Hydrodynamics with spin — pseudo-gauge transformations, semi-classical expansion, and Pauli-Lubański vector

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Abstract

Recent progress in the formulation of relativistic hydrodynamics for particles with spin one-half is reviewed. We start with general arguments advising introduction of a tensor spin chemical potential that plays a role of the Lagrange multiplier coupled to the spin angular momentum. Then, we turn to a discussion of spin-dependent distribution functions that have been recently proposed to construct a hydrodynamic framework including spin and serve as a tool in phenomenological studies of hadron polarization. Distribution functions of this type are subsequently used to construct the equilibrium Wigner functions that are employed in the semi-classical kinetic equation. The semi-classical expansion elucidates several aspects of the hydrodynamic approach, in particular, shows the ways in which different possible versions of hydrodynamics with spin can be connected by pseudo-gauge transformations. These results point out at using the de Groot - van Leeuwen - van Weert versions of the energy-momentum and spin tensors as the most natural and complete physical variables. Finally, a totally new method is proposed to design hydrodynamics with spin, which is based on the classical treatment of spin degrees of freedom. Interestingly, for small values of the spin chemical potential the new scheme brings the results that coincide with those obtained before. The classical approach also helps us to resolve problems connected with the normalization of the spin polarization three-vector. In addition, it clarifies the role of the Pauli-Lubański vector and the entropy current conservation.
Keywords: relativistic heavy-ion collisions, relativistic hydrodynamics, spin polarization, pseudo-gauge transformations, semi-classical expansion, Pauli-Lubański vector

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1. Introduction

It is commonly expected that non-central heavy-ion collisions at low and intermediate energies create strongly-interacting systems with large global angular momenta. This may produce a spin polarization of the hot and dense matter in a way similar to the magnetomechanical effects of Einstein and de Haas [1] and Barnett [2]. Interestingly, the longitudinal polarization of the Λ hyperons was discussed as early as in 1980s by Jacob and Rafelski in connection with the formation of a quark-gluon plasma [3]. However, the first heavy-ion experiments that measured the Λ spin polarization in Dubna [4], as well as at CERN [5] and BNL [6], reported negative results. Consequently, it is no wonder that the first positive measurements of the Λ–hyperon spin polarization [7, 8] initiated vast theoretical studies analyzing the spin polarization and vorticity formation in heavy-ion collisions. In particular, they have explored the importance of the spin-orbit coupling [9, 10, 11, 12, 13] and global equilibrium with a rigid rotation [14, 15, 16, 17, 18]. Related to these works are: kinetic models of spin dynamics [19, 20, 21, 22], the works on anomalous hydrodynamics [23, 24], and the Lagrangian formulation of hydrodynamics [25, 26, 27]. In the experimental context, it has been suggested that the global angular momentum should be indeed reflected in the polarization of observed hadrons, for example, in the case of Λ hyperons and vector mesons [7, 10, 11, 28].

Despite these efforts, the present works say little about the changes of spin polarization during the heavy-ion collision process, in particular, if the latter is described with the help of fluid dynamics which has become now the basic ingredient of heavy-ion models (the latest developments within relativistic hydrodynamics have recently been reviewed in Refs. [29, 30]). This
is quite surprising, since the studies of fluids with spin have a long history initiated in 1930s [31, 32, 33]. Recent works have contributed mainly to our understanding of global equilibrium states which exhibit interesting features of vorticity-spin alignment [14, 17], and polarization effects present at the final, kinetic freeze-out stage of collisions [16, 21, 34]. This situation has changed very recently with the formulation of a hydrodynamic framework [35, 36] that allows for studies of space-time evolution of the spin polarization [37, 38].

The formulation of hydrodynamics with spin proposed in Refs. [35, 36] is based, however, on a particular choice of the forms of energy-momentum and spin tensors. More recent works have clarified the use of different forms of such tensors by the analysis of their physical significance and by explicit constructions of the pseudo-gauge transformations that relate different frameworks [39, 40]. In the first sections of this work we review some of the most important findings of Refs. [39, 40] and introduce the concept of a spin chemical potential. In the following sections we advocate future applications of the de Groot - van Leeuwen - van Weert (GLW) formalism [41]. A totally new aspect of our presentation is the introduction of a framework that treats spin classically [42]. We show that this approach agrees with the quantum GLW results for small values of the spin chemical potential. Moreover, this framework: i) indicates how one can avoid problems with the normalization of the average polarization three-vector for large values of the spin potential, ii) helps to define microscopic conditions that validate the use of the proposed equilibrium functions, and iii) can be used to define entropy current and prove its conservation within the perfect-fluid approach with spin. The results obtained with the classical treatment of spin can be helpful for future construction of a dynamic quantum description based on the spin density matrix and not restricted to small values of the spin chemical potential.

Conventions and notation: We use the following conventions and notation for the metric tensor, the four-dimensional Levi-Civita tensor, and the scalar product in flat Minkowski space: \( g_{\mu\nu} = \text{diag}(+1,-1,-1,-1) \), \( \epsilon^{0123} = -\epsilon_{0123} = 1 \), \( a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu \). Three-vectors are shown in bold font and a dot is used to denote the scalar product of both four- and three-vectors, hence, \( a^\mu b_\mu = a \cdot b = a^0 b^0 - a \cdot b \). The symbol \( 1 \) is used for a \( 4 \times 4 \) unit matrix. The traces in spin and spinor spaces are distinguished by using the symbols \( \text{tr}_2 \) and \( \text{tr}_4 \), respectively. The symbol \( \text{tr} \) denotes the trace in the Hilbert space. The Lorentz invariant measure in the momentum space is denoted as
\[ dP = \frac{d^3p}{(2\pi)^3E_p}, \]  

where \( E_p = \sqrt{m^2 + p^2} \) is the on-mass-shell particle energy, and \( p^\mu = (E_p, \mathbf{p}) \).

The particle momenta which are not necessarily on the mass shell and appear as arguments of the Wigner functions are denoted by the four-vector \( k^\mu \).

The square brackets denote antisymmetrization, \( t^{[\mu\nu]} = (t^{\mu\nu} - t^{\nu\mu})/2 \). The symbol of tilde is used to denote dual tensors, which are obtained from the rank-two antisymmetric tensors \( a_{\mu\nu} \) by contraction with the Levi-Civita symbol and division by a factor of two. For example,

\[ \tilde{a}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} a^{\alpha\beta}. \]

The inverse transformation is

\[ d^{\rho\sigma} = -\frac{1}{2} \epsilon^{\rho\sigma\mu\nu} a_{\mu\nu}. \]

Throughout the paper we use natural units, \( c = \hbar = k_B = 1 \), except for the parts where we discuss semi-classical expansions and it is important to display the Planck constant \( \hbar \) explicitly.

Our considerations are restricted to hydrodynamics of spin-1/2, massive particles. We distinguish between the particle rest frame PRF (connected with a single particle) and the local fluid rest frame LFRF (connected with a group of particles forming a fluid element). The quantities defined in PRF are denoted by an asterisk, while unlabeled quantities refer to the laboratory frame LAB. For a particle with four-momentum \( p^\mu \) in the laboratory frame, the particle rest frame is obtained by boosting \( p^\mu \) by the three-velocity \(-\mathbf{p}/E_p\). The boosts considered in this work are all canonical (also known as pure) boosts \([36, 43]\).

2. Spin chemical potential

Constructions of the hydrodynamic frameworks rely on the local conservation laws. In standard situations one includes the conservation of energy, linear momentum, baryon number, and possibly of other conserved quantities.
like electric charge or strangeness. While dealing with particles with spin, it is necessary to include also the conservation of angular momentum, which in this case becomes an independent condition. In the similar way as the conservation of energy, linear momentum and baryon number are connected with the introduction of temperature $T$, fluid velocity $u^\mu$, and baryon chemical potential $\mu_B$, the conservation of the angular momentum for particles with spin requires introduction of a new spin chemical potential $\mu_s$. As a matter of fact, the spin chemical potential is not a scalar but an antisymmetric rank-two tensor.

Inclusion of the conservation of angular momentum in field-theoretical frameworks is connected with a well-known ambiguity of the localization of energy and spin densities. For any conserved energy-momentum and spin tensors, $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$, it is possible to construct a new pair of such quantities, $T'+^{\mu\nu}$ and $S'+^{\lambda,\mu\nu}$, that are also conserved and yield the same total values of the energy, linear momentum, and angular momentum as the original tensors. The conversion rules that relate such energy-momentum and spin tensors are known as pseudo-gauge transformations. The most known example of such a transformation is the Belinfante construction that starts with the canonical expressions for $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$, and ends up with a symmetric energy-momentum tensor, $T_\text{Bel}^{\mu\nu} = T_\text{Bel}^{\nu\mu}$, and a vanishing spin tensor $S_\text{Bel}^{\lambda,\mu\nu} = 0$. In this case the total angular momentum, $J_\text{Bel}^{\lambda,\mu\nu}$, has the form of an orbital one and can be expressed solely by $T_\text{Bel}^{\mu\nu}$.

The formalism of relativistic hydrodynamics with spin necessarily uses the concept of the spin tensor. Hence, the problems connected with the energy and spin localization naturally appear in such an approach. The results presented below shed new light on many of them. In particular, a semi-classical expansion (in powers of $\hbar$) of the Wigner function shows that the leading-order term of the canonical energy-momentum tensor is in fact symmetric. Hence, it can be used in General Relativity which is a classical theory. Moreover, we construct an explicit form of the pseudo-gauge transformation which connects the canonical framework with the formalism of de Groot, van Leeuwen, and van Weert (GLW). Both the canonical and GLW approaches include spin dynamics, and the results obtained in one framework can be translated to the other one. From this point of view, the Belinfante

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1In the case where the electric charge conservation is independent from the conservation of the baryon number or if the flavor-changing weak interactions may be neglected.
approach seems to be an effective description that might be valid only in special situations; for example, in the case where the spin dynamics is solely determined by other physical quantities like thermal vorticity. In general, the information about spin cannot be encoded in the form of orbital angular momentum, since the latter can be eliminated by changing the Lorentz frame. We note that similar issues are currently discussed in the context of the nucleon spin [45].

2.1. Pseudo-gauge transformations

As the hydrodynamic frameworks are based on the local conservation laws that are used to define local thermodynamic equilibrium, let us start our discussion with general considerations about locally conserved quantities.

In relativistic quantum field theory, according to the Noether theorem, for each continuous symmetry of the action there is a corresponding conserved current. The currents associated with the translational symmetry and the Lorentz symmetry\(^2\) are the canonical energy-momentum and angular momentum tensors,

\[
\hat{T}^{\mu\nu}_{\text{can}} = \sum_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \hat{\psi}^a)} \partial^\nu \hat{\psi}^a - g^{\mu\nu} \mathcal{L},
\]

\[
\hat{J}^{\mu,\lambda\nu}_{\text{can}} = x^\lambda \hat{T}^{\mu\nu}_{\text{can}} - x^\nu \hat{T}^{\mu\lambda}_{\text{can}} + \hat{S}^{\mu,\lambda\nu}_{\text{can}} \equiv \hat{L}^{\mu,\lambda\nu}_{\text{can}} + \hat{S}^{\mu,\lambda\nu}_{\text{can}}.
\]

Here \(\mathcal{L}\) is the Lagrangian density, the hat symbol denotes operators, \(\hat{L}_{\text{can}}\) is the orbital part of the total angular momentum, while \(\hat{S}_{\text{can}}\) is its spin part called the spin tensor and obtained from the expression

\[
\hat{S}^{\mu,\lambda\nu}_{\text{can}} = -i \sum_{a,b} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \hat{\psi}^a)} D_a (J^\lambda_{\nu})^b \hat{\psi}^b,
\]

where \(D^a_b\) is the irreducible representation matrix of the Lorentz group specific for the considered field. The above tensors fulfill the following equations,

\[
\partial_\mu \hat{T}^{\mu\nu}_{\text{can}} = 0,
\]

\[
\partial_\mu \hat{J}^{\mu,\lambda\nu}_{\text{can}} = \hat{T}^{\lambda\nu}_{\text{can}} - \hat{T}^{\nu\lambda}_{\text{can}} + \partial_\mu \hat{S}^{\mu,\lambda\nu}_{\text{can}} = 0,
\]

\(^2\)By Lorentz symmetry transformations we understand here Lorentz boosts and rotations.
which also gives
\[ \partial_\mu \hat{S}_{\text{can}}^{\mu,\nu} = \hat{T}_{\text{can}}^{\nu\lambda} - \hat{T}_{\text{can}}^{\lambda\nu}. \]  
At this point it is important to stress that the canonical energy-momentum tensor is not symmetric, hence the spin part of the angular momentum is not conserved separately. Only the total angular momentum is conserved, which is expressed above by Eq. (8).

