq 0$, $\operatorname{supp} \varphi_0 \subset B(0,2)$ and $\operatorname{supp} \varphi_k \subset \{ \xi \in \mathbb{R}^n : 2^{k-1} < |\xi| < 2^{k+1} \}$ for all $k \in \mathbb{N}$,

(ii) $\sum_{k \in \mathbb{N}_0} \varphi_k(\xi) = 1$ for all $\xi \in \mathbb{R}^n$,

(iii) for all $\alpha \in \mathbb{N}_0^n$ there exists a $c_\alpha > 0$ with

$$|\xi|^{\alpha} |\partial^\alpha \varphi_k(\xi)| \leq c_\alpha \ (\xi \in \mathbb{R}^n, \ k \in \mathbb{N}_0).$$

It is easy to define a dyadic decomposition by scaling a fixed function $\varphi_1$ (see [Tri95], Section 2.3). By the theorem of Paley-Wiener, for every $u \in \mathcal{S}'(\mathbb{R}^n)$ the distribution $\operatorname{op}[\varphi_k]u$ is a regular distribution and even a smooth function. Therefore, $(\operatorname{op}[\varphi_k]u)(x)$ is well-defined. In the following, let $X$ be a Banach space.

Definition 4.2. a) For $s \in \mathbb{R}$, $p, q \in [1, \infty)$, the Besov space $B^s_{pq}(\mathbb{R}^n; X)$ is defined by $B^s_{pq}(\mathbb{R}^n; X) := \{ u \in \mathcal{S}'(\mathbb{R}^n; X) : \|u\|_{B^s_{pq}(\mathbb{R}^n; X)} < \infty \}$, where

$$\|u\|_{B^s_{pq}(\mathbb{R}^n; X)} = \left[ \sum_{k \in \mathbb{N}_0} 2^{skq} \left( \int_{\mathbb{R}^n} \|\operatorname{op}[\varphi_k]u(x)\|_X^p \right)^{q/p} dx \right]^{1/q}.$$ 

b) For $s \in \mathbb{R}$ and $p, q \in [1, \infty)$ the Triebel-Lizorkin space $F^s_{pq}(\mathbb{R}^n; X)$ is defined by $F^s_{pq}(\mathbb{R}^n; X) := \{ u \in \mathcal{S}'(\mathbb{R}^n; X) : \|u\|_{F^s_{pq}(\mathbb{R}^n; X)} < \infty \}$, where

$$\|u\|_{F^s_{pq}(\mathbb{R}^n; X)} = \left[ \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{N}_0} 2^{skq} \|\operatorname{op}[\varphi_k]u(x)\|_X^q \right)^{p/q} dx \right]^{1/p}.$$ 

c) If $p = \infty$ or $q = \infty$, the above definitions hold with the standard modification.

By an application of Fubini’s theorem, we immediately see that for $p = q$ the definitions of Besov spaces and Triebel-Lizorkin spaces coincide, but in general we have two different scales of Sobolev space type. For the third scale, the Bessel potential spaces, we consider the function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \to \mathbb{R}$, $\xi \mapsto \langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. For the following definition, we refer to [HvNVW16], Definition 5.6.2.
Definition 4.3. Let $s \in \mathbb{R}$ and $p \in [1, \infty]$. Then the Bessel potential space $H^s_p(\mathbb{R}^n; X)$ is defined as the space of all $u \in \mathcal{S}(\mathbb{R}^n; X)$ for which $\text{op}[\langle \cdot \rangle^s]u \in L^p(\mathbb{R}^n; X)$. The corresponding norm is defined as

$$
\|u\|_{H^s_p(\mathbb{R}^n; X)} := \|\text{op}[\langle \cdot \rangle^s]u\|_{L^p(\mathbb{R}^n; X)}.
$$

Remark 4.4. a) Many classical Sobolev spaces can be found as special cases of the above definition.

- Let $X$ be a UMD space, $k \in \mathbb{N}$, and $p \in (1, \infty)$, and let $W^k_p(\mathbb{R}^n; X)$ denote the classical Sobolev space,

$$
W^k_p(\mathbb{R}^n; X) := \{ u \in L^p(\mathbb{R}^n; X) : \forall |\alpha| \leq k : \partial^\alpha u \in L^p(\mathbb{R}^n; X) \}.
$$

Then $W^k_p(\mathbb{R}^n; X) = H^k_p(\mathbb{R}^n; X)$ with equivalent norms ([HvNVW16], Theorem 5.6.11).

- Let $p \in (1, \infty)$ and $s \in \mathbb{R}$. Then the equality $H^s_p(\mathbb{R}^n; X) = F^s_{pq}(\mathbb{R}^n; X)$ holds if and only if $X$ is isomorphic to a Hilbert space ([HM96], Theorem 1.2).

- Let $p \in [1, \infty)$ and $s \in (0, \infty) \setminus \mathbb{N}$. Then the Sobolev-Slobodeckii space $W^s_p(\mathbb{R}^n; X)$ is given as $W^s_p(\mathbb{R}^n; X) = B^s_{pp}(\mathbb{R}^n; X)$ ([Ama19], Remark 3.6.4).

- Let $s \in (0, \infty) \setminus \mathbb{N}$. Then the classical Hölder space is given as $C^s(\mathbb{R}^n; X) = B^s_{\infty, \infty}(\mathbb{R}^n; X)$ ([Ama19], Remark 3.6.4).

b) Let $G \subset \mathbb{R}^n$ be a domain. Then the space $B^s_{pq}(G; X)$ is defined by restriction, i.e.

$$
B^s_{pq}(G; X) := \{ u \in \mathcal{D}'(G; X) : \exists \tilde{u} \in B^s_{pq}(\mathbb{R}^n; X) : u = \tilde{u}|_G \}
$$

with canonical norm

$$
\|u\|_{B^s_{pq}(G; X)} := \inf \{ \|\tilde{u}\|_{B^s_{pq}(\mathbb{R}^n; X)} : u = \tilde{u}|_G \}.\n$$

Note here that the restriction of a distribution is defined as $\tilde{u}|_G := \tilde{u}|_{\mathcal{D}(G)}$. In the same way, the other scales are defined on domains.

The following result can be shown with the theory of interpolation spaces and is the basis for the description of the trace spaces. We refer to [Ama19], Theorem 2.7.4, for a proof (with $G = \mathbb{R}^n$, the case of a domain can be handled by a retraction-coretraction argument if the domain is smooth enough).

Theorem 4.5. Let $G \subset \mathbb{R}^n$ be a sufficiently smooth domain, and let $p, q \in (1, \infty)$, $k \in \mathbb{N}$, and $s \in (0, k)$. Then

$$
B^s_{pq}(G; X) = (L^p(G; X), W^k_p(G; X))_{s/k,q}.
$$

From this theorem and the description of the trace spaces as real interpolation space, one can easily obtain $\gamma_0 W^k_p(G; X) = B^{k-1/p}_{pp}(G; X)$, where $\gamma_0 u := u|_{\partial G}$ stands for the trace on the boundary of the domain. This typical loss of derivatives of order $1/p$ leads to non-integer Sobolev spaces for inhomogeneous boundary data. For parabolic equations, we also have to consider time and boundary traces of the solution space:

Corollary 4.6. Let $G \subset \mathbb{R}^n$ be a sufficiently smooth domain, $J = (0, T)$ with $T \in (0, \infty]$, $k \in \mathbb{N}$, and let $X = W^1_p(J; L^p(G) \cap L^p(J; W^k_p(G)))$ (the typical parabolic solution space).

a) For the time trace $\gamma_t : u \mapsto u|_{t=0}$, we obtain the trace space

$$
\gamma_t X = B^{k-1/p}_{pp}(G).
$$
b) For the boundary trace $\gamma_0: u \mapsto u|_{\partial G}$, we obtain the trace space

$$\gamma_0 X = B_{pp}^{1-1/(kp)}(J; L^p(\partial G)) \cap L^p(J; B_{pp}^{k-1/p}(\partial G)).$$

**Proof.** We only give the main ideas for a proof and refer to [DHP07], Section 3, for a complete version.

a) By Lemma 2.4 a), we have $\gamma_t X = (L^p(G), W^k_p(G))_{1-1/p,p}$ which equals $B_{pp}^{k-1/p}(G)$ due to Theorem 4.5 with $p = q$.

b) Locally, we can choose a coordinate system such that the inner normal vector is the $x_n$-variable. Then we have to take the trace with respect to $x_n$ instead of $t$ which gives by Lemma 2.4 a real interpolation space again. Computing the real interpolation space of the intersection then gives a Besov space both with respect to time and with respect to the other space variables. □

**Remark 4.7.** In the above corollary, we have considered functions which are $L^p$ in time and $L^q$ in space. If one considers functions which are $L^p$ in time and $L^q$ in space with $p \neq q$, a result similar to Corollary 4.6 holds, but now also Triebel-Lizorkin spaces appear. More precisely, for $X := W^1_p(J; L^q(G)) \cap L^q(J; W^k_q(G))$ we obtain (see [DHP07], Section 6, and [MV14], Section 4)

$$\gamma_t X = B_{pq}^{k-1/p}(G),$$

$$\gamma_0 X = F_{pq}^{1-1/(kp)}(J; L^p(\partial G)) \cap L^q(J; B_{pq}^{k-1/q}(\partial G)).$$

5. **Parabolic PDE systems in the whole space**

As a first application of the previous results, we now consider parabolic systems of partial differential equations in the whole space $\mathbb{R}^n$. In the following, let $1 < p < \infty$ and $\Sigma_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\} = \Sigma_{n/2}$. We assume that we have a linear differential operator $A = A(x, D)$ of the form

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$$

with $m \in \mathbb{N}$ and matrix-valued coefficients $a_\alpha: \mathbb{R}^n \to \mathbb{C}^{N \times N}$. Recall that $D := -i\partial$. The definition of parabolicity below is based on the concept of parameter-ellipticity which was developed by Agmon [Agm62] and Agranovich-Vishik [AV64].

For the formal differential operator $A = A(x, D)$, we define its symbol

$$a(x, \xi) := \sum_{|\alpha| \leq 2m} a_\alpha(x) \xi^\alpha$$

and the principal symbol

$$a_0(x, \xi) := \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha.$$ 

Both symbols map $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}^{N \times N}$. The $L^p$-realization $A_p$ of $A(x, D)$ is defined as the unbounded linear operator $A_p: L^p(\mathbb{R}^n; \mathbb{C}^N) \supset D(A_p) \to L^p(\mathbb{R}^n; \mathbb{C}^N)$ with

$$D(A_p) := W^m_p(\mathbb{R}^n; \mathbb{C}^N), \quad A_p u := A(x, D) u \quad (u \in W^m_p(\mathbb{R}^n; \mathbb{C}^N)).$$
Definition 5.1. The operator $A(x, D)$ is called parameter-elliptic with angle $\varphi \in (0, \pi]$ if
\begin{equation}
|\det(a_0(x, \xi) - \lambda)| \geq C_P(|\xi|^{2m} + |\lambda|)^N \quad (x \in \mathbb{R}^n, (\xi, \lambda) \in (\mathbb{R}^n \times \Sigma_\varphi) \setminus \{0\}).
\end{equation}
If this holds for $\varphi = \frac{\pi}{2}$ (i.e., $\Sigma_\varphi = \mathbb{C}_+^2$), then $\partial_t - A$ is called parabolic.

Remark 5.2. a) For every fixed $x \in \mathbb{R}^n$, the map $(\xi, \lambda) \mapsto p(x, \xi, \lambda) := \det(a_0(x, \xi) - \lambda)$ is quasi-homogeneous in the sense that
\begin{equation*}
p(x, r\xi, r^{2m}\lambda) = r^{2mN}p(x, \xi, \lambda) \quad (r > 0, (\xi, \lambda) \in (\mathbb{R}^n \times \Sigma_\varphi) \setminus \{0\}).
\end{equation*}
Therefore, it is sufficient to consider the compact set $\{(\xi, \lambda) : |\xi|^{2m} + |\lambda| = 1\}$. The operator $A(x, D)$ is parameter-elliptic if and only if
\begin{equation*}
\inf \{ |\det(a_0(x, \xi) - \lambda)| : x \in \mathbb{R}^n, (\xi, \lambda) \in \mathbb{R}^n \times \Sigma_\varphi \text{ with } |\xi|^{2m} + |\lambda| = 1 \} > 0.
\end{equation*}

b) If $a_\alpha \in L^\infty(\mathbb{R}^n)$ for all $|\alpha| < 2m$, then the lower-order terms of the symbol can be estimated uniformly in $x$. Therefore, $A(x, D)$ is parameter-elliptic if and only if there exist constants $C, R > 0$ with
\begin{equation*}
|\det(a(x, \xi) - \lambda)| \geq C(|\xi|^{2m} + |\lambda|)^N \quad (x \in \mathbb{R}^n, \lambda \in \Sigma_\varphi, |\xi| \geq R).
\end{equation*}
This is one possible definition of parameter-ellipticity and parabolicity for pseudodifferential operators. We remark that the principal symbol of a pseudodifferential operator is defined only for so-called classical symbols.

Remark 5.3. If $\partial_t - A(x, D)$ is parabolic in the sense of parameter-ellipticity in the closed sector $\mathbb{C}_+$, then $A(x, D)$ is also parameter-elliptic in some larger sector $\mathbb{C}_\theta$ with $\theta > \frac{\pi}{2}$. In fact, it is easily seen that the set of all angles of rays with respect to $\lambda$, in which condition (5-1) holds, is open.

Following a standard approach in elliptic theory, we first consider the so-called model problem and then use perturbation results for variable coefficients. The remainder of this section is based on [DHP03], Sections 5 and 6, and [KW04], Sections 6 and 7.

Theorem 5.4. Let $A(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ with constant coefficients $a_\alpha \in \mathbb{C}^{N \times N}$ ($|\alpha| = 2m$) and without lower-order terms. If $\partial_t - A(D)$ is parabolic with parabolicity constant $C_P$ in (5-1), then $\rho(A_p) \supset \mathbb{C}_+ \setminus \{0\}$, and the set
\begin{equation*}
\{ \lambda(\lambda - A_p)^{-1} : \lambda \in \mathbb{C}_+ \setminus \{0\}\}
\end{equation*}
is $\mathcal{R}$-bounded. Here, the $\mathcal{R}$-bound only depends on $p, n, m, N, C_P$ and
\begin{equation*}
M := \sum_{|\alpha|=2m} |a_\alpha|_{\mathbb{C}^{N \times N}}.
\end{equation*}
In particular, $A_p$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle larger than $\frac{\pi}{4}$, and $A_p$ has maximal $L^q$-regularity for all $q \in (1, \infty)$.

Proof. Note that because of (5-1), for $\lambda \in \mathbb{C}_+ \setminus \{0\}$ and $\xi \in \mathbb{R}^n$ the symbol $(\lambda - a_0(\xi))^{-1}$ is well defined. We show that the family $\{m_\lambda : \lambda \in \mathbb{C}_+ \setminus \{0\}\}$ with $m_\lambda(\xi) := \lambda(\lambda - a_0(\xi))^{-1}$ satisfies the assumptions of Corollary 3.29.

For any $r > 0$ we have $r^{2m}\lambda - a_0(r\xi) = r^{2m}(\lambda - a_0(\xi))$. Therefore, the map $(\xi, \lambda) \mapsto \frac{1}{2}(\lambda - a_0(\xi))$ is quasi-homogeneous in $(\xi, \lambda)$ of degree 0, and the same holds for its inverse $(\xi, \lambda) \mapsto \lambda(\lambda - a_0(\xi))^{-1}$. By Lemma 3.23, $m_\lambda$ satisfies the Mikhlin condition uniformly
with respect to $\lambda$. Now we can apply Corollary 3.29 to obtain the $R$-boundedness of 
\[ \{ \text{op}[m_\lambda] : \lambda \in \mathbb{T}_+ \setminus \{0\} \} \subset L(L^p(\mathbb{R}^n)). \]
Because of $\frac{1}{\lambda} \text{op}[m_\lambda](\lambda - A_p) = \text{id}_{W^2_p(\mathbb{R}^n)}$ and $\frac{1}{\lambda} (\lambda - A_p) \text{op}[m_\lambda] = \text{id}_{L^p(\mathbb{R}^n)}$, we see that $\text{op}[m_\lambda] = (\lambda - A_p)^{-1}$. By Corollary 3.29, $A_p$ is $R$-sectorial with angle larger than $\frac{\pi}{2}$, and Theorem 3.32 implies that $A_p$ has maximal $L^q$-regularity for all $q \in (1, \infty)$. To show the statement on the $R$-bound, we have to quantify the Mikhlin constant.

