VIRTUAL CYCLES OF GAUGED WITTEN EQUATION

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Abstract

We construct virtual cycles on moduli spaces of perturbed gauged Witten equation over a fixed smooth r-spin curve, under the framework of [TX15]. We also prove a wall-crossing formula for variations of strongly regular perturbations. This completes the construction of the correlation function for the gauged linear $\sigma$-model announced in [TX14] as well as the proof of its invariance.

Keywords: gauged linear $\sigma$-model, gauged Witten equation, virtual cycle

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1. INTRODUCTION

This is the third input in a series of papers aiming at constructing rigorously a mathematical theory of the gauged linear $\sigma$-model (GLSM) as Witten proposed, following the two papers [TX15, TX14]. Since its introduction in [Wit93], GLSM has become a fundamental framework in many studies of mathematics and physics related to string theory. It is of crucial importance in the physical proof of mirror symmetry [HV00]. However, despite many efforts, a mathematical foundation of GLSM has not been established and is needed for many important applications, for example, a mathematical proof of the LG/CY correspondence, and a geometric understanding of mirror symmetry.

In [TX15, TX14], we initiated our project on constructing a mathematical theory of GLSM. The central objects we consider are the so-called gauged Witten equation and its moduli space. The gauged Witten equation is a system combining both the Witten equation, which was used to construct the Landau-Ginzburg model mathematically [FJR08, FJR13, FJR11], and the symplectic vortex equation, which was used to construct the Hamiltonian-Gromov-Witten invariants [Mun12, Mun03], [CGS00] and [MT09]. Roughly speaking, given a noncompact Kähler manifold
X, an action on X by a reductive Lie group \( G^C \) with a moment map \( \mu : X \to \mathfrak{g} \), and a holomorphic function \( W : X \to \mathbb{C} \) which is homogeneous with respect to the \( G^C \)-action, the gauged Witten equation reads

\[
\begin{aligned}
\bar{\partial}_A u + \nabla W(u) &= 0, \\
\ast F_A + \mu(u) &= 0,
\end{aligned}
\tag{1.1}
\]

where \( A \) is a connection on a \( G \)-bundle \( P \) over a Riemann surface \( \Sigma \), and \( u \) is a section \( u : P \to X \).

This system is the classical equation of motion with respect to the action

\[ E(A, u) = \frac{1}{2} (\|d_A u\|^2_{L^2} + \|F_A\|^2_{L^2} + \|\mu(u)\|^2_{L^2} + \|\nabla W(u)\|^2_{L^2}). \]

The moduli space of gauge equivalence classes of solutions contains the information of GLSM that can be extracted mathematically. In [TX15] we have obtained crucial analytical results about the gauged Witten equation and its moduli space. These results are needed in constructing GLSM mathematically and will be recalled in due course in the main body of this paper.

As in Gromov-Witten theory, the correlation function of GLSM is the most important numerical output. The (symplectic) Gromov-Witten invariants, which can naively be interpreted as “curve counting”, were constructed on any symplectic manifolds (cf. [LT98b], [FO99], also see [RT95], [Rua96], [RT97] on semi-positive symplectic manifolds). One major difficulty of the construction is the lack of transversality, namely, the moduli space of holomorphic curves may not be smooth or of expected dimension. The moduli space can be regarded as the zero locus of certain Fredholm section \( F \) of a Banach orbibundle \( \mathcal{E} \to \mathcal{B} \). If \( \mathcal{F} \) intersects the zero section transversely, then the moduli space gives rise to a well-defined fundamental class, which leads to the curve counting. In the non-transverse situation, we need to construct perturbations of \( \mathcal{F} \) which are transverse to the zero section. But it is a highly non-trivial problem. In fact, the perturbation scheme becomes more sophisticated when involved with stable map compactification and nontrivial automorphisms of stable maps.

The method of constructing a system of consistent perturbations for the stratified moduli space and extracting topological information (the virtual cycle) is often called the virtual technique. Since they were introduced in [LT98b] and [FO99], different versions of the virtual technique have been developed (in the non-algebraic case). We refer the readers to [LT98c], [HWZ07, HWZ09a, HWZ09b], [CLW13, CLW15], [Joy07, Joy14], [Par15], [MW15b, MW15a, MW15c], [FOOO12, FOOO14, FOOO15] for more recent developments.

In [TX14] we defined the correlation function assuming the existence and good properties of a virtual cycle. In this paper we give a detailed construction of such a virtual cycle and a proof of the properties we need. Here we briefly explain our results. Let \( X \) be a noncompact Kähler manifold with a \( \mathbb{C}^* \)-action, \( Q : X \to \mathbb{C} \) be a holomorphic function, homogeneous of degree \( r \). \((X, Q)\) is required to satisfy Hypothesis 2.2 and 2.3. Let \( \tilde{X} = X \times \mathbb{C} \), which is acted by another \( \mathbb{C}^* \)-action. Let \( W : \tilde{X} \to \mathbb{C} \) be \( W(p, x) = pQ(x) \). There is also a moment map \( \mu : \tilde{X} \to \text{Lie}(S^1 \times S^1) \). Then one can write down the gauged Witten equation \((1.1)\) over a so-called \( r \)-spin curve, which is an ordinary punctured Riemann surface \( \Sigma \) with an orbifold line bundle \( L \) whose \( r \)-th tensor power is isomorphic to the log-canonical line bundle. Moreover, we assume in Hypothesis 2.7 that there is large enough vector space \( V \) of holomorphic functions with nice properties in order to properly perturb the gauged Witten equation. Our first main theorem is the following (the same as Theorem 3.1).

**Theorem 1.1.** Let \( C \) be an \( r \)-spin curve with broad punctures \( z_1, \ldots, z_b \). For \( B \in H^2_S(\tilde{X} ; \mathbb{Z}[1/r]) \), a strongly regular perturbation \( P = (P_1, \ldots, P_b) \), and a choice \( \kappa = (\kappa_1, \ldots, \kappa_b) \) of asymptotic constrains at broad punctures, the moduli space \( \mathcal{M}_P(C, B, \kappa) \) of perturbed gauged Witten equation over \( C \) admits an oriented virtual orbifold atlas of dimension \( \chi(C, B) \) (which is defined in \((3.8)\)).
In particular, when $\chi(C, B) = 0$, there is a well-defined virtual counting

$$\#\mathcal{M}_P(C, B, \kappa) \in \mathbb{Q}.$$ 

Further, when there is at least one puncture on $C$, $\#\mathcal{M}_P(C, B, \kappa)$ is an integer.

Here a perturbation is strongly regular if for each broad puncture $z_j \in \Sigma$, a perturbed function $\widetilde{W}_j : X \to \mathbb{C}$ is Morse and all its critical values have distinct imaginary parts. These type of perturbations form a subset of the space $V$ which is a complement of a real analytic hypersurface $Wall \subset V$. In order to define a correlation function which is independent of the choice of perturbation, one needs to study the wall-crossing phenomenon when varying strongly regular perturbations. The most crucial result regarding the wall-crossing is Theorem 4.6. This implies the following well-definedness of the correlation function (the same as Theorem 4.1).

**Theorem 1.2.** For a fixed $r$-spin curve $C$ (with $k$ punctures) and $B \in H^2_S(S^1 \times S^1(\tilde{X}; \mathbb{Z}[r^{-1}])$, the correlation function $\langle \cdot \rangle^B_C : (\mathcal{H}_Q)^\otimes k \to \mathbb{Q}$ defined by (3.6) and linear extension is independent of the choice of strongly regular perturbations at broad punctures.

Both of the above two main theorems rely on the construction of corresponding virtual cycle/chain in moduli spaces. Our construction uses a version of the virtual technique which originated in [LT98b]. The method from [LT98b] is topological, in the sense that the charts are only topological and the smoothness of coordinate changes is not needed. This differs from other methods, such as, the Kuranishi method or polyfold method, for which certain weak smoothness of coordinate changes has to be established. Usually, it is rather technical to establish the smoothness property required in those methods. On the other hand, in a recent work [Par15], J. Pardon introduced a virtual technique which does not require smoothness of coordinate changes. In contrast, we get the virtual cycles by constructing perturbed sections in a more classical and topological way, while he has the virtual cycles constructed in Cech cohomology.

This paper is organized as follows. In Section 2 we recall the basic set-up of gauged Witten equation and perturbations in [TX15]. In Section 3 we recall the linear Fredholm theory and then construct the virtual cycle in the case of strongly regular perturbations. This completes the definition of the correlation function. Sections 4–8 provide a proof of the invariance of the correlation function. In Section 4 we consider variations of strongly regular perturbations and state the bifurcation formula. In Section 5, 6 and 7 we construct the virtual orbifold atlas and the virtual cycle on the universal moduli space over the family of perturbations. In Section 8 and Section 9 we prove Theorem 4.10 and 4.11, which identifies the bifurcation term in the wall-crossing formula with the contribution of BPS solitons. In the appendix, we provide an abstract theory on virtual technique following [LT98b].

During the preparation of this paper, there appeared [CLLL15] and [FJR15] which use algebraic methods to construct virtual cycles for GLSM in the absence of broad punctures.

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2. Gauged Witten Equations and Perturbations

In this section we recall the basic setup of gauged Witten equation given in [TX15].

2.1. The target space and the domain.
2.1.1. The LG space. Let \((X, \omega, J)\) be a Kähler manifold and \(Q : X \to \mathbb{C}\) be a holomorphic function, with a single critical point \(\star \in X\). Suppose there exists a Hamiltonian \(S^1\)-action with moment map \(\mu_0 : X \to i\mathbb{R}\). We denote this group by \(G_0\) instead of \(S^1\); we also identify its Lie algebra \(g_0\) with \(i\mathbb{R}\). For any \(\xi_0 \in g_0\), denote its infinitesimal action by \(L_{\xi_0} \in \Gamma(TX)\).

We assume that the \(G_0\)-action extends to a holomorphic \(\mathbb{C}^*\)-action, and \(Q\) is homogeneous of degree \(r \geq 1\) with respect to this \(\mathbb{C}^*\)-action. This means for \(x \in X\) and \(\xi \in \mathbb{C}^*\),

\[Q(\xi x) = \xi^r Q(x)\.

Then necessarily \(Q(\star) = 0\). Let \(X_Q := Q^{-1}(0)\), which is smooth away from \(\star\). For any \(\gamma \in \mathbb{Z}_r\), let \(X_\gamma \subset X\) be the fixed point set of \(\gamma\). Denote

\[\tilde{X} = X \times \mathbb{C}, \quad \tilde{X}_\gamma = X_\gamma \times \mathbb{C}.

Definition 2.1. \(\gamma \in \mathbb{Z}_r\) is called broad (resp. narrow) if \(X_\gamma \neq \{\star\}\) (resp. \(X_\gamma = \{\star\}\)).

We make the following assumptions on the structures, which are satisfied by nondegenerate homogeneous and a large class of quasi-homogeneous polynomials on \(\mathbb{C}^n\) (of degree at least 2).

Hypothesis 2.2.

(X1) \((X, \omega)\) is symplectically aspherical.

(X2) The Riemannian curvature of \(X\) is uniformly bounded; the complex structure \(J\) is uniformly continuous on \(X\) with respect to the Kähler metric.

(X3) The moment map \(\mu_0\) is proper and there exists \(c > 0\) such that for any \(x \in X\),

\[\frac{1}{c} \mu_0(x) - c \leq |L_x(x)|^2 \leq c|\mu_0(x)| + c.

(X4) As a real quadratic form on \(TX\), we have

\[0 \leq \nabla^2 (i\mu_0) \leq r.

Hypothesis 2.3.

(Q1) \(Q\) has a unique critical point \(\star \in X\) and \(|\nabla Q|\) is a proper function on \(X\).

(Q2) There are a constant \(c_Q > 1\), a \(G_0\)-invariant compact subset \(K_Q \subset X\) and a \(G_0\)-invariant neighborhood \(U_Q\) of \(X_Q\) such that

\[x \notin K_Q \implies |\nabla^3 Q(x)| + |\nabla^2 Q(x)| \leq c_Q |\nabla Q(x)|.

\[x \notin K_Q \cup U_Q \implies |\nabla Q(x)| \leq c_Q |Q(x)|.

2.1.2. The GLSM space. The GLSM space is a triple \((\tilde{X}, W, G)\) specified as follows.

(1) \(\tilde{X} = X \times \mathbb{C}\), whose coordinates are denoted by \(\tilde{x} = (x, p)\). The Kähler structure on \(\tilde{X}\), which is still denoted by \((\omega, J)\), is the product of the two factors, where on \(\mathbb{C}\) we have the standard structure.

(2) \(G\) is the compact Lie group \(G_0 \times S^1\), where the \(G_0\)-factor acts on the \(x\) coordinate and the (second) \(S^1\)-factor acts by

\[\xi_1(x, p) = (\xi_1 x, (\xi_1)^{-1} p).

The \(G\)-action is Hamiltonian with a moment map

\[\mu(x, p) = (\mu_0, \mu_1) =: \left(\mu_0(x), \mu_0(x) + \frac{ir}{2} |p|^2 - \tau\right).

Here \(\tau \in i\mathbb{R}\) is a constant, which we fix from now on. We denote the second \(S^1\)-factor by \(G_1\) and let \(g_1\) be its Lie algebra. For any \(\xi = (\xi_0, \xi_1) \in g\), we denote by \(L_\xi \in \Gamma(TX)\) the infinitesimal action of \(\xi\).
(3) \( W : \overline{X} \to \mathbb{C} \) is defined as \( W(x,p) = pQ(x) \), which is \( G_1^C \)-invariant where \( G_1^C \) is the complexification of \( G_1 \).

To define the vortex equation, one needs to specify a metric on \( \mathfrak{g} \). Indeed, depending on the value of the constant \( c \) in (X3) of Hypothesis 2.2, one can choose a sufficiently large \( \lambda_0 > 0 \) and define a metric on \( \mathfrak{g} \) as

\[
| (\xi_0, \xi_1) |^2 = \lambda_0^{-1} | \xi_0 |^2 + | \xi_1 |^2.
\]

This specific choice was used to establish certain convexity condition and to prove compactness in [TX15], but we won’t make explicit use of it in the current paper. From now on we will use primarily this metric on \( \mathfrak{g} \simeq (i\mathbb{R})^2 \).

### 2.1.3. Rigidified \( r \)-spin curves

We recall the notion of rigidified \( r \)-spin curves following [FJR13, Section 2.1]. Let \( \Sigma \) be a compact Riemann surface and \( \{z_1, \ldots, z_k\} \) is a finite subset of punctures (marked points). We denote \( \Sigma^\ast := \Sigma \setminus \{z_1, \ldots, z_k\} \). We can attach orbifold charts near each puncture to obtain an orbicurve \( \mathcal{C} \). Suppose the local group of orbifold chart near each \( z_j \) is \( \Gamma_j \), which is canonically isomorphic to a cyclic group \( \mathbb{Z}_r \). Then \( \Sigma \) can be viewed as the “desingularization” of \( \mathcal{C} \). There is a projection \( \pi_{\mathcal{C}} : \mathcal{C} \to \Sigma \). The orbicurve \( \mathcal{C} \) has the log-canonical bundle \( K_{\log} \simeq \pi_{\mathcal{C}}^* K_{\log} \), where \( K_{\log} \to \Sigma \) is the bundle

\[
K_{\log} = K_{\Sigma} \otimes \mathcal{O}(z_1) \otimes \cdots \otimes \mathcal{O}(z_k).
\]

**Definition 2.4.** Fix \( r \in \mathbb{Z} \), \( r \geq 3 \). An \( r \)-spin curve is a triple \((\mathcal{C}, \mathcal{L}, \varphi)\) where \( \mathcal{C} \) is an orbicurve, \( \mathcal{L} \to \mathcal{C} \) is an orbibundle, and \( \varphi : \mathcal{L} \otimes^r \to K_{\log} \) is an isomorphism of orbibundles.

A rigidification of the \( r \)-spin structure \((\mathcal{L}, \varphi)\) at \( z_j \) is an element \( e_j \) of \( \mathcal{L}|_{z_j} \) such that

\[
\varphi(e_{j}^{\otimes r}) = \frac{dw}{w}.
\]

Here \( w \) is a smooth holomorphic coordinate centered at \( z_j \). We choose a rigidification for each \( j \).

From now on, we fix a rigidified \( r \)-spin curve and abbreviate it by \( \mathcal{C} \).

At each marked point \( z_j \), the orbibundle \( \mathcal{L} \) has its local monodromy, which is a faithful representation \( \Gamma_j \to S^1 \). \( \Gamma_j \) can be viewed as a subgroup of \( \mathbb{Z}_r \), with generator \( \exp \left( \frac{2\pi i m_j}{r} \right) \) \((m_j \in \{0, 1, \ldots, r-1\})\). Then the \( r \)-spin structure induces an isomorphism

\[
|\varphi| : |\mathcal{L}| \otimes^r \to K_{\log} \otimes \mathcal{O}
\left(-\sum_{j=0}^{k} m_j z_j \right)
\]

as usual line bundles over \( \Sigma \), where \( |\mathcal{L}| \to \Sigma \) is the desingularization of \( \mathcal{L} \). Therefore, for any choice of local coordinate \( w \) around \( z_j \), a rigidification induces a choice of local frame \( e_j \) of \( |\mathcal{L}| \) near \( z_j \) such that

\[
|\varphi|(e_{j}^{\otimes r}) = w^{m_j} \frac{dw}{w}.
\]

We denote \( \lambda_j = im_j/r \) (resp. \( \gamma_j = \exp(2\pi i \lambda_j) \)) and call it the residue (resp. monodromy) of the \( r \)-spin structure at \( z_j \).

**Definition 2.5.** A puncture \( z_j \) is narrow (resp. broad) if \( \gamma_j \in \mathbb{Z}_r \) is narrow (resp. broad).

We assume the first \( b \) punctures \( z_1, \ldots, z_b \) are broad and the other ones \( z_{b+1}, \ldots, z_{b+n} \) are narrow.

**Remark 2.6.** In the LG A-model of [FJR08, FJR13, FJR11], one also considers the case that \( Q \) has larger symmetry group \( \Gamma \) than \( S^1 \). Using a larger symmetry group results in generalizations of the current setting for the LG space, in two directions. The first is that there are more possibilities than having an \( r \)-th root of \( K_{\log} \) in order to lift \( Q \) to a vector bundle over \( \Sigma \) (the so-called \( W \)-structure with \( W = Q \) in Fan-Jarvis-Ruan’s literature). The second is that the theory depends
on the choice of an “admissible subgroup” of $\Gamma$. The current setting means that we restrict to some special $W$-structures and always take the admissible subgroup to be the minimal one $\mathbb{Z}_r$.

Nevertheless, such a generality doesn’t bring in extra difficulty in the analytic side of the theory and we will stay within this somewhat special situation. On the other hand, for a generic quasi-homogeneous polynomial $Q$ on the weighted $\mathbb{C}^N$, the general setting coincides with our special one because there is no extra discrete symmetry of $Q$ and the admissible subgroup can only be $\mathbb{Z}_r$.

2.1.4. Perturbation of $W$.

**Hypothesis 2.7.** For each broad puncture $z_j$, there exists a nonzero, finite dimensional complex vector space $V_j$ parametrizing holomorphic functions $F_j : X \to \mathbb{C}$ satisfying the following conditions.

1. There is a basis $\{f_i \mid j = 2, 3, \ldots, s\}$ of $V_j$ where $f_i$ is homogeneous of degree $r_i$ with $\gamma_j^{r_i} = 1$. Moreover, for each $l$,
   $$ \sup_X \frac{|f_i|}{\sqrt{1 + |\mu_0|}} < \infty, \sup_X |\nabla^{(i)} f_i| < \infty, \ (i = 1, 2, \ldots). $$

2. For each $a \in \mathbb{C}^*$, there is a complex analytic subset $V_j^{\text{sing}}(a) \subset V_j$ such that for each $F \in V_j \setminus V_j^{\text{sing}}(a)$, the restriction of the function
   $$ \tilde{W}_j(x, p) = W(x, p) - ap + F(x) $$
   to $\tilde{X}_j$ is a holomorphic Morse function. Moreover, $\tilde{W}_j|_{\tilde{X}_j}$ has no critical point at infinity in the following sense. There exist $\epsilon_{a,F} > 0, c_{a,F} > 0$ and a compact subset $\tilde{K}_{a,F} \subset \tilde{X}_j$ such that
   $$(x, p) \in B_{\epsilon_{a,F}}(\tilde{X}_j \setminus \tilde{K}_{a,F}) \implies |\nabla \tilde{W}_j(x, p)| \geq c_{a,F}. $$

3. There is a locally finite union of real analytic hypersurfaces $\text{Wall}_j(a) \subset V_j \setminus V_j^{\text{sing}}(a)$, defined by the coincidence of the imaginary parts of two different critical values of the function $\tilde{W}_j|_{\tilde{X}_j}$.

We define $\tilde{V}_j = \mathbb{C} \times V_j$ where the $\mathbb{C}$ factor parametrizes $a$. Denote

$$ \tilde{V}_j^{\text{sing}} = \{(F_j, a) \mid F_j \in V_j^{\text{sing}}(a)\}, \quad \text{Wall}_j = \{(F_j, a) \mid F_j \in \text{Wall}_j(a)\}. $$

**Definition 2.8.** $P_j = (a_j, F_j) \in C^* \times V_j$ is called a perturbation of $W$ at the $j$-th broad puncture. It is called regular (resp. strongly regular) if $P_j \notin \tilde{V}_j^{\text{sing}}$ (resp. $P_j \notin \text{Wall}_j$). It is called small if it is close to the origin of $\tilde{V}_j$.

A perturbation to the gauged Witten equation over $C$ is a collection

$$ P = (P_j)^b_{j=1} = (a, F) = (a_j, F_j)^b_{j=1} $$

where $P_j$ is a perturbation at the $j$-th broad puncture. It is called regular (resp. strongly regular) if each $P_j$ is regular (resp. strongly regular).

Let $P_j = (a_j, F_j)$ be a perturbation at $z_j$. We write

$$ F_j = \sum_{l=2}^s F_{j,l} := \sum_{l=2}^s b_{j,l} f_{j,l} $$

where $\{f_{j,l}\}_{l=2}^s$ is the basis of $V_j$ given by Hypothesis 2.7. We write $F_{j,0} = W$, $F_{j,1} = -a_j p$ and

$$ \tilde{W}_j = \sum_{l=0}^s F_{j,l}. $$
We introduce the following notations. For \( t \in \mathbb{R}_+^\ast = \mathbb{C}^\ast \), we denote
\[
\tilde{W}^{(t)}_j(x,p) = t^r \tilde{W}_j(t^{-1} x, p) = W(x, p) - t^r a_{j,p} + \sum_{l=1}^{s-1} t^{r-l} F_{j,l}(x) =: \sum_{l=0}^{s} F_{j,l}^{(t)}.
\] (2.3)
Most of the time we will consider the case \( t \in \mathbb{R}_+ \) and use \( \delta \) instead of \( t \). We have
\[
(x, p) \in \text{Crit}(\tilde{W}_j^s) \iff (tx, p) \in \text{Crit} \left( \tilde{W}_j^s \right) ; \ W_j^s(tx, p) = t^r \tilde{W}_j(x, p).
\]

**Remark 2.9.** The purpose of using perturbation is the same as in finite dimensional Morse theory of functions with degenerate singularities. It is possible to develop the mathematical GLSM without using this type of perturbations. Namely, since \( W \) is analytic, the flow lines (Morse or Floer type) ending at degenerate locus of \( \text{Crit} W \) can be stratified, as discussed in a preprint of Jake Solomon [Sol]. The stratification may essentially give the same topological information as we need in our approach. We will consider this alternative approach elsewhere.

### 2.2. Gauged Witten equation

The gauged Witten equation is the combination of the Witten equation and the symplectic vortex equation, where the latter requires an area form on the Riemann surface. Choose a smooth area form \( \nu \in \Omega^2(\Sigma) \). Together with the complex structure, \( \nu \) determines a Riemannian metric on \( \Sigma \), to which will be referred to as the “smooth metric”. On the other hand, for each \( z_j \), we fix a holomorphic coordinate patch \( w : B_1 \to \Sigma \) with \( w(0) = z_j \) and use the log function to identify the punctured \( C_j = w(B_1) \setminus \{ z_j \} \) with the cylinder \( \Theta_+ := [0, +\infty) \times S^1 \). The latter has coordinates \( s + it = -\log w \). We can choose a different area form \( \Omega \) such that \( \nu = \nu \Omega \) where the conformal factor \( \nu : \Sigma^\ast \to \mathbb{R}_+ \) is a smooth function whose restriction to each \( \Theta_+ \) is equal to \( e^{-2s} \). The metric determined by \( \Omega \) and the complex structure is called the “cylindrical metric” on \( \Sigma^\ast \).

From now on, for each puncture \( z_j \), we fix the coordinate \( w \) centered at \( z_j \), the cylindrical end \( C_j \), and its identification with \( \Theta_+ \). For any \( S \geq 0 \), we denote by \( C_j(S) \subset \Sigma^\ast \) the subset identified with \([S, +\infty) \times S^1\). We define certain weighted Sobolev spaces, \( W_{k,p}^* \) to be the Banach space completions of \( C_0^\infty(\Sigma^\ast) \) with respect to the following norm
\[
\|f\|_{W_{k,p}^*(\Sigma^\ast)} := \|\nu^{-\delta/2} f\|_{W_{k,p}(\Sigma^\ast)}
\]
where the latter Sobolev norm is taken with respect to the cylindrical metric on \( \Sigma^\ast \). It is similar to define the Sobolev space \( W_{k,p}(C_j) \) for each cylindrical end \( C_j \). Define
\[
G_{0,\delta}^{2,p} := \left\{ g_0 \in W_{2,loc}^{2,p}(\Sigma^\ast, G_0) \mid g_0|_{C_j} = \exp(\pm i\xi_j), \xi_j \in W_{2,\delta}^{2,p}(C_j) \right\}.
\]
From now on we fix \( p > 2 \).

**Definition 2.10.** A \( W_{loc}^{2,p} \) metric \( H \) on \( |\mathcal{L}|_{\Sigma^\ast} \) is called **adapted** if there is \( \delta > 0 \) such that
\[
\log \left( |\nu|^{-m_j/r} |e_j|_H \right) \in W_{\delta}^{2,p}(C_j), \quad j = 1, \ldots, b+n.
\]
Here \( m_j \) and \( e_j \) are the ones in (2.2).

Let \( \mathcal{H} \) be the space of all adapted metrics on \( |\mathcal{L}|_{\Sigma^\ast} \). There is an \( \mathbb{R}_+\)-action on \( \mathcal{H} \) by rescaling a metric. Fix a reference metric \( H_0 \in \mathcal{H} \) and denote \( P_0 = P_0(H) \to \Sigma^\ast \) the unit circle bundle. Then each \( H \in \mathcal{H} \) induces a \( U(1) \)-connection \( A(H) \) on \( P_0 \). Moreover, the map
\[
\mathcal{H} \ni H \mapsto A(H)
\]
is one-to-one modulo the rescaling. Denote
\[
\mathcal{H}_+ := \left\{ e^h H_0 \mid h \in W_{2,\delta}^{2,p}(\Sigma^\ast), \quad \delta > 0 \right\}.
\]
which is a slice of the rescaling. Define
\[
G_0 = \bigcup_{\delta > 0} G_{0,\delta}^{2,p}, \quad A_0 = \left\{ g_0 A(H) \mid H \in \mathcal{H}_+, \quad g_0 \in G_0 \right\}.
\]
On each cylindrical end $C_j$, let $e_j$ be the local holomorphic frame of $L$ satisfying (2.2), which is unique up to the $\mathbb{Z}_r$-action. Then we trivialize $P_0|_{C_j}$ by the local unitary frame

$$
\epsilon_j := \frac{e_j}{\|e_j\}_H_0
$$

This trivialization is denoted by

$$
\varphi_{j,0} : C_j \times S^1 \to P_0|_{C_j}.
$$

2.2.1. The $G_1$-bundle. We used $G_1$ to denote another copy of the group $S^1$ to distinguish from the structure group of $P_0$. Fix a smooth $G_1$-bundle $P_1 \to \Sigma$. For each broad puncture, let

$$
Fr(P_1, z_j) := \left\{ \varphi_j : G_1 \to P_1|_{z_j} \right\}
$$

be the set of trivializations of the fibre at $z_j$, or “frames”, which has a left $G_1$-action

$$
e^{i\rho} \cdot \varphi_j = \varphi_j \circ e^{i\rho}.
$$

Moreover, we extend each $\varphi_j \in Fr(P_1, z_j)$ to a trivialization $\tilde{\varphi}_j : G_1 \times C_j \to P_1|_{C_j}$ such that $e^{i\rho} \varphi_j = \tilde{\varphi}_j \circ e^{i\rho}$. So we identify a frame of $P_1$ at $z_j$ with a trivialization over $C_j$.

We denote its restriction to $\Sigma^*$ still by $P_1$ and denote by

$$
P = P_0 \times_{\Sigma^*} P_1 \to \Sigma^*
$$

the fibre product, which is a $G_1$-bundle.

We denote $A_1$ to be the space of $W^{1,p}_{\text{loc}}$-connections on $P_1|_{\Sigma^*}$ such that for each $C_j$, with respect to $\varphi_j \in Fr(P_1, z_j)$, any $A_1 \in A_1$ can be written as

$$
A_1 = d + \alpha_1, \quad \alpha_1 \in W^{1,p}_{\text{loc}}(\Sigma^*, \Lambda^1 \otimes \mathfrak{g}_1) \text{ for some } \delta > 0.
$$

Denote $A = A_0 \times A_1$, which is a set of $G$-connections on $P$. For any $\delta > 0$, denote by $\mathcal{G}^{2,p}_{1,\delta}$ the group of $G_1$-gauge transformations on $\Sigma^*$ of class $W^{2,p}_{\delta}$ and denote

$$
\mathcal{G}_1 = \bigcup_{\delta > 0} \mathcal{G}^{2,p}_{1,\delta}, \quad \mathcal{G} = \mathcal{G}_0 \times \mathcal{G}_1.
$$

Then $\mathcal{G}$ acts on $A$.

In [TX15, Section 2] we also constructed a system of functions $h_{j,A} : C_j \to \mathfrak{g} \otimes \mathbb{C}$. For simplicity we don’t recall their definitions but the properties are listed as below.

**Condition 2.11.** $h_{j,A}$ satisfies the following conditions.

1. $h_{j,A}$ only depends on the restriction of $A$ on $C_j$ and is affine linear in $A$.
2. With respect to the trivialization $(\varphi_{j,0}, \varphi_j)$, one has

$$
A = d + \phi_j ds + (\psi_j + \lambda_j) dt, \quad (\phi_j ds + \psi_j dt)^{0,1} = \overline{\partial} h_{j,A}.
$$

3. If $A \in A_0$, then $h_{j,A} \in W^{2,p}_{\delta}(C_j, \mathfrak{g}_\mathbb{C})$.
4. If $g = e^{h_j}$ for $h_j \in W^{2,p}_{\delta}(C_j, \mathfrak{g} \otimes \mathbb{C})$ and $h_j|_{\partial C_j} = 0$, then

$$
h_{j,g^* A} = h_{j,A} + h_j.
$$

Here $g^* A = A + \overline{\partial} h_j + \partial(\overline{h_j})$ is the complex gauge transformation.
2.2. The fibre bundle and the lift of the superpotential. We have the associated fibre bundle

$$\pi : \tilde{Y} := \tilde{P} \times_G \tilde{X} \to \Sigma^*.$$  

The vertical tangent bundle $T\tilde{Y}^\perp \subset T\tilde{Y}$ consists of vectors tangent to a fibre. Then since the $G$-action is Hamiltonian and preserves $J$, the Kähler structure on $\tilde{X}$ induces a Hermitian structure on $T\tilde{Y}^\perp$. On the other hand, for any continuous connection $A$, the tangent bundle $T\tilde{Y}$ splits as the direct sum of $T\tilde{Y}^\perp$ and the horizontal tangent bundle. The latter is isomorphic to $\pi^*T\Sigma^*$, therefore the connection induces an almost complex structure on $\tilde{Y}$. Moreover, the almost complex structure is integrable and $\tilde{Y}$ becomes a holomorphic fibre bundle over $\Sigma^*$.

A general smooth section is denoted by $u \in \Gamma(\tilde{Y})$; more generally, we will consider sections $u \in \Gamma^1_{\text{loc}}(\tilde{Y})$ of class $W^1_{\text{loc}}$. The group $\mathcal{G}$ also acts on the space of sections.

We describe how to lift the superpotential $W$ to $\tilde{Y}$. Around each point $q \in \Sigma^*$, there exists local coordinate $z_q$ and local frame $e_q$ of $\mathcal{L}$ such that

$$\phi(e^{\text{loc}}_q) = dz_q.$$  

Let $e'_q$ be an arbitrary local frame of $P_1$. Define

$$\mathcal{W}_A([e_q, e'_q, x]) = W(x)dz_q.$$  

It is easy to check that $\mathcal{W}_A \in \Gamma(\tilde{Y}, \pi^*K_{\Sigma})$ is well-defined and is holomorphic with respect to the holomorphic structure on $\tilde{Y}$ induced from $A = (A_0, A_1) \in \mathcal{A}$. Moreover, over each cylindrical end $C_j$, if we use the unitary trivialization $(\tilde{\varphi}_j, 0, \tilde{\varphi}_j)$, then

$$\mathcal{W}_A(z, x) = e^{\rho_0(h_j, A(z)) + \lambda_j t}W(x)\frac{dw}{w}.$$  

Notice that $\mathcal{W}_A$ only depends on $A_0$.

Under our framework, we need to perturb $\mathcal{W}_A$. Instead of stating the full definition of a family of perturbations $\tilde{\mathcal{W}}_A \in \Gamma(\tilde{Y}, \pi^*K_{\Sigma})$, for the purpose of the current paper, we only need the expression of $\tilde{\mathcal{W}}_A$ over each cylindrical end $C_j$. Firstly, we have a smooth functional

$$\delta_A \in (0, 1], \quad \forall A \in \mathcal{A}. \quad (2.6)$$  

Then choose a cut-off function $\beta_j$ supported on $C_j$. Then

$$\tilde{\mathcal{W}}_A(z, x) = \mathcal{W}_A(z, x) + \beta_j(z) \sum_{l=1}^s F_j^{(\delta_A)}(z, x). \quad (2.7)$$  

Here $F_j^{(\delta_A)} \in \Gamma(\tilde{Y}|_{C_j}, \pi^*K_{\Sigma})$ is given by

$$F_j^{(\delta_A)} = \sum_{l=1}^s e^{\rho_1(h_j, A(z)) + \lambda_j t}F_j^{(\delta_A)}(x).$$  

2.2.3. The gauged Witten equation. The vertical differential of $\tilde{\mathcal{W}}_A$ is a section

$$d\tilde{\mathcal{W}}_A \in \Gamma(\tilde{Y}, \pi^*K_{\Sigma} \otimes (T\tilde{Y}^\perp)^*).$$  

The vertical Hermitian metric on $T\tilde{Y}^\perp$ induces a conjugate linear isomorphism

$$T\tilde{Y}^\perp \simeq (T\tilde{Y}^\perp)^*.$$  

On the other hand, the complex structure on $\Sigma^*$ induces a conjugate linear isomorphism

$$K_{\Sigma^*} \simeq \Lambda_{\Sigma^*}^{0,1}.$$  

Therefore we have a conjugate linear isomorphism

$$\pi^*K_{\Sigma^*} \otimes (T\tilde{Y}^\perp)^* \simeq \pi^*\Lambda_{\Sigma^*}^{0,1} \otimes T\tilde{Y}^\perp.$$
The image of $d\tilde{\mathcal{W}}_A$ under this map is called the vertical gradient of $\tilde{\mathcal{W}}_A$, denoted by

$$\nabla \tilde{\mathcal{W}}_A \in \Gamma(Y, \pi^* \Lambda^0_{\Sigma^*} \otimes T\tilde{Y}^\perp).$$

Then the (perturbed) gauged Witten equation is

$$\begin{cases}
\bar{\partial}_A u + \nabla \tilde{\mathcal{W}}_A(u) = 0; \\
* F_A + \mu^*(u) = 0.
\end{cases} \tag{2.8}$$

Each term in the system is defined as follows: the connection $A$ induces a continuous splitting $T\tilde{Y} \cong T\tilde{Y}^\perp \oplus \pi^* T\Sigma^*$ and $d_A u \in \mathcal{W}^{1,0}_{\text{loc}}(T^*\Sigma^* \otimes u^* T\tilde{Y}^\perp)$ is the covariant derivative of $u$; the $G$-invariant complex structure $J$ induces a complex structure on $T\tilde{Y}^\perp$ and $\bar{\partial}_A u$ is the $(0,1)$-part of $d_A u$ with respect to this complex structure. $\nabla \tilde{\mathcal{W}}_A(u)$ is the pull-back of $\nabla \mathcal{W}_A$ by $u$, which lies in the same vector space as $\bar{\partial}_A u$. $F_A \in \Omega^2(\Sigma^*) \otimes \mathfrak{g}$ is the curvature form of $A$, $* : \Omega^2(\Sigma^*) \rightarrow \Omega^0(\Sigma^*)$ is the Hodge-star operator with respect to the smooth metric on $\Sigma$; the moment map $\mu$ lifts to a $\mathfrak{g}$-valued function on $\tilde{Y}$ and $\mu^*(u)$ is the dual of $\mu(u)$ with respect to the metric defined by (2.1).

Using the local frame $\tilde{\varphi}$ over $C_j$, (2.8) can be written in coordinates as

$$\begin{cases}
\frac{\partial u}{\partial s} + L_\varphi(u) + \frac{\partial u}{\partial \lambda} \left( \frac{\partial u}{\partial \lambda} + L_{\psi+j}(u) \right) + 2 \nabla \mathcal{W}(u) = 0, \\
\frac{\partial \psi}{\partial s} - \frac{\partial \varphi}{\partial \lambda} + \nu \mu^*(u) = 0.
\end{cases}$$

The term $\nabla \tilde{\mathcal{W}}_A$ can be written as

$$\nabla \tilde{\mathcal{W}}_A(z, x) = e^{\tilde{\varphi}(h_+, a(z)+\lambda_j t)} \nabla \mathcal{W}(x) + \beta_j(z) \sum_{i=1}^{s} e^{\tilde{\varphi}(h_+, a(z)+\lambda_j t)} \nabla F_i(h_j, \lambda_j)(x).$$

Notation 2.12. By abuse of notation, we use $\varphi_j \in Fr(P_1, z_j)$ to denote a local trivialization of $P|_{C_j}$ and the induced trivialization of $\tilde{Y}|_{C_j}$. It contains the trivialization $\tilde{\varphi}_{j,0}$ of $P_0$ (2.4), which is fixed, and the trivialization $\tilde{\varphi}_j$. Moreover, for any section $u \in \Gamma(\tilde{Y}|_{C_j})$, we denote

$$u_{\varphi_j} : C_j \rightarrow \tilde{X}, \quad u_{\varphi_j}(z) = \varphi_j^{-1} u(z).$$

2.3. Asymptotic behavior and compactness.

2.3.1. Solitons and BPS solitons. Let $\gamma_j = \exp(2\pi \lambda_j)$ be the monodromy at $z_j$ and $\tilde{W}_j$ be the perturbed superpotential. For $\delta \in (0, 1]$, consider the following equation for maps $\sigma_j : \Theta \rightarrow \tilde{X}.$

$$\frac{\partial \sigma_j}{\partial s} + J \left( \frac{\partial \sigma_j}{\partial \lambda}, L_{\lambda_j}(\sigma_j) \right) + \nabla \mathcal{W}_j^{(\delta)}(e^{\lambda_j t} \sigma_j) = 0. \tag{2.9}$$

Since $\tilde{W}_j$ is $\gamma_j$-invariant, the inhomogeneous term $\nabla \mathcal{W}_j^{(\delta)}(e^{\lambda_j t} \sigma_j)$ is well-defined over $\Theta$. A bounded solution to (2.9) is called a soliton. It is not hard to show that for a soliton $\sigma_j$, there exists $\kappa_{j, \pm} \in \text{Crit}(\tilde{W}_j^{(\delta)}|_{\tilde{X}_j})$ such that

$$\lim_{s \rightarrow \pm \infty} \sigma_j(s, t) = e^{-\lambda_j t} \kappa_{j, \pm}.$$

Then we have the energy identity

$$\left\| \frac{\partial \sigma_j}{\partial s} + J \left( \frac{\partial \sigma_j}{\partial \lambda}, L_{\lambda_j}(\sigma_j) \right) \right\|_{L^2(\Theta)}^2 = 2\pi \left( \tilde{W}_j(\kappa_{j, -}) - \tilde{W}_j(\kappa_{j, +}) \right). \tag{2.10}$$

Since $\tilde{X}$ is also symplectically aspherical, nontrivial solitons exist only when $\tilde{W}_j$ is not strongly regular, i.e., $\tilde{W}_j|_{\tilde{X}_j}$ has two different critical values with identical imaginary part.

We call a soliton $\sigma_j$ a BPS soliton if $e^{\lambda_j t} \sigma_j(s, t)$ is independent of $t$ for all $s \in \mathbb{R}$. If $\sigma_j$ is a BPS soliton, then $e^{\lambda_j t} \sigma_j(s, t) \in \tilde{X}_j$ and (2.9) becomes

$$\frac{d}{ds} \left( e^{\lambda_j t} \sigma_j(s, t) \right) + \nabla \mathcal{W}_j^{(\delta)}(e^{\lambda_j t} \sigma_j) = 0.$$
So \(e^{\lambda t}\sigma_{j}(s,t)\) is a negative gradient line of the real part of \(\tilde{W}^{(\delta)}_{j}|_{\tilde{X}}\).

2.3.2. Soliton solutions. For general regular perturbations \(P = (P_1, \ldots, P_b)\), a soliton solution to the perturbed gauged Witten equation over \(C\) is a tuple
\[
\left(A, u, \varphi; \{\sigma_{j}^{b}\}_{j=1}^{b}\right)
\]
where

1. \(X = (A, u, \varphi)\) is a bounded solution over \(\hat{C}\);
2. \(\sigma_{j} = (\sigma_{j,1}, \ldots, \sigma_{j,k_{j}})\), where \(k_{j} \geq 0\) and each \(\sigma_{j,i}\) is a nontrivial solution to (2.9) for \(\delta = \delta_{X}\).
3. When \(k_{j} \geq 1\), we require that
\[
\lim_{s \to +\infty} u_{\varphi}(s,t) = \lim_{s \to -\infty} \sigma_{j,1}(s,t), \ldots, \lim_{s \to +\infty} \sigma_{j,k_{j}-1}(s,t) = \lim_{s \to -\infty} \sigma_{j,k_{j}}(s,t).
\]

Definition 2.13. Suppose \((X; \{\sigma_{j}^{b}\}_{j=1}^{b})\) and \((X'; \{\sigma'_{j}^{b}\}_{j=1}^{b})\) are two soliton solutions having the same combinatorial type (i.e., for each \(j\), \(\sigma_{j}\) and \(\sigma'_{j}\) have the same number of components). An isomorphism from \((X; \{\sigma_{j}^{b}\}_{j=1}^{b})\) to \((X'; \{\sigma'_{j}^{b}\}_{j=1}^{b})\) is a tuple
\[
\left(g, \{T_{j,i} + i\theta_{j,i} \mid 1 \leq j \leq b, 1 \leq i \leq k_{j}\}\right),
\]
where \(g \in G\) is a gauge transformation and \(T_{j,i} + i\theta_{j,i} \in \Theta\). They should satisfy the following conditions

1. \(X' = g^*X\);
2. For each \(j \in \{1, \ldots, b\}\), each \(i \in \{1, \ldots, k_{j}\}\), we have
\[
\sigma'_{j,i}(s,t) = e^{\lambda t\theta_{j,i}} \sigma_{j,i}(s + T_{j,i}, t + \theta_{j,i}).
\]

The automorphism group of a soliton solution is a subgroup of a torus. A soliton solution is called a BPS soliton solution if the automorphism group is infinite, or equivalently, there is at least one BPS soliton in the soliton solution. Let \(\mathcal{M}_{P}^{\#}(B)\) be the set of isomorphism classes of soliton solutions representing the class \(B\) and denote
\[
\overline{\mathcal{M}}_{P}(B) = \mathcal{M}_{P}(B) \cup \mathcal{M}_{P}^{\#}(B).
\]

2.3.3. Compactness. In [TX15] we defined the following notion of convergence. Let
\[
X^{(\nu)} = (A^{(\nu)}, u^{(\nu)}, \varphi^{(\nu)}) \in \widetilde{\mathcal{M}}_{P}(B)
\]
be a sequence of smooth solutions. We say that this sequence converges modulo gauge to a soliton solution \((A, u, \varphi; \{\sigma_{j}^{b}\}_{j=1}^{b})\), if there exist a sequence of gauge transformations \(g^{(\nu)} \in \mathcal{G}\) and a collection of sequences of real numbers \(s^{(\nu)}_{j,i}\) such that

1. \(\varphi^{(\nu)}\) converges to \(\varphi\) and \((g^{(\nu)})^* (A^{(\nu)}, u^{(\nu)})\) converges to \((A, u)\) uniformly with all derivatives on compact subsets of \(\Sigma^*\);
2. For \(j = 1, \ldots, b\) and \(i = 1, \ldots, k_{j}\), \(s^{(\nu)}_{j,i}\) diverges to \(+\infty\) and
\[
i < i' \implies \lim_{\nu \to +\infty} \left(s^{(\nu)}_{j,i'} - s^{(\nu)}_{j,i}\right) = +\infty.
\]
3. For \(j = 1, \ldots, b\) and \(i = 1, \ldots, k_{j}\), \((\varphi^{(\nu)} - s^{(\nu)}_{j,i}) \cdot (-s^{(\nu)}_{j,i})\) converges to \(\sigma_{j,i}\) uniformly with all derivatives on compact subsets of \(\Theta\).
4. The energy of \(X^{(\nu)}\) converge to the energy of \((A, u, \varphi; \{\sigma_{j}^{b}\}_{j=1}^{b})\).

Theorem 2.14. (cf. [TX15]) The above notion of convergence induces a topology on \(\overline{\mathcal{M}}_{P}(C, B, \kappa)\) which is compact and metrizable. In particular, when the perturbation \(P\) is strongly regular, \(\mathcal{M}_{P}^{\#}(C, B, \kappa) = \emptyset\) and \(\mathcal{M}_{P}(C, B, \kappa)\) is compact and metrizable.
Proof. The above notion of sequential convergence clearly extends to the case that $\lambda^{(c)}$ are allowed to be soliton solutions. The sequential compactness is proved in [TX15]. It induces a topology on $\overline{M}_P(C, B, \kappa)$ as in [MS04, Section 5.6], which is first countable and Hausdorff. Moreover, one can approach in the same way as in [MS04, Section 5.6] that the topology has a countable dense subset. Hence it is second countable, and compact. Urysohn’s metrization theorem implies that $\overline{M}_P(C, B, \kappa)$ is metrizable. \hfill $\square$

3. Definition of the Correlation Function

The definition of the GLSM correlation function given in [TX14, Section 3] relies on the following theorem.

Theorem 3.1. Let $C$ be an $r$-spin curve with broad punctures $z_1, \ldots, z_b$. For $B \in H^2_{\text{et}}(\bar{X}; \mathbb{Z}[1/r])$, a strongly regular perturbation $P = (P_1, \ldots, P_b)$, and a choice $\kappa = (\kappa_1, \ldots, \kappa_b)$ of asymptotic constraints at broad punctures, the moduli space $M_P(C, B, \kappa)$ of perturbed gauged Witten equation over $C$ admits an oriented, metrizable virtual orbifold atlas of virtual dimension $\chi(C, B)$ (which is defined in (3.8)). In particular, when $\chi(C, B) = 0$, there is a well-defined virtual counting

$$\#M_P(C, B, \kappa) \in \mathbb{Q}.$$ 

Further, when there is at least one puncture on $C$, $\#M_P(C, B, \kappa)$ is an integer.

For the precise meanings of virtual orbifold atlas and the virtual counting, see the appendix.

Theorem 3.1 will be proved momentarily, in Subsections 3.2–3.4. Before doing that we first recall the definition of the correlation function.

3.1. Definition of the correlation function.

3.1.1. The state space.

Definition 3.2. Let $\gamma$ be an element of $\mathbb{Z}_r$.

1. If $\gamma$ is narrow, then the $\gamma$-sector of the GLSM state space $\mathcal{H}_\gamma$ is a 1-dimensional $\mathbb{Q}$-vector space, generated by an element $e_\gamma$.
2. If $\gamma$ is broad, then the $\gamma$-sector of the GLSM state space is

$$\mathcal{H}_\gamma := H^{p_\gamma-1}(X_{Q, \gamma}^c/\mathbb{C}^*; \mathbb{Q}) \quad (n_\gamma = \dim_{\mathbb{C}} X_\gamma).$$

3. The total GLSM state space is

$$\mathcal{H}_{\text{GLSM}} := \mathcal{H}_Q := \bigoplus_{\gamma \in \mathbb{Z}_r} \mathcal{H}_\gamma.$$

Suppose $\gamma$ is broad. Since $Q_\gamma := Q|_{X_{\gamma}} : X \to \mathbb{C}$ is homogeneous of degree $r$, for any $a \in \mathbb{C}^*$, the inclusion $Q_\gamma^c := Q_\gamma^{-1}(a) \to X_{Q, \gamma}^c$ induces a diffeomorphism

$$Q_\gamma^c/\mathbb{Z}_r \simeq X_{Q, \gamma}^c/\mathbb{C}^*$$

as orbifolds. By the basic property of cohomology of orbifolds and equivariant cohomology, we know that

$$\mathcal{H}_\gamma \simeq H^{p_\gamma-1}(Q_\gamma^c; \mathbb{Q})^\mathbb{Z}_r. \quad (3.1)$$

The monodromy action on the cohomology $H^*(Q_\gamma^c; \mathbb{Q})$ induced from the locally trivial fibration $Q_\gamma : X_{Q, \gamma}^c \to \mathbb{C}^*$ is equivalent to operator on $H^*(Q_\gamma^c; \mathbb{Z})$ induced from the $\mathbb{Z}_r$-action on $Q_\gamma^c$. Therefore, $\mathcal{H}_\gamma$ is the monodromy invariant part of the middle dimensional rational cohomology of a regular fibre of $Q_\gamma$.

We will use certain natural perfect pairing on the state space.

Notation 3.3. For each $\gamma \in \mathbb{Z}_r$, we denote by $\langle \cdot, \cdot \rangle_\gamma : \mathcal{H}_\gamma \otimes \mathcal{H}_\gamma \to \mathbb{Q}$ a perfect pairing such that with respect to the obvious isomorphism $\varepsilon : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma-1}$, $\langle \cdot, \cdot \rangle_\gamma = \langle \varepsilon, \varepsilon \rangle_{\gamma-1}$. 

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This pairing defines an isomorphism
\[ \theta \mapsto \theta^*: \mathcal{H}_\gamma \mapsto \mathcal{H}_\gamma^* \simeq H_{n\gamma-1}(Q^\omega_{\gamma}; \mathbb{Q})^{\mathbb{Z}_r}. \] (3.2)

**Remark 3.4.** There are many situations where one has a naturally defined perfect pairing. For example, we have the following \( \mathbb{Z}_r \)-equivariant exact sequence
\[ H^{*-1}(X_\gamma) \longrightarrow H^{*-1}(Q^\omega_{\gamma}) \longrightarrow H^*(X_\gamma, Q^\omega_{\gamma}) \longrightarrow H^*(X_\gamma). \]

If the arrow in the middle is an isomorphism in \( \mathbb{Q} \)-coefficients for \( * = n\gamma \) (it is the case when \( X = \mathbb{C}^N \)), then one can define a pairing between \( H^{n\gamma-1}(Q^\omega_{\gamma}; \mathbb{Q}) \) and \( H^{n\gamma-1}(Q^{-\omega}_{\gamma}; \mathbb{Q}) \) as in [FJR13, Page 36].

Following the degree shifting convention of [FJR13, Page 37] (and general shifting scheme of Chen-Ruan orbifold cohomology theory), we define the following quantities. For each \( j \in \mathbb{Z}_r \), we write \( \gamma = \exp(2\pi im_\gamma/r) \) for \( m_\gamma \in \{0, 1, \ldots, r-1\} \). The normal bundle \( N_\gamma \to X_\gamma \) splits as the direct sum of line bundles
\[ N_\gamma = \bigoplus_{i=1}^{n-n_\gamma} L^{(i)}_\gamma \]
such that for each factor \( L^{(i)}_\gamma \), \( \mathbb{Z}_r \) acts by weight \( \nu^{(i)}_\gamma \in \{0, 1, \ldots, r-1\} \) such that \( m_\gamma \nu^{(i)}_\gamma / r \notin \mathbb{Z} \). Here \( n = \dim_{\mathbb{C}} X \). We define
\[ \ell_\gamma = \sum_{i=1}^{n-n_\gamma} \frac{(m_\gamma - 1)\nu^{(i)}_\gamma}{r}. \] (3.3)

Then we define the grading of \( \mathcal{H}_\mathbb{Q} \) such that the sector \( \mathcal{H}_\gamma \) is contained in \( \mathcal{H}^{n\gamma+2\nu_\gamma}_\mathbb{Q} \).

### 3.1.2. Vanishing cycles in \( Q^\omega_{\gamma} \)

The use of Lagrange multiplier requires us to consider the complex Morse theory of the hypersurfaces \( Q^\omega_{\gamma} \). If we have a holomorphic Morse function \( F \) defined on \( Q^\omega_{\gamma} \), then the unstable or stable manifolds of the critical points of \( F \) (with respect to the negative gradient flow of the real part of \( F|_{Q^\omega_{\gamma}} \)) represent certain \( \infty \)-relative cycles.