It turns out, however, that the energy-momentum and angular momentum tensors are not defined uniquely. Different pairs can be obtained by either changing the Lagrangian density or by means of the so-called pseudo-gauge transformations [44],
\[ \hat{T}'^{\mu\nu} = \hat{T}^{\mu\nu} + \frac{1}{2} \partial_\lambda \left( \hat{\Phi}^{\lambda,\mu\nu} - \hat{\Phi}^{\mu,\lambda\nu} - \hat{\Phi}^{\nu,\lambda\mu} \right), \]
\[ \hat{S}'^{\lambda,\mu\nu} = \hat{S}^{\lambda,\mu\nu} - \hat{\Phi}^{\lambda,\mu\nu}, \]
where \( \hat{\Phi} \) is a rank-three tensor field antisymmetric in the last two indices. It is called and below referred to as a superpotential. The newly defined tensors preserve the total energy, linear momentum, and angular momentum,
\[ \hat{P}^{\nu} = \int_{\Sigma} d\Sigma_\mu \hat{T}^{\mu\nu} = \int_{\Sigma} d\Sigma_\mu \hat{T}'^{\mu\nu}, \]
\[ \hat{J}^{\lambda\nu} = \int_{\Sigma} d\Sigma_\mu \hat{J}^{\mu,\lambda\nu} = \int_{\Sigma} d\Sigma_\mu \hat{J}'^{\mu,\lambda\nu}, \]
as well as the conservation equations (7) and (8). In Eqs. (12) and (13) the symbol \( d\Sigma_\mu \) specifies the element of a space-like hypersurface. The latter can be determined, for example, by the condition \( t = \text{const.} \), which implies in this case that \( d\Sigma_\mu = (dV, 0, 0, 0) \) with \( dV \) being a standard three-volume element.

A special case of the pseudo-gauge transformation is that starting with the canonical definitions \( \hat{T}_{\text{can}}^{\mu\nu} \) and \( \hat{S}_{\text{can}}^{\lambda,\mu\nu} \), and using the spin tensor itself as a

\[ \text{The pseudo-gauge transformation defined by Eqs. (10) and (11) can be still generalized by adding the term } \partial_\alpha Z^{\alpha\lambda\mu\nu} \text{ (with the properties } Z^{\alpha\lambda\mu\nu} = -Z^{\lambda\alpha\mu\nu} \text{ and } Z^{\alpha\lambda\mu\nu} = -Z^{\alpha\lambda\nu\mu} \text{) to the right-hand side of Eq. (11) [44]. However, in our considerations it is enough to consider the case } Z^{\alpha\lambda\mu\nu} = 0. \]
superpotential, i.e., $\hat{\Phi} = \hat{S}_{\text{can}}^{\lambda,\mu\nu}$ [46]. In this case, the new spin tensor vanishes, while the new energy-momentum tensor has the form

$$\hat{T}_{\text{Bel}}^{\mu\nu} = \hat{T}_{\text{can}}^{\mu\nu} + \frac{1}{2} \partial_{\lambda} \left( \hat{S}_{\text{can}}^{\lambda,\mu\nu} - \hat{S}_{\text{can}}^{\mu,\lambda\nu} - \hat{S}_{\text{can}}^{\nu,\lambda\mu} \right).$$  \hspace{1cm} (14)

In the following we shall refer to $\hat{T}_{\text{Bel}}^{\mu\nu}$ defined by Eq. (14) as to the Belinfante energy-momentum tensor.

2.2. Local-equilibrium density operator

The local-equilibrium density operator $\hat{\rho}_{\text{LEQ}}$ is obtained by maximizing the entropy with the constraints specifying mean densities of the conserved currents over the hyper-surface $\Sigma$ [47, 48, 49]. In particular, the projections of the mean energy-momentum tensor and charge current onto the normalized vector perpendicular to $\Sigma$ must be equal to the prescribed values $T_{0}^{\mu\nu}$ and $j_{0}^{\mu}$,

$$n_{\mu} \text{tr} \left( \hat{\rho}_{\text{LEQ}} \hat{T}^{\mu\nu} \right) = n_{\mu} T_{0}^{\mu\nu}, \quad n_{\mu} \text{tr} \left( \hat{\rho}_{\text{LEQ}} \hat{j}^{\mu} \right) = n_{\mu} j_{0}^{\mu}.  \hspace{1cm} (15)$$

In addition to the energy, linear momentum, and charge densities, one should a priori include the angular momentum density among the constraints listed above, namely,

$$n_{\mu} \text{tr} \left( \hat{\rho}_{\text{LEQ}} \hat{J}^{\mu,\lambda\nu} \right) = n_{\mu} \text{tr} \left[ \hat{\rho}_{\text{LEQ}} \left( x^{\lambda} \hat{T}^{\mu\nu} - x^{\nu} \hat{T}^{\mu\lambda} + \hat{S}_{\text{can}}^{\mu,\lambda\nu} \right) \right] = n_{\mu} j_{0}^{\mu,\lambda\nu}.  \hspace{1cm} (16)$$

If one uses the Belinfante versions of the energy-momentum and spin tensors, the latter vanishes and Eq. (16) becomes redundant — it is already included in Eq. (15). In this case the local-equilibrium density operator reads

$$\hat{\rho}_{\text{LEQ}} = \frac{1}{Z} \exp \left[ - \int_{\Sigma} d\Sigma_{\mu} \left( \hat{T}_{\text{Bel}}^{\mu\nu} \beta_{\nu} - \xi \hat{j}_{\mu}^{\nu} \right) \right],  \hspace{1cm} (17)$$

where $\beta_{\nu}$ and $\xi$ are the Lagrange multiplier functions, whose meaning is the ratio between the local four-velocity $u^{\mu}$ and temperature $T$ (a four-temperature vector) and the ratio between local chemical potential $\mu$ and $T$, respectively.

On the contrary, if one works with the canonical tensors, the spin tensor does not vanish and Eq. (16) contains a non-trivial part

$$n_{\mu} \text{tr} \left( \hat{\rho}_{\text{LEQ}} \hat{S}^{\mu,\lambda\nu} \right) = n_{\mu} S_{0}^{\mu,\lambda\nu}.  \hspace{1cm} (18)$$
Since this is an independent constraint, one has to introduce an antisymmetric tensor field $\omega_{\lambda\nu}$. It is dubbed the spin chemical potential or the spin polarization tensor $^4$. In analogy with the variable $\xi$, the components of $\omega$ play a role of Lagrange multipliers coupled to the spin tensor. Including Eq. (18) in the construction of local equilibrium leads to the form

$$\hat{\rho}_{\text{LEQ}} = \frac{1}{Z} \exp \left[ - \int_{\Sigma} d\Sigma_{\mu} \left( \hat{T}_{\text{can}}^{\mu\nu} \beta_{\nu} - \frac{1}{2} \omega_{\lambda\nu} \hat{S}_{\text{can}}^{\mu,\lambda\nu} - \xi \hat{j}^{\mu} \right) \right].$$

(19)

It is interesting now to compare Eqs. (17) and (19). Substituting Eq. (14) into Eq. (19) and using the property that the canonical spin tensor is invariant under cyclic changes of the indices, we find

$$\hat{\rho}_{\text{LEQ}} = \frac{1}{Z} \exp \left[ - \int_{\Sigma} d\Sigma_{\mu} \left( \hat{T}_{\text{Bel}}^{\mu\nu} \beta_{\nu} - \frac{1}{2} (\omega_{\lambda\nu} - \varpi_{\lambda\nu}) \hat{S}_{\text{can}}^{\mu,\lambda\nu} - \xi \hat{j}^{\mu} \right) \right],$$

(20)

where $\varpi_{\lambda\nu}$ is the thermal vorticity $^5$

$$\varpi_{\lambda\nu} = -\frac{1}{2} (\partial_\lambda \beta_\nu - \partial_\nu \beta_\lambda).$$

(21)

Hence, Eqs. (17) and (19) are equivalent only if $\omega_{\lambda\nu} = \varpi_{\lambda\nu}$. This fact indicates that the description based on the Belinfante tensors is reduced compared to the description employing the canonical tensors.

2.3. Global-equilibrium density operator

In global equilibrium the integral in the argument of the exponential function in Eq. (19) should be independent of the choice of the space-like hypersurface $\Sigma$. To check when this happens, we rewrite Eq. (19) introducing the total angular momentum operator. In this way we obtain

$$\hat{\rho}_{\text{LEQ}} = \frac{1}{Z} \exp \left[ - \int_{\Sigma} d\Sigma_{\mu} \left( \hat{T}_{\text{can}}^{\mu\nu} (\beta_\nu - \omega_{\nu\lambda} x^\lambda) - \frac{1}{2} \omega_{\lambda\nu} \hat{J}_{\text{can}}^{\mu,\lambda\nu} - \xi \hat{j}^{\mu} \right) \right].$$

(22)

$^4$Below we use more often the name spin polarization tensor, as it has been used in our earlier publications and we try to avoid proliferation of names. Moreover, one may try to keep the name “spin chemical potential” for the tensor $\Omega_{\mu\nu} = T \omega_{\mu\nu}$, in analogy with the notation used for the charge chemical potential which is defined as $\mu = T \xi$.

$^5$Different possible definitions of the relativistic vorticity and their physical significance is discussed, for example, in Ref. [50].
The operator $\hat{\rho}_{LEQ}$ is constant if the divergence of the integrand in Eq. (22) vanishes. Since the energy-momentum and angular-momentum tensors are conserved, $\partial_\mu \hat{T}^\mu_\nu = 0$ and $\partial_\mu \hat{J}^\mu_\nu = 0$, we obtain three conditions for the Lagrange multipliers $^6$:

$$\partial_\mu \omega_\nu = 0, \quad \partial_\mu \beta_\nu = \omega_\nu, \quad \partial_\mu \xi = 0,$$

(23)

which imply that $\xi$ and the spin polarization tensor $\omega_\nu$ should be constant, whereas the $\beta_\nu$ field should have the form

$$\beta_\nu = b_\nu + \omega_\nu x^\lambda.$$

(24)

Equations (23) and (24) imply that the $\beta_\nu$ field satisfies the Killing equation

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$

(25)

and the spin polarization tensor is equal to thermal vorticity, $\omega_\lambda = \varpi_\lambda$. Thus, the global-equilibrium statistical operator has the form

$$\hat{\rho}_{GEQ} = \frac{1}{Z} \exp \left[ - \int d\Sigma_\mu \left( \hat{T}^\mu_\nu \beta_\nu - \frac{1}{2} \varpi_\lambda \hat{S}^\mu_\lambda - \xi \hat{j}^\mu \right) \right],$$

(26)

where $\varpi_\lambda = -\frac{1}{2} (\partial_\lambda \beta_\nu - \partial_\nu \beta_\lambda) = \text{const.}$

2.4. General concept of perfect-fluid hydrodynamics with spin

If the conditions (23) are not satisfied, the integral over the hypersurface $\Sigma$ depends on its choice. The two such integrals, over the hypersurfaces $\Sigma_1$ and $\Sigma_2$ differ by the volume integral. One can show, however, that such a volume integral describes dissipative phenomena [47, 48], hence, if we neglect dissipation we may treat the local-equilibrium operator (19) as constant. In this case we define the expectation values of the conserved currents through the expressions:

$$T^\mu_\nu = \text{tr} \left( \hat{\rho}_{LEQ} \hat{T}^\mu_\nu \right), \quad S^\mu_\lambda = \text{tr} \left( \hat{\rho}_{LEQ} \hat{S}^\mu_\lambda \right), \quad j^\mu = \text{tr} \left( \hat{\rho}_{LEQ} \hat{j}^\mu \right).$$

(27)

They are all functions of the hydrodynamic variables $\beta_\mu$, $\omega_\mu$, and $\xi$, namely,

$$T^\mu_\nu = T^\mu_\nu[\beta, \omega, \xi], \quad S^\mu_\lambda = S^\mu_\lambda[\beta, \omega, \xi], \quad j^\mu = j^\mu[\beta, \omega, \xi],$$

(28)

$^6$We use here the fact that $\hat{T}^\mu_\nu$ is not symmetric and $\hat{J}^\mu_\nu$ has no additional properties than asymmetry in the last two indices.
and satisfy the conservation laws
\[ \partial_\mu T^{\mu\nu} = 0, \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}, \quad \partial_\mu j^\mu = 0. \] (29)

In general, these are 11 equations for 11 unknown functions, which represent a generalization of the standard perfect-fluid hydrodynamics to the case including spin. We also stress that below we shall deal with the expectation values defined by Eqs. (27) rather than with operators.