For this, we write
\[ (\lambda - a_0(\xi))^{-1} = \frac{1}{\det(\lambda - a_0(\xi))} b(\xi, \lambda) \]
with the adjunct matrix $b(\xi, \lambda)$. The coefficients of $b(\xi, \lambda)$ are determinants of $(N - 1) \times (N - 1)$-matrices which are constructed by omitting one row and one column of the matrix $\lambda - a_0(\xi)$. Therefore, we obtain
\[ ||b(\xi, \lambda)||_{C^{N \times N}} \leq C(m, n, M, N)(|\xi|^{2m} + |\lambda|)^{N-1}. \]
Due to (5-1), we get
\[ ||\lambda(\lambda - a_0(\xi))^{-1}||_{C^{N \times N}} \leq C(m, n, M, N, C_p) \frac{|\lambda|}{|\xi|^{2m} + |\lambda|} \leq C(m, n, M, N, C_p). \]

For the derivatives, we note that
\[
\left\| \xi_k \frac{\partial}{\partial \xi_k} a_0(\xi) \right\|_{C^{N \times N}} = \left\| \xi_k \frac{\partial}{\partial \xi_k} \sum_{|\alpha|=2m} a_\alpha \xi^\alpha \right\|_{C^{N \times N}} \leq \sum_{|\alpha|=2m} ||a_\alpha||_{C^{N \times N}} \left| \xi_k \frac{\partial}{\partial \xi_k} \xi^\alpha \right|
\leq 2mM|\xi|^{2m}.
\]
Iteratively, we obtain $||\xi^\alpha \partial^\alpha a_0(\xi)||_{C^{N \times N}} \leq C(m, n, M, N)|\xi|^{2m}$ for all $\alpha \in \{0,1\}^n$. In the same way, the derivative of $b(\xi, \lambda)$ can be estimated. This yields
\[ ||\xi^\alpha \partial^\alpha b(\xi, \lambda)||_{C^{N \times N}} \leq C(m, n, M, N)(|\xi|^{2m} + |\lambda|)^{N-1}. \]

With the product rule (Leibniz rule) we have for the inverse matrix the inequality
\[ ||\xi^\alpha D^\alpha (\lambda - a_0(\xi))^{-1}||_{C^{N \times N}} \leq C(m, n, M, N, C_p)(|\xi|^{2m} + |\lambda|)^{-1}, \]
and therefore $||\xi^\alpha D^\alpha m_\lambda(\xi)||_{C^{N \times N}} \leq C(m, n, M, N, C_p)$ for all $\alpha \in \{0,1\}^n, \lambda \in \mathbb{T}_+ \setminus \{0\}$ and all $\xi \in \mathbb{R}^n$. By Corollary 3.29, the $R$-bound of $\{\lambda(\lambda - A_p)^{-1} : \lambda \in \mathbb{T}_+ \setminus \{0\}\}$ only depends on $m, n, M, N, C_p$, and $p$. 

To generalize the above result to operators with variable coefficients, we need perturbation results for $R$-boundedness. For this, we define for an $R$-sectorial operator $A$ with $R$-angle $\varphi_R(A)$ and for $\theta \in (0, \varphi_R(A))$:
\[ M_\theta(A) := \sup \{ ||(\lambda(\lambda - A)^{-1}|| : \lambda \in \Sigma_\theta \}, \]
\[ \tilde{M}_\theta(A) := \sup \{ ||A(\lambda - A)^{-1}|| : \lambda \in \Sigma_\theta \}, \]
\[ R_\theta(A) := \mathcal{R}(\{ \lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\theta \}) , \]
\[ \tilde{R}_\theta(A) := \mathcal{R}(\{ A(\lambda - A)^{-1} : \lambda \in \Sigma_\theta \}). \]

Note that $\tilde{M}_\theta(A)$ is finite because of $A(\lambda - A)^{-1} = \lambda(\lambda - A)^{-1} - 1$, and the same holds for $\tilde{R}_\theta(A)$. 

Theorem 5.5. Let $X$ be a Banach space and $A$ be an $\mathcal{R}$-sectorial operator in $X$ with angle $\varphi_\mathcal{R}(A) > 0$. Further, let $\theta \in (0, \varphi_\mathcal{R}(A))$, and let $B$ be a linear operator in $X$ with $D(B) \supset D(A)$ and
\begin{equation}
\|Bx\| \leq a\|Ax\| \quad (x \in D(A)).
\end{equation}
If $a < \frac{1}{R_\theta(A)}$, then $A + B$ is $\mathcal{R}$-sectorial, too, with angle larger or equal to $\theta$ and
\[ R_\theta(A + B) \leq \frac{R_\theta(A)}{1 - aR_\theta(A)}. \]

Proof. For $\lambda \in \Sigma_\theta \setminus \{0\}$ one obtains
\[ \|B(\lambda - A)^{-1}x\| \leq a\|A(\lambda - A)^{-1}x\| \leq a\widetilde{M}_\theta(A)\|x\| \quad (x \in X). \]
Because of $a < \frac{1}{R_\theta(A)}$, the operator $1 + B(\lambda - A)^{-1}$ is invertible, and we get
\begin{align*}
(\lambda - (A + B))^{-1} &= (\lambda - A)^{-1}[1 + B(\lambda - A)^{-1}]^{-1} \\
&= (\lambda - A)^{-1}\sum_{n=0}^{\infty}(-B(\lambda - A)^{-1})^n.
\end{align*}
In particular, $\rho(A + B) \supset \Sigma_\theta$. By definition of $\mathcal{R}$-boundedness and due to the assumption, we get
\[ \mathcal{R}(\{B(\lambda - A)^{-1} : \lambda \in \Sigma_\theta\}) \leq a\mathcal{R}(\{A(\lambda - A)^{-1} : \lambda \in \Sigma_\theta\}) = a\widetilde{R}_\theta(A). \]
Inserting this into the above series yields
\[ R_\theta(A + B) \leq \frac{R_\theta(A)}{1 - aR_\theta(A)}. \]
This shows that also $A + B$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\geq \theta$. 

The second perturbation result deals with the case where we have an additional term $\|x\|$ on the right-hand side of (5-2). However, now the $\mathcal{R}$-sectoriality of the operator holds only with an additional shift in the operator.

Theorem 5.6. Let $A$ be $\mathcal{R}$-sectorial with angle $\varphi_\mathcal{R}(A) > 0$, and let $\theta \in (0, \varphi_\mathcal{R}(A))$. Let $B$ be a linear operator satisfying $D(B) \supset D(A)$ and
\[ \|Bx\| \leq a\|Ax\| + b\|x\| \quad (x \in D(A)) \]
with constants $b \geq 0$ and $0 \leq a < \left[\widetilde{M}_\theta(A)\widetilde{R}_\theta(A)\right]^{-1}$. Then $A + B - \mu$ is $\mathcal{R}$-sectorial for
\[ \mu > \frac{bM_\theta(A)\widetilde{R}_\theta(A)}{1 - aM_\theta(A)\widetilde{R}_\theta(A)}. \]
For the $\mathcal{R}$-angle, we have $\varphi_\mathcal{R}(A + B - \mu) \geq \theta$.

Proof. For $\mu > 0$, the following inequalities hold
\begin{align*}
\|B(A - \mu)^{-1}x\| &\leq a\|(A - \mu)^{-1}x\| + b\|(A - \mu)^{-1}x\| \\
&\leq \left(a\widetilde{M}_\theta(A) + \frac{b}{\mu}M_\theta(A)\right)\|x\| \quad (x \in X).
\end{align*}
Therefore, \( B \) satisfies the assumption of Theorem 5.5 with \( A \) being replaced by \( A - \mu \). In Theorem 5.5, the condition for the constants is given by \( c(\mu)\widetilde{R}_\theta(A) < 1 \), where \( c := a\widetilde{M}_\theta(A) + \frac{b}{\mu}M_\theta(A) \). Because of \( a\widetilde{M}_\theta(A) < 1 \), this is the case if
\[
\mu > \frac{bM_\theta(A)\widetilde{R}_\theta(A)}{1 - a\widetilde{M}_\theta(A)\widetilde{R}_\theta(A)}.
\]
The above perturbation results allow us to treat small perturbations in the principal part of the differential operator.

**Lemma 5.7.** Let \( A(x, D) = \sum_{|\alpha| = 2m} a_\alpha D^\alpha \) with \( a_\alpha \in \mathbb{C}^{N \times N} \) for \( |\alpha| = 2m \), and assume \( \partial_t - A(x, D) \) to be parabolic with constant \( C_P \). Then there exists some \( \theta > \frac{\pi}{2} \) such that \( A(x, D) \) is parameter-elliptic in \( \Sigma_\theta \), and there exist \( \varepsilon > 0 \) and \( K > 0 \) such that for all operators \( B(x, D) = \sum_{|\alpha| = 2m} b_\alpha(x)D^\alpha \) with \( b_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{N \times N}) \) and
\[
\sum_{|\alpha| = 2m} \|b_\alpha\|_\infty < \varepsilon
\]
the inequality
\[
\mathcal{R}\left( \left\{ \lambda(\lambda - (A_p + B_p))^{-1} : \lambda \in \Sigma_\theta \setminus \{0\} \right\} \right) \leq K
\]
holds. Here, \( \varepsilon \) and \( K \) only depend on \( n, p, m, N, C_P \).

**Proof.** Let \( \varepsilon > 0 \) and \( f \in W^{2m}_p(\mathbb{R}^n; \mathbb{C}^N) \). Then the inequality
\[
\|Bf\|_{L^p(\mathbb{R}^n; \mathbb{C}^N)} \leq \sum_{|\alpha| = 2m} \|b_\alpha\|_\infty \|D^\alpha f\|_{L^p(\mathbb{R}^n; \mathbb{C}^N)} \leq \varepsilon \max_{|\alpha| = 2m} \|D^\alpha f\|_{L^p(\mathbb{R}^n; \mathbb{C}^N)}
\]
holds if \( B \) satisfies the above condition. We write
\[
D^\alpha f = (\mathcal{F}^{-1}m_\alpha \mathcal{F})A(D)f
\]
with
\[
m_\alpha(\xi) := \xi^{\alpha} \left( \sum_{|\beta| = 2m} a_\beta \xi^\beta \right)^{-1}.
\]
Then \( m_\alpha \in C^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{N \times N}) \), and \( m_\alpha \) is homogeneous of degree 0 and therefore satisfies the Mikhlin condition. Consequently, there exists some \( C_1 > 0 \) such that we have
\[
\|\text{op}[m_\alpha]\|_{L(L^p(\mathbb{R}^n; \mathbb{C}^N))} \leq C_1 \quad (|\alpha| = 2m).
\]
Choose \( \varepsilon < \left[ C_1(\widetilde{R}_\theta(A) + 1) \right]^{-1} \). Then
\[
\|Bf\|_{L^p(\mathbb{R}^n; \mathbb{C}^N)} \leq \varepsilon C_1 \|Af\|_{L^p(\mathbb{R}^n; \mathbb{C}^N)} \leq a \|Af\|_{L^p(\mathbb{R}^n; \mathbb{C}^N)}
\]
with \( a = \frac{1}{\widetilde{R}_\theta(A) + 1} \). By Theorem 5.5, the operator \( A_p + B_p \) is \( \mathcal{R} \)-sectorial with angle \( \geq \theta \), and
\[
\widetilde{R}_\theta(A + B) \leq \frac{\widetilde{R}_\theta(A)}{1 - a\widetilde{R}_\theta(A)} =: K.
\]
In the next step, we consider an operator \( A \) whose coefficients in the principal part are bounded and uniformly continuous. We can reduce this situation to the small perturbation from the last lemma by introducing an infinite partition of unity. This is done in the following lemma.
Lemma 5.8. For every \( r > 0 \) there exists \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) with \( 0 \leq \varphi \leq 1 \), supp \( \varphi \subset (-r, r)^n \) and
\[
\sum_{\ell \in \mathbb{Z}^n} \varphi^2_\ell(x) = 1 \quad (x \in \mathbb{R}^n).
\]
Here, \( \varphi_\ell(x) := \varphi(x - \ell) \).

Proof. a) We first consider the case \( r = 1 \) and \( n = 1 \). Choose some \( \varphi_1 \in \mathcal{D}(\mathbb{R}) \) with \( \varphi_1 > 0 \) in \( (-\frac{3}{4}, \frac{3}{4}) \), supp \( \varphi_1 = [-\frac{3}{4}, \frac{3}{4}] \), and \( \varphi_1(x) = \varphi_1(-x) \) for all \( x \in \mathbb{R} \). We set
\[
\varphi(x) := \begin{cases} \sqrt{\frac{\varphi_1^2(x)}{\varphi_1^2(x) + \varphi_1^2(1-x)}} & \text{if } x \in [0, \frac{3}{4}], \\ 0 & \text{if } x \in (\frac{3}{4}, \infty), \end{cases}
\]
and \( \varphi(x) := \varphi(-x) \) for \( x < 0 \). Then supp \( \varphi \subset (-1, 1) \), and for \( x \in [0, 1] \) we obtain
\[
\sum_{\ell \in \mathbb{Z}} \varphi^2_\ell(x) = \varphi^2(x) + \varphi^2(x - 1) = \varphi^2(x) + \varphi^2(1 - x) = 1.
\]
As \( \sum_{\ell \in \mathbb{Z}} \varphi^2_\ell \) is periodic with period 1, we have \( \sum_{\ell \in \mathbb{Z}} \varphi^2_\ell = 1 \) in \( \mathbb{R} \).

b) In the general case, define \( \varphi^{(n)}(x) := \prod_{j=1}^n \varphi(\frac{x_j}{r}) \) with \( \varphi \) from part a). Then
\[
\sum_{\ell \in \mathbb{Z}^n} (\varphi^{(n)}(x - \ell))^2 = \sum_{\ell \in \mathbb{Z}^n} \prod_{j=1}^n \varphi^2 \left( \frac{x_j - \ell_j}{r} \right) = \prod_{\ell \in \mathbb{Z}^n} \varphi^2(y_j - \ell_j) = \prod_{j=1}^n \sum_{\ell_j \in \mathbb{Z}} \varphi^2(y_j - \ell_j) = 1
\]
for \( y := \frac{x}{r} \).

We now come to the main result of this section. Here, BUC(\( \mathbb{R}^n \)) stands for the space of all bounded and uniformly continuous functions.

Theorem 5.9. Let \( A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha \) with
\[
a_\alpha \in \text{BUC}(\mathbb{R}^n; \mathbb{C}^{N \times N}) \quad (|\alpha| = 2m),
\]
\[
a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{N \times N}) \quad (|\alpha| < 2m).
\]
Let \( 1 < p < \infty \). If \( \partial_t - A(x, D) \) is parabolic, then there exist \( \theta > \frac{\pi}{2} \) and \( \mu > 0 \) such that \( A_p - \mu \) is \( \mathcal{R} \)-sectorial with angle \( \theta \). In particular, \( A_p - \mu \) has maximal \( L^q \)-regularity for all \( 1 < q < \infty \).

Proof. As \( A(x, D) \) is parameter-elliptic in \( \mathbb{C}^+ \) by assumption, there exists a \( \theta > \frac{\pi}{2} \) such that \( A(x, D) \) is still parameter-elliptic in \( \Sigma_\theta \) (Remark 5.3). The proof of the theorem uses localization and is done in several steps. We first explain the ideas.

1. We fix the coefficients of \( A \) at some point \( \ell \in \Gamma \), where the grid \( \Gamma \subset \mathbb{R}^n \) is chosen fine enough such that in each cube the localized operator \( A^\ell \) is a small perturbation of the model problem with frozen coefficient. Here, we apply Lemma 5.7 to see that \( A^\ell \) is still \( \mathcal{R} \)-sectorial.
(2) We consider the sequence \( A := (A^\ell)_{\ell \in \Gamma} \) of all localized operators and show that this defines an \( R \)-sectorial operator in some suitably chosen sequence space \( X_0 \).

(3) The \( L^p \)-realization \( A_p \) and the operator \( A \) have the same properties up to lower-order perturbations. More precisely, we have \( JA_p = AJ \) and \( A_p P = PA \) modulo lower order operators, where \( J \) and \( P \) are the localization and the patching operator, respectively.

