An approach to the classification of \( p \)-brane solitons

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ABSTRACT

We give a review of some recent work on the construction and classification of \( p \)-brane solutions in maximal supergravity theories in all dimensions \( 4 \leq D \leq 11 \). These solutions include isotropic elementary and solitonic \( p \)-branes, dyonic \( p \)-branes, and multi-scalar \( p \)-branes. These latter two categories include massless strings and black holes as special cases. For all the solutions, we analyse their residual unbroken supersymmetry by means of an explicit construction of the eigenvalues of the Bogomol’nyi matrix, defined as the anticommutator of the conserved supercharges.
1 Introduction

The study of \( p \)-brane solitons in low-energy superstring theories has been the subject of detailed investigation recently [1–12]. With the growing realisation that the duality symmetries of string theories can bring about a unification of theories that were previously thought to have been distinct, and that the non-perturbative spectrum of string theory is within grasp, it becomes more and more important to achieve a thorough understanding of the solitonic states, since they will have an important rôle to play in the emerging theory. Many solitonic \( p \)-brane solutions have been found, and in this review we shall not attempt to cover all of the ground. Instead, we shall focus on an approach to the problem of classifying the solutions that we have been developing, which seems to have an advantage of simplicity. We also find a simple way of determining the supersymmetry of the solutions.

The class of low-energy effective actions that we shall discuss are those obtained by dimensional reduction of the effective action of the ten-dimensional type IIA string theory. Equivalently, and more simply, these maximal supergravity theories can be viewed as coming from eleven-dimensional supergravity. As we shall see, all the resulting lower-dimensional supergravity theories can be discussed in a unified way, and the supersymmetry of the \( p \)-brane solutions in any dimension can easily be determined by simply using the properties of the eleven-dimensional transformation rules. In particular, there is no need to examine the supersymmetry transformation rules of the lower-dimensional theories themselves, and there is no need to decompose the eleven-dimensional gamma matrices into products of lower-dimensional ones.

The bosonic sector of the Lagrangian for \( D = 11 \) supergravity is

\[
\mathcal{L} = eR - \frac{1}{48} e \hat{A}_3^2 + \frac{1}{8} \hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3,
\]

where \( \hat{A}_3 \) is the 3-form potential for the 4-form fields strength \( \hat{F}_4 \). Upon Kaluza-Klein dimensional reduction to \( D \) dimensions, this yields the following Lagrangian:

\[
\mathcal{L} = eR - \frac{1}{2} e (\partial \tilde{\phi})^2 - \frac{1}{3} e e \tilde{\phi} F_4^2 - \frac{1}{2} e \sum_i e \tilde{\phi} (F_3^i)^2 - \frac{1}{4} e \sum_{i<j} e \tilde{\phi} (F_2^{ij})^2
\]

\[
- \frac{1}{4} e \sum_i e \tilde{\phi} (F_2^i)^2 - \frac{1}{2} e \sum_{i<j<k} e \tilde{\phi} (F_1^{ijk})^2 - \frac{1}{4} e \sum_{i<j} e \tilde{\phi} (F_1^{ij})^2 + \mathcal{L}_{FFA},
\]

(1.1)

where \( F_4, F_3^i, F_2^i \) and \( F_1^{ijk} \) are the 4-form, 3-forms, 2-forms and 1-forms coming from the dimensional reduction of \( \hat{F}_4 \) in \( D = 11 \); \( F_2^i \) are the 2-forms coming from the dimensional reduction of the vielbein, and \( F_1^{ij} \) are the 1-forms coming from the dimensional reduction of these 2-forms. The quantity \( \tilde{\phi} \) denotes an \((11-D)\)-component vector of scalar fields, which we shall loosely refer to as dilatonic scalars that arise from the dimensional reduction of the elfbein. These scalars appear undifferentiated, via the exponential prefactors for the antisymmetric-tensor kinetic terms, and they should be distinguished from the remaining spin-0 quantities in \( D \) dimensions, namely the 0-form
potentials $A^{ijk}_0$ and $A^{ij}_{0}$. These fields have constant shift symmetries, under which the action is invariant, and are thus properly thought of as 0-form potentials rather than true scalars. In particular, their 1-form field strengths can adopt topologically non-trivial configurations, corresponding to magnetically-charged sources for $p$-brane solitons.

There are two subtleties that need to be addressed with regard to the $p$-brane solutions following from (1.1). Firstly, the dimensional reduction of the $\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3$ term in $D = 11$ gives rise to a term in $D$ dimensions, which we denote by $L_{FFA}$. This gives contributions to the equations of motion which, in general, will be non-zero. The solutions that we wish to discuss, in common with most of those in the literature, are ones for which this term can be ignored. Thus we must check that our solutions are such that the extra terms in the equations of motion following from $L_{FFA}$ vanish.

The second complication is that the various field strengths occurring in (1.1) are not, in general, simply given by the exterior derivatives of potentials. There are extra terms, which we refer to loosely as “Chern-Simons terms,” coming from the process of dimensional reduction. Again, the solutions of interest will be ones where these extra terms vanish, and so again, this means that certain constraints must be satisfied in order for this to be true. Both of these issues are discussed in detail in [12] and so we shall not consider them further here.

The only aspect of the bosonic Lagrangians (1.1) that remains unexplained is the constant vectors $\vec{a}_{i\cdots j}$ and $\vec{b}_{i\cdots j}$ appearing in the exponential prefactors of the kinetic terms for the antisymmetric tensors. As was shown in [12], these “dilatonic vectors” can be expressed as follows:

\begin{align*}
F_{MNPQ} & \quad \text{vielbein} \\
4 - \text{form} : & \quad \vec{a} = -\vec{g} , \\
3 - \text{forms} : & \quad \vec{a}_i = \vec{f}_i - \vec{g} , \\
2 - \text{forms} : & \quad \vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g} , \quad \vec{b}_i = -\vec{f}_i \\
1 - \text{forms} : & \quad \vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g} , \quad \vec{b}_{ij} = -\vec{f}_i + \vec{f}_j ,
\end{align*}

(1.2)

where the vectors $\vec{g}$ and $\vec{f}_i$ have $(11 - D)$ components in $D$ dimensions, and are given by

\begin{align*}
\vec{g} & = 3(s_1, s_2, \ldots, s_{11-D}) , \\
\vec{f}_i & = \left( 0, 0, \ldots, 0, (10 - i)s_i, s_{i+1}, s_{i+2}, \ldots, s_{11-D} \right) ,
\end{align*}

