Polymeric Quantization and Black Hole Thermodynamics

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Abstract

Polymer quantization is a non-standard representation of the quantum mechanics that inspired by loop quantum gravity. To study the associated statistical mechanics, one needs to find microstates’ energies which are eigenvalues of the Hamiltonian operator in the polymer framework. But, this is not an easy task at all since the Hamiltonian takes a nonlinear form in polymer picture. In this paper, we introduce a semiclassical method in which it is not necessary to solve the eigenvalue problem. Instead, we work with the classical Hamiltonian function and the deformed density of states in the polymeric phase space. Implementing this method, we obtain the canonical partition function for the polymerized systems and we show that our results are in good agreement with those arising from full quantum considerations. Using the partition function, we study the thermodynamics of quantum Schwarzschild black hole and we obtain corrections to the Bekenstein-Hawking entropy due to loop quantum gravity effects.

PACS numbers: 04.60.Pp, 04.60.Bc, 04.70.Dy

1 Introduction

Our current understanding of gravity is based on the Einstein general theory of relativity. The essential feature of General Relativity (GR) is that gravity forms the geometry and geometry tells the particles how to move in spacetime. In this sense, while GR seems to be a purely classical theory, in its most important applications, say, cosmology and Black Hole (BH) theories, the system under consideration essentially obeys the quantum mechanical rules. Quantization of gravity is therefore the main challenge of theoretical physics community. In the absence of a full theory of quantum gravity (QG), however, there have been a variety of approaches to the issue; among them is the canonical quantum theory of gravity first introduced by De Witt [1]. Also, string theory, Loop Quantum Gravity (LQG) and noncommutative geometry, as other approaches to QG proposal, have revealed some unknown aspects of ultimate QG scenario [2,3]. One common feature of all approaches to QG proposal is the existence of a minimal measurable length (preferably of the order of the Planck length) [4]. In recent years, some phenomenological aspects of effective QG candidates have been introduced in quantum mechanics through deformation of algebraic structure of ordinary quantum mechanics. For instance the generalized uncertainty principle [5], and noncommutative geometry [6] are the most well-known deformations that impose the ultraviolet and infrared cutoffs for the physical systems [7]. Among the models that deal with the idea of the existence of a minimal measurable length scale, the so-called polymer quantization of a dynamical system uses a method much similar to the effective

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models of LQG [8, 9]. The minimal length scale, here known as the polymer length scale, is encoded in the Hamiltonian of the system and thus instead of a deformed algebraic structure coming from the noncommutative phase space variables, deformation shows itself in the Hamiltonian function. With this motivations, polymer quantization has attracted some attentions in recent years in the fields dealing with the quantum gravitational effects in a physical system. Specially, in the spirit of quantum features of cosmological models, the polymeric effects provide a modified Friedmann equation which is very similar to the one coming from loop quantization of the model [9, 10, 11]. Also, through the lines sketched by the polymer quantum mechanics, it is expected that applying the polymer quantization approach to the quantum gravitational systems, including thermodynamics of BHs and other statistical systems, leads one to some results which may be comparable with those obtained from more fundamental LQG and string theories (see [12, 13, 14, 15, 16, 17] for some investigations of BH thermodynamics in the polymer picture).

Indeed, formulation of BH thermodynamics is based on quantum field theoretical considerations in the curved spacetime leading to the well-known result of BH thermal emission through Hawking process [18]. The discovery of BH evaporation makes this system to be a thermodynamical system and consequently one can extract the thermodynamical quantities from the statistical mechanics considerations. This issue was first studied by Bekenstein that showed the relation between the horizon area of BH and the number of accessible microstates [19]. To implement the standard statistical mechanics methods, one needs the Hamiltonian formulation of the BH that is not generally formulated even in the classical framework of GR. Nevertheless, it is possible to define Hamiltonian for the some particular cases, such as Schwarzschild BH [20, 21]. Through this procedure, quantization of the Schwarzschild mass is realized which coincides with Bekenstein proposal [19]. Now, polymer quantization provides a new framework to extract all these results satisfactorily with the advantage that it contains quantum gravity effects from very beginning. Nevertheless, some technical difficulties arise due to the nonlinear form of the polymeric Hamiltonian [9, 22]. In this paper, we formulate a semiclassical statistical mechanics in polymer framework and we show that our results coincide with those obtained from full quantum considerations in the limit of high temperature. This method is applicable to study the thermodynamics of the quantum BHs. The advantage of this semiclassical procedure is that it is not necessary to solve the Hamiltonian eigenvalue problem. Implementing this method, we study the thermodynamics of the quantum Schwarzschild BH in polymer framework and we find corrections to the Hawking temperature and entropy.