(4) With the help of the interpolation inequality for Sobolev spaces, the lower-order operators can be seen as a small perturbation, and therefore the \( R \)-sectoriality of \( A \) implies the \( R \)-sectoriality of \( A_p \).

In detail, these steps can be done in the following way.

(1) Choose \( \varepsilon = \varepsilon(n, p, m, N, C_p) \) as in Lemma 5.7 for the operator \( \sum_{|\alpha| = 2m} a_\alpha(\ell) D^\alpha \) with \( \ell \in \mathbb{Z}^n \). As \( a_\alpha \in \text{BUC}(\mathbb{R}^n; \mathbb{C}^{N \times N}) \), there exists a \( \delta > 0 \) with

\[
\sum_{|\alpha| = 2m} |a_\alpha(x) - a_\alpha(y)| < \varepsilon \quad (|x - y| \leq \delta).
\]

Now choose \( r \in (0, \delta) \) and \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) as in Lemma 5.8. We write \( Q := (-r, r)^n \) and \( Q_\ell := Q + \ell \) for \( \ell \in r\mathbb{Z}^n =: \Gamma \). Choose \( \psi \in \mathcal{D}(\mathbb{R}^n) \) with \( \psi \subset Q \), \( 0 \leq \psi \leq 1 \), \( \psi = 1 \) on \( \text{supp} \varphi \), and set \( \psi_\ell(x) := \psi(x - \ell) (\ell \in \mathbb{Z}) \). Define the coefficients

\[
a^\alpha_\ell(x) := \begin{cases} a^\alpha_\ell(x), & x \in Q_\ell, \\ a_\alpha, & x \not\in Q_\ell \quad (\ell \in \Gamma, |\alpha| = 2m) \end{cases}
\]

and the operator \( A^\ell(x, D) := \sum_{|\alpha| = 2m} a^\alpha_\ell(x) D^\alpha \). For the principal part, we obtain \( A_0(x, D) = A^\ell(x, D) (x \in Q_\ell) \) and therefore \( A_0(x, D) u = A^\ell(x, D) u \) for all \( u \in W^{2m}_p(\mathbb{R}^n; \mathbb{C}^N) \) with \( \text{supp} u \subset Q_\ell \).

(2) Define \( X_k := \ell_p(\Gamma; W^k_p(\mathbb{R}^n; \mathbb{C}^N)) \) for \( k \in \mathbb{N}_0 \) and the operator \( A : X_0 \supset D(A) \to X_0 \) by \( D(A) := X_{2m} \) and \( A(u_\ell)_{\ell \in \Gamma} := (A^\ell u_\ell)_{\ell \in \Gamma} \).

By Lemma 5.7, the operator \( A^\ell \) is \( R \)-sectorial with \( R_\theta(A^\ell) \leq K \), where \( K \) does not depend on \( \ell \). We show that the same holds for \( A \). For this, let \( T_j = \lambda_j(A - \lambda_j)^{-1} \) with \( \lambda_j \in \Sigma_\theta \) and \( x_j = (f_\ell(j))_{\ell \in \Gamma} \in X_0 \) for \( j = 1, \ldots, J \). Then we obtain

\[
\left\| \sum_{j=1}^J r_j T_j x_j \right\|_{L^p([0,1];X_0)} = \\
= \left( \int_0^1 \left\| \sum_{j=1}^J r_j(t) T_j x_j \right\|_{X_0}^p dt \right)^{1/p} \\
= \left( \int_0^1 \left( \sum_{\ell \in \Gamma} \left\| \sum_{j=1}^J r_j(t) \lambda_j(A^\ell - \lambda_j)^{-1} f_{\ell,j} \right\|_{L^p(\mathbb{R}^n; \mathbb{C}^N)}^p dt \right)^{1/p} \\
= \left( \sum_{\ell \in \Gamma} \int_0^1 \left( \sum_{j=1}^J r_j(t) \lambda_j(A^\ell - \lambda_j)^{-1} f_{\ell,j} \right\|_{L^p(\mathbb{R}^n; \mathbb{C}^N)}^p dt \right)^{1/p} \\
= \left( \sum_{\ell \in \Gamma} \left\| \sum_{j=1}^J r_j \lambda_j(A^\ell - \lambda_j)^{-1} f_{\ell,j} \right\|_{L^p([0,1];L^p(\mathbb{R}^n; \mathbb{C}^N))}^p \right)^{1/p}
\]
Analogously, we obtain for $u$

$$
\left(\sum_{\ell \in \Gamma} [R_\theta(A^\ell)]^p \right)^{1/p} \left(\sum_{j=1}^J r_j f_j^{(j)} \right)^p \leq K \left(\sum_{j=1}^J r_j x_j \right)_{LP([0,1]; L^p(\mathbb{R}^n; \mathbb{C}^N))}^{1/p},
$$

i.e. $R_\theta(A) \leq K$.

Now we consider the localization operator $J: L^p(\mathbb{R}^n; \mathbb{C}^N) \rightarrow X_0, f \mapsto (\varphi_\ell f)_\ell$. As we have

$$
\sum_{\ell \in \Gamma} \|\varphi_\ell f\|^p_{L^p(\mathbb{R}^n; \mathbb{C}^N)} \leq \sum_{\ell \in \Gamma} \|\chi_{Q_\ell} f\|^p_{L^p(\mathbb{R}^n; \mathbb{C}^N)} = 2^N \|f\|^p_{L^p(\mathbb{R}^n; \mathbb{C}^N)},
$$

the operator $J$ is continuous. In the same way, one sees that $J \in L(W^{2m}_p(\mathbb{R}^n; \mathbb{C}^N), X_{2m})$.

Analogously, the patching operator $P$ is defined by

$$
P: X_0 \rightarrow L^p(\mathbb{R}^n; \mathbb{C}^N), (f_\ell)_{\ell \in \Gamma} \mapsto \sum_{\ell \in \Gamma} \varphi_\ell f_\ell.
$$

Note here that the sum is locally finite. We obtain $P \in L(X_0, L^p(\mathbb{R}^n; \mathbb{C}^N))$ and $PJ = \text{id}_{L^p(\mathbb{R}^n; \mathbb{C}^N)}$ because of $PJ = \sum_{\ell \in \Gamma} \varphi_\ell^2 f = f$.

(3) Now let $A_p$ be the $L^p(\mathbb{R}^n; \mathbb{C}^N)$-realization of $A(x, D)$ and $A_{p,0}$ the $L^p(\mathbb{R}^n; \mathbb{C}^N)$-realization of $A_0(x, D)$. Then for $u \in W^{2m}_p(\mathbb{R}^n; \mathbb{C}^N)$ and $\ell \in \Gamma$ the following equality holds:

$$
\varphi_\ell A_p u = A_p(\varphi_\ell u) + (\varphi_\ell A_p - A_p \varphi_\ell) u
= A^\ell(\varphi_\ell u) + (A_p - A_{p,0}) \psi_\ell u + \sum_{k: Q_\ell \cap Q_k \neq \emptyset} (\varphi_\ell A_p - A_p \varphi_\ell) \varphi_k u_k.
$$

Thus, $J A_p = AJ + BJ$ with

$$
B((\varphi_\ell u_\ell)_{\ell \in \Gamma}) := \left( (A_p - A_{p,0}) \psi_\ell u_\ell + \sum_{k: Q_\ell \cap Q_k \neq \emptyset} (\varphi_\ell A_p - A_p \varphi_\ell) \varphi_k u_k \right)_{\ell \in \Gamma}.
$$

Writing $B((u_\ell)_{\ell \in \Gamma}) = (\sum_{k \in \Gamma} B_{k \ell} u_\ell)_{k \in \Gamma}$, we see that $B_{k \ell}$ is a differential operator of order not greater than $\leq 2m - 1$, and the number of elements in each row of the infinite matrix $(B_{k \ell})_{k, \ell}$ is bounded. As $a_{\alpha} \in L^\infty(\mathbb{R}^n; \mathbb{C}^N)$, this yields $B \in L(X_{2m-1}, X_0)$.

Analogously, we obtain for $(u_\ell)_{\ell \in \Gamma} \in X_{2m}$ the equality

$$
(A_p P - PA)(u_\ell)_{\ell \in \Gamma} = A_p \left( \sum_{\ell \in \Gamma} \varphi_\ell u_\ell \right) - \sum_{\ell \in \Gamma} \varphi_\ell A^\ell u_\ell
= \sum_{\ell \in \Gamma} \varphi_\ell (A_p - A_{p,0}) u_\ell + \sum_{k \in \Gamma} (A_p \varphi_k - \varphi_k A_p) u_k
= \sum_{\ell \in \Gamma} \varphi_\ell (A_p - A_{p,0}) u_\ell + \sum_{k \in \Gamma, \ell: Q_\ell \cap Q_k \neq \emptyset} \varphi_\ell^2 (A_p \varphi_k - \varphi_k A_p) u_k
= \sum_{\ell \in \Gamma} \varphi_\ell \left[ (A_p - A_{p,0}) u_\ell + \sum_{k: Q_\ell \cap Q_k \neq \emptyset} \varphi_\ell (A_p \varphi_k - \varphi_k A_p) u_k \right]
= PD(u_\ell)_{\ell \in \Gamma}.
$$
with
\[
D(u_\ell)_{\ell \in \Gamma} := \left((A_p - A_{p,0})u_\ell + \sum_{k: Q_k \cap Q_\ell \neq \emptyset} (A_p \varphi_k - \varphi_k A_p)u_k\right)_{\ell \in \Gamma}.
\]
In the same way as before, we see that \( D \in L(X_{2m-1}, X_0) \).

(4) We apply the interpolation inequality for Sobolev spaces and obtain for every \( \varepsilon > 0 \)
the inequality
\[
\|B(u_\ell)_{\ell \in \Gamma}\|_{X_0} + \|D(u_\ell)_{\ell \in \Gamma}\|_{X_0} \leq C\|u_\ell\|_{X_{2m-1}} \leq \varepsilon\|u_\ell\|_{X_{2m-1}} + C\varepsilon\|D(u_\ell)_{\ell \in \Gamma}\|_{X_0} \quad (u \in X_{2m}).
\]
Due to Theorem 5.6, there exists a \( \mu > 0 \) such that \( A + B - \mu \) and \( A + D - \mu \) are both \( \mathcal{R} \)-sectorial with angle \( \geq \theta \).

Let \( u \in W^{2m}_p(\mathbb{R}^n; \mathbb{C}^N) \) and \( f := (\lambda + \mu - A_p)u \in L^p(\mathbb{R}^n; \mathbb{C}^N) \). Then
\[
Jf = J(\lambda + \mu - A_p)u = (\lambda + \mu - (A + B))Ju,
\]
and therefore
\[
u = PJu = P(\lambda + \mu - (A + B))^{-1}Jf.
\]
In particular, \( \lambda + \mu - A_p \) is injective.

On the other hand, for \( f \in L^p(\mathbb{R}^n; \mathbb{C}^N) \) we get
\[
f = PJf = P(\lambda + \mu - (A + D))(\lambda + \mu - (A + D))^{-1}Jf = (\lambda + \mu - A_p)P(\lambda + \mu - (A + D))^{-1}Jf \in R(\lambda + \mu - A_p),
\]
i.e., \( \lambda + \mu - A_p \) is surjective, too. Therefore, \( \lambda + \mu \in \rho(A_p) \) and
\[
(\lambda + \mu - A_p)^{-1} = P(\lambda + \mu - (A + D))^{-1}J.
\]
Because of \( P \in L(X_0, L^p(\mathbb{R}^n; \mathbb{C}^N)) \), \( J \in L(L^p(\mathbb{R}^n; \mathbb{C}^N), X_{2m}) \), and \( R_\theta(A + D - \mu) < \infty \),
it follows that \( R_\theta(A_p - \mu) < \infty \), and \( A_p - \mu \) is \( \mathcal{R} \)-sectorial with angle greater or equal to \( \theta \). \( \square \)

6. PARABOLIC BOUNDARY VALUE PROBLEMS

In the last section, we considered parabolic systems in the whole space. Now we want
to show that similar results also hold for boundary value problems in sufficiently smooth
domains. In addition to the parameter-ellipticity of the operator \( A \), we now have to impose a condition on the boundary operators called Shapiro-Lopatinskii condition. For a reference for this condition, we mention, e.g., [Wlo87], §11.

6.1. The Shapiro-Lopatinskii condition. In the following, let \( p \in (1, \infty) \), and let \( G \subset \mathbb{R}^n \) be a bounded domain. We consider a linear partial differential operator \( A = A(x, D) \) of the form
\[
A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha
\]
with \( m \in \mathbb{N}, a_\alpha : \overline{G} \to \mathbb{C} \) and boundary operators \( B_1, \ldots, B_m \) of the form
\[
B_j(x', D) = \sum_{|\beta| \leq m_j} b_{j,\beta}(x') \gamma_0 D^\beta
\]
with \( m_j < 2m, b_{j\beta} : \partial G \to \mathbb{C} \). Here, \( \gamma_0 \) stands for the boundary trace \( u \mapsto u|_{\partial G} \), which is a bounded linear map

\[
\gamma_0 : W_p^k(\Omega) \to W_p^{k-1/p}(\partial \Omega),
\]

\( k = 1, \ldots, 2m \) if \( G \) is a \( C^{2m} \)-domain. Note here that \( W_p^{k-1/p}(\partial \Omega) = B_p^{k-1/p}(\partial \Omega) \) is the Sobolev-Slobodeckii space (see Section 4).

The \( L^p \)-realization \( A_{B,p} \) of the boundary value problem \((A, B) = (A, B_1, \ldots, B_m)\) is defined by

\[
D(A_{B,p}) := \{ u \in W_p^{2m}(G) : B_1(x, D)u = \cdots = B_m(x, D)u = 0 \}
\]

and \( A_{B,p}u := A(x, D)u \) \((u \in D(A_{B,p}))\). We will assume the following smoothness:

(i) The domain \( \Omega \) is bounded and of class \( C^{2m} \).

(ii) For the coefficients \( a_\alpha \) of \( A(x, D) \) we have

\[
a_\alpha \in C(\overline{G}) \quad (|\alpha| = 2m),
\]

\[
a_\alpha \in L^\infty(G) \quad (|\alpha| < 2m).
\]

(iii) For the coefficients \( b_{j\beta} \) of \( B_j(x', D) \) we have

\[
b_{j\beta} \in C^{2m-m_j}(\partial G) \quad (|\beta| \leq m_j, j = 1, \ldots, m).
\]

By trace results on Sobolev spaces, we immediately see the following continuity:

**Lemma 6.1.** The operator

\[
(A, B) : W_p^{2m}(G) \to L^p(G) \times \prod_{j=1}^m W_p^{2m-m_j-1/p}(\partial G)
\]

is continuous.

As usual, we define the principal symbols \( a_0(x, \xi) := \sum_{|\alpha|=2m} a_\alpha(x) \xi^\alpha \) and \( b_{j0}(x', \xi) := \sum_{|\beta|=m_j} b_{j\beta}(x') \xi^\beta \).

**Definition 6.2.** The boundary value problem \((A, B)\) is called parameter-elliptic in the sector \( \Sigma_\varphi \) if:

(a) We have \( a_0(x, \xi) - \lambda \neq 0 \) for all \( x \in \overline{G} \) and all \( (\xi, \lambda) \in (\mathbb{R}^n \times \Sigma_\varphi) \setminus \{0\} \).

(b) The following Shapiro-Lopatinskii condition is satisfied: for all \( x' \in \partial G \) and all \( (\xi', \lambda) \in (\mathbb{R}^{n-1} \times \Sigma_\varphi) \setminus \{0\} \) the ordinary differential equation

\[
(a_0(x', \xi', D_n) - \lambda)v(x_n) = 0 \quad (x_n > 0),
\]

\[
b_{j0}(x', \xi', D_n)v(x_n)|_{x_n=0} = 0 \quad (j = 1, \ldots, m),
\]

\[
v(x_n) \to 0 \quad (x_n \to \infty)
\]

has only the trivial solution. Here, the boundary value problem is written in coordinates corresponding to \( x' \). These coordinates arise from the original ones by translation and rotation in such a way that the \( x_n \)-direction in the new coordinates is the direction of the inner normal at the point \( x' \).

If this holds for the sector \( \Sigma_{\pi/2} = \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq 0 \} \), the instationary problem \((\partial_t - A, B)\) is called parabolic.
Note that (a) implies inequality (5.1) from Definition 5.1, as \( \overline{G} \) is compact and \( a_0 \) is continuous in \( x \) and homogeneous in \( \xi \).