(1.3)

where $s_i = \sqrt{2/((10 - i)(9 - i))}$. It is easy to see that they satisfy

\begin{align*}
\vec{g} \cdot \vec{g} & = \frac{2(11-D)}{D-2} , \\
\vec{g} \cdot \vec{f}_i & = \frac{6}{D-2} , \\
\vec{f}_i \cdot \vec{f}_j & = 2\delta_{ij} + \frac{2}{D-2} .
\end{align*}

(1.4)

Note that the definitions in (1.2) are given for $i < j < k$, and that the vectors $\vec{a}_{ij}$ and $\vec{a}_{ijk}$ are antisymmetric in their indices. The 1-forms $F^{(j)}_{\tilde{M}i}$ and hence the vectors $b_{ij}$ are only defined for $i < j$,
but it is sometimes convenient to regard them as being antisymmetric too, by defining $\vec{b}_{ij} = -\vec{b}_{ji}$ for $i > j$. Eqn. (1.3), together with (1.4), contains all the necessary information about the dilaton vectors in $D$-dimensional maximal supergravity.

In the absence of a $p$-brane, the equations of motion following from (1.1) admit a $D$-dimensional Minkowski vacuum, with $SO(1, D-1)$ Lorentz symmetry. The $p$-brane solutions we shall consider break this to $SO(1, d-1) \times SO(D-d)$, where $d = p + 1$ is the dimension of the world volume of the $p$-brane. The metric tensor, compatible with this residual symmetry, is given by

$$ds^2 = e^{2A} dx^\mu dx^n \eta_{\mu n} + e^{2B} dy^m dy^n,$$  \hspace{1cm} (1.5)

where $A$ and $B$ are functions only of $r = \sqrt{y^m y^m}$. The coordinates $x^\mu$ lie in the $d$-dimensional world volume, and $y^m$ lie in the $(D-d)$-dimensional transverse space.

The solutions that we shall consider involve a subset of the dilatonic scalars $\vec{\phi}$, and a subset of the antisymmetric tensor field strengths $F$ and $\mathcal{F}$. The rank $n$ of all the participating fields strengths in a given solution are all the same, although in dimensions $D \leq 7$ certain field strengths of higher rank will be dualised to field strengths of lower rank. Each participating field strength can have non-vanishing components given by either

$$F_{m\mu_1...\mu_{n-1}} = \epsilon_{\mu_1...\mu_{n-1}} (e^C)' \frac{y^m}{r} \quad \text{or} \quad F_{m_1...m_n} = \lambda \epsilon_{m_1...m_n p} \frac{y^p}{r^{n+1}},$$  \hspace{1cm} (1.6)

where a prime denotes a derivative with respect to $r$. The first case gives rise to an elementary $(d-1)$-brane with $d = n - 1$, and the second gives rise to a solitonic $(d-1)$-brane with $d = D - n - 1$. These two field configurations have electric charge $u = \frac{1}{4\omega_{D-n}} \int_{\partial \Sigma} *F$ or magnetic charge $v = \frac{1}{4\omega_n} \int_{\partial \Sigma} \mathcal{F}$ respectively, where $\partial \Sigma$ is the boundary $(D-d-1)$-sphere of the transverse space. The first case corresponds to an elementary, or fundamental, $p$-brane solution, and the second case corresponds to a solitonic $p$-brane solution. In a case where the degree $n$ of the field strength is such that $D = 2n$, the possibility arises that field strengths might have both electric and magnetic contributions, involving both of the ansätze in (1.6) simultaneously. In such cases, the configuration describes a dyonic $p$-brane. It is useful to distinguish two different kinds of dyonic configuration.

In a dyonic solution of the first type, each participating field strength has only electric or magnetic non-vanishing components, but some of the field strengths are electric while others are magnetic. In a dyonic solution of the second type, each participating field strength has both electric and magnetic non-vanishing components. We shall encounter both kinds of dyonic solution later.

In order to determine the supersymmetry properties of the various $p$-brane solutions, it suffices to study the transformation laws of $D = 11$ supergravity. In particular, from the commutator of the conserved supercharges $Q_\epsilon = \int_{\partial \Sigma} \bar{\epsilon} \Gamma^{ABC} \psi_C d\Sigma_{AB}$, we may read off the $32 \times 32$ Bogomol’nyi matrix

\begin{align*}
&32 \times 32
\end{align*}
\( \mathcal{M} \), defined by \([Q_{e_1}, Q_{e_2}] = e_1^\dagger \mathcal{M} e_2 \), whose zero eigenvalues correspond to unbroken components of \( D = 11 \) supersymmetry. The expression for \( \mathcal{M} \) for maximal supergravity in an arbitrary dimension \( D \) then follows by dimensional reduction of the expression in \( D = 11 \). A straightforward calculation shows that it is given by

\[
\mathcal{M} = m \mathbf{1} + u \Gamma_{012} + u_i \Gamma_{0i1} + \frac{1}{2} u_{ij} \Gamma_{0ij} + p_i \Gamma_{0i} + v \Gamma_{i246} + v_i \Gamma_{i246} + \frac{1}{2} v_{ijk} \Gamma_{12ij} + q_i \Gamma_{i3} + \frac{1}{2} q_{ij} \Gamma_{12ij} .
\] (1.7)

The indices 0, 1, \ldots run over the dimension of the \( p \)-brane worldvolume, \( \hat{1}, \hat{2}, \ldots \) run over the transverse space of the \( y^m \) coordinates, and \( i, j, \ldots \) run over the dimensions that were compactified in the Kaluza-Klein reduction from 11 to \( D \) dimensions. The quantities \( u, u_i, u_{ij} \) and \( p_i \) are the electric Page charges associated with the field strengths \( F_4, F_3^i, F_2^{ij} \) and \( \mathcal{F}_2^i \) respectively, while \( v, v_i, v_{ijk}, q_i \) and \( q_{ij} \) are the magnetic Page charges associated with \( F_4, F_3^i, F_2^{ij}, F_1^{ijk}, \mathcal{F}_2^i \) and \( \mathcal{F}_1^{ij} \) respectively. The quantity \( m \) is the mass per unit \( p \)-volume for the solution, given by \( \frac{1}{2}(A' - B')e^{-B\tilde{r}^{\tilde{d}+1}} \) in the limit \( r \to \infty \), where \( \tilde{d} \equiv D - d - 2 \).