2 Polymer representation of quantum mechanics

Almost all candidates for QG predict that the spacetime, when it is considered in the high energy limit, has a discrete structure. This is due to the existence of natural cutoffs such as the minimal measurable length. In ordinary Schrödinger representation of quantum mechanics, the position and momentum operators have continuous spectrum and therefore one can work in position or momentum space representations. However, when one considers the quantum gravitational effects, these representations are no longer applicable. For instance, position space representation fails to be applicable due to fuzzy nature of spacetime manifold in the presence of a minimal measurable length [23]. In other words, existence of a minimum measurable length of the order of Planck length prevents complete resolution of spacetime points, leading naturally to generalization of the Heisenberg uncertainty principle [5, 7].

Polymer quantization provides an alternative framework to study the mathematical and physical aspects of the discrete spaces [3, 9]. In what follows we present a brief review of this issue and its semiclassical outcomes.

Suppose an abstract ket $|\nu\rangle$, labeled by a real number $\nu$, which belongs to the nonseparable Hilbert space $\mathcal{H}_{\text{poly}}$. Considering a finite collections of vectors $|\nu_i\rangle$ where $\nu_i \in \mathbb{R}$ and $i = 1, 2, ..., N$, one can
construct an appropriate state by taking a linear combination of these vectors as

\[ |\psi\rangle = \sum_{i=1}^{N} a_i |\mu_i\rangle. \]  

(1)

After introduction of the abstract Hilbert space of the model, we can consider the wave function in this framework. Suppose a system with a phase space with coordinates \(x\) and conjugate momenta \(p\). In momentum representation, states will be denoted by \(\psi(p) = \langle p|\psi\rangle\) in a nonseparable Hilbert space \(\mathcal{H}_{\text{poly}}\) as

\[ \psi_p(p) = \langle p|\nu\rangle = e^{ip\hat{p}/\hbar}. \]

One can define one-parameter family of exponential shift operator \(\hat{V} = e^{i\lambda p/\hbar}\) in the Hilbert space \(\mathcal{H}_{\text{poly}}\) which acts on the states as

\[ \hat{V}(\lambda). \psi_p(p) = e^{i\lambda p/\hbar} e^{ip\hat{p}/\hbar} = e^{i(\nu+\lambda)p/\hbar} = \psi_{p+\lambda}(p). \]

(2)

The next step is to consider the dynamics of the system in the polymer representation which needs the Hamiltonian of the corresponding underlying quantum theory. We start with the classical Hamiltonian

\[ H = \frac{p^2}{2m} + U(x), \]

(3)

which is a smooth function of the phase space variables \(x\) and \(p\). The standard method of quantization is to replace the classical phase space variables with corresponding operators. In polymer representation, however, this argument fails to be applicable, because the corresponding momentum operator does not exist. The main task now is to define momentum operator and the square of the momentum in polymer framework. The standard prescription is to define the lattice \(\gamma_{\mu_0}\) on the configuration space as \(\gamma_{\mu_0} = \{ x \in \mathbb{R} | x = n\mu_0, \forall n \in \mathbb{Z}\}\), which ensures the discreteness of the position \(x\) with discreteness parameter \(\mu_0\). The states in Hilbert space \(\mathcal{H}_{\gamma_{\mu_0}}\) are given by

\[ |\psi\rangle = \sum_n b_n |\mu_n\rangle, \]

(4)

where coefficients \(b_n\) satisfy \(\sum_n |b_n|^2 < \infty\). Since in QG scales the space becomes discrete, a natural question then arises: Is it possible to recover the ordinary Schrödinger representation in low energy regime? This problem, known as the continuum limit of the theory, is studied in [9, 10, 24] and it is shown that the theory should be formulated in such a way that the standard Schrödinger representation is recovered in the limit of small values of the discreteness parameter \(\mu_0 \to 0\). Furthermore, note that since momentum is the generator of translation we should have a momentum operator that generates discrete displacement according to the relation \(4\), such that the results remain in the lattice \(\gamma_{\mu_0}\). How can we apply this setup to form the momentum operator? This may be done by the exponential shift operator \(\hat{V}(\mu_0)\) as

\[ \hat{V}(\mu_0) |\mu_n\rangle = e^{i\mu_0 p/\hbar} e^{i\mu_n p/\hbar} = e^{i(\mu_0 + \mu_n)p/\hbar} = |\mu_n + \mu_0\rangle = |\mu_{n+1}\rangle. \]

