Numerical evidence of breaking of vortex lines in an ideal fluid

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Abstract Emergence of singularity of vorticity at a single point, not related to any symmetry of the initial distribution, has been demonstrated numerically for the first time. Behavior of the maximum of vorticity near the point of collapse closely follows the dependence $(t_0 - t)^{-1}$, where $t_0$ is the time of collapse. This agrees with the interpretation of collapse in an ideal incompressible fluid as of the process of vortex lines breaking.

1. Introduction

The problem of collapse in hydrodynamics, i.e. of a process of singularity formation in a finite time, is essential for understanding of the physical nature of developed turbulence. Despite a progress in construction of statistical theory of Kolmogorov spectra within both diagram and functional approaches (see, e.g., Monin & Yaglom 1992; L’vov 1991 and references therein), so far the question whether the Kolmogorov spectrum is a solution to the statistical equations of hydrodynamics remains open. Another important problem, as yet unsolved, is the one of intermittency. In statistical sense intermittency can be interpreted as a consequence of a strongly non-Gaussian distribution of turbulent velocity, resulting in deviation of exponents for higher correlation functions from their Kolmogorov values (Frisch 1995). Non-Gaussian behavior implies
that odd correlation functions do not vanish; this indicates the presence of strong correlations between velocity fluctuations, suggesting existence of coherent structures in turbulence. Analysis of both numerical and experimental data reveals (see Frisch 1995 and references therein) that in the regime of fully developed turbulence distribution of vorticity is strongly inhomogeneous in space – it is concentrated in relatively small regions. What is the reason of this? Can such a high concentration be explained by formation of singularity of vorticity in a finite time? How can one derive from this hypothesis the Kolmogorov spectrum? This question is not rhetoric: it is well known that any singularity results in a power-law kind of spectrum in the short-scale region. Thus, the problem of collapse is of ultimate importance in hydrodynamics.

The most popular object in the studies of collapse in hydrodynamics is a system of two anti-parallel vortex tubes, inside which vorticity is continuously distributed (Kerr 1993), or in a more general setup – flows with a higher spatial symmetry (Boratav & Pelz 1994; Pelz 1997). It is well known, that two anti-parallel vortex filaments undergo the so-called Crow instability (Crow 1970) leading to stretching of vortex filaments in the direction normal to the plane of the initial distribution of vortices and to reduction of their mutual distance. It was demonstrated in numerical experiments (Kerr 1993) that point singularities are formed in cores of each vortex tubes at the nonlinear stage of this instability, and $|\omega|$ near the point of collapse increases like $(t_0 - t)^{-1}$, $t_0$ being the time of collapse (see also Grauer, Marliani & Germaschewski 1998).

2. Basic equations

In this paper we present results\(^1\) of a numerical experiment, which can be interpreted as emergence of singularity of vorticity at a single point in a three-dimensional ideal hydrodynamic system, where initial data lacks any symmetry. The representation of the Euler equation for vorticity $\omega(r,t)$ in terms of vortex lines is employed, which was introduced in Kuznetsov & Ruban 1998:

$$\omega(r,t) = (\omega_0(a) \cdot \nabla a)R(a,t)/J.$$  \hfill (1)

Here the mapping

$$r = R(a,t)$$  \hfill (2)

represents transition to a new curvilinear system of coordinates associated with vortex lines, so that $b = (\omega_0(a) \cdot \nabla a)R(a,t)$ is a tangent

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\(^1\)Preliminary results were communicated in Zheligovsky, Kuznetsov & Podvigina 2001.
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vector to a given vortex line, $J = \det \| \partial \mathbf{R} / \partial \mathbf{a} \|$ is the Jacobian of the mapping (2). Dynamics of the vector $\mathbf{R}(\mathbf{a},t)$ satisfies

$$\partial_t \mathbf{R} = \hat{\Pi} \mathbf{v}(\mathbf{R},t),$$

(3)

where $\mathbf{v}(\mathbf{R},t)$ is the flow velocity at a point $\mathbf{r} = \mathbf{R}$ and $\hat{\Pi}$ is the transverse projection to the vortex line at this point:

$$\Pi_{\alpha\beta} = \delta_{\alpha\beta} - \xi_\alpha \xi_\beta, \quad \xi = b / |b|. \quad (4)$$