3. Local equilibrium functions

3.1. Spin dependent phase-space distribution functions

In this section we put forward the phase-space distribution functions for spin-1/2 particles and antiparticles in local equilibrium, which have been introduced by Becattini et al. in Ref. [16]. In this approach, to include spin degrees of freedom, the standard scalar functions are generalized to 2×2 Hermitian matrices in spin space for each value of the space-time position \( x \) and four-momentum \( p \),

\[
\begin{align*}
[f^+(x,p)]_{rs} &\equiv f^+_{rs}(x,p) = \bar{u}_r(p)X^+u_s(p), \\
[f^-(x,p)]_{rs} &\equiv f^-_{rs}(x,p) = -\bar{v}_s(p)X^-v_r(p).
\end{align*}
\] (30)

Here \( u_r(p) \) and \( v_r(p) \) are Dirac bispinors (with the spin indices \( r \) and \( s \) running from 1 to 2), and the normalization \( \bar{u}_r(p)u_s(p) = \delta_{rs} \) and \( \bar{v}_r(p)v_s(p) = -\delta_{rs} \).

Note the minus sign and different ordering of spin indices in Eq. (31) compared to Eq. (30).

The 4×4 matrices \( X^\pm \) in Eqs. (30) and (31) are defined as products of the relativistic Boltzmann distributions (Jüttner distributions [51]) and matrices \( M^\pm \), namely

\[ X^\pm = \exp \left[ \pm \xi(x) - \beta_\mu(x)p^\mu \right] M^\pm, \] (32)

where

\[ M^\pm = \exp \left[ \pm \frac{1}{2} \omega_{\mu\nu}(x) \Sigma^{\mu\nu} \right]. \] (33)

In Eqs. (32) and (33) \( \beta^\mu \) denotes, as above, the ratio between the fluid four-velocity \( u^\mu \) and the local temperature \( T \), while \( \xi \) is the ratio between the
chemical potential $\mu$ and $T$. The quantity $\omega_{\mu\nu}$ is the spin polarization tensor introduced in the previous section. For the sake of simplicity, we restrict ourselves herein to classical Boltzmann statistics. The quantum statistics may be included by replacing the exponential functions in Eqs. (32) and (33) by the Fermi-Dirac distribution. This approach was proposed and worked out in Ref. [16].

In analogy to the Faraday tensor $F_{\mu\nu}$ used in classical electrodynamics, the antisymmetric spin polarization tensor $\omega_{\mu\nu}$ can always be defined in terms of electric- and magnetic-like three-vectors in LAB, $e = (e^1, e^2, e^3)$ and $b = (b^1, b^2, b^3)$. In this case we write (following the electrodynamic sign conventions of Ref. [52])

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}. \quad (34)$$

To switch from $\omega_{\mu\nu}$ to the dual tensor $\tilde{\omega}_{\mu\nu}$, one replaces $e$ by $b$ and $b$ by $-e$. Hence, the parametrization of the dual spin polarization tensor reads

$$\tilde{\omega}_{\mu\nu} = \begin{bmatrix} 0 & b^1 & b^2 & b^3 \\ -b^1 & 0 & e^3 & -e^2 \\ -b^2 & -e^3 & 0 & e^1 \\ -b^3 & e^2 & -e^1 & 0 \end{bmatrix}. \quad (35)$$

One also finds that

$$\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} = -\frac{1}{2}\tilde{\omega}_{\mu\nu}\tilde{\omega}^{\mu\nu} = b \cdot b - e \cdot e, \quad \tilde{\omega}_{\mu\nu}\omega^{\mu\nu} = -4e \cdot b. \quad (36)$$

Before we continue our discussion, it is important to stress that the forms of the equilibrium distributions with spin given by Eqs. (30) and (31) have not been derived yet from any underlying microscopic model or theory. In particular, it is not known at the moment if they can be obtained from a kinetic-theory approach. To large extent, Eqs. (30) and (31) represent an “educated guess” for the form of equilibrium phase-space distributions with spin-1/2. In the present work, by discussing the quantum kinetic equation in the semi-classical expansion and by developing the hydrodynamic framework with a classical treatment of spin, we eventually conclude on the applicability range of Eqs. (30) and (31). Nonetheless, before concluding on this issue,
we develop a formalism based on Eqs. (30) and (31), as this allows us to build a necessary theoretical scheme and to introduce relevant definitions and concepts.

3.2. Important special cases

The exponential dependence of the distribution function on the Dirac spin operator $\Sigma^{\mu\nu}$ given in Eq. (33), which is defined in terms of a power series, can be resummed. This results in the following expression for $M^\pm$ [36],

$$M^\pm = \mathbb{1} \left[ \Re(\cosh z) \pm \Re \left( \frac{\sinh z}{2z} \right) \omega^{\mu\nu} \Sigma^{\mu\nu} \right]$$

$$+ i\gamma_5 \left[ \Im(\cosh z) \pm \Im \left( \frac{\sinh z}{2z} \right) \omega^{\mu\nu} \Sigma^{\mu\nu} \right],$$

(37)

where $\mathbb{1}$ is a unit $4 \times 4$ matrix and

$$z = \frac{1}{2\sqrt{2}} \sqrt{\omega^{\mu\nu}\omega^{\mu\nu} + i\omega^{\mu\nu}\tilde{\omega}^{\mu\nu}}.$$  

(38)

In a series of papers [35, 36, 37], the special case was analyzed where elements of the spin polarization tensor fulfill the conditions

$$e \cdot b = 0, \quad b \cdot b - e \cdot e \geq 0.$$  

(39)

If Eqs. (39) are satisfied, the variable $z$ becomes a real number $\zeta$ and Eq. (37) simplifies to the expression linear in the operator $\Sigma^{\mu\nu}$, namely

$$M^\pm = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta} \omega^{\mu\nu} \Sigma^{\mu\nu},$$

(40)

where

$$\zeta = \frac{1}{2\sqrt{2}} \sqrt{\omega^{\mu\nu}\omega^{\mu\nu}}.$$  

(41)

Another interesting case corresponds to small polarization. If $\omega^{\mu\nu} \ll 1$ we can use the expression

$$M^\pm = 1 \pm \frac{1}{2} \omega^{\mu\nu} \Sigma^{\mu\nu},$$

(42)

---

7The conditions (39) have been relaxed in a series of recent papers, for example, see Refs. [53, 54, 55].
where the elements of the polarization tensor are not constrained by the conditions (39). One can easily notice that Eq. (42) follows directly from the expansion of the exponential function (33) up to the first order in $\omega_{\mu\nu}$ and does not constrain otherwise the coefficients $\omega_{\mu\nu}$. Below, most of the calculations are done with $M^\pm$ defined by Eq. (40) as it reproduces Eq. (42) as a special case.

3.3. Spin polarization three-vector

The spin observables are represented by the Pauli matrices $\sigma$ and the expectation values of $\sigma$ provide information on the polarization of spin-$1/2$ particles in their rest frame. Since we consider Dirac bispinors obtained by the canonical Lorentz boosts applied to states with zero three-momentum, we refer to the resulting spin distributions and particle rest frames as the canonical ones (they differ from other definitions by a rotation).

Using Eq. (40), the spin dependent distribution functions defined by Eqs. (30) and (31) can be rewritten in a form linear in the Dirac spin tensor,

$$f_{rs}^+(x,p) = e^{\xi-p \cdot \beta} \left[ \cosh(\zeta) \delta_{rs} + \frac{\sinh(\zeta)}{2\zeta} \bar{u}_r(p) \omega_{\alpha\beta} \Sigma^\alpha_{\beta} u_s(p) \right],$$

(43)

$$f_{rs}^-(x,p) = e^{-\xi-p \cdot \beta} \left[ \cosh(\zeta) \delta_{rs} + \frac{\sinh(\zeta)}{2\zeta} \bar{v}_s(p) \omega_{\alpha\beta} \Sigma^\alpha_{\beta} v_r(p) \right].$$

(44)

We then find a compact expression [36]

$$f^\pm(x,p) = e^{\pm \xi-p \cdot \beta} \left[ \cosh(\zeta) - \frac{\sinh(\zeta)}{2\zeta} P \cdot \sigma \right],$$

(45)

where we have introduced a polarization vector

$$P = \frac{1}{m} \left[ E_p b - p \times e - \frac{p \cdot b}{E_p + m} p \right].$$

(46)

The three-vector $P$ can be interpreted as a spatial part of the polarization four-vector $P^\mu$, with a vanishing zeroth component, $P^\mu(x,p) = (0, P(x,p))$. The average polarization vector is defined by the formula

$$\langle P(x,p) \rangle = \frac{1}{2} \text{tr}_2 \left[ \frac{(f^+ + f^-) \sigma}{f^+ + f^-} \right] = -\frac{1}{4\zeta} \tanh(\zeta) P.$$ 

(47)
We note that the expression on the right-hand side of Eq. (46) defines the field $b$ in the particle rest frame [52]. We summarize this finding by writing

$$ P = b_* . $$

(48)

Thus, the polarization is determined by the value of the field $b$ in the particle canonical rest frame. Moreover, using Eq. (41), we may obtain an alternative expression for the average polarization vector

$$ \langle P(x,p) \rangle = -\frac{1}{2} \tanh \left[ \frac{1}{2} \sqrt{b_* \cdot b_* - e_* \cdot e_*} \right] \frac{b_*}{\sqrt{b_* \cdot b_* - e_* \cdot e_*}} , $$

(49)

where we have used the fact that the quantity $b \cdot b - e \cdot e$ is independent of the choice of the Lorentz frame.

Equation (49) indicates a potential problem with the application of the present formalism for particles with the momentum $p$, which are placed at the space-time point $x$ and for which the condition

$$ | \langle P(x,p) \rangle | \leq \frac{1}{2} $$

(50)

is violated. The condition (50) holds, for example, if for all particles under consideration the magnetic-like components dominate $|e_*| < |b_*|$ or if the spin polarization components are sufficiently small, $\zeta < 1$, and $|b_*| < 1$.

Such constraints can be always checked for systems under investigations. We shall come back to a discussion of the condition (50) in Sec. 6.6, where we reexamine it in the context of classical description of spin.

4. Semi-classical expansion of Wigner function

In this section we study consequences of using Eqs. (30) and (31) as an input for construction of the equilibrium Wigner function. Our approach closely follows recent investigations presented in Ref. [40] — we first recapitulate the steps leading to the kinetic equations satisfied by the coefficients

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8Strictly speaking, there are always particles in a system, which have large momenta and violate the condition $|b_*| < 1$. However, their abundances are usually negligible, since suppressed by a thermal factor. The only exception are particles that move fast together with a fluid element. In this case it is reasonable to replace the condition $|b_*| \ll 1$ by a more precise requirement that $|b_*| \gamma < 1$, where $\gamma$ is the Lorentz factor connected with a fluid velocity.
of the equilibrium Wigner function (in the Clifford-algebra representation) and, subsequently, argue that simple moments of the kinetic equations lead to the GLW hydrodynamic picture.

