RIEMANN PROBLEMS FOR A CLASS OF COUPLED HYPERBOLIC SYSTEMS OF CONSERVATION LAWS WITH A SOURCE TERM

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Abstract. The Riemann problems for a class of coupled hyperbolic systems of conservation laws with a source term are studied. The Riemann solutions exactly include two kinds: delta-shock solutions and vacuum solutions. In order to see more clearly the influence of the source term on Riemann solutions, the generalized Rankine-Hugoniot relations of delta shock waves are derived in detail, and the position, propagation speed and strength of delta shock wave are given. It is also shown that, as the source term vanishes, the Riemann solutions converge to the corresponding ones of the homogeneous system, which is just the generalized zero-pressure flow model, and contains the one-dimensional zero-pressure flow as a prototypical example. Furthermore, the generalized balance relations associated with the generalized mass and momentum transportation are established for the delta-shock solution. Finally, two typical examples are presented to illustrate the application of our results.

1. Introduction. In this paper, we consider the Riemann problem for the coupled hyperbolic system of conservation laws with a source term

\[
\begin{align*}
vt + (vf(u))_x &= 0, \\
(vu)_t + (vuf(u))_x &= \beta v,
\end{align*}
\]

with the following initial data

\[
(v, u) (0, x) = (v_\pm, u_\pm) \quad (\pm x > 0),
\]

where \(f(u)\) is given to be a smooth and strictly monotone function and the sign of \(v\) is assumed to be unchanging, \(\beta\) is a constant coefficient.

If \(\beta = 0\), namely, as the source term vanishes, then the system (1) becomes the so called generalized zero-pressure flow model, or generalized pressureless gas

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dynamics model, whose Riemann problem was constructively solved by Yang [36] in 1999, and all of the existence, uniqueness, and stability of solutions to viscous perturbations were proved under the generalized Rankine-Hugoniot relation and entropy condition. Yang and Sun [38] solved the Riemann problem with the initial data containing Dirac delta functions and obtained four kinds of different structures for solutions. What is more, Huang [17] considered the Cauchy problem, and established the existence of an entropy solution by studying the interaction of the elementary waves and the generalized characteristics. Mitrovic and Nedeljkov [25] discussed the existence of delta shock waves obtained as a limit of two shock waves.

While if $\beta = 0$ and $f(u) \equiv u, v \geq 0$, then (1) transforms into the noted one-dimensional zero-pressure flow model, which is also called the transport equations, or pressureless gas dynamics model, and has been extensively investigated since 1994 by a large number of scholars. The zero-pressure flow model is derived from Boltzmann equations [1] or the flux-splitting scheme of the full compressible Euler equations [24] and can be used to describe the motion process of free particles sticking together under collision [2], or explain the formation of large-scale structures in the universe [28, 41]. An interesting feature for the pressureless gas dynamics model is that the delta shock waves and vacuum states do appear in Riemann solutions. As for the delta shock waves and zero-pressure flow model, there have been rich results and we refer to [4, 5, 18, 21, 22, 23, 29, 32, 34, 35, 36, 37, 39, 40] and the references cited therein.

Motivated by [17, 25, 36, 38], in which the generalized zero-pressure flow model has been systematically studied, it is natural to study the Riemann problem for the generalized zero-pressure flow system with a source term, such as the friction, damping and relaxation effect, etc. Here, we concentrate on the source term as presented in (1). This kind of source term, sometimes called the Coulomb-like friction term, was earlier introduced by Savage and Hutter [27] to describe granular flow behaviour. Specifically, when $f(u) \equiv u$ and $v \geq 0$, the system (1) becomes formally the zero-pressure flow model with a Coulomb-like friction term, whose Riemann problem was solved constructively by Shen [30] and the delta-shock solution and vacuum states were obtained. Furthermore, if $\beta = -1$, then it models the violent discontinuities in shallow flows with large Froude number studied by Edwards et al. in the paper [10]. Currently, the mathematical study of Euler equations with Coulomb-like term is active and the interested reader may refer to Shen [31] for the Riemann problem for the Chaplygin gas equations and Sun [33] for the generalized Chaplygin gas equations, respectively. Based on them, Guo, Li and Yin [14, 15] recently studied the vanishing pressure limits of Riemann solutions to the Chaplygin and generalized Chaplygin gas equations with a source term, respectively. For more related works about the pressureless Euler system with source term, we refer to [3, 8, 9, 13, 16, 26], etc.

In the present paper, we study the Riemann problem of the system (1), which can be regarded as the generalized zero-pressure flow model with a source term satisfying the Coulomb-like friction law with a constant friction coefficient. One of our main goals is to explore how the delta-shock solution of the generalized zero-pressure flow system develops under the influence of the source term. However, it is not easy to deal with the Riemann problem for the inhomogeneous hyperbolic conservation laws (or called as hyperbolic balance laws) (1) because the characteristics are curved and the Riemann solutions of (1) and (2) are not self-similar any more. Moreover,
influenced by the source term, the state variable $u$ changes linearly at the rate $\beta$ with respect to $t$.

Nevertheless, noting that (1) can be rewritten in a conservative form that is linearly degenerate, so it is possible for us to construct the the exact Riemann solutions of (1) and (2) by contact discontinuities, vacuum state or delta shock wave connecting two states $(v_\pm, u_\pm + \beta t)$. To this end, we derive in detail the generalized Rankine-Hugoniot conditions of the delta shock waves for both the system (1) and its modified conservative form. Then, by proving the generalized balance relations connected with the generalized mass and momentum transportation for the system (1), we show that the generalized total area mass is conserved and does not rely on the time $t$ and while the generalized total area momentum is not conserved any more and depends on the time $t$ because of the appearance of the source term. Finally, we present two typical examples to illustrate the application of our results and proofs.

As to solving the Riemann problem for the generalized zero-pressure flow system with a source term, one of the main difficulties for mathematical analysis is that the characteristics are curved and the Riemann solutions are not self-similar any more. Another stiff difficulty is that since the expression of the Riemann solution can not be concretely and explicitly formulated, so both the existence and uniqueness of delta-shock solutions can only be checked and proved qualitatively and abstractly. To overcome them, we use some new ideas and skills to achieve our goal by virtue of analytical analysis. From this point of view, our work, to a certain degree, extends the results and proofs in [30]. Moreover, with the aid of the obtained knowledge on (1), we can study the nonlinear geometric optic system with a source term conveniently. In what follows, we outlook the context of each section of this paper.

In Section 2, by introducing a new state variable $\tilde{u}(x,t) = u(x,t) - \beta t$, we transform (1) into a conservative form. Then, we construct the exact solutions of the Riemann problem for the conservative form based on discussing the general properties of the conservative form involving jump relations, entropy conditions and elementary waves, and two kinds of solutions containing delta shock waves and vacuum states are obtained. Particularly, for the delta shock wave, we rigorously derive the generalized Rankine-Hugoniot conditions so that the position, strength and propagation speed of which are uniquely determined under the entropy condition.

Section 3 solves the Riemann problem (1) and (2). The generalized Rankine-Hugoniot relations are also proposed, and the existence and uniqueness of solutions are established under the generalized Rankine-Hugoniot relation and entropy condition. In addition, it is verified that the delta-shock solution is indeed a weak one of the Riemann problem (1) and (2) in the distributional sense.

Finally, we derive the generalized mass and momentum balance relations for the delta-shock solution of the system (1) in Section 4, and present two typical examples to show the application of our results and proofs in Section 5.

2. Riemann problem for a modified conservative system. Similar to that in Faccanoni and Mangeney [12] for the shallow water equations, we now introduce a new state variable $\tilde{u}(x,t)$ and do the transformation $\tilde{u}(x,t) = u(x,t) - \beta t$. Then, the system (1) and initial data (2) are reduced into the following conservative form

$$\begin{align*}
v_t + (vf(\tilde{u} + \beta t))_x &= 0, \\
(v\tilde{u})_t + (v\tilde{f}(\tilde{u} + \beta t))_x &= 0
\end{align*}$$

(3)
with initial data
\[(v, \tilde{u})(0,x) = (v_{\pm}, u_{\pm}) \quad (\pm x > 0). \quad (4)\]

This kind of transformation for variables is of effect to study the balance law with the source term like (1) for the reason that a system in the conservative form can be obtained under the transformation. This skill is also introduced in [14, 15, 30, 31, 33] etc. In what follows, we are planing to deal with the Riemann problem (3) and (4).

It is clear that the Riemann solutions of the original Riemann problem (1) and (2) will be achieved by \((v, u)(x,t) = (v, \tilde{u} + \beta t)(x,t)\) from the ones of (3) and (4) directly.