Equations (1)-(3) are closed by the relation between vorticity and velocity:

$$\mathbf{\omega}(\mathbf{r},t) = \nabla \times \mathbf{v}(\mathbf{r},t); \quad \nabla \cdot \mathbf{v} = 0. \quad (5)$$

The system of equations (1)-(5) can be regarded as a result of partial integration of the Euler equation

$$\partial_t \mathbf{\omega} = \nabla \times [\mathbf{v} \times \mathbf{\omega}], \quad \nabla \cdot \mathbf{\omega} = 0. \quad (6)$$

A vector field $\mathbf{\omega}_0(\mathbf{a})$ incorporated in (1), $\nabla_\mathbf{a} \cdot \mathbf{\omega}_0(\mathbf{a}) = 0$, is the Cauchy invariant, manifesting frozenness of vorticity into the fluid. If $\mathbf{R}(\mathbf{a},0) = \mathbf{a}$, $\mathbf{\omega}_0$ is the initial distribution of vorticity.

The Jacobian $J$ can take arbitrary values because the description under consideration is a mixed, Lagrangian-Eulerian one (Kuznetsov & Ruban 1998; Kuznetsov & Ruban 2000). In particular, $J$ can vanish at some point, which by virtue of (1) implies a singularity of vorticity. It was demonstrated by Kuznetsov & Ruban 2000 that collapses of this type are possible in the three-dimensional integrable hydrodynamics (Kuznetsov & Ruban 1998), where in the Euler equation (6) a modified relation between vorticity and velocity (both generalized) is assumed:

$$\mathbf{v} = \nabla \times (\delta \mathcal{H} / \delta \mathbf{\omega}), \quad \mathcal{H} = \int |\mathbf{\omega}| d\mathbf{r}. \quad (7)$$

Emergence of singularity of vorticity at a point, where $J = 0$, means that a vortex line touches at this point another vortex line. This is the process of breaking of vortex lines. Being analogous to breaking in a gas of dust particles (dynamics of a gas with a zero pressure), this process is completely determined by the mapping (2).
3. Breaking of vortex lines

Let us assume now that collapse in the Euler hydrodynamics occurs due to breaking of vortex lines. Denote by \( \tilde{t}(a) > 0 \) a solution to the equation \( J(a, t) = 0 \), and let \( t_0 = \min_a \tilde{t}(a) \), where the minimum is achieved at \( a = a_0 \). Near the point of the minimum \((t_0, a_0)\) the Jacobian can be expanded (cf. Kuznetsov & Ruban 2000):

\[
J = \alpha(t_0-t) + \gamma_{ij} \Delta a_i \Delta a_j + ..., \tag{8}
\]

where \( \alpha > 0, \gamma \) is a positive definite matrix and \( \Delta a = a - a_0 \). The Taylor expansion (8) is obtained under the assumption that \( J \) is smooth, which is conceivable up to the moment of singularity formation. At \( t = t_0 \) the numerator in (1), i.e. the vector \( b \), does not vanish: the condition \( J = 0 \) is satisfied when the three vectors \( \partial R/\partial a_i \) \((i = 1, 2, 3)\) lie in a plane, but generically none of them equals zero (that were a degeneracy) so that near the point of singularity

\[
\omega(r, t) \approx \frac{b(t_0, a_0)}{\alpha(t_0 - t) + \gamma_{ij} \Delta a_i \Delta a_j}. \tag{9}
\]

Furthermore, \( J = 0 \) implies that an eigenvalue of the Jacoby matrix (say, \( \lambda_1 \)) vanishes, and generically the other two eigenvalues (\( \lambda_2 \) and \( \lambda_3 \)) are non-zero. Therefore, there exist one “soft” direction associated with \( \lambda_1 \), and two “hard” directions associated with \( \lambda_2 \) and \( \lambda_3 \). It follows from (8), that in the auxiliary \( a \)-space the self-similarity \( \Delta a \sim (t_0 - t)^{1/2} \) is uniform in all directions. However, in the physical space the scales are different. Following Kuznetsov & Ruban 2000, we show how an anisotropic self-similarity emerges in the flow. The analysis for the Euler equation coincides with that for the integrable hydrodynamics (7).