4.1. Equilibrium Wigner functions

The functions \( f^{\pm}_{rs}(x,p) \) can be used to determine explicit expressions for the corresponding equilibrium (particle and antiparticle) Wigner functions \( W_{\text{eq}}^{\pm}(x,k) \). We construct them using the relations derived in Ref. [41],

\[
W_{\text{eq}}^{+}(x,k) = \frac{1}{2} \sum_{r,s=1}^{2} \int dP \delta^{(4)}(k - p) u^{r}(p) \bar{u}^{s}(p) f^{+}_{rs}(x,p),
\]

(51)

\[
W_{\text{eq}}^{-}(x,k) = -\frac{1}{2} \sum_{r,s=1}^{2} \int dP \delta^{(4)}(k + p) v^{s}(p) \bar{v}^{r}(p) f^{-}_{rs}(x,p).
\]

(52)

Here \( k \) is the four-momentum of particles or antiparticles, and \( dP \) is the invariant integration measure defined by Eq. (1). The total Wigner function is a simple sum of the particle and antiparticle contributions

\[
W_{\text{eq}}(x,k) = W_{\text{eq}}^{+}(x,k) + W_{\text{eq}}^{-}(x,k).
\]

(53)

Using Eqs. (30) and (31) we find

\[
W_{\text{eq}}^{\pm}(x,k) = \frac{1}{4m} \int dP \delta^{(4)}(k \mp p)(\not{p} \pm m) X^{\pm}(\not{p} \pm m).
\]

(54)

With the help of Eq. (40) we can further rewrite this equation as

\[
W_{\text{eq}}^{\pm}(x,k) = \frac{e^{\pm \xi}}{4m} \int dP e^{-\beta \not{p}} \delta^{(4)}(k \mp p)
\times \left[ 2m(\not{p} \pm m) \cosh(\xi) \pm \frac{\sinh(\xi)}{2\xi} \omega_{\mu\nu}(\not{p} \pm m) \Sigma^{\mu\nu}(\not{p} \pm m) \right].
\]

(55)

The presence of the Dirac delta functions in the definitions of the equilibrium Wigner functions (55) indicates that, to large extent, they describe classical motion — the energy of particles is always on the mass shell. This suggests that the functions (55) cannot be regarded as complete, quantum-mechanical equilibrium distributions. In fact, we shall see below that the
expressions (55) can be identified only with the leading order terms in \( \hbar \) of
the “true” equilibrium Wigner functions which satisfy the quantum kinetic
equation. For our approach it is important, however, that the functions (55)
icorporate spin degrees of freedom and may serve to construct the formalism
of hydrodynamics with spin (in the leading order in \( \hbar \)).

4.2. Clifford-algebra expansion

The Wigner functions \( W^\pm(x, k) \) are 4\times4 matrices satisfying the conju-
gation relation \( W^\pm(x, k) = \gamma_0 W^\pm(x, k) \gamma_0 \). Consequently, they can be expanded in terms of 16 independent generators of the Clifford algebra with real coefficients [56]. Such an expansion method was used very successfully in the past to formulate the transport equations for abelian plasmas [57, 58], the quark-gluon plasma [59, 60], and chiral models [61]. More recently, it has been used, for example, in Refs. [19, 21, 22, 62].

In our case, we start with the decomposition of the equilibrium Wigner function (55) in the form

\[
W^\pm_{\text{eq}}(x, k) = \frac{1}{4} \left[ F^\pm_{\text{eq}}(x, k) + i \gamma_5 P^\pm_{\text{eq}}(x, k) + \gamma^\mu V^\pm_{\text{eq}, \mu}(x, k) \right.
\]
\[
+ \gamma_5 \gamma^\mu A^\pm_{\text{eq}, \mu}(x, k) + \Sigma^{\mu\nu} S^\pm_{\text{eq}, \mu\nu}(x, k) \right].
\]

The coefficient functions in the expansion (56) can be obtained in the straight-
forward way by calculating the trace of \( W^\pm_{\text{eq}}(x, k) \) multiplied first by the ma-
trices: \( 1, -i \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5 \) and \( 2 \Sigma_{\mu\nu} \). In this way we find the expressions [40]:

\[
F^\pm_{\text{eq}}(x, k) = 2m \cosh(\zeta) \int dP \ e^{-\beta \cdot p \pm \xi} \delta(4)(k \mp p),
\]

(57)

\[
P^\pm_{\text{eq}}(x, k) = 0,
\]

(58)

\[
V^\pm_{\text{eq}, \mu}(x, k) = \pm 2 \cosh(\zeta) \int dP \ e^{-\beta \cdot p \pm \xi} \delta(4)(k \mp p) p_\mu,
\]

(59)

\[
A^\pm_{\text{eq}, \mu}(x, k) = -\frac{\sinh(\zeta)}{\zeta} \int dP \ e^{-\beta \cdot p \pm \xi} \delta(4)(k \mp p) \bar{\omega}_{\mu\nu} p^\nu,
\]

(60)

\[
S^\pm_{\text{eq}, \mu\nu}(x, k) = \pm \frac{\sinh(\zeta)}{m\zeta} \int dP \ e^{-\beta \cdot p \pm \xi} \delta(4)(k \mp p)
\]
\[
\times \left[ (p_\mu \omega_{\nu\alpha} - p_\nu \omega_{\mu\alpha}) p^\alpha + m^2 \omega_{\mu\nu} \right].
\]

(61)

One can easily check that the functions defined by Eqs. (57)–(61) satisfy the
following set of constraints:

\[
k^\mu V^\pm_{\text{eq}, \mu}(x, k) = m F^\pm_{\text{eq}}(x, k), \quad k_\mu F^\pm_{\text{eq}}(x, k) = m V^\pm_{\text{eq}, \mu}(x, k),
\]

(62)
\[ P_{\text{eq}}^\pm(x,k) = 0, \quad k^\mu A_{\text{eq},\mu}^\pm(x,k) = 0, \quad k^\mu S_{\text{eq},\mu\nu}^\pm(x,k) = 0, \quad (63) \]

\[ k^\beta S_{\text{eq},\mu\beta}^\pm(x,k) + m A_{\text{eq},\mu}^\pm(x,k) = 0, \quad (64) \]

\[ \epsilon_{\mu\nu\alpha\beta} k^\alpha A_{\text{eq}}^{+\beta}(x,k) + m S_{\text{eq},\mu\nu}^\pm(x,k) = 0. \quad (65) \]

We note that such constraints are fulfilled also by the total Wigner function given by the sum of particle and antiparticle contributions, see Eq. (53). We also note that Eqs. (62)–(65) follow from the algebraic structure of the equilibrium Wigner functions and are satisfied for any form of the fields: \( \beta(x), \xi(x), \) and \( \omega_{\mu\nu}(x) \).

4.3. Global equilibrium

Generally speaking, the Wigner function \( \mathcal{W}(x,k) \) satisfies the kinetic equation that can be schematically written as

\[ (\gamma_\mu K^{\mu} - m) \mathcal{W}(x,k) = C[\mathcal{W}(x,k)]. \quad (66) \]

Here \( K^{\mu} \) is the operator defined by the expression

\[ K^{\mu} = k^{\mu} + \frac{i\hbar}{2} \partial^{\mu}, \quad (67) \]

whereas \( C[\mathcal{W}(x,k)] \) is the collision term. We tacitly assume that \( C[\mathcal{W}(x,k)] \) vanishes if \( \mathcal{W}(x,k) \) describes any form of equilibrium, global or local.

In the case of global equilibrium, with the vanishing collision term, the Wigner function \( \mathcal{W}(x,k) \) exactly satisfies the equation \(^{9}\)

\[ (\gamma_\mu K^{\mu} - m) \mathcal{W}(x,k) = 0. \quad (68) \]

The standard treatment of this equation is based on the semi-classical expansion in powers of \( \hbar \) of the coefficient functions defining \( \mathcal{W}(x,k) \). Such an expansion shows that one can choose, as two independent functions, the coefficients \( F_{(0)}(x,k) \) and \( A_{(0)}^{\nu}(x,k) \) — the other coefficient functions are defined in terms of these two functions only. Moreover, one can check that algebraic

\(^{9}\) Here we neglect the presence of the electromagnetic and other mean fields. Their inclusion is straightforward and left for future studies generalizing the present framework.
relations imposed by Eq. (68) in the leading order (LO) in $\hbar$ are satisfied by the equilibrium functions (57)–(61). Hence, one can assume that:

$$F^{(0)} = F_{\text{eq}},$$

$$P^{(0)} = 0,$$

$$V_{\mu}^{(0)} = V_{\text{eq},\mu},$$

$$A_{\mu}^{(0)} = A_{\text{eq},\mu},$$

$$S_{\mu\nu}^{(0)} = S_{\text{eq},\mu\nu}.$$  

(69) \hspace{2cm} (70) \hspace{2cm} (71) \hspace{2cm} (72) \hspace{2cm} (73)

The next-to-leading-order (NLO) terms in $\hbar$ coming from Eq. (68) yield the dynamic equations

$$k^\mu \partial_\mu F_{\text{eq}}(x,k) = 0,$$

$$k^\mu \partial_\mu A_{\text{eq}}^\nu(x,k) = 0,$$

$$k^\mu \partial_\mu A_{\text{eq}}^\nu(x,k) = 0,$$

(74) \hspace{2cm} (75)

which are nothing else but the kinetic equations that should be obeyed by the coefficient functions $F_{\text{eq}}(x,k)$ and $A_{\text{eq}}^\nu(x,k)$.

One can easily check that Eqs. (74) and (75) are satisfied if $\beta_\mu$ is a Killing vector, see Eq. (25), while $\xi$ and $\omega_{\mu\nu}$ are constant. However, the thermal vorticity obtained with the field $\beta_\mu$ is not necessarily equal to $\omega_{\mu\nu}$, although the two tensors should be constant. Our general arguments presented earlier, see Sec. 2.3, indicated that thermal vorticity should be equal to the spin polarization tensor in global thermodynamic equilibrium, so how these two facts can be reconciled? The most likely answer to this question is that the equality of the spin polarization tensor and thermal vorticity is a result of dissipative phenomena that are neglected in the present approach.

In view of the present discussion we may distinguish between global and extended global equilibria [40]: In global equilibrium the $\beta_\mu$ field is a Killing vector satisfying Eq. (25), $\varpi_{\mu\nu} = -\frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) = \text{const}$, the spin polarization tensor is constant and agrees with thermal vorticity, $\omega_{\mu\nu} = \varpi_{\mu\nu}$, in addition $\xi = \text{const}$. In an extended global equilibrium $\beta_\mu$ field is a Killing vector, $\varpi_{\mu\nu} = -\frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) = \text{const}$, the spin polarization tensor is constant but $\omega_{\mu\nu} \neq \varpi_{\mu\nu}$, $\xi = \text{const}$. Correspondingly, we also differentiate between local and extended local equilibria. In local equilibrium the $\beta_\mu$ field is not a Killing vector but we still have $\omega_{\mu\nu}(x) = \varpi_{\mu\nu}(x)$, $\xi$ is allowed to depend on space-time coordinates, $\xi = \xi(x)$. In extended local equilibrium
the $\beta_\mu$ field is not a Killing vector and $\omega_{\mu\nu}(x) \neq \overline{\omega}_{\mu\nu}(x)$, moreover $\xi = \xi(x)$. We note that Eqs. (62)–(65) follow from the algebraic structure of the equilibrium Wigner functions and are satisfied for any form of the fields: $\beta_\mu(x)$, $\xi(x)$, and $\omega_{\mu\nu}(x)$. Thus, they hold for four different types of equilibrium specified above.

4.4. Perfect-fluid hydrodynamics with spin from kinetic theory

Equations of the perfect-fluid hydrodynamics with spin can be obtained by approximate treatment of Eqs. (74) and (75). In this case we do not require that they are exactly satisfied but demand instead that certain moments of them (in the momentum space) vanish, allowing for space-time dependence of the hydrodynamic variables $\beta_\mu$, $\xi$, and $\omega_{\mu\nu}$.

Dealing with Eq. (74) is quite well established — one has to include its zeroth and first moments, which leads to the conservation of charge, energy, and linear momentum. Indeed, the four-dimensional integration of Eq. (74) over $k$ yields

$$\partial_\alpha N_\text{eq}^\alpha(x) = 0,$$

where

$$N_\text{eq}^\alpha = 4 \cosh(\zeta) \sinh(\xi) \int \frac{d^3p}{(2\pi)^3} p^\alpha e^{-\beta \cdot p}.$$

Doing the integral over the three-momentum in Eq. (77), one finds that the charge current is proportional to the flow vector,

$$N_\text{eq}^\alpha = n u^\alpha,$$

where

$$n = 4 \cosh(\zeta) \sinh(\xi) n_{(0)}(T)$$

is the charge density [35]. The quantity $n_{(0)}(T) = \langle (u \cdot p) \rangle_0$ is the number density of spin-0, neutral Boltzmann particles, obtained using the thermal average

$$\langle \cdots \rangle_0 \equiv \int \frac{d^3p}{(2\pi)^3} E_p \cdot \cdots e^{-\beta \cdot p}.$$
If we multiply Eq. (74) first by $k^\nu$ and then perform the four-dimensional integration over $k$, we obtain the conservation of energy and momentum in the form

$$\partial_\alpha T^{\alpha\beta}_{\text{GLW}}(x) = 0,$$

where the energy-momentum tensor is defined by the perfect-fluid formula

$$T_{\text{GLW}}^{\mu\nu}(x) = (\varepsilon + P)u^\mu u^\nu - P g^{\mu\nu},$$

with

$$\varepsilon = 4 \cosh(\zeta) \cosh(\xi) \varepsilon_{(0)}(T)$$

and

$$P = 4 \cosh(\zeta) \cosh(\xi) P_{(0)}(T),$$

respectively [35]. In analogy to the density $n_{(0)}(T)$, we define the auxiliary quantities $\varepsilon_{(0)}(T) = ((u \cdot p)^2)_0$ and $P_{(0)}(T) = -(1/3)((p \cdot p - (u \cdot p)^2))_0$. We have added the subscript GLW to the $T^{\mu\nu}$ symbol in Eq. (82) because, as we shall see below, this formula can be obtained with the expressions provided in Ref. [41]. We note that GLW denotes always the (extended) local equilibrium result — for simplicity of notation we skip the symbol “eq” in this case.