**Definition 6.3.** Assume that in the situation of Definition 6.2, (a) holds. Then \( A(x, D) - \lambda \) is called proper parameter-elliptic if for all \( (x', \xi', \lambda) \in \partial G \times (\mathbb{R}^{n-1} \setminus \{0\}) \times \overline{\mathbb{C}}_+ \), the polynomial \( a_0(x', \xi', \cdot) - \lambda \) has exactly \( m \) roots (including multiplicities) \( \tau_j = \tau_j(x', \xi', \lambda), \quad j = 1, \ldots, m \) with positive imaginary part. In this case, define

\[
a_+(\tau) := a_+(x', \xi', \lambda, \tau) := \prod_{j=1}^{m} (\tau - \tau_j(x', \xi', \lambda)) \in \mathbb{C}[\tau].
\]

We consider the equivalence class \( \bar{b}_{j0} = \bar{b}_{j0}(x', \xi', \lambda, \cdot) \in \mathbb{C}[\tau]/(a_+) \) of \( b_{j0} \) modulo \( a_+ \), and write \( \bar{b}_{j0} \) with respect to the canonical basis \( \bar{1}, \bar{\tau}, \ldots, \bar{\tau}^{m-1} \in \mathbb{C}[\tau]/(a_+) \), i.e.

\[
\begin{pmatrix}
\bar{b}_{10} \\
\vdots \\
\bar{b}_{m0}
\end{pmatrix} = L \begin{pmatrix}
\bar{1} \\
\vdots \\
\bar{\tau}^{m-1}
\end{pmatrix} \quad \text{with} \quad L = L(x', \xi', \lambda) \in \mathbb{C}^{m \times m}.
\]

Then \( L \) is called the Lopatinskii matrix of \( (A, B) \) at the point \( x \).

**Lemma 6.4.** Let \( A \) be properly parameter-elliptic in \( \overline{G} \). Then the Shapiro-Lopatinskii holds if and only if

\[
\det L(x', \xi', \lambda) \neq 0 \quad (x' \in \partial G, (\xi', \lambda) \in (\mathbb{R}^{n-1} \times \overline{\mathbb{C}}_+) \setminus \{0\}).
\]

**Proof.** Let \( v_j \) \((j = 1, \ldots, m)\) be the solution of

\[
(a_+(x', \xi, D_n) - \lambda)v(x_n) = 0 \quad (x_n > 0),
\]

\[
D_n^{k-1}v(x_n)|_{x_n=0} = \delta_{kj} \quad (k = 1, \ldots, m).
\]

Then \( \{v_1, \ldots, v_m\} \) is a basis of the space \( \mathcal{M}_+ \) of all stable solutions of the ordinary differential equation \( (a_+(x', \xi, D_n) - \lambda)v(x_n) = 0 \). Therefore, for all \( v \in \mathcal{M}_+ \) we have the representation \( v = \sum_{j=1}^{m} \lambda_j v_j \) and

\[
\begin{pmatrix}
\bar{b}_{10}(D_n) \\
\vdots \\
\bar{b}_{m0}(D_n)
\end{pmatrix} v(x_n)|_{x_n=0} = \begin{pmatrix}
\bar{b}_{10}(D_n) \\
\vdots \\
\bar{b}_{m0}(D_n)
\end{pmatrix} (v_1(x_n), \ldots, v_m(x_n))|_{x_n=0} = \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_m
\end{pmatrix} = L \begin{pmatrix}
\bar{1} \\
\vdots \\
\bar{\tau}^{m-1}
\end{pmatrix} (v_1(x_n), \ldots, v_m(x_n))|_{x_n=0} = \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_m
\end{pmatrix}.
\]

Note that \( \bar{b}_{k0}(D_n)v_j(x_n)|_{x_n=0} = \bar{b}_{k0}(D_n)v_j(x_n)|_{x_n=0} \) holds because \( a_+(D_n)v_j(x_n) = 0 \). Therefore, (6-1) has only the trivial solution if and only if \( \det L \neq 0 \). \( \square \)
Remark 6.5. a) The condition of Lemma 6.4 can be formulated in the following way: The boundary conditions are linearly independent modulo \( a_+ \), i.e., \( \overline{b}_{10}, \ldots, \overline{b}_{m0} \) are linearly independent in \( \mathbb{C}[\tau]/(a_+) \).

b) The boundary conditions \( B_1, \ldots, B_m \) are called completely elliptic if for every proper parameter-elliptic \( A \) the boundary value problem \( (A, B) \) is parameter-elliptic. This is the case for

(i) \( B_j(x', D) = \gamma_0(\frac{\partial}{\partial x_n})^{j-1} (j = 1, \ldots, m) \) (general Dirichlet boundary conditions),

(ii) \( B_j(x', D) = \gamma_0(\frac{\partial}{\partial x_n})^{m+j-1} (j = 1, \ldots, m) \) (general Neumann boundary conditions).

More general, this holds for all boundary conditions of the form

\[
B_j(x', D) = \gamma_0 \left( \frac{\partial}{\partial x_n} \right)^{s+j-1} + \text{lower order terms} \ (j = 1, \ldots, m),
\]

where \( s \in \{0, \ldots, m\} \) is fixed. To see this, we have to show that \( \{\tau^{s+j-1} : j = 1, \ldots, m\} \) is linearly independent in \( \mathbb{C}[\tau]/(a_+) \). If this is not the case, there exist \( c_j \in \mathbb{C} \) and \( p \in \mathbb{C}[\tau] \) with

\[
\sum_{j=1}^{m} c_j \tau^{s+j-1} = p(\tau) a_+(\tau).
\]

Because of \( a_+(0) \neq 0 \), it follows that \( \tau^s \) is a divisor of \( p(\tau) \). Therefore, \( \sum_{j=1}^{m} c_j \tau^j = \tilde{p}(\tau) a_+(\tau) \) with some polynomial \( \tilde{p} \), in contradiction to \( \deg a_+ = m \).

c) If the domain and the coefficients of \( (A, B) \) are infinitely smooth, then for every fixed \( \lambda \in \mathbb{T}_+ \), the coefficients of \( L(x', \xi', \lambda) \) are symbols of pseudodifferential operators on the closed \((n - 1)\)-dimensional manifold \( \partial \mathbb{G} \).

6.2. The main result on parameter-elliptic boundary value problems. Under the condition of parameter-ellipticity, one can construct the solution operators for boundary value problems. We follow the exposition in [ADF97], Section 2, and [DHP03], Sections 6 and 7. We start with a remark on ordinary differential equations.

Theorem 6.6. Let \( (A, B) \) be parameter-elliptic in some sector \( \Sigma_\varphi \), and let \( (x', \xi', \lambda) \in \partial \mathbb{G} \times ((\mathbb{R}^n - 1 \times \Sigma_\varphi) \setminus \{0\}) \). Choose a closed curve \( \gamma = \gamma(x', \xi', \lambda) \) in \( \{z \in \mathbb{C} : \text{Im } z > 0\} \), enclosing all roots \( \tau_1, \ldots, \tau_m \) of \( a_+ \). We define \( p_k \) by

\[
a_+(x', \xi', \lambda, \tau) = \sum_{\ell=0}^{m} p_{\ell}(x', \xi', \lambda) \tau^{m-\ell},
\]

and set \( N_k(\tau) := N_k(x', \xi', \lambda, \tau) := \sum_{\ell=0}^{m-k} p_{\ell}(x', \xi', \lambda) \tau^{m-k-\ell} \) and

\[
(M_1(\tau), \ldots, M_m(\tau)) := (N_1(\tau), \ldots, N_m(\tau)) L^{-1}.
\]

Let \( w_k(x_n) = w_k(x', \xi', \lambda, x_n) \) \( (x_n > 0) \) be defined by

\[
w_k(x_n) := \frac{1}{2\pi i} \int_{\gamma} \frac{M_k(\tau)}{a_+(\tau)} e^{ix_n \tau} d\tau \quad (k = 1, \ldots, m).
\]

Then \( \{w_1, \ldots, w_m\} \) is a basis of the stable solution space of \( a_0(D_n)w = 0 \), \( w(x_n) \rightarrow 0 \ (x_n \rightarrow \infty) \) and satisfies the initial conditions

\[
b_{jk}(x', \xi', \lambda, D_n)w_k(x_n)|_{x_n=0} = \delta_{jk} \quad (j, k = 1, \ldots, m).
\]
Proof. (i) We first show that
\[ \frac{1}{2\pi i} \int_{\gamma} \frac{N_k(\tau)\tau^{j-1}}{a_+(\tau)} d\tau = \delta_{kj} \quad (j, k = 1, \ldots, m). \]

For this, we replace \( \gamma \) by a large ball \( \{ \tau \in \mathbb{C} : |\tau| = R \} \). For \( j < k \) we have deg \( N_k(\tau)^{j-1} \) = \( m - k + j - 1 \leq m - 2 \). Therefore, the integrand is of order \( O(R^{-2}) \) for \( R \to \infty \) which shows that the integral vanishes.

For \( j = k \), the integrand equals \( \frac{\tau^{m-1} + O(\tau^{m-2})}{(\tau - r_1) \cdots (\tau - r_m)} \). By the residue’s theorem, the integral has the value 1.

For \( j > k \) we consider
\[ Q(\tau) := -a_+(\tau)\tau^{j-k-1} + N_k(\tau)\tau^{j-1} \]
\[ = - \sum_{\ell=0}^{m} p_{\ell} \tau^{m-\ell+j-k-1} + \sum_{\ell=0}^{m-k} p_{\ell} \tau^{m-\ell+j-k-1}. \]

We obtain \( \text{deg } Q = j - 2 \leq m - 2 \), and therefore
\[ \int_{B(0,R)} \frac{N_k(\tau)\tau^{j-1}}{a_+(\tau)} d\tau = \int_{B(0,R)} \frac{a_+(\tau)\tau^{j-k-1} + Q(\tau)}{a_+(\tau)} d\tau = \int_{B(0,R)} \frac{Q(\tau)}{a_+(\tau)} d\tau = 0. \]

(ii) We have modulo \( a_+ \), i.e., as equality in \( \mathbb{R}[\tau]/(a_+) \):
\[ \left( \begin{array}{c} \mathcal{B}_{10}(\tau) \\ \vdots \\ \mathcal{B}_{m0}(\tau) \end{array} \right) \left( \begin{array}{c} \mathcal{M}_1(\tau), \ldots, \mathcal{M}_m(\tau) \end{array} \right) \]
\[ = \left( \begin{array}{c} \mathcal{T}_{01}(\tau) \\ \vdots \\ \mathcal{T}_{m0}(\tau) \end{array} \right) \left( \begin{array}{c} \mathcal{N}_1(\tau), \ldots, \mathcal{N}_m(\tau) \end{array} \right) \mathcal{L}^{-1} \]
\[ = \mathcal{L} \left( \begin{array}{c} \mathcal{M}_1(\tau), \ldots, \mathcal{M}_m(\tau) \end{array} \right) \mathcal{L}^{-1}. \]

Therefore,
\[ \left( \frac{1}{2\pi i} \int_{\gamma} \frac{b_{j0}(\tau)M_k(\tau)}{a_+(\tau)} d\tau \right)_{j,k=1,\ldots,m} = \mathcal{L} \cdot \left( \frac{1}{2\pi i} \int_{\gamma} \frac{\tau^{j-1}N_k(\tau)}{a_+(\tau)} d\tau \right)_{j,k=1,\ldots,m} \cdot \mathcal{L}^{-1} \]
\[ = \mathcal{L} \cdot \mathcal{I}_m \cdot \mathcal{L}^{-1} = \mathcal{I}_m. \]

This yields
\[ \frac{1}{2\pi i} \int_{\gamma} \frac{b_{j0}(\tau)M_k(\tau)}{a_+(\tau)} d\tau = \delta_{jk} \quad (j, k = 1, \ldots, m). \]

(iii) Define \( w_k \) as in the theorem. Because of \( \gamma \subset \{ z \in \mathbb{C} : \text{Im } z > 0 \} \), we see that \( w_k(x_n) \to 0 \) for \( x_n \to \infty \). Further,
\[ a_0(D_n)w(x_n) = \frac{1}{2\pi i} \int_{\gamma} \frac{M_k(\tau)}{a_+(\tau)} a(\tau)e^{ix_n\tau} d\tau = 0, \]
as the integrand is holomorphic. Finally,
\[ b_j \partial_\gamma w_{\alpha}(x_n) \big|_{x_n=0} = \frac{1}{2\pi i} \int \frac{b_j(\tau) M_k(\tau)}{a_+^{\gamma}(\tau)} e^{i\gamma x_n \tau} \big|_{x_n=0} \, d\tau = \delta_{jk} \quad (j, k = 1, \ldots, m), \]
which finishes the proof. \(\square\)

**Remark 6.7.** a) With the above notation, the following expressions are quasi-homogeneous in \((\xi', \lambda, \tau)\), more precisely, positively homogeneous in \((\xi', \lambda^{1/2m}, \tau)\):

- \(a_+(x', \xi', \lambda, \tau)\) of degree \(m\),
- \(\tau_j(\xi', \lambda)\) of degree 1,
- \(p_\tau(x', \xi', \lambda)\) of degree \(\ell\),
- \(N_k(x', \xi', \lambda, \tau)\) of degree \(m - k\),
- \(b_{j0}(x', \xi', \tau)\) of degree \(m_j \quad (j = 1, \ldots, m)\),
- \(L_{ij}(x', \xi', \lambda)\) of degree \(m_i - j + 1\),
- \(M_k(x', \xi', \lambda, \tau)\) of degree \(m - m_k - 1\),
- \(\gamma(x', \xi', \lambda)\) of degree 1,
- \(\frac{M_k(\tau)}{a_+^{\gamma}(\tau)}\) of degree \(-m_k - 1\).

b) In the following, let
\[ (\xi')_\lambda := |\xi'| + |\lambda|^{1/2m}. \]
By a), the length of \(\gamma(x', \xi', \lambda)\) can be estimated by \(C(\xi')_\lambda\). For \(\tau \in \gamma\), one gets
\[ \text{Im} \, \tau \geq C(\xi')_\lambda, \]
\[ |\tau - \tau_j(x', \xi', \lambda)| \geq C(\xi')_\lambda, \]
\[ |e^{i\tau x_n}| \leq \exp(-C(\xi')_\lambda x_n). \]

For \(\gamma' \in \mathbb{N}_{0}^{n-1}\) and \(\alpha_n \in \mathbb{N}_0\), we obtain
\[ \left| D_n^{\alpha_n} D_{\xi'} w_k(x', \xi', \lambda, x_n) \right| \leq C(\xi')_\lambda^{-m_k + |\alpha| - |\gamma'|} e^{-C(\xi')_\lambda x_n}. \]
In the smooth situation, these estimates show that \(w_k\) is the symbol of a Poisson operator. Such operators belong to the pseudodifferential calculus of boundary value problems which is also known as the Boutet de Monvel calculus (see, e.g., [Gru96]).