Once the mass per unit \( p \)-volume and the Page charges have been determined for a given \( p \)-brane solution, it becomes a straightforward algebraic exercise to calculate the eigenvalues of the Bogomol’nyi matrix, given by (1.7). The fraction of \( D = 11 \) supersymmetry that is preserved by the solution is then equal to \( k/32 \), where \( k \) is the number of zero eigenvalues of the 32 \( \times \) 32 Bogomol’nyi matrix.

2 Single-scalar solutions

In the single-scalar \( p \)-brane solutions, the Lagrangian (1.1) is consistently truncated to the simple form

\[
\mathcal{L} = eR - \frac{1}{2} e (\partial \phi)^2 - \frac{1}{2n!} e^{a\phi} F_n^2 ,
\] (2.1)

where the scalar field \( \phi \) is some linear combination of the dilatonic scalars \( \bar{\phi} \) of the \( D \)-dimensional theory, and \( F_n \) is a single canonically-normalised \( n \)-index field strength, to which all of the \( N \) original field strengths that participate in the solution are proportional. The constant \( a \) appearing in the exponential prefactor can conveniently be parameterised as

\[
a^2 = \Delta - \frac{2d\tilde{d}}{D - 2} ,
\] (2.2)

where \( \tilde{d} \equiv D - d - 2 \) and \( d\tilde{d} = (n - 1)(D - n - 1) \). The quantity \( \Delta \), unlike \( a \) itself, is preserved under Kaluza-Klein dimensional reduction.
If we write $\vec{\phi} = \vec{n}\phi + \vec{\phi}_\perp$, where $\vec{n}$ is a constant unit vector and $\vec{n} \cdot \vec{\phi}_\perp = 0$, then it is easy to see that the conditions that must be satisfied in order for the truncation to (2.1) to be consistent with the equations of motion for $\vec{\phi}_\perp$ are that

$$\sum_\alpha \vec{a}_\alpha F^2_\alpha = a\vec{n} \sum_\alpha F^2_\alpha , \text{ and hence } \sum_\alpha M_{\alpha\beta} F^2_\alpha = a^2 \sum_\alpha F^2_\alpha ,$$

where $M_{\alpha\beta} \equiv \vec{a}_\alpha \cdot \vec{a}_\beta$. If the matrix $M_{\alpha\beta}$ is invertible, we therefore have

$$F^2_\beta = a^2 \sum_\alpha (M^{-1})_{\alpha\beta} \sum_\gamma F^2_\gamma , \quad a^2 = \left( \sum_\alpha (M^{-1})_{\alpha\beta} \right)^{-1} ,$$

which gives rise to a p-brane solution with $\Delta = \left( \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \right)^{-1} + 2\tilde{d}/(D-2)$. In this case, we can easily see that $a \neq 0$, and hence the unit vector $\vec{n}$ can be read off from eqn (2.3). If $M_{\alpha\beta}$ is non-invertible, then it is clear from (2.3) that one solution is when $F^2_\alpha$ is a zero eigenvector of $M_{\alpha\beta}$, with the constant $a$ being zero. It turns out that this is the only solution in the singular case that does not simply reduce to an already-considered non-singular case with a smaller number $N$ of participating field strengths [12]. It is also clear that if the number of participating field strengths exceeds the dimension $(11 - D)$ of the dilaton vectors, then the associated matrix $M_{\alpha\beta}$ will be singular, and in fact it turns out that in all such cases, there is no new solution [12]. Thus in any dimension $D$, it follows that the number $N$ of participating field strengths must always satisfy $N \leq 11 - D$.

Having reduced the Lagrangian (1.1) to (2.1) by the above procedure, it is now a simple matter to obtain solutions for the equations of motion that follow from (2.1). An important point is that one can reduce the second-order equations of motion for $A, B$ and $\phi$ to first-order equations, by making an ansatz where they are all proportional, and in which the exponential factor in these variables, coming from the $F^2$ source term on the right-hand sides of their equations of motion, is required to be proportional to $A'$ [9]. In the case of solutions where some supersymmetry remains unbroken, these requirements are dictated by the supersymmetry transformation rules. It also provides a way to obtain solutions of the same general form, even in cases where there is no residual unbroken supersymmetry. The solution is

$$ds^2 = \left( 1 + \frac{k}{r^d} \right)^{-\frac{4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + \left( 1 + \frac{k}{r^d} \right)^{\frac{4d}{\Delta}} dy^m dy^m ,$$

$$e^\phi = \left( 1 + \frac{k}{r^d} \right)^{\frac{2a}{c\Delta}} ,$$

where $\epsilon = 1$ and $-1$ for the elementary and the solitonic solutions respectively, and $k = -\sqrt{\Delta} \lambda / (2\tilde{d})$. 

5
In the elementary case, the function $C$ satisfies the equation

$$e^C = \frac{2}{\sqrt{\Delta}} \left( 1 + \frac{k}{r^d} \right)^{-1}. \quad (2.6)$$

The mass of the solution is given by $\lambda/(2\sqrt{\Delta})$. Note that the dual of the solution for the field strength in the elementary case is identical to the field strength of the solitonic case, and vice versa. For this reason, we shall only consider solutions for field strengths with $n \leq D/2$.

In order to enumerate all the single-scalar $p$-brane solutions of this type, one simply has to consider, for each dimension $D$ and each degree $n$ for the field strengths, all possible choices of the associated $N \leq 11 - D$ dilaton vectors, and then calculate the values of $a$, and the corresponding ratios of participating field strengths, using the above equations. This is easily done for $n = 4$ (where there is always only one field strength) and $n = 3$ (where the number of field strengths is small). For $n = 2$ and $n = 1$, where the numbers of field strengths grow significantly with decreasing dimension $D$, the enumeration is most conveniently carried out by computer. Substituting the results for the field strengths into (1.7), it is then straightforward, for each solution, to determine the fraction of the original supersymmetry that is preserved. Most of the solutions turn out to break all of the supersymmetry, but in certain cases, some of the supersymmetry survives. Details may be found in [12]; here, we summarise the results:

4-Form solutions

Since there is only one 4-form field strength in any dimension, there is a unique solution for each dimension $D$, in which the scalar field $\phi$ appearing in (2.1) is taken to be the canonically-normalised scalar proportional to the entire exponent of the prefactor for the 4-form’s kinetic term. The value of $a$ for this solution corresponds to $\Delta = 4$ for all $D$. The eigenvalues of the Bogomol’nyi matrix (1.7) turn out to be $\mu = m\{0_{16}, 2_{16}\}$, where the subscripts denote the degeneracies of each eigenvalue. Thus these solutions preserve $\frac{1}{2}$ of the $D = 11$ supersymmetry.