The system (3) can be written as a quasi-linear form as follows
\[
\begin{pmatrix}
1 & 0 \\
\tilde{u} & v
\end{pmatrix}
\begin{pmatrix}
v \\
\tilde{u}
\end{pmatrix}_t + \begin{pmatrix}
f(\tilde{u} + \beta t) & vf'(\tilde{u} + \beta t) \\
\partial f(\tilde{u} + \beta t) & \partial f(\tilde{u} + \beta t)
\end{pmatrix}
\begin{pmatrix}
v \\
\tilde{u}
\end{pmatrix}_x = 0,
\]
from which we can computer that the eigenvalue of (3) is \(\lambda = f(\tilde{u} + \beta t)\) with only one right eigenvector \(r = (1, 0)^T\). Since \(\nabla \lambda \cdot r \equiv 0\), where \(\nabla = (\partial / \partial v, \partial / \partial \tilde{u})\), so the system (3) is non-strictly hyperbolic and the characteristic is linearly degenerated. Thus, the elementary waves involve only contact discontinuities, denoted by \(J\). For the bounded discontinuity \(x = x(t)\), let us denote by \(\sigma(t) = x'(t)\) as its speed, then the Rankine-Hugoniot relations for (3) are
\[
\begin{cases}
-\sigma(t)[v] + [vf(\tilde{u} + \beta t)] = 0, \\
-\sigma(t)[v\tilde{u}] + [v\tilde{u}f(\tilde{u} + \beta t)] = 0,
\end{cases}
\]
where the jump across the discontinuous is \([h] = h_r - h_l\) with \(h_l = h(x(t) - 0, t)\) and \(h_r = h(x(t) + 0, t)\). One can clearly see that the Rankine-Hugoniot relations (5) for the system (3) is obviously different from those of the generalized zero-pressure flow proposed in [36], because the propagation speed of discontinuity in (5) has dependence on \(t\).

If \(\sigma(t) \neq 0\), then we get from (5) that
\[
(v_r - v_l)(v_r \tilde{u}_r f(\tilde{u}_r + \beta t) - v_l \tilde{u}_l f(\tilde{u}_l + \beta t)) = (v_r \tilde{u}_r - v_l \tilde{u}_l)(v_r f(\tilde{u}_r + \beta t) - v_l f(\tilde{u}_l + \beta t)),
\]
from which we have
\[
v_l v_r (\tilde{u}_r - \tilde{u}_l)(f(\tilde{u}_r + \beta t) - f(\tilde{u}_l + \beta t)) = 0.
\]
Hence, the two non-vacuum constant states \((v_l, \tilde{u}_l)\) and \((v_r, \tilde{u}_r)\) can be connected by a contact discontinuity \(J\) if and only if \(\sigma(t) = f(\tilde{u}_l + \beta t)(= \lambda_l) = f(\tilde{u}_r + \beta t)(= \lambda_r)\), namely, \(\tilde{u}_l = \tilde{u}_r\) because of the monotonicity of \(f(u)\).

We now start to construct the solution of the Riemann problem (3) and (4). Without loss of generality, we always assume that
\[
f'(u) > 0, \quad v \geq 0 \quad (6)
\]
through out the paper just for convenience, since the rest cases can be discussed in a similar way. Then, with constants, vacuum and contact discontinuity, one can construct the solution of (3) and (4) by two cases.

For the case \(u_- < u_+\), the Riemann solution consisting of two contact discontinuities with a vacuum state (denoted by \(Vac\)) in between can be expressed as
follows

\[(v, \bar{u})(x, t) = \begin{cases} (v_-, u_-), & x < x_1(t), \\ Vac, & x_1(t) < x < x_2(t), \\ (v_+, u_+), & x > x_2(t), \end{cases} \tag{7} \]

where the locations of contact discontinuities \(J_1 \) and \(J_2 \) are given by \(x_1(t) = \int_0^t f(u_- + \beta t)dt\) and \(x_2(t) = \int_0^t f(u_+ + \beta t)dt\), and the propagation speeds of them are \(\sigma_1(t) = x'_1(t) = f(u_- + \beta t)\) and \(\sigma_2(t) = x'_2(t) = f(u_+ + \beta t)\). Particularly, for the case \(u_- = u_+\), the two constant states \((v_\pm, u_\pm)\) can be directly connected by a contact discontinuity, which is a trivial case.

Let us pay more attention to the case \(u_- > u_+\). At this time, the singularity must happen due to the superposition of linearly degenerate characteristics in some specific initial data. Since the Cauchy problem can not possess a weak \(L^\infty\) solution, so the non-classical situation occurs. In order to construct the Riemann solution with such a non-classical situation, a solution involving a weighted \(\delta\)-measure supported on a curve is adopted, as done in [4, 23, 32, 34, 35, 36] etc. To this end, we give the following definition introduced by Danilov and Shelkovich [6, 7] in 2005, and improved by Kalisch and Mitrovic [19, 20] in 2012.

**Definition 2.1.** A two-dimensional weighted delta function \(p(s)\delta_S\) supported on a smooth curve \(S\) parameterized as \(x = x(s), t = t(s) \ (a \leq s \leq b)\) is defined by

\[\langle p(s)\delta_S, \varphi(x(s), t(s)) \rangle = \int_a^b p(s)\varphi(x(s), t(s))\sqrt{x'(s)^2 + t'(s)^2}ds \tag{8} \]

for all test functions \(\varphi(x, t) \in C_c^\infty (R^1 \times R^+)\).

Just for simplicity, we chose the parameter \(s = t\) so that the strength of the delta shock wave can be denoted by \(w(t) = \sqrt{1 + (x'(t))^2}p(t)\) hereafter. Noticing that the exact definition of the delta shock wave solution is based on the concepts of a vanishing family of distributions and weak asymptotic solutions, so we need to deliver these concepts for completeness.

**Definition 2.2.** Assume that \(f_\varepsilon(x) \in D'(R)\) is a family of distributions which has the dependence on \(\varepsilon \in (0, 1)\); then we say that \(f_\varepsilon(x) = o_D(1)\) if the estimate \(\langle f_\varepsilon, \varphi \rangle = o(1)\) as \(\varepsilon \rightarrow 0\) is true in the usual Landau sense for \(\varphi \in D(R)\). For arbitrarily fixed \(t \in R^+\), if it is a pair of distributions for the limit \(\varepsilon \rightarrow 0\) of \((v_\varepsilon, \bar{u}_\varepsilon)\), then the family of function pairs \((v_\varepsilon, \bar{u}_\varepsilon)\) is named as a weak asymptotic solution of the system (3) where

\[\begin{cases} (v_\varepsilon)_t + (v_\varepsilon f(\bar{u}_\varepsilon + \beta t))_x = o_D(1), \\ (v_\varepsilon \bar{u}_\varepsilon)_t + (v_\varepsilon \bar{u}_\varepsilon f(\bar{u}_\varepsilon + \beta t))_x = o_D(1). \end{cases} \tag{9} \]

Suppose that \(\Gamma = \{\gamma_i | i \in I\} \) is a set of curves in \(\{(x, t)| (x, t) \in (-\infty, \infty) \times [0, \infty]\}\), where \(I\) is a finite index set and \(\gamma_i\) are smooth arcs. Also assume that \(I_0\) is the subset of \(I\) including all the indices for arcs emitting from the x-axis and \(\Gamma_0 = \{x_0^j | j \in I_0\}\) is the set of initial points of \(\gamma_j\) when \(j \in I_0\). Nextly, we show the definition of solutions of the Cauchy problem for the system (3) with measure-valued initial data in the sense of distributions.

**Definition 2.3.** Let \((v, \bar{u})\) be a pair of distributions in which \(v\) may be expressed as

\[v(t, x) = \hat{v}(t, x) + w(x, t)\delta(\Gamma), \tag{10} \]

where
in which \( \hat{v}, \hat{u} \in (R \times R^+) \) and the singular term may be given by
\[
w(x, t)\delta(\Gamma) = \sum_{i \in I} w_i(x, t)\delta(\gamma_i). \tag{11}
\]