For any compact subset \( Z \subset X \), consider the relative homology \( H_*(Q^\omega_{\gamma}, Q^\omega_{\gamma} \setminus Z) \). The inverse limit with respect to the direct system of compact subsets under inclusion is denoted by
\[ H_*(Q^\omega_{\gamma}, \infty). \]

This is the dual space of \( H^*_c(Q^\omega_{\gamma}) \). Then we have the intersection pairing
\[ \cap: H_*(Q^\omega_{\gamma}) \otimes H_*(Q^\omega_{\gamma}, \infty) \to \mathbb{Z}. \] (3.4)

Now take strongly regular perturbation \( P = (P_1, \ldots, P_b) \) at broad punctures of \( C \), where \( P_j = (a_j, F_j) \). Abbreviate \( Q_{\gamma j} \) by \( Q_j \) and \( Q^{-1}_{\gamma j}(a_j) \) by \( Q^-_j \). Consider the negative gradient flow of the real part of \( F \) restricted to \( Q^\omega_j \). For each critical point \( \kappa_j \) of \( F_j|_{Q^\omega_j} \), denote by
\[ [W^\omega_{\kappa_j}] \in H_{n_j-1}(Q^\omega_j, F^-_j; \mathbb{Z}) \quad \text{(resp. } [W^\omega_{\kappa_j}] \in H_{n_j-1}(Q^\omega_j, F^+_j; \mathbb{Z}) \text{)} \]
the class of the unstable (resp. stable) manifold of this flow. Here
\[ F^\omega_j = Q^\omega_j \cap (\text{Re}F_j)^{-1}([M, +\infty)), \quad F^-_j = Q^\omega_j \cap (\text{Re}F_j)^{-1}((\infty, -M]) \]
for some \( M >> 0 \). We still use \( [W^\omega_{\kappa_j}/s] \) to denote their images under the map
\[ H_{n_j-1}(Q^\omega_j, F^\pm_j; \mathbb{Z}) \to H_{n_j-1}(Q^\omega_j, \infty; \mathbb{Z}). \]
3.1.3. The correlation function. The correlation function is a multilinear map

\[ \langle \cdot \rangle_C^B : \bigotimes_{j=1}^k \mathcal{H}_{\gamma_j} \to \mathbb{Q}. \]  

To define (3.5), we choose the last \( n \) inputs (narrow states) to be the generators of the corresponding sectors \( \theta_j = e_{\gamma_j} \in \mathcal{H}_{\gamma_j}, \ j = b+1, \ldots, b+n \). Suppose the first \( b \) inputs (the broad states) are \( \theta_j \in \mathcal{H}_{\gamma_j}, \ j = 1, \ldots, b \). Then define

\[ \langle \theta_1, \ldots, \theta_b, \theta_{b+1}, \ldots, \theta_m \rangle_{C,B,P} := \sum_{\kappa} \#\mathcal{M}_P(C,B,\kappa) \prod_{j=1}^b \theta_j^* \cap [W_{\kappa_j}^u]. \]  

Here \( \theta_j^* \in H_{a_j-1}(Q^\partial_j; \mathbb{Q})^{|\mathbb{Q}|} \) is the image of \( \theta_j \) under (3.2) and the \( \cap \) is the intersection (3.4). In general, (3.5) is defined by taking linear extension of the above values.

3.2. Review the linear theory.

3.2.1. Banach manifold, Banach bundle and the Fredholm section. Let \( \mathcal{C} \) be a rigidified \( r \)-spin curve with underlying Riemann surface \( \Sigma \). Let \( z = \{ z_1, \ldots, z_{b+n} \} \) be the set of punctures and the monodromy of \( \mathcal{C} \) at \( z_j \) is \( \gamma_j \). We assume \( \gamma_j \) is broad for \( j = 1, \ldots, b \) and narrow for \( j = b+1, \ldots, b+n \).

We slightly modify the weighted Sobolev spaces we used in [TX15], but will give equivalent theories. Let \( \tau > 0 \). We use \( \tilde{W}^{k,p}(\Sigma^*, E) \) to denote the space of sections \( s \) of a vector bundle \( E \) over \( \Sigma^* \) satisfying

1. \( s \) is of class \( W^{k,p}_{\text{loc}}; \)
2. \( s|_{C_j} \) for \( z_j \) narrow is of class \( W^{k,p}_\tau \) and \( s|_{C_j} \) for \( z_j \) broad is of class \( W^{k,p} \).

Here the Sobolev norms is taken with respect to some fixed choice of connection on \( E \). We will omit the domain \( \Sigma^* \) in this section and abbreviate the space by \( \tilde{W}^{k,p}(E) \). When \( k = 0 \) we also denote it by \( L^p(E) \).

If \( z_j \) is broad, then \( \tilde{W}_j|_{\tilde{X}_j} \) is a holomorphic Morse function with finitely many critical points \( \kappa_j^{(i)}, i = 1, \ldots, m_j \). Moreover, we regard \( \delta \in (0,1] \) as an element of \( G^0_{\delta} \) and we use \( m_\delta \) to denote the action by applying \( \delta \). Then

\[ \kappa_j^{(i)} := m_\delta \kappa_j^{(i)} \in \text{Crit}(\tilde{W}_j|_{\tilde{X}_j}). \]

For each \( A \in \mathcal{A} \), we have \( \delta_\lambda \) given by (2.6). Then for each \( A \in \mathcal{A} \), denote \( \kappa_{j,A}^{(i)} = \kappa_j^{(i)} \).

For any \( x_j \in \tilde{X}_j \), define \( \tilde{x}_j : S^1 \to \tilde{X}_j \) by \( \tilde{x}_j(t) = e^{-\lambda t} x_j \). The pull-back of \( \tilde{x}_j \) to any cylinder \( [a, b] \times S^1 \) via the projection \( [a, b] \times S^1 \to S^1 \) is still denoted by \( \tilde{x}_j \).

Now we state the definition of the Banach manifold of fields. Choose a small \( \tau > 0 \) and \( B \in H^{\partial}_2(X; \mathbb{Z}[1/r]) \). For \( j = 1, \ldots, b \), choose \( \iota_j \in \{ 1, \ldots, m_j \} \) and denote

\[ \kappa_j = \kappa_j^{(\iota_j)}, \ \kappa_{j,A} = \kappa_j^{(\iota_j)}, \ \kappa = (\kappa_j^{(\iota_j)})_{j=1}^b. \]

**Definition 3.5.** We define

\[ \mathcal{B} := \mathcal{B}^{1,p}_r(B, \kappa) \subset A^{1,p}_r \times W^{1,p}_{\text{loc}}(\Sigma^*, \tilde{Y}) \times \prod_{j=1}^b \text{Fr}(P_j, z_j), \]

which consists of tuples \( \mathcal{X} = (A, u, \varphi) \) such that

1. There is an \( S > 0 \) satisfying the following conditions.
   - For each broad \( z_j \), there is a section \( \tilde{\eta}_j \in W^{1,p}(C_j(S), (\kappa_{j,A})^* T\tilde{X}) \) such that
     \[ u|_{C_j(S)} = \varphi_j\left( \exp_{\kappa_{j,A}} \tilde{\eta}_j \right). \]
• For each narrow $z_j$, there is $p_j \in \tilde{X}_j$ and $\tilde{\eta}_j \in W^{1,p}_\tau(C_j(S), T_{p_j} \tilde{X})$ such that
  \[ u|_{C_j(S)} = \phi_j(\exp_{p_j} \tilde{\eta}_j). \]

(2) The above two conditions implies that $u$ extends to an orbifold section over $C$ and we require that $[A, u] = B$.

**Definition 3.6.** $E^-$ is the vector bundle $E^- \to B$ whose fibre over $\mathcal{X} = (A, u, \varphi)$ is

\[ E|_{\mathcal{X}} = L^p_\tau(L^{0,1} \oplus u^*T\tilde{Y}^\perp) \oplus L^p(g). \]

Denote $G_\tau = G \cap G^{2,p}_\tau$, which acts on $B$ on the first two components $A$ and $u$ by gauge transformations. The $G_\tau$-action lifts to $E^-$, making it an equivariant vector bundle. The perturbed gauged Witten equation induces a smooth, $G_\tau$-equivariant section of $E^- \to B$. More precisely, for $\mathcal{X} = (A, u, \varphi) \in B$, the left-hand-side of (2.8) defines

\[ \mathcal{F}^-(A, u, \varphi) = \left( W(A, u, \varphi), V^-(A, u, \varphi) \right) \in E^-|_{\mathcal{X}}. \]

Here the letter $W$ represents the Witten equation and $V$ represents the vortex equation.

We summarize our linear model as the following proposition.

**Proposition 3.7.**

(1) $B$ has a Banach manifold structure, whose tangent space at $(A, u, \varphi)$ is isomorphic to

\[ T_{\mathcal{X}} B \simeq T \mathcal{A}^{1,p}_\tau \oplus \tilde{W}^{1,p}_\tau(u^*T\tilde{Y}^\perp) \oplus \mathbb{C}^n \oplus \mathbb{R}^p. \quad (3.7) \]

(2) $G_\tau$ acts smoothly on $B$ such that the isomorphism (3.7) is equivariant in a natural way. Moreover, when there is at least one puncture on $C$, the $G_\tau$-action is free.

(3) $E^- \to B$ is a $G_\tau$-equivariant, smooth Banach vector bundle. $\mathcal{F}^- : B \to E^-$ is a $G_\tau$-equivariant section.

(4) For each $B \in H^2_\ell(\tilde{X}, \mathbb{Z}[1/r])$, there exists $\tau = \tau(B) > 0$ such that for any bounded solution $(A, u, \varphi)$ to the perturbed gauged Witten equation over $C$ satisfying: 1) $u_*(\Sigma) = B$; 2) for each broad puncture $z_j$, $ev_j(A, u, \varphi) = \kappa_j$, there is a gauge transformation $g \in \mathcal{G}$ such that $g^*(A, u, \varphi) \in B^{1,p}_\tau(B, \kappa)$.

**Proof.** It is partially proved in [TX15] so here we only explain what was missing there. For (1), the difference from the case of [TX15] is that we shall vary the framing at broad punctures. This gives the additional $\mathbb{R}^p$-factor in (3.7). For (2), we know that an automorphism of a connection is a constant gauge transformation. When there are punctures, $\mathcal{G}$ contains no constants other than the identity. □

### 3.2.2. Deformation theory and gauge fixing.

The linearization of $\mathcal{F}^-$ at $\mathcal{X} \in B$ is a bounded linear operator

\[ D_{\mathcal{X}} \mathcal{F}^- : T_{\mathcal{X}} B \to E^-|_{\mathcal{X}}. \]

On the other hand, the linearization of infinitesimal gauge transformation is a linear operator

\[ D_{\mathcal{X}} G : \ W^{2,p}_\tau(g) \to (W^{1,p}_\tau(A^1 \otimes g)) \oplus \tilde{W}^{1,p}_\tau(u^*T\tilde{Y}^\perp) \]

\[ \xi \mapsto (d\xi, -L\xi). \]

Then the deformation complex at $\mathcal{X}$ is the following complex of Banach spaces

\[ \operatorname{Lie} G \xrightarrow{D_{\mathcal{X}} G} T_{\mathcal{X}} B \xrightarrow{D_{\mathcal{X}} \mathcal{F}^-} E^-|_{\mathcal{X}}. \]

A usual way of considering problems with gauge symmetry is to take special slices of the action by the group of gauge transformations. In our case the gauge group is abelian, so one can
transform all connections to a common slice. More precisely, choose from now on a smooth reference connection \(A_0 \in \mathcal{A}_r\). Consider the operator
\[
d^* : T\mathcal{A}_r \to L^p_r(g).
\]
One tends to use the gauge-fixing condition \(d^*(A - A_0) = 0\). However, it may change the index of the problem since \(d^*\) may not be surjective. Instead, choose smooth, compactly supported elements \(s_1, \ldots, s_k \in L^p_r(g)\) such that
\[
\text{Im}(\Delta : W^{2,p}_r(g) \to L^p_r(g)) + \text{Span}\{s_1, \ldots, s_k\} = L^p_r(g).
\]
Here \(-k\) is the index of the operator \(\Delta : W^{2,p}_r(g) \to L^p_r(g)\). Denote \(\Lambda_{ob} = \text{Span}\{s_1, \ldots, s_k\}\) and
\[
\overline{d}^* : T\mathcal{A}_r \to \frac{L^p_r(g)}{\Lambda_{ob}}.
\]

**Definition 3.8.** \(A\) is said to be in Coulomb gauge (relative to \(A_0\)) if \(\overline{d}^*(A - A_0) = 0\).

**Lemma 3.9.** Any connection in \(\mathcal{A}_r\) can be transformed via a gauge transformation in \(\mathcal{G}^{2,p}_r\) to a unique connection in Coulomb gauge relative to \(A_0\).

**Proof.** Take \(A \in \mathcal{A}_r\). Consider gauge transformations of the form \(g = e^h\) for \(h \in W^{2,p}_r(g)\). The Coulomb gauge condition relative to \(A_0\) for \((e^h)^*A\) reads
\[
\Delta h + d^*(A - A_0) \in \Lambda_{ob}.
\]
This is uniquely soluble modulo the obstruction space \(\Lambda_{ob}\). \(\square\)

3.2.3. **The linearized operator and the index formula.** Denote \(\mathcal{E}|_X = \mathcal{E}^-|_X \oplus L^p_r(g)/\Lambda_{ob}\). Then \(\mathcal{E} \to \mathcal{B}\) is a Banach space bundle. Define
\[
\mathcal{F} : \mathcal{B} \to \mathcal{E}, \quad \mathcal{F}(X) = (\mathcal{F}^-(X), \overline{d}^*(A - A_0)).
\]
We also denote
\[
\mathcal{V}(X) = (\mathcal{V}^-(X), \overline{d}^*(A - A_0)) \in L^p_r(g) \oplus \frac{L^p_r(g)}{\Lambda_{ob}}.
\]
The linearized operator at \(X\) is a bounded linear operator
\[
\mathcal{D}_X \mathcal{F} : T_X \mathcal{B} \to \mathcal{E}|_X.
\]
Finally, we state the index formula proved in [TX15]. Define
\[
b_j = \dim_{\mathbb{C}} N_j.
\]
For each \(j\), the normal bundle \(N_j := \tilde{N}_j \to \tilde{X}_j\) splits as
\[
\tilde{N}_j = \bigoplus_{i=1}^{\text{codim}_X} \tilde{N}_j^{(i)}
\]
where each line bundles \(\tilde{N}_j^{(i)}\) has associated weight \(\nu_j^{(i)} \in \mathbb{Z}\) such that \((\gamma_j)\nu_j^{(i)} \neq 1\). We define
\[
m_j = -\hat{i} \sum_i \left(\nu_j^{(i)} \lambda_j - \lfloor \nu_j^{(i)} \lambda_j \rfloor\right) \in \mathbb{Q}_{\geq 0}.
\]
Here \(|a| \in \mathbb{Z}\) is the greatest integer which is no greater than \(a \in \mathbb{R}\).

**Theorem 3.10.** For any \(X \in \mathcal{M}_r(B)\), the operator \(d^*_X\mathcal{F}\) is Fredholm with index
\[
\text{ind}(\mathcal{D}_X \mathcal{F}) = \chi(C, B) := (2 - 2g)\dim_{\mathbb{C}} X + 2e_1^r(B) - \sum_{j=1}^b b_j - 2 \sum_{j=1}^k m_j. \quad (3.8)
\]
Here \(e_1^r(B)\) is the equivariant Chern number of the class \(B\).
3.3. The orientation of $D_XF$. By an orientation of a Fredholm operator $F : E \to E'$ we mean an orientation of the determinant line

$$\det F := \det \ker F \otimes (\det \text{coker} F)'.$$

An orientation of a continuous family of Fredholm operators

$$F_x : E_x \to E_x', \ x \in N$$

is a continuous trivialization of the determinant line bundle $\det F \to N$. Hence if there is a homotopy of Fredholm operators $F_{x,t} : E_x \to E_x', t \in [0,1]$, then the orientability problem of the family $F_{x,0}$ is equivalent to that of $F_{x,1}$.

In our case, consider the family of linearized operators

$$D_XF : T_XB \to \xi_X, \ X \in B.$$ 

We also have the family of linearizations

$$D_XW : \bar{W}_{1,p}^j(S^*, \Lambda^* \bar{\nabla}^{-1}) \to L_p^j(S^*, \Lambda^{0,1} \otimes \Lambda^* \bar{\nabla}^{-1}), \ X = (A, u, \varphi).$$

Lemma 3.11. The family $\{D_XF\}_{X \in B}$ is oriented if and only the family $\{D_XW\}_{X \in B}$ is oriented, and there is a canonical identification between their orientations.

Proof. Firstly, the finite rank part of $D_XF$ on the variables $\zeta$ and $\rho$ doesn’t affect the orientability problem. So one only needs to consider following operator at $a = 1$.

$$\left( \begin{array}{c} \beta \\ v \end{array} \right) \mapsto \left( \begin{array}{c} D_XW(v) + aD'_XW(\beta) \\ *d\beta + avd\mu(u) \cdot v \\ \overline{d\beta} \end{array} \right).$$

Here $a \in [0,1]$ and $D'_XW$ is the other component in the linearization of $W$. By varying $a$ from 1 to 0, this family of operators remain Fredholm and hence the orientation is reduced to the case of $a = 0$. Moreover, the operator $\beta \mapsto (*d\beta, \overline{d\beta})$ is independent of $X$, hence is naturally orientable. Therefore, the orientability is reduced to the family $\{D_XW\}_{X \in B}$. □

Next, notice that $B$ is of a fibre bundle structure, while the condition on the section $u$ depends on the connection $A$ and the framing $\varphi$. We denote by $B_{A,\varphi}(\kappa) \subset B(\kappa)$ be the subset consisting of $(A, u, \varphi)$ with fixed $(A, \varphi)$. Moreover, since $A$ lies in a contractible space, which doesn’t affect the orientability. Hence we omit the dependence of $A$ and denote by $B(\varphi, \kappa)$ be the space of all such $(A, u, \varphi)$’s.

Then the orientability of $\{D_XW\}_{X \in B(\varphi, \kappa)}$ is similar to the case of ordinary Gromov-Witten and Floer theory. The orientation can be chosen in a coherent way in the sense of [FH93]. More precisely, it means the following. For each critical point $\kappa_j \in \text{Crit} W_j|_{X_j}$, one can choose an orientation on the unstable manifold $W^u_{\kappa_j}$. Then for two critical points $\kappa_j, \kappa'_j$, one chooses a curve $l_j : (-\infty, +\infty) \to \tilde{X}_j$ such that $l_j|_{(-\infty, -1]} = \kappa_j$ and $l_j|_{[1, +\infty)} = \kappa'_j$. Define

$$\tilde{l}_j : \Theta \to \tilde{X}, \ \tilde{l}(s,t) = e^{-\lambda s}l(s).$$

Then one can glue each $u \in B(\varphi, \kappa)$ with $\tilde{l}_j$ in a consistent way and obtain a continuous map

$$l : B(\varphi, \kappa) \to B(\varphi, \kappa').$$

Let the determinant line bundle be $L \to B(\varphi, \kappa)$ and $L' \to B(\varphi, \kappa')$ respectively.

On the other hand, the choice of orientations on the unstable manifolds of $\kappa_j$ and $\kappa'_j$ induces an orientation of the operator

$$Dl_j : W^{1,p}_j(\mathbb{R}, l_j^* T\tilde{X}_j) \to L^p(\mathbb{R}, l_j^* T\tilde{X}_j), \ Dl_j(\xi) = \nabla_\xi \xi + \nabla^2 {\bar{W}}_j(l_j) \cdot \xi$$

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and hence an orientation of the linearization of the soliton equation along $\tilde{t}_j$ (for details see Section 9). The gluing process induces an isomorphism (well-defined up to homotopy)

$$l_* : \mathcal{L} \to t^*\mathcal{L}'$$.

**Definition 3.12.** An orientation of $\{D_X\mathcal{W}\}_{X \in \mathcal{B}(\varphi_\kappa)}$ and an orientation of $\{D_{X'}\mathcal{W}\}_{X' \in \mathcal{B}(\varphi_{\kappa'})}$ are coherent with respect to the orientations on the unstable manifolds of all $\kappa_j$ and $\kappa'_j$, if they are consistent with the isomorphism $l_*$.

The following proposition follows by applying the well-known results of Floer-Hofer [FH93].

**Proposition 3.13.** Let $\varphi \in \prod_{j=1}^b Fr(P_j, z_j)$ and $\kappa$ be a choice of asymptotic constrains.

1. The family $\{D_X\mathcal{W}\}_{X \in \mathcal{B}(\varphi_\kappa)}$ is orientable.
2. If we choose orientations on the unstable manifolds $W^u_{\kappa_j}$ of the critical points $\kappa_j$, then there is a canonically induced orientation on $\{D_X\mathcal{W}\}_{X \in \mathcal{B}(\varphi_\kappa)}$. Moreover, if we change the orientation on one $W^u_{\kappa_j}$, then the orientation of $\{D_X\mathcal{W}\}_{X \in \mathcal{B}(\varphi_\kappa)}$ changes.
3. Let $\kappa = (\kappa_1, \ldots, \kappa_b)$ and $\kappa' = (\kappa'_1, \ldots, \kappa'_b)$ be two choices of asymptotic constrains and we choose orientations on $W^u_{\kappa_j}$ and $W^u_{\kappa'_j}$ for all $j$. Then the induced orientation on $\{D_X\mathcal{W}\}_{X \in \mathcal{B}(\varphi_\kappa)}$ and the induced orientation on $\{D_{X'}\mathcal{W}\}_{X' \in \mathcal{B}(\varphi_{\kappa'})}$ are coherent.

Lastly, we need to show that the orientation bundle over the space of all $\varphi$’s is trivial. It suffices to consider the variation of one $\varphi_j$. It is easy to see that changing $\varphi_j$ to $e^{i\theta}\varphi_j$ is the same as fixing $\varphi_j$ while changing $(a_j, F_j)$ to $(e^{i\theta}a_j, F_j \circ e^{-i\theta})$. On the other hand, the orientation of $\{D_X\mathcal{W}\}_{X \in \mathcal{B}(\varphi_\kappa)}$ only depends on the orientation on $W^u_{\kappa_j}$. As we vary $\theta$, the critical point $\kappa_j = (p, x_j)$ is changed to $e^{i\theta}\kappa_j := (e^{-i\theta}p, e^{i\theta}x_j)$. Moreover, $e^{i\theta}$ will map the unstable manifold of $\kappa_j$ to that of $e^{i\kappa_j}$. It is also easy to see that as $\theta$ moves from 0 to $2\pi$, the orientation on the unstable manifold doesn’t alter. Therefore, the orientation bundle is trivial over the space of $\varphi$’s.

### 3.4 Virtual cycle on $\mathcal{M}_p(B)$.

Since $P$ is strongly regular, $\mathcal{M}_p(B) = \overline{\mathcal{M}}_p(B)$ because there is no nontrivial solitons. Hence $\mathcal{M}_p(B)$ is compact with respect to the weak topology. We remark that the weak topology on $\mathcal{M}_p(B)$ coincides with the topology induced from $\mathcal{B}$ (i.e., the strong topology). This is due to the common feature of $\epsilon$-regularity in elliptic theories; in our case, it was proved in [TX15, Section 4] that when the energy on a cylindrical end is sufficiently small, the energy density should decay exponentially. Hence if a sequence in $\mathcal{M}_p(B)$ converges in the weak topology, since there cannot be any energy escape at infinity to form a nontrivial soliton in the limit, the sequence also converges strongly.

Then one can construct a virtual orbifold atlas (and hence a virtual cycle). There is no gluing procedure to be performed. (In fact one can cover $\mathcal{M}_p(B)$ by a single chart, similar to the case of [Rua98], but we don’t need this fact here.) Indeed, take an arbitrary $\mathcal{X} \in \mathcal{M}_p(B)$. If $\Sigma$ has at least one puncture, then the $\mathcal{G}$-action on $\mathcal{A}$ is free. We proceed with this assumption and the special case in which $\mathcal{C}$ is a smooth surface is commented in Remark 3.15. Moreover, there is no nontrivial automorphisms of solutions to $\mathcal{F}(\mathcal{X}) = 0$. If $\Sigma$ has no puncture (see the remark below), then $\mathcal{X}$ possibly has a nontrivial finite automorphism group $\Gamma_\mathcal{X}$. In any case, the linearization $D_X \mathcal{F}$ is $\Gamma_\mathcal{X}$-equivariant, and one can find an obstruction space $E_X \subset \mathcal{E}|_\mathcal{X}$, which is $\Gamma_\mathcal{X}$-invariant, and which satisfies

$$\text{Im}(D_X \mathcal{F}) + E_X = \mathcal{E}_X$$.

Then for any element $\mathcal{X}' \in \mathcal{B}$ in a small neighborhood $\mathcal{U}$ of $\mathcal{X}$, one can use parallel transport to embed $E_X$ into $\mathcal{E}|_{\mathcal{X}'}$. For $e \in E_X$, we denote by $e(\mathcal{X}') \in \mathcal{E}|_{\mathcal{X}'}$ be the embedding image of $e$. This induces a smooth section

$$\mathcal{F}_X : \mathcal{U} \times E_X \to \mathcal{E}|_{\mathcal{U}}, \mathcal{F}_X(\mathcal{X}') = \mathcal{F}(\mathcal{X}') + e(\mathcal{X}')$$.
If we take $\mathcal{U}$ sufficiently small, then $\tilde{U}_X = \mathcal{F}^{-1}_X(0)$ is a smooth manifold of dimension $\chi(B) + \dim E_X$. Denote $U_X = \tilde{U}_X / \Gamma_X$ and with abuse of notation, denote $E_X = (\tilde{U}_X \times E_X) / \Gamma_X$ as an orbibundle over $U_X$. There is the natural section $S_X : U_X \to E_X$ induced by the projection $\mathcal{U} \times E_X \to E_X$. $S_X^{-1}(0)$ naturally embeds into $\mathcal{M}_P(B)$, via a map $\psi_X$ and with image $E_X$ an open neighborhood of $\mathcal{X}$ in $\mathcal{M}_P(B)$. This constructs a local chart $K_X = (U_X, E_X, S_X, \psi_X, F_X)$.

To construct a virtual atlas, one takes finitely many $X_i \in \mathcal{M}_P(B)$, $i = 1, \ldots, N$ such that $\{F_X \mid i = 1, \ldots, N\}$ is an open cover of $\mathcal{M}_P(B)$. One can construct the so-called “sum charts”, indexed by elements of the partially ordered set
\[
\mathcal{I} = \{I \subset \{1, \ldots, N\} \mid \bigcap_{i \in I} F_{X_i} \neq \emptyset\},
\]
where the partial order is defined by $I \leq J$ if $I \subset J$. We can choose an order on the set $\mathcal{I}$ as $\{I_1, I_2, \ldots, I_m\}$ such that

- For $k = 1, \ldots, m$, if $J \in \mathcal{I}$ and $J \geq I_k$, then $J \in \{I_1, \ldots, I_m\}$.

We introduce a notation $\sqsubseteq$ which will be used frequently in this paper. If $X$ is a topological space and $Y_1, Y_2 \subset X$, then we denote by $Y_1 \sqsubseteq Y_2$ if $\overline{Y_1}$ is compact and is contained in $Y_2$. Since $\mathcal{M}_P(B)$ is metrizable, for $i = 1, \ldots, N$ and $k = 1, \ldots, m$, we can choose open subsets $F_i^{(k)}, G_i^{(k)}$ of $F_X$, such that
\[
F_i^{(1)} \sqsubseteq G_i^{(1)} \sqsubseteq \cdots \sqsubseteq G_i^{(m)} \sqsubseteq F_i \sqsubseteq F_X,
\]
and such that $\{F_i^{(1)} \mid i = 1, \ldots, N\}$ is still an open cover of $\mathcal{M}_P(B)$.

**Lemma 3.14.** For $k = 1, \ldots, m$, if we define
\[
F_{I_k} := \left( \bigcap_{i \in I_k} F_i^{(k)} \right) \setminus \left( \bigcup_{j \notin I_k} G_j^{(k)} \right),
\]
then $\{F_{I_k} \mid k = 1, \ldots, m\}$ is an open cover of $\mathcal{M}_P(B)$ and satisfies the overlapping condition (see Definition A.15), namely,
\[
\overline{F_{I_k}} \cap \overline{F_{I_l}} \neq \emptyset \implies I_k \leq I_l \text{ or } I_l \leq I_k.
\]

**Proof.** We first check the overlapping condition. Suppose $k < l$ and $\overline{F_{I_k}} \cap \overline{F_{I_l}} \neq \emptyset$, we need to show that $I_k \leq I_l$. Suppose it is not the case, then there exists $i \in I_k \setminus I_l$. However, for any $x \in \overline{F_{I_k}} \cap \overline{F_{I_l}}$, $x \in F_i^{(k)} \subset G_i^{(l)}$; moreover,
\[
x \in \overline{F_{I_k}} \subset \mathcal{M}_P(B) \setminus \bigcup_{j \notin I_k} G_j^{(l)} \subset \bigcap_{j \notin I_k} \left( \mathcal{M}_P(B) \setminus G_j^{(l)} \right) \subset \mathcal{M}_P(B) \setminus G_i^{(l)}.
\]
which is a contradiction. Hence the overlapping condition holds.

Now we prove that all $F_{I_k}$ form an open cover. Since all $F_i^{(1)}$ form an open cover, for each $x \in \mathcal{M}_P(B)$, there is some $I_k = \{i_1, \ldots, i_s\} \in \mathcal{I}$ such that
\[
x \in \bigcap_{i \in I_k} F_i^{(1)} \subset \bigcap_{i \in I_k} F_i^{(k)}.
\]
Choose the largest $k$ such that $x \in \bigcap_{i \in I_k} F_i^{(k)}$ holds. If $x \notin F_{I_k}$, then there is some $j \notin I_k$ such that $x \in G_j^{(k)} \subset F_X$. Then there is some $l > k$ such that $I_l = I_k \cup \{j\} \in \mathcal{I}$. So $x \in G_j^{(l)} \subset F_j^{(l)}$. Then
\[
x \in \left( \bigcap_{i \in I_k} F_i^{(k)} \right) \cap F_j^{(l)} \subset \bigcap_{i \in I_l} F_i^{(l)},
\]
which contradicts the maximality of $k$. Therefore $x \in F_{I_k}$. This proves that all $F_{I_k}$ form an open cover of $\mathcal{M}_P(B)$. \qed
3.4.1. The local charts. If $F_I = \emptyset$, then we remove $I$ from the set $\mathcal{I}$.

Next, we can construct charts of $\mathcal{M}_P(B)$ whose footprints are $F_I$ (we are using the terminology of Definition A.10). Indeed, each $F_I$ can be viewed as subsets of the Banach manifold $B$. Denote

$$E_I := \bigoplus_{x \in I} E_x,$$

and let $E'_{I}$ be its $\varepsilon$-ball centered at the origin. Then for a small neighborhood of $\mathcal{U}_I \subset B$ of $\overline{F'_{I}}$, there is a homomorphism $\text{Emb}_I : E_I \to \mathcal{E}_{|\mathcal{U}_I}$, which is also denoted by $e_I \mapsto e_I(\mathcal{X})$ (for $\mathcal{X} \in \mathcal{U}_I$). Consider the section

$$\mathcal{F}_I : E_I \times \mathcal{U}_I \to \mathcal{E}_{|\mathcal{U}_I}, \quad \mathcal{F}_I(e_I, \mathcal{X}) = e_I(\mathcal{X}) + \mathcal{F}(\mathcal{X}).$$

By transversality, for $\mathcal{U}_I$ and $\varepsilon > 0$ sufficiently small, $U'_{I} := \mathcal{F}_I^{-1}(0) \cap (E'_{I} \times \mathcal{U}_I)$ is a smooth manifold of dimension $\chi(B) + \dim E_I$. This gives a local chart

$$K'_I := (U'_{I}, U'_I \times E_I, S_I, \psi_I, F'_I)$$

where $S_I$ is the restriction of the projection $E_I \times \mathcal{U}_I \to E_I$ to $U'_I$, $\psi_I : S_I^{-1}(0) \to \mathcal{M}_P(B)$ is the natural map and $F'_I$ is the footprint which contains $U'_I$. We can shrink the chart $K'_I$ such that its footprint is exactly $F_I$. Denote the shrunk chart by

$$K_I = (U_I, E_I, S_I, \psi_I, F_I).$$

3.4.2. The weak coordinate changes. For each pair $I, J \in \mathcal{I}$, if $\overline{F'_{I}} \cap \overline{F'_{J}} \neq \emptyset$, then (assuming $I \leq J$ by Lemma 3.14) one can construct weak coordinate changes $T_{IJ} = (U_{IJ}, \phi_{IJ}, \pi_{IJ})$ (see Definition A.13). Indeed, for each $I \in \mathcal{I}$, choose $F'_I \subset F_I$ such that $\{F'_I \mid I \in \mathcal{I}\}$ is still an open cover of $\mathcal{M}$. Then there is an open neighborhood $\mathcal{U}_{IJ} \subset \mathcal{U}_I \cap \mathcal{U}_I \subset B$ of $\overline{F'_I} \cap \overline{F'_J}$. By the inclusion $E_I \subset E_J$ and transversality, there is a natural embedding

$$\phi_{IJ} : U_I \cap \mathcal{U}_{IJ} \to U_J,$$

which is lifted to a bundle embedding.

$$\hat{\phi}_{IJ} : E_I|_{U_I \cap \mathcal{U}_{IJ}} \to E_J.$$

Then define $U_{IJ} = U_I \cap \mathcal{U}_{IJ}$. Moreover, the finite-dimensional implicit function theorem implies that there is a tubular neighborhood $N_{IJ} \subset U_J$ of $\phi_{IJ}(U_{IJ})$ together with a projection

$$\nu_{IJ} : N_{IJ} \to \phi_{IJ}(U_{IJ})$$

such that the following holds. If we denote by $\tilde{\pi}_{IJ} : E_J \to \bigoplus_{x \in I} E_x$ and denote the composition

$$N_{IJ} \xrightarrow{S_J} E_J \xrightarrow{1 - \tilde{\pi}_{IJ}} \bigoplus_{x \in I \setminus J} E_x =: \ker \tilde{\pi}_{IJ}$$

by $S_{IJ}$, then $S_{IJ}$ is a fibrewise homeomorphism.

3.4.3. The strong coordinate changes. To construct strong coordinate changes, we need to shrink the charts $U_I$. We choose $F'_I \subset F_I$ such that $\{F'_I \mid I \in \mathcal{I}\}$ is still an open cover of $\mathcal{M}_P(B)$. By the above construction of the weak coordinate changes, we know that $F_{IJ} := \psi_I(S_I^{-1}(0) \cap U_{IJ}) \supset F'_I \cap F'_J$. Then for each pair $I, J \in \mathcal{I}$ with $I \leq J$, we have $F'_I \cap F'_J \subset F_{IJ}$. By applying Proposition A.24, we find shrinkings (see Definition A.10) $K'_I \subset K_I$, $K'_J \subset K_J$ such that

1. The induced coordinate change $T'_{IJ} : K'_I \to K'_J$ is a strong coordinate change.
2. The footprint $F'_I$ of $K'_I$ contains $F'_I$. 

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For all pairs $I, J \in \mathcal{I}$ with $I \leq J$, we can do the above shrinking process successively. More precisely, let the partial order on $\mathcal{I}$ be the finite set

$$\{(I_\nu, J_\nu) \mid \nu = 1, \ldots, m\}.$$  

For $k = 2, \ldots, m$, suppose we have replaced all the charts $K_I$ by $K_I'$, which is either a shrinking of $K_I$ or the same as $K_I$, and modified the coordinate changes to $T_{JJ}^{II}$ by restriction, such that

1. $F_{I}^{I'} \subset F_{I}^{I}$, where $F_{I}^{I'}$ is the footprint of $K_I'$;
2. $F_{I}^{I} \cap F_{J}^{I} \subset F_{JJ}^{II}$, where $F_{JJ}^{II}$ is the footprint of $T_{JJ}^{II}$;
3. $T_{JJ}^{II} : K_{I_\nu}' \to K_{J_\nu}'$ is a strong coordinate change for $\nu = 1, \ldots, k - 1$.

Then by Proposition A.24, for the pair $I_k \leq J_k$, we can find a shrinking $K_{I_k}'' \subset K_{I_k}'$, $K_{J_k}'' \subset K_{J_k}'$ such that the induced coordinate change $T_{I_k I_k}^{J_k J_k} : K_{I_k}'' \to K_{J_k}''$ is strong and the footprint $F_{I_k}''$ (resp. $F_{J_k}''$) still contains $\overline{F_{I_k}'}$ (resp. $\overline{F_{J_k}'}$). Moreover, we have to modify all coordinate changes which involve $I_k$ or $J_k$. Let the modified coordinate changes be $T_{JJ}^{II}$ with footprints $F_{JJ}^{II}$. It is easy to see that $F_{I}^{I} \cap F_{J}^{I} \subset F_{JJ}^{II}$ holds for all $I \leq J$. Therefore the induction continues and one can finish the process so that eventually all coordinate changes become strong.

Lastly, we see that the collection $\mathfrak{A} = (\{K_I \mid I \in \mathcal{I}\}, \{T_{JJ} \mid I \leq J\})$ on $\mathcal{M}_P(B)$ satisfy the cocycle condition, the filtration condition (by the direct sum construction of $E_I$) and the overlapping condition (by Lemma 3.14) of Definition A.15, hence is a strong virtual orbifold atlas equipped with an orientation. By Lemma A.19 and Lemma A.20, an appropriate shrinking of $\mathfrak{A}$ produces a metrizable atlas. This finishes the proof of Theorem 3.1.

Remark 3.15. We comment on a special case. When there is no puncture, i.e., there is a smooth $r$-th root of $K_S$, which is only possible when $r$ divides $2g - 2$, the $G_r$-action on $B$ may not be free because $G_r$ contains constant gauge transformations now. On the other hand, by the energy identity, if $(A, u)$ is a solution in this case, then $u(\Sigma) \subset \text{Crit} W$ and $(A, u)$ is also a solution to the symplectic vortex equation.

In this case, the first thing we should do is to choose the moment map (by adding to $\mu$ a constant in $g$) so that for solutions $(A, u)$, $u(\Sigma^c)$ is not contained in $\mu^{-1}(c)$ for a singular value $c$ of $\mu$. Then the automorphism group of a solution $(A, u)$ is at most finite. After that, the construction of the virtual orbifold atlas and the virtual count is the same as the other case. One difference is that the virtual count is in general a rational number.

Remark 3.16. There have been various choices made in the construction of the good virtual orbifold atlas. We have to prove that the virtual count is independent of these choices. The two major choices are: the basic charts $K_a$, whose footprints cover the moduli space; the various enlargings we made in order to achieve a good virtual orbifold atlas. The independence from the second set of choices can be proved in an abstract setting. On the other hand, if we have two collections of basic charts $\{K_{a_i}\}$ and $\{K_{b_i}\}$, then one can construct a virtual orbifold atlas with boundary on the product $\mathcal{M}_P(B) \times [0, 1]$, such that the two boundary components have the corresponding virtual orbifold atlases constructed out of these two collection of basic charts. Using this cobordism one can prove that the virtual count is well-defined.

4. INVARIANCE OF THE CORRELATION FUNCTION

In this section we prove the invariance of the correlation function under changes of strongly regular perturbations, though most of the technical burden is thrown to the next few sections. The basic idea has been sketched in [TX14]. Here is the main theorem.

Theorem 4.1. The correlation function defined by (3.6) is independent of the choice of the strongly regular perturbation $P$. 

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We need to compare the correlation functions defined for two different strongly regular perturbations $P^-$ and $P^+$. This comparison is obviously reduced to the case that $P^-$ and $P^+$ only differ for a single broad puncture, say $z_j$. Let $P^-_j = (a_j^-, F^-_j)$, $P^+_j = (a_j^+, F^+_j)$ be the corresponding perturbations at $z_j$.

The proof consists of several steps. First we reduce the comparison to the case that $a_j^+ = a_j^- = a_j$. This reduction is obtained in the next subsection. Then to compare $(a_j, F^+_j)$ and $(a_j, F^-_j)$, we use a cobordism argument, connecting $F^+_j$ and $F^-_j$ by a smooth path. The path may cross the wall where the perturbation is not strongly regular, and we have to prove a wall-crossing formula (Theorem 4.6). Using the wall-crossing formula the invariance of the correlation function is proved.

4.1. Independence on $a_j$. Take $P_j = (a_j, F_j)$ where $a_j \in \mathbb{C}^*$ and $F_j \in V_j \setminus (V_j^{sing}(a_j) \cup Wall_j(a_j))$. For $a_j^\lambda = \lambda a_j$ with $\lambda > 0$, we define $F_j^\lambda(x) = \lambda F_j(\lambda^{-1}x)$ where we view $\lambda$ as an element of $G_0^\ast$. Denote $P_j^\lambda = (a_j^\lambda, F_j^\lambda)$ and $\tilde{W}^\lambda_j = W - a_j^\lambda p + F_j^\lambda$. Then $\tilde{W}^\lambda_j = \tilde{W}^{(\lambda)}_j$ using the notation (2.3), and

$$\begin{cases} Q_j(x_j) = a_j \quad \iff \quad Q_j(\lambda x_j) = a_j^\lambda \\ pdQ_j(x_j) + dF_j(x_j) = 0 \end{cases}$$

If we denote by $\kappa_j = (p, x_j)$ a general critical point of $\tilde{W}^\lambda_j|_{\tilde{X}_j}$, then we use $\kappa_j^\lambda = (p, \lambda x_j)$ to denote the corresponding critical point of $\tilde{W}^{(\lambda)}_j|_{\tilde{X}_j}$. Moreover, $\tilde{W}^{(\lambda)}_j(\kappa_j^\lambda) = \lambda \tilde{W}^{(\lambda)}_j(\kappa_j)$. So $P_j$ is strongly regular if and only if $P_j^\lambda$ is strongly regular.

We use a homotopy argument to compare $P_j$ and $P_j^\lambda$. For $t \in [0, 1]$, consider the gauged Witten equation for a family of strongly regular perturbations $\widehat{P} := (P^t)_{t \in [0, 1]} := (P^t, P_j^\lambda_t)$. For any $\kappa_j \in Crit(\tilde{W}^\lambda_j|_{\tilde{X}_j})$, $\kappa_j^\lambda$ is a family of critical points of $\tilde{W}^{(\lambda)}_j|_{\tilde{X}_j}$. Choosing critical points $\kappa' = (\kappa_1, \kappa_2, \ldots, \kappa_b)$ for other broad punctures where the perturbations don’t change, we can consider a universal (parametrized) moduli space

$$\widetilde{M}_P(C, B, \kappa) = \left\{ (t, [\mathcal{X}]) \mid t \in [0, 1], \ [\mathcal{X}] \in M_{P^t}(C, B, \kappa', \kappa_j^\lambda) \right\}.$$

Theorem 4.2. There is an oriented virtual orbifold atlas on $\widetilde{M}_{P}(C, B, \kappa)$ with oriented virtual boundary

$$\partial \widetilde{M}_P(C, B, \kappa) = \left( \mathcal{M}_{(P^t, P_j^\lambda)}(C, B, \kappa', \kappa_j^\lambda) \right) \cup \left( - \mathcal{M}_{(P^t, P_j)}(C, B, \kappa', \kappa_j) \right).$$

Therefore

$$\# \mathcal{M}_{(P^t, P_j)}(C, B, \kappa', \kappa_j^\lambda) = \# \mathcal{M}_{(P^t, P_j)}(C, B, \kappa', \kappa_j).$$

Proof. The proof is a standard homotopy argument and we omit the details. \hfill \Box

Therefore, when comparing the two choices of perturbations, we may assume that the axial parts of $a_j^\pm$ are equal. Then we consider $a_j' = e^{i\theta}a_j$ for $\theta \in [0, 2\pi)$. Take $F'_j(x) = F_j(e^{-i\theta/r}x)$ and $P_j' = (a_j', F'_j)$, then

$$\begin{cases} Q_j(x_j) = a_j \quad \iff \quad Q_j(e^{i\theta/r}x_j) = a_j' \\ pdQ_j(x_j) + dF_j(x_j) = 0 \end{cases}$$

If we denote by $\kappa_j = (p, x_j)$ a general critical point of $\tilde{W}_j|_{\tilde{X}_j}$, then we use $\kappa_j' = (e^{-i\theta}p, e^{i\theta/r}x_j)$ to denote the corresponding critical point of $\tilde{W}_j'|_{\tilde{X}_j}$. Moreover, $\tilde{W}_j(\kappa_j') = \tilde{W}_j(\kappa_j)$ so $P_j$ is strongly regular if and only if $P_j'$ is strongly regular. As before, take $P = (P_1, \ldots, P_j, \ldots, P_b)$ and $P' = (P_1', \ldots, P_j', \ldots, P_b)$ be the two sets of perturbations. Choose a set of critical points $\kappa = (\kappa_1, \ldots, \kappa_j, \ldots, \kappa_b)$ for all broad punctures except the $j$-th one.
Theorem 4.3. There exist a homeomorphism
\[ \mathcal{M}_P(C, B; \kappa, \kappa_j) \simeq \mathcal{M}_P'(C, B; \kappa, \kappa'_j) \]
and an identification between their oriented virtual orbifold atlases, compatible with this homeomorphism. Therefore
\[ \# \mathcal{M}_P(C, B; \kappa, \kappa_j) = \# \mathcal{M}_P'(C, B; \kappa, \kappa'_j). \]

Proof. We define a gauge transformation \( g_\theta : \Sigma^* \to G_1 \) which is equal to \( e^{i\theta/r} \) over \( C_j \) and, by using a cut-off function, extended to the identity away from \( C_j \).

Indeed, for any solution \( \mathcal{X}' = (A', u', \varphi, \varphi_j) \) to the gauged Witten equation with perturbation \( P' \), we define \( g_\theta \) as the frame at \( z_j \) and \( \varphi \) represents frames at other broad punctures, define
\[ \mathcal{X}' = (A', u', \varphi, \varphi_j) = (g_\theta^* A, g_\theta^* u, \varphi, e^{-i\theta/r} \varphi_j). \]
We would like to show that \( \mathcal{X}' \) solves the gauged Witten equation with perturbation \( P' \). Firstly, away from \( C_j \) where the equation is independent of the perturbation, \( \mathcal{X}' \) still solves the equation. It remains to show that \( \mathcal{X}' \) solves the equation over \( C_j \). It is obvious for the vortex equation. Hence it suffices to check for the Witten equation for \( \mathcal{X}' \) over \( C_j \).

By the definition of the perturbation term (2.7), we have
\[ \tilde{W}_{A, \varphi_j}(\varphi_j(z, x)) = e^{p_i(h_A(z))}W(x) + \beta_j \sum_{l=1}^{s_j} e^{p_i(h_A(z))}(\delta_A)^{-r_l}F_{j,l}(x). \]
Since \( \delta_{A'} = \delta_A \) and \( h_{j, A'} = h_{j, A} \), we have
\[ \tilde{W}_{A', \varphi_j'}(\varphi_j'(z, x)) = \tilde{W}_{A', \varphi_j'}(\varphi_j'(e^{i\theta/r}x)) = e^{p_i(h_A(z))}(e^{i\theta/r}x) + \beta_j \sum_{l=1}^{s_j} e^{p_i(h_A(z))}(\delta_A)^{-r_l}F_{j,l}(e^{i\theta/r}x) = \tilde{W}_{A, \varphi_j}(\varphi_j(z, x)). \]
Therefore, \( \tilde{W}_{A, \varphi_j} = \tilde{W}_{A', \varphi_j} \) and \( \nabla \tilde{W}_{A, \varphi_j} = \nabla \tilde{W}_{A', \varphi_j} \). Moreover, by the gauge invariance of the perturbed gauged Witten equation, over \( C_j \) where \( g_\theta \) is constant,
\[ (g_\theta^{-1})^*(\nabla A' + \nabla \tilde{W}_{A', \varphi_j}(u')) = \nabla A + \nabla \tilde{W}_{A, \varphi_j}(g_\theta^* u) = 0. \]
Therefore, \( \mathcal{X}' \) solves the gauged Witten equation with perturbation \( P' \). Therefore, \( g_\theta \) induces a homeomorphism \( \mathcal{M}_P(C, B, \kappa, \kappa_j) \simeq \mathcal{M}_P'(C, B, \kappa, \kappa'_j) \), and obviously the virtual orbifold atlases on the two moduli spaces can be identified. \( \square \)

Corollary 4.4. The correlation functions defined for \( P \) and \( P' \) coincide.

Proof. The map \( x \mapsto e^{i\theta/r}x \) induces a isomorphism \( f^* : H^{n-1}(Q^a_j, Z)^{\mathbb{Z}_r} \simeq H^{n-1}(Q^a_j, Z)^{\mathbb{Z}_r} \) such that the following diagram commutes.

\[
\begin{array}{ccc}
H^{n-1}(Q^a_j, Z)^{\mathbb{Z}_r} & \xrightarrow{f^*} & H^{n-1}(Q^a_j, Z)^{\mathbb{Z}_r} \\
\phi_j & \downarrow & \phi_j' \\
\mathcal{M}_{\mathcal{X}_j} & \xrightarrow{\phi_j'} & \mathcal{M}_{\mathcal{X}_j}'
\end{array}
\]  
(4.1)
Moreover, the gradient flow of $F_j|_{Q_j^{-1}(a_j)}$ is mapped to the gradient flow of $F'_j|_{Q_j^{-1}(a_j)}$. Therefore, with respect to the diagram (4.1), for $\theta_j \in \mathcal{H}_j$, 

$$(\phi_j^{-1}(\theta_j))^* \cap [W^n_{\alpha_j}] = \left( (\phi'_j)^{-1}(\theta_j) \right)^* \cap [W^n_{\alpha_j}].$$

By the definition of the correlation function (3.6) and Theorem 4.3, the correlation functions defined for $P$ and $P'$ coincide.}

4.2. Wall-crossing formula. Now we reduce the comparison between correlation functions for the two strongly regular perturbations to the case that $a_j^+ = a_j^- = a_j$. Consider $P_j^\pm = (a_j, F_j^\pm)$ with $F_j^\pm \in V_j \setminus (V_j^{\text{sing}}(a_j) \cup \text{Wall}_j(a_j))$. Since $V_j^{\text{sing}}(a_j)$ is a proper complex analytic set, $V_j \setminus V_j^{\text{sing}}(a_j)$ is path-connected. So we can choose a smooth path $F_j = (F_j^\alpha)_{\alpha \in [-1,1]} \subset V_j \setminus V_j^{\text{sing}}(a_j)$.

Denote $$P_j^\alpha = (a_j, F_j^\alpha), \quad P^\alpha = (P, P^\alpha).$$

Moreover, we can perturb the path such that it only intersects $\text{Wall}_j(a_j)$ through the smooth locus of the wall transversely. Therefore without loss of generality we may assume that $$F_j^{-1}(\text{Wall}_j(a_j)) = \{0\}.$$ Since the wall is defined by the coincidence of the imaginary parts of two different critical values and the bifurcation is transverse, there exists smooth paths $v^\alpha, \kappa^\alpha \in \text{Crit}(\tilde{W}_j^\alpha|_{\tilde{X}_j}) \simeq \text{Crit}(F_j^\alpha|_{Q_j^{-1}(a_j)})$ such that

$$\text{Re} \tilde{F}_j^0(v^0) > \text{Re} \tilde{F}_j^0(\kappa^0), \quad \text{Im} \tilde{F}_j^0(v^0) = \text{Im} \tilde{F}_j^0(\kappa^0),$$

$$\frac{d}{d\alpha} \bigg|_{\alpha = 0} \left( \text{Im} \tilde{F}_j^\alpha(v^\alpha) - \text{Im} \tilde{F}_j^\alpha(\kappa^\alpha) \right) \neq 0. \quad (4.2)$$

The other critical values all have distinct imaginary parts for all $\alpha \in [-1,1]$. Denote $$v^\pm = v^{\pm 1}, \quad \kappa^\pm = \kappa^{\pm 1}.$$

**Definition 4.5.** We define $(-1)^{\bar{F}_j} := \text{sign}(\tilde{F}_j)$ to be +1 (resp. -1) if the left hand side of (4.2) is positive (resp. negative).

**Theorem 4.6.** Let $\kappa' = (\kappa_1, \ldots, \kappa_j, \ldots, \kappa_0)$ be any choice of asymptotic conditions at all broad punctures except the $j$-th one. Let $\kappa_j^\alpha \in \text{Crit}(\tilde{W}_j^\alpha|_{\tilde{X}_j})$ be a continuous path of critical points. Then in the situation described above, the following holds.

1. If $\kappa_j^\alpha \neq \kappa^\alpha$, then $$\# \mathcal{M}_P^+ (C, B; \kappa^\alpha, \kappa_j^+) - \# \mathcal{M}_P^- (C, B; \kappa', \kappa_j^-) = 0.$$  

2. If $\kappa_j^\alpha = \kappa^\alpha$, then $$\# \mathcal{M}_P^+ (C, B; \kappa', \kappa_j^+) - \# \mathcal{M}_P^- (C, B; \kappa', \kappa_j^-) = -(-1)^{\bar{F}_j} \left( \# \mathcal{N}(v^0, \kappa^0) \right) \left( \# \mathcal{M}_P^- (C, B; \kappa', v^-) \right).$$

Here $\# \mathcal{N}(v^0, \kappa^0)$ is the (algebraic) count of the number of BPS solitons in $Q_j^\alpha$, for the function $F_j^\alpha|_{Q_j^\alpha}$ between the two critical points $v^0$ and $\kappa^0$.