3-Form solutions

From (1.2) and (2.4), it is easy to see that there exist solutions involving $N$ participating field strengths, with values of $\Delta$ given by $\Delta = 2 + 2/N$, where, as always, $N \leq 11 - D$. Substituting the solutions for the ratios of the field strengths, and hence the ratios of the Page charges, into (1.7), one finds [12] that all the supersymmetry is broken except when $N = 1$, and hence $\Delta = 4$, and the eigenvalues are the same as for the case of 4-form solutions, namely $\mu = m\{0_{16}, 2_{16}\}$. In this supersymmetric case, any one of the 3-form field strengths can be used in constructing the solution. In $D = 6$, there are further solutions, associated with the possibility of truncating the theory to self-dual supergravity. We shall discuss these further in the next section.
2-Form solutions

As one can see from (1.2), the set of possible choices for dilaton vectors is considerably larger for 2-form field strengths, and correspondingly there are many more solutions. If only a single 2-form field strength participates in the solution, then as usual we find \( \Delta = 4 \). For \( N = 2 \), we can have \( \Delta = 3 \) or \( \Delta = 2 \). For \( N = 3 \), there are always solutions with \( \Delta = \frac{8}{3} \) and \( \frac{12}{7} \). In addition, if \( D \leq 5 \), there is a further solution with \( \Delta = \frac{4}{3} \). As \( N \) increases to its maximum allowed value of \( N = 7 \) in \( D = 4 \), more and more solutions appear. Full details may be found in [12]. Here, we shall just give details for the supersymmetric solutions. These occur for \( N = 1, 2, 3 \) and 4 participating field strengths, with \( \Delta = \frac{4}{N} = 4, 2, \frac{4}{3}, \frac{4}{3} \) and 1 respectively. The eigenvalues of the Bogomol’nyi matrix are as follows:

\[
\begin{align*}
\Delta = 4 : & \quad \mu = 2m\{0_{16}, 1_{16}\} , \quad D \leq 10 , \\
\Delta = 2 : & \quad \mu = m\{0_8, 1_{16}, 2_8\} , \quad D \leq 9 , \\
\Delta = \frac{4}{3} : & \quad \mu = \frac{2}{3}m\{0_4, 1_{12}, 2_{12}, 3_{12}\} , \quad D \leq 5 , \\
\Delta = 1' : & \quad \mu = m\{0_4, 1_{24}, 2_{4}\} , \quad D = 4 ,
\end{align*}
\]

where we also indicate the maximum dimension \( D \) in which each solution can occur. We see that the \( p \)-brane solutions preserve \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8} \), and \( \frac{1}{8} \) of the \( D = 11 \) supersymmetry respectively. The prime on the value of \( \Delta \) for the fourth case is for later convenience, to distinguish it from another \( \Delta = 1 \) configuration that will arise in the discussion of 1-form solutions. It should also be noted that the equations (2.4) only determine the relations between the squares of the Page charges, and thus the signs can be arbitrarily chosen. For the configurations with \( N \leq 3 \) participating field strengths that give rise to supersymmetric solutions, the eigenvalues of the Bogomol’nyi matrix are insensitive to these sign choices. For the \( N = 4 \) case, on the other hand, the choice of relative signs does matter, and it turns out that there are exactly two possible sets of eigenvalues that can result. One of these is the supersymmetric solution with \( \Delta = 1' \) listed in (2.7); for the inequivalent choice of relative signs, the eigenvalues are \( \mu = m\{1_{16}, 3_{16}\} \), and thus there is no supersymmetry. Note that any one of the 2-form field strengths can be used to obtain the \( \Delta = 4 \) solution. For \( \Delta = 2 \), there are various possible pairs of field strengths that can be chosen, the number of such choices increasing with decreasing \( D \). An example, which can be used in all dimensions \( D \leq 9 \), is to choose \( F^{12}_2 \) and \( F^1_2 \), with Page charges given by \( u_{12} = p_1 = \lambda/(4\sqrt{2}) \). The non-supersymmetric \( \Delta = 3 \) solution is obtained for other choices of two participating field strengths, for example \( p_1 = p_2 = \lambda/(4\sqrt{2}) \).

Examples with \( N = 3 \) and \( N = 4 \) field strengths, yielding the \( \Delta = \frac{4}{3} \) and \( \Delta = 1 \) supersymmetric solutions, are \( u_{12} = u_{34} = u_{56} = \lambda/8 \), and \( u_{23} = u_{46} = u_{57} = p_1^e = \lambda/8 \) respectively. In the latter case, \( p_1^e \) denotes an electric Page charge for the dual of the field strength \( F^1_2 \). As mentioned above,
if the signs of the charges in this $\Delta = 1$ case are changed, then eight of the sign choices give the same solution whilst for the other eight we still get a solution, with the same metric, but with no supersymmetry preserved.

1-Form solutions

The situation for 1-form solutions is more complicated again; further details may be found in [12]. As $N$ increases up to 7, there is a considerable proliferation of solutions, mostly non-supersymmetric. Here, we shall just describe the supersymmetric $p$-brane solitons. There are in total eight inequivalent field configurations that can give rise to supersymmetric solutions, namely one for each value of $N$ in the range $1 \leq N \leq 7$, together with a second inequivalent possibility for $N = 4$. In all cases, the value of $\Delta$ is given by $\Delta = 4/N$. Four of the eight solutions have Bogomol’nyi matrices with identical eigenvalues to those given in (2.7) for 2-form solutions (although the Bogomol’nyi matrices themselves are of course different). These again arise for $N = 1, 2, 3$ and 4 participating field strengths. Examples of the Page charges that can give rise to these four case are $q_{12}, \{q_{12}, v_{123}\}, \{q_{12}, q_{45}, v_{123}\}$ and $\{q_{12}, q_{45}, v_{123}, v_{345}\}$ respectively, where in each case all the listed charges are equal. The remaining four solutions, which cannot occur above $D = 4$ dimensions, have eigenvalues as follows:

$$
\begin{align*}
\Delta = 1 : & \quad \{q_{12}, q_{34}, q_{56}, v_{127}\}, \quad \mu = \frac{1}{2} m \{0_2, 1_8, 2_{12}, 3_8, 4_2\}, \\
\Delta = \frac{4}{5} : & \quad \{q_{12}, q_{34}, q_{56}, v_{127}, v_{347}\}, \quad \mu = \frac{2}{5} m \{0_2, 1_2, 2_{12}, 3_{12}, 4_2, 5_2\}, \\
\Delta = \frac{2}{3} : & \quad \{q_{12}, q_{34}, q_{56}, v_{127}, v_{347}, v_{567}\}, \quad \mu = \frac{1}{3} m \{0_2, 2_6, 3_{16}, 4_6, 6_2\}, \\
\Delta = \frac{4}{7} : & \quad \{q_{12}, q_{34}, q_{56}, v_{127}, v_{347}, v_{567}, v_7^*\}, \quad \mu = \frac{2}{7} m \{0_2, 3_{14}, 4_{14}, 7_2\},
\end{align*}
$$

where in each case the listed set of Pages charges provides an example of a set that gives rise to the solution. In each case, the Page charges are all equal. All of these four cases correspond to solutions that preserve $\frac{1}{16}$ of the $D = 11$ supersymmetry. In the first of these four solutions, the eigenvalues are insensitive to the relative sign choices for the Page charges, but in the last three cases, we again find the phenomenon that there are precisely two inequivalent sets of eigenvalues for each $\Delta$, depending on the relative signs of the Page charges. We have given the choices that include zero eigenvalues. The other choices, for which there is no supersymmetry, give rise to the sets of eigenvalues

$$
\begin{align*}
\Delta = \frac{4}{3} : & \quad \mu = \frac{2}{3} m \{1_8, 2_8, 3_8, 4_8\}, \\
\Delta = \frac{2}{3} : & \quad \mu = \frac{1}{3} m \{1_4, 2_8, 3_8, 4_8, 5_4\}, \\
\Delta = \frac{4}{7} : & \quad \mu = \frac{2}{7} m \{1_2, 2_6, 3_8, 4_8, 5_6, 6_2\},
\end{align*}
$$

(2.9)
To conclude this section, we give a table that summarises the dimensions in which the various supersymmetric $p$-brane solutions first occur:

| Dimension $D$ | 4-form | 3-form | 2-form | 1-form |
|---------------|--------|--------|--------|--------|
| $D = 11$      | $\Delta = 4$ |        |        |        |
| $D = 10$      | $\Delta = 4$ | $\Delta = 4$ |        |        |
| $D = 9$       | $\Delta = 2$ | $\Delta = 4$ |        |        |
| $D = 8$       | $\Delta = 2$ |        |        |        |
| $D = 7$       |        |        |        |        |
| $D = 6$       | $\Delta = 2$ |        | $\Delta = \frac{4}{3}, 1'$ |        |
| $D = 5$       | $\Delta = \frac{4}{3}$ |        |        |        |
| $D = 4$       | $\Delta = 1'$ | $\Delta = 1, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}$ |        |        |

Table 1: Supersymmetric $p$-brane solutions

3 Dyonic solutions

In dimensions $D = 2n$, the field strength $F_n$ can in principle have components given by both the elementary and solitonic ansätze (1.6) simultaneously. In this case, the equations of motion can be reduced to the two independent differential equations

$$
\phi'' + n \frac{\phi'}{r} = \frac{1}{2} a(S_1^2 - S_2^2) , \quad A'' + n \frac{A'}{r} = \frac{1}{4}(S_1^2 + S_2^2) ,
$$

(3.1)

together with the relations $B = -A$, $(e^{C})' = \lambda_1 e^{a\phi + 2(n-1)A} r^{-n}$, where $S_1$ and $S_2$ are given by

$$
S_1 = \lambda_1 e^{\frac{1}{2} a\phi + (n-1)A} r^{-n} , \quad S_2 = \lambda_2 e^{\frac{1}{2} a\phi + (n-1)A} r^{-n} .
$$

(3.2)

By making a suitable ansatz of the kind discussed in the previous section, which reduces the equations to first-order equations, one finds [12] that the equations (3.1) admit a simple solution either when $a^2 = n - 1$, and hence $\Delta = 2n - 2$, given by

$$
e^{-\frac{1}{2} a\phi - (n-1)A} = 1 + \frac{\lambda_1}{a\sqrt{2}} r^{-n+1} , \quad e^{\frac{1}{2} a\phi - (n-1)A} = 1 + \frac{\lambda_2}{a\sqrt{2}} r^{-n+1} ,
$$

(3.3)

or when $a = 0$, and hence $\Delta = n - 1$, given by

$$
\phi = 0 , \quad e^{-(n-1)A} = 1 + \frac{1}{2} \frac{\lambda_1^2 + \lambda_2^2}{(n-1)} r^{-n+1} .
$$

(3.4)

The possible dimensions for dyonic solutions are $D = 8, 6$ and $4$. As discussed in [12], solutions of the kind we are discussing, where the contributions from the $L_{FFA}$ term are assumed to vanish,
cannot occur in $D = 8$. Furthermore, the solutions (3.3) and (3.4) would require that the 4-form have a dilaton prefactor with $\Delta = 6$ or $\Delta = 3$, whereas in fact it has $\Delta = 4$. Thus we are left with the cases $D = 6$ and $D = 4$.