If we take the initial data
\[
(v, \hat{u})(x, 0) = \left( \hat{v}_0(x) + \sum_{j \in \Lambda_0} w_j(x_j^0, 0)\delta(x - x_j^0), \hat{u}_0(x) \right), \tag{12}
\]
where \( \hat{v}_0, \hat{u}_0 \in L^\infty(R) \), then \((v, \hat{u})\) is named as a generalized delta shock wave solution of the Cauchy problem (3) and (12) when the following integral identities hold for all \( \Gamma \) and the Riemann problem (3) and (4) in the form
\[
\sum_{i \in I} w_i(x, t)\frac{\partial \varphi(x, t)}{\partial l} \bigg|_{\gamma_i} = \frac{\partial \varphi(x, t)}{\partial x} \bigg|_{\gamma_i} = \frac{S_{it}}{S_{tx}} \bigg|_{\gamma_i} \]
\[
\int_{\gamma_i} (\hat{v}\varphi_t + \hat{v}f(\hat{u} + \beta t)\varphi_x) \, dx \, dt + \sum_{i \in I} \int_{\gamma_i} w_i(x, t)\frac{\partial \varphi(x, t)}{\partial l} \, dl
\]
\[
+ \int_{R} \hat{v}_0(x)\varphi(x, 0) \, dx + \sum_{k \in \Lambda_0} \int_{\gamma_i} w_k(x_k^0, 0)\varphi(x_k^0, 0) = 0, \tag{13}
\]

\[
\int_{\gamma_i} (\hat{v}\varphi_t + \hat{v}f(\hat{u} + \beta t)\varphi_x) \, dx \, dt + \sum_{i \in I} \int_{\gamma_i} w_i(x, t)\hat{u}_\delta(x, t)\frac{\partial \varphi(x, t)}{\partial l} \, dl
\]
\[
+ \int_{R} \hat{v}_0(x)\hat{u}_0(x)\varphi(x, 0) \, dx + \sum_{k \in \Lambda_0} \int_{\gamma_i} w_k(x_k^0, 0)\hat{u}_\delta(x_k^0, 0)\varphi(x_k^0, 0) = 0 \tag{14}
\]
hold for all \( \varphi \in C^\infty_c(R_1 \times R^+) \), in which \( \frac{\partial \varphi(x, t)}{\partial l} \bigg|_{\gamma_i} \) denotes the tangential derivative of \( \varphi \) on \( \gamma_i \) and \( \int_{\gamma_i} \) is the line integral along \( \gamma_i \). More precisely, let us denote the arcs with \( \gamma_i = \{(x, t) : S_i(x, t) = 0, S_i \in C^1, S_{tx} \neq 0 \} \); then
\[
\frac{d\varphi(x, t)}{dt} \bigg|_{\gamma_i} = \left( \frac{\partial \varphi}{\partial t} - \frac{S_{it}}{S_{tx}} \frac{\partial \varphi}{\partial x} \right) \bigg|_{S_i(x, t) = 0} \tag{15}
\]
is a \( \delta \)-derivative with respect to \( t \), which is the tangential derivative on the curve \( \gamma_i \). The velocity of the delta shock wave on the curve \( \gamma_i \) can be calculated by
\[
\sigma(x, t)|_{\gamma_i} = f(\hat{u}_\delta(x, t) + \beta t)|_{\gamma_i} = -\frac{S_{it}}{S_{tx}} \bigg|_{\gamma_i}. \tag{16}
\]

The concept of generalized delta shock wave solution in Definition 2.3 should be understood based on that of weak asymptotic solutions in Definition 2.2. That is, the family of function pairs \((v_\varepsilon, \hat{u}_\varepsilon)\) is regarded as a weak asymptotic solution of the Cauchy problem (3) and (12) if it is a pair of distributions for the limit \( \varepsilon \to 0 \) of \((v_\varepsilon, \hat{u}_\varepsilon)\), which satisfies the system (9) and the initial data
\[
(v_\varepsilon, \hat{u}_\varepsilon)|_{\varepsilon = 0} - (v, \hat{u})(x, 0) = o_D(1) \quad \text{as} \quad \varepsilon \to 0. \tag{17}
\]
The weak asymptotic solution is constructed such that the terms not involving a distributional limit cancel each other in the limit \( \varepsilon \to 0 \) situation and the problem about multiplication of distributions is automatically eliminated.

Now, for the case \( u_- > u_+ \), we propose to look for a piecewise smooth solution of the Riemann problem (3) and (4) in the form
\[
(v, \hat{u})(x, t) = \begin{cases}
(v_-, u_-), & x < x(t), \\
w(t)\delta(x - x(t)), & x = x(t), \\
(v_+, u_+), & x > x(t),
\end{cases} \tag{18}
\]
where \( x = x(t) \) means the delta shock wave curve, \( w(t) = \sqrt{1 + \sigma^2(t)}p(t) \) and \( \sigma(t) = \frac{dx}{dt} \)  denote the strength and propagation speed of delta shock wave, respectively, \( \tilde{u}_\delta \) is the assignment of \( \tilde{u} \) on this delta shock wave curve. In fact, the delta-shock solution (18) to the Riemann problem (3) and (4) is the simplest example that the graph contains only one arc.

In what follows, we prove that the delta-shock solution of the form (18) for the Riemann problem (3) and (4) satisfy the following generalized Rankine-Hugoniot relations

\[
\begin{align*}
\frac{dx}{dt} &= \sigma(t), \\
\frac{dw(t)}{dt} &= [v]\sigma(t) - [vf(\tilde{u} + \beta t)], \\
\frac{d(w(t)\tilde{u}_\delta)}{dt} &= [v]\sigma(t) - [v\tilde{u}f(\tilde{u} + \beta t)],
\end{align*}
\]

and

\[
\sigma(t) = f(\tilde{u}_\delta + \beta t).
\]

To prove it, we assume that the delta shock wave curve \( \Gamma : \{ (x,t) | x = x(t) \} \) is a smooth curve in the upper-half \((x,t)\)-plane across which \( (v,\tilde{u}) \) is a jump discontinuity. Then, the delta-shock solution should satisfy the system (3) in any compactly supported set in the upper-half \((x,t)\)-plane in the sense of distributions, which means that

\[
\begin{align*}
\langle v_t + (vf(\tilde{u} + \beta t))_x, \varphi \rangle &= 0, \\
\langle (v\tilde{u})_t + (v\tilde{u}f(\tilde{u} + \beta t))_x, \varphi \rangle &= 0
\end{align*}
\]

holds for any test function \( \varphi(x,t) \in C^\infty_c(\Omega) \), in which \( \Omega \) is a small ball centered at the point \( P \) that is any point on \( \Gamma \). Then, we make a step further to assume that the intersection points of \( \Omega \) and \( \Gamma \) are \( P_1 = (x(t_1),t_1) \) and \( P_2 = (x(t_2),t_2) \) where \( t_1 < t_2 \), and \( \Omega_- \) and \( \Omega_+ \) are the left-hand and right-hand parts of \( \Omega \) cut by \( \Gamma \) respectively. Then, for any test function \( \varphi(x,t) \in C^\infty_c(\Omega) \), we have

\[
I_\gamma = \int_{\Omega} \int_{t_1}^{t_2} \left( v\tilde{u}^\gamma\varphi_t + v\tilde{u}^\gamma f(\tilde{u} + \beta t)\varphi_x \right) dx dt
\]

\[
= \int_{\Omega_-} \int_{t_1}^{t_2} v_- u_\gamma^- \left( \varphi_t + f(u_- + \beta t)\varphi_x \right) dx dt
\]

\[
+ \int_{\Omega_+} \int_{t_1}^{t_2} v_+ u_\gamma^+ \left( \varphi_t + f(u_+ + \beta t)\varphi_x \right) dx dt
\]

\[
+ \int_{t_1}^{t_2} w(t)\tilde{u}_\delta \left( \varphi_t(x(t),t) + f(\tilde{u}_\delta + \beta t)\varphi_x(x(t),t) \right) dt,
\]

which can be expressed as

\[
I_\gamma = \int_{\Omega_-} \int_{t_1}^{t_2} \left( v_- u_-^\gamma \varphi_t + v_- u_-^\gamma f(u_- + \beta t)\varphi_x \right) dx dt
\]

\[
+ \int_{\Omega_+} \int_{t_1}^{t_2} \left( v_+ u_+^\gamma \varphi_t + v_+ u_+^\gamma f(u_+ + \beta t)\varphi_x \right) dx dt + \int_{t_1}^{t_2} w(t)\tilde{u}_\delta d\varphi(x(t),t).
\]
By applying the divergence theorem and the first equation of (19) and (20), one has
\[
I_\gamma = \int_{\partial \Omega_-} -v_- u_-^\gamma \varphi dx + v_- u_-^\gamma f(u_- + \beta t) \varphi dt \\
+ \int_{\partial \Omega_+} -v_+ u_+^\gamma \varphi dx + v_+ u_+^\gamma f(u_+ + \beta t) \varphi dt + \int_{t_1}^{t_2} w(t) \tilde{u}_3^\gamma d\varphi(x(t), t) \\
= \int_{t_1}^{t_2} \left( (v_+ u_+^\gamma - v_- u_-^\gamma) \frac{dx}{dt} - v_+ u_+^\gamma f(u_+ + \beta t) + v_- u_-^\gamma f(u_- + \beta t) \right) \varphi(x(t), t) dt \\
- \int_{t_1}^{t_2} \frac{d(w(t) \tilde{u}_3^\gamma)}{dt} \varphi(x(t), t) dt,
\]
in which \( \partial \Omega_\pm \) denotes the boundary of \( \Omega_\pm \). Thereby, the second and third equalities of the generalized Rankine-Hugoniot relations (19) can be obtained when \( I_0 \) and \( I_1 \) vanish for all test functions \( \varphi(x, t) \in C_\infty^r(\Omega) \).