The above theorem is referred to as the bifurcation or wall-crossing formula. The remaining of this paper is devoted primarily to proving this theorem. This most difficult part of the proof is the construction of a virtual chain over a universal moduli space (4.3) below, as outlined in the remaining of this section. Because of the appearance of soliton solutions for the non-strongly regular perturbation $P_j^0$, the universal moduli space is not compact. One can compactify it by adding soliton solutions. When constructing the virtual chain, an important fact is that non-BPS soliton solutions are interior points and BPS soliton solutions are boundary points of the universal moduli space. This phenomenon already appeared in [FJR11] in Landau-Ginzburg A-model. This
is why the counting of BPS solitons appears in the wall-crossing formula. The construction of virtual orbifold atlas and the virtual chain of the universal moduli space occupies Sections 5, 6 and 7.

There is another step we need to consider in order to derive Theorem 4.6. On one hand, the moduli space of BPS soliton solutions for \( P_j^\pm \) is not the product of two moduli spaces, due to our way of perturbing the gauged Witten equation. On the other hand, the BPS solitons are not contained in the hypersurface \( Q_j^{\alpha_j} \), but are gradient lines of the Lagrange multiplier \( W_j \).

To identify the contribution of BPS soliton solutions, we need to carry out another homotopy argument. This is done in Section 8.

Using Theorem 4.6 one can easily derive the invariance of the correlation function, as we explained in [TX14]. Without loss of generality, we assume that \( C \) has only one puncture with broad monodromy \( \gamma \) and we choose a generic path \( \tilde{F} = (F^\alpha)_{\alpha \in [-1,1]} \) connecting two strongly regular perturbations \( F^- \) and \( F^+ \). For each path of critical points \( \kappa^\alpha \) of \( \tilde{W}^{\alpha}\gamma|_{\tilde{X}_\gamma} \), we denote by \( n^\pm \) the virtual count of the moduli space \( \mathcal{M}_{P^\pm}(C, B, \kappa^\pm) \) of gauged Witten equation for the perturbation \( F^\pm \). Then the correlation functions for \( P^\pm \) are linear maps \( f^\pm : H_{\kappa,-1}(|Q_j^\alpha; Q)|_{\mathbb{Z}_2} \rightarrow \mathbb{Q} \) defined by

\[
f^\pm(\theta) = \sum_{\kappa^\pm \in \text{Crit}W^\pm} n^\pm \left( \theta^* \cap [(\kappa^\pm)^-] \right).
\]

Suppose the wall-crossing happens between the two paths of critical points \( \kappa^\alpha_a, \kappa^\alpha_b \) such that

\[
\text{Re} \tilde{W}^{\alpha}_\gamma(\kappa^\alpha_a) < \text{Re} \tilde{W}^{\alpha}_\gamma(\kappa^\alpha_b), \quad \frac{d}{d\alpha} \bigg|_{\alpha = 0} \left( \text{Im} \tilde{W}^{\alpha}_\gamma(\kappa^\alpha_a) - \text{Im} \tilde{W}^{\alpha}_\gamma(\kappa^\alpha_b) \right) \neq 0.
\]

Then by Theorems 4.6 and the Picard-Lefschetz theorem (), we have \( n^+_\kappa = n^-_{\kappa} \) for \( \kappa \neq \kappa_b \), \( |W^u_\kappa| = |W^-_\kappa| \) for \( \kappa \neq \kappa_a \), and

\[
f^+(\theta) - f^-(\theta) = n^+_\kappa_a (\theta^* \cap |W^u_\kappa|) + n^+_\kappa_b (\theta^* \cap |W^u_\kappa|) - n^-_{\kappa_b} (\theta^* \cap |W^u_\kappa|) = (-1)\tilde{F} \#N(\kappa^\alpha_a, \kappa^\alpha_b) n^+_\kappa_a (\theta^* \cap |W^u_\kappa|) + n^+_\kappa_b (\theta^* \cap (-1)\tilde{F} \#N(\kappa^\alpha_a, \kappa^\alpha_b)|W^u_\kappa|) = -1.
\]

Therefore it completes the proof of Theorem 4.1.

4.3 Virtual orbifold atlas. From now on we simplify the notations as

\[
\mathcal{M}_{P^\pm}(C, B; \kappa^\pm) = \mathcal{M}_{\kappa^\pm}, \quad \mathcal{M}_{P(\alpha)}(C, B; \kappa^\alpha) = \mathcal{M}_{\kappa^\alpha}.
\]

In order to compare this pair of moduli spaces, one considers the universal moduli space

\[
\mathcal{M}_{\kappa}([-1,1]) := \left\{ (X, \alpha) \mid \alpha \in [-1,1], \ X \in \mathcal{M}_{\kappa^\alpha} \right\}.
\]

\( \mathcal{M}_{\kappa}([-1,1]) \) is not compact in general due to the soliton degeneration at \( \alpha = 0 \). To compactify it we consider the union

\[
\mathcal{M}_\kappa([-1,1]) := \mathcal{M}^\#([-1,1]) \cup \mathcal{M}_\kappa^\# = \mathcal{M}^\#([-1,1]) \cup (\mathcal{M}_{\kappa_0}^\# \times \{0\}).
\]

Its topology is defined in an obvious fashion, slightly generalized from the topology of the moduli space for a fixed regular perturbation.

**Proposition 4.7.** \( \mathcal{M}_\kappa([-1,1]) \) is compact and metrizable.

**Proof.** Consider a sequence of elements in \( \mathcal{M}_\kappa([-1,1]) \) and we need to prove that a subsequence converges in the weak topology. Indeed, \( \mathcal{M}_\kappa^\# \) itself is sequentially compact hence we only need to consider the case that the sequence is a sequence \( X^{(i)} = (X^{(i)}, \alpha^{(i)})) \in \mathcal{M}_\kappa^\#([-1,1]) \).
Suppose \( \mathcal{A}^{(i)} = (A^{(i)}, u^{(i)}, \varphi^{(i)}) \in \mathcal{B}_{\kappa,0}^{\kappa,0} \). If one has uniform \( C^0 \)-bound on \( u^{(i)} \), then the sequential compactness follows from standard method. Therefore we only need to check if the \( C^0 \)-bound established for a fixed regular perturbation can be extended to the current situation.

The crucial point is the \textit{a priori} energy quantization ([TX15, Theorem 7.1]). Going through the (long) proof of [TX15, Theorem 7.1], one finds that, since the perturbations can be uniformly controlled in the sequence, it holds uniformly for all the perturbations in this sequence. It is also easy to check that [TX15, Corollary 7.2, Theorem 8.3] still hold uniformly for the given sequence. Hence we have uniform \( C^0 \)-bound on \( u^{(i)} \). The remaining part of the proof of sequential compactness follows easily.

Lastly, the compactness and metrizability can be proved in the same way as Theorem 2.14. \( \square \)

The main theorem of this section is the following. For the precise definition of virtual orbifold atlases and virtual boundary, see Appendix A.

**Theorem 4.8.** There exists a virtual orbifold atlas with boundary on the moduli space \( \mathcal{M}_\kappa([-1,1]) \), of virtual dimension \( \chi(C, B) + 1 \), whose virtual boundary is the disjoint union of three pieces

\[
\partial \mathcal{M}_\kappa([-1,1]) = \left( \{-1\} \times \mathcal{M}_{\kappa,-} \right) \cup \left( \{+1\} \times \mathcal{M}_{\kappa,+} \right) \cup \left( \{0\} \times \mathcal{M}_{\kappa,0}^b \right).
\]

(4.4)

Here \( \mathcal{M}_{\kappa,0}^b \) is the moduli space of BPS soliton solutions.

**Remark 4.9.** Various choices will be made in the construction of the virtual orbifold atlas of \( \mathcal{M}_\kappa([-1,1]) \) to be given in the next section, making the atlas less canonical. However it is not an issue because the atlas is only used to prove the invariance of the correlation functions.

To prove Theorem 4.6, it remains to identify the contribution of \( \mathcal{M}_{\kappa,0}^b \) and well as the induced orientation as a piece of boundary of \( \mathcal{M}_\kappa([-1,1]) \). In Section 8 we prove the following theorems.

**Theorem 4.10.** For any choice of orientations of the unstable manifolds \( W^u \), there is a canonical orientation on the boundary virtual orbifold atlas of \( \mathcal{M}_{\kappa,0}^b \). Moreover, its virtual count is

\[
\#\mathcal{M}_{\kappa,0}^b = \left( \Delta_{\nu_0} \cdot \Delta_{\kappa_0} \right) \left( \#\mathcal{M}_{(0)}(C, B; \kappa', \nu_0) \right).
\]

(4.5)

**Theorem 4.11.** If we orient \( \mathcal{M}_{\kappa,-}, \mathcal{M}_{\kappa,+} \) and \( \mathcal{M}_{\kappa,0}^b \) in the coherent way in the sense of [FH93], then in the oriented sense,

\[
\partial \mathcal{M}_\kappa([-1,1]) \simeq \left( \mathcal{M}_{\kappa,+} \right) \cup \left( - \mathcal{M}_{\kappa,-} \right) \cup \left( (-1)\hat{F}_j, \mathcal{M}_{\kappa,0}^b \right).
\]

Theorem 4.6 then follows from Theorems 4.8, 4.10 and Corollary A.23.

5. Constructing Local Charts

We first make certain simplifying assumptions, under which we don’t lose the generality essentially and the notations in our argument can be greatly simplified.

**Hypothesis 5.1.** Let \( C \) be the rigidified \( r \)-spin curve.

1. \( C \) has only one puncture, which is broad, denoted by \( z_j \).
2. The monodromy at \( z_j \) is trivial.

Notice that under the above assumption \( \hat{W}^{k,p} = W^{k,p} \).

We explain why this hypothesis is legitimate. Since the bifurcation formula has been reduced to the case that the family of perturbations only change at one single broad puncture, the other punctures are irrelevant in the proof. Therefore we can assume (1). Moreover, the existence of the nontrivial monodromy will only bring in extra notations and no essential feature will be used in the proof (though we do need to consider maps into the submanifold \( X_j \) instead of \( \hat{X} \)). Hence (2) won’t essentially affect the effectiveness of the proof.
Then we have a natural map $U$ for the tangent space $T_E$. We will also denote it by $X$. It consists of tuples $(\tilde{\Sigma}, \mathbf{a}, \mathbf{b})$. $E$ is a smooth manifold of dimension $\text{dim} E = 27$. We use the same notations as Definition 3.5. Let $\mathcal{B}_{\kappa_a}$ be the Banach manifold of triples $X = (A, u, \varphi) \in \mathcal{A} \times W^1_{loc}(\Sigma^*) \times \mathbb{F}(P_1, z_j)$, where $\kappa_a$ indicates the asymptotic constrain of $u$ at the only (broad) puncture $z_j$. Let $\mathcal{E}_{\kappa_a} \to \mathcal{B}_{\kappa_a}$ be the corresponding Banach space bundle. The augmented gauged Witten equation for the perturbed superpotential $\tilde{W}^\alpha$ gives a Fredholm section and moduli space

$$\mathcal{F}^\alpha : \mathcal{B}_{\kappa_a} \to \mathcal{E}_{\kappa_a}, \quad \mathcal{M}_{\kappa_a} := (\mathcal{F}^\alpha)^{-1}(0).$$

We describe a Banach manifold (with boundary) $\mathcal{B}_\kappa$ which contains $\mathcal{M}_{\kappa_a} \times \{\alpha\}$ for all $\alpha \in [-1,1]$. It consists of tuples $X := (X, \alpha) = (A, u, \varphi, \alpha)$ where $X \in \mathcal{B}_{\kappa_a}$ and $\alpha \in [-1,1]$. A description of the tangent space $T_X \mathcal{B}_\kappa$ and the definition of an exponential map is given in Subsection 6.1.

One also has a Banach space bundle $\mathcal{E}_\kappa \to \mathcal{B}_\kappa$ whose fibre over $X = (A, u, \varphi, \alpha)$ is

$$\mathcal{E}_\kappa|_X := \mathcal{E}_{\kappa_a}|_X = L^p(\Sigma^*, u^*T\tilde{Y}) \oplus L^p(\Sigma^*, g \oplus g).$$

The augmented gauged Witten equation defines a smooth Fredholm section

$$\mathcal{F} : \mathcal{B}_\kappa \to \mathcal{E}_\kappa, \quad \mathcal{F}(A, u, \varphi, \alpha) = \mathcal{F}^\alpha(A, u, \varphi).$$

The expression of the linearized operator $D_X \mathcal{F} : T_X \mathcal{B}_\kappa \to \mathcal{E}_\kappa|_X$ is provided in Subsection 6.1. The zero locus of $\mathcal{F}$ is the moduli space $\mathcal{M}_\kappa$.

### 5.1. Local charts for non-soliton solutions.

For an arbitrary $a = X = (X, \alpha) \in \mathcal{M}_\kappa$, we construct a simple chart around $a$ as follows. Choose $E_a \subset \mathcal{E}_\kappa|_a = \mathcal{E}_{\kappa_a}|_X$ generated by finitely many linearly independent smooth sections $s_1, \ldots, s_k$, supported away from punctures such that

$$\text{Im}(D_X \mathcal{F}^\alpha) + E_a = \mathcal{E}_{\kappa_a}|_X = \mathcal{E}_\kappa|_X.$$

Choose a neighborhood $\mathcal{U}_X$ of $X$ in $\mathcal{B}_\kappa$, which is small enough; for every $X' \in \mathcal{U}_X$, $X'$ is $C^0$ close to $X$ so $u'(z) = \exp_{u(z)}(t \xi'(z))$ for $\xi' \in W^1_{loc}(u^*T\tilde{Y})$. Then parallel transport along the family of short geodesics $\exp_{u(z)}(t \xi'(z))$ provides a canonical inclusion

$$\text{Emb}_X^\alpha : E_a \to \mathcal{E}_\kappa|_{X'}.$$

We will also denote it by $e \mapsto e(X')$. This induces a section $\mathcal{F}_a : \mathcal{U}_X \times E_a \to \mathcal{E}_\kappa|_{U_a}$ given by

$$\mathcal{F}_a(X', e) = e(X') + \mathcal{F}(X').$$

For $\mathcal{U}_X$ and $\|e\|$ small enough, the linearization of $\mathcal{F}_a$ at $(X', e) \in \mathcal{U}_X \times E_a$, which reads

$$D_{(e,X')} \mathcal{F}_a(v, \chi) = v(X') + \frac{\partial e(X')}{\partial X'}(\chi) + D_{X'} \mathcal{F}(\chi), \quad (v, \chi) \in E_a \times T_{X'} \mathcal{B}$$

is always surjective. Let $E_a^\epsilon$ be the $\epsilon$-ball of $E_a$ centered at the origin. Then

$$U_a := \mathcal{F}_a^{-1}(0) \cap (\mathcal{U}_X \times E_a^\epsilon),$$

is a smooth manifold of dimension $\text{dim} E_a + \chi$. By abusing the notations, let $E_a \to U_a$ be the trivial bundle with fibre $E_a$, and let $S_a : U_a \to E_a$ be induced by the projection $\mathcal{U}_X \times E_a^\epsilon \to E_a$. Then we have a natural map $\psi_a : S_a^{-1}(0) \to \mathcal{M}_\kappa$, which is homeomorphic onto an open subset $F_a \subset \mathcal{M}_\kappa$. A simple local chart of $a$ is the tuple

$$K_a = (U_a, E_a, S_a, \psi_a, F_a).$$

Moreover, when $\alpha \neq \pm 1$, $K_a$ is an interior chart; when $\alpha = \pm 1$, $K_a$ is a boundary chart.
5.2. Local charts for non-BPS soliton solutions.

5.2.1. Linear theory for soliton solutions. For a soliton solution \( X = (A, u, \varphi, \sigma) \), we regard \( \sigma \) as a map \( \sigma : \Theta \to \check{Y}_j \), where \( \check{Y}_j \) be the fibre of \( Y \) at the puncture \( z_j \). Using the frame \( \varphi : \check{X} \to \check{Y}_j \), we identify \( \sigma \) with a map \( \sigma_{\varphi} := \varphi^{-1}\sigma \) from \( \Theta \) to \( \check{X} \).

For any \( \kappa, \nu \in \check{X} \), consider the Banach manifold \( B_{\kappa}(\varphi \nu, \varphi \kappa) \) consisting of maps \( \sigma \in W_{1,p}^{1,1}(\Theta, \check{Y}_j) \) which are asymptotic to \( \varphi \kappa \) (resp. \( \varphi \nu \)) at \( +\infty \) (resp. \( -\infty \)) in the \( W^{1,1,p} \)-sense. We have a Banach space bundle \( E_{\kappa}(\varphi \nu, \varphi \kappa) \to \check{Y}_j \) whose fibre over \( \sigma \) is \( L^p(\sigma^* T\check{Y}_j) \).

Take \( X = (X, \alpha) \in \mathcal{B}_v \). We have the number \( \delta_X \in (0, 1] \) and the frame \( \varphi_X : \check{X} \to \check{Y}_j \). View \( \delta_X \) as an element in \( G_0^C \). For \( v_{\alpha}, \kappa_{\alpha} \in \text{Crit}\check{W}_\alpha \), denote

\[
\nu_X = \varphi(\delta_X v_{\alpha}), \quad \kappa^X = \varphi(\delta_X \kappa_{\alpha}) \in \check{Y}_j.
\]

Define

\[
\mathcal{B}_{\kappa}^p := \left\{ X^\# = (X, \sigma, \alpha) \mid X = (X, \alpha) \in \mathcal{B}_v, \sigma \in B_{\kappa}(v^X, \kappa^X) \right\}.
\]

which is a Banach manifold. Its tangent space at \( X^\# = (X, \sigma, \alpha) \) can be identified as

\[
T_{X^\#} \mathcal{B}_{\kappa}^p \simeq T_X \mathcal{B}_v \oplus T_{\sigma} B_{\kappa}(v^X, \kappa^X).
\]

(5.1)

Define the Banach bundle \( E_{\kappa}^p \to \mathcal{B}_{\kappa}^p \) as

\[
E_{\kappa}^p|_{(X, \sigma)} := E_{\nu}|_{X} \oplus E_{\sigma}^{\kappa}(v^X, \kappa^X)|_{\sigma}.
\]

(5.2)

Define a section \( f^\#: \mathcal{B}_{\kappa}^p \to E_{\kappa}^p \) by

\[
f^\#(X^\#) = (f(X), S^X(\sigma)).
\]

Here \( S^X(\sigma) \) is the soliton operator

\[
S^X(\sigma) = \varphi \left( \bar{\partial}_{\varphi}^+ + \nabla_{\check{Y}_j}(\delta_X(\sigma_{\varphi})) \right).
\]

(5.3)

The linearization of \( f^\# \) at \( X^\# \), written in a matrix form, is denoted by

\[
D_{X^\#} f^\# = \begin{pmatrix} D_L & 0 \\ T^R_L & D_R \end{pmatrix}.
\]

(5.4)

Here the block decomposition is taken with respect to (5.1) and (5.2). The notations indicates that the principal component is viewed as on the left and the soliton component is viewed on the right, as we will always presume. The concrete expressions of \( D_L, D_R \) and \( T^R_L \) are provided in Subsection 6.1.

5.2.2. The stabilization. In order to deal with nontrivial automorphisms, we also need to consider the moduli space of gauged Witten equation with one additional marked point, denoted by \( \widetilde{\mathcal{M}}_k \).

It can be partially compactified by adding \( \widetilde{\mathcal{M}}_k^\#, \) the moduli space of soliton solutions with an additional marked point on the soliton component.

We first describe the topology of the corresponding partial compactification on the level of Banach manifolds. Let \( \check{B}_k \) consist of objects \( (X, w) \in \check{B}_k \times \Sigma^* \), which is a Banach manifold. It can be partially compactified by adding objects with one cylindrical component which has the marked point. We denote this stratum by \( \check{B}_k^\# \), which has a free action by the cylindrical translation group \( \Theta \). Then a natural topology on the union

\[
\overline{\check{B}_k} := \check{B}_k \cup \left( \check{B}_k^\# / \Theta \right)
\]

is well-defined. Let \( \check{a} \in \overline{\check{B}_k} / \Theta \) be represented by \( \check{X}_j^\# = (X_a, \sigma_a, w_a) = (A_a, u_a, \varphi_a, \alpha_a, \sigma_a, w_a) \).

A neighborhood of \( \check{a} \) can be described by the following lemma.

**Lemma 5.2.** \( \check{a} \) has a neighborhood \( \overline{\mathcal{V}}(\check{a}) \subset \overline{\check{B}_k} \) satisfying the following condition.

---

1 If the monodromy is nontrivial, \( \check{Y} \) is an orbifold bundle and we need to take a finite cover of its fibre over \( z_j \).
(1) If $\hat{Y} := (A', u', \varphi', \alpha'; w) \in \bar{\mathcal{V}}(\hat{a}) \cap \bar{\mathcal{B}}_k$, then for $\varphi = \exp(w - w_a)$, we have
\[
\sup_{z \in \Sigma \setminus C_j(2\pi T)} d(u'(z), u_a(z)) + \sup_{z \in C_j(2\pi T)} d(u'(z), \text{Twist}_\varphi(\sigma_a(z)) \leq \min \{ \text{Inj}(X), 1 \}.
\]

(2) If $\hat{Y}' := (A', u', \varphi', \sigma', \alpha'; w) \in \bar{\mathcal{B}}_k'$ represents a point in $\bar{\mathcal{V}}(\hat{a})$, then for $\varphi = \exp(w - w_a)$, we have
\[
\sup_{z \in \Sigma \setminus C_j} d(u'(z), u_a(z)) + \sup_{z \in \Theta} d(\sigma'(z), \text{Twist}_\varphi \sigma_a(z)) \leq \min \{ \text{Inj}(X), 1 \}
\]
The above description of $\bar{\mathcal{V}}(\hat{a})$ is independent of the representative $\hat{X}_a' = (X_a', w_a)$.

In the statement of the above lemma, $\text{Twist}_\varphi$ is defined by (6.23).

Let $a \in \mathcal{M}_k$ be represented by a non-BPS soliton solution $X_a' = (X, 0, \sigma) = (A, u, \varphi, 0, \sigma)$. Below is a well-known result in Floer theory\(^2\).

**Lemma 5.3.** There is an open dense subset $\Theta^{reg} \subset \Theta$ such that for all $z \in \Theta^{reg},$
\[
V_s(z) := \frac{\partial \sigma_{\varphi}}{\partial s}(z), \quad V_t(z) := \frac{\partial \sigma_{\varphi}}{\partial t}
\]
are linearly independent in $T_{\sigma_a(z)} \hat{X}$.

Choose $w_a \in \Theta^{reg}$ and denote $x_a = \sigma_{\varphi}(w_a)$; choose a hyperplane $H_a \subset T_{x_a} \hat{X}$ which is complementary to the plane generated by $V_s(w_a)$ and $V_t(w_a)$. Take $D_a = \exp_{x_a}(H_a \cap B(T_{x_a} \hat{X}))$, a codimension 2 submanifold to which $\sigma$ intersects transversely at $w_a$. Moreover, choose a linear functional $T_{x_a} \hat{X} \to \mathbb{C}$ with kernel equal to $H_a$, which is pushed forward to a smooth function
\[
f_a : \exp_{x_a}(B(\tau(T_{x_a} \hat{X}))) \to \mathbb{C}
\]
which vanishes transversely along $D_a$.

**Definition 5.4.**

(1) The triple $(w_a, H_a, f_a)$ is called a normalization datum for $X_a'$.

(2) $\hat{Y} = (\gamma', w) = (A, u, \varphi, \alpha, w) \in \bar{\mathcal{B}}_k$ is said to be normal to $(X_a', w_a, H_a, f_a)$, if
(a) $w$ is in the cylindrical end $C_j$.
(b) For each $\gamma = \exp(2\pi mi/d_a) \in \Gamma_a$, define
\[
\gamma w = w + \frac{2\pi mi}{d_a}.
\]
Then $u_{\varphi}(\gamma w)$ lies in the domain of $f_a$ and
\[
\sum_{\gamma \in \Gamma_a} f_a(u_{\varphi_a}(\gamma w)) = 0.
\]

(3) $\hat{Y}' = (\gamma', w) = (A, u, \varphi, \alpha, \sigma, w) \in \bar{\mathcal{B}}_k'$ is said to be normal to $(X_a', w_a, H_a, f_a)$, if for each $\gamma \in \Gamma_a$ (with $\gamma w$ defined by (5.6)), $\sigma_{\varphi_a}(\gamma w)$ lies in the domain of $f_a$ and
\[
\sum_{\gamma \in \Gamma_a} f_a(\sigma_{\varphi_a}(\gamma w)) = 0.
\]

For each $a \in \mathcal{M}_k$, choosing a representative $X_a'$ and a normalization data $(w_a, H_a, f_a)$, the object $\hat{X}_a' = (X_a', w_a)$ represents a point $\hat{a} \in \mathcal{M}_k$. Notice that $\hat{X}_a'$ has no nontrivial automorphism. To construct a local chart of $\mathcal{M}_k$, we will first construct a local chart of $\hat{a}$ in $\mathcal{M}_k$.

\(^2\)When the monodromy is nontrivial, we can take a multiple cover of $\sigma$ which can be identified with a Floer trajectory.
5.2.3. The basic charts. Fix the representative $X^\#_a = (X^a, 0, \sigma)$ for $a \in \mathcal{M}_a$ and normalization data $(w_a, H_a, f_a)$. Let $\hat{X}^\#_a = (\hat{X}^\#_a, w_a) \in \hat{B}^\#_a$. For the principal component $X = (a, u, \varphi)$, there is a finite dimensional subspace $E_{L,a} \subset \mathcal{E}_{v_0} = \mathcal{E}_{v_0|X}$ such that
\[ \text{Im}(D_{L,a}) + E_{L,a} = \mathcal{E}_{v_0|X}. \]
Here $D_{L,a}$ is the operator $D_L$ in (6.6) for $X = X_a$, i.e., the restriction of $D_L$ to the space of deformations $(\beta_L, \nu_L, \rho_L) \in T_X \mathcal{B}_a$ with $\rho_L = 0$. For the soliton component $\sigma$, there is a finite-dimensional subspace $E_{R,a} \subset \mathcal{E}_J(v^X, \kappa^X)|_\sigma$, spanned by smooth sections $s_1, \ldots, s_l$ with compact support such that $E_{R,a}$ is $\Gamma_a$-invariant and
\[ \text{Im}(D_{R,a}) + E_{R,a} = \mathcal{E}_J(v^X, \kappa^X)|_\sigma. \]
Here $D_{R,a}$ is the block in (5.4). Denote $E_a = E_{L,a} \oplus E_{R,a} \subset \mathcal{E}_J|_{X_a^\#}$, for which we used the identification (5.2). Then $\text{Im}(D_{(X,a), \sigma}^\#) + E_a = \mathcal{E}_J|_{X_a^\#}$. Define
\[ M_a := \{ (\xi, e) \in T_{(X,a)} \mathcal{B}_a \oplus E_a \mid e + D_{(X,a), \sigma}^\#(\xi) = 0 \}. \]
Because there is a positive dimensional group of automorphisms of the domain of the soliton component, we need to take a slice inside $M_a$ which is transverse to the direction of the infinitesimal domain symmetries. Take
\[ N_a := \{ (\xi, e) \in M_a \mid \xi = (\beta_L, \nu_L, \rho_L, \nu_R), \sum_{\gamma \in \Gamma_a} (\nu_R)_{\varphi_a} (\gamma w_a) \in H_a \}. \]
Moreover, $M_a$ and $N_a$ have the induced metric and we denote by $M^\epsilon_a$ or $N^\epsilon_a$ their $\epsilon$-balls centered at the origins. Let $T_a \cong \mathbb{R}^2$ be the tangent space of $w_a$, and $T^\epsilon_a$ be its $\epsilon$-ball centered at 0. Define
\[ \hat{M}^\epsilon_a = M^\epsilon_a \times (-\epsilon, \epsilon) \times T^\epsilon_a, \quad \hat{N}^\epsilon_a = N^\epsilon_a \times (-\epsilon, \epsilon) \times T^\epsilon_a. \]
For convenience, introduce
\[ \hat{\mathcal{B}}^\#_a = \hat{\mathcal{B}}_a \times E_a, \quad \hat{\mathcal{B}}^\#_{\sigma} = \hat{\mathcal{B}}^\#_a \times E_a. \]
Then $\hat{M}^\epsilon_a \subset T_{X_a^\#} \hat{\mathcal{B}}_a$. We denote an element of $\hat{M}^\epsilon_a$ by $\xi = (\chi, w, e)$ where $\chi = (\xi, t) \in T_{X^\#_a} \hat{\mathcal{B}}_a$, $w \in T_a$ and $e \in E_a$. Lastly, let $R^\epsilon_a$ be the $\epsilon$ ball of $C$ centered at the origin, which parametrizes the gluing parameter $\varpi = e^{-4T^\epsilon - i\theta}$. Denote $R^\epsilon_a := R^\epsilon_a \setminus \{0\}$. There is a $\Gamma_a$-action on the product $R^\epsilon_a \times M_a$ while $R^\epsilon_a \times N_a$ is invariant.

**Proposition 5.5.** For each $a \in \mathcal{M}^\#_a$, choose a representative $X^\#_a$ and a set of normalization data $(w_a, H_a, f_a)$. Then there exist $\epsilon_a > 0$ and two families of objects
\[ \{ \hat{\gamma}_{\varpi, \xi} = (\gamma_{\varpi, \xi}, w_{\varpi, \xi}, e_{\varpi, \xi}) \in \hat{\mathcal{B}}_a \mid (\varpi, \xi) \in R^\epsilon_a \times \hat{M}^\epsilon_a \}, \]
\[ \{ \hat{\gamma}_{#, \xi} = (\gamma_{#, \xi}, w_{#, \xi}, e_{#, \xi}) \in \hat{\mathcal{B}}^\#_{\sigma} \mid \xi \in \hat{M}^\epsilon_a \}. \]
which satisfy the following conditions.

1. For $\xi = 0 \in \hat{M}^\epsilon_a$, $\hat{\gamma}_{0, \xi} = \hat{X}^\#_a = (X^\#_a, w_a, 0)$.
2. For $\xi \in \hat{N}^\epsilon_a$, if the soliton component of $\gamma_{#, \xi}$ is $\sigma_{\xi}$, then we have
\[ \sum_{\gamma \in \Gamma_a} f_a((\sigma_{\xi})_{\varphi_a}(\gamma w_a)) = 0. \]  

3. For $(\varpi, \xi) \in R^\epsilon_a \times \hat{N}^\epsilon_a$, if $\gamma_{\varpi, \xi} = (A_{\varpi, \xi}, u_{\varpi, \xi}, \varphi_{\varpi, \xi}, \alpha_{\varpi, \xi})$ and $\varpi = e^{-4T^\epsilon - i\theta}$, then
\[ \sum_{\gamma \in \Gamma_a} f_a\left( (u_{\varpi, \xi})_{\varphi_a}(\gamma w_a + 4T + i\theta) \right) = 0. \]
(3a) If we define \( \tilde{S}^\#_a : \tilde{M}^\#_a \to E_a \) by \( \tilde{S}^\#_a(\xi) = e_{\#, \xi} \) and define \( \hat{\psi}^\#_a : \tilde{M}^\#_a \cap (\tilde{S}^\#_a)^{-1}(0) \to \tilde{M}^\#_a \) by \( \hat{\psi}^\#_a(\xi) = [\gamma_{\#, \xi}, w_{\#, \xi}] \), then the restriction of \( \hat{\psi}^\#_a \) onto \( \tilde{N}^\#_a \times (\tilde{S}^\#_a)^{-1}(0) \) is a homeomorphism onto a neighborhood \( \tilde{F}^\#_a \) of \( \hat{a} \) inside \( \tilde{M}^\#_a \). Moreover, the tuple 
\[
\tilde{K}^\#_a := (\tilde{N}^\#_a, \tilde{N}^\#_a \times E_a, \tilde{S}^\#_a, \hat{\psi}^\#_a, \tilde{F}^\#_a)
\]
is a simple local chart of \( \tilde{M}^\#_a \) around \( \hat{a} \).

(3b) If we define
\[
\hat{S}_a : (R_{e_a} \times \tilde{M}^\#_a) \to E_a, \quad \hat{S}_a(\varpi, \xi) = e_{\varpi, \xi},
\]
then \( \hat{S}_a \) (resp. \( \hat{\psi}_a \)) extends continuously to \( \tilde{S}^\#_a \) (resp. \( \hat{\psi}^\#_a \)) as \( \varpi \to 0 \). If we denote the extension of \( \hat{S}_a \) (resp. \( \hat{\psi}_a \)) by the same symbol, then the restriction of \( \hat{\psi}_a \) onto \( R_{e_a} \times \tilde{N}^\#_a \times \tilde{S}^\#_a \) is a homeomorphism onto a neighborhood \( \hat{F}_a \) of \( \hat{a} \) inside \( \hat{M}_a \) and the tuple
\[
\hat{K}_a := (R_{e_a} \times \tilde{N}^\#_a, R_{e_a} \times \tilde{N}^\#_a \times E_a, \hat{S}_a, \hat{\psi}_a, \hat{F}_a)
\]
is a chart of \( \hat{M}_a \) around \( \hat{a} \).

(4) The normalization map \( n_a : R_{e_a} \times \tilde{M}^\#_a \to \mathbb{C} \), defined by
\[
n_a(\varpi, \xi) = \begin{cases} 
\sum_{\gamma \in \Gamma_a} f_a((u_{\varpi, \xi})_{\varpi}(\gamma_{\varpi, \xi})), & \varpi \neq 0, \\
\sum_{\gamma \in \Gamma_a} f_a((\sigma_{\#, \xi})_{\varpi}(\gamma_{\#, \xi})), & \varpi = 0,
\end{cases}
\]
is continuous and transverse to zero (in the topological sense).

(5) For \( (\varpi, \xi) \in R_{e_a} \times \tilde{M}^\#_a \) and \( \gamma \in \Gamma_a \), we have
\[
(\gamma_{\varpi, \xi}, \gamma_{\#}, w_{\varpi, \xi}, \gamma_{\varpi, \xi}, e_{\gamma_{\varpi, \xi}}) = \begin{cases} 
(\varpi_{\varpi, \xi}, \gamma_{\varpi, \xi}, \gamma_{\varpi, \xi}, \gamma_{\varpi, \xi}), & \varpi \neq 0, \\
(\gamma_{\#}, w_{\#}, \gamma_{\#}, \gamma_{\#}), & \varpi = 0.
\end{cases}
\]
Here \( \gamma w_{\varpi, \xi} \) is defined by (5.6) and \( \gamma_{\#} \varpi, \xi \) is defined by reparameterizing the soliton component by the domain automorphism \( \gamma \).

### 5.2.4 Inducing a chart of \( \hat{M}_a \).

We define \( \tilde{\psi}_a : (R_{e_a} \times \tilde{M}^\#_a) \cap \tilde{S}^\#_a \to \hat{M}_a \) by
\[
\tilde{\psi}_a(\varpi, \xi) = \begin{cases} 
[\gamma_{\#, \xi}], & \varpi = 0, \\
[\gamma_{\#, \xi}], & \varpi \neq 0.
\end{cases}
\]

### Corollary 5.6

For \( \epsilon \in (0, \epsilon_a) \), we define
\[
U_{a} = \frac{(R_{e} \times \tilde{N}^\#_a) \cap n^{-1}_{a}(0)}{\Gamma_a}, \quad E_{a} = \frac{((R_{e} \times \tilde{N}^\#_a) \cap n^{-1}_{a}(0)) \times E_{a}}{\Gamma_a}.
\]

Let \( S_a : U_a \to E_a \) be induced from \( \tilde{S}_a \) of (5.9), and let \( \psi_a : S^{-1}_a \to \hat{M}_a \) be induced from \( \tilde{\psi}_a \) of (5.12). Then for \( \epsilon \) small enough, \( \psi_a \) is a homeomorphism onto an open neighborhood \( \hat{F}_a \subset \hat{M}_a \) and the tuple \( K_a := (U_a, E_a, S_a, \psi_a, F_a) \) is a local chart of \( \hat{M}_a \) around \( a \).

**Proof.** Firstly, (5.11) implies that \( \tilde{S}_a \) is \( \Gamma_a \)-equivariant, so it induces a section \( S_a : U_a \to E_a \). For \( \gamma = (A, u, \varphi, \alpha) \in \hat{M}_a \) sufficiently close to \( a \), there exists \( w \in \Sigma^* \) such that
\[
\sum_{\gamma \in \Gamma_a} f_a((u)_{\varphi}(\gamma w)) = 0.
\]
Therefore, the object \( (\gamma_{\#}, w_{\#}) \) is in the image of \( \hat{\psi}_a \) restricted to \( (R_{e} \times \tilde{N}^\#_a) \cap n^{-1}_{a}(0) \). Therefore, \( \psi_a \) surjects onto a neighborhood \( \hat{F}_a \subset \hat{M}_a \) of \( a \). \( \psi_a \) is one-to-one because the point \( w \) is unique up to rotating by an element \( \gamma \in \Gamma_a \). 

\[\square\]
5.2.5. **Extension of obstruction spaces.** Before we prove Proposition 5.5, we would like to construct a canonical family of linear embeddings

\[ \text{Emb}_{\tilde{a}}^\#: E_a \to \mathcal{E}_a|_\gamma^\# \quad \text{(resp. Emb}_{\tilde{a}}^{\#} : E_a \to \mathcal{E}_a^\#|_\gamma^\#) } \]  

(5.13)

whenever \( \tilde{\gamma} \) (resp. \( \tilde{\gamma}^\# \)) represents a point in \( \mathcal{V}(\tilde{a}) \). First of all, by Lemma 5.2, one can use the parallel transport to embed \( E_{L,a} \) to \( \mathcal{E}_a|_\gamma_L \). To embed \( E_{R,a} \) into \( \mathcal{E}_j|_\gamma_R \), take \( \varpi = \exp(w - w_a) \). Then we can define an inclusion

\[ \text{Emb}_{\tilde{a}}^{\#} := \text{Paral}^w_{\mathcal{V}} \circ \text{Twist}_{\varpi} : E_{R,a} \to \mathcal{E}_j|_\gamma_R. \]

This defines the map (5.13). For \( e \in E_a \), and \( \tilde{\gamma} = (\gamma', w) \) (resp. \( \tilde{\gamma}^\# = (\gamma^\#, w) \)), we also denote

\[ e(\gamma', w) = \text{Emb}_{\tilde{a}}^\#(e) \quad \text{(resp. } e(\gamma^\#, w) = \text{Emb}_{\tilde{a}}^{\#}(e)). \]  

(5.14)

Then it is easy to see that for \( \gamma \in \Gamma_a \),

\[ (\gamma e)(\gamma', w) = e(\gamma', w) \quad \text{(resp. } (\gamma e)(\gamma^\#, w) = e(\gamma^\#, w)). \]  

(5.15)

5.3. **Proof of Proposition 5.5: part I.** To prepare for the gluing process, we first construct the family \( \tilde{\gamma}^{\#}_\xi \) in the lower stratum. For \( \epsilon > 0 \), take \( \xi = (\chi, w, v) \in M^*_a \times T^*_a \times E^*_a \). Consider the family of elements

\[ \tilde{\gamma}^{\#}_{\chi, w, v} := \left( \gamma^{\#}_{\chi, w, v}, \xi \right) := \left( \exp^\#_{\chi, w, v} (\xi), w_a + w, e \right) \in \tilde{\mathcal{B}}^\# \times E_a, \]

(5.16)

where \( \exp^\# \) is the exponential map in \( \tilde{\mathcal{B}}^\#. \) If we choose a right inverse \( \tilde{Q}_{\chi, w, v}^\# \) of the operator

\[ \tilde{D}_{\chi, w, v} \tilde{\gamma}^{\#} : T_{\chi, w, v}^{\#} \tilde{\mathcal{B}}^\# \to \mathcal{E}_a|_{\chi, w, v}, \]

(5.17)

then by the standard implicit function theorem for Banach spaces, for \( \epsilon \) small enough, the family (5.16) can be corrected by adding a unique element \( (\chi', w', \xi') \in \text{Im} \tilde{Q}_{\chi, w, v}^\# \) such that for

\[ \tilde{\gamma}^{\#}_{\chi, w, v} := \left( \gamma^{\#}_{\chi, w, v}, \xi \right) := \left( \exp^\#_{\chi, w, v} (\xi + \chi', w', \xi), w_a^{\#} + w, e \right), \]

one has

\[ \tilde{Q}_{\chi, w, v}^\# (\tilde{\gamma}^{\#}_{\chi, w, v}) = e_{\chi, w, v} (\gamma^{\#}_{\chi, w, v}, w_a^{\#}, e_{\chi, w, v}), \quad (\gamma^{\#}_{\chi, w, v}(\chi, w, V) = V(\chi, w, V) + D_{\chi, w, v} \tilde{\gamma}^{\#}(\chi). \]

In order for the family to satisfy (2a) of Proposition 5.5, we need to impose a condition on \( \tilde{Q}_{\chi, w, v}^\# \). Indeed, the vector fields \( V_a \) and \( V_a \) (see Lemma 5.3) are contained in the kernel of \( \tilde{D}_{\chi, w, v} \tilde{\gamma}^{\#} \). Hence we can choose \( \tilde{Q}_{\chi, w, v}^\# \) to satisfying the following infinitesimal normalization condition.

- For any \( \xi = (\chi, w, V) = (\beta_L, v_L, \rho_L, v_R, w, V) \in \text{Im} \tilde{Q}_{\chi, w, v}^\#, \) one has

\[ \sum_{\gamma \in \Gamma_a} (v_R)_{\phi_a}(\gamma w_a) \in H_a. \]

Then for \( \xi \in \tilde{\mathcal{N}}^*_a \), the approximate solution \( \tilde{\gamma}^{\#}_{\chi, w, v} \) satisfies the normalization condition. After corrected by adding an element in the image of \( \tilde{Q}_{\chi, w, v}^\# \), it still satisfies the normalization condition. Hence (2a) of Proposition 5.5 is satisfied. (3a) of Proposition 5.5 is merely a restatement of the implicit function theorem.

5.4. **Gluing.**
5.4.1. Pregluing. Let \( \varpi = e^{-4T - i\theta} \in R^*_\epsilon \) be the gluing parameter and let \( \hat{X}^\# = (A, u, \varphi, \sigma, w) \in B^\#_\kappa \). We cut \( \Sigma^\# \) into the following pieces
\[
\Sigma^\# = (\Sigma^\# \setminus C_j(T)) \cup (C_j(T) \setminus C_j(3T)) \cup C_j(3T).
\]
Then for \( T \) sufficiently large, there are \( \zeta^L \in W^{1,p}(C_j(T), T_{\varpi}, \hat{X}) \) such that over \( C_j(T), u_\varphi = \exp_{\delta \chi_{\varpi}} \zeta^L \). There also exists \( \zeta^R \in W^{1,p}(C_j(-T), T_{\varpi}, \hat{X}) \) such that over \( C_j(-T), \sigma_\varphi = \exp_{\delta \chi_{\varpi}} \zeta^R \). Then
\[
\text{Twist}(\sigma_\varphi)(s, t) = \exp_{\delta \chi_{\varpi}} \left( \text{Twist}(\zeta^R)(s, t) \right), \quad s \leq 3T.
\]
Choose a cut-off function \( \beta : \mathbb{R} \to [0, 1] \) such that \( \beta(s) = 0 \) for \( s \leq 0 \), \( \beta(s) = 1 \) for \( s \geq 1/2 \) and \( \beta \) is non-increasing. Let \( \beta^H(s) = 1 - \beta((s - T)/T), \beta^R(s) = \beta((s - 2T)/T) \). Define
\[
\Gamma(\Sigma^\#, \hat{X}) \ni u_{\varpi, \kappa} = \left\{ \begin{array}{ll}
u, & \text{if } \varphi \exp_{\delta \chi_{\varpi}} \left( \beta^E_1 \zeta^L + \beta^R \text{Twist}(\zeta^R) \right), \\
\Sigma^\# \setminus C_j(T), & \varphi(\text{Twist}(\sigma_\varphi)), \\
C_j(T) \setminus C_j(3T), & C_j(3T).
\end{array} \right.
\]
(5.18)
Define \( X^\#_{\varpi, \kappa} = (\hat{X}^\#, u_{\varpi, \kappa}, \varphi) \in B_\kappa \). Then for each \( \xi \in \hat{M}^\#_\kappa \) and the corresponding \( \hat{Y}^\#_{\#., \xi} = (\mathcal{Y}^\#_{\#., \xi}, w_{\#., \xi, e., \#., \xi}) \), define
\[
\hat{Y}^\#_{\#., \xi} = (\hat{Y}^\#_{\#., \xi}, w^\#_{\#., \xi, e., \#., \xi})
\]
as follows. \( \mathcal{Y}^\#_{\#., \xi} \in B_\kappa \) is constructed as above by gluing the two components of \( \mathcal{Y}^\#_{\#, \xi} \), and
\[
\hat{Y}^\#_{\#., \xi} = \mathcal{Y}^\#_{\#, \xi} + 4T + i\theta, \quad e^\#_{\#, \xi} = e_{\#, \xi}.
\]
In particular, denote \( \hat{X}^\#_{\#., \xi} : (X^\#_{\#., \xi}, w_{\#., \xi, 0}) \equiv \hat{Y}^\#_{\#., \xi} \).

For \( \epsilon \) and \( \|\varpi\| \) small enough, all \( \mathcal{Y}^\#_{\#, \xi} \) are in a small neighborhood of \( X^\#_{\#., \xi} \) inside \( B_\kappa \). Hence we can write
\[
\mathcal{Y}^\#_{\#, \xi} = \exp_{\hat{X}^\#_{\#, \xi}} (\hat{x}_{\#, \xi}), \quad \hat{x}_{\#, \xi} \in T_{\hat{X}^\#_{\#, \xi}} B_\kappa.
\]
(5.19)
5.4.2. Estimates. We look for a family of solutions to the equation
\[
\hat{F}_\epsilon(\hat{X}) = 0
\]
(5.20)
which are close to the family \( \hat{Y}^\#_{\#., \xi} \). We first state a few technical results.

**Lemma 5.7.** There exist \( \epsilon_1 > 0 \) and \( c_1 > 0 \) such that for \( (\varpi, \xi) \in R^*_\epsilon \times \hat{M}^\#_\kappa \),
\[
\|\hat{F}_\xi(\hat{Y}^\#_{\#., \xi})\| \leq c_1\|\varpi\|^{\tau_0}.
\]
(5.21)
**Proof.** See Subsection 6.2.

**Lemma 5.8.** There exist \( \epsilon_2 \in [0, \epsilon_1], c_2 > 0 \) and a family of bounded linear operators
\[
\hat{Q}_{\varpi, \kappa} : B_\kappa|_{X^\#_{\varpi, \kappa}} \to T_{X^\#_{\varpi, \kappa}} B_\kappa, \quad \varpi \in R^*_\epsilon
\]
satisfying the following conditions.

1. For each \( \varpi \), \( \|\hat{Q}_{\varpi, \kappa}\| \leq c_2; \)
2. \( \hat{Q}_{\varpi, \kappa} \) is a right inverse of \( D_{now} \hat{F}_\epsilon; \)
3. For each \( \varpi = e^{-4T - i\theta} \) and each \( \xi = (\chi, w, e) = (\beta, v, e, w, e) \in \text{Im}(\hat{Q}_{\varpi, \kappa}), \) we have
\[
\sum_{\gamma \in \Gamma_\kappa} (v)(\gamma w + 4T + i\theta) \in H_{\kappa}.
\]
(5.22)
**Proof.** See Subsection 6.3.
5.4.3. Exact solutions. Recall the following version of the implicit function theorem.

**Proposition 5.9** (Implicit Function Theorem). [MS04, Proposition A.3.4]
Consider \((X, Y, U, F, x_0, Q, \delta, c)\) be as follows. \(X, Y\) are Banach spaces, \(U \subset X\) is an open neighborhood of the origin and \(F : U \rightarrow Y\) is a continuously differentiable map, \(x_0 \in U\) is such that \(DF(x_0) : X \rightarrow Y\) is surjective and \(Q : Y \rightarrow X\) is a bounded right inverse to \(DF(x_0)\). Moreover,

\[
\|Q\| \leq c, \quad B_\delta(x_0) \subset U, \tag{5.23}
\]

\[
\|x - x_0\| < \delta \implies \|DF(x) - DF(x_0)\| \leq \frac{1}{2c}. \tag{5.24}
\]

Then, if \(x' \in X\) satisfies

\[
\|F(x')\| < \frac{\delta}{4c}, \quad \|x' - x_0\| < \frac{\delta}{8}. \tag{5.25}
\]

there exists a unique \(x \in X\) such that

\[
F(x) = 0, \quad x - x' \in \text{Im}Q, \quad \|x - x_0\| \leq \delta. \tag{5.26}
\]

Moreover,

\[
\|x - x'\| \leq 2c\|F(x')\|. \tag{5.27}
\]

Let \( \varepsilon_2 \) be the one of Lemma 5.8 and fix \( \varpi \in R^*_\varepsilon \). Define \( X_{\varpi,a} = T_{X_{\varpi,a}} \hat{B}_a \). We identify points in \( \hat{B}_a \) near \( \hat{X}_{\hat{\varpi},a} \) with tangent vectors in \( X_{\varpi,a} \), using the exponential map of \( \hat{B}_a = \hat{B}_a \times E_a \). This gives a small neighborhood \( U_{\varpi,a} \subset X_{\varpi,a} \) of the origin. For each \( x \in U_{\varpi,a} \), let \( \hat{\gamma}_{\varpi} \in \hat{B}_a \) be the corresponding point in the Banach manifold. On the other hand, define \( \mathcal{Y}_{\varpi,a} := \mathcal{E}_{\varpi,a} \hat{X}_{\hat{\varpi},a} \). Parallel transport between nearby objects induces a continuous trivialization \( \mathcal{E}_{\varpi,a} \mid U_{\varpi,a} \simeq U_{\varpi,a} \times \mathcal{Y}_{\varpi,a} \). Then consider the map \( \mathcal{F}_{\varpi,a} : U_{\varpi,a} \rightarrow \mathcal{Y}_{\varpi,a} \) given by \( \mathcal{F}_{\varpi,a}(x) = \hat{\mathcal{F}}_{\varepsilon}(\hat{\gamma}_{\varpi}) \). Let \( Q_{\varpi,a} : \mathcal{Y}_{\varpi,a} \rightarrow X_{\varpi,a} \) be the \( \hat{Q}_{\varpi,a} \) of Lemma 5.8, which is a right inverse to \( \mathcal{D}\mathcal{F}_{\varpi}(0) \). One takes \( c = c_2 \) where \( c_2 \) is the one in Lemma 5.8.

**Lemma 5.10.** There exist \( \delta > 0, \epsilon_3 \in (0, \epsilon_2) \) such that for \( \varpi \in R^*_\varepsilon \) and \( x \in U_{\varpi,a} \) with \( \|x\| \leq \delta \), one has

\[
\|DF_{\varpi,a}(x) - DF_{\varpi,a}(0)\| \leq \frac{1}{10c_2}. \tag{5.28}
\]

**Proof.** Proved by straightforward calculation. \( \square \)

The following lemma is also as obvious as the pre-gluing in Gromov-Witten theory.

**Lemma 5.11.** For any \( \delta > 0 \), there exists \( \epsilon(\delta) > 0 \) such that for \( \varpi \in R^*_\varepsilon \) and \( \xi \in \hat{M}_{\varepsilon}, \hat{Y}_{\varpi,\xi} \) lies in the \( \delta \)-neighborhood of \( \hat{X}_{\varpi,a} \).

Then the tuple \((X_{\varpi,a}, \mathcal{Y}_{\varpi,a}, U_{\varpi,a}, F_{\varpi,a}, 0, Q_{\varpi,a}, \delta, c_2)\) satisfies the hypothesis of Proposition 5.9. Moreover, by Lemma 5.7 and Lemma 5.11, there exists \( \epsilon_4 \in (0, \epsilon_3) \) such that for \((\varpi, \xi) \in R^*_\varepsilon \times \hat{M}^*_{\varepsilon}\), \( \hat{Y}_{\varpi,\xi} \) is identified with a point \( \hat{x}_{\varpi,\xi} \in X_{\varpi,a} \) satisfying (5.25). Then by the implicit function theorem, there exists a unique \( x_{\varpi,\xi} \) satisfying (5.26), i.e.,

\[
\mathcal{F}_{\varpi,a}(x_{\varpi,\xi}) = 0, \quad x_{\varpi,\xi} - \hat{x}_{\varpi,\xi} \in \text{Im}Q_{\varpi,\xi}, \quad \|x_{\varpi,\xi}\| \leq \delta.
\]

So we have actually proved

**Proposition 5.12.** There exist \( \epsilon_\alpha > 0 \) such that for every \( \varpi \in R^*_\varepsilon \) and \( \xi \in \hat{M}^*_{\varepsilon} \), there exists a unique \( n'_{\varpi,\xi} = (x_{\varpi,\xi}, w_{\varpi,\xi}, e'_{\varpi,\xi}) \in \text{Im}(\hat{Q}_{\varpi,\xi}) \), such that if we denote

\[
\hat{Y}_{\varpi,\xi}^{\text{exact}} = \left( \exp_{X_{\varpi,a}}(x_{\varpi,\xi} + x'_{\varpi,\xi}), w_{\varpi,\xi} + w'_{\varpi,\xi}, e_{\varpi,\xi} + e'_{\varpi,\xi} \right),
\]

then \( \hat{\mathcal{F}}_{\varepsilon}(\hat{Y}_{\varpi,\xi}^{\text{exact}}) = 0. \)
5.5. Proof of Proposition 5.5: part II. We need to prove that the family \( \hat{\gamma}_{\varepsilon, \xi} := \hat{\gamma}_{\varepsilon, \xi}^{\text{exact}} \) satisfies conditions (2b), (3b), (4), and (5) of Proposition 5.5.