Decompose the Jacoby matrix \( \tilde{J} \) in the bases of eigenvectors of the direct \((\tilde{J}\psi^{(n)} = \lambda_n \psi^{(n)})\) and conjugate \((\tilde{\psi}^{(n)} \tilde{J} = \lambda_n \tilde{\psi}^{(n)})\) spectral problems:

\[
J_{ik} = \frac{\partial x_k}{\partial a_i} = \sum_{n=1}^{3} \lambda_n \psi_i^{(n)} \tilde{\psi}_k^{(n)}. \tag{10}
\]

The two sets of eigenvectors are mutually orthogonal:

\[(\tilde{\psi}^{(n)} \cdot \psi^{(m)}) = \delta_{nm}.\]

In a vicinity of the point of collapse the eigenvectors can be regarded as approximately constant.

Decompose the vectors \( x \) and \( \nabla a \) in (10) in the respective bases, denoting their components by \( X_n \) and \( A_n \):

\[
X_n = (x \cdot \psi^{(n)}), \quad \frac{\partial}{\partial A_n} = (\tilde{\psi}^{(n)} \cdot \nabla_a).
\]
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The vector $\Delta \mathbf{a}$ can be represented in terms of $A_n$ as follows:

$$\Delta a_\alpha = \sum_n \psi^{(n)}_\alpha \overline{\psi}^{(n)} |2 A_n.$$

As a result, (10) can be expressed as

$$\frac{\partial X_1}{\partial A_1} = \tau + \Gamma_{mn} A_m A_n, \quad (11)$$
$$\frac{\partial X_2}{\partial A_2} = \lambda_2, \quad \frac{\partial X_3}{\partial A_3} = \lambda_3, \quad (12)$$

where

$$\Gamma_{mn} = \gamma_{\alpha\beta} (\lambda_2 \lambda_3)^{-1} \psi^{(n)}_\alpha \overline{\psi}^{(m)}_\beta |\overline{\psi}^{(n)}|2 |\overline{\psi}^{(m)}|^2$$

and $\tau = \alpha (t_0 - t)/ (\lambda_2 \lambda_3)$ is assumed to be small. Consequently, size reduction along the directions $\psi^{(2)}$ and $\psi^{(3)}$ is the same as in the auxiliary $a$-space, i.e., $\tau^{1/2}$, but in the soft direction, $\psi^{(1)}$, the spatial scale is $\sim \tau^{3/2}$. Therefore, in terms of new self-similar variables $\zeta_1 = X_1/\tau^{3/2}$, $\zeta_2 = X_2/\tau^{1/2}$ and $\zeta_3 = X_3/\tau^{1/2}$, integration of the system yields for $\zeta_2$ and $\zeta_3$ a linear dependence on $\eta = \Delta \mathbf{a}/\tau^{1/2}$, and for $\zeta_1$ a cubic one:

$$\zeta_1 = (1 + \Gamma_{ij} \eta_i \eta_j) \eta_1 + \frac{1}{2} \Gamma_{11} \eta_1^2 + \frac{1}{3} \Gamma_{11} \eta_1^3, \quad i, j = 2, 3 \quad (13)$$
$$\zeta_2 = \lambda_2 \eta_2, \quad \zeta_3 = \lambda_3 \eta_3. \quad (14)$$

Together with (9), relations (13) and (14) implicitly define the dependence of $\omega$ on $r$ and $t$. The presence of two different self-similarities shows, that the spatial vorticity distribution becomes strongly flattened in the first direction, and a pancake-like structure is formed for $t \rightarrow t_0$. Due to (1) and the degeneracy of the mapping ($J = 0$), vorticity $\omega$ lies in the plane of the pancake. Near the singularity behavior of $\omega$ is defined by the following self-similar asymptotics:

$$\omega = \tau^{-1} \Omega(\zeta_1, \zeta_2, \zeta_3). \quad (15)$$

In essence, in the above analysis one is concerned with the behavior of the mapping near a fold, and thus breaking of vortex lines can be naturally explained within the classical catastrophe theory (Arnold 1981; Arnold 1989).
4. Super-weak collapse

According to the collapse classification of Zakharov & Kuznetsov 1986, breaking of vortex lines is not a weak collapse but a super-weak one, because already a contribution from the singularity to the enstrophy $I = \int |\omega|^2 \, dr$ characterizing the energy dissipation rate due to viscosity is small, $\sim \tau^{1/2}$; a contribution to the total energy is $\sim \tau^{3/2}$. However, the integral $\int |\nabla \omega|^2 \, dr$ is divergent as $t \to t_0$. Thus, the breaking solution $v = v(r, t)$ cannot be continued beyond $t = t_0$ in the Sobolev space $H^2(\mathbb{R}^3)$ with the norm $\| f \|_q = (\sum_{q \leq 2} \int |\nabla^q f|^2 \, dr)^{1/2}$. According to the theorem proved by Beale, Kato & Majda 1984, this suffices for

$$\int_0^{t_0} \sup_r |\omega| \, dt = \infty$$

(16)

to hold. The condition (16) is necessary and sufficient for collapse in the Euler equation, and it is satisfied for (15).