In addition to the generalized Rankine-Hugoniot relations (19) and (20), to guarantee uniqueness, the discontinuity must satisfy
\[
\lambda(v_+, u_+) < \sigma(t) < \lambda(v_-, u_-),
\]
namely,
\[
f(u_+ + \beta t) < \sigma(t) = f(\tilde{u}_\delta + \beta t) < f(u_- + \beta t).
\]
Condition (22) is called the entropy condition which means that all the characteristics on both sides of the discontinuity are in-coming. Under the assumption (6), it is equivalent to
\[
u_+ < \tilde{u}_\delta < u_-.
\]

In what follows, the generalized Rankine-Hugoniot relation will be applied in particular to the Riemann problem (3) and (4) for the case \( u_- > u_+ \). In this situation, the Riemann problem is reduced to solving (19) and (20) with the initial data
\[
t = 0 : \quad x(0) = 0, \quad w(0) = 0.
\]

By virtue of the knowledge concerning delta shock waves in [30, 32, 34, 35, 36], we find that \( \tilde{u}_\delta \) is a constant and \( w(t) \) is a linear function of \( t \). Thus, a delta-shock solution of (3) and (4) can be assumed to take the form
\[
\delta : \quad x(t) = \int_0^t \sigma(\bar{t}) d\bar{t} = \int_0^t f(\tilde{u}_\delta + \beta \bar{t}) d\bar{t}, \quad w(t) = w_0 t, \quad \tilde{u}_\delta(t) = \tilde{u}_\delta
\]
satisfying (24), in which \( w_0 \) and \( \tilde{u}_\delta \) are to be determined constants. It yields from (25) and (19)-(20) that
\[
\begin{align*}
\sigma(t) &= f(\tilde{u}_\delta + \beta t), \\
w_0 \tilde{u}_\delta &= [v] \sigma(t) - [vf(\tilde{u} + \beta t)], \\
w_0 \tilde{u}_\delta &= [v\tilde{u}] \sigma(t) - [v\tilde{u}f(\tilde{u} + \beta t)], \\
[w_0] &= [v] \sigma(t) - [vf(\tilde{u} + \beta t)] \tilde{u}_\delta + [v\tilde{u}f(\tilde{u} + \beta t)] = 0.
\end{align*}
\]
Under the entropy condition (23), we consider the function equation (27). Let \( G(\tilde{u}_\delta) \) be the left side of (27), then we can calculate that
\[
G(u_+) = ([v]u_+ - [v\tilde{u}])f(u_+ + \beta t) - [v f(\tilde{u} + \beta t)]u_+ + [v\tilde{u} f(\tilde{u} + \beta t)]
\]
\[
= v_-(u_- - u_+) \left( f(u_+ + \beta t) - f(u_- + \beta t) \right)
\]
\[
= -v_- [u] [f(u + \beta t)],
\]
and we have, similarly,
\[
G(u_-) = v_+ [u] [f(u + \beta t)].
\]
With the help of the assumption (6), one can easily check that
\[
G(u_+) \cdot G(u_-) = -v_- v_+ [u]^2 [f(u + \beta t)]^2 < 0.
\]
Moreover, differentiating \( G(\tilde{u}_\delta) \) with respect to \( \tilde{u}_\delta \) yields that
\[
G'(\tilde{u}_\delta) = [v] f(\tilde{u} + \beta t) + ([v] \tilde{u}_\delta - [v\tilde{u}]) f'(\tilde{u}_\delta + \beta t) - [v\tilde{u} f(\tilde{u} + \beta t)]
\]
\[
= v_+ \left( f(\tilde{u}_\delta + \beta t) - f(u_+ + \beta t) \right) + v_- \left( f(u_- + \beta t) - f(\tilde{u}_\delta + \beta t) \right)
\]
\[
+ \left( v_+(\tilde{u}_\delta - u_+) + v_-(u_- - \tilde{u}_\delta) \right) f'(\tilde{u}_\delta + \beta t)
\]
\[
> 0
\]
by remembering (6) in mind. Hence, by using zero point theorem, we conclude that there exists one and only one zero point of function \( G(\tilde{u}_\delta) \) in the interval \((u_+, u_-)\). That is to say, the equation (27) owns a unique solution, denoted by \( \tilde{u}_\delta \), under the entropy condition (23). Returning to the relation (25), then \( x(t) \) and \( w_0 \) will be solved uniquely. Thereby, the following result can be summarized.

**Theorem 2.4.** Under the assumption (6), for the case \( u_- > u_+ \), the Riemann problem (3) and (4) admits one and only one entropy solution of the form (18), where
\[
x(t) = \int_0^t \sigma(t) d\tilde{t} = \int_0^t f(\tilde{u}_\delta + \beta t) d\tilde{t}, \quad w(t) = w_0 t,
\]
and the constants \( w_0 \) and \( \tilde{u}_\delta \) are determined uniquely by (23) and (26).

It is obviously that Theorem 2.4 is true for the case \( f'(u) > 0, v \leq 0 \). What is more, by the same arguments as used in the above proofs with only few modifications, we can obtain the results similar to Theorem 2.4 for the case \( f'(u) < 0, v \geq 0 \) and \( f'(u) < 0, v \leq 0 \). Thus, we have the following theorem.

**Theorem 2.5.** Assume that \( f(u) \) is a smooth and strictly monotone function, and assume that the sign of \( v \) is unchanging. Then there exists a unique entropy solution of the Riemann problem (3) and (4), which contains a vacuum state in the case \( f(u_- + \beta t) < f(u_+ + \beta t) \) and a delta shock wave in the case \( f(u_- + \beta t) > f(u_+ + \beta t) \).

**Remark 1.** From the discussions above, one can find that the solutions of the Riemann problem (3) and (4) when \( \beta = 0 \) are identical with those for the homogeneous pressureless Euler system, namely, the generalized zero-pressure flow. Besides, when \( f(u) \equiv u, v \geq 0 \), the results obtained here are identical with the results in Shen [30]. It is clear that both the strength \( w(t) \) and the assignment \( \tilde{u}_\delta \) for the delta shock wave have no dependence on the parameter \( \beta \) for the inhomogeneous situation \( (\beta \neq 0) \). The difference between the homogeneous \( (\beta = 0) \) and inhomogeneous \( (\beta \neq 0) \) situations only lies in that all the characteristics, contact
discontinuities and the delta shock wave curve transform from straight lines into parabolas in the \((x,t)\)-plane under the influence of the source term.

**Remark 2.** Compared with the generalized zero-pressure flow system \((\beta = 0)\), the delta shock wave curve \(x = x(t)\) is a parabola instead of a straight line in the \((x,t)\)-plane for the solution of the Riemann problem \((3)\) and \((4)\) with the inhomogeneous situation \((\beta \neq 0)\). Thus, the inverse function of \(x = x(t)\) does not exist globally in some cases, as will be discussed in next section. Therefore, a localization method has to be adopted to derive the corresponding generalized Rankine-Hugoniot conditions \((19)\), which is obviously different with the classical method for the homogeneous situation \((\beta = 0)\) presented in Tan, Zhang and Zheng \([34, 35]\), Sheng and Zhang \([32]\), Yang \([36]\).

3. **Riemann problem for the original system** \((1)\). In this section, we solve the Riemann problem \((1), (2)\).

If \(u_- < u_+\), then the Riemann solution of \((1), (2)\) can be expressed as

\[
(v, u)(x, t) = \begin{cases} 
(v_-, u_- + \beta t), & x < x_1(t), \\
\text{Vac}, & x_1(t) < x < x_2(t), \\
(v_+, u_+ + \beta t), & x > x_2(t),
\end{cases}
\tag{28}
\]

in which the locations and propagation speeds of contact discontinuities \(J_1\) and \(J_2\) are identical with those given in the Riemann solution of \((3), (4)\).

Analogy, in the distributional sense, we define the weak solutions of the Cauchy problem for the system \((1)\) with delta measure initial data as follows.