5.5.1. Normalization. By (2a) of Proposition 5.5, for \( \xi \in \hat{N}_a, \hat{\gamma}_{\varepsilon, \xi} \) is normal to \( (X^\#, w_a, H_a, f_a) \). So is \( \hat{\gamma}_{\varepsilon, \xi}^{\text{app}} \) by the pre-gluing construction. Since \( \hat{\gamma}_{\varepsilon, \xi}^{\text{exact}} \) is corrected from \( \hat{\gamma}_{\varepsilon, \xi}^{\text{app}} \) by adding an element in the image of \( \hat{Q}_{\varepsilon,a} \), by condition (3) of Lemma 5.8, the normalization condition persists. This proves (2b).

5.5.2. Injectivity. Suppose there exist two elements \( (\varpi, \xi_1) \in R^*_a \times \hat{N}_a^{\varepsilon}, \nu = 1, 2 \), such that
\[
\hat{M}_a \ni \psi_{\nu}(\varpi_1, \xi_1) = \psi_{\nu}(\varpi_2, \xi_2) \iff \hat{\gamma}_{\varepsilon, \xi_1}^{\text{exact}} = \hat{\gamma}_{\varepsilon, \xi_2}^{\text{exact}} = (A', u', \varphi', \alpha', w).
\]
Suppose in \( \xi_\nu \), the deformation of the marked point is \( w_\nu \in T_a \), which is small. Then by the coincidence of the marked points, we know
\[
w = w_a + w_1 + 4T_1 + i\theta_1 = w_a + w_2 + 4T_2 + i\theta_2.
\]
So \( w_1 - w_2 = 4T_2 + i\theta_2 - (4T_1 + i\theta_1) \). On the other hand, since \( \xi_\nu \in \hat{N}_a^{\varepsilon} \), we have
\[
\sum_{\gamma \in \Gamma_a} f_\gamma((u')_{\varphi'}/(\gamma w_a + 4T_\nu + i\theta_\nu)) = 0, \quad \nu = 1, 2.
\]
Therefore, \( \varpi_1 \) and \( \varpi_2 \) differ by an element in \( \Gamma_a \). However, \( w_1 - w_2 = 4T_1 + i\theta_1 - 4T_2 - i\theta_2 \) is very small. So the only possibility is that \( \varpi_1 = \varpi_2 \). Therefore, by the implicit function theorem, \( \xi_1 = \xi_2 \). This proves that \( \psi_{\nu} \) restricted to \( R^*_a \times \hat{N}_a^{\varepsilon} \) is injective.

5.5.3. Surjectivity. Now we prove the surjectivity of \( \hat{\psi}_{\alpha} \). Using the notations of Subsection 5.6, let \( g_{\varepsilon} : M^*_a \to X_{\varepsilon} \) be the map
\[
g_{\varepsilon}(\xi) = x_{\varepsilon, \xi} = n_{\varepsilon, \xi} + n_{\varepsilon, \xi}.
\]
By shrinking \( \varepsilon_3 \) a little, we may assume that \( g_{\varepsilon} \) is defined over the closure of \( \hat{M}^{\varepsilon}_a \).

**Lemma 5.13.** There exist \( \delta'_a \in (0, \delta_a] \) and \( \epsilon'_a \in (0, \epsilon_a] \) such that for each \( \varpi \in R^*_a \), \( B_{\delta'_a} \cap F_{\varepsilon,a}^{-1}(0) \) is contained in \( g_{\varepsilon}(\hat{M}^{\varepsilon}_a) \).

**Proof.** Suppose the lemma is not true. Then there is a sequence \( \varpi_i \to 0 \) and a sequence of points \( y_i \in F_{\varepsilon,a}^{-1}(0) \) which is not in the image of \( g_{\varepsilon} \), and \( \|y_i\| \to 0 \). Then consider the segment \( s y_i \) for \( s \in [0, 1] \). By using Lemma 5.10, one can show that as \( i \to \infty \), \( F_{\varepsilon,a}(s y_i) \) converges to zero uniformly in \( s \). Hence we can apply the implicit function theorem: for each \( s \), there exists a unique \( w_i(s) \) such that
\[
F_{\varepsilon,a}(s y_i + w_i(s)) = 0, \quad w_i(s) \in \text{Im} Q_{\varepsilon,a}, \quad \|w_i(s)\| \leq \delta_a.
\]
Moreover, by (5.26),
\[
\|w_i(s)\| \leq 2\epsilon\|F_{\varepsilon,a}(s y_i)\| \to 0.
\]
Denote \( y_i(s) = s y_i + w_i(s) \), whose norm converges uniformly to zero as \( i \to \infty \). Note that for each \( i \), \( y_i(0) \) lies in the image of \( g_{\varepsilon} \). Suppose \( s_i \) is the largest number in \( [0, 1] \) such that \( y_i([0, s_i]) \subset \text{Im}(g_{\varepsilon}) \). Then
\[
y_i(s_i) \in g_{\varepsilon}(\partial \hat{M}^{\varepsilon}_a).
\]
Take \( \xi_i \in \partial(\hat{M}^{\varepsilon}_a) \) such that \( y_i(s_i) = n_{\varepsilon, \xi_i} + n_{\varepsilon, \xi_i}' \). However, by Lemma 5.7 and (5.27),
\[
\|n_{\varepsilon, \xi_i}'\| \leq 2\epsilon_1\|F_{\varepsilon.a}(\hat{\gamma}^{\text{app}}_{\varepsilon, \xi_i})\| \to 0.
\]
Then
\[
\liminf_{i \to \infty} \|y_i(s_i)\| = \liminf_{i \to \infty} \|n_{\varepsilon, \xi_i}\| > 0.
\]
This contradicts the fact that \( \|y_i(s)\| \) converges uniformly to zero. Hence the lemma is proved. \( \square \)
Suppose the surjectivity doesn’t hold, then there exists a sequence of points $\hat{a}_k \in \hat{\mathcal{M}}_\alpha$ converging to $\hat{a}$ and

$$\hat{a}_k \notin \hat{\psi}_a(R_{e_a} \times \hat{M}_a^{\topo}).$$

It suffices to consider the case that all $\hat{a}_k$ are in the top stratum, i.e., they are genuine objects

$$\hat{\mathcal{Y}}_k = (\mathcal{Y}_k, w_k, 0) = (A_k, u_k, \varphi_k, w_k, 0) \in \hat{\mathcal{B}}_a$$

converging in weak topology to $\hat{\mathcal{Y}}_a^\#$. Then, if we define $\varpi'_k = \exp(w_k - w_k)$, then we have

$$\lim_{k \to \infty} \sup_{z \in \Sigma} \text{dist}(u_k(z), u_{\varpi'_k}(z)) = 0.$$

Moreover, the above convergence is in $C^\infty$ locally, hence there exists a unique point $z_k$, close to $w_k$, such that

$$\sum_{\gamma \in \Gamma_a} f_a((u_k)_{\varpi_a}(\gamma z_k)) = 0. \tag{5.28}$$

Define $\varpi_k = \exp(w_k - z_k)$, $w_k = z_k - w_k$. Hence for $k$ large enough, we can write

$$\mathcal{Y}_k = \exp_{\varpi_k} \cdot \mathcal{X}, \quad \mathcal{X} \in T_{\varpi_k} \mathcal{B}_a.$$

By elliptic regularity, $\lim_{k \to \infty} \|\mathcal{X}\| = 0$. Then for $k$ large, $\underline{x}_k := (x_k, w_k, 0) \in \underline{\mathcal{B}}_a$, where $\underline{\mathcal{B}}_a$ is given by Lemma 5.13. Hence $x_k = \underline{\mathcal{X}}(\xi_k)$ for some $\xi_k \in \hat{\mathcal{M}}^{\topo}_a$. Moreover, (5.28) implies that $\xi_k$ lies in the slice $\hat{\mathcal{N}}^{\topo}_a$. This contradicts our assumption. Hence we proved the surjectivity.

5.5.4. The normalization. The normalization map (5.10) is only shown to be continuous. Indeed, the smoothness of the evaluation maps is a crucial point in constructing smooth virtual atlases in the Kuranishi approach. However we don’t need smoothness in our approach.

By the definition of topological transversality (Definition A.7 (1)), we first need to show that $n^{-1}_a(0)$ is itself a topological manifold. Indeed, by (2a) and (2b) of Proposition 5.5, $n^{-1}_a(0) = R_{e_a} \times N^{\topo}_a \times (-\epsilon, \epsilon) \times \{0\}$, i.e., the set of infinitesimal deformations in which the marked point doesn’t deform. So $n^{-1}_a(0)$ itself is a manifold. On the other hand, the deformation of the marked point gives the normal bundle of the submanifold $n^{-1}_a(0)$. Hence $n^{-1}_a(0)$ is embedded in $R_{e_a} \times \hat{\mathcal{N}}^{\topo}_a$ in a locally flat fashion. Moreover, since each $u_{\varpi, \xi}$ or $u_{\varpi, \xi}$ is locally immersive near $w_a + 4T + i\theta$, the last condition of Definition A.7 (1) is satisfied.

5.5.5. The $\Gamma_a$-action. Lastly we prove (5) of Proposition 5.5. Take $\gamma \in \Gamma_a$. If $\varpi = 0$, then the approximate solution $\hat{\mathcal{Y}}^{\text{app}}_{\varpi, \xi}$ of (5.16) satisfies (5.11). Since the exact solution is obtained by adding a correction term in the image of $\hat{\mathcal{Q}}^{\text{app}}_{\varpi, \xi}$, which is $\Gamma_a$-equivariant, $\hat{\mathcal{Q}}^{\text{exact}}_{\varpi, \xi}$ also satisfies (5.11). If $\varpi \neq 0$, then the approximate solution $\hat{\mathcal{Q}}^{\text{app}}_{\varpi, \xi}$ also satisfies (5.11). On the other hand, $\hat{\mathcal{Y}}^{\text{app}}_{\varpi, \xi} = \hat{\mathcal{Q}}^{\text{app}}_{\varpi, \xi}$ and the right inverse $\hat{\mathcal{Q}}^{\text{app}}_{\varpi, \xi}$ of Lemma 5.8 satisfies $\hat{\mathcal{Q}}^{\text{app}}_{\varpi, \xi} = \hat{\mathcal{Q}}^{\text{app}}_{\varpi, \xi}$. Since the exact solution is obtained by an correction from the image of the right inverse, (5.11) is satisfied.

5.6. Boundary charts for BPS soliton solutions. Suppose $\mathcal{X}_a^\# := (\mathcal{X}_a, \sigma_a)$ is a BPS soliton solution representing a point $a \in \mathcal{M}_a^{\topo}$. In this subsection we will construct a local chart of $a$. The difference from the case of a non-BPS soliton solution is that $\sigma$ has no nontrivial automorphisms, and we don’t need to add extra marked point.

By definition, $\sigma_a : \mathbb{R} \to \hat{\mathcal{Y}}_j$ is a nonconstant Morse flow line, i.e., a solution to the equation

$$\frac{d\sigma_a}{ds} + \nabla W(\delta\sigma_a)(\sigma_a) = 0, \quad \lim_{s \to -\infty} \sigma_a = \nu^X_a, \quad \lim_{s \to +\infty} \sigma_a(s) = \kappa^X_a. \tag{5.29}$$

We also view $\sigma_a$ as a map from $\Theta$ which is independent of $t$. Let $w_a$ be $(0, 0) \in \Theta$ and $x_a$ be $(\sigma_a, \varphi_a)(0, 0)$. Then there is a real hyperplane $H_a \subset T_{x_a} \hat{\mathcal{X}}$ which is transverse to $\sigma$. Choose a function $f_a : \hat{\mathcal{X}} \to \mathbb{R}$ which is locally a linear function near $x_a$ vanishing on $\exp_{x_a} H_a$. We still call $(w_a, H_a, f_a)$ a set of normalization data.
Lemma 5.14. If the perturbation is small enough, then there exists a finite-dimensional subspace \( E_{R,a} \subset L^p(\mathbb{R}, \sigma_a^* T\bar{Y}_j) \) generated by smooth sections with compact support such that
\[
\text{Im}(\mathcal{D}_{R,a}) + E_{R,a} = \mathcal{E}_j|_{\sigma_a}.
\]

Proof. \( \mathcal{D}_{R,a} \) can be written as
\[
\mathcal{D}_{R,a}(v) = \frac{1}{2} \left( \nabla_a v + \nabla^2 \bar{W}(\delta_x)(\sigma_a) \cdot v \right) + \frac{1}{2} J \partial v. \tag{5.30}
\]
Here the first piece is induced from the linearized operator of (5.29) along \( \sigma_a \). We view (5.30) as an unbounded operator on \( L^2(\Theta, \sigma_a^* T\bar{Y}_j) \), which can be decomposed as
\[
L^2(\Theta, \sigma_a^* T\bar{Y}_j) \simeq L^2(\mathbb{R}, \sigma_a^* T\bar{Y}_j) \oplus L^2(\Theta, \sigma_a^* T\bar{Y}_j), \tag{5.31}
\]
where the latter consists of sections whose average over each circle is zero. Then (5.30) decomposes with respect to (5.31) as a direct sum. When the zero-th order term \( \nabla^2 \bar{W}(\delta_x) \) has small \( C^0 \) norm, the second part is invertible. Hence in such a situation the obstruction only comes from \( L^2(\mathbb{R}, \sigma_a^* T\bar{Y}_j) \). It is indeed the case when the perturbation is small enough, which we explain as follows. Because the energy of the soliton is proportional to the difference of the critical values of the two critical points connected by this soliton. If the perturbation is small, then the critical values of the perturbed superpotential are very close to zero. On the other hand, small energy of the soliton \( \sigma \) implies that the image of \( \sigma \) is very close to \((\ast,0) \in \bar{X}\), where \( \nabla^2 \bar{W} \) vanishes. Hence \( \nabla^2 \bar{W} \) will also be sufficiently small. \( \Box \)

Choose a finite dimensional obstruction space \( E_{R,a} \subset L^p(\mathbb{R}, \sigma_a^* T\bar{Y}_j) \) as Lemma 5.14. On the other hand, as before, one can find an obstruction space \( E_{L,a} \subset \mathcal{E}_v|_{X_a} \). Then we define
\[
E_a := E_{L,a} \oplus E_{R,a} \subset \mathcal{T}_a^\#|_{X_a^\#},
\]
which has a trivial \( S^1 \)-action. Then take
\[
M_a := \left\{ (\xi, e) \in T_{X_a^\#} \mathcal{B}_a \oplus E_a \mid e + \mathcal{D}_{X_a^\#} \mathcal{F}^\#(\zeta) = 0 \right\},
\]
\[
N_a := \left\{ (\xi, e) = (\beta, v_L, \rho_L, \varepsilon_L, v_R; e) \in M_a \mid \int_{S^2} (v_R)_{\varphi_a}(0, \theta) d\theta \in H_a \right\}.
\]
Then \( N_a \) is a real hyperplane in \( M_a \), and the \( S^1 \)-actions on \( M_a \) and \( N_a \) are trivial. For \( \varepsilon > 0 \), let \( M_a^\varepsilon \) (resp. \( N_a^\varepsilon \)) be its \( \varepsilon \)-ball centered at the origin. Note that there is a homeomorphism
\[
(R_{\varepsilon} \times N_a^\varepsilon)/S^1 \simeq [0, \varepsilon] \times N_a^\varepsilon.
\]
This is the local model of the boundary charts. More precisely, we have the following proposition analogous to Proposition 5.5. For simplicity, we will omit the part corresponding to the lower stratum.

Proposition 5.15. There exist \( \varepsilon > 0 \) and two families of objects
\[
\left\{ \vec{\mathcal{Y}}_{0,\xi} = (\mathcal{Y}_{0,\xi}, e_{0,\xi}) \in \mathcal{B}_a \times E_a \mid (t, \xi) \in (0, \varepsilon) \times M_a^\varepsilon \right\},
\]
\[
\left\{ \vec{\mathcal{Y}}_{+,\xi} = (\mathcal{Y}_{+,\xi}, e_{+,\xi}) \in \mathcal{B}_a^\# \times E_a \mid \xi \in M_a^\varepsilon \right\}.
\]

which satisfy the following conditions.

1. \( \vec{\mathcal{Y}}_{+,0} = \vec{X}_a^\# = (X_a^\#, 0) \).
2. For \( \xi \in N_a^\varepsilon \), if the soliton component of \( \mathcal{Y}_{+,\xi} \) is \( \sigma_\xi \), then we have
\[
\int_{S^2} f_a((\sigma_\xi)_{\varphi_a}(w_a + i\theta)) d\theta = 0. \tag{5.32}
\]
Lemma 5.16. If we define $S_a^\# : M_a^r \to E_a$ by $S_a^\#(\xi) = e_{\#\xi}$ and define $\psi_a^\# : M_a^r \cap (S_a^\#)^{-1}(0) \to M_a^\#$ by $\psi_a^\#(\xi) = [\gamma_{\#\xi}]$, then the restriction of $\psi_a^\#$ onto $N_a^r \times (S_a^\#)^{-1}(0)$ is a homeomorphism onto a neighborhood $F_a^\#$ of an inside $M_a^\#$. Moreover, the tuple

$$K_a^\# := \left(N_a^r, N_a^r \times E_a, S_a^#, \psi_a^#, F_a^#\right)$$

is a simple local chart of $M_a^\#$ around $\hat{a}$.

(3b) If we define

$$S_a : (0, \epsilon) \times M_a^r \to E_a, \quad S_a(t, \xi) = e_{t, \xi}$$

then $S_a$ (resp. $\psi_a$) extends continuously to $S_a^\#$ (resp. $\psi_a^\#$) as $t \to 0$. If we denote the extension of $S_a$ (resp. $\psi_a$) by the same symbol, then the restriction of $\psi_a$ onto $([0, \epsilon) \times N_a^r) \cap S_a^{-1}(0)$ is a homeomorphism onto a neighborhood $F_a$ of an inside $M_a^\#$ and the tuple

$$K_a := ([0, \epsilon) \times N_a^r, [0, \epsilon) \times N_a^r \times E_a, S_a, \psi_a, F_a)$$

is a boundary chart of $M_a^\#$ around $\hat{a}$.

Proof. By the pregluing construction and applying the implicit function theorem, one can obtain two families of exact solutions

$$\left\{\gamma_{\#\xi}, \xi \in M_a^r\right\} \cup \left\{\gamma_{\#\xi}, (\varpi, \xi) \in (R^r_\# \times M_a^r)\right\}.$$

The $S^1$-equivariance can be checked during each step of the construction so that the family actually only depends on $t = |\varpi|$. The only thing we want to emphasize for the construction is that, one needs to choose a right inverse $Q_{R,a}$ to $D_{R,a}$ satisfying the following condition. For any $\nu R \in \text{Im}Q_{R,a},$

$$\int_{S^1} (v_R)_{\varphi_a}(0, \theta)d\theta \in H_a.$$  \hfill (5.34)

It remains to prove the injectivity and surjectivity. To prove injectivity, we need the following result, whose proof is left to the reader.

Lemma 5.16. There exist a neighborhood $U_a$ of an inside $M_a^\#$ satisfying the following condition.

1. For any $\gamma'' = (A', u', \varphi'; \alpha') \in U_a \cap M_a^\#$, there exists a unique $s_0 \in \mathbb{R}$ such that

$$\int_{S^1} f_a((u')_{\varphi_a}(s_0, \theta))d\theta = 0.$$  \hfill (5.35)

2. For any $\gamma'^\# = (A', u', \varphi', \sigma'; 0)$ representing $a' \in U_a \cap M_a^\#$, there exists a unique $s_0 \in \mathbb{R}$ such that

$$\int_{S^1} f_a((\sigma')_{\varphi_a}(s_0, \theta))d\theta = 0.$$

Having this lemma, suppose we have $(t_\nu, \xi_\nu) \in (0, \epsilon) \times \tilde{N}_a^r$ such that

$$\overline{\gamma}^\text{exact}_{t_1, \xi_1} = \overline{\gamma}^\text{exact}_{t_2, \xi_2} = (A', u', \varphi'; \alpha') \in \hat{U}_a.$$  

Then by Lemma 5.16, there exists a unique $s_0 \geq 0$ such that (5.35) holds. On the other hand, by (5.34), we have

$$\int_{S^1} f_a((u')_{\varphi_a}(4T_\nu + i\theta))d\theta = 0, \quad \nu = 1, 2.$$

Hence by the uniqueness part of Lemma 5.16, $t_1 = t_2$. Then $\chi_1, \chi_2$ follows from the implicit function theorem. The injectivity in the lower stratum follows in a similar way.
The surjectivity of $\psi_a$ can be proved in a similar manner as the case of non-BPS soliton solutions, and is left to the reader.

**Corollary 5.17.** When the perturbation $P_j(0) \in \tilde{V}_j$ is small enough, $\mathcal{M}^\#_k$ is the disjoint union of $\mathcal{M}^\#_k$ and $\mathcal{M}^\#_k$, both of which are compact.

**Proof.** For each $a \in M^\#_k$ and $(0, \xi) \in (S_\alpha)^{-1}(0)$, $\psi_a(0, \xi)$ is represented by a soliton solution $\mathcal{Y}_{\#}^\#$. Moreover, since the $S^1$-action on $N_a$ is trivial, by the $S^1$-equivariance,

$$\tilde{\mathcal{Y}}_{\#}^\# \circ \gamma = \tilde{\mathcal{Y}}_{\#, \gamma}^\#.$$

Therefore $\mathcal{Y}_{\#}^\#$ is a BPS soliton solution. Hence the image of $\psi_a$ is disjoint from $\mathcal{M}^\#_k$. So $\mathcal{M}^\#_k$ is closed. On the other hand $\mathcal{M}^\#_k$ is also closed. Therefore the corollary holds.

6. Technical Details of Gluing Analysis

Now we prove the technical results in the gluing construction.

6.1. The linearized operators.

6.1.1. The tangent space of $\mathcal{B}$ and the linearization $D_X \mathcal{F}$. We need to identify a neighborhood of $0$ of the right hand side of (3.7) with a neighborhood of $X = (A, u, \varphi)$ in $\mathcal{B}$, in order to express the linearization of $\mathcal{F}$. Take $\beta \in T_X A$, $v \in W^1 \rho(u^* T Y^J, \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n, \rho = (\rho_1, \ldots, \rho_b) \in \mathbb{R}^b$, whose norms are small. We identify it with some $X' = (A', u', \varphi') \in \mathcal{B}$ near $X$ where naturally $A'$ should be the connection $A + \beta$ and $\varphi'$ should be the collection of frames $(e^{i \rho_1}, \ldots, e^{i \rho_b})$. What we would like to clarify here is how to express $u'$, since the asymptotic constrains of $u'$ will be different from $u$. More precisely, denote

$$A_t = A + t \beta, \quad \delta_t = \delta_{\Delta t} = \delta_{\Delta t} e^{m_t}, \quad t \in [0, 1]; \quad m_\beta := m_1.$$

Using the cut-off functions $\beta_j$ to define a complex gauge transformation $g$ on $P$, i.e.,

$$g_{\beta, \rho}(z) = \exp \left( \sum_{j=1}^b \beta_j i \rho_j \right) \exp \left( \sum_{j=1}^b \beta_j m_\beta \right). \quad (6.1)$$

Denote its linearization as

$$\eta_{\beta, \rho}(x) := \frac{d}{dt} \bigg|_{t=0} g_{\beta, t \rho}(x), \quad x \in \tilde{X}. \quad (6.2)$$

Then we write

$$u' = g_{\beta, \rho} \exp_u \left( v + \sum_{j=b+1}^{b+n} \beta_j \xi_j \right) \quad (6.3)$$

whose restriction to a broad $C_j$ is equal to (we also identify $\xi$ with a vector field along $u_{\phi_j}$)

$$\varphi_j \left( e^{i \rho_j + m_\beta} \exp_{u_{\phi_j}} \right) = \varphi_j' \left( e^{m_\beta} \exp_{u_{\phi_j}} \right).$$

We see that it is asymptotic to $\varphi_j'(\tilde{v}_{j-A'})$. Hence $(A', u', \varphi') \in \mathcal{B}$. Now we look at the linearization of $\mathcal{F} : \mathcal{B} \rightarrow \mathcal{E}$. Take $X = (A, u, \varphi) \in \mathcal{B}$. Recall that

$$\mathcal{F}(X) = \left( \bar{\partial}_A u + \nabla \bar{W}_{A, \varphi} (u) \right) \quad \left( * F_A + \nu \mu (u) \right) \quad \left( \frac{d}{dt} (A - A_0) \right).$$

We denote by $\mathcal{W}(X)$ the first component (i.e., the Witten equation) and by $\mathcal{V}(X)$ the second and the third (i.e., the augmented vortex equation). The linearization of $\mathcal{V}'$ is simple, namely

$$D_{\mathcal{V}, X}(\beta, v, \xi) = \left( * d_\beta + \nu d_\mu (u) \cdot (v + \eta_{\beta, \rho} + \sum_{j=b+1}^{b+n} \beta_j \xi_j) \right) \quad \left( \frac{d}{dt} \right).$$

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We look more carefully at \( D_{W,X} : T_X B \to L^p(\Sigma^*, u^* T\tilde{Y}^\perp) \). We write
\[
D_{W,X}(\beta, v, \zeta, \rho) = D_{W,X}^{\text{loc}}(\beta, v, \zeta) + D_{W,X}^{\text{glob}}(\beta, \rho).
\]

The “local” part is
\[
D_{W,X}^{\text{loc}}(\beta, v, \zeta) = D_A^{0,1}(v + \sum_{j=b+1}^{b+n} \beta_j \zeta_j) + L_{\beta_{0,1}}(u) + \nabla^2 \tilde{W}_A(u) \cdot \left(v + \sum_{j=b+1}^{b+n} \beta_j \zeta_j\right).
\]

The “semi-local” part is
\[
D_{W,X}^{\text{semi-loc}}(\beta) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \beta_j \varphi_j \sum_{l=1}^s e^{\tilde{\eta}_l(\lambda_j t + h_A + \epsilon h_j)} \nabla F^{(\delta_l)}_{l}(u_{\varphi_j}).
\]

Here \( h_j \) is the Cauchy primitive of \( \beta \) on \( C_j \). Hence the dependence of \( D_{W,X}^{\text{semi-loc}}(\beta) \) on \( \beta \) is not local.

The third part \( D_{W,X}^{\text{glob}} \) is more “global”, coming from the variation of the function \( \delta_A \) and \( \varphi \), hence is of finite rank. It is only nonzero on the supports of \( \beta_j \). Namely,
\[
D_{W,X}^{\text{glob}}(\beta, \rho)|_{C_j} = \frac{1}{2} \varphi_j \left( \nabla_s L_{\rho_j}(u_{\varphi_j}) + J \nabla_t L_{\rho_j}(u_{\varphi_j}) \right)
+ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \beta_j \varphi_j \sum_{l=1}^s e^{\tilde{\eta}_l(\lambda_j t + h_A)} \nabla F^{(\delta_l)}_{l}(u_{\varphi_j})
+ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \beta_j (\epsilon i \varphi_j \varphi_j) \sum_{l=1}^s e^{\tilde{\eta}_l(\lambda_j t + h_A)} \nabla F^{(\delta_l)}_{l}(u_{\varphi_j}).
\]

6.1.2. The tangent space of \( B_\kappa \) and the exponential map. We need to define an exponential map \( \exp \) identifying a small neighborhood of the origin of the tangent space with a neighborhood of \((\lambda, \alpha)\). Indeed, for \( \epsilon \) sufficiently small with \( \alpha + \epsilon \in [-1, 1] \), there exists a unique \( \zeta_\alpha \in T_{\kappa_{\alpha}} \tilde{X} \) such that \( \exp_{\kappa_{\alpha}} \zeta_\alpha = \kappa_{\alpha + \epsilon} \). Since \( u_{\varphi}(s, t) \) converges to \( \kappa_\alpha \) as \( s \to +\infty \), for \( s \) sufficiently large, one can use parallel transport along the shortest geodesic connecting \( \kappa_\alpha \) and \( u_{\varphi}(s, t) \) to identify \( \zeta_\alpha \) with a vector tangent to \( u_{\varphi}(s, t) \), and then multiply it by a cut-off function \( \beta_u \). This defines a vertical vector field \( \beta_u \zeta_\alpha \) along \( u \). One uses the notations above to define
\[
\exp_X(y, \epsilon) := \exp_X(\beta, v, \rho, \epsilon) := (A + \beta, g_{\beta, \rho} \exp_u(v + \beta_u \zeta_\alpha), e^{i\epsilon \varphi}, \alpha + \epsilon).
\]

Here \( g_{\beta, \rho} \in C^\infty \) is defined by (6.1). Moreover, by the smooth dependence of the perturbations on \( \alpha \), one has the infinitesimal deformation
\[
\zeta_\alpha = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \zeta_\alpha \in \Gamma(u^* T\tilde{Y}^\perp).
\]

The linearization \( D_{f,X} \), besides in the direction of \( T_X B_{\kappa_{\alpha}} \), has another derivative in the direction of \( \alpha \). Nevertheless this is a finite rank operator.

6.1.3. The tangent space of \( B^\#_\kappa \). Let \( X^\# = (X, \sigma) = (A, u, \varphi, \alpha, \sigma) \) and denote \( X = (A, u, \varphi) \). Then the tangent space \( T_{X^\#} B^\#_\kappa \) is identified as (5.1). One can define an exponential map on \( B^\#_\kappa \); however, we don’t need its explicit expression. The linearization \( D_{X^\#} f^\# \) can be written in the block form as (5.4). Here
\[
D_L(\beta_L, v_L, \rho_L, \epsilon) = D_{f,X}(\beta_L, v_L, \rho_L, \epsilon).
\]

\( \iota^R_L : T_X B_\kappa \to E_j|_{\sigma} \) is finite rank. On the other hand, \( D_R : T_{\sigma} B_j \to E_j|_{\sigma} \) is
\[
D_R(v_R) = \bar{\partial}_\sigma v_R + \nabla^2 \tilde{W}_\sigma(X) \cdot v_R.
\]

Here \( \bar{\partial}_\sigma \) is the Cauchy-Riemann operator on \( \Gamma(\Theta, \sigma^* T\tilde{X}) \).
We will primarily consider the derivatives in the direction of deformations \((β_L, v_L, ρ_L, ε)\) of \(X\) with \(ρ_L = 0, ε = 0\). Let \(T'_X B_{v_o}\) be the space of deformations of the type \((β_L, v_L, 0)\). The restriction of \(D_L\) to \(T'_X B_{v_o}\) is denoted by

\[ D'_L : T'_X B_{v_o} \to E_{v_o}\vert_K. \tag{6.6} \]

**Lemma 6.1.** For \(y_L = (β_L, v_L, ρ_L) \in T'_X B_{v_o}\), we have

\[ \|T^R_L(y_L)\|_{L^p(C_{(−1)}(S)∪C_j(S))} \leq c_5 e^{-τ_0 S}∥y_L∥. \]

**Proof.** By definition,

\[ S^X(σ) = \partial_σ + \nabla \tilde{W}(δX)(σ). \]

By definition, \(T^R_L\) only depends on the parameter \(β_L\) through the variation of \(δ_X\). Denote

\[ δ_X δ_t = δ_{A+tβ_L}, \ t ∈ [0, 1]. \]

Then by the definition of \(\tilde{W}(δ)\) (2.3),

\[ \nabla \tilde{W}(δ_X δ_t)(δ_t σ) = \left[ d\tilde{W}(δ_X δ_t)(δ_t σ) \right]^W = (δ_X δ_t)^r \left[ (δ_X^{-1} δ_t)^* d\tilde{W}(δ_t σ) \right]^W \]

\[ = (δ_X δ_t)^r \left[ (δ_X^{-1} δ_t)^*(d\tilde{W}(δ_X^{-1} σ)) \right]^W. \]

Since \(δ_X^{-1} σ\) approaches \(v\) (resp. \(κ\)) exponentially as \(s \to −∞\) (resp. as \(s \to +∞\)), we have

\[ \|d\tilde{W}(δ_X^{-1} σ(s, t))\| \leq ce^{-τ_0 s}. \]

Therefore,

\[ T^R_L(y_L) = \frac{d}{dt}_{|t=0} \left( \frac{∂}{∂σ} (δ_t σ) + \nabla \tilde{W}(δ_X δ_t)(δ_t σ) \right) \]

\[ = (D_σ \partial_σ)(η_{β_L}) + \left( \frac{d}{dt}_{|t=0} (δ_X δ_t)^r d\tilde{W} \right)^W. \]

Here \(D_σ \partial_σ\) is the linearization of \(\partial_σ\) along \(σ\) and \(η_{β_L} = η_{β_L, 0}\) is defined by (6.2) for \(ρ = 0\). Therefore it is easy to see that the \(L^p\)-norm of \(T^R_L(y_L)\) outside \([-S, S] × S^1\) is controlled by \(e^{-τ_0 S}\|y_L\|\). \(\square\)

**6.2. Proof of Lemma 5.7.** Let \(V^‘\) be the augmented vortex equation operator, i.e.,

\[ V^‘(A, u, φ) = \left( * F_A + ν u, (A - A_0) \right) \in L^p_+(Σ^*, C_j(T)) \]
Write the differential of the exponential map as $d \exp_z V = E_1(x, \exp_z V) dx + E_2(x, \exp_z V) \nabla V$. Then by (6.10), over $C_j(T) \setminus C_j(2T)$, for some $c_2 > 0$, one has

\begin{align}
2|\mathcal{W}(\gamma_{\text{app}}_{\xi \xi})| &= |\partial_x u' + L_\phi(u') + J (\partial_y u' + L_\phi(u')) + \nabla \hat{W}_\rho(u')| \\
&\leq |L_\phi(u') + J L_\phi(u')| + |\nabla \hat{W}_\rho(u') - \nabla \hat{W}_\rho(v_a)| \\
&+ |E_2(v_a, u') \cdot \nabla (\beta^L_L |\xi|) + |E_2(v_a, u') \cdot \nabla (\beta^L_L |\xi|)| \\
&\leq c_2 (|\phi| + |\psi| + |\xi| + |\nabla \xi|). \tag{6.11}
\end{align}

In a similar way, over $C_j(2T) \setminus C_j(3T)$, one has

\begin{align}
2|\mathcal{W}(\gamma_{\text{app}}_{\xi \xi})| &\leq c_2 (|\phi| + |\psi| + |\text{Twist}_x(\zeta^R)| + |\nabla \text{Twist}_x(\zeta^R)|). \tag{6.12}
\end{align}

Notice that $\zeta^L, \zeta^R$ decays like $e^{-70s}$. So by (6.8), (6.9), (6.11) and (6.12), for some $c_3 > 0$,

\begin{align}
\|e_{\xi \xi}(\gamma_{\text{app}}_{\xi \xi}, \omega_{\text{app}}_{\xi \xi}) + \mathcal{W}(\gamma_{\text{app}}_{\xi \xi})\|_{L^p} &= \|\mathcal{W}(\gamma_{\text{app}}_{\xi \xi})\|_{L^p(C_j(T) \setminus C_j(3T))} \\
&\leq c_2 (\|(|\phi, \psi|)\|_{L^p(C_j(T) \setminus C_j(3T))} + \|\xi\|_{L^1(C_j(T) \setminus C_j(2T))} + \|\nabla \xi\|_{L^{1,p}(C_j(T) \setminus C_j(2T))}) \\
&\leq c_3 e^{-70T}. \tag{6.13}
\end{align}

Lemma 5.7 follows from (6.7) and (6.13).

6.3. Proof of Lemma 5.8. The values of the numerous constants $c_k$ and $\varepsilon_k$ for $k = 1, 2, \cdots$ are reset within this subsection.

6.3.1. Some de Rham and Dolbeault theory on the cylinder. Let $RW^{1,p}_t(\Theta, \Lambda^1 \otimes g)$ be the Banach space consisting of $g$-valued 1-forms $\beta$ on the cylinder $\Theta$ such that there are $\Phi, \Psi \in g$ such that

$$\beta|_{\Theta_+} \in RW^{1,p}_t(\Theta_+, \Lambda^1 \otimes g), \quad \beta - \Phi ds - \Psi dt|_{\Theta_-} \in RW^{1,p}_t(\Theta_-, \Lambda^1 \otimes g).$$

Consider the Fredholm operator

$$dR : RW^{1,p}_t(\Theta, \Lambda^1 \otimes g) \to L^p_t(\Theta, g \otimes g) \tag{6.14}$$
defined by $dR(\alpha) = (\ast d\beta, d^* \beta)$. It is easy to see that $dR$ is an isomorphism, with the inverse

$$Q_{dR} : L^p_t(\Theta, g \otimes g) \to RW^{1,p}_t(\Theta, g). \tag{6.15}$$

On the other hand, consider the operator

$$dB := \overline{\theta} : W^{2,p}_t(\Theta, g^\mathbb{C}) \to W^{1,p}_t(\Theta, \Lambda^{0,1} \otimes g^\mathbb{C}).$$

It is injective and has complex Fredholm index $-2$. Choose a subspace $E_{dB} \subset W^{1,p}_t(\Theta, \Lambda^{0,1} \otimes g)$ complementary to the image of $dB$, spanned by smooth compactly supported forms $\nu_1, \nu_2$, which are independent of the angular coordinate $t$.

Let $\omega$ be a complex number with small modulus and $\omega = e^{-4T-t^0}$. Define $\nu^T_4(s) = \nu_4(s + 4T)$ and let $E^T_{dB}$ be spanned by $\nu^T_1, \nu^T_2$. Without loss of generality, assume that $\nu^T_1$ is supported in $(-\infty, -4T] \times S^1$. Then by the translation invariance of $dB$, we have the decomposition

$$W^{1,p}_t(\Theta, \Lambda^{0,1} \otimes g^\mathbb{C}) \simeq \text{Im} dB \oplus E^T_{dB}.$$

So for any $\beta_R \in W^{1,p}_t(\Theta, \Lambda^1 \otimes g)$, define the decomposition

$$\beta_R^{0,1} = dB(h_R^T) + \Omega^T(\beta_R^{0,1}) \in \text{Im} dB + E^T_{dB}. \tag{6.16}$$
6.3.2. Extensions of $D_R$ and right inverses. Abbreviate $X^{\text{app}} = X'_\omega = (X'_\omega, 0) \in \mathcal B_\omega$. Let
\[
\mathcal D\omega := \mathcal D X'_\omega \partial_\omega : T X'_\omega \mathcal B_{\kappa_0} \oplus E_a \to \mathcal E_{\kappa_0} \big| X'_\omega
\] (6.17)
be the linearized operator at $X'_\omega$. Let $T X'_\omega \mathcal B_{\kappa_0} \subset T X'_\omega \mathcal B_{\kappa_0}$ be the subspace parametrizing deformations of the form $(\beta, v, 0)$, i.e., those deformations which don’t change the frame $\varphi$; let $\mathcal D\omega$ be the restriction of $\mathcal D\omega$ to this subspace. To construct a right inverse of $\mathcal D\omega$, for sufficiently small gluing parameter $\omega$, it suffices to consider $\mathcal D\omega$.

One is tempted to use a right inverse of (6.4) (modulo obstruction) to construct the right inverse. However, the standard procedure of constructing the right inverse is through a cut-and-glue process. This requires that the linearized operators on both components of the node are of similar type, which is the case in the current situation. We need to do the following modification.

We abbreviate $\mathcal B_j := \mathcal B_j (v_0^X, \kappa_0^X)$ and $\mathcal E_j := \mathcal E_j (v_0^X, \kappa_0^X)$. We enlarge $T_{\sigma_a} \mathcal B_j$ and $\mathcal E_j | \sigma_a$, by
\[
T^+_{\sigma_a} \mathcal B_j := T_{\sigma_a} \mathcal B_j \oplus RW_{\tau}^{1,p}(\Theta, \Lambda^1 \otimes g), \quad T^+_j | \sigma_a := \mathcal E_j | \sigma_a \oplus L^p(\Theta, g \oplus g).
\]
We redefine certain Banach norms. On the summands $L^p(\Theta, g \otimes g) \subset \mathcal E_j^+ | \sigma_a$ and $RW_{\tau}^{1,p}(\Theta, \Lambda^1 \otimes g) \subset T^+_{\sigma_a} \mathcal B_j$, the original Sobolev norms are essentially weighted by the function $\varphi(s, t) = e^{\tau \sqrt{1 + s^2}}$. We redefine the norms by using the weight function $\varphi^T(s, t) = \varphi(s + 4T, t)$. The new norms are labelled with an extra index $\omega$.

Let $D_R : T_{\sigma_a} \mathcal B_j | \sigma_a$ be the linearization of the soliton operator along $\sigma_a$. We would like to extend it to an operator $D^+_R : T^+_{\sigma_a} \mathcal B_j | \sigma_a$. It is defined as follows. Choose a cut-off function $\chi : (-\infty, +\infty) \to [0, 1]$, which is supported on $[0, +\infty)$ and is equal to 1 on $[1, +\infty)$. Choose a large number $R_0 > 0$ and denote $\chi^T(s) = \chi(s + 4T + R_0)$. Then for any $\beta \in RW_{\tau}^{1,p}(\Theta, \Lambda^1 \otimes g)$, we denote
\[
\beta^T := \chi^T \beta \in W_{\tau}^{1,p}(\Theta, \Lambda^1 \otimes g).
\]
Then by the decomposition from (6.16), we can write
\[
\beta^{0,1}_{\sigma_a} = \Theta h^T + \Omega^T(\beta^{0,1}_{n_T}) + \Omega^T(\beta^{0,1}_{n_T}), \quad h^T_{\sigma_a} \in W^{2,p}(\Theta, g^\sigma).
\] (6.18)

Then we define
\[
ds^{T}_{\sigma_a}(\beta_R) = L_{\beta_R,T}^{0,1}(\sigma_a) + \frac{d}{dx} \bigg|_{x=0} \sum_{l=0}^{s} e^{\tau (\chi h^T)} \nabla F^T_l(s_{\kappa_a}(\sigma_a)) \in L^p(\Theta, \sigma_a^T \bar X).
\] (6.19)

\[
\mathcal D^+_R (v_R, \beta_R) = \left(D_R(v_R) + ds^{T}_{\sigma_a}(\beta_R), dR(\beta_R)\right)
\] (6.20)
where $dR$ is the de Rham operator (6.14). Because $dR$ is an isomorphism, we have
\[
\text{Ind}(D_R) = \text{Ind}(D^+_R).
\]
Let $\bar D_R : T_{\sigma_a} \mathcal B_j \oplus E_{\omega,a} \to \mathcal E_j | \sigma_a$ and $\bar D^+_R : T^+_{\sigma_a} \mathcal B_j \oplus E_{\omega,a} \to \mathcal E_j^+ | \sigma_a$ be the linearization with the obstruction addition and its extension. If we have a right inverse $\bar Q_{\omega,a} : \mathcal E_j | \sigma_a \to T_{\sigma_a} \mathcal B_j \oplus E_{\omega,a}$ to $\bar D_R$, then it is easy to see that the operator $\bar Q^+_{\omega,a} : \mathcal E_j^+ | \sigma_a \to T^+_{\sigma_a} \mathcal B_j \oplus E_{\omega,a}$, defined by
\[
\bar Q^+_{\omega,a}(\eta_R, \zeta_R, \zeta'_R) = \left(\bar Q_{\omega,a}(\eta_R, \zeta_R, \zeta'_R), Q_{HR}(\zeta_R, \zeta'_R)\right),
\] (6.21)
is a right inverse to $\bar D^+_R$. Here $Q_{HR}$ given in (6.15) is the inverse of $dR$.

On the other hand, keep in mind that we can choose $\bar Q_{\omega,a}$ to be $\Gamma_a$-equivariant, and such that
\[
(v_R, e_R) \in \text{Im } \bar Q_{\omega,a} \implies \sum_{\gamma \in \Gamma_a} v_R(\gamma w_a) \in H_a \subset T_{\omega_a} \bar X.
\] (6.22)
6.3.3. The auxiliary operations. Let $f$ be a map defined on a cylinder $[a, b] \times S^1$ and take $\varpi = e^{-4T-i\theta}$. We define $\text{Twist}(f) = \text{Twist}_\varpi(f)$ to be the reparametrization

$$
\text{Twist}_\varpi(f)(s, t) = f(s - 4T, t - \theta), \ (s, t) \in [a + 4T, b] \times S^1.
$$

We see that $(u'_\varpi)|_{C_j(3T)} = \text{Twist}_\varpi(\sigma_a)$. Moreover, by construction, $(u'_\varpi)|_{C_j(T)}$ is exponentially close to $\text{Twist}_\varpi(\sigma_a)$. Therefore one can define the parallel transport

$$
\text{Paral}^R : W^{1,p}([-3T, +\infty) \times S^1, \sigma'_a T \tilde{X}) \to W^{1,p}(C_j(T), (u'_\varpi)^* T \tilde{Y}^\perp).
$$

Similarly we define

$$
\text{Paral}^L : W^{1,p}(\Sigma^* \setminus C_j(3T), u'_a T \tilde{Y}^\perp) \to W^{1,p}(\Sigma^* \setminus C_j(3T), (u'_\varpi)^* T \tilde{Y}^\perp).
$$

Choose $h \in (0, 1/2)$ and two cut-off functions $\chi^L_{hT} : \mathbb{R} \to [0, 1]$ such that when $s \leq 2T$, $\chi^L_{hT}(s) = 1$; when $s \geq (2 + h)T$, $\chi^L_{hT}(s) = 0$; when $s \geq -2T$, $\chi^R_{hT}(s) = 1$; when $s \leq -(2 + h)T$, $\chi^R_{hT}(s) = 0$ (see the picture). Moreover,

$$
|\nabla \chi^L_{hT}| \leq \frac{2}{hT}, \quad |\nabla^2 \chi^L_{hT}| \leq \frac{10}{(hT)^2}.
$$

Abbreviate them by $\chi^L$ and $\chi^R$ respectively.

Let $T^-_{X_\varpi} B_{v_0} \subset T^+_{X_\varpi} B_{v_0}$ be the subspace parametrizing deformations of the form $(\beta_L, v_L, 0)$, i.e., fixing the frame. We define $\text{Glue} : T^-_{X_\varpi} B_{v_0} \oplus T^+_{X_\varpi} B_j \to T^-_{X_\varpi} B_{v_0}$ as follows. Take $y_L = (\beta_L, v_L) \in T^-_{X_\varpi} B_{v_0}$ and $y_R = (\beta_R, v_R) \in T^+_{\sigma_a} B_j$. Recall the map $A \mapsto h_{j,A}$ specified by Condition 2.11. Denote $h_L = h_{jL}(\beta_L) = h_{jL} + h_{j,A} - h_{j,A}$ which is linear in $\beta_L$ such that $\beta_L = \overline{\partial} h_L + \overline{\partial} h_{j,A}$. We also recall (6.18), by which we can write

$$
\chi^T \beta_R = \overline{\partial} h_R^T + \overline{\partial} h^T_R + \Omega^T(\chi^T \beta^0_R) + \Omega^T(\chi^T \beta^1_R).
$$

Then define

$$
\chi^L * y_L = \left( \overline{\partial}(\chi^L h_L) + \partial(\chi^L h_L), \chi^L v_L \right) \in T^-_{X_\varpi} B_{v_0},
$$

$$
\chi^R * y_R = \left( \overline{\partial}(\chi^R h_R^T), \chi^R v_R \right) \in T^+_{\sigma_a} B_j.
$$

We define $\text{Glue}(y_L, y_R) = (\beta_{L,R}, v_{L,R})$, where

$$
\beta_{L,R} = \overline{\partial}(\chi^L h_L) + \partial(\chi^L h_L) + \text{Twist} \left( \overline{\partial}(\chi^R h_R^T) + \partial(\chi^R h_R^T) \right) \in W^{1,p}(\Sigma^*, \Lambda^1 \otimes \mathfrak{g}),
$$

$$
v_{L,R} = \chi^L \text{Paral}^L(v_L) + \text{Paral}^R(\chi^R(v_R)) \in W^{1,p}(\Sigma^*, (u'_\varpi)^* T \tilde{Y}^\perp).
$$

On the other hand, we define $\text{Cut}$ and $\text{Paste}$ as linear maps

$$
\text{Cut} : \mathcal{E}_{x_\varpi}|_{x_\varpi} \to \mathcal{E}_{v_0}|_{x_\varpi} \oplus \mathcal{E}^+_{j}|_{\sigma_a}, \quad \text{Paste} : \mathcal{E}_{v_0}|_{x_\varpi} \oplus \mathcal{E}^+_{j}|_{\sigma_a} \to \mathcal{E}_{x_\varpi}|_{x_\varpi},
$$

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such that \( \text{Paste} \) is a left inverse of \( \text{Cut} \). More precisely, let \( \mathbf{z} = (\eta, \varsigma, \varsigma') \in \mathcal{E}_\kappa \big|_{\mathcal{X}_\varpi} \), where \( \eta \in L^p(\Lambda^{0,1} \otimes (u')^*T^*Y^\perp), \varsigma \in L^p(\hat{g}), \varsigma' \in L^p(\hat{g})/\Lambda_{ob} \). Then define \( \text{Cut}(\eta, \varsigma) = (\mathbf{z}_L, \mathbf{z}_R) \) where

\[
\mathbf{z}_L = \begin{cases} (\text{Paral}_L^{-1}(\eta), \varsigma, \varsigma'), & \Sigma^* \setminus C_j(2T), \\ 0, & \mathbb{C}(-2T), \end{cases} \quad \mathbf{z}_R = \begin{cases} 0, & \mathbb{C}(-2T), \\ \text{Twist}^{-1}(\text{Paral}_R^{-1}(\eta), \varsigma, \varsigma'), & \Sigma^* \setminus C_j(2T), \end{cases}
\]

(6.26)

On the other hand, if \( \mathbf{z}_L = (\eta_L, \varsigma_L, \varsigma'_L) \in \mathcal{E}_\nu \big|_{\mathcal{X}_\varpi}, \mathbf{z}_R = (\eta_R, \varsigma_R) \in \mathcal{E}_j \big|_{\mathcal{X}_\varpi} \), then define

\[
\text{Paste}(\mathbf{z}_L, \mathbf{z}_R) = \begin{cases} (\text{Paral}_L(\eta_L), \varsigma_L, \varsigma'_L), & \Sigma^* \setminus C_j(2T), \\ (\text{Paral}_R(\eta_R), \text{Twist}_\varpi(\varsigma_R), \text{Twist}_\varpi(\varsigma'_R)), & \Sigma^* \setminus C_j(2T) \end{cases}
\]

6.3.4. The approximate right inverse and estimates. Recall the operator \( \mathcal{T}_L^R : T_{\mathcal{X}_\varpi} B_{\nu_0} \to \mathcal{E}_j \big|_{\mathcal{X}_\varpi} \) in (5.4), which is of rank one and only depends on the variable \( \beta_L \). Hence we can regard it as defined on \( T_{\mathcal{X}_\varpi} B_{\nu_0} \). The operator

\[
y_L \mapsto (\mathcal{T}_L^R(y_L), 0) \in \mathcal{E}_j \big|_{\mathcal{X}_\varpi} \oplus \mathcal{W}_1^\varpi(\Theta, g \oplus \hat{g}) = \mathcal{E}_j \big|_{\mathcal{X}_\varpi}
\]

is still denoted by \( \mathcal{T}_L^R \). Let \( \mathcal{D}_{\#, a}^+ := \mathcal{D}_{\#, a}^+ \mathcal{F}_\#^0 \) be the operator

\[
\mathcal{D}_{\#, a}^+ = \begin{pmatrix} \mathcal{D}_L^- & 0 \\ \mathcal{T}_L^R & \mathcal{D}_R^+ \end{pmatrix} : T_{\mathcal{X}_\varpi} B_{\nu_0} \oplus T_{\mathcal{X}_\varpi}^a B_j \to \mathcal{E}_\nu \big|_{\mathcal{X}_\varpi} \oplus \mathcal{E}_j \big|_{\mathcal{X}_\varpi}.
\]

Including the obstruction addition, we obtain

\[
\tilde{\mathcal{D}}_{\#, a}^+ := \begin{pmatrix} \mathcal{D}_L^- & 0 \\ \mathcal{T}_L^R & \mathcal{D}_R^+ \end{pmatrix} : (T_{\mathcal{X}_\varpi} B_{\nu_0} \oplus E_{L,a}) \oplus (T_{\mathcal{X}_\varpi}^a B_j \oplus E_{R,a}) \to \mathcal{E}_\nu \big|_{\mathcal{X}_\varpi} \oplus \mathcal{E}_j \big|_{\mathcal{X}_\varpi}.
\]

Let \( \mathcal{Q}_{L,a} : \mathcal{E}_\nu \big|_{\mathcal{X}_\varpi} \to T_{\mathcal{X}_\varpi} B_{\nu_0} \oplus E_{L,a} \) be a chosen right inverse to \( \mathcal{D}_L^- \). Together with \( \mathcal{Q}_{R,a} \) given in (6.21), one can construct a right inverse to \( \tilde{\mathcal{D}}_{\#, a}^+ \) defined as follows. For \( \mathbf{z}_L \in \mathcal{E}_\nu \big|_{\mathcal{X}_\varpi}, \mathbf{z}_R \in \mathcal{E}_j \big|_{\mathcal{X}_\varpi} \), denote \( \overline{y}_L = \mathcal{Q}_L(z_L) \). Then define

\[
\mathcal{Q}_{\#, a}^+(z_L, z_R) = \left( \overline{y}_L, \mathcal{Q}_{R,a}(z_R - \mathcal{T}_L^R(\overline{y}_L)) \right).
\]

(6.27)

Moreover, \( \text{Glue} \) naturally extends to

\[
\text{Glue} : \left( T_{\mathcal{X}_\varpi} B_{\nu_0} \oplus E_{L,a} \right) \oplus \left( T_{\mathcal{X}_\varpi}^a B_j \oplus E_{R,a} \right) \to T_{\mathcal{X}_\varpi} B_{\kappa_0} \oplus E_a.
\]

Then we define the \textbf{approximate right inverse} to \( \mathcal{D}_{\#, a}^+ \) by

\[
\mathcal{Q}_{\#, a}^{app} = \text{Glue} \circ \mathcal{Q}_{\#, a}^+ \circ \text{Cut} : \mathcal{E}_\kappa \big|_{\mathcal{X}_\varpi} \to T_{\mathcal{X}_\varpi} B_{\kappa_0} \oplus E_a.
\]

\textbf{Lemma 6.2.} There exist constants \( c_1 > 0 \) and \( \varepsilon_1 > 0 \), such that for any \( \varpi \in R^{*}_{\varepsilon_1}, \| \mathcal{Q}_{\#, a}^{app} \| \leq c_1. \)

\textbf{Proof.} Firstly, for \((\eta, \varsigma, \varsigma') \in \mathcal{E}_\kappa \big|_{\mathcal{X}_\varpi}, \) by the definition (6.26) and the new norm \( \| \cdot \|_{\mathcal{W}_\varpi} \), we have \( \| (z_L, z_R) \|_{\mathcal{W}_\varpi} \leq \| (\eta, \varsigma, \varsigma') \|. \) Secondly, we consider the norm of \( \mathcal{Q}_{\#, a}^+ \). Indeed, since \( \mathcal{Q}_{L,a} \) and \( \mathcal{T}_L^R \) are uniformly bounded, by (6.27) it suffices to consider \( \mathcal{Q}_{R,a}^+ \). In (6.21), \( \mathcal{T}_{\nu_0} \) and \( \mathcal{T}_{R,a} \) are uniformly bounded with respect to the \( \varpi \)-norms. Hence it reduces to the operator \( \mathcal{D}_{\#, a}^+ \). Indeed, notice that the map \( \mathcal{D}_{\#, a}^+ : h_{\mathcal{R}_L} \) is uniformly bounded with respect to \( \varpi \)-norm; it is routine to check that \( \mathcal{D}_{\#, a}^+ \) is uniformly bounded. Therefore \( \mathcal{Q}_{\#, a}^+ \) is uniformly bounded by a constant. Lastly, \( \text{Glue} \) is uniformly bounded because when \( |\varpi| \) is small, the derivatives of \( \text{Paral}_L/R \) can be controlled; moreover, the cut-off functions used in the definition of \( \text{Glue} \) are sufficiently tame. Therefore the lemma holds for some appropriate \( c_1. \) \( \square \)
Lemma 6.3. There exist \( \varepsilon_2 \in (0, \varepsilon_1) \) and \( c_2 > 0 \) such that for every \( \varpi \in R^{*}_{\varepsilon_2} \),
\[
\left\| \widetilde{D}_{\varpi,a}^{-1} \circ \widetilde{Q}^{\text{app}}_{\varpi,a} - \text{Id} \right\| \leq \frac{c_2}{T}.
\]

Lemma 6.3 will be proved in a moment. It implies that for \( \varpi \) sufficiently close to zero, one can construct an exact right inverse of the operator \( \widetilde{D}_{\varpi} : T_{X_{\varpi}} B_{\kappa_0} + E_a \rightarrow \mathcal{E}_{\kappa_0} | x_{\varpi} \) given by
\[
\widetilde{Q}_{\varpi,a} := \widetilde{Q}^{\text{app}}_{\varpi,a} \circ (\widetilde{D}_{\varpi} \circ \widetilde{Q}^{\text{app}}_{\varpi,a})^{-1}.
\]

Lemma 6.2 and Lemma 6.3 imply that \( \widetilde{Q}_{\varpi,a} \) is uniformly bounded. Then one obtains a right inverse to the full linearized operator \( \widetilde{D}_{\varpi,a} : T_{X_{\varpi}} \mathcal{B}_a \rightarrow \mathcal{E}_{\kappa_0} | \mathcal{X}_{\varpi,a} \) defined by
\[
\widetilde{Q}_{\varpi,a}(z) := (\widetilde{Q}_{\varpi,a}(z), 0, 0) \in (T_{X_{\varpi}} B_{\kappa_0} \oplus E_a) \oplus \mathbb{R} \oplus T_a.
\]
Furthermore, since \( \widetilde{Q}_{\varpi,a} \) has the same image as \( \widetilde{Q}^{\text{app}}_{\varpi,a} \), by (6.22), we see that (5.22) holds for all \( \xi = (\beta, v, \rho, w, e) \in \text{Im} \widetilde{Q}_{\varpi,a} \).