The treatment of Eq. (75) is less obvious. In Ref. [40] it was demonstrated how one can connect it with the conservation of the spin tensor defined in Ref. [41]. We first rewrite Eq. (75) as

$$0 = k^\alpha \partial_\alpha \int dP e^{-\beta p} \frac{\sinh(\zeta)}{\zeta} \left[ \delta^{(4)}(k - p)e^\xi + \delta^{(4)}(k + p)e^{-\xi} \right] \tilde{\omega}_{\mu\nu} p^\nu$$

and then multiply Eq. (85) by the four-vector $k_\beta$, contract it with the Levi-Civita tensor $\epsilon^{\mu\beta\gamma\delta}$ and, finally, integrate the resulting equation over $k$. In this way we obtain the conservation of the spin tensor in the GLW version,

$$\partial_\lambda S_{\text{GLW}}^{\lambda,\mu\nu}(x) = 0,$$

with

$$S_{\text{GLW}}^{\lambda,\mu\nu} = \frac{\hbar \sinh(\zeta) \cosh(\xi)}{m^2 \zeta} \int dP e^{-\beta p} p^\lambda \left( m^2 \omega_{\mu\nu} + 2 p^\alpha p^{[\mu} \omega_{\nu]} \right)$$

$$= \frac{\hbar v}{4\zeta} u^\lambda \omega_{\mu\nu} + S_{\Delta}^{\lambda,\mu\nu}.$$
In the last line we have used the spin density \( w \) defined in [35]

\[ w = 4 \sinh(\zeta) \cosh(\xi) n_{(0)}(T) \]  

and the auxiliary tensor

\[ S^{\lambda,\mu\nu}_\Delta = \frac{2\hbar \sinh(\zeta) \cosh(\xi)}{m^2 \zeta} s^{\lambda,\mu\nu}_{\text{GLW}}, \]

where

\[ s^{\lambda,\mu\nu}_{\text{GLW}} = A u^{\lambda}_\alpha u^{\mu\nu}_\alpha + B \left( \Delta^{\lambda\alpha}_\mu u^{\alpha\nu}_\alpha + u^{\lambda\Delta\alpha\mu}_\nu + u^{\nu\Delta\lambda\alpha}_\mu \right), \]

with the thermodynamic coefficients

\[ A = \frac{1}{\beta} \left[ 3 \varepsilon_{(0)} + \left( 3 + \frac{m^2}{T^2} \right) P_{(0)} \right] = -3B + \frac{m^2}{T} P_{(0)}, \]

\[ B = -\frac{1}{\beta} \left( \varepsilon_{(0)} + P_{(0)} \right). \]

It is important to note that for dimensional reasons, we have implemented the Planck constant \( \hbar \) in the definition (87).

5. Conserved currents

5.1. NLO corrections to Wigner function

In the last section we have analyzed the moments of the kinetic equations considered in the leading order of \( \hbar \), which have led us to the hydrodynamic equations. Since the spin tensor enters with an extra power of \( \hbar \), it is important to reexamine the Wigner function up to the next-to-leading order (NLO) in \( \hbar \), where the functions \( F^{(1)} \) and \( A_{(1)}^{\mu} \) may be treated again as independent variables, and other coefficients are expressed in terms of these two functions and other NLO terms [40]:

\[ \mathcal{P}^{(1)} = -\frac{1}{2m} \partial^\mu A^{(0)}_{\mu}, \]

\[ \gamma^{(1)}_{\mu} = \frac{1}{m} \left( k_{\mu} F^{(1)} - \frac{1}{2} \partial^\nu S^{(0)}_{\nu\mu} \right), \]

23
\[ S^{(1)}_{\mu\nu} = \frac{1}{2m} \left( \partial_\mu \mathcal{V}^{(0)}_\nu - \partial_\nu \mathcal{V}^{(0)}_\mu \right) - \frac{1}{m} \epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}_\beta^{(1)}. \]  

Similarly to the leading order, the functions \( \mathcal{F}^{(1)} \) and \( \mathcal{A}_\nu^{(1)} \) satisfy the kinetic equations

\[ k^\mu \partial_\mu \mathcal{F}^{(1)}(x, k) = 0, \]

\[ k^\mu \partial_\mu \mathcal{A}_\nu^{(1)}(x, k) = 0, \quad k_\nu \mathcal{A}_\nu^{(1)}(x, k) = 0. \]

If \( \mathcal{F}^{(1)} \) and \( \mathcal{A}_\nu^{(1)} \) are determined, the quantities \( \mathcal{P}^{(1)}, \mathcal{V}_\mu^{(1)}, \) and \( S^{(1)}_{\mu\nu} \) are obtained from Eqs. (93)–(95).

An important aspect of the equations above is that the NLO coefficients gain non-trivial contributions from the leading order. Hence, the equilibrium LO terms generate non-trivial corrections. Herein, we assume that \( \mathcal{F}^{(1)} = \mathcal{A}_\nu^{(1)} = 0 \), and the functions \( \mathcal{P}^{(1)}, \mathcal{V}_\mu^{(1)}, \) and \( S^{(1)}_{\mu\nu} \) are generated solely by the leading-order equilibrium terms:

\[ \mathcal{P}^{(1)} = -\frac{1}{2m} \partial^\mu \mathcal{A}_{\text{eq},\mu}, \]

\[ \mathcal{V}_\mu^{(1)} = -\frac{1}{2m} \partial^\nu \mathcal{S}_{\text{eq},\nu\mu}, \]

\[ S^{(1)}_{\mu\nu} = \frac{1}{2m} \left( \partial_\mu \mathcal{V}_{\text{eq},\nu} - \partial_\nu \mathcal{V}_{\text{eq},\mu} \right). \]

5.2. Charge current

Expressing the charge current in terms of the Wigner function \( \mathcal{W}(x, k) \) we obtain [41]

\[ N_{\text{tot}}^\alpha(x) = \text{tr}_4 \int d^4k \gamma^\alpha \mathcal{W}(x, k) = \int d^4k \gamma^\alpha \mathcal{W}(x, k). \]

In the equilibrium case, to include the NLO corrections, besides Eq. (71) we also use Eq. (99) to have a complete expression for \( \mathcal{V}^\alpha(x, k) \). In this way we find

\[ N_{\text{tot}}^\alpha(x) = N_{\text{eq}}^\alpha(x) + \delta N_{\text{eq}}^\alpha(x), \]

10Note that Eq. (66) with (67) couples expressions that differ by one power of \( \hbar \).
where

\[ N_{\text{eq}}^\alpha(x) = \frac{1}{m} \int d^4k \, k^\alpha F_{\text{eq}}(x, k) \]  

and

\[ \delta N_{\text{eq}}^\alpha(x) = -\frac{\hbar}{2m} \int d^4k \, \partial_\lambda S_{\text{eq}}^{\lambda\alpha}(x, k). \] 

One can easily check that Eq. (103) agrees with Eq. (77), and \( \partial_\alpha \delta N_{\text{eq}}^\alpha(x) = 0 \), which follows from the antisymmetry of the tensor \( S_{\text{eq}}^{\lambda\alpha}(x, k) \). Hence, the conservation of the charge current (101) agrees with the results obtained from the LO kinetic theory, discussed in Sec. 4.4.

5.3. GLW energy-momentum and spin tensors

Adopting the kinetic-theory framework derived by de Groot, van Leeuwen, and van Weert in Ref. [41], where the energy-momentum tensor is expressed directly by the trace of the Wigner function, we can use the following expression

\[ T_{\mu\nu}^{\text{GLW}}(x) = \frac{1}{m} \int d^4k \, k_\mu k_\nu W(x, k) = \frac{1}{m} \int d^4k \, k_\mu k_\nu F(x, k). \] 

In the equilibrium case, we consider Eq. (105) up to the first order in \( \hbar \) using Eq. (69) and setting \( F^{(1)}(x, k) = 0 \). In this way, we obtain the expressions that agree with Eqs. (82)–(84).

Let us turn now to the discussion of the spin tensor. In the GLW approach, it has the following form [41]

\[ S_{\text{GLW}}^{\lambda\mu\nu} = \frac{\hbar}{4} \int d^4k \, \text{tr}_4 \left[ \left( \{ \sigma^{\mu\nu}, \gamma^\lambda \} + \frac{2i}{m} (\gamma^{[\mu} k^{\nu]} - \gamma^\lambda \gamma^{[\mu} k^{\nu]}) \right) W(x, k) \right]. \] 

For dimensional reasons, we have implemented here the Planck constant \( \hbar \). Its presence implies that in equilibrium we may take the leading order expression for the Wigner function and assume \( W(x, k) = W_{\text{eq}}(x, k) \). Using Eq. (55) in Eq. (106), performing the appropriate traces over spinor indices, and then carrying out the integration over \( k \) we obtain Eq. (87).

The results obtained in this section are very much instructive, since they show us that the GLW formulation appears naturally in the context of the
kinetic theory — the conservation laws are obtained as the zeroth and first moments of the kinetic equations fulfilled by the scalar and axial-vector coefficients. Moreover, since the GLW energy-momentum tensor is symmetric, the GLW spin tensor should be conserved separately. Hence, the perfect-fluid hydrodynamics can be obtained from the equations

\[ \partial_\alpha N_{eq}^\alpha(x) = 0, \quad \partial_\alpha T^{\alpha\beta}_{\text{GLW}}(x) = 0, \quad \partial_\lambda S^\lambda_{\text{GLW}}(x) = 0. \] (107)

Equations (107) can be interpreted as a specific realization of the general framework based on Eqs. (29).

5.4. Canonical tensors

The canonical forms of the energy-momentum and spin tensors, \( T^\mu_\text{can}_\nu(x) \) and \( S^\lambda_\text{can}_\mu_\nu(x) \), can be obtained directly from the Dirac Lagrangian by applying the Noether theorem [56]:

\[ T^\mu_\text{can}_\nu(x) = \int d^4k k^\nu \mathcal{V}^\mu(x, k) \] (108)

and

\[ S^\lambda_\text{can}_\mu_\nu(x) = \frac{\hbar}{4} \int d^4k \text{tr}_4 \left[ \{ \sigma^\mu_\nu, \gamma^\lambda \} \mathcal{W}(x, k) \right] = \frac{\hbar}{2} \epsilon^{\kappa\lambda_\mu_\nu} \int d^4k A_\kappa(x, k). \] (109)

Here we have used the anticommutation relation \( \{ \sigma^\mu_\nu, \gamma^\lambda \} = -2\epsilon^{\mu_\nu\lambda_\kappa_\gamma_5} \) to express directly the canonical spin tensor by the axial-vector coefficient function \( A_\kappa(x, k) \).

Including the components of \( \mathcal{V}^\mu(x, k) \) up to the first order in the equilibrium case we obtain

\[ T^\mu_\text{can}_\nu(x) = T^\mu_\nu_{\text{GLW}}(x) + \delta T^\mu_\nu_\text{can}(x) \] (110)

where

\[ \delta T^\mu_\text{can}_\nu(x) = -\frac{\hbar}{2m} \int d^4k k^\nu \partial_\lambda S^\lambda_\text{eq}^\mu(x, k) = -\partial_\lambda S^\lambda_\nu_\text{GLW}^\mu(x). \] (111)

The canonical energy-momentum tensor should be exactly conserved, hence, in analogy to Eq. (81) we require

\[ \partial_\alpha T^\alpha_\text{can}_\beta(x) = 0. \] (112)
It is interesting to observe that Eqs. (81) and (112) are consistent, since \( \partial_{\mu} \delta T_{\text{can}}^{\mu \nu} (x) = 0 \). The latter property follows directly from the definition of \( \delta T_{\text{can}}^{\mu \nu} (x) \), see Eq. (111).