**Dyonic solutions in $D = 6$**

In $D = 6$, the constraints implied by the requirement that $\mathcal{L}_{FPA}$ not contribute imply that there cannot be dyonic solutions of the first type, where each individual field strength has only electric or magnetic components. However, dyonic solutions of the second type, where each field strength has electric and magnetic components, can occur, for $N \leq 5$ participating field strengths, with $\Delta = 2 + 2/N$. The simple dyonic solutions (3.3) and (3.4) require $\Delta = 4$ and $\Delta = 2$ respectively, and thus we see that there exist dyonic solutions given by (3.3) when $N = 1$. The mass per unit length of this dyonic string is $m = \frac{1}{4}(\lambda_1 + \lambda_2)$, and the Page charges are $u = \frac{1}{4}\lambda_1$ and $v = \frac{1}{4}\lambda_2$. The eigenvalues of the Bogomol’nyi matrix are given by

$$\mu = m \pm u \pm v = \{0, (\frac{1}{2}\lambda_1)_8, (\frac{1}{2}\lambda_2)_8, (\frac{1}{2}(\lambda_1 + \lambda_2))_8\},$$

where, as usual, the subscripts on the eigenvalues indicate their degeneracies. Thus the solution preserves $\frac{1}{4}$ of the supersymmetry $\mathbb{R}^8$. When either $\lambda_1 = 0$ or $\lambda_2 = 0$, the solution reduces to the previously-discussed purely solitonic and purely elementary solutions, which preserve $\frac{1}{2}$ of the supersymmetry. When $\lambda_1 = \lambda_2$, in which case the field strength becomes self-dual and the dilaton vanishes, the solution is equivalent to the self-dual string in $D = 6$ self-dual supergravity, which we shall discuss below. When $\lambda_1 = -\lambda_2$, the field strength is anti-self-dual, and we have a massless string which preserves $\frac{1}{2}$ of the supersymmetry; however, the eigenvalues, given by (3.5), for such a solution are not positive semi-definite. In this case, the dilaton field does not vanish, and hence the solution is distinct from the anti-self-dual string in $D = 6$ anti-self-dual gravity. It is worth remarking that the eigenvalues (3.5) for these dyonic solutions of the second type are quite different from those for all the solutions we have discussed previously. In those cases, the eigenvalues are non-negative as long as the mass per unit $p$-volume is positive. However, for the dyonic solutions of the second type, we see that the Page charges can be chosen so that eigenvalues (3.5) of the Bogomol’nyi matrix take both signs, even when the mass is positive.

In the above discussion, we saw that the field strength of the solution could be chosen to be either self-dual or anti-self-dual. In fact, one can alternatively truncate the theory so as to retain a single 3-form field strength on which a self-dual (or anti-self-dual) condition is imposed [17]. In this case, the dilatonic fields are all consistently truncated from the theory, implying that $a = 0$ and hence $\Delta = 2$. The metric is given by (3.4) with $\lambda_1 = \lambda_2 = \lambda$. The mass per unit length is given
by \( m = \frac{1}{2} \sqrt{(\lambda_1^2 + \lambda_2^2)/\Delta} = \frac{1}{2} \lambda \); the Page charges of the solution comprise an electric charge \( u \) and a magnetic charge \( v \), with \( u = v = \frac{1}{4} \lambda \) (\( u = -\frac{1}{4} \lambda \) for the anti-self-dual case). The eigenvalues of the Bogomol’nyi matrix are given by \( \mu = \frac{1}{4} (2 \pm 1 \pm 1) = m \{0, 1, 2\} \), and so a quarter of the \( D = 11 \) supersymmetry is preserved in this (anti)-self-dual case. Note that the mass per unit \( p \)-volume of the self-dual (anti-self-dual) solution in the previous paragraph is given by \( m = u + v \), whilst the mass of these solution in \( D = 6 \) self-dual (anti-self-dual) supergravity is given by \( m = \sqrt{u^2 + v^2} \).

**Dyonic solutions in \( D = 4 \)**

In \( D = 4 \), the solutions \((3.3)\) and \((3.4)\) occur when \( \Delta = 2 \) and \( \Delta = 1 \), corresponding to \( a = 1/\sqrt{3} \) and \( a = 0 \). As we discussed for the elementary and solitonic solutions in the previous section, in the case \( \Delta = 1 \), arising when there are 4 participating field strengths, the relative signs of the Page charges, which are undetermined by the field equations, affect the eigenvalues of the Bogomol’nyi matrix, giving two inequivalent outcomes, one with supersymmetry and the other without. We can therefore now have three inequivalent sets of eigenvalues, depending on whether the relative signs in each of the electric and the magnetic sectors are chosen both to reduce to the previous supersymmetric choices, or else one supersymmetric and the other not, or finally both non-supersymmetric. (The occurrence of non-supersymmetric solutions for certain sign choices for the Page charges has also been seen recently in [22].) Accordingly, we list the three possibilities for \( \Delta = 1 \) below, in this order:

\[
\begin{align*}
\Delta = 1 : & \quad m = \frac{1}{2} \lambda , \quad \{u_{57}, u_{46}, u_{23}, p_{1}^*, v_{57}, v_{46}, v_{23}, q_{1}^*\} = \frac{1}{8} \{\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_2, -\lambda_2\} , \\
& \quad \mu = \frac{1}{4} \lambda \{0, 1, 24, 2\} , \\
\Delta = 1 : & \quad m = \frac{1}{2} \lambda , \quad \{u_{57}, u_{46}, u_{23}, p_{1}^*, v_{57}, v_{46}, v_{23}, q_{1}^*\} = \frac{1}{2} \{\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2\} , \\
& \quad \mu = \frac{1}{4} \{(\lambda - \frac{1}{2} \lambda_2)4, (\lambda + \frac{1}{2} \lambda_2)4, (\lambda - \sqrt{\lambda_1^2 + \frac{1}{4} \lambda_2^2})12, (\lambda + \sqrt{\lambda_1^2 + \frac{1}{4} \lambda_2^2})12\} , \\
\Delta = 1 : & \quad m = \frac{1}{2} \lambda , \quad \{u_{57}, u_{46}, u_{23}, p_{1}^*, v_{57}, v_{46}, v_{23}, q_{1}^*\} = \frac{1}{2} \{\lambda_1, \lambda_1, -\lambda_1, \lambda_2, \lambda_2, \lambda_2, \lambda_2\} , \\
& \quad \mu = \frac{1}{4} \lambda \{1, 16, 316\} , \\
\Delta = 2 : & \quad m = \frac{\lambda_1 + \lambda_2}{2 \sqrt{2}} , \quad \{p_1, u_{12}, q_1, v_{12}\} = \frac{1}{4\sqrt{2}} \{\lambda_1, \lambda_1, \lambda_2, \lambda_2\} , \\
& \quad \mu = \sqrt{2} \{(\lambda_1 + \lambda_2 - \lambda)8, (\lambda_1 + \lambda_2 + \lambda)8, (\lambda_1 + \lambda_2)16\} , \\
\end{align*}
\]

where \( \lambda \equiv \sqrt{\lambda_1^2 + \lambda_2^2} \). The first \( \Delta = 1 \) solution always preserves \( \frac{1}{8} \) of the supersymmetry, regardless of the values of \( \lambda_1 \) and \( \lambda_2 \). The second \( \Delta = 1 \) solution breaks all the supersymmetry for generic
values of the Page charges, but gives rise to a supersymmetric elementary solution when $\lambda_2 = 0$. The last $\Delta = 1$ solution breaks all the supersymmetry for all values of the Page charges. For $\Delta = 2$, we can have zero eigenvalues only for the following three cases: $\lambda_1 = 0$, $\lambda_2 = 0$ or $\lambda_1 = -\lambda_2$. The first two cases correspond to the purely solitonic and purely elementary solutions which preserve $\frac{1}{4}$ of the supersymmetry. The third case gives rise to a massless black hole (which has been discussed in [19]), which preserves $\frac{1}{2}$ of the supersymmetry. However, some of the eigenvalues are negative in this case.