(5)

So, the momentum operator can be defined by means of the shift operator as

\[ \hat{p}_{\mu_0} |\mu_n\rangle = \frac{\hbar}{2i\mu_0} \left[ \hat{V}(\mu_0) - \hat{V}(-\mu_0) \right] |\mu_n\rangle = \frac{i\hbar}{2\mu_0} \left( |\mu_{n+1}\rangle - |\mu_{n-1}\rangle \right). \]

(6)

To consider the kinetic term in the Hamiltonian \(3\), one needs the squared momentum which can be obtained as

\[ \hat{p}_{\mu_0}^2 |\mu_n\rangle = \frac{\hbar^2}{\mu_0^2} \left[ 2 - \hat{V}(\mu_0) - \hat{V}(-\mu_0) \right] |\mu_n\rangle = \frac{\hbar^2}{\mu_0^2} \left( 2|\mu_n\rangle - |\mu_{n+1}\rangle - |\mu_{n-1}\rangle \right). \]

(7)
The quantum polymeric Hamiltonian operator can be deduced from the classical Hamiltonian (3) through the relation (7) as

\[ \hat{\mathcal{H}}_{0} = \frac{\hbar}{2m\mu_0^2} \left[ 2 - \hat{V}(\mu_0) - \hat{V}(-\mu_0) \right] + \hat{U}(x). \]  

(8)

Now one can replace the shift operator in the relations (6) and (7) with the exponential function through the relation (2) as \( \hat{V}(\mu_0) \rightarrow e^{i\mu_0 p/\hbar} \), in order to get an approximation for the momentum and its square [9, 10]

\[ \hat{p} \rightarrow e^{i\mu p} - e^{-i\mu p} = \frac{\sin(\mu p)}{\mu}, \]  

(9)

\[ \hat{p}^2 \rightarrow \frac{1}{\mu^2} \left( 2 - e^{i\mu p} - e^{-i\mu p} \right) = \frac{2}{\mu^2} \left[ 1 - \cos(\mu p) \right], \]  

(10)

where \( \mu = \mu_0/\hbar \) and the approximations are reliable for \( \mu p \ll 1 \) to recover the standard continuum theory in the limit of small discreteness parameter [9]. By substituting from the relation (10) into the relation (8), the polymeric quantum Hamiltonian can be approximated as

\[ H_\mu = \frac{1}{m\mu^2} \left[ 1 - \cos(\mu p) \right] + \hat{U}(x). \]  

(11)

The replacements (9) and (10) suggest the idea that a classical polymeric theory may be obtained via the process that is dubbed usually as Polymerization in literature [25]

\[ \mathcal{P}[x] = x, \quad \mathcal{P}[p] = \frac{\sin(\mu p)}{\mu}, \quad \mathcal{P}[p^2] = \frac{2}{\mu^2} \left[ 1 - \cos(\mu p) \right], \]  

(12)

where now \( (x, p) \) are the classical phase space variables. Clearly, the momenta are bounded as \( -\frac{\pi}{\mu} < p < \frac{\pi}{\mu} \) in the polymer framework. Applying the transformations (12) to the classical Hamiltonian (3) gives

\[ H_\mu = \frac{1}{m\mu^2} \left[ 1 - \cos(\mu p) \right] + U(x), \]  