For the equilibrium spin tensor it is enough to consider the axial-vector component in Eq. (109) in the zeroth order, \( A_{\kappa}^{(0)}(x, k) = A_{\text{eq}, \kappa}(x, k) \). Then, using Eq. (60) in Eq. (109) and carrying out the integration over the four-momentum \( k \) we get

\[
S_{\text{can}}^{\lambda, \mu \nu} = \frac{\hbar \sinh(\zeta) \cosh(\xi)}{\zeta} \int dP e^{-\beta \cdot p} \left( \omega^{\mu \nu} p^\lambda + \omega^{\nu \lambda} p^\mu + \omega^{\lambda \mu} p^\nu \right)
\]

\[
= \frac{\hbar w}{4 \zeta} \left( u^\lambda \omega^{\mu \nu} + u^\nu \omega^{\lambda \mu} + u^\mu \omega^{\nu \lambda} \right) = S_{\text{GLW}}^{\lambda, \mu \nu} + S_{\text{GLW}}^{\mu, \lambda \nu} + S_{\text{GLW}}^{\nu, \lambda \mu}. \tag{113}
\]

It is interesting to notice that the energy-momentum tensor (110) is not symmetric. In such a case, the spin tensor is not conserved and its divergence is equal to the difference of the energy-momentum components. For the case discussed in this section we obtain

\[
\partial_{\lambda} S_{\text{can}}^{\lambda, \mu \nu} (x) = T_{\text{can}}^{\mu \nu} - T_{\text{can}}^{\nu \mu} = -\partial_{\lambda} S_{\text{GLW}}^{\mu, \lambda \nu} (x) + \partial_{\lambda} S_{\text{GLW}}^{\nu, \lambda \mu} (x). \tag{114}
\]

One can immediately check, using the last line of Eq. (113), that Eq. (114) is consistent with the conservation of the spin tensor in the GLW approach.

5.5. Connecting GLW and canonical formulations

In the last section we have discussed the energy-momentum and spin tensors obtained from the canonical formalism and related them to the expressions introduced by de Groot, van Leeuven, and van Weert. In this section we demonstrate that the two versions of the energy-momentum and spin tensors are connected by a pseudo-gauge transformation. Indeed, if we introduce the tensor \( \Phi_{\text{can}}^{\lambda, \mu \nu} \) defined by the relation

\[
\Phi_{\text{can}}^{\lambda, \mu \nu} \equiv S_{\text{GLW}}^{\mu, \lambda \nu} - S_{\text{GLW}}^{\nu, \lambda \mu}, \tag{115}
\]

we can write

\[
S_{\text{can}}^{\lambda, \mu \nu} = S_{\text{GLW}}^{\lambda, \mu \nu} - \Phi_{\text{can}}^{\lambda, \mu \nu} \tag{116}
\]

and

\[
T_{\text{can}}^{\mu \nu} = T_{\text{GLW}}^{\mu \nu} + \frac{1}{2} \partial_{\lambda} \left( \Phi_{\text{can}}^{\lambda, \mu \nu} + \Phi_{\text{can}}^{\mu, \nu \lambda} + \Phi_{\text{can}}^{\nu, \mu \lambda} \right). \tag{117}
\]
Here, we have used the property that both $S_{\text{GLW}}^{\lambda,\mu\nu}$ and $\Phi_{\text{can}}^{\lambda,\mu\nu}$ are antisymmetric with respect to exchange of the last two indices. Equations (10) and (11) are an example of the pseudo-gauge transformation introduced and discussed in Sec. 2.1.

The most common use of such a transformation is connected with a change from the canonical formalism to the Belinfante one [46] — it provides a symmetric energy-momentum tensor and eliminates completely the spin tensor, see Eq. (14). Our calculations show that the leading-order, classical expressions for the canonical, GLW, and the Belinfante energy-momentum tensors are all the same. The main difference between the Belinfante and the other two approaches is that no spin tensor is available in the Belinfante case, hence, the conservation of the Belinfante energy-momentum tensor is not sufficient to determine the space-time evolution of the polarization tensor $\omega_{\mu\nu}$. It should be determined by other physical conditions, for example, by relating $\omega_{\mu\nu}$ to the thermal vorticity. Within the processes discussed in this work we cannot select a mechanism responsible for the identification of $\omega_{\mu\nu}$ with $\zeta_{\mu\nu}$. Most likely, these two tensors become equal as a result of dissipative phenomena that remain beyond the perfect-fluid picture analyzed herein.

It has been recently argued that the use of tensors that differ by the pseudo-gauge transformation leads to different predictions for measurable quantities such as spectrum and polarization of particles [39]. We expect that the results presented in this work can be useful to study such effects in more detail within explicitly defined hydrodynamic models.

5.6. FFJS hydrodynamic model

Connections between different hydrodynamic models with spin, realized by the pseudo-gauge transformations, can be used to clarify the role of the FFJS hydrodynamic model [35] which introduced for the first time the concept of the spin chemical potential as the Lagrange multiplier that can play a role of a hydrodynamic variable. The FFJS approach employs the energy-momentum tensor in the GLW version and the phenomenological spin tensor of the form

$$S_{\text{ph}}^{\lambda,\mu\nu} = \frac{h\nu}{4\zeta} \omega^{\lambda,\mu\nu}. \quad (118)$$
It can be related to the canonical and GLW spin tensors through the following relations

\[ S^\lambda_{\mu \nu}^\text{can} = S^\lambda_{\mu \nu}^\text{ph} + S^\mu_{\lambda \nu}^\text{ph} + S^\nu_{\lambda \mu}^\text{ph} \]  
(119)

\[ S^\lambda_{\mu \nu}^\text{GLW} = S^\lambda_{\mu \nu}^\text{ph} + S^\lambda_{\nu \mu}^\Delta , \]  
(120)

where the tensor \( S^\lambda_{\nu \mu}^\Delta \) is defined by Eq. (87).

Equation (120) can be interpreted as an element of the pseudo-gauge transformation leading from the canonical formalism to the phenomenological FFJS model [35],

\[ S^\lambda_{\mu \nu}^\text{ph} = S^\lambda_{\mu \nu}^\text{can} - \Phi_{\mu \nu}^\lambda \]  
(121)

with

\[ \Phi_{\mu \nu}^\lambda = S^\mu_{\nu \lambda}^\text{ph} + S^\nu_{\lambda \mu}^\text{ph} . \]  
(122)

It is interesting to check now the form of the energy-momentum tensor, which is induced by the superpotential defined above (in the transition from the canonical forms). According to Eq. (10) we use

\[ T^\mu_{\nu}^\text{ph} = T^\mu_{\nu}^\text{can} + \frac{1}{2} \partial_{\lambda} \left( \Phi_{\mu \nu}^\lambda + \Phi_{\nu \mu}^\lambda + \Phi_{\nu \lambda}^\mu \right) \]  
(123)

and obtain

\[ T^\mu_{\nu}^\text{ph} = T^\mu_{\nu}^\text{GLW} - \partial_{\lambda} S^\nu_{\mu \lambda}^\Delta . \]  
(124)

Since \( S^\nu_{\mu \lambda}^\Delta = -S^\nu_{\lambda \mu}^\Delta \), the conservation of the phenomenological tensor \( T^\mu_{\nu}^\text{ph} \) coincides with the conservation of \( T^\mu_{\nu}^\text{GLW} \), which validates the use of the equation \( \partial_{\mu} T^\mu_{\nu}^\text{GLW} = 0 \) in Ref. [35].

Since \( T^\mu_{\nu}^\text{ph} \) defined by Eq. (124) is not symmetric, the spin tensor (118) is in general not conserved. Similarly to (9) we should have

\[ \partial_{\lambda} S^\lambda_{\mu \nu}^\text{ph} = T^\mu_{\nu}^\text{ph} - T^\mu_{\nu}^\text{ph} , \]  
(125)

which after simple manipulations yields

\[ \partial_{\lambda} S^\lambda_{\mu \nu}^\text{ph} = -\partial_{\lambda} S^\lambda_{\nu \mu}^\Delta . \]  
(126)
One can easily notice that the equation above defines the conservation of the GLW spin tensor.

In the hydrodynamic model \[35\] it was assumed that the spin tensor $S_{\lambda\mu\nu}^{\text{ph}}$ is conserved, which is equivalent to neglecting the term on the right-hand side of Eq. \((126)\) \[^{11}\] The improvement of the FFJS model suggested by this work is straightforward and indicates using of the GLW framework in a consistent way.

5.7. Pauli-Lubański vector

Starting from the definition of the Pauli-Lubański (PL) four-vector in the form $\Pi_\mu = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}J^{\nu\alpha}p^\beta$ (where $J^{\nu\alpha}$ is the total angular momentum), and following Ref. \[16\], we introduce the phase-space density of $\Pi_\mu$ defined by the following expression

$$E_p \frac{d\Delta \Pi_\mu(x, p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta} \Delta \Sigma(x) E_p \frac{dJ^{\lambda\nu\alpha}(x, p) p^\beta}{d^3p} \frac{m}{p}.$$  \((127)\)

Here $\Delta \Sigma(x)$ denotes, as above, an element of the hyper-surface containing the particles of interest, and $E_p dJ^{\lambda\nu\alpha}(x, p)/d^3p$ denotes the invariant angular momentum phase-space density of particles with four-momentum $p$. The density $E_p dJ^{\lambda\nu\alpha}/d^3p$ may be replaced by $E_p dS^{\lambda\nu\alpha}/d^3p$, since these two differ by terms proportional to four-momenta that do not contribute to Eq. \((127)\). Using either the GLW or the canonical definition we find

$$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta} E_p \frac{dS^{\lambda\nu\alpha}(x, p)}{d^3p} = \frac{h \cosh(\xi)}{(2\pi)^3} \sinh(\zeta) \frac{e^{-p^\beta p^\lambda \tilde{\omega}_{\mu\beta}}}{\zeta}. \quad \text{(128)}$$

Since we are interested in the polarization effect per particle, it is necessary to introduce the particle density in the volume $\Delta \Sigma$. It is described by the expression

$$E_p \frac{d\Delta N}{d^3p} = \frac{4}{(2\pi)^3} \Delta \Sigma \cdot p e^{-p^\beta \cosh(\xi) \cosh(\zeta)}. \quad \text{(129)}$$

\[^{11}\] The term $S^{\lambda\nu\alpha}_{\Delta}$ may be interpreted as a relativistic correction to the term $S^{\lambda\nu\alpha}_{\text{ph}}$. In the case of heavy particles, such as the $\Lambda$ hyperon, this could be a reasonable approximation (to be verified by future numerical calculations).
The PL vector per particle is then obtained by dividing Eq. (127) by Eq. (129),
\[ \pi_\mu(x, p) \equiv \frac{\Delta \Pi_\mu(x, p)}{\Delta N(x, p)} = -\frac{\hbar \tanh(\zeta)}{4m\zeta} \tilde{\omega}_{\mu\beta} p^\beta. \] (130)

In order to transform the four-vector \( \pi^\mu \) to the local rest frame of a particle with momentum \( p \), we use the canonical boost. In this way, we can express the time and space components of \( \pi^\mu = (\pi^0, \pi^\alpha) \) in the LAB frame in the three-vector notation:
\[ \pi^0 = 0 \] (131)

and
\[ \pi^\alpha = -\frac{\hbar \tanh(\zeta)}{4\zeta} P. \] (132)

This is an important result showing that the space part of the PL vector in PRF agrees with the mean polarization three-vector obtained in Sec. 3.3.