To show maximal regularity for parabolic boundary value problems, we again start with the model problem related to \((A, B)\) acting in \(\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}\) with boundary \(\partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}\). For this, we fix \(x'_0 \in \partial G\) and choose the coordinate system corresponding to \(x'_0\). We obtain the boundary value problem
\[ (A_0(D) - \lambda)u = f \quad \text{in} \ \mathbb{R}^n_+, \]
\[ B_{j0}(D)u = 0 \quad (j = 1, \ldots, m) \quad \text{on} \ \mathbb{R}^{n-1}. \]

Here we have set
\[ A_0(D) := \sum_{|\alpha| = 2m} a_\alpha(x'_0) D^\alpha, \]
\[ B_{j0}(D) := \sum_{|\beta| = m_j} b_{j\beta}(x'_0) \gamma_0 D^\beta. \]
In the following result, we construct the solution operators for the model problem.
Theorem 6.8. Let the boundary value problem \((A, B)\) be parameter-elliptic in the sector \(\Sigma_{\varphi}\), and let \(x_0' \in \partial \Omega\) be fixed. Then the model problem (6-2) has for every \(f \in L^p(\mathbb{R}^n_+)^e\) and \(\lambda \in \Sigma_{\varphi} \setminus \{0\}\) a unique solution \(u \in W^{2m}(\mathbb{R}^n_+)\). This solution is given by

\[
u = R_+(\lambda)E_0 f - \sum_{j=1}^{m} T_j(\lambda)\Lambda_{2m-m_j}(\lambda)\tilde{B}_j(D)R_+(\lambda)E_0 f - \sum_{j=1}^{m} \tilde{T}_j(\lambda)\Lambda_{2m-m_j-1}(\lambda)\tilde{B}_j(D)R_+(\lambda)E_0 f.
\]

Here, the operators are defined in the following way:

a) \(E_0 : L^p(\mathbb{R}^n_+) \to L^p(\mathbb{R}^n), f \mapsto E_0 f\) with

\[
E_0 f := \begin{cases} f, & \text{for } x_n > 0, \\ 0, & \text{for } x_n \leq 0 \end{cases}
\]

(trivial extension by 0).

b) \(R(\lambda) := (A_p - \lambda)^{-1} \in L(L^p(\mathbb{R}^n)), \) where \(A_p\) is the \(L^p(\mathbb{R}^n)\)-realization of \(A_0(D)\).

c) \(R_+ : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n_+), u \mapsto u|_{\mathbb{R}^n_+}, \) the restriction to \(\mathbb{R}^n_+\).

d) \(\tilde{B}_j(D) := \sum_{|\beta|=m_j} b_j(x_0')D^\beta, \) the boundary operators without taking the trace \(\gamma_0\) on the boundary.

e) \(\Lambda_s(\lambda) := (\mathcal{F}')^{-1}(\lambda + |\xi'|^{2m})^{s/2m}\mathcal{F}' \in L(W^{s}(\mathbb{R}^n_+), L^p(\mathbb{R}^n_+))\) for \(s \in \mathbb{N}_0\), where \(\mathcal{F}'\) denotes the Fourier transform in the tangential variables \(x' = (x_1, \ldots, x_{n-1})\).

f) \(T_j(\lambda)\) is given by

\[
(T_j(\lambda)\varphi)(x', x_n) := \int_0^\infty (\mathcal{F}')^{-1}(\partial_{\xi_j}w_j)(x_0', \xi', \lambda, x_n + y_n)\mathcal{F}'(\Lambda_{-2m+m_j}(\lambda)\varphi)(\xi', y_n)dy_n
\]

for \(\varphi \in L^p(\mathbb{R}^n_+).\)

g) \(\tilde{T}_j(\lambda)\) is given by

\[
(\tilde{T}_j(\lambda)\varphi)(x', x_n) := \int_0^\infty (\mathcal{F}')^{-1}w_j(x_0', \xi', \lambda, x_n + y_n)\mathcal{F}'(\Lambda_{-2m+m_j+1}(\lambda)\varphi)(\xi', y_n)dy_n
\]

for \(\varphi \in L^p(\mathbb{R}^n_+).\)

The functions \(w_j(x_0', \xi', \lambda, x_n)\) are defined in Theorem 6.6.

Proof. Here we only show the solution formula for \(u\), as the property \(u \in W^{2m}(\mathbb{R}^n_+)\) will be included in the proof of the \(\mathcal{R}\)-boundedness of the solution operators below.

Let \(u_1 \in W^{2m}(\mathbb{R}^n_+)\) be the unique solution of

\[
(A_0(D) - \lambda)u_1 = E_0 f \quad \text{in } \mathbb{R}^n
\]

which exists due to Theorem 5.4. So we have \(u_1 = R(\lambda)E_0 f\). For \(u\), we choose the ansatz \(u = u_1 + u_2\). Then \(u\) is a solution of (6-2) if and only if \(u_2\) is a solution of the boundary value problem

\[
(A_0(D) - \lambda)u_2 = 0 \quad \text{in } \mathbb{R}^n_+.
\]
\[ B_{j0}(D)u_2 = g_j \ (j = 1, \ldots, m) \text{ on } \mathbb{R}^{n-1} \]

with
\[ g_j := -B_{j0}(D)R_+u_1. \]

Taking partial Fourier transform \( \mathcal{F}' \) with respect to \( x' \), we obtain
\[
\begin{align*}
(a_0(x'_0, \xi', D_n) - \lambda)v(x_n) &= 0 \ (x_n > 0), \\
b_{j0}(x'_0, \xi', D_n)v(x_n)|_{x_n=0} &= h_j(\xi') \ (j = 1, \ldots, m). 
\end{align*}
\]

Here, \( v(x_n) := v(\xi', x_n) := (\mathcal{F}'u_2(\cdot, x_n))(\xi') \) and \( h_j(\xi') := (\mathcal{F}'g_j)(\xi') \). By Theorem 6.6, the unique solution of (6-3) is given by
\[
v(\xi', x_n) = \sum_{j=1}^{m} w_j(x'_0, \xi', \lambda, x_n)h_j(\xi').
\]

Note that \( g_j \) is first defined only on the boundary \( \mathbb{R}^{n-1} \). By
\[
\tilde{g}_j := \sum_{|\beta|=m_j} b_{j(\beta)}(x'_0)D^\beta u_1 = \tilde{B}_{j0}(D)u_1
\]
we define an extension \( g_j \) to \( \mathbb{R}^n_+ \). Then \( \tilde{h}_j := \mathcal{F}'\tilde{g}_j(\cdot, x_n) \) is an extension of \( h_j \).

For \( j = 1, \ldots, m \) we write (this is sometimes called the “Volevich trick”)
\[
w_j(x'_0, \xi', \lambda, x_n)h_j(\xi')
\]
\[
= - \int_0^\infty \partial_n[w_j(x'_0, \xi', \lambda, x_n + y_n)\tilde{h}_j(\xi', y_n)]dy_n
\]
\[
= - \int_0^\infty (\partial_nw_j)(x'_0, \xi', \lambda, x_n + y_n)\tilde{h}_j(\xi', y_n)dy_n
\]
\[
- \int_0^\infty w_j(x'_0, \xi', \lambda, x_n + y_n)(\partial_n\tilde{h}_j)(\xi', y_n)dy_n.
\]

For \( \lambda \in \mathbb{C}_+ \setminus \{0\} \) it holds that \( \Lambda_-s(\lambda)\Lambda_+(\lambda) = \text{id}_{L^p(\mathbb{R}^n)} \) for all \( s \in \mathbb{R} \). Therefore, we can write \( \tilde{g}_j = \Lambda_{-2m+m_j}(\lambda)\Lambda_{2m-m_j}(\lambda)\tilde{g}_j \) and \( \partial_n\tilde{g}_j = \Lambda_{-2m+m_j+1}(\lambda)\Lambda_{2m-m_j+1}(\lambda)\partial_n\tilde{g}_j \), respectively. This yields
\[
u_2(x', x_n) = ((\mathcal{F}')^{-1}v(\cdot, x_n))(x')
\]
\[
= \sum_{j=1}^{m} (T_j(\lambda)\Lambda_{2m-m_j}(\lambda)\tilde{g}_j + \tilde{T}_j(\lambda)\Lambda_{2m-m_j+1}(\lambda)\partial_n\tilde{g}_j).
\]

Inserting \( \tilde{g}_j = \tilde{B}_{j0}(D)R_+u_1 \) and \( u = u_1 + u_2 \) into this formula, the solution formula of the theorem follows. As both the whole space problem as well as (6-3) is uniquely solvable and as the Fourier transform is a bijection in \( \mathcal{S}'(\mathbb{R}^{n-1}) \), we obtain unique solvability with the unique solution \( u = u_1 + u_2 \). \( \square \)

**Lemma 6.9.** The one-sided Hilbert transform
\[
(Hf)(x) := \int_0^\infty \frac{f(y)}{x + y} \, dy
\]
defines a bounded linear operator \( H \in L(L^p(\mathbb{R}^n_+)) \).
The following result shows that the solution operators are indeed $\mathcal{R}$-bounded. Therefore, the one-dimensional Fourier transform.

\[ (\mathcal{F}^{-1}_{1} m_{\varepsilon} \mathcal{F} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \text{sign}(\xi) e^{-\varepsilon|\xi|} (\mathcal{F}_{1}(\xi)) d\xi \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \left[ e^{ix\xi - \varepsilon \xi} \mathcal{F}_{1} f(\xi) - e^{-ix\xi - \varepsilon \xi} \mathcal{F}_{1} f(-\xi) \right] d\xi \]

\[ = \frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} \left( e^{ix\xi - \varepsilon \xi - iy\xi} - e^{-ix\xi - \varepsilon \xi + iy\xi} \right) f(y) dy d\xi \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( e^{i(x-y)\xi - \varepsilon \xi} - e^{-i(x-y)\xi - \varepsilon \xi} \right) f(y) dy \]

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x-y}{(x-y)^2 + \varepsilon^2} f(y) dy. \]

Define for $\varepsilon \in (0,1]$

\[ (H_{\varepsilon} f)(x) := \int_{0}^{\infty} \frac{x+y}{(x+y)^2 + \varepsilon^2} f(y) dy \quad (f \in L^p(\mathbb{R}^+)). \]

Then $H_{\varepsilon} f(x) = (-\frac{\pi}{i}) (\mathcal{F}^{-1}_{1} m_{\varepsilon} \mathcal{F}_{1} E_0 f)(-x)$ for $x \geq 0$, where $E_0 : L^p(\mathbb{R}^+) \rightarrow L^p(\mathbb{R})$ again stands for the trivial extension. We obtain

\[ \|H_{\varepsilon} f\|_{L^p(\mathbb{R}^+)} \leq \pi \|\mathcal{F}^{-1}_{1} m_{\varepsilon} \mathcal{F}_{1} E_0 f\|_{L^p(\mathbb{R})} \leq C \|E_0 f\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^p(\mathbb{R}^+)}. \]

The sequence $H_{1/n}(|f|)$ is monotonously increasing and converges pointwise to $H(|f|)$. By monotone convergence, we see that

\[ \|H f\|_{L^p(\mathbb{R}^+)} \leq \|H(|f|)\|_{L^p(\mathbb{R}^+)} = \lim_{n \rightarrow \infty} \|H_{1/n}(|f|)\|_{L^p(\mathbb{R}^+)} \]

\[ \leq C \|f\|_{L^p(\mathbb{R}^+)} = C \|f\|_{L^p(\mathbb{R}^+)}. \]

Therefore, $H \in L(L^p(\mathbb{R}^+))$. \qed

The following result shows that the solution operators are indeed $\mathcal{R}$-bounded.

**Theorem 6.10.** Let $\delta > 0$ be fixed. In the situation of Theorem 6.8, the following operator families in $L(L^p(\mathbb{R}^+_+))$ are $\mathcal{R}$-bounded:

a) $\{A_{2m-j}(\lambda)\bar{B}_{j0}(D) R_{+}(\lambda) E_0 : j = 1, \ldots, m, \lambda \in \mathbb{C}_+, |\lambda| \geq \delta\}$,

b) $\{A_{2m-j-1}(\lambda)\partial_u \bar{B}_{j0}(D) R_{+}(\lambda) E_0 : j = 1, \ldots, m, \lambda \in \mathbb{C}_+, |\lambda| \geq \delta\}$,

c) $\{\lambda T_{j}(\lambda) : j = 1, \ldots, m, \lambda \in \mathbb{C}_+, |\lambda| \geq \delta\}$,

d) $\{\lambda \bar{T}_{j}(\lambda) : j = 1, \ldots, m, \lambda \in \mathbb{C}_+, |\lambda| \geq \delta\}$. 

Proof. a) We have \( \Lambda_{2m-m_j}(\lambda) \tilde{B}_{j0}(D) R_+ = R_+ \Lambda_{2m-m_j}(\lambda) \tilde{B}_{j0}(D) \). As the operators \( R_+ \in L(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n_+)) \) and \( E_0 \in L(L^p(\mathbb{R}^n_+), L^p(\mathbb{R}^n)) \) are bounded, it suffices to show the \( \mathcal{R} \)-boundedness of

\[
\{ \Lambda_{2m-m_j}(\lambda) \tilde{B}_{j0}(\lambda) R(\lambda) : j = 1, \ldots, m, \lambda \in \mathbb{T}_+, |\lambda| \geq \delta \}.
\]

The corresponding family of symbols (with respect to the Fourier transform in \( \mathbb{R}^n \)) is given by

\[
m(\xi, \lambda) := (\lambda + |\xi'|^{2m})^{2m-m_j} b_{j0}(x_0', \xi) (a_0(x_0', \xi) - \lambda)^{-1}.
\]

As \( m(\xi, \lambda) \) is quasi-homogeneous of degree 0 in \( (\xi, \lambda) \) and bounded on \( |\lambda| + |\xi'|^{2m} = 1 \), it follows that

\[
|D^\alpha m(\xi, \lambda)| \leq C|\xi|^{-|\alpha|} \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \lambda \in \mathbb{T}_+, |\lambda| \geq \delta).
\]

By Corollary 3.29, the operator family in a) is \( \mathcal{R} \)-bounded.

b) can be shown analogously.

c) For \( \varphi \in L^p(\mathbb{R}^n_+) \), we write

\[
\lambda T_j(\lambda) \varphi = \int_0^\infty k_\lambda(x_n, y_n) \psi(y_n) dy_n
\]

with \( \psi \in L^p(\mathbb{R}^n_+; L^p(\mathbb{R}^{n-1})) \), \( \psi(y_n) := \varphi(\cdot, y_n) \), and the operator valued integral kernel

\[
k_\lambda(x_n, y_n) := (\mathcal{F})^{-1} \tilde{m}(\xi', \lambda, x_n + y_n) \mathcal{F}'
\]

\[= (\mathcal{F})^{-1} \varphi_0 w_j(x_0', \xi', \lambda, x_n + y_n) (\lambda + |\xi'|^{2m})^{-2m-m_j} \mathcal{F}'.
\]

By Remark 6.7 b), the inequalities

\[
|D^\gamma \tilde{m}(\xi', \lambda, x_n + y_n) \leq C(|\xi'| + |\lambda|^{1/2m}) \exp \left( -C(|\xi'| + |\lambda|^{1/2m})(x_n + y_n) |\xi'|^{-|\gamma|} \right)
\]

\[\leq C \frac{1}{x_n + y_n} |\xi'|^{-|\gamma|}
\]

hold, where in the last step we again used the elementary estimate \( te^{-t} < 1 \) (\( t > 0 \)). Again by Corollary 3.29, it follows that \( k_\lambda(x_n, y_n) \in L(L^p(\mathbb{R}^{n-1})) \) with

\[
\mathcal{R} \{ k_\lambda(x_n, y_n) : \lambda \in \mathbb{T}_+, |\lambda| \geq \delta \} \leq \frac{C}{x_n + y_n}.
\]

The scalar integral operator with kernel \( k_0(x_n, y_n) := \frac{1}{x_n + y_n} \), given by

\[
(K_0 g)(x_n) := \int_0^\infty \frac{g(y_n)}{x_n + y_n} dy_n \quad (g \in L^p(\mathbb{R}_+))
\]

is the one-sided Hilbert transform in \( L^p(\mathbb{R}_+) \) and, due to Lemma 6.9, a bounded linear operator \( K_0 \in L(L^p(\mathbb{R}_+)) \). By Theorem 3.18 we get

\[
\mathcal{R} \{ \lambda T_j(\lambda) : \lambda \in \mathbb{T}_+, |\lambda| \geq \delta \} \leq C \| K_0 \|_{L(L^p(\mathbb{R}_+))} < \infty.
\]

d) follows in the same way as c). \( \square \)

Now maximal regularity for the model problem is an immediate consequence of the previous results.
Theorem 6.11. Let the boundary value problem \((\partial_t - A, B)\) be parabolic, and let \(x_0' \in \partial G\). Choose the coordinate system corresponding to \(x_0'\), and consider the \(L^p\)-realization \(A^{(0)}_B\) of the model problem \((A_0(x_0', D), B(x_0', D))\). Then \(\rho(A^{(0)}_B) \supset \overline{\mathbb{C}}_+ \setminus \{0\}\), and for every \(\delta > 0\) the operator family
\[
\{ \lambda (\lambda - A^{(0)}_B)^{-1} : \lambda \in \overline{\mathbb{C}}_+, |\lambda| \geq \delta \} \subset L(L^p(\mathbb{R}^n_+))
\]
is \(\mathcal{R}\)-bounded. In particular, \(A^{(0)}_B - \delta\) has for every \(\delta > 0\) maximal \(L^q\)-regularity for all \(1 < q < \infty\) (and generates a bounded holomorphic \(C_0\)-semigroup).