(13)

which coincides with Hamiltonian (11) in the semiclassical regime. So, a one-parameter \( \mu \)-dependent classical theory can be obtained from the polymerization process without any direct reference to the quantum ones. Indeed, in this approach instead of first finding the quantum solutions in the polymer quantization framework based on the Hamiltonian operator (8), one modifies the classical Hamiltonian (3) according to the polymerization process (12) to get an effective polymer-deformed Hamiltonian (13) and then deals with classical dynamics of the system with this deformed Hamiltonian. In the corresponding classical system the discreteness parameter \( \mu \) plays an important role since its existence supports the idea that the \( \mu \)-correction to the classical theory is a signal from QG. In this sense, we emphasize that the classical Hamiltonian and also the classical equations of motion should be recovered in the continuum limit \( \mu \rightarrow 0 \). In summary, to study the quantum effects in a given classical system there is a simple strategy: one constructs a suitable one parameter family of an effective Hamiltonian \( H_\mu \), with correct continuum limit based on which the resulting classical theory shows the quantum effects, with no direct reference to the corresponding quantum theory. This strategy seems to work well specially on some cosmological models that are treated, for example in [17] and [25], and the resulting equations of motion are in good agreement with the one coming from the quantization of the system by LQG methods. In the following, we will use this strategy to study the thermodynamics of the physical systems in polymer framework.

### 3 Polymeric statistical mechanics

Quantum mechanics determines the microstates of the physical systems, and all thermodynamical quantities attributed to a given system can be extracted from its statistical properties through the
quantum partition function

\[ Z = \sum_\varepsilon e^{-\beta \varepsilon}, \]  

where \( \varepsilon \) are the energy eigenvalues of the microstates. In our case, these eigenvalues of the polymer-deformed Hamiltonian operator \( \hat{H}_0 \) are given by the relation

\[ \hat{H}_0 |\psi\rangle = \varepsilon |\psi\rangle, \]  

where the corresponding physical states are defined by the relation \( |\psi\rangle \) on the polymeric Hilbert space \( H_{\text{poly}} \). In fact, solving the eigenvalue problem \( (15) \) is not an easy task at all (see for example \[9, 22\]). Nevertheless, inspired by the idea of polymerization we can seek for an alternative approach to consider the quantum effects, that is, to work with the classical Hamiltonian \( (13) \) and associated density of states in the semiclassical regime. The advantage of this method is that it does not need to solve the eigenvalue problem \( (15) \) in the polymer picture. Furthermore, we introduce the transformation in the polymeric phase space which, as we will see, is more understandable from the statistical point of view.

Consider the noncanonical transformation

\[ (x, p) \rightarrow \left(X = x, P = \frac{2}{\mu} \sin \left(\frac{\mu p}{2}\right)\right), \]  

in the polymeric phase space with Hamiltonian \( (13) \). In the transformation \( (16) \), \( (x, p) \) are the usual canonical phase space variables and \( (X, P) \) are the new noncanonical variables that we call them polymer variables. From the transformation \( (16) \), it is clear that the polymer momentum should be bounded as \( -\frac{2}{\mu} < P < \frac{2}{\mu} \), since the canonical momentum is bounded as \( -\frac{\pi}{\mu} < p < \frac{\pi}{\mu} \) in polymer phase space. More precisely, the topology of the momentum part of the polymer phase space is \( S^1 \) rather than usual \( \mathbb{R} \) and there is always a maximal momentum for the system under consideration in this framework \[9\]. This result of the polymeric systems is comparable to the predictions of other theories such as the ones studied in \[26\] and \[27\]. The reason for introducing the transformation \( (16) \) becomes clear if one notes that the effective Hamiltonian \( (13) \) gets the standard form

\[ H_\mu = \frac{P^2}{2m} + U(X), \]  

where one adopts the polymer variables \( (X, P) \) and consequently all the polymeric effects are summarized in the density of states. To obtain the corresponding density of states, one needs to the Jacobian of the transformation \( (16) \) which is given by \[28\]

\[ J = \frac{\partial(X, P)}{\partial(x, p)} = \cos(\mu p/2) = \sqrt{1 - (\mu P/2)^2}. \]  

Having the transformation’s Jacobian, the polymer-deformed density of states can be obtained as

\[ \frac{1}{\hbar} \int_{|P| < \frac{2}{\mu}} dx dp \rightarrow \frac{1}{\hbar} \int \frac{dX dP}{J} = \frac{1}{\hbar} \int_{|P| < \frac{2}{\mu}} \sqrt{1 - (\mu P/2)^2} dX dP, \]  