6.3.5. Proof of Lemma 6.3. Let \( (\eta, \varsigma, \varsigma') \in \mathcal{E}_{\kappa_0} | x_L \), \( \text{Cut}(\eta, \varsigma, \varsigma') = (z_L, z_R) \) and
\[
(\tilde{y}_L, \tilde{y}_R) = \widetilde{Q}_{\#}^{+} (z_L, z_R) \in \left( T_{X_{\varpi}} B_{\kappa_0} \oplus E_{L,a} \right) \oplus \left( T_{X_{\varpi}} B_{\kappa_0} \oplus E_{R,a} \right).
\]

Using the notations of (6.25),
\[
(\widetilde{D}_{\varpi} \circ \text{Glue} \circ \widetilde{Q}_{\#}^{+} \circ \text{Cut}) (\eta, \varsigma, \varsigma')
\]
\[
= (\widetilde{D}_{\varpi} (\beta L \# R, v L \# R, e_L + e_R))
\]
\[
= (\widetilde{D}_{\varpi} \circ \text{Paral}^L - \text{Paste} \circ \widetilde{D}_{\varpi} - \text{Paste} \circ \mathcal{T}_L^R (\chi^L \ast y_L)
\]
\[
+ (\widetilde{D}_{\varpi} \circ \text{Paral}^R - \text{Paste} \circ \widetilde{D}_{\varpi}^R (\chi^R \ast y_R)
\]
\[
- \text{Paste} \left( \widetilde{D}_{\varpi} (y_L - \chi^L \ast y_L) + \widetilde{D}_{\varpi}^R (y_R - \chi^R \ast y_R) \right)
\]
\[
+ \text{Paste} \left( \widetilde{D}_{\varpi} (y_L) + \mathcal{T}_L^R (\chi^L \ast y_L) + \widetilde{D}_{\varpi}^R (y_R) + e_L + e_R \right)
\]
\[
=: I + II + III + IV.
\]

Note that by the definition of \( \text{Paste} \) and the property of \( \mathcal{T}_L^R \),
\[
IV = \text{Paste} \left( \widetilde{D}_{\varpi} (y_L) + \mathcal{T}_L^R (\chi^L \ast y_L) + \widetilde{D}_{\varpi}^R (y_R) + e_L + e_R \right)
\]
\[
= \text{Paste} \left( \widetilde{D}_{\varpi} (y_L) + \mathcal{T}_L^R (y_L) + \widetilde{D}_{\varpi}^R (y_R) + e_L + e_R \right) = \text{Paste} \circ \text{Cut} (\eta, \varsigma, \varsigma') = (\eta, \varsigma, \varsigma').
\]

Then Lemma 6.3 follows from the three lemmata below.

Lemma 6.4. For some constant \( c_3 > 0 \), we have
\[
\left\| I \right\| = \left\| (\widetilde{D}_{\varpi} \circ \text{Paral}^L - \text{Paste} \circ \widetilde{D}_{\varpi} - \text{Paste} \circ \mathcal{T}_L^R (\chi^L \ast y_L)) \right\| \leq c_3 \left( \frac{1}{hT} + e^{-\tau_0 (2h)^T} \right) \left\| (\eta, \varsigma, \varsigma') \right\|.
\]

Lemma 6.5. For some constant \( c_4 > 0 \), we have
\[
\left\| II \right\| = \left\| (\widetilde{D}_{\varpi} \circ \text{Paral}^R - \text{Paste} \circ \widetilde{D}_{\varpi}^R (\chi^R \ast y_R)) \right\| \leq c_4 \left( \frac{1}{hT} + e^{-\tau_0 (2h)^T} \right) \left\| (\eta, \varsigma, \varsigma') \right\|.
\]

Lemma 6.6.
\[
III = \text{Paste} \left( \widetilde{D}_{\varpi} (y_L - \chi^L \ast y_L) + \widetilde{D}_{\varpi}^R (y_R - \chi^R \ast y_R) \right) = 0
\]
Lastly, on the support of $\lambda$, the distance between $u_i$ and $u'$ is controlled by $e^{-\tau_0(2-h)T}$. Hence for some $c_{22} > 0$,

$$\left\| \chi^L \left( D_{W,x_a^e}^0 (\beta_L, Paral R^T (v_L)) - Paral^L D_{W,x_a^e}^0 (\beta_L, v_L) \right) \right\|_{L^p} \leq c_{22} e^{-\tau_0(2-h)T}. \quad (6.32)$$

Similarly, for some $c_{22} > 0$,

$$\left\| \chi^L \left( D_{W,x_a^e}^0 (\beta_L) - Paral^L D_{W,x_a^e}^0 (\beta_L) \right) \right\|_{L^p} + \left\| \chi^L \left( D_{W,x_a^e}^0 (\beta_L) - Paral^L D_{W,x_a^e}^0 (\beta_L) \right) \right\|_{L^p} \leq c_{22} e^{-\tau_0(2-h)T}. \quad (6.33)$$

Lastly, on the support of $(1 - \chi^L)$, by (6.5),

$$D_{W,x_a^e}^0 (\beta_L) = \frac{d}{d\epsilon} \left|_{\epsilon=0} \sum_{j=0}^{s} e^\epsilon (h_A + \epsilon h_L) \nabla F_j (\delta_{a} \delta_i) (u_{\varphi}), \right.$$  

$$T_{L}^R (y_L) = \frac{d}{d\epsilon} \left|_{\epsilon=0} \sum_{j=0}^{s} e^\epsilon (h_A + \epsilon h_L) \nabla F_j (\delta_{a} \delta_i) (\sigma_a \varphi). \right.$$  

Since the distance between $u_{\varphi}$ and $Twist(\sigma_a \varphi)$ is exponentially small, $h_A$ is exponentially small, we see that for some $c_{23} > 0$,

$$\left\| (1 - \chi^L) \left( D_{W,x_a^e}^0 (\beta_L) - Paste T_{L}^R (y_L) \right) \right\|_{L^p} \leq c_{23} e^{-2\tau_0 T} \left\| y_L \right\|. \quad (6.34)$$

Therefore, by (6.30)-(6.33), for an appropriate $c_{24} > 0$, we have

$$\left\| \pi_1 (I) \right\|_{L^p} \leq c_{24} \left( \frac{1}{K^T} + e^{-\tau_0(2-h)T} \right) \| (\eta, \zeta, \zeta') \|. \quad (6.35)$$
Now we estimate \( \pi_2(I) \). Over \( \Sigma^* \backslash C_j(2T) \),

\[
\pi_2(I) = \pi_2 \left( (\mathcal{D}_{X^+} \mathcal{F}^0 \circ \text{Paral}^{L} - \text{Paste} \circ \mathcal{D}_{L} - \text{Paste} \circ \mathcal{T}_{L}^{R}) (\chi^L \ast y_L) \right)
\]

\[
= \left( * d \beta_L + \nu d \mu(u') \cdot \text{Paral}^{L}(v_L) \cdot \overline{\mathcal{F}} \beta_L - \left( * d \beta_L + \nu d \mu(u) \cdot v_L, \overline{\mathcal{F}} \beta_L \right) \right)
\]

\[
= \left( \nu d \mu(u') \cdot \text{Paral}^{L}(v_L) - \nu d \mu(u) \cdot v_L, 0 \right).
\]

Using the fact that over \( C_j(2T) \), \( \pi_2(I) = (\ast d \beta_L + \nu d \mu(u_i) \cdot v_L, \overline{\mathcal{F}} \beta_L) = \pi_2 \text{Cut} (\eta, \varsigma, \varsigma') = 0 \), one obtains

\[
\pi_2(I)_{C_j(2T)} = \left( * d (\mathcal{F} \chi^L h_L) + \partial (\chi^L h_L) + \nu d \mu(u') \cdot \text{Paral}^{L}(\chi^L v_L), \overline{\mathcal{F}} (\mathcal{F} \chi^L h_L) + \partial (\chi^L h_L) \right)
\]

\[
= \Phi^L_1(\nabla \chi^L, \nabla h_L) + \Phi^L_2(\nabla^2 \chi^L, h_L) + \chi^L \left( * d \beta_L + \nu d \mu(u') \cdot \text{Paral}^{L}(v_L), \overline{\mathcal{F}} \beta_L \right)
\]

\[
= \Phi^L_1(\nabla \chi^L, \nabla h_L) + \Phi^L_2(\nabla^2 \chi^L, h_L) + \chi^L \left( \nu d \mu(u') \cdot \text{Paral}^{L}(v_L) - \nu d \mu(u_i) \cdot v_L, 0 \right)
\]

Here \( \Phi^L_1, \Phi^L_2 \) are certain bilinear functions. Comparing with (6.36), (6.37) is a unifying expression of \( \pi_2(I) \). Then for certain \( c_{25} > 0 \) and \( c_{26} > 0 \), we have

\[
\| \pi_2(I) \|_{L^\infty} \leq \| \Phi^L_1 \|_{L^\infty} + \| \Phi^L_2 \|_{L^\infty} + \| \chi^L \nu (d \mu(u') \cdot \text{Paral}^{L}(v_L) - d \mu(u) \cdot v_L) \|_{L^\infty}
\]

\[
\leq c_{25} \left( \| \nabla \chi^L \|_{L^\infty} + \| \nabla^2 \chi^L \|_{L^\infty} \right) \| \beta_L \|_{W^{1,p}} + c_{25} e^{-2(2-T)^r} \| v_L \|_{W^{1,p}}
\]

\[
\leq c_{26} \left( \frac{1}{h^T} + \frac{1}{(h^T)^2} + e^{-2(2-T)^r} \right) \|(\eta, \varsigma, \varsigma')\|.
\]

(6.35) and (6.38) conclude this Lemma. \( \square \)

**Proof of Lemma 6.5.** Over \( C_j(2T) \), one has

\[
\pi_1(II) = \pi_1 \left( (\mathcal{D}_{X^+} \circ \text{Paral}^R - \text{Paste} \circ \mathcal{D}_{L}) (\chi^R \ast y_R) \right)
\]

\[
= \mathcal{G}_A (\text{Paral}^R(v_R)) + L_{\text{Twist}_X^R}(\mathcal{G}^R h_R^T) (u') + \mathcal{D}_{W^L X^+} (\text{Twist}^*_X \mathcal{G}^R h_R^T) (u')
\]

\[
- \text{Paral}^R (\mathcal{D}_R(v_R) + L_{\mathcal{G}^R h_R^T}(\sigma_R) + d \mathcal{G}^R_{\mathcal{G}^R h_R^T}(\beta_R))
\]

Over \( \Sigma^* \backslash C_j(2T) \), by the definition of Paste,

\[
\pi_1(II) = \mathcal{G}_A (\chi^R (\text{Paral}^R(v_R))) + L_{\text{Twist}_X^R} \mathcal{G}(\chi^R h_R^T) (u')
\]

\[
+ \mathcal{D}_{W^L X^+} (\text{Twist}^*_X (\mathcal{G}(\chi^R h_R^T) + \partial (\chi^R h_R^T)))
\]

\[
= \text{Twist}^*_X (\mathcal{G}(\chi^R h_R^T) + \partial (\chi^R h_R^T)) + \left( \text{Twist}^*_X \chi^R \text{Paral}^R(v_R) \right) + L_{\text{Twist}_X^R} (\mathcal{G}(\chi^R h_R^T) + \partial (\chi^R h_R^T))
\]

\[
+ \left( \text{Twist}^*_X \chi^R (\text{Paral}^R(v_R)) \right) + L_{\text{Twist}_X^R} (\mathcal{G}(\chi^R h_R^T) + \partial (\chi^R h_R^T))
\]

\[
- \left( \text{Twist}^*_X \chi^R \text{Paral}^R (\mathcal{D}_R(v_R) + L_{\mathcal{G}^R h_R^T}(\sigma_R) + d \mathcal{G}^R_{\mathcal{G}^R h_R^T}(\beta_R)) + \eta_T \mathcal{G}^R_{\mathcal{G}^R h_R^T}(y_R) \right).
\]

Here \( \eta_T : \Theta \rightarrow \mathbb{R} \) is the characteristic function of \( (-\infty, -2T] \times S^1 \). Note that in the last equality we used the fact that over \( (-\infty, -2T] \times S^1 \),

\[
\mathcal{D}_R^{\mathcal{G}}(y_R) + \mathcal{T}^{\mathcal{G}}_L(y_L) = \mathcal{D}_R(v_R) + L_{\mathcal{G}^R h_R^T}(\sigma_R) + d \mathcal{G}^R_{\mathcal{G}^R h_R^T}(\beta_R) + \mathcal{T}^{\mathcal{G}}_L(y_L) = \pi_1 \text{Cut} (\eta, \varsigma, \varsigma') = 0.
\]
Hence (6.39) is a unifying expression of $\pi_1(II)$. For some $c_{41} > 0$ we have
\[
\begin{align*}
\|\pi_1(II)\|_{L^p} &\leq \|\text{Twist}^*_w(\bar{\sigma}^R)(\text{Paral}^R(v_R))\|_{L^p} + \|L_{\text{Twist}^*_w}(\bar{\sigma}^R h^T_R)(u'_R)\|_{L^p} + \|\chi^R \eta^T_R T^R_L(y_L)\|_{L^p} \\
&+ \|\left(\text{Twist}^*_w(\bar{\sigma}^R)(\text{Paral}^R(v_R)) - \text{Paral}^R(D_R(v_R))\right)\|_{L^p} \leq c_{41}\left(\frac{1}{\eta^T} + e^{-\tau_0(2-h)T}\right)\|y_L, y_R\|.
\end{align*}
\]

In deriving the last inequality, we used Lemma 6.1, (6.24), and the fact that $u'_w$ and $\text{Twist}^*_w \sigma_a$ are exponentially closed over $C_j(T)$.

On the other hand, over $C_j(2T)$, by (6.18),
\[
\begin{align*}
\pi_2(II) &= \left( * d(\text{Twist}^*_w(\bar{\sigma}^R h^T_R) + \partial(\chi^R h^T_R)), \text{Paral}^R(v_R), \bar{\sigma}^R(\text{Twist}^*_w(\bar{\sigma}^R h^T_R)) \right) \\
&- \left( * d(\text{Twist}^*_w(\bar{\sigma}^R h^T_R) + \partial(\chi^R h^T_R)), \bar{\sigma}^R(\text{Twist}^*_w(\bar{\sigma}^R h^T_R)) \right), \\
&= (\nu d\mu(u'_w) \cdot \text{Paral}^R(v_R), 0).
\end{align*}
\]

Using a $\pi_2$-version of (6.40), i.e., over $\overline{C}_j(-2T)$,
\[
\begin{align*}
\pi_2(D^+_R(y_R)) &= dR(\beta_R) = (\nu d\beta_R, 0) = 0,
\end{align*}
\]
we see that over $\Sigma_{2T}$,
\[
\begin{align*}
\pi_2(II) &= \text{Twist}^*_w \left( * d(\bar{\sigma}^R h^T_R + \partial(\chi^R h^T_R)), \nu \text{Paral}^R(v_R) \right) \\
&= \Psi_1(\nabla^R \chi_R, \nabla^R h^T_R) + \Psi_2(\nabla^2 \chi^R, h^T_R) \\
&+ \text{Twist}^*_w \left( \chi^R(\nabla^T \beta_R), \bar{\sigma}^R(\nabla^T \beta_R) \right) - \chi^R(\nu d\beta_R, \bar{\sigma}^R(\nabla^T \beta_R)), \theta_R(\text{Paral}^R(v_R), 0).
\end{align*}
\]

The last equality follows from the fact that the supports of $\chi^R$ and $\chi^T - 1$ are disjoint. Comparing to (6.41), (6.42) is a unifying expression of $\pi_2(II)$. Then for some $c_{42}, c_{43} > 0$, we have
\[
\begin{align*}
\|\pi_2(II)\| &\leq \|\Psi_1\| + \|\Psi_2\| + \|\nu d\mu(u'_w) \cdot \text{Paral}^R(v_R)\| \\
&\leq c_{42}\left(\|\nabla^R \chi\|_{L^\infty} + \|\nabla^2 \chi^R\|_{L^\infty}\right)\|\beta_R\| + c_{42}e^{-2T}\|v_R\|_{L^p} \\
&\leq c_{43}\left((hT)^{-1} + e^{-2T}\right)\|y_L\|.
\end{align*}
\]
Proof of Lemma 6.6. By definition,
\[ y_L - \chi^L \ast y_L = \left( \partial((1 - \chi^L)h_L) + \partial((1 - \chi^L)h_L), (1 - \chi^L)v_L \right). \]

By the definition of \( D_L \) (see Subsection 6.1), \( D_L(y_L - \chi^L \ast y_L) \) is the sum of local terms depending on \( (1 - \chi^L)h_L \) and \( (1 - \chi^L)v_L \), hence vanishes after applying \textit{Paste}. On the other hand, by (6.16),
\[ y_R - \chi^R \ast y_R = \left( \beta_R - \partial(\chi^R h^T_R), v_R - \chi^R v_R \right). \]

Since \( v_R - \chi^R v_R \) is supported in \((-\infty, -2T]\), by the definition of \textit{Paste},
\[
\text{Paste}\left(D_R^+(0, v_R - \chi^R v_R)\right) = 0.
\]

On the other hand, by (6.18) and (6.19),
\[
\begin{align*}
\pi_1\left(D_R^+\left(\beta_R - \partial(\chi^R h^T_R) - \partial(\chi^R h^T_R)\right)\right) \\
= \pi_1\left(D_R^+\left(\chi^T \beta_R - \partial(\chi^R h^T_R) - \partial(\chi^R h^T_R)\right)\right) \\
= d\tilde{s}_a\left(\partial h^T_R + \partial h^T_R - \partial(\chi^R h^T_R) - \partial(\chi^R h^T_R)\right) \\
= d\tilde{s}_a\left(\partial((1 - \chi^R)h^T_R) + \partial(1 - \chi^R)h^T_R\right).
\end{align*}
\]

It vanishes after applying \textit{Paste}. Moreover
\[
\pi_2\left(D_R^+\left(\beta_R - \partial(\chi^R h^T_R) - \partial(\chi^R h^T_R)\right)\right) = dR(\beta_R - \partial(\chi^R h^T_R) - \partial(\chi^R h^T_R)).
\]

By (6.18), it vanishes after applying \textit{Paste}. Therefore \( III = 0. \)

7. The Virtual Orbifold Atlas on the Universal Moduli Space

The local charts constructed in Section 5 don’t have coordinate changes between each other so they don’t form an atlas. In this section we construct from these local charts an atlas which satisfies the conditions of Definition A.15. The basic idea is the same as Section 3, namely, to select a finite cover of the given collection of local charts, and built the “sum charts” by the overlapping information of the chosen finite cover. This section is organized as follows. In Subsection 7.1 we construct the sum charts for the lower stratum and state the main propositions. In Subsection 7.2 we perform the corresponding gluing procedure and prove the main propositions. In Subsection 7.3 we construct the transition functions between these charts. In Subsection 7.4 we complete the atlas construction on the whole moduli space \( M_\kappa \), which finishes the proof of Theorem 4.8.

7.1. The sum charts. In Section 5, for each \( a \in M_\kappa^# \) we have constructed a local chart \( K_a \) for \( M_\kappa \). From now on we fix the choice of \( \left\{ a_i \right\}_{i=1}^N \) such that the union of \( F_a \), contains \( M_\kappa^# \) and abbreviate the corresponding charts by \( K_i = (U_i, E_i, S_i, \psi_i, F_i) \) and let \( \Gamma_i \) be the automorphism group of the representative of \( a_i \). We can shrink all \( F_i \) to precompact open subsets \( F_i^\# \subset F_i \) such that the union of \( F_i^\# \) still cover \( M_\kappa^# \). Denote
\[
F_i^\# := F_i \cap M_\kappa^# , \quad F_i^{\#\downarrow} := F_i^\# \cap M_\kappa^# .
\]

Define
\[
\mathcal{I}^\# := \left\{ I \subset \left\{ 1, \ldots, N \right\} \mid I \neq \emptyset, \bigcap_{i \in I} F_i^\# \neq \emptyset \right\} .
\]

Corollary 5.17 implies that \( \mathcal{I}^\# \) is the disjoint union of \( \mathcal{I}^s \) and \( \mathcal{I}^b \), where
\[
i \in I \in \mathcal{I}^s/b \implies F_i^\# \subset M_\kappa^#/b .
\]
Moreover, $I^\#$ is partially ordered: $I \leq J$ if $I \subset J$. For each $I \in I^\#$, define

$$F^\#_I = \bigcap_{i \in I} F^\#_i, \quad F^\#_I = \bigcap_{i \in I} F^\#_i,$$

$$\Gamma_I := \{ \Pi_{i \in I} \Gamma_i, \quad I \in I^I, \quad \{1\}, \quad I \in I^\#.$$

Hence we obtain a new open cover

$$M^\#_k = \bigcup_{I \in I} F^\#_I.$$

7.1.1. Sum charts for $M^\#_k$. In order to construct a sum chart indexed by $I \in I^I$, we need to introduce the Banach manifold $\overline{B}_I$ of objects with $|I|$-tuple of marked points, and its partial compactification

$$\overline{B}_I = B_I \cup (\overline{B}^\#_I / \Theta).$$

Here $\overline{B}^\#_I$ consists of objects $(X^#, \tilde{w})$ where $X^# \in B^\#_k$ and where $\tilde{w} = (w_i)_{i \in I} \in \Theta^I$ (not necessarily distinct). $\Theta$ acts on $\overline{B}^\#_I$ by translating both the soliton components as well as the marked points. The zero locus of the gauged Witten equation gives the moduli space $\overline{M}_I \cup \overline{M}^\#_I$.

We will use the following notations

$$\tilde{\gamma}_I = (\gamma_i)_{i \in I}, \quad \tilde{w}_I = (w_i)_{i \in I} \in \Theta^I, \quad \tilde{e}_I = (e_i)_{i \in I} \in E_I.$$

Consider the set

$$H^\#_{I,i} := \{ \tilde{\gamma}^\#_{I,i} = (\gamma^\#_{I,i}, w_{I,i}, 0) \mid \xi \in \hat{N}_{\xi;0} \cap n_i^{-1}(0) \cap S_i^{-1}(0), \quad [\gamma^\#_{I,i}] \in F^\#_{I,i} \}$$

where the notation $\hat{N}_{\xi;0}$ comes from Proposition 5.5. For each element $\tilde{\gamma}^\#_{I,i} \in H^\#_{I,i}$ and each $j \in I$, the object $\gamma^\#_{I,i}$ is close to $X^#_{a_j}$ (up to $\Theta$-translation). Namely, there exists a gluing parameter $\varpi = \exp(-4T - i\theta)$ such that

$$\sup_{z \in C_j(\epsilon T)} d(\sigma_{a_j}(z), \gamma^\#_{I,i}(z + 4T + i\theta))$$

is small. Then there exist exactly $d_j$ points $w_{I,i;\xi;j} \in \Theta$ (on the same $\Gamma_j$-orbit) such that $w_{I,i;\xi;j} - 4T - i\theta$ is close to $\gamma_{I,i}$. Then define

$$\tilde{H}^\#_{I,i} := \{ \tilde{\gamma}^\#_{I,i} = (\gamma^\#_{I,i}, \tilde{w}_{I,i}, 0) \mid \xi \in \hat{N}_{\xi;0} \cap n_i^{-1}(0) \cap S_i^{-1}(0), \quad [\gamma^\#_{I,i}] \in F^\#_{I,i} \} \subset \overline{B}^\#_I.$$

There is a $d_j/d_{i'}$-to-one map $\tilde{H}^\#_{I,i} \to H^\#_{I,i}$ sending $(\gamma^\#_{I,i}, \tilde{w}_{I,i}, 0) \to (\gamma^\#_{I,i}, \tilde{w}_{I,i}, 0)$. If replacing $F^\#_{I,i}$ by $F^\#_{I,i}$, then we denote the subsets by $\tilde{H}^\#_{I,i}$. $\tilde{H}^\#_{I,i}$ can be viewed as a finite dimensional subset of $T_{X^#_{a_j}} B^\#_k \times \Theta^I \times E_I$. The latter has a natural $\Gamma^I$-action on the factor $\Theta^I \times E_I$ by

$$\tilde{\gamma}_I \cdot (\tilde{w}_I, \tilde{e}_I) = ((\gamma_i w_i)_{i \in I}, (\gamma_i e_i)_{i \in I}).$$

For $\epsilon > 0$, let

$$\mathcal{U}^\#_{I,i,\epsilon} \subset T_{X^#_{a_j}} B^\#_k \times \Theta^I \times E_I$$

be the $\epsilon$-neighborhood of $\tilde{H}^\#_{I,i}$, which is $\Gamma^I$-invariant. Moreover, there is a natural $\Gamma^I$-invariant map sending $(\gamma^\#_{I,i}, \tilde{w}_I, \tilde{e}_I) \in \mathcal{U}^\#_{I,i}$ to $\tilde{e}_I(\gamma^\#_{I,i}, \tilde{w}) = \sum_{j \in I} e_j(\gamma^\#_{I,i}, w_j) \in T^\#_{\gamma^\#_{I,i}} \gamma^\#_{I,i}$. Then we have the section $\tilde{f}^\#_I : \mathcal{U}^\#_{I,i} \to E^\#_k$ defined by

$$\tilde{f}^\#_I(\gamma^\#_{I,i}, \tilde{w}_I, \tilde{e}_I) = \tilde{e}_I(\gamma^\#_{I,i}, \tilde{w}_I) + \tilde{f}^\#(\gamma^\#_{I,i}).$$

Moreover, for a small neighborhood $\overline{V}_I \subset \overline{B}_I$ of $\tilde{H}^\#_{I,i}/\Theta$, one can define

$$\tilde{f}_I : (\overline{V}_I \cap \overline{B}_I) \times E_I \to E^\#_k,$$

which extends (7.1).
Denote $\tilde{V}^{\#}_{I,i} := \{ \xi \in U^{\#}_{I,i} \mid \tilde{S}^\#(\xi) = 0 \}$ which contains $\tilde{H}^\#_{I,i}$ and which is still $\Gamma_I$-invariant. We also denote the corresponding object by $\tilde{Y}^\#_{I,i} := (Y^\#_{I,i}, \tilde{w}^\#_{I,i}, \tilde{c}^\#_{I,i})$ and denote

$$\tilde{S}^\#_{I,i} : \tilde{V}^{\#}_{I,i} \to E_I, \quad \tilde{S}^\#_{I,i}(\xi) = \tilde{c}^\#_{I,i};$$

(7.2)

$$\tilde{\psi}_{I,i} : (\tilde{S}^\#_{I,i})^{-1}(0) \to \tilde{M}^\#, \quad \tilde{\psi}_{I,i}(\xi) = [Y^\#_{I,i}, \tilde{w}^\#_{I,i}].$$

(7.3)

By transversality, for $\epsilon$ small enough, $\tilde{V}^{\#}_{I,i}$ is a smooth manifold of dimension $\chi(B) + \dim E_I + 2|I|$. Denote

$$\tilde{W}^{\#}_{I,i} := \{ \tilde{Y}^\#_{I,i} \in \tilde{V}^{\#}_{I,i} \mid (Y^\#_{I,i}, w^\#_{I,i}, c^\#_{I,i}) \text{ is normal to } (X^\#_{I,i}, w_{a_i}, H_{a_i}, f_{a_i}) \}.$$

If we replace $\hat{H}^\#_{I,i}$ by $\hat{H}^\#_{I,i}$, then we replace the notations of $\tilde{V}^{\#}_{I,i}$ or $\tilde{W}^{\#}_{I,i}$ by $\tilde{V}^{\#\prime}_{I,i}$ or $\tilde{W}^{\#\prime}_{I,i}$.

**Proposition 7.1.** There exist $\epsilon_I > 0$ and a family of objects

$$\tilde{V}^{\epsilon}_{I,i} := \{ \tilde{Y}^\#_{I,i} := (Y^\#_{I,i}, \tilde{w}^\#_{I,i}, \tilde{c}^\#_{I,i}) \in \mathcal{B}_I \times E_I \mid (\varpi, \xi) \in R^{\ast}_{\epsilon_I} \times \tilde{V}^{\#\prime}_{I,i} \}.$$

This family satisfies the following conditions.

1. $\tilde{V}^{\epsilon}_{I,i} \subset \tilde{F}^{-1}(0) \subset (V_I \cap \mathcal{B}_I) \times E_I$.

2. For $(\varpi, \xi) \in R^{\ast}_{\epsilon_I} \times \tilde{W}^{\#\prime}_{I,i}$, $(Y^\#_{I,i}, \tilde{w}^\#_{I,i}, \tilde{c}^\#_{I,i})$ is normal to $(X^\#_{a_i}, w_{a_i}, H_{a_i}, f_{a_i})$.

3. If we define

$$\tilde{S}_{I,i} : R^{\ast}_{\epsilon_I} \times \tilde{V}^{\#\prime}_{I,i} \to E_I, \quad \tilde{S}_{I,i}(\varpi, \xi) = \tilde{c}^\#_{I,i};$$

$$\tilde{\psi}_{I,i} : (R^{\ast}_{\epsilon_I} \times \tilde{V}^{\#\prime}_{I,i}) \cap \tilde{S}^{-1}_{I,i}(0) \to \tilde{M}_I, \quad \tilde{\psi}_{I,i}(\varpi, \xi) = [Y^\#_{I,i}, \tilde{w}^\#_{I,i}].$$

then they extend continuously to the maps $\tilde{S}^\#_{I,i}$ in (7.2) and $\tilde{\psi}_{I,i}$ in (7.3) as $\varpi \to 0$. We denote their extensions still by $\tilde{S}_{I,i}, \tilde{\psi}_{I,i}$, respectively.

4. The restriction of $\tilde{\psi}_{I,i}$ to $(R^{\ast}_{\epsilon_I} \times \tilde{W}^{\#\prime}_{I,i}) \cap \tilde{S}^{-1}_{I,i}(0)$ is a homeomorphism onto an open neighborhood $\tilde{F}_{I,i}^{\ast}$ of $\tilde{H}^{\prime\ast}_{I,i}/\Theta$ inside $\tilde{M}_I$ and the tuple

$$\tilde{K}_I = (R^{\ast}_{\epsilon_I} \times \tilde{W}^{\#\prime}_{I,i}, U^{\prime}_{I,i} \times E_I, \tilde{S}_{I,i}, \tilde{\psi}_{I,i}, \tilde{F}_{I,i})$$

is a local chart of $\tilde{M}_I$.

5. The map $n_{I,i} : \tilde{W}^{\epsilon}_{I,i} \to \mathbb{C}^I$ defined by the same formula as (5.10) is transverse to the origin.

This proposition is proved in the next subsection. Now we need to induce a local chart whose footprint contains $F^{\#\prime}_{I,i} \subset M_c$. The normalization map $n_{I,i}$ is transverse, so its zero locus

$$U^{\epsilon}_{I,i} := (R_{\epsilon_I} \times \tilde{W}^{\#\prime}_{I,i}) \cap n_{I,i}^{-1}(0)$$

is a topological manifold of dimension $\chi(B) + \dim E_I$. It has the natural action by $\Gamma_I$, defined by

$$\gamma_I(\varpi, \xi, \tilde{w}^\#_{I,i}, \tilde{c}^\#_{I,i}) = (\varpi, \gamma_I \tilde{w}^\#_{I,i}, \gamma_I \tilde{c}^\#_{I,i}).$$

We also use the $\Gamma_I$-action on $E_I$ to define the orbifold bundle

$$\frac{U^{\epsilon}_{I,i} \times E_I}{\Gamma_I} \to \tilde{F}^{\#\ast}_{I,i}.$$

So $\tilde{S}_{I,i}$ induces a section $S_{I,i}$ of this bundle, and $\tilde{\psi}_{I,i}$ induces a map $\psi_{I,i} : S_{I,i}^{-1}(0) \to M_c$, whose image is denoted by $F_{I,i}$.

**Corollary 7.2.** The tuple $K^{\#\ast}_{I,i} := \left( \frac{U^{\epsilon}_{I,i} \times E_I}{\Gamma_I}, S_{I,i}, \psi_{I,i}, F_{I,i} \right)$ is a local chart of $M_c$ whose footprint contains an open neighborhood of $F^{\#\prime}_{I,i}$.

**Proof.** Same as Corollary 5.6. \qed
7.1.2. Sum charts for $\mathcal{M}_\kappa^\#$. For $I \in \mathcal{I}$ and $i \in I$, consider the family
\[ \tilde{\mathcal{Y}}_{\#, \xi,i} = (\mathcal{Y}_{\#, \xi,i}; e_{\#, \xi,i}), \quad \xi \in \mathcal{M}_{\nu_i}^e, \]
given by Proposition 5.15. We consider
\[ H_{I, i}^\# = \{ \tilde{\mathcal{Y}}_{\#, \xi,i} = (\mathcal{Y}_{\#, \xi,i}; 0) \mid \xi \in N_{\nu_i}^e \cap (S_I^e)^{-1}(0), \ [\mathcal{Y}_{\#, \xi}] \in F_I^\# \}. \]
Let $\mathcal{U}_{I, i}^\#$, $e$ be the $\epsilon$-neighborhood of $H_{I, i}^\#$ inside $T_{X_{\nu_i}} \mathcal{B}_\kappa^\# \times E_I$. For $\epsilon$ small enough, each $\mathcal{Y}_{\#, \xi} \in \mathcal{U}_{I, i}^\#$ is close to $X_{\nu_i}$ for all $j \in I$ (modulo $\Theta$-translation). Hence using the normalization condition, one can embed $E_I$ into $\mathcal{E}_\# |_{\mathcal{Y}_{\#}}$. Denote the image of $\tilde{e}_I \in E_I$ with respect to this embedding by $\tilde{e}_I(\mathcal{Y}_{\#})$. Then one has a section
\[ \tilde{f}_I^\# : \mathcal{U}_{I, i}^\# \to \mathcal{E}_\#, \quad \tilde{f}_I^\#(\mathcal{Y}_{\#, \xi}; \tilde{e}_I) = \tilde{e}_I(\mathcal{Y}_{\#}) + \mathcal{F}_I^\#(\mathcal{Y}_{\#}). \]
Moreover, this section extends to a small neighborhood $\overline{V}_I \subset \overline{\mathcal{B}}_\kappa$ of $H_{I, i}^\#$, whose restriction on the top stratum is denoted as
\[ \tilde{f}_I : (\overline{V}_I \cap \mathcal{B}_\kappa) \times E_I \to E_\kappa. \]
Denote
\[ \tilde{V}_{I, i}^\#, e = \{ \xi \in \mathcal{U}_{I, i}^\# \mid \tilde{f}_I^\#(\xi) = 0 \}. \]
Then we have natural maps analogous to those in (7.2) and (7.3)
\[ S_I^\# : \tilde{V}_{I, i}^\#, e \to E_{I, i}, \quad \psi_I^\# : \tilde{V}_{I, i}^\#, e \cap (S_I^e)^{-1}(0) \to \mathcal{M}_\kappa^e. \]
For $\epsilon$ small enough, $\tilde{V}_{I, i}^\#, e$ is a manifold of dimension $\chi(B) + \dim E_I$, containing a real hypersurface $\tilde{W}_{I, i}^\#, e$ consisting of those solitons which are normal to $(X_{\nu_i}, H_{\nu_i}, f_{\nu_i})$.

**Proposition 7.3.** There exist $\epsilon_I > 0$ and a family of objects
\[ \tilde{V}_{I, i}^\#, \epsilon_I := \{ \tilde{Y}_{\#, \xi; I} := (\mathcal{Y}_{\#, \xi; I}; \tilde{e}_{I, \xi; I}) \in \mathcal{B}_\kappa \times E_I \mid (t, \xi) \in (0, \epsilon_I) \times \tilde{V}_{I, i}^{1, \epsilon_I} \}. \]
This family satisfies the following conditions.

1. $\tilde{V}_{I, i}^\#, \epsilon_I \subset \tilde{f}_I^{-1}(0) \subset (\overline{V}_I \cap \mathcal{B}_\kappa) \times E_I$.
2. For $(t, \xi) \in (0, \epsilon_I) \times \tilde{W}_{I, i}^{1, \epsilon_I}$, $\tilde{Y}_{\#, \xi; I}$ is normal to $(X_{\nu_i}, w_{\nu_i}, H_{\nu_i}, f_{\nu_i})$.
3. If we define
\[ S_I : (0, \epsilon_I) \times \tilde{V}_{I, i}^{1, \epsilon_I} \to E_{I, i}, \quad S_I(t, \xi) = \tilde{e}_{I, \xi; I}, \]
\[ \psi_I : (0, \epsilon_I) \times \tilde{V}_{I, i}^{1, \epsilon_I} \cap S_I^{-1}(0) \to \mathcal{M}_\kappa, \quad \psi_I(t, \xi) = \tilde{Y}_{\#, \xi; I}, \]
then they extend continuously to the maps $S_I^\#$ in (7.4) and $\psi_I^\#$ in (7.5) as $t \to 0$. We denote their extensions still by $S_I$, $\psi_I$, respectively.
4. The restriction of $\psi_I$ to $[0, \epsilon_I] \times \tilde{W}_{I, i}^{1, \epsilon_I} \cap S_I^{-1}(0)$ is a homeomorphism onto an open neighborhood $F_I$ of $\tilde{f}_I^{\#}$ inside $\mathcal{M}_\kappa$ and the tuple
\[ K_I = \left( [0, \epsilon_I] \times \tilde{W}_{I, i}^{1, \epsilon_I}, [0, \epsilon_I] \times \tilde{W}_{I, i}^{1, \epsilon_I} \times E_I, S_I, \psi_I, F_I \right) \]
is a local chart with boundary of $\mathcal{M}_\kappa$. 

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7.2. Gluing. To prove Proposition 7.1 and Proposition 7.3, we have to do the gluing construction again. Let \( \epsilon \) be smaller than the \( c_a \) of Proposition 5.5. For each \( \hat{\mathbb{V}}^\#_{\xi, i} = (\mathbb{V}^\#_{\xi, i}, \mathbb{V}^\#_{\xi, i}, \mathbb{V}^\#_{\xi, i}) \) in \( \hat{\mathbb{V}}^\#_{\xi, i} \) and \( \varpi \in R^*_\epsilon \), one has the pre-glued object
\[
\hat{\mathbb{V}}^\#_{\xi, i} = (\mathbb{V}^\#_{\xi, i}, \mathbb{V}^\#_{\xi, i}, \mathbb{V}^\#_{\xi, i}) = (\mathbb{V}^\#_{\xi, i} + 4T + i\theta, \mathbb{V}^\#_{\xi, i}) \in \hat{\mathbb{B}}_I \times E_I,
\]
constructed in the same way as in Section 5. Recall that \( X^\#_{\varpi, a} \in \mathcal{B}_\kappa \) is the object obtained by gluing \( X^\#_{\varpi, a} \) which represents \( a_i \).

Let \( \delta_a \) be the \( \delta_a \) of Lemma 5.10. By the gluing construction of Section 5, for \( \epsilon_1 \) small enough,
\[
\varpi \in R^*_\epsilon, \xi \in \hat{\mathbb{V}}^\#_{\xi, i} = (\mathbb{V}^\#_{\xi, i}, \mathbb{V}^\#_{\xi, i}, \mathbb{V}^\#_{\xi, i}) \in B_{\delta_a}, \hat{\mathbb{V}}^\#_{\xi, i} \in \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa.
\]
(7.6)

Therefore, by choosing \( \epsilon_1, \delta_1 \) small enough, by Lemma 5.11, we have
\[
\varpi \in R^*_\epsilon, \xi \in \hat{\mathbb{V}}^\#_{\xi, i} = (\mathbb{V}^\#_{\xi, i}, \mathbb{V}^\#_{\xi, i}, \mathbb{V}^\#_{\xi, i}) \in B_{\delta_1, \hat{\mathbb{V}}^\#_{\xi, i} \in \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa}.
\]

We can identify the tangent spaces
\[
T_{\hat{\mathbb{V}}^\#_{\xi, i}} \mathcal{B}_I \times E_I = \mathbb{X}^{I, i} := T_{X^\#_{\varpi, a}} \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa.
\]

Then the linearization of \( \hat{f}_I \) at \( \hat{\mathbb{V}}^\#_{\xi, i} \) becomes
\[
\hat{D}_{\hat{\mathbb{V}}^\#_{\xi, i}} f_I : \mathbb{X}^{I, i} \to \mathcal{Y}^{I, i} := \mathcal{E}_\kappa|_{X^\#_{\varpi, a}}.
\]

Lemma 7.4. There exist \( \epsilon_2 \in (0, \epsilon_1) \) and \( c_2 > 0 \) such that for \( \varpi \in R^*_\epsilon \) and \( \xi \in \hat{\mathbb{V}}^\#_{\xi, i} \), one has
\[
\| \hat{f}_I(\hat{\mathbb{V}}^\#_{\xi, i}) \| \leq c_2|\varpi|^\alpha.
\]

Proof. The lemma can be proved in the same way as proving Lemma 5.7. □

Lemma 7.5. There exist \( \epsilon_3 \in (0, \epsilon_2) \) and a family of bounded linear maps
\[
Q_{\varpi, \xi, i} : \mathcal{Y}^{I, i} \to \mathbb{X}^{I, i}, \xi \in \hat{\mathbb{V}}^\#_{\xi, i} \cap (\hat{\mathbb{S}}^\#)^{-1}(0), \varpi \in R^*_\epsilon
\]
satisfying the following conditions.

1. For all \( (\varpi, \xi) \in R^*_\epsilon \times (\hat{\mathbb{V}}^\#_{\xi, i} \cap (\hat{\mathbb{S}}^\#)^{-1}(0)) \), \( \|Q_{\varpi, \xi, i}\| \leq 2c_{a_i} \), where \( c_{a_i} > 0 \) is the one from Lemma 5.8 and 5.10.

2. Each \( Q_{\varpi, \xi, i} \) is a right inverse to the operator (7.7).

3. For each \( \varpi \in R^*_\epsilon \), the images of \( Q_{\varpi, \xi, i} \) are independent of \( \xi \).

Proof. Let
\[
\hat{Q}_{\varpi, a_i} : \mathcal{E}_\kappa|_{X^\#_{\varpi, a_i}} \to \mathbb{X}^{\#_{\varpi, a_i}} = T_{X^\#_{\varpi, a_i}} \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa, \mathcal{B}_\kappa
\]
be the right inverse we used when constructing the basic charts in Lemma 5.8, which is uniformly bounded by the constant \( c_{a_i} \). For \( \xi \in \hat{\mathbb{V}}^\#_{\xi, i} \), the identification \( \mathcal{E}_\kappa|_{X^\#_{\varpi, a_i}} \simeq \mathcal{E}_\kappa|_{X^\#_{\varpi, a_i}} \) is defined by parallel transport, so is an isometry. Then \( \hat{Q}_{\varpi, a_i} \) of (7.8) gives an approximate right inverse
\[
\hat{Q}^\#_{\varpi, a_i} : \mathcal{E}_\kappa|_{X^\#_{\varpi, a_i}} \to \mathbb{X}^{\#_{\varpi, a_i}}, \xi \in \hat{\mathbb{V}}^\#_{\xi, i}.
\]

By (7.6), the distance between \( (\mathbb{V}^\#_{\varpi, a_i}, \mathbb{V}^\#_{\varpi, a_i}) \) and \( \hat{\mathbb{V}}^\#_{\varpi, a_i} \) in \( \mathbb{X}^{\#_{\varpi, a_i}} \) is bounded by \( \delta_a \), where \( \delta_a \) is the one of Lemma 5.10. So by Lemma 5.10
\[
\left\| DF_{\varpi, a}(\hat{\mathbb{V}}^\#_{\varpi, a_i}) - DF_{\varpi, a}(\mathbb{V}^\#_{\varpi, a_i}, \mathbb{V}^\#_{\varpi, a_i}) \right\| \leq \frac{1}{10c_{a_i}}.
\]

If we denote
\[
\hat{Q}_{\varpi, a_i} := \hat{Q}_{\varpi, a_i} \circ \left( DF_{\varpi, a}(\mathbb{V}^\#_{\varpi, a_i}, \mathbb{V}^\#_{\varpi, a_i}) \circ \hat{Q}_{\varpi, a_i} \right)^{-1},
\]
then
\[
\left\| \hat{Q}_{\varpi, a_i} \right\| \leq \frac{11}{10} \left\| \hat{Q}_{\varpi, a_i} \right\| \leq 2c_{a_i}.
\]
Using the inclusion $\mathcal{X}_{\varpi, a} \to \mathcal{X}^{L,i}_{\varpi}$, one obtain a family of maps $\mathcal{Q}_{\varpi, i} : \mathcal{E}_{\varpi} |_{\mathcal{Y}^{app}_{\varpi, i}} \to \mathcal{X}^{L,i}_{\varpi}$ which, for fixed $\varpi$ and different $\xi$'s, all have the same image.

**Proposition 7.6.** There exist $\varepsilon_4 \in (0, \varepsilon_3)$, $\delta_1 \in (0, \delta_1]$ such that for $\varpi \in R^*_\varepsilon_4$, $\xi \in \overline{H^{#_i}}_{\varpi, i}$ and $\hat{\gamma}_I \in B_{\delta_I}(\hat{\gamma}^{app}_{\varpi, \xi, i}, \hat{\gamma}_I \times E_I)$ one has

$$\| \hat{D}_{\gamma}^{app}_{\varpi, \xi, i} \hat{\gamma}_I - \hat{D}_{\gamma}^{app}_{\varpi, \xi, i} \hat{\gamma}_I \| \leq \frac{1}{\delta \varepsilon_i}.$$  \hspace{1cm} (7.9)

Here the domain and the target of the two linearized operators are identified with $\mathcal{X}^{L,i}_{\varpi}$ and $\mathcal{E}_{\varpi} |_{\mathcal{Y}^{app}_{\varpi, i}}$ respectively.

**Proof.** We use a compactness argument. For each $\eta \in \overline{H^{#_i}}_{\varpi, i}$, there is $\varepsilon_4(\eta) > 0$ and $\delta_1(\eta) > 0$ such that (7.9) holds for all $\varpi \in R^*_\varepsilon_4$, $\xi \in B_{\varepsilon_4}(\eta, \hat{\gamma}^{app}_{\varpi, \xi, i})$ and $\hat{\gamma}_I \in B_{\delta_1}(\eta, \hat{\gamma}^{app}_{\varpi, \xi, i})$. This can be proved in the same way as Lemma 5.10 by straightforward calculation. Then by compactness, one can find finitely many $\eta_k \in \overline{H^{#_i}}_{\varpi, i}$, $k = 1, \ldots, s$ such that

$$\{ B_{\varepsilon_4}(\eta_k, \hat{\gamma}^{app}_{\varpi, \xi, i}) \ | \ k = 1, \ldots, s \}$$

is an open cover of $\overline{H^{#_i}}_{\varpi, i}$. Then $\varepsilon_4 = \min_{1 \leq k \leq s} \varepsilon_4(\eta_k)$ and $\delta_I = \min_{1 \leq k \leq s} \delta_I(\eta_k)$ satisfy the condition of this proposition. \hspace{1cm} \Box

### 7.2.1. Proof of Proposition 7.1.

Choose an arbitrary element $\xi_0 \in \overline{H^{#_i}}_{\varpi, i}$ which, via the pregluing, gives an object $\hat{\gamma}^{app}_{\varpi, \xi_0, i}$. Denote the corresponding point in $\mathcal{X}^{L,i}_{\varpi}$ by $\varpi_{\varpi, \xi_0, l}$. Then apply the implicit function theorem (Proposition 5.9) to the tuple

$$\big( \mathcal{X}, \mathcal{Y}, \mathcal{F}, \varpi_0, \mathcal{Q} \big) := \big( \mathcal{X}^{L,i}_{\varpi}, \mathcal{Y}^{\alpha}, \hat{\gamma}_I, \varpi_{\varpi, \xi_0, l}, \mathcal{Q}_{\varpi, i} \big)$$

with constants $c = 2 \delta \varepsilon_i$, from Lemma 7.5 and $\delta = \delta_I$ from Proposition 7.6. Then for all $\eta \in \mathcal{X}^{L,i}_{\varpi}$ satisfying

$$\| \mathcal{F}(\eta) \| \leq \frac{\delta_I}{8 \varepsilon_i}, \| \eta - \varpi_{\varpi, \xi_0, l} \| \leq \frac{\delta_I}{8},$$

there exists a unique $\eta' \in \text{Im} \mathcal{Q}$ such that $\mathcal{F}(\eta + \eta') = 0$, $\| \eta' \| \leq \delta_I$.