Proof. Replacing in the proof of Theorem 6.10 the operators \(\lambda T_j(\lambda)\) by \(D^\alpha T_j(\lambda)\) (and analogously for \(\tilde{T}_j(\lambda)\)) with \(|\alpha| = 2m\), we see that the solution operators in fact define a solution \(u \in W^2_{\rho}(\mathbb{R}^n_+)\). Therefore, the solution coincides with the resolvent. Now the \(\mathcal{R}\)-boundedness follows directly from the resolvent description in Theorem 6.8 and the statements on \(\mathcal{R}\)-boundedness from Theorem 6.10.

To deal with variable coefficients, we first study small perturbations in the principal part.

Theorem 6.12. Let \(A^0(x, D) = \sum_{|\alpha| = 2m} a^0_\alpha D^\alpha\) and \(B^0_j(x, D) = \sum_{|\beta| = m_j} b^0_{j\beta} D^\beta\) with \(a^0_\alpha \in \mathbb{C}\) and \(b^0_{j\beta} \in \mathbb{C}\). Assume the boundary value problem \((\partial_t - A, B)\) to be parabolic in the domain \(\mathbb{R}^n_+\). Then there exists an \(\varepsilon > 0\) such that the following statement holds: Let \(A(x, D) = A^0(x, D) + \tilde{A}(x, D)\) and \(B(x, D) = B^0(x, D) + \tilde{B}(x, D)\) with
\[
\tilde{A}(x, D) = \sum_{|\alpha| = 2m} \tilde{a}_\alpha(x) D^\alpha,
\]
\[
\tilde{B}_j(x, D) = \sum_{|\beta| = m_j} \tilde{b}_{j\beta}(x) D^\beta \quad (j = 1, \ldots, m).
\]
Here, \(\tilde{a}_\alpha \in L^\infty(\mathbb{R}^n_+)\) and \(\tilde{b}_{j\beta} \in \text{BUC}^{2m - m_j}(\mathbb{R}^{n - 1})\). Assume further that
\[
\sum_{|\alpha| = 2m} \|\tilde{a}_\alpha\|_{L^\infty(\mathbb{R}^n_+)} \leq \varepsilon,
\]
\[
\sum_{|\beta| = m_j} \|\tilde{b}_{j\beta}\|_{L^\infty(\mathbb{R}^{n - 1})} \leq \varepsilon \quad (j = 1, \ldots, m).
\]
Let \(A_{B,p}\) be the \(L^p\)-realization of the boundary value problem \((A(x, D), B(x, D))\). Then there exists a \(\mu > 0\) such that the operator family
\[
\{ \lambda (A_{B,p} - \lambda)^{-1} : \lambda \in \overline{\mathbb{C}}_+, |\lambda| \geq \mu \} \subset L(L^p(\mathbb{R}^n_+))
\]
is \(\mathcal{R}\)-bounded. Here, \(\varepsilon\) and the \(\mathcal{R}\)-bound only depend on \((A^0(x, D), B^0(x, D))\), and \(\mu\) additionally depends on the norms \(\|b_{j\beta}\|_{\text{BUC}^{2m - m_j}(\mathbb{R}^{n - 1})}\) for \(|\beta| = m_j, j = 1, \ldots, m\).

Proof. We indicate the main steps of the proof, for a more elaborated version, see [DHP03], Subsection 7.3.

Without loss of generality, we may assume that the coefficients of \(\tilde{B}(x, D)\) are defined on all of \(\mathbb{R}^n_+\). We write the boundary value problem
\[
(A(x, D) - \lambda)u = f \quad \text{in} \ \mathbb{R}^n_+,
\]
\[
B_j(x, D)u = 0 \quad (j = 1, \ldots, m) \quad \text{on} \ \mathbb{R}^{n - 1}
\]
in the form
\[(A^0(x, D) - \lambda)u = f - \tilde{A}(x, D)u \quad \text{in } \mathbb{R}_+^n, \]
\[B_j^0(x, D)u = -\tilde{B}_j(x, D)u \quad (j = 1, \ldots, m) \quad \text{on } \mathbb{R}^{n-1}.\]

Let \((A_{B,p}^0 - \lambda)^{-1}\) be the resolvent of the \(L^p\)-realization of \((A^0(x, D), B^0(x, D))\), which exists due to Theorem 6.11. Applying the solution operators from Theorem 6.8, we obtain
\[
u = (A_{B,p}^0 - \lambda)^{-1} f - (A_{B,p}^0 - \lambda)^{-1} \tilde{A}(x, D)u
- \sum_{j=1}^m T_j(\lambda)\Lambda_{2m-m_j}(\lambda)\tilde{B}_j(x, D)u
- \sum_{j=1}^m \tilde{T}_j(\lambda)\Lambda_{2m-m_j-1}(\lambda)\partial_n\tilde{B}_j(x, D)u
=: (A_{B,p}^0 - \lambda)^{-1} f - S(\lambda)u.
\]

We estimate the norm of \(S(\lambda)u\). For the term \((A_{B,p}^0 - \lambda)^{-1} \tilde{A}(x, D)u\), we use
\[
\|(A_{B,p}^0 - \lambda)^{-1}\|_{L(L^p(\mathbb{R}_+^n), W^{2m}(\mathbb{R}_+^n))} \leq C_1
\]
and obtain
\[
\|(A_{B,p}^0 - \lambda)^{-1} \tilde{A}(x, D)u\|_{W^{2m}(\mathbb{R}_+^n)} \leq C_1 \|\tilde{A}(x, D)u\|_{L^p(\mathbb{R}_+^n)} \leq C_1 \|u\|_{W^{2m}(\mathbb{R}_+^n)}.
\]

For the other terms, we use the fact that the operator families
\[
\{\lambda^{(2m-|\alpha|)/2m} D^\alpha T(\lambda) : |\alpha| \leq 2m, \lambda \in \mathbb{T}_+, |\lambda| \geq \lambda_0\} \subset L(L^p(\mathbb{R}_+^n))
\]
are \(\mathcal{R}\)-bounded and therefore bounded, which can be seen as in the proof of Theorem 6.10. This yields
\[
\|T_j(\lambda)\Lambda_{2m-m_j}(\lambda)\tilde{B}_j(x, D)u\|_{W^{2m}(\mathbb{R}_+^n)} \leq C \|\Lambda_{2m-m_j}(\lambda)\tilde{B}_j(x, D)u\|_{L^p(\mathbb{R}_+^n)}
\leq C \|\tilde{B}_j(x, D)u\|_{W^{2m-m_j}(\mathbb{R}_+^n)}.
\]

The terms of the form \(\tilde{b}_{j\beta}D^\beta u\) can be estimated, using the Leibniz rule, by
\[
\|\tilde{b}_{j\beta}D^\beta u\|_{W^{2m-m_j}(\mathbb{R}_+^n)} \leq C \sum_{|\gamma| \leq 2m-m_j} \sum_{\delta+\delta' = \gamma} \|D^\delta \tilde{b}_{j\beta}(D^{\delta'} u)\|_{L^p(\mathbb{R}_+^n)}
\leq C_2 \|u\|_{W^{2m}(\mathbb{R}_+^n)} + C_3 \|u\|_{W^{2m-1}(\mathbb{R}_+^n)}.
\]

Here, the constant \(C_3\) depends on the norm \(\|b_{j\beta}\|_{\text{BUC}^{2m-m_j}(\mathbb{R}_+^{n-1})}\). With the interpolation inequality, we see that for some constants \(C_1, C_2\) we have
\[
\|S(\lambda)u\|_{W^{2m}(\mathbb{R}_+^n)} + |\lambda| \|S(\lambda)u\|_{L^p(\mathbb{R}_+^n)} \leq C_1 \|u\|_{W^{2m}(\mathbb{R}_+^n)} + C_2 \|u\|_{L^p(\mathbb{R}_+^n)}.
\]

Now we endow \(W^{2m}(\mathbb{R}_+^n)\) with the parameter-dependent norm \(\|u\| := \|u\|_{W^{2m}(\mathbb{R}_+^n)} + |\lambda| \|u\|_{L^p(\mathbb{R}_+^n)}\). Note that for every fixed \(\lambda\), this norm is equivalent to the standard norm.

For \(|\lambda| \geq 2C_2\) and \(C_1 \varepsilon \leq \frac{1}{2}\), it follows that
\[
\|S(\lambda)u\| \leq \frac{1}{2} \|u\|.
\]

Therefore, \((1 + S(\lambda)) \in L(W^{2m}(\mathbb{R}_+^n))\) is invertible (with respect to the new norm, and therefore also with respect to the standard norm). Thus, we have seen that the above boundary value problem is uniquely solvable and that the resolvent \((A_{B,p} - \lambda)^{-1}\) exists for all \(\lambda \in \mathbb{T}_+\) with \(|\lambda| \geq 2C_2\).
To obtain an estimate on the $\mathcal{R}$-bounds, we can argue similarly. Starting from the identity

$$(A_{B,p} - \lambda)^{-1} = (A_{B,p}^0 - \lambda)^{-1} - S(\lambda)(A_{B,p} - \lambda)^{-1},$$

one can show for sufficiently large $\mu > 0$

$$\mathcal{R}\{ \tilde{A}(x,D)(A_{B,p} - \lambda)^{-1} : \lambda \in \mathbb{C}_+, |\lambda| \geq \mu \}$$

$$\leq \sum_{|\alpha|=2m} \| \tilde{a}_\alpha \|_{L^\infty(\mathbb{R}_+^n)} \mathcal{R}\{ D^\alpha (A_{B,p} - \lambda)^{-1} : \lambda \in \mathbb{C}_+, |\lambda| \geq \mu \}$$

$$\leq C \varepsilon \mathcal{R}\{ D^\alpha (A_{B,p} - \lambda)^{-1} : \lambda \in \mathbb{C}_+, |\lambda| \geq \mu \}.$$ 

Similarly, the other terms in $S(\lambda)(A_{B,p} - \lambda)^{-1}$ can be estimated. Consider the operator family

$$\mathcal{T} := \{ \lambda^{2m-|\alpha|}/(2m) D^\alpha (A_{B,p} - \lambda)^{-1} : |\alpha| \leq 2m, \lambda \in \mathbb{C}_+, |\lambda| \geq \mu \}.$$ 

The above calculations show that for every finite subset $\mathcal{T}_0$ of $\mathcal{T}$, we get the inequality

$$\mathcal{R}(\mathcal{T}_0) \leq R_1 + (C_1 \varepsilon + C_2(\mu)) \mathcal{R}(\mathcal{T}_0).$$

Here,

$$C_1 := \mathcal{R}\{ \lambda^{2m-|\alpha|}/(2m) D^\alpha (A_{B,p}^0 - \lambda)^{-1} : |\alpha| \leq 2m, \lambda \in \mathbb{C}_+, |\lambda| \geq \mu \} < \infty$$

and

$$C_2(\mu) \to 0 \text{ for } \mu \to \infty.$$ 

Choosing $\varepsilon$ small enough and $\mu$ large enough, we have

$$C_1 \varepsilon + C_2(\mu) < \frac{1}{2},$$

and therefore $\mathcal{R}(\mathcal{T}_0) < 2R_1 < \infty$. As this holds for every finite subset $\mathcal{T}_0$ of $\mathcal{T}$, with $R_1$ being independent of $\mathcal{T}_0$, we get the same estimate for $\mathcal{T}$, i.e., $\mathcal{R}(\mathcal{T}) \leq 2R_1$. \hfill $\square$

The last result deals with small perturbations of the top-order coefficients. As before, lower-order terms of the operators can be handled by the interpolation inequality. For a proof of maximal regularity in the situation of a bounded domain and under the above smoothness assumptions, the method of localization can be used. We mention some main ideas in the following remark.

**Remark 6.13** (Localization). Let $(\partial_t - A, B)$ be a parabolic boundary value problem in the bounded domain $G$, and assume the smoothness assumptions from the beginning of this section to hold. To prove $\mathcal{R}$-sectoriality of the $L^p$-realization of $(A, B)$, one can use the following steps:

a) For every fixed $x_0 \in \partial G$, by definition of a $C^{2m}$-domain, there exists a neighbourhood $U(x_0) \subset \mathbb{R}^n$ and a $C^{2m}$-diffeomorphism $\Phi_{x_0} : U(x_0) \to V(x_0) := \Phi_{x_0}(U(x_0)) \subset \mathbb{R}^n$ with

$$\Phi_{x_0}(U(x_0) \cap G) = V(x_0) \cap \mathbb{R}_+^n.$$ 

We denote by $(\tilde{A}, \tilde{B})$ the transformed boundary value problem in the domain $V(x_0)$. The coefficients $\tilde{a}_\alpha$ of $\tilde{A}$ are defined in $V(x_0) \cap \mathbb{R}^n_+$ and satisfy the same smoothness assumptions as $a_\alpha$. In the same way, this holds for the transformed coefficients $\tilde{b}_{j\beta}$ of $\tilde{B}$. Moreover, it is possible to show that the transformed problem is parabolic in $V(x_0) \cap \mathbb{R}_+^n$.

The coefficients $\tilde{a}_\alpha$ and $\tilde{b}_{j\beta}$ can be extended to the half space $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}$, respectively, in such a way that both the smoothness and the parabolicity is preserved. For $\tilde{a}_\alpha$, we can choose an appropriate continuous extension. For the coefficients on the boundary $\tilde{b}_{j\beta}$, we have to preserve higher smoothness. For this, one can, e.g., define

$$\tilde{b}_{j\beta}(y) := \tilde{b}_{j\beta}(y_0 + \chi \left( \frac{y - y_0}{r} \right) (y - y_0)) \quad (y \in \mathbb{R}^{n-1}),$$
where $\chi \in C^\infty(\mathbb{R}^{n-1})$ satisfies $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Here, $y_0 := \Phi_{x_0}(x_0)$, and $r > 0$ is chosen sufficiently small.

For an eventually even smaller $r = r(x_0)$, the following inequalities hold true for a given $\varepsilon > 0$:

$$\sum_{|\alpha| = 2m} \| \tilde{a}_\alpha(\cdot) - \tilde{a}_\alpha(y_0) \|_{L^\infty(\mathbb{R}^n)} < \varepsilon,$$

$$\sum_{|\beta| = m_j} \| \tilde{b}_{j\beta}(\cdot) - \tilde{b}_{j\beta}(y_0) \|_{L^\infty(\mathbb{R}^{n-1})} < \varepsilon \quad (j = 1, \ldots, m).$$

Therefore, the localized boundary value problems satisfy the conditions of Theorem 6.12.

For fixed $\varepsilon > 0$, this construction yields an open cover of the form

$$\partial G \subset \bigcup_{x_0 \in \partial G} \Phi_{x_0}^{-1}(B(y_0, r(x_0))).$$

By compactness of $\partial G$, there exists a finite subcover $\partial G \subset \bigcup_{k=1}^N U_k$, where we have set $U_k := \Phi_{x_k}^{-1}(B(y_k, r(x_k)))$.

b) In the same way, in the interior of the domain, we obtain for every $x_0 \in G$ a small neighbourhood $U(x_0) \subset \mathbb{R}^n$ and an extension $\tilde{a}_\alpha$ of $a_\alpha|_{U(x_0)}$ such that

$$\sum_{|\alpha| = 2m} \| \tilde{a}_\alpha(\cdot) - \tilde{a}_\alpha(x_0) \|_{L^\infty(\mathbb{R}^n)} < \varepsilon$$

holds. In this way, we obtain an open cover

$$G \setminus \bigcup_{k=1}^N U_k \subset \bigcup_{x_0 \in G} B(x_0, r(x_0)).$$

Note that no boundary operator and no diffeomorphism is involved. As $G \setminus \bigcup_{k=1}^N U_k$ is compact, there exists a finite subcover

$$G \setminus \bigcup_{k=1}^N U_k \subset \bigcup_{k=N+1}^M U_k$$

with $U_k = B(x_k, r(x_k))$. Altogether, this yields a finite open cover $\mathcal{G} \subset \bigcup_{k=1}^M U_k$.

c) With this construction, one obtains finitely many operators $(\tilde{A}^{(k)}, \tilde{B}^{(k)})$ for $k = 1, \ldots, N$ and $\tilde{A}^{(k)}$ for $k = N + 1, \ldots, M$, which satisfy the assumptions of Theorem 6.12 and Lemma 5.7, respectively. Now we can use the resolvents of the $L^p$-realization of these operators to show $\mathcal{R}$-sectoriality of $A_{B,p} - \mu$ for large $\mu$. This can be done similarly as in the proof of Theorem 5.9, using a partition of unity and estimating the commutators with help of the interpolation inequality.