**Definition 3.1.** If we consider the initial data of the form

\[
(v, u)(x, 0) = \left(\hat{v}_0(x) + \sum_{j \in I_0} w_j(x_j^0, 0)\delta(x - x_j^0), u_0(x)\right),
\tag{29}
\]

where \(\hat{v}_0, u_0 \in L^\infty(R)\), then the pair of distributions \((v, u)\) are called as a generalized delta-shock solution of the Cauchy problem of the Riemann problem \((1)\) and \((29)\) if the following integral identities

\[
\int_{R^+} \int_{R} \left(\hat{v}\phi_t + \hat{v}f(u)\phi_x\right)dxdt + \sum_{i \in I} \int_{\gamma_i} w_i(x, t)\frac{\partial \phi(x, t)}{\partial l} dl \\
+ \int_{R} \hat{v}_0(x)\phi(x, 0)dx + \sum_{k \in I_0} w_k(x_k^0, 0)\varphi(x_k^0, 0) = 0,
\tag{30}
\]

\[
\int_{R^+} \int_{R} \left(\hat{u}\psi_t + \hat{u}f(u)\psi_x\right)dxdt + \sum_{i \in I} \int_{\gamma_i} w_i(x, t)u_\delta(x, t)\frac{\partial \phi(x, t)}{\partial l} dl \\
+ \int_{R} \hat{u}_0(x)\varphi(x, 0)dx + \sum_{k \in I_0} w_k(x_k^0, 0)u_\delta(x_k^0, 0)\varphi(x_k^0, 0) \\
= \int_{R^+} \int_{R} \beta\hat{v}\phi dxdt + \sum_{i \in I} \int_{\gamma_i} \beta w_i(x, t)\varphi(x, t) dl
\tag{31}
\]

are true for all test function \(\varphi \in C_c^\infty(R_1 \times R^+)\). The velocity of the delta shock wave on the curve \(\gamma_i\) is given by

\[
\sigma(x, t)|_{\gamma_i} = f(u_\delta(x, t))|_{\gamma_i} = -\left.\frac{S_i}{S_{ex}}\right|_{\gamma_i}. \tag{32}
\]
Therefore, when $u_- > u_+$, we search the Riemann solution of (1) and (2) in the form

$$
(v, u)(x, t) = \begin{cases} 
(v_-, u_- + \beta t), & x < x(t), \\
(w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\
(v_+, u_+ + \beta t), & x > x(t),
\end{cases}
$$

in which $u_\delta(t)$ is the assignment of $u$ on this delta shock wave curve and $u_\delta(t) - \beta t$ is assumed to be a constant. As we have done before, one can prove that the delta-shock solution of the Riemann problem (1) and (2) must obey the following generalized Rankine-Hugoniot conditions

$$
\begin{align*}
\frac{dx}{dt} &= \sigma(t) = f(u_\delta(t)), \\
\frac{dv}{dt} &= [v]\sigma(t) - [vf(u)], \\
\frac{dw}{dt} &= [vu]\sigma(t) - [vvf(u)] + \beta w(t),
\end{align*}
$$

and the over-compressive entropy condition (21), namely, at this time,

$$
f(u_+ + \beta t) < \sigma(t) = f(u_\delta(t)) < f(u_- + \beta t),
$$

which is just equivalent to

$$
u_+ + \beta t < u_\delta(t) < u_- + \beta t
$$

under the assumption (6). In (34), it should be noticed that the jumps across the discontinuity are

$$
\begin{align*}
[vu] &= v_+(u_+ + \beta t) - v_-(u_- + \beta t), \\
[vf(u)] &= v_+f(u_+ + \beta t) - v_-f(u_- + \beta t), \\
[vuf(u)] &= v_+(u_+ + \beta t)f(u_+ + \beta t) - v_- (u_- + \beta t)f(u_- + \beta t).
\end{align*}
$$

Similarly, one can prove that $\sigma(t)$ and $w(t)$ are determined by the generalized Rankine-Hugoniot condition (34) and the entropy condition (36). In a word, for the case $u_- > u_+$, we have the following result.

**Theorem 3.2.** Under the assumption (6), if $u_- > u_+$, the Riemann problem (1) and (2) admits one and only one entropy solution of the form (33), where

$$
x(t) = \int_0^t \sigma(\bar{t})d\bar{t} = \int_0^t f(u_\delta(\bar{t}))d\bar{t}, \quad w(t) = w_0t, \quad u_\delta(t) = \tilde{u}_\delta + \beta t,
$$

in which the constants $\tilde{u}_\delta$ and $w_0$ are uniquely determined by (23) and (26).

**Proof.** When $u_- > u_+$, we should firstly establish the existence and uniqueness of solutions of (1) and (2) in the form (33). Considering that $u_\delta(t) - \beta t$ is a constant, so one has

$$
\begin{align*}
\frac{d(w(t)u_\delta(t))}{dt} &= w(t)\frac{du_\delta(t)}{dt} + u_\delta(t)\frac{dw(t)}{dt} = \beta w(t) + u_\delta(t)\frac{dw(t)}{dt}.
\end{align*}
$$

Thus, we immediately get from the third equation of (34) that

$$
u_\delta(t)\frac{dw(t)}{dt} = [vu]\sigma(t) - [vvf(u)],
$$

in which the jump relations (37) across the discontinuity have been used.
Multiplying \(u_\delta(t)\) on both sides of the second equation of (34) and together with (39), it yields that
\[
([v]u_\delta(t) - [vu])f(u_\delta(t)) - [vf(u)]u_\delta(t) + [vu]f(u) = 0,
\]
(40)
in which \([vu], [vf(u)], [vu]f(u)\] should be understood in the manner of (37). Noting that \(u_\delta(t) - \beta t\) is a constant again, we set \(\tilde{u}_\delta = u_\delta(t) - \beta t\) without loss of generality, and rewrite \([vu], [vf(u)]\) and \([vu]f(u)\) in (37) as \([v(\tilde{u} + \beta t)], [vf(\tilde{u} + \beta t)]\) and \([v(\tilde{u} + \beta t)]]\), respectively, as we used in Section 2. Then, (40) is reduced into
\[
([v](\tilde{u}_\delta + \beta t) - [v(\tilde{u} + \beta t)])f(\tilde{u}_\delta + \beta t) - [vf(\tilde{u} + \beta t)](\tilde{u}_\delta + \beta t)
\]
\[
+ [v(\tilde{u} + \beta t)]f(\tilde{u} + \beta t)] = 0,
\]
that is,
\[
([v]\tilde{u}_\delta - [v\tilde{u}])f(\tilde{u}_\delta + \beta t) - [vf(\tilde{u} + \beta t)]\tilde{u}_\delta + [v\tilde{u}]f(\tilde{u} + \beta t)] = 0,
\]
which is nothing but the equation (27). This fact implies that the delta-shock solution of (1) and (2) can be achieved by performing the substitution \(u_\delta(t) = \tilde{u}_\delta + \beta t\) from the one of (1) and (2) directly. Therefore, from (25), (19) and (34), we conclude that the location, weight and propagation speed of the delta shock wave of (1) and (2) can be expressed as (38), where the constants \(\tilde{u}_\delta\) and \(w_0\) are determined by (23) and (26).

In the following, we need to prove that the delta-shock solution of the form (33) satisfies the system (1) in the sense of distributions. For this purpose, we prove that
\[
\left\{ \begin{array}{l} \langle v_t + (vf(u))_x, \varphi \rangle = 0, \\
\langle (vu)_t + (vu) f(u)_x, \varphi \rangle = \langle \beta v, \varphi \rangle
\end{array} \right.
\]
holds for any test function \(\varphi \in C_0^\infty(R_+^2)\), which is equivalent to prove that
\[
\left\{ \begin{array}{l} I_1 = \langle v, \varphi_t \rangle + \langle v (f(u)), \varphi_x \rangle = 0, \\
I_2 = \langle vu, \varphi_t \rangle + \langle vu f(u), \varphi_x \rangle = -\langle \beta v, \varphi \rangle.
\end{array} \right.
\]
(41)

Noting that \(x'(t) = f(\tilde{u}_\delta + \beta t)\), so if \(\beta \cdot (\tilde{u}_\delta - f^{-1}(0)) < 0\), then there exists a critical point \((x^*, t^*) = \left( \int_0^{t^*} f(\tilde{u}_\delta + \beta t)dt, \frac{f^{-1}(0) - \tilde{u}_\delta}{2} \right)\) on the curve of the delta shock wave, in which \(f^{-1}(0)\) is the inverse image of \(f\) at 0, such that the inverse function of \(x(t)\) should be solved for \(t \leq t^*\) and \(t > t^*\), respectively. In this situation, we need to divide the integral region \((-\infty, +\infty) \times [0, +\infty)\) into two parts \((-\infty, +\infty) \times [0, t^*)\) and \((-\infty, +\infty) \times (t^*, +\infty)\) and then to check that (41) holds in the two parts respectively. Otherwise, if \(\beta \cdot (\tilde{u}_\delta - f^{-1}(0)) > 0\), then the curve of the delta shock wave is strictly monotonic with respect to the time \(t\) such that there exists an inverse function of \(x = x(t)\) for all the time.