By Lemma 7.4, if $\varepsilon_4 > 0$ small enough, for each $(\varpi, \xi) \in R^*_\varepsilon_4 \times \hat{\gamma}^{#_i, \varepsilon_4}_I$,

$$\| \hat{\gamma}_I(\hat{\gamma}^{app}_{\varpi, \xi, i}) \| \leq \frac{\delta_I}{8 \varepsilon_i};$$

We write $\hat{\gamma}^{app}_{\varpi, \xi, i}$ as an element $\hat{\gamma}^{app}_{\varpi, \xi, i} \in \mathcal{X}^{L,i}_{\varpi}$. On the other hand, by Lemma 5.11, for $\varepsilon_4 = \varepsilon_4(\xi_0) > 0$ small enough,

$$(\varpi, \xi) \in R^*_\varepsilon \times B_{\varepsilon_4}(\xi_0, \hat{\gamma}^{#_i, \varepsilon_4}_I) \implies \| \hat{\gamma}^{app}_{\varpi, \xi, i} - \varpi_{\varpi, \xi_0, l} \| \leq \frac{\delta_I}{8}.$$ 

Hence there exists a unique $\varpi_{\varpi, \xi_0, l} \in \text{Im} \mathcal{Q}$ such that

$$\mathcal{F}(\varpi_{\varpi, \xi_0, l} + \eta', \varpi_{\varpi, \xi_0, l}) = 0, \| \varpi_{\varpi, \xi_0, l} + \eta' - \varpi_{\varpi, \xi_0, l} \| \leq \delta_I.$$

Denote $\varpi_{\varpi, \xi_0, l} := \varpi_{\varpi, \xi_0, l} + \eta' \in \mathcal{X}^{L,i}_{\varpi}$, which corresponds to

$$\eta' : (\varpi, \xi, \xi) = (\varpi, \xi_0, l, \xi_0, l, \xi_0, l) \in T_{\varpi_{\varpi, \xi_0, l}} \mathcal{B}_\nu \oplus T \mathcal{O}_l \oplus E_I$$

Then define

$$\hat{\gamma}^{exact}_{\varpi, \xi, l} := \hat{\gamma}^{exact}_{\varpi, \xi, l} = \left( \mathcal{F}^{exact}_{\varpi, \xi, l}, \varpi^{exact}_{\varpi, \xi, l}, \varpi^{exact}_{\varpi, \xi, l} \right)$$

$$= \left( \exp_{\gamma^{app}_{\varpi, \xi, l}}(\gamma^{app}_{\varpi, \xi, l} + \gamma^{app}_{\varpi, \xi, l}), \varpi^{app}_{\varpi, \xi, l} + \varpi^{app}_{\varpi, \xi, l}, \varpi^{app}_{\varpi, \xi, l} + \varpi^{app}_{\varpi, \xi, l} \right).$$

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The above procedure constructed, for $\xi_0 \in \overline{H^{\#^1}_{I,i}}$ and $\varpi \in R^*_e(\xi_0)$, a map
\[
\varpi = \xi_0 : B_{\epsilon_I' I}(\xi_0, \hat{V}^{\#}, e_4) \to \mathcal{X}^{l, i}_{\varpi}
\]
(7.10)
which assigns to $\xi$ the element $\hat{\gamma}^{exact}_{\varpi, \xi, I}$. By the compactness of $\overline{H^{\#^1}_{I,i}}$, we can choose finitely many $\xi_i \in \overline{H^{\#^1}_{I,i}}$ such that
\[
\overline{H^{\#^1}_{I,i}} \subset \bigcup_{i=1}^k B_{\epsilon_I' I}(\xi_i, \hat{V}^{\#}, e_4).
\]
Then we can find a small $\epsilon_I > 0$ such that $\epsilon_I \leq \epsilon_I'(\xi_i)$ and
\[
B_{\epsilon_I} \left(\overline{H^{\#^1}_{I,i}}, \hat{V}^{\#}, e_4\right) \subset \bigcup_{i=1}^k B_{\epsilon_I'(\xi_i)}(\xi_i, \hat{V}^{\#}, e_4).
\]
Moreover, since $\varpi = \xi_0$ is defined by adding to $\hat{\gamma}^{app}_{\varpi, \xi, I}$ a unique small element in the image of $\overline{\hat{\gamma}}_{\varpi, \xi, I}$, while the images of these right inverses are the same (Lemma 7.5 (3)). Hence for $\varpi \in R^*_e$, all $\varpi = \xi_0$’s define a continuous map
\[
\varpi = \xi_0 : B_{\epsilon_I} \left(\overline{H^{\#^1}_{I,i}}, \hat{V}^{\#}, e_4\right) \to \mathcal{X}^{l, i}_{\varpi}.
\]

So we have constructed the family $\hat{\gamma}^{\varpi}_{\varpi, \xi, I} = \varpi = \xi_0(\xi)$ which is supposed to satisfy Proposition 7.1. Now we check these conditions. (1)–(3) of Proposition 7.1 are easy to check. To verify (4), it suffices to prove that $\psi_I$ is both injective and surjective onto a neighborhood of $\overline{H^{\#^1}_{I,i}}/\Theta$ inside $\hat{\mathcal{M}}_I$, when restricted to $(R_{s_I} \times \hat{W}^{\#, e_4}_{I,i}) \cap \hat{S}^{-1}(0)$. The injectivity can be proved in the way as Proposition 5.5, making use of the normalization condition.

To prove surjectivity, we need the following lemma on the property of the map $\varpi = \xi_0$ of (7.10), which can be proved in the same way as, Lemma 5.13.

**Lemma 7.7.** For each $\xi_0 \in \overline{H^{\#^1}_{I,i}}$, there exists $\delta_I(\xi_0) > 0$ and $\epsilon_I(\xi_0) > 0$ such that for all $\varpi \in R^*_e(\xi_0)$ and any $\xi \in B_{\delta_I(\xi_0)}(\xi_0, \hat{V}^{\#}, e_4)$,
\[
B_{\delta_I(\xi_0)} \left(\hat{\gamma}^{app}_{\varpi, \xi, I}, \mathcal{X}^{l, i}_{\varpi}\right) \cap \hat{\gamma}^{-1}_I(0) \subset \varpi = \xi_0 \left(B_{\epsilon_I}(\xi_0), \hat{V}^{\#}, e_4\right).
\]

Now we can prove the surjectivity. Suppose it doesn’t hold. Then there exists a sequence of points $a_k \in \mathcal{M}_I$ converging to some point $b \in \overline{H^{\#^1}_{I,i}}/\Theta$ and $a_k \notin \text{Im}(\psi_I)$. It suffices to consider the case that $a_k$ are in the top stratum and are objects $\gamma_k = (A_k, u_k, \varphi_k, \alpha_k, w_{b,k}, 0) \in \hat{\mathcal{Y}}_I \times \mathbb{E}_I$. Let $b$ be represented by $\gamma_k, I = (A, u, \varphi, 0, w_{b,k}, I) \in \overline{\hat{H}^{\#^1}_{I,i}}$. Then define $\varpi_k = \text{exp}(w_{b,k} - w_{b,k})$, then one can prove in the same way as when proving Proposition 5.5 that for $k$ large, as points in $\mathcal{X}^{l, i}_{\varpi_k, I}$,
\[
\|\gamma^{app}_{\varpi_k, \xi, I} - \gamma_k\| \leq \delta_I(\xi_0)
\]
This contradicts Lemma 7.7. Therefore, the image of $\psi_I$ contains a neighborhood of $\overline{H^{\#^1}_{I,i}}/\Theta$.

To finish the proof of Proposition 7.1, it remains to prove the statement about the normalization map $\mathbf{n}_I$. Indeed, let $\mathbf{n}^\#_I : \hat{W}^{\#}, e_4 \to \overline{\mathbb{C}^{|I|}}$ be the restriction of $\mathbf{n}_I$ to the subset where $\varpi = 0$, which is smooth and transverse to zero. For each $\xi \in (\mathbf{n}^\#_I)^{-1}(0)$ and $\varpi \in R^*_e$, by Lemma 7.4, we have
\[
\|\hat{\gamma}_I(\gamma^{app}_{\varpi, \xi, I})\| \leq c_2|\varpi|^\sigma.
\]
By (5.27) of the implicit function theorem, $\gamma^{app}_{\varpi, \xi, I}$ is corrected from $\gamma^{app}_{\varpi, \xi, I}$ by a term controlled by $|\varpi|^\sigma$. Hence for certain positive number $c$ independent of $\varpi$, we have
\[
\|\mathbf{n}_I(\varpi, \xi)\| \leq c|\varpi|^\sigma.
\]
One can choose a right inverse $Q$ to $\frac{\partial n}{\partial \xi} : T \hat{\mathcal{W}}_{i,j}^{\#} \to \mathbb{C}^{|I|}$ which is uniformly bounded. Applying Newton iteration (implicit function theorem), there exists a unique $\xi'(\varpi, \xi) \in \hat{\mathcal{W}}_{i,j}^{\#}$ which is close to $\xi$, such that
\[ n_i(\varpi, \xi'(\varpi, \xi)) = 0, \quad \xi'(\varpi, \xi) - \xi \in \text{Im} Q \subset T \hat{\mathcal{W}}_{i,j}^{\#}. \]
Therefore, $R_{\xi} \times (n_i^\#)^{-1}(0)$ gives a local chart of $n_i^{-1}(0)$ by $(\varpi, \xi) \mapsto (\varpi, \xi'(\varpi, \xi))$. Therefore, $n_i^{-1}(0) \subset R_{\xi} \times \hat{\mathcal{W}}_{i,j}^{\#}$ is a topological manifold near $\varpi = 0$. It is locally flat and the variation of the $I$-tuple of marked points gives the normal bundle. By the definition of topological transversality (Definition A.7), this finishes the proof of (5) of Proposition 7.1.

The proof of Proposition 7.3 follows in a very similar way. All the necessary ingredients have been provided when proving Proposition 7.1. Moreover, here we don’t need to bypass to $\hat{\mathcal{M}}_I$ and adding extra marked points to perform the gluing. Hence we leave the details to the reader.

7.3. The transition functions. Now we construct coordinate changes and meanwhile we will shrink the charts $\hat{K}_I$. We start with an element $J \in \mathcal{I}^\#$. For any $I \subseteq J$ and $i \in I$, consider the following subsets of $\hat{\mathcal{H}}_{J,i}^{\#} \subset B_I^\# \times \Theta^I \times E_I$

\[
\hat{\mathcal{H}}_{J,i}^{\#} = \{ (\hat{\mathcal{Y}}_{J,i,I}, \hat{\mathcal{W}}_{J,i,I,0}) \in \hat{\mathcal{H}}_{J,i}^{\#} \mid |\hat{\mathcal{Y}}_{J,i,I}| \in F_j^\# \},
\]

\[
\hat{\mathcal{H}}_{J,i}^{\#} = \{ (\hat{\mathcal{Y}}_{J,i,I}, \hat{\mathcal{W}}_{J,i,I,0}) \in \hat{\mathcal{H}}_{J,i}^{\#} \mid |\hat{\mathcal{Y}}_{J,i,I}| \in F_j^{\#} \}.
\]

For $\epsilon \in (0, \epsilon_I]$ where $\epsilon_I$ is from Proposition 7.1, denote
\[
\hat{\mathcal{W}}_{J,i,I}^{\#,\epsilon} := B_\epsilon \left( \hat{\mathcal{H}}_{J,i}^{\#} \right), \quad \hat{\mathcal{W}}_{J,i,I}^{\#} := B_\epsilon \left( \hat{\mathcal{H}}_{J,i}^{\#} \right).
\]

Then
\[
U_{J,I} := \frac{(R_\epsilon \times \hat{\mathcal{W}}_{J,i,I}^{\#}) \cap n_j^{-1}(0)}{\Gamma_j} \subset U_I
\]
is an open suborbifold, which induces a subchart of $\hat{K}_I$.

Lemma 7.8. For $I \subseteq J \in \mathcal{I}^\#$, there exist $\epsilon_{J,I} \in (0, \epsilon_I]$ and a weak coordinate change
\[
T_{J,I}^{\#} = (U_{J,I}^{\#}, \phi_{J,I}) : K_I \to K_J,
\]
whose footprint contains $\overline{F_j^{\#}} \subset F_j^{\#}$.

Proof. By our construction, there is $j_0 \in J$ and $\epsilon_{J,I} > 0$ such that
\[
U_I = \frac{(R_\epsilon \times \hat{\mathcal{W}}_{J,j_0,I}^{\#}) \cap n_j^{-1}(0)}{\Gamma_j}.
\]

Take an arbitrary $\hat{\mathcal{Y}}_{J_0,I_0} = (\mathcal{Y}_{J_0,I_0}, \sigma_{J_0}, \hat{\mathcal{W}}_{J_0,I_0}) \in \hat{\mathcal{H}}_{J_0,I_0}^{\#} \subset B_K^\# \times E_I$. For any $j \in J \setminus I$, since $(\mathcal{Y}_{J_0,I_0}, \sigma_{J_0}) \in B_K^\#$ is close to $X_{a_j}$, there is a point $\hat{\mathcal{W}}_{J_0,I_0,j} \in \Theta$, unique up to $\Gamma_j$-actions, such that
\[
\sum_{a_j \in \Gamma_j} f_{a_j} (|\sigma_{J_0}|_{\hat{\mathcal{W}}_{J_0,I_0,j}} \gamma_{\hat{\mathcal{W}}_{J_0,I_0,j}}) = 0, \quad j \in J \setminus I.
\]

Then denote $\hat{\mathcal{W}}_{J_0,I_0,j} = (\hat{\mathcal{W}}_{J_0,I_0})_{a_j} \in J$ and $\hat{\mathcal{Y}}_{J_0,I_0,j} = (\mathcal{Y}_{J_0,I_0}, \sigma_{J_0}, \hat{\mathcal{W}}_{J_0,I_0,j}) \in B_K^\# \times E_J$. Denote
\[
\vartheta_{J_0} = \exp(\hat{\mathcal{W}}_{J_0,I_0,j} - \hat{\mathcal{W}}_{J_0,I_0,J}^+).
\]

Then element
\[
\text{Twist}_{J_0,\hat{\mathcal{Y}}_{J_0,I_0,j}} : = (\mathcal{Y}_{J_0,I_0}, \text{Twist}_{\hat{\mathcal{Y}}_{J_0,I_0,j}} \sigma_{J_0}, \text{Twist}_{\hat{\mathcal{Y}}_{J_0,I_0,j}} \hat{\mathcal{W}}_{J_0,I_0,j})
\]
is normal to $(X_{a_{J_0}}, \hat{\mathcal{W}}_{a_{J_0}}, H_{a_{J_0}}, f_{a_{J_0}})$ and hence belongs to the family $H_{J_0,J_0}^{\#}$. We can denote it by
\[
(\mathcal{Y}_{J_0,I_0}, \text{Twist}_{\hat{\mathcal{Y}}_{J_0,I_0,j}} \sigma_{J_0}, \text{Twist}_{\hat{\mathcal{Y}}_{J_0,I_0,j}} \hat{\mathcal{W}}_{J_0,I_0,j}) =: \hat{\mathcal{Y}}_{J_0,\sigma_{J_0},\hat{\mathcal{W}}_{J_0,I_0,j}}.
\]
Then, let \( \epsilon(\xi_0) > 0 \) be sufficiently small. For each \( \varpi \in R_c(\xi_0) \) and \( \xi \in B_c(\xi_0)(\xi_0, \tilde{W}_{f_{ij}}^\varpi, \epsilon(\xi_0)) \), the object

\[
\tilde{\gamma}(\varpi, \xi, \eta, \varpi', \xi', \eta') = (\gamma(\varpi, \xi), \tilde{w}(\varpi, \xi, \varpi', \xi'), \tilde{w}(\varpi, \xi, \varpi', \xi'))
\]

is in a sufficiently small neighborhood of \( \tilde{\gamma}(\varpi, \xi, \eta, \varpi', \xi', \eta') \). Hence for each \( j \in J \setminus I \), there exists \( w_{\varpi, \xi, j} \in \Sigma^\varpi \) which is close to \( \tilde{\gamma}(\varpi, \eta, \xi, \varpi', \eta', \xi') \), such that

\[
\sum_{\gamma \in \Gamma_{IJ}} f_{j_0}( \gamma w_{\varpi, \xi, j} ) = 0.
\]

Define \( \tilde{w}^+_{\varpi, \xi, j} = (w_{\varpi, \xi, j})_{\xi, j, \varpi', \xi', \eta, \eta'} \). Then, by compactness of \( \tilde{H}^{\#+}_{j, i} \), one can find \( \epsilon_{JJ} > 0 \) such that for all \( \varpi \in R_{c, j} \) and \( \xi \) in the \( \epsilon_{JJ} \)-neighborhood of \( \tilde{H}^{\#+}_{j, i} \), there exists \( \varpi' \in R_{c, j} \) and \( \eta \in \tilde{W}_{\varpi', \xi, j}^{\#+, \xi} \) such that

\[
\tilde{\gamma}(\varpi, \xi, \varpi', \xi, \eta) = \tilde{\gamma}(\varpi, \xi, \varpi', \xi, \eta).
\]

The ambiguity of extending \( \tilde{\gamma}(\varpi, \xi, \varpi', \xi, \eta) \) to \( \tilde{\gamma}(\varpi, \xi, \varpi', \xi, \eta) \) is up to an element in \( \prod_{j \in J \setminus I} \Gamma_{ij} \). So this naturally induces an orbifold map

\[
\phi_{JJ} : U_{JJ} \to U_j.
\]

The accompanying orbibundle map \( \hat{\phi}_{JJ} : E_l |_{U_{JJ}} \to E_j \) and the germ of tubular neighborhood \( \hat{\phi}_{JJ}^+ \) are easy to construct, by the finite dimensional implicit function theorem. Hence this gives a weak coordinate change whose footprint contains \( F_{j^+}^{\#} \cap F_{j^+}^{\#} \).

For each pair \( I \leq J \in \mathcal{I}^\# \), one can construct a weak coordinate change from \( K_I \) to \( K_J \) whose footprint contains \( F_{j^+}^{\#} \). The cocycle conditions among all such weak coordinate changes are routine to check. Hence we obtain the following corollary.

**Corollary 7.9.** There exist a family of weak coordinate changes

\[
\{ T_{JJ} = (U_{JJ}, \phi_{JJ}) \mid I \leq J \in \mathcal{I}^\# \}
\]

such that the following conditions are satisfied.

1. They satisfy the cocycle condition.
2. The footprint of \( K_I |_{U_{JJ}} \) contains \( F_{j^+}^{\#} \).

Next we have to shrink the charts such that the weak coordinate changes becomes strong ones.

**Proposition 7.10.** For each \( I \in \mathcal{I}^\# \), there exist a subchart \( K_I' := K_I |_{U_I} \) satisfying the following conditions.

1. The footprint of \( K_I' \) contains \( F_{j^+}^{\#} \).
2. If we denote \( U_{JJ}^* = U_J \cap \hat{\phi}_{JJ}^+ |_{U_J} \cap U_{JJ} \), and denote by \( T_{JJ}^* \) the restriction of \( T_{JJ} \) onto \( U_{JJ}^* \), then \( T_{JJ}^* \) is strong.

**Proof.** The same as the case of Section 3.
7.4. The virtual orbifold atlas. We have constructed the system of local charts of \( \mathcal{M}_k \)
\[
K^*_i = (U^*_i, E^*_i, S^*_i, \psi^*_i, F^*_i), \quad I \in \mathcal{I}^#
\]
and the system of strong coordinate changes
\[
T^*_j = (U^*_j, \phi^*_j), \quad I \leq J \in \mathcal{I}^#
\]
that satisfy the conditions of Proposition 7.10. The collection \( \{F^*_i\} \) covers an open neighborhood \( \mathcal{V} \subset \mathcal{M}_k \) of \( \mathcal{M}_k^# \).

On the other hand, for each \( i \), we can choose shrink the chart \( K_a \), such that its footprint \( F_a \), satisfies the following conditions
(1) For each \( I \in \mathcal{I}^# \),
\[
\bigcap_{i \in I} F_{a_i} \subset F^*_i.
\]
(2) There is a precompact open neighborhood \( \mathcal{V}^{(1)} \subset \mathcal{V} \) of \( \mathcal{M}_k^# \) such that
\[
\overline{\mathcal{V}^{(1)}} \subset \bigcup_{i=1}^{N} F_{a_i} \subset \mathcal{V}.
\]

Then, for each \( b \in \mathcal{M}_k \setminus \mathcal{V}^{(1)} \), one can construct a local chart \( K_b := (U_b, E_b, S_b, \psi_b, F_b) \) as in Subsection 5.1. Then there exist finitely many local charts \( K_{b_j}, j = 1, \ldots, N' \) such that
\[
\mathcal{M}_k \setminus \mathcal{V}^{(1)} \subset \bigcup_{j=1}^{N'} F_{b_j}.
\]

Now we obtain an open cover of \( \mathcal{M}_k \):
\[
\left\{ F_{a_i} \mid i = 1, \ldots, N \right\} \cup \left\{ F_{b_j} \mid j = 1, \ldots, N' \right\}.
\]

Abbreviate these footprints by \( F_i, i = 1, \ldots, N + N' \). Choose precompact open subsets \( F^i \subset F_i \) the collection of \( F^i \)'s is still an open cover of \( \mathcal{M}_k \). Denote
\[
\mathcal{I} = \left\{ I = I^# \cup I^c \subset \{1, \ldots, N\} \cup \{N + 1, \ldots, N + N'\} \mid \bigcap_{i \in I^#} F_i \neq \emptyset \right\}.
\]

Order the set \( \mathcal{I} \) as \( \{I_1, \ldots, I_m\} \) such that
- For \( k = 1, \ldots, m \), if \( J \geq I_k \), then \( J \in \{I_1, \ldots, I_m\} \).

For each \( i \) choose precompact shrinkings
\[
F^i = : G^{(1)}_i \sqsubset G^{(1)}_i \sqsubset \cdots \sqsubset G^{(m)}_i \sqsubset F^{(m)}_i \sqsubset F_i.
\]

Then for \( I_k \in \mathcal{I} \), with abusive use of notations, define
\[
F_{I_k} := \left( \bigcap_{i \in I} F^{(k)}_i \right) \setminus \left( \bigcup_{i \notin I} \overline{G^{(k)}_i} \right).
\]

Then for the same reason as Lemma 3.14, the collection \( \{F_{I_k} \mid I_k \in \mathcal{I}\} \) is an open cover of \( \mathcal{M}_k \) and satisfies the overlapping condition
\[
\overline{F_I} \cap \overline{F_J} \neq \emptyset \implies I \leq J \text{ or } J \leq I.
\]

Now we construct local charts \( K_I \) whose footprints are the \( F_I \)'s above. If \( I = I^# \cup I^c \notin \mathcal{I}^# \), or equivalently, \( I^c \neq \emptyset \), then \( F_I \) is disjoint from the lower stratum, so such a local chart \( K_I \) is easy to construct. If \( I^c = \emptyset \), then (7.11) implies that \( F_I \subset F^*_I \), and \( K_I \) can be obtained by taking a subchart of \( K^*_I \) whose footprint is \( F_I \).

Next we construct coordinate changes. For each \( I \in \mathcal{I} \), choose a precompact subset \( F^i_I \subset F_I \) such that the collection \( \{F^i_I\} \) is still an open cover of \( \mathcal{M}_k \). Then we claim that for any pair \( I, J \in \mathcal{I}, I \leq J \), there is a weak coordinate change \( T_{IJ} : K_I \rightarrow K_J \) whose footprint \( F_{IJ} \) contains
Indeed, if $I$ or $J$ is not in $\mathcal{I}^\#$, then it is the same situation as in Section 3. If $I, J \in \mathcal{I}^\#$, then $F_I^* \cap F_J^* \subset F_I^* \cap F_J^*$. By Proposition 7.10, the latter is the footprint of the strong coordinate change $T_{IJ}$. Then $T_{IJ}$ can be obtained by restricting $T_{IJ}^*$ to a subchart.

The above procedure produces a collection of weak coordinate changes $T_{IJ} := (U_{IJ}, \phi_{IJ})$ with footprints $F_{IJ} \supset F_I^* \cap F_J^*$. The cocycle condition among this collection is as easy to check as the case of Section 3. Hence we obtained a weak virtual orbifold atlas on $\mathcal{M}_a$. Next, using the condition $F_{IJ} \subset F_I^* \cap F_J^*$ as well as Proposition A.24, one can shrink each $K_I$ to a subchart $K_I' = (U_I', E_I', S_I', \psi_I', F_I')$ satisfying the following conditions

1. $F_I' \subset F_I^*$;
2. The induced coordinate change $T_{IJ}' : K_I' \to K_J'$ is strong.

Hence $\{K_I', \{T_{IJ}'\}\}$ is a strong virtual orbifold atlas (with boundary) of $\mathcal{M}_a$. Together with the known information about the boundary charts, this finishes the proof of Theorem 4.8.

8. Identifying the Wall-Crossing Term

In this section we evaluate the oriented counting of the moduli space of BPS soliton solutions $\mathcal{M}_a^\delta$, namely, to prove Theorem 4.10. Here we used the same notations as the last few sections. Moreover, we drop the unnecessary indices, such as the $j$ which labels the broad puncture, and abbreviate $\upsilon_0, \kappa_0$ by $\upsilon, \kappa$ respectively.

Essentially, Theorem 4.10 is a result of finite dimensional Morse theory. Before going into details let us look at the problem from an intuitive perspective. Note that the soliton component $\sigma$ of each element $(X, \sigma) \in \mathcal{M}_a^\delta$ is a solution to

$$\frac{d\sigma}{ds} + \nabla \tilde{W}(\delta_X)(\sigma(s)) = 0, \quad \lim_{s \to -\infty} \sigma(s) = \upsilon_X, \quad \lim_{s \to +\infty} \sigma(s) = \kappa_X,$$

which is an equation depending on object $X$ on the principal component. Hence $\mathcal{M}_a^\delta$ is not a product. To relate the counting with intersection numbers, we use a homotopy to deform the parameter $\delta_X$ to the constant 1. Then $\mathcal{M}_a^\delta$ is cobordant (in certain “virtual” sense) to a product, denoted by

$$\mathcal{S}_a^\delta = \mathcal{M}_\upsilon \times \mathcal{S}_\tilde{W}(\upsilon, \kappa).$$

Here $\mathcal{S}_\tilde{W}(\upsilon, \kappa)$ is the moduli space of solutions to

$$\frac{dy}{ds} + \nabla \tilde{W}(y(s)) = 0, \quad \lim_{s \to -\infty} y(s) = \upsilon, \quad \lim_{s \to +\infty} y(s) = \kappa,$$

modulo time translation.

Theorem 4.10 follows if one can show in addition that the counting of $\mathcal{S}_\tilde{W}(\upsilon, \kappa)$ is the intersection number $\Delta_\upsilon \cdot \Delta_\kappa$. In Section 9, we will show that the intersection number is equal to the counting of gradient flow lines of $\text{Re}(F|_{Q\upsilon})$, where $(a, F)$ is the strongly regular perturbation. Abbreviate $F^a := F|_{Q\upsilon}$. Let $\mathcal{S}_{F^a}(\upsilon, \kappa)$ be the space of solutions to

$$\frac{dy}{ds} + \nabla F^a(y(s)) = 0, \quad \lim_{s \to -\infty} y(s) = \upsilon, \quad \lim_{s \to +\infty} y(s) = \kappa,$$

modulo time translation.

Therefore proving Theorem 4.10 reduces to showing that $\#\mathcal{S}_\tilde{W}(\upsilon, \kappa) = \#\mathcal{S}_{F^a}(\upsilon, \kappa)$. This follows from the following adiabatic limit argument, which has been carried out in [SX14] in a similar context. The gradient flow equation of $\tilde{W}$ can be written in components as

$$\begin{align*}
\frac{dx}{ds} + p(s)\nabla Q(x(s)) + \nabla F(x(s)) &= 0, \\
\frac{dp}{ds} + Q(x(s)) - a &= 0.
\end{align*}$$
We introduce a real parameter $\lambda > 0$ and consider

$$\begin{cases}
\frac{dx}{ds} + p(s)\nabla Q(x(s)) + \nabla F(x(s)) = 0, \\
\frac{dp}{ds} + \lambda^2 (Q(x(s)) - a) = 0, \\
\lim_{s \to -\infty} (x(s), p(s)) = \nu, \\
\lim_{s \to +\infty} (x(s), p(s)) = \kappa.
\end{cases} \quad (8.4)$$

In finite dimensional Morse theory, it is convenient to consider the energy functional

$$E(x, p) = \frac{1}{2} \left( \|p(s)\nabla Q(x) + \nabla F(x)\|_{L^2}^2 + \lambda^2 \|Q(x) - a\|_{L^2}^2 + \|x'|^2_{L^2} + \frac{1}{\lambda^2} \|p'|^2_{L^2} \right). \quad (8.5)$$

The energy of a solution to (8.4) is equal to $\tilde{W}(\nu) - \tilde{W}(\kappa)$, which is independent of $\lambda$. Hence as $\lambda \to +\infty$, solutions are approximately negative gradient lines of the restriction $Re F|_{Q^\pm}$. If we denote by $S_{\tilde{W}}^\lambda(\nu, \kappa)$ the moduli space of solutions to (8.4) (modulo translation), then we will construct an orientation-preserving homeomorphism

$$S_{\tilde{W}}^\lambda(\nu, \kappa) \simeq S_{F^\prime}(\nu, \kappa), \quad \forall \lambda > 0.$$

This will complete the proof of Theorem 4.10.

In the remaining of this section, we carry out the details of the above arguments. In Subsection 8.1 we prove that $M^\delta_\kappa$ is cobordant to $S^\delta_\kappa$. In Subsection 8.2 we discuss certain issues about transversality. In Subsection 8.3 we prove the identification $S_{\tilde{W}}^\lambda(\nu, \kappa) \simeq S_{F}(\nu, \kappa)$.

8.1. The first homotopy. Let $\epsilon \in [0, 1]$. Let $\delta^\epsilon : \mathcal{B}_{v_0} \to \mathbb{R}_+^+$ be the functional

$$\delta_X^\epsilon = (\delta_X)^\epsilon.$$

Let $M_{\kappa, \epsilon}^\delta$ be the moduli space of BPS soliton solutions to the gauged Witten equation over $\mathcal{C}$ where the soliton components satisfy (8.1) with $\delta_X$ replaced by $\delta_X^\epsilon$. Then consider the universal moduli space

$$\widetilde{M}_{\kappa}^\delta := \{ (X^\#, \epsilon) \mid \epsilon \in [0, 1], X^\# \in M_{\kappa, \epsilon}^\delta \}.$$

**Proposition 8.1.** There exists an oriented virtual manifold atlas with boundary on $\widetilde{M}_{\kappa}^\delta$ such that its oriented virtual boundary is

$$\partial\widetilde{M}_{\kappa}^\delta = (M_{\kappa}^\delta) \cup (-S_{\kappa}^\delta).$$

**Proof.** The construction of local charts is similar to the situation in Section 4, since no gluing is needed and the automorphism groups are always trivial. \qed

8.2. On transversality. It is well-known that in real Morse theory, the index of a flow line connecting two nondegenerate critical points $p$ and $q$ of a Morse function is equal to $\text{ind}(p) - \text{ind}(q)$. In particular, generically, there is no flow line connecting two distinct critical points with equal index. On the other hand, for a holomorphic Morse function $f$ on an $n$-dimensional complex manifold, critical points of $f$, viewed as critical points of $Re f$, all have index $n$. Therefore, a BPS soliton, viewed as a negative gradient flow line of $Re f$, is not transverse. For any solution $\rho : \mathbb{R} \to X$ to the equation

$$\rho'(s) + \nabla f(\rho(s)) = 0, \quad \lim_{s \to -\infty} \rho(s) = p, \quad \lim_{s \to +\infty} \rho(s) = q, \quad (8.6)$$

its linearization $D_{\rho} : W^{1,2}(a^*TX) \to L^2(a^*TX)$ is not transverse; a necessary condition for having one solution is that \[ \text{Im} f(p) = \text{Im} f(q). \]
We introduce the equation on pairs \( \hat{\rho} = (a, \rho) \) where \( a \in \mathbb{R} \) and \( \rho : \mathbb{R} \to X \),
\[
\rho'(s) + \nabla f(\rho(s)) - aJ\nabla f(\rho(s)) = 0, \quad \lim_{s \to -\infty} \rho(s) = p, \quad \lim_{s \to +\infty} \rho(s) = q.
\] (8.7)
If \( (a, \rho) \) solves the solution, then \( \rho \) is a negative gradient flow line of \( \text{Re}(f + af) \), hence
\[
\text{Im}(f(p) + af(p)) = \text{Im}(f(q) + af(q)),
\] which implies that \( a = 0 \). Therefore, solutions to (8.6) and solutions to (8.7) are in one-to-one correspondence, by identifying \( \rho \) with \( \hat{\rho} = (0, \rho) \).

**Definition 8.2.** A solution \( \rho \) to (8.6) is called **maximally transverse** (with respect to the Hermitian metric) if the corresponding linearization \( \hat{D}_\rho \) is surjective, or equivalent, \( \text{coker} D_\rho \) is one-dimensional (spanned by \( J\nabla f(\rho) \)).

Now we specialize to the family of equations (8.4). Consider the Banach manifold \( \mathcal{B} \) of \( W^{1,2}_{\text{loc}} \) maps from \( \mathbb{R} \) to \( \mathbb{C} \times X \) which are asymptotic to \( u \) and \( \kappa \) at \(+\infty\) and \(-\infty\) respectively, both in a \( W^{1,2}\)-sense. Consider the Banach space bundle \( \mathcal{E} \to \mathcal{B} \) whose fibre over \( \rho = (x, p) \in \mathcal{B} \) is
\[
\mathcal{E}_\rho = L^2(\rho^*TX) = L^2(x^*TX \oplus \mathbb{C}).
\]
There is a family of sections
\[
\mathcal{F}_W^\lambda(\rho) = \left( \begin{array}{c} \dot{x}_\rho + \mathcal{F}_\rho \nabla Q(x) + \nabla F(x) \\ \dot{p}_\rho + \lambda^2 (Q(x) - a) \end{array} \right).
\]
Its linearization at \( \rho \) reads
\[
D_\rho^\lambda(V, h) = \left( \begin{array}{cc} \nabla_v V + \nabla_v (\mathcal{F} \nabla Q + \nabla F) + h \nabla Q \\ h' + \lambda^2 \nabla dQ \end{array} \right).
\]
Denote
\[
\mathcal{S}_W^\lambda(v, \kappa) := (\mathcal{F}_W^\lambda)^{-1}(0), \quad \mathcal{S}^\lambda(v, \kappa) = \mathcal{S}_W^\lambda(v, \kappa)/\mathbb{R}.
\]
Consider
\[
\widetilde{\mathcal{S}}_W := \bigcup_{\lambda \geq 1} \{ \lambda \} \times \mathcal{S}_W^\lambda, \quad \mathcal{S}_W := \bigcup_{\lambda \geq 1} \mathcal{S}_W^\lambda(v, \kappa)
\]
The former can be viewed as the zero locus of \( \mathcal{F}_W^\lambda : [1, +\infty) \times \mathcal{B} \to \mathcal{E} \) with \( \mathcal{F}_W^\lambda(\lambda, \rho) = \mathcal{F}_W^\lambda(\rho) \).
It is not hard to show that by perturbing the Hermitian metric on \( X \), one can assume the following:
1. \( \widetilde{\mathcal{S}}_W \) is transverse, i.e., for any \( \hat{\rho} := (\lambda, \rho) \in \widetilde{\mathcal{S}}_W \), the linearized operator \( D_\rho \mathcal{F}_W^\lambda : \mathbb{R} \times T_\rho \mathcal{B} \to \mathcal{E}_\rho \)

is surjective.
2. Each \( \rho \in \mathcal{S}_W^\lambda \) is maximally transverse.
3. Any \( \mathcal{F} \in \mathcal{S}_{F^\lambda}(v, \kappa) \) is maximally transverse with respect to the restricted Hermitian metric on \( Q^\kappa \).

**8.3. Adiabatic limit.** Recall that \( \star \in X \) is the unique critical point of \( Q \). There is a well-defined function \( q : X \setminus \{ \star \} \to \mathbb{C} \) defined by
\[
(q(x) dQ(x) + dF(x)) \cdot \nabla Q(x) = 0.
\]
Notice that for \( x \in Q^\kappa \),
\[
q(x) \nabla Q + \nabla F(x) = \nabla F^\kappa(x).
\]
Proposition 8.3. Let \( \lambda_i \) be a sequence of positive numbers such that \( \lim_{i \to \infty} \lambda_i = +\infty \). Suppose \( \bar{x}_i(s) = (x_i(s), p_i(s)) \) is a sequence of solutions to (8.4) with \( \lambda = \lambda_i \). Then there exist a subsequence (still indexed by \( i \)), a sequence of numbers \( s_i \in \mathbb{R} \), and a solution to the equation

\[
\hat{x}(s) + \nabla F_a(x(s)) = 0, \quad x : \mathbb{R} \to Q^a, \quad \lim_{s \to -\infty} x(s) = v, \quad \lim_{s \to +\infty} x(s) = \kappa
\]
such that the following conditions hold.

1. \( x_i(s + s_i) \) converges to \( x(s) \) uniformly on compact subsets of \( \mathbb{R} \).
2. \( p_i(s + s_i) \) converges to \( q(x(s)) \) uniformly.

Proof. The key of the proof is the \( C^0 \)-bound, i.e., there exists a compact subset \( \bar{K} \subset \bar{X} \) such that \( \bar{x}_i(s) \in \bar{K} \). Once this is proved, this proposition can be proved in the same way as in [SX14].

Let \( 3d_a = \text{dist}(\ast, Q^a) > 0 \). First, we claim that for \( i \) large enough, \( x_i(s) \notin B_{d_a}(\ast) \) for all \( s \in \mathbb{R} \) (recall that \( \ast \in X \) is the unique critical point of \( Q \)). We prove this claim by contradiction. Suppose \( d(\ast, x_i(s_i)) \leq d_a \). Let \( [s_i - a_i, s_i + b_i] \) be the maximal interval containing \( s_i \) such that \( x_i([s_i - a_i, s_i + b_i]) \subset B_{d_a}(\ast) \). Then we have

\[
d_a \leq \text{dist}(x_i(s_i - a_i), x_i(s_i)) \leq \int_{s_i - a_i}^{s_i} |x_i'(s)| ds \leq \sqrt{a_i} \|x_i'\|_{L^2} \leq \sqrt{a_i} E.
\]

Therefore, \( a_i \geq d_a^2 / E \). Similarly \( b_i \geq d_a^2 / E \). On the other hand, there is \( \epsilon_a > 0 \) such that \( |Q(x_i(s)) - a| \geq \epsilon_a \) for all \( s \in [s_i - a_i, s_i + b_i] \). Then

\[
E \geq \lambda_i^2 \|Q(x_i) - a\|_{L^2(\mathbb{R})}^2 \geq \lambda_i^2 \|Q(x_i) - a\|_{L^2([s_i - a_i, s_i + b_i])} \geq \lambda_i^2 \epsilon_a^2 / E.
\]

This is in possible for \( i \) sufficiently large. Therefore the claim is proved.

Second, by the definition of \( q(x) \), the properness of \( \nabla Q \) and the boundedness of \( \nabla F \), it is easy to see that \( q \) and \( \nabla q \) are uniformly bounded away from \( B_{d_a}(\ast) \). By (8.4), we have

\[
0 = \frac{d}{ds} Q(x_i) - dq : x_i'
\]

\[
= \frac{d}{ds} Q(x_i) + dQ \cdot (\nabla Q(x_i) + \nabla F(x_i))
\]

\[
= \frac{d}{ds} Q(x_i) + dQ \cdot (p_i - q(x_i)) \nabla Q(x_i)
\]

\[
= \frac{d}{ds} Q(x_i) + \lambda_i |\nabla Q(x_i)|^2 \left( \lambda_i^{-1}(p_i - q(x_i)) \right).
\]

We also have

\[
\frac{d}{ds} \left( \lambda_i^{-1}(p_i - q(x_i)) \right) + \lambda_i \left( Q(x_i) - a \right)
\]

\[
= \lambda_i^{-1}(p_i' - dq \cdot x_i') + \lambda_i \left( Q(x_i) - a \right)
\]

\[
= -\lambda_i^{-1}dq \cdot x_i'.
\]

Consider the operator

\[
D_{\lambda_i} : W^{1,2}(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)
\]

\[
(f_1, f_2) \mapsto \left( \frac{df_1}{ds} + \lambda_i |\nabla Q(x_i)|^2 f_2, \frac{df_2}{ds} + \lambda_i f_1 \right).
\]

Notice that \( |\nabla Q(x_i)| \) is uniformly bounded from below. Then as an unbounded operator from \( L^2 \) to \( L^2 \), \( D_{\lambda_i} \) is bounded from below by \( c\lambda_i \) for some constant \( c > 0 \). On the other hand, it has a right inverse from \( L^2 \) to \( W^{1,2} \) which is uniformly bounded. Hence by (8.8) and (8.9), we have

\[
\|Q(x_i) - a\|_{L^2(\mathbb{R})} + \lambda_i^{-1} \|p_i - q(x_i)\|_{L^2(\mathbb{R})} \leq c\lambda_i^{-2} \|dq \cdot x_i'\|_{L^2(\mathbb{R})} \leq c\lambda_i^{-2} \|x_i'\|_{L^2(\mathbb{R})}.
\]

\[
\|Q(x_i) - a\|_{W^{1,2}(\mathbb{R})} + \lambda_i^{-1} \|p_i - q(x_i)\|_{W^{1,2}(\mathbb{R})} \leq c\lambda_i^{-1} \|x_i'\|_{L^2(\mathbb{R})}.
\]

In particular, \( \|p_i\|_{L^\infty} \) and \( \|p_i'\|_{L^2} \) are uniformly bounded.
Now we prove that $x_i(s)$ is uniformly bounded. Consider the moment map $\mu_0 : \tilde{X} \to i\mathbb{R}$ of the $G_0$-action. For some constant $c > 0$, we have
\[
\frac{d^2}{ds^2} i\mu_0(x_i) = \partial_s \langle \nabla i\mu_0(x_i), x'_i \rangle \\
= \partial_s \langle \nabla i\mu_0(x_i), -\overline{\nabla Q(x_i)} - \nabla F(x_i) \rangle \\
= -\partial_s \left( \Re \left( rW(\tilde{x}_i) + F_b(x_i) \right) \right) \\
= -\partial_s \left( \Re \left( r\overline{W}(\tilde{x}_i) + F_b(x_i) - rF(x_i) - r\mu_0 \right) \right) \\
= -r\nabla^2 \overline{W}(\tilde{x}_i) \cdot x'_i - \nabla (F_b - rF) \cdot x'_i + r\Re(p'_i) \\
\geq r|x'_i|^2 - c|x'_i| - c|p'_i| \\
\geq -c|x'_i| - c|p'_i|.
\]

Now we explain how to obtain the above estimate. $F_b : X \to \mathbb{C}$ is the function $F_b(x) = \sum_{i=1}^s r_i F_i$ and the third equality follows from the homogeneity of $F_i$. The second last inequality follows from (1) of Hypothesis 2.7.

Notice that $x'_i$ and $p'_i$ are both bounded in $L^2$. If there exists $s_i \in \mathbb{R}$ such that
\[
\lim_{i \to \infty} i\mu_0(x_i(s_i)) = \lim_{i \to +\infty} \sup_{r \in \mathbb{R}} i\mu_0(x_i) = +\infty,
\]
then it is easy to derive that $i\mu_0(x_i(\cdot + s_i))$ diverges to infinity uniformly on compact sets. However, if this is true, then by (2) of Hypothesis 2.7, the energy of $\tilde{x}_i$ can be infinitely large, which is impossible. Hence $i\mu_0(x_i)$ is uniformly bounded. Since $\mu_0$ is proper, $x_i$ is uniformly bounded.

Therefore, since $x_i$ is in a small neighborhood of $Q^a$, one can write $x_i(s) = (\overline{\tau}_i(s), Q(x_i(s)))$ with $\overline{\tau}_i(s) \in Q^a$. Moreover,
\[
|\overline{\tau}_i(s) + \nabla F_a(\overline{\tau}_i(s))| = |\overline{\tau}_i(s) + q(\overline{\tau}_i(s))\nabla Q(\overline{\tau}_i(s)) + \nabla F(\overline{\tau}_i(s))| \leq c|Q(x_i(s)) - a|
\]
which converges uniformly to zero. Hence a subsequence of $\overline{\tau}_i$ (still indexed by $i$) converges (modulo reparametrization) to a flow line of $F_a$. Moreover, the flow line cannot break because it must connect $\nu$ and $\kappa$. \hfill \Box

Now we define a compactification of $\overline{S_W}$, i.e.,
\[
\overline{S_W} := S_W \cup \{ +\infty \} \times S_{F^a}(\nu, \kappa).
\]
We say that a sequence $[\overline{\rho}_i] = [\lambda_i, \rho_i] \in \overline{S_W}$ converges to $[+\infty, \overline{\nu}]$ if up to translation, the two conditions of Proposition 8.3 hold.

In order to prove Theorem 4.10, we need construct a boundary chart near the $+\infty$ side of $\overline{S_W}$. We have

**Proposition 8.4.** Suppose the negative gradient line $\overline{\nu} : \mathbb{R} \to Q^a$ is maximally transverse. Then there exist $\Lambda = \Lambda_{\overline{\nu}} > 0$, and a continuous map
\[
\Phi_{\overline{\nu}} : [\Lambda, +\infty] \to \overline{S_W}
\]
which is a homeomorphism onto a neighborhood of $[+\infty, \overline{\nu}]$.

This proposition will be proved shortly. It puts a boundary chart on $\overline{S_W}$ and hence implies an oriented cobordism from $S_{\overline{W}}(\nu, \kappa)$ to $S_{F^a}(\nu, \kappa)$. Therefore,
\[
\# S_{\overline{W}}(\nu, \kappa) = \# S_{F^a}(\nu, \kappa) = \Delta_{\nu} \cdot \Delta_{\kappa}. \tag{8.10}
\]
Together with Proposition 8.1, it implies Theorem 4.10.
8.4. Proof of Proposition 8.4. Choose a small $\epsilon > 0$. Let $Q^{\epsilon} = \{ y \in X | d(y,\partial X) < \epsilon \}$ be the $\epsilon$-neighborhood of $X^\rho$. There is a line bundle $L \subset TX|_{Q^{\epsilon}}$ spanned by $\nabla Q$ (over $\mathbb{C}$). Let $L^\perp \subset TX|_{Q^{\epsilon}}$ be its orthogonal complement.

Then for those $\rho \in B$ which are contained in $Q^{\epsilon}$, we can decompose the domain and the target space of the linearization $D^\rho_{\rho}$ as

\[
T_{\rho}B \approx W_L(\rho) \oplus W_T(\rho), \quad W_L(\rho) = W^{1,2}(x^*L \oplus \mathbb{C}), \quad W_T(\rho) = W^{1,2}(x^*L^\perp),
\]

(8.11)

We rescale the norms on $W_L(\rho)$ and $\mathcal{E}_L(\rho)$ as follows. We identify $(h_1, h_2) \in W_L(\rho)$ with $(h_1, h_2) \in W^{1,2}(x^*L' \oplus W)$ and define

\[
\|(h_1, h_2)\|_{W_\lambda} = \lambda\|h_1\|_{L^2} + \|h_1\|_{L^2} + \|h_2\|_{L^2} + \lambda^{-1}\|h_2\|_{L^2}.
\]

(8.12)

For $(h_1, h_2) \in \mathcal{E}_L(\rho)$, we define

\[
\|(h_1, h_2)\|_{L_\lambda} = \|h_1\|_{L^2} + \lambda^{-1}\|h_2\|_{L^2}.
\]

(8.13)

The norms on the tangential components are unchanged, and we use $W_\lambda$ and $L_\lambda$ to denote the modified norms on $T_{\rho}B$ and $\mathcal{E}_\rho$ respectively.

Let $\mathbb{R} \to Q^\rho$ be a negative gradient line of $F^\rho$. Then the linearization along $\mathbb{R}$ reads

\[
\mathbb{R} : \mathbb{R} \oplus W^{1,2}(\mathbb{R} Q^\lambda) \to L^{2}(\mathbb{R} Q^\lambda), \quad \mathbb{R}(\mathbb{R}) = \nabla_{\mathbb{R}} + \nabla_{\mathbb{R}}(\nabla F + q(\mathbb{R})\nabla Q).
\]

We define

\[
\mathbb{R} : \mathbb{R} \oplus W^{1,2}(\mathbb{R} Q^\lambda) \to L^{2}(\mathbb{R} Q^\lambda), \quad \mathbb{R}(\mathbb{R}) = \nabla_{\mathbb{R}}(\nabla F + q(\mathbb{R})\nabla Q).
\]

Since $\mathbb{R}$ is maximally transverse, $\mathbb{R}$ is surjective.

Now we define $\rho_{\mathbb{R}} \in B$ by $\rho_{\mathbb{R}}(s) = (\mathbb{R}(s), q(\mathbb{R}(s)))$. This is going to be our approximate solution to (8.4) for $\lambda$ large. To apply the implicit function theorem, we have a few estimates to make.

Firstly, by straightforward calculation,

\[
\|S^\lambda(\rho_{\mathbb{R}})\|_{L_\lambda} = \lambda^{-1}\|dQ \cdot \mathbb{R}\|_{L^2}.
\]

(8.14)

Secondly, let $D^\lambda_{\rho_{\mathbb{R}}}: T_{\rho_{\mathbb{R}}}B \to \mathcal{E}_{\rho_{\mathbb{R}}}$ be the linearization along $\rho_{\mathbb{R}}$. For $\mathbb{R} \in W_T(\rho_{\mathbb{R}}) \subset T_{\rho_{\mathbb{R}}}B$, we have

\[
D^\lambda_{\rho_{\mathbb{R}}}(\mathbb{R}) = (\nabla_{\mathbb{R}} + \nabla_{\mathbb{R}}(\nabla F + q(\mathbb{R})\nabla Q), \lambda^2dQ \cdot \nabla_{\mathbb{R}})(\mathbb{R}, 0).
\]

Hence with respect to the decompositions in (8.11), $D^\lambda_{\rho_{\mathbb{R}}}$ can be written as

\[
D^\lambda_{\rho_{\mathbb{R}}} = \begin{pmatrix}
\mathbb{R} & A_{\mathbb{R}} \\
0 & D^\lambda_{\mathbb{R}}
\end{pmatrix}
\]

where for $H = (h_1 \nabla Q, h_2) \in W_L(\rho_{\mathbb{R}})$, $A_{\mathbb{R}}(H)$ is real linear in $h_1$ and is independent of $\lambda$. Therefore,

\[
\|A_{\mathbb{R}}(h_1 \nabla Q, h_2)\|_{L_\lambda} \leq c_1\|h_1\|_{L^2} \leq c_1\lambda^{-1}\|H\|_{W_\lambda}.
\]

On the other hand, it is easy to see that $D^\lambda_{\mathbb{R}}: W_L(\rho_{\mathbb{R}}) \to \mathcal{E}_L(\rho_{\mathbb{R}})$ has an inverse $Q^+_{\mathbb{R}}$ whose operator norm with respect to the modified Banach norms (8.12) and (8.13) is bounded by a number independent of $\lambda$. Therefore, we construct the operator

\[
Q^+_{\mathbb{R}} = \begin{pmatrix}
\mathbb{R} & 0 \\
0 & Q^+_{\mathbb{R}}
\end{pmatrix} : (\mathbb{R} \oplus \mathcal{E}_T(\rho_{\mathbb{R}})) \oplus \mathcal{E}_T(\rho_{\mathbb{R}}) \to W_T(\rho_{\mathbb{R}}) \oplus W_L(\rho_{\mathbb{R}}).
\]

Moreover, for certain constant $c > 0$,

\[
\|\text{Id} - D^+_{\rho_{\mathbb{R}}}Q^+_{\mathbb{R}}\|_{\lambda} = \|Q^+_{\mathbb{R}}A^\lambda_{\mathbb{R}}\|_{\lambda} \leq c\lambda^{-1}.\]

Therefore, for $\lambda$ sufficiently large, $Q^+_{\mathbb{R}}$ is approximately a right inverse to $D^+_{\rho_{\mathbb{R}}}$.
Thirdly, we need to estimate the variation of the linearized operator near $ρ_Ψ$. One can give a local chart of the Banach manifold $B$ near $ρ_Ψ$ and trivialize the bundle $E \to B$ over this chart as follows. Using the identification $Q^n \simeq Q^2 \times B_+$, for any $V = (\nabla, h_1 \nabla Q, h_2) \in T_ρB$ with small norm, we identify it with the map $\Phi^B(V) := (\exp_\nabla \nabla, h_1, q(\nabla) + h_2)$ into $Q^n \times B_+ \subset \mathbb{C}$, where $\exp_\nabla$ is the exponential map inside $Q^n$. We trivialize $E$ over the image of $\Phi^B$. Let $D_ρ^{λ^+} : \mathbb{R} \oplus T_ρB \to E_ρ$ be the linearization of $Φ^E \circ F^{λ^+} \circ Φ^B$ at $V \in T_ρB$.

Lemma 8.5. There exist, $e, c, Λ > 0$ such that for $λ ≥ Λ$ and $∥V∥_{W_λ} ≤ e$, we have

$$∥D_ρ^{λ^+} - D_ρ^{λ^+}∥ ≤ c∥V∥_{W_λ}.$$ 

Proof. This can be proved similarly as [SX14, Lemma 9] and the detail is left to the reader. □

Therefore, one can apply the implicit function theorem and one derives that for each sufficiently large $λ$, there is a unique $V^{λ^+}_λ = (a_λ, V_λ) \in \text{Im} Q^n_{ρ_Ψ} \subset \mathbb{R} \oplus T_ρB$ such that

$$(a_λ, \exp_ρ \xi_λ) ∈ (F^{λ^+}_{ρ_W})^{-1}(0).$$

We also know that $a_λ = 0$. Hence it defines a map $\hat{Ψ}_Ψ : [Λ, +\infty] \to \overline{S_ρ}$ by

$$λ \mapsto [λ, Φ_ρ(V_λ)].$$

Moreover, one can show that it is a homeomorphism onto a neighborhood of $[+\infty, \overline{ρ}]$ inside $\overline{S_ρ}$. The details are left to the reader. Hence $\hat{Ψ}_Ψ$ provides a boundary chart on $\overline{S_ρ}$ and we have finished the proof of Proposition 8.4.

9. Orientations

In this section we specify the conventions about orientations and finish the proof of Theorem 4.11, which is essentially to identify the relevant signs according to our conventions.

9.1. Linear algebra. Let $F : X \to Y$ be a Fredholm operator between Banach spaces $X, Y$. The determinant line det $F$ is defined as

$$\det F := \det \ker F \otimes (\det \text{coker} F)^\vee.$$ 

An orientation of $F$ is a homotopy class of trivializations of this space.

Let $F(X, Y)$ be the space of Fredholm operators from $X$ to $Y$. $\{\det F\}_{F \in F(X, Y)}$ defines the so-called determinant line bundle det $F(X, Y)$. For any continuous map $f : A \to F(X, Y)$ defined on a contractible space $A$, $f^* \det F(X, Y) \to A$ should be trivial. A trivialization $φ_A : f^* \det F \to A \times \mathbb{R}$ can be easily write down if $\dim(\ker f(a))$ is a constant for $a \in A$. In this case, we say $θ_1 \in \det f(a_1)$ and $θ_2 \in \det f(a_2)$ are in the same orientation in $f^* \det F(X, Y)$ if the second factors of $φ_A(θ_1), φ_A(θ_2)$ are of the same sign.