where \( \hbar = 2\pi\hbar \) is the Planck constant. As we said before, from the relations \( (17) \) and \( (19) \) it is clear that all the polymeric effects are summarized in the density of states when one implements the polymer variables \( (X, P) \). The density of states determines the number of microstates for the statistical systems and consequently the polymeric effects change only the number of accessible microstates for the statistical system. Also, it is important to note that the measure \( (19) \) is nonsingular since the momentum is bounded as \( |P| < \frac{2}{\mu} \). From another perspective, this can be viewed as a redefinition of the Planck constant as an effective momentum dependent quantity \( \hbar^{(\text{poly})} = \hbar \sqrt{1 - (\mu P/2)^2} \). This result is in some sense similar to the results coming from the generalized uncertainty principle and noncommutative phase space as have been obtained in \[29\].
Now one can find immediately the polymeric partition function from the polymer-modified states density \([19]\) and the associated Hamiltonian function \([17]\) as

\[
Z^{(\text{poly})} = \frac{1}{\hbar} \int \exp[-\beta U(X)] dX \times \int_{-\frac{\mu}{2}}^{\frac{\mu}{2}} \exp \left[ -\frac{\beta P^2}{2m} \right] dP.
\]  

(20)

For the one-dimensional harmonic oscillator with the potential \(U(X) = \frac{1}{2}m\omega^2 X^2\), where \(m\) and \(\omega\) denote the mass and frequency respectively, the corresponding partition function can be obtained from the relation \([20]\) as

\[
Z^{(\text{poly})} = \frac{1}{\hbar \omega \mu} \left( \frac{2\pi}{m \beta} \right)^{1/2} I_0 \left( \frac{\beta}{m \mu^2} \right) \exp \left[ -\frac{\beta}{m \mu^2} \right],
\]  

(21)

where \(I_0\) denotes the first kind of the modified Bessel functions. In the limit of the small discreteness parameter, \(\mu \to 0\), the above partition function gives

\[
Z^{(\text{poly})} = (\beta \hbar \omega)^{-1} \left( 1 + \frac{\mu^2}{8} \left( \frac{m}{\beta} \right) + O[\mu^4] \right),
\]  

(22)

where the first term is the usual semiclassical partition function for the one-dimensional harmonic oscillator and the second term is the polymeric correction.

At this step, it is interesting to compare the result \([22]\) with the corresponding result coming from full quantum considerations. In quantum picture, one has to solve the eigenvalue problem \([15]\) when the system under study is a one-dimensional harmonic oscillator. The energy eigenvalues for the one-dimensional harmonic oscillator in the limit of \(\mu \to 0\) is obtained in the Ref. \([22]\) that are given by

\[
\epsilon_n = \left( n + \frac{1}{2} \right) \hbar \omega - \left( \frac{2n^2 + 2n + 1}{32} \right) m \mu^2 \hbar^2 \omega^2.
\]  

(23)

Substituting the above result into relation \([14]\) gives the quantum partition function of the polymeric one dimensional harmonic oscillator as \([15]\)

\[
Z^{(\text{poly})} = \frac{e^{-\beta \hbar \omega/2}}{1 - e^{-\beta \hbar \omega}} \left( 1 + \frac{1}{32} m \mu^2 \hbar^2 \omega^2 \beta \left( \frac{1 + e^{-\beta \hbar \omega}}{1 - e^{-\beta \hbar \omega}} \right)^2 + O[\mu^4] \right).
\]  

(24)

In the limit of high temperature, \(\beta \to 0\), where the semiclassical and quantum statistics are going to be equivalent, relation \([24]\) exactly coincides with the relation \([22]\). It is important to note that, one should doing some tedious calculation to obtain the result \([23]\) (see \([22]\)) and then performing the summation over energy eigenvalues \([23]\) in quantum partition function \([14]\) to obtain the result \([24]\) which is not an easy task at all (see \([15]\)). However, the method we have introduced gives the same results but now in a more simpler manner. While the full quantum results preserve their importance for the other goals \([8, 9, 22]\), our method is more suitable when one concerns for the statistical considerations of the polymeric systems. In the next section, we shall see that the quantum Schwarzschild BH can be modeled with a one-dimensional harmonic oscillator and hence, the relation \([22]\) may be used to study its thermodynamical properties in the polymer quantization scheme.