With the above techniques, it is possible to show the following main theorem on parabolic boundary value problems:

**Theorem 6.14.** Assume the boundary value problem $(\partial_t - A, B)$ to be parabolic and to satisfy the smoothness assumptions above. Let $1 < p < \infty$. Then there exist $\theta > \frac{\pi}{2}$ and $\mu > 0$ such that $\rho(A_{B,p} - \mu) \supset \mathcal{C}_\theta$ and the operator $A_{B,p} - \mu$ is $\mathcal{R}$-sectorial with angle $\theta$. In particular, $A_{B,p} - \mu$ has maximal $L^q$-regularity for all $q \in (1, \infty)$. 
7. Quasilinear parabolic evolution equations

We have seen in the previous sections that, under appropriate parabolicity and smoothness assumptions, the $L^p$-realization of linear boundary value problems have maximal regularity. This is the basis for the analysis of nonlinear problems, which will be described in the present section.

7.1. Well-posedness for quasilinear parabolic evolution equations. We consider nonlinear evolution equations which can be written in the abstract form

$$\partial_t u(t) - A(t, u(t))u(t) = F(t, u(t)) \quad \text{in } (0, T_0),$$

$$u(0) = u_0.$$  

Here, $T_0 \in (0, \infty)$. We fix the following situation: Let $p \in (1, \infty)$, and let $X_1 \subset X_0$ be Banach spaces with $X_1$ being dense in $X_0$. With $T \in (0, T_0]$, the spaces for the right-hand side and the solution are

$$\mathbb{F} := \mathbb{F}_T := L^p((0, T); X_0) \quad \text{and} \quad \mathbb{E} := \mathbb{E}_T := H^1_p((0, T); X_0) \cap L^p((0, T); X_1),$$

respectively. The time trace space, and therefore the space for the initial value $u_0$, is given by $\gamma_t \mathbb{E} = (X_0, X_1)_{1-1/p, p}$ (cf. Lemma 2.4). We again set $\mathfrak{n} \mathbb{E} := \{u \in \mathbb{E} : \gamma_t u = 0\}$. Here and in the following, we consider the operator $A$ as a map $A : (0, T_0) \times \gamma_t \mathbb{E} \to L(X_1, X_0)$. For each $t \in (0, T_0)$ and $v \in \gamma_t \mathbb{E}$, the operator $A(t, v) \in L(X_1, X_0)$ is identified with the unbounded operator $A(t, v)$ acting in $X_0$ with domain $X_1$, and $A(t, v) \in M(\mathfrak{n} \mathbb{E})$ has to be understood in this sense.

Example 7.1. We recall the example of the graphical mean curvature flow (Example 2.1), which has the form

$$\partial_t u - \left(\Delta u - \sum_{i,j=1}^n \frac{\partial_i u \partial_j u}{1 + |\nabla u|^2} \partial_i \partial_j u\right) = 0 \quad \text{in } (0, T_0),$$

$$u(0) = u_0.$$  

This quasilinear equation can be written in the form (7-1), where

$$A(t, u(t)) = \Delta - \sum_{i,j=1}^n \frac{\partial_i u(t) \partial_j u(t)}{1 + |\nabla u(t)|^2} \partial_i \partial_j$$

and $F = 0$. Here we have $X_0 = L^p(\mathbb{R}^n)$, $X_1 = W^2_p(\mathbb{R}^n)$, and $\gamma_t \mathbb{E} = B^{2-2/p}_{pp}(\mathbb{R}^n) = W^2_p(\mathbb{R}^n)$.

For the nonlinearities $A$ and $F$ in (7-1), we assume:

(A1) We have $A \in C([0, T_0] \times \gamma_t \mathbb{E}, L(X_1, X_0))$, and for all $R > 0$ there exists a Lipschitz constant $L(R) > 0$ with

$$||A(t, w)v - A(t, \overline{w})v||_{X_0} \leq L(R)||w - \overline{w}||_{\gamma_t \mathbb{E}}||v||_{X_1}$$

for all $t \in [0, T_0]$, $v \in X_1$ and all $w, \overline{w} \in \gamma_t \mathbb{E}$ with $||w||_{\gamma_t \mathbb{E}} \leq R$ and $||\overline{w}||_{\gamma_t \mathbb{E}} \leq R$.

(A2) For the mapping $F : [0, T_0] \times \gamma_t \mathbb{E} \to X_0$ we assume:

(i) $F(\cdot, w)$ is measurable for every $w \in \gamma_t \mathbb{E}$,

(ii) $F(t, \cdot) \in C(\gamma_t \mathbb{E}, X_0)$ for almost all $t \in [0, T_0]$,

(iii) $f(\cdot) := F(\cdot, 0) \in L^p((0, T_0); X_0)$,
(iv) For every $R > 0$, there exists a $\varphi_R \in L^p((0,T_0))$ with
\[ \|F(t,w) - F(t,\overline{w})\|_{X_0} \leq \varphi_R(t)\|w - \overline{w}\|_{\gamma_tE} \]
for almost all $t \in [0,T_0]$ and all $w, \overline{w} \in \gamma_tE$ with $\|w\|_{\gamma_tE} \leq R$, $\|\overline{w}\|_{\gamma_tE} \leq R$.

Apart from standard conditions on measurability and continuity, the above conditions essentially mean that the functions $A(t,\cdot)v$ and $F(t,\cdot)$ are locally Lipschitz, i.e., they are Lipschitz on bounded subsets of $\gamma_tE$. The following result is based on [Pru02, Section 3 (see also [CL94]).

**Theorem 7.2.** Assume (A1) and (A2) as well as $A_0 := A(0,u_0) \in \text{MR}(X_0)$. Then there exists a $T \in (0,T_0]$ such that (7-1) has a unique solution $u \in E_T$ in the interval $(0,T)$.

**Proof.** (i) We use the maximal regularity of $A_0 := A(0,u_0)$ in the time interval $(0,T)$ with $T \leq T_0$ to obtain estimates for the solutions of the linearized equation. For this, we first consider the equation with initial value 0,
\[
\begin{align*}
\partial_tw(t) - A_0w(t) &= g(t) \quad (t \in (0,T)), \\
w(0) &= 0.
\end{align*}
\]
(7-3)

As $A_0 \in \text{MR}(X_0)$, for every $g \in F$ there exists a unique solution $w \in E$, and we obtain the estimate
\[ \|w\|_E \leq C_0\|g\|_F \]
with a constant $C_0 > 0$ which does not depend on $T$ or $w$ (Lemma 4.7). By Lemma 4.4 b), there exists a constant $C_1$ (again independent of $T > 0$ and $w$) with
\[ \|w\|_{C([0,T],\gamma_tE)} \leq C_1\|w\|_E. \]

Note here that $w(0) = 0$ holds.

In the following, we consider the reference solution $u^* \in E$ which is defined as the unique solution of
\[
\begin{align*}
\partial_tw(t) - A_0w(t) &= f(t) \quad (t \in (0,T)), \\
w(0) &= u_0.
\end{align*}
\]
(7-4)

Here, $f := F(\cdot,0) \in F$ due to condition (A2) (iii).

(ii) For $r \in (0,1]$ set
\[ B_r := \{ v \in E : v - u^* \in \gamma_0E, \|v - u^*\|_E \leq r \}. \]

For each $v \in B_r$, define $\Phi(v) := u$ as the unique solution of
\[
\begin{align*}
\partial_tu(t) - A_0u(t) &= F(t,v(t)) - (A(0,u_0) - A(t,v(t)))v(t) \quad (t \in (0,T)), \\
u(0) &= u_0.
\end{align*}
\]
(7-5)

We will show that $\Phi(B_r) \subset B_r$ holds and that $\Phi$ is a contraction in $B_r$, given that both $T$ and $r$ are sufficiently small.

(iii) In this step, we show that $\Phi(B_r) \subset B_r$ holds for sufficiently small $T$ and $r$. For this, we write
\[
\|\Phi(v) - u^*\|_E = \|u - u^*\|_E \leq C_0\left(\|F(\cdot,v) - f(\cdot)\|_F + \|(A(0,u_0) - A(\cdot,v))v\|_F\right).
\]
(7-6)
Let $m_T := \sup_{t \in [0, T]} \|A(0, u_0) - A(t, u_0)\|_{L(X_1, X_0)}$. By condition (A1) with fixed $R := C_1 + \|u^*\|_{L^{\infty}(0, T); \gamma E}$, we obtain
\[
\|A(0, u_0)v - A(\cdot, v)v\|_F = \|A(0, u_0)v - A(\cdot, v)v\|_{L^p((0, T); X_0)} \\
\leq \|A(0, u_0) - A(\cdot, v)\|_{L^{\infty}((0, T); L(X_1, X_0))}\|v\|_{L^p((0, T); X_1)} \\
\leq \left(\|A(0, u_0) - A(\cdot, u_0)\|_{L^{\infty}((0, T); L(X_1, X_0))} \\
+ \|A(\cdot, u_0) - A(\cdot, v)\|_{L^{\infty}((0, T); L(X_1, X_0))}\right)\|v\|_E \\
\leq \left(m_T + L(R)\|v - u_0\|_{L^{\infty}((0, T); \gamma E)}\right)\|v\|_E \\
\leq \left(m_T + L(R)C_1\|v - u_0\|_E\right)\|v\|_E.
\]
For $r \leq 1$, we can estimate
\[
\|v - u_0\|_E \leq \|v - u^*\|_E + \|u^* - u_0\|_E \leq r + \|u^* - u_0\|_E
\]
and
\[
\|v\|_E \leq \|v - u^*\|_E + \|u^*\|_E \leq r + \|u^*\|_E.
\]
Therefore, we obtain
\[
\|A(0, u_0)v - A(\cdot, v)v\|_F \leq \left(m_T + L(R)C_1(r + \|u^* - u_0\|_E)\right)(r + \|u^*\|_E).
\]
In a similar way, using (A2), we see that
\[
\|F(\cdot, v) - F(\cdot, u^*)\|_F \leq \|F(\cdot, v) - F(\cdot, u^*)\|_F + \|F(\cdot, u^*) - F(\cdot, 0)\|_F \\
\leq \|\varphi_R\|_{L^p((0, T); E)}\left(\|v - u^*\|_{L^{\infty}((0, T); \gamma_E)} + \|u^*\|_{L^{\infty}((0, T); \gamma_E)}\right) \\
\leq \|\varphi_R\|_{L^p((0, T); E)}\left(C_1\|v - u^*\|_E + \|u^*\|_{L^{\infty}((0, T); \gamma_E)}\right) \\
\leq \|\varphi_R\|_{L^p((0, T); E)}C_1\left(r + \|u^*\|_{L^{\infty}((0, T); \gamma_E)}\right).
\]
Inserting this into (7-6), we get
\[
\|\varphi(v) - u^*\|_E \leq C_0\left[\|\varphi_R\|_{L^p((0, T); E)}\left(C_1r + \|u^*\|_{L^{\infty}((0, T); \gamma_E)}\right) \\
+ \left(m_T + L(R)C_1(r + \|u^* - u_0\|_E)\right)\|u^*\|_E\right] \\
\leq C_0\left(C_1 + \|u^*\|_{L^{\infty}((0, T); \gamma_E)}\right)\|\varphi_R\|_{L^p((0, T); E)} \\
+ C_0\left(r + \|u^*\|_E\right)\left(m_T + L(R)C_1r + L(R)C_1\|u^* - u_0\|_E\right).
\]
In the limit $T \to 0$, we obtain the following convergences:

- $m_T \to 0$, as $A(\cdot, u_0)$ is continuous,
- $\|\varphi_R\|_{L^p((0, T); E)} \to 0$, as $\varphi_R \in L^p((0, T_0))$,
- $\|u^* - u_0\|_{E_T} \to 0$, as $u^* - u_0 \in E_{T_0}$,
- $\|u^*\|_{E_T} \to 0$, as $u^* \in E_{T_0}$.

First, choose $r > 0$ small enough such that
\[
C_0L(R)C_1r < \frac{1}{8}
\]
holds. Then, choose $T > 0$ small enough such that the following inequalities hold:
\[
\|u^*\|_E < r
\]
\[ C_0(C_1 + \|u^*\|_{L^\infty([0,T);\gamma_t\mathbb{E})})\|F_R\|_{L^p([0,T))} < \frac{r}{T}, \]
\[ C_0(m_T + L(R)C_1\|u^* - u_0\|_{\mathbb{E}}) < \frac{1}{8}. \]
Inserting this into (7-7), we obtain
\[ \|\Phi(v) - u^*\|_{\mathbb{E}} \leq \frac{r}{T} + (r + r)(\frac{1}{8} + \frac{1}{2}) = r, \]
which shows that \( \Phi(B_r) \subset B_r. \)

(iv) In the same way as in (iii), one sees that for sufficiently small \( r > 0 \) and \( T > 0 \) the inequality
\[ \|\Phi(v) - \Phi(\overline{v})\|_{\mathbb{E}} \leq \frac{1}{2}\|v - \overline{v}\|_{\mathbb{E}} \]
holds for all \( v, \overline{v} \in B_r. \) Therefore, \( \Phi: B_r \to B_r \) is a contraction, and with the Banach fixed point theorem (contraction mapping principle), there exists a unique fixed point \( u \) of \( \Phi. \) By definition of \( \Phi, \) its fixed points are exactly the solutions of the nonlinear equation (7-1), which finishes the proof. \( \square \)

**Theorem 7.3.** Assume (A1) and (A2) to hold, and assume \( A(t,v) \in \text{MR}(X_0) \) for all \( t \in [0,T_0) \) with \( T_0 \in (0,\infty]. \) Then for every \( u_0 \in \gamma_t\mathbb{E} \) there exists a unique maximal solution of (7-1) with maximal existence interval \([0,T^+(u_0)) \subset [0,T_0). \) If \( T^+(u_0) < T_0 \) (i.e., if there is no global solution), then \( T^+(u_0) \) is characterized by each of the following conditions.

1. \( \lim_{t \to T^+(u_0)} u(t) \) does not exist in \( \gamma_t\mathbb{E}, \)
2. \( \int_0^{T^+(u_0)} (\|u(t)\|_{X_1}^p + \|\partial_t u(t)\|_{X_0}^p) dt = \infty. \)

**Proof.** Assume \( u \in \mathbb{E}_T \) to be a local solution on the interval \((0,T). \) Then \( u \in C([0,T];\gamma_t\mathbb{E}). \) Therefore, we can apply Theorem 7.2 in the interval \((T,T_0) \) with initial condition \( u_1 = u(T) \in \gamma_t\mathbb{E}, \) and obtain an extension of \( u \) to some interval \((0,T') \) with \( T' > T. \) Continuing in this way, we obtain a unique maximal solution which exists in some time interval \([0,T^+(u_0)). \)

If \( \lim_{t \to T^+(u_0)} u(t) \in \gamma_t\mathbb{E} \) exists, this can be taken as initial value at time \( T^+(u_0). \) By the above arguments, we see that \( u \) can be extended to a small time interval \((T^+(u_0), T^+(u_0) + \varepsilon) \), which is a contradiction to the maximality of \( T^+(u_0). \) Therefore, \( T^+(u_0) \) is characterized by condition (i).

For each \( T < T^+(u_0) \) we have, by definition of a solution, \( \int_0^T (\|u(t)\|_{X_1}^p + \|\partial_t u(t)\|_{X_0}^p) dt < \infty. \) If this also holds for \( T = T^+(u_0), \) then \( u \in \mathbb{E}_{T^+(u_0)}(X_1,X_0) \subset C([0,T^+(u_0)];\gamma_t\mathbb{E}). \) Therefore, \( \lim_{t \to T^+(u_0)} u(t) \) exists in \( \gamma_t\mathbb{E} \) in contradiction to (i). \( \square \)

As an application of the above theorems, we obtain a result on lower-order perturbation (the map \( B \) in the following lemma) for linear non-autonomous problems.