However, we need to specify our conventions of comparing the orientations when the dimensions of the kernels jump. For a Fredholm operator $F : X \to Y$, let $F^{0(k)} : \mathbb{R}^k \oplus X \to \mathbb{R}^k \oplus Y$ be the operator $F^{0(k)}(v, x) = (0, F(x))$. We define $ψ^{0(k)} : \det F \to \det F^{0(k)}$ by

$$ψ^{0(k)} \left( \bigwedge_{i=1}^m x_i \wedge \bigwedge_{j=1}^n y_j \right) = \left( \bigwedge_{i=1}^k (e_i, 0) \wedge \bigwedge_{j=1}^m (0, x_i) \right) \otimes \left( \bigwedge_{i=1}^k (e^*_i, 0) \wedge \bigwedge_{j=1}^n (0, y^*_j) \right).$$

Here $x_1, \ldots, x_m$ is a basis of $\ker F$, $y_1, \ldots, y_n$ is a basis of $\text{coker} F$ and $y^*_1, \ldots, y^*_n$ is the dual basis; $e_1, \ldots, e_k$ is a basis of $\mathbb{R}^k$ and $e^*_1, \ldots, e^*_k$ is its dual basis. We also define $F^{1(k)}(v, x) = (v, F(x))$ and $ψ^{1(k)} : \det F \to \det F^{1(k)}$ by

$$ψ^{1(k)} \left( \bigwedge_{i=1}^m x_i \wedge \bigwedge_{j=1}^n y_j \right) = \bigwedge_{i=1}^m (0, x_i) \otimes \bigwedge_{j=1}^n (0, y^*_j).$$

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The basic convention is that we regard the orientation of \( \theta \in \det F \) as “the same” as those of \( \psi_0^{(k)}(\theta) \in F_0^{(k)} \) and \( \psi_1^{(k)}(\theta) \in \det F_1^{(k)} \).

More generally, let \( G^{(k)}(X,Y) \) be the space of finite rank operators \( G: \mathbb{R}^k \oplus X \to \mathbb{R}^k \oplus Y \) of the form

\[
G = \begin{pmatrix} G_1 & G_2 \\ G_3 & 0 \end{pmatrix}.
\]

Then let \( F_G : \mathbb{R}^k \oplus X \to \mathbb{R}^k \oplus Y \) be \( F_0^{(k)} + G \). This gives a family of Fredholm operators over the contractible space \( G^{(k)}(X,Y) \) and \( \det F_G \) is trivial over \( G^{(k)}(X,Y) \).

Therefore, given a continuous family of Fredholm operators \( F_T : X \to Y, \ T \in [0,1] \) such that

\[
k = \dim(\ker F_0) - \dim(\ker F_1) = \max\{\dim(\ker F_{T_1}) - \dim(\ker F_{T_2}) \mid 0 \leq T_1, T_2\},
\]

we can extend the family to a family

\[
F_{T,G} : \mathbb{R}^k \oplus X \to \mathbb{R}^k \oplus Y, \ T \in [0,1], \ G \in G^{(k)}(X,Y).
\]

Two nonzero elements \( \theta_1 \in \det F_1 \) and \( \theta_0 \in \det F_0 \) are called in the same orientation if there is a curve \( G: [0,1] \to G^{(k)}(X,Y) \) such that the following conditions hold.

1. \( \dim(\ker F_{T,G(T)}) \) is a constant.
2. Define \( f : [0,1] \to F(\mathbb{R}^k \times X, \mathbb{R}^k \times Y) \) by \( f(T) = F_{T,G(T)} \). Then \( \psi_0^{(k)}(\theta_0) \in \det f(0) \) and \( \psi_1^{(k)}(\theta_1) \in \det f(1) \) are in the same orientation in \( f^* \det F(\mathbb{R}^k \times X, \mathbb{R}^k \times Y) \to [0,1] \).

With this in section, we denote \( \theta_1 \sim \theta_2 \) if they are in the same orientation.

We examine a special situation which is used in this paper.

**Lemma 9.1.** Let \( F_T : X \to Y, \ T \geq 0 \) be a family of Fredholm operators of the form

\[
F_T = F + TP
\]

where \( P : X \to Y \) is of finite rank. Assume the following.

1. For all \( T > 0 \), \( \{v_1, \ldots, v_m\} \) is a basis of \( \ker F_T \), and \( \{w_1, \ldots, w_n\} \subset Y \) represents a basis of \( \text{coker} F_T \). Let \( \{w_1^*, \ldots, w_n^*\} \) be the dual basis of \( (\text{coker} F_T)^\vee \).
2. For \( T = 0 \), \( \{v_1, \ldots, v_m; u_1, \ldots, u_k\} \) is a basis of \( \ker F_0 \) and \( \{w_1, \ldots, w_n; P(u_1), \ldots, P(u_k)\} \) represents a basis of \( \text{coker} F \). Let \( \{w_1^*, \ldots, w_n^*, P(u_1)^*, \ldots, P(u_k)^*\} \) be the dual basis.

Denote

\[
\theta_T := \bigwedge_{i=1}^m v_i \otimes \bigwedge_{j=1}^n \omega_j^* \in \det F_T, \ (T > 0);
\]

\[
\theta_0 := \left( \bigwedge_{l=1}^k u_l \wedge \bigwedge_{i=1}^m v_i \right) \otimes \left( \bigwedge_{l=1}^k P(u_1)^* \wedge \bigwedge_{j=1}^n w_j^* \right) \in \det F_0.
\]

Then \( \theta_1 \) and \( \theta_0 \) are in the same orientation.

**Proof.** Without loss of generality, assume that \( n = 0 \). Define \( G_{1,T} : \mathbb{R}^k \to \mathbb{R}^k \), \( G_{2,T} : X \to \mathbb{R}^k \) and \( G_{3,T} : \mathbb{R}^k \to Y \) by

\[
G_{1,T}(a) = (1-T)(a_1, \ldots, a_k);
\]

\[
G_{2,T}(x) = (P(u_1)^*(F_T(x)), \ldots, P(u_k)^*(F_T(x)));
\]

\[
G_{3,T}(a) = (1-T)(a_1 P(u_1), \ldots, a_k P(u_k)).
\]

Define \( F_{T,G(T)} := F_T + G_{1,T} + G_{2,T} + G_{3,T} \). We can check that its kernel is spanned by \( \{(Te_l, -(1-T)u_l) \mid l = 1, \ldots, k\} \cup \{(0, v_i) \mid i = 1, \ldots, m\} \) and its cokernel is spanned by \( \{(e_l^*, -P(u_i)^*) \mid l = 1, \ldots, k, i = 1, \ldots, m\} \).
Consider a holomorphic function $f : X \to \mathbb{C}$. For each nondegenerate critical point $v \in \text{Crit} f$, its unstable manifold $W^u_v$ can be identified with the solution space of the ODE

$$
\dot{x}(s) + \nabla f(x(s)) = 0, \quad s \in (-\infty, 0], \quad \lim_{s \to -\infty} x(s) = v. \tag{9.1}
$$

It is an $N$-dimensional smooth manifold homeomorphic to an $N$-disk. Choosing an orientation on $W^u_v$ is the same as choosing, for each solution $\sigma_-(s)$ of (9.1), an orientation of the linearized operator

$$
D_{\sigma_-} : W^{1,p}(\mathbb{R}_-, \sigma_-^* TX) \to L^p(\mathbb{R}_-, \sigma_-^* TX), \quad D_{\sigma_-}(\xi) = \nabla_\xi \xi + \nabla^2 f(\sigma_-) \cdot \xi. \tag{9.2}
$$

Here $p \geq 2$ and the orientation of $D_{\sigma_-}$ is consistent among all choices of Sobolev regularity. An orientation on $\ker D_{\sigma_-}$ induces an orientation on the corresponding vanishing cycle. Namely, for $p \in W^u_v \setminus \{v\}$ corresponding to $\sigma_-$, if $\tilde{\sigma}_-((0), v_1, \ldots, v_{N-1})$ is an oriented basis of $T_p W^u_v$, then $v_1, \ldots, v_{N-1}$ is an oriented basis of the vanishing cycle $\Delta_v$ in $f^{-1}(f(p))$.

Similarly, consider $\kappa \in \text{Crit} f$ and its stable manifold $W^s_{\kappa}$. It is the solution space of the ODE

$$
\dot{x}(s) + \nabla f(x(s)) = 0, \quad s \in [0, +\infty), \quad \lim_{s \to +\infty} x(s) = \kappa. \tag{9.3}
$$

An orientation on $W^s_{\kappa}$ is the same as an orientation of the linearized operator of each solution $\sigma_+$ to (9.3), which reads

$$
D_{\sigma_+} : W^{1,p}(\mathbb{R}_+, \sigma_+^* TX) \to L^p(\mathbb{R}_+, \sigma_+^* TX), \quad D_{\sigma_+}(\xi) = \nabla_\xi \xi + \nabla^2 f(\sigma_+) \cdot \xi. \tag{9.4}
$$

For $p \in W^s_{\kappa} \setminus \{\kappa\}$, let $\Delta_\kappa \subset f^{-1}(f(p))$ be the vanishing cycle corresponding to $\kappa$. Then an orientation on $W^s_{\kappa}$ induces an orientation on $\Delta_\kappa$ as follows: $w_1, \ldots, w_{N-1}$ is an oriented basis of $T_p \Delta_\kappa$ if and only if $-\tilde{\sigma}_+(s), w_1, \ldots, w_{N-1}$ is an oriented basis on $T_p W^s_{\kappa}$. Notice the difference in signs for stable and unstable manifolds.

Consider two nondegenerate critical points $v, \kappa$ with identical imaginary parts of their critical values. Suppose $\text{Re} f(v) > \text{Re} f(\kappa)$. Then the equation for BPS soliton connecting $v$ and $\kappa$ is

$$
\dot{x}(s) + \nabla f(x(s)) = 0, \quad s \in (-\infty, +\infty), \quad \lim_{s \to +\infty} x(s) = v, \quad \lim_{s \to -\infty} x(s) = \kappa. \tag{9.5}
$$

Let $\tilde{S}(v, \kappa)$ be the space of solutions and $S(v, \kappa) = \tilde{S}(v, \kappa)/\mathbb{R}$. We choose orientations on $W^u_v$ and $W^s_{\kappa}$ (independently). This determines an orientation on the linearized operator $D_{\sigma}$ along any solution $\sigma(s)$ as follows. Let $\sigma_- = \sigma|_{\mathbb{R}_-}$. Then consider the family of operators

$$
\tilde{D}_T^\sigma : W^{1,p}(\mathbb{R}_-, \sigma_-^* TX) \oplus W^{1,p}(\mathbb{R}_+, \sigma_+^* TX) \to L^p(\mathbb{R}, \sigma^* TX) \oplus T_{\sigma(0)}X
$$

$$
\tilde{D}_T^\sigma(\xi_-, \xi_+) = (D_{\sigma_-} \xi_- + D_{\sigma_+} \xi_+, T(\xi_-(0) - \xi_+(0))).
$$
Lemma 9.2. For any $T > 0$, there are canonical isomorphisms

$$
\ker \tilde{D}_T^* \simeq \ker D_\sigma, \tag{9.6}
$$

$$
(\text{coker } \tilde{D}_T^*)^\perp \simeq (\text{Im } \tilde{D}_T^*)^\perp \simeq (\text{Im } D_\sigma)^\perp \simeq (\text{coker } D_\sigma)^\vee. \tag{9.7}
$$

Proof. The first item follows by straightforward calculation. When $T > 0$, $(\xi_-, \xi_+)$ are elements of $\ker \tilde{D}_T^*$ if and only if $\xi_-$ and $\xi_+$ are the restriction of some $\xi \in W^{1,p}(\mathbb{R}, \sigma^* T X)$. This gives the isomorphism $\ker \tilde{D}_T^* \simeq \ker D_\sigma$. Moreover, if $\eta \in (\text{Im } D_\sigma)^\perp$, then for $\xi_\pm \in W^{1,p}(\mathbb{R}, \sigma^1_T X)$,

$$
(D_{\sigma \pm} \xi_\pm, \eta) = \mp \langle \xi_\pm(0), \eta(0) \rangle.
$$

Therefore $\eta \mapsto (T\eta, -\eta(0))$ induces an isomorphism $(\text{Im } D_\sigma)^\perp \simeq (\text{Im } \tilde{D}_T^*)^\perp$. \hfill \Box

Remark 9.3. If we regard $\sigma$ as a map from the cylinder $\Theta = \mathbb{R} \times S^1$ which is independent of the second coordinate, then an orientation of $D_\sigma$ also induces an orientation of the linearized operator over the cylinder. By the result of [FH93], this means choosing orientations on the unstable and stable manifolds induces the so-called consistent orientations on the moduli space of solitons.

It is convenient to assume that $X$ is Kähler and $\sigma$ is maximally transverse and this assumption doesn’t affect the generality when discussing orientations. Then for each $\sigma \in \mathcal{S}(v, \kappa)$, $\dot{\sigma}$ spans $(\text{Im } D_\sigma)^\perp$ and we have a distinguished element

$$
\dot{\sigma} \otimes (J \dot{\sigma}) \in \det D_\sigma.
$$

We define $\text{sign}_\infty(\sigma) = \text{sign}(\dot{\sigma} \otimes J \dot{\sigma}(s)) \in \{\pm 1\}$ with respect to the orientation of $D_\sigma$ induced by (9.8). Then we define

$$
\# \mathcal{S}(v, \kappa) = \sum_{[\sigma] \in \mathcal{S}(v, \kappa)} \text{sign}_\infty(\sigma).
$$

Choose $\beta = \frac{1}{2}(f(v) + f(\kappa))$. Since $f^{-1}(\beta)$ is a complex submanifold, which has a canonical complex orientation, there is a well-defined intersection number

$$
\Delta_v \cdot \Delta_\kappa \in \mathbb{Z}.
$$

On the other hand, $H = (\text{Im } f)^{-1}(\text{Im } f(\kappa))$ is a real hypersurface away from $\kappa$ and $v$. $H$ can be oriented as follows: $\omega_H \in \Lambda^{top} T^* H$ is a positive volume form if and only if $-d(\text{Im } f) \wedge \omega_H$ is a positive volume form of $X$. Then $W^0_v$ and $W^s_\kappa$ intersect transversely in $H$ and $W^0_v \cap W^s_\kappa$ has an induced orientation. We define $\text{sign}_H(\sigma) = 1$ (resp. $-1$) if $\sigma$ with the flow orientation is negative (resp. positive).

Lemma 9.4. For every $\sigma \in \mathcal{S}(v, \kappa)$, $\text{sign}_\infty(\sigma) = \text{sign}_H(\sigma)$; further, $\# \mathcal{S}(v, \kappa) = \Delta_v \cdot \Delta_\kappa$.

Proof. For each $p \in \Delta_v \cap \Delta_\kappa$, if $\omega_v$ and $\omega_\kappa$ are positive volume forms on $T_p \Delta_v$ and $T_p \Delta_\kappa$, then we define $\text{sign}_{top}(p)$ to be $1$ (resp. $-1$) if $\omega_v \wedge \omega_\kappa$ is positive (resp. negative) with respect to the complex orientation of $T_p f^{-1}(f(p))$. Each $p$ corresponds to a unique $[\sigma] \in \mathcal{S}(v, \kappa)$.

We have

$$
\text{sign}_{top}(p) = \text{sign}(\omega_v \wedge \omega_\kappa)
$$

$$
= \text{sign}(d(\text{Re } f) \wedge d(\text{Im } f) \wedge \omega_v \wedge \omega_\kappa)
$$

$$
= \text{sign}(d(\text{Re } f) \wedge \omega_v \wedge \omega_\kappa)
$$

$$
= \text{sign}_H(\sigma).
$$
The last line follows from the fact that along the flow $\text{Re} f$ decreases. Therefore

$$
\Delta_v \cdot \Delta_{v,\kappa} = \sum_{p \in \Delta_v \cap \Delta_{v,\kappa}} \text{sign}_{\text{top}}(p) = \sum_{[\sigma] \in \mathcal{S}(v,\kappa)} \text{sign}_H(\sigma).
$$

On the other hand, $\text{sign}_{\infty}(\sigma)$ depends on whether $\hat{\sigma} \otimes (J\hat{\sigma})$ is positive or negative in $\det D_\sigma$. Via the isomorphisms (9.6) and (9.7), for $T \in (0,1]$, this element is identified with

$$(\hat{\sigma}_-, \hat{\sigma}_+) \otimes (T \hat{\sigma}_0 \cap -J\hat{\sigma} \cap 0) \in \det \ker \tilde{D}_\sigma^T \otimes (\det \coker \tilde{D}_\sigma^T)^{\vee} \quad (9.9)$$

Moreover, $(\hat{\sigma}_-, \hat{\sigma}_+) \subset \ker \tilde{D}_\sigma^T$, and it can be extended to a basis of $\ker \tilde{D}_\sigma^T$ by

$$(\hat{\sigma}_-, \hat{\sigma}_+), (-\hat{\sigma}_-, \hat{\sigma}_+), (v_1, 0), \ldots, (v_{N-1}, 0), (0, w_1), \ldots, (0, w_{N-1}),$$

such that $\hat{\sigma}_-, v_1, \ldots, v_{N-1}$ (resp. $-\hat{\sigma}_+, w_1, \ldots, w_{N-1}$) form an oriented basis of $T_p \mathcal{W}^u = \ker D_{\sigma_-}$ (resp. $T_p \mathcal{W}^s = \ker D_{\sigma_+}$). Then by Lemma 9.1, for $T \in [0,1]$, the sign of (9.9) in $\det \tilde{D}_\sigma^T$ is the same as

$$
\begin{align*}
\left( \left( \hat{\sigma}_- \cap \hat{\sigma}_+ \right) \bigwedge_{i=1}^{N-1} (v_i, 0) \bigwedge_{i=1}^{N-1} (0, w_i) \bigwedge (\hat{\sigma}_-, \hat{\sigma}_+) \right) \\
\otimes \left( \tilde{D}_\sigma^T (\hat{\sigma}_- \cap \hat{\sigma}_+) \bigwedge_{i=1}^{N-1} \tilde{D}_\sigma^T (v_i, 0) \bigwedge_{i=1}^{N-1} \tilde{D}_\sigma^T (0, w_i) \bigwedge (T J\hat{\sigma} \cap -J\hat{\sigma} \cap 0) \right) \\
\sim \left( (-1)^{N-1} \hat{\sigma}_- \bigwedge_{i=1}^{N-1} v_i \bigwedge (-\hat{\sigma}_+) \bigwedge_{i=1}^{N-1} w_i \right) \otimes \left( (-J\hat{\sigma}(0)) \bigwedge \hat{\sigma}(0) \bigwedge \left( \bigwedge_{i=1}^{N-1} v_i \bigwedge \bigwedge_{i=1}^{N-1} (-w_i) \right) \right) \\
\sim \left( (-J\hat{\sigma}(0)) \bigwedge \hat{\sigma}(0) \bigwedge \left( \bigwedge_{i=1}^{N-1} v_i \bigwedge \bigwedge_{i=1}^{N-1} w_i \right) \right)
\end{align*}
$$

Since $\hat{\sigma}(0) = -\nabla \text{Re} f, J\hat{\sigma}(0) = -\nabla \text{Im} f$, we see that the last sign is the same as $\text{sign}_H(\sigma)$. \qed

Now consider a family of holomorphic functions $f_\alpha : X \to \mathbb{C}$ with $\alpha \in (-\epsilon, \epsilon)$. Let the critical points and critical values of $f_\alpha$ be

$$
\kappa_{k,\alpha} \in \text{Crit} f_\alpha, \ \beta_{k,\alpha} = f_\alpha(\kappa_{k,\alpha}).
$$

We make the following assumptions.

1. For each $\alpha$, $f_\alpha$ is Morse.
2. For $\alpha \neq 0$, $f_\alpha$ is strongly regular, i.e., the imaginary parts $\text{Im} \beta_{k,\alpha}$ are distinct.
3. At $\alpha = 0$, there are two critical points $v_0, \kappa_0 \in \text{Crit} f_0$ such that

$$
\text{Im} f_0(v_0) = \text{Im} f_0(\kappa_0), \ \text{Re} f_0(v_0) > \text{Re} f_0(\kappa_0), \ \left. \frac{d}{d\alpha} \right|_{\alpha=0} (\text{Im} f_\alpha(v_\alpha) - \text{Im} f_\alpha(\kappa_\alpha)) \neq 0.
$$

One can identify the unstable manifold of $v_\alpha$ with the space of solutions to the ODE

$$
\dot{x}(s) + \nabla f_\alpha(x(s)) = 0, \ \lim_{s \to -\infty} x(s) = v_\alpha, \ s \in (-\infty, 0].
$$

We make a consistent choice of orientations of $\mathcal{W}^u_{v_\alpha}$ for all $\alpha$. Then the union $\mathcal{W}^u = \bigcup_{-1 \leq \alpha \leq 1} \mathcal{W}^u_{v_\alpha}$ has an induced orientation. It is not compact because the flow lines can break at $\alpha = 0$. One can compactify it by adding broken ones.

**Proposition 9.5.** Suppose the moduli space of BPS solitons connecting $v_0$ and $\kappa_0$ is regular. Then the space $\mathcal{W}^u_{v_0}$ is an oriented manifold with boundary and the boundary is the disjoint union

$$
\left( W^u_{v_{-1}} \right) \cup \left( - W^u_{v_{-1}} \right) \cup \left( - (1)^T \mathcal{S}(v_0, \kappa_0) \times W^u_{\kappa_0} \right).
$$
Here $S(v_0, \kappa_0)$ and $W^u_{\kappa_0}$ have their own orientations.

This proposition is proved right away. It implies the following Picard-Lefschetz formula.

**Corollary 9.6 (Picard-Lefschetz formula).** For $M$ sufficiently large, we have

$$[W^u_{v_{i+1}}] - [W^u_{v_{i-1}}] = (-1)^i \# S(v_0, \kappa_0) \left[ W^u_{\kappa_0} \right] \in H_N(X, (Rf)^{-1}(\mathbb{R}^d)).$$

### 9.3. Proof of Proposition 9.5.

Consider a Banach manifold $\tilde{B}$ consisting of pairs $(\alpha, \psi)$ with $\alpha \in [-1, 1]$ and $\psi \in W^{1,p}_0(\mathbb{R}_+, X)$ such that $\psi(s)$ is asymptotic to $v_\alpha$ in the $W^{1,p}$-sense. Consider the Banach bundle $\tilde{E} \to \tilde{B}$ whose fibre over $\psi = (\alpha, \psi)$ is $L^p(\mathbb{R}_+, \psi^*TX)$. Let $\tilde{F} : \tilde{B} \to \tilde{E}$ be the section $\tilde{F}(\alpha, \psi) = \dot{\psi} + \nabla f_\alpha(\psi)$. Then $W^u_\psi \simeq \tilde{F}^{-1}(0)$ and the orientation on $W^u_\psi$ is given by the orientation on the linearized operator

$$\tilde{D}_{\alpha, \psi} : \mathbb{R} \times W^{1,p}(\mathbb{R}, \psi^*TX) \to L^p(\mathbb{R}, \psi^*TX).$$

We need to construct a local chart near any singular flow line. Let $\sigma \in \hat{S}(v_0, \kappa_0)$ and $b \in W^u_{\kappa_0}$. For $\alpha$ sufficiently small, there are families of vectors

$$\eta^-_\alpha \in T_{v_0}X, \quad \eta^+_\alpha \in T_{\kappa_0}X$$

such that

$$\exp_{v_0} \eta^-_\alpha = v_\alpha, \quad \exp_{\kappa_0} \eta^+_\alpha = \kappa_\alpha.$$

For $S > 0$ sufficiently large, we can extend $\eta^\pm_\alpha$ to vector fields

$$\tilde{\eta}^-_\alpha \in \Gamma((\mathbb{R}^d, -S], \sigma^*TX), \quad \tilde{\eta}^+_\alpha \in \Gamma([S, +\infty), \sigma^*TX),$$

which are asymptotic to $\eta^\pm_\alpha$ in the $W^{1,p}$-sense. Choosing cut-off functions $\beta_-$ (supported on $(-\infty, -S + 1]$) and $\beta_+$ (supported in $[S - 1, +\infty)$), define

$$\tilde{\eta}^\alpha = \beta_- \tilde{\eta}^-_\alpha + \beta_+ \tilde{\eta}^+_\alpha.$$

Denote

$$\tilde{\eta}^\pm_\alpha = \frac{d}{d\alpha} \bigg|_{\alpha = 0} \tilde{\eta}^\pm_\alpha, \quad \tilde{\eta} = \frac{d}{d\alpha} \bigg|_{\alpha = 0} \tilde{\eta}^\alpha.$$

Then we have the linearized operator

$$\tilde{D}_\sigma : \mathbb{R}\{\partial_{\alpha}\} \times W^{1,p}(\mathbb{R}, \sigma^*TX) \to L^p(\mathbb{R}, \sigma^*TX),$$

$$\tilde{D}_\sigma(\alpha \partial_{\alpha}, \xi) = D_\sigma(\xi) + L_\sigma(\alpha \partial_{\alpha})$$

where $L_\sigma(\alpha \partial_{\alpha}) = D_\sigma(a \hat{\eta})$. Since we assumed that $\sigma$ is maximally transverse, it is easy to see that $\tilde{D}_\sigma$ is surjective.

We can perform the gluing construction in the standard way. The following lemma is left to the reader.

**Lemma 9.7.** There exists $\epsilon > 0$, such that for each $t \in (0, \epsilon)$ and $\sigma' \in B_t(\sigma, \hat{S}(v_0, \kappa_0))$, $b' \in B_t(b, W^u_{\kappa_0})$, there is a solution

$$\hat{\psi}_{t, \sigma', b'} = (\psi_{t, \sigma', b'}, \alpha_{t, \sigma', b'}) \in \tilde{F}^{-1}(0).$$

1. The map $\Phi : (0, \epsilon) \times \{\sigma\} \times B_t(b, W^u_{\kappa_0}) \to W^u_\sigma$ defined by

$$\Phi(t, \sigma, b') := \hat{\psi}_{t, \sigma, b'}$$

extends continuously as $t \to 0$ to $(\sigma, b')$ and the extension $\Phi : [0, \epsilon) \times \{\sigma\} \times B_t(b, W^u_{\kappa_0}) \to W^u_\sigma$ is a homeomorphism onto a neighborhood of the singular object $(\sigma, b)$.

2. Fix $t \in (0, \epsilon)$. The map $\Psi : \{t\} \times B_t(\sigma, \hat{S}(v_0, \kappa_0)) \times B_t(b, W^u_{\kappa_0}) \to W^u_\sigma$ defined by

$$\Psi(t, \sigma', b') := \hat{\psi}_{t, \sigma', b'}$$

is a homeomorphism onto a neighborhood of $\hat{\psi}_{t, \sigma, b}$.
(3) The orientation on $W^u_\psi$ induced from the chart $\Phi$ and that induced from $\Psi$ are the same.

We call the orientation on $W^u_\psi$ in Lemma 9.7 “induced from the boundary chart”. We need to see if it is consistent with the interior orientation.

Indeed, define $\hat{D}_\sigma^T = D_\sigma + TL_\sigma$ and we have isomorphisms
\[
\det \hat{D}_\sigma \simeq \det \hat{D}_\sigma^T \simeq \det \hat{D}_\sigma^0.
\]
Using Lemma 9.1, the direction represented by $(0, \dot{\sigma}) \in \ker \hat{D}_\sigma$ is identified with
\[
(\partial_\alpha, 0) \wedge (0, \dot{\sigma}) \otimes L_\sigma(\partial_\alpha, 0) \sim \text{sign}\langle L_\sigma(\partial_\alpha), J\dot{\sigma}\rangle(\partial_\alpha, 0) \wedge (0, \dot{\sigma}) \otimes (J\dot{\sigma}) =: \theta^0 \in \det \hat{D}_\sigma^0.
\]
Choose a positive volume form $\omega_\psi$ on $\ker D_\psi$ for any $(\psi, \alpha)$ in the image of the $\Phi$ of (9.10). Since have the canonical identification $\det D_\sigma \otimes \det D_\sigma \simeq \det D_\psi$, $\theta^0 \wedge \omega_\psi \in \det \hat{D}_\sigma^0 \otimes \det D_\psi$ is identified canonically with
\[
\left(\text{sign}\langle L_\sigma(\partial_\alpha), J\dot{\sigma}\rangle\right)\left(\text{sign}_\infty(\sigma)\right)(\partial_\alpha, 0) \otimes \omega_\psi \in \det \hat{D}_\alpha,\psi.
\]
Hence the oriented boundary of $\overline{W^u_\psi}$ is
\[
\overline{\partial W^u_\psi} = \left(W^u_{v_+}\right) \cup \left(-W^u_{v_-}\right) \cup \left(-\text{sign}\langle L_\sigma(\partial_\alpha), J\dot{\sigma}\rangle(\partial_\alpha, 0) \otimes \omega_\psi\right).
\]
Lastly, we need to evaluate the sign of $\langle L_\sigma(\partial_\alpha), J\dot{\sigma}\rangle$. We assume that
\[
\frac{d}{d\alpha}\bigg|_{\alpha=0} (\text{Im} f_a(v_\alpha) - f_a(\kappa_\alpha)) > 0 \iff (-1)^{\hat{f}} = +1.
\]

since the other case is opposite. Then by definition,
\[
L_\sigma(\partial_\alpha) = \nabla_s \bar{\eta} + \nabla^2 f(\sigma)\bar{\eta} - \nabla_s (\beta_- \bar{\eta}_- + \beta_+ \bar{\eta}_+) + \nabla^2 f(\sigma)(\beta_- \bar{\eta}_- + \beta_+ \bar{\eta}_+) = \beta_- \bar{\eta}_- + \beta_+ \bar{\eta}_+.
\]
Moreover, we know that $J\dot{\sigma}(s) = -\nabla \text{Im} f(\sigma(s))$. Therefore,
\[
\langle L_\sigma(\partial_\alpha), J\dot{\sigma}\rangle = \langle \beta_- \bar{\eta}_- - \nabla \text{Im} f, \beta_+ \bar{\eta}_+ \rangle > 0.
\]

9.4. Boundary orientation of soliton solutions. Now we prove Theorem 4.11, which is essentially to identify the boundary orientation on $\mathcal{M}_a^{\beta}$. Let $a \in \mathcal{M}_a^{\beta}$ be represented by a soliton solution $X = (X, \sigma, 0)$. We assume that $X$ is regular and $\sigma$ is maximally transverse, which is enough to study orientation. Hence as in Section 5, we can constructed a boundary chart
\[
K_a = (U_a, E_a, S_a, \psi_a, F_a).
\]
Here $U_a = [0, \epsilon) \times B_{\infty}((\ker(D_X F)) \times (-\epsilon, \epsilon)$, and $E_a$ is spanned by $J\dot{\sigma}$. Let the coordinates of $U_a$ be $(t, v_L, \alpha)$. Then for $t > 0$, there is a family of smooth solutions
\[
Y_{t, v_L, \alpha} \in \mathcal{M}_\kappa.
\]

Similar to Lemma 9.7, one can fix $t$ and vary $\sigma$ to obtain a different local coordinates of the moduli space, i.e., we take
\[
U'_a := \{t_0\} \times B_{\infty}\left((\ker(D_X F)) \times B_{\infty}\left(\sigma, S_{\overline{W}}(\sigma)(v^X, \kappa^X)\right) \times (-\epsilon, \epsilon)\right).
\]
But contrary to Lemma 9.7, the $\partial_t$-direction is replaced not by $\dot{\sigma}$, but by $-\dot{\sigma}$ (which is a tangent vector in the third factor above). Hence the sign appeared in Theorem 4.11 is exactly opposite to that appeared in Proposition 9.5.

Figure 1 shows the wall-crossing phenomenon for the unstable manifolds $W^u_{v_\alpha}$ and the moduli space $\mathcal{M}_{\kappa_\alpha}$. 

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Appendix A. Topological Virtual Orbifolds and Virtual Cycles

In this appendix we provide a general framework of constructing virtual fundamental cycles associated to moduli problems. Such constructions, usually called “virtual technique”, has a long history since it firstly appeared in algebraic Gromov-Witten theory by [LT98a]. The current method is based on the topological approach of [LT98b].

A.1. Topological orbifolds and orbibundles. We only consider effective orbifolds. We use Satake’s convention of V-manifolds [Sat56] instead of groupoids to treat orbifolds, and only discuss it in the topological category.

Let $X$ be a paracompact Hausdorff topological space and $x \in X$. A topological orbifold chart (with boundary) of $x$ consists of a triple $(V_x, \Gamma_x, \varphi_x)$, where $V_x$ is a topological manifold with possibly empty boundary $\partial V_x$, $\Gamma_x$ is a finite group acting continuously and effectively on $(V_x, \partial V_x)$ and $\varphi_x : V_x/\Gamma_x \to X$ is a continuous map which is a homeomorphism onto an open neighborhood of $x$. If $p \in V_x$, take $\Gamma_p = (\Gamma_x)_p \subset \Gamma_x$ the stabilizer of $p$. Let $V_p \subset V_x$ be a $\Gamma_p$-invariant neighborhood of $p$. Then there is an induced chart (which we call a subchart) $(V_p, \Gamma_p, \varphi_p)$, where $\varphi_p$ is the composition

$$\varphi_p : V_p/\Gamma_p \to V_x/\Gamma_x \to X.$$ 

Two charts $(V_x, \Gamma_x, \varphi_x)$ and $(V_y, \Gamma_y, \varphi_y)$ are compatible if for any $p \in V_x$ and $q \in V_y$ with $\varphi_x(p) = \varphi_y(q) \in X$, there exist an isomorphism $\Gamma_p \to \Gamma_q$, subcharts $V_p \ni p$, $V_q \ni q$ and an equivariant homeomorphism $\varphi_{pq} : (V_p, \partial V_p) \simeq (V_q, \partial V_q)$.

A topological orbifold atlas of $X$ is a set $\{(V_\alpha, \Gamma_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of topological orbifold charts of $X$ such that $X = \bigcup_{\alpha \in A} \text{Im} \varphi_\alpha$ and for each pair $\alpha, \beta \in A$, $(V_\alpha, \Gamma_\alpha, \varphi_\alpha), (V_\beta, \Gamma_\beta, \varphi_\beta)$ are compatible. Two atlases are equivalent if the union of them is still an atlas. A structure of topological orbifold (with boundary) is an equivalence class of atlases. A topological orbifold (with boundary) is a paracompact Hausdorff space with a structure of topological orbifold (with boundary). We will often skip the term “topological” in the remaining of this appendix as well as in the main body of this paper.
Embeddings. Recall that, if $N, M$ are topological manifolds and $f : N \to M$ is a continuous map which is a homeomorphism onto its image, then $f$ is called an **embedding** if: 1) for each $p \in N \setminus \partial N$, there is a chart $\varphi : U \to \mathbb{R}^{n+k}$ of $f(p)$ such that $\varphi(f(N) \cap U) \subset \mathbb{R}^n \times \{0\}$; 2) for each $p \in \partial N$, there is a boundary chart $\varphi : U \to \mathbb{R}^{n+k-1} \times [0, \epsilon)$ of $f(p)$ such that $\varphi(f(N) \cap U) \subset \mathbb{R}^{n-1} \times [0, \epsilon)$.  

**Definition A.1.** Let $N, M$ be orbifolds and $f : N \to M$ is a continuous map which is a homeomorphism onto its image. $f$ is called an embedding if for any pair of orbifold charts, $(V, \Gamma, \varphi)$ of $N$ and $(V', \Gamma', \varphi')$ of $M$, any pair of points $p \in V, p' \in V'$ with $f(\varphi(p)) = \varphi'(p')$, there are subcharts $(V_p, \Gamma_p, \varphi_p) \subset (V, \Gamma, \varphi)$ and $(V_{p'}, \Gamma_{p'}, \varphi_{p'}) \subset (V', \Gamma', \varphi')$, an isomorphism $\Gamma_p \simeq \Gamma_{p'}$ and an equivariant embedding $\tilde{f}_{pp'} : V_p \to V_{p'}$ such that the following diagram commutes.

$$
\begin{array}{ccc}
V_p & \xrightarrow{\tilde{f}_{pp'}} & V_{p'} \\
\varphi_p & \downarrow & \varphi_{p'} \\
N & \xrightarrow{f} & M
\end{array}
$$

**Orbifold bundles.** Let $E, N$ be orbifolds and $\pi : E \to N$ is a continuous map.

**Definition A.2.** A vector bundle chart (resp. disk bundle chart) of $\pi : E \to N$ (of rank $n$) is a tuple $(V, \mathbb{R}^n, \Gamma, \varphi)$ (resp. $(V, \mathbb{D}^n, \Gamma, \varphi)$), where $(V, \Gamma, \varphi)$ is a chart of $N$ and $(V \times \mathbb{R}^n, \Gamma, \varphi)$ (resp. $(V \times \mathbb{D}^n, \Gamma, \varphi)$) is a chart of $E$, where $\Gamma$ acts on the $\mathbb{R}^n$ (resp. $\mathbb{D}^n$) via a linear representation $\Gamma \to GL(\mathbb{R}^n)$ (resp. $\Gamma \to O(n)$). The compatibility condition is required, namely, the following diagram commutes.

$$
\begin{array}{ccc}
V \times \mathbb{R}^n / \Gamma & \xrightarrow{\theta} & E \\
\downarrow & & \downarrow \pi \\
V / \Gamma & \xrightarrow{\varphi} & N
\end{array}
\quad \text{resp.} \quad
\begin{array}{ccc}
V \times \mathbb{D}^n / \Gamma & \xrightarrow{\theta} & E \\
\downarrow & & \downarrow \pi \\
V / \Gamma & \xrightarrow{\varphi} & N
\end{array}
$$

If $(V_p, \Gamma_p, \varphi_p)$ is a subchart of $(V, \Gamma, \varphi)$, then one can restrict the bundle chart to $\tilde{\pi}^{-1}(V_p)$.

We can define the notion of compatibility between bundle charts, the notion of orbifold bundle structures and the notion of orbifold bundles in a similar fashion as in the case of orbifolds. We skip the details.

Suppose $f : N \to M$ is an orbifold embedding. Let $E \to N, F \to M$ be orbifold vector bundles. A bundle map covering $f$ is a continuous map $\tilde{f} : E \to F$ such that for any bundle chart $(V, \mathbb{R}^n, \Gamma, \varphi)$ of $E$ and any bundle chart $(V', \mathbb{R}^m, \Gamma', \varphi')$ of $F$, any pair of points $p \in V$, $p' \in V'$ with $f(\varphi(p)) = \varphi'(p')$, there exist subcharts $V_p \subset V, V_{p'} \subset V'$, an isomorphism $\Gamma_p \simeq \Gamma_{p'}$, and an equivariant map $\tilde{f}_{pp'} : V_p \times \mathbb{R}^n \to V_{p'} \times \mathbb{R}^m$ such that the following diagram commutes.

$$
\begin{array}{ccc}
V_p \times \mathbb{R}^n & \xrightarrow{\tilde{f}_{pp'}} & V_{p'} \times \mathbb{R}^m \\
\pi & \downarrow & \pi' \\
V_p & \xrightarrow{f_{pp'}} & V_{p'}
\end{array}
$$

Of course we require that $\tilde{f}_{pp'}$ on each fibre is an injective linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We also regard $\tilde{f}(E)$ as a subbundle of $F$. 

---

3This is also referred to as a “locally flat embedding”
Normal bundles of topological embeddings. Unlike the smooth case, embedded topological submanifold/orbifold doesn’t necessarily have a tubular neighborhood.

**Definition A.3.** Let $f : Y \to X$ be a topological embedding.

1. A tubular neighborhood is a map $\pi : N \to Y$ where $N$ is a neighborhood of $f(Y)$ inside $X$ such that it contains another neighborhood $N'$ of $f(Y)$ such that $\pi|_{N'} : N' \to Y$ is a disk bundle over $Y$.

2. Two tubular neighborhoods $\pi_1 : N_1 \to Y$ and $\pi_2 : N_2 \to Y$ are equivalent if there is another tubular neighborhood $\pi_3 : N_3 \to Y$ with $N_3 \subset N_1 \cap N_2$ and $\pi_1|_{N_3} = \pi_2|_{N_3} = \pi_3$.

3. A germ of tubular neighborhoods is an equivalence class of tubular neighborhoods.

**Remark A.4.** By a “disk bundle”, we always mean a fibre bundle which can be identified with a disk bundle of some vector bundle.

A **strong embedding** from $Y$ to $X$ is a topological embedding $f : Y \to X$ together with a germ of tubular neighborhoods $f^+$. An open embedding is automatically a strong embedding. Moreover, compositions of strong embeddings are still strong embeddings.

**Lemma A.5.** If $(g, g^+) : Z \to Y$, $(f, f^+) : Y \to X$ are strong embeddings, then there is a strong embedding $(f \circ g, (f \circ g)^+)$ where the germ $(f \circ g)^+$ is naturally induced from $f^+$ and $g^+$.

**Proof.** Obviously $f \circ g$ is a topological embedding. On the other hand, let $g^+$ be represented by a tubular neighborhood $\pi_g : N_g \to Z$ and $f^+$ be represented by $\pi_f : N_f \to Y$. Then consider $\pi_f^{-1}(N_g)$, which is a neighborhood of $f(g(Z))$ inside $X$. Define $\pi : \pi_f^{-1}(N_g) \to X$ by the composition $\pi_g \circ \pi_f$. It is a tubular neighborhood of $f \circ g$, and the germ it represents only depends on $f^+$ and $g^+$.

We consider the embedding of orbifold vector bundles. Let $E_1 \to U_1$, $E_2 \to U_2$ be orbifold vector bundles. A bundle embedding of $E_1 \to U_1$ into $E_2 \to U_2$ is a pair $(\phi, \hat{\phi})$ where $\phi : U_1 \to U_2$ is a topological embedding, and $\hat{\phi} : E_1 \to E_2$ is an injective bundle map which covers $\phi$. A tubular neighborhood of the bundle embedding $(\phi, \hat{\phi})$ is a tubular neighborhood $\nu : N \to \phi(U_1)$ of $\phi : U_1 \to U_2$ as well as a projection

$$\hat{\nu} : E_2|_{N} \to E_2|_{\phi(U_1)}$$

whose restriction to $\phi(U_1)$ is the identity. Two tubular neighborhoods $(\nu : N \to \phi(U_1), \hat{\nu})$ and $(\nu' : N' \to \phi(U_1), \hat{\nu}')$ are equivalent if there is another tubular neighborhood $\nu'' : N'' \to \phi(U_1)$ such that $N'' \subset N \cap N'$ and

$$\nu|_{N''} = \nu'|_{N''}, \ \hat{\nu}|_{E_2|_{N''}} = \hat{\nu}'|_{E_2|_{N''}}.$$

An equivalence class of tubular neighborhoods is called a germ. A strong embedding of orbifold vector bundles is an embedding together with a germ of tubular neighborhoods.

**Multisections and perturbations.** We recall the definition of multisections. Let $A$, $B$ be sets, $l \in \mathbb{N}$, and $S^l(B)$ be the $l$-fold symmetric product of $B$. An $l$-multimap $f$ from $A$ to $B$ is a map $f : A \to S^l(B)$. For another $l' \in \mathbb{N}$, there is a natural map

$$m_{l'} : S^l(B) \to S^{l'}(B). \quad (A.1)$$

Hence $f$ can be identified with an $ll'$-multimap $m_{l'} \circ f$. If $\Gamma = \{g_1, \ldots, g_s\}$ acts on $A$ and $B$, then it acts on $S^l(B)$ for any $l \in \mathbb{N}$ and we can talk about $\Gamma$-equivariant multimeaps from $A$ to $B$.

Let $X$ be a topological orbifold and $E \to X$ be a vector bundle. A **representative** of a multisection $s$ of $E$ is a collection

$$\{ (V_\alpha \times \mathbb{R}^k, \Gamma_\alpha, \hat{\varphi}_\alpha, \varphi_\alpha; s_\alpha, t_\alpha) \mid \alpha \in \mathcal{A} \}$$
where \( \{(V_α × \mathbb{R}^k, \Gamma_α, \hat{ϕ}_α, ϕ_α) \mid α ∈ \mathcal{A}\} \) is an atlas of \((E, X)\) and \(s_α : V_α → S^{l_α}(\mathbb{R}^k)\) is a \(\Gamma_α\)-equivariant continuous \(l_α\)-multimap, satisfying the following compatibility condition.

- For any \(p ∈ V_α\) and \(p' ∈ V_{α'}\) with \(ϕ_α(p) = ϕ_{α'}(p') ∈ X\), there exist subcharts \(V_p ⊂ V_α\), \(V_{p'} ⊂ V_{α'}\), an isomorphism \((\hat{ϕ}_{pp'}, ϕ_{pp'})\) of subcharts, a common multiple \(l\) of \(l_α\) and \(l_{α'}\), such that
  \[
  \hat{ϕ}_{pp'} \circ m_{l/l_{α'}} \circ s_α|_{V_p} = m_{l/l_{α'}} \circ s_{α'}|_{V_{p'}} \circ ϕ_{pp'}.
  \]

Two representatives are equivalent if their union is also a representative. An equivalence class is called a multisection of \(E\).

On the other hand, for a multisection \(s\) of \(E → X\) and \(p ∈ X\), there exists a \textbf{local representative} \((V_p × \mathbb{R}^k, \Gamma_p, φ_p, ϕ_p; s_p, l_p)\) where \((V_p × \mathbb{R}^k, \Gamma_p, φ_p, ϕ_p)\) is a local chart around \(p\) and \(s_p : V_p → S^{l_p}(\mathbb{R}^k)\) is a \(\Gamma_p\)-equivariant continuous \(l_p\)-multimap. \(s\) is called \textbf{locally liftable} if for any \(p ∈ X\), there is a local representative \(s_p : V_p → S^{l_p}(\mathbb{R}^k)\) and continuous maps \(s_p^{(1)}, \ldots, s_p^{(l_p)} : V_p → \mathbb{R}^k\) such that
  \[
  s_p(x) = [s_p^{(1)}(x), \ldots, s_p^{(l_p)}(x)].
  \]

We call such a local representative \textbf{liftable} and \(s_p^{(1)}, \ldots, s_p^{(l_p)}\) are its \textbf{branches}.

**Lemma A.6.** Let \(V\) and \(\mathbb{R}^k\) be acted by a finite group \(Γ = \{g_1, \ldots, g_n\}\). For a liftable \(Γ\)-equivariant \(l\)-multimap \(s : V → S^{l}(\mathbb{R}^k)\), written as
  \[
  s(x) = [s^{(1)}(x), \ldots, s^{(l)}(x)],
  \]
define \(s_Γ : V → S^{nl}(\mathbb{R}^k)\) by
  \[
  s_Γ(x) = [g_β^{-1} s^{(j)}(g_β x)]_{β=1,\ldots,n}^{j=1,\ldots,s}.
  \]
Then \(s_Γ\) is equivariant and \(s_Γ = m_α \circ s\), where \(m_α : S^{l}(\mathbb{R}^k) → S^{nl}(\mathbb{R}^k)\) is the map \((A.1)\).

**Proof.** It is obvious that \(s_Γ\) is equivariant. For any \(x ∈ V\), \(g_β ∈ Γ\),
  \[
  [g_β^{-1} s^{(1)}(g_β x), \ldots, g_β^{-1} s^{(l)}(g_β x)] = [s^{(1)}(x), \ldots, s^{(l)}(x)].
  \]
So \(g_β^{-1} s^{(1)}(g_β x), \ldots, g_β^{-1} s^{(l)}(g_β x)\) is a permutation of the values \(s^{(1)}(x), \ldots, s^{(l)}(x)\). So there are \(n\) permutations, indexed by \(g_β ∈ Γ\), which altogether give the same element in \(S^{nl}(\mathbb{R}^k)\) as repeating the values \(s^{(1)}(x), \ldots, s^{(l)}(x)\) for \(n\) times.

**Definition A.7.**

1. Let \(U ⊂ \mathbb{R}^m\) be an open subset and \(f : U → \mathbb{R}^n\) be a continuous map. We say that \(f\) is \textbf{transverse} to 0 if \(f^{-1}(0)\) is a locally flat submanifold of \(\mathbb{R}^m\) and for each \(p ∈ f^{-1}(0)\), each locally flat chart \(ϕ : U_p → \mathbb{R}^m\) such that \(ϕ(f^{-1}(0) ∩ U_p) ⊂ \mathbb{R}^m−n\), the restriction
  \[
  f|_{ϕ^{-1}(\mathbb{R}^n)} : ϕ^{-1}(\mathbb{R}^n) → \mathbb{R}^n
  \]
is a local homeomorphism.

2. Let \(X\) be a topological orbifold with boundary and \(E → X\) is an orbibundle. A multisection \(s\) of \(E\) is called \textbf{transverse}, if it is locally liftable and for each liftable local representative \(s_p : V_p → S^{l_p}(\mathbb{R}^k)\), each branch of \(s_p\) is transverse to 0. Here being transverse means both transverse in the interior and transverse on the boundary.

**Proposition A.8.** Suppose \(M\) is an orbifold with boundary, \(E → M\) is an orbifold vector bundle, and \(s\) is a continuous section of \(E\). Let \(Z ⊂ K\) be compact subsets of \(M\).

Suppose \(s_{Z,i}\) is a sequence of multisections of \(E\) defined over an open neighborhood \(U_Z\) of \(Z\) such that \(\lim_{i→∞} s_{Z,i} = s\) uniformly and for each \(i\), \(s_{Z,i}\) is transverse over \(U_Z\). Then there exists a sequence of multisections \(s_{K,i}\) of \(E\) defined over an open neighborhood \(U_K\) of \(K\) such that

1. \(s_{K,i}|_{Z} = s_{Z,i}|_{Z}\);
2. \(\lim_{i→∞} s_{K,i} = s\) uniformly over \(U_K\);
(3) for each $i$, $s_{K,i}$ is transverse over $U_K$.

Proof. First, we choose a cut-off function $\chi_Z$ supported in $U_Z$ and is identically one over a smaller open neighborhood $U'_Z$ of $Z$. By replacing $s_{Z,i}$ by $(1 - \chi_Z)s + \chi_Zs_{Z,i}$ and replacing $U_Z$ by $U'_Z$, we can assume that $s_{Z,i}$ is defined everywhere on $M$ and is locally liftable.

Next, for each point $x \in K$, there is a local chart $(V_x \times \mathbb{R}^k, \Gamma_x, \bar{\varphi}_x, \varphi_x)$ (which may depend on $i$) of $E$ such that $s_{Z,i}$ is liftable over $\varphi(V_x)$. By the compactness of $K$, one can find finitely many such charts $\{(V_\alpha \times \mathbb{R}^k, \Gamma_\alpha, \bar{\varphi}_\alpha, \varphi_\alpha) \mid \alpha = 1, \ldots, m\}$ of $E$ such that

$$K \subset U_Z \cup \bigcup_{\alpha=1}^m \varphi_\alpha(V_\alpha), \ Z \cap \bigcup_{\alpha=1}^m \varphi_\alpha(V_\alpha) = \emptyset.$$  

If we can prove that, for the fixed $i$, in the case $m = 1$, we can construct a transverse multisection $s_{K,i}$ on $U_Z \cup \varphi_1(V_1)$ whose restriction to $Z$ coincides with $s_{Z,i}$ and whose difference from $s$ is arbitrarily small, then by an induction argument, the proposition can be proved. When $m = 1$, we remove the index $\alpha$ in notations. Assume $\Gamma = \{g_1, \ldots, g_n\}$.

Since $s_{Z,i}$ is liftable over $\varphi(V)$, one can write

$$s_{Z,i} = [s_{Z,i}^{(1)}, \ldots, s_{Z,i}^{(j)}]$$

where $s_{Z,i}^{(j)} : V \to \mathbb{R}^k$ is a single valued continuous function. By the main theorem of [Qui88], there exist continuous functions $\rho_i^{(j)} : V \to \mathbb{R}^k$ which is zero near $\varphi^{-1}(U_Z^i \cap \varphi(V))$ for some smaller $U_Z^i \subset U_Z$, such that $s_{Z,i}^{(j)} + \rho_i^{(j)}$ is transverse to zero and we can make $\rho_i^{(j)}$ as small as possible. Then we define a multimap $s_{K,i} : V \to S^s(\mathbb{R}^k)$ by

$$s_{K,i}(x) = \left[g_{\beta}^{-1}(s_{Z,i}^{(j)} + \rho_i^{(j)})(g_{\beta}x)\right]_{\beta=1, \ldots, n}^{j=1, \ldots, l}.$$  

This is close to the multimap defined by

$$x \mapsto \left[g_{\beta}^{-1}s_{Z,i}^{(j)}(g_{\beta}x)\right]_{\beta=1, \ldots, n}^{j=1, \ldots, l}$$

which, by Lemma A.6, is the same as $m_n \circ s_{Z,i}$. Therefore $s_{K,i}$ extends to $U_Z \cup \varphi(V)$ and satisfies the requirement of this proposition.

A.2. Virtual orbifold atlases. In this subsection, all orbifolds can be realized as effective global quotients as $M/\Gamma$ where $M$ is a topological manifold (with boundary) and $\Gamma$ is a finite group.

Definition A.9. A virtual orbifold patch (patch for short) is a tuple $P = (U, E, S)$ where

(1) $U$ is an orbifold (with boundary).
(2) $E \to U$ is an orbifold vector bundle.
(3) $S : U \to E$ is a continuous section.

Definition A.10. Let $X$ be a metrizable space.