Without loss of generality, we suppose that \(\tilde{u}_\delta - f^{-1}(0) > 0\) and \(\beta > 0\) for conciseness. Then, one can solve \(t = t(x)\) from \(x(t) = \int_0^t f(\tilde{u}_\delta + \beta t)dt\) uniquely. Moreover, we can calculate from (38) that
\[
\frac{d\varphi(x(t), t)}{dt} = \varphi_t(x(t), t) + f(u_\delta(t))\varphi_x(x(t), t),
\]
(42)
so we have

\[ I_1 = \int_0^\infty \int_{-\infty}^\infty (v\varphi_t + v f(u)\varphi_x)dxdt \]
\[ = \int_0^\infty \int_{-\infty}^{\varepsilon(t)} (v_- \varphi_t + v_- f(u_- + \beta t)\varphi_x)dxdt \]
\[ + \int_0^\infty \int_{\varepsilon(t)}^{\infty} (v_+ \varphi_t + v_+ f(u_+ + \beta t)\varphi_x)dxdt \]
\[ + \int_0^\infty w(t)(\varphi_t(x(t), t) + f(u_\delta(t))\varphi_x(x(t), t))dt. \]

By exchanging the integral orders, one has

\[ I_1 = \int_0^\infty \int_{-\infty}^\infty v_- \varphi_t dtdx + \int_0^\infty \int_{-\infty}^{\varepsilon(t)} v_- f(u_- + \beta t)\varphi_x dtdx + \int_0^\infty \int_{\varepsilon(t)}^{\infty} v_+ \varphi_t dtdx \]
\[ + \int_0^\infty \int_{\varepsilon(t)}^{\infty} v_+ f(u_+ + \beta t)\varphi_x dtdx + \int_0^\infty w(t)d\varphi(x(t), t) \]
\[ = -\int_0^\infty v_- \varphi(x(t), t)dx + \int_0^\infty \int_{\varepsilon(t)}^{\infty} v_- f(u_- + \beta t)\varphi(x(t), t)dt + \int_0^\infty \int_{\varepsilon(t)}^{\infty} v_+ \varphi(x(t), t)dx \]
\[ - \int_0^\infty v_+ f(u_+ + \beta t)\varphi(x(t), t)dt - \int_0^\infty \frac{dw(t)}{dt}\varphi(x(t), t)dt. \]

By applying the variable substitution \( dx = f(\tilde{u}_\delta + \beta t)dt \) and following the notations in (37), one can deduce that

\[ I_1 = -\int_0^\infty \int_{\varepsilon(t)}^{\infty} (v_+ - v_-)f(\tilde{u}_\delta + \beta t) - (v_+ f(u_+ + \beta t) - v_- f(u_- + \beta t) - \frac{dw(t)}{dt}) \varphi(x(t), t)dt \]
\[ = \int_0^\infty \left( [v]f(u_\delta(t)) - [v f(u)] - \frac{dw(t)}{dt} \right) \varphi(x(t), t)dt \]
\[ = 0. \]

For the second equation of (41), one has

\[ I_2 = \int_0^\infty \int_{-\infty}^\infty (v u \varphi_t + v u f(u)\varphi_x)dxdt \]
\[ = \int_0^\infty \int_{-\infty}^{\varepsilon(t)} (v_- (u_- + \beta t)\varphi_t + v_- (u_- + \beta t)f(u_- + \beta t)\varphi_x)dxdt \]
\[ + \int_0^\infty \int_{\varepsilon(t)}^{\infty} (v_+ (u_+ + \beta t)\varphi_t + v_+ (u_+ + \beta t)f(u_+ + \beta t)\varphi_x)dxdt \]
\[ + \int_0^\infty w(t)u_\delta(t)(\varphi_t(x(t), t) + f(u_\delta(t))\varphi_x(x(t), t))dxdt. \]
By exchanging the integral orders as before, we get that

\[
I_2 = \int_0^\infty \int_{x(t)}^\infty v_-(u_+ + \beta t) \varphi_t dt dx + \int_0^\infty \int_{-\infty}^{x(t)} v_-(u_+ + \beta t) f(u_+ + \beta t) \varphi dx dt \\
+ \int_0^\infty \int_0^{t(x)} v_+(u_+ + \beta t) \varphi_t dt dx + \int_0^\infty \int_{x(t)}^\infty v_+(u_+ + \beta t) f(u_+ + \beta t) \varphi dx dt \\
+ \int_0^\infty w(t)(\tilde{\alpha}_t + \beta t) d\varphi(x(t), t),
\]

in which we have

\[
\int_{t(x)}^\infty v_-(u_+ + \beta t) \varphi_t dt = -v_-(u_+ + \beta t(x)) \varphi(x(t), t(x)) - \int_{t(x)}^\infty \beta v_+ \varphi dt, \tag{43}
\]

\[
\int_0^{t(x)} v_+(u_+ + \beta t) \varphi_t dt = v_+(u_+ + \beta t(x)) \varphi(x(t), t(x)) - \int_0^{t(x)} \beta v_- \varphi dt, \tag{44}
\]

\[
\int_0^\infty w(t)(\tilde{\alpha}_t + \beta t) d\varphi(x(t), t) = -\int_0^\infty \frac{d(w(t)u_+(t))}{dt} \varphi(x(t), t) dt. \tag{45}
\]

Substituting (43)-(45) into \(I_2\) yields that

\[
I_2 = -\int_0^\infty v_-(u_+ + \beta t(x)) \varphi(x, t(x)) dx - \int_0^\infty \int_{t(x)}^\infty \beta v_- \varphi dt dx \\
+ \int_0^\infty v_-(u_+ + \beta t) f(u_+ + \beta t) \varphi(x(t), t) dt + \int_0^\infty v_+(u_+ + \beta t(x)) \varphi(x(t), t(x)) dx \\
- \int_0^\infty \int_0^{t(x)} \beta v_+ \varphi dt dx - \int_0^\infty v_+(u_+ + \beta t) f(u_+ + \beta t) \varphi(x(t), t) dt \\
- \int_0^\infty \frac{d(w(t)u_+(t))}{dt} \varphi(x(t), t) dt.
\]

Using the variable substitution and exchanging the integral orders again, one can obtain that

\[
I_2 = \int_0^\infty A(t) \varphi(x(t), t) dt - \int_0^\infty \int_{-\infty}^{x(t)} \beta v_- \varphi dx dt - \int_0^\infty \int_{x(t)}^\infty \beta v_+ \varphi dx dt, \tag{46}
\]

where

\[
A(t) = -v_-(u_+ + \beta t) f(\tilde{\alpha}_t + \beta t) + v_-(u_+ + \beta t) f(u_+ + \beta t) \\
+ v_+(u_+ + \beta t) f(\tilde{\alpha}_t + \beta t) - v_+(u_+ + \beta t) f(u_+ + \beta t) - \frac{d(w(t)u_+(t))}{dt},
\]

which can be written as, by virtue of the third equation of (34) and (37),

\[
A(t) = [v_0] \sigma(t) - [vu f(u)] - \frac{d(w(t)u_+(t))}{dt} = -\beta w(t).
\]

Thereby, the second equation of (41) is true in the sense of distributions. The proof is finished.

We now make a further study of the Riemann problem (1) and (2). Since the characteristic equation of the system (1) is

\[
\frac{dx}{dt} = f(u), \quad \frac{du}{dt} = \beta, \tag{47}
\]
so for the given initial point \((x_0, 0)\), the characteristic curve of \((1)\) starting from this point and the value of \(u\) along the characteristic curve before intersection can be expressed, respectively, by

\[
\begin{align*}
x &= \int_0^t f(u)\,dt + x_0, \quad u = u_- + \beta t \quad \text{for} \quad x_0 < 0, \\
x &= \int_0^t f(u)\,dt + x_0, \quad u = u_+ + \beta t \quad \text{for} \quad x_0 > 0.
\end{align*}
\]

One can conclude from the above discussions that, for any given constant states \((v_\pm, u_\pm + \beta t)\), the the Riemann solution of \((1)\) and \((2)\) under the assumption \((6)\) can be constructed by contact discontinuities, vacuum states or delta shock waves. Precisely, if \(u_- < u_+\), then the Riemann solution consists of two contact discontinuities with the vacuum state in between, as shown in \((28)\). Specifically, in this case, both the characteristics and contact discontinuities are curved towards the righthand (or left-hand) side for \(\beta > 0\) (or \(\beta < 0\)), as displayed in Fig.1. While if \(u_- > u_+\), then the Riemann solution of \((1)\) and \((2)\) involves a delta shock wave in the form of \((33)\), which connects two states \((v_\pm, u_\pm + \beta t)\). Similarly, if \(\beta > 0\) (or \(\beta < 0\)), then both the delta shock wave and contact discontinuities are curved toward the right-hand (or left-hand) side, please see Fig.2 bellow.

![Figure 1](image_url)

**Figure 1.** The Riemann solution of \((1)\) and \((2)\) when \(u_- < 0 < u_+\) and \(\beta > 0\) for a given time \(t\) before the time \((f^{-1}(0) - u_-)/\beta\). The left is the \((u, v)\)-phase plane, and the right is the corresponding \((x, t)\)-characteristic plane.