4 Multi-scalar solutions

The third class of solutions that we shall describe here are ones where the individual Page charges of the field strengths participating in a solution are independent parameters (unlike the solutions discussed in the previous two sections, where they are fixed in a specific set of ratios in any given solution). This is achieved by allowing more than one independent combination of the $(11-D)$ dilatonic scalar fields to be excited in the solution. In fact, the number of independent combinations is precisely equal to the number $N$ of independent Page charges. These combinations can be expressed as $\varphi_\alpha = \vec{a}_\alpha \cdot \vec{\phi}$, where as usual $\vec{a}_\alpha$ denotes the set of $N$ dilaton vectors for the $N$ participating field strengths. One can easily verify from the equations of motion following from (1.1) that the remaining orthogonal combinations of the $\vec{\phi}$ fields can be consistently set to zero. A natural choice for the solutions turns out to be to take $dA + \tilde{d}B = 0$ and $A = -\frac{4d}{D-2} \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi_\beta$. The remaining equations can then be solved by making an ansatz that is analogous to the one discussed in section 2, which reduces the second-order equations of motion to first-order equations [20]. For this ansatz to be consistent, it turns out that the matrix $M_{\alpha\beta}$ of dot products of dilaton vectors must take the special form $M_{\alpha\beta} = 4\delta_{\alpha\beta} - 2dd/(D - 2)$. In fact this is precisely the same as the form that $M_{\alpha\beta}$ takes in the various supersymmetric single-scalar solutions that we discussed in section 2. Thus we conclude that multi-scalar solutions of the type we are discussing here have the interpretation of being generalisations of the supersymmetric single-scalar solutions, in which the Page charges of the individual participating field strengths, whose magnitudes were previously required to be equal, become independent free parameters. The solutions are given by [20]

$$e^{\frac{1}{2}\ell \varphi_\alpha - dA} = 1 + \frac{\lambda_\alpha}d r^{-\tilde{d}},$$

$$ds^2 = \prod_{\alpha=1}^N \left( 1 + \frac{\lambda_\alpha}d r^{-\tilde{d}} \right)^{-\frac{d}{(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + \prod_{\alpha=1}^N \left( 1 + \frac{\lambda_\alpha}d r^{-\tilde{d}} \right)^{-\frac{d}{(D-2)}} dy^m dy^m. \quad (4.1)$$
We may now calculate the mass per unit $p$-brane volume and the Page charges for the solution, finding

\[ m = \frac{1}{4} \sum_{\alpha=1}^{N} \lambda_\alpha, \quad P_\alpha = \frac{1}{4} \lambda_\alpha. \]

(4.2)

Note that in our derivation of the solutions, we assumed that the matrix $M_{\alpha\beta}$ is non-singular, which in general is the case. However, it can be singular in two relevant cases, namely $D = 5$, $N = 3$ and $D = 4$, $N = 4$ for the 2-form field strengths. In these cases, the analysis requires modification; however, it turns out that (4.1) continues to solve the equations of motion.

We have seen that multi-scalar solutions arise as generalisations of the previous supersymmetric single-scalar $p$-brane solutions, in which the Page charges that were previously equal become independent. Thus we may view the supersymmetric single-scalar solutions with $N \geq 2$ participating field strengths as starting points for such generalisations. In the light of the findings described in section 2, this means that multi-scalar solutions arise for 2-form field strengths, with $2 \leq N \leq 4$, and for 1-form field strengths with $2 \leq N \leq 7$. The supersymmetry of these multi-scalar solutions can determined using the same approach as for the single-scalar solutions, by calculating the eigenvalues of the Bogomol’nyi matrix. A full analysis of all these solutions and their supersymmetry is contained in [20], and we shall only summarise the results here.

For 2-form multi-scalar solutions, the eigenvalues of the Bogomol’nyi matrix turn out to be given by

\[ N = 2 : \quad \{ p_1, u_{12} \} = \frac{1}{4} \{ \lambda_1, \lambda_2 \}, \quad \text{for } D \leq 9, \]
\[ \mu = \frac{1}{2} \{ 0, \lambda_1, \lambda_2, \lambda_1 + \lambda_2 \}, \]
\[ N = 3 : \quad \{ u_{12}, u_{34}, u_{56} \} = \frac{1}{4} \{ \lambda_1, \lambda_2, \lambda_3 \}, \quad \text{for } D \leq 5, \]
\[ \mu = \frac{1}{2} \{ 0, \lambda_1, \lambda_2, \lambda_3, \lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_3 \}, \]
\[ N = 4 : \quad \{ u_{12}, u_{34}, u_{56}, p_7^* \} = \frac{1}{4} \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \}, \quad \text{for } D = 4, \]
\[ \mu = \frac{1}{2} \{ 0, \lambda_1 + \lambda_4, \lambda_2 + \lambda_4, \lambda_3 + \lambda_4, \lambda_1 + \lambda_4, \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \}. \]

(4.3)

Here a * on a Page charge indicates that the associated field strength is dualised. Thus $p_7^*$ is the electric charge of the dualised field strength $+F^{(7)}_{MN}$, and so it is the magnetic charge in terms of the original undualised field strength $F^{(7)}_{MN}$. The degeneracies of each eigenvalue in each set are equal.