Another restriction follows from the theorem by Constantin, Feferman & Majda 1996, stating that there is no collapse for any $t \in [0, t_0]$ if

$$\int_0^{t_0} \sup \nabla^2 |\xi|^2 \, dt < \infty,$$

(17)

where the supremum is over a region near the maximum of vorticity $|\omega|$. Occurrence of collapse implies divergence of the integral (17) for $\tau \to 0$. Consequently, $\sup |\nabla \xi|$ has to increase at least like $\tau^{-1/2}$. It is evident that, due to solenoidality of $\omega$, either the derivative $(\xi \cdot \nabla) \xi$ in the direction along the vector $\omega$ in the pancake-like region should have no a singularity at the scales of the order of $\tau^{1/2}$ or larger, or the singularity should be weaker than $\tau^{-1/2}$. However, this does not rule out large gradients of $\xi$ in a region separated in the soft direction from the pancake-like region, for instance, with the behavior $\partial \xi / \partial X_1 \sim \tau^{-\alpha}$ with $1/2 \leq \alpha < 3/2$. This conjecture is plausible, since transition from the $a$-space to the physical one involves a significant contraction in the soft direction of the region near the point of breaking: a sphere of radius $\sim \tau^{1/2}$ is mapped into the pancake-like region. Thus, a sphere in the $r$-space of radius $\sim \tau^{1/2}$ containing the pancake includes a large preimage of the region outside the sphere in the $a$-space of radius $\sim \tau^{1/2}$ (the shape of the preimage is governed by higher order terms in the expansion (8)). Hence in the process of breaking of vortex lines three scales can appear: $l_1 \sim \tau^{3/2}$, $l_\perp \sim \tau^{1/2}$ and an intermediate scale $l_{in} \sim \tau^\alpha$ with $1/2 \leq \alpha < 3/2$ (the presence of which assures that there are no contradictions with the theorem of Constantin, Feferman & Majda 1996).
5. Numerical results

To verify the hypothesis that formation of singularity in the solutions to the Euler equation can be due to vortex line breaking, we performed a numerical experiment for the system of equations (1-5). Two features of this system are notable. First, in contrast with the original Euler equation, possessing an infinite number of integrals of motion – the Cauchy invariants – the system (1-5) is partially integrated and therefore contains the Cauchy invariants explicitly. Hence, while the invariants are guaranteed to be conserved when (1-5) is solved numerically, it is necessary to test to which extent they are conserved in the course of direct numerical integration of the Euler equation (6). Second, in the system (1-5) integration in time (in (3)) is separated from integration over space (in (5)), i.e. from inversion of the operator curl.

The system (1-5) is considered under the periodicity boundary conditions and inversion of the operator curl can be performed by the standard spectral techniques with the use of Fast Fourier Transform. The main difficulty in numerical integration of the system stems from the necessity of transition (both direct and inverse) between the variables $r$ and $a$ at each time step. It was circumvented by the use of two independent grids in the $r$-space: a moving one (the $R$-grid), the motion of whose points is governed by (3), and a steady regular one (the $r$-grid), which coincides with the $a$-grid. The numerical algorithm consists of the following steps:

(i) by integrating (3) in time, find new positions of the $R$-grid points;
(ii) compute new values (1) of $\omega$ on the $R$-grid by finite differences;
(iii) by linear interpolation from the values of vorticity at nearby points of the $R$-grid, determine $\omega$ on the $r$-grid (for that, for each point of the regular grid it is necessary to find a tetrahedron, containing the point, whose vertices are the nearest points of the $R$-grid);
(iv) solve the problem (5) to determine flow velocity $v$ on the $r$-grid;
(v) by linear interpolation, determine $v$ on the $R$-grid.