4 Thermodynamics of quantum black holes

4.1 Quantization of the Schwarzschild black hole

As a canonical system, a Schwarzschild BH can be described by a single canonical pair \((m, p_m)\), where \(m\) may be identified with its mass. In Ref. \([20]\) it is shown, under some conditions, that the numerical value of the Hamiltonian of such a system is equal to its mass, that is, \(H = m\). However, there is a
canonical transformation (again introduced in [20]) from \((m, p_m)\) to a new canonical pair \((a, p_a)\) such that

\[
|p_m| = \sqrt{2ma - a^2} + m \arcsin \left(1 - \frac{a}{m}\right) + \frac{\pi}{2} m, \quad p_a = sgn(p_m) \sqrt{2ma - a^2},
\]

which in terms of the new variables the classical Hamiltonian takes the following form

\[
H = \frac{p_a^2}{2a} + \frac{1}{2} a.
\]

Therefore, according to the canonical quantization formalism, the Wheeler-De Witt (WDW) equation for the Schwarzschild BH can be written in the form \(\hat{H} \Psi = m \Psi\), which in ordinary units reads

\[
\hbar^2 \frac{G^2}{c^6} a^{-s-1} \frac{d}{da} \left(a^s \frac{d}{da} \Psi(a)\right) = \left(a - \frac{2GM}{c^2}\right) \Psi(a),
\]

in which we have identified \(p^2_a = -\hbar^2 \frac{G^2}{c^6} a^{-s} \frac{d}{da} \left(a^s \frac{d}{da} \right)\), where the parameter \(s\) represents the ambiguity in the ordering of factors \(a\) and \(p_a\) in the first term of the Hamiltonian and \(\Psi(a)\) is the BH wave function. By setting \(R_s = \frac{2GM}{c^2}\), one gets

\[
\hbar^2 \frac{G^2}{c^6} \frac{1}{a} \left(\frac{d^2}{da^2} + \frac{2}{a} \frac{d}{da}\right) \Psi(a) = (a - R_s) \Psi(a),
\]

in which we have set \(s = 2\).\(^2\) Using the following transformations

\[
\Psi(a) = \frac{1}{a} U(a) \quad \text{and} \quad y = a - R_s,
\]

and bearing in mind that the energy of excited state associated with the variable \(a\) is not positive (this can be explained by the physical argument that the total energy of the BH is included and the ADM energy is equal to zero), the WDW equation takes the form

\[
\left(-\frac{1}{2} l_p^2 E_s \frac{d^2}{dy^2} + \frac{E_p}{2 l_p^2 y^2}\right) U(y) = \frac{R_s}{4 l_p} U(y),
\]

where \(E_s = M c^2\) is the ADM energy of the Schwarzschild BH, \(l_p = \sqrt{\frac{G}{c^5}}\) and \(E_p = \sqrt{\frac{\hbar c}{G}}\) are the Planck’s length and energy respectively. Relation [30] is the same as the Schrödinger equation for a one-dimensional harmonic oscillator with mass \(m = M_p = \sqrt{\frac{\hbar c}{G}}\) and frequency \(h\omega = \sqrt{\frac{3}{2\pi}} E_p\). So, the quantized energy eigenvalues of the Schwarzschild BH are given by

\[
\frac{R_s(n)}{4 l_p} E_s = (n + \frac{1}{2}) E_p.
\]

The mass of the Schwarzschild BH is also quantized with respect to the Planck mass as

\[
M^2(n) = (2n + 1) M_p^2,
\]

in agreement with the Bekenstein’s proposal in the BHs thermodynamics [19]. We use these results in our forthcoming arguments.

\(^2\)Indeed, there are various possibilities for ordering. Since the factor-ordering parameter will not affect the semiclassical calculations in minisuperspace models, one usually chooses a special value for it in a given model.
4.2 Hawking temperature and entropy

In this section we discuss the Feynman’s path integral method to study thermodynamics of the Schwarzschild BH. In this method, one deals with a semiclassical partition function instead of the quantum one, and all of the quantum effects can be summarized in a modified potential \[30\]. By the path integral method applied to the harmonic oscillator, its potential energy and classical partition function will change as follows \[31\]

\[
U(x) = \frac{3E_p}{4\pi l_p^2} \left( x^2 + \frac{\beta l_p^2 E_p}{12} \right),
\]

and

\[
Z = \sqrt{\frac{2\pi}{3}} \exp \left[ -\frac{\beta E_p^2}{16\pi} \right].
\]