**Remark 3.** One can see from \((28)\), \((33)-(36)\) that the solutions of the Riemann problem \((1)\) and \((2)\) when \(\beta = 0\) are coincident with those for the homogeneous generalized zero-pressure flow system with the same Riemann initial data. However, the state variable \(u\) supported on both the characteristics and the delta shock wave curve varies linearly at the same rate \(\beta\) with respect to the time \(t\). In fact, as pointed out in [30] that, the contact discontinuity lines \(J_1\) and \(J_2\) translate with the same speed \(\beta\) in the phase plane. What is more, the strength \(w(t)\) is the same for the inhomogeneous \((\beta \neq 0)\) and homogeneous \((\beta = 0)\) situations because both the characteristics and the delta shock wave curve are curved into parabolas and have the same degree of curvature for the inhomogeneous \((\beta \neq 0)\) situation, such that the mass accumulation on the delta shock wave curve has the same rate for the inhomogeneous \((\beta \neq 0)\) and homogeneous \((\beta = 0)\) situations.
Remark 4. The rest cases \( f'(u) > 0, v \leq 0 \), or \( f'(u) < 0, v \geq 0 \), or \( f'(u) < 0, v \leq 0 \) can be analogously discussed with only few modifications, and the similar result like Theorem 2.5 will be obtained for the Riemann problem (1) and (2), so we omit it here.

4. Generalized balance relations for the delta shock wave. In view of that the generalized zero-pressure flow is a natural generalization of the traditional zero-pressure flow modelling the conservation of mass and momentum, so in this section, we briefly discuss the geometrical and physical sense of generalized Rankine-Hugoniot condition of the delta shock wave for the system (1), and derive the generalized balance relations connected with the area transportation, which include the generalized mass and momentum transportation.

Nevertheless, for the delta-shock solution to the system (1), the classical conservation laws usually do not hold. Thus, motivated by [30, 31, 33], the generalized conservation laws for the system (1) is proposed as follows.

Denoted by

\[
S_v(t) = \int_{-\infty}^{x(t)} v(x,t)dx + \int_{x(t)}^{\infty} v(x,t)dx, \\
S_{vu}(t) = \int_{-\infty}^{x(t)} v(x,t)u(x,t)dx + \int_{x(t)}^{\infty} v(x,t)u(x,t)dx,
\]

Then we prove the following result to describe the generalized conservation laws for (1).

**Theorem 4.1.** Suppose that \( (v(x,t), u(x,t)) \) is a generalized delta shock wave type solution of Cauchy problem for the system (1) when the initial data belong to the delta shock wave type. Let us also suppose that \( v(x,t) = \hat{v}(x,t) + w(t)\delta(\Gamma) \) and \( \Gamma : \{(x,t)|x=x(t)\} \) is the discontinuity, in which both \( \hat{v}(x,t) \) and \( u(x,t) \) are compactly supported functions with respect to \( x \). Then, we have the following relations

\[
\frac{d(S_v(t) + w(t))}{dt} = 0, \\
\frac{d^2(S_{vu}(t) + w(t)u_\delta(t))}{dt^2} = 0.
\]
Proof. Differentiating (50) with respect to $t$ leads to
\[
\frac{dS_v(t)}{dt} = \sigma(t)v(x(t)^-, t) + \int_{x(t)}^{x(t)^+} v_x dx - \sigma(t)v(x(t)^+, t) + \int_{x(t)}^{\infty} v_x dx \\
= -\sigma(t)[v] - \int_{-\infty}^{x(t)^+} (vu)_x dx - \int_{x(t)}^{\infty} (vu)_x dx \\
= -\sigma(t)[v] + [vu],
\]
where $\sigma(t) = f(u_\delta(t))$. Thus, (52) is obtained from (54) and the second equation of (34).

By using a similar discussion, differentiating (51) with respect to $t$, we have
\[
\frac{dS_{vu}(t)}{dt} = \sigma(t)(vu)(x(t)^-, t) + \int_{x(t)^-}^{x(t)^+} (vu)_t dx - \sigma(t)(vu)(x(t)^+, t) + \int_{x(t)^+}^{\infty} (vu)_t dx \\
= -\sigma(t)[vu] - \int_{-\infty}^{x(t)^+} (vuf(u))_x dx - \int_{x(t)}^{\infty} (vuf(u))_x dx + \beta S_v(t) \\
= -\sigma(t)[vu] + [vuf(u)] + \beta S_v(t).
\]
Together with the third equation of (34) and (55), we get that
\[
\frac{d(S_v(t) + w(t)u_\delta(t))}{dt} = \beta(S_v(t) + w(t)).
\]
Thereby, the relation (53) is established by combining (52) and (56) together. We finish the proof of Theorem 4.1.

Remark 5. The balance relation (52) in Theorem 4.1 indicates that the generalized total area mass $S_v(t) + w(t)$ has no dependence on time $t$, because it is conserved and unchanged with respect to the time $t$. By contrast, it is observed from (56) that the generalized total area momentum $S_{vu}(t) + w(t)u_\delta(t)$ increases or decreases linearly with respect to the time $t$ due to the present of the source term.

5. Two typical examples. In this section, we present two typical examples to illustrate the application of our results and proofs in the above sections. In these two systems, we mainly focus our attention on the delta shock wave, the most interesting topic, of the modified conservative system. In fact, the discussion on the modified conservative system will pave the way for the study of the delta-shock solution of the original inhomogeneous system, since the Riemann solutions of the original system can be directly obtained by a variable substitution from the ones of the modified conservative system.

Example 5.1. Consider the Riemann problem of the zero-pressure flow system with the Coulomb-like friction law with a constant coefficient
\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
\rho u_t + (\rho u^2)_x &= \beta \rho,
\end{aligned}
\]
with the initial data
\[
(\rho, u)(0, x) = (\rho_\pm, u_\pm) \quad (\pm x > 0),
\]
with $u_- > u_+$, where $\rho, u$ denote the gas density and velocity, respectively, and $\rho_\pm, u_\pm$ are given constants satisfying $\rho_\pm > 0$. In this case, $f(u) = u, f'(u) = 1 > 0.$
By introducing a new state variable \( \tilde{u}(x,t) = u(x,t) - \beta t \), the system (57) and the initial data (58) are reduced into
\[
\begin{aligned}
\rho_t + (\rho(\tilde{u} + \beta t))_x &= 0, \\
(\rho \tilde{u})_t + (\rho \tilde{u}(\tilde{u} + \beta t))_x &= 0
\end{aligned}
\] (59)
and
\[
(\rho, \tilde{u}) (0, x) = (\rho_{\pm}, u_{\pm}) (\pm x > 0).
\] (60)
The eigenvalue of (59) is \( \lambda = \tilde{u} + \beta t \) with only one right eigenvector \( r = (1, 0)^T \) satisfying \( \nabla \lambda \cdot r \equiv 0 \), so the system (59) is degenerated.

When \( u_- > u_+ \), we look for the delta-shock solution of the Riemann problem (59) and (60) of the form (18). Then the following generalized Rankine-Hugoniot relation holds,
\[
\begin{aligned}
\frac{d}{dt} w_0 &= \sigma(t), \\
\frac{d}{dt} \tilde{u} \delta &= [\rho] \sigma(t) - [\rho(\tilde{u} + \beta t)], \\
\frac{d}{dt} \rho(t) \tilde{u} \delta &= [\rho \tilde{u}] \sigma(t) - [\rho \tilde{u}(\tilde{u} + \beta t)],
\end{aligned}
\] (61)
and
\[
\sigma(t) = \tilde{u} \delta + \beta t,
\] (62)
which is equivalent to
\[
\begin{aligned}
w_0 &= [\rho] \sigma(t) - [\rho(\tilde{u} + \beta t)], \\
w_0 \tilde{u} \delta &= [\rho \tilde{u}] \sigma(t) - [\rho \tilde{u}(\tilde{u} + \beta t)], \\
\sigma(t) &= \tilde{u} \delta + \beta t,
\end{aligned}
\] (63)
in which \( \tilde{u} \delta \) is a constant and \( w(t) = w_0 t \), where \( w_0 \) is a constant to be determined. One can get from (63) that
\[
([\rho] \tilde{u} \delta - [\rho \tilde{u}]) (\tilde{u} \delta + \beta t) - [\rho(\tilde{u} + \beta t)] \tilde{u} \delta + [\rho \tilde{u}(\tilde{u} + \beta t)] = 0,
\]
namely,
\[
[\rho] \tilde{u} \delta^2 - 2[\rho \tilde{u}] \tilde{u} \delta + [\rho \tilde{u}^2] = 0,
\] (64)
which is a quadratic equation of one variable. Let \( G(\tilde{u} \delta) = [\rho] \tilde{u} \delta^2 - 2[\rho \tilde{u}] \tilde{u} \delta + [\rho \tilde{u}^2] \), then under the entropy condition
\[
u_+ < \tilde{u} \delta < \nu_-,
\] (65)
one can check that
\[
G(u_+) \cdot G(u_-) = -\rho_- \rho_+ |u|^4 < 0,
\]
\[
G'(\tilde{u} \delta) = 2(\rho_-(u_- - \tilde{u} \delta) + \rho_+(\tilde{u} \delta - u_+)) > 0.
\]
Thus, the equation (64) owns a unique solution \( \tilde{u} \delta \in (u_+, u_-) \). Of course, one can directly solve (64) to obtain that
\[
\tilde{u} \delta = \frac{\sqrt{\rho_- u_-} + \sqrt{\rho_+ u_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}} = \sigma(t) - \beta t.
\] (66)
Then, one has
\[
w(t) = w_0 t = \sqrt{\rho_- - \rho_+} (u_- - u_+) t, \quad x(t) = \tilde{u} \delta t + \frac{1}{2} \beta t^2.
\] (67)
Therefore, the unique delta-shock solution of (59) and (60) is obtained as follows