(1) A virtual orbifold chart (chart for short) is a tuple $K := (U, E, S, \psi, F)$ where $P := (U, E, S)$ is a patch, $\psi : S^{-1}(0) \to X$ is a homeomorphism onto its image $F \subset X$. $F$ is called the footprint of the chart $K$. The integer $\dim^{\text{vir}} K := \dim U - \text{rank} E$ is called the virtual dimension of $K$.
(2) If $P = (U, E, S)$ is a patch and $U' \subset U$ is a precompact open subset, then we can restrict $P$ to $U'$ in the obvious way, denoted by $P|_{U'}$. $P|_{U'}$ is called a sub-patch of $P$. If $P$ belongs to a chart $K$, then the induced chart, denoted by $K|_{U'}$ is called a sub-chart, or a shrinking of $K$.
**Definition A.11.** Let \( K_i := (U_i, E_i, S_i, \psi_i, F_i), \ i = 1, 2 \) be two charts. An embedding of \( K_1 \) into \( K_2 \) consists of a strong embedding \( \phi_{21} := (\phi_{21}, \hat{\phi}_{21}, \phi^+_{21}) \) of orbifold vector bundles and a projection \( \hat{\pi}_{21} : E_2|_{\phi_{21}(U_i)} \to \hat{\phi}_{21}(E_1) \), which make the following diagrams commute:

\[
\begin{array}{c}
E_1 \xrightarrow{\hat{\pi}_{21}} E_2, \quad S_1^{-1}(0) \xrightarrow{\phi_{21}} S_2^{-1}(0).
\end{array}
\]

Moreover, denote \( E_{21} := \ker(\hat{\pi}_{21}) \subset E_j|_{\phi_{21}(U_i)} \) which is complementary to \( \hat{\phi}_{21}(E_1) \). For any tubular neighborhood \( (\nu_{21} : N_{21} \to \phi_{21}(U_1), \nu_{21}) \) representing the germ \( (\phi^+_{21}, \hat{\phi}^+_{21}) \), consider the composition

\[
N_{21} \xrightarrow{S_{21}} E_{21}|_{N_{21}} \xrightarrow{\nu_{21}} E_2|_{\phi_{21}(U_1)} \xrightarrow{\text{Id}-\hat{\pi}_{21}} E_{21}
\]

which preserves fibres. We require that its restriction to each fibre is a homeomorphism near the origin.

**Lemma A.12.** The composition of two embeddings is still an embedding.

*Proof.* Left to the reader. \( \square \)

**Definition A.13.** Let \( K_i := (U_i, E_i, S_i, \psi_i, F_i), \ (i = 1, 2) \) be two charts. A weak coordinate change from \( K_1 \) to \( K_2 \) is a triple \( T_{21} := (U_1, \phi_{21}, \hat{\pi}_{21}) \), where

1. \( U_2 \subset U_1 \) is an open subset.
2. \( (\phi_{21}, \hat{\pi}_{21}) \) is an embedding from \( K_1|_{U_2} \) to \( K_2 \).
   It is called a strong coordinate change if in addition
3. \( \psi_1 ((S_1)^{-1}(0) \cap U_2) = F_1 \cap F_2 \subset X \).

**Lemma A.14.** Let \( K_i := (U_i, E_i, S_i, \psi_i, F_i), \ (i = 1, 2) \) be two charts and let \( T_{21} := (U_2, \phi_{21}) \) be a weak coordinate change from \( K_1 \) to \( K_2 \). Suppose \( K_1' := K_1|_{U_1'} \) be a shrinking of \( K_1 \). Then the restriction \( T'_{21} := T_{21}|_{U_1' \cap \phi_{21}^{-1}(U_2')} \) is a weak coordinate change from \( K_1' \) to \( K_2' \). Moreover, if \( T_{21} \) is strong, then \( T'_{21} \) is also strong.

*Proof.* Left to the reader. \( \square \)

**Definition A.15.** Let \( X \) be a compact metrizable space. A weak (resp. strong) virtual orbifold atlas of virtual dimension \( k \) on \( X \) is a collection

\( \mathfrak{A} := \{ \{ K_I := (P_I, \psi_I, F_I) \mid I \in \mathcal{I} \}, \{ T_{IJ} = (P_{IJ}, \phi_{IJ}) \mid I \subset J \} \}, \)

where

1. \( (\mathcal{I}, \leq) \) is a finite, partially ordered set.
2. For each \( I \in \mathcal{I} \), \( K_I \) is a virtual orbifold chart of virtual dimension \( k \) on \( X \).
3. For each pair \( I \leq J, T_{IJ} = (U_{IJ}, \phi_{IJ} = (\phi_{IJ}, \hat{\phi}_{IJ}, \phi^+_{IJ})) \) is a weak (resp. strong) coordinate change from \( K_I \) to \( K_J \).

They are subject to the following conditions.

- **(Cocycle Condition)** For \( I \leq J \leq K \in \mathcal{I} \), denote \( U_{KIJ} = U_{KIJ} \cap \phi_{IJ}^{-1}(U_{KJ}) \subset U_I \), then we require
  \[
  \phi_{KIJ}|_{U_{KIJ}} = \phi_{KJ} \circ \phi_{IJ}|_{U_{KIJ}}.
  \]
  More explicitly, denote \( E_{I,KIJ} = E_I|_{U_{KIJ}} \). Then the cocycle condition means
  \[
  \phi_{KIJ}|_{U_{KIJ}} = \phi_{KJ} \circ \phi_{IJ}|_{U_{KIJ}},
  \]
  \[
  \hat{\phi}_{KIJ}|_{E_{I,KIJ}} = \hat{\phi}_{KJ} \circ \hat{\phi}_{IJ}|_{E_{I,KIJ}}.
  \]
(Filtration Condition) For $I \leq J \leq K \in \mathcal{I}$, we have,
\[ \hat{\pi}_K \mid _{\phi_K(U_{KJI})} = \hat{\pi}_I \mid _{\phi_I(U_{KJI})} \circ \hat{\pi}_J \mid _{\phi_J(U_{KJI})} \]
Equivalently, if we denote $E_{JI} = \ker \hat{\pi}_{JI}$, then
\[ E_{KJ} \mid _{\phi_K(U_{KJI})} = E_{KI} \mid _{\phi_K(U_{KJI})} \oplus \hat{\phi}_J(E_{JI} \mid _{\phi_J(U_{KJI})}) \quad \text{(A.2)} \]

(Overlapping Condition) For $I, J \in \mathcal{I}$, we have
\[ \mathcal{F}_I \cap \mathcal{F}_J \neq \emptyset \implies I \leq J \text{ or } J \leq I. \]

All virtual orbifold atlases considered in this paper have definite virtual dimensions, although sometimes we do not explicitly mention it.

**Notation** A.16. If $\tilde{\varphi}_{JI} : E_J \mid _{N_{JI}} \rightarrow E_J \mid _{\phi_{JI}(U_{JI})}$ is a representative of $\tilde{\varphi}_{JI}^+$, then it induces an isomorphism
\[ \hat{\varphi}_{JI} : E_J \mid _{N_{JI}} \cong \nu_J^* (E_J \mid _{\phi_{JI}(U_{JI})}) \]
On the other hand, the projection $\hat{\pi}_{JI} : E_J \mid _{\phi_{JI}(U_{JI})} \rightarrow \hat{\varphi}_{JI}(E_I \mid _{U_{JI}})$ induces a splitting
\[ E_J \mid _{\phi_{JI}(U_{JI})} \cong \hat{\varphi}_{JI}(E_I \mid _{U_{JI}}) \oplus E_{JI}. \]
It can be extended to the tubular neighborhood $N_{JI}$, denoted by
\[ \hat{\varphi}_{JI} : E_J \mid _{N_{JI}} \cong \nu_J^* \hat{\varphi}_{JI}(E_I \mid _{U_{JI}}) \oplus \nu_J^* E_{JI} \quad \text{(A.3)} \]

**A.2.1. Orientations.** Now we discuss orientation. Over a topological manifold $M$ one has a topological vector bundle $\mathfrak{o}_M \rightarrow M$ which is a $\mathbb{Z}_2$-covering. $M$ is called orientable if $\mathfrak{o}_M$ is trivial. A continuous vector bundle $E \rightarrow M$ also has its orientation bundle $\mathfrak{o}_E \rightarrow M$.

**Definition** A.17. (1) A patch $P = (U, E, S)$ is **locally orientable** if $(U, E) = (\tilde{U}/\Gamma, \tilde{E}/\Gamma)$ and for any $\tilde{x} \in S^{-1}(0) \subset \tilde{U}$, the stabilizer $\Gamma_{\tilde{x}}$ acts trivially on the fibre $\mathfrak{o}_{\tilde{U}/\tilde{x}} \otimes \mathfrak{o}_{\tilde{E}/\tilde{x}}$.
(2) If $P$ is locally orientable, then $\mathfrak{o}_{\tilde{U}/\tilde{x}} \otimes \mathfrak{o}_{\tilde{E}/\tilde{x}}$ induces a $\mathbb{Z}_2$-covering over $U$, denoted by $\mathfrak{o}_{P}$. If $\mathfrak{o}_{P}$ is trivial (resp. trivialized), then we say that $P$ is orientable (resp. oriented).
(3) A coordinate change $T_{21} = (U_{21}, \varphi_{21})$ between two oriented patches $P_1 = (U_1, E_1, S_1)$ and $P_2 = (U_2, E_2, S_2)$ is called oriented if $\varphi_{21}$ is compatible with $\mathfrak{o}_{P_1}$ and $\mathfrak{o}_{P_2}$ in the obvious sense.
(4) An atlas $\mathcal{A}$ is oriented if all charts are oriented and all coordinate changes are oriented.

**A.2.2. Boundary.** We consider the induced atlas on the “virtual” boundary. Let $X$ be equipped with a virtual orbifold atlas $\mathcal{X} = \{(K_I \mid I \in \mathcal{I}), (T_{JI} \mid I \leq J \in \mathcal{I})\}$. Denote
\[ \partial F_I = \psi_I(S_I^{-1}(0) \cap \partial U_I), \quad \partial X := \bigcup_{I \in \mathcal{I}} \partial F_I \subset X. \]
It is easy to see that $\partial X$ is closed in $X$ and hence compact. Take
\[ \partial K_I = (\partial U_I, E_I \mid _{\partial U_I, S_I \mid _{\partial U_I}}, \psi_I \mid _{\partial U_I \cap S_I^{-1}(0)}, \partial F_I). \]
This is a virtual orbifold chart of $\partial X$ (which may be empty). Denote $\mathcal{I} = \{I \in \mathcal{I} \mid \partial F_I \neq \emptyset\}$, with the induced partial order $\leq$. Moreover, the strong coordinate changes can be restricted to the boundary. Namely, for each pair $I \leq J \in \partial \mathcal{I}$, denote
\[ \partial T_{JI} := (\partial U_{JI}, \partial \varphi_{(JI)}) := (\partial U_{JI}, \varphi_{JI} \mid _{\partial U_{JI}}). \]
This is a strong coordinate change from $\partial K_I$ to $\partial K_J$. Hence we obtain a strong virtual orbifold atlas
\[ \partial \mathcal{A} := \left( \{\partial K_I \mid I \in \partial \mathcal{I}\}, \{\partial T_{JI} \mid I \leq J \in \partial \mathcal{I}\} \right). \]
If $\mathcal{X}$ is oriented, then it is routine to check that $\partial \mathcal{A}$ has an induced orientation.
A.3. The Virtual Fundamental Chain and Cycle. In order to construct the virtual chain/cycle, we need some more technical preparations.

**Definition A.18.** Let \( \mathfrak{A} := \{ (K_I = (U_I, E_I, S_I, \psi_I, F_I) \mid I \in \mathcal{I} \}, \{ T_{IJ} = (U_{IJ}, \phi_{IJ}) \mid I \leq J \} \) be a strong virtual orbifold atlas on \( X \). A **shrinking** of \( \mathfrak{A} \) is another virtual orbifold atlas \( \mathfrak{A}' = \{ (K'_I = (U'_I, E'_I, S'_I, \psi'_I, F'_I) \mid I \in \mathcal{I} \}, \{ T'_{IJ} = (U'_{IJ}, \phi'_{IJ}) \mid I \leq J \} \) indexed by elements of the same set \( \mathcal{I} \) such that

1. For each \( I \in \mathcal{I} \), \( K'_I \) is a shrinking \( K_I \) of \( K_I \).
2. For each pair \( I \leq J \), \( T'_{IJ} \) is the induced shrinking of \( T_{IJ} \) given by Lemma A.14.

Given a strong virtual orbifold atlas \( \mathfrak{A} := \{ (K_I = (U_I, E_I, S_I, \psi_I, F_I) \mid I \in \mathcal{I} \}, \{ T_{IJ} = (U_{IJ}, \phi_{IJ}) \mid I \leq J \} \), we define a relation \( \gamma \) on the disjoint union \( \bigsqcup_{I \in \mathcal{I}} U_I \) as follows. \( U_I \ni x \gamma y \in U_J \) if one of the following holds.

1. \( I = J \) and \( x = y \);
2. \( I \leq J \), \( x \in U_{IJ} \) and \( y = \phi_{IJ}(x) \);
3. \( J \leq I \), \( y \in U_{IJ} \) and \( x = \phi_{IJ}(y) \).

If \( \mathfrak{A}' \) is a shrinking of \( \mathfrak{A} \), then it is easy to see that the relation \( \gamma' \) on \( \bigsqcup_{I \in \mathcal{I}} U'_I \) defined as above is induced from the relation \( \gamma \) for \( \mathfrak{A} \) via restriction.

**Lemma A.19.** For each virtual orbifold atlas \( \mathfrak{A} \), there exists a shrinking \( \mathfrak{A}' \) of \( \mathfrak{A} \) such that \( \gamma' \) is an equivalence relation.

This lemma is proved in the next subsection. The above condition for the shrinking \( \mathfrak{A}' \) was used in Li-Tian’s approach. Its explicit formulation was pointed out in [Joy07] in the Kuranishi approach.

For a strong atlas \( \mathfrak{A} \), if \( \gamma \) is an equivalence relation, we can form the quotient space

\[ \bigsqcup_{I \in \mathcal{I}} U_I / \gamma. \]

with the quotient topology. \( X \) is then a compact subset of \( \bigsqcup_{I \in \mathcal{I}} U_I \).

**Lemma A.20.** There is a shrinking \( \mathfrak{A}' \) of \( \mathfrak{A} \) such that \( \bigsqcup_{I \in \mathcal{I}} U_I / \gamma \) is metrizable and for each \( I \in \mathcal{I} \), the natural map \( U'_I \to \bigsqcup_{I \in \mathcal{I}} U_I / \gamma \) is a homeomorphism onto its image.

This is also proved in the next subsection (see also discussions in [FOOO14], [MW15c]).

**Definition A.21.** A strong virtual orbifold atlas satisfying the conditions of \( \mathfrak{A}' \) in Lemma A.19 and Lemma A.20 is called a **metrizable** atlas.

**Theorem A.22.** Let \( X \) have a metrizable virtual orbifold atlas

\[ \mathfrak{A} = \left\{ (K_I = (P_I, \psi_I, F_I) \mid I \in \mathcal{I} \}, \{ T_{IJ} = (P_{IJ}, \phi_{IJ}, \pi_{IJ}) \mid I \leq J \} \right\}. \]

Then for any \( \epsilon > 0 \), there exist multisections \( t_I^j \) of \( E_I \to U_I \) for all \( I \in \mathcal{I} \) satisfying the following conditions.

1. For \( I \leq J \), there is a tubular neighborhood \( \tilde{\nu}_{IJ} : E_I|_{N_{IJ}} \to E_J|_{\phi_{IJ}(U_{IJ})} \) representing the germ \( \tilde{\nu}^+_{IJ} \) such that for \( z_j \in N_{IJ} \),

\[ \tilde{\nu}_{IJ}(z_j) \left( t^j_I(z_j) \right) = \left( \tilde{\nu}_{IJ}(t^j_I(\phi^{-1}_{IJ}(\nu_{IJ}(z_j)))) \right), 0_{IJ}. \] \hspace{1cm} (A.4)

Here we used the notation (A.3) and \( 0_{IJ} \in \nu^*_{IJ} E_{IJ} \) is the zero section.

2. For every \( I \in \mathcal{I} \), \( \| t^j_I \|_{C^0} \leq \epsilon \).

3. For every \( I \in \mathcal{I} \), \( S^j_I := S_I + t^j_I \) is a transverse multisection.

Moreover, for \( \epsilon \) small enough, \( Z(\{ S^j_I \}) := \left( \bigsqcup_{I \in \mathcal{I}} (S^j_I)^{-1}(0) \right) / \gamma \subset \bigsqcup_{I \in \mathcal{I}} U_I \) is compact.
Then we define the multiplicity of \( x \) by the cocycle conditions, we have
\[
(\text{2}) \quad \text{and } (3).
\]
Next we construct \( t'_j \) for a minimal \( J \) in \( \mathcal{I} \setminus \mathcal{I}' \).

For each pair \( I \leq J \) with \( I \in \mathcal{I}' \), consider a tubular neighborhood
\[
u_{IJ} : N_{IJ} \rightarrow \phi_{IJ}(U_{IJ}), \quad \hat{\nu}_{IJ} : E_{IJ} \rightarrow E_{IJ} |_{\phi_{IJ}(U_{IJ})}
\]
representing the germ \((\phi_{IJ}, \hat{\phi}_{IJ})\). Then for \( z_j \in N_{IJ} \subset U_J \), we define \( t'_j(z_j) \) by (A.4). By Definition A.11, by taking \( N_{IJ} \) sufficiently small, \( s_j + t'_j \) is transverse inside \( N_{IJ} \).

We need to prove that for \( I, L \leq J \) with \( I, L \in \mathcal{I}' \) and \( z_j \in N_{IJ} \cap N_{IL} \),
\[
\hat{\varphi}_{IJ}(z_j)^{-1}\left(\hat{\varphi}_{IJ}(t'_j(\nu_{IJ}(z_j))), 0_{IJ}\right) = \hat{\varphi}_{IL}(z_j)^{-1}\left(\hat{\varphi}_{IL}(t'_j(\nu_{IL}(z_j))), 0_{IL}\right).
\]
(A.5)

Indeed, if \( N_{IJ} \cap N_{IL} = \emptyset \), then \( \phi_{IJ}(U_{IJ}) \cap \phi_{IL}(U_{IL}) = \emptyset \). Since the relation \( \gamma \) is an equivalence relation, we must have either \( I \leq L \) or \( L \leq I \). Suppose \( I \leq L \), then denote
\[
z_I = \hat{\varphi}_{IJ}(\nu_{IJ}(z_j)) \in U_{IJ}, \quad z_L = \hat{\varphi}_{IL}(\nu_{IL}(z_j)) \in U_{IL}.
\]

Then by the cocycle conditions, we have
\[
\hat{\varphi}_{IJ}(z_j)^{-1}\left(\hat{\varphi}_{IJ}(t'_j(\nu_{IJ}(z_j))), 0_{IJ}\right) = \hat{\varphi}_{IL}(z_I)^{-1}\left(\hat{\varphi}_{IL}(t'_j(\nu_{IL}(z_I))), 0_{IL}\right) = \hat{\varphi}_{IL}(z_L)^{-1}\left(\hat{\varphi}_{IL}(t'_j(\nu_{IL}(z_L))), 0_{IL}\right).
\]
(A.5) holds and (A.4) defines a multisection \( t'_j \) of \( E_J \) in a neighborhood of \( \bigcup_{K \leq J} \phi_{JK}(U_{JK}) \) inside \( U_J \), which satisfies (2) and (3).

Then by Proposition A.8, we can extend \( t'_j \) to (an appropriately shrunk) \( U_J \) without changing its value near each \( \phi_{JK}(U_{JK}) \) for all \( K \leq J \), such that (2) and (3) hold for \( J \). This finishes the induction process. Therefore, we have constructed the family of perturbations that satisfy (1)–(3).

Lastly, viewing \( X \) as a compact subset of \( \mathfrak{A} \). Since \( \mathfrak{A} \) is metrizable, using a metric, one can guarantee that for \( \epsilon > 0 \) small enough, \( Z(\{S'_I\}) \) is contained in a compact neighborhood \( U \subset \mathfrak{A} \) of \( X \). Moreover, \( Z(\{S'_I\}) \) is closed, hence it is compact.

In particular, if \( \mathfrak{A} \) is oriented, and the virtual dimension of \( \mathfrak{A} \) is zero, one can define the \textbf{virtual count} of \( \mathfrak{A} \) as follows. Take a family of multisection perturbations \( \{S'_I = S_I + t'_I\} \). For each point of \( Z(\{S'_I\}) \) represented by \( x_I \in (S'_I)^{-1}(0) \), since \( S'_I \) is locally liftable, one can write
\[
S'_I(x) = \left[ s_i^{(1)}(x), \ldots, s_i^{(l)}(x) \right],
\]
such that each branch \( s_i^{(j)} \) is continuous and transverse to zero. For each \( j \), if \( s_i^{(j)}(x_I) = 0 \), one has an associated sign \( \epsilon_j = \pm 1 \) according to the orientation; if \( s_i^{(j)}(x_I) \neq 0 \), we define \( \epsilon_j = 0 \).

Then we define the multiplicity of \( x_I \) to be
\[
m(x_I) = \frac{1}{l} \sum_{j=1}^{l} \epsilon_j.
\]
It is straightforward to check that \( m(x_I) \) only depends on the equivalence class \( [x_I] \in |\mathfrak{A}| \). Then we define the virtual count to be

\[
\#\mathfrak{A} = \sum_{[x_I] \in |\mathfrak{S}|(\{S_I\})} m(x_I).
\]

Using a homotopy argument one can prove that the virtual count is independent of the choice of multisection perturbations.

**Proposition A.23.** If \( \mathfrak{A} \) is an oriented virtual orbifold atlas of dimension 1 with boundary on \( X \) and let \( \partial X \) be equipped with the induced virtual orbifold atlas \( \partial \mathfrak{A} \), then \( \#(\partial \mathfrak{A}) = 0 \).

**A.4. Technical Results.** The following proposition explains how to shrinking two charts such that the weak coordinate changes between them becomes a strong one.

**Proposition A.24.** Let \( K_i = (U_i, E_i, S_i, \psi_i, F_i) \), \( i = 1, 2 \) be two local charts \( X \) and let \( T_21 = (U_{21}, \phi_{21}) \) be a weak coordinate change from \( K_1 \) to \( K_2 \) with footprint \( F_{21} \). Let \( Z_i \subset F_i \) be compact subsets such that \( Z_1 \cap Z_2 \subset F_{21} \). Then there exist shrinkings \( K'_{i} = K_{i}[U'_{i}] \) with footprints \( F'_i \supset Z_i \) such that the induced shrinking of \( T_{21} \) is a strong coordinate change.

**Proof.** Define the equivalence relation \( \sim_\phi \) on the disjoint union \( U_1 \sqcup U_2 \) generated by

\[
x = \phi_{21}(y) \in U_2 \text{ or } y = \phi_{21}(x) \in U_2.
\]

Let \( U_1 \cup_\phi U_2 \) be the quotient space, which is not necessarily Hausdorff. In the same way we define \( E_1 \cup_\phi E_2 \), which is a fibre bundle over \( U_1 \cup_\phi U_2 \) with the zero section and a section \( S : U_1 \cup_\phi U_2 \to E_1 \cup_\phi E_2 \). There is also a continuous map

\[
\psi : S^{-1}(0) \to X
\]

whose image is the union of the footprints \( F = F_1 \cup F_2 \). It is easy to see that \( T_{21} \) is strong if and only if \( \psi \) is injective. Then the proposition follows from the lemma below. \( \Box \)

**Lemma A.25.** For compact subsets \( Z_1 \subset F_1, Z_2 \subset F_2 \) such that \( Z_1 \cap Z_2 \subset F_{21} \), there exists open subsets \( U'_1 \subset U_1, U'_2 \subset U_2 \), containing \( \psi_1^{-1}(Z_1) \) and \( \psi_2^{-1}(Z_2) \) respectively, such that the restriction of \( \psi \) to \( (U'_1 \cup_\phi U'_2) \cap S^{-1}(0) \) is injective.

**Proof.** For \( i = 1, 2 \), since \( U_i \) is an orbifold, there exist a sequences of open subsets \( U_i^{(n)} \subset U_i \), such that

\[
U_i^{(n+1)} \subset U_i^{(n)}, \quad \bigcap_{n} U_i^{(n)} = \psi_i^{-1}(Z_i).
\]

We claim that for \( n \) sufficiently large, \( U'_1 = U_1^{(n)} \) and \( U'_2 = U_2^{(n)} \) satisfy the required condition.

Suppose the claim is not true, then there exist two sequences of points \( x_1^{(n)} \in U_1^{(n)} \cap S_1^{-1}(0), x_2^{(n)} \in U_2^{(n)} \cap S_2^{-1}(0) \) with \( \psi_1(x_1^{(n)}) = \psi_2(x_2^{(n)}) \) but \( x_1^{(n)} \notin U_{21} \). By taking a subsequence, we may assume that \( x_i^{(n)} \) converges to \( z_i \in \psi_i^{-1}(Z_i) \). It is easy to see that \( \psi_1(z_1) = \psi_2(z_2) \in Z_1 \cap Z_2 \).

Since \( F_{21} \) is an open neighborhood of \( Z_1 \cap Z_2 \), for \( n \) sufficiently large, we know that \( x_1^{(n)} \in \psi_1^{-1}(F_{21}) \subset U_{21} \), which is a contradiction. Hence the claim is true. \( \Box \)

Lastly, we prove Lemma A.19 and Lemma A.20.

**Proof of Lemma A.19.** By definition, the relation \( \Uparrow \) is reflexive and symmetric. We need to shrink the charts to make the induced relation transitive. It suffices to guarantee that the restriction of \( \Uparrow \) onto \( U_I \sqcup U_J \sqcup U_K \) transitive, for any three distinct elements \( I, J, K \leq \mathcal{I} \). Let \( x \in U_I, y \in U_J, z \in U_K \) be general elements.

Choose precompact open subset \( F'_I \subset F_I, F'_J \subset F_J, F'_K \subset F_K \) such that

\[
X = F'_I \cup F'_J \cup F'_K \cup \bigcup_{L \neq I, J, K} F_L.
\]

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Since $U_I$ (resp. $U_J$ resp. $U_K$) is an orbifold and hence metrizable, we can choose a sequence of open subsets $U_I^{(n)} \subset U_I$ (resp. $U_J^{(n)} \subset U_J$ resp. $U_K^{(n)} \subset U_K$) containing $F_I$ (resp. $F_J$ resp. $F_K$) such that

$$U_I^{(n+1)} \subset U_I^{(n)}, \quad \bigcap_n U_I^{(n)} = F_I,$$

$$U_J^{(n+1)} \subset U_J^{(n)}, \quad \bigcap_n U_J^{(n)} = F_J,$$

$$U_K^{(n+1)} \subset U_K^{(n)}, \quad \bigcap_n U_K^{(n)} = F_K.$$

Then for each $n$, $U_I^{(n)}, U_J^{(n)}, U_K^{(n)}$ induce a shrinking of the atlas $\mathcal{A}$, denoted by $\mathcal{A}^{(n)}$. Let the induced binary relation by $\gamma^{(n)}$. Consider the domains of the shrunk coordinate changes

$$U_{JI}^{(n)} = U_I^{(n)} \cap U_J, \quad U_{KJ}^{(n)} = U_J^{(n)} \cap \phi_{KJ}^{-1}(U_K^{(n)}), \quad U_{KI}^{(n)} = U_I^{(n)} \cap \phi_{KI}^{-1}(U_K^{(n)}).$$

We claim that for $n$ large enough, $\gamma^{(n)}$ is an equivalence relation on $U_I^{(n)} \cup U_J^{(n)} \cup U_K^{(n)}$.

If this is not true, then for all large $n$, there exist points $x^{(n)} \in U_I^{(n)}$, $y^{(n)} \in U_J^{(n)}$, $z^{(n)} \in U_K^{(n)}$ such that

$$(x^{(n)}, y^{(n)}, (y^{(n)}, z^{(n)}) \in \gamma^{(n)}, \quad (x^{(n)}, z^{(n)}) \notin \gamma^{(n)}.$$  \tag{A.6}

or

$$(x^{(n)}, y^{(n)}, (x^{(n)}, z^{(n)}) \in \gamma^{(n)}, \quad (y^{(n)}, z^{(n)}) \notin \gamma^{(n)}.$$  \tag{A.6}

or

$$(x^{(n)}, z^{(n)}, (y^{(n)}, z^{(n)}) \in \gamma^{(n)}, \quad (x^{(n)}, y^{(n)}) \notin \gamma^{(n)}.$$

Without loss of generality, assume that (A.6) happens for infinitely many $n$, then a subsequence of $x^{(n)}$ converges to a limit

$$x^{(\infty)} \in \psi_I^{-1}(F_I \cap F_J \cap F_K) \subset \psi_I^{-1}(F_I \cap F_J \cap F_K).$$

By the overlapping condition of Definition A.15, $\{I, J, K\}$ is totally ordered.

(1) If $J$ is the middle element, without loss of generality assume that $I \leq J \leq K$. Then $x^{(n)} \in U_{JI}^{(n)} \cap \phi_{JI}^{-1}(U_K^{(n)})$. Moreover, since $x^{(\infty)} \in \psi_I^{-1}(F_I \cap F_K) \subset \psi_I^{-1}(F_I) \subset U_{KI}$, for $n$ large enough, $x^{(n)} \in U_{KI}$. Then by the cocycle condition of Definition A.15,

$$\phi_{KI}(x^{(n)}) = \phi_{KI}(\phi_{JI}(x^{(n)})) = \phi_{KJ}(y^{(n)}) = z^{(n)} \in U_{KJ} \subset U_K^{(n)}.$$

So $x^{(n)} \gamma^{(n)} z^{(n)}$, which contradicts (A.6).

(2) If $J$ is the least element, without loss of generality assume that $J \leq I \leq K$. Then we still have that for large $n$, $x^{(n)} \in U_{KI}$. Then by the cocycle condition,

$$z^{(n)} = \phi_{KJ}(y^{(n)}) = \phi_{KI}(\phi_{IJ}(y^{(n)})) = \phi_{KI}(x^{(n)}).$$

So $x^{(n)} \gamma^{(n)} z^{(n)}$, which contradicts (A.6).

(3) If $J$ is the largest element, without loss of generality assume that $I \leq K \leq J$. Then we have for large $n$, $x^{(n)} \in U_{KJ}$. Moreover, since $\phi_{KI}(x^{(\infty)}) \in \psi_K^{-1}(F_J \cap F_K) \subset U_{JK}$, for $n$ large enough, $x^{(n)} \in \phi_{KI}^{-1}(U_{IK})$. Then by the cocycle condition,

$$y^{(n)} = \phi_{JI}(x^{(n)}) = \phi_{JK}(\phi_{KI}(x^{(n)})) = \phi_{JK}(z^{(n)}).$$

Since $\phi_{JK}$ is an embedding, we have $\phi_{KI}(x^{(n)}) = z^{(n)}$. Therefore, $x^{(n)} \gamma^{(n)} z^{(n)}$, which contradicts (A.6).

Therefore, $\gamma^{(n)}$ is an equivalence relation on $U_I^{(n)} \cup U_J^{(n)} \cup U_K^{(n)}$ for large enough $n$. We can perform the shrinking for any triple of elements of $\mathcal{I}$, which eventually makes $\gamma$ an equivalence relation. $\Box$
Before proving Lemma A.20, we need some preparations. Suppose \( \gamma \) is already an equivalence condition and \( |\mathfrak{A}| \) is the quotient space. Let

\[
\pi_k : \bigcup_{I \in \mathcal{I}} U_I \to |\mathfrak{A}|
\]

the quotient map. It induces for each \( I \in \mathcal{I} \) a continuous map \( \pi_I : U_I \to |\mathfrak{A}| \). A point of \( |\mathfrak{A}| \) is denoted by \( |x| \), which can be represented by a point \( x \in U_I \) for some \( I \).

Order the finite set \( \mathcal{I} \) as \( \{I_1, \ldots, I_m\} \) such that for \( k = 1, \ldots, m \),

\[
J \geq I_k \implies J \in \{I_k, I_{k+1}, \ldots, I_m\}.
\]

For each \( k \), \( \gamma \) induces an equivalence relation on \( \bigcup_{i \geq k} U_{I_i} \), and denote the quotient space by \( |\mathfrak{A}_k| \). There is a natural continuous map

\[
|\iota_k| : |\mathfrak{A}_k| \to |\mathfrak{A}|.
\]

**Proof of Lemma A.20.** For each \( k \), we would like to construct shrinkings \( U_{I_i}^\prime \subset U_{I_i} \) for all \( i \geq k \) such that \( |\mathfrak{A}_k'| \) is Hausdorff. Our construction is based on a top-down induction. Firstly, for \( k = m \), \( |\mathfrak{A}_m| \simeq U_{I_m} \) and hence is Hausdorff. Suppose \( |\mathfrak{A}_{k+1}| \) is Hausdorff. Choose open subsets \( F'_{I_i} \subset F_{I_i} \) for all \( i \geq k \) such that

\[
X = \bigcup_{i \geq k} F'_{I_i} \cup \bigcup_{i \leq k-1} F_{I_i}.
\]

Choose sequences of shrinkings \( U_{I_i}^{(n)} \subset U_{I_i} \) for \( i \geq k \) such that

\[
U_{I_i}^{(n+1)} \subset U_{I_i}^{(n)}, \cap_n U_{I_i}^{(n)} = F_{I_i}.
\]

For \( l = k, k+1 \), denote

\[
|\mathfrak{A}_l^{(n)}| := \left( \bigcup_{i \geq l} U_{I_i}^{(n)} \right) / \gamma^{(n)}.
\]

By the induction hypothesis, \( |\mathfrak{A}_{k+1}^{(n)}| \) is Hausdorff for all \( n \). We claim that for sufficiently large \( n \), \( |\mathfrak{A}_{k+1}^{(n)}| \) is Hausdorff.

Suppose this claim is not true, then for each \( n \), there exist a pair of points \( |x^{(n)}|, |y^{(n)}| \in |\mathfrak{A}_k^{(n)}| \) which cannot be separated by open sets. There are the following three possibilities which we treat separately.

1. Suppose \( |x^{(n)}|, |y^{(n)}| \in |\mathfrak{A}_{k+1}^{(n)}| \). Then by the induction hypothesis, there exist open neighborhoods \( |\mathcal{V}_{I_i}^{(n)}|, |\mathcal{W}_{I_i}^{(n)}| \subset |\mathfrak{A}_{k+1}^{(n)}| \) of \( |x^{(n)}|, |y^{(n)}| \) such that

\[
|\mathcal{V}_{I_i}^{(n)}| \cap |\mathcal{V}_{I_i}^{(n)}| = \emptyset.
\]

By the definition of the quotient topology on \( |\mathfrak{A}_{k+1}^{(n)}| \), we have

\[
\pi_{I_i}^{-1} |\mathfrak{A}_{k+1}^{(n)}| \left( |\mathcal{V}_{I_i}^{(n)}| \right) = |\mathcal{V}_{I_i}^{(n)}| = \bigcup_{i \geq k+1} V_i^{(n)},
\]

\[
\pi_{I_i}^{-1} |\mathfrak{A}_{k+1}^{(n)}| \left( |\mathcal{W}_{I_i}^{(n)}| \right) = |\mathcal{W}_{I_i}^{(n)}| = \bigcup_{i \geq k+1} W_i^{(n)}.
\]

where for each \( i \geq k+1 \) and all \( n \), \( V_i^{(n)} \) and \( W_i^{(n)} \) are disjoint open sets of \( U_{I_i}^{(n)} \). Then define

\[
V_k^{(n)} = \bigcup_{i \geq k+1, \, I_i \geq I_k} \phi_I^{-1} \left( V_i^{(n)} \cap U_{I_k}^{(n)} \right), \quad W_k^{(n)} = \bigcup_{i \geq k+1, \, I_i \geq I_k} \phi_I^{-1} \left( W_i^{(n)} \cap U_{I_k}^{(n)} \right).
\]

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which are disjoint open subsets of $U_{I_k}^{(n)}$. Then $\bigcup_{i \geq k} V_i^{(n)}$, $\bigcup_{j \geq k} W_j^{(n)}$ are preimages of two disjoint open subsets of $|\mathcal{A}_k^{(n)}|$, which contain $|x^{(n)}|$ and $|y^{(n)}|$ respectively. This contradicts our hypothesis.

(2) Suppose $|x^{(n)}|$ and $|y^{(n)}|$ are represented by $x^{(n)}, y^{(n)} \in U_{I_k}^{(n)}$. Then by taking a subsequence, we know that there exist limits

$$\lim_{n \to \infty} x^{(n)} = x^{(\infty)} \in \psi_{I_k}^{-1}(F_{I_k}^n), \quad \lim_{n \to \infty} y^{(n)} = y^{(\infty)} \in \psi_{I_k}^{-1}(F_{I_k}^n).$$

Define $I_{k+1}', J_{k+1}', I_{k+1}''$, $J_{k+1}''$ by

$$I_{k+1}' = \{ i \geq k + 1 \mid x^{(\infty)} \in \psi_{I_k}^{-1}(F_{I_k}^n \cap F_{I_k}^i) \}, \quad I_{k+1}'' = \{ k + 1, \ldots, m \} \setminus I_{k+1}';$$

$$J_{k+1}' = \{ j \geq k + 1 \mid y^{(\infty)} \in \psi_{I_k}^{-1}(F_{I_k}^n \cap F_{I_k}^j) \}, \quad J_{k+1}'' = \{ k + 1, \ldots, m \} \setminus J_{k+1}'.$$

For each $i \geq k + 1$, we choose an open neighborhood $V_{k,i} \subset U_{I_k}$ (resp. $W_{k,i} \subset U_{I_k}$) of $x^{(\infty)}$ (resp. $y^{(\infty)}$) as follows.

- If $i \in I_{k+1}'$ (resp. $j \in J_{k+1}'$), then for some sufficiently big $N_i$ (resp. $M_j$),

$$x^{(\infty)} \notin U_{I_k}^{(N_i)} \quad \text{resp.} \quad y^{(\infty)} \notin U_{I_k}^{(M_j)}$$

(here $U_{I_k}^{(N_i)}$ or $U_{I_k}^{(M_j)}$ may be empty). Choose $V_{k,i}$ (resp. $W_{k,j}$) such that

$$\overline{V_{k,i} \cap U_{I_k}^{(N_i)}} = \emptyset \quad \text{resp.} \quad \overline{W_{k,j} \cap U_{I_k}^{(M_j)}} = \emptyset.$$ 

- Take $N \geq 1$ such that

$$N \geq \max_{i \in I_{k+1}'} N_i, N \geq \max_{j \in J_{k+1}'} M_j.$$ 

Then for $i \in I_{k+1}'$ (resp. $j \in J_{k+1}'$), then we choose $V_{k,i}$ (resp. $W_{k,j}$) such that

$$\overline{V_{k,i} \cap U_{I_k}^{(N_i)}} \subset U_{I_k}^{(N)} \quad \text{resp.} \quad \overline{W_{k,j} \cap U_{I_k}^{(M_j)}} \subset U_{I_k}^{(N)}.$$ 

Take

$$V_k = \bigcap_{i \geq k + 1} V_{k,i} \subset U_{I_k}, \quad W_k = \bigcap_{i \geq k + 1} W_{k,i} \subset U_{I_k}.$$ 

Then for $n$ sufficiently large, $x^{(n)} \in V_k$ and $y^{(n)} \in W_k$. Choose open neighborhoods

$$V_k^{(n)} \subset V_k \cap U_{I_k}^{(n)}, \quad W_k^{(n)} \subset W_k \cap U_{I_k}^{(n)}$$

of $x^{(n)}$ and $y^{(n)}$ respectively such that $\overline{V_k^{(n)} \cap W_k^{(n)}} = \emptyset$. If $i \in I_{k+1}'$ (resp. $i \in J_{k+1}'$), then $\phi_{I_k,i}(V_k)$ (resp. $\phi_{I_k,i}(W_k)$) is compact inside $U_{I_k}$.

Consider the following two compact subsets of $|\mathcal{A}_k^{(n)}|$:

$$|P_{k+1}^{(n)}| := \pi_{\mathcal{A}_k^{(n)}} \left( \bigcup_{i \in I_{k+1}'} \phi_{I_k,i} \left( \overline{V_k^{(n)}} \right) \right), \quad |Q_{k+1}^{(n)}| := \pi_{\mathcal{A}_k^{(n)}} \left( \bigcup_{i \in J_{k+1}'} \phi_{I_k,i} \left( \overline{W_k^{(n)}} \right) \right).$$

Claim. $|P_{k+1}^{(n)}|$ and $|Q_{k+1}^{(n)}|$ are disjoint in $|\mathcal{A}_k^{(n)}|$. 

Proof of the claim. Suppose this claim is not true, then there exists $\phi_{I_k,i}(a_k^{(n)}) \gamma^{(N)} \phi_{I_k,i}(b_k^{(n)})$ for some $i \in I_{k+1}'$, $a_k^{(n)} \in V_k^{(n)}$ and $j \in J_{k+1}'$, $b_k^{(n)} \in W_k^{(n)}$. Since $\gamma^{(N)}$ is an equivalence relation, we have $a_k^{(n)} = b_k^{(n)}$, which contradicts the disjointness between $\overline{V_k^{(n)}}$ and $\overline{W_k^{(n)}}$. 

\[ \square \]
Then, by the Hausdorff property of $|\mathfrak{A}^{(n)}_{k+1}|$ (the induction hypothesis), there exist disjoint open subsets $|\mathcal{Y}^{(n)}_{k+1}|, |\mathcal{W}^{(n)}_{k+1}| \subset |\mathfrak{A}^{(n)}_{k+1}|$ containing $|P^{(n)}_{k+1}|$ and $|Q^{(n)}_{k+1}|$ respectively. Denote

$$\mathcal{Y}^{(n)}_{k+1} := \pi^{-1}_{\mathfrak{A}^{(n)}_{k+1}} \left( |\mathcal{Y}^{(n)}_{k+1}| \right) = \bigcup_{i \geq k+1} V^{(n)}_i,$$

$$\mathcal{W}^{(n)}_{k+1} := \pi^{-1}_{\mathfrak{A}^{(n)}_{k+1}} \left( |\mathcal{W}^{(n)}_{k+1}| \right) = \bigcup_{i \geq k+1} W^{(n)}_i.$$

Define

$$\tilde{V}^{(n)}_k = V^{(n)}_k \cup \bigcup_{i \geq k+1, i \not\in I_k} \phi^{-1}_{I_i, I_k}(V^{(n)}_i), \quad \tilde{W}^{(n)}_k = W^{(n)}_k \cup \bigcup_{i \geq k+1, i \not\in I_k} \phi^{-1}_{I_i, I_k}(W^{(n)}_i).$$

Then $\tilde{V}^{(n)}_k \cup \mathcal{Y}^{(n)}_{k+1}$ and $\tilde{W}^{(n)}_k \cup \mathcal{W}^{(n)}_{k+1}$ are disjoint preimages of open subsets of $|\mathfrak{A}^{(n)}_{k+1}|$ which contains $|x^{(n)}|$ and $|y^{(n)}|$ respectively. Since the topology of $|\mathfrak{A}^{(n)}_{k+1}|$ is the same as the one induced as a subset of $|\mathfrak{A}^{(n)}_{k}|$, $|x^{(n)}|$ and $|y^{(n)}|$ are separated by two open subsets, which contradicts our hypothesis.

(3) Now suppose $|x^{(n)}|$ is represented by $x^{(n)} \in U^{(n)}_k$ and $|y^{(n)}| \in |\mathfrak{A}^{(n)}_{k+1}|$. Then a subsequence of $x^{(n)}$ converges to $x^{(\infty)} \in \overline{U^{(n)}_k}$. Then define $I_k, I_{k+1}$ by (A.7). Similar to the above situation, one can find a sufficiently large $N$ and an open neighborhood $V_k \subset U_k$ of $x^{(\infty)}$ such that

$$i \in I_k \implies V_k \subset U^{(n)}_{I_i, I_k}; \quad i \in I_{k+1} \implies V_k \cap \overline{U^{(n)}_{I_i, I_k}} = \emptyset.$$

Then for $n$ sufficiently large, $x^{(n)} \in V_k$. Choose an open neighborhood $x^{(n)} \supset V^{(n)}_k \subset V_k \cap U^{(n)}_k$ such that for any $i \in I_{k+1}$. Moreover, since $|x^{(n)}|$ and $|y^{(n)}|$ are different as points in $|\mathfrak{A}^{(n)}_k|$, they are also different as points in $|\mathfrak{A}^{(n)}_k|$. Hence we can choose $V^{(n)}_k$ in such a way that, if $|y^{(n)}|$ can be represented by $y^{(n)}_i \in U^{(n)}_i$, then

$$y^{(n)}_i \notin \phi_{I_i, I_k}(V^{(n)}_k).$$

Now consider the compact subset

$$|P^{(n)}_{k+1}| := \pi^{-1}_{\mathfrak{A}^{(n)}_{k+1}} \left( \bigcup_{i \in I_{k+1}} \phi_{I_i, I_k}(\overline{V^{(n)}_i}) \right) \subset |\mathfrak{A}^{(n)}_{k+1}|.$$

By the last condition, it doesn’t contain $|y^{(n)}|$. Then by the induction hypothesis, there exist an open neighborhood $|\mathcal{Y}^{(n)}_{k+1}| \supset |P^{(n)}_{k+1}|$ and an open neighborhood $|\mathcal{W}^{(n)}_{k+1}| \supset |y^{(n)}|$ which are disjoint. Suppose

$$\mathcal{Y}^{(n)}_{k+1} := \pi^{-1}_{\mathfrak{A}^{(n)}_{k+1}} \left( |\mathcal{Y}^{(n)}_{k+1}| \right) = \bigcup_{i \geq k+1} V^{(n)}_i \subset \bigcup_{i \geq k+1} U^{(n)}_i,$$

$$\mathcal{W}^{(n)}_{k+1} := \pi^{-1}_{\mathfrak{A}^{(n)}_{k+1}} \left( |\mathcal{W}^{(n)}_{k+1}| \right) = \bigcup_{i \geq k+1} W^{(n)}_i \subset \bigcup_{i \geq k+1} U^{(n)}_i.$$

So for each $i \geq k+1$, $V^{(n)}_{i, I_i, I_k} \cap W^{(n)}_{i, I_i, I_k} = \emptyset$. Then define

$$\tilde{V}^{(n)}_k = V^{(n)}_k \cup \bigcup_{i \geq k+1, i \not\in I_k} \phi^{-1}_{I_i, I_k}(V^{(n)}_i), \quad \tilde{W}^{(n)}_k = \bigcup_{i \geq k+1, i \not\in I_k} \phi^{-1}_{I_i, I_k}(W^{(n)}_i),$$

which are disjoint open subsets of $U^{(n)}_k$ and $x^{(n)} \in \tilde{V}^{(n)}_k$. Then $\tilde{V}^{(n)}_k \cup \mathcal{Y}^{(n)}_{k+1}$ and $\tilde{W}^{(n)}_k \cup \mathcal{W}^{(n)}_{k+1}$ are two disjoint preimages of open sets of $|\mathfrak{A}^{(n)}_{k+1}|$, which contains $|x^{(n)}|$ and $|y^{(n)}|$. Since the topology of $|\mathfrak{A}^{(n)}_{k+1}|$ is the same as the one induced as a subset of $|\mathfrak{A}^{(n)}_{k}|$, $|x^{(n)}|$ and $|y^{(n)}|$ can also be separated by open sets in $|\mathfrak{A}^{(n)}_{k+1}|$. This contradicts with our hypothesis.
Therefore, we can finish the induction and construct a shrinking such that $|\mathfrak{A}'|$ is Hausdorff. A further shrinking can guarantee that $|\mathfrak{A}'|$ is metrizable and for each $I$, the natural map $\pi_I : U'_I \to |\mathfrak{A}'|$ is a homeomorphism onto its image.

References

[CGS00] Kai Cieliebak, Ana Gaio, and Dietmar Salamon, *J*-holomorphic curves, moment maps, and invariants of Hamiltonian group actions, International Mathematical Research Notices 16 (2000), 831–882.

[CLL15] Huai-Liang Chang, Jun Li, Wei-Ping Li, and Chiu-Chu Melissa Liu, *Mixed-Spin-P fields of Fermat quintic polynomials*, http://arxiv.org/abs/1505.07532, 2015.

[CLW13] Bohui Chen, An-Min Li, and Bai-Ling Wang, Virtual neighborhood technique for pseudoholomorphic spheres, http://arxiv.org/abs/1508.01556, 2013.

[CLW15] Gluing principle for orbifold stratified spaces, http://arxiv.org/abs/1502.05103, 2015.

[FJR11] Andreas Floer and Helmut Hofer, Coherent orientations for periodic orbit problems in symplectic geometry, Mathematische Zeitschrift 212 (1993), 13–38.

[FJR08] Huijun Fan, Tyler Jarvis, and Yongbin Ruan, Geometry and analysis of spin equations, Communications on Pure and Applied Mathematics 61 (2008), no. 6, 745–788.

[FJR15] The Witten equation, mirror symmetry and quantum singularity theory, Annals of Mathematics 178 (2013), 1–106.

[FO99] Kenji Fukaya and Kaoru Ono, A mathematical theory of the gauged linear sigma model, http://arxiv.org/abs/1506.02109, 2015.

[FOOO12] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, A general Fredholm theory. I. A splicing-based differential geometry., Journal of the European Mathematical Society (JEMS) 9 (2007), 841–876.

[FOOO14] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, A general Fredholm theory. III. Fredholm functors and polyfolds., Geometry & Topology 13 (2009), 2279–2387.

[FOOO15] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, A general Fredholm theory. II. Implicit function theorems, Geometric and Functional Analysis 19 (2009), 206–293.

[HV00] Kentaro Hori and Cumrun Vafa, Mirror symmetry, http://arxiv.org/abs/hep-th/0002222, 2000.

[HWZ07] Helmut Hofer, Krzysztof Wysocki, and Eduard Zehnder, A general Fredholm theory. I. A splicing-based differential geometry., Journal of the European Mathematical Society (JEMS) 9 (2007), 841–876.

[HWZ09a] A general Fredholm theory. II. Implicit function theorems, Geometric and Functional Analysis 19 (2009), 206–293.

[HWZ09b] A general Fredholm theory. III. Fredholm functors and polyfolds., Geometry & Topology 13 (2009), 2279–2387.

[Joy07] Dominic Joyce, Kuranishi homology and Kuranishi cohomology, http://arxiv.org/abs/0707.3572, 2007.

[Joy14] A new definition of Kuranishi space, http://arxiv.org/abs/1409.6908, 2014.

[LT98a] Jun Li and Gang Tian, Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties, Journal of American Mathematical Society 11 (1998), no. 1, 119–174.

[LT98b] Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds, Topics in symplectic 4-manifolds (Irvine, CA, 1996), First International Press Lecture Series., no. I, International Press, Cambridge, MA, 1998, pp. 47–83.

[LT98c] Gang Liu and Gang Tian, Floer homology and Arnold conjecture, Journal of Differential Geometry 49 (1998), 1–74.

[MS04] Dusa McDuff and Dietmar Salamon, *J*-holomorphic curves and symplectic topology, Colloquium publications, vol. 52, American Mathematical Society, 2004.

[MT99] Iganci Mundet i Riera and Gang Tian, A compactification of the moduli space of twisted holomorphic maps, Advances in Mathematics 222 (2009), 1117–1196.

[Mun03] Iganci Mundet i Riera, Hamiltonian Gromov-Witten invariants, Topology 43 (2003), no. 3, 525–553.

[Mun12] Yong-Mills-Higgs theory for symplectic fibrations, Ph.D. thesis, Universidad Autónoma de Madrid, 19912.

[MW15a] Dusa McDuff and Katrin Wehrheim, The fundamental class of smooth kuranishi atlases with trivial isotopy, http://arxiv.org/abs/1508.01560, 2015.

[MW15b] Smooth Kuranishi atlases with isotropy, http://arxiv.org/abs/1508.01556, 2015.

[MW15c] The topology of kuranishi atlases, http://arxiv.org/abs/1508.01844, 2015.
[Par15] John Pardon, *An algebraic approach to virtual fundamental cycles on moduli space of pseudo-holomorphic curves*, Geometry and Topology **To appear** (2015).

[Qui88] Frank Quinn, *Topological transversality holds in all dimensions*, American Mathematical Society. Bulletin. New Series **18** (1988), 145–148.

[RT95] Yongbin Ruan and Gang Tian, *A mathematical theory of quantum cohomology*, Journal of Differential Geometry **42** (1995), 259–367.

[RT97] , *Higher genus symplectic invariants and sigma models coupled with gravity*, Inventiones Mathematicae **130** (1997), 455–516.

[Rua96] Yongbin Ruan, *Topological sigma model and Donaldson-type invariants in Gromov theory*, Duke Mathematical Journal **83** (1996), 461–500.

[Rua98] , *Virtual neighborhoods and the monopole equations*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), First International Press Lecture Series., vol. I, International Press, Cambridge, MA, 1998.

[Sat56] Ichirō Satake, *On a generalization of the notion of manifold*, Proceedings of the National Academy of Sciences of the United States of America **42** (1956), 359–363.

[Sol] Jake Solomon, *The Witten complex for non-Morse functions*, Preprint.

[SX14] Stephen Schecter and Guangbo Xu, *Morse theory for Lagrange multipliers and adiabatic limits*, Journal of Differential Equations **257** (2014), 4277–4318.

[TX14] Gang Tian and Guangbo Xu, *Correlation functions in gauged linear σ-model*, http://arxiv.org/abs/1406.4253, 2014.

[TX15] , *Analysis of gauged Witten equation*, Journal für die Reine und Angewandte Mathematik (2015), DOI: 10.1515/crelle–2015–0066., http://arxiv.org/abs/1405.6352.

[Whit93] Edward Witten, *Phases of N = 2 theories in two dimensions*, Nuclear Physics **B403** (1993), 159–222.

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