**Lemma 7.4.** Let \( A \in C([0,T],L(X_1,X_0)) \) with \( A(t) \in \text{MR}(X_0) \) \( (t \in [0,T]), \) and let \( B \in L^p((0,T);L(\gamma_t\mathbb{E},X_0)). \) Then the initial value problem
\[ \partial_t u(t) - A(t)u(t) = B(t)u(t) + f(t) \quad (t \in [0,T]), \]
\[ u(0) = u_0 \]
has for each \( f \in \mathbb{F}_T \) and each \( u_0 \in \gamma_t\mathbb{E} \) a unique solution \( u \in \mathbb{E}_T. \)
Proof. We set \( A(t, u(t)) = A(t) \) and \( F(t, u(t)) = B(t)u(t) + f(t) \). Obviously, the conditions (A1) and (A2) are satisfied with \( \varphi_R(t) := \|B(t)\|_{L(\gamma_t E, X_0)} \). The proof of Theorem 7.2 shows that the length of the existence interval only depends on \( u_0 \) and the constants \( L(R), C_0, C_1 \) and \( \gamma_T \). Because of \( A \in C([0, T], L(X_1, X_0)) \) and the continuity of \( A \rightarrow \|A^{-1}\|_{L(F, E)} = C_0(A) \), all these constants can be chosen globally in the time interval \([0, T]\). Therefore, we have global existence of the solution. \( \square \)

7.2. Higher regularity. We consider the same situation as in the last subsection and study the autonomous quasilinear differential equation

\[
\partial_t u(t) - A(u(t))u(t) = F(u(t)) \quad (t \in (0, T)),
\]

\( u(0) = u_0. \)

Here, \( T \in (0, \infty) \), \( u_0 \in \gamma_t E(X_1, X_0) \), \( A: \gamma_t E \to L(X_1, X_0) \) and \( F: \gamma_t E \to F. \)

It is well known that parabolic equations are smoothing, and the solution is even – in many applications – real analytic. We start with a definition.

**Definition 7.5.** Let \( X, Y \) be Banach spaces, \( U \subset X \) open, and \( T: U \to Y \) be a function. Then \( T \) is called real analytic if for all \( u_0 \in U \) there exists an \( r > 0 \) with \( B(u_0, r) \subset U \) and

\[
T(u) = \sum_{k=0}^{\infty} \frac{D^k T(u_0)}{k!} (u - u_0, \ldots, u - u_0) \quad (u \in B(u_0, r)).
\]

Here, \( D^k T(u_0) \in L(X \times \ldots \times X, F) \) denotes the \( k \)-th Fréchet derivative of \( T \) at \( u_0 \). In this case, we write \( T \in C^\omega(U, Y). \)

The main step in the proof of smoothing properties for parabolic equations is the implicit function theorem in Banach spaces.

**Theorem 7.6** (Implicit function theorem). Let \( X, Y, Z \) be Banach spaces, \( U \subset X \times Y \) be open, and \( T \in C^1(U, Z) \). Further, let \( (x_0, y_0) \in U \) with \( T(x_0, y_0) = 0 \) and \( D_y T((x_0, y_0)) \in L_{1\text{hom}}(Y, Z) \), where \( D_y T \) stands for the Fréchet derivative with respect to the second component. Then there exist neighbourhoods \( U_X \) of \( x_0 \) and \( U_Y \) of \( y_0 \) with \( U_X \times U_Y \subset U \) and a unique function \( \psi \in C^1(U_X, U_Y) \) such that

\[
T(x, \psi(x)) = 0 \quad (x \in U_X)
\]

and \( \psi(x_0) = y_0. \) Therefore, the equation \( T(x, y) = 0 \) is locally solvable with respect to \( y. \) The function \( \psi \) has the same regularity as \( T, \) i.e., if \( T \in C^k(U, Z) \) for \( k \in \mathbb{N} \cup \{\infty, \omega\} \), then also \( \psi \in C^k(U_X, U_Y). \)

With the help of the implicit function theorem, one can prove smoothing properties with respect to the time variable. As references, we mention [Ang90], [Prü02], Section 5, and [PS16], Section 5.2.

**Theorem 7.7.** Let \( k \in \mathbb{N} \cup \{\infty, \omega\} \), and let \( A \in C^k(\gamma_t E; L(X_1, X_0)) \) and \( F \in C^k(\gamma_t E, X_0). \) Assume \( u \in E_T(X_1, X_0) \) to be a solution of (7-8), and assume that \( A(u(t)) \in \text{MR}(X_0) \) for all \( t \in [0, T]. \) Then

\[
t \mapsto \partial_t^j u(t) \in W^k_p(J; X_0) \cap L^p(J; X_1)
\]

holds for all \( j \in \mathbb{N}_0 \) with \( j \leq k. \) In particular,

\[
u \in W^{k+1}_p((\varepsilon, T); X_0) \cap W^k_p((\varepsilon, T); X_1)
\]
for every $\varepsilon > 0$ as well as
\[ u \in C^k((0, T); \gamma_t(E)) \cap C^{k+1-1/p}((0, T); X_0) \cap C^{k-1/p}((0, T); X_1). \]
Here, $C^{k+1-1/p}$ and $C^{k-1/p}$ stand for the Hölder spaces of order $k+1-1/p$ and $k-1/p$, respectively. If $k = \infty$, then $u \in C^\infty((0, T); X_1)$, and if $k = \omega$, then $u \in C^\omega((0, T); X_1)$.

**Proof.** We fix $\varepsilon \in (0, 1)$ and set $T(\varepsilon) := \frac{T}{1-\varepsilon}$. For $\lambda \in (1-\varepsilon, 1+\varepsilon)$ we define the function $u_\lambda : [0, T(\varepsilon)] \rightarrow \gamma_t(E)$ by $u_\lambda(t) := u(\lambda t) \ (t \in [0, T(\varepsilon)])$. Then $\partial_t u_\lambda(t) = \lambda(\partial_t u)(\lambda t)$, and therefore
\[ \partial_t u_\lambda(t) - \lambda A(u_\lambda(t)) u_\lambda(t) = \lambda F(u_\lambda(t)) \ (t \in (0, T(\varepsilon))), \]
and $u_\lambda(0) = u_0$.

Now consider the function
\[ H : (1 - \varepsilon, 1 + \varepsilon) \times E(\varepsilon) \rightarrow \mathbb{E}_T(\varepsilon) \times \gamma_t(E) \]
defined by
\[ H(\lambda, w)(t) := \left( \partial_t w(t) - \lambda A(w(t)) w(t) - \lambda F(w(t)) \right) \ (t \in (0, T(\varepsilon))) \]
for $\lambda \in (1-\varepsilon, 1+\varepsilon)$ and $w \in E_T(\varepsilon)$. As $A$ and $F$ are both of class $C^k$, the same holds for $H$. Moreover, $H(1, w) = 0$ and
\[ D_\lambda H(\lambda, w) = \begin{pmatrix} -A(w) w - F(w) \\ 0 \end{pmatrix}, \]
\[ D_w H(\lambda, w) = \begin{pmatrix} \partial_t h - \lambda A(w) h - \lambda A'(w) h + \lambda F'(w) h \\ h(0) \end{pmatrix} \]
for $h \in E_T(\varepsilon)$. Here $A'(u)$ stands for the Fréchet derivative of $A$ at $u$. In particular, we obtain for $\lambda = 1$ and $w = u$
\[ D_w H(1, u) = \begin{pmatrix} \partial_t h + A(u) h + A'(u) h - F'(u) h \\ h(0) \end{pmatrix}. \]
For $t \in [0, T(\varepsilon)]$ and $v \in \gamma_t(E)$, we define $B(t)v := -A'(u(t))v(t) + F'(u(t))v$. As $A \in C^1(\gamma_t(E), L(X_1, X_0))$ and $F \in C^1(\gamma_t(E), X_0)$, we get $B \in L^p((0, T); L(\gamma_t(E), X_0))$. Therefore, we can apply Lemma 7.4 (replacing $A(t)$ in this lemma by $A(u(t))$). Note that $t \mapsto A(u(t)) \in C([0, T(\varepsilon)], L(X_1, X_0))$ holds because of $t \mapsto u(t) \in C([0, T(\varepsilon)]; \gamma_t(E))$. By assumption, $A(u(t)) \in \text{MR}(X_0)$ for every $t \in [0, T]$, and we can apply Lemma 7.4. This yields
\[ D_w H(1, u) \in L_{\text{Isom}}(E_T(\varepsilon), \mathbb{E}_T(\varepsilon) \times \gamma_t(E)). \]
Now the implicit function theorem, Theorem 7.6, tells us that there exists a $\delta > 0$ and a $C^k$-function $\psi : (1-\delta, 1+\delta) \rightarrow E_T(\varepsilon)$ with $H(\lambda, \psi(\lambda)) = 0 \ (\lambda \in (1-\delta, 1+\delta))$ and $\psi(1) = u$.

By definition of $H$ and the uniqueness of the solution, we obtain $\psi(\lambda) = u_\lambda$, i.e., $\lambda \mapsto u_\lambda \in C^k((1-\delta, 1+\delta), E_T(\varepsilon))$. Because of $E_T(\varepsilon) \subset C([0, T(\varepsilon)], \gamma_t(E)$, we obtain $\lambda \mapsto u_\lambda(t) = u(\lambda t) \in C^k((1-\delta, 1+\delta), \gamma_t(E)$. But this means $u \in C^k((0, T(\varepsilon)), \gamma_t(E)$.

Now we use $\frac{d}{dt}u_\lambda(t)|_{\lambda=1} = \partial_t u(t) \ (t \in (0, T(\varepsilon)))$. As $\psi \in C^k((1-\delta, 1+\delta), E_T(\varepsilon)$, we get $t \mapsto t\partial_t u(t) \in E_T(\varepsilon)$. An iteration shows that $t \mapsto t^k\partial_t^k u(t) \in E_T(\varepsilon)$, and therefore
\[ u \in W^{k+1}_p((-\delta, T(\varepsilon)); X_0) \cap W^k_p((-\delta, T(\varepsilon)); X_1) \]
for every $\delta > 0$ and $\varepsilon > 0$. Now we apply Sobolev’s embedding theorem which tells us that $W^k_p((\delta, T(\varepsilon))] \subset C^{k-1/p}([\delta, T(\varepsilon)])$. With this we obtain, as $\varepsilon > 0$ and $\delta > 0$ can be chosen arbitrary,

$$u \in C^{k+1-1/p}(0, T); X_0) \cap C^{k-1/p}((0, T); X_1).$$

In the case $k = \infty$, we get $u \in C^\infty((0, T); X_1)$. If $k = \omega$, then the function $\psi$ is real analytic. The above embeddings are linear and therefore real analytic, too, which yields $u \in C^\omega((0, T), X_1)$.

**Remark 7.8.** This method of proof is known as parameter trick or method of Angenent [Ang90]. Note that the two main ingredients are the implicit function theorem in Banach spaces and the fact that $D_w H(1, u)$ is an isomorphism. The latter is exactly the maximal regularity of the linearization, and it can also be seen as one of the main ideas of the maximal regularity approach to show that the implicit function theorem can be applied to the nonlinear equation.

As an example, we consider the quasilinear autonomous second order equation in $\mathbb{R}^n$

$$\partial_t u(t, x) - \text{tr} \left( a(u(t, x), \nabla u(t, x)) \nabla^2 u(t, x) \right) = f(u(t, x), \nabla u(t, x))$$

((t, x) \in (0, T) \times \mathbb{R}^n),

$$u(0, x) = u_0(x)$$

To solve the nonlinear problem, we need the following result from the linear theory, which can be shown by the methods of Section 5.

**Lemma 7.9.** Let $b \in \text{BUC}(\mathbb{R}^n; \mathbb{R}^{n \times n})$ with $b(x) \geq cI_n$ ($x \in \mathbb{R}^n$) for some constant $c > 0$. Define the operator $B$ by $D(B) := W^2_p(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$,

$$(Bu)(x) := \text{tr} \left( b(x) \nabla^2 u(x) \right) = \sum_{i,j=1}^n b_{ij}(x) \partial_i \partial_j u(x) \quad (x \in \mathbb{R}^n, u \in D(B)).$$

Then $B \in \text{MR}(L^p(\mathbb{R}^n))$.

For the nonlinear equation, we obtain the following result (see [Prü02], Theorem 5.1).

**Theorem 7.10.** Let $p \in (n+2, \infty)$ and $k \in \mathbb{N} \cup \{\infty, \omega\}$. Assume that $a \in C^k(\mathbb{R}^{n+1}, \mathbb{R}^{n \times n})$ and $f \in C^k(\mathbb{R}^{n+1}, \mathbb{R})$ with $f(0) = 0$, and assume that for all $(r, p) \in \mathbb{R} \times \mathbb{R}^n$ the matrix $a(r, p)$ is positive definite. Then equation (7-9) has for all $u_0 \in W^{2-2/p}_p(\mathbb{R}^n)$ a unique maximal solution $u \in L^p((0, T^+); W^2_p(\mathbb{R}^n)) \cap W^1_p((0, T^+); L^p(\mathbb{R}^n))$ in the existence interval $J = (0, T^+)$ with $T^+ = T^+(u_0) > 0$. Moreover,

$$u \in C^k(J; W^{2-2/p}_p(\mathbb{R}^n)) \cap C^{k+1-1/p}(J; L^p(\mathbb{R}^n)) \cap C^{k-1/p}(J; W^2_p(\mathbb{R}^n)).$$

**Proof.** For $X_0 := L^p(\mathbb{R}^n)$ and $X_1 := W^2_p(\mathbb{R}^n)$, the trace space is given by $\gamma_1 \mathbb{E}(X_0, X_1) = (X_0, X_1)_{1-1/p,p} = W^{2-2/p}_p(\mathbb{R}^n)$. An application of Sobolev’s embedding theorem yields

$$\gamma_1 \mathbb{E} = W^{2-2/p}_p(\mathbb{R}^n) \subset C^0_0(\mathbb{R}^n) := \{ u \in C^1(\mathbb{R}^n) : \lim_{|x| \to \infty} |\partial^\alpha u(x)| = 0 \ (|\alpha| \leq 1) \}. $$

Now define the mappings $A : \gamma_1 \mathbb{E} \to L(X_0, X_1)$ and $F : \gamma_1 \mathbb{E} \to X_0$ by

$$(A(v)w)(x) := \text{tr} \left( a(v(x), \nabla v(x)) \nabla^2 w(x) \right),$$

$$F(v)(x) := f(v(x), \nabla v(x))$$
for $x \in \mathbb{R}^n$, $v \in \gamma_t \mathbb{E}$, and $w \in W_p^2(\mathbb{R}^n)$.

Let $v \in \gamma_t \mathbb{E}$. Because of $v \in C^1_0(\mathbb{R}^n)$, the set $\{(v(x), \nabla v(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ is bounded. As $a$ is continuous by assumption, we see that

$$b_v := a(v(\cdot), \nabla v(\cdot)) \in \text{BUC}(\mathbb{R}^n)$$

and $b_v(x) \geq c_v I_n$ $(x \in \mathbb{R}^n)$ with $c_v > 0$. By Lemma 7.9, we obtain $A(v) \in \text{MR}(X_0)$ for all $v \in \gamma_t \mathbb{E}$.

To show that assumptions (A1) and (A2) are satisfied, we use the fact that $a$ is a $C^1$-function and therefore Lipschitz on bounded sets. Therefore, we get for all $v, \tau \in \gamma_t \mathbb{E}$ and $w \in X_1$ with $\|v\|_{\gamma_t \mathbb{E}} \leq R$, $\|w\|_{\gamma_t \mathbb{E}} \leq R$ the inequality

$$\|A(v)w - A(\tau)w\|_{L^p(\mathbb{R}^n)} = \|\text{tr} (a(v, \nabla v)w - a(\tau, \nabla \tau)w)\|_{L^p(\mathbb{R}^n)}$$

$$\leq C \|a(v, \nabla v) - a(\tau, \nabla \tau)\|_{L^p(\mathbb{R}^n)} \|\nabla^2 w\|_{L^p(\mathbb{R}^n)}$$

$$\leq CL(R) \|v - \tau\|_{C^1(\mathbb{R}^n)} \|w\|_{X_1}$$

This shows assumption (A1) and, in particular, the continuity of $A : \gamma_t \mathbb{E} \rightarrow L(X_0, X_1)$. Similarly, assumption (A2) can be shown. Here, we have to show the continuity of $F : \gamma_t \mathbb{E} \rightarrow X_0$. For this we use the fact that $F$ is a variant of the so-called Nemyckii operators, i.e.,

$$F : W_p^{2-2/p}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n), \quad F(v) := f(v(\cdot), \nabla v(\cdot)) \quad (v \in W_p^{2-2/p}(\mathbb{R}^n)).$$

For this, we also use $f(0) = 0$. By known results on the Nemyckii operator, one obtains $A \in C^k(\gamma_t \mathbb{E}, L(X_1, X_0))$ and $F \in C^k(\gamma_t \mathbb{E}, X_0)$. Therefore, all assumptions of Theorem 7.7 are satisfied, and we obtain higher regularity for the solution $u$ as stated in the theorem. □

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