6. Classical treatment of spin

6.1. Internal angular momentum tensor

In the classical treatment of spin one introduces the internal angular momentum tensor \( s^{\alpha\beta} \) [63] defined here in terms of the particle four-momentum \( p_\gamma \) and spin four-vector \( s_\delta \) [56],
\[ s^{\alpha\beta} = \frac{1}{m} \epsilon^{\alpha\beta\gamma\delta} p_\gamma s_\delta. \] (133)

Equation (133) implies that \( s^{\alpha\beta} = -s^{\beta\alpha} \) and
\[ p_\alpha s^{\alpha\beta} = 0. \] (134)

The last condition is known in the literature as the Frenkel (or Weyssenhoff) condition. The spin four-vector is orthogonal to four-momentum
\[ s \cdot p = 0, \] (135)

hence, we can invert Eq. (133) to obtain
\[ s^\alpha = \frac{1}{2m} \epsilon^{\alpha\beta\gamma\delta} p_\beta s_\gamma s_\delta. \] (136)
In PRF, where \( p^\mu = (m, 0, 0, 0) \), the four-vector \( s^\alpha \) has only space components, \( s^\alpha = (0, s_\alpha) \), with the normalization \( |s_\alpha| = \sigma \). For spin one-half particles we use the value of the Casimir operator,

\[
\sigma^2 = \frac{1}{2} \left( 1 + \frac{1}{2} \right) = \frac{3}{4}.
\] (137)

6.2. Invariant measure in spin space

A straightforward generalization of the phase-space distribution function \( f(x, p) \) is a spin dependent distribution \( f(x, p, s) \). Different orientations of spin can be integrated out with the help of a covariant measure

\[
\int dS \ldots = \frac{m}{\pi \sigma} \int d^4s \delta(s \cdot s + \sigma^2) \delta(p \cdot s) \ldots .
\] (138)

The two delta functions in Eq. (138) take care of the normalization and orthogonality conditions, while the prefactor \( m/(\pi \sigma) \) is chosen to obtain the normalization

\[
\int dS = \frac{m}{\pi \sigma} \int d^4s \delta(s \cdot s + \sigma^2) \delta(p \cdot s) = 2,
\] (139)

that accounts for the spin degeneracy factor of spin one-half particles.

6.3. Equilibrium distribution

To construct the equilibrium function for particles with spin we have to identify the so-called collisional invariants of the Boltzmann equation. In addition to four-momentum and conserved charges, for particles with spin one can include the total angular momentum

\[
\hat{j}_{\alpha \beta} = l_{\alpha \beta} + s_{\alpha \beta} = x_{\alpha}p_{\beta} - x_{\beta}p_{\alpha} + s_{\alpha \beta}.
\] (140)

If we change the reference point for the calculation of the orbital part by replacing \( x^\mu \) by \( x^\mu + \delta^\mu \) we obtain the new orbital angular momentum \( l'_{\alpha \beta} = l_{\alpha \beta} + \delta_{\alpha}p_{\beta} - \delta_{\beta}p_{\alpha} \). At face value, this change could be absorbed into a new definition of the spin part, \( s'_{\alpha \beta} = s_{\alpha \beta} - \delta_{\alpha}p_{\beta} + \delta_{\beta}p_{\alpha} \), leaving the total angular momentum \( \hat{j}_{\alpha \beta} \) unchanged. However, the Frenkel condition (134) applied to the old and new spin parts implies that \( \delta_{\beta} = p_{\beta}(p \cdot \delta)/m^2 \) in this case, hence \( s'_{\alpha \beta} = s_{\alpha \beta} \). Consequently, for massive particles the changes of the orbital
part of the angular momentum cannot be compensated by a redefinition of the spin part if Eqs. (133) and (134) are used.

The locality of the standard Boltzmann equation \(^\text{12}\) suggests that the orbital part in Eq. (140) can be eliminated, and the spin part can be considered separately. For elastic binary collisions of particles 1 and 2 going to 1' and 2', this suggests that \[s_1^{\alpha\beta} + s_2^{\alpha\beta} = s_{1'}^{\alpha\beta} + s_{2'}^{\alpha\beta}.\] (141)

The tensors \(s^{\alpha\beta}\) appearing in Eq. (141) depend on the four-momenta \(p\) and spin four-vectors \(s\) of colliding particles. To examine more closely the properties which a collision process should have in order to satisfy Eq. (141), we switch to the center-of-mass frame of the colliding particles 1 and 2 (CMS). In this frame the incoming three-momenta are \(p_1 = p\) and \(p_2 = -p\). After the collision, the three-momenta of outgoing particles are \(p_{1'} = p'\) and \(p_{2'} = -p'\). Since we deal with an elastic collision, \(|p| = |p'|\) and \(E_p = E_{p'} = E\). The three-velocities before and after the collision are denoted as \(v = p/E\) and \(v' = p'/E\). Using this notation, the six independent equations included in the tensor equation (141) may be written as two vector equations:

\[E (s_1 + s_2) - p (s_1^0 - s_2^0) = E (s_{1'} + s_{2'}) - p' (s_{1'}^0 - s_{2'}^0), \quad (142)\]

\[p \times (s_1 - s_2) = p' \times (s_{1'} - s_{2'}), \quad (143)\]

Here we have introduced the time and space components of the spin four-vectors, \(s_1^\mu = (s_1^0, s_1), s_2^\mu = (s_2^0, s_2), s_{1'}^\mu = (s_{1'}^0, s_{1'}),\) and \(s_{2'}^\mu = (s_{2'}^0, s_{2'}).\)

The orthogonality condition (135) implies that: \(s_1^0 = v \cdot s_1, s_2^0 = -v \cdot s_2, s_{1'}^0 = v' \cdot s_{1'},\) and \(s_{2'}^0 = -v' \cdot s_{2'}\.\) Hence, Eqs. (142) and (143) are equivalent to

\[s_1 + s_2 - v (v \cdot (s_1 + s_2)) = s_{1'} + s_{2'} - v' (v' \cdot (s_{1'} + s_{2'})), \quad (144)\]

\[v \times (s_1 - s_2) = v' \times (s_{1'} - s_{2'}). \quad (145)\]

It is interesting to notice that Eqs. (144) and (145) admit two types of simple solutions; they correspond to the cases where either the sum of two spin three-vectors or their difference (before and after the collision) vanishes. With a

\(^\text{12}\)By this we mean here the dependence of the collision term on a single space-time component \(x\) that may be set equal to zero by a translation. A non-local version of the Boltzmann equation was proposed, for example, in Ref. \([64]\) where non-local effects are included through gradients of the phase-space distribution function.
grain of salt, they may be interpreted as collisions in the spin singlet and triplet states.

1. If the sum of initial and final spin three-vectors is zero, we may write that \( s_1 = -s_2 = s \) and \( s_{1'} = -s_{2'} = s' \). In this case Eq. (144) is automatically fulfilled, while Eq. (145) becomes

\[
v \times s = v' \times s'.
\]  

(146)

The structure of this equation implies that all vectors appearing in (146) lie in the same plane, and a natural solution for \( s' \) is a vector obtained from \( s \) by exactly the same rotation that transforms \( v \) into \( v' \). In this case the time component of the spin vector is the same for all four particles participating in the collision, \( s_0 = v \cdot s \), and the normalization of \( s \) is obtained from the equation

\[
|s| = \frac{g}{\sqrt{1 - v^2 \cos^2 \phi}},
\]  

(147)

where \( \phi \) is the angle between the three-vectors \( v \) and \( s \). Note that this case includes a situation in which the spin vectors are parallel to velocities (\( \phi = 0 \) and \( |s| = s/\sqrt{1 - v^2} \)).

2. If the difference of initial and final spin three-vectors is zero, we may write that \( s_1 = s_2 = s \) and \( s_{1'} = s_{2'} = s' \). In this case Eq. (145) is automatically fulfilled, whereas Eq. (144) takes the form

\[
s - v(v \cdot s) = s' - v'(v' \cdot s').
\]  

(148)

To solve this equation we assume that \( s' = s \), and the vector \( s \) is perpendicular to both \( v \) and \( v' \). In this case \( s_0 = 0 \) and

\[
|s| = g.
\]  

(149)

If the collision integral allows for processes such as discussed above, the conservation law (141) should be definitely included among the other, more common, conservation laws. This implies that we can introduce a spin-dependent, equilibrium distribution functions for particles and antiparticles in the form

\[
f_{eq}^\pm(x, p, s) = \exp \left( -p \cdot \beta(x) \pm \xi(x) + \frac{1}{2} \omega_{\alpha\beta}(x) s^\alpha s^\beta \right).
\]  

(150)
As in the previous sections, the tensor $\omega_{\alpha\beta}(x)$ plays a role of the chemical potential conjugated to the spin angular momentum and we keep calling it the spin polarization tensor. We note that a similar procedure was applied before in Ref. [65] to introduce colored chemical potentials into the kinetic theory of quark-antiquark plasma with classical description of the color charge. We also note at this point that the Frenkel condition (134) imposed on the tensor $s^{\alpha\beta}$ has no effect on the spin polarization tensor $\omega_{\alpha\beta}$. In particular we do not demand that $u_\mu \omega^{\mu\nu} = 0$. This constraint is sometimes also called the Frenkel condition but, in general, it is not fulfilled (see, for example, the case of global thermodynamic equilibrium studied in Ref. [14]).

Using Eq. (133) we write

$$\frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta} = \frac{p_\gamma}{m} \tilde{\omega}^{\gamma\delta} s_\delta.$$  

(151)

In PRF, the components of the four-vector $(p_\gamma/m)\tilde{\omega}^{\gamma\delta}$ reproduce the elements of the polarization three-vector, see Eqs. (48), (130) and (132), hence, we can write

$$\frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta} = b_\ast \cdot s_\ast = P \cdot s_\ast.$$  

(152)

This formula will be frequently used below.

6.4. Charge current and energy-momentum tensor

The charge current is obtained from the straightforward generalization of the standard definition

$$N_{\text{eq}}^\mu = \int dP \int dS \ p^\mu \ \left[ f_+^{\text{eq}}(x, p, s) - f_-^{\text{eq}}(x, p, s) \right],$$  

(153)

which after using the forms of the equilibrium functions (150) leads to the expression

$$N_{\text{eq}}^\mu = 2 \sinh(\xi) \int dP \ p^\mu \ \exp(-p \cdot \beta) \int dS \ \exp\left(\frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta}\right).$$  

(154)

For small values of the polarization tensor $\omega$, one can expand the exponential function, which gives

$$N_{\text{eq}}^\mu = 2 \sinh(\xi) \int dP \ p^\mu \ e^{-p \cdot \beta} \int dS \ \left[ 1 + \frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta}\right]$$

$$= 4 \sinh(\xi) \int dP \ p^\mu \ e^{-p \cdot \beta}.$$  

(155)
Here we used the normalization (139) and the fact that \(\int dS s^\mu = 0\). Equation (155) agrees up to the first order in \(\omega\) with Eq. (78).

The treatment of the energy-momentum tensor is analogous. Repeating the same steps we find

\[
T_{\text{eq}}^{\mu\nu} = \int dP \int dS \, p^\mu p^\nu \left[ f_{\text{eq}}^+(x, p, s) + f_{\text{eq}}^-(x, p, s) \right]
\]

\[= 2 \cosh(\xi) \int dP \, p^\mu p^\nu \exp(-p \cdot \beta) \int dS \exp\left(\frac{1}{2} \omega_{\alpha\beta} s^\alpha s^\beta\right)\]  

which agrees with Eq. (82) again up to the first order in \(\omega\).

In order to obtain the last integral in Eq. (156) for arbitrary value of \(\omega\) we switch to PRF,

\[
\int dS \exp\left(\frac{1}{2} \omega_{\alpha\beta} s^\alpha s^\beta\right) = \frac{m}{\pi g} \int ds_0 \int |s_*|^2 d|s_*| \int d\Omega \delta(|s_*|^2 - g^2) \delta(m s_0)e^{P \cdot s_*}. (157)
\]

Here \(d\Omega = \sin(\theta) d\theta d\phi\) denotes the integration over the solid angle (two independent directions of the three-vector \(s_*\)). Doing the integrals over \(s_0, |s_*|\) and \(\phi\), and using the substitution \(x = \cos(\theta)\), we obtain

\[
\int dS \exp\left(\frac{1}{2} \omega_{\alpha\beta} s^\alpha s^\beta\right) = \int_{-1}^{+1} e^{sP x} dx = \frac{2 \sinh(\beta P)}{g P}, \quad (158)
\]

where \(P = |P| = |b_*|\). As expected, for small values of \(P\) we reproduce the normalization result (139). Interestingly, since \(P\) depends on momentum, the energy-momentum tensor for large values of \(\omega\) has no longer a simple perfect-fluid form. The presence of \(P\) induces momentum anisotropy, which is an expected result, since polarization defines a privileged direction in space.