For 1-form field strengths, there exist multi-scalar generalisations for all the supersymmetric solutions described in section 2 with $2 \leq N \leq 7$. Rather than present them all here, we shall just give the results for the case $N = 7$ here, from which all the lower-$N$ cases can in fact be derived by successively setting Page charges to zero. The full details can be found in [20]. Thus we have

\[ \{ q_{12}, q_{34}, q_{56}, v_{127}, v_{347}, v_{567}, v_7^* \} = \frac{1}{4} \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \}, \]
\[
\mu = \{0, \lambda_{126}, \lambda_{135}, \lambda_{234}, \lambda_{147}, \lambda_{257}, \lambda_{367}, \lambda_{456}, \lambda_{1245}, \lambda_{1346}, \lambda_{1237}, \lambda_{1234567}\},
\]

where we define \(\lambda_{\alpha\beta\cdots\gamma} = \lambda_{\alpha} + \lambda_{\beta} + \cdots + \lambda_{\gamma}\). Note that we presented only one representative set of Page charges among many possibilities, since they all give identical eigenvalues. The degeneracy of each eigenvalue is the same, with the total number of eigenvalues being 32.

For generic values of the Page charges, we see that the numbers of zero eigenvalues of the Bogomol’nyi matrices for all the multi-scalar solutions given above are the same as for their single-scalar limits in section 2. At certain special values of the Page charges, however, some additional zero eigenvalues can arise, corresponding to an enhancement of the supersymmetry. For example when \(N = 2\) for 2-form solutions, we see from (4.3) that by setting \(\lambda_2 = -\lambda_1\), the generic 8 zero eigenvalues are enlarged to 16, implying that \(\frac{1}{2}\) the supersymmetry is now preserved instead of just \(\frac{1}{4}\). Since the final eigenvalue in the set, \(\frac{1}{2}(\lambda_1 + \lambda_2)\), is equal to twice the mass, it follows that the solution becomes massless in this special case. In \(D = 4\), this corresponds to a massless black hole. The solution appears to suffer from two pathologies, however. Firstly, since one of \(\lambda_1\) or \(\lambda_2\) must be negative in this special case, we can see from (4.1) that there is a naked singularity. More seriously, perhaps, we see from (4.3) that some of the eigenvalues of the Bogomol’nyi matrix must also now be negative. The non-negativity of the Bogomol’nyi matrix can be proved for classical solutions (subject to some conditions that are evidently violated in this massless example), and is required at the quantum level for all acceptable states, since it arises as the square of the Hermitean supercharge operators. Thus classical solutions with negative eigenvalues cannot form part of the true quantum spectrum.

There are other examples of supersymmetry enhancements that can occur while maintaining the non-negativity of the Bogomol’nyi matrix. For example, the 4-scalar solution for 2-forms can lead to three inequivalent enhancements, given by

\[
\begin{align*}
\lambda_1 &= -\lambda, \quad \lambda_2 = \lambda, \quad \mu = \frac{1}{2}\{0_8, (\lambda_3 \pm \lambda)_4, (\lambda_4 \pm \lambda)_4, (\lambda_3 + \lambda_4)_8\}, \\
\lambda_1 &= -\lambda, \quad \lambda_2 = \lambda_3 = \lambda, \quad \mu = \frac{1}{2}\{0_{12}, (2\lambda)_4, (\lambda_4 - \lambda)_4, (\lambda_4 + \lambda)_12\}, \\
\lambda_1 &= -\lambda, \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda, \quad \mu = \lambda\{0_{16}, 1_{16}\},
\end{align*}
\]

where as usual the subscripts denote the degeneracies of each eigenvalue. Thus these three cases preserve \(\frac{1}{4}, \frac{3}{8}\) and \(\frac{1}{2}\) of the supersymmetry respectively, in contrast to \(\frac{1}{8}\) for generic values of the charge parameters. Note that in these cases, although the Bogomol’nyi matrices have no negative eigenvalues when the supersymmetry enhancements occur, the metrics of the solutions still seem to have naked singularities since one of the Page charges \(\lambda_\alpha\) is negative. If we relax the condition that the eigenvalues of the Bogomol’nyi matrix should be non-negative, then further
enhancements are possible, in which $\frac{5}{8}$ or $\frac{3}{4}$ of the supersymmetry is preserved. (This does not violate the classification of supermultiplets given in [21], since non-negativity of the commutator of supercharges was assumed there.) Similar supersymmetry enhancements can occur for the 1-form solutions, as can easily be derived from (4.4).

Finally, we remark that the non-supersymmetric $p$-brane solutions that are obtained by reversing the signs of certain Page charges in the $\Delta = 1'$, $\frac{5}{8}, \frac{3}{4}$ and $\frac{1}{4}$ single-scalar solutions also admit multi-scalar generalisations. These can be summarised by presenting the eigenvalues of the Bogomol’nyi matrix for $N = 4$ scalars for 2-form solutions, and $N = 7$ scalars for 1-form solutions:

\[
N = 4 \quad \mu = \frac{1}{2}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_{123}, \lambda_{124}, \lambda_{134}, \lambda_{234}\}, \\
N = 7 \quad \mu = \frac{1}{2}\{\lambda_7, \lambda_{14}, \lambda_{25}, \lambda_{36}, \lambda_{123}, \lambda_{156}, \lambda_{246}, \lambda_{345}, \lambda_{1267}, \lambda_{1357}, \lambda_{2347}, \lambda_{4567}, \\
\lambda_{12457}, \lambda_{13467}, \lambda_{23567}, \lambda_{123456}\}.
\]

(4.6)

Note that in these cases none of the eigenvalues is proportional to the mass, for generic $\lambda_\alpha$. A detailed discussion of the various cases that arise by choosing special values for the Page charges is given in [20].

To conclude, we present a table summarising the various multi-scalar $p$-brane solutions that we have been discussing.

| Dim. | 2-Forms | 1-Forms |
|------|---------|---------|
| $D = 10$ | $N = 1$ | $p = 0, 6$ |
| $D = 9$ | $N = 2$ | $p = 0, 5$ | $N = 1$ | $p = 6$ |
| $D = 8$ | $p = 0, 4$ | $N = 2$ | $p = 5$ |
| $D = 7$ | $p = 0, 3$ | $p = 4$ |
| $D = 6$ | $p = 0, 2$ | $N = 3, 4'$ | $p = 3$ |
| $D = 5$ | $N = 3$ | $p = 0, 1$ | $p = 2$ |
| $D = 4$ | $N = 4$ | $p = 0$ | $N = 4, 5, 6, 7$ | $p = 1$ |

Table 2: Multi-scalar $p$-brane solutions

Here we list the highest dimensions where $p$-brane solutions with the indicated numbers $N$ of field strengths first occur. They then occur also at all lower dimensions.

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