\[
(\rho, \tilde{u})(x, t) = \begin{cases} 
(\rho_-, u_-), & x < \tilde{u}_d + \frac{1}{2} \beta t^2, \\
(w(t)\delta(x - (\tilde{u}_d + \frac{1}{2} \beta t^2)), \tilde{u}_d), & x = \tilde{u}_d + \frac{1}{2} \beta t^2, \\
(\rho_+, u_+), & x > \tilde{u}_d + \frac{1}{2} \beta t^2,
\end{cases}
\]

where \(\sigma(t) = \tilde{u}_d + \beta t\) and \(w(t) = w_0 t\) are uniquely obtained from (66) and (67), respectively.

**Example 5.2.** Consider the Riemann problem

\[
\begin{cases} 
v_t + \left( \frac{vU}{\sqrt{1 + U^2}} \right)_x = 0, \\
(vU)_t + \left( \frac{vU^2}{\sqrt{1 + U^2}} \right)_x = \beta v,
\end{cases}
\]

and

\[
(v, U)(0, x) = (v_\pm, U_\pm) \quad (\pm x > 0),
\]

with \(U_- > U_+, v_\cdot v_+ > 0\).

The system (68) can be obtained by performing the transformation \(U = u/v\), i.e., \(u = vU\) for the nonlinear geometric optics system with a source term

\[
\begin{cases} 
u_t + \left( \frac{u^2}{\sqrt{u^2 + v^2}} \right)_x = 0, \\
v_t + \left( \frac{uv}{\sqrt{u^2 + v^2}} \right)_x = \beta v.
\end{cases}
\]

If \(\beta = 0\), then the system (70) was systematically studied in [11, 36, 38, 40].

To obtain a modified conservative form of (68), we introduce a new variable \(\tilde{U}(x, t) = U(x, t) - \beta t\), then the Riemann problem (68) and (69) are reduced into

\[
\begin{cases} 
v_t + \left( \frac{v(\tilde{U} + \beta t)}{\sqrt{1 + (\tilde{U} + \beta t)^2}} \right)_x = 0, \\
(v\tilde{U})_t + \left( \frac{v\tilde{U}(\tilde{U} + \beta t)}{\sqrt{1 + (\tilde{U} + \beta t)^2}} \right)_x = 0,
\end{cases}
\]

and

\[
(v, \tilde{U})(0, x) = (v_\pm, U_\pm) \quad (\pm x > 0).
\]

For the system (71),

\[
\lambda = f(\tilde{U} + \beta t) = \frac{\tilde{U} + \beta t}{\sqrt{1 + (\tilde{U} + \beta t)^2}}, \quad f'(U) = \frac{1}{(\sqrt{1 + U^2})^3} > 0.
\]
Let \((v, U, \sigma(t), w(t), \tilde{U}_\delta)\) donate the delta-shock solution of (71) and (72). Then the generalized Rankine-Hugoniot relation

\[
\begin{align*}
\frac{dx}{dt} &= \sigma(t), \\
\frac{dw(t)}{dt} &= [v] \sigma(t) - \left[ \frac{v\tilde{U}}{\sqrt{1 + (\tilde{U} + \beta t)^2}} \right], \\
\frac{d(w(t)\tilde{U}_\delta)}{dt} &= [v\tilde{U}] \sigma(t) - \left[ \frac{v\tilde{U}(\tilde{U} + \beta t)}{\sqrt{1 + (\tilde{U} + \beta t)^2}} \right],
\end{align*}
\]

holds, where

\[
\sigma(t) = \frac{\tilde{U}_\delta + \beta t}{\sqrt{1 + (\tilde{U}_\delta + \beta t)^2}},
\]

which can be reduced to

\[
\begin{align*}
w_0 &= [v] \sigma(t) - \left[ \frac{v(\tilde{U} + \beta t)}{\sqrt{1 + (\tilde{U} + \beta t)^2}} \right], \\
\tilde{w}_0\tilde{U}_\delta &= [v\tilde{U}] \sigma(t) - \left[ \frac{v\tilde{U}(\tilde{U} + \beta t)}{\sqrt{1 + (\tilde{U} + \beta t)^2}} \right], \\
\sigma(t) &= \frac{\tilde{U}_\delta + \beta t}{\sqrt{1 + (\tilde{U}_\delta + \beta t)^2}},
\end{align*}
\]

in which \(\tilde{U}_\delta\) is a constant and \(w(t) = w_0 t\), where \(w_0\) is a constant to be determined.

What is more, the entropy condition is, in this situation,

\[
\frac{U_+ + \beta t}{\sqrt{1 + (U_+ + \beta t)^2}} < \frac{\tilde{U}_\delta + \beta t}{\sqrt{1 + (\tilde{U}_\delta + \beta t)^2}} < \frac{U_- + \beta t}{\sqrt{1 + (U_- + \beta t)^2}},
\]

which is equivalent to

\[
U_+ < \tilde{U}_\delta < U_-.
\]

It follows from (75) that

\[
([v]\tilde{U}_\delta - [v\tilde{U}]) \frac{\tilde{U}_\delta + \beta t}{\sqrt{1 + (\tilde{U}_\delta + \beta t)^2}} - \left[ \frac{v\tilde{U}(\tilde{U} + \beta t)}{\sqrt{1 + (\tilde{U} + \beta t)^2}} \right] \tilde{U}_\delta \\
+ \left[ \frac{v\tilde{U}(\tilde{U} + \beta t)}{\sqrt{1 + (\tilde{U} + \beta t)^2}} \right] = 0.
\]

Let \(G(\tilde{U}_\delta)\) be the left side of (78), then one can calculate that, under the entropy conditions (76) and (77),

\[
G(U_+) \cdot G(U_-) = -v_- v_+ [U]^2 \left[ \frac{U + \beta t}{\sqrt{1 + (U + \beta t)^2}} \right]^2 < 0,
\]
and
\[ G'(\tilde{U}_\delta) = v_+ \left( \frac{\tilde{U}_\delta + \beta t}{\sqrt{1 + (\tilde{U}_\delta + \beta t)^2}} - \frac{U_+ + \beta t}{\sqrt{1 + (U_+ + \beta t)^2}} \right) + v_- \left( \frac{U_- + \beta t}{\sqrt{1 + (U_- + \beta t)^2}} - \frac{\tilde{U}_\delta + \beta t}{\sqrt{1 + (U_\delta + \beta t)^2}} \right) \]
\[ + \left( v_+(\tilde{U}_\delta - U_+) + v_-(U_- - \tilde{U}_\delta) \right) \frac{1}{\sqrt{1 + (\tilde{U}_\delta + \beta t)^2}}. \]

which is positive for \( v_-, v_+ > 0 \), and negative for \( v_-, v_+ < 0 \). Therefore, the equation (78) possesses a unique solution \( \tilde{U}_\delta \in [U_+, U_-] \). Owing to (75), we can solve \( \sigma(t) \) and \( w_0 \) uniquely. So the unique delta-shock solution of (71) and (72) can be expressed as

\[ (v, \tilde{U})(x, t) = \begin{cases} 
(v_-, U_-), & x < x(t), \\
(w(t) \delta(x - x(t)), \tilde{U}_\delta), & x = x(t), \\
(v_+, U_+), & x > x(t),
\end{cases} \quad (79) \]

where \( \sigma(t) \) and \( w(t) = w_0 t \) are uniquely determined by (75) and (77), respectively.

The formula (79) indicates that there is a weighted Dirac delta function only in \( v \) for the system (68) and (71). With the aid of the transformation \( u = Uv \), we can conjecture that weighted Dirac delta functions may appear simultaneously in the state variables \( u \) and \( v \) for the system (70), which will be left for our future study.

In fact, the theory of delta shock waves with Dirac delta functions developing in both state variables has been established by Yang and Zhang [39, 40] for a class of nonstrictly hyperbolic systems of conservation laws. See also Zhang and Zhang [42] for more discussions.

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