Such systems can be analyzed in the future with the methods worked out in the context of anisotropic hydrodynamics [66, 67].
6.5. Spin tensor

The spin tensor is defined as an expectation value of the internal angular momentum tensor,

\[ S_{\lambda,\mu\nu}^{eq} = \int dP \int dS p^\lambda s^{\mu\nu} \left[ f^{+}_{eq}(x, p, s) + f^{-}_{eq}(x, p, s) \right] \]

\[ = 2 \cosh(\xi) \int dP p^\lambda \exp(-p \cdot \beta) \int dS s^{\mu\nu} \exp \left( \frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta} \right). \] (159)

It is interesting first to check the leading-order approximation in \( \omega \) for \( S_{\lambda,\mu\nu}^{eq} \). Expanding the exponential function we find

\[ \int dS s^{\mu\nu} \exp \left( \frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta} \right) = \int dS s^{\mu\nu} \left[ 1 + \frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta} \right] \]

\[ = \frac{1}{2} \omega_{\alpha\beta} \int dS s^{\mu\nu} s^{\alpha\beta} = \frac{\omega_{\alpha\beta}}{2m^2} \epsilon^{\mu
u\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} p_\rho p_\gamma \int dS s_\delta s_\sigma. \] (160)

The last integral in the expression above is a function of momentum and, as a symmetric tensor, should have the following decomposition

\[ \int dS s_\delta s_\sigma = a g_\delta \sigma + b p_\delta p_\sigma, \] (161)

where \( a \) and \( b \) are scalar functions obtained by the contractions:

\[ \int dS (p \cdot s)^2 = a m^2 + b m^4 = 0, \]

\[ \int dS s^2 = 4a + b m^2 = -2 s^2. \] (162)

We find that \( b = -a/m^2 \) and \( a = -(2/3) s^2 \), which gives

\[ \int dS s_\delta s_\sigma = -\frac{2}{3} s^2 \left( g_\delta \sigma - \frac{p_\delta p_\sigma}{m^2} \right). \] (163)

Substituting Eq. (163) into Eq. (160) and performing the contraction of the Levi-Civita tensors we find

\[ \int dS s^{\mu\nu} \exp \left( \frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta} \right) = \frac{2}{3m^2} s^2 \left( m^2 \omega^{\mu\nu} + 2p^\alpha p^{[\mu} \omega^{\nu]} \right). \] (164)
Using subsequently Eq. (164) in the definition of the spin tensor (159) we find

\[ S_{\text{eq}}^{\lambda,\mu\nu} = \frac{4}{3m^2} \sigma^2 \cosh(\xi) \int dP P^\lambda e^{-p^\beta} \left( m^2 \omega^{\mu\nu} + 2P^\alpha p^{[\mu} \omega^{\nu]}_\alpha \right). \] (165)

It is striking to observe that with the value (137) used for \( \sigma \) we reproduce the result (87) in the case of small \( \omega \).

In the case of arbitrary large \( \omega \) or \( P \), a similar calculation to that outlined above leads to the results

\[ \int dS s^{\mu\nu} \exp \left( \frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta} \right) = \frac{\chi(P \sigma)}{m^2} \left( m^2 \omega^{\mu\nu} + 2P^\alpha p^{[\mu} \omega^{\nu]}_\alpha \right) \] (166)

and

\[ S_{\text{eq}}^{\lambda,\mu\nu} = \frac{2\chi(P \sigma)}{m^2} \cosh(\xi) \int dP P^\lambda e^{-p^\beta} \left( m^2 \omega^{\mu\nu} + 2P^\alpha p^{[\mu} \omega^{\nu]}_\alpha \right), \] (167)

where the function \( \chi(P \sigma) \) is defined by the formula

\[ \chi(P \sigma) = \frac{2 [P \sigma \cosh(P \sigma) - \sinh(P \sigma)]}{P^3 \sigma}. \] (168)

For small values of \( P \), we may use the approximation

\[ \chi(P \sigma) \approx \frac{2 \sigma^2}{3} + \frac{\sigma^4 P^2}{15}. \] (169)

We thus see that in the leading order of the expansion in \( P \), Eq. (167) is reduced to Eq. (165).

6.6. Pauli-Lubański vector

We define the particle number current for both particles and antiparticles as

\[ \mathcal{N}_{\text{eq}}^\mu = \int dP \int dS p^\mu \left[ f_{\text{eq}}^+(x, p, s) + f_{\text{eq}}^-(x, p, s) \right]. \] (170)

Using this expression we obtain the momentum density of the total number of particles

\[ E_p \frac{d\Delta \mathcal{N}}{d^3p} = \frac{\cosh(\xi)}{4\pi^3} \Delta \Sigma \cdot p e^{-p^\beta} \int dS \exp \left( \frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta} \right). \] (171)
Similarly, the spin density defined by the spin tensor is
\[ E_p \frac{d\Delta S^{\mu\nu}}{d^3 p} = \frac{\cosh(\xi)}{4\pi^3} \Delta \Sigma \cdot p e^{-\eta p} \int dS s^{\mu\nu} \exp \left( \frac{1}{2} \omega_{\alpha\beta} s^{\alpha\beta} \right) \] (172)

The phase-space density of the PL vector is then obtained as the ratio
\[ \pi_\mu(x, p) = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \int dS s^{\nu\alpha} \exp \left( \frac{1}{2} \omega_{\rho\sigma} s^{\rho\sigma} \right) \frac{p^\beta}{m} \] (173)

Using Eqs. (158) and (166) we find that the PL vector can be expressed by the simple expression
\[ \pi_\mu = -\frac{\omega_{\mu\beta}}{P} \frac{P^\beta}{m} L(P \theta), \] (174)

where \( L(x) \) is the Langevin function defined by the formula
\[ L(x) = \coth(x) - \frac{1}{x}. \] (175)

It has the following expansions for large and small arguments: \( L \approx 1 \) for \( x \gg 1 \) and \( L \approx \frac{x}{3} \) for \( x \ll 1 \). In the particle rest frame we find
\[ \pi_0 = 0, \quad \pi_* = -\frac{\theta}{P} P L(P \theta). \] (176)

Hence, in PRF the direction of the PL vector agrees with that of the polarization vector \( P \). For small and large \( P \) we obtain two important results:
\[ \pi_* = -\frac{\theta}{P} P, \quad |\pi_*| = \theta = \sqrt{3} \frac{1}{4}, \quad \text{if} \quad P \gg 1 \] (177)

and
\[ \pi_* = -\frac{\theta^2}{3} P, \quad |\pi_*| = \frac{\theta^2}{3} = \frac{P}{4}, \quad \text{if} \quad P \ll 1. \] (178)

Equation (177) demonstrates that the normalization of the PL vector cannot exceed the value of \( \theta \). On the other hand, Eq. (178) shows that for small values of \( P \) the classical treatment of spin reproduces the quantum mechanical result.
6.7. Entropy conservation

Classical treatment of spin allows for explicit derivation of the entropy current conservation. For the latter we adopt the Boltzmann definition

$$H^\mu = - \int dP \int dS p^\mu \left[ f_{eq}^+ (\ln f_{eq}^+ - 1) + f_{eq}^- (\ln f_{eq}^- - 1) \right]. \quad (179)$$

Using Eq. (150) and the conservation laws for energy, linear and angular momentum, and charge, we obtain

$$H^\mu = \beta_\alpha T_{eq}^{\mu\alpha} - \frac{1}{2} \omega_{\alpha\beta} S_{eq}^{\mu,\alpha\beta} - \xi N_{eq}^\mu + N_{eq}^\mu \quad (180)$$

and

$$\partial_\mu H^\mu = (\partial_{\mu} \beta_\alpha) T_{eq}^{\mu\alpha} - \frac{1}{2} (\partial_{\mu} \omega_{\alpha\beta}) S_{eq}^{\mu,\alpha\beta} - (\partial_{\mu} \xi) N_{eq}^\mu + \partial_{\mu} N_{eq}^\mu. \quad (181)$$

With the help of the relation

$$N_{eq}^\mu = \frac{\cosh(\xi)}{\sinh(\xi)} N_{eq}^\mu \quad (182)$$

that is valid for classical statistics, and the conservation of charge one can easily show that

$$\partial_\mu H^\mu = 0. \quad (183)$$

One can notice that the contributions to the entropy current (180), connected with the polarization tensor, start with quadratic terms in $\omega$. Hence, to account for the substantial polarization effects in a consistent matter, one should include also (at least) the second-order terms in the expressions for other thermodynamic quantities, such as the energy density or pressure. Nevertheless, if we restrict ourselves to linear terms in $\omega$, all thermodynamic quantities (including entropy) become independent of $\omega$, while the conservation of the angular momentum determines the polarization evolution in a given hydrodynamic background. This is important from the practical point of view, since the latter can be obtained from any hydrodynamic code without polarization and the spin effects can be studied on top of this.
7. Conclusions

We close the paper with the following series of comments:

1. The arguments collected in this work suggest using the de Groot - van Leeuwen - van Weert (GLW) forms of the energy-momentum and spin tensors, together with their conservation laws, as the building blocks for construction of hydrodynamics with spin. The GLW framework follows naturally from the kinetic-theory considerations which directly show that the GLW hydrodynamic equations can be obtained as the zeroth and first moments of the Boltzmann equation — both in the case of a semi-classical treatment of the Wigner function and in the case of classical description of spin. One has to admit, however, that no hydrodynamic solutions within the GLW hydrodynamic scheme have been obtained so far (analytic or numerical). A construction of such solutions needs much further research.

2. The GLW framework can be connected with the canonical one (obtained with the help of the Noether theorem from the underlying Lagrangian) through a pseudo-gauge transformation that has been explicitly constructed. Both, the GLW and canonical frameworks include spin degrees of freedom, hence can be used at the same footing to describe spin polarization phenomena. An advantage of the GLW formalism (i.e., the symmetric energy-momentum tensor and the spin tensor strictly conserved) is that it allows for a simple physics interpretation of the hydrodynamic variables.

3. The pseudo-gauge transformation from the canonical to the Belinfante forms neglects the spin degrees of freedom and leads to a formalism that is not satisfactory for description of the polarization phenomena — the total angular momentum in the Belinfante approach has the form of the orbital angular momentum which can be always set equal to zero by a Lorentz transformation.

4. In the canonical, GLW, and Belinfante approaches, the semi-classical expansion of the Wigner function leads to the same, symmetric expression for the energy-momentum tensor in the leading-order in $\hbar$. This expression, with the spin chemical potential consistently neglected, can be used in General Theory of Relativity which is a classical theory.
5. In the Belinfante case one can directly connect the spin polarization with thermal vorticity. This has now become a common procedure followed by different phenomenological analyzes of the $\Lambda$-hyperon polarization. Nevertheless, it is not obvious how to construct a hydrodynamic picture in this case, which would allow for space-time studies of polarization. The dependence of equilibrium distributions on the thermal vorticity is a reflection of non-locality, since thermal vorticity depends not only on the hydrodynamic variables but also on their gradients.

6. Different frameworks described in this work lead to a separate conservation of the spin tensor. This behavior may be traced back to the locality of the collision term, which implies the conservation of internal angular momentum in binary collisions. Departure from the locality assumption is most likely a necessary condition to include dissipative interactions which eventually could make the spin polarization tensor $\omega_{\mu\nu}$ equal to the thermal vorticity $\varpi_{\mu\nu}$ (as discussed in the point above).

7. It was already pointed out by Hess and Waldmann in their seminal paper on the kinetic theory of particles with spin, Sec. 12 of Ref. [68], that locality of the collision operator makes their model unable to describe the Barnett effect, i.e., the orientation of spins by a local or uniform rotation of the system. Recent developments obtained within the Chiral Kinetic Theory [69, 70] may give some insights how to improve the frameworks discussed in this work.

8. The magnitude of the polarization three-vectors obtained with the quantum distribution functions may be not bounded from above, which suggests that such distributions are appropriate only for small values of the spin chemical potential. In this case the GLW framework coincides with the approach where spin is treated classically. Within these two frameworks the normalization of the polarization and Pauli-Lubański vectors is fixed, hence, any additional schemes to renormalize their magnitude seem to be not justified.

9. Using the classical concept of spin one can formulate a consistent framework of hydrodynamics with spin, which for small values of the polarization agrees with the approach using relativistic spin-density matrices. The classical-spin approach is free from the problems connected
with normalization of the polarization three-vector and indicates that the hydrodynamic system becomes anisotropic if the spin densities are large. The classical approach allows also for the explicit definition of a conserved entropy current.

10. In the future practical applications it would be useful to consider first the expressions linear in the polarization tensor. In this case one can solve first the standard hydrodynamic equations and the spin evolution can be later analyzed as a problem of the polarization dynamics in a given hydrodynamic background.

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