One can obtain all the thermodynamical quantities of the Schwarzschild BH through the partition function (34). The internal energy of the BH can be obtained by the standard definition as

\[
\bar{E} = -\frac{\partial \ln[Z]}{\partial \beta} = Mc^2,
\]

where we have set explicitly \(\bar{E} = Mc^2\). We note that in the canonical ensemble the temperature is an external parameter. Substituting the partition function (34) into the relation (35), gives the relation between the temperature and the mass of the BH

\[
\beta^2 - \frac{8\pi Mc^2}{E_p^2} \beta + \frac{8\pi}{E_p^2} = 0.
\]

Solving the above equation for \(\beta\), we arrive at the following expression for the BH temperature in terms of its mass

\[
\beta = \beta_H \left( 1 - \frac{1}{\beta_H Mc^2} \right),
\]

where \(\beta_H = \frac{8\pi Mc^2}{E_p^2}\) is the inverse of the Hawking temperature and we have used the fact that \(E_p \ll Mc^2\). The entropy of the BH can be obtained from the partition function by the definition

\[
\frac{S}{k} = \ln[Z] - \beta \frac{\partial \ln[Z]}{\partial \beta}.
\]

Considering the partition function (34) in this relation and substituting the Hawking temperature, the entropy of the BH becomes

\[
\frac{S}{k} = \frac{A_s}{4l_p^2} \left( 1 - \frac{1}{8\pi} \left( \frac{E_p}{Mc^2} \right)^2 \right)^2 - \frac{1}{2} \ln \left[ \frac{A_s}{4l_p^2} \left( 1 - \frac{1}{8\pi} \left( \frac{E_p}{Mc^2} \right)^2 \right)^2 \right] - \frac{1}{2} \ln[24] + 1,
\]

where \(A_s = 4\pi R_s^2\) is the horizon area of the BH. Defining the area of the BH in terms of the horizon area as \(A_{BH} = A_s \left( 1 - \frac{1}{8\pi} \left( \frac{E_p}{Mc^2} \right)^2 \right)^2\), the above relation can be rewritten in the following form

\[
\frac{S}{k} = \frac{A_{BH}}{4l_p^2} - \frac{1}{2} \ln \left[ \frac{A_{BH}}{4l_p^2} \right] + O\left( \frac{M_p}{M} \right)^2.
\]

As it is clear, the well-known logarithmic correction to the entropy appears naturally in this method. This kind of correction term also appears in the other approaches such as string theory \[32\], canonical QG \[33\], LQG \[34\], GUP and noncommutative frameworks \[35, 36, 37, 38, 39, 40, 41\].
5 Hawking temperature and entropy in the polymer framework

In the previous section we applied the Feynman’s path integral method to the semiclassical partition function in order to consider the quantum effects in the thermodynamics of the Schwarzschild BH. On the other hand, we obtained the modified semiclassical polymeric partition function for a one-dimensional harmonic oscillator. So, one can implement the partition function (22) to study the thermodynamics of the Schwarzschild BH as a harmonic oscillator as we have reviewed in the previous section. It is also important to note that our results are applicable only to Planck scale BH, since the result (22) coincides with quantum ones in the high temperature limit.

One can include the LQG effects in the path integral approach by replacing the partition function (34) with the polymeric ones (22) as

\[ Z^{(\text{poly})} = \sqrt{\frac{2\pi}{3}} \exp \left( -\frac{\beta^2 E_p^2}{16\pi^2} \right) \left( 1 + \frac{\mu^2 M_p}{8\beta} + \ldots \right), \tag{41} \]

where we have substituted \( m = M_p \) for Schwarzschild BH when we used the relation (22). Therefore, relation (41) shows the partition function of the quantum Schwarzschild BH in the polymer framework and we may investigate the thermodynamics of the BH using this relation. As usual, the definition (35) gives the relation between temperature and mass of the Schwarzschild BH in polymer framework. Substituting the polymeric partition function (41) into the relation (35), we obtain the following expression for the BH temperature in the limit of \( \mu \to 0 \),

\[ \beta^3 - \beta H \beta^2 + \frac{\beta \mu}{M c^2} \beta = -\frac{\mu^2 \beta M_p}{8M c^2}, \tag{42} \]

which upon solving this equation we find

\[ \beta = \beta_H \left( 1 - \frac{(1 + \mu^2 M_p/8)}{\beta H M c^2} \right) + O \left( \frac{E_p^2}{M c^2} \right). \tag{43} \]