Computations are performed with the resolution of $128^3$ grid points. In order to check numerical stability of the algorithm test runs are made for several initial conditions, which are ABC flows. Any ABC flow is an eigenfunction of the curl and hence it is a steady solution to the force-free Euler equation (6). They are found to remain steady in computations with the time step $dt = 10^{-3}$ up to $t = 4$ with the relative error of the solution being within the $10^{-14}$ threshold, and the Jacobian $J$ being reproduced with the $10^{-13}$ accuracy.

An initial vorticity which we consider is a solenoidal field comprised of random-amplitude Fourier harmonics with an exponentially decaying spectrum; the decay is by 6 orders of magnitude from the first to the
last spherical shell in the Fourier space, the cut-off being at wavenumber 8. It satisfies $\omega \neq 0$ everywhere in the box of periodicity (this enables one to perform the projection (4); it is checked that this condition is not violated at all times during the run). This field does not possess any symmetry. In the course of numerical integration we monitor energy conservation: kinetic energy of the flow remains constant with the accuracy better than 1%. For such an initial condition we observe formation of a peak of $|\omega|$ at a single point. At this point the Jacobian $J$ and $|\omega|^{-1}$ are minimal over space at all times close to the time of collapse, and the minimal values decrease in time to a high precision linearly (Fig. 1). (In this run the time step is $dt = 10^{-4}$ for $t \leq 0.08$, and $dt = 5 \cdot 10^{-6}$ afterwards.) In this run the maximum of vorticity increased almost 20 times before integration was terminated. The final width of the peak of $|\omega|$ is 2-3 times the length of the interval of spatial discretization (Fig. 2 shows a strong localization of $|\omega|$ at the end of the run). Figure 3 illustrates concentration of vorticity lines and formation of a fold near the point of singularity. Formation of similar peaks of vorticity accompanied by

![Graph showing the spatial minimum of $|\omega|^{-1}$ as a function of time.](image)

*Figure 1.* The spatial minimum of $|\omega|^{-1}$ (vertical axis) as a function of time (horizontal axis) at the saturated regime close to the time of collapse. Pluses show the values obtained in computations.
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Figure 2. Vorticity $|\omega|$ (vertical axis) as a function of the coordinates $R_1$ and $R_2$ at the plane $R_3=\text{const}$, containing the point of minimum of $J$ at $t=0.08055$ (close to the time of collapse).

A decrease of $J$ to zero is also observed in several runs for other initial conditions of vorticity in the same class.

At different times the global (over space) minima of $J$ and of $|\omega|^{-1}$ are achieved at four different points. Behavior of the Jacobian in time at one of these points (short-dashed line on Fig. 4) suggests that the second singularity can also be developing; it is not traced down to the time of its collapse, because this is prevented by formation of the first singularity.

The peak of vorticity turns out to be narrow from the moment of its birth. In order to verify that it is not spurious (i.e. it emerges not due to a numerical instability of our algorithm) we have reproduced its formation in computations by a modified algorithm, with different interpolation techniques employed for linear interpolation at step $(iv)$. These techniques introduce some smoothing intended to inhibit formation of a spurious singularity. However, in the new run all numerical data has been reproduced with the relative precision $10^{-6}$.
Figure 3. Isolines of restriction of the function $R_1(a)$ on the plane $a_1 = \text{const}$ through the point $a = (7\pi/32; 41\pi/32; 13\pi/8)$, where the collapse occurs. Small dashes show downhill directions.

To check that the given process can be considered as breaking of vortex lines we compute time dependencies of the Hessian of the Jacobian $\partial^2 J / \partial a_\alpha \partial a_\beta$ at the point of the minimum of $J$. At the final stage of the saturated asymptotic linear behavior of the minimum we did not find any essential temporal variation of its eigenvalues. This agrees qualitatively with the expansion (8). Figure 3 illustrates final positions of vortex lines near the point of collapse. Some anisotropy is observed in the spatial distribution of $\omega(r,t)$ near the maximum of vorticity. However,
due to an apparent lack of spatial resolution we cannot claim that two essentially different scales emerge. The following questions also remain open: Why is the time of occurrence of collapse small compared to the turnover time? Why is the peak of vorticity quite narrow basically from the very moment of its appearance?

The obtained results can be interpreted as the first evidence of the vortex line breaking; the collapse, which is observed numerically, is not related to any symmetry of the initial vorticity distribution and in particular the collapse occurs at a single point.

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