It is easy to see that in the limit of \( \mu \to 0 \), the above solution reduces to the non-deformed case (37). The next step is to calculate the entropy of the BH from partition function (41) through the definition (38), which yields

\[ \frac{S^{(\text{poly})}}{k} = \frac{\beta^2 E_p^2}{16\pi} - \frac{1}{2} \ln \left( \frac{\beta^2 E_p^2}{16\pi} \right) + \ln \left( 1 + \frac{\mu^2 M_p}{8\beta} \right) + \left( 1 + \frac{8\beta}{\mu^2 M_p} \right)^{-1} - \frac{1}{2} \ln[24] + 1. \tag{44} \]

Substituting from the relation (43) into this relation we find

\[ \frac{S^{(\text{poly})}}{k} = \frac{A_s}{4\ell_p^2} \left( 1 - \frac{(1 + \mu^2 M_p/8)}{8\pi} \left( \frac{E_p}{M c^2} \right)^2 \right)^2 - \frac{1}{2} \ln \left[ \frac{A_s}{4\ell_p^2} \left( 1 - \frac{(1 + \mu^2 M_p/8)}{8\pi} \left( \frac{E_p}{M c^2} \right)^2 \right) \right] \]

\[ + \frac{\mu^2 E_p^3}{32\pi M c^4} \left( 1 - \frac{(1 + \mu^2 M_p/8)}{8\pi} \left( \frac{E_p}{M c^2} \right)^2 \right)^{-1} - \frac{1}{2} \ln[24] + 1. \tag{45} \]

Defining the polymeric area of the BH in terms of the horizon area as

\[ A_{BH}^{(\text{poly})} = A_s \left( 1 - \frac{(1 + \mu^2 M_p/8)}{8\pi} \left( \frac{E_p}{M c^2} \right)^2 \right)^2, \tag{46} \]

and notifying that \( \frac{A_{BH}^{(\text{poly})}}{A_s} < 1 \), we obtain the entropy of the quantum Schwarzschild BH in polymer picture as

\[ \frac{S^{(\text{poly})}}{k} = \frac{A_{BH}^{(\text{poly})}}{4\ell_p^2} - \frac{1}{2} \ln \left[ \frac{A_{BH}^{(\text{poly})}}{4\ell_p^2} \right] + M c^2 \left( 1 - \frac{A_{BH}^{(\text{poly})}}{A_s} \right) + O \left[ A_{BH}^{(\text{poly})} \right]^{-1}. \tag{47} \]
The above results are obtained in the high temperature limit since our semiclassical consideration coincides with the full quantum approach in this limit. As an important consequence, the semiclassical, high-temperature result (47) is applicable only for Planck scale BHs such as the quantized Schwarzschild BH that we studied previously in this paper. We see that the relation (47) is compatible with calculation of entropy in other approaches such as GUP and noncommutative geometry frameworks. Specially, the existence of logarithmic correction term is a common feature in all these approaches (see [32, 33, 34, 35, 36, 37, 38, 39, 40, 41]). The sign of this correction is also compatible with previous studies.

6 Conclusions

The polymer representation of quantum mechanics is an effective model which has been investigated in a symmetric sector of LQG. In this paper, we have developed statistical mechanics in the polymer framework. The energy eigenvalues of microstates in statistical mechanics are usually given by the Schrödinger equation in the standard Schrödinger representation of the quantum mechanics. However, solving the corresponding eigenvalue problem in the polymer representation it is not an easy task due to the nonlinear form of the Hamiltonian operator in this picture. Therefore, we introduce a semiclassical method in which one does not need to solve the eigenvalue problem, instead one works with the Hamiltonian function and the density of states in the classical phase space. Moreover, we have defined a particular transformation in the polymeric phase space which transforms the polymeric nonlinear Hamiltonian to the standard forms and consequently all the polymeric effects summarized in the density of states when one working with these new phase space variables. Implementing this method, we obtain the polymeric partition function and we have shown that our results coincide with full quantum mechanical ones in the limit of high temperature. Finally, we studied thermodynamics of quantum Schwarzschild BH in the polymer framework by using of the Feynman path integral approach in the semiclassical regime. We found the corrections to the Hawking temperature and entropy due to the LQG effects. These corrections are compatible with results obtained through other approaches such as generalized uncertainty principle or noncommutative geometry considerations.

Acknowledgement

We would like to thank an anonymous referee for very insightful comments.

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